Metric on state space of Markov chain

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Abstract

We consider finite irreducible Markov chains. It was shown that mean hitting time from one state to another satisfies the triangle inequality. Hence, sum of mean hitting time between couple of states in both directions is a metric on the space of states.

1 Paths, sets of paths and probabilities

Basic definitions and all used properties of Markov chains could be found in [3].

Consider Markov chain $M$ with finite state space $Z = \{z_1, \ldots, z_n\}$ and matrix of transition probabilities $P = (p_{i,j})_{i,j=1,…,n}$; $P$ is non-negative and for all $i = 1, \ldots, n$

$$\sum_{j=1}^{n} p_{i,j} = 1 \quad (1)$$

Let $\lambda^m = (\lambda_i^m, \ldots, \lambda_n^m)$ is distribution of probabilities of states on step $m$ ($\lambda^0$ is initial distribution). It is known that for all $m \geq 0$

$$\lambda^m = \lambda^0 P^m$$

Let $P^m = (p_{i,j}^{(m)})_{i,j=1,…,n}$. Coefficient $p_{i,j}^{(m)}$ is the probability that chain with initial state $z_i$ will be in the state $z_j$ after $m$ steps.

Any sequence $x = (x_0, \ldots x_m)$ of states we call a path of length $m$. Denote length of path and its first and last elements as $\text{Len}(x) = m$, $\text{Begin}(x) = x_0$ and $\text{End}(x) = x_m$.

If $\text{End}(x) = \text{Begin}(y)$, then $x \otimes y$ is concatenation of paths $x$ and $y$.

Path $y$ is extension of path $x$, if

$$(\text{Len}(x) \leq \text{Len}(y)) \quad \& \quad (\forall i \leq \text{Len}(x))(x_i = y_i)$$

Let $S$ is the set of all paths and, for any $m \geq 0$, $a,b \in Z$,

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\[ S^m_{(a,b)} = \{ s \in S | (\text{Len}(s) = m) \& (\text{Begin}(s) = a) \& (\text{End}(s) = b) \} \]

Asterisk instead of some index in \( S^m_{(a,b)} \) means dropping of corresponding condition. For example, \( S^*_{(a,*)} = \{ s \in S | \text{Begin}(s) = a \} \).

For any path \( x \) denote by \( Ext^k(x) \) set of all extensions of path \( x \), having length \( k \):
\[ Ext^k(x) = \{ s \in S | (\text{Len}(s) = k) \& (s \text{ is extension of } x) \} \]

For any set of paths \( X \subseteq S \), let
\[ Ext^k(X) = \bigcup_{x \in X} Ext^k(x) \]

Sometimes we will write \( p(z_i, z_j) \) instead of \( p_{i,j} \). For any \( a \in Z, m > 0, x \in S^m_{(a,*)} \) define
\[ P^m_a(x) = \prod_{i=1}^{m} p(x_{i-1}, x_i) \quad (2) \]

Evidently, \( P^m_a(x) \) is the probability that Markov chain \( M \), being at initial moment in state \( a \), will follow path \( x \) on first \( m \) steps.

Similarly, for any \( X \subseteq S^m_{(a,*)} \)
\[ P^m_a(X) = \sum_{x \in X} P^m_a(x) \]

is the probability that Markov chain \( M \), being at initial moment in state \( a \), will follow on first \( m \) steps to some path from \( X \). Note that
\[ P^{(m)}_{a,b} = P^m_a(S^m_{(a,b)}) \]

and
\[ P^m_a(S^m_{(a,*)}) = \sum_{b \in Z} P^m_a(S^m_{(a,b)}) = 1 \quad (3) \]

For any \( X \subseteq S \) define function \( P \) on sets of paths \( X \) as
\[ P(X) = \sum_{x \in X} P(x) = \sum_{x \in X} P^{\text{Len}(x)}_{\text{Begin}(x)}(x) \]

**Lemma 1.** If \( X \subseteq S^m_{(a,*)} \) and \( k \geq m \), then \( P^k_a(Ext^k(X)) = P^m_a(X) \)
Proof. Case \( k = m \) is trivial. For \( k = m + 1 \)

\[
P^m_{a + 1}(Ext^{m+1}(X)) = \sum_{y \in Ext^{m+1}(X)} P^m_{a + 1}(y) = \sum_{x \in X, z \in Z} P^m_{a + 1}(x \otimes (x_m, z)) =
\]

\[
\sum_{x \in X} \sum_{z \in Z} (\prod_{i=1}^{m} p(x_{i-1}, x_i)) p(x_m, z) = \sum_{x \in X} (\prod_{i=1}^{m} p(x_{i-1}, x_i)) \sum_{z \in Z} p(x_m, z)
\]

According to (1),

\[
\sum_{z \in Z} p(x_m, z) = 1
\]

therefore

\[
P^m_{a + 1}(Ext^{m+1}(X)) = \sum_{x \in X} \prod_{i=1}^{m} p(x_{i-1}, x_i) = \sum_{x \in X} P^m_{a}(x) = P^m_{a}(X)
\]

For any \( k \geq m \) extension \( Ext^k(X) \) could be obtained by successive one-step extensions, therefore lemma hold for all \( k \geq m \).

\[\square\]

2 Irreducible finite Markov chains

Markov chain \( M \) is irreducible, if there is positive probability of transition from any state to any other (possibly, in more than one step), i.e.

\[
(\forall i, j)(\exists m)(p_{i,j}^{(m)} > 0)
\]

For any set of paths \( X \) and any \( W \subseteq Z \) denote by \( X_{[-W]} \) the set of paths from \( X \) that does not contain states from \( W \), and by \( X_{[+W]} \) the set of paths from \( X \) that contain at least one element from \( W \):

\[
X_{[-W]} = \{ x \in X \mid (\forall w \in W)(w \notin X) \}
\]

\[
X_{[+W]} = \{ x \in X \mid (\exists w \in W)(w \in X) \}
\]

For one-element sets we will use shortcuts \( X_{[\pm w]} = X_{[\pm \{w\}]} \).

Lemma 2. If finite Markov chain is irreducible, then for any \( a, w \in Z \)

\[
\lim_{m \to \infty} P^m_{a}(S_{(a,*)[-w]}^{cm}) = 0
\]
Proof. For any \( u, v \in Z \) expression \( P^m_a(S^m_{(u,*)}[+v]) \) is monotonously non-strictly increasing function of \( m \) or, equivalently, \( P^m_a(S^m_{(u,*)}[-v]) \) is monotonously non-strictly decreasing. Actually,

\[
m_1 < m_2 \implies Ext^{m_2}(S^{m_1}_{(u,*)}[+v]) \subseteq S^{m_2}_{(u,*)}[+v] \implies P^{m_2}_u(Ext^{m_2}(S^{m_1}_{(u,*)}[+v])) \leq P^{m_2}_u(S^{m_2}_{(u,*)}[+v])
\]

According to Lemma 1, \( P^{m_1}_u(S^{m_1}_{(u,*)}[+v]) = P^{m_2}_u(Ext^{m_2}(S^{m_1}_{(u,*)}[+v])) \), therefore

\[
P^{m_1}_u(S^{m_1}_{(u,*)}[+v]) \leq P^{m_2}_u(S^{m_2}_{(u,*)}[+v])
\]

Due to irreducibility, for all \( u, v \in Z \) exists index \( t(u,v) \) that satisfies condition \( p_{u,v}^{(t(u,v))} > 0 \). Let \( t = \max\{t(u,v)\} \) and \( q = \min\{p_{u,v}^{(t(u,v))}\} \). Then, for any \( u, v \in Z \),

\[
q \leq p_{u,v}^{(t(u,v))} = P^{t(u,v)}_u(S^{t(u,v)}_{(u,v)}) \leq P^{t(u,v)}_u(S^{t(u,v)}_{(u,*)}[+v]) \leq P^{t} U_u(S^{t} (u,*)[+v])
\]

Hence, for some \( q > 0 \) and any \( u, w \in Z \)

\[
P^{t}_u(S^{t}_{(u,*)}[-v]) \leq 1 - q \quad \text{(4)}
\]

Inequality (4) means that for any sequence of \( t \) steps, probability of event “all states in the sequence differ from \( w \)” does not exceed \( 1 - q \). Hence, for any sequence, containing \( kt \) steps, probability of avoiding state \( w \) does not exceed \((1 - q)^k\), and for any \( a, w \in Z \)

\[
P^m_a(S^m_{(a,*)}[-w]) \leq (1 - q)^{m/t} \xrightarrow{m \to \infty} 0
\]

\[ \square \]

Corollary 1. If finite Markov chain is irreducible, then exist \( t_0, \alpha > 0, 1 > \beta > 0 \) such that for any \( u, v \in Z \) and any \( t \geq t_0 \)

\[
P^{t}_u(S^{t}_{(u,*)}[-v]) \leq \beta^a \quad \text{(5)}
\]

Proof. Proof is evident from the proof of lemma 2. \[ \square \]

Let \( X \) is a set of paths and \( b \in W \subseteq Z \). Denote by \( X_{[+W:b]} \) the set of paths from \( X \) that contain elements from \( W \), and \( b \) is the first element on the path that belongs to \( W \):

\[
X_{[+W:b]} = \{ x \in X_{[+W]} \mid (\forall c \in W) (x \in X_{[+c]} \implies x \in X_{[+b]}) \}
\]

Lemma 3. If finite Markov chain is irreducible, \( \varnothing \subseteq W \subseteq Z \), and \( a \in Z \), then for any \( b \in W \) exist limit

\[
\overline{P}_{W,a,b} = \lim_{m \to \infty} P^m_a(S^m_{(a,*)}[+W:b])
\]

and

\[
\sum_{b \in W} \overline{P}_{W,a,b} = 1 \quad \text{(5)}
\]
Proof. If $a \in W$, statement is trivial and

$$\mathcal{P}_{W,a,b} = \begin{cases} 1 & \text{if } b = a \\ 0 & \text{if } b \neq a \end{cases}$$

For any $a \notin W$, $b \in W$, expression $P^m_a(S^m_{(a_*,|+W;b)})$ is monotonously non-strictly increasing function of $m$. Really, if $m_1 < m_2$, then $Ext^{m_2}(S^m_{(a_*,|+W;b)}) \subseteq S^m_{(a_*,|+W;b)}$ and, according to Lemma 1, $P^m_a(S^m_{(a_*,|+W;b)}) = P^m_a(Ext^{m_2}(S^m_{(a_*,|+W;b)})) \leq P^m_a(S^m_{(a_*,|+W;b)})$.

Evidently, sets $\{S^m_{(a_*,|+W;b)}\}_{b \in W}$ are pairwise disjoint subsets of $S^m_{(a_*,|)}$ and hence, according to (3),

$$\sum_{b \in W} P^m_a(S^m_{(a_*,|+W;b)}) \leq 1 \quad (6)$$

Because addends in (6) are non-negative monotonously non-decreasing functions of $m$, limits

$$\mathcal{P}_{W,a,b} = \lim_{m \to \infty} P^m_a(S^m_{(a_*,|+W;b)})$$

exist and

$$\sum_{b \in W} \mathcal{P}_{W,a,b} \leq 1$$

Note that

$$S^m_{(a_*,|)} = \bigcup_{b \in W} S^m_{(a_*,|+W;b)} \cup S^m_{(a_*,|-W)}$$

therefore

$$\sum_{b \in W} P^m_a(S^m_{(a_*,|+W;b)}) = 1 - P^m_a(S^m_{(a_*,|-W)}) \quad (7)$$

Select some element $c$ in non-empty set $W$. From $c \in W$ follows $S^m_{(a_*,|-W)} \subseteq S^m_{(a_*,|-c)}$ and $P^m_a(S^m_{(a_*,|-W)}) \leq P^m_a(S^m_{(a_*,|-c)})$.

According to lemma 2, $\lim_{m \to \infty} P^m_a(S^m_{(a_*,|-c)}) = 0$, therefore

$$\lim_{m \to \infty} P^m_a(S^m_{(a_*,|-W)}) = 0 \quad (8)$$

From (7) and (8) follows

$$\lim_{m \to \infty} \sum_{b \in W} P^m_a(S^m_{(a_*,|+W;b)}) = 1$$

and (5) proved. \qed

Corollary 2. If finite Markov chain is irreducible, $\emptyset \subset W \subseteq Z$, and $a \in Z$, then for any $b \in W$ and any $m \geq 1$

$$\mathcal{P}_{W,a,b} \geq P^m_a(S^m_{(a_*,|+W;b)})$$

Proof. Proof is evident from the proof of lemma 3. \qed
Let $W \subseteq Z$, $|W| > 1$, $a, b \in W$ and $a \neq b$. We say that path $x$ is $W$-arrow from $a$ to $b$, if

$$(\text{Begin}(x) = a) \land (\text{End}(x) = b) \land (\forall i < \text{Len}(x))(x_i \in W \Rightarrow x_i = a)$$

i.e. path $x$ leads from $a$ till the first hitting $b$, and contains no states from $W$ besides $a$ and $b$. Denote by $R_{a,b}^W$ the set of all $W$-arrows from $a$ to $b$.

**Lemma 4.** If finite Markov chain is irreducible, $W \subseteq Z$, $|W| > 1$, $a, b \in Z$ and $a \neq b$, then

$$P_{W \setminus \{a\}, a, b} = P(R_{a,b}^W)$$

**Proof.** By definition,

$$P_{W \setminus \{a\}, a, b} = \lim_{m \to \infty} P_a^m(S^m_{(a,*)[+W \setminus \{a\}:b]\}}$$

Note that $S^m_{(a,*)[+W \setminus \{a\}:b]}$ is the set of paths of length $m$ that start in $a$, hit some others states from $W \setminus \{a\}$, and $b$ is the first one among them. This means that

$$S^m_{(a,*)[+W \setminus \{a\}:b]} = \bigcup_{x \in R_{a,b}^W \& (\text{Len}(x) \leq m)} \text{Ext}^m(x)$$

All sets of paths $\text{Ext}^m(x)$ in the union on the right part are pairwise disjoint, therefore according to lemma 1,

$$P_a^m(S^m_{(a,*)[+W \setminus \{a\}:b]\}} = \sum_{x \in R_{a,b}^W \& (\text{Len}(x) \leq m)} P_a^m(\text{Ext}^m(x)) = \sum_{x \in R_{a,b}^W \& (\text{Len}(x) \leq m)} P_{\text{Len}(x)}(x)$$

Hence

$$\lim_{m \to \infty} P_a^m(S^m_{(a,*)[+W \setminus \{a\}:b]\}} = \lim_{m \to \infty} \sum_{x \in R_{a,b}^W \& (\text{Len}(x) \leq m)} P_{\text{Len}(x)}(x) = \sum_{x \in R_{a,b}^W} P_{\text{Len}(x)}(x) = P(R_{a,b}^W)$$

and proof completed. \qed

According to lemmas 2,3,4 $P(R_{a,b}^W)$ is conditional probability of event “$b$ will be the first touched state from $W \setminus \{a\}$, under condition that initial state is $a$”.

### 3 Factor chain

Using lemma 3 introduce, for finite irreducible Markov chains, notion of factor chain. Let $W \subseteq Z$ and $|W| > 1$. Consider new Markov chain $\overline{M}$ with the state space $W$ and transition matrix $\overline{P} = (\overline{p})_{a,b \in W}$, where

$$\overline{p}_{a,b} = \begin{cases} P(R_{a,b}^W) & \text{if } a \neq b \\ 0 & \text{if } a = b \end{cases}$$
We will say that $\overline{M}$ is *factor chain of* $M$ *by set of states* $W$ and write $\overline{M} = M/W$. Intuitively, moving from $M$ to $\overline{M} = M/W$ means

- Ignore all states that not belong to $W$,
- Consider as single step any sequence of steps from some element of $W$ till first hitting some other element of $W$.

Matrix $\overline{P}$ is non-negative and, according to lemmas 2, 3, 4 and definition (9), for any $a \in W$ satisfies condition

$$\sum_{b \in W} \overline{p}_{a,b} = 1 \quad (10)$$

**Lemma 5.** If finite Markov chain $M$ is irreducible, $W \subseteq Z$ and $|W| > 1$, then factor-chain $\overline{M} = M/W$ is irreducible too.

**Proof.** Consider any $a, b \in W$, $a \neq b$. Because $M$ is irreducible, there exists some path $x = (x_0, \ldots, x_m)$ that leads (in $M$) from state $a$ to state $b$ and has positive probability, i.e. $p(x_{i-1}, x_i) > 0$ for all $i \leq m$.

Select in the sequence of states $x$ sub-sequence $y = (y_0, \ldots, y_k) = (x_{h(0)}, \ldots, x_{h(k)})$ in the following way. Let $H(0) = 0$. If $h(j)$ already selected and $h(j) < m$, then select $h(j + 1) = Min\{i | (i > h(j)) \& (x_i \in W) \& (x_i \neq x_{h(j)})\}$

Evidently, $y$ is subsequence of sequence $x_0, \ldots, x_m$ that contains only elements of $W$, each element in $y$ differs from the previous one, $y_0 = x_{h(0)} = a$ and $y_k = x_{h(k)} = b$.

According to definition of $\overline{M}$, corollary 2 and lemma 4 for any $j \leq k$

$$\overline{p}(y_{j-1}, y_j) = P(R_{y_{j-1},y_j}^W) = \overline{p}_{W \setminus \{y_{j-1}\}, y_{j-1}, y_j} \geq P_{y_{j-1}}^t \left(S_{(y_{j-1}^t)}^t[+W \setminus \{y_{j-1}\}; y_j]\right)$$

where $t = h(j) - h(j - 1)$.

On the other hand,

$$S_{(y_{j-1}^t)}^t[+W \setminus \{y_{j-1}\}; y_j] = S_{(x_{h(j-1)}^t)}^t[+W \setminus \{x_{h(j-1)}\}; x_{h(j)}]$$

and

$$(x_{h(j-1)}, \ldots, x_{h(j)}) \in S_{(x_{h(j-1)}^t)}^t[+W \setminus \{x_{h(j-1)}\}; x_{h(j)}]$$

Hence,

$$\overline{p}(y_{j-1}, y_j) \geq P_{x_{h(j-1)}}^t ((x_{h(j-1)}, \ldots, x_{h(j)})) = \prod_{j < i \leq h(j)} p(x_{i-1}, x_i) > 0$$

and $y = (y_0, \ldots, y_k)$ is a path in $\overline{M}$ that leads from $a$ to $b$ and has positive transition probability from each its state to the next one. Proof completed. $\square$
Note that if \( p_{z,z} = 0 \) for all \( z \in Z \) and \( W = Z \), then chains \( M \) and \( \overline{M} \) coincide. From this evident statement follows that transitivity matrix \( \overline{P} \) of factor chain could have zero elements outside main diagonal. We could use as a sample \( M/Z \) for any irreducible Markov chain \( M \), having matrix \( P \) with the same property.

We will use dashed letters for all notations, related to factor chain. For example, \( \overline{S} \) is the set of all paths in \( \overline{M} \). For \( a = b \), according to (9), \( \overline{P}_{a,b} = 0 \), therefore paths with two coinciding adjacent states have zero probability. In following sections we will consider only paths from \( \overline{S}^+ \):

\[
\overline{S}^+ = \{ x \in \overline{S} | (\text{Len}(x) > 0) \& (\forall i \leq \text{Len}(i))(x_{i-1} \neq x_i) \}
\]

4 Weighted transitions and weighted hitting time

Hitting time in Markov chain \( M \) defined for hitting some set of states (see [3], p.12), but we will consider hitting time only for sets that contain one state. For any \( a, b \in Z \), hitting time from \( a \) to \( b \) is random value \( \tau_{a,b} \) - minimal index of step when \( M \) achieved state \( b \), under condition that initial state is \( a \).

Subject of our interest is the mean hitting time, i.e. function

\[
H(a,b) = E(\tau_{a,b}) = \sum_{i=0}^{\infty} i \cdot \text{Probability}(\tau_{a,b} = i)
\]

where \( E \) denotes mathematical expectation.

Note that some authors (e.g. [1], p.29) use a little different definition of hitting time \( \tau'_{a,b} \): minimal positive index of step when \( M \) achieved state \( b \), under condition that initial state is \( a \). Evidently, \( \tau'_{a,b} = \tau_{a,b} \) for all \( a \neq b \). While \( \tau_{a,a} = 0 \) for all \( a \), value of \( \tau'_{a,a} \) could be positive.

For any set of paths \( X \) and any \( a, b \in Z \) denote by \( X_{<a,b>} \) the set of paths from \( X \) that leads from \( a \) to \( b \), and contains \( b \) only as its end state:

\[
X_{<a,b>} = \{ x \in X | (x \in S_{(a,b)}) \& (\forall i < \text{Len}(x))(x_i \neq b) \}
\]

Evidently, \( X_{<a,b>} \) is the set of all \( \{a,b\}\)-arrows from \( a \) to \( b \) that belongs to \( X \), i.e.

\[
X_{<a,b>} = X \cap R_{a,b}^{(a,b)}
\]

If initial state of Markov chain is \( a \), then the sequence of states till first hitting \( b \) will follow some random path from \( S_{<a,b>} \), and \( \tau_{a,b} \) is the length of this path,

\[
\text{Probability}(\tau_{a,b} = i) = P^i_{a}(S^i_{<a,b>})
\]
and
\[ H(a, b) = E(\tau_{a,b}) = \sum_{x \in S_{<a,b>}^{<a,b>}} \text{Len}(x) \cdot P_{a}^{\text{Len}(x)}(x) \]

For proving some properties of function \( H(a, b) \) we need to expand considered notion and introduce weighted hitting time.

Select some positive matrix \( V = (v_{i,j})_{i,j=1,...,n} \) \((\forall i, j = 1, \ldots, n)(v_{i,j} > 0)\). Similar to matrix \( P \), notations \( v(z_j, z_j) \) and \( v_{i,j} \) considered as equivalent. For any path \( x \in S \) define \textit{weight of path} \( x \) \textit{with weight matrix} \( V \) as
\[ \text{Weight}(V, x) = \sum_{i=1}^{\text{Len}(x)} v(x_{i-1}, x_i) \]

Define \( \tau_{V,a,b} \) as weight of random sequence of states of Markov chain with initial state \( a \) till first hitting the state \( b \). Mathematical expectation of \( \tau_{V,a,b} \) is \textit{mean weighted hitting time from} \( a \) \textit{to} \( b \) \textit{with weight matrix} \( V \):
\[ H(V, a, b) = \sum_{x \in S_{<a,b>}^{<a,b>}} \text{Weight}(V, x) \cdot P_{a}^{\text{Len}(x)}(x) \]

Let \( E \) is a trivial weight matrix, with all elements equal to 1. Evidently,
\[ \text{Weight}(E, x) = \text{Len}(x) \]

and
\[ H(E, a, b) = H(a, b) \]

\textbf{Lemma 6.} If finite Markov chain is irreducible, then for any positive weight matrix \( V \) and states \( a, b \in Z \) mean weighted hitting time from \( a \) to \( b \) is finite, i.e. the sum
\[ H(V, a, b) = \sum_{x \in S_{<a,b>}^{<a,b>}} \text{Weight}(V, x) \cdot P_{a}^{\text{Len}(x)}(x) \]
is convergent.

\textit{Proof.} Order addends of sum according to length of paths:
\[ H(V, a, b) = \sum_{x \in S_{<a,b>}^{<a,b>}} \text{Weight}(V, x) P_{a}^{\text{Len}(x)}(x) = \sum_{m=0}^{\infty} \sum_{x \in S_{<a,b>}^{m}} \text{Weight}(V, x) P_{a}^{\text{Len}(x)}(x) \]
Let \( v_{\text{max}} = \max\{v_{i,j} | i, j = 1, \ldots, n\} \). Note that for \( x \in S_{<a,b>}^{m} \) holds
\[ \text{Weight}(V, x) \leq v_{\text{max}} \cdot \text{Len}(x) \]
Evidently, \( S^m_{<a,b>} \subseteq Ext^m(S^{m-1}_{(a,*)[-b]}) \). Using lemma 1, obtain

\[
P^m_a(S^m_{<a,b>}) \leq P^m_a(Ext^m(S^{m-1}_{(a,*)[-b]})) = P^{m-1}_a(S^{m-1}_{(a,*)[-b]})
\]

and

\[
\sum_{x \in S^m_{<a,b>}} Weight(V, x) P^m_a(x) = \sum_{x \in S^m_{<a,b>}} v_{\text{max}} \cdot P^m_a(x) = v_{\text{max}} m \cdot P^m_a(S^{m-1}_{(a,*)[-b]})
\]

According to corollary 1, there are \( t_0, \alpha > 0, 1 > \beta > 0 \) such that for any \( t \geq t_0 \)

\[
P^t_a(S^t_{(a,*)[-b]}) \leq \beta^\alpha t
\]

Hence, for any \( t \geq t_0 \)

\[
\sum_{x \in S^m_{<a,b>}} Weight(V, x) P^m_a(x) = \sum_{x \in S^m_{<a,b>}} v_{\text{max}} m \cdot \beta^\alpha (m-1)
\]

Sum

\[
\sum_{m=t_0}^{\infty} m \cdot \beta^\alpha (m-1)
\]

is convergent, and this proves the lemma.

\[\square\]

## 5 Direct sum and concatenation on sets of paths

Concatenation operation \( \otimes \), considered above as operation on paths, could be considered as partial operation on sets of paths. If \( a \in Z, X \subseteq S^*_<(a,a) \) and \( Y \subseteq S^*_<(a,*), \) define **concatenation of sets** \( X \text{ and } Y \) as \( X \otimes Y = \{x \otimes y | (x \in X) \& (y \in Y)\} \).

Besides, we need another partial operation on sets of paths: **direct sum** \( \oplus \), or union of disjoint sets. Formal definitions of these partial operations are as follows:

\[
W = X \oplus Y \iff (X \cap Y = \emptyset) \& (Z = X \cup Y)
\]

(12)

\[
W = X \otimes Y \iff \exists a \in Z((X \subseteq S^*_<(a,a)) \& (Y \subseteq S^*_<(a,*)) \& (Z = \{x \otimes y | (x \in X) \& (y \in Y)\}))
\]

(13)

For any positive weight matrix \( V \) and any path \( x \) define function

\[
H^V(x) = Weight(V, x) \cdot P^m_a(x) = \left( \sum_{i=1}^{\text{Len}(x)} v(x_{i-1}, x_i) \right) \cdot \prod_{i=1}^{\text{Len}(x)} p(x_{i-1}, x_i)
\]

(14)
Expand function $H^V(x)$ on sets of paths: for any set of paths $X \subseteq S$ define

$$H^V(X) = \sum_{x \in X} H^V(x) \quad (15)$$

Using definition (2) of function $P^m_a(x)$, define function $P$ on any path $x$

$$P(x) = P^{\text{Len}(x)}_{\text{Begin}(x)}(x) = \prod_{i=1}^{m} p(x_{i-1} \cdot x_i) \quad (16)$$

and expand function $P$ on sets of paths:

$$P(X) = \sum_{x \in X} P(x) \quad (17)$$

We consider partial function $P$ (partial function $H^V$) defined only on sets $X \subseteq S$, for which sum, used in definition, is convergent.

Note that

$$H(V, a, b) = H^V(S_{<a,b>}) \quad (18)$$

**Lemma 7.** If finite Markov chain is irreducible, then for any positive weight matrix $V$ partial operations $\oplus$, $\otimes$ and partial functions $P(\cdot), H^V(\cdot)$ satisfy the following properties:

(a) $X \oplus Y = Y \oplus X$
(b) $(X \oplus Y) \oplus W = X \oplus (Y \oplus W)$
(c) $(X \otimes Y) \otimes W = X \otimes (Y \otimes W)$
(d) $X \otimes (Y \oplus W) = (X \otimes Y) \oplus (X \otimes W)$
(e) $(X \oplus Y) \otimes W = (X \otimes W) \oplus (Y \otimes W)$
(f) $P(X \oplus Y) = P(X) + P(Y)$
(g) $H^V(X \oplus Y) = H^V(X) + H^V(Y)$
(h) $P(X \otimes Y) = P(X) \cdot P(Y)$
(i) $H^V(X \otimes Y) = H^V(X) \cdot P(Y) + P(X) \cdot H^V(Y)$

As usual, equality of partial functions means that
- Left and right parts of equality defined or undefined simultaneously and
- If left and right parts are both defined, their values coincide.

**Proof.** Almost all equalities immediately follow from definition of considered partial operations and partial functions. Verify statements (h) and (i). Let $a \in Z$, $X \subseteq S^*_{(s,a)}$ and $Y \subseteq S^*_{(a,s)}$. 

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Check (h):

\[ P(X \otimes Y) = \sum_{w \in X \otimes Y} P(w) = \sum_{x \in X} \sum_{y \in Y} P(x \otimes y) = \sum_{x \in X} \sum_{y \in Y} \left( \prod_{i=1}^{\text{Len}(x)} p(x_{i-1}, x_i) \prod_{j=1}^{\text{Len}(y)} p(y_{j-1}, y_j) \right) = \sum_{x \in X} \left( \prod_{i=1}^{\text{Len}(x)} p(x_{i-1}, x_i) \right) \sum_{y \in Y} \left( \prod_{j=1}^{\text{Len}(y)} p(y_{j-1}, y_j) \right) = \sum_{x \in X} \left( \prod_{i=1}^{\text{Len}(x)} p(x_{i-1}, x_i) \right) \sum_{y \in Y} \left( \prod_{j=1}^{\text{Len}(y)} p(y_{j-1}, y_j) \right) = P(Y) \sum_{x \in X} P(x) = P(Y) \cdot P(X) \]

Check (i):

\[ H^V(X \otimes Y) = \sum_{w \in X \otimes Y} H^V(w) = \sum_{x \in X} \sum_{y \in Y} H^V(x \otimes y) = \sum_{x \in X} \sum_{y \in Y} \left( \left( \sum_{i=1}^{\text{Len}(x)} v(x_{i-1}, x_i) \right) \prod_{i=1}^{\text{Len}(x)} p(x_{i-1}, x_i) \prod_{j=1}^{\text{Len}(y)} p(y_{j-1}, y_j) \right) + \sum_{x \in X} \sum_{y \in Y} \left( \left( \sum_{j=1}^{\text{Len}(y)} v(y_{j-1}, y_j) \right) \prod_{i=1}^{\text{Len}(x)} p(x_{i-1}, x_i) \prod_{j=1}^{\text{Len}(y)} p(y_{j-1}, y_j) \right) = \sum_{x \in X} \sum_{y \in Y} \left( \left( \sum_{i=1}^{\text{Len}(x)} v(x_{i-1}, x_i) \right) \prod_{i=1}^{\text{Len}(x)} p(x_{i-1}, x_i) \right) + \sum_{x \in X} \sum_{y \in Y} \left( \left( \sum_{j=1}^{\text{Len}(y)} v(y_{j-1}, y_j) \right) \prod_{j=1}^{\text{Len}(y)} p(y_{j-1}, y_j) \right) = \sum_{x \in X} \sum_{y \in Y} \left( \sum_{i=1}^{\text{Len}(x)} v(x_{i-1}, x_i) \right) \prod_{i=1}^{\text{Len}(x)} p(x_{i-1}, x_i) + \sum_{x \in X} \sum_{y \in Y} \left( \sum_{j=1}^{\text{Len}(y)} v(y_{j-1}, y_j) \right) \prod_{j=1}^{\text{Len}(y)} p(y_{j-1}, y_j) = \sum_{y \in Y} \prod_{j=1}^{\text{Len}(y)} p(y_{j-1}, y_j) \sum_{x \in X} \left( \sum_{i=1}^{\text{Len}(x)} v(x_{i-1}, x_i) \right) \prod_{i=1}^{\text{Len}(x)} p(x_{i-1}, x_i) + \sum_{x \in X} \prod_{i=1}^{\text{Len}(x)} p(x_{i-1}, x_i) \sum_{y \in Y} \left( \sum_{j=1}^{\text{Len}(y)} v(y_{j-1}, y_j) \right) \prod_{j=1}^{\text{Len}(y)} p(y_{j-1}, y_j) = \sum_{y \in Y} \prod_{j=1}^{\text{Len}(y)} p(y_{j-1}, y_j) \cdot H^V(X) + \sum_{x \in X} \prod_{i=1}^{\text{Len}(x)} p(x_{i-1}, x_i) \cdot H^V(Y) = P(Y) \cdot H^V(X) + P(X) \cdot H^V(Y) \]

Proof completed. \(\square\)
Partial operation of direct sum could be expanded on several disjoint sets:

\[ Y = \bigoplus_{i=1}^{m} X_i \iff \left( Y = \bigcup_{i=1}^{m} X_i \right) \land (\forall i, j \leq m) (i \neq j \Rightarrow X_i \cup X_j = \emptyset) \]

According to associate property of concatenation (lemma 7c), concatenation of several sets

\[ Y = \bigotimes_{i=1}^{m} X_i = X_1 \otimes (X_2 \otimes \ldots \otimes X_m) \ldots \]

does not depends on order of parentheses. Similar to concatenation of two sets,

\[ Y = \bigotimes_{i=1}^{m} X_i \iff (\forall i = 1, \ldots, m - 1)(\exists a_i \in Z)((X_i \subseteq S^*_{(x, a_i)}) \land (X_{i+1} \subseteq S^*_{(a_i, a)}) \land (Z = \{x_1 \otimes x_2 \ldots \otimes x_m | (x_1 \in X_1) \land \ldots \land (x_m \in X_m)\})

**Lemma 8.** If finite Markov chain is irreducible, then for any positive weight matrix \( V \) and any \( m \geq 2 \) the following properties hold:

(a) \[ P\left( \bigoplus_{i=1}^{m} X_i \right) = \sum_{i=1}^{m} P(X_i) \]

(b) \[ H^V\left( \bigoplus_{i=1}^{m} X_i \right) = \sum_{i=1}^{m} H^V(X_i) \]

(c) \[ P\left( \bigotimes_{i=1}^{m} X_i \right) = \prod_{i=1}^{m} P(X_i) \]

(d) \[ H^V\left( \bigotimes_{i=1}^{m} X_i \right) = \sum_{i=1}^{m} \left( H^V(X_i) \cdot \prod_{j=1}^{m} P(X_j) \right) \]

**Proof.** Statements (a),(b),(c) immediately follow from lemma 7 (f), (g), (h) respectively. Prove (d) by induction by \( m \). For \( m = 2 \) statement coincides with lemma 7(i). If (d) true for some value of \( m \), then it is true for \( m + 1 \) too, because

\[ H^V\left( \bigotimes_{i=1}^{m+1} X_i \right) = H^V\left( \left( \bigotimes_{i=1}^{m} X_i \right) \otimes X_{m+1} \right) = H^V\left( \bigotimes_{i=1}^{m} X_i \right) P(X_{m+1}) + H^V(X_{m+1}) P\left( \bigotimes_{j=1}^{m} X_j \right) = \]
\[
\sum_{i=1}^{m} \left( H^V(X_i) \prod_{j=1 \atop j \neq i}^m P(X_j) \right) P(X_{m+1}) + H^V(X_{m+1}) \prod_{j=1}^{m+1} P(X_j) = \\
\sum_{i=1}^{m} \left( H^V(X_i) \prod_{j=1 \atop j \neq i}^m P(X_j) \right) + H^V(X_{m+1}) \prod_{j=1}^{m+1} P(X_j) = \sum_{i=1}^{m+1} \left( H^V(X_i) \prod_{j=1 \atop j \neq i}^{m+1} P(X_j) \right)
\]

Proof completed. \(\square\)

6 Relation between chain and its factor chain

Consider finite irreducible Markov chain \(\mathbf{M}\) with set of states \(Z\), some subset \(W \subseteq Z\), \(|W| > 1\) and factor chain \(\overline{\mathbf{M}} = \mathbf{M}/W\). We will use dashed letters for all notations, related to factor chain.

Let \(S_W\) is the set of all paths \(x\) in \(\mathbf{M}\) that
- Have non-zero length,
- Connect states from \(W\) and
- Last two elements of \(x\) that belong to \(W\) are different.

Formally,
\[
S_W = \{x | (x \in S) \& (\text{Len}(x) > 0) \& (\text{Begin}(x) \in W) \& (\text{End}(x) \in W) \& (x_{\max\{i < \text{Len}(x) | x_i \in W\}} \neq \text{End}(x))\}
\]

Note that \(S_W\) is closed under partial operation \(\otimes\), i.e.
\[
(v = x \otimes y) \& (x, y \in S_W) \implies (v \in S_W)
\]

Define mapping \(\varphi : S_W \rightarrow \overline{S}\) of \(S_W\) in the set \(\overline{S}\) of all paths in \(\overline{\mathbf{M}}\).

Let \(x = (x_0, \ldots, x_m) \in S_W\), \(a = \text{Begin}(x) \in W\) and \(b = \text{End}(x) \in W\). Like in the proof of lemma 5, select in the sequence of states \(x\) subsequence \(y = (y_0, \ldots, y_k) = (x_{h(0)}, \ldots, x_{h(k)})\) as follows:

Let \(h(0) = 0\).
If \(h(j)\) already selected, \(h(j) < m\) and
\[
(\exists i)(i > h(j)) \& (x_i \in W) \& (x_i \neq x_{h(j)})
\]
then
\[
h(j + 1) = \text{Min}\{i | (i > h(j)) \& (x_i \in W) \& (x_i \neq x_{h(j)})\}
\]
Sequence \( y \) satisfies properties:
- \( y \) contains only elements of \( W \);
- Each element in \( y \) differs from the previous one;
- \( y_0 = x_{h(0)} = a \);
- \( y_k = x_{h(k)} = b \).

Define \( \varphi(x) = y \). Here \( y \subseteq W \) considered as path in \( \overline{M} \).

Immediately from definition follow that the range of \( \varphi \) is the set \( \overline{S}^+ \) of all paths from \( \overline{S} \), having non-zero length and different adjacent states:

\[
\text{Range}(\varphi) = \overline{S}^+ = \{ \pi \in \overline{S} | (\text{Len}(\pi) > 0) \& (\forall i \leq \text{Len}(\pi)) (\pi_{i-1} \neq \pi_i) \}
\]

It is also evident that \( \varphi \) preserves concatenation:

\[
(\forall v, x, y \in S_W) (v = x \otimes y \implies \varphi(v) = \varphi(x) \otimes \varphi(y))
\]

Hence, \( \varphi \) is homomorphism of algebraic system \( < S_W, \otimes > \) on system \( < \overline{S}^+, \otimes > \).

Using homomorphism \( \varphi \), we could define on \( S_W \) equivalence relation \( =_{\varphi} \):

\[
[x]_{\varphi} = \{ x' \in S_W | x' =_{\varphi} x \} \quad x \in S_W
\]

Consider again subsequence \( y = (y_0, \ldots, y_k) = (x_{h(0)}, \ldots, x_{h(k)}) \), constructed in definition of \( \varphi \). According to definition of function \( h \), for all \( i = 1, \ldots, k \) segment \( (x_{h(i-1)}, \ldots, x_{h(i)}) \) of sequence \( x \) satisfy properties:
- Its begin state is \( x_{h(i-1)} \in W \);
- Its end state is \( x_{h(i)} \in W \);
- \( x_{h(i-1)} \neq x_{h(i)} \);
- All elements of segment, besides end one, belong to \( (Z \setminus W) \cup \{ x_{h(i-1)} \} \).

This means that \( (x_{h(i-1)}, \ldots, x_{h(i)}) \) is \( W \)-arrow from \( x_{h(i-1)} \) to \( x_{h(i)} \):

\[
(x_{h(i-1)}, \ldots, x_{h(i)}) \in R^W_{x_{h(i-1)}, x_{h(i)}}
\]

and

\[
x \in \bigotimes_{i=1}^{k} R^W_{x_{h(i-1)}, x_{h(i)}}
\]

Note that \( \varphi((x_{h(i-1)}, \ldots, x_{h(i)})) = (x_{h(i-1)}, x_{h(i)}) \) and \( R^W_{x_{h(i-1)}, x_{h(i)}} \) is equivalence class of relation \( =_{\varphi} \):

\[
R^W_{x_{h(i-1)}, x_{h(i)}} = [(x_{h(i-1)}, \ldots, x_{h(i)})]_{\varphi} = \varphi^{-1}((x_{h(i-1)}, x_{h(i)}))
\]
Here \( (x_{h(i-1)}, x_{h(i)}) \) in the rightmost expression considered as a path of length 1 in \( \overline{M} \).

Now we will fix weight matrices for chains \( M \) and \( \overline{M} \). For \( M \) we will use trivial weight matrix \( E \), all elements of \( E \) equal to 1. Weight matrix \( V = (v_{a,b})_{a,b \in W} \) for \( \overline{M} \) define as follows: for all couples of states \( a, b \in W \) in \( \overline{M} \)

\[
\overline{v}_{a,b} = \begin{cases} 
H^E(R^W_{a,b})/P(R^W_{a,b}) & \text{if } (a \neq b) \& (P(R^W_{a,b}) > 0) \\
1 & \text{otherwise}
\end{cases}
\]  

(20)

This means that for all \( a, b \in W \), \( a \neq b \), if \( P(R^W_{a,b}) > 0 \), then

\[
\overline{v}_{a,b} = \left( \sum_{x \in R^W_{a,b}} \text{Len}(x)P^\text{Len}(x)(x) \right)/\left( \sum_{x \in R^W_{a,b}} P^\text{Len}(x)(x) \right)
\]

Note that if \( a, b \in W \), \( a \neq b \), and \( P(R^W_{a,b}) = 0 \) then \( P^\text{Len}(x)(x) = 0 \) for all \( x \in R^W_{a,b} \) and hence \( H^E(R^W_{a,b}) = 0 \) too.

Let \( H^E(\cdot) \) is the function, defined on paths and set of paths in \( M \) according to (14) and (15) with usage of trivial weight matrix \( E \), and \( H^\overline{V}(\cdot) \) is the similar function in \( \overline{M} \), using weight matrix \( \overline{V} \).

**Lemma 9.** If finite Markov chain \( M \) is irreducible, \( \overline{M} = M/W \) is factor chain, \( |W| > 1 \), and functions \( \varphi, H^E(\cdot), \overline{H}^V(\cdot) \) are as defined above, then

(a) For all \( a, b \in W, a \neq b \)

\[
\overline{H}^V((a, b)) = H^E(\varphi^{-1}((a, b)))
\]

(b) For all \( a, b, c \in W, a \neq b, b \neq c \)

\[
\overline{H}^V((a, b) \otimes (b, c)) = H^E(\varphi^{-1}((a, b)) \otimes \varphi^{-1}((b, c)))
\]

(c) For any path \( y = (y_0, \ldots, y_k) \in \mathcal{S}, (\forall i \leq k)(y_{i-1} \neq y_i) \)

\[
\overline{H}^V(y) = H^E \left( \bigotimes_{i=1}^{k} \varphi^{-1}((y_{i-1}, y_i)) \right)
\]

**Proof.** (a) For any \( a, b \in W, a \neq b \) if \( P(R^W_{a,b}) > 0 \), then

\[
\overline{H}^V((a, b)) = \overline{v}_{a,b} \cdot \overline{p}_{a,b} = (H^E(R^W_{a,b})/P(R^W_{a,b})) \cdot P(R^W_{a,b}) = H^E(R^W_{a,b}) = H^E(\varphi^{-1}((a, b)))
\]

If \( P(R^W_{a,b}) = 0 \), then \( \overline{H}^V((a, b)) = \overline{v}_{a,b} \cdot \overline{p}_{a,b} = 0 \) and \( H^E(\varphi^{-1}((a, b))) = H^E(R^W_{a,b}) = 0 \).
(b) Note that for any \(a, b \in W\), \(a \neq b\)
\[
\overline{p}_{a,b} = P(R^{W}_{a,b}) = P(\varphi^{-1}((a, b)))
\]

Let \(a, b, c \in W\), \(a \neq b\), \(b \neq c\). Applying lemma 7(i) to concatenation of one-element sets of paths \((a, b) \otimes (b, c)\) in \(\overline{M}\), receive
\[
\overline{H}^{V}((a, b) \otimes (b, c)) = \overline{H}^{V}((a, b)) \cdot \overline{p}(a, b) + \overline{p}(a, b) \cdot \overline{H}^{V}((b, c)) = \overline{H}^{E}(\varphi^{-1}((a, b))) \cdot \overline{p}_{b,c} + \overline{p}_{a,b} \cdot \overline{H}^{E}(\varphi^{-1}((b, c)))
\]
Now (b) follows from lemma 7(i) for sets of paths \(\varphi^{-1}((a, b))\) and \(\varphi^{-1}((b, c))\) in \(M\).

(c) Similarly to proof of (b), for any \(y = (y_0, \ldots, y_k) \in S\), \((\forall i \leq k)(y_{i-1} \neq y_i)\) use lemma 8(d):
\[
\overline{H}^{V}(y) = \overline{H}^{V}\left(\bigotimes_{i=1}^{k}((y_{i-1}, y_i))\right) = \sum_{i=1}^{k} \left(\overline{H}^{V}((y_{i-1}, y_i)) \prod_{j=1}^{m} \overline{p}((y_j, y_{j+1}))\right) = \sum_{i=1}^{k} \left(\overline{H}^{E}(\varphi^{-1}(y_{i-1}, y_i)) \prod_{j \neq i}^{m} P(\varphi^{-1}(y_{j-1}, y_j))\right) = \overline{H}^{E}\left(\bigotimes_{i=1}^{k} \varphi^{-1}(y_{i-1}, y_i)\right)
\]
Proof completed.

Recall that if weight matrix \(V\) is given, weighted hitting time from \(a\) to \(b\) is
\[
H(V, a, b) = \sum_{x \in S_{<a,b>}} H^{V}(x)
\]
where \(S_{<a,b>} = R^{(a,b)}\) is the set of paths that leads from \(a\) to \(b\) and contain \(b\) only as end state.

**Lemma 10.** If finite Markov chain \(M\) is irreducible, \(\overline{M} = M/W\) is factor chain, \(|W| > 1\), \(a, b \in W\) and \(a \neq b\), then
\[
\overline{H}^{V}(a, b) = H(E, a, b)
\]

Here \(\overline{H}^{V}(a, b)\) is weighted mean hitting time in factor chain \(\overline{M}\), calculated using matrix \(\overline{V}\), defined above (20).
Proof. In equality
\[
\overline{H}(V, a, b) = \sum_{x \in \overline{S}_{<a,b>}} \overline{H}^T(x)
\]
drop from the sum addends for those paths \( x \) that have coinciding adjacent states. This does not affect the sum, because these paths have zero probability. Hence, instead of \( \overline{S}_{<a,b>} \) we could use
\[
\overline{S}^+_{<a,b>} = \overline{S}_{<a,b>} \cap \overline{S}^+ = \{ x \in \overline{S} | (\text{Len}(x) > 0) \& (\text{Begin}(x) = a) \& (\text{End}(x) = b) \& (\forall i \leq \text{Len}(i))(x_{i-1} \notin \{x_i, b\}) \}\]

According to lemma 9(c),
\[
\overline{H}(V, a, b) = \sum_{x \in \overline{S}^+_{<a,b>}} \overline{H}^T(x) = \sum_{x \in \overline{S}^+_{<a,b>}} H^E \left( \bigotimes_{i=1}^{\text{Len}(x)} \varphi^{-1}((x_{i-1}, x_i)) \right)
\]

Note that if \( x', x'' \in \overline{S}^+_{<a,b>} \) and \( x' \neq x'' \), then sets
\[
\bigotimes_{i=1}^{\text{Len}(x')} \varphi^{-1}((x'_{i-1}, x'_i)) \quad \text{and} \quad \bigotimes_{i=1}^{\text{Len}(x'')} \varphi^{-1}((x''_{i-1}, x''_i))
\]
are disjoint. Actually, because \( \varphi \) preserves concatenation, any path \( u \) that belongs to intersection of these two sets must satisfy inconsistent system of equalities: \( \varphi(u) = x' \) and \( \varphi(u) = x'' \).

Applying to (23) lemma 8(b), obtain
\[
\overline{H}(V, a, b) = H^E \left( \bigoplus_{x \in \overline{S}^+_{<a,b>}} \bigotimes_{i=1}^{\text{Len}(x)} \varphi^{-1}((x_{i-1}, x_i)) \right)
\]

For completion proof it is sufficient verify equality
\[
\bigoplus_{x \in \overline{S}^+_{<a,b>}} \bigotimes_{i=1}^{\text{Len}(x)} \varphi^{-1}((x_{i-1}, x_i))
\]
or
\[
\bigoplus_{x \in \overline{S}^+_{<a,b>}} R^W_{x_{i-1}, x_i}
\]

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If \( x = (x_0, \ldots, x_k) \in S_{<a,b>\dagger} \), then all elements in \( x = (x_0, \ldots, x_k) \) belong to \( W \), \( a = x_0 \), each element differ from the previous one, and \( x_k \) is the only element that coincide with \( b \). This means that any path from

\[
\bigotimes_{i=1}^{k} R_{x_{i-1},x_i}^W
\]

leads (in chain \( M \)) from \( a \) to \( b \), and contains \( b \) only at its end. Hence, \( x \in S_{<a,b>} \).

On the other hand, if \( x = (x_0, \ldots, x_m) \in S_{<a,b>} \) then, using subsequence \( y = (y_0, \ldots, y_k) = (x_{h(0)}, \ldots, x_{h(k)}) \) constructed for definition \( \varphi \), receive

\[
x = \bigotimes_{i=1}^{k} (x_{h(i-1)}, \ldots, x_{h(i)})
\]

and

\[
(x_{h(i-1)}, \ldots, x_{h(i)}) \in R_{x_{h(i-1)},x_{h(i)}}^W
\]

Subsequence \( y \) was selected in such a way that \( (y_0, \ldots, y_k) \in S_{<a,b>\dagger} \), therefore

\[
x \in \bigotimes_{i=1}^{k} R_{y_{i-1},y_i} \subseteq \bigoplus_{x \in S_{<a,b>\dagger}} \bigotimes_{i=1}^{\text{Len}(x)} R_{x_{i-1},x_i}^W
\]

Proof completed. \( \square \)

7 The triangle inequality for mean hitting time

**Theorem 1.** In finite irreducible Markov chain, mean hitting time \( H \) satisfies triangle inequality

\[
H(a,c) \leq H(a,b) + H(b,c)
\]

for any states \( a, b, c \).

**Proof.** If at least two of states \( a, b, c \) coincide, the statement is trivial. Consider non-trivial case of mutually different \( a, b, c \).

Let \( W = \{a, b, c\} \). Consider factor chain \( \overline{M} = M/W \) with weight matrix \( \overline{V} \), defined above \( (20) \). According to lemma \( (10) \) \( H(a,b) = H(E,a,b) = \overline{H}(\overline{V},a,b) \), \( H(b,c) = \overline{H}(\overline{V},b,c) \) and \( H(a,c) = \overline{H}(\overline{V},a,c) \). Hence, instead of \( (24) \) we could prove equivalent inequality

\[
\overline{H}(\overline{V},a,c) \leq \overline{H}(\overline{V},a,b) + \overline{H}(\overline{V},b,c)
\]

in very simple factor chain \( \overline{M} \): it has only three states \( a, b, c \), and \( \overline{p}_{a,a} = \overline{p}_{b,b} = \overline{p}_{c,c} = 0 \).
Calculate $H(V, a, c)$, $H(V, a, b)$ and $H(V, b, c)$.

$$H(V, a, c) = \sum_{x \in S_{<a,c>}} Weight(V, x) \cdot P_{a}^{\text{Len}(x)}(x)$$

Here $S_{<a,c>}$ is the set of all paths in $W = \{a, b, c\}$ that leads from $a$ to $c$ and contains $c$ only as its end state.

Note that

$$\overline{P}(S_{<a,c>}) = 1 \quad (26)$$

Formula (26) asserts that chain $\overline{M}$, being in state $a$, will hit (after one or several steps) state $c$ with probability 1. Actually,

$$\overline{P}(S_{<a,c>}) = \sum_{x \in S_{<a,c>}} P_{a}^{\text{Len}(x)}(x) = \lim_{m \to \infty} \sum_{x \in S_{<a,c>}, \text{Len}(x) \leq m} P_{a}^{\text{Len}(x)}(x)$$

And, according to lemma 1

$$\overline{P}(S_{<a,c>}) = \lim_{m \to \infty} \sum_{x \in S_{<a,c>}, \text{Len}(x) \leq m} P_{a}^{\text{Ext}^m(x)}(x) = \lim_{m \to \infty} \left( 1 - \sum_{x \in S_{<(a,*)[-c]}} P_{a}^{\text{Ext}^m(x)}(x) \right) = 27$$

According to lemma 2

$$\lim_{m \to \infty} \left( \sum_{x \in S_{<(a,*)[-c]}} P_{a}^{\text{Ext}^m(x)}(x) \right) = 0$$

and (26) proved.

Evidently, in $\overline{M}$ any path that leads from $a$ till hitting $c$, either coincides with $(a, c)$, or starts from $(a, b)$. This means that

$$S_{<a,b>} = (a, c) \oplus ((a, b) \otimes S_{<b,c>}) \quad (28)$$

According to (18),

$$H(V, a, b) = H(V) (S_{<a,b>}) \quad (29)$$

decay from (28) obtain

$$H(V, a, c) = H(V) (S_{<a,c>}) = H(V) ((a, c) \oplus ((a, b) \otimes S_{<b,c>})) \quad (30)$$

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Applying to (30) lemma (g,i), obtain
\[ H(V, a, c) = H(V, (a, c)) + H(V, (a, b) \otimes S_{<b,c>} ) = \]
\[ H(V, (a, c)) + H(V, (a, b)) \cdot P(S_{<b,c>}) + P((a, b)) \cdot H(V, S_{<b,c>}) \]

Using (26) and (29), receive
\[ H(V, a, c) = H(V, (a, c)) + H(V, (a, b)) + P((a, b)) \cdot P(S_{<b,c>}) + P(S_{<b,c>}) \cdot H(V, b, c) \]

Values of function $H$ on paths of length 1 directly expressed according (14):
\[ H(V, (a, c)) = v_{a,c} \cdot p_{a,c} \]
\[ H(V, (a, b)) = v_{a,b} \cdot p_{a,b} \]
therefore
\[ H(V, a, c) = v_{a,c} \cdot p_{a,c} + v_{a,b} \cdot (v_{a,b} + H(V, b, c)) \]

Swapping roles of $a$ and $b$, receive
\[ H(V, b, c) = v_{b,c} \cdot p_{b,c} + v_{b,a} \cdot (v_{b,a} + H(V, a, c)) \]

Substitute (33) into (32) and get
\[ H(V, a, c) = v_{a,c} \cdot p_{a,c} + v_{a,b} \cdot (v_{a,b} + v_{b,c} \cdot p_{b,c} + v_{b,a} \cdot p_{b,a}) \]

and
\[ (1 - p_{a,b} \cdot p_{b,a}) \cdot H(V, a, c) = v_{a,c} \cdot p_{a,c} + v_{a,b} \cdot (v_{a,b} + v_{b,c} \cdot p_{b,c} + v_{b,a} \cdot p_{b,a}) \]

Recall that
\[ \begin{cases} p_{a,b} + p_{a,c} = 1 \\ p_{b,a} + p_{b,c} = 1 \\ p_{c,a} + p_{c,b} = 1 \end{cases} \]

Note that
\[ 1 - p_{a,b} \cdot p_{b,a} > 0 \]

Actually, otherwise $p_{a,b} = p_{b,a} = 1$, and according to (36), $p_{a,c} = 0$ and $p_{b,c} = 0$. These two equalities contradict to irreducibility of $M$, asserted in lemma 5.

Hence, from (35) follow
\[ H(V, a, c) = \frac{v_{a,c} \cdot p_{a,c} + v_{a,b} \cdot v_{a,b} + p_{a,b} \cdot (v_{b,c} \cdot p_{b,c} + v_{b,a} \cdot p_{b,a})}{1 - p_{a,b} \cdot p_{b,a}} \]
Swapping roles of $b, c$ and $a, b$, receive two similar formulas:

$$\bar{H}(V, a, b) = \frac{\overline{v}_{a,b} \cdot \overline{p}_{a,b} + \overline{p}_{a,c} \cdot \overline{v}_{a,c} + \overline{v}_{a,c} \cdot (\overline{v}_{c,b} \cdot \overline{p}_{c,b} + \overline{p}_{c,a} \cdot \overline{v}_{c,a})}{1 - \overline{p}_{a,c} \cdot \overline{p}_{c,a}} \quad (38)$$

$$\bar{H}(V, b, c) = \frac{\overline{v}_{b,c} \cdot \overline{p}_{b,c} + \overline{p}_{b,a} \cdot \overline{v}_{b,a} + \overline{v}_{b,a} \cdot (\overline{v}_{a,c} \cdot \overline{p}_{a,c} + \overline{p}_{a,b} \cdot \overline{v}_{a,b})}{1 - \overline{p}_{b,a} \cdot \overline{p}_{b,a}} \quad (39)$$

Substituting these three expressions in (25) (and dropping overlines for simpler notations) we receive that (25) is equivalent to inequality

$$\frac{v_{a,c}p_{a,c} + p_{a,b}v_{a,b} + p_{a,b}(v_{b,c}p_{b,c} + p_{b,a}v_{b,a})}{1 - p_{a,b}p_{b,a}} \leq \frac{v_{b,c}p_{b,c} + p_{b,a}v_{b,a} + p_{b,a}(v_{a,c}p_{a,c} + p_{a,b}v_{a,b})}{1 - p_{b,a}p_{a,b}} \quad (40)$$

Because all denominators are positive, for proving (40) it is sufficient to verify that expression $g$ is non-negative:

$$g = (1 - p_{a,c}p_{c,a})(v_{b,c}p_{b,c} + p_{b,a}v_{b,a} + p_{b,a}(v_{a,c}p_{a,c} + p_{a,b}v_{a,b}) - v_{a,c}p_{a,c} - p_{a,b}v_{a,b} - p_{a,b}(v_{b,c}p_{b,c} + p_{b,a}v_{b,a})) + (1 - p_{a,b}p_{b,a})(v_{a,b}p_{a,b} + p_{a,c}v_{a,c} + p_{a,c}(v_{c,b}p_{c,b} + p_{c,a}v_{c,a})) \quad (41)$$

Make a series of transformations:

$$g = (1 - p_{a,c}p_{c,a})(v_{a,b}(p_{b,a}p_{c,a} - p_{a,b}) + v_{a,c}(p_{b,a}p_{a,c} - p_{a,c}) + v_{b,a}(p_{b,c} - p_{a,b}) - p_{a,b}v_{a,b} - p_{a,b}(v_{b,c}p_{b,c} + p_{b,a}v_{b,a})) + (1 - p_{a,b}p_{b,a})(v_{a,b}p_{a,b} + p_{a,c}v_{a,c} + p_{a,c}(v_{c,b}p_{c,b} + p_{c,a}v_{c,a}))$$

$$g = (1 - p_{a,c}p_{c,a}) \cdot (v_{a,b}(p_{b,a}p_{c,a} - p_{a,b}) + v_{a,c}(p_{b,a}p_{a,c} - p_{a,c}) + v_{b,a}(p_{b,c} - p_{a,b}) - p_{a,b}v_{a,b} - p_{a,b}(v_{b,c}p_{b,c} + p_{b,a}v_{b,a})) + (1 - p_{a,b}p_{b,a}) \cdot (v_{a,b}p_{a,b} + v_{a,c}p_{a,c} + v_{c,a}p_{a,c}p_{c,a} + v_{c,b}p_{a,c}p_{c,b})$$

$$g = (1 - p_{a,c}p_{c,a}) \cdot (v_{a,b}(p_{b,a}p_{c,a} - p_{a,b} - 1) + v_{a,c}(p_{b,a}p_{a,c} - p_{a,c} - 1) + v_{b,a}(p_{b,c} - p_{a,b} - 1) - p_{a,b}v_{a,b} - p_{a,b}(v_{b,c}p_{b,c} + p_{b,a}v_{b,a} - 1)) + (1 - p_{a,b}p_{b,a}) \cdot (v_{a,b}p_{a,b} + v_{a,c}p_{a,c} + v_{c,a}p_{a,c}p_{c,a} + v_{c,b}p_{a,c}p_{c,b})$$

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\[ g = (1 - p_{a,c} p_{a,c}) \cdot (-v_{a,b} p_{a,b} p_{b,c} - v_{a,c} p_{a,c} p_{b,c} + v_{b,a} p_{b,a} p_{a,c} + v_{b,c} p_{b,c} p_{a,c}) + (1 - p_{a,b} p_{b,a}) \cdot (v_{a,b} p_{a,b} + v_{a,c} p_{a,c} + v_{c,a} p_{c,a} + v_{c,b} p_{c,b}) \]

Consider \( g \) as linear form of weights:

\[ g = v_{a,b} (1 - p_{b,c} + p_{a,c} p_{c,a} p_{b,c} - p_{a,b} p_{b,a}) p_{a,b} + v_{a,c} (1 - p_{a,c} p_{c,a} p_{b,c} - p_{a,b} p_{b,a}) p_{a,c} + v_{b,a} (1 - p_{a,c} p_{c,a} p_{b,c} - p_{a,b} p_{b,a}) p_{b,c} + v_{b,c} (1 - p_{a,c} p_{c,a} p_{b,c} - p_{a,b} p_{b,a}) p_{a,c} \]

and do some transformations, using (24) (36):

\[ g = v_{a,b} (p_{b,a} + p_{a,c} p_{c,a} p_{b,c} - p_{a,b} p_{b,a}) p_{a,b} + v_{a,c} (p_{c,a} + p_{a,c} p_{c,a} p_{b,c} - p_{a,b} p_{b,a}) p_{a,c} + v_{b,a} (p_{b,a} + p_{a,c} p_{c,a} p_{b,c} - p_{a,b} p_{b,a}) p_{b,c} + v_{b,c} (p_{b,a} + p_{a,c} p_{c,a} p_{b,c} - p_{a,b} p_{b,a}) p_{a,c} \]

\[ g = v_{a,b} (p_{b,a} (1 - p_{a,b}) + p_{a,c} p_{c,a} p_{b,c}) p_{a,b} + v_{a,c} (p_{c,a} (1 - p_{a,b}) + p_{a,c} p_{c,a} p_{b,c}) p_{a,c} + v_{b,a} (p_{b,a} (1 - p_{a,b}) + p_{a,c} p_{c,a} p_{b,c}) p_{b,c} + v_{b,c} (p_{b,a} (1 - p_{a,b}) + p_{a,c} p_{c,a} p_{b,c}) p_{a,c} \]

\[ g = v_{a,b} (p_{b,a} p_{a,c} + p_{a,c} p_{c,a} p_{b,c}) p_{a,b} + v_{a,c} (p_{c,a} p_{a,c} + p_{a,c} p_{c,a} p_{b,c}) p_{a,c} + v_{b,a} (p_{b,a} p_{a,c} + p_{a,c} p_{c,a} p_{b,c}) p_{b,c} + v_{b,c} (p_{b,a} p_{a,c} + p_{a,c} p_{c,a} p_{b,c}) p_{a,c} \]

Now \( g \) is linear form on non-negative weights with non-negative coefficients for each weight. Hence, \( g \geq 0 \) and proof completed.
8 Metric on space of states

**Theorem 2.** If $M$ is a finite irreducible Markov chain with state space $Z$, then function

$$\rho(a, b) = H(a, b) + H(b, a)$$

is a metric on $Z$.

**Proof.** The theorem asserts that for all $a, b, c \in Z$ the following properties hold:

(a) $\rho(a, b) \geq 0$
(b) $\rho(a, b) = 0 \iff a = b$
(c) $\rho(a, b) = \rho(b, a)$
(d) $\rho(a, c) \leq \rho(a, b) + \rho(b, c)$

Properties (a)-(c) are evident from definition of $\rho$. Property (d) follows from the theorem 1:

$$\rho(a, c) = H(a, c) + H(c, a) \leq (H(a, b) + H(b, c)) + (H(c, b) + H(b, a)) =$$

$$(H(a, b) + H(b, a)) + (H(b, c) + H(c, b)) = \rho(a, b) + \rho(b, c)$$

\[\square\]

Note that $\rho(a, b)$ has alternative equivalent definition as mean length of minimal loops that include both $a$ and $b$. Set of all such loops is $(S_{<a,b>} \otimes S_{<b,a>}) \cup (S_{<b,a>} \otimes S_{<a,b>})$.

Let $Z = \{z_1, \ldots, z_n\}$ is the state space of irreducible Markov chain $M$ with transition matrix $P = (p_{i,j})_{i,j=1,\ldots,n}$. Matrix of mean hitting times $h = (h_{i,j})_{i,j=1,\ldots,n}$ (and hence matrix of distances $\rho = (\rho_{i,j})_{i,j=1,\ldots,n}$) could be calculated, using theorem for calculation mean hitting time of arbitrary subset $A \subseteq Z$ (3, Theorem 1.3.5).

This theorem asserts that vector $(k_i^A)_{i=1,\ldots,n}$ of mean hitting time (form state $z_i$ to some element of $A$) is the minimal non-negative solution of the system of linear equations

$$\begin{cases}
  k_i^A = 0 & \text{if } z_i \in A \\
  k_i^A = 1 + \sum_{z_j \notin A} p_{i,j} k_j^A & \text{if } z_i \notin A
\end{cases}$$

Here “minimal non-negative solution” means that $(\forall i = 1, \ldots, n)(k_i^A \leq t_i)$ for any other non-negative solution $(t_i)_{i=1,\ldots,n}$.

For any fixed value of index $j$, apply this theorem to one-element set $A = \{z_j\}$, and receive that vector $(h_{i,j})_{i=1,\ldots,n}$ is the minimal non-negative solution of the system of linear equations

$$\begin{cases}
  h_{j,j} = 0 \\
  h_{i,j} = 1 + \sum_{k \neq j} p_{i,k} h_k^j & \text{for } i \neq j
\end{cases}$$

Simplified procedures for calculation mean hitting times proposed by Hunter [2].
References

[1] Olle Häggström, Finite Markov Chains and Algorithmic Applications, Cambridge University Press, 2002

[2] Jeffrey J. Hunter, Finite Markov Chains and Algorithmic Applications, Simple Procedures for Finding Mean First Passage Times in Markov Chains, Asia-Pacific Journal of Operational Research, Vol. 24, No. 6(2007), 813-829

[3] J.R. Norris, Markov Chains, Cambridge University Press, 1997