General Birkhoff’s Theorem

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Abstract

Space-time is spherically symmetric if it admits the group of SO(3) as a group of isometries, with the group orbits spacelike two-surfaces. These orbits are necessarily two-surface of constant positive curvature. One commonly chooses coordinate \( \{t, r, \theta, \phi\} \) so that the group orbits become surfaces \( \{t, r = \text{const}\} \) and the radial coordinate \( r \) is defined by the requirement that \( 4\pi r^2 \) is the area of these spacelike two-surfaces with the range of zero to infinity. According to the Birkhoff’s theorem upon the above assumptions, Schwarzschild metric is the only solution of the vacuum Einstein field equations. Our aim is to reconsider the solution of the spherically symmetric vacuum Einstein field equations by regarding a weaker requirement. We admit the evident fact that in the completely empty space the radial coordinate \( r \) may be defined so that \( 4\pi r^2 \) becomes the area of spacelike two-surfaces \( \{t, r = \text{const}\} \) with the range of zero to infinity. This is not necessarily to be true in the presence of a material point mass \( M \). It turns out that in spite of imposing asymptotically flatness and staticness as initial conditions the equations have general classes of solutions which Schwarzschild metric is the only member of them which has an intrinsic singularity at the location of the point mass \( M \). The area of \( \{t, r = \text{const}\} \) is \( 4\pi(r + \alpha M)^2 \) in one class and \( 4\pi(r^2 + a_1 Mr + a_2 M^2) \) in the other class while the center of symmetry is at \( r = 0 \).

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Introduction

Every spherically symmetric vacuum solution of Einstein’s field equations is part of the Schwarzschild solution according to the Birkhoff’s theorem. The solution possesses a four dimensional isometry group at least locally. Historically Jebsen (1921) was the first to formulate it and Birkhoff (1923) was the first to prove it [1, 2, 3]. A popular proof has been presented by Hawking and Ellis [4]. Usually the method which applied to prove this theorem is under a relatively stringent assumption. The radial coordinate \( r \) is defined by the requirement that \( 4\pi r^2 \) is the area of the spacelike two-surfaces \( \{ t, r = \text{const} \} \) which are invariant under the group of isometries \( \text{SO}(3) \) operating on them. This assumption which is certainly an evident fact in the completely empty space, physically is not necessarily always to be true even in the presence of a material point mass particle \( M \). Our aim in this article is to reconsider the solution of the spherically symmetric vacuum Einstein’s field equations i.e. Birkhoff’s theorem by regarding a weaker assumption. Let assume only in completely empty space radial coordinate \( r \) may be defined so that \( 4\pi r^2 \) to be the area of spacelike two-surfaces with the range of zero to infinity. It results in that Schwarzschild metric is not the unique solution but general classes of Schwarzschild form solutions exist which most of them are geodesically complete. The Schwarzschild metric is the only member of them which possesses an intrinsic singularity at the location of the point particle.

Spherically Symmetric Space

Spherical symmetry requires the existence of a special coordinate system \((t, r, \theta, \phi)\) so that in which the line element has the form [5, 6]

\[
ds^2 = B(r, t)dt^2 - A(r, t)dr^2 - 2C(r, t)dt\,dr - D(r, t)(d\theta^2 + \sin^2\theta\,d\phi^2) \tag{1}
\]

As mentioned above \( r \) is defined so that the area of spacelike two-surfaces \( \{ t, r = \text{const} \} \) becomes \( 4\pi r^2 \). There is a common belief that the line element can be transformed to the following standard form by a suitable coordinate transformation. Some texts start from here e.g. [7].

\[
ds^2 = B'(r', t')dt'^2 - A'(r', t')dr'^2 - r'^2(d\theta'^2 + \sin^2\theta'\,d\phi'^2) \tag{2}
\]

Taking advantage of the mentioned form to compute the field equations it can be shown that it is necessarily static and has a unique Schwarzschild solution, as required by Birkhoff’s theorem. This means that the other solutions are just different forms of this metric which are related by coordinate transformation. What indeed
is flawing the reasoning, is that the change of the coordinate \( r \rightarrow r' = \sqrt{D(r, t)} \) with new parameter having the same range of \( r \), while it is true in completely empty space is not necessarily to be generally true in the presence of material particle \( M \). Accordingly we believe that the steps which follow to arrive at the standard form are not justified. Since it is a hard task to solve the vacuum field equations with the general form of the metric (1) we restrict our investigation to asymptotically flat and static space-time by convention i.e.

\[
ds^2 = B(r)dt^2 - A(r)dr^2 - D(r)(d\theta^2 + \sin^2\theta \, d\varphi^2)
\]

This metric tensor has the nonvanishing components

\[
g_{tt} = -B(r), \quad g_{rr} = A(r), \quad g_{\theta\theta} = D(r), \quad g_{\varphi\varphi} = D(r) \sin^2\theta
\]

with functions \( A(r) \), \( B(r) \) and \( D(r) \) that are to be determined by solving the field equations. The nonvanishing contravariant components of the metric are:

\[
g^{tt} = -B^{-1}, \quad g^{rr} = A^{-1}, \quad g^{\theta\theta} = D^{-1}, \quad g^{\varphi\varphi} = D^{-1} \sin^{-2}\theta
\]

The metric connection can be computed by the use of (4) and (5) from the usual definition. Its only nonvanishing components are:

\[
\Gamma^r_{rr} = \frac{A'}{2A}, \quad \Gamma^r_{\theta\theta} = -\frac{B'}{2A}, \quad \Gamma^r_{\varphi\varphi} = -\sin^2\theta \frac{D'}{2A}, \quad \Gamma^r_{tt} = \frac{B'}{2A}
\]

\[
\Gamma^\theta_{r\theta} = \frac{B'}{2B}, \quad \Gamma^\theta_{\varphi\varphi} = -\sin \theta \cos \theta
\]

\[
\Gamma^\varphi_{r\varphi} = \frac{B'}{2D}, \quad \Gamma^\varphi_{\theta\theta} = \cot \theta
\]

\[
\Gamma^t_{tr} = \frac{B'}{2B}
\]

where primes stand for differentiation with respect to \( r \). With these connections the Ricci tensor can be obtained.

\[
R_{rr} = \frac{B''}{2B} - \frac{B'}{4B}(\frac{A'}{A} + \frac{B'}{B}) - \frac{A'D'}{2AD} + \frac{D''}{D} - \frac{D'^2}{2D^2}
\]

\[
R_{\theta\theta} = -1 + \frac{D'}{4A}(\frac{A'}{A} + \frac{B'}{B}) + \frac{D''}{2A}
\]

\[
R_{\varphi\varphi} = \sin^2\theta \, R_{\theta\theta}
\]

\[
R_{tt} = -\frac{B''}{2A} + \frac{B'}{2A}(\frac{A'}{A} + \frac{B'}{B}) - \frac{B'D'}{2AD}
\]

\[
R_{\mu\nu} = 0 \quad \text{for } \mu \neq \nu
\]
The Einstein field equations for vacuum are $R_{\mu\nu} = 0$. Dividing $R_{rr}$ by $A$ and $R_{tt}$ by $B$ and putting them together we get

$$- \frac{D'}{2AD}(\frac{A'}{A} + \frac{B'}{B}) + \frac{1}{A}(\frac{D''}{D} - \frac{D'^2}{2D^2}) = 0 \quad (10)$$

Multiplying (10) by $\frac{2AD}{D'}$ gives

$$\frac{A'}{A} + \frac{B'}{B} = \frac{2D''}{D'} - \frac{D'}{D} \quad (11)$$

Now let integrate (11) with respect to $r$ and find

$$AB = C_1 \frac{D'^2}{D} \quad (12)$$

where $C_1$ is a constant of integration which can be fixed by requiring that $D$ asymptotically approaches to $r^2$ and $A$ and $B$ to one. This will fix $C_1$ to $\frac{1}{4}$ by (12), thus

$$AB = \frac{D'^2}{4D} \quad (13)$$

Now using (9) and dividing the field equation $R_{tt} = 0$ by $\frac{B'}{2A}$ we obtain

$$- \frac{B''}{B'} + \frac{1}{2}(\frac{A'}{A} + \frac{B'}{B}) - \frac{D'}{D} = 0 \quad (14)$$

Substituting (11) in (14) we get

$$\frac{B''}{B'} + \frac{3D'}{2D} = \frac{D''}{D'} \quad (15)$$

The next step is to integrate (15) with respect to $r$ which gives

$$B'D^{\frac{3}{2}} = C_2 D' \quad (16)$$

where $C_2$ is a constant of integration. Dividing (16) by $D^{\frac{3}{2}}$ and taking another integration with respect to $r$ we get

$$B = -2C_2 D^{-\frac{1}{2}} + C_3 \quad (17)$$

where $C_3$ is another constant of integration. $C_2$ and $C_3$ can be fixed by considering the Newtonian limit of $B$ which gives $C_3 = 1$ and $C_2 = M$ ($G = c = 1$). Thus

$$B = 1 - 2MD^{-\frac{1}{2}} \quad (18)$$

$A$ may be computed by (12) and (18). It is
\[ A = \frac{D'^2}{1 - 2MD^{-\frac{3}{2}}} \] (19)

It is a natural expectation that the functional form of \( D \) to be fixed by using the \( \theta\theta \) component of the field equation, that is \( R_{\theta\theta} = 0 \). But using (18), (19) and (11) it turns out that the equation \( R_{\theta\theta} = 0 \) will become an identical relation of zero equal to zero for any analytic function of \( D \) which is only restricted to the following constraints

\[ D \rightarrow r^2 \quad if \quad r \rightarrow \infty \cup M \rightarrow 0 \] (20)

This means that \( D \) has the functional form

\[ D(r, M) = r^2 f\left(\frac{M}{r}\right) \] (21)

where \( f(0) = 1 \). The appearance of square root of \( D \) in the final solution reveals that \( D \) cannot adapt negative values otherwise the metric becomes complex which physically is not acceptable. Let’s assume \( D \) to be an analytic monotonic non-negative increasing function of \( r \) from zero to infinity and satisfies (21). There is no obligation that \( D(0, M) \) becomes zero. Since the functional form of \( f \) is not fixed then generally \( D(0, M) \) may admit any positive value. Thus we may have the transformation \( r \rightarrow r' = \sqrt{D(r, M)} = r\sqrt{f\left(\frac{M}{r}\right)} \) but this does not specify the range of \( r' \) at all. It is clear that the Schwarzschild solution is a special case of the general solution by choosing \( f = 1 \), indeed the simplest case. This does not based on any physical fact. This unjustified choice which is not even stand on any fundamental reasoning, is the source of the trouble of singularity in some parts of space-time and of course can be avoided. Then we may perform the following coordinate transformation

\[ r \] with the range \((0, +\infty) \rightarrow r' = D(r, M)^{\frac{1}{2}} \] with the range \((D(0, M)^{\frac{1}{2}}, \infty) \) (22)

where \( D(0, M) \) may be any arbitrarily real non-negative number. Since \( D \) has \([L^2]\) dimension and \( D(0, 0) \) is equal to zero and also \( M \) is the only natural parameter of the system which has length dimension we may conclude that the general form of \( D(0, M) \) is

\[ D(0, M)^{\frac{1}{2}} = \alpha(M)M \] (23)

where \( \alpha \) is a dimensionless parameter which for simplicity may take it as a constant independent of \( M \). Now we make another coordinate transformation
The range of new radial coordinate is from zero to infinity. Dropping primes the final form of the metric becomes

\[ ds^2 = (1 - \frac{2M}{r + \alpha M}) dt^2 - \frac{dr^2}{1 - \frac{2M}{r + \alpha M}} - (r + \alpha M)^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]  

(25)

Thus \( \alpha \) is an arbitrary constant which its different values define the members of our general solutions family [3, 4]. The only condition on \( \alpha \) is that it should not be much bigger than one, otherwise it would contradict with Newtonian mechanics predictions. The corresponding metrics related to the values of \( \alpha \) bigger than 2 are all regular in the whole space. The actual metric of the spacetime of course is merely a special member of this class but not necessarily the Schwarzschild metric (\( \alpha = 0 \)). Specifying this requires further information about the actual properties of spacetime at Schwarzschild scales which at this time there is no access to such data.

Test for Equivalence

Someone may be taken in by the apparent form of the general solution that these are just the usual Schwarzschild solution with \( r \) replaced by \( r + \alpha M \) and conclude that they are not new. In the following we clarify this point and show that it is not so.

1- In Schwarzschild solution the range of radial coordinate \( r \) is between 0 to \( \infty \) and the center of symmetry is at \( r = 0 \). Replacing \( r \) by \( r + \alpha M \) does not lead to the new solution because the range of the transformed radial coordinate is between \( -\alpha M \) to \( \infty \). Of course negative values for conventional radial coordinate is meaningless and the center of symmetry is located at \( \dot{r} = -\alpha M \). This is not identical to the new solution because the range of radial coordinate is between 0 to \( \infty \) and the center of symmetry is at \( \dot{r} = 0 \). On the other hand in the new metric replacing \( r + \alpha M = \dot{r} \) does not lead to Schwarzschild solution because the range of \( r \) here is between 0 to \( \infty \) and the center of symmetry is at \( r = 0 \). While the range of the transformed radial coordinate is between \( \alpha M \) to \( \infty \) and the center of symmetry is at \( \dot{r} = \alpha M \). This evidently is different from Schwarzschild solution. Though the extension of \( \dot{r} \) to values smaller than \( \alpha M \) is physically meaningless because a point by definition has no internal structure to be extended inside of it, let us consider such a mathematical hypothetical spacetimes. Even these are essentially different from Schwarzschild spacetime because in Schwarzschild the point mass \( M \) is at \( r = 0 \) while in these spacetimes it is located at \( r = \alpha M \).

2- The center of spherical symmetry that is the position of point mass \( M \) is a common point between the Schwarzschild spacetime and the presented spherically symmetric vacuum spacetimes in this manuscript. As it has been shown the field equations
and the given boundary conditions are not sufficient to fix $\alpha$. Thus if we take $\alpha \neq 0$, then these solutions will not be singular at the center of symmetry while the Schwarzschild spacetime possesses an intrinsic singularity at the center of symmetry. If these new metrics were isometric to Schwarzschild metric they should be singular too, because coordinate transformation cannot change the intrinsic properties of spacetime. This clearly shows that the presented metrics are not Kottler solutions of Schwarzschild spacetime.

3- The presented general solution and the Schwarzschild solution have exactly the same space extension. Making use of Cartesian coordinate system as frame of reference will elucidate this fact. It turns out that all components have the same range $(-\infty, +\infty)$.

4- Let us consider hypothetically spacetimes which possesses different lower bound for the surface area of a sphere. Obviously they have different geometrical structures and present different physics.

5- The zone of $r$ of the order of Schwarzschild radius is the domain in which gravitational field is tremendously strong and conventionally we have to give up our common sense and replace the character of $r$ and $t$. So it is not surprising to have a geometry completely different.

6- Schwarzschild solution is a special case of our general solutions Eq.(25) for the case of $\alpha = 0$. Therefore both Schwarzschild and general solutions are in the same coordinate system which manifestly have different form. Any transformation to new radial coordinate $r' = r + am$ requires the Schwarzschild metric be written in this new coordinate too which means we should replace $r$ by $r' - am$. Thus in the new coordinate they will have different form too. This means general solutions are not isometric to any piece of the Schwarzschild metric.

Completeness

For $\alpha > 2$ in (25) $t$ remains time coordinate everywhere. This means the hypersurfaces $t = const.$ become spacelike positive definite Riemannian 3-dimensional manifolds. These are metrically complete because every Cauchy sequence with respect to the distance function converges to a point in the manifold. It is well known that metric completeness and geodesically completeness are equivalent for a positive definite metric [10].

The Most General Case

What we have assumed in the previous sections for $D(r, M)$ i.e. to be an analytic monotonic non-negative increasing function is not generally a necessary requirement. It is merely necessary to be analytic and non-negative. To see how this may be
possible let us consider a series expansion of \( f\left(\frac{M}{r}\right) \) as follows:

\[
f\left(\frac{M}{r}\right) = \sum_{n=-\infty}^{+\infty} a_n\left(\frac{M}{r}\right)^n
\]  
(26)

The condition, \( f(0) = 1 \) implies that

\[
a_n = 0, \quad n = -1, -2, \ldots, -\infty \quad a_0 = 1
\]  
(27)

Since \( D(r, M) \) is supposed to be analytic everywhere including \( r = 0 \), this leads to

\[
a_n = 0, \quad n = 3, 4, \ldots, +\infty
\]  
(28)

Thus the most general form of \( D(r, M) \) is

\[
D(r, M) = r^2 f\left(\frac{M}{r}\right) = r^2 + a_1 M r + a_2 M^2
\]  
(29)

where \( a_1 \) and \( a_2 \) are dimensionless numbers which should be fixed by comparison with observation. The non-negativeness of \( D(r, M) \) would be guaranteed if

\[
a_2 \geq 0 \quad \text{and} \quad a_1^2 - 4a_2 \leq 0
\]
\[
or \quad -2\sqrt{a_2} \leq a_1 \leq 2\sqrt{a_2}
\]  
(30)

The minimum value of \( D(r, M) \) is at

\[
r = -\frac{a_1 M}{2}
\]  
(31)

For \( a_1 \geq 0 \) (29) is a non-negative monotonic and increasing function of \( r \) from zero to infinity as the same case which was discussed previously. But for \(-2\sqrt{a_2} \leq a_1 \leq 0 \), \( D(r, M) \) decreases as \( r \) goes from \( +\infty \) to \(-\frac{a_1 M}{2}\) and then increases as \( r \) goes from \(-\frac{a_1 M}{2}\) to zero. Thus for the case \( a_2 \geq 0 \) and \(-2\sqrt{a_2} \leq a_1 \leq 0 \) it is not possible to have a transformation like \( r \to r' = \sqrt{D} \). Because there is not one to one correspondence between \( r \) and \( r' \) in this case.

We may write the most general form of the solution as

\[
ds^2 = (1 - 2MD^{-\frac{3}{2}})dt^2 - \frac{d\theta^2}{1 - 2MD^{-\frac{3}{2}}} - D(d\theta^2 + \sin^2\theta d\phi^2)
\]  
(32)

\[
D = r^2 + a_1 Mr + a_2 M^2 \quad \text{and} \quad a_2 \geq 0, \quad -2\sqrt{a_2} \leq a_1 \leq 0
\]

This metric would be free of any coordinate singularity if we have

\[
a_2 - \frac{a_1^2}{4} > 4
\]  
(33)
(33) will restrict the allowed values of \(a_1\) and \(a_2\) to the range
\[
a_2 > 4 \quad \text{and} \quad -2\sqrt{a_2 - 4} < a_1 < 0
\]
(34)
The proof of completeness which was presented for (25) does hold for (32).

Conclusion

We may conclude the discussion that Birkhoff’s theorem to be modified with this statement that the vacuum Einstein field equation spherical solutions are uniquely of general Schwarzschild-like form (25) or (32) which may be called general Birkhoff’s theorem. Spacetimes (25) for \(\alpha > 2\) and (32) for (34) are nonsingular and maximally extended.

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