APPROXIMATION THEOREMS CONNECTED WITH
DIFFERENTIAL-DIFFERENCE OPERATOR

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ABSTRACT. In the present paper, we propose to give an extension to the context of Dunkl theory of the notion of translation and in connection with this a corresponding extension of Taylor’s formula. More precisely, we prove some properties and estimates of the integral remainder in the generalized Taylor formula associated to the Dunkl operator on the real line and we describe the Besov-type spaces for which the remainder has a given order.

1. INTRODUCTION

Delsartes gave in [10, 11] a certain extension of the notion of translation and in connection with this a corresponding extension of Taylor’s formula. Generalized translation have later been considered from various points of view by many authors (see Levitan [13], Bochner [8, 9]). Löfström and Peetre in [14] estimated the remainder in the generalized Taylor’s formula and they described the space of functions for which the remainder has a given order.

Our aim in this paper is to extend the results obtained in [14] to the context of Dunkl theory. More precisely, we prove some properties and estimates of the integral remainder of order $k$ associated to the Dunkl operator on the real line and we establish the coincidence between two characterizations of Besov-type spaces related to this integral remainder.

For a real parameter $\alpha > -\frac{1}{2}$, the Dunkl operator on the real line denoted by $\Lambda_\alpha$, is a differential-difference operator introduced in 1989 by C. Dunkl in [12]. This operator is associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$ and is given by

$$\Lambda_\alpha f(x) = \frac{df}{dx}(x) + \frac{2\alpha + 1}{x} \left[ f(x) - f(-x) \right], f \in C^1(\mathbb{R}).$$

The Dunkl operator can be considered as a perturbation of the usual derivative by reflection part. This operator plays a major role in the study of quantum harmonic oscillators governed by Wigner’s commutation rules (see [18]). The Dunkl kernel $E_\alpha$ related to $\Lambda_\alpha$ is used to define the Dunkl transform which enjoys properties similar to those of the classical Fourier transform. The Dunkl kernel $E_\alpha$ satisfies a product formula (see [19]). This allows us to define the Dunkl translation $\tau_x$, $x \in \mathbb{R}$ (see next section). If the parameter $\alpha = -\frac{1}{2}$, then the operator $\Lambda_\alpha$ reduces to the

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differential operator $\frac{d^k}{dx^k}$. Therefore Dunkl analysis can be viewed as a generalization of the classical Fourier analysis on $\mathbb{R}$.

In 2003, the classical Taylor formula with integral remainder was extended in [16] to the one dimensional Dunkl operator $\Lambda_\alpha$:

For $k = 1, 2, \ldots, f \in \mathcal{E}(\mathbb{R})$ and $a \in \mathbb{R}$, we have

$$\tau_x(f)(a) = \sum_{p=0}^{k-1} b_p(x) \Lambda_\alpha^p f(a) + R_k(x, f)(a), \quad x \in \mathbb{R}\setminus\{0\},$$

with $R_k(x, f)(a)$ is the integral remainder of order $k$ given by

$$R_k(x, f)(a) = \int_{|x|}^{[x]} \Theta_{k-1}(x, y) \tau_y(\Lambda_\alpha^k f)(a) A_\alpha(y) dy,$$

where $\mathcal{E}(\mathbb{R})$ is the space of infinitely differentiable functions on $\mathbb{R}$ and $(\Theta_p)_{p \in \mathbb{N}}$, \( (b_p)_{p \in \mathbb{N}} \) are two sequences of functions constructed inductively from the function $A_\alpha$ defined on $\mathbb{R}$ by $A_\alpha(x) = |x|^{2\alpha+1}$ (see next section).

There are many ways to define the Besov spaces (see [6, 7, 17]) and the Besov-Dunkl spaces (see [1, 2, 3, 4]). It is well known that Besov spaces can be described by means of differences using the modulus of continuity of functions. These spaces defined by the modulus of smoothness occur more naturally in many areas of analysis including approximation theory.

In this paper we define the following weighted function spaces:

Let $0 < \beta < 1$, $1 \leq p < +\infty$, $1 \leq q \leq +\infty$ and $k$ a positive integer ($k = 1, 2, \ldots$).

- We denote by $L^p(\mu_\alpha)$ the space of complex-valued functions $f$, measurable on $\mathbb{R}$ such that

$$\|f\|_{p, \alpha} = \left( \int_{\mathbb{R}} |f(x)|^p d\mu_\alpha(x) \right)^{1/p} < +\infty,$$

where $\mu_\alpha$ is a weighted Lebesgue measure associated to the Dunkl operator given by

$$d\mu_\alpha(x) = \frac{A_\alpha(x)}{2^{\alpha+1}\Gamma(\alpha+1)} dx,$$

with $A_\alpha$ is the function defined on $\mathbb{R}$ by

$$A_\alpha(x) = |x|^{2\alpha+1}, \quad x \in \mathbb{R}.$$

- The Besov-Dunkl space of order $k$ denoted by $\mathcal{B}^k_{p, \alpha}$ is the subspace of functions $f$ in $\mathcal{E}(\mathbb{R}) \cap L^p(\mu_\alpha)$ such that $\Lambda_\alpha^{k-1} f \in L^p(\mu_\alpha)$ and satisfying

$$\int_0^{+\infty} \frac{\omega_{p, \alpha}^k(x, f)}{x^{\alpha+k-1}} \frac{q \, dx}{x} < +\infty \quad \text{if} \quad q < +\infty$$

and

$$\sup_{x>0} \frac{\omega_{p, \alpha}^k(x, f)}{x^{\alpha+k-1}} < +\infty \quad \text{if} \quad q = +\infty,$$

with $\omega_{p, \alpha}^k(x, f) = \sup_{y \leq x} \|R_{k-1}(y, f) - b_{k-1}(y) \Lambda_\alpha^{k-1} f\|_{p, \alpha}$, where we put $A_\alpha^0 f = f$ and $R_0(x, f) = \tau_x(f)$, for $k = 1$.

- Put $\mathcal{D}^k_{p, \alpha}$ the subspace of functions $f$ in $\mathcal{E}(\mathbb{R}) \cap L^p(\mu_\alpha)$ such that $\Lambda_\alpha^k f$ are in
We consider the subspace $K^k D^{\beta,\alpha}_{p,q}$ of functions $f \in D^{k-1}_{\mu,\alpha} + D^k_{\mu,\alpha}$ satisfying
\[\int_0^{+\infty} \left( \frac{K^k_{p,\alpha}(x, f)}{x^\beta} \right)^q \frac{dx}{x} < +\infty \quad \text{if} \quad q < +\infty \]
and \[\sup_{x > 0} \frac{K^k_{p,\alpha}(x, f)}{x^\beta} < +\infty \quad \text{if} \quad q = +\infty,\]
where $K^k_{p,\alpha}$ is the Peetre K-functional given by
\[K^k_{p,\alpha}(x, f) = \inf_{f = f_0 + f_1} \left\{ \| \Lambda_0 f_0 \|_{p,\alpha} + x \| \Lambda_0 f_1 \|_{p,\alpha}, f_0 \in D^{k-1}_{\mu,\alpha}, f_1 \in D^k_{\mu,\alpha} \right\}.\]

The contents of the present paper are as follows.
In section 2, we collect some basic definitions and results about harmonic analysis associated with the Dunkl operator $\Lambda_\alpha$.
In section 3, we give some properties and estimates of the integral remainder of order $k$. Finally, we establish that $B^k D^{\beta,\alpha}_{p,q} = K^k D^{\beta,\alpha}_{p,q}$.

Along this paper, we use $c$ to represent a suitable positive constant which is not necessarily the same in each occurrence.

2. Preliminaries

In this section, we recall some notations and results in Dunkl theory on $\mathbb{R}$ and we refer for more details to \cite{5,12,19}.

For $\lambda \in \mathbb{C}$, the initial problem
\[\Lambda_\alpha(f)(x) = \lambda f(x), \quad f(0) = 1, \quad x \in \mathbb{R},\]
has a unique solution $E_\alpha(\lambda)$ called Dunkl kernel given by
\[E_\alpha(\lambda x) = j_\alpha(i\lambda x) + \frac{\lambda x}{2(\alpha + 1)} j_{\alpha+1}(i\lambda x), \quad x \in \mathbb{R},\]
where $j_\alpha$ is the normalized Bessel function of the first kind and order $\alpha$.

The Dunkl kernel $E_\alpha$ satisfies the following product formula
\[E_\alpha(izt)E_\alpha(iyt) = \int_{\mathbb{R}} E_\alpha(iz\gamma) d\gamma_{x,y}(z), \quad x, y, t \in \mathbb{R},\]
where $\gamma_{x,y}$ is a signed measure on $\mathbb{R}$ with compact support.

For $x, y \in \mathbb{R}$ and $f$ a continuous function on $\mathbb{R}$, the Dunkl translation operator $\tau_x$ is given by
\[\tau_x(f)(y) = \int_{\mathbb{R}} f(z) d\gamma_{x,y}(z).\]

The Dunkl translation operator satisfies the following properties:
(1) $\tau_x$ is a continuous linear operator from $E(\mathbb{R})$ into itself.
(2) For all $f \in E(\mathbb{R})$, we have
\[\tau_x(f)(y) = \tau_y(f)(x) \quad \text{and} \quad \tau_0(f)(x) = f(x)\]
\[\tau_x \circ \tau_y = \tau_y \circ \tau_x \quad \text{and} \quad \Lambda_\alpha \circ \tau_x = \tau_x \circ \Lambda_\alpha.\]
Proposition 2.1. For $k = 1, 2, ..., f \in \mathcal{E}(\mathbb{R})$ and $a \in \mathbb{R}$, we have
\[
\tau_{x} f(a) = \sum_{p=0}^{k-1} b_{p}(x) \Lambda_{\alpha}^{p} f(a) + R_{k}(x, f)(a), \quad x, f \in \mathbb{R} \setminus \{0\},
\]
with $R_{k}(x, f)(a)$ is the integral remainder of order $k$ given by
\[
R_{k}(x, f)(a) = \int_{-|x|}^{|x|} \Theta_{k-1}(x, y) \tau_{y}(\Lambda_{\alpha}^{k} f)(a) A_{\alpha}(y) dy.
\]

Remark 3.1. Let $k = 1, 2, ..., f \in \mathcal{E}(\mathbb{R})$ and $x \in \mathbb{R} \setminus \{0\}$.

(1) From Proposition 2.1, we have
\[
R_{k}(x, f) = \tau_{x}(f) - f - b_{1}(x) \Lambda_{\alpha} f - ... - b_{k-1}(x) \Lambda_{\alpha}^{k-1} f
\]
\[
= R_{k-1}(x, f) - b_{k-1}(x) \Lambda_{\alpha}^{k-1} f,
\]
where we put for $k = 1$, $R_{0}(x, f) = \tau_{x}(f)$. Observe that
\[
R_{1}(x, f) = R_{0}(x, f) - b_{0}(x) \Lambda_{\alpha}^{0} f = \tau_{x}(f) - f.
\]

(2) According to [16, p.352] and Proposition 2.1, (i), we have
\[
\int_{-|x|}^{|x|} |\Theta_{k-1}(x, y)| A_{\alpha}(y) dy \leq b_{k}(|x|) + |x| b_{k-1}(|x|)
\]
\[
\leq c |x|^{k}.
\]
(3) Note that the function \( y \mapsto \tau_y(f) - f \) is continuous on \( \mathbb{R} \) (see [15], Lemma 1, (ii)), which implies that the same is true for the function \( y \mapsto R_k(y, f) \).

**Lemma 3.1.** Let \( k = 1, 2, \ldots, \) then there exists a constant \( c > 0 \) such that for all \( f \in E(\mathbb{R}) \) satisfying \( \Lambda_{\alpha}^{k-1}f \in L^p(\mu_{\alpha}) \), we have

\[
\|R_{k-1}(x, f)\|_{p, \alpha} \leq c |x|^{k-1}\|\Lambda_{\alpha}^{k-1}f\|_{p, \alpha}, \quad x \in \mathbb{R}\setminus\{0\}.
\]  

(3.3)

**Proof.** Let \( k = 1, 2, \ldots, \) \( f \in E(\mathbb{R}) \) such that \( \Lambda_{\alpha}^{k-1}f \in L^p(\mu_{\alpha}) \) and \( x \in \mathbb{R}\setminus\{0\} \). For \( k = 1 \), by (2.2), it’s clear that \( \|R_0(x, f)\| = \|\tau_x(f)\|_{p, \alpha} \leq c \|f\|_{p, \alpha} \). Using Minkowski’s inequality for integrals, (2.2) and (2.4), we have for \( k \geq 2 \)

\[
\|R_{k-1}(x, f)\|_{p, \alpha} \leq \int_{-|x|}^{|x|} |\Theta_{k-2}(x, y)| \|\tau_y(\Lambda_{\alpha}^{k-1}f)\|_{p, \alpha} A_\alpha(y) dy
\]

\[
\leq c \|\Lambda_{\alpha}^{k-1}f\|_{p, \alpha} \int_{-|x|}^{|x|} |\Theta_{k-2}(x, y)| A_\alpha(y) dy.
\]

From (3.2), we deduce our result. □

**Remark 3.2.** Let \( k = 1, 2, \ldots, \) \( f \in E(\mathbb{R}) \) such that \( \Lambda_{\alpha}^{k-1}f \in L^p(\mu_{\alpha}) \). Then for \( x \in \mathbb{R}\setminus\{0\} \), we have by (3.1), (3.3) and Proposition 2.1,

\[
\|R_k(x, f)\|_{p, \alpha} \leq \|R_{k-1}(x, f)\|_{p, \alpha} + \|b_{k-1}(x)\Lambda_{\alpha}^{k-1}f\|_{p, \alpha}
\]

\[
\leq c |x|^{k-1}\|\Lambda_{\alpha}^{k-1}f\|_{p, \alpha}.
\]  

(3.4)

**Lemma 3.2.** For \( x \in \mathbb{R}\setminus\{0\} \) and \( p \in \mathbb{N} \), we have

\[
\int_{-|x|}^{|x|} \Theta_0(x, y)b_p(y)A_\alpha(y) dy = b_{p+1}(x).
\]  

(3.5)

**Proof.** Let \( x \in \mathbb{R}\setminus\{0\} \). Using Proposition 2.1, we have:

- If \( p = 2m, m \in \mathbb{N}, \)

\[
\int_{-|x|}^{|x|} \Theta_0(x, y)b_{2m}(y)A_\alpha(y) dy
\]

\[
= \int_{-|x|}^{|x|} \frac{\text{sgn}(x)|y|^{2\alpha+1}}{2|x|^{2\alpha+1}} b_{2m}(y) dy + \int_{-|x|}^{|x|} \frac{\text{sgn}(y)}{2} b_{2m}(y) dy
\]

\[
= \frac{x}{2^{2m}|x|^{2\alpha+2}(\alpha + 1)m!} \int_0^{|x|} y^{2\alpha+2m+1} dy
\]

\[
= \frac{x}{2^{2m}(\alpha + 1)m! 2(\alpha + m + 1)}
\]

\[
= b_{2m+1}(x).
\]
Lemma 3.4. Let 

By induction, we deduce our result. □

Hence the Lemma is proved. □

Lemma 3.3. Let \( k = 1, 2, \ldots, f \in \mathcal{E}(\mathbb{R}), x \in \mathbb{R}\setminus\{0\} \) and \( a \in \mathbb{R} \). Then we have,

\[
R_k(x, f)(a) = \int_{-|x|}^{|x|} \Theta_0(x, y) R_{k-1}(y, \Lambda_f)(a) A_\alpha(y) dy.
\]  

(3.6)

Proof. Let \( k = 1, 2, \ldots, f \in \mathcal{E}(\mathbb{R}), x \in \mathbb{R}\setminus\{0\} \) and \( a \in \mathbb{R} \). We have from (2.3), (2.4) and the fact that \( R_0(y, \Lambda_f)(a) = \tau_y(\Lambda_f) \),

\[
R_1(x, f)(a) = (\tau_x f - f)(a) = \int_{-|x|}^{|x|} \Theta_0(x, y) \tau_y(\Lambda_f)(a) A_\alpha(y) dy,
\]

hence the property (3.6) is true for \( k = 1 \).

Suppose that

\[
R_k(x, f)(a) = \int_{-|x|}^{|x|} \Theta_0(x, y) R_{k-1}(y, \Lambda_f)(a) A_\alpha(y) dy,
\]

then by (3.1) and (3.5), we get

\[
\int_{-|x|}^{|x|} \Theta_0(x, y) R_k(y, \Lambda_f)(a) A_\alpha(y) dy
\]

\[
= \int_{-|x|}^{|x|} \Theta_0(x, y) R_{k-1}(y, \Lambda_f)(a) A_\alpha(y) dy
\]

\[
- \int_{-|x|}^{|x|} \Theta_0(x, y) b_k(y) \Lambda_\alpha^k f(a) A_\alpha(y) dy
\]

\[
= R_k(x, f)(a) - \Lambda_\alpha^k f(a) \int_{-|x|}^{|x|} \Theta_0(x, y) b_k(y) A_\alpha(y) dy
\]

\[
= R_k(x, f)(a) - b_k(x) \Lambda_\alpha^k f(a)
\]

\[
= R_{k+1}(x, f)(a).
\]

By induction, we deduce our result. □

Lemma 3.4. Let \( k = 1, 2, \ldots, f \in \mathcal{E}(\mathbb{R}), x \in \mathbb{R}\setminus\{0\} \) and \( a \in \mathbb{R} \). We denote by

\[
I_1(x, f)(a) = \int_{-|x|}^{|x|} \Theta_0(x, y) \tau_y(f)(a) A_\alpha(y) dy,
\]
and for $k \geq 2$

$$I_k(x, f)(a) = \int_{-|x|}^{|x|} \Theta_0(x, y)I_{k-1}(y, f)(a)A_{\alpha}(y)dy.$$  

Then, we have

$$\Lambda_{\alpha}^{k+1}(I_k(x, f))(a) = \Lambda_{\alpha}^k(I_k(x, \Lambda_{\alpha}f))(a) \quad (3.7)$$

and

$$\Lambda_{\alpha}^k I_k(x, f)(a) = R_k(x, f)(a). \quad (3.8)$$

**Proof.** Let $k = 1, 2, ..., f \in \mathcal{E}({\mathbb{R}})$, $x \in \mathbb{R} \setminus \{0\}$ and $a \in \mathbb{R}$.

- Using (2.1), we have

$$\Lambda_{\alpha}^2(I_1(x, f))(a) = \int_{-|x|}^{|x|} \Theta_0(x, y)\Lambda_{\alpha}\tau_y(\Lambda_{\alpha}f)(a)A_{\alpha}(y)dy = \Lambda_{\alpha}(I_1(x, \Lambda_{\alpha}f))(a).$$

Suppose that

$$\Lambda_{\alpha}^{k+1}(I_k(x, f))(a) = \Lambda_{\alpha}^k(I_k(x, \Lambda_{\alpha}f))(a),$$

this gives

$$\Lambda_{\alpha}^{k+2}(I_{k+1}(x, f))(a) = \int_{-|x|}^{|x|} \Theta_0(x, y)\Lambda_{\alpha}(\Lambda_{\alpha}^{k+1}I_k(y, f))(a)A_{\alpha}(y)dy = \int_{-|x|}^{|x|} \Theta_0(x, y)\Lambda_{\alpha}(\Lambda_{\alpha}^kI_k(y, \Lambda_{\alpha}f))(a)A_{\alpha}(y)dy = \Lambda_{\alpha}^{k+1}(I_{k+1}(x, \Lambda_{\alpha}f))(a).$$

Then, we obtain (3.7) by induction.

- From (2.1) and (2.4), we can write

$$\Lambda_{\alpha}(I_1(x, f))(a) = \int_{-|x|}^{|x|} \Theta_0(x, y)\Lambda_{\alpha}(\Lambda_{\alpha}f)(a)A_{\alpha}(y)dy = \int_{-|x|}^{|x|} \Theta_0(x, y)\tau_y(\Lambda_{\alpha}f)(a)A_{\alpha}(y)dy = R_1(x, f)(a).$$

Suppose that

$$\Lambda_{\alpha}^k(I_k(x, f))(a) = R_k(x, f)(a),$$

then by (3.6) and (3.7), we have

$$\Lambda_{\alpha}^{k+1}(I_{k+1}(x, f))(a) = \int_{-|x|}^{|x|} \Theta_0(x, y)\Lambda_{\alpha}^{k+1}(I_k(y, f))(a)A_{\alpha}(y)dy = \int_{-|x|}^{|x|} \Theta_0(x, y)\Lambda_{\alpha}^k(I_k(y, \Lambda_{\alpha}f))(a)A_{\alpha}(y)dy = \int_{-|x|}^{|x|} \Theta_0(x, y)R_k(y, \Lambda_{\alpha}f)(a)A_{\alpha}(y)dy = R_{k+1}(x, f)(a).$$

By induction, this gives (3.8). \qed
Before establishing that $B^kD^{\beta,\alpha}_{p,q} = K^kD^{\beta,\alpha}_{p,q}$, we give a remark, a proposition containing sufficient conditions and an example.

**Remark 3.3.** For $k = 1, 2, \ldots, f \in \mathcal{E}(\mathbb{R}) \cap L^p(\mu_\alpha)$ such that $\Lambda_\alpha^{k-1} f$ is in $L^p(\mu_\alpha)$ and $x \in (0, +\infty)$, we can assert from (3.1) that

1. $\omega^k_{\alpha,\alpha}(x, f) = \sup_{|y| \leq x} ||R_k(y, f)||_{p,\alpha}$.
2. For $k = 1$, we have $\omega^{k}_{\alpha,\alpha}(x, f) = \sup_{|y| \leq x} ||\tau_y(f) - f||_{p,\alpha}$.

**Proposition 3.1.** Let $k = 1, 2, \ldots, 1 \leq p < +\infty, 1 \leq q \leq +\infty, 0 < \beta < 1$ and $f \in \mathcal{E}(\mathbb{R}) \cap L^p(\mu_\alpha)$. If $\Lambda_\alpha^{k-1} f$ and $\Lambda_\alpha^k f$ are in $L^p(\mu_\alpha)$. For $x \in (0, +\infty)$, we obtain by (3.3) and (3.4),

$$\omega^k_{\alpha,\alpha}(x, f) \leq c x^k ||\Lambda_\alpha^k f||_{p,\alpha} \text{ and } \omega^k_{\alpha,\alpha}(x, f) \leq c x^{k-1} ||\Lambda_\alpha^{k-1} f||_{p,\alpha}.$$ 

Then we can write,

$$\int_0^{+\infty} \left(\frac{\omega^k_{\alpha,\alpha}(x, f)}{x^{\beta+k-1}}\right) \frac{q dx}{x} \leq c \int_0^1 \left(\frac{||\Lambda_\alpha^k f||_{p,\alpha}}{x^{\beta-k+1}}\right) \frac{q dx}{x} + c \int_1^{+\infty} \left(\frac{||\Lambda_\alpha^{k-1} f||_{p,\alpha}}{x^{\beta-k+1}}\right) \frac{q dx}{x},$$

giving two finite integrals. Here when $q = +\infty$, we make the usual modification. □

**Example 3.1.** From (2.5) and Proposition 3.1, we can assert that the spaces $C^\infty_c(\mathbb{R})$ and $S(\mathbb{R})$ are included in $B^kD^{\beta,\alpha}_{p,q}$.

**Theorem 3.1.** Let $0 < \beta < 1$, $k = 1, 2, \ldots, 1 \leq p < +\infty$ and $1 \leq q \leq +\infty$, then

$$B^kD^{\beta,\alpha}_{p,q} = K^kD^{\beta,\alpha}_{p,q}.$$

**Proof.** We start with the proof of the inclusion $K^kD^{\beta,\alpha}_{p,q} \subset B^kD^{\beta,\alpha}_{p,q}$. Let $f$ a function in $K^kD^{\beta,\alpha}_{p,q}$. If $f = f_0 + f_1, f_0 \in D^{k-1}_{p,q}$ and $f_1 \in D^k_{p,q}$ is any decomposition of $f$, we have by (3.3)

$$\omega^k_{\alpha,\alpha}(x, f_1) = \sup_{|y| \leq x} ||R_k(y, f_1)||_{p,\alpha} \leq c x^k ||\Lambda_\alpha^k f_1||_{p,\alpha}, \quad x \in (0, +\infty). \tag{3.9}$$

Using (3.4), we obtain

$$\omega^k_{\alpha,\alpha}(x, f_0) \leq \sup_{|y| \leq x} ||R_{k-1}(y, f_0)||_{p,\alpha} + \sup_{|y| \leq x} ||b_{k-1}(y)\Lambda_\alpha^{k-1} f_0||_{p,\alpha} \leq c x^{k-1} ||\Lambda_\alpha^{k-1} f_0||_{p,\alpha}, \quad x \in (0, +\infty). \tag{3.10}$$

Hence by (3.9) et (3.10), we deduce that

$$\omega^k_{\alpha,\alpha}(x, f) \leq c x^{k-1} K^k_{\alpha,\alpha}(x, f),$$

then, $f \in B^kD^{\beta,\alpha}_{p,q}$.

To prove the inclusion $B^kD^{\beta,\alpha}_{p,q} \subset K^kD^{\beta,\alpha}_{p,q}$, we have to make for $f \in B^kD^{\beta,\alpha}_{p,q}$ a proper choice of $f_0$ and $f_1$. We take for $x \in (0, +\infty)$

$$f_1 = \frac{1}{b_k(x)} I_k(x, f).$$
Using (3.8), we obtain
\[ x\| \Lambda_k f_1 \|_{p,\alpha} \leq x \left( b_k(x) \right)^{-1} \omega_{p,\alpha}^k(x, f) \]
\[ \leq c \frac{\omega_{p,\alpha}^k(x, f)}{x^{k-1}}. \]  
(3.11)

On the other hand, put \( f_0 = f - f_1 \), we can write by (3.5)
\[ f_0 = -\frac{1}{b_k(x)} \int_{-x}^{x} \Theta_0(x, y) \left( I_{k-1}(y, f) - b_{k-1}(y) f \right) A_\alpha(y) dy. \]

From (3.1) and (3.8), we obtain
\[ \Lambda_k^{-1} f_0 = -\frac{1}{b_k(x)} \int_{-x}^{x} \Theta_0(x, y) R_k(y, f) A_\alpha(y) dy. \]

By Minkowski’s inequality for integrals and (3.2), we get
\[ \| \Lambda_k^{-1} f_0 \|_{p,\alpha} \leq (b_k(x))^{-1} \int_{-x}^{x} |\Theta_0(x, y)| \| R_k(y, f) \|_{p,\alpha} A_\alpha(y) dy \]
\[ \leq c x^{-k} \omega_{p,\alpha}^k(x, f) \int_{-x}^{x} |\Theta_0(x, y)| A_\alpha(y) dy \]
\[ \leq c \frac{\omega_{p,\alpha}^k(x, f)}{x^{k-1}}. \]  
(3.12)

By (3.11) et (3.12), we deduce that
\[ K_{p,\alpha}^k(x, f) \leq c \frac{\omega_{p,\alpha}^k(x, f)}{x^{k-1}}, \]
then, \( f \in K^k \mathcal{D}_{p,\alpha} \) which completes the proof of the theorem. \( \square \)

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