Statistical properties of the attendance time series in the minority game

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We study the statistical properties of the attendance time series corresponding to the number of agents making a particular decision in the minority game (MG). We focus on the analysis of the probability distribution and the autocorrelation function of the attendance over a time interval in the efficient phase of the game. In this regime both the probability distribution and the autocorrelation function are shown to have similar behaviour for time differences corresponding to multiples of $2 \cdot 2^m$, which is twice the number of possible history bit strings in a MG with agents making decisions based on the most recent $m$ outcomes of the game.

The minority game (MG), introduced by Challet and Zheng \cite{1}, represents a simplified version of the El Farol bar-attendance problem \cite{2,3} proposed by Arthur. It gives perhaps the simplest model of a complex adaptive system \cite{4} in which agents of similar capability are competing to be in the minority based on some globally shared information. The model may be useful in investigating many of the features observed in financial markets \cite{5,6}. In its basic form, the MG describes a system (market) in which an odd number $N$ of agents are allowed to make two possible choices in each turn. At each step, agents have to choose either to be in side 0 (buying) or side 1 (selling). After every agent has independently chosen a side, the side with fewer agents (the minority side) is identified as the winning side. The “output” of each time step can be represented by a single binary digit: 0 for side 0 winning, and 1 for side 1 winning. The record of the outputs for the last $m$ steps is the only information given to all agents based on which each agent is to decide which side to take in the next time step. Therefore, there are a total of $2^m$ possible history $m$-bit strings. The whole strategy space thus consists of $2^m$ strategies. The number of strategies grows rapidly with $m$. Fortunately, it has been pointed out \cite{7,8} that a reduced strategy space consisting of $2^m$ pairs of mutually anti-correlated strategies, thus having a total of $2 \cdot 2^m$ strategies, is sufficient to represent the full strategy space. In the beginning of the game, each agent randomly picks $s$ strategies, with repetitions allowed. After each step, each agent assigns (deducts) one (virtual) point to each of his strategies which would have predicted the correct (incorrect) output. At a given time step, each agent decides based on the prediction of the most successful strategy in his bag of $s$ strategies. The past history thus creates a feedback mechanism that leads to adaptivity in the population.

The MG and some of its variations have been the subject of much recent attention \cite{9,10}. One of the most important features in the MG is that the average standard deviation $\sigma$ of the attendance time series in any one of the two sides over different runs in computer simulations could drop to values better than the case in which agents’ decisions are made randomly. For small $s$, $\sigma$ has a minimum as a function of $m$ \cite{11,12}. If the number of strategies in the reduced strategy space is small compared to the total number of agents, i.e., $2 \cdot 2^m \ll N \cdot s$, many agents tend to use the same strategy at any given time step and hence make the same decision. They form a “crowd” that leads to the large standard deviation in the statistics of the attendance. This regime is refereed to as the efficient phase since the agents cannot make use of the information hidden in the history bit-string as hidden information lies in strings longer than $m$ \cite{13,14}. In the large $m$ limit, the strategy pool is much larger than the number of strategies actually in play. The agents are essentially making independent decisions randomly, leading to $\sigma \sim \sqrt{N}/2$ in this random-coin-toss limit. The minimum in $\sigma$ occurs near $2 \cdot 2^m \sim N \cdot s$ in which there are about the same numbers of agents playing a strategy (forming a crowd) and its anticorrelated partner (forming an anti-crowd) \cite{15,16}. It has been found that the MG has very rich statistical features in the efficient regime \cite{17}.

In this work, we focus on the changes in the attendance time series of the MG in the efficient regime. Analogous to the statistical analysis of the log-return in financial time series, we study the statistics of the logarithmic changes in the attendance time series of MG. We also study the autocorrelation function of the logarithmic change time series.

For a given time series of the number of agents $N_0(t)$ making a particular choice, say taking side 0, in the MG, we construct the time series of the successive differences of the natural logarithm of the number of attendance as \cite{18}

$$G_{\Delta t}(t) = \ln N_0(t + \Delta t) - \ln N_0(t),$$

(1)
where $\Delta t$ is the sampling time interval. Such time series of logarithmic changes in prices have been extensively studied in the context of econophysics [21]. A time series can be constructed for a given $\Delta t$. First we look into the statistics of the values of $G_{\Delta t}(t)$ for given $\Delta t$. Figure 1 shows the probability distribution $P$ of $G_{\Delta t}(t)$ for $\Delta t = 1, 5, \text{ and } 8$ for the case of $N = 1001, s = 2$ and $m = 2$. All results are averaged over 32 independent runs. Note that the probability distributions for $\Delta t = 1$ and $\Delta t = 5$ show very clear multi-peak structures, while for $\Delta t = 8$, the peak corresponding to zero-return dominates. Our results for $\Delta t = 1$ are consistent with those in Ref. [21]. In Ref. [21], a plot of $N_0(t+1)$ against $N_0(t)$ was constructed. It was found that the plot consists of patches reminiscent of attractors in a map. The existence of patches implies the existence of dominant values in $G_{\Delta t=1}(t)$ as shown in Fig.1(a). It is related to the virtual point assignments through which the performance of the strategies are rated and the limited adaptability of the agents imposed by a small value of $s$. The present work hence represents an extension of the study in Ref. [21] to other values of $\Delta t$ and investigate the interesting phenomena of a periodicity of $2 \cdot 2^m$ in time series in MG. The results in Fig.1(b) imply that similar patches also appear in the $N_0(t+5)$ against $N_0(t)$ plot, which is given in Fig.2(a). The plot is averaged over different runs and different return maps with different initial time $t$. For $\Delta t = 8$, the sharp peak for zero returns implies the formation of patches concentrated among the diagonal in a plot of $N_0(t+8)$ against $N_0(t)$ (see Fig. 2(b)). We note that the separate return map plotting non-overlapping data given by $N_0(t), N_0(t+8), N_0(t+M), \cdots$ cover only a few patches in Fig.2(b) for given initial value $t$. After investigating the distribution for higher values of $\Delta t$, we found that the probability distribution corresponding to a given $\Delta t$ is identical to that corresponding to $\Delta t+2 \cdot 2^m$. Thus the probability distributions show identical features for different values of $\Delta t$ which are separated by multiples of $2 \cdot 2^m$. For $m = 2$, it implies that $P$ for $\Delta t = 9$ is identical to that in Fig.1 (a) for $\Delta t = 1$. Similar properties are also observed for $m = 3$ and other values of $m$ in the efficient phase for small values of $s$. Figure 3 shows the probability of zero return $R$, which corresponds to the value of the zero-return peak in Fig.1, as a function of $\Delta t$ for $m = 2$ and $m = 3$. It is observed that values for different values of $\Delta t$ which are multiples of $2 \cdot 2^m$ apart are nearly identical. The sharp peaks at $\Delta t = n \cdot 2 \cdot 2^m$, where $n$ is a positive integer, reflect the dominant peak in the probability distribution for $\Delta t = 8$ (see Fig.1(c)). Similarly for $m = 3$, sharp peaks appear at $\Delta t$ equal to multiples of $2 \cdot 2^m = 16$ (see Fig.3(b)).

It is also interesting to look at the autocorrelation function of the log-return time series. For $\Delta t = 1$, the autocorrelation $C(k)$ of $G_1(t)$ is

$$C(k) = \frac{< G_1(t)G_1(t+k) > - < G_1(t) >^2}{< G_1(t)^2 > - < G_1(t) >^2},$$

where $k$ is called the time lag. The averages $< \cdots >$ are taken over the time series. Figure 4 shows the autocorrelation function for $m = 2$ and $m = 3$. Again, periodicity of $2 \cdot 2^m$ is evident.

The statistics of the time series $G_{\Delta t}(t)$ of logarithmic changes in the number of agents making a particular decision in the MG show interesting statistically periodic features with period $T = 2 \cdot 2^m$ in the efficient regime. The period $T = 2 \cdot 2^m$ turns out to be twice the number of possible history bit-strings in a game of given $m$. The observation can be understood qualitatively as the system goes back to a similar situation only after multiples of $2 \cdot 2^m$ time steps [21]. Imagine in the beginning of the game, each agent when encountering a particular history will choose a strategy at random in making the decision as the virtual points are all equal in the beginning. After the outcome is made known, the strategies which gave the correct prediction gain one point. Assuming the game visits each possible $m$-bit history with equal probability, the particular history bit-string will be encountered about $2^m$ time steps later as there are $2^m$ possible history bit-strings. At this time, the agents holding the strategies which predicted correctly in the previous occurrence of the history would make the same decision and due to overcrowding they will lose. Virtual points will then be deducted from these strategies and virtual points will be assigned to the other strategies. After about another $2^m$ time steps, the history will be encountered again. This time all the strategies have similar virtual points and the situation is back to that in the beginning of the game, hence leading to the observation of similar behaviour in the statistics of $G_{\Delta t}(t)$ for $\Delta t$ differs by multiples of $2 \cdot 2^m$. The periodic behaviour hence is a result of the interplay between the memory $m$ of the agents and the limited adaptability when $s$ is small. Difference in the behaviour of the game for a particular history having occurred an odd number and an even number of times has been discussed within the context of strategy selections in MG [22].

The behaviour is also related to the features in the statistics of the occurrence of bit-strings of various lengths in the history, which is a binary series. Close observation on the occurrence of history bit-strings of length $m$ reveals the same periodic behaviour in that the history occurrence almost repeats itself every $2 \cdot 2^m$ turns when $s$ is small, i.e. when the agents have limited adaptability. The periodicity is, however, too long for the agents with memory $m$ to spot it. For larger values of $s$, the periodic behaviour become less obvious. Similarly, it has been pointed out that [22] for MG in the efficient regime, bit-strings of length $m+1$ or shorter occurs evenly while bit-strings of length $k > m+1$ occurs unevenly. For $m = 2, s = 2$, and $N = 1001$, uneven bit-strings distribution arises for $k \geq 4$. Interestingly, for $k = 2 \cdot 2^m = 8$, it is found that the string 11101000 and 11100010 and their permutations dominate the bit-string distribution. It is consistent with the even distribution for $k \leq 3$ as these particular 8-bit strings contain all the possible 3-bit strings. In fact, given that the bit-strings must occur
uniformly up to $k_{\text{max}} = 3$ and there is some periodicity in longer bit-string, the shortest length that is compatible with the requirement is $2^{k_{\text{max}}} = 8$. For $k_{\text{max}} = 3$, the total number of different 3-bit history strings is 8. The bit-string with the shortest length that contains all the eight 3-bit strings once must be of length 8. This result reveals one more time the embedded periodicity of $2 \cdot 2^m$ in the MG in the efficient regime when the agents’ adaptability is limited. For large $m$, the strategy pool is huge and the game approaches the random coin-toss limit in which the agents are efficiently choosing a decision at random in each turn. In this case, the multi-peak structure in the probability distribution of $G_{\Delta t}(t)$ will no longer persist.

In summary, we have studied the statistics of the time series of the successive differences of the natural logarithm of the number of attendance in MG. Interesting behaviour with a period doubling that of the number of possible history bit-strings in the efficient regime are pointed out.

ACKNOWLEDGMENTS

DFZ acknowledges the support from the Natural Science Foundation of Guangdong Province, China. BHW acknowledges the support from the Special Funds for Major State Basic Research Projects in China (973 Project), the National Basic Research Climbing-up Project “Nonlinear Science”, and the National Natural Science Foundation in China (NNSFC) under Key Project Grant No. 19932020 and General Project Grants Nos. 19974039 and 59876039. We would like to thank Dr. P.M. Hui for useful discussions and for a critical reading of the manuscript. This work was initiated during our visit to the Department of Physics at CUHK. The visits were supported in part by a Grant (CUHK4129/98P) from the Research Grants Council of the Hong Kong SAR Government.

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Figure Captions

Figure 1: Probability distribution of $G_{\Delta t}(t)$ for $N = 1001$, $m = 2$ and $s = 2$ with (a) $\Delta t = 1$; (b) $\Delta t = 5$; and (c) $\Delta t = 8$.

Figure 2: (a) The plot of $N_0(t + 5)$ against $N_0(t)$. (b) The plot of $N_0(t + 8)$ against $N_0(t)$.

Figure 3: The probability of zero return $R$ as a function of $\Delta t$ for $N = 1001$ and $s = 2$ with (a) $m = 2$ and (b) $m = 3$.

Figure 4: The autocorrelation function $C(k)$ as a function of $k$ for $N = 1001$ and $s = 2$, with (a) $m = 2$ and (b) $m = 3$. 
Figure 1
Figure 2
Figure 3

(a) $m=2$

(b) $m=3$
Figure 4