MIXING TIMES OF MARKOV CHAINS ON A CYCLE WITH ADDITIONAL LONG RANGE CONNECTIONS

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Abstract. We develop Markov chain mixing time estimates for a class of Markov chains with restricted transitions. We assume transitions may occur along a cycle of $n$ nodes and on $n^\gamma$ additional edges, where $\gamma < 1$. We find that the mixing times of reversible Markov chains present a link between the mixing times of the cycle with no added edges and of the cycle with $cn$ added edges (which is in turn a Small World Network model). In the case of non-reversible Markov-chains, we may experience a significant speedup compared to the reversible case. It is remarkable that such a speedup is not possible in the extreme cases of cycles with no edges or the Small World Network model using $cn$ additional edges.

1. Introduction

Mixing time is an important quantity arising in numerous applications. In Markov Chain Monte Carlo (MCMC) simulations, as described in the now classic papers Metropolis et al. [17], Hastings [10] mixing time can be interpreted as the time needed to generate a single sample of a given distribution with prescribed accuracy. See Jerrum [12] for a modern exposition on the subject.

Another current, hot area of application is the theory of distributed average consensus algorithms. The flow of such an algorithm can be viewed as the evolution of a distributions on some states according to a Markov chain. For details see Olshesky, Tsitsiklis [12] or Boyd et al. [5]. Here the time needed to get within a certain neighborhood of the average value can be quantified by the mixing time. Motivated by these and other applications, the estimation of mixing time is in the center of interest.

The present paper is connected to two previous results. First, a simple random walk on a cycle of $n$ nodes has a mixing time of $cn^2$. This is the result of the central limit theorem which tells us that we need $ck^2$ steps to move away a distance $k$ from the starting point. Second, when $cn$ random edges are added to the cycle, the mixing time drops to $c \log^2 n$, see Durrett [7], Addario-Berry and Lei [1].

Our goal is to investigate the case in between. This time we add some edges to the cycle, but we keep their number negligible compared to the size of the cycle, and develop mixing time estimates for such graphs.

2. Problem formulation

For the sake of completeness we present the definition of mixing time together with some dependent concepts. We will work with discrete time Markov chains on a finite state space $X$ which has size $n = |X|$. The set of probability distributions on the state space $X$ will be denoted by $\mathcal{P}(X)$. In order to define mixing time we need a metric to measure the distance of probability distributions.

One of the widely used options is the total variation distance defined as follows:

**Definition 1.** Given two probability measures $\mu, \nu$ on $X$, the total variation distance is defined as

$$\|\mu - \nu\|_{TV} = \max_{A \subseteq X} |\mu(A) - \nu(A)|.$$  

When considering Markov chain the transition matrix is denoted by $P = (p_{ij})$, with $p_{ij}$ referring to the probability of moving from state $i$ to state $j$, while $\pi$ stands for the unique stationary distribution (if it exists). We can now define the central notion of this paper.

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**Definition 2.** For a Markov chain having a unique stationary distribution we define the mixing time of the chain for any \( \varepsilon > 0 \) as

\[
\tmix(P, \varepsilon) = \max_{\sigma \in P(\mathcal{X})} \min \left\{ k : \| \sigma P^k - \pi \|_{TV} \leq \varepsilon \right\}.
\]

For the rest of the paper \( \varepsilon \) is considered to be fixed thus we use the simplified notation \( \tmix(P) \) or even \( \tmix \) when the argument \( P \) is obvious from the context.

**Definition 3.** The connectivity graph of a Markov chain is a graph on the states of the Markov chain. We connect nodes \( i \neq j \) if either \( p_{ij} > 0 \) or \( p_{ji} > 0 \).

We restrict ourselves to those cases where the unique stationary distribution is uniform. For the transition matrix this translates to the condition of being doubly stochastic.

**Definition 4.** A Markov chain is reversible if starting from the stationary distribution \( \pi \), the probability of the consecutive pair \((i, j)\) is the same as the probability of the consecutive pair \((j, i)\). Formally:

\[
\pi_i p_{ij} = \pi_j p_{ji} \quad \forall i, j.
\]

The usefulness of the separation of reversible and non-reversible Markov chains is widely recognized in the literature. Often it is more convenient to prove certain properties for reversible chains, and there are tighter general bounds on the mixing time for them. The reason to consider also non-reversible chains is the fact that they may deliver much faster mixing than similar reversible chains.

The possible gap between the mixing times of reversible and non-reversible Markov chains is indicated by the following proposition.

**Proposition 5.** For some fixed connectivity graph let \( P \) and \( \tilde{P} \) be the doubly stochastic transition matrices of the best reversible and non-reversible chains, respectively, yielding the smallest mixing times. Then for the respective mixing times we have

\[
\tmix(P) \leq c \tmix(\tilde{P}) \log n.
\]

A simple proof is given in Section 4.

As noted in the introduction one of the starting points is the case when the connectivity graph is a cycle with \( n \) nodes. The mixing time of the symmetric random walk is of the order of \( n^2 \). It is far more complicated to deal with the case when we consider any Markov chain, including non-reversible ones. Still, the order of magnitude of the mixing time does not decrease, as shown by the author [9]:

**Theorem 6.** Consider a Markov chain on a cycle with \( n \) nodes having a doubly stochastic transition matrix \( P \). Then, with some global constant \( C > 0 \) we have

\[
\tmix(P, 1/8) \geq Cn^2.
\]

Our goal is to decrease the mixing time by adding a few more edges to the connectivity graph. It is far from trivial to choose the edges that help the most. We performed numerical optimization to get the best setting of the new edges, but none of the resulting graphs did show any symmetry or structure, but looked random to the human eye.

This drove us to choose the edges randomly. This choice turned out to be fruitful as we get consistently low mixing times with high probability. To put into context, let us mention the case where the number of added edges is large, approximately \( cn \) for some \( c > 0 \) constant. This way we get a model of Small World Networks (SWN), Namely if we add an Erdős-Rényi random graph with edge density \( c/n \) to the cycle we get the model of Newman et al. [18]. This and other similar models were built to model large real networks, see Watts, Strogatz [22], Bollobás, Chung [3]. There is an intensive research activity on SWNs, in particular the mixing time of random walks on them has been widely investigated, see Tahbaz-Salehi and Jadbabaie [2] or Hovareshti, Baras and Gupta [11]. The following result is due to Durrett [7], Addario-Berry and Lei [1]:
Theorem 7. Consider an $n$ node graph from the model of Newman et al. [18]. Then for the symmetric random walk on this graph we have

$$c_1 \log^2 n < t_{\text{mix}} < c_2 \log^2 n$$

asymptotically almost surely (a.a.s.) with some global constants $c_1, c_2 > 0$.

This is a huge gain in speed compared to the mixing time of $n^2$ for the cycle alone.

In this paper we consider graphs where the cycle is modified only by a much smaller number of new edges. Clearly for a random graph we cannot completely exclude some pathological cases. Therefore, we are interested in the typical behavior, and we look for properties that are true asymptotically almost surely (a.a.s.) as the size of the graph goes to infinity. This approach allows us to work with claims that only hold when the graph is large enough. We are interested in the order of the mixing time as $n$ increases but we do not care about constant factors. For that reason, we use $c$ or $c_i$ for constants whose value is unimportant. They might represent different values in each expression. We investigate a general class of Markov chains exhibiting such connectivity graphs, however, we will get sharp bounds only for reversible chains, see Corollary 20 through 23.

The rest of the paper is structured as follows. In Section 3 we describe the connectivity graphs we work with. Section 4 presents the tools used for obtaining the mixing time bounds. In Section 5 we work out the specific estimates required by the preceding section. The main results are deduced in Section 6. We close with conclusions in Section 7.

3. Long range connections on the cycle

We call the newly added edges long range edges to distinguish them from the original ones (which connect nodes that are “close”). Let the target edge density of the added long range edges be $2n^{-\alpha}$ for some parameter $\alpha \in (1, 2)$. We therefore expect $n^{2-\alpha}$ extra edges. We introduce three models for randomly choosing the long range edges:

M1: We take the $2 \lceil n^{2-\alpha} \rceil$ almost equidistant nodes $\{i n^{\alpha-1}/2, \ 0 \leq i < 2 \lceil n^{2-\alpha} \rceil\}$, and add edges corresponding to a random matching on them.

M2: From all possible long range edges we draw a subset of size $\lceil n^{2-\alpha} \rceil$ randomly, uniformly.

M3: For all possible long range edge we randomly decide to include it or not. Each edge is included independently with probability $2n^{-\alpha}$.

In the models M2 and M3 we allow original edges of the cycle to be chosen as long range edges to simplify our discussion.

The coming results depend on the asymptotic growth rate of the number of long range edges, but not on whether we have exactly $\lceil n^{2-\alpha} \rceil$ or $\lfloor n^{2-\alpha} \rfloor$ of them. In this spirit we omit integer rounding operations from now on, this only introduces an asymptotically vanishing multiplicative error, but relieves unnecessary complexity from our formulas.

We mainly consider the case of homogeneous chains which have a simple transition probability structure. At every node, we take a clockwise step with probability $q_c + r$, a counterclockwise one with probability $q_c - r$, use long range edges with probability $q_l/d(\alpha)$ each. Otherwise, we stay put. A random graph may occasionally have nodes with high degree. The role of $d(\alpha)$ is to prevent such nodes from having outgoing transition probabilities summing over 1. The following theorem ensures that this way we define meaningful Markov chains.

Theorem 8. For every $\alpha \in (1, 2)$ there is a $d(\alpha)$ such that for all 3 classes of random graphs M1, M2, M3 there is no node with more than $d(\alpha)$ long range edges a.a.s.

Consequently, assuming $2q_c + q_l \leq 1$, homogeneous chains will be feasible Markov chains a.a.s.

Proof. For graphs from model M1 the statement is straightforward: every node has 0 or 1 long range edge.

Let us now consider a graph from model M3. Take a single node and denote the number of its long range edges by $X$. Clearly it follows a binomial distribution $\text{Binom}(n - 1, 2n^{-\alpha})$. To get an upper bound on
Let us choose with arbitrary $t > 0$. The moment generating function of $X$ is
\[
\mathbb{E}(e^{tX}) = (1 + 2n^{-\alpha}(e^t - 1))^{n-1}.
\]
Let us choose $t = (\alpha - 1) \log n$ to get the following:
\[
\mathbb{E}(e^{tX}) = (1 + 2n^{-\alpha}(n^{\alpha - 1} - 1))^{n-1} \rightarrow e^2
\]
as $n \to \infty$. Now let us fix $d(\alpha) = 2/(\alpha - 1)$ and any $c > e^2$. For large enough $n$ we get
\[
P(X > d(\alpha)) \leq \frac{c}{e^{(\alpha - 1) \log n \cdot d(\alpha)}} \leq \frac{c}{n^{\alpha}}.
\]
The probability of the event that some node has more than $d(\alpha)$ long range edges can be bounded from above by a simple union bound resulting in $c/n$ which tends to 0. With this the claim is proven.

For graphs from the model M2 the number of long range edges of a single node follows a hypergeometric distribution: out of $\binom{n}{2}$ possible long range edges $n^{2-\alpha}$ are marked, we count the number of those within the $n-1$ possible edges of the current node. This is less convenient to estimate than the binomial distribution before. We will use our previously obtained bounds for M3 graphs by showing a special way of generating an M2 graph.

We start with a modified “heavy” M3 graph where the edge probability is increased to $4n^{-\alpha}$. Let the total number of long range edges be $m$. Depending on whether $m$ exceeds $n^{2-\alpha}$ or not, we either discard some edges chosen uniformly from the selected ones, or add some edges chosen uniformly from the unselected ones. This way we get the prescribed number of edges and by symmetry arguments it follows that the final subset is chosen uniformly from all subsets of size $n^{2-\alpha}$.

We know that there is a $d(\alpha)$ such that the initial “heavy” M3 graph has at most $d(\alpha)$ long range edges at every node a.a.s. If we have to discard edges from this graph then this property remains true. We might increase the degree of a node only in the case when we have to add edges. The probability of this to happen is:
\[
P(m < n^{2-\alpha}) < P\left(\left| m - 4n^{-\alpha} \frac{n(n-1)}{2} \right| > \frac{1}{2} n^{2-\alpha}\right) < e^{-\frac{4n^{-\alpha}(1 - 4n^{-\alpha})}{n^{4-2\alpha}}} < cn^{\alpha-2}.
\]
The first inequality is based on the inclusion of the events. To control the deviation of $m$ from its expected value we use Chebyshev’s inequality.

In the end, the probability on the left hand side also vanishes as $n \to \infty$, consequently using the value of $d(\alpha)$ we got for “heavy” M3 graphs the statement of the theorem holds true for M2 graphs.

Once we properly understand the case of homogeneous chains we will check how the mixing time bounds extend to other Markov chains on the graph (whose stationary distribution is still uniform).

4. Tools

A tool we heavily rely on as a proxy to the mixing time is the conductance of a Markov chain, introduced by Jerrum and Sinclair [13]. This is a quantity indicating the capacity of the worst bottleneck of the chain when the state space is split into two parts.

**Definition 9.** The conductance of a Markov chain is
\[
\Phi = \min_{\emptyset \neq S \subseteq \mathcal{X}} \Phi(S) = \min_{\emptyset \neq S \subseteq \mathcal{X}} \frac{Q(S, S^C)}{\pi(S)\pi(S^C)} = \min_{\emptyset \neq S \subseteq \mathcal{X}} \frac{\sum_{i \in S, j \in S^C} \pi_i p_{ij}}{\pi(S)\pi(S^C)},
\]
where $S^C = \mathcal{X} \setminus S$, the complement of the set $S$.

This neat concept has evolved since its birth into different refined variants such as average conductance (see Lovász, Kannan [15]), and blocking conductance (see Kannan, Lovász, Montenegro [14]). This quantity is strongly related to mixing times. The simplest universal inequality relating the conductance of a Markov chain with its mixing time is given in Borgs [4]:
**Proposition 10.** There is a constant $c$ such that for any discrete time Markov chain we have
\[
\frac{c}{\Phi} \leq t_{\text{mix}}.
\]

The constant $c$ depends only on $\varepsilon$, showing up in the definition of the mixing time.

Thus the conductance provides a lower bound for the mixing time. The conductance can also be used to get an upper bound, however, for this we need some additional conditions for the Markov chain. The following classical result is due to Jerrum and Sinclair [20]:

**Theorem 11.** There is a constant $c$ such that for any aperiodic, irreducible, reversible Markov chain the following upper bound for the mixing time holds:
\[
t_{\text{mix}} \leq c \frac{1}{\Phi^2} \log \left( \frac{1}{\pi_*} \right).
\]

Again the constant $c$ depends only on $\varepsilon$, showing up in the definition of the mixing time. The variable $\pi_*$ refers to the lowest value of the stationary distribution, $\pi_* = \min \pi_i$. Currently the stationary distribution is uniform, thus $\pi_* = 1/n$.

Let us also cite the following version of the above theorem due to Lovász and Simonovits [16]. This theorem does not require reversibility, but it assumes that the Markov chain is lazy, i.e. $p_{ii} \geq 1/2$ for all $i$:

**Theorem 12.** There is a universal constant $c$ such that for any aperiodic, irreducible, lazy Markov chain the following upper bound for the mixing time holds:
\[
t_{\text{mix}} \leq c \frac{1}{\Phi^2} \log \left( \frac{1}{\pi_*} \right).
\]

The constant $c$ depends only on $\varepsilon$, showing up in the definition of the mixing time.

The same way we could ensure the feasibility of the random homogeneous chains we can provide laziness. If we choose $q_c, q_i$ such that $2q_c + q_i \leq 1/2$, Theorem 8 shows that the remaining probability to stay put is at least $1/2$ at every node a.a.s.

Now we are ready to prove the proposition comparing the best reversible and non-reversible Markov chains on a graph.

**Proof of Proposition 10** Let us define $P' = (\tilde{P} + \tilde{P}^T)/2$. It is easy to see that $P'$ is the transition matrix of a reversible Markov chain with the same connectivity graph (here we use that the stationary distribution is uniform). Moreover, observe that $\Phi_{P'}(S) = \Phi_{\tilde{P}}(S)$ for any $S \subseteq X$ thus $\Phi_{P'} = \Phi_{\tilde{P}}$. Using Theorem 11 and Proposition 10 this implies
\[
t_{\text{mix}}(P') \leq c_1 \frac{1}{\Phi_{P'}} \log n = c_1 \frac{1}{\Phi_{\tilde{P}}} \log n \leq c_2 t_{\text{mix}}(\tilde{P}) \log n.
\]

The matrix $P'$ might not be the best choice for a reversible transition matrix, but substituting it with a better $P$ just further decreases the left hand side. \hfill \Box

5. Conductance estimates

The next step is bounding the conductances of the Markov chains. First we present a technical tool to simplify the minimization occurring at the calculation of the conductance.

**Lemma 13.** Suppose that $\emptyset \neq S_1, S_2 \subset X$, $S_1 \cap S_2 = \emptyset$ and there is no edge between them. Then we have
\[
\Phi(S_1 \cup S_2) > \min(\Phi(S_1), \Phi(S_2)).
\]

**Proof.**
\[
\Phi(S_1 \cup S_2) = \frac{Q(S_1 \cup S_2, (S_1 \cup S_2)^c)}{\pi(S_1 \cup S_2)\pi((S_1 \cup S_2)^c)} = \frac{Q(S_1, S_2^c) + Q(S_2, S_1^c)}{\pi(S_1) + \pi(S_2)} \frac{1}{\pi((S_1 \cup S_2)^c)}.
\]

The first term is between $Q(S_1, S_2^c)/\pi(S_1)$ and $Q(S_2, S_1^c)/\pi(S_2)$. The second term is strictly greater than both $1/\pi(S_1)$ and $1/\pi(S_2)$, thus the lemma follows. \hfill \Box

We immediately get the following property for the minimizing set.
Theorem 14. The set $S$ minimizing $\Phi(S)$ must be connected.

Proof. Let $S \subset X$ be a disconnected set, $S_1$ one of its connected components. We may use the previous lemma with $S_1$ and $S_2 = S \setminus S_1$ to obtain that $S$ is not be minimizing $\Phi(S)$. $\square$

Let us now present three theorems to determine the exact order of magnitude of the conductance for all three models.

Theorem 15. Consider a graph from model M1. The conductance of the homogeneous chain on this graph satisfies the following inequality a.a.s.:

$$c_1 d(\alpha)^{-1} n^{1-\alpha} < \Phi < c_2 n^{1-\alpha}.$$  

Proof. The upper bound is simple: Let $A$ be one of the $n^{\alpha-1}/2$ long arcs without a long range edge. We can use $\Phi(A)$ to bound the conductance:

$$\Phi = \min_{\emptyset \neq S \subseteq V} \Phi(S) = \Phi(A) = \frac{Q(A, A^C)}{\pi(A)\pi(A^C)} \leq \frac{2n^{-1}}{n^{\alpha-2}/2 \cdot 1/2} = c\alpha^{1-\alpha}.$$

The lower bound is a bit more intricate. Using Corollary we have to minimize over connected subsets to find $\Phi$. Connected subgraphs are composed of a collection of arcs which are connected by long range edges. Let us define a new chain with nodes $\tilde{X}$ as shown in Figure 1. For every node of $X$ with a long range edge there is one node in $\tilde{X}$. Two nodes of $\tilde{X}$ are connected if they are connected in $X$ or if they follow each other on the cycle. In other words, we reduce all long empty arcs to single edges. Clearly the new chain has $2n^{2-\alpha}$ nodes. We use the same homogeneous transition probabilities as before.

**Figure 1.** Reducing M1 graphs

We want to compare the conductance $\Phi$ of the original chain with the conductance $\tilde{\Phi}$ of the new one. For any connected $S \subset X$ we may naturally define $\tilde{S} \subset \tilde{X}$ by keeping only the nodes in $\tilde{X}$. When we want to bound $\Phi(S)$, we may freely swap $S$ with $S^C$ as $\Phi(S) = \Phi(S^C)$. If $|\tilde{S}| > |\tilde{X}|/2$, let us swap $S$ for $S^C$ (and pick one of its connected components if needed). This way we can ensure $|\tilde{S}| \leq |\tilde{X}|/2$. We need to estimate the expressions appearing in $\Phi(S)$. The transition probabilities are the same, the stationary measure changes, thus

$$Q(S, S^C) \geq \frac{2n^{2-\alpha}}{n} Q(\tilde{S}, \tilde{S}^C).$$

For any node in $\tilde{S}$ there are at most the two adjacent empty arcs present in $\tilde{S}$, consequently

$$\pi(S) < 2\pi(\tilde{S}).$$

For the complement set, we made sure $\tilde{S}$ is “small” before so we have

$$\pi(S^C) < 1 \leq 2\pi(\tilde{S}^C).$$
Combining these inequalities we arrive at

\[ \Phi(S) \geq cn^{1-\alpha}\tilde{\Phi}(\tilde{S}), \]

\[ \Phi \geq cn^{1-\alpha}\tilde{\Phi}. \]

(1)

The reduced graph is a cycle with \(2n^{2-\alpha}\) nodes with a random matching added, which is exactly the Bollobás-Chung small world model [3]. The conductance of the symmetric random walk on the Bollobás-Chung model is already known, see e.g. Durrett [7] p. 163-164., where it is shown that it is bounded below by a positive constant. Our reduced chain is slightly different as the long range edges have transition probabilities \(q_l/d(\alpha)\) instead of a global constant. The conductance scales with the transition probabilities, hence for our reduced chain we have

(2) \[ \tilde{\Phi} \geq cd(\alpha)^{-1}. \]

Using this bound together with Equation 1 completes the proof. \(\Box\)

**Theorem 16.** Consider a graph from model M2. The conductance of the homogeneous chain on this graph satisfies the following inequality a.a.s.:

\[ c_1 d(\alpha)^{-1} \frac{n^{1-\alpha}}{\log n} < \Phi < c_2 \frac{n^{1-\alpha}}{\log n}. \]

Proof. To establish an upper bound, we search again for a long arc \(A\) without a long range edge. In this context, adding \(n^{2-\alpha}\) random edges means we cut the cycle into arcs at \(k = 2n^{2-\alpha}\) random points. Asymptotically this is equivalent to splitting the unit interval by \(k-1\) i.i.d. uniform variables (in terms of the resulting lengths). For length \(l\) of the largest gap it is known that

\[ c_1 \log(k-1)/(k-1) < l < c_2 \log(k-1)/(k-1) \]

a.a.s. See Slud [21] or Devroye [6] for details. Therefore the number of nodes in the longest empty arc \(A\) is a.a.s. at least

\[ nl \geq cn \frac{\log k}{k} = cn \frac{(2-\alpha)\log n + \log 2}{2n^{2-\alpha}} = cn^{\alpha-1} \log n + O(n^{\alpha-1}). \]

Consequently we can use a similar estimate as before:

\[ \Phi \leq \frac{Q(A,A^c)}{\pi(A)\pi(A^c)} \leq \frac{2n^{-1}}{cn^{\alpha-2} \log n \cdot 1/2} = \frac{n^{1-\alpha}}{\log n}. \]

For the proof of the lower bound we intend to follow the same idea as for Theorem 15 but a few things have to be updated. First of all, there might be nodes which have multiple long range edges. For the graph on \(\tilde{X}\) we want the long range edges to form a random matching. Thus we include multiple copies of such a node and randomly distribute the long range edges among them, see Figure 2.
We use similar inequalities to those in the proof of Theorem 15. There are still $n^{1-\alpha}$ long range edges, thus the reduced graph has $2n^{1-\alpha}$ nodes again.

$$Q(S, S^C) > 2n^{1-\alpha}Q(\tilde{S}, \tilde{S}^C).$$

This time, the collapsed arcs are not necessarily of the same length. Still, we can use $cn^{\alpha-1}\log n$ as an upper bound as we have shown before. This results in a weakened version of the second inequality:

$$\pi(S) < c\log n \tilde{\pi}(\tilde{S}).$$

For the third inequality we use the same trick as before, swapping $S$ with $S^C$ if necessary to ensure $\tilde{S}^C$ is large. We get again

$$\pi(S^C) < 2\pi(\tilde{S}^C).$$

Joining these inequalities yields

$$\Phi_S \geq 8c\frac{n^{1-\alpha}}{\log n} \tilde{\Phi}_{S^C} \geq c2n^{1-\alpha}\log n.$$ 

We use Equation 2 again for $\tilde{\Phi}$ to conclude the proof. □

**Theorem 17.** Consider a graph from model $M3$. The conductance of the homogeneous chain on this graph satisfies the following inequality a.a.s.:

$$c_1d(\alpha)^{-1}\frac{n^{1-\alpha}}{\log n} < \Phi < c_2\frac{n^{1-\alpha}}{\log n}.$$ 

**Proof.** For this proof we use a more direct approach, partially based on ideas from Durrett [7]. Let us start with the lower bound. For any $S \subset \mathcal{X}$, $|S| \leq n/2$ we have

$$\Phi(S) = \frac{Q(S, S^C)}{\pi(S)\pi(S^C)} \geq \frac{cd(\alpha)^{-1}\frac{|\partial S|}{n}}{\frac{|S|}{n}} = cd(\alpha)^{-1}\frac{|\partial S|}{|S|},$$

where $\partial S$ is the set of edges between $S$ and $S^C$. We have to ensure this is large enough for all possible subsets $S$. Let us fix $s = |S| \leq n/2$ and the number of disjoint intervals $l$ it consists of. We focus at only these subsets at once.

We can estimate the number of possible subsets $k$ in the following way:

$$k \leq \binom{n}{l}\binom{s-1}{l-1} \leq \binom{n}{l}\binom{s}{l}.$$ 

The first binomial coefficient counts how we can choose the starting points of the intervals, the second distributes the total length of $s$ among them. To continue, we use the following inequality:

$$\binom{m}{t} \leq \left(\frac{me}{t}\right)^t.$$

For $k$ this gives us

$$k \leq \left(\frac{me}{l}\right)^l \left(\frac{se}{l}\right)^l \leq \exp\left(l\left(\log \frac{n}{l} + \log \frac{s}{l} + 2\right)\right) < \exp(4l\log n).$$

The outgoing edges from $S$ are partially edges of the cycle at interval boundaries and partially long range edges. We have $2l$ edges at the interval boundaries and the number of long range edges $L$ follows a $\text{Binom}(s(n-s), 2n^{-\alpha})$ distribution. According to Equation 5 a subset violates the conductance bound we proposed if

$$\frac{n^{1-\alpha}}{\log n} \frac{|\partial S|}{|S|} = \frac{L + 2l}{s}.$$
We introduce the new notation \( \zeta \) because its value is important, as we will see. The probability of this violation to happen can be written in the following way:

\[
p = P \left( L < s\frac{n^{1-\alpha}}{\log n} - 2l \right).
\]

Let us introduce the temporary notation \( r = s\frac{n^{1-\alpha}}{\log n} - 2l \). If \( r \leq 0 \), then this probability is 0, that’s the best we can get. If not, then we have the implied inequality

\[
s\frac{n^{1-\alpha}}{\log n} > 2l \log n.
\]

In this case we have to find an upper bound on \( p \). First we refer to basic properties of the Binomial distribution. We use \( s(n-s) > sn/2 \) and the fact that \( r \) is smaller than the expected value of any of the Binomial distributions appearing.

\[
p = P(\text{Binom}(s(n-s), 2n^{-\alpha}) < r) \leq P(\text{Binom}(sn/2, 2n^{-\alpha}) < r)
\]

\[
\leq rP(\text{Binom}(sn/2, 2n^{-\alpha}) = r).
\]

Now we use the bound on binomial coefficients.

\[
p \leq rP(\text{Binom}(sn/2, 2n^{-\alpha}) = r) =
\]

\[
r \left( \frac{sn/2}{r} \right) (2n^{-\alpha})^r \left( 1 - 2n^{-\alpha} \right)^{sn/2-r} \leq r \left( \frac{sne}{2r} \right)^r (2n^{-\alpha})^r (1 - 2n^{-\alpha})^{sn/2-r} \leq
\]

\[
\exp \left( \log r + r \log \frac{sne}{2r} - \alpha r \log n + r \log 2 - \frac{sn}{n^\alpha} + \frac{2r}{n^\alpha} \right).
\]

We would like to find out the asymptotic behavior of this expression and show that the exponent goes to \(-\infty \) fast enough. The term \( \alpha r \log n \) asymptotically dominates \( \log r, 2r/n^\alpha \), and \( r \log 2 \), so these four terms add up to something negative. The second term can be bounded the following way:

\[
r \log \frac{sne}{2r} < r \log (2n^2) < \frac{sn^{1-\alpha}}{\log n} 3 \log n = (3\zeta)sn^{1-\alpha}.
\]

This is dominated by the fifth term \(-sn^{1-\alpha} \) if \( \zeta \) is small enough. In the end we get the following upper bound on \( p \):

\[
p \leq \exp \left( (3\zeta - 1) sn^{1-\alpha} \right).
\]

Now let us collect all subsets \( S \) of \( s \) nodes and \( l \) intervals. The probability that there is one which violates the conductance is at most \( kp \). Using Equation \( 3 \) we have an upper bound for \( k \),

\[
\log k < 4l \log n < (2\zeta)sn^{1-\alpha}.
\]

Let us join our previous estimates. For \( n \) large enough we have

\[
\log(kp) < (2\zeta)sn^{1-\alpha} + (3\zeta - 1) sn^{1-\alpha} = (5\zeta - 1) sn^{1-\alpha}.
\]

For \( \zeta \leq 1/6 \) we get a coefficient at most \(-1/6 \). From Equation \( 3 \) again,

\[
-\frac{1}{6} sn^{1-\alpha} < -\frac{l}{3^\zeta} \log n.
\]

Here we need \( \zeta \leq 1/9 \) to get at most \(-3 \log n \). After all, with the proper \( \zeta \) we end up with

\[
k p < \frac{1}{n^2}.
\]

It is only left to sum over all possible \( s \) and \( l \) values. This introduces an extra \( n^2 \) term, but the probability remains asymptotically 0. In the end we see the lower bound on the conductance is false only with asymptotically vanishing probability.

Let us now turn our attention to the upper bound. If we find an arc \( A \) that is at least \( cn^{\alpha-1} \log n \) long with no long range edges going out of it then we can use the same estimate as before:

\[
\Phi \leq \frac{Q(A, A^C)}{\pi(A)\pi(A^C)} \leq \frac{2n^{-1}}{rn^{\alpha-2} \log n \cdot \frac{1}{2}} = \frac{n^{1-\alpha}}{\log n}.
\]
Again, we have to be careful with the constants. We will search for an arc at least \( \tau n^{\alpha - 1} \log n \) long, and we will specify \( \tau \) later. To do this, let us split the cycle into arcs of length \( b = \tau n^{\alpha - 1} \log n \). We define a graph on these arcs, we connect two of them if there is any long range edge between them. According to the independence of the edges this is in fact an Erdős-Rényi random graph. Our goal translates to finding an isolated node in it.

For a sequence of Erdős-Rényi graphs on \( m \) nodes with edge probability \( q \) it is known [8] that they have isolated nodes a.a.s. if \( m \to \infty \) but \( q < (1 - \varepsilon) \frac{\log m}{m} \) for some fixed \( \varepsilon > 0 \). In our case the number of nodes is

\[
m = \frac{n}{b} = \frac{n^{2 - \alpha}}{\tau \log n}.
\]

We can bound the edge probability in the new graph by adding up the appropriate edge probabilities in the original graph:

\[
q \leq b^2 2^{-2n^{-\alpha}} = 2\tau^2 n^{\alpha - 2} \log^2 n.
\]

We have to compare this quantity with the following:

\[
\frac{\log m}{m} = \tau n^{\alpha - 2} \log n((2 - \alpha) \log n - \log \tau - \log \log n).
\]

The major term is the first one, which is fortunately of the same order as \( q \). In order to have an isolated node a.a.s. we simply need

\[
2\tau^2 < \tau(2 - \alpha),
\]

\[
\tau < 1 - \frac{\alpha}{2}.
\]

There was no other restriction on \( \tau \) apart from being positive so we can choose it to satisfy this last inequality. This concludes the proof. \( \square \)

6. Mixing time bounds

Let us now turn to estimating the mixing time itself. The first result is a lower bound based on a previous result for cycles without added edges.

**Proposition 18.** Consider the graph model M1, let us also assume the nodes with long range edges are equidistant from each other. Then for any homogeneous chain,

\[
 cn^{2n-2} \leq t_{\text{mix}}.
\]

**Proof.** Observe that we can “wind up” the chain around a cycle of \( n^{\alpha - 1}/2 \) nodes so that long range edges become loop edges, see Figure 3. Let us choose any starting distribution on the original chain. It is easy to see that we get the same if we project the starting distribution on the reduced graph and run the Markov

![Figure 3. Reducing M1 graphs](image-url)
chain there or if we run the original Markov chain and project the resulting distribution. Consequently the lower bound from Theorem 6 for the mixing time of the reduced graph is also valid for the mixing time of the original chain.

The same claim is true if the nodes with long range edges are not exactly equidistant, but the Markov chain is reversible.

**Proposition 19.** Consider the graph model M1. Then for any reversible homogeneous chain,
\[ cn^{2\alpha-2} < t_{\text{mix}}. \]

**Proof.** By the definition of the graph model M1 there are arcs at least \( cn^{\alpha-1} \) long without a long range edge. Let us now focus only on one of these arcs. If the Markov chain is initialized at the center of the arc, it stays within the arc for at least \( cn^{2\alpha-2} \) steps with probability \( 1 - \varepsilon \). For small \( \varepsilon > 0 \), this ensures mixing did not yet happen, consequently \( cn^{2\alpha-2} \) is a lower bound on the mixing time.

Using Theorem 12 together with the conductance bound Theorem 15 for M1 graphs we can complement this result with an upper bound.

**Corollary 20.** Consider the graph model M1. The mixing time of the homogeneous lazy chain on such graphs satisfies the following inequality a.a.s.:
\[ c_1 n^{\alpha-1} < t_{\text{mix}} < c_2 d(\alpha)^2 n^{2\alpha-2} \log n. \]

For graphs from the model M2 we formulate bounds separately for reversible and non-reversible Markov chains.

**Corollary 21.** Consider the graph model M2. The mixing time of the reversible homogeneous chain on such graphs satisfies the following inequality a.a.s.:
\[ c_1 n^{2\alpha-2} \log^2 n < t_{\text{mix}} < c_2 d(\alpha)^2 n^{2\alpha-2} \log^3 n. \]

In the case of lazy non-reversible homogeneous chains this changes to
\[ c_1 n^{\alpha-1} \log n < t_{\text{mix}} < c_2 d(\alpha)^2 n^{2\alpha-2} \log^3 n. \]

**Proof.** The upper bounds and the weaker lower bounds follow by combining Theorem 12 with the conductance bounds Theorem 16. The sharper bounds for reversible chains follow the same way as for Proposition 19. This time the longest arc without a long range edge is at least \( cn^{\alpha-1} \log n \) long a.a.s. as shown during the proof of Theorem 16.

In a similar way we can acquire mixing time bounds for graphs from the model M3.

**Corollary 22.** Consider the graph model M3. The mixing time of the reversible homogeneous chain on such graphs satisfies the following inequality a.a.s.:
\[ c_1 n^{2\alpha-2} \log^2 n < t_{\text{mix}} < c_2 d(\alpha)^2 n^{2\alpha-2} \log^3 n. \]

In the case of lazy non-reversible homogeneous chains this changes to
\[ c_1 n^{\alpha-1} \log n < t_{\text{mix}} < c_2 d(\alpha)^2 n^{2\alpha-2} \log^3 n. \]

**Proof.** Again, we use Theorem 12 and the appropriate conductance estimate, now Theorem 17. We also want to reuse the claim based on the existence of a long arc without long range edges. However, during the proof of Theorem 17 we only showed that there is an arc of length \( cn^{\alpha-1} \log n \) such that there is no long range edge going out of it. The long range edges going within the arc are independent from the ones going out, the probability of having none within the arc is
\[ (1 - 2n^{-\alpha})(cn^{\alpha-1} \log n)^2 = (1 - 2n^{-\alpha}) n^\alpha e^{\log^2 n/n^{2-\alpha}} > e^{-3\log^2 n/n^{2-\alpha}}. \]
This is 1 in the limit, consequently the arc we have chosen does not have any long range edge at all a.a.s.
Therefore we can apply the same reasoning as before.

All the previous results are about homogeneous chains. However, we are more interested in the best mixing time we can achieve on the same graphs using different chains. The following Corollary answers this question.

**Corollary 23.** The bounds for non-reversible chains in Corollary 20, 21, 22 also hold for the fastest lazy Markov chains for all models M1, M2, M3.

**Proof.** The mixing time of the fastest chain can not be higher than the mixing time of any specific chain on the same graph. This ensures the upper bound.

To get the lower bound, observe that the long empty arc is still present in the graph. This limits the conductance and thus the mixing time as before.

The bounds we got for reversible chains provide a reasonably tight estimates. For non-reversible chains it is still unclear where the mixing time really is between these bounds.

Still, simulations suggest that there is a considerable gain for non-reversible chains. In Figure 4 we plot the mixing times of homogeneous reversible and non-reversible chains on several graphs coming from model M2 with $\alpha = 1.5$.

![Figure 4. Log-log plot for mixing times of homogeneous M2 chains](image)

This is a log-log scaled histogram using > 70000 random graphs in total. We aim to understand the typical behavior, so we discarded the lowest and highest 5% for each graph size $n$. We see two clusters, the upper one contains the mixing times of all the reversible chains, while the lower one contains that of all non-reversible chains. For comparison, we include two solid lines corresponding to $cn \log^2 n$ and two dashed lines indicating $c\sqrt{n}\log n$.

It is interesting to experience this speedup as it was not observable in the extreme cases. For the cycle without added edges, there is no speedup at all as shown in [9], for Small World Networks the mixing times of reversible Markov chains are already subpolynomial.
7. Conclusions

In the case of reversible chains we have obtained tight bounds on the mixing times for the random graphs of models M1, M2, M3. All these bounds are of the form \( cn^{2\alpha-2} \log^\delta n \) with \( \delta \) differing by one between the lower and upper bound for each specific choice of model parameters. Closing this gap is probably far from obvious. We note that in the limiting case of Small World Networks the final log \( n \) factor between the lower and upper bound has been successfully removed by Addario-Berry and Lei [1] using an involved reasoning. In the dominant part \( n^{2\alpha-2} \) the exponent may take on all possible values between 0 and 2. The limiting case of \( n^0 \) is known to correspond to the case of Small World Network of Newman et al., see [7]. [1], the case of \( n^2 \) corresponds to the case of a plain cycle, see [9].

The situation is more diverse for non-reversible Markov chains. For homogeneous M1 chains with some additional restrictions, we have shown that the mixing time does not decrease compared to reversible Markov chains, having the same \( n^{2\alpha-2} \) lower bound as before. On the other hand, in general the lower bounds drop to \( cn^{\alpha-1} \log^\delta n \). This indicates the possibility of having significantly lower mixing times. This is indeed in line with our experimental findings for M2 chains. This is demonstrated in Figure [4].

A challenging open problem is the clarification of this decrease of mixing times of non-reversible M2 chains. In general, it would be interesting to find other connectivity graphs, where introducing non-reversible Markov chains offers strong speedup compared to reversible ones, without changing the underlying graph.

A further interesting research direction might be to extend the results for time-inhomogeneous Markov chains. In the case of the cycle, when every transition matrix is doubly stochastic and also reversible, it is easy to show that the mixing time is at least of the order of \( n^2 \). However, the case of doubly stochastic but non-reversible transition matrices is still open, it is unclear if the result of [9] can be extended to this case. On the other hand, if we relax the condition on the transition matrices by not requiring them to be doubly stochastic, we can significantly improve the mixing time. In particular it is known that the mixing time can be as low as \( n \).

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