Non-Hermitian skin effect and quantum criticality in bosonic Kitaev-Hubbard models

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Motivated by recent work on the non-Hermitian skin effect in the bosonic Kitaev-Majorana model, we study the interacting bosonic Kitaev-Hubbard models in the hard-core limit for a chain and a two-leg ladder. We show exactly that the non-Hermitian skin effect disappears by transforming hard-core bosonic models to spin-1/2 models. Importantly, we show that hard-core bosons can engineer the Kitaev interaction, the Dzyaloshinskii-Moriya interaction and the compass interaction in the presence of the complex hopping and pairing terms. Quantum phase transitions and quantum criticalities are investigated via the density matrix renormalization group method. This work reveals the effect of many-body interactions on the non-Hermitian skin effect and highlights the power of bosons with pairing terms as a probe for the engineering of interesting models.

I. INTRODUCTION

The engineering of exotic quantum phases is one of major goals in modern physics. The Kitaev chain is a well-known spinless fermionic quadratic model with p-wave superconducting pairing [1], which exhibits Majorana zero modes localized at the ends of the chain [2]. In contrast, for a bosonic single-particle Hamiltonian, Majorana bosons are forbidden by the no-go theorems [3]. Instead, another important quantum phase, the Bose-Einstein condensation, appears near absolute zero for free bosons [4, 5]. Recently, the non-Hermitian skin effect [6] was proposed and investigated in a one-dimensional bosonic quadratic Hamiltonian with pairing terms [3, 7–13] analogous to the fermionic Kitaev chain. This phenomenon is particularly interesting as the original bosonic quadratic Hamiltonian is Hermitian, while the physics of free bosons is characterized by a non-Hermitian Bogoliubov-de Gennes (BdG) Hamiltonian [7, 12] that leads to the non-Hermitian skin effect.

The non-Hermitian skin effect that corresponds to the localization of bulk states at the boundary reveals the breakdown of the bulk-boundary correspondence and leads to the non-Bloch band theory [6]. The non-Hermitian skin effect has attracted much attention as a unique phenomenon of non-Hermitian systems without a counterpart in conventional Hermitian models [6, 14–33]. The understanding of the interplay between the non-Hermiticity and many-body interactions is becoming an important research area for many-body systems. A key issue is whether the non-Hermitian skin effect remains under the many-body interactions. The study on the non-Hermitian skin effect has previously been considered in a few non-Hermitian many-body systems [34–61]. To this end, we investigate the non-Hermitian skin effect in the presence of strong interactions in a Hermitian bosonic model with pairing terms [7].

The Kitaev chain is a noninteracting spinless fermionic model that can be exactly solved featuring a quantum phase transition in the Ising universality class [1]. Moreover, the Kitaev chain with interactions has been investigated via various methods as well [62–65]. On the other hand, low dimensional bosonic systems have attracted great interest during last decades [66, 67]. The competition between the hopping, interactions and the chemical potential leads to a large variety of interesting quantum states of matter [68]. However, interacting Bose-Hubbard models with pairing terms are less explored [69].

In this paper, we introduce a hard-core bosonic Kitaev-Hubbard model in a one-dimensional chain and a two-leg ladder. The non-Hermitian skin effect is exactly proved to vanish for the chain by mapping the hard-core bosons to fermions via the Jordan-Wigner transformation. Surprisingly, a quantum compass model is found as a multi-critical point from the merging the Ising transitions. Moreover, the quantum criticality is studied by the density matrix renormalization group (DMRG) for the hard-core bosonic Kitaev-Hubbard ladder. An interesting continuous phase transition between two ordered

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phases are discovered in addition to the ferromagnet to paramagnet phase transition.

The paper is organized as follows. In Sec.II, we introduce the bosonic Kitaev-Hubbard model in a chain. In Sec.III, we discuss the non-Hermitian skin effect of the bosonic Kitaev-Hubbard chain for free bosons. In Sec.IV, we study the bosonic Kitaev-Hubbard chain in the hard-core limit. In Sec.V, we numerically investigate the hard-core bosonic Kitaev-Hubbard model in a two-leg ladder. In Sec.VI, we summarize our main results.

II. BOSONIC KITAEV-HUBBARD CHAIN

The Kitaev chain is a simple, one dimensional tight-binding model of spinless fermions with both the nearest-neighbor hopping and the pairing term on each bond [1]. The corresponding bosonic Kitaev-Hubbard Hamiltonian as shown in Fig.1(a) with complex amplitudes and on-site interactions can be defined by [7, 69]

\[ H = \sum_{r=1}^{L} (-t e^{i \theta} b_{r} b_{r+1} + \Delta e^{i \phi} b_{r} b_{r+1} + \text{H.c.}) + \sum_{r=1}^{L} U n_{r} (n_{r} - 1) - \sum_{r=1}^{L} \mu n_{r}, \]

where, \( t \geq 0 \) is the hopping amplitude with a gauged phase \( \theta \), \( \Delta \geq 0 \) is the pairing amplitude with a phase \( \phi \) that can be gauged out. \( U \) and \( \mu \) are the strengths of the on-site interaction and the chemical potential, respectively. Here, \( b_{r}^{\dagger} \) (annihilation) operator and the local particle number operator at the \( r \)th site. The periodic boundary condition is imposed as \( b_{L+1} = b_{1} \), where \( L \) is the system size. For \( U = 0 \), the system is a free bosonic model, in which the ground state is the Bose-Einstein condensation at \( \Delta = 0 \). For \( U \rightarrow \infty \), the system is a hard-core bosonic model that can be mapped to a spinless free fermionic model in terms of the Jordan-Wigner transformation. In the following, we will discuss this model in detail.

III. NON-HERMITIAN SKIN EFFECT

Let us firstly review how the non-Hermitian skin effect can appear for free bosons. In this section, we will consider \( t > \Delta \) as the opposite regime for \( t < \Delta \) is dynamically unstable [7]. For \( U = 0 \) and \( \mu = 0 \), Eq.(1) can be explicitly written as

\[ H = \sum_{r=1}^{L} (-t e^{i \theta} b_{r} b_{r+1} + \Delta e^{i \phi} b_{r} b_{r+1} + \text{H.c.}) \]

Using the Fourier transformation,

\[ b_{r} = \frac{1}{\sqrt{L}} \sum_{k} e^{i k r} b_{k}, \]

the model in Eq.(2) can be transformed into momentum space with the Nambu notations,

\[ H = \frac{1}{2} \sum_{k} (b_{k}^{\dagger} b_{-k} - b_{k} b_{-k}^{\dagger}) H_{\text{BdG}}(k), \]

in which the BdG Hamiltonian \( H_{\text{BdG}}(k) \) is,

\[ H_{\text{BdG}}(k) = -2 t \cos \theta \cos k \sigma_{0} + 2 t \sin \theta \sin k \sigma_{0} - 2 \Delta \cos \phi \cos k \sigma_{x} - 2 \Delta \sin \phi \sin k \sigma_{y}. \]

Here \( \sigma_{x}, \sigma_{y}, \sigma_{z} \) are Pauli matrices and \( \sigma_{0} \) is the identity matrix. In this case, the energy spectrum of the Hamiltonian in Eq.(2) is derived by solving the characteristic equation,

\[ \det [\sigma_{x} H_{\text{BdG}}(k) - E(k)] = 0, \]

which yields

\[ E(k) = 2 t \sin \theta \sin k \pm 2 \sqrt{t^{2} \cos^{2} \theta - \Delta^{2} \cos k}. \]

As expected, the energy eigenvalues presented in Eq.(7) are independent on superconducting phase \( \phi \) similar to that in the fermionic Kitaev chain. Note that the energy eigenvalues of the bosonic Kitaev model in Eq.(2) are characterized by the modified BdG Hamiltonian \( H_{\text{BdG}}(k) = \sigma_{x} H_{\text{BdG}}(k) \) instead of the original BdG Hamiltonian \( H_{\text{BdG}}(k) \) because of the bosonic commutation relation [7]. Hence, \( H_{\text{BdG}}(k) \) can in principle become a non-Hermitian matrix, in which the non-hermitian physics may occur in the Hermitian bosonic model with pairing terms [7].

To perceive it, we take the hopping phase \( \theta = \pi/2 \). The corresponding energy eigenvalues \( E(k) \) become complex values \( 2 t \sin k \pm 2 \Delta \cos k \) as shown in Fig.(2), indicating the occurrence of the non-Hermitian skin effect [7].
In contrast, when \( \theta = 0 \), the system in Eq.(2) has a trivial energy spectrum \( E(k) = \pm \sqrt{\mu^2 - \Delta^2 \cos k} \). Consequently, the hopping phase \( \theta \) that cannot be gauged out because of the pairing term \( \Delta \) plays a key role in the emergence of the non-Hermitian skin effect [7]. By comparison, the hopping phase \( \theta \) is less considered in the fermionic Kitaev chain as it may merely shift the phase boundaries of the Ising transition. We will explore it in detail in the next section.

IV. HARD-CORE BOSONIC KITAEV-HUBBARD CHAIN

In the strong-interaction limit (hard-core bosons), the Hilbert space of bosons at each local site can be truncated into two states \(|0\rangle \) and \(|1\rangle \), which can be treated as effective spin-1/2 states. The transformations between bosonic operators \( b_1^\dagger, b_r \) and spin operators \( \sigma^+_r, \sigma^-_r, \sigma^z_r \), can be written as \( \sigma^+_r = b_r^\dagger, \sigma^-_r = b_r, \sigma^z_r = 1 - 2b_r^\dagger b_r \), by identifying \(|0\rangle \rightarrow |\uparrow\rangle \) and \(|1\rangle \rightarrow |\downarrow\rangle \). Then the Hamiltonian in Eq.(1) can be mapped onto a transverse-field spin chain as

\[
H_S = -\frac{1}{2} \sum_{r=1}^{L} \left[ (t \cos \theta + \Delta) \sigma^x_r \sigma^x_{r+1} + (t \cos \theta - \Delta) \sigma^y_r \sigma^y_{r+1} \right] + \frac{1}{2} \sum_{r=1}^{L} t \sin \theta (\sigma^x_r \sigma^y_{r+1} - \sigma^y_r \sigma^x_{r+1}) - \frac{1}{2} \sum_{r=1}^{L} \mu (1 - \sigma^z_r), \tag{8}
\]

where the raising (lowering) operators \( \sigma^\pm_r = \frac{1}{2} (\sigma^x_r \pm i \sigma^y_r) \) have been used. Here, we obtain the anisotropic XY model with the Dzyaloshinskii-Moriya (DM) interaction. This effective spin model in Eq.(8) is transformed into the fermionic Kitaev chain,

\[
H_F = \sum_r (-t e^{i \theta} c_r^\dagger c_{r+1} - \Delta c_r^\dagger c_{r+1} + \text{H.c.}) - \sum_r \mu c_r^\dagger c_r \tag{9}
\]

under the Jordan-Wigner transformation,

\[
\sigma^+_r = \prod_{i=1}^{r-1} (1 - 2c_i^\dagger c_i)c_r, \quad \sigma^-_r = \prod_{i=1}^{r-1} (1 - 2c_i^\dagger c_i)c_r^\dagger, \quad \sigma^z_r = 1 - 2c_r^\dagger c_r. \tag{10}
\]

When \( \theta = 0 \), the fermionic Kitaev chain exhibits topological Majorana bound states in the regime \(-2 < \mu/t < 2\). The phase transition between the topological phase and the trivial phase occurs at \( \mu/t = \pm 2 \) belonging to the Ising universality class.

As the hopping phase \( \theta \) in the fermionic Kitaev chain cannot be gauged out in the presence of the pairing term \( \Delta \), it is expected that the phase \( \theta \) would modify the phase diagram in the fermionic Kitaev chain. We thus derive the energy spectrum of the fermionic Kitaev chain for an arbitrary phase \( \theta \),

\[
E(k) = 2t \sin \theta \sin k \pm \sqrt{(\mu + 2t \cos \theta \cos k)^2 + (2\Delta \sin k)^2}, \tag{11}
\]

The critical point is obtained as \( \mu_c = \pm 2t \cos \theta \) for \( \Delta > t \sin \theta \) by solving the equation \( E(k) = 0 \) [70, 71]. As the energy eigenvalues \( E(k) \) are always real, the non-Hermitian skin effect cannot occur for hard-core bosons. In order to confirm our analysis, we perform the DMRG [72, 73] calculations for the spin chain in Eq.(8) with \( t = \Delta \) under periodic boundary conditions. The critical values \( \mu_c \) and the correlation-length critical exponent \( \nu \) are obtained by the ground-state fidelity susceptibility [74–76],

\[
\chi_{F} = \lim_{\delta \lambda \to 0} \frac{-2 \ln F(\lambda, \lambda + \delta \lambda)}{(\delta \lambda)^2}. \tag{12}
\]

Here \( F(\lambda, \lambda + \delta \lambda) = \langle | \psi(\lambda) | \psi(\lambda + \delta \lambda) \rangle \rangle \) is the ground-state fidelity with a control parameter \( \lambda = t_1, \mu \). For second-order phase transitions, the fidelity susceptibility per site \( \chi_L \equiv \chi_{F}/L \) near the critical point in one dimension scales as [74–76],

\[
\chi_L \propto L^{2/\nu - 1}. \tag{13}
\]

The phase diagram as shown in Fig.3(a) is computed from the fidelity susceptibility [cf. Fig.3(c)] for \( t = \Delta = 1 \) with periodic boundary conditions. Numerical results
are consistent with the analytical values. The phase transition remains the Ising transition from the ferromagnetic (FM) phase to the paramagnetic (Para) phase [cf. Fig.3(b)] with the correlation-length critical exponent $\nu = 1$ [cf. Fig.3(d)] for a finite $\theta$. We note that the hard-core bosonic Kitaev-Hubbard chain with periodic boundary conditions is exactly equivalent to the fermionic counterpart with antiperiodic boundary conditions on a finite system. Interestingly, two Ising transition lines merge at $\mu = 0$, $\theta = \pi/2$, implying that this point might be a multi-critical point, which corresponds to the celebrated compass model with $2L/2-1$ fold degenerate ground states $[77–80]$. To be more explicit, the Hamiltonian of the hard-core bosonic Kitaev-Hubbard chain with $t = \Delta$, $\theta = \pi/2$ and $\mu = 0$ is given by

$$H_S' = -\frac{\Delta}{2} \sum_{r=1}^{L} (\sigma^x_{r} + \sigma^y_{r})(\sigma^x_{r+1} - \sigma^y_{r+1})$$

$$= -\Delta \sum_{r=1}^{L} \tilde{\sigma}^x_{2r} \tilde{\sigma}^x_{2r+1}.$$  \hspace{1cm} (14)

Here, we introduce new spin operators $\tilde{\sigma}^x_{r} = (\sigma^x_{r} - \sigma^y_{r})/\sqrt{2}$ and $\tilde{\sigma}^y_{r} = (\sigma^x_{r} + \sigma^y_{r})/\sqrt{2}$. Performing a transformation only for the even sites $\tilde{\sigma}^x_{2r} \rightarrow \tilde{\sigma}^y_{2r}$, $\tilde{\sigma}^y_{2r} \rightarrow \tilde{\sigma}^x_{2r}$ and $\tilde{\sigma}^x_{2r} \rightarrow \tilde{\sigma}^x_{2r}$, we arrive at the following compass model $[78, 79]$, 

$$H_S' = -\Delta \sum_{r=1}^{L} (\tilde{\sigma}^y_{2r-1} \tilde{\sigma}^y_{2r} + \tilde{\sigma}^y_{2r} \tilde{\sigma}^x_{2r+1})$$  \hspace{1cm} (15)

The high ground-state degeneracy of the hard-core bosonic Kitaev-Hubbard chain at $t = \Delta$, $\theta = \pi/2$ and $\mu = 0$ is numerically verified as well for small systems using exact diagonalization.

V. HARD-CORE BOSONIC KITAEV-HUBBARD LADDER

Having discussed the strongly interacting bosonic Hubbard chain with pairing terms, we now turn to a two-leg ladder as shown in Fig.1(b). Without loss of generality, we will only consider the real hopping matrix for simplicity as the hopping phase $\theta$ would not bring new phases in the regime we discussed except for the special point $\theta = \pi/2$. The corresponding Hamiltonian of the hard-core bosonic Kitaev-Hubbard model on a two-leg ladder is then given by

$$H_L = -\sum_{l=1,2,r=1}^L (tb^\dagger_{l,r}b_{l,r+1} + \Delta b_{l,r}b_{l,r+1} + H.c.)$$

$$-\sum_{r=1}^L (t_{\perp}b^\dagger_{1,r}b^\dagger_{2,r} + H.c.) - \sum_{l=1,2,r=1}^L \mu n_{l,r},$$  \hspace{1cm} (16)

where $b^\dagger_{l,r}$ is the bosonic creation operator at the $l$th leg and the $r$th rung. Here $t$ and $t_{\perp}$ are hopping matrix elements along legs and rungs. The bosons are assumed to be paired only on the legs. In order to understand the phase diagram, we map the hard-core bosonic Kitaev-Hubbard ladder onto the spin ladder, whose Hamiltonian is described by

$$H_{LS} = -\frac{1}{2} \sum_{l=1,2,r=1}^L \left[ (t + \Delta)\sigma^x_{l,r} \sigma^x_{l,r+1} + (t - \Delta)\sigma^y_{l,r} \sigma^y_{l,r+1} \right]$$

$$-\frac{1}{2} \sum_{r=1}^L t_{\perp}(\sigma^x_{1,r} \sigma^x_{2,r} + \sigma^y_{1,r} \sigma^y_{2,r}) - \frac{1}{2} \sum_{r=1}^L \mu (1 - \sigma^z_{1,r}).$$  \hspace{1cm} (17)

When $t_{\perp} = 0$, the system consists of two decoupled bosonic Kitaev chain as we discussed in Sec.IV. The ground state is an XY phase for the $\Delta = 0$ with a U(1) symmetry. When a nonzero $\Delta > 0$ is included, the U(1) symmetry is reduced to the $\mathbb{Z}_2$ symmetry. The ground state becomes a FM phase along the $x$-direction. In the opposite limit $t_{\perp} \rightarrow \infty$, the ground state of the system evolves into a rung-singlet (RS) phase $[81–83]$. In be-
tween, a direct continuous second-order phase transition will take place between the FM phase and the RS phase [81–83]. To verify it, we perform the DMRG calculation with open boundary conditions up to $L = 192$ rungs. The full phase diagram is presented in the Fig.4(a), which is obtained from the fidelity susceptibility in Fig.4(c). In addition to the phase transitions between the FM phase and the Para phase, the system exhibits another second-order phase transition that can be described by bond order parameters $B = (B_{\text{leg}}, B_{\text{rung}})$ presented in Fig.4(b) between FM phase and the RS phase. The nature of the phase transition as shown in Fig.4(d) belongs to the Ising universality class.

Furthermore, when many-body interactions $H_V = V \sum_{r=1}^{L}(n_{1,r} - \frac{1}{2})(n_{2,r} - \frac{1}{2})$ along rungs is considered, the system can be mapped to a spin-1/2 Ising ladder,

$$H_{L1} = H_L + \frac{1}{4}V \sum_{r=1}^{L} \sigma_{1,r}^z \sigma_{2,r}^z.$$  \hspace{1cm} (18)

In the case of $t_\perp = 0$, the system is decoupled into two interacting fermionic Kitaev chains [84]. Especially, the system reduces to the quantum compass ladder that can be solved exactly at $t_\perp = 0$, $\Delta = 1$ and $\mu = 0$ [85]. Hence, bosons with pairing terms offer a simple way to mimic various interesting many-body models.

VI. CONCLUSION

In summary, we have studied the phase diagram and the associated phase transitions for the bosonic Kitaev-Hubbard model on a chain and a two-leg ladder. We show that the one-dimensional interacting bosonic Hubbard chain with pairing terms is identical to the fermionic Kitaev chain and hereby declare that the non-Hermitian skin effect should vanish in the hard-core boson limit.

Moreover, we reveal that in the presence of the hopping phase the Dzyaloshinskii-Moriya interactions can be engineered for hard-core bosons. The ground state of the system remains the doubly degenerate Ising phase with the $Z_2$ symmetry for any $\theta \neq \pi/2$ in the case $\Delta > t \sin \theta$, while the non-Hermitian skin effect can emerge for free bosons by tuning the phase $\theta$. It is surprising that the effective compass model can be realized characteristic of a $2L/2-1$-fold degenerate ground state at $\theta = \pi/2$. In the two-leg ladder, we find a continuous phase transition between the ordered FM phase and the RS phase falling into the Ising universality class as ever. More relevant interesting models are finally discussed. It would be more interesting to investigate two-dimensional many-body systems to explore unusual quantum phases in the future.

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