A Diophantine inequality with four prime variables

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Abstract: Let \(N\) be a sufficiently large real number. In this paper, it is proved that, for \(1 < c < \frac{1093}{889}\), the following Diophantine inequality
\[
|p_1^c + p_2^c + p_3^c + p_4^c - N| < \log^{-1} N
\]
is solvable in prime variables \(p_1, p_2, p_3, p_4\), which improves the result of Mu [14].

Keywords: Diophantine equation; Waring–Goldbach problem; prime variables; exponential sum

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1 Introduction and main result

Let \(k \geq 1\) be a fixed integer and \(N\) a sufficiently large integer. The famous Waring–Goldbach problem is to study the solvability of the following Diophantine equality
\[
N = p_1^k + p_2^k + \cdots + p_r^k
\]
in prime variables \(p_1, p_2, \ldots, p_r\). For linear case, in 1937, Vinogradov [22] proved that every sufficiently large odd integer \(N\) can be written as the sum of three primes. For \(k = 2\), in 1938, Hua [11] proved that the equation (1.1) is solvable for \(r = 5\) and sufficiently large integer \(N\) satisfying \(N \equiv 5 \pmod{24}\).

In 1952, Piatetski-Shapiro [15] studied the following analogue of the Waring–Goldbach problem: Suppose that \(c > 1\) is not an integer, \(\varepsilon\) is a small positive number, and \(N\) is a sufficiently large real number. Denote by \(H(c)\) the smallest natural number \(r\) such that the following Diophantine inequality
\[
|p_1^c + p_2^c + \cdots + p_r^c - N| < \varepsilon
\]

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is solvable in primes $p_1, p_2, \ldots, p_r$, then it was proved in [15] that
\[ \limsup_{c \to +\infty} \frac{H(c)}{c \log c} \leq 4. \]

Also, in [15], Piatetski–Shapiro considered the case $r = 5$ in (1.2) and proved that $H(c) \leq 5$ for $1 < c < 3/2$. Later, the upper bound 3/2 for $H(c) \leq 5$ was improved successively to
\[ \frac{14142}{8923}, \frac{1 + \sqrt{5}}{2}, \frac{81}{40}, \frac{108}{53}, 2.041, \frac{665756}{319965} \]
by Zhai and Cao [23], Garaev [8], Zhai and Cao [24], Shi and Liu [18], Baker and Weingartner [1], Zhang and Li [25], respectively.

From these results and Goldbach–Vinogradov theorem, it is reasonable to conjecture that if $c$ is near to 1, then the Diophantine inequality (1.2) is solvable for $r = 3$. This conjecture was first established by Tolev [20] for $1 < c < \frac{27}{26}$. Since then, the range of $c$ was enlarged to
\[ \frac{15}{14}, \frac{13}{12}, \frac{11}{10}, \frac{237}{214}, \frac{61}{55}, \frac{10}{9}, \frac{43}{36} \]
by Tolev [21], Cai [3], Cai [4] and Kumchev and Nedeva [12] independently, Cao and Zhai [6], Kumchev [13], Baker and Weingartner [2], Cai [5], successively and respectively.

Combining Tolev’s method and the techniques of estimates on exponential sums of Fouvry and Iwaniec, in 2003, Zhai and Cao [23] proved that $H(c) \leq 4$ for $1 < c < \frac{81}{68}$. Later, the range of $c$ for $H(c) \leq 4$ was enlarged to $1 < c < \frac{97}{81}$ by Mu [14].

In this paper, motivated by [5], we shall continue to improve the result of Mu and establish the following theorem.

**Theorem 1.1** Suppose that $1 < c < \frac{1193}{889}$, then for any sufficiently large real number $N$, the following Diophantine inequality
\[ |p_1^c + p_2^c + p_3^c + p_4^c - N| < \log^{-1} N \]
(1.3)
is solvable in primes $p_1, p_2, p_3, p_4$.

**Remark** In order to compare our result with the results of Mu [14] and Zhai and Cao [23], we list the numerical result as follows
\[ \frac{1193}{889} = 1.341957255 \cdots ; \quad \frac{97}{81} = 1.197530864 \cdots ; \quad \frac{81}{68} = 1.191176471 \cdots . \]

**Notation.** Throughout this paper, we suppose that $1 < c < \frac{1193}{889}$. Let $p$, with or without subscripts, always denote a prime number. $\eta$ always denotes an arbitrary
small positive constant, which may not be the same at different occurrences; \(N\) always denotes a sufficiently large real number. As usual, we use \(\Lambda(n)\) to denote von Mangoldt’s function; \(e(x) = e^{2\pi ix}\); \(f(x) \ll g(x)\) means that \(f(x) = O(g(x))\); \(f(x) \asymp g(x)\) means that \(f(x) \ll g(x) \ll f(x)\).

We also define

\[
X = \frac{1}{2} \left(\frac{2N}{5}\right)^{1/4}, \quad \varepsilon = \log^{-2} X, \quad K = \log^{10} X, \quad \tau = X^{1-c-\eta},
\]

\[
S(x) = \sum_{X < p \leq 2X} (\log p) e(p^c x), \quad I(x) = \int_{X}^{2X} e(t^c x) dt, \quad T(x) = \sum_{X < n \leq 2X} e(n^c x).
\]

2 Preliminary Lemmas

In this section, we shall give some preliminary lemmas, which are necessary in the proof of Theorem 1.1.

**Lemma 2.1** Let \(a, b\) be real numbers, \(0 < b < a/4\), and let \(r\) be a positive integer. Then there exists a function \(\phi(y)\) which is \(r\) times continuously differentiable and such that

\[
\begin{cases}
\phi(y) = 1, & \text{if } |y| \leq a - b, \\
0 < \phi(y) < 1, & \text{if } a - b < |y| < a + b, \\
\phi(y) = 0, & \text{if } |y| \geq a + b,
\end{cases}
\]

and its Fourier transform

\[
\Phi(x) = \int_{-\infty}^{+\infty} e(-xy) \phi(y) dy
\]

satisfies the inequality

\[
|\Phi(x)| \leq \min \left(2a, \frac{1}{\pi|x|}, \frac{1}{\pi|x|} \left(\frac{r}{\pi|x|b}\right)^r\right).
\]

**Proof.** See Piatetski–Shapiro [15] or Segal [17].

**Lemma 2.2** Let \(\mathcal{L}, Q \geq 1\) and \(z_\ell\) be complex numbers. Then we have

\[
\left| \sum_{\mathcal{L} < \ell \leq 2\mathcal{L}} z_\ell \right|^2 \leq \left(2 + \frac{\mathcal{L}}{Q}\right) \sum_{|q| < Q} \left(1 - \frac{|q|}{Q}\right) \sum_{\mathcal{L} < \ell + q, \ell - q \leq 2\mathcal{L}} z_{\ell + q} z_{\ell - q}.
\]

**Proof.** See Lemma 2 of Fouvry and Iwaniec [7].
Lemma 2.3 Let \( f(x) \) be a real differentiable function such that \( f'(x) \) is monotonic, and \( |f'(x)| \geq m > 0 \), throughout the interval \([a, b]\). Then we have

\[
\left| \int_a^b e^{f(x)} \, dx \right| \ll \frac{4}{m}.
\]

Proof. See Lemma 4.2 of Titchmarsh [19].

Lemma 2.4 Suppose that \( f(x) : [a, b] \to \mathbb{R} \) has continuous derivatives of arbitrary order on \([a, b]\), where \( 1 \leq a < b \leq 2a \). Suppose further that

\[
|f^{(j)}(x)| \asymp \lambda_j a^{1-j}, \quad j \geq 1, \quad x \in [a, b].
\]

Then for any exponential pair \((\kappa, \lambda)\), we have

\[
\sum_{a < n \leq b} e(f(n)) \ll \lambda_1^\kappa a^\lambda + \lambda_1^{-1}.
\]

Proof. See (3.3.4) of Graham and Kolesnik [9].

Lemma 2.5 For \( 1 < c < 2 \), we have

\[
\begin{align*}
\int_{\tau < |x| < K} |S^2(x)\Phi(x)| \, dx &\ll X^{1+\eta}, \\
\int_{\tau < |x| < K} |S^4(x)\Phi(x)| \, dx &\ll X^{4-c+\eta}, \\
\int_{-\tau}^{+\tau} |S(x)|^2 \, dx &\ll X^{2-c} \log^3 X, \\
\int_{-\tau}^{+\tau} |I(x)|^2 \, dx &\ll X^{2-c} \log^3 X.
\end{align*}
\]

Proof. For (2.1) and (2.2), one can see Lemma 2.6 of Mu [14]. For (2.3) and (2.4), one can see Lemma 7 of Tolev [21].

Lemma 2.6 For \( 1 < c < 2 \), then for \( |x| \leq \tau \) we have

\[
S(x) = I(x) + O\left( X \exp \left( - (\log X)^{1/5} \right) \right).
\]

Proof. See Lemma 4 of Zhai and Cao [23].

Lemma 2.7 For \( 1 < c < 2 \), we have we have

\[
\int_{-\infty}^{+\infty} I^4(x)\Phi(x)e(-Nx) \, dx \gg \varepsilon X^{4-c}.
\]
Proof. See Lemma 8 of Zhai and Cao [23].

Lemma 2.8 Let \( \alpha, \beta \in \mathbb{R} \) with \( \alpha\beta(\alpha - 1)(\beta - 1)(\alpha - 2)(\beta - 2) \neq 0, F > 0, M \geq 1, L \geq 1, |a_m| \leq 1, |b_{\ell}| \leq 1 \). Then we have

\[
(FML)^{-\eta} \sum_{M < m \leq 2M} \sum_{L < \ell \leq 2L} a_m b_{\ell} e\left(F \frac{m^\alpha \ell^\beta}{M^\alpha L^\beta}\right) \\
\ll (F^4 M^{31} L^{34})^{\frac{1}{32}} + (F^6 M^{33} L^{51})^{\frac{1}{36}} + (F^6 M^{46} L^{41})^{\frac{1}{36}} + (F^2 M^{38} L^{29})^{\frac{1}{36}} + (F M^9 L^6)^{\frac{1}{36}} \\
+ (F^2 M^7 L^6)^{\frac{1}{36}} + (F^3 M^{43} L^{32})^{\frac{1}{36}} + (F M^6 L^6)^{\frac{1}{36}} + M^2 L + ML^2 + F^{-\frac{1}{2}} ML.
\]

Proof. See Theorem 9 of Sargos and Wu [16].

Lemma 2.9 Let \( 3 < U < V < Z < X \) and suppose that \( Z - \frac{1}{2} \in \mathbb{N}, X \gg Z^2 U, Z \gg U^2, V^3 \gg X \). Assume further that \( F(n) \) is a complex–valued function such that \( |F(n)| \leq 1 \). Then the sum

\[
\sum_{X < n \leq 2X} \Lambda(n) F(n)
\]

can be written into \( O(\log^{10} X) \) sums, each of which either of Type I:

\[
\sum_{M < m \leq 2M} a(m) \sum_{L < \ell \leq 2L} F(m\ell)
\]

with \( L \gg Z \), where \( a(m) \ll m^\eta, ML \asymp X \), or of Type II:

\[
\sum_{M < m \leq 2M} a(m) \sum_{L < \ell \leq 2L} b(\ell) F(m\ell)
\]

with \( U \ll M \ll V \), where \( a(m) \ll m^\eta, b(\ell) \ll \ell^\eta, ML \asymp X \).

Proof. See Lemma 3 of Heath–Brown [10].

Lemma 2.10 Suppose that \( \tau < |x| < K, M \ll X^{\frac{2071}{10068}}, a(m) \ll m^\eta, ML \asymp X \), then we have

\[
S_I(M, L) := \sum_{M < m \leq 2M} \sum_{L < \ell \leq 2L} a(m) e(xm^\ell L^\epsilon) \ll X^{\frac{2071}{10068} + \eta}.
\]

Proof. If \( M \ll X^{\frac{4961}{10068}} \), then by Lemma 2.4 with the exponential pair \( (\kappa, \lambda) = A^2 B(0, 1) = (\frac{1}{14}, \frac{1}{14}) \), we deduce that

\[
S_I(M, L) \ll X^\eta \sum_{M < m \leq 2M} \left| \sum_{L < \ell \leq 2L} e(xm^\ell L^\epsilon) \right| \\
\ll X^\eta \sum_{M < m \leq 2M} \left| \left( |x| X^\epsilon L^{-1} \right)^{\frac{1}{14}} L^{\frac{1}{14}} + \frac{1}{|x| X^\epsilon L^{-1}} \right|
\]

5
\[
\ll X^\eta \left( K \frac{1}{\pi} X \frac{1}{\pi} ML \frac{\tau}{\phi} + \tau^{-1} X^{1-c} \right) \\
\ll X^{\frac{1}{\tau^2} + \eta} \frac{1}{\phi} \ll X^{\frac{2911}{2667} + \eta}.
\]

If \( X^{\frac{4961}{10668}} \ll M \ll X^{\frac{2071}{8337}} \), then by Lemma 2.8 with \((m, \ell) = (m, \ell)\), we obtain

\[
S_I(M, L) \ll X^{\frac{2911}{2667} + \eta},
\]

which completes the proof of Lemma 2.10.

**Lemma 2.11** Suppose that \( \tau < |x| < K, X^{\frac{304}{2667}} \ll M \ll X^{\frac{1147}{2667}}, a(m) \ll m^\eta, b(\ell) \ll \ell^\eta, ML \ll X \). Then we have

\[
S_{II}(M, L) := \sum_{M < m \leq 2M} \sum_{L < \ell \leq 2L} a(m)b(\ell)e(xm^\ell c^\ell) \ll X^{\frac{2911}{2667} + \eta}.
\]

**Proof.** Taking \( Q = X^{\frac{304}{2667}}(\log X)^{-1} \), if \( X^{\frac{304}{2667}} \ll M \ll X^{\frac{1147}{2667}} \), by Cauchy’s inequality and Lemma 2.2, we deduce that

\[
S_{II}(M, L) \ll \left( \sum_{L < \ell \leq 2L} |b(\ell)|^2 \right)^{\frac{1}{2}} \left( \sum_{M < m \leq 2M} a(m)e(xm^\ell c^\ell) \right)^2 \ll L^{\frac{1}{\tau^2} + \eta} \left( \sum_{L < \ell \leq 2L} \frac{M}{Q} \sum_{0 \leq q < Q} \left( 1 - \frac{q}{Q} \right) \right) \ll L^{\frac{1}{\tau^2} + \eta} \left( \frac{M}{Q} \sum_{L < \ell \leq 2L} \left( M^{1+\eta} + \sum_{1 \leq q < Q} \left( 1 - \frac{q}{Q} \right) \right) \right) \ll L^{\frac{1}{\tau^2} + \eta} \left( \frac{X^2}{Q} + \frac{X}{Q} \sum_{1 \leq q < Q} \sum_{M < m \leq 2M} \sum_{L < \ell \leq 2L} e(x\ell^c((m+q)^c - (m-q)^c)) \right)^{\frac{1}{2}}. \tag{2.5}
\]

Therefore, it is sufficient to estimate the inner sum

\[
\mathcal{S}_0 := \sum_{L < \ell \leq 2L} e(x\ell^c((m+q)^c - (m-q)^c)).
\]

From Lemma 2.4 with the exponential pair \((\kappa, \lambda) = AB(0, 1) = (\frac{1}{6}, \frac{2}{3})\), we have

\[
\mathcal{S}_0 \ll (|x|X^{c-1}q)^{\frac{1}{4}} L^{\frac{1}{2}} + \frac{1}{|x|X^{c-1}q}. \tag{2.6}
\]
Putting the estimate (2.6) into (2.5), we deduce that
\[
S_{II}(M, L) \ll X^\nu \left( \frac{X^2}{Q} + \frac{X}{Q} \sum_{1 \leq q < Q} \sum_{M < m \leq 2M} \left( \left( |x|X^{c-1}q \right)^{\frac{\nu}{2}} + \frac{1}{|x|X^{c-1}q} \right) \right)^{1/2}
\]
\[
\ll X^\nu \left( \frac{X^2}{Q} + \frac{X}{Q} \left( K^{1/2} \frac{X}{\log X} L^{\frac{\nu}{2}} M Q^{\frac{\nu}{2}} + \tau^{-1} X^{1-c} M \log Q \right) \right)^{1/2}
\]
\[
\ll (X^{2+\eta} Q^{-1})^{\frac{1}{2}} \ll X^{\frac{2515}{2667} + \eta},
\]
which completes the proof of Lemma 2.11.

\[\boxed{-}\]

Lemma 2.12 For \(1 < c < \frac{1193}{889}\) and \(\tau < |x| < K\), we have
\[
S(x) \ll X^{\frac{2515}{2667} + \eta}.
\]

\[\textbf{Proof.}\] Trivially, we have
\[
S(x) = \Omega(x) + O(X^{1/2}), \tag{2.7}
\]
where
\[
\Omega(x) = \sum_{X < n \leq 2X} \Lambda(n) e(n^c x).
\]

Taking \(U = X^{\frac{104}{2667}}, V = X^{\frac{1147}{2667}}, Z = \left[X^{\frac{104}{2667}}\right] + \frac{1}{2}\) in Lemma 2.9, it is not difficult to see that the sum
\[
\sum_{X < n \leq 2X} \Lambda(n) e(n^c x)
\]
can be written into \(O(\log^{10} X)\) sums, each of which either of Type I:
\[
S_I(M, L) = \sum_{M < m \leq 2M} \sum_{L < \ell \leq 2L} a(m) e(xm^c \ell^c)
\]
with \(L \gg Z, a(m) \ll m^\eta, ML \asymp X\), or of Type II:
\[
S_{II}(M, L) = \sum_{M < m \leq 2M} \sum_{L < \ell \leq 2L} a(m)b(\ell) e(xm^c \ell^c)
\]
with \(U \ll M \ll V, a(m) \ll m^\eta, b(\ell) \ll \ell^\eta, ML \asymp X\). For the sums of Type I, since \(L \gg Z\) and \(ML \asymp X\), we get \(M \ll X^{\frac{2515}{2667}}\). By Lemma 2.10, we have \(S_I(M, L) \ll X^{\frac{2515}{2667} + \eta}\). For the sums of Type II, by Lemma 2.11, we get \(S_{II}(M, L) \ll X^{\frac{2515}{2667} + \eta}\). Thus, we deduce that
\[
\sum_{X < n \leq 2X} \Lambda(n) e(n^c x) \ll X^{\frac{2515}{2667} + \eta}. \tag{2.8}
\]

From (2.7) and (2.8), we finish the proof of Lemma 2.12.

\[\boxed{-}\]
3 Proof of Theorem 1.1

In this section, we use $\Phi(x)$ and $\phi(y)$ to denote the functions which appear in Lemma 2.1 with parameter $a = \frac{9}{10}$, $b = \frac{10}{11}$, $r = \lfloor \log X \rfloor$. Define

$$B_4(N) = \sum_{X < p_1, p_2, p_3, p_4 \leq 2X} \prod_{j=1}^{4} \log p_j, \quad \text{Subject to } |p_1 + \cdots + p_4 - N| < \varepsilon$$

From the property of $\phi(y)$, we get $B_4(N) \geq C_4(N)$, where $C_4(N) = \sum_{X < p_1, p_2, p_3, p_4 \leq 2X} \prod_{j=1}^{4} \log p_j \cdot \phi(p_1^c + \cdots + p_4^c - N)$.

From the Fourier transformation formula, we derive that

$$C_4(N) = \sum_{X < p_1, \ldots, p_4 \leq 2X} \left( \prod_{j=1}^{4} \log p_j \right) \int_{-\infty}^{+\infty} e^{(p_1^c + \cdots + p_4^c - N)y} \Phi(y) dy = \int_{-\infty}^{+\infty} S^4(x) \Phi(x) e(-Nx) dx$$

$$= \left( \int_{|x| \leq \tau} + \int_{\tau < |x| < K} + \int_{|x| \geq K} \right) S^4(x) \Phi(x) e(-Nx) dx$$

$$= C_4^{(1)}(N) + C_4^{(2)}(N) + C_4^{(3)}(N), \quad \text{say.} \quad (3.1)$$

3.1 The Estimate of $C_4^{(1)}(N)$

Define

$$H_4(N) = \int_{-\infty}^{+\infty} I^4(x) \Phi(x) e(-Nx) dx, \quad H_\tau(N) = \int_{-\tau}^{+\tau} I^4(x) \Phi(x) e(-Nx) dx.$$  

From Lemma 2.1 and Lemma 2.3, we derive that

$$|H_4(N) - H_\tau(N)| \ll \int_{\tau}^{+\infty} |I(x)|^4 |\Phi(x)| dx \ll \varepsilon \int_{\tau}^{+\infty} \left( \frac{1}{|x|X^{c-1}} \right)^4 dx \ll \varepsilon X^{4-c-\eta}. \quad (3.2)$$

From Lemma 2.5, Lemma 2.6 and the trivial estimate $S(x) \ll X$, we get

$$|C_4^{(1)}(N) - H_\tau(N)| \ll \int_{-\tau}^{+\tau} |S^4(x) - I^4(x)||\Phi(x)| dx$$

$$\ll \varepsilon \cdot \int_{-\tau}^{+\tau} |S(x) - I(x)||(S(x)|^3 + |I(x)|^3) dx$$
\[ \ll \varepsilon \cdot X \exp \left( - \frac{1}{3} \log X \right) \left( \int_{-\tau}^{+\tau} |S(x)|^3 \, dx + \int_{-\tau}^{+\tau} |I(x)|^3 \, dx \right) \]
\[ \ll \varepsilon X^{4-c} \exp \left( - \frac{1}{6} \log X \right). \] (3.3)

It follows from Lemma 2.7, (3.2) and (3.3) that
\[ \mathcal{C}_4^{(1)}(N) + \mathcal{H}_r(N) + (\mathcal{H}_r(N) - \mathcal{H}_4(N)) \gg \varepsilon X^{4-c}. \] (3.4)

### 3.2 The Estimate of \( \mathcal{C}_4^{(2)}(N) \)

According to the definition of \( \mathcal{C}_4^{(2)}(N) \), we obtain
\[
|\mathcal{C}_4^{(2)}(N)| = \left| \sum_{X < p < 2X} (\log p) \int_{|\tau| < |x| < K} e(p^\varepsilon x) S^3(x) \Phi(x) e(-N x) \, dx \right|
\leq \sum_{X < p < 2X} (\log p) \left| \int_{|\tau| < |x| < K} e(p^\varepsilon x) S^3(x) \Phi(x) e(-N x) \, dx \right|
\ll (\log X) \sum_{X < n < 2X} \left| \int_{|\tau| < |x| < K} e(n^\varepsilon x) S^3(x) \Phi(x) e(-N x) \, dx \right|.
\]

By Cauchy’s inequality, we deduce that
\[
|\mathcal{C}_4^{(2)}(N)| \ll X^{\frac{1}{2}} (\log X) \left( \sum_{X < n < 2X} \left| \int_{|\tau| < |x| < K} e(n^\varepsilon x) S^3(x) \Phi(x) e(-N x) \, dx \right| \right)^{\frac{1}{2}}
\]
\[
= X^{\frac{1}{2}} (\log X) \left( \sum_{X < n < 2X} \int_{|\tau| < |x| < K} e(n^\varepsilon x) S^3(x) \Phi(x) e(-N x) \, dx \right)
\times \left( \int_{|\tau| < |y| < K} e(n^\varepsilon y) S^4(y) \Phi(y) e(-N y) \, dy \right)^{\frac{1}{2}}
\]
\[
= X^{\frac{1}{2}} (\log X) \left( \left[ \int_{|\tau| < |y| < K} S^3(y) \Phi(y) e(-N y) \, dy \right] \left[ \int_{|\tau| < |x| < K} S^3(x) \Phi(x) e(-N x) T(x-y) \, dx \right] \right)^{\frac{1}{2}}
\ll X^{\frac{1}{2}} (\log X) \left( \left[ \int_{|\tau| < |y| < K} S^3(y) \Phi(y) \, dy \right] \left[ \int_{|\tau| < |x| < K} S^3(x) \Phi(x) T(x-y) \, dx \right] \right)^{\frac{1}{2}}.
\] (3.5)

For the inner integral in (3.5), we get
\[
\int_{|\tau| < |x| < K} |S^3(x) \Phi(x) T(x-y)| \, dx
\ll \int_{|\tau| < |x| < K} |S^3(x) \Phi(x) T(x-y)| \, dx + \int_{|x-y| < 2K} |S^3(x) \Phi(x) T(x-y)| \, dx.
\] (3.6)
From Lemma 2.12 and the trivial estimate $T(x - y) \ll X$, we get
\[
\int_{\tau < |x| < K} |S^3(x)\Phi(x)T(x - y)| \, dx \\
\ll \varepsilon X \times \sup_{\tau < |x| < K} |S(x)|^3 \times \int_{\tau < |x| < K} \frac{1}{|x - y|^{c - \varepsilon}} \, dx \\
\ll \varepsilon X \cdot X^{\frac{2515}{8899} - c + \eta} \ll \varepsilon X^{\frac{3494}{8899} - c + \eta}.
\] (3.7)

According to Lemma 2.4, for $X^{-c} < |x - y| \leq 2K$, we get
\[
T(x - y) \ll (|x - y|X^{-c})^6 X^\lambda + \frac{1}{|x - y|^{c - 1}} \\
\ll |x - y|^\kappa X^{\kappa + \lambda - \kappa} + \frac{1}{|x - y|^{c - 1}}.
\] (3.8)

By choosing
\[
(\kappa, \lambda) = BA^2BA^2BABABA^2BABAB(0, 1) = \left(\frac{1731}{4492}, \frac{591}{1123}\right)
\]
in (3.8), we deduce that
\[
T(x) \ll |x - y|^{\frac{1731}{4492}X^{\frac{1731c + 633}{4492}}} + \frac{1}{|x - y|^{c - 1}}.
\] (3.9)

On the other hand, by Lemma 2.5 and Cauchy’s inequality, we obtain
\[
\int_{\tau < |x| < K} |S^3(x)\Phi(x)| \, dx \\
\ll \left(\int_{\tau < |x| < K} |S^2(x)\Phi(x)| \, dx\right)^{\frac{1}{2}} \left(\int_{\tau < |x| < K} |S^4(x)\Phi(x)| \, dx\right)^{\frac{1}{2}} \\
\ll (X^{1 + \eta})^{\frac{1}{2}} \cdot (X^{4 - c + \eta})^{\frac{1}{2}} \ll X^{\frac{5}{2}c + \eta}.
\] (3.10)

By (3.9), (3.10) and Lemma 2.12, we have
\[
\int_{\tau < |x| < K} |S^3(x)\Phi(x)T(x - y)| \, dx \\
\ll \int_{X^{-c} < |x - y| \leq 2K} |S^3(x)\Phi(x)| \left(|x - y|^{\frac{1731}{4492}X^{\frac{1731c + 633}{4492}}} + \frac{1}{|x - y|^{c - 1}}\right) \, dx \\
\ll X^{\frac{1731c + 633}{4492} + \eta} \int_{\tau < |x| < K} |S^3(x)\Phi(x)| \, dx \\
+ \varepsilon X^{1 - c} \times \sup_{\tau < |x| < K} |S(x)|^3 \times \int_{X^{-c} < |x - y| \leq 2K} \frac{1}{|x - y|} \, dx \\
\ll X^{\frac{1731c + 633}{4492} + \eta} \cdot X^{\frac{5}{2}c + \eta} + \varepsilon X^{1 - c} \cdot X^{\frac{2515}{8899} + \eta} \\
\ll X^{\frac{11868}{4492} + \eta} + \varepsilon X^{\frac{2404}{8899} - c + \eta} \ll \varepsilon X^{\frac{3494}{8899} - c + \eta}.
\] (3.11)
From (3.6), (3.7) and (3.11), we get
\[ \int_{r<|x|<K} |S^3(x)\Phi(x)|T(x-y)|dx \ll \varepsilon X^{\frac{1}{3}+\eta}, \]
from which and (3.10), we can conclude that
\[ |C^{(2)}_4(N)| \ll X^\frac{1}{7}(\log X)\left(X^{\frac{1}{3}+\eta} \cdot \varepsilon X^{\frac{1}{3}+\eta}\right)^\frac{1}{2} \ll \varepsilon X^{4-c-\eta}. \quad (3.12) \]

### 3.3 The Estimate of \( C^{(3)}_4(N) \)

According to Lemma 2.1, we have
\[
|C^{(3)}_4(N)| \ll \int_{K} \left|S(x)\right|^4|\Phi(x)|dx \ll X^4 \int_{K} \frac{1}{\pi |x|} \left(\frac{r}{2\pi |x|b}\right)^r dx \\
\ll X^4 \left(\frac{r}{2\pi b}\right)^r \int_{K} \frac{dx}{x^{r+1}} \ll \frac{X^4}{r^r} \left(\frac{r}{2\pi K b}\right)^r \\
\ll \frac{X^4}{\log X} \left(\frac{1}{2\pi \log^2 X}\right)^{\log X} \ll \frac{X^4}{X^7 \log \log X + \log(2\pi)} \ll 1. \quad (3.13)\]

### 3.4 Proof of Theorem 1.1

From (3.1), (3.4), (3.12) and (3.13), we deduce that
\[ C_4(N) = C^{(1)}_4(N) + C^{(2)}_4(N) + C^{(3)}_4(N) \gg \varepsilon X^{4-c}, \]
and thus
\[ B_4(N) \gg C_4(N) \gg \varepsilon X^{4-c} \gg \frac{X^{4-c}}{\log^2 X}, \]
which completes the proof of Theorem 1.1.

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