GLOBAL MAGNETOHYDRODYNAMICAL SIMULATIONS OF ACCRETION TORI

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ABSTRACT

Global time-dependent simulations provide a means to investigate time-dependent dynamic evolution in accretion disks. This paper seeks to extend previous local simulations by beginning a systematic effort to develop fully global three-dimensional simulations. The nonlinear development of the magnetorotational instability is investigated using a time-explicit finite difference code written in cylindrical coordinates. The equations of ideal magnetohydrodynamics are solved with the assumption of an adiabatic equation of state. Both a Newtonian potential and a pseudo-Newtonian potential are used. Two simplifications are also explored: a cylindrical gravitational potential (the “cylindrical disk”) and axisymmetry.

The results from those simulations are compared with fully three-dimensional global simulations. The global simulations begin with equilibrium pressure-supported accretion tori. Two different initial field geometries are investigated: poloidal fields that are constant along initial equidensity surfaces and toroidal fields with a constant ratio of gas to magnetic pressure. In both cases the magnetorotational instability rapidly develops and the torus becomes turbulent. The resulting turbulence transports angular momentum, and the torus develops an angular momentum distribution that is near Keplerian. A comparison with axisymmetric simulations shows that in three dimensions the magnetorotational instability can act as a dynamo and regenerate poloidal field, thereby sustaining the turbulence. As previously observed in local simulations, the stress is dominated by the Maxwell component. The total stress in the interior of the disk is ≈0.1–0.2 times the thermal pressure. At late time the disks are characterized by relatively thick configurations, with rapid time dependence and tightly wrapped, low-m spiral structures.

Subject heading: accretion, accretion disks — instabilities — MHD

1. INTRODUCTION

Accretion powers a wide range of energetic objects, systems ranging in size from low-mass X-ray binary systems to the disks surrounding supermassive black holes in active galaxies and quasars. Theoretical models of these systems must be, necessarily, highly simplified. One such simplification that is often employed is to assume a time-stationary disk. However the rapid time variability seen in space- and ground-based observations emphasizes that accretion disks are highly dynamic systems. These observations highlight the need to solve the fully time-dependent equations describing accretion disks, and that, in turn, requires numerical techniques.

Recently, time-dependent numerical simulations have been applied to the problem of accretion disk transport. The discovery of the magnetorotational instability (MRI) in accretion disks (Balbus & Hawley 1991) has elucidated the physical basis for this transport: magnetohydrodynamic (MHD) turbulence. At the same time, the development of practical three-dimensional MHD codes brought the study of disk MHD turbulence, transport, and evolution into the computational domain.

To date, most of the numerical simulations of the MRI have been carried out in a local disk approximation known as the shearing box (Hawley, Gammie, & Balbus 1995; hereafter HGB). In the shearing box model, one considers a frame corotating with the angular velocity at a fiducial radius $R_*$. By restricting the computational domain to small excursions from this fiducial radius, one can reduce the geometry (but not the dynamics) to a simple Cartesian system while retaining tidal and Coriolis forces. Local shearing box simulations (e.g., HGB; Hawley, Gammie, & Balbus 1996, hereafter HGB2; Brandenburg et al. 1995; Matsumoto & Tajima 1995; Stone et al. 1996) have demonstrated that MHD turbulence is the nonlinear outcome of the MRI and that outward angular momentum transport is its natural consequence. The total stress $T_{k\phi}$ is dominated by the Maxwell, or magnetic, component rather than by the kinematic, or Reynolds, stress. In disks the amplitude of this stress is often parameterized with the Shakura & Sunyaev (1973) $\alpha$ formulation, $T_{k\phi} = \alpha P$, where $P$ is the pressure. In the local simulations typical stress values range from $\alpha \approx 0.1$ to 0.01. These simulations are more fully reviewed in Balbus & Hawley (1998).

An important step in increasing the realism of numerical disk studies is to move from this local approximation to fully global disk simulations. The difficulties of three-dimensional global simulations and, in particular, the demands such simulations place on computer hardware make this step challenging. In general, the timescale and lengthscale differences between the extended radial scales of an accretion disk and the scale of the MHD turbulence are too great to be fully resolved. Nevertheless, global simulations are now possible for somewhat restricted ranges in radius, height, and angle.

Several global MHD disk simulations have already been done. One of the first, by Matsumoto & Shibata (1997), followed the evolution of a thick torus embedded in a weak vertical magnetic field. What transpires is highly dynamic: the outer layers of the torus slough inward in a process referred to as avalanche accretion. This avalanche accretion is the global consequence of the radial streaming motion (the “channel solution”), which is the nonlinear manifestation of the vertical field instability in the local limit (Hawley...
turbulence. Magnetic energy is amplified and approaches a toroidal field. The nonaxisymmetric MRI rapidly leads to turbulence and significant angular momentum transport. The vertical field is with the sine function \( z = \sin(P) \) and the cylindrical approximation is used. This initial state is unstable, and turbulence rapidly develops along with significant angular momentum transport; \( \alpha \) values are of order 0.1. As in local simulations, the Maxwell stress is several times the Reynolds stress. The magnetic field in the turbulent disk is dominated by the toroidal component, and the turbulence itself is characterized by structures with small azimuthal wavenumbers \( m \). Hawley & Balbus (1999) similarly computed cylindrical models of Keplerian disks and constant angular momentum tori. They found in all cases that the MRI rapidly developed, leading directly to turbulence and significant angular momentum transport.

Global simulations such as these have become possible only recently with advances in available computer speed and memory. As computer capacities continue to increase, ever more ambitious simulations will become possible. The aim of this paper is to begin a systematic effort to develop such global simulations and to model the relativistic effects associated with a Schwarzschild metric (e.g., minimum stable orbit). This inner boundary permits supersonic (and super-Alfvénic) accretion off the inner radial grid, reducing the likelihood of unphysical influences from the boundary conditions. The pseudo-Newtonian potential has the form

\[
\Phi = -\frac{GM}{r - r_g},
\]

where \( r_g \) is the “gravitational radius” (akin to the black hole horizon). For this potential, the Keplerian specific angular momentum (corresponding to a circular orbit) is

\[
\ell_{\text{Kep}} = (GMr)^{1/2} \frac{r}{r - r_g},
\]

and \( \Omega R^2 = \ell \). Here we set \( r_g = 1 \), and, for both gravitational potentials, \( GM = 1 \). This determines the units of time in the simulation with \( \Omega = 1 \) at \( R = 1 \) for a Newtonian gravitational potential. All times given will be reported in these units; the corresponding orbital periods at locations in the tori will also be given where appropriate.

Equations (1)–(4) are solved using time-explicit Eulerian finite differencing. The global disk code is written in cylindrical coordinates, \(( R, \phi, z)\). The center of the coordinate system is excised; i.e., the radial coordinate begins at a

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0, \\
\frac{\partial \mathbf{v}}{\partial t} + (\rho \mathbf{v} \cdot \nabla)\mathbf{v} &= -\nabla P + \frac{B^2}{8\pi} - \rho \nabla \Phi + \left( \frac{B}{4\pi} \cdot \mathbf{v} \right) B, \\
\frac{\partial \rho e}{\partial t} + \nabla \cdot (\rho e \mathbf{v}) &= -P \nabla \cdot \mathbf{v}, \\
\frac{\partial B}{\partial t} &= \nabla \times (\mathbf{v} \times \mathbf{B}),
\end{align*}
\]
nonzero minimum $R_{\text{min}}$; this avoids the coordinate singularities associated with the axis. In some simulations, the gravitational potential is assumed to be cylindrical; this removes the vertical component of gravity and reduces the number of $z$ grid zones required. When the vertical component of gravity is included, the $z$ coordinate is centered on the equatorial plane. The angle $\phi$ is periodic and covers the full $2\pi$ or some integer fraction of $2\pi$. In some simulations, only the two-dimensional axisymmetric system $(R, z)$ will be computed. A schematic of the computational domain is shown in Figure 1.

Most of the three-dimensional global simulations are run on parallel computer systems with a version of the global code that uses the Message Passing Interface (MPI) library for interprocess communication. The full computational domain is divided into overlapping subgrids, one for each processor. Boundary zone values are passed between processors using explicit MPI routines. In the description of what follows, however, this subdivision will be ignored; the physical domain size and the number of grid zones discussed will be the totals.

The MHD algorithm now has a long history of use. The magnetic field is evolved with the constrained transport (CT) approach of Evans & Hawley (1988), which preserves the constraint $\nabla \cdot \mathbf{B} = 0$. The CT scheme was designed to work in any general curvilinear coordinate system, and here the addition of appropriate geometric terms easily adapts the procedure to cylindrical coordinates. The algorithm uses information propagated along Alfven characteristics to solve a restricted set of characteristic equations for time-advanced fields and electromotive forces within the CT framework. This approach is known as the method of characteristics constrained transport (MOCCT) algorithm, and its first implementation is described in detail in Stone & Norman (1992b); the current version is described in Hawley & Stone (1995).

Considerable effort has gone into determining the most satisfactory scheme for the individual numerical terms in the present application. Local conservation of angular momentum is particularly important, and a number of tests using hydrodynamic Keplerian disks on a variety of grids suggest that some numerical approaches are better than others. Specifically, it is best to evolve $\rho \ell$, where $\ell$ is the specific angular momentum, as a fundamental variable (as opposed to angular velocity, or specific angular momentum alone) and to use consistent advection (Norman, Wilson, & Barton 1980) where the advection of all variables is tied to the advection of the density. To minimize errors associated with the coordinate singularity located at $R = 0$, we make use of a regularized form of the operator $R^{-1} \partial R$ (Evans & Hawley 1986).

A variety of test simulations were used to verify code performance. The hydrodynamic implementation was tested with Keplerian disks and a series of simulations of constant angular momentum tori similar to those studied by Hawley (1991). Constant and near-constant angular momentum tori are subject to particularly vigorous growth of the Papaloizou-Pringle instability (Papaloizou & Pringle 1984), especially when such tori are “slender,” meaning that their cross sectional radius is small compared to the radius of the pressure maximum. Slender tori are unstable to the principal mode of the Papaloizou-Pringle instability, with growth rates near the maximum theoretical value. The results of these tests were consistent with past simulations.

For an MHD test, the grid was moved out to large radius and an isolated vertical magnetic field flux tube was embedded in a Keplerian flow. This test is a three-dimensional version of the axisymmetric simulations described in Hawley & Balbus (1991). Again the results were fully consistent with the earlier work.

2.2. Accretion Tori

Most of the simulations discussed below consider the problem of the nonlinear evolution of the magneto-rotational instability in accretion tori. Tori have significant internal pressure gradients that balance the gravitational and centrifugal accelerations in hydrostatic equilibrium. This pressure gives the torus a vertical thickness $H$ that can be comparable to its radius $R$. Significant departures from Keplerian angular momentum distributions are possible as the torus becomes ever thicker; the limit is the constant angular momentum torus. A particularly useful feature of these tori for numerical simulations is that they are well-defined equilibria that can be completely contained on the finite domain of a computational simulation. This allows their evolution to proceed (at least initially) independent of the grid boundary conditions.

For an adiabatic equation of state, a gravitational potential $\Phi$, and an assumed rotation law $\Omega \propto R^{-q}$, the density in the torus is determined by

$$\frac{\Gamma KP}{(\Gamma - 1)\rho} = C - \Phi - \frac{1}{(2q - 2)} \frac{\ell^2}{R^{2q - 2}},$$

where $\ell_k$ is the Keplerian angular momentum at the pressure maximum and $C$ is a constant of integration that establishes the zero pressure surface of the torus. A given torus is specified by the angular velocity gradient $q$ (with $q = 2$ corresponding to constant angular momentum), the radial location of the pressure maximum $R_{\text{Kep}}$, and the radial location of the inner edge of the torus, $R_{\text{in}}$ (which determines $C$). Outside of the torus the grid is filled with a cold gas with low density. A floor value of the density and internal energy is applied to ensure that these zones do not become evacuated or develop extremely high sound speeds. The floor value of density should be set low enough that its contribu-
tion to the total mass or energy is not significant at any place on the grid. Here the floor value is $10^{-3}$ and the typical torus initial density maximum is $\sim 10$.

Tori are known to be subject to the purely hydrodynamic global instability discovered by Papaloizou & Pringle (1984). The Papaloizou-Pringle instability has been extensively investigated both with perturbation analyses and with several numerical simulations. The mechanism of the instability was elucidated by Goldreich, Goodman, & Narayan (1986) and involves the establishment of a global wave that gains energy through the exchange of angular momentum between the inner and outer region of the disk. Amplification requires wave reflection at least one disk boundary. The nonlinear evolution of the Papaloizou-Pringle instability has been followed for both slender (Hawley 1987) and relatively wide (Blaes & Hawley 1988; Hawley 1991) tori. In the slender-torus case, the instability saturates as a highly nonaxisymmetric orbiting fluid ellipse (the “planet” solution; Hawley 1987) that itself proves to be an equilibrium solution (Goodman, Narayan, Goldreich 1987). Wider tori saturate in spiral pressure waves (Blaes & Hawley 1988; Hawley 1991).

Torus linear stability and, as we shall see, nonlinear dynamics are quite different when a magnetic field is present. The largest growth rate of the MRI greatly exceeds those typical of the hydrodynamic Papaloizou-Pringle instability. Linear stability analyses bear this out. Curry & Pudritz (1996) and Ogilvie & Pringle (1996) have done global linear MHD stability analyses of cylindrical systems ($1/R$ gravitational potential) with a variety of initial angular momentum distributions. They find rapidly growing unstable modes for a wide range of equilibria and field strengths.

Although global linear analyses are most appropriate for strong magnetic fields whose most unstable wavelengths are comparable to the size of the torus, weak fields have unstable wavelengths that are much smaller than the torus dimensions. The essential physics of the MRI in these circumstances is local, and the nonlinear evolution should be similar to that seen in the local shearing box simulations. This is, of course, a testable proposition. In any case, global simulations will provide information regarding the ultimate consequences of the MRI for the evolution, and, indeed, existence of thick tori. Obviously, the initial equilibrium torus cannot persist in the face of vigorous instability and the resulting angular momentum transport. Its overall global evolution, however, may present novel features, and the end state of its evolution may represent a more realistic accretion disk structure.

2.3. Diagnostics

A global simulation will involve a grid with $10^6-10^9$ zones, run for possibly $10^5-10^6$ time steps. A major task is to develop useful diagnostics that concisely and adequately characterize the essential behavior of an evolving disk in such a simulation.

The simplest type of diagnostic is the volume integral of a quantity over the entire grid. For example, the total mass is

$$M = \int \rho R dR d\phi dz.$$  \hspace{1cm} (8)

Time histories are computed of this quantity and others such as total kinetic and magnetic energies (by component), and the radial mass flux

$$\langle M \rangle = \int \rho v R d\phi dz$$  \hspace{1cm} (9)

through both the inner and outer boundaries.

Standard steady state thin disk models are often expressed in terms of vertically averaged quantities. Here such values can be computed by integrating density-weighted quantities over $z$ and $\phi$. The vertically and azimuthally averaged mass density is

$$\Sigma(R) = \frac{\int \rho R d\phi dz}{\int R d\phi dz}.$$  \hspace{1cm} (10)

The averaged mass flux, $\langle M \rangle$, is defined by equation (9), and one can construct similar averages for other quantities. For non-Keplerian disks the radial dependence of the average specific angular momentum will be of interest, and this can be computed from

$$\langle \ell \rangle = \frac{\int \rho \ell R d\phi dz}{\int \rho R d\phi dz}.$$  \hspace{1cm} (11)

The stress $\langle T_{R\phi} \rangle$ consists of a magnetic component (the Maxwell stress),

$$\langle T_{R\phi}^{\text{Max}} \rangle = \frac{\int (B_R B_\phi/4\pi) R d\phi dz}{\int R d\phi dz},$$  \hspace{1cm} (12)

and a kinematic component (the Reynolds stress),

$$\langle T_{R\phi}^{\text{Rey}} \rangle = \frac{\int \rho v_R v_\phi R d\phi dz}{\int R d\phi dz}.$$  \hspace{1cm} (13)

The value of the perturbed angular velocity $\delta v_\phi$ is defined for a Keplerian disk by $\delta v_\phi = v_\phi - R\Omega_{Kep}$. For other angular momentum distributions, including average angular momentum distributions that change with time, we adopt an alternative definition for the perturbed orbital velocity $\delta v_\phi$ in terms of the difference between the total instantaneous angular momentum flux and the mass flux times the average angular momentum. The difference then represents the excess or deficit angular momentum transport due to orbital velocity fluctuations compared to the mean. Specifically,

$$\langle \rho v_R \delta v_\phi \rangle = \langle \rho v_R v_\phi \rangle - \langle \rho v_R \rangle \langle \ell \rangle / R.$$  \hspace{1cm} (14)

Together the Maxwell and Reynolds stresses make up the total stress, which, in the Shakura & Sunyaev (1973) parameterization, is set equal to $aP$. The stresses observed in the simulations can be similarly scaled using the vertically averaged pressure to derive an $a$ as a function of $R$. In the simulations, $a$ varies with both time and space. Its utility is mainly to provide a familiar point of reference for characterizing the observed stress. The local shearing box simulations (HGB; HGB2) found that the Maxwell stress, which dominates the total transport, is highly correlated with the total magnetic pressure. This suggests an alternate stress parameter, the “magnetic $a$ value,” defined as $a_m \equiv \langle B_R B_\phi \rangle / \langle B^2 \rangle$.

In a steady state disk, the total stress, Maxwell plus Reynolds, would be equal to the radial angular momentum flux carried inward by the net radial drift velocity minus the angular momentum flux at the inner boundary of the disk, where the stress is presumed to vanish. The latter term is generally ignored for radii large compared to the inner disk.
boundary. In the simulations none of these assumptions hold a priori: the typical torus radius is comparable to the inner disk boundary, the stress does not vanish at the inner disk boundary, and the disk will not be in steady state. In fact, the simulations will prove to be highly dynamic with rapid variations in both time and space. This is best appreciated from time-dependent animations, which also have been used as diagnostics for these global simulations. While reporting time- and space-averaged quantities tends to obscure this feature, such averages are more practical for a written summary.

3. RESULTS

In this section the results from a range of three-dimensional simulations are presented. The simulations are listed in Table 1, which gives a model designation, the grid resolution used, the gravitational potential, the initial torus angular momentum distribution \( q \), the initial magnetic field topology, and the time to which the simulation was run.

3.1. Cylindrical Disks

The first set of simulations make use of a cylindrical gravitational potential \(-1/R\) (the cylindrical disk limit). This is a significant simplification, as it permits the use of periodic boundary conditions in the vertical direction. Further, with this potential there is only one important vertical lengthscale, namely, that of the MRI; this reduces the number of \( z \) grid zones required. Finally, in the cylindrical limit one can study the evolution of models with vertical fields without the stringent Courant limits due to the high Alfvén speeds associated with strong fields passing through the low-density region above the disk.

The first two cylindrical simulations assume a Keplerian initial disk. Keplerian simulations provide an immediate comparison with the local shearing box results, which also, for the most part, assumed Keplerian flow. Consider first the run, labeled “CK1” in Table 1; this is a cylindrical computational domain \((R, \phi, z)\) running from 1 to 4 in \( R \), from 0 to 1 in \( z \), and from 0 to \( \pi/2 \) in \( \phi \). The grid has 90 \( \times \) 80 \( \times \) 24 zones. An outflow boundary condition is used at both the outer and inner radial boundaries, and periodic boundary conditions are used for \( \phi \) and \( z \). The initial disk has a constant mass density from \( r = 1.5 \) to the outer boundary. The adiabatic sound speed is constant and equal to 5% of the orbital velocity at the inner edge of the disk. The initial magnetic field is vertical and proportional to \( \sin(R)/R \) between \( R = 1.5 \) and 3.5, with a maximum strength corresponding to \( \beta = 400 \). The strength of the field, and hence the Alfvén speed \( v_A \), was chosen so that the characteristic wavelength of the MRI was \( \lambda_{MRI} \equiv 2\pi v_A/\Omega = 0.38 \) at the location of the field strength maximum. This ensures that the \( z \) domain size and the vertical grid resolution will be adequate to resolve the fastest growing modes. Random nonaxisymmetric pressure fluctuations are added to create a full range of low-amplitude initial perturbations.

As the evolution proceeds, the MRI sets in, field energy is amplified, and the characteristic radial streaming structures (referred to as the channel solution) of the vertical field instability appear, much as they do in the local shearing box models. In the present simulation, these structures develop first at the inner part of the disk, where the rotation frequency is the highest. The amplitude of the MRI becomes nonlinear by 3 orbits at the center of the grid \((P_{orb} = 24.8)\), and filaments of strong magnetic field are carried inward and outward by fluid elements well out of Keplerian balance. These reach the outer part of the disk even before the local MRI in that region becomes fully nonlinear. Thus there are two immediate global effects not seen in local simulations: linear growth rates that vary strongly with radius \((\omega_{ MRI} \propto \Omega \propto R^{-3/2})\) and extended radial motions of significantly non-Keplerian fluid.

In local simulations with initial vertical fields that vary sinusoidally with radius, the initial phase of the instability promptly breaks down into MHD turbulence. This happens in the global simulation as well. After the onset of turbulence, the disk displays many tightly wrapped (i.e., large radial wavenumber) trailing spiral features with low azimuthal wavenumber, \( m \). The magnetic field energy saturates at about \( \beta = 4 \), with the toroidal component dominant: \( B_T^2/B_R^2 = 0.075 \) and \( B_T^2/B_R^2 = 0.34 \). Similar ratios were found in the shearing box simulations with zero net initial magnetic field (HGB2).

The MHD turbulence produces rapid angular momentum transport and mass accretion, with \( \dot{z} \) peaking at 0.21 at \( t = 90 \) and declining beyond this point, dropping to 0.06 at the end of the simulation. The Maxwell stress displays a high correlation with the total magnetic pressure. After an early peak at 0.68, \( \sigma_m \) declines with time; at the end of the simulation \( \sigma_m = 0.35 \). The ratio of Maxwell to Reynolds stress is about 3 at the time of the largest total stress and rises to about 9 at the end of the simulation. The overall angular momentum distribution remains nearly Keplerian throughout the simulation, becoming slightly sub-Keplerian outside of the original inner disk edge and slightly super-Keplerian in the region between this point and the inner grid boundary.

The mass flux, \( \langle M \rangle \), varies both in space and time. After 8 orbits of evolution, over one-half of the initial disk mass has been lost through the boundaries, particularly the outer

| Model  | Grid   | Potential | \( q \) | Initial Field | End Time |
|--------|--------|-----------|--------|--------------|---------|
| CK1... | 90 \( \times \) 80 \( \times \) 24 | 1/R       | 1.5    | \( B_A \sin(R)/R^2 \) | 188     |
| CK2... | 90 \( \times \) 80 \( \times \) 24 | 1/R       | 1.5    | \( B_A \phi \)       | 285     |
| CT1... | 90 \( \times \) 80 \( \times \) 24 | 1/R       | 2      | \( B_A z \)         | 102     |
| CT2... | 90 \( \times \) 80 \( \times \) 24 | 1/R       | 2      | \( B_A \phi \)       | 403     |
| CT3... | 128 \( \times \) 64 \( \times \) 32 | 1/(R - R_g) | 2      | \( A_g \propto \rho(R) \) | 420     |
| GT1... | 128 \( \times \) 64 \( \times \) 128 | 1/(r - r_s) | 2      | \( A_g \propto \rho(R, z) \) | 780     |
| GT2... | 120 \( \times \) 64 \( \times \) 90 | 1/(r - r_s) | 2      | \( A_g \propto \rho(R, z) \) | 727     |
| GT3... | 128 \( \times \) 128 \( \times \) 128 | 1/(r - r_s) | 2      | \( B_A \phi \)       | 727     |
| GT4... | 128 \( \times \) 128 \( \times \) 128 | 1/(r - r_s) | 1.68   | \( A_g \propto \rho(R, z) \) | 1283    |
with zero net vertical field. The larger azimuthal wavelengths available to the MRI. Further, the tent with what is seen in local simulations with net vertical aB is slowly accreting. The magnetic field is subthermal and outer boundary); what remains is piled up near the mass has been lost from the grid (much of it through the accretion begin, and by 12 orbits over one-half of the disk linear analysis (Balbus & Hawley 1992). Turbulence and same Keplerian disk, but with an initial toroidal field of total stress.

With a toroidal initial field, the disk evolves at a slower rate that CK1. The instability grows over the first few orbits, with the fastest rate of growth associated with the innermost radius. The growing modes of the instability have the same appearance as seen in the local simulations; high m structures appear first, building to lower m with time. The vertical and radial structure also features high wavenumbers. The rapid growth phase of the instability ceases after about 8 orbits at the grid center. At this point the field exhibits a low-m, tightly wrapped structure, with rapid variations in R and z. This behavior is consistent with the linear analysis (Balbus & Hawley 1992). Turbulence and accretion begin, and by 12 orbits over one-half of the disk mass has been lost from the grid (much of it through the outer boundary); what remains is piled up near R = 1.7 and is slowly accreting. The magnetic field is subthermal and dominated by the toroidal component. The component magnetic energies at late time are quite similar to those seen in the vertical field run CK1.

Angular momentum transport is again mainly due to the Maxwell stress. Compared with CK1, the toroidal field model has lower overall stress. This is partly because the vertical field model has a very vigorous initial phase associated with the saturation of the linear instability. At the end of both CK1 and CK2, \( \alpha_m = 0.3 - 0.4 \). The mass accretion rate, magnetic field strength, and \( \alpha \) fluctuate considerably both in time and in space. The total \( \alpha \) value rises to 0.12 and then declines to 0.06 by the end of CK2. The average ratio of the Maxwell to Reynolds stress is 6 at late times. The value of \( \alpha \) never rises as high as in local vertical field models, including CK1. It is comparable, however, to the typical value found for weak-field toroidal field models in the local shearing box simulations that began with comparable initial toroidal field strengths (HGB).

These Keplerian disk models suffer from significant mass loss through the outer boundary. In contrast, tori can be completely and self-consistently contained initially within the computational domain. We next turn to cylindrical models of constant angular momentum (\( q = 2 \)) tori, beginning with a torus with \( R_{in} = 2.0 \) and \( R_{out} = 2.5 \). The torus outer boundary is at \( R = 3.3 \). Two simulations are done, models CT1 with an initial vertical field and CT2 with an initial toroidal magnetic field; both use the same computational grid as above.

Model CT1 begins with a vertical field with constant \( \beta = 100 \) from \( r = 2.1 \) to 3.1. This gives \( \lambda_{MRI} = 0.25 \) at the pressure maximum. As the evolution proceeds, the magnetorotational instability rapidly develops, again with the characteristic channel. Early on, the beginnings of the Papaloizou-Pringle principal mode can also be seen as \( k_c = 0 \) oscillations at the edges of the torus. However long before this global instability can develop, the torus is dramatically altered by the local MRI. The torus does not endure as a torus: it expands rapidly outward as the angular momentum distribution shifts from constant toward Keplerian. After 4 orbits at the initial torus pressure maximum, the system has evolved to a nearly Keplerian disk that fills the computational domain.

The magnetic field is amplified to an overall average value of \( \beta = 2 \). The toroidal field dominates: \( B_2^2/B_3^2 = 0.1 \) and \( B_2^2/B_4^2 = 0.3 \). As always, the Maxwell stress exceeds the Reynolds stress by a factor of several. The value of \( \alpha_m \) peaks at 0.7 and declines to 0.4 at the end of the simulation \( (t = 102) \). The overall \( \alpha \) value varies throughout the disk; at \( t = 85 \) it varies between 0.3 and 0.4.

Next take the same torus and apply an initial toroidal field with \( \beta = 100 \) (model CT2). With this strength field, the critical azimuthal wavenumber at the pressure maximum is \( m_{crit} = 63 \), so the fastest growing modes are underresolved on this grid. As with the Keplerian simulation CK2, the initial toroidal field model becomes turbulent at a later time compared with an initial vertical field model. The total poloidal magnetic field amplification is also considerably smaller than seen in the vertical field model. Despite this, the qualitative outcome of the instability for the torus is largely the same. The slower onset of the MRI allows the principal mode of the Papaloizou-Pringle instability to appear, but soon the transport of angular momentum brings this to a halt. The disk spreads outward with the bulk of the mass slowly moving inward. The overall angular
momentum distribution changes from constant to Keplerian from the inner boundary to $R = 2.5$, and sub-Keplerian but increasing beyond this point. As in the previous simulations, the disk exhibits large local fluctuations in density, stress and field strength, and many low azimuthal, high radial wavenumber features. Similar time and space variations were observed in the local models as well; they are characteristic of the MHD turbulence.

At late times the magnetic energy has risen to $\beta = 15$, with the toroidal field dominant by a large factor: $B_r^2/B_\phi^2 = 0.04$. The ratio of the vertical to radial field energy is 0.16. The value of $\alpha_m$ rises to $\sim 0.3$, close to, but below, the value in the Keplerian toroidal field simulation above. The total stress corresponds to an $\alpha = 0.02$–0.03 at late time. Again the Maxwell stress exceeds the Reynolds component by a factor of 3–4. These properties are consistent with toroidal field shearing box simulations (HGB).

Finally, consider a constant angular momentum torus, CT3, with an initial field constructed by setting the azimuthal component of the vector potential equal to the density in the torus, $A_\phi = \rho(R)$. The resulting field is normalized to an average $\beta$ of 100. In the cylindrical limit, the density depends only on radius $R$, and the resulting field is vertical with zero net value integrated over the disk. This model also adopts the pseudo-Newtonian potential $\sim 1/(R - R_s)$ with $R_s = 1$. The computation domain runs from $R = 1.5$ to $13.5$, $\phi = 0$ to $\pi/2$ and $z = 0$ to 2. The grid resolution is $128 \times 64 \times 32$. The torus has an inner edge at $r = 4.5$ and a pressure maximum at $R_{\text{max}} = 6.5$ where $P_{\text{orb}} = 88$; the orbital period at the outer boundary is 289. A new feature in this model is the presence of the marginally stable orbit at $R_{\text{ms}} = 3$. Inside of this radius, matter will rapidly accelerate inward.

The MRI grows rapidly in the inner regions of the disk, again with the characteristic radial channel appearance. Accretion through the inner boundary begins at about $t = 100$. The magnetic energy rises to peak at $\beta = 8$ at $t = 150$. The magnetic energy grows more slowly after that.

Fig. 2.—Vertically and azimuthally averaged velocities as a function of radius in simulation CT3 at time $t = 420$ (4.8 orbits at the initial pressure maximum). The curves trace the toroidal speed $v_r$, the adiabatic sound speed $c_s$, the toroidal and poloidal Alfvén speeds $v_{A_t}$ and $v_{A_p}$, and the radial speed $v_r$. The vertical dashed line indicates the location of the marginally stable orbit in the pseudo-Newtonian potential. The short-dashed line corresponds to the Keplerian velocity.

Fig. 3.—Gray-scale plots of $\log(\rho)$ in a two-dimensional axisymmetric simulation of a constant angular momentum torus containing poloidal field loops. Each image is labeled by time. At $t = 50$ the torus has expanded due to shear amplification of the toroidal field. At $t = 180$ the poloidal field MRI has set in. A period of turbulence follows ($t = 250, 350$), which dies out by the end of the simulation ($t = 850$, 17 orbits at the initial pressure maximum).
The overall Maxwell stress in the MHD turbulence reaches a peak at $t = 150$, drops off, and then slowly climbs again. At late times $\alpha_m = 0.5$ and the total stress parameter $\alpha$ varies with radius inside the disk, from 0.1 near the initial pressure maximum to 0.2 at the outer regions. The angular momentum distribution rapidly evolves away from constant, toward Keplerian. By $t = 420$ it is nearly Keplerian from the pressure maximum inward and sub-Keplerian out beyond this point. Inside of $R_{ms} = 3$ the Maxwell and Reynolds stresses decline with radius, although not as quickly as the pressure. This makes $\alpha$ rise sharply.

By the end of the simulation ($t = 420$) over one-quarter of the disk mass has been lost, and 70% of it has gone inward past the marginally stable orbit. An examination of the various vertically and azimuthally averaged velocities at the end of the run (Fig. 2) reveals that the inflow velocity $v_r$ accelerates rapidly inside of $R = 4$. By the time the inner grid boundary is encountered, the inflow speed strongly supersonic and super-Alfvénic. Inside the disk the radial speed is an order of magnitude smaller than the sound speed and smaller than the toroidal or poloidal Alfvén speeds, which themselves remain subthermal. Going inward from the marginally stable orbit, the toroidal Alfvén speed rises more slowly than the poloidal Alfvén speed. The radial field strength is roughly constant, but the fluid density drops inside of $R_{ms}$ as the flow accelerates inward. The specific angular momentum drops 3% from $R = 3$ to the inner boundary at $R = 1.5$, indicating that there is still some net stress inside $R_{ms}$.

3.2. Axisymmetry: Simulations in the $(R, z)$ Plane

The cylindrical disk limit represents a useful way of simplifying the full global problem. Another potentially useful simplification is the axisymmetric limit. In this series of simulations, the torus evolution problem is considered in the axisymmetric $(R, z)$ plane. These simulations use a pseudo-Newtonian potential and begin with a pressure-supported torus that is fully contained on the grid. The initial magnetic field is chosen to satisfy two requirements: it must have a poloidal component to allow for the development of the MRI, and, as a practical matter, it should be contained completely within the torus to avoid the Courant limitations caused by high Alfvén speeds due to strong fields in low-density regions. A suitable initial configuration consists of magnetic field loops lying along equidensity surfaces in the torus. This initial setup will develop strong toroidal field due to shearing of the initial radial field. However, experience has shown that strong toroidal fields are the outcome of all initial field choices, so this should not represent too great an idealization. All of the two-dimensional simulations considered here have also been run in three dimensions.

The first simulation has a radial grid running from $R = 1.5$ to an outer boundary at $R = 11.5$ and a vertical grid running from $z = -5$ to 5. The grid resolution is $128 \times 128$. A constant-$\zeta$ torus is placed on the grid with an inner boundary at the marginally stable orbit, $R_{in} = 3$, and a pressure maximum at $R_{sep} = 4.7$. The orbital period at the pressure maximum is 50. The initial field is obtained from an azimuthal vector potential $A_{\phi} = \rho(R, z)$ and is normalized to $\beta = 100$ using the total integrated magnetic and thermal energies.

Density gray-scale plots from this simulation are presented in Figure 3. The initial period of evolution is dominated by the growth of toroidal magnetic field due to shear. As the magnetic pressure increases, the torus expands, particularly at the inner edge. At first, low-density material is driven outward perpendicularly to the torus surface; subsequently, it flows radially out and around the torus. Because of the initial reflection symmetry across the equator, the toroidal field changes sign there, and a strong current sheet forms. This current sheet proves to be unstable and oscillates around the equator. This is an important part of the evolution and indicates that it is necessary to simulate the full $(R, z)$ plane rather than adopt the equator as an explicit boundary.

As the toroidal field pressure increases, it drives inflow through the marginally stable orbit; this initial accretion flow peaks at $t = 50$ then begins to decline. In the meantime, the poloidal field MRI grows within the torus and begins to manifest itself visibly in the typical form of radial channels by $t = 180$ (3.6 orbits). There follows a period of violent readjustment within the disk, featuring strong mass inflow punctuated by episodic accretion events. This phase lasts until $t \sim 350$ (7 orbits), beyond which the poloidal magnetic energy, and with it the level of turbulence in the disk and the accretion rate, declines. At the end of the simulation ($t = 850$, 17 orbits) about 60% of the initial disk mass has been lost. Most of this mass is lost by $t = 500$; after this time the inflow accretion rate is very small.

Thus there are three distinct phases to the two-dimensional torus evolution: expansion due to the shear-amplified toroidal magnetic pressure, strong nonlinear evolution of the poloidal field MRI, and finally a more quiescent turbulent state with declining poloidal magnetic field strengths. Angular momentum transport occurs in all three phases at different rates. In the first phase there is a growing Maxwell stress from the amplified $B_{\phi}$ field, mainly in the inner region of the disk where the orbital frequency is highest. The Reynolds stress is negligible. The initially constant angular momentum distribution is unchanged within the disk except in this inner region. With the onset of the MRI, angular momentum transport occurs everywhere, and the specific angular momentum begins to increase with radius. During the middle of this second phase, angular momentum transport peaks, with $\alpha$ ranging between 0.2 and 0.5 through the disk. As the turbulence subsides, the rate of transport declines, with $\alpha$ varying between 0.02 and 0.1 within the main part of the disk. Since this is an axisymmetric simulation, the strength of the poloidal magnetic field is limited topologically by the antidynamo theorem, and the MHD turbulence must eventually die out.

The next model is a torus located initially at a larger radius, which increases the amount of time that it can evolve prior to reaching the marginally stable orbit. The grid runs from 1.5 to 13.5 in $R$ and from $-4.5$ to 4.5 in $z$. The grid resolution is $120 \times 90$. The constant-$\zeta$ torus has an inner edge at $R_{in} = 4.5$ and a pressure maximum at $R_{sep} = 6.5$. The orbital period at the pressure maximum is 88. Again, magnetic field loops are placed along the equidensity
Fig. 4.—Gray-scale plots of log (ρ) in simulation GT1, initially a constant angular momentum torus containing poloidal field loops. Each image consists of a side view and an equatorial view, and is labeled by time. At $t = 80$ the torus has expanded due to shear amplification of the toroidal field, which has become visibly unstable in the inner regions. Full turbulence sets in by $t = 200$ (4 orbits at the initial pressure maximum) and continues for the remainder of the simulation. The total disk mass drops steadily due to accretion. These images should be compared with the axisymmetric model in Fig. 3.
surfaces within the torus, with a total magnetic energy corresponding to $\beta = 100$.

As before, this torus undergoes three phases of evolution: toroidal field amplification, development of the nonlinear poloidal MRI, and subsequent turbulence. The MRI saturates around $t = 300$ (3.4 orbits) with total magnetic energy $\beta = 2$. The phase of strong MRI turbulence is over by $t = 450$ (5.1 orbits); when the average $\beta$ rises to 6; fairly steady accretion follows for the remainder of the simulation, which runs to time $t = 830$ (9.5 orbits).

In the third simulation, the initial torus has an angular momentum distribution closer to Keplerian, specifically $q = 1.68$. The torus inner boundary is at 4 and its pressure maximum at $R_{Kep} = 10$ (orbital period = 179). The computational domain runs from 1.5 to 21.5 in $R_a$ and from $-10$ to 10 in $z$; the grid resolution is $128 \times 128$.

Although it began with a much different initial angular momentum distribution, the evolution is quite similar to the two previous cases. In the early stage, toroidal field is amplified by shear. The poloidal MRI soon comes into play, producing the characteristic radial streams. The total magnetic energy peaks at about $t = 800$ (4.5 orbits). After this the poloidal field energy declines steadily, as do the Maxwell stress and the mass accretion rate through the inner radial boundary. The toroidal field dominates, with $B_r^2/B_\phi^2 = 0.017$. Beyond $t = 2000$, the toroidal field energy is essentially constant, with $\beta = 1.3$.

Angular momentum transport begins almost immediately. Initially it is confined to the inner regions of the torus, where shear is strongest, and dominated by the Maxwell stress associated with the growing toroidal field. However, the growth of the MRI produces stress throughout the torus. By $t = 1000$ the averaged $\alpha$ value is between 0.1 and 0.2. At the end of the simulation at $t = 2840$, $\alpha$ ranges between 0.01 and 0.04 within the torus. The ratio of the Maxwell stress to the magnetic pressure, $\alpha_m$, begins small, rises to a value of 0.1 during the initial saturation of the MRI, then steadily declines to 0.03 by the end of the simulation as poloidal field is preferentially destroyed. Since this torus began with an angular momentum distribution similar to the end state of the initially $q = 2$ tori, there is only a slight change in the slope of the average angular momentum. Once the torus expands, this slope is extended in range all the way from $R_{min}$ to the outer radial boundary.

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**Fig. 5.—** Mass flux through the inner grid radius $R_{min}$ as a function of time for global models (solid lines) and their axisymmetric counterparts (dashed lines), where applicable. In each case, the mass accretion rate is normalized by the initial mass of the torus. In models GT1, GT2, and GT4, the initial accretion is driven mainly by the growth of the toroidal field due to shear. Subsequently the poloidal MRI drives strong accretion. The coherent channel solution in axisymmetry produces particularly strong accretion events. At late times, accretion in the axisymmetric models dies down, while in the global models accretion continues at a reduced, but significant, level. Note that, despite their quite different initial field strengths and topologies, models GT2 and GT3 have similar accretion rates at late times.
characteristic large-wavenumber structures associated with the instability quickly appear in the inner regions of the disk (by \( t = 80 \)). This creates turbulence and momentarily increases the mass accretion rate over that seen in the axisymmetric simulation (Fig. 5). By \( t = 200 \) the disk contains tightly wrapped, low-\( m \) trailing spiral waves. The presence of this nonaxisymmetric structure affects the development of the poloidal field instability, eliminating the organized radial streaming flows seen in axisymmetry. The flow is more turbulent, and the mass accretion rate is steadier, without the large impulsive spikes in \( \dot{M} \) associated with the radial streaming in two dimensions. This overall turbulence declines after saturation of the MRI but continues through the end of the run. This is the stage when the accretion rate in the two-dimensional simulation drops to very small values. While in three dimensions one might hope that the disk could achieve a nearly steady state mass accretion rate, there is, of course, only a finite amount of disk mass available to accrete. At the end of the run \( 85\% \) of the initial total mass has been lost from the grid, most of it through the innermost stable orbit. The amount of mass ejected off the outer boundary is about \( 30\% \) of that accreted through the inner boundary.

Although the total volume-averaged magnetic energy drops with time beyond \( t = 400 \), most of this decline is due to loss from accretion. The average value of \( \beta \) after \( t = 200 \) is \( \approx 4 \) and remains fairly constant thereafter. The magnetic energy is dominated by the toroidal component: after \( t = 200 \) the poloidal field \( \beta \) value has an average value of \( \sim 30 \) and also remains nearly constant. In the axisymmetric simulation the poloidal \( \beta \) value attains a value of 15 at \( t = 280 \) but rises steadily to \( \beta = 75 \) by \( t = 800 \), indicating a poloidal magnetic energy loss rate exceeding that due to accretion alone.

Angular momentum transport results in rapid restructuring of the disk. The angular momentum distribution in the inner half of the torus steepens to a nearly Keplerian value by \( t = 50 \) (Fig. 6). The angular momentum in the outer part of the torus increases over the remainder of the simulation. At the end time the angular velocity \( \Omega \propto R^{-4} \) has a slope near \( q = 1.55 \). This is essentially the same as in the axisymmetric run above (short-dashed line in Fig. 6). This slope adequately characterizes the disk since the angular momentum and hence the angular velocity are nearly constant on cylinders. After turbulence has developed, the average \( z \) values in the main part of the disk are between 0.1 and 0.2. The ratio of the Maxwell to Reynolds stress varies but lies between 1 and 4 throughout the main part of the disk. The time- and space-averaged value of \( \alpha_m \) after \( t = 200 \) is 0.40.

Figure 6 shows that the specific angular momentum continues to decline even inside the marginally stable orbit. This indicates that even here there remains a significant net stress. In fact, inside the marginally stable orbit \( \alpha \) rapidly increases because the gas pressure drops while the Maxwell stress remains roughly constant. The presence of this stress means there is no maximum in the epicyclic frequency, \( \kappa^2 \equiv 2(\Omega/R)(\partial \Omega/\partial R) \). In the pseudo-Newtonian potential (and, of course, in the relativistic potential that the pseudo-Newtonian potential was designed to imitate), the Keplerian value of \( \kappa^2 \) has a maximum that occurs at about \( 3.7R_\star \). More generally, the epicyclic frequency will go to zero at a stress-free inner edge of a disk where \( \partial \Omega/\partial R = 0 \), ensuring a \( \kappa \) maximum somewhere in the disk. Here, however, the inner boundary of the disk is not characterized

**3.3. Three-dimensional Tori: Full Global Simulations**

Now we turn to the evolution of fully three-dimensional tori using a pseudo-Newtonian potential. The simulations described in this section are three-dimensional versions of the tori considered above in the axisymmetric limit. The first, model GT1, is the constant angular momentum torus with \( R_{\text{kep}} = 4.7 \) and \( R_{\text{in}} = 3 \). The orbital period at the pressure maximum is \( t_{\text{orb}} = 50 \). The computational domain runs from 1.5 to 11.5 in radius, from \(-5\) to 5 in \( z \), and from \( \phi = 0 \) to \( \pi/2 \). The initial magnetic field is constructed by setting the toroidal component of the vector potential equal to the density inside the disk, \( A_\phi = \rho(R, z) \), for all \( \rho \) greater than a minimum value. The total magnetic energy is then normalized to a value of \( \beta = 100 \), using the total integrated gas pressure of the torus. Different regions within the torus will, of course, have larger or smaller values of \( \beta \). The strongest initial fields are found between the pressure maximum and the inner edge of the torus. The simulation is run to time \( t = 780 \) (15.6 orbits).

The evolution of this torus is illustrated with a series of gray-scale plots in \( \log(\rho) \) (Fig. 4) of vertical \((R, z)\) and equatorial \((R, \phi)\) slices. As with the axisymmetric torus (Fig. 3), the initial phase of evolution is controlled by the shear amplification of the toroidal field. However, in three dimensions this toroidal field is itself unstable to the MRI. The
by a zero-stress condition, the slope of $\partial \epsilon / \partial R$ is nearly constant, and the epicyclic frequency does not turn over but continues to rise. It has been proposed that potentially observable oscillatory modes might be trapped near the disk inner edge where the epicyclic frequency turns over (e.g., Nowak & Wagoner 1993). The present result must be regarded as preliminary, but if significant stress inside the marginally stable orbit proves to be a generic property of magnetic turbulence, trapped disk oscillations may not be present.

In some respects, the two- and three-dimensional simulations are similar. They both have shear amplification of toroidal field, they both evolve because of the resulting increase in toroidal magnetic pressure, and they both develop turbulence because of the poloidal field MRI. Both rapidly evolve from constant to nearly Keplerian specific angular momentum distributions (Fig. 6). The three-dimensional simulation, however, permits the development of the nonaxisymmetric MRI, which increases and sustains turbulence and mass accretion. In two dimensions the organized poloidal field channel solution produces an impulsive accretion rate greater than that seen in three dimensions during the initial saturation of the poloidal MRI. However axisymmetry causes the two-dimensional poloidal field, and with it the Maxwell stress, to decline. The contrast between $x_m$ in the three- and the two-dimensional simulations is instructive (Fig. 7). In three dimensions $x_m$ remains relatively constant and near the value typically seen in the local shearing box simulations. In two dimensions $x_m$ declines steadily with time following the saturation of the MRI. Maintenance of the poloidal field through dynamo action is possible only in three dimensions.

Although two-dimensional simulations cannot capture these essential features of global evolution, they do have one clear advantage: they are considerably easier to compute. Two-dimensional simulations are useful for searching a wide range of initial conditions in support of the more challenging three-dimensional models. To test the idea of using an evolved two-dimensional simulation as an initial condition, consider next a constant-$\ell$ torus with an initial inner edge at $R_{in} = 4.5$ and a pressure maximum at $R = 6.5$ where $P_{obs} = 88$. This same initial torus was run both in the cylindrical limit and in the axisymmetric $(R, z)$ limit above. Here the computational domain runs from $R = 1.5$ to 13.5, $\phi = 0$ to $\pi/2$, and $z = -4.5$ to 4.5. The grid resolution is $120 \times 64 \times 90$. The three-dimensional torus GT2 is initial-

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**Fig. 7.** Time evolution of volume-integrated value of $x_m = -B_r B_r/4\pi P_{mag}$ in the global torus models (solid lines) and the axisymmetric version (dashed lines), where appropriate. Individual graphs are labeled by global torus model number. In three dimensions, poloidal field amplitudes are maintained relative to the toroidal field and the Maxwell stress remains appreciable compared to the total magnetic pressure. In two dimensions the poloidal fields die out and the stress drops.
ized by taking the output from the axisymmetric simulation at time \( t = 233 \) and expanding it outward in the \( \phi \) direction. This time corresponds to shortly before the nonlinear aspects of the MRI first manifest themselves noticeably in the density plots. Random pressure perturbations are added to break the axisymmetry.

While this initialization procedure reduces the total computational time required, it does not allow the toroidal field instability the opportunity to grow during the initial phase. This means that the strong coherent poloidal field instability develops as it does in two dimensions. Nevertheless, significant nonaxisymmetric structure appears by \( t = 400 \), and the overall turbulence is increased over that seen in axisymmetry. Accretion inflow at the inner boundary is about twice what it is in two dimensions after this point in time. After \( t = 400 \) the average magnetic field strength is \( \beta = 4.7 \), and the poloidal field strength \( \beta_p = 24 \). At late times, \( \alpha \approx 0.1 \), and \( \alpha_m = 0.37 \), a level similar to that seen in GT1. In two dimensions the Maxwell stress drops and the poloidal \( \beta \) increase. A the end of the simulation, \( \alpha_m = 0.17 \) and \( \beta_p = 55 \). The toroidal field \( \beta \) is fairly constant with time in both two- and three-dimensional runs after \( t = 450 \).

This run can also be compared with the cylindrical run CT3. Figure 8 is a plot of the radial dependence of the azimuthally and vertically averaged speeds. These curves are very similar to those in Figure 2 from CT3. GT2 shows a stronger initial field amplification phase and a stronger MRI saturation (created by initializing from the axisymmetric run). However the magnetic field amplitudes and stress levels are comparable at late times in both simulations. These similarities indicate that the cylindrical limit provides a useful approximation for investigating the nonlinear evolution of MHD turbulence near the midplane of a disk.

To summarize, GT2 shows that fully three-dimensional turbulence can be rapidly produced and sustained in a simulation initialized from the output of an axisymmetric simulation. The late-time properties of such a simulation, essentially one with complicated nonlinear initial perturbations, are quite similar to those obtained from a simulation evolved from a formal equilibrium and linear perturbations. The two-dimensional simulations, therefore, serve as just another type of initial condition, albeit more complicated than usually adopted.

One characteristic of all these simulations is that at late times the magnetic field is predominantly toroidal. The next simulation considers a torus that begins with only a toroidal field. Model GT3 is of a \( q = 2 \) torus with an inner boundary at 4.5 and a pressure maximum at \( R_{\text{Kep}} = 6.5 \) (orbital period = 88). The initial toroidal field has \( \beta = 4 \) everywhere within the torus. The MRI that will result is nonaxisymmetric; an axisymmetric \((R, z)\) version of this model does not develop turbulence or transport angular momentum. The computational domain runs from 1.5 to 13.5 in \( R_{\text{in}} \), from \(-6 \) to \( 6 \) in \( z \), and over the full \( 2\pi \) in angle; the grid resolution is \( 128 \times 128 \times 128 \).

No attempt is made to keep the initial torus in pressure balance; the initial magnetic field simply adds to the equilibrium hydrodynamic pressure. As a consequence, the torus undergoes an axisymmetric readjustment because of this additional magnetic pressure. The toroidal field MRI develops rapidly at the inner edge of the disk where \( \Omega \) is largest, before spreading throughout the disk. The poloidal field energy grows steadily with time until about \( t = 250 \), when it reaches a value of \( \beta_p = 34 \). After \( t = 400 \), \( \beta_p \) increases with time to 65 at the end of the run. Gray-scale images of GT3 are given in Figure 9.

Models GT2 and GT3 began with the same hydrodynamic torus; they differ in their initial magnetic fields, in the size of the \( \phi \) domain, and in resolution. Despite these differences, the late-time evolution in these two runs is very similar. They have comparable mass accretion rates through \( R_{\text{min}} \) (Fig. 5). Both have \( \alpha_m = 0.36 \) at \( t = 720 \). Figure 10 illustrates the redistribution of mass and angular momentum as a function of time. The slope of the angular momentum distribution at late time is the same in GT3 as in both GT1 and GT2.

Figure 11 shows the radial run of density \( \Sigma \), velocity \( \langle v_R \rangle \), and mass flux \( \langle M \rangle \) at time \( t = 720 \). Density is normalized by the maximum \( \Sigma \) at \( t = 0 \). Within the torus the radial drift velocity is small compared to the orbital velocity; \( \langle v_R \rangle \) reverses outside \( R = 8 \), beyond which there is net outflow. Inside \( R_{\text{in}} \) the density decreases rapidly as the inflow accelerates inward. The accretion rate increases slightly from \( R = 7 \) inward. Figure 12 shows gas and magnetic pressures at this same end time (normalized by the initial pressure maximum value), as well as the Maxwell stress. The magnetic energy remains subthermal throughout; although it drops less rapidly than the gas pressure inside of \( R_{\text{in}} \). The ratio of the Maxwell stress to the gas pressure has a minimum of 0.03 between \( R = 4 \) and \( 5 \), rising sharply inside \( R_{\text{in}} \) and more slowly toward the outer boundary.

The final global simulation, GT4, is the \( q = 1.68 \) torus with an inner boundary at 4 and a pressure maximum at \( R_{\text{Kep}} = 10 \). This angular momentum distribution yields a torus that extends over a large radial distance without becoming too thick in the vertical direction. The computational domain runs from 1.5 to 21.5 in \( R_{\text{in}} \), from \(-10 \) to \( 10 \) in \( z \), and over the full \( 2\pi \) in angle; the grid resolution is
128 x 128 x 128. The simulation is run to time t = 1280. This is longer than the other simulations, although it amounts to fewer orbits at the pressure maximum: t = 1280 is 7.2 orbits at R_{Kep}. This evolution time is sufficient to observe all of the stages seen in the other three-dimensional simulations, although not long enough for the torus to have settled into a quasi-steady state.

Density plots from GT4 are presented in Figure 13. At the beginning, the toroidal field grows by shearing out the radial field, but as it does so the toroidal MRI sets in. This soon leads to turbulence. The poloidal MRI develops rapidly in the inner regions of the disk and subsequently spreads throughout. The inner edge of the disk moves slowly inward until it reaches the marginally stable orbit at t = 145, after which gas plunges inward. Initially the inflowing fluid is confined largely to the equatorial plane. As time passes, however, this region fills with gas and becomes thicker.

As with the previous global simulations, the mass accretion rate is larger in three dimensions than in two. Near the end of the simulation, \langle M \rangle at the inner radial boundary is approximately 2.5 times that seen in axisymmetry. The value of \alpha_m is rising with time, indicating that the poloidal field strength is still increasing at the end of the simulation.

Figure 14 shows the radial mass and angular momentum distributions at the initial and final times in GT4. Also shown are curves from two times in the equivalent axisymmetric calculation. The average angular velocity parameter \beta decreases by a very small amount over the course of the evolution. The torus mass, on the other hand, has been substantially redistributed.

4. DISCUSSION

In this paper, we have carried out global MHD simulations of accretion tori. We have also made use of several different limits or approximations: two-dimensional axisymmetry, three-dimensional cylindrical gravitational potential, and the full three-dimensional global model. In addition to the intrinsic interest in the results, these simulations begin to map out what is currently possible with the present hardware and algorithms.
First, how useful are the two-dimensional and cylindrical approximations? While neither should be relied on exclusively, they both have appropriate applications, and they greatly reduce the computational difficulty of the simulations. Two-dimensional axisymmetric torus simulations demonstrate effects from the generation of toroidal field due to shear, and from the development of the poloidal field MRI. The latter leads to turbulence, rapid angular momentum transport, and evolution toward a nearly Keplerian angular momentum distribution. A significant limitation of the axisymmetric system, however, is embodied in Cowling's antidynamo theorem. The component of the vector potential is conserved, except for losses due to dissipative processes. Poloidal fields can grow from axisymmetric stretching and folding, but this is ultimately limited. In the simulations, after the nonlinear saturation of the MRI, the poloidal magnetic field energies decline and the turbulence begins to die out.

Despite this, considerable similarity is seen between the two- and three-dimensional simulations. This is due to the dominance in the initial stages of the torus evolution by what is largely axisymmetric dynamics: increase in toroidal field pressure due to shear, the development of the channel solution, and the rapid redistribution of angular momentum by large stresses. During phases of a disk's evolution when such effects are most important, two-dimensional simulations are a good approximation. Their utility must be limited, however, since genuine nonaxisymmetric effects, including the development of the toroidal field MRI, dynamo amplification and maintenance of poloidal fields, and nonaxisymmetric spiral waves, are generally of dominant importance over the long term.

Cylindrical disks are a natural extension of the shearing box model. The cylindrical disk, like the shearing box simulations of HGB and HGB2, does not include the effects of vertical gravity. Cylindrical Keplerian disk simulations initialized with vertical fields or toroidal fields show rapid development of the MRI consistent with the shearing box results. Field amplification and stress levels are comparable.
Fig. 13.——Gray-scale plots of log ($\rho$) in simulation GT4, initially a torus with $q = 1.68$ and poloidal field loops. Each frame consists of a side and an equatorial view and is labeled by time. The orbital period at the initial pressure maximum is 179. At $t = 320$ the torus has expanded due to shear amplification of the toroidal field, which has, in turn, become unstable, with $m = 4$ dominating in the equatorial plane. Beyond this time the poloidal field MRI begins and turbulence follows.
between the two types of simulation as well. With the cylindrical disk, however, we are able to observe the direct consequences of the stress: redistribution of angular momentum toward Keplerian and mass accretion. Cylindrical disk simulations can provide detailed information about the radial dependence of the growth and saturation of the MRI, the resulting MHD turbulence, the transport of angular momentum, and the net accretion flow.

Simulations of cylindrical constant angular momentum tori with vertical or toroidal initial fields illustrate many aspects of a full global evolution. Regardless of the initial field configuration, the initially constant angular momentum tori evolve toward radial angular momentum distributions that are nearly, but not quite, Keplerian. The final outcome is largely the same for either initial vertical or toroidal fields, although the early growth of the MRI is more rapid for vertical fields and dominated by small-scale structure for toroidal fields. This is entirely consistent with the local linear analysis. Of course, we want to simulate disks fully globally and with as few approximations or geometric constraints as possible. Indeed, full three dimensionality is essential. Cylindrical simulations can provide little, if any, information about the vertical structure of a disk, energy transport, or the possible formation of winds or jets.

As with the cylindrical disk simulations, the overall evolution of fully global tori are consistent with the intuitions developed from shearing box simulations. The MRI grows rapidly and produces MHD turbulence that transports angular momentum. In all cases, toroidal fields dominate, followed by radial and then vertical fields. One difference is that the shearing box calculations with zero net field (e.g., HGB2) typically have a total stress value of $\alpha \approx 0.01$, whereas here $\alpha \approx 0.1$ in the heart of the turbulent disks. This is more a matter of the magnetic pressure that is sustained in the torus versus the shearing box, rather than some qualitative difference in the behavior of the MRI. In both the global and local simulations, the stress is directly proportional to $B^2/8\pi$; $\alpha_m \approx 0.4$–0.5. All of the present global simulations began with relatively strong fields, either in the form of poloidal field loops, which immediately generated strong toroidal fields through shear, or from the presence of initially strong toroidal fields. This field strength is by and large sustained, and thus the observed $\alpha$ values are consistent with $\beta \lesssim 10$. It is suggestive that Matsumoto (1999) also obtained $\beta \approx 10$ for his toroidal field simulations, even one that began with $\beta = 1000$. Global (rather than local) lengthscales make it natural to have larger magnetic field strengths in the saturated state.

Computational problems associated with the inner boundary are greatly reduced through the use of the pseudo-Newtonian potential. This provides a physical inner radius for the disks and ensures that gas and field will flow smoothly off the inner radial grid. The accretion flow rapidly accelerates inward near $R_{\text{in}}$, with the radial speed quickly exceeding the sound speed. Interestingly, the angular momentum continues to drop inside of $R_{\text{in}}$, indicating the continuing presence of Maxwell stress. The absence of a stress-free inner disk boundary is one of the ways in which these disks differ from standard analytic models.

The most important conclusion from these simulations is that constant or nearly constant angular momentum tori are remarkably unstable in the presence of weak magnetic fields. The MHD turbulence resulting from the MRI simply transports angular momentum too efficiently. Within a few orbits the angular momentum distribution changes from $q = 2$ to $q \approx 1.6$. The tori considered here did not remain maintain their initial constant angular momentum distribution long enough to develop the coherent structures associated with the nonlinear outcome of the Papaloizou-Pringle instability. This does not, however, necessarily imply that accretion disks must be thin. Moderately thick configurations, as seen in the gray-scale plots, are still possible even for nearly Keplerian angular momentum distributions if there is significant internal pressure.

Averaged properties of the torus such as density, pressure or angular momentum distribution fail to convey the impression of disorder seen in animations of the evolution. Low-density, high magnetic field filaments entwine themselves throughout the torus. Regions of strong field develop and rise through the torus into the surrounding low-density atmosphere. Low-density material flows outward around the bulk of the torus. Since angular momentum transport is by MHD turbulence, it follows that the disk should be highly dynamic. The effect of this on observed properties of disks must be considered in subsequent, more sophisticated simulations. It is not premature, however, to question the relevance of the traditional image of a quiescent steady state disk.

There remain several limitations to the present simula-
tions to be addressed in subsequent work. First, greater grid resolution is always welcome, particularly where the accretion inflow is squeezed into a narrow equatorial flow, or for following the growth of the MRI from substantially weaker initial field strengths. Because the disks develop oscillations across the equatorial plane, particularly in the simulations that begin with poloidal field loops, the full \((R, z)\) plane must be simulated in global models without applying explicit (e.g., reflecting) equatorial boundary conditions. This doubles the number of grid zones required but appears to be necessary. Higher resolution simulations are feasible with the addition of more processors to the parallel computation. The simulations described here used a maximum of 64. Indeed, a great many issues can be profitably investigated using simulations with more grid zones. For example, the global simulations presented here featured tori that began close to the marginally stable orbit. Little evolution is required to accrete through the central hole. Simulating a greater dynamic range in disk radii, for longer times and with larger numbers of radial grid zones, is an immediate next goal.

The present simulations focus only on the dynamical properties of MHD in tori. As the thermodynamic properties of the disk are of obvious importance in establishing an overall global disk structure, further algorithm development is desirable. In the present simulations the simplified equation of state and lack of explicit resistivity mean that the only sources of gas heating are artificial shock viscosity and adiabatic compression. Some amount of energy is necessarily lost through the numerical reconnection of magnetic field. Further, neither radiative transport nor simple radiative losses were included. These and other issues of global disk evolution must be deferred to subsequent work.

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