Recurrence relations for Apostol-Bernoulli, -Euler and -Genocchi polynomials of higher order

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Abstract

In [14, 18], Luo and Srivastava introduced some generalizations of the Apostol-Bernoulli polynomials and the Apostol-Euler polynomials. The main object of this paper is to extend the result of [25] to these generalized polynomials. More precisely, using the Padé approximation of the exponential function, we obtain recurrence relations for Apostol-Bernoulli, Euler and also Genocchi polynomials of higher order. As an application we prove lacunary relation for some particular cases.

Keywords: generalized Apostol-Bernoulli polynomials, generalized Apostol-Euler polynomials, generalized Apostol-Genocchi polynomials, Padé approximants.

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1. Introduction, definition and notations

The generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{C}$, the generalized Euler polynomials $E_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{C}$ and the generalized Genocchi polynomials $G_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{C}$, each of degree $n$, are respectively defined by the following generating functions (see [8], vol. III, p.253 and seq., [13], Section 2.8 and [20]):

\[ \left( \frac{t}{e^t - 1} \right)^\alpha e^{tx} = \sum_{k=0}^{\infty} B_k^{(\alpha)}(x) \frac{t^k}{k!}, \quad (|t| < 2\pi; 1^\alpha := 1, \alpha \in \mathbb{C}), \quad (1.1) \]

\[ \left( \frac{2}{e^t + 1} \right)^\alpha e^{tx} = \sum_{k=0}^{\infty} E_k^{(\alpha)}(x) \frac{t^k}{k!}, \quad (|t| < \pi; 1^\alpha := 1, \alpha \in \mathbb{C}), \quad (1.2) \]

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The classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials are given by
\[ B_n(x) := B_n^{(1)}(x), \quad E_n(x) := E_n^{(1)}(x), \quad G_n(x) := G_n^{(1)}(x), \quad (n \in \mathbb{N}) \text{ respectively.} \]
Moreover, the Bernoulli $B_n$'s numbers, Euler $E_n$'s numbers and Genocchi $G_n$'s numbers are given by:
\[ B_n := B_n(0), \quad E_n := 2^n E_n^\left(\frac{1}{2}\right), \quad G_n := G_n^\left(0\right). \]

These polynomials and numbers play a fundamental role in various branches of mathematics including combinatorics, number theory and special functions.

Q.M. Luo and Srivastava introduced the Apostol-Bernoulli polynomials of higher order (also called generalized Apostol-Bernoulli polynomials):

**Definition 1.** (Luo and Srivastava, [18]) The Apostol-Bernoulli polynomials \( B_k^{(\alpha)}(x; \lambda) \) of order \( \alpha \) in the variable \( x \) are defined by means of the generating function:
\[ \left( \frac{2t}{e^t + 1} \right)^\alpha e^{xt} = \sum_{k=0}^{\infty} G_k^{(\alpha)}(x) \frac{t^k}{k!}, \quad (|t| < \pi; 1^\alpha := 1, \alpha \in \mathbb{C}), \quad (1.3) \]

\( G_k^{(\alpha)}(x) \)

The Apostol-Bernoulli polynomials $B_k(x; \lambda)$ are given by $B_k := B_k(0; \lambda)$. Moreover, we call $B_k(\lambda) := B_k^{(1)}(\lambda)$ the Apostol-Bernoulli numbers.

Explicit representation of $B_k^{(\alpha)}(x; \lambda)$ in terms of a generalization of the Hurwitz-Lerch zeta function can be found in [9].

In [14], Luo introduced the Apostol-Euler polynomials of higher order $\alpha$.

**Definition 2.** The Apostol-Euler polynomials $E_k^{(\alpha)}(x; \lambda)$ of order real or complex $\alpha$ in the variable $x$ are defined by means of the following generating function:
\[ \left( \frac{t}{\lambda e^t - 1} \right)^\alpha e^{xt} = \sum_{k=0}^{\infty} E_k^{(\alpha)}(x; \lambda) \frac{t^k}{k!}, \quad (|t| < 2\pi \text{ when } \lambda = 1; |t| < |\log(\lambda)|, 1^\alpha := 1). \quad (1.5) \]

The Apostol-Euler polynomials $E_k(x; \lambda)$ are given by $E_k := E_k^{(1)}(x; \lambda)$. The Apostol-Euler numbers $E_k(\lambda)$ are given by $E_k(\lambda) := 2^k E_k^{(1)}(\frac{1}{2}; \lambda)$.

Some relations between Apostol-Bernoulli and Apostol-Euler polynomials of order $\alpha$ can be found in [19]. For more results on these polynomials, the readers are referred to [7, 16, 17].

In [15] Luo introduced and investigated the Apostol-Genocchi polynomials of order $\alpha$, which are defined as follows.
Definition 3. The Apostol-Genocchi polynomials $G^{(\alpha)}_k(x; \lambda)$ of order $\alpha$ in the variable $x$ are defined by means of the following generating function:

$$
\left( \frac{2t}{\lambda e^t + 1} \right)^\alpha e^{xt} = \sum_{k=0}^{\infty} G^{(\alpha)}_k(x; \lambda) \frac{t^k}{k!} \quad (|t| < |\log(-\lambda)|, 1^\alpha := 1).
$$

The Apostol-Genocchi polynomials $G_k(x; \lambda)$ are given by $G_k(x; \lambda) := G^{(1)}_k(x; \lambda)$. The Apostol-Genocchi numbers $G_k(\lambda)$ are given by $G_k(\lambda) := G^{(0)}_k(0; \lambda)$.

When $\lambda = 1$ in (1.4) and when $\lambda = -1$ in (1.6), the order $\alpha$ of the generalized Apostol-Bernoulli polynomial $B^{(\alpha)}_k(x; \lambda)$ and the order $\alpha$ of the generalized Apostol-Genocchi polynomial $G^{(\alpha)}_k(x; \lambda)$ should tacitly be restricted to non negative values.

In this paper, we consider Apostol-type polynomials of order $\alpha$, $B^{(\alpha)}_k(x; \lambda)$, $E^{(\alpha)}_k(x; \lambda)$, and $G^{(\alpha)}_k(x; \lambda)$.

The aim of this paper is to apply Padé approximation to $e^t$ in the generating functions (1.4), (1.5) and (1.6) to get relations between Apostol-type polynomials of order $\alpha$ depending on three parameters $n$, $m$, $p$ where $n$ and $m$ are respectively the degree of the numerator and the degree of the denominator of the Padé approximant used to approximate the function $e^t$ and $p$ is some positive integer.

The paper is organized as follows. In the next section, we recall the definition of Padé approximant to a general series and its expression for the case of the exponential function. In Section 3, we apply Padé approximation to prove the recurrence relations (Theorem 1 and 2) for the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials of higher order.

2. Padé approximant

In this section, we recall the definition of Padé approximation to general series and their expression in the case of the exponential function. Given a function $f$ with a Taylor expansion

$$f(t) = \sum_{i=0}^{\infty} c_i t^i$$

in a neighborhood of the origin, a Padé approximant denoted $[n, m]_f$ to $f$ is a rational fraction of degree $n$ (resp. $m$) for the numerator (resp. the denominator):

$$[n, m]_f(t) = \frac{\alpha_0 + \alpha_1 t + \cdots + \alpha_n t^n}{\beta_0 + \beta_1 t + \cdots + \beta_m t^m},$$

whose Taylor expansion agrees with (2.1) as far as possible:

$$\sum_{i=0}^{\infty} c_i t^i - \frac{\alpha_0 + \alpha_1 t + \cdots + \alpha_n t^n}{\beta_0 + \beta_1 t + \cdots + \beta_m t^m} = O(t^{m+n+1}).$$
In the general case, the resulting linear system has unique solutions \( \alpha_i, \beta_i \) (see, e.g., [2]).

Padé approximation is related with convergence acceleration [6, 21, 23, 24], continued fractions [3, 11], orthogonal polynomials, quadrature formulas [10] and number theory. Moreover the denominators of Padé approximants satisfy a three terms recurrence [4, 5] and this property allows finding another proof of the irrationality of \( \zeta(2) \) and \( \zeta(3) \) [22].

If \( f(t) = e^t \) then

\[
[n, m]_f(t) := \frac{P^{(n,m)}(t)}{Q^{(n,m)}(t)} = \frac{1}{1} F_1(-n, -m - n, t) - P^{(n,m)}(t)
\]

and satisfies

\[
R^{(n,m)}(t) := e^t - [n, m](t) = e^t - \frac{P^{(n,m)}(t)}{Q^{(n,m)}(t)}
\]

\[
R^{(n,m)}(t) = t^{m+n+1} \frac{e^t}{Q^{(n,m)}(t)} \int_0^1 (x - 1)^m x^n e^{-x} t \, dx
\]

where the Pochhammer symbol \((a)_j\) is defined as

\[
(a)_j = a(a+1) \cdots (a+j-1) \text{ if } j \geq 1,
\]

\[
= 1 \text{ if } j = 0
\]

and the hypergeometric series \( 1 F_1(a, b, z) \) is defined as \( 1 F_1(a, b, z) := \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!} \). In the sequel we write \([n, m](t)\) for the Padé approximant to \( e^t \). The remainder term is defined by

In the sequel, let
\[ \alpha_j^{(n,m)} := \binom{n}{j} (n + m - j)!, \quad 0 \leq j \leq n, \]
\[ \beta_j^{(n,m)} := \binom{m}{j} (n + m - j)!(-1)^j, \quad 0 \leq j \leq m, \]
\[ \gamma_j^{(n,m)} := (-1)^m \frac{n! (m + j)!}{(m + n + 1 + j)! j!}, \quad j \geq 0. \]

The polynomials \( P^{(n,m)} \), \( Q^{(n,m)} \) and the product \( R^{(n,m)} Q^{(n,m)} \) are then given by
\[
P^{(n,m)}(t) = \sum_{j=0}^{n} \alpha_j^{(n,m)} t^j, \\
Q^{(n,m)}(t) = \sum_{j=0}^{m} \beta_j^{(n,m)} t^j, \\
R^{(n,m)}(t) Q^{(n,m)}(t) = \sum_{j=0}^{\infty} \gamma_j^{(n,m)} t^{j+m+n+1}.
\]

3. Recurrence relations for Apostol-type polynomials

Let us recall a classical method to derive a basic formula for Apostol-Bernoulli polynomials of order \( \alpha \).

From the definition,
\[
\left( \frac{t}{\lambda e^t - 1} \right)^\alpha e^{(x+y)t} = \sum_{k=0}^{\infty} B_k^{(\alpha)}(x + y; \lambda) \frac{t^k}{k!}, 
\]
we write the left handside member of the previous equation as
\[
\left( \frac{t}{\lambda e^t - 1} \right)^\alpha e^{(x+y)t} = \left( \frac{t}{\lambda e^t - 1} \right)^\alpha e^x t e^y t
\]
and substitute \( e^y t \) by its Taylor expansion around 0. It arises
\[
\left( \sum_{k=0}^{\infty} B_k^{(\alpha)}(x; \lambda) \frac{t^k}{k!} \right) \left( \sum_{k=0}^{\infty} \frac{y^k}{k!} \right) = \sum_{k=0}^{\infty} B_k^{(\alpha)}(x + y; \lambda) \frac{t^k}{k!}, 
\]
By identification, the following formula is proved:
\[
B_n^{(\alpha)}(x + y; \lambda) = \sum_{k=0}^{\infty} \binom{n}{k} B_k^{(\alpha)}(x; \lambda) y^k. 
\]
In this section, the main idea is to replace the exponential function $e^{yt}$ in the generating functions by some Padé approximant of Apostol-type polynomials (1.4), (1.5) and (1.6), $e^{yt}$ not by its Taylor expansion around 0 but by its Padé approximant $[n,m]$. This gives the following main result.

**Theorem 1.** For all integers $m \geq 0, n \geq 0$, for arbitrary real or complex parameter $\lambda$, the Apostol-type polynomials $\Lambda_k^{(\alpha)}(x;\lambda)$ of order $\alpha$ satisfy

$$
\sum_{k=0}^{n} \frac{(n+m-k)!}{(p-k)!} \binom{n}{k} y^k \Lambda_{p-k}^{(\alpha)}(x;\lambda) = \sum_{k=0}^{m} (-y)^k \frac{(n+m-k)!}{(p-k)!} \binom{m}{k} \Lambda_{p-k}^{(\alpha)}(x+y;\lambda)
$$

for $p \geq m+n+1$,

$$
\sum_{k=0}^{n} \frac{(n+m-k)!}{(p-k)!} \binom{n}{k} y^k \Lambda_{p-k}^{(\alpha)}(x;\lambda) = \sum_{k=0}^{m} (-y)^k \frac{(n+m-k)!}{(p-k)!} \binom{m}{k} \Lambda_{p-k}^{(\alpha)}(x+y;\lambda) - (-1)^m n! m! \sum_{k=0}^{p-m-n-1} \binom{p-k}{k} \binom{p-k-n-1}{m} \Lambda_k^{(\alpha)}(x;\lambda)
$$

where the $\Lambda_k^{(\alpha)}(x;\lambda)$’s are the Apostol-Bernoulli, -Euler or -Genocchi polynomials of order $\alpha$.

We use the convention $\Lambda_k^{(\alpha)}(x;\lambda) = 0$ for $k \leq -1$.

Particular cases:

1) If $p = m + n$, then

$$
\sum_{k=0}^{n} \binom{n}{k} y^k \Lambda_{p-k}^{(\alpha)}(x;\lambda) = \sum_{k=0}^{m} (-y)^k \binom{m}{k} \Lambda_{p-k}^{(\alpha)}(x+y;\lambda).
$$

2) If $m = 0$, then

$$
\Lambda_p^{(\alpha)}(x+y;\lambda) = \sum_{k=0}^{p} \binom{p}{k} y^k \Lambda_{p-k}^{(\alpha)}(x;\lambda).
$$

Formula (3.4) is then recovered.

3) If $n = 0$,

$$
\Lambda_p^{(\alpha)}(x;\lambda) = \sum_{k=0}^{p} \binom{p}{k} (-y)^k \Lambda_{p-k}^{(\alpha)}(x+y;\lambda).
$$

which is the dual formula of the previous one.

**Proof of Theorem 1**
We display the proof only for the case of Apostol-Bernoulli polynomials of order \( \alpha \). Let us start from the generating functions (1.4), (1.5),(1.6)

\[
\Phi(t, \lambda, \alpha) e^{x \, t} = \sum_{k=0}^{\infty} \Lambda_k^{(\alpha)}(x; \lambda) \frac{t^k}{k!}
\]

where

\[
\Phi(t, \lambda, \alpha) = \left( \frac{t}{\lambda e^t - 1} \right)^\alpha, \text{ for Apostol - Bernoulli polynomials of order } \alpha,
\]

\[
\Phi(t, \lambda, \alpha) = \left( \frac{2}{\lambda e^t + 1} \right)^\alpha, \text{ for Apostol - Euler polynomials of order } \alpha,
\]

\[
\Phi(t, \lambda, \alpha) = \left( \frac{2t}{\lambda e^t + 1} \right)^\alpha, \text{ for Apostol - Genocchi polynomials of order } \alpha,
\]

and for some integers \( n, m \), we replace \( e^{y \, t} \) by its Padé approximant previously defined:

\[
e^{y \, t} = \frac{[n, m](y \, t) + R^{(n,m)}(y \, t)}{Q^{(n,m)}(y \, t) + R^{(n,m)}(y \, t)}.
\]

We get

\[
\Phi(t, \lambda, \alpha) e^{x \, t} e^{y \, t} = \Phi(t, \lambda, \alpha) e^{x \, t} \left( \frac{P^{(n,m)}(y \, t)}{Q^{(n,m)}(y \, t)} + R^{(n,m)}(y \, t) \right) = \sum_{k=0}^{\infty} \Lambda_k^{(\alpha)}(x + y; \lambda) \frac{t^k}{k!}
\]

with

\[
P^{(n,m)}(y \, t) = \sum_{k=0}^{n} \alpha_k^{(n,m)} y^k t^k = \sum_{k=0}^{n} \binom{n}{k} (m + n - k)! \frac{t^k}{k!},
\]

\[
Q^{(n,m)}(y \, t) = \sum_{k=0}^{m} \beta_k^{(n,m)} y^k t^k = \sum_{k=0}^{m} \binom{m}{k} (-1)^k (m + n - k)! \frac{t^k}{k!},
\]

and

\[
R^{(n,m)}(y \, t) Q^{(n,m)}(y \, t) = \sum_{k=0}^{\infty} \gamma_k^{(n,m)}(y \, t)^{m+n+1+k} = \sum_{k=0}^{\infty} \frac{n! (m+k)!}{(m+n+1+k)! k!} (y \, t)^{m+n+1+k}.
\]

This leads to

\[
\sum_{k=0}^{\infty} \Lambda_k^{(\alpha)}(x; \lambda) \frac{t^k}{k!} \times \left( P^{(n,m)}(y \, t) + R^{(n,m)}(y \, t) Q^{(n,m)}(y \, t) \right) = Q^{(n,m)}(y \, t) \sum_{k=0}^{\infty} \Lambda_k^{(\alpha)}(x + y; \lambda) \frac{t^k}{k!}
\]

which gives, applying the Cauchy product for series, the following identity:

\[
\sum_{p=0}^{\infty} \sum_{k+j=p \atop k \geq 0,j \geq 0} \alpha_{j}^{(n,m)} \Lambda_k^{(\alpha)}(x; \lambda) \frac{t^k}{k!} = \sum_{p=0}^{\infty} \sum_{k+j=p \atop k \geq 0,j \geq 0} \beta_{j}^{(n,m)} \Lambda_k^{(\alpha)}(x + y; \lambda) \frac{t^k}{k!} = \sum_{p=0}^{\infty} \sum_{k+j=p \atop k \geq 0,j \geq 0} \gamma_{j}^{(n,m)} \Lambda_k^{(\alpha)}(x + y; \lambda) \frac{t^k}{k!}
\]
Comparing the coefficient of \( t^p \) on both sides of the previous equation, we obtain the assertions (3.5) and (3.6) of Theorem 1.

Theorem 1 provides a recurrence relation of length \( \max(n,m) \) from \( B^{(\alpha)}_{p-\max(n,m)}(x; \lambda) \) (resp. \( E^{(\alpha)}_{p-\max(n,m)}(x; \lambda) \), \( G^{(\alpha)}_{p-\max(n,m)}(x; \lambda) \)) to \( B^{(\alpha)}_p(x; \lambda) \) (resp. \( E^{(\alpha)}_p(x; \lambda) \), \( G^{(\alpha)}_p(x; \lambda) \)) for \( p \) less than \( m+n \). On the other hand, if \( p \) is greater than \( m+n \), we obtain also a recurrence relation for the same Apostol-type polynomials of higher order, but with supplementary first terms from \( B^{(\alpha)}_0(x; \lambda) \) (resp. \( E^{(\alpha)}_0(x; \lambda) \), \( G^{(\alpha)}_0(x; \lambda) \)) to \( B^{(\alpha)}_{p-m-n-1}(x; \lambda) \) (resp. \( E^{(\alpha)}_{p-m-n-1}(x; \lambda) \), resp. \( G^{(\alpha)}_{p-m-n-1}(x; \lambda) \)).

Of course, Theorem 1 is valid for classical Bernoulli, Euler and Genocchi polynomials and also for Bernoulli, Euler and Genocchi numbers which are particular Apostol-type polynomials. By this mean we recover known results in the literature as shown in [25].

By using the same method as in the proof of Theorem 1, we establish now a recurrence formula for \( \Lambda^{(\alpha+1)}_k(x; \lambda) \) and \( \Lambda^{(\alpha)}_k(x; \lambda) \).

**Theorem 2.** For \( n, m, p \in \mathbb{N} \),

if \( 0 \leq p \leq m+n \), then the following relations for Apostol-type polynomials of consecutive order \( \alpha \) and \( \alpha+1 \),

\[
\sum_{j=0}^{\max(n,m)} (\lambda \alpha^{(n,m)}_j - \beta^{(n,m)}_j) B^{(\alpha+1)}_{p-j}(x; \lambda) \frac{(p-j)!}{(p-j-1)!} = \sum_{j=0}^{m} \beta^{(n,m)}_j B^{(\alpha)}_{p-j-1}(x; \lambda) \frac{(p-j)!}{(p-j-1)!},
\]

(3.10)

\[
\sum_{j=0}^{\max(n,m)} (\lambda \alpha^{(n,m)}_j + \beta^{(n,m)}_j) E^{(\alpha+1)}_{p-j}(x; \lambda) \frac{(p-j)!}{(p-j-1)!} = 2 \sum_{j=0}^{m} \beta^{(n,m)}_j E^{(\alpha)}_{p-j-1}(x; \lambda) \frac{(p-j)!}{(p-j-1)!},
\]

(3.11)

\[
\sum_{j=0}^{\max(n,m)} (\lambda \alpha^{(n,m)}_j + \beta^{(n,m)}_j) G^{(\alpha+1)}_{p-j}(x; \lambda) \frac{(p-j)!}{(p-j-1)!} = 2 \sum_{j=0}^{m} \beta^{(n,m)}_j G^{(\alpha)}_{p-j-1}(x; \lambda) \frac{(p-j)!}{(p-j-1)!},
\]

(3.12)

hold,

where \( \alpha^{(n,m)}_j = \binom{n}{j} (n+m-j)! \) and \( \beta^{(n,m)}_j = (-1)^j \binom{m}{j} (n+m-j)! \).

For \( p \geq m+n+1 \),

\[
\sum_{j=0}^{\max(n,m)} (\lambda \alpha^{(n,m)}_j - \beta^{(n,m)}_j) B^{(\alpha+1)}_{p-j}(x; \lambda) \frac{(p-j)!}{(p-j-1)!} - \lambda \sum_{j=0}^{p-m-n-1} \gamma^{(n,m)}_j B^{(\alpha+1)}_{p-m-n-j-1}(x; \lambda) \frac{(p-j)!}{(p-j-1)!}.
\]
\[
\sum_{j=0}^{\max(m,n)} \left( \lambda \alpha_j^{(n,m)} + \beta_j^{(n,m)} \right) \frac{\gamma_p(x; \gamma)}{(p-j)!} = 2 \sum_{j=0}^{m} \beta_j^{(n,m)} \frac{\gamma_p(x; \lambda)}{(p-j)!} - \lambda \sum_{j=0}^{p-m-n-1} \gamma_j^{(n,m)} \frac{\gamma_p(x; \lambda)}{(p-m-n-j-1)!}
\]

where \( \gamma_j^{(n,m)} = (-1)^m \frac{n!(m+j)!}{(m+n+1+j)!j!} \)

**Proof**

We only display the proof for the Apostol-Bernoulli polynomials of order \( \alpha \).

We start from the generating function (1.4)

\[
\left( \frac{t}{\lambda e^t - 1} \right)^\alpha e^{xt} = \sum_{k=0}^{\infty} B_k^{(\alpha)}(x; \lambda) \frac{t^k}{k!}
\]

We multiply the two hand-side members of this equation by \( \frac{t}{\lambda e^t - 1} \) to get

\[
\sum_{k=0}^{\infty} B_k^{(\alpha+1)}(x; \lambda) \frac{t^k}{k!} = \left( \frac{t}{\lambda e^t - 1} \right) \sum_{k=0}^{\infty} B_k^{(\alpha)}(x; \lambda) \frac{t^k}{k!}
\]

and

\[
(\lambda e^t - 1) \sum_{k=0}^{\infty} B_k^{(\alpha+1)}(x; \lambda) \frac{t^k}{k!} = \sum_{k=0}^{\infty} B_k^{(\alpha)}(x; \lambda) \frac{t^{k+1}}{k!},
\]

in which we replace \( e^t \) by its Padé approximant. This leads to

\[
(\lambda P^{(n,m)}(t) - Q^{(n,m)}(t) + \lambda R^{(n,m)}(t) Q^{(n,m)}(t)) \sum_{k=0}^{\infty} B_k^{(\alpha+1)}(x; \lambda) \frac{t^k}{k!} = Q^{(n,m)}(t) \sum_{k=0}^{\infty} B_k^{(\alpha)}(x; \lambda) \frac{t^{k+1}}{k!},
\]

where

\[
P^{(n,m)}(t) = \sum_{k=0}^{n} \alpha_k^{(n,m)} t^k, \quad Q^{(n,m)}(t) = \sum_{k=0}^{m} \beta_k^{(n,m)} t^k \quad \text{and} \quad R^{(n,m)}(t) = \sum_{k=0}^{m} \gamma_k^{(n,m)}(t) t^k.
\]

Theorem 2 concerned with Apostol-Bernoulli polynomials of order \( \alpha \) is obtained by equating the coefficient of \( t^p \).
4. Applications

In this section, we will consider the value of the parameter $p$ with respect to $n, m$.

Let us first consider the particular values $p = m + n$ and $\alpha = 0$. Then we can prove the following corollary.

**Corollary 1.** For $m \geq 0, n \geq 0$,

$$\lambda \sum_{k=0}^{n} \binom{n}{k} B_{m+k}(x; \lambda) - (-1)^m \sum_{k=0}^{m} (-1)^k \binom{m}{k} B_{n+k}(x; \lambda) = ((m+n)x-n)x^{n-1}(x-1)^m \quad (4.1)$$

$$\lambda \sum_{k=0}^{n} \binom{n}{k} E_{m+k}(x; \lambda) + (-1)^m \sum_{k=0}^{m} (-1)^k \binom{m}{k} E_{n+k}(x; \lambda) = 2x^n(x-1)^m, \quad (4.2)$$

$$\lambda \sum_{k=0}^{n} \binom{n}{k} G_{m+k}(x; \lambda) + (-1)^m \sum_{k=0}^{m} (-1)^k \binom{m}{k} G_{n+k}(x; \lambda) = 2((m+n)x-n)x^{n-1}(x-1)^m \quad (4.3)$$

**Proof**

The two formulas $(4.1), (4.2)$ have been proved in [25]. We prove the formula $(4.3)$.

If $p = m + n$ and $\alpha = 0$ in (3.12), then

$$\sum_{j=0}^{\max(m,n)} (\lambda \alpha_j^{(n,m)} + \beta_j^{(n,m)}) G_{m+n-j}(x; \lambda) = \frac{\sum_{j=0}^{m} \beta_j^{(n,m)} x^{m+n-j-1}}{(m+n-j-1)!}$$

since $G_k^{(1)}(x; \lambda) = G_k(x; \lambda)$ and $G_k^{(0)}(x; \lambda) = x^k$.

Thus replacing $\alpha_j^{(n,m)}, \beta_j^{(n,m)}$ by their expression, we obtain the following relation:

$$\lambda \sum_{j=0}^{n} \binom{n}{j} G_{m+n-j}(x; \lambda) + \sum_{j=0}^{m} (-1)^j \binom{m}{j} G_{m+n-j}(x; \lambda) = \frac{\sum_{j=0}^{m} (-1)^j \binom{m}{j} (m+n-j)x^{m+n-j-1}}{(m+n-j-1)!}$$

$$= 2((m+n)x-n)x^{n-1}(x-1)^m$$

Let $s \geq 1$. Setting $p = m + n - s$ in Theorem 2, we will immediately get Corollary 2.

**Corollary 2.** For $n \in \mathbb{N}, m \in \mathbb{N}, s \in \mathbb{N}$ such that $1 \leq s \leq n + m$,

$$\lambda \sum_{k=0}^{n} \binom{n}{k} (m-s+k+1)sB_{m-s+k}^{(\alpha+1)}(x; \lambda) - (-1)^m \sum_{k=0}^{m} (-1)^k \binom{m}{k} (n-s+k+1)sB_{n-s+k}^{(\alpha+1)}(x; \lambda) =$$

$$(-1)^m \sum_{k=0}^{m} (-1)^k \binom{m}{k} (n-s+k)s+1B_{n-s+k-1}^{(\alpha)}(x; \lambda),$$
\[
\lambda \sum_{k=0}^{n} \binom{n}{k} (m-s+k+1)_{s} \mathcal{E}_{m-s+k}^{(a+1)}(x;\lambda) + (-1)^{m} \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} (n-s+k+1)_{s} \mathcal{E}_{n-s+k}^{(a+1)}(x;\lambda) = \\
2(-1)^{m} \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} (n-s+k+1)_{s} \mathcal{E}_{n-s+k}^{(a)}(x;\lambda),
\]

where the Pochhammer symbol \((a)_{j}\) is defined as

\[
(a)_{j} = a(a+1) \cdots (a+j-1) \text{ if } j \geq 1, \\
= 1 \text{ if } j = 0
\]

Remark 1. For \(\lambda = 1\) and \(s = 1\), Corollary 2 reduces to the following formulas,

\[
\sum_{k=0}^{n} \binom{n}{k} (m+k) B_{m+k-1}^{(a+1)}(x) - (-1)^{m} \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} (n+k) B_{n+k-1}^{(a+1)}(x) = \\
(-1)^{m} \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} (n+k-1)(n+k) B_{n+k-2}^{(a)}(x),
\]

\[
\sum_{k=0}^{n} \binom{n}{k} (m+k) E_{m+k-1}^{(a+1)}(x) + (-1)^{m} \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} (n+k) E_{n+k-1}^{(a+1)}(x) = \\
2(-1)^{m} \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} (n+k) E_{n+k-1}^{(a)}(x),
\]

\[
\sum_{k=0}^{n} \binom{n}{k} (m+k) G_{m+k-1}^{(a+1)}(x) + (-1)^{m} \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} (n+k) G_{n+k-1}^{(a+1)}(x) = \\
2(-1)^{m} \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} (n+k-1)(n+k) G_{n+k-2}^{(a)}(x),
\]

which extend Kaneko’s formula [12] to Bernoulli, Euler and Genocchi polynomials of higher order \(\alpha\).
Remark 2. If \( p = m + n + r \), with \( r \geq 1 \), similar relations exist. We only display the case \( r = 1 \):

\[
\lambda \sum_{j=0}^{n} \binom{n}{j} \frac{\mathcal{B}_{m+j+1}^{(\alpha+1)}(x; \lambda)}{m+j+1} + (-1)^{m+1} \sum_{j=0}^{m} (-1)^{j} \binom{m}{j} \frac{\mathcal{B}_{n+j+1}^{(\alpha+1)}(x; \lambda)}{n+j+1}
\]

\[
= (-1)^{m} \sum_{j=0}^{m} (-1)^{j} \binom{m}{j} \mathcal{B}_{n+j}^{(\alpha)}(x; \lambda) - \lambda (-1)^{m} \frac{n!m!}{(n+m+1)!} \mathcal{B}_{n+1}^{(\alpha+1)}(x; \lambda)
\]

\[
\lambda \sum_{j=0}^{n} \binom{n}{j} \mathcal{E}_{m+j+1}^{(\alpha+1)}(x; \lambda) + (-1)^{m} \sum_{j=0}^{m} (-1)^{j} \binom{m}{j} \frac{\mathcal{E}_{n+j+1}^{(\alpha+1)}(x; \lambda)}{n+j+1}
\]

\[
= 2(-1)^{m} \sum_{j=0}^{m} (-1)^{j} \binom{m}{j} \mathcal{E}_{n+j}^{(\alpha)}(x; \lambda) - \lambda (-1)^{m} \frac{n!m!}{(n+m+1)!} \mathcal{E}_{n+1}^{(\alpha+1)}(x; \lambda)
\]

\[
\lambda \sum_{j=0}^{n} \binom{n}{j} \mathcal{G}_{m+j+1}^{(\alpha+1)}(x; \lambda) + (-1)^{m} \sum_{j=0}^{m} (-1)^{j} \binom{m}{j} \frac{\mathcal{G}_{n+j+1}^{(\alpha+1)}(x; \lambda)}{n+j+1}
\]

\[
= 2(-1)^{m} \sum_{j=0}^{m} (-1)^{j} \binom{m}{j} \mathcal{G}_{n+j}^{(\alpha)}(x; \lambda) - \lambda (-1)^{m} \frac{n!m!}{(n+m+1)!} \mathcal{G}_{n+1}^{(\alpha+1)}(x; \lambda)
\]

5. Limit case: \( m = \rho n, n \to \infty \).

In formulas (3.10, 3.11, 3.12) of Theorem 2, let us assume that \( m = \rho n \) with \( n \) going to infinity. After dividing (3.10) by \((n + m - p)!\), we get

\[
\max(m,n) \sum_{j=0}^{\rho n} \left( \lambda \binom{n}{j} - (-1)^{j} \binom{m}{j} \right) \binom{n + m - j}{n + m - p} \mathcal{B}_{p-j}^{(\alpha+1)}(x; \lambda) = \sum_{j=0}^{m} (-1)^{j} \binom{m}{j} \binom{n + m - j}{n + m - p} (p-j) \mathcal{B}_{p-j-1}^{(\alpha)}(x; \lambda)
\]

Making use of

\[
\binom{n}{j} \sim \frac{n^j}{j!}, \quad (j \geq 0)
\]

and

\[
\binom{n + \rho n - j}{n + \rho n - p} \sim \frac{(1 + \rho)^{p-j}}{(p-j)!} n^{p-j}, \quad (0 \leq j \leq p),
\]

we obtain

\[
\max(m,n) \sum_{j=0}^{\rho n} \left( \lambda \frac{n^j}{j!} - (-\rho)^{j} \frac{n^j}{j!} \right) \frac{(1 + \rho)^{p-j}}{(p-j)!} n^{p-j} \mathcal{B}_{p-j}^{(\alpha+1)}(x; \lambda) = \sum_{j=0}^{m} (-\rho)^{j} \frac{n^j}{j!} \frac{(1 + \rho)^{p-j}}{(p-j)!} \mathcal{B}_{p-j-1}^{(\alpha)}(x; \lambda).
\]
after multiplying the two members by \( \frac{p!}{n^p(1 + \rho)^p} \) it arises

\[
\sum_{j=0}^{p} \binom{p}{j} (\lambda - (-\rho)^j)(1 + \rho)^{-j} B^{(\alpha+1)}_{p-j}(x; \lambda) = \sum_{j=0}^{p} p \binom{p-1}{j} (-\rho)^j(1 + \rho)^{-j} B^{(\alpha)}_{p-j-1}(x; \lambda),
\]

Of course in the previous equation, \( \rho \) is a rational number. But it is also valid if \( \rho \) is any complex number as proved in the following Theorem.

**Theorem 3.** For all complex number \( \rho \) and for all integer \( p \), the following formulas hold:

\[
\sum_{j=0}^{p} \binom{p}{j} (\lambda - (-\rho)^j)(1 + \rho)^{-j} B^{(\alpha+1)}_{p-j}(x; \lambda) = \sum_{j=0}^{p} p \binom{p-1}{j} (-\rho)^j(1 + \rho)^{-j} B^{(\alpha)}_{p-j-1}(x; \lambda)
\]

\[= p B^{(\alpha)}_{p-1} \left( x - \frac{\rho}{1 + \rho}; \lambda \right), \tag{5.1} \]

\[
\sum_{j=0}^{p} \binom{p}{j} (\lambda + (-\rho)^j)(1 + \rho)^{-j} \mathcal{E}^{(\alpha+1)}_{p-j}(x; \lambda) = \sum_{j=0}^{p} 2 \binom{p}{j} (-\rho)^j(1 + \rho)^{-j} \mathcal{E}^{(\alpha)}_{p-j}(x; \lambda)
\]

\[= 2 \mathcal{E}^{(\alpha)}_{p} \left( x - \frac{\rho}{1 + \rho}; \lambda \right), \tag{5.2} \]

\[
\sum_{j=0}^{p} \binom{p}{j} (\lambda + (-\rho)^j)(1 + \rho)^{-j} \mathcal{G}^{(\alpha+1)}_{p-j}(x; \lambda) = \sum_{j=0}^{p} 2 p \binom{p-1}{j} (-\rho)^j(1 + \rho)^{-j} \mathcal{G}^{(\alpha)}_{p-j-1}(x; \lambda)
\]

\[= 2p \mathcal{G}^{(\alpha)}_{p-1} \left( x - \frac{\rho}{1 + \rho}; \lambda \right) \tag{5.3} \]

**Proof.** It can be found that the generating functions of both sides of \((5.1)\) are

\[
\frac{(t(1 + \rho))^{\alpha+1}}{(\lambda e^{t(1 + \rho)} - 1)^\alpha} e^{(\alpha(1 + \rho) - \rho)t}, 
\]

of both sides of \((5.2)\) are

\[
\frac{2^{\alpha+1}}{\lambda e^{t(1 + \rho)} + 1} e^{(\alpha(1 + \rho) - \rho)t}
\]

and of both sides of \((5.3)\) are

\[
\frac{(2t(1 + \rho))^{\alpha+1}}{(\lambda e^{t(1 + \rho)} + 1)^\alpha} e^{(\alpha(1 + \rho) - \rho)t}
\]

**Remark 3.** Set \( X = x - \frac{\rho}{(1 + \rho)} \), the previous identities turn to

\[
\lambda B^{(\alpha+1)}_{p}(X + 1; \lambda) - B^{(\alpha+1)}_{p}(X; \lambda) = p B^{(\alpha)}_{p-1}(X; \lambda)(p \geq 0),
\]

\[
\lambda \mathcal{E}^{(\alpha+1)}_{p}(X + 1; \lambda) + \mathcal{E}^{(\alpha+1)}_{p}(X; \lambda) = 2 \mathcal{E}^{(\alpha)}_{p}(X; \lambda)(p \geq 0),
\]

and

\[
\lambda \mathcal{G}^{(\alpha+1)}_{p}(X + 1; \lambda) + \mathcal{G}^{(\alpha+1)}_{p}(X; \lambda) = 2p \mathcal{G}^{(\alpha)}_{p-1}(X; \lambda)(p \geq 0),
\]

which can be found in [1, 14, 20] for the first two relations.
6. Lacunary recurrence relation

In this section we consider the particular case $\lambda = 1$ and find lacunary relations for Bernoulli, Euler and Genocchi polynomials of higher order.

If $\rho = i$ ($i^2 = -1$) then formulas (5.1), (5.2), (5.3) reduce to the following lacunary relations of length 4.

Corollary 3.

$$\sum_{k=2(4)}^{p} \binom{p}{k} 2^{1-k/2}(-1)^{(k+2)/4} B_{p-k}^{(\alpha+1)}(x) = \Im \left( p B_{p-1}^{(\alpha)} \left( x - \frac{1+i}{2} \right) \right)$$  \hspace{1cm} (6.1)

$$\sum_{k=0(4)}^{p} \binom{p}{k} 2^{1-k/2}(-1)^{k/4} E_{p-k}^{(\alpha+1)}(x) = \Re \left( 2 E_{p}^{(\alpha)} \left( x - \frac{1+i}{2} \right) \right)$$ \hspace{1cm} (6.2)

$$\sum_{k=0(4)}^{p} \binom{p}{k} 2^{1-k/2}(-1)^{k/4} G_{p-k}^{(\alpha+1)}(x) = \Re \left( 2 p G_{p}^{(\alpha)} \left( x - \frac{1+i}{2} \right) \right).$$ \hspace{1cm} (6.3)

Proof Formula (5.1) with $\rho = i$ becomes

$$\sum_{j=0}^{p} \binom{p}{j} (1 - (-i)^j)(1 + i)^{-j} B_{p-j}^{(\alpha+1)}(x) = p B_{p-1}^{(\alpha)} \left( x - \frac{i}{1+i} \right).$$

The coefficients $c_j := (1 - (-i)^j)(1 + i)^{-j} = 2^{-j/2}(e^{-ij\pi/4} - (-1)^j e^{ij\pi/4})$, satisfy:

$$j \equiv 0(4) \quad c_j = 0,$$
$$j \equiv 1(4) \quad c_j = (-1)^{(j-1)/2} (1-j)/2,$$
$$j \equiv 2(4) \quad c_j = (-1)^{(j+2)/2} (2-j)/2 i,$$
$$j \equiv 3(4) \quad c_j = (-1)^{(j+1)/2} (1-j)/2.$$

So, if we consider only the imaginary part of both sides of this equation, it provides (6.1) which is a recurrence relation with a gap of length 4. The others formulas are proved in the same manner.

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