Implications on the cosmic coincidence by a dynamical extrinsic curvature

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Abstract

In this work, we apply the smooth deformation concept in order to obtain a modification of Friedmann’s equations. It is shown that the cosmic coincidence can be at least alleviated using the dynamical properties of the extrinsic curvature. We investigate the transition from nucleosynthesis to the coincidence era obtaining a very small variation of the ratio \( r = \frac{\rho_m}{\rho_{\text{ext}}} \) that compares the matter energy density to extrinsic energy density, compatible with the known behavior of the deceleration parameter. We also show that the calculated ‘equivalence’ redshift matches the transition redshift from a deceleration to accelerated phase and the coincidence ceases to be. The dynamics on \( r \) is also studied based on Hubble parameter observations as the latest Baryons Acoustic Oscillations/Cosmic Microwave Background Radiation (BAO/CMBR) + SNIa.

Keywords: comological constant, geometry, Nash theorem

(Some figures may appear in colour only in the online journal)

1. Introduction

Modifications of gravity at very large scales associated with the proposal of extra dimensions have been revealed to be an alternative route to deal with the problem of accelerated expansion of the Universe in various theories beyond the standard model of particle physics. As proposed in [1], these extra dimensions may also provide a possible explanation for the
huge difference between the two fundamental energy scales in nature, namely, the electro-
weak and Planck scales $[M_{Pl}/m_{Pl} \sim 10^{6}]$. Essentially, if our four-dimensional space-time is embedded in a higher dimensional space, then the gravitational field can also propagate along the extra dimensions, while the standard gauge interactions remain confined to the four-dimensional space-time. The majority of contributions on this theme are string (or brane) inspired models in which the brane-world is generated by the motion of a three-dimensional brane in a larger space (the bulk), e.g., [1–9]. On the other hand, non-inspired string models have been proposed with the embedding of geometries as a cornerstone of a new description of physics, e.g., [10–15]. In this regard, Rubakov and Shaposhnikov’s seminal work [16] marked the conceptual foundation of the embedding in physics considering that the standard gauge interactions and ordinary matter remain confined to the four-dimensional space-time acting as a domain wall, in the context of the non-Abelian Kaluza–Klein’s theory [17, 18].

Particularly, motivated by the possibility of finding a mechanism of investigating a more underlying physics in the embedding context, we have been investigating the physical applications and implications of the use of the concept of smooth deformations of Nash’s embedding theorem [19] where the bulk geometry is defined by the Einstein–Hilbert principle of smooth curvature and only gravity necessarily propagates in the extra dimensions. In that respect, it should be emphasized that the four-dimensionality of the embedded space-time is regarded as a consequence of the structure of the gauge field equations, which is applicable to all embedded space-times (and not just to a fixed boundary). Such extended notion of confinement is consistent with the Einstein–Hilbert dynamics for the bulk geometry and also to the extrinsic curvature satisfying the Gupta equations [20, 21] for spin-2 fields in space-time.

In the reference [22], in the context of the accelerated expansion problem, we essentially tested the proposed theoretical scheme against the data from BAO/CMB + SNIa constraints, SNLS SNIa, x-ray Galaxy clusters and the gold sample (SNIa) with the cosmic statefinder analysis (Hubble, deceleration and jerk parameters). This allowed us to verify if the cosmological models were in accordance with the conceived accelerated cosmic scenarios (e.g., $\Lambda$CDM, phantom models, quintessence, etc). As a result, we obtained a well-accommodated and constrained cosmological models against the data. In what follows, we use those previous results as a background to investigate basically the evolution of energy densities of the Universe to try to shed light on another current fundamental problem in modern cosmology. This paper is focused on the coincidence problem, sometimes referred as the ‘new’ cosmological constant (CC) problem [23–27], which, in short, refers to the lack of a proper explanation of the present-day contributions of vacuum energy density $\Omega^v_0$ and the matter energy density $\Omega^m_0$ that are roughly the same order, i.e, $\Omega^m_0 \sim \Omega^v_0$. Accordingly, if CC is considered the main cause of the accelerated expansion of the Universe, then the coincidence problem emerges. Hence, as commonly stated, it seems we are living in a very special phase to observe it.

In addition, we study a process to at least alleviating the cosmic coincidence through the introduction of the dynamics of the extrinsic curvature $k_{\mu
u}$, interpreted as an additional component to the gravitational field. Then, CC is totally uncorrelated to this process and is not considered here as a dynamical quantity. We also analyze the necessary conditions of our model to do not jeopardize the nucleosynthesis from the standard cosmological model and the compatibility with the deceleration parameter in different eras. Moreover, using a modified Friedmann equations we look for an expression in order to relate the ratio $r = \frac{\rho_c}{\rho_m}$ to the Hubble parameter $H$, where $\rho_m$ denotes the total matter energy density (including cold dark matter contribution) and the $\rho_c$ density denotes the extrinsic energy density, in order to get information of the dynamics on the ratio $r$. In addition, final remarks are presented in the
conclusion section. Finally, in the appendix section we summarize the essentials of Nash’s embedding theorem and derive a simplified procedure to obtain its main result.

2. Essentials of smooth embedding

The traditional gravitational perturbation mechanisms in cosmology are essentially plagued by coordinate gauges, mostly inherited from the group of diffeomorphisms of general relativity. Fortunately there are some very successful criteria to filter out the latter perturbations [28–30], but they still depend on a choice of a perturbative model. A lesser known, but far more general approach to gravitational perturbation can be derived from a theorem due to John Nash, showing that any Riemannian geometry can be generated by a continuous sequence of local infinitesimal increments of a given geometry [19, 31].

Nash’s theorem solves an old dilemma of Riemannian geometry, namely that the Riemann tensor is not sufficient to make a precise statement about the local ‘shape’ (curvature) of a geometrical object or a manifold. The simplest example is given by a two-dimensional Riemannian manifold, where the Riemann tensor has only one component $R_{1212}$, which coincides with the Gaussian curvature. Thus, a flat Riemannian two-manifold defined by $R_{2121} = 0$ may be interpreted as a plane, a cylinder or even a helicoid, in the sense of Euclidean geometry. The same situation can be applied to cones, ruled hyperboloids, or even to three-dimensional helicoidal space-times, which also present a vanishing curvature tensor.

Riemann regarded his concept of curvature as defining an equivalent class of manifolds instead of a specific one [32]. While such equivalence of forms is mathematically interesting, it is less than adequate to derive physical conclusions from today’s sophisticated astronomical observations. In other words, the shape (or local curvature) of an object becomes a relative concept, instead of the ‘absolute shape’ as originally intended by Riemann and as adopted by Einstein in his gravitational theory. Since our cosmological observations are made essentially in an Euclidean three-space, but gravity is expressed by a four-dimensional Riemann geometry, we need to make an agreement between the Euclidean-observed shapes and the Riemannian-evaluated shapes, notably in the theory of structure formations in cosmology.

In this regard, using only differentiable functions, Nash showed that any embedded Riemannian geometry (in the sense of positive and negative signatures) can be generated by differentiable perturbations by means of a perturbed metric $g_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}$, where

$$ \delta g_{\mu\nu} = -2k_{\mu\nu}^a dy^a, $$

where $dy^a$ is an infinitesimal displacement in the extra dimensions and $k_{\mu\nu}$ is the non-perturbed extrinsic curvature. From this new metric, we obtain a new extrinsic curvature $k_{\mu\nu}$ and the procedure can be repeated indefinitely:

$$ \tilde{g}_{\mu\nu} = g_{\mu\nu} + \delta y^a k_{\mu\nu}^a + \delta y^a \delta y^b g^{\alpha\beta} k_{\mu\alpha}^a k_{\beta\nu}^b + \cdots, $$

and gives the possibility to generate new geometries by smooth deformations. Interestingly, since this theorem sits at the very foundation of how geometrical structures are formed and compared, it may be somehow related to the formation of structures in cosmology [33] and may provide an interesting route to understand perturbations in the early Universe. In addition, in the appendix section we provide a short derivation of the main result of Nash’s embedding theorem.
3. Modified Friedmann equations

In our description of the Universe, we use the standard Friedmann–Lemaître–Robertson–Walker (FLRW) line element, which is given by

\[ ds^2 = -dt^2 + a^2 [dr^2 + f_r^2 (r)(d\theta^2 + \sin^2 \theta d\varphi^2)], \]

where the set of functions \( f_r (r) = \sin r, r, \sinh r \) corresponds to spatial curvatures \( \kappa = (1, 0, -1) \), and the function \( a = a(t) \) is the expansion parameter. This geometry can be regarded as a four-dimensional hypersurface dynamically evolving in a five-dimensional bulk space. It is important to point out that the \( y \)-coordinate, commonly used in rigid models \([2, 3]\), is omitted in the line element (3). Once the FLRW geometry is completely embedded in five-dimensions \([34–36]\), it follows the dynamics of the embedding process given by Nash’s theorem.

The bulk geometry is actually defined by the Einstein–Hilbert principle, which leads to the Einstein equations

\[ \mathcal{R}_{AB} - \frac{1}{2} \mathcal{R}_G = \alpha \kappa T^g_{AB}, \]

where \( T^g_{AB} \) denotes the energy-momentum tensor of the known sources. For the present application, capital Latin indices run from 1 to 5. Small case Latin indices refer to the only one extra dimension. All Greek indices refer to the embedded space-time counting from 1 to 4.

The confinement of gauge fields and ordinary matter are a standard assumption specially in what concerns the brane-world program as a part of the solution of the hierarchy problem of the fundamental interactions: the four-dimensionality of space-time is a consequence of the invariance of Maxwell’s equations under the Poincaré group. Such condition was latter seen to be proper to all gauge fields expressed in terms of differential forms and their duals. However, in spite of many attempts, gravitation, in the sense of Einstein, does not fit in such scheme. Thus, while all known gauge fields are confined to the four-dimensional submanifold, gravitation, as defined in the whole bulk space by the Einstein–Hilbert principle, propagates in the bulk. The proposed solution of the hierarchy problem says that gravitational energy scale is somewhere within TeV scale. We adopt the confinement as a condition based on experimentally high-energy tests indicate \([37]\). The most general expression of this confinement is that the confined components of \( T_{AB} \) are proportional to the energy-momentum tensor of general relativity: \( \alpha \kappa T_{\mu \nu} = -8\pi GT_{\mu \nu} \). On the other hand, since only gravity propagates in the bulk we have the vector and scalar components \( T_{\mu \nu} = 0 \) and \( T_{0b} = 0 \), respectively.

Since we are dealing with embedded space-times, we need to write the induced field equations of the embedded geometry. These equations result from the geometrical features of the bulk space by the integration of the Gauss–Codazzi equations defined in the Gaussian frame \( \{ Z^a, \eta^A \} \) as

\[ \mathcal{R}_{ABCD} Z^A_{, a} Z^B_{, b} Z^C_{, c} Z^D_{, d} = R_{\alpha \beta \gamma \delta} + (k_{\alpha \gamma} k_{\beta \delta} - k_{\alpha \delta} k_{\beta \gamma}), \]

\[ \mathcal{R}_{ABCD} Z^A_{, a} Z^B_{, b} Z^C_{, c} \eta^D = k_{\alpha \beta \gamma \delta}, \]

where \( \mathcal{R}_{ABCD} \) is the five dimensional Riemann tensor for the bulk, \( \{ Z^A \} \) is a perturbed coordinate originated from the Lie transport of the Gaussian frame \( \{ \chi_{\mu A}, \eta^A \} \) that gives \( Z^A = \chi^A + \eta^a (\xi^a_{\mu A}) \). The vector \( \eta^A \) denotes the components of the normal vectors of \( V_4 \) and the four dimensional Riemann tensor is denoted by \( R_{\alpha \beta \gamma \delta} \). In other words, these equations
show that the dynamics of the bulk is transferred to the embedded space serving as a reference for the Riemann curvature of the embedded space-time.

To find the field equations, we can define a five-dimensional local embedding with an embedding map $Z : V_4 \rightarrow V_5$. We admit that $Z^A$ is a regular and differentiable map with $V_4$ and $V_5$ being the embedded space-time and the bulk, respectively. The components $Z^A = f^A(x^1, \ldots, x^4)$ associate with each point of $V_4$ a point in $V_5$ with coordinates $Z^A$. These coordinates are the components of the tangent vectors of $V_4$. Moreover, taking the tangent, vector and scalar components of equation (4) defined in the Gaussian frame vielbein $\{Z^A_{\mu}, \eta^A\}$, one can obtain the following equations in the embedded space-time

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - Q_{\mu\nu} = -8\pi G T_{\mu\nu},$$

$$k^\rho_{\mu, \rho} - h_{\mu} = 0,$$

where $T_{\mu\nu}$ is the four-dimensional energy-momentum tensor of the perfect fluid expressed in co-moving coordinates as

$$T_{\mu\nu} = (p + \rho) U_{\mu} U_{\nu} + \rho g_{\mu\nu}, \quad U_{\mu} = \delta^4_{\mu}.$$

We notice from the set of equations (7) and (8) how the embedding process changes the dynamics of the geometries with the appearance of the extrinsic curvature, which is not an abrupt process, but a smooth construction in the sense of Nash’s. Another important point is that when the extrinsic curvature vanishes, one obtains only the Einstein field equations. More importantly, these equations must be understood in the context of embeddings (the principle of general covariance does not apply to the bulk geometry) and with the additional confinement conditions for ordinary matter and gauge fields. In order to recover general relativity the principle of general covariance must be restated. The higher dimensional case for these equations can be found in [38].

The quantity $Q_{\mu\nu}$ is a geometrical term defined as

$$Q_{\mu\nu} = g^{\sigma\tau} k_{\mu\sigma} k_{\nu\tau} - k_{\mu\nu} h - \frac{1}{2} (K^2 - h^2) g_{\mu\nu},$$

where $h = g^{\mu\nu} k_{\mu\nu}$, $h^2 = h.h$ and $K^2 = k^{\mu\nu} k_{\mu\nu}$. It follows that the quantity $Q_{\mu\nu}$ is conserved in the sense that

$$Q^{\mu\nu}_{\rho \nu} = 0,$$

where the symbol $(\cdot)$ denotes the four-dimensional covariant derivative.

The general solution for equation (8) using the FLRW metric is

$$k_{ij} = \frac{b}{a^2} g_{ij}, \quad k_{44} = -\frac{1}{a} \frac{b}{a} \frac{d}{dt} a$$

in this case $i, j = 1, 2, 3$, where the bending function $b(t) = k_{ij}$ is an arbitrary function of time, resulting from the Codazzi homogeneous equations in equation (8).

From the calculations of equations (7) and (8), one can obtain

$$k_{44} = -\frac{b}{a^2} \left( \frac{B}{H} - 1 \right) g_{44}, \quad h = \frac{b}{a^2} \left( \frac{B}{H} + 2 \right),$$

$$K^2 = \frac{b^2}{a^3} \left( \frac{B^2}{H^2} - 2 \frac{B}{H} + 4 \right).$$
\[ Q_{ij} = \frac{h^2}{a^3} \left( 2\frac{B}{H} - 1 \right) g_{ij}, \quad Q_{44} = -\frac{3b^2}{a^2}, \quad (13) \]

\[ Q = -(K^2 - h^2) = \frac{6b^2 B}{a^2 H} \quad (14) \]

In the case of equation (13), consider \( i, j = 1, 2, 3 \). The usual Hubble parameter in terms of the expansion scaling factor \( a(t) = a \) is denoted by \( H = \dot{a}/a \) and the extrinsic parameter \( B = b/\dot{b} \), where the dot holds for the ordinary time derivative. It is important to point out that the determination of the bending function \( b(t) \) comes from the determination of dynamical equations for the extrinsic curvature.

The study of linear massless spin-2 fields in Minkowski space-time by Fierz and Pauli dates back to late 1930’s [39]. In 1954, Gupta [20] noted that the Fierz-Pauli equation has a remarkable resemblance with the linear approximation of Einstein’s equations for the gravitational field, suggesting that such equation could be just the linear approximation of a more general, nonlinear equation for massless spin-2 fields. In reality, he found that any spin-2 field in Minkowski space-time must satisfy an equation that has the same formal structure as Einstein’s. This amounts to saying that, in the same way as Einstein’s equations can be obtained by an infinite sequence of infinitesimal perturbations of the linear gravitational equation, it is possible to obtain a non-linear equation for any spin-2 field by applying an infinite sequence of infinitesimal perturbations to the Fierz-Pauli equations. The result obtained by Gupta is an Einstein-like system of equations [20]. In the following, we apply Gupta’s equations for the specific case of the extrinsic curvature of the FLRW cosmology embedded in a space of five-dimensions using Nash’s embedding. To this end, we used the ‘vacuum’ Gupta equations

\[ F_{\mu\nu} = 0, \quad (15) \]

where, unlike the case of Einstein’s equations, we do not have the equivalent to the Newtonian weak field limit and we cannot tell about the nature of the source term of equation (15) based on current experience and observations. Accordingly, the ‘f-Ricci tensor’ and the ‘f-Ricci scalar’, defined with \( f_{\mu\nu} \) are, respectively,

\[ F_{\mu\nu} = f^{\alpha\lambda} F_{\alpha\lambda\mu} \text{ and } F = f^{\mu\nu} F_{\mu\nu} \]

and also the ‘f-Riemann tensor’

\[ F_{\alpha\lambda\mu\nu} = \partial_{\alpha} Y_{\lambda\mu\nu} - \partial_{\lambda} Y_{\alpha\mu\nu} + Y_{\alpha\sigma\mu} Y_{\lambda\nu} - Y_{\lambda\sigma\mu} Y_{\alpha\nu} \]

constructed from a ‘connection’ associated with \( k_{\mu\nu} \). We stress that the geometry of the embedded space-time has been previously defined by \( g_{\mu\nu} \). Hence, we define the tensors

\[ f_{\mu\nu} = 2 \frac{f_{\mu\nu}}{K}, \quad \text{and} \quad f^{\mu\nu} = 2 \frac{K_{\mu\nu}}{K}, \quad (16) \]

so that \( f^{\mu\nu} f_{\mu\nu} = \delta^\mu_\nu \). In the sequence we construct the ‘Levi-Civita connection’ associated with \( f_{\mu\nu} \), based on the analogy with the ‘metricity condition’ \( f_{\mu\nu} \parallel_\nu = 0 \), where \( \parallel \) denotes the covariant derivative with respect to \( f_{\mu\nu} \) (while keeping the usual (:) notation for the covariant derivative with respect to \( g_{\mu\nu} \)). With this condition we obtain the ‘f-connection’

\[ \nabla_{\mu} \lambda = \frac{1}{2} (\partial_{\mu} f_{\lambda\nu} + \partial_{\nu} f_{\lambda\mu} - \partial_{\lambda} f_{\mu\nu}) \]

and

\[ \nabla_{\nu} \lambda = f^{\lambda\sigma} \nabla_{\nu} \lambda_{\sigma\mu}. \]
Replacing these results in equation (7), we obtain the Friedman equation modified by the extrinsic curvature as
\[
\left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} = \frac{4}{3} \pi G \rho + \frac{b^2}{a^2},
\] (17)
where the general expression for \( b(t) \) is given by
\[
b(t) = \alpha_0 a^0 e^{\pm \gamma(t)},
\] (18)

where \( \alpha_0 = \frac{b_0}{a_0^0} \) denoting \( a_0 \) by the present value of the expansion scaling factor and \( b_0 \) is an integration constant representing the present-day warp of the Universe. The \( \gamma \)-exponent in the exponential function is given by
\[
\gamma(t) = \sqrt{4 \eta_0 a^4 - 3} - \frac{3}{2} \arctan \left( \frac{2}{\sqrt{4 \eta_0 a^4 - 3}} \right).
\]
The two signs represent two possible signatures of the evolution of the bending function \( b(t) \) in which we denote for simplicity \( \gamma^+ \) and \( \gamma^- \) solutions. The \( \beta_0 \) parameter affects the magnitude of the deceleration parameter \( q \) and the \( \eta_0 \) parameter measures the width of the transition phase \( z_t \) from a decelerating to accelerating regime. These two parameters were generated essentially from equation (10) and as a solution of equation (15), respectively, as shown in [21].

Accordingly, using equations (17) and (18) we can write Friedmann equations in a form
\[
H(z) = H_0 \sqrt{\Omega_m (1 + z)^3 + \Omega_{\text{ext}} (1 + z)^{4 - 2 \beta_0} e^{2 \gamma(t)}},
\] (19)
where \( H(z) \) is the Hubble parameter in terms of redshift \( z \) and \( H_0 \) is the current Hubble constant. The matter density parameter is denoted by \( \Omega_m \) and the term \( \Omega_{\text{ext}} \) stands for the density parameter associated with the extrinsic curvature.

### 4. Alleviating the coincidence

The coincidence problem basically results in the explanation of the why matter density contribution \( \rho_m \) and the vacuum contribution \( \rho_v \) are about the same order at present time. The \textit{alleviation} occurs when the contributions \( \Omega_m / \Omega_v < (1 + z)^3 \) [40, 41] and is used to select cosmological models. Since we are not attributing any dynamical property to CC, we use the extrinsic curvature to make the appropriate correction adding an extra-information to this framework with the ‘extrinsic’ contribution \( \Omega_{\text{ext}} \). Moreover, using ‘fluid analogy’ and equation (17) we denote \( \Omega_{\text{ext}} \) as
\[
\Omega_{\text{ext}} = \frac{b^2}{a^2} = \frac{8 \pi G}{3} \rho_{\text{ext}},
\]
where \( \rho_{\text{ext}} \) denotes the current extrinsic energy density. It is important to notice that matter energy density and extrinsic energy density are conserved independently in the sense that \( T^\mu_{\mu ; \nu} = 0 \) and, according to equation (10), \( Q_{\mu ; \nu} = 0 \). Thus, we can write the conservation equation for matter as
\[
\dot{\rho}_m + 3H \rho_m = 0,
\]
where the dot symbol denotes the time derivative. Moreover, using equation (17), we denote
\[
\rho_{\text{ext}} = \rho_{\text{ext}}^0 (1 + z)^{4 - 2 \beta_0},
\]
where \( \rho_{\text{ext}}^0 \) is the current extrinsic energy density, one can get the conservation equation for the extrinsic contribution as
\[
\dot{\rho}_{\text{ext}} + (4 - 2 \beta_0) H \rho_{\text{ext}} = 0.
\]
At first, due to its dynamical characteristics, we regard the extrinsic curvature as the main cause of the current accelerated expansion rather than CC in accordance with (21). Since they are two different quantities, the coincidence tends to vanish once the dynamics of extrinsic curvature can be understood. To this end, we define the ratio \( r_{\text{ext}} = \frac{\rho_{\text{ext}}}{\rho_{\text{en}}} \) and study its behavior seeking a relation with \( r \) to the Hubble parameter as we do not have an independent observational data of the ratio \( r \). As a starting point, we adopt the current value as \( r_{\text{370}} = \frac{42}{40} \) as an input parameter. Bearing in mind that the modified Friedmann equations as shown in equation (17) they can be written in terms of energy density as

\[ 3H^2 = \kappa^2 (\rho_m + \rho_{\text{ext}} \exp(\pm \gamma)), \]

and one can obtain

\[ \dot{r} = (1 - 2\beta_0) \dot{r} H. \]  

(20)

Moreover, from the analysis of the critical point from the direct derivation of equation (20), one can obtain

Figure 1. We show the Hubble parameter as a function of redshift for two models [22]. The model 1 is defined by the \( \gamma^{(+)} \) and \( \gamma^{(-)} \) solutions with the values for \( \beta_0 = 2 \) and \( \eta_0 = 0.001 \), the curves coincide with the curve from \( \Lambda \text{CDM} \) (solid line). Moreover, in the model II the curves are shown with \( \gamma^{(+)} \) (triangles) and \( \gamma^{(-)} \) (circles) solutions for the values \( \beta_0 = 2 \) and \( \eta_0 = 0.25 \). Error bar points were extracted from [46] supplemented with additional data from [47, 48] at \( H(z = 2.3) \).
\[
\left( \frac{\mathcal{L}}{\mathcal{F}} \right)_{0} = (1 - 2\beta_0)^2 r H_0^2 > 0. 
\]  
(21)

As it happens, the minima points occur at \( \beta_0 < 1/2 \), which means that \( r \) is bounded from below.

In addition, equation (20) can be easily integrated and gives the solution
\[
r(z) = r_0 (1 + z)^{2(\beta_0 - 1)},
\]
in terms of redshift, where we have used the relation \( \frac{d}{dz} = -H(1 + z) \frac{d}{dH} \).

It is important to notice that the pair of parameters \((\beta_0, \eta_0)\) in equation (17) was already constrained in the model presented in [22] in a different way and different values were obtained as compared to those in [21], where it was applied the \( \chi^2 \)-test in the parametric space with values \( \Omega_m \times \beta_0 \), and the two parameters \((\eta_0, \beta_0)\) of the model were constrained to be compatible with cosmic acceleration based on the observation of 115 events of Supernova Ia in \( z \approx 1.01 \) at 68.3\% C.L.

According to [22], based on the fact that equation (17) can provide different solutions with the term \( \gamma \), for the special case when \( \pm \gamma(z) = 0 \), one can obtain XCDM-like patterns as shown in [21] with a correspondence
\[
4 - 2\beta_0 = 3(1 + w),
\]
(23)
where \( w \) is a dimensionless parameter of the X-fluid equation of state \( w = \frac{\mathcal{L}}{\mathcal{F}} \) [43] as a ratio between its pressure \( p \) and density \( \rho \). This consideration can avoid a larger error propagation from observational data concerning cosmography parameters [45]. It is important to notice that equation (23) provides a more fundamental meaning since it can transfer to \( \beta_0 \) also the characteristic of phase transition of \( w \) cosmology, which means that cosmological eras can be established with a certain value of the parameter \( \beta_0 \) from \( w \) parameter. Moreover, we adopt the current value of Hubble constant \( H_0 \) as \( H_0 = 67.8 \pm 0.9 \text{ km s}^{-1} \text{ Mpc}^{-1} \) based on the latest observations [44].

For instance, we adopt the values \( \beta_0 = 2, \eta_0 = 0.001 \) and \( \eta_0 = 0.25 \) that passed through cosmokinetics tests. For accelerated expansion it was shown that \( 2 \leq \beta_0 \leq 3 \) [22]. Moreover, the solution with \( \beta_0 = 2 \) and \( \eta_0 = 0.25 \) can lead to a collapsing Universe with a total density parameter \( \Omega > 1 \). In figure 1, we have used equation (19) and present the evolution of the Hubble parameter for the two models \((\gamma^\perp, \gamma^\parallel)\) with adopted values of \((\beta_0, \eta_0)\). The graph also shows that those solutions are very close to \( \Lambda \text{CDM} \) prediction (solid line).

In order to do not jeopardize the nucleosynthesis, we use \( \rho_{\text{nuc}} = r(z \sim 10^3) \leq 10\% \) [49, 50] that gives a constraint \( \beta_0 \lesssim 0.465 \). Moreover, from equation (19), we can write the deceleration parameter conveniently written in terms of the redshift \( z \) as
\[
q(z) = \frac{1}{H(z)} \frac{dH(z)}{dz}(1 + z) - 1.
\]
(24)

Hence, we can write
\[
q(z) = \frac{3}{2} \frac{\Omega_m (1 + z)^3 + \gamma^* \Omega_{\text{ext}} (1 + z)^4 - 2 \beta_0 e^{\gamma^* z}}{\Omega_m (1 + z)^3 + \Omega_{\text{ext}} (1 + z)^4 - 2 \beta_0 e^{\gamma^* z}} - 1,
\]
(25)
where \( \gamma^* = \frac{4}{3} \left( 4 - 2\beta_0 \pm 2 \sqrt{\frac{4\eta_0}{1 + 3\beta_0} - 3} \right) \).

Using the related value of \( \beta_0 \) for nucleosynthesis era, the baryon contribution \( \Omega_b = 0.022 \pm 0.00023 \) and Cold dark matter contribution \( 0.1197 \pm 0.0022 \) with 68\% C.L. [44], we obtain the deceleration parameter \( q \sim 0.535 \) with the expected ratio \( r \sim 0.1 \). Moreover, for \( \beta_0 = 1/2 \) (that corresponds to the matter dominated era with \( w = 0 \) in
In accordance with equation (23) and \( \eta_0 = 0.001 \), we obtain the predicted value \( q = 1/2 \) expected for the matter domination and putting \( \beta_0 = 1/2 \) in equation (22) it converges to the value \( r_0 = 3/7 \). Actually, the same value for the deceleration parameter is obtained even considering the highest value of \( \eta_0 = 0.5 \). Moreover, taking equation (25) with the previous values for \( \beta_0 = 1/2 \) and \( q = 1/2 \) related to matter-dominated era, we can obtain an estimate for the magnitude of the ‘equivalence redshift’ \( z_e \) given by

\[
z_e = \left( \frac{4}{3} |\eta_0|^{1/4} - 1 \right).
\]

(26)

In order to get close to a description of the matter dominated era we know that the variation of the \( \eta_0 \) parameter has a constrained small value in the accelerated expansion, where was found that \( 0 \leq \eta_0 \leq 0.5 \) that gives the range \( 1 \geq z_e \geq 0.09 \). Interestingly, if we consider a tighter range for the value of \( \eta_0 \), i.e., \( 0.01 \leq \eta_0 \leq 0.25 \), we have \( 0.66 \geq z_e \geq 0.24 \), which means that the beginning of the ‘coincidence’ happens in the end of the matter dominated era, since the value of \( z_e \) matches the transition redshift from a deceleration to accelerated phase, as several different data sets indicate, e.g., \( z_t = 0.56^{+0.13}_{-0.10} \) as the latest Baryons Acoustic Oscillations/Cosmic Microwave Background (BAO/CMBR) + SNIa [51] with MLCS2K2 light-curve fitter or using the SALT2 fitter \( z_t = 0.64^{+0.13}_{-0.07} \). This leads us to the interesting conclusion that the apparent ‘coincidence’ began in the matter-dominated era and the current ‘coincidence’ observed is far from being special.

It is important to point out that the apparent variation of the \( \beta_0 \) parameter is from its inner relation to the deceleration parameter \( q \) and any change of \( \beta_0 \) is related to phase transitions in the Universe. As previously commented, the current accelerated expansion (at \( z = 0 \)) regime can be obtained with \( 2 \leq \beta_0 \leq 3 \) and with the expected \( q_0 < 0 \). In short, based on the fact that we have minima points such that \( \beta_0 < 1/2 \), the \( \beta_0 \) parameter can be constrained as \( 0.465 \leq \beta_0 \leq 3 \) from nucleosynthesis until the present-day accelerated expansion regime.

In addition, we try to obtain more information on the evolving \( r \) looking for a relation between Hubble parameter and the ratio \( r \). At first, one can write the following relation

\[
\dot{r} = H \frac{dr}{dH}. \tag{27}
\]

Starting from calculating the first derivative of the Friedman equation in equation (17) in terms of densities \( \rho_m \) and \( \rho_{m,\text{ext}} \), one can obtain

\[
H = \frac{k^2}{6} \left( (2\beta_0 - 4)\rho_{m,\text{ext}} \exp(\pm \gamma) - 3\rho_m \pm \rho_{m,\text{ext}} \exp(\pm \gamma) \right).
\]

After a long algebra, one can write

\[
\dot{H} = -\frac{3}{2} \left[ \frac{r + \Theta \exp(\pm \gamma)}{r + \exp(\pm \gamma)} \right] H^2.
\]

Interestingly, if we consider a particular case (i.e, to mimic XCDM), one can set the term \( \gamma = 0 \) and the appropriate correspondence \( \beta_0 = \frac{1}{2}(1 - 3w) \), where \( w < -1/3 \). So we have \( \Theta = 1 + w \). Hence,

\[
\dot{H} = -\frac{3}{2} \left[ \frac{1 + w + r}{r + 1} \right] H^2,
\]

which is the same equation obtained in [42].

Moreover, the relation in equation (27) can be readily integrated and after a long algebra, one can write
where the Hubble parameter $H$ is a function of $r$. The function $Q(r)$ is denoted by

\[
Q(r) = \pm \sqrt{4\eta_0 \left( \frac{r}{r_0} \right)^{\frac{4}{3}} - 3},
\]

with two signs $Q^+$ and $Q^-$. These two possibilities induce to four possible behaviors of $H(r)$ which we denote $H(r)^{+-}$, $H(r)^{--}$, $H(r)^{++}$, $H(r)^{--}$ as shown below.

\[
H(r)^{+-} = H_0 \left[ \frac{r + \exp(+\gamma)}{r_0 + \exp(+\gamma)} \right]^{\frac{4}{3}} \frac{r}{r_0}^{-\frac{4}{3}},
\]

\[
H(r)^{--} = H_0 \left[ \frac{r + \exp(-\gamma)}{r_0 + \exp(-\gamma)} \right]^{\frac{4}{3}} \frac{r}{r_0}^{-\frac{4}{3}},
\]

\[
H(r)^{++} = H_0 \left[ \frac{r + \exp(+\gamma)}{r_0 + \exp(+\gamma)} \right]^{\frac{4}{3}} \frac{r}{r_0}^{-\frac{4}{3}},
\]

\[
H(r)^{--} = H_0 \left[ \frac{r + \exp(-\gamma)}{r_0 + \exp(-\gamma)} \right]^{\frac{4}{3}} \frac{r}{r_0}^{-\frac{4}{3}}.
\]

In figure 2 we present some results of different models with an evolving Hubble parameter as the ratio $r$ increases (and the coincidence ceases to be). As a particular solution of our model, one can get the curves in figure 2 that mimics XCDM, as previously commented. The thin line mimics the $\Lambda$CDM model with $w = -1$, the thick dashed line mimics a quintessence model with $w = -0.73$. Phantom models with $w = -1.2$ are represented by the thick dotted–dashed line. The dashed lines and thick lines are solutions for $\pm \gamma \neq 0$. 

**Figure 2.** The relation of Hubble parameter to ratio $r$ ranging from the input parameter $r_0 = 3/7$ to $r = 15.4$ where is present the solutions of the model, including a XCDM-like solutions for $\pm \gamma = 0$. The thin line mimics the $\Lambda$CDM model with $w = -1$, the thick dashed line mimics a quintessence model with $w = -0.73$. Phantom models with $w = -1.2$ are represented by the thick dotted–dashed line. The dashed lines and thick lines are solutions for $\pm \gamma \neq 0$. 


dashed line. In the left panel, the dashed line and the thick line represent the functions $H(r)^{++}$ and $H(r)^{+-}$, respectively. In the right panel, the dashed line and the thick line represent the functions $H(r)^{-+}$ and $H(r)^{-}$, respectively.

We note that both left and right panels present a general similar behavior. Starting with $H(r)^{++}$ and $H(r)^{-+}$ one finds curves very close to a quintessence model. Moreover, the $H(r)^{+-}$ and $H(r)^{-}$ are close to phantom. The solution $H(r)^{-+}$ presents a smooth decaying at $r < 1$ differently from $H(r)^{-}$ that mimics a more negative equation of state with $w = -1.3$. It is worth noting to say that the evolution of $H(z)$ in terms of redshift was shown in [22] and the values of parameters ($\eta_0 = 0.25$, $\beta_0 = 2$) passed through cosmokinetics tests confronted to observational data from [46] based on observations of red-enveloped galaxies [52] and BAO peaks [53] being also supplemented with the observational data on Hubble parameter (OHD) and BAO in Lyα [47, 48] with $H(z = 2.3) = (224 \pm 8)$ km s$^{-1}$ Mpc$^{-1}$. As it may seem, the apparent coincidence is caused by the effect of extrinsic curvature on the dynamics of the Universe mainly on the variation of the deceleration parameter.

5. Final remarks

Until recently the problem of classes of equivalence of manifolds defined by the same Riemann curvature, but with different topological properties, was ignored in general relativity because the Minkowski space-time, postulated as the ground state of gravitational field, is uniquely and well defined as a flat-plane manifold, characterized by the existence of Poincaré translations in all directions in all of its points. However, recent astrophysical experiments such as WMAP, SDSS and Planck Mission, indicate that exists a small CC. Within the so called $\Lambda$CDM model, CC is regarded the main cause to the accelerated expansion of the Universe. In this case, we no longer have the Minkowski space-time as a solution of Einstein’s equations, so that the ground state of the gravitational field, or equivalently the Minkowski standard of Riemann curvature set by Einstein is ambiguous: either we have Minkowski without CC or else we have de Sitter space-times with CC. In order to solve that dilemma, the embedding between geometries has been revealed to be an appropriate underlying mechanism to obtain a gravitational theory and a possible way to get to a quantum-gravity theory [12–14] in the future. The relevant detail in Nash’s theorem is that it provides coordinate gauge free way to construct any Riemannian geometry (i.e, with positive and negative signatures), and in particular any space-time structures by a continuous sequence of infinitesimal perturbations generated by the extrinsic curvature along the extra dimensions of the bulk space. Due to its perturbative characteristics, Nash’s geometric smooth perturbative process can be an important principle to cosmological applications.

In the context of the dark energy problem, an important cosmological test was performed in a previous communication [22], where we tested the model against the data from BAO/CMB + SNIa constraints, SNLS SNIa, x-ray Galaxy clusters and the gold sample(SNIa). As a result, we obtained constrained cosmological models by the analysis of the main cosmography parameters (Hubble parameter, deceleration parameter and jerk parameter), which are regarded as the fundamental pillars for the kinematical study of the current accelerated expansion phase of the Universe. Accordingly, we also obtained different landscapes of kinematical models from a more accelerated regime (mimicking phantom models behaviors) to the $\Lambda$CDM pattern. For instance, in the present investigation, we have used models in such a way that they are related to the $\Lambda$CDM behavior as shown in figure 1 defined by the values for the $\beta_0$ and $\eta_0$ parameters.
In this paper, we have taken a different direction with the analysis on the evolution of the contribution of energy densities of matter and, as we have defined, the extrinsic contribution, since we are not attributing any dynamical characteristic to CC, which deepens the previous results shown in [22]. For any cosmological model, this defines a solid structure to advance in a future analysis in the high redshift scale. We have focused our present work on the ‘new’ CC problem, commonly referred as the coincidence problem on the values of the contributions of matter and the vacuum and the why they seem to have the same order.

Motivated by the former geometrical dilemma on Riemann’s geometry, we have studied the contribution of the extrinsic curvature as a dynamical quantity and its influence to the coincidence problem as we replace the CC contribution by the extrinsic one by means of using the essentials of the main cosmography parameters. It was shown that the coincidence problem can be at least alleviated with the presence of the extrinsic curvature. Interestingly, the $\beta_0$ parameter has revealed to be very promising term since it can be related to the magnitude of the deceleration parameter and small changes on values of $\beta_0$ may be related to transition phases of the Universe. It has helped us to understand that the current coincidence ceases to be since it began close to the end of the matter dominated or even in the passage from the decelerated to accelerated regime, where we found the ‘equivalence’ redshift around $0.66 \geq z_c \geq 0.24$ compatible with the transition phase. Moreover, the parameter $\beta_0$ could be highly constrained as $0.465 \leq \beta_0 \leq 3$ from nucleosynthesis until the present-day accelerated expansion era showing how the ratio $r$ evolved to its present value $r_0 = 3/7$. An explicit relation of the ratio of the densities $r = \frac{\rho_m}{\rho_v}$ to the Hubble parameter was obtained to understand in more details the dynamics of the ratio $r$. This analysis can be improved as the observational data of the ratio $r$ can be available in future observations.

We believe that the mechanism presented in the paper deals with the question ‘why now the coincidence?’ and our answer is focused on the understanding of the presented dynamics of the ratio $r$ that eventually converges to a coincidence point, as verified today and could not have happened before, as shown in the presented mechanism. On the other hand, in order to build a Universe landscape, in principle these (geometrical) deformations should be associated with the shapes or forms of the large structures and also the physics of the early Universe, which is far from the scope of the present paper. As a geometrical process, Nash’s perturbations may occur anywhere at any time, which suggest that the primordial perturbations in the early Universe can be investigated in a quantum theory of gravity along with Nash’s embedding concept.

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Appendix. A short view on Nash’s embedding

Based on the argument of using differentiable functions, Nash’s theorem [19] solved the embedding problem between Riemannian manifolds. In 1954, Nash showed that a $C^1$ manifold can be embedded in a $2n$-dimensional Euclidean space and in 1956, he dealt with the case of $C^k$ manifold in which $3 \leq k \leq \infty$ by using the regularity and differentiability to the embedding functions. Thus, the dimension $D$ of the bulk for a local and isometric embedding for a $V_n$ manifold depends on the embedding functions. Nash’s results were later extended by Greene to negative signatures [31]. For instance, if ones uses analytic functions
as in the Janet–Cartan theorem \cite{[54, 55]}, the bulk will have the dimension \(D \leq n(n + 1)/2\), where \(n\) denotes the dimension of the embedded space. On the other hand, if one uses the Nash–Greene theorem, the dimension of the bulk will increase by the formulae
\[
D \leq n(n + 3)/2.
\]
The central idea of Nash’s embedding theorem can summarized as

Let an embedding function \(X: \mathcal{V}_n \rightarrow \mathcal{V}_D\) be regular and differentiable, then one can construct an embedding of a manifold \(\mathcal{V}_n\) in a manifold \(\mathcal{V}_D\) sufficiently large by continuous deformation on a normal direction at a point onto the manifold \(\mathcal{V}_n\).

We start with a local and isometric embedding with the map \(X^A: \mathcal{V}_A \rightarrow \mathcal{V}_D\), where \(\mathcal{V}_n\) denotes a non-perturbed manifold to be embedded in a larger space, the bulk \(\mathcal{V}_D\). Concerning conventions \(A = 1, \ldots, D\) and \(D = 4 + n\), where \(n\) represents the number of extra dimension.

The embedding functions \(X^A = f^A(x^1, x^2, \ldots, x^D)\) associate any point of \(\mathcal{V}_n\) to \(\mathcal{V}_D\). Moreover, the coordinates \(X^A\) must satisfy the embedding equations
\begin{align}
X_{,\alpha}X_{,\beta}G_{AB} &= g_{\alpha\beta}, \quad (28) \\
X_{,\alpha}\eta_{,\beta}G_{AB} &= 0, \quad (29) \\
\eta_{,\alpha}\eta_{,\beta}G_{AB} &= g_{\alpha\beta}, \quad (30)
\end{align}
where we denote the symbols (,) for common derivative and covariant derivative, respectively. The metric \(G_{AB}\) denotes the metric of the bulk and the metric \(\bar{g}_{\alpha\beta}\) denotes the metric of the non-perturbed embedded space \(\mathcal{V}_n\). The vectors \(\bar{\eta}_{,\alpha}\), \(\bar{\eta}_{,\beta}\) are the components of normal vectors on \(\mathcal{V}_n\) with a metric \(g_{\alpha\beta}\) of an inner space \(B_n\) of these vectors, where \(a, b \rightarrow n + 1, \ldots, D\).

Given a geometric object \(\mathcal{O}\) in \(\mathcal{V}_n\), its Lie transport along the flow for a small distance \(\delta y\) is given by \(\Omega = \Omega + \delta y\mathcal{L}_\eta\Omega\), where \(\mathcal{L}_\eta\) denotes the Lie derivative with respect to \(\eta\). In particular, the Lie transport of the Gaussian frame \(\{X^\mu, \eta^A\}\), defined on \(\mathcal{V}_n\) gives
\begin{align}
Z^A &= X^A + \delta y^\mu(\mathcal{L}_\eta)^A, \quad (31) \\
Z_{,\mu}^A &= X_{,\mu}^A + \delta y \mathcal{L}_\eta^A = X_{,\mu}^A + \delta y \eta_{,\mu}^A, \quad (32) \\
\eta^A &= \bar{\eta}^A + \delta y [\bar{\eta}, \bar{\eta}]^A = \bar{\eta}^A. \quad (33)
\end{align}
We note that in general \(\eta_{,\mu} = \bar{\eta}_{,\mu}\). Moreover, in order to describe a new manifold the new coordinates \(Z^A\) they have to satisfy the similar embedding equations
\begin{align}
Z_{,\mu}Z_{,\nu}G_{AB} &= g_{\mu\nu}, \quad (34) \\
Z_{,\mu}\eta_{,\nu}G_{AB} &= g_{\mu\nu}, \quad (35) \\
\eta_{,\mu}\eta_{,\nu}G_{AB} &= g_{\mu\nu} = \epsilon_{ab}\delta_{ab}, \quad (36)
\end{align}
where \(g_{\mu\nu}\), \(g_{\mu\nu}\), and \(g_{\mu\nu}\) are perturbed metric. In addition, using equation (29), we can write equation (35) as
\[
g_{\mu\nu} = Z_{,\mu}\eta_{,\nu}G_{AB} = \delta y^\mu A_{\mu
u},
\]
where \(A_{\mu\nu}\) is the torsion vector on the perturbed embedded geometry. Using the fact that \(\eta^A = \bar{\eta}^A\), we obtain
\[ A_{\mu\nu} = \eta^A_{a,\mu} \eta^B_{a,\nu} G_{AB} = \tilde{A}_{\mu\nu}, \quad (38) \]

which shows that the torsion vector does not change under deformations.

In order to express the metric \( g_{\mu\nu} \) in terms of perturbed quantities, we take equations (31) and (34), and write

\[ g_{\mu\nu} = Z^A_{\mu,\nu} Z^B_{\nu,\mu} G_{AB} = (\chi^A_{\mu,\nu} + \delta y^a \eta^A_{a,\mu})(\chi^B_{\nu,\mu} + \delta y^b \eta^B_{b,\nu}) G_{AB}. \]

Conversely, we have

\[ g_{\mu\nu} = \tilde{g}_{\mu\nu} - 2y^a \tilde{K}_{\mu,\nu} + \delta y^a \delta y^b \eta^A_{a,\mu} \eta^B_{b,\nu} G_{AB}. \quad (39) \]

In addition, we can work on the term \( \eta^A_{a,\mu} \eta^B_{b,\nu} G_{AB} \) and express it in terms of extrinsic quantities, i.e., the extrinsic curvature \( k_{\mu\nu} \) and the torsion vector \( A_{\mu\nu} \). To do so, we use the Gauss-Weingarten equations

\[ \eta^A_{a,\mu} = A_{\mu a} g^{bc} \chi^A_{b,\nu} - \tilde{K}_{\mu,\nu} g^{bc} \chi^A_{b,\nu}. \quad (40) \]

Hence the term \( \eta^A_{a,\mu} \eta^B_{b,\nu} G_{AB} \) can be written as

\[ \eta^A_{a,\mu} \eta^B_{b,\nu} G_{AB} = (A_{\mu a} g^{bc} \chi^A_{b,\nu} - \tilde{K}_{\mu,\nu} g^{bc} \chi^A_{b,\nu})(A_{\nu b} g^{de} \chi^A_{d,\mu} - \tilde{K}_{\nu,\mu} g^{de} \chi^A_{d,\mu}) G_{AB}, \]

and using equations (28), (29) and (30) we can write

\[ \eta^A_{a,\mu} \eta^B_{b,\nu} G_{AB} = g^{cd} A_{\mu\nu\rho} A_{\rho\mu\nu} + g^{cd} \tilde{K}_{\mu\nu\rho} \tilde{K}_{\rho\mu\nu}. \quad (41) \]

Conversely, one can get the expression for the deformed geometry as

\[ g_{\mu\nu} = \tilde{g}_{\mu\nu} - 2y^a \tilde{K}_{\mu,\nu} + \delta y^a \delta y^b \eta^A_{a,\mu} \eta^B_{b,\nu} G_{AB}. \quad (42) \]

Similarly, one can obtain a deformed extrinsic curvature. To do so, we can write the extrinsic curvature in terms of the perturbed coordinate \( Z^B_{\nu,\mu} \) as

\[ k_{\mu,\nu} = -\eta^A_{a,\mu} Z^b_{a,\nu} \tilde{G}_{AB}, \quad (43) \]

and using equation (31), one obtains

\[ k_{\mu,\nu} = \tilde{k}_{\mu,\nu} - \delta y^b \eta^A_{a,\mu} \eta^B_{b,\nu} G_{AB}. \]

thus,

\[ k_{\mu,\nu} = \tilde{k}_{\mu,\nu} - \delta y^b (g^{cd} A_{\muca} A_{\nu db} + g^{cd} \tilde{K}_{\muca} \tilde{K}_{\nu db}). \quad (44) \]

Extending the analysis, Nash’s perturbative process can be understood by the relation of the metric and extrinsic curvature with a propagation of the extrinsic curvature to the extra dimensions \( y^a \). In doing so, we take the derivative of equation (42) with respect to the perturbative parameter \( y^a \) and obtain

\[ \frac{\partial g_{\mu\nu}}{\partial y^a} = -2 \tilde{k}_{\mu,\nu} + 2 \delta y^b \delta_{a} g^{cd} A_{\muca} A_{\nu db} + g^{cd} \tilde{K}_{\muca} \tilde{K}_{\nu db}. \]
and according to equation (43) we can write
\[
\frac{\partial g_{\mu \nu}}{\partial \gamma^a} = -2k_{\mu \nu a},
\]
thus,
\[
k_{\mu \nu a} = -\frac{1}{2} \frac{\partial g_{\mu \nu}}{\partial \gamma^a},
\]
(45)
or, alternatively,
\[
\delta g_{\mu \nu} = -2k_{\mu \nu a} \delta \gamma^a,
\]
(46)
that shows the Nash perturbations of geometry can be generated by small increments associated to the metric. In a different approach, the similar expression was obtained years later by Choquet-Bruhat and York [56] that was used in the ADM formulation of general relativity.

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