FIBERS OF CHARACTERS IN GELFAND-TSETLIN CATEGORIES

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Abstract. We solve the problem of extension of characters of commutative subalgebras in associative (noncommutative) algebras for a class of subrings (Galois orders) in skew group rings. These results can be viewed as a noncommutative analogue of liftings of prime ideals in the case of integral extensions of commutative rings. The proposed approach can be applied to the representation theory of many infinite dimensional algebras including universal enveloping algebras of reductive Lie algebras, Yangians and finite \( W \)-algebras. In particular, we develop a theory of Gelfand-Tsetlin modules for \( \mathfrak{gl}_n \). Besides classification results we characterize their categories in the generic case extending the classical results on \( \mathfrak{gl}_2 \).

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1. Introduction

The functors of restriction onto subalgebras and induction from subalgebras are important tools in the representation theory. The effectiveness of these tools depends upon the choice of a subalgebra. Denote by Specm\ A (Spec A) the space of maximal (prime) ideals in A, endowed with the Zarisky topology. In the classical commutative algebra setup, an integral extension A \subset B of two commutative rings induces a map \varphi : Spec B \to Spec A, whose fibers are non-empty for every point of Spec A (e.g. A = B^G, where G is a finite subgroup of the automorphism group of B). In particular, every character of A can be extended to a character of its integral extension B. Moreover, if B is finite over A then all fibers \varphi^{-1}(I), I \in Spec A are finite, and hence, the number of extensions of a character of A is finite. The Hilbert-Noether theorem provides an example of such situation with B being the symmetric algebra on a finite-dimensional vector space V and A = B^G, where G is a finite subgroup of GL(V).

The primary goal of this paper is to generalize these results to "semi-commutative" pairs \Gamma \subset U where U is an associative (noncommutative) algebra over a base field \k and \Gamma is an integral domain. The canonical embedding \Gamma \subset U induces a functor from the category of U-modules which are direct sums of finite dimensional \Gamma-modules (Gelfand-Tsetlin modules with respect to \Gamma) to the category of torsion \Gamma-modules. This functor induces a “multivalued function” from Specm\ \Gamma associating to an ideal \mathfrak{m} \in Specm\ \Gamma the fiber \Phi(\mathfrak{m}) of left maximal ideals of U that contain \mathfrak{m}. Our goal is to find natural sufficient conditions for the fibers to be non-empty and finite for any point in Specm\ \Gamma. On the other hand, for a maximal left ideal \mathfrak{I} \subset U such that U/\mathfrak{I} is a Gelfand-Tsetlin module, it is interesting to investigate its support in Specm\ \Gamma (i.e. the set of \mathfrak{m} \in Specm\ \Gamma, such that \Gamma/\mathfrak{m} is a subfactor of U/\mathfrak{I} as a \Gamma-module) and find the multiplicity of \Gamma/\mathfrak{m} in U/\mathfrak{I}.

A motivation for the study of such pairs (U, \Gamma) comes from the representation theory. The classical framework of Harish-Chandra modules ([D], Ch. 9) is related to a pair of a reductive Lie algebra \mathcal{F} and its reductive subalgebra \mathcal{F}', where U and \Gamma are their universal enveloping algebras respectively. A more general concept of Harish-Chandra modules (related to a pair (U, \Gamma)) was introduced in [DFO2].

The case when U is the universal enveloping algebra of a reductive finite dimensional Lie algebra and \Gamma is the universal enveloping algebra of a Cartan subalgebra leads to the theory of Harish-Chandra modules with respect to this Cartan subalgebra, commonly known as generalized weight modules. Classification of such simple modules is well known for \text{gl}_2 and for any simple finite-dimensional Lie algebra for modules with finite-dimensional weight spaces, due to Fernando [Fe] and Mathieu [Ma]. It remains an open problem in general. To approach this classification problem, the full subcategory of weight Gelfand-Tsetlin U(gl_n)-modules with respect to the Gelfand-Tsetlin subalgebra was introduced in [DFO1]. This class is based on natural properties of a Gelfand-Tsetlin basis for finite-dimensional representations of simple classical Lie algebras [GTs], [Zh], [M].

Gelfand-Tsetlin subalgebras were considered in [FM] in connection with the solutions of the Euler equation, in [Vi] in connection with subalgebras of maximal Gelfand-Kirillov dimension in the universal enveloping algebra of a simple Lie algebra, in [KW1], [KW2] in connection with classical mechanics, and also in [Gr] in connection with general hypergeometric functions on the Lie group GL(n, \mathbb{C}). A similar approach was used by Okunkov and Vershik in their study of the representations of the symmetric group S_n [OV], with U being the group algebra of S_n and \Gamma being the maximal commutative subalgebra generated by the Jucys-Murphy elements

\[(1i) + \ldots + (i - 1i), \quad i = 1, \ldots, n.\]
In this case the elements of Spec\(m\) \(\Gamma\) parametrize basis of irreducible representations of \(U\). Recent advances in the representation theory of Yangians ([FMO]) and finite \(W\)-algebras ([FMO1]) are also based on similar techniques.

What is the intrinsic reason of the existence of Gelfand-Tsetlin formulae and of the successful study of Gelfand-Tsetlin representations of various classes of algebras? This question led us to the introduction in [FO1] of concepts of Galois rings and Galois orders in invariant skew (semi)group rings.

For the rest of the paper we assume that \(\Gamma\) is a commutative domain, \(K\) the field of fractions of \(\Gamma\), \(K \subset L\) a finite Galois extension, \(\mathcal{M} \subset \text{Aut} L\) a submonoid closed under conjugation by the elements of the Galois group \(G = G(L/K)\). We will always assume that if for any \(m_1, m_2 \in \mathcal{M}\) from\(m_1|_K = m_2|_K\) follows \(m_1 = m_2\). Such monoid \(\mathcal{M}\) we call separating with respect to \(K\).

The group \(G\) acts on the skew semigroup ring \(L \ast \mathcal{M}\) via the action \((lm)^g = l^g m^g\), where \(m^g = g^{-1}mg\). By \(\mathcal{K}\) we denote the subring of \(G\)-invariants \((L \ast \mathcal{M})^G \subset L \ast \mathcal{M}\).

**Definition 1.** A Galois ring \(U\) over \(\Gamma\) is a finitely generated over \(\Gamma\) subring in \(K\) such that \(KU = UK = \mathcal{K}\). A Galois ring \(U\) over \(\Gamma\) is called right (respectively left) order if for any finite dimensional right (respectively left) \(K\)-subspace \(W \subset \mathcal{K}\) (respectively \(W \subset \mathcal{K}\)), \(W \cap U\) is a finitely generated right (respectively left) \(\Gamma\)-module. A Galois ring is order if it is both right and left order ([FO1]).

Galois orders are natural versions of “noncommutative orders” in skew semigroup rings of invariants. In comparison with the classical notion of an order we note, that \(\Gamma \subset U\) is not central, but maximal commutative subalgebra.

A class of Galois orders includes in particular the following subrings in the corresponding skew group rings ([FO1]): Generalized Weyl algebras over integral domains with infinite order automorphisms, e.g. \(n\)-th Weyl algebra \(A_n\), quantum plane, \(q\)-deformed Heisenberg algebra, quantized Weyl algebras, Witten-Woronowicz algebra among the others ([Ba], [BavO]); the universal enveloping algebra of \(\mathfrak{gl}_n\) over the Gelfand-Tsetlin subalgebra ([DFO1], [DFO2]), associated shifted Yangians and finite \(W\)-algebras ([FMO], [FMO1]); certain rings of invariant differential operators on torus. In Section 2 we present some necessary facts about Galois orders.

In this paper we develop a representation theory of Galois orders. The main tool for our investigation of categories of representation is a technique from [DFO2]. In Section 3 we give a detailed exposition of the results from [DFO2] adapted for case of a commutative subalgebra considered in this paper.

Last two sections are devoted to the representation theory of Galois orders. We emphasize that the theory of Galois orders unifies the representation theories of universal enveloping algebras and generalized Weyl algebras. Our main result establishes sufficient conditions for the fiber \(\Phi(m)\) to be nontrivial and finite. Let \(\ell_m\) be any lifting of \(m\) to the integral closure of \(\Gamma\) in \(L\), \(M_m\) the stabilizer of \(\ell_m\) in \(M\). Note that the group \(M_m\) is defined uniquely up to \(G\)-conjugation, hence its cardinality is correctly defined.

Our main result is the following

**Theorem A.** Let \(\Gamma\) be a commutative domain which is finitely generated as a \(k\)-algebra, \(U\) a Galois ring over \(\Gamma\), \(m \in \text{Spec} \Gamma\). Suppose that \(M_m\) is finite.

- If \(U\) is a right Galois order over \(\Gamma\) then the fiber \(\Phi(m)\) is non-empty.
- If \(U\) is a Galois order over \(\Gamma\) then the fiber \(\Phi(m)\) is finite.
The methods we use in the proof of these results allow us to show that Gelfand-Tsetlin modules for a large class of Galois orders over \( \Gamma \) have similar “nice” properties. For any \( m \in \text{Spec}_m \Gamma \) with finite \( M_m \) we obtain an effective estimate for the number of isomorphism classes of simple Gelfand-Tsetlin modules \( M \) whose support contains \( m \) and for the dimension of generalized weight spaces \( M(m) \) (Theorem 5.2). In particular, for \( U = U(gl_n) \) these numbers are limited by \( 2! \ldots (n-1)! \).

In Section 4 we collected the results related to Gelfand-Tsetlin representations in “general position”. The corresponding blocks in module category have a unique simple representation, and such a block is equivalent to the category of finite dimensional representations of a completed commutative algebra. As an application we obtain a version of the Harish-Chandra theorem for Galois orders, (Theorem B, (3)).

**Theorem B.** Let \( \Gamma \) a commutative domain which is finitely generated and normal as a \( k \)-algebra, \( U \) a Galois order over \( \Gamma \) with \( M \) being a group.

1. There exists a massive subset \( W \subset \text{Spec}_m \Gamma \) such that for any \( m \in W \), \( |\Phi(m)| = 1 \) and hence there exists a unique simple \( U \)-module \( L_m \) whose support contains \( m \).
2. The extension category generated by \( L_m \) contains all indecomposable modules whose support contains \( m \) and it is equivalent to the category of modules over the completion \( \Gamma_m \).
3. If \( M \) is a group, then for any nonzero \( u \in U \) there exists a massive set of non-isomorphic simple Gelfand-Tsetlin \( U \)-modules on which \( u \) acts non-trivially.

Note, that in the case \( U = U(sl_2) \) the structure of the category of Gelfand-Tsetlin modules is well known (see, e.g. [Dr]). This category splits into a direct sum of blocks, and each block is equivalent to the category of finite dimensional representations of complete algebras over \( k[c]_m \), where \( c \in U \) is the Casimir element and \( m \in \text{Spec}_m k[c] \) is a maximal ideal which acts on the corresponding block nilpotently. These algebras are presented by the quivers

![Diagram](image)

The first quiver corresponds to the case of generic blocks, while the second case corresponds to the blocks which contain Verma module, but does not contain a finite dimensional module. The third quiver corresponds to the blocks, containing a finite dimensional \( sl_2 \)-module. Besides, in this last case holds the relation \( ab = cd \).

In the situation of an arbitrary Galois order (even in the case of the universal enveloping algebra of \( sl_n \)) we are far from such detailed description of the blocks of the category of Gelfand-Tsetlin modules. Nevertheless we are able to characterize the structure of generic blocks. Theorem B above describes the generic blocks of the category of Gelfand-Tsetlin modules, the situation is analogous to the case of \( sl_2 \), namely generic blocks are equivalent to module categories for the algebra of formal power series.

A step in the direction of a characterization of the structure of the blocks of the category of Gelfand-Tsetlin modules is Proposition 5.1. In Section 5 we establish sufficient conditions under which a block of a Gelfand-Tsetlin category contains a finite number of non-isomorphic simple modules (Corollary 5.5). Note that an analogue of this statement (see [DFO2], Theorem 32) allows to prove the statement,

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1The problem of classification of finite dimensional modules over this algebra is a famous “Gelfand problem”, [Ge].
that the "subgeneric" blocks of Gelfand-Tsetlin modules for $U(gl_n)$ are described by a finite quiver with relations. The structure of the blocks, and closely related question of the finiteness of length of the left $U$-module $U/Um$ we will discuss in a subsequent paper.

We note here an important connection which arose in the case when $U = U(gl_n)$ and $\Gamma \subset U$ is a Gelfand-Tsetlin subalgebra. In this case an important role is played by the variety of so-called strongly nilpotent matrices ([O]). In was shown in [O] that this variety is a complete intersection. In particular, it implies that $U$ is free both as a right and as a left $\Gamma$-module ([FO2]). Kostant and Wallach ([KW1], [KW2]) introduced a generalization of the variety of strongly nilpotent matrices and revealed a deep relation between this variety and the hamiltonian mechanics. A connection between Gelfand-Tsetlin representations of $U$ and the structure of the Kostant-Wallach variety is evidently important and should be a topic of a further study.

In this paper we apply the theory only to Lie algebras of type $A$, but we believe it can be extended to other types. This technique was used in [FMO1] to address the classification problem of irreducible Gelfand-Tsetlin modules for finite $W$-algebras and shifted Yangians associated with $gl_n$ and to prove an analogue of the Gelfand-Kirillov conjecture for these algebras.

2. Preliminaries

2.1. Notations. All fields in the paper contain the base field $k$, which is algebraically closed of characteristic 0. All algebras in the paper are $k$-algebras. If $A$ is an associative ring then by $A$-mod we denote the category of finitely generated left $A$-modules. Let $\mathcal{C}$ be a $k$-category, i.e. all Hom$_{\mathcal{C}}$-sets are endowed with a structure of a $k$-vector space and all the compositions maps are $k$-bilinear. The category of $\mathcal{C}$-modules $\mathcal{C}$-Mod is defined as the category of $k$-linear functors $M: \mathcal{C} \rightarrow k$-Mod, where $k$-Mod is the category of $k$-vector spaces. The category of locally finitely generated $\mathcal{C}$-modules we denote by $\mathcal{C}$-mod. Let $G$ be a group, $X$ a $G$-set, then by $X/G$ we denote the corresponding factor-set and by $X^G$ the set of $G$-invariants.

For a set $X$ by $|X|$ we denote the cardinality of $X$. Let $G$ be a group, $H \subset G$ a subgroup. Then the notation $\sum_{g \in G/H} F(g)$ will mean, that element $g$ runs a set of representatives of $G/H$ under the assumption that the sum does non depend on the choice of these representatives.

2.2. Integral extensions. Let $A$ be an integral commutative domain, $K$ its field of fractions and $\bar{A}$ the integral closure of $A$ in $K$. The ring $A$ is called normal if $A = \bar{A}$. The following is standard

Proposition 2.1. Let $A$ be a normal noetherian ring, $K \subset L$ a finite Galois extension, $\bar{A}$ is the integral closure of $A$ in $L$. Then $\bar{A}$ is a finite $A$-module.

Let $i: A \hookrightarrow B$ be an integral extension. Then it induces a surjective map $\text{Spec} B \rightarrow \text{Spec} A$ (Spec $B \rightarrow$ Spec $A$). In particular, for any character $\chi: A \rightarrow k$ there exists a character $\tilde{\chi}: B \rightarrow k$ such that $\tilde{\chi}|_A = \chi$. If, in addition, $B$ is finite over $A$, i.e. finitely generated as an $A$-module, then the number of different characters of $B$ which correspond to the same character of $A$, is finite.

Corollary 2.1. ([S], Ch. III, Prop. 11, Prop. 16) If $A$ is a finitely generated $k$-algebra then for any character $\chi: A \rightarrow k$ there exists finitely many characters $\tilde{\chi}: \bar{A} \rightarrow k$ such that $\tilde{\chi}|_A = \chi$.

The following statement is probably well known but in [FO1] we include the proof for the convenience of the reader. Note that we consider Proposition 2.2 as a motivation for introducing the notion of a Galois order.
Proposition 2.2. ([FO1]) Let \( i : A \hookrightarrow B \) be an embedding of integral domains with a regular \( A \). Assume the induced morphism of varieties \( i^* : \text{Specm} \, B \rightarrow \text{Specm} \, A \) is surjective (e.g. \( A \subset B \) is an integral extension). If \( b \in B \) and \( ab \in A \) for some nonzero \( a \in A \) then \( b \in A \).

2.3. Skew (semi)group rings. Let \( R \) be a ring, \( M \) a submonoid of \( \text{Aut} \, R \). The skew semigroup ring, \( R \ast M \), is a free left \( R \)-module with a basis \( M \) and with the multiplication

\[
(r_1 m_1) \cdot (r_2 m_2) = (r_1 r_2^{-1})(m_1 m_2), \quad m_1, m_2 \in M, \ r_1, r_2 \in R.
\]

If \( x \in R \ast M \) and \( m \in M \) then denote by \( x_m \) the element in \( R \) such that \( x = \sum_{m \in M} x_m m. \) Set

\[
\text{supp} \, x = \{ m \in M | x_m \neq 0 \}.
\]

If a finite group \( G \) acts by automorphisms on \( R \) and by conjugations on \( M \) then \( G \) acts on \( R \ast M \). Denote by \( (R \ast M)^G \) the invariants under this action. Then \( x \in (R \ast M)^G \) if and only if \( x_{m^g} = x_m^g \) for \( m \in M, \ g \in G. \)

For \( \varphi \in \text{Aut} \, R \) and \( a \in R \) set \( H_\varphi = \{ h \in G | \varphi^h = \varphi \} \) and

\[
[a \varphi] := \sum_{g \in G/H_\varphi} a^g \varphi^g \in (R \ast M)^G. \tag{1}
\]

Then

\[
(R \ast M)^G = \bigoplus_{\varphi \in M/G} (R \ast M)^G_\varphi, \quad \text{where}
\]

\[
(R \ast M)^G_\varphi = \{ [a \varphi] | a \in R^{H_\varphi} \}.
\]

We will use the following formulae:

\[
\gamma \cdot [a \varphi] = [(a \gamma) \varphi], \quad [a \varphi] \cdot \gamma = [(a \varphi^\gamma)] \varphi, \quad \gamma \in R^G,
\]

and \( [a \varphi] = [\varphi a^{\varphi^{-1}}]. \)

Remark 2.1. The formulae (3) means, that as left \( R \)-module \( R[a \varphi]R \) is isomorphic to \( RR^\varphi \) and as a right \( R \)-module \( R[a \varphi]R \) is isomorphic to \( RR^{\varphi^{-1}}. \)

2.4. Galois rings. The notations and results Subsection 2.3 we will use in the case when \( R = L \) is a field, \( K \subset L \) is a finite Galois extension of fields, \( G = G(L/K) \) its Galois group. We will denote by \( \iota \) the canonical embedding \( K \hookrightarrow L. \) Recall, that we will use the notations introduced before Definition 1.

Let \( S \subset M \) be a finite \( G \)-invariant subset, \( V \subset K \) a \( G \)-submodule. Then we introduce the following \( G \)-submodule in \( V \)

\[
V(S) = \{ x \in V | \supp \, x \subset S \} \tag{4}
\]

We need the following simple fact.

Lemma 2.1. In the assumption above the right and the left dimensions of \( K(S) \) coincides with \( |S/G|. \)

In particular, for any \( \varphi \in M \) holds

\[
\dim_K K_\varphi = [L^{H_\varphi} : K] = [G : H_\varphi] = |G_\varphi|. \tag{5}
\]

\( K_\varphi \) is irreducible as a \( K \)-bimodule. Such bimodule is denoted in [FO1] by \( V(\varphi). \)

\footnote{Recall, that the notation \( g \in G/H_\varphi \) means that \( g \) runs over a set of representatives of cosets from \( G/H_\varphi \) and the result does not depend on a choice of these representatives.}
Note also, that since $\Gamma$ acts torsion-free on $\mathcal{K}$ both from the left and from the right, we obtain that the canonical maps
\begin{align}
e_r : U \otimes_\Gamma K &\longrightarrow \mathcal{K}, \ u \otimes x \mapsto ux, u \in U, x \in K; \\
e_l : K \otimes_\Gamma U &\longrightarrow \mathcal{K}, \ x \otimes u \mapsto xu, u \in U, x \in K
\end{align}
isomorphism of $\Gamma - K$ and $K - \Gamma$-bimodules correspondingly.

Recall, that the monoid $\mathcal{M} \subset \text{Aut} \, L$ from definition of a Galois ring we assume to be separating (with respect to $K$), i.e. if for any $m_1, m_2 \in \mathcal{M}$ the equality $m_1|_K = m_2|_K$ implies $m_1 = m_2$. An automorphism $\varphi : L \longrightarrow L$ is called separating (with respect to $K$) if the monoid generated by $\{\varphi^g | g \in G\}$ in $\text{Aut} \, L$ is separating.

**Lemma 2.2.** ([FO1], Proposition 2.3) Let $\mathcal{M}$ be a separating monoid with respect to $K$. Then
\begin{enumerate}
\item $\mathcal{M} \cap G = \{e\}$.
\item For any $m \in \mathcal{M}, m \neq e$ there exists $\gamma \in K$ such that $\gamma^m \neq \gamma$.
\item If $Gm_1G = Gm_2G$ for some $m_1, m_2 \in \mathcal{M}$, then there exists $g \in G$ such that $m_1 = m_2^g$.
\item If $\mathcal{M}$ is a group, then the statements (1), (2), (3) are equivalent and each of them implies that $\mathcal{M}$ is separating.
\end{enumerate}

Let $U$ be a Galois $\Gamma$-ring over $\Gamma$ (cf. Introduction). Denote by $i : \Gamma \hookrightarrow U$.

**Lemma 2.3.** (compare [FO1], Lemma 4.1) Let $u \in U$ be nonzero element,
\[ T = \text{supp} \, u, \ u = \sum_{m \in T} [a_m \cdot m]. \]
Then
\[ K(\Gamma u \Gamma) = (\Gamma u \Gamma)K = KuK = \bigoplus_{m \in T} V(a_m \cdot m), \]
where $V(a_m \cdot m) = K[a_m \cdot m]K$ is an irreducible $K$-bimodule.

In particular it shows that for every $m \in \mathcal{M}$, $U$ contains the elements $[b_1 \cdot m], \ldots, [b_k \cdot m]$ where $b_1, \ldots, b_k$ is a $K$-basis in $L^{H_m}$. Let $e \in \mathcal{M}$ be the unit element and $U_e = U \cap Ke$.

**Theorem 2.1.** ([FO1], Theorem 4.1) Let $U$ be a Galois ring over $\Gamma$. Then
\begin{enumerate}
\item $U_e \subset K$.
\item $U \cap K$ is a maximal commutative $k$-subalgebra in $U$.
\item The center $Z(U)$ of $U$ equals $U \cap K^\mathcal{M}$.
\end{enumerate}

2.5. Galois orders and Harish-Chandra subalgebras. In this section we recall the basic properties of Galois orders following [FO1]. For simplicity we only consider right Galois orders. Let $M$ be a right $\Gamma$-submodule of a Galois order $U$ over $\Gamma$. Set
\[ \mathbb{D}_r(M) = \{u \in U \mid \text{there exists } \gamma \in \Gamma, \gamma \neq 0 \text{ such that } u \cdot \gamma \in M\}. \]
We have the following characterization of Galois orders.

**Proposition 2.3.** ([FO1], Proposition 5.1) A Galois ring $U$ over a noetherian $\Gamma$ is a right order if and only if for every finitely generated right $\Gamma$-module $M \subset U$, the right $\Gamma$-module $\mathbb{D}_r(M)$ is finitely generated.

In particular, if $U$ is right integral then $\Gamma \subset U_e$ is an integral extension and $U_e$ is a normal ring.

Recall that $\Gamma$ is called a Harish-Chandra subalgebra in $U$ if $\Gamma u \Gamma$ is finitely generated both as a left and as a right $\Gamma$-module for any $u \in U$ [DFO2]. We will also say that $\Gamma$ is a right (left) Harish-Chandra subalgebra if $\Gamma u \Gamma$ is finitely generated.
as a right (left) $\Gamma$-module for any $u \in U$. Note, that these property is enough to check for some set of generators of the ring $U$ over $\Gamma$. Harish-Chandra subalgebras of Galois rings have the following properties.

**Proposition 2.4.** If $U$ is a right (left) Galois order over a noetherian $\Gamma$ then for any $m \in M$ holds $m^{-1}(\Gamma) \subset \Gamma$ $(m(\Gamma) \subset \Gamma)$.

**Proposition 2.5.** Assume $\Gamma$ is finitely generated algebra over $k$, $U$ is a Galois ring. Then $\Gamma$ is a Harish-Chandra subalgebra in $U$ if and only if $m \cdot \Gamma = \Gamma$ for every $m \in M$.

**Proposition 2.6.** ([FO1], Corollary 5.3) If $U$ is a Galois order over $\Gamma$ and $\Gamma$ is a noetherian $k$-algebra then $\Gamma$ is a Harish-Chandra subalgebra in $U$.

The next Lemma is a main technical tool in our investigation of representations of Galois orders. Let $S \subset M$ be a finite $G$-invariant subset. Denote

$$U(S) = \{ u \in U \mid \text{supp } u \subset S \}.$$

For every $f \in \Gamma$ consider $f^S \subset \Gamma \otimes_k K$ as follows

$$f^S = \prod_{s \in S} (f \otimes 1 - 1 \otimes f^{-1}) = \sum_{i=0}^{\vert S \vert} f^{\vert S \vert-i} \otimes T_i, \ (T_0 = 1).$$

Similarly we define $f^I = \prod_{s \in S} (f \otimes 1 - 1 \otimes f) \in K \otimes_k \Gamma$.

**Lemma 2.4.** ([FO1], Lemma 5.2) If $m^{-1}(\Gamma) \subset \Gamma$ $(m(\Gamma) \subset \Gamma$ respectively) for all $m \in M$, then for any $G$-invariant subset $X \subset M$ and $f_X = f_X^I$. Besides for a $G$-invariant subset $S \subset M$ holds the following.

1. $u \in U(S)$ if and only if $f_S \cdot u = 0$ for every $f \in \Gamma$.
2. If $T = \text{supp } u \setminus S$ then $f_T \cdot u \in U(S)$ for every $f \in \Gamma$.
3. If $f_S = \sum_{i=1}^n f_i \otimes g_i, \ [am] \in L \ast M$ then

$$f_S \cdot [am] = \{ [\sum_{i=1}^n f_i g_i^m u] = [\prod_{s \in S} (f - f^{ms^{-1}})am] .$$

4. Let $S$ be a $G$-orbit and $T$ an $G$-invariant subset in $M$. The $G$-bimodule homomorphism $P^T_S(= P^T_S(f)) : U(T) \rightarrow U(S), \ u \mapsto f_{T\setminus S} \cdot u, \ f \in \Gamma$ is either zero or $\text{Ker } P^T_S = U(T \setminus S)$.

5. Let $S = S_1 \sqcup \cdots \sqcup S_n$ be the decomposition of $S$ in $G$-orbits and $P^S_i : U(S) \rightarrow U(S_i), \ i = 1, \ldots , n$ are defined in (4) nonzero homomorphisms. Then the homomorphism

$$P^S : U(S) \rightarrow \bigoplus_{i=1}^n U(S_i), \ P^S = (P^S_{S_1}, \ldots , P^S_{S_n}) ,$$

is a monomorphism.

We have the following equivalent conditions for Galois ring to be a Galois order.

**Theorem 2.2.** (cf. [FO1], Theorem 5.1) Let $U$ be a Galois ring over a noetherian $\Gamma$ and assume that $\Gamma$ is a right (left) Harish-Chandra $k$-subalgebra of $U$. Then the following statements are equivalent.

1. $U$ is right (respectively left) Galois order.
2. $U(S)$ is finitely generated right (respectively left) $\Gamma$-module for any finite $G$-invariant $S \subset M$. 

(3) $U(G \cdot m)$ is finitely generated right (respectively left) $\Gamma$-module for any $m \in M$.

**Theorem 2.3.** (cf. [FO1], Theorem 5.2) Let $U$ be a Galois ring over a noetherian $\Gamma$ and $M$ a subgroup of $\text{Aut} L$.

1. If $U_e$ is integral extension of $\Gamma$ and $m^{-1}(\Gamma) \subset \bar{\Gamma}$ (respectively $m(\Gamma) \subset \bar{\Gamma}$), then $U$ is a right (respectively left) Galois order if and only if $U_e$ is an integral extension of $\Gamma$.

2. If $U_e$ is an integral extension of $\Gamma$ and $\Gamma$ is a Harish-Chandra $k$-subalgebra in $U$, then $U$ is Galois order over $\Gamma$.

The corollary below gives us a converse statement to Theorem B.

**Corollary 2.2.** Let $U \subset L \ast M$ be a Galois ring over a noetherian $\Gamma$. If every character $\chi : \Gamma \to k$ extends to a representation of $U$ then $U_e \subset \bar{\Gamma} \cap K$. If in addition $M$ is a group and $\Gamma$ is a Harish-Chandra subalgebra then $U$ is a Galois order.

**Proof.** If $\chi$ extends to a representation of $U$, then it extends to a representation of $U_e \subset K$ in particular. Proposition 2.2 implies that $U_e$ belongs to the integral closure of $\Gamma$ in $K$. The second statement follows immediately from Theorem 2.3. □

The next corollary sets a bridge between the theory of (noncommutative) Galois orders and commutative case in Proposition 2.2.

**Corollary 2.3.** ([FO1], Corollary 5.6) Let $U \subset L \ast M$ be a Galois ring over noetherian $\Gamma$, $M$ a group and $\Gamma$ a normal $k$-algebra. Then the following statements are equivalent

1. $U$ is a Galois order.

2. $\Gamma$ is a Harish-Chandra subalgebra and, if for $u \in U$ there exists a nonzero $\gamma \in \Gamma$ such that $\gamma u \in \Gamma$ or $u \gamma \in \Gamma$, then $u \in \Gamma$.

### 3. Gelfand-Tsetlin categories

**3.1. Motivation.** The constructions of this section are the main tools we will use to investigate the class of Gelfand-Tsetlin $U$-modules. First such constructions appeared in [DFO2] in general setting, but for our purposes we consider here a special case of a commutative subalgebra $\Gamma$ and present it in details. In this section we just assume that $\Gamma$ is a commutative Harish-Chandra subalgebra in a finitely generated associative algebra $U$.

Before going into details we give some motivation of the constructions below. Let $U$ be a finite dimensional associative algebra over $k$, $R \subset U$ its Levi subalgebra, $\Gamma \subset U$ the center of $R$. Then $\Gamma = \bigoplus_{i=1}^{n} k e_i$, where $\{e_1, \ldots, e_n\}$ is the complete (i.e. $e_1 + \cdots + e_n = 1$) family of mutually orthogonal idempotents. Obviously, $\Gamma \subset U$ is a Harish-Chandra subalgebra.

This data allows to present $U$ as a ring of the $n \times n$ matrices of the form

$$
(e_j U e_i)_{i,j=1,...,n}, \quad u \mapsto \begin{pmatrix}
e_1 u e_1 & \ldots & e_1 u e_n \\
\vdots & \ddots & \vdots \\
e_n u e_1 & \ldots & e_n u e_n
\end{pmatrix}
$$

with the standard multiplication. This presentation is called the **two-sided Pierce decomposition of $U$**. Besides, we can associate with $U$ a $k$-linear category

$A = A(U; \Gamma)$ where $\text{Ob} A = \{1, \ldots, n\}$, $A(i,j) = e_j U e_i$,
and the composition of morphism is defined by the multiplication in $U$. One simple but important remark is the existence of the canonical isomorphism

$$U - \text{Mod} \cong \mathcal{A} - \text{Mod}. \quad (10)$$

If $\Gamma = R$, equivalently, $U$ is a basic (or Morita reduced) algebra, then the category $\mathcal{A}$ is presented usually as a quiver with relations. This presentation is the key feature in the study of finite dimensional representations of $U$ (see e.g. [DK], [GR] for details).

The definition of $\mathcal{A}$ can be rewritten as follows. As objects of $\mathcal{A}$ one can consider $\text{Specm} \Gamma = \{m_1, \ldots, m_n\}$, where $m_i$ is the kernel of the projection of $\Gamma$ onto $\kappa e_i$. Besides, set $\mathcal{A}(m_i, m_j) = \Gamma / m_j \otimes_{\Gamma} U \otimes_{\Gamma} \Gamma / m_i$ with the composition of morphisms induced by the multiplication in $U$. The construction of the category $\mathcal{A}$ in [DFO2] can be considered as a generalization of the two sided Pierce decomposition. The construction below (see Definition 3) is a special case of a commutative Harish-Chandra subalgebra $\Gamma \subset U$, where $U$ is not necessary finite dimensional. As above, we associate with the pair $\Gamma \subset U$ a category $\mathcal{A}$ with $\text{Ob} \mathcal{A} = \text{Specm} \Gamma$. Unfortunately, there is no equivalence of categories of $U$-modules and $\mathcal{A}$-modules. Instead we have a weaker result for the full subcategory of Gelfand-Tsetlin $U$-modules (see Theorem 3.2).

3.2. Gelfand-Tsetlin modules. We assume $U$ an algebra over $\kappa$ and $\Gamma \subset U$ is a commutative finitely generated subalgebra. The following is the key notion of the paper.

**Definition 2.** A finitely generated $U$-module $M$ is called Gelfand-Tsetlin module (with respect to $\Gamma$) provided that the restriction $M|_\Gamma$ is a direct sum of $\Gamma$-modules

$$M|_\Gamma = \bigoplus_{m \in \text{Specm} \Gamma} M(m), \quad (11)$$

where

$$M(m) = \{v \in M | m^k v = 0 \text{ for some } k \geq 0\}.$$

When for all $m \in \text{Specm} \Gamma$ and all $x \in M(m)$ holds $mx = 0$ such Gelfand-Tsetlin module $M$ is called weight module (with respect to $\Gamma$). More generally, for a left (right) $\Gamma$-module $X$ and $m \in \text{Specm} \Gamma$ we call an element $x \in X$ left (right) $m$-nilpotent, provided that $m^k x = 0$ ($xm^k = 0$) for some $k \geq 1$.

All Gelfand-Tsetlin modules form a full abelian and closed with respect to extensions subcategory $\mathcal{H}(U, \Gamma)$ in $U - \text{mod}$. A full subcategory of $\mathcal{H}(U, \Gamma)$ consisting of weight Gelfand-Tsetlin we denote by $\mathcal{H}W(U, \Gamma)$.

The support of a Gelfand-Tsetlin module $M$ is a set

$$\text{supp } M = \{m \in \text{Specm} \Gamma | M(m) \neq 0\}.$$

For $D \subset \text{Specm} \Gamma$ denote by $\mathcal{H}(U, \Gamma, D)$ the full subcategory in $\mathcal{H}(U, \Gamma)$ formed by $M$ such that $\text{supp } M \subset D$. For a given $m \in \text{Specm} \Gamma$ we denote by $\chi_m : \Gamma \rightarrow \Gamma / m$ the corresponding character of $\Gamma$. Conversely, for a character $\chi : \Gamma \rightarrow \kappa$ denote $m_\chi = \text{Ker } \chi$, so we will identify the set of all characters of $\Gamma$ with $\text{Specm} \Gamma$. If there exists a simple Gelfand-Tsetlin module $M$ with $M(m) \neq 0$ then we say that the character $\chi_m$ lifts to $M$.

From now on we assume that $\Gamma$ is a Harish-Chandra subalgebra in $U$. For a $\Gamma$-bimodule $V$ any pair $(m, n) \in \text{Specm } \Gamma \times \text{Specm } \Gamma$ and $m, n \geq 0$ we will use the following notations:

$$n^n V = V / n^n V, \quad V_m^m = V / V m^m, \quad n^n V_m^m = V / (n^n V + V m^m). \quad (12)$$

For $a \in U$ and $V = \Gamma a \Gamma$ denote the first two bimodules above by $L_{a, m}$ and $R_{a, n}$ respectively.
Lemma 3.1. In the assumption above holds the following.

1. The modules $L_{a,m}, R_{a,n}$ are finite dimensional.
2. Any from the following three conditions
   (a) $\Gamma/n$ is the subquotient of $L_{a,1}$ as a left $\Gamma$-module.
   (b) $\Gamma/m$ is the subquotient of $R_{a,1}$ as a right $\Gamma$-module.
   (c) $\Gamma/n \otimes_{\Gamma} \Gamma \otimes_{\Gamma} \Gamma/m \neq 0$.
   defines the same set $X_a$ of pairs $(m, n) \in \text{Specm} \Gamma \times \text{Specm} \Gamma$.
3. For $m, n \in \text{Specm} \Gamma$ define the set
   $$X_a(m) = \{ n \in \text{Specm} \Gamma \mid (m, n) \in X_a \}$$
   and the set
   $$X^n(a) = \{ m \in \text{Specm} \Gamma \mid (m, n) \in X_a \}.$$
   Then both $X_a(m)$ and $X^n(a)$ are finite. Besides, the kernels of simple subquotients of all $L_{a,m}$ (respectively $R_{a,n}$) belong to $X_a(m)$ (respectively $X^n(a)$).
4. Let $M$ be a Gelfand-Tsetlin $U$-module, $m \in \text{Specm} \Gamma$. Then
   $$(12) \quad aM(m) \subset \sum_{n \in X_a(m)} M(n).$$
5. Let $M$ be a Gelfand-Tsetlin $U$-module, $\pi_n : M \rightarrow M(n)$, $n \in \text{Specm} \Gamma$ the canonical projection and $m \notin X^n(a)$. Then
   $$(13) \quad \pi_n(aM(m)) = 0.$$
6. If $X$ is a finite-dimensional $\Gamma$-module then $U \otimes_{\Gamma} X$ is a Gelfand-Tsetlin module.

Proof. We will prove the statements for $L_{a,m}$. The case of $R_{a,n}$ is analogous.

Since $\Gamma$ is finitely generated, $\dim_k \Gamma/m^m < \infty$. Then $L_{a,m} \cong \Gamma \alpha \Gamma \otimes_{\Gamma} \Gamma/m^m$ is finite dimensional, since $\Gamma \alpha \Gamma$ is finitely generated as a right $\Gamma$-module. This proves (1). If $L_{a,m} = \bigoplus_{k=1}^t L_k$ is a decomposition into a direct sum of indecomposable left $\Gamma$-modules, then for every $k = 1, \ldots, t$ there exists $n_k \in \text{Specm} \Gamma$ and $n_k \geq 1$, such that $n_k \Gamma L_k = 0$. In particular, the subquotients $L_k$ are isomorphic to $\Gamma/n_k \Gamma$. On the other hand, $\Gamma/n \otimes_{\Gamma} \Gamma \cong L_0/n L_0$ is nonzero if and only if $n = n_k$, which, together with (1), proves (2) and (3). To prove (4) consider any $x \in M(m)$. Then there exists $m > 1$, such that $m^m x = 0$. It follows that the left $\Gamma$-submodule $\Gamma \alpha \Gamma x \subset M$ is the factor of $L_{a,m^m}$. Then the statement follows from (3). The statement (5) is proved analogously. To show (6) it is enough to consider the case $\dim_k V = 1$. But then the statement follows from (1).

Denote by $\Delta$ the minimal equivalence on $\text{Specm} \Gamma$ containing all $X_a$, $a \in U$ and by $\Delta(U, \Gamma)$ the set of the $\Delta$–equivalence classes on $\text{Specm} \Gamma$.

Lemma 3.2. Let $X, X'$ be Gelfand-Tsetlin modules, $\text{supp} X \subset D$, $\text{supp} X' \subset D'$, where $D$ and $D'$ are different classes of $\Delta$-equivalence. Then

$$\text{Hom}_U(X, X') = 0, \text{Ext}^1_U(X, X') = 0.$$  

Proof. Obviously, $\text{Hom}_U(X, X') = 0$ all the moreover $\text{Hom}_U(X, X') = 0$. It is enough to prove that every exact sequence in $U - \text{mod}$

$$0 \rightarrow X' \xrightarrow{\alpha} Y \xrightarrow{\beta} X \rightarrow 0$$

splits. Since $D \cap D' = \emptyset$, this sequence splits uniquely as a sequence of $\Gamma$-modules, thus we can assume

$$Y(m) = X'(m) \oplus X(m), m \in \text{supp} Y.$$
Since $D \cap D' = \varnothing$, in this sum either $X'(m) = 0$ or $X(m) = 0$. For $a \in U$ holds $aX'(m) \subseteq X'$ and $aX(m) \subseteq X$, by Lemma 3.1, (4). Hence $X'$ and $X$ are $U$-submodules in $Y$.

Immediately from Lemma 3.1, (4), (5) and Lemma 3.2 we obtain the following

**Corollary 3.1.**

$$\mathbb{H}(U, \Gamma) = \bigoplus_{D \in \Delta(U, \Gamma)} \mathbb{H}(U, \Gamma, D).$$

The subcategory $\mathbb{H}(U, \Gamma, D)$, where $D \in \Delta(U, \Gamma)$ will be called a block of $\mathbb{H}(U, \Gamma)$.

### 3.3. The category $A$

For a $\Gamma$-bimodule $V$ denote by $\hat{\mathcal{V}}_n$ the $I$-adic completion of $\Gamma \otimes_k \Gamma$-module $V$, where $I \subseteq \Gamma \otimes \Gamma$ is a maximal ideal $I = n \otimes \Gamma + \Gamma \otimes m$, in other words

$$\hat{\mathcal{V}}_n = \lim_{\substack{\longrightarrow \\n, m}} V_{n^m}.$$  

Let $B$ be a $\Gamma$-bimodule, satisfying the Harish-Chandra condition: every finite generated bimodule in $B$ is finitely generated both from the left and from the right.

Denote by $F(B)$ the set of finitely generated $\Gamma$-subbimodules in $B$. Note, that if $V, W \in F(B)$ and $V \subseteq U$, then the canonical embedding induces a monomorphism $\hat{\mathcal{V}}_n \hookrightarrow \hat{\mathcal{V}}_W$. It allows us to give the following definition.

The finitary completion $\hat{\mathcal{B}}_m$ of the bimodule $B$ by the ideal $I = n \otimes \Gamma + \Gamma \otimes m$ is a $(\Gamma \otimes \Gamma)_I$ module

$$\hat{\mathcal{B}}_m = \lim_{\substack{\longrightarrow \\n, m}} V_{n^m}.$$  

For any $V \in F(B)$, $m, n \in \text{Specm} \Gamma$, $m, n > 0$ we have a family of $\Gamma$-bimodule morphisms

$$\hat{\mathcal{V}}_n \longrightarrow V_{n^m} \longrightarrow \hat{\mathcal{B}}_m,$$

where the first is the canonical map from the projective limit and the second is induced by the embedding $V \subseteq B$. This family defines a homomorphism

$$\Psi : \hat{\mathcal{B}}_m = \lim_{\substack{\longrightarrow \\n, m}} V_{n^m} \longrightarrow \hat{\mathcal{V}}_n \longrightarrow \hat{\mathcal{B}}_m = \hat{\mathcal{B}}_m.$$

If $B$ is finitely generated as $\Gamma$-bimodule then $\Psi$ is an isomorphism.

For $m \in \text{Specm} \Gamma$ denote by $\hat{\Gamma}_m = \lim_{\longrightarrow \\Gamma/m^m}$ the completion of $\Gamma$.

**Definition 3.** Define a category $A = A_{U, \Gamma}$ with objects $\text{Ob} A = \text{Specm} \Gamma$. The space of morphisms from $m$ to $n$ is defined as the completion of $U$, i.e.

$$A(m, n) = \hat{\mathcal{U}}_n \left( = \lim_{\substack{\longrightarrow \\n, m}} \lim_{\substack{\longrightarrow \\n, m}} V_{n^m} \right).$$

The spaces $A(m, n)$ are endowed with the standard topology defined by the limits. Besides, any $A(m, n)$ is endowed with a canonical structure of a completed $\hat{\Gamma}_n - \hat{\Gamma}_m$-bimodule. For $l \in \text{Specm} \Gamma$ set

$$\hat{V}_{n^m} = \{ \bar{a} \in V_{n^m} \mid l \bar{a} = 0 \text{ for some } l \geq 1 \},$$

$$\hat{V}^l = \{ \bar{b} \in \hat{V} \mid \bar{b}^l = 0 \text{ for some } l \geq 1 \}.$$  

For $V \subseteq U$ consider $V_{n^m}$ as a left $\Gamma$-module. By Lemma 3.1, (3) for every $a \in V$ there exists a finite set $X_a(m) = \{ n_1, \ldots, n_k \}$ and $N \geq 1$, $N = N(a, m)$, such that
Lemma 3.4. Let $m, n \in \text{Specm} \Gamma$ the map $U$ in (1). The statements (3) and (4) are proved analogously to (1).

Proof. Let $m$ in $\text{Specm} \Gamma$ and $m \geq 1$. Then

$$V_m^m = \bigoplus_{t \in X_m} ^1V_m^m, \text{ where } X_m = \bigcup_{a \in V} X_a(m)$$

(15)

$$n^nV = \bigoplus_{t \in X^n} n^nV^t, \text{ where } X^n = \bigcup_{a \in V} X^n(a).$$

Lemma 3.3. (1) Let $D, D' \subset \text{Specm} \Gamma$ be two different classes of $\Delta$-equivalence, $m \in D, n \in D'$. Then $A(m, n) = 0$.

(2) We have a decomposition of $A$ into a direct sum of its full subcategories,

$$A = \bigoplus_{D \in \Delta(U, T)} A_D,$$

where $A_D$ is the restriction of $A$ on $D$.

(3) If for $a \in U$ holds $n \notin X_a(m)$, then the class of $a$ in $A(m, n)$ equals 0.

(4) If for $a \in U$ holds $m \notin X^n(a)$, then the class of $a$ in $A(m, n)$ equals 0.

Let $V$ and $W$ be finitely generated $\Gamma$-submodules in $U$. Then the bimodule $T \subset U$, spanned by all products $vw, \ v \in V, \ w \in W$, is finitely generated submodule in $U$, since $\Gamma$ is a Harish-Chandra subalgebra in $U$. Denote by $\mu : V \otimes_{\Gamma} W \rightarrow T$ the map $\mu(v \otimes w) = vw, \ v \in V, \ w \in W$.

For a $\Gamma$-bimodule $B$ denote by $n^n\mathbb{1}_m^m (\ n^n\mathbb{1}_m^m (B))$ the canonical epimorphism $n^n\mathbb{1}_m^m : B \rightarrow B/(n^nB + Bm^m)$.

Lemma 3.4. Let $V, W \subset U$ be finitely generated $\Gamma$-bimodules, $T = VW, \ m, n \in \text{Specm} \Gamma, \ m, n \geq 0$.

(1) For $p, p' \in \text{Specm} \Gamma, \ p \neq p'$ holds $n^n V^p \otimes_{\Gamma} p' W_m^m = 0$.

(2) The induced by the decomposition (15) homomorphism

$$\Phi : \bigoplus_{p \in X^n \cap X_m} (n^n V^p \otimes_{\Gamma} p W_m^m) \rightarrow n^n V \otimes_{\Gamma} W_m^m$$

is an isomorphism of $\Gamma$-bimodules.

(3) There exists $P > 0 (= P(m, m, n, V, W))$, such that for any $p \geq P$ the canonical projections

$$n^n \pi_{P^p} : n^n V \rightarrow n^n V^{P^p}, \ p^p \pi_{m^m} : W_m^m \rightarrow p W_m^m,$$

induce isomorphisms

$$n^n \pi_{P^p} : n^n V^p \rightarrow n^n V^{P^p}, \ p^p \pi_{m^m} : p W_m^m \rightarrow p W_m^m.$$

(4) There exists $P > 0$, such that for $p \geq P$ there exists a unique homomorphism

$$\hat{\mu}(m^m, n^n) : \bigoplus_{p \in X^n \cap X_m} (n^n V^p \otimes_{\Gamma} p W_m^m) \rightarrow n^n T_m^m,$$

which makes the diagram
The statement (1) obviously follows from the Chinese remainder theorem. The statement (2) follows from the decomposition (15) and from the statement (1) above. To prove the statement (3) note first that the sequence of finite dimensional
modules \( n^n V_p \) stabilizes for some \( P \) and all \( p \geq P \). Hence \( n^n V_p \) is a factor of \( n^n V_{p^p} \), \( p \geq P \). On the other hand every \( n^n \pi_p \) factorizes through \( n^n V_p \), which proves (3) for \( V \). The case of \( W \) is considered analogously.

Note that for sufficiently large \( p \) all \( c_p \) are isomorphisms. In fact, the first map in the definition of \( c_p \) is an isomorphism due to (3), and the second map is an isomorphism, since both multipliers in the tensor product are finite dimensional \( p \)-torsion modules. Hence the third vertical arrow in the diagram is an isomorphism. Besides, \( \Phi \) is an isomorphism by (2), that proves (4). Independence on \( p \) is obvious.

Let \( \bar{v} = \sum_l \bar{v}_l, \bar{w} = \sum_l \bar{w}_l \), be the decompositions from (15). Then for \( \gamma', \gamma'' \in \Gamma \), satisfying (16) for large enough \( s \) holds

\[
\bar{v} \gamma' = \bar{v}_p + \sum_{l \notin X'' \cap X_n} \bar{v} \gamma' \gamma'' \bar{w} = p \bar{w} + \sum_{l \notin X'' \cap X_n} \gamma'' \bar{w}.
\]

Then following (1), \( v \gamma' \otimes \gamma'' w = v_p \otimes p w \), which competes the proof of (5). The statements (6) and (7) follow from (5). In these diagrams both images of \( \bar{v} \otimes \bar{w} \) from the left upper corner of the diagram equal to the class \( v \gamma w \) in the right lower corner for a suitable \( \gamma \in \Gamma \).

A composition in the category \( \mathcal{A} \) is defined as follows. Since direct limits commute with tensor product, we can write

\[
\mathcal{A}(l, n) \otimes_{\mathcal{F}_l} \mathcal{A}(m, l) \simeq \lim_{V \in \mathcal{F}(U)} \lim_{W \in \mathcal{F}(U)} n V_l \otimes_{\mathcal{F}_l} W_m
\]

Then we have the following composition

\[
n V_l \otimes_{\mathcal{F}_l} W_m \xrightarrow{\mu_{l(m', n')}} n T_{m,m'} \xrightarrow{\nu_{m,n}} \mathcal{A}(m, n),
\]

where the first homomorphism is constructed above and the second is the canonical map in the direct limit. From the commutative diagrams in Lemma 3.4, (6), we have a well defined map

\[
\mathcal{A}(p, n) \times \mathcal{A}(m, p) \to \mathcal{A}(m, n).
\]

3.4. **Generalized Pierce decomposition.** We start from the following categorical statement. Assume \( \mathcal{C} \) is a \( k \)-category with sums and products and \( \{ C_i \mid i \in \mathcal{I} \} \) be a family of objects of \( \mathcal{C} \). Denote by \( (*) \) the following properties of this family:

\( (*) \) for every \( j \in \mathcal{J} \) there exist finitely many \( i \in \mathcal{I} \), such that \( \mathcal{C}(C_i, C_j) \) and \( \mathcal{C}(C_j, C_i) \) are nonzero.

Consider the vector space

\[
\Pi_{\mathcal{I}} = \prod_{(i,j) \in \mathcal{I} \times \mathcal{I}} \mathcal{C}(C_j, C_i),
\]

written as \( \mathcal{I} \times \mathcal{I} \) matrices, provided that \( j \)'s correspond to columns and \( i \)'s corresponds to rows. In general the standard "column-by-row" product of such matrices is not well defined. By \( M_j \) denote the subspaces of \( \Pi_{\mathcal{I}} \), formed by the matrices with finitely many nonzero elements in any column and in any row. Then "column-by-row" product turns it into a \( k \)-algebra.

**Lemma 3.5.** Assume the family \( \{ C_i \mid i \in \mathcal{I} \} \) of objects of the \( k \)-category \( \mathcal{C} \) satisfies the property \( (*) \).

(1) There exists a canonical isomorphism of \( k \)-algebras

\[
M_2 \simeq \text{End}_{\mathcal{C}}(\bigoplus_{i \in \mathcal{I}} C_i),
\]

where \( \text{End}_{\mathcal{C}} \) denotes the endomorphisms ring in the category \( \mathcal{C} \).
(2) Let \( \mathcal{C}_3 \) be the restriction of the category \( \mathcal{C} \) on \( J \), \( M_3 - \text{Mod}_r \), the full subcategory in \( M_3 - \text{Mod} \) consisting of modules \( M \), such that \( M = \bigoplus_{i \in J} e_{ii} M \), where \( e_{ii} \) is a matrix unit corresponding \( i \in J \). Then there exists the canonical equivalence

\[
F : \mathcal{C}_3 - \text{Mod} \simeq M_3 - \text{Mod}_r,
\]

where for \( N \in \mathcal{C}_3 - \text{Mod} \) holds \( F(N) = \bigoplus_{i \in J} N(i) \).

Proof. Every element \( (f_{ij} \mid i, j \in J) \in \Pi_J \) defines canonically a homomorphism \( f : \bigoplus C_i \longrightarrow \prod_{i \in J} C_i \). By the condition (*) the image of \( f \) belongs to \( \bigoplus C_i \subset \prod_{i \in J} C_i \).

The second statement is standard. \( \square \)

The following statement is an analogue of two-sided Pierce decomposition for the pair \( \Gamma \subset U \).

**Theorem 3.1.** For \( u \in U \) denote by \( [u] \) the matrix from \( M_A \), such that \( [u]_{m,n} = (u_{m,n}), m, n \in \text{Specm} \Gamma \), where \( u_{m,n} \) is the image of \( u \) in \( A(m,n) \).

1. The mapping \( i_A = \lceil \rceil : U \longrightarrow M_A \), which sends \( u \in U \) to \([u]\) is a homomorphism of \( k \)-algebras.
2. Endow \( \Pi_A \) with the topology of direct product and \( M_D \subset \Pi_A \) with the induced topology. Then the image of \( \pi_D i_A \) is dense in \( M_D \).

Proof. Following Lemma 3.3, (3), (4), the matrix \([u]\) has finitely many nonzero elements in any row and any column, hence \([u] \in M_A \).

Fix \( u, v \in U, m, n \in \text{Specm} \Gamma, m, n \geq 1 \). Also fix \( V, W, T \) and \( P > 0 \), satisfying the conditions of Lemma 3.4. Then from

\[
\hat{\mu}(m^n, n^m) = \sum_{p \in X^n \cap X_m} \hat{\mu}_p(m^n, n^m)
\]

follows (as in the proof of Lemma 3.4, (5))

\[
\overline{uv} = \sum_{p \in X^n \cap X_m} \hat{\mu}_p(m^n, n^m)(\overline{u} \otimes \overline{v}),
\]

where \( \overline{uv} \) is a class of \( uv \in n^m T_m \). Taking the limits we obtain that

\[
[uv]_{m,n} = \sum_{p \in X^n \cap X_m} [u]_{n,p} [v]_{p,m},
\]

that proves the first statement. To prove the second statement it is enough to show that if \( A(m,n) \subset M_D \), then \( A(m,n) \subset \overline{\text{Im} \pi_D i_A} \). First note that the image of \( U \) is dense in the image of \( A(m,n) \subset M_D \), that is, matrices from \( M_D \), which are zero in all positions except \((n,m)\). Let \( \{u_i \in U \mid i = 1, \ldots \} \) be a sequence in \( U \), which converges to an element \( f \) from \( A(m,n) \). Note that by definition any \( D \) is at most countable. Consider an increasing sequence of finite subsets \( S_i \subset D, i = 1, \ldots \) such that \( \bigcup_{i=1}^{\infty} S_i = D \), strictly increasing sequence of integers \( k_i > 0 \) and the elements \( \mu_i, \nu_i \in \Gamma \), such that

\[
\begin{cases}
\mu_i \equiv 1 \mod m^{k_i} \\
\mu_i \equiv 0 \mod m^{k_i}, \ m' \in S_i, m' \neq m.
\end{cases}
\]

and

\[
\begin{cases}
\nu_i \equiv 1 \mod n^{k_i} \\
\nu_i \equiv 0 \mod n^{k_i}, \ n' \in S_i, n' \neq n.
\end{cases}
\]

Then \( i_D(\mu_i) \) (respectively \( i_D(\nu_i) \)) tends in \( M_D \) to the diagonal matrix unit in the position \( m \) (respectively \( n \)), hence the sequence \( i_D(\nu_i u_i \mu_i) = i_D(\nu_i) i_D(\mu_i) \)
converges to \( f \) since it tends to 0 in all positions except \((n, m)\) and tends to \( f \) in the position \((n, m)\).

\(\square\)

3.5. Gelfand-Tsetlin modules as \(A\)-modules. We consider the category of \(k \rightarrow \text{Mod} \) endowed with the discrete topology and consider the category \(A \rightarrow \text{Mod} \), of continuous functors \(M : A \rightarrow k \rightarrow \text{Mod} \), which in \([\text{DFO2]}\) are called discrete modules. It means, for every \(m \in \text{Specm} \Gamma \) there exists \(m(= m(m)) \geq 0\), such that \(m^n M(m) = 0\). For a discrete \(A\)-module \(M\) define a Gelfand-Tsetlin \(U\)-module \(\mathbb{F}(M)\) by setting

\[
\mathbb{F}(M) = \bigoplus_{m \in \text{Specm} \Gamma} M(m) \quad \text{and for} \quad x \in M(m), a \in U
\]

\[
\text{set} \quad ax = \sum_{n \in \text{Specm} \Gamma} a_{m, n} x,
\]

where \(a_{m, n}\) is the image of \(a\) in \(A(m, n)\). If \(f : M \rightarrow N\) is a morphism in \(A \rightarrow \text{mod}_d\) then define \(\mathbb{F}(f) = \bigoplus_{m \in \text{Specm} \Gamma} f(m)\).

**Theorem 3.2.** ([DFO2], Theorem 17) The defined above functor \(\mathbb{F}\) is an equivalence of categories

\[
\mathbb{F} : A \rightarrow \text{mod}_d \rightarrow \mathbb{H}(U, \Gamma).
\]

Moreover it induces a functorial isomorphism

\[
\text{Ext}^1_A(\mathbb{F}(X), \mathbb{F}(Y)) \simeq \text{Ext}^1(U, Y), \quad X, Y \in \mathbb{H}(U, \Gamma).
\]

**Proof.** Following Lemma 3.5, (2) there exists a canonical equivalence

\[
A \rightarrow \text{mod}_d \simeq M_A \rightarrow \text{mod}_r,
\]

that together with the functor, induced by \(i_A : U \rightarrow M_A\) (Theorem 3.1, (1)) gives us the functor \(A \rightarrow \text{mod}_d \rightarrow \mathbb{U} \rightarrow \text{mod}\), besides the image of this functor belongs to \(\mathbb{H}(U, \Gamma)\). The statement on \(\text{Ext}^1\) follows from the fact that the functor \(\mathbb{F}\) preserves the exact sequences.

Let \(u \in U, m, n \in \text{Specm} \Gamma, u_{m, n}\) the image of \(u\) in \(A(m, n)\). Let

\[
M = \sum_{m \in \text{Specm} \Gamma} M(m)
\]

be a Gelfand-Tsetlin module. The action of \(u_{m, n}\) on \(M\) is described in the next lemma.

**Lemma 3.6.** Let \(m_u : M(m) \rightarrow M\) be the map \(x \mapsto ux, x \in M(m)\),

\[
u_{m, n} = \pi_n m_u : M(m) \rightarrow M(n),
\]

where \(\pi_n\) is the canonical projection of \(M\) onto \(M_n\). If \(n \notin X_u(m), \text{ then } u_{m, n} = 0\).

**Proof.** Condition \(n \notin X_u(m)\) holds if and only if \(\Gamma u \Gamma = n u \Gamma + \Gamma u m, \text{ equivalently, } u \in n u \Gamma + \Gamma u m\). But then \(u \in n^n u \Gamma + \Gamma u m^n\) for every \(m, n > 0\) and hence \(u_{m, n} = 0\). On the other hand in this case \(\pi_n m_u = 0\) by (4). By the Chinese remainder theorem, there exists a sequence \(\gamma_{n} \in \Gamma, n \geq 1, \text{ such that } \gamma_{n} = 1(\mod n^n), \gamma_{n} = 0(\mod l^n)\) for any \(l \in X_u(m), l \neq n\). Then in \(A(m, n)\) holds \(\lim_{n \rightarrow \infty} \gamma_{n} u = u_{m, n}\). On the other hand the operator \(m_{u} = \lim_{n \rightarrow \infty} m_{\gamma_{n} u} : M(m) \rightarrow M\) is well defined, \(\pi_{n} m_{u} = \pi_{m} m_{u}\) and \(\pi_{n'} m_{u} = 0\) for any \(n' \neq n\). Then (18) completes the proof. \(\square\)
We will identify a discrete $\mathcal{A}$-module $N$ with the corresponding Gelfand-Tsetlin module $\mathbb{F}(N)$. For $m \in \text{Spec}_m \Gamma$ denote by $m\hat{}$ a completion of $m$. Consider a two-sided ideal $I \subset A$ generated by $m\hat{}$ for all $m \in \text{Spec}_m \Gamma$ and set $A(W) = A/I$. Then Theorem 3.2 implies the following corollary.

**Corollary 3.2.** The categories $\mathbb{W}(U, \Gamma)$ and $A(W) - \text{mod}_d$ are equivalent.

The following standard statement shows usefulness of the category $\mathcal{A}$ for the study of simple Gelfand-Tsetlin modules over $U$.

**Lemma 3.7.** Let $M$ be a simple $\mathcal{A}$-module, $m \in \text{Spec}_m \Gamma$, $M(m) \neq 0$, $N$ a simple $\mathcal{A}(m, m)$-module. Then the correspondences

$$M \mapsto M|_{\mathcal{A}(m, m)} \text{ and } N \mapsto (A \otimes_{\mathcal{A}(m, m)} N)/J,$$

where $J$ is a unique maximal $\mathcal{A}$-submodule in the induced module, realizes a bijection between the sets of isomorphism classes of simple $\mathcal{A}(m, m)$-modules and isomorphism classes of simple $\mathcal{A}$-modules $M$ such that $M(m) \neq 0$.

The subalgebra $\Gamma$ is called **big** in $m \in \text{Spec}_m \Gamma$ if $\mathcal{A}(m, m)$ is finitely generated as a right $\Gamma_m$-module. The importance of this concept is described in the following statement.

**Lemma 3.8.** ([DFO2], Corollary 19) If $\Gamma$ is big in $m \in \text{Spec}_m \Gamma$ then there exists finitely many non-isomorphic irreducible Gelfand-Tsetlin $U$-modules $M$ such that $M(m) \neq 0$. For any such module $M$ holds $\dim_k M(m) < \infty$.

**3.6. Examples of computation of $\mathcal{A}$.** First example is given in the beginning of the section, namely, the presentation of a basic associative algebra as a quiver with relations.

Now we illustrate our techniques applying it to the study of representations of skew group algebras over a commutative ring and to obtain the classical result on the construction of irreducible representations of a finite group $G = N \rtimes H$ with abelian $N$.

The case of a skew group algebra is summarized in the following statement.

**Proposition 3.1.** Let $\Lambda$ be a commutative ring, $M$ a monoid, acting on $\Lambda$, $U = \Lambda \ast M$ the skew-group algebras. For $m, n \in \text{Spec}_m \Lambda$ set

$$M(m, n) = \{\varphi \in M \mid \varphi \cdot m = n\}.$$

Then $\Lambda \subset U$ is a Harish-Chandra subalgebra and

1. The blocks of the category $\mathcal{A} = \mathcal{A}(\Lambda)$ correspond to the orbits $\text{Spec}_m \Lambda/M$.
   
   For $D \in \text{Spec} \Lambda/M$ the set of objects of $\mathcal{A}_D$ coincides with $D$. Moreover, $\mathcal{A}_D$ itself is a groupoid such that for $m \in D$ holds $\mathcal{A}(m, m) \simeq \Lambda_m \ast M(m, m)$.
2. In every block there exists a simple Gelfand-Tsetlin module.
3. If the action of $M$ on $D$ is free, then $\mathcal{A}_D - \text{mod}_d$ is equivalent to the category of finite dimensional modules over $\Lambda_m$ for any $m \in \text{Spec}_m \Lambda$.

**Proof.** To compute $\mathcal{A}(m, n)$ there is enough to consider in Definition 3 the bimodules $V$ of the form $V = \bigoplus_{\varphi \in S} \Lambda \varphi$ for some $S \subset M$. In this case

$$V/(n^nV + Vm^n) \simeq \bigoplus_{\varphi \in \mathcal{S} \cap M(m, n)} (\Lambda/n^k)\varphi, \text{ where } k = \min\{m, n\}.$$

Since in the definition of $\mathcal{A}(m, n)$ we can consider only $V$ containing $M(m, n)$, we obtain that

$$\mathcal{A}(m, n) \simeq \bigoplus_{\varphi \in M(m, n)} \Lambda_n \varphi \simeq \bigoplus_{\varphi \in M(m, n)} \varphi \Lambda_m.$$
The last isomorphism \((\Lambda_n \varphi \simeq \varphi \Lambda_m)\) is defined using the isomorphism of \(\Lambda\)-bimodules
\[
(\Lambda/n^k) \varphi \simeq \varphi (\Lambda/m^k), \quad k \geq 1.
\]

This isomorphism allows to define the composition in \(A\) as in Theorem 3.1. In particular \(A(m, m)\) is isomorphic to \(\bigoplus_{\varphi \in M(m, m)} \Lambda_m \varphi\) as \(\Lambda\)-bimodule. It is also isomorphic to \(\Lambda_m \ast M(m, m)\) as an algebra. Any pair \((m, n)\) of the objects in a block \(A\) are isomorphic by application of an element from \(M(m, n)\), which proves (1).

By Lemma 3.7 the statement (1) reduces the problem of classification of simple Harish-Chandra \(U\)-modules to the problem of classification of simple \(\Lambda_m \ast M(m, m)\)-modules, \(m \in \text{Specm } A\). But the \(\Lambda\)-bimodule \(\Lambda_m\) is a direct summand of \(\Lambda_m \ast M(m, m)\) as a \(\Lambda_m\)-bimodule, hence \(U/U\) \(\neq 0\). It proves (2).

The statement (3) is obvious. \(\square\)

Next we will show that the classical Mackey construction can be interpreted as a special case of the category of Gelfand-Tsetlin modules ([FO3]). We assume that the base field \(k\) is algebraically closed and its characteristic is coprime with \(n = |G|\). We set \(U = k[G]\) and \(\Gamma = k[N]\). Obviously, \(\Gamma\) is a Harish-Chandra subalgebra in \(U\). Denote by \(\tilde{N}\) the set of characters of \(N\). Then \(k[N] \simeq \prod_{\chi \in \tilde{N}} k\chi\), where \(k\chi\) corresponds to \(\chi\). By \(m_\chi\) denote the kernel the character \(\chi\).

The group \(G\) acts by conjugations from the left on the group \(N\), \(x \mapsto gxg^{-1}, x \in N, g \in G\), and it also acts on \(\tilde{N}\) from the right, \(\chi \mapsto \chi^g, \chi \in \tilde{N}, g \in G\), such that \(\chi^g(x) = \chi(gx), g \in G\). Denote by \(\text{St}_G x\) and \(\text{St}_G \chi\) the stabilizers of the corresponding actions.

We give a construction of the category \(A = A_{U, \Gamma}\). In [FO3] the following statement is shown:

**Proposition 3.2.** Let \(Y = \mathcal{Y}(G, N)\) be a groupoid, such that
\[
\text{Ob } Y = \tilde{N}, \quad Y(\chi_1, \chi_2) = \{g \in G \mid \chi_1 = \chi_2^g\}, \quad \chi_1, \chi_2 \in \tilde{N},
\]
\(N\) be a subgroupoid of \(Y\), such that \(N(\chi, \chi) = N\) for any \(\chi \in \tilde{N}\) and empty otherwise, \(X = Y/N\) and \(kX\) be its \(k\)-linear envelope. Then there exists a canonical isomorphism of categories \(\Phi : kX \rightarrow A\), identical on the objects. Besides
1. \(A(\chi, \chi) \simeq k[\text{St}_\chi]\).
2. \(\chi_1, \chi_2 \in \text{Ob } A\) are isomorphic if and only if \(\chi_1^g = \chi_2^j\) for some \(g \in G\).

4. **Generic representations**

In this section we will assume that \(M\) is finitely generated, in particular, \(M\) is countable. Assume \(k\) is uncountable. A non-empty set \(X \subset \text{Specm } \Gamma\) is called **massive**, provided that \(X\) contains an intersection of countable many dense open subsets. In this case \(X\) is dense in \(\text{Specm } \Gamma\) ([Ga], Lemma 4.2). Assume \(s : \Gamma \rightarrow \Gamma'\) is a homomorphism of algebras, such that for the induced \(s^* : \text{Specm } \Gamma' \rightarrow \text{Specm } \Gamma\), \(\text{Im } s^*\) is massive in \(\text{Specm } \Gamma\). In this case if \(\mathcal{X}' \subset \text{Specm } \Gamma'\) is a massive subset, then \(s^*(\mathcal{X}')\) is a massive subset in \(\text{Specm } \Gamma\).

Let \(U_\chi = U/(U_m \chi)\). We call \(U_\chi\) the **universal module generated by a \(\chi\)-eigenvector**. If \(\Gamma\) is a Harish-Chandra subalgebra then \(U_\chi \subset H(U, \Gamma)\). The character \(\chi\) lifts to a simple \(U\)-module if and only if \(U_\chi\) is nonzero.

Let \(\Lambda_0\) be the algebra generated by \(\bigcup_{m \in M} m(\Gamma)\) (if \(\Gamma_0 \subset \Gamma\) then \(U\) is a right Galois order). Every nonzero element \(u \in U\) can be presented (not uniquely) in the form
\[
u = \sum_{\varphi \in M/G} [a(u) \varphi], \quad \text{where } a(u) \varphi \neq 0.
\]
Denote by $\Lambda_1$ the algebra, generated by $\Lambda_0$ and all possible $m \cdot a(u)_\varphi$, where $u$ runs $U \setminus \{0\}$ and $m$ runs $\mathbb{M}$. Analogously by $\Lambda_2$ the algebra generated by $\Lambda_0$ and all $(ma(u)_\varphi)^{\pm 1}$. Also denote $\mathcal{L}_i = \text{Specm} \Lambda_i$, $\pi_0$ the canonical map from $\mathcal{L}_0$ to $\text{Specm} \Gamma$, $\Omega_i = \pi_0(\mathcal{L}_i)$, $i = 1, 2$. Then

$$A_2 \supset A_1 \supset A_0, \mathcal{L}_2 \subset \mathcal{L}_1 \subset \mathcal{L}, \quad \Omega_2 \subset \Omega_1 \subset \Omega.$$

By $\mathcal{L}_r \subset \mathcal{L}$ denote the set of $\ell$, such that $\mathcal{M}$ acts on $\ell$ freely and $\mathcal{M} \cdot \ell \cap G \cdot \ell = \{\ell\}$ (in terms of 5.1 for $m = \pi_0(\ell) \in \text{Specm} \Gamma$ holds $\mathcal{S}(m, m) = \{e\}$) and $\Omega_r = \pi_0(\mathcal{L}_r)$.

**Lemma 4.1.** The sets $\mathcal{L}_i \subset \mathcal{L}$ and $\Omega_i \subset \text{Specm} \Gamma$, $i = 1, 2$, are massive. Besides $\mathcal{L}_i$, $i = 1, 2$ are invariant with respect to the actions of $G$ and $\mathcal{M}$. If $\mathcal{M} \cdot \Gamma \subset \bar{\Gamma}$ then $\mathcal{L}_r$ and $\Omega_r$ are massive.

**Proof.** Every element $a \in L$ defines a rational function on $\mathcal{L}$ with a non-empty open domain of $X(a) \subset \mathcal{L}$. Then $\mathcal{L}_i, i = 1, 2$ by definition are intersections of countably many sets of the form $X(a)$, hence are massive. The properties of $\mathcal{M}$- and $G$-invariance follows from the facts, that $gm\Gamma = m^g\Gamma$ and $[a\varphi] = [a^g\varphi^g]$, $g \in G$, $m \in \mathcal{M}$.

For any $m \in \mathcal{M}$, $m \neq e$, denote $X_m = \{\ell \in \mathcal{L} \mid m \cdot \ell \in G \cdot \ell\}$ and for $m \in \mathcal{M}$, $g \in G$ define a closed in $\mathcal{L}$ subset $\mathcal{L}(m, g) = \{\ell \in \mathcal{L} \mid m \cdot \ell = g \cdot \ell\}$. Then $X_m = \bigcup \mathcal{L}(m, g)\}$ is closed subset in $\mathcal{L}$, since $G$ is finite. Assume that $\mathcal{L} = X_m$ for some $m \neq e$. Since $\mathcal{L}$ is irreducible, we conclude that $\mathcal{L}(m, g) = \mathcal{L}$ for some $g \in G$, and hence $m = g$. This is a contradiction since $\mathcal{M}$ is separating. But $\bigcup_{m \in \mathcal{M}, m \neq e} X_m$ is the complement of $e\mathcal{L}_r$ in $\mathcal{L}$ and $\mathcal{L}_r$ is massive. The sets $\Omega_i$, $i = 1, 2, r$, are massive as the images of massive subset $\mathcal{L}_i$. 

The following useful fact follows from the separating of $\mathcal{M}$-action (Lemma 2.2).

**Lemma 4.2.** Let $\bar{\Gamma}$ be the integral closure on $\Gamma$ in $L$, $\ell_{m_k}, \ldots, \ell_{m_k} \in \text{Specm} \bar{\Gamma}$ belong to different orbits of $G$. Then there exists $\gamma \in \Gamma$, such that $\gamma(\ell_{m_1}), \ldots, \gamma(\ell_{m_k})$ are distinct, that is $\Gamma$ distinguishes the orbits of $G$.

**Theorem 4.1.** Suppose that the field $\mathbb{k}$ is uncountable, $\chi$ a character of $\Gamma$, $m_\chi \in \text{Specm} \bar{\Gamma}$ its kernel.

1. If $\chi \in \Omega_1$, then $U_\chi$ is nonzero Gelfand-Tsetlin module and $\text{supp} U_\chi \subset \mathcal{M}m$.
2. If $\mathcal{M}$ is a group, then for any $\chi \in X_2$ the module $U_\chi$ is a unique $\mathcal{O}_m$-graded irreducible $U$-module generated by a $\chi$-eigenvector and $\text{supp} U_\chi = \mathcal{O}_m$.
3. If $\mathcal{M}$ is a group and $\mathcal{M} \cdot \Gamma \subset \bar{\Gamma}$, $\chi \in \Omega_2 \cap \Omega_r$, then $U_\chi$ is irreducible weight $U$-module with 1-dimensional components. In this case there is a canonical isomorphism of $\mathbb{k}$-vector spaces $k\mathcal{M} \simeq U_\chi$.

**Proof.** Note that the canonical embedding $U \hookrightarrow L \ast \mathcal{M}$ factorizes through a skew semigroup algebra $\Lambda_1 \ast \mathcal{M} \subset L \ast \mathcal{M}$. A character $\bar{\chi} : \Lambda_1 \rightarrow \mathbb{k}$ induces a $\Lambda_1 \ast \mathcal{M}$-module $M(\bar{\chi})$ which is isomorphic to $\bigoplus_{\varphi \in \mathcal{M}} \Lambda_1(\bar{\varphi})$, as a $\mathbb{k}$-vector space, where $\bar{\varphi} = \text{Ker } \bar{\chi}$.

Since the restriction of $M(\bar{\chi})$ on $U$ is nontrivial then for every $\chi \in X_1$, $U_\chi \neq 0$. Clearly, $U_\chi$ is a Gelfand-Tsetlin module. Moreover, since $U_\chi$ is Specm $\mathcal{L}$-graded, it has an irreducible quotient with a nonzero $\chi$-eigenvector. This implies (1).

Let $\chi \in \Omega_2$, $U = \sum_{h \in \mathcal{M}} [x_h, h] \in U$. By assumption, for every $n \in \mathcal{M} \cdot m$ holds $x_h(n) \neq 0$, hence every graded component of $U_\chi$ generates the whole $U_\chi$. Therefore, $U_\chi$ is irreducible as $\mathcal{O}_m$-graded $U$-module. Moreover, since $U_\chi$ is the universal module generated by a $\chi$-eigenvector, it is a unique such graded irreducible module, implying statement (2).

Note that if $\mathcal{M}$ acts on $m$ with a nontrivial stabilizer then $U_\chi$ is not irreducible. Since $\mathcal{M} \cdot \Gamma \subset \bar{\Gamma}$, the set $\Omega_r$ is massive by Lemma 4.1. Consider a subset $\Omega_2 \cap \Omega_r$. 

Since $\Gamma$ distinguishes $G$-orbits by Lemma 4.2, it implies the irreducibility of $U_\chi$ for any $\chi \in X_\tau$. The basis elements of $U_\chi$ in this case are labeled by the elements of $\kappa M$ which completes the proof of (3).

Lemma 4.3. Let $M$ be a finitely generated right $\Gamma$-module. Then the set of $m \in \text{Specm} \, \Gamma$ such that $\text{Tor}_1^\Gamma(M, \Gamma/m) = 0$ contains an open dense subset $X \subset \text{Specm} \, \Gamma$.

Proof. Let $R^* : \cdots \to d^2 \to \Gamma^{n_2} \to d^1 \to \Gamma^{n_1} \to d^0 \to \Gamma^{n_0} \to 0 \cdots$ be a free resolution of $M$. It induces the resolution $R^* \otimes_\Gamma K$ of $M \otimes_\Gamma K$. Denote $r = \dim_K \text{Im}(d^1 \otimes 1)$. Let $d^0(m)$ be the specialization of $d^0$ in $m \in \text{Specm} \, \Gamma$. Then all those $m$ satisfying the conditions $\dim \text{Im}(d^1(m)) = n_1 - r$ and $\dim \text{Im}(d^2(m)) = r$, form an open dense set $X$. Then for $m \in X$ the first cohomology of $R^* \otimes_\Gamma \Gamma/m$ equals 0, that completes the proof.

Proposition 4.1. For a nonzero $u \in U, u \neq 0$ there exists a massive set $\Omega_u \subset \text{Specm} \, \Gamma$ such that for every $m \in \Omega_u$ the image $\bar{u}$ of $u$ in $U/m$ is nonzero.

Proof. Let $m \in \text{Specm} \, \Gamma$, $N = u\Gamma$. Then $\bar{u} = 0$ if and only if $u = \sum_{i=1}^n u_i m_i$, for some $u_i \in U, m_i \in m$, $i = 1, \ldots, n$. Denote $S = \bigcup_{i=1}^n \text{supp} u_i$ and $M = U(S)$. If $\bar{u} = 0$, then the exact sequence of right $\Gamma$-modules

$$0 \to N \to M \to M/N \to 0$$

becomes non-exact after tensoring with $\Gamma/m$, i.e. $\text{Tor}_1^\Gamma(M/N, \Gamma/m) \neq 0$. Denote $X(u, S) = \{m \in \text{Specm} \, \Gamma \mid \text{Tor}_1^\Gamma(M/N, \Gamma/m) = 0\}$. Following Lemma 4.3 we can set $\Omega_u = \bigcap_{S \subset \mathcal{M}} X(u, S)$.

5. Representations of Galois orders

5.1. Extension of characters for Galois orders. Given $m \in \text{Specm} \, \Gamma$ fix $\ell_m \in \text{Specm} \, \Gamma$ and denote $M_{\ell_m} \subset \mathcal{M}$ ($G_{\ell_m} \subset G$) the stabilizer of $\ell_m$ in $\mathcal{M}$ (in $G$). The ideal $m$ defines $M_m \subset \mathcal{M}$ and $G_m$ uniquely up to $G$-conjugation. It allows us to use the notation $M_m$ instead of $M_{\ell_m}$ and $G_m$ instead of $G_{\ell_m}$.

Denote by $S(m, n)$ the following $G$-invariant subset in $\mathcal{M}$

$$S(m, n) = \{m \in \mathcal{M} \mid \ell_n \in Gm \cdot \ell_m\} = \{m \in \mathcal{M} \mid g_2 \ell_n = m g_1 \ell_m \text{ for some } g_1, g_2 \in G\}.
$$

(20)

Obviously this definition does not depend on the choice of $\ell_m$ and $\ell_n$.

Lemma 5.1. Let $m \in \text{Specm} \, \Gamma$ and $\mathcal{M}$ is a group. Then

(1) $|\mathcal{M}_m| < \infty$ if and only if for some $n \in \text{Specm} \, \Gamma$ at least one from the sets $S(m, n), S(n, m)$ is nonempty and finite. If $|\mathcal{M}_m| < \infty$ and $S(m, n) \neq \emptyset$, $S(n, m) \neq \emptyset$, then $|\mathcal{M}_{\ell_n}| < \infty$ and $|S(m, n)| = |S(n, m)|$.

(2)

$$|S(m, n)| \leq \frac{|G|^2 |\mathcal{M}_m|}{|G_{\ell_m}|}. $$

(3)

$$|S(m, n)/G| \leq \{|m \in \mathcal{M} \mid \pi(m \ell_m) = n\}.$$ 

Proof. $S(m, n)$ is infinite if and only if for some fixed $g \in G$, $\ell_m$ and $\ell_n$ the set $M(\ell_m, \ell_n, g) = \{m \in \mathcal{M} \mid m g \ell_m = \ell_n\}$ is infinite. In this case for any $m' \in \mathcal{M}(\ell_m, \ell_n, g)$ holds

$$(m'g)^{-1} M(\ell_m, \ell_n, g) g \subset \mathcal{M}_{m}, \quad \mathcal{M}(\ell_m, \ell_n, g)m'^{-1} \subset \mathcal{M}_n.$$
To prove the equality note, that $m \in S(m, n)$ if and only if $m^{-1} \in S(n, m)$. It proves (1). If $|M_m| < \infty$ then there exists at most $|M_m|$ elements $\varphi \in M$, such that $\ell_n = \varphi \ell_m$. Considering the $G$-orbits of $\ell_m$ and $\ell_n$, which have the lengths $\frac{|G|}{|G_m|}$ and respectively, we obtain the inequality (2). Assume $mg_1 \ell_m = g_2 \ell_n$, $g_1, g_2 \in G$, $m \in M$. Then $(g_1^{-1}mg_1)\ell_m = g_1^{-1}g_2 \ell_n$, which proves (3).

The property of the Galois ring $U$ to be a Galois order has the following immediate impact on the representation theory of $U$. We will consider right Galois orders. The case of left orders is analogous.

**Lemma 5.2.** Let $U$ be a Galois ring over $\Gamma$, $\Gamma$ a noetherian algebra which is a right Harish-Chandra subalgebra of $U$, $m \in \text{Spec}\Gamma$, such that $M_m$ is finite, $S = S(m, m)$. If $U = Um$ then for every $k \geq 1$ there exist $\gamma_k \in \Gamma \setminus m$, $\nu_j \in U$, $\nu_j \in m^k$, $j = 1, \ldots, N$ such that

$$\gamma_k = \sum_{j=1}^{N} \nu_j \nu_j$$

and $\text{supp} \nu_j \in S$, $j = 1, \ldots, n$.

**Proof.** The condition $U = Um$ is equivalent to the condition $1 \in Um$, i.e. holds the equality

$$1 = \sum_{i=1}^{n} w_i u_i, \text{ where } w_i \in U, u_i \in m.$$

We use induction in $k$ to prove the statement of the lemma without the condition $\text{supp} \nu_i \in S$, $i = 1, \ldots, n$. The base of induction $k = 1$ follows immediately from (22). The induction step is obtained by substitution of the right side of (22) in (21): if

$$1 = \sum_{j=1}^{N} w_j \nu_j, \text{ where } w_j \in U, \nu_j \in m^k, j = 1, \ldots, N,$$

then

$$1 = \sum_{j=1}^{N} w_j \nu_j = \sum_{j=1}^{N} w_j \cdot 1 \cdot \nu_j =$$

$$\sum_{j=1}^{N} w_j \left( \sum_{i=1}^{n} u_i \mu_i \right) \nu_j = \sum_{j=1}^{N} \sum_{i=1}^{n} w_j u_i \mu_i \nu_j.$$ 

It proves the induction step, since all $\mu_i \nu_j \in m^{k+1}$.

Denote

$$T = \bigcup_{i=1}^{N} \text{supp} w_i \setminus S.$$

Since $T \cap S(m, m) = \emptyset$, the ideals $t^e_m$ and $\ell_m$ for every $t \in T$ belong to different $G$-orbits. By Lemma 4.2 there exists $f \in \Gamma$ such that $f(\ell_m) \neq f(\ell_m')$ for every $t \in T$. Without loss of generality we can assume that (Lemma 2.4, (3))

$$f_T(m, m) = \prod_{t \in T} (f(\ell_m) - f^t(\ell_m)) = \prod_{t \in T} (f(\ell_m) - f(\ell_m')) = 1.$$
In particular, it implies that \( f_T \in 1 + m \otimes \Gamma + \Gamma \otimes m \). Then by Lemma 2.4, (3), \( \gamma_k = f_T \cdot 1 \in 1 + m \). Besides, by Lemma 2.4, (2), \( v_i = f_T \cdot w_i \) belongs to \( U(S) \). Applying \( f_T \) to the equality (23) we obtain

\[
\gamma_k = \sum_{i=1}^N v_i \nu_i, \text{ where } \gamma_k \in \Gamma \setminus m, v_i \in U(S), \nu_i \in m^k.
\]

\[\square\]

**Corollary 5.1.** Let \( \Gamma \) be a noetherian algebra, \( U \) a right Galois order over \( \Gamma \), \( m \in \text{Specm} \Gamma \), such that \( |M_m| < \infty \). Then \( U m \neq U \). In particular, there exists a simple left Gelfand-Tsetlin \( U \)-module \( M \), generated by \( x \in M \), such that \( m \cdot x = 0 \).

**Proof.** By Lemma 5.1, (1) the set \( S = S(m, m) \) is finite. Since \( U \) is a Galois order, by Theorem 2.2, (2) \( U(S) \) is finitely generated as a right \( \Gamma \)-module. Then applying the Artin-Rees Lemma (Theorem 8.5. [Mat]) for a right \( \Gamma \)-module \( U(S) \) and its submodule \( \Gamma \) we conclude that there exists \( c \geq 0 \) such that for every \( k \geq c \)

\[U(S)m^k \cap \Gamma = (U(S)m^c \cap \Gamma)m^{k-c}.
\]

But by Lemma 5.2 for every \( k > c \) there exists \( \gamma_k \in U(S)m^k \cap \Gamma \), such that \( \gamma_k \not\in m \). Hence \( \gamma_k \not\in (U(S)m^c \cap \Gamma)m^{k-c} \), provided that \( k-c > 0 \). The obtained contradiction shows that \( U \neq U m \).

Denote by \( v \) the image of 1 in \( U/U m \neq 0 \). Then \( m v = 0 \) which defines a structure of Gelfand-Tsetlin module on \( U/U m \) (Lemma 3.1, (6)). We can choose by \( M \) a simple quotient of \( U/U m \) satisfies the statement. Therefore, as the module \( M \) we can choose any simple quotient of \( U/U m \) and set \( x \) the class of \( v \).

\[\square\]

### 5.2. Finiteness of extensions of characters for Galois orders

In this subsection we assume that \( U \) is a Galois order over \( \Gamma \) where \( \Gamma \) is normal noetherian over \( k \). In particular, \( \Gamma = \Gamma = U \) and \( \Gamma \) is finite over \( \Gamma \) by Proposition 2.1. Also \( \Gamma \) is a Harish-Chandra subalgebra by Proposition 2.3. If \( \ell \in \text{Specm} \Gamma \) projects to \( m \in \text{Specm} \Gamma \) then we will write \( \ell = \ell_m \) and say that \( \ell_m \) is lying over \( m \). Let \( \ell_m \) and \( \ell_n \) be some maximal ideals of \( \Gamma \) lying over \( m \) and \( n \) respectively. Note that given \( m \in \text{Specm} \Gamma \) the number of different \( \ell_m \) is finite due to Corollary 2.1.

**Lemma 5.3.** Let \( m, n \in \text{Specm} \Gamma \), \( S = S(m, n) \), \( m, n \geq 0 \). Then \( U = U(S) + n^k U + U m^n \). Moreover, for every \( u \in U, k \geq 0 \) there exists \( u_k \in U(S) \), such that \( u = u_k + n[k/2] u \Gamma + \Gamma u m^{k/2} \).

**Proof.** Fix \( u \in U \) and denote \( T = \text{ supp } u \setminus S \). If \( T = \emptyset \) then \( u \in U(S) \). Let \( T \neq \emptyset \).

We show by induction in \( k \), that there exists \( u_k \in U(S) \), such that

\[
(24) \quad u = u_k + \sum_{i=0}^k n^{k-i}u m^{i} , \quad u_k \in U(S) \quad (\text{hence } u_{k+1} - u_k = \sum_{i=0}^k n^{k-i}u m^{i}).
\]

Since \( \ell_n \) and \( \ell_m \) belong to different \( G \)-orbits if \( t \not\in S \), then by Lemma 4.2 there exists \( f \in \Gamma \) such that \( f(\ell_n) \neq f(\ell_m) \) for every \( t \in T \). Without loss of generality we can assume that \( f_T(n, m) = \prod_{t \in T} (f(\ell_n) - f^{t^{-1}}(\ell_m)) = 1 \), which implies \( f_T \in 1 + n \otimes \Gamma + \Gamma \otimes m \). Set \( u_1 = f_T \cdot u \). Then \( u_1 \) belongs to \( u + n u \Gamma + \Gamma u m \) and, hence, \( u \in u_1 + n u \Gamma + \Gamma u m \). Moreover, \( u_1 \in U(S) \) by Lemma 2.4, (2). We prove the induction step \( k \Rightarrow k + 1 \). Writing in (24) the expression for \( u \) in the right hand
side we obtain
\[ u \in u_k + \sum_{i=0}^k n^{k-i}(u_k + \sum_{j=0}^k n^{k-j}um^j)m^i \subset u_k + \sum_{i=0}^k n^{k-i}u_km^i + \sum_{i=0}^{k+1} n^{k+1-i}um^i, \]

that proofs of the induction step, since \( u_k + \sum_{i=0}^k n^{k-i}u_km^i \subset U(S) \).

**Theorem 5.1.** For any \( m, n \in \text{Specm} \Gamma \) such that \( S = S(m, n) \) is finite, the completed \( \Gamma_n - \Gamma_m \)-bimodule \( A(m, n) \) (see (3)) is finitely generated. Moreover, \( A(m, n) \) coincides with the image of \( U(S) \) in \( A \). Besides, \( A(m, n) \) is finitely generated both as a right \( \hat{\Gamma}_m \)-module and as a left \( \hat{\Gamma}_n \)-module.

**Proof.** In view of Lemma 5.3 and formula (3) we have an embedding
\[ (25) \quad A(m, n) \subset \lim_{\leftarrow n,m} U(S)/(n^nU + Um^m \cap U(S)). \]

Since \( U(S) \) is a noetherian \( \Gamma \)-bimodule by Theorem 2.2, the generators of \( U(S) \) as a \( \Gamma \)-bimodule generate any \( U(S)/(n^nU + Um^m \cap U(S)) \) as a \( \Gamma \)-bimodule, and hence generate \( A(m, n) \) as a complete \( \Gamma_n - \Gamma_m \)-bimodule ([Mat], Theorem 8.7). The statement, that \( A(m, n) \) is finitely generated both from the left and from the right, follows from Theorem 2.2, (2) and from Theorem 8.7, [Mat]. This completes the proof. \( \square \)

Note that Theorem 5.1 and Definition 3 imply that the embedding (25) is an isomorphism.

The following is obvious (see Lemma 3.8).

**Corollary 5.2.** In assumptions of Theorem 5.1, \( \Gamma \) is big in \( m \). In particular there exists only finitely many non-isomorphic extensions of \( \chi \) to simple \( U \)-modules.

**5.3. Proof of Theorem A.** Corollary 5.1 implies Theorem A.(1). The condition \( |M_m| < \infty \) implies the finiteness of \( S(m, m) \) (Lemma 5.1, (1)). Consider \( \chi : \Gamma \to k \) such that \( m = \text{Ker} \chi \). If \( \Gamma \) is not normal then \( \hat{\Gamma} \) is a finite \( \Gamma \)-module and \( \chi \) admits finitely many extensions to \( \hat{\Gamma} \), by Corollary 2.1. Hence, it is enough to prove the statement for normal \( \Gamma \). But in this case the statement follows from Corollary 5.2, which completes the proof of Theorem A.

Combining Theorem A.(1) and Corollary 2.2 we obtain the following module-theoretic characterization of left and right Galois orders.

**Corollary 5.3.** Let \( U \) be a Galois ring with respect to a noetherian algebra \( \Gamma \), \( M \) a group and \( m^{-1}(\Gamma) \subset \hat{\Gamma} (m(\Gamma) \subset \hat{\Gamma}) \) for any \( m \in M \). Then every character \( \chi : \Gamma \to k \) extends to a simple left (right) \( U \)-module if and only if \( U \) is right (left) Galois order.

**5.4. Bounds for dimensions and blocks of the category of Gelfand-Tsetlin modules.** Denote by \( r(m, n) \) the minimal number of generators of \( U(S(m, n)) \) as a right \( \Gamma \)-module. Since \( \Gamma \) is a Harish-Chandra subalgebra from \( |S(m, m)| < \infty \) follows \( r(m, n) < \infty \) and, by Theorem 5.1, \( A(m, m) \) is finitely generated as a right \( \hat{\Gamma}_m \)-module (in terms of [DFO2], \( \Gamma \) is big in \( m \)). In particular there exists finitely many non-isomorphic simple \( A \)-modules \( M \) such that \( M(m) \neq 0 \), besides \( M(m) \) is finite dimensional (Lemma 3.8).

**Lemma 5.4.** Let \( m, n \in \text{Specm} \Gamma, S = S(m, n), M = U \otimes_\Gamma \Gamma/m, x = 1 \otimes (1 + m) \in M(m), \pi_n : M \to M(n) \) the canonical projection. Then
\[ A(m, n) \cdot x = \pi_n(Ux) = \pi_n(U(S)x), \]
and
\[ \dim_k M(n) \leq \dim_k(U(S)x), \dim_k M(n) \leq r(m,n). \]
Analogous statements hold for any \( U \)-module \( N \) generated by a nonzero \( y \in N(m) \) such that \( my = 0 \).

**Proof.** The first equality follows from Lemma 3.6. To prove the second equality consider some \( u \in U \). Then by the Chinese remainder theorem there exists \( \gamma \in \Gamma \) such that \( \gamma ux = \pi_n(ux) \) and we replace \( u \) by \( \gamma u \). Then in the notations of Lemma 5.3, set \( k = 2t \), where \( n^tux = 0 \). Then \( \pi_n(ux) = \pi_n(u_kx) \), where \( u_k \in U(S) \).

The second statement is obvious. \( \square \)

**Theorem 5.2.** Let \( U \) be a Galois order over \( \Gamma \) where \( \Gamma \) is a normal noetherian \( \mathbb{K} \)-algebra, \( M \) is a \( U \)-module.

1. If for some \( m \in D \), \( M_m \) is finite and \( M \in \mathbb{H}(U, \Gamma, D) \) is simple then \( M(m) \) is finite dimensional. Both \( \dim_k M(m) \) and the number of isomorphism classes of simple modules \( N \), such that \( N(m) \neq 0 \), are bounded by \( r(m,n) \).
2. If in addition \( M \) is a group then for any \( n \in D \)
   \[ \dim_k M(n) \leq r(m,n) < \infty. \]
3. Let \( U \) be free as a right \( \Gamma \)-module. If \( M \) is generated over \( U \) by \( x \in M \), \( mx = 0 \), \( m \in \text{Spec} \Gamma \), then
   \[ \dim_k M(n) \leq |S(m,n)/G| \]

**Proof.** The statements (1) and (2) follow from Lemma 5.4 and Lemma 5.1. To prove (3) consider a free right \( \Gamma \)-module \( F \), which covers \( U(S) \), \( p : F \rightarrow U(S) \), \( q : F \rightarrow U \) is the composition of \( p \) with the canonical embedding, where \( S = S(m,n) \).

Then the image of
\[ q \otimes \Gamma \text{id}_K : F \otimes \Gamma K \rightarrow U \otimes \Gamma K \simeq \mathbb{K} \], see (6)
coincides with \( KU(S) = \mathbb{K}(S) \subset \mathbb{K} \).

Following Lemma 2.1 \( \dim_k \mathbb{K}(S) = |S/G| \).

From semicontinuity of the dimension of image of mapping between free modules we obtain, that for
\[ q \otimes \Gamma \text{id}_{\Gamma/m} : F \otimes \Gamma \text{id}_m \rightarrow U \otimes \Gamma \text{id}_m \simeq U/Um \]
holds \( \dim_k \text{Im}(q \otimes \Gamma \text{id}_m) \leq \dim_k \mathbb{K}(S) = |S/G| \). \( \square \)

Let \( D \) be an equivalence class in \( \Delta \), \( m, n \in D \) and \( \varphi \in S(m,n) \). We say, that \( m \) and \( n \) are connected by \( \varphi \) if the following two conditions hold

1. There exist \( [a_\varphi^{-1}], [a_\varphi] \in U \), such that \( a_\varphi^{-1}a_\varphi \neq 0 \) and \( a_\varphi^{-1}a_\varphi \) do not belong to \( \{g\ell_m, g \in G\} \).
2. The number of elements in the set \( \{\pi(g^0(\ell_m)) : g \in G\} \) equals \( |G/H_\varphi| \).

Endow \( D \) with a structure of a non-oriented graph as follows. The vertices are the elements of \( D \). An edge between \( m \) and \( n \) exists if and only if there exists some \( \varphi \in M \) that connects \( m \) and \( n \). The following statement is a generalization of Theorem 32, [DFO1].

**Proposition 5.1.** If \( m \) and \( n \) in \( D \) are connected by \( \varphi \in M \), then \( m \simeq n \in A_D \).

**Proof.** As in Lemma 4.2, for every \( n \geq 0 \) there exists a function \( f_n \in \Gamma \), such that
\[ \gamma_n \equiv 1 \mod \ell_n^m, \gamma_n \equiv 0 \mod (\varphi^0 \cdot \ell_m)^n \quad \text{for } g \in G, \ell_n \neq \varphi^0 \cdot \ell_m. \]

Consider the element \( x_n = [a_\varphi^{-1}][a_\varphi^2] \) whose coefficient by the element \( e \in M \) equals
\[ \tau_n = \sum_{g \in G/H_\varphi} a_{\varphi^{-1}}^g a_\varphi^g (\gamma_n^g)^2. \]
By the construction $\tau_n \not\in \mathfrak{m}$, and without loss of generality we may assume that $\tau_n \in 1 + \mathfrak{m}$. Consider two sequences in $\Gamma$, $y_n$, and $z_n$, $n \geq 1$, which converge in $A$ to $1_\mathfrak{m}$ and $1_n$ respectively. Then the sequences

$$g_n = \gamma_n[a_+\varphi]y_n, f_n = z_n[a_-\varphi^{-1}]\gamma_n$$

converge to $g : m \rightarrow n$ and $f : n \rightarrow m$ respectively, such that $fg = 1_m + \mu$, where $\mu$ belongs to the image of $m$ in $A(m, m)$. Since $1_m + \mu$ is invertible, we can assume $fg = 1_m$. Analogously one constructs $f' : n \rightarrow m, g' : m \rightarrow n$, such that $g'f' = 1_n$. \hfill \Box

Immediately from Proposition 5.1 follows

**Corollary 5.4.** If $m$ and $n$ belong to the same connected component of the graph $D$ then they are isomorphic in $A_D$.

**Theorem 5.3.** Let $D$ be a class of $\Delta$-equivalence. Suppose $\mathcal{M}$ is a group, $D$ has a finite number of connected components and for some $m \in D$ the group $\mathcal{M}_m$ is finite. Then the module $U/Um$ is of finite length.

**Proof.** By Corollary 5.4 the skeleton $\mathcal{B}_D$ of the category $A_D$ contains a finite number of objects, $\text{Ob} \mathcal{B}_D = \{n_1, \ldots, n_k\}$. Consider $U/Um$ as an element in $A - \text{mod}_d$ and denote the $\mathcal{B}_D$-module $M = (U/Um)|_{\mathcal{B}_D}$. Then by Theorem 5.2, (2), $\dim_k M(n_i) \leq r(m, n_i)$ for any $i = 1, \ldots, k$. Since the categories $A - \text{mod}_d$ and $\mathcal{B}_D - \text{mod}_d$ are equivalent, it completes the proof. \hfill \Box

From the proof above we obtain

**Corollary 5.5.** In assumptions of Theorem 5.3 the length of $U/Um$ and the number of simple objects in the block $\mathbb{H}(U, \Gamma, D)$ does not exceed $\sum_{i=1}^k r(m, n_i)$, where $n_i$ runs the set of representatives of the connected components of $D$.

5.5. **Further properties of Gelfand-Tsetlin modules.** In this subsection we assume that $U$ is a Galois order over noetherian $\Gamma$. In the conditions of Theorem 5.1 we are able to prove the following generalization of Corollary 5.1.

**Theorem 5.4.** Let $U$ be a Galois order over a noetherian algebra, $m, n \in \text{Spec}_m \Gamma$ such that $S(m, n), |M_m|, |M_n|$ are finite and $S(m, n) \neq \emptyset$. Then the image of $U(S)$ in $A(m, n)$ (Definition (3)) is nonzero.

**Proof.** Note that for a nonempty $G$-invariant $S \subseteq M$ the $\Gamma$-bimodule $U(S)$ is nonzero, since $KU(S) \subset \mathcal{X}$ is nonzero. Consider $U$ as a $\Gamma \otimes \Gamma$-module and denote by $I$ the ideal $n \otimes \Gamma + \Gamma \otimes n$ in $\Lambda \otimes \Lambda$. Then the class of $u \in U$ in $A(m, n)$ is 0 if for any $N \geq 0$ holds $u \in I^NU$. We prove the following statement: if for some $n \geq 0$ holds $U(S) \subset I^NU$, then $U(S) \subset I^NU(S)$. Assume $U(S) \subset I^NU$, equivalently, if $v_1, \ldots, v_k$ are the generators of $U(S)$ as $\Gamma$-bimodule, then

$$v_i = \sum_{j=1}^k v_{ij}u_{ij}, \text{ for some } v_{ij} \in I^N \subset \Gamma \otimes \Gamma, u_{ij} \in U, i = 1, \ldots, k. \quad (26)$$

Set $T = \bigcup_{i=1}^n \text{supp } u_i \setminus S$. As in the proof of Lemma 5.3 construct the element

$$f_T = 1 - F \in 1 + n \otimes \Gamma + \Gamma \otimes m$$

such that for all $v_{ij} = f_T \cdot u_{ij}$ holds $\text{supp } v_{ij} \subset S$. Applying $f_T$ to both sides of the equality (26) we obtain
\[ v_i = F \cdot v_i + \sum_{j=1}^{k} \nu_{ij} v_{ij}, \quad \nu_{ij} \in I^N, v_{ij} \in U(S), i = 1, \ldots, k. \]

Substituting the value \( v_i \) into the right hand side of (27) we obtain
\[ v_i = F^2 \cdot v_i + (F + 1) \sum_{j=1}^{k} \nu_{ij} v_{ij}. \]

Iterating this procedure \( N - 1 \) times we obtain
\[ v_i = F^N \cdot v_i + (F^N - 1 + \cdots + F + 1) \cdot \sum_{j=1}^{k} \nu_{ij} v_{ij}, \]
for some \( \nu_{ij} \in I^N, v_{ij} \in U(S), i = 1, \ldots, k, \)
which shows \( v_i \in I^N U(S) \), since \( v_i \) itself and all \( v_{ij} \) belongs to \( U(S) \) and \( F^N \in I^N \).

In particular, it means that \( U(S) = \bigcap_{n=1}^{\infty} I^N U(S) \).

Then by the Krull Theorem (Theorem 8.9, [Mat]), there exists \( a \in 1 + I \), such that \( a \cdot U(S) = 0 \).

Since \( \Gamma \otimes \Gamma \) acts on \( X \) the action of \( a \) is defined on \( V = U(S)K \subset X \) and \( a \cdot V = 0 \) as well. But, following Lemma 2.3, all irreducible summands of \( V \) as a \( K \)-bimodule are of the form \( V(\varphi) \) for some \( \varphi \in M \), and since \( \text{supp} \ V = S(m, n) \), there exist coimages \( \ell_m \) and \( \ell_n \), such that \( \ell_n = \ell_{m^\varphi} \). Note, that the \( K \)-bimodule \( V(\varphi) \) is isomorphic to \( L_{H^\varphi} \) endowed with the structure of \( K \)-bimodule. Then \( a \in 1 + I \subset \Gamma \otimes \Gamma \) and it can be written in the form
\[ a = 1 + \sum_{i=1}^{m} \nu_i \otimes \alpha_i + \sum_{j=1}^{n} \beta_j \otimes \mu_j, \alpha_i, \beta_j \in \Gamma, \mu_i \in m, \nu_j \in n. \]

Write the action of \( a \) on \( 1 \in L_{H^\varphi} : \)
\[ 0 = a \cdot 1 = (1 + \sum_{i=1}^{m} \nu_i \otimes \alpha_i + \sum_{j=1}^{n} \beta_j \otimes \mu_j) \cdot 1 = \]
\[ 1 + \sum_{i=1}^{m} \nu_i^\varphi \alpha_i + \sum_{j=1}^{n} \beta_j^\varphi \mu_j \in 1 + \ell_n, \]
because all the elements in the formulas above belong to \( \Gamma \), \( \nu_i \in \ell_n \) and \( \mu_j^\varphi \in \ell_n \), since \( m^\varphi \subset \ell_{m^\varphi} = \ell_n \). But \( 0 \not\in 1 + \ell_n \), which completes the proof. \( \square \)

Note that this theorem in the case \( m = n \) together with Theorem 3.2 gives another proof of Corollary 5.1.

Let \( \Gamma \) be the integral closure of \( \Gamma \) in \( L \), \( \varphi \in M \) and \( i : \Gamma \to \bar{\Gamma}, i_\varphi : \varphi(\Gamma) \to \bar{\Gamma} \) the canonical embeddings, \( \pi \) and \( \pi_\varphi \) the induced mappings of the maximal spectra, \( \text{Spec} \Gamma, \pi^{-1}(m) = \{\ell_1, \ldots, \ell_k\} \). The following lemma describes the sets \( X_a, a \in U \) (see Lemma 3.1).
Lemma 5.5. Let \( a \in L^H_\varphi \) and \( V = \Gamma[a\varphi]\Gamma \). Then the set of simple factors of the left \( \Gamma \)-module \( V \otimes_G \Gamma/\mathfrak{m} \) coincides with the set of simples of the form \( \Gamma/\mathfrak{n} \), \( \mathfrak{n} \in \pi(\varphi^{-1}(\mathfrak{m})) = \{\pi(\ell_1^\varphi), \ldots, \pi(\ell_k^\varphi)\} \). Besides, for \( u \in U \) and \( \mathfrak{m} \in \text{Specm} \Gamma \) holds \( X_u(\mathfrak{m}) = \pi(\text{supp } u \cdot \varphi^{-1}(\mathfrak{m})) \).

Proof. By Remark 2.1 \( V/V\mathfrak{m} \cong \Gamma^\varphi/\Gamma\mathfrak{m}^\varphi \). Since \( \Gamma^\varphi \subseteq \Gamma\mathfrak{m}^\varphi \) is a finite integral extension, holds \( (\Gamma\mathfrak{m})\mathfrak{m}^\varphi = \Gamma\mathfrak{m}^\varphi \neq \Gamma^\varphi \subseteq \Gamma \). The kernels of homomorphisms \( p: \hat{\Gamma} \rightarrow k \) extending \( p^\varphi: \Gamma^\varphi \rightarrow \Gamma^\varphi/\mathfrak{m}^\varphi \) form a \( G^\varphi \)-orbit \( \{\pi(\ell_1^\varphi), \ldots, \pi(\ell_k^\varphi)\} \), whose restrictions on \( \Gamma \) uniquely define all characters of \( \Gamma^\varphi \), extending \( p^\varphi \). This proves the first statement.

Let \( V = \Gamma u \Gamma \). Then Lemma 2.4, (5) reduces the second statement to the case of \( \Gamma \)-bimodule \( V \) generated by elements of the form \( [a_1\varphi], \ldots, [a_k\varphi] \). Hence the second statement follows from the first one. \( \square \)

We assume that \( \Gamma \) is normal noetherian \( k \)-algebra and \( \Omega_2 \) and \( \Omega_r \) are as in Section 4. For \( \mathfrak{m} \in \text{Specm} \Gamma \) denote by \( D(\mathfrak{m}) \) the class of \( \Delta \)-equivalence of \( \mathfrak{m} \), where \( \Delta \) is defined in 3.2.

Theorem 5.5.

1. If \( S(\mathfrak{m}, \mathfrak{n}) = \emptyset \) for some \( \mathfrak{m}, \mathfrak{n} \in \text{Specm} \Gamma \), then \( A(\mathfrak{m}, \mathfrak{n}) = 0 \).
2. Let \( \Delta' \) be the minimal equivalence, containing all \( (\mathfrak{m}, \mathfrak{n}) \in \text{Specm} \Gamma \times \text{Specm} \Gamma \) such that \( S(\mathfrak{m}, \mathfrak{n}) \neq \emptyset \). Then \( \Delta = \Delta' \). Besides, under the assumption of Theorem 5.1 the category \( A(D) \) does not split into a non-trivial direct sum and acts faithfully on \( \mathbb{H}(U, \Gamma, D) \).
3. If \( \mathfrak{m} \in \Omega_r \), then \( A(\mathfrak{m}, \mathfrak{m}) \) is a homomorphic image of \( \Gamma_m \). In particular, there exists a unique up to isomorphism simple \( U \)-module \( M \), extending the character \( \chi: \Gamma \rightarrow \Gamma/\mathfrak{m} \).
4. Let \( \mathfrak{m} \in \Omega_r \), \( D = D(\mathfrak{m}), M_\mathfrak{m} = A_D/A_D\mathfrak{m}, \) where \( \mathfrak{m} \subseteq \mathfrak{m} \) is the completed ideal. Then \( U/\mathfrak{m} \) is canonically isomorphic to \( \mathbb{F}(M_\mathfrak{m}) \). In particular, if \( \mathfrak{m} \in \Omega_r \), then \( U/\mathfrak{m} \) is simple.
5. Let \( M \) be a group, \( \mathfrak{m} \in \Omega_r \cap \Omega_2 \). Then for every \( \mathfrak{n} \in D(\mathfrak{m}) \) all objects of \( A_D \) are isomorphic and \( A(\mathfrak{n}, \mathfrak{n}) \simeq \hat{\Gamma}_\mathfrak{n} \).

Proof. The statement (1) follows from Lemma 5.3. By Lemma 5.5 \( \varphi \in S(\mathfrak{m}, \mathfrak{n}) \) if and only if \( \Gamma/\mathfrak{n} \) is a right subfactor of \( \Gamma[a\varphi]\Gamma/\Gamma[a\varphi]\mathfrak{m} \). It proves the first statement from (2). To prove the second statement, note that \( U(\mathfrak{m}, \mathfrak{n}) \neq 0 \) and only if \( S(\mathfrak{m}, \mathfrak{n}) \neq \emptyset \) and, following Theorem 5.4 and (1), if and only if \( A(\mathfrak{m}, \mathfrak{n}) \neq 0 \). On other hand, if \( a \in A(\mathfrak{m}, \mathfrak{n}), a \neq 0 \), then there exists \( N \geq 1 \), such that \( a \notin A(\mathfrak{m}, \mathfrak{n})m^N \), hence \( a \) acts nontrivially on \( U/Um^N \). It proves statement on \( A(D) \).

To prove (3) we note that, for \( \mathfrak{m} \in \Omega_r \) holds \( |S(\mathfrak{m}, \mathfrak{m})| = 1 \) hence by Theorem 5.1 \( A(\mathfrak{m}, \mathfrak{m}) \) is generated as \( \Gamma_\mathfrak{m} \)-bimodule by the class of \( e \in U \), which is central element in \( A(\mathfrak{m}, \mathfrak{m}) \). On other hand, there exists the canonical complete algebra homomorphism \( i: \hat{\Gamma}_\mathfrak{m} \rightarrow A(\mathfrak{m}, \mathfrak{m}), i(1) = \bar{e}, \) which is surjective. By Theorem 3.2 \( A_D - \text{mod}_d \) is equivalent to the full subcategory \( F(A_D - \text{mod}_d) \subseteq \mathbb{H}(\Gamma, U, D) \subseteq U - \text{mod} \).

For \( \mathfrak{m} \in D \) consider the functor \( W_\mathfrak{m}: \mathbb{H}(\Gamma, U, D) \rightarrow k - \text{Mod} \)

\[
W_\mathfrak{m}(M) = \{x \in M \mid \mathfrak{m} \cdot x = 0\}.
\]

This is a representable functor, namely \( W_\mathfrak{m} \simeq \text{Hom}_U(U/Um, -) \).

For \( N = F(N') \) we have \( \text{Hom}_U(F(M_\mathfrak{m}), F(N')) \simeq \text{Hom}_A(M_\mathfrak{m}, N') \simeq W_\mathfrak{m}(F(N')) = W_\mathfrak{m}(N) \),
where all isomorphisms are functorial, i.e., $U/Um \simeq \mathbb{F}(M_m)$. It implies (4). Consider in $\mathbb{F}(A_{D(m)} - \text{mod})$ a $U$-module:

$$M_{X,n} = U \otimes_{\Gamma} \Gamma/m^n \simeq \bigoplus_{n \in \mathbb{N}_0} \Gamma/n^n.$$

Any nonzero element from $\hat{\Gamma}_m$ acts nontrivially on $M_{X,n}$ for any $n$. Thus $A(m, m) \simeq \hat{\Gamma}_m$ by the Krull intersection theorem ([Mat], Theorem 8.10, (II)).

**Remark 5.1.** Recall that if $m$ is nonsingular point of $\text{Specm } \Gamma$, then $\hat{\Gamma}_m$ is isomorphic to the algebra of formal power series in $\text{GKdim } \Gamma$ variables.

**Corollary 5.6.** Let $M$ be a group, $D = D(m) \subset \text{Specm } \Gamma$ a $\Delta(U, \Gamma)$-equivalence class of a maximal ideal $m \in \Omega_r \cap \Omega_2$. Then the category $\mathbb{H}(U, \Gamma, D)$ is equivalent to the category $\hat{\Gamma}_m - \text{mod}$.

**Proof.** Since all objects in $A_D$ are isomorphic by Theorem 4.1, the categories $A_D - \text{mod}$ and $A(m, m) - \text{mod}$ are equivalent. Note that the functors of restriction $\text{res} : \mathbb{H}(U, \Gamma, D) \rightarrow A(m, m) - \text{mod}$ and of induction $\text{ind} : A(m, m) \rightarrow \mathbb{H}(U, \Gamma, D)$ are quasi-inverse.

**5.6. Proof of Theorem B.** First statement of Theorem B follows from Theorem 4.1.

Theorem 4.1, Theorem 5.5 and Corollary 5.6 immediately imply the second part of Theorem B.

The third statement of Theorem B is an analogue of the Harish-Chandra theorem for the universal enveloping algebras ([D]). In particular it shows that the subcategories in $U - \text{mod}$, described in Corollary 5.6, contain enough modules. Suppose that conditions of Theorem B are satisfied. Consider the massive set $\Omega_u$ constructed in Proposition 4.1 and set

$$\Omega'_u = \Omega_u \cap \Omega_2 \cap \Omega_r.$$

Then for any $m \in \Omega'_u$ the element $u$ acts nontrivially on $U/U'm$ which is simple by Theorem 5.5. This completes the proof of Theorem B.

6. GELFAND-TSETLIN MODULES FOR $\mathfrak{gl}_n$

Consider the general lineal Lie algebra $\mathfrak{gl}_n$ with the standard basis $e_{ij}, i, j = 1, \ldots, n$. Set $U = U(\mathfrak{gl}_n), U_m = U(\mathfrak{gl}_m), 1 \leq m \leq n$. Let $Z_m$ be the center of $U_m$. We identify $\mathfrak{gl}_m$ for $m \leq n$ with a Lie subalgebra of $\mathfrak{gl}_n$ spanned by $\{e_{ij} | i, j = 1, \ldots, m\}$, so that we have the following chain of inclusions

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \ldots \subset \mathfrak{gl}_n.$$

It induces the inclusions $U_1 \subset U_2 \subset \ldots \subset U_n$ of the universal enveloping algebras. The Gelfand-Tsetlin subalgebra ([DFO1]) $\Gamma$ in $U$ is generated by $\{Z_m | m = 1, \ldots, n\}$, where $Z_m$ is a polynomial algebra in $m$ variables $\{c_{mk} | k = 1, \ldots, m\}$,

$$c_{mk} = \sum_{(i_1, \ldots, i_k) \in \{1, \ldots, m\}^k} c_{i_1i_2} c_{i_2i_3} \cdots c_{i_{k-1}i_k}.$$

Hence $\Gamma$ is a polynomial algebra in $\frac{n(n + 1)}{2}$ variables $\{c_{ij} | 1 \leq j \leq i \leq n\}$ ([Zh]).

Denote by $K$ be the field of fractions of $\Gamma$. Let $M \subset \mathcal{L}, M \simeq \mathbb{Z}_{\mathbb{Z}}^{\frac{n(n-1)}{2}}$ be a free abelian group generated by $\delta^{ij}, 1 \leq j \leq i \leq n - 1, (\delta^{ij})_{kl} = 1$ if $i = k, j = l$ and 0 otherwise. For $i = 1, \ldots, n$ denote by $S_i$ the $i$-th symmetric group and set
Proposition 6.1. (see [FO1], Proposition 7.2) Let $S = \Gamma \setminus \{0\}$. Then

- $t$ is an embedding;
- $UK = KU \simeq (L * \mathbb{Z}^m)^G$, $m = n(n - 1)/2$;
- $U$ is a Galois order over $\Gamma$.

To estimate the number of isomorphism classes of simple Gelfand-Tsetlin modules for $U(\text{gl}_n)$ we need the following statement.

Theorem 6.1. ([FO2]) $U(\text{gl}_n)$ is free both as a left and as a right $\Gamma$-module.

Set

$$Q_n = \prod_{i=1}^{n-1} i!.$$

Corollary 6.1. Let $U = U(\text{gl}_n)$, $\Gamma \subset U$ is the Gelfand-Tsetlin subalgebra, $D$ a $\Delta$-class, $m \in D$. Then

1. For a $U$-module $M$, such that $M(m) \neq 0$ and $M$ is generated by some $x \in M(m)$ (in particular for a simple module) holds

$$\dim_k M(m) \leq Q_n.$$

2. The number of isomorphism classes of simple $U$-modules $N$, such that $N(m) \neq 0$ is always nonzero and does not exceed $Q_n$.

Proof. Note, that a simple $M$ such that $M(m) \neq 0$, is generated by any nonzero vector from $M(m)$. Since $U$ is a free $\Gamma$-module, we can apply Theorem 5.2, (3) and obtain as a boundary $\dim_k M(m) \leq |S(m,m)/G|$. On the other hand, by Lemma 5.1,(3), the right hand side here is bounded by the cardinality of the set

$$B = \{m \in \mathbb{Z}^{(n-1)/2} | \pi(m + \ell_m) = \ell_m\}.$$

Equivalently, $m \in B$ if and only if the $i$th rows of $\ell_m$ and $\ell_m + m$ differ by a permutation from $S_i$, $i = 1, \ldots, n-1$. It gives us at most $|S_1| \cdot |S_2| \cdot \ldots \cdot |S_{n-1}|$
possibilities for $m \in M$ and implies (1). By Lemma 3.7, the number of isomorphism classes of simple $U$-modules $N$, such that $N(m) \neq 0$, equals the number of isomorphism classes of simple $A(m,m)$-modules, and the correspondence is given by $M \leftrightarrow M(m)$. Therefore, if $X = U/U_m$, then the $A(m,m)$-module $X(m)$ covers any simple $A(m,m)$-module. Together with (1) it proves (2). □

Remark 6.1. We believe that the bound $Q_n$ in (1) can not be improved. It is known to be exact for $n = 2$ and $n = 3$ [DFO1].

Remark 6.2. Applying Theorem B to the case of $U(gl_n)$ and the Gelfand-Tsetlin subalgebra $\Gamma$, we note that $\Gamma$ has a simple spectrum not only on finite-dimensional modules (which is well known) but also on generic Gelfand-Tsetlin modules. On the other hand, it is known not to be the case in general (see [DFO1]).

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