Impact of high-order effects on soliton explosions in the complex cubic–quintic Ginzburg–Landau equation

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We investigate the impact of higher-order nonlinear and dispersive effects on the onset of soliton explosions in the complex cubic–quintic Ginzburg–Landau equation. We show how the interplay of the high order effects (HOEs) results in the splitting of symmetric explosion modes and to the formation of right- or left-side periodic explosions. In addition, we demonstrate that HOEs induce a series of pulsating instabilities, leading to a significant reduction of the stability region of the single soliton solution.

Soliton explosions are among the most striking and fascinating nonlinear phenomena studied in mode-locked lasers. They were first predicted theoretically in a one-dimensional complex cubic-quintic Ginzburg–Landau equation (CQGLE) \cite{2} for anomalous linear dispersion and then verified experimentally in a Kerr lens mode-locked Ti:sapphire laser \cite{2}. In this regime, a localized pulse circulating in the cavity experiences an abrupt structural collapse at certain points of its time evolution and subsequently recovers its original shape. A large number of numerical studies have been reported in this framework \cite{3–8}. Among the reported features is the stable existence of symmetric and asymmetric explosive localized states (LSs) over a wide range of system parameters. Further away from the explosion threshold, the exploding LSs experience a complex dynamics and exhibit spatiotemporal chaos; The LSs conserve an almost identical shape after each explosion cycle and the times between explosions appear to be randomly distributed. In two spatial dimensions, it was shown that the center of mass of asymmetrically exploding LS undergoes a subdiffusive continuous-time random walk despite the deterministic character of the underlying model, while in the case dominated by only asymmetric explosions, it becomes characterized by normal diffusion \cite{9, 10}. Recently, exploding LSs have been observed in an all-normal-dispersion Yb-doped mode-locked fiber laser operated in a transition regime between stable and noise-like emission \cite{11}. The resulting experimental evidence has been successfully compared to realistic numerical simulations based on an envelope function approach \cite{12}. There, the observed explosions manifest themselves as abrupt temporal shifts in the output pulse train. In \cite{13}, the connection between the pulse propagation model in optical fibers developed in \cite{12} and the CQGLE with additional higher-order nonlinear and dispersive effects was established. The latter leads to a formation of periodic, non-chaotic one-side explosions. The influence of these higher-order effects (HOEs) on the exploding LSs in the CQGLE was studied numerically in \cite{14–17}. In particular it was shown that that a proper combination of the three higher-order effects can provide a shape stabilization of an exploding LS. However, despite significant theoretical interest, the impact of HOEs on the onset of soliton explosions have not been extensively studied so far and the full bifurcation study of the explosion regime is still lacking.

In this letter we investigate the impact of the HOEs, namely, self-frequency shift, self-steepening, and third-order dispersion on the onset of soliton explosions in the CQGLE. Using path following techniques applied to partial differential equations (PDEs), we show that HOEs split the symmetric and asymmetric explosion modes, leading to the formation of left- and right-side periodic explosions close to the onset. Furthermore, we show how the interplay of the HOEs can result in non-trivial interaction of the explosion modes resulting in the selection of right- or left-explosions for certain sets of system parameters. Finally we study the impact of HOEs terms on the stable LSs profile and we disclose new, HOEs induced, pulsating instabilities, leading to a significant reduction of the stability region of the single LSs.

In nonlinear optics, the CQGLE is considered as one of the paradigms for LS formation in mode-locked lasers and it is also widely used to describe such phenomena as short pulse propagation in optical transmission lines, dynamics of multimode lasers, parametric oscillators, and transverse pattern formation in nonlinear optical media \cite{18–26}. The CQGLE also plays a preponderant role in in the modeling of various non-equilibrium phenomena such as convection instabilities, binary fluid convection, and phase transitions \cite{27} and is an amplitude equation describing the onset of an Andronov–Hopf bifurcation in dynamical systems \cite{28}. The one-dimensional CQGLE with HOEs reads

\[ i \partial_t A + \frac{D}{2} \partial_x^2 A + |A|^2 A + \nu |A|^4 A = i \delta A + i |A|^2 A + i \beta \partial_x^2 A + i \mu |A|^4 A + \text{H.O.E.} , \]
where
\[
\text{H.O.E.} = i\beta_3 \partial_t^3 A - i s \partial_t (|A|^2 A) + \tau_R \partial_t |A|^2.
\]

Here, \(z\) is the normalized propagation distance (or the cavity round-trip number when used to describe passively mode-locked lasers), \(t\) is the retarded time (or a transversal spatial coordinate), \(A\) is the normalized envelope of the field, \(D = \pm 1\) is the group velocity dispersion coefficient corresponding to an anomalous or normal regime, \(\delta > 0\) (\(< 0\)) is the linear gain (loss) coefficient, \(\beta > 0\) accounts for spectral filtering, \(\mu < 0\) represents the saturation of the nonlinear gain, \(\nu\) corresponds to the saturation of the nonlinear refractive index and \(\epsilon\) is the nonlinear gain parameter. Further, \(\beta_3\) accounts for the third order dispersion (TOD), \(s\) corresponds to the self-stepping (SST) and \(\tau_R\) is a coefficient related to the intrapulse Raman scattering (IRS), which determines the soliton self-frequency shift. While the classical cubic CGLE describes a supercritical bifurcation, in the case of subcritical instability this equation is augmented with a fifth-order terms to allow the existence of stable pulse-like localized solutions if \(\delta < 0\) and \(\epsilon > 0\).

Stationary localized solutions of (1) can be found using the ansatz \(A(t, z) = A(t - v z) e^{-i \omega z}\), where \(\omega\) is the spectral parameter and \(v\) is the propagation speed which adds a contribution \((v \partial_t + i \omega) A\) to the right-hand side of (1). We choose the parameters of (1) in a range where soliton explosions exist [1] and can now track the solutions of the resulting equation in parameter space using the path following technique within the pde2path framework [29]. During the continuation, both \(\omega\) and \(v\) become two additional free parameters that are automatically adapted during the continuation. In order to determine them, we impose additional auxiliary conditions, accounting for the translational and phase-shift symmetries of (1) and preventing the continuation algorithm to trivially follow solutions along the corresponding neutral degree of freedom. Note, that in contrast to direct numerical simulations, continuation algorithms are able to track both stable and unstable solutions of the underlying system and make the reconstruction of the whole solution branch, including the information of possible instabilities, feasible. The latter can be followed in the parameter space so that bifurcation diagrams containing the important information about, i.e., a stability region of the studied solution can be created. Note that path continuation techniques are widely employed to obtain different types of solutions of nonlinear ordinary or delay differential equations. However, continuation tools for multidimensional partial differential equations are still scarce. In nonlinear optics, the bifurcation analysis can be quite involved because of the presence of complex fields and different continuous symmetries. In particular, for the CQGLE (1) the frequency shift \(\omega\) which is connected to the phase-shift invariance is very large, demanding the high accuracy calculations: In order to obtain the LS properly we track \(N_t = 4 \times 2048\) degrees of freedom (real and imaginary part of the field and their gradients) together with 550 eigenvalues to resolve the explosion modes.

Figure 1. (a) A maximal intensity of a single LS of (1) as a function of the loss parameter \(\delta\) for \(\beta_3 = s = \tau_\nu = 0\). A LS is stable between a AH point \(H_0\) and the threshold of explosions, given by a double AH point \(H_L, H_R\) at \(\delta = -0.5537\). (b) Space time representation of the time evolution of a LS calculated at \(\delta = -0.3\). (c,d) Real parts of the critical eigenfunctions (red) at \(\delta = -0.6\), corresponding to asymmetrical and symmetrical explosions, respectively. Cyan lines represent the Re(A) field, whereas the black line corresponds to the intensity profile. Parameters are: \(\epsilon = 1.0188, \beta = 0.125, \mu = -0.1, \nu = -0.6\).

In the absence of HOE terms, the velocity \(v\) remains zero and a branch of a stationary LSs emerges when changing the loss parameter \(\delta\). Figure 1 (a) shows the maximal intensity \(I = |A|^2\) of the field as a function of \(\delta\). One can see that a single LS bifurcates subcritically from a homogeneous zero state at \(\delta = 0\) and experiences a fold \(F\) (red point) at some negative value of \(\delta\). The LS’s stability is governed by the Andronov-Hopf (AH) point \(H_0\), where a pair of complex eigenvalues corresponding to the symmetrical pulsation becomes stable. The LS remains stable for the increasing \(\delta\) (thick blue line) until a double AH bifurcation point \((H_L, H_R)\), corresponding to the symmetric (even) and asymmetric (odd) perturbations. The corresponding eigenmode for Re(A) is shown in cyan together with the intensity profile (black line) in Fig. 1 (c,d). One notices that the critical symmetric (asymmetric) eigenfunctions are localized on both flanks of the LS in phase (antiphase). At the double AH point, two branches of periodic solutions emerge leading to the formation of symmetric and asymmetric explosions at these branches. Figure 1 (b) shows an example of the symmetric-asymmetric exploding LS obtained from a direct numerical simulation of (1) on the domain of \(L_t = 100\) with \(N_t = 1024\) grid points for \(\delta = -0.3\) far above from the bifurcation point. Note that in [3] the double AH point and the critical eigenfunctions were found for fixed values of \(\delta\) using numerical linear stability.
analysis. In contrast, the continuation algorithms allows to track the eigenvalue spectrum along the whole solution branch giving the complete information about the LS stability.

Figure 2. (a,b) A branch of a single LS of (1) as a function δ for \( \beta_3 = 0.016, \tau_R = 0.032, s = 0.009 \). The splitting of AH points \( H_R, H_L \) can be seen. (a) A maximal intensity and (b) the drift velocity are shown. A LS is stable at an AH point \( H_0 \) and the threshold of right-side explosions \( H_R \). (c,d) Real parts of the critical eigenfunctions (red), \( \text{Re}(A) \) (cyan) and \( I = |A|^2 \) (black) at \( \delta = -0.6 \) corresponding to the right- and left-side explosions, respectively.

However, the presence of HOEs significantly changes the behavior of the LS: Each of the TOD, SST and IRS terms in (1) break the parity symmetry of the system and affect the leading and falling edges of LSs differently (cf. black dashed lines in Fig. 2 (c,d)) and the LSs start to drift as presented in Fig. 2 (b), where the dependence of the LSs velocity \( v \) on \( \delta \) is shown for non-vanishing fixed values of \( \beta_3, s \) and \( \tau \). One can see, that the unstable part of the branch is a nonlinear function of \( \delta \). However, after a fold, the velocity \( v \) monotonously increases with \( \delta \). In Fig. 2 (a) the peak intensity of a single LS as a function of \( \delta \) is presented. Note that the overall shape of the branch remains the same as in the case of vanishing HOEs, namely, a single LS emerges from the homogeneous background at \( \delta = 0 \). The solution branch has a fold at a certain \( \delta \) value (cf. the red circle) and gains stability at the AH point \( H_0 \). However, at the high power branch the double AH point \( H_{L,R} \) breaks in the presence of HOEs and two distinct AH points \( H_R \) and \( H_L \) form. The eigenfunctions corresponding to the perturbation of \( \text{Re}(A) \) are not located symmetrically anymore and are localized at left- or right-flanks of the LS as shown in Fig. 2 (c,d) (red lines).

This splitting of the explosion modes leads to the emergence of two branches of periodic solutions, where periodic one-side left- and right- explosions start to exist. The splitting of symmetric explosion modes is a general feature of any of the HOEe presented in the system and is the result of the breaking of the parity symmetry. However, the position of the selected left or right modes strongly depends on the amount of HOE coefficients \( \beta_3, s \) and \( \tau \). In order to study the selection of the one-side explosion modes and to determine the region of stability of a LS in the presence of HOE, we perform a fold and AH point continuation and reconstruct a bifurcation diagram in the plane spanned by \( \delta \) and HOE parameters. Figure 3 (a) shows the impact of the TOD coefficient \( \beta_3 \) on the evolution of the fold \( F \), as well as the AH points \( H_0, H_L \) and \( H_R \) in the \( (\delta, \beta_3) \) plane for \( s = \tau_R = 0 \). One can see that the position of the fold \( F \) as well as of the AH point \( H_0 \) remain almost unaffected by the TOD. However, even a small amount of positive \( \beta_3 \) induces a splitting of the double AH point \( H_{L,R} \), making the right-explosions mode \( H_R \) to be selected first. That is, for any fixed small \( \beta_3 > 0 \), a LS is stable between \( H_0 \) and \( H_R \) lines, and right-side explosions set in first. Increasing \( \delta \), the left-side explosion curve \( H_L \) can also be crossed and a combination of right- and left- side explosions can be found. Note that a similar behavior and selection order can be achieved by changing the SST coefficient \( s \) (or the IRS \( \tau_R \)) keeping the other HOEs to zero. Note also that in all these cases the order of the selected explosions modes can be changed by changing the sign of the corresponding HOE term. However, the selection of the \( H_L, H_R \) modes can also be tuned by choosing a nonzero amount of all three HOE coefficients as shown in Fig. 3 (b), where the bifurcation diagram in the \( (\delta, \beta_3) \)
plane is presented for non-vanishing values of $\tau_R$ and $s$. One can see that as in the case of zero SST and IRS terms, the positions of the fold $F$ and the AH point $H_0$ remain almost the same in $\delta$ and the double AH points split again. However, a presence of nonzero $s$ and $\tau_R$ shifts the $H_L$, $H_R$ crossing point in the direction of the positive TOD coefficient $\beta_3$. This leads to the formation of two regions in the parameter space: Whereas after the crossing point, the right-side explosions are selected (red dashed line), a new region emerges, where the left-side explosion mode wins for small values of $\beta_3$ (green dotted line). That is, depending on the loss parameter $\delta$ one can select left- or right-side explosions depending on the values of the TOD. Two examples of the direct numerical simulations of (1) showing the left- and right-side periodic explosions of a single LS are presented in Fig. 3 (c, d). Note that in this case both left- and right-side explosions keep the positive propagation direction.

Figure 4. (a) The intensity profile (black dashed line) of a single LS of (1) calculated at $\beta_3 = 0.068$ and $\delta = -1$ together with the real part of the critical eigenfunction $H_1$ (red), whereas the cyan line stands for the Re$(A)$ field. (b) The drift velocity $v$ as a function of the TOD coefficient $\beta_3$. The points $H_{1,2,3,4}$ correspond to the TOD induced AH bifurcations, whereas $H_0$ corresponds to a symmetric explosion, whereas $H_0$ corresponds to a symmetric explosion. Further, we demonstrated how the interplay of different HOEs can result in the selection of right- or left- explosions for certain sets of system parameters. Finally we found that the third-order dispersion leads to new pulsating instabilities, which result in the significant reduction of the stability region of the single LSs.

In conclusion, we studied the impact of self-frequency shift, self-steepening, and third-order dispersion on the onset of soliton explosions in the CQGLE. Using path continuation techniques, we showed, how HOEs induce a splitting of the symmetric and asymmetric explosion modes, leading to the formation of left- and right- one-side periodic explosions. Further, we demonstrated how the interplay of different HOEs can result in the selection of right- or left- explosions for certain sets of system parameters. Finally we found that the third-order dispersion leads to new pulsating instabilities, which result in the significant reduction of the stability region of the single LSs.

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