CATEGORIZATION OF QUANTUM GENERALIZED KAC-MOODY ALGEBRAS AND CRYSTAL BASES

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ABSTRACT. We construct and investigate the structure of the Khovanov-Lauda-Rouquier algebras $R$ and their cyclotomic quotients $R^\lambda$ which give a categorification of quantum generalized Kac-Moody algebras. Let $U_q(g)$ be the integral form of the quantum generalized Kac-Moody algebra associated with a Borcherds-Cartan matrix $A = (a_{ij})_{i,j \in I}$ and let $K_0(R)$ be the Grothendieck group of finitely generated projective graded $R$-modules. We prove that there exists an injective algebra homomorphism $\Phi : U_{-A}(g) \rightarrow K_0(R)$ and that $\Phi$ is an isomorphism if $a_{ii} \neq 0$ for all $i \in I$. Let $B(\infty)$ and $B(\lambda)$ be the crystals of $U_q(g)$ and $V(\lambda)$, respectively, where $V(\lambda)$ is the irreducible highest weight $U_q(g)$-module. We denote by $B(\infty)$ and $B(\lambda)$ the isomorphism classes of irreducible graded modules over $R$ and $R^\lambda$, respectively. If $a_{ii} \neq 0$ for all $i \in I$, we define the $U_q(g)$-crystal structures on $B(\infty)$ and $B(\lambda)$, and show that there exist crystal isomorphisms $B(\infty) \cong B(\infty)$ and $B(\lambda) \cong B(\lambda)$. One of the key ingredients of our approach is the perfect basis theory for generalized Kac-Moody algebras.

INTRODUCTION

In [24, 25] and [31], Khovanov-Lauda and Rouquier independently introduced a new family of graded algebras $R$ which gives a categorification of quantum groups associated with symmetrizable Kac-Moody algebras. More precisely, let $U_q(g)$ be the quantum group associated with a symmetrizable Kac-Moody algebra and let $U_A(g)$ be the integral form of $U_q(g)$, where $A = \mathbb{Z}[q, q^{-1}]$. Then it was shown that the Grothendieck group $K_0(R)$ of finitely generated graded projective $R$-modules is isomorphic to $U_A^-(g)$, the negative part of $U_A(g)$. Furthermore, for symmetric Kac-Moody algebras, Varagnolo and Vasserot proved that the isomorphism classes of principal indecomposable $R$-modules correspond to Lusztig’s canonical basis (or Kashiwara’s lower global basis) under this isomorphism [35]. The algebra $R$ is called the Khovanov-Lauda-Rouquier algebra associated with $g$.

For each dominant integral weight $\lambda \in P^+$, the algebra $R$ has a special quotient $R^\lambda$ which is called the cyclotomic quotient. It was conjectured that the cyclotomic quotient $R^\lambda$ gives a categorification of the irreducible highest weight module $V(\lambda)$ [24]. For type $A_\infty$ and $A^{(1)}_n$, this conjecture was proved...
in \cite{4,5}. In \cite{14}, Kang and Kashiwara proved Khovanov-Lauda categorification conjecture for all symmetrizable Kac-Moody algebras. Webster also gave a proof of this conjecture by a completely different method \cite{36}. In \cite{27}, the crystal version of this conjecture was proved. That is, in \cite{27}, Lauda and Vazirani investigated the crystal structure on the set of isomorphism classes of irreducible graded modules over $R$ and $R^{\lambda}$, and showed that these crystals are isomorphic to the crystals $B(\infty)$ and $B(\lambda)$, respectively.

The purpose of this paper is to extend the study of Khovanov-Lauda-Rouquier algebras to the case of \textit{generalized Kac-Moody algebras}. The generalized Kac-Moody algebras were introduced by Borcherds in his study of Monstrous Moonshine \cite{2}, and they form an important class of algebraic structure behind many research areas such as algebraic geometry, number theory and string theory (see, for example, \cite{3, 6, 7, 10, 18, 29, 30, 32, 33}). In particular, the \textit{Monster Lie algebra}, a special example of generalized Kac-Moody algebras, played a crucial role in proving the Moonshine conjecture \cite{3}. Moreover, the generalized Kac-Moody algebras draw more and more attention among mathematical physicists due to their connection with string theory and other related topics. The quantum deformations of generalized Kac-Moody algebras and their integrable highest weight modules were constructed in \cite{13} and the crystal basis theory for quantum generalized Kac-Moody algebras was developed in \cite{11,12}. In \cite{21}, the canonical bases for quantum generalized Kac-Moody algebras were realized as certain semisimple perverse sheaves, and in \cite{10,17}, a geometric construction of crystals $B(\infty)$ and $B(\lambda)$ was given using Lusztig’s and Nakajima’s quiver varieties, respectively.

In this paper, we construct and investigate the structure of Khovanov-Lauda-Rouquier algebras $R$ and their cyclotomic quotients $R^{\lambda}$ which give a categorification of quantum generalized Kac-Moody algebras. Let $U_q(\mathfrak{g})$ be the quantum generalized Kac-Moody algebra associated with a Borcherds-Cartan matrix $A = (a_{ij})_{i,j \in I}$. We first define the Khovanov-Lauda-Rouquier algebra $R$ in terms of generators and relations. A big contrast with the case of Kac-Moody algebras is that the nil Hecke algebras corresponding to the imaginary simple roots with norm $\leq 0$ may have nonconstant twisting factors for commutation and braid relations. In this work, we choose any homogeneous polynomials $P_i(u,v)$ of degree $1 - a_{ii}^2$ and their variants $\overline{P}_i$ and $\overline{P}'_i$ ($i \in I$) as these twisting factors (see Definition \ref{2.1}). When $a_{ii} = 2$, we are reduced to the case of Kac-Moody algebras. The role of these twisting factors is still mysterious. For convenience, we also give a diagrammatic presentation of the algebra $R$.

Next, we show that there exists an injective algebra homomorphism $\Phi : U^-_q(\mathfrak{g}) \rightarrow K_0(R)$, where $K_0(R)$ is the Grothendieck group of finitely generated graded projective $R$-modules (Theorem \ref{3.4}). Thus $\text{Im} \Phi$ gives a categorification of $U^-_q(\mathfrak{g})$. To do this, we need to show that the quantum Serre relations are preserved by the map $\Phi$. In general, $\Phi$ is not surjective even for the case $A = (0)$. The whole Grothendieck group seems rather large and nontrivial. However, if $a_{ii} \neq 0$ for all $i \in I$, we can show that $\Phi$ is an isomorphism (Theorem \ref{3.15}). As in the case of Kac-Moody algebras, we conjecture that, if the Borcherds-Cartan matrix $A = (a_{ij})_{i,j \in I}$ is symmetric and $a_{ii} \neq 0$ for all $i \in I$, then the isomorphism classes of graded projective indecomposable $R$-modules correspond to canonical
basis elements under the isomorphism $\Phi$. We will investigate this conjecture in a forthcoming paper following the framework given in $[21, 35]$.

Now we focus on the crystal structures. We would like to emphasize that one of the key ingredients of our approach is the perfect basis theory for generalized Kac-Moody algebras and it can be applied to the Kac-Moody algebras setting as well. Our work is different from $[27]$ in this respect. In $[1]$, Berenstein and Kazhdan introduced the notion of perfect bases for integrable highest weight modules $V(\lambda)$ ($\lambda \in P^+$) over Kac-Moody algebras. They showed that the colored oriented graphs arising from perfect bases are all isomorphic to the crystal $B(\lambda)$. Their work was extended to the integrable highest weight modules over generalized Kac-Moody algebras in $[19]$. In this work, we define the notion of perfect bases for $U_q(g)$ as a module over the quantum boson algebra $B_q(g)$. The existence of perfect basis for $U_q(g)$ is provided by constructing the upper global basis (or dual canonical basis) of $U_q(g)$. We also show that the crystal arising from any perfect basis of $U_q(g)$ is isomorphic to the crystal $B(\infty)$ (Theorem 4.19).

With perfect basis theory at hand, we construct the crystal $B(\infty)$ as follows. Let $G_0(R)$ be the Grothendieck group of finite-dimensional graded $R$-modules and set $G_0(R)_{Q(q)} = Q(q) \otimes_A G_0(R)$. We denote by $B(\infty)$ the set of isomorphism classes of irreducible graded $R$-modules and define the crystal operators using induction and restriction functors. Moreover, we show that $G_0(R)_{Q(q)}$ has a $B_q(g)$-module structure and that if $a_{ii} \neq 0$ for all $i \in I$, then $B(\infty)$ is a perfect basis of $G_0(R)_{Q(q)}$. Therefore, by the main theorem of perfect basis theory, we obtain a crystal isomorphism (Theorem 5.4):

$$B(\infty) \simeq B(\infty).$$

For a dominant integral weight $\lambda \in P^+$, we define the cyclotomic Khovanov-Lauda-Rouquier algebra $R^\lambda$ to be the quotient of $R$ by a certain two-sided ideal depending on $\lambda$. Let $B(\lambda)$ denote the set of isomorphism classes of irreducible graded $R^\lambda$-modules and define the crystal operators using induction/restriction functors and projection/inflation functors. It was shown in $[12]$ that there exists a strict crystal embedding

$$B(\lambda) \hookrightarrow B(\infty) \otimes T_\lambda \otimes C.$$

If $a_{ii} \neq 0$ for all $i \in I$, using the above crystal embedding, we construct a crystal isomorphism (Theorem 5.14):

$$B(\lambda) \simeq B(\lambda).$$

In $[15]$, after this work was completed, Khovanov-Lauda cyclotomic conjecture was proved for all symmetrizable generalized Kac-Moody algebras.

This paper is organized as follows. Section 1 contains a brief review of quantum generalized Kac-Moody algebras and crystal bases. In Section 2, we define the Khovanov-Lauda-Rouquier algebra $R$ associated with a Borcherds-Cartan matrix $A = (a_{ij})_{i,j \in I}$, and investigate its algebraic structure and representation theory. We construct a faithful polynomial representation of $R(\alpha)$ and prove the Khovanov-Lauda-Rouquier algebra version of the quantum Serre relations. In Section 3, we show that the algebra $R$ gives a categorification of $U_q^- (g)$. We define a twisted bialgebra structure on $K_0(R)$ using induction and restriction functors, and show that there exists an injective algebra homomorphism.
\[ \Phi : U_q^- (g) \rightarrow K_0 (R) \]. In particular, we prove that \( U_q^- (g) \cong K_0 (R) \) when \( a_{ii} \neq 0 \) for all \( i \in I \). Section 4 is devoted to the theory of perfect bases. We define the notion of perfect bases for \( U_q^- (g) \) as a \( B_q (g) \)-module and show that \( U_q^- (g) \) has a perfect basis by constructing the upper global basis of \( U_q^- (g) \). The main theorem in Section 4 asserts that the crystals arising from perfect bases are all isomorphic to \( B(\infty) \). In Section 5, we study the crystal structures on \( \mathcal{B}(\infty) \) and \( \mathcal{B}(\lambda) \). Using the theory of perfect bases, we prove that there exists a crystal isomorphism \( \mathcal{B}(\infty) \cong B(\infty) \) when \( a_{ii} \neq 0 \) for \( i \in I \). Furthermore, we define the cyclotomic quotient \( R^\lambda \) of \( R \), and investigate the basic properties of irreducible \( R^\lambda \)-modules. Combining the isomorphism \( \mathcal{B}(\infty) \cong B(\infty) \) with the strict embedding \( B(\lambda) \rightarrow B(\infty) \otimes T_\lambda \otimes C \), we obtain a crystal isomorphism \( \mathcal{B}(\lambda) \cong B(\lambda) \).

1. Quantum Generalized Kac-Moody Algebras

Let \( I \) be a countable (possibly infinite) index set. A matrix \( A = (a_{ij})_{i,j \in I} \) with \( a_{ij} \in \mathbb{Z} \) is called an even integral Borcherds-Cartan matrix if it satisfies (i) \( a_{ii} = 2 \) or \( a_{ii} \in 2\mathbb{Z}_{\geq 0} \), (ii) \( a_{ij} \leq 0 \) for \( i \neq j \), (iii) \( a_{ij} = 0 \) if and only if \( a_{ji} = 0 \). For \( i \in I \), \( i \) is said to be real if \( a_{ii} = 2 \) and is said to be imaginary otherwise. We denote by \( I^{re} \) the set of all real indices and by \( I^{im} \) the set of all imaginary indices. In this paper, we assume that \( A \) is symmetrizable; i.e., there is a diagonal matrix \( D = \text{diag}(s_i \in \mathbb{Z}_{>0} | i \in I) \) such that \( DA \) is symmetric.

A Borcherds-Cartan datum \( (A, P, \Pi, \Pi^\vee) \) consists of

(1) a Borcherds-Cartan matrix \( A \),
(2) a free abelian group \( P \), the weight lattice,
(3) \( \Pi = \{ \alpha_i \in P \mid i \in I \} \), the set of simple roots,
(4) \( \Pi^\vee = \{ h_i \mid i \in I \} \subset P^\vee := \text{Hom}(P, \mathbb{Z}) \), the set of simple coroots,

satisfying the following properties:

(a) \( \langle h_i, \alpha_j \rangle = a_{ij} \) for all \( i, j \in I \),
(b) \( \Pi \) is linearly independent,
(c) for any \( i \in I \), there exists \( \Lambda_i \in P \) such that \( \langle h_j, \Lambda_i \rangle = \delta_{ij} \) for all \( j \in I \).

Let \( \mathfrak{h} = \mathbb{Q} \otimes \mathbb{Z} P^\vee \). Since \( A \) is symmetrizable, there is a symmetric bilinear form \( (\ , \ ) \) on \( \mathfrak{h}^* \) satisfying

\[
(\alpha_i | \alpha_j) = s_i a_{ij} \quad (i, j \in I).
\]

We denote by \( P^+ := \{ \lambda \in P| \langle \lambda(h_i) \rangle \in \mathbb{Z}_{\geq 0}, i \in I \} \) the set of dominant integral weights. The free abelian group \( Q = \oplus_{i \in I} \mathbb{Z} a_i \) is called the root lattice. Set \( Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} a_i \). For \( \alpha = \sum k_i a_i \in Q^+ \), we denote by \( |\alpha| \) the height of \( \alpha \): \( |\alpha| = \sum k_i \).

Let \( q \) be an indeterminate and \( m, n \in \mathbb{Z}_{\geq 0} \). Set \( c_i = -\frac{1}{2} a_{ii} \) and \( q_i = q^{c_i} \) for \( i \in I \). If \( i \in I^{re} \), define

\[
[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^{n} [k]_i, \quad \frac{m}{n}_i = \frac{[m]_i!}{[m-n]_i! [n]_i!}.
\]
If $a_{ii} < 0$, we define
\[
\{n\}_i = \frac{q_i^{e_i n} - q_i^{-e_i n}}{q_i^{e_i} - q_i^{-e_i}}, \quad \{n\}_i! = \prod_{k=1}^{n} \{k\}_i, \quad \binom{m}{n}_i = \frac{\{m\}_i!}{\{m-n\}_i! \{n\}_i!}.
\]
If $a_{ii} = 0$, we define
\[
\{n\}_i = n, \quad \{n\}_i! = n!, \quad \binom{m}{n}_i = \binom{m}{n}.
\]

**Definition 1.1.** The quantum generalized Kac-Moody algebra $U_q(g)$ associated with a Borcherds-Cartan datum $(A, P, \Pi, \Pi')$ is the associative algebra over $\mathbb{Q}(q)$ with 1 generated by $e_i, f_i$ ($i \in I$) and $q^h$ ($h \in P'$) satisfying following relations:

1. $q^0 = 1, q^h q^{h'} = q^{h+h'}$ for $h, h' \in P'$,
2. $q^h e_i q^{-h} = q^{(h, \alpha_i)} e_i$, $q^h f_i q^{-h} = q^{-(h, \alpha_i)} f_i$ for $h \in P', i \in I$,
3. $e_i f_j - f_j e_i = \delta_{ij} K_i - K_i^{-1}$, where $K_i = q_i^h$,
4. $\sum_{r=0}^{1-a_{ij}} (-1)^r \left[ \begin{array}{c} 1 - a_{ij} \\ r \end{array} \right] e_i^{1-a_{ij}-r} e_j e_i^r = 0$ if $i \in I^+$ and $i \neq j$,
5. $\sum_{r=0}^{1-a_{ij}} (-1)^r \left[ \begin{array}{c} 1 - a_{ij} \\ r \end{array} \right] f_i^{1-a_{ij}-r} f_j f_i^r = 0$ if $i \in I^+$ and $i \neq j$,
6. $e_i e_j - e_j e_i = 0, f_i f_j - f_j f_i = 0$ if $a_{ij} = 0$.

Let $U_q^+(g)$ (resp. $U_q^-(g)$) be the subalgebra of $U_q(g)$ generated by the elements $e_i$ (resp. $f_i$), and let $U_q^0(g)$ be the subalgebra of $U_q(g)$ generated by $q^h$ ($h \in P'$). Then we have the triangular decomposition

\[
U_q(g) \cong U_q^-(g) \otimes U_q^0(g) \otimes U_q^+(g),
\]

and the root space decomposition

\[
U_q(g) = \bigoplus_{\alpha \in \mathbb{Q}} U_q(g)_\alpha,
\]

where $U_q(g)_\alpha := \{ x \in U_q(g) \mid q^h x q^{-h} = q^{(h, \alpha)} x \text{ for any } h \in P' \}$. Define a $\mathbb{Q}$-algebra automorphism $^\gamma: U_q^-(g) \rightarrow U_q^-(g)$ by

\[
ed_i \mapsto e_i, \quad f_i \mapsto f_i, \quad q^h \mapsto q^{-h}, \quad q \mapsto q^{-1}.
\]

Let $\mathbb{A} = \mathbb{Z}[q, q^{-1}]$. For $n \in \mathbb{Z}_{>0}$, set

\[
e_i^{(n)} = \begin{cases} \frac{q^n}{[n]^!} & \text{if } i \in I^+, \\ q^n & \text{if } i \in I^-, \end{cases} \quad f_i^{(n)} = \begin{cases} \frac{q^n}{[n]^!} & \text{if } i \in I^+, \\ f_i^n & \text{if } i \in I^-, \end{cases}
\]

and denote by $U_{q, \mathbb{A}}$ (resp. $U_{q, \mathbb{A}}^+(g)$) the $\mathbb{A}$-algebra of $U_q^-(g)$ generated by $f_i^{(n)}$ (resp. $e_i^{(n)}$).

Define a twisted algebra structure on $U_q^-(g) \otimes U_q^-(g)$ as follows:

\[
(x_1 \otimes x_2)(y_1 \otimes y_2) = q^{-(\beta_2 \gamma_1)}(x_1 y_1 \otimes x_2 y_2),
\]
where \( x_i \in U_q^- (\mathfrak{g}) \), \( y_i \in U_q^- (\mathfrak{g}) \), \( i = 1, 2 \). Then there is an algebra homomorphism \( \Delta_0 : U_q^- (\mathfrak{g}) \to U_q^- (\mathfrak{g}) \otimes U_q^- (\mathfrak{g}) \) satisfying
\[
\Delta_0 (f_i) := f_i \otimes 1 + 1 \otimes f_i \quad (i \in I).
\]

Fix \( i \in I \). For any \( P \in U_q^- (\mathfrak{g}) \), there exist unique elements \( Q, R \in U_q^- (\mathfrak{g}) \) such that
\[
e_i P - P e_i = \frac{K_i Q - K_i^{-1} R}{q_i - q_i^{-1}}.
\]
We define the endomorphisms \( e'_i, e''_i : U^- q (\mathfrak{g}) \to U^- q (\mathfrak{g}) \) by
\[
e'_i (P) = R, \quad e''_i (P) = Q.
\]
Consider \( f_i \) as the endomorphism of \( U_q^- (\mathfrak{g}) \) defined by left multiplication by \( f_i \). Then we have
\[
e'_i f_j = \delta_{ij} + q_i^{-a_{ij}} f_j e'_i.
\]

**Definition 1.2.** The quantum boson algebra \( B_q (\mathfrak{g}) \) associated with a Borcherds-Cartan matrix \( A \) is the associative algebra over \( \mathbb{Q}(q) \) generated by \( e'_i, f_i \) \( (i \in I) \) satisfying the following relations:
\[
\begin{align*}
(1) & \quad e'_i f_j = q_i^{-a_{ij}} f_j e'_i + \delta_{ij}, \\
(2) & \quad \sum_{r=0}^{1-a_{ij}} (-1)^r \left[ 1 - a_{ij} \right] \left[ 1 - a_{ij} - r \right] e'_i f_j e'_i = 0 \quad \text{if} \quad i \in I^e, \quad i \neq j, \\
(3) & \quad \sum_{r=0}^{1-a_{ij}} (-1)^r \left[ 1 - a_{ij} \right] f_i f_j e'_i = 0 \quad \text{if} \quad i \in I^e, \quad i \neq j, \\
(4) & \quad e'_i e'_j - e'_j e'_i = 0, \quad f_i f_j - f_j f_i = 0 \quad \text{if} \quad a_{ij} = 0.
\end{align*}
\]

The algebra \( U_q^- (\mathfrak{g}) \) has a \( B_q (\mathfrak{g}) \)-module structure from the equation \((1.3) \quad (11) \quad (22)\).

**Proposition 1.3.**
\[
\begin{align*}
(1) & \quad \text{If} \quad x \in U_q^- (\mathfrak{g}) \quad \text{and} \quad e'_i x = 0 \quad \text{for all} \quad i \in I, \quad \text{then} \quad x \quad \text{is a constant multiple of} \quad 1. \\
(2) & \quad U_q^- (\mathfrak{g}) \quad \text{is a simple} \quad B_q (\mathfrak{g}) \quad \text{-module}.
\end{align*}
\]

**Proof.** The proof is almost the same as in \([22] \quad \text{Lemma 3.4.7, Corollary 3.4.9}. \quad \square
\]

Consider the anti-automorphism \( \varphi \) on \( B_q (\mathfrak{g}) \) defined by
\[
\varphi (e'_i) = f_i \quad \text{and} \quad \varphi (f_i) = e'_i.
\]
We define the symmetric bilinear forms \( ( , )_K \) and \( ( , )_L \) on \( U_q^- (\mathfrak{g}) \) as follows (cf. \([22] \quad \text{Proposition 3.4.4}, \quad [28] \quad \text{Chapter 1} \)):
\[
(1, 1)_K = 1, \quad (x, y)_K = (x, \varphi (b) y)_K, \\
(1, 1)_L = 1, \quad (f_i, f_j)_L = \delta_{ij} (1 - q_i^2)^{-1}, \quad (x, yz)_L = (\Delta_0 (x), y \otimes z)_L
\]
for \( x, y, z \in U_q^- (\mathfrak{g}) \) and \( b \in B_q (\mathfrak{g}) \).

**Lemma 1.4.**
\[
(1) \quad \text{The bilinear form} \quad ( , )_K \quad \text{on} \quad U_q^- (\mathfrak{g}) \quad \text{is nondegenerate}.
\]
(2) For homogeneous elements $x \in U_q^- (g)_{-\alpha}$ and $y \in U_q^- (g)_{-\beta}$, we have
\[
(x, y)_L = \prod_{i \in I} \frac{1}{(1 - q_i^2)} (x, y)_K,
\]
where $\alpha = \sum_{i \in I} k_i \alpha_i \in \mathbb{Q}^+$. Hence $(\ , \ )_{L}$ is nondegenerate.

(3) For any $x, y \in U_q^- (g)$, we have
\[
(e'_i x, y)_L = (1 - q_i^2) (x, f_i y)_L.
\]

Proof. The assertion (1) is proved in [11].

It was shown in [11] (2.4) that the bilinear form $(\ , \ )_K$ satisfies
\[
(x, y)_K = \sum_n (x^{(1)}_n, y)_{L} (x^{(2)}_n, z)_K,
\]
where $\Delta_0 (x) = \sum_n x^{(1)}_n \otimes x^{(2)}_n$. Then the assertion (2) can be proved by induction on $|\alpha|$.

To prove the assertion (3), without loss of generality, we may assume that $x \in U_q^- (g)_{-\alpha}$, where $\alpha = - \sum_{i} k_i \alpha_i \in -\mathbb{Q}^+$. Then by (2) and the definition of $(\ , \ )_K$, we have
\[
(e'_i x, y)_L = \frac{1}{(1 - q_i^2)} \prod_{j \neq i} (1 - q_j^2)^{k_j} (e'_i x, y)_K
= \frac{1 - q_i^2}{(1 - q_i^2)} \prod_{j \neq i} (1 - q_j^2)^{k_j} (x, f_i y)_K
= (1 - q_i^2) (x, f_i y)_L,
\]
which proves the assertion (3). □

We now briefly review the crystal basis theory of quantum generalized Kac-Moody algebras which was developed in [11] [12]. For any homogeneous element $u \in U_q^- (g)$, $u$ can be expressed uniquely as
\[
(1.5) \quad u = \sum_{l \geq 0} f_i^{(l)} u_l,
\]
where $e_i^{(l)} u_l = 0$ for every $l \geq 0$ and $u_l = 0$ for $l \gg 0$. We call it the $i$-string decomposition of $u$ in $U_q^- (g)$. We define the lower Kashiwara operators $\hat{\epsilon}_i, f_i$ ($i \in I$) of $U_q^- (g)$ by
\[
\hat{\epsilon}_i u = \sum_{k \geq 1} f_i^{(k-1)} u_k, \quad \hat{f}_i u = \sum_{k \geq 0} f_i^{(k+1)} u_k.
\]

Let $\mathcal{A}_0 = \{ f/g \in \mathbb{Q}(q) \mid f, g \in \mathbb{Q}[q], g(0) \neq 0 \}$.

**Definition 1.5.** A lower crystal basis of $U_q^- (g)$ is a pair $(L, B)$ satisfying the following conditions:

1. $L$ is a free $\mathcal{A}_0$-module of $U_q^- (g)$ such that $U_q^- (g) = \mathbb{Q}(q) \otimes_{\mathcal{A}_0} L$ and $L = \bigoplus_{\alpha \in \mathbb{Q}^+} L_{-\alpha}$, where $L_{-\alpha} := L \cap U_q^- (g)_{-\alpha}$,

2. $B$ is a $\mathbb{Q}$-basis of $L/qL$ such that $B = \bigsqcup_{\alpha \in \mathbb{Q}^+} B_{-\alpha}$, where $B_{-\alpha} := B \cap (L_{-\alpha}/qL_{-\alpha})$,

3. $\hat{\epsilon}_i B \subset B \sqcup \{0\}$, $\hat{f}_i B \subset B$ for all $i \in I$,

4. For $b, b' \in B$ and $i \in I$, $b' = \hat{f}_i b$ if and only if $b = \hat{\epsilon}_i b'$.
Proposition 1.6. [11] Theorem 7.1] Let $L(\infty)$ be the free $\mathbb{A}_0$-module of $U_q^{-}(\mathfrak{g})$ generated by $\{f_i \cdots f_i 1 \mid r \geq 0, i_k \in I\}$ and let
\[ B(\infty) = \{f_i \cdots f_i 1 + qL(\infty) \mid r \geq 0, i_k \in I\} \setminus \{0\}. \]
Then the pair $(L(\infty), B(\infty))$ is a unique lower crystal basis of $U_q^{-}(\mathfrak{g})$.

Let $\mathcal{O}_{\text{int}}$ be the abelian category of $U_q(\mathfrak{g})$-modules defined in [11] Definition 3.1. For each $\lambda \in P^+$, let $V(\lambda)$ denote the irreducible highest weight $U_q(\mathfrak{g})$-module with highest weight $\lambda$. It is generated by a unique highest weight vector $v_\lambda$ with defining relations:
\[
q^h v_\lambda = q^{(h,\lambda)} v_\lambda \text{ for all } h \in P^+, \\
e_i v_\lambda = 0 \text{ for all } i \in I, \\
f_i^{(h,\lambda)+1} v_\lambda = 0 \text{ for } i \in I^{\text{re}}, \\
f_i v_\lambda = 0 \text{ for } i \in I^{\text{im}} \text{ with } \langle h_i, \lambda \rangle = 0.
\]

It was proved in [11] Theorem 3.7 that the category $\mathcal{O}_{\text{int}}$ is semisimple and that all the irreducible objects have the form $V(\lambda)$ for $\lambda \in P^+$.

Let $M$ be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}_{\text{int}}$. For any $i \in I$ and $u \in M_\mu$, the element $u$ can be expressed uniquely as
\[ u = \sum_{k \geq 0} f_i^{(k)} u_k, \]
where $u_k \in M_{\mu+k\alpha_i}$ and $e_i u_k = 0$. We call it the $i$-string decomposition of $u$. We define the lower Kashiwara operators $\tilde{e}_i, \tilde{f}_i$ ($i \in I$) by
\[ \tilde{e}_i u = \sum_{k \geq 1} f_i^{(k-1)} u_k, \quad \tilde{f}_i u = \sum_{k \geq 0} f_i^{(k+1)} u_k. \]

Definition 1.7. A lower crystal basis of $U_q(\mathfrak{g})$-module $M$ is a pair $(L, B)$ satisfying the following conditions:

1. $L$ is a free $\mathbb{A}_0$-module of $M$ such that $M = \mathbb{Q}(q) \otimes_{\mathbb{A}_0} L$ and $L = \bigoplus_{\lambda \in P} L_{\lambda}$, where $L_{\lambda} := L \cap M_{\lambda}$,
2. $B$ is $\mathbb{Q}$-basis of $L/qL$ such that $B = \bigsqcup_{\lambda \in P} B_{\lambda}$, where $B_{\lambda} := B \cap L_{\lambda}/qL_{\lambda}$,
3. $\tilde{e}_i B \subset B \cup \{0\}$, $\tilde{f}_i B \subset B \cup \{0\}$ for all $i \in I$,
4. For $b, b' \in B$ and $i \in I$, $b' = \tilde{f}_i b$ if and only if $b = \tilde{e}_i b'$.

Proposition 1.8. [11] Theorem 7.1] For $\lambda \in P^+$, let $L(\lambda)$ be the free $\mathbb{A}_0$-module of $V(\lambda)$ generated by $\{f_i \cdots f_i v_\lambda \mid r \geq 0, i_k \in I\}$ and let
\[ B(\lambda) = \{f_i \cdots f_i v_\lambda + qL(\lambda) \mid r \geq 0, i_k \in I\} \setminus \{0\}. \]
Then the pair $(L(\lambda), B(\lambda))$ is a unique lower crystal basis of $V(\lambda)$.
2. Khovanov-Lauda-Rouquier algebra \( R \)

In this section, we construct the Khovanov-Lauda-Rouquier algebra \( R \) associated with a Borcherds-Cartan matrix \( A \), and investigate its algebraic structure and representation theory.

2.1. The algebras \( R(\alpha) \).

Let \( \mathbb{F} \) be a field. For \( \alpha \in Q^+ \) with \( |\alpha| = d \), set

\[
\text{Seq}(\alpha) = \{ \mathbf{i} = (i_1 \ldots i_d) \in I^d \mid \alpha_{i_1} + \cdots + \alpha_{i_d} = \alpha \},
\]

\[
\text{Seqd}(\alpha) = \{ \mathbf{i} = (i_1^{(d_1)} \ldots i_r^{(d_r)}) \in I^d \mid d_1 \alpha_{i_1} + \cdots + d_r \alpha_{i_r} = \alpha \}.
\]

Then the symmetric group \( S_d = \langle r_i \mid i = 1, \ldots, d - 1 \rangle \) acts naturally on \( \text{Seq}(\alpha) \). For \( \mathbf{i} = (i_1 \ldots i_d) \in \text{Seq}(\alpha) \), \( \mathbf{j} = (j_1 \ldots j_d) \in \text{Seq}(\beta) \), we denote by \( \mathbf{i} \ast \mathbf{j} \) the concatenation of \( \mathbf{i} \) and \( \mathbf{j} \):

\[
\mathbf{i} \ast \mathbf{j} := (i_1 \ldots i_d j_1 \ldots j_d) \in \text{Seq}(\alpha + \beta).
\]

The symmetric group \( S_d \) acts on the polynomial ring \( \mathbb{F}[x_1, \ldots, x_d] \) by

\[
w \cdot f(x_1, \ldots, x_d) = f(x_{w(1)}, \ldots, x_{w(d)}) \quad \text{for } w \in S_d \text{ and } f(x_1, \ldots, x_d) \in \mathbb{F}[x_1, \ldots, x_d].
\]

For \( t = 1, \ldots, d - 1 \), define the operator \( \partial_t \) on \( \mathbb{F}[x_1, \ldots, x_d] \) by

\[
\partial_t(f) = \frac{r_t f - f}{x_t - x_{t+1}}
\]

for \( f \in \mathbb{F}[x_1, \ldots, x_d] \). We take a matrix \( (Q_{i,j}(u, v))_{i,j} \) in \( \mathbb{F}[u, v] \) such that \( Q_{i,j}(u, v) = Q_{j,i}(v, u) \) and \( Q_{i,j}(u, v) \) has the form

\[
Q_{i,j}(u, v) = \begin{cases} 
\sum_{p,q} t_{i,j;p,q} u^p v^q & \text{if } i \neq j, \\
0 & \text{if } i = j,
\end{cases}
\]

where the summation is taken over all \( p, q \in \mathbb{Z}_{\geq 0} \) such that \( (\alpha_i \alpha_j) + s_i p + s_j q = 0 \) and \( t_{i,j;p,q} \in \mathbb{F} \). In particular, \( t_{i,j;-a_{ij},0} \in \mathbb{F}^x \). For each \( i \in I \), choose a nonzero polynomial \( P_i(u, v) \in \mathbb{F}[u, v] \) having the form

\[
P_i(u, v) = \sum_{p,q} h_{i;p,q} u^p v^q,
\]

where the summation is taken over all \( p, q \in \mathbb{Z}_{\geq 0} \) such that \( 2 - a_{ii} - 2p - 2q = 0 \) and \( h_{i;p,q} \in \mathbb{F} \). In particular, \( h_{i; -\frac{s_i}{2}, 0}, h_{i; 0, -\frac{s_i}{2}} \in \mathbb{F}^x \).

**Definition 2.1.** Let \( (A, P, \Pi, \Pi') \) be a Borcherds-Cartan datum. For \( \alpha \in Q^+ \) with height \( d \), the *Khovanov-Lauda-Rouquier algebra* \( R(\alpha) \) of weight \( \alpha \) associated with the data \( (A, P, \Pi, \Pi') \), \( (P_i)_{i \in I} \) and \( (Q_{i,j})_{i,j \in I} \) is the associative graded \( \mathbb{F} \)-algebra generated by \( 1_i \) (\( i \in \text{Seq}(\alpha) \)), \( x_k \) (\( 1 \leq k \leq d \)),
where

\[ \tau t = \begin{cases} \partial t \mathcal{P} t(x_t, x_{t+1}) \tau t_1 & \text{if } i_t = i_{t+1}, \\ \mathcal{Q}_{i_t, i_{t+1}}(x_t, x_{t+1}) \tau t_1 & \text{if } i_t \neq i_{t+1}, \end{cases} \]

(\tau t x_k - x_{r_t(k)} \tau t)_1 = \begin{cases} -\mathcal{P} t(x_t, x_{t+1}) \tau t_1 & \text{if } k = t \text{ and } i_t = i_{t+1}, \\ \mathcal{P} t(x_t, x_{t+1}) \tau t_1 & \text{if } k = t + 1 \text{ and } i_t = i_{t+1}, \\ 0 & \text{otherwise,} \end{cases}

(\tau t_1 + \tau t_{t+1} - \tau t_1 \tau t_{t+1}) \tau t_1

(2.2)

where

\[ \mathcal{P}'_i(u, v, w) := \frac{\mathcal{P}_i(v, u) \mathcal{P}_i(u, w)}{(u - v)(u - w)} + \frac{\mathcal{P}_i(u, w) \mathcal{P}_i(v, w)}{(u - v)(v - w)} - \frac{\mathcal{P}_i(u, v) \mathcal{P}_i(v, w)}{(u - v)(v - w)}, \]

\[ \mathcal{P}''_i(u, v, w) := -\frac{\mathcal{P}_i(u, v) \mathcal{P}_i(u, w)}{(u - v)(u - w)} + \frac{\mathcal{P}_i(u, w) \mathcal{P}_i(v, w)}{(u - v)(v - w)} + \frac{\mathcal{P}_i(u, v) \mathcal{P}_i(v, w)}{(u - v)(v - w)}, \]

\[ \mathcal{Q}_{i, j}(u, v, w) := \frac{\mathcal{Q}_{i, j}(u, v) - \mathcal{Q}_{i, j}(u, v)}{u - w}. \]

Let \( R := \bigoplus_{\alpha \in Q^+} R(\alpha) \). The \( \mathbb{Z} \)-grading on \( R(\alpha) \) is given by

\[ \deg(1_i) = 0, \quad \deg(x_k 1_i) = 2s_{it}, \quad \deg(\tau t 1_i) = -(\alpha_{it} | \alpha_{i_{t+1}}). \]

Note that \( \mathcal{P}'_i, \mathcal{P}''_i \) and \( \mathcal{Q}_{i, j} \) are polynomials. If \( i \in I^* \), then \( \mathcal{P}_i(u, v) \) is a nonzero constant, which will be normalized to be 1 in this paper. If \( I \) is finite and \( a_{ii} = 2 \) for all \( i \in I \), then the algebra \( R \) coincides with the Khovanov-Lauda-Rouquier algebra introduced in [24, 25, 31].

The algebra \( R \) can be defined by using planar diagrams with dots and strands. For simplicity, we assume that \( \mathcal{P}_i \) are symmetric and \( t_{i, j = a_{ij}, 0} = t_{i, j = 0, -a_{ij}} = 1 \) and \( t_{i, j; p, q} = 0 \) for other \( p, q \). Note that \( \partial_t \mathcal{P}_t(x_t, x_{t+1}) = 0 \). We denote by \( R \) the \( \mathbb{F} \)-vector space spanned by braid-like diagrams, considered up to planar isotropy, such that all strands are colored by \( I \) and can carry dots. The multiplication \( D \cdot D' \) of two diagrams \( D \) and \( D' \) is given by stacking of the diagram \( D \) on the diagram \( D' \) if the color on the top of \( D' \) matches with the color at the bottom of \( D \) and defined to be 0 otherwise. It is obvious that the following elements are generators of \( R(\alpha) \) \( (\alpha \in Q^+, i = (i_1, \ldots, i_d) \in \text{Seq}(\alpha)) \):

\[
\begin{align*}
1_i := & \begin{array}{cccc}
\cdots & \cdots & \cdots & \\
\cdots & \cdots & 1_i & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
1_i & \cdots & \cdots & \cdots \\
\end{array}, \\
x_k 1_i := & \begin{array}{cccc}
\cdots & \cdots & \cdots & \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\end{array}, \\
\tau t 1_i := & \begin{array}{cccc}
\cdots & \cdots & \cdots & \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\end{array}.
\end{align*}
\]
The local relations are given as follows:

\[ i \rightarrow j = \begin{cases} 
0 & \text{if } i = j, \\
-\alpha_{ij} & \text{if } (\alpha_i | \alpha_j) = 0, \\
-\alpha_{ji} & \text{if } (\alpha_i | \alpha_j) \neq 0,
\end{cases} \tag{2.5} \]

\[ i \rightarrow i - j = \begin{cases} 
\mathcal{P}_1(x, y) & \text{if } i = j, \\
0 & \text{otherwise},
\end{cases} \tag{2.6} \]

\[ i \rightarrow i - j = \begin{cases} 
\mathcal{P}_1(x, y) & \text{if } i = j, \\
0 & \text{otherwise},
\end{cases} \tag{2.7} \]

(Here, \( x := \begin{array}{c} i \\
i \end{array} \) and \( y := \begin{array}{c} i \\
i \end{array} \))

\[ i \rightarrow j - k = \begin{cases} 
\mathcal{P}_1(x, z) - a_{ij} - a_{ik} - s & \text{if } i = k \neq j, a_{ij} \neq 0, \\
\overline{\mathcal{P}}_1(x, y, z) \left( \begin{array}{c} i \\
i \end{array} \right) & \text{if } i = j = k,
\end{cases} \tag{here, } x := \begin{array}{c} i \\
i \end{array}, \ y := \begin{array}{c} i \\
i \end{array}, \text{ and } z := \begin{array}{c} i \\
i \end{array} \]

For \( t = (t_1 \ldots t_d) \in \mathbb{Z}_{\geq 0}^d \) and a reduced expression \( w = r_{i_1} \cdots r_{i_s} \in S_d \), set
\[ x^t = x_{i_1}^{t_1} \cdots x_{i_s}^{t_s} \] and \( \tau_w = \tau_{r_{i_1}} \cdots \tau_{r_{i_s}} \).

It follows from the defining relations that
\[ \{ \tau_w x^t i_1 \mid t \in \mathbb{Z}_{\geq 0}^d, \ i \in \text{Seq}(\alpha), \ w : \text{reduced in } S_d \} \]
is a spanning set of \( R(\alpha) \).
Consider the graded anti-involution $\psi : R(\alpha) \to R(\alpha)$ which is the identity on generators. For a graded left $R(\alpha)$-module $M$, let $M^*$ be the graded right $R(\alpha)$-module whose underlying space is $M$ with $R(\alpha)$-action given by

$$v \cdot r = \psi(r)v \quad \text{for } v \in M^*, \ r \in R(\alpha).$$

We will investigate the structure of $R(m\alpha_i)$ ($m \geq 0$) in more detail. If $a_{ii} = 2$, then the defining relations for $R(m\alpha_i)$ reduce to

$$x_k x_l = x_l x_k, \quad \tau_t^2 = 0,$$

$$\tau_t \tau_{t+1} \tau_t = \tau_{t+1} \tau_t \tau_{t+1}, \quad \tau_t \tau_s = \tau_s \tau_t \quad \text{if } |t - s| > 1,$$

$$\tau_t x_t = x_t + 1, \quad \tau_t x_{t+1} = x_t + 1,$$

$$\tau_t x_k = x_k \tau_t \quad \text{if } k \neq t, t+1.$$

Hence the algebra $R(m\alpha_i)$ is isomorphic to the nil Hecke algebra $NH_m$, which is the associative algebra generated by $x_k$ $(1 \leq k \leq m)$ and $\partial_t$ $(1 \leq t \leq m-1)$ satisfying the following relations:

$$x_k x_l = x_l x_k, \quad \partial_t^2 = 0,$$

$$\partial_t \partial_{t+1} \partial_t = \partial_{t+1} \partial_t \partial_{t+1}, \quad \partial_t \partial_s = \partial_s \partial_t \quad \text{if } |t - s| > 1,$$

$$\partial_t x_t = x_t + 1, \quad \partial_t x_{t+1} = x_t \partial_t + 1,$$

$$\partial_t x_k = x_k \partial_t \quad \text{if } k \neq t, t+1.$$

Therefore, as was shown in [23], the algebra $R(m\alpha_i)$ has a primitive idempotent $\tau_{w_0} x_1^{m-1} \cdots x_{m-2} x_{m-1}$, where $w_0$ is the longest element in $S_m$, and has a unique (up to isomorphism and degree shift) irreducible module $L(i^m)$. The irreducible module $L(i^m)$ is isomorphic to the one induced from the trivial $\mathbb{F}[x_1, \ldots, x_m]$-module of dimension 1 over $\mathbb{F}$.

If $a_{ii} < 0$, then $P_i(u, v)$ is a homogeneous polynomial with degree $1 - \frac{a_{ii}}{2} > 1$, and $\overline{P}_i(u, v, w)$ and $\overline{P}''_i(u, v, w)$ have positive degree. By (2.4), $R(m\alpha_i)$ has positive grading and hence it has a unique idempotent $1_{(i, \ldots, i)}$. Thus there exists a unique irreducible $R(m\alpha_i)$-module $L(i^m) = \mathbb{F}v$ defined by

$$(2.8) \quad 1_{(i, \ldots, i)} \cdot v = v, \quad x_k \cdot v = 0, \quad \tau_t \cdot v = 0.$$

If $a_{ii} = 0$, then in general, $R(m\alpha_i)$ has many primitive idempotents, which means that there are many irreducible $R(m\alpha_i)$-modules. For example, if $m = 3$ and $P_3(u, v) = u - v$, then $\tau_1 \tau_2, \tau_2 \tau_1$ and $1 - \tau_1 \tau_2 - \tau_2 \tau_1$ are orthogonal primitive idempotents. The algebra $R(m\alpha_i)$ itself, not principal indecomposable modules, will serve as one of the projective modules that give our categorification. The whole Grothendieck group of the category of finitely generated projective $R(m\alpha_i)$-modules seems rather large and nontrivial. We hope to investigate it in a later work.

We now construct a faithful polynomial representation of $R(\alpha)$. First, we define an $R(m\alpha_i)$-module structure on $\mathbb{F}[x_1, \ldots, x_m]$ by

$$x_k \cdot f(x_1, \ldots, x_m) = x_k f(x_1, \ldots, x_m),$$

$$\tau_t \cdot f(x_1, \ldots, x_m) = P_i(x_t, x_{t+1}) \partial_t (f(x_1, \ldots, x_m)).$$
for \( x_k, \tau_l \in R(m\alpha_i), f(x_1, \ldots, x_m) \in \mathbb{F}[x_1, \ldots, x_m] \).

**Lemma 2.2.** \( \mathbb{F}[x_1, \ldots, x_m] \) is a faithful representation of \( R(m\alpha_i) \).

**Proof.** If \( i \in I^\omega \), our assertion was shown in [24, Example 2.2]. Assume that \( i \in I^{im} \) and let \( x_k \) be the endomorphism of \( \mathbb{F}[x_1, \ldots, x_m] \) defined by

\[
x_k(f(x_1, \ldots, x_m)) = x_kf(x_1, \ldots, x_m)
\]

for \( f(x_1, \ldots, x_m) \in \mathbb{F}[x_1, \ldots, x_m] \). Note that

\[
\{ \partial_{j_1} \cdots \partial_{j_k} x^t \mid t \in \mathbb{Z}_{\geq 0}^m, \ r_{j_1} \cdots r_{j_k} \text{ is a reduced expression in } S_m(k \geq 0) \}
\]

is a linearly independent subset of \( \text{End}(\mathbb{F}[x_1, \ldots, x_m]) \). Let

\[
\iota : R(m\alpha_i) \rightarrow \text{End}(\mathbb{F}[x_1, \ldots, x_m])
\]

be the map defined by \( \iota(x_k) = x_k \) and \( \iota(\tau_l) = P_l(x_k, x_{k+1}) \cdot \partial_{l} \).

We first show that \( \iota \) is well-defined. Since \( P_l(u, v) \) is a homogeneous polynomial, it is easy to verify that the relations \([2.1]\) hold. To check the relations in \([2.2]\), for simplicity, we assume that \( m = 3 \) and let \( x = x_1, y = x_2, z = x_3, P(u, v) = P_l(u, v) \).

Set

\[
P(u, v) = \frac{P(u, v)}{u - v}.
\]

By a direct computation, we have

\[
\iota(\tau_1 \tau_2 \tau_3) = P(x, y)P(y, z)P(x, z)(r_2 r_1 r_2 - r_2 r_1 - r_1 r_2 + r_1) - P(y, z)P(x, y)P(z, x)(1 - r_2) + P(x, y)P(y, z)^2(r_2 - 1),
\]

\[
\iota(\tau_1 \tau_2 \tau_1) = P(x, y)P(y, z)P(x, z)(r_1 r_2 r_1 - r_2 r_1 - r_1 r_2 + r_2) - P(x, y)P(y, x)P(z, x)(1 - r_1) + P(x, y)^2P(y, z)(r_1 - 1).
\]

As \( \iota(\tau_k) = P(x_k, x_{k+1})(r_k - 1) \) for \( k = 1, 2, \)

\[
\iota(\tau_2 \tau_1 \tau_2) - \iota(\tau_1 \tau_2 \tau_1) = (-P(x, y)P(x, z) + P(y, z)P(x, z) - P(x, y)P(y, z))\iota(\tau_1)
\]

\[
+ (P(x, y)P(y, z) + P(z, y)P(x, z) - P(x, y)P(x, z))\iota(\tau_2),
\]

which shows that the relation \([2.2]\) holds. It remains to show that \( \iota \) is injective. Take a nonzero element

\[
y = \tau_{w_1} f_1 + \cdots + \tau_{w_t} f_t \quad (0 \neq f_k \in \mathbb{F}[x_1, \ldots, x_m], \ w_k \text{ is a reduced expression in } S_m)
\]

of \( R(m\alpha_i) \) such that \( w_i \neq w_j \) if \( i \neq j \) and \( \ell(w_1) \geq \ell(w_k) \) for \( 0 \leq k \leq t \). Write the reduced expression of \( w_1 \) as \( w_1 = r_{i_1} \cdots r_{i_t} \). Then, \( \iota(y) \) can be written as

\[
\iota(y) = \partial_{i_1} \cdots \partial_{i_t} f' + \cdots \text{ lower terms} \cdots
\]

for some nonzero polynomial \( f' \), which implies that \( \iota(y) \) is nonzero. Therefore \( \iota \) is injective. \( \square \)
Now we consider the general case $R(\alpha)$ with $\alpha \in Q^+$. Take a total order $\prec$ on $I$. Let
\[
\mathcal{P}(\alpha) = \bigoplus_{i \in \text{Seq}(\alpha)} \mathbb{F}[x_1(i), \ldots, x_d(i)].
\]
For any polynomial $f \in \mathbb{F}[u_1, \ldots, u_d]$, let $f(i)$ be the polynomial in $\mathbb{F}[x_1(i), \ldots, x_d(i)]$ obtained from $f$ by replacing $u_k$ by $x_k(i)$. We define an $R(\alpha)$-module structure on $\mathcal{P}(\alpha)$ as follows: for $i \in \text{Seq}(\alpha)$ and $f \in \mathbb{F}[u_1, \ldots, u_d]$, we define
\[
\begin{align*}
1_i \cdot f(i) &= \delta_{ij}f(i) \quad ( j \in \text{Seq}(\alpha)), \\
x_k \cdot f(i) &= x_k(i)f(i), \\
\tau_t \cdot f(i) &= \left\{ \begin{array}{ll}
\mathcal{P}_i(x_1(r_t i), x_{t+1}(r_t i))\partial_t f(r_t i) & \text{if } i_t = i_{t+1}, \\
\mathcal{Q}_{i_{t+1}, i_t}(x_t(r_t i), x_{t+1}(r_t i))\tau_t f(r_t i) & \text{if } i_t \neq i_{t+1}, i_t \succ i_{t+1}, \\
\tau_t f(r_t i) & \text{if } i_t \neq i_{t+1}, i_t \prec i_{t+1}.
\end{array} \right.
\end{align*}
\]
(2.9)

\textbf{Lemma 2.3.} $\mathcal{P}(\alpha)$ is a well-defined $R(\alpha)$-module.

\textit{Proof.} We verify the defining relations of $R(\alpha)$. The relations (2.1) can be verified in a straightforward manner. In the proof of Lemma 2.2, we already proved our assertion when $i_t = i_{t+1} = i_{t+2}$. Thus it suffices to consider the following three cases in (2.2):
(i) $i_t = i_{t+2} \neq i_{t+1}$, (ii) $i_t, i_{t+1}, i_{t+2}$ are distinct, (iii) $i_t = i_{t+1}$ and $i_t \neq i_{t+2}$. For simplicity, let $d = 3$, $i = (i, j, k)$ and $f(u, v, w) = u^a v^b w^c$. Set $x = x_1(i)$, $y = x_2(i)$ and $z = x_3(i)$.

Case (i): Let $i = (i, j, i)$ with $i \neq j$. Without loss of generality, we may assume $i < j$. Then, by a direct computation, we have
\[
\begin{align*}
\tau_1 \tau_2 \tau_1(x^a y^b z^c) &= \mathcal{P}_i(x, z)\mathcal{Q}_{ij}(x, y)\frac{x^cy^b z^a - x^ay^b z^c}{x - z}, \\
\tau_2 \tau_1 \tau_2(x^a y^b z^c) &= \mathcal{P}_i(x, z)\mathcal{Q}_{ij}(x, y)\frac{x^cy^b z^a - x^ay^b z^c}{x - z},
\end{align*}
\]
which yield
\[
(\tau_2 \tau_1 \tau_2 - \tau_1 \tau_2 \tau_1)(x^a y^b z^c) = \mathcal{P}_i(x, z)\frac{\mathcal{Q}_{ij}(x, y) - \mathcal{Q}_{ij}(z, y)}{x - z}x^ay^b z^c.
\]

Case (ii): Let $i = (i, j, k)$ such that $i, j, k$ are distinct. Since the other cases are similar, we will only prove our assertion when $i \succ j \succ k$. Then we have
\[
\begin{align*}
\tau_1 \tau_2 \tau_1(x^a y^b z^c) &= \mathcal{Q}_{ij}(y, z)\mathcal{Q}_{jk}(x, y)\mathcal{Q}_{ik}(x, z)x^cy^b z^a, \\
\tau_2 \tau_1 \tau_2(x^a y^b z^c) &= \mathcal{Q}_{ij}(y, z)\mathcal{Q}_{jk}(x, y)\mathcal{Q}_{ik}(x, z)x^cy^b z^a,
\end{align*}
\]
which implies that $(\tau_2 \tau_1 \tau_2 - \tau_1 \tau_2 \tau_1)(x^a y^b z^c) = 0$.

Case (iii): Similarly as above, we consider $i = (i, i, j)$ with $i \succ j$ only. Then
\[
\begin{align*}
\tau_1 \tau_2 \tau_1(x^a y^b z^c) &= \mathcal{Q}_{ij}(x, y)\mathcal{Q}_{ij}(x, z)\mathcal{P}_i(y, z)\frac{x^cy^b z^a - x^cy^a z^b}{y - z}, \\
\tau_2 \tau_1 \tau_2(x^a y^b z^c) &= \mathcal{Q}_{ij}(x, y)\mathcal{Q}_{ij}(x, z)\mathcal{P}_i(y, z)\frac{x^cy^b z^a - x^cy^a z^b}{y - z}.
\end{align*}
\]
Hence we have $(\tau_2 \tau_1 \tau_2 - \tau_1 \tau_2 \tau_1)(x^a y^b z^c) = 0$, which completes the proof. $\square$
Note that $R(\alpha) = \bigoplus_{i,j \in \text{Seq}(\alpha)} jR(\alpha)_i$, where $jR(\alpha)_i := 1_jR(\alpha)_1$. Given each $w \in S_d$, fix a minimal representative $\underline{w}$ of $w$. For $i,j \in \text{Seq}(\alpha)$, let

$$jS_i = \{\underline{w} \mid w \in S_d, \ w(i) = j\}.$$ 

It follows from the defining relations that

$$jB(\alpha)_i := \{\tau_{\underline{w}}x^t1_1 \mid t \in \mathbb{Z}_{\geq 0}^d, \ \underline{w} \in jS_i\}$$

is a spanning set of $jR(\alpha)_i$. Moreover, we have the following proposition.

**Proposition 2.4.**

1. The set $jB(\alpha)_i$ is a homogeneous basis of $jR(\alpha)_i$.
2. $\mathfrak{pol}(\alpha)$ is a faithful representation of $R(\alpha)$.

**Proof.** Let $<$ be the lexicographic order of $\text{Seq}(\alpha)$ arising from the order $<$ of $I$, and let $j_1w_1$, be the minimal element in $j_2S_{j_1}$ for $j_1, j_2 \in \text{Seq}(\alpha)$. Let

$$\Upsilon : R(\alpha) \rightarrow \text{End}(\mathfrak{pol}(\alpha))$$

be the algebra homomorphism given in (2.9). We will show that $\Upsilon(jB(\alpha)_i)$ is linearly independent, which would imply the set $jB(\alpha)_i$ is linearly independent. The injectivity of $\Upsilon$ would also follow immediately. We prove our claim using induction on the lexicographic order $<$ on $\text{Seq}(\alpha)$.

Let $i \in \text{Seq}(\alpha)$, and let

$$j = (j_1 \ldots j_1 j_2 \ldots j_2 \ldots j_r) \in \text{Seq}(\alpha)$$

such that $j_1 > j_2 > \cdots > j_r$. Note that $j$ is a maximal element in $\text{Seq}(\alpha)$.

Let $m$ be a linear combination of $jB(\alpha)_i$ such that $\Upsilon(m) = 0$. Note that $m$ can be expressed as

$$m = \sum_s \tau_{w_s}1_j x^{k_s}1_1$$

for some $k_s \in \mathbb{Z}_{\geq 0}^d$ and some $w_s \in S_{d_1} \times \cdots \times S_{d_r}$. It follows from (2.9) that $\Upsilon(\tau_{jw_s}1_1)$ can be viewed as a linear map from $F[x_1(i), \ldots, x_d(i)]$ to $F[x_1(j), \ldots, x_d(j)]$ sending $1_1$ to $1_j$. Hence,

$$\Upsilon(m) = 0 \quad \text{if and only if} \quad \Upsilon(\sum_s \tau_{w_s}x^{jw_s(k_s)}1_1) = 0.$$

Since $\Upsilon(\sum_s \tau_{w_s}x^{jw_s(k_s)}1_1)$ can be regarded as a linear map in $\bigoplus_{k=1}^d \text{End}(F[x_1, \ldots, x_{d_k}])$, by Lemma 2.2 we have

$$\Upsilon(\sum_s \tau_{w_s}x^{jw_s(k_s)}1_1) = 0 \quad \text{if and only if} \quad \sum_s \tau_{w_s}x^{jw_s(k_s)}1_1 = 0,$$

which implies $m = 0$. Therefore, $\Upsilon(jB(\alpha)_i)$ is linearly independent.

We now consider the case when $j$ is an arbitrary sequence in $\text{Seq}(\alpha)$. This step can be proved by a similar induction argument as in [24, Theorem 2.5], which completes the proof. \hspace{1cm} \square
For any \( \alpha, \beta \in Q^+ \), let
\[
1_\alpha = \sum_{i \in \text{Seq}(\alpha)} 1_i, \\
1_{\alpha, \beta} = \sum_{i \in \text{Seq}(\alpha), j \in \text{Seq}(\beta)} 1_{i+j}.
\]
Then \( 1_{\alpha, \beta} R(\alpha + \beta) \) has a natural graded left \( R(\alpha) \otimes R(\beta) \)-module structure.

**Corollary 2.5.** \( 1_{\alpha, \beta} R(\alpha + \beta) \) is a free graded left \( R(\alpha) \otimes R(\beta) \)-module.

**Proof.** Let \( d := |\alpha|, d' := |\beta| \), and \( S_d \times S_{d'} \setminus S_{d+d'} \) be the set of minimal right \( S_d \times S_{d'} \)-coset representatives of \( S_{d+d'} \). For \( w \in S_d \times S_{d'} \setminus S_{d+d'} \), set
\[
\tilde{r}_w = \sum_{i \in \text{Seq}(\alpha), j \in \text{Seq}(\beta)} 1_{i+j} \tau_w 1_{w^{-1}(i+j)}.
\]
Then, it follows from Proposition 2.4 that
\[
\{ \tilde{r}_w \mid w \in S_d \times S_{d'} \setminus S_{d+d'} \}
\]
is a basis of \( 1_{\alpha, \beta} R(\alpha + \beta) \) as a left \( R(\alpha) \otimes R(\beta) \)-module. \( \square \)

For a graded \( R(\alpha) \)-module \( M = \bigoplus_{i \in \mathbb{Z}} M_i \), let \( M(k) \) denote the graded \( R(\alpha) \)-module obtained from \( M \) by shifting the grading by \( k \); i.e., \( M(k) := \bigoplus_{i \in \mathbb{Z}} M_{i+k} \). Given \( \alpha, \alpha', \beta, \beta' \in Q^+ \) with \( \alpha + \beta = \alpha' + \beta' \), let
\[
\alpha_{\beta} R_{\alpha', \beta'} := 1_{\alpha, \beta} R(\alpha + \beta) 1_{\alpha', \beta'}.
\]
We write \( \alpha_{\beta} R_{\alpha', \beta'} \) (resp. \( \alpha_{\beta} R_{\alpha', \beta'} \)) for \( \alpha_{\beta} R_{\alpha', \beta'} \) if \( \beta = 0 \) (resp. \( \beta' = 0 \)). Note that \( \alpha_{\beta} R_{\alpha', \beta'} \) is a graded \( (R(\alpha) \otimes R(\beta), R(\alpha') \otimes R(\beta')) \)-bimodule. Now we obtain the Mackey’s Theorem for Khovanov-Lauda-Rouquier algebras.

**Proposition 2.6.** The graded \( (R(\alpha) \otimes R(\beta), R(\alpha') \otimes R(\beta')) \)-bimodule \( \alpha_{\beta} R_{\alpha', \beta'} \) has a graded filtration with graded subquotients isomorphic to
\[
\alpha_{\beta} R_{\alpha, \gamma} \otimes \beta_{\rho, \gamma'} \otimes \beta_{\gamma, \beta'} \otimes R'_{\alpha', \rho, \gamma, \rho'} \otimes \gamma_{\alpha', \gamma, \beta'} R_{\beta} ((\gamma | \beta + \gamma - \beta')),
\]
where \( R' = R(\alpha - \gamma) \otimes R(\gamma) \otimes R(\beta + \gamma - \beta') \otimes R(\beta' - \gamma) \) for all \( \gamma \in Q^+ \) such that every term above lies in \( Q^+ \).

**Proof.** The proof is almost identical to that of [24] Proposition 2.18. \( \square \)

For \( \alpha = \sum_{i \in I} k_i \alpha_i \in Q^+ \) with \( |\alpha| = d \), we define
\[
\text{Pol}(\alpha) = \prod_{i \in \text{Seq}(\alpha)} \mathbb{F}[x_{1,i}, \ldots, x_{d,i}].
\]
Then the symmetric group \( S_d \) acts on \( \text{Pol}(\alpha) \) by \( w \cdot x_{k,i} := x_{w(k),i} \) for \( w \in S_d \). Let
\[
\text{Sym}(\alpha) = \text{Pol}(\alpha)^{S_d}.
\]
Note that \( \text{Sym}(\alpha) \cong \bigotimes_{i \in I} \mathbb{F}[x_1, \ldots, x_{k_i}]^{S_{k_i}} \). Considering \( \text{Sym}(\alpha) \) as a subalgebra of \( \text{R}(\alpha) \) via the natural inclusion \( \text{Pol}(\alpha) \hookrightarrow \text{R}(\alpha) \) sending \( x_{k,i} \) to \( x_k 1_i \), we have the following lemma.
Lemma 2.7.

(1) $\text{Sym}(\alpha)$ is the center of $R(\alpha)$.
(2) $R(\alpha)$ is a free module of rank $(d!)^2$ over its center $\text{Sym}(\alpha)$.

Proof. We first consider the case when $\alpha = m\alpha_i$ for $i \in I$. If $i \in I^\text{re}$, it follows from $R(m\alpha_i) \simeq NH_m$ that $\text{Sym}(\alpha)$ is the center of $R(m\alpha_i)$. Suppose that $i \in I^\text{im}$. By Lemma 2.2, $R(\alpha)$ can be considered as a subalgebra of $\text{End}(\mathbb{F}[x_1, \ldots, x_d])$. Let $x_k$ be the endomorphism of $\mathbb{F}[x_1, \ldots, x_m]$ defined by multiplication by $x_k$. It is obvious that $\text{Sym}(\alpha)$ is contained in the center of $R(\alpha)$ and $\mathbb{F}[x_1, \ldots, x_m] \subset R(\alpha)$.

For $f \in \mathbb{F}[x_1, \ldots, x_m]$, from the defining relations, we have
\[
f_{\tau_{i_1} \cdots \tau_{i_k}} = \tau_{i_1} \cdots \tau_{i_k} (r_{i_k} \cdots r_{i_1} f) + \cdots \text{ lower terms } \cdots
\]
with respect to the Bruhat order. Let $y = \sum \tau_{w_i} f_i$ be an element in the center of $R(\alpha)$. We assume $\ell(w_1) \geq \ell(w_k)$ for all $k$. Take $j$ such that $w_j(j) \neq j$. Then
\[
yx_j - x_j y = y(x_j - x_{w_1(j)}) + \cdots \text{ lower terms } \cdots,
\]
which implies $\tau_{w_i} = 1$ for all $i$. Since $y$ commutes with all $\tau_i$, $y$ should be a symmetric polynomial. Therefore, the center of $R(\alpha)$ is $\text{Sym}(\alpha)$.

We now deal with the general case when $\alpha \in Q^+$. In this case, using the fact that $\text{Sym}(m\alpha_i)$ is the center of $R(m\alpha_i)$ for $i \in I$, our assertion can be proved in the same manner as in [24, Theorem 2.9, Corollary 2.10].

\[\square\]

2.2. Quantum Serre relations.

Let $R(\alpha)$-mod (resp. $R(\alpha)$-pmod, $R(\alpha)$-fmod) be the category of arbitrary (resp. finitely generated projective, finite-dimensional) graded left $R(\alpha)$-modules. The morphisms in these categories are homogeneous homomorphisms. Let
\[
K_0(R) = \bigoplus_{\alpha \in Q^+} K_0(R(\alpha)\text{-pmod}) \quad \text{and} \quad G_0(R) = \bigoplus_{\alpha \in Q^+} G_0(R(\alpha)\text{-fmod}),
\]
where $K_0(R(\alpha)\text{-pmod})$ (resp. $G_0(R(\alpha)\text{-fmod})$) is the Grothendieck group of $R(\alpha)$-pmod (resp. $R(\alpha)$-fmod). Then $K_0(R)$ and $G_0(R)$ have the $A$-module structure given by $q[M] = [M(-1)]$, where $[M]$ is the isomorphism classes of an $R(\alpha)$-module $M$. For $M, N \in R(\alpha)$-mod, let $\text{Hom}(M, N)$ be the $F$-vector space of homogeneous homomorphisms of degree 0, and let $\text{Hom}(M(k), N) = \text{Hom}(M, N(-k))$ be the $F$-vector space of homogeneous homomorphisms of degree $k$. Define
\[
\text{HOM}(M, N) = \bigoplus_{k \in \mathbb{Z}} \text{Hom}(M, N[k]).
\]

Let $\text{Sym}^+(\alpha)$ be the maximal ideal of $\text{Sym}(\alpha)$. Since $\text{Sym}^+(\alpha)$ acts on any irreducible graded $R(\alpha)$-module trivially, the isomorphism classes of irreducible graded modules over $R(\alpha)$ are in 1-1 correspondence with the isomorphism classes of irreducible graded modules over the quotient
\( R(\alpha)/\text{Sym}^+(\alpha)R(\alpha) \). It follows from Lemma 2.7 that there are only finitely many irreducible \( R(\alpha) \)-modules, and all irreducible \( R(\alpha) \)-modules are finite-dimensional. Note that \( R(\alpha) \) has the Krull-Schmidt unique direct sum decomposition property for finitely generated modules since each graded part of \( R(\alpha) \) is finite-dimensional. Hence irreducible \( R(\alpha) \)-modules form a basis of \( G_0(R(\alpha)\text{-fmod}) \) as an \( \mathbb{A} \)-module, which implies that the projective covers of irreducible \( R(\alpha) \)-modules form a basis of \( K_0(R(\alpha)\text{-pmod}) \) as an \( \mathbb{A} \)-module.

Let us consider the \( \mathbb{A} \)-bilinear pairing \((\ ,\ ) : K_0(R(\alpha)) \times G_0(R(\alpha)) \to \mathbb{A}\) defined by

\[
(\lfloor P \rfloor, \lfloor M \rfloor) = \dim_q (P^* \otimes_{R(\alpha)} M),
\]

where \( \dim_q(N) := \sum_{i \in \mathbb{Z}} (\dim_{\mathbb{F}} N_i)q^i \) for a \( \mathbb{Z} \)-graded module \( N = \bigoplus_{i \in \mathbb{Z}} N_i \). Then, the paring \((\ ,\ )\) is perfect. Thus \( K_0(R(\alpha)) \) and \( G_0(R(\alpha)) \) are dual to each other with respect to the pairing \((\ ,\ )\). By Lemma 2.7 the pairing \((2.10)\) can be extended to an \( \mathbb{A} \)-bilinear form \((\ ,\ ) : K_0(R(\alpha)) \times K_0(R(\alpha)) \to \mathbb{Q}(q)\) given by

\[
(\lfloor P \rfloor, \lfloor Q \rfloor) = \dim_q (P^* \otimes_{R(\alpha)} Q).
\]

Since the pairing \((2.10)\) is perfect and \( P^* \otimes_{R(\alpha)} Q \simeq Q^* \otimes_{R(\alpha)} P \), we conclude that the pairing \((2.11)\) is a nondegenerate symmetric bilinear form on \( K_0(R(\alpha)) \).

For a finite-dimensional \( R(\alpha) \)-module \( M \), we define the \textit{character} \( \chi_q(M) \) of \( M \) to be

\[
\chi_q(M) = \sum_{i \in \text{Seq}(\alpha)} (\dim_q(1_i M)) \iota_i.
\]

For \( i = (i_1^{(d_1)}, \ldots, i_r^{(d_r)}) \in \text{Seq}(\alpha) \), let

\[
i_i := 1_{i_1, d_1} \otimes \cdots \otimes 1_{i_r, d_r},
\]

where

\[
i_{i, d} := \begin{cases} \tau_{w_0} x_1^{d-1} \cdots x_{d-1} 1_{(i \ldots, i)} & \text{if } i \in I^r, \\ 1_{(i \ldots, i)} & \text{if } i \in I^m,
\end{cases}
\]

and \( w_0 = r_1 r_2 r_1 \cdots r_{d-1} \cdots r_1 \) is the longest element in \( S_d \). Since each \( 1_{i_k, d_k} \) is an idempotent in \( R(d_k \alpha_{i_k}) \) \((k = 1, \ldots, r)\), \( 1_i \) is an idempotent. Define an \( R(\alpha) \)-module \( P_i \) corresponding to \( i = (i_1^{(d_1)}, \ldots, i_r^{(d_r)}) \in \text{Seq}(\alpha) \) by

\[
P_i := R(\alpha) 1_i \left( \sum_{k=1}^{r} \frac{d_k(d_k - 1)(\alpha_{i_k} | \alpha_{i_k})}{4} \right).
\]

Note that \( P_i \) is a projective graded \( R(\alpha) \)-module. By construction, if \( i \in I^m \), then

\[
P_{(i^{(d)})} = P_{(i \ldots, i)}. \]

For a finitely generated graded projective \( R(\alpha) \)-module \( P \), define

\[
\mathcal{P} = \text{HOM}(P, R(\alpha))^*.
\]
Note that $P$ is a graded projective left $R(\alpha)$-module and that $P_1(\alpha) \simeq P_1(-\alpha)$ for $i \in \text{Seqd}(\alpha)$. Hence we get a $\mathbb{Z}$-linear involution $- : K_0(R) \to K_0(R)$.

We now prove the quantum Serre relations on $K_0(R)$. Suppose that $i \in I^e, j \in I$ and $a_{ij} \neq 0$. Let $N = 1 - a_{ij}$ and take nonnegative integers $a, b \geq 0$ with $a + b = N$. Define the homogeneous elements

Choose a pair of sequences $i_1$ and $i_2$ such that $i_1 \ast (i(a) j^i(b)) \ast i_2 \in \text{Seqd}(\alpha)$, and write $P_{\ast(i(a) j^i(b)) \ast i_2}$ for $P_{i_1 \ast(i(a) j^i(b)) \ast i_2}$. Then these elements give rise to homomorphisms of graded projective modules

$$d^+_{a,b} : P_{\ast(i(a) j^i(b))} \longrightarrow P_{\ast(i(a+1) j^i(b-1))},$$

$$m \longmapsto m \cdot 1_i \otimes \alpha^+_{a,b} \otimes 1_{i_2},$$

$$d^-_{a,b} : P_{\ast(i(a) j^i(b))} \longrightarrow P_{\ast(i(a-1) j^i(b+1))},$$

$$m \longmapsto m \cdot 1_i \otimes \alpha^-_{a,b} \otimes 1_{i_2}.$$ 

Set $d^+_{N,0} = 0$ and $d^-_{0,N} = 0$. Then we have

$$0 \overset{\text{Seqd}}{\longrightarrow} P_{\ast(i(0) j^i(N))} \overset{\text{Seqd}}{\longrightarrow} \cdots \overset{\text{Seqd}}{\longrightarrow} P_{\ast(i(a-1) j^i(b+1))} \overset{\text{Seqd}}{\longrightarrow} \cdots \overset{\text{Seqd}}{\longrightarrow} P_{\ast(i(N) j^i(0))} \overset{\text{Seqd}}{\longrightarrow} 0.$$

**Lemma 2.8.**

1. $d^+_{a,b} \circ d^-_{a-1,b+1} = 0, \quad d^-_{a,b} \circ d^+_{a+1,b-1} = 0$ for $a, b > 0$.
2. $d^-_{1,N-1} \circ d^-_{N,0} = t_{i,j} - a_{ij} \cdot \text{id}, \quad d^-_{1,N-1} \circ d^+_{0,N} = (-1)^{N-1} t_{i,j} - a_{ij} \cdot \text{id}$.
3. For $1 < a, b < N$, we have

$$d^-_{a-1,b+1} \circ d^+_{a,b} - d^+_{a-1,b+1} \circ d^-_{a,b} = (-1)^{b-1} t_{i,j} - a_{ij} \cdot \text{id}.$$

**Proof.** If $j \in I^e$, this lemma was proved in [25 31]. We will prove our lemma when $j \in I^{im}$.

Let $d = 2 - a_{ij}$ and let $e_{a,b} = 1_i \otimes 1_j \otimes 1_k$ for $a, b \geq 0$. Since $i \in I^e$ and $P_i(u, v) = 1$, it follows from [25 31] that

$$\alpha^+_{a,b} = \tau_d - 1 \cdots \tau_{a+1} \cdot e_{a+1,b-1} = e_{a,b} \tau_d - 1 \cdots \tau_{a+1} \cdot e_{a+1,b-1},$$

$$\alpha^-_{a,b} = \tau_1 \cdots \tau_{a-1} \cdot e_{a-1,b+1} = e_{a,b} \tau_1 \cdots \tau_{a-1} \cdot e_{a-1,b+1}.$$
By a direct computation, we have
\[ \alpha_{a-1,b+1}^+ \alpha_{a,b}^+ = \alpha_{a-1,b+1} \tau_{d-1} \cdots \tau_a \epsilon_{a,b} \epsilon_{a,b} \tau_{d-1} \cdots \tau_a + 1 \epsilon_{a+1,b-1} = \alpha_{a-1,b+1} \tau_{d-1} \cdots \tau_a \epsilon_{a+1,b-1} = 0. \]
In the same manner, we get \( \alpha_{a+1,b-1}^+ \alpha_{a,b}^- = 0. \)

On the other hand, using the same argument as in [25], for \( a, b > 0 \), we obtain
\[ \alpha_{a,b}^+ \alpha_{a-1,b-1}^+ = \tau_1 \cdots \tau_{a-1} \tau_{d-1} \cdots \tau_a \epsilon_{a+1,b} \epsilon_{a,b} \tau_{d-1} \cdots \tau_a + 1 \epsilon_{a+1,b}, \]
\[ \alpha_{a,b}^- \alpha_{a-1,b+1}^- = \tau_1 \cdots \tau_{a-1} \tau_{d-1} \cdots \tau_a \epsilon_{a-1,b+1} \epsilon_{a,b} \tau_{d-1} \cdots \tau_a + 1 \epsilon_{a+1,b}, \]
which implies
\[ \alpha_{N,0}^+ \alpha_{N-1,1}^- = t_{i,j,-a_{ij},0} \epsilon_{N,0}, \quad \alpha_{0,N}^+ \alpha_{1,N-1}^- = (-1)^N t_{i,j,-a_{ij},0} \epsilon_{0,N}, \]
and
\[ \alpha_{a,b}^+ \alpha_{a-1,b-1}^- - \alpha_{a,b}^- \alpha_{a-1,b+1}^+ = \tau_1 \cdots \tau_{a-1} \tau_{d-1} \cdots \tau_a + 2 (\tau_a + 1 \tau_a + 1) \epsilon_{a,b} \]
\[ = \tau_1 \cdots \tau_{a-1} \tau_{d-1} \cdots \tau_a + 2 (\tilde{Q}_{i,j} (x_a, x_{a+1}, x_{a+2})) \epsilon_{a,b} = (-1)^{b-1} t_{i,j,-a_{ij},0} \epsilon_{a,b}. \]
Therefore, we obtain
\[ \alpha_{a-1,b+1}^+ \alpha_{a,b}^+ = 0, \quad \alpha_{a+1,b-1}^- \alpha_{a,b}^- = 0, \]
\[ \alpha_{N,0}^+ \alpha_{N-1,1}^- = t_{i,j,-a_{ij},0} \epsilon_{N,0}, \quad \alpha_{0,N}^+ \alpha_{1,N-1}^- = (-1)^N t_{i,j,-a_{ij},0} \epsilon_{0,N}, \]
\[ \alpha_{a,b}^+ \alpha_{a+1,b-1}^- - \alpha_{a,b}^- \alpha_{a+1,b+1}^+ = (-1)^{b-1} t_{i,j,-a_{ij},0} \epsilon_{a,b}, \]
as desired. \( \square \)

**Theorem 2.9.**

1. If \( a_{ij} = 0 \), then \([P_{\cdots ij \cdots}] = [P_{\cdots ji \cdots}].\)
2. If \( i \in I^* \) and \( j \in I \) with \( i \neq j \), then
\[ \sum_{k=0}^{1-a_{ij}} (-1)^k [P_{\cdots i(k)j i^{(1-a_{ij}-k)} \cdots}] = 0. \]

**Proof.** If \( a_{ij} = 0 \), let \( \tau^- \) (resp. \( \tau^+ \)) be the element in \( R \) changing \( (ij) \) to \( (ji) \) (resp. \( (ji) \) to \( (ij) \)) and define
\[ d^- : P_{\cdots ij \cdots} \to P_{\cdots ji \cdots} \] (resp. \( d^+ : P_{\cdots ji \cdots} \to P_{\cdots ij \cdots} \)) to be the map given by right multiplication by \( t_{i,j;0,0} \tau^- \) (resp. \( t_{j,i;0,0} \tau^+ \)). From the defining relation \([2.1]\), we see that \( d^+ \) and \( d^- \) are inverses to each other. Hence
\[ [P_{\cdots ij \cdots}] = [P_{\cdots ji \cdots}]. \]
Suppose that $a_{ij} \neq 0$ and $i \in I^e$. By Lemma 2.8, the complex $(P_{(\ldots ;i^{(a)} j^{(b)} \ldots )}, d^+_a,b)$ becomes an exact sequence with the splitting maps $(-1)^{b-1} t_{ij-a_{ij},0} d_{a,b}^-$. Therefore, our assertion follows from the Euler-Poincarè principle.

\[ \square \]

3. Categorification of $\mathbb{C}^\gamma (\mathfrak{g})$

In this section, we show that the Khovanov-Lauda-Rouquier algebra $R$ gives a categorification of $\mathbb{C}^\gamma (\mathfrak{g})$.

3.1. Induction and restriction.

For $\alpha, \beta \in \mathbb{Q}^+$, consider the natural embedding

\[ \iota_{\alpha, \beta} : R(\alpha) \otimes R(\beta) \hookrightarrow R(\alpha + \beta), \]

which maps $1_\alpha \otimes 1_\beta$ to $1_{\alpha, \beta}$. For $M \in R(\alpha) \otimes R(\beta)$-mod and $N \in R(\alpha + \beta)$-mod, we define

\[ \text{Ind}_{\alpha, \beta} M = R(\alpha + \beta) \otimes_{R(\alpha) \otimes R(\beta)} M, \]
\[ \text{Res}_{\alpha, \beta} N = 1_{\alpha, \beta} N. \]

Then it is straightforward to verify that the Frobenius reciprocity holds:

\begin{equation}
\text{HOM}_{R(\alpha + \beta)}(\text{Ind}_{\alpha, \beta} M, N) \cong \text{HOM}_{R(\alpha) \otimes R(\beta)}(M, \text{Res}_{\alpha, \beta} N).
\end{equation}

When there is no ambiguity, we will simply write Ind and Res for $\text{Ind}_{\alpha, \beta}$ and $\text{Res}_{\alpha, \beta}$, respectively.

Given $i \in \text{Seq}(\alpha)$ and $j \in \text{Seq}(\beta)$, a sequence $k \in \text{Seq}(\alpha + \beta)$ is called a string of $i$ and $j$ if $k$ is a permutation of $i * j$ such that $i$ and $j$ are subsequences of $k$. For a string $k$ of $i \in \text{Seq}(\alpha)$ and $j \in \text{Seq}(\beta)$, let

\[ \deg(i, j, k) = \deg(\tau w 1_{i,j}), \]

where $w$ is the element in $S_{|\alpha|+|\beta|}/S_{|\alpha|} \times S_{|\beta|}$ corresponding to $k$. Given $X = \sum x_i i$ and $Y = \sum y_j j$, the shuffle product $X \star Y$ of $X$ and $Y$ is defined to be

\[ X \star Y = \sum_k \left( \sum_{i,j} q^{\deg(i,j,k)} x_i y_j \right) k, \]

where $k$ runs over all the shuffles of $i$ and $j$. Then, by Proposition 2.4, we have

\begin{equation}
\text{ch}_q(\text{Ind}_{\alpha, \beta} M \boxtimes N) = \text{ch}_q(M) \star \text{ch}_q(N)
\end{equation}

for $M \in R(\alpha)$-fmod and $N \in R(\beta)$-fmod.

By Corollary 2.8, $\text{Ind}_{\alpha, \beta}$ and $\text{Res}_{\alpha, \beta}$ take projective modules to projective modules. Since $1_{\alpha, \beta}$ is an idempotent, $\text{Ind}_{\alpha, \beta}$ and $\text{Res}_{\alpha, \beta}$ can be viewed as exact functors between the categories of projective modules. Hence we obtain the linear maps

\[ \text{Ind}_{\alpha, \beta} : K_0(R(\alpha)) \otimes K_0(R(\beta)) \longrightarrow K_0(R(\alpha + \beta)), \]
\[ \text{Res}_{\alpha, \beta} : K_0(R(\alpha + \beta)) \longrightarrow K_0(R(\alpha)) \otimes K_0(R(\beta)). \]
It follows from Proposition 2.6 that
\[ \text{Ind}_{\alpha,\beta}(P_i \otimes P_j) \simeq P_{i+j} \quad \text{for } i \in \text{Seq}(\alpha), \ j \in \text{Seq}(\beta), \]
\[ \text{Res}_{\alpha,\beta}P_k \simeq \bigoplus_{i,j} P_i \otimes P_j(- \deg(i,j,k)) \quad \text{for } k \in \text{Seq}(\alpha + \beta), \]
where the sum is taken over all \( i \in \text{Seq}(\alpha), \ j \in \text{Seq}(\beta) \) such that \( k \) can be expressed as a shuffle of \( i \) and \( j \). We extend the linear maps \( \text{Ind}_{\alpha,\beta} \) and \( \text{Res}_{\alpha,\beta} \) to the whole space \( K_0(R) \) by linearity:
\[ \text{Ind} : K_0(R) \otimes K_0(R) \to K_0(R) \quad \text{given by} \quad ([M], [N]) \mapsto [\text{Ind}_{\alpha,\beta}M \otimes N], \]
\[ \text{Res} : K_0(R) \to K_0(R) \otimes K_0(R) \quad \text{given by} \quad [L] \mapsto \sum_{\alpha',\beta' \in Q^+} [\text{Res}_{\alpha',\beta'}L]. \]
We denote by \([M][N]\) the product \( \text{Ind}([M],[N]) \) of \([M]\) and \([N]\) in \( K_0(R) \).

**Proposition 3.1.**

1. The pair \((K_0(R), \text{Ind})\) becomes an associative unital \(\mathbb{A}\)-algebra.
2. The pair \((K_0(R), \text{Res})\) becomes a coassociative counital \(\mathbb{A}\)-coalgebra.

**Proof.** Our assertions on associativity and coassociativity follow from the transitivity of induction and restriction. Define
\[ \iota : \mathbb{A} \to K_0(R) \quad \text{by} \quad \iota(\sum_k a_k q^k) = \sum_k a_k q^k 1, \]
\[ \epsilon : K_0(R) \to \mathbb{A} \quad \text{by} \quad \epsilon(M) = \dim_q(M_0), \]
where \(M_0\) is the image of \(M\) under the natural projection \(K_0(R) \to K_0(R(0))\). Then one can verify that \(\iota\) (resp. \(\epsilon\)) is the unit (resp. counit) of \(K_0(R)\).

We define the algebra structure on \(K_0(R) \otimes K_0(R)\) by
\[ ([M_1] \otimes [M_2]) \cdot ([N_1] \otimes [N_2]) = q^{-\beta_2\gamma_1}[M_1][N_1] \otimes [M_2][N_2] \]
for \(M_i \in K_0(R(\beta_i)), \ N_i \in K_0(R(\gamma_i)) \) \((i = 1, 2)\). Using Proposition 2.6 we prove:

**Proposition 3.2.** \(\text{Res} : K_0(R) \to K_0(R) \otimes K_0(R)\) is an algebra homomorphism.

Let us recall the bilinear paring \((\ , \ ) : K_0(R) \otimes K_0(R) \to \mathbb{Q}(q)\) given in (2.11) and the projective modules \(P_i\) for \(i \in \text{Seq}(\alpha)\) defined in (2.12). We denote by \(1\) the 1-dimensional \(R(0)\)-module of degree 0.

**Proposition 3.3.** The bilinear paring \((\ , \ ) : K_0(R) \otimes K_0(R) \to \mathbb{Q}(q)\) satisfies the following properties:

1. \((1, 1) = 1\),
2. \([P_i][P_j]\) \(= \delta_{ij}(1 - q^2)^{-1}\) for \(i, j \in I\),
3. \([L][M][N] = (\text{Res}[L], [M] \otimes [N])\) for \([L], [M], [N] \in K_0(R)\),
4. \([L][M], [N] = ([L] \otimes [M], \text{Res}[N])\) for \([L], [M], [N] \in K_0(R)\).
Proof. The assertions (1) and (2) follow from the $\mathbb{Z}$-grading on $R(\alpha)$. Suppose that $L \in R(\alpha+\beta)$-pmod, $M \in R(\alpha)$-pmod and $N \in R(\beta)$-pmod. Then we have

$$([L], [M][N]) = \dim_q(L^* \otimes_{R(\alpha+\beta)} \text{Ind}_{\alpha, \beta} M \boxtimes N)$$

$$= \dim_q((\text{Res}_{\alpha, \beta} L)^* \otimes_{R(\alpha)} M \boxtimes N) = (\text{Res}_{\alpha, \beta} L, M \boxtimes N),$$

which yields that $([L], [M][N]) = (\text{Res}[L], [M] \otimes [N])$.

The assertion (4) can be proved in the same manner.

Define a map $\Phi : U^-_\Delta (\mathfrak{g}) \longrightarrow K_0(R)$ by

$$f^{(d_1)}_{i_1} \cdots f^{(d_r)}_{i_r} \mapsto [P_{(i^{(d_1)}_1) \cdots i^{(d_r)}_r}].$$

Theorem 3.4. The map $\Phi$ is an injective algebra homomorphism.

Proof. By Theorem 2.5, $\Phi$ is an algebra homomorphism. Since both of $\Delta_0$ and $\text{Res}$ are algebra homomorphisms and

$$\Delta_0(f_i) = f_i \otimes 1 + 1 \otimes f_i, \quad \text{Res}(P_{(i)}) = P_{(i)} \otimes 1 + 1 \otimes P_{(i)} (i \in I),$$

by (1.3) and Proposition 3.3 we have

$$(x, y)_L = (\Phi(x), \Phi(y)) \text{ for all } x, y \in U^-_\Delta (\mathfrak{g}).$$

Hence $\text{Ker}\Phi$ is contained in the radical of the bilinear form $(\ , \ )_L$, which is nondegenerate. The assertion follows immediately.

Therefore, $\text{Im}\Phi$ gives a categorification of $U^-_\Delta (\mathfrak{g})$. In general, the homomorphism $\Phi$ is not surjective. However, if $a_{ii} \neq 0$ for all $i \in I$, then $\Phi$ is an isomorphism as will be shown in the next subsection.

3.2. Surjectivity of $\Phi$.

In this subsection, we assume that $a_{ii} \neq 0$ for all $i \in I$. We have seen in Section 2 that the algebra $R(ma_i)$ has a unique irreducible graded module $L(i^m)$. If $i \in I^c$, we have

$$L(i^m) \simeq \text{Ind}^R_{R[\mathbb{F}[x_1, \ldots, x_m]]} 1,$$

where $1$ is the trivial $\mathbb{F}[x_1, \ldots, x_m]$-module of dimension 1 over $\mathbb{F}$. Note $\dim_q(1) = 1$. If $i \in I^m$, then $L(i^m)$ is isomorphic to the trivial graded $R(ma_i)$-module with defining relations given in (2.8). We know $\text{ch}_q(L(i^m)) = (i \cdots i)$.

For $M \in R(\alpha)$-mod and $i \in I$, define

$$\Delta_i M = 1_{\delta_i, n-\delta_i}, M \in R(\alpha) \otimes R(\delta_i - \alpha_i)-\text{mod},$$

$$\varepsilon_i(M) = \max\{k \geq 0 \mid \Delta_i M \neq 0\},$$

$$\hat{\mathcal{C}}_i(M) = \text{soc}((\text{Res}_{\alpha_i, \delta_i} \circ \Delta_i(M)) \in R(\alpha - \alpha_i)-\text{mod},$$

$$\hat{f}_i(M) = \text{hdInd}_{\alpha_i, \alpha_i}(L(i) \boxtimes M) \in R(\alpha + \alpha_i)-\text{mod}. $$

Note that they are defined in the opposite manner to (2.3) and (2.4). By the Frobenius reciprocity, we have

$$\text{HOM}_{R(\alpha)}(\text{Ind}_{\alpha_i, \alpha-ma_i} L(i^m) \boxtimes N, M) \simeq \text{HOM}_{R(ma_i) \otimes R(\alpha-ma_i)}(L(i^m) \boxtimes N, \Delta_i M).$$
for \( N \in R(\alpha - m\alpha_i)\)-mod and \( M \in R(\alpha)\)-mod.

**Lemma 3.5.** For \( i \in \Pi^m \), take \( m_1, \ldots, m_k \in \mathbb{Z}_{>0} \) and set \( m = m_1 + \cdots + m_k \). Then the following statements hold.

1. \( \text{Res}_{m_1\alpha_1, \ldots, m_k\alpha_k} L(i^m) \) is isomorphic to \( L(i^{m_1}) \otimes \cdots \otimes L(i^{m_k}) \).
2. \( \text{Ind}_{m_1\alpha_1, \ldots, m_k\alpha_k} (L(i^{m_1}) \otimes \cdots \otimes L(i^{m_k})) \) has an irreducible head, which is isomorphic to \( L(i^m) \).

**Proof.** The assertion (1) follows from the definition \( 2.8 \). To prove (2), for simplicity, we assume \( k = 2 \). Let \( i = (i, \ldots, i) \) and \( L = \text{Ind}L_1 \otimes L_2 \), where \( L_j := L(i^{m_j}) \) \((j = 1, 2)\). Set \( L' = \{ x \in L | \deg(x) > 0 \} \).

Then, since \( 1 \otimes (L_1 \otimes L_2) \not\subset L' \), \( L' \) is a unique maximal submodule of \( L \); i.e., \( L/L' \simeq L(i^m) \) as a graded module. We will show that \( \text{hd}L \) is irreducible. By a direct computation,

\[
\text{ch}_q(L) = \sum_{w \in S_{m_1+m_2}/S_{m_1} \times S_{m_2}} q^{-\ell(w)(\alpha_1|\alpha_2)} i
\]

\[
= i + (\ldots \text{other terms with } q^t \ldots) \ (t \in \mathbb{Z}_{>0}).
\]

Note that \( \text{ch}_q(L_1 \otimes L_2) = i \). For any quotient \( Q \) of \( L \), by the Frobenius reciprocity \( 3.11 \), we have an injective homomorphism of degree 0

\[
L_1 \otimes L_2 \hookrightarrow \text{Res}_{m_1\alpha_1, m_2\alpha_2} Q,
\]

which yields

\[
\text{ch}_q(Q) = i + (\ldots \text{other terms with } q^t \ldots) \quad \text{for } t \in \mathbb{Z}_{>0}.
\]

Therefore, \( \text{hd}L \) has only one summand, and hence it is irreducible. \( \square \)

**Lemma 3.6.** Let \( M \) be an irreducible \( R(\alpha)\)-module and let \( L(i^m) \otimes N \) be an irreducible submodule of the \( R(m\alpha_1) \otimes R(\alpha - m\alpha_1)\)-module \( \Delta_m M \). Then \( \varepsilon_i(N) = \varepsilon_i(M) - m \).

**Proof.** If \( i \in \Pi^m \), then the proof is the same as that of \( 24 \text{ Lemma 3.6} \). If \( i \in \Pi^m \), by the definition, we have \( \varepsilon_i(N) \leq \varepsilon_i(M) - m \). From the equation \( 3.5 \), we obtain

\[
0 \to K \to \text{Ind}L(i^m) \otimes N \to M \to 0
\]

for some submodule \( K \) of \( \text{Ind}L(i^m) \otimes N \). It follows from \( 3.2 \) and the exactness of \( \Delta_\alpha \) that \( \varepsilon_i(N) \geq \varepsilon_i(M) - m \), which yields our assertion. \( \square \)

**Lemma 3.7.** Let \( N \) be an irreducible \( R(\alpha)\)-module with \( \varepsilon_i(N) = 0 \) and let \( M = \text{Ind}L(i^m) \otimes N \). Then we have

1. \( \Delta_m M \simeq L(i^m) \otimes N \),
2. \( \text{hd}M \) is an irreducible module with \( \varepsilon_i(\text{hd}M) = m \),
3. for all other composition factors \( L \) of \( M \), we have \( \varepsilon_i(L) < m \).

**Proof.** Our assertion can be proved in the same manner as in \( 24 \text{ Lemma 3.7} \). \( \square \)
Lemma 3.8. Let $M$ be an irreducible $R(\alpha)$-module and let $\varepsilon = \varepsilon_i(M)$. Then $\Delta_i M$ is isomorphic to $L(i^\varepsilon) \boxtimes N$ for some irreducible $R(\alpha - \varepsilon \alpha_i)$-module $N$ with $\varepsilon_i(N) = 0$.

Proof. Our assertion can be proved in the same manner as in [26, Lemma 5.1.4] (cf. [24, Lemma 3.8]). □

Lemma 3.9. Suppose that $i \in I^\text{im}$ and $N$ is an irreducible $R(\alpha)$-module with $\varepsilon_i(N) = 0$. Let

$$M = \text{Ind}_L(i^{m_1}) \boxtimes \cdots \boxtimes L(i^{m_k}) \boxtimes N$$

for some positive integers $m_1, \ldots, m_k \in \mathbb{Z}_{>0}$ and set $m = m_1 + \cdots + m_k$. Then

1. $\text{hd} M$ is irreducible,
2. $\varepsilon_i(\text{hd} M) = m$.

Proof. By the definition, we have

$$\Delta_{i^m} M = (\text{Ind}_L(i^{m_1}) \boxtimes \cdots \boxtimes L(i^{m_k})) \boxtimes N.$$ 

In the Grothendieck group $G_0(R(m \alpha_i) \otimes R(\alpha - m \alpha_i))$ of the category of finite-dimensional graded $R(m \alpha_i) \otimes R(\alpha - m \alpha_i)$-modules, we have

$$[\Delta_{i^m} M] = \sum_w q^{-f(w)(\alpha_i|\alpha_i)}[L(i^m) \boxtimes N],$$

where $w$ runs over all the elements in $S_m/S_{m_1} \times \cdots \times S_{m_k}$. By the Frobenius reciprocity, for any quotient $Q$ of $M$, there is a nontrivial homomorphism of degree 0

$$\Delta_{i^m} M = (\text{Ind}_L(i^{m_1}) \boxtimes \cdots \boxtimes L(i^{m_k})) \boxtimes N \rightarrow \Delta_{i^m} Q.$$ 

By Lemma 3.9 (2), we have

$$[\Delta_{i^m} Q] = [L(i^m) \boxtimes N] + ( \cdots \text{other terms with } q^t \cdots ),$$

in the Grothendieck group $G_0(R(m \alpha_i) \otimes R(\alpha - m \alpha_i))$. Therefore, by the same argument as in Lemma 3.9, $\text{hd} M$ is irreducible and $\varepsilon_i(\text{hd} M) = m$. □

Lemma 3.10. Let $N$ be an irreducible $R(\alpha)$-module and let $M = \text{Ind}_L(i^m) \boxtimes N$.

1. $\text{hd} M$ is an irreducible module with $\varepsilon_i(\text{hd} M) = \varepsilon_i(N) + m$.
2. If $i \in I^\text{re}$, then for all other composition factors $L$ of $M$, we have $\varepsilon_i(L) < \varepsilon_i(N) + m$.

Proof. If $i \in I^\text{im}$, then the proof is identical with that of [26, Lemma 5.1.5] (cf. [24, Lemma 3.9]). Suppose that $i \in I^\text{im}$. Let $\varepsilon = \varepsilon_i(N)$. By Lemma 3.8, we have

$$\Delta_{i^\varepsilon} N = L(i^\varepsilon) \boxtimes K$$

for some irreducible $R(\alpha - \varepsilon \alpha_i)$-module $K$ with $\varepsilon_i(K) = 0$. By the Frobenius reciprocity, there is a surjective homomorphism

$$\text{Ind}_L(i^\varepsilon) \boxtimes K \twoheadrightarrow N,$$
which yields

$$\text{Ind} L(i^m) \boxtimes L(i^2) \boxtimes K \rightarrow \text{Ind} L(i^m) \boxtimes N.$$  

Therefore, our assertion follows from Lemma 3.9 \(\square\)

**Lemma 3.11.** Let \(M\) be an irreducible \(R(\alpha)\)-module. Then, for \(0 \leq m \leq \varepsilon_i(M)\), the submodule \(\text{soc}\Delta_{i^{-m}}M\) of \(M\) is an irreducible module of the form \(L(i^m) \boxtimes L\) with \(\varepsilon_i(L) = \varepsilon_i(M) - m\) for some irreducible \(R(\alpha - ma_i)\)-module \(L\).

**Proof.** If \(i \in \mathcal{I}^e\), then the proof is the same as that of [26 Lemma 5.1.6] (cf. [24 Lemma 3.10]). If \(i \in \mathcal{I}^m\), let \(\varepsilon = \varepsilon_i(M)\). Note that every summand of \(\text{soc}\Delta_{i^{-m}}M\) has the form \(L(i^m) \boxtimes L\) for some irreducible \(R(\alpha - ma_i)\)-module \(L\). It follows from Lemma 3.10 that

$$\varepsilon_i(L) = \varepsilon - m,$$

so \(L(i^m) \boxtimes \Delta_{i^{-m}}(L) \neq 0\). It is clear that \(\text{Res}_{\Delta_{i^{-m}}(L)}^{\Delta_{\alpha,\alpha - \varepsilon}} \Delta_{i^{-m}}(L)\) has \(L(i^m) \boxtimes L\) as a submodule. On the other hand, by Lemma 3.5 and Lemma 3.8, there exists an irreducible \(R(\alpha - \varepsilon)\)-module \(N\) such that

$$\text{Res}_{\Delta_{i^{-m}}(L)}^{\Delta_{\alpha,\alpha - \varepsilon}} \Delta_{i^{-m}}(L) \simeq L(i^m) \boxtimes L(i^{-m}) \boxtimes N,$$

which is irreducible. Hence \(\text{soc}\Delta_{i^{-m}}M\) is irreducible and isomorphic to \(L(i^m) \boxtimes L\). \(\square\)

By Lemma 3.10 and Lemma 3.11 the operators \(\tilde{e}_i\) and \(\tilde{f}_i\) take irreducible modules to irreducible modules or 0, and

$$\varepsilon_i(M) = \max\{k \geq 0 \mid \tilde{e}_i^k M \neq 0\}, \quad \varepsilon_i(\tilde{f}_i M) = \varepsilon_i(M) + 1.$$

**Lemma 3.12.** Let \(M\) be an irreducible \(R(\alpha)\)-module. Then we have

1. \(\text{soc}\Delta_{i^{-m}}M \simeq L(i^m) \boxtimes (\tilde{e}_i^m M),\)
2. \(\text{Ind} \text{Ind}(L(i^m) \boxtimes M) \simeq \tilde{f}_i^m M.\)

**Proof.** If \(i \in \mathcal{I}^e\), then the proof is the same as in [26 Lemma 5.2.1]. Suppose that \(i \in \mathcal{I}^m\). We first focus on the assertion (1). Since the case \(m > \varepsilon_i(M)\) is trivial, we may assume that \(m \leq \varepsilon_i(M)\). Since \(L(i) \boxtimes \tilde{e}_i M \hookrightarrow \Delta_i M\), we have

$$\underbrace{L(i) \boxtimes \cdots \boxtimes L(i)}_{m} \boxtimes \tilde{e}_i^m M \hookrightarrow \text{Res}_{\alpha,0,\alpha - \varepsilon}^{\alpha,\alpha - \varepsilon} \Delta_{i^{-m}} M,$$

which implies there is a nontrivial homomorphism

$$\text{Ind}(L(i) \boxtimes \cdots \boxtimes L(i)) \boxtimes \tilde{e}_i^m M \rightarrow \Delta_{i^{-m}} M.$$

Since any quotient of \(\text{Ind}(L(i) \boxtimes \cdots \boxtimes L(i))\) has a 1-dimensional submodule, \(\Delta_{i^{-m}} M\) has a submodule which is isomorphic to \(L(i^m) \boxtimes \tilde{e}_i^m M\). Hence the assertion (1) follows from Lemma 3.11.

For the assertion (2), by the definition of \(\tilde{f}_i\), there is a nontrivial homomorphism

$$\text{Ind}(\text{Ind}(L(i) \boxtimes \cdots \boxtimes L(i)) \boxtimes M) \rightarrow \tilde{f}_i^m M.$$
Using the same argument in the proof of Lemma 3.10 we have
\[ \text{hdInd}(\text{Ind}(L(i) \boxtimes \cdots \boxtimes L(i)) \boxtimes M) \simeq \tilde{f}_i^m M. \]

On the other hand, the nontrivial homomorphism
\[ \text{Ind}(L(i) \boxtimes \cdots \boxtimes L(i)) \rightarrow L(i^m) \]
induces a nontrivial homomorphism
\[ \text{Ind}(\text{Ind}(L(i) \boxtimes \cdots \boxtimes L(i)) \boxtimes M) \rightarrow \text{Ind}(i^m) \boxtimes M. \]

Therefore, we conclude \( \text{hdInd}(L(i^m) \boxtimes M) \simeq \tilde{f}_i^m M. \)

**Lemma 3.13.** Let \( M \) be an irreducible \( R(\alpha) \)-module and let \( N \) be an irreducible \( R(\alpha + \alpha_i) \)-module. Then we have
\[ \tilde{f}_i M \simeq N \text{ if and only if } M \simeq \tilde{e}_i N. \]

**Proof.** Using Lemma 3.12 our assertion can be proved in the same manner as in [26, Lemma 5.2.3] \( \square \)

Let \( \mathbb{A}\text{Seq}(\alpha) \) (resp. \( \mathbb{Q}(q)\text{Seq}(\alpha) \)) be the free \( \mathbb{A} \)-module (resp. \( \mathbb{Q}(q) \)-module) generated by \( \text{Seq}(\alpha) \). For an irreducible \( R(\alpha) \)-module \( M \), the character \( \text{ch}_q(M) \) can be viewed as an element in \( \mathbb{A}\text{Seq}(\alpha) \). Using the above lemmas, one can prove the following proposition in the same manner as in [26, Theorem 5.3.1].

**Proposition 3.14.** The character map
\[ \text{ch}_q : G_0(R(\alpha)) \rightarrow \mathbb{A}\text{Seq}(\alpha) \]
is injective.

Let \( \mathcal{F} \) be the free associative algebra over \( \mathbb{Q}(q) \) generated by \( f_i \) \((i \in I)\) and consider the natural projection \( \pi : \mathcal{F} \rightarrow U_q^-(\mathfrak{g}) \) given by \( f_i \mapsto f_i \) \((i \in I)\). Then the vector space \( \mathbb{Q}(q)\text{Seq}(\alpha) \) can be regarded as the dual space of \( \mathcal{F}_\alpha := \pi^{-1}(U_q^-(\mathfrak{g})_\alpha) \) for \( \alpha \in Q^+ \). Set
\[ K_0(R)_{\mathbb{Q}(q)} = \mathbb{Q}(q) \otimes_{\mathbb{A}} K_0(R), \quad K_0(R(\alpha))_{\mathbb{Q}(q)} = \mathbb{Q}(q) \otimes_{\mathbb{A}} K_0(R(\alpha)), \]
\[ G_0(R)_{\mathbb{Q}(q)} = \mathbb{Q}(q) \otimes_{\mathbb{A}} G_0(R), \quad G_0(R(\alpha))_{\mathbb{Q}(q)} = \mathbb{Q}(q) \otimes_{\mathbb{A}} G_0(R(\alpha)), \]
and denote by \( \Phi_{\mathbb{Q}(q)} : U_q^-(\mathfrak{g}) \rightarrow K_0(R)_{\mathbb{Q}(q)} \) the algebra homomorphism induced by \( \Phi : U_q^-(\mathfrak{g}) \rightarrow K_0(R) \). Then \( \text{ch}_q \) is the dual map of \( \Phi_{\mathbb{Q}(q)} \circ \pi \), which yields the following diagram:

\[
\begin{array}{ccc}
\mathcal{F}_\alpha & \xrightarrow{\pi} & U_q^-(\mathfrak{g})_\alpha \\
\downarrow\text{dual} & & \downarrow\text{dual w.r.t. ( , )} \\
\mathbb{Q}(q)\text{Seq}(\alpha) & \xrightarrow{\text{ch}_q} & G_0(R(\alpha))_{\mathbb{Q}(q)} \\
\end{array}
\]

Combining Theorem 3.3 with Proposition 3.14 we conclude
\[ \Phi_{\mathbb{Q}(q)} : U_q^-(\mathfrak{g}) \rightarrow K_0(R)_{\mathbb{Q}(q)} \]
is an isomorphism.
Theorem 3.15. The map $\Phi : U^-_\Lambda(g) \to K_0(R)$ is an isomorphism if $a_{ii} \neq 0$ for all $i \in I$.

Proof. It suffices to show that $\Phi_{Q(q)}(U^-_\Lambda(g)) = K_0(R)$. Choose a sequence $(i_k)_{k \geq 0}$ of $I$ such that, for each $i \in I$, $i$ appears infinitely many times in $(i_k)_{k \geq 0}$. Let $B_\alpha$ be the set of all isomorphism classes of irreducible $R(\alpha)$-modules. We fix a representative $S_b$ for each $b \in B_\alpha$. To each $b \in B_\alpha$, we assign the sequence $p_b := p_0p_1 \cdots$ given as follows: if $M_0 := S_b$, define

$$p_k = \varepsilon_{i_k}(M_k)$$

and

$$M_{k+1} = e_{i_k}^p(M_k) \quad (k \geq 0)$$

inductively. For $b \in B_\alpha$, let

$$P_b = P_{i_b},$$

where $i_b := (i_k)_{k \geq 0}$. Note that $P_b$ is well-defined since $i_b$ has only finitely many nonnegative integers. Define a total order $\prec$ on $B_\alpha$ by

$$b \prec c$$

if and only if $p_b <_{\text{lex}} p_c,$

where $<_{\text{lex}}$ is the lexicographic order. Then it follows from the definition of the pairing (2.10) that

$$(P_b, S_c) = 0 \text{ if } b \succ c \quad \text{and} \quad (P_b, S_b) = q^t$$

for some $t \in \mathbb{Z}$. Hence, any projective module $[P]$ in $K_0(R(\alpha))$ can be written as an $\Lambda$-linear combination of $\{P_b \mid b \in B_\alpha\}$, which implies $\Phi_{Q(q)}(U^-_\Lambda(g)) = K_0(R)$. \hfill $\Box$

4. Crystals and Perfect bases

In this section, we develop the theory of perfect bases for $U^-_q(g)$ as a $B_q(g)$-modules. We prove that the negative part $U^-_q(g)$ has a perfect basis by constructing the upper global basis. We also show that the crystals arising from perfect bases of $U^-_q(g)$ are all isomorphic to the crystal $B(\infty)$.

4.1. Crystals.

We review the basic theory of abstract crystals for quantum generalized Kac-Moody algebras introduced in [12].

Definition 4.1. An abstract crystal is a set $B$ together with the maps $\text{wt} : B \to \mathbb{P}$, $\varphi_i, \varepsilon_i : B \to \mathbb{Z} \sqcup \{-\infty\}$ and $\tilde{e}_i, \tilde{f}_i : B \to B \cup \{0\}$ ($i \in I$) satisfying the following conditions:

1. $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \text{wt}(b) \rangle$,
2. $\text{wt}(\tilde{e}_i b) = \text{wt}(b) + \alpha_i$, $\text{wt}(\tilde{f}_i b) = \text{wt}(b) - \alpha_i$ if $\tilde{e}_i b, \tilde{f}_i b \in B$,
3. for $b, b' \in B$ and $i \in I$, $b' = \tilde{e}_i b$ if and only if $b = \tilde{f}_i b'$,
4. for $b \in B$, if $\varphi_i(b) = -\infty$, then $\tilde{e}_i b = \tilde{f}_i b = 0$,
5. if $b \in B$ and $\tilde{e}_i b \in B$, then

$$\varepsilon_i(\tilde{e}_i b) = \begin{cases} \varepsilon_i(b) - 1 & \text{if } i \in I^w, \\ \varepsilon_i(b) & \text{if } i \in I^m, \end{cases} \quad \varphi_i(\tilde{e}_i b) = \begin{cases} \varphi_i(b) + 1 & \text{if } i \in I^w, \\ \varphi_i(b) + a_{ii} & \text{if } i \in I^m, \end{cases}$$
(6) if \( b \in B \) and \( \tilde{f}_i b \in B \), then
\[
\varepsilon_i(\tilde{f}_i b) = \begin{cases}
\varepsilon_i(b) + 1 & \text{if } i \in \mathbb{R}^e, \\
\varepsilon_i(b) & \text{if } i \in \mathbb{R}^m, \\
\varphi_i(\tilde{f}_i b) = \begin{cases}
\varphi_i(b) - 1 & \text{if } i \in \mathbb{R}^e, \\
\varphi_i(b) - a_{ii} & \text{if } i \in \mathbb{R}^m.
\end{cases}
\end{cases}
\]

**Example 4.2.**

(1) For \( b \in B(\infty) \), define \( \text{wt}, \varepsilon_i, \) and \( \varphi_i \) as follows:
\[
\text{wt}(b) = -\langle \alpha_i, \epsilon_i \rangle + qL(\infty),
\]
\[
\varepsilon_i(b) = \begin{cases}
\max\{k \geq 0 | \varepsilon_i^k b \neq 0\} & \text{for } i \in \mathbb{R}^e, \\
0 & \text{for } i \in \mathbb{R}^m.
\end{cases}
\]
\[
\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \text{wt}(b) \rangle.
\]

Then \( (B(\infty), \text{wt}, \varepsilon_i, \tilde{f}_i, \varepsilon_i, \varphi_i) \) becomes an abstract crystal.

(2) For \( b \in B(\lambda) \), define \( \text{wt}, \varepsilon_i, \) and \( \varphi_i \) as follows:
\[
\text{wt}(b) = \lambda - \langle \alpha_i, \epsilon_i \rangle + qL(\lambda),
\]
\[
\varepsilon_i(b) = \begin{cases}
\max\{k \geq 0 | \tilde{e}_i^k b \neq 0\} & \text{for } i \in \mathbb{R}^e, \\
0 & \text{for } i \in \mathbb{R}^m.
\end{cases}
\]
\[
\varphi_i(b) = \begin{cases}
\max\{k \geq 0 | f_i^k b \neq 0\} & \text{for } i \in \mathbb{R}^e, \\
\langle h_i, \text{wt}(b) \rangle & \text{for } i \in \mathbb{R}^m.
\end{cases}
\]

Then \( (B(\lambda), \text{wt}, \varepsilon_i, \tilde{f}_i, \varepsilon_i, \varphi_i) \) becomes an abstract crystal.

(3) For \( \lambda \in P \), let \( T_\lambda = \{\lambda\} \) and define
\[
\text{wt}(t_\lambda) = \lambda, \quad \varepsilon_i(t_\lambda) = \tilde{f}_i t_\lambda = 0 \quad \varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty \text{ for all } i \in I.
\]

Then \( (T_\lambda, \text{wt}, \varepsilon_i, \tilde{f}_i, \varepsilon_i, \varphi_i) \) is an abstract crystal.

(4) Let \( C = \{c\} \) and define
\[
\text{wt}(c) = 0, \quad \varepsilon_i c = \tilde{f}_i c = 0 \quad \varepsilon_i(c) = \varphi_i(c) = 0 \text{ for all } i \in I.
\]

Then \( (C, \text{wt}, \varepsilon_i, \tilde{f}_i, \varepsilon_i, \varphi_i) \) is an abstract crystal.

**Definition 4.3.**

(1) A crystal morphism \( \phi \) between abstract crystals \( B_1 \) and \( B_2 \) is a map from \( B_1 \) to \( B_2 \cup \{0\} \) satisfying the following conditions:

(a) if \( b \in B_1 \) and \( \phi(b) \in B_2 \), then \( \text{wt}(\phi(b)) = \text{wt}(b), \varepsilon_i(\phi(b)) = \varepsilon_i(b) \) and \( \varphi_i(\phi(b)) = \varphi_i(b) \),

(b) if \( b \in B_1 \) and \( i \in I \) with \( \tilde{f}_i b \in B_1 \), then we have \( \tilde{f}_i \phi(b) = \phi(\tilde{f}_i b) \).

(2) A crystal morphism \( \phi : B_1 \to B_2 \) is called strict if
\[
\phi(\varepsilon_i b) = \tilde{e}_i \phi(b) \quad \text{and} \quad \phi(\tilde{f}_i b) = \tilde{f}_i \phi(b)
\]
for all \( i \in I \) and \( b \in B_1 \).
The tensor product of two crystals is defined as follows: for given two crystals $B_1$ and $B_2$, their tensor product $B_1 \otimes B_2$ is the set $\{b_1 \otimes b_2 \mid b_1 \in B_1, b_2 \in B_2\}$ with the maps $\text{wt}, \varepsilon_i, \varphi_i, \tilde{e}_i$ and $\tilde{f}_i$ given by

$$\begin{align*}
\text{wt}(b_1 \otimes b_2) &= \text{wt}(b_1) \otimes \text{wt}(b_2), \\
\varepsilon_i(b_1 \otimes b_2) &= \max \{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \text{wt}(b_1) \rangle\}, \\
\varphi_i(b_1 \otimes b_2) &= \max \{\varphi_i(b_1) + \langle h_i, \text{wt}(b_2) \rangle, \varphi_i(b_2)\}, \\
\tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \\
\tilde{f}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\
b_1 \otimes \tilde{f}_i(b_2) & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2),
\end{cases} \\
(4.1) & \quad \text{for } i \in I^e, \\
\tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \\
\tilde{e}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\
b_1 \otimes \tilde{e}_i(b_2) & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2),
\end{cases} \\
& \quad \text{for } i \in I^e, \\
\tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \\
\tilde{e}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) - \alpha_{ii}, \\
0 & \text{if } \varepsilon_i(b_2) - \varphi_i(b_1) \leq \varepsilon_i(b_2) - \alpha_{ii}, \\
b_1 \otimes \tilde{e}_i(b_2) & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2).
\end{cases} \\
& \quad \text{for } i \in I^m.
\end{align*}$$

It was proved in [12] Lemma 3.10 that $B_1 \otimes B_2$ becomes an abstract crystal. Moreover, they proved the recognition theorem of $B(\lambda)$ ($\lambda \in P^+$) using the abstract crystal structure of $B(\infty)$.

**Proposition 4.4.** [12] Theorem 5.2] For $\lambda \in P^+$, the crystal $B(\lambda)$ is isomorphic to the connected component of $B(\infty) \otimes T_\lambda \otimes C$ containing $1 \otimes t_\lambda \otimes c$.

### 4.2. Perfect bases.

We revisit the algebra $U_q^-(\mathfrak{g})$. We analyze $U_q^-(\mathfrak{g})$ as a $B_q(\mathfrak{g})$-module and develop the perfect basis theory for $U_q^-(\mathfrak{g})$. The crystal structure is revealed when $e_i'$ acts on a perfect basis.

Let

$$e_i'^{(n)} = \begin{cases} \\
(e_i')^n & \text{if } i \in I^e, \\
\frac{(e_i')^n}{\{n\}_{ii}!} & \text{if } i \in I^m.
\end{cases}$$

Then we obtain the following commutation relations:

$$
\begin{align*}
\tilde{e}_i'^{(n)} f_j'^{(m)} &= \begin{cases} \\
\sum_{k=0}^{n} q_i^{-2nm+(n+m)k-k(k-1)/2} \binom{n}{k} f_i^{(m-k)} e_i'^{(n-k)} & \text{if } i = j \text{ and } i \in I^e, \\
\sum_{k=0}^{m} q_i^{-c_i(-2nm+(n+m)k-k(k-1)/2)} \binom{m}{k} f_i^{(m-k)} e_i'^{(n-k)} & \text{if } i = j \text{ and } i \in I^m, \\
q_i^{-n\alpha_{ii}} f_j'^{(m)} e_i'^{(n)} & \text{if } i \neq j.
\end{cases}
\end{align*}
$$

For $i \in I$ and $v \in U_q^-(\mathfrak{g})$, let

$$\ell_i(v) = \min\{n \in \mathbb{Z}_{\geq 0} \mid e_i'^{n+1} v = 0\}.$$
Note that \( \ell_i \) is well-defined since \( e'_i \) is locally nilpotent (see [13]). Then, for \( i \in I \) and \( k \in \mathbb{Z}_{\geq 0} \),

\[
U_q(\mathfrak{g})_{<k} := \{ v \in U_q(\mathfrak{g}) \mid \ell_i(v) < k \}
\]

becomes a \( \mathbb{Q}(q) \)-vector space.

**Definition 4.5.** A basis \( B \) of \( U_q(\mathfrak{g}) \) is said to be perfect if

1. \( B = \bigsqcup_{\mu \in Q^-} B_\mu \), where \( B_\mu := B \cap U_q(\mathfrak{g})_\mu \),
2. for any \( b \in B \) and \( i \in I \) with \( e'_i(b) \neq 0 \), there exists a unique \( e_i(b) \in B \) such that
   \[
e_i'(b) \in e_i(b) + U_q(\mathfrak{g})_{<\ell_i(b)-1} \text{ for some } c \in \mathbb{Q}(q)^\times,
   \]
3. if \( e_i(b) = e_i(b') \) for \( b, b' \in B \), then \( b = b' \) (\( i \in I \)).

Now, we define the upper Kashiwara operators for the \( B_q(\mathfrak{g}) \)-module \( U_q(\mathfrak{g}) \). Let \( u \in U_q(\mathfrak{g}) \) such that \( e'_i u = 0 \). Then, for \( n \in \mathbb{Z}_{\geq 0} \), we define the upper Kashiwara operators \( \tilde{E}_i, \tilde{F}_i \) by

\[
\tilde{E}_i(f_i^{(n)}u) = \begin{cases} 
q_i^{-(n-1)}f_i^{(n-1)}u & \text{if } i \in I^\text{re}, \\
\{n\}q_i^{c_i(n-1)}f_i^{(n-1)}u & \text{if } i \in I^\text{im},
\end{cases}
\]

\[
\tilde{F}_i(f_i^{(n)}u) = \begin{cases} 
q_i^n[n+1]f_i^{(n+1)}u & \text{if } i \in I^\text{re}, \\
1f_i^{(n)}u & \text{if } i \in I^\text{im}.
\end{cases}
\]

From the \( i \)-string decomposition (1.10), the upper Kashiwara operators \( \tilde{E}_i \) and \( \tilde{F}_i \) can be extended to the whole space \( U_q(\mathfrak{g}) \) by linearity.

**Definition 4.6.** An upper crystal basis of \( U_q(\mathfrak{g}) \) is a pair \((L^\vee, B^\vee)\) satisfying the following conditions:

1. \( L^\vee \) is a free \( \mathbb{A}_0 \)-module of \( U_q(\mathfrak{g}) \) such that \( U_q(\mathfrak{g}) = \mathbb{Q}(q) \otimes_{\mathbb{A}_0} L^\vee \) and \( L^\vee = \bigoplus_{\alpha \in Q^+} L^\vee_\alpha \), where \( L^\vee_{-\alpha} := L^\vee \cap U_q(\mathfrak{g})_{-\alpha} \),
2. \( B^\vee \) is a \( \mathbb{Q} \)-basis of \( L^\vee / qL^\vee \) such that \( B^\vee = \bigsqcup_{\alpha \in Q^+} B^\vee_{-\alpha} \), where \( B^\vee_{-\alpha} := B^\vee \cap (L^\vee_{-\alpha} / qL^\vee_{-\alpha}) \),
3. \( \tilde{E}_i B^\vee \subset B^\vee \cup \{0\} \), \( \tilde{F}_i B^\vee \subset B^\vee \) for all \( i \in I \),
4. For \( b, b' \in B^\vee \) and \( i \in I \), \( b^\vee = \tilde{F}_i b^\vee \) if and only if \( b^\vee = \tilde{E}_i b^\vee \).

We have the following lemma which is the \( U_q(\mathfrak{g}) \)-version of [19] Lemma 4.3.

**Lemma 4.7.** For any \( u, v \in U_q(\mathfrak{g}) \), we have

\[
(f_i u, v)_K = (u, \tilde{E}_i v)_K, \quad (\tilde{e}_i u, v)_K = (u, \tilde{F}_i v)_K.
\]

**Lemma 4.8.** Let \( u \in U_q(\mathfrak{g}) \), and \( n \) be the smallest integer such that \( e'_i u = 0 \). Then we have

\[
e_i^n u = \begin{cases} 
n! \tilde{E}_i^n u & \text{if } i \in I^\text{re}, \\
\tilde{E}_i^n u & \text{if } i \in I^\text{im}.
\end{cases}
\]
Definition 4.10. A triple \((A,q,L^-)\) is a balanced triple if
\begin{enumerate}
\item \(U_q^-(\mathfrak{g}) \cong K_q \otimes K \cong K \otimes K_{\infty} \cong K_{\infty} \otimes K \) as \(K_q\)-vector spaces,
\item the natural \(K_q\)-linear map \(E \rightarrow L/qL\) is an isomorphism, where \(E := U_q^\wedge \cap L \cap L^\wedge\).
\end{enumerate}

It was shown in \cite{22} that the condition (2) is equivalent to saying that there are natural isomorphisms \(U_q^\wedge \cong A_0 \otimes E, \ L \cong A_0 \otimes E, \ L^\wedge \cong A_{\infty} \otimes E\).

Let \(U_q^0(\mathfrak{g})\) be the \(A\)-subalgebra of \(U_q(\mathfrak{g})\) generated by \(q^h, \prod_{k=1}^m \frac{1-q^kq^{-h}}{1-q^k}\) for all \(m \in \mathbb{Z}_{\geq 0}, \ h \in P^\vee\) and let \(U_q^\wedge(\mathfrak{g})\) be the \(A\)-algebra generated by \(U_q^0(\mathfrak{g}), \ U_q^+(\mathfrak{g})\) and \(U_q^-(\mathfrak{g})\).

Proposition 4.11 \cite{11}. \((U_q^\wedge(\mathfrak{g}), L(\infty), L(\infty)^\wedge)\) is a balanced triple for \(U_q^-(\mathfrak{g})\).

Recall the \(K_q\)-algebra automorphism \(\sim: U_q^-(\mathfrak{g}) \rightarrow U_q^-(\mathfrak{g})\) given in \cite{22}. Define
\begin{align*}
U_q^\wedge(\mathfrak{g}) &= \{ u \in U_q^-(\mathfrak{g}) \mid (u, U_q^-(\mathfrak{g}))_K \subset A \}, \\
L(\infty) &= \{ u \in U_q^-(\mathfrak{g}) \mid (u, L(\infty))_K \subset A_0 \}, \\
L(\infty)^\wedge &= \{ u \in U_q^-(\mathfrak{g}) \mid (u, L(\infty)^\wedge)_K \subset A_{\infty} \}.
\end{align*}
By the same argument as in [23], one can verify that \( (U_q^{-}(g)^\vee, L(\infty)^\vee, \overline{L(\infty)^\vee}) \) is a balanced triple for \( U_q^{-}(g) \). Hence there is a natural isomorphism
\[
E^\vee := U_q^{-}(g)^\vee \cap L(\infty)^\vee \cap \overline{L(\infty)^\vee} \xrightarrow{\sim} L(\infty)^\vee / qL(\infty)^\vee.
\]

Let \( G^\vee \) denote the inverse of this isomorphism and set
\[
\mathcal{B}(\infty) = \{ G^\vee(b^\vee) \mid b^\vee \in B(\infty)^\vee \}.
\]

**Lemma 4.12.** Let \( b^\vee \in L(\infty)^\vee / qL(\infty)^\vee \) and \( n \in \mathbb{Z}_{\geq 0} \).

1. If \( \tilde{E}_i^{n+1}b^\vee = 0 \), then \( e_i^nG(b^\vee) = \begin{cases} [n]!G(\tilde{E}_i^n b^\vee) & \text{if } i \in \mathcal{I}^e, \\ G(\tilde{E}_i^n b^\vee) & \text{if } i \in \mathcal{I}^m. \end{cases} \)

2. \( e_i^{n+1}G^\vee(b^\vee) = 0 \) if and only if \( \tilde{E}_i^{n+1}b^\vee = 0 \).

**Proof.** We first prove the assertion (1). Let \( i \in \mathcal{I}^e \). Since \( \varphi\left(\frac{1}{[n]!}e_i^n\right) = f_i^{(n)} \), by Lemma 4.8 we obtain
\[
\frac{1}{[n]!}e_i^nG^\vee(b^\vee) = \tilde{E}_i^nG^\vee(b^\vee) \in U_q^{-}(g)^\vee \cap L(\infty)^\vee \cap \overline{L(\infty)^\vee},
\]
which yields \( \frac{1}{[n]!}e_i^nG^\vee(b^\vee) = G^\vee(\tilde{E}_i^n b^\vee) \).

Similarly, for \( i \in \mathcal{I}^m \), it follows from \( \varphi(e_i^n) = f_i^{(n)} \) that
\[
e_i^nG^\vee(b^\vee) = \tilde{E}_i^nG^\vee(b^\vee) \in U_q^{-}(g)^\vee \cap L(\infty)^\vee \cap \overline{L(\infty)^\vee}.
\]
Thus we have \( e_i^nG^\vee(b^\vee) = G^\vee(\tilde{E}_i^n b^\vee) \).

For the assertion (2), it is obvious that \( e_i^{n+1}G^\vee(b^\vee) = 0 \) implies \( \tilde{E}_i^{n+1}b^\vee = 0 \). To prove the converse, suppose \( e_i^{n+1}G^\vee(b^\vee) \neq 0 \) and take the smallest \( m > n \) such that \( e_i^{m+1}G^\vee(b^\vee) = 0 \). By (1), we have
\[
e_i^mG^\vee(b^\vee) = \begin{cases} [m]!G^\vee(\tilde{E}_i^m b^\vee) = 0, & \text{if } i \in \mathcal{I}^e, \\ G^\vee(\tilde{E}_i^m b^\vee) = 0, & \text{if } i \in \mathcal{I}^m, \end{cases}
\]
which is a contradiction to the choice of \( m \). Hence we conclude \( e_i^{n+1}G^\vee(b^\vee) = 0 \). \( \square \)

For \( b^\vee \in B(\infty)^\vee \), we define
\[
e_i^\alpha(b^\vee) = \min\{ n \in \mathbb{Z}_{\geq 0} \mid \tilde{E}_i^{n+1}b^\vee = 0 \},
\varphi_i^\alpha(b^\vee) = \min\{ n \in \mathbb{Z}_{\geq 0} \mid \tilde{E}_i^{n+1}b^\vee = 0 \}.
\]
Proposition 4.13. For $b^\vee \in B(\infty)^\vee$, we have

\begin{align*}
\epsilon_i G^\vee (b^\vee) &= \begin{cases} \left[ e_i^{\sigma}(b^\vee) \right] \sum_{k \leq i} E_{b^\vee, b^\vee} G^\vee (b^\vee) & \text{if } i \in I^{re}, \\
G^\vee (\tilde{E}_i b^\vee) + \sum_{i < b^\vee} E_{b^\vee, b^\vee} G^\vee (b^\vee) & \text{if } i \in I^{im}, \\
\end{cases} \\
f_i G^\vee (b^\vee) &= \begin{cases} 
q_i^{-\epsilon_i^{\sigma}(b^\vee) \epsilon_i^{\sigma}(b^\vee) + 1} G^\vee (\tilde{F}_i b^\vee) + \sum_{i \leq b^\vee} F_{b^\vee, b^\vee} G^\vee (b^\vee) & \text{if } i \in I^{re}, \\
\{ q_i^{2} e_i^{(n+1)} \} \sum_{i \leq b^\vee} G^\vee (\tilde{F}_i b^\vee) + \sum_{i \leq b^\vee} F_{b^\vee, b^\vee} G^\vee (b^\vee) & \text{if } i \in I^{im}.
\end{cases}
\end{align*}

for some $E_{b^\vee, b^\vee}^{i}, F_{b^\vee, b^\vee}^{i} \in \mathbb{Q}(q)$.

Proof. If $i \in I^{re}$, our assertions were proved in [23]. We will prove the case when $i \in I^{im}$. Set $n = \epsilon_i^{\sigma}(b^\vee)$. By Lemma 4.12 and Definition 4.6 (4), we have

\[ e_i^n G^\vee (b^\vee) = G^\vee (\tilde{E}_i b^\vee) = G^\vee (\tilde{E}_i^{n-1} \tilde{E}_i b^\vee) = e_i^{n-1} G^\vee (\tilde{E}_i b^\vee), \]

which implies

\[ e_i G^\vee (b^\vee) - G^\vee (\tilde{E}_i b^\vee) \in \text{Ker}(e_i^{n-1}). \]

Using the equation (1.2), we get

\[ e_i G^\vee (b^\vee) = (q_i^{2} e_i^{(n+1)} f_i G^\vee (b^\vee) + q_i^{2} e_i^{(n+1)} + q_i^{2} e_i^{(n+1)} G^\vee (b^\vee). \]

Hence Lemma 4.12 yields

\[ e_i^{(n+1)} f_i G^\vee (b^\vee) = \frac{1}{\{ n \}_1} q_i^{2} e_i^{(n+1)} G^\vee (b^\vee) = \frac{1}{\{ n \}_1} q_i^{2} e_i^{(n+1)} G^\vee (\tilde{E}_i b^\vee). \]

Using Lemma 4.12 again, we obtain

\[ \frac{1}{\{ n \}_1} q_i^{2} e_i^{(n+1)} G^\vee (\tilde{E}_i^{n+1} \tilde{F}_i b^\vee) = \frac{1}{\{ n \}_1} q_i^{2} e_i^{(n+1)} G^\vee (\tilde{F}_i b^\vee) = \{ n + 1 \}_1 q_i^{2} e_i^{(n+1)} G^\vee (\tilde{F}_i b^\vee). \]

Thus we have

\[ f_i G^\vee (b^\vee) - \{ n + 1 \}_1 q_i^{2} e_i^{(n+1)} G^\vee (\tilde{F}_i b^\vee) \in \text{Ker}(e_i^{n+1}) \]

as desired. \hfill \square

Combining Proposition 4.9 and Proposition 4.13, we obtain the existence of perfect basis for $U_q^{-} (\mathfrak{g})$. 

Proposition 4.14. $\mathbb{B}(\infty)$ is a perfect basis of the $B_q(\mathfrak{g})$-module $U_q^{-} (\mathfrak{g})$.

Let $B$ be a perfect basis of $U_q^{-} (\mathfrak{g})$. For $b \in B$, define $\text{wt}(b) = \mu$ if $b \in B_\mu$ and

\[ f_i (b) = \begin{cases} \delta_{i} (b') & \text{if } e_i (b') = b \\
0 & \text{otherwise}, \end{cases} \]

\[ \varphi_i (b) = \epsilon_i (b) + \langle h_i, \text{wt}(b) \rangle. \]
Then it is straightforward to verify that \((B, \text{wt}, e_i, f_i, \varepsilon_i, \varphi_i)\) is an abstract crystal. The graph obtained from the crystal \((B, \text{wt}, e_i, f_i, \varepsilon_i, \varphi_i)\) is called a perfect graph of \(U_q^- (g)\). The following proposition asserts that the perfect basis \(B(\infty)\) yields the crystal \(B(\infty)\).

**Proposition 4.15.** There exist crystal isomorphisms

\[
\mathbb{B}(\infty) \cong B(\infty)^\vee \cong B(\infty).
\]

**Proof.** Let \(\vee : B(\infty) \to B(\infty)^\vee\) defined by \(b \mapsto b^\vee\). Then

\[
\tilde{f}_b = b' \iff (\tilde{f}_b, b^\vee)_K = 1 \iff (b, \tilde{E}_i b^\vee)_K = 1 \iff b^\vee = \tilde{E}_i b^\vee \iff \tilde{F}_i b^\vee = b'^\vee.
\]

Hence we have \(B(\infty)^\vee \cong B(\infty)\) from Lemma 4.4 and Lemma 4.7.

By Proposition 4.13, we have \(\tilde{E}_i b^\vee = b'^\vee \iff e_i G^\vee (b^\vee) = G^\vee (\tilde{E}_i b^\vee) = G^\vee (b'^\vee)\).

Hence the map \(G^\vee\) gives a crystal isomorphism between \(\mathbb{B}(\infty)\) and \(B(\infty)^\vee\). \(\square\)

In the rest of this section, we will show that the perfect graph arising from any perfect basis of \(U_q^- (g)\) is isomorphic to the crystal \(B(\infty)\). Our argument follows the outline given in [19, Section 6].

Let \(B\) be a perfect basis of \(U_q^- (g)\). For each sequence \(i = (i_1, \ldots, i_m) \in I^m (m \geq 1)\), we define a binary relation \(\preceq_i\) on \(U_q^- (g) \setminus \{0\}\) as follows:

\[
\text{if } i = (i), \text{ then } v \preceq_i v' \iff \ell_i (v) \leq \ell_i (v')
\]

\[
\text{if } i = (i; i') \text{, then } v \preceq_i v' \iff \begin{cases} \ell_i (v) < \ell_i (v') & \text{or} \\ \ell_i (v) = \ell_i (v') & e_i^\text{top} (v) \leq e_i^\text{top} (v')(v'). \end{cases}
\]

We write \(v \equiv_i v'\) if \(v \preceq_i v'\) and \(v' \preceq_i v\). For a given \(i = (i_1, \ldots, i_m) \in I^m\), define the maps \(e_i^\text{top} : U_q^- (g) \to U_q^- (g)\) and \(e_i^\text{top} : B \to B \cup \{0\}\) as follows:

\[
e_i^\text{top} (v) = e_i^\text{top} (v) \text{ for } m = 1 \quad \text{and} \quad e_i^\text{top} = e_i^\text{top} \circ \cdots \circ e_i^\text{top} \text{ for } m > 1,
\]

\[
e_i^\text{top} (b) = e_i^\text{top} (b) \text{ for } m = 1 \quad \text{and} \quad e_i^\text{top} = e_i^\text{top} \circ \cdots \circ e_i^\text{top} \text{ for } m > 1.
\]

By Proposition 4.3, we identify \(\mathbb{Q}(q)\) with \(\{v \in U_q^- (g) | e_i (v) = 0 \text{ for all } i \in I\}\). Note that \(\mathbb{Q}(q) \cap B = \{1\}\). For each \(v \in U_q^- (g)\), there exists a sequence \(i\) such that \(e_i^\text{top} (v) \in \mathbb{Q}(q)\). From [19], one can check that the following statements hold.

**Lemma 4.16.** For any sequence \(i = (i_1, \ldots, i_m) \in I^m (m \geq 1)\), we have

1. \(e_i^\text{top} (b) \in \mathbb{Q}(q)^\times e_i^\text{top} (b)\) for any \(b \in B\),
2. if \(e_i^\text{top} (b) \in \mathbb{Q}(q)^\times \) for some \(b \in B\), then \(e_i^\text{top} (b) \in \mathbb{Q}(q)^\times\),
3. if \(b \equiv_i b'\) and \(e_i^\text{top} (b) = e_i^\text{top} (b')\), then \(b = b'\) for all \(b, b' \in B\).

**Definition 4.17.** Let \(B, B'\) be perfect bases of \(U_q^- (g)\). A perfect morphism \([\phi, \tilde{\phi}, c] : (U_q^- (g), B) \to (U_q^- (g), B')\) is a triple \((\phi, \tilde{\phi}, c)\), where

1. \(\phi : U_q^- (g) \to U_q^- (g)\) is a \(B(\mathfrak{g})\)-module endomorphism such that \(0 \notin \phi (B)\),
2. \(\tilde{\phi} : B \to B'\) is a map satisfying \(\tilde{\phi}(1) = \phi(1)\).
graded modules over $R$.

The crystal we define a crystal structure on $B$ similar argument in [19, Lemma 6.5], for a given proof.

Since the proof is almost the same as [19, Theorem 6.6], we only give a sketch of proof. By a construction of crystals $B$.

Let $\tilde{\phi} : B \to B'$ and a unique map $c : B \setminus \{1\} \to \mathbb{Q}(q)^\times$ satisfying $\tilde{\phi}(\mathbf{1}) = 1$ and

\[ b - c(b)\tilde{\phi}(b) \prec_i b \]

for each $b \in B \setminus \{1\}$ and any sequence $i = (i_1, \ldots, i_m)$ such that $e_i^{\text{top}}(b) \in \mathbb{Q}(q)$.

**Lemma 4.18.** Let $\phi$ be a $B_q(\mathfrak{g})$-endomorphism of $U_q^{-}(\mathfrak{g})$.

1. If a perfect morphism $[\phi, \tilde{\phi}, c]$ exists, then $\tilde{\phi}$ and $c$ are uniquely determined.
2. For a given perfect morphism $[\phi, \tilde{\phi}, c] : (U_q^{-}(\mathfrak{g}), B) \to (U_q^{-}(\mathfrak{g}), B')$, the map $\tilde{\phi}$ is a crystal morphism.

**Proof.** This lemma is essentially the same as [19] Lemma 6.3, Lemma 6.4. However, since our algebra $U_q^{-}(\mathfrak{g})$ is considered as a $B_q(\mathfrak{g})$-module, Proposition [1.3] plays a key role in proving this lemma. Then our assertions follow by a similar argument in [19]. □

Now we state and prove the main result of this section.

**Theorem 4.19.** Let $B$ and $B'$ be two perfect bases of $U_q^{-}(\mathfrak{g})$. Then the identity map $\text{id} : U_q^{-}(\mathfrak{g}) \to U_q^{-}(\mathfrak{g})$ induces a perfect isomorphism from $(U_q^{-}(\mathfrak{g}), B)$ to $(U_q^{-}(\mathfrak{g}), B')$. That is, there exists a unique crystal isomorphism $\tilde{\phi} : B \to B'$ and a unique map $c : B \setminus \{1\} \to \mathbb{Q}(q)^\times$ satisfying $\tilde{\phi}(\mathbf{1}) = 1$ and

\[ b - c(b)\tilde{\phi}(b) \prec_i b \]

for each $b \in B \setminus \{1\}$ and any sequence $i = (i_1, \ldots, i_m)$ with $e_i^{\text{top}}(b) = 1$.

**Proof.** Since the proof is almost the same as [19] Theorem 6.6, we only give a sketch of proof. By a similar argument in [19] Lemma 6.5, for a given $b \in B \setminus \{1\}$, one can show that there exist unique $b' \in B'$, $v \in U_q^{-}(\mathfrak{g})$ and $k \in \mathbb{Q}(q)^\times$ satisfying

1. $b \equiv_i b'$,
2. $b = v + kb'$,
3. $v = 0$ or $v \prec_i b$, $v \prec_i b'$

for any sequence $i$ with $e_i^{\text{top}}(b) \in \mathbb{Q}(q)^\times$. Then the maps $\text{id} : U_q^{-}(\mathfrak{g}) \to U_q^{-}(\mathfrak{g})$, $\tilde{\phi} : B \to B'$ and $c : B \setminus \{1\} \to \mathbb{Q}(q)^\times$ defined by $b \mapsto b'$ and $b \mapsto k$ give rise to a perfect isomorphism. □

5. Construction of crystals $\mathfrak{B}(\infty)$ and $\mathfrak{B}(\lambda)$

In this section, we investigate the crystal structures on the sets of isomorphism classes of irreducible graded modules over $R$ and its cyclotomic quotient $R^\lambda$. We assume that $a_{ii} \neq 0$ for all $i \in I$.

5.1. The crystal $\mathfrak{B}(\infty)$.

Let $\mathfrak{B}(\infty)$ be the set of isomorphism classes of irreducible graded $R$-modules. In this subsection, we define a crystal structure on $\mathfrak{B}(\infty)$ and show that it is isomorphic to the crystal $B(\infty)$ using the perfect basis theory given in Section [12]
Let $\alpha \in Q^+$. For any $P \in R(\alpha)$-mod and $M \in R(\alpha)$-fmod, we define

$$f_i(P) = \text{Ind}_{\alpha_i, \alpha}(P_i \boxtimes P), \quad e'_i(P) = P^* \otimes_{R(\alpha)} L(i),$$

$$F_i(M) = \text{Ind}_{\alpha_i, \alpha}(L(i) \boxtimes M), \quad E'_i(M) = \text{Res}^{\alpha_i \ominus \alpha_i}_\alpha \circ \Delta_i M,$$

where $R'(\alpha_i) := R(\alpha_i) \otimes 1_{\alpha - \alpha_i} \hookrightarrow R(\alpha_i) \otimes R(\alpha - \alpha_i) \subset R(\alpha)$. Here, the $(R(\alpha_i), R(\alpha - \alpha_i))$-bimodule structure of $P^*$ is given as follows: for $v \in P^*$, $r \in R(\alpha - \alpha_i)$ and $s \in R(\alpha_i)$,

$$r \cdot v := (1_{\alpha_i} \otimes r) v, \quad v \cdot s := \psi(s \otimes 1_{\alpha - \alpha_i}) v.$$

Since $f_i$ and $e'_i$ (resp. $F_i$ and $E'_i$) take projective modules to projective modules (resp. finite-dimensional modules to finite-dimensional modules), they induce the linear maps

$$f_i : K_0(R) \rightarrow K_0(R), \quad e'_i : K_0(R) \rightarrow K_0(R),$$

$$F_i : G_0(R) \rightarrow G_0(R), \quad E'_i : G_0(R) \rightarrow G_0(R).$$

Then we have the following lemma, which is the Khovanov-Lauda-Rouquier algebra version of the equation [1,3].

**Lemma 5.1.**

1. $e'_i f_j = \delta_{ij} + q_i^{-\alpha_i} f_j e'_i$ on $K_0(R)$.
2. $E'_i F_j = \delta_{ij} + q_i^{-\alpha_i} F_j E'_i$ on $G_0(R)$.

**Proof.** (1) Fix $i \in \text{Seq}(\alpha)$ and let $i' = (j) \ast i \in \text{Seq}(\alpha + \alpha_j)$. By the equation [1,2],

$$\Delta_i P_i \simeq \sum_{\text{shuffles of } (i) \text{ and } j} P_{(i)} \boxtimes P_{(j)(\deg((i), j, i'))}$$

$$\simeq \delta_{ij} P_{(i)} \boxtimes P_i + \sum_{\text{shuffles of } (i) \text{ and } k} P_{(i)} \boxtimes P_{(j) \ast k}(- \deg((i), k, i) + (\alpha_i | \alpha_j)),$$

which yields

$$e'_i f_j [P_i] = e'_i [P_i]$$

$$= [P_i^* \otimes_{R'(\alpha_j)} L(i)]$$

$$= [((\Delta_i P_i)^* \otimes_{R'(\alpha_j)} L(i))$$

$$= \delta_{ij} [P_i] + q^{-(\alpha_i | \alpha_j)} f_j [\text{Res}^{\alpha_i \ominus \alpha_i}_\alpha (P_i^* \otimes_{R'(\alpha_j)} L(i)))]$$

$$= \delta_{ij} [P_i] + q^{-(\alpha_i | \alpha_j)} f_j e'_i [P_i].$$

(2) For an irreducible $R(\alpha)$-module $M$, it follows from Proposition 2.6 that

$$E'_i F_j [M] = E'_i ([\text{Ind}_{\alpha_j, \alpha} L(j) \boxtimes M])$$

$$= [E'_i(L(j))][M] + [\text{Ind}_{\alpha_j, \alpha - \alpha_i} L(j) \boxtimes E'_i(M)((\alpha_j | \alpha_i))]$$

$$= \delta_{ij} [M] + q^{-(\alpha_i | \alpha_j)} F_j E'_i [M].$$

We also have analogues of the equation [1,3] and Lemma 1.4 (3).
Lemma 5.2.

1. For $[P], [Q] \in K_0(R)$, we have
   \[(e'_i[P], [Q]) = (1 - q^2)([P], f_i[Q]).\]

2. For $[P] \in K_0(R)$ and $[M] \in G_0(R)$, we have
   \[(f_i[P], [M]) = ([P], E'_i[M]), \quad (e'_i[P], [M]) = ([P], F_i[M]).\]

Proof. (1) Let $P, Q \in R(\alpha)$-mod. Then
   \[
   ([P], f_i[Q]) = \text{dim}_q(P^* \otimes_{R(\alpha + \alpha_i)} \text{Ind}_{P(i)}(Q)) \\
   = \text{dim}_q((\Delta_i P)^* \otimes_{R(\alpha)} (P(i) \boxtimes Q)) \\
   = (1 - q^2)^{-1} \text{dim}_q((e'_i P)^* \otimes_{R(\alpha)} Q) \\
   = (1 - q^2)^{-1}(e'_i[P], [Q]).
   \]

(2) Let $P \in R(\alpha)$-pmod and $M \in R(\alpha + \alpha_i)$-fmod. By definition, we have the first assertion:
   \[
   (f_i[P], [M]) = \text{dim}_q((\text{Ind}_{P(i)}(P) \boxtimes P)^* \otimes_{R(\alpha + \alpha_i)} M) \\
   = \text{dim}_q((P(i) \boxtimes P)^* \otimes_{R(\alpha)} \Delta_i M) \\
   = \text{dim}_q(P^* \otimes_{R(\alpha)} \text{Res}_{\alpha^*}^\alpha \Delta_i M) \\
   = ([P], E'_i[M]).
   \]

In a similar manner, we have
   \[
   (e'_i[P], [M]) = \text{dim}_q \left( ((P^* \otimes_{R(\alpha)} L(i)) \otimes_{R(\alpha)} M) \right) \\
   = \text{dim}_q \left( (\Delta_i P)^* \otimes_{R(\alpha)} L(i) \boxtimes M \right) \\
   = \text{dim}_q \left( P^* \otimes_{R(\alpha + \alpha_i)} \text{Ind}_{L(i)} M \right) \\
   = ([P], F_i[M]).
   \]

We now define a $B_q(\mathfrak{g})$-module structure on $K_0(R)_{Q(q)}$ and $G_0(R)_{Q(q)}$ as follows:
   \[
   e'_i \cdot [P] := e'_i[P], \quad f_i \cdot [P] := f_i[P] \quad \text{for } [P] \in K_0(R)_{Q(q)}, \]
   \[
   e'_i \cdot [M] := E'_i[M], \quad f_i \cdot [M] := F_i[M] \quad \text{for } [M] \in G_0(R)_{Q(q)}.
   \]

By the same argument as in the proof of \cite{22} Lemma 3.4.2, it follows from Lemma 5.1 Lemma 5.2 and Theorem 2.9 that $K_0(R)_{Q(q)}$ and $G_0(R)_{Q(q)}$ are well-defined $B_q(\mathfrak{g})$-modules. Consider the $B_q(\mathfrak{g})$-module homomorphism
   \[
   \Phi_{Q(q)}^\vee : U_q^{-}(\mathfrak{g}) \longrightarrow G_0(R)_{Q(q)}
   \]
   given by
   \[
   \Phi_{Q(q)}^\vee(f_i) = L(i) \quad \text{for } i \in I.
   \]
Then, by Theorem 3.15, we obtain the following diagram.

\[
\begin{array}{ccc}
\Phi_{\mathbb{Q}(q)} & : & U_q^{-}(\mathfrak{g}) \\
\downarrow_{\text{dual w.r.t. } (,)_K} & & \downarrow_{\text{dual w.r.t. } (,)} \\
\Phi_{\mathbb{Q}(q)}' & : & U_q^{+}(\mathfrak{g}) \\
\end{array}
\]

\[K_0(R)_{\mathbb{Q}(q)} \sim \leftarrow \leftarrow \]

\[G_0(R)_{\mathbb{Q}(q)} \sim \leftarrow \leftarrow \]

Therefore, \(K_0(R)_{\mathbb{Q}(q)}\) and \(G_0(R)_{\mathbb{Q}(q)}\) are well-defined \(B_q(\mathfrak{g})\)-modules, which are isomorphic to \(U_q^{-}(\mathfrak{g})\).

The following lemma is the Khovanov-Lauda-Rouquier algebra version of Proposition 1.13.

**Lemma 5.3.** Let \(M\) be an irreducible \(R(\alpha)\)-module and \(\varepsilon = \varepsilon_i(M)\). Then we have

\[E_i' [M] = \begin{cases} q_i^{-\varepsilon+1} [\varepsilon_i M] + \sum_k c_k [N_k] & \text{if } i \in I^e, \\ [\varepsilon_i M] + \sum_k c'_k [N'_k] & \text{if } i \in I^m, \end{cases}\]

where \(c_k, c'_k \in \mathbb{Q}(q)\) and \(\varepsilon_i(N_k), \varepsilon_i(N'_k) < \varepsilon - 1\).

**Proof.** If \(i = I^e\), then the assertion can be proved in the same manner as \([20, \text{Lemma } 3.9]\). Suppose that \(i \in I^m\). By Lemma 3.38,

\[\Delta_i M \simeq L(i^\varepsilon) \boxtimes N\]

for some irreducible module \(N\) with \(\varepsilon_i(N) = 0\). Then, from (3.5), we have an exact sequence

\[0 \to K \to \text{Ind}_{\alpha_i,\alpha-\varepsilon_i} L(i^\varepsilon) \boxtimes N \to M \to 0\]

for some \(R(\alpha)\)-module \(K\). Note that \(\varepsilon_i(K) < \varepsilon\).

On the other hand, it follows from \(\varepsilon_i(N) = 0\) and Lemma 3.38 that

\[[\Delta_i \text{Ind}_{\alpha_i,\alpha-\varepsilon_i} L(i^\varepsilon) \boxtimes N] = [\text{Ind}_{\alpha_i,\alpha-\varepsilon_i,\alpha-\varepsilon_i} L(i) \boxtimes L(i^{\varepsilon-1}) \boxtimes N].\]

By Lemma 3.37, Lemma 3.32, and Lemma 3.38, we have

\[\text{hd} (\text{Ind}_{\alpha_i,\alpha-\varepsilon_i} L(i) \boxtimes L(i^{\varepsilon-1}) \boxtimes N) \simeq L(i) \boxtimes (\tilde{\varepsilon}_i^{-1} N) \simeq L(i) \boxtimes \varepsilon_i M\]

and all the other composition factors of \(\text{Ind}_{\alpha_i,\alpha-\varepsilon_i} L(i) \boxtimes L(i^{\varepsilon-1}) \boxtimes N\) are of the form \(L(i) \boxtimes L\) with \(\varepsilon_i(L) < \varepsilon - 1\). Moreover, since \(\varepsilon_i(K) < \varepsilon\), all composition factors of \(\Delta_i(K)\) are of the form \(L(i) \boxtimes L'\) with \(\varepsilon_i(L') < \varepsilon - 1\). Therefore, applying the exact functor \(\Delta_i\) to (5.2), we have

\[E_i' [M] = [\varepsilon_i M] + \sum_k c'_k [N'_k]\]

for some \(R(\alpha)\)-modules \(N'_k\) with \(\varepsilon_i(N'_k) < \varepsilon - 1\). \(\square\)

For an element \([M] \in \mathfrak{B}(\infty)\), we define

\[\text{wt}([M]) = -\alpha \quad \text{if } M \in R(\alpha)\text{-mod},\]

\[\varepsilon_i([M]) = \begin{cases} \max \{ k \geq 0 \mid \varepsilon_i^{k} M \neq 0 \} & \text{if } i \in I^e, \\ 0 & \text{if } i \in I^m, \end{cases}\]

\[\varphi_i([M]) = \varepsilon_i(b) + \langle h_i, \text{wt}([M]) \rangle.\]
Then we have the following theorem.

**Theorem 5.4.** The sextuple \((\mathcal{B}(\infty), \text{wt}, \hat{e}_i, \hat{f}_i, \varepsilon_i, \varphi_i)\) becomes an abstract crystal, which is isomorphic to the crystal \(B(\infty)\) of \(U_q^- (\mathfrak{g})\).

**Proof.** It follows from Lemma 3.13 and Lemma 5.3 that the pair \((\mathcal{B}(\infty), \{\hat{e}_i\}_{i \in I})\) is a perfect basis for the \(B_q(\mathfrak{g})\)-module \(G_0(R)\mathbb{Q}(q)\). Hence by Theorem 4.19, \(\mathcal{B}(\infty)\) is isomorphic to \(B(\infty)\). □

### 5.2. Cyclotomic quotients \(R^\lambda\) and their crystals \(\mathcal{B}(\lambda)\).

In this subsection, we define the cyclotomic quotient \(R^\lambda\) of \(R\) for \(\lambda \in P^+\), and investigate the crystal structure on the set of isomorphism classes of irreducible \(R^\lambda\)-modules.

For \(\alpha \in Q^+\) with \(|\alpha| = d\) and \(\lambda \in P^+\), let \(I^\lambda(\alpha)\) denote the two-sided ideal of \(R(\alpha)\) generated by

\[
\{x_d^{(h_{i\alpha}, \lambda)} I_i | i = (i_1, \ldots, i_d) \in \text{Seq}(\alpha)\}
\]

Note that it is defined in the opposite manner to [24, Section 3.4]. We define \(R^\lambda(\alpha) = R(\alpha)/I^\lambda(\alpha)\).

The algebra \(R^\lambda := \bigoplus_{\alpha \in Q^+} R^\lambda(\alpha)\) is called the **cyclotomic Khovanov-Lauda-Rouquier algebra** of weight \(\lambda\). For an irreducible \(R(\alpha)\)-module \(M\), let

\[
\varepsilon_i^\gamma(M) = \max\{k \geq 0 | 1_{\alpha-k\alpha, \lambda} M \neq 0\}.
\]

This definition is also the opposite to [27, (5.6)]. Combining Lemma 3.7 and Lemma 3.12 with (2.8) and the fact that \(x_m^k L(i_m) = 0\) for \(k \geq m, \ i \in I^e\), we obtain

\[
I^\lambda(\alpha) \cdot M = 0 \text{ if and only if } \begin{cases} 
\varepsilon_i^\gamma(M) \leq \langle h_i, \lambda \rangle & \text{for } i \in I^e, \\
\varepsilon_i^\gamma(M) = 0 & \text{for } i \in I^m \text{ with } \langle h_i, \lambda \rangle = 0,
\end{cases}
\]

where \(M\) is an irreducible \(R(\alpha)\)-module.

**Lemma 5.5.** Let \(M\) be an irreducible \(R(\alpha)\)-module.

1. For \(i \in I\), either \(\varepsilon_i^\gamma(\hat{f}i M) = \varepsilon_i^\gamma(M)\) or \(\varepsilon_i^\gamma(M) + 1\).
2. For \(i, j \in I\) with \(i \neq j\), we have \(\varepsilon_i^\gamma(\hat{f}_j M) = \varepsilon_i^\gamma(M)\).

**Proof.** The proof is the same as that of [27, Proposition 6.2]. □

For \(M \in R^\lambda(\alpha)\)-fmod and \(N \in R(\alpha)\)-fmod, let \(\text{infl}^\lambda M\) be the inflation of \(M\), and \(\text{pr}^\lambda N\) be the quotient of \(N\) by \(I^\lambda(\alpha) N\). Let \(\mathcal{B}(\lambda)\) denote the set of isomorphism classes of irreducible graded
$R^\lambda$-modules. For $M \in R^\lambda(\alpha)$-fmod, define

$$
\text{wt}^\lambda(M) = \lambda - \alpha,
\tilde{e}_i^\lambda M = \text{pr}^\lambda \circ \tilde{e}_i \circ \text{infl}^\lambda M,
\tilde{f}_i^\lambda M = \text{pr}^\lambda \circ \tilde{f}_i \circ \text{infl}^\lambda M,
$$

(5.5)

$$
\varepsilon_i^\lambda(M) = \begin{cases} 
\max\{k \geq 0 \mid (\tilde{e}_i^\lambda)^k M \neq 0\} & \text{for } i \in \check{I}^e, \\
0 & \text{for } i \in \check{I}^m, 
\end{cases}
$$

$$
\varphi_i^\lambda(M) = \begin{cases} 
\max\{k \geq 0 \mid (\tilde{f}_i^\lambda)^k M \neq 0\} & \text{for } i \in \check{I}^e, \\
\langle h_i, \text{wt}^\lambda(M) \rangle & \text{for } i \in \check{I}^m.
\end{cases}
$$

We will show that $(\mathcal{B}(\lambda), \text{wt}^\lambda, \tilde{e}_i^\lambda, \tilde{f}_i^\lambda, \varepsilon_i^\lambda, \varphi_i^\lambda)$ is an abstract crystal. For this purpose, we need several lemmas.

**Lemma 5.6.** Let $i \in \check{I}^e$ and $\lambda, \mu \in P^+$. For $[M], [N] \in \mathcal{B}(\infty)$ with $\text{pr}^\lambda M \neq \emptyset$, $\text{pr}^\lambda N \neq \emptyset$, $\text{pr}^\mu M \neq \emptyset$, $\text{pr}^\mu N \neq \emptyset$, we have

$$
\varphi_i^\lambda(M) - \varphi_i^\lambda(N) = \varphi_i^\mu(M) - \varphi_i^\mu(N).
$$

**Proof.** The assertion can be proved in the same manner as in [27] Proposition 6.6, Remark 6.7. □

**Lemma 5.7.** Let $i \in \check{I}^e$ and $j \in I$ with $a_{ij} < 0$.

1. If $m \leq -a_{ij}$, then for each $0 \leq k \leq m$, there exists a unique irreducible $R(m\alpha_i + \alpha_j)$-module $L(i^k j^{m-k})$ with

$$
\varepsilon_i(L(i^k j^{m-k})) = k \quad \text{and} \quad \varepsilon_i^\vee(L(i^k j^{m-k})) = m - k.
$$

2. If $0 \leq k \leq -a_{ij}$, then the module

$$
\text{Ind}L(i^s) \boxtimes L(i^k j^{-a_{ij}}) \simeq \text{Ind}L(i^k j^{-a_{ij}}) \boxtimes L(i^s)
$$

is irreducible for all $s \geq 0$.

3. If $0 \leq k \leq -a_{ij} \leq c$ and $N$ is an irreducible $R(c\alpha_i + \alpha_j)$-module with $\varepsilon_i(N) = k$, then we have $c + a_{ij} \leq k \leq c$ and

$$
N \simeq \text{Ind}L(i^{c+a_{ij}}) \boxtimes L(i^{k-c-a_{ij}}).
$$

**Proof.** To prove (1), we consider the induced module $\text{Ind}L(i^k) \boxtimes L(j) \boxtimes L(i^{m-k})$ for $0 \leq k \leq m$. Let

$$
K = \text{Span}_F \{ \tau_w \otimes (t \otimes u \otimes v) \mid w \in S_{m+1}, \ell(w) > 0, t \in L(i^k), u \in L(j), v \in L(i^{m-k}) \}.
$$

By the same argument as in [27] Proposition 6.11, we deduce that $K$ is a proper maximal submodule of $\text{Ind}L(i^k) \boxtimes L(j) \boxtimes L(i^{m-k})$, and that $\text{hd} \text{Ind}L(i^k) \boxtimes L(j) \boxtimes L(i^{m-k})$ is the quotient module $\text{Ind}L(i^k) \boxtimes L(j) \boxtimes L(i^{m-k})/K$ which is irreducible. We denote it by $L(i^k j^{m-k})$. By the Frobenius reciprocity (8.1), we have

$$
\varepsilon_i(L(i^k j^{m-k})) = k \quad \text{and} \quad \varepsilon_i^\vee(L(i^k j^{m-k})) = m - k.
$$

(5.6)
On the other hand, there is a surjective homomorphism of degree 0
\[ \text{Ind} L(i^k) \otimes L(j) \otimes L(i^{m-k}) \rightarrow \tilde{e}_i^k \tilde{e}_j \tilde{e}_i^{m-k} 1, \]
which implies that \( L(i^k j i^{m-k}) \simeq \tilde{e}_i^k \tilde{e}_j \tilde{e}_i^{m-k} 1 \). By Theorem 5.3
\[ \{ \tilde{e}_i^k \tilde{e}_j \tilde{e}_i^{m-k} 1 \mid 0 \leq k \leq m \} \]
is a complete set of irreducible \( R(m \alpha_i + \alpha_j) \)-module. Therefore, \( L(i^k j i^{m-k}) \) is a unique irreducible \( R(m \alpha_i + \alpha_j) \)-module satisfying (5.6).

The assertion (2), (3) can be proved by the same argument as in [27, Theorem 6.10]. □

Fix \( i \in I^e \) and \( j \in I \) with \( i \neq j, a_{ij} \neq 0 \) and let
\[ \mathcal{L}(k) = L(i^k j i^{a_{ij} - k}) \] for \( 0 \leq k \leq -a_{ij} \).

**Lemma 5.8.** Let \( c, d \in \mathbb{Z}_{\geq 0} \) with \( c + d \leq -a_{ij} \).

1. We have
\[ \text{hdInd}(i^m) \otimes L(i^c j i^d) \simeq \tilde{f}_i^m L(i^c j i^d) \simeq \tilde{f}_i^{m+c} L(j)^d \]
\[ \simeq \begin{cases} 
\text{Ind} L(i^{m+a_{ij}+c+d}) \otimes \mathcal{L}(-a_{ij} - d) & \text{if } m \geq -a_{ij} - c - d, \\
\mathcal{L}(i^{m+c} j i^d) & \text{if } m < -a_{ij} - c - d.
\end{cases} \]

2. Suppose that there is a nonzero homomorphism
\[ \text{Ind} L(i^m) \otimes \mathcal{L}(c_1) \otimes \cdots \otimes \mathcal{L}(c_r) \rightarrow Q \]
where \( Q \) is irreducible. Then
\[ \varepsilon_i(Q) = m + \sum_{t=1}^r c_t \text{ and } \varepsilon_i^\vee(Q) = m + \sum_{t=1}^r (-a_{ij} - c_t). \]

3. Let \( M \) and \( Q \) be irreducible. Suppose that there is a nonzero homomorphism \( \text{Ind}\mathcal{L}(k) \otimes M \rightarrow Q \). Then \( \varepsilon_i(Q) = \varepsilon_i(M) + k. \)

**Proof.** The proof is identical to that of [27, Lemma 6.13]. □

**Lemma 5.9.**

1. If \( N \) is an irreducible \( R(c \alpha_i + a \alpha_j) \)-module with \( \varepsilon_i(N) = 0 \), then there exist \( r \in \mathbb{Z}_{>0} \) and \( b_t \leq -a_{ij} \) for \( 1 \leq t \leq r \) such that
\[ \text{Ind} L(j i^{b_1}) \otimes \cdots \otimes L(j i^{b_r}) \rightarrow N. \]

2. Let \( a := -a_{ij} \). Suppose that we have a surjective homomorphism
\[ \text{Ind} L(j^b) \otimes L(j i^{b_1}) \otimes \cdots \otimes L(j i^{b_r}) \rightarrow Q, \]
where \( Q \) is irreducible.
(a) If $h \geq \sum_{i=1}^{r}(a - b_i)$, then we have a surjective homomorphism

$$\text{Ind} L(i^9) \otimes L(a - b_1) \otimes \cdots \otimes L(a - b_r) \to Q,$$

where $g := h - \sum_{i=1}^{r}(a - b_i)$.

(b) Otherwise, we have

$$\text{Ind} L(a - b_1) \otimes \cdots \otimes \text{Ind} L(a - b_{s-1}) \otimes L(j^i j^b_1) \otimes L(j^{s+1}) \otimes \cdots \otimes L(j^{b_1}) \to Q,$$

where $g' = h - \sum_{i=1}^{s-1}(a - b_i)$ and $s$ is such that

$$\sum_{i=1}^{s-1}(a - b_i) \leq h < \sum_{i=1}^{s}(a - b_i).$$

Proof. The assertions can be proved in the same manner as in [27, Lemma 6.14, Lemma 6.15].

**Proposition 5.10.** Let $i \in I^e$ and $j \in I$ with $i \neq j$. Let $M$ be an irreducible $R(c_{a_j} + d\alpha_j)$-module, and $\lambda \in P^+$ such that $\text{pr}^\lambda(M) \neq 0$ and $\text{pr}^\lambda(f_j M) \neq 0$. Then we have

$$\varepsilon_i^\lambda(f_j M) = \varepsilon_i^\lambda(M) + a_{ij} + k, \quad \varphi_i^\lambda(f_j M) = \varphi_i^\lambda(M) + k$$

for some $0 \leq k \leq -a_{ij}$.

Proof. Using the argument in [27, Theorem 6.19] with Lemma 5.6, Lemma 5.7, Lemma 5.8 and Lemma 5.9 our assertion follows.

**Proposition 5.11.** Let $i \in I^e$, and $M$ be an irreducible $R(\alpha)$-module with $\text{pr}^\lambda(M) \neq 0$.

1. For $j \in I$ with $i \neq j$, we have

$$\varphi_i^\lambda(f_j M) - \varepsilon_i^\lambda(f_j M) = -\langle h_i, \alpha_j \rangle + \varphi_i^\lambda(M) - \varepsilon_i^\lambda(M).$$

2. Moreover, we have

$$\varphi_i^\lambda(M) = \varepsilon_i^\lambda(M) + (h_i, \text{wt}^\lambda(M)).$$

Proof. Combining [27, Proposition 6.20] with Proposition 5.10 we obtain the assertion (1). Since $\varphi_i^\lambda(1) = \varepsilon_i^\lambda(1) + \langle h_i, \lambda \rangle$, the assertion (2) follows by induction on $|\alpha|$ combined with the assertion (1).

Combining Proposition 5.11 with (5.5), we obtain the following proposition.

**Proposition 5.12.** The sextuple $(\mathfrak{B}(\lambda), \text{wt}^\lambda, \varepsilon_i^\lambda, f_i^\lambda, \varphi_i^\lambda)$ is an abstract crystal.

We would like to show that $\mathfrak{B}(\lambda)$ is isomorphic to the crystal $B(\lambda)$. For this purpose, we first prove the following lemma.

**Lemma 5.13.** Let $i \in I^e$ and $M$ be an irreducible $R^\lambda(\alpha)$-module. Then

$$\langle h_i, \text{wt}^\lambda(M) \rangle \leq 0 \quad \text{if and only if} \quad f_i^\lambda M = 0.$$
Proof. Let $\alpha = \sum_{j \in I} k_j \alpha_j$ with $|\alpha| = d$. For simplicity, we identify $M$ with $\text{infl}^\lambda M$.

We first assume that $\langle h_i, \text{wt}^\lambda(M) \rangle \leq 0$. Since $\langle h_i, \lambda \rangle \geq 0$ and $\langle h_i, -\alpha_j \rangle \geq 0$ for all $j \in I$, we have
\[ \langle h_i, \lambda \rangle = 0 \quad \text{and} \quad k_j = 0 \text{ for } j \in I \text{ with } a_{ij} \neq 0. \]

Take an element $\mathbf{j} = (j_1 \ldots j_d) \in \text{Seq}(\alpha)$ such that $1_{\mathbf{j}} M \neq 0$. Note that $a_{ij_k} = 0$ for all $k = 1, \ldots, d$. By the Frobenius reciprocity (3.5), we have an embedding
\[ L(i) \boxtimes M \hookrightarrow \Delta_i \tilde{f}_i M, \]
which implies that $1_{(i)\mathbf{j}} (\tilde{f}_i M) \neq 0$. Since $a_{ij_1} = 0$, it follows from the quantum Serre relations that
\[ 1_{(i)j_2 \ldots \mathbf{j}d} (\tilde{f}_i M) \neq 0. \]

Repeating this process, we have
\[ 1_{\mathbf{j}_0} (\tilde{f}_i M) \neq 0, \]
which yields that $I^\lambda(\alpha + \alpha_i) \tilde{f}_i M \neq 0$ since $1_{\mathbf{j}_0} \in I^\lambda(\alpha + \alpha_i)$. Therefore, we have the only if part of our assertion.

We now prove the converse. We will actually prove the contrapositive:
\[ \langle h_i, \text{wt}^\lambda(M) \rangle > 0 \quad \Rightarrow \quad \tilde{f}_i^\lambda M \neq 0. \]

Assume that $\langle h_i, \text{wt}^\lambda(M) \rangle > 0$.

First consider the case $\langle h_i, -\alpha \rangle = 0$. In this case, $\langle h_i, \lambda \rangle > 0$ and $k_j = 0$ for $j \in I$ with $a_{ij} \neq 0$. Take a nonzero element $v \in L(i)$. By definition, we have
\[ \text{Ind}L(i) \boxtimes M = \text{Span}_{\mathbb{F}}\{r_1 \cdots r_t \otimes (v \otimes m) \mid m \in M, \ 0 \leq t \leq d\}. \]

Since $I^\lambda(\alpha) M = 0$ and $k_j = 0$ for $j \in I$ with $a_{ij} \neq 0$, it follows from the definition (5.3) that
\[ I^\lambda(\alpha + \alpha_i)(\text{Ind}L(i) \boxtimes M) = 0. \]

Hence we have $\tilde{f}_i^\lambda M \neq 0$.

Now we suppose that $\langle h_i, -\alpha \rangle > 0$. Take a nonzero element $v$ in $L(i)$, and define $N$ to be the submodule of $\text{Ind}L(i) \boxtimes M$ generated by
\[ N = \{ x_{d+1}^{(h_i, \lambda)} \tau_d \cdots \tau_1 1_{(i)\mathbf{k}} \otimes (v \otimes m) \mid 0 \leq t \leq d, \ m \in M, \ \mathbf{k} \in \text{Seq}(\alpha) \}. \]

As $\langle h_i, -\alpha \rangle > 0$, we have
\[ \deg(x_{d+1}^{(h_i, \lambda)} \tau_d \cdots \tau_1 1_{(i)\mathbf{k}} \otimes (v \otimes m)) > \deg(1 \otimes v \otimes m). \]

Then, as $M$ is $R^\lambda(\alpha)$-module, we have $I^\lambda(\alpha + \alpha_i)(\text{Ind}L(i) \boxtimes M) \subset N$. Hence $(\text{Ind}L(i) \boxtimes M)/N$ is $R^\lambda(\alpha + \alpha_i)$-module. To prove $\tilde{f}_i^\lambda M \neq 0$, it suffices to show that $(\text{Ind}L(i) \boxtimes M)/N$ is nontrivial; i.e., $N$ is proper.
Take $m_0 \in M$ such that $\deg(m_0) \leq \deg(m)$ for all $m \in M$. We claim that $1 \otimes (v \otimes m_0) \notin N$. Suppose that $1 \otimes (v \otimes m_0) \in N$. Since $I^\lambda(\alpha)M = 0$, it follows from the defining relations \[ \eqref{eq:2.1} \] and \[ \eqref{eq:2.2} \] that
\[
x_r(x_{d+1}^{(h_i, \lambda)} \tau_d \cdots \tau_1 1_{(i)k} \otimes (v \otimes m)) = x_{d+1}^{(h_i, \lambda)} \tau_d \cdots \tau_1 (x_{r+1}^{(h_i, \lambda)} \tau_d \cdots \tau_1 (x_{d+1}^{(h_i, \lambda)} \tau_d \cdots \tau_1 1_{(i)k} \otimes (v \otimes m)),
\]
\[
x_s(x_{d+1}^{(h_i, \lambda)} \tau_d \cdots \tau_1 1_{(i)k} \otimes (v \otimes m)) = x_{d+1}^{(h_i, \lambda)} \tau_d \cdots \tau_1 (x_{d+1}^{(h_i, \lambda)} \tau_d \cdots \tau_1 1_{(i)k} \otimes (v \otimes m))
\]
for $m \in M$, $1 \leq r \leq d$ and $1 \leq s \leq d - 1$. So, the element $1 \otimes (v \otimes m_0)$ can be written as
\[
1 \otimes (v \otimes m_0) = \sum_j \tau_{i_j} \tau_{i_j+1} \cdots \tau_d x_{d+1}^k n_j,
\]
for some $n_j \in \mathbb{N}$, $t_j, k \in \mathbb{Z}_{\geq 0}$. Since $\langle h_i, -\alpha \rangle > 0$ and $m_0$ is minimal, we have
\[
\deg(1 \otimes (v \otimes m_0)) < \deg(n_j) \leq \deg(\tau_{i_j} \tau_{i_j+1} \cdots \tau_d x_{d+1}^k n_j),
\]
which gives a contradiction. Therefore, $1 \otimes (v \otimes m_0)$ is not contained in $N$ and $N$ is proper. □

We are now ready to state and prove the crystal version of categorification of $V(\lambda)$. Define a map $\Psi_\lambda : \mathcal{B}(\lambda) \to \mathcal{B}(\infty) \otimes T_\lambda \otimes C$ by
\[
[M] \mapsto [\text{infl}^M] \otimes t_\lambda \otimes c.
\]

**Theorem 5.14.**

1. $\Psi_\lambda$ is a strict crystal embedding.
2. The crystal $\mathcal{B}(\lambda)$ is isomorphic to the crystal $B(\lambda)$.

**Proof.** To prove (1), let $M$ be an irreducible $R^\lambda(\alpha)$-module and let $M_0 = \text{infl}^\lambda M$. Note that
\[
\varepsilon_i^\lambda(M) = \varepsilon_i(M_0), \quad \varphi_i(M_0) = \varepsilon_i(M_0) + \langle h_i, \lambda \rangle = \varphi_i^\lambda(M) \geq 0.
\]
By the tensor product rule \[ \eqref{eq:4.1} \] and Proposition \[ \ref{prop:5.12} \] we have
\[
\text{wt}(\Psi_\lambda(M)) = \text{wt}(M_0 \otimes t_\lambda \otimes c) = \lambda - \alpha = \text{wt}^\lambda(M),
\]
\[
\varepsilon_i(\Psi_\lambda(M)) = \varepsilon_i(M_0 \otimes t_\lambda \otimes c) = \max\{\varepsilon_i(M_0), -\langle h_i, \lambda - \alpha \rangle\} = \varepsilon_i^\lambda(M),
\]
\[
\varphi_i(\Psi_\lambda(M)) = \varphi_i(M_0 \otimes t_\lambda \otimes c) = \max\{\varphi_i(M_0) + \langle h_i, \lambda \rangle, 0\} = \varphi_i^\lambda(M).
\]
On the other hand, it follows from Lemma \[ \ref{lem:5.13} \] that
\[
\langle h_i, \lambda - \alpha + \alpha_i \rangle \leq 0 \implies \hat{e}_i^\lambda M = 0.
\]
By a direct computation, we have
\[
\hat{f}_i(M_0 \otimes t_\lambda \otimes c) = \begin{cases} 
    (\hat{f}_iM_0) \otimes t_\lambda \otimes c & \text{if } \varphi_i^\lambda(M) > 0, \\
    0 & \text{if } \varphi_i^\lambda(M) \leq 0,
\end{cases}
\]
\[
\hat{e}_i(M_0 \otimes t_\lambda \otimes c) = \begin{cases} 
    (\hat{e}_iM_0) \otimes t_\lambda \otimes c & \text{if } i \in I^m, \ \varphi_i^\lambda(M) \geq 0, \\
    (\hat{e}_iM_0) \otimes t_\lambda \otimes c & \text{if } i \in I^m, \langle h_i, \lambda - \alpha + \alpha_i \rangle > 0, \\
    0 & \text{if } i \in I^m, \langle h_i, \lambda - \alpha + \alpha_i \rangle \leq 0.
\end{cases}
\]
By (5.7) and Lemma 5.13 we get
\[
\tilde{e}_i(\Psi_\lambda(M)) = \Psi_\lambda(\tilde{e}_i^\lambda(M)) \quad \text{and} \quad \tilde{f}_i(\Psi_\lambda(M)) = \Psi_\lambda(\tilde{f}_i^\lambda(M)),
\]
which completes the proof of (1).

Since \(\Psi_\lambda\) takes 1 to \(1 \otimes t_\lambda \otimes c\), the assertion (2) follows from (1) and Proposition 4.4.

\[\square\]

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