Better upper bounds on the Füredi–Hajnal limits of permutations

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Abstract

A binary matrix is a matrix with entries from the set \{0, 1\}. We say that a binary matrix \(A\) contains a binary matrix \(S\) if \(S\) can be obtained from \(A\) by removal of some rows, some columns, and changing some 1-entries to 0-entries. If \(A\) does not contain \(S\), we say that \(A\) avoids \(S\). A k-permutation matrix \(P\) is a binary \(k \times k\) matrix with exactly one 1-entry in every row and one 1-entry in every column.

The Füredi–Hajnal conjecture, proved by Marcus and Tardos, states that for every permutation matrix \(P\), there is a constant \(c_P\) such that for every \(n \in \mathbb{N}\), every \(n \times n\) binary matrix \(A\) with at least \(c_P n\) 1-entries contains \(P\).

We show that \(c_P \leq 2^{O(k^2/3 \log^{7/3} k / (\log \log k)^{1/3})}\) asymptotically almost surely for a random \(k\)-permutation matrix \(P\). We also show that \(c_P \leq 2^{(4+o(1))k}\) for every \(k\)-permutation matrix \(P\), improving the constant in the exponent of a recent upper bound on \(c_P\) by Fox.

We also consider a higher-dimensional generalization of the Stanley–Wilf conjecture about the number of \(d\)-dimensional \(n\)-permutation matrices avoiding a fixed \(d\)-dimensional \(k\)-permutation matrix, and prove almost matching upper and lower bounds of the form \((2^k)^{O(n)} \cdot (n!)^{d-1/(d-1)}\) and \(n^{-O(k) \cdot (n!)^{d-1/(d-1)}}\), respectively.

1 Introduction

A binary matrix is a matrix with entries from the set \{0, 1\}. We say that an \(n \times n\) binary matrix \(A\) contains a \(k \times k\) binary matrix \(B\) if \(B\) can be obtained from \(A\) by removing some rows, some columns and by changing some 1-entries to 0-entries. If \(A\) does not contain \(B\), we say that \(A\) avoids \(B\).

For every \(n \in \mathbb{N}\), we abbreviate the set \(\{1, 2, \ldots, n\}\) as \([n]\). A k-permutation \(\pi\) is a permutation on \([k]\), that is, a bijective function \(\pi : [k] \to [k]\). We will also sometimes represent a permutation by the sequence of the function values, that is, as \((\pi(1), \pi(2), \ldots, \pi(k))\). A permutation matrix is a square binary matrix with exactly one 1-entry in every row and in every column. A \(k \times k\) permutation matrix is also called a k-permutation matrix. A k-permutation matrix \(P\) corresponds to the k-permutation \(\pi\) satisfying, for every \(i, j \in [k]\), \(\pi(i) = j\) if and only if \(P_{ij} = 1\). Note that by this definition, a graph of \(\pi\) as a function is obtained by rotating \(P\) by 90 degrees counterclockwise.

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The **restriction** of an $n$-permutation $\rho$ on a set \{${s_1, s_2, \ldots, s_l}$\} of positions, where $1 \leq s_1 < s_2 < \cdots < s_l \leq n$, is the $l$-permutation $\pi$ where $\pi(i) < \pi(j)$ if and only if $\rho(s_i) < \rho(s_j)$ for every $i, j \in [l]$. If $\pi$ is not a restriction of $\rho$ on any set of positions, then we say that $\rho$ **avoids** $\pi$. By definition, a permutation $\pi$ is a restriction of a permutation $\rho$ if and only if the permutation matrix $Q$ corresponding to $\rho$ contains the permutation matrix $P$ corresponding to $\pi$.

For a binary matrix $A$ and $n \in \mathbb{N}$, let $\text{ex}(n, A)$ be the maximum number of 1-entries in an $n \times n$ binary matrix avoiding $A$. The Füredi–Hajnal conjecture [12], proved by Marcus and Tardos [19], states that for every permutation matrix $P$, there is a constant $c$ such that for every $n \in \mathbb{N}$, we have $\text{ex}(n, P) \leq cn$.

For a permutation matrix $P$ and $n \in \mathbb{N}$, let $S_P(n)$ be the number of $n$-permutation matrices that avoid $P$. In other words, $S_P(n)$ is the number of $n$-permutations avoiding $\pi$, where $\pi$ is the permutation corresponding to $P$. The Stanley–Wilf conjecture states that for every permutation matrix $P$, there is a constant $s$ such that $|S_P(n)| \leq s^n$ for every $n \in \mathbb{N}$. The validity of the conjecture follows from the validity of the Füredi–Hajnal conjecture by an earlier result of Klazar [16].

Fix a permutation matrix $P$. Arratia [2] showed by the supermultiplicativity of $\text{ex}(n, P)$ [20, Lemma 1(ii)] together with the Füredi–Hajnal conjecture imply the same conclusion for the limit

$$s_P = \lim_{n \to \infty} |S_P(n)|^{1/n}$$

exists and is finite. Similarly, the superadditivity of $\text{ex}(n, P)$ [20, Lemma 1(ii)] together with the Stanley–Wilf limit of the permutation $\pi$ corresponding to $P$.

The Marcus–Tardos proof [19] of the Füredi–Hajnal conjecture implies the upper bound $c_P \leq 2k^2(\log k)^2$ for every $k$-permutation matrix $P$. Klazar’s reduction shows that $s_P \leq 15^{c_P}$ for every permutation matrix $P$, thus showing $s_P \leq 2^{2^{O(k \log(k))}}$ for every $k$-permutation matrix $P$. The first author [8] showed that the values of the two limits are close to each other, in particular, $\Omega(c_P^{2/9}) \leq s_P \leq 2.88c_P^2$ for every permutation matrix $P$. Fox [11] improved the upper bound on the Füredi–Hajnal limit of $k$-permutation matrices to $c_P \leq 3k2^{8k}$ (which can be easily lowered to $c_P \leq k^{O(1)}2^{8k}$). Thus both $s_P$ and $c_P$ are in $2^{O(k)}$, where $k$ is the size of $P$.

The Stanley–Wilf limit of the identity $k$-permutation is $(k - 1)^2$ [21]. By a result of Valtr published in [15], for every $k$ and every $k$-permutation matrix $P$, $s_P \geq (k - 1)^2/e^3$. Let $s_{1324}$ be the Stanley–Wilf limit of the permutation $(1, 3, 2, 4)$. Albert et al. [1] proved the lower bound $s_{1324} \geq 9.47$. Bóna [5] proved that there are infinitely many permutation matrices $P$ with $s_P \geq s_{1324} \cdot (k - 1)^2/9$. Bevan [3] increased the lower bound on $s_{1324}$ to 9.81, thus increasing the lower bound for infinitely many permutation matrices $P$ to $s_P \geq 9.81(k - 1)^2/9$.

Until recently, no $k$-permutation has been known to have the Stanley–Wilf limit larger than quadratic in $k$, which lead to the widely believed conjecture that the Stanley–Wilf limits are always at most quadratic in $k$; see the survey by Steingrímsson [23]. This belief was further supported by the fact that the Stanley–Wilf limit of every layered $k$-permutation is bounded
from above by $4k^2$ [9] (a layer permutation is a concatenation of decreasing sequences $S_1, S_2, \ldots, S_t$ such that for every $i \leq l - 1$, all elements of $S_i$ are smaller than all elements of $S_{i+1}$).

The situation concerning the Füredi–Hajnal limit was similar. A simple observation gives the lower bound $c_P \geq 2(k - 1)$ for all $k$-permutation matrices and this lower bound is attained by the $k \times k$ unit matrix and several other $k$-permutation matrices [12]. The best lower bound on the Füredi–Hajnal limit of some class of permutations was quadratic in the size of the permutations [8].

A breakthrough occurred when Fox [11] gave a randomized construction showing that for every $k$, there are $k$-permutation matrices $P$ with $c_P \geq 2^{\Omega(k^{1/2})}$ and thus $s_P \geq 2^{\Omega(k^{1/2})}$. He additionally showed that as $k$ goes to infinity, almost all $k$-permutation matrices satisfy $c_P \geq 2^{\Omega((k/\log k)^{1/2})}$.

**Contractions and interval minors.** Contracting rows, columns and blocks is a crucial technique for studying permutation avoidance. Let $A$ be an $n \times n$ binary matrix with rows $r_1, r_2, \ldots, r_n$, in this order. A partition $\mathcal{I} = \{I_1, I_2, \ldots, I_t\}$ of the set of rows of $A$ is called an interval decomposition of the rows of $A$ if each of the sets $I_j$ consists of a nonzero number of consecutive rows, and $i < i'$ whenever $j < j'$, $r_i \in I_j$ and $r_i' \in I_{j'}$. The sets $I_j$ are called the intervals of the decomposition. An interval decomposition of the columns is defined analogously.

A block decomposition of $A$ is determined by a row decomposition $\mathcal{I} = \{I_1, I_2, \ldots, I_t\}$ and a column decomposition $\mathcal{T} = \{I'_1, I'_2, \ldots, I'_{t'}\}$ as follows. For every $i \in [t]$ and $j \in [t']$, the $(i, j)$-block of $A$ is the submatrix of $A$ on the intersection of $I_i$ and $I'_j$. To contract the blocks means to create a $t \times t'$ binary matrix $B$ such that $B_{i,j} = 0$ if and only if the $(i, j)$-block of $A$ contains only zeros. To contract by an interval decomposition of rows means to create a matrix with one row for each interval where each row has 0-entries exactly in those columns where the corresponding interval has only zeros. Contraction by an interval decomposition of columns is defined analogously.

A binary matrix $B$ is an interval minor of a binary matrix $A$ if $B$ can be obtained from $A$ by the contraction of blocks of some block decomposition followed possibly by replacing some 1-entries with 0-entries. Although contractions were used earlier, the interval minors were defined only recently by Fox [11].

The matrix $J_{r,k}$ is the $r \times k$ matrix with 1-entries only. The matrix $J_{k,k}$ is abbreviated as $J_k$.

Given a binary matrix $B$, let $\operatorname{exm}(n, B)$ be the maximum number of 1-entries in an $n \times n$ matrix $A$ such that $B$ is not an interval minor of $A$. Clearly, if $P$ is a permutation matrix, then for every binary matrix $A$, $A$ contains $P$ if and only if $P$ is an interval minor of $A$, and thus $\operatorname{exm}(n, P) = \operatorname{ex}(n, P)$. Furthermore, if $M$ is an interval minor of $A$, then every interval minor $B$ of $M$ is also an interval minor of $A$.

Marcus and Tardos [19] actually proved that $\operatorname{exm}(n, J_k) \leq 2k^4(k^2)^n$, which implies the same upper bound on $\operatorname{ex}(n, P)$ for every $k$-permutation matrix $P$. Fox [11] improved the upper bound to $\operatorname{exm}(n, J_k) \leq 3k2^{8k}n$.

**Higher-dimensional matrices.** Similar questions can be asked for higher-dimensional permutation matrices.

We call $M \in \{0, 1\}^{[n_1] \times \cdots \times [n_d]}$ a $d$-dimensional binary matrix of size $n_1 \times \cdots \times n_d$. A
\textbf{d-dimensional binary matrix} $P$ of size $k \times \cdots \times k$ is a \textit{d-dimensional $k$-permutation matrix} if $P$ contains $k$ 1-entries and the positions of every pair of 1-entries of $P$ differ in all coordinates. We say that a \textit{d-dimensional binary matrix} $P = (p_{i_1,\ldots,i_d})$ of size $k_1 \times \cdots \times k_d$ is contained in a \textit{d-dimensional binary matrix} $A = (a_{i_1,\ldots,i_d})$ of size $n_1 \times \cdots \times n_d$ if there exist $d$ increasing injections $f_i : [k_i] \rightarrow [n_i]$, $i = 1, 2, \ldots, d$ such that for all $i_1, i_2, \ldots, i_d \in [k]$, if $p_{i_1,\ldots,i_d} = 1$ then $a_{f_1(i_1),\ldots,f_d(i_d)} = 1$. If $P$ is not contained in $A$, we say that $A$ avoids $P$.

When $P$ is a \textit{d-dimensional permutation matrix}, we let $\text{exp}_P(n)$ be the maximum number of 1-entries in a $P$-avoiding $n \times \cdots \times n$ \textit{d-dimensional binary matrix}. Klazar and Marcus [17] proved an analogue of the Füredi–Hajnal conjecture for higher-dimensional matrices. For any given \textit{d-dimensional permutation matrix} $P$, they showed that $\text{exp}_P(n) \leq 2^{O(k \log k) n^{d-1}}$.

Geneson and Tian [13, Equation (4.5)] improved the upper bound to $\text{exp}_P(n) \leq 2^{O(k) n^{d-1}}$, generalizing the upper bound for 2-dimensional permutation matrices by Fox [11].

For a \textit{d-dimensional permutation matrix} $P$, let $S_P(n)$ be the set of \textit{d-dimensional} $n \times \cdots \times n$ permutation matrices avoiding $P$. The first author [8] proved that for every fixed forbidden matrix $P$, we have

$$2^\Omega(n) \cdot (n!)^{d-2} \leq |S_P(n)| \leq 2^{O(n \log \log n)} \cdot (n!)^{d-1-1/(d-1)}.$$

\subsection{1.1 New results}

A 1-entry in a matrix is identified by the pair $(i, j)$ of the row index $i$ and the column index $j$. The \textit{distance vector} between the entries $(i_1, j_1)$ and $(i_2, j_2)$ is $(i_2 - i_1, j_2 - j_1)$. We say that a vector $(d, d')$ is \textit{r-repeated} in a permutation matrix $P$ if $(d, d')$ occurs as the distance vector of at least $r$ pairs of 1-entries. If some vector is \textit{r-repeated} in a permutation matrix $P$, then $P$ has an \textit{r-repetition}; otherwise, $P$ is \textit{r-repetition-free}.

The following theorem shows that the Füredi–Hajnal limit (and hence the Stanley–Wilf limit) is subexponential\footnote{A function $f : \mathbb{R} \rightarrow \mathbb{R}$ grows exponentially if $f(n) \in 2^{o(n)}$. Notice that Fox [11] uses the less restrictive definition where all functions $f(n) \in 2^{o(n)}$ are exponential.} for $k$-permutation matrices with no $\Omega(k/\log^6(k))$-repetition.

\textbf{Theorem 1.1.} Let $k \geq 9$, $r \geq 3$ and let $P$ be an \textit{r-repetition-free} $k$-permutation matrix. The Füredi–Hajnal limit of $P$ satisfies

$$c_P \leq 2^{O(r^{1/3} k^{2/3} \log^2 k)}.$$

We say that a $k$-permutation matrix $P$ is \textit{scattered} if $P$ is $r$-repetition-free for every $r \geq 4 \log_2 k / \log_2 \log_2 k$. In Section 2, we show that as $k$ goes to infinity, almost all $k$-permutation matrices are scattered. This immediately implies the following upper bound on the Füredi–Hajnal limit of asymptotically almost all permutation matrices.

\textbf{Corollary 1.2.} For every $k \geq 9$ and a random $k$-permutation matrix $P$, the Füredi–Hajnal limit of $P$ satisfies

$$c_P \leq 2^{O(k^{2/3} \log^{7/3} (k^{1/3} \log k))}$$

asymptotically almost surely.

We also show upper bounds for some permutation matrices that are far from being scattered.
Let $k$ be a square of an integer and let $G_k$ be the $k \times k$ binary matrix with 1-entries at positions $(a + b\sqrt{k} + 1, b + a\sqrt{k} + 1)$ for every pair $a, b \in \{0, \ldots, \sqrt{k} - 1\}$. Fox [11] used $G_k$ as an example of a permutation matrix for whose Füredi–Hajnal limit he proved the $2\Omega(\sqrt{k}^{1/2})$ lower bound. We show an upper bound that differs only by a $\log^2(k)$ multiplicative factor in the exponent.

**Theorem 1.3.** For every $k \geq 1$ such that $\sqrt{k} \in \mathbb{N}$, we have

$$c_{G_k} \leq 2^{O(\sqrt{k}\log^2 k)}.$$

In fact, we show a slightly more general upper bound for matrices obtained by so-called grid products; see Theorem 5.2.

Let $k \geq 2$ be an even integer. Let $X_k$ be the $k \times k$ matrix with 1-entries on both diagonals; that is, at positions $(i, j)$ where $i, j \in [k]$ and $i + j = k + 1$ or $i - j = 0$.

Given an odd integer $k$, let $\text{Cross}_k$ be the $k$-permutation matrix corresponding to the permutation $\pi$ satisfying $\pi(i) = i$ for $i$ odd and $\pi(i) = k + 1 - i$ for $i$ even. Notice that $\text{Cross}_k$ is contained in $X_{k+1}$. By a result of the first author [8], the Füredi–Hajnal limit of $\text{Cross}_k$ is at least $\Omega(k^2)$. We show a quasipolynomial upper bound.

**Theorem 1.4.** Let $k$ be an even integer and let $Q$ be a permutation matrix. If $Q$ is contained in $X_k$, then

$$c_Q \leq 2^{O(\log^2 k)}.$$

The density of a matrix is the ratio of the number of 1-entries to the total number of entries of the matrix. Our general strategy for proving the upper bounds on $c_P$ is first to prove an upper bound on the density of small $P$-avoiding matrices (see Theorem 4.3 and Lemmas 5.1 and 5.3) and then use the following theorem.

**Theorem 1.5.** Let $u \in \mathbb{N}$ and $q \in (1/u, 1)$. If a permutation matrix $P$ satisfies

$$\exp(u) < qu^2,$$

then

$$c_P \leq 2u^3u[-\log u/\log q].$$

Fox [11] proved that for every $k \in \mathbb{N}$, $\text{exm}(J_k, n) \leq 3k2^{8k}n$. The constant in the exponent can be easily decreased from 8 to 6. We further improve it to 4.

**Theorem 1.6.** Let $k \in \mathbb{N}$. The extremal function for the forbidden $J_k$-minor satisfies

$$\text{exm}(J_k, n) \leq \frac{8}{3}(k + 1)^2 \cdot 2^{4k}n.$$

Since every $k$-permutation matrix is contained in $J_k$, we have the following corollary.

**Corollary 1.7.** For every $k \in \mathbb{N}$ and for every $k$-permutation matrix $P$, the Füredi–Hajnal limit of $P$ satisfies

$$c_P \leq \frac{8}{3}(k + 1)^2 \cdot 2^{4k}.$$

We extend the Stanley–Wilf conjecture to higher dimensions, and prove asymptotically matching lower and upper bounds, improving previous much weaker bounds [8].
Theorem 1.8. For every \( d, k \geq 2 \) and every \( d \)-dimensional \( k \)-permutation matrix \( P \), we have
\[
n^{-O(k)} \left( \Omega \left( k^{1/(2^d-1)} \right) \right)^n \cdot (n!)^{d-1-1/(d-1)} \leq |S_P(n)| \leq \left( 2^{O(k)} \right)^n \cdot (n!)^{d-1-1/(d-1)},
\]
where the constants hidden by the \( O \)-notation and \( \Omega \)-notation do not depend on \( k \) and \( n \).

We prove Theorem 1.5 in Section 3, Theorem 1.1 in Section 4, Theorems 1.3 and 1.4 in Section 5, Theorem 1.6 in Section 6 and Theorem 1.8 in Section 7.

All logarithms in this paper are base 2.

2 Almost all permutation matrices are scattered

Lemma 2.1. Let \( k \in \mathbb{N} \), \( r \in [k-1] \) and \( d, d' \in \{-k + 1, -k + 2, \ldots, k-1\} \). The number of \( k \)-permutation matrices where \((d, d')\) is \( r \)-repeated is at most \( k! / r! \).

Proof. Let \( P \) be a permutation matrix where \((d, d')\) is \( r \)-repeated and let \( \pi \) be its corresponding permutation. Using symmetries, we can assume, without loss of generality, that \( d, d' > 0 \).

For every pair \( P_{i,j}, P_{i+d,j+d'} \) of 1-entries of \( P \) with distance vector \((d, d')\), we say that \( P_{i,j} \) is a starting entry and \( P_{i+d,j+d'} \) is an ending entry. Notice that an entry can be both a starting entry and an ending entry. An ending row in \( P \) is a row containing an ending entry.

We map \( P \) to the pair \((S, \sigma)\) where

- \( S \) is the set of the \( r \) ending rows of \( P \) and
- \( \sigma \) is the restriction of \( \pi \) on the set of indices of the non-ending rows of \( P \).

Clearly, there are at most \( \binom{k}{r}(k-r)! = \frac{k!}{r!} \) such pairs.

We now prove that the mapping is injective by showing that if two permutation matrices \( P \) and \( P' \) are mapped to the same pair \((S, \sigma)\), then \( P = P' \). For contradiction, let \( j \) be the leftmost column in which \( P \) and \( P' \) differ.

First, consider the case that the 1-entry in column \( j \) in \( P \) is in an ending row \( i \). This implies that \( P_{i-d,j-d'} = 1 \) and since the column \( j - d' \) is to the left of the column \( j \), we have \( P'_{i-d,j-d'} = 1 \). Since \( P \) and \( P' \) have the same sets of starting rows and \( i - d \) is a starting row of \( P \), we have \( P'_{i,j} = 1 \), a contradiction. By a symmetrical reasoning, we obtain a contradiction in the case when the 1-entry in column \( j \) in \( P' \) is in an ending row.

In the remaining case the 1-entries in column \( j \) in \( P \) and \( P' \) are in different non-ending rows \( i_1 \) and \( i_2 \), respectively. Let \( j' \) be the number of 1-entries in non-ending rows to the left of the \( j \)th column. Let \( i'_1 \) and \( i'_2 \) be the number of non-ending rows above row \( i_1 \) and \( i_2 \), respectively. Since \( i_1 \) and \( i_2 \) are different non-ending rows, we have \( i'_1 \neq i'_2 \). But we also have \( \sigma(i'_1 + 1) = j' + 1 \) and \( \sigma(i'_2 + 1) = j' + 1 \), which is a contradiction with the choice of \( j \).

\[ \square \]

Theorem 2.2. Let \( k \in \mathbb{N} \) and let \( r \in [k] \). The number of \( k \)-permutation matrices with an \( r \)-repetition is at most
\[ 2k^2 \frac{k!}{r!} \]

Consequently, the number of \( k \)-permutation matrices that are not scattered is in \( o(k!) \).
Proof. The distance vector of a pair of 1-entries in a $k$-permutation matrix can attain $(2k-2)^2$ different values of the form $(d, d')$, where $|d|, |d'| \in \{1, 2, \ldots, k-1\}$. For every $(d, d')$ the vector $(d, d')$ occurs in a matrix the same number of times as the vector $(-d, -d')$. Therefore to get an upper on the number of $k$-permutation matrices with an $r$-repetition, it is enough to consider only the $2(k-1)^2$ values of the distance vector where $d, |d'| \in \{1, 2, \ldots, k-1\}$. The first part of Theorem 2.2 now follows from Lemma 2.1.

The second part of Theorem 2.2 follows by using the formula to bound the number of permutation matrices with a $\lceil 4 \log k / \log \log k \rceil$-repetition. We have

$$2k^2 \frac{k!}{(4 \log k / \log \log k)!} \leq 2^{1+2 \log k} \cdot k! \left( \frac{e \log \log k}{4 \log k} \right)^{4 \log k / \log \log k} \leq k! \cdot 2^{1+2 \log k} \cdot 2^{(\log \log k - \log \log k) \cdot 4 \log k / \log \log k} \leq k! \cdot 2^{1-2 \log k+o(\log k)} = o(k!) .$$

\[ \square \]

3 Trade-off between size and density of $P$-avoiding matrices

In this section we prove Theorem 1.5.

The density of a row of a matrix is the ratio of the number of 1-entries in this row to the number of columns. For a permutation matrix $P$, let $f_P(z, y)$ be the maximum possible number of rows of a binary $P$-avoiding matrix with $z$ columns and at least $y$ 1-entries in every row. That is, we are interested in matrices where the density of every row is at least $q = y/z$.

Marcus and Tardos [19] bounded $f_P(k^2, k)$ for every $k$-permutation matrix $P$, to show that $c_P$ is finite. Fox used an upper bound on $f_P(2^{2k}, 2k-1)$ to show that $c_P \leq 2^{8k}$ for every $k$-permutation matrix $P$. In general, Fox’s generalization of the Marcus–Tardos recursion [11, Lemma 12] requires an upper bound on $f_P(z, y)$ where $(y-1)^2 < z$, in order to prove a linear upper bound on $\exp_P(n)$. The next lemma allows us to deduce upper bounds on $c_P$ from bounds on $f_P(z, y)$ where $y$ is close to $z$.

By $P^T$ we denote the transpose of $P$. The following proposition is the heart of Theorem 1.5.

**Proposition 3.1.** Let $P$ be a permutation matrix, $u, h \in \mathbb{N}$ and $q \in (0, 1)$. Suppose that for every $z \geq u$, we have

1. $f_P(z, qz) < h$ and
2. $f_{PT}(z, qz) < h$.

Then

$$c_P \leq 2u^3 h \lceil -\log u / \log q \rceil .$$

We break the proof of Proposition 3.1 into a sequence of statements.

Notice that if $q \leq 1/u$, then condition 1 with $z = u$ implies that every $h \times u$ matrix with one 1-entry in every row contains $P$. This is satisfied only when $P$ is the 1-permutation matrix. Then $c_P = 0$ and the conclusion of the proposition is valid. We therefore further assume that $q > 1/u$.

We define a sequence $q_i$ of densities of 1-entries as follows. For every $i \geq 1$, let

$$q_i = \max\{1/u, q^i\} .$$
Since $q > 1/u$, we have $q_1 = q$. Since $q < 1$, there is some $i_0 > 1$ such that $q_i = q^i$ whenever $i < i_0$ and $q_i = 1/u$ for $i \geq i_0$. We thus have

$$\frac{q_i}{q_{i-1}} \geq q \quad \text{for every } i \geq 2. \quad (1)$$

**Lemma 3.2.** Under the conditions of Proposition 3.1, for every $i \geq 1$, we have

$$f_P(u^2, q_i u^2) < h^i.$$

**Proof.** We proceed by induction on $i$. The case $i = 1$ follows from condition 1 of Proposition 3.1.

Given $i \geq 2$, suppose for contradiction that $A_i$ is an $h^i \times u^2$ binary $P$-avoiding matrix with at least $q_i u^2$ 1-entries in every row. We split the matrix $A_i$ into $h^i - 1$ intervals of consecutive $h$-tuples of rows. For every $j \in \{1, 2, \ldots, h^i - 1\}$, let $A_{i,j}$ be the matrix formed by the $j$th interval of rows.

First, assume that for some $j$, the matrix $A_{i,j}$ has at most $\lfloor q_{i-1} u^2 \rfloor$ columns with at least one 1-entry. Then we consider an $h \times \lfloor q_{i-1} u^2 \rfloor$ matrix $A'_{i,j}$ created from $A_{i,j}$ by removing some columns with 0-entries only. Since $A'_{i,j}$ contains all 1-entries of $A_{i,j}$, it has at least $q_i u^2$ 1-entries in every row, which is at least $qq_{i-1} u^2$ by (1). Condition 1 of Proposition 3.1 with $z = \lfloor q_{i-1} u^2 \rfloor$ implies that $A'_{i,j}$ contains $P$.

Now assume that for every $j$, the matrix $A_{i,j}$ has at least $q_{i-1} u^2$ columns with at least one 1-entry. Let $B_i$ be the matrix formed from $A_i$ by contracting the intervals of rows forming the matrices $A_{i,j}$. Then $B_i$ is an $h^{i-1} \times u^2$ binary matrix with at least $q_{i-1} u^2$ 1-entries in every row. By the induction hypothesis, $B_i$ contains $P$ and consequently $A_i$ contains $P$. \hfill \Box

**Corollary 3.3.** Under the conditions of Proposition 3.1, we have

$$f_P(u^2, u) \leq h^{\lceil -\log u/\log q \rceil}.$$  

**Proof.** We use Lemma 3.2 with $i = \lceil -\log u/\log q \rceil$. Then we have

$$q^i = q^{\lceil -\log u/\log q \rceil} \leq 1/u,$$

and so $q_i = 1/u$. By Lemma 3.2,

$$f_P(u^2, u) < h^i. \quad \Box$$

Let $g_P(z, y)$ be the maximum number of columns of a binary $P$-avoiding matrix with $z$ rows and at least $y$ 1-entries in every column. Since $g_P(z, y) = f_{P_T}(z, y)$ for every $z, y \in \mathbb{N}$, we have the following corollary.

**Corollary 3.4.** Under the conditions of Proposition 3.1, we have

$$g_P(u^2, u) \leq h^{\lceil -\log u/\log q \rceil}.$$  

**Proof of Proposition 3.1.** By Fox’s generalized Marcus–Tardos recursion [11, Lemma 12], for every permutation matrix $P$ and for all positive integers $n, s, t$ with $s \leq t$, we have

$$\text{ex}(tn, P) \leq \text{ex}(s - 1, P) \cdot \text{ex}(n, P) + \text{ex}(t, P) \cdot n \cdot (f_P(t, s) + g_P(t, s)).$$
Marcus and Tardos [19] used the recursion with parameters \( t = k^2 \) and \( s = k \). We choose the parameters \( t = u^2 \) and \( s = u \).

Let
\[
\tilde{h} = h\left\lceil -\frac{\log u}{\log q} \right\rceil.
\]

By Corollaries 3.3 and 3.4 and by the trivial estimates \( \text{ex}(s-1, P) \leq (s-1)^2 \) and \( \text{ex}(t, P) \leq t^2 \), we have
\[
\text{ex}(u^2 n, P) \leq (u - 1)^2 \cdot \text{ex}(n, P) + u^4 n \cdot 2\tilde{h}. \tag{2}
\]

Arratia [2] proved that \( |S_P(n)| \) is supermultiplicative in \( n \) for every fixed permutation matrix \( P \). An analogous proof shows that for every permutation matrix \( P \), the extremal function \( \text{ex}(n, P) \) is superadditive [20, Lemma 1(ii)]; that is, for every \( m, n \in \mathbb{N} \), we have
\[
\text{ex}(m + n, P) \geq \text{ex}(m, P) + \text{ex}(n, P).
\]

Consequently, for every permutation matrix \( P \) and \( n, \alpha \in \mathbb{N} \), we have
\[
\text{ex}(\alpha n, P) \geq \alpha \cdot \text{ex}(n, P). \tag{3}
\]

By combining inequality (3) for \( \alpha = u^2 \) with inequality (2), we obtain
\[
u^2 \cdot \text{ex}(n, P) \leq \text{ex}(u^2 n, P) \leq (u - 1)^2 \cdot \text{ex}(n, P) + u^4 n \cdot 2\tilde{h}
\]
\[
(2u - 1) \cdot \text{ex}(n, P) \leq u^4 n \cdot 2\tilde{h}
\]
\[
\text{ex}(n, P) \leq 2u^3 \tilde{h} \left\lceil -\frac{\log u}{\log q} \right\rceil n.
\]

Proof of Theorem 1.5. For every \( z \geq u \), if a \( u \times z \) matrix \( A \) contains at least \( qz \) 1-entries in every row, then \( A \) contains at least \( quz \) 1-entries. Thus, we can select \( u \) columns having together at least \( qu^2 \) 1-entries. Consequently, the condition \( \text{ex}_P(u) < qu^2 \) implies condition 1 of Proposition 3.1 with \( h = u \). The validity of condition 2 of Proposition 3.1 follows from the fact that \( \text{ex}_P(u) = \text{ex}_{P^T}(u) \).

\[\square\]

4 Repetition-free permutation matrices

In this section we prove Theorem 1.1. We first show that for given \( k \) and \( r \) and an \( r \)-repetition-free permutation matrix \( P \), every \( 3k \times 3k \) matrix with a sufficiently small number of 0-entries in every row and every column contains \( P \) (see Lemma 4.2). We then show that every \( 4k \times 4k \) matrix with a sufficiently small total number of 0-entries contains \( P \) (see Theorem 4.3). Theorem 1.1 then follows by Theorem 1.5.

We analyse a straightforward greedy algorithm for finding an occurrence of a \( k \)-permutation matrix \( P \) on a given \( k \)-tuple of rows of a binary matrix \( B \). In this setting, every 1-entry of \( P \) has a prescribed row of \( B \) in which it is to be mapped. For every \( j \), let \( r_j \) be the row of \( B \) in which the 1-entry from the \( j \)th column of \( P \) is to be mapped. Figure 1 shows an example of the execution of the algorithm.

In every step of the algorithm, one entry of \( B \) is inspected. The entry inspected in the \( i \)th step of the algorithm is always in the \( i \)th column of \( B \). The entry of \( B \) inspected in the first step lies in the row \( r_1 \). In every step, the algorithm does the following. If the inspected entry is 0, the algorithm stays in the same row for the next step, and we say that it stalls. If the inspected entry is 1 and the current row is \( r_j \) for some \( j \leq k - 1 \), the algorithm goes to
the row $r_{j+1}$ for the next step, and we say that the algorithm moves. If the inspected entry is 1 and the current row is $r_k$, then an occurrence of $P$ has been found and the algorithm terminates.

We note that if the algorithm fails to find an occurrence of $P$, then the given $k$-tuple of rows does not contain an occurrence of $P$. This fact is, however, not used in the proof.

Let $B$ be a $3k \times 3k$ matrix. We simultaneously run $2k + 1$ instances of the algorithm, one for every $k$-tuple of consecutive rows of $B$. If an instance of the algorithm does not find an occurrence of $P$, then at most $k - 1$ of its steps are moves. Hence, if at least one of the instances makes at least $k$ moves, $B$ contains $P$.

Given integers $k \geq 9$ and $r \geq 3$, let

$$w = \left\lceil \frac{35}{24} \left( \frac{k}{r} \right)^{1/3} \right\rceil$$

and

$$v = \frac{1}{3} \left( \frac{k}{r} \right)^{1/3}.$$

For every $k \geq 9$ and $r \geq 3$, we have

$$w \leq \frac{k}{3}. \quad (4)$$

Indeed, if $k \leq 11$, then $w \leq 3 \leq k/3$. Otherwise, $w < \left\lceil 1.1k^{1/3} \right\rceil < 1.1k^{1/3} + 1 < k/3$.

The following claim is the main part of the proof.

**Lemma 4.1.** Let $k \geq 9$ and let $B$ be a $3k \times 3k$ binary matrix with at most $v$ 0-entries in every row and in every column. Let $r \geq 3$ and let $P$ be an $r$-repetition-free $k$-permutation matrix. For every $j \in \{1, 2, \ldots, 3k - w\}$, either at least $3k/4$ instances of the algorithm make a move in the $j$th step or the sum of the numbers of moves made by the instances in steps $j, j + 1, \ldots, j + w - 1$ is at least $3kw/4$.

**Proof.** If an instance of the algorithm stalls (moves) after inspecting $B_{i,j}$, then we say that the instance stalls (moves) on $B_{i,j}$.

Assume that at most $3k/4$ of the instances make a move in the $j$th step. Consider the instances stalled on some $B_{i,j} = 0$. Since the $i$th row of $B$ contains at most $v$ 0-entries, all the instances stalled on $B_{i,j}$ will move simultaneously in $j'$th step for some $j' \in \{j + 1, j + 2, \ldots, j + \lceil v \rceil \}$.

For every $l \geq 1$, let $M_l$ be the set of instances that stall on $B_{i,j}$ and make a move in each of the steps $j', j' + 1, \ldots, j' + l - 1$. Thus $M_1$ is the set of all instances stalled on $B_{i,j}$. For every $l \geq 1$, the set $M_l \setminus M_{l+1}$ is the set of instances from $M_l$ that are stalled on a 0-entry in the $(j' + l)$th column of $B$. 
We now use the fact that $P$ is $r$-repetition-free to bound the size of $M_l \setminus M_{l+1}$. By the selection of the $k$-tuples of rows on which the instances are running, every instance in $M_l$ made a different number of moves before the $j$th step. Consider a 0-entry $B_{i',j'+l}$. There are at least as many occurrences of the distance vector $(i' - i, l)$ between two 1-entries of $P$ as there are instances from $M_l$ stalled on $B_{i',j'+l}$. Thus, on each of the 0-entries in the $(j'+l)$th column of $B$, at most $r$ of the instances from $M_l$ are stalled. Since every column of $B$ contains at most $v$ 0-entries, we have

$$|M_l \setminus M_{l+1}| \leq vr$$

and consequently

$$|M_1 \setminus M_{w-[v]}| \leq (w - v)vr.$$  

All instances in $M_{w-[v]}$ were stalled on $B_{i,j}$ and made at least $w - v$ moves in steps $j, j+1, \ldots, j + w - 1$.

The $j$th column of $B$ contains at most $v$ 0-entries. Each of the more than $5k/4$ instances stalled in the $j$th step is stalled on one of these 0-entries. We thus conclude that the number of instances that made at least $w - v$ moves in steps $j, j+1, \ldots, j + w - 1$ is at least

$$\frac{5}{4} k - v(w - v)vr \geq \frac{5}{4} k - \frac{1}{9} \left(\frac{k}{r}\right)^{2/3} r \left(\frac{35}{24} - \frac{1}{3}\right) \frac{k}{r}^{1/3}$$

$$= \frac{5}{4} k - \frac{1}{9} \frac{9}{8} k$$

$$= \frac{9}{8} k.$$

We have

$$\frac{w - v}{w} \geq 1 - \frac{8}{35} = \frac{27}{35}$$

and thus the number of moves in steps $j, j+1, \ldots, j + w - 1$ is at least

$$\frac{9}{8} k(w - v) \geq \frac{9}{8} \cdot \frac{27}{35} k w > \frac{3}{4} kw.$$

A **tight occurrence** of a $k \times k$ binary matrix $P$ in a matrix $B$ is an occurrence of $P$ on some $k$ consecutive rows of $B$.

**Lemma 4.2.** Let $k \geq 9$ and $r \geq 3$. Let $B$ be a $3k \times 3k$ binary matrix and let $P$ be an $r$-repetition-free $k$-permutation matrix. If $B$ has at most $(1/3)(k/r)^{1/3}$ 0-entries in every row and in every column then $B$ contains $P$. Moreover, the occurrence of $P$ in $B$ is tight.

**Proof.** We assign types to some of the steps of the algorithm. The assignment starts with the step 1. Let $j$ be a step considered during the assignment procedure. If $j > 3k - w$, we finish the assignment procedure. If at least $3k/4$ instances make a move in the $j$th step, then we say that the $j$-th step is of *type 1* and proceed to the $(j+1)$st step, otherwise the steps $j, j+1, \ldots, j + w - 1$ are of *type 2* and we proceed to the $(j+w)$th step. The number of moves in every step of type 1 is at least $3k/4$. By Lemma 4.1, the average number of moves in steps of type 2 is at least $3k/4$. The total number of moves is thus at least

$$(3k - w) \cdot \frac{3}{4} k \geq \left(3 - \frac{1}{3}\right) : k \cdot \frac{3}{4} k = 2k^2 > (2k + 1)(k - 1)$$

and so at least one of the instances made $k$ moves and found an occurrence of $P$. □
Theorem 4.3. Let $k \geq 9$ and $r \geq 3$. Let $A$ be a $4k \times 4k$ binary matrix and let $P$ be an $r$-repetition-free $k$-permutation matrix. If $A$ has at most $(k/3)(k/r)^{1/3}$ 0-entries then $A$ contains $P$.

Proof of Theorem 4.3. The matrix $A$ has at most $k$ rows with more than $v_0$ 0-entries and at most $k$ columns with more than $v_0$ 0-entries. Thus, after removing $k$ rows and $k$ columns with the largest number of 0-entries, we obtain a matrix $B$ satisfying the requirements of Lemma 4.2, thus containing $P$. 

Proof of Theorem 1.1. Let $k \geq 9$, $r \geq 3$ and let $P$ be an $r$-repetition-free $k$-permutation matrix. By Theorem 4.3, we can use Theorem 1.5 with $u = 4k$ and $q = 1 - \frac{1}{48r^{1/3}k^{2/3}}$. We have $\log u \in O(\log k)$ and $\log q \leq -\frac{\log e}{48r^{1/3}k^{2/3}} \leq -\frac{1}{34r^{1/3}k^{2/3}}$.

By Theorem 1.5, $c_P \leq 2u^{3+\lceil -\log u/\log q \rceil} \leq 2u^{4+34r^{1/3}k^{2/3} \log u} = 2^{1+4 \log u + 34r^{1/3}k^{2/3} \log^2 u}$. Thus $c_P \leq 2^{O(r^{1/3}k^{2/3} \log^2 k)}$. 

5 Some additional upper bounds

In this section, we show subexponential upper bounds on $c_P$ for a few special matrices that are far from being scattered.

5.1 Grid products

Consider a $k$-permutation matrix $P$ with 1-entries at positions $(i, \pi(i))$ for every $i \in [k]$ and an $l$-permutation matrix $Q$ with 1-entries at positions $(j, \rho(j))$ for every $j \in [l]$. We define the grid product $R = P \# Q$ to be the $(kl)$-permutation matrix with 1-entries at positions $((j-1) \cdot k + i, (\pi(i) - 1) \cdot l + \rho(j))$ for every $i \in [k]$ and $j \in [l]$. See Figure 2 for an example.

Lemma 5.1. Let $k, l \geq 2$ and $m \geq 1$. Let $P$ be a $k$-permutation matrix, $Q$ an $l$-permutation matrix, $t = kl$ and $R = P \# Q$. We have $\operatorname{ex}_R(mt) < (mt)^2 - k \cdot ((ml - 1)^2 - \operatorname{ex}_Q(ml - 1))$.

Proof. Let $z = (ml - 1)^2 - \operatorname{ex}_Q(ml - 1)$, that is, the minimum number of 0-entries in an $(ml - 1) \times (ml - 1)$ $Q$-avoiding matrix. Let $A$ be an $mt \times mt$ matrix with at most $zk - 1$ 0-entries. We show that $A$ contains $R$. 

12
We cut $A$ into $k$ rectangles of width $ml$, rearrange them on top of each other with a small vertical displacement determined by $P$, and form their “superposition” matrix $A'$. See Figure 3. Formally, let $A'$ be the $(mt - k) \times ml$ matrix such that for every $i \in mt - k$ and $j \in [ml]$, $A'_{ij} = 1$ if and only if for every $\alpha \in [k]$, $a_{i+\alpha-1,j+ml(\pi(\alpha)-1)} = 1$. Since every element of $A$ is used to define at most one element of $A'$, the number of 0-entries in $A'$ is at most $(zk - 1)$. Notice that if $A'$ contains the matrix $Q'$ obtained from $Q$ by inserting $k - 1$ rows full of zeros between every pair of consecutive rows of $Q$, then $A$ contains $R$.

Let $B$ be the $(ml - 1) \times ml$ matrix formed by the set of rows $\{p + \alpha k : \alpha \in \{0, \ldots, ml - 2\}\}$ of $A'$ where $p$ is chosen from $[k]$ so as to minimize the number of 0-entries of $B$. Thus $B$ has at most $z - 1$ 0-entries and so it contains $Q$. An occurrence of $Q$ in $B$ implies an occurrence of $Q'$ in $A'$. Consequently, $A$ contains $R$.

**Theorem 5.2.** Let $k, l \geq 2$. Let $Q$ be an $l$-permutation matrix with Füredi–Hajnal constant $c_Q \geq 3$ and let $P$ be a $k$-permutation matrix. Let $R$ be the $kl$-permutation matrix $P \# Q$. Then

$$c_R \leq 2^{O(k \log^2 (c_Q k))}.$$

**Proof.** We use Theorem 1.5 with $u = mkl$, where $m = \lceil 2c_Q/l \rceil$. Since $c_Q \geq 2(l - 1)$ (see e.g. [8, Claim 1]) and $l \geq 2$, we have $m < (2c_Q + l)/l \leq 3c_Q/l$ and $u \leq 3c_Q k$.

Since $ex_Q(ml - 1) \leq c_Q(ml - 1)$, Lemma 5.1 implies

$$ex_R(u) < u^2 - k \cdot ((ml - 1)^2 - c_Q(ml - 1))$$

$$= u^2 - k \cdot (ml - 1)(ml - 1 - c_Q)$$

$$\leq u^2 - k \cdot (2c_Q - 1)(c_Q - 1)$$

$$\leq u^2 - kc_Q^2$$

$$\leq u^2 - km^2l^2/9$$

$$= u^2(1 - 1/(9k)).$$
That is, \( \text{ex}_R(u) \leq u^2 q \), where \( q = 1 - 1/(9k) \). We estimate
\[
\log q = \log \left( 1 - 1/(9k) \right) < -\log(e)/(9k).
\]
By Theorem 1.5, we have
\[
c_R \leq 2u^3 u^{[-\log u / \log q]} \leq u^{O(k \log u)} \leq 2^{O(k \log^2(c_Q k))}.
\]

**Proof of Theorem 1.3.** We have \( G_k = I_{\sqrt{k}} \# I_{\sqrt{k}} \), where \( I_{\sqrt{k}} \) is the \( \sqrt{k} \times \sqrt{k} \) identity matrix. It is known that \( c_{I_{\sqrt{k}}} = 2(\sqrt{k} - 1) \) (see e.g. [8, Claim 1]). Thus, using Theorem 5.2 with \( P = Q = I_{\sqrt{k}} \), we get
\[
c_{G_k} \leq 2^{O(\sqrt{k} \log^2(2\sqrt{k} \sqrt{k}))} \leq 2^{O(\sqrt{k} \log^2 k)}.
\]

**Remark.** Guillemot and Marx [14] define the canonical \( r \times s \) grid permutation as \( I_r \# J_s \), where \( J_s \) is the reversal matrix with 1-entries at positions \( (i, j) \) satisfying \( i + j = s + 1 \). Theorem 5.2 thus gives the upper bound \( c_{I_r \# J_s} \leq 2^{O(r \log^2(rs))} \). In general, the same asymptotic upper bound is obtained for any grid product \( P \# Q \) where \( P \) is an \( r \)-permutation matrix and \( Q \) is an \( s \)-permutation matrix with \( c_Q \) polynomial in \( s \).

### 5.2 The cross matrix

**Lemma 5.3.** For every integer \( k \geq 6 \) that is a multiple of \( 6 \), we have
\[
\text{ex}_{X_k}(2k) < (2k)^2 - k^2/18.
\]

**Proof.** Let \( A \) be a \( 2k \times 2k \) matrix with at most \( k^2/18 \) 0-entries.

Given \( d \in \{-2k+1, -2k+2, \ldots, 2k-2, 2k-1\} \), the \( d \)-diagonal of \( A \) is the set of entries at positions \( (i, j) \) satisfying \( i, j \in [2k] \) and \( i - j = d \). Given \( c \in \{2, 3, \ldots, 4k\} \), the \( c \)-antidiagonal of \( A \) is the set of entries at positions \( (i, j) \) satisfying \( i, j \in [2k] \) and \( i + j = c \).

Since \( A \) has at most \( k^2/18 \) 0-entries, there exists \( d \in \{-k/6, -k/6 + 1, \ldots, k/6\} \) such that the \( d \)-diagonal contains at most \( k/6 \) 0-entries. Analogously, there is \( c \in \{2k - k/6 + 1, 2k - k/6 + 2, \ldots, 2k + k/6 + 1\} \) such that the \( c \)-antidiagonal contains at most \( k/6 \) 0-entries.

Let \( r = (c + d)/2 \) and \( s = (c - d)/2 \). Note that if \( c \) and \( d \) have the same parity, then the entry at position \( (r, s) \) is the intersection of the \( d \)-diagonal and the \( c \)-antidiagonal. We have
\[
k - \frac{k}{6} + \frac{1}{2} \leq r, s \leq k + \frac{k}{6} + \frac{1}{2}.
\]
If \( c \) and \( d \) have the same parity, for every \( i \in \{1, \ldots, 5k/6\} \), let \( S_i \) be the set of entries at positions \( (r - i, s - i), (r - i, s + i), (r + i, s - i) \) and \( (r + i, s + i) \). Similarly, if \( c \) and \( d \) have the opposite parity, for every \( i \in \{1/2, 3/2, \ldots, 5k/6 - 1/2\} \), let \( S_i \) be the set of entries at positions \( (r - i, s - i), (r - i, s + i), (r + i, s - i) \) and \( (r + i, s + i) \). Note that by (5), the entries of each such \( S_i \) lie in \( A \). Additionally, all these entries lie in the union of the \( d \)-diagonal and the \( c \)-antidiagonal, thus there are only at most \( 2k/6 \) 0-entries among them.

Let \( I \) be the set of at least \( k/2 \) indices \( i \) such that all the four entries of \( S_i \) are 1-entries. The set \( \bigcup_{i \in I} S_i \) forms an occurrence of \( X_k \) in \( A \).

**Proof of Theorem 1.4.** We use Theorem 1.5 with \( u = 2k \) and \( q = 1 - 1/72 \). In particular, \( \log q \) is a negative constant. By Lemma 5.3, \( \text{ex}_{X_k}(u) < qu^2 \) and thus by Theorem 1.5, we have
\[
c_Q \leq 2u^3 u^{[-\log u / \log q]} \leq 2^{O(\log^2 k)}.
\]
for every permutation matrix \( Q \) contained in \( X_k \).
6 General permutation matrices

In this section, we prove Theorem 1.6.

Given \( r, k, s, t \in \mathbb{N} \), let \( f_{r,k}(t, s) \) be the maximum number of rows in a \( J_{r,k} \)-minor-free binary matrix with \( t \) columns where each row contains at least \( s \) 1-entries. Notice that if \( s > t \) then \( f_{r,k}(t, s) = 0 \) since a matrix cannot have more 1-entries in a row than the number of columns.

Fox [11] proved the following recurrence for every \( r, k, s, t \in \mathbb{N} \):

\[
f_{r,k}(t, s) \leq 2f_{r,k}(t/2, s) + 2f_{r,k-1}(t/2, s/2).
\]

We use it to prove an upper bound on \( f_{r,k}(t, s) \) that is slightly different from the upper bound used by Fox [11].

**Lemma 6.1.** For every \( r, k, s, t \in \mathbb{N} \) where \( t \) and \( s \) are powers of 2 and \( t \geq s \geq 2^k - 1 \), we have

\[
f_{r,k}(t, s) \leq r2^{2k-2}(t/s)^2.
\]

**Proof.** The claim is trivially true when \( s > t \), because then

\[
f_{r,k}(t, s) = 0 \leq r2^{2k-2}(t/s)^2.
\]

The claim is also true when \( k = 1 \) and \( t \geq s \geq 1 \):

\[
f_{r,1}(t, s) = r - 1 \leq r2^{2k-2}(t/s)^2.
\]

We proceed by induction on \( k + \log(t/s) \), which is an integer since \( s \) and \( t \) are powers of 2. We have

\[
f_{r,k}(t, s) \leq 2f_{r,k}(t/2, s) + 2f_{r,k-1}(t/2, s/2)
\]

\[
\leq 2r2^{2k-2}(t/2s)^2 + 2r2^{2k-4}(t/s)^2
\]

\[
\leq r2^{2k-2}(t/s)^2.
\] □

**Proof of Theorem 1.6.** Let \( t = 2^{2k} \) and let \( s_i = 2^i \) for every \( i \in \{k - 1, k, \ldots, 2k\} \). Let \( A \) be an \( n \times n \) binary matrix. We discard \( n \) mod \( t \) rightmost columns and bottommost rows of \( A \) and split the rest of \( A \) into \( \lfloor n/t \rfloor \times \lfloor n/t \rfloor \) blocks of size \( t \times t \).

We say that a \( t \times t \) block of \( A \) is \( s_i \)-wide (\( s_i \)-tall) if it has more than \( s_i \) nonempty columns (rows).

By contracting the blocks, we form an \( \lfloor n/t \rfloor \times \lfloor n/t \rfloor \) matrix that does not contain \( J_k \) as an interval minor, and thus it has at most \( \text{exm}(\lfloor n/t \rfloor, J_k) \) 1-entries. The number of 1-entries in the blocks that are neither \( s_{k-1} \)-wide nor \( s_{k-1} \)-tall is thus at most

\[
\text{exm}(s_{k-1}, J_k) \cdot \text{exm}(\lfloor n/t \rfloor, J_k).
\]

If \( A \) has \( \lfloor n/t \rfloor f_{k,k}(t, s_i) \) \( s_i \)-wide blocks then some \( f_{k,k}(t, s_i) \) of them are on the same columns of \( A \). This implies that \( J_k \) is an interval minor of \( A \). An \( s_i \)-wide block that is neither \( s_{i+1} \)-wide nor \( s_{i+1} \)-tall contains at most \( \text{exm}(s_{i+1}, J_k) \) 1-entries. The total number of 1-entries in blocks that are \( s_i \)-wide but neither \( s_{i+1} \)-wide nor \( s_{i+1} \)-tall in an \( n \times n \) \( J_k \)-avoiding matrix is thus at most

\[
\text{exm}(s_{i+1}, J_k) \cdot \frac{n}{t} \cdot f_{k,k}(t, s_i).
\]
The same bound holds for blocks that are $s_i$-high but neither $s_{i+1}$-wide nor $s_{i+1}$-high.

The number of entries in the discarded rows and columns is together smaller than $2tn$.

The claim of Theorem 1.6 is clearly true whenever $n \leq 2^k$. We then proceed by induction on $n$ with $k$ fixed.

$$\text{exm}(n, J_k) \leq \text{exm}(s_{k-1}, J_k) \cdot \text{exm}([n/t], J_k) + 2tn + \sum_{i=k-1}^{2k-1} 2 \cdot \text{exm}(s_{i+1}, J_k) \cdot \frac{n}{t} \cdot f_{k,k}(t, s_i)$$

$$\leq 2^{2(k-1)} \cdot \frac{8}{3} (k+1)^2 \cdot 2^{4k} \frac{n}{2^k} + 2^{2k+1}n + 2 \cdot \frac{n}{2^k} \cdot \sum_{i=k-1}^{2k-1} (s_{i+1})^2 \cdot k \cdot 2^{2k-2}(t/s_i)^2$$

$$\leq \frac{2}{3} (k+1)^2 \cdot 2^{4k}n + 2^{2k+1}n + 2n \cdot 2^{-2k} \cdot 2^{2k-2}k \cdot \sum_{i=k-1}^{2k-1} 2^{2i+2} \cdot 2^{4k-2i}$$

$$\leq \frac{8}{3} (k+1)^2 \cdot 2^{4k}n.$$

$$\square$$

7 Higher-dimensional matrices

Let $P$ be a given $d$-dimensional $k$-permutation matrix $P$. Recall that $S_P(n)$ denotes the set of all $d$-dimensional $n$-permutation matrices avoiding $P$. Let $T_P(n)$ be the set of all $d$-dimensional matrices of size $n \times \cdots \times n$ that avoid $P$.

7.1 Upper bound

The proof of the upper bound in Theorem 1.8 uses the following high-dimensional generalization of a lemma by Fox [11, Lemma 11], which he used in his simplified proof of the bound $s_P \in O(c_P)^2$.

Lemma 7.1. Let $P$ be a $d$-dimensional permutation matrix. Let $t, u \in \mathbb{N}$ and let $n = tu$. Then

$$|S_P(n)| \leq t^{|n|} |T_P(u)|$$

Proof. Let $A$ be a $d$-dimensional $n$-permutation matrix that avoids $P$. We split $A$ into blocks of size $t \times t \times \cdots \times t$ by hyperplanes orthogonal to the coordinate axes. Let $B$ be the $u \times u \times \cdots \times u$ matrix formed by contracting these blocks. The number of choices for $B$ is at most $T_P(u)$.

Let the $i$th slice of a $d$-dimensional matrix be the set of entries with first coordinate equal to $i$. Since $A$ is a permutation matrix, each slice of $A$ contains exactly one 1-entry and each slice of $B$ contains at most $t$ 1-entries.

We fix one such $B$ and count the number of $d$-dimensional $n$-permutation matrices $A$ whose contraction is $B$. For every $i$, the 1-entry in the $i$th slice of $A$ can correspond to one of the at most $t$ 1-entries in the $[i/t]$th slice of $B$. After selecting this 1-entry of $B$, we have $t^{d-1}$ positions for the 1-entry in the $i$th slice of $A$. Hence, the number of $d$-dimensional $n$-permutation matrices $A$ whose contraction is $B$ is at most $(t^d)^n$.

$$\square$$

Proof of the upper bound in Theorem 1.8. We assume without loss of generality that $n = 2^m$ for some $m \in \mathbb{N}$. Our plan is to show an upper bound on $|T_P(u)|$ for a suitable $u = 2^i$ and
then use Lemma 7.1. Every $2^i \times \cdots \times 2^i$ $d$-dimensional $P$-avoiding binary matrix can be built by a sequence of expansions from smaller $d$-dimensional $P$-avoiding matrices, reversing the contraction operation of $2 \times \cdots \times 2$ blocks. We start with $A_0$, the $1 \times \cdots \times 1$ $d$-dimensional matrix containing one 1-entry. In each step, we transform the matrix $A_i$ of size $2^i \times \cdots \times 2^i$ into a matrix $A_{i+1}$ of size $2^{i+1} \times \cdots \times 2^{i+1}$ by replacing each 0-entry of $A_i$ by a $2 \times \cdots \times 2$ block containing only 0-entries and each 1-entry of $A_i$ by a $2 \times \cdots \times 2$ block containing at least one 1-entry. There is a single possibility of replacing a 0-entry and 2 possibilities of replacing a 1-entry.

We use the high-dimensional generalization of the Füredi–Hajnal conjecture, that is, the estimate $\exp(n) = \Theta(n^{d-1})$ [17]. Thus, $\exp(2^i) \leq cP 2^{i(d-1)}$ for some constant $cP$ and so

$$|T_P(2^i)| \leq 2^{2d-1} \cdot cP \cdot 2^{(i-1)(d-1)} \cdot |T_P(2^{i-1})| \leq \cdots \leq 2^{2d} \cdot cP \cdot 2^{i(d-1)}.$$

We select

$$i = \left\lfloor \frac{1}{d-1} \log \left( \frac{n}{2^d \cdot cP} \right) \right\rfloor,$$

so that $|T_P(2^i)| \leq 2^n$. We have

$$2^i \geq \frac{1}{2} \left( \frac{n}{2^d \cdot cP} \right)^{1/(d-1)} = \left( \frac{n}{2^{2d-1} \cdot cP} \right)^{1/(d-1)}.$$  

By Lemma 7.1 with $u = 2^i$ and $t = n/2^i$, we have

$$|S_P(n)| \leq t^n |T_P(2^i)| \leq n^{dn} \cdot 2^{-idn} \cdot 2^n \leq n^{dn} \cdot (2^{2d-1} \cdot cP/n)^{dn/(d-1)} \cdot 2^n \leq (n^{dn/(1-1/(d-1))} \cdot (2^{2d-1} \cdot cP)^{dn/(d-1)} \cdot 2^n \leq (e^n n!)^{d-1-1/(d-1)} \cdot (2^{2d-1} \cdot cP)^{dn/(d-1)} \leq (n!)^{d-1-1/(d-1)} \cdot e^d \cdot (2^{2d-1} \cdot cP)^{dn/(d-1)}.$$  

Using the upper bound $cP \leq 2^{O(k)}$ implied by the result of Geneson and Tian [13, Equation (4.5)], we obtain

$$|S_P(n)| \leq \left( 2^{O(k)} \right)^n (n!)^{d-1-1/(d-1)},$$

where the constant hidden by the $O$-notation does not depend on $k$ and $n$. \hfill \square

### 7.2 Lower bound

A partial order $\prec$ on $[n]$ is an intersection of $d$ linear orders $\prec_1, \prec_2, \ldots, \prec_d$ on $[n]$ if $\forall a, b \in [n] \ (a \prec b \Leftrightarrow \forall i \in [d] \ a \prec_i b)$. A partial order $\prec$ has dimension $d$ if $d$ is the smallest positive integer such that $\prec$ is an intersection of $d$ linear orders. A random $d$-dimensional partial order on $[n]$ is the intersection of $d$ linear orders on $[n]$ taken uniformly and independently at random. A partial order is an antichain if no two elements are comparable by the partial order. A linear order $\prec$ on $[n]$ is a linear extension of $\prec$ if $\forall a, b \in [n] \ (a \prec b \Rightarrow a < b)$.

Brightwell [6] showed the following lower bound on the number of linear extensions of almost all partial orders of a given dimension.
Theorem 7.2 (Brightwell [6, Corollary 4]). Almost every \((d - 1)\)-dimensional partial order on \([n]\) has at least \((e^{-2} n^{1-1/(d-1)})^n\) linear extensions.

Let \(Q_d(n)\) be the probability that a random \(d\)-dimensional partial order on \([n]\) is an antichain.

Corollary 7.3. We have

\[ Q_d(n) \geq \frac{(e^{-2} n^{1-1/(d-1)})^n}{n!}. \]

Proof. The reverse \(>\) of a linear order \(<\) on \([n]\) is the linear order satisfying for every distinct \(a, b\) from \([n]\) that \(a > b\) if and only if \(b < a\). By a well-known observation (see e.g. the introduction of the Brightwell’s paper [6]), a linear order \(<\) is a linear extension of \(\prec\) if and only if the intersection of \(\prec\) and the reverse of \(<\) is an antichain. Therefore, the expected number of linear extensions of a random \((d - 1)\)-dimensional partial order is \(n!\) times larger than \(Q_d(n)\).

Let \(I^d_k\) be the \(d\)-dimensional \(k\)-permutation matrix with 1-entries at positions \((i, i, \ldots, i)\) for every \(i \in [k]\).

Theorem 7.4. We have

\[ |S_{I^d_k}(n)| \geq (1/(en)) \cdot e^{-(1+1/(d-1))n} \cdot (n!)^{d-1-1/(d-1)}. \]

Proof. We consider the uniform probability space of \(d\)-dimensional \(n\)-permutation matrices, that is, each of the \((n!)^{d-1}\) matrices has probability \(1/(n!)^{d-1}\). A random \(d\)-dimensional \(n\)-permutation matrix from this space can be formed by taking \(d\) permutations \(\pi_1, \pi_2, \ldots, \pi_d\) of \([n]\) independently and uniformly at random, and placing 1-entries to positions \((\pi_1(a), \pi_2(a), \ldots, \pi_d(a))\) for every \(a \in [n]\).

Consider a \(d\)-dimensional \(n\)-permutation matrix \(R\) with 1-entries at positions \((\pi_1(a), \pi_2(a), \ldots, \pi_d(a))\), for every \(a \in [n]\). We define the partial order \(\prec_R\) as the intersection of the linear orders \(\prec_1, \prec_2, \ldots, \prec_d\) where \(a \prec_i b\) if and only if \(\pi_i(a) < \pi_i(b)\). Thus, if \(R\) is a random \(d\)-dimensional \(n\)-permutation matrix, then \(\prec_R\) is a random \(d\)-dimensional partial order on \([n]\). An occurrence of \(I^d_2\) in \(R\) corresponds to a pair of elements of \([n]\) comparable in \(\prec_R\), and so \(R\) avoids \(I^d_2\) if and only if \(\prec_R\) is an antichain.

Consequently, by Corollary 7.3, the probability that a random \(d\)-dimensional \(n\)-permutation matrix avoids \(I^d_2\) is at least \((e^{-2} n^{1-1/(d-1)})^n/n!\) and thus

\[ |S_{I^d_k}(n)| \geq e^{-2n \cdot n^{(1-1/(d-1))} \cdot (n!)^{d-2}} \geq e^{-2n \cdot (n! \cdot e^{n/(en)})^{1-1/(d-1)}} \cdot (n!)^{d-2} \]

\[ \geq (1/(en)) \cdot e^{-(1+1/(d-1))n} \cdot (n!)^{d-1-1/(d-1)}. \]

\[ \Box \]

Theorem 7.5. We have

\[ |S_{I^{d}_{l+k}}(n)| \geq n^{-O(k)} \left(\Omega(k^{1/(d-1)})\right)^n \cdot (n!)^{d-1-1/(d-1)}, \]

where the constants hidden by \(\Omega\) and \(O\) do not depend on \(n\) and \(k\).

Proof. All permutations in this proof are \(d\)-dimensional. Let \(A\) be an \(lm\)-permutation matrix. If we can split the 1-entries of \(A\) into \(m\) \(m\)-tuples such that each of these \(m\)-tuples forms an occurrence of an \(I^d_2\)-avoiding matrix, then \(A\) avoids \(I^d_{l+1}\). We now count how many permutation
matrices we obtain by the reverse process, that is, by merging \( I_2^d \)-avoiding \( m \)-permutation matrices to form an \( I_{l+1}^d \)-avoiding matrix.

The number of ways to choose an ordered \( l \)-tuple of matrices from \( S_{I_2^d}(m) \) is

\[ |S_{I_2^d}(m)|^l. \]

Given an \( l \)-tuple of \( m \)-permutation matrices, the number of ways to form an \( lm \)-permutation matrix whose 1-entries can be split into \( l m \)-tuples forming the occurrences of the \( l \) selected permutations is

\[ \binom{lm}{m, m, \ldots, m}. \]

The number of ways to split the 1-entries of an \( lm \)-permutation matrix into \( l m \)-tuples is

\[ \binom{lm}{m, m, \ldots, m}. \]

We thus have

\[ |S_{I_{l+1}^d}(lm)| \geq |S_{I_2^d}(m)|^l \cdot \binom{lm}{m, m, \ldots, m}^d \cdot \binom{lm}{m, m, \ldots, m}^{-1} \geq \]

\[ \geq \left( \frac{1}{em} \right)^l \cdot e^{-(1+1/(d-1))lm} \cdot (ml)^{d-1-1/(d-1)} \cdot \left( \frac{lm!}{(ml)!} \right)^{d-1} = \]

\[ = \left( \frac{1}{em} \right)^l \cdot \frac{1}{e^{1/(d-1)lm}} \cdot \left( \frac{lm!}{(ml)!} \right)^{1/(d-1)} \cdot (lm!)^{d-1-1/(d-1)}. \]

We have

\[ \frac{(lm)!}{(ml)!} \geq \left( \frac{lm}{em(m/e)^m} \right)^l = \frac{l^m}{(em)^l} \]

and so

\[ |S_{I_{l+1}^d}(lm)| \geq \left( \frac{1}{em} \right)^l \cdot \frac{1}{e^{1/(d-1)lm}} \cdot \left( \frac{l^m}{(em)!} \right)^{1/(d-1)} \cdot (lm!)^{d-1-1/(d-1)} \]

\[ \geq \left( \frac{1}{em} \right)^{2l} \cdot \left( \frac{l^{1/(d-1)}}{e^{1/(d-1)}} \right)^l \cdot (lm!)^{d-1-1/(d-1)}. \]

That is, when \( k = l + 1 \) and \( n = lm = (k - 1)m \), we have

\[ |S_{I_k^d}(n)| \geq \left( \frac{1}{em} \right)^{2k} \cdot \left( \frac{(k - 1)^{1/(d-1)}}{e^{1/(d-1)}} \right)^n \cdot (n!)^{d-1-1/(d-1)}. \]

\( \square \)

**Proof of the lower bound in Theorem 1.8.** A \( d \)-dimensional permutation matrix \( M \) is monotone if its 1-entries can be ordered in such a way that for every \( i \in [d] \), the \( i \)th coordinates of the 1-entries are either increasing or decreasing. Observe that by symmetry, \( |S_{M}(n)| = |S_{I_k^d}(n)| \) for every \( n, k \in \mathbb{N} \) and every monotone \( k \)-permutation matrix \( M \). By applying the Erdős–Szekeres lemma on monotone subsequences \([10]\) \( d-1 \) times, every \( d \)-dimensional \( k \)-permutation matrix \( P \) contains a monotone \( d \)-dimensional \([k^{1/2^{d-1}}]\)-permutation (see also [18]). Therefore the lower bound in Theorem 1.8 is a corollary of Theorem 7.5. \( \square \)
8 Concluding remarks

8.1 Specific permutation matrices

There are two types of permutation matrices \( P \) for which we have a subexponential upper bound on their Füredi–Hajnal limit \( c_P \). The first type are the scattered matrices, which have generally very little structure. The second type includes practically all previously known examples of matrices with subexponential Füredi–Hajnal limit, and consists of matrices obtained from the identity matrix by a few elementary operations, like the direct sum. The direct sum of a \( k \times k \) matrix \( A \) and an \( l \times l \) matrix \( B \) is the \( (k+l) \times (k+l) \) block matrix \( \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \). Similarly, the skew sum of \( A \) and \( B \) is the block matrix \( \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \). Layered matrices, obtained as a multiple direct sum of identity matrices, form the most natural class for which a polynomial upper bound on \( c_P \) is known. The upper bound follows from the upper bound \( s_P \leq 4k^2 \) on the Stanley–Wilf limit of every layered \( k \)-permutation \( P \) [9], since \( c_P \) and \( s_P \) are polynomially related [8]. More recently, the first author [7] has shown directly that \( c_P \) is at most linear in \( k \) for every layered \( k \)-permutation \( P \). The matrices of the second type have generally a lot of structure; in particular, they are far from being scattered. We have added the cross matrix and certain grid products to the second type of matrices, but there are still many matrices that do not belong to any of these types. For example, we do not have any subexponential upper bound on the Füredi–Hajnal limit of a permutation matrix whose 1-entries in the odd columns lie on the diagonal and even columns induce a scattered matrix.

Question 8.1. Is \( c_P \) polynomial in \( k \) for \( k \)-permutation matrices \( P \) contained in a two-diagonal matrix?

Decomposable permutation matrices generalize layered matrices and are defined as the smallest class of matrices closed under direct sum and skew sum, and containing all identity matrices. The cross matrix \( \text{Cross}_k \) is decomposable but our current upper bound on its Füredi–Hajnal limit is slightly superpolynomial.

Question 8.2. Is \( c_{\text{Cross}_k} \) polynomial in \( k \)?

8.2 Higher-dimensional matrices

In the 2-dimensional case, it was shown by Arratia [2] that the limit \( \lim_{n \to \infty} |S_P(n)|^{1/n} \) exists for every permutation matrix \( P \). Analogously, by the super-additivity shown by Pach and Tardos [20], the limit \( \lim_{n \to \infty} \exp_P(n)/n \) always exists. Geneson and Tian [13, Lemma 4.7] showed that for every \( d > 2 \) and every \( d \)-dimensional permutation matrix \( P \) with at least one 1-entry in a corner, there exists a constant \( K \) such that \( \exp_P(sn) \geq Ks^{d-1}\exp_P(n) \) for every
integer $n$. They asked whether this holds with $K = 1$ for every $P$. A positive answer would imply the existence of the Füredi–Hajnal limit $\lim_{n \to \infty} \frac{\exp_P(n)}{n}$ for every $d$-dimensional permutation matrix $P$.

We pose an analogous question about the higher-dimensional Stanley–Wilf limit.

**Question 8.3.** Does the limit

$$\lim_{n \to \infty} \left( \frac{|S_P(n)|}{(n!)^{d/(d-1)}} \right)^{1/n}$$

exist for every $d > 2$ and every $d$-dimensional permutation matrix $P$?

Let

$$\overline{s}_P = \limsup_{n \to \infty} \left( \frac{|S_P(n)|}{(n!)^{d/(d-1)}} \right)^{1/n} \quad \text{and} \quad \underline{s}_P = \liminf_{n \to \infty} \left( \frac{|S_P(n)|}{(n!)^{d/(d-1)}} \right)^{1/n}.$$ 

We have seen in Section 7.2 that the number of $d$-dimensional $n$-permutation matrices avoiding the $d$-dimensional 2-permutation matrix $I_d^2$ with 1-entries at $(1,1,\ldots,1)$ and $(2,2,\ldots,2)$ is equal to $(n!)^{d-1}$ times the probability that a random $d$-dimensional partial order on $[n]$ is an antichain. Thus the following are the best known bounds on the limit superior and limit inferior of $I_d^2$:

$$\overline{s}_{I_d^2} \geq 1 \ [22]$$

$$\overline{s}_{I_d^2} \leq \sqrt{\pi}/2 \ [4]$$

$$e^{-1-1/(d-1)} \leq \underline{s}_{I_d^2} \leq \overline{s}_{I_d^2} \leq 2(d-1)e^{1-1/(d-1)} \quad \text{for } d \geq 4 \ [6].$$

In the general case, Theorem 1.8 gives the following. For every $d,k \geq 2$ and every $d$-dimensional $k$-permutation matrix $P$,

$$\Omega\left( k^{1/(2^{d/(d-1)})} \right) \leq \underline{s}_P \leq \overline{s}_P \leq 2^{O(k)},$$

where the constants hidden by the $O$-notation and the $\Omega$-notation do not depend on $k$.

From the proof of the upper bound of Theorem 1.8 in Section 7.1 we know that $\overline{s}_P$ is bounded from above by $O((\overline{s}_P)^{d/(d-1)})$ for every $d$-dimensional permutation matrix. In the case $d = 2$, $c_P$ is also bounded from above by a polynomial in $s_P$ [8], but it is not known whether the following is true.

**Question 8.4.** Is $c_P$ bounded from above by a polynomial in $s_P$ for all $d$-dimensional permutation matrices $P$?

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