Solutions of deformed three-dimensional gravity

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Abstract

We investigate the gauging of a three-dimensional deformation of the anti-de Sitter algebra, which accounts for the existence of an invariant energy scale. By means of the Poisson sigma model formalism, we obtain explicit solutions of the field equations, which reduce to the BTZ black hole in the undeformed limit.

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1 Introduction

Theories of gravitation based on deformations of the Poincaré algebra have been recently considered in two dimensions [1, 2] as a possible implementation of ideas of deformed special relativity (DSR) [3]-[5] to the domain of gravitational physics. DSR aims at an effective description of the Planck scale physics based on the hypothesis that the Planck energy is a fundamental constant on the same footing as the speed of light, and must therefore be left invariant under transformations of the frames of reference. This implies that the Poincaré algebra is deformed at microscopic scales [3].

Deformed Poincaré algebras can be considered as special instances of nonlinear algebras [6]. Although two-dimensional gravity theories based on nonlinear algebras have been largely studied [7], the same cannot be said about higher-dimensional models. This is because in more than two dimensions one cannot define a natural action for the theory without introducing auxiliary fields [8], whose physical interpretation is not evident.

A useful tool for solving the field equations of nonlinear gauge theories is the Poisson sigma model formalism [9]. This has been employed till now only in the study of two-dimensional gravity [10], but can be easily generalized for the investigation of higher-dimensional models [11].

In this paper, we consider an example of three-dimensional gravity based on the deformation of the anti-de Sitter algebra introduced in [2], which generalizes the deformed Poincaré algebra of [5] to the case of nonvanishing cosmological constant. We discuss black hole and cosmological solutions of the model by applying Poisson sigma model techniques.

The field equations of nonlinear gauge theories are assumed to be of topological type [7], since they require the vanishing of the field strength. It is known that in three-dimensional riemannian geometry, the condition of vanishing (constant) curvature is equivalent to the Einstein equations with (non)vanishing cosmological constant and hence, in the limit where the algebra is not deformed we recover the known solutions of three-dimensional general relativity. In this way we show the possibility of using the Poisson sigma model formalism to solve also higher-dimensional gravity.

The paper is organized as follows: In sect. 2 we review the formalism of nonlinear gauge theories and its application to gravity. In sect. 3 we recover known results about 3D anti-de Sitter gravity in the Poisson sigma model formalism. In sect. 4 we extend these results to deformed anti-de Sitter gravity. A problem arises in the interpretation of the results due to the lack
of gauge invariance. We shall briefly discuss a possible interpretation of this in the context of DSR, but a more detailed discussion will be given elsewhere [12]. In sect. 5, we consider the limit $\lambda \to 0$ of Poincaré gravity.

2 Nonlinear gauge theories

In this section, we review the definition of nonlinear gauge theories and their application to gravity.

An algebra with generators $Q_A$ and commutation relations

$$[Q_A, Q_B] = W_{AB}(Q),$$

(1)

where the structure functions $W_{AB}(Q)$ are regular functions of the generators, antisymmetric in the two indices, that obey the generalized Jacobi identities

$$\frac{\partial W_{[AB}}^{\quad C]}{\partial Q_D} W_{CD} = 0,$$

(2)

is called nonlinear [6].

A gauge theory for this algebra [7] can be defined by introducing a gauge field $A^A$ and a coadjoint multiplet of scalar fields $\Phi_A$, which under infinitesimal transformation of parameter $\xi^A$ transform as

$$\delta A^A = d\xi^A + U^{A}_{BC}(\Phi)A^B \xi^C,$$

$$\delta \Phi_A = -W_{AB}(\Phi)\xi^B,$$

(3)

where the $W_{AB}$ are now functions of the fields $\Phi^A$, and the $U^A_{BC}$ are defined as

$$U^A_{BC} = \frac{\partial W_{BC}}{\partial \Phi_A}.$$  

(4)

One can then define the covariant derivative of the scalar multiplet

$$D\Phi_A = d\Phi_A + W_{AB}A^B,$$

(5)

and the curvature of the gauge fields

$$F^A = dA^A + \frac{1}{2} U^A_{BC}A^B \wedge A^C.$$  

(6)
The variation of $F^A$ under a gauge transformation is

$$\delta F^A = \left( \frac{\partial W_{BC}}{\partial \Phi^A} F^B - \frac{1}{2} \frac{\partial^2 W_{BC}}{\partial \Phi^A \partial \Phi^D} A^A \wedge D \Phi^D \right) \xi^C. \tag{7}$$

Note that the gauge fields do not transform covariantly, due to the second term. One can nevertheless define a covariant derivative of the gauge fields as

$$DF^A = dF^A + \frac{\partial W_{BC}}{\partial \Phi^A} A^B \wedge F^C - \frac{1}{2} \frac{\partial^2 W_{BC}}{\partial \Phi^A \partial \Phi^D} A^B \wedge A^C \wedge D \Phi^D, \tag{8}$$

This is easily checked to obey the Bianchi identity

$$DF^A = 0. \tag{9}$$

In order for the representation of the algebra to close, one must impose the vanishing of the covariant derivative of the $\Phi$ fields,

$$D \Phi_A = 0. \tag{10}$$

Moreover, we require that the theory be of topological type, and hence obey the field equations

$$F^A = 0. \tag{11}$$

In three dimensions, the field equations (10,11) can be derived from a BF-type lagrangian [8], by introducing two auxiliary fields $C^A$ and $B_A$

$$\mathcal{L} = *C^A \wedge D \Phi_A + *B_A \wedge F^A, \tag{12}$$

where $C^A$ and $B_A$ are 1- and 2-forms, respectively and the star denotes the Hodge dual.

Deformed gravity on a three-dimensional manifold $M^3$ can be defined starting from a six-dimensional nonlinear algebra, (e.g. a deformation of the three-dimensional Poincaré algebra), and identifying three of the generators, which we shall denote as $P_a$, $a = 0, 1, 2$, with the generators of translations and the other three, $M_\varphi$, with the generators of Lorentz rotations. One then identifies the components $A^a$ of the gauge fields with the dreibeins $e^a$ and the components $A^\varphi$ with the spin connection $\omega^a$. The components of the gauge field strength can then be written in terms of the geometric quantities of $M^3$. We are mainly interested in the special case in which the Lorentz group
is undeformed, while the momenta transform nonlinearly, as in DSR models. This implies

\[ W_{\pi a} = \epsilon_{ab} \Phi_c, \quad W_{ab} = V_{ab}(\Phi_A), \quad W_{\pi b} = Y_{ab}(\Phi_A), \]

for arbitrary functions \( V_{ab} \) and \( Y_{ab} \).

The components of the gauge field strength are then

\[ F^\pi = R^a + \frac{\partial V_{bc}}{\partial \Phi^\pi} e^b \wedge e^c + \frac{\partial Y_{bc}}{\partial \Phi^\pi} \omega^b \wedge e^c, \quad (13) \]

and

\[ F^a = T^a + \frac{\partial V_{bc}}{\partial \Phi^a} e^b \wedge e^c + \left( \frac{\partial Y_{bc}}{\partial \Phi^a} - \epsilon^a_{bc} \right) \omega^b \wedge e^c, \quad (14) \]

where \( R^a = d\omega^a + \epsilon^a_{bc} \omega^b \wedge \omega^c \) is the curvature and \( T^a = de^a + \epsilon^a_{bc} \omega^b \wedge e^c \) the torsion of \( \mathcal{M}^3 \).

The geometric quantities so defined in general are not covariant under the full gauge group, even when \( D\Phi_A = 0 \), but only under a subgroup. For example, in the case of undeformed Poincaré invariance, curvature and torsion transform covariantly only under the Lorentz subalgebra, while under translations they transform one into each other.

As in ordinary gauge theories of gravity, one can also show that general coordinate transformations are equivalent to gauge transformations on shell. In fact, writing \( A^A = A^A_\mu dx^\mu \), under an infinitesimal change of coordinates of parameter \( \zeta^\nu \), the gauge fields transform as standard vectors, \( \delta_C A^A_\mu = \partial_\mu \zeta^\nu A^A_\nu + \zeta^\nu \partial_\nu A^A_\mu \), while under an infinitesimal gauge transformation of parameter \( \xi^A \), \( \delta_G A^A_\mu = D_\mu \xi^A \), see (3). Simple algebraic manipulations permit then to write \( \delta_C A^A_\mu \) as \( \delta_C A^A_\mu = D_\mu (\zeta^\nu A^A_\nu) + \zeta^\nu F^A_{\nu \mu} \). On shell, where \( F^A_{\nu \mu} = 0 \), general coordinate transformations are therefore equivalent to gauge transformations with parameter \( \xi^A = \zeta^\nu A^A_\nu \).

In order to solve the field equations, it is useful to adopt the Poisson sigma model formalism [9]. Essentially, this formalism is based on the identification of the fields \( \Phi_A \) with the coordinates of a Poisson manifold \( \mathcal{N} \) with Poisson structure given by the functions \( W_{BC}(\Phi_A) \). The gauge fields are then 1-forms on \( \mathcal{N} \) and one can perform a change of coordinates on \( \mathcal{N} \) to a Darboux basis where the field equations assume an almost trivial form.
3 Anti-de Sitter gravity

Let us first consider the undeformed anti-de Sitter algebra $so(2, 4)$. It satisfies
\[ \{M_a, M_b\} = \epsilon_{ac} M_c, \quad \{M_a, P_b\} = \epsilon_{ac} P_c, \quad \{P_a, P_b\} = \lambda^2 \epsilon_{ac} M_c, \]
and admits two quadratic Casimir invariants:
\begin{align*}
C_1 &= h^{ab}(P_a P_a + \lambda^2 M_a M_a), \\
C_2 &= \lambda h^{ab} M_a P_a,
\end{align*}
(15)
where $h_{ab} = (-1, 1, 1)$ is the flat metric on $M^3$. It is convenient to define new generators
\[ N^\pm_a = M_a \pm \frac{1}{\lambda} P_a, \]
(16)
which satisfy the $so(1, 2) \times so(1, 2)$ algebra
\[ \{N^\pm_a, N^\pm_b\} = \epsilon_{ac} N^\pm_c, \quad \{N^+_a, N^-_b\} = 0. \]
(17)
In this basis the Casimir invariants are given by
\[ C_\pm = h^{ab} N^\pm_a N^\pm_b. \]
(18)
Likewise, one can define scalar fields
\[ \eta^\pm_a = \Phi_a \pm \frac{1}{\lambda} \Phi_2. \]
(19)

The choice of a Darboux basis is complicated by the existence of different representations of $so(1, 2) \times so(1, 2)$, which give rise to different solutions of the field equations. The representations can be classified according to the sign of $(\eta^0_\pm)^2 - (\eta^-_1)^2$ and of $\eta^\pm_0$. Of course a great number of subcases is possible, so we shall discuss only the most interesting.

3.1 Black hole solutions: $(\eta^0_\pm)^2 > (\eta^-_1)^2$.

In the case $(\eta^0_\pm)^2 > (\eta^-_1)^2$, $\eta^\pm_0 > 0$, a Darboux basis is given by
\[ X^\pm_1 = h^{ab} \eta^\pm_a \eta^\pm_b, \quad X^\pm_2 = \eta^\pm_2, \quad X^\pm_3 = \text{arccosh} \frac{\eta^\pm_0}{\sqrt{(\eta^0_\pm)^2 - (\eta^-_1)^2}}. \]
(20)
The new fields obey
\[ \{X^\pm_1, X^\pm_2\} = \{X^\pm_1, X^\pm_3\} = 0, \quad \{X^\pm_2, X^\pm_3\} = 1, \quad \{X^+_a, X^-_b\} = 0. \]
(21)
The relations (20) can be inverted, to obtain
\[ \eta_0^\pm = \sqrt{(X_2^\pm)^2 - X_1^\pm} \cosh X_3^\pm, \quad \eta_1^\pm = \sqrt{(X_2^\pm)^2 - X_1^\pm} \sinh X_3^\pm. \]

In the basis (20) the field equations take an elementary form [9] \((\alpha = 1, 2, 3)\):
\[ dA_\alpha^\pm = 0, \quad dX_1^\pm = 0, \quad dX_2^\pm = -A_3^\pm, \quad dX_3^\pm = A_2^\pm, \quad (22) \]
and are solved by
\[ X_1^\pm = \text{const} = Q^\pm, \quad X_2^\pm = \eta^\pm, \quad X_3^\pm = \theta^\pm, \]
\[ A_1^\pm = \frac{1}{2} d\psi^\pm, \quad A_2^\pm = d\theta^\pm, \quad A_3^\pm = -d\eta^\pm, \quad (23) \]
where \( \eta^\pm, \theta^\pm \) and \( \psi^\pm \) are arbitrary functions. Hence, \( \eta_2^\pm = \eta^\pm \) and
\[ \eta_0^\pm = \sqrt{(\eta^\pm)^2 - Q^\pm \cosh \theta^\pm}, \quad \eta_1^\pm = \sqrt{(\eta^\pm)^2 - Q^\pm \sinh \theta^\pm}. \]

Defining \( E^a_\pm \equiv \frac{1}{2} (\omega^a \pm \lambda e^a) \), one has
\[ E_\pm^0 = \frac{\partial X_\alpha^\pm}{\partial \eta_0^\pm} A_\alpha^\pm = -2\eta_0^\pm A_1^\pm - \frac{\eta_1^\pm}{(\eta_0^\pm)^2 - (\eta_1^\pm)^2} A_3^\pm, \]
\[ E_\pm^1 = \frac{\partial X_\alpha^\pm}{\partial \eta_1^\pm} A_\alpha^\pm = 2\eta_1^\pm A_1^\pm + \frac{\eta_0^\pm}{(\eta_0^\pm)^2 - (\eta_1^\pm)^2} A_3^\pm, \]
\[ E_\pm^2 = \frac{\partial X_\alpha^\pm}{\partial \eta_2^\pm} A_\alpha^\pm = 2\eta_2^\pm A_1^\pm + A_2^\pm. \]

One must now perform a gauge choice. Since the gauge algebra is six-dimensional, one must fix three of the free functions in order to obtain the coordinates of a three-dimensional spacetime. Hence we set \( \theta^+ = \theta^- = 0 \); the third condition is imposed by introducing a new variable \( r \) such that \( \eta^\pm = \mp \lambda r + \frac{J}{2r} \). Moreover, we define \( Q^\pm = M \mp \lambda J \). It follows that
\[ \sqrt{(\eta^\pm)^2 - Q^\pm} = \sqrt{(\eta^-)^2 - Q^-} = \sqrt{\lambda^2 r^2 - M + \frac{J^2}{4r^2}} \equiv \Gamma(r). \quad (24) \]

In this gauge the scalar fields read \( N_0^\pm = \Gamma, N_1^\pm = 0, N_2^\pm = \mp \lambda r + J/2r \), and then
\[ E_\pm^0 = -\Gamma d\psi^\pm, \quad E_\pm^1 = \left( \frac{J}{2r^2} \pm \lambda \right) \frac{d\Gamma}{\Gamma}, \quad E_\pm^2 = \left( \pm \lambda r + \frac{J}{2r} \right) d\psi^\pm. \]
It is easy to check that, if one puts $\psi^\pm = -(\phi \pm \lambda t)$, this coincides with the BTZ solution [13]. In fact, one has
\begin{align*}
e^0 &= \Gamma dt, \quad e^1 = \frac{dr}{\Gamma}, \quad e^2 = r d\phi - \frac{J}{2r} dt, \\
\omega^0 &= \Gamma d\phi, \quad \omega^1 = \frac{J}{2r^2} dr, \quad \omega^2 = \lambda^2 r dt - \frac{J}{2r} d\phi.
\end{align*}
(25)

As is well known, this solution corresponds to a spacetime of constant curvature, with a conical singularity at the origin, and two horizons at $r_\pm^2 = (M \pm \sqrt{M^2 - \lambda^2 J^2})/2\lambda^2$. For $J = 0$, it reduces to the anti-de Sitter black hole with a single horizon at $r_+^2 = M/\lambda^2$.

Another interesting possibility occurs for $\eta_0^+ < 0$, $\eta_0^- > 0$. In this case, the definition of $X_3^\pm$ in (20) is modified according to
\begin{align*}
X_3^\pm &= \mp \arccosh \frac{\pm \eta_0^\pm}{\sqrt{(\eta_0^\pm)^2 - (\eta_1^\pm)^2}}.
\end{align*}
The Poisson brackets (21) are still satisfied. Going through the same steps as before, one obtains $\eta_2^\pm = \eta_1^\pm$ and
\begin{align*}
\eta_1^\pm &= \mp \sqrt{(\eta_1^\pm)^2 - Q^\pm \cosh \theta_3^\pm}, \quad \eta_0^\pm = \mp \sqrt{(\eta_0^\pm)^2 - Q^\pm \sinh \theta_3^\pm}.
\end{align*}
Imposing the gauge conditions $\theta^+ = \theta^- = 0$, $\eta_1^\pm = \lambda r \mp \frac{J}{2r}$ and defining $Q^\pm = M \mp \lambda J$, one gets $N_0^\pm = \mp \Gamma$, $N_1^\pm = 0$, $N_2^\pm = \lambda r \mp J/2r$, and
\begin{align*}
E_0^\pm &= \pm \Gamma d\psi^\pm, \quad E_1^\pm = \left( \frac{J}{2r^2} \pm \lambda \right) \frac{dr}{\Gamma}, \quad E_2^\pm = \left( \lambda r \mp \frac{J}{2r} \right) d\psi^\pm.
\end{align*}
Putting $\psi^\pm = \lambda t \pm \phi$, one recovers (25), although in this case the scalar fields are different. This will have important implications in the deformed case.

### 3.2 Cosmological solutions: $(\eta_0^\pm)^2 < (\eta_1^\pm)^2$.

The solutions with $(\eta_0^\pm)^2 < (\eta_1^\pm)^2$ can be interpreted either as the region between the horizons of the black hole solutions of the previous section, or as cosmological solutions.

They can be obtained in the same way as before, except that one has to modify the definition of $X_3^\pm$ in (20). For example in the case $\eta_0^+ > 0$, one must define
\begin{align*}
X_3^\pm &= \arcsinh \frac{\eta_0^\pm}{\sqrt{(\eta_1^\pm)^2 - (\eta_0^\pm)^2}}.
\end{align*}
A straightforward calculation yields

\[ \eta_0^\pm = \sqrt{Q^\pm - (\eta^\pm)^2} \sinh \theta^\pm, \quad \eta_1^\pm = \sqrt{Q^\pm - (\eta^\pm)^2} \cosh \theta^\pm. \]

In the gauge \( \theta^\pm = 0, \eta^\pm = \mp \lambda r + \frac{J}{2r} \), the dreibein read

\[ e^0 = \frac{dr}{r}, \quad e^1 = \Gamma dt, \quad e^2 = r d\phi - \frac{J}{2r} dt. \]  

The coordinate \( t \) is now spacelike, while \( r \) is timelike, and the solution can then be interpreted as the interior of the black hole (25).

However, a more interesting interpretation is as a cosmological solution. In particular, taking for simplicity \( \lambda = 1, M = 1, J = 0 \) and defining \( \tau = \arccos \eta, \psi^\pm = \phi \pm \chi \), one has

\[ e^0 = d\tau, \quad e^1 = \sin \tau d\chi, \quad e^2 = \cos \tau d\phi, \quad \omega^0 = 0, \quad \omega^1 = \sin \tau d\phi, \quad \omega^2 = \cos \tau d\chi. \]  

This is the anti-de Sitter cosmological solution in unusual coordinates.

4 Deformed anti-de Sitter gravity

We pass now to consider the deformed anti-de Sitter algebra introduced in [2]. This is a generalization to the case of nonvanishing cosmological constant of the deformed Poincaré algebra introduced in [5].

We split the indices \( a, b \) in one timelike, 0, and two spacelike indices, \( i, j = 1, 2 \). The Lorentz subalgebra is undeformed

\[ \{ M_a, M_b \} = \epsilon_{ab}^c M_c, \]

while

\[ \{ M_0, P_0 \} = 0, \quad \{ P_i, P_j \} = -\lambda^2 \left( 1 - \frac{P_0}{\kappa} \right)^2 \left( \epsilon_{ij} M_0 + \frac{\epsilon_{ik} M_k P_j - \epsilon_{jk} M_k P_i}{\kappa} \right), \]

\[ \{ M_i, P_j \} = -\left( \epsilon_{ij} P_0 - \epsilon_{ik} \frac{P_k P_j}{\kappa} \right), \quad \{ P_0, P_i \} = \lambda^2 \left( 1 - \frac{P_0}{\kappa} \right)^3 \epsilon_{ij} M_j, \]

\[ \{ M_i, P_0 \} = \left( 1 - \frac{P_0}{\kappa} \right) \epsilon_{ij} P_j, \quad \{ M_0, P_i \} = \epsilon_{ij} P_j. \]
The algebra admits two quadratic Casimir invariants:

\[ C_1 = h^{ab} \left[ \frac{P_a P_a}{(1 - P_0/\kappa)^2} + \lambda^2 M_a M_a \right], \quad C_2 = \frac{\lambda h^{ab} M_a P_a}{1 - P_0/\kappa}. \]

Similarly to before, it is convenient to define new generators

\[ N_a^\pm = M_a \pm \frac{1}{\lambda} \frac{P_a}{1 - P_0/\kappa}, \tag{28} \]

which satisfy the algebra (17).

Likewise, one can define scalar fields

\[ \eta_a^\pm = \Phi_a^\pm \pm \frac{1}{\lambda} \frac{\Phi_a^0}{1 - \Phi_a^0/\kappa}. \tag{29} \]

Since the algebra satisfied by the \( \eta_a^\pm \) is identical to that of the previous section, one can proceed in the same way, but the relations between with fields \( \Phi_A \) are now more complicated:

\[ \Phi_{\pi} = \eta_a^+ + \eta_a^-, \quad \Phi_\omega = \frac{\lambda(\eta_a^+-\eta_a^-)}{1 + \lambda(\eta_a^+-\eta_a^-)/\kappa}. \tag{30} \]

### 4.1 Deformed black hole solutions.

As we have seen, black hole solutions occur when \((\eta_0^\pm)^2 > (\eta_1^\pm)^2\). In this case, assuming \(\eta_0^\pm > 0\), one can use the Darboux coordinates (20) obtaining

\[
e^0 = \sum_{\pm} \frac{\partial X_\pm}{\partial \Phi_0} A_\alpha^\pm = \frac{1}{\lambda} \sum_{\pm} \frac{\mp 1}{(1 - \Phi_0/\kappa)^2} \left( 2\eta_0^\pm A_1^\pm + \frac{\eta_1^\pm}{(\eta_0^\pm)^2 - (\eta_1^\pm)^2} A_3^\pm \right),
\]

\[
e^1 = \sum_{\pm} \frac{\partial X_\pm}{\partial \Phi_1^\pm} A_\alpha^\pm = \frac{1}{\lambda} \sum_{\pm} \frac{\pm 1}{1 - \Phi_0/\kappa} \left( 2\eta_1^\pm A_1^\pm + \frac{\eta_0^\pm}{(\eta_0^\pm)^2 - (\eta_1^\pm)^2} A_3^\pm \right),
\]

\[
e^2 = \sum_{\pm} \frac{\partial X_\pm}{\partial \Phi_2^\pm} A_\alpha^\pm = \frac{1}{\lambda} \sum_{\pm} \frac{\pm 1}{1 - \Phi_0/\kappa} \left( 2\eta_2^\pm A_1^\pm + A_2^\pm \right),
\]

and for the connection,

\[
\omega^0 = \sum_{\pm} \frac{\partial X_\pm}{\partial \Phi_0} A_\alpha^0 = - \sum_{\pm} \left( 2\eta_0^\pm A_1^\pm + \frac{\eta_1^\pm}{(\eta_0^\pm)^2 - (\eta_1^\pm)^2} A_3^\pm \right),
\]

\[
\omega^1 = \sum_{\pm} \frac{\partial X_\pm}{\partial \Phi_1^\pm} A_\alpha^0 = \sum_{\pm} \left( 2\eta_1^\pm A_1^\pm + \frac{\eta_0^\pm}{(\eta_0^\pm)^2 - (\eta_1^\pm)^2} A_3^\pm \right),
\]
\[ \omega^2 = \sum_{\pm} \frac{\partial X^\pm_\alpha}{\partial \Phi_\pm} A_\pm^\alpha = \sum_{\pm} \left( 2\eta_1^\pm A_1^\pm + A_2^\pm \right). \]

Using the solutions (23) of the field equations and setting as before \( \theta^+ = \theta^- = 0 \), one defines \( \eta^\pm = \mp \lambda r + \frac{J}{2r} \), \( \psi^\pm = -(\phi \pm \lambda t) \), \( Q^\pm = M \mp \lambda J \). Substituting, one has

\[ \Phi_0 = \Phi_1 = 0, \quad \Phi_2 = -\lambda^2 r, \]
\[ \Phi_\mp = \Gamma, \quad \Phi_\mp = 0, \quad \Phi_\mp = J/2r. \]

It is then easy to see that the vielbein and the connection are identical to (25). Hence, in this gauge the solution are not deformed.

A less trivial situation occurs when \( \eta_0^+ < 0, \eta_0^- > 0 \). Now, setting \( \theta^+ = \theta^- = 0 \) and defining \( \eta^\pm = \lambda r \pm \frac{J}{2r}, \psi^\pm = \lambda t \pm \phi \), \( Q^\pm = M \mp \lambda J \), one has

\[ \Phi_0 = -\lambda \Gamma / \Delta, \quad \Phi_1 = 0, \quad \Phi_2 = -\lambda J / 2r, \]
\[ \Phi_\mp = \Phi_\mp = 0, \quad \Phi_\mp = \lambda r, \]

where \( \Delta = 1 - \lambda \Gamma / \kappa \) and \( \Gamma \) has been defined previously. Moreover,

\[ e^0 = \Delta^2 \Gamma \, dt, \quad e^1 = \frac{\Delta}{\Gamma} \, dr \quad e^2 = \Delta \left( r \, d\phi - \frac{J}{2r} \, dt \right), \]

and

\[ \omega^0 = \Gamma \, d\phi, \quad \omega^1 = \frac{J}{2r^2} \, \frac{dr}{\Gamma}, \quad \omega^2 = \lambda^2 r \, dt - \frac{J}{2r} \, d\phi. \]

Hence, while the connection takes the same expression as in the undeformed case, the vielbeins are deformed by factors of \( \Delta \). It follows that the solutions still have constant curvature, but nontrivial torsion. In the limit \( J = 0 \), for example, the torsion reads

\[ T^0 = -\frac{3\lambda r}{\kappa \Delta^2} e^1 \wedge e^0, \quad T^1 = 0, \quad T^2 = -\frac{\lambda r}{\kappa \Delta^2} e^1 \wedge e^2. \]

It is evident that the solutions have a singularity of the torsion at \( \Delta = 0 \), i.e. \( r^2_+ = \left( M + \kappa / \lambda + \sqrt{(M + \kappa / \lambda)^2 - \lambda^2 J^2} \right) / 2\lambda^2 \). A coordinate singularity (horizon) is located at \( r^2_+ = (M + \sqrt{M^2 - \lambda^2 J^2}) / 2\lambda^2 \). Since \( r_+ > r_- \) a naked singularity is always present. These solution resemble the two-dimensional solutions of ref. [2].
4.2 Deformed cosmological solutions.

The solutions with \((\eta_0^\pm)^2 > (\eta_i^\pm)^2\) can be discussed as in the undeformed case. When \(\eta_0^\pm > 0\), in the gauge \(\theta^\pm = 0\) they take again the form (26) and can then be interpreted either as the interior of the BTZ black hole or as a cosmological solution.

When \(\eta_0^+ < 0, \eta_0^- > 0\), the cosmological solution becomes more involved. Proceeding as in section 2.2, one obtains for the dreibein

\[
e^0 = \Delta^2 d\tau, \quad e^1 = \Delta \sin \tau \, dx, \quad e^2 = \Delta \cos \tau \, d\phi,
\]

where \(\Delta = 1 - \sin \tau / \kappa\), and again (27) for the connection. The solution has constant curvature, but nontrivial torsion. The nonvanishing torsion components in an orthonormal frame are proportional to \(1/\Delta^3\) and hence singular at \(\sin \tau = \kappa\). The solution displays therefore a big bang singularity with the universe beginning with a size of the order of the Planck length.

4.3 Interpretation.

Till now we have implicitly assumed that the properties of the solutions of deformed gravity can be discussed in the gauge \(\theta^\pm = 0\). However, as has been noticed in sect. 2, the geometric quantities do not transform covariantly under the full gauge group. This implies for example that in general it is not possible to define a spacetime metric invariant under all the gauge transformations\(^1\). In particular, in our case the standard metric is invariant under rotations, but not under boosts. This fact requires a physical interpretation compatible with the postulates of DSR. Such interpretation must necessarily be different from that of general relativity, and share some analogy with that of Finsler geometry [15]. A possible approach is to assume that test particles at rest in a given reference system experience a metric \(ds_0^2 = h_{ab}\epsilon^a e^b\), where \(h_{ab}\) is the usual flat metric, while test particle moving with respect to the observer’s frame are seen to experience a different metric, given by the gauge transformed one. This topic will be discussed at length elsewhere [12].

\(^1\)A metric invariant under the action of the Lorentz group can actually be constructed using also the scalar multiplet \(\Phi_A\), at the cost of introducing an additional structure in the theory [11].
5 The Poincaré limit

The solutions for the gauge theory in the Poincaré limit \( \lambda = 0 \) of the anti-de Sitter algebra can be obtained either from the previous results, taking the limit \( \lambda \to 0 \) after a suitable rescaling of the scalar fields \( \eta^\pm \), or repeating from the beginning the steps that led to the solution in the anti-de Sitter case.

Finding Darboux coordinates for the Poincaré algebra is not trivial. A possible choice is

\[
X_+^1 = \Phi_2 \Phi_4, \quad X_+^2 = \Phi_2, \quad X_+^3 = \Phi_4, \quad X_-^1 = \Phi_1 \Phi_4, \quad X_-^2 = \Phi_1, \quad X_-^3 = \Phi_4,
\]

\[
X_+^+ = \frac{\Phi_1 \Phi_0 - \Phi_2 \Phi_3}{\Phi_1^2 - \Phi_2^2}, \quad X_-^- = \arccosh \frac{\Phi_0}{\sqrt{\Phi_1^2 - \Phi_2^2}},
\]

which satisfy the algebra (20). The field equations have therefore the form (22), with solutions (23). However, the relations between the original fields and the \( X_0^\pm \) are more involved:

\[
\Phi_2 = \gamma \cosh X_-^3, \quad \Phi_0 = \gamma X_+^+ \cosh X_-^3 + \gamma^{-1} (X_-^1 - X_+^+ X_-^2) \sinh X_-^3,
\]

\[
\Phi_1 = \gamma \sinh X_-^3, \quad \Phi_4 = \gamma X_+^+ \sinh X_-^3 + \gamma^{-1} (X_-^1 - X_+^+ X_-^2) \cosh X_-^3,
\]

where \( \gamma = \sqrt{X_+^+ - (X_+^2)^2} \).

In the gauge \( \theta^\pm = 0, \eta^+ \eta^- = J/2 \), setting \( Q^+ = M, Q^- = J, \eta^+ = r, \omega^+ = t, \omega^- = \phi \), one recovers solutions of the form (25), but with

\[
\Gamma(r) = \sqrt{M + \frac{J}{2r^2}}. \tag{35}
\]

These are the solution of ref. [14] in different coordinates, and describe flat spacetime with a conical singularity at the origin.

In the deformed case, for \( \lambda \to 0 \), the deformed algebra reduces to the deformed Poincaré algebra of ref. [5]. Again, the solutions can have the form (25) or (31), but with \( \Gamma \) given by (35).

We also remark that solutions in the case of positive cosmological constant can be obtained by analytic continuation \( \lambda \to i \lambda \) of the anti-de Sitter solutions, but we shall not discuss them here.
6 Conclusions

We have shown how to obtain the solutions of deformed three-dimensional models of gravity by using the Poisson sigma model formalism. In the case of anti-de Sitter or Poincaré invariance we recover the results obtained by standard methods. The discussion of the physical properties in the deformed case is not straightforward, since the geometry does not display the same invariance as the gauge theory, and depends on an interpretation of the model, which will be discussed in detail elsewhere [12].

The same techniques could be applied to more standard deformations of the gauge algebra that preserve Lorentz invariance, analogous to the two-dimensional models studied in [10]. A drawback of this formalism is however that there is no natural way to fix the gauge conditions on the fields. We have chosen them in such a way to obtain the known results in the undeformed limit, but it is difficult to give a general rule that clarifies the physical meaning of specific gauge choices and their relation with coordinate transformations.

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