Gauge Theory of the String Geodesic Field

ANTONIO AURILIA

Department of Physics
California State Polytechnic University
Pomona, CA 91768

ANAI S SMALI AGIC

International Center for Theoretical Physics, Trieste, Italy

and

EURO SPALUCCI

Dipartimento di Fisica Teorica
Università di Trieste,
INFN, Sezione di Trieste
Trieste, Italy 34014

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* E-Mail address: AURILIA@CSUPOMONA.EDU
† E-Mail address: ANAIS@ITSICTP.BITNET
‡ E-Mail address: SPALUCCI@TRIESTE.INFN.IT
ABSTRACT

A relativistic string is usually represented by the Nambu-Goto action in terms of the extremal area of a 2-dimensional timelike submanifold of Minkowski space. Alternatively, a family of classical solutions of the string equation of motion can be globally described in terms of the associated geodesic field.

In this paper we propose a new gauge theory for the geodesic field of closed and open strings. Our approach solves the technical and conceptual problems affecting previous attempts to describe strings in terms of local field variables. The connection between the geodesic field, the string current and the Kalb-Ramond gauge potential is discussed and clarified.

A non-abelian generalization and the generally covariant form of the model are also discussed.
1. Introduction, motivations and technical background

The dynamics of a relativistic point-particle of mass $m$ is encoded into an action which is essentially the proper length of the particle world-line $x^\mu = X^\mu(\tau)$, i.e.

$$S = -m \int d\tau \sqrt{-\dot{X}^\mu \dot{X}_\mu} \equiv \int d\tau L(X, \dot{X}),$$

where $\dot{X}^\mu(\tau)$ is the tangent 4-vector. Besides its geometrical meaning, this action also exhibits invariance under time reparametrization $\tau \rightarrow \tau'(\tau)$. With such a symmetry, this elementary system represents the “prototype” of any theory invariant under general coordinate redefinitions. The dynamical variables commonly used to describe point-like particles are either the Hamiltonian pair $(X^\mu, P_\mu)$, where $P_\mu$ is the linear momentum conjugate to the tangent vector $\dot{X}^\mu$ according to $P_\mu = \partial L/\partial \dot{X}^\mu$, or the Lagrangian coordinates $(X^\mu, \dot{X}^\mu)$. The two (dynamically equivalent) descriptions are related to each other by the Legendre transform.

A string is the simplest generalization of a point-like particle: it extends in one spatial dimension and spans, evolving in time, a two dimensional world-surface. If $\mathcal{H}$ is a domain in the space of the parameters $\xi^a = (\tau, \sigma)$ which represent local coordinates on the lorentzian string manifold, and $\Omega$ is an embedding of $\mathcal{H}$ in the Minkowski space $M$, then $\Omega : \xi \in \mathcal{H} \rightarrow \Omega(\xi) = X^\mu(\xi) \in M$. In particular, the tangent bi-vector in parameter space

$$\frac{\partial}{\partial \tau} \wedge \frac{\partial}{\partial \sigma}$$ (1.1)

is mapped by $\Omega$ into the tangent bi-vector

$$\dot{X}^{\mu \nu} = \frac{\partial X^\mu}{\partial \tau} \wedge \frac{\partial X^\nu}{\partial \sigma} = \delta^{[a b]} \partial_a X^\mu \partial_b X^\nu$$ (1.2)

at each point of the embedded sub-manifold $x^\mu = X^\mu(\xi)$ representing the string.
history \( \mathcal{W} \) in Minkowski spacetime. In the analogy with the point-particle case, a reparametrization invariant action, which is proportional to the world-sheet area, can be assigned to the string according to:

\[
S_{\text{NG}} = -\frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{-\frac{1}{2} \dot{X}^{\mu\nu} \dot{X}_{\mu\nu}} \equiv \int d\tau d\sigma L_{\text{NG}}(X, \dot{X}). \quad (1.3)
\]

In the canonical approach based on the analogy with the point-particle case, one defines two linear momenta, \( P_\mu \) and \( P'_\mu \) canonically conjugated to \( \dot{X}^\mu = \partial X^\mu / \partial \tau \) and \( X'^\mu = \partial X^\mu / \partial \sigma \) respectively, and then one develops the corresponding hamiltonian formalism. This is the usual starting point to string normal modes decomposition and subsequent quantization. However, this approach breaks reparametrization invariance on the world-sheet from the very beginning, so one of the main features of the model is lost. In order to preserve this symmetry at all times, one has to treat \( \tau, \sigma \) on an equal footing, and this requires a non-canonical formulation of string dynamics, i.e. the introduction of new non-canonical variables. In this connection, the key remark is that in going from a point-like object to an extended system, quantities like \( \partial_a X^\mu \) lose their physical meaning of “velocities”; rather, they become projectors on the string world-sheet. To maintain the analogy with the point-like particle case and with the geometric meaning of the velocity as tangent element to the world-trajectory of the physical object, it is preferable to define the string velocity as the tangent bi-vector (1.2). In this case, the conjugate dynamical variable is the area momentum [1,2]

\[
\Pi_{\mu\nu} \equiv \frac{\partial L_{\text{NG}}}{\partial \dot{X}^{\mu\nu}} = \frac{1}{2\pi\alpha'} \frac{\dot{X}_{\mu\nu}}{\sqrt{-\frac{1}{2} \dot{X}^{\mu\nu} \dot{X}_{\mu\nu}}}, \quad (1.4)
\]
which involves both $P_\mu$ and $P_\mu^\star$, and, according to eq.(1.4), is proportional to the unit norm tangent element to the string world-sheet. Then, $\Pi_{\mu\nu}$ satisfies the generalized mass-shell condition

$$-\frac{1}{2}\Pi_{\mu\nu}\Pi^{\mu\nu} = \frac{1}{(2\pi\alpha')^2}$$

(1.5)

which corresponds to the relativistic particle momentum constraint $p_\mu p^\mu = -\mu^2$.

Similarly, in terms of the dynamical variables $(X^\mu, \Pi_{\mu\nu})$, the string equations of motion take on the compact form

$$\delta^{[ab]}\partial_a\Pi_{\mu\nu}\partial_b X^\nu = 0$$

$$\left(\Pi_{\mu\nu}\partial_b X^\nu\right)|_{\partial\mathcal{W}} = 0.$$  

(1.6)

The embedding $\Omega$ maps the boundary of $\mathcal{H}$ into the world-lines of the the string end-points $x^\mu = X^\mu(\tau, \sigma = \sigma_1(\tau)) = X^\mu_1(\tau)$, $x^\mu = X^\mu(\tau, \sigma = \sigma_2(\tau)) = X^\mu_2(\tau)$.

Thus the first equation in (1.6) represents the conservation of the area momentum along the string world-sheet, while the second provides boundary condition at the string end points.

**Motivation and objectives**

* The area momentum is simply related to the canonical momenta; in fact

$$\Pi_{\mu\nu}\partial_c X^\nu = \frac{1}{2\pi\alpha'}\delta^{[ab]}\frac{\partial_a X_\mu\partial_b X_\nu}{\sqrt{-\frac{1}{2}X^\rho\sigma X_\rho\sigma}} \partial_c X^\nu$$

$$= \frac{1}{2\pi\alpha'}\delta^{[ab]}\frac{\partial_a X_\mu}{\sqrt{-\frac{1}{2}X^\rho\sigma X_\rho\sigma}} \gamma_{bc}$$

$$= \frac{1}{2\pi\alpha'}\delta^{[ab]}\gamma_{bc}\frac{\partial_a X_\mu}{\sqrt{-\frac{1}{2}X^\rho\sigma X_\rho\sigma}},$$

where $\partial_a X_\mu/\sqrt{-\frac{1}{2}X^\rho\sigma X_\rho\sigma} \equiv 2\pi\alpha' P_\alpha^{a\mu}$, and $P_{0\mu} \equiv P_\mu, P_{1\mu} \equiv P_\mu'$. 

5
To reiterate the main point of our introductory remarks: the choice of dynamical variables \((X^\mu, \Pi_{\mu\nu})\) offers several distinct advantages over the conventional choice of canonical variables. First, it preserves covariance on the world-sheet at any stage in the formulation of string dynamics. This property is particularly desirable since it suggests a novel approach to the quantum theory of extended systems [2]; second, it lends itself to a straightforward generalization to the case of submanifolds of higher dimensionality and, in the process, it enlightens the close correspondence between the theory of extended systems and the Hamilton-Jacobi formulation of the mechanics of point particles [3,4]. Against this background, our immediate objective is to show that the new choice of variables \((X^\mu, \Pi_{\mu\nu})\) enables one to cast the first equation of motion in (1.6) in the form of a Bianchi Identity. This property, in turn, opens the way to the formulation of string dynamics as the gauge theory of an antisymmetric tensor field. Such a gauge formulation for strings is the primary purpose of the paper. In physical terms, the payoff of this new gauge formulation is a mechanism of mass generation for antisymmetric tensor fields (in this specific instance, the Kalb-Ramond field). The resulting equations describe massive spin-1 particles and represent the relativistic counterpart of the London equations of superconductivity. Elsewhere we have speculated that this new mechanism of mass generation, when applied to an antisymmetric tensor field of rank-3, may play an important role in cosmology in connection with the problem of production of dark matter in the early universe [5]. An equally interesting application of the gauge formulation of string dynamics is briefly discussed in section-3 in connection with the induced gravity program pioneered by Zel’dovich and Sakharov [6] as a way to get around the long standing problem of quantizing General Relativity.
There we will argue that the Einstein and Kalb-Ramond terms are generated in the effective action for the background fields as induced quantum terms describing the low energy behavior of the underlying quantum string theory.

**Technical background**

Before we embark on a detailed discussion of the gauge formulation of string dynamics, it may be helpful to address the main technical difficulty that we need to overcome. Suppose that one is able to invert the relation between $x^\mu$ and $\xi^a \equiv (\tau, \sigma)$ so that $\partial_a$ in (1.6) is expressed through the chain rule as $\partial_a = \frac{\partial X^\lambda}{\partial \xi^a} \frac{\partial}{\partial X^\lambda}$. Then, the first equation in (1.6) can be written as

$$\delta^{[ab]} \partial_a X^\lambda \partial_b X^\nu \partial_\lambda \Pi_{\mu\nu} = 0 \implies \dot{X}^{\lambda\nu} \partial_\lambda \Pi_{\mu\nu} = 0.$$  

(1.7)

However, from the condition (1.5) we deduce that $\Pi_{\mu\nu} \partial_\alpha \Pi_{\mu\nu} = 0$. Then, eq.(1.7) can be fully antisymmetrized, yielding

$$\dot{X}^{\lambda\nu} \partial_{[\lambda} \Pi_{\mu\nu]} = 0.$$  

(1.8)

Thus, whenever the matrix $\dot{X}^{\lambda\nu}$ is non-degenerate, eq.(1.8) implies that $\Pi_{\mu\nu}$ satisfies a Bianchi-type Identity and therefore can be written, at least locally, in terms of a gauge potential $B_\mu(x)$. *The above remarks are quite general and apply both to closed and open strings.*

Unfortunately, this result hinges on two assumptions which are at least questionable. In fact:

i) in order to invert the relation $x^\mu = X^\mu(\tau, \sigma)$ one has to consider a two-parameter family of classical solutions $x^\mu = X^\mu(\tau, \sigma, \phi_1, \phi_2)$ and assign to $\phi_1, \phi_2$ the role of
additional coordinates in parameter space [7]. Then, the embedding functions establish a mapping between spacetime and parameter space which, however, may not be one-to-one and is thus beset with integrability problems. In any case, this kind of approach which describes strings in terms of a pair of scalar fields, i.e. \( \phi_1, \phi_2 \), is hard to interpret as a genuine gauge theory.

ii) If the matrix \( \dot{X}^{\mu\nu} \) represents the tangent element to the string world-sheet, then it is degenerate. In fact

\[
\det \dot{X}^{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} \epsilon_{\mu'\nu'\rho'\sigma'} \dot{X}^{\mu\mu'} \dot{X}^{\nu\nu'} \dot{X}^{\rho\rho'} \dot{X}^{\sigma\sigma'} \\
= \delta^{[ab]} \delta^{[cd]} \delta^{[ef]} \delta^{[gh]} \epsilon_{\mu\nu\rho\sigma} \partial_a X^\mu \partial_c X^\nu \partial_e X^\rho \partial_g X^\sigma \epsilon_{\mu'\nu'\rho'\sigma'} \partial_b X^{\mu'} \partial_d X^{\nu'} \partial_f X^{\rho'} \partial_h X^{\sigma'} \\
\equiv 0 ,
\]

since any four index totally antisymmetric tensor in two dimensions is identically zero. Therefore, eq. (1.8) does not imply the existence of a gauge potential for the string, even locally.

With the above observations in mind, in the next sections we shall discuss a lagrangian field theory which provides a consistent description of string dynamics in terms of local gauge fields. Presently, we shall briefly introduce the formal apparatus required to realize this programme.

The mathematical object needed to give \( \Pi_{\mu\nu}(\xi) \) the status of local variable, i.e. defined at any spacetime point rather than on the string world-sheet alone, is the slope field or sheet field [3,2] \( \Phi^{\mu\nu}(x) \). The slope field is a totally anti-symmetric tensor which assigns a tangent plane at any spacetime point. More precisely, due to its geometrical meaning, the slope-field is characterized by the following properties:
a) when evaluated on the string world-sheet, the slope field coincides with the tangent element, i.e. \( \Phi^{\mu\nu}(x = X) = \dot{X}^{\mu\nu} \);

b) it is orthogonal to its dual, i.e. \( \tilde{\Phi}^{\mu\nu}\Phi_{\mu\nu} = 0^* \);

c) if it satisfies the Bianchi Identities, i.e. \( \partial_{[\lambda}\Phi_{\mu\nu]} = 0 \), then it is called geodesic field [3].

Properties b) and c) imply that the 2-form \( \Phi \equiv \frac{1}{2}\Phi_{\mu\nu}(x)dx^\mu \wedge dx^\nu \) has rank 2 in four dimensions, rather than four, i.e. there exist a coordinate system where \( \Phi_{\mu\nu} \) can be written in terms of a pair of Clebsch [1] potentials \( S^1(x) \) and \( S^2(x) \) as \( \Phi(x) = dS^1 \wedge dS^2 \). This property is essential to solve the problem i) [8].

The role of the slope field in connection with string dynamics has already been discussed by some authors, but mainly from a kinematical point of view as a useful device to describe a family of minimal surfaces solving the classical string equation of motion [1-3].

Property c) led Nielsen and Olesen [9] to identify the string field strength as the dual of a closed world-sheet

\[
F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} \int_{D: \partial D = 0} d^2\sigma \dot{X}^{\rho\sigma} \delta^4 [x - X(\sigma)] \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} J^{\rho\sigma}. \tag{1.10}
\]

Here, the absence of a boundary guarantees that \( F_{\mu\nu} \) defined by (1.10) satisfies the Bianchi Identity so that, at least locally, a gauge potential can be defined:

\[
\epsilon^{\alpha\beta\mu\nu} \partial_\beta F_{\mu\nu} = 2\partial_\mu J^{\mu\alpha} = 0 \Rightarrow F_{\mu\nu} = \partial_{[\mu} A_{\nu]} . \tag{1.11}
\]

Therefore, in the Nielsen Olesen formulation, \( F_{\mu\nu} \) acquires the meaning of string

\* Sometimes in the literature this property is referred to as the Plücker condition [3].
geodesic field. However, the ansatz (1.10) has several drawbacks: first, it appears that only closed strings can be given a gauge type description; second, it is not possible to prove that (1.10) is a solution of the field equations of $A_\mu$; third, the degeneracy of the matrix $\dot{X}^{\mu\nu}$ forbids a frame-independent derivation of the string equation of motion from the $A_\mu$ field equation.

Our main purpose is to derive a general gauge description both for closed and open strings, solving the above technical and conceptual problems. The way-out is to give up the ansatz (1.10) and to consider the slope and the geodesic fields as basically distinct objects, which are related to each other by a set of classical field equations derived from a suitable Lagrangian density. The slope field accounts for the geometric properties a) and b); instead, the geodesic field represents the gauge partner of the slope field, and satisfies the Bianchi Identity by definition. Then, the solutions of the field equations provides the link between them, and relates geometric features to gauge properties of the string. However, in order to implement this program, one has to revise, from the very beginning, the relation between the slope field and the string current; then, one has to discuss the connection between the slope field, the string current and the Kalb-Ramond potential which mediates the gauge interaction between string elements.

Basically, the slope field is nothing but the generalization of the notion of velocity field in a continuous medium. Suppose that a spacetime region is completely filled with a fluid of point-like particles, each moving along non-intersecting worldlines. Rather than describing the fluid dynamics in terms of the motion of each microscopic constituent, one defines a regular vector field which matches at any point the 4-velocity of the corresponding particle. This idea can be applied to the
string theory as well, in which case $\Phi^{\mu\nu}$ can be related to the string current

$$J^{\mu\nu}(x) = \int \frac{d^2 \xi \delta^4}{\mathcal{H}} (x - Y(\xi)) \dot{Y}^{\mu\nu}.$$  \hspace{1cm} (1.12)

Indeed

$$J^{\mu\nu}(x) = \int d^2 \xi' \delta^4 (x - Y(\xi')) \dot{Y}^{\mu\nu}$$

$$= \text{const.} \frac{1}{a^2} \int d^2 \xi' \delta^2 (\xi - \xi') \frac{X^{\mu\nu}}{\sqrt{\frac{1}{2} \dot{X}_{\rho\sigma} \dot{X}^{\rho\sigma}}}$$  \hspace{1cm} (1.13)

$$= \text{const.} \Pi^{\mu\nu}(\xi)$$

where we have explicitly regularized the distribution $J^{\mu\nu}$ by assigning a physical width $a$ to the string. Notice that the string current satisfies the condition

$$\tilde{J}^{\mu\nu}(x)J_{\mu\nu}(x') =$$

$$\int d^2 \xi \int d^2 \xi' \delta^4 (x - X(\xi)) \delta^4 (x - Y(\xi')) \delta^{[ab]} \delta^{[cd]} \epsilon_{\mu\nu\rho\sigma} \partial_a X^\mu \partial_b X^\nu \partial_c Y^\rho \partial_d Y^\sigma \equiv 0$$  \hspace{1cm} (1.14)

again, because there is no totally antisymmetric four index tensor in two dimensions. The only relevant difference between the string current and the slope field is that $J^{\mu\nu}(x)$ is a distribution different from zero only along the string world-sheet, while $\Phi^{\mu\nu}(x)$ is a field defined over the whole spacetime manifold. The above results are concisely expressed by the relationship

$$J^{\mu\nu}(x) = \Phi^{\mu\nu}(x) \int d^2 \xi \delta^4 (x - X(\xi)),$$  \hspace{1cm} (1.15)

which is our own definition of the slope field in terms of the string current. By comparing eq.(1.12) with eq.(1.15) one would be tempted to say that $\Phi^{\mu\nu}(x)$ is nothing but $\dot{Y}^{\mu\nu}$ with $Y(\xi)$ replaced by $x^\mu$ [9]. This identification, however, must
be made with caution. As a matter of fact, the geodesic field resulting from a given family of minimal surfaces which are solutions of the classical string equation of motion, should be constructed according to the following procedure: given a classical solution $X^\mu(\xi)$, compute the corresponding area momentum, then use both the embedding equations $x^\mu = X^\mu(\xi)$ and the string equation of motion to write the area momentum as a function of $x$. Then, the resulting field is a smooth function of the coordinates and represents the “canonical” extension of $\Pi_{\mu\nu}$ in the sense that the 2-form $\Pi(x) \equiv \frac{1}{2} \Pi_{\mu\nu} dx^\mu \wedge dx^\nu$ has rank two [3].

The rest of the paper is organized as follows:

in Sect.2 we study a non-linear Lagrangian for the geodesic field of both open and closed strings interacting with Kalb-Ramond and electromagnetic potentials. The on-shell equivalence of this model with ordinary string theory is shown.

In Sect.3 we discuss the coupling of the string geodesic field to gravity, and its non-abelian generalization.

2. The String Gauge Field Strength

In this section we shall introduce a non-linear Lagrangian for the string geodesic field which is partly suggested by the physical interpretation of strings as extended solitons of an underlying local field theory, and partly by earlier investigation of string non-linear electrodynamics by Nambu [1,3], Nielsen and Olesen [9].

The two cases of open and closed strings have to be discussed separately.

Closed strings
Let us consider the following action

\[ S^{cl} = -\tilde{g}^2 \int d^4 x \sqrt{-\frac{1}{2} W_{\mu\nu} W^{\mu\nu}} + \frac{1}{2} \int d^4 x W^{\mu\nu} \partial_{[\mu} B_{\nu]} \]

\[ F_{\mu\nu}(x) \equiv \partial_{[\mu} B_{\nu]}(x) , \]

where \( W_{\mu\nu}(x) \) is a totally antisymmetric tensor and \( \tilde{g} \) is a dimensional constant. Physical dimensions are assigned as follows: \([W_{\mu\nu}] = [F_{\mu\nu}] = [\tilde{g}^2] = M^2\). An action of this type was proposed by Nambu as an effective abelian theory interpolating between QCD and classical string dynamics [3].

At this stage, the \( B \)-field appearing in eq.(2.1) is simply a Lagrange multiplier enforcing a "transversality" condition for the \( W_{\mu\nu} \) field; we shall see that on-shell the \( B_\mu \)-field becomes the string gauge potential. In fact, by varying the action (2.1) with respect to \( B_\mu \) and \( W_{\mu\nu} \), we get the following set of field equations

\[ \partial_\mu W^{\mu\nu} = 0 , \]

\[ \tilde{g}^2 \frac{W_{\mu\nu}}{\sqrt{-\frac{1}{2} W_{\alpha\beta} W^{\alpha\beta}}} + F_{\mu\nu} = 0 . \]

The closed string appears as a special solution of (2.2), namely

\[ \hat{W}^{\mu\nu}(x) = c \int \frac{\delta^4(x - X(\xi))}{\mathcal{H}} X^{\mu\nu} = c J^{\mu\nu}(x) , \]

\[ c = \text{dimensionless const.} . \]

The general solution of (2.2), i.e., \( W^{\mu\nu} = c J^{\mu\nu} + \epsilon^{\mu\nu\rho\sigma} \partial_\rho V_\sigma \), includes a "radiation part " described by a vector field \( V_\lambda(x) \), which we set equal to zero everywhere in the following discussion since our present purpose is to identify string-like solutions of the action (2.1).
The right hand side of (2.4) is, except for a multiplicative constant, the current distribution associated with the two-dimensional manifold \( W \) representing the string history. If the string is spatially closed, then \( \partial W = \emptyset \) and \( J^{\mu\nu} \) has vanishing divergence. Eq.(2.3) is an algebraic relation linking the slope field \( W^{\mu\nu}(x) \) to the string field strength \( F^{\mu\nu} \):

\[
F^{\mu\nu}(x) = -\bar{g}^2 \frac{\dot{W}^{\mu\nu}(x)}{\sqrt{-\frac{1}{2}W_{\alpha\beta}W^{\alpha\beta}}} = -\bar{g}^2 \frac{J^{\mu\nu}(x)}{\sqrt{-\frac{1}{2}J_{\alpha\beta}J^{\alpha\beta}}} = -\bar{g}^2 \frac{\Phi^{\mu\nu}(x)}{\sqrt{-\frac{1}{2}\Phi_{\alpha\beta}\Phi^{\alpha\beta}}} \tag{2.5}
\]

from which it follows that the Hamilton-Jacobi (H-J) equation

\[
-\frac{1}{2} F^{\mu\nu}(x) F_{\mu\nu}(x) = \bar{g}^4 \tag{2.6}
\]

holds everywhere. Thus, \( F = \frac{1}{2} F^{\mu\nu} dx^\mu \wedge dx^\nu \) (which is closed by definition) is a 2-form which satisfies (on-shell) the generalized H-J equation (2.6).

The net result of these manipulations is that while \( W^{\mu\nu}(x) \) is a singular field having support only along the string history, \( F^{\mu\nu} \) is defined over the whole space-time manifold. Furthermore, when evaluated on the string world-tube, \( F^{\mu\nu} \) is proportional to the area conjugate momentum. In fact,

\[
F^{\mu\nu}(x = X(\xi)) = -\bar{g}^2 \frac{\dot{X}^{\mu\nu}}{\sqrt{-\frac{1}{2}\dot{X}_{\alpha\beta}\dot{X}^{\alpha\beta}}} \equiv -\frac{1}{c} \Pi^{\mu\nu} . \tag{2.7}
\]

Conversely, eq.(2.5) defines the area field \( \Pi^{\mu\nu}(x) \) which is the canonical(=rank-2) extension of the volume momentum:

\[
\Pi^{\mu\nu}(x) \equiv -c F^{\mu\nu}(x) = \frac{1}{2\pi \alpha'} \frac{\Phi^{\mu\nu}(x)}{\sqrt{-\frac{1}{2}\Phi_{\alpha\beta}\Phi^{\alpha\beta}}} . \tag{2.8}
\]

Note that eq.(2.7) implies that we identify the term \( \bar{g}^2 c \) with the string tension. That this is indeed the case can be verified directly by inserting the solution (2.4)
into the action (2.1). This operation yields the classical effective action for \(X(\xi)\),

\[
S_{\text{eff}} = -c\bar{g}^2 \int d^4x \left( \frac{1}{2} \Phi_{\alpha\beta} \Phi^{\alpha\beta} \left( \int_{\mathcal{H}} H d^3\xi \delta^4(x - X(\xi)) \right) \right)^2
\]

\[
= -c\bar{g}^2 \int d^4x \int_{\mathcal{H}} d^2\xi \delta^4(x - X(\xi)) \sqrt{-\frac{1}{2} \dot{X}_{\alpha\beta} \dot{X}^{\alpha\beta}}
\]

\[
= -\frac{1}{2\pi \alpha'} \int_{\mathcal{H}} d^2\xi \sqrt{-\frac{1}{2} \dot{X}_{\alpha\beta} \dot{X}^{\alpha\beta}}.
\]

which represents the action for a free string with an effective string tension \(1/2\pi \alpha' \equiv c\bar{g}^2\).

Finally, as a consistency check, we wish to show that the gauge field representation of the string in terms of \(F_{\mu\nu}\) leads to the classical equations of motion (1.6). To this end, we recall that \(F_{\mu\nu}\) satisfies the Bianchi identities everywhere, so that in view of eq. (2.8)

\[
\partial_{[\lambda} \Pi_{\mu\nu]}(x) = 0
\]

(2.10)

at each spacetime point. Then we can project eq.(2.10) along the string history, that is, we evaluate \(\Pi_{\mu\nu}(x)\) at \(x = X(\xi)\) and take the interior product with \(\dot{X}^{\lambda\mu}\):

\[
\dot{X}^{\lambda\mu} \partial_{[\lambda} \Pi_{\mu\nu]}(\xi) = \\
\delta^{[ab]} \partial_a X^{\lambda} \partial_b X^{\mu} \partial_{[\lambda} \Pi_{\mu\nu]}(\xi) = \\
\delta^{[ab]} \partial_b X^{\mu} \partial_a \Pi_{\mu\nu}(\xi) = 0.
\]

(2.11)

The last line in (2.11) is just the classical equation of motion (1.6) of the string.

\* Note that second term in (2.1) does not contribute to (2.9) since \(\partial_\mu J^{\mu\nu} = 0\)
The gauge interaction

Closed inter-string interaction is known to be mediated by the Kalb-Ramond gauge potential $A_{\mu\nu}(x)$ [10]. Thus, it seems natural to ask what is the relationship, if any, between the geodesic field associated to the string and its gauge partner $A_{\mu\nu}(x)$. In other words, once it is accepted that the action (2.1) describes a theory of closed strings, the next step is to study how to introduce the interaction with the Kalb-Ramond field.

Let us consider the following model:

$$S^{\text{cl.}} = \int d^4x \left[ -\bar{g} \sqrt{-\frac{1}{2} W_{\mu\nu} W^{\mu\nu} + \frac{1}{2} W^{\mu\nu} \partial_{[\mu} B_{\nu]} + \frac{\kappa}{2} W^{\mu\nu} A_{\mu\nu} - \frac{1}{2 \cdot 3!} H_{\mu\nu\rho} H^{\mu\nu\rho} \right]$$

(2.12)

where the coupling constant $\kappa$ has the dimension of mass, and $H_{\mu\nu\rho} = \partial_{[\mu} A_{\nu\rho]}$ is the Kalb-Ramond field strength. The action (2.12) is invariant under the set of transformations

$$\delta A_{\mu\nu} = -\frac{1}{\kappa} \partial_{[\mu} \Lambda_{\nu]} ,$$

$$\delta B_{\mu} = \Lambda_{\mu} + \partial_{\mu} \phi ,$$

$$\delta W^{\mu\nu} = 0 ,$$

(2.13)

which shows that under the generalized gauge transformation of the Kalb-Ramond potential, the $B$-field transforms as the corresponding Goldstone Boson. From this viewpoint, $B_{\mu}$ can be seen as the gauge part of $A_{\mu\nu}$ or as a Stueckelberg compensating field.
Now the field equations become

\[ \partial_\mu W^{\mu\nu} = 0, \quad (2.14) \]

\[ \frac{g^2}{\sqrt{-\frac{1}{2}W_{\alpha\beta}W^{\alpha\beta}}} \partial_{[\mu}B_{\nu]} + \kappa A_{\mu\nu} = 0, \quad (2.15) \]

\[ \partial_\mu H^{\nu\rho} + \kappa W^{\nu\rho} = 0. \quad (2.16) \]

Again, the closed string appears as a special solution of the type (2.4) of eq.(2.14), while the field strength of the string, i.e. the geodesic field of the string, can be absorbed into a redefinition of the Kalb-Ramond field \( \tilde{A}_{\mu\nu} \) which is gauge invariant under (2.13), and therefore can be interpreted as the physical string field strength

\[ \kappa \tilde{A}_{\mu\nu} \equiv \kappa A_{\mu\nu} + \partial_{[\mu}B_{\nu]} = \frac{g^2}{\sqrt{-\frac{1}{2}W_{\alpha\beta}W^{\alpha\beta}}} W^{\mu\nu}, \quad (2.17) \]

\[ -\frac{1}{2} \tilde{A}_{\mu\nu} \tilde{A}^{\mu\nu} = \frac{g^4}{\kappa^2}, \quad (2.18) \]

\[ \tilde{A}_{\mu\nu}(x = X(\xi)) = \frac{\tilde{g}^2}{\kappa} \frac{X^{\mu\nu}}{\sqrt{-\frac{1}{2}X_{\alpha\beta}X^{\alpha\beta}}} \equiv -\frac{1}{ck} \Pi_{\mu\nu}(\xi). \quad (2.19) \]

If we insert the solutions of eqs.(2.14-2.16) in the action (2.12), and take into account that the field strength \( H_{\mu\nu\rho} \) of \( A_{\mu\nu} \) and \( \tilde{H}_{\mu\nu\rho} \) of \( \tilde{A}_{\mu\nu} \) are the same, then we find the effective action

\[ S_{\text{eff.}} = -\frac{1}{2\pi\alpha'} \int_{\mathcal{H}} d^2\xi \sqrt{-\frac{1}{2}X_{\alpha\beta}X^{\alpha\beta}} + \frac{\bar{\kappa}}{2} \int d^4x J^{\mu\nu} \tilde{A}_{\mu\nu} - \frac{1}{2\cdot3!} \int d^4x \tilde{H}_{\mu\nu\rho} \tilde{H}^{\mu\nu\rho} \]

(2.20)

which is just the usual Kalb-Ramond classical action coupled to a closed string,
where, $1/2\pi \alpha' \equiv e g^2$ is the effective string tension, and $\tilde{\kappa} \equiv c\kappa$ is the effective coupling constant.

By inverting eq.(2.19) one obtains for the string geodesic field $\Pi_{\mu \nu}(x) = -\tilde{\kappa}\tilde{A}_{\mu \nu}(x)$. Then, the curl of $\Pi_{\mu \nu}(x)$ turns out to be proportional to the Kalb-Ramond field strength

$$\partial_{[\mu}\Pi_{\nu \rho]}(x) = -\tilde{\kappa}\tilde{H}_{\mu \nu \rho}(x)$$

Finally, by projecting eq.(2.21) on the string-world sheet, we obtain

$$\dot{X}^{\mu \nu}\partial_{[\mu}\Pi_{\nu \rho]}(x) = -\tilde{\kappa}\tilde{H}_{\mu \nu \rho}(x)\dot{X}^{\mu \nu} \Rightarrow \delta^{ab}\partial_a\Pi_{\mu \nu}\partial_b X^\nu = -\frac{\tilde{\kappa}}{2}\tilde{H}_{\rho \nu \mu}(x)\dot{X}^{\nu}, \quad (2.22)$$

which is the “Lorentz force” equation for the string. Furthermore, by substituting the solution (2.4) into eq.(2.16), we obtain the Kalb-Ramond field equation coupled to the string current

$$\partial_{\mu}H^{\mu \nu \rho} = \tilde{\kappa}J^{\nu \rho}.$$

**Open string**

The open case requires some further discussion. In fact, the current of an open string has a non vanishing divergence, and thus it does not satisfy eq.(2.2). More precisely

$$\partial_{\mu}J^{\mu \nu} = \int d\tau \sum_{i=1,2} (-1)^i \delta^4(x - X(\tau, \sigma = \sigma_i(\tau))) \left( \frac{dX^\nu}{d\tau} \right)_{\sigma = \sigma_i} \equiv J^{\nu}(x) \quad (2.24)$$

where $x^\mu = X^\mu(\tau, \sigma = \sigma_i(\tau))$ represent the world-lines of the two string end points $\sigma = \sigma_1, \sigma = \sigma_2$. Read from right to left, eq.(2.24) represents the old “trick”
used by Dirac to describe the electrodynamics of a pair of opposite point-charges in terms of string variables [11]. The boundary current $J^\nu$ has vanishing divergence as consequence of the identity $\partial_\mu \partial_\nu J^{\mu\nu} \equiv 0$. The above remarks suggest the rationale to modify our gauge field formalism in order to be able to describe the open string as well. The action (2.12) has to be supplemented with terms describing the motion of the string end points and with an abelian vector gauge potential to compensate for the “leakage” of symmetry through the boundary.

\[
S^{\text{op}} = \int d^4x \left[ -\frac{g^2}{3!} W^{\mu\nu}(x) W^{\mu\nu}(x) + \frac{1}{2} W^{\mu\nu}(x) \partial_\mu B_\nu(x) + f B_\mu(x) J^\mu(x) \\
+ \frac{\kappa}{2} W^{\mu\nu}(x) A_{\mu\nu} - \frac{1}{2 \cdot 3!} H_{\mu\nu\rho} H^{\mu\nu\rho} \\
+ e A_\mu J^\mu - \frac{1}{4} \left( F^{\mu\nu} - \frac{f \kappa}{e} A^{\mu\nu} \right) \left( F^{\mu\nu} - \frac{f \kappa}{e} A^{\mu\nu} \right) \\
- \mu_0 \int d\tau \sum_{i=1,2} \sqrt{-\dot{X}_i^{\mu}(\dot{X}_i^{\mu})} ,
\]

(2.25)

where $\kappa$ is the Kalb-Ramond coupling constant, and $\mu_0$ is the mass of the particles located at the string boundaries. Note that the action depends now explicitly on $B$ because of the coupling to the boundary current, whereas the action (2.12) depends on $B$ only through its field strength. With hindsight one realizes that the action (2.25) is designed to ensure that a special solution of the type (2.4) still exists.

The model is now invariant under the extended gauge transformations.
\[ \delta A_{\mu\nu} = \partial_{[\mu} \Lambda_{\nu]} , \]
\[ \delta B_{\mu} = \partial_{\mu} \theta - \kappa \Lambda_{\mu} , \]
\[ \delta A_{\mu} = \partial_{\mu} \phi + \frac{\kappa f}{e} \Lambda_{\mu} , \]
\[ \delta W^{\mu\nu} = 0 , \]
\[ \delta X^{\mu}(\tau) = 0 . \]

From the above gauge transformations it follows that \( \delta H^{\mu\nu\rho} = 0 \), but
\[ \delta F_{\mu\nu} = \frac{\kappa f}{e} \partial_{[\mu} \Lambda_{\nu]} \quad (2.27) \]
so that \( F \) does not represent a physical quantity. Gauge invariant field strengths can be assembled as follows:

\[ \delta \left( F_{\mu\nu} - \frac{f}{e} A_{\mu\nu} \right) = 0 , \]
\[ \delta \left( eF_{\mu\nu} + f \partial_{[\mu} B_{\nu]} \right) = 0 , \quad (2.28) \]
\[ \delta \left( \partial_{[\mu} B_{\nu]} + \kappa A_{\mu\nu} \right) = 0 . \]

The set of eqs. (2.28) displays the mixing of various fields to form physical (=gauge invariant ) quantities; specifically, the first equation in (2.28) justifies, \textit{a posteriori}, the non-standard choice for the \( A_{\mu} \) kinetic term in (2.25). Varying the action (2.25) with respect to \( A_{\mu\nu} \), \( A_{\mu} \), \( B_{\mu} \), \( W_{\mu\nu} \) and \( X^{\mu}(\tau) \) we get the following set of field equations

\[ \delta A_{\mu\nu} S^{\text{op.}} = 0 : \quad \partial_{\mu} H^{\mu\nu\rho} - \frac{f}{e} \left( F^{\nu\rho} - \frac{f}{e} A^{\nu\rho} \right) = -\kappa W^{\nu\rho} , \quad (2.29) \]
\[ \delta B_{\mu} S^{\text{op.}} = 0 : \quad \partial_{\mu} W^{\mu\nu}(x) = f J^{\nu}(x) , \quad (2.30) \]
\[
\delta W_{\mu\nu} S^{\text{op.}} = 0 : \quad \hat{g}^2 \frac{W_{\mu\nu}(x)}{\sqrt{\frac{1}{2} W_{\alpha\beta} W^{\alpha\beta}}} + \partial_{[\mu} B_{\nu]} + \kappa A_{\mu\nu} = 0 , \tag{2.31}
\]

\[
\delta X_{i(\tau)} S^{\text{op.}} = 0 : \quad \frac{dP_i}{d\tau} = (-1)^i \left( f \partial_{[\mu} B_{\nu]} + e F_{\mu\nu} \right) \dot{X}_{i\nu} , \tag{2.32}
\]

\[
\delta A_{\mu} S^{\text{op.}} = 0 : \quad \partial_{\mu} \left( F^{\mu\nu} - \frac{f}{e} A^{\mu\nu} \right) = e J^\nu ; \tag{2.33}
\]

where

\[
P_i = -\mu_0 \frac{\dot{X}_i}{\sqrt{-\dot{X}_i^\nu \dot{X}_i^\nu}} . \tag{2.34}
\]

At this point several comments seem appropriate: a) the coupled field equations above involve only gauge invariant combinations of the various field variables. Eq.(2.31) is the same as (2.15), and again relates \( A_{\mu\nu} \) and \( B_{\mu\nu} \) to \( W_{\mu\nu} \); as promised, eq.(2.30) admits a special solution \( \hat{W}^{\mu\nu} \) which is proportional to the current of an open string having the two world-lines \( x = X_{1(\tau)} \) and \( x = X_{2(\tau)} \) as its only boundary:

\[
\hat{W}^{\mu\nu}(x) = f J^{\mu\nu}(x) ,
\]

\[
J^{\mu\nu}(x) = \int_H d\tau d\sigma \delta^4 \left( x - X(\tau, \sigma) \right) \dot{X}^{\mu\nu} , \quad \partial_{\mu} J^{\mu\nu}(x) = J^\nu(x) . \tag{2.35}
\]

b) Equation(2.32) describes the motion of the boundary under the combined action of the string geodesic field \( \partial_{[\mu} B_{\nu]} \) and the Lorentz force acting on the charged end points.

c) Equation (2.31) relates \( B_{\mu} \) and \( A_{\mu\nu} \) to the slope field:

\[
\partial_{[\mu} B_{\nu]} + \kappa A_{\mu\nu} = -\hat{g}^2 \frac{\Phi^{\mu\nu}(x)}{\sqrt{-\frac{1}{2} \Phi_{\alpha\beta} \Phi^{\alpha\beta}}} = -\frac{1}{f} \Pi_{\mu\nu}(x) , \tag{2.36}
\]

where the effective string tension now is \( 1/2\pi \alpha' \equiv \hat{g}^2 f \).
d) Using the string solution (2.35), the above system of field equations can be written in the form:

\[ \partial_{\mu}H^{\mu\nu\rho} - \frac{f_{\kappa}}{e}(F^{\nu\rho} - \frac{f_{\kappa}}{e}A^{\nu\rho}) = -f_{\kappa}J^{\nu\rho}, \tag{2.37} \]

\[ \frac{dP_{i\mu}}{d\tau} = (-1)^{i}(\Pi_{\mu\nu} - f_{\kappa}A_{\mu\nu} + eF_{\mu\nu})\dot{X}_{i}^{\nu} \quad i = 1, 2 \tag{2.38} \]

\[ \partial_{\nu}(F^{\nu\rho} - \frac{f_{\kappa}}{e}A^{\nu\rho}) = eJ^{\rho}, \tag{2.39} \]

\[ \partial_{[\mu}\Pi_{\nu\rho]} = -f_{\kappa}H^{\mu\nu\rho} . \tag{2.40} \]

The interpretation of the above equations is as follows. First, equation (2.40) which is the covariant curl of eq (2.36), provides the actual link with string dynamics. Indeed, by projecting eq.(2.40) on the world-sheet, one recovers the equation of motion (2.22) for the “body” of the string. Instead, eq.(2.38) describes the dynamics of the string end-points acted upon by the “internal force” \( \Pi_{\mu\nu}\dot{X}^{\nu} \), and by the “extended Lorentz force” \((f_{\kappa}A_{\mu\nu} - eF_{\mu\nu})\dot{X}^{\nu}\). Thus, eqs.(2.38) and (2.40) are the equations that actually govern the dynamics of the string.

The other two field equations tell us something new: first we note that eq.(2.39) guarantees that \( H^{\mu\nu\rho} \) is a regular function. In fact, from equation (2.37)

\[ \partial_{\nu}\partial_{\mu}H^{\mu\nu\rho} = \frac{f_{\kappa}}{e}\partial_{\nu}\left(F^{\nu\rho} - \frac{f_{\kappa}}{e}A^{\nu\rho}\right) - f_{\kappa}\partial_{\nu}J^{\nu\rho} \]

\[ = f_{\kappa}J^{\rho} - f_{\kappa}J^{\rho} \equiv 0 . \tag{2.41} \]

Moreover, if one looks at \( \frac{f_{\kappa}}{e}\tilde{A}^{\nu\rho} = -F^{\nu\rho} + \frac{f_{\kappa}}{e}A^{\nu\rho} \) as a “gauge transformation” of \( A^{\nu\rho} \), leaving the Kalb-Ramond field strength \( H^{\mu\nu\rho} \) unmodified, then eq.(2.37) can
be written as a *London-type equation*

$$\partial_{\mu} H^{\mu\nu\rho} - m^2 \tilde{A}^{\nu\rho} = -\bar{\kappa} J^{\nu\rho}, \quad m^2 \equiv \frac{f\kappa}{e}, \quad \bar{\kappa} \equiv f\kappa , \quad (2.42)$$

describing the propagation of a *massive, spin 1* field coupled to its source $J^{\nu\rho}(x)$. So, the initially *massless, spinless* field $A_{\mu\nu}$, because of its mixing with the vector gauge potential $A_{\mu}$ acquires mass and spin: this is a peculiar mechanism through which tensor gauge potentials become massive [12]. Notice that the count of degrees of freedom is the same as in the conventional Higgs mechanism: a spin-1 gauge field with two degrees of freedom combines with a massless spin-0 field thereby acquiring the three degrees of freedom necessary to describe a massive spin-1 particle. The difference here lies in the fact that the massless spin-0 field is itself a gauge field, i.e., the rank-2 antisymmetric Kalb-Ramond potential $A_{\mu\nu}$. On physical grounds, equations (2.37-2.40) evoke the familiar picture of Abrikosov vortices existing in type-2 superconductors. What is at work here is a kind of relativistic Meissner effect: we have a massive (superconducting) medium embedded into which are strings (vortices) compressed by the pressure of the surrounding medium into lines of electromagnetic flux. From here two scenarios come to mind. The first is the familiar one of quark confinement (electric or magnetic) originally advocated by Nambu: as flux tubes squeezed by the pressure of a mass inducing medium, such strings constitute very suitable traps for the hadron constituents (string end points). Alternatively, one may be tempted to associate such lines with “cosmic strings” as seeds of material structures in an otherwise featureless sea of dark matter.* In either scenario, eqs.(2.37) and (2.39) or eq. (2.42) apply to

* A similar picture can be constructed in terms of higher dimensional extended objects, i.e. membranes or bags and higher rank antisymmetric tensor gauge fields [5].
the surrounding background medium, whereas equations (2.38) and (2.40) govern
the evolution of the string in such a background. Clearly, if there is any element of
truth in the above scenarios, then two questions arise immediately: i) what is the
effect of gravity on the gauge formulation of string dynamics?, and ii) with an eye
on the Standard Model of particle physics, is it possible to attach internal indices
to the string geodesic field $W^{\mu\nu}$?

We will briefly examine both questions in the next section. However, for com-
pleteness, we shall close this section with the observation that the equations of
motion (2.37)-(2.40) can be derived from the effective action

$$
S_{\text{eff.}} = - \frac{1}{2\pi\alpha'} \int \sqrt{-G} \left[ \frac{1}{2} X_{\alpha\beta} \partial_{\alpha} X_{\beta} + \int d^4x \left[ \frac{f \kappa}{2} J^{\mu\nu}(x) A_{\mu\nu} - \frac{1}{2 \cdot 3!} H_{\mu\nu\rho} H^{\mu\nu\rho} \right] + \int d^4x \left[ e A_{\mu} J^{\mu} - \frac{1}{4} \left( F^{\nu\rho} + \frac{f \kappa}{e} A^{\nu\rho} \right) \left( F_{\nu\rho} + \frac{f \kappa}{e} A_{\nu\rho} \right) \right] - \mu_0 \sum_{i=1,2} \int d\tau \sqrt{-\dot{X}_i} \right]
$$

(2.43)

which is obtained by inserting the solution (2.35) in the action (2.12), and repre-
sents an open string with massive charges at the end points. “Neutral” strings
with massive end-points have been studied by several authors [13], [14], mainly in
connection with hadron dynamics [15]. The novel feature of $S_{\text{eff.}}$ is the “residual
gauge symmetry” [12]

$$
\delta A_{\mu\nu} = \partial_{[\mu} \Lambda_{\nu]} \\
\delta A_{\mu} = \partial_{\mu} \phi + \frac{f \kappa}{e} \Lambda_{\mu},
$$

(2.44)

surviving after the elimination of $B_{\mu\nu}$ in favor of the string variables.
3. Generally covariant and non-abelian formulation

As we anticipated in the previous section, here we consider the possibility of extending the formalism described so far in two directions: i) coupling the system to gravity, and ii) non-abelian string geodesic fields.

**Generally Covariant Formulation.**

For the sake of simplicity, we shall consider the interaction of closed strings with the gravitational field. Accordingly, we substitute in (2.1) the Minkowski metric with \( g_{\mu\nu}(x) \), replace ordinary derivatives with generally covariant ones \( \nabla_{\mu} \), and add the Einstein action. Thus, we are led to consider

\[
S_{cl.} = -\bar{g}^2 \int d^4x \sqrt{-g} \left[ \frac{1}{2} g_{\mu\rho} g_{\nu\sigma} W^{\mu\nu} W^{\rho\sigma} + \frac{1}{2} \int d^4x \sqrt{-g} W^{\mu\nu} \nabla_{[\mu} B_{\nu]} \right] - \frac{1}{16\pi G} \int d^4x \sqrt{-g} R
\]

(3.1)

\( F_{\mu\nu}(x) \equiv \partial_{[\mu} B_{\nu]}(x) \equiv \nabla_{[\mu} B_{\nu]} \).

Here, \( g \equiv \det g_{\mu\nu} \). The corresponding set of field equations represents the generally covariant generalization of eqs.(2.2-2.3)

\[
\nabla_{\mu} W^{\mu\nu} = 0 \Rightarrow \frac{1}{\sqrt{-g}} \partial_{\mu} \sqrt{-g} W^{\mu\nu} = 0 \quad (3.2)
\]

\[
-g^2 \frac{W^{\mu\nu}}{\sqrt{-W_{\alpha\beta} W^{\alpha\beta}}} = \bar{F}_{\mu\nu} , \quad W^{\alpha\beta} W_{\alpha\beta} \equiv \frac{1}{2} g^{\alpha\mu} g^{\beta\nu} W_{\alpha\beta} W^{\mu\nu} , \quad (3.3)
\]

supplemented by the Einstein equations:

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \quad (3.4)
\]
where the energy-momentum tensor in the r.h.s. is given by

\[
T_{\mu\nu} = -2 \left[ \frac{g^2}{2} \frac{W_{\mu\alpha} W_{\nu}^\alpha}{\sqrt{-W_{\alpha\beta} W_{\alpha\beta}}} + W_{\mu\alpha} F_{\nu}^\alpha \right] + g_{\mu\nu} L. \tag{3.5}
\]

Then, a closed string appears as a special solution of (3.2), namely

\[
\hat{W}^{\mu\nu}(x) = \frac{c}{\sqrt{-g}} \frac{1}{\dot{t}} \int d^2\xi \, \delta^4(x - X(\xi)) \hat{X}^{\mu\nu} = c J^{\mu\nu}(x), \tag{3.6}
\]

\(c = \) dimensionless const.,

which is nothing but the general covariant form of the string-current. To demonstrate the equivalence with gravity coupled to a closed string we still have to show that (3.4) is the Einstein field equation with a string source in the r.h.s. To do that, we notice that the Lagrangian in (3.5) vanishes on-shell, i.e. \(L(F; W) = 0\). Then

\[
T_{\mu\nu} = g^2 c^2 \frac{J_{\mu\alpha} J_\nu^\alpha}{\sqrt{-\frac{1}{2} J_{\alpha\beta} J_{\alpha\beta}}} = g^2 c^2 \frac{1}{\sqrt{-g}} \int d^3\xi \frac{\hat{X}_{\mu\alpha} \hat{X}_{\nu}^\alpha}{\sqrt{-\frac{1}{2} \hat{X}_{\alpha\gamma} \hat{X}_{\beta\gamma}}} \delta^4(x - X(\xi)), \tag{3.7}
\]

and the equivalence with General Relativity becomes manifest once we show that the energy-momentum tensor has vanishing divergence. This is indeed the case if \(X^\mu(\xi)\) represents a classical solution of the string equation of motion. In fact
\[ \nabla_\mu T^{\mu\nu} = \frac{g^2 c^2}{\sqrt{-g}} \int d^2 \xi \frac{\dot{X}^{\mu} \dot{X}^{\nu}}{-\frac{1}{2} \dot{X}^{\beta\gamma} \dot{X}_{\beta\gamma}} \delta^4 \nabla_\mu (x - X(\xi)) \]

\[ = -\frac{g^2 c^2}{\sqrt{-g}} \int d^2 \xi \delta^{[ma]} \frac{\partial_a X_\alpha \dot{X}^{\nu\alpha}}{-\frac{1}{2} \dot{X}^{\beta\gamma} \dot{X}_{\beta\gamma}} \nabla_m \delta^4 (x - X(\xi)) \]

\[ = -\frac{g^2 c^2}{\sqrt{-g}} \int d^2 \xi \delta^{[ma]} \left[ \nabla_m \left( \frac{\partial_a X_\alpha \dot{X}^{\nu\alpha}}{-\frac{1}{2} \dot{X}^{\beta\gamma} \dot{X}_{\beta\gamma}} \right) \delta^4 (x - X(\xi)) \right] . \tag{3.8} \]

The first term in the last line of equation (3.8) is a pure surface integral, which is zero for a closed string. Furthermore, anti-symmetrized covariant derivatives can be replaced with ordinary partial derivatives, and this yields the desired result,

\[ \nabla_\mu T^{\mu\nu} = \frac{g^2 c^2}{\sqrt{-g}} \int d^2 \xi \delta^{[ma]} \partial_m \left( \frac{\partial_a X_\alpha \dot{X}^{\nu\alpha}}{-\frac{1}{2} \dot{X}^{\beta\gamma} \dot{X}_{\beta\gamma}} \right) \delta^4 (x - X(\xi)) \]

\[ = \frac{1}{\sqrt{-g}} \int d^2 \xi \left[ \delta^{[ma]} \partial_m \Pi^{\nu\alpha} \partial_a X_\alpha \right] \delta^4 (x - X(\xi)) = 0 . \tag{3.9} \]

Notice how the consistency condition (3.9) immediately gives the equation of motion of the string as an alternative to projecting the Bianchi identity for \( \tilde{F}_{\mu\nu} \) on the string world-sheet. This property represents a distinct advantage of the generally covariant formulation of our model. Ultimately, of course, the consistency condition for the Einstein equations coupled to matter, i.e., that both the Ricci tensor and the energy momentum tensor must have vanishing covariant divergence, can be traced back to the Bianchi Identity for the Riemann tensor.
At this point one might elect to investigate the nature of the solutions of the classical system (3.1). However, in view of the equivalence that we have just established, a discussion of some such solutions already exists in the literature, especially in connection with cosmic strings [16]. Presently, what interests us is a deeper conceptual question: with an eye on quantum gravity and on its attending difficulties, is it really necessary to include from the very beginning the Einstein term in the action (3.1)?

We are partly led to this question by the result communicated in a previous article [17] where we have shown that General Relativity may arise as the low energy limit of a quantum theory of relativistic membranes. Indeed, the current attitude towards ultra short distance physics is to replace local fields with extended objects, mainly strings, as fundamental constituents of matter and to treat particle physics below some (string) energy scale as a local limit of the fundamental theory. In the following we shall argue briefly that our gauge formulation of string dynamics is perfectly consistent with this point of view.

Suppose we start from the action

\[ S^{\text{cl.}} = -\bar{g}^2 \int d^4x \sqrt{-g} \left\{ -\frac{1}{2} g_{\mu\rho} g_{\nu\sigma} W^{\mu\nu} W^{\rho\sigma} + \frac{1}{2} \int d^4x \left[ \sqrt{-g} W^{\mu\nu} \nabla_{[\mu} B_{\nu]} + W^{\mu\nu} K_{\mu\nu} \right] \right\}, \]

(3.10)

where \( g_{\mu\nu}(x) \) and \( K_{\mu\nu}(x) \) represent respectively a symmetric and an anti-symmetric arbitrary external source, i.e., they are background fields implementing invariance under general coordinate transformations and extended gauge invariance,
\[ g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^{\mu}} \frac{\partial x'^\sigma}{\partial x^\nu} g_{\rho\sigma}(x) , \]
\[ W'_{\mu\nu}(x') = \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} W_{\rho\sigma}(x) , \]
\[ B'_{\mu}(x') = \frac{\partial x^\rho}{\partial x'^{\mu}} B_{\rho}(x) , \]
\[ \delta B_{\mu}(x) = \Lambda_{\mu}(x) + \partial_{\mu} \phi(x) , \]
\[ \delta K_{\mu\nu}(x) = \partial_{[\mu} \Lambda_{\nu]} . \]

At this stage there is no relationship between \( g_{\mu\nu}(x) \) and the physical spacetime metric, nor between \( K_{\mu\nu}(x) \) and the Kalb-Ramond tensor potential. However, once \( W_{\mu\nu}(x) \) is eliminated from (3.10) by means of the formal solution (2.4), we obtain the action for a string non-linear \( \sigma \)-model [18]:

\[
S^{\text{cl.}} = -\bar{g}^2 c \int d^2\xi \sqrt{-\frac{1}{2} g_{\mu\rho}(X) g_{\nu\sigma}(X) \dot{X}^\mu \dot{X}^\nu \dot{X}^\rho \dot{X}^\sigma + \frac{c}{2} \int d^2 \xi K_{\mu\nu}(X) \dot{X}^{\mu\nu} . \tag{3.12}
\]

Then, an effective action for the background fields is induced at the two-loop quantum level [19], and can be computed in a perturbative expansion in powers of the inverse effective string tension:

\[
\Gamma = -\bar{g}^2 c \int d^4 x \sqrt{-g} \left[ ag'^{\mu\nu} R_{\mu\nu} - dH^{\mu\nu\rho} H_{\mu\nu\rho} + O \left[ (1/\bar{g}^2 c)^{3/2} \right] \right] . \tag{3.13}
\]

Thus, in such an approach, the Einstein and Kalb-Ramond terms are recovered as induced quantum terms, describing the low-energy behavior of the underlying quantum string theory.

**Non-Abelian geodesic field**

Spatially extended objects were introduced into hadronic physics after the recognition that the energy spectrum of the dual resonance model could be interpreted in terms of the vibrating modes of a relativistic string [20]. Further
elaboration of this idea led to models of the meson as a pair of colored quarks, or monopoles, joined by a thin flux-tube [21,22].

Therefore, if we believe that the geodesic field approach discussed above is general enough to provide a consistent description of elementary strings as well as gauge strings, then it should be possible to embody non-abelian symmetries into the proposed approach. It turns out, however, that there are severe restrictions on the feasibility of this program. The rest of this section is devoted to discuss this problem.

Again, we shall follow a line of reasoning similar to the one proposed in the previous section, i.e., we start from the non-abelian current associated with a pair of “colored” point-like objects

\[ J^{a \mu} = \int d\tau \sum_{i=1,2} (-1)^i \delta^4(x - X_i(\tau)) \rho^a[X_i(\tau)] \left( \frac{dX_i^\mu}{d\tau} \right), \]  
(3.14)

which acts as the source of an SU(3) gauge field \( A^a_\mu \) governed by the Yang-Mills action. \( \rho^a[X] \) is the a-th component of the Yang-Mills charge carried by the “quark” located at \( x^\mu = X_1^\mu(\tau) \).

By implementing again the “Dirac trick”, we write \( J^{a \mu} \) as the gauge covariant divergence of a singular Yang-Mills field

\[ G^{a \mu \nu}(x) = \int d\tau d\sigma \rho^a[X] \delta^4(x - X(\xi)) \dot{X}^{\mu \nu}, \]  
(3.15)

where \( x^\mu = X^\mu(\tau, \sigma) \) describe a world-sheet having the two world-lines \( x^\mu =

---

\* What we have in mind is the QCD string, so we consider \( SU(3)_c \) as the underlying symmetry group of strong interactions, and let the color index \( a = 1, \ldots, 8 \).

In order to distinguish internal indices we shall write them always as upper indices. Repeated indices are traced over with an euclidean metric.
$X_i^\mu(\tau)$ as its only boundary, i.e.

$$x^\mu = X^\mu(\tau, \sigma = \sigma_i(\tau)) = X_i^\mu(\tau), \quad i = 1, 2.$$  \hspace{1cm} (3.16)

In order to implement the Dirac relationship between $G^{a\mu\nu}$ and $J^a\mu$ we require that $\rho^a[X]$ be covariantly constant [23], i.e.

$$D^{ab\mu}_{\mu}\rho^b[X] = 0 \Rightarrow D^{ab\mu}_{\mu}G^{b\mu\nu} = J^a\nu.$$  \hspace{1cm} (3.17)

This constraint is reasonable since $\rho^b[X]$ is not a dynamical variable, but rather an external source which can be suitably chosen. It also follows from (3.17) that

$$(\rho^a)^2 \equiv \rho^a\rho^a$$

is independent of the world-sheet coordinates

$$\partial_\tau(\rho^a)^2 = \partial_\sigma(\rho^a)^2 = 0.$$  \hspace{1cm} (3.18)

It is also worthwhile to remark that the regularized form of $G^{a\mu\nu}$ evaluated on the string world-sheet factorizes in the product of the color distribution times the (abelian) volume momentum

$$G^{a\mu\nu}(x = X) = \int d^2\xi' \rho^a[Y]\delta^4(\xi - Y(\xi')) \hat{Y}^{\mu\nu}$$

$$= \text{const.} \frac{1}{\alpha^2} \int d^2\xi' \rho^a[Y]\delta^2(\xi - \xi') \frac{\hat{X}^{\mu\nu}}{\sqrt{-\frac{1}{2} \hat{X}_{\rho\sigma} \hat{X}^{\rho\sigma}}}$$

$$= \text{const.'} \rho^a[X] \Pi^{\mu\nu}(\xi).$$  \hspace{1cm} (3.19)

Accordingly, we establish the following relation between the string field $G^{a\mu\nu}(x)$
and the slope field:

\[ G^{a\mu\nu}(x) = \Phi^{\mu\nu}(x) \int d^2\xi \rho^a[X] \delta^4(x - X(\xi)) \] . \quad (3.20)

From the above equation one derives a formal expression for \( G^{a\mu\nu} \) squared:

\[ G^{a\mu\nu}(x)G^{a\mu\nu}(x) = \Phi^{\mu\nu}(x)\Phi_{\mu\nu}(x) \int d^2\xi d^2\xi' \rho^a[X] \rho^a[X] \delta^4(x - X(\xi)) \delta^4(x - X(\xi')) \]

\[ = \Phi^{\mu\nu}(x)\Phi_{\mu\nu}(x)(\rho^a)^2 \left( \int d^2 \delta^4(x - X(\xi)) \right)^2 , \]

which we shall use later on.

The above remarks suggest we can write an \( SU(3) \) invariant action for the string geodesic field as follows

\[ S_{\text{c}} = \int d^4x \left[ -g^2 \sqrt{-\frac{1}{3!} W_a^{\mu\nu} W_a^{\mu\nu}} + \frac{1}{2} W_a^{\mu\nu}(x) D^{ab}_{\mu\nu} B^b(x) + f B^a_{\mu}(x) J^a_{\mu}(x) \right] \]

\[ + \epsilon A^a_{\mu} J^a_{\mu} - \frac{1}{4} F^{a\mu\nu} F^{a\mu\nu} - \mu_0 \int d\tau \sum_{i=1,2} \sqrt{\dot{X}_i^\mu \dot{X}_i^\mu} , \]

(3.22)

where \( J^a_{\mu}(x) \) is given by (3.14) and \( D^{ab}_{\mu\nu} B^b_{\nu}(x) \) is the gauge covariant derivative of the \( B \)-field. The colored objects in (3.22) transform under color internal rotations as

\[ \delta W_a^{\mu\nu}(x) = f^{a}_{bc} \Lambda^b W_c^{\mu\nu}(x) , \]

\[ \delta B^a_{\mu}(x) = D^{ab}_{\mu} \Lambda^b , \]

\[ \delta \rho^a[X] = f^{a}_{bc} \Lambda^b \rho^c[X] , \]

(3.23)

where \( f^{a}_{bc} \) are the \( SU(3) \) structure constants. The corresponding field equations
are
\begin{align}
\delta B_\mu S^c &= 0 : \quad D^{ab}_\mu G^{b\mu\nu}(x) = f J^a\nu(x) , \\
\delta W_{\mu\nu} S^c &= 0 : \quad g^2 \frac{G^a_{\mu\nu}(x)}{\sqrt{-\frac{1}{2} G^b_{\alpha\beta} G^b_{\alpha\beta}}} + D^{ac}{}_{[\mu} B^c_{\nu]} = 0 , \\
\delta X^{\alpha(\tau)} S^c &= 0 : \quad \frac{dP_i^\mu}{d\tau} = (-1)^i \rho^a [X_i] \left( f D^{ab}_{[\mu} B^b_{\nu]} + e F^{a\mu\nu} \right) \dot{X}_i^\nu , \\
\delta A_\mu S^c &= 0 : \quad D^{ab}_\mu F^{b\mu\nu} = e J^a\nu .
\end{align}

From the first two equations we derive the following formal solutions in terms of string variables
\begin{align}
W^{a\mu\nu}(x) &= f G^{a\mu\nu}(x) \quad \Rightarrow \quad W^{a\mu\nu}(x = X) = \text{const} \rho^a [X] \Pi^{a\mu\nu}(\xi) ,
\end{align}
and
\begin{align}
B^{a\mu\nu}(x) &\equiv D^{ac}{}_{[\mu} B^c_{\nu]} = -g^2 \frac{G^a_{\mu\nu}(x)}{\sqrt{-\frac{1}{2} G^b_{\alpha\beta} G^b_{\alpha\beta}}} \quad \Rightarrow \quad -\frac{1}{2} B^{a\mu\nu}(x) B^{a\mu\nu}(x) = g^2 .
\end{align}

Equation (3.29) allows us to introduce a colored geodesic field
\begin{align}
\Pi^{a\mu\nu}(x) &\equiv -f B^{a\mu\nu}(x)
\end{align}
which on the string world-sheet reads
\begin{align}
\Pi^{a\mu\nu}(x = X) = \frac{\bar{g}^2 f}{\sqrt{\rho^2}} \frac{\rho^a [X] \dot{X}_{\mu\nu}}{\sqrt{-\frac{1}{2} \dot{X}_{\mu\nu} X_{\mu\nu}}} \equiv \frac{1}{2\pi \alpha'} \frac{\rho^a [X] \dot{X}_{\mu\nu}}{\sqrt{-\frac{1}{2} \dot{X}_{\mu\nu} X_{\mu\nu}}} ,
\end{align}
and represents the area-momentum of a colored string with an effective tension

$$\frac{1}{2\pi \alpha'} \equiv \frac{\bar{g}^2 f}{\sqrt{(\rho a)^2}} . \quad (3.32)$$

From (3.29), (3.30) and the constraint (3.17), we deduce that the color-singlet

$$\rho^a[X]B^a_{\mu\nu}(x)$$

satisfies the Bianchi Identity

$$\partial_{[\lambda} \rho^a B^a_{\mu\nu]} = 0 . \quad (3.33)$$

Once evaluated along the world-sheet, eq.(3.33) yields the equations of motion for

$$X(\tau, \sigma):$$

$$\dot{X}^\lambda \partial_{[\lambda} \rho^a [X]B^a_{\mu\nu]}(\xi) = 0 \Rightarrow \bar{g}^2 \sqrt{(\rho a)^2} \delta^{[jk]} \partial_j X^\mu \partial_k \frac{\dot{X}_{\mu\nu}}{\sqrt{-\frac{1}{2} \dot{X}_{\mu\nu} \dot{X}^{\mu\nu}}} = 0 . \quad (3.34)$$

Therefore, the system of eqs. (3.24)-(3.27) can now be written in the familiar “string” form:

$$\delta^{[jk]} \partial_j X^\mu \partial_k \Pi_{\mu\nu}(\xi) = 0 , \quad (3.35)$$

$$\frac{d\Pi^\mu_i}{d\tau} = (-1)^i (\Pi_{\mu\nu} + e\rho^a [X_i] F^a_{\mu\nu}) \dot{X}_{[i]}^\nu , \ i = 1, 2 \quad (3.36)$$

$$D^a_{\mu} F_{b\mu\nu} = eJ^a_{\nu} . \quad (3.37)$$

The first equation of this system is nothing but (3.34) written by taking into account eq.(3.32), and describes a minimal surface traced in spacetime by the evolution of the body of a string. Equations(3.36) represent a generalization of the Wong equation [24], and describe the motion of the string end-points under the influence
of both the Yang-Mills field and the "internal Lorentz force" represented by $\Pi^{\mu\nu}$.

Finally, the colored string boundary enters eq.(3.37) as the source of the Yang-Mills field itself. The whole system of eqs.(3.35)-(3.37) can be derived through variation of the effective action functional obtained by inserting (3.28),(3.29) into (3.22):

$$S^{\text{eff.}} = \int d^4x \left[ -\hat{g}^2 f (\rho^a)^2 \sqrt{-\frac{1}{2} \Phi_{\mu\nu}(x) \Phi^{\mu\nu}(x)} \int d^2\xi \delta^4(x - X(\xi)) 
+ e A^a_{\mu} J^a_{\mu} - \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} \right] - \mu_0 \int d\tau \sum_{i=1,2} \sqrt{-\dot{X}_i^{\mu} \dot{X}_i^{\mu}}$$

$$= - \int d\tau \left[ \frac{1}{2\pi\alpha'} \int d\sigma \sqrt{-\frac{1}{2} \dot{X}_i^{\mu} \dot{X}_i^{\mu}} + \mu_0 \sum_{i=1,2} \sqrt{-\dot{X}_i^{\mu} \dot{X}_i^{\mu}} \right]$$

$$- \frac{1}{4} \int d^4x F^a_{\mu\nu} F^a_{\mu\nu} + e \int d\tau \sum_{i=1,2} (-1)^i \rho^a [X_i] \dot{X}_i^{\mu} A^a_{\mu}(X_i).$$

The functional (3.38) is a generalization of the "massive ends" string action suggested by Chodos and Thorne [13], including color and interacting with a Yang-Mills field.

*Notice how the color degree of freedom disappear from the Nambu-Goto action, having been completely re-absorbed into the definition of the string tension.* In fact, non-abelian gauge symmetry is incompatible with reparametrization invariance for any kind of spatially extended object [25], so that a true realization of a colored string seems to be impossible. In this connection, observe that in eqs.(3.17) and (3.29), the color charge $\rho^a[X]$ simply multiplies the ordinary (=abelian) string variables, and thus remains localized at the pointlike string boundaries. Accordingly, the coupling of the string with the Yang-Mills field occurs only along the world-lines of the pointlike boundaries.
Now, it would be tempting to inquire if a Kalb-Ramond interaction can be added in a sensible way, or in other words, if the gauge transformations (2.27) admit a non-abelian generalization. The main problem is to introduce a suitable kinetic term for the internal algebra valued Kalb-Ramond tensor potential $A_{\mu\nu}(x)$. In fact, the naive extension $H^a_{\mu\nu\rho}H^{a\mu\nu\rho}$ breaks the vector gauge invariance. Possible solutions to this problem are currently under investigation along two main lines of thought: one is to implement the equivalence of a certain class of non-abelian Kalb-Ramond theories and chiral, non-linear $\sigma$-models [26]; the second consists in compensating the lack of vector gauge symmetry by means of suitable matter fields [27]. The final goal is to introduce a new mass generating mechanism arising from the mixing between tensor and vector gauge bosons.

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