High-$T_c$ superconductivity of electron systems with flat bands pinned to the Fermi surface

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The phenomenon of flat bands pinned to the Fermi surface is analyzed on the basis of the Landau-Pitaevskii relation, which is applicable to electron systems of solids. It is shown that the gross properties of normal states of high-$T_c$ superconductors, frequently called strange metals, are adequately explained within the flat-band scenario. Most notably, we demonstrate that in electron systems moving in a two-dimensional Brillouin zone, superconductivity may exist in domains of the Lifshitz phase diagram lying far from lines of critical antiferromagnetic fluctuations, even if the effective electron-electron interaction in the Cooper channel is repulsive.

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A central issue confronting present-day condensed matter theory is that of unambiguous determination of the mechanisms governing the rich non-Fermi-liquid (NFL) behavior revealed by intensive experimental studies of strongly correlated electron systems of solids and liquid $^3$He films. Prominent on the scene of this seminal area of condensed matter physics are numerous versions of the Hertz-Millis-Moriya (HMM) description. These scenarios ascribe such NFL behavior to quantum critical fluctuations. Proponents of the HMM approach claim that accounting for such fluctuations also allows one to explain the phenomenon of high-$T_c$ superconductivity, which entails substantial enhancement of the critical temperatures $T_c$ of superconducting phase transitions relative to standard BCS values.

In BCS theory, a typical $T-x$ phase diagram of a superconducting system (x being a relevant control parameter such as doping or pressure) consists of an “island” of superconductivity, surrounded by a Fermi-liquid (FL) “sea,” a domain where the resistivity varies as $\rho_{FL}(T) = \rho_0 + A_2 T^2$. Real phase diagrams of strongly correlated metals exhibiting superconductivity look different, as exemplified by the phase diagram of the LCCO family of electron-doped high-$T_c$ superconductors. This phase diagram, established empirically in Refs. 1, 2, is reproduced in Fig. 1. A prominent feature is the presence of two distinct regimes of NFL temperature behavior of the resistivity $\rho(T)$ at $T > T_c$, which separate the “blue FL sea” from the “yellow island” of superconductivity. In the interval $T_2(x) < T < T_1(x)$, the resistivity $\rho(T)$ changes linearly with $T$, thus $\rho(T) = \rho_0 + A_1 T$. Above $T_1(x)$ a different NFL regime appears, in which $\rho(T)$ varies as $T^n$ with $n \approx 1.6$.

In the conventional HMM approach, the NFL linearity of the resistivity $\rho(T)$ observed in normal states of many high-$T_c$ compounds is attributed to antiferromagnetic critical fluctuations with wave vector $Q = (\pi/a, \pi/a)$. In two-dimensional (2D) systems of electrons moving in the external field of a square lattice, this explanation works provided saddle points are located close to the Fermi lines; otherwise it fails. Significantly, the additional NFL regime in the LCCO phase diagram of Fig. 1 having $n \approx 1.6$ in $\rho(T) = \rho_0 + A_2 T^n$ terminates at the same critical doping as the linear-in-$T$ regime. This behavior cannot plausibly be associated with antiferromagnetic fluctuations. Further, there are numerous examples of 3D systems whose low-$T$ resistivity also varies linearly with $T$, in contrast to predictions of the spin-fluctuation scenario for 3D systems. In addition, fluctuation-induced phenomena such as critical opalescence, exhibited as a huge enhancement in absorption of light by a liquid that is ordinarily transparent, emerge only in the immediate vicinity of points (or lines) of second-order phase transitions. The range $\Delta T$ of the interval $T_N - T$ impacted by critical fluctuations on the ordered side of the transition is determined by equating the mean-field value of the order parameter, behaving as $\sqrt{T_N - T}$, to the corresponding fluctuation contribution. Consequently, the fluctuation scenarios become irrelevant when $|T - T_N| > \Delta T$.

In fact, experiments on the systems of interest often furnish clear evidence for the persistence of NFL behavior far from such lines of criticality. In the well-studied heavy-fermion metal YbRh$_2$Si$_2$ 3, which provides the classic example of NFL linear behavior of $\rho(T)$ in the disordered phase and a $T^2$ resistivity regime on the ordered side of the posited antiferromagnetic phase transition, the latter behavior is found to prevail almost up to the transition temperature $T_N = 70$ mK. Given that the FL regime of $\rho(T)$ holds in the antiferromagnetic state of YbRh$_2$Si$_2$ almost up to $T_N$, the critical fluctuations must...
To be specific, consider the compound. The crossover from the NFL linear-in-$T$ regime is the enhancement of the electron effective mass $\rho$ from $\rho_0$ to its FL regime, occurring in the normal state $T > T_N$ to its FL regime, which drops by factor around ten on the FL side of the crossover [5]. Such behavior is inconceivable within the textbook understanding of kinetic phenomena in Fermi liquids.

All these facts and many others portend that in some strongly correlated electron systems of solids, it is the single-particle degrees of freedom that are the real playmakers in the observed NFL behavior, rather than critical fluctuations. By implication the HHM approach is fallacious when applied to these systems, since the single-particle degrees are integrated out. It has become clear that the distinctive signature of the underlying physics of the Lifshitz phase diagram is the appearance of a so-called quantum critical point (QCP) where the density of states $N(T = 0)$, associated in homogeneous matter with the effective mass $M^*$, is divergent. Accordingly, the Landau state becomes unstable beyond the QCP and necessarily undergoes rearrangement [6]. The relevance of the QCP to high-$T_c$ superconductivity has been confirmed in recent experimental work of Ramshaw et al. [2] on YBa$_2$Cu$_3$O$_{6+x}$, one of the most prominent high-$T_c$ superconductors, with $T_c$ around 90 K. In documenting "A quantum critical point at the heart of high-$T_c$ superconductivity," they have studied the enhancement of the electron effective mass $M^*$ in dHvA oscillations of the normal metallic state, subjecting the sample to huge magnetic fields $B$ of more than 90 K that terminate superconductivity. The measured effective mass $M^*(B)$ is found to diverge at the optimal doping $x_o$, where the critical temperature $T_c(x, B = 0)$ attains its maximum.

It is noteworthy that the presence of a QCP in the Lifshitz phase diagram of homogeneous 3D electron systems was predicted in microscopic parameter-free calculations [8] prior to its experimental discovery in strongly correlated heavy-fermion metals [4]. Application of the approach of Ref. [3] to the 2D problem [9] demonstrated that in this case the QCP lies at realistic electron densities, corresponding to $r_s \simeq 8$.

Absent any change of symmetry, the anticipated rearrangement of the Landau state occurring at the QCP is naturally attributed to some topological transition. The earliest topological scenario for NFL behavior of strongly correlated Fermi systems, advanced more than 20 years ago [10,12], traced this behavior to an interaction-induced rearrangement of the Landau state, often called fermion condensation (FC). This phenomenon, described more vividly as a swelling of the Fermi surface, is associated with the occurrence of a flat band pinned to the Fermi surface.

In a significant formal development, the FC phenomenon was rediscovered in 2009 within the framework of the adS-CFT duality [13]. More to the point, the formation of flat bands has been demonstrated both analytically and numerically for the Hubbard model, one of the most popular models of strongly correlated electron systems [14]. Therefore the question of principle whether the FC rearrangement exists and is relevant to condensed-matter theory already has a positive answer. The remaining issue, addressed herein and elsewhere, is whether or not the FC phenomenon can actually provide the basis for a satisfactory explanation of the experimentally observed NFL behavior of such systems. That the FC scenario competes favorably with other attempts to explain the salient experimental data has been established in many studies, notably Refs. [15–18]. Additionally, invocation of FC theory has recently resolved a long-standing puzzle associated with the disappearance of a specific set of Shubnikov-de Haas magnetic oscillations in the 2D electron gas of MOSFETs, which results in the doubling of oscillation periods near a quantum critical point [14,20].

The paramount objective of this communication is to apply the flat-band scenario to the elucidation of high-$T_c$ superconductivity, discovered 30 years ago and still a challenge to theoretical understanding. The adequacy of any theory of this phenomenon depends largely on how well it reproduces the properties of strange metals – normal states of high-$T_c$ superconductors. The Landau FL theory of normal states provides the basis for BCS theory, which, however, fails to describe high-$T_c$ superconductivity. With this in mind, we present the essential elements of FC theory, focusing on those departures from FL theory of conventional Fermi liquids, this require-
ment is always met toward \( T = 0 \), since the damping is proportional to \( T^2 \). The situation is more complicated in the flat-band scenario, because systems having flat bands belong in fact to the class of marginal Fermi liquids, in which the damping changes linearly with \( T \), but with a prefactor proportional to the ratio \( \eta \) of the volume occupied in momentum space by the flat bands to the total Fermi volume (see below). Thus, the above requirement for applicability is met provided \( \eta \) is small.

Importantly, in electron systems of solids where translational invariance breaks down, the single-particle states are identified by quasimomentum \( \mathbf{p} \), and the FL relation

\[
n(\mathbf{p}) = (1 + e^{\varepsilon(\mathbf{p})/T})^{-1}, \tag{1}
\]

between the quasiparticle momentum distribution \( n(\mathbf{p}) \) and the single-particle spectrum \( \varepsilon(\mathbf{p}) \) (measured from the chemical potential \( \mu \)), continues to apply. As shown in Ref. [22], the Landau-Pitaevskii (LP) identity can be employed as a second relation between these quantities that holds for the electron system moving in the external field of the crystal lattice. In the notation adopted here, the LP equation takes the form

\[
\frac{\partial \varepsilon(\mathbf{p})}{\partial \mathbf{p}} = \frac{\partial \varepsilon_0(\mathbf{p})}{\partial \mathbf{p}} + \int f(\mathbf{p}, \mathbf{p}') \frac{\partial n_\ast(\mathbf{p}')}{\partial \mathbf{p}'} \, d\mathbf{p}', \tag{2}
\]

where the quantity \( \partial \varepsilon_0(\mathbf{p})/\partial \mathbf{p} \) contains only regular contributions coming from domains far from the Fermi surface.

Like the corresponding set of equations for homogeneous matter the set [11] and [2] also possesses a class of flat-band solutions for which the electron group velocity vanishes in a finite domain \( \mathbf{p} \in \Omega \). Accordingly, one is required to solve the reduced equation

\[
0 = \frac{\partial \varepsilon(\mathbf{p})}{\partial \mathbf{p}} + \int f(\mathbf{p}, \mathbf{p}') \frac{\partial n_\ast(\mathbf{p}')}{\partial \mathbf{p}'} \, d\mathbf{p}', \quad \mathbf{p}, \mathbf{p}' \in \Omega. \tag{3}
\]

in this region. The function \( n_\ast(\mathbf{p}) \) is introduced to represent a nontrivial FC solution of this equation. Outside the FC region, the quasiparticle spectrum obeys the familiar Landau-type relation

\[
\frac{\partial \varepsilon(\mathbf{p})}{\partial \mathbf{p}} = \frac{\partial \varepsilon_0(\mathbf{p})}{\partial \mathbf{p}} + \int f(\mathbf{p}, \mathbf{p}') \frac{\partial n(\mathbf{p}')}{\partial \mathbf{p}'} \, d\mathbf{p}', \quad \mathbf{p} \notin \Omega. \tag{4}
\]

Such a solution is characterized by its topological charge (TC), an invariant expressed in terms of a contour integral constructed from the single-particle Green function \( G(\mathbf{p}, \varepsilon) \) and its derivatives [11, 24]. The TC of a state exhibiting a flat band takes a half-odd-integral value, whereas the TC assigned to a Lifshitz state featuring a multi-connected Fermi surface (or “Lifshitz pockets”) is always integral, since this state has standard quasiparticle occupation numbers \( n(\mathbf{p}) = 0, 1 \).

Proceeding to low \( T \neq 0 \), one may insert the solution \( n_\ast(\mathbf{p}) \) of Eq. (3) as a zeroth approximation for \( n(\mathbf{p}, T) \) to obtain [12]

\[
\varepsilon(\mathbf{p}, T \to 0) = T \ln \frac{1 - n_\ast(\mathbf{p})}{n_\ast(\mathbf{p})}, \quad \mathbf{p} \in \Omega. \tag{5}
\]

In the FC momentum region \( \mathbf{p} \in \Omega \), the resulting dispersion \( \nu_n = \partial \varepsilon(\mathbf{p})/\partial p_n \) of the single-particle spectrum is then found to be proportional to \( T \).

An essential feature of the flat-band scenario is the presence of a residual entropy [10, 17, 18]

\[
S_\ast = - \int_{\Omega} (1 - n_\ast(\mathbf{p})) \ln (1 - n_\ast(\mathbf{p})) + n_\ast(\mathbf{p}) \ln n_\ast(\mathbf{p}) \, d\mathbf{p} \tag{6}
\]

associated with the FC region, where the occupation numbers \( n_\ast(\mathbf{p}) \) differ from 0 and 1. This residual entropy does not contribute at all to the specific heat \( C(T) = TdS/dT \). However, in normal states of systems with flat bands, \( S_\ast(T \to 0) \) retains a finite value and makes a huge \( T \)-independent contribution to the thermal expansion \( \beta \propto \partial S/\partial T \) of these states. This conclusion is in agreement with the thermal expansion of the strongly correlated heavy-fermion metal CeCoIn\(_5\) measured at \( T > T_c \approx 23 \, \text{K} \) [25].

To avoid contradiction with the Nernst theorem mandating \( S(0) = 0 \), the residual entropy \( S_\ast \) must be released by means of some first- or second-order phase transitions, or with the aid of crossovers to a state having an additional \( \beta \) feature.

Let us examine more closely the NFL behavior of the resistivity \( \rho(T) \) at \( T > T_c \) in superconducting materials exhibiting flat bands, recognizing that this is the foundation of the phase diagrams presented in Figs. [1] and [2]. Ideally, the collision term differs from zero only due to Umklapp processes. However, strongly correlated electron systems of solids have open Fermi surfaces where these processes work in full force, such that the detailed structure of the kernel can be ignored, and we are left with the integral

\[
I(n) \propto \int [n_1 n_2 (1 - n_1')(1 - n_2') - n_1' n_2 (1 - n_1)(1 - n_2)]
\times \delta(p_1 + p_2 - p_1' - p_2') \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_1' - \varepsilon_2') \, dv_1 dv_1' dv_2 dv_2'. \tag{7}
\]

Eq. (7) exhibits several factors \( 1/\nu_n(\mathbf{p}, T) \) upon making the standard replacement \( d^3p \to dS(d\varepsilon/n_\ast(\mathbf{p})) \), where \( dS \) is an element of the isoenergetic surface. In conventional Fermi liquids, these factors are \( T \)-independent and yield the FL result. Contrariwise, in systems with flat bands, it is seen from Eq. (5) that the behavior \( \nu_n(\mathbf{p}, T) \propto T \) applies in the “hot spot” associated with the corresponding FC region. Any momentum integration over this region
FIG. 1: (color online) Temperature-doping $T - x$ phase diagram of $\text{La}_{2-x}\text{Ce}_x\text{CuO}_4$ [1], reprinted with authors’ permission. The yellow region is the superconducting dome. The resistivity $\rho(T)$ in the different normal phases has the form $\rho(T) = \rho_0 + A_n T^n$, with $n = 2$ for the FL domain (blue), $n = 1$ for the NFL FC domain (red), and $n = 1.6$ for the NFL QCP domain (white) (see the text). The temperatures $T_1$ (triangles) and $T_{FL}$ (inverted triangles), traced by dashed curves, indicate the crossover temperatures to the linear-in-$T$ regime of $\rho(T)$ and the FL regimes, respectively.

FIG. 2: (color online) Temperature-pressure $T - P$ phase diagram of CeCoIn$_5$ [2], constructed based on the phase diagram published in Ref. [3]. Unlike the phase diagram of Fig. 1, there exists a small pseudogap (PG) region in orange where superconductivity is precluded, but a gap $\Delta$ persists. The occurrence of a PG region in this compound will be discussed in detail in a future article.

then contributes a factor $\eta/T$ to the collision integral (and to the resistivity as well), where $\eta$ is the dimensionless FC density. This causes a damping of single-particle excitations, rendering systems with flat bands marginal Fermi liquids [26, 27]. Such a conclusion is in agreement with the results of experimental studies of kinetic properties of the heavy-fermion metal CeCoIn$_5$ [3, 28].

Ordinarily $\eta$ is very small, so we need only retain the leading terms of order $\eta$ and $\eta^2$ to arrive at the resistivity expression

$$\rho(T) = \rho_0(P, x) + A_1(P, x)T$$

in the presence of a flat band, with

$$\rho_0(P, x) = \rho_0^i + a_0\eta^2(P, x), \quad A_1(P, x) = a_1\eta(P, x),$$

wherein $\rho_0^i$ denotes an impurity-induced contribution to the residual resistivity $\rho_0$ and $a_0, a_1$ are constants. The total residual resistivity $\rho_0$ becomes dependent on pressure $P$ and doping $x$. The cleaner the metal, the greater the magnitude of the jump of the residual resistivity $\rho_0$ in the rapid crossover from the NFL regime [9] to the standard FL regime. In the limit $\rho_0 \to 0$, the ratio of the $\rho_0$ value on the NFL side of the crossover to that on the FL side tends to infinity. This observation serves to explain the rapid variation of $\rho_0$ found in the especially clean samples of CeCoIn$_5$ [5]. Such otherwise puzzling behavior has also been reported in measurements of the residual resistivity $\rho_0(P)$ of the metal CeAgSb$_2$, where the aforesaid ratio attains huge values of order $10^2$ [29].

As will be demonstrated below (cf. Eq. (16)), the critical temperature $T_c$ for termination of superconductivity in high-$T_c$ superconductors is proportional to the FC density $\eta$, so that

$$A_1(x) \propto T_c(x) \propto \eta(x).$$

Hence the theoretical ratio $T_c/A_1$ is independent of the FC density. While $\eta$ changes somehow with doping $x$, the ratio $T_c/A_1$ turns out to be independent of $x$, in agreement with the experimental behavior uncovered in the electron-doped materials LCCO and PCCO, as well as in the Bechgaard class of organic superconductors (TMTSF)$_2$PF$_6$ [1, 30]. In addition, the relation (10) implies that the factor $A_1$, which specifies the linear-in-$T$ NFL regime of resistivity, vanishes at the same doping $x_c$ where high-$T_c$ superconductivity terminates, again in agreement with experiment (see Fig. 1). Thus we infer that three different resistivity regimes come into play in the immediate vicinity of FC onset: (i) the FL regime $\rho(T) \propto T^2$, (ii) the FC regime $\rho(T) \propto T$, and (iii) the high-$T_c$ superconducting regime with $\rho = 0$.

On the other hand, for heavy-fermion superconductors such as CeCoIn$_5$ in which the critical temperature $T_c$ is extremely low, the BCS logarithmic term cannot be ignored. In this situation, the onset of FC is disconnected from the occurrence of superconductivity, and merging of different resistivity regimes in the corresponding normal states does not take place. This conclusion is seen to be in agreement with the $T - P$ phase diagram of CeCoIn$_5$ displayed in Fig. 2.
In the systems under study, still another profound NFL contribution to $\rho(T)$ is possible, specific to what can be called the QCP resistivity regime. Elucidation of its emergence is especially simple in the homogeneous electron liquid where, at the QCP, the Fermi velocity vanishes to yield

$$\epsilon(p \to p_F) \propto (p - p_F)^2 |p - p_F|$$

and hence $v(p) = de(p)/dp \propto \sqrt{\epsilon(p)}$. One easily verifies that the leading contribution to $\rho(T)$ now increases as $T^{3/2}$. This distinctive NFL regime has its onset at the critical doping $x_c$ as well. It exists in those electron systems that possess a QCP, e.g. in the LCCO family \([1]\) and in CeCoIn$_5$ [31]. There a minor difference between our theoretical predictions and experiment. In the white NFL regime of Fig. [11] the measured resistivity $\rho(T)$ varies as $T^{1.6}$, and in the corresponding regime of Fig. [2] as $\rho(T) \propto T^{1.5\pm0.1}$. Our analysis leads to the relation $\rho(T) = A_3/2T^{3/2} + A_2T^2$.

Having confirmed that the flat-band-scenario can successfully explain gross properties of the strange metals, we may now turn to the primary aim of this article: explanation of the occurrence of D-pairing and dramatic enhancement of its critical temperature $T_c$. In doing so we will proceed within the standard BCS theory, ignoring an enigmatic pseudogap phenomenon. In this case, the structure of the gap function and the magnitude of $T_c$ are revealed with the aid of the linearized BCS gap equation

$$\Delta(p) = -\int \mathcal{V}(p, p') \tanh \left( \frac{\epsilon(p', T_c)}{2T_c} \right) \frac{\Delta(p')}{2|\epsilon(p', T_c)|} dv'.$$  

Upon defining the functions

$$X(p, T_c) = \sqrt{\tanh \left( \frac{\epsilon(p, T_c)}{2T_c} \right) / \epsilon(p, T_c)},$$

$$\zeta(p) = \Delta(p)X(p, T_c), \quad \text{and} \quad H(p, p', T_c) = X(p, T_c)X(p', T_c)\mathcal{V}(p, p'),$$

Eq. (12) is conveniently rewritten in the form of a linear integral equation with a symmetric kernel,

$$\zeta(p) = -\frac{1}{2} \int H(p, p', T_c)\zeta(p')dv'.$$  

Employing Eq. (15), we may obtain the result

$$X(p, T_c) = T_c^{-1/2} \sqrt{\frac{1 - 2n_s(p)}{\ln \left( (1 - n_s(p))/n_s(p) \right)}}, \quad p \in \Omega$$

and observe that the function $X(p, T_c)$ is greatly enhanced in FC domains.

In view of the inverse proportionality to $T_c$ exhibited by the kernel $H$, we infer that the overwhelming contributions to the right side of Eq. (15) come from the FC domains, provided the ratio $\eta/T_c$ exceeds unity. In that case, solution of this equation is obviated, accounting for the minor change of the block $\mathcal{V}$ in a FC region. Upon neglecting other contributions to Eq. (15) and introducing a reduced kernel $h(p, p') = (T_c/2)H(p, p')$, we are left with the simple equation

$$T_c\zeta(p) = -\int_\Omega h(p, p')\zeta(p')dv',$$  

in which the kernel $h$ is practically $T$-independent. We thus arrive at the pivotal conclusion that $T_c$ changes linearly with the strength of the interaction $\mathcal{V}$, a distinctive fingerprint of high-$T_c$ superconductivity. Further, since the right side of Eq. (15) is, in fact, proportional to the FC volume, we affirm that $T_c$ does vary linearly with the FC density $n_f$, in agreement with Eq. (10).

In what follows we address the case of a 2D quadratic Brillouin zone where a single small FC pocket resides in each quadrant of the zone. Eq. (15) then reduces to the set of algebraic equations

$$T_c\zeta_i = -\sum_{j,k} h_{ik}\zeta_k,$$  

where we have introduced quantities $\zeta_i = \zeta(p_i)$ and $h_{ik} = h(p_i, p_k)$, with indexes $i, k$ running from 1 to 4. Obviously, $h_{ll} = h_0$, $h_{l,l+1} = h_1$, and $h_{l,l+2} = h_2$ are $T$-independent. Their signs and magnitudes depend largely on the interplay between phonon attraction and Coulomb repulsion. To solve the system (16) it is advantageous to make the substitution $\zeta_k = e^{i\alpha_k}$, with the requirement $e^{i\alpha} = 1$. Analysis demonstrates that there are 4 different high-$T_c$ solutions of the problem. The first solution, corresponding to $\alpha = 0$, where all $\zeta_i$ are the same, occurs provided $h_0 + 2h_1 + h_2 < 0$. It describes $S$-pairing, with respective critical temperature $T_c = -(h_0 + 2h_1 + h_2)$.

Another interesting solution, with

$$\alpha = \pi, \quad T_c = 2h_1 - h_0 - h_2,$$

has the usual $D$-pairing form, with $\zeta_1 = \zeta_3 = -\zeta_2 = -\zeta_4$. It occurs in the region of the Lifshitz phase diagram where $2h_1 - h_0 - h_2 > 0$. Near the line of critical antiferromagnetic fluctuations where conventionally $h_0 = h_2 = 0$ while $h_1 > 0$, the solution (17) coincides with the standard one. However, as seen, the presence of critical fluctuations is not a necessary condition for the occurrence of high-$T_c$ $D$-pairing. In electron systems of solids hosting flat bands, this solution can exist far from the critical line $T_N(x)$, even if the $e-e$ interaction in the Cooper channel is repulsive.

A remaining pair of solutions, corresponding to $\alpha = \pi/2$ and $\alpha = 3\pi/2$ and having $\zeta_4 = i\zeta_2 = -\zeta_2 = -i\zeta_4$, describes $P$-pairing [32], with the same critical temperatures given by $T_c = h_2 - h_0$.

In summary, we have generalized the Landau quasi-particle approach to determine observable properties of
systems possessing flat bands. Specifically, we have motivated and introduced a set of two Landau-like equations that replace the basic equation of Fermi liquid theory connecting the single-particle spectrum and quasiparticle momentum distribution. Analyzing a typical phase diagram of the LCCO family of electron-doped high-$T_c$ compounds, we have demonstrated that gross properties of strange metals are incisively and economically interpreted within the flat-band scenario developed here. Importantly, we have shown that in a quadratic Brillouin zone, the BCS gap equation has nontrivial solutions even if the interaction between quasiparticles in the Cooper channel is of repulsive character. The successful description of key features of the phase diagrams and other properties of high-$T_c$ materials should provide ample incentive for a change of theoretical course in the search for deeper understanding of non-Fermi-liquid phenomena as well as the mechanism of high-$T_c$ superconductivity.

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