THREE-LOOP CALCULATION OF THE ANYONIC FULL CLUSTER EXPANSION

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Abstract

We calculate the perturbative correction to every cluster coefficient of a gas of anyons through second order in the anyon coupling constant, as described by Chern-Simons field theory.
It has been known for some years that two dimensional physics offers the possibility of fractional statistics (anyons) interpolating between bosons and fermions [1]. Either for its possible relevance in some areas of condensed matter physics, or for the deep implications it has on the concept of identical particles, the study of the thermodynamical properties of the anyonic system has deserved great effort since the past decade. However, until recently, the low density, high temperature expansion could only provide the second virial coefficient [2]. Some restricted information about the third virial coefficient was obtained in refs. [3,4]. Numerical computations have also been performed [5-7]. The root of the problem is the unsolvability of the $N$-anyon quantum mechanical system beyond $N = 2$.

This obstacle can be overcome making use of second quantized theory. The price one has to pay is that perturbative field theory only provides an expansion in powers of $\theta$ (the statistical parameter) near bosonic or fermionic statistics.

In this approach, a problem is posed by the singular nature of the short range statistical interaction. In refs. [8-11] a solution is proposed introducing a two-body non-hermitian interaction. Divergences must be carefully controlled with harmonic regulators or boundary conditions. In this way, Dasières de Veigy and Ouvry [11] have been able to compute the partition function to order $\theta^2$, expanded to all orders in the fugacity $z = \exp(\mu/T)$.

Recently another model has been considered to solve the problem [12]. There, fractional statistics is implemented in standard way through non-relativistic matter interacting with a Chern-Simons gauge field. UV divergences are cancelled with a two-body $\delta$-function potential when the starting statistics is bosonic. Starting from fermionic statistics does not require such interaction. This model is used for to the computation of the second virial coefficient, to three-loop order. In this letter we develop further the model and obtain the solution to all orders in $z$, to the same loop order. Remarkably, our results are in full agreement with those of refs. [10,11].†

† The relation between both approaches is discussed in ref. [12].
The setting of our model, as proposed in ref. [12] consists of a spinless non-relativistic self-coupled matter field $\psi$ interacting with Chern-Simons gauge bosons. The Lagrangian density will be given by

$$\mathcal{L}(t, x) = -\frac{1}{2\kappa} \partial_t a \times a + \frac{1}{\kappa} a_0 B - \frac{1}{2\rho} (\nabla \cdot a)^2 + \psi \psi^\dagger i D_0 - \frac{1}{2m} |D\psi|^2 + \mu \psi \psi^\dagger - \frac{\alpha}{4} (\psi \psi^\dagger)^2,$$

$$D_0 = \partial_t + i a_0, \quad D = \nabla - i a.$$  \hspace{1cm} (1)

Here $B = \nabla \times a$ is the magnetic field, $\mu$ is the chemical potential and $\rho$ is a gauge fixing parameter. The Coulomb gauge to be used refers to the choice $\rho = 0$. The anyon coupling $\kappa$ can be related to the statistical angle parameter $\theta$ as $\kappa = 2\theta$. Also, the strength of the contact interaction $\alpha$ must be taken [12] as

$$\alpha = (1 + \zeta) \frac{k}{m},$$  \hspace{1cm} (2)

($\zeta = 1$ for bosons; $\zeta = -1$ for fermions). With this choice, the divergences appearing for fiducial bosonic statistics are exactly cancelled; for fermions, $\alpha$ vanishes, not needing additional interactions.

In the imaginary-time formalism of thermal field theory [13], the functional integral expression for the grand partition function involves an integration over imaginary time from 0 to $\beta = T^{-1}$,

$$Z(\mu, T, V) = \int_{(\text{anti})\text{periodic}} D a_0 D a D \psi D \psi^\dagger \exp \left( \int_0^\beta d\tau \int d^2 x \mathcal{L}(t = -i\tau, x) \right).$$ \hspace{1cm} (3)

Here (anti)periodic means that the integration over fields is constrained so that $\psi(x, \beta) = \pm \psi(x, 0)$ where the (lower) upper sign refers to (fermions) bosons.

Now we can proceed to expand in a power series in $\kappa$, by using a set of diagrammatic rules which follows from (1). These are listed in Table 1. The diagrams describing the perturbative series for $\ln Z$ have the form of connected closed loops. It should be noted that there will be a factor of $\beta A$ left over for each graph, corresponding to the extensivity of $\ln Z$. This factor cancels out in expressions for the pressure.
The contribution to $\ln Z$ up to order $\kappa^2$ involves a set of three-loop diagrams. The non-zero graphs are shown in fig.1†. Note that all graphs with only one gauge boson line vanish due to the index summation associated with vertices. In order to keep the correct order of the operators according to their $\tau$-values, one must insert a factor $e^{i\omega_n\eta}$ whenever a particle line either closes on itself or is joined by the same instantaneous interaction line. We take $\eta \to 0^+$ at the end of the calculation. This implies the following basic frequency sums:

1. \[
\frac{1}{\beta} \sum_n \frac{e^{i\omega_n \eta}}{i\omega_n + \mu - q^2/(2m)} = -\zeta \exp \left[ \frac{\beta (q^2/(2m) - \mu)}{2} \right] - \zeta = -\zeta n_q , \tag{4}
\]

2. \[
\frac{1}{\beta} \sum_n \frac{1}{i\omega_n + \mu - q^2/(2m)} = -\frac{1}{2} - \zeta n_q . \tag{5}
\]

As a consequence, graphs (c), (g) and (h) are of order $z^3$.

The above explained formalism allows us to derive the perturbative corrections to the pressure

\[
P = \frac{T}{A} \ln Z . \tag{6}
\]

To each order in $\kappa$ our results will give the pressure as a function of the fugacity $z = \exp(\mu/T)$. This corresponds to the cluster expansion

\[
P(\mu, T) = \frac{1}{\lambda_T^2} \sum_{l=1}^{\infty} b_l z^l , \tag{7}
\]

($\lambda_T = (2\pi/mT)^{1/2}$ is the thermal wavelength). However, the statistical dependence is more clearly manifest in the virial expansion

\[
P(n, T) = \frac{1}{\lambda_T^2} \sum_{l=1}^{\infty} a_l (n\lambda_T^2)^l , \tag{8}
\]

with

\[
n \equiv N/A = z \frac{\partial}{\partial z} \left( \frac{P}{T} \right) . \tag{9}
\]

Both expansions can be related easily order by order:

\[
a_1 = 1 ,
\]

\[
a_2 = \frac{b_2}{b_1} ,
\]

\[
a_3 = 4 a_2^2 - 2 \frac{b_3}{b_1} , \tag{10}
\]

...† Graph (h) was missing in ref.[12]; since its contribution is of order $z^3$ the results of ref.[12] remain unaffected.
We proceed now to calculate the first two orders of the perturbative expansion in κ.

First order

As argued before, to this order, diagrams with one gauge boson line vanish. This leaves only diagram (a). As a result, fermionic statistics receives no corrections of order κ. The contribution to the pressure can be written

\[ P^{(1)} = -\frac{\alpha}{2} \left[ \frac{1}{\beta} \sum_{\omega_1} \int \frac{d^2p}{(2\pi)^2} G^0(\omega_1, p) e^{i\omega_1 \eta} \right]^2 \]

\[ = -\frac{\alpha}{2} \frac{1}{\lambda_T^2} \ln^2(1 - z). \] (11)

This result agrees with that of refs. [9,11]. It is easy to show from eqs. (10) and (11) that only the second virial coefficient is corrected to this order.

Second order

Diagrams (b), (e) and (f) are hardest to compute. They are given by

\[ P^b = \frac{\alpha^2}{8\beta^3} \sum_{\omega_1 \omega_2 \nu} \int \frac{d^2p}{(2\pi)^2} \frac{d^2k}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} G^0(\omega_1, p) G^0(\omega_1, p + q) \]

\[ \times G^0(\omega_2, k) G^0(\omega_2, k + q) \]

\[ = (1 + \zeta) \frac{\kappa^2}{2m} \int \frac{d^2p}{(2\pi)^2} \frac{d^2k}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \left[ n_p n_{k+q} + 2 \zeta n_k n_p n_{k+q} \right] \frac{q \cdot (p - k)}{q \cdot (p - k)^2 q^2}, \] (12)

\[ P^e = -\zeta \frac{\kappa^2}{4m^2\beta^3} \sum_{\omega_1 \omega_2 \nu} \int \frac{d^2p}{(2\pi)^2} \frac{d^2k}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} G^0(\omega_1, p) G^0(\omega_1, p + q) \]

\[ \times G^0(\omega_2, k) G^0(\omega_2 + \nu, k + q) \frac{[(p - k) \times q]^2}{(p - k)^2 q^2} \]

\[ = -\zeta \frac{\kappa^2}{2m} \int \frac{d^2p}{(2\pi)^2} \frac{d^2k}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \left[ n_p n_{k+q} + 2 \zeta n_k n_p n_{k+q} \right] \frac{[(p - k) \times q]^2}{(p - k)^2 q^2}, \] (13)
\[
P^f = \zeta \frac{\kappa^2}{m_\beta^3} \sum_{\omega_1 \omega_2 \nu} \int \frac{d^2 p}{(2\pi)^2} \frac{d^2 k}{(2\pi)^2} \frac{d^2 q}{(2\pi)^2} \mathcal{G}^0(\omega_1, p) e^{i\omega_1 \eta} \mathcal{G}^0(\omega_2, k) e^{i\omega_2 \eta} \times \mathcal{G}^0(\omega_2 + \nu, k + q) \frac{(p - k) \cdot q}{(p - k)^2 q^2} \]

\[
= -\zeta \frac{\kappa^2}{2m} \int \frac{d^2 p}{(2\pi)^2} \frac{d^2 k}{(2\pi)^2} \frac{d^2 q}{(2\pi)^2} \left( n_p n_{k+q} + 2\zeta n_k n_p n_{k+q} \right) \frac{(p - k) \cdot q}{(p - k)^2 q^2}. \tag{14}
\]

From these equations it is straightforward to notice the following relation

\[
P^b = -(1 + \zeta)(P^e + P^f), \tag{15}
\]
reflecting the cancellation of exchange diagrams in the bosonic case. Identity (15) implies a substantial reduction of calculations; to find \(P^b + P^e + P^f\), the only integral to compute is (12).

There are two different terms in (12): one involving the product of two \(n\)'s and another with three \(n\)'s. They give quite distinct contributions:

\[
P^b + P^e + P^f = \frac{\kappa^2}{4\pi m \lambda_T^4} \ln^2(1 - \zeta) \int_0^{x^*} dx \Phi(x) + \frac{\kappa^2}{2\pi m \lambda_T^4} \sum_{l \geq 3} (\zeta z)^l \sum_{n=1}^{l-2} \sum_{s=1}^{l-n-1} \frac{1}{s(l-n-s)} \int_0^{x^*} dx \Phi(x) e^{-\frac{ln}{s(l-n-s)}x^2}. \tag{16}
\]

Here \(x^*\) is a UV cutoff and \(\Phi(x)\) is the plasma dispersion function [14],

\[
\Phi(x) = 2 \int_0^x dt \frac{e^{-t^2} t}{\sqrt{x^2 - t^2}} = 2x e^{-x^2} \int_0^x dt e^{t^2}, \tag{17}
\]
its derivative being given by

\[
\Phi'(x) = 2 - 2x \Phi(x). \tag{18}
\]
The UV divergence comes from the loop with the seagull vertex and from the divergence in the second Born approximation for scattering with a \(\delta\)-function interaction [15]. These divergences will cancel when we consider the whole set of diagrams.

We evaluate now the ring contribution from graph (d) of fig. 1,

\[
P^d = \frac{\kappa^2}{2\beta} \sum_{\nu} \int \frac{d^2 q}{(2\pi)^2} \frac{\Pi^0_{00}(\nu, q) \Xi^0(\nu, q)}{q^2}. \tag{19}
\]
where $\Pi_{00}^0(\nu, q)$ and $\Xi^0(\nu, q)$ represent the density-density correlator and the transverse component of the current-current correlator respectively at the lowest order,

$$
\Pi_{ij}^0(\nu, q) = -\frac{\zeta}{m^2} \frac{1}{\beta} \sum_{\omega_1} \int \frac{d^2p}{(2\pi)^2} \left[ (p + \frac{q}{2})_i (p + \frac{q}{2})_j G^0(\omega_1, p) G^0(\omega_1 + \nu, p + q) \right.
\left. + m \delta_{ij} e^{i\omega_1} G^0(\omega_1, p) \right] = -\Pi_{00}^0(\nu_n, q) \nu_n^2 \frac{q_i q_j}{q^2} + \Xi^0(\nu_n, q) (\delta_{ij} - \frac{q_i q_j}{q^2}).
$$

Performing the sums and integrals, it yields (the first part comes from two-$n$ terms; the second, from three-$n$ terms)

$$
P^d = \frac{\kappa^2}{8\pi m\lambda_T^4} \ln^2(1 - \zeta z) \int_0^{x^*} dx \left( \frac{\Phi(x)}{x^2} - \frac{2}{x} \right)
+ \frac{\kappa^2}{4\pi m\lambda_T^4} \sum_{l \geq 3} (\zeta z)^l \sum_{n=1}^{l-2} \sum_{s=1}^{l-n-1} \frac{l-n-s}{s(l-s)} \int_0^{x^*} dx \left( \frac{\Phi(x)}{x^2} - \frac{2}{x} \right) e^{-\ln s (l-n-s) x^2}.
$$

The ring loop with a seagull vertex has yielded a UV logarithmic divergence. This exactly cancels the infinities we have found before. To see how this happens, we quote some results about the plasma dispersion function, namely

$$
\int_0^\infty dx \, e^{-ax^2} \left( \frac{\Phi(x)}{x^2} - \frac{2}{x} + 2(1 + a)\Phi(x) \right) = 2,
$$

$$
\int_0^\infty dx \, e^{-ax^2} \Phi(x) = \frac{1}{2\sqrt{1+a}} \ln \left( \frac{\sqrt{1+a} + 1}{\sqrt{1+a} - 1} \right).
$$

Both of them follow from eqs. (17) and (18). Now, first use eq. (22) (with $a = 0$) to compute the two-$n$ contribution from eqs. (16) and (19):

$$
\frac{\kappa^2}{4\pi m\lambda_T^4} \ln^2(1 - \zeta z).
$$

Split the three-$n$ term in eq. (16) in two parts: one is what must be added to the three-$n$ term in eq. (21) so that eq. (22) can be used. The sum in $z$ can be performed:

$$
-\frac{\kappa^2}{4\pi m\lambda_T^4} \ln(1 - \zeta z) \left( \frac{\zeta z}{1 - \zeta z} + \ln(1 - \zeta z) \right).
$$
The remaining part of eq. (16) is integrated using eq. (23):

\[ \frac{\kappa^2}{4\pi m\lambda^4} \sum_{l \geq 4} (\zeta)^l \sum_{n=1}^{l-2} \sum_{s=1}^{l-n-1} \frac{\alpha_{\text{lns}}}{\beta_{\text{lns}}} \ln \left( \frac{\beta_{\text{lns}} + 1}{\beta_{\text{lns}} - 1} \right), \]

where

\[ \alpha_{\text{lns}} \equiv \frac{n(2s - l + n)}{s^2(l - s)(l - n - s)}, \quad \beta_{\text{lns}} \equiv \sqrt{\frac{(l - s)(n + s)}{s(l - n - s)}}. \]

Due to the logarithms, it does not seem possible to perform the sums. Notice also that these terms give no correction to the first three cluster (and virial) coefficients.

It only remains to evaluate diagrams (c), (g), (h). These are the easiest to compute. They yield finite results:

\[ P^c = \frac{1 + \zeta}{2} \frac{\kappa^2}{\pi m\lambda^4} \frac{z}{1 - \zeta} \ln^2(1 - \zeta z), \]

\[ P^g = -\frac{\kappa^2}{48\pi m\lambda^4} \frac{z}{1 - \zeta} \ln^2(1 - \zeta z), \]

\[ P^h = \frac{\kappa^2}{4\pi m\lambda^4} \ln(1 - \zeta) \left[ \frac{\zeta z}{1 - \zeta} + \ln(1 - \zeta z) \right]. \]

The solution we have found consists of a summable part (eqs. (24), (25), (28), (29) and (30)) and a non-summable logarithmic part, eq. (26). Collecting the summable terms we get

\[ \frac{\kappa^2}{2\pi m\lambda^4} \left( \frac{1}{2} + (\zeta + 1 - \frac{1}{24}) \frac{z}{1 - \zeta z} \right) \ln^2(1 - \zeta z). \]

We find complete agreement with the results of ref. [11]. The third virial coefficient obtained from these results does not depend on \( \zeta \); hence it satisfies the ‘mirror symmetry’ discovered by Sen [3]. For higher virial coefficients, this symmetry does not hold.

To conclude, we have shown that the model based on standard perturbative Chern-Simons field theory coupled to non-relativistic matter accounts for the thermodynamic properties of the anyon gas. As an illustration, we have derived a closed formula for the n-cluster coefficient through second order in the anyon coupling constant.

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Figure Captions

Fig. 1. Non zero diagrams contributing to the pressure to the second order in $\kappa$. The sign $\pm$ refers to Bose or Fermi propagators. Graphs (c), (g) and (h) do not contribute to the second virial coefficient. Combinatoric factors are shown in the diagram.