NONCONVEX LIPSCHITZ FUNCTION IN PLANE WHICH IS
LOCALLY CONVEX OUTSIDE A DISCONTINUUM

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Abstract. We construct a Lipschitz function on \( \mathbb{R}^2 \) which is locally convex
on the complement of some totally disconnected compact set but not convex.
Existence of such function disproves a theorem that appeared in a paper by
L. Pasqualini and was also cited by other authors.

1. Introduction

In his work from 1938 L. Pasqualini presents a theorem (see [3, Theorem 51, p.
43]) of which the following statement is a reformulation:

Let \( f : \mathbb{R}^d \to \mathbb{R} \) be a continuous function and \( M \subset \mathbb{R}^d \) a set not containing any
continuum of topological dimension \((d - 1)\). Suppose that \( f \) is locally convex on the
complement of \( M \). Then \( f \) is convex on \( \mathbb{R}^d \).

The proof however contains a gap. This result also appeared in the survey paper
[1], where the (incorrect) proof was shortly repeated. Also V.G. Dmitriev mentions
this result in [2], although he provides a wrong reference.

As a counterexample to the theorem of Pasqualini we present the following theo-
rem:

Theorem 1.1. There is a Lipschitz function \( f : \mathbb{R}^2 \to \mathbb{R} \) and \( M \subset \mathbb{R}^2 \) such that

\begin{itemize}
  \item \( f \) is locally convex on \( \mathbb{R}^2 \setminus M \),
  \item \( f \) is not convex on \( \mathbb{R}^2 \),
  \item \( M \) is compact and totally disconnected,
  \item \( f \) has compact support.
\end{itemize}

Note that it is simple observation that such set \( M \) cannot be of one dimen-
sional Hausdorff measure 0 (this fact actually essentially follows from the original
argument by Pasqualini).

In this situation it seems natural to call a compact set \( M \) convex nonremovable
if there is a nonconvex say Lipschitz function \( f \) which is locally convex on the
complement of \( M \). Note that in such context it may be relevant that the function from
Theorem 1.1 is Lipschitz (or continuous) or that it has a compact support or that
it is defined on whole \( \mathbb{R}^2 \), since it is possible that such notion of nonremovability
might differ if we a priori assume some of those conditions to hold for \( f \). In some
sense the set \( M \) from Theorem 1.1 may be considered as nonremovable in one of
the strongest ways possible.

2. Preliminaries

In the paper we will use the following more or less standard notation and defi-
nitions:

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For $a, b \in \mathbb{R}^d$ and $r > 0$ we will denote by $B(a, r)$ the closed ball with center $a$ and radius $r$ and $[a, b]$ will denote the closed line segment with endpoints $a$ and $b$. For $A \subset \mathbb{R}^d$ the symbol co$A$ will mean the convex hull of $A$ and $A^c$ will mean the complement of $A$. If $l \subset \mathbb{R}^2$ is a line and $\varepsilon > 0$ then we define $l(\varepsilon) = \{x \in \mathbb{R}^2 : \text{dist}(x, l) < \varepsilon\}$.

A function $f$ defined on a set $A \subset \mathbb{R}^2$ is called $L$-Lipschitz, if for every $x, y \in A$, $x \neq y$, we have $|f(x) - f(y)| \leq L|x - y|$. We will call $f$ locally convex on $A$ if for every $x, y$ such that $[x, y] \subset A$ and $\alpha \in [0, 1]$ we have $f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$.

Finally, $f$ will be called piecewise affine on $A$ if there is a locally finite triangulation $\Delta$ of $A$ such that $f$ is affine on every triangle from $\Delta$.

3. Construction of the function

Definition 3.1. Let $Q$ be a system of all unions of finite systems of (closed) polytopes in $\mathbb{R}^2$. Let $L > 0$, $f : \mathbb{R}^2 \to \mathbb{R}$ and $P \in Q$. We say that a pair $(P, f)$ is $L$-good if

1. $f$ is $L$-Lipschitz,
2. $f$ is piecewise affine on $P^c$,
3. $f$ is locally convex on $P^c$.

The key technical result is the following:

Lemma 3.2. Let $\varepsilon, L > 0$, $l$ line in $\mathbb{R}^2$ let $(P, g)$ be a $L$-good. Then there is an $(L + \varepsilon)$-good pair $(Q, h)$ such that

1. $Q \subset P$,
2. $h = g$ on $P^c$,
3. if $x, y \in Q$ belong to the different component of $\mathbb{R}^2 \setminus l(\varepsilon)$ then they belong to the different component of $Q$.

We first prove Theorem 1.1 using Lemma 3.2.

Proof of Theorem 1.1. Choose a sequence $\{x_n\}_{n=1}^{\infty}$ dense in the plane and consider any sequence of lines $\{l_n\}_{n=1}^{\infty}$ with the property that for any $i, j \in \mathbb{N}$ there is some $k \in \mathbb{N}$ such that $x_i, x_j \in l_k$. Choose a sequence $\{\varepsilon_n\}_{n=1}^{\infty} \subset (0, \infty)$ such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. Then the sequence $\{l_n(\varepsilon_n)\}_{n=1}^{\infty}$ has the property that for every $x, y \in \mathbb{R}^2$, $x \neq y$, there is some $k \in \mathbb{N}$ such that $x$ and $y$ belong to the different component of $\mathbb{R}^2 \setminus l_k(\varepsilon_k)$.

We will proceed by induction and construct a sequence of functions $f_i : \mathbb{R}^2 \to \mathbb{R}$ and a sequence $P_i \subset Q$, $i = 0, 1, ..., $ such that for every $i$ the following conditions hold:

1. pair $(P_i, f_i)$ is $(1 + \sum_{n=1}^{i} \varepsilon_n)$-good,
2. if $i > 0$ then $P_i \subset P_{i-1}$,
3. if $i > 0$ then $f_i = f_{i-1}$ on $(P_{i-1})^c$,
4. if $i > 0$ and if $x, y \in P_i$ belong to the different component of $\mathbb{R}^2 \setminus l_i(\varepsilon_i)$ then they belong to the different component of $P_i$.

To do this let $f_0$ be an arbitrary 1-Lipschitz function on $\mathbb{R}^2$ which is equal to 0 on $(-3, 3)^2$ and equal to 1 on $[-1, 1]^2$ and put $P_0 := [-3, 3]^2 \setminus (-1, 1)^2$. Validity of conditions (1) – (4) is obvious.

Now, if we have $f_{i-1}$ and $P_{i-1}$ constructed we obtain $f_i$ and $P_i$ simply by applying lemma 3.2 with $\varepsilon = \varepsilon_i$, $L = (1 + \sum_{n=1}^{i-1} \varepsilon_n)$, $l = l_i$, $P = P_{i-1}$ and $g = f_{i-1}$. The function $f_i$ will be then equal to $h$ from the statement of lemma 3.2 and $P_i$ will be equal to the corresponding $Q$. Validity of conditions (1) – (4) follows directly from lemma 3.2.
Put \( M := \cap P_i \). Due to property (2) \( M \) is compact and nonempty. To prove that \( M \) is totally disconnected consider \( x, y \in M, x \neq y \). By the choice of the sequences \( \{l_n\}_{n=1}^{\infty} \) and \( \{\varepsilon_n\}_{n=1}^{\infty} \subset \mathbb{R}^+ \) there is some \( i \) such that \( x \) and \( y \) belong to the different component of \( \mathbb{R}^2 \setminus l_i(\varepsilon_1) \). By property (3) we have that \( x \) and \( y \) belong to the different component of \( P_i \). Using property (2) again we then obtain that \( x \) and \( y \) belong to the different component of \( M \) as well.

Define \( \tilde{f} : M^c \to \mathbb{R} \) in such a way that \( \tilde{f}(x) = f_i(x) \) whenever \( x \in (P_i)^c \). It is easy to see that the definition of \( \tilde{f} \) is correct due to properties (2) and (3) and the definition of \( M \), and also that by property (1) the function \( \tilde{f} \) is \( (1 + \sum_{\varepsilon_n}^{\infty} -\varepsilon_n) \)-Lipschitz and locally convex on \( M^c \). By Kirszbraun’s theorem there is a \( (1 + \sum_{\varepsilon_n}^{\infty} -\varepsilon_n) \)-Lipschitz function \( f : \mathbb{R}^2 \to \mathbb{R} \) such that \( f = \tilde{f} \) on \( M^c \). Therefore \( f \) is locally convex on \( M^c \) as well. Also, \( f \) has compact support due to properties (2) and (3), the fact that \( P_0 \) is compact and that \( f_0 \) is supported in \( P_0 \).

It remains to show that \( f \) is not convex on \( \mathbb{R}^2 \), but this is easy since

\[
\frac{f(-3,0) + f(3,0)}{2} = 0 < 1 = f(0,0).
\]

\( \square \)

The proof of Lemma 3.2 is divided into several lemmae.

**Lemma 3.3.** Let \( H \subset \mathbb{R}^2 \) be a closed halfplane, \( x \in \mathbb{R}^2 \setminus H \) and \( L > 0 \). If \( f : H \cup \{x\} \to \mathbb{R} \) is \( L \)-Lipschitz and affine on \( H \), then for every \( y \in \partial H \) the function

\[
g_y(z) = \begin{cases} f(z), & \text{if } z \in H, \\ \alpha f(x) + (1 - \alpha)f(y), & \text{for } z = \alpha x + (1 - \alpha)y, \alpha \in [0,1]. \end{cases}
\]

is \( L \)-Lipschitz as well.

**Proof.** Without any loss of generality we can suppose that \( H = \{(x, y) \in \mathbb{R}^2 : x \leq 0\} \), \( f(y) = 0 \) and that \( y = (0,0) \). This means that \( g_y \) is in fact linear on both \( H \) and \( [x, y] \). Choose \( a \in H \) and \( b = \alpha x \) for some \( \alpha \in [0,1] \). Now,

\[
|g_y(a) - g_y(b)| = \alpha \left| g_y \left( \frac{1}{\alpha} a \right) - g_y \left( \frac{1}{\alpha} b \right) \right| = \alpha \left| g_y \left( \frac{1}{\alpha} a \right) - g_y \left( \frac{1}{\alpha} \alpha x \right) \right|
\]

\[
= \alpha \left| g_y \left( \frac{1}{\alpha} x \right) - g_y (x) \right| \leq \alpha L \left| \frac{1}{\alpha} a - x \right| = \alpha L \left| \frac{1}{\alpha} a - \frac{1}{\alpha} \alpha x \right|
\]

\[
= L|a - \alpha x| = L|a - b|.
\]

Similarly, if \( a = \alpha x \) and \( b = \beta x \) for some \( \alpha, \beta \in [0,1] \) \( \alpha \neq \beta \) we have

\[
|g_y(a) - g_y(b)| = |\alpha f(x) - \beta f(x)| \leq |\alpha - \beta|f(x) \leq |\alpha - \beta|L.
\]

\( \square \)

**Lemma 3.4.** Let \( \varepsilon, L, K > 0 \). Let \( f \) be a \( L \)-Lipschitz function on \([-K, K]^2 \), which is equal to an affine function \( f_1 \) on \([-K, 0] \times [-K, K] \), and \( z \in (0, K) \times (-K, K) \). Then there is an \( x \in [(0,0), z] \) and \( \gamma > 0 \) such that for every \( y \in B(x, \gamma) \) and every \( w \in B((0,0), \gamma) \cap (\{0\} \times (-K, K)) \) the function

\[
g_{y,w}(u) = \begin{cases} f(u), & \text{if } u \in [-K, 0] \times [-K, K], \\ \alpha f(w) + (1 - \alpha)f(y), & \text{for } u = \alpha w + (1 - \alpha)y, \alpha \in [0,1]. \end{cases}
\]

is \((L + \varepsilon)\)-Lipschitz and \( |g_{y,w} - f| < \varepsilon \) on \([-K, 0] \times [-K, K] \cup [w, y] \).
Proof. Without any loss of generality we can suppose that $K = L = 1$ and that $f(0, 0) = 0$. Since $f$ is 1-Lipschitz we can find a sequence $\{x_i\}_{i=1}^{\infty} \subset [(0, 0), z]$ converging to $(0, 0)$ such that for some $s \in [-1, 1]$

$$s_i := \frac{f(x_i)}{|x_i|} \to s \quad \text{as} \quad i \to \infty. \quad (3.1)$$

Consider now the sequence of functions $h_i : [-\frac{1}{|x_i|}, 0] \times [-\frac{1}{|x_i|}, \frac{1}{|x_i|}] \cup \{x_i = 0\} \to \mathbb{R}$ defined as

$$h_i(u) := \frac{1}{|x_i|}f (|x_i|u).$$

Then $h_i$ is 1-Lipschitz for every $i$. Since $f$ is equal to an affine function $f_1$ on $[-1, 0] \times [-1, 1]$ and $f(0, 0) = 0$ we have $h_i = f_1$ on $[-\frac{1}{|x_i|}, 0] \times [-\frac{1}{|x_i|}, \frac{1}{|x_i|}]$.

Also $h_i(\tilde{z}) = s_i$. Therefore by (3.3) the function $h : (-\infty, 0] \times (-\infty, \infty) \cup \{\tilde{z}\} \to \mathbb{R}$ which is equal to $f_1$ on $(-\infty, 0] \times (-\infty, \infty)$ and such that $h(\tilde{z}) = s$ is also 1-Lipschitz.

Consider $\tilde{\gamma} > 0$ such that $\tilde{\gamma} < \frac{\gamma^2}{2}$ (here by $\tilde{\gamma}$ we mean the first coordinate of $\tilde{z}$) and such that

$$\frac{|v - z|}{|v - \tilde{z}| - \tilde{\gamma}} < 1 + \frac{\gamma}{2}$$

for every $v \in (-\infty, 0] \times (-\infty, \infty)$.

Now, for every $\tilde{s} \in [s - \tilde{\gamma}, s + \tilde{\gamma}]$, $v \in (-\infty, 0) \times (-\infty, \infty)$, and $u \in B(\tilde{z}, \tilde{\gamma})$

$$\frac{f_1(v) - \tilde{s}}{|v - u|} \leq \frac{f_1(v) - s}{|v - u|} + \frac{|s - \tilde{s}|}{|v - u|} \leq \frac{|f_1(v) - s|}{|v - \tilde{z}| - \tilde{\gamma}} + \frac{\tilde{\gamma}}{|v - \tilde{z}| - \tilde{\gamma}} \leq \frac{|f_1(u) - s|}{|v - \tilde{z}|} \cdot \frac{|v - \tilde{z}|}{|v - \tilde{z}| - \tilde{\gamma}} + \frac{2\tilde{\gamma}}{2} \leq \left(1 + \frac{\gamma}{2}\right) + \frac{\gamma}{2} = 1 + \frac{\gamma}{2}.$$

Therefore, by lemma 3.3 for every $\tilde{s} \in [s - \tilde{\gamma}, s + \tilde{\gamma}]$, $v \in \{0\} \times (-\infty, \infty)$ and $t \in B(\tilde{z}, \tilde{\gamma})$ the function

$$\tilde{h}_{v, t, \tilde{s}}(u) = \begin{cases} f_1(u), & \text{if } u \in (-\infty, 0) \times (-\infty, \infty), \\ (1 - \alpha)\tilde{s} + \alpha f_1(v), & \text{for } u = (1 - \alpha)t + \alpha v, \alpha \in [0, 1]. \end{cases}$$

is $(1 + \frac{\gamma}{2})$-Lipschitz as well.

Choose $i$ such that $s_i \in [s - \tilde{\gamma}, s + \tilde{\gamma}]$ and put $x = x_i$ and $\tilde{\gamma} = \frac{|x_i|}{2}$. Now, consider some $y \in B(x, \gamma)$ and some $w \in B((0, 0), \gamma) \cap \{0\} \times (-1, 1)$ and let $g_{y, w}$ be as in the statement on the lemma. First we will prove that $g_{y, w}$ is $(1 + \frac{\gamma}{2})$-Lipschitz. To do this we first observe that $\frac{1}{|x_i|}g_{y, w}(\frac{x_i}{|x_i|})$ is equal to $\tilde{h}_{\frac{y}{|x_i|}, \tilde{z}, \tilde{s}}(\frac{x_i}{|x_i|})$, where the first function is defined. Now, we have $\frac{|y|}{|x_i|} \in \{0\} \times (-\infty, \infty)$,

$$\frac{|y - x|}{|x_i|} = \frac{|y|}{|x|} - \frac{|x|}{|x|} = \frac{|y - x|}{|x|} \leq \frac{|x|\tilde{\gamma}}{|x|} \leq \tilde{\gamma},$$

which means $\frac{|y|}{|x|} \in B(\tilde{z}, \tilde{\gamma})$ and finally

$$\frac{|f(y)|}{|x|} - s = \frac{|f(y) - f(x) + f(x)|}{|x|} - s \leq \frac{|f(y) - f(x)|}{|x|} + \frac{|f(x)|}{|x|} - s \leq \frac{|y - x|}{|x|} + \frac{\tilde{\gamma}}{2} \leq \frac{|y|}{|x|} + \frac{\tilde{\gamma}}{2} = \frac{\tilde{\gamma}}{2} + \frac{\tilde{\gamma}}{2} = \tilde{\gamma},$$

which means that $\frac{f(y)}{|x|} \in [s - \tilde{\gamma}, s + \tilde{\gamma}]$ and we are done since $\frac{1}{|x_i|}g_{y, w}(\frac{x_i}{|x_i|})$ and $g_{y, w}$ have the same Lipschitz constant.

To finish the proof it is now sufficient to observe that if we additionally choose $x_i$ small enough we obtain also $|g_{x_i} - f| < \varepsilon$ on $[-1, 0] \times [-1, 1] \cup [w, y]$. □

Lemma 3.5. Let $L, \varepsilon, \delta > 0$, $a < b$ and $c < d$ be given. Let

$$P = \text{co}\{(-1, a), (-1, b), (1, c), (1, d)\}$$

and

$$P^\varepsilon = \text{co}\{(-1, a - \varepsilon), (-1, b + \varepsilon), (1, c - \varepsilon), (1, d + \varepsilon)\}.$$
Suppose that $f$ is a $L$-Lipschitz function defined on $\mathbb{R}^2$ which is locally affine on $P^c \setminus P$. Then there are
\[
\frac{a + c}{2} = a_0 < a_1 < \ldots < a_{n-1} < a_n := \frac{b + d}{2}
\]
and $\frac{1}{2} > \kappa > 0$ such that, using the notation defined below, the function $g_\kappa : P^c \setminus (P^0 \setminus [-\kappa, \kappa] \times \mathbb{R}) \to \mathbb{R}$ defined as $g_\kappa(z_i^+ = f(z_i^+)$ for $i = 0, n$, $g_\kappa(z_i^- = f(z_i)$ for $i = 1, \ldots, n - 1$ and

$$g_\kappa(x) = \begin{cases} f(x), & \text{if } x \in P^c \setminus P, \\ \alpha g(z_i^+) + \beta g(z_i^-) + \gamma g(z_{i+1}^+), & \text{for } x = \alpha z_i^+ + \beta z_i^- + \gamma z_{i+1}^+, \\ \alpha g(z_i^-) + \beta g(z_{i+1}^+) + \gamma g(z_i^+), & \text{for } x = \alpha z_i^- + \beta z_{i+1}^+ + \gamma z_i^+, \\ \end{cases}$$

is $(L + \delta)$-Lipschitz and such that $|f - g_\kappa| < \delta$ on $\mathbb{R}^2$.

Here we denoted $z_0^+ := (\pm \kappa, a + c \pm \frac{\kappa(a-c)}{2})$, $z_n^+ := (\pm \kappa, b + d \pm \frac{\kappa(b-d)}{2})$, $z_i^+ := (\pm \kappa, a_i)$ for $i = 1, \ldots, n - 1$ and $z_i := (0, a_i)$ for $i = 0, \ldots, n$.

**Proof.** Without any loss of generality we can suppose $L = 1$. Denote $P_i$ the connectivity component of $P^c \setminus P^c$ containing $z_i$, $i = 0, n$. When we have $a_i$ found we will denote $P_i = \text{co}\{c_i^+, c_i^-\}$ for $i = 0, \ldots, n - 1$.

First use lemma 3.4 to find $a_1 \in B(a_0, \min(\{|a_0-a_1|, 1\}))$ and $a_{n - 1} \in B(a_n, \min(\{|a_n-a_{n-1}|, 1\}))$ and $\kappa_1 > 0$ such that for every $\kappa > 0$ the function $g|_{P_0^c \cup P_0}$ and $g|_{P_0^c \cup P_{n-1}}$ are both $(L + \delta)$-Lipschitz and such that $|f - g_\kappa| < \delta$ on $P^c \cup P_0 \cup P_{n-1}$.

Observe that for every $u_0 \in P_0^c \cup P_0$ and every $u_n \in P_n^c \cup P_n$ we have
\[
\frac{|g_\kappa(u_0) - g_\kappa(u_n)|}{|u_0 - u_n|} \leq \frac{|g_\kappa(u_0) - g_\kappa(z_0)|}{|u_0 - u_n|} + \frac{|g_\kappa(z_0) - g_\kappa(z_n)|}{|u_0 - u_n|} + \frac{|g_\kappa(z_n) - g_\kappa(u_n)|}{|u_0 - u_n|}
\]
and since the last formula can be smaller than $1 + \delta$ when we assume $|a_0 - a_1|$ and $|a_{n-1} - a_n|$ to be small enough, we can additionally assume that $g|_{P^c \cup P_0 \cup P_{n-1}}$ is $(1 + \delta)$-Lipschitz.

Next, note that the function $g_\kappa|_{\{z_1, z_{n-1}\}}$ is actually independent on $\kappa$ and that it is 1-Lipschitz for any choice of $a_2, \ldots, a_{n-2}$ (this is because in one dimension the affine extension never increases the Lipschitz constant). This also means that for $S = \text{co}\{c_i^+, c_i^-\}$ we have $g_\kappa|_S$ is 1-Lipschitz for any choice of $a_2, \ldots, a_{n-2}$ as well. Put $\alpha = \text{dist}(S, P^c \setminus P)$, we can assume $\kappa_2$ to be small enough that $1 + \alpha > 0$ (here we used the fact that $|a_0 - a_1|, |a_{n-1} - a_n| \leq \frac{1}{2}$). Consider $n$ big enough such that $\frac{|a_1-a_{n-1}|}{n-1} \leq \frac{x}{n}$, put $a_i = a_1 + \frac{\delta(n-1)}{n-1}$ and pick $\kappa_3 < \min(\kappa_2, \frac{\delta}{n})$. Then for $\kappa < \kappa_3$ and $a \in S$
\[
|g_\kappa(a) - f(a)| \leq |g_\kappa(a) - g_\kappa(z_i)| + |g_\kappa(z_i) - f(z_i)| + |f(z_i) - f(a)|
\]
(3.2)
\[
|a - z_i| + 0 + |a - z_i| \leq \frac{\delta}{2} < \delta,
\]
where $i$ is chosen such that $a \in P_i$.

To finish the proof we need to observe that for $\kappa < \kappa_3$ the function $g_\kappa$ is $(1 + \delta)$-Lipschitz. Since $S \cup P_0 \cup P_{n-1}$ is convex, the remaining case we have to consider is $a \in S$ and $b \in P^c \setminus P$. Find $i$ such that $a \in P_i$. With this choice we have $|a - z_i| \leq \frac{\alpha \delta}{2}$ and therefore
\[
|b - z_i| \leq |a - b| + |a - z_i| \leq |a - b| + \frac{\alpha \delta}{2} \leq (1 + \delta) |a - b|.
\]
Now,
\[|g_\varepsilon(a) - g_\varepsilon(b)| \leq |g_\varepsilon(a) - g_\varepsilon(z_i)| + |g_\varepsilon(z_i) - g_\varepsilon(b)| \]
\[\leq \frac{\delta a}{2} + |f(z_i) - f(b)| \leq \frac{\delta}{2}|a - b| + |b - z_i| \]
\[\leq \frac{\delta}{2}|a - b| + \left(1 + \frac{\delta}{2}\right)\cdot|a - b| \leq (1 + \delta)|a - b|.
\]

**Lemma 3.6.** Let \(1 > \varepsilon > 0\) and \(\alpha, L > 0\). Let \(f\) be a \(L\)-Lipschitz function on \([-1, 1]^2\) which is affine on both \([-1, 0] \times [-1, 1]\) and \([0, 1] \times [-1, 1]\) (and equal to affine functions \(f_1\) and \(f_2\), respectively). Put
\[A_1 = [-1, 0] \times [-1, -1/2], A_2 = [0, 1] \times [1/2, 1],\]
\[B_1 = [0, \varepsilon] \times [-1, \varepsilon], B_2 = [-\varepsilon, 0] \times [-\varepsilon, 1]\]
and
\[A = A_1 \cup A_2 \cup B_1 \cup B_2.\]
Then either \(f\) is convex on \([-1, 1]^2\) or the function \(g_\varepsilon : A \rightarrow \mathbb{R}\) defined as
\[g(x) = \begin{cases} f_1(x), & \text{if } x \in A_1 \cup B_1, \\ f_2(x), & \text{if } x \in A_2 \cup B_2. \end{cases}\]
is locally convex on \(A\). Moreover, if \(\varepsilon\) is small enough, \(g_\varepsilon\) is \((L + \alpha)\)-Lipschitz and \(|g_\varepsilon - f| < \alpha\) on \(A\).

**Proof.** Direct computation.

**Lemma 3.7.** Let \(L, \alpha > 0\) and \(1 > \gamma > \varepsilon > 0\). Let \(f\) be a \(L\)-Lipschitz function on \([-4, 4]^2 \cup [4, 5] \times [1, 2]\) which is affine on both \([-4, 0] \times [-4, 4]\) and \([0, 4] \times [-4, 4] \cup [4, 5] \times [1, 2]\) (and equal to affine functions \(f_1\) and \(f_2\), respectively). Put
\[A_1 = [0, \gamma] \times [-3, -2], A_2 = [\gamma, \gamma + \varepsilon] \times [-3, 0], A_3 = [\gamma - \varepsilon, \gamma] \times [-1, 2],\]
\[A_4 = [\gamma, 4] \times [1, 2], B_1 = [-4, 0] \times [-4, 4], B_2 = [4, 5] \times [1, 2],\]
and
\[A = A_1 \cup A_2 \cup A_3 \cup A_4 \cup B_1 \cup B_2.\]
Then either \(f\) is locally convex on \([-4, 4]^2 \cup [4, 5] \times [1, 2]\) or the function
\[g(x) = \begin{cases} f_1(x), & \text{if } x \in A_1 \cup A_2 \cup B_1, \\ f_2(x) + \frac{f_1(\gamma, 0) - f_1(0, 0) - f_2(\gamma, 0) + f_1(0, 0)}{\gamma - 4}\cdot(x \cdot (1, 0) - 4), & \text{if } x \in A_3 \cup A_4, \\ f_2(x), & \text{if } x \in B_2, \end{cases}\]
is \((L + \alpha)\)-Lipschitz, locally convex on \(A\) and \(|f - g| < \alpha\) on \(A\), if \(\varepsilon\) and \(\gamma\) are small enough.

**Proof.** Without any loss of generality we can suppose \(L = 1\). First we prove that \(g\) is continuous on \(A\). To do this we need to prove that
\[(3.3) \quad f_1(\gamma, a) = f_2(\gamma, a) + \frac{f_1(\gamma, 0) - f_1(0, 0) - f_2(\gamma, 0) + f_1(0, 0)}{\gamma - 4}\cdot((\gamma, a) \cdot (1, 0) - 4)\]
whenever \((\gamma, a) \in A\) and that
\[(3.4) \quad f_2(4, a) = f_2(4, a) + \frac{f_1(\gamma, 0) - f_1(0, 0) - f_2(\gamma, 0) + f_1(0, 0)}{\gamma - 4}\cdot((4, a) \cdot (1, 0) - 4)\]
whenever \((4, a) \in A\). Define an affine function \(f_3\) on \(\mathbb{R}^2\) as
\[f_3(u, v) = \frac{f_1(\gamma, 0) - f_1(0, 0) - f_2(\gamma, 0) + f_1(0, 0)}{\gamma - 4}\cdot((u, v) \cdot (1, 0) - 4).\]
To prove (3.3) we can write
\[ g(\gamma, a) = f_2(\gamma, a) + f_3(\gamma, a) \]
\[ = f_2(\gamma, a) + f_1(\gamma, 0) - f_1(0, 0) - f_2(\gamma, 0) + f_1(0, 0) \frac{(\gamma - 4)}{\gamma - 4} \cdot (\gamma - 4) \]
\[ = f_2(\gamma, a) + f_1(\gamma, 0) - f_1(0, 0) - f_2(\gamma, 0) + f_2(0, 0) \]
\[ = f_2(\gamma, a) + f_1(\gamma, a) - f_1(0, a) - f_2(\gamma, a) + f_2(0, a) \]
\[ = f_2(\gamma, a) + f_1(\gamma, a) - f_1(0, a) - f_1(0, a) - f_2(\gamma, a) + f_1(0, a) = f_1(\gamma, a). \]

To prove (3.4) we can write
\[ g(4, a) = f_2(4, a) + f_3(4, a) \]
\[ = f_2(4, a) + f_1(4, 0) - f_1(0, 0) - f_2(4, 0) + f_1(0, 0) \frac{(4 - 4)}{(4 - 4)} = f_2(4, a). \]

Next note that since both \( f_1 \) and \( f_2 \) are 1-Lipschitz we have
\[ g \] is 1-Lipschitz on \( B_1 \cup A_1 \cup A_2 \),
and
\[ g \] is 1-Lipschitz on \( B_2 \),
also since additionally \( f_3 \) is constant on all lines parallel to \( y \)-axis and since
\[ \frac{f_3(\gamma, 0) - f_3(4, 0)}{4 - \gamma} \leq \frac{f_1(\gamma, 0) - f_1(0, 0) - f_2(\gamma, 0) + f_2(0, 0)}{3} \leq \frac{2\gamma}{3} \leq \gamma. \]

we have
\[ g \] is \( (1 + \gamma) \)-Lipschitz on \( A_4 \cup A_3 \).
and
\[ |g - f_2| \leq 4\gamma \] on \( A_4 \cup A_3 \).

Now, if \( x \in B_1 \) and \( y \in A_3 \) then \( g(x) = f_1(x) \), \( |g(y) - f_1(y)| \leq 3\varepsilon \) and \( |x - y| \geq \gamma - \varepsilon \)
and therefore
\[ |g(x) - g(y)| \leq |g(x) - f_1(y)| + |f_1(y) - g(y)| \leq |x - y| + 3\varepsilon \leq \frac{\gamma + 2\varepsilon}{\gamma - \varepsilon}. \]

So
\[ g \] is \( \frac{\gamma + 2\varepsilon}{\gamma - \varepsilon} \)-Lipschitz on \( B_1 \cup A_3 \).
If \( x \in B_1 \) and \( y \in A_4 \) then \( g(x) = f_1(x) \), \( f(y) \leq g(y) \leq f_1(y) \) and therefore
\[ g \] is 1-Lipschitz on \( B_1 \cup A_4 \).
Using (3.6) and (3.7) and continuity of \( g \) we obtain that
\[ g \] is \( (1 + \gamma) \)-Lipschitz on \( A_2 \cup A_3 \) and on \( B_2 \cup A_4 \).
Finally, if \( x \in A_1 \cup A_2 \) and \( y \in A_4 \cup B_2 \) or \( x \in A_1 \) and \( y \in A_3 \cup A_4 \cup B_2 \) we have
\[ |g(x) - f_2(x)| \leq 2(\gamma + \varepsilon) \leq 4\gamma, \quad |g(y) - f_2(y)| \leq 4\gamma \]
and \( |x - y| \geq 1 \). This implies
\[ |g(x) - g(y)| \leq |g(x) - f_2(x)| + |f_2(x) - f_2(y)| + |f_2(y) - g(y)| \leq 4\gamma + |x - y| + 4\gamma \leq (1 + 8\gamma)|x - y|. \]

Now, according to \( 3.3, 3.6, 4.7, 4.8, 5.11 \) and \( 5.12 \) it is sufficient to choose \( \frac{\gamma}{\gamma - \varepsilon} > 1 + \gamma > \varepsilon > 0 \) small enough such that
\[ \max \left( 1 + 8\gamma, \frac{\gamma + 2\varepsilon}{\gamma - \varepsilon} \right) < 1 + \alpha \]
to obtain that $g$ is $(1 + \alpha)$-Lipschitz on $A$ and $|f - g| < \alpha$ on $A$. \hfill \Box

**Lemma 3.8.** Under the assumptions of Lemma 3.5 there is a $\frac{1}{2} > \kappa > 0$, $R \subset P^o \cap \mathbb{R} \times (-\kappa, \kappa)$ and a function $h : (P^c \setminus P) \cup R \to \mathbb{R}$ such that:

(a) $R \in \mathcal{Q}$,
(b) $h = f$ on $P^c \setminus P^o$,
(c) $h$ is locally convex on $(P^c \setminus P^o) \cup R$,
(d) $(P^c \setminus P) \cup R$ is connected,
(e) $h$ is piecewise affine on $(P^c \setminus P^o) \cup R$,
(f) $h$ is $(L + \delta)$-Lipschitz.

**Proof.** Without any loss of generality we can suppose $L = 1$. Let $\kappa, z_i g_\kappa$ as in Lemma 3.5, but with $\frac{1}{2}$ in the place of $\delta$. Consider the sets $X = [-4, 4]^2 \cup [4, 5] \times [1, 2]$ and $Y = [-1, 1]^2$.

Find similarities $\Psi_i : \mathbb{R}^2 \to \mathbb{R}^2$, $i = 0, \ldots, n$ such that if we put $M_i = \Psi_i(X)$, $i = 0, n$ and $M_i = \Psi_i(Y)$, $i = 1, \ldots, n - 1$ we have

(A) $M_i \cap M_j = \emptyset$ if $i \neq j$,

(B) $\Psi_0([-4, 0] \times [-4, 4]) \subset P^c \setminus P^o$,

(C) $\Psi_n([-4, 0] \times [-4, 4]) \subset P^c \setminus P^o$,

(D) $M_i \subset \mathbb{R} \times (-\kappa, \kappa)$,

(E) $[z_i^-, z_i^+] \subset \Psi_i([0] \times \mathbb{R})$,

(F) $\Psi_i$ preserves orientation for $i = 1, \ldots, n - 1$.

Put $\Omega = \min_{i \neq j} \text{dist}(M_i, M_j)$, note that $\Omega > 0$ due to property (A). Define

$$T_i := \text{co}\{\Psi_i((1, \frac{1}{2}), \Psi_i)(1, 1), \Psi_{i+1}((-1, -\frac{1}{2}), \Psi_{i+1})(-1, -1)\},$$

for $i = 1, \ldots, n - 2$,

$$T_0 := \text{co}\{\Psi_0(5, 1), \Psi_0(5, 2), \Psi_1(-1, -\frac{1}{2}), \Psi_1(-1, -1)\}$$

and

$$T_{n-1} := \text{co}\{\Psi_n(5, 1), \Psi_n(5, 2), \Psi_{n-1}(1, \frac{1}{2}), \Psi_{n-1}(1, 1)\}.$$

and put

$$R := \left( \bigcup_{i=0}^{n-1} T_i \right) \cup \left( \bigcup_{i=0}^{n} M_i \right).$$

Let $\rho_i$ be scaling ratio of $\Psi_i$. Let $g_i$, $i = 1, \ldots, n - 1$ be the function $g$ from Lemma 3.6, but with $\frac{1}{2}$ in the place of $\delta$ (and corresponding $\varepsilon$) and with $f_1(x) = \rho_i \kappa \circ \Psi_i$ and $f_2(x) = \rho_i \kappa \circ \Psi_i$ (with the exception if $g_\kappa$ is already convex on $M_i$, in which case we put $g_i = g_\kappa|_{M_i}$), let $g_0$ be the function $g$ from Lemma 3.7, with $\gamma = \frac{1}{2\delta}$ (and corresponding $\varepsilon$ and $\gamma$) and with $f_1 = \rho_0 \kappa \circ \Psi_0$ and $f_2 = \rho_0 \kappa \circ \Psi_0$ and finally, let $g_n$ be the function $g$ from Lemma 3.7, with $\gamma = \frac{1}{2\delta}$ (and corresponding $\varepsilon$ and $\gamma$) and with $f_1 = \rho_n \kappa \circ \Psi_n$ and $f_2 = \rho_n \kappa \circ \Psi_n$.

Consider now the function $h$ defined by the formula

$$h = \begin{cases} \frac{1}{\rho_i} g_i \circ \Psi_i^{-1} & \text{on } M_i \\ g_\kappa & \text{otherwise.} \end{cases}$$

Property (a) follows from (3.14) and the fact that every $M_i$ and every $T_i$ is a polygon. Properties (b), (c) and (e) follow directly from the construction and corresponding properties of the functions $g_i$ and property (d) is obvious. We will now finish the proof by proving property (f).
So suppose that \( a, b \in (P^e \setminus P) \cup R \). We need to prove that \(|h(a) - h(b)| \leq (1 + \delta)|a - b|\). We can additionally suppose that either \( a \) or \( b \) belongs to some \( M_i \), since otherwise there is nothing to prove. We will prove only the case \( a \in M_i \), \( b \in M_j \), \( i \neq j \), the other cases can be proved following the same lines. By Lemma 3.8 (for \( i = 1, \ldots, n - 1 \)) and Lemma 3.7 (for \( i = 0, n \)) we can now write

\[
|h(a) - h(b)| \leq |h(a) - g_\kappa(a)| + |g_\kappa(a) - g_\kappa(b)| + |g_\kappa(b) - h(b)|
\]

\[
\leq \frac{1}{\rho_i} \frac{\Omega \delta \rho_i}{4} + \left(1 + \frac{\delta}{2}\right)|a - b| + \frac{\Omega \delta \rho_j}{4}
\]

\[
\leq \frac{\delta}{2}|a - b| + \left(1 + \frac{\delta}{2}\right)|a - b| = (1 + \delta)|a - b|
\]

which is what we need.

\[\square\]

**Proof of Lemma 3.2** Without any loss of generality we can suppose \( L = 1 \). Let \( V \) be the set of all points \( v \in \partial P \) with the property that there is some \( \varepsilon > 0 \) such that \( P \cap B(v, \varepsilon) \) is similar to \( \{(x, y) : x \geq 0\} \cap B(0, 1) \) and that \( f \) is affine on \( P \cap B(v, \varepsilon) \). Since \( P \subset Q \), the set \( \partial P \setminus V \) is finite and we can without any loss of generality assume that \( l(\varepsilon) \cap (\partial P \setminus V) = \emptyset \).

This means that the closure of every bounded component \( C_i \) of \( P \cap l(\varepsilon) \) is a similar copy of

\[
\text{co}\{(1, a_i), (-1, b_i), (1, c_i), (1, d_i)\} =: P_i
\]

for some \( a_i < b_i, c_i < d_i \) and such that for some \( \varepsilon_i > 0 \) \( f \) is locally affine on \( P_i^\varepsilon \setminus P \), where

\[
P_i^\varepsilon := \text{co}\{(-1, a_i - \varepsilon_i), (-1, b_i + \varepsilon_i), (1, c_i - \varepsilon_i), (1, d_i + \varepsilon_i)\}.
\]

Then

\[
\alpha = \min_{i \neq j} (C_i, C_j) > 0
\]

Let \( \Psi_i \) be a similarity between \( C_i \) and \( S_i \) and let \( \kappa_i, R_i \) and \( h_i \) be \( \kappa, R \) and \( h \) as obtained from Lemma 3.8 for \( \varepsilon = \varepsilon_i \), \( P = P_i \), \( f = \rho_i g \circ \Psi_i \) and \( \delta = \frac{\min(\varepsilon_i, 1) \rho_i \varepsilon_i}{4} \), where \( \rho_i \) is the similarity ratio on \( \Psi_i \).

Put \( Q = P \setminus (\bigcup R_i) \) and define \( \tilde{h} : Q^e \to \mathbb{R} \) by

\[
\tilde{h} = \begin{cases} \frac{1}{\rho_i} h_i \circ \Psi_i^{-1} & \text{on } R_i \\ g & \text{otherwise.} \end{cases}
\]

Let \( K \) be the Lipschitz constant of \( \tilde{h} \), the using the Kirszbraun theorem on extensions of Lipschitz functions we can find a \( K \)-Lipschitz function \( h o \) on \( \mathbb{R}^2 \) such that \( h = \tilde{h} \) on \( P^e \).

Now, property (1) follows directly from the definition of \( Q \) and \( (a) \) in Lemma 3.8 property (2) from the definition of \( h \) and \( (b) \) in Lemma 3.8 and property (3) from (d) in Lemma 3.8.

It remains to prove that the pair \( (Q, h) \) is \((1 + \varepsilon)\)-good. The local convexity and piecewise affinity of \( h \) on \( Q^e \) follows from \((c) \) and \((e) \) in Lemma 3.8 and the corresponding properties of \( g \), so the proof will be finished, if we verify that \( K \leq (1 + \varepsilon) \).

To do this pick \( a, b \in \mathbb{R}^2 \), we need to prove that \(|h(a) - h(b)| \leq (1 + \varepsilon)|a - b|\).

We can additionally suppose that either \( a \) or \( b \) belongs to some \( R_i \) since otherwise there is nothing to prove. We will prove only the case \( a \in R_i, b \in R_j, i \neq j \), the other cases can be proved following the same lines.
Using the definition of $h$, namely property $(f)$ from Lemma 3.8, we can now write
\[ |h(a) - h(b)| = |h_i(a) - h_j(b)| \leq |h_i(a) - f(a)| + |f(a) - f(b)| + |f(b) - h_j(b)| \]
\[ \leq \frac{1}{\rho_i} \cdot \min(\alpha, \varepsilon_i) \rho_i \varepsilon + \left( 1 + \frac{\varepsilon}{4} \right) \cdot |a - b| + \frac{1}{\rho_j} \cdot \min(\alpha, \varepsilon_j) \rho_j \varepsilon \]
\[ \leq \frac{2\varepsilon}{4} |a - b| + \left( 1 + \frac{\delta}{2} \right) \cdot |a - b| < (1 + \delta) |a - b|. \]

\[ \square \]

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