OPTIMALITY OF A CLASS OF ENTANGLEMENT WITNESSES FOR 3 \otimes 3 SYSTEMS

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Abstract. Let \( \Phi_{t,\pi} : M_3(\mathbb{C}) \rightarrow M_3(\mathbb{C}) \) be a linear map defined by \( \Phi_{t,\pi}(A) = (3-t) \sum_i E_i A E_i + t \sum_{i,j} E_{i,\pi(j)} A E_{i,\pi(j)} - A \), where \( 0 \leq t \leq 3 \) and \( \pi \) is a permutation of \( (1,2,3) \). We show that the Hermitian matrix \( W_{\Phi_{t,\pi}} \) induced by \( \Phi_{t,\pi} \) is an optimal entanglement witness if and only if \( t = 1 \) and \( \pi \) is cyclic.

1. Introduction

Let \( H \) be a separable complex Hilbert space. Recall that a quantum state on \( H \) is a density operator \( \rho \in B(H) \) which is positive and has trace 1. Denote by \( S(H) \) the set of all states on \( H \). If \( H \) and \( K \) are finite dimensional, a state in the bipartite composition system \( \rho \in S(H \otimes K) \) is said to be separable if \( \rho \) can be written as \( \rho = \sum_i p_i \rho_i \otimes \sigma_i \), where \( \rho_i \) and \( \sigma_i \) are states on \( H \) and \( K \) respectively, and \( p_i \) are positive numbers with \( \sum_i p_i = 1 \). Otherwise, \( \rho \) is entangled.

Entanglement is an important physical resource to realize various quantum information and quantum communication tasks such as teleportation, dense coding, quantum cryptography and key distribution \cite{10, 11}. It is very important but also difficult to determine whether or not a state in a composite system is separable. One of the most general approaches to characterize quantum entanglement for bipartite composition systems is based on the notion of entanglement witnesses (see \cite{4}). A Hermitian matrix \( W \) acting on \( H \otimes K \) is an entanglement witness (briefly, EW) if \( W \) is not positive and \( \text{Tr}(W\sigma) \geq 0 \) holds for all separable states \( \sigma \). Thus, if \( W \) is an EW, then there exists an entangled state \( \rho \) such that \( \text{Tr}(W\rho) < 0 \) (that is, the entanglement of \( \rho \) can be detected by \( W \)). It was shown that, a state is entangled if and only if it is detected by some entanglement witness \cite{4}. Constructing entanglement witnesses is a hard task, too. There was a considerable effort in constructing and analyzing the structure of entanglement witnesses \cite{1, 3, 7, 8, 15}. However, complete characterization and classification of EWs is far from satisfactory.

Due to the Choi-Jamiołkowski isomorphism \cite{2, 9}, a Hermitian matrix \( W \in B(H \otimes K) \) with \( \dim H \otimes K < \infty \) is an EW if and only if there exists a positive linear map which is not completely positive (NCP) \( \Phi : B(H) \rightarrow B(K) \) and a maximally entangled state \( P^+ \in \) PACS. 03.67.Mn, 03.65.Ud, 03.65.Db.

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The entanglement witness is that the necessary and sufficient condition for the Hermitian matrix $X$ where $|\psi^+\rangle = \frac{1}{\sqrt{n}}(|11\rangle + |22\rangle + \cdots |nn\rangle)$, where $n = \dim H$ and $\{|i\rangle\}_{i=1}^n$ is an orthonormal basis of $H$. Thus, up to a multiple by positive scalar, $W_\Phi$ can be written as the matrix $W_\Phi = (\Phi(E_{ij}))_{n \times n}$, where $E_{ij} = |i\rangle\langle j|$. For a positive linear map $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$, we always denote $W_\Phi$ the Choi-Jamiolkowski matrix of $\Phi$ with respect to a given basis of $H$, that is $W_\Phi = (\Phi(E_{ij}))_{n \times n}$, and we say that $W_\Phi$ is the witness induced by the positive map $\Phi$. Conversely, for an EW $W$, we denote $\Phi_W$ for the associated positive map so that $W = W_{\Phi_W}$.

For any entanglement witness $W$, let $\mathcal{D}_W = \{\rho : \rho \in S(H \otimes K), \text{Tr}(W\rho) < 0\}$, that is, $\mathcal{D}_W$ is the set of all entangled states that detected by $W$. For entanglement witnesses $W_1, W_2$, we say that $W_1$ is finer than $W_2$ if $\mathcal{D}_{W_2} \subset \mathcal{D}_{W_1}$, denoted by $W_2 \prec W_1$. While, an entanglement witness $W$ is optimal if there exists no other witness finer than it. Obviously, a state $\rho$ is entangled if and only if there is some optimal EW such that $\text{Tr}(W\rho) < 0$. In [10], Lewenstein, Kraus, Cirac and Horodecki proved that: (1) $W$ is an optimal entanglement witness if and only if $W - Q$ is no longer an entanglement witness for arbitrary positive operator $Q$; (2) $W$ is optimal if $\mathcal{P}_W = \{|e,f\rangle \in H \otimes K : \langle e,f|W|e,f\rangle = 0\}$ spans the whole $H \otimes K$ (in this case, we say that $W$ has spanning property). However, the criterion (2) is only a sufficient condition. There are known optimal witnesses that have no spanning property, for example, the entanglement witnesses induced by the Choi maps. Recently, Qi and Hou in [12] gave a necessary and sufficient condition for the optimality of entanglement witnesses in terms of positive linear maps.

**Theorem 1.1.** ([12, Theorem 2.2]) *Let $H$ and $K$ be finite dimensional complex Hilbert spaces. Let $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ be a positive linear map. Then $W_\Phi$ is an optimal entanglement witness if and only if, for any $C \in \mathcal{B}(H,K)$, the map $X \mapsto \Phi(X) - CXC^\dagger$ is not a positive map.*

This approach is practical for some situations, especially when the witnesses have no spanning property. Applying it, Qi and Hou [12] showed that the entanglement witnesses arising from some positive maps in [13] are indecomposable optimal witnesses.

If $\dim H = n$, by fixing an orthonormal basis, one may identify $\mathcal{B}(H)$ with $M_n(\mathbb{C})$, the $n \times n$ complex matrix algebra. In this note, we will consider the linear maps $\Phi_{t,\pi}$ defined by

$$
\Phi_{t,\pi}(X) = \begin{pmatrix}
(2 - t) x_{11} + tx_{\pi(1),\pi(1)} & -x_{12} & -x_{13} \\
-x_{21} & (2 - t) x_{22} + tx_{\pi(2),\pi(2)} & -x_{23} \\
-x_{31} & -x_{32} & (2 - t) x_{33} + tx_{\pi(3),\pi(3)}
\end{pmatrix},
$$

where $X = (x_{ij}) \in M_3(\mathbb{C})$, $0 \leq t \leq 3$ and $\pi$ is any permutation of $(1,2,3)$. We will show that the necessary and sufficient condition for the Hermitian matrix $W_{\Phi_{t,\pi}}$ to be an optimal entanglement witness is that $t = 1$ and $\pi$ is cyclic (Theorem 2.2).

## 2. Main result and proof

In this section, we give the main result and its proof.
Let \( \pi \) be a permutation of \( (1, 2, \ldots, n) \) and \( 0 \leq t \leq n \). For a subset \( F \) of \( \{1, 2, \ldots, n\} \), if \( \pi(F) = F \), we say \( F \) is an invariant subset of \( \pi \). Let \( F \) be an invariant subset of \( \pi \). If both \( G \subseteq F \) and \( G \) is invariant under \( \pi \) imply \( G = F \), we say \( F \) is a minimal invariant subset of \( \pi \). It is obvious that a minimal invariant subset is a loop of \( \pi \) and \( \{1, 2, \ldots, n\} = \bigcup_{s=1}^{r} F_s \), where \( \{F_s\}_{s=1}^{r} \) is the set of all minimal invariant subsets of \( \pi \). Denote by \( \#F_s \) the cardinal number of \( F_s \). Then \( \sum_{s=1}^{r} \#F_s = n \). We call \( \max\{\#F_s : s = 1, 2, \ldots, r\} \) the length of \( \pi \), denoted by \( l(\pi) \). In the case that \( l(\pi) = n \), we say that \( \pi \) is cyclic.

The following lemma was shown in [13].

**Lemma 2.1.** For any permutation \( \pi \) of \( \{1, 2, 3\} \), let \( \Phi_{t, \pi} : M_3(\mathbb{C}) \rightarrow M_3(\mathbb{C}) \) be a map defined by Eq. (1.1). Then \( \Phi_{t, \pi} \) is positive if and only if \( 0 \leq t \leq \frac{2}{l(\pi)} \).

The following is our main result in this note, which states that \( W_{\Phi_{t, \pi}} \) is an optimal entanglement witness if and only if \( t = 1 \) and \( \pi \) is cyclic.

**Theorem 2.2.** For any permutation \( \pi \) of \( \{1, 2, 3\} \), let \( \Phi_{t, \pi} : M_3(\mathbb{C}) \rightarrow M_3(\mathbb{C}) \) be the map defined by Eq. (1.1). Then \( W_{\Phi_{t, \pi}} \) is an optimal entanglement witness if and only if \( t = 1 \) and \( l(\pi) = 3 \).

Before stating the main results in this section, let us recall some notions and give two lemmas that we needed.

Let \( l, k \in \mathbb{N} \) (the set of all natural numbers), and let \( A_1, \cdots, A_k \), and \( C_1, \cdots, C_l \in \mathcal{B}(H, K) \). If, for each \( |\psi\rangle \in H \), there exists an \( l \times k \) complex matrix \( (\alpha_{ij}(|\psi\rangle)) \) (depending on \( |\psi\rangle \)) such that

\[
C_i|\psi\rangle = \sum_{j=1}^{k} \alpha_{ij}(|\psi\rangle)A_j|\psi\rangle, \quad i = 1, 2, \ldots, l,
\]

we say that \( (C_1, \cdots, C_l) \) is a locally linear combination of \( (A_1, \cdots, A_k) \), \( (\alpha_{ij}(|\psi\rangle)) \) is called a local coefficient matrix at \( |\psi\rangle \). Furthermore, if a local coefficient matrix \( (\alpha_{ij}(|\psi\rangle)) \) can be chosen for every \( |\psi\rangle \in H \) so that its operator norm \( \|\alpha_{ij}(|\psi\rangle)\| = \sup\{\|\alpha_{ij}(|\psi\rangle)x\| : x \in \mathbb{C}^k, \|x\| \leq 1\} \leq 1 \), we say that \( (C_1, \cdots, C_l) \) is a contractive locally linear combination of \( (A_1, \cdots, A_k) \); if there is a matrix \( (\alpha_{ij}) \) such that \( C_i = \sum_{j=1}^{k} \alpha_{ij}A_j \) for all \( i \), we say that \( (C_1, \cdots, C_l) \) is a linear combination of \( (A_1, \cdots, A_k) \) with coefficient matrix \( (\alpha_{ij}) \).

The following characterization of positive linear maps was obtained in [5], also, see [6].

**Lemma 2.3.** Let \( H \) and \( K \) be complex Hilbert spaces of any dimension, \( \Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K) \) be a linear map defined by \( \Phi(X) = \sum_{i=1}^{k} C_i X C_i^\dag - \sum_{j=1}^{l} D_j X D_j^\dag \) for all \( X \). Then \( \Phi \) is positive if and only if \( (D_1, \cdots, D_l) \) is a contractive locally linear combination of \( (C_1, \cdots, C_k) \).

Furthermore, \( \Phi \) is completely positive if and only if \( (D_1, \cdots, D_l) \) is a linear combination of \( (C_1, \cdots, C_k) \) with a contractive coefficient matrix, and in turn, if and only if there exist \( E_1, E_2, \ldots, E_r \) in \( \text{span}\{C_1, \cdots, C_k\} \) such that \( \Phi = \sum_{i=1}^{r} E_i (\cdot) E_i^\dag \).

**Lemma 2.4.** Let \( t \) be a fixed number with \( 0 < t < 1 \) and let \( x_1, x_2, x_3 \) be any positive numbers with \( x_1 x_2 x_3 = 1 \) and \( (x_1, x_2, x_3) \neq (1, 1, 1) \). Then we have

\[
\sum_{i=1}^{2} \frac{1}{(3-t)+tx_i} - \frac{1}{(3-t)+tx_1} \geq (1-t).
\]
Proof. Let \( f \) be the function in 3-variables defined by
\[
f(x_1, x_2, x_3) = \frac{1}{\sum_{i=1}^{3} \frac{1}{(3-t)+tx_i} - \frac{1}{(3-t+tx_1)(3-t+tx_2)}},
\]
where \( t \) is fixed with \( 0 < t < 1 \) and \( x_1, x_2, x_3 \) are any positive numbers with \( x_1x_2x_3 = 1 \) and \((x_1, x_2, x_3) \neq (1, 1, 1)\). Since the denominator of \( f(x_1, x_2, x_3) \) is not zero whenever \((x_1, x_2, x_3) \neq (1, 1, 1)\), a computation shows that
\[
f(x_1, x_2, x_3) \geq (1-t)
\]
\[
\Leftrightarrow 1 - \sum_{i=1}^{3} \frac{1}{(3-t)+tx_i} \geq (\sum_{i=1}^{3} \frac{1}{(3-t)+tx_i} - \frac{4}{(3-t+tx_1)(3-t+tx_2)}) (1-t)
\]
\[
\Leftrightarrow g(x_1, x_2, x_3) \geq 0,
\]
where
\[
g(x_1, x_2, x_3) = (2t^2 - 2t - 3) + (1-t)x_1 + (1-t)x_2 + (1-t^2)x_3 + (2t - t^2)x_1x_2 + tx_2x_3 + tx_1x_3.
\]
Thus, to complete the proof of the lemma, we only need to check that the minimum of the 3-variable function \( g \) is zero on the region \( x_i > 0 \) with \( x_1x_2x_3 = 1, i = 1, 2, 3 \).

To do this, let
\[
L(x_1, x_2, x_3, \lambda) = g(x_1, x_2, x_3) + \lambda(x_1x_2x_3 - 1).
\]
By the method of Lagrange multipliers, we have the system
\[
\begin{aligned}
L'_{x_1} &= (1-t) + (2t - t^2)x_2 + tx_3 + \lambda x_2x_3 = 0, \\
L'_{x_2} &= (1-t) + (2t - t^2)x_1 + tx_3 + \lambda x_1x_3 = 0, \\
L'_{x_3} &= (1-t^2) + tx_2 + tx_1 + \lambda x_1x_2 = 0, \\
L'_\lambda &= x_1x_2x_3 - 1 = 0.
\end{aligned}
\]
(2.1)
Solving this system, one obtains
\[
(x_2 - x_1)(2t - t^2 + \lambda x_3) = 0,
\]
which implies that
\[
either x_1 = x_2 \text{ or } 2t - t^2 + \lambda x_3 = 0.
\]
If \( 2t - t^2 + \lambda x_3 = 0 \), by Eq.(2.1), one gets \( x_3 = \frac{t-1}{t} < 0 \), a contradiction. Hence we must have \( x_1 = x_2 \). Thus, by Eq.(2.1) again, we have
\[
(2t - t^2)x_1^4 + (1-t)x_1^3 - tx_1 + (t^2 - 1) = 0,
\]
that is,
\[
(x_1 - 1)[(2t - t^2)x_1^3 + (1+t-t^2)x_1^2 + (1+t-t^2)x_1 + (1-t^2)] = 0. \tag{2.2}
\]
Note that \((2t - t^2)x_1^3 + (1+t-t^2)x_1^2 + (1+t-t^2)x_1 + (1-t^2) > 0 \) for all \( x_1 > 0 \) and \( 0 < t < 1 \). So Eq.(2.2) holds if and only if \( x_1 = 1 \), which forces \( x_2 = x_3 = 1 \). It follows that the function \( g(x_1, x_2, x_3) \) takes its extremum at the point \((1,1,1)\). Moreover, it is easy
to check that \((1,1,1)\) is the minimal point of \(g(x_1, x_2, x_3)\). Hence \(g(x_1, x_2, x_3) \geq g(1,1,1) = 0\) for all \(x_i > 0\) with \(x_1x_2x_3 = 1\), \(i = 1, 2, 3\).

Therefore, the inequality in Lemma 2.4 holds for all \(x_i > 0\), \(i = 1, 2, 3\), with \(x_1x_2x_3 = 1\) and \((x_1, x_2, x_3) \neq (1,1,1)\). The proof is finished. \(\square\)

Now we are in a position to give the proof of Theorem 2.1.

Proof of Theorem 2.1. By Lemma 2.1, \(\Phi_{t,\pi}\) is positive whenever \(0 \leq t \leq \frac{2}{l(\pi)}\). We will prove the theorem by considering several cases. Note that, \(\Phi_{0,\pi}\) is completely positive; so we may assume that \(t > 0\).

Case 1. \(l(\pi) = 1\).

If \(l = 1\), then \(\pi = id\) (the identical permutation). In this case, \(\Phi_{t,\pi}\) is a completely positive linear map for all \(0 < t \leq 3\) (see [13, Proposition 2.7]), and so \(W_{\Phi_{t,\pi}} \geq 0\), which is not an EW.

Case 2. \(l(\pi) = 2\).

If \(l = 2\), then \(\pi^2 = id\). Without loss of generality, assume that \(\pi(1) = 2\), \(\pi(2) = 1\) and \(\pi(3) = 3\). Since \(\Phi_{t,\pi}(E_{11}) = (2-t)E_{11} + tE_{22}\), \(\Phi_{t,\pi}(E_{22}) = (2-t)E_{22} + tE_{11}\), \(\Phi_{t,\pi}(E_{33}) = 2E_{33}\) and \(\Phi_{t,\pi}(E_{ij}) = -E_{ij}\) with \(1 \leq i \neq j \leq 3\), the Choi matrix of \(\Phi_{t,\pi}\) is

\[
W_{\Phi_{t,\pi}} = \sum_{i=1}^{3}(2-t)E_{ii} \otimes E_{ii} + tE_{22} \otimes E_{11} + tE_{11} \otimes E_{22} + tE_{33} \otimes E_{33} - \sum_{i \neq j} E_{ij} \otimes E_{ij}
\]

\[
= (2-t)E_{11} \otimes E_{11} + (2-t)E_{22} \otimes E_{22} + 2E_{33} \otimes E_{33}
\]

\[+ tE_{22} \otimes E_{11} + tE_{11} \otimes E_{22} - \sum_{i \neq j} E_{ij} \otimes E_{ij}.\]

If \(1 \leq t \leq \frac{3}{2}\), then let

\[
C_1 = (2-t)E_{11} \otimes E_{11} + (2-t)E_{22} \otimes E_{22} + 2E_{33} \otimes E_{33} - \sum_{i \neq j; \pi(i) \neq j} E_{ij} \otimes E_{ij}
\]

and

\[
C_2 = tE_{22} \otimes E_{11} + tE_{11} \otimes E_{22} - E_{12} \otimes E_{12} - E_{21} \otimes E_{21}.
\]

It is easily checked that \(C_1 \geq 0\). As \(C_2^T = tE_{22} \otimes E_{11} + tE_{11} \otimes E_{22} - E_{12} \otimes E_{21} - E_{21} \otimes E_{12} \geq 0\), we see that \(C_2\) is PPT. It is clear that \(C_1 \neq 0\) and \(W_{\Phi_{t,\pi}} = C_1 + C_2\). Hence \(W_{\Phi_{t,\pi}}\) is decomposable and not optimal.

If \(0 < t < 1\), then let

\[
D_1 = (2-t)E_{11} \otimes E_{11} + (2-t)E_{22} \otimes E_{22} + 2E_{33} \otimes E_{33}
\]

\[\quad - \sum_{i \neq j; \pi(i) \neq j} E_{ij} \otimes E_{ij} - (1-t)E_{12} \otimes E_{12} - (1-t)E_{21} \otimes E_{21}
\]

and

\[
D_2 = tE_{22} \otimes E_{11} + tE_{11} \otimes E_{22} - tE_{12} \otimes E_{12} - tE_{21} \otimes E_{21}.
\]

It is also clear that \(D_2\) is PPT and \(D_1 \geq 0\). We still have \(D_1 \neq 0\) and \(W_{\Phi_{t,\pi}} = D_1 + D_2\). Hence \(W_{\Phi_{t,\pi}}\) is decomposable and not optimal.

Case 3. \(l(\pi) = 3\), i.e., \(\pi\) is cyclic.

If \(l(\pi) = 3\) and \(t = 1\), then \(\pi\) is a cyclic permutation, and by [13, Theorem 3.2], \(W_{\Phi_{1,\pi}}\) is optimal.

In the sequel we always assume that \(l(\pi) = 3\). Our aim is to prove that \(W_{\Phi_{t,\pi}}\) is not optimal for any \(0 < t < 1\). Without loss of generality, let \(\pi(i) = (i+1) \mod 3\), \(i = 1, 2, 3\). By Theorem
1.1, to prove that $W_{\Phi_t,\pi}$ is not optimal, we have to prove that there exists a matrix $C \in M_3(\mathbb{C})$ such that the linear map $A \mapsto \Phi_{t,\pi}(A) - CAC^\dagger$ is positive. Indeed, we will show that, for any positive number $0 < c \leq \sqrt{1 - t}$, let $C_0 = \text{diag}(c, -c, 0)$; then the map $A \mapsto \Phi_{t,\pi}(A) - C_0AC^\dagger$ is positive.

To do this, let $C_0 = \text{diag}(c, -c, 0)$ with $c > 0$ and let $\Psi_{C_0}$ be the map defined by

$$
\Psi_{C_0}(A) = \Phi_{t,\pi}(A) - C_0AC^\dagger = (3 - t) \sum_{i=1}^{3} E_{ii}AE^\dagger_{ii} + \sum_{i=1}^{3} E_{i,i+1}AE^\dagger_{i,i+1} - A - C_0AC^\dagger
$$

for all $A \in M_3(\mathbb{C})$.

If $\Psi_{C_0}$ is positive, then by Lemma 2.3, for any unit $|x\rangle \in \mathbb{C}^3$, there exist scalars $\{\alpha_i(|x\rangle)\}_{i=1}^{3}$, $\{\beta_i(|x\rangle)\}_{i=1}^{3}$, $\{\delta_i(|x\rangle)\}_{i=1}^{3}$ and $\{\gamma_i(|x\rangle)\}_{i=1}^{3}$ such that

$$
|x\rangle = I|x\rangle = \sum_{i=1}^{3} \alpha_i(|x\rangle)(\sqrt{3 - t}E_{ii})|x\rangle + \sum_{i=1}^{3} \beta_i(|x\rangle)\sqrt{t}E_{i,i+1}|x\rangle,
$$

$$
C|x\rangle = \sum_{i=1}^{3} \delta_i(|x\rangle)(\sqrt{3 - t}E_{ii})|x\rangle + \sum_{i=1}^{3} \gamma_i(|x\rangle)\sqrt{t}E_{i,i+1}|x\rangle,
$$

and the matrix

$$
F_x = \begin{pmatrix}
\alpha_1(|x\rangle) & \alpha_2(|x\rangle) & \alpha_3(|x\rangle) \\
\beta_1(|x\rangle) & \beta_2(|x\rangle) & \beta_3(|x\rangle) \\
\delta_1(|x\rangle) & \delta_2(|x\rangle) & \delta_3(|x\rangle) \\
\gamma_1(|x\rangle) & \gamma_2(|x\rangle) & \gamma_3(|x\rangle)
\end{pmatrix}
$$

is contractive.

Note that $\|F_x\| \leq 1$ if and only if $\|F_xF_x^\dagger\| \leq 1$.

In the sequel, for any unit $|x\rangle \in \mathbb{C}^3$, we write $|x\rangle = (|x_1|e^{i\theta_1}, |x_2|e^{i\theta_2}, |x_3|e^{i\theta_3})^T$. Then $|x_1|^2 + |x_2|^2 + |x_3|^2 = 1$.

**Subcase 1.** $|x_1| = |x_2| = |x_3| = \frac{1}{\sqrt{3}}$.

In Eqs.(2.3)-(2.4), by taking

$$(\alpha_1, \alpha_2, \alpha_3) = \left(\frac{\sqrt{3 - t}}{3}, \frac{\sqrt{3 - t}}{3}, \frac{\sqrt{3 - t}}{3}\right)$$

and

$$(\delta_1, \delta_2, \delta_3) = \left(\frac{\sqrt{3 - tc}}{3}, \frac{\sqrt{3 - tc}}{3}, 0\right),$$

we get

$$(\beta_1, \beta_2, \beta_3) = \left(\frac{\sqrt{tx_1}}{3x_2}, \frac{\sqrt{tx_2}}{3x_3}, \frac{\sqrt{tx_3}}{3x_1}\right)$$

and

$$(\gamma_1, \gamma_2, \gamma_3) = \left(\frac{\sqrt{tx_1}}{3x_2}, \frac{\sqrt{tx_2}}{3x_3}, 0\right).$$

So $\sum_{i=1}^{3}(|\alpha_i|^2 + |\beta_i|^2) = 1$, $\sum_{i=1}^{3}(|\delta_i|^2 + |\gamma_i|^2) = \frac{6c^2}{3}$ and $\sum_{i=1}^{3}(\alpha_i\delta_i + \beta_i\gamma_i) = 0$. It follows that

$$F_xF_x^\dagger = \begin{pmatrix}
\sum_{i=1}^{3}(\alpha_i^2 + |\beta_i|^2) & \sum_{i=1}^{3}(\beta_i\gamma_i) \\
\sum_{i=1}^{3}(\alpha_i\delta_i + \beta_i\gamma_i) & \sum_{i=1}^{3}(\gamma_i^2 + |\beta_i|^2)
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{6c^2}{9} \end{pmatrix},$$

which implies that $\|F_xF_x^\dagger\| \leq 1 \Leftrightarrow c^2 \leq \frac{9}{6}$. Hence

$$c^2 \leq 1 - t \Rightarrow \|F_xF_x^\dagger\| \leq 1.$$

**Subcase 2.** $x_i \neq 0$ for all $i = 1, 2, 3$ and $(|x_1|, |x_2|, |x_3|) \neq (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. 

Let \( r_i = \left| \frac{x_i}{x_{i+1}} \right|^2 \) for \( i = 1, 2, 3 \). Then \( r_i > 0 \), \( i = 1, 2, 3 \), and \( r_1 r_2 r_3 = 1 \). Take

\[
(\alpha_1, \alpha_2, \alpha_3) = \left( \frac{\sqrt{3 - t} r_1}{t + (3 - t)r_1}, \frac{\sqrt{3 - t} r_2}{t + (3 - t)r_2}, \frac{\sqrt{3 - t} r_3}{t + (3 - t)r_3} \right)
\]

and

\[
(\delta_1, \delta_2, \delta_3) = \left( \frac{\sqrt{3 - t} r_1 c}{t + (3 - t)r_1}, -\frac{\sqrt{3 - t} r_2 c}{t + (3 - t)r_2}, 0 \right).
\]

By Eqs. (2.3)-(2.4), we get

\[
(\beta_1, \beta_2, \beta_3) = \left( \frac{\sqrt{r_1}}{t + (3 - t)r_1} e^{i(\theta_1 - \theta_2)}, \frac{\sqrt{r_2}}{t + (3 - t)r_2} e^{i(\theta_2 - \theta_3)}, \frac{\sqrt{r_3}}{t + (3 - t)r_3} e^{i(\theta_3 - \theta_1)} \right)
\]

and

\[
(\gamma_1, \gamma_2, \gamma_3) = \left( \frac{\sqrt{r_1 c}}{t + (3 - t)r_1}, -\frac{\sqrt{r_2 c}}{t + (3 - t)r_2}, 0 \right).
\]

So

\[
f(\alpha_1, \alpha_2, \alpha_3) = \sum_{i=1}^3 |\alpha_i|^2 + \sum_{i=1}^3 |\beta_i|^2 = \sum_{i=1}^3 \frac{r_i}{t + (3 - t)r_i},
\]

\[
f_{C_0}(\delta_1, \delta_2, \delta_3) = \sum_{i=1}^3 |\delta_i|^2 + \sum_{i=1}^3 |\gamma_i|^2 = \frac{r_1 c^2}{t + (3 - t)r_1} + \frac{r_2 c^2}{t + (3 - t)r_2}
\]

and

\[
\sum_{i=1}^3 (\alpha_i \delta_i + \beta_i \gamma_i) = \frac{r_1 c}{t + (3 - t)r_1} - \frac{r_2 c}{t + (3 - t)r_2}.
\]

It follows that

\[
F_x F_x^{\dagger} = \left( \begin{array}{ccc} \sum_{i=1}^3 (|\alpha_i|^2 + |\beta_i|^2) & \sum_{i=1}^3 (\alpha_i \delta_i + \beta_i \gamma_i) \\ \sum_{i=1}^3 (\alpha_i \delta_i + \beta_i \gamma_i) & \sum_{i=1}^3 (|\delta_i|^2 + |\gamma_i|^2) \end{array} \right) = \left( \begin{array}{ccc} \frac{r_1 c}{t + (3 - t)r_1} & \frac{r_1 c}{t + (3 - t)r_1} - \frac{r_2 c}{t + (3 - t)r_2} \\ \frac{r_1 c}{t + (3 - t)r_1} - \frac{r_2 c}{t + (3 - t)r_2} & \frac{r_1 c}{t + (3 - t)r_1} + \frac{r_2 c}{t + (3 - t)r_2} \end{array} \right).
\]

Note that \( \| F_x F_x^{\dagger} \| \leq 1 \) if and only if its maximal eigenvalue \( \lambda_{\text{max}} \leq 1 \). By a calculation, it is easily checked that

\[
\lambda_{\text{max}} \leq 1
\]

holds if and only if

\[
c^2 \leq \frac{1 - \sum_{i=1}^3 \frac{r_i}{t + (3 - t)r_i}}{\left( 1 - \sum_{i=1}^3 \frac{r_i}{t + (3 - t)r_i} \right)^2 + \left( \frac{r_1}{t + (3 - t)r_1} - \frac{r_2}{t + (3 - t)r_2} \right)^2}, \tag{2.5}
\]

where \( r_1, r_2, r_3 > 0 \) with \( r_1 r_2 r_3 = 1 \) and \( (r_1, r_2, r_3) \neq (1, 1, 1) \). Let

\[
g(r_1, r_2, r_3) = \frac{1 - \sum_{i=1}^3 \frac{r_i}{t + (3 - t)r_i}}{\left( 1 - \sum_{i=1}^3 \frac{r_i}{t + (3 - t)r_i} \right)^2 + \left( \frac{r_1}{t + (3 - t)r_1} - \frac{r_2}{t + (3 - t)r_2} \right)^2}.
\]

Replacing \( r_i \) by \( \frac{1}{r_i} \) in the above function \( g(r_1, r_2, r_3) \), we have

\[
g(r_1, r_2, r_3) = \frac{1 - \sum_{i=1}^3 \frac{1}{(3 - t + tr_i)}}{\sum_{i=1}^2 \frac{1}{(3 - t + tr_i)} - \frac{4}{(3 - t + tr_1)(3 - t + tr_2) - (3 - t + tr_1)(3 - t + tr_3) - (3 - t + tr_2)(3 - t + tr_3)}}.
\]
Now applying Lemma 2.4, we see that

\[ g(r_1, r_2, r_3) \geq 1 - t \]

holds for all positive numbers \( r_1, r_2, r_3 \) with \( r_1r_2r_3 = 1 \) and \( (r_1, r_2, r_3) \neq (1, 1, 1) \). This and Eq.(2.5) imply

\[ c^2 \leq (1 - t) \Rightarrow \lambda_{\text{max}} \leq 1 \Rightarrow \|F_xF_x^\dagger\| \leq 1. \]

\textbf{Subcase 3.} \( x_1 = 0 \) and \( x_i \neq 0 \) for \( i = 2, 3 \).

In this case, by Eqs.(2.3)-(2.4), one may choose \( \beta_1 = \delta_3 = \gamma_1 = 0, \alpha_3 = \frac{1}{\sqrt{3-t}}, \beta_2 = \frac{(1-\sqrt{3-t})x_2}{\sqrt{tx_3}} \) and \( \gamma_2 = \frac{(-c-\sqrt{3-t})x_2}{\sqrt{tx_3}} \). Write \( r_2 = \frac{|x_2|^2}{x_3} = \frac{|x_2|^2}{1-|x_2|^2} \). Then by taking

\[ (\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) = (0, \frac{\sqrt{3-t}r_2}{t+(3-t)r_2}, \frac{1}{\sqrt{3-t}}, 0, \frac{\sqrt{r_2}}{t+(3-t)r_2}, e^{i(\theta_2-\theta_3)}, 0) \]

and

\[ (\delta_1, \delta_2, \delta_3, \gamma_1, \gamma_2, \gamma_3) = (0, -\frac{\sqrt{r_2}c}{t+(3-t)r_2}e^{i(\theta_2-\theta_3)}, 0), \]

which meet Eqs.(2.3)-(2.4), we get

\[ f(\alpha_1, \alpha_2, \alpha_3) = 3 \sum_{i=1}^{3} |\alpha_i|^2 + 3 \sum_{i=1}^{3} |\beta_i|^2 = \frac{r_2}{t+(3-t)r_2} + \frac{1}{3-t}, \]

\[ f_{C_0}(\delta_1, \delta_2, \delta_3) = 3 \sum_{i=1}^{3} |\delta_i|^2 + 3 \sum_{i=1}^{3} |\gamma_i|^2 = \frac{r_2c^2}{t+(3-t)r_2} \]

and

\[ \sum_{i=1}^{3} (\alpha_i \delta_i + \beta_i \gamma_i) = -\frac{r_2c}{t+(3-t)r_2}. \]

Hence

\[ F_xF_x^\dagger = \left( \begin{array}{ccc} \frac{r_2}{t+(3-t)r_2} + \frac{1}{3-t} & -\frac{r_2c}{t+(3-t)r_2} \\ -\frac{r_2c}{t+(3-t)r_2} & \frac{r_2c^2}{t+(3-t)r_2} \end{array} \right). \]

Still, by a calculation, one can easily obtain

\[ \|F_xF_x^\dagger\| \leq 1 \Leftrightarrow c^2 \leq \frac{1 - \frac{r_2}{t+(3-t)r_2} - \frac{1}{3-t}}{\frac{r_2}{t+(3-t)r_2} - \frac{r_2c}{(t+(3-t)r_2)(3-t)}} \]

(2.6)

where \( r_2 > 0 \) is any positive number. Let

\[ g(r_2) = \frac{1 - \frac{r_2}{t+(3-t)r_2} - \frac{1}{3-t}}{\frac{r_2}{t+(3-t)r_2} - \frac{r_2c}{(t+(3-t)r_2)(3-t)}}. \]

A direct calculation yields that,

\[ g(r_2) \geq 1 - t \Leftrightarrow r_2 \geq \frac{t(2-t)}{t-1}. \]

(2.7)

Note that \( r_2 > 0 \) and \( \frac{t(2-t)}{t-1} < 0 \) as \( 0 < t < 1 \). So we always have \( r_2 \geq \frac{t(2-t)}{t-1} \). Thus by Eq.(2.7), we have proved that \( g(r_2) \geq 1 - t \) holds for all positive numbers \( r_2 > 0 \). It follows from Eq.(2.6) that

\[ c^2 \leq 1 - t \Rightarrow \|F_xF_x^\dagger\| \leq 1. \]

\textbf{Subcase 4.} \( x_2 = 0 \) and \( x_i \neq 0 \) for \( i = 1, 3 \).
Let $r_3 = \frac{|x_3|^2}{t^2 + (3-t)r_3^2}$. By Eqs.(2.3)-(2.4), we can choose $\beta_2 = \gamma_2 = 0$, $\alpha_1 = \frac{1}{\sqrt{3-t}}$, $\alpha_3 = \frac{\sqrt{t-r_3}}{t + (3-t)r_3^2}$, $\beta_3 = \frac{(1 - \sqrt{3-t}r_3)x_1}{\sqrt{t^2}}$ and $\gamma_3 = \frac{\sqrt{t-r_3}e^{x_1}}{\sqrt{t^2}}$. Now take

$$(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) = \left( \frac{1}{\sqrt{3-t}}, 0, \frac{\sqrt{3-t}r_3}{r_1}, 0, 0, \frac{\sqrt{t}r_1}{t + (3-t)r_1} e^{i(\theta_3 - \theta_1)} \right)$$

and

$$(\delta_1, \delta_2, \delta_3, \gamma_1, \gamma_2, \gamma_3) = \left( \frac{c}{\sqrt{3-t}}, 0, 0, 0, 0, 0 \right).$$

It follows that

$$f(\alpha_1, \alpha_2, \alpha_3) = \sum_{i=1}^{3} |\alpha_i|^2 + \sum_{i=1}^{3} |\beta_i|^2 = \frac{r_3}{t + (3-t)r_3} + \frac{1}{3-t},$$

$$f_{C_0}(\delta_1, \delta_2, \delta_3) = \sum_{i=1}^{3} |\delta_i|^2 + \sum_{i=1}^{3} |\gamma_i|^2 = \frac{c^2}{3-t}$$

and

$$\sum_{i=1}^{3} (\alpha_i \bar{\delta}_i + \beta_i \bar{\gamma}_i) = \frac{c}{3-t}.$$

Hence

$$F_x F_x^\dagger = \left( \frac{r_3}{t + (3-t)r_3} + \frac{1}{3-t}, \frac{c}{3-t}, \frac{c^2}{3-t} \right).$$

Still, one can easily checked that

$$\|F_x F_x^\dagger\| \leq 1 \iff c^2 \leq \frac{1 - \frac{r_3}{t + (3-t)r_3} - \frac{1}{3-t}}{\frac{r_3}{t + (3-t)r_3} - \frac{1}{3-t}}$$

and

$$\frac{1}{3-t} - \frac{r_3}{t + (3-t)r_3} \geq 1 - t \iff t \geq (t-1)r_3,$$

where $r_3 > 0$ is any positive number and $0 < t < 1$. Note that $t \geq (t-1)r_3$ as $0 < t < 1$ and $r_3 > 0$. Thus we see that we still have

$$c^2 \leq 1 - t \Rightarrow \|F_x F_x^\dagger\| \leq 1.$$

Subcase 5. $x_3 = 0$ and $x_i \neq 0$ for $i = 1, 2$.

Let $r_1 = \frac{|x_1|^2}{x_2^2 + |x_2|^2}$. By Eqs.(2.3)-(2.4), one may choose $\beta_3 = \gamma_3 = 0$, $\alpha_2 = \frac{1}{\sqrt{3-t}}$, $\beta_1 = \frac{(1 - \sqrt{3-t}r_3)x_3}{\sqrt{t^2}}$, $\delta_2 = \frac{-c}{\sqrt{3-t}}$ and $\gamma_1 = \frac{\sqrt{3-t}o_3}{\sqrt{t^2}}$. Then for choice

$$(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) = \left( \frac{\sqrt{3-t}r_3}{t + (3-t)r_3}, \frac{1}{\sqrt{3-t}}, 0, \frac{\sqrt{t}r_1}{t + (3-t)r_1} e^{i(\theta_3 - \theta_2)}, 0, 0 \right)$$

and

$$(\delta_1, \delta_2, \delta_3, \gamma_1, \gamma_2, \gamma_3) = \left( \frac{\sqrt{3-t}r_3+c}{t + (3-t)r_1}, \frac{-c}{\sqrt{3-t}}, 0, \frac{\sqrt{t}r_1}{t + (3-t)r_1} e^{i(\theta_3 - \theta_2)}, 0, 0 \right),$$

we get

$$f(\alpha_1, \alpha_2, \alpha_3) = \sum_{i=1}^{3} |\alpha_i|^2 + \sum_{i=1}^{3} |\beta_i|^2 = \frac{r_1}{t + (3-t)r_1} + \frac{1}{3-t}.$$
We obtain
\[ f_{C_3}(\delta_1, \delta_2, \delta_3) = \sum_{i=1}^{3} |\delta_i|^2 + \sum_{i=1}^{3} |\gamma_i|^2 = \frac{r_1c^2}{t + (3-t)r_1} + \frac{c^2}{3-t} \]
and
\[ \sum_{i=1}^{3} (\alpha_i \overline{\delta}_i + \beta_i \overline{\gamma}_i) = \frac{r_1c}{t + (3-t)r_1} - \frac{c}{3-t}. \]
So
\[ F_x F_x^\dagger = \left( \frac{r_1}{t + (3-t)r_1} + \frac{1}{3-t} \right) \left( \frac{r_1c}{t + (3-t)r_1} - \frac{c}{3-t} \right). \]
It is easily checked that
\[ \|F_x F_x^\dagger\| \leq 1 \iff c^2 \leq \frac{1 - \frac{r_1}{t + (3-t)r_1} - \frac{1}{3-t}}{(1 - \frac{r_1}{t + (3-t)r_1}) (1 + \frac{1}{3-t}) + (\frac{r_1}{t + (3-t)r_1} - \frac{1}{3-t})^2}. \] (2.8)
where \( r_1 > 0 \) is any positive number and \( 0 < t < 1 \). Let
\[ g(r_1) = \frac{1 - \frac{r_1}{t + (3-t)r_1} - \frac{1}{3-t}}{(1 - \frac{r_1}{t + (3-t)r_1}) (1 + \frac{1}{3-t}) + (\frac{r_1}{t + (3-t)r_1} - \frac{1}{3-t})^2}. \]
By a direct calculation, one gets
\[ g(r_1) \geq 1 - t \iff 1 - t^2 + tr_1 \geq 0. \]
Hence we always have \( g(r_1) \geq 1 - t \). This and Eq.(2.8) yield again
\[ c^2 \leq 1 - t \Rightarrow \|F_x F_x^\dagger\| \leq 1. \]

**Subcase 6.** \( x_1 = x_2 = 0 \) and \( x_3 \neq 0 \).
By Eqs.(2.3)-(2.4), we have \( \beta_2 = \delta_3 = \gamma_2 = 0 \) and \( \alpha_3 = \frac{1}{\sqrt{3-t}} \). Then take
\[ (\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) = (0, 0, 0, \frac{1}{\sqrt{3-t}}, 0, 0, 0) \] and \( (\delta_1, \delta_2, \delta_3, \gamma_1, \gamma_2, \gamma_3) = (0, 0, 0, 0, 0, 0) \).
We obtain \( F_x F_x^\dagger = \left( \frac{1}{3-t} \right) \), which is contractive.

**Subcase 7.** \( x_1 = x_3 = 0 \) and \( x_2 \neq 0 \).
By Eqs.(2.3)-(2.4), we have \( \beta_1 = \gamma_1 = 0, \alpha_2 = \frac{1}{\sqrt{3-t}} \) and \( \gamma_2 = \frac{c}{\sqrt{3-t}} \). Then by taking
\[ (\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) = (0, \frac{1}{\sqrt{3-t}}, 0, 0, 0, 0) \] and \( (\delta_1, \delta_2, \delta_3, \gamma_1, \gamma_2, \gamma_3) = (0, 0, 0, 0, \frac{c}{\sqrt{3-t}}, 0) \),
we obtain \( F_x F_x^\dagger = \left( \frac{1}{3-t} \right) \left( \frac{c}{\sqrt{3-t}} \right) \). It is easy to check that
\[ \|F_x F_x^\dagger\| \leq 1 \iff c^2 \leq (2 - t). \]
So \( c^2 \leq 1 - t \) implies \( \|F_x F_x^\dagger\| \leq 1. \)

**Subcase 8.** \( x_2 = x_3 = 0 \) and \( x_1 \neq 0 \).
The case is the same as Case 7.
Thus, by combining Subcases 1-8 and applying Lemma 2.3, we have proved that, for any matrix $C_0 = \text{diag}(c, -c, 0)$ with $0 < c^2 \leq 1 - t$, the map $A \mapsto \Phi_{t, \pi}(A) - C_0 A C_0^\dagger$ is positive. Then, by Theorem 1.1, we see that $W_{\Phi_{t, \pi}}$ is not optimal whenever $l(\pi) = 3$ and $0 < t < 1$.

The proof is finished. □

3. Conclusions

Every entangled state can be detected by an optimal entanglement witness. So, it is important to construct as many as possible optimal EWs. A natural way of constructing optimal EWs is through NCP positive maps by Choi-Jamiołkowski isomorphism $\Phi \leftrightarrow W_{\Phi}$. In [14], for $0 \leq t \leq n$, a class of new $D$-type positive maps $\Phi_{t, \pi} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ induced by an arbitrary permutation $\pi$ of $(1, 2, \ldots, n)$ was constructed, where $\Phi_{t, \pi}$ is defined by

$$\Phi_{t, \pi}(A) = (n - t) \sum_{i=1}^{n} E_{ii} A E_{ii} + t \sum_{i=1}^{n} E_{i, \pi(i)} A E_{i, \pi(i)}^\dagger - A. \quad (3.1)$$

It was shown in [14] that $\Phi_{t, \pi}$ in NCP positive if and only if $0 < t \leq \frac{n}{l(\pi)}$. In [12], by using Theorem 1.1, we proved that $W_{\Phi_{1, \pi}}$ is optimal if $l(\pi) = n$ and $\pi^2 \neq \text{id}$. But it is not clear that whether or not there exist other optimal $W_{\Phi_{t, \pi}}$s. We guess there are no.

**Conjecture.** For $n \geq 3$, $W_{\Phi_{t, \pi}}$ is an optimal entanglement witness if and only if $t = 1$, $l(\pi) = n$ and $\pi^2 \neq \text{id}$.

The case $n = 2$ is simple. It is easily checked that $W_{\Phi_{t, \pi}}$ is optimal if and only if $t = 1$ and $l(\pi) = 2$. Note that, $\pi^2 = \text{id}$ if $n = 2$.

The present note gives an affirmative answer to the above conjecture for the case $n = 3$.

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