On a topology property for moduli space of Kapustin-Witten equations

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Abstract

In this article, we study the Kapustin-Witten equations on a closed, simply-connected, four-manifold. We using a compactness theorem due to Taubes [23] to prove that if \((A, \phi)\) is a solution of Kapustin-Witten equations and the connection \(A\) is closed to a generic ASD connection \(A_\infty\), then \((A, \phi)\) must be a trivial solution. We also prove that the moduli space of the solutions of Kapustin-Witten equations is non-connected if the connections on the compactification of moduli space of ASD connections are all generic. As one application, we extend the ideas of Kapustin-Witten equations to other equations on gauge theory—Hitchin-Simpson equations and Vafa-Witten on compact Kähler surface with a Kähler metric \(g\).

1 Introduction

Let \(X\) be an oriented 4-manifold with a given Riemannian metric \(g\). On a 4-manifold \(X\) the Hodge star operator \(*\) takes 2-forms to 2-forms and we have \(*^2 = Id_{\Omega^2}\). The self-dual and anti-self-dual forms, we denoted \(\Omega^+\) and \(\Omega^-\) are defined to be the \(\pm\) eigenspace of \(*\): \(\Omega^2 T^*X = \Omega^+ \oplus \Omega^-\). Let \(P\) be a principal bundle over \(X\) with structure group \(G\). Supposing that \(A\) is the connection on \(P\), then we denote by \(F_A\) its curvature 2-form, which is a 2-form on \(X\) with values in the bundle associated to \(P\) with fiber the Lie algebra of \(G\) denoted by \(g\). We define by \(d_A\) the exterior covariant derivative on section of \(\Lambda^\bullet T^*X \otimes (P \times_G g)\) with respect to the connection \(A\).

The Kapustin-Witten equations are defined on a Riemannian 4-manifold given a principle bundle \(P\). For most present considerations, \(G\) can be taken to be \(SU(2)\) or \(SO(3)\). The equations require a pair \((A, \phi) \in \mathcal{A}_P \times \Omega^1(X, \mathfrak{g}_P)\) satisfies

\[
(F_A - \phi \wedge \phi)^+ = 0,
\]

\[
(d_A \phi)^- = d^*_A \phi = 0.
\]

(1.1)
These equations were introduced by Kapustin-Witten [14] at first time. The motivation is from the viewpoint of $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions to study the geometric Langlands program [8, 10, 14] and [27, 28, 29, 30]. One also can see Gagliardo–Uhlenbeck’s article[7].

In mathematics, the analytic properties of solutions of Kapustin-Witten equations were discussed by Taubes [23, 24, 25] and Tanaka [20]. In [23], Taubes studied the Uhlenbeck style compactness problem for $SL(2, \mathbb{C})$ connections, including solutions to the above equations, on four-manifolds, see [24, 25]. In [20], Tanaka observed that equations on a compact Kähler surface are the same as Hitchin-Simpson’s equations [11, 17] and proved that the singular set introduced by Taubes for the case of Simpson’s equations has a structure of a holomorphic subvariety. In [12], the author proved that there exist a lower bounded for the $L^2$-norm of extra fields under some mild conditions on $X, P, g, G$.

One always using continuous method to construct the solutions of same PDE. For example, Freed-Uhlenbeck [5] used this way to constructed the ASD connections over some four-manifolds. The ASD connections was constructed by Taubes [21] at first time. In this article, if we suppose there is a anti-self-dual connection $A_\infty$ on $P$. We suppose the pair $(A_\infty + a, \phi)$ also satisfies the Kapustin-Witten equations, i.e.

\[
\begin{align*}
&d_{A_\infty}^+ a + (a \wedge a)^+ - (\phi \wedge \phi)^+ = 0, \\
&(d_{A_\infty} \phi + [a, \phi])^- = 0
\end{align*}
\]

But unfortunately, we will show there is non-existence trivial solutions of Kapustin-Witten equations on a four-manifold when the connections on a neighbourhood of a generic anti-self-dual connection.

**Theorem 1.1.** Let $X$ be a closed, oriented, simply-connected, four-dimensional manifold with a smooth Riemannain metric $g$, $P \to X$ be a principal $SU(2)$ or $SO(3)$-bundle with $p_1(P)$ negative. Suppose there exist a generic ASD connection $A_\infty$ on $P$, then there is a positive constant $\delta = \delta(g, P, A_\infty)$ with following significance. If $(A, \phi)$ is a solution of Kapustin-Witten equations over $X$ then one of following must hold:

1. $F_A^+ = 0$ and $\phi = 0$;
2. the pair $(A, \phi)$ satisfies

\[
\text{dist}(A, A_\infty) \geq \delta,
\]

where $\text{dist}(\cdot, \cdot)$ defined by

\[
\text{dist}(A, B) := \inf_{g \in \mathcal{G}(P)} \|g^*(A) - B\|_{L^2(X)}, \text{ for } A, B \in \mathcal{A}_P.
\]

**Remark 1.2.** There is another result that every solution $(A, \phi)$ of the Kapustin-Witten equation, which is close to a solution $(A_\infty, 0)$ with $A_\infty$ a generic connection, then $(A, \phi)$ must be the form $(B, 0)$ with $B$ ASD. It assume that the extra fields with a small $L^2$-norm,
it’s a small variation of the result in [12]. We should point out that the result is different to Theorem 1.1. On theorem 1.1, we don’t make any assumptions on extra fields. The following case may happen: if \((A, \phi)\) is a smooth solution of Kapustin-Witten equations, when \(A\) tends to \(A_\infty\), the \(L^2\)-norm of extra fields \(\|\phi\|_{L^2(X)}\) may tend to infinite. But thanks for the compactness theorem of Taubes’, we observed that if \(\{(A_i, \phi_i)\}\) is a sequence of smooth solutions of Kapustin-Witten equations, then there exist a subsequence \(\Xi \subset \mathbb{N}\) such that the sequence \(\{\|\phi_i\|_{L^2(X)}\}\) is a bounded subsequence under some conditions, see Lemma 3.9 Using this useful observe, we could prove Theorem 1.1.

We denote the moduli space of solutions of Kapustin-Witten by

\[ M_{KW}(P, g) := \{(A, \phi) \mid (A_\infty, \phi) \in \mathcal{G}_P \}. \]

The moduli space \(M_{ASD}\) of all ASD connections can be embedded into \(M_{KW}\) via \(A_\infty \mapsto (A_\infty, 0)\), \(A_\infty\) is an ASD connection on \(P\). We also denote \(\tilde{M}_{ASD}\) by the compactification of moduli space of ASD connection.

Following the idea of Donaldson on [2] Section 4.2.1, we write \(([A], [\phi])\) for the equivalence class of a pair \((A, \phi)\), a point in \(M_{KW}\). We set,

\[ \|(A_1, \phi_1) - (A_2, \phi_2)\|^2 = \|A_1 - A_2\|^2_{L^2(X)} + \|\phi_1 - \phi_2\|^2_{L^2(X)}, \]

is preserved by the action of \(\mathcal{G}_P\), so descends to define a distance function on \(M_{KW}\):

\[ \text{dist} \left( ([A_1], [\phi_1]) - ([A_2], [\phi_2]) \right) := \inf_{g \in \mathcal{G}} \|(A_1, \phi_1) - g^*(A_2, \phi_2)\|. \]

At first, we observe that if the pair \((A, \phi)\) is the solution of Decoupled Kapustin-Witten equations over a compact simply-connected four-manifold, the extra field \(\phi\) is vanish when the connection \(A\) is irreducible, see Theorem 3.4. Hence, we can denote

\[ \text{dist}(\langle A, \phi \rangle, M_{ASD}) := \inf_{g \in \mathcal{G}, A_\infty \in M_{ASD}} \|g^*(A, \phi) - (A_\infty, 0)\| \]

\[ = \inf_{g \in \mathcal{G}, A_\infty \in M_{ASD}} (\|g^*(A) - A_\infty\|^2_{L^2(X)} + \|\phi\|^2_{L^2(X)})^{\frac{1}{2}}, \]

by the distance between \(M_{ASD}\) and \(M_{KW} \setminus M_{ASD}\).

**Theorem 1.3.** Let \(X\) be a closed, oriented, simply-connected, four-dimensional manifold with a smooth Riemannain metric \(g\), \(P \to X\) be a principal \(SU(2)\) or \(SO(3)\)-bundle with \(p_1(P)\) negative. Suppose the connections in \(\tilde{M}_{ASD}(P, g)\) are all generic, then there is a positive constant \(\delta = \delta(g, P)\) with following significance. If \((A, \phi)\) is a solution of Kapustin-Witten equations, then one of following must hold:

1. \(F_A^+ = 0\) and \(\phi = 0\);
2. the pair \((A, \phi)\) satisfies

\[ 2\|\phi\|^2_{L^2} \geq \|F_A^+\|_{L^2(X)} \geq \delta. \]
In particular, \((A, \phi)\) satisfies
\[
\text{dist}(A, M_{\text{ASD}}) := \inf_{g \in G, A_\infty \in M_{\text{ASD}}} \|g^*(A) - A_\infty\|_{L^2(X)} \geq \tilde{\delta},
\]
for a positive constant \(\tilde{\delta} = \tilde{\delta}(g, P)\), unless \(A\) is anti-self-dual with respect to \(g\).

Next we suppose \(X\) is a compact Kähler surface, \(E\) is a principal \(G\)-bundle over \(X\), the Kapustin-Witten equations are the same as Hitchin-Simpson equations. If \(A\) is an \(SU(N)\)-ASD connection on \(E\), then
\[
\ker d^+_A d^*_A |_{\Omega^{2,+(X,adE)}} \cong \ker (\bar{\partial}_A \bar{\partial}_A^* + \bar{\partial}_A^* \bar{\partial}_A) |_{\Omega^0,2(X,adE)} \oplus \ker d_A |_{\Omega^0(X,adE)},
\]
see [13] Proposition 2.3 or [6] Chapter IV. It’s difficult to addtion certain mild conditions to ensure \(\ker (\bar{\partial}_A \bar{\partial}_A^* + \bar{\partial}_A^* \bar{\partial}_A) |_{\Omega^{0,2}(X,adE)}\) and \(\ker d_A |_{\Omega^0(X,adE)} = 0\) vanish at some time. But one can see that \(\ker d_A |_{\Omega^0(X,adE)} = 0\) is equivalent to the connection \(A\) is irreducible. It is convenient to introduce the \(c\text{-generic}\) metric which ensures the connections on the compactification of moduli space of ASD connections on \(E\), \(\bar{M}(E, g)\), are irreducible, one also can see [6] Chapter IV.

**Definition 1.4.** Let \(X\) be a compact, connected, Kähler surface, \(c\) is a positive integer. We say that a Kähler metric \(g\) on \(X\) is \(c\text{-generic}\) if for every \(SU(2)\)-bundle \(E\) over \(X\) with \(c_2(E) \leq c\), there are no reducible ASD connections on \(E\).

If we suppose the Kähler metric \(g\) is \(c\text{-generic}\), we could deformation a connection \(A \in A^{1,1}\) which obeying \(\|\Lambda_\omega F_A\|_{L^2(X)} \leq \varepsilon\) for a suitable sufficiently small positive constant to an other connection \(A_\infty\) satisfies \(\Lambda_\omega F_{A_\infty} = 0\), see Theorem 4.8. Even though, the connection \(A_\infty\) may be not an ASD connection, the \((0, 2)\)-part \(F_{A_\infty}^{0,2}\) of the curvature \(F_{A_\infty}\) would estimated by \(\Lambda_\omega F_A\). Hence we would extend the ideas of Kapustin-Witten equations to the case of Hitchin-Simpson equations.

**Theorem 1.5.** Let \(X\) be a compact, Kähler surface with a Kähler metric \(g\), \(E\) be a principal \(SU(2)\)-bundle with \(c_2(E) = c\) positive. Suppose \(g\) is a \(c\text{-generic}\) metric in the sense of Definition 1.4, there is a positive constant \(C = C(g, E)\) with following significance. If the Higgs pair \((A, \theta) \in A^{1,1}_E \times \Omega^{1,0}(X, adE)\) is a solution of Hitchin-Simpson equations, then one of following must hold:

1. \(\Lambda_\omega F_A = 0\), or
2. the Higgs field \(\theta\) satisfies
\[
\|\theta\|_{L^2(X)} \geq C.
\]

Furthermore, if \(X\) is simply-connected, the curvature \(F_A\) obeying
\[
\|\Lambda_\omega F_A\|_{L^2(X)} \geq \tilde{C}
\]
for a positive constant \(\tilde{C} = \tilde{C}(g, P)\), unless \(\Lambda_\omega F_A = 0\).
We denote the moduli space of solutions of Hitchin-Simpson equations by
\[ M_{HS} := \{(A, \phi) \in \mathcal{A}_E^{1,1} \times \Omega^{1,0}(X, \text{ad}E) \mid \Lambda_\omega(F_A + [\theta, \theta^\ast]) = 0, \bar{\partial}_A \theta = \theta \wedge \theta = 0\}/G_E. \]

In [11], Hitchin proved that the moduli space of stable Higgs bundle is connected and simply connected, see [11] Theorem 7.6, if the bundle \( E \) is a rank-2 bundle of odd degree over a Riemannian surface of genus \( g > 1 \). We would show the topology property of moduli space of stable Higgs bundle on a simply-connected Kähler surface is differential to the case of Riemannian surface, see Corollary 4.12.

**Remark 1.6.** If the principal \( G \)-bundle \( P \to X \) with \( p_1(P) = 0 \), the solutions \((A, \phi)\) of the Kapustin-Witten equations are flat \( G_\mathbb{C} \)-connections with moment map condition:
\[
F_A - \phi \wedge \phi = 0,
\]
\[
d_A \phi = d_A^\ast \phi = 0.
\]

The [2] Proposition 2.2.3 shows that the gauge-equivalence classes of flat \( G \)-connections over a connected manifold, \( X \), are in one-to-one correspondence with the conjugacy classes of representations \( \rho : \pi_1(X) \to G \). If we suppose \( X \) is a simply-connected manifold, i.e. \( \pi_1(X) \) is trivial, then the representations \( \rho \) must be a trivial representation. It is no sense to consider Kapustin-Witten equations on a principal \( G \)-bundle with \( p_1(P) = 0 \) over a simply-connected four-manifolds.

The organization of this paper is as follows. In section 2, we first recall gauge theory in 4-dimensional manifolds. In section 3, we recall a vanish theorem for the extra fields. Using an optimal inequality proved by Donaldson, we prove that the extra fields of non-trivial solutions of Kapustin-Witten equations have a positive lower bounded. Thanks to Taubes’ compactness theorem [23], we observe that if \((A_i, \phi_i)\) is a sequence solutions of Kapustin-Witten equations and the connections \( \{A_i\}_{i \in \mathbb{N}} \) obeying \( \text{dist}(A_i, A_\infty) \to 0, A_\infty \) is an ASD connection, then the sequence \( \{\|\phi_i\|_{L^2(X)}\} \) has a bounded subsequence. At last, we obtain our main result: there is non-existence non-trivial solution on a neighbourhood of a general ASD connection. In section 4, we extends the results to the global situation, we would prove that the \( L^2 \)-norm of self-dual part of curvature \( \|F_A^+\|_{L^2(X)} \) has a uniform positive lower bounded if the connections on the compactification of moduli space of ASD connections, \( \tilde{M}_{ASD} \), are generic. In particular, we could proved that the moduli space of the solutions of Kapustin-Witten equations is non-connected. We also given some 4-manifolds \( X \) with Riemannian metric \( g \) and principle \( SO(3) \)-bundles \( P \to X \) ensure the connections on \( \tilde{M}_{ASD} \) are generic. At last section, we extend the ideas of Kapustin-Witten equations to other equations on gauge theory– Hitchin-Simpson equations and Vafa-Witten on compact Kähler surface with a Kähler metric \( g \).
2 A neighbourhood of an ASD connection

2.1 Yang-Mills theory on 4-manifolds

Let $X$ be an oriented Riemannian 4-manifold, $P \to X$ be a principal $G$-bundle with $G$ being a compact Lie group. The Hodge star operator gives an endomorphism of $\Omega^2$ with property $\ast^2 = Id$. We denote by $\Omega^{2+}$ and $\Omega^{2-}$ the eigenvalues of $+1$ and $-1$. A 2-form in $\Omega^{2+}$ (or in $\Omega^{2-}$) is called self-dual (or anti-self-dual). Decomposing the curvature $F_A$ of a connection $A$ according to the decomposition $\Omega^2 = \Omega^{2+} \oplus \Omega^{2-}$ of the 2-forms into self-dual and anti-self-dual parts. An ASD connection $A$ on $P$ naturally induces the Yang-Mills complex

$$
\Omega^0(X, g_P) \xrightarrow{d_A} \Omega^1(X, g_P) \xrightarrow{d_A^*} \Omega^{2+}(X, g_P).
$$

The $i$-th cohomology group $H^i_A$ of this complex if finite dimensional and the index $d = h^0 - h^1 + h^2$ ($h^i = \dim H^i_A$) is given by $c(G)\kappa(P) - \dim G(1 - b_1 + b^+)$, $H^0_A$ is the Lie algebra of the stabilizer $\Gamma_A$, the group of gauge transformation of $P$ fixing by $A$. We called a connection generic when $H^0_A = 0$ and $H^2_A = 0$. We denote $M_{\text{gen}}$ the subset of

$$
M_{\text{ASD}} := \{ A \in \mathcal{A}_P : F_A + \ast F_A = 0 \}/G_P
$$

of generic ASD connections on $P$. $M_{\text{gen}}$ becomes a smooth manifold whose tangent space is $H^1_A$, $M_{\text{gen}}$ consists exactly of all singular points and we have two types according to either case (1) in which $A$ is irreducible $H^0_A = 0$ but $H^2_A \neq 0$ or case (2) in which $A$ is reducible $H^0_A \neq 0$. So if the anti-self-dual connections $[A] \in M_{\text{ASD}}$ are all generic, the moduli space $M_{\text{ASD}}$ is a smooth manifold. Furthermore, We called an ASD connection is regular when $H^2_A = 0$.

2.2 An inequality for the connections near an ASD connection

Let $A_\infty$ be a fixing ASD connection on $P$, any connection $A$ can be written uniquely as

$$
A = A_\infty + a \text{ with } a \in \Omega^1(g_P).
$$

In [1], Donaldson proved that the connection $A$ can be written as

$$
A = \tilde{A}_\infty + d_{A_\infty}^{+*} u,
$$

where $\tilde{A}_\infty$ is also an ASD connection and $u \in \Omega^{2+}(X, g_P)$, i.e., the connection $A$ satisfies

$$
-d_{A_\infty}^{+*} d_{A_\infty}^{+*} u + (d_{A_\infty}^{+*} u \wedge d_{A_\infty}^{+*} u)^+ - ([a \wedge d_{A_\infty}^{+*} u]^+) + F_A^+ = 0 \tag{2.1}
$$

when the connection $A_\infty$ is regular and $a$ is small enough in $L^2$-norm. The operator $d_{A_\infty}^{+*} d_{A_\infty}^{+*}$ is an elliptic self-adjoint operator on the space of $L^2$ sections of $\Omega^{2+}TX \otimes g_P$. It is a standard result that the spectrum of $d_{A_\infty}^{+*} d_{A_\infty}^{+*}$ is discrete, and the lowest eigenvalue is nonnegative.
Definition 2.1. For $A_\infty \in M_{ASD}$, define

$$\mu(A_\infty) := \inf_{v \in \Omega^{2,+}(X, adE) \setminus \{0\}} \frac{\|d_{A_\infty}^+ v\|^2}{\|v\|^2}.$$ 

is the lowest eigenvalue of $d_{A_\infty}^+ d_{A_\infty}^{+,*}$.

The Sobolev norms $L^p_{k,A}$, where $1 \leq p < \infty$ and $k$ is an integer, with respect to the connections defined as:

$$\|u\|_{L^p_{k,A}(X)} := \left( \sum_{j=0}^k \int_X |\nabla_A^j u|^p dvol_g \right)^{1/p}, \forall u \in L^p_{k,A}(X, ad(E)),$$

where $\nabla_A^j := \nabla_A \circ \cdots \circ \nabla_A$ (repeated $j$ times for $j \geq 0$).

One can see that an ASD connection $A_\infty$ is regular, i.e. $\mu(A_\infty) > 0$.

Theorem 2.2. ([1] Proposition 22) Let $X$ be a closed, four-dimensional, smooth Riemannian manifold with a smooth Riemannian metric, $G$ be a compact Lie group, $P$ be a smooth principal $G$-bundle over $X$. If there is a $C^\infty$ ASD connection $A_\infty$ on $P$ that is regular, then there is constant $\sigma = \sigma(\mu(A_\infty), g, G) \in (0,1]$ with the following significance. If $A := A_\infty + b$ is a smooth connection on $P$ obeying

$$\|\nabla_A^0\|_{L^2(X)} + \|b\|_{L^2(X)} \|F_{A_\infty}\|_{L^4(X)} \leq \sigma,$$

then there exist a solution $a := d_{A_\infty}^{+,*} u \in \Omega^1(X, g_P)$ where $u \in \Omega^{2,+}(X, g_P)$ to Equation (2.1). In fact, the connection $\tilde{A}_\infty := A - a$ is an anti-self-dual connection on $P$. Further, there exist a constant $C = C(\mu(A_\infty), g, G) \in (0, \infty)$ such that

$$\|\nabla_A a\|_{L^2(X)} \leq C \|F_{\tilde{A}}^+\|_{L^2(X)} + \|F_{A_\infty}\|_{L^4} \|F_{\tilde{A}}^+\|_{L^{4/3}(X)},$$

$$\|a\|_{L^2(X)} \leq C \|F_{\tilde{A}}^+\|_{L^{4/3}(X)}.$$ 

This theorem 2.2 by follows the method of proof of [21] Theorem 2.2 applied to $F^+(A + d_{A_\infty}^{+,*} u) = 0$. One also can see [1] Proposition 22.

3 Non-existence solutions on a neighbourhood of a generic ASD connection

3.1 Decoupled Kapustin-Witten equations

Definition 3.1. Let $G$ be a compact Lie group, $P$ be a $G$-bundle over a closed, smooth four-manifold $X$ and endowed with a smooth Riemannian metric, $g$. The decoupled Kapustin-Witten equations on $P$ over $X$ are the equations require a pair $(A, \phi) \in \mathcal{A}_P \times \Omega^1(X, g_P)$ satisfies

$$F_{\tilde{A}}^+ = 0,$$

$$(\phi \wedge \phi)^+ = 0, \quad (d_A^+ \phi)^- = d_A\phi = 0.$$
Lemma 3.2. If a pair \((A, \phi)\) is a solution of decoupled Kapustin-Witten equations, the extra \(\phi\) also obeys
\[
\phi \wedge \phi = 0, \quad d_A \phi = 0.
\]

Proof. At first, we observe that
\[
\int_X \text{tr}(\phi \wedge \phi \wedge \phi \wedge \phi) = 0,
\]
hence
\[
\|\phi \wedge \phi\|_{L^2(X)}^2 = 2\|\phi \wedge \phi\|_{L^2(X)}^2 = 0,
\]
i.e. \(\phi \wedge \phi = 0\). From \((d_A \phi)^- = 0\), i.e. \(d_A \phi = *d_A \phi\), then we obtain
\[
d^*d_A \phi = -*[F_A, \phi].
\]
Hence take the \(L^2\) inner of above identity with \(\phi\) and integrable by parts, we obtain
\[
\|d_A \phi\|_{L^2(X)} = -\int_X \text{tr}([F_A, \phi] \wedge \phi) = \int_X \text{tr}(F_A \wedge [\phi \wedge \phi]) = 0.
\]
Thus we have \(d_A \phi = 0\). \(\square\)

We recall a vanishing theorem on the extra fields of decoupled Kapustin-Witten equations. The prove is similar to Vafa-Witten equations [15] Theorem 4.2.1. At first, we recall a useful lemma proved by Donaldson [2]Lemma 4.3.21.

Lemma 3.3. If \(A\) is an irreducible \(SU(2)\) or \(SO(3)\) ASD connection on a bundle \(E\) over a simply connected four-manifold \(X\), then the restriction of \(A\) to any non-empty open set in \(X\) is also irreducible.

Theorem 3.4. ([12] Theorem 2.9) Let \(X\) be a closed, simply-connected, smooth, oriented, Riemannian four-manifold, \(P \to X\) be an \(SU(2)\) or \(SO(3)\) principal bundle. Let \((A, \phi)\) be a solution of the decoupled Kapustin-Witten equations on \(P\). If \(A\) is irreducible, then the extra fields \(\phi\) vanish.

Proof. We denote \(Z^c\) by the complement of the zero of \(\phi\). By unique continuation of the elliptic equation \((d_A + d_A^*)\phi = 0\), \(Z^c\) is either empty or dense. Since \(\phi \wedge \phi = 0\), \(\phi\) has at most rank one. The Lie algebra of \(SU(2)\) or \(SO(3)\) is three-dimensional, with basis \(\{\sigma^i\}_{i=1,2,3}\) and Lie brackets
\[
\{\sigma^i, \sigma^j\} = 2\varepsilon_{ijk}\sigma^k.
\]
In a local coordinate, we can set \(\phi = \sum_{i=1}^3 \phi_i \sigma^i\), where \(\phi_i \in \Omega^1(X)\). We then have
\[
0 = \phi \wedge \phi = 2(\phi_1 \wedge \phi_2)\sigma^3 + 2(\phi_3 \wedge \phi_1)\sigma^2 + 2(\phi_2 \wedge \phi_3)\sigma^1,
\]
i.e.

\[ 0 = \phi_1 \wedge \phi_2 = \phi_3 \wedge \phi_1 = \phi_2 \wedge \phi_3. \]  

(3.1)

On \( Z^c \), \( \phi \) is non-zero, then without loss of generality we can assume that \( \phi_1 \) is non-zero. From (3.1), there exist functions \( \mu \) and \( \nu \) such that

\[ \phi_2 = \mu \phi_1 \text{ and } \phi_3 = \nu \phi_1. \]

Hence we would re-written \( \phi \) to

\[ \phi = \phi_1 (\sigma^1 + \mu \sigma^2 + \nu \sigma^3) \]

\[ = \phi_1 (1 + \mu^2 + \nu^2)^{1/2} \left( \frac{\sigma^1 + \mu \sigma^2 + \nu \sigma^3}{\sqrt{1 + \mu^2 + \nu^2}} \right). \]

Then on \( Z^c \) write \( \phi = \xi \otimes \omega \) for \( \xi \in \Omega^0(Z^c, g_P) \) with \( \langle \xi, \xi \rangle = 1 \), and \( \omega \in \Omega^1(Z^c) \). We compute

\[ 0 = d_A (\xi \otimes \omega) = d_A \xi \wedge \omega - \xi \otimes d \omega, \]

\[ 0 = d_A * (\xi \otimes \omega) = d_A \xi \wedge * \omega - \xi \otimes d * \omega. \]

Taking the inner product with \( \xi \) and using the consequence of \( \langle \xi, \xi \rangle = 1 \) that \( \langle \xi, d_A \xi \rangle = 0 \), we get \( d \omega = d^* \omega = 0 \). It follows that \( d_A \xi \wedge \omega = 0 \) and \( d_A \xi \wedge * \omega = 0 \). Since \( \omega \) is nowhere zero along \( Z^c \), we must have \( d_A \xi = 0 \) along \( Z^c \). Therefore, \( A \) is reducible along \( Z^c \). However according to Lemma 3.3, \( A \) is irreducible along \( Z^c \). This is a contradiction unless \( Z^c \) is empty. Therefore \( Z = X \), so \( \phi \) is identically zero.

\[ \square \]

### 3.2 A bounded property of extra fields

In this section, we would proved that the extra fields have a lower positive bounded if the connections on neighbourhood of a regular ASD connection. The result has a direct proof by using the Kuranshi model for the moduli space of Kapustin-Witten solutions around the pair \((A_\infty, 0)\), where \( A_\infty \) is an ASD connection. Our proof employs a Weizenb"ock formula, the idea could extends to the case of Kapustin-Witten equations and Vafa-Witten equations on K"ahler surface. Now, we recall a bounded on \( \|\phi\|_{L^\infty} \) in terms of \( \|\phi\|_{L^2} \). The technique is similar to Vafa-Witten equations [15].

**Theorem 3.5.** ([12] Theorem 2.4). Let \( X \) be a closed, four-dimensional, smooth Riemannian manifold with a smooth Riemannian metric \( g \), \( P \rightarrow X \) be a principal \( G \)-bundle with \( G \) be a compact Lie group. Then there is a positive constant \( C = C(g) \) with following significance. If the pair \((A, \phi)\) is a solution of Kapustin-Witten equations, then

\[ \|\phi\|_{L^\infty(X)} \leq C \|\phi\|_{L^2(X)}. \]
We suppose the connection \( [A] \) on a neighbourhood of a regular ASD connection \( [A_\infty] \) then the Theorem \ref{theo:2.2} which provides existence of an other ASD connection \( \tilde{A}_\infty \) on \( P \) and a Sobolev norm estimate for the distance between \( A \) and \( \tilde{A}_\infty \). Then we have

**Proposition 3.6.** Let \( X \) be a closed, oriented, smooth, four-dimensional Riemannian manifold with Riemannian metric \( g \), \( P \to X \) be a principal \( G \)-bundle with \( G \) being a compact Lie group with \( p_1(P) \) negative. Suppose \( A_\infty \) is a regular ASD connection on \( P \), then there are positive constants \( \delta = \delta(g, A_\infty) \in (0, 1) \), \( C = C(g, A_0) \in [1, \infty) \) with following significance. If \( (A, \phi) \) is a solution of Kapustin-Witten equations over \( X \) and the connection \( A \) obeys
\[
\text{dist}(A, A_\infty) \leq \delta
\]
then either the extra field obeying
\[
\|\phi\|_{L^2} \geq C
\]
or \( A \) is anti-self-dual with respect to \( g \).

**Proof.** From Theorem \ref{theo:2.2} for a suitable constant \( \delta \), the connection \( A \) can be written as
\[
A = \tilde{A}_\infty + a,
\]
where \( \tilde{A}_\infty \) is also an ASD connection. Following the idea on \cite{12}, we have two integrable inequalities:
\[
\|\nabla_A \phi\|_{L^2(X)}^2 + \langle \text{Ric} \circ \phi, \phi \rangle_{L^2(X)} + 2\|F_A^+\|_{L^2(X)}^2 = 0,
\]
\[
\|\nabla_{\tilde{A}_\infty} \phi\|_{L^2(X)}^2 + \langle \text{Ric} \circ \phi, \phi \rangle_{L^2(X)} \geq 0.
\]
Combining the preceding inequalities gives
\[
0 \leq \|\nabla_{A_0} \phi\|_{L^2(X)}^2 + \langle \text{Ric} \circ \phi, \phi \rangle_{L^2(X)}
\leq \|\nabla_A \phi\|_{L^2(X)}^2 + \langle \text{Ric} \circ \phi, \phi \rangle_{L^2(X)} + \|\nabla_A \phi - \nabla_{\tilde{A}_\infty} \phi\|_{L^2(X)}^2
\leq c\|a\|_{L^2(X)} \|\phi\|_{L^\infty(X)} - 2\|F_A^+\|_{L^2(X)}^2
\leq (c\|\phi\|_{L^2(X)} - 2)\|F_A^+\|_{L^2(X)}.
\]
If \( \|F_A^+\|_{L^2(X)} \) is non-zero, thus \( \|\phi\|_{L^2(X)} \geq 2/c \). We complete the proof of this theorem. \( \square \)

### 3.3 Uhlenbeck type compactness of Kapustin-Witten equations

In this section, we recall a compactness theorem of Kapustin-Witten equations proved by Taubes \cite{23}.

**Theorem 3.7.** Let \( X \) be a closed, oriented, smooth Riemannian four-manifold with Riemannian metric \( g \), \( P \to X \) be a principal \( G \)-bundle over \( X \) with \( G \) being \( SU(2) \) or \( SO(3) \). Let \( \{ (A_i, \phi_i) \}_{i \in \mathbb{N}} \) being a sequence solutions of Kapustin-Witten equations and
the extra fields obeying $\int_X |\phi_i|^2 \leq C$.

(1) there exist a principal $P_\Delta \to X$ and a pair $(A_\Delta, \phi_\Delta) \in A_{P_\Delta} \times \Omega^1(X, g_{P_\Delta})$ obeys the Kapustin-Witten equations;

(2) a finite set $\Sigma \subset X$ of points, a subsequence $\Xi \in \mathbb{N}$ and a sequence $\{g_i\}_{i \in \Xi}$ of automorphisms of $P_\Delta|_{X-\Sigma}$ such that $\{(g_i^*(A_i), g_i^*(\phi_i))\}_{i \in \Xi}$ converges to $(A_\Delta, \phi_\Delta)$ in the $C^\infty$ topology on compact subsets in $X - \Sigma$.

Furthermore, Taubes also obtained a Uhlenbeck-type compactness theorem for sequence of solutions with the sequence $r_i := \|\phi_i\|_{L^2(X)}$ has no bounded subsequence to Kapustin-Witten equations, see [23] Theorem 1.1.

\textbf{Theorem 3.8.} Let $\{(A_i, \phi_i)\}_{i \in \mathbb{N}}$ be a sequence solutions of Kapustin-Witten equations, set $r_i$ to the $L^2$-norm of $\phi_i$. If the sequence $\{r_n\}_{n=1,2,\ldots}$ has no bounded subsequence. There exists in this case the following data,

(1) A finite set $\Theta \subset X$ and a closed, nowhere dense set $Z \subset X - \Theta$,

(2) a real line bundle $\mathcal{I} \to X - (Z \cup \Theta)$,

(3) a harmonic $\mathcal{I}$-form $v$ on $X - (Z \cup \Theta)$, the norm of $v$ extends over the whole of $X$ as a bounded $L^2$ function. In addition,

a) The extension of $|v|$ is a continuous on $X - \Theta$ and its zero locus is the set $Z$.

b) Let $U$ denote an open set in $X - \Theta$ with compact closure. The function $|v|$ is Hölder continuous on $U$ with Hölder exponent that is independent of $U$ and of the original sequence $\{(A_i, \phi_i)\}_{i=1,2,\ldots}$.

c) If $p$ is any given point in $X$, then the function $\text{dist}(-, p)^{-1}|\nabla v|$ extends to the whole of $X$ as an $L^2$ function.

(4) A principal $SO(3)$ bundle $P_\Delta \to X - (Z \cup \Theta)$ and a connection $A_\Delta$ on $P_\Delta$ with harmonic curvature.

(5) An isometric $A_\Delta$ covariantly constant homorphism $\sigma_\Delta : \mathcal{I} \to g_P$.

In addition, there exist a subsequence $\Lambda \subset \Xi$ and a sequence of automorphisms $g_i : P_\Delta \to P|_{X-(Z\cup\Theta)}$ such that

(i) $\{g_i^*(A_i)\}$ converges to $A_\Delta$ in the $L^2$ topology on compact subset in $X - (Z \cup \Theta)$ and

(ii) The sequence $\{r^{-1}g_i^*(\phi_i)\}$ converges to $v \otimes \sigma_\Delta$ in $L^2$ topology on compact subset in $X - (Z \cup \Theta)$ and $C^0$-topology on $X - \Theta$.

We following an idea used by Tanaka for Vafa-Witten equations, see [19] Theorem 1.3, to proved a useful lemma.

\textbf{Lemma 3.9.} Let $X$ be a closed, oriented, simply-connected, 4-dimensional manifold with a smooth Riemannian metric $g$, $P \to X$ be a principal $G$-bundle with $G$ being $SU(2)$ or $SO(3)$. If $\{(A_i, \phi_i)\}$ is a sequence of smooth solutions of Kapustin-Witten equations such that

$$\text{dist}(A_i, A_\infty) \to 0, \text{ as } i \to \infty,$$
where $A_\infty$ is an irreducible ASD connection on $P$. Then there exist a subsequence $\Xi \subset \mathbb{N}$ such that the sequence $\{r_i := \|\phi_i\|_{L^2(X)}\}_{i \in \Xi}$ is a bounded subsequence.

**Proof.** For the sequence of $\{A_i\}_{i \in \mathbb{N}}$ on $P$, one can see $S(\{A_i\}) = \emptyset$, where $S(A_i)$ defined as in [18] (1.1). From [18] Theorem 1.3, there exist a subsequence $\Xi \subset \mathbb{N}$ and a sequence of gauge transformations $g_i^* (A_i)_{i \in \Xi}$ converges weakly in the $L^2$-topology, we denote the limit connection by $\tilde{A}_\infty$. Under our assumption on the sequence $\{A_i\}_{i \in \mathbb{N}}$, there exist a gauge transformation $g_\infty$ such that $g_\infty^* (A_\infty) = \tilde{A}_\infty$, thus $\tilde{A}_\infty$ is also irreducible. Since on simply-connected manifold, the locally reducible is reducible, using [19] Theorem 1.3 again, there exist a positive number $C$ such that $\|\phi_i\|_{L^2(X)} \leq C$ for all $i \in \Xi$. \qed

**Proof of Theorem 1.1.** We suppose the pair $(A, \phi)$ is a solution of Kapustin-Witten equations and $A$ is not ASD connection. If we suppose that the constant $\delta$ does not exist, we may choose a sequence $\{(A_i, \phi_i)\}$ of solutions of Kapustin-Witten equations on $P$ such that $\text{dist}(A_i, A_\infty) \to 0$ as $i \to \infty$.

Then from Proposition 3.6 and Lemma 3.9 there exists a subsequence $\Xi \subset \mathbb{N}$ and two positive constants $C, c$, such that

$$c \leq \|\phi_i\|_{L^2(X)} \leq C.$$ 

From the compactness Theorem 3.7 there exist a pair $(A_\Delta, \phi_\Delta) \in \mathcal{A}_{P_\Delta} \times \Omega^1(X, g_{P_\Delta})$ that obeys the Kapustin-Witten equations and there has a subsequence $\Xi' \subset \Xi$ and a sequence $(g_i)_{i \in \Xi'}$ of automorphisms of $P_\Delta$ such that $\{(g_i^* (A_i), g_i^* (\phi_i))\}_{i \in \Xi'}$ converges to $(A_\Delta, \phi_\Delta)$ in the $C^\infty$ topology on $X$. Thus the extra field $\phi_\Delta$ also has a lower positive bounded:

$$\|\phi_\Delta\|_{L^2(X)} \geq \lim \inf \|\phi_i\|_{L^2(X)} \geq c.$$ 

But on the other hand, under our initial assumption regarding the sequence $\{A_i\}_{i \in \mathbb{N}}$, we have $[A_\Delta] \equiv [A_\infty]$. Thus the connection $A_\Delta$ is also irreducible, from vanishing Theorem 3.4 the extra field $\phi_\Delta = 0$. Its contradiction to $\|\phi_\Delta\|_{L^2(X)}$ has a uniform lower bound. The preceding argument shows that the desired constant $\delta$ exists. \qed

### 3.4 A bounded property of curvatures

In this section, we extends the method of proof of Theorem 1.1 to global situation. We suppose the connection $[A_\infty] \in \bar{M}_{ASD}$ are all regular, i.e. $\text{Coker}d_{A_\infty}^+ = 0$, following the way in [4] Section 3, we know that the connections on

$$B_\varepsilon(P, g) := \{A \in \mathcal{A}_P : \|F_A^+\|_{L^2(X)} \leq \varepsilon\}$$

also obey $\text{Coker}d_A^+ = 0$ for suitable constant $\varepsilon$. 
Proposition 3.10. Let \( G \) be a compact Lie group, \( P \rightarrow X \) a principal \( G \)-bundle over a closed, connected, four-dimensional manifold, \( X \) with Riemannian metric \( g \). Suppose the connections on \( \bar{M}_{\text{ASD}} \) are all regular, then there are positive constants \( \varepsilon = \varepsilon(P, g) \) and \( \mu = \mu(P, g) \) such that
\[
\mu(A) \geq \mu, \quad [A] \in \mathcal{B}_\varepsilon(P, g).
\]
where \( \mu(A) \) is as in Definition 2.1.

Then we recall a gap result of extra fields of Kapustin-Witten equations which proved by author [12].

Theorem 3.11. Assume the hypotheses of Proposition 3.10. Suppose the connections on \( \bar{M}_{\text{ASD}} \) are all regular. There is a positive constant \( \delta = \delta(P, g) \) with following significance. If \( (A, \phi) \) is a solution of Kapustin-Witten equations, then either extra filed \( \phi \) obeying
\[
\|\phi\|_{L^2(X)} \geq \delta,
\]
or \( A \) is anti-self-dual with respect to \( g \).

Now, we begin to consider a sequence smooth solutions \( \{(A_i, \phi_i)\}_{i \in \mathbb{N}} \) of Kapustin-Witten equations. If we suppose that \( \|\phi_i\|_{L^2(X)} \) has no bounded subsequence, the from the compactness theorem 3.8 due to Taubes, we only know the connection \( A_\Delta \) with harmonic curvature. Moreover if we suppose the curvatures \( F_{A_i} \) of the connection \( A_i \) obeying
\[
\|F_{A_i} \|_{L^2(X)} \rightarrow 0 \quad \text{as} \quad i \rightarrow \infty,
\]
Then we can claim \( A_\Delta \) is an anti-self-dual connection on the complement of \( Z \cup \Theta \cup \Sigma, \Sigma \) is a set of finite points on \( X \).

Corollary 3.12. Let \( \{(A_i, \phi_i)\} \) be a sequence solutions of Kapustin-Witten equations, set \( r_i \) to the \( L^2 \)-norm of \( \phi_i \). Suppose \( \{F_{A_i}^+\}_{i \in \mathbb{N}} \) converge to zero in \( L^2 \)-topology and the sequence \( \{r_n\}_{n=1,2,...} \) has no bounded subsequence. Let \( Z, \Theta \) and \( I \) be as described in Theorem 3.8 so that \( \sigma_\Delta \) and \( A_\Delta \) are defined over \( X - (Z \cup \Theta \cup \Sigma) \). Then the connection \( A_\Delta \) is anti-self-dual connection on \( P_\Delta \).

Before the proof of Corollary 3.12 we should recall a key a compactness theorem due to Sedlacek, see [16] Theorem 4.3 or [3] Theorem 35.17.

Theorem 3.13. Let \( G \) be a compact Lie group and \( P \) a principal \( G \)-bundle over a closed, smooth, oriented, four-dimensional Riemannian manifold \( X \) with a Riemannian metric \( g \). If \( \{A_i\}_{i \in \mathbb{N}} \) is a sequence \( C^\infty \) connection on \( P \) and the curvatures obeying
\[
\|F_{A_i}^+\|_{L^2(X)} \rightarrow 0 \quad \text{as} \quad i \rightarrow \infty,
\]
then there exists
(1) An integer \( L \) and a finite set of points, \( \Sigma = \{x_1, \ldots, x_L\} \subset X \),;
(2) A smooth anti-self-dual \( A_\infty \) on a principal \( G \)-bundle \( P_\infty \) over \( X \) with \( \eta(P_\infty) = \eta(P) \).

(3) A subsequence \( \Xi \subset \mathbb{N} \), we also denote by \( \{ A_i \} \), a sequence gauge transformation \( \{ g_i \} \) such that, \( g_i^*(A_i) \) weakly converges to \( A_\infty \) in \( L^2 \) on \( X \setminus \Sigma \), and \( g_i^*(F_{A_i}) \) weakly converges to \( F_{A_\infty} \) in \( L^2 \) on \( X \setminus \Sigma \).

**Proof Corollary 3.12** We can apply Theorem 3.8 to the sequence \( \{ A_i \} \) on a principal \( G \)-bundle \( P_\infty \) which is the weakly \( L^2 \) limit of \( \{ g_i^*(A_i) \} \) over \( X - \Sigma \), \( \Sigma \) is the set of some points on \( X \). If the sequence \( r_i \) don't has a bounded subsequence. Apply Theorem 3.8 to the subsequence \( \{ (A_i, \phi_i) \} \) of gauge-equivalent connections. Since weakly \( L^2 \) limits preserve \( L^2 \) gauge equivalence, it follows that there exists a Sobolev-class \( L^2 \) gauge transformation \( g_\Delta \) such that \( g_\Delta^*(A_\Delta) = A_\infty \). Thus \( A_\Delta \) is anti-self-dual on the complement of \( Z \cup \Theta \cup \Sigma \).

The result due to Tanaka (13) Theorem 1.3 for the sequence \( \{ A_i \} \) obeys the curvatures converge to zero in \( L^2 \)-topology will invalid, since \( S(A_i) \) may be not empty. But we would also prove a compactness theorem for a sequence solutions of Kapustin-Witten equations by using the useful Lemma 3.3.

**Proposition 3.14.** Let \( X \) be a closed, oriented, simply-connected, four-dimensional manifold with a smooth Riemannian metric \( g \), \( P \to X \) be a principal \( SU(2) \) or \( SO(3) \)-bundle with \( p_1(P) \) negative. Suppose the connections on \( \tilde{M}_{\text{ASD}}(P, g) \) are irreducible. If \( (A_i, \phi_i) \) is a sequence solutions of Kapustin-Witten equations and the curvatures obeying

\[
\| F_{A_i}^+ \|_{L^2(X)} \to 0 \text{ as } i \to \infty, 
\]

then there exist a there exist a subsequence \( \Xi \subset \mathbb{N} \), an anti-self-dual connection \( A_\infty \) on a principal \( P_\infty \), a finite point \( \{ x_1, \cdots, x_L \} \) on \( X \), and a sequence of gauge transformations \( \{ g_i \} \) such that \( \{ (g_i^*(A_i), g_i^*(\phi_i)) \} \) converges in \( C^\infty \)-topology to a pair \( (A_\infty, 0) \) over \( X - \{ x_1, \cdots, x_L \} \).

**Proof.** At first, we claim that the sequence \( r_i := \| \phi_i \|_{L^2(X)} \) has a bounded subsequence. If not, the sequence \( r_i \) don’t has a bounded subsequence, from Theorem 3.8 we have \( Z, \Theta \) and \( \sigma_\Delta, v \) which described in Theorem 3.8. We define \( \sigma_\infty := g_\Delta^*(\sigma_\Delta) \) over \( X - (Z \cup \Theta \cup \Sigma) \), where \( g_\Delta, \Sigma \) are as described in the proof of Corollary 3.12. From Theorem 3.8 we then have \( \nabla_{A_\infty} \sigma_\infty = \nabla_{A_\Delta} \sigma_\Delta = 0 \) on \( X - (Z \cup \Theta \cup \Sigma) \). Thus we have a section \( s := v \otimes \sigma_\infty \) on \( P_\infty |_{X - (Z \cup \Theta \cup \Sigma)} \) and one can see \( v \otimes \sigma_\infty \) is non-zero all over \( X - (Z \cup \Theta \cup \Sigma) \). We can rewritten \( s \) to \( s = \tilde{\sigma} \otimes \tilde{v} \), where \( \tilde{\sigma} \in \Gamma(X - (Z \cup \Theta \cup \Sigma, \mathfrak{g}_{P_\infty}) \) and \( \tilde{v} \in \Omega^1(X - (Z \cup \Theta \cup \Sigma)) \). We also setting \( \langle \tilde{\sigma}, \tilde{\sigma} \rangle = 1 \), thus \( \langle d_{A_\infty} \tilde{\sigma}, \tilde{\sigma} \rangle = 0 \) along \( X - (Z \cup \Theta \cup \Sigma) \). In a direct calculate, we have \( d_{A_\infty} \tilde{\sigma} \wedge \tilde{v} + \tilde{\sigma} \wedge dv = 0 \), thus \( dv = 0 \). It follows that \( d_{A_\infty} \tilde{\sigma} \wedge v = 0 \). Since \( \tilde{v} \) is nowhere
zero along \( X - (Z \cup \Theta \cup \Sigma) \), we must have \( d_{A_n} \tilde{\sigma} = 0 \). According to Lemma 3.3, \( A \) is irreducible along a open set of \( X - (Z \cup \Theta \cup \Sigma) \), then \( \tilde{\sigma} = 0 \). It is contradiction to \( s \) is non-zero on \( X - (Z \cup \Theta \cup \Sigma) \). Hence we prove the sequence \( \{r_n\}_{n=1,2,\ldots} \) must has a bounded subsequence. Then form the compactness theorem 3.7, there exist a pair \((A_\Delta, \phi_\Delta) \in \mathcal{A}_{P_\Delta} \times \Omega^1(X, g_{P_\Delta}) \) that obeys the Kapustin-Witten equations and there has a subsequence \( \Xi' \subseteq \Xi \) and a sequence \( \{g_i\}_{i \in \Xi} \) of automorphisms of \( P_\Delta \) such that \( \{(g_i^*(A_i), g_i^*(\phi_i))\}_{i \in \Xi} \) converges to \((A_\Delta, \phi_\Delta) \) in the \( C^\infty \) topology on \( X - \{x_1, \cdots, x_L\} \). Under the assumption of \( \{A_i\}_{i \in \mathbb{N}} \), the connection \( A_\Delta \) is an irreducible anti-self-dual connection and the vanish theorem ensures \( \phi_\Delta = 0 \).

**Proof of Theorem 1.3** Now we begin to proof Theorem 1.3. Suppose the constant \( \delta \) does not exist. We may choose a sequence of solutions \( \{(A_i, \phi_i)\}_{i \in \mathbb{N}} \) of Kapustin-Witten equations such that \( \|F_A^+\|_{L^2(X)} \to 0 \). From Proposition 3.14, there exist a subsequence \( \Xi \subseteq \mathbb{N} \) and a sequence transformation \( \{g_i\}_{i \in \Xi} \) such that \( (g^*(A_i), g^*(\phi_i)) \to (A_\infty, 0) \) in \( C^\infty \) over \( X - \{x_1, \cdots, x_L\} \). There also exist a positive constant \( C \), such that \( \|\phi_i\|_{L^2(X)} \leq C \). We then have

\[
\|\phi_i\|_{L^2(X)} \leq c \|\phi_i\|_{L^2(X)} \leq cC,
\]

where \( c = c(g) \) is a positive constant. We denote \( \Sigma = \{x_1, \cdots, x_k\} \),

\[
\lim_{i \to \infty} \int_X |\phi_i|^2 = \lim_{i \to \infty} \int_{X-\Sigma} |\phi_i|^2 + \lim_{i \to \infty} \int_{\Sigma} |\phi_i|^2 \leq cC \mu(\Sigma) = 0.
\]

It’s contradiction to \( \|\phi_i\|_{L^2(X)} \) has a uniform positive lower bound, see Theorem 3.11. The preceding argument shows that the desired constant \( \delta \) exists.

If we denote \( A_0 \) is an ASD on \( P \), then the curvature \( F_A \) of a connection \( A := A_0 + a \) has a estimate

\[
\|F_A^+\|_{L^2(X)} = \|(d_{A_0}a + a \wedge a)^+\|_{L^2(X)} \\
\leq \|d_{A_0}a\|_{L^2(X)} + \|a \wedge a\|_{L^2(X)} \\
\leq C(\|\nabla_{A_0}a\|_{L^2(X)} + \|a\|_{L^4(X)}^2) \\
\leq C(\|a\|_{L^4_\Sigma(X)} + \|a\|_{L^2_\Sigma(X)}^2),
\]

where \( C \) is a positive constant. If \( \|a\|_{L^2_\Sigma(X)} \leq 1 \), then

\[
\|F_A^+\|_{L^2(X)} \leq 2C\|a\|_{L^2_\Sigma(X)},
\]

then we have

\[
\|a\|_{L^2_\Sigma(X)} \geq \frac{\delta}{2C}.
\]

So we can set \( \tilde{\delta} := \min\{1, \frac{\delta}{2C}\} \), hence

\[
dist(A, M_{ASD}) := \inf_{g \in G, A_0 \in M_{ASD}} \|g^*(A) - A_0\|_{L^2(X)} \geq \tilde{\delta}.
\]
3.5 Some examples

In this section we give some conditions on the topology of manifolds, the metric of manifold and principle bundles to ensure the connections on $\tilde{M}_{ASD}$ are all generic. For a compact four-manifold $X$, the compactification $\tilde{M}_{ASD}(P, g)$ of $M_{ASD}(P, g)$ contained in the disjoint union

$$\tilde{M}_{ASD}(P, g) \subset \cup(M_{ASD}(P_l, g) \times \text{Sym}^l(X)).$$

(3.2)

We denote $\eta(P)$ is the element in $H^2(X, \mathbb{R})$ which defined as [16] Definition 2.1. From [16] Theorem 5.5, every principal $G$-bundle, $M(P_l, g)$ over $X$ appearing in (3.2) has the property that $\eta(P_l) = \eta(P)$.

Proposition 3.15. Let $X$ be a closed, oriented, simply-connected, four-dimensional manifold with a generic Riemannain metric $g$, $P \to X$ be a principal $SU(2)$ or $SO(3)$-bundle with $p_1(P)$ negative. If $b^+(X) > 0$, then the connection $[A] \in M_{ASD}$ is irreducible.

Proof. For $G = SU(2)$ or $SO(3)$ and $b^+(X) > 0$, $X$ is a simply-connected four-manifold, from [2] Corollary 4.3.15, the only reducible ansi-self-dual connection on a principal $SU(2)$ or $SO(3)$-bundle over $X$, is the product connection on the product bundle $P = X \times G$ if only if the anti-self-dual connection is flat connection, then $p_1(P) = 0$. Hence if we suppose the $p_1(P)$ is negative, then the anti-self-dual connection must be irreducible.

We mean by generic metric the metrics in the second category subset of the space of $C^k$ for some fixed $k > 2$ ([2] Section 4 and [4] Corollary 2). It may reassure the reader to know that for all practical purposes one can work with an open dense subset of the smooth metrics, or even real analytic metrics.

Proposition 3.16. Let $X$ be a closed, oriented, simply-connected, four-dimensional manifold with a generic Riemannain metric $g$, $P \to X$ be a principal $SO(3)$-bundle with $p_1(P)$ negative. If $b^+(X) > 0$ and the second Stiefel-Whitney class $w_2(P) \neq 0$, then the connections $[A] \in \tilde{M}_{ASD}$ are all generic.

Proof. Under the assumptions, the connections $[A]$ on $\tilde{M}_{ASD}$ are regular, i.e. $\mu(A) > 0$ where $\mu(A)$ is as in Definition [2.1] see [4] Corollary 3.9. For $G = SO(3)$, from [16] Theorem 2.4, we have $\eta(P) = w_2(P)$. Then in our condition, every principal $G$-bundle, $M(P_l, g)$ over $X$ appearing in (3.2) has the property that $w_2(P_l)$ is non-trivial. We would claim the ASD connections on $M(P_l, g)$ are irreducible. Since an reducible ASD connection on $P_l$ ensures the bundle $P_l$ is trivial bundle, it’s contradiction to $w_2(P_l) \neq 0$. By the similar method in Proposition [4.5] for any $[A] \in \tilde{M}_{ASD}(P, g)$, we have $\lambda(A) > 0$, where $\lambda(A)$ is as in Definition [4.2], i.e. $[A]$ is irreducible. We complete the proof of this proposition.
Thus from Theorem 1.3 we have

**Corollary 3.17.** Assume the hypotheses on Proposition 3.16. Suppose $b^+(X) > 0$ and the second Stiefel-Whitney class $w_2(P) \neq 0$. There is a positive constant $\delta = \delta(P, g)$ with following significance. If $(A, \phi)$ is a solution of Kapustin-Witten equations, then one of following must hold:
1. $F_A^+ = 0$ and $\phi = 0$;
2. the pair $(A, \phi)$ satisfies
   \[2\|\phi\|^2_{L^2} \geq \| F_A^+ \|_{L^2(X)} \geq \delta.\]

In particular, $(A, \phi)$ satisfies
   \[\text{dist}(A, M_{ASD}) := \inf_{g \in G_{P, A}} \| g^* (A) - A_{\infty} \|_{L^2_1(X)} \geq \tilde{\delta},\]
for a positive constant $\tilde{\delta} = \tilde{\delta}(g, P)$, unless $A$ is anti-self-dual with respect to $g$.

### 4 Kähler surfaces

We now take $X$ to be a compact Kähler surface with Kähler form $\omega$, we also set $d_A = \partial_A + \bar{\partial}_A$, $d_A^* = \partial_A^* + \bar{\partial}_A^*$ and $\phi = \sqrt{-1}(\theta - \theta^*)$, where $\theta \in \Omega^{1,0}(X, adE)$. Thus, Tanaka observed that Kapustin-Witten equations on a closed Kähler surface are the same as Hitchin-Simpson’s equations, see [20] Proposition 3.1.

**Proposition 4.1.** Let $X$ be a closed Kähler surface, the Kapustin-Witten equations have the following form that asks $(A, \theta) \in A_E \times \Omega^{1,0}(X, E)$ to satisfy

\[\bar{\partial}_A \theta = 0, \quad \theta \wedge \theta = 0,\]
\[F_A^{0,2} = 0, \quad \Lambda_\omega (F_A^{1,1} + [\theta \wedge \theta^*]) = 0.\]

Hence the bundle $E$ on $X$ is holomorphic and $\theta$ is a holomorphic section of $End(E) \otimes \Omega^{1,0}(X)$, i.e. the bundle $(E, \theta)$ is a Higgs bundle. From the Kobayashi-Hitchin corresponding for Higgs bundle, see [11] [17], we know that the Higgs bundle $(E, \theta)$ is stable since the pair $(A, \phi)$ satisfies the Hitchin-Simpson equations.

#### 4.1 Irreducible connections

In this section, we first recall a definition of irreducible connection on a principal $G$-bundle. Given a connection $A$ on a principal $G$-bundle $E$ over $X$. We can define the stabilizer $\Gamma_A$ of $A$ in the gauge group $G_E$ by

\[\Gamma_A := \{ u \in G_E | u^*(A) = A \},\]
A connection \( A \) called reducible if the connection \( A \) whose stabilizer \( \Gamma_A \) is larger than the centre \( C(G) \) of \( G \). Otherwise, the connections are irreducible, they satisfy \( \Gamma_A \cong C(G) \). For the cases \( G = SU(2) \) or \( SO(3) \), it’s easy to see that a connection \( A \) is irreducible when it admits no nontrivial covariantly constant Lie algebra-value 0-form, i.e.,

\[
\ker d_A|_{\Omega^0(X,\text{ad}E)} = 0.
\]

We can defined the least eigenvalue \( \lambda(A) \) of \( d_A^*d_A \) as follow.

**Definition 4.2.** For \( A \in \mathcal{A}_E \), define

\[
\lambda(A) := \inf_{v \in \Omega^0(X,\text{ad}E) \setminus \{0\}} \frac{\|d_A v\|^2}{\|v\|^2}.
\]  

(4.1) is the lowest eigenvalue of \( d_A^*d_A \).

By the similar method of the proof of [4] Proposition A.3 or [3] Proposition 35.14, we would also show that the least eigenvalue \( \lambda(A) \) of \( d_A^*d_A \) with respect to connection \( A \) is \( L^p_{\text{loc}} \)-continuity \( 2 \leq p < 4 \).

**Proposition 4.3.** Let \( X \) be a closed, connected, oriented, smooth four-manifold with Riemannian metric, \( g \). Let \( \Sigma = \{x_1, x_2, \ldots, x_L\} \subset X \) \((L \in \mathbb{N}^+) \) and \( \rho = \min_{i \neq j} \text{dist}_g(x_i, x_j) \), let \( U \subset X \) be the open subset give by

\[
U := X \setminus \bigcup_{l=1}^L \bar{B}_{\rho/2}(x_l).
\]

Let \( G \) be a compact Lie group, \( A_0, A \) are \( C^\infty \) connections on the principal \( G \)-bundles \( E_0 \) and \( E \) over \( X \) and \( p \in [2, 4) \). There is an isomorphism of principal \( G \)-bundles, \( u : E \mid X\backslash \Sigma \cong E_0 \mid X\backslash \Sigma \), and identify \( P \mid X\backslash \Sigma \) with \( E_0 \mid X\backslash \Sigma \) using this isomorphism. Then \( \lambda(A) \) satisfies upper bound

\[
\sqrt{\lambda(A)} \geq \sqrt{\lambda(A_0)} - c \sqrt{Lp^{1/6}(\lambda(A)+1)} - cL\rho(\sqrt{\lambda(A)+1}) - c_p \|A - A_p\|_{L^p(U)}(\lambda(A)+1),
\]

and the lower bound,

\[
\sqrt{\lambda(A)} \leq \sqrt{\lambda(A_0)} + c \sqrt{Lp^{1/6}(\lambda(A_0)+1)} + cL\rho(\sqrt{\lambda(A)+1}) + c_p \|A - A_p\|_{L^p(U)}(\lambda(A_0)+1),
\]

where \( c \) is a positive constant depends on \( g, p \).

We now have the useful

**Corollary 4.4.** Assume the hypotheses of Theorem [3,13] Then

\[
\lim_{i \to \infty} \lambda(A_i) = \lambda(A_\infty).
\]

where \( \lambda(A) \) is as in Definition [4,2].
Proof. The proof is similar to the least eigenvalue of operator $d^+_A d^+_{A^*}|_{Ω^{2,+}(X,adE)}$ with respect to connection $A$ (\cite{[5]} Corollary 35.16.).

For a compact Kähler surface $X$ we have a moduli space of ASD connections $M(E, g)$. The compactification $\bar{M}(E, g)$ of $M(E, g)$ contained in the disjoint union

$$\bar{M}(E, g) \subset \bigcup (M(E_l, g) \times Sym^l(X)). \quad (4.2)$$

From \cite{[2]} Theorem 4.4.3, the space $\bar{M}(E, g)$ is compact.

**Proposition 4.5.** Let $X$ be a compact, Kähler surface with a Kähler metric $g$, $E$ a principal $SU(2)$-bundle with $c_2(E) = c$ positive. If $g$ is a $c$-general Kähler metric. Then there are positive constants $\varepsilon = \varepsilon(E, g)$ and $\lambda_0 = \lambda_0(E, g)$. If $A$ is a connection on $E$ such that

$$\|F^+_A\|_{L^2(X)} \leq \varepsilon,$$

and $\lambda(A)$ is as in \cite{[4]}, then

$$\lambda(A) \geq \lambda_0.$$

**Proof.** If $g$ is $c$-generic metric, $\lambda(A) > 0$ for $[A] \in M(E_l, g)$ and $E_l$ is a principal $SU(2)$-bundle over $X$ appearing in the Uhlenbeck compactification. Since the function $\lambda(A)$, $A \in \bar{M}(E, g)$ is continuous by Corollary 4.4 and $\bar{M}(E, g)$ is compact, there is a uniform positive constant $\lambda$ such that $\lambda(A) \geq \lambda$ for $[A] \in \bar{M}(E, g)$. Suppose that the constant $\varepsilon$ does not exist. We may then choose a minimizing sequence $\{A_i\}_{i \in \mathbb{N}}$ of connections on $E$ such that $\|F^+_A\|_{L^2(X)} \to 0$ and $\lambda(A_i) \to 0$ as $i \to \infty$. According to Corollary 4.4 $A_i$ converges to $A_\infty$ a ASD connection on $L^2_{2,loc}(X)$. Then $\lim_{i \to \infty} \lambda(A_i) = \lambda(A_\infty) > 0$, contradicting our initial assumption regarding the sequence $\{A_i\}_{i \in \mathbb{N}}$.

Fix an algebraic surface $S$ and an ample line bundle $L$ on $S$. For every integer $c$ we have defined the moduli space $\mathcal{M}_c(S, L)$ of $L$-stable rank two holomorphic vector bundles $V$ on $S$ and such that $c_1(V) = 0$, $c_2(V) = c$. Taubes \cite{[22]} has shown that, if $X$ is an arbitrary closed 4-manifold and $g$ is any Riemannian metric on $X$, then the moduli space of all irreducible $g$-ASD connections on $P$ with $c_2(P) = c$ moduli gauge equivalence which denote by $\mathcal{M}(P_c, g)$ is nonempty if $c$ is sufficiently large. In the case of an algebraic surface $S$ and a Hodge metric corresponding to the ample line bundle $L$, similar existence results are due to Maruyama and Gieseker \cite{[9]}. Friedman-Morgan (\cite{[6]} Chapter IV, Theorem 4.7) given a very general result along these lines is the following:

**Theorem 4.6.** With $S$ and $L$ as above, for all $c \geq 2p_g(S) + 2$, the moduli space $\mathcal{M}_c(S, L)$ is nonempty.
Next let us determine when a Hodge metric with Kähler form $\omega$ admits reducible ASD connections. Corresponding to such a connection is an associated ASD harmonic 1-form $\alpha$, well-defined up to $\pm 1$, representing an integral cohomology class, which by the description of $\Omega^2 (\mathbb{C})$ is of type $(1, 1)$ and orthogonal to $\omega$. Thus Friedman-Morgan proved for integer $c > 0$, there exist Hodge metric $g$ over $S$ is $c$-generic in the sense of Definition 1.4 (see [6] Chapter IV, Proposition 4.8). For the convenience of readers, we give a detailed proof.

**Proposition 4.7.** Fix $c > 0$. Then there is an open dense subset $\mathcal{D}$ of the cone of ample divisors on $S$ such that if $g$ is a Hodge metric whose Kähler form lies in $\mathcal{D}$, $g$ is a $c$-generic metric in the sense of Definition 1.4.

**Proof.** By standard argument, the set of $\omega$ in the ample cone which are orthogonal to an integral class $\alpha$ with $0 < -\alpha^2 \leq c$ is the intersection of the ample cone with a collection of hyperplanes in $H^2 (S; \mathbb{R})$ which is locally finite on the ample cone. This result is immediate from this. 

### 4.2 Approximate ASD connections

In this section we will give a general criteria under which an approximate ASD connection $A \in \mathcal{A}_{E}^{1,1}$ can be deformation into an other approximate ASD connection $A_{\infty}$ which obeying $\Lambda_{\omega} F_{A_{\infty}} = 0$. Let $A$ be a connection on a principal $G$-bundle over $X$. The above equation for a second connection $A_{\infty} := A + a$, where $a \in \Omega^1 (X, adE)$ is a bundle valued 1-form, can be written:

$$\Lambda_{\omega} (d_{A} a + d_{A}a \wedge d_{A}) = -\Lambda_{\omega} F_{A}. \quad (4.3)$$

We seek a solution of the equation (4.3) in the form

$$a = d^{*}_{A} (s \otimes \omega) = \sqrt{-1} (\partial_{A}s - \bar{\partial}_{As})$$

where $s \in \Omega^{0} (X, adE)$ is a bundle value 0-form. Then (4.3) becomes the second order equation:

$$- d^{*}_{A} d_{As} + \Lambda_{\omega} (d_{As} \wedge d_{A}s) = -\Lambda_{\omega} F_{A}. \quad (4.4)$$

For convenience, we define a map

$$B(u, v) := \frac{1}{2} \Lambda_{\omega} [d_{A} u \wedge d_{A} v].$$

It’s easy to check, we have the pointwise bound:

$$|B(u, v)| \leq C |\nabla_{A} u| |\nabla_{A} v|,$$

where $C$ is a uniform positive constant. We want to prove that if $\Lambda_{\omega} F_{A}$ is small in an appropriate sense there is a small solution $s$ to equation (4.4).
Theorem 4.8. Let $X$ be a compact, Kähler surface with a Kähler metric $g$, $E$ a principal $SU(2)$-bundle over $X$. If $A \in \mathcal{A}_E^{1,1}$ satisfies
\[
\| \Lambda_\omega F_A \|_{L^2(X)} \leq \varepsilon, \\
\lambda(A) \geq \lambda,
\]
where $\varepsilon = \varepsilon(E, g) \in (0, 1)$ and $\lambda = \lambda(E, g) \in (0, \infty)$ are two positive constants, then there is a section $f \in \Omega^0(X, adE)$ such that the connection $A_\infty := A + \sqrt{-1} (\partial A s - \bar{\partial} A s)$ satisfies
\begin{enumerate}
\item $\Lambda_\omega F_{A_\infty} = 0$
\item $\| s \|_{L^2_2(X)} \leq C \| \Lambda_\omega F_A \|_{L^2_2(X)}$
\item $\| F_{A_\infty}^0 \|_{L^2_2(X)} \leq C \| \Lambda_\omega F_A \|_{L^2_2(X)}^2$
\end{enumerate}
where $C = C(\lambda, g) \in [1, \infty)$ is a positive constant.

Now, we being to prove Theorem 4.8, the method of proof above theorem is base on Taubes’ ideas [21]. At first, suppose $s$ and $f$ are sections of $adE$ with
\[
d^*_A d_A s = f, \text{ i.e. } \nabla^*_A \nabla_A s = f, \tag{4.5}
\]
the first observation is

Lemma 4.9. If $\lambda(A) \geq \lambda > 0$, then there exists a unique $C^\infty$ solution to equation (4.5). Furthermore, we have
\[
\| s \|_{L^2_2(X)} \leq c \| f \|_{L^2(X)}, \\
\| B(s, s) \|_{L^2(X)} \leq c \| f \|_{L^2(X)}^2,
\]
where $c = c(\lambda, g)$ is a positive constant.

Proof. Since $\nabla^*_A \nabla_A$ is an elliptic operator of order 2, then for each $k \geq 0$, there is a positive constant $C_k$ so that for all section $v$ of $adE$, see [2] (A8),
\[
\| v \|_{L^2_{k+2}(X)} \leq C_k (\| \nabla^*_A \nabla_A v \|_{L^2_k(X)} + \| v \|_{L^2_k(X)}) \\
\leq C_k (\| \nabla^*_A \nabla_A v \|_{L^2_k(X)} + \lambda^{-1} \| \nabla^*_A \nabla_A v \|_{L^2_k(X)}).
\]

We take $v = s$ and $k = 0$, then
\[
\| s \|_{L^2_2(X)} \leq c \| \nabla^*_A \nabla_A s \|_{L^2_2(X)} \\
\leq c \| f \|_{L^2_2(X)},
\]
where $c = c(\lambda, g)$ is a positive constant. By the Sobolev inequality in four dimension,
\[
\| B(s, s) \|_{L^2(X)} \leq C \| \nabla_A s \|_{L^2_4(X)}^2 \\
\leq C \| \nabla_A s \|_{L^2_2(X)}^2 \\
\leq c \| s \|_{L^2_2(X)}^2,
\]
where $C$ is a positive constant. Hence we complete the proof of this lemma. □
Lemma 4.10. If \( d_A^* d_A s_1 = f_1, \ d_A^* d_A s_2 = f_2, \) then
\[
\|B(s_1, s_2)\|_{L^2(X)} \leq c\|f_1\|_{L^2(X)}\|f_2\|_{L^2(X)}.
\]

We will prove the existence of a solution of (4.4) by the contraction mapping principle.

We write \( s = (d_A^* d_A)^{-1} f \) and (4.4) becomes an equation for \( f \) of the from
\[
f - S(f, f) = \Lambda_\omega F_A,
\]
where \( S(f, g) := B((d_A^* d_A)^{-1} f, (d_A^* d_A)^{-1} g) \).

By Lemma 4.10,
\[
\|S(f_1, f_1) - S(f_2, f_2)\|_{L^2(X)} = \|S(f_1 + f_2, f_1 - f_2)\|_{L^2(X)} \leq c\|f_1 + f_2\|_{L^2(X)}\|f_1 - f_2\|_{L^2(X)}.
\]

We denote \( g_k = f_k - f_{k-1} \) and \( g_1 = f_1 \), then
\[
g_1 = \Lambda_\omega F_A, \ g_2 = S(g_1, g_1)
\]
and
\[
g_k = S(\sum_{i=1}^{k-1} g_i, \sum_{i=1}^{k-1} g_i) - S(\sum_{i=1}^{k-2} g_i, \sum_{i=1}^{k-2} g_i), \ \forall \ k \geq 3.
\]

It is easy to show that, under the assumption of \( \Lambda_\omega F_A \), the sequence \( f_k \) defined by
\[
f_k = S(f_{k-1}, f_{k-1}) + \Lambda_\omega F_A,
\]
starting with \( f_1 = \Lambda_\omega F_A \), is Cauchy with respect to \( L^2 \), and so converges to a limit \( f \) in the completion of \( \Gamma(adE) \) under \( L^2 \).

Proposition 4.11. There are positive constant \( \varepsilon \in (0, 1) \) and \( C \in (1, \infty) \) with following significance. If
\[
\|\Lambda_\omega F_A\|_{L^2(X)} \leq \varepsilon,
\]
then each \( g_k \) exists and is \( C^\infty \).

Further for each \( k \geq 1 \), we have
\[
g_k\|_{L^2(X)} \leq C^{k-1}\|\Lambda_\omega F_A\|_{L^2(X)}^k.
\]

Proof. The proof is by induction on the integer \( k \). The induction begins with \( k = 1 \), one can see \( g_1 = \Lambda_\omega F_A \). The induction proof if completed by demonstrating that if (4.7) is satisfied for \( j < k \), then it also satisfied for \( j = k \). Indeed, since
\[
\|S(\sum_{i=1}^{k-1} g_i, \sum_{i=1}^{k-1} g_i) - S(\sum_{i=1}^{k-2} g_i, \sum_{i=1}^{k-2} g_i)\|_{L^2(X)} \leq c\|\sum_{i=1}^{k-1} g_i + \sum_{i=1}^{k-2} g_i\|_{L^2(X)}\|g_{k-1}\|_{L^2(X)} \leq 2c\|g_{k-1}\|_{L^2(X)}\|g_{k-1}\|_{L^2(X)} \leq \frac{2c}{1 - C\|\Lambda_\omega F_A\|_{L^2(X)}}C^{k-2}\|\Lambda_\omega F_A\|_{L^2(X)}^k.
\]
Now, we provide the constants \( \varepsilon \) sufficiently small and \( C \) sufficiently large to ensures
\[
\|\Lambda_\omega F_A\|_{L^2(X)} \leq C^{-2}(C - 2c), \ i.e. \ \frac{2c}{1 - C\|\Lambda_\omega F_A\|_{L^2(X)}} \leq C \], hence we complete the proof of this Proposition. \( \square \)
Proof of Theorem 4.8: The sequence $g_k$ is Cauchy in $L^2$, the limit

$$f := \lim_{i \to \infty} f_i$$

is a solution to (4.6). Using Lemma 4.9 and Proposition 4.11, we have

$$\|s\|_{L^2(X)} \leq C\|f\|_{L^2(X)} \leq \frac{\|\Lambda\omega F_A\|_{L^2(X)}}{1 - C\|\Lambda\omega F_A\|_{L^2(X)}},$$

we provide $\varepsilon$ and $C$ to ensure $C\varepsilon \leq \frac{1}{2}$, hence

$$\|s\|_{L^2(X)} \leq 2\|\Lambda\omega F_A\|_{L^2(X)}.$$  

We denote $A_\infty := A + \sqrt{-1}(\partial A_s - \bar{\partial}A_s)$, then

$$\|F_{A_\infty}^{0,2}\|_{L^2(X)} = \| - \sqrt{-1}\partial A_s \bar{\partial} A_s - \bar{\partial}A_s \wedge \partial A_s\|_{L^2(X)}$$

$$\leq 2\|\bar{\partial}A_s\|^2_{L^1(X)}$$

$$\leq c\|\nabla A_s\|^2_{L^4(X)}$$

$$\leq c\|\Lambda\omega F_A\|^2_{L^2(X)}.$$

where $c$ is a positive constant. We complete the proof of Theorem 4.8.

4.3 A topology property on stable Higgs bundles

In this section, we use the inequality in Theorem 4.8 to prove that the Higgs field has a positive lower bound if the Kähler metric $g$ is $c$-generic.

Proof Theorem 1.5: For $(A, \theta) \in A_{k+1}^{1,1} \times \Omega^{1,0}(X, g_E)$ is a solution of Hitchin-Simpson equations, we denote $\phi = \sqrt{-1}(\theta - \theta^*) \in \Omega^1(X, adE)$. Suppose that the constant $C$ does not exist. We may provide a positive constant $\varepsilon_0$ such that

$$0 < \|\Lambda\omega F_A\|_{L^2(X)} \leq c\|\theta\|^2_{L^2(X)} \leq C\varepsilon_0 < \varepsilon$$

where $c = c(g)$ is a positive constant and $\varepsilon$ is a constant as Proposition 4.5, then $\lambda(A) > 0$.

Form Theorem 4.8, there exist a connection $A_\infty$ such that

$$\|A - A_\infty\|_{L^2(X)} \leq c\|\Lambda\omega F_A\|_{L^2(X)}$$

for some positive constant $c = c(g, P)$. We have two integrable inequalities:

$$\|\nabla A\phi\|^2_{L^2(X)} + \langle Ric \circ \phi, \phi \rangle_{L^2(X)} + \|\Lambda\omega F_A\|^2_{L^2(X)} = 0,$$

$$\|\nabla A_\infty\phi\|^2_{L^2(X)} + \langle Ric \circ \phi, \phi \rangle_{L^2(X)} + \langle F_{A_\infty}^{0,2} + \Lambda\omega F_{A_\infty}^{0,2}, [\phi, \phi]\rangle_{L^2(X)} \geq 0.$$
We also observe that \( \langle F_{A_\infty}^{0,2} + F_{A_\infty}^{0,2}, [\phi, \phi]\rangle_{L^2(X)} = 0 \).

Combing the preceding inequalities gives
\[
0 \leq ||D_{A_\infty} \phi||^2_{L^2(X)} + \langle \text{Ric} \circ \phi, \phi \rangle_{L^2(X)}
\leq ||D_A \phi||^2_{L^2(X)} + \langle \text{Ric} \circ \phi, \phi \rangle_{L^2(X)} + ||D_{A_\infty} \phi||^2_{L^2(X)}
\leq (c ||\phi||^2_{L^2(X)} - 1)||\Lambda_\omega F_A||^2_{L^2(X)},
\]
for a positive constant \( c = c(g, P) \). If we provide \( C = 1/(2c) \) such that \( ||\phi||^2_{L^2(X)} \leq C \), then \( \Lambda_\omega F_A \equiv 0 \). It’s contradiction to our initial assumption regarding the \( \Lambda_\omega F_A \). The preceding argument shows that the desired constant \( C \) exists.

We also suppose \( X \) is simply-connected, if the constant \( C \) does not exist. We may choose a sequence \( \{(A_i, \theta_i)\}_{i \in \mathbb{N}} \) of Hitchin-Simpson equations such that \( ||\Lambda_\omega F_{A_i}||_{L^2(X)} \to 0 \). We set \( \phi_i := \frac{1}{\sqrt{-1}}(\theta_i - \theta^*_i) \). We following the idea of Theorem 1.3 From Proposition 3.7 there exist a subsequence \( \Xi \in \mathbb{N} \) and a sequence transformation \( \{g_i\}_{i \in \Xi} \) such that \( (g^*(A_i), g^*(\phi_i)) \to (A_\infty, 0) \) in \( C^\infty \) over \( X - \{x_1, \cdots, x_L\} \). There also exist a positive constant \( C \), such that \( ||\phi_i||_{L^2(X)} \leq C \). We then have
\[
||\phi_i||_{L^\infty(X)} \leq c||\phi_i||_{L^2(X)} \leq cC.
\]
where \( c = c(g) \) is a positive constant. We denote \( \Sigma = \{x_1, \cdots, x_k\} \).

\[
\lim_{i \to \infty} \int_X |\phi_i|^2 = \lim_{i \to \infty} \int_{X - \Sigma} |\phi_i|^2 + \lim_{i \to \infty} \int_{\Sigma} |\phi_i|^2 \leq cC \mu(\Sigma) = 0.
\]

It’s contradiction to \( ||\phi_i||_{L^2(X)} \) has a uniform positive lower bounded. The preceding argument shows that the desired constant \( C \) exists.

Corollary 4.12. Let \( X \) be a compact, simply-connected, Kähler surface with a Kähler metric \( g \), \( E \) be a principal \( SU(2) \)-bundle over \( X \). There is a positive integer \( c \) with following significance. If \( g \) is a \( c \)-generic metric in the sense of Definition 1.4 and \( (E, \theta) \) is a stable Higgs bundle with \( c_2(E) = c \), then the moduli space of Higgs bundle is non-connected.

Proof. Taubes [22] has shown that, if \( X \) is an arbitrary closed 4-manifold and \( g \) is any Riemannian metric on \( X \), \( P \) be a \( SU(2) \)-bundle with \( c_2(E) = c \) sufficiently large, then there exist an irreducible ASD connection on \( P \). The moduli space \( M(E, g) \) is non-empty under our conditions. Since the map \( (A, \theta) \mapsto ||\theta||_{L^2} \) is continuous, then the moduli space \( M_{HS} \) is not connected.

From Proposition 3.7 and Corollary 4.12, we have

Corollary 4.13. Let \( S \) is a compact, simply-connected, algebraic surface with a Hodge metric \( g \), \( (E, \theta) \) is a Higgs bundle over \( X \). There is an open dense subset \( \mathcal{D} \) of the cone of ample divisor on \( S \) and a positive integer \( c \) with following significance. If the Kähler form of \( g \) lies in \( \mathcal{D} \) and \( (E, \theta) \) is stable with \( c_2(E) = c \), then the moduli space of stable Higgs bundle is non-connected.
4.4 Vafa-Witten equations

We next consider analogue of the Hitchin-Simpson equations, called Vafa-Witten equations [26]. Let $X$ be a closed, oriented, smooth, four-manifold, $E$ be a principal $G$-bundle with $G$ is a compact Lie group over $X$. We denote $A$ a connection on $E$, $B$ a section of the associated bundle $\Omega^{2,+} \otimes adE$ and $C$ a section of $adE$. A triple $(A, B, C)$ is a solution of Vafa-Witten equations if $(A, B, C)$ satisfies

$$F^+_A + [B, B] + [B, \Gamma] = 0, \quad d^+_A B + d_A \Gamma = 0,$$

where $[B, B] \in \Gamma(X, \Omega^{2,+} \otimes adE)$ is defined through the Lie brackets of $adE$ and $\Omega^{2,+}$, see [15] Appendix A. We take $X$ to be a compact K"ahler surface with K"ahler form $\omega$. In a local orthonormal coordinates, we can write $B = \beta - \beta^* + B_0 \omega$, where $\beta \in \Omega^{2,0}(X, adE)$, $B_0 \in \Omega^0(X, adE)$ and $\gamma = C - \sqrt{-1}B_0$, one can see [15] Chapter 7 for details. We have

**Theorem 4.14.** ([15] Theorem 7.1.2) On a compact K"ahler surface, the Vafa-Witten equations (4.8) have the following form that asks $(A, \beta, \gamma) \in A^{1,1}_E \times \Omega^{2,0}(X, adE) \times \Omega^0(X, adE)$ to satisfies

$$\sqrt{-1} \omega \wedge F_A + \frac{1}{2} [\beta \wedge \beta^*] = 0,$$

$$\bar{\partial}_A \beta = d_A \gamma = 0,$$

$$[\gamma, \gamma^*] = [\gamma, \beta + \beta^*] = 0.$$ (4.9)

Mares also discussed a relation between the existence of a solution to the equations and a stability of vector bundles in as following, one also can see [15] Definition 7.2.3. [18] for details.

**Definition 4.15.** For any $\beta \in \Omega^{2,0}(X, \text{End} E) \cong \Omega^0(\text{Hom}(E, E \otimes K))$, we say that a subbundle $E' \subset E$ is $\beta$-invariant, if

$$\beta(E') \subset E' \otimes K.$$ 

A holomorphic bundle $E$ is $\beta$-stable (semi-stable) if all $\beta$-invariant holomorphic subbundle $E' \subset E$ satisfies

$$\mu(E') = \frac{\text{deg}(E')}{\text{rank} E'} < (\leq) \mu(E) = \frac{\text{deg}(E)}{\text{rank} E}.$$ 

We also recall a bound on $\|\beta\|_{L^\infty}$ in terms of $\|\beta\|_{L^2}$, one can see [15] for detail.

**Theorem 4.16.** Let $X$ be a compact, K"ahler surface with a K"ahler metric $g$, $E$ a principal $G$-bundle over $X$ with $G$ be a compact Lie group. If the pair $(A, \beta, \gamma)$ is a $C^\infty$ solution of Equations (4.9), there is a positive constant $C = C(g)$ such that

$$\|\beta\|_{L^\infty(X)} \leq C \|\beta\|_{L^2(X)}.$$
Theorem 4.17. Let \( X \) be a compact, Kähler surface with a Kähler metric \( g \). Let \( E \) be a principal \( SU(2) \)-bundle with \( c_2(E) = c \) positive. Suppose \( g \) is a \( c \)-generic metric in the sense of Definition (1.5), there is a positive constant \( C = C(g, P) \) with following significance. If the triple \( (A, \beta, \gamma) \in A_1^{1} \times \Omega^{2,0}(X, adE) \times \Omega^0(X, adE) \) is satisfies equations (4.9), then one of following must hold:

1. \( \Lambda_\omega F_A = 0 \), or
2. the extra field \( \beta \) satisfies

\[
\left\| \beta \right\|_{L^2(X)} \geq C.
\]

Furthermore, if \( X \) is simply-connected, the curvature \( F_A \) obeying

\[
\left\| \Lambda_\omega F_A \right\|_{L^2(X)} \geq \tilde{C}
\]

for a positive constant \( \tilde{C} = \tilde{C}(g, P) \), unless \( \Lambda_\omega F_A = 0 \).

Proof. For \((A, \beta, \gamma)\) is a solution of equations (4.9), we denote \( \beta_0 := \beta - \beta^* \in \Omega^{2+}(X, adE) \), hence \( d_A^{+} \beta_0 = *(\bar{\partial}_A \beta - \partial_A \bar{\beta}) = 0 \). Suppose that the constant \( C \) does not exist. We may provide a positive constant \( \varepsilon > 0 \) such that

\[
0 < \left\| \Lambda_\omega F_A \right\|_{L^2(X)} \leq C \left\| \beta \right\|_{L^2(X)} \leq c \varepsilon < \varepsilon
\]

where \( c = c(g) \) is a positive constant and \( \varepsilon \) is a constant as Proposition 4.5, thus \( \lambda(A) > 0 \). From Theorem 4.8 there exist a connection \( A_\infty \) such that

\[
\left\| A - A_\infty \right\|_{L^2(X)} \leq c \left\| \Lambda_\omega F_A \right\|_{L^2(X)}
\]

for a positive constant \( c = c(g, P) \). Using the Weizenböck formula in [5], we have an integrable identity:

\[
\left\| \nabla_A \beta_0 \right\|_{L^2(X)} + \left\langle \left( \frac{1}{3} s - 2\omega^+ \right) \beta_0, \beta_0 \right\rangle_{L^2(X)} + \left\| \Lambda_\omega F_A \right\|_{L^2(X)}^2 = 0,
\]

\[
\left\| \nabla_{A_\infty} \beta_0 \right\|_{L^2(X)} + \left\langle \left( \frac{1}{3} s - 2\omega^+ \right) \beta_0, \beta_0 \right\rangle_{L^2(X)} + \left\langle F_{A_\infty}^{2,0} + F_{A_\infty}^{0,2}, [\beta_0, \beta_0] \right\rangle \geq 0.
\]

We also observe that

\[
\left\langle F_A^{0,2} + F_A^{2,0}, [\beta_0, \beta_0] \right\rangle_{L^2(X)} \leq c \left\| F_A^{0,2} \right\|_{L^2(X)} \left\| \beta_0 \right\|_{L^2(X)}^2 \leq c \left\| \Lambda_\omega F_A \right\|_{L^2(X)}^2 \left\| \beta_0 \right\|_{L^2(X)}^2,
\]

for a positive constant \( c = c(g, P) \). Combining the preceding inequalities gives

\[
0 \leq \left\| \nabla_{A_\infty} \beta_0 \right\|_{L^2(X)} + \left\langle \left( \frac{1}{3} s - 2\omega^+ \right) \beta_0, \beta_0 \right\rangle_{L^2(X)} + \left\langle F_{A_\infty}^{2,0} + F_{A_\infty}^{0,2}, [\beta_0, \beta_0] \right\rangle
\]

\[
\leq \left\| \nabla_A \beta_0 \right\|_{L^2(X)} + \left\langle \left( \frac{1}{3} s - 2\omega^+ \right) \beta_0, \beta_0 \right\rangle_{L^2(X)}
\]

\[
+ \left\| \nabla_A \beta_0 - \nabla_{A_\infty} \beta_0 \right\|_{L^2(X)} + c \left\| \Lambda_\omega F_A \right\|_{L^2(X)}^2 \left\| \beta_0 \right\|_{L^2(X)}^2
\]

\[
\leq \left\| [A - A_\infty] \beta_0 \right\|_{L^2(X)} + c \left\| F_A^{0,2} \right\|_{L^2(X)} \left\| \beta_0 \right\|_{L^2(X)}^2 - \left\| \Lambda_\omega F_A \right\|_{L^2(X)}^2
\]

\[
\leq (c \left\| \beta_0 \right\|_{L^2(X)}^2 - 1) \left\| \Lambda_\omega F_A \right\|_{L^2(X)}^2.
\]
for a positive constant $c = c(g, P)$. We provide $\|\beta_0\|_{L^2(X)}^2 \leq 1/(2c)$, thus $\Lambda_\omega F_A = 0$. It’s contradiction to our initial assumption regarding the $\Lambda_\omega F_A$. The preceding argument shows that the desired constant $C$ exists.

We denote the moduli space of solutions of Hitchin-Simpson equations by

$$M_{VW} := \{ (A, \beta, \gamma) \in A_{E}^{1,1} \times \Omega^{2,0}(adE) \times \Omega^0(adE) \mid (A, \beta, \gamma) \text{satisfies (4.9)} \} / G_E.$$

We apply a vanishing theorem of extra fields of Vafa-Witten equations due to Mare [15] Theorem 4.2.1 to prove that

**Theorem 4.18.** Let $X$ be a simply-connected, Kähler surface with a Kähler metric $g$, $E$ be a principal $SU(2)$ or $SO(3)$-bundle. Let the triple $(A, \beta, \gamma)$ be a solution of the decoupled Vafa-Witten equations:

$$\Lambda_\omega F_A = 0,$$

$$\bar{\partial}_A \beta = [\beta \land \beta^*] = 0,$$

$$d_A \gamma = [\gamma, \gamma^*] = [\gamma, \beta + \beta^*] = 0.$$

If $A$ is an irreducible connection, then $\beta = \gamma = 0$.

Following the idea on Hitchin-Simpson equations, we also have

**Corollary 4.19.** Let $X$ be a compact, simply-connected, Kähler surface with a Kähler metric $g$, $E$ be a principal $SU(2)$-bundle. There is a positive integer $c$ with following significance. If $g$ is a $c$-generic metric in the sense of Definition (1.4) and $(E, \beta)$ is stable $\beta$-stable bundle with $c_2(E) = c$, then the moduli space of $\beta$-stable bundle is non-connected.

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