ON THE ACTION OF THE (2,3,7)-HOMOLOGY SPHERE GROUP ON ITS SPACE OF LEFT-ORDERS

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Abstract. We show that the action of the group $\Sigma = \langle a, b, c \mid a^2 = b^3 = c^7 = abc \rangle$ on its space of left-orders has exactly two minimal components.

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A left-order on a group $G$ is a total order $\prec$ on $G$ which is preserved by the left-multiplication of $G$ on itself. A left-order is uniquely determined by its cone of positive elements $P_\prec = \{ g \in G : g \succ \text{id} \}$, which forms a semi-group, and satisfies $G = P_\prec \sqcup P_\prec^{-1} \sqcup \{ \text{id} \}$. Conversely, any semi-group satisfying such properties is the positive cone for some left-order. Because of this, one can see the set $\text{LO}(G)$ of left-orders on $G$ as the set of positive cones on $G$, so that it is naturally a closed subset of $2^G$. A fundamental system of neighborhoods at a given left-order $\prec$ is given by the subsets $V_{\prec,F} = \{ \prec' \in \text{LO}(G) : P_\prec \cap F = P_{\prec'} \cap F \}$, where $F$ runs over finite subsets of $G$.

The conjugacy action of $G$ on itself induces an action by homeomorphisms on $\{0,1\}^G$ which preserves the subset of positive cones, and thus $G$ has a natural action by homeomorphisms on $\text{LO}(G)$. Given an element $g \in G$ and a left-order $\prec \in \text{LO}(G)$ we denote by $\prec_g$ the left-order obtained using this action. By definition, this is determined by the condition $P_{\prec_g} = g P_\prec g^{-1}$.

A general approach to understand the qualitative dynamical behavior of this action has been developed by Clay [3]. Notably he proved (see also Rivas [9]) that in the case of nonabelian finite rank free groups, the action is topologically transitive, namely there exists a dense orbit. When $G$ is a non-trivial group such that $\text{LO}(G)$ is finite (such groups have been classified by Tararin, and they are a particular class of polycyclic groups), the action of $G$ on $\text{LO}(G)$ has exactly two orbits. Using this, Clay gave an example of countable, not finitely generated, group for which the action is minimal, namely every orbit is dense. The problem of finding a non-trivial finitely generated group with this property is well-known to experts, see for instance Navas’s survey [8, Question 6]. One of the main difficulties is that there are often obstructions to make the orbit of a positive cone $P$ accumulate on $P^{-1}$ (see for instance [3, Proposition 6]).

The purpose of this note is to describe the action on the space of left-orders for the fundamental group of the homology 3-sphere $\Sigma(2,3,7)$, namely

$$\Sigma = \langle a, b, c \mid a^2 = b^3 = c^7 = abc \rangle.$$ 

We prove that the obstruction mentioned above is the only one for this group.

Theorem 1. For any $\prec, \prec' \in \text{LO}(\Sigma)$ such that $abc \in P_\prec \cap P_{\prec'}$, we have $\text{Orb}_{\Sigma}(\prec) = \text{Orb}_{\Sigma}(\prec')$. In particular, the action of $\Sigma$ on $\text{LO}(\Sigma)$ has exactly two minimal components.

Let us comment that $\Sigma$ gives the first example of finitely generated group with this property, and admitting uncountably many left-orders. The proof of this result is somehow close to the proof of [10, Proposition 2.12], and it relies on the tight relation between left-orders and orientation-preserving actions on the real line. Indeed, given any left-order $\prec$ on a countable group $G$, one can construct a faithful action of $G$ on the line, by the so-called dynamical realization. More precisely, there exists an action $\varphi_\prec : G \to \text{Homeo}_+(\mathbb{R})$ and a base point $p \in \mathbb{R}$ such that $\varphi_\prec(g)(p) > p$ if and only if $g \in P_\prec$. 

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Note that for any \( g \in G \), the action \( \varphi_\prec \) with base point \( \varphi_\prec(g)(p) \) provides a dynamical realization of the left-order \( \prec_g \).

The group \( \Sigma \) is the \( \mathbb{Z} \)-central extension in \( \widetilde{\text{PSL}}(2, \mathbb{R}) \) of the cocompact Fuchsian \((2, 3, 7)\)-triangle group \( \Delta \cong \langle a, b, c \mid a^2, b^3, c^7, abc \rangle \), where the center is generated by the element \( abc \), and as so, \( \Sigma \) admits a natural faithful action on the real line.

**Remark 2.** More precisely, the action of \( \Sigma \) is the lift of the action of \( \Delta \) on the boundary of the hyperbolic disk \( \partial \mathbb{H} \), and thus it has the property that every point stabilizer is either trivial or cyclic (because \( \Delta \) is a discrete subgroup of \( \text{PSL}(2, \mathbb{R}) \)), and in the latter case one can find a generator of the stabilizer which contracts any sufficiently small neighborhood of the point (because \( \Delta \) is cocompact in \( \text{PSL}(2, \mathbb{R}) \)).

The group \( \Sigma \) is a quite remarkable group acting on the line, for instance Thurston [11] remarked that this is a finitely generated, perfect group of homeomorphisms of the real line. We will refer to the action of \( \Sigma \) coming from the embedding on \( \widetilde{\text{PSL}}(2, \mathbb{R}) \), as to the standard action \( \rho : \Sigma \to \text{Homeo}_+(\mathbb{R}) \). The remarkable fact that we will use is that the standard action basically gives the unique way \( \Sigma \) can act on the line.

**Theorem 3.** Let \( \varphi : \Sigma \to \text{Homeo}_+(\mathbb{R}) \) be an action without global fixed points. Then there exists a continuous monotone map \( h : \mathbb{R} \to \mathbb{R} \) such that \( h \circ \varphi(g) = \rho(g) \circ h \) for every \( g \in \Sigma \).

The conclusion in the statement is often rephrased by saying that \( \varphi \) and \( \rho \) are semi-conjugate actions. We will say that they are positively semi-conjugate if the map \( h \) is increasing. When \( h \) is not injective, we also say that \( \varphi \) is a blow up of \( \rho \). The proof of Theorem 3 is well-known, and it relies on the fact that the group \( \Delta \) admits a unique non-trivial action on the circle up to semi-conjugacy, by a combination of the Milnor–Wood inequality and a theorem of Matsumoto (see for instance Calegari [2]).

**Remark 4.** In general, if \( \psi : G \to \text{Homeo}_+(\mathbb{R}) \) is a blow up of a dynamical realization \( \varphi_\prec \) with base point \( p \), with positive semi-conjugacy \( h : \mathbb{R} \to \mathbb{R} \), then for any \( q \in h^{-1}(p) \), the linear order of points in the orbit \( \psi(G)(q) \) agrees with that of \( \varphi_\prec(G)(p) \).

**Proof of Theorem 1.** Fix two left-orders \( \prec \) and \( \prec' \) on \( \Sigma \) and a finite subset \( F \subset \Sigma \). We write \( P \) and \( P' \) for the corresponding positive cones. Assume that \( abc \in P \cap P' \). We want to find an element \( g \in \Sigma \) such that \( \prec_g \) belongs to the neighborhood \( V_{\prec,F} \). Let \( \varphi : \Sigma \to \text{Homeo}_+(\mathbb{R}) \) and \( \varphi' : \Sigma \to \text{Homeo}_+(\mathbb{R}) \) be dynamical realizations of \( \prec \) and \( \prec' \) respectively, with corresponding base points \( p \) and \( p' \). As \( abc \in P \cap P' \), the dynamical realizations \( \varphi \) and \( \varphi' \) are positively semi-conjugate. As any two positively semi-conjugate actions admit a common blow up (see for instance [5, Theorem 2.2]), by Remark 4 we can actually assume \( \varphi = \varphi' \), and we will simply write \( g(x) \) instead of \( \varphi(g)(x) \) for any \( g \in \Sigma \), \( x \in \mathbb{R} \). We denote by \( \Lambda \subset \mathbb{R} \) the minimal invariant closed subset for the action \( \varphi \). When \( p \) or \( p' \) is not in \( \Lambda \), we let \( J \) (respectively, \( J' \)) be the closure of the connected component of \( \mathbb{R} \setminus \Lambda \) containing the base point \( p \) (respectively, \( p' \)), and we write \( K = \text{Stab}(J) \) (respectively, \( K' = \text{Stab}(J') \)). In case \( p \) or \( p' \) is in \( \Lambda \), we simply put \( J = \{ p \} \) (or \( J' = \{ p' \} \)), in which case \( K \) (respectively, \( K' \)) is trivial. Note that as for the standard action \( \rho \) point stabilizers are either trivial or cyclic (Remark 2), the subgroup \( K \) (and so \( K' \)) is either trivial or cyclic.

As an easy first case, assume to start that \( K \) is trivial. By continuity of the action, we can find a neighborhood \( V \) of \( J \), such that \( f(x) > x \) if and only if \( f(p) > p \) for every \( f \in F \) and \( x \in V \). Let \( g \in \Sigma \) be any element such that \( g(p') \in \mathbb{V} \), in which case we have \( \prec_g \in V_{\prec,F} \), as wanted.

When \( K \) is non-trivial, we need to take some care to choose such an element \( g \) to send \( p' \) to the correct side of \( J \). By Remark 2, we can take a generator \( k \) of \( K \) such that for every sufficiently small neighborhood \( U \) of \( J \), one has \( k(U) \subset U \). That is to say, if \( u \) is close to \( J \) but strictly to the left of \( J \), we have \( k(u) > u \), and if \( u \) is close to \( J \) but strictly to the right, then \( k(u) < u \). Suppose first that \( k \in P \). Then choose \( g \in \Sigma \) such that \( g(p') \) lies to the left of \( J \) in \( U - J \), chosen sufficiently close so
that \( f(x) > x \) if and only if \( f(p) > p \) for any \( f \in F \) (note that our choice of the left side ensures this holds even if \( K \cap F \neq \emptyset \)), and such that \( gK'g^{-1} \cap F = \emptyset \). The last condition may easily be satisfied because each side of \( U - J \) intersects the minimal set for the action. Then \( g' \in V_{\preceq,F} \), as desired. If instead we have \( k \in P^{-1} \), one makes the same argument choosing \( g(p') \) to be to the right. \( \square \)

As a final comment, let us mention that other finitely generated groups are known to admit a unique nontrivial action on the line, up to semi-conjugacy. One interesting example is provided by the central extension \( \tilde{T} \) of Thompson’s group \( T \) of dyadic piecewise linear circle homeomorphisms (see the work of Matte Bon and the second author [7]). Understanding this example was our original motivation, as we were wondering whether one could find examples of groups whose action on the space of orders is minimal by considering the groups of Hyde and Lodha [4] (or their more conceptual generalization presented by Le Boudec and Matte Bon in [6]), which are “quasi-periodic” versions of \( \tilde{T} \). However, point stabilizers for the standard action of \( \tilde{T} \) are very large, as they are always isomorphic to a subgroup of Thompson’s group \( F \) containing \([F,F]\), and the classification of actions of such groups (even up to semi-conjugacy, see the work of Brum, Matte Bon, Rivas and the second author [1]) is quite complex. Because of this, the classification of actions of \( \tilde{T} \) up to conjugacy is also complex, and there is little hope to understand the minimal components of the action of \( \tilde{T} \) on \( \text{LO}(\tilde{T}) \) (and even worse in the case of the groups of Hyde and Lodha and generalizations).

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