Leading Effect of $CP$ Violation with Four Generations

Wei-Shu Hou$^{1,2}$, Yao-Yuan Mao$^1$, and Chia-Hsien Shen$^1$

1 Department of Physics, National Taiwan University, Taipei, Taiwan 10617
2 National Center for Theoretical Sciences, North Branch, National Taiwan University, Taipei, Taiwan 10617

In the Standard Model with a fourth generation of quarks, we study the relation between the Jarlskog invariants and the triangle areas in the $4 \times 4$ CKM matrix. To identify the leading effects that may probe the $CP$ violation in processes involving quarks, we invoke small mass and small angle expansions, and show that these leading effects are enhanced considerably compared to the three generation case by the large masses of fourth generation quarks. We discuss the leading effect in several cases, in particular the possibility of large $CP$ violation in $b \to s$ processes, which echoes the heightened recent interest because of experimental hints.
I. INTRODUCTION AND MOTIVATION

The experimental discovery \[1\] of CP Violation (CPV) in 1964 came as a surprise, but it provided the clue that lead Sakharov to suggest \[2\] the three conditions that need to be satisfied to explain a long standing and profound puzzle: the disappearance of antimatter from the very early Universe. It was Kobayashi and Maskawa (KM) who proposed \[3\], in 1973, that CPV can arise from charged current weak interactions, if there is a third generation of quarks. This proposal became part of the Standard Model (SM). The $3 \times 3$ quark mixing matrix can describe all flavor physics measurements to date, the crowning glory being the measurement of the fundamental CPV phase in modes such as $B^0 \rightarrow J/\psi K_S$ by the B factory experiments \[4\]. However, even the Nobel committee noted \[5\] that the KM phase is insufficient for the Sakharov conditions, typically by a factor of $10^{-10}$ or worse.

The numerics for the $10^{-10}$ factor can be most easily seen by a dimensional analysis of the so-called Jarlskog invariant CPV measure \[6, 7\] of the 3-generation Standard Model (SM3),

$$J = (m_i^2 - m_j^2)(m_k^2 - m_l^2)(m_m^2 - m_n^2)(m_p^2 - m_q^2)(m_r^2 - m_s^2) \Delta A,$$  \hspace{1cm} (1)

and comparing with the baryon-to-photon ratio $n_B/\gamma$ of our Universe, which is of order $10^{-9}$. In Eq. (1), $A \approx 3 \times 10^{-5}$ \[4\] is the area of, e.g. the triangle formed by the three sides of the unitarity relation $V_{ud}V_{ub} + V_{cd}V_{cb} + V_{td}V_{tb} = 0$ of SM3. Note that all possible triangle areas have the same value in SM3. Since $J$ has 12 mass dimensions, normalizing by $\gamma$ the vacuum expectation value of electroweak symmetry breaking (EWSB), or more trivially the electroweak phase transition temperature $T_{EW} \approx 100$ GeV, one finds $J/T_{EW}^4 \lesssim 10^{-20}$ falls short of $n_B/\gamma$, by at least $10^{-10}$.

The main suppression factor of $J$ come mainly from the small masses, $m_i^2 m_j^2 m_k^2 (T_{EW}^3)$, rather than from $A$. Noting this, one of us had suggested \[8\] that, if one had an extra generation of quarks, i.e. 4-generation Standard Model (SM4), then an analog to $J$ in SM3,

$$J_{(2, 3, 4)}^{b} = (m_i^2 - m_j^2)(m_k^2 - m_l^2)(m_m^2 - m_n^2)(m_p^2 - m_q^2)(m_r^2 - m_s^2)A_{234}^{b},$$ \hspace{1cm} (2)

would be enhanced by $\sim 15$ orders of magnitude with respect to $J$. The staggering enhancement is brought about by the heavy $t'$, $b'$ quark masses, which are taken to be in the range of 300 to 600 GeV. A phenomenological analysis \[9\] for $A_{234}^{b} = \text{Im} \left( [V_{ts}V_{tb}^*]^3V_{tb}^2 \right)$ has been taken into account. It was further argued \[8\] that the proximity to the $m_d \approx m_s \approx 0$ degeneracy limit \[7\] on the scale $\nu$ implies an almost 3-generation world involving 2-3-4 generation quarks, in which Eq. (2) is indeed the leading term. Unlike the case for SM3, this large enhancement likely allows SM4 to provide sufficient CPV for the matter or baryon asymmetry of the Universe (BAU), which provides strong support for the possible existence of a 4th generation. The scenario can be directly searched for quite definitely \[10\] at the LHC. In fact, because of recent experimental activities at the Tevatron, be it direct search for the $t'$ \[11\] or $b'$ \[12\] quarks, or CPV studies in $B_s \rightarrow J/\psi \phi$ \[13, 14\] or recent hint of dimuon asymmetry \[15\], interest in the 4th generation has been steadily growing \[16, 17\]. Our interest here, however, is the more fundamental.

It may still be questioned whether the analogy of Eq. (2) and Eq. (1) between SM4 and SM3 covers the whole truth. Indeed, the expression of invariants probing the CP nonconservation become much more complicated when one adds a fourth generation, and one should inspect all the invariant quantities in SM4 more carefully. In the following section, we first review the various discussions of necessary and sufficient conditions for CPV. We then invoke the “natural ordering” — the apparent hierarchy of mass-mixing parameters in the quark sector — to make a small mass expansion. The fact that mixing angles involving the 4th generation cannot be large is less useful, since the pattern of mixing angles (e.g. $|V_{ts}|$ vs $|V_{tb}|$) is less clear at the moment, precisely because of the recent hints for possibly large CPV effects in $b \rightarrow s$ transitions. However, we are able to identify, from the phenomenological indication that $b \rightarrow d$ transitions appear consistent with SM3, the condition that simplifies the Jarlskog invariants further, and confirm that, in our world, Eq. (2) is indeed (close to) the leading effect for CPV. In the above process, we are also able to find the next-to-leading terms. We offer some discussion on the approximations made, before giving our conclusion. More tedious algebra and a discussion on the relations between triangle areas are given as appendices.

II. CONDITIONS FOR CP CONSERVATION

Many studies has been made on the necessary and sufficient conditions for CP conservation with 3 and 4 (or more) generations. In SM3, which is a very special case, we have only one condition for CP conservation:

$$\frac{1}{6} \text{Im tr} [S, S^*]^3 = - \text{Im tr} (S^2 S^* S^2 S^2) = J(1, 2, 3) = v(1, 2, 3) v'(1, 2, 3) A = 0,$$ \hspace{1cm} (3)

where $S^{(i)}$ is the up(down)-type Hermitian squared mass matrix, defined as $S = M M^\dagger$ with $M$ the quark mass matrix. All the primed symbols hereafter denote down-type quantities. In Eq. (3), $v$ is the Vandermonde determinant.
of squared masses,

\[ v(\alpha, \beta, \gamma) = (m_\alpha^2 - m_\beta^2)(m_\beta^2 - m_\gamma^2)(m_\gamma^2 - m_\alpha^2), \]

and \( J(1, 2, 3) \) (which is identical to \( J \) in Eq. (1)) is the Jarlskog invariant,

\[ J(\alpha, \beta, \gamma) = v(\alpha, \beta, \gamma) \text{Im tr} (P_\alpha S^a P_\beta S^b P_\gamma S^c), \]

where \( P_\alpha \) is the projection operator for the indicated flavor, \( S P_\alpha = m_\alpha^2 P_\alpha \).

That the number of conditions for \( CP \) conservation in SM3 is exactly one reflects the unique \( CP \)V phase in the quark mixing matrix. With fourth generations, we have two more CKM phases, hence more conditions are needed for \( CP \) conservation. The number of conditions may, however, be larger than 3 due to the complexity of, and interdependency between, invariants. For instance, Botella and Chau \[13\] showed that there are nine independent triangle areas in SM4, rather than the single area in SM3, and \( CP \) is conserved \emph{if and only if} all nine areas vanish. Note that these triangles are not “unitary triangles,” since in SM4 the unitarity relations give quadrangles. However, every two sides of these quadrangles still form triangles, and we refer to these triangles as “CKM triangles.”

\[ A_{d_1d_2}^{u_1u_2} = \text{Im tr} [(V_{u_1d_1} V_{u_1d_1}^* V_{u_2d_2} V_{u_2d_2}^*)] = 0, \quad \forall u_1 \neq u_2, d_1 \neq d_2, \]

is defined quite in the same way with the conventions in SM3. One can see the number of total possible triangles is \((C_2^4)^2 = (4!/2!)^2 = 36\), but the unitarity conditions reduce this number to \((C_2^4)^2 = (3!/2!)^2 = 9\) (the corresponding numbers for SM4 are therefore 9 and 1, respectively). These CKM triangles, though rephasing invariant, may not be fully independent of each other (see Appendix B for some discussion). Furthermore, quark masses do not appear explicitly, although we know that \( CP \)V would vanish under certain mass degeneracy conditions.

Eq. (5), which gives the Jarlskog invariant \( CP \)V measure for SM3 as in Eq. (11), is of course invariant under any change of flavor basis. Extending to SM4, Jarlskog showed \[7\] that it is the sum over four Jarlskog invariants of the form in Eq. (5), or three-cycles, that is

\[ \text{Im tr} (S^2 S^t S^2 S^t) = J(2, 3, 4) + J(1, 3, 4) + J(1, 2, 4) + J(1, 2, 3). \]

The Jarlskog proposal is that one would have \( CP \) conservation \emph{if and only if} all four invariants vanish. This proposal shows a transparent analogy between SM3 and SM4.

There are other basis-independent approaches, however, to the conditions for \( CP \) conservation. Gronau, Kfir, and Loewy (GKL) introduced \[14\] 5 more invariants in addition to Eq. (5), and proposed that \( CP \) is conserved in SM4 \emph{if and only if} all six invariants vanish,

\[ \text{Im tr} (S^2 S^t S^2 S^t) = \text{Im tr} (S^2 S^t S^2 S^t) = \text{Im tr} (S^2 S^2 S^2 S^2) = \text{Im tr} (S^2 S^2 S^2) = 0. \]

Whether these two sets of conditions are really sufficient for \( CP \) conservation has been debated \[20, 21\]. What is certain is that, if \( CP \) is violated, some of these quantities would be nonzero. Both sides do agree that the two sets of conditions are equivalent if there is no vanishing element in the quark mixing matrix \( V \) in SM4.

The pragmatic question is how these quantities appear in process amplitudes that give rise to observable \( CP \) violation. In SM3 we know that the Jarlskog invariant enters various \( CP \)V measures. It further encodes the notion that, if any two like-charge quarks are degenerate in mass, or if the CKM triangle area vanishes, there would be no \( CP \) violation. Taking this as a hint, it is clear that the CKM triangle areas should enter various \( CP \)V measures, together with some mass difference factors, so the GKL and Jarlskog invariants should play a role in these measures. The generation labels of the CKM triangles in the measure should tell us what are the related processes for the search of \( CP \) violation.

Note that the GKL and Jarlskog invariants are basis-independent, and thus more likely to appear in physical quantities.

The invariants in Eqs. (5) and (8) are, however, rather complicated, and it is not apparent how the fourth generation effect on \( CP \) violation emerges. In contrast, the suggested leading effect of Eq. (2) is much more intuitive, and rather similar to the SM3 result of Eq. (11), but with the emphasis placed clearly on \( b \to s \) transitions. We should try to express the invariants, whether the GKL \[19\] or Jarlskog \[7\] kind, in terms of mass difference factors and CKM triangle areas, just like in SM3, and then identify the leading terms by considering the physical limits, such as physical quark masses. An extensive work that expands the GKL invariants into the nine CKM triangles \[18\] has been done in Ref. \[22\], but no comparison between different terms were made. For example, one has,

\[ \text{Im tr} (S^2 S^t S^2 S^t) = \sum v(1, i, j)v'(1, a, b)A_{ij}^a, \]

(9)
where the summation is over \((i, j)\) and \((a, b) \in \{(2, 3), (3, 4), (4, 2)\}\), with similar expansions for the other invariants in Eq. (8). Comparing different GKL invariants would be less meaningful, however, as there are several different mass dimensions. Even if they are shown to enter some CPV measure, it would be difficult to tell whether the fourth generation enhances the CP violation or not. On the other hand, all the Jarlskog invariants have the same dimension, and the general form of Eq. (8) is maintained as one extends from SM3 to SM4.

In the following, we will express the Jarlskog invariants in terms of the nine CKM triangles, then identify the leading terms in these invariants, by adopting proper physical limits. We find that the suggestion of Eq. (2) is indeed the leading term, but various next-to-leading terms are only smaller by roughly a factor of 10.

### III. SMALL MASS AND ANGLE EXPANSION OF JARLSKOG INVARIANTS

#### A. Jarlskog Invariants and CKM triangles

Let us express the Jarlskog invariants of Eq. (3) in terms of the nine CKM triangles of Eq. (9). For convenience, we choose to decompose the down-type Jarlskog invariants into CKM triangle areas. The up-type relations can be obtained analogously. Since the invariants can be evaluated in any basis, we are allowed to choose \(S'\) to be diagonal and write \(S = V^\dagger M V\), where \(V\) is the familiar quark mixing matrix. The Jarlskog invariants become

\[
J'(a, b, c) = -v'(a, b, c) \text{Im tr} \left( P'_a S'_b S'_c S' \right) = -v'(a, b, c) \text{Im} \left( \tilde{S}_{ab} \tilde{S}_{bc} \tilde{S}_{ca} \right)
\]

where the indices \(a, b, c\) are chosen all differently within \(\{1, 2, 3, 4\}\), but not summed. With an eye towards Eq. (2), we use the unitarity relation

\[
V_{ia} V_{ib} = -V_{2a} V_{2b} - V_{3a} V_{3b} - V_{4a} V_{4b} + \delta_{ab},
\]

to eliminate all 1’s for the indices that are summed over. We factor out any real factor that appears in \(\text{Im} \left( \cdots \right)\) of Eq. (10), and the remaining four CKM matrix elements then form a CKM triangle area.

After some algebra, which is rendered to Appendix A, the four down-type Jarlskog invariants are reduced to

\[
\begin{align*}
J'(2, 3, 4) &= -v'(2, 3, 4) \Lambda_{234}, \\
J'(1, 3, 4) &= v'(1, 3, 4) \left[ \Lambda_{234} - v(1, 3, 4) A_{234} - v(1, 2, 3) A_{23} - v(1, 4, 2) A_{42} - v(1, 4, 2) A_{42}^* \right], \\
J'(1, 2, 3) &= v'(1, 2, 3) \left[ \Lambda_{234} - v(1, 3, 4) A_{23} - v(1, 2, 3) A_{23}^* - v(1, 4, 2) A_{42} - v(1, 4, 2) A_{42}^* \right], \\
J'(1, 4, 2) &= v'(1, 4, 2) \left[ \Lambda_{234} - v(1, 3, 4) A_{23} - v(1, 2, 3) A_{23}^* - v(1, 4, 2) A_{42} - v(1, 4, 2) A_{42}^* \right],
\end{align*}
\]

where

\[
\Lambda_{234} = \left[ (m_2^2 - m_1^2)(m_3^2 - m_1^2) \right] \left[ \left( m_1^2 - m_2^2 \right) \left( |V_{22}|^2 A_{34}^2 + |V_{32}|^2 A_{23}^2 \right) \right.
\]

\[
\left. -\left( m_1^2 - m_3^2 \right) \left( |V_{32}|^2 A_{34}^2 + |V_{42}|^2 A_{43}^2 \right) \right]
\]

\[
+ \left( m_2^2 - m_3^2 \right) \left( m_4^2 - m_3^2 \right) \left[ \left( m_2^2 - m_3^2 \right) \left( |V_{22}|^2 A_{43}^2 + |V_{34}|^2 A_{43}^2 \right) \right.
\]

\[
\left. -\left( m_2^2 - m_4^2 \right) \left( |V_{34}|^2 A_{43}^2 + |V_{44}|^2 A_{43}^2 \right) \right]
\]

\[
+ \left( m_3^2 - m_4^2 \right) \left( m_4^2 - m_1^2 \right) \left[ \left( m_3^2 - m_4^2 \right) \left( |V_{22}|^2 A_{43}^2 + |V_{34}|^2 A_{43}^2 \right) \right.
\]

\[
\left. -\left( m_3^2 - m_2^2 \right) \left( |V_{34}|^2 A_{43}^2 + |V_{44}|^2 A_{43}^2 \right) \right].
\]

This rather compact form, though still quite complicated, should be compared with the many terms of Ref. [22], obtained by expanding the 6 GKL invariants [18] of Eq. (8) in terms of the 9 CKM triangle areas [18] of Eq. (9). We note again that the GKL invariants are of several different mass dimensions, and it is not easy to compare the relative importance of the numerous possible terms. In contrast, all four Jarlskog invariants have the same mass dimension, hence are more readily compared with one another. As remarked already, for SM4 with no vanishing CKM mixing matrix elements, GKL and Jarlskog approaches are equivalent, but the latter is clearly more convenient for our purpose.
B. Small Mass and Angle Expansions

One of our goals is to identify the leading effects in the Jarlskog invariants. We now depart from generality by noting the fact of a clear hierarchy of physical quark masses in Nature, namely $m_d^2 \gg m_u^2 \gg m_s^2 \gg m_b^2$. This implies that the last two factors in Eq. (12) are much more suppressed than the first two. Likewise, the up-type hierarchy $m_u^2 < m_c^2 < m_t^2$ further suppresses the first and third terms in $A_{234}$, as well as the terms with factors $v(1, 2, 3)$ and $v(1, 2, 4)$ in $J'(1, 3, 4)$. Dropping these $m_u^2$ and $m_c^2$ suppressed terms, Eqs. (12) and (13) become

$$J'(2, 3, 4) \simeq -(m_d^2 - m_u^2)(m_c^2 - m_t^2)(m_u^2 - m_s^2)A_{234},$$

$$J'(1, 3, 4) \simeq (m_u^2 - m_d^2)(m_c^2 - m_t^2)(m_u^2 - m_s^2) \left[ A_{234} - (m_d^2 - m_u^2)(m_u^2 - m_t^2)(m_u^2 - m_s^2)A_{sb}^{tt'} \right].$$

where we have returned all indices to physical labels, and kept the small suppressed masses in the explicit mass differences in these remaining terms, for sake of correspondence with Eq. (2).

As there are still quite a few terms in Eq. (15), we expand further in the strength of $|V_{ij}|^2$. Phenomenologically, we now know (16) that the rotation angles in the CKM matrix are small, i.e. $|V_{ij}|^2 \ll |V_{kk}|^2 \sim 1$, which holds not only in SM3, but seems to extend into SM4 as well. The Cabibbo angle appears to be the largest rotation angle, while $|V_{ub}|$ cannot be much different from the SM3 value of $\approx 0.04$, and $|V_{ts}|^2$ and $|V_{tb}|^2$ should be of order $10^{-2}$ or less (17).

Assuming small rotation angles, we drop the off-diagonal $|V_{ij}|^2$ terms in $A_{234}$, and get

$$J'(2, 3, 4) \sim -(m_t^2 - m_d^2)(m_c^2 - m_t^2)(m_d^2 - m_s^2)(m_u^2 - m_s^2)$$

$$\times \left[ (m_t^2 - m_c^2)|V_{tb}|^2 A_{bb'}^{tt'} + (m_d^2 - m_t^2)|V_{tb'}|^2 A_{bb'}^{tt'} \right],$$

$$J'(1, 3, 4) \sim (m_c^2 - m_u^2)(m_c^2 - m_t^2)(m_u^2 - m_s^2)(m_u^2 - m_s^2)$$

$$\times \left[ (m_u^2 - m_c^2)|V_{ts}|^2 A_{bb'}^{tt'} + (m_u^2 - m_t^2)|V_{ts'}|^2 A_{bb'}^{tt'} - (m_d^2 - m_t^2)A_{sb'}^{tt'} \right].$$

If one looks at the order of magnitude of quark masses, Eq. (16) is similar to Eq. (2). But they are not exactly the same: more than one CKM area carry the heaviest mass factor $m_t^2m_u^2m_d^2m_b^2$, and the masses of $u, d$ quarks also enter the expression.

In Eqs. (16) and (17), we have used “$\sim$” rather than “$\simeq$”, because we have treated the CKM triangle areas as “free parameters” while dropping the off-diagonal $|V_{ij}|^2$ terms. These areas are characterized not only by the strength of CKM matrix elements, but also their relative phases, which makes clear that these two equations are for illustration purpose only. Note that we have not assumed further hierarchical structure in the mixing elements. This is because of the possible indication of large CPV effect involving $b \to s$ transitions, hence we do not know whether the hierarchy structure of $|V_{ub}|^2 \ll |V_{cb}|^2 \ll |V_{us}|^2 \ll 1$ would extend to elements involving the 4th generation. We remark that, without assuming further structure in the CKM elements, unlike the application of mass hierarchies that lead to Eqs. (14) and (15), had we applied $|V_{ij}|^2 \ll |V_{kk}|^2 \sim 1$ (where $i \neq j$) first to Eqs. (12) and (13), not much of Eq. (15) would have been gained, and there would still be four Jarlskog invariants.

What CKM angle pattern could be noteworthy for further simplifications?

C. Leading Effect of Jarlskog Invariants

For our purpose of finding the leading effect of CPV, what pattern of small off-diagonal elements in the CKM matrix could provide additional approximate relations between triangle areas as defined in Eq. (15)? Taking note of the three specific triangle areas that enter Eq. (15), we note that the unitarity triangle

$$V_{td}V_{td}^* + V_{ts}V_{ts}^* + V_{tb}V_{tb}^* + V_{tb'}V_{tb'}^* = 0,$$

a special case of our starting Eq. (11), would approach a triangle, if $|V_{td}V_{td}^*|$ is small compared to the other terms. If this is the case, then any two of the other three sides would form the same triangle area, i.e.

$$A_{sb}^{tt'} \simeq A_{sb'}^{tt'} \simeq A_{bb'}^{tt'}, \quad \text{for } |V_{td}V_{td}^*| \ll 1,$$
which relates the three CKM triangles appearing in $A_{234}$ of Eq. (15). Applying Eq. (19) to Eq. (15), we get

$$A_{234} \simeq (m_{t'}^2 - m_{u}^2)(m_{t}^2 - m_{b}^2)[1 - (m_{t}^2 - m_{c}^2)(1 - |V_{td}|^2)A_{tb}^{tt'} + (m_{t'}^2 - m_{c}^2)(1 - |V_{td}|^2)A_{tb}^{tt'}],$$

where we have assumed the smallness of both $|V_{td}|^2$ and $|V_{td}|^2$ in the second step, which is somewhat stronger than what is needed for Eq. (19) to hold. Substituting Eq. (20) into Eq. (14), we obtain

$$J'(2, 3, 4) \simeq -(m_{b'}^2 - m_{s}^2)(m_{b}^2 - m_{s}^2)(m_{t}^2 - m_{c}^2)(m_{t'}^2 - m_{c}^2)(m_{t}^2 - m_{b}^2)(m_{t'}^2 - m_{b}^2)A_{tb}^{tt'},$$

$$J'(1, 3, 4) \simeq 0 + \text{subleading terms.}$$

With $A_{234}^{tt'} = A_{tb}^{tt'}$, $J'(2, 3, 4)$ in Eq. (21) is the same as $J_{(2,3,4)}^{tt'}$ of Eq. (2), except that $m_{s}^2$ is replaced by $m_{s}^2$, which makes little difference since $m_{s}^2$ and $m_{s}^2$ are negligible compared with $m_{t}^2$ and $m_{t}^2$. What seems a little curious is that the starting point of Eq. (18) is more relevant for $t \rightarrow t$ transitions, but it leads to the result in Eq. (21) (through Eq. (19)), which seems more relevant to $b \rightarrow s$ transitions.

In Ref. [22], the authors also showed that $A_{tb}^{tt'}$ and $A_{tb}^{tt'}$ have similar area through a phenomenological study. Such a study started [9] from the hint of New Physics CPV in $b \rightarrow s$ transitions, while $b \rightarrow d$ transitions (including $B_d$ mixing-dependent CPV) mimic SM3, as first noted by Ref. [24]. If we take the approximation in Eq. (19), it implies that $|V_{ts}|$ and $|V_{ts}|$ are stronger than the counterparts involving $d$, which implies interesting CPV effects in $b \rightarrow s$ processes, including in $B \rightarrow J/\psi \phi$ [9,14,25]. This is precisely where we are finding several experimental hints [13,14]! We have thus clarified the phenomenological link and reasoning behind Eq. (21), which echoes quite well those given in Ref. [8] for Eq. (2), but in a more hand-waving way.

One may question whether our choice to eliminate the index “$t'$” via Eq. (11) can keep its generality, once we introduce the hierarchy of quark masses. To check this, we note that one could apply small mass expansion to Eq. (10) directly and obtain the same results without passing any algebra, or even using Eq. (11). Taking Eq. (2) as a guide, by collecting terms with factor $m_{t}^4 m_{s}^2$ in the summation in $J'(2, 3, 4)$ in Eq. (10), one has

$$J'(2, 3, 4) \sim m_{t}^4 m_{t}^2 m_{s}^2 m_{b}^2 |V_{t'b'}|^2 A_{tb}^{tt'} - |V_{t'b'}|^2 A_{tb}^{tt'} + |V_{t's}|^2 A_{tb}^{tt'},$$

$$J'(1, 3, 4) \sim -m_{t}^4 m_{t}^2 m_{s}^2 m_{b}^2 |V_{t'b'}|^2 A_{tb}^{tt'} - |V_{t'b'}|^2 A_{tb}^{tt'} + (|V_{ts}|^2 - 1)A_{tb}^{tt'},$$

which is exactly what we have in Eqs. (14) and (15) when neglecting any terms of order equal to or smaller than $m_{t}^2 m_{c}^2$. If we invoke $|V_{td}|$ to be much smaller than the other three elements, and Eq. (19) is satisfied, then we again get a formula for $J'(2, 3, 4)$ that is in line with Eq. (21), while $J'(1, 3, 4)$ cancels away as in Eq. (22). One may then question the utility of the algebra, which constitutes the bulk of the paper. Note that Eq. (23) does not satisfy the requirements for the $t \rightarrow t'$ degeneracy limit. This can be remedied by collecting the $m_{t}^2 m_{s}^2$ terms as well. But if one wishes to explore other subleading effects, then Eq. (23) offers no guidance, and to explore these, one might as well resort to Eqs. (12) and (13). As we will show in Discussion below, it is possible, through the structure of the CKM matrix, that Eq. (21) (hence Eq. (23)) is in fact absent. Thus, Eqs. (12) and (13) offer the general starting point, independent of the quark mass hierarchy. It provided an easy way to evaluate the leading effects with the hierarchy taken into account, and can always be used in discussing special CKM structures. Our formulas show the complete mass factors in front of each CKM triangle area, which all have the form of difference of mass squares. This feature allows us to explore some more general cases, as we will discuss in the next section.

**IV. DISCUSSION**

At the end of the previous section, we have seen the implications for very small $|V_{td}|$ and $|V_{td}|$ and quark mass hierarchy. Only one of the four Jarlskog invariants, $J'(2,3,4)$ of Eq. (21), remains nonzero, while the other three are subleading, and hence could in principle vanish if we have exact $|V_{td}| = |V_{td}| = 0$ and $m_{u} = m_{c}$. In this case, it would be an effective three-generation world, where the first generation decouples from the other three heavier generations, when taking the extra freedom in $|V_{td}|$ provided by $u-c$ degeneracy. This seems to be in contrast to the assertion by Jarlskog in Ref. [7] that when three of these invariants vanish exactly, the fourth would also vanish. However, as Jarlskog mentioned, this assertion is not valid in the present case. In fact, this assertion is not valid whenever one generation decouples from the other three generations (zeros in the CKM matrix).

Using the small mass expansion is quite different from taking mass degeneracy limits mathematically. If one has mass degeneracy, extra freedom in the quark mixing matrix must be taken into account. In Ref. [7], the author also
One sees that if the fourth generation does not totally decouple from the other three, it will leave its fingerprint on the mass-squared difference appears in the denominator in defining the projection operators, and the second equality cannot apply in the $t$-$t'$ degeneracy limit. To address this issue, rather than flipping the definition of primed versus unprimed objects, we could inspect the behavior of $b$-$b'$ degeneracy limit instead of $t$-$t'$. One immediately sees that, if one maintains $m_d - m_u > m_s - m_d$ while letting $m_s \to 0$, then taking $b'$-$b$ degeneracy limit, all four Jarlskog invariants would properly vanish, hence the previous paradox is an artefact of choosing to decompose down-type Jarlskog invariants. But we then see that "$t$-$t'$ degeneracy" (that mimic true $b$-$b'$ degeneracy) indeed cannot be applied. We therefore gain an insight that, while massless degeneracy of the first two generations can simultaneously be applied for up and down type quarks (because of vanishing mass protection in reaching second equality of Eq. (10)), this is not so for the degeneracy of the massive 3rd and 4th generations.

In the previous section, we considered only the case with small $|V_{td}|$ and $|V_{td}|$. Now let us consider more scenarios when some of the elements in the CKM matrix are extremely small, leading to some vanishing triangle areas. First, let us consider the case where the fourth generation is totally decoupled from the first three generations. One then expects an effective three-generation theory, and the CPV effect should be the same as in SM3. Due to the decoupling, all rotation angles which link the fourth generation and lower generations are zero. That is,

$$V_{14} = V_{24} = V_{34} = V_{41} = V_{42} = V_{43} = 0,$$

and it follows that any $A_{d_1d_2}^{u_1u_2}$ that contains 4 in its label is zero. $A_{234}$ is also zero because every term in $A_{234}$ contains at least one zero factor. The four invariants then become

$$J'(2,3,4) = J'(1,3,4) = J'(1,4,2) = 0, \quad J'(1,2,3) = -v' (1,2,3)v(1,2,3) A_{234}^{12},$$

which is exactly what we have in SM3, and there is no CPV effect induced by fourth generation, as expected.

But if the fourth generation exists, it is hard to conceive that it decouples from all other generations. Consider the case where the fourth generation decouples from the first two generations, that is,

$$V_{14} = V_{24} = V_{41} = V_{42} = 0,$$

then the only non-vanishing triangle areas in Eq. (12) are $A_{33}^{23}$ and $A_{34}^{24}$. One can show from Eqs. (B1) and (B2) that one must have either $A_{33}^{23} = A_{34}^{24} = 0$ or $V_{34} = V_{43} = 0$. The first solution means there is no CPV at all, and the third generation also decouples from the first two generation, which contradicts experimental observation. The second solution, on the other hand, means that the fourth generation decouples also from the third generation, which is the previous scenario we have just discussed. This result shows that if the fourth generation does exist, it must either couple with at least 2 lower generations, or must fully decouple from all 3 lower generations.

However, even if the fourth generation is present and couples to all other generations, it is still possible that we have only one CPV phase. Consider, for instance, having $b'$ decoupled from $u$ and $c$, which gives $V_{14} = V_{24} = 0$. Then any triangle areas $A_{d_1d_2}^{u_1u_2}$ with a "4" in lower indices and an "1" or a "2" in upper indices will vanish. In addition, there are other triangle areas that would also vanish by using the unitarity condition,

$$A_{14,23}^{24} = -A_{14,23}^{24} = 0.$$  

The only non-vanishing triangle areas used in Eq. (12) are $A_{33}^{23}$, $A_{34}^{24}$, and $A_{34}^{24}$. But Eq. (B1) with $(u_1, u_2, u_3, u_4) = (1, 2, 3, 4)$ and $(1, 3, 2, 4)$ gives

$$(|V_{34}|^2 - |V_{44}|^2) A_{23}^{24} = 0, \quad |V_{34}|^2 A_{23}^{24} = -|V_{44}|^2 A_{23}^{24}.$$  

Provided that $|V_{34}|^2 \neq 0$ and $|V_{34}|^2 \neq |V_{44}|^2$, there is only one degree of freedom in triangle areas, hence only one CPV phase. All the Jarlskog invariants are then proportional to this area, and the leading effect is

$$J'(2,3,4) \sim -(m_u^2 - m_d^2)^2 (m_u^2 - m_d^2) (m_u^2 - m_s^2) (m_u^2 - m_c^2) |V_{ub}|^2 A_{sh}^{24},$$

$$J'(1,3,4) \sim (m_s^2 - m_d^2)^2 (m_s^2 - m_u^2) (m_s^2 - m_d^2) (m_s^2 - m_c^2) |V_{ub}|^2 A_{sh}^{24},$$

which is smaller than Eq. (10) due to the factor $m_s^2$, but it is still enhanced by $\sim 10^{10}$ when compared with Eq. (11). One sees that if the fourth generation does not totally decouple from the other three, it will leave its fingerprint on some CPV process(es).

For other possible scenarios, one can follow the same recipe we used. First, the dependence of triangle areas are determined by the relations in Eqs. (B1) and (B2). Then, inserting these relations into Eqs. (12) and (13), one can identify the leading effect in the corresponding scenario. Finally, one should note that it is very unlikely to have any exact zero in the quark mixing matrix from theoretical perspective, and certainly not experimentally either. These cases allow us to see the asymptotic behavior of the leading effect.
The formula in Eq. (2), as if involving just 2-3-4 generations in a 4-generation world, would be enhanced above the 3-generation Jarlskog invariant $J$ of Eq. (14) by an astounding $10^{15}$ or so. This is because the dependence on the small mass squared differences between the two lightest generations get replaced by heavier masses at the weak scale. The purpose of our study is to check to what extent Eq. (2) is the leading CPV effect in the 4-generation Standard Model.

We chose the more convenient starting point of Jarlskog’s extension to 4 invariants in SM4. As we always maintain physical finite values for quark masses and CKM mixing elements, this is equivalent in SM4 to the more complicated Gronau, Kfir and Loewy approach. Through algebraic manipulations, the more tedious of which are relegated to the appendices, we arrive at the general results of Eq. (12), which depend on an algebraic function $\Lambda_{234}$ defined in Eq. (13). With full generality, this does not offer too much insight. We then invoked the hierarchy of physical quark masses, notably for mixing-dependent CPV in $J/\psi K_S$ transitions seem to conform with 3 generation Standard Model. The resulting Eq. (21) largely confirms the suggestion of Eq. (2). In fact, one could have taken a much more efficient approach, for the purpose of identifying the leading effect, by making small mass expansion from the outset in Eq. (10), and arrive at Eq. (23). This retains all features of the proposed $J_{2,3,4}^{sb}$ in Eq. (2), keeping to $m_i^2/m_4^2$ order, as well as order of CKM elements, which could be as large as 0.1. We therefore see that though $J_{2,3,4}^{sb}$ in Eq. (2) does seem to be the leading term in the presence of quark mass hierarchies and small rotation angles — which is our world — there should be a myriad of subleading terms that are perhaps only 10 times smaller.

In the course of our study, we also uncovered the apparent phenomenological condition for $J_{2,3,4}^{sb}$ in Eq. (2) to be the leading term. Current data suggest that CPV in $b \rightarrow s$ transitions, notably for mixing-dependent CPV in $B_s \rightarrow J/\psi \phi$, could be sizable, despite the B factory confirmation of consistency with a 3-generation source for $b \rightarrow d$ transitions (notably for mixing-dependent CPV in $B_d \rightarrow J/\psi K_S$). Thus, $V_{td}$ and $V_{t'd}$ seem subdued compared with $V_{ts}$ and $V_{t's}$ in strength, respectively. In this case, we were able to derive Eq. (21) which is extremely close to Eq. (2), except for very minor differences. We thus conclude that, in general the claim of a large enhancement by 4th generation masses is true, although there would be several terms comparable to $J_{2,3,4}^{sb}$ in Eq. (2). If, however, we do discover sizable CPV effect in $B_s \rightarrow J/\psi \phi$ that is much enhanced over SM3 expectations, then indeed $J_{2,3,4}^{sb}$ of Eq. (2), or $J'(2,3,4)$ of Eq. (21), is the single leading term. But there would still be subleading terms that could be just an order of magnitude less in strength, depending on the strength of associated CKM elements.

### Appendix A: Some Algebra

We start from Eq. (10)

$$J'(a, b, c) = v'(a, b, c) \sum_{i,j,k=1}^{4} m_i^2 m_j^2 m_k^2 \text{Im} (V_{ia} V_{ib}^* V_{jb} V_{jc}^* V_{kc} V_{ka}).$$

(A1)

Replace every term which contains $m_i^2$ in the above summation by the unitarity condition, the summation becomes

$$\sum_{i,j,k=2}^{4} (m_i^2 - m_1^2)(m_j^2 - m_1^2)(m_k^2 - m_1^2) \text{Im} (V_{ia} V_{ib}^* V_{jb} V_{jc}^* V_{kc} V_{ka}),$$

(A2)

where now the sum is over all possible $i, j, k$ in the set $\{2, 3, 4\}$. Note that since $a, b, c$ are all different, there exists no term like $\delta_{aa} = 1$ in Eq. (A2).

We can apply the similar trick to the down-type indices of $V$. Consider the case $a = 1$ and $b, c$ are chosen differently
We note that the indices of \( A \) where we have defined the mass prefactor matrix elements \( \omega \) thus the four Jarlskog invariants in SM4 can be written as
\begin{equation}
\text{Im} (V_{i4} V_{i5} V_{j6} V_{kc} V_{k4} V_{k5}) = -\text{Im} (V_{i2} V_{i6} V_{j6} V_{jc} V_{kc} V_{k6}) - \text{Im} (V_{i3} V_{i6} V_{j6} V_{jc} V_{kc} V_{k3})
- \text{Im} (V_{i4} V_{i6} V_{j6} V_{kc} V_{k4} V_{k6}) + \text{Im} (V_{i6} V_{j6} V_{kc} V_{k6} V_{k6} \delta_{ik})
= -\text{Im} (V_{i4} V_{i6} V_{j6} V_{kc} V_{k6} V_{k6}) - \text{Im} (V_{i6} V_{j6} V_{kc} V_{k6} V_{k6} \delta_{ik})
- \text{Im} (V_{i4} V_{i6} V_{j6} V_{kc} V_{k6} V_{k6}) + \text{Im} (V_{i6} V_{j6} V_{kc} V_{k6} V_{k6} \delta_{ik})
+ |V_{ib}|^2 A_{dc}^j - |V_{kc}|^2 A_{dc}^i.
\end{equation}

where \( d \) is taken to be different from \( a, b, c \), and the second equality follows from replacing 2, 3, 4 by \( b, c, d \), by reordering the first three terms. Real factors are taken out in the third equality, and we also used
\begin{equation}
A_{be}^j = \text{Im} \left[ (V_{ib} V_{ie} V_{j6} V_{jc} V_{kc} V_{k6}) \right].
\end{equation}

Substituting Eqs. (A2) and (A3) back into Eq. (A1), we have
\begin{align}
J'(1, b, c) &= v'(1, b, c) \left\{ \sum_{i,j=2}^4 (m_i^2 - m_j^2)^2 (m_k^2 - m_l^2) A_{bc}^{ij} 
- \sum_{i,j,k=2}^4 (m_i^2 - m_j^2)(m_k^2 - m_l^2)(m_k^2 - m_l^2) \text{Im} (V_{id} V_{ib} V_{jc} V_{kc} V_{kd}) 
+ \sum_{i,j,k=2}^4 (m_i^2 - m_j^2)(m_k^2 - m_l^2)(m_k^2 - m_l^2) (|V_{ib}|^2 A_{dc}^j - |V_{kc}|^2 A_{dc}^i) \right\}.
\end{align}

The third summation would vanish, since the upper indices of \( A \) are anti-symmetric hence each component will cancel one another.
Define now
\begin{equation}
A_{bc} = \sum_{i,j,k=2}^4 (m_i^2 - m_j^2)(m_k^2 - m_l^2)(m_k^2 - m_l^2) \text{Im} (V_{id} V_{ib} V_{jc} V_{kc} V_{kd}).
\end{equation}

We note that the indices of \( A \) and \( \Lambda \) are antisymmetric, and hence we have
\begin{equation}
(m_i^2 - m_j^2)^2 (m_k^2 - m_l^2) A_{bc}^{ij} + (m_i^2 - m_j^2)^2 (m_k^2 - m_l^2) A_{bc}^{ji} = -v(1, i, j) A_{bc}^{ij}.
\end{equation}

Thus the four Jarlskog invariants in SM4 can be written as
\begin{align}
J'(2, 3, 4) &= -v'(2, 3, 4) A_{234},
J'(1, 3, 4) &= v'(1, 3, 4) [A_{234} - v(1, 3, 4) A_{342} - v(1, 2, 3) A_{324} - v(1, 4, 2) A_{423}],
J'(1, 2, 3) &= v'(1, 2, 3) [A_{234} - v(1, 3, 4) A_{342} - v(1, 2, 3) A_{324} - v(1, 4, 2) A_{423}],
J'(1, 4, 2) &= v'(1, 4, 2) [A_{234} - v(1, 3, 4) A_{342} - v(1, 2, 3) A_{324} - v(1, 4, 2) A_{423}].
\end{align}

\( A_{234} \) can be expressed further in terms of triangle areas. From Eq. (A6), we consider the following quantity,
\begin{equation}
-(m_i^2 - m_j^2)(m_k^2 - m_l^2) \text{Im} (V_{ib} V_{ic} V_{jc} V_{jd} V_{kd} V_{kb}) = -M_{ijk}^l \omega_{bcd}.
\end{equation}

where we have defined the mass prefactor \( M_{ijk}^l \equiv (m_i^2 - m_j^2)(m_k^2 - m_l^2)(m_i^2 - m_j^2) \) and the imaginary part of six CKM matrix elements \( \omega_{bcd} = \text{Im} (V_{ib} V_{ic} V_{jc} V_{kd} V_{kb}) \). Note that \( \omega_{bcd} \) has the following properties,
\begin{align}
\sum_{i=2}^4 \omega_{bcd}^{ij} &= -\omega_{bcd}^{jik},
\omega_{bcd}^{ij} &= \omega_{cde}^{jki} = \omega_{bdc}^{kij},
\omega_{bcd}^{ij} &= 0,
\omega_{bcd}^{ij} &= |V_{ib}|^2 A_{cd}^{ij}.
\end{align}
\( \Lambda_{bcd} \) could be written as the following summation,

\[
\Lambda_{bcd} = \sum_{i,j,k=2}^{4} -M_{i}^{ijk} \omega_{bcd}^{ijk}.
\]  
(A11)

Note that from the second property of Eq. (A10), \( \Lambda_{bcd} = \Lambda_{dbc} = \Lambda_{cda} \), while the first property of Eq. (A10) implies,

\[
\sum_{i,j,k=2}^{4} \omega_{bcd}^{ijk} = \sum_{j,k=2}^{4} -\omega_{bcd}^{ijk} = \sum_{k=2}^{4} \omega_{bcd}^{11k} = -\omega_{bcd}^{111} = 0.
\]  
(A12)

Combining Eqs. (A11) and (A12), we have

\[
\Lambda_{bcd} = \sum_{i,j,k=2}^{4} -(M_{1}^{ijk} - M_{1}^{234}) \omega_{bcd}^{ijk}.
\]  
(A13)

Since the upper indices of \( M \) are symmetric and \( \omega_{bcd}^{iii} = 0, (M_{1}^{ijk} - M_{1}^{234}) \omega_{bcd}^{ijk} = 0 \) if \( i, j, k \) are taken to be all different or all the same from \{2, 3, 4\}. Thus, Eq. (A13) could be written as

\[
\Lambda_{bcd} = [(M_{1}^{234} - M_{1}^{232})(\omega_{bcd}^{232} + \omega_{bcd}^{323} + \omega_{bcd}^{232}) + (M_{1}^{324} - M_{1}^{234})(\omega_{bcd}^{323} + \omega_{bcd}^{233} + \omega_{bcd}^{323})]
\]  
\[
+ \text{other two cyclic permutations of (2, 3, 4),}
\]  
(A14)

where we have used the symmetric property of the upper indices of \( M \) and also the second property of Eq. (A10). Using now the fourth property of Eq. (A10) and express terms like \( \omega_{bcd}^{232} \) in terms of triangle areas with real factors, \( \Lambda_{bcd} \) becomes

\[
\Lambda_{bcd} = \left[ (m_{2}^{2} - m_{1}^{2})(m_{3}^{2} - m_{1}^{2})(m_{4}^{2} - m_{1}^{2})(|V_{2b}|^{2} A_{cd}^{23} + |V_{2c}|^{2} A_{db}^{23} + |V_{2d}|^{2} A_{bc}^{23})
\]
\[
- (m_{2}^{2} - m_{1}^{2})(m_{3}^{2} - m_{1}^{2})(m_{4}^{2} - m_{1}^{2})(|V_{3d}|^{2} A_{cd}^{23} + |V_{3c}|^{2} A_{db}^{23} + |V_{3d}|^{2} A_{bc}^{23}) \right]
\]  
\[
+ \text{other two cyclic permutations of (2, 3, 4).}
\]  
(A15)

Substituting \( (2, 3, 4) \) for \( (b, c, d) \) one obtains Eq. (B3).

**Appendix B: Relations between Triangle Areas**

Since there are only three independent phases in the CKM matrix in SM4, one expects there exists some relations among the nine triangle areas. For example, one can obtain \( J'(1, 3, 4) \) directly from \( J'(2, 3, 4) \) by exchanging the indices 1 and 2, rather than using the approach we presented. In this case, one would get two different expressions for \( J'(1, 3, 4) \). By comparing the two, one would find non-trivial relations of some of the triangle areas.

There are in fact 6 relations, consisting of three up-type relations,

\[
\left[(|V_{u11}|^{2} - |V_{u22}|^{2}) A_{lm}^{u_{1}u_{2}} + (|V_{u33}|^{2} - |V_{u44}|^{2}) A_{lm}^{u_{3}u_{4}} \right] \text{ cyclic perm. of (l, m, n) = 0},
\]  
(B1)

and three down-type relations,

\[
\left[(|V_{d11}|^{2} - |V_{d22}|^{2}) A_{dl}^{d_{1}d_{2}} + (|V_{d33}|^{2} - |V_{d44}|^{2}) A_{dl}^{d_{3}d_{4}} \right] \text{ cyclic perm. of (l, m, n) = 0},
\]  
(B2)

where \( u_{1} \) to \( u_{4} \) (or \( d_{1} \) to \( d_{4} \) for down-type) are taken all differently from \{1, 2, 3, 4\}, and \( l, m, n \) are also taken all differently from \{1, 2, 3, 4\}. Different choices of \( (l, m, n) \) would in fact give equivalent relations, so throughout this appendix we will regard \( (l, m, n) \) as given labels, say \( (2, 3, 4) \), without loss of generality.

Let us present a direct proof of these 6 relations. First, we consider the up-type relations. We define the LHS of Eq. (B1) to be

\[
\mathcal{R}_{u_{1}u_{2}u_{3}u_{4}} = -\text{Im} \left[ (V_{u11} V_{u22}^{*} V_{u33} V_{u44}^{*}) - (V_{u22} V_{u33}^{*} V_{u44} V_{u11}^{*}) - (V_{u33} V_{u44}^{*} V_{u11} V_{u11}^{*}) \right] + \text{other two cyclic permutations of (l, m, n),}
\]  
(B3)
where we put back the real factors into Im (\cdots). With the unitarity condition, we can replace $V_{u_1n}V_{u_1l}^*$ by $-V_{u_2n}V_{u_2l}^* - V_{u_3n}V_{u_3l}^* - V_{u_4n}V_{u_4l}^*$, and substitute this into Eq. (B3). For example, the first term in Eq. (B3) would become

$$-\text{Im} (V_{u_1n}V_{u_1l}^*V_{u_1m}V_{u_1m}V_{u_1m}V_{u_1m}V_{u_1m}V_{u_1m}V_{u_1m}V_{u_1m}) = |V_{u_2l}|^2 \text{Im} (V_{u_1n}V_{u_1m}V_{u_2m}V_{u_2n})$$

$$+ \text{Im} (V_{u_3l}V_{u_3m}V_{u_4m}V_{u_2n}) + \text{Im} (V_{u_4l}V_{u_4m}V_{u_2m}V_{u_2n})$$

$$= |V_{u_2l}|^2 A_{mn}^l + \omega_{imn}^l + \omega_{imn}^l,$$

where $\omega_{imn}^l$ is defined in Eq. (A9). Applying the unitarity condition to every term in Eq. (B3), we have

$$\mathcal{R}_{u_1u_2u_3u_4} = \left[ |V_{u_2l}|^2 A_{mn}^l - |V_{u_1l}|^2 A_{mn}^l + |V_{u_3l}|^2 A_{mn}^l - |V_{u_4l}|^2 A_{mn}^l \right]$$

$$+ \omega_{imn}^l + \omega_{imn}^l + \omega_{imn}^l + \omega_{imn}^l + \omega_{imn}^l + \omega_{imn}^l$$

$$+ \text{other two cyclic permutations of } (l, m, n).$$

The first line in the brackets in the RHS together with the cyclic permutations amounts to $-\mathcal{R}_{u_1u_2u_3u_4}$. And due to the second property of Eq. (A10), the cyclic permutations on $\omega$’s lower labels can be moved to its upper labels when all cyclic permutations are summed, so the left 24 $\omega$’s can be written as $\sum_{(i,j,k)} \omega_{imn}^{ijk}$ where $(i, j, k)'s$ are taken all differently from $\{u_1, u_2, u_3, u_4\}$, which is equivalent to the set $\{1, 2, 3, 4\}$. So now we have

$$2\mathcal{R}_{u_1u_2u_3u_4} = \sum_{(i,j,k)'} \omega_{imn}^{ijk}$$

$$= \sum_{i,j,k=1}^{4} \omega_{imn}^{ijk} - \sum_{i,j=1}^{4} (\omega_{imn}^{ij} + \omega_{imn}^{ji}).$$

In the second equality, we allow all possible $i, j, k$ in the summation and subtract back the extra terms. However, from the first property of Eq. (A10), $\sum_{a=1}^{4} \omega_{imn}^{abc} = 0$, so all terms in the RHS of Eq. (B6) vanish when summing over $j$ from 1 to 4. Finally we have

$$\mathcal{R}_{u_1u_2u_3u_4} = 0,$$

which proves Eq. (B7). The down-type relations Eq. (B2) can be derived similarly.

As mentioned in [13], in SM4 there are nine independent CKM triangle areas, but the relations shown in this appendix seem to reduce the number of independent CKM triangle areas to at most three. There is in fact no conflict. Provided the magnitude of each CKM mixing matrix element is known, the degree of freedom of the CKM triangles can be further reduced to at most three, which equals the number of physical phases, as shown in this appendix. This result satisfies our intuition because one needs rotation angles and phases to describe the mixing matrix. So if one does not know the magnitude of matrix elements, the degree of freedom of triangle areas would be larger than the number of physical phases, but when those rotation angles are known, one should be able to express some CKM triangle areas in terms of others like what we proposed in Eqs. (B1) and (B2).

\begin{thebibliography}{99}
\bibitem{1} J.H. Christenson, J. W. Cronin, V.L. Fitch, and R. Turlay, Phys. Rev. Lett. 13, 138 (1964).
\bibitem{2} A.D. Sakharov, Pis'ma Zh. Eksp. Teor. Fiz. 5, 32 (1967) [JETP Lett. 5, 24 (1967)].
\bibitem{3} M. Kobayashi and T. Maskawa, Prog. Theor. Phys. 49, 652 (1973).
\bibitem{4} C. Amsler et al. (Particle Data Group), Phys. Lett. B667, 1 (2008), and 2009 partial update for the 2010 edition.
\bibitem{5} See http://nobelprize.org/nobel_prizes/physics/
\bibitem{6} C. Jarlskog, Phys. Rev. Lett. 55, 1039 (1985); Z. Phys. C 29, 491 (1985).
\bibitem{7} C. Jarlskog, Phys. Rev. D 36, 2128 (1987).
\bibitem{8} W.-S. Hou, Chin. J. Phys. 47, 134 (2009).
\bibitem{9} W.-S. Hou, M. Nagashima, and A. Soddu, Phys. Rev. Lett. 95, 141601 (2005).
\bibitem{10} A. Arhrib and W.-S. Hou, JHEP 0607, 009 (2006).
\bibitem{11} T. Aaltonen et al. [CDF Collaboration], Phys. Rev. Lett. 100, 161803 (2008); see CDF/PUB/TOP/PUBLIC/10110 (J. Conway et al.) for the latest result with 4.6 fb$^{-1}$ data.
\bibitem{12} T. Aaltonen et al. [CDF Collaboration], Phys. Rev. Lett. 104, 091801 (2010).
\end{thebibliography}
[13] T. Aaltonen et al. [CDF Collaboration], Phys. Rev. Lett. 100, 161802 (2008); V.M. Abazov et al. [D0 Collaboration], Phys. Rev. Lett. 101, 241801 (2008); the combined result is given by G. Punzi, arXiv:1001.4886 [hep-ex], presented at Europhysics Conference on High Energy Physics, Krakow, Poland, July 2009.

[14] Talk by L. Oakes (CDF Collaboration) at Flavor Physics and CP Violation conference, Torino, Italy, May 2010.

[15] V.M. Abazov et al. [D0 Collaboration], arXiv:1005.2757 [hep-ex].

[16] For a succinct recent summary, see B. Holdom, W.-S. Hou, T. Hurth, M.L. Mangano, S. Sultansoy and G. Ünel, PMC Phys. A 3, 4 (2009).

[17] For recent phenomenological work on the flavor and CPV side, see, for example, A. Soni, A.K. Alok, A. Giri, R. Mohanta and S. Nandi, arXiv:1002.0595 [hep-ph]; A.J. Buras, B. Duling, T. Feldmann, T. Heidsieck, C. Promberger and S. Recksiegel, arXiv:1002.2126 [hep-ph]; W.-S. Hou and C.-Y. Ma, arXiv:1004.2186 [hep-ph]; O. Eberhardt, A. Lenz and J. Rohrwild, arXiv:1005.3505 [hep-ph].

[18] F.J. Botella and L.-L. Chau, Phys. Lett. B 168, 97 (1986).
[19] M. Gronau, A. Kfir, and R. Loewy, Phys. Rev. Lett. 56, 1538 (1986).
[20] M. Gronau and R. Loewy, Phys. Rev. D 39, 986 (1989).
[21] C. Jarlskog, Phys. Rev. D 39, 988 (1989).
[22] F. del Aguila, J.A. Aguilar-Saavedra, Phys. Lett. B 386, 241 (1996).
[23] A. Arhrib and W.-S. Hou, Phys. Rev. D 80, 076005 (2009).
[24] W.-S. Hou, M. Nagashima, and A. Soddu, Phys. Rev. D 72, 115007 (2005).
[25] W.-S. Hou, M. Nagashima, and A. Soddu, Phys. Rev. D 76, 016004 (2007).