SPACETIME SINGULARITIES: RECENT DEVELOPMENTS

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Recent developments concerning oscillatory spacelike singularities in general relativity are taking place on two fronts. The first treats generic singularities in spatially homogeneous cosmology, most notably Bianchi types VIII and IX. The second deals with generic oscillatory singularities in inhomogeneous cosmologies, especially those with two commuting spacelike Killing vectors. This paper describes recent progress in these two areas: in the spatially homogeneous case focus is on mathematically rigorous results, while analytical and numerical results concerning generic behavior and so-called recurring spike formation are the main topic in the inhomogeneous case. Unifying themes are connections between asymptotic behavior, hierarchical structures, and solution generating techniques, which provide hints for a link between the nature of generic singularities and a hierarchy of hidden asymptotic symmetries.

Keywords: spacetime singularities, spikes, BKL, cosmology,

PACS numbers: 04.20.Dw, 04.20.Ha, 04.20.-q, 05.45.-a, 98.80.Jk

1. Introduction

This article describes and elaborates on the remarkable progress that has taken place during the last few years regarding the nature of generic spacelike singularities in general relativity. The developments have been taking place on two fronts: (i) the nature of generic singularities in spatially homogeneous (SH) Bianchi type VIII and IX models, and (ii) the nature of generic spacelike singularities in inhomogeneous models, especially in models with two commuting spacelike Killing vectors, the so-called $G_2$ models. To keep the account reasonably short, considerations is restricted to 4-dimensional vacuum spacetimes, which is a less restrictive assumption than one might initially think. In their work, Belinski, Khalatnikov and Lifshitz (BKL) provided heuristic evidence that some sources, like perfect fluids with sufficiently soft equations of state such as dust or radiation, lead to models that generically are asymptotically ‘vacuum dominated’, i.e., in the asymptotic approach to a generic spacelike singularity, the spacetime geometry is not influenced by the matter content, even though, e.g., the energy density blows up. However, arguably the most central, and controversial, assumption of BKL is their ‘locality’
conjecture. According to BKL, asymptotic dynamics toward a generic spacelike singularity in inhomogeneous cosmologies is ‘local,’ in the sense that each spatial point is assumed to evolve toward the singularity individually and independently of its neighbors as a spatially homogeneous model. It is no understatement to say that this conjecture has set the stage for much of subsequent investigations about the detailed nature of generic spacelike singularities.

A common unifying ingredient underlying recent progress at the two front lines is the recasting of Einstein’s field equations into scale invariant asymptotically regularized dynamical systems (first order systems of autonomous ordinary differential equations (ODEs) and partial differential equations (PDEs) in cases (i) and (ii), respectively) in the approach towards a generic spacelike singularity. One of the advantages of such dynamical systems, if appropriately defined, is that they lead to a state space picture with a hierarchy of invariant subsets, where simpler invariant subsets constitute boundaries of more complex ones. This hierarchical structure is especially relevant in the context of generic spacelike singularities since solutions of certain invariant ‘building block’ subsets at and near the ‘bottom’ of a hierarchy can be joined into ‘concatenated chains of solutions’ that describe the asymptotic evolution along time lines toward the singularity. In the case of inhomogeneous models this turns out to hold for ‘local BKL-like’ behavior, for which spatial derivatives can be neglected, as well as for ‘non-local recurring spike behavior’ along certain time lines where spatial derivatives cannot be neglected. Remarkably, all types of behavior are linked to the solutions at the lowest invariant subset level of the hierarchy by means of iterations of a solution generating algorithm, which in turn suggests that there exist asymptotic hidden symmetries, yet to be discovered.

The outline of the paper is as follows: section 2 focuses on recent mathematically rigorous results concerning vacuum Bianchi type VIII and IX spacetimes, while section 3 describes recent progress in the study of generic singularities in inhomogeneous $G_2$ spacetimes. The concluding section relates the material in these two sections to the context of generic singularities in general models without any symmetries. In particular links are discussed between the nature of generic singularities and the hierarchy of invariant subsets, which provide the building blocks for the asymptotic construction of generic ‘temporally oscillatory’ solutions toward generic spacelike singularities.

2. Bianchi type VIII and IX vacuum models

The type VIII and IX vacuum models belong to the ‘class A’ Bianchi vacuum models for which the metric can be written as

$$\begin{align*}
4g &= -dt \otimes dt + g_{11}(t) \omega^1 \otimes \omega^1 + g_{22}(t) \omega^2 \otimes \omega^2 + g_{33}(t) \omega^3 \otimes \omega^3, \\
\omega^1 &= -\hat{n}_1 \omega^2 \wedge \omega^3, \quad \omega^2 = -\hat{n}_2 \omega^3 \wedge \omega^1, \quad \omega^3 = -\hat{n}_3 \omega^1 \wedge \omega^2,
\end{align*}$$

where $\{\omega^1, \omega^2, \omega^3\}$ is a symmetry-adapted (co)frame and where the constant parameters $\hat{n}_1, \hat{n}_2, \hat{n}_3$ describe the structure constants of the different class A Bianchi
Much of the recent mathematically rigorous progress is based on the scale invariant ‘Hubble-normalized’ dynamical system formulation of Einstein’s field equations for the SH ‘diagonal’ class A Bianchi models, introduced by Wainwright and Hsu\(^2\) and generalized and elaborated on in the book “Dynamical Systems in Cosmology”\(^2\)\(^3\)\(^4\). In this approach scale invariant dimensionless variables are introduced by quotienting out the Hubble variable \(H\), which is related to the expansion \(\theta\) of the normal congruence of the SH symmetry surfaces and \(\text{tr} \, k\), the trace of the second fundamental form \(k_{\alpha \beta}\), according to \(H = \theta/3 = -\text{tr} \, k/3\). This yields the dimensionless variables\(^5\)

\[
\begin{align*}
\Sigma_1 &:= \frac{k_1}{\frac{1}{3} \text{tr} \, k}, \\
\Sigma_2 &:= \frac{k_2}{\frac{1}{3} \text{tr} \, k}, \\
\Sigma_3 &:= \frac{k_3}{\frac{1}{3} \text{tr} \, k}, \\
N_1 &:= -\frac{\hat{n}_1}{\frac{1}{3} \text{tr} \, k} \sqrt{\frac{g_{11}}{g_{22}g_{33}}}, \\
N_2 &:= -\frac{\hat{n}_2}{\frac{1}{3} \text{tr} \, k} \sqrt{\frac{g_{22}}{g_{33}g_{11}}}, \\
N_3 &:= -\frac{\hat{n}_3}{\frac{1}{3} \text{tr} \, k} \sqrt{\frac{g_{33}}{g_{11}g_{22}}},
\end{align*}
\]

and hence \(\Sigma_1 + \Sigma_2 + \Sigma_3 = 0\) (\(k_1 = -\frac{1}{2}g_{11}^{-1}dg_{11}/dt\), and similarly for \(k_2\) and \(k_3\))\(^\dagger\). In addition a new dimensionless time variable \(\tau\) is defined according to \(d\tau/ dt = \hat{H}\).

When the (vacuum) Einstein field equations are reformulated in terms of the dimensional Hubble variable \(H\) and the dimensionless ‘Hubble-normalized’ variables \((\Sigma_\alpha, N_\alpha)\) it follows from dimensional reasons that the equation for \(H\),

\[
H^\prime = -(1 + 2\Sigma^2)H,
\]

decouples from the remaining equations, which form the following coupled system of ODEs:\(^5\)

\[
\begin{align*}
\Sigma_\alpha' &= -2(1 - \Sigma^2)\Sigma_\alpha - \frac{1}{3} \left[ N_\alpha (2N_\alpha - N_\beta - N_\gamma) - (N_\beta - N_\gamma)^2 \right], \\
N_\alpha' &= 2(\Sigma^2 + \Sigma_\alpha) N_\alpha \\
&\text{(no sum over } \alpha), \\
1 &= \Sigma^2 + \frac{1}{12} \left[ N_1^2 + N_2^2 + N_3^2 - 2 (N_1 N_2 + N_2 N_3 + N_3 N_1) \right],
\end{align*}
\]

where \((\alpha \beta \gamma) \in \{(123), (231), (312)\}\) in (4a), and where a prime denotes the derivative \(d/d\tau\).

It follows from the Gauss constraint (4c) that \(H\) remains positive if \(H\) is positive initially for all vacuum class A models except type IX. In Bianchi type IX, however,

\(^{\dagger}\)The dynamical systems approach and methods used in Ref.\(^4\) has many precursors: notably work by Collins, Novikov, Bogoyavlensky, Rosquist, Jantzen, Wainwright, CU, Coley, and many more, for references, see Refs.\(^4\) and \(^6\).

\(^\dagger\)In Bianchi types VIII and IX there is a one-to-one correspondence between \(g_{11}, g_{22}, g_{33}\) and \(N_1, N_2, N_3\). For the lower Bianchi types I–VII\(_0\), some of the variables \(N_1, N_2, N_3\) are zero, cf. \(^2\); in this case, the other variables, i.e., \(H, \Sigma_\alpha\), are needed as well to reconstruct the metric; see Ref.\(^7\) for a group theoretical approach.

\(^4\)It is common to globally solve \(\Sigma_1 + \Sigma_2 + \Sigma_3 = 0\) by introducing new variables according to \(\Sigma_1 = -2\Sigma_+, \Sigma_2 = \Sigma_- - \sqrt{3}\Sigma_-, \Sigma_3 = \Sigma_+ + \sqrt{3}\Sigma_-, \) or a permutation thereof, which yields \(\Sigma^2 = \Sigma_+^2 + \Sigma_-^2\), but this unfortunately breaks the permutation symmetry in Bianchi type IX.
a theorem by Lin and Wald implies that all vacuum models first expand ($H > 0$), reach a point of maximum expansion ($H = 0$), and then re-collapse ($H < 0$). In this case the variables $(\Sigma_\alpha, N_\alpha)$ break down at the point of maximum expansion, but they give a correct description of the dynamics in the expanding phase which we will focus on henceforth.

Since $\Sigma_1 + \Sigma_2 + \Sigma_3 = 0$, equations (2) and (4) imply that the dimensionless state space of the Bianchi type IX and VIII vacuum models is 4-dimensional (all $N_\alpha$ are non-zero and have the same sign in type IX, while in type VIII one $N_\alpha$ has an opposite sign compared to the others, which is due to the signs of $\hat{n}_\alpha$, which determine the symmetry group type). Each lower Bianchi type than types IX and VIII in class A can be obtained by means of Lie contractions, i.e., by setting one, two, and eventually all three of the constants $\hat{n}_1, \hat{n}_2, \hat{n}_3$ to zero. The associated dimensionless state spaces are described by setting the corresponding $N_\alpha$ to zero (recall that $N_\alpha \propto \hat{n}_\alpha$), which corresponds to invariant boundary subsets of (4). The system (4) thus exhibits a hierarchical invariant boundary subset structure: In Bianchi types VII$_0$ and VI$_0$ two of the variables $N_\alpha$ are non-zero, and the dimensionless state space is therefore 3-dimensional; in Bianchi type II only one $N_\alpha$ is non-zero, which leads to a 2-dimensional state space, while in type I all $N_\alpha$ are zero, which hence results in a 1-dimensional state space.

The hierarchical ‘Lie contraction boundary subset structure’ of (4) is central for the asymptotic dynamics toward the initial singularity. Every time a constant $\hat{n}_\alpha$ is set to zero the dimension of the automorphism group increases by one (in this context automorphisms are linear constant transformations of the symmetry adapted spatial frame $\{\hat{\omega}^1, \hat{\omega}^2, \hat{\omega}^3\}$ that leave the structure constants unchanged). The kinematical consequence of this is that a given Lie contracted boundary subset of (4) describes the true degrees of freedom of the associated Bianchi type. Due to this, the metric of that Bianchi type can be explicitly constructed from the solution to the equations on the subset, and the quadrature for the decoupled variable $H$ by integrating (3), whose decoupling is a consequence of the scale invariance symmetry, see Ref. 7.

More importantly, however, are the dynamical implications of the group of automorphisms and scale transformations. As explicitly shown in Ref. 11 on each level in the ‘Lie contraction boundary subset hierarchy’ the combined scale-automorphism group induces monotone functions, and even constants of the motion at the bottom of the hierarchy. The resulting hierarchy of monotone functions pushes the dynamics towards the past singularity to boundaries of boundaries in the hierarchy, where the solutions on the simplest subsets, i.e., those of Bianchi types I and II, are completely determined by scale and automorphism symmetries. The dynamics towards the initial singularity hence turns out to be governed to a large extent by structures induced by the scale-automorphism groups on the different levels in the

\[\text{For bounded variables that do not break down at the point of maximum expansion and allows for a global description of the dynamics, see Ref. 10}\]
Lie contraction boundary subset hierarchy. Since the automorphisms in the present context correspond to the spatial diffeomorphism freedom that respects the symmetries of the various Bianchi models, it therefore follows that the dynamics towards the initial singularity is partly determined by physical first principles, namely scale invariance and general covariance.

As a consequence of the hierarchical structure, it is both natural and necessary to describe the dynamics from the bottom up, i.e., from Bianchi type I and upwards in the Lie contraction hierarchy. This is conveniently done by projecting the dynamics onto \((\Sigma_1, \Sigma_2, \Sigma_3)\) space.

One of the advantages of scale invariant approaches such as the Hubble-normalized approach is that self-similar, i.e., scale invariant, models appear as fixed points in the dimensionless state space. This is the case for the Bianchi type I vacuum models, the ‘Kasner’ solutions, which in the Hubble-normalized state space picture form the so-called Kasner circle \(K^O\) of fixed points, characterized by \(N_1 = N_2 = N_3 = 0, \Sigma^2 = 1,\) and constant \(\Sigma_1, \Sigma_2, \Sigma_3.\) Although points on \(K^O\) are characterized by \(\Sigma_\alpha\) variables via the relation \(\Sigma_\alpha = 3p_\alpha - 1.\) Due to permutations of the axes, \(K^O\) is naturally divided into six equivalent sectors, denoted by permutations of the triple \((123)\) where sector \((\alpha\beta\gamma)\) is defined by \(\Sigma_\alpha < \Sigma_\beta < \Sigma_\gamma,\) or, equivalently, \(p_\alpha < p_\beta < p_\gamma,\) see Fig. 1. The boundaries of the sectors are six special points that are associated with locally rotationally symmetric (LRS) solutions, \(T_\alpha: (\Sigma_\alpha, \Sigma_\beta, \Sigma_\gamma) = (+2, -1, -1),\) or, equivalently, \((p_\alpha, p_\beta, p_\gamma) = (1, 0, 0),\) \(Q_\alpha: (\Sigma_\alpha, \Sigma_\beta, \Sigma_\gamma) = (-2, +1, +1),\) or, equivalently, \((p_\alpha, p_\beta, p_\gamma) = (-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}),\) \((5a)\) \((5b)\)

The Taub points \(T_\alpha (\alpha = 1, 2, 3)\) correspond to the Taub (LRS) representations of the Minkowski spacetime, while \(Q_\alpha\) yield three equivalent LRS solutions with non-flat geometry.

It is useful to parameterize \(K^O\) with extended Kasner parameters\(^{15,16}\) To each cyclic permutation of \((123)\) there exists a natural parameter, \(\bar{u}_\alpha \in (-\infty, \infty),\) defined by

\[ p_\alpha = -\bar{u}_\alpha/f(\bar{u}_\alpha), \quad p_\beta = (1 + \bar{u}_\alpha)/f(\bar{u}_\alpha), \quad p_\gamma = \bar{u}_\alpha(1 + \bar{u}_\alpha)/f(\bar{u}_\alpha), \quad (6) \]

In the case of matter sources, hierarchies become even more important than in the vacuum case. Then, in addition to Lie contractions, one also have source contractions, where the vacuum is at the bottom of the source hierarchy. For each level of the source contraction hierarchy the scale-automorphism group yields different structures, such as monotone functions, leading to restrictions on asymptotic dynamics; see Ref.\(^{11}\) where this general feature is exemplified explicitly by a perfect fluid in the case of diagonal class A Bianchi models.

The 1-parameter family of Kasner solutions is often given in terms of the line element \(ds^2 = -dt^2 + t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2.\)
where \((\alpha\beta\gamma) = (123)\) and cycle, and where
\[
f = f(x) := 1 + x + x^2.
\]  
(7)

Factoring out the permutation freedom leads to the usual frame (gauge) invariant Kasner parameter
\[u \in (1, \infty)\] (note the difference in interval compared to that for \(\hat{u}_\alpha\)), which is defined according to
\[2, 17\]
\[
p_\alpha = -u/f(u), \quad p_\beta = (1 + u)/f(u), \quad p_\gamma = u(1 + u)/f(u),
\]  
(8)

where the boundary points of sector \((\alpha\beta\gamma)\), \(Q_\alpha\) and \(T_\gamma\), are characterized by
\[u = 1\] and \(u = \infty\), respectively.\[5\]

Comparing (8) and (6) gives a transformation between \(u\) and \(\hat{u}_\alpha\) for each sector:
\[
(\alpha\beta\gamma) : (1, \infty) \ni \hat{u}_\alpha = u, \\
(\alpha\gamma\beta) : (0, 1) \ni \hat{u}_\alpha = u^{-1}, \\
(\gamma\alpha\beta) : (-1, 0) \ni \hat{u}_\alpha = -\frac{1}{u+1}, \\
(\gamma\beta\alpha) : (-\infty, -1) \ni \hat{u}_\alpha = -\frac{u+1}{u}, \\
(\beta\gamma\alpha) : (-2, -1) \ni \hat{u}_\alpha = -\frac{u+1}{u}, \\
(\beta\alpha\gamma) : (-\infty, -2) \ni \hat{u}_\alpha = -(u+1),
\]  
(9a)

where \((\alpha\beta\gamma) = (123)\) and cycle, which yields
\[
\hat{u}_1 = -\frac{\hat{u}_2 + 1}{\hat{u}_2}, \quad \hat{u}_2 = -\frac{\hat{u}_3 + 1}{\hat{u}_3}, \quad \hat{u}_3 = -\frac{\hat{u}_1 + 1}{\hat{u}_1},
\]  
(10a)

\[
\hat{u}_1 = -\frac{1}{\hat{u}_3 + 1}, \quad \hat{u}_2 = -\frac{1}{\hat{u}_1 + 1}, \quad \hat{u}_3 = -\frac{1}{\hat{u}_2 + 1}.
\]  
(10b)

\[8\]The Kasner parameter \(u\) can be related to a function that is constructed entirely from Weyl scalars in a one-to-one manner, and hence \(u\) is a gauge invariant quantity.
The next level in the Lie contraction hierarchy consists of the Bianchi type II subsets $\mathcal{B}_{N_{\alpha}}$ given by $N_{\alpha} \neq 0$, $N_{\beta} = N_{\gamma} = 0$. The solutions of these subsets are also, as in the Bianchi type I case, completely determined by the scale-automorphism group. Projected onto $(\Sigma_1, \Sigma_2, \Sigma_3)$ space they form families of straight lines (see e.g. Refs. 18 and 8) where each straight line connects two fixed points on $K^O$, see Fig. 2.

It follows that each solution trajectory is a so-called heteroclinic orbit\cite{heteroclinic} i.e., a solution trajectory that starts ($\alpha$-limit) and ends ($\omega$-limit) at two different fixed points. In Refs. 8 and 18 these orbits were denoted as Bianchi type II transitions, $T_{N_{\alpha}}$ ($\alpha = 1, 2, 3$), because each orbit can be viewed as representing a transition from one Kasner state to another. It follows that the type II vacuum models are past and future asymptotically self-similar, since all orbits begin and end at two different Kasner fix points that correspond to self-similar Kasner solutions.

In the context of Bianchi types VIII and IX, it follows from the properties of a $T_{N_{\alpha}}$ transition on the $\mathcal{B}_{N_{\alpha}}$ boundary that it gives rise to a 'Mixmaster map' between two fixed points on $K^O$ in the direction towards the past initial singularity according to\cite{mixmaster}

$$T_{N_{\alpha}}: \tilde{u}_{\alpha} = \tilde{u}_{\alpha}^i \in (0, +\infty) \mapsto \tilde{u}_{\alpha}^f = -\tilde{u}_{\alpha}^i \in (-\infty, 0).$$

Quoting out the gauge dependence, the Mixmaster map yields the Kasner map (also known as the BKL map)\cite{kasner, bklinv}

$$u^f = \begin{cases} 
    u^i - 1 & \text{if } u^i \in [2, \infty), \\
    (u^i - 1)^{-1} & \text{if } u^i \in [1, 2].
\end{cases}$$

Based on work reviewed and developed in Ref. 4 and by Rendall\cite{rendall} Ringström, in 2000 and 2001, produced the first major proofs about asymptotic Bianchi type IX dynamics.\cite{ringstrom1, ringstrom2} In particular Ringström managed to prove that the past attractor in Bianchi type IX resides on a subset that consists of the union of the Bianchi

Fig. 2. Projections of Bianchi type II transitions $T_{N_{1}}, T_{N_{2}}, T_{N_{3}}$ on the Bianchi type II subsets $\mathcal{B}_{N_{1}}, \mathcal{B}_{N_{2}}, \mathcal{B}_{N_{3}}$ (from left to right in the figure) onto $(\Sigma_1, \Sigma_2, \Sigma_3)$ space. The arrows indicate the direction of time towards the past.
type I and II vacuum subsets (for a shorter proof that uses the complete structure induced by the scale-automorphism group, see Ref. 10), i.e.,

\[ \mathcal{A} = K^0 \cup B_{N_1} \cup B_{N_2} \cup B_{N_3}. \]  

This achievement is impressive, but Ringström’s ‘attractor theorem’ does not say if all of \( \mathcal{A} \) is the attractor nor if the Kasner map is relevant for dynamics asymptotic to the initial singularity in Bianchi type IX, and the theorem says nothing about Bianchi type VIII; for further discussion, see Ref. 8. The Kasner map, when iterated, turns out to be associated with chaotic properties that has attracted considerable attention, see Ref. 8 for references. Taken together with BKL’s locality conjecture, these properties are often said to imply that generic spacelike singularities, and hence also Einstein’s equations, are chaotic. But up until recently there were no rigorous results, including Ringström’s theorems, that in any way tied the Mixmaster and Kasner maps to asymptotic dynamics. For example, there was nothing that excluded that some periodic sequence(s) of orbits (further discussed below), associated with a particular value (values) of \( u \), could not be the past attractor, which then would lead to a simple analytic asymptotic description that in no way could be called chaotic. Nor is it possible that any numerical experiment can shed any light on this since it follows from continuity, and the transversal hyperbolicity of \( K^0 \), that there are solution trajectories that shadow orbits associated with any sequence of \( u \) obtained by iterations of (12) for arbitrarily long finite \( \tau \) intervals. Fortunately, during the last couple of years there has been substantial progress that has begun to rigorously asymptotically connect the structures on \( \mathcal{A} \) with the Mixmaster and Kasner maps.

To understand these new results we need to focus on the heteroclinic structure associated with \( \mathcal{A} \). The heteroclinic orbits \( T_{N_1}, T_{N_2}, T_{N_3} \) can be concatenated on \( \mathcal{A} \) to yield heteroclinic ‘Mixmaster’ chains by identifying the ‘final’ fixed point (\( \omega \)-limit point) of one transition with the ‘initial’ fixed point (\( \alpha \)-limit point) of another transition (in dynamical systems theory a heteroclinic chain is defined as a sequence of heteroclinic orbits such that the \( \omega \)-limit point of one orbit is the \( \alpha \)-limit point of the subsequent orbit), see Fig. 3.

The Mixmaster chains induce iterations of the maps (11) and (12). The iterations of the Mixmaster map can be analytically described by combining (11) with (10), which allows one to describe the sequence of Kasner fixed points obtained via Bianchi type II transitions by means of a single extended Kasner parameter, e.g. \( \hat{u}_1 \). In terms of the Kasner parameter \( u \), a sequence of transitions corresponds to an iteration of (12). Let \( l = 0, 1, 2, \ldots \) and let \( u_l \) denote the initial Kasner state of the \( l \)th transition, then the iterated Kasner map is given by:

\[ u_l \xrightarrow{\text{\( l \)th transition}} u_{l+1} : \quad u_{l+1} = \begin{cases} 
    u_l - 1 & \text{if } u_l \in [2, \infty), \\
    (u_l - 1)^{-1} & \text{if } u_l \in [1, 2]. 
\end{cases} \]  

Since each value of the Kasner parameter \( u \in (1, \infty) \) represents an equivalence class of six Kasner fixed points, the Kasner map can be regarded as the map induced by
the Mixmaster map on these equivalence classes via the equivalence relation.

In a sequence \((u_l)_{l=0,1,2,...}\) that is generated by the Kasner map \((14)\), each Kasner state \(u_l\) is called a Kasner epoch. Every sequence \((u_l)_{l=0,1,2,...}\) possesses a natural partition into pieces called Kasner eras with a finite number of epochs. An era begins with a maximal value \(u_{in}\) (where \(u_{in}\) is generated from \(u_{in-1}\) by \(u_{in} = [u_{in-1} - 1]^{-1}\)), and continues with a sequence of Kasner parameters obtained via \(u_l \mapsto u_{l+1} = u_l - 1\); it ends with a minimal value \(u_{out}\) that satisfies \(1 < u_{out} < 2\), so that \(u_{out+1} = [u_{out} - 1]^{-1}\) begins a new era as exemplified by

\[
3.45 \rightarrow 2.45 \rightarrow 1.45 \rightarrow 2.23 \rightarrow 1.23 \rightarrow 4.33 \rightarrow 3.33 \rightarrow 2.33 \rightarrow \ldots
\]  

Denoting the initial and maximal value of the Kasner parameter \(u\) in era number \(s\) (where \(s = 0, 1, 2, \ldots\)) by \(u_s\), and decomposing \(u_s\) into its integer \(k_s = \lfloor u_s \rfloor\) and fractional \(x_s = \{u_s\}\) parts, gives

\[
u_s = k_s + x_s,\tag{16}\]

where \(k_s\) represents the (discrete) length of era \(s\), and hence the number of Kasner epochs it contains. The final and minimal value of the Kasner parameter in era \(s\) is given by \(1 + x_s\), which implies that era number \((s + 1)\) begins with

\[
u_{s+1} = \frac{1}{x_s} = \frac{1}{\{u_s\}}.\tag{17}\]

The map \(u_s \mapsto u_{s+1}\) is the so-called ‘era map.’ Starting from \(u_0 = u_0\) it recursively determines \(u_s, s = 0, 1, 2, \ldots\), and thereby the complete Kasner sequence \((u_l)_{l=0,1,\ldots\ldots}\).
The era map admits an interpretation in terms of continued fractions. Applying the Kasner map to the continued fraction representation of the initial value $u_0$, 

$$u_0 = k_0 + \frac{1}{k_1 + \frac{1}{k_2 + \cdots}} = [k_0; k_1, k_2, k_3, \ldots], \quad (18)$$

gives

$$u_0 = u_0 = [k_0; k_1, k_2, \ldots] \rightarrow [k_0 - 1; k_1, k_2, \ldots] \rightarrow \ldots \rightarrow [1; k_1, k_2, \ldots] \rightarrow u_1 = [k_1; k_2, k_3, \ldots] \rightarrow [k_1 - 1; k_2, k_3, \ldots] \rightarrow \ldots \rightarrow [1; k_2, k_3, \ldots] \rightarrow u_2 = [k_2; k_3, k_4, \ldots] \rightarrow [k_2 - 1; k_3, k_4, \ldots] \rightarrow \ldots, \quad (19)$$

and hence the era map is simply a shift to the left in the continued fraction expansion,

$$u_s = [k_s; k_{s+1}, k_{s+2}, \ldots] \rightarrow u_{s+1} = [k_{s+1}; k_{s+2}, k_{s+3}, \ldots]. \quad (20)$$

Some of the era and Kasner sequences are periodic, notably $u_0 = [(1)] = [1; 1, 1, 1, \ldots] = (1 + \sqrt{5})/2$, which is the golden ratio, gives $u_s = (1 + \sqrt{5})/2 \forall s$, and hence the Kasner sequence is also a sequence with period 1,

$$(u_t)_t \in \mathbb{N} : \frac{1}{2}(1 + \sqrt{5}) \rightarrow \frac{1}{2}(1 + \sqrt{5}) \rightarrow \frac{1}{2}(1 + \sqrt{5}) \rightarrow \frac{1}{2}(1 + \sqrt{5}) \rightarrow \ldots,$$

while this yields two heteroclinic cycles of period 3 in the state space picture (since the axis permutations are not quotiented out in the state space), see the figures in Ref. 8.

Although BKL,\textsuperscript{2,3,17} as well as Misner,\textsuperscript{22,23} conjectured that the asymptotic dynamics of Bianchi type IX is governed by the Kasner and era maps, it was only recently that rigorous results were obtained relating these maps to asymptotic dynamics in Bianchi types VIII and IX. To describe these results, it is convenient to use a modified version of the classification scheme of Kasner sequences and the associated Mixmaster chains that was introduced in Ref. 8:

(i) $u_0 = [k_0; k_1, k_2, \ldots, k_n]$, i.e., $u_0 \in \mathbb{Q}$. The associated Kasner sequence is finite with $n$ eras and have an associated Mixmaster chain that terminates at one of the Taub points. It has been proven that these sequences are not asymptotically realized in the generic non-LRS case since a Taub point is not the $\omega$-limit set of any non-LRS solution.\textsuperscript{19,20,21}

(ii) $u_0 = [k_0; k_1, \ldots]$ such that the sequence of partial quotients of its continued fraction representation is bounded, with or without periodicity. As a consequence the associated Mixmaster chains avoid a (small) neighborhood of the Taub points. In the case of no periodicity and no cycles, Béguin proved that a family of solutions of codimension one converges to each associated chain,\textsuperscript{23} (for the proof to work, cycles must be excluded to avoid resonances). By using different techniques, and different differentiability conditions, Liebscher et al\textsuperscript{25} proved explicitly that a family of solutions of
codimension one converges to the 3-cycle associated with \( u_0 = [(1)] \). The authors also gave arguments about how their methods could be extended to the present general case. This was explicitly proved in Ref. 26, where the authors introduced a new technique that involves the invariant Bianchi type I and II subset structure. This is a quite promising development since this structure is related to the Lie contraction hierarchy, which in turn is tied to basic physical principles that characterize the problem at hand.

(iii) \( u_0 = [k_0; k_1, \ldots] \) is an unbounded sequence of partial quotients, which is the generic case. The associated Kasner sequence is unbounded and the associated Mixmaster chain enters every neighborhood of the Taub points infinitely often. As argued in Ref. 27, a subset of these chains is relevant to the description of the asymptotic dynamics of actual solutions: for each \( u_0 \) such that the sequence \( (k_n)_{n \in \mathbb{N}} \) can be bounded by a function of \( n \) with a prescribed growth rate, there exists an actual solution that converges to the chain determined by \( u_0 \). On the other hand, chains associated with initial values \( u_0 = [k_0; k_1, \ldots] \) with rapidly increasing partial quotients \( k_n, n \in \mathbb{N} \) are perhaps less relevant for the description of the asymptotic dynamics of actual solutions; if a solution shadows a finite part of such a chain it may be thrown off course at the point where the chain enters a (too) small neighborhood of the Taub points. The prescribed growth rate given in Ref. 27 is weak enough to not destroy genericity; a generic real number has a continued fraction representation compatible with the required boundedness condition. Note, however, that these results do not say anything about how many solutions actually converge to a given chain, nor if the asymptotic dynamics of a generic initial data set is represented by a heteroclinic chain.

The above results imply that \( \mathcal{A} \) is indeed the global past attractor for Bianchi type IX, but \( \mathcal{A} \) is not necessarily the global past attractor for type VIII, since it still has not been excluded that the type VIII attractor also involves the vacuum Bianchi type VI\(_0\) subset. It is also worth mentioning that the general Bianchi type VI\(_{1/3}\) models are as general as those of types VIII and IX, but arguably they are more relevant for generic singularities.\(^\text{18}\) They also have an oscillatory singularity, and hence asymptotic 'self-similarity breaking'\(^\text{28}\) but instead of being characterized by Mixmaster chains, the singularity is conjectured to be characterized by so-called Iwasawa chains, see Ref. 18. Unfortunately, there exist no rigorous mathematical results concerning their past asymptotic dynamics.

\(^{18}\)Ref. 27 uses quite different mathematical techniques than the other rigorous papers in this area. As a consequence the results seem to be somewhat controversial in the research community, although the claims are arguably quite plausible. Due to this, and due to an intrinsic value, it would be of interest if the results could be confirmed, or preferably even extended, with some other independent methods.
3. Inhomogeneous vacuum models

The central assumption of BKL in the general \textit{inhomogeneous} context is their locality conjecture. A physical justification of asymptotic locality may heuristically be attempted in terms of the following scenario: ultra strong gravity increasingly affects the causal structure as the singularity is approached, and as a consequence particle horizons shrink to zero size toward the singularity along each timeline. This prohibits communication between different time lines in the asymptotic limit, and the causal feature of asymptotically shrinking particle horizons along time lines may hence be referred to as \textit{asymptotic silence}, while the associated singularity is said to be \textit{asymptotically silent}.

In order to construct the solution in a sufficiently small spacetime neighborhood of a generic spacelike singularity, Uggla \textit{et al.} attempted to:

(i) capture asymptotic silence and locality in a rigorous manner, and
(ii) contextualize the results obtained in SH cosmology in terms of a general state space picture.

The tool needed to satisfy (i) and (ii) while pursuing the goal of constructing an asymptotic solution in a sufficiently small spacetime neighborhood of a generic spacelike singularity in Refs. 18, 29, 30 is a reformulation of Einstein’s field equations according to the following prescription. Assume that a small neighborhood near the singularity can be foliated with a family of spacelike surfaces such that the singularity ‘occurs’ simultaneously. Then ‘factor out’ the expansion $\theta$ of the normal congruence to the assumed foliation from Einstein’s field equations by first performing a conformal transformation (thereby respecting the expected key causal structure),

$$ g = H^{-2} G, \quad (21) $$

where $H$ is the Hubble variable associated with the expansion ($H = \frac{1}{3} \theta$) and $g$ is the physical metric; since $g$ has dimensional weight $[\text{length}]^2$ and $H$ $[\text{length}]^{-1}$ it follows that the unphysical metric $G$ is dimensionless. As the next step, introduce an orthonormal frame for $G$, or equivalently, a corresponding conformal orthonormal frame for $g$, where the unit normal of the reference foliation is chosen to be the timelike vector field of the orthonormal frame, and set the shift vector to zero so that the time lines are tangential to that vector field. Finally, calculate Einstein’s vacuum field equations in terms of $H$ and the dimensionless frame and commutator functions (or, equivalently, the connection) associated with the orthonormal frame of $G$. The conformal Hubble variable turns out to be minus the deceleration parameter $q$ of the physical spacetime; moreover, it follows from dimensional reasons that $q$ is algebraically determined by the Raychaudhuri equation in terms of the other Hubble-normalized dimensionless variables.

In the special case of spatial homogeneity, and a frame that is chosen so that the spatial frame is tangential to the symmetry surfaces, the evolution equations for the spatial Hubble-normalized frame variables (which essentially are the spatial metric variables associated with $G$) decouple from the remaining equations. This follows
from the fact that the spatial frame derivatives of $H$ and the conformally Hubble-normalized commutator functions are zero, since these quantities only depend on time due to the symmetry assumption. Moreover, the evolution equation for $H$ also decouples from the remaining equations since $H$ is the only variable that carries dimension. This leaves a system of ODEs that coincides with the usual Hubble-normalized dynamical system discussed in the previous section (when restricted to the diagonalized class A vacuum case, otherwise it gives the general Hubble-normalized SH equations), i.e., the above procedure provides a general geometric setting for producing the usual Hubble-normalized dynamical system that has been so successful in SH cosmology.

More importantly, however, is the following feature: the system of PDEs for the general inhomogeneous case admits an invariant unphysical boundary subset that is obtained by setting all spatial frame variables associated with $G$ to zero. Since this leads to that all spatial frame derivatives are set to zero, this results in that the equations on this boundary subset form a set of ODEs that coincides with the decoupled equation for $H$ and the Hubble-normalized dynamical system of the SH case. The difference is that now constants of integrations are only temporal constants, i.e., they now depend on the spatial coordinates. Since the original expectation was that BKL locality is due to asymptotic silence, this boundary was originally called the silent boundary. Presumably asymptotic silence is a necessary condition for BKL locality, but it turns out that asymptotic silence also admits other possibilities, and for this reason the name of the invariant boundary subset has been changed from the silent boundary to ‘the local boundary’.

It follows that the BKL assumption of locality, i.e., that the dynamics of an individual timeline is asymptotically described by a SH model, therefore corresponds to the statement that the asymptotic past dynamical evolution is described by the local boundary, i.e., the BKL scenario obtains a precise state space setting in the conformal Hubble-normalized approach. Since BKL is about generic behavior, it is the past attractor on the local boundary that is of interest, i.e., the asymptotic evolution along a generic timeline should be described by the attractor for generic SH cosmology. The most general of the SH models are those of Bianchi types IX, VIII and VI_{1/9}. The expected attractor for these models resides on a subset that consists of the union of the vacuum Bianchi type I (Kasner) and II subsets. This, together with setting the Hubble-normalized spatial frame variables to zero, yields the same set of ODEs as those of the dimensionless coupled system in the SH case. An advantage of the Hubble-normalized variables is that the dimensionless variables are bounded on the expected past attractor subset, i.e., even though the expected singularity is a scalar curvature singularity with associated blow ups, the conformal Hubble-normalization leads to an asymptotically regularized and bounded dimensionless state space. Furthermore, if one uses $H^{-1}$ as the dimensional variable, $H^{-1}$ tends to zero towards the singularity, where $H^{-1} = 0$ is an invariant subset, associated with the fact that the singularity is a crushing singularity.

The above approach and picture was used in Ref. [18] in order to establish the
consistency of the BKL scenario as well as the cosmological billiards discussed by Damour and coworkers\textsuperscript{31} in the general inhomogeneous context. However, the consistency of BKL locality and cosmological billiards does not exclude other types of behavior. To gain further insights about general oscillatory singularities in inhomogeneous spacetimes it is natural to restrict investigations to models with two commuting spacelike Killing vectors, so-called $G_2$ models. This was done in Ref.\textsuperscript{32} where heuristical and numerical support was gained for the BKL scenario in the Hubble-normalized state space context for an open set of time lines, but this work also yielded evidence for ‘recurring oscillatory spike formation’ for time lines forming 2D spatial surfaces, and for time lines in their neighborhoods, for which it is \textit{not} possible to neglect the Hubble-normalized spatial frame derivatives asymptotically, i.e., ‘asymptotic locality’ is broken.

Oscillatory BKL behavior arises because certain ‘trigger’ variables destabilizes the Kasner circle on the local boundary, which leads to transitions between different fixed points on $K^0$ on the local boundary. If such a variable goes through zero at a spatial coordinate, i.e., at a spatial surface, then the ‘BKL transition’ on the local boundary cannot take place. Instead spatial derivatives grow and give rise to \textit{spike transitions} between two different points on $K^0$ on the local boundary. This had already been noticed in special $G_2$ models such as the so-called Gowdy $T^3$ models, for reviews and references, see Refs.\textsuperscript{34} and \textsuperscript{16} but the asymptotic results of this phenomenon are quite different in such special models, where the end result is that the evolution point wise approaches $K^0$ on the local boundary in a non-uniform way, leading to ‘permanent spikes’. This is \textit{not} what happens in the general case. Instead infinitely oscillating recurring and transient spike formation, leading to spike transitions from one Kasner state to another, take place, which results in quite different non-uniform features.

The scenario in Ref.\textsuperscript{32} was also supported in Ref.\textsuperscript{33} but more importantly, based on earlier work by Rendall and Weaver\textsuperscript{35}, Lim managed to produce explicit inhomogeneous $G_2$ solutions by means of a solution generating algorithm\textsuperscript{36} that analytically describes spike transitions to high accuracy\textsuperscript{37} indeed the explicit solutions were essential in order to accurately describe numerically several successive spike transitions since this is quite challenging numerically. This work in turn led to Ref.\textsuperscript{16} which yielded further analytic insights and numerical progress concerning BKL and spike oscillations, as well as providing a context for previous work on non-oscillatory models such as the $T^3$ Gowdy models. It is this work we take as starting point for the discussion below (for further details, see Ref.\textsuperscript{16}).

The description of $G_2$ models as well as the special $G_2$ spike solutions naturally leads to the use of so-called Iwasawa frames. Since we choose the reference time lines to be orthogonal to the assumed foliation, the shift vector is set to zero which allows the line element to be written in the form:

$$ds^2 = -N^2(d\tau^0)^2 + \sum_{\alpha=1}^{3} \exp(-2b^\alpha) \omega^\alpha \otimes \omega^\alpha, \quad \omega^\alpha = \Lambda^\alpha_i \, dx^i,$$

(22)
where \( i = 1, 2, 3 \) is summed over. In the general case without symmetries, \( N = N(x^\mu) \), \( b^\alpha = b^\alpha(x^\mu) \), \( N^\alpha_i = N^\alpha_i(x^\mu) \) (\( \mu = 0, 1, 2, 3 \)), and where an Iwasawa parametrization of the spatial metric is implemented by setting

\[
\left( N^\alpha_i \right)_{\alpha,i} = \begin{pmatrix} 1 & n_1 & n_2 \\ 0 & 1 & n_3 \\ 0 & 0 & 1 \end{pmatrix},
\]

(23)

where \( n_1(x^\mu), n_2(x^\mu), n_3(x^\mu) \) (in the case of the \( G_2 \) models with two commuting Killing vector fields \( \partial_{x_1} \) and \( \partial_{x_2} \), the use of a symmetry adapted frame leads to that all variables depend on \( x^0 \) and \( x^3 \) only).

It follows that the diagonal conformally Hubble-normalized shear variables, \( \Sigma_\alpha \equiv \Sigma_{\alpha\alpha} \), of the timelike reference congruence are given by:

\[
\Sigma_\alpha := -(1 + (HN)^{-1}\partial_{x_0}b^\alpha), \quad \Sigma_1 + \Sigma_2 + \Sigma_3 = 0.
\]

(24)

Other central variables are

\[
N_1 := \frac{1}{2}H^{-1}N_1 \exp(b_2 + b_3 - b_1),
\]

(25a)

\[
R_1 := -\frac{1}{2}\exp(b^3 - b^2)(HN)^{-1}\partial_x(n_3),
\]

(25b)

\[
R_3 := -\frac{1}{2}\exp(b^3 - b^1)(HN)^{-1}\partial_x(n_1),
\]

(25c)

where \( N_1 \) is a Hubble-normalized spatial commutator function (\( \tilde{N}_1 \) is a rather complicated expression that involves spatial derivatives of \( n_1, n_2, n_3 \), see Ref. [13], \( R_1 = \Sigma_{23}, R_3 = \Sigma_{12} \), while \( R_2 = -\Sigma_{31} \) can be set to zero for the vacuum \( G_2 \) models. The Kasner circle \( K_0 \) on the local boundary is described in the same way as in the spatially homogeneous case, so that the only non-zero variables on that circle are the temporally constant, but spatially dependent, Hubble-normalized diagonal shear variables; define the total shear quantity \( \Sigma^2 := \frac{1}{6}(\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2) \). The variables \( N_1, R_1 \) and \( R_3 \) are ‘trigger variables’ that destabilize \( K_0 \) leading to transitions between different fixed points on \( K_0 \); \( N_1 \) yields \( T_{N_1} \) Bianchi type II transitions on the local boundary (see Fig. [2]) while \( R_1 \) and \( R_3 \) result in so-called \( T_{R_1} \) and \( T_{R_3} \) ‘frame transitions’ (trajectories that form straight lines in \( (\Sigma_1, \Sigma_2, \Sigma_3) \) space with \( \Sigma_1 = \text{const} \) and \( \Sigma_3 = \text{const} \), respectively), i.e., rotations of the spatial frame that transfer one Fermi propagated (i.e., gyroscopically fixed) Kasner state on \( K_0 \), i.e., a fixed point, to another, keeping the gauge invariant (spatially dependent) Kasner parameter \( \kappa \) fixed, see Fig. [4].

In the general case one of the trigger variables might become zero at a value of a spatial coordinate. In this case the role of the trigger is replaced by increasing spatial gradients which act as a ‘spike trigger’ that changes the dynamical state. If \( R_1 \) or \( R_3 \) goes through zero this asymptotically leads to so-called false spikes, which are gauge features which we will refrain from discussing, while if \( N_1 \) goes through zero this results in recurring ‘true spikes.’

In the case the dynamics along a timeline in a general \( G_2 \) model approaches the local boundary in the state space and \( N_1 R_1 R_3 \neq 0 \), the dynamics is BKL-like and can be described asymptotically by concatenation of \( T_{N_1}, T_{R_1} \) and \( T_{R_3} \) transitions.
Fig. 4. Projections onto $(\Sigma_1, \Sigma_2, \Sigma_3)$ space of the two types of frame transitions, $T_{R_1}$ and $T_{R_3}$, that exist in the $G_2$ case. As usual the direction of time indicated by the arrows is towards the past.

into heteroclinic (BKL) ‘Iwasawa chains’ on the local boundary (one for each spatial point), see Fig. 5.

Fig. 5. An Iwasawa chain is an (in general) infinite heteroclinic chain on the local boundary consisting of $T_{N_1}$, $T_{R_1}$ and $T_{R_3}$ transitions. The paths are not unique; e.g., at the point $X$ two continuations are possible, a $T_{R_1}$ or a $T_{N_1}$ transition (dashed line). However, the different paths rejoin at the point $Y$ where the chains continue with a (not shown) $T_{R_3}$ transition.

However, if $N_1 = 0$ at a value of $x^3$ this gives rise to so-called spike chains. In contrast to the BKL scenario, if $N_1$ goes through zero it affects a whole family of time lines whose dynamics differ, which is illustrated by a family of curves in

1In contrast to Bianchi types IX and VIII there exist several triggers in sectors (132) and (321) on $K^0$. These might be simultaneously activated, giving rise to a ‘multiple’ transition that results in a final state on $K^0$ that coincides with the fixed point obtained by successively applying the present ‘single’ frame transitions that constitute the boundary of the multiple transition. However, as discussed in Ref. [18] multiple transitions are not expected to play a generic role in the asymptotic dynamics of solutions towards a generic spacelike singularity, and we therefore only discuss the ‘single’ transitions $T_{N_1}$, $T_{R_1}$ and $T_{R_3}$.
the state space picture, conveniently projected onto \((\Sigma_1, \Sigma_2, \Sigma_3)\) space, in Fig. 6. Nevertheless, the trajectories of all ‘spike’ time lines rejoin at a common fixed point on \(K^O\), and it is therefore these instances that define the natural ‘concatenation blocks’. These turn out to consist of either so-called ‘high-velocity’ solutions, which hence are referred to as high-velocity transitions \(T_{Hi}\), and so-called low-velocity solutions combined with \(T_{R1}\) transitions and part of high-velocity solutions that form a \(joint\ low/high\ velocity\ spike\ transition,\ \(T_{Jo}\) \(\text{Fig. 6}\). In addition, there are so-called Bianchi type II spiky features, which we will not discuss, but instead refer to Ref. 16.

![Fig. 6](image)

The effect of \(T_{Hi}\) as well as \(T_{Jo}\) for a family of time lines is to transform a Kasner state described by the Kasner parameter \(u^i\) to another Kasner state \(u^f\) (the time direction is towards the singularity) according to a map that is obtained by applying \((12)\) twice, which results in

\[
u^f = \begin{cases} 
\frac{u^i - 2}{(u^i - 2)^{-1}} & u^i \in [3, \infty), \\
\frac{(u^i - 1)^{-1} - 1}{(u^i - 1)^{-1} - 1} & u^i \in [2, 3], \\
\frac{(u^i - 1)^{-1} - 1}{(u^i - 1)^{-1} - 1} & u^i \in [3/2, 2], \\
\frac{(u^i - 1)^{-1} - 1}{(u^i - 1)^{-1} - 1} & u^i \in [1, 3/2]. 
\end{cases}
\]  

Iteration of spike concatenation blocks lead to oscillations of different Kasner states, common to all time lines affected by the recurring spike and frame transitions, see Fig. 7 for concatenation of spike surface trajectories. As shown and estimated in Ref. 16 the spatial size of the spike shrinks towards the singularity...
and asymptotically this is expected to lead to a new type of non-uniformity. The dynamics along the time lines that form the spike surface, at the value of $x^3$ for which $N_1(x^3) = 0$, is given by the dynamics at that $x^3$ of an invariant subset that consists of the exact inhomogeneous $G_2$ solutions found by Lim (the so-called high and low velocity spike solutions) and the Kasner subset on the local boundary, while the dynamics along the surrounding time lines is described by the past (billiard) attractor (see Ref. [18] on the local boundary (a completely different part of the state space than the ‘Lim solution subset’). This is in stark contrast to the asymptotic non-uniformities found in special $G_2$ models for which the ‘trigger variable’ $R_1$ is identically zero as occurs for the $T^3$ Gowdy models. There non-uniformities are associated with ‘permanent’ spikes that occur because the natural concatenation block $T_{Jo}$ is ‘interrupted’ in its evolution, thereby being ‘cut in half,’ due to the absence of $R_1$, which leads to a picture that is quite misleading regarding generic spacelike singularities, see Ref. [16] for details.

4. Discussion

The concatenation blocks of ‘BKL’ chains and spike chains are all closely related in a hierarchical manner. Firstly, the Kasner circle $K^O$ on the local boundary is the simplest and most basic common ingredient. Secondly, BKL chains consisting of frame and Bianchi type II transitions on the local boundary describe the dynamical evolution of spike transitions far from the spike surface of recurring spikes. Thirdly, all solutions that form the concatenation blocks are related to each other in a hierarchical manner via a solution generating algorithm. The Kasner solutions in the form associated with the Kasner circle $K^O$ in the spatially homogeneous...
Bianchi type I model act as the initial seed solutions that yield the Kasner solutions in a rotating frame which describe the Kasner frame transitions. That form for the Kasner solutions is subsequently the seed for the Bianchi type II vacuum solutions, which in turn act as the seed yielding a frame rotated version of the Bianchi type II solutions, which form the so-called false spike solutions. Using these solutions as the seed then results in the spike solutions.

The hierarchical structures discussed in the previous paragraph are not the only ones; we have seen earlier that the Bianchi Lie contraction hierarchy plays an essential role for the asymptotics of the Bianchi models, and that this to a large extent determines the asymptotics on the local boundary. This is in turn part of a greater hierarchy. The primary importance of the $G_2$ models regarding generic singularities is not the models themselves but the following important property: in the $G_2$ models certain Hubble-normalized spatial frame variables decouple due to the imposed symmetries. Setting these to zero in the general case without symmetries yields an invariant boundary subset, called the partially local $G_2$ boundary subset, with equations that are identical to the coupled set of equations describing the $G_2$ models, but the solutions on this subset involve all the coordinates. The situation is therefore completely analogous to the relationship between the local boundary and the spatially homogeneous models.

The partially local $G_2$ boundary subset was previously referred to as a partially silent boundary subset, since it is associated with some solutions that have singularities that break asymptotic silence, which corresponds to the existence of directions in which particle horizons extend to infinity, see Ref. [38]. However, since it is also associated with recurring spikes, which, due to the prominence of oscillating Kasner states, seem likely to be connected with asymptotic silence, it is motivated to refer to it as the partially local $G_2$ boundary subset, rather than a partially silent boundary subset.

Going beyond the $G_2$ assumption and looking at models with fewer isometries not only shifts attention to the partially local $G_2$ boundary, it leads to potentially new phenomena as well. In the general case spike surfaces are no longer planes and they can intersect in curves that in turn can intersect at points, which lead to new spike dynamics. At present it is not known whether such intersections persist or recur, although weak numerical evidence suggests that intersections only occur momentarily. If this is correct, it follows that spike intersections are irrelevant asymptotically. This would then imply that the BKL picture in combination with $G_2$ spike oscillations may capture the essential features of generic spacelike singularities [16].

The presently discussed recurring spikes are located at fixed spatial locations due to the choice of initial data. In general recurring spikes are moving in space. It may be that they asymptotically freeze, but this is an open issue. If they do freeze, then our present knowledge about recurring spikes represents the first step in understanding general recurring spike behavior, otherwise perhaps not. Apart from this, there are many other unresolved questions pertaining to recurring spike be-
behavior and generic spacelike singularities. For example, how many spikes can form? Can spikes annihilate? Are there generic singularities without recurring spikes? Are there generic singularities with a dense set of recurring spikes? Are there boundary conditions associated with special physical conditions that explain the existence of recurring spikes? Are oscillatory singularities not cosmological in nature but instead the spacelike part of generic black hole singularities? These issues, together with those mentioned for the spatially homogeneous case, illustrate that we are only at the beginning of understanding generic singularities, even though considerable progress has been accomplished during the last few years.

Acknowledgements

It is a pleasure to thank Mark Heinzle and Woei Chet Lim for joint work in this area, and many helpful and stimulating discussions that have made this article possible. I would also like to thank Bob Jantzen for help with the manuscript and especially for his support and friendship.

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