K-subadditive and K-superadditive set-valued functions bounded on “large” sets

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Dedicated to Professor Dr. Ludwig Reich on his 80th birthday.

Abstract. We prove that every K-subadditive set-valued map weakly K-upper bounded on a “large” set (e.g. not null-finite, not Haar-null or not a Haar-meager set), as well as any K-superadditive set-valued map K-lower bounded on a “large” set, is locally K-lower bounded and locally weakly K-upper bounded at every point of its domain.

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1. Introduction and preliminaries

The classical subadditive functions, i.e. functions $f : X \to \mathbb{R}$ satisfying

$$f(x_1 + x_2) \leq f(x_1) + f(x_2), \quad x_1, x_2 \in X,$$

have many remarkable properties of boundedness discussed, among others, in [11,16,17,19], and recently in [3–6]. For instance, it is known that if $f : \mathbb{R}^n \to \mathbb{R}$ is subadditive and upper bounded on a set $T \subset \mathbb{R}^n$ which is of positive Lebesgue measure or is of the second category with the Baire property, then $f$ is locally bounded at every point of $\mathbb{R}^n$ (see [16, Theorem 16.2.3]). This classical result was generalized by Bingham et al. in [3] to the case of others “large” sets in abelian Polish groups, e.g. not null-finite, not Haar-meager, not Haar-null sets.

Recall also that a function $f$ is called superadditive, if $-f$ is subadditive.

In this paper we extend the notions of subadditive and superadditive functions to $K$-subadditive and $K$-superadditive set-valued maps. Next, we prove theorems which are far-reaching generalizations of the results mentioned above.
For the concept of $K$–subadditivity and $K$–superadditivity we refer to the paper [18].

Let $X$ and $Y$ be abelian metric groups (both with invariant metrics). Assume that $K$ is a subsemigroup of $Y$ (i.e. $K + K \subseteq K$). Denote by $n(Y)$ the family of all nonempty subsets of $Y$.

A set–valued map $F : X \to n(Y)$ is called $K$–subadditive, if
\[ F(x_1) + F(x_2) \subseteq F(x_1 + x_2) + K \]
for all $x_1, x_2 \in X$, and $K$–superadditive, if
\[ F(x_1 + x_2) \subseteq F(x_1) + F(x_2) + K \]
for all $x_1, x_2 \in X$.

Note that if $F$ is $K$–subadditive and single–valued and moreover $Y$ is endowed with the relation $\leq_K$ of partial order defined by
\[ x \leq_K y \iff y - x \in K, \]
then conditions (1) and (2) reduce to the following conditions:
\[ F(x_1 + x_2) \leq_K F(x_1) + F(x_2) \]
and
\[ F(x_1) + F(x_2) \leq_K F(x_1 + x_2), \]
respectively. In particular, if $Y = \mathbb{R}$ and $K = [0, \infty)$, we obtain the standard definitions of subadditive and superadditive functions.

For $K = \{0\}$ $K$–subadditivity ($K$–superadditivity) means that
\[ F(x_1) + F(x_2) \subseteq F(x_1 + x_2) \quad (F(x_1 + x_2) \subseteq F(x_1) + F(x_2)) \]
for all $x_1, x_2 \in X$, which is the definition of the superadditivity (subadditivity) of set–valued functions introduced and investigated by Smajdor in [20,21].

Note, however, that if $F$ is single–valued, then each of the above inclusions means that $F$ is an additive function (i.e. $F(x_1 + x_2) = F(x_1) + F(x_2)$). Thus subadditive and superadditive set–valued maps are extensions of additive functions, whereas $K$–subadditive and $K$–superadditive set–valued maps generalize subadditive and superadditive functions, respectively.

Now, let us recall that a subset $B$ of a complete abelian metric group $X$ with an invariant metric is called:

- **Universally Baire** if for each continuous function $f : K \to X$ mapping a compact metric space $K$ into $X$ the set $f^{-1}(A+x)$ has the Baire property for every $x \in X$ (see [10]);
- **Haar–meager** if there exist a universally Baire set $A \supset B$, a compact metric space $K$ and a continuous function $f : K \to X$ such that $f^{-1}(A+x)$ is meager in $K$ for each $x \in X$ (see [8] and also [2]);
- **Universally measurable** if it is measurable with respect to each complete Borel probability measure on $X$ (see [7]);
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• Haar-null if there exist a universally measurable set \( A \supset B \) and a \( \sigma \)-additive probability Borel measure \( \mu \) on \( X \) such that \( \mu(A + x) = 0 \) for each \( x \in X \) (see [7]).

It was proved in [7] and [8] that in each locally compact abelian Polish group the notions of a Haar–meager set and a Haar–null set are equivalent to the notions of a meager set and a set of Haar measure zero, respectively. Moreover, Haar–meager sets and Haar–null sets have many analogous properties (see, e.g., [1,9,14]).

In [2] a new concept of “small” sets was introduced, generalizing (to some extent) the notions of a Haar–meager set and a Haar–null set.

Definition 1. A subset \( A \) of an abelian metric group \( X \) is called null–finite if there exists a sequence \((x_n)_{n \in \mathbb{N}}\) convergent to 0 in \( X \) such that the set \( \{n \in \mathbb{N} : x + x_n \in A\} \) is finite for every \( x \in X \).

The following crucial property of null–finite sets was proved in [2].

Theorem 1. [2][Theorems 5.1 and 6.1] In a complete abelian metric group with an invariant metric:

• Each universally Baire null–finite set is Haar–meager,
• Each universally measurable null–finite set is Haar–null.

In the same paper [2] the authors applied the above result to show that every real-valued additive (midpoint convex) function upper bounded on a set which is universally measurable non-Haar-null or Borel non-Haar-meager in a complete abelian metric group (linear space) with an invariant metric is continuous. Next, Bingham et al. [3] showed that every subadditive real valued function upper bounded on a set which is “large” in the same sense is locally bounded at each point of the domain.

In this paper we generalize results from [3] to \( K \)-subadditive and \( K \)-superadditive set-valued maps. Our results are also counterparts of some results from [15] concerning \( K \)-midconvex and \( K \)-midconcave set-valued maps bounded on “large” sets.

2. Main results

Let \( X \) and \( Y \) be abelian metric groups with invariant metrics. Denote by \( B_X(r) \) and \( B_Y(r) \) open balls with center 0 and radius \( r \) in \( X \) and \( Y \), respectively.

A set \( B \subset Y \) is called bounded in \( Y \), if \( B \) is contained in an open ball \( B_Y(r) \) for some \( r > 0 \). This notion generalizes the well-known notion of bounded sets.

\footnote{Actually, the mentioned definitions have been introduced in an abelian Polish group, but the separability is not necessary. Hunt et al. highlighted it in [12]–[13].}
in a real topological vector space. Clearly, if $B_1, B_2$ are bounded sets in $Y$, then the set $B_1 + B_2$ is also bounded in $Y$.

Denote by $\mathcal{B}(Y)$ the family of all nonempty bounded subsets of $Y$. A set-valued map $F : X \to \mathcal{B}(Y)$ is called:

- **$K$-upper bounded** on a set $A \subset X$, if there exists a set $B \in \mathcal{B}(Y)$ such that
  $$F(x) \subset B - K \quad \text{for all } x \in A;$$

- **Weakly $K$-upper bounded** on a set $A \subset X$, if there exists a set $B \in \mathcal{B}(Y)$ such that
  $$F(x) \cap (B - K) \neq \emptyset \quad \text{for all } x \in A;$$

- **$K$-lower bounded** on a set $A \subset X$, if there exists a set $B \in \mathcal{B}(Y)$ such that
  $$F(x) \subset B + K \quad \text{for all } x \in A;$$

- **Weakly $K$-lower bounded** on a set $A \subset X$, if there exists a set $B \in \mathcal{B}(Y)$ such that
  $$F(x) \cap (B + K) \neq \emptyset \quad \text{for all } x \in A.$$

Clearly, $K$-upper ($K$-lower) bounded set-valued maps are weakly $K$-upper ($K$-lower) bounded. Moreover, in the case when $K = \{0\}$, weak $K$-upper boundedness and weak $K$-lower boundedness are equivalent assumptions, called simply weak boundedness.

A set-valued map $F : X \to \mathcal{B}(Y)$ is called *locally weakly $K$-upper (K-lower) bounded at $x \in X$, if it is weakly $K$-upper (K-lower) bounded on some neighborhood of $x$.*

First we prove a result which generalizes Theorem 2.2 from [3].

**Theorem 2.** Let $X$ and $Y$ be abelian metric groups with invariant metrics. Assume that $A \subset X$ is a set which is not null-finite and $K$ is a subsemigroup of $Y$. If a set-valued map $F : X \to \mathcal{B}(Y)$ is $K$-subadditive and weakly $K$-upper bounded on $A$, then $F$ is locally weakly $K$-upper bounded and locally $K$-lower bounded at each point of $X$.

**Proof.** Let $F$ be $K$-subadditive and weakly $K$-upper bounded on $A$. Then there exists a set $B \in \mathcal{B}(Y)$ such that

$$F(x) \cap (B - K) \neq \emptyset, \quad x \in A. \quad (3)$$

First we prove that $F$ is locally weakly $K$-upper bounded at 0. So, suppose that it is not true and $F$ is not weakly $K$-upper bounded on any neighborhood of 0. Consequently, for every $n \in \mathbb{N}$,

$$n \in \mathbb{N}, \quad U_n := B_X \left( \frac{1}{2^n} \right) \quad \text{and} \quad B_n := B + B_Y(n) \in \mathcal{B}(Y),$$

there exists $x_n \in U_n$ such that

$$F(x_n) \cap (B_n - K) = \emptyset.$$
Hence
\[(F(x_n) + K) \cap B_n = \emptyset. \tag{4}\]
Moreover, by the definition of $U_n$ the sequence $(x_n)_{n \in \mathbb{N}}$ is convergent to 0 in $X$. The set $A$ is not null-finite, so there exists $a \in X$ such that the set $\mathbb{N}_0 := \{n \in \mathbb{N} : a + x_n \in A\}$ is not finite. By (3) we have
\[F(a + x_n) \cap (B - K) \neq \emptyset, \quad n \in \mathbb{N}_0. \tag{5}\]
In view of $K$-subadditivity,\[F(x_n) + K \supseteq F(-a) + F(a + x_n), \quad n \in \mathbb{N}_0. \tag{6}\]Since $F(-a) \in \mathcal{B}(Y)$, we can find $n_0 \in \mathbb{N}_0$, such that $F(-a) \subset B_Y(n_0)$. Hence $F(-a) \subset B_Y(n)$ for every $n \geq n_0$. Fix $n \in \mathbb{N}_0$, $n \geq n_0$. By (5) there exist $b_n \in B$ and $k_n \in K$ such that
\[b_n - k_n \in F(a + x_n). \tag{7}\]Then, in view of (6) we have
\[b_n - k_n + F(-a) \subset F(x_n) + K, \tag{8}\]and hence
\[b_n + F(-a) \subset F(x_n) + K. \tag{9}\]On the other hand
\[b_n + F(-a) \subset B + B_Y(n) = B_n. \tag{10}\]From (7) and (8) we obtain
\[(F(x_n) + K) \cap B_n \neq \emptyset, \tag{11}\]which contradicts (4).
Thus $F$ is locally weakly $K$-upper bounded at 0, i.e. there are a neighborhood $U_0$ of 0 and a set $B_0 \in \mathcal{B}(Y)$ such that
\[F(x) \cap (B_0 - K) \neq \emptyset, \quad x \in U_0. \tag{12}\]
In the second step we prove that $F$ is locally weakly $K$-upper bounded at every point of the domain. For a proof by contradiction suppose that $F$ is not weakly $K$-upper bounded on any neighborhood $U_{x_0}$ of some point $x_0$. Consequently, for
\[U_{x_0} := U_0 + x_0 \quad \text{and} \quad B_0 + F(x_0) \in \mathcal{B}(Y), \tag{13}\]there exists $x_1 \in U_{x_0}$ such that
\[F(x_1) \cap (B_0 + F(x_0) - K) = \emptyset, \tag{14}\]and thus
\[(F(x_1) + K) \cap (B_0 + F(x_0)) = \emptyset. \tag{15}\]Since $K$-subadditivity implies
\[F(x_1) + K \supseteq F(x_1 - x_0) + F(x_0), \tag{16}\]
where \( x_1 - x_0 \in U_0 \), in view of (9) we can find \( b \in B_0 \) and \( k \in K \) such that \( b - k \in F(x_1 - x_0) \). Then, by (11),
\[
F(x_1) + K \ni b - k + F(x_0)
\]
and consequently,
\[
F(x_1) + K \ni b + F(x_0),
\]
which contradicts (10) and proves that \( F \) is locally weakly \( K \)-upper bounded at \( x_0 \).

Finally, we will show that \( F \) is locally \( K \)-lower bounded at every point \( x \in X \). We have already proved that there exist a neighborhood \( U_0 \) of 0 and a set \( B_0 \in \mathcal{B}(Y) \) such that (9) holds. We may assume that \( U_0 \) is symmetric with respect to 0. Fix \( x \in X \) arbitrarily and take \( U_x := U_0 + x \). If \( y \in U_x \), then \( x - y \in U_0 \) and by (9)
\[
F(x - y) \cap (B_0 - K) \neq \emptyset, \ y \in U_x.
\]
Hence there exist \( z \in F(x - y) \), \( b \in B_0 \) and \( k \in K \) such that \( z = b - k \). By the \( K \)-subadditivity of \( F \) we have
\[
F(y) + b - k = F(y) + z \subset F(y) + F(x - y) \subset F(x) + K, \ y \in U_x
\]
and hence
\[
F(y) \subset F(x) - b + K \subset (F(x) - B) + K, \ y \in U_x.
\]
Since \( F(x) - B \in \mathcal{B}(Y) \), this shows that \( F \) is \( K \)-lower bounded on \( U_x \) and finishes the proof. \( \square \)

In Theorem 2 a stronger assumption like \( K \)-upper boundedness of \( F \) on a “large” set \( A \) does not strengthen the statement. More precisely, a set-valued map \( F : X \to \mathcal{B}(Y) \) which is \( K \)-subadditive and \( K \)-upper bounded on \( A \) need not be locally \( K \)-upper bounded at each point of \( X \).

**Example 1.** Let \( K = [0, \infty) \) and \( F : \mathbb{R} \to \mathcal{B}(\mathbb{R}) \) be given by
\[
F(x) = \begin{cases} 
[0, \frac{1}{|x|}], & x \neq 0, \\
\{0\}, & x = 0.
\end{cases}
\]
Such a set-valued mapping is \( K \)-subadditive and \( K \)-upper bounded e.g. on the set \([1, 2]\) (it is enough to choose \( B = [0, 1] \)). But \( F \) is not \( K \)-upper bounded at 0.

Now, we will prove an analogous result for \( K \)-superadditive set-valued maps. Note, however, that this result can not be obtained as a consequence of Theorem 2. Namely, the \( K \)-superadditivity of a set-valued map \( F \) does not imply the \( K \)-subadditivity of \(-F\).
Example 2. Let \( \mathbb{Q} \) be the set of all rational numbers. Let \( F : \mathbb{R} \to n(\mathbb{R}) \) be given by \( F(x) = [0, |x|] \) for \( x \in \mathbb{R} \). Clearly, \( F \) is \( \mathbb{Q} \)-superadditive. Moreover, the set-valued map \(-F\) given by \(-F(x) = [-|x|, 0]\) for \( x \in \mathbb{R} \) is not \( \mathbb{Q}\)-subadditive (but is \( \mathbb{Q}\)-superadditive).

Similarly, we can find an example where \( K \) is not a group.

Example 3. Let \( K = [0, \infty) \) and \( F : \mathbb{R} \to n(\mathbb{R}) \) be given by \( F(x) = [\sin x - 3, \sin x + 3] \) for \( x \in \mathbb{R} \). Clearly, \( F \) is \( K\)-superadditive, but the set-valued map \(-F\) given by \(-F(x) = [-3 - \sin x, 3 - \sin x]\) for \( x \in \mathbb{R} \) is not \( K\)-subadditive (but is \( K\)-superadditive).

Theorem 3. Let \( X \) and \( Y \) be abelian metric groups with invariant metrics. Assume that \( A \subset X \) is a set which is not null-finite and \( K \) is a subsemigroup of \( Y \). If a set-valued map \( F : X \to B(Y) \) is \( K\)-superadditive and \( K\)-lower bounded on \( A \), then \( F \) is locally \( K\)-lower bounded and locally weakly \( K\)-upper bounded at each point of \( X \).

Proof. Since \( F \) is \( K\)-lower bounded on \( A \), there exists a set \( B \in B(Y) \) such that
\[
F(x) \subset B + K, \quad x \in A. \quad (12)
\]

First we will prove that \( F \) is locally \( K\)-lower bounded at 0. So, suppose that it is not true and \( F \) is not \( K\)-lower bounded on any neighborhood of 0. Consequently, for every \( n \in \mathbb{N} \),
\[
U_n := B_X \left( \frac{1}{2^n} \right) \quad \text{and} \quad B_n := B + B_Y(n) \in B(Y),
\]
there exists \( x_n \in U_n \) such that
\[
F(x_n) \not\subset B_n + K. \quad (13)
\]

Moreover, by the definition of \( U_n \) the sequence \((x_n)_{n \in \mathbb{N}}\) is convergent to 0 in \( X \). The set \( A \) is not null-finite, so there exists \( a \in X \) such that the set \( N_0 := \{ n \in \mathbb{N} : a + x_n \in A \} \) is not finite. Then, by (12) we have
\[
F(a + x_n) \subset B + K, \quad n \in N_0. \quad (14)
\]

Since \( F(-a) \in B(Y) \), we can find \( n_0 \in N_0 \), such that \( F(-a) \subset B_Y(n_0) \). Consequently \( F(-a) \subset B_Y(n) \) for every \( n \geq n_0 \). Fix \( n \in N_0, n \geq n_0 \). In view of \( K\)-superadditivity and (12),
\[
F(x_n) \subset F(-a) + F(a + x_n) + K \subset B_Y(n) + B + K = B_n + K,
\]
which contradicts (13).

Thus \( F \) is locally \( K\)-lower bounded at 0, i.e. there are a neighborhood \( U_0 \) of 0 and a set \( B_0 \in B(Y) \) such that
\[
F(x) \subset B_0 + K, \quad x \in U_0. \quad (15)
\]
Now, we will prove that $F$ is locally $K$-lower bounded at every point of the domain. For a proof by contradiction suppose that $F$ is not $K$-lower bounded on any neighborhood $U_{x_0}$ of some point $x_0$. Consequently, for

$$U_{x_0} := U_0 + x_0 \quad \text{and} \quad B_0 + F(x_0) \in \mathcal{B}(Y),$$

there exists $x_1 \in U_{x_0}$ such that

$$F(x_1) \not\subset B_0 + F(x_0) + K.$$  \hspace{1cm} (16)

Since $x_1 - x_0 \in U_0$, $K$-superadditivity and (15) implies

$$F(x_1) \subset F(x_1 - x_0) + F(x_0) + K \subset B_0 + F(x_0) + K,$$

which contradicts (16) and proves that $F$ is locally $K$-lower bounded at $x_0$.

Finally, we will show that $F$ is locally weakly $K$-upper bounded at each point $x \in X$. Since $F$ is locally $K$-lower bounded at 0, there are a symmetric neighborhood $U_0$ of 0 and a set $B_0 \in \mathcal{B}(Y)$ such that (15) holds. Fix an arbitrary $x \in X$ and take $U_x := U_0 + x$. Take any $y \in U_x$. Then $y = z + x$, where $z \in U_0$. Since $U_0$ is symmetric, also $-z \in U_0$. By the $K$-superadditivity of $F$ and (15) we have

$$F(x) \subset F(x + z) + F(-z) + K \subset F(y) + B_0 + K.$$ \hspace{1cm} (17)

Fix any $z_0 \in F(x)$. By (17) there exist $v \in F(y)$, $b \in B_0$ and $k \in K$ such that $z_0 = v + b + k$. Hence $v = z_0 - b - k$ and, consequently,

$$F(y) \cap ((z_0 - B) - K) \neq \emptyset.$$ 

Since this condition holds for any $y \in U_x$, it proves that $F$ is locally weakly $K$-upper bounded at $x$. The proof is finished. \hfill \Box

The next example shows that in Theorem 3 a weaker assumption like weakly $K$-lower boundedness of $F : X \to \mathcal{B}(Y)$ on a “large” set $A$ does not imply even a weaker conclusion. More precisely, a set-valued map $F$ which is $K$-superadditive and weakly $K$-lower bounded on $A$ need not be locally weakly $K$-lower bounded at each point of $X$.

**Example 4.** Let $K = [0, \infty)$, $a : \mathbb{R} \to \mathbb{R}$ be a discontinuous additive function and $F : \mathbb{R} \to \mathcal{B}(\mathbb{R})$ be given by

$$F(x) = \begin{cases} 
-2|a(x)|, & |x| < 1, \\
-2|a(x)|, & |x| \geq 1.
\end{cases}$$

Such a set-valued mapping is $K$-superadditive and weakly $K$-lower bounded e.g. on the set $[1, 2]$. Moreover, $F$ is not weakly $K$-lower bounded at 0. Indeed, if $F(x) \cap (B + K) \neq \emptyset$ for each $x \in U_0$ a neighbourhood $U_0$ and a bounded set $B$, then $|a(x)| \leq -\inf B$ for each $x \in U_0$ which contradicts the discontinuity of $a$. 
Remark 1. In Theorems 2 and 3 we can not obtain that $F$ is locally $K$-upper bounded (instead of locally weakly $K$-upper bounded) at each point. Consider, for example, the set-valued map $F : \mathbb{R} \to \mathcal{B}(\mathbb{R})$ defined by

$$F(x) = [0, |a(x)|], \quad x \in \mathbb{R},$$

where $a : \mathbb{R} \to \mathbb{R}$ is an additive discontinuous function, and take $K = [0, \infty)$. Then $F$ is $K$-subadditive as well as $K$-superadditive. Moreover, $F$ is locally weakly $K$-upper bounded at every point (it is even weakly $K$-upper bounded on the whole $\mathbb{R}$ because $F(x) \cap ([0, 1] - K) \neq \emptyset$ for every $x \in \mathbb{R}$). $F$ is also locally $K$-lower bounded at every point (it is even $K$-lower bounded on the whole $\mathbb{R}$ because $F(x) \subset [0, 1] + K$ for every $x \in \mathbb{R}$). However, $F$ is not locally $K$-upper bounded at any point. Indeed, if for some open set $U \subset \mathbb{R}$ and some $[m, M] \subset \mathbb{R}$ we had $F(x) \subset [m, M] - K$ for $x \in U$, then $a(x) \leq M$ for $x \in U$, which is impossible because $a$ is discontinuous (by [16][Lemma 9.3.1]).

As an immediate consequence of Theorems 2, 3 and 1, we obtain the following generalization of [3][Corollary 2.4].

Corollary 4. Let $X$ be a complete abelian metric group and $Y$ be an abelian metric group, both with invariant metrics. Assume that $A \subset X$ is a universally measurable non-Haar-null or universally Baire non-Haar-meager set and $K$ is a subsemigroup of $Y$. If a set-valued map $F : X \to \mathcal{B}(Y)$ is $K$-subadditive and weakly $K$-upper bounded on $A$ or $K$-superadditive and $K$-lower bounded on $A$, then $F$ is locally $K$-lower bounded and locally weakly $K$-upper bounded at each point of $X$.

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References

[1] Banakh, T., Głąb, S., Jabłońska, E., Swaczyna, J.: Haar-$\mathcal{I}$ sets: looking at small sets in Polish groups through compact glasses. Diss. Math. 564, 1–105 (2021)
[2] Banakh, T., Jabłońska, E.: Null-finite sets in metric groups and their applications. Israel J. Math. 230, 361–386 (2019)
[3] Bingham, N.H., Jabłońska, E., Jabłoński, W., Ostaszewski, A.J.: On subadditive functions bounded above on a large set. Results Math. 75, 58 (2020)
[4] Bingham, N.H., Ostaszewski, A.J.: Generic subadditive functions. Proc. Am. Math. Soc. 136(12), 4257–4266 (2008)
[5] Bingham, N.H., Ostaszewski, A.J.: Automatic continuity: subadditivity, convexity, uniformity. Aequationes Math. 78(3), 257–270 (2009)
[6] Bingham, N.H., Ostaszewski, A.J.: Additivity, subadditivity and linearity: automatic continuity and quantifier weakening. Indag. Math. (N.S.) 29, 687–713 (2018)
[7] Christensen, J.P.R.: On sets of Haar measure zero in abelian Polish groups. Israel J. Math. 13, 255–260 (1972)
[8] Darji, U.B.: On Haar meager sets. Topology Appl. 160, 2396–2400 (2013)
[9] Elekes, M., Nagy, D.: Haar null and Haar meager sets: a survey and new results. Bull. London Math. Soc. 52, 561–619 (2020)
[10] Feng, Q., Magidor, M., Woodin, H.: Universally baire sets of reals. In: Judah, H., Just, W., Woodin, H. (eds.) Set Theory of the Continuum, pp. 203–242. Springer, New York (1992)
[11] Hille, E., Philips, R.S.: Functional Analysis and Semigroups. American Mathematical Soc, vol. 31. (1957)
[12] Hunt, B.R., Sauer, T., Yorke, J.A.: Prevalence: a translation-invariant almost every on infinite-dimensional spaces. Bull. Am. Math. Soc. 27, 217–238 (1992)
[13] Hunt, B.R., Sauer, T., Yorke, J.A.: Prevalence: an addendum. Bull. Am. Math. Soc. 28, 306–307 (1993)
[14] Jabłońska, E.: Some analogies between Haar meager sets and Haar null sets in abelian Polish groups. J. Math. Anal. Appl. 421, 1479–1486 (2015)
[15] Jabłońska, E., Nikodem, K.: K-midconvex and K-midconcave set-valued maps bounded on large sets. J. Convex Anal. 26, 563–572 (2019)
[16] Kuczma, M.: An Introduction to the Theory of Functional Equations and Inequalities. Cauchy’s Equation and Jensen’s Inequality, PWN – Uniwersytet Śląski, Warszawa–Kraków–Katowice, 1985, 2nd edn. Birkhäuser, Basel–Boston–Berlin (2009)
[17] Matkowski, J.: Subadditive functions and a relaxation of the homogeneity condition of semigroups. Proc. Am. Math. Soc. 117, 991–1001 (1993)
[18] Nikodem, K.: K-Convex and K-Concave Set-valued Functions, Zeszyty Nauk. Politechniki Łódzkiej Mat. 559; Rozprawy Mat. 114, Łódź (1989)
[19] Rosenbaum, R.A.: Sub-additive functions. Duke Math. J. 17, 227–242 (1950)
[20] Smajdor, W.: Subadditive and subquadratic set-valued functions. Uniwersytet Śląski, Katowice (1987)
[21] Smajdor, W.: Superadditive set-valued functions and Banach-Steinhaus theorem. Radovi Mat. 3, 203–214 (1987)

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