On subgraphs of $C_{2k}$-free graphs and a problem of Kühn and Osthus

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Abstract

Let $c$ denote the largest constant such that every $C_6$-free graph $G$ contains a bipartite and $C_4$-free subgraph having $c$ fraction of edges of $G$. Győri et al. showed that $\frac{3}{8} \leq c \leq \frac{2}{5}$. We prove that $c = \frac{3}{8}$. More generally, we show that for any $\varepsilon > 0$, and any integer $k \geq 2$, there is a $C_{2k}$-free graph $G_1$ which does not contain a bipartite subgraph of girth greater than $2k$ with more than $(1 - \frac{1}{2^{k-1}}) \frac{2}{2k-1} (1 + \varepsilon)$ fraction of the edges of $G_1$. There also exists a $C_{2k}$-free graph $G_2$ which does not contain a bipartite and $C_4$-free subgraph with more than $(1 - \frac{1}{2^{k-1}}) \frac{1}{2k-1} (1 + \varepsilon)$ fraction of the edges of $G_2$.

One of our proofs uses the following statement, which we prove using probabilistic ideas, generalizing a theorem of Erdős: For any $\varepsilon > 0$, and any integers $a, b, k \geq 2$, there exists an $a$-uniform hypergraph $H$ of girth greater than $k$ which does not contain any $b$-colorable subhypergraph with more than $(1 - \frac{1}{2^{k-1}}) \frac{1}{2k-1} (1 + \varepsilon)$ fraction of the hyperedges of $H$. We also prove further generalizations of this theorem.

In addition, we give a new and very short proof of a result of Kühn and Osthus, which states that every bipartite $C_{2k}$-free graph $G$ contains a $C_4$-free subgraph with at least $1/(k-1)$ fraction of the edges of $G$. We also answer a question of Kühn and Osthus about $C_{2k}$-free graphs obtained by pasting together $C_{2l}$’s (with $k > l \geq 3$).

1 Introduction

Let $e(H)$ denote the number of (hyper)edges in a (hyper)graph $H$. For a family of graphs $\mathcal{F}$, let $\text{ex}(n, \mathcal{F})$ denote the maximum number of edges in an $n$-vertex graph which does not contain any $F \in \mathcal{F}$ as a subgraph. (In the case when $\mathcal{F} = \{F\}$, we write simply $\text{ex}(n, F)$.) The girth of a graph is defined as the length of a shortest cycle if it exists, and infinity otherwise. In [6], Győri proved that every bipartite, $C_6$-free graph contains a $C_4$-free subgraph with at least half as many edges. Later Kühn and Osthus [9] generalized this result by showing

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Theorem 1.1 (Kühn and Osthus [9]). Let $k \geq 3$ be an integer and $G$ a $C_{2k}$-free bipartite graph. Then $G$ contains a $C_4$-free subgraph $H$ with $e(H) \geq \frac{e(G)}{k-1}$.

In Section 2 we give a new short proof of their result. The complete bipartite graphs $K_{k-1,m}$ (for large enough $m$) show that the factor $\frac{1}{k-1}$ cannot be replaced by anything larger (see Proposition 5 in [9]).

Füredi, Naor and Verstraëte [5] gave another generalization of Győri’s theorem by showing that every $C_6$-free graph $G$ has a subgraph of girth larger than 4 with at least half as many edges as $G$. Again, $K_{2,m}$ shows that this factor cannot be improved. It follows that $\text{ex}(n, C_6) \leq 2 \cdot \text{ex}(n, \{C_4, C_6\})$. Since any graph has a bipartite subgraph with at least half as many edges, Theorem 1.1 shows that $\text{ex}(n, C_{2k}) \leq 2(k-1) \cdot \text{ex}(n, \{C_4, C_{2k}\})$. These results confirm special cases of the compactness conjecture of Erdős and Simonovits [4] which states that for every finite family $\mathcal{F}$ of graphs, there exists an $F \in \mathcal{F}$ such that $\text{ex}(n, F) = O(\text{ex}(n, \mathcal{F}))$.

Since any $C_6$-free graph contains a bipartite subgraph with at least half as many edges, using any of the results above it is easy to show that any $C_6$-free graph $G$ has a bipartite, $C_4$-free subgraph with at least $\frac{1}{4}$ of the edges of $G$. Győri, Kensell and Tompkins [7] improved this factor by showing that

Theorem 1.2 (Győri, Kensell and Tompkins [7]). If $c$ is the largest constant such that every $C_6$-free graph $G$ contains a $C_4$-free and bipartite subgraph $B$ with $e(B) \geq c \cdot e(G)$, then $\frac{3}{8} \leq c \leq \frac{2}{5}$.

The complete graph $K_5$ (as well as a graph consisting of vertex disjoint $K_5$’s) gives that $c \leq \frac{2}{5}$. To show that $\frac{3}{8} \leq c$ they use a probabilistic deletion procedure where they first randomly two-color the vertices, and then delete some additional edges carefully in order to remove the remaining $C_4$’s. In this paper we show that $c = \frac{3}{8}$. In fact, we prove the following two general results; putting $k = 3$ in either of the statements below gives that $c = \frac{3}{8}$. To prove these theorems we will construct graphs by replacing the hyperedges of certain (probabilistically constructed) hypergraphs with fixed small graphs.

Theorem 1.3. For any $\varepsilon > 0$, and any integer $k \geq 2$, there is a $C_{2k}$-free graph $G$ which does not contain a bipartite subgraph of girth greater than $2k$ with more than $(1 - \frac{1}{2k-1})\frac{2}{2k-1}e(G)(1 + \varepsilon)$ edges.

Note that $K_{2k-1}$ is $C_{2k}$-free, and the only subgraphs with girth greater than $2k$ are forests, giving an upper bound of $\frac{2}{2k-1}e(G)(1 + \varepsilon)$. The factor $1 - \frac{1}{2k-1}$ is the probability that a random two-coloring of $K_{2k-1}$ is not monochromatic.

Theorem 1.4. For any $\varepsilon > 0$, and any integer $k \geq 2$, there is a $C_{2k}$-free graph $G$ which does not contain a bipartite and $C_4$-free subgraph with more than $(1 - \frac{1}{2k-1})\frac{1}{k-1}e(G)(1 + \varepsilon)$ edges.

Theorem 1.4 improves the upper bound of $\frac{1}{k-1}e(G)(1 + \varepsilon)$, which is given by the complete bipartite graphs $K_{k-1,m}$. Take a random bipartition of the vertices of $K_{k-1,m}$ and consider the bipartite subgraph $B$ between the colour classes of this bipartition. The factor $(1 - \frac{1}{2k-1})\frac{1}{k-1}$ in the above theorem is the limit of the expected value of the fraction of
edges of \( K_{k-1,m} \) in the biggest \( C_4 \)-free subgraph of \( B \) as \( m \to \infty \). (Note that because any graph has a bipartite subgraph with at least half of its edges, Theorem 1.1 implies that every \( C_{2k} \)-free graph contains a bipartite and \( C_4 \)-free subgraph with at least \( \frac{1}{2(k-1)} \) fraction of its edges.) Interestingly, our proofs use theorems about hypergraphs that are generalizations of the following theorem of Erdős [2].

Every graph \( G \) has a bipartite subgraph with at least \( \frac{1}{2} \) as many edges as \( G \), and the complete graph \( K_n \) shows that the factor \( \frac{1}{2} \) cannot be improved. Interestingly, Erdős showed that even if one requires girth to be large, the factor \( \frac{1}{2} \) still cannot be improved.

More precisely,

**Theorem 1.5 (Erdős [2]).** For any \( \varepsilon > 0 \), and any integer \( k \geq 2 \), there exists a graph \( G \) with girth greater than \( k \) which does not contain a bipartite subgraph with more than \( \frac{1}{2}e(G)(1 + \varepsilon) \) edges.

In Section 3, we prove a series of lemmas about hypergraphs which are broad generalizations of Theorem 1.5, and which may be of independent interest. These lemmas have the theme that for most hypergraphs, every fixed coloring behaves like a random coloring with color classes of the same sizes as in the fixed coloring. Our proof of Theorem 1.4 uses these general lemmas directly. The proof of Theorem 1.3 uses a more direct analogue of the above statement for hypergraphs: Theorem 1.7, which we present below. We will prove Theorem 1.7 from the more general lemmas.

A Berge-cycle of length \( l \) in a hypergraph \( H \) is a subhypergraph consisting of \( l \geq 2 \) distinct hyperedges \( e_1, \ldots, e_l \) and containing \( l \) distinct vertices \( v_1, \ldots, v_l \) (called its defining vertices), such that \( v_i \in e_i \cap e_{i+1}, \ i = 1, \ldots, l \), where addition in the indices is taken modulo \( l \). The girth of a hypergraph \( H \) is the length of a shortest Berge-cycle if it exists, and infinity otherwise. (Note that having girth greater than \( 2 \) implies that no two hyperedges share more than one vertex.) A hypergraph is \( b \)-colorable if there is a coloring of its vertices using \( b \) colors so that none of its hyperedges are monochromatic. Erdős and Hajnal [3] showed the existence of hypergraphs of any uniformity, arbitrarily high girth and arbitrarily high chromatic number. Lovász [10] gave a constructive proof for this; several newer proofs exist as well. The following simple proposition is easy to see. We include its proof for completeness.

**Proposition 1.6.** For any integers \( a, b \geq 2 \), every \( a \)-uniform hypergraph \( H \) contains a \( b \)-colorable subhypergraph with at least \( (1 - \frac{1}{b^a-1}) e(H) \) hyperedges.

**Proof.** Color each vertex of \( H \) randomly and independently, using \( b \) colors with equal probability. For each hyperedge \( f \) of \( H \), the probability that \( f \) is monochromatic is \( \frac{b}{b^a} = \frac{1}{b^{a-1}} \). Therefore, the expected number of monochromatic hyperedges in \( H \) is \( \frac{e(H)}{b^{a-1}} \). So there exists a coloring of the vertices of \( H \) such that there are at most \( \frac{e(H)}{b^{a-1}} \) monochromatic hyperedges in that coloring. Thus, the subhypergraph of \( H \) consisting of all the non-monochromatic hyperedges of \( H \) contains at least \( (1 - \frac{1}{b^{a-1}}) e(H) \) hyperedges and is \( b \)-colorable, as desired. \( \square \)

Again the complete \( a \)-uniform hypergraph shows that the factor \( (1 - \frac{1}{b^{a-1}}) \) cannot be improved in the above proposition. We show that (as in case of graphs), this factor cannot be improved even if one requires the girth to be large.
Theorem 1.7. For any $\varepsilon > 0$, and any integers $a, b, k \geq 2$, there exists an $a$-uniform hypergraph $H$ of girth more than $k$ which does not contain a $b$-colorable subhypergraph with more than $(1 - \frac{1}{e^a}) e(H) (1 + \varepsilon)$ hyperedges.

Clearly, letting $a = b = 2$ in the above theorem, we get Theorem 1.5. The hypergraph lemmas in Section 3 can be used to prove statements analogous to Theorem 1.7 with different notions of colorability. As an example application, we will prove the analogous Proposition 3.8 about strong (or rainbow) colorable subhypergraphs. More generally, a graph $G$ is called $H$-colorable (where $H$ is a fixed graph) if there is a homomorphism $G \to H$. Our Lemma 3.4 can be said to generalize the notion of $H$-coloring to hypergraphs, and allow for proving statements similar to Theorem 1.7 for $H$-colorability or analogous hypergraph conditions.

In Section 5, we answer a question of Kühn and Osthus in [9]. A graph is said to be pasted together from $C_2l$’s if it can be obtained from a $C_2l$ by successively adding new $C_2l$’s which have at least one edge in common with the previous ones.

**Question 1.8** (Kühn, Osthus [9]). Given integers $k > l \geq 2$, does there always exist a number $d = d(k)$ such that every $C_{2k}$-free graph which is pasted together from $C_2l$’s has average degree at most $d$?

Kühn and Osthus show in [9] that an affirmative answer to the above question, even when restricted to bipartite graphs, would imply that any $C_{2k}$-free graph contains a $C_{2l}$-free subgraph containing a constant fraction of the edges of $G$. They gave a positive answer to the question when $l = 2$ and the graph is bipartite: they showed that if $k \geq 3$ is an integer and $G$ is a bipartite $C_{2k}$-free graph which is obtained by pasting together $C_4$’s, then the average degree of $G$ is at most $16k$.

We answer Question 1.8 negatively by showing two different pastings of $C_6$’s to form a $C_8$-free graph with high average degree. These two examples show (in two very different ways) that many $C_6$’s can be packed into a graph while still keeping it $C_8$-free. We will show that the first example can be easily generalized to any pair $k, l$ with $k > l \geq 3$, showing that $l = 2$ is the only case when any $C_{2k}$-free graph obtained by pasting together $C_2l$’s has average degree bounded by a constant $d = d(k)$.

**Our paper is organized as follows:** In Section 2, we give a short proof of Theorem 1.1. In Section 3, we prove a series of hypergraph lemmas and Theorem 1.7. Our proofs in Section 3 use counting arguments and probabilistic ideas very similar to Erdős’s proof. In Section 4 we prove Theorem 1.3 and Theorem 1.4. In Section 5, we give two examples of pasting together $C_6$’s to form a $C_8$-free graph with high average degree, answering Question 1.8.

2 A simple proof of a theorem of Kühn and Osthus (Theorem 1.1)

**Proof of Theorem 1.1.** Let $G$ be a $C_{2k}$-free bipartite graph with color classes $A := \{a_1, a_2, \ldots, a_m\}$ and $B := \{b_1, b_2, \ldots, b_n\}$ for some $m, n \geq 1$. Order the vertices in $A$ and $B$
as \( a_1 < a_2 < \ldots < a_m \) and \( b_1 < b_2 < \ldots < b_n \) respectively. An edge \( ab \in E(G) \) with \( a \in A \) and \( b \in B \) is denoted by the ordered pair \((a, b)\).

We define a partial order \( \mathcal{P} = (E(G), \leq_p) \) on the edge set of \( G \) as follows. For any two edges \( (a, b), (a', b') \in E(G) \), we say that \( (a, b) \leq_p (a', b') \) if and only if there exists an integer \( r \geq 1 \) and edges \((p_i, q_i) \in E(G), i = 1, 2, \ldots, r\) such that \( a = p_1, b = q_1 \) and \( a' = p_r, b' = q_r \), and the following conditions hold: \( p_i < p_{i+1} \) and \( q_i < q_{i+1} \) and the vertices \( p_i, q_i, p_{i+1}, q_{i+1} \) induce a \( C_4 \) for all \( 1 \leq i < r - 1 \).

It is easy to see that if there is a chain of length \( k \) in \( \mathcal{P} \) then \( G \) contains a cycle of length \( 2k \), a contradiction (see Figure 1). So the length of a longest chain in \( \mathcal{P} \) is at most \( k - 1 \) which implies that the size of a largest antichain in \( \mathcal{P} \) is at least \( \frac{1}{k-1}|E(G)| \) by Mirsky’s theorem [11]. Since \( G \) is bipartite, any \( C_4 \) in \( G \) contains two edges \((p_1, q_1), (p_2, q_2) \in E(G)\) such that \((p_1, q_1) <_p (p_2, q_2)\), so the subgraph \( H \) of \( G \) consisting of the edges in this largest antichain is \( C_4 \)-free, completing the proof of the theorem. \( \square \)

### 3 Hypergraph lemmas and proof of Theorem 1.7

Let \( \mathcal{H}(a, n, m) \) denote the family of all \( a \)-uniform hypergraphs with \( n \) vertices and \( m \) hyperedges for some \( a \geq 2 \). \(|\mathcal{H}(a, n, m)| = \begin{pmatrix} n \end{pmatrix}_m^a \). Given a coloring \( C : [n] \to [b] \) of the vertex set \([n]\) with \( b \) colors (with \( b \geq 2 \)), let \( n^e_j \) be the number of vertices of color \( j \). The multiset of the colors of the vertices of a hyperedge \( e \) (with the multiplicity with which they occur in \( e \)) is called the color multiset of \( e \) (with respect to \( C \)). For an \( a \)-element multiset of colors \( T \), let \( p^C(T) \) be the probability that the color multiset of a random hyperedge of the complete \( a \)-uniform hypergraph on \( n \) vertices, with the coloring \( C \), is \( T \). (Note that in this paper, when we mention a coloring, we mean an arbitrary coloring of the vertex set, not necessarily a proper coloring of a hypergraph, unless indicated.)

**Proposition 3.1.** For \( n \to \infty \), asymptotically

\[
p^C(T) \sim \frac{\prod_{j=1}^b \binom{n^e_j}{I_T(j)}}{\binom{n}{a}} \cdot \frac{a!}{\prod_{j=1}^b I_T(j)!}
\]

where \( I_T(j) \) denotes the multiplicity of \( j \) in the multiset \( T \).

We will also use the following tail bound on the binomial and the hypergeometric distributions. Hoeffding proves this bound in a more general setting, see Section 2 in [8] for the binomial distribution and Section 6 for the hypergeometric distribution. If a random
variable $X$ has binomial distribution with $m$ trials and success probability $p$, we write $X \sim \text{Binomial}(m, p)$. If $X$ has hypergeometric distribution with a population of size $N$ containing $pN$ successes, and with $m$ draws, we write $X \sim \text{Hypergeometric}(pN, N, m)$.

**Proposition 3.2.** Let $m, N \in \mathbb{N}$ and $p, \varepsilon \in [0, 1]$, and let $X$ be a random variable with $X \sim \text{Binomial}(m, p)$ or $X \sim \text{Hypergeometric}(pN, N, m)$. Then

$$P(|X - pm| > \varepsilon m) \leq 2e^{-2 \varepsilon^2 m}.$$

**Lemma 3.3.** Let $n \to \infty$ and $\frac{m}{n} \to \infty$. For any fixed $\varepsilon > 0$, for every hypergraph $H$ in $\mathcal{H}(a, n, m)$, with the exception of $o\left(\binom{n}{m}\right)$ hypergraphs, the following holds:

For any coloring $C$ of the vertex set $[n]$ with $b$ colors, and any $a$-element multiset of colors $T$, the number of hyperedges of $H$ whose color multiset is $T$ is

$$P(C(T) \leq n^a \pm \varepsilon) (p^C(T) \pm \varepsilon) m. \quad (1)$$

(Note: In this paper, whenever we write $X = Y \pm \varepsilon$, we mean $X \in [Y - \varepsilon, Y + \varepsilon]$.)

**Proof.** Let $u$ be the number of hypergraphs in $\mathcal{H}(a, n, m)$ for which (1) does not hold. Corresponding to each such hypergraph $H$ there is at least one $b$-coloring $C$ of its vertices, and a multiset of colors $T$, such that (1) does not hold for $C$ and $T$. Therefore

$$u \leq |\{(H, C, T) : H \in \mathcal{H}(a, n, m), (1) \text{ does not hold for } H, C \text{ and } T\}|.$$

The number of $b$-colorings of $n$ vertices with $b$ fixed colors is $|C| = b^n$. The number of multisets of $a$ elements of $b$ colors is $\binom{a+b-1}{a}$. Therefore

$$u \leq b^n \binom{a+b-1}{a} \max_{\text{multiset of colors } T} \left\{ \left. \frac{H \in \mathcal{H}(a, n, m)}{(1) \text{ does not hold for } H, C \text{ and } T} \right\} \right|.$$

Fix a $b$-coloring $C$ and a multiset of colors $T$. A hypergraph $H \in \mathcal{H}(a, n, m)$ consists of $m$ hyperedges, out of $\binom{n}{a}$ possibilities. Out of all possible hyperedges, $p^C(T) \binom{n}{a}$ have $T$ as their color multiset. So $|\{e \in H : C(e) = T\}| \sim \text{Hypergeometric}(p^C(T) \binom{n}{a}, \binom{n}{a}, m)$. (1) fails to hold for $H, C$ and $T$ if

$$\left| \left| \{e \in H : C(e) = T\} - p^C(T)m \right| > \varepsilon m. \right|$$

By the tail bound for the hypergeometric distribution in Proposition 3.2, the number of hypergraphs $H \in \mathcal{H}(a, n, m)$ for which this holds is at most

$$\binom{n}{m} \cdot 2e^{-2 \varepsilon^2 m}, \text{ so}$$

$$u \leq 2b^n \binom{a+b-1}{a} e^{-2 \varepsilon^2 m} \binom{n}{m} = o \left( \binom{n}{m} \right)$$

as $\frac{m}{n} \to \infty.$
The following lemma is a corollary of Lemma 3.3.

**Lemma 3.4.** Let $\mathcal{T}$ be a family of multisets of $a$ elements which are in $[b]$. Let $n \to \infty$ and $\frac{a}{n} \to \infty$. For a $b$-coloring of $n$ vertices $C$, let $p^C(\mathcal{T}) = \sum_{T \in \mathcal{T}} p^C(T)$ (that is, the probability that the color multiset of a random hyperedge of the complete hypergraph is in $\mathcal{T}$); and let $C_M$ be a $b$-coloring for which $p^C(\mathcal{T})$ takes its maximum. For a hypergraph $H \in \mathcal{H}(a, n, m)$, let $q(H)$ be the number of hyperedges in the biggest subhypergraph of $H$ which is colorable in such a way that the color multiset of every hyperedge of $H$ is in $\mathcal{T}$. For any fixed $\epsilon > 0$, for every hypergraph $H$ in $\mathcal{H}(a, n, m)$, with the exception of $a\binom{n}{m}$ hypergraphs,

$$q(H) \leq p^{C_M}(\mathcal{T})m(1 + \epsilon). \quad (2)$$

**Proof.** If $\mathcal{T} = \emptyset$, then $q(H) = 0$ for any $H$. From now we assume that $\mathcal{T} \neq \emptyset$. We show that we may also assume that $p^{C_M}(\mathcal{T}) > \frac{|\mathcal{T}|}{2b^a}$ when $n$ is sufficiently large. Let $T \in \mathcal{T}$, and let $C_E$ be a $b$-coloring in which every color class has a size $\approx \frac{a}{n}$. Then, by Proposition 3.1, asymptotically

$$p^{C_E}(T) = \frac{a!}{b^a \prod_{j=1}^b I_T(j)!} \geq \frac{1}{b^a}.$$

Since $p^{C_M}(\mathcal{T}) \geq p^{C_E}(T) = \sum_{T \in \mathcal{T}} p^{C_E}(T)$, for sufficiently large $n$, $p^{C_M}(\mathcal{T}) \geq \frac{|\mathcal{T}|}{2b^a}$.

An equivalent definition of the function $q$ is

$$q(H) = \max_{b\text{-coloring } C} |\{ e \in H : c^C(e) \in \mathcal{T} \}|.$$

We use Lemma 3.3 with $\frac{\epsilon}{2b^a}$ in place of $\epsilon$. For almost every hypergraph $H \in \mathcal{H}(a, n, m)$, for every coloring $C$,

$$|\{ e \in H : c^C(e) \in \mathcal{T} \}| = \sum_{T \in \mathcal{T}} |\{ e \in H : c^C(e) = T \}| \leq \sum_{T \in \mathcal{T}} \left( p^C(T) + \frac{\epsilon}{2b^a} \right) m = \left( p^C(T) + \frac{|\mathcal{T}|}{2b^a} \right) m \leq p^{C_M}(\mathcal{T})m(1 + \epsilon)$$

using that $p^{C_M}(\mathcal{T}) \geq \frac{|\mathcal{T}|}{2b^a}$. \qed

We define an **oriented hypergraph** as a set of ordered sequences without repetition (called hyperedges) over a vertex set, such that two hyperedges are not allowed to differ only in their order. (The order of the vertices on different hyperedges is independent of each other.) An oriented hypergraph is thus equivalent to a hypergraph along with a total order on the vertices of each hyperedge. Let $\mathcal{O}(a, n, m)$ denote the family of all $a$-uniform oriented hypergraphs with $n$ vertices and $m$ hyperedges. (Note that other meanings of the term “oriented hypergraph” exist in the literature.)

Let $C : [n] \to [b]$ be a coloring of the vertex set $[n]$ with $b$ colors ($b \geq 2$). We call the color sequence (with respect to $C$) of an $a$-tuple of vertices $e = (v_1, \ldots, v_a)$ the sequence $c^C(e) = (C(v_1), \ldots, C(v_a))$. If we choose a random $a$-tuple of the vertex set $V$
without repetition, the probability that its color sequence is a given sequence of colors \( s = (s_1, \ldots, s_a) \) is

\[
\frac{1}{\binom{n}{a}} \frac{n_a!}{\prod_{j=1}^{b} (n_j^c - |\{ i \in [a] : s_i = j \}|)!} \sim \prod_{i=1}^{a} \frac{n_i^c}{n} \]

if \( n \to \infty \).

The following lemma is a variant of Lemma 3.3 for oriented hypergraphs.

**Lemma 3.5.** Let \( n \to \infty \) and \( \frac{m}{n} \to \infty \). For any fixed \( \varepsilon > 0 \), for every oriented hypergraph \( O \in \mathcal{O}(a, n, m) \), with the exception of \( o(\mathcal{O}(a, n, m)) \) hypergraphs, the following holds:

For any coloring \( C \) of the vertex set \([n]\) with \( b \) colors, and any \( a \)-tuple of colors \( s \), the number of hyperedges of \( O \) whose color sequence is \( s \) is \((\prod_{i=1}^{a} \frac{n_i^c}{n} \pm \varepsilon) \cdot m\). (3)

**Proof.** We use Lemma 3.3 with \( \frac{1}{a} \) in the place of \( \varepsilon \), i.e. that (1) holds (with \( \frac{1}{a} \)) for almost every hypergraph \( H \in \mathcal{H}(a, n, m) \). In every hypergraph in \( \mathcal{H}(a, n, m) \), the hyperedges can be ordered in the same number of ways: \( (a!)^m \). So for almost every \( O \in \mathcal{O}(a, n, m) \), (1) holds for the corresponding hypergraph (obtained by forgetting the orders on the hyperedges).

Let \( \hat{\mathcal{O}}(a, n, m) \subset \mathcal{O}(a, n, m) \) be the family of oriented hypergraphs for which (1) holds (forgetting the orders) with \( \frac{1}{a} \) in the place of \( \varepsilon \). Let \( u \) be the number of oriented hypergraphs in \( \hat{\mathcal{O}}(a, n, m) \) for which (3) does not hold. Corresponding to each such oriented hypergraph \( O \in \hat{\mathcal{O}}(a, n, m) \), there is at least one \( b \)-coloring \( C \) of its vertices, and an \( a \)-tuple of colors \( s \), such that (3) does not hold for \( C \) and \( s \). Therefore

\[
u \leq |\{(O, C, s) : O \in \hat{\mathcal{O}}(a, n, m), (3) \text{ does not hold for } O, C \text{ and } s\}|.
\]

The number of \( b \)-colorings of \( n \) vertices with \( b \) fixed colors is \( |C| = b^n \). The number of \( a \)-tuples of \( b \) colors is \( b^a \). Therefore

\[
u \leq b^{n+a} \max_{\text{coloring } C} \max_{\text{a-tuple of colors } s} |\{(O, C, s) : O \in \hat{\mathcal{O}}(a, n, m), (3) \text{ does not hold for } O, C \text{ and } s\}|.
\]

Fix a \( b \)-coloring \( C \) and an \( a \)-tuple of colors \( s \). Let \( T \) be the multiset consisting of the elements of \( s \) with the multiplicity with which they occur in \( s \) (that is, \( T \) is \( s \) forgetting the order). If (1) holds for a \( H \in \mathcal{H}(a, n, m) \) with \( \frac{1}{a} \), the number of hyperedges whose color multiset is \( T \) is

\[
M_H := \left( p^C(T) + \frac{\varepsilon}{4} \right) m = \left( \prod_{j=1}^{b} \left( \frac{n_j^c}{n} \right)^{I_T(j)} \cdot \frac{a!}{\prod_{j=1}^{b} I_T(j)!} \right) \frac{\varepsilon}{2} m
\]

using the Proposition 3.1 for large enough \( n \). (Changing \( \frac{1}{a} \) to \( \frac{1}{a} \) accounts for the fact that Proposition 3.1 is asymptotic.) We can obtain an oriented hypergraph from \( H \) by
ordering its hyperedges in one of the $a!$ possible ways, independently from each other. If we take a hyperedge whose color multiset is $T$, some of these orders yield the color sequence $s$. The number of such orders is $\prod_{j=1}^{b} I_T(j)!$, so if we take a random ordering of a hyperedge whose color multiset is $T$, the probability that it has color sequence $s$ is

$$\frac{\prod_{j=1}^{b} I_T(j)!}{a!}.$$  

So if we obtain an oriented hypergraph $O$ by randomly ordering every hyperedge of $H$, then $|\{e \in O : c^T(e) = s\}| \sim \text{Binomial}(\sum_{j=1}^{a} \frac{n_j^C}{a} \pm \frac{\varepsilon}{2})m$. And the expected value of the number of hyperedges whose color sequence is $s$ is

$$E_H := \frac{\prod_{j=1}^{b} I_T(j)!}{a!} \sum_{j=1}^{a} \frac{n_j^C}{a} \pm \frac{\varepsilon}{2}.$$  

If the number of hyperedges whose color sequence is $s$ is in the range $[E_H - \frac{\varepsilon}{2}m, E_H + \frac{\varepsilon}{2}m]$, then (3) holds for $O$, $C$ and $s$, since

$$E_H \pm \frac{\varepsilon}{2}m = \left(\prod_{i=1}^{a} \frac{n_i^C}{n} \pm \varepsilon\right)m.$$  

We want to bound the probability that in a randomly selected oriented hypergraph obtained from $H$, the number of hyperedges whose color sequence is $s$ is not in the range $[E_H - \frac{\varepsilon}{2}m, E_H + \frac{\varepsilon}{2}m]$. By the tail bound for the binomial distribution in Proposition 3.2, this probability is at most

$$2 \cdot e^{-\frac{\varepsilon^2}{2}} \sum_{j=1}^{a} \frac{n_j^C}{a} \pm \frac{\varepsilon}{2} \leq 2e^{-\frac{(\varepsilon/2)^2}{2}m}, \text{ so}$$

$$u \leq b^n + 2e^{-\frac{(\varepsilon/2)^2}{2}m} |\mathcal{O}(a, n, m)| = o(|\mathcal{O}(a, n, m)|)$$

as $\frac{m}{n} \to \infty$.  

\begin{lemma}
Let $n \to \infty$, $k \geq 2$ and $m = o\left(n^{1+\frac{1}{k}}\right)$. Every hypergraph $H$ in $\mathcal{H}(a, n, m)$, with the exception of $o\left(\binom{n}{m}\right)$ hypergraphs, has at most $n$ Berge-cycles with $k$ or fewer hyperedges.
\end{lemma}

\begin{proof}
A Berge-cycle of length $l$ has $l$ defining vertices, and each of its $l$ hyperedges contains $a - 2$ additional vertices. So the number of Berge-cycles of length $l$ is less than $n^{(a-1)l}$. The number of hypergraphs in $\mathcal{H}(a, n, m)$ which contain a fixed Berge-cycle of length $l$ is $\binom{C}{m-l}$, since the $l$ hyperedges of the Berge-cycle can be arbitrarily extended to a hypergraph of $m$ hyperedges. Therefore the number of pairs $(H, B)$ where $H \in \mathcal{H}(a, n, m)$ and $B$ is any Berge-cycle of length $l$ in $H$, is less than

$$n^{(a-1)l} \binom{n}{a-l} \binom{m}{m-l} \leq O\left(\frac{m}{n}\right)^l \left(\frac{n}{m}\right)\binom{m}{a}.$$

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Let \( f_k(H) \) denotes the number of Berge-cycles of length \( k \) or less in \( H \). Using \( m = o\left(\frac{n^{1+\frac{1}{k}}}{\varepsilon} \right) \), we have

\[
\sum_{H \in \mathcal{H}(a,n,m)} \frac{f_k(H)}{n^{k+1}} = \sum_{l=2}^{k} O\left(\frac{m^l}{n^l} \binom{n}{m}^l \right) = O\left(\frac{m^k}{n^k} \binom{n}{m} \right) = o\left(\frac{n^{k+1}}{\varepsilon} \right).
\]

The number of hypergraphs \( H \in \mathcal{H}(a,n,m) \) with more than \( n \) Berge-cycles of length \( k \) or less is clearly

\[
\frac{o\left(n^{\binom{k}{m}} \right)}{n} = o\left(\binom{n}{m} \right),
\]

proving Lemma 3.6.

**Proposition 3.7.** For any \( \varepsilon > 0 \) and \( k \geq 2 \), there exists an \( a \)-uniform hypergraph \( H \) of girth more than \( k \) for which (1) in Lemma 3.3 and (2) in Lemma 3.4 hold. There also exists an \( a \)-uniform oriented hypergraph \( O \) of girth more than \( k \) (using the usual meaning of girth, not taking the orders on the hyperedges into consideration) for which (3) in Lemma 3.5 holds.

**Proof.** Take a sufficiently large \( n \), and \( m = o\left(\frac{n^{1+\frac{1}{k}}}{\varepsilon} \right) \) but such that \( \frac{m}{n} \to \infty \) as \( n \to \infty \).

Then there is a hypergraph \( H \in \mathcal{H}(a,n,m) \) such that (1) in Lemma 3.3 holds with \( \frac{\varepsilon}{2m} \) in place of \( \varepsilon \), and \( H \) contains at most \( n \) Berge-cycles with \( k \) or fewer hyperedges (indeed, all but \( o\left(\frac{n^{\binom{a}{m}}}{m} \right) \) hypergraphs have both properties). Now remove a hyperedge from every Berge-cycle of length \( k \) or smaller in \( H \). The resulting hypergraph \( H' \) has \( m - n \) hyperedges. Fix any coloring \( C \) and an \( a \)-element multiset of colors \( T \). In \( H \), the number of hyperedges whose color multiset with respect to \( C \) is \( T \) is \( (p^C(T) \pm \frac{\varepsilon}{2m}) m \). The number of such hyperedges in \( H' \) is at least \( (p^C(T) - \frac{\varepsilon}{2m}) m - n \) and at most \( (p^C(T) + \frac{\varepsilon}{2m}) m \), so it is in the range \( (p^C(T) \pm \frac{\varepsilon}{2m}) (m - n) \) for big enough \( n \) because \( \frac{m}{n} \to \infty \). So (1) in Lemma 3.3 holds for \( H' \), even with \( \frac{\varepsilon}{2m} \) in the place of \( \varepsilon \). From the proof of Lemma 3.4 it is clear that if (1) holds with \( \frac{\varepsilon}{2m} \), then (2) holds.

In every hypergraph in \( \mathcal{H}(a,n,m) \), the hyperedges can be ordered in the same number of ways, so Lemma 3.6 holds for oriented hypergraphs too. The proof in the previous paragraph works similarly for oriented hypergraphs, proving the existence of \( O \).

Now we use Proposition 3.7 to prove Theorem 1.7.

**Proof of Theorem 1.7.** By Proposition 3.7, there is an \( a \)-uniform hypergraph \( H \) of girth more than \( k \) for which (2) in Lemma 3.4 holds. We use (2) with \( T \) consisting of those multisets which contain at least two different colors, and with \( \frac{\varepsilon}{2} \) in the place of \( \varepsilon \).

With the notation of Lemma 3.4,

\[
q(H) < p^C(T) m \left(1 + \frac{\varepsilon}{2}\right) = \left(\sum_{T \in \mathcal{T}} p^C(T)\right) m \left(1 + \frac{\varepsilon}{2}\right)
\]

\[
= \left(1 - \sum_{j=1}^{b} p^C(T)\left(\frac{a}{j, j, \ldots, j}\right)\right) m \left(1 + \frac{\varepsilon}{2}\right) \leq \left(1 - \sum_{j=1}^{b} \frac{n^{C(T)}}{m} \right) m \left(1 + \varepsilon\right)
\]

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using the asymptotic Proposition 3.1 for large enough \( n \). \( \sum_{j=1}^{b} n_{j}^{C_{M}} = n \), and using the power mean inequality we get that

\[
\left( \frac{1}{b} \sum_{j=1}^{b} \left( \frac{n_{j}^{C_{M}}}{n} \right)^{a} \right)^{\frac{1}{a}} \geq \frac{1}{b}.
\]

So \( \sum_{j=1}^{b} \left( \frac{n_{j}^{C_{M}}}{n} \right)^{a} \geq \frac{1}{b^{1-a}} \), which implies the statement. \( \square \)

We show another example application of Lemma 3.4 and Proposition 3.7. A \( b \)-coloring of the vertices of a hypergraph is called a rainbow (or strong) coloring if all the vertices have different colors in every hyperedge. (For \( a \)-uniform hypergraphs, this is only possible if \( a \leq b \).)

**Proposition 3.8.** Let \( n \to \infty \) and \( \frac{m}{n} \to \infty \). For any fixed \( \varepsilon > 0 \) and integers \( 2 \leq a \leq b \), every hypergraph \( H \) in \( \mathcal{H}(a,n,m) \), with the exception of \( o \left( \left( \frac{n}{a} \right)^{m} \right) \) hypergraphs, contains no subhypergraph that is rainbow colorable with \( b \) colors with more than \( \binom{b}{a} \frac{a!}{b^{a}} e(H) (1 + \varepsilon) \) hyperedges. Furthermore, for any \( \varepsilon > 0 \) and integers \( k \geq 2 \) and \( 2 \leq a \leq b \), there exists an \( a \)-uniform hypergraph \( H \) of girth more than \( k \) which does not contain a subhypergraph that is rainbow colorable with \( b \) colors with more than \( \binom{b}{a} \frac{a!}{b^{a}} e(H) (1 + \varepsilon) \) hyperedges.

**Proof.** A hypergraph coloring is a rainbow coloring if the color multiset of every hyperedge is a conventional set (i.e., every color appears at most once in the multiset). Let \( T = \binom{[b]}{a} \).

We will prove that if (2) in Lemma 3.4 holds for a hypergraph \( H \) with this \( T \) and with \( \varepsilon' = \varepsilon \), then it does not contain a subhypergraph that is rainbow colorable with \( b \) colors with more than \( \binom{b}{a} \frac{a!}{b^{a}} e(H) (1 + \varepsilon) \) hyperedges. The first statement of the proposition then follows directly from Lemma 3.4, while the second statement follows from Lemma 3.7.

With the notation of Lemma 3.4, and using the asymptotic Proposition 3.1 for large enough \( n \),

\[
q(H) < p^{C_{M}}(T) m \left( 1 + \frac{\varepsilon}{2} \right) = \left( \sum_{T \in \mathcal{T}} p^{C_{M}}(T) \right) m \left( 1 + \frac{\varepsilon}{2} \right)
\leq \left( \sum_{T \in \mathcal{T}} \prod_{j \in T} \frac{n_{j}^{C_{M}}}{n} \right)^{a! m (1 + \varepsilon)}. \tag{4}
\]

We claim that, under the assumption that \( \sum_{j=1}^{b} n_{j}^{C_{M}} = n \), (4) takes its maximum when \( n_{1}^{C_{M}} = \ldots = n_{b}^{C_{M}} = \frac{n}{b} \). Let us assume that the \( n_{j}^{C_{M}} \)'s are not all equal – then there is a
j_1 and j_2 such that n_{j_1}^{C_M} < \frac{n}{b} < n_{j_2}^{C_M}. Rewriting the first factor in (4), we have

$$
\sum_{T \in \mathcal{T}} \prod_{j \in T} \frac{n_{j_2}^{C_M}}{n} = \sum_{T \in \binom{[a]}{a_1,j_1,j_2}} \prod_{j \in T} \frac{n_{j_2}^{C_M}}{n} + (n_{j_2}^{C_M} + n_{j_2}^{C_M}) \sum_{T \in \binom{[a]}{a_2,j_1,j_2}} \prod_{j \in T} \frac{n_{j_2}^{C_M}}{n}.
$$

If we replace $n_{j_1}^{C_M}$ with $\frac{n}{b}$, and $n_{j_2}^{C_M}$ with $n_{j_2}^{C_M} - \frac{n}{b} + n_{j_1}^{C_M}$, (4) does not decrease: the first two terms do not change, while in the third term, $n_{j_1}^{C_M} n_{j_2}^{C_M}$ is replaced by $\frac{n}{b} \left( n_{j_2}^{C_M} - \frac{n}{b} + n_{j_1}^{C_M} \right) = n_{j_1}^{C_M} n_{j_2}^{C_M} + (n_{j_2}^{C_M} - \frac{n}{b}) \left( \frac{n}{b} - n_{j_1}^{C_M} \right) > n_{j_1}^{C_M} n_{j_2}^{C_M}$.

Repeating this step, we can increase the number of $n_{j_1}^{C_M}$’s which equal $\frac{n}{b}$ without decreasing (4), until all of them equal $\frac{n}{b}$.

So

$$
q(H) \leq \left( \sum_{T \in \mathcal{T}} \prod_{j \in T} \frac{1}{b} \right) a! m (1 + \varepsilon) = \binom{b}{a} \frac{a!}{b^a} m (1 + \varepsilon).
$$

\[ \square \]

4 Subgraphs of $C_{2k}$-free graphs – Proof of Theorems 1.3 and 1.4

**Proof of Theorem 1.3.** Fix \( \varepsilon > 0 \). By Theorem 1.7, there exists a $2k-1$-uniform hypergraph $H$ with girth more than $2k$ which does not contain a 2-colorable subhypergraph having more than \( (1 - \frac{1}{2^{2k-2}}) e(H) (1 + \varepsilon) \) hyperedges. We produce a graph $G_H$ from the hypergraph $H$ by replacing each hyperedge of $H$ with a complete graph (i.e. a clique) on $2k-1$ vertices. We refer to these complete graphs as $2k-1$-cliques. It is easy to check that the resulting graph $G_H$ is $C_{2k}$-free.

Notice that since the girth of $H$ is more than $2k \geq 4$, no two hyperedges of $H$ intersect in more than one vertex. Therefore, the $2k-1$-cliques of $G_H$ are edge-disjoint, and by definition every edge of $G_H$ is in some $2k-1$-clique. We show that $G_H$ does not have a bipartite subgraph with girth more than $2k$ which has more than \( (1 - \frac{1}{2^{2k-2}}) \frac{2k-2}{(2k-2)!} e(G_H) (1 + \varepsilon) = (1 - \frac{1}{2^{2k-2}}) \frac{2}{2k-1} e(G_H) (1 + \varepsilon) \) edges. Assume that $B$ is a bipartite subgraph of $G_H$ with girth more than $2k$. Notice that any set of more than $2k - 2$ edges from a clique on $2k - 1$ vertices must contain a cycle of length at most $2k - 1$. Therefore $B$ can contain at most $2k - 2$ edges from each $2k - 1$-clique of $G_H$. Furthermore, since $B$ is bipartite, there is a 2-coloring of the vertices so that the edges of $B$ are properly colored. If an edge of $B$ is contained in a $2k-1$-clique of $G_H$, then the corresponding hyperedge of $H$ contains two vertices with different colors in this 2-coloring. By our assumption on $H$, at most \( (1 - \frac{1}{2^{2k-2}}) (1 + \varepsilon) \) fraction of the hyperedges are not monochromatic in this 2-coloring of the vertices. So $B$ has at most \( (1 - \frac{1}{2^{2k-2}}) (2k - 2) e(H) (1 + \varepsilon) \) edges. Since $e(G_H) = \binom{2k-1}{2} e(H)$, $B$ has at most \( (1 - \frac{1}{2^{2k-2}}) \frac{2(2k-1)}{2k-1} e(G_H) (1 + \varepsilon) \) edges, as desired. \[ \square \]
In the proof of Theorem 1.4, we use the following proposition. For a proof, see the proof of Proposition 5 in [9]. (Note that the bound can be attained when \( w \geq \binom{k}{2} \).)

**Proposition 4.1.** In the complete bipartite graph \( K_{u,w} \), a \( C_4 \)-free subgraph has at most \( w + \binom{w}{2} \) edges.

**Proof of Theorem 1.4.** Let \( l \) be a large integer. By Proposition 3.7, there exists a \( k - 1 + \lceil \frac{2}{24} \cdot 2^{k-1} \rceil \) uniform oriented hypergraph \( O \) with girth more than \( 2k \) for which (3) in Lemma 3.5 holds with \( \frac{1}{24} \cdot 2^{k-1} \) in place of \( \varepsilon \). Let \( n \) be the number of vertices of \( O \). We produce a graph \( G_O \) from the oriented hypergraph \( O \) by replacing each hyperedge of \( O \) with a copy of \( K_{k-1,m} \) the following way: in a hyperedge \( (v_1, \ldots, v_{k-1 + l}) \), we connect every vertex in \( \{v_1, \ldots, v_{k-1}\} \) with every vertex in \( \{v_k, \ldots, v_{k-1+l}\} \) with an edge. The resulting graph \( G_O \) is \( C_{2k} \)-free.

Since the girth of \( O \) is more than \( 2k \geq 4 \), no two hyperedges of \( O \) intersect in more than 1 vertex. Therefore the copies of \( K_{k-1,l} \) in \( G_O \) are edge-disjoint, and by definition every edge of \( G_O \) is in one of the copies of \( K_{k-1,l} \). We show that \( G_O \) does not have a bipartite and \( C_4 \)-free subgraph which has more than \((1 - \frac{1}{24}) \cdot \frac{1}{24} n \) edges. Assume that \( B \) is a bipartite and \( C_4 \)-free subgraph of \( G_O \), its classes being \( pn \) red vertices and \((1-p)n \) blue vertices. Now consider a random hyperedge \( e = (v_1, \ldots, v_{k-1}, v_k, \ldots, v_{k-1+l}) \) of \( O \). How many edges of \( B \) can there be between the vertices of \( e \)? Each such edge has a red and a blue endpoint; also, each such edge has an endpoint in \( \{v_1, \ldots, v_{k-1}\} \) and an endpoint in \( \{v_k, \ldots, v_{k-1+l}\} \). Let \( u \) and \( w \) be the number of red vertices among \( \{v_1, \ldots, v_{k-1}\} \) and \( \{v_k, \ldots, v_{k-1+l}\} \) respectively. The restriction of \( B \) to the vertices of \( e \) (which we will denote \( B|_e \)) is thus a \( C_4 \)-free subgraph of the union of a \( K_{u,l-w} \) and a \( K_{k-1-u,w} \) on disjoint vertex sets. We have three possibilities:

- \( u \in \{0, k-1\} \). Then, by Proposition 4.1, \( B|_e \) consists of at most \( l - w + \binom{w}{2} + \binom{k-1-u}{2} \leq l + \binom{k-1}{2} \) edges.
- \( u = k - 1 \). Then \( K_{k-1-u,w} \) is degenerate (as \( k - 1 - u = 0 \)), and \( B|_e \) has at most \( l - w + \binom{k-1}{2} \) edges.
- \( u = 0 \). Then \( K_{u,l-w} \) is degenerate, and \( B|_e \) has at most \( w + \binom{k-1}{2} = l + \binom{k-1}{2} - (l-w) \) edges.

Let \((C_1, \ldots, C_{k-1+l})\) be the color sequence of \( e \) (with \( C_i \in \{\text{red, blue}\} \)). For any color sequence \((c_1, \ldots, c_{k-1+l})\) (with \( c_i \in \{\text{red, blue}\} \)), the probability that \((C_1, \ldots, C_{k-1+l}) = (c_1, \ldots, c_{k-1+l})\) is \( p^{\left|\{i: C_i = \text{red}\}\right|} \cdot (1-p)^{\left|\{i: C_i = \text{blue}\}\right|} \leq \frac{\varepsilon}{24} \cdot 2^{k-1} \), since (3) in Lemma 3.5 holds for \( O \) with \( \frac{1}{24} \cdot 2^{k-1} \). (Note that \( e \) was chosen as a random hyperedge of \( O \).) Let \( \tilde{C}_1, \ldots, \tilde{C}_{k-1+l} \) be independent random variables which take the value “\( \text{red} \)” with probability \( p \) and the value “\( \text{blue} \)” with probability \( 1 - p \). Let \( f(C_1, \ldots, C_{k-1+l}) \) be a real valued function of a color sequence. We claim that

\[
\left| E(f(C_1, \ldots, C_{k-1+l})) - E(f(\tilde{C}_1, \ldots, \tilde{C}_{k-1+l})) \right| \leq \frac{\varepsilon}{24} \max \left| f \right|.
\]

(5)

Indeed,

\[
E(f(\tilde{C}_1, \ldots, \tilde{C}_{k-1+l})) = \sum_{(c_1, \ldots, c_{k-1+l}) \in \{\text{red, blue}\}^{k-1+l}} p^{\left|\{i: C_i = \text{red}\}\right|} \cdot (1-p)^{\left|\{i: C_i = \text{blue}\}\right|} f(c_1, \ldots, c_{k-1+l}),
\]

and
\[
E(f(C_1, \ldots, C_{k-1+l})) = \sum_{(c_1, \ldots, c_{k-1+l}) \in \{\text{red, blue}\}^{k-1+l}} \left( p^{\lfloor \varepsilon |c_i| = \text{red} \rfloor} + (1-p)^{\lfloor \varepsilon |c_i| = \text{blue} \rfloor} \right) \pm \frac{\varepsilon}{24 \cdot 2^{k-1+l}} \cdot f(c_1, \ldots, c_{k-1+l})
+ \sum_{(c_1, \ldots, c_{k-1+l}) \in \{\text{red, blue}\}^{k-1+l}} \left( \pm \frac{\varepsilon}{24 \cdot 2^{k-1+l}} \right) f(c_1, \ldots, c_{k-1+l})
= E(f(\tilde{C}_1, \ldots, \tilde{C}_{k-1+l})) \pm \frac{\varepsilon}{24} \max |f|.
\]

Using (5) with \( f(C_1, \ldots, C_{k-1+l}) = \begin{cases} 1 & \text{if } C_1 = \ldots = C_{k-1} = \text{red} \\ 0 & \text{otherwise} \end{cases} \), we have \( P(u = k-1) = E(I_{u=k-1}) = p^{k-1} + \frac{\varepsilon}{24} \), with \( f(C_1, \ldots, C_{k-1+l}) = \begin{cases} 1 & \text{if } C_1 = \ldots = C_{k-1} = \text{blue} \\ 0 & \text{otherwise} \end{cases} \),
we have \( P(u = 0) = E(I_{u=0}) = (1-p)^{k-1} + \frac{\varepsilon}{24} \); and with \( f(C_1, \ldots, C_{k-1+l}) = \{i \in \{k, \ldots, k-1+l\} : C_i = \text{red}\} \), we have \( E(w) = pl \pm \frac{\varepsilon l}{24} \). So
\[
E(e(B|e)) = P(u \notin \{0, k-1\}) \left( l + \left( \frac{k-1}{2} \right) \right) + P(u = k-1)E \left( l - w + \left( \frac{k-1}{2} \right) \right)
+ P(u = 0)E \left( l + \left( \frac{k-1}{2} \right) - (l - w) \right)
\leq l + \left( \frac{k-1}{2} \right) - \left( p^{k-1} + \frac{\varepsilon}{24} \right) \left( p \pm \frac{\varepsilon}{24} \right) l - \left( (1-p)^{k-1} + \frac{\varepsilon}{24} \right) \left( 1 - p \pm \frac{\varepsilon}{24} \right) l
\leq l + \left( \frac{k-1}{2} \right) - pl - (1-p)^{k-1} + \frac{\varepsilon}{4} \leq \left( 1 - \frac{1}{2^{k-1}} \right) l + \left( \frac{k-1}{2} \right) + \frac{\varepsilon}{4} l
\]
assuming \( \varepsilon \leq 1 \).

That is, if \( O \) has \( m \) hyperedges, \( e(B) = \left( \left( 1 - \frac{1}{2^{k-1}} \right) l + \left( \frac{k-1}{2} \right) + \frac{\varepsilon l}{4} \right) m \), while \( e(G_O) = m(k-1)l \). Let \( l \geq \frac{k(k-2)}{2} \), then
\[
e(B) \leq \left( \left( 1 - \frac{1}{2^{k-1}} \right) \frac{1}{k-1} + \frac{k-2}{2l} + \frac{\varepsilon}{4(k-1)} \right) e(G_O)
\leq \left( \left( 1 - \frac{1}{2^{k-1}} \right) \frac{1}{k-1} + \frac{\varepsilon}{k} \right) e(G_O) \leq \left( 1 - \frac{1}{2^{k-1}} \right) \frac{1}{k-1} e(G_O)(1 + \varepsilon). \]

5 Pasting \( C_6 \)'s to produce a \( C_8 \)-free graph

We will make use of the following proposition of Nešetřil and Rödl [12] in the second example, and in the general version of the first example.

**Proposition 5.1** (Nešetřil, Rödl [12]). For any positive integers \( r \geq 2 \) and \( s \geq 3 \), there exists an \( n_0 \in \mathbb{N} \) such that for any integer \( n \geq n_0 \) there is a \( r \)-uniform hypergraph with girth at least \( s \) and more than \( n^{1+1/s} \) hyperedges.
5.1 First example

For our construction here we will need a bipartite graph of girth at least 10 with many edges (and with degree at least 2 in every vertex). We will derive such a graph from the following construction of Benson [1].

**Theorem 5.2** (Benson [1]). Let $q$ be an odd prime power. There is a $(q+1)$-regular, bipartite, girth 12 graph $Q$ with $2(q^5 + q^4 + q^3 + q^2 + q + 1)$ vertices.

First, let us notice that since $Q$ is a regular bipartite graph, it has color classes of equal size. Moreover, we may assume that $Q$ is connected, for otherwise we may add some edges to make it connected without creating cycles. So we have the following corollary.

**Corollary 5.3.** There exists a connected bipartite graph of girth at least 10 with $n/2$ vertices in each color class such that every vertex has degree at least $(n/2)^{1/5}(1-o(1))$. (So it contains at least $(1-o(1))(n/2)^{6/5}$ edges.)

**Theorem 5.4.** There exists a $C_6$-free graph $G$ on $4n$ vertices with average degree at least $4n^{1/5}$ which is pasted together from $C_6$’s.

To prove Theorem 5.4, let us take a connected, bipartite graph $G_1$ of girth at least 10 on $2n$ vertices such that each vertex has degree at least $n^{1/5}(1-o(1))$ (and having $n$ vertices in each color class). The existence of such a graph is guaranteed by Corollary 5.3. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be the two color classes of $G_1$. Now let $G_2$ be a copy of $G_1$ with vertices $a'_1, a'_2, \ldots, a'_n$ and $b'_1, b'_2, \ldots, b'_n$ and edge set $E(G_2) = \{a'_ib'_j \mid a_ib_j \in E(G_1)\}$.

Finally, the graph $G$ is defined to have the vertex set $V(G) = V(G_1) \cup V(G_2)$ and the edge set $E(G) = E(G_1) \cup E(G_2) \cup \{b_ib'_i \mid 1 \leq i \leq n\}$ (see Figure 2). So $G$ has $4n$ vertices and $2n^{6/5}(1-o(1)) + n$ edges.

To show that $G$ is pasted together from $C_6$’s, we have to show that every edge is contained in a $C_6$, and for any two edges $e_1, e_2 \in E(G)$, there is a sequence $O_1, O_2, \ldots, O_m$ of $C_6$’s in $G$ such that for any $1 \leq i \leq m - 1$, $O_i$ and $O_{i+1}$ share at least one edge, and $e_1$ and $e_2$ are contained in $O_m$.
$e_2$ are edges of $O_1$ and $O_m$ respectively. The graph can then be built starting from an arbitrary fixed edge. It is easy to see that every edge is contained in some $C_6$ of the form $a_ob_ia'_kb_k$, and so we can assume that both $e_1$ and $e_2$ are of the form $a_ob_j$. Let $(a_{i_0}b_{i_1}a_{i_2}b_{i_3}a_{i_4}b_{i_5} \ldots a_{i_{t-1}}b_{i_t}(a_{i_{t+1}})$ be a path starting with $e_1$ and ending with $e_2$, with $e_1 = a_{i_0}b_{i_1}$ or $e_1 = b_{i_1}a_{i_2}$, and $e_2 = a_{i_{t-1}}b_{i_t}$ or $e_2 = b_{i_t}a_{i_{t+1}}$ (such a path exists since $G_1$ is connected). Then the path $b_{i_1}a'_{i_2}b'_{i_3}a'_{i_4}b'_{i_5} \ldots a'_{i_{t-1}}b'_{i_t}$ is contained in $G_2$. These two paths together with the edges $b_{i_1}b'_{i_1}, b_{i_2}b'_{i_2}, \ldots, b_{i_t}b'_{i_t}$ give the desired sequence of $C_6$'s (together with an arbitrary $C_6$ of the form $a_{i_0}b_ia'_kb_j$ if $e_1 = a_{i_0}b_{i_1}$, and a $C_6$ of the form $a_{i_{t+1}}b_{i_t}a'_{i_{t+1}}b'_{i_k}$ if $e_2 = b_{i_t}a_{i_{t+1}}$).

It remains to show that $G$ is $C_8$-free. Suppose for a contradiction that it has a $C_8$. Since the graph $G_1$ is of girth at least 10, this $C_8$ cannot be completely in $G_1$ or $G_2$. So it has to contain at least one edge from each of the three sets $E(G_1), E(G_2)$ and $\{b_ib'_i \mid 1 \leq i \leq n\}$. Moreover, it is easy to see that it contains exactly two edges from $\{b_ib'_i \mid 1 \leq i \leq n\}$, say $b_{i_1}'$ and $b_{i_2}'$. So there is a path of $q$ edges between $b_{i_1}$ and $b_{i_2}$ in $G_1$ and a path of $r$ edges between $b'_{i_1}$ and $b'_{i_2}$ in $G_2$ such that $q + r = 6$. Let these paths be $b_{i_1}a_{i_1}b_{i_2} \ldots a_{i_{q-1}}b_{i_q}$ and $b'_{i_1}a'_{i_1}b'_{i_2} \ldots a'_{i_{r-1}}b'_{i_r}$ respectively. By construction, the second path in $G_2$ implies that $G_1$ contains the path $b_{i_1}a_{i_1}b_{i_2} \ldots a_{i_{q-1}}b_{i_q}$, which, together with $b_{i_1}a_{i_1}b_{i_2} \ldots a_{i_{q-1}}b_{i_q}$, would produce a cycle of length 4 or 6 in $G_1$, a contradiction.

Remark 5.5. We may modify the above construction as described below to find a pasting of $C_{2l}$'s to produce a $C_{2k}$-free graph $G$ for any given integers $k > l \geq 3$ and having average degree at least $\Omega\left(n^{1/(2k+2)}\right)$.

Proof. A graph of girth $2k + 1$ and having $\Omega\left(n^{1+1/(2k+1)}\right)$ edges exists by applying Proposition 5.1 with $r = 2$. So it has average degree $\Omega\left(n^{1/(2k+1)}\right)$. It is easy to find a bipartite subgraph of such a graph, with equal color classes and having a constant fraction of all the edges. Then we can delete vertices of degree 1 without decreasing its average degree, so we can assume it has minimum degree at least 2, and as usual, we can assume it is connected, because otherwise we can add edges without creating a cycle to make it connected. Let $G_1$ be this bipartite, connected graph of girth greater than $2k$ on $2n$ vertices with average degree $\Omega\left(n^{1/(2k+1)}\right)$. Then let $G_2$ be defined in the same way as in the above proof (based on $G_1$). However, now, for each $i$ we connect the vertices $b_i \in V(G_1)$ and $b'_i \in V(G_2)$ by a path containing $l - 2$ edges and let the resulting graph be $G$. Using the same argument as in the above proof, we can see that this gives a pasting of $C_{2l}$'s and that $G$ is $C_{2k}$-free.

5.2 Second example

A hypergraph $H$ is connected if for any two vertices $u, v \in V(H)$, there exist hyperedges $h_i \in E(H)$, $1 \leq i \leq m$, such that $u \in h_1, v \in h_m$ and $h_i \cap h_{i+1} \neq \emptyset$ for all $1 \leq i \leq m - 1$. A minimal collection of such hyperedges is called a path between $u$ and $v$ in $H$. We may assume that the hypergraph provided by Proposition 5.1 is connected, for otherwise we can simply take a connected component of it containing the biggest number of hyperedges.

Theorem 5.6. There exists a $C_8$-free graph $G$ on $2n$ vertices with average degree at least $6 \cdot n^{1/9}$ which is pasted together from $C_6$'s.
To prove Theorem 5.6, we apply Proposition 5.1 to find a (connected) 3-uniform hypergraph $H_1$ on $n$ vertices with girth at least 9 and more than $n^{1+1/9}$ hyperedges. Let $V(H_1) = \{u_1, u_2, \ldots, u_n\}$. Replace each vertex $u_i \in V(H_1)$ with a pair of vertices $u_i, u_i'$ so that every hyperedge containing $u_i$ now contains both $u_i$ and $u_i'$. This produces a 6-uniform hypergraph which we denote by $H_2$.

Now we construct the desired graph $G$ from $H_2$ in the following fashion. If $\{u_i, u_i', u_j, u_j', u_k, u_k'\}$ is a hyperedge in $H_2$ with $1 \leq i \leq j \leq k \leq n$, then we add the edges $u_iu_i', u_iu_j, u_ju_j', u_j'u_k, u_k'u_i$ to $E(G)$. We repeat this procedure for every hyperedge of $H_2$. Let us call the edges $u_iu_i' \in E(G)$ (1 \leq i \leq n) fat edges and the rest of the edges of $G$ thin edges.

Note that two fat edges never share a vertex. We claim that a thin edge cannot be added by two different hyperedges of $H_2$. Suppose by contradiction that $u_i'u_j$ is a thin edge added by two different hyperedges $h_1, h_2$ of $H_2$. Then since a hyperedge of $H_2$ either contains both vertices $u_r, u_r'$ or neither of them for any given $1 \leq r \leq n$, it follows that $\{u_i, u_i', u_j, u_j'\} \subset h_1$ and $\{u_i, u_i', u_j, u_j'\} \subset h_2$. Consider the two hyperedges in $H_1$ which correspond to $h_1$ and $h_2$. They both contain $u_i$ and $u_j$; so they intersect in at least two vertices, which is a contradiction since $H_1$ is a linear hypergraph. (Notice, on the other hand, that a fat edge may have been added by several hyperedges.) So each hyperedge in $H_2$ adds precisely 3 new thin edges to $E(G)$. Therefore the number of thin edges in $G$ is three times the number of hyperedges in $H_2$. Since the number of fat edges is $n$, we have $|E(G)| = 3 \cdot n^{1+1/9} + n$. Thus it has the desired average degree.

Since $H_1$ is connected, we can construct it by adding hyperedges one by one, in such a way that each hyperedge intersects one of the previous hyperedges in at least one vertex. We can construct $H_2$ by adding the $C_6$’s corresponding to the hyperedges of $H_1$ in the same order; this shows that $G$ is pasted together from $C_6$’s.

It only remains to show that $G$ is $C_8$-free. We say an edge is between two edges $e_1, e_2$ if one of its end vertices is in $e_1$ and the other is in $e_2$.

**Claim 5.7.** There is at most one thin edge between any two fat edges of $G$.

**Proof.** Consider any two fat edges $u_iu_i'$ and $u_ju_j'$ of $G$. As noted earlier, any thin edge between them is added by a unique hyperedge $h$ of $H_2$, and $h$ contains all four vertices $u_i, u_i', u_j, u_j'$. Because of the linearity of $H_1$, no hyperedge of $H_2$ other than $h$ may contain all four vertices $u_i, u_i', u_j, u_j'$. Now note that in our procedure, any hyperedge of $H_2$ adds at most one thin edge between any two fat edges contained in it, proving the claim. □

Now suppose for a contradiction that $G$ contains a $C_8$. Since no two fat edges in $G$ share a vertex, there can be at most four fat edges in this $C_8$. Contract every pair of vertices $u_i, u_i'$ in $G$ into a single vertex $u_i$. Then this $C_8$ would become a closed walk $C''$ of length at most 8 and at least 4 (only thin edges remain after contraction). While this closed walk may have repeated vertices, we show that it cannot have repeated edges (i.e., it is actually a circuit). Suppose that after contracting every pair of vertices $u_i, u_i'$ to $u_i$, some two thin edges $xy$ and $zw$ coincide. Then, for some $i$ and $j$, we have $x, z \in \{u_i, u_i'\}$ and $y, w \in \{u_j, u_j'\}$. Between the fat edges $u_iu_i'$ and $u_ju_j'$, there are two thin edges (namely $xy$ and $zw$), contradicting Claim 5.7.

The 2-shadow of a hypergraph $H$ is the graph which contains an edge $uv$ if and only if there is a hyperedge of $H$ which contains $u$ and $v$. $C''$ must be contained in the 2-shadow.
of $H_1$. Since $H_1$ has girth at least 9, it is not difficult to see that the only possible length of a circuit in its 2-shadow that is between 4 and 8 is 6 and it must be of the form $ab, bc, ce, ed, dc, ca$ (notice that $c$ is a repeated vertex). Therefore, the original $C_8$ in $G$ must be contained in the set of edges added by the hyperedges $\{a, a', b, b', c, c'\}$ and $\{c, c', d, d', e, e'\}$ of $H_2$, but this is impossible as these edges consist of two $C_6$'s sharing exactly one edge.

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