Dynamic and static density-density correlations in the one-dimensional Bose gas:
exact results and approximations

Alexander Yu. Cherny¹ and Joachim Brand²

¹Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980, Dubna, Moscow region, Russia
²Centre for Theoretical Chemistry and Physics and Institute of Fundamental Sciences, Massey University, Auckland, New Zealand
(Dated: December 17, 2008)

We discuss approximate formulas for the dynamic structure factor of the one-dimensional Bose gas in the Lieb-Liniger model that appear to be applicable over a wide range of the relevant parameters such as the interaction strength, frequency, and wavenumber. The suggested approximations are consistent with the exact results known in limiting cases. In particular, we encompass exact edge exponents as well as Luttinger liquid and perturbation theoretic results. We further discuss derived approximations for the static structure factor and the pair distribution function \(g(x)\). The approximate expressions show excellent agreement with numerical results based on the algebraic Bethe ansatz.

PACS numbers: 03.75.Kk, 03.75.Hh, 05.30.Jp

I. INTRODUCTION

Correlations in ultracold atomic gases arise from the interplay of quantum statistics, interactions and thermal and quantum fluctuations. Recently, a lot of progress has been made experimentally to probe and characterize these correlations [1, 2, 3, 4]. The one-dimensional Bose gas is a particularly interesting system as quantum correlations generally play a larger role compared to three dimensional Bose-Einstein condensates and regimes with very different correlation properties can be probed experimentally [5, 6]. In these experiments, elongated “spaghetti” traps are created by optical lattices, which confine the atomic motion in the transverse dimensions to zero-point quantum oscillations [7]. Thus, the systems become effectively one-dimensional.

Theoretically, interactions of the rarefied atoms in one-dimensional waveguides are well described by effective δ-function interactions [8]. The resulting model of a one-dimensional Bose gas is an archetypal of an integrable but non-trivial many-body system that has been receiving long standing interest from physicists and mathematicians alike. The model was first solved with Bethe ansatz by Lieb and Liniger [9, 10], who calculated the ground-state and excitation energies. Depending on the value of the dimensionless coupling strength, the Lieb-Liniger model describes various regimes with the corresponding correlations. Being exactly solvable, the model, however, does not admit complete analytic solution for the correlation functions. Up-to-now, this is complicated and challenging problem in 1D physics [11, 12].

Dynamical density-density correlations can be measured in cold atoms by the two-photon Bragg scattering [1, 13]. Theoretically, they are described by the dynamic structure factor (DSF) [14]

\[
S(k, \omega) = \int \frac{dtdx}{2\pi\hbar} e^{i(\omega t-kx)} \langle 0 | \hat{\rho}(x,t) \hat{\rho}(0,0) | 0 \rangle,
\]

(1)

where \(\hat{\rho}(x,t) \equiv \hat{\rho}(x,t) - n\) is the operator of density fluctuations and \(n = N/L\) is the equilibrium density of particles. We consider zero temperature, where \(\langle 0 | ... | 0 \rangle\) denotes the ground-state expectation value. The DSF is proportional to the probability of exciting a collective mode from the ground state with the transfer of momentum \(k\) and energy \(\hbar\omega\), as can be seen from the energy representation of Eq. (1)

\[
S(k, \omega) = \sum_m |\langle 0 | \delta \hat{p}_k | n \rangle|^2 \delta(\hbar\omega - E_m + E_0),
\]

(2)

where \(\delta \hat{p}_k = \sum_j e^{-ikx_j} - N\Delta(k)\) is the Fourier component of \(\delta \hat{\rho}(x)\), \(\Delta(k) = 1\) at \(k = 0\) and \(\Delta(k) = 0\) otherwise. Once the DSF is known, the static structure factor \(S(k)\) and the pair distribution function \(g(x)\) can be calculated by integration as is discussed in Sec. III.B.

Previously known results for the DSF of the one-dimensional Bose gas come from Luttinger liquid theory, which predicts a power-law behavior of the DSF at low energies in the vicinity of the momenta \(k = 0, 2\pi n, 4\pi n, ...\) and yields universal values for the exponents [13, 14, 15]. In the regime of strong interactions, we have previously derived perturbatively valid expressions covering arbitrary energies and momenta at zero [15] and finite temperature [16]. For finite systems, it is possible to compute the correlation functions numerically, using the results of the algebraic Bethe ansatz calculations [20, 21]. Finally, the exact power-law behavior along the limiting dispersion curve of the collective modes has recently been calculated in Refs. [22, 23]. These exponents differ from those predicted by Luttinger liquid theory raising the question whether the different results are compatible with each other. We address this question in Sec. III.A of this paper, where we show that the results can be reconciled by taking appropriate limits. The apparent difference between the edge exponents valid along the dispersion curves and the Luttinger liquid result in the limit of vanishing energy can be traced back to the fact that the dispersion relations are curved and not straight, as is presumed by Luttinger liquid theory.
The exact values of the exponents found in Refs. \[22,23\] are of interest; however, they are not sufficient for practical estimations of the DSF as long as the prefactors are not known. In this paper we construct an approximate formula for the DSF \[24\] based on the exponents of Refs. \[22,23\]. Within the proposed scheme, the prefactor can be found using the well-known f-sum rule (see, e.g., \[13\]). The result turns out to be consistent with numerical results by Caux and Calabrese \[20\]. Besides, it is compatible with the results of Luttinger liquid theory \[13,16,17\] and perturbation theory \[13\]. The approximate formula, in effect, takes into account single quasiparticle-quasihole excitations but neglects multiparticle excitations. We also present an approximate expression for the static structure factor and for the density-density correlation function, which is derived from the approximation for the DSF.

II. EXACT RESULTS FOR DYNAMIC STRUCTURE FACTOR IN LIEB-LINIGER MODEL

We model cold bosonic atoms in a waveguide-like micro trap by a simple 1D gas of \(N\) bosons with point interactions of strength \(g_a > 0\)

\[
H = \sum_{i=1}^{N} \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_i^2} + g_B \sum_{1 \leq i < j \leq N} \delta(x_i - x_j)
\]

and impose periodic boundary conditions on the wave functions. The strength of interactions can be measured in terms of dimensionless parameter \(\gamma = mg_B/(\hbar^2 n)\). In the limit of large \(\gamma\), the model is known as the Tonks-Girardeau (TG) gas. In this limit, it can be mapped onto an ideal Fermi gas since infinite contact repulsions emulate the Pauli principle. In the opposite limit of small \(\gamma\), we recover the Bogoliubov model of weakly interacting bosons.

A. DSF expansion in \(1/\gamma\)

For finite \(\gamma\), the model can also be mapped onto a Fermi gas \[27\] with local interactions, inversely proportional to \(g_B\). Using the explicit form of the interactions, one can develop the time-dependent Hartree-Fock scheme \[18,19\] in the strong-coupling regime with small parameter \(1/\gamma\). The scheme yields the correct expansion of the DSF up to the first order \[18,19\]

\[
S(k,\omega) = \frac{\varepsilon_F}{N} = \frac{k_F}{4k} \left( 1 + \frac{8}{\gamma} \right) + \frac{1}{2\gamma} \ln \frac{\omega^2 - \omega_-^2}{\omega^2 - \omega_+^2} + O\left( \frac{1}{\gamma^2} \right),
\]

for \(\omega_- < \omega \leq \omega_+\), and zero elsewhere \[28\]. The symbol \(O(x)\) denotes terms of order \(x\) or even smaller. Here \(\omega_{\pm}(k)\) are the limiting dispersions \[29\] that bound quasiparticle-quasihole excitations (see Fig. \[1\]); in the strong-coupling regime they take the form

\[
\omega_{\pm}(k) = \hbar^2 k_F k \pm k^2/(1 - 4/\gamma)/(2m) + O(1/\gamma^2).
\]

By definition, \(k_F \equiv \pi n\) and \(\varepsilon_F \equiv \hbar^2 k_F^2/(2m)\) are the Fermi wave vector and energy of TG gas, respectively.

B. Link to Luttinger liquid theory

Luttinger liquid theory describes the behavior of the DSF at low energies for arbitrary strength of interactions \[13,17\]. In particular, one can show \[13,17\] that in the vicinity of “umklapp” point \((k = 2\pi n, \omega = 0)\) it is given by

\[
S(k,\omega) = \frac{n c}{\hbar \omega} \left( \frac{\hbar \omega}{mc} \right)^{2K} A(K) \left( 1 - \frac{\omega^2(k)}{\omega^2} \right)^{K-1}
\]

for \(\omega \geq \omega_-(k)\), and zero otherwise. Within the Luttinger-liquid theory, the dispersion is linear near the umklapp point: \(\omega_-(k) \simeq c|k - 2\pi n|\). By definition,

\[
K \equiv \hbar \pi n/(mc)
\]

and \(c\) is the sound velocity. For the repulsive bosons, the value of parameter \(K\) lies between 1 (TG gas) and +\(\infty\) (ideal Bose gas). In the strong-coupling regime, the linear behavior of the dispersions \[3\] at small momentum determines the sound velocity, which allows us to calculate the value of the Luttinger parameter

\[
K = 1 + 4/\gamma + O(1/\gamma^2).
\]

The coefficient \(A(K)\) is model-dependent; in the Lieb-Liniger model, it is known in two limiting cases: \(A(K) = \pi/4K = 1\) and \(A(K) \simeq 8^{1/2} K \exp(-2 \gamma c K) \pi^2 / \Gamma^2(K)\) for \(K \gg 1\), where \(\gamma_c = 0.5772\ldots\) is the Euler constant and \(\Gamma(K)\) is the gamma function.

By comparing the first-order expansion \[3\] in the vicinity of the umklapp point with Eq. \[9\] and using the expansion \[8\], one can easily obtain the model-dependent coefficient at large but finite interactions when \(K - 1 < 1\)

\[
A(K) = \frac{\pi}{4} \left[ 1 - (1 + 4 \ln 2) (K - 1) \right] + O((K - 1)^2).
\]

Note that the relation \[8\] leads to different exponents precisely at the umklapp point and outside of it:

\[
S(k,\omega) \sim \begin{cases} \omega^{2(K-1)}, & \omega = \omega_+ k \equiv 2\pi n, \\ (\omega - \omega_-)^{K-1}, & \omega \neq 2\pi n. \end{cases}
\]

C. Exact edge exponents from the Lieb-Liniger solutions

As was shown in Refs. \[22,23\] (see also \[30\]), within the Lieb-Liniger model the DSF exhibits the following power-law behavior near the borders of the spectrum \(\omega_{\pm}(k)\)

\[
S(k,\omega) \sim |\omega - \omega_{\pm}(k)|^{-\mu_{\pm}(k)}.
\]
The positive exponents $\mu_{\pm}$ are related to the quasiparticle scattering phase and can be calculated in the thermodynamic limit by solving a system of integral equations. In particular, Imambekov and Glazman found the following right limit

$$\lim_{k \to 2\pi n} \mu_-(k) = 2\sqrt{K}(\sqrt{K} - 1),$$

which is different from the Luttinger liquid exponent. However, Imambekov’s and Glazman’s result is accurate in the immediate vicinity of $\omega_-$ provided that the finite curvature of $\omega_-(k)$ is taken into consideration. Thus the difference in the exponents can be attributed to the linear spectrum approximation within the Luttinger liquid theory. Note, however, that the thin “strip” in $\omega$-$k$ plane, where the exponents are different, vanishes in the point $k = 2\pi n$; hence, the Luttinger exponent $2(K - 1)$ becomes exact there.

A typical behavior of the exponents is shown in Fig. 2. As described in Appendix A, the exponents can be easily evaluated by solving equation (A6) for the shift function and using Eq. (A9).

D. Algebraic Bethe ansatz

Recent progress in the computation of correlation functions within the Lieb-Liniger model and other 1D models has been achieved through the algebraic Bethe ansatz. In this method, matrix elements of the density operator involved in Eq. (2) were calculated with the algebraic Bethe ansatz. They are given by the determinant of a matrix, which can be evaluated numerically for a finite number of particles. So, this method is based on combining integrability and numerics. The results of the numerical calculations of Ref. [20] are shown in Figs. 1 and 2.

III. APPROXIMATE EXPRESSION FOR DYNAMIC STRUCTURE FACTOR

A. Approximate expression for arbitrary values of interaction strength

Here we suggest a phenomenological expression, which is consistent with all the above-mentioned results. It reads

$$S(k, \omega) = C(\omega^\alpha - \omega^-)^{\mu_-}/(\omega^\alpha - \omega^+)^{\mu_+}$$

for $\omega_-(k) \leq \omega \leq \omega_+(k)$, and zero otherwise. It follows from energy and momentum conservation that $S(k, \omega)$ is exactly equal to zero below $\omega_-(k)$ for $0 \leq k \leq 2\pi n$. In the other regions of $\omega > \omega_+$ and $\omega < \omega_-$ (for $k > 2\pi n$), possible contributions can arise due to coupling to multiparticle excitations. However, these contributions are known to vanish in the Tonks-Girardeau ($\gamma \to \infty$) and Bogoliubov ($\gamma \to 0$) limits and are found to be very small numerically for finite interactions.

In Eq. (13) $C$ is a normalization constant, $\mu_+(k)$ and $\mu_-(k)$ are the exponents of Eq. (11), and $\alpha \equiv 1 + 1/\sqrt{K}$. From the definition of $K$ (7), one can see that for repulsive spinless bosons $K \geq 1$, and, hence, $1 < \alpha \leq 2$. The normalization constant depends on the momentum but not the frequency and can be determined from the $s$-sum rule

$$\int_0^{+\infty} d\omega \omega S(k, \omega) = N_k^2 k^2/2m$$

In Eq. (13), we assume that the value of the exponent $\mu_-(k = 2\pi n)$ coincides with its limiting value $\mu_-(\infty)$ in vicinity of the umklapp point.
The most general way of obtaining \( \omega^{\pm}(k) \), \( \mu^{\pm}(k) \), and \( K \) is to solve numerically the corresponding integral equations (see Appendix A).

Note that the sum rule for the isothermal compressibility \[14\]

\[
\lim_{k \to 0} \int_{0}^{+\infty} \frac{S(k, \omega) \, d\omega}{N \omega} = \frac{1}{2mc^2} \tag{15}
\]
is satisfied by virtue of Eq. \[A10\] and the phonon behavior of the dispersions at small momentum: \( \omega^{\pm}(k) \simeq ck \) (see Fig. 1).

Now one can see from \[13\] that

\[
S(k, \omega) \sim \left\{ \begin{array}{ll}
\omega^{2(K-1)}, & k = 2\pi n, \\
(\omega - \omega_{-})^{\mu-(k)}, & k \neq 2\pi n.
\end{array} \right. \tag{16}
\]

Thus, the suggested formula \[13\] is consistent with both the Luttinger liquid behavior at the umklapp point and Imambekov’s and Glazman’s power-law behavior in vicinity of it, as it should be.

In the strong-coupling regime, Eq. \[13\] yields the correct first order expansion \[13\]. In order to show this, it is sufficient to use the strong-coupling values of \( K \) \[8\], the exponents \[A12\], and the frequency dispersions \[9\].

Comparison with the numerical data by Caux and Calabrese \[21\] (Fig. 3) shows that the suggested formula works well in the regimes of both weak and strong coupling.

Let us discuss how the Bogoliubov approximation arises in the weak-coupling regime in spite of the absence of the Bose-Einstein condensation in one dimension even at zero temperature \[31\] \[32\]. At small \( \gamma \), the upper dispersion curve \( \omega^{+}(k) \) is described well \[10\] by the Bogoliubov relation \[13\]:

\[
\hbar \omega_{k} = \sqrt{T_{k}^{2} + 4T_{k}\epsilon_{F}\gamma/\pi^{2}}, \tag{17}
\]

where \( T_{k} = \hbar^{2}k^{2}/(2m) \) denotes the usual one-particle kinetic energy. Besides, when \( \gamma \) is finite and \( \gamma \rightarrow 0 \), the associated exponents \( \mu_{+} \) approach the limiting value \[A11\], which in turn is very close to one. This implies that the DSF has a strong singularity near \( \omega_{+} \), and, hence, it is localized almost completely within a small vicinity of the upper branch (see Fig. 3). Thus, the behavior of the DSF simulates the \( \delta \)-function spike. One can simply put \( S_{Bog}(k, \omega) = C\delta(\omega - \omega_{k}) \) and determine the constant \( C \) from the \( f \)-sum rule \[14\]

\[
S_{Bog}(k, \omega) = N\frac{T_{k}}{\hbar \omega_{k}}\delta(\omega - \omega_{k}). \tag{18}
\]

B. Simplified analytic approximation for intermediate and large strength of interactions

One can further simplify the expression for the DSF and replace the parameter \( \alpha \) in Eq. \[13\] by its limiting value \( \alpha = 2 \) for the Tonks-Girardeau gas, which turns out to be a good approximation even for intermediate coupling strength \( \gamma \gtrsim 1 \). This replacement allows us to write down the normalization constant explicitly. From the \( f \)-sum rule we obtain

\[
S(k, \omega) = N\frac{k^{2}}{m} \frac{\Gamma(2 + \mu_{+} - \mu_{-})}{\Gamma(1 + \mu_{-})\Gamma(1 - \mu_{+})} \left( \omega^{2} - \omega_{-}^{2} \right)^{\mu_{-}} \times \left( \omega^{2} - \omega_{+}^{2} \right)^{\mu_{+} - 1} \tag{19}
\]
for \( \omega_- (k) \leq \omega \leq \omega_+ (k) \), and zero otherwise. This approximation ensures all the properties of the DSF mentioned in Sec. II except for the Luttinger liquid theory predictions in the close vicinity of the umklapp point (see discussion in Sec. II B). However, outside the umklapp point, it agrees well with the Caux and Calabrese numerical data (see Figs. 4 and 6).

From the explicit formula \[ \psi \] one can find analytic expressions for the static structure factor and the dynamic polarizability. The static structure factor \( S(k, \omega) \simeq \omega^2 (K - 1) \) with \( K = 3.425 \ldots \) at \( \gamma = 1 \), see the insert. However, the difference in absolute values is negligible due to a strong suppression of the DSF outside the close vicinity of the upper brunch.

The dynamic polarizability determines the linear response of the density to an external field \[ \gamma \] and \[ \omega \] yield

\[
S(k) = F_1 \left( \frac{3}{2} + \mu_+ - \mu_-, 1 + \mu_+ - \mu_-, 1 - \frac{\omega_+^2}{\omega_-^2} \right) \\
\times \frac{\hbar k^2}{2m\omega_+} \left( \frac{\omega_-}{\omega_+} \right)^{1+2\mu_+},
\]

where \( F_1 \) is the hypergeometric function. The results for the static structure factor are plotted in Fig. 5. One can see that the formula for the static structure function works well even for weak coupling. This is due to the smallness of the DSF contribution to the static structure function at the umklapp point for small \( \gamma \). Thus, the approximate formula provides a good accuracy for arbitrary strength of interactions. In the weak-coupling regime, one can obtain a good approximation for the static structure function from the Bogoliubov formula \[ \psi \] for the DSF

\[
S(k) = \frac{T_k}{\hbar \omega_k}.
\]

The behavior of the pair distribution function \[ \psi \] in the Lieb-Liniger model was studied at large \[ \gamma \] and short distances \[ \psi \] in various regimes. For \( \gamma \ll 1 \), one can obtain from Eqs. \[ \psi \] and \[ \psi \] the analytical expression \[ \psi \]

\[
g(x) = 1 - \sqrt{\frac{\omega}{\gamma}} \left[ L_{-1} (2\sqrt{\gamma} k_F x / \pi) - I_1 (2\sqrt{\gamma} k_F x / \pi) \right],
\]

where \( L_{-1} \) is the modified Struve function and \( I_1 \) is a Bessel function. In the opposite limit \( \gamma \gg 1 \), one can directly use the strong-coupling expression \[ \psi \] for the DSF and obtain \[ \psi \]

\[
g(x) = 1 - \sin^2 z \frac{z^2}{x^2} - \frac{2\pi}{\gamma} \frac{\partial}{\partial z} \frac{\sin^2 z}{\gamma} \left( \frac{4 \sin^2 z}{\gamma} \right)
\]

\[
+ \frac{2}{\gamma} \frac{\partial}{\partial z} \left[ \sin z \int_{-1}^{1} d \eta \sin(\eta z) \ln \frac{1 + \eta}{1 - \eta} \right] + O(\gamma^{-2}),
\]

where \( z = k_F x = \pi n x \). The last equation implies that \( g(x = 0) \) vanishes not only in the TG limit but also in the first order of \( \gamma^{-1} \), which is consistent with the results of Refs. \[ \psi \] and \[ \psi \].

The behavior of \( g(x, \gamma) \), obtained from the formula \[ \psi \] is shown in Fig. 6. It is consistent with both the weak- and the strong-coupling limits.

The dynamic polarizability determines the linear response of the density to an external field \[ \psi \]. It can be calculated using the DSF

\[
\chi(k, z) = \int_{-\infty}^{\infty} 2\omega' S(k, \omega') d\omega'
\]

On substituting Eq. \[ \psi \] in \[ \psi \], we get

\[
\chi(k, z) = F_1 \left( 1, 1 + \mu_+ - \mu_-, 1 + \mu_+ - \mu_-, 1 - \frac{\omega_+^2}{\omega_-^2} \right)
\]

\[
\times \frac{1}{m} \frac{k^2}{\omega_-^2 - \omega_+^2}.
\]
IV. CONCLUSION

We have discussed an approximate formula [Eq. (13)] for the DSF of the one-dimensional Bose gas at zero temperature, which can be used for a wide range of momenta, energies, and coupling strengths. It neglects, in effect, only the multiparticle excitations, whose contribution is small, anyway, outside the bounds given by the dispersion curves $\omega_{\pm}$. Our formula is consistent with the predictions of the Luttinger liquid theory. It gives the exact exponents at the edge of the spectrum, the correct first-order expansion in the strong-coupling regime, and shows good agreement with the available numerical data. For intermediate and large values of the interaction strength $\gamma \gtrsim 1$ and outside the close vicinity of the umklapp point ($\omega = 0$, $k = 2\pi n$), the further simplified analytic formulas for the DSF (19) and the dynamic polarizability (27) provide excellent accuracy. The analytic expression (22) for the static structure factor works well even for weak interactions.

Our results provide a reference against which experimental measurements of static and dynamic density correlations in the one-dimensional Bose gas can be tested. They further provide a basis for future work on the consequences of correlations in this interesting system.

Acknowledgments

The authors are grateful to Lev Pitaevskii for valuable discussions and to Jean-Sebastien Caux for making the data of numerical calculations of Ref. [20] available to us and to Thomas Ernst for checking our numerical results. JB is supported by the Marsden Fund Council (contract MAU0706) from Government funding, administered by the Royal Society of New Zealand. AUCh thanks Massey University for hospitality.
APPENDIX A: THE LIEB-LINIGER SOLUTIONS AND THE EDGE EXPONENTS

For the Lieb-Liniger model [3] of N bosons embedded in the circuit of length L, the Bethe ansatz yields [4, 11, 37] the following system of equations for N quasimomenta \( \lambda_j \)

\[
L \lambda_j + \sum_{k=1}^{N} \theta(\lambda_j - \lambda_k) = 2\pi n_j \quad (A1)
\]

where \( \theta(\lambda) \equiv 2\arctan[h^2\lambda/(g_{Bm})] \), and \( n_j \) are integer for odd \( N \) and half-integer for even \( N \). In the same way as for the gas of non-interacting fermions, eigenstates of Hamiltonian [3] are completely described by a certain set of \( N \) numbers \( n_j \); the total energy and momentum of the states are given by

\[
E_N = \frac{\hbar^2}{2m} \sum_{j=1}^{N} \lambda_j^2, \quad P_N = \hbar \sum_{j=1}^{N} \lambda_j = \frac{2\pi \hbar}{L} \sum_{j=1}^{N} n_j. \quad (A2)
\]

One can also write down explicitly [4, 11, 37] rather cumbersome expression for the eigenfunctions in terms of quasimomenta. One can prove the following properties of the Lieb-Liniger solutions [11, 38]:

i) The Bethe ansatz solutions form a complete set of eigenfunctions.

ii) There is one-to-one correspondence between \( n_j \) and \( \lambda_j \); if \( n_j > n_k \) then \( \lambda_j > \lambda_k \).

iii) The set of numbers \( n_j = j - 1 - (N-1)/2 \), \( 1 \leq j \leq N \) provides the ground state; for the associated quasimomenta we have \( |\lambda_j| \leq q_0 \leq \pi n \).

iv) In the thermodynamic limit, when \( N \to \infty \) and \( n = N/L = \text{const} \), the distance between the consecutive quasimomenta tends to zero: \( \lambda_j - \lambda_k \sim 1/L \).

In order to pass to thermodynamic limit, it is convenient to introduce the density of quasimomenta \( \rho(\lambda) \equiv \lim l/[L(\lambda+1 - \lambda)] \). From Eq. (A1) and property iii) one can obtain a system of linear integral equations for the density in the ground state

\[
2\pi \rho(\nu) = 1 + \int_{-q_0}^{q_0} K(\nu, \mu) \rho(\mu) \, d\mu, \quad (A3)
\]

\[
n = \int_{-q_0}^{q_0} \rho(\nu) \, d\nu. \quad (A4)
\]

We denote

\[
K(\nu, \mu) \equiv \frac{2g_{Bm}}{\hbar^2} \left[ \left( \frac{g_{Bm}}{\hbar^2} \right)^2 + (\nu - \mu)^2 \right]^{-1} \quad (A5)
\]

and assume that quasimomenta fill “Fermi segment” with bounds \( \pm q_0 \). The density at the bounds can be related [11] to the Luttinger liquid parameter by equation \( \rho(\pm q_0) = \sqrt{K}/(2\pi) \).

Classification of all the excitations can be done in the same manner as for the Tonks-Girardeau gas of non-interacting fermions \( g_{Bm} \to +\infty \). Thus, in order to create elementary particle-like excitation, one needs to add a quasimomentum \( \lambda_p \) beyond “Fermi segment” \([-q_0, q_0] \).

By contrast, for a hole-like excitation, one needs to remove a quasimomentum lying inside the “Fermi segment”. All the excitations can be constructed from the above elementary excitations. Adding or removing a particle with quasimomentum \( \lambda_p \) leads to a new set of quasimomenta \( \hat{\nu}_j \). One can conveniently define a shift function \( F_B(\nu|\lambda_p) \equiv \pm (\nu_j - \hat{\nu}_j)/(\nu_{j+1} - \nu_j) \), where we take plus for a particle-like excitation and minus for a hole-like excitation. It follows from (A1) that the shift function obeys the integral equation in the thermodynamic limit [11, 23]

\[
F_B(\nu|\lambda) - \frac{1}{2\pi} \int_{-q_0}^{q_0} K(\nu, \mu) F_B(\mu|\lambda) \, d\mu = \frac{\pi + \theta(\nu - \lambda)}{2\pi}. \quad (A6)
\]

Using Eq.1 (A2) and the definitions of the density and shift function, one can evaluate the frequency and the wavenumber of elementary single excitation in the thermodynamic limit [11]

\[
\omega_{p,h}(\nu) = \pm \frac{\hbar}{2m} \left[ \nu^2 - \int_{-q_0}^{q_0} 2\mu F_B(\mu|\nu) \, d\mu \right], \quad (A7)
\]

\[
k_{p,h}(\nu) = \pm \left[ \nu + \int_{-q_0}^{q_0} \theta(\nu - \mu) \rho(\mu) \, d\mu \right]. \quad (A8)
\]

For the particle excitation, as discussed above, we should take \( \nu > q_0 \), and for the hole excitation \( |\nu| < q_0 \). Note that Eq. (A7) yields \( k_{h}(\pm q_0, \mp 0) = \pm k_F \) and \( k_{p}(\pm q_0, \pm 0) = \pm k_F \), as it should be.

Once equations (A3), (A4), and (A5) are solved, we can write down the dispersions \( \omega_{\pm}(k) \) [20] bounding single-particle-hole excitations (see Fig. 1). The upper \( \omega_+ \) branch is constructed by creating hole with quasimomentum \( q_0 \) and particle with quasimomentum \( \lambda > q_0 \), while the lower \( \omega_- \) branch by creating hole inside the “Fermi segment” \((|\lambda| < q_0) \) and particle with quasimomentum lying just above \( q_0 \). Explicitly, the branches and associated momenta corresponds to the pair excitations \( \lambda_h = \lambda_0 \) and \( \lambda_p = \lambda + \lambda_0 \) (upper, or “+” branch), \( \lambda_h = \lambda_0 - \lambda \) and \( \lambda_p = \lambda_0 \) (lower, or “−” branch). Summing energies (A4) and wavevectors (A8) of the corresponding elementary excitations yields the dependencies \( \omega(\lambda) \) and \( k(\lambda) \) for the upper branch

\[
\omega_+(\lambda) = \omega_h(\lambda_0) + \omega_p(\lambda + \lambda_0),
\]

\[
k_+(\lambda) = k_h(\lambda_0) + k_p(\lambda + \lambda_0),
\]

and the lower branch

\[
\omega_-(\lambda) = \omega_h(\lambda_0 - \lambda) + \omega_p(\lambda_0),
\]

\[
k_-(\lambda) = k_h(\lambda_0 - \lambda) + k_p(\lambda_0).
\]

After excluding quasimomentum, we obtain the dispersions as functions of \( k \).

It turns out [11] that the shift function in the thermodynamic limit is directly related to the scattering phase of two particles (holes) in the presence of filled “Fermi
In the strong-coupling limit, one can show \(^22\) that

\[
\mu_\pm(k) = 2k/(\pi n\gamma) + O(1/\gamma^2). \quad \text{(A12)}
\]

The exponents obey \(^22\) the limiting relations

\[
\mu_\pm(0) = 0, \quad \text{(A10)}
\]

\[
\mu_+(+\infty) = 1 - 1/(2K). \quad \text{(A11)}
\]

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[1] J. Stenger, S. Inouye, A. P. Chikkatur, D. M. Stamper-Kurn, D. E. Pritchard, and W. Ketterle, Phys. Rev. Lett. \textbf{82}, 4569 (1999).
[2] E. Altman, E.Demler, and M. D. Lukin, Phys. Rev. A \textbf{70}, 013603 (2004).
[3] I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. \textbf{80}, 885 (2008).
[4] T. Gericke, P. Würtz, D. Reitz, T. Langen, and H. Ott, I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. \textbf{87}, 130402 (2005).
[5] B. Paredes, A. Widera, V. Murg, O. Mandel, S. Fölling, I. Cirac, G. Shlyapnikov, T. W. Hänsch, and I. Bloch, Nature Phys. \textbf{2}, 277 (2006).
[6] T. Kinoshita, T. Wenger, and D. S. Weiss, Science \textbf{305}, 1125 (2004).
[7] A. Görlich, J. M. Vogels, A. E. Leanhardt, C. Raman, T. L. Gustavson, J. R. Abo-Shaeer, A. P. Chikkatur, S. Gupta, S. Inouye, T. Rosenband, et al., Phys. Rev. Lett. \textbf{87}, 130402 (2001).
[8] M. Olshanii, Phys. Rev. Lett. \textbf{81}, 938 (1998).
[9] E. H. Lieb and W. Liniger, Phys. Rev. \textbf{130}, 1605 (1963).
[10] E. H. Lieb, Phys. Rev. \textbf{130}, 1616 (1963).
[11] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, \textit{Quantum Inverse Scattering Method and Correlation Functions} (University, Cambridge, 1993).
[12] T. Giamarchi, \textit{Quantum Physics in One Dimension} (Clarendon, Oxford, 2004).
[13] R. Ozeri, N. Katz, J. Steinhauer, and N. Davidson, Rev. Mod. Phys. \textbf{77}, 187 (2005).
[14] L. Pitaevskii and S. Stringari, \textit{Bose-Einstein Condensation} (Clarendon, Oxford, 2003).
[15] F. D. M. Haldane, Phys. Rev. Lett. \textbf{47}, 1840 (1981).
[16] A. H. Castro Neto, H. Q. Lin, Y.-H. Chen, and J. M. P. Carmelo, Phys. Rev. B \textbf{50}, 14302 (1994).
[17] G. E. Astrakharchik and L. P. Pitaevskii, Phys. Rev. A \textbf{70}, 013608 (2004).
[18] J. Brand and A. Yu. Cherny, Phys. Rev. A \textbf{72}, 033619 (2005).
[19] A. Yu. Cherny and J. Brand, Phys. Rev. A \textbf{73}, 023612 (2006).
[20] J.-S. Caux and P. Calabrese, Phys. Rev. A \textbf{74}, 031605(R) (2006).
[21] J.-S. Caux, P. Calabrese, and N. A. Slavnov, J. Stat. Mech. \textbf{2007}, P01008 (2007).
[22] M. Khodas, M. Pustilnik, A. Kamenev, and L. I. Glazman, Phys. Rev. Lett. \textbf{99}, 110405 (2007).
[23] A. Imambekov and L. I. Glazman, Phys. Rev. Lett. \textbf{100}, 206805 (2008).
[24] The approximate formula \(^33\) was first presented at the conference “Dubna-Nano 2008” \(^39\).
[25] T. Cheon and T. Shigehara, Phys. Rev. Lett. \textbf{82}, 2536 (1999).
[26] M. D. Girardeau and M. Olshanii, Phys. Rev. A \textbf{70}, 023608 (2004).
[27] B. E. Granger and D. Blume, Phys. Rev. Lett. \textbf{92}, 133202 (2004).
[28] It follows from the symmetry considerations that \(S(k, \omega) = S(-k, \omega)\). For this reason, in this paper we assume that in the formulas for the dynamic and static structure factors \(k \geq 0\).
[29] We slightly change the notations: ours \(\omega_\pm\) and \(\pm \mu_\pm\) correspond to \(\omega_{1,2}\) and \(\mu_{1,2}\) in Ref. \(^22\), respectively. We also denote the density of particles \(n\) and the Fermi wavevector \(q_0\) instead of \(D\) and \(q\) used in Refs. \(^11\) \(^22\), respectively.
[30] V. V. Cheianov and M. Pustilnik, Phys. Rev. Lett. \textbf{100}, 126403 (2008).
[31] N. N. Bogoliubov, \textit{Lectures on Quantum Statistics}, Vol. 2 (Gordon and Breach, New York, 1970).
[32] P. C. Hohenberg, Phys. Rev. \textbf{158}, 383 (1967).
[33] N. N. Bogoliubov, J. Phys. USSR \textbf{11}, 38 (1947), reprinted in \textit{The many-body problem}, edited by D. Pines (Benjamin, New York, 1961).
[34] D. Pines and P. Nozières, \textit{The Theory of Quantum Liquids: Normal Fermi Liquids} (W. A. Benjamin, N.Y., 1966).
[35] D. M. Gangardt and G. V. Shlyapnikov, Phys. Rev. Lett. \textbf{90}, 010401 (2003).
[36] A. G. Sykes, D. M. Gangardt, M. J. Davis, K. Viering, G. R. Raizen, and K. V. Kheruntsyan, Phys. Rev. Lett. \textbf{100}, 160406 (2008).
[37] C. N. Yang and C. P. Yang, J. Math. Phys. \textbf{10}, 1115 (1969).
[38] M. Gaudin, \textit{La fonction d’onde de Bethe}. (Collection du Commissariat à l’Energie Atomique. Série Scientifique. Paris etc.: Masson. XVI, 330 p., 1983).
[39] A. Yu. Cherny and J. Brand, J. Phys.: Conf. Ser. \textbf{129}, 012051 (2008).