Asymptotically Optimal Tree-based Group Key Management Schemes

Hideyuki Sakai
Hitachi, Ltd., Systems Development Laboratory
Asao-ku, Kawasaki-shi, Kanagawa 215–0013, Japan
Email: sakai@sdl.hitachi.co.jp

Hirosuke Yamamoto
School of Frontier Sciences, University of Tokyo
Kashiwa-shi, Chiba 277–8561, Japan
Email: Hirosuke@ieee.org

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Abstract

In key management schemes that realize secure multicast communications encrypted by group keys on a public network, tree structures are often used to update the group keys efficiently. Selçuk and Sidhu have proposed an efficient scheme which updates dynamically the tree structures based on the withdrawal probabilities of members. In this paper, it is shown that Selçuk-Sidhu scheme is asymptotically optimal for the cost of withdrawal. Furthermore, a new key management scheme, which takes account of key update costs of joining in addition to withdrawal, is proposed. It is proved that the proposed scheme is also asymptotically optimal, and it is shown by simulation that it can attain good performance for nonasymptotic cases.

Index Terms

Multicast communication, Key management schemes, Logical key hierarchy scheme, Selçuk-Sidhu scheme

I. INTRODUCTION

In the multicast communication of a group on a public network, a group secret key is often used to realize secure communication. But, when a member joins and/or withdraws from the group, a new
group key must be redistributed.

The Logical Key Hierarchy (LKH) scheme, which was independently proposed by Wallner-Harder-Agee [1] and Wong-Gouda-Lam [2] in 1997, is a scheme with a tree structure that can renew the group key securely and efficiently when a member changes. Poovendran and Baras [3] analyzed the LKH scheme information-theoretically by considering the withdrawal probability of members in the scheme. Furthermore, Selçuk and Sidhu [4] have proposed a more efficient scheme such that a tree structure is dynamically updated based on the withdrawal probabilities of members. They analyzed the performance of their scheme information-theoretically. But, their evaluation is very loose.

In this paper, we derive an asymptotically tight upper bound of the key update cost in Selçuk-Sidhu scheme. More precisely, the key update cost is $O(\log n)$ when a group has $n$ members, and our upper bound is tight within a constant factor which does not depend on $n$. Furthermore, we propose a new dynamical key management scheme, which takes account of key update costs for joining in addition to withdrawal. We show that the proposed scheme is also asymptotically optimal. Moreover, it is shown by simulation that in nonasymptotic cases, the proposed scheme is more efficient than Selçuk-Sidhu scheme for joining while it is almost as efficient as Selçuk-Sidhu scheme for withdrawal.

In this paper, we assume that channels are noiseless and public. Hence, any information sent over the channels may be wiretapped by adversaries who may be inside or outside of the group. Each member has a private key and several subgroup keys in addition to a group key. The subgroup key and group key are shared by the members of a subgroup and the group, respectively. The group key is used to encrypt secret messages to communicate among the group. On the other hand, the private key and subgroup keys are used when the keys must be updated by the change of members.

Furthermore, we suppose the following in this paper. A reliable server, who has all the keys in the group, updates and distributes new keys when a member changes. The number of members in the group is sufficiently large, and the frequency of joining and withdrawal is relatively large. The key update cost is evaluated by the number of keys that must be updated when a member changes. To keep the security of communication, the key management scheme needs to meet the so-called Forward Security and Backward Security, which are defined as follows.

- [Forward Security] A member who withdraws from a group cannot decrypt any data that will be sent in the group after the withdrawal.
- [Backward Security] A member who joins a group cannot decrypt the data that were sent in the group before the joining.

In Section II, Selçuk-Sidhu scheme is reviewed, and the performance of the scheme is evaluated precisely in Section III. Furthermore, in Section IV, Selçuk-Sidhu scheme is extended to consider the cost of joining. Finally, some simulation results are shown in Section V.
II. SELÇUK-SIDHU SCHEME

The LKH scheme [1], [2] can be represented by a binary tree such that each member of a group corresponds to each leaf of the tree while the root, each internal node, and each leaf also correspond to the group key, a subgroup key, and a private key, respectively. Each member holds all the keys on the path from the root to the leaf of the member in the tree. Each internal node makes a subgroup which consists of the descendants of the node, and the subgroup can communicate securely against any other members not included in the subgroup by using the subgroup key. In the multicast communication of the group, the group key is used to realize secure communication. But, when a member joins or withdraws from the group, the subgroup keys and private keys are used to update the keys. Note that in order to keep security, it is necessary to update all the keys on the path from the root to the leaf of the member.

For the LKH scheme, Poovendran and Baras [3] introduced the withdrawal probabilities of members to analyze information-theoretically the average cost of key update in the case of the withdrawal. Let \( G \) be a group and let \( P_M \) be the probability that a member \( M \in G \) withdraws from the group within a certain period\(^1\). \( P_M \) is assumed to be given since it can be often estimated from the statistics and the personal data of the member. \( P_M \) satisfies \( 0 < P_M \leq 1 \). But, note that

\[
P_G \equiv \sum_{M \in G} P_M, \tag{1}
\]

is usually not equal to one. Hence, we use the normalized withdrawal probability distribution \( P = \{ P_M / P_G : M \in G \} \) to evaluate the performance.

When a member withdraws from the group, the average withdrawal cost \( L \) and the average normalized withdrawal cost \( l \) are defined by

\[
L \equiv \sum_{M \in G} P_M d_M, \tag{2}
\]

\[
l \equiv \sum_{M \in G} \frac{P_M}{P_G} d_M, \tag{3}
\]

respectively, where \( d_M \) is the number of keys that must be updated when member \( M \) withdraws. We note that \( d_M \) is equal to the depth of member \( M \) in the key tree of the LKH scheme.

In the case of lossless source coding, \( l \) given by (3) corresponds to the average code length for a fixed-to-variable length code (FV code) with probability distribution \( P \) and codeword length \( \{ d_M : M \in G \} \), and it is well known that the Huffman tree [5] is the best tree to minimize the average code length under the prefix condition. Furthermore, if the group is incremented and the probability distribution changes as the coding progresses, the optimal code tree can be kept by the dynamic Huffman coding algorithm [6][7].

\(^1\) In order to keep the system securely, all keys are usually renewed periodically. Hence, the period is finite and \( P_M < 1 \) for many members.
In the case of key management, the prefix condition is also required because the set of keys of each member must be different from that of others to keep security. Based on this observation, Poovendran and Baras have shown that in the case of key management, the Huffman tree is the best tree to minimize the average normalized withdrawal cost. However, if the key tree is updated by the dynamic Huffman coding algorithm to keep the key tree optimally, the key update cost cannot be minimized usually because the algorithm often changes the tree structure for many members besides a withdrawn member, and this causes additional key update costs. Hence, in the case of key management, it is better to keep the tree structure as unchanged as possible for non-withdrawn members. Based on this idea, Selçuk and Sidhu [4] have proposed two key tree updating algorithms.

In order to explain Selçuk-Sidhu algorithms, we first define an operation $\text{Insert}(M, X)$, which represents the insertion of a new member $M$ at node $X$, i.e. a new node $N$ is inserted between $X$ and its parent node $Y$ as shown in Fig. 1 and $M$ is linked as a child of $N$.

For node $X$, let $P_X$ be the weight that is given by the sum of the withdrawal probabilities of all members included in the descendants of node $X$. Then, the first algorithm to update a key tree is described as follows.

**Algorithm 1**

Let $M$ be a new member and let $X$ be the root of a given key tree.

1. If $X$ is a leaf, then operate $\text{Insert}(M, X)$ and exit.
2. Let $X_l$ and $X_r$ be the left and right children of $X$, respectively. If it holds that $P_M \geq P_{X_l}$ and $P_M \geq P_{X_r}$, then operate $\text{Insert}(M, X)$ and exit.
3. If $P_{X_l} \geq P_{X_r}$, then let $X \leftarrow X_r$. Otherwise, let $X \leftarrow X_l$. Go back to Step 1.

$^2$If $X$ is the root, $P_X$ is equal to $P_G$. 

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Figure 1. Insertion of $M$ by $\text{Insert}(M, X)$. 

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In order to describe the second algorithm, we first define the cost increase $C_{M,X}$ for a new member $M$ and a node $X$ as follows [4].

$$C_{M,X} \equiv (d_X + 1)P_M + P_X,$$

which represents the increase of cost $L$ for the case that a new member $M$ is inserted at node $X$.

Let $C_{\text{min}}$ be the minimum cost increase that is given by

$$C_{\text{min}} \equiv \min_X C_{M,X}.$$  (5)

Then, the second algorithm inserts a new member $M$ at the node that can attain $C_{\text{min}}$. Formally, the second algorithm to update a tree key is defined as follows.

**Algorithm 2**

Let $M$ be a new member.

1) First calculate $C_{M,X}$ for every node $X$, and obtain $C_{\text{min}}$. Let $X_{\text{min}}$ be the node that attains $C_{\text{min}}$.

2) Operate Insert$(M, X_{\text{min}})$.

It is shown by simulation in [4] that Algorithm 2 can attain less average withdrawal cost $L$ than Algorithm 1. But, although Algorithm 1 can be implemented with $O(\log n)$ time complexity when $|G| = n$, i.e. the size of a group is $n$, Algorithm 2 requires $O(n)$ time complexity in the search of $X_{\text{min}}$.

Selçuk and Sidhu evaluated the average normalized withdrawal cost $l$ for the case of Algorithm 1 as follows$^3$ [4].

$$d_M \leq K_1(- \log P_M + \log P_G) + K_2,$$  \hspace{1cm} (6)

$$l \leq K_1 H(\mathcal{P}) + K_2,$$  \hspace{1cm} (7)

where $H(\mathcal{P})$ is the entropy of the probability distribution $\mathcal{P} = \{P_M/P_G : M \in G\}$, and it is defined by

$$H(\mathcal{P}) \equiv - \sum_{M \in G} \frac{P_M}{P_G} \log \frac{P_M}{P_G}.$$  \hspace{1cm} (8)

$K_1$ and $K_2$ are constants given by

$$K_1 \equiv \frac{1}{\log \alpha} \approx 1.44,$$  \hspace{1cm} (9)

$$K_2 \equiv \frac{1}{\log \alpha} \log \frac{\sqrt{5}}{\alpha} \approx 0.672,$$  \hspace{1cm} (10)

where $\alpha = \frac{1 + \sqrt{5}}{2}$.

We note from the source coding theorem for FV codes [8] that the average normalized withdrawal cost $l$ must satisfy

$$l \geq H(\mathcal{P}).$$  \hspace{1cm} (11)

$^3$In this paper, the base of $\log$ is 2.
Furthermore, it holds from Theorem 1 shown below that $H(\mathcal{P}) = O(\log n)$ for $|\mathcal{G}| = n$. Hence, the upper bound of $l$ given by (11) is not asymptotically tight as $n$ becomes large. This result means that Algorithm 1 is not efficient or the upper bound is loose. In the next section, we will show that Algorithm 1 is asymptotically optimal by deriving an asymptotically tight upper bound.

**Theorem 1** Assume that the maximum and minimum probabilities of withdrawal defined by

$$
P_{\text{max}} \equiv \max_{M \in \mathcal{G}} P_M \leq 1, \quad (12)$$

$$
P_{\text{min}} \equiv \min_{M \in \mathcal{G}} P_M > 0 \quad (13)
$$

are fixed. Then, for $n = |\mathcal{G}|$, $H(\mathcal{P})$ defined by (8) satisfies

$$
H(\mathcal{P}) = O(\log n). \quad (14)
$$

*Proof:* Let $\epsilon_{\text{min}} = P_{\text{min}}/P_G$, $\epsilon_{\text{max}} = P_{\text{max}}/P_G$, and $k = \epsilon_{\text{min}}/\epsilon_{\text{max}} = P_{\text{min}}/P_{\text{max}}$. Then, $H(\mathcal{P})$ can be bounded as follows.

$$
H(\mathcal{P}) = -\sum_{M \in \mathcal{G}} P_M \log \frac{P_M}{P_G}
\geq -\sum_{M \in \mathcal{G}} \epsilon_{\text{min}} \log \epsilon_{\text{min}}
= n \epsilon_{\text{min}} \log \frac{1}{\epsilon_{\text{min}}}
\geq \frac{\epsilon_{\text{min}}}{\epsilon_{\text{max}}} \log n
= k \log n, \quad (15)
$$

where inequalities (a) and (b) hold because of the following reasons.

(a): $-t \log t$ is monotonically increasing when $t > 0$ is small. Furthermore, when $n$ is sufficiently large and $P_G \gg 1$, we have that $\epsilon_{\text{min}} = \frac{P_{\text{min}}}{P_G} \leq \frac{1}{P_G} \ll 1$.

(b): From the relation $nP_{\text{max}} \geq P_G \geq nP_{\text{min}}$, it holds that

$$
\frac{1}{\epsilon_{\text{min}}} = \frac{P_G}{P_{\text{min}}} \geq n \geq \frac{P_G}{P_{\text{max}}} = \frac{1}{\epsilon_{\text{max}}}. \quad (16)
$$

Similarly, we can easily show that

$$
H(\mathcal{P}) \leq \frac{1}{k} \log n. \quad (17)
$$

Therefore, (14) is obtained from (15) and (17).
III. DETAILED ANALYSIS OF SELÇUK-SIDHU SCHEME

In order to derive a tight upper bound for the key tree constructed by Algorithm 1, we use the following lemma.

**Lemma 1** Let $X$ and $S$ be sibling nodes each other in the key tree constructed by Algorithm 1. Then, it holds that

$$|P_X - P_S| \leq P_{\text{max}},$$

where $P_{\text{max}}$ is defined in (12).

**Proof:** The lemma can be proved by mathematical induction for the key tree with $|G| = n$. Let $P_X^{(n)}$, $P_S^{(n)}$, and $P_{\text{max}}^{(n)}$ be $P_X$, $P_S$, and $P_{\text{max}}$ in the case of $|G| = n$, respectively.

1. When $n = 2$, it holds that $P_X^{(2)} \leq P_S^{(2)} = P_{\text{max}}^{(2)}$ or $P_S^{(2)} < P_X^{(2)} = P_{\text{max}}^{(2)}$. In the former case, we have $0 \leq P_{\text{max}}^{(2)} - P_X^{(2)} < P_{\text{max}}^{(2)}$. Otherwise, $0 < P_X^{(2)} - P_S^{(2)} < P_{\text{max}}^{(2)}$. Hence, (18) holds.

2. Supposed that

$$|P_X^{(n)} - P_S^{(n)}| \leq P_{\text{max}}^{(n)}$$

holds for every pair of sibling nodes $(X, S)$ in the key tree with $|G| = n$, and the key tree is incremented to $|G| = n + 1$ by inserting a new member $M$ with probability $P_M$ according to Algorithm 1. Then, we have

$$P_{\text{max}}^{(n+1)} = \max\{P_{\text{max}}^{(n)}, P_M\} \geq P_{\text{max}}^{(n)}.$$  

We assume, without loss of generality, that $P_S^{(n)} \geq P_X^{(n)}$. Then, from Algorithm 1 there may occur the following three cases.

Case 1: $M$ is inserted outside nodes $X$, $S$, and their descendants.

In this case, it holds obviously that $P_X^{(n+1)} = P_X^{(n)}$ and $P_S^{(n+1)} = P_S^{(n)}$. Hence, we obtain from (19) and (20) that $|P_X^{(n+1)} - P_S^{(n+1)}| \leq P_{\text{max}}^{(n+1)}$.

Case 2: $M$ is inserted at node $X$ as shown in Fig. 1.

In this case, we have the new pairs of sibling nodes, $(X, M)$ and $(N, S)$, where $N$ was the new parent node of $X$, and it holds from Step 2 of Algorithm 1 that $[P_X^{(n)} \leq P_M < P_S^{(n)}]$ or $P_M < P_X^{(n)} \leq P_S^{(n)}$ and $[P_M \geq P_X^{(n)}$, $P_M \geq P_X^{(n)}]$, where $X_l$ and $X_r$ are the children of $X$.

Hence, from $P_X^{(n+1)} = P_X^{(n)}$ and $P_S^{(n+1)} = P_S^{(n)}$, we have that $[P_X^{(n+1)} \leq P_M < P_S^{(n+1)}]$ or $P_M < P_X^{(n+1)} \leq P_S^{(n+1)}$ and $P_X^{(n+1)} \leq 2P_M$.

In the case of $P_X^{(n+1)} \leq P_M$, it holds that $0 \leq P_M - P_X^{(n+1)} < P_S^{(n+1)} - P_X^{(n+1)} = P_S^{(n)} - P_X^{(n)} \leq P_{\text{max}}^{(n)} \leq P_{\text{max}}^{(n+1)}$. Furthermore, in the case of $P_X^{(n+1)} > P_M$, it holds that $0 < P_X^{(n+1)} - P_M \leq 2P_M - P_M = P_M \leq P_{\text{max}}^{(n+1)}$. 

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For the pair \((N, S)\), we have that \(|P_S^{(n+1)} - P_N^{(n+1)}| = |P_S^{(n+1)} - P_X^{(n+1)} - P_M| = \max\{|P_S^{(n)} - P_X^{(n)}|, P_M\} = P_{max}^{(n+1)}
\]

Case 3: \(M\) is inserted at a descendant node of \(X\).

In this case, we have that \(P_S^{(n+1)} = P_S^{(n)}\) and \(P_X^{(n+1)} = P_X^{(n)} + P_M\). Hence, it holds that

\[
|P_S^{(n+1)} - P_X^{(n+1)}| = |(P_S^{(n)} - P_X^{(n)})| - P_M| \leq \max\{P_{max}^{(n)}, P_M\} = P_{max}^{(n+1)}.
\]

Now, we evaluate the weight of the ancestors of an arbitrarily given node \(X\) in the key tree generated by Algorithm 1. Let nodes \(F\) and \(G\) be the parent and grandparent of \(X\), respectively, and let \(U\) be the sibling of \(F\). Then, we have from Lemma 1 that

\[
P_F = P_X + P_S \geq 2P_X - P_{max}.
\]

Furthermore, we have that

\[
P_G = P_F + P_U \geq 2P_F - P_{max} \geq 2^2P_X - 2P_{max} - P_{max},
\]

where the first and second inequalities hold from Lemma 1 and (21), respectively. By repeating the same procedure, we obtain that

\[
P_G \geq 2^{d_X}P_X - (2^{d_X-1} + 2^{d_X-2} + \cdots + 2 + 1)P_{max}
\]

\[
= 2^{d_X}(P_X - P_{max}) + P_{max},
\]

where \(d_X\) is the depth of node \(X\).

Therefore, the following theorem holds.

**Theorem 2** In the key tree constructed by Algorithm 1, the following relation holds for any node \(X\) and any leaf \(M_X\) that is a descendant of \(X\).

\[
d_X \leq \log P_G + \log \left(1 - \frac{P_{max}}{P_G}\right) - \log(P_X - P_{max})
\]

\[
d_M^{(X)} \leq K_1(-\log P_M + \log P_X) + K_2,
\]

where \(d_M^{(X)}\) is the depth from node \(X\) to leaf \(M\).

**Proof:** (24) and (25) hold from (23) and (6), respectively.

Next, we evaluate \(P_X\).
Lemma 2 Let $X_l$ and $X_r$ be the children of node $X$. Assume that the weight of $X$ is larger than a real number $t$ but the weight of $X_l$ is not larger than $t$, i.e. $P_X > t \geq P_{X_l}$. Then, the following inequalities hold.

$$t < P_X \leq 2t + P_{\text{max}} \quad (26)$$

Proof: From (18), we obtain that

$$P_X = P_{X_l} + P_{X_r} \leq 2P_{X_l} + P_{\text{max}} \leq 2t + P_{\text{max}}. \quad (27)$$

Let $t(> P_{\text{max}})$ be a parameter which will be optimized later. Now, for a given leaf $M$, we consider the node $X$ that is the nearest ancestor of $M$ under the condition $P_X > t$. Then, from (24), (25), and (27), the depth $d_M$ of leaf $M$ can be bounded as follows.

$$d_M = d_X + D^{(X)}_M$$

$$\leq \log P_G + \log \left(1 - \frac{P_{\text{max}}}{P_G}\right) - \log(P_X - P_{\text{max}}) + K_1(- \log P_M + \log P_X) + K_2$$

$$< \log P_G + \log \left(1 - \frac{P_{\text{max}}}{P_G}\right) - \log(t - P_{\text{max}})$$

$$+ K_1(- \log P_M + \log(2t + P_{\text{max}})) + K_2 \quad (28)$$

We can easily show for $f(t) = -\log(t - P_{\text{max}}) + K_1 \log(2t + P_{\text{max}})$ that $f(t)$ can be minimized at $t = t_m$ given by

$$t_m = \frac{2 + \log \alpha}{2(1 - \log \alpha)} \approx 4.405, \quad (29)$$

where $\alpha = \frac{1 + \sqrt{5}}{2}$. Note that if a key tree is sufficient large and efficiently constructed, there exists the node $X$ that satisfies $P_X > t_m > P_{\text{max}}$. Hence, by substituting $t = t_m$ into (28) and some calculations, we obtain the following theorem.

Theorem 3 When the key tree constructed by Algorithm 1 is sufficiently large, the depth $d_M$ of a leaf $M$ in the key tree is upper bounded as follows.

$$d_M < \log P_G - K_1 \log P_M + (K_1 - 1) \log P_{\text{max}} + \log \left(1 - \frac{P_{\text{max}}}{P_G}\right) + K_3 \quad (30)$$

where $K_3$ is defined by

$$K_3 = -\log \frac{3 \log \alpha}{2(1 - \log \alpha)} + \frac{1}{\log \alpha} \left(\log \frac{3}{1 - \log \alpha} + \log \frac{\sqrt{5}}{\alpha}\right) \approx 3.65. \quad (31)$$

By averaging $d_M$ for all member in $G$, the following theorem holds for the average normalized withdrawal cost $l$. 
**Theorem 4** When a key tree constructed by Algorithm 1 is sufficiently large, the average normalized withdrawal cost \( l \) of the key tree satisfies that

\[
l < H(P) + (K_1 - 1) \log \frac{P_{\text{max}}}{P_{\text{min}}} + \log \left(1 - \frac{P_{\text{max}}}{P_{G}}\right) + K_3.
\]

**Proof:** \( l \) can be evaluated as follows.

\[
l = \sum_{M \in G} \frac{P_M}{P_{G}} d_M
\]

\[
< \log P_{G} - K_1 \sum_{M \in G} \frac{P_M}{P_{G}} \log P_M + (K_1 - 1) \log P_{\text{max}} + \log \left(1 - \frac{P_{\text{max}}}{P_{G}}\right) + K_3
\]

\[
= \log P_{G} - K_1 \sum_{M \in G} \frac{P_M}{P_{G}} \log P_M - K_1 \sum_{M \in G} \frac{P_M}{P_{G}} \log P_{G}
\]

\[
+ (K_1 - 1) \log P_{\text{max}} + \log \left(1 - \frac{P_{\text{max}}}{P_{G}}\right) + K_3
\]

\[
(a) \quad \log P_{G} + K_1 H(P) - K_1 \log P_{G} + (K_1 - 1) \log P_{\text{max}} + \log \left(1 - \frac{P_{\text{max}}}{P_{G}}\right) + K_3
\]

\[
= K_1 H(P) - (K_1 - 1) \log P_{G} + (K_1 - 1) \log P_{\text{max}} + \log \left(1 - \frac{P_{\text{max}}}{P_{G}}\right) + K_3
\]

\[
(b) \quad K_1 H(P) + (K_1 - 1) \left(\log \frac{1}{P_{\text{min}}} - H(P)\right) + (K_1 - 1) \log P_{\text{max}} + \log \left(1 - \frac{P_{\text{max}}}{P_{G}}\right) + K_3
\]

\[
= H(P) + (K_1 - 1) \log \frac{P_{\text{max}}}{P_{\text{min}}} + \log \left(1 - \frac{P_{\text{max}}}{P_{G}}\right) + K_3
\]

\[
(32)
\]

where equality \((a)\) and inequality \((b)\) hold from (8) and the following lemma, respectively.

**Lemma 3** \( H(P), P_{G}, \) and \( P_{\text{min}} \) satisfy that

\[
- \log P_{G} \leq \log \frac{1}{P_{\text{min}}} - H(P).
\]

**Proof:** It is well known that the entropy \( H(P) \) is bounded by \( \log n \) for \( |G| = n \), and it holds obviously that \( P_{G} \geq n P_{\text{min}} \). Hence, we obtain that

\[
H(P) - \log P_{G} \leq \log n - \log (n P_{\text{min}}) = \log \frac{1}{P_{\text{min}}}
\]

\[
(34)
\]

We finally note that in (32), the coefficient of \( H(P) = O(\log n) \) is one and the second and third terms are constants. Hence, Theorem 4 gives an asymptotically tight bound of \( l \).

**IV. EXTENSION OF SELÇUK-SIDHU SCHEME**

In Selçuk-Sidhu scheme [4], only the withdrawal cost of a new member is considered. But, the withdrawal cost is an expected cost in the future, which may not be occur. On the other hand, it is always necessary to update a key tree when a new member joins. Hence, in this section, we propose extended schemes of Algorithms 1 and 2 to consider the joining cost in addition to the withdrawal cost.
When a new member is inserted at the node $X$ with depth $d_X$, the withdrawal cost $L$ increases by $C_{M,X}$, which is given by (4). But, at the same time, $d_X + 1$ keys in the tree must be updated with probability one for the joining. Hence, the cost increase including the joining cost, say $C^*_M$, can be given by

$$C^*_M = (d_X + 1)P_M + P_X + 1 \cdot (d_X + 1)$$

$$= (d_X + 1)(P_M + 1) + P_X. \tag{37}$$

Comparing $C^*_M$ with $C_M$, we note that $P_M$ in $C_M$ is changed to $P_M + 1$ in $C^*_M$. Hence, by substituting $P_M + 1$ into $P_M$ in Algorithms 1 and 2, we can obtain the following algorithms which consider the joining cost.

**Algorithm 3**

Let $M$ be a new member and let $X$ be the root of a given key tree.

1) If $X$ is a leaf, then operate Insert($M, X$) and exit.

2) Let $X_l$ and $X_r$ be the left and right children of $X$, respectively. If it holds that $P_M + 1 \geq P_{X_l}$ and $P_M + 1 \geq P_{X_r}$, then operate Insert($M, X$) and exit.

3) If $P_{X_l} \geq P_{X_r}$, then let $X \leftarrow X_r$. Otherwise, let $X \leftarrow X_l$. Go back to Step 1.

**Algorithm 4**

Let $M$ be a new member.

1) First calculate $C^*_M$ for every node $X$, and obtain $C^*_M$, where $C^*_M \equiv \min_X C^*_M$. Let $X_{min}^*$ be the node that attains $C^*_M$.

2) Operate Insert($M, X_{min}^*$).

For Algorithm 3, the following theorem holds in the same way as Theorem 4.

**Theorem 5** When the key tree constructed by Algorithm 3 is sufficiently large, the average normalized withdrawal cost $l$ of the key tree satisfies that

$$l < H(P) + \log P_{max} + (K_4 - 1) \log(3P_{max} + 5)$$

$$-K_1 \log P_{min} + \frac{P_{max} + 4}{P_{min}} + \log \left(1 - \frac{P_{max} + 2}{P_{G}}\right) + K_4, \tag{38}$$

where $K_4$ is defined as follows.

$$K_4 = - \left(\frac{1}{\log \alpha} - 1\right) \log \left(\frac{1}{\log \alpha} - 1\right) + \frac{1}{\log \alpha} \log \frac{2\sqrt{5}e}{\alpha \log e} \approx 3.95 \tag{39}$$

**Proof:** (The proof is given in the appendix.)

We note from Theorem 5 that the coefficient of $H(P)$ in (38) is also one although the constant terms are larger than (32). This means that Algorithm 3 can also attain asymptotically optimal key tree for the withdrawal cost in addition to decreasing the joining cost.
TABLE I
AVERAGE CASES FOR JOINING

| n   | 100  | 10,000 | 100 | 10,000 |
|-----|------|--------|-----|--------|
| m   | 100  | 10,000 | 100 | 10,000 |
| Alg. 1 | 7.42 | 7.32   | 13.19 | 14.14 |
| Alg. 2 | 7.53 | 7.35   | 14.23 | 14.20 |
| Alg. 3 | 6.50 | 6.23   | 12.85 | 13.12 |
| Alg. 4 | 6.51 | 6.26   | 13.02 | 13.14 |

TABLE II
AVERAGE CASES FOR WITHDRAWAL

| n   | 100  | 10,000 | 100 | 10,000 |
|-----|------|--------|-----|--------|
| m   | 100  | 10,000 | 100 | 10,000 |
| Alg. 1 | 5.46 | 5.51   | 12.11 | 12.19 |
| Alg. 2 | 5.39 | 5.45   | 12.03 | 12.14 |
| Alg. 3 | 5.75 | 5.76   | 12.26 | 12.33 |
| Alg. 4 | 5.57 | 5.71   | 12.13 | 12.28 |

V. SIMULATION RESULTS

In the previous sections, we showed that Algorithms 1 and 3 are asymptotically optimal and Algorithms 2 and 4 are expected to achieve more efficient performance than Algorithms 1 and 3, respectively, in the case of withdrawal. In this section, we evaluate the performances of Algorithms 1–4 by simulation.

We first construct the optimal tree, i.e., Huffman tree for a group with n members. Then, a new member joins the group each after a member withdraws from the group. Such joining and withdrawal are repeated m times. It is assumed that the withdrawal probability of a new member $P_M$ is uniformly distributed in $[0, 0.9]$. For this case, the average costs of joining and withdrawal are shown in Tables I and II respectively.

We note from the tables that Algorithms 3 and 4 can improve the cost of joining at a little increased cost of withdrawal. Algorithms 2 and 4 are more efficient than Algorithms 1 and 3 respectively, in the case of withdrawal. But the difference is not large, and Algorithms 2 and 4 require $O(n)$ time complexity although Algorithms 1 and 3 can be implemented with $O(\log n)$ time complexity. Therefore, Algorithms 3 and 4 should be used in the cases of large n and small n, respectively.

If the backward security described in section I is not required for a group, we don’t need change any
group and subgroup keys when a new member joins the group. Hence, it is preferable to use Algorithms 1 or 2 in such a case.

APPENDIX

A. The proof of Theorem 5

For the key tree constructed by Algorithm 3, the following lemma holds.

**Lemma 4** Let \( X \) and \( S \) be sibling nodes each other in the key tree constructed by Algorithm 3. Then, it holds that

\[
|P_X - P_S| \leq P_{\text{max}} + 2,
\]

where \( P_{\text{max}} \) is defined in (12).

**Proof:** The lemma can be proved in the same way as Lemma 1.

Now, for a given leaf \( M \), let nodes \( X \) and \( Y \) be ancestors of \( M \) such that \( Y \) is an ancestor of \( X \), \( P_X > 1 \), and \( P_Y > P_{\text{max}} + 2 \). When \( |G| = n \) is sufficiently large, there always exist such nodes \( X \) and \( Y \). We represent the depths from the root to node \( Y \), from node \( Y \) to node \( X \), and from node \( X \) to leaf \( M \) by \( d_Y \), \( d_X^{(Y)} \), \( d_M^{(X)} \), respectively, which satisfy that

\[
d_M = d_Y + d_X^{(Y)} + d_M^{(X)}.
\]

Then, by using Lemma 3, we can prove in the same way as (24) and (25) that

\[
d_Y \leq \log P_Y + \log \left(1 - \frac{P_{\text{max}} + 2}{P_Y}\right) - \log(P_Y - P_{\text{max}} - 2),
\]

\[
d_X^{(Y)} \leq K_1[-\log(P_X - 1) + \log(P_Y - 1)] + K_2.
\]

Furthermore, \( d_M^{(X)} \) obviously satisfies that

\[
d_M^{(X)} \leq \frac{P_X}{P_{\text{min}}} - 1.
\]

Let real numbers \( t > P_{\text{max}} + 2 \) and \( s > 1 \) be parameters which will be optimized later. For given \((t, s)\), we select nodes \( X \) and \( Y \) such that \( X \) is the nearest ancestor of \( M \) under the condition \( P_X > s \) and \( Y \) is the nearest ancestor of \( X \) under the condition \( P_Y > t \). Then, in the same way as (26), we can show that

\[
s < P_X \leq 2s + P_{\text{max}} + 2,
\]

\[
t < P_Y \leq 2t + P_{\text{max}} + 2.
\]
By combining (42)–(47), we obtain the following bound of $d_M$.

$$d_M \leq \log P_G + \log \left(1 - \frac{P_{\text{max}} + 2}{P_G}\right) - \log (P_Y - P_{\text{max}} - 2)$$

$$+ K_1[-\log(P_X - 1) + \log(P_Y - 1)] + K_2 + \frac{P_X}{P_{\text{min}}} - 1$$

$$< \log P_G + \log \left(1 - \frac{P_{\text{max}} + 2}{P_G}\right) - \log (t - P_{\text{max}} - 2)$$

$$+ K_1[-\log(s - 1) + \log(2t + P_{\text{max}} + 1)] + K_2 + \frac{2s + P_{\text{max}} + 2}{P_{\text{min}}} - 1$$

(48)

Letting

$$g(t) = -\log(t - P_{\text{max}} - 2) + K_1 \log(2t + P_{\text{max}} + 1),$$

(49)

$$h(s) = -K_1 \log(s - 1) + \frac{2s}{P_{\text{min}}},$$

(50)

we can easily show that $g(t)$ and $h(s)$ are minimized at $t = \hat{t}_m$ and $s = \hat{s}_m$, respectively, which are given by

$$\hat{t}_m = \frac{(2 + \log \alpha)P_{\text{max}} + 4 + \log \alpha}{2(1 - \log \alpha)}$$

$$\approx t_m P_{\text{max}} + 7.676,$$

(51)

$$\hat{s}_m = \frac{\log e}{2 \log \alpha} P_{\text{min}} + 1$$

$$\approx 1.040P_{\text{min}} + 1,$$

(52)

where $t_m \approx 4.405$ is defined in (29). By substituting $t = \hat{t}_m$ and $s = \hat{s}_m$ into (48), we can obtain after some calculations that

$$d_M < \log P_G + (K_1 - 1) \log(3P_{\text{max}} + 5) - K_1 \log P_{\text{min}}$$

$$+ \frac{P_{\text{max}} + 4}{P_{\text{min}}} + \log \left(1 - \frac{P_{\text{max}} + 2}{P_G}\right) + K_4,$$

(53)

where $K_4$ is defined in (39).

Since the average normalized withdrawal cost $l$ is the average of $d_M$, $l$ is bounded as follows.

$$l = \sum_{M \in G} \frac{P_M}{P_G} d_M$$

$$< \log P_G + (K_1 - 1) \log(3P_{\text{max}} + 5) - K_1 \log P_{\text{min}} + \frac{P_{\text{max}} + 4}{P_{\text{min}}} + \log \left(1 - \frac{P_{\text{max}} + 2}{P_G}\right) + K_4$$

$$\leq H(P) + \log P_{\text{max}} + (K_1 - 1) \log(3P_{\text{max}} + 5) - K_1 \log P_{\text{min}}$$

$$+ \frac{P_{\text{max}} + 4}{P_{\text{min}}} + \log \left(1 - \frac{P_{\text{max}} + 2}{P_G}\right) + K_4,$$

(54)

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where the last inequality holds because we have from (8) that

\[
\log P_G = H(P) + \sum_{M \in G} \frac{P_M}{P_G} \log P_M \\
\leq H(P) + \sum_{M \in G} \frac{P_M}{P_G} \log P_{\max} \\
= H(P) + \log P_{\max}.
\]  

(55)

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