The Bifurcation Analysis of Food Web Prey- Predator Model with Toxin

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Abstract: Local and global bifurcations of food web model consists of immature and mature preys, first predator, and second predator with the current of toxicity and harvesting was studied. It is shown that a trans-critical bifurcation occurs at the equilibrium point $E_0$, and it revealed the existence of saddle-node bifurcation occurred at equilibrium points $E_1$, $E_2$, and $E_3$. At any point, the occurrence of bifurcation of the pitch fork does not occur. However, Hopf bifurcation is carried out around positive equilibrium point only.

Keywords: Equilibrium point, Sotomayor’s theorem, Local and Hopf bifurcation.

1. Introduction

The predatory system plays an essential and main part in the relations between the biological species. The analysis of the predator system was carried out with different functional response; a lot of work has been conducted on the prey-predator dynamic with specific functional responses [6, 9, 18]. Prey predator models with different functional responses and harvesting are developed to better reflect the specific characteristics of different species, ecological models focused on age are more logical than models without stage structure for example see [2],[5].

The Food web concept and method of analysis was first suggested in the 1930s by Elton, who started investigating the social structure of ecosystems from a network point of view [7]. Subsequently, food webs were developed extensively analyzed in different facets of their characteristics, including complexity, stability, dynamics, etc. By studying the properties of the food webs, a variety of investigation studies have been achieved. Many scientific studies played a major role in studying natural environments characteristics, see [8, 17]. Food web systems usually consist of various types of interacting organism relationships, of which the predation relationship is an essential one [12, 15].

Currently, Work into food web dynamics has fallen into cases where four organisms or more live and communicate with each other [10, 16]. Wide complex interactions in food webs of four or more kinds, such as balances, bifurcations, periodic orbits, etc., have led to a greater understanding of the nature between coexistence between organisms and ecosystem stability. Gakkhar et al. [16] are developed for a four-dimensional food-web system the coexistence of all four organisms is formed when the relationships
between middle predators and other organisms are symmetric. But in asymmetric situations the weaker of them would end up extinction.

System of normal differential equations is one of the best essential mathematical constructs used to model processes in the natural sciences. In general, these models lead to non-linear systems that are dependent on parameters. Its behavior can change dramatically based on the parameters. A bifurcation happens when there is a small, smooth changing in parameter values causes a sudden 'qualitative' change in the behavior of the system. For discrete as well as continuous systems bifurcations occur.

It is essential to classify bifurcations in two major classes: bifurcations local and global. Local bifurcations that can be fully studied by adjusting the local stability assets of the equilibrium. Periodic orbits or other invariants are defined as parameters that reach critical thresholds; and global bifurcations that sometimes occur when the system has larger sets of invariants colliding with one another or when the function is stable them cannot be found purely by the equilibrium study of the equilibrium (fixed points).

The Hopf bifurcation is a critical point where the stability of the system changes and periodic solutions occur. More specifically, This is a local bifurcation where the fixed points of the dynamic system losses equilibrium. As a complex pair of uniquely linearized values conjugate the complex plane from the imaginary axis to the fixed point. Under reasonably generic assumptions of a dynamic system, the small amplitude limits the cycle of the branches at a fixed point. The term "bifurcation" was first coined by Henri Poincare in 1885. [11] Henri Poincare has later identified and classified different forms of stationary points. In prey predator with refuge and age structures in [4] Majeed introduced the impact the bifurcation analysis of an ecological model. While Majeed and Ali [1] represented bifurcation of a predator-structured stage food chain model with shelter. The presence of the Hopf bifurcation and local bifurcation for ecological system consisting of predators and structured prey researched by Majeed in [3].

In this work, we examined the local bifurcation analysis and evaluate the conditions for Hopf bifurcation of food web focused on a single μ parameter near to positive equilibrium point for the system, including harvest, age and toxicity.

1. Model formulation [14]

Consider the model which is given in [14]

\[
\begin{align*}
\frac{dX_1}{dT} &= rX_2 \left(1 - \frac{X_2}{k}\right) - \eta_1X_1 - \delta_1X_1^2 - \varphi_1X_1 - \frac{\theta_1X_1}{b_1 + X_1}Y_1, \\
\frac{dX_2}{dT} &= \eta_1X_1 - \delta_2X_2^2 - \varphi_2X_2 - \frac{\theta_2X_2}{b_2 + X_2}Y_1, \\
\frac{dY_1}{dT} &= e_1\frac{\theta_1X_1}{b_1 + X_1}Y_1 + e_2\frac{\theta_2X_2}{b_2 + X_2}Y_1 - \theta_3Y_1Y_2 - \delta_3Y_1 - \varphi_3Y_1 - \gamma_1Y_1, \\
\frac{dY_2}{dT} &= e_3\theta_3Y_1Y_2 - \delta_4Y_2 - \varphi_4Y_2 - \gamma_2Y_2.
\end{align*}
\]

Where the variables and parameters are illustrated in the following table:
### Table 1: The positive variables and parameters through a mathematical model sign:

| Variable | Description |
|----------|-------------|
| $X_1(T)$ | The immature prey population at time $T$. |
| $X_2(T)$ | The mature prey population at time $T$. |
| $Y_1(T)$ | The population size of the first predator at the time $T$. |
| $Y_2(T)$ | The population size of the second predator at the time $T$. |
| $r > 0$ | The growth rate of immature prey. |
| $k > 0$ | The carrying capacity. |
| $\theta_i > 0$, $i=1, 2$. | The maximum attack rates of first predator on its prey. |
| $b_i > 0$, $i=1,2$ | The half saturations rates of first predator. |
| $0 < e_i < 1$, $i=1,2$ | The rates of conversion of food to first predator. |
| $\gamma_1 > 0$ | The death rate of first predator in the absence of its prey. |
| $\theta_i > 0$ | The maximum attack rate of second predator on first predator. |
| $0 < e_3 < 1$ | The conversion rate of food to second predator. |
| $\gamma_2 > 0$ | The death rate of second predator in the absence of its first predator. |
| $\delta_i > 0$ $i=1,2,3,4$ | The toxicity rates of the matured prey, immature prey, the first predator and the second predator respectively. |
| $\phi_i > 0$ $i=1,2,3,4$ | The catch ability coefficient rates of the matured prey, immature prey, the first predator and the second predator respectively. |

The dimensional of system (1) which is given in [14]:

\[
\frac{dx_1}{dt} = x_1 \left[ \frac{x_2(1-x_2)}{x_1} - (\alpha_1 + h_1) - u_1 x_1 - \frac{B_1}{b + x_1} y_1 \right] = x_1 f_1(x_1,x_2,y_1,y_2),
\]

\[
\frac{dx_2}{dt} = x_2 \left[ \frac{e_1 x_1}{x_2} - u_2 x_2 - h_2 - \frac{B_2}{c + x_2} y_1 \right] = x_2 f_2(x_1,x_2,y_1,y_2),
\]

\[
\frac{dy_1}{dt} = y_1 \left[ \frac{f_1 x_1}{b + x_1} + \frac{f_2 x_2}{c + x_2} - B_3 y_2 - (u_3 + h_3 + d_4) \right] = y_1 f_3(x_1,x_2,y_1,y_2),
\]

\[
\frac{dy_2}{dt} = y_2 \left[ f_3 y_1 - (u_4 + h_4 + d_2) \right] = y_2 f_4(x_1,x_2,y_1,y_2).
\]
Where:
\[ t = rT, \quad x_1 = \frac{y_1}{k}, \quad x_2 = \frac{y_2}{k}, \quad y_1 = \frac{y_1}{k}, \quad y_2 = \frac{y_2}{k}, \quad a_1 = \frac{\gamma_1}{r}, \quad a_2 = \frac{\gamma_2}{r}, \quad u_1 = \frac{\delta_1}{r}, \quad h_j = \frac{\psi_j}{r}. \]

\[ B_i = \frac{\theta_i}{r}, \quad B = \frac{\theta_1}{r}, \quad B_3 = \frac{\theta_3}{r}, \quad F_i = \frac{\epsilon_2 \theta_i}{r}, \quad c = \frac{\gamma_2}{r}, \quad d_i = \frac{\psi_i}{r}, \quad u_i = \frac{\delta_i}{r}, \quad F_3 = \epsilon_3 B_3. \]

Where \( i = 1, 2, \) \( j = 1, 2, 3, 4 \) and \( l = 3, 4. \)

2. Local bifurcation analysis

The application of Sotomayor hypothesis [13] is appropriate for local bifurcation in the provided theorems.

The Jacobian matrix of the system (2) which is given in [14] is:
\[ J = [a_{ij}]_{4 \times 4}. \]  

Where:
\[ a_{11} = -\left(a_1 + h_1 + 2u_1 x_1 + \frac{b_1 b}{(b + x_1)^2} y_1 \right), \quad a_{12} = 1 - 2x_2, \quad a_{13} = -\frac{b_1 x_1}{(b + x_1)}, \quad a_{14} = 0, \]
\[ a_{21} = a_1, \quad a_{22} = -\left(2u_2 x_2 + h_2 + \frac{b_2 c}{(c + x_2)^2} y_1 \right), \quad a_{23} = -\frac{b_2 x_2}{(c + x_2)}, \quad a_{24} = 0, \]
\[ a_{31} = \frac{\epsilon_2 b}{(b + x_1)^2} y_1, \quad a_{32} = \frac{\epsilon_2 c}{(c + x_2)^2} y_1, \quad a_{33} = \frac{\epsilon_2 v_2}{(b + x_1)} + \frac{\epsilon_2 v_2}{(c + x_2)} - B_3 y_2 - (u_3 + h_3 + d_1), \quad a_{34} = -B_3 y_1, \]
\[ a_{41} = 0, \quad a_{42} = 0, \quad a_{43} = F_3 y_2, \quad a_{44} = F_3 y_1 - (u_4 + h_4 + d_2). \]

It can be verity for any nonzero vector that \( V = (v_1, v_2, v_3, v_4)^T \) we have
\[ D^3 f_\mu(X, \mu)(V, V) = [b_{ij}]_{4 \times 4}. \]  

Where:
\[ b_{11} = -2 \left[ u_1 v_1^2 + v_1^2 + \frac{b_1 b}{(b + x_1)^2} v_1 v_3 \right], \]
\[ b_{21} = -2 \left[ u_2 v_2^2 + \frac{b_2 c}{(c + x_2)^2} v_2 v_3 \right], \]
\[ b_{31} = 2 \left[ \frac{\epsilon_2 b}{(b + x_1)^2} v_1 v_3 + \frac{\epsilon_2 c}{(c + x_2)^2} v_2 v_3 - B_3 v_3 v_4 \right], \]
\[ b_{41} = 2 \left[ F_3 v_3 v_4 \right]. \]

and,
\[ D^3 f_\mu(X, \mu)(V, V, V) = [0, 0, 0, 0]^T. \]
**Theorem 1**: Assume that the condition below is achieved:

\[ h_2 < 1. \]  \hspace{1cm} (6)

Then, at the point of equilibrium \( E_0 = (0, 0, 0, 0) \), with value of the parameter \( h_1^* = \frac{a_2(1-h_2)}{h_2} \), system (2) Possesses transcritical bifurcation at \( E_0 \).

**Proof**: By substituting \( E_0 \) in the Jacobian matrix given in eq. (3) and compute characteristic equation of \( f_0 \) we get that \( f_0 \) has zero eigenvalue (say \( \lambda_{ax} = 0 \)) at \( h_1 = h_1^* \).

Now, let \( V^{[0]} = \left(v_1^{[0]}, v_2^{[0]}, v_3^{[0]}, v_4^{[0]}\right)^T \) be the eigenvector which corresponds to the eigenvalue \( \lambda_{ax} = 0 \).

So, \( (f_0 - \lambda_{ax} I) V^{[0]} = 0 \) this gives:

\[ v_2^{[0]} = \frac{\alpha_1}{h_2} v_1^{[0]}, \quad v_3^{[0]} = 0, \quad v_4^{[0]} = 0 \]

and \( v_1^{[0]} \) any number that is nonzero real.

Let \( \psi^{[0]} = \left(\psi_1^{[0]}, \psi_2^{[0]}, \psi_3^{[0]}, \psi_4^{[0]}\right)^T \) be the eigenvector correlated with the eigenvalue \( \lambda_{ax} = 0 \) of the matrix \( f_0^T \).

Then we get:

\[ (f_0^T - \lambda_{ax} I) \psi^{[0]} = 0. \]

By resolving this equation for \( \psi^{[0]} \), we get, \( \psi^{[0]} = \left(\psi_1^{[0]}, \frac{1}{h_2} \psi_4^{[0]}, 0, 0\right)^T \),

where, \( \psi_1^{[0]} \) any number that is nonzero real.

Now, consider that: \( \frac{\partial f}{\partial h_1} = f_{h_1}(X, h_1) = \left(\frac{\partial f_1}{\partial h_1}, \frac{\partial f_2}{\partial h_1}, \frac{\partial f_3}{\partial h_1}, \frac{\partial f_4}{\partial h_1}\right)^T = (-x_1, 0, 0, 0)^T \).

So, \( f_{h_1}(E_0, h_1^*) = (0, 0, 0, 0)^T \) and So \( (\psi^{[0]})^T f_{h_1}(E_0, h_1^*) = 0. \)

According to Sotomayor’s theorem, no bifurcation of a saddle-node will occur at \( E_0 \), whereas the first condition for a transcritical bifurcation is fulfilled.

Now, since \( Df_{h_1}(X, h_1) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \)

Where: \( Df_{h_1}(X, h_1) \); denotes the derivative of \( Df_{h_1}(X, h_1) \) with regards to \( X = (x_1, x_2, y_1, y_2)^T \).

Furthermore, it has been noted that:
\[
D f_n(X, h_2)^{[0]} = \begin{bmatrix}
-1 & 0 & 0 & 0
& 0 & 0 & 0 & u_1^{[0]}
& 0 & 0 & 0 & 0
& 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
v_1^{[0]}
\end{bmatrix}
= \begin{bmatrix}
-v_1^{[0]}
\end{bmatrix},
\]

So, we obtain that,
\[
(\psi^{[0]})^T [D f_n(X, h_2)^{[0]}] = -v_1^{[0]} \psi_1^{[0]} \neq 0.
\]

Now, by substituting \( V^{[0]} \) in (4), we get:
\[
D^2 f(E_0, h_1)^{[0]}(V^{[0]}, V^{[0]}) = \begin{bmatrix}
-2u_1(v_1^{[0]})^2 + \left(\frac{a_1}{a_2} v_1^{[0]}\right)^2
& -2u_2\left(\frac{a_1}{a_2} v_1^{[0]}\right)^2

0

0
\end{bmatrix}.
\]

Hence, it is obtained that:
\[
(\psi^{[0]})^T D^2 f(E_0, h_1)^{[0]}(V^{[0]}, V^{[0]}) = -2 \psi_1^{[0]} Q \psi_1^{[0]} \neq 0,
\]
where:
\[
Q = \frac{a_1 h_2 + \alpha_1 h_2}{h_2^2}.
\]

Therefore, the system (2) has a transcritical bifurcation according to the Sotomayor theorem at \( E_0 \) with the parameter \( h_1 = h_1'. \)

**Theorem 2:** Suppose conditions are met as follows:
\[
\begin{align*}
\bar{x}_2 & < \frac{1}{2}, \\
\alpha_1 & > \frac{2u_2 \bar{x}_2 (\alpha_1 + h_1 + 2u_1 \bar{x}_1)}{(1 - 2\bar{x}_2)},
\end{align*}
\]

Then, at the equilibrium point \( E_1 = (\bar{x}_1, \bar{x}_2, 0, 0) \) with value of the parameter:
\[
\hat{h}_2 = \frac{\alpha_1 (1 - 2\bar{x}_2) - 2u_2 \bar{x}_2 (\alpha_1 + h_1 + 2u_1 \bar{x}_1)}{(\alpha_1 + h_1 + 2u_1 \bar{x}_1)}.
\]

System (2) possesses saddle-node bifurcation at \( E_1 = (\bar{x}_1, \bar{x}_2, 0, 0) \).

**Proof:** By substituting \( E_1 \) in the Jacobian matrix given in eq. (3) and compute characteristic equation of \( J_1 \) we get that \( J_1 \) has zero eigenvalue \( (\lambda_{1x_2} = 0) \) at \( h_2 = \hat{h}_2 \).

Now, let \( V^{[1]} = (v_1^{[1]}, v_2^{[1]}, v_3^{[1]}, v_4^{[1]})^T \) be the eigenvector equivalent to an eigenvalue \( \lambda_{1x_2} = 0 \).

Thus \( (\hat{J}_1 - \lambda_{1x_2} I) V^{[1]} = 0 \), this gives: \( V^{[1]} = (\hat{L}_1 v_2^{[1]}, v_2^{[1]}, 0, 0) \), where \( \hat{L}_1 = -\frac{\hat{C}_{12}}{\hat{C}_{11}} \) and \( v_2^{[1]} \) any non-zero real number.
Let $\Psi^{[1]} = \left(\psi_1^{[1]}, \psi_2^{[1]}, \psi_3^{[1]}, \psi_4^{[1]}\right)^T$ be the eigenvector related with the eigenvalue $\lambda_{1x_2} = 0$ of the matrix $\tilde{J}_2^T$.

Then we have:

$$\left(\tilde{J}_2^T - \lambda_{1x_2} I\right)\Psi^{[1]} = 0.$$

for $\Psi^{[1]}$ by getting a solution to this equation will get:

$$\Psi^{[1]} = \left(\tilde{L}_2\psi_1^{[1]}, \psi_2^{[1]}, \tilde{L}_3\psi_2^{[1]}, 0\right)^T,$$

where, $\tilde{L}_2 = -\left(\tilde{c}_{11} \tilde{c}_{11} + \tilde{c}_{11}\right)$.

Now, consider:

$$\frac{\partial f}{\partial h_2} = f_{h_2}(X, h_2) = \left(\frac{\partial f_1}{\partial h_2}, \frac{\partial f_2}{\partial h_2}, \frac{\partial f_3}{\partial h_2}, \frac{\partial f_4}{\partial h_2}\right)^T = (0, -\tilde{x}_2, 0, 0)^T.$$

So, $f_{h_2}(E_1, \tilde{h}_2) = (0, -\tilde{x}_2, 0, 0)^T$.

Therefore $(\Psi^{[1]})^T f_{h_2}(E_1, \tilde{h}_2) = -\tilde{x}_2 \psi_2^{[1]} \neq 0$.

According to Sotomayor’s theorem, therefore, there will be no transcritical bifurcation at $E_1$, while the first condition of the saddle-node bifurcation is satisfied.

Now, by replacing $V^{[1]}$ in (4), we get:

$$D^2 f_{h_2}(E_1, \tilde{h}_2)(V^{[1]}, V^{[1]}) = \begin{bmatrix}
-2 \left[u_1 \tilde{L}_1^2 + 1\right] \left(\psi_2^{[1]}\right)^2 \\
-2 u_2 \left(\psi_2^{[1]}\right)^2 \\
0 \\
0
\end{bmatrix}.$$

Hence, it is obtained that: $(\Psi^{[1]})^T D^2 f_{h_2}(E_1, \tilde{h}_2)(V^{[1]}, V^{[1]}) = -2 \tilde{Q} \left(\psi_2^{[1]}\right)^2 \psi_2^{[1]} \neq 0$

Where $\tilde{Q} = \tilde{L}_2 \left(u_1 \tilde{L}_1^2 + 1\right) + u_2$.

Consequently, according to the Sotomayor theorem, the system (2) has a bifurcation of the saddle node $E_1$ with the parameter $h_2 = \tilde{h}_2$. 
**Theorem 3:** Suppose conditions are met as follows:

\[
\hat{x}_2 < \frac{1}{2},
\]

\[
\alpha > \frac{(a_1 + h_1 + 2u_1 \hat{x}_1 + \frac{b_1 b}{(b + \hat{x}_1)^2} \hat{y}_1)(2u_2 \hat{x}_2 + h_2 + \frac{b_2 c}{(c + \hat{x}_2)^2} \hat{y}_1)}{(1 - 2 \hat{x}_2)}, \tag{9}
\]

\[
\frac{f_1 \hat{x}_1}{(b + \hat{x}_1)} + \frac{f_2 \hat{x}_2}{(c + \hat{x}_2)} > (u_3 + d_3), \tag{10}
\]

\[
\left[ \frac{f_1 b}{(b + x_1)^2} L_1 + \frac{f_2 c}{(c + x_2)^2} L_2 \right] \neq \left[ L_1 \left( u_1 \hat{x}_1 \hat{L}_3 + \frac{b_1 b}{(b + x_1)^2} \hat{L}_3 \right) + L_2 \hat{L}_4 \left( u_2 \hat{x}_2 + \frac{b_2 c}{(c + x_2)^2} \hat{L}_3 \right) + \hat{L}_2 \hat{L}_3 \right]. \tag{11}
\]

Then the system (2) at the equilibrium point \( E_2 = (\hat{x}_1, \hat{x}_2, \hat{y}_1, 0) \) with the parameter value:

\[
\hat{h}_3 = \frac{\hat{Q}_1}{(d_{11}d_{22} - d_{12}d_{21})}, \text{ where:}
\]

\[
\hat{Q}_1 = -\left( \mu_5 + \mu_6 + \left( \frac{f_1 \hat{x}_1}{(b + x_1)} + \frac{f_2 \hat{x}_2}{(c + x_2)} - (u_3 + d_3) \right) \mu_3 \right) + \hat{d}_{11} \mu_2 + \hat{d}_{22} \mu_4.
\]

Possesses saddle-node bifurcation, but a transcritical bifurcation cannot occur at \( E_2 \).

**Proof:** The characteristic equation of \( f_2 \) which is given in [14] having zero eigenvalues (say \( \lambda_{2y_1} = 0 \)), if and only if \( \hat{A}_3 = 0 \) and then \( E_2 \) turns into a non-hyperbolic equilibrium point.

The Jacobian matrix of the system (2) at the equilibrium point \( E_2 \) with parameter \( h_3 = \hat{h}_3 \) becomes:

\[
\hat{f}_2 = f_2( h_3 = \hat{h}_3 ) = [\hat{d}_{ij}]_{4 \times 4},
\]

where \( \hat{d}_{ij} = d_{ij} \) for all \( i, j = 1, 2, 3, 4 \) except \( \hat{d}_{33} = \hat{f}_1 \hat{x}_1 + \hat{f}_2 \hat{x}_2 + (u_3 + \hat{h}_3 + d_3), \) where \( d_{ij} \) given in [14].

Note that, \( \hat{h}_3 > 0 \) provided that conditions (9) – (11) hold.

Now, let \( \nu^{[2]} = (v_1^{[2]}, v_2^{[2]}, v_3^{[2]}, v_4^{[2]})^T \) be the eigenvector which corresponds to the eigenvalue \( \lambda_{2y_1} = 0 \).
Thus \((f_2 - \lambda_{2y})V^{[2]} = 0\), which gives: 
\[ V^{[2]} = \begin{pmatrix} \tilde{L}_1v_3^{[2]} \end{pmatrix}, \tilde{L}_2v_3^{[2]}, \begin{pmatrix} v_3^{[2]} \end{pmatrix}, \begin{pmatrix} 0 \end{pmatrix} \] , where \(v_3^{[2]}\) any nonzero real number.

With \(\tilde{L}_1, \tilde{L}_2\) which was mentioned in state theorem.

Let \(\psi^{[2]} = \begin{pmatrix} \psi_1^{[2]} \psi_2^{[2]} \psi_3^{[2]} \psi_4^{[2]} \end{pmatrix}^T\) be the eigenvector related with the eigenvalue \(\lambda_{2y} = 0\) of the matrix \(f_2^T\).

Then we have: 
\[ (f_2^T - \lambda_{2y})\psi^{[2]} = 0. \]
By finding the solution to the equation, it obtain
\[ \psi^{[2]} = \begin{pmatrix} \tilde{L}_3\tilde{\psi}_3^{[2]} \tilde{L}_4\tilde{\psi}_3^{[2]} \tilde{\psi}_3^{[2]} \tilde{\psi}_3^{[2]} \end{pmatrix}^T \]

Where \(\tilde{\psi}_3^{[2]}\) any nonzero real number, with \(\tilde{L}_3, \tilde{L}_4\) and \(\tilde{L}_5\) which are indicated in the theorem state.

Now,
\[ \frac{df}{dh_3} = f_{h_3}(x, \tilde{h}_3) = \begin{pmatrix} \frac{df_1}{dh_3} \frac{df_2}{dh_3} \frac{df_3}{dh_3} \frac{df_4}{dh_3} \end{pmatrix} = (0, 0, -y_1, 0)^T. \]

So,
\[ f_{h_3}(E_2, \tilde{h}_3) = (0, 0, -y_1, 0)^T, \]

hence, \(\psi^{[2]} = \begin{pmatrix} \tilde{\psi}_3^{[2]} \end{pmatrix} \neq 0.\)

Thus, no transcritical bifurcation will occur with the Sotomayor theorem at \(E_2\), while the first condition of a saddle-node bifurcation is fulfilled.

Also, by replacing \(V^{[2]}\) in (4), we get:
\[ D^2f_{h_3}(E_2, \tilde{h}_3)(V^{[2]}, V^{[2]}) = \begin{bmatrix} -2 \left( u_1(\tilde{L}_1)^2 + (\tilde{L}_2)^2 + \frac{\beta_1 b}{(b + x_1)^2} \tilde{L}_1 \right) \left( v_3^{[2]} \right)^2 \\ -2 \left( u_2(\tilde{L}_2)^2 + \frac{\beta_2 c}{(c + x_2)^2} \tilde{L}_2 \right) \left( v_3^{[2]} \right)^2 \\ 2 \left( \frac{F_3 b}{(b + x_1)^2} \tilde{L}_1 + \frac{F_3 c}{(c + x_2)^2} \tilde{L}_2 \right) \left( v_3^{[2]} \right)^2 \\ 0 \end{bmatrix}. \]

Hence, it is obtained that:
\[ (\psi^{[2]})^T D^2f_{h_3}(E_2, \tilde{h}_3)(V^{[2]}, V^{[2]}) = 2 \tilde{Q}_2 \tilde{\psi}_3^{[2]} \tilde{\psi}_3^{[2]}, \]

where:
\[ \tilde{Q}_2 = \begin{bmatrix} \frac{F_3 b}{(b + x_1)^2} \tilde{L}_1 + \frac{F_3 c}{(c + x_2)^2} \tilde{L}_2 - \tilde{L}_1 \left( u_1 \tilde{L}_1 \tilde{L}_3 + \frac{\beta_1 b}{(b + x_1)^2} \tilde{L}_3 \right) + \tilde{L}_2 \left( u_2 \tilde{L}_2 \tilde{L}_3 + \frac{\beta_2 c}{(c + x_2)^2} \tilde{L}_3 \right) + (\tilde{L}_2)^2 \tilde{L}_3 \end{bmatrix} \]

With \(\tilde{L}_i\) ; \(i = 1, 2, 3, 4\) which are associated in the theorem state.

So, if the condition (12), is achieved, we get that:
\[ (\psi^{[2]})^T D^2f_{h_3}(E_2, \tilde{h}_3)(V^{[2]}, V^{[2]}) \neq 0, \]

So, the system (2) has a saddle-node bifurcation at \(E_2\) with the parameter \(h_3 = \tilde{h}_3\), according to the Sotomayor theorem.
Theorem 4: Suppose conditions are met as follows:

\[
\frac{F_i \dot{x}_1}{b + x_1} + \frac{F_2 \dot{x}_2}{c + x_2} > B_3 y_2 + (u_3 + h_3),
\]

(13)

\[
F_i y_1 > (u_4 + h_4 + d_2),
\]

(14)

\[
\frac{-a_1 b_1 x_1^2}{(1-\delta_s) x_2} > \frac{F_i b_1 (b+x)}{c (c+x_2)}
\]

(15)

\[
\begin{bmatrix}
F_i b \\
F_2 c
\end{bmatrix} \dot{x}_1 + \begin{bmatrix}
F_i c \\
F_2 b
\end{bmatrix} \dot{x}_2 + F_3 \dot{y}_3 \dot{x}_6 \neq \begin{bmatrix}
\dot{L}_1 (u_1 \dot{L}_1 \dot{L}_4 + \frac{b_1 b_1}{b+x} \dot{L}_3) + \dot{L}_2 (u_2 \dot{L}_2 + \frac{b c}{c+x_2}) + \dot{L}_3 (x_3 \dot{L}_3) \\
\end{bmatrix}
\]

(16)

Where:

\[
\dot{L}_1 = -\left(\frac{e_{12} e_{22} e_{23}}{e_{11}}\right), \quad \dot{L}_2 = \left(\frac{e_{21} e_{12} e_{22}}{e_{11} e_{22} - e_{12} e_{21}}\right),
\]

\[
\dot{L}_3 = -\left(\frac{e_{32} e_{33}}{e_{34}}\right),
\]

\[
\dot{L}_4 = -\left(\frac{e_{21} e_{22} e_{23}}{e_{22}}\right), \quad \dot{L}_5 = \left(\frac{e_{21} e_{12} e_{23}}{e_{21} e_{12} e_{23} - e_{22} e_{21}}\right), \quad \dot{L}_6 = -\left(\frac{e_{32} e_{33}}{e_{34}}\right).
\]

Then the system (2) at the equilibrium point \( E_3 = (x_1, x_2, y_1, y_2) \) with the parameter value:

\[
d_4 = \left(\frac{Q_1}{e_{44} e_{12} e_{22} - e_{12} e_{21}}\right),
\]

where:

\[
Q_1 = e_{44} (f_5 + f_8 + \left(\frac{F_2 x_1}{b+x} + \frac{F_2 x_2}{c+x_2} - B_3 y_2 - (u_3 + h_3)\right) f_3) - f_8 f_11 - f_9 f_12.
\]

Possesses saddle-node bifurcation, but a transcritical bifurcation cannot occur at \( E_3 \).

Proof: The characteristic equation of \( f_3 \) which is assumed in [14] having zero eigenvalues (say \( \lambda_{3y_1} = 0 \)), if and only if \( B_4 = 0 \) and then \( E_3 \) becomes a non-hyperbolic equilibrium points.

The Jacobian matrix of the system (2) at the equilibrium point \( E_3 \) with parameter \( d_4 = d_4 \) becomes:

\[
J_{d_1} = f_3 (d_4 = d_4) = \left[\begin{array}{c}
\dot{e}_{ij}
\end{array}\right]_4 \times_4,
\]

where \( \dot{e}_{ij} = \dot{e}_{ij} \) for all \( i, j = 1, 2, 3, 4 \) except \( \dot{e}_{33} = F_i \dot{x}_1 + F_2 \dot{x}_2 - B_3 \dot{y}_2 - (u_3 + h_3 + d_4) \), where \( e_{ij} \) given in [14].

Note that, \( d_4 > 0 \) provided that conditions (13) – (15) hold.

Now, let \( V^{[3]} = \left( v_1^{[3]}, v_2^{[3]}, v_3^{[3]}, v_4^{[3]} \right)^T \) be the eigenvector equivalent to the eigenvalue \( \lambda_{3y_1} = 0 \).

Thus \( \left( J_d - \lambda_3 v_1^T \right) V^{[3]} = 0 \), this gives:

\[
V^{[3]} = \left( L_1 v_1^{[3]}, L_2 v_2^{[3]}, L_3 v_3^{[3]}, L_4 v_4^{[3]} \right)^T,
\]
where $\nu_3^{\ast}$ any nonzero number, with $\dot{\nu}_4, \dot{\nu}_5$ and $\dot{\nu}_6$ which are associated to in the state of theorem.

Let $\psi^{\ast} = (\psi_1^{\ast}, \psi_2^{\ast}, \psi_3^{\ast}, \psi_4^{\ast})^T$ be the eigenvector related with the eigenvalue $\lambda_{3\nu_3} = 0$ of the matrix $J_3$.

Then we have: \( (J_3 - \lambda_{3\nu_3}I) \psi^{\ast} = 0 \). By finding the solution to the equation, it obtain

\[
\psi^{\ast} = (L_4\psi_4^{\ast}, L_5\psi_5^{\ast}, L_6\psi_6^{\ast})^T, \quad \text{where } \psi_3^{\ast} \text{ any nonzero number, with } \dot{\nu}_4, \dot{\nu}_5 \text{ and } \dot{\nu}_6 \text{ which are stated in the theorem situation.}
\]

Now, \[
\frac{\partial f}{\partial \dot{d}_1} = \frac{\partial f_1}{\partial \dot{d}_1} + \frac{\partial f_2}{\partial \dot{d}_1} + \frac{\partial f_3}{\partial \dot{d}_1} + \frac{\partial f_4}{\partial \dot{d}_1} \cdot \psi_3^{\ast} = (0, 0, -y_1, 0)^T.
\]

So, $f_{d_1}(E_3, d_1^{\ast}) = (0, 0, -y_1, 0)^T$.

Hence $(\psi^{\ast})^T f_{d_1}(E_3, d_1^{\ast}) = -y_1 \psi_3^{\ast} \neq 0$.

So, according to Sotomoyar’s theorem, no transcritical bifurcation will occur at $E_3$, while the first condition of a saddle-node bifurcation is fulfilled.

Moreover, by substituting $V^{\ast}$ in (4), we get:

\[
D^2f_{d_1}(E_3, d_1^{\ast})(V^{\ast}, \nu^{\ast}) = \begin{bmatrix}
-2 \left( u_1 \left( \dot{L}_1 \right)^2 + \left( \dot{L}_2 \right)^2 + \frac{B_1b}{(b + x_1)^2} \dot{L}_1 \right) \psi_3^{\ast}^2 \\
-2 \left( u_2 \left( \dot{L}_2 \right)^2 + \frac{B_2c}{(c + x_2)^2} \dot{L}_2 \right) \psi_3^{\ast}^2 \\
2 \left( \frac{F_1b}{(b + x_1)^2} \dot{L}_1 + \frac{F_2c}{(c + x_2)^2} \dot{L}_2 - B_3 \dot{L}_3 \right) \psi_3^{\ast}^2 \\
2 \left( F_3 \dot{L}_3 \right) \psi_3^{\ast}^2
\end{bmatrix}.
\]

Hence, it is obtained that: $(\psi^{\ast})^T D^2f_{d_1}(E_3, d_1^{\ast})(V^{\ast}, \nu^{\ast}) = 2 \nu_3 \psi_3^{\ast}$,

Where: $Q_2 = \begin{bmatrix}
\frac{F_1b}{(b + x_1)^2} \dot{L}_1 + \frac{F_2c}{(c + x_2)^2} \dot{L}_2 + F_3 \dot{L}_3 L_6 \\
\left( \dot{L}_2 \right)^2 L_4 + B_3 \dot{L}_3
\end{bmatrix} - \begin{bmatrix}
\dot{L}_1 \left( u_1 \dot{L}_4 + \frac{B_1b}{(b + x_1)^2} \dot{L}_4 \right) + \dot{L}_2 \dot{L}_3 \left( u_1 \dot{L}_4 + \frac{B_1b}{(b + x_1)^2} \dot{L}_4 \right) + \\
\left( \dot{L}_2 \right)^2 L_4 + B_3 \dot{L}_3
\end{bmatrix}.

So, if the condition (16) is achieved, we get that: $(\psi^{\ast})^T D^2f_{d_1}(E_2, d_2^{\ast})(V^{\ast}, \nu^{\ast}) \neq 0$.

Thus, according to the Sotomoyar theorem, the system (2) has a saddle node bifurcation at $E_3$ with the parameter $d_1 = d_1^{\ast}$. 
4. Hopf bifurcation analysis

In this section, the possibility of a Hopf bifurcation of near to Positive equilibrium of the system (2) is explored as seen in the following theorem.

**Theorem 6**: Suppose that the local condition it’s referenced in [14] with the following condition is satisfied:

\[ N_1 > N_2 . \]  
\[ N_3 > N_4 . \]  
\[ N_4 + N_5 > \left( a_1 + h_1 + 2u_1 x_1^* + \frac{B_1 b}{b + x_1^*} y_1^* \right) (G_2 + N_3 ) + N6 \]  
\[ N_7 < N_8 . \]

\[ \frac{4B_1B_2 - B_1^3}{4} < B_3 < \min \left\{ \frac{B_1B_2}{2} , \frac{e_{44}f_6 - e_{22}f_2 B_1}{f_1 + e_{22}} \right\} \]

where:

\[ N_1 = e_{22}f_6 - f_1f_2 - e_{22}f_1^2 - e_{22}^2 f_1. \]

\[ N_2 = e_{44}f_6 - f_6. \]

\[ N_3 = (f_1 + e_{22})((f_1 + f_6) - f_6f_10) + f_4(f_6 - e_{22}f_1 - f_2) + N_1(1 - f_1) - (f_1 + e_{22})^2 f_6 + e_{12}e_{21}(f_6 - 2e_{22}f_1 - f_1^2) \]

\[ N_4 = e_{44}(f_7 + f_8) + 2e_{22}f_6 - (f_6 + f_6) + f_6(f_2 - f_6) - f_1(N_2 - 2f_6). \]

\[ N_5 = f_6(f_3 + f_10 - 2e_{44}f_6 + 3f_1f_12 - 2f_12) + e_{22}^2 f_1 + f_6 + (2f_6 - e_{22}f_1 - f_2)(f_7 + f_8 - f_6f_10) - 2(e_{44}(f_7 + f_8) - f_1f_12)(f_1 + e_{22}) + e_{12}e_{21}(f_6(e_{22} - 1) + 2f_1(f_2 - e_{22}^2)) + (e_{22}^3 + f_1^3)N_2. \]

\[ N_6 = e_{22}f_1 (e_{22}f_6 - f_1f_2 - e_{22}^2 f_1 - e_{22}^2 f_2 + e_{12}e_{21}) + f_4f_5(2f_2 + e_{22}f_1 + f_2^2) + e_{22}(f_4 - f_1^2 f_2 - f_6 - e_{22}f_2 + f_6f_12) - f_6(2f_4 + f_1 + e_{22}^2 f_2 - e_{44}f_1 + f_1^2 f_2) - f_1^2 f_6 - ((e_{22} + f_1)^2 - f_6 - e_{12}e_{21})(f_7 + f_8 - f_6f_10) - e_{12}e_{21}(e_{44}f_6 + f_1(2e_{22}f_1 + f_4 - 2e_{22}f_1 - f_1^2)). \]
\[ N_7 = f_1 f_2 (e_{22} (f_2 - f_4 + e_{22}^2 + e_{22} f_1) - e_{44} f_6) + (f_4 - f_3^2) f_6 f_{12} + e_{12} e_{21} f_6 f_{12} - e_{22} f_1 (f_1^2 - f_4 + e_{22} f_1 - e_{12} e_{21}) + f_1^2 f_6 \]
\[ + \left( e_{22} f_1^2 (1 + e_{22} f_4 - e_{22} f_1 + f_1 (f_2 - f_3)) \right) ((f_7 + f_6) - e_{22} f_{10}) + f_6 (e_{44} f_1 f_6 - e_{22}^2 f_2). \]

\[ N_8 = \left( e_{22} + 2 f_1 f_{12} + f_1^2 e_{44} \right) (f_7 + f_6) - (2 e_{44} f_6 - e_{22} f_2 + e_{12} e_{21} f_1 - e_{12} e_{21} e_{22}) \]
\[ (f_7 + f_6) - e_{22}^2 (f_9 f_{12} - e_{44} f_6) - f_{12} (f_2 f_6 + f_1 f_6 f_{22} + f_1) \]
\[ - e_{12} e_{21} f_6 (2 e_{22} f_2 - e_{44} f_6) + e_{44}^2 f_6^2. \]

Then at parameter value \( h_1 = h_4 \), the system (2) has a Hopf bifurcation close to point \( E_3 \).

**Proof:** The characteristic equation of system (2) at \( E_3 \) which is given by:

\[ [\lambda^4 + B_1 \lambda^3 + B_2 \lambda^2 + B_3 \lambda + B_4] = 0, \quad (22) \]

Then, by the Hopf bifurcation theory, for \( n = 4 \), we need to find a parameter to say \( \left( h_1^* \right) \) to confirm necessary and sufficient conditions for the Hopf bifurcation to achieve:

\[ B_i \left( h_1^* \right) > 0; \quad i = 1,3,4, \quad \Delta_1 \left( h_1^* \right) > 0, \quad B_i^2 \left( h_1^* \right) - 4 \Delta_1 \left( h_1^* \right) > 0 \text{ and } \Delta_2 \left( h_1^* \right) = 0, \]

where \( B_i; \quad i = 1,3,4 \), denotes characteristic coefficients of eq.(22).

Direct calculation provides that:

\[ B_i \left( h_1^* \right) > 0; \quad i = 1,3,4 \text{ and } \Delta_1 \left( h_1^* \right) > 0, \text{ Under the local conditions, see [14].} \]

While, \( B_i^3 \left( h_1^* \right) - 4 \Delta_1 \left( h_1^* \right) > 0 \), provided the condition holds (21).

It is examined that \( \Delta_2 = 0 \) gets that \( B_3 (B_1 B_2 - B_3) - B_1^2 B_4 = 0, \)

Straightforward computation we get:

\[ G_1 h_1^* + G_2 h_1^* + G_3 h_1^* + G_4 = 0, \quad (23) \]

Where:

\[ G_1 = N_1 - N_2 \]
\[ G_2 = 3 \left( a_1 + 2 u_1 x_1^* + \frac{B_1}{b + x_1^*} y_1^* \right) G_3 + (N_3 - N_4), \]
\[ G_3 = \left( \alpha_1 + 2u_1 \dot{x}_1 + \frac{b_1}{(b + s_1)} \right) \left( G_2 + (N_3 - N_4) \right) - (N_5 - N_6). \]

\[ G_4 = \left( \alpha_1 + 2u_1 \dot{x}_1 + \frac{b_1}{(b + s_1)} \right) \left( G_3 - 2 \left( \alpha_1 + 2u_1 \dot{x}_1 + \frac{b_1}{(b + s_1)} \right)^2 G_1 - (N_3 - N_4) \right) + (N_7 - N_8). \]

with, \( N_i \); \( i = 1 - 8 \), which have been stated in the state of theorem.

Clearly, \( G_i > 0 \), \( i = 1,2 \), and \( G_j < 0 \), \( j = 3,4 \) providing that under the local conditions, see [14], in addition to the conditions \((17 − 20)\) hold.

Observe that, when utilizing Descartes law of sign eq.(23), has a unique positive root \( h_1^* \).

Now, at \( h_1 = h_1^* \) the characteristic equation is given by eq. (22). It can be written as:

\[ \left( \lambda^2 + \frac{b_1}{b_t} \right) \left( \lambda^2 + B_1 \lambda + \frac{B_1}{b_t} \right) = 0, \] which has four roots:

\[ \lambda_{1,2} = \pm i \sqrt{\frac{b_1}{b_t}} \] \text{ and } \[ \lambda_{3,4} = \frac{1}{2} \left( -B_1 \pm \sqrt{B_1^2 - 4 \frac{B_1}{b_t}} \right). \]

Clearly, at \( h_1 = h_1^* \) there are two pure imaginary eigenvalues (\( \lambda_1 \) and \( \lambda_2 \)) and two real and negative eigenvalues. Now for all those values of \( h_1 \) in the neighbourhood of \( h_1^* \), In general, the roots of the following form:

\[ \lambda_1 = \omega_1 + i \omega_2 \] \text{, } \[ \lambda_2 = \omega_1 - i \omega_2 \] \text{, } \[ \lambda_{3,4} = \frac{1}{2} \left( -B_1 \pm \sqrt{B_1^2 - 4 \frac{\Delta_1}{B_1}} \right). \]

Clearly, \( \text{Re}(\lambda_M(h_{1}^{*})) \mid_{h_{1} = h_1^{*}} = \omega_1 \left( h_1^{*} \right) = 0 \), \( M = 1,2 \), this implies that the first condition of the necessary and sufficient Hopf bifurcation is achieved at \( h_1 = h_1^{*} \).

Now, that the transversality condition is verified, we have to prove that:

\[ \hat{\theta} \left( h_1^{*} \right) \hat{\psi} \left( h_1^{*} \right) + \hat{\Gamma} \left( h_1^{*} \right) \Phi \left( h_1^{*} \right) \neq 0, \]

Note that for \( h_1 = h_1^{*} \) we have \( \omega_1 (h_1^{*}) = 0 \) and \( \omega_2 (h_1^{*}) = \frac{b_1}{\sqrt{b_t}} \), thus give the following simplifications:

\[ \hat{\psi} \left( h_1^{*} \right) = -2 B_3 \left( h_1^{*} \right), \] \text{ and } \[ \Phi \left( h_1^{*} \right) = 2 \frac{\omega_2 (h_1^{*})}{b_t} \left( B_1 B_2 - 2 B_3 \right), \]

\[ \hat{\theta} \left( h_1^{*} \right) = B_4 \left( h_1^{*} \right) - \frac{B_2}{B_1} B_4^* \left( h_1^{*} \right), \] \text{ and } \[ \hat{\Gamma} \left( h_1^{*} \right) = \omega_2 \left( h_1^{*} \right) \left( B_3^* \left( h_1^{*} \right) - \frac{B_3}{B_1} B_4^* \left( h_1^{*} \right) \right). \]
Where:  
\[ B'_1 = \frac{dB_1}{dh_1} \bigg|_{h_1=h_1^*} = -1, \quad B'_2 = \frac{dB_2}{dh_1} \bigg|_{h_1=h_1^*} = -\left(f_1 + \hat{e}_{22}\right), \]
\[ B'_3 = \frac{dB_3}{dh_1} \bigg|_{h_1=h_1^*} = f_2 + \hat{e}_{22}f_2, \quad B'_4 = \frac{dB_4}{dh_1} \bigg|_{h_1=h_1^*} = -\hat{e}_{22}f_2 + \hat{e}_{44}f_6. \]

Well, then we get the:  
\[ \hat{\theta}(h_1^*) \hat{\psi}(h_1^*) + \hat{\Gamma}(h_1^*) \hat{\phi}(h_1^*) = N_0 + N_{10} \neq 0. \]

Where:
\[ N_0 = 2B_3 \left( \hat{e}_{22}f_2 - \hat{e}_{44}f_6 - \frac{b_1}{h_1} \left(f_1 + \hat{e}_{22}\right) \right), \]
\[ N_{10} = 2 \frac{b_2}{h_1^*} \left( f_2 + \hat{e}_{22}f_2 + \frac{b_1}{h_1} \right) \left(B_1B_2 - 2B_3\right). \]

Now, by condition (21), we have:
\[ \hat{\theta}(h_1^*) \hat{\psi}(h_1^*) + \hat{\Gamma}(h_1^*) \hat{\phi}(h_1^*) \neq 0. \]

Therefore, we get that the Hopf bifurcation happens at the point of equilibrium \( E_3 \) at the parameter \( h_1 = h_1^* \).

5. Numerical simulation

In this part, the behavior of the system (2) is analyzed numerically to validate our analytical outcome. The time series and the phase image of the system solutions (2) are developed for the next set of parameters.

\[ \alpha_1 = 0.01, \quad u_1 = 0.5, \quad u_2 = 0.05, \quad u_i = 0.001; \quad i = 3, 4, \quad h_1 = 0.001, \quad h_j = 0.01; \quad j = 2, 3, 4 \]
\[ B_i = 0.1, \quad F_i = 0.09; \quad i = 1, 2, 3, \quad d_1 = 0.0001, \quad d_2 = 0.001, \quad b = 0.3, \quad c = 0.3. \] (24)

Fig. (1): (a) with \( h_1 = 0.005 \), time series for the data set on the equation (24), (b) with \( h_1 = 0.99 \), time series for the data set on the equation (24).
Table 2: Numerical behavior with the data it provides (24) with changing the behavior of the solution of the system (2).

| Range of parameter | The bifurcation point |
|--------------------|-----------------------|
| $0.001 \leq h_1 < 0.129$ | $h_1 = 0.129$ |
| $0.89 \leq h_1 < 2$ | $h_1 = 0.89$ |
| $0.01 \leq h_2 < 0.103, 0.65 \leq h_2 < 1.$ | $h_2 = 0.103, h_2 = 0.65$ |
| $0.001 \leq u_3, d_4 < 0.054$ | $u_3, d_4 = 0.054$ |
| $0.054 \leq u_3, d_4 < 0.093$ | $u_3, d_4 = 0.093$ |
| $0.093 \leq u_3, d_4 < 1$ | |
| $0.01 \leq h_3 < 0.063$ | $h_3 = 0.063$ |
| $0.063 \leq h_3 < 1$ | |

6. Conclusions and discussion

This paper suggests and analyzes a model of food web made up of immature and mature prey, the first predator, and the second predator. Investigating this system's dynamic behavior shows the system has at most four points of equilibrium. This method of study is important because of the availability of various forms of organisms in the ecosystem that include specific species factors at the same time, like harvesting and toxic substances and others can affect the persistence of all organisms. Mathematical modeling is also used to consider the dynamics of such a food web model and the effects of various values of parameters that reflect different environmental factors. However, bifurcation research is used to explain the consequences of various system parameters. During the analysis and the investigation, transcritical bifurcation is observed to occur at the $E_0$, balance point, just though the seat core bifurcation happens at the $E_1$, $E_2$ and $E_3$ harmony points. It needs referencing; the pitch fork bifurcation has no plausibility case at either point. Finally, some numerical simulation was used to delineate local incidents. Clearly, the system (2) does not have periodic dynamic, and the parameters $u_i, i = 1,2,4, \ F_i, B_i, i = 1,2,3, \ \alpha_i, h_4, d_2, b$ and $c$ have no impact on the dynamic behavior of the system (2) and the solution of the system is still approaching at this point $E_3 = (\hat{x}_1, \hat{x}_2, \hat{y}_1, \hat{y}_2)$. 
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