Point symmetries of the heat equation revisited

Serhii D. Koval† and Roman O. Popovych‡

†Department of Mathematics and Statistics, Memorial University of Newfoundland,
St. John’s (NL) A1C 5S7, Canada
Department of Mathematics, Kyiv Academic University, 36 Vernads’koho Blvd, 03142 Kyiv, Ukraine
‡Mathematical Institute, Silesian University in Opava, Na Rybníčku 1, 746 01 Opava, Czech Republic
Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivska Str., 01024 Kyiv, Ukraine

E-mails: skoval@mun.ca, rop@imath.kiev.ua

We derive a nice representation for point symmetry transformations of the (1+1)-dimensional linear heat equation and properly interpret them. This allows us to show that the pseudogroup of these transformations has exactly two connected components. That is, the heat equation admits a single independent discrete symmetry, which can be chosen to be alternating the sign of the dependent variable. The classification of subalgebras of the essential Lie invariance algebra of the heat equation is enhanced as well.

1 Introduction

The (1+1)-dimensional (linear) heat equation

\[ u_t = u_{xx} \]  

(1)

is one of the simplest but fundamental equations of mathematical physics. This equation became a test example in a number of branches within the theory of differential equations, including symmetry analysis of such equations. Studying symmetries and related objects of the equation (1) was initiated by Sophus Lie himself [4]. In particular, he computed its maximal Lie invariance algebra and showed that it gives a unique (modulo the point equivalence) maximal Lie-symmetry extension in the class of linear (1+1)-dimensional second-order evolution equations. In the present, the heat equation is the first standard equation for testing packages for symbolic computation of symmetries of various kinds and related objects for differential equations. It was the equation (1) that was used as the only example for introducing the concept of nonclassical reduction in [1]. Such reductions of (1) were first completely described only in [2]. To gain an impression about the state of the art in symmetry analysis of the equation (1), see, e.g., [2, 5, 6, 7], [9, Section A] and [10, p. 531–535].

In spite of the rich and diverse history of studying the equation (1), a number of basic problems related to it even within the framework of classical group analysis still require refinement. Thus, a neat description of the complete point symmetry pseudogroup \( G \) of (1) and an accurate classification of subalgebras of the essential Lie invariance algebra \( \mathfrak{g}^{\text{ess}} \) of (1) have not yet been presented in the literature. Improper interpretations of continuous and discrete symmetries of the equation (1) led to the inconsistency between the action of the essential point symmetry group \( G^{\text{ess}} \) on \( \mathfrak{g}^{\text{ess}} \) and the inner automorphism group of \( \mathfrak{g}^{\text{ess}} \). In the present work, we successfully solve the above problems using an approach from [3].

The structure of the paper is as follows. In Section 2, we present the maximal Lie invariance algebra of the equation (1) and describe its key properties. Using the direct method, in Section 3 we compute the complete point symmetry group of (1) and analyze its structure, including its decomposition and the description of its discrete elements. Section 4 is devoted to the classification of subalgebras of \( \mathfrak{g}^{\text{ess}} \).
\section{Lie invariance algebra}

The maximal Lie invariance algebra \( \mathfrak{g} \) of the equation (1) is spanned by the vector fields

\begin{align*}
\mathcal{P}^t &= \partial_t, \quad \mathcal{D} = 2t \partial_t + x \partial_x - \frac{1}{2} u \partial_u, \quad \mathcal{K} = t^2 \partial_t + tx \partial_x - \frac{1}{4} (x^2 + 2t) u \partial_u, \\
\mathcal{G}^x &= t \partial_x - \frac{1}{2} x u \partial_u, \quad \mathcal{P}^x = \partial_x, \quad \mathcal{I} = u \partial_u, \quad \mathcal{Z}(f) = f(t, x) \partial_u,
\end{align*}

where the parameter function \( f \) depends on \((t, x)\) and runs through the solution set of the equation (1).

The vector fields \( \mathcal{Z}(f) \) constitute the infinite-dimensional abelian ideal \( \mathfrak{g}^{\text{lin}} \) of \( \mathfrak{g} \), associated with the linear superposition of solutions of (1), \( \mathfrak{g}^{\text{lin}} := \{ \mathcal{Z}(f) \} \). Thus, the algebra \( \mathfrak{g} \) can be represented as a semi-direct sum, \( \mathfrak{g} = \mathfrak{g}^{\text{ess}} \oplus \mathfrak{g}^{\text{lin}} \), where

\begin{equation}
\mathfrak{g}^{\text{ess}} = (\mathcal{P}^t, \mathcal{D}, \mathcal{K}, \mathcal{G}^x, \mathcal{P}^x, \mathcal{I})
\end{equation}

is a (six-dimensional) subalgebra of \( \mathfrak{g} \), called the \textit{essential Lie invariance algebra} of (1).

Up to antisymmetry of the Lie brackets, the nonzero commutation relations between vector fields spanning \( \mathfrak{g}^{\text{ess}} \) are exhausted by

\begin{align*}
[\mathcal{D}, \mathcal{P}^t] &= -2\mathcal{P}^t, \quad [\mathcal{D}, \mathcal{K}] = 2\mathcal{K}, \quad [\mathcal{P}^t, \mathcal{K}] = \mathcal{D}, \\
[\mathcal{P}^t, \mathcal{G}^x] &= \mathcal{P}^x, \quad [\mathcal{D}, \mathcal{G}^x] = \mathcal{G}^x, \quad [\mathcal{D}, \mathcal{P}^x] = -\mathcal{P}^x, \quad [\mathcal{K}, \mathcal{P}^x] = -\mathcal{G}^x, \\
[\mathcal{G}^x, \mathcal{P}^x] &= \frac{1}{2} \mathcal{I},
\end{align*}

The algebra \( \mathfrak{g}^{\text{ess}} \) is nonsolvable. Its Levi decomposition is given by \( \mathfrak{g}^{\text{ess}} = \mathfrak{d} \oplus \mathfrak{r} \), where the radical \( \mathfrak{r} \) of \( \mathfrak{g}^{\text{ess}} \) coincides with the nilradical of \( \mathfrak{g}^{\text{ess}} \) and is spanned by the vector fields \( \mathcal{G}^x, \mathcal{P}^x \) and \( \mathcal{I} \). The Levi factor \( \mathfrak{d} = (\mathcal{P}^t, \mathcal{D}, \mathcal{K}) \) of \( \mathfrak{g}^{\text{ess}} \) is isomorphic to \( \text{sl}(2, \mathbb{R}) \), the radical \( \mathfrak{r} \) of \( \mathfrak{g}^{\text{ess}} \) is isomorphic to the rank-one Heisenberg algebra \( h(1, \mathbb{R}) \), and the real representation of the Levi factor \( \mathfrak{d} \) on the radical \( \mathfrak{r} \) coincides, in the basis \( (\mathcal{G}^x, \mathcal{P}^x, \mathcal{I}) \), with the real representation \( \rho_1 \oplus \rho_0 \) of \( \text{sl}(2, \mathbb{R}) \). Here \( \rho_n \) is the standard real irreducible representation of \( \text{sl}(2, \mathbb{R}) \) on \( \mathbb{R}^{n+1} \). More specifically, \( \rho_n(\mathcal{P}^t)_{ij} = (n-j) \delta_{i,j+1}, \rho_n(\mathcal{D})_{ij} = (n-2j) \delta_{ij}, \rho_n(-\mathcal{K})_{ij} = j \delta_{i+1,j} \), where \( i, j \in \{0, \ldots, n\} \), \( n \in \mathbb{N} \cup \{0\} \), and \( \delta_{kl} \) is the Kronecker delta, i.e., \( \delta_{kl} = 1 \) if \( k = l \) and \( \delta_{kl} = 0 \) otherwise, \( k, l \in \mathbb{Z} \). Thus, the entire algebra \( \mathfrak{g}^{\text{ess}} \) is isomorphic to the so-called special Galilei algebra \( \text{sl}(2, \mathbb{R}) \oplus \rho_1 \oplus \rho_0 \text{h}(1, \mathbb{R}) \), which is denoted by \( L_{6,2} \) in the classification of indecomposable Lie algebras of dimensions up to eight with nontrivial Levi decompositions from [8].

Another basis of \( \mathfrak{g}^{\text{ess}} \), which stems from the Iwasawa decomposition of \( \text{SL}(2, \mathbb{R}) \) and is thus more convenient in many aspects, is \((Q^+, \mathcal{D}, \mathcal{P}^t, \mathcal{G}^x, \mathcal{P}^x, \mathcal{I})\), where \( Q^+ := \mathcal{P}^t \pm \mathcal{K} \).

\section{Complete point symmetry pseudogroup}

The equation (1) belongs to the class \( \mathcal{E} \) of linear (1+1)-dimensional second-order evolution equations of the general form

\begin{equation}
u_t = A(t, x)u_{xx} + B(t, x)u_x + C(t, x)u + D(t, x) \quad \text{with} \quad A \neq 0.\end{equation}

Here the tuple of arbitrary elements of \( \mathcal{E} \) is \( \theta := (A, B, C, D) \in \mathcal{S}_E \), where \( \mathcal{S}_E \) is the solution set of the auxiliary system consisting of the single inequality \( A \neq 0 \) with no constraints on \( B, C \) and \( D \).

To find the complete point symmetry pseudogroup \( G \) of the equation (1), we start with considering the equivalence groupoid of the class \( \mathcal{E} \), which in its turn is a natural choice for a (normalized) superclass for the equation (1). We use the papers [6, 7] as reference points for known results on admissible transformations of the class \( \mathcal{E} \).
Proposition 1 ([7]). The class $\mathcal{E}$ is normalized in the usual sense. Its usual equivalence pseudogroup $G_\mathcal{E}$ consists of the transformations of the form
\begin{align}
\dot{t} &= T(t), \quad \dot{x} = X(t, x), \quad \dot{u} = U^1(t, x)u + U^0(t, x), \\
\bar{A} &= \frac{X_x^2}{T_t} A, \quad \bar{B} = \frac{X_x}{T_t} \left( B - 2 \frac{U^1}{U^1} A \right) - \frac{X_t - X_{xx} A}{T_t}, \quad \bar{C} = -\frac{U^1}{T_t} E \frac{1}{U^1},
\end{align}
where $T, X, U^0$ and $U^1$ are arbitrary smooth functions of their arguments with $T_t X_t U^1 \neq 0$, and $E := \partial_t - A \partial_{xx} - B \partial_x - C$.

The Theorem 2. The complete point symmetry pseudogroup $G$ of the (1+1)-dimensional linear heat equation (1) consists of the point transformations of the form
\begin{align}
\dot{t} &= \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \dot{x} = \frac{x + \lambda_1 t + \lambda_0}{\gamma t + \delta}, \\
\dot{u} &= \sigma \sqrt{\gamma t + \delta} \exp \left( \frac{2(x + \lambda_1 t + \lambda_0)^2}{4(\gamma t + \delta)} - \frac{\lambda_1}{2} x - \frac{\lambda_1^2}{4} t \right) (u + h(t, x)),
\end{align}
where $\alpha, \beta, \gamma, \delta, \lambda_1, \lambda_0$ and $\sigma$ are arbitrary constants with $\alpha \delta - \beta \gamma = 1$ and $\sigma \neq 0$, and $h$ is an arbitrary solution of (1).

Proof. The linear heat equation (1) corresponds to the value $(1, 0, 0, 0) =: \theta^0$ of the arbitrary-element tuple $\theta = (A, B, C, D)$ of class $\mathcal{E}$. Its vertex group $G_{\theta^0} := G_\mathcal{E}(\theta^0, \theta^0)$ is the set of admissible transformations of the class $\mathcal{E}$ with $\theta^0$ as both their source and target, $G_{\theta^0} = \{ (\theta^0, \Phi, \theta^0) \mid \Phi \in G \}$. This argument allows us to use Proposition 1 in the course of computing the pseudogroup $G$.

We should integrate the equations (4), where both the source value $\theta$ of the arbitrary-element tuple and its target value $\bar{\theta}$ coincide with $\theta^0$, with respect to the parameter functions $T, X, U^1$ and $U^0$. After a simplification, the equations (4b) take the form
\begin{align}
X_x^2 &= T_t, \quad \frac{U^1}{U^1} = -\frac{X_t}{2 X_x}, \quad 0 = \frac{U^1}{T_t} E \frac{1}{U^1},
\end{align}
where $E := \partial_t - \partial_{xx}$. The first equation in (6) implies that $T_t > 0$, and the first two equations in (6) can be easily integrated to
\begin{align}
X &= \varepsilon \sqrt{T_t} x + X^0(t), \quad U^1 = \phi(t) \exp \left( -\frac{T_t}{8 T_t} x^2 - \frac{\varepsilon X^0_t}{2 \sqrt{T_t}} x \right),
\end{align}
where $\varepsilon$ takes values in $\{-1, 1\}$, and $\phi$ is a nonvanishing smooth function of $t$. Substituting these expressions for $X$ and $U^1$ into the third equation from (6) and subsequently splitting the obtained equation with respect to powers of $x$, we derive three equations, $T_{tt}/T_t - \frac{3}{2} (T_{tt}/T_t)^2 = 0$, $X^0_t T_t - X^0 T_{tt} = 0$ and $4T_t \phi_t + (T_{tt} + (X^0_t)^2) \phi = 0$, respectively considering as equations for $T, X^0$ and $\phi$. The first equation means that the Schwarzian derivative of $T$ is zero. Therefore, $T$ is a linear fractional function of $t$, $T = (\alpha t + \beta)/(\gamma t + \delta)$. Since the constant parameters $\alpha, \beta, \gamma$ and $\delta$ are defined up to a constant nonzero multiplier and $T_t > 0$, i.e., $\alpha \delta - \beta \gamma > 0$, we can assume that $\alpha \delta - \beta \gamma = 1$. Then these parameters are still defined up to a multiplier in $\{-1, 1\}$, and hence we can choose them in such a way that $\varepsilon = \text{sgn}(\gamma t + \delta)$. The equation for $X^0$ simplifies to the equation $(\gamma t + \delta) X^0_{tt} + 2 \gamma X^0_t = 0$, whose general solutions is $X^0 = (\lambda_1 t + \lambda_0)/(\gamma t + \delta)$. The equation for $\phi$ takes the form $4((\gamma t + \delta)^2 \phi_t - 2\gamma (\gamma t + \delta) \phi + (\lambda_1 \delta - \gamma \lambda_0)^2 \phi = 0$ and integrates, in view of $\phi \neq 0$, to $\phi = \sigma \sqrt{|\gamma t + \delta|} \exp \left( -\frac{1}{4} (\lambda_1 \delta - \gamma \lambda_0) X^0 \right)$ with $\sigma \in \mathbb{R} \setminus \{0\}$. Finally, the equation (4c) takes the form $((\partial_t - \partial_{xx})(U^0/U^1)) = 0$. Therefore, $U^0 = U^1 h$, where $h = h(t, x)$ is an arbitrary solution of (1).
To avoid complicating the structure of the pseudogroup \( G \), we should properly interpret transformations of the form (5) and their composition. Given a fixed transformation \( \Phi \) of the form (5), it is natural to assume that its domain \( \text{dom} \Phi \) coincides with the relative complement of the set \( M_{\gamma \delta} := \{ (t, x, u) \in \mathbb{R}^3 \mid \gamma t + \delta = 0 \} \) with respect to \( \text{dom} h \times \mathbb{R}_u, \text{dom} \Phi = (\text{dom} h \times \mathbb{R}_u) \setminus M_{\gamma \delta} \). Here \( \text{dom} F \) denotes the domain of a function \( F \). Recall that \((\gamma, \delta) \neq (0, 0)\), and note that the set \( M_{\gamma \delta} \) is the hyperplane defined by the equation \( t = -\delta/\gamma \) in \( \mathbb{R}^3_{t, x, u} \) if \( \gamma \neq 0 \), and \( M_{\gamma \delta} = \emptyset \) otherwise. Instead of the standard transformation composition, we use a modified composition for transformations of the form (5). More specifically, the domain of the standard composition \( \Phi_1 \circ \Phi_2 := \tilde{\Phi} \) of transformations \( \Phi_1 \) and \( \Phi_2 \) is usually defined as the preimage of the domain of \( \Phi_1 \) with respect to \( \Phi_2 \), \( \text{dom} \tilde{\Phi} = \Phi^{-1}_2(\text{dom} \Phi_1) \). For transformations \( \Phi_1 \) and \( \Phi_2 \) of the form (5), we have \( \tilde{\Phi} = (\text{dom} \tilde{h} \times \mathbb{R}_u) \setminus (M_{\gamma_2 \delta_2} \cup M_{\gamma_1 \delta_1}) \) where \( \gamma = \gamma_1 \alpha_2 + \delta_1 \gamma_2, \delta = \gamma_1 \beta_2 + \delta_1 \delta_2, \text{dom} \tilde{h} = ((\pi_4 \Phi_2)^{-1} \text{dom} h^1) \cap \text{dom} h^2, \pi \) is the natural projection onto \( \mathbb{R}^3_{t, x, u} \), and the parameters with indices 1 and 2 and tildes correspond \( \Phi_1, \Phi_2 \) and \( \tilde{\Phi} \), respectively. As the modified composition \( \Phi_1 \circ^m \Phi_2 \) of transformations \( \Phi_1 \) and \( \Phi_2 \), we take the continuous extension of \( \Phi_1 \circ \Phi_2 \) to the set
\[
\text{dom}^m \tilde{\Phi} := (\text{dom} \tilde{h} \times \mathbb{R}_u) \setminus M_{\gamma \delta},
\]
i.e., \( \text{dom} (\Phi_1 \circ^m \Phi_2) = \text{dom}^m \tilde{\Phi} \). In other words, we set \( \Phi_1 \circ^m \Phi_2 \) to be the transformation of the form (5) with the same parameters as in \( \Phi_1 \circ \Phi_2 \) and with natural domain. It is obvious that we redefine \( \Phi_1 \circ \Phi_2 \) on the set \( (\text{dom} \tilde{h} \times \mathbb{R}_u) \cap M_{\gamma_2 \delta_2} \) if \( \gamma_1 \gamma_2 \neq 0 \); otherwise \( \text{dom}^m \tilde{\Phi} = \text{dom} \tilde{\Phi} \) and the extension is trivial. A disadvantage of the above interpretation is that it is then common for elements of \( G \) to have different signs of their Jacobians on different connected components of their domains, but the benefits we receive due to it are more essential.

Now we can analyze the structure of \( G \). The point transformations of the form
\[
\mathcal{Z}(f): \quad \tilde{t} = t, \quad \tilde{x} = x \quad \tilde{u} = u + f(t, x),
\]
where the parameter function \( f = f(t, x) \) is an arbitrary solution of the equation (1), are associated with the linear superposition of solutions of this equation and, thus, can be considered as trivial. They constitute the normal pseudosubgroup \( G^{\text{lin}} \) of the pseudogroup \( G \). The pseudogroup \( G \) splits over \( G^{\text{lin}} \), \( G = G^{\text{ess}} \ltimes G^{\text{lin}} \), where the subgroup \( G^{\text{ess}} \) of \( G \) consists of the transformations of the form (5) with \( f = 0 \) and thus is a six-dimensional Lie group. We call the subgroup \( G^{\text{ess}} \) the essential point symmetry group of the equation (1). This subgroup itself splits over \( R, G^{\text{ess}} = F \ltimes R \). Here \( R \) and \( F \) are the normal subgroup and the subgroup of \( G^{\text{ess}} \) that are singled out by the constraints \( a = \delta = 1, \beta = \gamma = 0 \) and \( \lambda_1 = \lambda_0 = 0, \sigma = 1 \), respectively. They are isomorphic to the groups \( H(1, \mathbb{R}) \times \mathbb{Z}_2 \) and \( \text{SL}(2, \mathbb{R}) \), and their Lie algebras coincide with \( \mathfrak{g} \simeq h(1, \mathbb{R}) \) and \( \mathfrak{g} \simeq \text{sl}(2, \mathbb{R}) \). Here \( H(1, \mathbb{R}) \) denotes the rank-one real Heisenberg group. The normal subgroups \( R_e \) and \( R_d \) of \( R \) that are isomorphic to \( H(1, \mathbb{R}) \) and \( \mathbb{Z}_2 \) are constituted by the elements of \( R \) with parameter values satisfying the constraints \( \sigma > 0 \) and \( \lambda_0 = \lambda_1 = 0, \sigma \in \{-1, 1\} \), respectively. The isomorphisms of \( F \) to \( \text{SL}(2, \mathbb{R}) \) and of \( R_e \) to \( H(1, \mathbb{R}) \) are established by the correspondences
\[
\varrho_1 = (\alpha, \beta, \gamma, \delta)_{\alpha \delta - \beta \gamma = 1} \mapsto \begin{pmatrix} \alpha & \beta \\ -\gamma & \delta \end{pmatrix}, \quad (\lambda_1, \lambda_0, \sigma), \sigma > 0 \mapsto \begin{pmatrix} 1 & -\frac{1}{2} \lambda_1 \ln \sigma \\ 0 & 1 \end{pmatrix}.
\]
The isomorphism \( \varrho_1 \) is in fact the standard two-dimensional representation of \( \text{SL}(2, \mathbb{R}) \). Thus, \( F \) and \( R_e \) are connected subgroups of \( G^{\text{ess}} \), but \( R_d \) is not. The natural conjugacy action of the group \( F \) on the normal subgroup \( R \) is given by \( (\lambda_0, \lambda_1, \bar{\sigma})^T = A (\lambda_0, \lambda_1, \sigma)^T \) in the parameterization (5) of \( G \), where \( A = \varrho_1 (\alpha, \beta, \gamma, \delta) \oplus (1) \). Denoting the trivial one-dimensional representation of \( \text{SL}(2, \mathbb{R}) \) by \( \varrho_0, \varrho_0 (\alpha, \beta, \gamma, \delta) = (1) \), we can sum up that \( G^{\text{ess}} \simeq (\text{SL}(2, \mathbb{R}) \ltimes \varrho_1 \times \varrho_0 H(1, \mathbb{R})) \times \mathbb{Z}_2 \).
Transformations from the one-parameter subgroups of $G^{\text{ess}}$ that are generated by the basis elements of $g^{\text{ess}}$ given in (2) are of the following form:

$$P(t): \begin{cases} \hat{t} = t + \epsilon, & \hat{x} = x, & \hat{u} = u, \\ \mathbb{D}(\epsilon): \hat{t} = e^{2\epsilon} t, & \hat{x} = e^\epsilon x, & \hat{u} = e^{-\frac{1}{2}\epsilon} u, \\ \mathbb{K}(\epsilon): \hat{t} = \frac{t}{1 - \epsilon^2}, & \hat{x} = \frac{x}{1 - \epsilon^2}, & \hat{u} = \sqrt{|1 - \epsilon^2|} e^{\frac{\epsilon^2 t}{1 - \epsilon^2} u}, \\ \mathbb{G}(\epsilon): \hat{t} = t, & \hat{x} = x + \epsilon, & \hat{u} = e^{-\frac{1}{2}\epsilon} u, \\ \mathbb{J}(\epsilon): \hat{t} = t, & \hat{x} = x, & \hat{u} = e^\epsilon u, \end{cases}$$

where $\epsilon$ is an arbitrary constant. At the same time, using this basis of $g^{\text{ess}}$ in the course of studying the structure of the group $G^{\text{ess}}$ hides some of its important properties and complicates its study.

Although the pushforward of the pseudogroup $G$ by the natural projection of $\mathbb{R}^3_{t,x,u}$ onto $\mathbb{R}_t$ coincides with the group of linear fractional transformations of $t$ and is thus isomorphic to the group $\text{PSL}(2, \mathbb{R})$, the subgroup $F$ of $G$ is isomorphic to the group $\text{SL}(2, \mathbb{R})$, and its Iwasawa decomposition is given by the one-parameter subgroups of $G$ respectively generated by the vector fields $\mathbb{P}^+: = P^t + \mathbb{K}$, $\mathbb{D}$ and $\mathbb{P}^t$. The first subgroup, which is associated with $\mathbb{Q}^+$, consists of the point transformations

$$\Omega^+(\epsilon): \begin{cases} \hat{t} = \frac{t \cos \epsilon - \sin \epsilon}{t \sin \epsilon + \cos \epsilon}, & \hat{x} = \frac{x}{t \sin \epsilon + \cos \epsilon}, & \hat{u} = \sqrt{|t \sin \epsilon + \cos \epsilon|} e^{\frac{\epsilon^2 \sin \epsilon}{t \sin \epsilon + \cos \epsilon}} u, \end{cases}$$

where $\epsilon$ is an arbitrary constant parameter, which is defined by the corresponding transformation up to a summand $2\pi k$, $k \in \mathbb{Z}$. The Jacobian of $\Omega^+(\epsilon)$ is positive and negative for all values of $(t, x, u)$ if $\epsilon = 0$ and $\epsilon = \pi$, respectively. For $\epsilon \in (0, \pi) \cup (\pi, 2\pi)$, the transformation $\Omega^+(\epsilon)$ is not defined if $t = -\cot \epsilon$, and for the other values of $(t, x, u)$ the sign of its Jacobian coincides with $\text{sgn}(t \sin \epsilon + \cos \epsilon)$.

The equation (1) is invariant with respect to the involution $\mathcal{J}$ alternating the sign of $x$ and the transformation $\mathcal{K}'$ inverting $t$,

$$\mathcal{J}: (t, x, u) \mapsto (t, -x, u), \quad \mathcal{K}': \hat{t} = \frac{1}{t}, \quad \hat{x} = \frac{x}{t}, \quad \hat{u} = \sqrt{|t|} e^{\frac{\epsilon^2 t}{1 - \epsilon^2}} u.$$  

Note that $(\mathcal{K}')^2 = \mathbb{I}$. In the context of the one-parameter subgroups of $G^{\text{ess}}$ (resp. of $G$) that are generated by the basis elements of $g^{\text{ess}}$ listed in (2), both the transformations $\mathcal{J}$ and $\mathcal{K}'$ look like discrete point-symmetry transformations of (1), but in fact this is not the case under the above interpretation of the group multiplication in $G$. Even though the Jacobian of the involution $\mathcal{J}$ is equal to $-1$ for all values of $(t, x, u)$, this involution belongs to the one-parameter subgroup $\{\Omega^+(\epsilon)\}$ of $G$, $\mathbb{J} = \Omega^+(\pi)$, and thus it belongs to the identity component of the pseudogroup $G$. A similar claim is true for the transformation $\mathcal{K}' = \Omega^+(\frac{1}{2}\pi)$, the sign of whose Jacobian coincides with $\text{sgn}t$.

Corollary 3. A complete list of discrete point symmetry transformations of the linear $(1+1)$-dimensional heat equation (1) that are independent up to combining with each other and with continuous point symmetry transformations of this equation is exhausted by the single involution $\mathcal{J}'$ alternating the sign of $u$, $\mathcal{J}': (t, x, u) \mapsto (t, x, -u)$. Thus, the factor group of the complete point symmetry pseudogroup $G$ with respect to its identity component is isomorphic to $\mathbb{Z}_2$.

Proof. It is obvious that the entire pseudosubgroup $G^{\text{lin}}$ is contained in the connected component of the identity transformation in $G$. The same claim holds for the subgroups $F$ and $R_c$ in view of their isomorphisms to the groups $\text{SL}(2, \mathbb{R})$ and $H(1, \mathbb{R})$, respectively. Therefore, without loss of generality a complete list of independent discrete point symmetry transformations of (1) can
be assumed to consist of elements of the subgroup \( R_d \). Thus, the only discrete point symmetry transformation of (1) that is independent in the above sense is the transformation \( \mathcal{T}' \), and the identity component \( G_{id} \) of \( G \) is constituted by transformations of the form (5) with \( \sigma > 0 \).

In the notation of Theorem 2, the most general form of the transformed counterpart of a given solution \( u = f(t, x) \) under action of \( G \) (resp. \( G_{id} \)) is

\[
\tilde{u} = \frac{\sigma}{\sqrt{|\gamma t - \alpha|}} \exp\left( \frac{\gamma x^2}{4(\alpha - \gamma t)} - \frac{\lambda_1 x}{2(\alpha - \gamma t)} + \frac{\lambda_1^2 \delta t - \beta}{4 \alpha - \gamma t} + \frac{\lambda_0 \lambda_1}{2} \right)
\times f\left( \frac{\delta t - \beta}{\alpha - \gamma t} \frac{x}{\alpha - \gamma t} - \frac{\lambda_1 \delta t - \beta}{\alpha - \gamma t} - \lambda_0 \right) + h(t, x),
\]

where in addition \( \sigma > 0 \) for \( G_{id} \), cf. [5, p. 120].

4 Subalgebras of essential Lie invariance algebra

In spite of an unusualness of the above claims and Corollary 3, they are well consistent with the structure of the abstract Lie group that is isomorphic to \( G^{\text{ess}} \) and with the inner automorphism group \( \text{Inn}(g^{\text{ess}}) \) of \( g^{\text{ess}} \). More specifically, the algebra \( g^{\text{ess}} \) is the Lie algebra of the group \( G^{\text{ess}} \), and the adjoint action of \( G^{\text{ess}} \) on \( g^{\text{ess}} \) coincides with \( \text{Inn}(g^{\text{ess}}) \). In particular, under the suggested interpretation we have \( \text{Ad}(\exp(\epsilon Q^+)) = Q^+(\epsilon)_* \),

\[
Q^+(\epsilon)_*Q^- = \cos(2\epsilon)Q^- - \sin(2\epsilon)D, \quad Q^+(\epsilon)_*P^x = \cos(\epsilon)P^x - \sin(\epsilon)G^x, \\
Q^+(\epsilon)_*D = \sin(2\epsilon)Q^- + \cos(2\epsilon)D, \quad Q^+(\epsilon)_*G^x = \sin(\epsilon)P^x + \cos(\epsilon)G^x,
\]

and the inner automorphisms associated with \( \beta = Q^+(\pi) \) and \( \mathcal{K}' = Q^+(\frac{1}{2}\pi) \) allow one to map \( P^t - G^x \) and \( D - \mu \mathcal{I} \) to \( P^t + G^x \) and \( D + \mu \mathcal{I} \), respectively. Retaining these facts, we refine the classification of subalgebras of \( g^{\text{ess}} \) or, equivalently, the special Galilei algebra \( \mathfrak{sl}(2, \mathbb{R}) \mathfrak{d} \mathfrak{p}_1 \oplus \mathfrak{p}_0 \) \( h(1, \mathbb{R}) \), cf. [5, Example 3.13] and [10, p. 531–535]. The detailed proof of the classification will be presented elsewhere.

Lemma 4. A complete list of inequivalent proper subalgebras of the algebra \( g^{\text{ess}} \) is exhausted by the following subalgebras, where \( \delta \in \{-1, 0, 1\}, \mu \in \mathbb{R}_{\geq 0} \) and \( \nu \in \mathbb{R} \):

1D: \( \langle P^t + G^x \rangle, \langle P^t + \delta I \rangle, \langle D + \mu I \rangle, \langle P^t + K + \nu I \rangle, \langle P^x \rangle, \langle I \rangle \)

2D: \( \langle P^t, D + \nu I \rangle, \langle P^t + G^x, I \rangle, \langle P^t + \delta I, P^x \rangle, \langle P^t, I \rangle, \langle D + \nu I, P^x \rangle, \langle D, I \rangle, \langle P^t + K, I \rangle, \langle P^x, I \rangle \)

3D: \( \langle P^t, D, K \rangle, \langle P^t, D + \nu I, P^x \rangle, \langle P^t, D, I \rangle, \langle P^t + G^x, P^x, I \rangle, \langle P^t, P^x, I \rangle, \langle D, P^x, I \rangle, \langle G^x, P^x, I \rangle \)

4D: \( \langle P^t, D, K, I \rangle, \langle P^t, D, P^x, I \rangle, \langle P^t, G^x, P^x, I \rangle, \langle D, G^x, P^x, I \rangle, \langle P^t + K, G^x, P^x, I \rangle \)

5D: \( \langle P^t, D, G^x, P^x, I \rangle \)

Acknowledgements

The authors are grateful to Alex Bihlo, Vyacheslav Boyko and Dmytro Popovych for valuable discussions. This research was undertaken thanks to funding from the Canada Research Chairs program, the InnovateNL LeverageR&D program and the NSERC Discovery Grant program. This research was also supported in part by the Ministry of Education, Youth and Sports of the Czech Republic (MŠMT ČR) under RVO funding for IC47813059. The authors express deepest thanks to the Armed Forces of Ukraine and the civil Ukrainian people for their bravery and courage in defense of peace and freedom in Europe and in the entire world from Russiam.
References

[1] Bluman G.W. and Cole J.D., The general similarity solution of the heat equation, *J. Math. Mech.* 18 (1969), 1025–1042.

[2] Fushchych W.I., Shtelen W.M., Serov M.I. and Popovych R.O., Q-conditional symmetry of the linear heat equation, *Dopov. Nats. Akad. Nauk Ukr.* (1992), no. 12, 28–33.

[3] Koval S.D., Bihlo A. and Popovych R.O., Extended symmetry analysis of remarkable (1+2)-dimensional Fokker-Planck equation, 2022, arXiv:2205.13526.

[4] Lie S., Über die Integration durch bestimmte Integrale von einer Klasse linear partieller Differentialgleichungen, *Arch. for Math.* 6 (1881), 328–368; translation by N.H. Ibragimov: Lie S. On integration of a class of linear partial differential equations by means of definite integrals, in *CRC Handbook of Lie group analysis of differential equations*, vol. 2, CRC Press, Boca Raton, FL, 1995, pp. 473–508.

[5] Olver P.J., *Application of Lie groups to differential equations*, Springer, New York, 2000.

[6] Opanasenko S. and Popovych R.O., Mapping method of group classification, *J. Math. Anal. Appl.* 513 (2022), 126209, arXiv:2109.11490.

[7] Popovych R.O., Kunzinger M. and Ivanova N.M. Conservation laws and potential symmetries of linear parabolic equations, *Acta Appl. Math.* 2 (2008), 113–185, arXiv:math-ph/0706.0443

[8] Turkowski P., Low-dimensional real Lie algebras, *J. Math. Phys.* 29 (1988), 2139–2144.

[9] Vaneeva O.O., Popovych R.O. and Sophocleous C., Extended symmetry analysis of two-dimensional degenerate Burgers equation. *J. Geom. Phys.* 169 (2021), 104336, arXiv:1908.01877.

[10] Winternitz P., Group theory and exact solutions of partially integrable differential systems, in Conte R. and Bocca N. (eds.), *Partially integrable evolution equations in physics*, Kluwer Acad. Publ., Dordrecht, 1990, pp. 515–567.