Polar Prékopa–Leindler Inequalities

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Abstract

We prove a new family of Prékopa–Leindler-type inequalities, corresponding to the linear structure induced by the polarity transform \( \mathcal{A} \).

1 Introduction and results

The well known Prékopa–Leindler inequality \[ \text{[12, 14]} \] states that, if three measurable functions \( f, g, h : \mathbb{R}^n \to \mathbb{R}^+ \) satisfy for some \( \lambda \in (0, 1) \) that for any \( x, y \)

\[
    h((1 - \lambda)x + \lambda y) \geq f(x)^{1-\lambda}g(y)^{\lambda},
\]

then

\[
    \int h \geq \left( \int f \right)^{1-\lambda} \left( \int g \right)^{\lambda}.
\]

In this note we prove a theorem of similar spirit, which we call polar Prékopa–Leindler, which is that if measurable functions \( f, g, h : \mathbb{R}^n \to \mathbb{R}^+ \) satisfy for a given \( \lambda \in (0, 1) \) that for every \( t \in (0, 1) \) and \( x, y \in \mathbb{R}^n \) one has

\[
    h((1 - t)x + ty) \geq \min \left\{ f(x)^{\frac{1-t}{1-\lambda}}, g(y)^{\frac{t}{\lambda}} \right\},
\]

then

\[
    \int_{\mathbb{R}^n} h \geq \left( 1 - \lambda \right) \left( \int_{\mathbb{R}^n} f \right)^{-1} + \lambda \left( \int_{\mathbb{R}^n} g \right)^{-1}.
\]

We also prove a more general version of this theorem when the integration is with respect to any log-concave measure, not necessarily the standard Lebesgue measure.

To put our theorem in context, let us provide some background. The smallest function \( h \) which satisfies the condition (1) in the Prékopa-Leindler Theorem, for a given \( \lambda \), is the function

\[
    h_{\lambda}(z) = \sup \left\{ f(x)^{1-\lambda}g(y)^{\lambda} : (1 - \lambda)x + \lambda y = z \right\}.
\]
This expression is sometimes referred to as the \textit{sup-convolution average} of the two functions $f$ and $g$, with weights $(1-\lambda)$ and $\lambda$. In the case where $f = e^{-\varphi}$, $g = e^{-\psi}$ for convex functions $\varphi, \psi : \mathbb{R}^n \to \mathbb{R}$, the function $h_\lambda$ has a nice representation in terms of the Legendre transform namely $h_\lambda = e^{-\xi_\lambda}$ with $\xi_\lambda = \mathcal{L}((1-\lambda)\mathcal{L}\varphi + \lambda\mathcal{L}\psi)$, where the Legendre transform of a function $\phi$ is given by

$$ (\mathcal{L}\phi)(y) = \sup_x \{ \langle x, y \rangle - \phi(x) \}. $$

Even without the convexity assumption on $\varphi$ and $\psi$, the function $\xi_\lambda$ can be geometrically described in that its epi-graph is the Minkowski average of the epi-graphs of $\varphi$ and $\psi$. More precisely, denoting the epi-graph of a function $\phi$ by

$$ \text{epi}(\phi) = \{(x, z) \in \mathbb{R}^n \times \mathbb{R} : \phi(x) < z\}, \quad (3) $$

one may easily check that

$$ \text{epi}(\xi_\lambda) = (1-\lambda)\text{epi}(\varphi) + \lambda\text{epi}(\psi). $$

We denote this type of $\lambda$-average by $\xi_\lambda = \varphi \square \lambda \psi$. The Prékopa–Leindler inequality then reads

$$ \int e^{-\varphi \square \lambda \psi} \geq \left( \int e^{-\varphi} \right)^{1-\lambda} \left( \int e^{-\psi} \right)\lambda. \quad (4) $$

Here the integral on the left hand side should be interpreted properly, as the function $h_\lambda$ might not be measurable, which is the reason behind considering three functions (rather than just $\varphi, \psi$, and $\varphi \square \lambda \psi$) in the original inequality.

In the class of convex functions, we may view the Prékopa–Leindler inequality as a log-concavity result for the “volume” functional $\phi \mapsto \int e^{-\phi}$ with respect to the linear structure which is the pullback of the standard, pointwise addition of functions, under the Legendre transform. In the general (not necessarily convex) case we are considering the linear structure induced by Minkowski addition of epi-graphs.

In this paper we consider non-negative functions. We were motivated by some recent results regarding the class $\text{Cvx}_0(\mathbb{R}^n)$ of \textit{geometric convex functions}, namely non-negative lower semi continuous convex functions vanishing at the origin. This class is invariant under the Legendre transform. In [2] the authors show that, considering the partial order of pointwise inequality in this class, there exist only two essentially different order reversing bijections on $\text{Cvx}_0(\mathbb{R}^n)$. The first is the Legendre transform, which is actually a bijection on the larger class $\text{Cvx}(\mathbb{R}^n)$ of all lower semi continuous convex functions. The second is the so called \textit{polarity transform} $\mathcal{A}$, defined by

$$ (\mathcal{A}\phi)(y) = \begin{cases} \sup_{\{x \in \mathbb{R}^n : \phi(x) > 0\}} \frac{(x,y)-1}{\phi(x)} \quad &\text{if } 0 \neq y \in \{ \phi^{-1}(0) \}^\circ \\
0 \quad &\text{if } y = 0 \\
+\infty \quad &\text{if } y \notin \{ \phi^{-1}(0) \}^\circ \end{cases} $$

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with the convention $\sup \emptyset = 0$. The transform $\mathcal{A}$ appears in [15] and a similar transform appears in [13], but it seems to have been left virtually untouched until recently, when it was discovered that the polarity transform and the Legendre transform are the only order reversing involutions on $\text{Cvx}_0(\mathbb{R}^n)$. For properties of $\mathcal{A}$ and different geometric interpretations, see [2]. For differential analysis concerning $\mathcal{A}$, and applications to solving families of differential equations, see [3]. We mention in this context that our main theorems can be interpreted as volume estimates for solutions of certain partial differential equations (those linearized by $\mathcal{A}$, as portrayed in [3]) in terms of the boundary or initial conditions.

It is natural to ask whether a concavity result, similar to the Prékopa–Leindler inequality, holds when one induces a linear structure on $\text{Cvx}_0(\mathbb{R}^n)$ via $\mathcal{A}$. Given two geometric convex functions $\varphi, \psi \in \text{Cvx}_0(\mathbb{R}^n)$, their geometric $\lambda$-inf-convolution is defined by

$$\varphi \square_\lambda \psi := \mathcal{A}((1 - \lambda)\mathcal{A}\varphi + \lambda \mathcal{A}\psi).$$

This yields yet another geometric convex function, and one of the main results in this note is that the “volume” functional $\phi \mapsto \int e^{-\phi}$ has some concavity property with respect to the geometric $\lambda$-inf-convolution. Our results imply that

$$\int_{\mathbb{R}^n} e^{-\varphi \square_\lambda \psi} \geq \left(1 - \lambda \right) \left( \int_{\mathbb{R}^n} e^{-\varphi} \right)^{-1} + \lambda \left( \int_{\mathbb{R}^n} e^{-\psi} \right)^{-1},$$

whenever $\lambda \in (0, 1)$, $\varphi, \psi \in \text{Cvx}_0(\mathbb{R}^n)$.

In fact we will prove (5) under weaker conditions, namely for measurable non-negative functions which are not necessarily in $\text{Cvx}_0(\mathbb{R}^n)$, but for this we need to properly extend the operation $\square_\lambda$ to this larger class of functions. In Section 2 we shall show the validity of the following formula for the geometric $\lambda$-inf-convolution of two geometric convex functions

$$(\varphi \square_\lambda \psi)(z) = \inf_{0 < t < 1} \inf_{z = (1-t)x + ty} \max \left\{ \frac{1-t}{1-\lambda} \varphi(x), \frac{t}{\lambda} \psi(y) \right\}.$$  

A geometric interpretation of (6) will also be given in Section 2. We can then restate (5) as follows. Fix $\lambda \in (0, 1)$. If three geometric log-concave functions (meaning $e^{-\phi}$ for $\phi \in \text{Cvx}_0(\mathbb{R}^n)$) satisfy

$$h((1 - t)x + ty) \geq \min \left\{ f(x)^{\frac{1-t}{1-z}}, g(y)^{\frac{t}{\lambda}} \right\},$$

whenever $t \in (0, 1)$ and $x, y \in \mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} h \geq \left(1 - \lambda \right) \left( \int_{\mathbb{R}^n} f \right)^{-1} + \lambda \left( \int_{\mathbb{R}^n} g \right)^{-1}.$$  

In this formulation, however, the statement does not actually require the functions to be geometric log-concave, and we prove:
Theorem 1.1. Let \( f, g, h : \mathbb{R}^n \to \mathbb{R}^+ \) be measurable functions, and \( \lambda \in (0, 1) \). If for any \( t \in (0, 1) \) and \( x, y \in \mathbb{R}^n \), one has
\[
h((1 - t)x + ty) \geq \min \left\{ f(x)^{\frac{1}{1-t}}, g(y)^{\frac{1}{t}} \right\},
\]
then
\[
\int_{\mathbb{R}^n} h \geq \left( (1 - \lambda) \left( \int_{\mathbb{R}^n} f \right)^{-1} + \lambda \left( \int_{\mathbb{R}^n} g \right)^{-1} \right)^{-1}.
\]

In the second part of the paper we extend Theorem 1.1 to integration with respect to a general log-concave measure. Note that while in the classical Prékopa-Leindler theory multiplying \( f, g \) and \( h \) by a log-concave density does not affect the validity of the pointwise inequality the functions satisfy, here this is no longer the case, and one must provide an independent proof.

To do this we utilize the fact that the geometric \( \lambda \)-inf-convolution operation is intimately related to an operation defined and used in the classical Busemann Theorem, regarding the convexity of the intersection body of a convex body. We discuss this relation in detail in Section 4, and use methods from known Busemann-type theorems to prove the following theorem.

Theorem 1.2. Let \( \mu \) be a log-concave measure on \( \mathbb{R}^n \) and let \( f, g, h : \mathbb{R}^n \to \mathbb{R}^+ \) be measurable functions which have finite integral with respect to \( \mu \). Let \( \lambda \in (0, 1) \) and assume that
\[
h((1 - t)x + ty) \geq \min \left\{ f(x)^{\frac{1}{1-t}}, g(y)^{\frac{1}{t}} \right\},
\]
whenever \( t \in (0, 1) \) and \( x, y \in \mathbb{R}^n \). Then
\[
\int_{\mathbb{R}^n} h \, d\mu \geq \left( (1 - \lambda) \left( \int_{\mathbb{R}^n} f \, d\mu \right)^{-1} + \lambda \left( \int_{\mathbb{R}^n} g \, d\mu \right)^{-1} \right)^{-1}.
\]

Our methods turn out to be quite general, and may be used to prove concavity results for various other “volume” functionals on functions. We illustrate this by proving yet another result of this type. Denoting for any \( p > 0 \) and measurable \( \phi : \mathbb{R}^n \to \mathbb{R}^+ \), \( \|\phi\|_p = (\int \phi^p)^{\frac{1}{p}} \) (clearly this is a norm only when \( p \geq 1 \)), we have

Theorem 1.3. Let \( p > 0 \) and let \( f, g, h : \mathbb{R}^n \to \mathbb{R}^+ \) be measurable functions such that \( \|f\|_p, \|g\|_p, \) and \( \|h\|_p \) are finite. Assume that
\[
h((1 - t)x + ty) \geq \min \left\{ \frac{f(x)}{1 - t}, \frac{g(y)}{t} \right\},
\]
whenever \( t \in (0, 1) \) and \( x, y \in \mathbb{R}^n \). Then
\[
\|h\|_p \geq \|f\|_p + \|g\|_p.
\]
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2 Extending the geometric inf-convolution

In this section we extend the operation of the geometric inf-convolution (which we abbreviate by $\text{ginf-convolution}$) to act between any two non-negative functions $\varphi$ and $\psi$. We then present a new formula for the geometric $\lambda$-inf-convolution.

In [2], it was shown that on $\text{Cvx}_0(\mathbb{R}^n)$ one has $A \circ L = L \circ A = J$, where $J$ is the order preserving bijection called the gauge transform and is given by

$$J(\varphi)(x) = \inf \left\{ r > 0 : \varphi \left( \frac{x}{r} \right) \leq \frac{1}{r} \right\}. \quad (8)$$

Thus, given $\varphi, \psi \in \text{Cvx}_0(\mathbb{R}^n)$ one has

$$J((\varphi \boxtimes_\lambda \psi))(x) = J((1 - \lambda)A\varphi + \lambda A\psi) = L((1 - \lambda)A\varphi + \lambda A\psi)$$

$$= L((1 - \lambda)LJ\varphi + \lambda LJ\psi) = (J\varphi)\boxtimes_\lambda (J\psi).$$

Considering the epi-graphs we get the following relation

$$\text{epi} \left( J(\varphi \boxtimes_\lambda \psi) \right) = (1 - \lambda)\text{epi} \left( J\varphi \right) + \lambda \text{epi} \left( J\psi \right). \quad (9)$$

This is, the ginf-convolution operation corresponds to the pullback of Minkowski addition of epi-graphs under the $J$ transform. In the same paper [2] it was shown that $J$ is induced by a point map on $\mathbb{R}^n \times \mathbb{R}^+$, given by $F(x,z) = \left( \frac{x}{z}, \frac{1}{z} \right)$. More precisely, for $\varphi \in \text{Cvx}_0(\mathbb{R}^n)$ we have that

$$F(\text{epi} \left( \varphi \right)) = \text{epi} \left( J\varphi \right). \quad (10)$$

Thus we could equivalently define the ginf-convolution, or the $\lambda$-ginf-convolution by

$$\text{epi} \left( \varphi \boxtimes \psi \right) = F(\text{epi} \left( \varphi \right) + \text{epi} \left( \psi \right)), \quad \text{epi} \left( \varphi \boxtimes_\lambda \psi \right) = F((1 - \lambda)\text{epi} \left( \varphi \right) + \lambda F(\text{epi} \left( \psi \right))).$$

This definition may be extended to the set of non-negative functions.

**Proposition 2.1.** Let $\lambda \in (0, 1)$ and let $\varphi, \psi : \mathbb{R}^n \to [0, \infty]$. The sets

$$F(\text{epi} \left( \varphi \right) + \text{epi} \left( \psi \right)), \quad F((1 - \lambda)\text{epi} \left( \varphi \right) + \lambda F(\text{epi} \left( \psi \right)),$$

are epi-graphs of functions from $\mathbb{R}^n$ to $[0, \infty]$. We denote these functions by $\varphi \boxtimes \psi$ and $\varphi \boxtimes_\lambda \psi$ respectively.
Proof. Note that $F$ is an involution on $\mathbb{R}^n \times \mathbb{R}^+$ that maps vertical fibers to intervals with one endpoint at the origin. More precisely, for $x \in \mathbb{R}^n$ and $c > 0$ we have $F(\{x\} \times [c, \infty)) = \frac{1}{c}(0, (x, 1)]$. Thus $F$ maps epi-graphs to sets in $\mathbb{R}^n \times \mathbb{R}^+$ that are star shaped about the origin, and such star shaped sets are mapped to epi-graphs. Since the class of star shaped sets is closed under Minkowski addition, the statement follows.

Next, we turn to proving formula (6).

**Proposition 2.2.** Let $\lambda \in (0, 1)$ and let $\varphi, \psi : \mathbb{R}^n \to [0, \infty]$. Then we have

\[
(\varphi \boxminus \lambda \psi)(z) = \inf_{0 < t < 1} \inf_{z = (1-t)x + ty} \max \left\{ (1-t)\varphi(x), t\psi(y) \right\},
\]

and

\[
(\varphi \boxplus \psi)(z) = \inf_{0 < t < 1} \inf_{z = (1-t)x + ty} \max \left\{ \frac{1-t}{1-\lambda}\varphi(x), \frac{t}{\lambda}\psi(y) \right\}.
\]

**Remark 2.3.** We mention that the usefulness of such a formula goes beyond its usage in this paper. Indeed, for the case of $\varphi, \psi \in \text{Cvx}_0(\mathbb{R}^n)$ we get by means of this formula an expression for the polar of a sum:

\[
A(\varphi + \psi)(y) = (A\varphi \boxminus A\psi)(y) = \inf_{0 < t < 1} \inf_{y = (1-t)y_1 + ty_2} \max \left\{ (1-t)A\varphi(y_1), tA\psi(y_2) \right\}.
\]

Here one clearly sees that the domain of $A(\varphi + \psi)(y)$ is the convex hull of the domains of $A\varphi$ and $A\psi$. In fact, one may figure out where the infimum is attained (say, in the differentiable case) as was shown in [3, Lemma 8.2].

**Proof of Proposition 2.2.** Note that since $\lambda F(\text{epi} (\varphi)) = F(\text{epi} (\varphi/\lambda))$, it follows

\[
\varphi \boxminus \lambda \psi = \frac{\varphi}{1 - \lambda} \boxminus \frac{\psi}{\lambda},
\]

so it suffices to prove (11). We have

\[
\text{epi} (\varphi \boxminus \psi) = F( F(\text{epi} (\varphi)) + F(\text{epi} (\psi))) = F \left( \left\{ \left( \frac{x}{s} + \frac{y}{t}, \frac{1}{s} + \frac{1}{t} \right) : \varphi(x) < s, \psi(y) < t \right\} \right) = \left\{ \left( \frac{x}{s} + \frac{y}{t}, \frac{1}{s} + \frac{1}{t} \right) : \varphi(x) < s, \psi(y) < t \right\}.
\]

Therefore,

\[
(\varphi \boxminus \psi)(z) = \inf \left\{ \frac{1}{\frac{1}{s} + \frac{1}{t}} : z = \frac{x}{s} + \frac{y}{t}, \varphi(x) < s, \psi(y) < t \right\}.
\]
Rewriting we get
\[(\varphi \square \psi)(z) = \inf \left\{ \frac{st}{s+t} : z = (1-a)x + ay, a = \frac{s}{s+t}, \varphi(x) < s, \psi(y) < t \right\}.\]

We take the infimum in two steps, first over all choices of \(x, s, y, t\) which satisfy the conditions for a fixed \(a\), and then over all \(a \in (0, 1)\). We claim that for any fixed \(a \in (0, 1)\) we have
\[
\inf \left\{ \frac{st}{s+t} : z = (1-a)x + ay, a = \frac{s}{s+t}, \varphi(x) < s, \psi(y) < t \right\} = \inf \{\max\{(1-a)\varphi(x), a\psi(y)\} : z = (1-a)x + ay\}. \tag{13}
\]

Indeed we have \(\frac{st}{s+t} = (1-a)s > (1-a)\varphi(x)\) and \(\frac{st}{s+t} = at > a\psi(y)\) for any \(x, s, y, t\) participating in the first infimum, thus the left hand side is not smaller than the right hand side.

For the other direction, assume that for a given \(x, y\) which satisfy \(z = (1-a)x + ay\) we have \((1-a)\varphi(x) \geq a\psi(y)\). Let us take \(s = \varphi(x) + \varepsilon\) and choose \(t\) so that \(a = \frac{s}{s+t}\), that is, \(at = (1-a)s = (1-a)(\varphi(x) + \varepsilon) \geq a\psi(y) + (1-a)\varepsilon > a\psi(y)\). Since \(t > \psi(y)\), we get that \(t\) participates in the infimum of the left hand side, and we have
\[
\frac{st}{s+t} = \frac{\varphi(x)t + \varepsilon t}{s+t} < \frac{\varphi(x)t}{s+t} + \varepsilon = (1-a)\varphi(x) + \varepsilon = \max\{(1-a)\varphi(x), a\psi(y)\} + \varepsilon.
\]

Since \(\varepsilon\) was arbitrary, we get that fixing \(z, a, x, y\)
\[
\inf \left\{ \frac{st}{s+t} : z = (1-a)x + ay, a = \frac{s}{s+t}, \varphi(x) < s, \psi(y) < t \right\} \leq (1-a)\varphi(x),
\]
in the case where \(z = (1-a)x + ay\), and \(\max\{(1-a)\varphi(x), a\psi(y)\} = (1-a)\varphi(x)\). The exact same reasoning works when the maximum of the two is \(a\psi(y)\). Therefore, the two expressions in (13) are the same for any fixed \(a \in (0, 1)\), and we get that
\[
(\varphi \square \psi)(z) = \inf_{0 < a < 1} \inf \{\max\{(1-a)\varphi(x), a\psi(y)\} : z = (1-a)x + ay\},
\]
which completes the proof.

\[\square\]

**Remark 2.4.** For \(m\) functions one easily checks the validity of the formulas
\[
(\varphi_1 \square \cdots \square \varphi_m)(z) = \inf_{\sum_{i=1}^{m} t_i=1} \inf_{z=\sum_{i=1}^{m} t_i x_i} \max \left\{ t_i \varphi_i(x_i) \right\},
\]
\[
\mathcal{A}(\varphi_1 + \cdots + \varphi_m)(z) = \inf_{\sum_{i=1}^{m} t_i=1} \inf_{z=\sum_{i=1}^{m} t_i x_i} \max \left\{ t_i \mathcal{A} \varphi_i(x_i) \right\}.
\]

7
3 Two Prékopa-Leindler type theorems

In this section we present the proofs of Theorem 1.1 and Theorem 1.3. To present these proofs, and also a very simple proof for the Prékopa-Leindler classical inequality (2), we will use C. Borell’s theorem on concavity of measures. It will be useful to introduce the following notation for the \( p \)-average of non-negative numbers: Given \( x, y > 0, \lambda \in [0, 1], \) and \( p \in \mathbb{R}, \) denote

\[
M_{\lambda}^p(x, y) := ((1 - \lambda)x^p + \lambda y^p)^{\frac{1}{p}}.
\]

The cases \( p = 0, \pm \infty \) are interpreted by limits, so that

\[
M_{0}^\lambda(x, y) = \frac{x}{1 - \lambda} + \frac{\lambda y}{1}, \quad M_{\infty}^\lambda(x, y) = \max\{x, y\}, \quad M_{-\infty}^\lambda(x, y) = \min\{x, y\}.
\]

Recall that a measure \( \mu \) on \( \mathbb{R}^n \) is called \( \kappa \)-concave if for all non-empty Borel sets \( A, B, C \) such that \((1 - \lambda)A + \lambda B \subseteq C,\) one has

\[
\mu(C) \geq M_{\lambda}^\kappa(\mu(A), \mu(B)). \tag{14}
\]

A function \( f \) on \( \mathbb{R}^n \) is called \( \kappa \)-concave if for all \( x, y \in \mathbb{R}^n:\)

\[
f((1 - \lambda)x + \lambda y) \geq M_{\lambda}^\kappa(f(x), f(y)).
\]

In particular, if \( \kappa > 0 \) then \( \kappa \)-concavity means that \( f^\kappa \) is concave whereas if \( \kappa < 0 \) then \( \kappa \)-concavity means that \( f^\kappa \) is convex. When \( \kappa = 0 \) we call \( \kappa \)-concavity log-concavity.

In [7], Borell proved the following classical result connecting the concavity of a measure with the concavity of its density function:

**Theorem 3.1 (Borell).** Let \( \mu \) be an absolutely continuous measure on \( \mathbb{R}^n \) with density \( f, \) and \( n \)-dimensional support set. Then \( \mu \) is \( \kappa \)-concave if and only if \( f \) is \( \kappa_n \)-concave, where

\[
\kappa_n = \frac{\kappa}{1 - n\kappa}.
\]

In particular, Borell’s result implies that a measure is log-concave if and only if its density is. Before proving our main theorem, let us demonstrate how Borell’s theorem easily implies the classical Prékopa–Leindler inequality.

Consider the measure \( \mu \) on \( \mathbb{R}^{n+1} = \{(x, z) : x \in \mathbb{R}^n, z \in \mathbb{R}\} \) with density \( d\mu(x, z) = e^{-z}dx dz. \) Since this is a log-concave density, by Borell’s theorem \( \mu \) is a log-concave measure. On the other hand we have that for any \( \phi, \mu(\text{epi}(\phi)) = \int e^{-\phi}, \) and also,

\[
\text{epi} (\phi \diamond_{\lambda} \psi) = (1 - \lambda)\text{epi} (\phi) + \lambda \text{epi} (\psi).
\]

Therefore, using the log concavity of \( \mu \) we have that

\[
\int e^{-\phi \diamond_{\lambda} \psi} = \mu((1 - \lambda)\text{epi} (\phi) + \lambda \text{epi} (\psi)) \geq \mu(\text{epi} (\phi))^{1-\lambda} \mu(\text{epi} (\psi))^\lambda = \left(\int e^{-\phi}\right)^{1-\lambda} \left(\int e^{-\psi}\right)^\lambda,
\]

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which is the Prékopa–Leindler inequality [2].

Our Theorem 1.1 is a concavity property of the measure $\mu$, with power $(-1)$, but with respect to a different addition operation on the epi-graphs, the one that is induced by the $A$ transform. We shall make use of the following lemma regarding the pullback of $\mu$ via $F$.

**Lemma 3.2.** Let $\nu$ be the measure on $\mathbb{R}^n \times \mathbb{R}^+$ given by $d\nu(x, z) = e^{-1/z}z^{-(n+2)}dzdx$. Then $\nu$ is $(-1)$-concave and for any measurable $\phi : \mathbb{R}^n \to \mathbb{R}^+$ we have

$$\int_{\mathbb{R}^n} e^{-\phi} = \nu(F(epi(\phi))).$$

**Proof.** The differential of $F$ is an upper triangular matrix, with diagonal entries $1/z_1, \ldots, 1/z, -1/z_2^2$, thus $|\det(D_F(x,z))| = z^{-(n+2)}$. It follows that the pullback of $\mu$ under $F$ has density $e^{-1/z}z^{-(n+2)}$, and

$$\int_{\mathbb{R}^n} e^{-\phi} = \int_{epi(\phi)} e^{-z}dzdx = \mu(epi(\phi)) = \nu(F(epi(\phi))).$$

It is easy to check that the density of $\nu$ is $-\frac{1}{n+2}$ concave. Theorem 3.1 thus implies that $\nu$ is $(-1)$-concave. \qed

**Proof of Theorem 1.1.** Given $f, g, h$ satisfying the conditions of the theorem, we may assume by approximation that they are bounded, and by rescaling we may further assume that their image is contained in $[0, 1]$. Let $\varphi, \psi, \eta : \mathbb{R}^n \to [0, \infty]$ be defined by $f = e^{-\varphi}, g = e^{-\psi}, h = e^{-\eta}$. By Proposition 2.2 it follows that $\eta \leq \varphi \boxplus_\lambda \psi$ so $epi(\varphi \boxplus_\lambda \psi) \subseteq epi(\eta)$ i.e.

$$(1 - \lambda)F(epi \varphi) + \lambda F(epi \psi) = F(epi (\varphi \boxplus_\lambda \psi)) \subseteq F(epi \eta).$$

By Lemma 3.2 the measure $\nu$ is $(-1)$-concave and thus

$$\mu(epi \eta) = \nu(F(epi \eta)) \geq M_{\lambda}^\Lambda(\nu(F(epi \varphi)), \nu(F(epi \psi)))$$

$$= M_{\lambda}^\Lambda(\mu(epi \varphi), \mu(epi \psi)).$$

This means

$$\int_{\mathbb{R}^n} e^{-\eta} \geq \left((1 - \lambda)\left(\int_{\mathbb{R}^n} e^{-\varphi}\right)^{-1} + \lambda\left(\int_{\mathbb{R}^n} e^{-\psi}\right)^{-1}\right)^{-1},$$

as required. \qed

Using the same methods, we give a proof of Theorem 1.3.
Lemma 3.3. Let $\nu_p$ be the measure on $\mathbb{R}^n \times \mathbb{R}^+$ given by $d\nu_p(x, z) = pz^{p-(n+1)}dzdx$. Then $\nu_p$ is $\frac{1}{p}$-concave and for any measurable $\phi : \mathbb{R}^n \to \mathbb{R}^+$ we have

$$\int_{\mathbb{R}^n} \frac{1}{\phi^p} = \nu_p(F(epi(\phi))).$$

Proof. Consider the measure on $\mathbb{R}^n \times \mathbb{R}^+$ given by

$$d\mu_p = \frac{p}{z^{p+1}}dzdx.$$ 

For a measurable function $\phi : \mathbb{R}^n \to \mathbb{R}^+$ we have that

$$\mu_p(epi(\phi)) = \int_{\mathbb{R}^n} \int_{\phi(x)}^{\infty} \frac{p}{z^{p+1}}dzdx = \int_{\mathbb{R}^n} \frac{1}{\phi(x)^p}dx.$$ 

We note that the measure $\nu_p$ is the pullback of $\mu_p$ under $F$, and its density $pz^{p-(n+1)}$ is $\left(\frac{1}{p-(n+1)}\right)$-concave, so by Borell’s Theorem 3.1 $\nu_p$ is $\frac{1}{p}$-concave. We get

$$\int_{\mathbb{R}^n} \frac{1}{\phi^p} = \mu_p(epi(\phi)) = \nu_p(F(epi(\phi))),$$

as required. \qed

Proof of Theorem 1.3. Fix some $\lambda \in (0, 1)$ e.g. $\lambda = \frac{1}{2}$. Given $f, g, h$ satisfying the conditions of the theorem, let $\varphi, \psi, \eta$ be defined by $\varphi = \frac{1-\lambda}{f}$, $\psi = \frac{\lambda}{g}$, and $\eta = \frac{1}{h}$. The conditions on $f, g, h$ imply that

$$\eta((1-t)x + ty) \leq \max \left\{ \frac{1-t}{1-\lambda} \varphi(x), \frac{t}{\lambda} \psi(y) \right\}.$$ 

By Proposition 2.2 it follows that $\eta \leq \varphi \Box_{\lambda} \psi$ so that $epi(\varphi \Box_{\lambda} \psi) \subseteq epi(\eta)$, and

$$(1-\lambda)F(epi \varphi) + \lambda F(epi \psi) = F(epi(\varphi \Box_{\lambda} \psi)) \subseteq F(epi \eta).$$ 

By Lemma 3.3 the measure $\nu_p$ is $\frac{1}{p}$-concave and thus

$$\mu_p(epi(\eta)) = \nu_p(F(epi(\eta))) \geq M^\lambda_{1/p}(\nu_p(F(epi(\varphi))), \nu_p(F(epi(\psi))))$$

$$= M^\lambda_{1/p}(\mu_p(epi(\varphi)), \mu_p(epi(\psi))).$$

This means that

$$\left(\int_{\mathbb{R}^n} \frac{1}{\eta^p}\right)^{\frac{1}{p}} \geq (1-\lambda)\left(\int_{\mathbb{R}^n} \frac{1}{\varphi^p}\right)^{\frac{1}{p}} + \lambda \left(\int_{\mathbb{R}^n} \frac{1}{\psi^p}\right)^{\frac{1}{p}}.$$ 

Rewriting the latter in terms of $f, g, h$ we get

$$\left(\int_{\mathbb{R}^n} h^p\right)^{\frac{1}{p}} \geq \left(\int_{\mathbb{R}^n} f^p\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^n} g^p\right)^{\frac{1}{p}},$$

as claimed. \qed
4 Relation to Busemann’s theorem

Busemann’s convexity theorem states that given a centrally symmetric convex body \( K \), its intersection body, whose radial function is given by \( r(u) = \text{Vol}_{n-1}(K \cap u^\perp) \), is convex. The proof appears in [8], see also [10] Theorem 8.1.10. In fact, Busemann proves more; without assuming central symmetry, his proof deals with the volume of intersections of \( K \) with half-spaces. One possible way for stating his theorem is as follows:

**Theorem 4.1** (Busemann). Let \( E \) be an \( n - 2 \) dimensional subspace of \( \mathbb{R}^n \). For every \( u \in E^\perp \), denote by \( H_u \) the closed \( n - 1 \) dimensional half-space \( E + \mathbb{R}^+ u \). Let \( x_0, x_1 \in E^\perp \), and let \( K_0, K_1 \) be compact convex subsets of \( H_{x_0}, H_{x_1} \) respectively. For \( \lambda \in (0, 1) \), let \( x_\lambda = (1 - \lambda)x_0 + \lambda x_1 \) and \( K_\lambda = \text{conv}(K_0, K_1) \cap H_{x_\lambda} \). Then

\[
\frac{|x_\lambda|}{\text{Vol}(K_\lambda)} \leq (1 - \lambda)\frac{|x_0|}{\text{Vol}(K_0)} + \lambda\frac{|x_1|}{\text{Vol}(K_1)}.
\]

In [5] Barthel and Franz offer a generalization of Busemann’s theorem is obtained where the convexity assumptions on the bodies is relaxed. In [11] Kim, Yaskin, and Zvavitch give a different extension of Busemann’s theorem. They show that when instead of volume, one considers some even log-concave measure, with respect to which the hyperplane intersections of some centrally symmetric convex body \( K \) are measured, the same conclusion holds, that is, the radial function \( 1/\mu(K \cap u^\perp) \) defines a norm (here \( \mu \) on \( u^\perp \) is understood via the restriction of its density function). Their argument is based on Ball’s result on convexity of certain bodies associated with log-concave functions, see [4], and [11] Chapter 10 for a discussion of these bodies and their important role in asymptotic convex geometry. These bodies were generalized by Bobkov [6] to measures with weaker concavity assumptions. In [8] Busemann’s theorem is extended to this larger class of measures, and a very short and elegant proof for the convexity of these bodies is given. They prove

**Theorem 4.2** (Cordero, Fradelizi, Paouris, Pivovarov). Let \( \psi : \mathbb{R}^n \to \mathbb{R}^+ \) be an even function which is \((-1/n)\)-concave, that is, it satisfies

\[
\psi^{-1/n}((1 - \lambda)x + \lambda y) \leq (1 - \lambda)\psi^{-1/n}(x) + \lambda\psi^{-1/n}(y).
\]

Then the function \( \Phi \) defined by \( \Phi(0) = 0 \) and for \( z \neq 0 \)

\[
\Phi(z) = \frac{z}{\int_{z^\perp} \psi(x) dx}
\]

is a norm.

These theorems are strongly related to our main theorems, as we shall see below. However, in order to prove our main Theorem 1.1 we shall need a version of Busemann’s Theorem which combines several of the above generalizations and seems not
to have appeared in the literature. Namely, we need a log-concave measure rather than Lebesgue volume, we work with half-spaces rather than even measures and centrally symmetric bodies, and we do not assume any kind of convexity on the sets involved. We prove

**Theorem 4.3.** Let $e^{-\psi} : \mathbb{R}^{n+2} \to \mathbb{R}^+$ be a log-concave density and let $E$ be an $n$-dimensional subspace of $\mathbb{R}^{n+2}$. For every $u \in E^\perp$, denote by $H_u$ the closed $(n+1)$-dimensional half-space $E + \mathbb{R}^+ u$. Let $x_0, x_1 \in E^\perp$ be linearly independent, $x_\lambda = (1 - \lambda)x_0 + \lambda x_1$ for some $\lambda \in (0, 1)$, and let $K_0, K_\lambda, K_1$ be subsets of $H_{x_0}, H_{x_\lambda}, H_{x_1}$ respectively, such that for any $t \in (0, 1)$

$$(1-t)K_0 + tK_1 \subseteq K_\lambda.$$ 

Then

$$\frac{|x_\lambda|}{\int_{K_\lambda} e^{-\psi}} \leq (1 - \lambda)\frac{|x_0|}{\int_{K_0} e^{-\psi}} + \lambda \frac{|x_1|}{\int_{K_1} e^{-\psi}}.$$

We shall prove Theorem 4.3 in Section 5 and devote the rest of this section to show how it implies Theorem 1.2.

**Proof of Theorem 1.2.** We may assume by approximation that the functions are bounded, and by rescaling we assume without loss of generality that $f, g, h : \mathbb{R}^n \to [0, 1]$. They satisfy

$$h((1-t)x + ty) \geq \min \left\{ f(x) \frac{(1-t)}{1}, g(y) \frac{t}{1} \right\},$$

whenever $t \in (0, 1)$ and $x, y \in \mathbb{R}^n$. Recall that $\mu$ is a log-concave measure on $\mathbb{R}^n$, and $\lambda \in (0, 1)$. Define $\varphi_0, \varphi_1, \varphi_\lambda$ by $f = e^{-\varphi_0}$, $g = e^{-\varphi_1}$, $h = e^{-\varphi_\lambda}$. The assumption on $f, g, h$ implies that

$$\varphi_\lambda \leq \varphi_0 \ominus \lambda \varphi_1.$$ 

(15)

Fix some $0 < s_0 < s_1$ and set $s_\lambda = (1 - \lambda)s_0 + \lambda s_1$. We identify the epi-graphs of the three functions $\varphi_0, \varphi_\lambda, \varphi_1$ with the sets $K_i \subseteq H_i := \mathbb{R}^n \times \mathbb{R}^+ \cdot x_i \subset \mathbb{R}^{n+2}$ where $x_i = (0, s_i, 1)$, for $i = 0, \lambda, 1$, as follows.

$$K_i = \{(x, s_i z, z) : z > \varphi_i(x)\} \subseteq H_i.$$ 

Assume that the measure $\mu$ has a log-concave density $e^{-\alpha} : \mathbb{R}^n \to \mathbb{R}$, and define the density $\psi : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ by $\psi(x, s, z) = \alpha(x) + z$. Note that

$$\int_{\mathbb{R}^n} e^{-\varphi_i} d\mu = \frac{1}{\sqrt{1 + s_i^2}} \int_{K_i} e^{-\psi},$$

where the integration on the right hand side is with respect to the $(n+1)$-dimensional Hausdorff measure $H_{n+1}$. In this terminology we need to show that

$$\frac{|x_\lambda|}{\int_{K_\lambda} e^{-\psi}} \leq (1 - \lambda)\frac{|x_0|}{\int_{K_0} e^{-\psi}} + \lambda \frac{|x_1|}{\int_{K_1} e^{-\psi}}.$$
which is exactly the statement of Theorem 4.3. We are thus left with showing that the conditions of the theorem are met. Clearly \( \psi \) is convex so we must show that for any \( t \in (0, 1) \)
\[
((1 - t)K_0 + tK_1) \cap H_\lambda \subseteq K_\lambda.
\] (16)

To this end we define \( \tilde{F} : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^+ \) by
\[
\tilde{F}(x, s, z) = \left( \frac{x}{z}, \frac{s}{z}, \frac{1}{z} \right).
\]
The map \( \tilde{F} \) is an involution, which maps segments to segments (it is a fractional linear map). It is closely related to the map \( F \) from Section 2. Indeed,
\[
\tilde{F}(K_i) = \left\{ \left( \frac{x}{z}, s_i, \frac{1}{z} \right) : z > \varphi_i(x) \right\} = \{(x, s_i, z) : (x, z) \in F(\text{epi } \varphi_i)\},
\] (17)
and by Proposition 2.1 and the inclusion (15), we have
\[
(1 - \lambda)F(\text{epi } (\varphi_0)) + \lambda F(\text{epi } (\varphi_1)) = F(\text{epi } (\varphi_0 \sqcup_\lambda \varphi_1)) \subseteq F(\text{epi } (\varphi_\lambda)).
\] (18)

Let \( H'_i = \tilde{F}(H_i) \) and \( A_i = \tilde{F}(K_i) \subseteq H'_i \), for \( i = 0, \lambda, 1 \). The half-spaces \( H'_i \) are parallel, and \( H'_\lambda = (1 - \lambda)H'_0 + \lambda H'_1 \). We have
\[
\bigcup_{k_0 \in K_0, k_1 \in K_1} [k_0, k_1] \cap H_\lambda = \tilde{F} \left( \tilde{F} \left( \bigcup_{k_0 \in K_0, k_1 \in K_1} [k_0, k_1] \right) \cap \tilde{F}(H_\lambda) \right)
= \tilde{F} \left( \left( \bigcup_{k_0 \in K_0, k_1 \in K_1} \tilde{F}([k_0, k_1]) \right) \cap H'_\lambda \right)
= \tilde{F} \left( \left( \bigcup_{a_0 \in A_0, a_1 \in A_1} [a_0, a_1] \right) \cap H'_\lambda \right)
= \tilde{F} \left( \left( \bigcup_{0 \leq \beta \leq 1} (1 - \beta)A_0 + \beta A_1 \right) \cap H'_\lambda \right)
= \tilde{F} \left( (1 - \lambda)A_0 + \lambda A_1 \right)
= \tilde{F} \left( (1 - \lambda)\tilde{F}(K_0) + \lambda \tilde{F}(K_1) \right) \subseteq K_\lambda,
\]
where the last inclusion follows from (17) and (18). We have established the condition (16) and thus the proof is complete.

5 Generalized Busemann Theorem

In this section we prove Theorem 4.3. Recall that we are given an \( n \)-dimensional subspace \( E \) of \( \mathbb{R}^{n+2} \), which we assume without loss of generality is spanned by the
first $n$ coordinates $E = \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^2$, and $K_0, K_\lambda, K_1$, which are subsets of $H_0 = H_{x_0}, H_\lambda = H_{x_\lambda}$ and $H_1 = H_{x_1}$ respectively, and $x_\lambda = (1 - \lambda)x_0 + \lambda x_1$ with $x_0$ and $x_1$ linearly independent. These satisfy that for any $t \in (0, 1)$:

$$((1 - t)K_0 + tK_1) \cap H_\lambda \subseteq K_\lambda. \quad (19)$$

Geometrically this means that the “one step convex hull” of $K_0$ and $K_1$ (obtained by taking all segments connecting the two sets), when intersected with $H_\lambda$ (also known as “harmonic linear combination” in [10]), is contained in $K_\lambda$. Denote $u_i = \frac{x_i}{|x_i|}$, and for $r > 0$ set

$$m_i(r) = \int_{K_i \cap (\mathbb{R}^n + ru_i)} e^{-\psi} dH_n, \quad \rho(u_i) = \int_0^\infty m_i(r) dr = \int_{K_i} e^{-\psi} dH_{n+1},$$

where the integration is with respect to Hausdorff measures. Our aim is to show that

$$\frac{|x_\lambda|}{\rho(u_\lambda)} \leq (1 - \lambda) \frac{|x_0|}{\rho(u_0)} + \lambda \frac{|x_1|}{\rho(u_1)}.$$  

To this end we define the percentile functions $p_0, p_1 : [0, 1] \to \mathbb{R}^+$ as follows. For $\theta \in [0, 1]$ we let

$$\theta = \frac{\int_0^{p_0(\theta)} m_0(r) dr}{\int_0^\infty m_0(r) dr} = \frac{\int_0^{p_1(\theta)} m_1(r) dr}{\int_0^\infty m_1(r) dr}.$$  

For $p_i$ to be well defined one may assume, for example, that $m_i$ are positive and continuous, which may be assumed by approximation. Differentiation with respect to $\theta$ yields:

$$\frac{\int_0^\infty m_0(r) dr}{m_0(p_0(\theta))} = p'_0(\theta), \quad \text{and} \quad \frac{\int_0^\infty m_1(r) dr}{m_1(p_1(\theta))} = p'_1(\theta).$$

Define $p_\lambda : [0, 1] \to \mathbb{R}^+$ by $p_\lambda(\theta)u_\lambda = (1 - \beta(\theta))p_0(\theta)u_0 + \beta(\theta)p_1(\theta)u_1$, where

$$\beta(\theta) = \frac{\lambda p_0(\theta)|x_1|}{(1 - \lambda)p_1(\theta)|x_0| + \lambda p_0(\theta)|x_1|}.$$  

One computes that

$$\frac{p_\lambda(\theta)}{|x_\lambda|} = \frac{p_0(\theta)p_1(\theta)}{(1 - \lambda)p_1(\theta)|x_0| + \lambda p_0(\theta)|x_1|} = M_\lambda^{-1} \left( \frac{p_0(\theta)}{|x_0|}, \frac{p_1(\theta)}{|x_1|} \right). \quad (20)$$

Differentiation of (20) with respect to $\theta$ (after taking reciprocals) gives:

$$\frac{|x_\lambda|}{p'_\lambda(\theta)} p'_\lambda(\theta) = \frac{(1 - \lambda)|x_0|}{p'_0(\theta)} p'_0(\theta) + \frac{\lambda|x_1|}{p'_1(\theta)} p'_1(\theta).$$

Let us next show that

$$m_\lambda(p_\lambda(\theta)) \geq m_0(p_0(\theta))^{1 - \beta(\theta)} m_1(p_1(\theta))^{\beta(\theta)}. \quad (21)$$
Indeed, by (19) we have in particular that
\[(1 - \beta(\theta))K_0 + \beta(\theta)K_1 \cap H_\lambda \subseteq K_\lambda.\] (22)

We may intersect this inclusion with the \((n+1)\)-dimensional affine subspace \(M\) containing \(E_0 = \mathbb{R}^n + p_0(\theta)u_0\) and \(E_1 = \mathbb{R}^n + p_1(\theta)u_1\). Note that \(M \cap H_i = E_i\), for \(i = 0, \lambda, 1\), where we denoted \(E_\lambda = \mathbb{R}^n + p_\lambda(\theta)u_\lambda\) (and, in particular, \(K_i \cap M = K_i \cap E_i\)). Intersecting with \(M\), the inclusion (22) implies
\[(1 - \beta(\theta)) (K_0 \cap M) + \beta(\theta) (K_1 \cap M) \subseteq ((1 - \beta(\theta))K_0 + \beta(\theta)K_1) \cap M \cap H_\lambda \subseteq K_\lambda \cap M.\]

Finally, we use the log-concavity of the density \(e^{-\psi}\) (on \(M\)) together with Prékopa’s theorem [14] on the log-concavity of the marginal of a log-concave density (which follows from [2], for example), the density here being \(e^{-\psi}\) restricted to \(\text{conv}(K_0 \cap M, K_1 \cap M)\). We get that
\[
\int_{K_\lambda \cap E_\lambda} e^{-\psi} \geq \left( \int_{K_0 \cap E_0} e^{-\psi} \right)^{1 - \beta(\theta)} \cdot \left( \int_{K_1 \cap E_1} e^{-\psi} \right)^{\beta(\theta)},
\]
which is (21). The rest of the argument follows closely the classical Busemann argument.

\[
\frac{\rho(u_\lambda)}{|x_\lambda|} = \frac{\int_{K_\lambda} e^{-\psi}}{\int_{|x_\lambda|}} = \frac{\int_0^\infty m_\lambda(r)dr}{|x_\lambda|} = \frac{\int_0^1 m_\lambda(p_\lambda(\theta))p'_\lambda(\theta)}{|x_\lambda|} d\theta = \frac{\int_0^1 m_\lambda(p_\lambda(\theta))p'_\lambda(\theta)}{|x_\lambda|^2} \frac{M_1^\lambda}{M_1} \left( \frac{|x_0|p'_0(\theta)}{p_0(\theta)} - \frac{|x_1|p'_1(\theta)}{p_1(\theta)} \right) d\theta
\]

\[
= \frac{\int_0^1 m_\lambda(p_\lambda(\theta))p'_\lambda(\theta)}{|x_\lambda|^2} M_1^\lambda \left( \frac{|x_0|\rho(u_0)}{m_0(p_0(\theta))p'_0(\theta)} - \frac{|x_1|\rho(u_1)}{m_1(p_1(\theta))p'_1(\theta)} \right) d\theta
\]

\[
= \frac{\int_0^1 m_\lambda(p_\lambda(\theta))M_\lambda^{-1}}{|x_\lambda|^2} \left( \frac{p_0(\theta)}{|x_0|} - \frac{p_1(\theta)}{|x_1|} \right)^2 M_1^\lambda \left( \frac{|x_0|\rho(u_0)}{m_0(p_0(\theta))p'_0(\theta)} - \frac{|x_1|\rho(u_1)}{m_1(p_1(\theta))p'_1(\theta)} \right) d\theta.
\]

Using (21) and denoting \(w_i(\theta) = \frac{|x_i|}{\rho(\theta)}\) we get

\[
\frac{\rho(u_\lambda)}{|x_\lambda|} \geq \int_0^1 m_0(p_0(\theta))^{1 - \beta(\theta)} m_1(p_1(\theta))^{\beta(\theta)} \left( \frac{\rho(u_0)w_0^2(\theta)}{|x_0| m_0(p_0(\theta))} - \frac{\rho(u_1)w_1^2(\theta)}{|x_1| m_1(p_1(\theta))} \right) d\theta
\]

\[
= \int_0^1 m_0(p_0)^{(1 - \lambda)w_0 + \lambda w_1} m_1(p_1)^{(1 - \lambda)w_0 + \lambda w_1} \left( \frac{(1 - \lambda)w_0 + \lambda w_1}{(1 - \lambda)w_0 + \lambda w_1} \right) \frac{\lambda w_1}{(1 - \lambda)w_0 + \lambda w_1} d\theta.
\]

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where $a_i(\theta) = \frac{w_i(\theta)\rho(u_i)}{|x_i| m_i(p_i(\theta))}$. Applying the arithmetic-geometric means inequality we get

$$\frac{\rho(u_\lambda)}{|x_\lambda|} \geq \int_0^1 \left( \frac{(a_0 m_0(p_0))^{(1-\lambda)w_0}}{(1-\lambda)w_0 + \lambda w_1} \left( a_1 m_1(p_1) \right)^{\frac{\lambda w_1}{(1-\lambda)w_0 + \lambda w_1}} \right) \left( \frac{|x_0|}{w_0} \right)^{(1-\lambda)w_0 + \lambda w_1} \left( \frac{|x_1|}{w_1} \right)^{\frac{\lambda w_1}{(1-\lambda)w_0 + \lambda w_1}} \frac{1}{(1-\lambda)w_0 + \lambda w_1} \, d\theta.$$  

Applying the geometric-harmonic means inequality in the last integral we get

$$\frac{\rho(u_\lambda)}{|x_\lambda|} \geq \int_0^1 \left( \frac{\rho(u_0)/|x_0| \left( a_0 m_0(p_0) \right)^{(1-\lambda)w_0} \left( a_1 m_1(p_1) \right)^{\frac{\lambda w_1}{(1-\lambda)w_0 + \lambda w_1}}}{(1-\lambda)w_0 + \lambda w_1} \right) \left( \frac{|x_0|}{w_0} \right)^{(1-\lambda)w_0 + \lambda w_1} \left( \frac{|x_1|}{w_1} \right)^{\frac{\lambda w_1}{(1-\lambda)w_0 + \lambda w_1}} \, d\theta = \int_0^1 \left( \frac{\rho(u_0)/|x_0| \left( a_0 m_0(p_0) \right)^{(1-\lambda)w_0} \left( a_1 m_1(p_1) \right)^{\frac{\lambda w_1}{(1-\lambda)w_0 + \lambda w_1}}}{(1-\lambda)w_0 + \lambda w_1} \right) \left( \frac{|x_0|}{w_0} \right)^{(1-\lambda)w_0 + \lambda w_1} \left( \frac{|x_1|}{w_1} \right)^{\frac{\lambda w_1}{(1-\lambda)w_0 + \lambda w_1}} \, d\theta = \frac{1}{(1-\lambda)w_0 + \lambda w_1} \left( \frac{|x_0|}{w_0} \right)^{(1-\lambda)w_0 + \lambda w_1} \left( \frac{|x_1|}{w_1} \right)^{\frac{\lambda w_1}{(1-\lambda)w_0 + \lambda w_1}}.$$  

That is,

$$\frac{|x_\lambda|}{\rho(u_\lambda)} \leq (1-\lambda) \frac{|x_0|}{\rho(u_0)} + \lambda \frac{|x_1|}{\rho(u_1)},$$  

as required.

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