On Non-commutative Geodesic Motion

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Abstract

In this work we study the geodesic motion on a noncommutative space-time. As a result we find a non-commutative geodesic equation and then we derive corrections of the deviation angle per revolution in terms of the non-commutative parameter when we specify the problem of Mercury's perihelion. In this way, we estimate the noncommutative parameter based in experimental data.

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I. INTRODUCTION

In 1845 the French astronomer Le Verrier observed that the perihelion of the planet Mercury precess faster than can be accounted for by the Newtonian mechanics with the distribution of masses of the solar system well-known until then. The calculations from Newtonian mechanics present a discrepancy of $43'11'' \pm 0,45''$ in comparison with experimental data. This discovery began different lines of investigation to explain the new phenomena. One of the explanations was the existence of a new planet that would explain the anomaly in Mercury’s orbit within the context of Newton’s laws [1, 2]. From then the problem of Mercury perihelion became a topic of hard discussions in the scientific community. However, with the advent of the General Relativity Theory the problem was solved and is considered one of triumphs of the new theory. Recently, questions arising in the study of quantum gravity regained interest in reconsidering these issues in the framework of non-commutative space-time. In this sense, we can cite recent studies that investigated the modifications introduced by a generalized uncertainty principle in classical orbit of particle [3–6]. The main consequence of these researches is impose a constraint on the minimal observable length and non-commutativity parameter in comparison with observational data of Mercury. In this work our aim is write the geodesic equation that arises from the metric tensor which have been corrected using non-commutative product between tetrads. In addition, we hope to obtain corrections in terms of the non-commutative of the deviation angle per revolution.

In this way, non-commutative geometry has its origin in the Weyl and Moyal works, studying quantization procedures in phase space [7]. Snyder [8, 9] was the first to develop a consistent theory for non-commutative space coordinates, which was based on representations of Lie algebras. Over the last decades there is a revival of non-commutative physics, motivated by some results coming from gravity [10–12], standard model [13–15], string theory [16, 17] and in the understanding of the quantum Hall effect [18]. One particular interest in this context is the development of representation theories for non-commutative fields [19]. So, as direct consequence of the non-commutative feature of the operators correspondent to space-time coordinates, is the impossibility to precisely measure a particle position. From the mathematical viewpoint, the simplest algebra of the operators $\hat{\mathbf{x}}^\mu$, that represents the hermitian operators correspondent to space-time coordinates, is given on anti-symmetric
tensor constant $\alpha^{\mu\nu}$,

$$[\hat{x}^\mu, \hat{x}^\nu] = i\alpha^{\mu\nu}. \quad (1)$$

The relations above imply in these uncertain relations

$$\Delta \hat{x}^{\mu} \Delta \hat{x}^{\nu} \geq \frac{1}{2} |\alpha^{\mu\nu}|. \quad (2)$$

These last relations suggest that, at distances of $\sqrt{|\alpha^{\mu\nu}|}$ order, effects of the non-commutative space-time turns out to be relevant, showing the end of the classical model of space-time and the beginning of a new geometric structure. Usually the non-commutativity is introduced by means the use of the Moyal product [8] defined as

$$f(x) \star g(x) \equiv \exp \left( \frac{i}{2} \alpha^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \right) f(x)g(y) |_{y\rightarrow x}$$

$$= f(x)g(x) + \frac{i}{2} \alpha^{\mu\nu} \partial_\mu f \partial_\nu g + \frac{1}{2!} \left( \frac{i}{2} \right)^2 \alpha^{\mu_1\nu_1} \alpha^{\mu_2\nu_2} (\partial_{\mu_1} \partial_{\nu_1} f)(\partial_{\nu_2} \partial_{\mu_2} g) + \cdots \quad (3)$$

where $\alpha^{\mu\nu}$ is the non-commutative parameter. This is accomplished replacing the usual product, in the classical lagrangian density, by the Moyal product. Such a product is defined in an arbitrary coordinates system $x^\mu$, here the quantity $y^\mu$ is just an auxiliary variable. The article is organized as follows. In section II we present the noncommutative corrections to metric field. In section III we calculate the noncommutative geodesic equation and we present an estimative for noncommutative parameter. Finally in the last section we present our concluding remarks. We use natural units such that $G = c = 1$.

II. NON-COMMUTATIVE CORRECTIONS OF METRIC TENSOR

Recently it was shown how to obtain the corrections on metric tensor components due to the non-commutativity of tetrad field [20]. We also have used such procedure to calculate such corrections for Schwarzschild spacetime [21] which can be described by the following line element

$$ds^2 = - \left( 1 - \frac{2M}{r} \right) dt^2 + \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right). \quad (4)$$
From the tetrad formulation of General Relativity we have the following relation

$$g^{\mu\nu} = e^a_{\mu} e^a_{\nu},$$

where $e^a_{\mu}$ is the tetrad or vierbein components. The tetrad field is related to the reference frame, thus using such a concept and the above relation, it is possible to completely determine the tetrad field. Then we use

$$e^a_{\mu} = \begin{bmatrix} \sqrt{-g_{00}} & 0 & 0 & 0 \\ 0 & \sqrt{g_{11}} \sin \theta \cos \phi & \sqrt{g_{22}} \cos \theta \cos \phi & -\sqrt{g_{33}} \sin \phi \\ 0 & \sqrt{g_{11}} \sin \theta \sin \phi & \sqrt{g_{22}} \cos \theta \sin \phi & \sqrt{g_{33}} \cos \phi \\ 0 & \sqrt{g_{11}} \cos \theta & -\sqrt{g_{22}} \sin \theta & 0 \end{bmatrix}, \quad (5)$$

this tetrad field is adapted to a stationary observer at spatial infinity.

In order to introduce the Moyal product into the metric tensor, we follow the approach in ref. [20], this will induce the settlement of a new metric tensor $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu}$. Such a non-commutative metric tensor reads

$$\tilde{g}_{\mu\nu} = \frac{1}{2} \left( e^a_{\mu} \ast e^a_{\nu} + e^a_{\nu} \ast e^a_{\mu} \right),$$

where the Moyal product is defined by the relation (3). It is important to note that the non-commutative metric tensor preserves the SO(3,1) symmetry. Let us consider a change in the reference frame which is implemented by the following transformation in the tetrad field $e'^a_{\mu} = \Lambda^a_{\ b} e^b_{\mu}$, where $\Lambda^a_{\ b}$ is the Lorentz matrix. Since $\Lambda^a_{\ b}$ do not depend on the coordinates, it is not affected by the Moyal product. As a consequence the quantity $\tilde{\eta}^{ab} = \frac{1}{2} \left( e^a_{\mu} \ast e^b_{\mu} + e^b_{\mu} \ast e^a_{\mu} \right)$ behaves like a tensor under SO(3,1) transformations. However the same thing is not true for the Diffeomorphism group, yet it is possible to get a tensorial behavior for $\tilde{g}_{\mu\nu}$. To this purpose it is introduced a deformed diffeomorphism group which is obtained by a suitable mapping of the original differential manifold. Thus, by the appropriate choice of the representation of such functions on the deformed manifold, $\tilde{g}_{\mu\nu}$ transforms like a tensor [20]. It has been showed that corrections in the metric tensor appears up to second order in the non-commutative parameter $\alpha^{\mu\nu}$ [22, 23].

We intent to analyze a geodesic movement over a plane $\theta = \frac{\pi}{2}$, thus the new metric tensor $\tilde{g}_{\mu\nu}$ will assume a much simpler form, which reads
$$
\tilde{g}_{00} = - \left( 1 - \frac{2M}{r} \right), \\
\tilde{g}_{11} = \left( 1 - \frac{2M}{r} \right)^{-1} \left[ 1 + \left( 1 - \frac{2M}{r} \right)^{-2} \left( \frac{\alpha^2 M}{2r^3} \right) \right], \\
\tilde{g}_{22} = r^2, \\
\tilde{g}_{33} = r^2 + \frac{\alpha^2}{8},
$$

(6)

where \( \alpha = \alpha^{13} \). We also have considered that the only non-vanishing component of the non-commutative parameter is \( \alpha^{13} \).

### III. NON-COMMUTATIVE GEODESIC EQUATION

Let us consider a system composed by a central mass distribution \( M \) and a point particle of mass \( m \), as stated before we assume that the movement takes place in a plane \( \theta = \frac{\pi}{2} \). Then the Hamilton-Jacobi equation is

$$
\tilde{g}^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} + m^2 = 0,
$$

which, after using the corrected metric tensor, reads

$$
- \left( 1 - \frac{2M}{r} \right)^{-1} \left( \frac{\partial S}{\partial t} \right)^2 + \left( 1 - \frac{2M}{r} \right) \left[ 1 + \left( 1 - \frac{2M}{r} \right)^{-2} \left( \frac{\alpha^2 M}{2r^3} \right) \right]^{-1} \left( \frac{\partial S}{\partial r} \right)^2 + \\
+ \left( r^2 + \frac{\alpha^2}{8} \right)^{-1} \left( \frac{\partial S}{\partial \phi} \right)^2 = -m^2.
$$

(7)

Therefore we assume an action given by \( S = -E_0 t + L \phi + S_r(r) \), where \( E_0 \) and \( L \) are the energy and angular momentum respectively. Then the function \( S_r(r) \) is given by

$$
S_r = \int \left[ 1 + \left( 1 - \frac{2M}{r} \right)^{-2} \left( \frac{\alpha^2 M}{2r^3} \right) \right]^{1/2} \left\{ \frac{E_0^2}{(1 - \frac{2M}{r})^2} - \left[ \frac{m^2 + \frac{L^2}{m^2 (1 + \alpha^2/8r^2)}}{(1 - \frac{2M}{r})^2} \right] \right\}^{1/2} dr.
$$

(8)

Thus it is possible to obtain equations for \( t \) and \( \phi \) by means the action above, \(-t + \frac{\partial S_r}{\partial E_0} = const. \) and \( \phi + \frac{\partial S_r}{\partial L} = const. \). Then, after some algebraic manipulations it yields

$$
t = \frac{E_0}{m} \int \frac{\left[ 1 + \left( 1 - \frac{2M}{r} \right)^{-2} \left( \frac{\alpha^2 M}{2r^3} \right) \right]^{1/2}}{(1 - \frac{2M}{r}) \left\{ \frac{E_0^2}{m^2} - \left( 1 - \frac{2M}{r} \right) \left[ 1 + \frac{\frac{L^2}{m^2 (1 + \alpha^2/8r^2)}}{(1 - \frac{2M}{r})^2} \right] \right\}^{1/2} dr.
$$

(9)
\[
\phi = \int \frac{\left(\frac{L}{mr^2}\right) \left[1 + \left(\frac{1}{r} - \frac{2M}{r}\right)^2 \left(\frac{\alpha^2 M}{2r^3}\right)^2\right]^{1/2}}{(1 + \frac{\alpha^2}{8r^2}) \left\{ \frac{E^2}{m^2} - \left(\frac{1}{r} - \frac{2M}{r}\right) \left[1 + \frac{\left(\frac{L^2}{m^2r^2}\right)}{1 + \frac{\alpha^2}{8r^2}}\right]\right\}^{1/2} dr. \tag{10}
\]

From equation (9) it is possible to get the conditions that lead to a circular orbit of \(m\) around \(M\) while formally the relation (10) defines the orbit itself. We would like to show the trajectory equation in a more familiar form. For such a purpose we change variables to \(U = \frac{1}{r}\) in (10), where \(U = U(\phi)\), and take the derivative with respect to \(\phi\). Hence we obtain

\[
(U')^2 = \left(1 + \frac{\alpha^2 U^2}{4}\right) \left\{ \left[E^2 - m^2 \left(1 - 2MU\right) \right] - U^2 \left(1 - 2MU\right) \right\} \left[1 - \frac{\alpha^2 MU^3}{2 \left(1 - 2MU\right)}\right], \tag{11}
\]

with \(U' = \frac{dU}{d\phi}\). If we again perform a derivative of above equation with respect to \(\phi\) and use \(\alpha U << 1\), then it yields

\[
U'' + U = \frac{m^2 M}{L^2} + 3MU^2 + \alpha^2 g(U), \tag{12}
\]

where \(g(U)\) is given by

\[
g(U) = \left[\frac{E_0^2 U + m^2 MU^2 - m^2 U \left(1 - 2MU\right)}{4L^2}\right] - \frac{MU^3}{2 \left(1 - 2MU\right)^2} \left[\frac{m^2 M}{L^2} - U \left(1 - 3MU\right)\right] - \frac{U^3}{8 \left(2 - 5MU\right)} - \frac{MU^2}{4 \left(1 - 2MU\right)^3} \left\{ \left[\frac{E_0^2 - m^2 \left(1 - 2MU\right)}{L^2}\right] - U^2 \left(1 - 2MU\right) \right\}. \tag{13}
\]

Equation (12) is the non-commutative geodesic equation. If we consider \(MU << 1\), then the last equation simplifies to

\[
U'' + U = \frac{m^2 M}{L^2} + 3MU^2 + \alpha^2 \left[\frac{(E_0^2 - m^2) U}{4L^2} - \frac{U^3}{4}\right]. \tag{14}
\]

In the reference [24] is given a prescription to calculate the deviation angle after one revolution. Let’s recall some ideas. First we consider a perturbation of Keplerian’s trajectory equation in the form

\[
U'' + U = \frac{M}{h^2} + \frac{f(U)}{h^2},
\]
where $h^2 = L^2/m^2$. In our case $\frac{f(U)}{h^2} = 3MU^2 + \alpha^2 g(U)$. Then it is possible to show that, after one revolution, there will be an angle deviation given by

$$\Delta \phi = \frac{\pi f_1}{h^2},$$

where $f_1 = \frac{d^2 f(U)}{dU^2} \bigg|_{U=1/l}$, the distance $l$ is defined by $l = a(1 - e^2)$, with $a$ denoting the major semi-axis and $e$ the eccentricity of the movement.

Hence if we compare our non-commutative geodesic equation to what is given above, then we get the following angle deviation

$$\Delta \phi = \frac{6M\pi}{a(1 - e^2)} + \alpha^2 \pi \left[ \frac{E_0^2/m^2 - 1}{4Ma(1 - e^2)} - \frac{3}{4a^2(1 - e^2)^2} \right],$$

(15)

it is possible to see that it comprises the well known general relativity prediction and a correction which is given in terms of the non-commutative parameter.

A. Estimating the parameter $\alpha$

In this subsection we will give a estimative of the parameter $\alpha$ based on experimental measurements of $\Delta \phi$ for the system Mercury-Sun. In such a system $m$ denotes the Mercury’s mass while $M$ stands for the Solar mass. Thus we will look for experimental value for the precession of Mercury’s perihelion to compare with our expression (15). In reference [25] we find the following form for the precession

$$\Delta \phi_{\text{exp}} = \frac{6\pi M(1 + \beta)}{a(1 - e^2)},$$

which is a way to determine deviations from what is predicted by general relativity. Such a deviation is incorporated by the parameter $\beta$, some experiments [25] has bounded its value as $\beta < 10^{-4}$. Therefore we find an expression for $\alpha$ of the form

$$\alpha \simeq \frac{2M\sqrt{\beta}}{v},$$

(16)

where $v$ is the planet’s velocity, thus using orbital data of Mercury and the experimental range of $\beta$ we finally find a range to settle the non-commutative parameter, in standard units it reads

$$\alpha < 10^2 Km,$$
which is a small value when compared to astronomical distances involved in the system Mercury-Sun. This is consistent with the approximation used to obtain eq. (15).

IV. CONCLUSION

In this paper we have obtained the geodesic equation that arises from the metric tensor which have been corrected using non-commutative product between tetrads. Then we find corrections of the deviation angle per revolution in terms of the non-commutative parameter. Comparing our expression to experimental data it is possible to estimate the range in which such parameter should lay. We have find \( \alpha \leq 10^2 \text{Km} \), to put this result in context we recall that the solar radius is approximatively 700000 \( \text{Km} \), thus the distance established by the non-commutative parameter would lay inside the sun. However the Schwarzschild solution (which was used to get our non-commutative geodesic equation) is only valid outside the solar mass distribution. Then the parameter \( \alpha \) could be felt outside the Sun by experiments such as the precession of Mercury’s perihelion. We hope that experimentalists would refine the accuracy of the tests to detect deviations from general relativity predictions in what concerns the precession of Mercury’s perihelion. Indeed the search for more accurate experiments would help to decide if the spacetime has a non-commutative structure.

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