Global solutions to stochastic Volterra equations driven by Lévy noise

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Abstract

In this paper we investigate the existence and uniqueness of semilinear stochastic Volterra equations driven by multiplicative Lévy noise of pure jump type. In particular, we consider the equation

\[
du(t) = \left( A \int_0^t b(t-s)u(s)\,ds \right) dt + F(t, u(t))\,dt \\
+ \int_Z G(t, u(t), z) \tilde{\eta}(dz, dt) + \int_{Z_L} G_{L}(t, u(t), z) \eta_{L}(dz, dt) ; \quad t \in (0, T],
\]

where \( Z \) and \( Z_L \) are Banach spaces, \( \tilde{\eta} \) is a time-homogeneous compensated Poisson random measure on \( Z \) with intensity measure \( \nu \) (capturing the small jumps), and \( \eta_L \) is a time-homogeneous Poisson random measure on \( Z_L \) independent to \( \tilde{\eta} \) with finite intensity measure \( \nu_L \) (capturing the large jumps). Here, \( A \) is a selfadjoint operator on a Hilbert space \( H \), \( b \) is a scalar memory function and \( F, G \) and \( G_L \) are nonlinear mappings. We provide conditions on \( b, F, G \) and \( G_L \) under which a unique global solution exists. We also present an example from the theory of linear viscoelasticity where our result is applicable.

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1. Introduction

In this work we analyse the existence and uniqueness of a class of stochastic Volterra equations driven by Lévy noise. To be more precise, let $Z$ and $Z_L$ be two Banach spaces, $\tilde{\eta}$ be a compensated Poisson random measure on $Z$ with intensity measure $\nu$, and $\eta_L$ be a Poisson random measure on $Z_L$ independent to $\tilde{\eta}$ with finite intensity measure $\nu_L$. With given mappings $G, G_L$ and $F$ and initial data $u_0$, satisfying certain conditions specified later, we are interested in the solution of the equation

$$
\begin{aligned}
&du(t) = 
\left(A \int_0^t b(t-s)u(s) \, ds\right) \, dt + F(t, u(t)) \, dt \\
&+ \int_Z G(t, u(t), z) \tilde{\eta}(dz, dt) + \int_{Z_L} G_L(t, u(t), z) \eta_L(dz, dt); \quad t \in (0, T], \\
&u(0) = u_0.
\end{aligned}
$$

(1)

The examples that we have in mind that can be described via (1) are linear models of viscoelastic materials perturbed by random abrupt forcing. Therefore, we impose conditions on the memory kernel $b$ that are typical in that setting, see [46, Chapter 5] for more details. We also would like to mention that Volterra equations, especially with the specific kernel $b(t) = c_\rho t^{\rho-2}$, $1 < \rho < 2$, are an important modeling tool in many fields of sciences, such as mechanical engineering (including viscoelasticity) [3, 13, 36, 37], chemistry [1], physics (heat conduction) [44, 45], neurology [38, 40] and several other fields [20, 30].

Gaussian perturbations are not always appropriate for interpreting real data in a reasonable way. This is the case when, for example, one wants to model abrupt pulses by a random perturbation process or when the time scale of the random process is much finer then the time scale of the deterministic process or when extreme events occur relatively frequently. In all these cases, a natural mathematical modeling framework could be based on Lévy processes or general semimartingales with jumps.
Existence and uniqueness of solutions for Gaussian noise driven Volterra equations in various frameworks have been treated by several authors, see, for example, [4, 5, 6, 7, 8, 9, 14, 19, 26, 27, 28, 29] for an incomplete list of papers and the references therein. For an additive fractional Brownian motion driven linear parabolic stochastic Volterra equation we refer to [50] while for linear additive square-integrable local martingale driven stochastic Volterra equations we refer [49]. Finally, we mention [18] where the regularity (but not pathwise) of a class of linear stochastic Volterra equations were investigated in an additive Poisson random measure setting but with different smoothing properties of the deterministic solution operator than the one in the present paper (as subdiffusion, rather than dissipative waves in viscoelastic materials as in the present paper, was considered). As far as we know nonlinear stochastic Volterra equations with multiplicative non-Gaussian noise have not been analysed in the literature.

The main results of the paper are Theorem 3.3 and Theorem 4.3. In Theorem 3.3 we prove, using a fixed point argument, that (1) has a unique mild solution with certain càdlàg path regularity when $G_L = 0$ and $G$ satisfy some integrability conditions with respect to $\nu$. In Theorem 4.3 we show that a global mild solution with the same càdlàg path regularity to (1) still exists when $G_L \neq 0$ and the finite measure $\nu_L$ satisfy a mild condition at infinity.

The paper is organized as follows. First, in Section 2 we introduce some necessary preliminary material and also state our main assumption on the memory kernel and on the operator $A$. Then, in Section 3 we investigate the existence and uniqueness of a mild solution of (1) with $G_L = 0$ and $G$ satisfying some integrability conditions with respect to $\nu$. In Section 4 we extend this result by allowing $G_L \neq 0$ but no integrability condition is assumed on $G_L$ as we take $\nu_L$ to be a finite measure. Here we will use the representation of the Lévy process or compound Poisson process associated with $\eta_L$ in terms of a sum over all its jumps. If there would be no memory term, one would simple glue together the solutions between the large jumps. However, if there is a non-trivial memory term, the solution is not a Markov process, hence, it is not possible to apply the gluing technique. We deal with this difficulty by introducing special Hilbert spaces, taking into account the jump times and sizes. In Section 5 we present an example describing the velocity field of a synchronous viscoelastic material in the presence of an abrupt force field modeled by space-time Lévy noise.

Finally, we note that we use Poisson random measures to describe the
stochastic perturbation for two reasons. Firstly, using a notation based on Poisson random measures simplify many calculations and, secondly, the use of Poisson random measures allow for a more general framework than Lévy jump processes. For a precise connection between Poisson random measures and Lévy processes and integration with respect to these we refer to [17, Appendix], [42, Chapters 6-8] and [48, Chapter 4].

2. Preliminaries

In this section we shortly introduce some preliminary results and state our main assumptions on the the operator $A$, memory kernel $b$. The operator $A$ is typically an elliptic differential operator. We will assume the following.

**Assumption 2.1.** The operator $A : D(A) \rightarrow H$ is an unbounded, densely defined, linear, self adjoint, negative-definite operator with compact inverse.

Throughout the paper we will use the fractional powers of the operator $-A$ and the spaces associated with them. Let $\alpha \in \mathbb{R}$. Then, for $\alpha < 0$ we define $H^A_\alpha$ to be the closure of $H$ under the norm $|x|_{H^A_\alpha} := |(-A)^\alpha x|_H$. If $\alpha \geq 0$, then $H^A_\alpha = D((-A)^\alpha)$ and $|x|_{H^A_\alpha} := |(-A)^\alpha x|_H$. For more details, we refer to, for example [41, Chapter 2].

Next we formulate our assumptions on the memory term typical in the theory of viscoelasticity so that the deterministic equation exhibits a parabolic behaviour, c.f. [32, Assumption 1] and [14, Hypothesis (b)]. The kernel $b$ is called $k$-monotone ($k \geq 2$) if $b$ is $(k-2)$-time continuously differentiable on $(0, \infty)$, $(-1)^n b^{(n)}(t) \geq 0$ for $t > 0$ and $0 \leq n \leq k-2$ and $(-1)^{k-2}b^{(k-2)}$ is nonincreasing and convex, see [46, Definition 3.4]. In particular, throughout the paper we assume the following.

**Assumption 2.2.** The kernel $0 \neq b \in L^1_{loc}(\mathbb{R}_+)$ is 4-monotone and

$$\lim_{t \to \infty} b(t) = 0.$$  

Furthermore,

$$\rho := 1 + \frac{2}{\pi} \sup\{|\arg \hat{b}(\lambda)|, \ \text{Re} \ \lambda > 0\} \in (1, 2).$$ (2)
It follows from [46, Proposition 3.10] that for 3-monotone and locally integrable kernels $b$, condition (2) is equivalent to
\[
\lim_{t \to 0} \frac{1}{t} \int_0^t sb(s) \, ds < +\infty.
\]

**Remark 2.3.** A simple, but important, example of a kernel $b$ that satisfies Assumption 2.2 is
\[
b(t) = \Gamma(\rho - 1)^{-1} t^{\rho - 2} e^{-\eta t},
\]
with $1 < \rho < 2$ and $\eta \geq 0$.

Under Assumptions 2.1 and 2.2 on $A$ and $b$ it follows that there exists a strongly continuous family $(S(t))_{t \geq 0}$ of resolvents such that the function
\[
u(t) = S(t)u_0, \quad u_0 \in H,
\]
is the unique solution of the Cauchy problem
\[
u(t) = A \int_0^t B(t - s)\nu(s) \, ds + u_0, \quad t \geq 0,
\]
with $B(t) = \int_0^t b(s) \, ds$, see, [46, Corollary 1.2]. Observe that the family of resolvents $(S(t))_{t \geq 0}$ does not have the semigroup property because of the presence of the memory term. However, the family of resolvents $(S(t))_{t \geq 0}$ satisfy certain smoothing properties as shown in [4, Lemma A.4], see also, [32, Proposition 2.5 and Remark 2.6].

**Lemma 2.4.** Suppose the operator $A$ satisfies Assumption 2.1 and the convolution kernel $b$ satisfy Assumptions 2.2. Then
(a) $\|S(t)\|_{L(H, H^d)} \leq Ct^{-\alpha \rho}, \quad t > 0, \quad \alpha \in [0, \frac{1}{\rho}]$;

(b) $\|\dot{S}(t)\|_{L(H, H^A)} \leq Ct^{-\alpha \rho - 1}, \quad t > 0, \quad \alpha \in [0, \frac{1}{\rho}]$.

In the remainder of the section we will investigate deterministic and stochastic convolutions. In order to proceed we first introduce some notation. Let $E$ be a Banach space and let $T > 0$ be fixed. For $1 \leq q < \infty$, and all $\lambda \geq 0$ let $L^q_\lambda(0, T; E)$ denote the space of all integrable functions $\xi : I \to E$ such that
\[
\|\xi\|_{L^q_\lambda(0, T; E)}^q := \int_0^T e^{-\lambda t} |\xi(t)|_E^q \, dt < \infty.
\]
2.1. Deterministic convolutions

Let us denote the convolution of an appropriate function \( \xi : [0, T] \to H \) and family of resolvents \((S(t))_{t \geq 0}\) by

\[
(C\xi)(t) := \int_0^t S(t-r)\xi(r) \, dr, \quad t \in [0, T].
\]

We have the following result.

**Proposition 2.5.** Under the Assumptions 2.1 and 2.2, for any \( 0 < \alpha < \frac{1}{p} \), the operator

\[
C : L^q_\lambda(0, T; H^A_{-\alpha}) \to L^q_\lambda(0, T; H)
\]

is a bounded linear operator. In particular, there exists a constant \( C = C(\alpha, \lambda) > 0 \), with

\[
\lim_{\lambda \to \infty} C(\lambda, \alpha) = 0,
\]

such that for any \( \xi \in L^q_\lambda(0, T; H^A_{-\alpha}) \) we have

\[
\|C\xi\|_{L^q_\lambda(0, T; H)} \leq C \|\xi\|_{L^q_\lambda(0, T; H^A_{-\alpha})}.
\]

**Proof 2.1.** Proposition 2.5 can be shown by straightforward calculations using the Minkowski inequality and the Young inequality for convolutions. Compare with [4, Proof of Theorem 3.3].

2.2. Stochastic convolutions

In the next paragraph we would like to show a similar result for stochastic convolutions. First, we introduce the setting we will use throughout the whole paper. Let \( \mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) \) be a complete filtered probability space with right continuous filtration \( \{\mathcal{F}_t\}_{t \geq 0} \), denoted by by \( \mathbb{F} \). The random perturbation we are considering will be specified via a compensated Poisson random measure.

**Definition 2.6.** Let \( Z \) be a separable Banach space and \( \nu \) be a \( \sigma \)-finite measure on \((Z, \mathcal{B}(Z))\). A (time-homogeneous) Poisson random measure \( \eta \) on \((Z \times \mathbb{R}_+, \mathcal{B}(Z) \otimes \mathcal{B}(\mathbb{R}_+)) \) over \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) (Poisson random measure on \( Z \) for short) with compensator \( \gamma := \nu \otimes \lambda \) is a family \( \eta := \{\eta(\omega, \cdot) : \omega \in \Omega\} \) of nonnegative measures \( \eta(\omega, \cdot) \) on \((Z \times \mathbb{R}_+, \mathcal{B}(Z) \otimes \mathcal{B}(\mathbb{R}_+)) \) such that
(i) the mapping $\omega \to \eta(\omega, \cdot)$ is measurable $(\Omega, \mathcal{F}) \to (M_f(Z \times \mathbb{R}_+), \mathcal{M}_f(Z \times \mathbb{R}_+))$, where $M_f(Z \times \mathbb{R}_+)$ is the set of nonnegative measures on $(Z \times \mathbb{R}_+, \mathcal{B}(Z) \otimes \mathcal{B}(\mathbb{R}_+))$ endowed with the $\sigma$-algebra $\mathcal{M}_f(Z \times \mathbb{R}_+)$ generated by all mappings

$$M_f(Z \times \mathbb{R}_+) \ni \rho \to \rho(\Gamma), \quad \Gamma \in \mathcal{B}(Z) \otimes \mathcal{B}(\mathbb{R}_+);$$

(ii) for each $B \in \mathcal{B}(Z) \otimes \mathcal{B}(\mathbb{R}_+)$, $\eta(B) := \eta(\cdot, B) : \Omega \to \bar{\mathbb{N}}$ is a Poisson random variable with parameter $\gamma(B)$;

(iii) $\eta$ is independently scattered, i.e. if the sets $B_j \in \mathcal{Z} \otimes \mathcal{B}(\mathbb{R}_+)$, $j = 1, \cdots, n$, are disjoint, then the random variables $\eta(B_j), j = 1, \cdots, n$, are independent.

The difference between a time homogeneous Poisson random measure $\eta$ and its compensator $\gamma$, i.e. $\tilde{\eta} = \eta - \gamma$, is called a compensated Poisson random measure. The measure $\nu$ is called intensity measure of $\eta$.

Since it would exceed the scope of the paper, we do not tackle here stochastic integration with respect to Poisson random measures. A short summary of stochastic integration is given e.g. in Breźniak and Hausenblas [11] or in [42], where the stochastic integral is defined for all progressive measurable processes, i.e. for all processes which can be approximated by simple predictable cáglág processes. The following inequality corresponds to the Itô isometry, compare e.g. [22]. Assume that $E$ is a Hilbert space. Then, for any $p \in [1, 2]$ and $q = pn$ for some $n \in \mathbb{N}$, there exists a constant $C > 0$ such that for all progressively measurable processes

$$\xi : \Omega \times [0, T] \to L^p(Z, \nu; E)$$

we have (see e.g. [22, 24])

$$\mathbb{E} \left| \int_0^T \int_Z \xi(s, z) \tilde{\eta}(dz, ds) \right|_E^q \leq C \left\{ \mathbb{E} \int_0^T \int_Z |\xi(s, z)|_E^q \nu(dz) ds + \mathbb{E} \left( \int_0^T \int_Z |\xi(s, z)|_E^p \nu(dz) ds \right)^{\frac{q}{p}} \right\}. \tag{4}$$

By interpolation techniques one can prove inequality (4) for all $q$ with $p \leq q < \infty$. We finish with the following version of the Stochastic Fubini Theorem (see [53]).

7
Theorem 2.7. Assume that $E$ is a Hilbert space and 

$$
\xi : \Omega \times [a, b] \times \mathbb{R}_+ \times Z \to E
$$

is a progressively measurable process. Then, for each $T \in [a, b]$, we have a.s.

$$
\int_a^b \left[ \int_0^T \int_Z \xi(s, r, z) \tilde{\eta}(dz, dr) \right] ds = \int_0^T \int_Z \left[ \int_a^b \xi(s, r, z) ds \right] \tilde{\eta}(dz, dr).
$$

We continue by introducing some notation. As before, fix $1 \leq q < \infty$, $0 < m < \infty$, and a Banach space $E$. Then, $\mathcal{M}_m^{m,q}(0, T; E)$ denotes the space of all progressively measure processes $\xi : \Omega \times [0, T] \to E$ such that

$$
\|\xi\|_{\mathcal{M}_m^{m,q}(0, T; E)} = \mathbb{E} \left( \int_0^T e^{-\lambda t} \|\xi(t)\|^q_E dt \right)^{\frac{m}{q}} < \infty.
$$

With this notation Proposition 2.5 immediately implies the following result.

Corollary 2.8. Under the Assumptions 2.1 and 2.2, for any $0 < \alpha < \frac{1}{p}$, the operator

$$
\mathcal{E} : \mathcal{M}_m^{m,q}(0, T; H_{-\alpha}) \to \mathcal{M}_m^{m,q}(0, T; H)
$$

is a bounded linear operator. In particular, there exists a constant $C = C(\alpha, \lambda) > 0$, with

$$
\lim_{\lambda \to \infty} C(\lambda, \alpha) = 0,
$$

such that for any $\xi \in \mathcal{M}_m^{m,q}(0, T; H_{-\alpha})$ we have

$$
\|\mathcal{E} \xi\|_{\mathcal{M}_m^{m,q}(0, T; H)} \leq C \|\xi\|_{\mathcal{M}_m^{m,q}(0, T; H_{-\alpha})}.
$$

Next, we investigate the properties of the stochastic convolution operator defined, for an appropriate process $\xi : \Omega \times [0, T] \times Z \to E$, by the formula,

$$
\mathcal{S}(\xi) = \left\{ \mathbb{R}_+ \ni t \mapsto \int_0^t \int_Z S(t - s) \xi(s, z) \tilde{\eta}(dz, ds) \right\}. \tag{5}
$$

For any $q, p \geq 1$ and $m > 0$ let $\mathcal{M}_m^{m,q}(0, T; L^p(Z, \nu; E))$ be the space of all progressively measure processes $\xi : \Omega \times [0, T] \to L^p(Z, \nu; E)$ such that

$$
\|\xi\|_{\mathcal{M}_m^{m,q}(0, T; L^p(Z, \nu; E))} := \mathbb{E} \left( \int_0^T e^{-\lambda t} \left( \int_Z |\xi(t, z)|^p_E \nu(dz) \right)^{\frac{m}{p}} dt \right)^{\frac{q}{m}} < \infty.
$$
Proposition 2.9. Let $\eta$ be a Poisson random measure with intensity measure $\nu$. Under the Assumptions 2.1 and 2.2, for any $\frac{1}{q\rho} > \alpha \geq 0$ and $\frac{1}{p\rho} > \alpha_1 \geq 0$ the stochastic convolution

$$\mathcal{G} : \mathcal{M}_{\lambda}^{q,q}(0, T; L^q(Z, \nu; H_{\alpha}^A)) \cap \mathcal{M}_{\lambda}^{q,q}(0, T; L^p(Z, \nu; H_{\alpha_1}^A)) \rightarrow \mathcal{M}_{\lambda}^{q,q}(0, T; H)$$

is linear and bounded. In particular, there exists a constant $C = C(\alpha, \lambda) > 0$, with

$$\lim_{\lambda \rightarrow \infty} C(\lambda, \alpha) = 0,$$

such that for any

$$\xi \in \mathcal{M}_{\lambda}^{q,q}(0, T; L^q(Z, \nu; H_{\alpha}^A)) \cap \mathcal{M}_{\lambda}^{q,q}(0, T; L^p(Z, \nu; H_{\alpha_1}^A))$$

we have

$$|\mathcal{G}\xi|_{\mathcal{M}_{\lambda}^{q,q}(0, T; H)}^q \leq C \left( \frac{1}{\lambda^{1-q\rho}} \int_0^T \int_Z \int_0^t |\xi(s, z)|^q_{H_{\alpha}^A} \nu(dz) ds dt \right)^{\frac{q}{p}}.$$

Proof 2.2. Applying inequality (4) we obtain

$$|\mathcal{G}\xi|_{\mathcal{M}_{\lambda}^{q,q}(0, T; H)}^q \leq C \left( \frac{1}{\lambda^{1-q\rho}} \int_0^T \int_Z \int_0^t |\xi(s, z)|^q_{H_{\alpha}^A} \nu(dz) ds dt \right)^{\frac{q}{p}}.$$

Using the estimates in Lemma 2.4 we get

$$|\mathcal{G}\xi|_{\mathcal{M}_{\lambda}^{q,q}(0, T; H)}^q \leq C \left( \frac{1}{\lambda^{1-q\rho}} \int_0^T \int_Z \int_0^t |\xi(s, z)|^q_{H_{\alpha}^A} \nu(dz) ds dt \right)^{\frac{q}{p}}.$$
Finally, it follows by Fubini’s Theorem and the Young inequality for convolutions, that

\[
|\mathbf{S}^q_{\lambda^q} S_{\lambda^q}(0, T; H) | \\
\leq \frac{C_1}{\lambda^{1-q} p} \mathbb{E} \int_0^T \int_Z e^{-\lambda |\xi(t, z)|^q_{H_{\lambda}}} \nu(dz) \, dt \\
+ \frac{C_2}{\lambda^{1-p} p} \mathbb{E} \left( \int_0^T e^{-\lambda t} \left( \int_Z |\xi(t, z)|^p_{H_{\lambda}} \nu(dz) \right)^{\frac{q}{p}} \, dt \right),
\]

which gives the assertion.

**Proposition 2.10.** Let $E$ be a Hilbert space and $\xi : \Omega \times [0, T] \to L^p(Z, \nu; E)$ be a progressively measurable processes. If $b$ satisfies Assumption 2.2, then the process $t \mapsto \Phi(\xi(t))$ has a càdlàg modification in $E$.

**Proof 2.3.** The statement follows the same way as the proof of [43, Proposition 3], see also, [23] for the original idea for using an unitary dilation. Note that, for any $\omega > 0$, the mapping $t \mapsto e^{-\omega |t|} S(|t|)$ is strongly continuous, positive definite, and $S(0) = I$. This follows from [43, Proposition 6] provided that the function $t \mapsto r_{\omega, \mu}(t) := e^{-\omega |t|} s_{\mu}(|t|), \mu > 0$ is fixed, is positive definite where $s_{\mu}$ is defined by

\[
\dot{s}_{\mu}(t) + \mu(b \ast s_{\mu})(t) = 0, \quad s_{\mu}(0) = 1.
\]

It is easily seen, using in particular the sector condition (2) on $b$, that $s_{\mu}$ is non-negative, continuous and bounded by 1, cf. [46, Corollary 1.2]. We calculate the Fourier transform of $r_{\omega, \mu}$ in terms of the Laplace transform $\hat{\mathbf{S}}_{\mu}$ of $s_{\mu}$ as

\[
\int_{-\infty}^{\infty} e^{-i\beta t} r_{\omega, \mu}(t) \, dt = 2 \Re \hat{s}_{\mu}(\omega + i\beta) = 2 \Re \frac{1}{\omega + i\beta + \hat{b}(\omega + i\beta)} \geq 0,
\]

where the non-negativity of the last term follows from the sector condition (2) on $b$. Hence, by Bochner’s theorem (more precisely, the distributional version, the Bochner-Schwarz theorem), it follows that $r_{\omega, \mu}$ is positive definite being continuous and the Fourier transform of a tempered non-negative measure. Hence $t \mapsto e^{-\omega |t|} S(|t|)$ has a unitary dilation $\{U(t)\}_{t \in \mathbb{R}}$, a strongly continuous
unitary group, on some Hilbert space $E_1 \leftrightarrow E$ with $e^{-\omega |t|} S(|t|) x = \Pi U(t) x$, $x \in H$, where $\Pi : E_1 \rightarrow E$ is the orthogonal projection. Then

$$G(\xi)(t) = e^{\omega t} \int_0^t \int_Z e^{-\omega s} e^{-\omega(t-s)} S(t-s) \xi(s, z) \tilde{\eta}(dz, ds)$$

$$= e^{\omega t} \int_0^t \int_Z e^{-\omega s} \Pi U(t-s) \xi(s, z) \tilde{\eta}(dz, ds)$$

$$= e^{\omega t} \Pi U(t) \int_0^t \int_Z e^{-\omega s} U(-s) \xi(s, z) \tilde{\eta}(dz, ds).$$

The statement then follows from the regularity of stochastic integrals, see, for example, [21, Theorems 1 and 2, pp 181-182].

3. Stochastic Volterra equations with integrable jumps

In this section we start to investigate the existence and uniqueness of a mild solution to the Volterra equation

$$du(t) = \begin{cases} 
A \int_0^t b(t-s) u(s) \, ds 
+ \int_Z G(t, u(t), z) \tilde{\eta}(dz, dt) 
+ \int_{Z_L} G_L(t, u(t), z) \eta_L(dz, dt); & t \in (0, T], \\
u_0 & u(0) = u_0.
\end{cases}$$

(SDE)

Here, $\tilde{\eta}$ is a compensated Poisson random measure on $Z$ with intensity measure $\nu$ and $\eta_L$ is Poisson random measure on $Z_L$ independent to $\tilde{\eta}$ with finite intensity measure $\nu_L$.

In case the stochastic perturbation is Gaussian we expect that the solution process is continuous. In case the stochastic perturbation is Lévy, or more generally given by a Poisson random measure, we do not necessarily expect this, but rather that the solution belongs to the space of Skorokhod functions. If $Y$ denotes a separable and complete metric space and $T > 0$ then the space $\mathbb{D}([0, T]; Y)$ denotes the space of all right continuous functions $x : [0, T] \rightarrow Y$ with left limits. The space of continuous function is usually equipped with the uniform topology. But, since $\mathbb{D}([0, T]; Y)$ is complete but not separable in the uniform topology, we equip $\mathbb{D}([0, T]; Y)$ with the Skorokhod topology.
in which $\mathbb{D}([0,T];Y)$ is both separable and complete. For more information about Skorokhod space and topology we refer to Billingsley’s book [10] or Ethier and Kurtz [23].

**Definition 3.1 (Mild Solution).** We say that $u$ is a mild solution of equation (SDE) in $H_0$, if $u$ belongs $\mathbb{P}$-a.s. to $\mathbb{D}(0,T;H_0)$, $u$ is adapted to $(\mathcal{F}_t)_{t \geq 0}$ and we have $\mathbb{P}$-a.s.,

$$
\int_0^t |S(t-s)F(s,u(s))|_{H_0} \, ds < \infty,
$$
and

$$
\int_0^t \int_Z |S(t-s)G(s,u(s),z)|_{H_0}^p \, \nu(dz) \, ds < \infty.
$$

Furthermore, we suppose that for all $z \in Z_L$, and $r \in [0,T]$, $\mathbb{P}$-a.s.,

$$
|S(t-r)G_L(r-,u(r-),z)|_{H_0} < \infty,
$$

and for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$
u(t) = S(t)u_0 + \int_0^t S(t-s)F(s,u(s)) \, ds
+ \int_0^t \int_Z S(t-s)G(s,u(s),z) \, \tilde{\eta}(dz,ds)
+ \int_0^t \int_{Z_L} S(t-s)G_L(s,u(s),z) \, \eta_L(dz,ds).
$$

First we will show existence and uniqueness of a mild solution when $G_L = 0$; that is, for the case where there are no big jumps, and therefore we consider the equation

$$
du(t) = \left( A \int_0^t b(t-s)u(s) \, ds \right) \, dt + F(t,u(t)) \, dt
+ \int_Z G(t,u(t),z) \, \tilde{\eta}(dz,dt); \quad t \in [0,T],
$$

(SDE1)

Our assumptions on $F$, $G$ and the initial condition are as follows.
Assumption 3.2. Fix $p < q$. We will assume that

(i) there exists a number $\alpha_G < \frac{1}{q \rho}$ such that $G$ considered as a mapping $G : [0, T] \times H \to L^p(Z, \nu; H_{-\alpha_G}^A) \cap L^q(Z, \nu; H_{-\alpha_G}^A)$ is continuous and for all $t \in [0, T]$ it is Lipschitz continuous with Lipschitz constant $L_G$ in the second variable;

(ii) there exists a number $\alpha_F < \frac{1}{\rho}$ such that $F : [0, T] \times H \to H_{-\alpha_F}^A$ is continuous and for all $t \in [0, T]$ it is Lipschitz continuous with Lipschitz constant $L_F$ in the second variable;

(iii) there exists a number $\alpha_I < \frac{1}{q \rho}$ such that $u_0 \in L^q(\Omega; H_{-\alpha_I}^A)$ is $\mathcal{F}_0$-measurable.

Now, we can formulate one of our main results.

Theorem 3.3. Suppose that Assumption 2.1 and Assumption 2.2 are satisfied. Let $\eta$ be a Poisson random measure with intensity measure $\nu$. In addition, suppose that the coefficients $F$ and $G$ and the initial condition $u_0$ and the numbers $\alpha_F, \alpha_G$ and $\alpha_I$ satisfy Assumption 3.2. If

\[ \alpha \geq \alpha_I, \quad \alpha \geq \alpha_G, \text{ and } (\alpha_F - \alpha) \rho < 1 - \frac{1}{q}, \]

then there exists a mild solution of (SDE1) in $H_{-\alpha}$. In particular,

\[ \mathbb{P} \left( u \in D([0, T]; H_{-\alpha}^A) \right) = 1. \]

If, in addition, also

\[ (\alpha_F - \alpha_I) \rho < 1 - \frac{1}{q} \quad \text{and} \quad \alpha_I \geq \alpha_G, \quad (7) \]

then, for each $t \in [0, T]$, we have $u(t) \in L^q(\Omega; H_{-\alpha_I}^A)$.

Proof 3.1. First we will prove the existence and uniqueness of a process $u$ satisfying (6) using Banach’s fixed point theorem. Then, we will investigate the regularity of the trajectories of $u$ in $H_{-\alpha}$; that is, the càdlàg property, to show that $u$ a mild solution in $H_{-\alpha}$ according to Definition 3.1. Finally, we will prove that $u(t) \in L^q(\Omega; H_{-\alpha_I}^A)$ whenever (7) holds.
Step I. Put
\[ \mathfrak{X}_\lambda := \mathcal{M}_\lambda^{q,q}(0,T;H). \]
For fixed \( u_0 \in L^q(\Omega;H^A_{-\alpha I}) \) let us define the integral operator
\[ (\mathfrak{F}u)(t) := (\mathfrak{T}u_0)(t) + (\mathfrak{S}u)(t) + (\mathfrak{G}u)(t), \quad t \in [0,T], \ u \in \mathfrak{X}_\lambda, \]
where \( \mathfrak{T} : H^A_{-\alpha I} \to \mathfrak{X} \) is defined by
\[ (\mathfrak{T}u_0)(t) := S(t)u_0, \quad u_0 \in H^A_{-\alpha I}, \]
\( \mathfrak{S} = \mathcal{C} \circ \mathfrak{F} \), where \( \mathcal{C} \) is defined in (3), and \( \mathfrak{G} = \mathfrak{S} \circ g \), where \( \mathfrak{S} \) is defined in (5).
We will show that there exists a \( \lambda > 0 \) such that for any \( u_0 \in L^q(\Omega;H^A_{-\alpha I}) \), \( \mathfrak{F} \) maps \( \mathfrak{X}_\lambda \) into \( \mathfrak{X}_\lambda \), and that \( \mathfrak{F} \) is a strict contraction on \( \mathfrak{X}_\lambda \).
The fact that \( \mathfrak{T} \) maps \( H^A_{-\alpha I} \) in \( \mathfrak{X}_\lambda \), follows from the Lemma 2.4. In particular, a straightforward calculations gives
\[ \| \mathfrak{T}u_0 \|_{L^q(0,T;H)} \leq \int_0^T e^{-\lambda t} |S(t)u_0|_H^q \ dt \leq C \int_0^T e^{-\lambda t} t^{-\rho\alpha_I^q} dt \ |u_0|_{H^A_{-\alpha_I}}^q, \]
and therefore,
\[ \| \mathfrak{T}u_0 \|_{L^q(0,T;H)} \leq CT(1 - \rho\alpha_I^q)^q \lambda^{\rho\alpha_I^q - \frac{1}{q}} |u_0|_{H^A_{-\alpha_I}}. \]  
(9)
Corollary 2.8 and the Lipschitz continuity of \( F \) imply that the operator
\[ \mathfrak{F} : \mathfrak{X}_\lambda \to \mathfrak{X}_\lambda. \]  
(10)
is Lipschitz continuous with Lipschitz constant \( L_F C(\alpha_F, \lambda) \) such that \( \lim_{\lambda \to \infty} C(\alpha_F, \lambda) = 0. \) Proposition 2.9 and the Lipschitz continuity of \( G \) implies that the operator
\[ \mathfrak{S} : \mathfrak{X}_\lambda \to \mathfrak{X}_\lambda. \]  
(11)
is Lipschitz continuous too with Lipschitz constant \( L_G C(\alpha_G, \lambda) \) where
\[ \lim_{\lambda \to \infty} C(\alpha_G, \lambda) = 0. \]
Hence, for all \( u_0 \in H^A_{-\alpha_I} \) and for all \( u \in \mathfrak{X}_\lambda \) we have
\[ \mathfrak{F}u = \mathfrak{T}u_0 + \mathfrak{S}u + \mathfrak{G}u \in \mathfrak{X}_\lambda. \]
In particular, the operator \( \mathfrak{F} \) is Lipschitz continuous with Lipschitz constant \( L_\lambda \), such that \( L_\lambda \to 0 \) as \( \lambda \to \infty \). Hence, for \( \lambda \) sufficiently large, there exists a fixed point in \( \mathfrak{X}_\lambda \).
Step II. Next, we show that $u$ has a càdlàg in modification in $H_{-\alpha}$. Notice that the resolvent family $S$ is strongly continuous in $H_{-\alpha}$; that is, for all $x \in H_{-\alpha}$ the mapping

$$[0, \infty) \ni t \mapsto S(t)x \in H_{-\alpha}$$

is continuous. As $u_0 \in H_{-\alpha}$ $\mathbb{P}$-a.s. we have that

$$\mathbb{P} \left( \lim_{h \to 0} |[S(t+h) - S(t)]u_0|_{H_{-\alpha}} = 0 \right) = 1. \quad (12)$$

Next we show that the process

$$[0, \infty) \ni t \mapsto \mathcal{F}u(t) = \int_0^t S(t-s)F(s, u(s))\, ds$$

has even a continuous modification in $H_{-\alpha}$. Let $0 \leq r < t \leq T$ and $\sigma > 1$ and $\delta > 1$ arbitrary. Then

$$E \left\| \mathcal{F}u(t) - \mathcal{F}u(r) \right\|_{H_{-\alpha}}^\delta \leq C \left( E \left\| \int_r^t S(t-s)F(s, u(s))\, ds \right\|_{H_{-\alpha}}^\delta + E \left\| \int_0^r (S(t-s) - S(r-s))F(s, u(s))\, ds \right\|_{H_{-\alpha}}^\delta \right) = (e_1)^\delta + (e_2)^\delta. \quad (13)$$

To estimate for $e_1$ we use Hölder’s inequality and, from Step I, the fact that $u \in \mathcal{M}_{q,q}^q(0, T; H)$ together with the Lipschitz continuity of $F$ and Jensen’s inequality, to calculate

$$e_1 \leq E \int_r^t \left\| S(t-s) \right\|_{L(H_{-\alpha}, H_{-\alpha})} \left\| F(s, u(s)) \right\|_{H_{-\alpha}} \, ds \quad (14)$$

$$\leq C E \int_r^t (t-s)^{(\alpha_F - \alpha)\rho} (1 + \|u(s)\|) \, ds \leq C \left( \int_r^t (t-s)^{(\alpha_F - \alpha)\rho \frac{1}{q} - \alpha \frac{1}{q} + \lambda_0} ds \right)^\frac{q-1}{q} \left( \int_0^r (1 + E\|u(s)\|^q) \, ds \right)^{\frac{1}{q}} \quad (15)$$

$$\leq C (t-r)^{(-\alpha_F - \alpha)\rho + \frac{q-1}{q}} \lambda_0^{\frac{2\alpha - 1}{q}}, \quad (16)$$

provided that $(-\alpha_F - \alpha)\rho \frac{q}{q-1} > -1$, that is, $\alpha_F - \alpha)\rho < 1 - \frac{1}{q}$. Next we bound $e_2$. We use Fubini’s Theorem, Hölder’s inequality and, from Step I,
the fact that \( u \in \mathcal{M}_\lambda^{q, q}(0, T; H) \) together with the Lipschitz continuity of \( F \) and Jensen’s inequality to get

\[
e_2 \leq \mathbb{E} \int_0^r \int_0^t \| \hat{S}(v - s) \|_{L(H_{-\alpha F}, H_{-\alpha})} dv \| F(s, u(s)) \|_{H_{-\alpha F}} ds
\]

\[
\leq C \mathbb{E} \int_0^r \int_0^r (v - s)^{(-\alpha F - \alpha) \rho - 1} (1 + \| u(s) \|) ds dv
\]

\[
\leq C \int_0^r \left( \int_0^r (v - s)^{(-\alpha F - \alpha) \rho - 1} \frac{q - 1}{q} - \frac{q - 1}{q} \right) ds \left( \int_0^r \left( 1 + \mathbb{E} \| u(s) \|^q \right) ds \right) \frac{1}{q} dv
\]

\[
\leq C \int_0^r v^{(-\alpha F - \alpha) \rho - 1} + \frac{q - 1}{q} \right) ds \left( 1 + \| u(s) \|^q \right) ds \right) \frac{1}{q} dv \leq C(t - r)^{-\alpha F - \alpha + \frac{q - 1}{q} \right),
\]

provided that \(-\alpha F - \alpha\) \( + \frac{q - 1}{q} > 0\), that is, \((\alpha F - \alpha) \rho < 1 - \frac{1}{q}\). Therefore, choosing \(\delta\) large enough so that \(\delta(-\alpha F - \alpha) \rho + \frac{q - 1}{q} > 1\), it follows from Kolomogorov continuity theorem, (see, e.g., [34, Theorem 1.4.1]), that the process \([0, \infty) \ni t \mapsto \mathfrak{y} u(t)\) has a continuous modification in \(H_{-\alpha}^A\).

It remains to show that the process

\[
[0, \infty) \ni t \mapsto \mathfrak{G} u(t) = \int_0^t \int_Z S(t - s) G(s, u(s), z) \eta(dz, ds)
\]

has a càdlàg modification in \(H_{-\alpha}^A\), but this follows from Proposition [2.10] as \(\alpha \geq \alpha_G\).

**Step III.** Finally, we have to show that under the additional condition \((3.2)\) the random variable \(u(t)\) is \(H_{-\alpha_t}^A\) -valued for any \(t \in [0, T]\). In fact, we will show that there exists a constant \(C(\lambda) > 0\) such that for any process \(\xi \in \mathcal{X}_\lambda\) and \(t \in (0, T)\)

\[
\mathbb{E} |(3 \xi)(t)|_{H_{-\alpha_t}^A}^q \leq C \left( 1 + |u_0|_{H_{-\alpha_t}^A} + C(\lambda) |\xi|_{\mathcal{X}_\lambda} \right).
\]

(17)

We have that

\[
\mathbb{E} |u(t)|_{H_{-\alpha_t}^A}^q \leq C \left( \mathbb{E} |S(t) u_0|_{H_{-\alpha_t}^A}^q + \mathbb{E} \left| \int_0^t S(t - s) F(s, u(s)) ds \right|_{H_{-\alpha_t}^A}^q \right. 
\]

\[
\left. + \mathbb{E} \left| \int_0^t \int_Z S(t - s) G(s, u(s), z) \eta(dz, ds) \right|_{H_{-\alpha_t}^A}^q \right).
\]

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Since $S(t) : H_{-\alpha}^A \to H_{-\alpha}^A$ is a bounded operator,
\[
\mathbb{E}|S(t)u_0|^q_{H_{-\alpha}^A} \leq C|u_0|^q_{H_{-\alpha}^A}.
\]

Next, we will treat the second summand. Here, Minkowski’s inequality, Lemma 2.4 and the Lipschitz property of $F$ taking into account that (7) holds together with Hölder’s inequality give
\[
\mathbb{E}
\left|
\int_0^t S(t-s)F(s,u(s)) \, ds
\right|^q_{H_{-\alpha}^A}
\leq \mathbb{E}
\left|
\int_0^t |S(t-s)F(s,u(s))|_{H_{-\alpha}^A} \, ds
\right|^q
\leq \mathbb{E}
\left|
\int_0^t (t-s)^{-((\alpha_F-\alpha)\wedge 0)\rho} |F(s,u(s))|_{H_{-\alpha}^A} \, ds
\right|^q
\leq C \left(1 + \mathbb{E}
\int_0^t |u(s)|^q_H \, ds\right).
\]

Since the RHS can be estimated by $C(1 + |u|^q_{L^q_H})$ we now bound the third summand. Inequality (4), gives
\[
\mathbb{E}
\left|
\int_0^t \int_Z S(t-s)G(s,u(s),z)\tilde{\eta}(dz,ds)
\right|^q_{H_{-\alpha}^A}
\leq \mathbb{E}
\left|
\int_0^t \int_Z S(t-s)G(s,u(s),z)\nu(dz) \, ds
\right|^q_{H_{-\alpha}^A}
\leq \mathbb{E}
\left|
\int_0^t \int_Z S(t-s)G(s,u(s),z)\nu(dz) \, ds
\right|^q_{H_{-\alpha}^A}
\leq C(t)\mathbb{E}
\int_0^t \left(\int_Z |S(t-s)G(s,u(s),z)|_{H_{-\alpha}^A}^p \, \nu(dz)\right)^{\frac{q}{p}} \, ds.
\]

Since $S(t) : H_{-\alpha}^A \to H_{-\alpha}^A$ is bounded, we get
\[
\mathbb{E}
\left|
\int_0^t \int_Z S(t-s)G(s,u(s),z)\tilde{\eta}(dz,ds)
\right|^q_{H_{-\alpha}^A}
\leq \mathbb{E}
\int_0^t |G(s,u(s),z)|_{L^q(Z,\nu;H_{-\alpha}^A)}^q \, ds + \mathbb{E}
\int_0^t |G(s,u(s),z)|_{L^p(Z,\nu;H_{-\alpha}^A)}^p \, ds.
\]
Finally, using the fact that $\alpha_I \geq \alpha_G$ we get, by the Lipschitz continuity of $G$, that

$$\mathbb{E} \left| \int_0^t \int_Z S(t-s)G(s,u(s),z)\tilde{\eta}(dz,ds) \right|_{H^\alpha_I}^q \leq C \left( 1 + \mathbb{E} \int_0^t |u(s)|_H^q \, ds \right) = C (1 + C(\lambda)|u|_{X^\lambda}),$$

which completes the proof of assertion (17) and also the proof of the theorem.

4. The stochastic Volterra equations with non-integrable jumps

As in the previous section, let $Z$ and $Z_L$ be two Banach spaces, $\tilde{\eta}$ be a compensated Poisson random measure on $Z$ with intensity measure $\nu$, and $\eta_L$ be a Poisson random measure on $Z_L$, independent to $\tilde{\eta}$ with finite intensity measure $\nu_L$. In applications, the first Poisson random measure will be of infinite activity capturing the small jumps, the second one is of finite activity capturing the large jumps.

In order to realize the independent random measures we consider $\tilde{\eta}$ be a compensated Poisson random measure on $(Z \times \mathbb{R}_+, \mathcal{B}(Z) \otimes \mathcal{B}(\mathbb{R}_+))$ over $\mathfrak{A}^S = (\Omega^S, \mathcal{F}^S, \{\mathcal{F}^S_t\}_{t \in [0,T]}, \mathbb{P}^S)$ with intensity measure $\nu$ where

$$\mathcal{F}^S = \sigma\{\eta(B, [0,s]) : B \in \mathcal{B}(Z), s \in [0,T]\}$$

and

$$\mathcal{F}^S_t = \sigma\{\eta(B, [0,s]) : B \in \mathcal{B}(Z), s \in [0,t]\}, 0 \leq t \leq T.$$  

Furthermore, let $\eta_L$ be a Poisson random measure on $(Z_L \times \mathbb{R}_+, \mathcal{B}(Z_L) \otimes \mathcal{B}(\mathbb{R}_+))$ over $\mathfrak{A}^L = (\Omega^L, \mathcal{F}^L, \{\mathcal{F}^L_t\}_{t \in [0,T]}, \mathbb{P}^L)$ with finite intensity measure $\nu_L$ where

$$\mathcal{F}^L = \sigma\{\eta(B, [0,s]) : B \in \mathcal{B}(Z_L), s \in [0,T]\}$$

and

$$\mathcal{F}^L_t = \sigma\{\eta(B, [0,s]) : B \in \mathcal{B}(Z_L), s \in [0,t]\}, 0 \leq t \leq T.$$  

Let $\Omega := (\Omega^S \times \Omega^L)$, $\mathcal{F} := \mathcal{F}^S \otimes \mathcal{F}^L$, $\mathcal{F}_t := \mathcal{F}^S_t \otimes \mathcal{F}^L_t$ and $P = P^S \otimes P^L$. With an abuse of notation we denote by $\tilde{\eta}$ the compensated Poisson random measure on $(Z \times \mathbb{R}_+, \mathcal{B}(Z) \otimes \mathcal{B}(\mathbb{R}_+))$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})$ defined by $\Omega \ni (\omega^S, \omega^L) \mapsto \tilde{\eta}(\omega^S, \cdot)$ and by $\eta_L$ the Poisson random measure on $(Z_L \times \mathbb{R}_+, \mathcal{B}(Z_L) \otimes \mathcal{B}(\mathbb{R}_+))$ over $(\Omega^L, \mathcal{F}^L, \{\mathcal{F}^L_t\}_{t \in [0,T]}, \mathbb{P}^L)$. 

Finally, using the fact that $\alpha_I \geq \alpha_G$ we get, by the Lipschitz continuity of $G$, that

$$\mathbb{E} \left| \int_0^t \int_Z S(t-s)G(s,u(s),z)\tilde{\eta}(dz,ds) \right|_{H^\alpha_I}^q \leq C \left( 1 + \mathbb{E} \int_0^t |u(s)|_H^q \, ds \right) = C (1 + C(\lambda)|u|_{X^\lambda}),$$

which completes the proof of assertion (17) and also the proof of the theorem. 

4. The stochastic Volterra equations with non-integrable jumps

As in the previous section, let $Z$ and $Z_L$ be two Banach spaces, $\tilde{\eta}$ be a compensated Poisson random measure on $Z$ with intensity measure $\nu$, and $\eta_L$ be a Poisson random measure on $Z_L$, independent to $\tilde{\eta}$ with finite intensity measure $\nu_L$. In applications, the first Poisson random measure will be of infinite activity capturing the small jumps, the second one is of finite activity capturing the large jumps.

In order to realize the independent random measures we consider $\tilde{\eta}$ be a compensated Poisson random measure on $(Z \times \mathbb{R}_+, \mathcal{B}(Z) \otimes \mathcal{B}(\mathbb{R}_+))$ over $\mathfrak{A}^S = (\Omega^S, \mathcal{F}^S, \{\mathcal{F}^S_t\}_{t \in [0,T]}, \mathbb{P}^S)$ with intensity measure $\nu$ where

$$\mathcal{F}^S = \sigma\{\eta(B, [0,s]) : B \in \mathcal{B}(Z), s \in [0,T]\}$$

and

$$\mathcal{F}^S_t = \sigma\{\eta(B, [0,s]) : B \in \mathcal{B}(Z), s \in [0,t]\}, 0 \leq t \leq T.$$  

Furthermore, let $\eta_L$ be a Poisson random measure on $(Z_L \times \mathbb{R}_+, \mathcal{B}(Z_L) \otimes \mathcal{B}(\mathbb{R}_+))$ over $\mathfrak{A}^L = (\Omega^L, \mathcal{F}^L, \{\mathcal{F}^L_t\}_{t \in [0,T]}, \mathbb{P}^L)$ with finite intensity measure $\nu_L$ where

$$\mathcal{F}^L = \sigma\{\eta(B, [0,s]) : B \in \mathcal{B}(Z_L), s \in [0,T]\}$$

and

$$\mathcal{F}^L_t = \sigma\{\eta(B, [0,s]) : B \in \mathcal{B}(Z_L), s \in [0,t]\}, 0 \leq t \leq T.$$  

Let $\Omega := (\Omega^S \times \Omega^L)$, $\mathcal{F} := \mathcal{F}^S \otimes \mathcal{F}^L$, $\mathcal{F}_t := \mathcal{F}^S_t \otimes \mathcal{F}^L_t$ and $P = P^S \otimes P^L$. With an abuse of notation we denote by $\tilde{\eta}$ the compensated Poisson random measure on $(Z \times \mathbb{R}_+, \mathcal{B}(Z) \otimes \mathcal{B}(\mathbb{R}_+))$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})$ defined by $\Omega \ni (\omega^S, \omega^L) \mapsto \tilde{\eta}(\omega^S, \cdot)$ and by $\eta_L$ the Poisson random measure on $(Z_L \times \mathbb{R}_+, \mathcal{B}(Z_L) \otimes \mathcal{B}(\mathbb{R}_+))$ over $(\Omega^L, \mathcal{F}^L, \{\mathcal{F}^L_t\}_{t \in [0,T]}, \mathbb{P}^L)$.
In this section, we will show that there exists a unique global mild solution over \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})\) even in the case of unbounded jumps; i.e., we consider the equation

\[
\begin{aligned}
du(t) &= \left( A \int_0^t b(t-s)u(s) \, ds \right) \, dt + F(t, u(t)) \, dt \\
&\quad + \int_Z G(t, u(t)z) \, \tilde{\eta}(dz, dt) + \int_{Z_L} G_L(t, u(t), z) \, \eta_L(dz, dt); \quad t \in (0, T],
\end{aligned}
\]

(18)

Here we will use the representation of the Lévy process or compound Poisson process associated with \(\eta_L\) in terms of a sum over all its jumps. If there would be no memory term, one would simply glue together the solutions between the large jumps. Since we have a non-trivial memory term, the solution does not generate a Markov process and it is not possible to glue together the solutions. Another possibility would be to adopt the approach taken in [42, Section 9.7] and truncate the intensity measure \(\nu\) and employ a stopping time argument. However, then the conditions on \(G_L\) would have to be strengthen significantly which we would like to avoid. Therefore, we take a different approach and we first prove existence and uniqueness of a solution given the large jumps and show thereafter that the argument remains valid when the jumps are the jumps of a Lévy process.

To this end, we suppose that the Poisson random measure \(\eta_L\) with finite intensity measure \(\nu_L\) on \(Z_L\) is constructed the following way (see, [42, Theorem 6.4]). Let \(\sigma = \nu_L(Z_L)\), and let \(\{\tau_n : n \in \mathbb{N}\}\) be a family of independent exponential distributed real-valued random variables on \(\Omega^L\) with parameter \(\sigma\). Consider

\[
T_n = \sum_{j=1}^{n} \tau_j, \quad n \in \mathbb{N},
\]

(19)

and let \(\{N(t) : t \geq 0\}\) be the counting process defined by

\[
N(t) := \sum_{j=1}^{\infty} 1_{[T_j, \infty)}(t), \quad t \geq 0.
\]
Observe that for any $t > 0$, $N(t)$ is a Poisson distributed random variable with parameter $\sigma t$. Let $\{Y_n : n \in \mathbb{N}\}$ be a family of independent, $\frac{1}{\sigma} \nu_L$-distributed, $Z_L$-valued random variables on $\Omega_L$. Then,

$$\eta_L = \sum_{j=1}^{\infty} \delta_{(Y_j,T_j)}$$

is a Poisson random measure with intensity measure $\nu_L$ and

$$\int_0^t \int_{Z_L} S(t-s) G_L(s,u(s),z) \eta_L(dz,ds) = \sum_{i=1}^{N(t)} 1_{[T_i,T_i]}(t) S(t-T_i) G_L(T_i-,u(T_i-),Y_i), \quad t \in [0,T].$$

In the following, we will use the same notation as in the Section 3. We introduce the following assumption on the finite intensity measure $\nu_L$.

**Assumption 4.1.** For some $C, \beta > 0$, the measure $\nu_L$ satisfies

$$\nu_L(\{z \in Z_L : |z| > x\}) \leq C x^{-\beta}$$

We make the following assumption on the mapping $G_L$.

**Assumption 4.2.** The mapping

$$G_L : [0,T] \times H^A_{-\alpha_I} \times Z_L \rightarrow H^A_{-\alpha_I}, \quad (t,x,z) \mapsto G_L(t,x,z),$$

is continuous and Lipschitz continuous in the second variable with Lipschitz constant $L_{G_L}(z)$, $z \in Z_L$, uniformly in $t \in [0,T]$, such that $L_{G_L}(z) \leq M(1 + \|z\|_{Z_L})$, $z \in Z_L$.

The main result of this section is as follows.

**Theorem 4.3.** Suppose that Assumption 2.1 and Assumption 2.2 are satisfied and that and suppose that $\nu_L$ satisfies Assumption 4.1. In addition, suppose that the data in (18) satisfy Assumption 3.2 and Assumption 4.2, and that

$$(\alpha_F - \alpha_I) \rho < 1 - \frac{1}{q} \quad \text{and} \quad \alpha_I \geq \alpha_G.$$
If
\[ \alpha \geq \alpha_I, \quad \alpha \geq \alpha_G, \quad \text{and} \quad (\alpha_F - \alpha) \rho < 1 - \frac{1}{q}, \]
then there exists a mild solution of (18) in \( H_{-\alpha} \). In particular,
\[ \mathbb{P}(u \in \mathcal{D}([0, T]; H^A_{-\alpha})) = 1. \]
Finally, for any \( t \in [0, T] \), \( u(t) \) is a \( H^A_{-\alpha_I} \)-valued random variable.

**Proof 4.1.** As indicated above, the proof proceeds in 2 steps.

**Step I.** In the first step, we will show that for any deterministic \( N < \infty \), \( \{T_1, \ldots, T_N\} \subset [0, T] \), \( T_i < T_{i+1}, \ i = 1, \ldots, N - 1 \), and \( \{Y_1, \ldots, Y_N\} \subset Z_L \) there exists a process \( u \) over \( \mathfrak{A}^S \) such that \( u \) solves \( \mathbb{P}^S \)-a.s. the integral equation

\[
(21) \quad u(t) = S(t)u_0 + \int_0^t S(t-s)F(s, u(s)) \, ds \\
+ \int_0^t \int Z S(t-s)G(s, u(s), z) \, \tilde{\eta}(dz, ds) \\
+ \sum_{i=1}^N 1_{[T_i, T_i]}(t) S(t-T_i)G_L(T_i^- , u(T_i^-), Y_i).
\]

In this step we will use the notation \( \mathbb{E} \) for the expectation over \( \Omega^S \) to shorten notation. Using the notation introduced in Step I of the proof of Theorem 3.3 we define the integral operator \( Z_0 \) by

\[
(22) \quad (Z_0(\xi; (x_i)_{i=1}^N))(t) := (\mathfrak{T} u_0)(t) + (\mathfrak{F} \xi)(t) \\
+ (\mathfrak{G} \xi)(t) + \sum_{i=1}^N 1_{[T_i, T_i]}(t) (\mathfrak{T} T_i G_L(T_i^- , x_i, Y_i))(t), \quad t \in [0, T], \xi \in \mathfrak{X}_\lambda.
\]

Here \( \mathfrak{T}_{T_i} : H^A_{-\alpha_I} \to \mathfrak{X}_\lambda \) is defined by

\[
(\mathfrak{T}_{T_i} x)(t) := S(t-T_i)1_{[T_i, T_i]}(t)x, \quad x \in H^A_{-\alpha_I}.
\]
Next, we define a space and an operator to apply a fixed point argument. In order to do this, let

$$L_{\lambda}(T_1, \ldots, T_N) := \left\{ \left( x_i \right)_{i=1}^N : x_i \in L^q(\mathfrak{A}^S; H^A_{-\alpha_i}) \right\},$$

for all $i = 1, \ldots, N$ the random variable $x_i$ is $\mathcal{F}_{T_i}^S$ measurable.

with norm

$$\left| \left( x_i \right)_{i=1}^N \right|_{L_{\lambda}} := \left( \sum_{i=1}^N e^{-\lambda T_i} \mathbb{E} \left| x_i \right|_{H^A_{-\alpha_i}}^q \right)^{\frac{1}{q}}.$$

Secondly, let

$$X(\lambda, K) := \mathfrak{X}_\lambda \times L_{\lambda}(T_1, \ldots, T_N),$$

with norm

$$\left| (\xi, (x_i)_{i=1}^N) \right|_{X(\lambda, K)} := K \left| \xi \right|_{X_{\lambda}} + \left| (x_i)_{i=1}^N \right|_{L_{\lambda}}, \quad (\xi, (x_i)_{i=1}^N) \in X(\lambda, K).$$

Let us define a operator $\Theta$ acting on $X(\lambda, K)$ by putting

$$\Theta(\xi; (x_i)_{i=1}^N) = \left( \mathfrak{F}_0(\xi, (x_i)_{i=1}^N); \left( \mathfrak{F}_0(\xi, (x_i)_{i=1}^N)(T_j^-) \right)_{j=1}^N \right),$$

for $(\xi, (x_i)_{i=1}^N) \in X(\lambda, K)$.

We will show in the first part, that for any $K > 0$ and $\lambda > 0$ the mapping $\Theta$ maps $X(\lambda, K)$ into itself, and, in the second part, that there exist numbers $K, \lambda > 0$ such that $\Theta$ is a contraction on $X(\lambda, K)$.

First note that there is a modification of $t \to \mathfrak{F}_0(\xi, (x_i)_{i=1}^N)(t)$ that is càdlàg in $H_{-\alpha}$ (in particular, we may take $\alpha = \alpha_I$) when $\xi \in \mathfrak{X}_\lambda$ and $(x_i)_{i=1}^N \in L_{\lambda}$ as Step II of Theorem 3.3 together with the assumption on $G_L$ show. Therefore, the one sided limits $\mathfrak{F}_0(\xi, (x_i)_{i=1}^N)(T_j^-)$ exist in $H_{-\alpha_I}$.

Similarly as before one can show that first $\mathfrak{F}_0(\cdot, (x_i)_{i=1}^N)$ maps $\mathfrak{X}_\lambda$ into $\mathfrak{X}_\lambda$ for given $(x_i)_{i=1}^N \in L_{\lambda}$. First, note, by estimate (9) we have

$$\left| \Sigma u_0 \right|_{X_{\lambda}} \leq C \lambda^{\frac{1}{q} - \alpha_I \rho} \left| u_0 \right|_{H^A_{-\alpha_I}}.$$

Next, Corollary 2.8, Proposition 2.9 and the Lipschitz continuity of $F$ and $G$ imply that the operators $\mathfrak{F}$ and $\mathfrak{G}$ map $\mathfrak{X}_\lambda$ into itself. Finally, from estimate (9) and Assumption 4.2 it follows that for each $i = 1, \ldots, N$ the terms
\( \mathcal{T}_iG_i(T_i^-, x_i, Y_i) \) belong to \( \mathcal{X}_\lambda \). Indeed, one can show by similar calculation as done for estimate (9), that for \( (y_i)^{N}_{i=1} \) the following estimate holds:

\[
\| \mathcal{T}_i y_i \|_{\mathcal{X}_\lambda}^q \leq \int_{T_i}^{T} e^{-\lambda t} |\mathcal{S}(t - T_i) y_i|_H^q \, dt \\
\leq \, C e^{-\lambda T_i} \int_{T_i}^{T} e^{-\lambda (t - T_i)} (t - T_i)^{-\rho_1 q} dt |y_i|_{H_{-\alpha_1}}^q.
\]

Therefore,

\[
E \| \mathcal{T}_i y_i \|_{\mathcal{X}_\lambda}^q \leq \, C \Gamma(1 - \rho_1 q) \lambda^{\rho_1 q - 1} e^{-\lambda T_i} E |y_i|_{H_{-\alpha_1}}^q.
\]

Taking the sum over \( i = 1, \ldots, N \) we get

\[
\sum_{i=1}^{N} E \| \mathcal{T}_i y_i \|_{\mathcal{X}_\lambda}^q \leq \, C \Gamma(1 - \rho_1 q) \lambda^{\rho_1 q - 1} \sum_{i=1}^{N} e^{-\lambda T_i} E |y_i|_{H_{-\alpha_1}}^q.
\]

Putting \( y_i = G_i(T_i^-, x_i, Y_i) \) and using Assumption 4.2 we get

\[
\left\| \sum_{i=1}^{N} 1_{[T_i, T]}(\cdot)(\mathcal{T}_iG_i(T_i^-, x_i, Y_i))(\cdot) \right\|_{\mathcal{X}_\lambda}^q \leq \, C \sum_{i=1}^{N} E \| \mathcal{T}_iG_i(T_i^-, x_i, Y_i) \|_{\mathcal{X}_\lambda}^q \\
\leq \, C \Gamma(1 - \rho_1 q) \lambda^{\rho_1 q - 1} \sum_{i=1}^{N} e^{-\lambda T_i} (1 + L_{G_i}(Y_i))^q E |x_i|_{H_{-\alpha_1}}^q,
\]

where \( C \) depends on \( N \). Thus, we have shown that

\[
\| \mathcal{Y}_0(\xi, (x_i)^{N}_{i=1}) \|_{\mathcal{X}_\lambda} \leq \, C \left( \|u_0\|_{H_{-\alpha_1}} + |\xi|_{\mathcal{X}_\lambda} + |(\xi, (x_i)^{N}_{i=1})|_{\mathcal{X}_\lambda} \right).
\]

It remains to show that

\[
\sum_{j=1}^{N} e^{-\lambda T_j} E \| \mathcal{Y}_0(\xi, (x_i)^{N}_{i=1})(T_j^-) \|_{H_{-\alpha_1}}^q < \infty.
\]

But this follows by estimate (17) in Step III of the proof of Theorem 3.3.

Next, we show that there exist numbers \( \lambda > 0 \) and \( K > 0 \) such that \( \Theta \) is a contraction on \( H(\lambda, K) \). First, we will analyse \( \mathcal{Y}_0(\xi, (x_i)^{N}_{i=1}) \). Let
Thus, setting $F$ of proof of Theorem 3.3 we know from Corollary 2.8 and the Lipschitz continuity of $F$ that the operator

$$\mathfrak{F}: \mathfrak{X}_\lambda \rightarrow \mathfrak{X}_\lambda.$$  \hfill (23)

is Lipschitz continuous with Lipschitz constant $L_F C(\alpha_F, \lambda)$ with

$$\lim_{\lambda \to \infty} C(\alpha_F, \lambda) = 0.$$  

Again, Proposition 2.9 and the Lipschitz continuity of $G$ implies that the operator

$$\mathfrak{G}: \mathfrak{X}_\lambda \rightarrow \mathfrak{X}_\lambda.$$  \hfill (24)

is Lipschitz continuous as well with Lipschitz constant $L_G C(\alpha_G, \lambda)$ with $\lim_{\lambda \to 0} C(\alpha_G, \lambda) = 0$. It remains to consider the mappings

$$t \mapsto 1_{[T_i, T]}(t)\Sigma T_i G_L(T_i^-, x_i, Y_i)(t), \quad t \in [0, T], \xi \in \mathfrak{X}_\lambda, i = 1, \ldots, N.$$  

Similar calculation as for (9) gives for $y_i \in \mathfrak{X}_\lambda$

$$\|\Sigma_T y_i\|_{\mathfrak{X}_\lambda}^q \leq \int_{T_i}^T e^{-\lambda t} |S(t-T_i) y_i|_H^q \, dt$$

$$\leq e^{-\lambda T_i} \int_{T_i}^T e^{-\lambda(t-T_i)} (t-T_i)^{-\rho \alpha_i q} \, dt \|y_i\|_{H_{-\alpha_i}}^q$$

and therefore,

$$\|\Sigma_T y_i\|_{\mathfrak{X}_\lambda}^q \leq \Gamma (1 - \rho \alpha_i q) \lambda^{\rho \alpha_i q - 1} e^{-\lambda T_i} \|y_i\|_{H_{-\alpha_i}}^q.$$  \hfill (25)

Thus, setting $y_i = G_L(T_i^-, x_i, Y_i)$, we get

$$\|1_{[T_i, T]}(\cdot)\Sigma_T G_L(T_i^-, x_i, Y_i)(\cdot) - 1_{[T_i, T]}(\cdot)\Sigma_T G_L(T_i^-, x_i, Y_i)(\cdot)\|_{\mathfrak{X}_\lambda}^q$$

$$\leq C \lambda^{\rho \alpha_i q - 1} e^{-\lambda T_i} \mathbb{E} \|G_L(T_i^-, x_i, Y_i) - G_L(T_i^-, x_i, Y_i)\|_{H_{-\alpha_i}}^q.$$  

The Lipschitz property of $G_L$ in the second variable gives

$$\|1_{[T_i, T]}(\cdot)\Sigma_T G_L(T_i^-, x_i, Y_i)(\cdot) - 1_{[T_i, T]}(\cdot)\Sigma_T G_L(T_i^-, x_i, Y_i)(\cdot)\|_{\mathfrak{X}_\lambda}^q$$

$$\leq C \lambda^{\rho \alpha_i q - 1} e^{-\lambda T_i} L_G(Y_i) \mathbb{E} |x_i - x_i|_{H_{-\alpha_i}}^q.$$
Taking the sum over $i = 1, \ldots, N$ gives
\[
\left\| \sum_{i=1}^{N} 1_{[T_i, T_j]}(\cdot) \mathbf{S}_{T_i} G_L(T_i^-, x_i^1, Y_i) - \sum_{i=1}^{N} 1_{[T_i, T_j]}(\cdot) \mathbf{S}_{T_i} G_L(T_i^-, x_i^2, Y_i) \right\|_{\mathcal{X}_\lambda}^{q}
\]
\[
\leq C \sum_{i=1}^{N} \left\| 1_{[T_i, T_j]}(\cdot) \mathbf{S}_{T_i} G_L(T_i^-, x_i^1, Y_i) - 1_{[T_i, T_j]}(\cdot) \mathbf{S}_{T_i} G_L(T_i^-, x_i^2, Y_i) \right\|_{\mathcal{X}_\lambda}^{q}
\]
\[
\leq \max(L_{G_L}(Y_i)^q, \ldots, L_{G_L}(Y_N)^q) C \lambda^{\rho_{\alpha q} - 1} \left| (x_1^1)_{i=1}^{N} - (x_1^2)_{i=1}^{N} \right|_{\mathcal{X}_\lambda}^{q},
\]
where $C$ depends on $N$. Therefore, we have shown that
\[
\left| 3_0(\xi_1, (x_1^1)_{i=1}^{N}) - 3_0(\xi_2, (x_1^2)_{i=1}^{N}) \right|_{\mathcal{X}_\lambda}^{q}
\]
\[
\leq C_0(\lambda) \left( \left| \xi_1 - \xi_2 \right|_{\mathcal{X}_\lambda} + \left| (x_1^1)_{i=1}^{N} - (x_1^2)_{i=1}^{N} \right|_{\mathcal{X}_\lambda} \right),
\]
with $C_0(\lambda) \to 0$ as $\lambda \to \infty$. It remains to consider
\[
\left| (3_0(\xi_1, (x_1^1)_{i=1}^{N})(T_j^-))_{j=1}^{N} - (3_0(\xi_2, (x_1^2)_{i=1}^{N})(T_j^-))_{j=1}^{N} \right|_{\mathcal{X}_\lambda}^{q}.
\]

Again, we begin with mimicking the calculations done in Step III of the proof of Theorem 3.3. In particular,
\[
e^{-\lambda T_j} \mathbb{E} \left| 3_0(\xi_1, (x_1^1)_{i=1}^{N})(T_j^-) - 3_0(\xi_2, (x_1^2)_{i=1}^{N})(T_j^-) \right|_{H_{-\alpha j}}^{q}
\]
\[
\leq C \left( e^{-\lambda T_j} \mathbb{E} \left| \int_{0}^{T_j^-} S(T_j^- - s) \left( F(s, \xi_1(s)) - F(s, \xi_2(s)) \right) ds \right|_{H_{-\alpha j}}^{q} + e^{-\lambda T_j} \mathbb{E} \left| \int_{0}^{T_j^-} \int_{\mathcal{Z}} S(T_j^- - s) \left( G(s, \xi_1(s), z) - G(s, \xi_2(s), z) \right) \tilde{\eta}(dz, ds) \right|_{H_{-\alpha j}}^{q} + e^{-\lambda T_j} \sum_{i=1}^{N} 1_{[T_i, T_j]}(T_j^-) \mathbb{E} \left| S(T_j^- - T_i) \left( G_L(T_i, x_i^1, Y_i) - G_L(T_i, x_i^2, Y_i) \right) \right|_{H_{-\alpha j}}^{q} \right).
\]

We are going to estimate the two summands separately. First, we get for a
fixed $T_j$ that

\[ e^{-\lambda T_j} \mathbb{E} \left[ \int_0^{T_j} S(T_j - s) \left( F(s, \xi^1(s)) - F(s, \xi^2(s)) \right) ds \right]^{\frac{q}{2}} \]

\[ \leq \mathbb{E} \left( \int_0^{T_j} e^{-\lambda(T_j - s)/q} (T_j - s)^{-((\alpha_F - \alpha_I)\wedge 0)R} e^{-\lambda s/q} \right. \]

\[ \left| F(s, \xi^1(s)) - F(s, \xi^2(s)) \right|_{H^\alpha} ds \]

\[ \leq C(\lambda) \mathbb{E} \int_0^{T_j} e^{-\lambda s} \left| \xi^1(s) - \xi^2(s) \right|_{H^\alpha} ds \leq C(\lambda) |\xi_1 - \xi_2|_X^{\lambda}, \]

with $C(\lambda) \to 0$ as $\lambda \to \infty$. Similarly we get for the second summand using that $\alpha_I \geq \alpha_G$,

\[ e^{-\lambda T_j} \mathbb{E} \left[ \int_0^{T_j} \int_Z S(T_j - s) \left( G(s, \xi^1(s); z) - G(s, \xi^2(s); z) \right) \tilde{\eta}(dz, ds) \right]^{\frac{q}{2}} \]

\[ \leq \left[ e^{-\lambda T_j} \mathbb{E} \left( \int_0^{T_j} \int_Z |S(T_j - s) \left( G(s, \xi^1(s); z) - G(s, \xi^2(s); z) \right) |_{H^\alpha} \nu(dz) ds \right]^{\frac{q}{p}} \right. \]

\[ + \mathbb{E} \left( \int_0^{T_j} \int_Z |S(T_j - s) \left( G(s, \xi^1(s); z) - G(s, \xi^2(s); z) \right) |_{H^\alpha} \nu(dz) ds \right]^{\frac{q}{p}} \]

\[ \leq C(1 + C(T)) \times \mathbb{E} \left[ \int_0^{T_j} e^{-\lambda(T_j - s)} e^{-\lambda s} |\xi^1(s) - \xi^2(s)|_{H^\alpha} ds \leq C |\xi^1 - \xi^2|_{X^\lambda}. \right\]
Finally, using the Lipschitz property of $G_L$, we have
\[
e^{-\lambda T} \mathbb{E} \sum_{i=1}^{N} 1_{[T_i, T_i]}(T_i^-) \sum_{j} G_L(T_i^-, x_i^1, Y_i)(T_j^-) - 1_{[T_i, T_i]}(T_i^-) \sum_{j} G_L(T_i^-, x_i^2, Y_i)(T_j^-) \bigg|_{H_{\alpha f}}^q
\]
\[
\leq C \mathbb{E} \sum_{i=j+1}^{N} e^{-\lambda (T_j^- - T_i)} e^{-\lambda T_i} \left| \sum_{j} \left( G_L(T_i^-, x_i^1, Y_i) - G_L(T_i^-, x_i^2, Y_i) \right)(T_j^-) \right|_{H_{\alpha f}}^q
\]
\[
\leq C e^{-\lambda \min_{i=j+1, \ldots, N}(T_j^- - T_i)} \mathbb{E} \sum_{i=j+1}^{N} e^{-\lambda T_i} L_{G_L}(Y_i)^q \left| x_i^1 - x_i^2 \right|_{H_{\alpha f}}^q,
\]
where $C$ depends on $N$.

Collecting the estimates and summing up over $j = 1, \ldots, N$ we get
\[
\left| (\mathbb{E} \sum_{i=1}^{N} e^{-\lambda (T_i^- - T_j^-)} e^{-\lambda T_j} \left| \sum_{j} \left( G_L(T_i^-, x_i^1, Y_i) - G_L(T_i^-, x_i^2, Y_i) \right)(T_j^-) \right|_{H_{\alpha f}}^q \right|_{2, \lambda}
\]
\[
= \sum_{j=1}^{N} e^{-\lambda \min_{i=j+1, \ldots, N}(T_j^- - T_i)} \mathbb{E} \sum_{i=j+1}^{N} e^{-\lambda T_i} L_{G_L}(Y_i)^q \left| x_i^1 - x_i^2 \right|_{H_{\alpha f}}^q
\]
\[
\leq C(\lambda, N, T) \left( |\xi_1 - \xi_2|_{\lambda}^q \right)
\]
\[
+ \sum_{j=1}^{N} e^{-\lambda \min_{i=j+1, \ldots, N}(T_j^- - T_i)} \mathbb{E} \sum_{i=j+1}^{N} e^{-\lambda T_i} L_{G_L}(Y_i)^q \left| x_i^1 - x_i^2 \right|_{H_{\alpha f}}^q
\]
\[
\leq C(\lambda, N, T) \left( |\xi_1 - \xi_2|_{\lambda}^q \right)
\]
\[
+ N e^{-\lambda \min_{i=j+1, \ldots, N}(T_j^- - T_i)} \max(L_{G_L}(Y_1)^q, \ldots, L_{G_L}(Y_N)^q)
\]
\[
\mathbb{E} \sum_{i=1}^{N} e^{-\lambda T_i} \left| x_i^1 - x_i^2 \right|_{H_{\alpha f}}^q
\]
Where $1 \leq C(\lambda, T, N) \leq C$ for all $\lambda > 1$. In summary, we obtain
\[
\left| \Theta(\xi_1, (x_i^1)_{i=1}^{N}) - \Theta(\xi_2, (x_i^2)_{i=1}^{N}) \right|_{H(\lambda, K)}
\]
\[
\leq \left[ C_0(\lambda) + \frac{C(\lambda, T, N)^{\frac{1}{2}}}{K} \right] K |\xi_1 - \xi_2|_{\lambda}
\]
\[
+ \left[ C_0(\lambda) K + C(\lambda, T, N)^{\frac{1}{2}} N^{\frac{1}{2}} e^{-\lambda \min_{i,j=1, \ldots, N}(T_j - T_i)} \max(L_{G_L}(Y_1), \ldots, L_{G_L}(Y_N)) \right] \times |(x_i^1)_{i=1}^{N} - (x_i^2)_{i=1}^{N}|_{2, \lambda}.
\]

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If $K$ is chosen such that \( \frac{C(\lambda, T, N)}{K} \frac{q}{N} \frac{1}{2} < \frac{1}{2} \) for all $\lambda > 1$ and then $\lambda > 1$ is chosen large enough so that $C_0(\lambda) \leq \frac{1}{2}$, then

\[
C_0(\lambda)K + C(\lambda, T, N)\frac{q}{N} \frac{1}{2} e^{-\frac{\lambda}{q} \min_{i,j=1,...,N} (T_j - T_i) \max(L_{GL}(Y_1), \ldots, L_{GL}(Y_N))} < 1,
\]

then there exists a number $0 < k < 1$ such that

\[
|\Theta(\xi_1, (x_1^1)_{i=1}^N) - \Theta(\xi_2, (x_2^2)_{i=1}^N)|_{H(\lambda, K)} \leq k|(\xi_1 - \xi_2, (x_1^1 - x_2^2)_{i=1}^N)|_{H(K, \lambda)}.
\] (26)

Thus, we have proven that there exist numbers $\lambda, K > 0$ such that $\Theta$ is a strict contraction on $H(\lambda, K)$. Applying Banach’s fixed point theorem, it follows that there exists a process $\xi^* \in X$ with a càdlàg modification in $H_{-\alpha}$ such that

\[
\xi^*(t) = S(t)u_0 + \int_0^t S(t-s)F(s, \xi^*(s)) \, ds + \int_0^t \int Z S(t-s)G(s, \xi^*(s), z) \tilde{\eta}(dz, ds) + \sum_{i=1}^N 1_{[T_i, T_i]}(t)S(t-T_i)G_L(T_i^--\xi^*(T_i^-), Y_i).
\] (27)

**Step II.** Fix $N, n \in \mathbb{N}$ and put

\[
A_n^N := \{\omega^L \in \Omega^L : N(T) = N; |Y_i|_{Z_L} \leq n, i = 1, \ldots, N; \\min_{1 \leq i, j \leq N} |T_i - T_j| \geq \frac{T}{n} \}\in \mathcal{F}^L,
\]

where $\{T_j : j = 1 \leq j \leq N\}$ and $\{Y_j : 1 \leq j \leq N\}$ are defined in (19) and (??). As the contraction constant $k$ in (26) only depends on $N$ and $n$ and $L_{GL}(z) \leq M(1 + |z|_{Z_L})$, Step I also show that, there is $0 < k < 1$, such that

\[
\sup_{\omega^L \in A_n^N} |\Theta(\xi_1, (x_1^1)_{i=1}^N) - \Theta(\xi_2, (x_2^2)_{i=1}^N)|_{H(\lambda, K)} \leq k|(\xi_1 - \xi_2, (x_1^1 - x_2^2)_{i=1}^N)|_{H(K, \lambda)}.
\]

where in the definition of $\Theta$ we allow $\{T_j : j = 1 \leq j \leq N\}$ and $\{Y_j : 1 \leq j \leq N\}$ be random variables from (19) and (??). Therefore, Banach’s fixed
point theorem on $L^\infty(A_n^N; H(\lambda, K))$ shows that there is a unique progressively measurable process $\xi_n^N$ on $\Omega^S \times A_n^N$ with a cádlág modification in $H_\alpha$ such that for $\mathbb{P}$ almost surely on $\Omega^S \times A_n^N$, the process $\xi_n^N$ satisfies $21$. We extend $\xi_n^N$ by setting it 0 on $\Omega^S \times \Omega^\infty \setminus \Omega^S \times A_n^N$. Let $\gamma$ be a number such that $0 < \gamma < \min(\frac{1}{2}, \beta)$, and put $g(n) = \lfloor n^\gamma \rfloor$. Let $B_n := \bigcup_{N=1}^{g(n)} A_n^N$ with $B_0 := \emptyset$. Note that for fixed $n$, the sets $\{A_n^N, N = 1, \ldots, g(n)\}$ are disjoint and that $B_n \subset B_{n+1}$. Define

$$u(t) := \sum_{n=1}^{\infty} 1_{B_n \setminus B_{n-1}} \sum_{N=1}^{g(n)} \xi_n^N(t)$$

Then $u$ satisfies $21$ for $\mathbb{P}$ almost all $\omega \in \Omega^S \times (\cup_{n=1}^\infty B_n)$. It remains to show that $\lim_{n \to \infty} \mathbb{P}^L(B_n) = 1$.

First, note that given $N(T) = N$, the times $\{T_i : i = 1, \ldots, N\}$ are uniformly distributed on the interval $[0, T]$ (see [15, Proposition 2.9]) and are independent of $\{Y_j : j \geq 0\}$. Fix $n \in \mathbb{N}$. To give an lower estimate of $\mathbb{P}^L(B_n)$, observe that

$$\mathbb{P}^L\left(|T_1 - T_2| \geq \frac{T}{n} \mid N(T) = N\right) \geq \left(1 - \frac{2}{n}\right).$$

Throwing $T_3$ into the interval $[0, T]$, by the Bayes formula, we get

$$\mathbb{P}^L\left(\min_{i,j=1,2,3} |T_i - T_j| \geq \frac{T}{n} \mid N(T) = N\right) = \mathbb{P}^L\left(|T_1 - T_2| \geq \frac{T}{n} \mid N(T) = N\right) \times \mathbb{P}^L\left(\min_{i=1,2} |T_3 - T_i| \geq \frac{T}{n} \mid |T_1 - T_2| \geq \frac{T}{n} \text{ and } N(T) = N\right) \geq (1 - \frac{2}{n})(1 - \frac{4}{n}).$$

Iterating, we get

$$\mathbb{P}^L\left(\min_{i,j=1,\ldots,N} |T_i - T_j| \geq \frac{T}{n} \mid N(T) = N\right) \geq \prod_{j=1}^{N} \left(1 - \frac{2j}{n}\right).$$
Since
\[- \log \left( \prod_{j=1}^{N} \left( 1 - \frac{2j}{n} \right) \right) = - \sum_{j=1}^{N} \log \left( 1 - \frac{2j}{n} \right) \leq \sum_{j=1}^{N} \frac{2j}{n} \approx \frac{N(N - 1)}{n}. \]

there exists a constant \( c > 0 \) such that for \( n \geq \sqrt{N} \)
\[P^L \left( \min_{i,j=1,...,N, i \neq j} \left| T_i - T_j \right| \geq \frac{T}{n} \mid N(T) = N \right) \geq e^{-c \frac{N(N-1)}{2n}}. \] (28)

As noted before, the sets \( \{ A^n_N, N = 1, ..., g(n) \} \) are disjoint for fixed \( n \) and therefore,
\[P^L (B_n) = \sum_{N=1}^{g(n)} \]
\[P^L \left( \left\{ \omega^L \in \Omega^L : N(T) = N; |Y_i|_{Z_L} \leq n, i = 1, ..., N; \min_{1 \leq i,j \leq N} \left| T_i - T_j \right| \geq \frac{T}{n} \right\} \right) \]
\[= \sum_{N=1}^{g(n)} P^L \left( \{ \omega^L \in \Omega^L : N(T) = N \} \right) \]
\[P^L \left( \left\{ \omega^L \in \Omega^L : |Y_i|_{Z_L} \leq n, i = 1, ..., N; \min_{1 \leq i,j \leq N} \left| T_i - T_j \right| \geq \frac{T}{n} \right\} \mid N(T) = N \right) \]

Under the condition \( N = N(T) \) the random variables \( T_i, i = 1, \ldots, N \) and \( Y_i, i = 1, \ldots, N, \) are mutually independent. In addition, \( N(T) \) is a Poisson distributed random variable with parameter \( \sigma T = \nu_L(Z_L) T \) and \( N(T) \) is independent of \( Y_i, i = 1, \ldots, N. \) Therefore, using Assumption 4.1, estimate
together with Bernoulli’s inequality, we get for \( n \) large enough that

\[
\mathbb{P}^L(B_n) \geq \sum_{N=1}^{g(n)} \mathbb{P}^L \left( \{ \omega^L \in \Omega^L : N(T) = N \} \right) e^{-c \frac{N(N-1)}{2n}} \left( 1 - Cn^{-\beta}N \right)
\]

\[
\geq \sum_{N=1}^{g(n)} e^{-\sigma T} \frac{(\sigma T)^N}{N!} e^{-\frac{N(N-1)}{2n}} \left( 1 - Cn^{-\beta} \right)
\]

\[
\geq \sum_{N=1}^{g(n)} e^{-\sigma T} \frac{(\sigma T)^N}{N!} e^{-\frac{g(n)^2}{2n}} \left( 1 - Cn^{-\beta} \right)
\]

\[
= e^{-\frac{2}{n}n^\gamma - 1} \left( 1 - Cn^{-\gamma} \right) e^{-\sigma T} \sum_{N=1}^{n^\gamma} \frac{(\sigma T)^N}{N!} \to 1
\]
as \( n \to \infty \). Thus \( \lim_{n \to \infty} \mathbb{P}^L(B_n) = 1 \) and the proof is complete.

5. Application to isotropic synchronous viscoelastic materials

One motivation for this work are the linear models of viscoelasticity. In the linear theory for homogeneous isotropic viscoelastic the velocity field \( v \) of the material occupying a bounded region \( \mathcal{O} \subset \mathbb{R}^d, d = 1, 2, 3 \), with \( C^1 \) boundary, which rests up to \( t = 0 \), is governed by

\[
\dot{v}(x, t) = \int_0^t \Delta v(x, t - s) da(s)
\]

\[
+ \int_0^t \nabla \nabla \cdot v(x, t - s) \left( c(s) + \frac{1}{3} a(s) \right) + g(x, t), \quad x \in \mathcal{O},
\]

(29)

where \( g \) denotes an external body force, see, for example, [46, Chapter 5]. For simplicity, we set the density of the material equal to constant 1. The functions \( a \) and \( c \) are called the shear modulus and the compression modulus, respectively. The material is called synchronous if for some \( \gamma > 0 \) it holds that \( c(t) = \gamma a(t) \). Suppose that \( a(t) = \int_0^t b(s) \, ds \). In this case (29) simplifies to

\[
\dot{v}(x, t) = \int_0^t \Delta v(x, t - s) b(s) \, ds
\]

\[
+ \int_0^t \nabla \nabla \cdot v(x, t - s) \left( \gamma + \frac{1}{3} b(s) \right) \, ds + g(x, t), \quad x \in \mathcal{O},
\]

(30)
We consider the external body force to be abrupt in irregular time and space instances and it is modelled by \( g(x, t) = \dot{L}(x, t) \) where \( L \) is a space time \( \beta \)-stable Lévy process with \( \beta < 1 \), having only positive jumps. For the definition of the space time Lévy noise, we refer to [12, Section A.3, Definition A.13]. We supplement (30) by initial and Dirichlet zero boundary conditions:

\[
v|_{\partial O} = 0, \quad v(0, x) = v_0(x).
\] (31)

Now, applying Theorem 4.3 to (30) with boundary conditions (31) with \( g(x, t) = L(t, x)y_0 \), following result can be stated. For simplicity, we take \( v_0 = 0 \).

**Corollary 5.1.** Under the assumption above, if the kernel \( b \) satisfies Assumptions 2.2 with \( \rho < \frac{4}{\alpha} \), then equation (30) has a unique mild solution \( u \) such that for any \( \alpha > \frac{d}{4} \), \( \mathbb{P}\text{-a.s.}, u \in \mathcal{D}([0, T]; H^{-2\alpha}(O)) \).

**Proof 5.1.** First, we split the noise into two noises, one with small, but bounded jumps, the other consisting of the large jumps. Let \( r > 0 \) be an arbitrary number, \( \eta \) be a space time Poisson random measure on \( \mathcal{O} \times \mathbb{R} \) with jump intensity

\[
\nu_{r}(U) := \int_{U \cap [0, r]} y^{-\beta-1} \, dy, \quad U \in \mathcal{B}(\mathbb{R})
\]

and \( \eta_{L} \) be a space time Poisson random measure \(^2\) on \( \mathcal{O} \times \mathbb{R} \) with jump intensity

\[
\nu_{L}(U) := \int_{U \cap (r, \infty)} y^{-\beta-1} \, dy, \quad U \in \mathcal{B}(\mathbb{R}).
\]

First we verify the assumptions of Theorem 3.3. Let us put \( H = L^2(\mathcal{O}) \) and let \( m = \text{Leb} \times \nu_{r} \) be a measure on \( \mathcal{O} \times \mathbb{R} \). Consider the function

\[
\Phi : (x, y) \in \mathcal{O} \times \mathbb{R} \mapsto \Phi(x, y) := (\delta_x) y
\] (32)

It is not hard to see that \( \Phi : \mathcal{O} \times \mathbb{R} \mapsto B^{\frac{d}{2}}_{2,\infty}(\mathcal{O}) \) is measurable. For the definition of the Besov spaces \( B^{s}_{2,\infty}(\mathbb{R}^d) \) and \( B^{s}_{2,\infty}(\mathcal{O}) \) see, for example, [12].

\(^2\)For the Definition of space time Poisson random measure we refer to [12, Section A.3, Definition A.16].
Appendix C]. Indeed, it follows from a straightforward calculation, see also [12, (C5)], that
\[
\sup_{x \in \mathbb{R}^d} \left| \delta_x \right|_{B_{2, \infty}^{-\frac{d}{2}}(\mathbb{R}^d)} < \infty. \tag{33}
\]
Note that \( |\mathcal{F}(\delta_a - \delta_b)(\xi)| \leq c \|\xi\| |a - b| \), where \( \mathcal{F} \) denotes the Fourier transform, and therefore, a short calculation yields that
\[
\|\delta_a - \delta_b\|_{B_{2, \infty}^{-\frac{d}{2}-\frac{1}{2}}(\mathbb{R}^d)} \leq C |a - b|.
\]
It follows from above by interpolation using in addition (33), that for any \( s > \frac{d}{2} \), \( a \to \delta_a \) is continuous in \( B_{2, \infty}^{-s}(\mathbb{R}^d) \). The continuity of the product \((x, y) \in \mathbb{R}^d \times \mathbb{R} \to (\delta_x)y \) in \( B_{2, \infty}^{-s}(\mathbb{R}^d) \) now easily follows. Finally if \( a, b \in \mathcal{O} \), then the support of \( y_1 \delta_a - y_2 \delta_b \) is in \( \mathcal{O} \) and hence, using the definition of \( B_{2, \infty}^{-s}(\mathcal{O}) \), see, for example [51, Definition 4.2.1], it follows that
\[
\|y_1 \delta_a - y_2 \delta_b\|_{B_{2, \infty}^{-s}(\mathcal{O})} \leq \|y_1 \delta_a - y_2 \delta_b\|_{B_{2, \infty}^{-s}(\mathbb{R}^d)}
\]
and thus \( \Phi \) defined in (32) is continuous from \( \mathcal{O} \times \mathbb{R} \) to \( B_{2, \infty}^{-s}(\mathcal{O}) \) hence measurable. Therefore, \( \Phi \) induces a (Lévy) measure \( \nu \) on the space \( B_{2, \infty}^{-s}(\mathcal{O}) \) (push forward measure) by setting
\[
\nu(U) := \int_{\mathcal{O} \times \mathbb{R}} 1_U(\Phi(x, y)) \, d\nu\mathbb{R}(dy), \quad U \in \mathcal{B}(B_{2, \infty}^{-s}(\mathcal{O})).
\]
Secondly, by Sobolev embedding theorems (see [47, p. 29, Chapter 2.2]) it follows that for any \( \theta > \frac{d}{2} \) there exists a \( \theta_1 > \frac{d}{2} \) such that we have
\[
B_{2, \infty}^{-\theta_1}(\mathcal{O}) \hookrightarrow H_{2}^{-\theta}(\mathcal{O}) \tag{34}
\]
continuously. In addition, since \( D(A) = H_{2,0}^2(\mathcal{O}) \) we have \( H_{2}^{-\theta}(\mathcal{O}) = H_{-\frac{d}{2}}^A \). Let \( \theta \) and \( \theta_1 > \frac{d}{2} \) be chosen such that (33) holds. Now, since the embedding from \( B_{2, \infty}^{-\theta}(\mathcal{O}) \) into \( H_{2}^{-\theta}(\mathcal{O}) \) is continuous, and \( H_{2}^{-\theta}(\mathcal{O}) = H_{-\frac{d}{2}}^A \), for \( Z = B_{2, \infty}^{-\theta_1}(\mathcal{O}) \) the mapping
\[
G : [0, T] \times H \times Z \rightarrow H_{-\frac{d}{2}}^A \tag{35}
\]
\[
(t, v, w) \mapsto w,
\]

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is Lipschitz continuous. Let us put $\alpha_G = \delta > \frac{d}{4}$. The mapping $G$ satisfies Assumption 3.2(i), if there exists some $q > p$ such that $\alpha_G < \frac{1}{q^\rho}$. Since $\nu^\mathbb{R}$ has bounded moments of all order and $\rho < \frac{4}{d}$, such a $q$ exists.

As we have only positive jumps, we have to add and subtract the compensator. Hence, defining a mapping

$$F : [0, T] \times H \rightarrow H^A_{\alpha_F}$$

by $F(t, v)(x) = \delta x v$, where $x v = \int_{(0, \delta]} y^{-\beta} dy$. Setting $\alpha_I = \alpha_G$, we know $\alpha_I = \alpha_F$ and therefore, condition (20) is trivially satisfied. Again, it is straightforward to show that for $\alpha_F = \alpha_G$ the mapping $F$ is Lipschitz continuous and Assumption 3.2 (ii) is satisfied.

As $v_0 = 0$, Assumption 3.2 (iii) is fulfilled with $\alpha_I = \alpha_G$.

Next, we have to show that Assumption 4.2 holds for bigger jumps. Let us put $Z_L = B^{\theta_1}_2(\mathcal{O})$ and $m_L = \text{Leb} \times \nu^\mathbb{R}_L$ be a measure on $\mathcal{O} \times \mathbb{R}$. By arguing as before, one can show that $\Phi$ defined in (32) induces a Lévy measure $\nu_L$ on the space $B^{\theta_1}_2(\mathcal{O})$ by setting

$$\nu_L(U) := \int_{\mathcal{O} \times \mathbb{R}} 1_U(\Phi(x, y)) \nu^\mathbb{R}_L(dy) dx, \quad U \in \mathcal{B}(B^{\theta_1}_2(\mathcal{O})).$$

Then, Assumption 4.1 is satisfied with $\beta$ being the index of the Lévy process.

The mapping

$$G_L : [0, T] \times H \times Z_L \rightarrow H^{-\theta}_2(\mathcal{O})$$

by $G_L(t, v, z) \mapsto z$, clearly satisfies Assumption 4.2. Let us denote by $\eta$ the space time Poisson random measure with intensity $\nu$, $\tilde{\eta}$ the compensated space time Poisson random measure and by $\eta_L$ the space time Poisson random measure with intensity $\eta_L$. In this way, equation (30) can be written in the following form

$$\left\{ \begin{array}{l}
    du(t) = \left( A \int_0^t b(t - s)u(s) ds + F(t, u(t)) dt \\
    + \int_Z G(t, u(t)z) \tilde{\eta}(dz, dt) + \int_{Z_L} G_L(t, u(t)z) \eta_L(dz, dt) \right) ; \quad t \in (0, T], \\
    u(0) = u_0,
\end{array} \right.$$  

where $G, G_L$ and $F$ are satisfying the assumptions of Theorem 4.3. Hence, we get existence and uniqueness of a mild solution with càdlàg paths in $H^{-2\alpha}_2(\mathcal{O})$ for $\alpha > \frac{d}{4}$. 

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References

[1] L. Albright, Albright’s Chemical Engineering Handbook. CRC Press, 2008.

[2] W. Arendt, C. J. K. Batty, M. Hieber and F. Neubrander, Vector-valued Laplace transforms and Cauchy problems, 2nd Edition, Vol. 96 of Monographs in Mathematics, Birkhäuser/Springer Basel AG, Basel, 2011.

[3] T. M. Atanacković, S. Pilipović, B. Stanković and D. Zorica, Fractional calculus with applications in mechanics. Mechanical Engineering and Solid Mechanics Series. London; John Wiley & Sons, 2014.

[4] B. Baeumer, M. Geissert and M. Kovács, Existence, uniqueness and regularity for a class of semilinear stochastic Volterra equations with multiplicative noise, J. Differential Equations 258 (2) (2015) 535–554.

[5] V. Barbu, S. Bonaccorsi, and L. Tubaro, Existence and Asymptotic Behavior for Hereditary Stochastic Evolution Equations, Appl. Math. Optim. 69(2) (2014), 273–314.

[6] S. Bonaccorsi, Volterra equations perturbed by a Gaussian noise, in: Seminar on Stochastic Analysis, Random Fields and Applications V, Vol. 59 of Progr. Probab., Birkhäuser, Basel, 2008, pp. 37–55.

[7] S. Bonaccorsi, G. Da Prato and L. Tubaro, Asymptotic Behavior of a Class of Nonlinear Stochastic Heat Equations with Memory Effects, SIAM J. Math. Anal. 44 (2012) 1562–158.

[8] S. Bonaccorsi and M. Fantozzi, Infinite dimensional stochastic Volterra equations with dissipative nonlinearity, Dynam. Systems Appl. (2006) 15 465–478.

[9] S. Bonaccorsi and E. Mastrogiacomo, An analytic approach to stochastic Volterra equations with completely monotone kernels, J. Evol. Equ. 9 (2) (2009) 315–339.

[10] P. Billingsley, Probability and measure, Wiley Series in Probability and Statistics, John Wiley & Sons, Inc., Hoboken, NJ, 2012.
[11] Z. Brzeźniak and E. Hausenblas, Maximal regularity for stochastic convolutions driven by Lévy processes, Probab. Theory Related Fields 145 (3-4) (2009) 615–637.

[12] Z. Brzeźniak, E. Hausenblas, and P. Razafimandimby, Martingale solutions for Stochastic Equation of Reaction Diffusion Type driven by Lévy noise or Poisson random measure, arXiv:1010.5933.

[13] A. Carpinteri and F. Mainardi, Fractals and fractional calculus in continuum mechanics. CISM Courses and Lectures. 378. Wien: Springer. 348 p., (1997).

[14] Ph. Clément, G. Da Prato, and J. Prüss, White noise perturbation of the equations of linear parabolic viscoelasticity, Rend. Inst. Mat. Univ. Trieste XXIX, (1997) 207–220.

[15] R. Cont and P. Tankov, Financial modelling with jump processes, Chapman & Hall/CRC Financial Mathematics Series, Chapman & Hall/CRC, Boca Raton, FL, 2004.

[16] R. C. Dalang, Level sets and excursions of the Brownian sheet, in: Topics in spatial stochastic processes (Martina Franca, 2001), Vol. 1802 of Lecture Notes in Math., Springer, Berlin, 2003, pp. 167–208.

[17] E. Dettweiler, A characterization of the Banach spaces of type p by Lévy measures, Math. Z. 157 (2) (1977) 121–130.

[18] G. Desch and S.-O. Londen, Regularity of stochastic integral equations driven by Poisson random measures, J. Evol. Equ. 17 (2017) 263–274.

[19] W. Desch and S. -O.Londen, Semilinear Stochastic Integral Equations in $L_p$, Progress in Nonlinear Differential Equations and Their Applications 80 (2011) 131–166.

[20] K. Diethelm, The analysis of fractional differential equations. Lecture Notes in Mathematics, 2004. Springer-Verlag, Berlin, 2010.

[21] N. Dinculeanu, Vector integration in Banach spaces and application to stochastic integration., Amsterdam: North-Holland, 2002.

[22] S. Dirksen, Itô isomorphisms for $L^p$-valued Poisson stochastic integrals, Ann. Probab. 42 (6) (2014) 2595–2643.
[23] S. N. Ethier and T. G. Kurtz, Markov processes, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons, Inc., New York, 1986.

[24] E. Hausenblas, Maximal inequalities of the Itô integral with respect to Poisson random measures or Lévy processes on Banach spaces, Potential Anal. 35 (3) (2011) 223–251.

[25] E. Hausenblas and J. Seidler, A note on maximal inequality for stochastic convolutions, Czechoslovak Math J. 51 (2001) 785–790.

[26] A. Karczewska, Convolution type stochastic Volterra equations, 101 pp., Lecture Notes in Nonlinear Analysis 10, Juliusz Schauder Center for Nonlinear Studies, Torun, 2007.

[27] A. Karczewska, On difficulties appearing in the study of stochastic Volterra equations, Quantum probability and related topics 27 (2011) 214–226.

[28] A. Karczewska and C. Lizama, Strong solutions to stochastic Volterra equations, J. Math. Anal. Appl 349 (2009) 301–310.

[29] D. N. Keck and M. A. McKibben, Abstract semilinear stochastic Itô-Volterra integrodifferential equations, J. Appl. Math. Stoch. Anal. (2006) 1–22.

[30] A. Kilbas, H. Srivastava and J. Trujillo, Theory and applications of fractional differential equations. North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[31] M. Kostić, Abstract Volterra integro-differential equations, CRC Press, Boca Raton, FL, 2015.

[32] M. Kovács and J. Printems, Strong order of convergence of a fully discrete approximation of a linear stochastic Volterra type evolution equation, Math. Comp. 83 (289) (2014) 2325–2346.

[33] M. Kovács, F. Lindner and R. L. Schilling, Weak convergence of finite element approximations of linear stochastic evolution equations with additive Lévy noise, SIAM/ASA J. Uncertain. Quantif. 3 (1) (2015) 1159–1199.
[34] H. Kunita, Stochastic Flows and Stochastic Differential Equations, Cambridge Studies in Advanced Mathematics, vol. 24. Cambridge University Press, Cambridge, 1990.

[35] W. Linde, Probability in Banach spaces: stable and infinitely divisible distributions. Second edition. A Wiley-Interscience Publication. John Wiley & Sons, Ltd., Chichester, 1986.

[36] F. Mainardi, Fractional calculus and waves in linear viscoelasticity. An introduction to mathematical models. Hackensack, NJ: World Scientific, 2010.

[37] F. Mainardi, Applications of fractional calculus in mechanics, P. Rusev (ed.) et al., Transform methods and special functions. Proceedings of the 2nd international workshop, Varna, Bulgaria, August 23–30, 1996. Sofia: Bulgarian Academy of Sciences, Institute of Mathematics and Informatics, 1998.

[38] R. Magin, Fractional Calculus in Bioengineering. Begell House Inc., Redding, CT, 2006.

[39] S. Monniaux and J. Prüss, A theorem of the Dore-Venni type for non-commuting operators, Trans. Amer. Math. Soc. 349 (1997) 4787–4814.

[40] H. Namazi and V. Kulish, Fractional Diffusion Based Modelling and Prediction of Human Brain Response to External Stimuli. Computational and Mathematical Methods in Medicine, (2015).

[41] A. Pazy, Semigroups of linear operators and applications to partial differential equations., Applied Mathematical Sciences, 44. New York etc.: Springer-Verlag. VIII, 279 pp., 1983.

[42] S. Peszat and J. Zabczyk, Stochastic partial differential equations with Lévy noise, Vol. 113 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2007.

[43] S. Peszat and J. Zabczyk, Time regularity for stochastic Volterra equations by the dilation theorem, J. Math. Anal. Appl. 409 (2014) 676–683.

[44] Y. Povstenko, Theories of thermal stresses based on space & time-fractional telegraph equations Computers and Mathematics with Applications 64 (2012) 3321–3328.
[45] Y. Povstenko, Linear fractional diffusion-wave equation for scientists and engineers. Cham: Birkhäuser/Springer, 2015.

[46] J. Prüss, Evolutionary integral equations and applications, Vol. 87 of Monographs in Mathematics, Birkhäuser Verlag, Basel, 1993.

[47] T. Runst and W. Sickel, Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations, Vol. 3 of de Gruyter Series in Nonlinear Analysis and Applications, Walter de Gruyter & Co., Berlin, 1996.

[48] K.-i. Sato, Lévy processes and infinitely divisible distributions, Vol. 68 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2013.

[49] R. Schnaubelt and M. Veraar, Regularity of stochastic Volterra equations by functional calculus methods, J. Evol. Equ. (2016). doi:10.1007/s00028-016-0365-z.

[50] S. Sperlich, On parabolic Volterra equations disturbed by fractional Brownian motions, Stoch. Anal. Appl. 27(1) (2009) 74–94.

[51] H. Triebel, Interpolation theory, function spaces, differential operators, 2nd Edition, Johann Ambrosius Barth, Heidelberg, 1995.

[52] W. Whitt and B. Sleeman, Some useful functions for functional limit theorems., Math. Oper. Res. 5 (1980) 67–85.

[53] J. Zhu, A Study of SPDES w.r.t. compensated Poisson random measures and related topics, Ph.D Thesis, University of York, (2010).