We consider the intrinsic stability of the vortex states of a pure Bose-Einstein condensate confined in a harmonic potential under the effects of coherent atom-atom interaction. We find that stable vortices can be supported, and that vortex stability can be controlled by changing the inter-particle interaction strength. At unstable regimes, a vortex will spontaneously disintegrate into states with different angular momenta even without external perturbations, with the lifetime determined by its imaginary excitation frequencies.

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I. INTRODUCTION

Vortices and their motions have long been an important branch of fluid mechanics. With the discovery of superfluid helium II, a new idea developed: that the circulation in a superfluid vortex must be quantized. The consequences of these quantized vortices is profound and the understanding of vortex dynamics plays a key role in the current understanding of superfluidity. Moreover, the detection of individual singly quantized vortices has vividly established the true macroscopic quantum nature of these remarkable degenerate fluids. Intimately related to the observation of the superfluid state of $^4$He is the evidence for the concurrent formation of a Bose-Einstein condensate (BEC). An important challenge is to clarify the link between these fundamental and important phenomena — Bose-Einstein condensation, superfluidity and the formation of a macroscopic quantum state.

The recent observation of Bose-Einstein condensation in dilute alkalai vapors has presented a striking new system for investigation; that of the dilute degenerate Bose gas. The alkali BECs differ fundamentally from the helium BEC in several crucial ways. BEC in both bulk liquid helium and the dilute helium “gas” are free systems (the “gas” BEC is created by introducing helium into a porous glass known as Vycor). By contrast, the alkali vapor BECs, although free of container walls (and/or the Vycor host), are created within the confines of a trapping potential. There is another major difference: in the trapped alkali condensates, samples can be prepared in which essentially all of the atoms are Bose condensed. By contrast, in bulk superfluid $^4$He, although the superfluid fraction can be near unity, momentum distribution measurements have shown that the bulk condensate fraction is closer to 0.1 with the remainder of the particles in finite momentum states. As researchers improve their ability to create and manipulate these new trapped gaseous condensates, a series of important questions naturally arise: Does the gaseous BEC support superflow? Is it indeed a superfluid? Are there stable vortices? This last question is the subject of this paper.

The problem of vortex state excitations has been recently treated by others. Sinha investigated the low-lying modes under Thomas-Fermi limit and Dodd et al. obtained the normal mode spectrum of a single quantized vortex state as a function of the number of condensate atoms for a BEC confined in a TOP trap. However, the important question of vortex stability was not addressed by these authors. More recently, Rokhsar studied the stability properties of the trapped vortices and argued that vortices are unstable due to the existence of a bound state inside the vortex core. However, throughout his analysis, the transition from a vortex state to a core state requires the presence of thermal atoms which serve as a reservoir to conserve the energy and angular momentum in the process. Hence, the more fundamental question concerning the intrinsic stability of an isolated vortex (i.e., without the external influence of thermal atoms) remains unanswered.

In this paper, we approach the problem by assuming that all atoms are in the condensate such that scattering with thermal background atoms can be neglected. This allows us to focus on the intrinsic coupling within and between different vortex states and on the effect of this coupling on vortex stability. (Here, we use the word “intrinsic” to emphasize coherent coupling between the condensate atoms.) We find that stable vortex states can in fact be supported and show that whether a vortex state is stable or not is determined by its angular momentum and the nonlinear inter-particle interaction strength. Furthermore, we point out that the lifetime of an unstable vortex can be directly determined from the frequencies of the collective excitations.

The paper is organized as follows. In Section II, we introduce our model and define the stability criterion. Our main results are presented in Section III, where the stable and unstable regions of trapped vortices are identified.
We also present a physical interpretation of the meaning of the instability. Finally, we give a summary and compare our work with others in Section IV.

II. PHYSICAL MODEL

To simplify our calculations, we consider a condensate confined in a 2d isotropic harmonic potential with trap frequency $\omega_0$ at zero temperature. In current experiments, condensates are achieved in 3d traps with cylindrical symmetry. A quasi-2d situation can be realized when $\omega_\perp \ll \omega_z$, where $\omega_\perp$ and $\omega_z$ are transverse and longitudinal trap frequencies respectively. In this limit, one can produce a “pancake”-shaped condensate with all the atoms lying exponentially in time and hence induce instability.

In the Gross-Pitaevskii treatment, the energy for $N$ condensed bosons of mass $m$ is given by the functional:

$$\frac{E(\Psi_\kappa)}{N} = \int d\mathbf{r} \left( \Psi_\kappa^* \hat{T} \Psi_\kappa + \hat{V} |\Psi_\kappa|^2 + \frac{1}{2} NU |\Psi_\kappa|^4 \right),$$

Here

$$\Psi_\kappa(\mathbf{r}) = \Phi_\kappa(r) e^{i\kappa \theta}, \quad \kappa = 0, \pm 1, \pm 2, \ldots$$

represents the wavefunction of the macroscopic vortex state with azimuthal angular momentum $\kappa \hbar$. $\hat{T} = -\hbar^2 \nabla^2 / 2m, \hat{V} = m \omega_0^2 r^2 / 2$ are the kinetic and potential energy operators, respectively, and the coupling constant $U$ describes the interactions between condensate atoms. In the quasi-2d situation considered here, the coupling constant takes the form $U = 4\sqrt{\pi \hbar \omega_z}$, where $\xi_z = \sqrt{\hbar / 2m \omega_z}$ is the harmonic oscillator length in $z$-dimension. In our analysis, the solution that minimizes Eq. (1) is found iteratively using a finite elements method (FEM). In our calculations, we normally used 20 elements, with 2 nodes and 3 degrees of freedom for each element. This numerical method is very efficient, and it typically took no more than a few minutes to find the wavefunction $\Psi_\kappa$ in a Cray-YMP2E/232.

With the solution of $\Psi_\kappa$ at our disposal, we can now calculate collective excitation frequencies by solving Bogoliubov equations:

$$\left( \mathcal{L} - \hbar \omega_\lambda - \mu_\kappa \right) u_\lambda(\mathbf{r}) + NU|\Psi_\kappa(\mathbf{r})|^2 v_\lambda^*(\mathbf{r}) = 0,$$

$$NU|\Psi_\kappa^*(\mathbf{r})|^2 u_\lambda(\mathbf{r}) + \left( \mathcal{L} + \hbar \omega_\lambda - \mu_\kappa \right) v_\lambda^*(\mathbf{r}) = 0,$$

(3a)

(3b)

where $\mu_\kappa$ is the chemical potential for state $\Psi_\kappa(\mathbf{r})$, $u_\lambda(\mathbf{r}), v_\lambda(\mathbf{r})$ are normal mode functions with mode frequency $\omega_\lambda$, and $\mathcal{L} = \hat{T} + \hat{V} + 2NU|\Psi_\kappa(\mathbf{r})|^2$. It is straightforward to show that if $\Psi_\kappa(\mathbf{r})$ is given by Eq. (2), then $u_\lambda(\mathbf{r}), v_\lambda(\mathbf{r})$ must have definite angular momentum compositions $\kappa_u \hbar$ and $\kappa_v \hbar$ respectively such that $u_\lambda(\mathbf{r}) = \tilde{u}_\lambda(\mathbf{r}) e^{i\kappa_u \theta}, v_\lambda(\mathbf{r}) = \tilde{v}_\lambda(\mathbf{r}) e^{i\kappa_v \theta}$, and $\kappa_u + \kappa_v = 2\kappa$.

III. RESULTS AND INTERPRETATION

Eqs. (3) were transformed to an eigenvalue problem for a finite-sized matrix and solved using the FEM. Our goal here is to find mode frequencies with non-zero imaginary part, in order to determine the vortex stability. As in the case of ground state, the vortex stability properties for a condensate with repulsive inter-particle interaction are drastically different from that for a condensate with attractive interaction. We will discuss these two cases separately.

**Repulsive interaction, i.e., $U > 0$** — When we calculate the collective excitation frequencies of a single quantized ($\kappa = 1$) vortex state $\Psi_1$, we find that all the excitation frequencies are real, which means that $\Psi_1$ is always stable. Next, we consider a double quantized ($\kappa = 2$) vortex state $\Psi_2$. Here we find that complex frequencies only exist for $\kappa_u = 0$ and $\kappa_v = 4$ (Without loss of generality, we assume $\kappa > 0$, and $\kappa_v > \kappa > \kappa_u$). We find that for any other pairs of $(\kappa_u, \kappa_v)$, the excitation frequencies are all real. Furthermore, for values of $NU$ for which the vortex is unstable, we find that there exists at most one complex frequency. Fig. 1(a) shows the imaginary part of the complex frequency as a function of interaction strength $NU$. As we can see, in this particular channel (i.e. choice of $(\kappa_u, \kappa_v)$), the parameter space of $NU$ is divided into alternating stable and unstable regions. In the unstable regions, the inverse of
Im(\( \omega \)) determines the lifetime of the unstable vortex. For the parameter range described in Fig. 1, the most unstable vortex state will decay after several periods of the harmonic trapping potential. We stress that the details of how the condensate will evolve under these instabilities is beyond the capability of the mean-field treatment and requires further investigation.

For a general state \( \Psi_\kappa \), our numerical calculations show that there are \((\kappa - 1)\) unstable channels that possess complex excitation frequencies; those with \( \kappa_u = 0, 1, ..., \kappa - 2 \) and \( \kappa_v = 2\kappa - \kappa_u \). Fig. 1(b) shows the imaginary part of the complex frequency for a triple quantized vortex state \( \Psi_3 \). We can see a similar pattern as in Fig. 1(a), but here, there are two unstable channels. Each channel shows its own quasi-periodic behavior as a function of \( NU \). The two channels have quite different “period” and characteristic width of unstable regions. At first look, this may appear rather unexpected. To interpret this behavior we will show that each unstable region in \( NU \)-space represents a decay channel in which two atoms from the given vortex state scatter into two new states, with angular momenta \( \kappa_u \hbar \) and \( \kappa_v \hbar \), respectively, thus inducing instability in that initial vortex state \( \Psi_\kappa \).

First, let us define a boson field operator as: \( \hat{\Psi}(r) \equiv \sqrt{N} \Psi_\kappa(r) + \hat{\psi}(r) \), where the c-number \( \Psi_\kappa(r) \) denotes the one-body wavefunction for the condensate and \( \hat{\psi}(r) \) is the field operator for the fluctuation part \[16\]. The second quantized Bogoliubov Hamiltonian reads:

\[
\hat{K}_B = \int dr \, \hat{\psi}^\dagger(r) [L - \mu_\kappa] \hat{\psi}(r) + \left[ \frac{1}{2} NU \int dr \, \hat{\psi}^\dagger(r) \hat{\psi}^\dagger(r) \psi_\kappa \psi_\kappa + h.c. \right] ,
\]

where the c-number part independent of \( \hat{\psi}(r) \) has been neglected. We can further decompose \( \hat{\psi}(r) \) as \( \hat{\psi}(r) = \sum_{n,\alpha} a_{n,\alpha} \phi_{n,\alpha}(r) \), where \( a_{n,\alpha} \) is an annihilation operator associated with a single-particle state \( \phi_{n,\alpha} \). The set of states \( \{ \phi_{n,\alpha} \} \) is defined as the eigenvectors of \( L \) with eigenvalues \( \epsilon_{n,\alpha} \), i.e., \( L \phi_{n,\alpha} = \epsilon_{n,\alpha} \phi_{n,\alpha} \), with subscripts \((n, \alpha)\) labeling the radial and angular quantum number respectively. Hamiltonian \( \hat{K}_B \) may then be rewritten as: \( \hat{K}_B = \hat{H}_0 + \hat{H}_1 \), where

\[
\hat{H}_0 = \sum_{n,\alpha} (\epsilon_{n,\alpha} - \mu_\kappa) a_{n,\alpha}^\dagger a_{n,\alpha},
\]

\[
\hat{H}_B = \sum_{n_u,\kappa_u} \sum_{n_v,\kappa_v} \Lambda \left(n_u,\kappa_u; n_v,\kappa_v\right) a_{n_u,\kappa_u}^\dagger a_{n_v,\kappa_v}^\dagger + h.c.,
\]

and
agreement between the two results. The agreement can be significantly improved if the contribution from states NU shown in Fig. 1, particularly for large different particle states. Further work would be necessary in order to understand all aspects of the complex structure is because a strong interaction can drastically change the frequencies of the oscillators and introduce mixing among our vortex stability predictions, the parametric resonance picture is valid only for the weak interaction regime. This along with the imaginary part of the complex excitation frequencies of vortex state Ψ

we can calculate the decay rate of the double quantized vortex state Ψ

condition (6), in support of our prediction that a single quantized vortex state is always stable for κ = 2 vortex, we find a pair of particle states (κ, φ = 0, κ, φ = 4) indeed satisfy inequality (6). In the weak coupling limit, we can calculate the decay rate of the double quantized vortex state Ψ using the Hamiltonian (4) by neglecting all the nonresonant terms (i.e., keeping only terms with κ = 0, κ = 0, ν = 4). The results are shown in Fig. 2 along with the imaginary part of the complex excitation frequencies of vortex state Ψ. We can see a clear qualitative agreement between the two results. The agreement can be significantly improved if the contribution from states (κ, φ = 0, φ = 4) is also included in calculating the decay rate (see Fig. 2). We remark that although useful for interpreting our vortex stability predictions, the parametric resonance picture is valid only for the weak interaction regime. This is because a strong interaction can drastically change the frequencies of the oscillators and introduce mixing among different particle states. Further work would be necessary in order to understand all aspects of the complex structure shown in Fig. 1, particularly for large NU.

\[ \Lambda(n_u, \kappa_u; n_v, \kappa_v) = \frac{1}{2} NU \int d\mathbf{r} \, \phi^*_{n_u, \kappa_u} (\mathbf{r}) \phi^*_{n_v, \kappa_v} (\mathbf{r}) \Psi_\kappa (\mathbf{r}) \Psi_\kappa (\mathbf{r}) . \]  

In the interaction picture, \[ a_{n,\alpha}^\dagger (t) = a_{n,\alpha}^\dagger e^{i(\epsilon_{n,\alpha} - \mu) t} , \] and the Hamiltonian is given by:

\[ \hat{H}_I(t) = \sum_{n_u,\alpha} \sum_{n_v,\kappa} \Lambda(n_u, \kappa_u; n_v, \kappa_v) e^{i(\epsilon_{n_u, \kappa_u} + \epsilon_{n_v, \kappa_v} - 2\mu) t} a_{n_u, \kappa_u}^\dagger a_{n_v, \kappa_v} + h.c. \]  

and hence the vortex is unstable under such resonance condition. We emphasize that the instability implied in this picture is purely quantum mechanical. The atoms in the vortex can spontaneously disintegrate into \[ \phi_{n_u, \kappa_u} \] and \[ \phi_{n_v, \kappa_v} \] states without the need of external (classical) perturbations, such as the interaction with the thermal background gases or perturbation of the trap. For a κ = 1 vortex, our numerical calculations show that there exists no particle states that satisfy the resonance condition (4), in support of our prediction that a single quantized vortex state is always stable for \( U > 0 \). For a κ = 2 vortex, we find a pair of particle states (\( \phi_{0,0}, \phi_{0,4} \)) indeed satisfy inequality (4). In the weak coupling limit, we can calculate the decay rate of the double quantized vortex state Ψ_2 using the Hamiltonian (4) by neglecting all the nonresonant terms (i.e., keeping only terms with \( n_u = n_v = 0, \kappa_u = 0, \kappa_v = 4 \)). The results are shown in Fig. 2 along with the imaginary part of the complex excitation frequencies of vortex state Ψ_2. We can see a clear qualitative agreement between the two results. The agreement can be significantly improved if the contribution from states (\( \phi_{1,0}, \phi_{1,4} \)) is also included in calculating the decay rate (see Fig. 2). We remark that although useful for interpreting our vortex stability predictions, the parametric resonance picture is valid only for the weak interaction regime. This is because a strong interaction can drastically change the frequencies of the oscillators and introduce mixing among different particle states. Further work would be necessary in order to understand all aspects of the complex structure shown in Fig. 1, particularly for large NU.

Attractive interaction, i.e., \( U < 0 \) — A condensate with strong attractive inter-particle interaction is known to be subject to collapse. However, a metastable condensate with a small number of atoms can still exist [4,17]. Fig. 3 shows the imaginary part of the complex excitation frequency for a single and a double quantized vortex state as functions of NU. Fig. 3(a) shows that Ψ_1 is stable for sufficiently small attractive interaction, but unstable for larger interaction strength. For Ψ_2, as we can see from Fig. 3(b), the channel (κ = 0, κ = 4) possesses complex frequency for all negative values of NU instead of showing a quasi-periodic pattern as in the case of repulsive interaction. Furthermore, we find that, similar to Ψ_1, other channels which are stable for NU > 0 become consistently unstable for sufficiently large NU. We only show two such channels in Fig. 3(b).]. Our calculations show that, for \( U < 0 \), stable vortices only exist for a single quantized vortex state in the weak interaction regime [see Fig. 3(a)]; multiple quantized vortex state
(i.e., \( \kappa > 1 \)) is always unstable. It has been speculated that the existence of vortices may help stabilize a condensate with negative scattering length \[18\]. However, as we show here, although such vortices may seem to be more stable against the collapse when compared to the ground state, they remain fundamentally unstable and small fluctuations will eventually destroy such vortices.

![Graph showing complex frequency](image)

**FIG. 3.** Imaginary part of the complex frequency of a single (a) and a double (b) quantized vortex state as a function of interaction strength \( NU \), with \( NU < 0 \).

**IV. SUMMARY AND DISCUSSION**

In summary, we have calculated the collective excitation frequencies of a Bose-Einstein condensate in a vortex state and have established intrinsic stability regions for these vortices. We have shown that, even without any perturbation, an unstable vortex can still decay spontaneously. For repulsive inter-particle interaction, we found that single quantized vortices are always stable, while imaginary excitation modes divide the interaction energy axis \( NU \) of multiple quantized vortices \( \kappa > 1 \) into alternating stable and unstable regions. Hence, one can control the vortex stability by varying the value of interaction strength, which in turn can be achieved by changing the scattering length \[19,20\], particle number or trap frequency. This provides us with the possibility of studying condensate evolution under the effect of imaginary modes.

For a condensate in vortex state, there may exist quasiparticle states with negative frequencies. One such negative frequency state was identified by Dodd et al. in Ref. \[8\]. The presence of negative frequencies implies that there exist states with lower energy. However, this does not necessarily mean that the condensate is unstable if no mechanism exists to drive the system to these lower energy states \[21\]. In Ref. \[9\], Rokhsar considered the instability arising from the incoherent interactions between condensate and thermal atoms, which induce the transition to the negative frequency core state. In contrast, in the present paper, we study the intrinsic stability of vortices in a pure condensate, excluding such incoherent processes while focusing on the coherent interactions within the condensate. In our work, instability occurs as a **coherent process** such that an unstable vortex state will disintegrate into different angular momenta states. We found that stable vortices can be supported in harmonic traps as long as the temperature is low enough such that the effects of thermal atoms are insignificant. At temperatures when thermal atoms cannot be neglected, both coherent and incoherent processes will be present and each will have its effect on vortex stability. It remains to be seen which process will be dominant. Further investigations should also include the possible influence of trap anisotropy and the dynamics of the disintegration processes.

Recently, vortex stability in 2d harmonic trap is studied by Caradoc-Davies et al. through a direct numerical simulation \[22\]. In that study, a blue detuned laser beam is applied to perturb the condensate in a vortex state. They found that the single quantized vortex is indeed stable, while a double quantized vortex can disintegrate into unit vortices under external perturbation. These results are consistent with ours presented in this paper.
Finally, as an example, let us consider a $^{23}$Na condensate (scattering length $a \approx 3$ nm) in a harmonic trap with $\omega_\perp = 2\pi \times 10$ Hz and $\omega_z = 2\pi \times 200$ Hz, in units of $\hbar \omega_\perp \xi_\perp^2$, $U \approx 0.02$. The plotted range of $NU$ from 0 to 4000 in Fig. 1 corresponds to particle number ranging from 0 to $2 \times 10^5$, well within the capability of current experiments. Recently, several methods on how to generate vortex states in alkali atomic BECs have been proposed [23]. With current technology and fast progress on this field, our study on vortex stability should be experimentally testable in the near future.

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