ON NONEXISTENCE OF SOLUTIONS TO SOME NONLINEAR
PARABOLIC INEQUALITIES

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ABSTRACT. We obtain sufficient conditions for nonexistence of positive solutions to some nonlinear parabolic inequalities with coefficients possessing singularities on unbounded sets.

1. Introduction. The problem of sufficient conditions for nonexistence of solutions to nonlinear parabolic differential equations and inequalities with singular coefficients was studied by many authors.

For the Laplacian and heat operator with a point singularity inside the domain, pioneering results in this direction were obtained by H. Brezis and X. Cabre [1] by means of comparison principles.

For higher order operators that do not satisfy the comparison principle, S. Pohozaev [6] suggested the nonlinear capacity method. Later it was developed in joint works with E. Mitidieri and other authors (see, in particular, monograph [5] and references therein). This method allowed one to obtain a number of new sharp sufficient conditions of non-solvability of differential inequalities in various functional classes. The method is based on deriving asymptotically optimal a priori estimates of the solutions by means of algebraic analysis of the integral form of the inequality under consideration with a special choice of test functions. Some applications of this method to nonlinear parabolic inequalities with singular coefficients at a single point or on the boundary can be found in [2] and [7, 8].

In the present paper, a modification of the nonlinear capacity method is used in order to obtain sufficient conditions of non-solvability for nonlinear parabolic inequalities both of second and of higher order, including those with nonlinear principal terms and, most importantly, with coefficients having singularities on some unbounded sets inside the domain of definition. This distinguishes the problem setting suggested here from the previous works in this field, where singularities appeared on certain bounded sets instead.

For the proof of nonexistence results by the nonlinear capacity method, test functions with different geometrical structure of the support are constructed, which takes into account the specific nature of problems under consideration. Our first results in this direction were published in [3, 4].

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The rest of the paper consists of four sections. In §2, we formulate geometrical assumptions on the set where the coefficients of the problems have singularities, and construct appropriate families of test functions. In §3, we establish nonexistence results for higher order semilinear parabolic inequalities, and in §4, for second order quasilinear ones.

**Remark on notation.** From here on, letter \( c \) denotes different positive constants, which may depend on the parameters of the problems under consideration. Letters \( c_0, c_1, \ldots \) denote absolute positive constants. For \( q > 1 \), we denote by \( q' \) the conjugate exponent defined by \( \frac{1}{q} + \frac{1}{q'} = 1 \) (that is, \( q' = \frac{q}{q-1} \)).

2. Assumptions on the singular set and test functions. We consider coefficients which have singularities on a closed possibly unbounded set \( S \subset \mathbb{R}^n \), which has geometrical structure characterized by the following assumptions.

Let \( \varepsilon > 0 \). Denote \( \rho(x) = \text{dist}(x, S) = \inf\{|x - y| : y \in S\} \quad (x \in \mathbb{R}^n) \) and \( S^\varepsilon = \{x \in \mathbb{R}^n : \rho(x, S) < \varepsilon\} \).

Assume that:

\( (H_1) \) There exist constants \( c_0 > 0 \) and \( \theta > 0 \) such that for sufficiently large \( R > 0 \) one has
\[
\mu((S^\frac{1}{2} \setminus S^\frac{1}{2}) \cap B_{2R}(0)) \leq c_0 R^{n-\theta}.
\]

\( (H_2) \) For some \( k \in \mathbb{N} \), there exist a family of functions \( \xi_R \in C^{2k}_0(\mathbb{R}^n \setminus S; [0,1]) \) such that
\[
\xi_R(x) = \begin{cases} 
0 & (x \in S^\frac{1}{4} \cup (\mathbb{R}^n \setminus S^{2R})), \\
1 & (x \in S^R \setminus S^\frac{1}{4})
\end{cases}
\]
and a constant \( c > 0 \) such that
\[
|D^\alpha \xi_R(x)| \leq c \rho^{-|\alpha|} \quad (x \in \mathbb{R}^n)
\]
for all multi-indices \( \alpha \) with \( 0 \leq |\alpha| \leq k \).

**Example 1.** As the set \( S \) one can consider a hyperplane \( S = \Pi_n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n = 0\} \). Then for all sufficiently large \( R > 0 \) one has
\[
\mu(\Pi_n^{1/R} \cap B_R(0)) = m_{n-1} R^{n-2} + o(R^{n-2}),
\]
where \( m_{n-1} \) is the measure of a unit ball in \( \mathbb{R}^{n-1} \). Thus \( \Pi_n \) satisfies assumption \( (H_1) \) with \( c_0 = 2m_{n-1} \) and \( \theta = 2 \). Similar estimates hold for any hyperplane in \( \mathbb{R}^n \).

For \( l \)-dimensional planes \( (1 \leq l \leq n-1) \), assumption \( (H_1) \) holds with \( \theta = n - l \).

To verify \( (H_2) \) for \( S = \Pi_n \), one can take functions \( \xi_R(x) = \tilde{\xi}_R(x_n) \), where
\[
\tilde{\xi}_R(x_n) = \begin{cases} 
0 & \left(\frac{1}{2R} \leq |x_n| \leq 2R \right), \\
1 & \left(\frac{1}{R} \leq |x_n| \leq R \right)
\end{cases}
\]
(see Fig. 1).
In some of our results, we will require that the following assumptions are valid:

\((H^*_1)\) There exist positive constants \(c_1, c_2, \theta\) such that for sufficiently large \(R > 0\) one has
\[
c_1 R^{n-\theta} \leq \mu((S^{3/R} \setminus S^{2/R}) \cap B_R(0)) \leq \mu((S^{4/R} \setminus S^{1/R}) \cap B_R(0)) \leq c_2 R^{n-\theta}.
\] (5)

\((H^*_2)\) There exists a family of functions \(\xi_{\mathbf{n}} \in C^0_0(\mathbb{R}^n \setminus S; [0, 1])\) such that
\[
\xi_{\mathbf{n}}(x) = \begin{cases} 
0 & (x \in S^{1/R} \cup (\mathbb{R}^n \setminus S^{4/R})) \\
1 & (x \in S^{3/R} \setminus S^{2/R})
\end{cases}
\] (6)
and (3) holds for all multi-indices \(\alpha\) with \(0 \leq |\alpha| \leq k\).

**Example 2.** Consider again the hyperplane \(S = \Pi_n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n; x_n = 0\}\). Then, by inequality (8), \(\Pi_n\) satisfies assumption \((H^*_1)\) with any \(c_1 \in (0, (3^n - 1 - 2^{n-1})m_{n-1})\), \(c_2 > (4^{n-1} - 1)m_{n-1}\) and \(\theta = 2\). Similar estimates hold for any hyperplane in \(\mathbb{R}^n\). For \(l\)-dimensional planes \((1 \leq l \leq n-1)\) assumption \((H^*_1)\) holds with \(\theta = n - l\).

To verify \((H^*_2)\) for \(S = \Pi_n\), one can take \(\xi_{\mathbf{n}}(x) = \tilde{\xi}_{\mathbf{n}}(x_n)\), where \(|x'| = \sqrt{\sum_{i=1}^{n-1} x_i^2}\) and
\[
\tilde{\xi}_{\mathbf{n}}(x_n) = \begin{cases} 
0 & (|x_n| \leq 1/R \text{ or } |x_n| \geq 4/R) \\
1 & (2/R \leq |x_n| \leq 3/R)
\end{cases}
\]
We will also use functions $\psi_R \in C^k_0(\mathbb{R}^n; [0, 1])$ such that
\[
\psi_R(x) = \begin{cases} 
1 & (|x| \leq R), \\
0 & (|x| \geq 2R)
\end{cases}
\] (7)
and a constant $c > 0$ such that
\[
|D^\alpha \psi_R(x)| \leq cR^{-|\alpha|} \quad (x \in \mathbb{R}^n)
\] (8)
holds for all multi-indices $\alpha$ with $0 \leq |\alpha| \leq k$ (see $\tilde{\psi}_R(x) = \tilde{\psi}_R(|x|)$, where
\[
\tilde{\psi}_R(x') = \begin{cases} 
1 & (x' \leq R), \\
0 & (x' \geq 2R)
\end{cases}
\] (7)
on Fig. 3).

3. Semilinear parabolic inequalities of higher order. A nonexistence result takes place for the semilinear parabolic inequality
\[
\frac{\partial u}{\partial t} + \sum_{|\alpha| \leq k} D^\alpha (A^\alpha(x, u)) \geq au^q |x|^\beta \rho^{-\gamma}(x) \quad (x \in \mathbb{R}^n \setminus S; t \in \mathbb{R}_+)
\] (9)
with the initial condition
\[
u(x, 0) = u_0(x) \geq 0 \quad (x \in \mathbb{R}^n \setminus S).
\] (10)

Define weak solutions of the Cauchy problem (9)–(10) in the following way.

**Definition 3.1.** A nonnegative function $u \in C(Q)$ is called a weak (local) solution of the Cauchy problem (9)–(10), if it satisfies the integral inequality
\[
a \int_{t_0}^{t_1} \int_{\mathbb{R}^n \setminus S} u^q |x|^\beta \rho^{-\gamma} \, dx \, dt - \int_{\mathbb{R}^n \setminus S} (u(x, t_1)\phi(x, t_1) - u(x, t_0)\phi(x, t_0)) \, dx
\leq \int_{t_0}^{t_1} \int_{\mathbb{R}^n \setminus S} \left( -u \frac{\partial \phi}{\partial t} + \sum_{|\alpha| \leq k} (-1)^{|\alpha|} A^\alpha(x, u) D^\alpha \phi \right) \, dx \, dt
\] (11)
for some $t^* > 0$, for all $t_0, t_1$ such that $0 \leq t_0 < t_1 \leq t^*$, and for any nonnegative function $\phi \in C^1((\mathbb{R}^n \setminus S) \times [t_0, t_1])$ such that for all $t \in [t_0, t_1]$ one has $\phi(\cdot, t) \in C^1_0(\mathbb{R}^n \setminus S)$, provided that all integrals exist and $\lim_{t \to 0^+} u(x, t) = u_0(x)$ for all $x \in \mathbb{R}^n \setminus S$. The supremum of all possible values $\tau = t_1 - t_0$ is called the life span of the solution $u$. If it is infinite, the solution is called *global*. 
Remark 1. If \( u \) and \( A^\alpha \) are sufficiently regular, inequality (11) can be derived from (9)–(10) by integration by parts.

Sufficient conditions for nonexistence of solutions to the Cauchy problem (9)–(10) in the given sense can be formulated as follows.

**Theorem 3.2.** Let the set \( S \) satisfy conditions \((H_1)\) and \((H_2)\). Suppose that the initial function \( u_0 \in C(\mathbb{R}^n \setminus S) \) is nonnegative, \( a > 0 \), and the coefficients \( A^\alpha : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) are Carathéodory functions such that for all \( \alpha : |\alpha| \leq k \) there exist functions

\[
a_\alpha : \mathbb{R}^n \to \mathbb{R}_+, \quad a_\alpha \in L^\infty_{\text{loc}}(\mathbb{R}^n_+)
\]

that satisfy the estimate

\[
|A^\alpha(x,t)| \leq a_\alpha(x)|t| \quad \text{for almost all } (x,t) \in \mathbb{R}^n \times \mathbb{R}_+
\]

with \( q > 1, \gamma \geq 0, \)

\[
2(kq - \gamma) < \theta(q - 1) \tag{12}
\]

and

\[
n + k + \frac{\gamma - kq - \beta}{q - 1} < 0. \tag{13}
\]

Then problem (9) – (10) has no global nonnegative solutions \( u \) in \((\mathbb{R}^n \setminus S) \times \mathbb{R}_+\) that are distinct from the identical zero.

**Proof.** For the Cauchy problem (9)–(10) we introduce test functions

\[
\phi(x,t) = \varphi_R(x)T_\tau(t), \quad \text{where } \varphi_R(x) \equiv (\xi_R(x)\psi_R(x))^\kappa \text{ depend only on spatial variables and } T_\tau(t) \text{ on time. Here } \xi_R \text{ are defined as in } (H_2), \psi_R \text{ satisfy (7)–(8), } \kappa > \frac{kq}{q - 1}, \text{ and } T_\tau \in C^1([0,\tau];[0,1]) \text{ with } \tau > 0 \text{ are such that}
\]

\[
T_\tau(t) = \begin{cases} 1 & (0 \leq t \leq \tau/2), \\ 0 & (3\tau/4 \leq t \leq \tau) \end{cases}
\]

and besides

\[
\int_{\tau/2}^{3\tau/4} \frac{|T_\tau'|^{q'}}{|T_\tau|^{q-1}} dt \leq c\tau^{1-q'} \tag{14}
\]

with some constant \( c > 0 \).

Multiplying both parts of (9) by \( \varphi_R(x)T_\tau(t) \) and integrating in parts, we get

\[
a \int_{\mathbb{R}_+} T_\tau dt \int_{\mathbb{R}^n} u^q|\varphi_R|^\beta dx + \int_{\mathbb{R}^n} u_0\varphi_R dx
\]

\[
\leq c \int_{\mathbb{R}_+} T_\tau dt \int_{\mathbb{R}^n} u \sum_{|\alpha| \leq k} D^\alpha \varphi_R dx - \int_{\mathbb{R}_+} T_\tau' dt \int_{\mathbb{R}^n} u\varphi_R dx
\]

\[
\leq c \int_{\mathbb{R}_+} T_\tau dt \int_{\mathbb{R}^n} u \sum_{|\alpha| \leq k} D^\alpha \varphi_R dx + \int_{\mathbb{R}_+} |T_\tau'| dt \int_{\mathbb{R}^n} u\varphi_R dx.
\]
Applying the parametric Young inequality to both terms on the right-hand side, we arrive at

\[
\frac{a}{2} \int_{\mathbb{R}^+} T_\tau dt \int_{\mathbb{R}^n} u^q \rho^{-\gamma} |x|^\beta \varphi_R dx + \int_{\mathbb{R}^n} u_0 \varphi_R dx
\]

\[
\leq c_1 \int_{\mathbb{R}^+} T_\tau dt \int_{\mathbb{R}^n} \left| \sum_{|\alpha| \leq k} D^\alpha \varphi_R \right| \rho^{\frac{\alpha}{\beta}} |x|^{-\frac{\beta}{\alpha}} \varphi_R^{\frac{1}{\beta}} dx
\]

\[
+ c_2 \int_{\mathbb{R}^+} \left| T_\tau \right|^{\frac{q}{\beta}} T_\tau^{\frac{1}{\beta}} dt \int_{\mathbb{R}^n} \rho^{\frac{\alpha}{\beta}} |x|^{-\frac{\beta}{\alpha}} \varphi_R dx
\]

with some constants \(c_1, c_2 > 0\).

Due to the choice of \(\varphi_R(x)\) and \(T_\tau(t)\), we can restrict integration to smaller domains on both sides of the inequality:

\[
\frac{a}{2} \int_{\mathbb{R}^+} T_\tau dt \int_{(S^R \setminus S^{2\tau}) \cap B_R(0)} u^q \rho^{-\gamma} |x|^\beta \varphi_R dx + \int_{(S^R \setminus S^{2\tau}) \cap B_R(0)} u_0 \varphi_R dx
\]

\[
\leq c_1 \int_{\mathbb{R}^+} T_\tau dt \int_{(S^R \setminus S^{2\tau}) \cap B_R(0)} \left| \sum_{|\alpha| \leq k} D^\alpha \varphi_R \right| \rho^{\frac{\alpha}{\beta}} |x|^{-\frac{\beta}{\alpha}} \varphi_R^{\frac{1}{\beta}} dx
\]

\[
+ c_2 \int_{\mathbb{R}^+} \left| T_\tau \right|^{\frac{q}{\beta}} T_\tau^{\frac{1}{\beta}} dt \int_{(S^R \setminus S^{2\tau}) \cap B_R(0)} \rho^{\frac{\alpha}{\beta}} |x|^{-\frac{\beta}{\alpha}} \varphi_R dx.
\]

Note that the second term on the left-hand side is nonnegative and \(\varphi_R(x) \equiv 1\) in the whole integration domain, and the first integral on the right-hand side is estimated by condition (12). Representing \(\varphi_R(x) \equiv \xi_R(x)\psi_R(x)\) and using (3), (8) and (14) with \(q = q'\), we get that

\[
\int_{\mathbb{R}^+} T_\tau dt \int_{(S^R \setminus S^{2\tau}) \cap B_R(0)} u^q \rho^{-\gamma} |x|^\beta dx
\]

\[
\leq c R^{n - \frac{\beta}{\gamma + \theta}} \left[ R^{-\frac{\beta q}{\gamma + \theta}} + R^{-\theta + \frac{\beta q}{\gamma + \theta}} \right] \left( R^\frac{\alpha}{\beta} + R^{-\theta - \frac{\beta q}{\gamma + \theta}} \right),
\]

which due to (12) for sufficiently large \(R\) implies

\[
\int_{\mathbb{R}^+} T_\tau dt \int_{(S^R \setminus S^{2\tau}) \cap B_R(0)} u^q \rho^{-\gamma} |x|^\beta dx \leq c R^{n - \frac{\beta q}{\gamma + \theta}} \left( R^{-\frac{\alpha}{\beta}} + R^{-\theta - \frac{\beta q}{\gamma + \theta}} \right). \quad (15)
\]

One can easily see that the right-hand side of (15) attains its minimum at

\[
\tau = c R^k. \quad (16)
\]

Substituting (16) into (15) and taking \(R \to \infty\), under assumption (13) we arrive at a contradiction, which proves the claim.

Under additional assumptions on the behavior of the initial function we can obtain sufficient conditions for nonexistence not only for global solutions of problem (9) − (10) but also for local ones. Namely, there holds
Theorem 3.3. Let the set \( S \) satisfy conditions \((H'_1)\) and \((H'_2)\), \( k \in \mathbb{N}, q > 1 \), and the initial function \( u_0 \in C(\mathbb{R}^n \setminus S) \) satisfies the inequality
\[
u_0(x) \geq c_0|x|^\beta \rho^\gamma(x) \quad (x \in \mathbb{R}^n \setminus S)
\] (17)
with some constants \( c_0 > 0 \) and \( \mu \in \mathbb{R} \), where
\[
\beta + \gamma > (\mu - \lambda)(q - 1) + k.
\] (18)

Then the Cauchy problem (9) – (10) has no positive solutions \( u \) in \((\mathbb{R}^n \setminus S) \times [0, T] \) for any arbitrarily small \( T > 0 \).

Proof. Repeating the previous arguments with functions \( \varphi_R(x) \equiv \xi_R(x)\psi_R(x) \), where \( \xi_R(x) \) are from \((H'_2)\), similarly to (15) we obtain the estimate
\[
\int_{(S^\# \setminus S^\#')\cap B_n(0)} u_0(x) \, dx \leq cR^{n-\theta - \frac{eta + \gamma}{q} T} \left( \tau^{\frac{q}{q-1}} + R^{\frac{kq}{q-1}} \right).
\] (19)
One can easily see that the right-hand side of (19) attains its minimum at
\[
\tau = cR^{-k}.
\] (20)

Substituting (20) into (19), estimating the left-hand side of the obtained inequality from below with the help of (17) and passing to the limit as \( R \to \infty \), under assumption (18) we reach a contradiction, which proves the claim.

4. Quasilinear parabolic inequalities. Let \( u_0 \in C(\mathbb{R}^n \setminus S), c > 0 \). Then one can formulate the Cauchy problem
\[
\begin{cases}
\frac{\partial u}{\partial t} - \text{div}(A(x, u, Du)Du) \geq au^q \rho^\gamma(x) & (x, t) \in Q = (\mathbb{R}^n \setminus S) \times \mathbb{R}_+), \\
u(x, 0) = u_0(x) & (x \in \mathbb{R}^n \setminus S).
\end{cases}
\] (21)

We define weak solutions of the Cauchy problem (21) in the following way.

Definition 4.1. A nonnegative function \( u \in C^1(Q) \) is called a weak (local) solution of the Cauchy problem (21) if it satisfies the integral inequality
\[
a \int_{t_0}^{t_1} \int_{\mathbb{R}^n \setminus S} u^q \rho^\gamma \phi \, dx \, dt - \int_{\mathbb{R}^n \setminus S} (u(x, t_1)\phi(x, t_1) - u(x, t_0)\phi(x, t_0)) \, dx \\
\leq \int_{t_0}^{t_1} \int_{\mathbb{R}^n \setminus S} \left( -\frac{\partial \phi}{\partial t} + A(x, u, Du)(Du, D\phi) \right) \, dx \, dt
\] (22)

for some \( t^* > 0 \), for all \( t_0, t_1 \) such that \( 0 \leq t_0 < t_1 \leq t^* \), and for any nonnegative function \( \phi \in C^1((\mathbb{R}^n \setminus S) \times [t_0, t_1]) \) such that for all \( t \in [t_0, t_1] \) one has \( \phi(\cdot, t) \in C^1_0(\mathbb{R}^n \setminus S) \), provided that all integrals exist and \( \lim_{t \to 0^+} u(x, t) = u_0(x) \) for all \( x \in \mathbb{R}^n \setminus S \). The supremum of all possible values \( \tau = t_1 - t_0 \) is called the life span of the solution \( u \). If it is infinite, the solution is called global.

Remark 2. If \( u \) and \( A \) are sufficiently regular, inequality (22) can be derived from (21) by integration by parts.

Sufficient conditions for the nonexistence of solutions to the Cauchy problem (21) in the given sense can be formulated as follows.
Theorem 4.2. Let the set $S$ satisfy conditions $(H_1)$ and $(H_2)$. Suppose that $A$ is a Carathéodory function such that

$$c_1|\eta|^{p-2} \leq A(x, z, \eta) \leq c_2|\eta|^{p-2}$$

for some constants $c_1, c_2 > 0$, the initial function $u_0 \in L^1_{\text{loc}}(\mathbb{R}^n \setminus S)$ is nonnegative, $a > 0$, $p > 1$, $q > \max(1, p - 1)$, $\gamma \geq 0$,

$$p(q - 1) - \gamma(p - 2) > 0,$$

$$2(pq - \gamma(p - 1)) < \theta(q - p + 1)$$

and

$$n + \frac{\gamma - p}{q - p + 1} < 0.$$  

Then the Cauchy problem (21) has no global nonnegative solutions $u$ in $(\mathbb{R}^n \setminus S) \times \mathbb{R}_+$ that are distinct from the identical zero.

Proof. For simplicity consider $A(x, u, Du) = |Du|^{p-2}$. Suppose that a solution $u$ of problem (21) does exist and consider the weak formulation (22) with test functions $\phi(x, t) = u^\nu(x, t)\varphi_R(x)T_\tau(t)$, where $\varphi_R = (\xi_R \cdot \psi_R)^n$, $\kappa > \frac{pq}{q-p+1}$, $\xi_R$ are as in $(H_2)$, $\psi_R$ satisfy assumptions (7)–(8), and for $T_\tau$ one has (14) with $r = \frac{q + \nu}{q - 1}$, where $\nu < 0$ and $|\nu|$ is sufficiently small.

In this case inequality (22) takes the form

$$a \int_0^\tau \int_{\supp \varphi_R} u^{q+\nu} \rho^{-\gamma} \varphi_RT_\tau \, dx \, dt + \int_{\supp \varphi_R} u_0^{\nu+1} \varphi_R \, dx \leq \nu \int_0^\tau \int_{\supp \varphi_R} |Du|^p u^{\nu-1} \varphi_RT_\tau \, dx \, dt + \int_0^\tau \int_{\supp |D\varphi_R|} u^{\nu}|Du|^{p-2} (Du, D\varphi_R)T_\tau \, dx \, dt$$

$$- \int_0^\tau \int_{\supp \varphi_R} u \frac{\partial(u^\nu \varphi_RT_\tau)}{\partial t} \, dx \, dt. \quad (26)$$

Integrating the last term in this inequality by parts twice, we obtain

$$\int_0^\tau \int_{\supp \varphi_R} u \frac{\partial(u^\nu \varphi_RT_\tau)}{\partial t} \, dx \, dt$$

$$= \int_{\supp \varphi_R} u^{\nu+1} \varphi_RT_\tau \, dx \bigg|_0^\tau - \int_0^\tau \int_{\supp \varphi_R} \frac{\partial u}{\partial t} u^{\nu} \varphi_RT_\tau \, dx \, dt$$

$$= \int_{\supp \varphi_R} u^{\nu+1} \varphi_RT_\tau \, dx \bigg|_0^\tau - \frac{1}{\nu + 1} \int_0^\tau \int_{\supp \varphi_R} \frac{\partial(u^{\nu+1})}{\partial t} \varphi_RT_\tau \, dx \, dt$$

$$= \left(1 - \frac{1}{\nu + 1}\right) \int_{\supp \varphi_R} u^{\nu+1} \varphi_RT_\tau \, dx \bigg|_0^\tau + \frac{1}{\nu + 1} \int_0^\tau \int_{\supp \varphi_R} u^{\nu+1} \varphi_R \frac{\partial T_\tau}{\partial t} \, dx \, dt$$

$$= \frac{1}{\nu + 1} \int_{\supp \varphi_R} u^{\nu+1} \varphi_R \, dx + \frac{1}{\nu + 1} \int_0^\tau \int_{\supp \varphi_R} u^{\nu+1} \varphi_R \frac{\partial T_\tau}{\partial t} \, dx \, dt.$$
Therefore (26) can be rewritten as
\[
a \int_0^\tau \int_0^{\supp \varphi_R} u^{q+p} \rho^{-\gamma} \varphi_R^\tau \, dx \, dt + \frac{1}{\nu + 1} \int_0^{\supp \varphi_R} u^{\nu+1} \varphi_R \, dx
\]
\[
\leq \nu \int_0^\tau \int_0^{\supp \varphi_R} |Du|^pu^{\nu-1} \varphi_R^\tau \, dx \, dt + \int_0^\tau \int_0^{\supp |D\varphi_R|} u^p |Du|^{p-2} (Du, D\varphi_R)T_\tau \, dx \, dt
\]
\[
+ \frac{1}{\nu + 1} \int_0^{\supp \varphi_R} u^{\nu+1} \varphi_R^\tau \, dx \, dt.
\]

Further we apply the Young inequality with appropriate parameters to the second and third terms on the right-hand side:
\[
\int_0^\tau \int_0^{\supp |D\varphi_R|} u^p |Du|^{p-2} (Du, D\varphi_R)T_\tau \, dx \, dt \leq \int_0^\tau \int_0^{\supp |D\varphi_R|} u^p |Du|^{p-1} |D\varphi_R| T_\tau \, dx \, dt
\]
\[
\leq |\nu| \int_0^\tau \int_0^{\supp \varphi_R} |Du|^pu^{\nu-1} \varphi_R^\tau \, dx \, dt + c \int_0^\tau \int_0^{\supp |D\varphi_R|} u^{p+\nu-1} |D\varphi_R|^{p^1-p} T_\tau \, dx \, dt
\]
\[
(27)
\]
\[
- \frac{1}{\nu + 1} \int_0^\tau \int_0^{\supp \varphi_R} u^{\nu+1} \varphi_R^\tau \, dx \, dt \leq \frac{1}{\nu + 1} \int_0^\tau \int_0^{\supp \varphi_R} u^{\nu+1} \varphi_R^\tau \, dx \, dt
\]
\[
\leq \frac{q}{4} \int_0^\tau \int_0^{\supp \varphi_R} u^{q+p} \rho^{-\gamma} \varphi_R^\tau \, dx \, dt + c \int_0^\tau \int_0^{\supp |D\varphi_R|} u^{p+\nu-1} |D\varphi_R|^{p^1-p} T_\tau \, dx \, dt
\]
\[
\leq \frac{q}{4} \int_0^\tau \int_0^{\supp \varphi_R} u^{q+p} \rho^{-\gamma} \varphi_R^\tau \, dx \, dt + c \int_0^\tau \int_0^{\supp |D\varphi_R|} u^{p+\nu-1} |D\varphi_R|^{p^1-p} T_\tau \, dx \, dt
\]
\[
(28)
\]
Similarly, the second term on the right-hand side of (28) can be estimated as
\[
c \int_0^\tau \int_0^{\supp |D\varphi_R|} u^{p+\nu-1} |D\varphi_R|^{p^1-p} T_\tau \, dx \, dt
\]
\[
\leq \int_0^\tau \int_0^{\supp |D\varphi_R|} \left( \frac{a}{4} u^{q+p} \rho^{-\gamma} \varphi_R + c |D\varphi_R|^{p^1-p} \rho^{\frac{(p+\nu-1)}{q-p+1}} \varphi_R^{\frac{(1-p)(p+\nu-1)-(q+\nu)}{q-p+1}} \right) T_\tau \, dx \, dt
\]
\[
(30)
\]
Since \( \varphi_R = (\xi_R \cdot \psi_R)^\kappa \), due to (3) and (8) for \( \kappa > \frac{pq}{q-p+1} \) we have
\[
|D\varphi_R(x)|^{\frac{p+\nu}{q-p+1}} \varphi_R^{\frac{(1-p)(p+\nu-1)-\nu}{q-p+1}} = \kappa^p |D(\xi_R \cdot \psi_R)(x)|^{\frac{p+\nu}{q-p+1}} |\xi_R \cdot \psi_R|^{\kappa - \frac{pq}{q-p+1} - \frac{pq}{q-p+1}}
\]
\[
\leq c \rho^{-\frac{p+\nu}{q-p+1}} (x) \quad (x \in \supp |D\xi_R|)
\]
\[
(31)
\]
and similarly
\[
|D\varphi_R(x)|^{\frac{p+\nu}{q-p+1}} \varphi_R^{\frac{(1-p)(p+\nu-1)-\nu}{q-p+1}} = c R^{-\frac{p+\nu}{q-p+1}} (x \in \supp |D\psi_R|).
\]
\[
(32)
\]
Combining inequalities (27)–(32), we obtain

\[
\frac{a}{2} \int_0^\tau \int_{\text{supp } \varphi_R} u^{q+p} \rho^{-\gamma} \varphi_R \, dx \, dt + \frac{1}{\nu + 1} \int_0^\nu u^{\nu+1} \varphi_R \, dx \\
\leq c \left( \int_0^\tau |T'_\tau|^{\frac{q+p}{q-1}} \int_{\text{supp } \varphi_R} \rho^{\frac{q+1}{q-1}} \, dx \\
+ \int_0^\tau \int_{\text{supp } \varphi_R} (\rho^{\frac{q+1-(q+p+1)}{q-1}} + R^{\frac{q+1-(q+p)}{q-1}}) \, dx \right).
\]

(33)

Here we take into account that \( \nu < 0 \). Since \( u_0 \) is nonnegative, we can omit the second term on the left-hand side. This and (14) imply

\[
\int_0^\tau \int_{\text{supp } \varphi_R} u^{q+p} \rho^{-\gamma} \varphi_R \, dx \, dt \leq c_1 R^n \left[ \tau^{\frac{q+1}{q-1}} R^{\frac{q+1}{q-1}} + \tau (R^{l_\nu} + R^{-\theta - l_\nu}) \right],
\]

(34)

where \( c > 0 \) and

\[
l_\nu = \frac{\gamma(q + p - 1) - p(q + \nu)}{q - p + 1}.
\]

(35)

By virtue of (24), the term \( R^{-\theta - l_\nu} \) can be omitted for sufficiently large \( R \) and small \( |\nu| \). Then one can easily see that the right-hand side of (34) attains its minimum at

\[
\tau_* = \left( \frac{\nu + 1}{q - 1} R^{1-\frac{q+1}{q-1}} \right)^{\frac{q-1}{q+1}} = cR^{\frac{\gamma(q+1)-p+1}{q-p+1}},
\]

(36)

where the exponent is positive by assumptions (23) and \( q > p - 1 \). Substituting (36) into (34), we reach a contradiction as \( R \to \infty \) due to assumption (25) if \( |\nu| \) is sufficiently small. This completes the proof.

Under additional assumptions on the behavior of the initial function we can obtain sufficient conditions for nonexistence not only for global solutions of problem (21) but also for local ones. Namely, there holds

**Theorem 4.3.** Let \( S \) satisfy \( (H^*_1) \) and \( (H^*_2) \). Suppose that \( a > 0, p > 1, q > \max(1, p-1), \) (23) holds, and the initial function \( u_0 \in C(\mathbb{R}^n \setminus S) \) satisfies the inequality

\[
u_0(x) \geq c_0 \rho^\mu(x) \quad (x \in \mathbb{R}^n \setminus S)
\]

(37)

with some constants \( c_0 > 0 \) and \( \mu \in \mathbb{R} \), where

\[
\gamma > \mu(q - p + 1) + p.
\]

(38)

Then problem (21) has no positive solutions \( u \) in \((\mathbb{R}^n \setminus S) \times [0, T]\) for any arbitrarily small \( T > 0 \).

**Proof.** The proof of Theorem 4.2 is based on a priori estimates for \( u \) with test functions \( \xi_R(x) \psi_R(x) T_t(t) \), where \( \xi_R \) are defined as in \( (H^*_1) \), and can be completed similarly to that of Theorem 3.2. 

\[\Box\]
To conclude, we consider a quasilinear parabolic problem with a gradient term
\[ \begin{aligned}
\frac{\partial u}{\partial t} - \text{div}(A(x, u, Du)Du) & \geq au^{q-\gamma} - b|Du|^s \\
u(x, 0) &= u_0(x) \quad (x \in \mathbb{R}^n \setminus S),
\end{aligned} \tag{39} \]
the solutions of which can be defined similarly to Definition 4.1.

By an appropriate modification of the proof of Theorem 4.2, one can obtain the following result.

**Theorem 4.4.** Let $S$ satisfy conditions $(H_1)$ and $(H_2)$. Let $q > \max(1, p - 1)$, $0 < s < \frac{pq}{q + 1}$, and let the parameters $\gamma, p, q$ and the initial function $u_0$ satisfy inequalities (23) and (37), as well as
\[ \gamma > \max\{\mu(q - p + 1) + p, \mu(q(p - s) - s)\}. \tag{40} \]
Then problem (39) has no positive solutions $u$ in $(\mathbb{R}^n \setminus S) \times [0, T]$ for any arbitrarily small $T > 0$.

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