Haldane’s asymptotics for Supercritical Branching Processes in an iid Random Environment

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April 19, 2024

Abstract

Branching processes in a random environment are natural generalisations of Galton-Watson processes. In this paper we analyse the asymptotic decay of the survival probability for a sequence of slightly supercritical branching processes in an iid random environment, where the offspring expectation converges from above to 1. We prove that Haldane’s asymptotics, known from classical Galton-Watson processes, turns up again in the random environment case, provided that one stays away from the critical/subcritical regime. A central building block is a connection to and a limit theorem for perpetuities with asymptotically vanishing interest rates.

1 Introduction and main result

In the early twentieth century Fisher [12], Haldane [17] and Wright [35] studied the survival probability of a beneficial mutant gene in large populations. They argued that, as long as the mutant is sufficiently rare and the selective advantage is small, the number of mutants should evolve like a slightly supercritical Galton-Watson process (GWP). As Haldane concluded, the probability \( \pi \) of ultimate survival should obey the asymptotics

\[
\pi \approx \frac{2\varepsilon}{\sigma^2}
\]

for small \( \varepsilon > 0 \), where \( 1 + \varepsilon \) is the offspring expectation and \( \sigma^2 \) the offspring variance. This approximation gained a lot of attention in the literature, see [30] for an overview. For GWPs this asymptotics was considered among others by Kolmogorov [27], Eshel [11], Athreya [3] and Hoppe [21].

In this paper we consider the asymptotic survival probability of a slightly supercritical branching process in a random environment. As it turns out Haldane’s asymptotics remains valid for such processes in case of an iid random environment, as long as one keeps away from the domain of subcritical behaviour. This might come as a surprise, since the accompanying Kolmogorov asymptotics for the survival probability of critical GWPs fails in the case of an iid random environment [13]. Close to subcriticality one observes a smooth adaption of Haldane’s formula.

Let us recall the notion of a branching process in random environment (BPRE). Denote by \( \mathcal{P}(\mathbb{N}_0) \) the space of all probability measures on \( \mathbb{N}_0 = \{0, 1, 2, \ldots\} \). Endow \( \mathcal{P}(\mathbb{N}_0) \) with the total variation metric and the induced Borel-\( \sigma \)-algebra. This allows to consider random probability

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measures on $\mathcal{P}(\mathbb{N}_0)$, namely random variables $P$ with values in $\mathcal{P}(\mathbb{N}_0)$. Relevant quantities related to a random measure $P$ are its (random) mean and (random) second factorial moment,

$$M = \sum_{z=1}^{\infty} z P[z], \quad M^{(2)} = \sum_{z=2}^{\infty} z(z-1) P[z],$$

where $P[z]$ denotes the (random) weight of $P$ at $z \in \mathbb{N}_0$. Next, a random environment $V = (P_1, P_2, \ldots)$ is a sequence of random probability measures. Given such an environment we call $Z = (Z_k, k \geq 0)$ a branching process in the random environment $V$, if it has the representation

$$Z_k = \sum_{i=1}^{Z_{k-1}} U_{i,k}, \quad Z_0 = 1,$$

where conditionally on $V$ the family $(U_{i,k}, i \geq 1, k \geq 1)$ is independent and for each $k \in \mathbb{N}$ the sequence $(U_{i,k}, i \geq 1)$ is identically distributed with distribution $P_k$. Here, we consider BPREs in an iid random environment, in this case the environment $V = (P_1, P_2, \ldots)$ consists of independent copies of a generic random measure $P$. Then, the random variables $U_{i,k}$ are (unconditionally) identically distributed, thus copies of a generic random variable $U$. For an account on BPREs we refer to [26] and the literature cited therein.

A BPRE in an iid environment is called subcritical, critical or supercritical, if $E[\log M]$ is less than, equal to or bigger than 0 (provided the existence of the expectation). If $E[\log M] \leq 0$, then the probability of ultimate survival

$$\pi := \lim_{k \to \infty} P(Z_k > 0)$$

vanishes [26, Theorem 2.1]. In particular $E[M] \leq 1$ implies a.s. ultimate extinction, as follows from Jensen’s inequality. Thus, we may observe a positive probability $\pi$ of survival only in the case of $E[M] > 1$.

In general, BPREs and GWPs differ a lot in their properties. However, this mainly concerns the subcritical and critical regime. In the supercritical range the random environment is less dominant, and both classes of processes share quite a few properties. Indeed, Tanny [33] derived the Kesten-Stigum theorem for GWPs in the random environment setup, see also [18]. This correspondence may be observed also for finer asymptotics of supercritical BPREs, as derived in [4, 16, 22]. Thus, one may wonder whether Haldane’s asymptotics for slightly supercritical GWPs, also transfers to BPREs. As we shall see, this is for iid environments largely, but not completely true.

Thus, let us consider a sequence $(Z_N, N \geq 1)$ of BPREs. All linked quantities are assigned to the index $N$, like the survival probabilities $\pi_N$, the random means $M_N$, the generic offspring number $U_N$ and so forth (we use sans-serif letters for random terms, which are designated to carry the index $N$, like generic variables). In particular, let the numbers $\varepsilon_N$ and $\nu_N$ be given by the equations

$$E[M_N] = 1 + \varepsilon_N, \quad \text{Var}(M_N) = \nu_N.$$

Following Haldane, we are concerned with the situation that the $\varepsilon_N$ form a positive null sequence and that the variance of $U_N$ stabilizes as $N \to \infty$, i.e.

$$\text{Var}(U_N) = \sigma^2 + o(1) \quad \text{with } \sigma^2 > 0. \quad (1)$$

Using $\varepsilon_N = o(1)$ this assumption may be equally expressed as

$$E[M_N^{(2)}] = \sigma^2 + o(1). \quad (2)$$

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Without further mentioning we require $M_N > 0$ a.s. for all $N$, since otherwise $\pi_N = 0$ trivially.

Within this scenario different behaviour may arise. There is a broad supercritical area, where Haldane’s asymptotics proves true. Here the conditional mean $M_N$ follows largely its expectation $1 + \varepsilon_N$, but deviations of higher magnitude than $\varepsilon_N$ may occur. This area will be left behind only, if $\nu_N$ attains the same order as $\varepsilon_N$, in other words, if the random fluctuations of $M_N$ are of order $\sqrt{\varepsilon_N}$, thus notably surpassing $\varepsilon_N$, the expected excess of $M_N$ above 1. In this area of transition $\pi_N$ falls below Haldane’s estimate, the subcritical range is adjacent and the random environment makes itself felt. Our result requires stronger assumptions than the classical moment conditions on $\log M_N$ for branching processes in iid random environments. This is no surprise, since we are addressing properties of $M_N$ itself.

**Theorem 1.** For branching processes $Z_N$, $N \geq 1$, in iid random environments, let $\varepsilon_N > 0$ for all $N$ and $\varepsilon_N \to 0$ as $N \to \infty$. Additionally to (1) assume

$$E[U_N^{4+\delta}] = O(1), \quad E[M_N^{4-\delta}] = O(1), \quad E[|M_N - E[M_N]|^{4+\delta}] = O(\nu_N^{2+\frac{\delta}{2}})$$

(3)

with some $\delta > 0$. Then we have:

i) If $\frac{\nu_N}{\varepsilon_N} \to 0$, then the survival probability obeys Haldane’s asymptotics

$$\pi_N \sim \frac{2\varepsilon_N}{\sigma^2} \quad \text{as } N \to \infty.$$  

ii) If $\frac{\nu_N}{\varepsilon_N} \to \rho$ with $0 < \rho < 2$, then

$$\pi_N \sim \frac{(2 - \rho)\varepsilon_N}{\sigma^2} \quad \text{as } N \to \infty.$$  

iii) If $\frac{\nu_N}{\varepsilon_N} \to 2$, then

$$\pi_N = o(\varepsilon_N) \quad \text{as } N \to \infty.$$  

iv) If $\lim\inf_{N \to \infty} \frac{\nu_N}{\varepsilon_N} > 2$, then for large $N$ the process $Z_N$ is subcritical, implying

$$\pi_N = 0.$$

Under items i) and ii) we are concerned with supercritical, and under iv) with subcritical processes (following from Lemma 4.1 ii) below). Only in the border case iii) this matter remains undecided, and we may have both $\pi_N > 0$ and $\pi_N = 0$. Here, a more precise handling of $\pi_N$ is a challenge. In case ii) the random environment $V_N$ becomes visible within

$$\pi(V_N) := \lim_{k \to \infty} P(Z_k > 0 \mid V_N),$$

the conditional probabilities of ultimate survival, given $V_N$. Other than under i), it stays asymptotically random, with a limiting $\Gamma$-distribution (see Proposition 4.6 below).

The theorem’s proof does not rely on the familiar fixed point characterization of the survival probability, but follows a new strategy. We use an explicit representation for the survival probability in terms of the shape function as well as the corresponding new techniques, which have been introduced in [25] for branching processes in varying environments. The derived expressions have a similar structure as perpetuities, known from a financial context. Thus, it is a major step of the proof to derive a limit theorem for perpetuities with asymptotically
vanishing interest rates. To the best of our knowledge, this is the first time that a relation
between branching processes in iid random environment and perpetuities occurs in the literature.
Alsmeyer’s publication [1] uses this connection in the case of iid linear fractional environments,
on arXiv it appeared almost simultaneously.

The paper is organized as follows. In Section 2 we recall the required notions on branching
processes in a varying environment and establish the representation of the survival probability.
Our result on perpetuities is given in Section 3, it might be of some interest on its own. There,
we also provide a more detailed discussion of the relevant literature concerning perpetuities.
The proof of the main theorem is given in Section 4.

2 An expression for the survival probability

In this section we derive an expression for the survival probability \( \pi \) of a BPRE, using ideas of
[25]. As to the notation, we find it convenient to write the probability generating function of a
probability measure \( p \in \mathcal{P}(\mathbb{N}_0) \) as

\[
p(s) = \sum_{z=0}^{\infty} s^z p[z], \quad 0 \leq s \leq 1,
\]

where \( p[z] \) are the weights of \( p \).

Let us first look at the special case of a deterministic environment \( v = (p_1, p_2, \ldots) \) of prob-
ability measures \( p_1, p_2, \ldots \in \mathcal{P}(\mathbb{N}_0) \) (which is called a varying environment). For \( k \leq n \) the probability of survival in generation \( n \), having one individual in generation \( k \), is given by

\[
\pi_{k,n}(v) := \mathbb{P}(Z_n > 0 \mid Z_k = 1).
\]

In order to express this probability more explicitly, define for \( k \leq n \) the probability generating
functions

\[
p_{k,n}(s) := p_{k+1} \circ \cdots \circ p_n(s), \quad 0 \leq s \leq 1,
\]

with the convention \( p_{k,k}(s) = s \). Then, similar to classical GWPs we have for \( k \leq n \) the formula

\[
\pi_{k,n}(v) = 1 - p_{k,n}(0).
\]

(4)

In order to expand the right hand term, we define as in [25] for any \( p \in \mathcal{P}(\mathbb{N}_0) \) with positive
finite mean \( m = p'(1) \) the shape function \( \varphi_p : [0, 1) \to \mathbb{R} \) via the equation

\[
\frac{1}{1 - p(s)} = \frac{1}{m(1-s)} + \varphi_p(s).
\]

Note that \( \varphi_p \) is a non-negative function and can be extended continuously to \([0, 1]\) by setting

\[
\varphi_p(1) := \frac{p''(1)}{2p'(1)^2}.
\]

Letting \( \varphi_k = \varphi_{p_k} \) be shape the function belonging to \( p_k \), we obtain iteratively

\[
\frac{1}{1 - p_{0,n}(s)} = \frac{1}{p'_1(1)(1 - p_{1,n}(s))} + \varphi_1(p_{1,n}(s)) = \cdots = \frac{1}{p'_1(1) \cdots p'_n(1)(1-s)} + \sum_{k=1}^{n} \frac{\varphi_k(p_{k,n}(s))}{p'_1(1) \cdots p'_{k-1}(1)},
\]
and for \( s = 0 \) by (4)
\[
\frac{1}{\pi_{0,n}(v)} = \frac{1}{m_1 \cdots m_n} + \sum_{k=1}^{n} \frac{\varphi_k(1 - \pi_{k,n}(v))}{m_1 \cdots m_{k-1}}
\] (5)

with \( m_k = p'_k(1) \).

For an iid random environment \( V = (P_1, P_2, \ldots) \), we write \( M_k, M_k^{(2)} \) and \( \Phi_k(s) = \varphi_{P_k}(s) \) for the mean, second factorial moment and shape function of the random probability measure \( P_k \). Note that \( P_k(S) = \sum_{z=0}^{\infty} P_k[z] S^z \) and, consequently, \( \Phi_k(S) \) are well-defined random variables for any random variable \( S \) taking values in \([0, 1]\). We set
\[
F_k^- := \Phi_k(0) = \frac{1}{1 - P_k[0]} - \frac{1}{M_k}, \quad F_k^+ := \Phi_k(1) = \frac{M_k^{(2)}}{2M_k^2}, \quad k \in \mathbb{N}.
\]

Again note that the \( F_k^\pm \) are iid copies of some \( F^\pm \). Throughout this work we make use of the fact that it holds
\[
\frac{1}{2} F_k^- \leq \Phi_k(S) \leq 2F_k^+ \quad \text{a.s.}
\] (6)

for any random variable \( 0 \leq S \leq 1 \). This estimate follows via conditioning on \( P_k \) from [25, Lemma 1], where it was derived for a varying environment. We set
\[
\mu_n = \prod_{k=1}^{n} M_k, \quad \mu_0 = 1.
\]

Now (5) reads
\[
\frac{1}{\pi_{0,n}(v)} = \frac{1}{\mu_n} + \sum_{k=1}^{n} \frac{\Phi_k(1 - \pi_{k,n}(v))}{\mu_k^{-1}} \quad \text{a.s.}
\] (7)

In this expression we are going to take the limit \( n \to \infty \), in order to proceed to
\[
\pi_{k,\infty}(V) := \lim_{n \to \infty} \pi_{k,n}(V),
\]
the probability of ultimate survival with 1 individual at generation \( k \), given the environment \( V \). We set
\[
\pi(V) := \pi_{0,\infty}(V).
\]

**Proposition 2.1.** Let \( Z = (Z_k, k \geq 0) \) be a branching process in the random environment \( V = (P_1, P_2, \ldots) \) consisting of independent copies of \( P \). Assume \( 0 < E[\log M] < \infty \) and \( E[\log^+ M^{(2)}] < \infty \). Then the conditional survival probability can be expressed as
\[
\pi(V) = \frac{1}{X},
\] (8)

where a.s.
\[
X := \sum_{k=1}^{\infty} \frac{\Phi_k(1 - \pi_{k,\infty}(V))}{\mu_k^{-1}} < \infty.
\]
Remark 2.2. In particular we have \( \pi = \mathbb{E}[1/X] > 0 \). The assumptions in Proposition 2.1 are slightly stronger than the classical requirements for a positive probability of ultimate survival, see [31, 32]. However, these publications contain no representation as \( \pi \) or \( \Phi \). We note that Proposition 2.1 holds equally for a stationary and ergodic random environment.

Proof. Observe that \( \mu_n \to \infty \) a.s., since
\[
\mu_n = \exp \left( \sum_{k=1}^{n} \log M_k \right) = \exp \left( n \mathbb{E} \left[ \log M \right] + o(n) \right),
\]
by the strong law of large numbers. Furthermore, we have in the limit \( n \to \infty \)
\[
\Phi_k(1 - \pi_{k,n}(V)) \to \Phi_k(1 - \pi_{k,\infty}(V)) \quad \text{a.s.}
\]
by continuity. Thus, (7) yields
\[
\frac{1}{\pi(V)} = \sum_{k=1}^{\infty} \frac{\Phi_k(1 - \pi_{k,\infty}(V))}{\mu_{k-1}},
\]
provided that we can justify the interchange of limits. By (8) we have
\[
\Phi_k(1 - \pi_{k,\infty}(V)) \leq 2F_k^+. \tag{9}
\]
Thus, letting
\[
\mathcal{Y} := \sum_{k=1}^{\infty} \frac{F_k^+}{\mu_{k-1}},
\]
it is sufficient in view of dominated convergence to prove \( \mathcal{Y} < \infty \) almost surely. Note that
\[
F_k^+ \leq \exp(\log^+ M_k^{(2)} - 2 \log M_k), \quad \text{and by our assumptions and the strong law of large numbers}
\]
we have a.s. \( \log M_k = o(k) \) and \( \log^+ M_k^{(2)} = o(k) \), therefore a.s. \( F_k^+ = e^{o(k)} \). Together with (9) it follows a.s. \( \mathcal{Y} < \infty \).

Later, we approximate the random variable \( X \) by \( \mathcal{Y} \). There, we rely on the following estimate.

Lemma 2.3. For an iid random environment \( V = (P_1, P_2, \ldots) \) we have for any \( 0 < \eta \leq 1/2 \)
\[
|\Phi_k(1) - \Phi_k(1 - \pi_{k,\infty}(V))| \leq \eta( (M_k^{(2)})^2 + M_k^{(2)} + \mathbb{E} [U_{1,k}^4 | V] ) + 2F_k^+ \mathbb{1}_{\{\pi_{k,\infty}(V) > \eta M_k \}} \quad \text{for} \quad k \geq 2. \tag{10}
\]

Proof. By conditioning on \( V \) we reduce our claim to the case of a varying environment. Hence, we may resort to [25, Lemma 2] yielding
\[
|\Phi_k(1) - \Phi_k(S)| \leq 2 \left( \frac{M_k^{(2)}}{M_k^3} \right)^2 (1 - S) + 2D \frac{M_k^{(2)}}{M_k^2} (1 - S) + \frac{2}{M_k^2} \mathbb{E} [U_{1,k}^2 | U_{1,k} \geq D + 1 | V],
\]
where \( 0 \leq S \leq 1 \) and \( D \in \{1, 2, \ldots\} \) are allowed to be any functions of \( V \). Choose \( S = 1 - \pi_{k,\infty}(V) \) and \( D = [\zeta^{-1}] \) with \( \zeta = \eta M_k \) and some \( 0 < \eta \leq 1/2 \). Then we obtain on the event \( \{\pi_{k,\infty}(V) \leq \zeta^3 \leq 1\} = \{\pi_{k,\infty}(V) \leq \zeta^3, D \geq 1\} \)
\[
|\Phi_k(1) - \Phi_k(1 - \pi_{k,\infty}(V))| \leq 2 \left( \frac{M_k^{(2)}}{M_k^3} \right)^2 \zeta^3 + 2 \frac{M_k^{(2)}}{M_k^2} \zeta^2 + 2 \frac{\zeta^2}{M_k^2} \mathbb{E} [U_{1,k}^4 | V] + \mathbb{E} [U_{1,k}^4 | V],
\]
On the complementary event \( \{1 - \pi_{k,\infty}(V) \leq \zeta^3 \leq 1\}^c \) we bound by means of (6) leading to (10).
3 On perpetuities with small interest rates

Let \((A_k, B_k), k \geq 1\), be independent copies of the random pair \((A, B)\), where \(A\) and \(B\) are non-negative random variables with finite means, and \(A\) is non-degenerate. Set \(C_k := A_1 A_2 \cdots A_k\) for \(k \geq 1\), \(C_0 = 1\), and consider the series

\[
Y := \sum_{k=1}^{\infty} B_k C_k - 1,
\]

(11)

which in a financial context is called a perpetuity, see [2] and the literature cited therein. We allow \(Y\) to take the value \(\infty\), see Remark 3.1. The random variable \(Y\) fulfills a distributional recursion, the annuity equation [8]

\[
Y \overset{d}{=} A Y + B,
\]

(12)

where on the right-hand side \((A, B)\) and \(Y\) are assumed to be independent.

In what follows we study the limiting behaviour of a sequence of perpetuities

\[
Y_N \overset{d}{=} A_N Y_N + B_N, \quad N \geq 1,
\]

as the expectation of \(A_N\) tends to 1 (which in a financial setting corresponds to asymptotically vanishing interest rates). Let the numbers \(\alpha_N < 1\), \(\upsilon_N > 0\) and \(\beta_N > 0\), \(N \geq 1\), be given by

\[
\mathbb{E}[A_N] = 1 - \alpha_N, \quad \text{Var}(A_N) = \upsilon_N, \quad \mathbb{E}[B_N] = \beta_N.
\]

The asymptotic behaviour of \(Y_N\) is dictated by the expectation and variance of \(A_N\), whereas the expectation of \(B_N\) acts just as a scaling factor. Depending on the asymptotic value of the ratio \(\alpha_N/\upsilon_N\) we observe different limiting distributions for rescaled versions of \(Y_N\).

**Theorem 2.** Assume that as \(N \to \infty\)

\[
\alpha_N \to 0, \quad \upsilon_N \to 0, \quad \beta_N \to \beta
\]

with \(0 < \beta < \infty\), furthermore \(\mathbb{E}[|A_N - 1|^{2+\delta}] = o(|\alpha_N| + \upsilon_N)\) and \(\mathbb{E}[B_N^{1+\delta}] = O(1)\) for some \(\delta > 0\). Additionally, assume

\[
\frac{\alpha_N}{\upsilon_N} \to \gamma \quad \text{with} \quad -\frac{1}{2} \leq \gamma \leq \infty.
\]

Then we have:

i) If \(\gamma = \infty\), then \(\alpha_N Y_N \to \beta\) in probability, as \(N \to \infty\).

ii) If \(-1/2 < \gamma < \infty\), then \(\upsilon_N Y_N\) is asymptotically inverse \(\Gamma\)-distributed, with density 

\[
\frac{c x^{a-2} e^{-b/x}}{x^{2+\delta}} dx,
\]

where \((a, b) = (2\gamma, 2\beta)\) and \(c = (2\beta)^{2\gamma+1}/\Gamma(2\gamma + 1)\).

iii) If \(\gamma = -1/2\), then \(\upsilon_N Y_N \to \infty\) in probability, as \(N \to \infty\).

Note that, as long as \(\gamma \neq 0\), also the \(|\alpha_N|\) may serve as scaling factors. This might appear more natural, since for positive \(\alpha_N\) we indeed have \(\mathbb{E}[Y_N] = \beta_N/\alpha_N\). However, for \(\gamma = 0\) they fail to be usable for this purpose.
Remark 3.1. The random variables $Y_N$ are well defined, since we allow the value $\infty$. Either $Y_N < \infty$ a.s. or $Y_N = \infty$ a.s., by a 01-law. The second case may occur in under iii) of Theorem 2. In both cases $Y_N$ fulfills the annuity equation, since $Y_N = \infty$ implies a.s. $A_N > 0$.

Our theorem does not require any assumption on the convergence or divergence of the series in (11), nevertheless there is a close connection. The classical criterion of Vervaat [24] states that in (11) a.s. convergence holds in case of $E[\log A] < 0$, and a.s. divergence in case of $E[\log A] > 0$ (for an ultimate criterion see [13]). In our case we have $A_N \approx 1$, thus $\log A_N \approx (A_N - 1) - (A_N - 1)^2/2$ and typically $E[\log A_N] \approx -\alpha_N - v_N/2$.

Hence, for $\gamma > -1/2$ we expect a.s. convergence for large $N$, which will indeed result from Lemma 3.4 below. Note that we may have a.s. convergence, even if $\alpha_N \leq 0$ for all $N$, provided that the variance $v_N$ stays sufficiently large. Here $E[\log A_N] < 0$, but $E[A_N] > 0$ and consequently $E[Y_N] = \infty$.

On the other hand, for $\gamma < -1/2$ we typically have $Y_N = \infty$ a.s. for large $N$. This range is of little interest in our context, thus we leave it aside in our theorem and focus on the border case $\gamma = -1/2$. Then a.s. convergence as well as a.s. divergence of (11) may occur.

Remark 3.2. In the literature several papers address the limiting behaviour of perpetuities. A result matching to our Theorem 2 ii) is contained in Dufresne’s seminal paper [10], see Proposition 4.4.4. therein. It requires stronger assumptions and aligns to ours only in case that $B$ is a.s. constant for all $N$. We note that Dufresne’s proof rests on an invalid argument (namely, that distributional convergence in the Skorohod sense of some stochastic processes $(Z_n(t), t \geq 0)$ to a process $(Z(t), t \geq 0)$ implies convergence of $Z_n(\infty) = \lim_t Z_n(t)$ to $Z(\infty) = \lim_t Z(t)$ in distribution, provided the a.s. existence of these limits. In order to validate Dufresne’s proof it would be necessary to show that the random variable $T$, introduced on top of page 62, does not depend on $n$.

Blanchet and Glynn [5] as well as Iksanov et al. [24] present results which belong to the range of part i) of our Theorem. Besides laws of large numbers they derive advanced approximations to the normal distribution. In another contribution Iksanov et al. [23] provide refined asymptotics for the scaled logarithm of perpetuities in a range where $\alpha_N$ and $v_N$ would not converge to 0.

Under the condition $E[\log A] \geq 0$, Hitczenko and Wesolowski [20] consider the convergence of rescaled partial sums of $Y$.

As a side remark, we note that equation (12) can be viewed as an equation for the stationary distribution of some real-valued Markov chain $(W_n, n \geq 0)$. General results in this spirit have been obtained by Borovkov and Korshunov in [21, 28], they show that the limiting stationary distribution of sequences of Markov chains is $\Gamma$-distributed in a certain asymptotic regime.

Thus, one might wonder, if our Theorem fits into this framework in the sense that the Markov chains $(1/W_n, n \geq 0)$ can be integrated therein. However, it is readily checked that there is no match of the respective asymptotic regimes. Indeed, our result might give rise to another general result on asymptotic stationary distributions of Markov chains.

Let us turn to the proof of Theorem 2. We shall establish convergence of the corresponding Laplace transforms and characterize the limit by a second order linear differential equation of singular type, related to the Bessel differential equation. This approach necessitates that the terms $A$ and $B$ are non-negative. We note that a corresponding differential equation for characteristic functions is hardly available. This would require the existence of the second moment, which for the inverse $\Gamma$-distribution is in general not at disposal. Thus, if one would like to overcome the assumption of non-negativity, a different approach seems to be needed.

We prepare the proof by three lemmata. Let $\tau_N$ denote either $|\alpha_N|$ or $v_N$, and let

$$\ell_N(\lambda) := E[\exp(-\lambda \tau_N Y_N)], \quad \lambda \geq 0,$$
be the Laplace transform of $\tau_N Y_N$. Recall, that $\ell_N$ is arbitrarily often differentiable at any $\lambda > 0$. We set $e^{-\infty} = 0$, thus $\ell_N(\lambda) = 0$ for all $\lambda > 0$ if a.s. $Y_N = \infty$. This case causes no problems in the following proofs.

**Lemma 3.3.** Under the assumptions of Theorem 2 we have for any $\lambda > 0$ as $N \to \infty$

$$
\frac{1}{2}E_N[\lambda N^2(\lambda) - \alpha_N\ell_N(\lambda) - \beta\tau_N\ell_N(\lambda) = o(|\alpha_N| + \nu_N)].
$$

**Proof.** Let $\ell(\lambda) = E[e^{-\lambda\tau Y}]$ be the Laplace transform of $\tau Y$ with $\tau > 0$ and with a single perpetuity $Y$. By means of equation (12) and independence of $(A, B)$ and $Y$ we have for any $\lambda \geq 0$

$$
\ell(\lambda) = E[E[e^{-\lambda\tau(AY+B)}] | (A, B)] = E[e^{-\lambda\tau B}\ell(\lambda A)].
$$

We approximate the right-hand expectation by means of Taylor expansions of its integrand, which is done in three steps. Let $\eta > 0$.

i) First, by restricting expectations onto the event $\{\tau B \leq \eta\}$ and its complement, we have

$$
|E[e^{-\lambda\tau B}\ell(\lambda A)] - E[(1 - \lambda\tau B)\ell(\lambda A)]| \\
\leq E[|e^{-\lambda\tau B} - (1 - \lambda\tau B)|\ell(\lambda A); \tau B \leq \eta] + E[|e^{-\lambda\tau B} - (1 - \lambda\tau B)|\ell(\lambda A); \tau B > \eta] \\
\leq E[\lambda^2\tau^2B^2; \tau B \leq \eta] + E[2 + \lambda\tau B; \tau B > \eta] \\
\leq \lambda^2\eta^2E[B] + \left(\frac{2}{\eta^{1+\delta}} + \frac{\lambda}{\eta^3}\right)\tau^{1+\delta}E[B^1+\delta].
$$

Now we take the dependence on $N$ into account. Let $\kappa > 0$. By the assumptions of Theorem 2 and by choosing $\eta$ sufficiently small, the first term in (15) becomes smaller than $\kappa\tau_N$. Since $\tau_N^2 \to 0$, also the second term in (15) becomes smaller than $\kappa\tau_N$, if only $N$ is large enough. This entails

$$
E[e^{-\lambda\tau_N B N^2}\ell_N(\lambda A_N)] = E[\ell_N(\lambda A_N)] - \lambda\tau_N E[B_N\ell_N(\lambda A_N)] + o(\tau_N).
$$

ii) The right-hand expectations in (16) are handled similarly, now by restricting them to the event $\{|A - 1| \leq \eta\}$ and its complement. For $0 < \eta \leq 1/2$, $\lambda > 0$ we have

$$
\lambda \sup_{\eta \leq |A - 1|} |\ell'(a\lambda)| \leq E[\lambda\tau Y e^{(1-\eta)\lambda\tau Y}] \leq \sup_{z \geq 0} z e^{-z/2}.
$$

Hence, with some random $A'$ between $A$ and $1$,

$$
|E[B\ell'(\lambda A)] - E[B\ell'(\lambda)]| \leq E[B|\ell'(a\lambda)|; |A - 1| \leq \eta] + E[2B; |A - 1| > \eta] \\
\leq \lambda\eta E[B] \sup_{\eta \leq |A - 1|} |\ell'(a\lambda)| + 2E[B^{1+\delta}][1 + \delta P(|A - 1| > \eta)]^{\frac{1}{1+\delta}} \\
\leq \eta E[B]|z e^{-z/2} + 2E[B^{1+\delta}][1 + \delta P(|A - 1| > \eta)]^{\frac{1}{1+\delta}}
$$

Regarding the dependence on $N$, by the theorem’s assumptions, by suitably adapting $\eta$ and by noting $E[(A_N - 1)^2] = \nu_N + \alpha_N^2 \to 0$, this expression can be, with increasing $N$, made smaller than any $\kappa > 0$. In other terms:

$$
E[B\ell_N(\lambda A_N)] = \beta\ell_N(\lambda) + o(1).
$$
Thus, theorem for any $t$

hand term becomes smaller than $κ$

For the stopping time $N$

The cases

Proof.

Regarding the dependence on $N$, the derivatives of $\ell$, including the preceding supremum, stay again bounded with $N$. Further, we have $E[(A_N - 1)^2] = O(v_N + |\alpha_N|)$ and $E[|A_N - 1|^{2+\delta}] = o(v_N + |\alpha_N|)$ by assumption. Thus, letting $κ > 0$ and adapting $η$ once more, the above right-hand term becomes smaller than $κ(|\alpha_N| + v_N)$ for large $N$. It follows

$$E[\ell_N(\lambda A_N)] = \ell_N(\lambda) + E[\ell'_N(\lambda)\lambda(A_N - 1) + \frac{1}{2} \ell''_N(\lambda)\lambda^2(A_N - 1)^2] + o(|\alpha_N| + v_N). \quad (18)$$

Combining equations (16), (17) and (18) we arrive at

$$E[e^{-\lambda t\tau_N B_N} \ell_N(\lambda A_N)] = \ell_N(\lambda) - \alpha_N \lambda \ell'_N(\lambda) + \frac{1}{2} v_N \lambda^2 \ell''_N(\lambda) + \beta \tau_N N \ell_N(\lambda) + o(|\alpha_N| + v_N).$$

Inserting this approximation into (14) yields our claim. \hfill \square

The next lemma applies under weaker assumptions on the sequence $(A_k, B_k)$, which will be of advantage later.

**Lemma 3.4.** Assume that the sequence $(A_k)$ consists of independent copies of the non-negative random variable $A$, and that $(B_k)$ contains non-negative random variables with identical finite mean $E[B] > 0$. Then, if $E[A^u] < 1$ for some $u > 0$, we have for any $c > 0$

$$P((1 - E[A^u])Y > cE[B]) \leq 2e^{-\frac{c}{1+a}}.$$

**Proof.** The cases $E[A] = 0$ and $E[B] = 0$ cause no problems, thus we assume $E[A] > 0$ and $E[B] > 0$. Consider the process $M_k = C_k^n/|E[A^u]|^k$, $k \geq 0$, which is a non-negative martingale. For the stopping time $T = \min\{k \geq 0 : M_k > a\}$ we obtain by means of the optional stopping theorem for any $t \in \mathbb{N}$

$$1 = M_0 = E[M_{T\wedge t}] \geq aP(T \leq t),$$

thus

$$P(T < \infty) \leq \frac{1}{a}.$$

On the event $\{T = \infty\}$ we have $C_k \leq a^{\frac{k}{D}}E[A^u]^\frac{k}{D}$ for all $k \geq 1$. This yields for any $τ > 0$

$$P(τY > cE[B]) \leq \frac{1}{a} + \frac{τ}{cE[B]}E[Y; T = \infty]$$

$$\leq \frac{1}{a} + \frac{τ}{cE[B]} \sum_{k=1}^{\infty} a^{\frac{k}{D}}E[B_kE[A^u]^\frac{k-1}{D}]$$

$$= \frac{1}{a} + \frac{τa^{\frac{1}{D}}}{c(1 - E[A^u]^\frac{1}{D})}.$$

Letting $a = e^{u/(1+u)}$ and $τ = 1 - E[A^u]^\frac{1}{D}$ yields the claim. \hfill \square
Remark 3.5. The previous inequality shows that the tail probabilities of the random variable $Y$ decrease at a polynomial rate. This issue attracted some attention in the literature. Sophisticated treatments by Goldie [14] and others [9, 8] show that under additional assumptions we indeed have $P(Y > c) \sim dc^{-\xi}$ with constants $d, \xi > 0$, where $\xi$ fulfills $E[A^\xi] = 1$. Moreover, Collamore and Vidyashankar presented an upper bound of the form $P(Y > c) \leq d'e^{-\xi}$ with some $d' > d$ [9, Proposition 2.1], or $P((d')^{-1/\xi}Y > c) \leq c^{-\xi}$. With our approach we underestimate the correct exponent $\xi$, however this is not our objective. We aim at suitable scaling factors for $Y$. Using the formula $P((d')^{-1/\xi}Y > c) \leq c^{-\xi}$ would require to sufficiently decrease $d'$, and at the same time get a handle on $\xi$, a formidable task. In contrast, our Lemma 3.4 provides the clear-cut, explicit scaling factor $1 - E[A^u]^{1/u}$. As we shall see, it has the right magnitude. (As to exponential tail bounds, compare [19]).

Lemma 3.6. Let $a \in \mathbb{R}$, $b > 0$. Then the linear differential equation
\[ \lambda^\mu(\lambda) = a\ell'(\lambda) + b\ell(\lambda), \quad \lambda > 0, \tag{19} \]
has the solution
\[ l(\lambda) = c \int_0^\infty e^{-\lambda x}x^{-a-2}e^{-b/x} \, dx, \tag{20} \]
which for $c > 0$ is the Laplace transform of the measure on $\mathbb{R}^+$ with the density $cx^{-a-2}e^{-b/x}$ with respect to the Lebesgue measure. The Laplace transform of any other non-vanishing measure on $\mathbb{R}^+$ fails to solve equation (19).

Proof. The differential equation can be solved by means of modified Bessel functions. Our approach is more direct and elementary. By partial integration we have
\[ \lambda \int_0^\infty e^{-\lambda x}x^{-a}e^{-b/x} \, dx = \int_0^\infty e^{-\lambda x}(-ax^{-a-1} + bx^{-a-2})e^{-b/x} \, dx, \]
which transforms directly into equation (19) for the function $l$ from (20).

For the second claim note that $l(\lambda) \to 0$ as $\lambda \to \infty$. Consider some non-vanishing measure $\mu$ on $\mathbb{R}^+$ with finite $h(\lambda) = \int_0^\infty e^{-\lambda x} \mu(dx)$ for all $\lambda > 0$. Suppose that $h$ solves (19). Since the $n$-th derivative of $h$ is equal to $\pm \int_0^\infty x^ne^{-\lambda x} \mu(dx)$, we have $h'(\lambda), h''(\lambda) \to 0$ as $\lambda \to \infty$. From (19) it follows that as well $h(\lambda) \to 0$, as $\lambda \to \infty$ (meaning that $\mu$ has no atom at zero).

Let $\lambda_1 > 0$. Since $h(\lambda_1)$ and $l(\lambda_1)$ are strictly positive, there is a $c > 0$ in (20) such that $h(\lambda_1) = l(\lambda_1)$. Thus the difference $d = h - l$ satisfies $d(\lambda_1) = 0$ and $d(\lambda) \to 0$ as $\lambda \to \infty$. Moreover $d$ fulfills the equation (19). Suppose that $d$ does not vanish on the interval $(\lambda_1, \infty)$. Then $d$ has at some point $\lambda_2 > \lambda_1$ a global maximum or minimum, in particular $d'(\lambda_2) = 0$. In case of a maximum we have $d(\lambda_2) > 0$ and $d''(\lambda_2) \leq 0$, which contradicts (19). The case of a minimum is analogue, thus $d$ has to vanish on the whole interval $(\lambda_1, \infty)$. Since $\lambda_1 > 0$ is arbitrary, we conclude that $h = l$, and our claim follows by uniqueness of Laplace transforms.

Proof of Theorem 2. We prove convergence of the Laplace transform of $\alpha_N Y_N$ or $v_N Y_N$ towards the Laplace transform of the limiting distributions. Note that convergence in distribution implies not only convergence of the Laplace transforms, but also convergence of their derivatives at points $\lambda > 0$, being of the form $\pm E[(\tau_N Y_N)^n] e^{-\lambda t_N Y_N}$. 

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In order to obtain tightness we shall apply Lemma 3.4. For $0 < u < 1$ and $0 < \eta < 1$ we have
\[
\left| \mathbb{E}[A^u - (1 + u(A - 1) + \frac{u(u-1)}{2}(A - 1)^2)] \right| \\
\leq \mathbb{E}[(A^u - 3|A - 1|^2; |A - 1| \leq \eta)] + \mathbb{E}[|A - 1|^u + |A - 1| + |A - 1|^2; |A - 1| > \eta] \\
\leq \eta(1 - \eta)^u - 3\mathbb{E}[(A^u - 1)^2] + \left( \frac{1}{\eta^{2+\delta-u}} + \frac{1}{\eta^{1+\delta}} + \frac{1}{\eta^2} \right) \mathbb{E}[|A - 1|^{2+\delta}].
\]

Turning to the sequence $(Y_N)$, let $\kappa > 0$. As in the proof of Lemma 3.3 (iii), by a suitable choice of $\eta$, the right-hand expression falls below $\kappa(\alpha_N + \nu_N)$ for large $N$. It follows
\[
\mathbb{E}[A_N^u] = 1 - u\left( \alpha_N + \frac{1-u}{2}\nu_N \right) + o(\alpha_N + \nu_N),
\]
consequently
\[
1 - \mathbb{E}[A_N^u] = \frac{1}{\kappa}(\alpha_N + \frac{1-u}{2}\nu_N + o(\alpha_N + \nu_N)).
\]

i) If $\gamma = \infty$, then $\nu_N = o(\alpha_N)$. Then (21) turns into $1 - \mathbb{E}[A_N^u]^{1/u} \sim \alpha_N$, and the sequence $(\alpha_N Y_N)$ is tight in view of Lemma 3.4. Let $\alpha_N' Y_{N'}$ be a subsequence converging in distribution and with limiting distribution $\ell'(\lambda)$. Dividing both sides by $\alpha_N'$ in (13) and taking $\nu_N = o(\alpha_N)$ into account we obtain the limiting differential equation
\[
\ell'(\lambda) + \beta\ell(\lambda) = 0, \quad \ell(0) = 1,
\]
which has the unique solution $\ell(\lambda) = e^{-\beta\lambda}$, the Laplace transform of the Dirac measure at point $\beta$. As is well-known, this implies part i) of Theorem 1.

ii) Next assume $-1/2 < \gamma < \infty$. Then, because of (21) $1 - \mathbb{E}[A_N^u]^{1/u} \sim (\gamma + \frac{1-u}{2})\nu_N$, which becomes positive for large $N$, if $u$ is chosen sufficiently small. Here the sequence $(\nu_N Y_N)$ is tight in view of Lemma 3.4. Consider a convergent subsequence $\nu_{N'} Y_{N'}$. Choosing $\tau_N = \nu_N$ in (13) we obtain the limiting differential equation
\[
\frac{1}{2}\lambda''(\lambda) + \gamma\ell'(\lambda) - \beta\ell(\lambda) = 0, \quad \ell(0) = 1.
\]

Lemma 3.6 yields that the limiting distribution is the inverse $\Gamma$-distribution, as stated.

iii) In case of $\gamma = 1/2$ we consider a convergent subsequence $\nu_{N'} Y_{N'}$ within the extended compactified range $[0, \infty]$. Then the limiting Laplace transform $\ell(\lambda)$ again solves (22), but the corresponding distribution may now result in a defective probability measure on $\mathbb{R}^+$. The measure given by (20) with $a = 2\gamma = -1$ is no longer finite and thus impractical. From Lemma 3.6 we see that there is no other non-vanishing measure on $\mathbb{R}^+$ at disposal. Therefore, the limiting distribution of $\nu_{N'} Y_{N'}$ has to be concentrated at the point $\infty$, which implies our claim.

4 Proof of Theorem 1

In view of Proposition 2.1 we determine the limit of $\mathbb{E}[1/X_N]$ as $N \to \infty$, recall
\[
X := \sum_{k=1}^{\infty} \frac{\Phi_k(1 - \pi_k(V))}{\mu_{k-1}} < \infty \quad \text{a.s.}
\]
To this end we shall prove that this expectation may be replaced by $\mathbb{E}[1/Y_N]$, where

$$Y := \sum_{k=1}^{\infty} \frac{F_k}{\mu k - 1}. \quad (24)$$

This random random variable has the form $\{1\}$, thus we may apply Theorem 2 to obtain the limiting distribution of the scaled $Y_N$, respectively $X_N$. In order to switch to expectations, we shall show uniform integrability of the scaled $1/X_N$. Then it remains to determine the corresponding expectation of the limiting distribution.

We prepare the proof by several lemmata.

**Lemma 4.1.** Under the assumptions of Theorem 4 if $\nu_N = o(1)$ then we have

i) $\mathbb{E} [M_{N}^{-u}] = 1 - u\varepsilon_N + \frac{u(u+1)}{2}\nu_N + o(\varepsilon_N + \nu_N)$, for $0 \leq u \leq 2$,

ii) $\mathbb{E} [\log M_N] = \varepsilon_N - \frac{1}{2}\nu_N + o(\varepsilon_N + \nu_N)$,

iii) $\mathbb{E} \left[ \frac{M_N^{(2)}}{M_N} \right] = \sigma^2 + o(1)$.

**Proof.** i) Let $0 < \eta < 1$. Similar as in the proof of Lemma 3.3 we split the expectation $\mathbb{E} [M^{-u}]$ into its parts on the event $\{|M - 1| \leq \eta\}$ and the complement. Thus

$$|\mathbb{E}[M^{-u}] - \mathbb{E}[1 - u(M-1) + \frac{u(u+1)}{2}(M-1)^2]|$$

$$\leq \mathbb{E}\left[ \frac{u(u+1)(u+2)}{6} (1-\eta)^{-u-\delta}|M - 1|^3; |M - 1| \leq \eta \right]$$

$$+ \mathbb{E}\left[ M^{-u} + u|M - 1| + \frac{u(u+1)}{2}(M-1)^2; |M - 1| > \eta \right]$$

$$\leq 4\eta(1-\eta)^{-u-\delta}\mathbb{E}\left[(M - 1)^2\right] + \mathbb{E}[M^{-2u}] \frac{2}{\eta} \mathbb{P}(|M - 1| > \eta)^{\frac{1}{2}} + \left( \frac{1}{\eta^2+\delta} + \frac{2}{\eta^{1+\delta} + \frac{3}{\eta^{\delta}}} \right) \mathbb{E}[|M - 1|^{2+\delta}] .$$

Turning to the sequence $(M_N)$ we obtain with $\delta > 0$ similar as in (3)

$$|\mathbb{E}[M^{-u}] - (1 - u\varepsilon_N + \frac{u(u+1)}{2}(\nu_N + \varepsilon_N^2))| \leq 4\eta(1-\eta)^{-u-\delta}(\nu_N + \varepsilon_N^2) + O(\nu_N^{1+\delta/2} + \varepsilon_N^{2+\delta}).$$

Let $\kappa > 0$. Then, because of $\varepsilon_N, \nu_N = o(1)$, there is an $\eta > 0$ such that the right-hand expression is smaller than $\kappa(\varepsilon_N + \nu_N)$ for large $N$. In other terms: The right-hand expression is of order $o(\varepsilon_N + \nu_N)$, and our claim follows.

ii) By the same line of argument, using $|\log x| \leq x + x^{-1}$ for all $x > 0$, we have for $0 < \eta < 1$

$$|\mathbb{E}[\log M_N] - \mathbb{E}[(M_N - 1) - \frac{1}{2}(M_N - 1)^2]|$$

$$\leq \mathbb{E}\left[ \frac{1}{3} (1-\eta)^{-3}|M_N - 1|^3; |M_N - 1| \leq \eta \right]$$

$$+ \mathbb{E}\left[(M_N + M_N^{-1}) + |M_N - 1| + \frac{1}{2}(M_N - 1)^2; |M_N - 1| > \eta \right],$$

and our claim is confirmed in much the same vein as under i).

iii) We have

$$\left| \mathbb{E} \left[ \frac{M_N^{(2)}}{M_N^2} \right] - \mathbb{E} \left[ M_N^{(2)} \right] \right| \leq \mathbb{E} \left[ \left| M_N^{(2)} - \frac{1}{M_N^2} \right| \right] \leq \mathbb{E} \left[ (M_N^{(2)})^2 \right] \frac{1}{2} \mathbb{E} \left[ \left( \frac{1}{M_N^2} - 1 \right)^2 \right] \frac{1}{2} .$$
Using $E_0 > \leq \nu$ where in the last inequality we used the fact that $0 > \nu$.

Proof. From $E = (1 - P[0])^{-1} - M^{-1} \geq 1 + P[0] - M^{-1}$ we obtain

$$P(F^- > \theta) \geq P(P[0] > \theta + M^{-1} - 1) \geq P(P[0] > 2\theta) - P(1 - M^{-1} > \theta) \geq \mathbb{E}[P(0)] - 2\theta - P(1 - M^{-1} > \theta),$$

where in the last inequality we used the fact that $0 \leq P[0] \leq 1$. On the other hand, for any $k > 0$

$$\mathbb{E}[(U - 1)^2; U \leq k] \leq \mathbb{E}(U = 0) + (k - 1)\mathbb{E}[U - 1; U \geq 2] = k\mathbb{E}(U = 0) + (k - 1)\mathbb{E}[M - 1].$$

Using $\mathbb{E}[P(0)] = \mathbb{P}(U = 0)$ we arrive at the estimate

$$P(F^- > \theta) \geq \frac{1}{k}\mathbb{E}[(U - 1)^2; U \leq k] - \frac{k - 1}{k}\mathbb{E}[M - 1] - 2\theta - P(1 - M^{-1} > \theta).$$

Turning to the sequence $(Z_N)$ of branching processes we note that because of (3) the sequence $(U_N)$ is uniformly integrable. Thus for $k$ sufficiently large we have for all $N$

$$\mathbb{E}[(U_N - 1)^2; U_N \leq k] \geq \frac{1}{2}\mathbb{E}[(U - 1)^2] \geq \frac{1}{2}\text{Var}(U_N) = \frac{\sigma^2}{2} + o(1).$$

Moreover, $\mathbb{E}[M_N - 1] = o(1)$ and $P(1 - M_N^{-1} > \theta) = o(1)$ because of $\varepsilon_N = o(1)$ and $\nu_N = o(1)$.

Thus we end up with

$$P(F^-_N > \theta) \geq \frac{\sigma^2}{2k} - 2\theta + o(1),$$

which implies our claim. \qed

Lemma 4.2. Under the assumptions of Theorem 1, if $\nu_N = o(1)$, then there exists $\theta > 0$ and $p_0 > 0$ such that for large $N$

$$P(F^-_N > \theta) \geq p_0.$$

Proof. From $F^- = (1 - P[0])^{-1} - M^{-1} \geq 1 + P[0] - M^{-1}$ we obtain

$$P(F^- > \theta) \geq P(P[0] > \theta + M^{-1} - 1) \geq P(P[0] > 2\theta) - P(1 - M^{-1} > \theta) \geq \mathbb{E}[P(0)] - 2\theta - P(1 - M^{-1} > \theta),$$

where in the last inequality we used the fact that $0 \leq P[0] \leq 1$. On the other hand, for any $k > 0$

$$\mathbb{E}[(U - 1)^2; U \leq k] \leq \mathbb{E}(U = 0) + (k - 1)\mathbb{E}[U - 1; U \geq 2] = k\mathbb{E}(U = 0) + (k - 1)\mathbb{E}[M - 1].$$

Using $\mathbb{E}[P(0)] = \mathbb{P}(U = 0)$ we arrive at the estimate

$$P(F^- > \theta) \geq \frac{1}{k}\mathbb{E}[(U - 1)^2; U \leq k] - \frac{k - 1}{k}\mathbb{E}[M - 1] - 2\theta - P(1 - M^{-1} > \theta).$$

Turning to the sequence $(Z_N)$ of branching processes we note that because of (3) the sequence $(U_N^2)$ is uniformly integrable. Thus for $k$ sufficiently large we have for all $N$

$$\mathbb{E}[(U_N - 1)^2; U_N \leq k] \geq \frac{1}{2}\mathbb{E}[(U - 1)^2] \geq \frac{1}{2}\text{Var}(U_N) = \frac{\sigma^2}{2} + o(1).$$

Moreover, $\mathbb{E}[M_N - 1] = o(1)$ and $P(1 - M_N^{-1} > \theta) = o(1)$ because of $\varepsilon_N = o(1)$ and $\nu_N = o(1)$.

Thus we end up with

$$P(F^-_N > \theta) \geq \frac{\sigma^2}{2k} - 2\theta + o(1),$$

which implies our claim. \qed

Lemma 4.3. Under the assumptions of Theorem 1

i) if $\varepsilon_N \rightarrow 0$ as $N \rightarrow \infty$, then $\varepsilon_N Y_N$ converges in probability to $\frac{\sigma^2}{2}$,

ii) if $\varepsilon_N \rightarrow \rho$ as $N \rightarrow \infty$ with $0 < \rho < 2$, then $\nu_N Y_N$ is asymptotically inverse $\Gamma$-distributed with density $c x^{a-2} e^{-b/x}$, where $(a, b) = \left(\frac{2(1-\rho)}{\rho}, \sigma^2\right)$.

iii) if $\varepsilon_N \rightarrow 2$ as $N \rightarrow \infty$, then $\nu_N Y_N \rightarrow \infty$ in probability, where $Y := \sum_{k=1}^{\infty} \frac{F^-_k}{\mu_{k+1}}$.

Proof. We are going to apply Theorem 2. Note that $\nu_N = o(1)$ because of $\varepsilon_N = o(1)$. The random variables $Y_N$ fulfill the distributional equation

$$Y_N \overset{d}{=} \frac{1}{M_N} Y_N + F_N^+,$$
which is the annuity equation \((12)\) with \(A_N = 1/M_N\) and \(B_N = F_N^+\). We have to verify the assumptions of Theorem 2. Regarding the random variables \(B_N\), Lemma 4.4 yields
\[
E[B_N] = \frac{\sigma^2}{2} + o(1),
\]
thus \(\beta = \frac{\sigma^2}{2}\). Additionally, we have for some \(\delta > 0\)
\[
E\left[ B_N^{1+\delta} \right] \leq E\left[ \left( M_N^{(2)} \right)^{2+2\delta} \right] \frac{1}{\delta} E\left[ M_N^{4-4\delta} \right]^{\frac{1}{2}} = O(1),
\]
by assumption \((3)\). For \(\alpha_N = 1 - E[A_N]\), Lemma 4.3 yields
\[
\alpha_N = \epsilon_N - \nu_N + o(\epsilon_N),
\]
and for \(\upsilon_N = \text{Var}(A_N)\)
\[
\upsilon_N = E\left[ M_N^{-2} - E\left[ M_N^{-1} \right]^2 \right] = \nu_N + o(\epsilon_N).
\]
Finally, we have to confirm \(E\left[ |A_N - 1|^{2+\delta} \right] = o(|\alpha_N| + \upsilon_N)\). It holds for \(\delta > 0\) sufficiently small by assumption \((3)\)
\[
E\left[ |M_N^{-1} - 1|^{2+\delta} \right] \leq E\left[ M_N^{-4-4\delta} \right]^{\frac{1}{2}} \frac{1}{\delta} E\left[ |1 - M_N|^{4+4\delta} \right]^{\frac{1}{2}} = O\left( \frac{\epsilon_N^{2+\delta} + \upsilon_N^{2+\delta}}{\epsilon_N^{2+\delta} + \upsilon_N^{2+\delta}} \right) = o(|\alpha_N| + \upsilon_N),
\]
since \(\nu_N = o(1)\). Thus all assumptions of Theorem 2 are fulfilled.

Regarding case i), the assumption \(\upsilon_N/\epsilon_N \to 0\) implies \(\alpha_N/\upsilon_N \to \infty\) and \(\alpha_N/\epsilon_N \to 1\), hence \(\epsilon_N Y_N \to \beta = \frac{\sigma^2}{2}\) in probability by Theorem 2(i).

In case ii), where \(\frac{\upsilon_N}{\epsilon_N} \to \rho\) with \(0 < \rho < 2\), we have
\[
\frac{\alpha_N}{\upsilon_N} = \frac{\epsilon_N - \upsilon_N + o(\upsilon_N)}{\upsilon_N} \to \gamma \text{ with } \gamma := \frac{1 - \rho}{\rho} > -\frac{1}{2}.
\]
Thus, by an application of Theorem 2(ii) \(\upsilon_N Y_N\) is asymptotically inverse \(\Gamma\)-distributed with parameters
\[
(a, b) = \left( \frac{2(1 - \rho)}{\rho}, \sigma^2 \right).
\]
In case iii), note that \(\frac{\upsilon_N}{\epsilon_N} \to 2\) entails \(\frac{\alpha_N}{\epsilon_N} \to -\frac{1}{2}\). Here we apply Theorem 2 with \(B_N = F_N^+\). By a subsequence argument we may assume convergence of \(E[B_N]\) to some constant \(0 \leq \beta \leq \infty\). Then Lemma 4.2 guarantees \(\beta > 0\), and the inequality \(F_N^- \leq 4F_N^+\) from \((6)\) yields \(\beta < \infty\). By Theorem 2(iii) we have \(\nu_N Y_N^- \to \infty\) in probability.

This concludes the proof. \(\square\)

In preparation for the subsequent lemma on uniform integrability, we insert a kind of maximal inequality.

**Lemma 4.4.** Let \(S_k := \zeta_1 + \cdots + \zeta_k, 1 \leq k \leq n,\) with iid summands fulfilling \(E[\zeta_1] = 0\) and \(E[|\zeta_1^r|] < \infty\) for some \(r \geq 2\). Denote \(\nu := E[\zeta_1^2]\). Then, for any \(x > 0\)
\[
P\left( \max_{1 \leq k \leq n} S_k > x \right) \leq 2 \exp\left( -\frac{x^2}{cn\nu} \right) + cn^{-r/2} \frac{E[|\zeta_1^r|]}{\nu^{r/2}} \]
with some \(c > 0\) depending only on \(r\).
Proof. Without loss we may assume $\nu = 1$. We use a bound by Fuk and Nagaev as presented in [29, Corollary 1.8] and saying that for some $c > 0$ depending only on $r$ we have

$$
\mathbb{P}(S_n > x) \leq \exp \left( -\frac{x^2}{cn} \right) + cnx^{-r}E[|\zeta_1|^r]
$$

(26)

for all $x > 0$. It is known that similar estimates hold as well for $T_n := \max_{1 \leq k \leq n} S_k$ instead of $S_n$ (see e.g. [6]). For our purpose the following short argument is sufficient. Note that Chebychev’s inequality implies $\mathbb{P}(S_k > x) \leq 1/2$ for $k \leq n$ and $x \geq \sqrt{2n}$. Thus for $x \geq 2\sqrt{2n}$ we have

$$
\mathbb{P}(T_n > x, S_n \leq x/2) \leq \frac{1}{2} \sum_{k=1}^{n} \mathbb{P}(T_{k-1} \leq x, S_k > x, S_n - S_k > x/2) 
\leq \frac{1}{2} \sum_{k=1}^{n} \mathbb{P}(T_{k-1} \leq x, S_k > x) = \frac{1}{2} \mathbb{P}(T_n > x),
$$

consequently $\mathbb{P}(T_n > x) \leq \mathbb{P}(T_n > x)/2 + \mathbb{P}(S_n > x/2)$ or

$$
\mathbb{P}(T_n > x) \leq 2\mathbb{P}(S_n > x/2).
$$

Combining this bound with (26) yields the lemma’s claim for any $x \geq 2\sqrt{2n}$ by suitably adjusting the constant $c > 0$. On the other hand, for $0 < x \leq 2\sqrt{2n}$ we have

$$
2 \exp \left( -\frac{x^2}{cn} \right) \geq 2 \exp \left( -\frac{8}{c} \right) \geq 1
$$

for sufficiently large $c$, and our claim is trivially fulfilled. \hfill \Box

Lemma 4.5. Under the assumptions of Theorem 1, if $\nu_N = O(\varepsilon_N)$ as $N \to \infty$, then the sequence $(1/\varepsilon_N X_N, N \geq 1)$ is uniformly integrable.

Proof. We allow that $X_N$ takes the value $\infty$, then the event $1/(\varepsilon_N X_N) = 0$ will occur. Essentially, this proof is concerned with bounding the distribution function of $X$.

Set $E[M] = 1 + \varepsilon$ with $\varepsilon > 0$ and let $n$ be the natural number satisfying $(n-1)\varepsilon < 1 \leq n\varepsilon$. Using (3) we have for $x > 0, y \geq 1$

$$
\mathbb{P}(X \leq x) \leq \mathbb{P}\left( \frac{1}{2} \sum_{k=1}^{n} \frac{F_k^-}{\mu_{k-1}} \leq x \right) \leq \mathbb{P}\left( \frac{1}{2} \sum_{k=1}^{n} F_k^- \leq xy \right) + \mathbb{P}\left( \max_{1 \leq k < n} \mu_k > y \right).
$$

Denoting $M_k - E[M_k] = \zeta_k$ we obtain

$$
\mu_k = \prod_{i=1}^{k} (1 + \varepsilon + \zeta_i) \leq \prod_{i=1}^{k} \exp(\varepsilon + \zeta_i) = \exp \left( k\varepsilon + \sum_{i=1}^{k} \zeta_i \right),
$$

hence

$$
\max_{1 \leq k < n} \mu_k \leq \exp \left( 1 + \max_{1 \leq k < n} \sum_{i=1}^{k} \zeta_i \right),
$$

and with $S_k = \zeta_1 + \cdots + \zeta_k$

$$
\mathbb{P}(X \leq x) \leq \mathbb{P}\left( \sum_{k=1}^{n} F_k^- \leq 2xy \right) + \mathbb{P}\left( \max_{1 \leq k < n} S_k > \log y - 1 \right).
$$
Applying Lemma 4.1 yields for \( y > e \) and \( r \geq 2 \)

\[
\mathbb{P}(X \leq x) \leq \mathbb{P}\left(\sum_{k=1}^{n} F_k^- \leq 2xy\right) + 2\exp\left(-\frac{(\log y - 1)^2}{c(n - 1)\nu}\right) + c(n - 1)^{1-r/2} \mathbb{E}[\zeta_1^r]\]

with \( \nu = \mathbb{E}[\zeta^2] \). Putting \( x = e^{-m}/\varepsilon \), \( y = qe^m \) with \( q > 0 \) and taking \((n - 1) < 1/\varepsilon \leq n\) into account we arrive for \( m > 1 - \log q \)

\[
\mathbb{P}(X \leq \frac{e^{-m}}{\varepsilon}) \leq \mathbb{P}\left(\sum_{k=1}^{n} F_k^- \leq 2qn\right) + 2\exp\left(-\frac{(m + \log q - 1)^2\varepsilon}{c\nu}\right) + c\mathbb{E}[|M - \mathbb{E}[M]|^r](n - 1)^{r/2-1}\nu^{r/2}.
\]

Now we exploit this formula for the sequence \((X_N)\). Note that \( \nu_N^{-r/2} \mathbb{E}[|M_N - \mathbb{E}[M_N]|^r] = O(1) \) for some \( r > 4 \) by \( [3] \). Thus, the rightmost term is of order \( O(n^{1-r/2}) = O(\varepsilon_N^{r/2-1}) \). Also, by Lemma 4.2 the term \( \sum_{k=1}^{n} F_k^- \) may be stochastically bounded from below by \( \theta B_{n} \) with \( \theta > 0 \) and a binomial random variable \( B_{n} \) with parameters \( n \) and \( p_0 \). Therefore, for \( q > 0 \) sufficient small the probability \( \mathbb{P}\left(\sum_{k=1}^{n} F_k^- \leq 2qn\right) \) decreases exponentially fast in \( n \), respectively in \( \varepsilon_N^{-1} \). Finally, by assumption there is a \( c > 0 \) such that \( \varepsilon_N \geq 4c\nu_N \) for all \( N \). With these ingredients our estimate shrinks to

\[
\mathbb{P}(X_N \leq \frac{e^{-m}}{\varepsilon}) \leq c'\varepsilon_N^{r/2-1} + \exp\left(-c'(m + \log q - 1)^2\right),
\]

for \( m > 1 - \log q \) and for \( N \) large enough, with some \( c' > 0 \).

We are ready to prove the lemma’s claim. Note that \( X_N \geq 1 \) a.s. by \( [5] \). Thus, we have

\[
\mathbb{E}\left[\frac{1}{\varepsilon_N X_N}: \frac{1}{\varepsilon_N X_N} \geq \epsilon^{m_0}\right] \leq \sum_{m=m_0}^{\lfloor \log 1/\varepsilon_N \rfloor} e^{m+1}\mathbb{P}\left(\frac{e^{-(m+1)}}{\varepsilon_N} \leq X_N \leq \frac{e^{-m}}{\varepsilon_N}\right)
\]

From \( [27] \) we obtain for \( m_0 > 1 - \log q \)

\[
\mathbb{E}\left[\frac{1}{\varepsilon_N X_N}: \frac{1}{\varepsilon_N X_N} \geq \epsilon^{m_0}\right] \leq \sum_{m=m_0}^{\lfloor \log 1/\varepsilon_N \rfloor} e^{m+1}\left(c'\varepsilon_N^{r/2-1} + 2e^{-c'(m + \log q - 1)^2}\right)
\]

\[
\leq c' c^2\varepsilon_N^{r/2-2} + 2\sum_{m=m_0}^{\infty} e^{m+1-c'(m + \log q - 1)^2}.
\]

Since the series is convergent and \( r > 4 \), the right-hand expression can be made arbitrarily small with increasing \( N \) and \( m_0 \), which confirms our claim. \( \square \)

**Proposition 4.6.** Under the assumptions of Theorem 1 and \( \nu_N \to 0 \) we have as \( N \to \infty \),

i) if \( \frac{\nu_N}{\varepsilon_N} \to 0 \), then \( \frac{\pi(V_N)}{\varepsilon_N} \to \frac{2}{\pi x} \) in probability,

ii) if \( \frac{\nu_N}{\varepsilon_N} \to \rho \) with \( 0 < \rho < 2 \), then \( \frac{\pi(V_N)}{\varepsilon_N} \) is asymptotically \( \Gamma \)-distributed with density \( c' x^{\rho - 1} e^{-b' x} \), where \( a' = \frac{2-\rho}{\rho} \), \( b' = \frac{\sigma^2}{\rho} \) and \( c' = (b')^{a'/\Gamma(a')} \),

iii) if \( \frac{\nu_N}{\varepsilon_N} \to 2 \), then \( \frac{\pi(V_N)}{\varepsilon_N} \to 0 \) in probability.
Proof. We begin with some considerations concerning the cases i) and ii). We are going to apply Proposition 2.1. Note that its assumption $E[\log^+ M_N^{(2)}] < \infty$ is satisfied because of (2). Moreover, $E[\log M_N] > 0$ for large $N$ because of Lemma 4.1 ii) and $0 \leq \rho < 2$. Also, taking Lemma 4.3 into account it is sufficient to show that $\varepsilon_N|Y_N - X_N|$ converges to 0 in probability. Using (23), (24) and Lemma 2.3 we have

$$|Y - X| \leq \sum_{k=1}^{N} \frac{B_k}{\mu_{k-1}},$$

here now with

$$B_k := \eta((M_k^{(2)})^2 + M_k^{(2)} + E[U_{1,k} | V]) + 2F_k^+ 1_{\{\pi_k, \infty(V) > (\eta M_k)^3 \text{ or } \eta M_k > 1\}}$$

and any $0 < \eta \leq 1/2$. Thus, we are once more in the setting of Lemma 3.4 with $A_k := 1/M_k$, $C_k = \mu_{k-1}$ and

$$E[A^u] = E[M^{-u}],$$
$$E[B] = \eta(E[(M^{(2)})^2] + E[M^{(2)}] + E[U^4]) + 2E[F^+ ; \pi(V) > (\eta M)^3 \text{ or } \eta M > 1].$$

(28)

Note that the sequence $A_k, k \geq 1$ is iid and that the $B_k$ have the same finite expectation. Now the conclusion of Lemma 3.4 reads

$$P\left((1 - E[A^u]^\frac{1}{2})|X - Y| > cE[B]\right) \leq 2e^{-\frac{c^2}{16}}$$

(29)

for any $c > 0$, provided that $E[A^u]^\frac{1}{2} < 1$.

In order to evaluate (29) we first show that $E[B_N] \to 0$. We have

$$P\left(\pi(V_N) > (\eta M_N)^3 \text{ or } \eta M_N > 1\right) \to 0,$$

since $M_N \to 1$ and $\pi(V_N) \to 0$ in probability, where we use $\pi(V_N) = X_N^{-1}$ by Proposition 2.1 and the fact that $X_N \to \infty$ in probability by Lemma 4.5. Since $F_N^+$ is uniformly integrable by (25), the right most term in (28) vanishes as $N \to \infty$. Further, by assumption (3) we have

$$E[(M_N^{(2)})^2] + E[M_N^{(2)}] + E[U_N^4] \leq C,$$

for some constant $C < \infty$ uniformly in $N$. Altogether, from (28) we obtain $E[B_N] \to 0$, since $\eta$ can be chosen arbitrarily small. Now we are ready to treat the cases i) to iii).

i) From $\nu_N = o(\varepsilon_N)$ by Lemma 4.1 i) we get

$$1 - E[A_N^u]^\frac{1}{2} = 1 - (1 - u\varepsilon_N + o(\varepsilon_N))^\frac{1}{2} = \varepsilon_N + o(\varepsilon_N).$$

Therefore, making use of (29) and $E[B_N] \to 0$ we get $\varepsilon_N|Y_N - X_N| \to 0$ in probability as $N \to \infty$. By Lemma 4.1 i), $\varepsilon_N X_N \to \frac{\pi^2}{4}$ in probability, hence our claim follows by Proposition 2.1.

ii) Lemma 4.1 i) yields

$$1 - E[A_N^u]^\frac{1}{2} = 1 - \left(1 - u\varepsilon_N + \frac{u(u+1)}{2} \nu_N + o(\varepsilon_N + \nu_N)\right)^\frac{1}{2} = \varepsilon_N - \frac{u + 1}{2} \nu_N + o(\varepsilon_N + \nu_N).$$

Here $\frac{\varepsilon_N}{\nu_N} \to \rho \in (0, 2)$, hence choosing $u$ small enough such that $\varepsilon_N - \frac{u+1}{2} \nu_N \geq u\nu_N$ for large $N$ and again by (29) and $E[B_N] \to 0$ we get that $\nu_N|Y_N - X_N| \to 0$ in probability. Consequently,
\( \nu_N X_N \) is asymptotically inverse \( \Gamma \)-distributed by Lemma 4.3 ii). Hence \( \frac{1}{\varepsilon_N X_N} \) is asymptotically \( \Gamma \)-distributed with the stated density. Therefore, the claim follows by Proposition 2.1.

iii) In this case equation (29) is no longer applicable. Here we distinguish two cases. By a subsequence argument we may assume that either \( E[\log M_N] \leq 0 \) for all \( N \), or \( E[\log M_N] > 0 \) for all \( N \). In the first case \( \pi_N = 0 \) by criticality or subcriticality. In the second case we may again apply Proposition 2.1. Then we have

\[
\frac{\pi(V_N)}{\varepsilon_N} = \frac{1}{\varepsilon_N X_N} \leq \frac{1}{2\varepsilon_N Y'_N} \to 0,
\]

in probability by (6) and by an application of Lemma 4.3 iii).

Proof of Theorem 1. i) Since \( \frac{\pi(V_N)}{\varepsilon_N} \) is uniformly integrable by Lemma 4.5, it follows

\[
\frac{\pi_N}{\varepsilon_N} = E \left[ \frac{\pi(V_N)}{\varepsilon_N} \right] \to \frac{2}{\sigma^2},
\]

by Proposition 4.6 i).

ii) Now we have \( \frac{\nu_N}{\varepsilon_N} \to \rho \in (0, 2) \). By the same line of arguments we obtain

\[
\frac{\pi_N}{\varepsilon_N} = E \left[ \frac{\pi(V_N)}{\varepsilon_N} \right] \to \frac{(b')^{a'}}{\Gamma(a')} \int_0^\infty x^{a'-1} e^{-b'x} dx = \frac{a'}{b'} = \frac{2 - \rho}{\sigma^2}.
\]

iii) This claim follows in the same vein using Lemma 4.5 and Proposition 4.6 iii).

iv) Again, in view of a subsequence argument, we may assume without loss of generality that the limits \( \nu_\infty = \lim_N \nu_N \) and \( l_\infty = \lim_N E[\log M_N] \) exist. Because of \( \varepsilon_N \to 0 \) it follows that

\[
E[f(M_N)] \to l_\infty,
\]

with \( f(x) := \log x - x + 1, x > 0 \). Since \( f(x) \leq 0 \) for all \( x \), we have \( l_\infty \leq 0 \) (possibly with value \( -\infty \)). Note that \( f \) is concave with a single zero at point 1. Thus the condition \( l_\infty = 0 \) entails that \( M_N \) converges to 1 and \( M_N - E[M_N] \to 0 \) in probability, as \( N \to \infty \).

Now, if \( \nu_\infty = 0 \), then we may resort to Lemma 4.4 ii), yielding under the present assumption \( E[\log M_N] < 0 \) for large \( N \), thus subcriticality.

On the other hand, if \( \nu_\infty > 0 \), then we have uniform integrability of \( (M_N - E[M_N])^2 \) because of (3), hence

\[
\liminf_{N} E[\min(\eta, (M_N - E[M_N])^2)] \geq \frac{\nu_\infty}{2} > 0,
\]

for some \( \eta > 0 \). Consequently, \( M_N - E[M_N] \) does not converge to 0 in probability. As just shown, this implies \( l_\infty < 0 \), which again implies the claimed subcriticality.

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