CONTINUED FRACTIONS AND $q$-SERIES GENERATING FUNCTIONS FOR THE GENERALIZED SUM-OF-DIVISORS FUNCTIONS

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Abstract. We construct new continued fraction expansions of Jacobi-type J-fractions in $z$ whose power series expansions generate the ratio of the $q$-Pochhammer symbols, $(a; q)_n/(b; q)_n$, for all integers $n \geq 0$ and where $a, b, q \in \mathbb{C}$ are non-zero and defined such that $|q| < 1$ and $|b/a| < |z| < 1$. If we set the parameters $(a, b) := (q, q^2)$ in these generalized series expansions, then we have a corresponding J-fraction enumerating the sequence of terms $(1-q)/(1-q^{n+1})$ over all integers $n \geq 0$. Thus we are able to define new $q$-series expansions which correspond to the Lambert series generating the divisor function, $d(n)$, when we set $z \mapsto q$ in our new J-fraction expansions. By repeated differentiation with respect to $z$, we also use these generating functions to formulate new $q$-series expansions of the generating functions for the sums-of-divisors functions, $\sigma_\alpha(n)$, when $\alpha \in \mathbb{Z}^+$. To expand the new $q$-series generating functions for these special arithmetic functions we define a generalized class of so-called Stirling-number-like “$q$-coefficients”, or Stirling $q$-coefficients, whose properties, relations to elementary symmetric polynomials, and relations to the convergents to our infinite J-fractions are also explored within the results proved in the article.

1. Introduction

1.1. Continued fraction expansions of ordinary generating functions.

Expansions of Jacobi-type J-fractions. Jacobi-type continued fractions, or J-fractions, correspond to power series defined by infinite continued fraction expansions of the form

$$J_\infty(z) = \frac{1}{1 - c_1 z - \frac{ab_2 z^2}{1 - c_2 z - \frac{ab_3 z^2}{\cdots}}} = 1 + c_1 z + \left(ab_2 + c_1^2\right) z^2 + \left(2ab_2 c_1 + c_1^3 + ab_2 c_2\right) z^3 + \left(ab_2^2 + 3ab_2 c_1 + 3ab_2 c_2 + ab_2 c_1^2 + c_1^3 + 2ab_2 c_1 c_2 + ab_2 c_2^2\right) z^4 + \cdots,$$

for arbitrary, application-specific implicit sequences $\{c_i\}_{i=1}^\infty$ and $\{ab_i\}_{i=2}^\infty$, and some typically formal series variable $z \in \mathbb{C}$ [15, cf. §3.10] [20]. The formal series enumerated by special cases of the truncated and infinite J-fraction series of this form include typically divergent ordinary

Date: 2017.05.04.
2010 Mathematics Subject Classification. 11J70; 11Y65; 40A30; 11B65; 11A25.
Key words and phrases. divisor function; sum of divisors function; continued fraction; J-fraction.

Conventions: We adopt a hybrid of the notation for the implicit continued fraction sequences $a_{b-1}b_b := ab_0$ from Flajolet’s article [7]. Our usage of $P/Q$ to denote the convergent function ratios is also consistent with the conventions from this reference.
Generalized properties of the convergents to infinite J-fractions. We define the $h^{th}$ convergent functions, $\text{Conv}_h(z) := P_h(z)/Q_h(z)$, to the infinite J-fraction in (1) recursively through the component numerator and denominator functions given by

\begin{align*}
P_h(z) &= (1 - c_h \cdot z) P_{h-1}(q, z) - ab_h \cdot z^2 P_{h-2}(q, z) + [h = 1] \delta \quad \text{(3)} \\
Q_h(z) &= (1 - c_h \cdot z) Q_{h-1}(q, z) - ab_h \cdot z^2 Q_{h-2}(q, z) + (1 - c_1 \cdot z) [h = 1] \delta + [h = 0] \delta.
\end{align*}

If we let $j_n := [z^n]J_\infty{(z)}$ in (1), the convergents to the full J-fraction defined as above provide 2$h$-order accurate truncated power series approximations to the infinite-order J-fraction generating functions in the following form for each $h \geq 1$:

\[
\text{Conv}_h(z) = j_0 + j_1 z + j_2 z^2 + \cdots + j_{2h-1} z^{2h-1} + \sum_{n \geq h} j_{h,n} z^n.
\]

The rationality of $\text{Conv}_h(z)$ for all $h \geq 1$ is a key property of these approximations to the infinite J-fraction expansion in (1) in that it allows us to prove finite difference equations of order less than or equal to $h$ for the coefficients generated by the power series expansions of these functions. We use these new $h$-order finite difference equations to prove our main theorem stated in the next subsection in Section 3. We now focus on the task of formulating specific cases of these generalized J-fraction expansions and convergent functions which lead to generating functions enumerating special number theoretic arithmetic functions we wish to study through these expansions.

1.2. Constructions of the J-fraction generating function for the divisor function.

Approach in the article. In this article we define the sequences implicit to the expansions of (1) to be functions of our primary series variable $q$ and then construct and prove new forms of convergent infinite J-fraction expansions whose power series expansions in $q$ after $z \mapsto q$ generate the divisor function, $d(n)$, or $\sigma_0(n) = \sum_{d|n} 1$. We prove that this infinite J-fraction expansion corresponds to an infinite $q$-series expansion depending on $z$ and $q$ which may be differentiated termwise with respect to $z$ to obtain new modified $q$-series forms of generating functions for the generalized sums-of-divisors functions, $\sigma_\alpha(n) = \sum_{d|n} d^n$ for $\alpha \in \mathbb{Z}^+$, when we again set $z \mapsto q$ in these expansions. These multiplicative functions are of great interest in number theory with applications in many famous open problems and number theoretic conjectures. We briefly touch on the significance, applications, and relations to recent research in number theory of our new results in Section 4.

Within the scope of this article, we study the expansions of generalized divisor function generating functions in the context of a much more broadly applicable method for enumerating sequences of special functions based on generalized continued fractions. In particular, we find formulas $ab_i(q)$ and $c_i(q)$ for the sequences, $\langle ab_i \rangle$ and $\langle c_i \rangle$, in (1) such that that the infinite J-fraction, denoted by $J_{\infty}(q, z)$ when the component sequences in this case are clear from context, generates the terms $[z^n]J_\infty{(z)} \equiv [z^n]J_\infty(q, z) = \frac{1-q^n}{1-q^m}$ for all $n \geq 0$. We then see from this construction that the power series expansions of this scaled J-fraction in $q$ when $z \mapsto q$ generates the divisor function, $\sigma_0(n) \equiv d(n)$ as

\[
\sum_{n \geq 1} \frac{q^n}{1 - q^n} = \sum_{m \geq 1} d(m) q^m, \quad |q| < 1.
\]

2 Special Notation: Iverson’s convention compactly specifies boolean-valued conditions and is equivalent to the Kronecker delta function, $\delta_{i,j}$, as $[n = k] \delta \equiv \delta_{n,k}$. Similarly, $[\text{cond} = \text{True}] \delta \equiv \delta_{\text{cond,True}} \in \{0, 1\}$, which is 1 if and only if $\text{cond}$ is true, in the remainder of the article.
Moreover, this procedure allows us to use the properties of more general continued fraction results, of which J-fractions are a special case, to formulate an infinite \( q \)-series in \( q \) and \( z \) for this generating function whose \( h^{th} \) partial sum is \( 2h \)-order accurate in enumerating the terms of \((1 - q^{n+1})^{-1}\) over \( z \).

**Complexity and structure of the new \( q \)-series generating function expansions.** We also notice that by the nature and complexity of the special arithmetic functions that we are able to generate with these constructions proved in the article, a priori these sums do (and should) correspond to complicated and rather involved combinatorial objects. We first demonstrate by a triple of examples the tricky nature inherent to directly expanding the symbolic sequences in the definition of (1) whose *Lambert series* generating functions in \( q \) when \( z \mapsto q \) generate the special functions, \( d(n), \sigma_1(n), \) and \( \sigma_\alpha(n) \). More precisely, for \([z^n]J_\infty(z) := 1/(1 - q^n)\), we see that

\[
\begin{align*}
 c_1(q) &= \frac{1}{1 - q} \\
c_2(q) &= \frac{1 + q + 4q^2}{2(q^3 - 1)} \\
c_3(q) &= \frac{1 + 5q + 14q^2 + 26q^3 + 34q^4 + 25q^5 + 9q^6}{2(1 + q + q^2)(1 + 2q + 3q^2)(1 + q + q^2 + q^3 + q^4)} \\
ab_2(q) &= -\frac{2q}{(1 - q)^2(1 + q)} \\
ab_3(q) &= -\frac{(1 - q)(1 + 2q + 3q^2)}{4(1 + q)(1 + q^2)(1 + q + q^2)^2},
\end{align*}
\]

and for \([z^n]J_\infty(z) := n/(1 - q^n)\), we similarly compute that

\[
\begin{align*}
 c_1(q) &= \frac{1}{1 - q} \\
c_2(q) &= \frac{q(-1 - q + 8q^2)}{(1 - q)(1 - 3q)(1 + q + q^2)} \\
c_3(q) &= -\frac{1 - 5q^2 - 16q^3 - 16q^4 + 40q^5 + 136q^6 + 144q^7 + 67q^8 + 8q^9 + q^{10}}{(1 - 3q)(1 + q + q^2)(1 + q + q^3 + q^4)(-1 + 4q^2 + 8q^3 + q^4)} \\
ab_2(q) &= \frac{1 - 3q}{(1 - q)^2(1 + q)} \\
ab_3(q) &= \frac{(1 - q)^3(-1 + 4q^2 + 8q^3 + q^4)}{(1 + q)(1 - 3q)^2(1 + q^2)(1 + q + q^2)^2}.
\end{align*}
\]

More generally, we can see that there is no apparent special formula for the expansions of the implicit sequences for \([z^n]J_\infty(z) := n^\alpha/(1 - q^n)\) by computing that

\[
\begin{align*}
 c_1(q) &= \frac{1}{1 - q} \\
c_2(q) &= \frac{3^\alpha(-1 + q)^2(1 + q) + (1 + 2^{1+\alpha}(-1 + q) + q)(1 + q + q^2)}{(1 + 2^\alpha(-1 + q) + q)(-1 + q^3)} \\
c_3(q) &= \left[\frac{z^{1+3\alpha}a^n}{(1-q)^3(1+q)^3(1+q+q^2)^3} - \frac{z^{2\alpha}}{(-1+q)^3(1-q)^2(1-q+q^2)} - \frac{z^\alpha}{(-1+q)^4(1-q)^2(1-q+q^2)^2} + \frac{z^{1+3\alpha}a^n}{(1+q)^3(1+q+q^2)^3} \right] + \left[\frac{z^{2\alpha}}{(-1+q)^3(1-q)^2(1-q+q^2)} - \frac{z^\alpha}{(-1+q)^4(1-q)^2(1-q+q^2)^2} + \frac{z^{1+3\alpha}a^n}{(1+q)^3(1+q+q^2)^3} \right] + \left[\frac{z^{2\alpha}}{(-1+q)^3(1-q)^2(1-q+q^2)} - \frac{z^\alpha}{(-1+q)^4(1-q)^2(1-q+q^2)^2} + \frac{z^{1+3\alpha}a^n}{(1+q)^3(1+q+q^2)^3} \right].
\end{align*}
\]
\[
\left[ \frac{1}{(-1+q)^2} + \frac{2^\alpha}{1-q^2} \left( \frac{8^\alpha}{(-1+q)^2} \right) \left( \frac{9^\alpha}{(-1+q)^2} \right) \left( \frac{4^\alpha}{(-1+q)^2(1-q^4)} \right) \frac{2^\alpha}{1-q^2} \right] \nabla^2 \frac{1}{(-1+q)^2} \frac{2^\alpha}{1-q^2} \left( \frac{3^\alpha}{(-1+q)^2} \right) \frac{4^\alpha}{1-q^2} \]
\]

\[
abla^2(q) = -\frac{1}{(-1+q)^2} + \frac{2^\alpha}{1-q^2}
\]

\[
abla^3(q) = \frac{8^\alpha}{(-1+q)^2} + \frac{8^\alpha}{(-1+q)^2(1-q^2)} - \frac{2^\alpha 3^\alpha}{(-1+q)^2(1-q^2)(1+q+q^2)} - \frac{9^\alpha}{(-1+q)^2} + \frac{4^\alpha}{(-1+q)^2(1+q+q^2+q^4)}.
\]

We can also compute numerical sequences directly generating the functions, \(\sigma_\alpha(n)\), over \(z\), i.e., in place of attempting to indirectly enumerate these special sequences by generating their corresponding Lambert series function expansions through (1). However, the resulting sequence expansions in these cases are similarly complicated and unreliable.

**An intermediate approach.** We observe that we may bypass the seemingly complex forms of the implicit J-fraction sequences characteristic of empirically computing the first terms of these sequences whose corresponding \(q\)-series expansion of (1) generates the divisor sum functions, \(d(n)\) and \(\sigma_\alpha(n)\) for \(\alpha \in \mathbb{Z}^+\), using a trick and a special case. That is, by constructing a new class of these J-fraction expansions generating the ratios of the \(q\)-Pochhammer symbols, \((a; q)_n/(b; q)_n\), according to Definition 1.1 below for some fixed non-zero \(a, b \in \mathbb{C}\), we may generate the divisor function generating function terms defined above in the particular case where \((a, b) := (q, q^2)\).

**Definition 1.1** (Sequence Definitions and Notation). We define the two sequences implicitly defined by (1) in the next theorem for some fixed non-zero \(a, b, q \in \mathbb{C}\) to be

\[
\begin{align*}
abla_i(a, b; q) &:= \frac{q^{2i-4}(1-bq^{i-3})(1-aq^{i-2})(a-bq^{i-2})}{(1-bq^{2i-5})(1-bq^{2i-4})^2(1-bq^{2i-3})}, \\
c_i(a, b; q) &:= \begin{cases} 
\frac{q^{i-2}(aq + abq^{i-3} + a(1-q^{i-1}) + q + b(-1-q+q^2))}{(1-bq^{2i-4})(1-bq^{2i-3})} & \text{if } i \geq 2; \\
\frac{q^{i-1}}{b-1} & \text{if } i = 1; \\
0 & \text{otherwise},
\end{cases}
\end{align*}
\]

where the \(h^{th}\) modulus component products are given by

\[
\lambda_h(a, b; q) := \ab_2(a, b; q) \cdots \ab_h(a, b; q)
\]

\[
= \frac{aq^{(h-1)^2}(b/q; q)_{h-1}(a; q)_{h-1}(b; q^{q^2})_{h-1}(a; q^{q^2})_{h-1}(q; q^2)_{h-1}}{(b/q; q^2)_{h-1}((b; q^{q^2})_{h-1}(b; q^{q^2})_{h-1})_{h-1}}.
\]

**Notation.** We formally introduce the notation of \(J_\infty(a, b; q, z)\) to denote the left-hand-side function in (1) and \(\Conv_h(a, b; q, z)\) for its \(h^{th}\) convergents to denote shorthands for the J-fraction expansions involving these sequences parametrized in the non-zero \(a, b, q \in \mathbb{C}\). Where it is clear from context that the particular J-fraction defined in this form is defined in terms of definite functions of \(q\), we drop the first two parameters in the previous notation for the generalized expansions to write \(J_\infty(q, z)\) for the infinite J-fraction and \(\Conv_h(q, z)\) for its convergents. Typically, we assume in these cases that \((a, b) := (q, q^2)\) when using this abbreviated notation.

There is “no free lunch” in that these comparatively simple sequence forms result in expansions of the convergent functions, denoted by \(\Conv_h(a, b; q, z)\) in these cases, which inherit a more rich and complex structure involving paired products of the \(\ab_i(a, b; q)\) functions interleaved with the expansions of a generalized set of Stirling \(q\)-coefficients whose expansions are explicitly formed by **elementary symmetric polynomials** over the sequence of \(q\)-functions, \(c_i(a, b; q)\). The resulting convergent-based series for these generating functions is considered to have the form of a “\(q\)-series expansion” since it is intertwined with finite \(q\)-Pochhammer...
symbols and reciprocal paired sums of the Stirling number $q$-coefficients weighted by individual products of the functions, $ab_i(a, b; q)$, which are parametrized in the choices of non-zero $a, b, q \in \mathbb{C}$. Choosing a construction of our new $q$-series results based on the trick to enumerate the special case of $[z^n]J_{\infty}(z) = (a; q)_n / (b; q)_n$ by these J-fractions, we do in fact obtain the desired simpler forms of the sequences, though again we see in turn in the next sections that the corresponding convergent denominator functions, $Q_h(a, b; q, z)$, in (3) satisfy the predicted more complicated and involved expansions which we define and prove in the next sections of the article.

Definitions and statement of the main theorem.

**Theorem 1.2** (J-fractions Generating a Special Ratio of $q$-Pochhammer Symbols). Let $a, b, q, z \in \mathbb{C}$ denote fixed non-zero parameters such that $|b/a| < |z| < 1$ and $|q| < 1$. We claim that for the special sequences defined in the notation of Definition 1.1 that for all $h \geq 2$ we have

$$\text{Conv}_h(a, b; q, z) = \frac{\lambda_i(a, b; q)z^{2i-2}}{Q_{i-1}(a, b; q, z) Q_i(a, b; q, z)},$$

where for each integer $h \geq 1$ and all $0 \leq n < 2h$

$$[z^n] \text{Conv}_h(a, b; q, z) = \frac{(a; q)_n}{(b; q)_n},$$

In our special case of interest where $(a, b) := (q, q^2)$ we have

$$J_{\infty}(q, z) = \frac{q(1 + q)}{1 + q - z} + \sum_{i \geq 2} \frac{q \cdot q^{(i-1)^2} (q; q)_i^4 z^{2i-2}}{(q; q)_{i-1}^2 (q^2; q^2)_{i-1}^2 (q^3; q^2)_{i-1}^2 \times Q_{i-1}(q, z) Q_i(q, z)}$$

provided that this infinite J-fraction is convergent for $|q|, |qz| < 1$ and our choices of $a, b \neq 0$.

We will prove this theorem and the convergence of the limiting case of the finite convergent function sums for $(a, b) := (q, q^2)$ as two of our main results in Section 3 below. We will first need some machinery for expanding the $h^{th}$ convergents to $J_{\infty}(z)$ for arbitrary sequences, $\langle ab_i \rangle$ and $\langle c_i \rangle$, developed by the results proved in Section 2.

1.3. A comparison of J-fraction expansions for known generating functions of the divisor and sum-of-divisors functions. The significance of Theorem 1.2 is that it allows us to formulate, as we will soon see, rich structured expansions of Lambert series generating functions for the divisor function, and by extension for $\sigma_\alpha(n)$ for all integers $\alpha \geq 1$. The reference [17] provides related convergent $q$-series expansions for the sum-of-divisors function, $\sigma_1(n)$, given through reciprocal sums over the Gaussian polynomials, or $q$-binomial coefficients in place of the sums we develop involving our so-termed generalized Stirling $q$-coefficients. These known special sequence generating functions are given by

$$\sum_{k=0}^{\infty} \sigma_1(2k+1) q^{2k+1} = \sum_{b=\pm 1} b \frac{b}{6} \left( 1 + 2bq \times \sum_{i=1}^{\infty} \frac{(-1)^{i-1} (bq)^{3i(i-1)} (q^2; q^2)_i^4}{\sum_{0 \leq j \leq n < 2i} [j]_q^2 [i-j]_q^2 [q^2]_i^2 (-bq^{2i-1})^n} \right)^4,$$

and

$$\sum_{n \geq 1} \sigma_1(n) q^n = -q \cdot d/dq \log (q; q)_\infty,$$
where for $|q| < 1$ we have that
\[
(q; q)_\infty = 1 - q \sum_{i=1}^{\infty} \frac{(-1)^{i-1}q^{(2i-2)(i-1)/2}(q^3; q^3)_{i-1}}{\sum_{0 \leq j \leq n < 2i} \frac{j}{q^i} \frac{[i-1]}{[n-j]} q^3} \cdot q^{(3i-2)n}.
\]

\[ - q^2 \sum_{i=1}^{\infty} \frac{(-1)^{i-1}q^{(2i+2)(i-1)/2}(q^3; q^3)_{i-1}}{\sum_{0 \leq j \leq n < 2i} \frac{j}{q^i} \frac{[i-1]}{[n-j]} q^3} \cdot q^{(3i-1)n}.
\]

We note that an example related to our methods constructed within this article, but that we do not explicitly cite closed-form sums for, provides another $q$-series generating functions for the divisor function in the form of [6] (cf. Table 4.1 on page 16)
\[
\sum_{n=1}^{\infty} d(n)q^n = (q; q)_\infty \sum_{n=1}^{\infty} \frac{nq^n}{(q; q)_n},
\]

where we can generate the terms of $n/(q; q)_n$ through first-order derivatives of the series for $1/(z; q)_\infty$ also identified in the reference. Additional identities of Dilcher from his article provide expansions of the series coefficients of
\[
U_{\alpha}(q) = \sum_{n \geq 1} n^\alpha q^n \times \prod_{j=n+1}^{\infty} (1 - q^j) = (q; q)_\infty \sum_{n} \frac{n^\alpha q^n}{(q; q)_n},
\]

when $\alpha \in \mathbb{Z}^+$. Unlike Dilcher’s article referenced above, our methods provide a generalizable construction that is employed to form explicit $q$-series generating functions that enumerate the sequence of $\sigma_\alpha(n)$ for any fixed integers $\alpha \geq 0$.

2. Expansions of Generalized Convergent Functions to Infinite J-Fractions

2.1. Definitions and statements of key lemmas.

**Definition 2.1** (Special Sums and Triangles of Stirling $q$-Coefficients). Let the function, $S_{h,m,s}(z)$, be defined by the nested sums in the next equation for non-negative integers $h \geq 2$, $m \leq h$, and $s \leq mh$.

\[
S_{h,m,s}(z) := \sum_{k_1=2}^{h-2(m-1)} \sum_{k_2=k_1+2}^{h-2(m-2)} \cdots \sum_{k_m=k_{m-1}+2}^{h} \left[ \frac{ab_{k_1}}{(1-c_{k_1}z)(1-c_{k_1-1}z)} \right] \cdots \left[ \frac{ab_{k_m}}{(1-c_{k_m}z)(1-c_{k_m-1}z)} \right] \times [k_1 + \cdots + k_m = s]_\delta.
\]

Next, we let the most general forms of the Stirling-number-like $q$-coefficient triangles be defined recursively by
\[
\begin{bmatrix} h \\ k \end{bmatrix}_c := \begin{bmatrix} h-1 \\ k \end{bmatrix}_c - c_h \begin{bmatrix} h-1 \\ k-1 \end{bmatrix}_c + [h = k = 0]_\delta.
\]

(9)

If we let the sequences implicit to the expansions of (1) correspond to the special $q$-Pochhammer ratio sequences defined in Definition 1.1, we define another auxiliary notation for these now so-termed Stirling $q$-coefficients where $[k]_c \mapsto [h]_{a,b,q}$ and these special case coefficients satisfy the corresponding triangular recurrence relation given by
\[
\begin{bmatrix} h \\ k \end{bmatrix}_{a,b,q} := \begin{bmatrix} h-1 \\ k \end{bmatrix}_{a,b,q} - c_h(a,b;q) \begin{bmatrix} h-1 \\ k-1 \end{bmatrix}_{a,b,q} + [h = k = 0]_\delta.
\]

(10)

**Lemma 2.2** (Products Generating the Stirling $q$-Coefficients). For all $h \geq 0$ and $0 \leq k \leq h$, we have that the most general forms of the Stirling $q$-coefficients defined above are generated by the products
\[
\begin{bmatrix} h \\ k \end{bmatrix}_c = [z^k](1-c_1z)(1-c_2z) \cdots (1-c_hz) + [h = k = 0]_\delta.
\]
Lemma 2.3 (Expansions of the Convergent Denominator Functions). For the most general forms of the convergent denominator functions defined by (3) and the corresponding most general forms of the Stirling \( q \)-coefficients, \( [h]_q \), defined in this section above, we have the following expansion identities for all \( 0 \leq n \leq h \) when \( h \geq 2 \):

\[
Q_h(z) = (1 - c_1z) \times \cdots \times (1 - c_hz) \left[ 1 + \sum_{m=1}^{[h/2]} \sum_{s=0}^{mh} (-z^2)^m S_{h,m,s}(z) \right] \quad (i)
\]

\[
[z^n]Q_h(z) = \begin{bmatrix} h \\ n \end{bmatrix}_c + \sum_{m=1}^{[h/2]} \sum_{s=0}^{mh} \sum_{k=0}^{n} (-1)^m \begin{bmatrix} h \\ n-k \end{bmatrix}_c [z^{k-2m}] S_{h,m,s}(z). \quad (ii)
\]

Remark 2.4. The intuition for the statement of (i) in Lemma 2.3, which we prove by induction on \( h \) in Section 2.2, is an interpretation of the successive expansions of the recurrence relation in (3) for \( Q_h(z) \). Namely, we have on one hand terms of the form \( (1 - c_i z) \) multiplied recursively by the inductive expansions of \( Q_h(z) \), and similarly on the other hand, we have terms of the form \( z^2 \cdot ab_i \) multiplied by inductive expansions of the expansion forms of the denominator functions with two indices of difference. This implies that we may pull out a factor of the product, \( (1 - c_1z) \cdots (1 - c_hz) \), from the overall expansion of \( Q_h(z) \), which leaves us with a sum of weighted terms in products of \( z^{2m} \cdot ab_i \cdots ab_{im} \) times corresponding paired reciprocals of \( (1 - c_i z)(1 - c_{i-1} z) \) in these expansions. The definition of the special sums, \( S_{h,m,s}(z) \), from Definition 2.1 makes the exact forms of these expansions for \( Q_h(z) \) precise according to the statement of the lemma.

Remark 2.5 (Exact Formulas for the Generalized \( q \)-Coefficients). We see immediately, and can easily verify by induction, that the particular first columns of the triangular \( q \)-coefficients defined in Definition 2.1 are given by the formulas

\[
\begin{bmatrix} h \\ 0 \end{bmatrix}_{q^2,q} = 1,
\]

\[
\begin{bmatrix} h \\ 1 \end{bmatrix}_{q^2,q} = -\frac{1}{1+q} + \sum_{k=0}^{h-2} \left\{ \frac{q}{2(1-q^{k+2})} - \frac{q^3 + 2q^2 - 3q - 2}{2(q^2 - 1)(1 + q^{k+2})} - \frac{1}{2(1+q)(1-q^{k+1})} - \frac{2q - 3}{2(1-q)(1+q^{k+1})} \right\},
\]

where the sums on the right-hand-side of the second equation are expanded through \( q \)-polygamma functions for each \( h \geq 1 \) [15, §5.18]. Formalizing exact formulas for the subsequent cases of columns indexed by \( k \geq 2 \) is a dicey proposition by inspection alone. That being said, the products generating these coefficients given by Lemma 2.2 do in fact give these coefficients another rich structure which inherits from the properties of elementary symmetric polynomials with respect to a single variable. For example, we have that

\[
\begin{bmatrix} h \\ 2 \end{bmatrix}_c = \sum_{1 \leq i_1 < i_2 \leq h} c_{i_1}c_{i_2},
\]

\[
\begin{bmatrix} h \\ 3 \end{bmatrix}_c = \sum_{1 \leq i_1 < i_2 < i_3 \leq h} c_{i_1}c_{i_2}c_{i_3},
\]

\[
\begin{bmatrix} h \\ k \end{bmatrix}_c = \sum_{1 \leq i_1 < \cdots < i_k \leq h} c_{i_1} \cdots c_{i_k}, \text{ for integers } k \geq 1,
\]

and in particular, if we define

\[
S_m(c_1, \ldots, c_h) := \sum_{j=1}^{h} c_j^m,
\]
then the Newton-Girard formulas, or Newton’s identities, imply that for $0 \leq k \leq h$

$$(-1)^k \binom{h}{k} + \sum_{m=1}^{k} (-1)^{k-m} s_m(c_1, \ldots, c_h) \binom{h}{m-k} = 0.$$ 

For other specific properties of these generalized Stirling-like coefficients, we may consult the results in the references [5, 11, 16].

2.2. Proofs of the key lemmas.

Proof of Lemma 2.2. The proof of this result is follows by defining a recurrence for the right-hand-side products and showing that it generates precisely the same triangular sequence as the generalized Stirling numbers we defined in Definition 2.1. In particular, for integers $h, k \geq 0$ we define the following sequence in terms of the products in the statement of the lemma:

$$f_{h,k} := [z^k](1 - c_1 z) \cdots (1 - c_h z) + [h = k = 0]_\delta.$$ 

Then by expanding the right-hand-side coefficients of the last equation recursively we obtain that

$$f_{h,k} = [z^k]((1 - c_1 z) \cdots (1 - c_{h-1} z) - c_h \cdot z(1 - c_1 z) \cdots (1 - c_{h-1} z)) + [h = k = 0]_\delta$$

$$= [z^k](1 - c_1 z) \cdots (1 - c_{h-1} z) - c_h \cdot [z^{k-1}](1 - c_1 z) \cdots (1 - c_{h-1} z) + [h = k = 0]_\delta$$

$$= f_{h-1,k} - c_h \cdot f_{h-1,k-1} + [h = k = 0]_\delta.$$ 

Thus we see that the product-wise coefficients, $f_{h,k}$, satisfy precisely the same recurrence relation as the generalized Stirling numbers defined above and have the same initial conditions. Namely, that $f_{h,k}, \binom{h}{k} = 0$ when $k, h < 0$ and $f_{h,k}, \binom{h}{k} = 1$ when $h = k = 0$. Hence, we conclude that the two sequences define the same triangle, and so the generalized Stirling numbers (or Stirling q-coefficients) are generated by the right-hand-side products in the statement of the lemma.

Proof of Lemma 2.3 (i). We prove part (i) of the lemma by induction on $h$. When $h = 1$, we have that $Q_h(z) = 1 - c_1 z$ and that $[h/2] = 0$ so that the double sum on the right-hand-side of (i) is zero, which implies that (i) is true in this case. Suppose that $h \geq 2$ and that (i) is true as a function of $k$ for all $k < h$. Then by expanding (3) according to our inductive hypothesis, we obtain

$$Q_h(z) = (1 - c_1 z) \cdots (1 - c_h z) \times \left[1 + \sum_{m=1}^{\lfloor h/2 \rfloor} \sum_{s=0}^{m(h-1)} (-1)^m z^{2m} s_{h-1,m,s}(z)\right]$$

$$- \frac{z^2 ab_h}{(1 - c_{h-1} z)(1 - c_h z)} - \frac{z^2 ab_h}{(1 - c_{h-1} z)(1 - c_h z)} \sum_{m=1}^{\lfloor h/2 \rfloor} \sum_{s=0}^{m(h-2)} (-1)^m z^{2m} s_{h-2,m,s}(z)$$

We first suppose that $h := 2j + 1$ is odd and will give an instructive proof of the formula in this case which similarly leads to a proof of the formula in the case where $h := 2j$ is even. We next define the auxiliary function, $T_h(z)$, and expand its terms as follows:

$$T_h(z) = \sum_{m=1}^{\lfloor h/2 \rfloor} \sum_{s=0}^{m h} (-1)^m z^{2m} s_{h,m,s}(z) - \sum_{m=1}^{\lfloor (h-1)/2 \rfloor} \sum_{s=0}^{m h} (-1)^m z^{2m} s_{h-1,m,s}(z)$$

$$= \sum_{m=1}^{\lfloor h/2 \rfloor} \sum_{s=0}^{m(h-1)} (-1)^m z^{2m} [s_{h,m,s}(z) - s_{h-1,m,s}(z)] + \sum_{m=1}^{\lfloor h/2 \rfloor} \sum_{s=0}^{m-1} (-1)^m z^{2m} s_{h,m,s+m-1+h}(z).$$

To simplify notation, let the shorthand $d_i := ab_i / ((1 - c_{i-1} z)(1 - c_i z))$. All terms with a factor of $ab_h$ in the expansion of $T_h(z)$ are of the form $d_1 \times \cdots \times d_{i-1} \times d_h$ where by the definition
of the special sums, \( S_{h,m,s}(z) \), we must have that \( i_{m-1} \leq h - 2 \) and \( i_j \in [i_{j-1} + 2, i_{j+1} - 2] \) for \( 1 \leq j < m \) and where by convention we set \( i_0 := 0 \). Equivalently, we see that

\[
\widetilde{T}_h(z) = \sum_{k_1=2}^{h-2} \cdots \sum_{k_{m-1} = k_{m-2} + 2}^{h} d_{k_1} \cdots d_{k_m} 
\]

\[
= \sum_{m=2}^{[h/2]} \sum_{s=0}^{m(h-2)} d_h \times (-1)^m z^{2m} S_{h-2,m-1,s}(z)
\]

\[
= - \sum_{m=1}^{[h/2]} \sum_{s=0}^{m(h-2)} z^2 d_h \times (-1)^m z^{2m} S_{h-2,m,s}(z),
\]

where the upper index \( m(h-2) \) of the second sum is formed by considering the maximum possible sum of \( k_1 + \cdots + k_m = s \). Then we finally see that

\[
\sum_{m=1}^{[h/2]} \sum_{s=0}^{m} (-1)^m z^{2m} S_{h,m,s}(z) = -z^2 d_h + \sum_{m=1}^{[h/2]} \sum_{s=0}^{m(h-1)} (-1)^m z^{2m} S_{h-1,m,s}(z)
\]

\[
- z^2 d_h \times \sum_{m=1}^{[h/2]} \sum_{s=0}^{m(h-2)} (-1)^m z^{2m} S_{h-2,m,s}(z),
\]

and so the result is proved. \( \square \)

Proof of Lemma 2.3 (ii). The result in (ii) of the lemma is not difficult to obtain from the statement of the first result in (i). In particular, we may first rewrite (i) in the following forms resulting from the expansions in Lemma 2.2:

\[
Q_h(z) = \sum_{k=0}^{h} \sum_{s=0}^{[h/2]} z^k + \sum_{m=1}^{[h/2]} \sum_{s=0}^{m} \sum_{s_0=0}^{h} (-1)^m z^{s_0-k} \left( \sum_{k=0}^{h} \frac{h}{k^c} z^k \right) \times \left[ z^{k-2m} S_{h,m,s}(z) \right]
\]

\[
= \sum_{k=0}^{h} \frac{h}{k^c} z^k + \sum_{m=1}^{[h/2]} \sum_{s=0}^{m} \sum_{s_0=0}^{h} (-1)^m z^{s_0} \left[ \sum_{k=0}^{h} \frac{h}{s_0-k^c} \right] \times \left[ z^{k-2m} S_{h,m,s}(z) \right].
\]

Hence we arrive at the coefficient formula in (ii) by removing the sum indexed by \( s_0 \) in the last equation and setting \( s_0 \mapsto n \) when \( 0 \leq n \leq h \). \( \square \)

2.3. Related expansions of the convergent numerator functions.

Roadmap for the proof of the theorem. To prove our main result stated in Theorem 1.2, it suffices to provide an argument inductively showing that for each \( 0 \leq n < h \) we have that

\[
\frac{1 - q}{1 - q^{n+1}} = [z^n]P_h(z) - \sum_{i=1}^{\min(n,h)} [z^i]Q_h(z) \cdot \frac{1 - q}{1 - q^{n+1-i}}
\]

\[
= [z^n]P_h(z) - \sum_{i=1}^{\min(n,h)} [z^{n+1-i}]Q_h(z) \cdot \frac{1 - q}{1 - q^i}.
\]

In this case we are concerned not only with the expansions of the \( h^{th} \) convergent functions defined in terms of the denominator functions by the theorem, but also with the polynomial expansions of the numerator functions, \( P_h(z) \), in \( z \). By the definitions of the two recurrence relations in (3), which are identical except for their initial conditions, we readily see by a simple two line proof that

\[
P_h(z) = Q_{h-1}(z) \quad \text{substituting} \quad c_i \mapsto c_{i+1}, \ ab_i \mapsto ab_{i+1}.
\]
Surprisingly, despite how seemingly closely related the expansions of these two convergent function subsequences are, the knowledge of Lemma 2.2 and Lemma 2.3 alone does not provide us with immediate working relations that allow us to combine like terms to simplify the right-hand-sides in (11). That is to say that we need to be slightly more inventive to arrive at the comparatively simple inductive proof of Theorem 1.2 given in the next section.

Statements of properties of the convergent numerator functions.

Definition 2.6 (Numerator Function Stirling q-Coefficients and Special Sums). In analogy to the product-based expansions of the generalized Stirling numbers we proved in Lemma 2.2 of the last subsection, we define the following equivalent modified forms of these coefficients to explore the properties of $P_h(z)$ in more structured detail:

\[
\begin{aligned}
\left[ \frac{h}{k} \right]_{c,P} &= \left[ \frac{h - 1}{k} \right]_c - c_{h+1} \left[ \frac{h - 1}{k} \right]_{c,P} \\
&= \left[ z^k \right] (1 - c_{2z})(1 - c_{3z}) \cdots (1 - c_{h,z}).
\end{aligned}
\]

We also define the next shifted forms of the special nested sums corresponding to the expansions of the numerator functions, $P_h(z)$, proved in the lemma below for

\[
S_{h,m,s}^{[P]}(z) = \sum_{i_1=2}^{h-2m} \sum_{i_2=i_1+2}^{h-2(m-1)} \cdots \sum_{k_m=k_{m-1}+2}^{h} \left( \prod_{p=1}^{m} \frac{ab_{k_p+1}}{(1 - c_{k_p}z)(1 - c_{k_{p+1}}z)} \right) [k_1 + \cdots + k_m = s - m]_d.
\]

We state the next claim and prove the next lemma each providing apparent first relations between the two sets of generalized Stirling q-coefficients and between the two distinct convergent function sequences. These results provide a more complicated approach to proving the result outlined in (11) above by attempting to relate the numerator and denominator convergent functions through the characteristic expansions of the denominator functions which we use to expand Conv$_h(a, b; q, z)$ in the theorem statement.

Claim 2.7. For all integers $1 \leq k \leq h$ we have the following relations between the two distinct index-shifted sets of generalized Stirling q-coefficients:

\[
\left[ \frac{h}{k} \right]_{c,P} = \left[ \frac{h - 1}{k} \right]_c + (c_1 - c_h) \times \sum_{1 \leq i_1 < \cdots < i_{k-1} < h} c_{i_1+1} \cdots c_{i_{k-1}+1}
\]

Moreover, for non-negative integers $h \geq 0$, $m \leq h/2$, and $s \leq h$, we have the next conjecture for a formula relating the differences of the two variants of the special nested sums we have defined above.

\[
S_{h-1,m,s}(z) - S_{h,m,s}^{[P]}(z) = \sum_{2 \leq i_1 < \cdots < i_m \leq h} \left( \prod_{k=1}^{m} \frac{ab_{i_k}}{(1 - c_{i_k-1}z)(1 - c_{i_k}z)} \right).
\]

Lemma 2.8 (Exact Expansions of the Convergent Numerator Functions). For all $h \geq 2$ and $0 \leq n < h$, we have the next exact expansions of the convergent numerator functions, $P_h(z)$.

\[
P_h(z) = (1 - c_{2z}) \cdots (1 - c_{h,z}) \left[ 1 + \sum_{m=1}^{[h/2]} \sum_{s=0}^{m(h+2) - 2} (-z^n)^m S_{h-1,m,s}^{[P]}(z) \right]
\]

\[
|z^n| P_h(z) = \left[ \frac{h}{n} \right]_{c,P} + \sum_{m=1}^{[h/2]} \sum_{s=0}^{m(h+2)-2} \sum_{k=0}^{h} (-1)^m \left[ \frac{h}{n-k} \right]_{c,P} [z^{k-2m}] S_{h-1,m,s}^{[P]}(z)
\]
Proof of (i) and (ii). We cite a short proof that follows from the proofs of Lemma 2.3 in the previous subsection by noticing that we have shown that (12) results from the recurrence relations defining the two convergent function sequences in (3). Thus we have already given an identical proof of these two related result in the previous subsection, and so we are done. □

The main expansion result for the numerator convergent functions. The next proposition provides a result that is relatively uncomplicated to prove by a nested, or double induction procedure. In the next section we then combine this result with the $h$-order finite difference equations phrased in (11) implied by the rationality of Conv$_h(q, z)$ for all $h \geq 1$ to complete our proof of Theorem 1.2.

**Proposition 2.9** (Polynomial Expansions in $z$). For each $h \geq 0$ and all $0 \leq n < h$, we have the following expansions of the polynomial coefficients of $P_h(q, z)$ in $z$:

$$[z^n]P_h(q, z) = \sum_{i=0}^{n} [z^i]Q_h(q, z) \cdot \frac{1 - q}{1 - q^{n+1-i}}.$$ 

Proof. We proceed to prove this result by a double induction procedure on $h$ with $n \in [0, h)$ for each fixed $h \geq 1$. Let the shorthand notation for the coefficients of the numerator functions with respect to $z$ be defined as $p_{h,n} := [z^n]P_h(z)$. We first show that this result is true in the first cases where $h := 1, 2$. More precisely, for $h := 1, 2$ we have the next expansions of the functions, $P_h(z)$, given by

$$P_1(q, z) = 1$$
$$P_2(q, z) = 1 - \frac{2q(1-q)z}{(1-q^2)},$$

which by computation agrees with the right-hand-side formula stated above. Next, we assume that the formula is correct for some $h \geq 1$ and all $0 \leq n < h$. We then proceed to expand the finite sum formula above using the recurrences in (3) as

$$\sum_{i=0}^{n} [z^i]Q_h(q, z) \cdot \frac{1 - q}{1 - q^{n+1-i}} = \sum_{i=0}^{n} \left( [z^i]Q_{h-1}(q, z) - c_h(q)[z^{i-1}]Q_{h-1}(q, z) ight) \cdot \frac{1 - q}{1 - q^{n+1-i}}$$

$$- ab_h(q)[z^{i-2}]Q_{h-2}(q, z) \cdot \frac{1 - q}{1 - q^{n+1-i}} - c_h(q)p_{h-1,n-1} - ab_h(q)p_{h-2,n-2},$$

which is equal to $[z^n]P_h(q, z)$ if the leftmost sum term is equal to $p_{h-1,n}$. We know that this occurs when $n < h - 1$, but we also see that when $n = h - 1$ that $p_{h-1,n} = 0$, so we may assume that this summation term equals $p_{h-1,n}$ for all $0 \leq n < h$. □

3. Proof of the main theorem and restatements of these results

3.1. Proof of the theorem and convergence as $h \rightarrow \infty$.

**Proof of Theorem 1.2 (Sketch).** The expansion in (6) follows from well-known recursive properties of the convergents to any J-fractions of the form in (1) [15, §1.12]. By the rationality of Conv$_h(q, z)$ for each $h \geq 1$, we have the following $h$-order finite difference equation satisfied exactly by the sequence of coefficients enumerated by Conv$_h(q, z)$ for all integers $n \geq 0$:

$$[z^n] \text{Conv}_h(q, z) = [z^n]P_h(q, z) - \sum_{i=1}^{\min(n,h)} [z^i]Q_h(q, z) \cdot [z^{n-i}] \text{Conv}_h(q, z).$$
For each fixed \( h \geq 2 \), we prove that the first form of (11) restated as

\[
\frac{1 - q}{1 - q^{k+1}} = [z^k]P_h(z) [0 \leq k < h] - \sum_{i=1}^{\min(k,h)} [z^i]Q_h(z) \cdot \frac{1 - q}{1 - q^{k+1-i}}, 
\]

holds for all \( k \in [0, h - 1] \) by induction on \( n \). Since \( \text{Conv}_h(q, z) \) is rational for all \( h \geq 1 \), this result suffices to prove (7). Then since we have the known property that \( [z^n] \text{Conv}_h(z) = [z^n]J_{\infty}(z) \) for all \( 0 \leq n < h \) (more precisely, for all \( 0 \leq n < 2h \)), this result combined with (6) suffices to prove that (8) holds in the limiting case as \( h \to \infty \) provided the infinite continued fraction converges. We prove the convergence of the infinite J-fraction expansion defined by our sequences in Definition 1.1 using Pringsheim’s theorem in the proposition stated immediately below.

\[ \Box \]

**Proof of (7) by Induction on \( n \).** Suppose that \( h \geq 2 \) is fixed. Since \([z^0]P_h(q, z) = 1 \) for all \( h \), it follows that (i) is true when \( k = 0 \). We next suppose that (i) is true for all \( k < n \) and proceed to show that this implies that the statement is true for \( k = n \). We first notice that the indices \( n-i \) when \( i \in [1, n] \) are in the range of \( \{ 0, 1, \ldots, n-1 \} \), so that we see that the right-hand-side of (i) is equal to \([z^n] \text{Conv}_h(q, z) \) by our inductive hypothesis. Then by Proposition 2.9, we see that (i) is true for this choice of \( k = n \). Hence our claim is true for all \( n \in [0, h - 1] \) when \( h \geq 2 \) is a fixed positive integer. \[ \Box \]

**Proposition 3.1** (Convergence of the Infinite J-Fraction Expansion, \( J_{\infty}(q, z) \)). For fixed non-zero \( q \in \mathbb{C} \) such that \( 0 < |q| < 0.206783 \), we have that the infinite J-fraction

\[
J_{\infty}(q, z) = (1 - q) \times \sum_{n \geq 0} \frac{z^n}{1 - q^{n+1}},
\]

generating the divisor function is convergent with \( \text{Conv}_h(q, z) \to J_{\infty}(q, z) \) in the limiting case as \( h \to \infty \). Moreover, the infinite sum for \( J_{\infty}(q, z) \) expanded in (8) converges uniformly as a function of \( z \). In particular, we may differentiate this sum representing the infinite J-fraction termwise with respect to \( z \).

**Proof of Convergence.** We define the sequences \( a_i, b_i \) as in the following equations and proceed to use Pringsheim’s theorem to prove convergence of \( \text{Conv}_h(q, z) \to J_{\infty}(z) \) when \( |q| < 0.206783 \) [15, §1.12(v)]:

\[
a_h(q) := \frac{z^2q^{2i-3} (1 - q^{i-1})^4}{(1 - q^{2i-1})(1 - q^{2i-2})^2(1 - q^{2i-1})},
\]
\[
b_h(q) := \frac{1 - qz^{i-2} (2q + q^{2i} - q^2 - q^3 + q^{i+2})}{(1 - q^{2i-2})(1 - q^{2i})}.
\]

In particular, if we can show that \(|b_h(q)| \geq |a_h(q)| + 1\) for all sufficiently large \( h \geq h_0 \), then we have that \( \text{Conv}_h(q, z) \to J_{\infty}(q, z) \) as \( h \to \infty \), that \(|\text{Conv}_h(q, z)| < 1\) for all \( h \), and that our infinite J-fraction satisfies \(|J_{\infty}(q, z)| < 1\). Since we will be setting \( z \mapsto q \) in the generating functions for the divisor and generalized sum-of-divisors functions in the results of the next subsection, we may assume that \( z \equiv q \), and in particular that \(|z| = |q|\).

We first expand several inequalities for \(|a_i| + 1\) when \( i \geq h_0 \) is taken to be a sufficiently large positive integer:

\[
|a_i| + 1 = \frac{|q|^{2i-1}|1 - q^{i-1}|^2}{|1 - q^{2i-3}|^2|1 - q^{2(i-1)}|} + 1 \leq \frac{|q|^{2i-1}(1 + |q|^{i-1})^2}{(1 - |q|^{2i-3})(1 - |q|^{2(i-1)})} + 1.
\]
Secondly, we expand a few corresponding inequalities for \(|b_i|\) in the following forms:

\[
|b_i| = \frac{|1 + q_i^{-1} (2q + q^{2i} + q^{i+2}) - q_i^{-1} (q'q^{i+1} + q^2 + q^3)|}{1 + q^{2(i-1)} (1 + q^{2i})} \\
\geq \frac{|1 + q_i^{-1} (2q + q^{2i} + q^{i+2}) - q_i^{-1} (q'q^{i+1} + q^2 + q^3)|}{1 + q^{2(i-1)} (1 + q^{2i})} \\
\geq \frac{C_{q,i}^2 \times |1 + q_i^{-1} (2q + q^{2i} + q^{i+2}) - q_i^{-1} (q'q^{i+1} + q^2 + q^3)|}{1 + q^{2(i-1)} (1 + q^{2i})},
\]

for some \(0 < C_{q,i} \leq \frac{(1 - |q|^{-1})^2}{1 + |q|^{2i-2}} \leq 1\).

Thus if we let \(t := q^{-i}\), it follows that if we can show that there is a positive constant, \(C_{q,i}\), depending on \(q\) and the \(i \geq h_0 \geq 1\) satisfying the above inequality and the condition that

\[
C_{q,i}^2 |1 + q(q^2 + q - 2)t + q(1 - q)t^2 - q^2 t^3| \geq (1 - |t|)^4 + |t|^2(1 + |t|)^2,
\]

then we have proven that \(|b_i| \geq |a_i| + 1\) for all sufficiently large \(i \geq 1\). The condition in (i) combined with the inequality defining \(C_{q,i}\) in the previous equations for \(|b_i|\) and a tentative technical requirement that \(|q^2 + q - 2|, |1 - q| \leq 1\), imply that we must find such a \(C_{q,i}\) satisfying

\[
\sqrt{(1 - |t|)^4 + |t|^2(1 + |t|)^2} \leq \sqrt{\frac{(1 - |t|)^4 + |t|^2(1 + |t|)^2}{1 + |q(q^2 + q - 2)t + q(1 - q)||t|^2 - |q|^2 t^3}} \\
\leq \sqrt{\frac{(1 - |t|)^4 + |t|^2(1 + |t|)^2}{1 + |q(q^2 + q - 2)t + q(1 - q)t^2 - q^2 t^3|}} \\
\leq C_{q,i} \leq \frac{(1 - |t|)^2}{1 + |t|^2} \leq 1,
\]

By numerical computations the requisite inequality

\[
\frac{(1 - |t|)^2}{1 + |t|^2} \geq \sqrt{\frac{(1 - |t|)^4 + |t|^2(1 + |t|)^2}{1 + |t|^2 + |t|^3}},
\]

is satisfied for \(|t|\) in the approximate range \(0 < |t| < 0.206783\). Then we finally see that since \(0 < |q|^{i+1} < |q|^i\) for all \(i \geq 1\), if \(0 < |q| < 0.206783\) we have convergence of our infinite J-fraction.

\[\square\]

**Proof of the Uniform Convergence of** (8). Moreover, we can prove easily that the infinite sum for \(J_\infty(q, z)\) defined in (8) is uniformly convergent as a function of \(z\). For the parameter setting of \((a, b) := (q, q^2)\) in the infinite J-fraction, \(J_\infty(a, b; q, z)\), let the inner terms in this sum be denoted by

\[
\text{\text{Conv}}_{i+1, q, z} := \frac{q(1 - q)(-1)^{i+1} q^{2i}(q; q^2)^{\frac{1}{2}} z^{2i}}{(q; q^2)^2(q^2; q^2)^{\frac{1}{2}} i Q_i(q, z) Q_{i+1}(q, z)},
\]

Then for some positive constant \(A\), a fixed function \(b_q\), and non-zero \(q, z \in \mathbb{C}\) such that \(|c_q z|, |q z| < 1\) where \(|q| < 0.206783\) as above, we can bound these inner sum terms by

\[
|\text{\text{Conv}}_{i, q, z}| \leq A \cdot b_q q^{i} z^{2i} < A,
\]

in which case the Weierstrass M-test implies our result. That is, the convergence of the sum in (8) is uniform in \(z\). \[\square\]
Corollary 3.2 (A Complete Generating Function for the Divisor Function). Let the generating function, \( D_{0,h}(q, z) \) be defined as

\[
D_{0,h}(q, z) := \frac{q(1 + q)}{(1 - q)(1 + q - z)} + \sum_{j=1}^{h-1} \frac{q \cdot q^{j^2}(q; q)_j^4}{(q; q)_j^2(q^2; q^2)_j^2} \times z^{2j} \times \widetilde{D}_{0,j}(q, z)^{-1},
\]

where the function, \( \widetilde{D}_{0,j}(q, z) \), is defined by the \( q \)-coefficient expansions

\[
\widetilde{D}_{0,j}(q, z) := \sum_{n=0}^{2j} \left[ \frac{j + 1}{n} \right]_{q,q,q^2} \left[ \frac{j}{2j - n} \right] z^n
\]

\[+ \sum_{n=0}^{2j+1} \left( \sum_{1 \leq s \leq \left\lfloor \frac{2j+1}{2} \right\rfloor} \sum_{1 \leq s_1 \leq m_j} \sum_{1 \leq k_1 \leq s_1} \left[ \frac{j + 1}{n - k} \right]_{q,q,q^2} \left[ \frac{j}{2j + 1 - n} \right]_{q,q,q^2} \times \right.
\]

\[\left. \times (-1)^{m_1 + m_2} \left[ z^{k_1 - 2m_1} \right] S_{j,m_1,s_1}(z) \cdot \left[ z^{k_2 - 2m_2} \right] S_{j+1,m_2,s_2}(z) \right) z^n \]

Then for all \( h \geq 2 \) and \( 0 \leq n < h \), we have that \( d(n) = [q^n]D_{0,h}(q, q) \) whenever \( 0 < |q| < 0.206783 \) and that the limiting case yields the convergent sum

\[
D_{0,\infty}(q, q) = \sum_{n \geq 1} \frac{q^n}{1 - q^n} = \sum_{m \geq 1} d(m) q^m.
\]

Proof. These two results are only a restatement of Theorem 1.2, whose convergence is guaranteed by Proposition 3.1, in light of the expansions of the convergent denominator functions, \( Q_h(q, z) \), given by Lemma 2.3.

\[\square\]

3.2. Modified generating functions for the integer-order sums-of-divisors functions. This section employs the uniform convergence of the right-hand-side sum in (8) from Theorem 1.2 to formulate new generating functions for the generalized sums-of-divisors functions, \( \sigma_\alpha(n) \) when \( \alpha \geq 1 \) is integer-valued. In particular, we borrow a more general result from the reference [18, §2] which states that given any \( m \in \mathbb{Z}^+ \) and any sequence, \( \langle f_n \rangle \), whose ordinary generating function (OGF), \( F_f(z) \), has higher-order derivatives of orders \( j \) for all \( 0 \leq j \leq m \), we have a transformation of this OGF into the OGF enumerating the modified sequence of \( \langle n^m f_n \rangle \) given by the following finite sum:\(^3\)

\[
\sum_{n \geq 0} n^m f_n z^n = \sum_{j=0}^{m} \left\{ \frac{m}{j} \right\} z^j F_f^{(j)}(z).
\]

Proposition 3.3 (Generating Functions for the Generalized sums-of-divisors Functions). Let the notation for the functions, \( \widetilde{D}_{0,j}(q, z) \), be defined as in the statement of Corollary 3.2 in the previous subsection. Then for each fixed integer \( \alpha \geq 1 \), we have the generating functions for

\[\text{\textsuperscript{3} The Stirling numbers of the second kind, } \{\binom{n}{k}\}, \text{ are also commonly denoted by } S(n, k) \text{ for integers } n, k \geq 0 \text{ such that } 0 \leq k \leq n \text{ [15, §26.8].}\]
the sums-of-divisors functions, \( \sigma_{\alpha}(n) \), expanded in terms of the next finite sums involving the Stirling numbers of the second kind.

\[
D_{\alpha,\infty}(q) = \sum_{m \geq 1} \sigma_{\alpha}(m)q^m = \sum_{n \geq 1} n^\alpha q^n
\]

\[
= \sum_{i=0}^{\alpha} \left\{ \frac{\alpha}{i} \right\} \left( \frac{z^i \cdot q(1 + q)}{(1 - q)(1 + q - z)^{i+1}} + \sum_{j \geq 1} \frac{q \cdot \eta_j^4}{(q; q^2)_j^2 (q^2; q^2)_j^2} \times \left( D^{(i)}( \frac{z^{2j}}{D_{0,j}(q, q)} ) \right) \right|_{z=q}
\]

**Proof.** This result is an immediate consequence of our new transformation result in (15) applied to the statement of Corollary 3.2, where we know by Proposition 3.1 that we may differentiate the sum in (13) termwise with respect to \( z \).

**Corollary 3.4** (Special Cases). For the special cases of \( \alpha := 1, 2 \), we have more explicit expansions of the generating functions for the sums-of-divisors functions, \( \sigma_1(n) \) and \( \sigma_2(n) \), given by

\[
D_{1,\infty}(q) = \sum_{m \geq 1} \sigma_1(m)q^m = \sum_{n \geq 1} nq^n
\]

\[
= \frac{q^2(1 + q)}{1 - q} + \sum_{j \geq 1} \frac{q \cdot \eta_j^4}{(q; q^2)_j^2 (q^2; q^2)_j^2} \times \left( \frac{2jq^2j}{D_{0,j}(q, q)} - \frac{q^{2j+1}\tilde{D}_{0,j}'(q, q)}{D_{0,j}(q, q)^2} \right)
\]

\[
D_{2,\infty}(q) = \sum_{m \geq 1} \sigma_2(m)q^m = \sum_{n \geq 1} n^2q^n
\]

\[
= \frac{q^2(1 + q)(1 + 2q)}{1 - q} + \sum_{j \geq 1} \frac{q \cdot \eta_j^4}{(q; q^2)_j^2 (q^2; q^2)_j^2} \times \left( \frac{4jq^2j}{D_{0,j}(q, q)} - \frac{(4j + 1)q^{2j+1}\tilde{D}_{0,j}'(q, q)}{D_{0,j}(q, q)^2} \right)
\]

\[
- \frac{q^{2j+1} \left( \tilde{D}_{0,j}(q, q)\tilde{D}_{0,j}'(q, q) - 2\tilde{D}_{0,j}'(q, q)^2 \right)}{D_{0,j}(q, q)^3},
\]

where the derivatives of \( \tilde{D}_{0,j}(q, q) \) are partial derivatives taken with respect to the second parameter \( z \) whose forms are easily expanded by differentiating the polynomial functions of \( z \) in (14).

**Proof.** These results follow from Proposition 3.3 combined with two computations with the quotient rule providing that for any non-zero function, \( G(z) \), and integers \( j \geq 1 \), the first and second derivatives of the quotient, \( z^{2j}/G(z) \), are expanded by

\[
z \cdot D \left[ \frac{z^{2j}}{G(z)} \right] = \frac{2jz^{2j+1}/G(z)}{G(z)} - \frac{z^{2j}G''(z)}{G(z)^2}
\]

\[
z^2 \cdot D^2 \left[ \frac{z^{2j}}{G(z)} \right] = \frac{2j(2j - 1)z^{2j} - 4jz^{2j+1}G'(z)}{G(z)^2} - \frac{z^{2j+2}(G(z)G''(z) - 2G'(z)^2)}{G(z)^3}.
\]

4. **Significance of the new results, applications, and relations to recent research**

4.1. **Constructions of generating functions enumerating other special \( q \)-series.** Our method of proof in Section 3 hints at, and in the most general cases applies to, the \( J \)-fraction expansions of the most general forms of (1) for arbitrary sequences, \( \langle a_i \rangle \) and \( \langle c_i \rangle \). In fact, when our so-called notions of the generalized Stirling \( q \)-coefficients in (14) of Corollary 3.2 are replaced with the coefficients, \( \left[ \begin{array}{c} h \\ k \end{array} \right] \cdot c \), defined as in (9) of Definition 2.1, then for any \( \text{convergent} \) \( J \)-fraction expansion of (1) we have new sums for the \( h^{th} \) convergents to \( J_{\infty}(z) \) expanded more generally by the corresponding formula in (6) of Theorem 1.2. Thus new infinite sums for
other known J-fraction expansions, such as those considered in the references and in Table 4.1 below, are formulated by an appeal to the generalized results for these continued fractions we have established within this article.

| \( [z^n] J_{\infty}(z) \) | \( c_1 \) | \( c_h \) for \( h \geq 2 \) | \( ab_h \) |
|-----------------|-----------|----------------|----------------|
| \( (a; q)_n \)  | 1 \(- a \) | \( q^{h-1} - aq^{h-2}(q^h + q^{h-1} - 1) \) | \( aq^{h-1}(aq^{h-2} - 1)(q^{h-1} - 1) \) |
| \( \frac{1}{(q; q)_n} \) | 1 \(- q \) | \( q^{h-1}(q^{h-1}[h-1] + q^{h-1} - [h-1]q) \) | \( \frac{1}{(q^{h-1} - 1)(q^{h-1} - 1)q} \) |
| \( (zq^{-n}; q)_n \) | \( q^{-z} \) | \( q^{h-1}(q^{h-1}[z-q^{h-1}] + q^{h-1} - q^{h-1}z) \) | \( q^{h-1}(q^{h-1}[z-q^{h-1}] + q^{h-1} - q^{h-1}z) \) |
| \( \frac{1}{(aq; q)_n} \) | 1 \(- a \) | \( q^{-z}(q + abq^{h-3} + a(1-q^{h-1}q + b(-1+q^h)) \) | \( (1-bq^{h-1})q \) |

Table 4.1. J-fraction parameters generating the terms in special q-series expansions

Examples of other notable q-series expansions derived from the generalized Jacobi-type J-fractions defined in Section 1.1 are summarized in Table 4.1 on page 16 [17, cf. §4]. We can use these continued fraction expansions combined with the definition of the more general Stirling q-coefficients in (9) and Lemma 2.3 to generate new q-series expansions for q-exponential functions, q-trigonometric functions, infinite q-Pochhammer symbol products and their reciprocals, and related series such as for the Rogers-Ramanujan continued fractions [3] [15, §17.3]. Generalized forms of the Stirling numbers of the first kind in the context of the product-based definitions we used to define our notions of the Stirling q-coefficients in the previous section are considered in the references [11, 12]. Other typical q-analogs to the Stirling numbers, or q-Stirling numbers, are defined in [5, 16] and in the particular form of [15, §17.3]

\[ a_{m,s}(q) = \frac{(1-q)^s}{q(2)(q; q)_s} \times \sum_{j=0}^{s} \left[ \begin{array}{c} s \\ j \end{array} \right] _q \frac{(-1)^j q^j(j)}{(1-q)^m} \]

Charalambides defines a q-analog to the Stirling numbers of the first kind by the products

\[ \langle [t]_q - [r]_q \rangle (1 + [t]_q) \cdots (1 + [t]_q - [r + n - 1]_q) = \sum_{k=0}^{n} s_q(n, k; r)[t]_q^k \]

where \( [r]_q := (1 - q^r)/(1 - q) \) denotes a q-real number, \( s_q(n, k; r) \) corresponds to a modified form of (9) with \( c_h := [n + r - 1]_q \), and where we have an explicit finite sum expanded in terms of the q-binomial coefficients by the formula

\[ s_q(n, k; r) = \frac{1}{(1-q)^{n-k}} \sum_{j=k}^{n} \left[ \begin{array}{c} n \\ j \end{array} \right] _q \frac{(-1)^j q^{j(n-j)+r(n-j)}}{a} \]

Additionally, we can extend the generating function results for the J-fraction expansions defined by Definition 1.1 to obtain new identities and generating function expansions of series involving ratios of two q-Pochhammer symbols. For example, we have the following q-series identities which are either direct applications of the new q-Pochhammer ratio series or that form special cases of our new results corresponding to other special q-series expansions where \( (a; q)_n - a = \prod_{k=1}^{n}(1 - a/q^{k-1}) = (a/q; 1/q)_n^{-1} \) [3]:

\[ \sum_{n=-\infty}^{\infty} \frac{(a; q)_n}{(b; q)_n} z^n = \frac{(az)_\infty(q/(az); q)_\infty(q; q)_\infty(b/a; q)_\infty}{(z; q)_\infty(b/(az); q)_\infty(b/a; q)_\infty(q/a; q)_\infty}; \quad | \frac{b}{a} | < |z| < 1, |q| < 1 \]
\[
\sum_{n \geq 0} \frac{(a; q)_n z^n}{(q; q)_n} = \frac{(az; q)_\infty}{(z; q)_\infty}, \quad |z|, |q| < 1
\]

\[
= \sum_{n \geq 0} \frac{(a; q)_n (az; q)_n (qz)_n q^{n(n-1)} (1 - azq^{2n})}{(q; q)_n (z; q)_{n+1}}
\]

\[
\sum_{n \geq 0} q^n z^n = \sum_{n \geq 0} \frac{(z; q)_n}{(zq; q^n)_n} z^n.
\]

Special cases of the last identity and the entries for the J-fraction expansions given in Table 4.1 similarly allow us to generate the infinite q-Pochhammer symbol product, \((z; q)_\infty\), and its reciprocal, which provides immediate applications to generating functions and rational approximate generating functions for partition functions.

### 4.2. Relations to recent research and some open problems

Recent work on the divisor function and the sum-of-divisors function using results obtained from Lambert series identities and the Lambert series generating functions for the generalized sum-of-divisors functions defined by

\[
L_\alpha(q) := \sum_{n \geq 1} \frac{n^\alpha q^n}{1 - q^n} = \sum_{m \geq 1} \sigma_\alpha(m) q^m,
\]

are considered in the references [14, 1, 13, 9]. The applications of these results are related to divisor sum convolutions, combinatorial interpretations of the explicit values of the divisor function, \(d(n) \equiv \sigma_0(n)\), and of course finding new generating functions for the two classical divisor functions, \(d(n)\) and \(\sigma(n)\), and the generalized sum-of-divisors functions, \(\sigma_\alpha(n)\). Our new results proved within the article provide new q-series expansions of the generating functions for the divisor functions and the generalized sum-of-divisors functions whose \(h\)-order accurate \(h\)-th convergents are rational for each finite \(h \geq 2\) (cf. equation (16) on page 18 below). Recent results on continued fraction expansions similar to the J-fraction expansions defined by Definition 1.1 and for other special q-series expansions are considered in [4, 19].

If we set \(q \mapsto 1^-\) in our expansions for the q-Pochhammer ratios from Definition 1.1, we obtain a special case of the generalized hypergeometric function, \(_1F_1(a; b; z)\). Several new results proved in [2] similarly provide asymptotic formulas for the sum-of-divisors generating functions near \(q = 1\). In particular, we may define the Lambert series, \(L_q(s, x)\), where \(L_q(s, 1)\) corresponds to the ordinary generating function for \(\sigma(n)\), as

\[
L_q(s, x) := \sum_{n \geq 1} \frac{n^s q^{nx}}{1 - q^n} = \frac{\Gamma(s + 1) \zeta(s + 1, x)}{(\log \frac{1}{q})^{s+1}} - \sum_{n \geq 0} \frac{\zeta(1 - s - n) B_n(x) (\log q)^{n-1}}{n!},
\]

where \(\zeta(s, a)\) denotes the Hurwitz zeta function and \(B_n(x)\) is a Bernoulli polynomial [15, cf. §24.2, §25.11]. For comparison, we may use the transformation identity in (15) and an expansion of the exponential generating function for the Bernoulli numbers in powers of \(n\) to obtain that

\[
\sum_{n \geq 1} \frac{n^\alpha q^n}{1 - q^n} = - (\log q) \times \sum_{n \geq 1} n^{\alpha-1} q^n - \sum_{k \geq 0} \frac{(-1)^k B_k (\log q)^{k-1}}{k!} \times \sum_{j=0}^{k+\alpha} \binom{k+\alpha}{j} \frac{q^j \cdot j!}{(1 - q)^{j+1}}.
\]

Relations of our new results to other open problems in number theory include the famous unresolved questions of whether there are infinitely-many perfect numbers, \(n\) for which \(\sigma(n) = 2n\), and whether there are any odd perfect numbers. It is beyond the scope of this article to give due attention to these famous unresolved problems, but we note that topics related to perfect numbers and their generalizations are an active research area for many researchers and mathematicians.
4.3. Generating functions and asymptotics. This section suggests several approaches to problems related to the generalized sum-of-divisors functions following as applications from our specific technique of an approach to these special function generating functions through J-fractions. First, we note that we may apply any number of known methods for obtaining asymptotic estimates of the coefficients in a formal power series to find new forms of asymptotic formulas for the generalized sum-of-divisors functions and the partial sums, \( \Sigma_n := \sum_{n \leq x} \sigma_n \), by considering the generating functions, \( \text{Conv}(q, z) \) and \( J_\infty(q, z) \), multiplied by a factor of \( 1/(1 - q) \). This consequence of using a generating-function-based approach to the generalized divisor functions is particularly relevant since we have \( h \)-order accurate power series approximations to the generating functions of these special divisor functions which are rational in \( q \) (and in \( z \)) for each finite case of \( h \geq 2 \). We compare the forms of these rational approximate generating functions to the known examples cited in Section 1.3.

4.4. Rational generating functions and new congruence results. One other consequence of utilizing the technique of using J-fraction expansions to generate the terms in our special Lambert series and \( q \)-series forms provides congruences for these functions modulo \( q \). More precisely, when \( h \geq 3 \) for any divisor, \( \hat{d}(q) \) of the \( \text{hth} \) modulus, \( M_h(a; b; q) := \lambda_1(a; b; q) \cdots \lambda_h(a; b; q) \), defined by (1.1) in Definition 1.1, we have that for \( n < 2h \) [8, §2]

\[
J_\infty(a; b; q; z) \equiv \text{Conv}_h(a; b; q; z) \pmod{M_h(a; b; q)}
\]

\[
\frac{(a; q)_n}{(b; q)_n} \equiv [q^n] \text{Conv}_h(a; b; q; z) \pmod{q^{2h}},
\]

and similarly for the special case series when \( (a, b) := (q, q^2) \) and \( n, x < 2h \) we have that

\[
d(n) \equiv [q^n] \frac{\text{Conv}_h(q, q^2; q; q)}{1 - q} \pmod{q^{2h}}
\]

\[
\Sigma_q(x) = [q^n] \frac{\text{Conv}_h(q, q^2; q; q)}{(1 - q)^2} \pmod{q^{2h}}.
\]

By considering the approximate generating functions, \( \text{Conv}_h(q, q^2; q; q)/(1 - q) \), modulo \( q^{2h} \) we are able to obtain simple rational functions in \( q \) which are \( 2h \)-order accurate in generating the divisor function:

\[
1 + 4q + 8q^2 + 11q^3 + 10q^4 \equiv 1 + \sum_{n=1}^{4} d(n)q^n + O(q^5)
\]

\[
-1 - 5q - 14q^2 - 29q^3 - 46q^4 - 62q^5 - 71q^6 \equiv 1 + \sum_{n=1}^{6} d(n)q^n + O(q^7)
\]

\[
\frac{1+6q+20q^2+50q^3+101q^4+175q^5+267q^6+369q^7+472q^8}{1+4q+10q^2+29q^3+57q^4+40q^5+38q^6+32q^7} \equiv 1 + \sum_{n=1}^{7} d(n)q^n + O(q^8).
\]

We can similarly form the \( 2h \)-order accurate rational approximate generating functions for the sum-of-divisors function, \( \sigma(n) \), by differentiating \( qz \cdot \text{Conv}_h(q, qz)/(1 - q) \) with respect to \( z \) which then implies the following analogous results:

\[
\frac{q (1 + 3q + 3q^2)}{(1 - q)(1 + q)} = \sum_{n=1}^{3} \sigma(n)q^n + O(q^4)
\]

\[
\frac{q (1 + 7q + 25q^2 + 62q^3 + 115q^4)}{(1 + q + 3q^2)(1 + 3q + 3q^2)} = \sum_{n=1}^{5} \sigma(n)q^n + O(q^6)
\]
\[
q \left( 1 + 9q + 44q^2 + 155q^3 + 430q^4 + 998q^5 + 2000q^6 \right) \\
1 + 6q + 22q^2 + 58q^3 + 120q^4 + 204q^5 + 290q^6 + 350q^7 = \sum_{n=1}^{7} \sigma(n)q^n + O(q^8)
\]

\[
q \left( 1 + 11q + 65q^2 + 276q^3 + 935q^4 + 2676q^5 + 6696q^6 + 14998q^7 + 30592q^8 \right) \\
1 + 8q + 34q^2 + 126q^3 + 347q^4 + 812q^5 + 1664q^6 + 30592q^7 + 56928q^8 + 7776q^9 = \sum_{n=1}^{9} \sigma(n)q^n + O(q^{10}).
\]

We can also reduce the coefficients of the numerator and denominator polynomials in \( q \) in these rational generating function approximations modulo any integer \( p \geq 2 \) to form congruences for the sums-of-divisors functions modulo \( p \). For example, when \( h := 4, 5 \) and \( p := 5 \), and when \( 1 \leq n < 2h \), we have that

\[
\frac{q + 4q^2 + 4q^3 + 3q^6}{1 + q + 2q^2 + 3q^3 + 4q^5} \equiv \sum_{n=1}^{7} [\sigma(n) \pmod{5}] q^n + O(q^8)
\]

\[
\frac{q + q^2 + q^4 + q^6 + q^7 + 3q^8 + 2q^9}{1 + 3q + 2q^2 + q^3 + 2q^4 + 2q^5 + 4q^6 + 4q^7 + 4q^8 + q^9} \equiv \sum_{n=1}^{9} [\sigma(n) \pmod{5}] q^n + O(q^{10})
\]

The identities in the previous several equations can be generalized to enumerate \( \sigma_{\alpha}(n) \) and \( \Sigma_{\alpha}(x) \) for \( \alpha \in \mathbb{Z}^+ \) by first differentiating with respect to \( z \) and then setting \( z \mapsto q \) as in the generating function constructions from Section 3.2.

5. Conclusions

5.1. Summary. We have defined the forms of infinite J-fractions in the form of (1) whose power series expansions in \( z \) generate the ratio of \( q \)-Pochhammer symbols, \( (a; q)_n/(b; q)_n \), for all \( n \geq 0 \). We focused on the special case of these expansions where \( (a, b) := (q, q^2) \), and subsequently proved the forms of new convergent infinite \( q \)-series involving the generalized Stirling \( q \)-coefficients we defined in (9) that enumerate the divisor function, \( d(n) \), and the generalized sums-of-divisors functions, \( \sigma_{\alpha}(n) \) for \( \alpha \in \mathbb{Z}^+ \). We cite comparisons of these results generating the divisor function and the positive integer-order sums-of-divisors functions according to the expansions of (14) with the known generating function expansions related to these special functions cited in Section 1.3 of the introduction. Further applications of the results we have proved in this article are derived from the subtlety that most of the expansions given for the convergent denominator functions, \( Q_h(z) \), involved in the infinite sums for these generalized J-fractions are stated and proved in the general case of the implicit sequences in (1).

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