Verifying Genuine High-order Entanglement

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High-order entanglement embedded in multipartite multilevel quantum systems (qudits) with many degrees of freedom (DOFs) plays an important role in quantum foundation and quantum engineering. Verifying high-order entanglement without the restriction of system complexity is a critical need in any experiments on general entanglement. Here, we introduce a scheme to efficiently detect genuine high-order entanglement, such as states close to genuine qudit Bell, Greenberger-Horne-Zeilinger, and cluster states as well as multilevel multi-DOF hyperentanglement. All of them can be identified with two local measurement settings per DOF regardless of the qudit or DOF number. The proposed verifications together with further utilities such as fidelity estimation could pave the way for experiments by reducing dramatically the measurement overhead.

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Introduction.—Entanglement is a remarkable property of multipartiy quantum systems. So far, a lot of effort has been given to investigate two-level entanglement. However, general quantum states could be multilevel, meaning that their entanglement can be more complicated but possess a high potential for quantum applications.

High-order entanglement is distinct from two-level entanglement. It offers a new insight into quantum mechanics by violating generic classical constraints \cite{1,2}. When qudits are entangled in more than one degree of freedom (DOF) \cite{4,5}, hyperentanglement (HE) is less affected by coherence than single-DOF multipartite entanglement \cite{6}. As resources for quantum technology, novel quantum strategies have shed light on high-performance and superior applications in quantum information \cite{6–14} and quantum metrology \cite{15}. In addition, more and more high-order entangled systems are being created and manipulated coherently in various implementations, e.g., high-order entangled photons \cite{4,9,11,14,15}. Efforts have also been devoted to create reliable superconducting qudits \cite{16} and scalable Greenberger-Horne-Zeilinger (GHZ) states \cite{20}.

Identifying high-order entanglement is crucial to both entanglement theories and experiments \cite{5,12,13,15,20}. There has been important progress in detecting two-level entanglement \cite{21–24}. However, how to identify genuine multiqubit entanglement and multilevel HE remains a crucial challenge. Genuine multipartite entanglement can be generated only when all the qudits strictly participate in the correlation creation \cite{22}. On the other hand, entanglement is truly multilevel if all the levels of qudit are involved in the quantum correlation \cite{22}. Although violations of classical constraint could reveal entanglement \cite{1,2}, it can not detect the genuineness of multilevel and multipartite nature. To detect genuine multipartite entanglement, a general criterion has been proposed recently by measuring identical two copies of multilevel systems \cite{25} with careful experimental design \cite{27}. While the criteria \cite{26} assisted by measuring quantum state fidelity could provide information about truly multilevel entanglement, the required experimental resources in terms of local measurement setting (LMS) increases exponentially with the qudit number in each DOF \cite{28}. There still lacks a unified and efficient method for verifying genuine multipartite multilevel entanglement and truly multilevel HE.

In this Letter, we will propose first a novel multilevel-dependent criterion for high-order entanglement identification. Afterwards, with the criterion, we will show that states close to genuine multilevel multipartite graph states, which constitute a large and highly significant class of multipartite entangled states in physics \cite{29}, can be efficiently identified regardless of the number of particles. In addition, we will provide a solution in all the related experiments necessary for detecting genuine multilevel HE \cite{25}. The first fidelity estimation of high-order entanglement without complete fidelity measurements will also be given \cite{30}. The entanglement verification and fidelity estimation can be readily applied to the present experiments \cite{5,3,11,17,20}. More instances for showing the power of the criterion such as verifying multiqudit supersinglets \cite{6} and constructing generic multilevel multipartite Bell inequalities will also be presented \cite{30}.

Our strategy for investigating correlations between qudits highlights the statistical dependence between outcomes of measurements performed on $d$-level $N$-partite systems. We assume that two possible measurements per DOF can be performed on each spatially separated particle and that each local measurement has $d$ possible outcomes: $v \in v = \{0, 1, \ldots, d−1\}$. As illustrated in Fig. 1, all the $N$ particles are assumed to be locally measured in parallel with a LMS. After sufficient runs of such
measures have been made, a joint probability distribution for any two subsystems could be derived from the experimental outcomes. If one of the subsystems, A, is composed of a particles and the other subsystem, B, is composed of b particles, then there are \(d^a\) for A and \(d^b\) for B possible measurement outcomes, \(v_{a,m}\) and \(v_{b,n}\), for \(m = 0, 1, ..., d^a - 1\) and \(n = 0, 1, ..., d^b - 1\).

**Multilevel-dependent criterion.** Independence identification is an important method for examining correlation between outcomes \([31]\). Two events \(v_{a,m}\) and \(v_{b,n}\) are statistically independent if and only if the joint probability \(p(v_{a,m}, v_{b,n})\) is a product of two individual marginal probabilities: \(p(v_{a,m})p(v_{b,n})\). In order to extend statistical independence to a multilevel-dependent criterion for entanglement verification, we propose complex polynomials of the form:

\[
R_n := \sum_{m=0}^{d^a-1} c_{mn} \exp(i\phi_{mn}) p(v_{a,m}, v_{b,n}),
\]

for \(n = 0, 1, ..., d^b - 1\). The phases \(\phi_{mn}\)'s are designed to satisfy the constraint: \(\sum_{n=0}^{d^b-1} c_{mn} \exp(i\phi_{mn}) = 0\), where \(c_{mn} \in \{0, 1\}\). The multilevel-dependent criterion is then described by the statement: If measurement results show that every \(R_n\) posses the same phase, then the outcomes of measurements for the two subsystems are statistically dependent.

The proof of the statement is straightforward. If the two subsystems A and B are independent with respect to the events of measurement, according to the definition of statistical independence we have \(R_n = S_n p(v_{b,n})\), where \(S_n = \sum_{m=0}^{d^a-1} c_{mn} \exp(i\phi_{mn}) p(v_{a,m})\). Since \(\sum_{n} S_n = 0\), every \(R_n\) should not have the same complex argument, whereas a contradiction reveals the dependence between the measurement results.

The property that all \(R_n\) are not parallel is not only a characteristic feature of two independent subsystems, but also a basis for a quantitative analysis of statistical dependence. Let us consider the quantity \(\left| \sum_n R_n \right|\) for independent systems. Since the variety of different phases, it is clear that if \(p(v_{a,m}) < 1\) and \(p(v_{b,n}) < 1\) for all events, then \(\left| \sum_n R_n \right| = \left| \sum_n S_n p(v_{b,n}) \right| < 1\), while \(\left| \sum_n S_n p(v_{b,n}) \right| = 1\) exists only when \(p(v_{a,m}) = p(v_{b,n}) = 1\) for some \(m^*\) and \(n^*\), which can be strictly identified by extracting the outcomes \(v_{a,m^*}\) and \(v_{b,n^*}\) with certainty. This quantitative discrimination between statistical dependence and independence is of the essence to signify truly multipartite multilevel entanglement.

**Detecting genuine many-qudit graph states.** In order to examine whether an experimental output state, \(\rho\), is a genuine high-order entangled state close to a graph state \(|G\rangle\) [see Fig. 2 (a)], the first step in our strategy is to specify the correlations between qudits. We feature the most typical correlation quality of \(|G\rangle\) by a sum of all related local operators for multilevel-dependent criterion. The correlation between the \(k\)th qudit and all qudits of its neighborhood is described by

\[
\hat{a}_k := \frac{t_d}{d-1} \left[ \frac{d}{\sum_{i=0}^{N(k)} |v_{i,k}(v)|^2} \sum_{i=0}^{N(k)} |v_{i,k}(v)| F_k \otimes |u_{i,k}(v)| - \hat{1} \right],
\]

where \(d\) denotes equality modulo \(d\), \(N(k)\) is the set of all \(i\)'s satisfying \((k, i) \in E\), and \(\hat{1}\) denotes the identity operator. As has been shown \([32]\), \(\hat{a}_k\) is constructed according to the multilevel-dependent criterion, and from which one can see that \(t_d\) is the value associated with...
tanglement with the nondegenerate eigenvectors ρ and ρ. It is difficult to realize it experimentally. Decomposing the higher than the proposed criterion [30], however, it is necessary in this method. For decomposition of |G⟩⟨G|, 2(dN − 1) settings [28] are heavily needed for further detection tasks. One can alternatively make a quantum state tomography for the multiqubit states and then invoke the projector-based criterion. Nevertheless, huge d2N − 1 LMSs [28] are necessary for the tomographic analysis. For the criterion by Huber et al. [27] to verify genuine multipartite entanglement, one needs identical two copies of states for each round of measurement and requires 2N(N − 1) LMSs for the most efficient detection in the illustrated criteria. Compared with these approaches, our scheme is rather efficient: two LMSs are sufficient for verifying genuine Bell, GHZ, cluster and any two-colorable graph states [Figs. 1(a) and 1(b)]. Furthermore, our criterion can be used to estimate the quality of the prepared state without full fidelity measurements [30].

Detecting genuine multilevel HE.—We continue to illustrate the scheme with verification of existing multilevel N-DOF HE [3]:

\[ |H⟩ = \bigotimes_{k=1}^{N} \frac{1}{d_k} \sum_{v_{k}=0}^{d_k-1} \sum_{v_{k}′=0}^{d_k-1} \omega^{v_{k}v_{k}′} |v⟩_A_k \otimes |v′⟩_B_k, \]

where A_k (B_k) denotes the kth DOF of the subsystem A (B) with d_k levels. The entangled state in each DOF is a two-qubit graph state. To identify genuine HE, it is crucial to recognize the difference between the state |H⟩ and a state with biseparable structure in the hyperentangled sense [24]:

\[ |h⟩ = |h_1⟩_{A_k,b_1} \otimes |h_2⟩_{B_k,b_2}. \]
where \( |h_1\rangle_A b_1 = \sum_{i=0}^{d_1-1} c_{ib_1} |i\rangle_A \otimes |u_i\rangle_{b_1} \) and \( |h_2\rangle_B b_2 = \sum_{i=0}^{d_2-1} c_{ib_2} |i\rangle_B \otimes |u_{i}\rangle_{b_2} \). \( \{A_k, b_1\} \) and \( \{B_k, b_2\} \) constitute the set of all DOFs, where the sets \( b_1 \) and \( b_2 \) are disjoint. The following criterion is introduced to distinguish genuine HE from correlations mimicked by biseparable states: \( \frac{N}{3} \sum_{k=1}^{N} \frac{\hat{a}_1^{(k)} + \hat{a}_2^{(k)} + \frac{d_k}{d}}{d_k} \geq 1 - \frac{D(d-1)}{3d(D-1)}, \) where \( \hat{a}_1^{(k)}, \hat{a}_2^{(k)} \) are of the form (2) for the \( k \)th DOF, \( d = \min\{d_k\} \), and \( D = \max\{d_k\} \). Refer to Appendix for the proof. This criterion can be efficiently implemented because each DOF needs only two LMSs. The measurement results also can be used to examine multilevel genuineness in each DOF by (3). A state mixing with white noise \( \rho_{\text{noise}} \) is detected as state with genuine HE if \( \rho_{\text{noise}} < \frac{1}{3(1-\frac{D}{d})} \). E.g., for \( D = d, \) the noise tolerance is at least 33% for any number of DOFs.

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Summary and outlook.—We have introduced a general criterion of multilevel dependence to demonstrate an efficient verification of high-order entanglement regardless of the number of particles and DOFs. The importance of our scheme are three fold. First, the detection scenario is applicable to any Bell-type experiment [2, 3] and the proposed criterion and the projector-based criterion, respectively, we prove that our criterion can be used to identify genuine HE.

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