Inherent color symmetry in quantum Yang-Mills theory

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We present the basic non-perturbative structure of the space of classical dynamical solutions and corresponding one particle quantum states in SU(3) Yang-Mills theory. It has been demonstrated that the Weyl group of su(3) algebra plays an important role in constructing non-perturbative solutions and leads to profound changes in the structure of the classical and quantum Yang-Mills theory. We show that the Weyl group as a non-trivial color subgroup of SU(3) admits singlet irreducible representations on a space of classical dynamical solutions which lead to strict concepts of one particle quantum states for gluons and quarks. The Yang-Mills theory is a non-perturbative theory and, in general, it is not possible to construct a Hilbert space of classical solutions and quantum states as a linear vector space, so, usually, a perturbative approach is applied. We propose a non-perturbative approach based on Weyl symmetric solutions to full non-linear equations of motion and construct a full space of dynamical solutions representing an infinite but countable solution space classified by a finite set of integer numbers. It has been proved that the Weyl singlet structure of classical solutions provides the existence of a stable non-degenerate vacuum which serves as a main precondition of the color confinement phenomenon. Some physical implications in quantum chromodynamics are considered.

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I. INTRODUCTION

Construction of a strict non-perturbative quantum theory of strong interaction on a basis of SU(3) quantum Yang-Mills theory represents one of the most fundamental problems in theoretical physics [1]. A primary task in constructing a quantum field theory is to determine a space of classical dynamical solutions defining the Hilbert space of one particle quantum states. In quantum chromodynamics (QCD) due to the presence of the confinement phenomenon the single color gluons and quarks are not observed, so such states should be excluded from the spectrum of physical quantum states. It is observed [2, 3] that origin of color confinement is conditioned by existence of a non-degenerate vacuum, which must be color invariant [4] and described by Abelian colorless gluons in the confinement phase [5]. With this one encounters a problem with construction of singlet classical solutions which could provide a color invariant and non-degenerate vacuum, and consistent definition of one particle quantum states. The problem is related with a simple mathematical fact, that color group SU(3) and its continuous subgroups do not admit non-trivial singlet irreducible representations. Selection of a vacuum belonging to the reducible representation leads to degenerate vacuum and spontaneous color symmetry breaking which is incompatible with the color confinement. On the other hand, after fixing the gauge color symmetry one needs some kind of residual color symmetry to find solutions with inherent symmetry which would provide non-degenerate color invariant vacuum and color attributes of fundamental particles, gluons and quarks. It is surprising, that the problem of searching solutions with inherent color symmetries in non-Abelian SU(3) Yang-Mills theory can be resolved due to the presence of the Weyl symmetry group of SU(3) which is the only color symmetry which survives after gauge fixing. The most important feature of the Weyl group is that it has singlet irreducible representations which allow to construct a deepest stable non-degenerate vacuum of QCD and describe a full space of singlet dynamical solutions leading to one particle quantum states.

In this Letter we present the basic construction of a space of Weyl symmetric classical stationary solutions realizing singlet irreducible representations of the Weyl group. Due to non-linear structure of Yang-Mills equations the space of solutions can not be supplied with a linear vector space structure as one has in Abelian theories like the electrodynamics. This leads to the absence of linear superposition rule in the theory, as a consequence, the space of quantum states does not represents a Hilbert vector space and it is not possible to define a complete set of basis solutions. Nevertheless, we demonstrate that all configuration space of regular finite energy stationary solutions is infinite and countable. This allows to construct a full space of discrete one particle quantum states classified by a finite set of integer quantum numbers. Note that direct canonical quantization of stationary solutions leads to quantum states which do not represent directly physical observable quantities. Due to generation of a non-trivial vacuum and non-zero vacuum gluon and quark condensates one has to take into account the vacuum polarization effect which leads to effective interaction of the singlet quantum states with vacuum condensates. We show that such interaction implies formation of localized bound states representing physical
observables in QCD, glueballs and mesons.

II. ANSATZ FOR SINGLET WEYL SYMMETRIC SOLUTIONS

Recently an ansatz for classical stationary solutions symmetric under Weyl group transformations has been proposed. It has been proved that Weyl symmetric solutions are stable against quantum gluon fluctuations, and the Abelian type solution provides a stable non-degenerate QCD vacuum [3] resolving a long-standing problem of vacuum instability in QCD [7,8]. In this section we prove that solutions defined by the Weyl symmetric ansatz possess a remarkable property: they describe only singlet irreducible representations of the Weyl group (classification of Weyl group representations is briefly described in the supplemental material [9]).

We start with a standard Lagrangian for $SU(3)$ Yang-Mills theory ($\mu, \nu = 0, 1, 2, 3; a = 1, 2, ..., 8$)

$$L_{YM} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}. \quad (1)$$

Our primary goal is to construct a space of regular stationary singlet Weyl symmetric solutions leading to one particle quantum states after standard quantization. After minor changing notations in the ansatz proposed in [6] we define first an extended Weyl symmetric ansatz by setting non-vanishing components of the gauge potential $A^a_\mu$ corresponding to $I, U, V$ type subgroups $SU(2)$ [10] as follows

$$I: A^I_1 = K_0, \quad A^I_2 = K_1, \quad A^I_3 = K_2, \quad A^I_4 = K_4, U: A^U_1 = -Q_0, \quad A^U_2 = -Q_1, \quad A^U_3 = -Q_2, \quad A^U_4 = Q_4, V: A^V_1 = S_0, \quad A^V_2 = S_1, \quad A^V_3 = S_2, \quad A^V_4 = S_4, A^\varphi_1 = A^\varphi_2 r^p, \quad A^\varphi_3 = K_3, \quad A^\varphi_4 = K_8, \quad (2)$$

where $r^p_\alpha$ ($\alpha = 3, 8$) are root vectors $r^1 = (1, 0)$, $r^2 = (-1/2, \sqrt{3}/2)$, $r^3 = (-1/2, -\sqrt{3}/2)$, (index $p = 1, 2, 3$ denotes $I, U, V$ sectors). To fix residual local color symmetry $U(1)$ in $I, U, V$ sectors of the Yang-Mills Lagrangian, we add Lorentz gauge fixing terms $\mathcal{L}_{gf}$ to the original Yang-Mills Lagrangian $\mathcal{L}_{YM}$

$$\mathcal{L}_{gf} = -\frac{1}{2} \sum_{\alpha=2,5,7} (\partial_\mu A^\alpha_\mu - \partial_\mu A^\alpha_\mu - \frac{1}{2} r^2 \partial_\mu A^\alpha_\mu)^2. \quad (3)$$

The gauge fixing procedure is a necessary step during quantization which removes all unphysical pure gauge field degrees of freedom. The gauge fixing Lagrangian fixes not only the local gauge symmetry, but also the global color symmetry $SU(3)$. So the ansatz is symmetric only under the Weyl group transformations, providing only Weyl representations for solutions.

It is suitable to choose representation of the Weyl group as a symmetric group $S_3$ acting on fields $A^a_\mu$ in $I, U, V$ sectors by permutations. So, the original octet $A^a_\mu$ realizes the eight dimensional vector reducible representation $\Gamma_8$ of the Weyl group. One can define an ansatz describing only singlet Weyl representations by imposing additional constraints providing consistency of the full ansatz with all Yang-Mills equations of motion ($i = 0, 1, 2$)

$$Q_i = S_i = K_i, \quad K_4 + Q_4 + S_4 = 0,$$

$$Q_4 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right) K_4, \quad S_4 = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right) K_4, K_3 = -\frac{\sqrt{3}}{2} K_4, \quad K_3 = K_8, \quad (4)$$

The ansatz [2,11] extracts three singlet representations $\{\Gamma_1\}_3$ of $S_3$ for fields $\{K_i, Q_i, S_i\}$ in color subspace spanned by generators $\{T^2, S_3\}$. Constraints (4) imply that representation acting on $(K_4, Q_4, S_4)$ is isomorphic to $S_3$ representation defined on $I, U, V$ fields $A^a_\mu$. Let us consider eigenvalues of a Lie algebra valued Abelian field, $A^\varphi = (A^3_\mu T^3, A_8^\mu T^8)$, acting in adjoint representation in the Cartan basis. Due to the constraint $K_3 = K_8$, (4), one can find

$$[A^p_\mu, T^p_r] = K_3 r^p T^p_r, \quad (5)$$

where the eigenvalues $K_3 r^p$ by module $K_3$ define color charges of $I, U, V$-components of the Abelian field. A total color charge of the Abelian field $A^p_\mu$ is zero, and eigenvalues $K_3 r^p$ match the symmetric system of three root vectors $r^p$ with a common field factor $K_3$. The Weyl group is defined as a symmetry group of root system, so that $K_3 r^p$ realize a singlet symmetry representation $\Gamma^1_3$ [9], and $K_3$ (or $K_4$ equivalently), representing a fixed point in the configuration space of fields under Weyl transformations, is a Weyl invariant Abelian field.

Applying ansatz [2,11] to $SU(3)$ Yang-Mills Lagrangian with gauge fixing terms, (3), one can write the total Lagrangian in an explicit Weyl symmetric form

$$\mathcal{L}^{Weyl}_{tot} = \mathcal{L}_{YM} + \sum_{I, U, V} \mathcal{L}^{I, U, V}_{gf},$$

$$= \sum_p \left\{ -\frac{1}{3} (\partial_\mu A^p_\mu)^2 - |D^p_\mu W^p_\nu|^2 -\frac{\sqrt{3}}{4} ((W^{\nu\mu} W^p_\nu)^2 - (W^{\nu\mu} W^{\nu p})(W^{\nu p} W^p_\nu)) \right\}, \quad (6)$$

with

$$W^I_\mu = \frac{1}{\sqrt{2}} (A^1_\mu + i A^8_\mu), \quad W^U_\mu = \frac{1}{\sqrt{2}} (A^1_\mu - i A^8_\mu),$$

$$W^V_\mu = \frac{1}{\sqrt{2}} (A^2_\mu + i A^7_\mu), \quad D^p_\mu = \partial_\mu + i A^p_\mu r^p_\alpha.$$

Note, that in general, a Weyl symmetric Lagrangian does not guarantee existence of singlet Weyl symmetric solutions. For instance, a one-loop effective potential with two Abelian fields $A^{3,8}_\mu \in \Gamma_2$ can be written in a manifest Weyl symmetric form, however, it has two degener-
erate vacuums. Contrary to this, an effective potential with one Weyl symmetric Abelian field under condition $A_3 = A_8$ has one deep non-degenerate vacuum [14] [15]. The Lagrangian $\mathcal{L}_{\text{Weyl}}^{\text{tot}}$ can be rewritten in terms of four independent fields $K_\mu$ (world index $\mu = 0, 1, 2, 3$) denotes space-time coordinates $(t, r, \theta, \varphi)$

$$\mathcal{L}_{\text{red}}(K) = \frac{3}{2r^2} \left[ r^2 (\partial_0 K_1 - \partial_0 K_0)^2 - (\partial_0 K_2)^2 \right] + \frac{3}{2} \left[ \left( \partial_0 K_1 \partial_2 K_3 - \partial_0 K_2 \partial_0 K_3 - \partial_0 K_3 \partial_1 K_2 + \partial_0 K_1 \partial_0 K_2 \right)^2 \right]$$

$$+ \frac{27}{16r^4 \sin^2 \theta} \left[ r^2 ((\partial_0 K_3)^2 - (\partial_0 K_3)^2) - (\partial_0 K_2)^2 \right]$$

$$- \frac{27}{16r^4 \sin^2 \theta} \left[ K_2^2 + r^2 (K_2^2 - K_0^2) \right]. \quad (7)$$

After substitution of the ansatz (2) into Euler equations corresponding to the total Lagrangian $\mathcal{L}_{\text{tot}}^{\text{Weyl}}$ equations of motion reduce to four non-degenerated second order partial differential equations and one constraint for four independent fields $K_\mu$ (9), (5-9).

Let us consider the structure of the space of Weyl symmetric solutions. It is instructive to consider first a simple case of Abelian reduced Weyl symmetric ansatz (2) obtained by setting to zero all off-diagonal components of the gauge potential $A_\mu^a$. Non-vanishing Abelian potentials $A_\mu^3, A_\mu^8$ define two constant Abelian chromomagnetic fields $H_{\mu}^{3,8}$. For external constant magnetic fields an explicit analytical expression for one-loop effective potential is known [14] [15]. The effective potential describes the vacuum energy which can be considered as a function of two number parameters $H^{3,8} = \sqrt{(H^{3,8}_{\mu\nu})^2}$. The potential has two degenerate vacuums located at the plane $(H^3, H^8)$: $H^3 = H_0, H^8 = 0$ and at $H^3 = \frac{1}{2} H_0, H^8 = \frac{\sqrt{3}}{2} H_0$. Since the effective potential is invariant under reflections $H^{3,8} \rightarrow \pm H^{3,8}$ one has six degenerate vacuums which form the Weyl sextet representation corresponding to the Weyl symmetry of root diagram of SU(3) Lie algebra. Note that six vacuum component pairs $(H^3, H^8)$ up to the common factor $K_0$ match exactly the roots, so that the vacuum sextet belongs to two-dimensional reducible Weyl representation $\Gamma_2$. In addition, the effective potential has an absolute minimum $H^3 = H^8 = 2^{-1/3} H_0$ which corresponds to a Weyl singlet representation $\Gamma_1$ due to the constraint $K^3 = K^8$ in the ansatz for Weyl singlet solutions [14]. One can verify [15] that local vacuums from the Weyl sextet are unstable (saddle points), and they are parameterized by an additional angle parameter, $\cos \theta = (H^3 \mu \nu \cdot H^8_{\mu \nu}) / H^3 H^8$. In the limit $\theta \to \pi/2$ all six local vacuums merge into the absolute vacuum. For the absolute vacuum the Weyl invariant variables $H^\mu = (H^3, H^U, H^V)$ [15]

$$H^\mu = \sqrt{(H^\mu_{\nu\rho})^2}, \quad H^\mu_{\nu\rho} = r^\rho_a H^a_{\nu\rho}, \quad \alpha = 3, 8; \quad (8)$$

take the same value, $H_I = H_U = H_V \equiv H_0$. So that the deepest vacuum is located at the point $(H_0, H_0, H_0)$ which represents a singlet standard representation $\Gamma_1$ of the permutation group $S_3$, i.e., the Weyl group.

Therefore, the Weyl symmetric non-singlet gluon solutions possess a higher symmetry to compare with non-Weyl symmetric solutions, and they form a degenerate vacuum sextet. The Weyl symmetric singlet solution reveals a highest inherent color symmetry which provides a deepest non-degenerate unique vacuum. This is the origin of vacuum stability against classical and quantum fluctuations what was proved first numerically in [6].

### III. SINGLET STRUCTURE OF NON-ABELIAN SOLUTIONS

Abelian solutions form a linear vector space, so the Hilbert space of the Abelian Weyl symmetric dynamical solutions is defined straightforward by a complete basis of transverse vector spherical harmonics of magnetic, $\vec{A}_{lm}^3$, and electric, $\vec{A}_{lm}^8$, type [11]. Let us return back to consideration of the structure of a space of non-Abelian Weyl symmetric solutions. Non-Abelian Weyl symmetric solutions are defined by four fields $K_\mu$ which satisfy a system of non-linear partial differential equations, and can be obtained only numerically. A numeric solution of magnetic type with the lowest energy density is presented in Fig. 1 in the leading order of Fourier series decomposition

$$K_{1,2,4}(r, \theta, t) = \hat{K}_{1,2,4}(Mr, \theta) \cos(Mt),$$

$$K_0(r, \theta, t) = \hat{K}_0(Mr, \theta) \sin(Mt), \quad (9)$$

where $M$ is a conformal mass scale parameter.

The non-Abelian solutions reveal the Abelian dominance effect for low energy solutions. Indeed, the Abelian numeric profile function $\hat{K}_{4}(r, \theta)$, Fig. 1(c), coincides with the lowest vector harmonic $A_{00}^3$ with a high accuracy, Fig. 1(e) and TABLE I. Moreover, one finds that a contribution of the Abelian field to the total energy inside finite space domain is near 95% ± 1.5%, which is very close to a known estimate established in the Wilson loop functional [12] [13]. The source of the Abelian dom-

| $\nu_{\text{num}}$ | $\nu_{\text{exact}}$ | $\mu_{\text{num}}$ | $\mu_{\text{exact}}$ |
|----------|-----------------|-----------------|-----------------|
| $\nu_{11}$ | 2.79 | 2.74 | $\mu_{11}$ | 4.32 | 4.49 |
| $\nu_{21}$ | 6.18 | 6.12 | $\mu_{21}$ | 7.81 | 7.73 |
| $\nu_{31}$ | 9.37 | 9.32 | $\mu_{31}$ | 10.97 | 10.90 |
| $\nu_{41}$ | 12.96 | 12.49 | $\mu_{41}$ | 14.14 | 14.07 |
| $\nu_{51}$ | 15.74 | 15.64 | $\mu_{51}$ | 17.29 | 17.22 |

TABLE I: Values of zeros and extremums of the numeric solution $\hat{K}_4$, and exact values of nodes $\nu_{\text{num}}$ and antinodes $\nu_{\text{exact}}$ of the radial part $r_{j}(r)$ of the vector harmonic $\vec{A}_{lm}^3$ (l = 1, m = 0).
K has only one Weyl singlet four-vector field. This implies that on a space of solutions one fixing terms contains a set of four independent fields. It is remarkable, all regular non-Abelian Weyl symmetric solutions defined by ansatz (2) represent singlet irreducible Weyl representations classified by one conformal mass parameter M and integer numbers (l, k). In applications to hadron physics one has to consider solutions defined in finite space domains, and the conformal parameter M takes only discrete values, M_{nl} = M_{nl}(v_{nl}). One has similar results for electric type solutions (2), section III).

IV. WEYL SYMMETRIC QUARK SOLUTIONS

Now we consider properties of the Weyl symmetric matter field described by the Lagrangian \( \mathcal{L}_q \) with one \( SU(3) \) fundamental quark triplet

\[
\mathcal{L}_q = \bar{\Psi} \left[ i\gamma^\mu (\partial_\mu - ig^2 A_\mu^a \lambda^a) - m \right] \Psi. \tag{10}
\]

The Euler equation for one flavor quark in the presence of gluon field \( A_\mu^a \) reads

\[
\left[ i\gamma^\mu (\partial_\mu - ig^2 A_\mu^a \lambda^a) - m \right] \Psi = 0, \tag{11}
\]
where the gluon field $A_\mu^a$ satisfies pure Yang-Mills equations without source $j^a = -\frac{g}{2} \bar{\Psi} \gamma_\mu \lambda^a \Psi$

$$(D^\mu \tilde{F}_{\mu\nu})^a = 0. \quad (12)$$

A usual simple Abelian projection with two independent Abelian fields $A_\mu^{3,\bar{8}}$ corresponding to two Cartan generators leads to Dirac equations for three independent $SU(3)$ color quarks

$$\left[i\gamma^\mu \partial_\mu - \frac{ig}{2} \sum_p \gamma^\mu A_\mu^a w^p_\alpha - m \right] \Psi_p = 0, \quad (13)$$

where $w^p_\alpha$ are the weight vectors $w^p = \{(1,1/\sqrt{3}), (-1,1/\sqrt{3}), (0,-2/\sqrt{3})\}$. The equations have three independent solutions for quarks forming an irreducible color triplet in the fundamental representation of the color group $SU(3)$. Note that the simple Abelian projection is not consistent with the Weyl symmetric structure defined by ansatz $(2,4)$. An important feature of the Weyl symmetric ansatz $(2,4)$ is that it implies a non-trivial Abelian projection with one Abelian field $K_3$ located in the extended color subspace spanned by generators $(T^{3,\bar{8},1,4,6})$. Substituting the Weyl symmetric ansatz into equation for quarks $(11)$ one obtains the following equation

$$\left[i\gamma^\mu \partial_\mu - m + \frac{g}{2} \gamma^\mu A_\mu G + \frac{g}{2} \gamma^\mu \hat{A}_\mu Q \right] \Psi = 0 \quad (14)$$

with color charge matrices $G$ and $Q$

$$G = \begin{pmatrix} w^1 & w^3 & \bar{w}^2 \\ \bar{w}^3 & w^2 & \bar{w}^1 \\ w^2 & \bar{w}^1 & \bar{w}^3 \end{pmatrix}, \quad (15)$$

$$Q = \begin{pmatrix} 0 & -i & i \\ i & 0 & -i \\ -i & i & 0 \end{pmatrix}, \quad (16)$$

where $w^p = w^p_\alpha + w^p_{\bar{8}}$, $A_\mu = \delta_{\mu\alpha} K_3$, $\hat{A}_\mu = \delta_{\mu\bar{8}} K_{\bar{8}}$. A Weyl symmetric structure of the charge matrix $G$ corresponding to interaction with vacuum gluon field implies that matrix $G$ has three eigenvectors $\tilde{u}^{0,\pm}$ with corresponding eigenvalues $\lambda^{0,\pm}$

$$\lambda^0 = 0, \quad \lambda^{\pm} = \pm \sqrt{\tilde{g}},$$

$$\tilde{u}^0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad (17)$$

$$\tilde{u}^\pm = \begin{pmatrix} w^1 w^3 + \tilde{w}^2 (\pm \tilde{g} - \tilde{w}^2) \\ \bar{w}^2 \tilde{w}^3 + \tilde{w}^1 (\pm \tilde{g} - \tilde{w}^1) \\ \tilde{w}^1 \tilde{w}^2 + \tilde{w}^3 (\pm \tilde{g} - \tilde{w}^3) + \tilde{g}^2 \end{pmatrix}, \quad (18)$$

$$\tilde{g}^2 = (\tilde{w}^1)^2 + (\tilde{w}^2)^2 + (\tilde{w}^3)^2 - \tilde{w}^1 \tilde{w}^2 - \tilde{w}^2 \tilde{w}^3 - \tilde{w}^3 \tilde{w}^1,$$

where $\tilde{g}$ is a Weyl invariant color charge. Three vectors $\tilde{u}^{0,\pm}$ form an orthogonal vector basis in the color space $R^3$. The vector $\tilde{u}^0$ is a common eigenvector for both charge matrices $G$ and $Q$. Substituting quark triplet $\Psi^0 = \psi(x) \tilde{u}^0$ in $(14)$ one obtains a free Dirac equation for the quark mode $\psi^0(x)$

$$(i\gamma^\mu \partial_\mu - m) \psi^0(x) = 0. \quad (19)$$

It is surprising, the system of equations $(14)$ admits an exact free quark solution, which is absent for the quark equation $(13)$ obtained by means of the usual simple Abelian projection. This implies, that one has a complete basis of vacuum fields containing the Weyl symmetric Abelian gluon field $K_3$ and the quark mode $\psi^0$ given by the vector and spinor spherical harmonics respectively. Free gluon and quark solutions with the lowest angular momentum describe a non-degenerate color invariant vacuum characterized by non-zero vacuum gluon and quark condensates which do not interact to each other in the leading order approximation.

Solutions $\Psi^\pm = \psi^\pm(x) \hat{u}^\pm$ belong to two-dimensional Weyl representation $\Gamma_3$ which contains two singlet non-vector symmetry representations for two color quarks $\psi^\pm$. Using the Weyl symmetric Abelian projection with vanished off-diagonal fields one obtains the following equations for Weyl invariant quark modes $\psi^\pm$ with opposite Weyl invariant charges $\pm \hat{g}$

$$\left[i\gamma^\mu \partial_\mu - m \pm g \tilde{g} \gamma^\mu \hat{A}_\mu \right] \psi^\pm = 0. \quad (20)$$

It is clear, that solutions $\psi^\pm$ can not be transformed into each other by Weyl transformations, and they form two separate Weyl representations. We conclude, the Weyl symmetric quark solutions form an orthogonal basis consisting of three singlet quarks $\Psi^{0,\pm}$ with color charges $(0, \pm \hat{g})$. Note that all three quarks $\psi^{0,\pm}$ are still color quarks since they interact at quantum level with all off-shell field components of the quantum (virtual) gluon $A_\mu^a$.

At first glance, the existence of exact free quark solution $\Psi^0$ seems to be in contradiction with the quark confinement. One should stress that the Weyl symmetric free gluons and quarks exist only in the classical theory. In quantum theory one has generation of non-trivial vacuum gluon and quark condensates. This changes drastically the structure of the space of quantum states. We show that interaction of the free gluon and quark with corresponding vacuum condensates leads to formation of localized bound states. To prove this we calculate first a one-loop effective action in the presence of quark condensate $(20)$ and demonstrate generation of a non-trivial vacuum. Our results imply the following expression for one-loop effective potential

$$V^{1,1}_{eff} = -\frac{g^2}{8\pi^2} \left\{-\frac{b}{4} \sqrt{-b^2 + 4c(\pi - 2\arctan \frac{b}{\sqrt{-b^2 + 4c}})} - \frac{\alpha_f^2}{y_0} - \frac{1}{4} (\hat{b}^2 - 2c) \log(\frac{c}{m^2} + y_0^2 \log(\frac{y_0}{m^2})) \right\}, \quad (21)$$

where

$$b = y_0 + m^2, \quad c = -\frac{\Phi^2}{y_0}, \quad \Phi = <0|\bar{\Psi}^0 \Psi^0|0>, \quad$$
and $y_0$ is a real root to cubic equation

$$c_g^2 \Phi^2 - 2c_g m \Phi y + m^4 y^2 + y^3 = 0,$$

where the group number factor $c_g$ takes values $(\frac{16}{9}, \frac{3}{2}, \frac{1}{2})$ for gauge groups $SU(3)$, $SU(2)$ and $U(1)$ respectively. The vacuum energy dependence on positive and negative values of vacuum quark condensate is depicted in Figs. 2(a), 2(b). It is clear that a non-trivial vacuum is generated at a finite negative value of the vacuum quark condensate. For small values of the quark condensate, $|\Phi| < 1$, the effective potential has a simple form

$$V_{eff}^{1-1} \approx m \Phi + \frac{3c_g g^2}{8\pi^2 m^2} \Phi^2 \left( \log \left( \frac{c_g g^2 |\Phi|}{m^3} \right) + \frac{3}{2} \right).$$

In quasiclassical approximation the quark condensate is described by a spinor spherical harmonic [19]

$$\psi_{jm}^\theta = \left( \phi(x)_{jm} \right) = \left( a(r) \Omega_{jm} \right),$$

A negative quark condensate is caused by the relativistic component $\chi$ with $l = 1$, Fig. 3.

It is well known and commonly accepted a simple mechanism of quark confinement provided by a gluon string which bounds the quark anti-quark pair in a meson. It is much less known on a possible mechanism of confinement of a single quark. It was proposed that origin of single quark confinement is related to vacuum polarization effects [17, 18]. We consider a possible mechanism of single quark confinement based on vacuum polarization effect leading to generation of a non-trivial vacuum characterized by a non-vanishing negative vacuum quark condensate.

A single quark in such a vacuum interacts to vacuum quark condensate and forms localized bound states. One can estimate the energy spectrum of bound states by solving a quantum mechanical problem defined by the

$$i \partial \psi \left( \frac{H + V}{\partial t} \right) = (H + V) \psi,$$

where $H = \bar{\psi} \gamma \beta m$ is a Dirac Hamiltonian. For qualitative estimate we approximate the effective potential by a “rectangular” spherically symmetric well

$$V(r) = \begin{cases} 0, & r > r_0 \\ -V_0, & r < r_0, \end{cases}$$

where the width $r_0$ of the well is defined by hadron size, $r_0 \approx 1 fm$. The depth $V_0$ is a free parameter since one loop approximation provides only a lower bound $V_0 \approx 0.5m$. Such a relativistic quantum mechanical problem has been solved in the analytical form in quantum electrodynamics in a case of the Dirac electron ([19], section 1.5). As a result, for values of the potential depth less than a critical value, $V_0 \leq V_{cr} \approx 3.5m$, one has a discrete energy spectrum for bound states with an exponentially decreasing radial wave function $\psi = \exp \left[ -\sqrt{m^2 - \epsilon^2} \right] (r \geq r_0)$ and discrete energy levels $|\epsilon| \leq m$. The ground state with orbital quantum number $l = 0$ describes $S$-state with zero angular momentum which can be treated as a meson state. In our consideration we did not take into account back reaction of the vacuum condensate to interacting single quark. We assume that after creating from the vacuum a single quark with baryon, electric and fermion charges, the condensate state has been changed taking opposite quantum numbers, so that a final bound state manifests quantum numbers of meson. This reminds the soft (long-time scale) mechanism of quark confinement proposed by Casher-Kogut-Susskind [17, 18] where quarks are present in deep-inelastic processes as free point-like particles, whereas on a longer time scale the polarization effects prevent the appearance of quarks in the final physical state.
Note that, a single quark can not be freed from the bound state, since the quark generates a non-trivial vacuum condensate where ever it goes, in other words, the quark stays permanently inside the potential well. For values of $V_0$ larger than critical value $V_{cr}$ the bound states become unstable causing creation of quark-antiquark pairs ([19], section 1.5). In this case the solution turns outside the framework of one particle quantum mechanical problem.

V. LOCALIZATION OF A SINGLET QUANTUM GLUON STATE

Now we consider interaction of a single gluon with vacuum gluon condensate. For qualitative estimate we apply a known expression for one-loop effective Lagrangian of SU(3) QCD [14, 15]

$$\mathcal{L}_{eff}^{1-l} = -\frac{1}{4} \bar{F}^2 - k_0 g^2 \bar{F}^2 \left( \log \left( \frac{g^2 \bar{F}^2}{\Lambda_{QCD}^2} \right) - c_0 \right),$$

(26)

where $\bar{F}^2 \equiv \bar{F}_{\mu \nu}^2$ is a squared Abelian field strength of magnetic type, and we treat $k_0, c_0$ as free parameters. A corresponding effective potential $V^{1-l} = -\mathcal{L}_{eff}^{1-l}$ has an absolute minimum at positive vacuum gluon condensate value

$$g^2 B^2_{\mu \nu} = \Lambda_{QCD} \exp \left( c_0 - 1 - \frac{1}{2 k_0 g^2} \right).$$

(27)

We split the field strength $\bar{F}_{\mu \nu}$ into two parts

$$\bar{F}_{\mu \nu} = B_{\mu \nu} + F_{\mu \nu},$$

(28)

where $B_{\mu \nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ describes the gluon condensate, and $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ contains the Abelian gauge potential of a single gluon. Decomposing the Lagrangian $\mathcal{L}^{1-l}$ around the vacuum condensate field $B_{\mu \nu}$ and using relation [27], one obtains an effective Lagrangian for a gluon in the field of vacuum gluon condensate

$$\mathcal{L}_{eff}^{(2)}[A] = -2k_0 g^2 \left( \frac{B_{\mu \nu} F_{\mu \nu}^2}{B^2} \right) \equiv -\kappa (B_{\mu \nu} F_{\mu \nu})^2,$$  

(29)

where we neglect terms corresponding to the absolute value of the vacuum energy, and $\kappa$ is a free parameter. The effective Lagrangian $\mathcal{L}_{eff}^{(2)}[A]$ is strikingly different from the classical Lagrangian of QCD. Note that this result is model independent, we could start with Ginsburg-Landau type Lagrangian and obtain the same result. The gauge potential of the vacuum gluon condensate is given by a magnetic vector harmonic $A_{\mu}^{m} = 1_{m=0}$, or explicitly,

$$B_{\phi}(r, \theta, t) = b(r) \sin^2 \theta \sin(t),$$

(30)

where $b(r) = r j_1(r)$. A single gluon is described by the Abelian potential which assumed to be time-coherent to the gluon condensate function with a phase shift $\phi_0$,

$$A_{\phi}(r, \theta, t) = a(r, \theta) \sin(t + \phi_0).$$

(31)

V. LOCALIZATION OF A SINGLET QUANTUM GLUON STATE

A time-averaged effective Lagrangian $\mathcal{L}_{eff}^{(2)}[A]$ leads to Euler equation for gluon field $a(r, \theta)$ which represents a second order partial differential equation. It turns out that equation for a gluon interacting to vacuum condensate is separable for a ground state with a zero angular momentum, $j = 0$. So the solution is spherically symmetric, $a(r, \theta) = f(r)$, and satisfies a simple ordinary differential equation

$$-r^2 b''(r) - 2b'(r) \left( b(r) - r^2 b''(r) \right) = 0,$$

(32)

where $\xi \equiv \cos(2\phi_0)/(2 + \cos(2\phi_0))$. The equation is solved numerically in dimensionless variables $x = r/\nu_1$, $\nu_1 = 2.74 \cdots$ is the first antinode of the radial function $b(r) = r j_1(r)$. The numeric solution implies localization of the single gluon inside a sphere with radius $r_0 = \nu_1$ (or $x = 1$), Fig. 4 (a).

![Image](image.png)

FIG. 4: (a) Solution $f(x)$; (b) a radial energy density $E/4\pi\xi$ (in red); (c) the first derivative of the energy density; (d) the second derivative of the energy density; $\phi_0 = \pm \pi/2$.

The point $r_0 = \nu_1$ represents a removable singularity, and the energy density has smoothly vanishing first and second radial derivatives at $r_0$, Fig. 4 (c, d). The solution has a minimal energy at phase shift value $\phi_0 = \pi/2$, and only at this value the time averaged effective Lagrangian vanishes, as in a case of Maxwell Lagrangian for photon plane waves, providing stability of the ground state. The ground state can be treated as a state of the lightest scalar glueball.

VI. SPECTRUM OF LIGHTEST GLUEBALLS

Explicit analytical expressions for quantum stable Abelian solutions are given in terms of vector spherical harmonics. This allows to perform standard quantization of Abelian fields defined in a finite space region constrained by a sphere of radius $a_0$ corresponding to an effective glueball size. It is suitable to introduce dimensionless units $M = M a_0, x = r/a_0, \tau = t/a_0$. To find proper boundary conditions we require that the Pointing
vector \( \hat{S} = \vec{E} \times \vec{B} \) vanishes on the sphere. This implies two possible types of boundary conditions [20]

\[
\begin{align*}
(\text{I}) & : \quad \hat{A}_{\text{in}}^m (\hat{M} x) |_{x=1} = 0, \quad \hat{M} n l = \mu_{n l}, \\
(\text{II}) & : \quad \partial_r (r \hat{A}_{\text{in}}^m (\hat{M} x) ) |_{x=1} = 0, \quad \hat{M} n l = \nu_{n l},
\end{align*}
\]

where \( \hat{M} n l \) stands for nodes \( \mu_{n l} \) or antinodes \( \nu_{n l} \) of the Bessel function \( j_l(r) \) \((n = 1, 2, 3, \ldots; l = 1, 2, 3, \ldots)\). We choose the following normalization condition for the vector harmonics \( \hat{A}_{\text{in}}^m (\hat{M} x) \)

\[
\frac{1}{4\pi} \int_0^1 dx \int d\theta d\varphi \ x^2 \sin(\theta) (\hat{A}_{\text{in}}^m (\hat{M} x))^2 = \frac{1}{\hat{M} n l}.
\]

The standard canonical quantization results in the following Hamiltonian expressed in terms of the creation and annihilation operators \( c_{n l m}^\pm \)

\[
H = \frac{1}{2} \sum_{n, l, m} \hat{M} n l (c_{n l m}^+ c_{n l m}^- + c_{n l m}^- c_{n l m}^+) \tag{35}
\]

One particle states \( \{c_{n l m}^\pm |0\} \) describe free gluons which are not observable quantities since we have to take into account their interaction to vacuum gluon condensate.

We apply a simple model based on one-loop effective Lagrangian of QCD which describes appearance of localized solutions corresponding to the lightest glueballs. Certainly, one loop effective potential is not a much appropriate tool for quantitative description of glueballs, nevertheless, it contains a non-perturbative part originated from summation of contributions from infinite number of one-loop quantum corrections. This provides qualitative description of formation of glueballs as a result of interaction of a single gluon with corresponding generated vacuum gluon condensate.

Potential allows to find analytical expressions for the radial density of the vacuum gluon condensate functions performing averaging over the time and polar angle. Averaged over the time and polar angle vacuum gluon condensate functions \( \alpha_s (\langle F_{\mu \nu}^a \rangle^2) \) corresponding to magnetic modes \( \hat{A}_{11}^\nu (\nu_{11} x) \) and \( \hat{A}_{11}^\nu (\mu_{11} x) \) are depicted in Fig. 5 \((\alpha_s = 0.5)\). The oscillating behavior of the vacuum gluon condensate density was obtained before within the instanton approach to QCD [21]. Integrating the radial density over the interval \((0 \leq x \leq 1)\) one can fit a value of the obtained vacuum gluon condensate parameter to the known value \( \alpha_s (\langle F_{\mu \nu}^a \rangle^2) = (540[\text{MeV}])^4 \), and obtain an explicit dependence of the glueball size on quantum number \( \hat{M} n l \)

\[
a_{n l}[\text{fm}] = \frac{197}{\nu_0} f_c^{-1/4} (\hat{M}) \approx \frac{107\alpha_s}{\nu_0} f_c^{-1/4} \sqrt{\hat{M} n l}, \tag{36}
\]

\[
f_c (\hat{M}) = \frac{N_{n l}^2}{12\pi \hat{M}^2} (3 - 4\hat{M}^2 + 2\hat{M}^4) \cos(2\hat{M}) - 3 - 2\hat{M}^2 + 2\hat{M} (3 - \hat{M}^2) \sin(2\hat{M}) + 4\hat{M}^5 \sin(2\hat{M})
\]

where \( v_0 = 540[1/\text{fm}] \), \( N_{n l} \) is the normalization factor of the vector harmonic, and \( \sin(2\hat{M}) \) is the sine integral function. With this one can find the energy spectrum of light scalar glueballs \( J^{PC} = 0^{++} \)

\[
E_{n l}[\text{MeV}] = \sqrt{\hat{M} n l} f_c^{-1/4} \approx \left( \frac{80}{7\alpha_s} \right)^{1/4} k v_0 \sqrt{\hat{M} n l}, \tag{37}
\]

where \( k \) is a free model parameter which can be fixed by fitting the energy value of the lightest glueball [22] [23]. The energy spectrum [37] agrees with the Regge theory.

\[\text{FIG. 5: Radial densities of the magnetic vacuum gluon condensates } \alpha_s (F^2) \text{ corresponding to modes } \nu_{11} = 2.74 \cdots, (a), \text{ and } \mu_{11} = 4.49 \cdots, (b); \ (n = l = 1, m = 0).\]

Qualitative estimates of the lightest scalar glueball spectrum can be performed in a model independent way assuming that vacuum gluon condensate is a universal order parameter for glueballs with different quantum numbers. The knowledge of explicit solutions for the vector potentials allows to find analytical expressions for the radial density of the vacuum gluon condensate functions performing averaging over the time and polar angle. Averaged over the time and polar angle vacuum gluon condensate functions \( \alpha_s (\langle F_{\mu \nu}^a \rangle^2) \) corresponding to magnetic modes \( \hat{A}_{11}^\nu (\nu_{11} x) \) and \( \hat{A}_{11}^\nu (\mu_{11} x) \) are depicted in Fig. 5 \((\alpha_s = 0.5)\). The oscillating behavior of the vacuum gluon condensate density was obtained before within the instanton approach to QCD [21]. Integrating the radial density over the interval \((0 \leq x \leq 1)\) one can fit a value of the obtained vacuum gluon condensate parameter to the known value \( \alpha_s (\langle F_{\mu \nu}^a \rangle^2) = (540[\text{MeV}])^4 \), and obtain an explicit dependence of the glueball size on quantum number \( \hat{M} n l \)

\[
a_{n l}[\text{fm}] = \frac{197}{\nu_0} f_c^{-1/4} (\hat{M}) \approx \frac{107\alpha_s}{\nu_0} f_c^{-1/4} \sqrt{\hat{M} n l}, \tag{36}
\]

\[
f_c (\hat{M}) = \frac{N_{n l}^2}{12\pi \hat{M}^2} (3 - 4\hat{M}^2 + 2\hat{M}^4) \cos(2\hat{M}) - 3 - 2\hat{M}^2 + 2\hat{M} (3 - \hat{M}^2) \sin(2\hat{M}) + 4\hat{M}^5 \sin(2\hat{M})
\]

where \( v_0 = 540[1/\text{fm}] \), \( N_{n l} \) is the normalization factor of the vector harmonic, and \( \sin(2\hat{M}) \) is the sine integral function. With this one can find the energy spectrum of light scalar glueballs \( J^{PC} = 0^{++} \)

\[
E_{n l}[\text{MeV}] = \sqrt{\hat{M} n l} f_c^{-1/4} \approx \left( \frac{80}{7\alpha_s} \right)^{1/4} k v_0 \sqrt{\hat{M} n l}, \tag{37}
\]

where \( k \) is a free model parameter which can be fixed by fitting the energy value of the lightest glueball [22] [23]. The energy spectrum [37] agrees with the Regge theory.

\[\text{VII. CONCLUSION}\]

In conclusion, we have demonstrated that a space of classical dynamical solutions for gluons and quarks contains a subspace of Weyl symmetric dynamical solutions which realize only singlet irreducible representations of the Weyl group. Such singlet solutions possess intrinsic color symmetry with respect to the Weyl color subgroup, as a consequence, they have special features due to more higher symmetry in comparison with other non-Weyl symmetric solutions. In a particular, a Weyl singlet Abelian classical vacuum solution provides a color invariant (under Weyl color transformations) non-degenerate deepest vacuum due to the highest inherent symmetry. The singlet structure of solutions with Abelian dominance phenomenon imply that the full space of dynamical singlet solutions and a corresponding space of one particle quantum states can be classified by a finite set of discrete quantum numbers \( M_{n l} = \mu_{n l}/\nu_{n l}, k \) \((n, l = 1, 2, 3, ..., k = 0, 1, 2, \ldots)\). This opens a new way towards non-perturbative formulation of the Yang-Mills theory based on exact gluon and quark solutions.
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