Global existence and exponential decay of the solution for a viscoelastic wave equation with a delay

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Abstract. In this paper, we consider initial-boundary value problem of viscoelastic wave equation with a delay term in the interior feedback. Namely, we study the following equation

\[ u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)\,ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0 \]

together with initial-boundary conditions of Dirichlet type in \( \Omega \times (0, +\infty) \) and prove that for arbitrary real numbers \( \mu_1 \) and \( \mu_2 \), the above-mentioned problem has a unique global solution under suitable assumptions on the kernel \( g \). This improves the results of the previous literature such as Nicaise and Pignotti (SIAM J. Control Optim 45:1561–1585, 2006) and Kirane and Said-Houari (Z. Angew. Math. Phys. 62:1065–1082, 2011) by removing the restriction imposed on \( \mu_1 \) and \( \mu_2 \). Furthermore, we also get an exponential decay results for the energy of the concerned problem in the case \( \mu_1 = 0 \) which solves an open problem proposed by Kirane and Said-Houari (Z. Angew. Math. Phys. 62:1065–1082, 2011).

Mathematics Subject Classification (2010). 35L05, 35L20, 35L70, 93D15.

Keywords. Viscoelastic wave equation · Global existence · Energy decay · Interior feedback.

1. Introduction

It is well known that the free vibration of membrane can be described by wave equation of the form

\[ u_{tt}(x,t) - \Delta u(x,t) = 0 \]  

in \( \Omega \times (0, +\infty), \) subjected to initial conditions and some boundary conditions. Here, \( \Omega \subset \mathbb{R}^2 \) is a bounded domain. For the case of high dimension, there are also many discussions in mathematical physics. For the better comprehension of our motivation, we appeal to readers to keep in mind that the system (1.1) is conservative.

To study the propagation mechanism of wave, two main factors, that is damping and time delay, are taken into consideration. When the damping occurs, the corresponding mathematical model becomes

\[ u_{tt}(x,t) - \Delta u(x,t) + \mu u_t = 0, \]  

and the energy of this system is dissipative. We here point out that the results about the models (1.1) and (1.2) are quite abundant. And what is regretful is that we cannot list these literatures one by one.

Moreover, time delay effects often appear in many daily life practical problems. These hereditary effects are sometime unavoidable, and it may turn a well-behaved system into a wild one. That is to say, they can induce some instabilities (see for instance [1–5] and the references therein). The stability issue of systems with delay is, therefore, of theoretical and practical importance.

For the wave equation with boundary or interior delay, S. Nicaise, C. Pignotti, M. Gugat and G. Leugering et al. obtained some profound results [6–11]. In [6], the authors examined a system of wave
equation with the damping and the time delay inside the domain. Namely, they considered the following system

\[
\begin{align*}
  u_{tt}(x, t) - \Delta u(x, t) + \mu_1 u_t(x, t) + \mu_2 u_t(x, t - \tau) &= 0, & x \in \Omega, & t > 0, \\
  u(x, t) &= 0, & x \in \partial\Omega, & t \geq 0, \\
  u(x, 0) &= u_0(x), & u_t(x, 0) &= u_1(x), & x \in \Omega, \\
  u_t(x, t - \tau) &= f_0(x, t - \tau), & x \in \Omega, & 0 < t < \tau,
\end{align*}
\] (1.3)

and proved that the energy of the above problem is exponentially decay (or the trivial solution is exponentially stable) provided that \(0 < \mu_2 < \mu_1\). Furthermore, it is also showed in the case \(\mu_2 \geq \mu_1\) that there exists a sequence of arbitrary small (or large) delays such that instabilities occur. The same results were showed when both the damping and the delay act on the boundary. Particularly, if \(\mu_1 = 0\), that is, if we have only the delay part in the boundary condition, the system becomes unstable (see [2]). From these results, we may figure out that the delay term lead to instability of the system. And to avoid this problem, Gugat [9] considered feedback laws where a certain delay is included as a part of the control law and not as a perturbation and obtained the exponential stability of the proposed feedback with retarded input. In addition, Gugat and Leugering [10] considered the feedback stabilization of quasilinear hyperbolic systems on star-shaped networks. They obtained the exponential stability of the system by introducing an \(L^2\)-Lyapunov function with delay terms. Recently, Pignotti [11] considered the wave equation with internal distributed time delay and local damping in a bounded and smooth domain and showed that an exponential stability result holds if the coefficient of the delay term is sufficiently small.

In this paper, we are concerned with the initial-boundary value problem (IBVP) of the viscoelastic wave equation

\[
\begin{align*}
  u_{tt}(x, t) - \Delta u(x, t) + \int_0^t g(t-s)\Delta u(x, s)ds + \mu_1 u_t(x, t) + \mu_2 u_t(x, t - \tau) &= 0, & x \in \Omega, & t > 0, \\
  u(x, t) &= 0, & x \in \partial\Omega, & t \geq 0, \\
  u(x, 0) &= u_0(x), & u_t(x, 0) &= u_1(x), & x \in \Omega, \\
  u_t(x, t - \tau) &= f_0(x, t - \tau), & x \in \Omega, & 0 < t < \tau,
\end{align*}
\] (1.4)

where \(\Omega \subset \mathbb{R}^n, n \geq 1\) be a regular and bounded domain with a boundary \(\partial\Omega\) of class \(C^2\). Moreover, \(\mu_1, \mu_2\) are two real numbers, \(\tau > 0\) represents the time delay and \(u_0, u_1, f_0\) are given initial data belonging to suitable spaces.

System (1.4) comes from linear models for propagation of viscoelastic wave in a compressible fluid. In mechanics, it is well known that solid and fluid materials exhibit not only elasticity but also hereditary properties. The hereditary properties are described by viscoelasticity, where the mechanical response of the materials is taken to be influenced by the previous behavior of the materials themselves. In other words, the viscoelastic materials exhibit memory effects. This leads to a constitutive relationship between the stress and strain involving the convolution of the strain with a relaxation function. This term is called a “memory term.” From the mathematical point of view, these hereditary properties are modeled by integro-differential operators.

To motivate our present work, let us recall some previous results regarding the viscoelastic wave equation. In the absence of time delay, the viscoelastic wave equation of the form

\[
u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x, s)ds + a(x)u_t = 0, \tag{1.5}\]

in \(\Omega \times (0, \infty)\), subjected to initial conditions and boundary conditions of Dirichlet type, has been considered by Cavalcanti et al. [12]. They showed an exponential decay result under some restrictions on \(a(x)\) and \(g(t)\). To be specific, they assumed that \(a : \Omega \to \mathbb{R}\) is a nonnegative and bounded function, and
the kernel $g$ in the memory term decays exponentially. This result has been improved later by Berrimi and Messaoudi [13] under weaker conditions on both $a$ and $g$. And then, more general problems than the one considered in [12] are studied in [14,15]. Some profound results about the relationship between the stability of solutions and the relaxation kernel $g$ are obtained. It is worthwhile to note that, if $a(x) \equiv 0$ in (1.5), some results about the energy decay are obtained by many researchers. See, for instance Tatar [16], Mustafa and Messaoudi [17].

Then a nature problem is that what would happen when a delay term occurs in (1.5). In this case, Kirane and Said-Houari [18] studied problem (1.4) with coefficients $\mu_1$ and $\mu_2$ positive. They showed that problem (1.4) has a unique weak solution (see [18], Theorem 3.1), and its energy is exponentially decay provided that $\mu_2 \leq \mu_1$. It is worth pointing out that the assumptions $\mu_1 > 0$ and $\mu_2 \leq \mu_1$ play a decisive role in proof of the above-mentioned results. Because the authors must use the damping term $\mu_1 u_t(x,t)$ to control the delay term $\mu_2 u_t(x,t-\tau)$ in the priori estimate of the solution and the decay estimate of the energy. This makes the authors arise an open problem that whether the decay property of the energy they have obtained are preserved in the case $\mu_1 = 0$? By the way, the results of [18] have been generalized recently by Liu [19] to a system with time-dependent delay. The method used by W.J. Liu is very similar to those used in [18].

The aims of the present paper are twofold. At first, we aim to prove an existence result of problem (1.4) without restrictions of $\mu_1, \mu_2 > 0$ and $\mu_2 \leq \mu_1$. Unlike in [18], our new idea is to control the delay term by the derivative of the energy instead of by the damping term in the priori estimate of solutions. In the second, we will give a positive answer to the open problem proposed by Kirane and Said-Houari [18]. That is we will prove a energy decay result for problem (1.4) in the case $\mu_1 = 0$. The main difficulty in handling this problem is that we have no damping term to control the delay term in the estimate of the energy decay. To overcome this difficulty, our basic idea is to control the delay term by making use of the viscoelasticity term. And in order to achieve this goal, a restriction of the size between the parameter $\mu_2$ and the kernel $g$ and a new Lyapunov functional, different from that one used in [18], is needed.

The paper is organized as follows. In Sect. 2, we give some preliminaries. In Sect. 3, we prove the existence and the uniqueness of the solution of problem (1.4). In Sect. 4, we prove the exponential decay of the energy of this problem in the case $\mu_1 = 0$.

2. Preliminaries

In this section, we shall prepare some materials needed in the proof of our result and state. We use the standard Lebesgue space $L^2(\Omega)$ and Sobolev space $H^1_0(\Omega)$ with their usual norms $\|\cdot\|_2$ and $\|\cdot\|_{H^1_0}$, respectively. And we will write $(\cdot,\cdot)$ to denote the inner product in $L^2(\Omega)$. Throughout this paper, $C$ and $C_i$ are used to denote generic positive constants.

We first state the general assumptions on the kernel function $g$ as follows:

(A1) $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a $C^1$ function satisfying

$$g(0) > 0, \quad 1 - \int_0^\infty g(s)ds = l > 0.$$  

(A2) There exists a positive constant $\zeta$ such that

$$g'(t) \leq -\zeta g(t), \quad \forall t > 0. \quad(2.1)$$

A typical example of such function is

$$g(t) = e^{-at}, \quad a > 1.$$
To simplify the notation, we set
\[
(\phi \circ \psi)(t) := \int_{0}^{t} \phi(t-s) \int_{\Omega} |\psi(t) - \psi(s)|^2 \, dx \, ds,
\]
where \( \psi \) may be a scalar or a vector valued function. For any \( \phi \in C^1(\mathbb{R}) \) and \( \psi \in H^1(0,T,L^2(\Omega)) \), the following identity is proved in \([18,20]\):
\[
\int_{\Omega} \psi_t(t) \int_{0}^{t} \phi(t-s) \psi(s) \, ds \, dx = \frac{1}{2} \phi' \circ \psi(\tau) - \frac{1}{2} \phi(\tau) \|\psi\|_2^2 - \frac{1}{2} \frac{d}{dt} \left\{ \int_{0}^{t} \phi(s) \, ds \right\} \|\psi\|_2^2.
\]

The following lemma will be used in Sect. 4 to prove the energy decay result for problem (1.4).

**Lemma 2.1.** (see Lemma 2.2 in \([18]\) or \([21]\)). For \( u \in L^2(0,T;H^1_0(\Omega)) \), we have
\[
\int_{\Omega} \left( \int_{0}^{t} g(t-s)(u(t) - u(s)) ds \right)^2 \, dx \leq (1-l) C^2_\ast (g \circ \nabla u)(t),
\]
where \( C_\ast \) is the Poincaré constant and \( l \) is given in (A1).

Let
\[
U = \{ u | u \in L^2(0,T;H^1_0(\Omega)), u_t \in L^2(0,T;L^2(\Omega)) \}
\]
and
\[
V = \{ v | v \in U, v(x,0) = 0, v_t(x,T) = 0 \}.
\]

Next, we give the definition of the weak solution of problem (1.4).

**Definition 2.2.** We say that a function \( u \in U \) is a weak solution of the initial-boundary value problem (1.4) if
\[
- \int_{0}^{T} (u_t, v) \, dt + \int_{0}^{T} (\nabla u(t), \nabla v(t)) \, dt = \int_{0}^{T} \int_{0}^{t} g(t-s)(\nabla u(s), \nabla v(s)) \, ds \, dt
\]
\[+ \mu_1 \int_{0}^{T} (u_t(t), v) \, dt + \mu_2 \int_{0}^{T} (u_t(t-\tau), v) \, dt + (v(x,0), u_1(x)) = 0
\]
for each \( v \in V \) and
\[
u(x,0) = u_0, \quad u_t(x,0) = u_1, \quad u_t(x,t-\tau) = f_0(x,t-\tau), \quad t \in (0,\tau).
\]

**Remark 2.3.** In view of Theorem 2 in §5.92 of [22], we know \( u \in C \left( 0,T;L^2(\Omega) \right) \) and \( u_t \in C \left( 0,T;H^{-1}(\Omega) \right) \). Consequently, the equalities (2.5) above make sense.

### 3. Existence of weak solutions

This section devotes to the study of well posedness of problem (1.4). The uniqueness result for weak solution of problem (1.4) will be deduced from an energy estimate given below in Theorem 3.1, and the existence result for weak solution of problem (1.4) will be proved in Theorem 3.3 by making use of Faedo–Galerkin’s method.
Theorem 3.1. (A priori estimate) For any weak solution \( u \in U \) of (1.4), there exists a constant \( C \), depending only on \( \Omega \) and \( T \), such that

\[
E(u(t)) \leq C,
\]

where

\[
E(u(t)) = \frac{1}{2} \left( \|u_t\|^2_2 + \left( 1 - \int_0^t g(s)ds \right) \|\nabla u\|^2_2 + (g \circ \nabla u)(t) \right). \tag{3.1}
\]

Proof. Since functions of the space \( C^\infty((0,T) \times \Omega) \) are dense in the space \( U \), we just need to prove the conclusion in the case of \( u \in C^\infty((0,T) \times \Omega) \). So, the following calculations are all in this space.

Multiplying the first equation in (1.4) by \( u_t \) and integrating on \( \Omega \), straightforward computations lead to

\[
E'(u(t)) = \frac{1}{2} \left( (g' \circ \nabla u)(t) - g(t) \|\nabla u(t)\|^2_2 \right) - \mu_1 \int_\Omega u^2_t(t)dx - \mu_2 \int_\Omega u_t(t-\tau)u_t(t)dx. \tag{3.2}
\]

By the assumptions (A1) and (A2) about \( g(t) \), we have

\[
(g' \circ \nabla u)(t) - g(t) \|\nabla u(t)\|^2_2 < 0. \tag{3.3}
\]

So, thanks to the Cauchy-Schwartz inequality, we deduce

\[
E'(u(t)) \leq \frac{\mu_2}{2} \int_\Omega u^2_t(t-\tau)dx + \left( \frac{\mu_2}{2} + \mu_1 \right) \int_\Omega u^2_t(t)dx. \tag{3.4}
\]

By integrating on \([0,t]\), we have

\[
E(u(t)) \leq \frac{\mu_2}{2} \int_0^t \int_\Omega u^2_s(s-\tau)dxds + \left( \frac{\mu_2}{2} + \mu_1 \right) \int_0^t \int_\Omega u^2_s(s)dxds + E(u(0)), \tag{3.5}
\]

where \( E(u(0)) \) is a nonnegative constant.

Now, using the past history values about \( u_t(t), \ t \in [-\tau,0] \), the first term in the right-hand side of (3.5) can be rewritten as follows

\[
\int_0^t \int_\Omega u^2_s(s-\tau)dxds = \int_\Omega \int_{-\tau}^t u^2_{\rho}(\rho)d\rho dx = \int_\Omega \int_{-\tau}^t u^2_{\rho}(\rho)d\rho dx + \int_\Omega \int_{0}^{t-\tau} u^2_{\rho}(\rho)d\rho dx
\]

\[
= \int_\Omega \int_{-\tau}^t f_0^2(\rho)d\rho dx + \int_\Omega \int_{0}^{t-\tau} u^2_{\rho}(\rho)d\rho dx \leq \int_\Omega \int_{-\tau}^t f_0^2(\rho)d\rho dx + \int_0^t \int_\Omega u^2_{\rho}(\rho)d\rho dx. \tag{3.6}
\]
From (3.5) and (3.6), we obtain
\[
\mathcal{E}(u(t)) \leq (|\mu_2| + |\mu_1|) \int_0^t \int_\Omega u^2(x) \rho dx + \frac{|\mu_2|}{2} \int_0^t \int_\Omega f_0^2(x) \rho dx + \mathcal{E}(u(0)).
\] (3.7)

Thus, we have
\[
\mathcal{E}(u(t)) \leq 2(|\mu_2| + |\mu_1|) \int_0^t \mathcal{E}(u(s)) ds + \frac{|\mu_2|}{2} \int_0^t \int_\Omega f_0^2(x) \rho dx + \mathcal{E}(u(0)).
\] (3.8)

By the Gronwall inequality, once \( T > 0 \) be given, \( \forall t \in [0, T] \), we have
\[
\mathcal{E}(u(t)) \leq \left( \frac{|\mu_2|}{2} \int_0^t \int_\Omega f_0^2(x) \rho dx + \mathcal{E}(u(0)) \right) e^{2(|\mu_2| + |\mu_1|)T}, \quad \forall t \in [0, T].
\] (3.9)

Denote
\[
C = \left( \frac{|\mu_2|}{2} \int_0^t \int_\Omega f_0^2(x) \rho dx + \mathcal{E}(u(0)) \right) e^{2(|\mu_2| + |\mu_1|)T}.
\]

Then we have \( \mathcal{E}(u(t)) \leq C \). \( \square \)

With the above energy estimate, we can prove a uniqueness result for weak solution of problem (1.4) as the following.

**Theorem 3.2.** (uniqueness of solution). There exists at most one solution of problem (1.4) in the sense of definition 2.2.

**Proof.** It suffices to show that the only weak solution of (1.4) with \( u_0 = u_1 = f_0 = 0 \) is
\[ u \equiv 0. \] (3.10)

According to the energy estimate (3.9) in Theorem 3.1, and noting that \( f_0 = 0, \; \mathcal{E}(u(0)) = 0 \), we obtain
\[ \mathcal{E}(u(t)) = 0, \; \forall t \in [0, T]. \] (3.11)

So, we have
\[ \|u(t)\|_2 = \|\nabla u(t)\|_2 = 0, \; \forall t \in [0, T]. \] (3.12)

And this implies (3.10). Thus, we conclude that problem (1.4) has at most one solution. \( \square \)

Next, we give an existence result for the weak solution of problem (1.4) by making use of the Faedo–Galerkin’s method in the following theorem.

**Theorem 3.3.** (Existence of weak solution). Assume that (A1) and (A2) hold. Then for any \( u_0 \in H^1_0(\Omega), u_1 \in L^2(\Omega), f_0(x, t) \in L^2(\Omega \times (-\tau, 0)) \) and \( T > 0 \), there exists a weak solution \( u(x, t) \) of problem (1.4) defined on \( \Omega \times (0, T). \)

**Proof.** We prove this theorem by Faedo–Galerkin’s method. To this end, we denote by \( \{\lambda_j\}_{j=1}^{\infty} \) the eigenvalue sequence of the following eigenvalue problem:
\[
\begin{align*}
\begin{cases}
-\Delta w = \lambda w, & x \in \Omega, \\
w = 0, & x \in \partial \Omega,
\end{cases}
\end{align*}
\] (3.13)

and \( \{w_j(x)\}_{j=1}^{\infty} \) the eigenfunction sequence associated with \( \{\lambda_j\}_{j=1}^{\infty} \) such that

\[
\int_{\Omega} w_j(x)w_k(x)dx = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases}
\] (3.14)

Hence,

\[
\int_{\Omega} \nabla w_j(x)\nabla w_k(x)dx = \begin{cases} \lambda_j, & j = k, \\ 0, & j \neq k. \end{cases}
\] (3.15)

It is well known that \( \{w_j\}_{j=1}^{\infty} \) is an orthogonal bases of \( L^2(\Omega) \) and of \( H_0^1(\Omega) \).

Since \( u_0 \in H_0^1(\Omega), u_1 \in L^2(\Omega) \) and \( f_0(x, t) \in L^2(\Omega \times (-\tau, 0)) \), we have the following expansion

\[
u_0(x) = \sum_{j=1}^{\infty} a_j w_j(x), \quad u_1(x) = \sum_{j=1}^{\infty} b_j w_j(x), \quad f_0(x, t) = \sum_{j=1}^{\infty} f_j(t) w_j(x)
\]

with

\[
a_j = (u_0(x), w_j(x)), \quad b_j = (u_1(x), w_j(x)), \quad f_j(t) = (f_0(x, t), w_j(x)).
\]

To construct an approximate solution of problem (1.4), for any fixed integer \( j \geq 1 \), we consider the following initial value problem of second-order ordinary differential equation

\[
\begin{align*}
\ddot{\gamma}_j(t) + \mu_1 \dot{\gamma}_j(t) + \mu_2 \gamma_j(t - \tau) + \lambda_j \left( 1 - \int_0^t g(s)ds \right) \gamma_j(t) &= 0, \\
\gamma_j(0) &= a_j, \quad \dot{\gamma}_j(0) = b_j,
\end{align*}
\] (3.16)

and prove that it has a unique \( H^1 \) solution on \( [0, T] \) for any given \( T > 0 \).

We first note that there exists a positive constant \( C \) independent of \( j \) such that any solution \( \gamma_j(t) \) of problem (3.16) defined on \( [0, T] \) satisfies

\[
\lambda_j \int_0^T \left( 1 - \int_0^{\sigma} g(s)ds \right) \gamma_j^2(\sigma)d\sigma \leq C, \quad \int_0^T \dot{\gamma}_j^2(\sigma)d\sigma \leq C,
\] (3.17)

which can be obtained by multiplying the differential equation in (3.16) by \( \dot{\gamma}_j(t) \) and then integrating the resulting equation on \( [0, T] \) (See the proof of Theorem 3.1 for a similar process).

By (A1), we have \( 1 - \int_0^\sigma g(s)ds > l \). In addition, as the eigenvalue sequence of (3.13), \( \{\lambda_j\}_{j=1}^{\infty} \) has a lower bound. Thus, we also have

\[
\int_0^T \gamma_j^2(s)ds \leq C.
\] (3.18)

And these estimates imply that \( \gamma_j(t) \in L^2(0, T) \) and \( \dot{\gamma}_j(t) \in L^2(0, T) \). In other words, \( \gamma_j(t) \in H^1(0, T) \).

Now, we divide the proof into two steps.

At first, we prove that problem (3.16) has a unique \( H^1 \) solution on some small interval \( [0, \delta_0] \). In fact, if we set \( \delta_0 = \min \left\{ \frac{1}{2\max\{1+|\mu_1|, \lambda_j\}}, \tau \right\}, \ y_1(t) = \gamma_j(t) \) and \( y_2(t) = \dot{\gamma}_j(t) \), then (3.16) is equivalent to the integral system on \( [0, \delta_0] \) as follows:
\begin{equation}
\begin{aligned}
y_1(t) &= \int_0^t y_2(s) ds + a_j, \\
y_2(t) &= -\lambda_j \int_0^t \left(1 - \int_0^\sigma g(s) ds\right) y_1(\sigma) d\sigma - \mu_1 \int_0^t y_2(s) ds + \tilde{y}(t),
\end{aligned}
\tag{3.19}
\end{equation}

where \( \tilde{y}(t) = -\mu_2 \int_0^t (f_0(s - \tau), w_j) ds + b_j. \)

Let \( X = L^2(0, \delta_0) \times L^2(0, \delta_0) \) be the Banach space endowed with the graph norm \( \| (\cdot, \cdot) \|_X = \| \cdot \|_2 + \| \cdot \|_2. \)

For any \( \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right) \in X, \) we define the following mapping

\[ \Psi \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right) := \left( \begin{array}{c} \int_0^t y_2(s) ds + a_j \\ \left( -\int_0^t \lambda_j \left(1 - \int_0^\sigma g(s) ds\right) y_1(\sigma) d\sigma - \mu_1 \int_0^t y_2(s) ds + \tilde{y}(t) \right) \end{array} \right). \]

It follows from the estimate (3.17) that \( \Psi \) maps \( X \) into \( X. \)

Next, we prove that \( \Psi \) is a contraction mapping. In fact, for any \( (y_1, y_2), (y_1^*, y_2^*) \in X, \) we have

\[ \left\| \Psi \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right) - \Psi \left( \begin{array}{c} y_1^* \\ y_2^* \end{array} \right) \right\|_X = \left\| -\lambda_j \int_0^t \left(1 - \int_0^\sigma g(s) ds\right) (y_1(\sigma) - y_1^*(\sigma)) d\sigma + \mu_1 \int_0^t (y_2(s) - y_2^*(s)) ds \right\|_2 \\
+ \left\| \int_0^t (y_2(s) - y_2^*(s)) ds \right\|_2 \\
\leq \int_0^t (y_2(s) - y_2^*(s)) ds \right\|_2 + \mu_1 \int_0^t (y_2(s) - y_2^*(s)) ds \right\|_2 \\
+ \lambda_j \int_0^t \left(1 - \int_0^\sigma g(s) ds\right) (y_1(\sigma) - y_1^*(\sigma)) d\sigma \right\|_2. \]

Since

\[ \left\| \int_0^t (y_2(s) - y_2^*(s)) ds \right\|_2 = \left\{ \int_0^t \left( \int_0^\sigma (y_2(s) - y_2^*(s))^2 ds \right) d\sigma \right\}^{\frac{1}{2}} \leq \left\{ \int_0^t \left( \int_0^\sigma 1^2 ds \right) \left( \int_0^\sigma (y_2(s) - y_2^*(s))^2 ds \right) d\sigma \right\}^{\frac{1}{2}} \leq t \|y_2(t) - y_2^*(t)\|_2, \]

we easily deduce that

\[ \left\| \mu_1 \int_0^t (y_2(s) - y_2^*(s)) ds \right\| \leq |\mu_1| t \|y_2(t) - y_2^*(t)\|_2. \]
and
\[ \left\| \lambda_j \int_0^t \left( 1 - \frac{\sigma}{j} \int_0^\sigma g(s) \, ds \right) (y_1(\sigma) - y_1^*(\sigma)) \, d\sigma \right\|_2 \leq \lambda_j t \|y_1(t) - y_1^*(t)\|_2. \]

Hence, we have
\[ \left\| \Psi \left( \frac{y_1}{y_2} \right) - \Psi \left( \frac{y_1^*}{y^*_2} \right) \right\|_X \leq \max \{1 + |\mu_1|, \lambda_j\} t \left\| \frac{y_1}{y_2} - \frac{y_1^*}{y^*_2} \right\|_X. \]

Since \( t \in [0, \delta_0] \) and \( \delta_0 \leq \frac{1}{2 \max\{1 + |\mu_1|, \lambda_j\}} \), we get
\[ \left\| \Psi \left( \frac{y_1}{y_2} \right) - \Psi \left( \frac{y_1^*}{y^*_2} \right) \right\|_X \leq \frac{1}{2} \left\| \frac{y_1}{y_2} - \frac{y_1^*}{y^*_2} \right\|_X. \]  

Hence, by the contraction mapping principle, \( \Psi \) has a unique fixed point in \( X \). Therefore, problem (3.16) has a unique \( H^1 \) solution on \([0, \delta_j]\).

In the second, we prove that the local solution of problem (3.16) obtained in the first step can be extended to \([0, T]\) for any given \( T > 0 \). To this end, we first note that \( \delta_0 \leq \tau \) and \( t - \tau \leq t - \delta_0 < \frac{5\delta_0}{3} - \delta_0 = \frac{2\delta_0}{3} \) for all \( t \in (\frac{2\delta_0}{3}, \frac{5\delta_0}{3}) \). Hence, the derivative \( \gamma_j(t - \tau) \) of the local solution \( \gamma_j \) of problem (3.16) obtained in the first step is well defined on \((\tau, \frac{5\delta_0}{3})\). Taking (3.17) and (3.18) into account, by a similar process used in the first step, we can easily prove that the following initial value problem
\[
\begin{cases}
\dot{\gamma}_j(t) + \mu_1 \dot{\gamma}_j(t) + \mu_2 \gamma_j(t - \tau) + \lambda_j \left( 1 - \int_0^t g(s) \, ds \right) \eta_j(t) = 0, \\
\eta_j(\frac{2\delta_0}{3}) = \gamma_j(\frac{2\delta_0}{3}), \\
\gamma_j(\frac{2\delta_0}{3}) = \dot{\gamma}_j(\frac{2\delta_0}{3}), \\
\gamma_j(t - \tau) = \dot{\gamma}_j(t - \tau), \quad t \in (\tau, \frac{5\delta_0}{3}),
\end{cases}
\]
has a unique solution \( \eta_j(t) \) in \( H^1(\frac{2\delta_0}{3}, \frac{5\delta_0}{3}) \). Moreover, by the uniqueness, we have \( \eta_j(t) = \gamma_j(t) \) on \([\frac{2\delta_0}{3}, \delta_0]\).

This implies that \( \gamma_j(t) \) can be extended to the interval \((0, \frac{5\delta_0}{3})\) as a \( H^1 \) solution of problem (3.16). By the compactness of \([0, T]\), we conclude that the local solution \( \gamma_j(t) \) of problem (3.16) obtained in the first step can be extended to the whole interval \([0, T]\) as a \( H^1 \) solution of problem (3.16) by repeating the above step finite times.

Now, let \( \gamma_j(t) \) be the solution of problem (3.16) defined on \([0, T]\). We set
\[ u_n(x, t) = \sum_{j=1}^n \gamma_j(t) w_j(x). \]  

Then, by (3.13), (3.14), (3.15) and (3.16), we can easily see that \( u_n(x, t) \in H^1(\Omega \times (0, T)) \) and satisfies
\[
\begin{cases}
\left\| u_n \right\|_2^2 \leq C, \\
\left\| u_n(t) \right\|_2^2 \leq C, \\
\left\| \nabla u_n \right\|_2^2 \leq C
\end{cases}
\]
for some positive constant \( C \) independent of \( n \). And from this, we see that the sequence \( \{u_n\}_{n=1}^\infty \) is bounded in \( L^2 \left( 0, T; H^1_0(\Omega) \right) \), \( \{u_n\}_{n=1}^\infty \) is bounded in \( L^2 \left( 0, T; L^2(\Omega) \right) \). As a consequence, there exists a subsequence \( \{u_k\}_{k=1}^\infty \subset \{u_n\}_{n=1}^\infty \) and \( u \in L^2 \left( 0, T; H^1_0(\Omega) \right) \), with \( u_n \to u \) weakly in \( L^2 \left( 0, T; H^1_0(\Omega) \right) \), as \( k \to \infty \) \( (3.24) \).
and

\[ u_{kt} \to u_t \text{ weakly in } L^2(0,T;L^2(\Omega)), \text{ as } k \to \infty. \]  

(3.25)

Next, arguing as Theorem 3.1 in [23], we will prove that \( u \) is a weak solution of problem (1.4). For this, we choose a function \( v \in \{ \eta | \eta \in C^1(0,T;H^1_0(\Omega)), \eta(x,T) = 0, \eta_t(x,T) = 0 \} \) of the form

\[ v(t) = \sum_{j=1}^{N} \gamma_j(t)w_j, \]  

(3.26)

where \( \{ \gamma_j \}_{j=1}^{N} \) are smooth functions, and \( N \) is a fixed integer.

We select \( k \geq N \), multiply the first equation of (3.23) by \( \gamma_j(t)w_j \), sum \( j = 1, \ldots, N \), and then integrate with respect to \( t \), to discover

\[ I + II = 0, \]  

(3.27)

where

\[ I = \int_{0}^{T} \left[ -(u_{tt},v_t) + (\nabla u_t(t), \nabla v(t)) - \int_{0}^{t} g(t-s)(\nabla u_k(s), \nabla v(s))ds \right] dt \]

and

\[ II = \int_{0}^{T} \left[ \mu_1(u_{tt}(t),v) + \mu_2(u_{tt}(t-\tau),v) \right] dt + (v(x,0),u_k(x,0)). \]

By (3.24), (3.25) and passing to the limit in (3.27) as \( k \to \infty \), we obtain

\[ \int_{0}^{T} \left[ -(u_{tt},v_t) + (\nabla u_t(t), \nabla v(t)) - \int_{0}^{t} g(t-s)(\nabla u_k(s), \nabla v(s))ds \right] dt \]

\[ + \int_{0}^{T} \left[ \mu_1(u_t(t),v) + \mu_2(u_t(t-\tau),v) \right] dt + (v(x,0),u_1(x)) = 0 \]

(3.28)

for all functions \( v \in V \), since functions of the form (3.26) are dense in this space. And thus, (2.4) holds.

We must now verify that the limit function \( u \) satisfies the initial conditions and the history value, i.e.,

\[ u(0) = u_0, \quad u_t(0) = u_1, \]  

(3.29)

\[ u_t(x,t-\tau) = f_0(x,t-\tau), \quad t \in (0,\tau). \]  

(3.30)

For this, choose any function \( v \in C^2(0,T;H^1_0(\Omega)) \), with \( v(T) = v_t(T) = 0 \). Then integrating by parts with respect to \( t \) in (3.28), we find

\[ \int_{0}^{T} \left[ (v_{tt},u) + (\nabla u(t), \nabla v(t)) - \int_{0}^{t} g(t-s)(\nabla u(s), \nabla v(s))ds \right] dt \]

\[ + \int_{0}^{T} \left[ \mu_1(u_t(t),v) + \mu_2(u_t(t-\tau),v) \right] dt = (u(0),v_t(0)) - (u_t(0),v(0)). \]

(3.31)
Similarly, from (3.27), we deduce
\[
\int_{0}^{T} \left[ (v_{tt}, u_k) + (\nabla u_k(t), \nabla v(t)) - \int_{0}^{t} g(t-s)(\nabla u_k(s), \nabla v(s))ds \right] dt \\
+ \int_{0}^{T} [\mu_1(u_{kt}(t), v) + \mu_2(u_{kt}(t-\tau), v)] dt = (u_k(0), v(0)) - (u_k(0), v(0)).
\] (3.32)

Recalling (3.24), (3.25) and the initial data in (3.16), we get
\[
\int_{0}^{T} \left[ (v_{tt}, u) + (\nabla u(t), \nabla v(t)) - \int_{0}^{t} g(t-s)(\nabla u(s), \nabla v(s))ds \right] dt \\
+ \int_{0}^{T} [\mu_1(u_t(t), v) + \mu_2(u_t(t-\tau), v)] dt = (u_0, v_t(0)) - (u_1, v(0)),
\] (3.33)
as \( k \to \infty \). Comparing identities (3.31) and (3.33), we conclude (3.29), since \( v(0), v_t(0) \) are arbitrary. In addition, \( \forall j, \int_{\Omega} u_{kt}(t-\tau)w_j dx \to \int_{\Omega} f_0(t-\tau)w_j dx \) weakly in \( L^2(0, T; L^2(\Omega)) \) as \( k \to \infty \). So we easily deduce \( u_t(x, t-\tau) = f_0(x, t-\tau) \). And then (3.30) holds. Hence \( u \) is a weak solution of (1.4) in the sense of Definition 2.2. \( \square \)

4. Exponential energy decay in the case \( \mu_1 = 0 \)

In this section, we will give the exponential energy decay result about the problem (1.4) in the case \( \mu_1 = 0 \). More precisely, we will study the following system
\[
\begin{aligned}
&u_{tt}(x, t) - \Delta u(x, t) + \int_{0}^{t} g(t-s)\Delta u(x, s)ds + \mu u_t(x, t-\tau) = 0, \quad x \in \Omega, \quad t > 0, \\
u(x, t) = 0, \quad x \in \partial \Omega, \quad t \geq 0, \\
u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \\
u_t(x, t-\tau) = f_0(x, t-\tau), \quad x \in \Omega, \quad t \in (0, \tau),
\end{aligned}
\] (4.1)
and deduce an exponential decay result for its modified energy.

We first define the classical energy by
\[
e(t) = \frac{1}{2} \left( \|u_t\|_2^2 + \|\nabla u\|_2^2 \right), \quad t \geq 0.
\]
Then by the first equation in (4.1), it is easy to see that
\[
e'(t) = \int_{\Omega} \nabla u_t \int_{0}^{t} g(t-s)\nabla u(s)dsdx - \mu \int_{\Omega} u_t(t)u_t(t-\tau)dx, \quad t \geq 0.
\]
As the integral item in the above formula is of an undefined sign, we will run into trouble in the process of decay estimate.
By (2.2) and inspired by [18], we define the new modified energy functional of problem (4.1) as follows:

\[
E(t) = E(u(x, t)) = \frac{1}{2} \left( \|u_t\|^2_2 + \left( 1 - \int_0^t g(s) ds \right) \|\nabla u\|^2_2 + (g \circ \nabla u)(t) \right) + \frac{\xi}{2} \int_{t-\tau}^t \int_\Omega e^{\sigma(s-t)} u_{2s}^2(x, s) dx ds,
\]

where \(\sigma\) and \(\xi\) are two positive constants to be determined later.

In order to achieve our goal, we need to differentiate \(E(t)\) with respect to \(t\). This requires that the derivatives \(u_{tt}\) and \(u_{tx}\) make sense in \(L^2(\Omega \times (0, T))\), for any \(T > 0\). Hence, we first state a regularity result for the weak solution of problem (1.4) obtained in Theorem 3.3.

**Theorem 4.1.** Assume that (A1) and (A2) hold. If \(u_0 \in H^1_0(\Omega) \cap H^2(\Omega), u_1 \in H^1_0(\Omega)\) and \(f_0(x, t) \in H^1((-\tau, 0); H^1_0(\Omega))\), then the weak solution \(u(x, t)\) of problem (1.4) obtained in Theorem 3.3 belongs to \(H^2(\Omega \times (0, T))\).

The above regularity result can be proved by making use of Faedo–Galerkin’s method in a similar way as that used in [24] (Theorem 4.1, Section 4, Chapter IV). Hence, we omit the detailed proof here.

Based on the above regularity result, we may assume that \(u_0(x), u_1(x)\) and \(f_0(x, t)\) smooth enough so that the solution \(u(x, t)\) of problem (1.4) belongs to \(H^2(\Omega \times (0, T))\) and deduce an exponential energy decay result for \(H^2\) solution \(u(x, t)\) of problem (1.4).

Once the exponential energy decay result for \(H^2\) solution is established, we can deduce a similar decay result for weak solution \(u(x, t)\) of problem (1.4) obtained in Theorem 3.3. In fact, if \(u_0(x), u_1(x), f_0(x, t)\) satisfy only \(u_0 \in H^1_0(\Omega), u_1 \in L^2(\Omega)\) and \(f_0(x, t) \in L^2(\Omega \times (-\tau, 0))\), we may choose \(u_{0j}(x) \in H^1_0(\Omega) \cap H^2(\Omega), u_{1j}(x) \in H^1_0(\Omega)\) and \(f_{0j}(x, t) \in H^1((-\tau, 0); H^1_0(\Omega))\) such that

\[
\begin{align*}
&u_{0j}(x) \to u_0(x) \text{ as } j \to +\infty \text{ in } H^1_0(\Omega), \\
&u_{1j}(x) \to u_1(x) \text{ as } j \to +\infty \text{ in } L^2(\Omega)
\end{align*}
\]

and

\[
\begin{align*}
&f_{0j}(x, t) \to f_0(x, t) \text{ as } j \to +\infty \text{ in } L^2(\Omega \times (-\tau, 0)).
\end{align*}
\]

Let \(u_j(x, t)\) be the \(H^2\) solution of problem (1.4) with initial data \(u_{0j}(x), u_{1j}(x)\) and \(f_{0j}(x, t)\), and \(u(x, t)\) be the weak solution of problem (1.4) with initial data \(u_0(x), u_1(x)\) and \(f_0(x, t)\). Then, \(u_j(x, t) - u(x, t)\) is a weak solution to a problem similar to problem (1.4) with initial data \(u_{0j} - u_0, u_{1j} - u_1\) and \(f_{0j} - f_0\). Thus, it follows from the a priori estimate obtained in Theorem 3.1 that

\[
\lim_{j \to +\infty} E(u_j(x, t)) = E(u(x, t)).
\]

This obviously implies that the energy decay result for weak solution can be deduced from the energy decay result for \(H^2\) solution.

Consequently, no loss of generality, we may assume that the solution \(u(x, t)\) of problem (1.4) belongs to \(H^2(\Omega \times (0, T))\) so that all the computation in the following paragraph make sense in \(L^2(\Omega \times (0, T))\).

Since the function \(g\) is positive, continuous and \(g(0) > 0\), then for any \(t \geq t_0 > 0\), we have

\[
\int_0^t g(s) ds \geq \int_0^{t_0} g(s) ds = g_0 > 0.
\]

And this fact will be used subsequently in the proof of our main result.

Our main result reads as follows:
Theorem 4.2. (Exponential energy decay) Assume that $g$ satisfies (A1) and (A2). Let $\alpha$ and $\zeta_0$ be the constants defined by (4.31) and (4.35), respectively. If $|\mu| < \alpha$ and $\zeta > \zeta_0$, then there exists two positive constants $K$ and $\lambda$ such that

$$E(t) \leq Ke^{-\lambda(t-t_0)}, \quad \forall t \geq t_0,$$  \tag{4.4}

where the constant $\zeta$ satisfies the assumption (A2).

Proof. We define the Lyapunov functional

$$L(t) := E(t) + \varepsilon_1 \Psi(t) + \varepsilon_2 \chi(t),$$  \tag{4.5}

where $\varepsilon_i, i = 1, 2$ are two positive real numbers which will be chosen later, and

$$\Psi(t) := \int_{\Omega} uu_t dx,$$  \tag{4.6}

$$\chi(t) := -\int_{\Omega} u \int_0^t g(t-s)(u(t) - u(s))ds dx.$$  \tag{4.7}

We first deduce that, for $\varepsilon_1$ and $\varepsilon_2$ small enough, the Lyapunov functional $L(t)$ and the energy $E(t)$ are equivalent in the sense that there exist two positive constants $\beta_1, \beta_2$ such that

$$\beta_1 E(t) \leq L(t) \leq \beta_2 E(t), \quad \forall t \geq 0.$$  \tag{4.8}

In fact, using Young’s inequality, Poincaré’s inequality and (2.3), we have

$$\varepsilon_2 |\chi(t)| \leq \frac{\varepsilon_2}{2} \|u_t\|^2_2 + \frac{\varepsilon_2 (1 - l) C^2_2}{2} (g \circ \nabla u)(t)$$

and

$$\varepsilon_1 |\Psi(t)| \leq \frac{\varepsilon_1}{2} \|u_t\|^2_2 + \frac{\varepsilon_1 C^*_2}{2} \|\nabla u\|^2_2.$$  

Clearly, by (4.2) and (4.5), for all $\varepsilon_i > 0, i = 1, 2$, there exists a positive constant $\beta_2$ such that $L(t) \leq \beta_2 E(t)$. And for $\varepsilon_i > 0$ small enough, there exists a positive constant $\beta_1$ such that $\beta_1 E(t) \leq L(t)$. Thus, (4.8) holds.

Now, we will estimate the derivative of $L(t)$ according to the following steps.

Step 1: Estimate of the derivative of $E(t)$.

Differentiating (4.2), we have

$$E'(t) = \int_{\Omega} \left( u_t u_{tt} + \left( 1 - \int_0^t g(s)ds \right) \nabla u \nabla u_t - \frac{1}{2} g(t) |\nabla u|^2 \right) dx$$

$$+ \int_0^t g(t-s) \int_{\Omega} \nabla u_t(t) (\nabla u(t) - \nabla u(s)) dx ds$$

$$+ \frac{1}{2} \int_0^t g'(t-s) \int_{\Omega} |\nabla u(t) - \nabla u(s)|^2 dx ds$$

$$+ \frac{\xi}{2} \int_{\Omega} u_1^2(x,t) dx - \frac{\xi}{2} \int_{\Omega} e^{-\sigma \tau} u_1^2(x,t-\tau) dx$$

$$- \frac{\sigma \xi}{2} \int_{t-\tau}^t \int_{\Omega} e^{-\sigma(t-s)} u_3^2(x,s) dx ds.$$  \tag{4.9}
Using the first equation of (4.1), we obtain

\[
E'(t) = \frac{1}{2} (g' \circ \nabla u)(t) - \frac{1}{2} g(t)\|\nabla u\|_2^2 - \mu \int_\Omega u_t(t)(t-\tau)dx + \frac{\xi}{2} \|u_t\|_2^2
\]

\[-\frac{\xi}{2}e^{-\sigma \tau} \int_\Omega u_t^2(x, t-\tau)dx - \frac{\sigma \xi}{2} \int_t^{t-\tau} \int_\Omega e^{-\sigma(t-s)}u_s^2(x, s)dxds. \tag{4.10}
\]

And then, by Cauchy inequalities, we get

\[
E'(t) \leq \frac{1}{2} (g' \circ \nabla u)(t) + \left( \frac{|\mu|}{2} + \frac{\xi}{2} \right) \|u_t\|_2^2 + \left( \frac{|\mu|}{2} - \frac{\xi}{2} e^{-\sigma \tau} \right) \int_\Omega u_t^2(x, t-\tau)dx
\]

\[-\frac{1}{2} g(t)\|\nabla u\|_2^2 - \frac{\sigma \xi}{2} \int_t^{t-\tau} \int_\Omega e^{-\sigma(t-s)}u_s^2(x, s)dxds. \tag{4.11}
\]

**Step 2:** Estimate of the derivative of \(\Psi(t)\).

Differentiating and integrating by parts, we obtain

\[
\Psi'(t) = \|u_t\|_2^2 + \int_\Omega u \left( \Delta u - \int_0^t g(t-s)\Delta u(s)ds - \mu u_t(t-\tau) \right) dx
\]

\[= \|u_t\|_2^2 + \int_\Omega \nabla u \left( \int_0^t g(t-s)(\nabla u(s) - \nabla u(t)) ds dx \right.
\]

\[+ \left( \int_0^t g(s)ds - 1 \right) \|\nabla u\|_2^2 - \mu \int_\Omega u(t)u_t(t-\tau)dx
\]

\[\leq \int_\Omega \nabla u \left( \int_0^t g(t-s)(\nabla u(s) - \nabla u(t)) ds \right.
\]

\[+ \|u_t\|_2^2 - l\|\nabla u\|_2^2 - \mu \int_\Omega u(t)u_t(t-\tau)dx. \tag{4.12}
\]

By Young’s inequality and Lemma 2.1, we get (see [18])

\[
\int_\Omega \nabla u \left( \int_0^t g(t-s)(\nabla u(s) - \nabla u(t)) ds \right.
\]

\[\leq \delta_1 \|\nabla u\|_2^2 + \frac{1}{4\delta_1} \int \left( \int_0^t g(t-s)|\nabla u(s) - \nabla u(t)|ds \right)^2 dx
\]

\[\leq \delta_1 \|\nabla u\|_2^2 + \left( \frac{1-l}{4\delta_1} \right) C^2 \int (g \circ \nabla u)(t) \tag{4.13}
\]

\((\forall \delta_1 > 0) \)

Also, using Young’s and Poincaré’s inequalities, we have

\[-\mu \int_\Omega u(t)u_t(t-\tau)dx \leq \delta_1 \|\nabla u\|_2^2 + C(\delta_1) \int_\Omega u_t^2(t-\tau)dx. \tag{4.14}
\]
Combining (4.13)–(4.14) and choosing $\delta_1$ small enough, the estimate
\[
\Psi'(t) \leq -\frac{1}{2} \|\nabla u\|^2_2 + C_1 \int_\Omega (u(t^2 + u(t - \tau)) \, dx + C_2 (g \circ \nabla u)(t) \tag{4.15}
\]
holds for some positive constants $C_i, i = 1, 2$.

**Step 3:** Estimate of the derivative of $\chi(t)$.

Differentiating (4.7) and integrating by parts, we have
\[
\chi'(t) = \left(1 - \int_0^t g(s) \, ds\right) \int_\Omega \nabla u \int_0^t g(t - s)(\nabla u(t) - \nabla u(s)) \, ds \, dx
\]
\[
+ \int_\Omega \left(\int_0^t g(t - s)(\nabla u(s) - \nabla u(t)) \, ds\right)^2 \, dx - \int_\Omega u_t \int_0^t g'(t - s) (u(t) - u(s)) \, ds \, dx
\]
\[
- \int_0^t g(s) ds \|u_t\|^2_2 + \mu \int_\Omega u_t(t - \tau) \int_0^t g(t - s)(u(t) - u(s)) \, ds \, dx. \tag{4.16}
\]

We first estimate the second item of (4.16) as follows:
\[
\int_\Omega \left(\int_0^t g(t - s)(\nabla u(s) - \nabla u(t)) \, ds\right)^2 \, dx
\]
\[
= \int_\Omega \left(\int_0^t \sqrt{g(t - s)} \left(\sqrt{g(t - s)}(\nabla u(s) - \nabla u(t)) \right) \, ds\right)^2 \, dx
\]
\[
\leq \int_\Omega \left(\int_0^t g(t - s) \, ds\right) \left(\int_0^t g(t - s)(\nabla u(s) - \nabla u(t))^2 \, ds\right) \, dx
\]
\[
\leq (1 - l)(g \circ \nabla u)(t).
\]

Then, using Young’s inequality and Lemma 2.1, we get (see [18,19])
\[
\left(1 - \int_0^t g(s) \, ds\right) \int_\Omega \nabla u \int_0^t g(t - s)(\nabla u(t) - \nabla u(s)) \, ds \, dx
\]
\[
\leq \delta_2 \|\nabla u\|^2_2 + \frac{C_3}{\delta_2} (g \circ \nabla u)(t), \quad (\forall \delta_2 > 0) \tag{4.17}
\]
\[
- \int_\Omega u_t \int_0^t g'(t - s) (u(t) - u(s)) \, ds \, dx \leq \delta_2 \|u_t\|^2_2 - \frac{C_4}{\delta_2} (g' \circ \nabla u)(t) \tag{4.18}
\]
and
\[
\mu \int_\Omega u_t(t - \tau) \int_0^t g(t - s)(u(t) - u(s)) \, ds \, dx \leq \frac{C_5}{\delta_2} (g \circ \nabla u)(t) + \delta_2 \int_\Omega u_t^2(t - \tau) \, dx \tag{4.19}
\]
where $C_i, i = 3, 4, 5$ are some positive constants.
Combining (4.16)–(4.19) and (2.3), we obtain

\[ \chi'(t) \leq \left( \delta_2 - \int_0^t g(s) ds \right) \left\| u_t \right\|^2 \| u \|^2 + \frac{C_6}{\delta_2} (g \circ \nabla u)(t) + \frac{C_7}{\delta_2} (g' \circ \nabla u)(t) + \delta_2 \int_\Omega u_t^2(t - \tau) d\tau, \]

where \( C_i, i = 6, 7 \) are some positive constants.

**Step 4:** Estimate of the derivative of \( L(t) \).

By using (4.3), (4.5), (4.11), (4.15) and (4.20), a series of computations yields, for \( t \geq t_0 \),

\[ L'(t) \leq \left( \frac{|\mu|}{2} + \frac{\xi}{2} + \varepsilon_1 C_1 + \varepsilon_2 (\delta_2 - g_0) \right) \left\| u_t \right\|^2 + \left( \varepsilon_2 \delta_2 - \frac{\varepsilon_1 l_2}{2} \right) \| \nabla u \|^2

+ \left( \frac{1}{2} - \frac{\varepsilon_2 C_7}{\delta_2} \right) (g' \circ \nabla u)(t) + \left( \varepsilon_1 C_1 + \varepsilon_2 \delta_2 + \frac{|\mu|}{2} - \frac{\xi}{2e^{\sigma \tau}} \right) \int_\Omega u_t^2(t - \tau) d\tau

+ \left( \varepsilon_1 C_2 + \frac{\varepsilon_2 C_6}{\delta_2} \right) (g \circ \nabla u)(t) - \frac{\sigma \xi}{2} \int_{t-\tau}^t \int_\Omega e^{-\sigma(t-\tau)} u_s^2(s) d\tau ds. \]

(4.21)

Now, we deduce that, for the positive constants \( \xi, \delta_2, \varepsilon_1, \varepsilon_2 \) and \( \sigma \), the following system of inequalities

\[ \left\{ \begin{array}{l}
\frac{|\mu|}{2} + \frac{\xi}{2} + \varepsilon_1 C_1 + \varepsilon_2 (\delta_2 - g_0) < 0, \\
\varepsilon_2 \delta_2 - \frac{\varepsilon_1 l_2}{2} < 0, \\
\frac{1}{2} - \frac{\varepsilon_2 C_7}{\delta_2} > 0, \\
\varepsilon_1 C_1 + \varepsilon_2 \delta_2 + \frac{|\mu|}{2} - \frac{\xi}{2e^{\sigma \tau}} < 0,
\end{array} \right. \]

(4.22)

is solvable only if we add some suitable conditions to \( |\mu| \).

In fact, we can find solutions of (4.22) according to the following steps.

**Step 1:** We first pick \( \delta_2 \) small enough such that

\[ \delta_2 < \min \left\{ \frac{g_0}{4}, \frac{l g_0}{16 C_1} \right\}. \]

(4.23)

Thus, we have

\[ \frac{g_0}{8 C_1} < \frac{g_0 - 2 \delta_2}{2 C_1}. \]

**Step 2:** As long as \( \delta_2 \) is fixed, we select \( \varepsilon_2 \) such that

\[ 0 < \varepsilon_2 < \frac{\delta_2}{2 C_7}. \]

(4.24)

Then we get

\[ \frac{1}{2} - \frac{\varepsilon_2 C_7}{\delta_2} > 0. \]

**Step 3:** Next, we choose \( \varepsilon_1 \) satisfies the relation

\[ \varepsilon_2 \cdot \frac{g_0}{8 C_1} < \varepsilon_1 < \varepsilon_2 \cdot \frac{g_0 - 2 \delta_2}{2 C_1}. \]

(4.25)

By (4.23) and (4.25), we get

\[ \varepsilon_2 (g_0 - \delta_2) - \varepsilon_1 C_1 > \varepsilon_1 C_1 + \varepsilon_2 \delta_2 > 0 \]

(4.26)
and
\[ \varepsilon_2 \delta_2 - \frac{\varepsilon_1 l}{2} < 0. \quad (4.27) \]

**Step 4:** Now, we must ensure that the first inequality and the fourth one in (4.22) hold. That is to say, for the positive constants $|\mu|, \sigma, \xi$, the following system of inequalities
\[
\begin{cases}
\frac{1}{2} |\mu| + \frac{1}{2} \xi < \varepsilon_2 (g_0 - \delta_2) - \varepsilon_1 C_1, \\
-\frac{1}{2} |\mu| + \frac{1}{e^{\sigma \tau}} \xi > \varepsilon_1 C_1 + \varepsilon_2 \delta_2,
\end{cases}
\quad (4.28)
\]
must be solvable.

Let $k_1 = \varepsilon_2 (g_0 - \delta_2) - \varepsilon_1 C_1$ and $k_2 = \varepsilon_1 C_1 + \varepsilon_2 \delta_2$. Thus, by (4.26), $k_1$ and $k_2$ are two positive constants depending on $g_0$. And then the system (4.28) becomes
\[
\begin{cases}
|\mu| + \xi < 2k_1, \\
-|\mu| + \frac{1}{e^{\sigma \tau}} \xi > 2k_2,
\end{cases}
\quad (4.29)
\]
By (4.26), we have $k_1 > k_2$. Note that $e^{\sigma \tau} \to 1$ as $\sigma \to 0$. Thus, if we choose $\sigma$ small enough, there exists a positive constant $\xi$ such that
\[ 2k_2 e^{\sigma \tau} < \xi < 2k_1. \quad (4.30) \]
And then, we have $2k_1 - \xi > 0$ and $\frac{\xi}{e^{\sigma \tau}} - 2k_2 > 0$.

Thus, the system of inequalities (4.29) is solvable if we choose
\[ |\mu| < \min\{2k_1 - \xi, \frac{\xi}{e^{\sigma \tau}} - 2k_2\} =: a. \quad (4.31) \]
Here, $a$ is only dependent on $g_0$.

Therefore, (4.22) is solvable under the condition (4.31).

Consequently, from (4.21), there exist two positive constants $\gamma_1$ and $\gamma_2$ such that
\[
\frac{dL(t)}{dt} \leq -\gamma_1 E(t) + \gamma_2 (g \circ \nabla u)(t), \quad \forall t \geq t_0. \quad (4.32)
\]

Similar to the steps in [18], the remaining part of the proof of inequality (4.4) can be finished. For reader’s convenience, we here write the details as follows:

By multiplying (4.32) by $\zeta$, we arrive at
\[
\zeta \frac{dL(t)}{dt} \leq -\gamma_1 \zeta E(t) + \gamma_2 \zeta (g \circ \nabla u)(t), \quad \forall t \geq t_0. \quad (4.33)
\]
Recalling (A2), (4.2), (4.11) and the first inequality in (4.29), we get
\[
\zeta \frac{dL(t)}{dt} \leq -\gamma_1 \zeta E(t) - \gamma_2 (g' \circ \nabla u)(t)
\leq -\gamma_1 \zeta E(t) - 2\gamma_2 E'(t) + 2k_1 \gamma_2 \|u_t\|_2^2
\leq -\gamma_1 \zeta E(t) - 2\gamma_2 E'(t) + 4k_1 \gamma_2 E(t), \quad \forall t \geq t_0. \quad (4.34)
\]
Now, we add a restriction condition on $\zeta$, that is, we suppose that
\[
\zeta > 4k_1 \frac{\gamma_2}{\gamma_1} := \zeta_0. \quad (4.35)
\]
Then, there exists a positive constant $\gamma_3$ such that
\[
E'(t) \leq -\gamma_3 \zeta E(t), \quad (4.36)\]
where
\[
F(t) = \zeta L(t) + 2\gamma_2 E(t) \sim E(t). \quad (4.37)
\]
And then, there exists a positive constant $\gamma_4$ such that

$$F'(t) \leq -\gamma_3 \zeta E(t) \leq -\gamma_4 \zeta F(t).$$  \hspace{1cm} (4.38)

A simple integration of (4.38) over $(t_0, t)$ leads to

$$F(t) \leq F(t_0) e^{-\gamma_4 \zeta (t-t_0)}, \quad \forall t \geq t_0.$$  \hspace{1cm} (4.39)

A combination of (4.37) and (4.39) leads to (4.4). The proof of Theorem 4.2 is thus completed. \hfill $\square$

**Remark 4.3.** Our results presented in this paper are under the assumption that $\zeta$ is a constant. However, we would like to point out here that these results can be easily extend to the case which $\zeta$ is a nonincreasing differentiable function $\zeta(t)$ with some suitable lower bound.

**Open problem**

What results can we hope to get in the case $0 < \mu_1 < \mu_2$? We first recall that, in the absence of time delay, the energy of the problem (1.2) is dissipative. That is to say, the damping item is a “good” item for the energy decay. So it is not difficult to imagine that, if $|\mu_1| < \mu_2 < a$ (Here, $a$ is defined by (4.31)), we also can obtain the energy decay result. But, if $a < |\mu_1| < \mu_2$, it could be instability. And this problem is still open.

**Acknowledgements**

The authors would like to thank the referees for their invaluable comments and suggestions which clarified and greatly improved the presentation of the paper. The first author was supported by NNSFC 11271120. And the Corresponding Author was partly supported by the key built disciplines of Hunan Province (Operations research and control theory, Hengyang Normal University, 2011).

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(Received: November 5, 2012; revised: August 22, 2013)