The problems of Classical Electrodynamics with the electron equation of motion and with non-integrable singularity of its self-field stress tensor are well known. They are consequences, we show, of neglecting terms that are null off the charge world line but that gives a non null contribution on its world line. The self-field stress tensor of a point classical electron is integrable, there is no causality violation and no conflict with energy conservation in its equation of motion, and there is no need of any kind of renormalization nor of any change in the Maxwell’s theory for this. (This is part of the paper hep-th/9510160, stripped, for simplicity, of its non-Minkowskian geometrization of causality and of its discussion about the physical meaning of the Maxwell-Faraday concept of field).

Three unsolved problems [1,2] make of Classical Electrodynamics of a point electron a non-consistent theory: 1) the field singularity in the Lienard-Wiechert solution; 2) the non-integrable singularities of its stress tensor; 3) the bizarre, causality violating behaviour of solutions of the Lorentz-Dirac equation [3,4].

The Lorentz-Dirac equation,

\[ ma = e F_{ext} \cdot V + \frac{2e^2}{3} (\dot{a} - a^2 V), \]  

is the greatest paradox of classical field theory as it cannot simultaneously preserve both the causality and the energy conservation, although there is nothing in the premisses for its derivation that justify such violations. The presence of the Schott term, \( \frac{2e^2}{3} \dot{a} \), is the cause of all pathological features of (1), like microscopic non-causality, runaway solutions, preacceleration, and other bizarre effects [5]. On the other hand the presence of this term is necessary for the maintenance of energy-momentum conservation; without it it would be required a null radiance for an accelerated charge, as \( \dot{a} . V + a^2 = 0 \). The argument, although correct, that such causality violations are not observable because they are outside the scope of classical physics [5] and are blurred [6] by quantum-mechanics effects is not enough compelling, because these same problems remain in a quantum formalism, just disguised in other apparently distinct problems [7].

The presence of non-integrable singularities in the electron self-field stress tensor is also a major problem. Previous attempts on taming these singularities have relied on modifications of the Maxwell’s theory with ad hoc addition of
extra-terms to the field stress tensor on the electron world-line (see for example the reviews [3,4,8]); it is particularly interesting that, as we will show here, instead of adding anything we should actually not drop out some null terms. Is their contribution (not null, in an appropriate limit) that avoid the infinities. The same problem happens in the derivations of the electron equation of motion: they are done with incomplete field expressions that do not contain these terms that are null only off the particle world-line. The Schott term in the Lorentz-Dirac equation is a consequence of this; it does not appear in the equation when the full field expression is correctly used.

The solution to the first problem, which is in essence the origin of the second and third problems, involves a discussion of the meaning of the Maxwell-Faraday concept of field and requires an entirely new formalism based on an anticipated recognition, yet in a classical context, of the discrete and localized character of the electromagnetic interaction, which is proper of a quantum formalism. The solution to the second and to the third problems is easier to describe and just requires the knowledge of the model of spacetime geometry behind this new formalism. We will present here a simplified Minkowskian version of it. The complete Riemannian version and its relevance to Quantum Field Theory and to the question of the origin and meaning of mass, is too long to be included in this letter. It is being discussed elsewhere [9,10]. The immediate difference in the outcomes of these two versions appears in the equation of motion for a point electron,

$$m \ddot{\alpha}^\mu - \frac{1}{4} \dot{\alpha}^2 \ddot{\alpha}^\mu = F^\mu_{\text{ext}} - \frac{2 \dot{\alpha}^2}{3} V^\mu >.$$  

The last term on the LHS corresponds to the energy associated to the Riemannian curvature of the electron manifold and it is not present in its Minkowskian counterpart [37]. This is not the Lorentz-Dirac equation as it does not contain the troublemaker third-order term. So, it does not violate causality, and it is, nonetheless, compatible with 4-momentum conservation, as we discuss below.

Equation (1) can be obtained from energy-momentum conservation in the Lienard - Wiechert solution [3,4,8],

$$A = \frac{V}{\rho}_{\text{ret}}, \text{ for } \rho > 0,$$  

in the limit of $\rho \to 0$. $V$ is the tangent to the particle world-line $z = z(\tau)$, parameterized by its proper time $\tau$, $(V = dz/d\tau$, and $V^2 = -1)$. It is expressed in terms of retarded coordinates [8], by which any spacetime point $x$ is constrained with a particle world-line point $z(\tau)$ by

$$R^2 = 0, \quad R^0 > 0,$$
\[ d\tau + K.d\mathbf{x} = 0, \quad (4) \]

where \( R \equiv x - z(\tau), \quad \rho \equiv -V.\eta.R \), where \( \eta \) is the Minkowski metric tensor, (with signature +2). \( \rho \) is the spatial distance between the point \( x \) where the electromagnetic field is observed and the point \( z(\tau) \), position of the charge, in the charge rest frame at its retarded time. \( R^2 = 0 \) implies on \( \rho = \Delta \tau \). \( K \), defined by

\[ K^\mu := \frac{\Delta x^\mu}{\rho} = \frac{\Delta x^\mu}{\Delta \tau}, \quad (5) \]

is a null 4-vector, \( K^2 = 0 \), and represents a light-cone generator, or the electromagnetic wave-front 4-vector.

The retarded Maxwell field, \( F_{\mu\nu,\text{ret}} := \partial_{[\mu}A_{\nu]} \), is given by

\[ F_{\mu\nu,\text{ret}} = \frac{1}{\rho^2}[K_\mu, V_\nu + \rho(\mathbf{a}_\nu + \mathbf{a}_K V_\nu)] = \frac{1}{\rho}[K_\mu, \mathbf{a}_\nu] + \frac{\mathbf{a}_K}{\rho}[K_\mu, V_\nu] + \frac{[K_\mu, V_\nu]}{\rho^2}, \quad (6) \]

where, for notational simplicity, we are using \( (A, B) := AB + BA \), \( [A, B] := AB - BA \), \( a_K := \mathbf{a}_K \), and we are making the electron charge and the speed of light equal to 1: \( e = c = 1 \).

Using (6) in \( 4\pi \Theta_{\mu\nu} = F_{\mu\beta} \eta^{\alpha\beta} F_{\alpha\nu} - \eta_{\mu\nu} \frac{F_{\alpha\beta} F_{\alpha\beta}}{4} \), for finding the electron self-field stress tensor,

\[ 4\pi \rho^4 \Theta = [K, \rho \mathbf{a} + V(1 + \rho \mathbf{a}_K)] \eta.[K, \rho \mathbf{a} + V(1 + \rho \mathbf{a}_K)] - \frac{\eta}{4}[K, \rho \mathbf{a} + V(1 + \rho \mathbf{a}_K)]^2, \quad (7) \]

or \( \Theta = \Theta_2 + \Theta_3 + \Theta_4 \), with

\[ 4\pi \rho^2 \Theta_2 = [K, \mathbf{a} + V \mathbf{a}_K] \eta.[K, \mathbf{a} + V \mathbf{a}_K] - \frac{\eta}{4}[K, \mathbf{a} + V \mathbf{a}_K]^2, \quad (8) \]

\[ 4\pi \rho^3 \Theta_3 = [K, V] \eta.[K, \mathbf{a} + V \mathbf{a}_K] + [K, \mathbf{a} + V \mathbf{a}_K] \eta.[V, f, V] - \frac{\eta}{2} \text{Tr}[K, V] \eta.[K, \mathbf{a}], \quad (9) \]

\[ 4\pi \rho^4 \Theta_4 = [K, V] \eta.[K, V] - \frac{\eta}{2}[K, V]^2. \quad (10) \]

It is worth to explicitly write (8-10) for \( \rho > 0 \) (then \( K^2 = 0 \)) and make some comments.

\[ 4\pi \rho^2 \Theta_{2,\mu\nu} = -K_\mu K_\nu \{\mathbf{a}^2 - \mathbf{a}_K^2\}, \quad (11) \]

\[ 4\pi \rho^3 \Theta_{3,\mu\nu} = 2K_\mu K_\nu \mathbf{a}_K - \left(K_\mu, (\mathbf{a} + V \mathbf{a}_K)_\nu\right). \quad (12) \]
\[ 4\pi \rho^{4} \Theta_{4\mu
u} = K_{\mu}K_{\nu} - (K_{\mu}, V_{\nu}) - \frac{\eta_{\mu
u}}{2}. \] (13)

\( \Theta_{2} \), although singular at \( \rho = 0 \), is nonetheless integrable. By that it is meant that \( \int d^{4}x \Theta_{2} \) exists [3], while \( \Theta_{3} \) and \( \Theta_{4} \) are not integrable; they generate, respectively, the problematic Schott term in the LDE and a divergent expression, the electron bound 4-momentum [8]. The most update prescription [3,4] is to redefine \( \Theta_{3} \) and \( \Theta_{4} \) at the electron world-line in order to make them integrable, but without changing them at \( \rho > 0 \), so to preserve the standard results of Classical Electrodynamics. This is possible with the use of distribution theory, but it is always an introduction of something strange to the system and in an \textit{ad hoc} way. The most unsatisfactory aspect of this procedure is that it regularizes the above integral but leaves an unexplained and unphysical discontinuity in the flux of 4-momentum from the charge world-line: \( \Theta(\rho = 0) \neq \Theta(\rho \sim 0) \).

We observe that

\[ K_{\mu} \Theta_{2}^{\mu\nu} \bigg|_{\rho > 0} = 0, \] (14)

which is important in the identification of \( \Theta_{2} \) with the radiated part of \( \Theta \), and that

\[ K_{\mu} \Theta_{3}^{\mu\nu} \bigg|_{\rho > 0} = 0 \] (15)

The important difference among the sets of equations (8-10) and (11-13) is that while the equations in the first one are complete, in the sense that they keep the terms proportional to \( K^{2} \), which are null (as \( K^{2} = 0 \)), in the last set of equations they have been dropped off. But these \( K^{2} \)-terms, even with \( K^{2} = 0 \), should not be dropped from the above equations, since they are necessary for producing the correct limits when \( \rho \to 0 \), or \( x \to z \). As \( K^{\mu} := \frac{\eta^{\mu}}{\rho} \), in this limit we have a 0/0-type of indeterminacy, which can be raised with the L’Hospital rule and \( \frac{\partial}{\partial \tau} \). This results in

\[ \lim_{R \to 0} K = V, \]

and

\[ \lim_{R \to 0} K_{\mu}K^{\mu} = \lim_{R \to 0} \frac{R \cdot \eta \cdot R}{\rho^{2}} = -1. \]

A Feynmann diagram, see the figure 1, helps in the understanding of these two results. In the limit of \( \rho \to 0 \), or at \( \tau = \tau_{ret} \) there are 3 distinct velocities: K, the photon 4-velocity, and \( V_{1} \) and \( V_{2} \), the electron initial and final 4-velocities. This is the reason for this indeterminacy at \( \tau = \tau_{ret} \). At \( \tau = \tau_{ret} + d\tau \) there is only \( V_{2} \), and only \( V_{1} \).
at $\tau = \tau_{ret} - d\tau$, or, back to the usual picture, $V(\tau)$, in general. We must observe that the classical picture of a continuous interaction cannot resolve this indeterminacy. The lesson one should learn from this is that even in a classical context, it is necessary to take into account the discrete and localized (quantum) character of the fundamental electromagnetic interaction in order to have a clear and consistent physical picture. This is the viewpoint adopted in [7,9], where the concept of a classical photon is introduced. It fulfills a requirement of an equal-foot treatment to the electron and to its self electromagnetic field, which is more in accordance with the experimental data that show both as equally fundamental physical objects of Nature.

To find the limit of something when $\rho \to 0$ will be done so many times in this letter that it is better to do it in a more systematic way. We want to find

$$\lim_{R \to 0} \frac{N(R)}{\rho^n},$$

where $N(R)$ is a homogeneous function of $R$, $N(R) \bigg|_{R=0} = 0$. Then, we have to apply the L’Hospital rule consecutively until the indeterminacy is resolved. As $\frac{\partial \rho}{\partial \tau} = -(1 + \mathbf{a} \cdot \mathbf{R})$, the denominator of (16) at $R = 0$ will be different of zero only after the $n^{th}$-application of the L’Hospital rule, and then, its value will be $(-1)^n n!$.

If $p$ is the smallest integer such that $N(R)_p \bigg|_{R=0} \neq 0$, where $N(R)_p := \frac{d^p}{d\tau^p} N(R)$, then

$$\lim_{R \to 0} \frac{N(R)}{\rho^n} = \begin{cases} \infty, & \text{if } p < n \\ (-1)^n \frac{N(0)}{n!}, & \text{if } p = n \\ 0, & \text{if } p > n \end{cases}$$

(17)

• Example 1: $\{K = R, K^2 = \frac{R \eta \cdot R}{\rho}, n = p = 1 \implies \lim_{R \to 0} K = V$

• Example 2: $\frac{[K_{\rho} A_{\rho}]_\rho}{\rho} = \frac{[K_{\rho} A_{\rho}]_\rho}{\rho^2}$, $n = p = 2 \implies \lim_{R \to 0} \frac{[K_{\rho} A_{\rho}]_\rho}{\rho}$ diverge

• Example 3: $\frac{[K_{\rho} V_{\rho}]}{\rho^2} = \frac{[K_{\rho} V_{\rho}]}{\rho^2}$, $n = p = 3 \implies \lim_{R \to 0} \frac{[K_{\rho} V_{\rho}]}{\rho^2} = 0$

• Example 4 $\frac{[K_{\rho} V_{\rho}]}{\rho^2} = [R_{\rho} V_{\rho}]$ $\implies p = 2 < n = 3 \implies \lim_{R \to 0} \frac{[K_{\rho} V_{\rho}]}{\rho^2}$ diverge

The second term of the RHS of (6) does not contribute to the electron self-field at $\rho = 0$, but the first and the third terms diverge, as expected, although they produce integrable contributions to $\Theta$. Let us find the integral of the electron self-field stress tensor at the electron causality-line: $\lim_{\rho \to 0} \int d^4 \Theta$, or $\lim_{\rho \to 0} \int d\tau \rho^2 d\rho d^2 \Omega \Theta$, in terms of retarded coordinates $x^\mu = z^\mu + \rho K^\mu$, where $d^2 \Omega$ is the element of solid angle in the charge rest-frame.
But, we will first prove a useful result to be used with (17), when the numerator has the form \( N_0 = A_0 \eta B_0 \), where \( A \) and \( B \) represent two possibly distinct homogeneous functions of \( R \), and the subindices indicate the order of \( \frac{d^k}{d \tau^k} \). Then

\[
N_p = \sum_{a=0}^{p} \binom{p}{a} A_{p-a} \eta B_a
\]  

(18)

So, for using (17) with (18), we just have to find the \( \tau \)-derivatives of \( A \) and \( B \) that produce the first non-null term at the limit of \( R \to 0 \).

Applying (17) and (18) for finding \( \lim_{\rho \to 0} \int dx^4 \Theta \) we just have to consider the first term of the RHS of (7); as the second one is the trace of the first, its behaviour under this limit can be inferred.

- Example 5

\[
\lim_{\rho \to 0} \rho^2 \left[ K, \rho \dot{a} + V(1 + \rho \ddot{a} K) \right] \eta \left[ K, \rho \dot{a} + V(1 + \rho \ddot{a} K) \right] =
\]

\[
= \lim_{\rho \to 0} \frac{R, \rho \dot{a} + V(1 \ddot{a}. R)}{\rho^4} \eta \left[ R, \rho \dot{a} + V(1 + \ddot{a} R) \right]
\]

So, \( A_0 = B_0 = [R, \rho \dot{a} + V(1 + \rho \ddot{a} R)] \implies A_2 = B_2 = [\dot{a}, V] + O(R) \). Therefore, according to (18), for producing a non null \( N_p \), \( a \) and \( p \) must be given by

\[
p - a = a = 2 \implies p = 4 = n \implies N_4 = 6[\dot{a}, V].\eta.[\dot{a}, V] + O(R).
\]

Then, we conclude from (17), that

\[
\lim_{\rho \to 0} \rho^2 \left[ K, \rho \dot{a} + V(1 + \rho \ddot{a} K) \right] \eta \left[ K, \rho \dot{a} + V(1 + \rho \ddot{a} K) \right] = \frac{1}{4} [\dot{a}, V].\eta.[\dot{a}, V]
\]

We have, therefore, from (7), that

\[
\lim_{\rho \to 0} \int dx^4 \Theta = [\dot{a}, V].\eta.[\dot{a}, V] - \frac{\eta}{4} [\dot{a}, V]^2
\]  

(19)

The flux of 4-momentum irradiated from the electron, which is the meaning of (19), is finite and depends only on its instantaneous velocity and acceleration. It is interesting that (19) comes entirely from the velocity term, \( \frac{[K,V]}{\rho^4} \), as we can see from the following example.
• Example 6

\[ \lim_{\rho \to 0} \frac{\rho^2[K, V].\eta.[K, V]}{\rho^4} = \lim_{\rho \to 0} \frac{[R, V].\eta.[R, V]}{\rho^4} = \frac{1}{4}[\bar{a}, V].\eta.[\bar{a}, V] \]

as \( A_2 = B_2 = [\bar{a}, V] + [R, \dot{a}] \implies p - a = a = 2 \implies N_4 = 6[\bar{a}, V] + \mathcal{O}(R) \) and \( p = n = 4 \). So,

\[ \lim_{\rho \to 0} \int dx^4 \Theta = \lim_{\rho \to 0} \int dx^4 \Theta_4 \]

The contribution from the other two terms just cancel to zero,

\[ \lim_{\rho \to 0} \int dx^4 \Theta_2 = - \lim_{\rho \to 0} \int dx^4 \Theta_3, \]

as can be easily verified. We must realize that the discontinuity,

\[ \Theta(\rho = 0) \neq \Theta(\rho \to 0), \]

still remains. It is a consequence of the problem 1; its solution requires an understanding of the physical meaning of the Maxwell-Faraday concept of field in this context. It is being discussed elsewhere [9].

The electron equation of motion

Let us now discuss the third problem. The electron equation of motion, can be obtained from

\[ \lim_{\varepsilon \to 0} \int dx^4 \partial_\mu T^{\mu\nu} \theta(\rho - \varepsilon) = 0, \]

where \( T^{\mu\nu} \) is the total electron energy-momentum tensor, which includes the contribution from the electron kinetic energy, from its interaction with external fields and from its self field. Let us move directly to the part that will produce novel results:

\[ m \int d\tau = \int F^\mu_{ext} d\tau - \lim_{\varepsilon \to 0} \int dx^4 \partial_\nu \Theta^{\mu\nu} \theta(\rho - \varepsilon), \quad (20) \]

where \( F^\mu_{ext} \) is the external forces acting on the electron, and the last term represents the impulse carried out by the emitted electromagnetic field in the Bhabha tube surrounding the electron world-line, defined by the Heaviside function, \( \theta(\rho - \varepsilon) \). Using the divergence theorem, we have that the last term of the RHS of (20) is transformed into

\[ \lim_{\varepsilon \to 0} \int dx^4 \Theta^{\mu\nu} \partial_\mu \rho \delta(\rho - \varepsilon). \quad (21) \]
This term represents the flux of 4-momentum through the cylindrical hypersurface $\rho = \varepsilon$. Let us denote it by $P^\mu$:

$$P^\mu = \lim_{\varepsilon \to 0} \int dx^4 \Theta^{\mu\nu} \partial_\nu \delta(\rho - \varepsilon),$$

As $\partial_\nu \rho = \rho a K_{\nu} + K_\nu - V_\nu$, and $\Theta = \Theta_2 + \Theta_3 + \Theta_4$, we can write $P^\mu := P^\mu_0 + P^\mu_1 + P^\mu_2$, with

$$P^\mu_2 = \lim_{\varepsilon \to 0} \int dx^4 \Theta^{\mu\nu}_2 (K - V)_\nu \delta(\rho - \varepsilon),$$

$$P^\mu_1 = \lim_{\varepsilon \to 0} \int dx^4 \{ \Theta^{\mu\nu}_3 K_\nu \rho a K + \Theta^{\mu\nu}_3 (K - V)_\nu \} \delta(\varepsilon - \rho),$$

$$P^\mu_0 = \lim_{\varepsilon \to 0} \int dx^4 \{ \Theta^{\mu\nu}_4 K_\nu \rho a K + \Theta^{\mu\nu}_4 (K - V)_\nu \} \delta(\rho - \varepsilon),$$

$P^\mu_1$ and $P^\mu_2$ are both null. In order to show this we need to apply (17) for a $N(R)$ with a generic form $N = A.\eta.B.C$, where A,B, and C are functions of R, such that $N(R = 0) = 0$. Then it is easy to show, from (18), that

$$N_\rho = \sum_{a=0}^p \sum_{c=0}^a \left( \begin{array}{c} p \\ a \end{array} \right) \left( \begin{array}{c} a \\ c \end{array} \right) A_{p-a}. \eta.B_{a-c}.C_c$$

From (19), the integrand of (23), produces (again, we do not need to consider the trace term)

$$\lim_{\rho \to 0} \frac{\rho^3[K,V].\eta.[K,V].(K - V)}{\rho^4} = \lim_{\rho \to 0} \frac{[R,V].\eta.[R,V].(R - \rho V)}{\rho^5},$$

or, schematically

$$\lim_{\rho \to 0} \frac{A.\eta.A.C}{\rho^5}$$

with $A_0 = B_0 = [R,V]$, and $C_0 = (R - V \rho)$. Then, $A_2 = [a,V] + \mathcal{O}(R)$, $C_2 = a + \mathcal{O}(R)$, and we have, from (26), the following restrictions on a and c for producing a $N(R = 0)_\rho \neq 0$:

$c = 2; \quad a - c = 2; \quad$ and $p - a = 2$ or $p = 6 > n = 5$. Therefore, according to (17)

$$P^\mu_2 = 0$$

From the second term of the integrand of (24) and from (19) we have

$$\lim_{\rho \to 0} \frac{\rho^2[K,V].\eta.[K,\alpha + V\alpha K].(K - V)}{\rho^3} = \lim_{\rho \to 0} \frac{[R,V].\eta.[R,\rho \alpha + V\alpha R].(R - \rho V)}{\rho^5},$$

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and then, from (26) with

\[ C_0 = R - V \rho \implies C_2 = a + \mathcal{O}(R) \implies c = 2. \]

\[ B_0 = [R, V] \implies B_2 = [a, V] + \mathcal{O}(R) \implies a = 4. \]

\[ A_0 = [R, a] \implies A_1 = [a, V] + \mathcal{O}(R) \implies p = 5 = n. \]

This produces \([a, V], \eta, [a, V].a = a^2a\), which is cancelled by the equal contribution from the trace term of (3). Therefore,

\[ \lim_{\rho \to 0} \rho^2 \Theta^{\mu\nu}_4 (K - V)_{\nu} = 0 \]

- From the first term of the integrand of (24) and from (10) we have

\[ \lim_{\rho \to 0} \rho^2 [K, V].\eta, [K, V]K\rho \cdot K \]

\[ = \lim_{\rho \to 0} \frac{[R, V].\eta, [R, V]R\cdot R}{\rho^5}. \]

Then, from (26), with \( C = C_0 = R \cdot a\cdot R \implies C_3 = -3V\dot{a} + a^2 + \mathcal{O}(R) \implies c = 3 \), and

\[ A = B = B_0 = [R, V] \implies A_2 = B_2 = [a, V] + \mathcal{O}(R) \implies a = 5 \text{ and } p = 7 > n = 5. \]

Therefore, from (17)

\[ \lim_{\rho \to 0} \rho^2 \Theta^{\mu\nu}_4 K_{\nu} \cdot a_K = 0 \]

Consequently

\[ P_1^\mu = 0 \quad (28) \]

and

\[ P^\mu = P_0^\mu \quad (29) \]

\( P_0^\mu \) is distinguished from \( P_1^\mu \) and \( P_2^\mu \) for not being \( \rho \)-dependent. Therefore, it is not necessary to use the L'Hospital rule on its determination, which, by the way, is not affected by the limit of \( \varepsilon \to 0 \). The physical meaning of this is that the flux of 4-momentum through the cylindrical surface \( \rho = \varepsilon \) comes entirely from the photon field.
It is convenient now to introduce the spacelike 4-vector $N$, defined by $N^\mu := (K - V)^\mu$ and such that $N \eta V = 0$ and $N \eta N = 1$. $N$ satisfies

\[ \frac{1}{4\pi} \int d\Omega N \cdots N = 0 \]  
(30)

\[ \frac{1}{4\pi} \int d\Omega NN = \frac{\eta + VV}{3}; \]  
(31)

as can be found, for example, in references [4,13] or in the appendix B of reference [10]. From (9) and (15), or from (30), we have

\[ \int d^4x \Theta^{\mu\nu} K_\nu \rho \alpha_K = 0, \]  
(32)

From (11), we have

\[ 4\pi \rho^2 \Theta_2^{\mu\nu} N_\nu = \left( (\alpha.N)^2 - \alpha^2 \right) \left( V^\mu + N^\mu \right), \]  
(33)

which, with (30) and (31), gives

\[ \lim_{\rho \to 0} \int d^4x \Theta_2^{\mu\nu} N_\nu = - \int d\tau \frac{2}{3} \alpha^2 V^\mu. \]  
(34)

Then, from (34), (32) and (28), we have

\[ P^\mu = - \int d\tau \frac{2}{3} \alpha^2 V^\mu, \]  
(35)

the Larmor term.

Finally, from (20), (21), (22) and (35), we can write the electron equation of motion, obtained from the Lienard-Wiechert solution, as

\[ m \dot{\alpha}^\mu = F_{ext}^\mu - \frac{2\alpha^2}{3} V^\mu, \]  
(36)

The external force provides the work for changing the charge velocity and for the energy dissipated by the radiation.

It is non-linear, like the Lorentz-Dirac equation, but it does not contain the Schott term, the responsible for its spuriously behaving solutions. This is good since it signals that there will be no problem with causality violation.

But the Schott term in the Lorentz-Dirac equation has also the role of giving the guaranty of energy conservation, which is obviously missing in (36). Assuming that the external force is of electromagnetic origin, ($F_{ext}^\mu = F_{ext}^{\mu\nu} V_\nu$), the
contraction of $V$ with eq. (36) would require a contradictory $a^2 \equiv 0$. But this is just an evidence that (36) cannot be regarded as a fundamental equation. It would be better represented as

$$m \dot{a}^\mu = F^\mu_{\text{ext}} - \left< \frac{2}{3} a^2 V^\mu \right>, \tag{37}$$

with

$$\left< \frac{2}{3} a^2 V^\mu \right> = P^\mu = \lim_{\varepsilon \to 0} \int dx^4 \Theta^{\mu\nu} \partial_\nu \rho \delta(\rho - \varepsilon),$$

It is just an effective or average result, in the sense that the contributions from the electron self field must be calculated, as in (20), by the electromagnetic energy-momentum content of a spacetime volume containing the charge world-line, in the limit of $\rho \to 0$:

$$m \int \dot{a}_\nu V d\tau = \int F_{\text{ext},\nu} V d\tau - \lim_{\varepsilon \to 0} \int dx^4 K_\mu \partial_\nu \Theta^{\mu\nu} \theta(\rho - \varepsilon). \tag{38}$$

Observe that in the last term, $V$, the speed of the electron, is replaced by $K$ the speed of the electromagnetic interaction; only in the limit of $R \to 0$ is that $K \to V$. We have to repeat the same steps from (20) to (36) in order to calculate this last term and to prove (done in the appendix) that it is null:

$$\lim_{\varepsilon \to 0} \int dx^4 K_\mu \partial_\nu \Theta^{\mu\nu} \theta(\rho - \varepsilon) = 0, \tag{39}$$

So, there is no contradiction anymore. Besides, it throws some light on the physical meaning of $A_\mu$, which is discussed in [9].

**I. APPENDIX**

Let us prove (39). Its LHS implies on

$$- \lim_{\varepsilon \to 0} \int dx^4 \left\{ (\partial_\nu K_\alpha) \Theta^{\alpha\nu} \theta(\rho - \varepsilon) + K_\mu \Theta^{\mu\nu} \partial_\nu \rho \delta(\rho - \varepsilon) \right\}. \tag{40}$$

To find how the first term of the integrand of (41) behaves in the limit of $R \to 0$, we need to find $\partial_\nu K_\mu$ from $K^\mu = \frac{R_\mu}{\Delta \tau}$ and from $\rho = \Delta \tau$. The difference between the derivatives of $\rho$ and of $\Delta \tau$ tends to zero in the limit of $\rho \to 0$. So, it is irrelevant if we use one or the other in the definition of $K$; we use the simplest one, $\Delta \tau$, and $\frac{\partial \Delta \tau}{\partial \tau} = -1$ to find:

$$\partial_\nu K_\mu = \frac{1}{\Delta \tau} \left\{ \eta_{\mu\nu} + K_\nu (V - K_\mu) \right\} = \frac{1}{(\Delta \tau)^3} \left\{ (\Delta \tau)^2 \eta_{\mu\nu} + \Delta \tau V_\mu R_\nu - R_\mu R_\nu \right\}.$$
Then, the first term in the integrand of (40) contains
\[
\lim_{R \to 0} \frac{[R, \rho \bar{a} + V(1 + \bar{a}.R)].\eta.[R, \rho \bar{a} + V(1 + \bar{a}.R)]\left\{ (\Delta \tau)^2 \eta_{\mu\nu} + \Delta \tau V_\mu R_\nu - R_\mu R_\nu \right\}}{\rho^5},
\] (41)

and, using the notation of (26),
\[
A_0 = B_0 = [R, \rho \bar{a} + V(1 + \rho \bar{a}.R)] \implies A_2 = B_2 = [\bar{a}, V] + O(R) \implies a = 4
\]

\[
C_0 = (\Delta \tau)^2 \eta_{\mu\nu} + \Delta \tau V_\mu R_\nu - R_\mu R_\nu \implies C_2 = 2\eta + O(R) \implies c = 2 \implies p = 6 < n = 7
\]

According to (17), this would produce a divergent result if \( N_6 \not= 0 \), but \( \Theta \) and its limit are traceless \( (\Theta^\mu_{\nu\eta} \eta_{\mu\nu} = 0) \), and so, \( N_6 = 0 \). Therefore, the indeterminacy \( 0/0 \) remains and a new application of the L’Hospital rule is demanded.

Then, from (18), for \( p = 6, \ a = 4, \ c = 2 \), we have
\[
N_6 = \binom{6}{2} \binom{2}{2} A_2 . \eta . B_2 . C_2,
\]

and then
\[
N_7 = \dot{N}_6 = \binom{6}{2} \binom{2}{2} \left\{ A_3 . \eta . A_2 C_2 + A_2 . \eta . A_2 C_2 + A_2 . \eta . A_2 C_2 + A_2 . \eta . A_2 C_2 \right\}.
\]

The terms containing \( C_2 \) will still give a null contribution (\( \Theta \) is traceless). With \( C_3 = \frac{3}{2}(V, a) + O(R) \),
\[
\left\{ A_2 . \eta . A_2 C_3 \right\} = \left\{ [V, \bar{a}].\eta.[V, \bar{a}].(9V\bar{a} + 6\bar{a}V) \right\} = 0,
\] (42)

and then,
\[
\lim_{\varepsilon \to 0} \int dx^4 \partial_\nu K_\alpha \Theta^\alpha_{\nu\eta}\delta(\rho - \varepsilon) = 0.
\] (43)

The last term in the integrand of (40) is related to (22-25). So, we can write
\[
<K.P_2> = \lim_{\varepsilon \to 0} \int dx^4 K_\mu \Theta^\mu_{4\nu} (K_\nu - V_\nu)\delta(\rho - \varepsilon),
\] (44)

which is null because
\[
\lim_{\rho \to 0} \frac{R.[[R,V].\eta.[R,V]].\eta.(R - V\rho)}{\rho^6} = 0;
\]

and

\[13\]
\[
<K.P_1> = \lim_{\varepsilon \to 0} \int dx^4 K_\mu \left\{ \Theta_{3}^{\mu \nu} (K - V)_{\nu} + \Theta_{2}^{\mu \nu} K_{\nu} \rho \theta_{K} \right\} \delta(\rho - \varepsilon),
\]
(45)

which is also null because

\[
\lim_{\rho \to 0} \frac{1}{\rho^5} R_{\{[R,V].\eta.\{R,\rho a + V a. R\}.(R - \rho V)\}} = \lim_{\rho \to 0} \frac{1}{\rho^5} R_{\eta.\{[R,V].\eta.\{R, V\}\}R a. R} = 0,
\]
as they can be easily verified. Finally,

\[
<K.P_0> = \int dx^4 K_\mu \left\{ \Theta_{2}^{\mu \nu} (K - V)_{\nu} + \Theta_{3}^{\mu \nu} K_{\nu} \rho \theta_{K} \right\} \delta(\rho - \varepsilon) = 0,
\]
(46)
as a consequence of (44).

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FIGURE 1 CAPTION

Fundamental process: the electron 4-velocity $V_1$ changes to $V_2$ after the emission/absorption of a classical photon of 4-velocity $K$. 
Figure 1