Oscillatory behavior of second-order nonlinear neutral differential equations

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Abstract

We shall consider a class of second-order nonlinear neutral differential equations. Some new oscillation criteria are established by using the Riccati transformation technique. One example is given to show the applicability of the main results.

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1 Introduction

In this paper, we study the oscillation of a class of second-order nonlinear differential equations,

\[(r(t)(z'(t))^{\alpha})' + f(t, x(\tau(t))) = 0, \quad t \geq t_0 > 0,\]

where \(z(t) = x(t) - p(t)x(\tau(t))\), \(\alpha > 0\), and \(\alpha\) is the ratio of two odd integers. The following assumptions are satisfied:

\((H_1)\) \(r, p \in C([t_0, \infty), R), r(t) > 0, 0 \leq p(t) \leq p_0 < 1\).

\((H_2)\) \(\tau \in C([t_0, \infty), R), \tau(t) \leq t, \lim_{t \to \infty} \tau(t) = \infty\).

\((H_3)\) \(\sigma \in C^1([t_0, \infty), R), \sigma(t) \leq t, \sigma'(t) > 0, \lim_{t \to \infty} \sigma(t) = \infty\).

\((H_4)\) \(f \in C(R, R), uf(t, u) > 0 \) for all \(u \neq 0\), and there exists a function \(q(t) \in C([t_0, \infty], [0, \infty))\) such that \(|f(t, u)| \geq q(t)|u^\alpha|\).

Second-order and third-order differential equations are widely used in population dynamics, physics, technology and other fields. Many scholars have studied the oscillation of second-order differential equations \([1–10]\). Similarly, many scholars have studied the oscillation of third-order differential equations \([11–14]\). On this basis, this paper studies the second-order neutral differential Eq. (1), Some new oscillation criteria are established by using the Riccati transformation technique.

2 Lemmas

In order to establish the oscillation criterion of Eq. (1), we will give three lemmas.
Lemma 2.1 Assume that
\[ \int_0^\infty r^{-\frac{1}{\alpha}}(t) \, dt = \infty \] (2)
and \( x(t) \) is an eventually positive solution of Eq. (1). Then \( z(t) \) has the following two possible cases:

(i) \( z(t) > 0 \), \( z'(t) > 0 \), \( (r(t)z'(t))^\alpha' \leq 0 \);
(ii) \( z(t) < 0 \), \( z'(t) > 0 \), \( (r(t)z'(t))^\alpha' \leq 0 \).

Proof Since \( x(t) \) is an eventually positive solution of (1), there exists a \( t_1 \geq t_0 \) such that \( x(t) > 0 \), for \( t \geq t_1 \). From (1), we have
\[ (r(t)(z'(t))^\alpha)' \leq 0 \]
hence \( r(t)(z'(t))^\alpha \) is decreasing function and of one sign, therefore \( z'(t) \) is also of one sign, that is, there exists a \( t_2 \geq t_1 \) such that, for \( t \geq t_2 \), \( z'(t) > 0 \) or \( z'(t) < 0 \).

If \( z'(t) > 0 \), we have (i) or (ii). Now, we prove that \( z'(t) < 0 \) will not happen.

If \( z'(t) < 0 \), we have
\[ r(t)(-z'(t))^\alpha \geq r(t_2)(-z'(t_2))^\alpha = K \geq 0, \]
where \( K = r(t_2)(-z'(t_2))^\alpha \geq 0 \), that is,
\[ z'(t) \leq -k^\frac{1}{\alpha} r^{-\frac{1}{\alpha}}(t). \]

Integrating this inequality from \( t_2 \) to \( t \), we have
\[ z(t) \leq z(t_2) - k^\frac{1}{\alpha} \int_{t_2}^t r^{-\frac{1}{\alpha}}(s) \, ds \]
by condition (2), \( \lim_{t \to \infty} z(t) = -\infty \). We will consider the following two cases.

Case 1. If \( x(t) \) is unbounded, then there exists a sequence \( \{t_m\} \), such that \( \lim_{m \to \infty} t_m = \infty \) and \( \lim_{m \to \infty} x(t_m) = \infty \), here \( x(t_m) = \max x(s) : t_0 \leq s \leq t_m \). Hence, we have
\[ x(\tau(t_m)) = \max \{ x(s) : t_0 \leq s \leq \tau(t_m) \} \leq \max \{ x(s) : t_0 \leq s \leq t_m \} = x(t_m). \]

We get
\[ z(t_m) = x(t_m) - p(t_m)x(\tau(t_m)) \geq \left[ 1 - p(t_m) \right] x(t_m) > 0. \]
This contradicts \( \lim_{t \to \infty} z(t) = -\infty \).

Case 2. If \( x(t) \) is bounded, then \( z(t) \) is bounded, this contradicts \( \lim_{t \to \infty} z(t) = -\infty \).

Hence, \( z(t) \) satisfies one of the cases (i) and (ii). \( \Box \)
Lemma 2.2 Assume that \( x(t) \) is a positive solution of Eq. (1) and \( z(t) \) satisfies case (i) of Lemma 2.1, then
\[
z(t) \geq R(t) r^{\frac{1}{\alpha}}(t) z'(t), \quad \left(\frac{z(t)}{R(t)}\right)' \leq 0,
\]
where \( R(t) = \int_t^T r^{-\frac{1}{\alpha}}(s) \, ds \), \( T \geq t_0 \).

Proof For \( t > T \geq t_0 \), we have
\[
z(t) = z(T) + \int_T^t r^{\frac{1}{\alpha}}(s) z'(s) ds \geq r^{\frac{1}{\alpha}}(t) z'(t) \int_T^t r^{-\frac{1}{\alpha}}(s) ds = R(t) r^{\frac{1}{\alpha}}(t) z'(t).
\]
Thus, we conclude that
\[
\left(\frac{z(t)}{R(t)}\right)' = z'(t) R(t) - z(t) R'(t) \leq z'(t) R(t) - R(t) r^{\frac{1}{\alpha}}(t) z'(t) r^{-\frac{1}{\alpha}}(t) = 0.
\]
\[\square\]

Lemma 2.3 Assume that \( x(t) \) is an eventually positive solution of (1) and
\[
\limsup_{t \to \infty} \int_{\tau^{-1}(\sigma(t))}^t \left(\frac{1}{r(s)} \int_s^t q(u) \, du\right)^{\frac{1}{\alpha}} \, ds > p_0.
\]
(3)

Then the impossibility for \( z(t) \) satisfies case (ii) of Lemma 2.1.

Proof Assume that \( z(t) \) satisfies case (ii) of Lemma 2.1, we have
\[
-z(t) = -x(t) + p(t) x(\tau(t)) < p(t) x(\tau(t)) \leq p_0 x(\tau(t)).
\]
That is,
\[
x(\tau(t)) \geq -\frac{1}{p_0} z(t).
\]
We deduce that
\[
x(t) \geq -\frac{1}{p_0} z(\tau^{-1}(t)), \quad x(\sigma(t)) \geq -\frac{1}{p_0} z(\tau^{-1}(\sigma(t))).
\]
From (1) and \((H_4)\), we have
\[
(r(t)(z'(t))^\alpha)' + q(t) (x(\sigma(t)))^\alpha \leq 0.
\]
We get
\[
(r(t)(z'(t))^\alpha)' + q(t) \left( -\frac{1}{p_0} \right)^\sigma z^\alpha (\tau^{-1}(\sigma(t))) \leq 0.
\]
Integrating this inequality from \( s \) to \( t \), we conclude that
\[
r(t)(z'(t))^\alpha - r(s)(z'(s))^\alpha - \frac{1}{p_0} \int_s^t q(u) z^\alpha (\tau^{-1}(\sigma(u))) \, du \leq 0.
\]
That is,
\[-z'(s) \leq \frac{1}{p_0} \left( \frac{1}{r(s)} \int_s^t q(u)z^{\alpha} (\tau^{-1}(\sigma(u))) \, du \right)^{\frac{1}{\alpha}}.
\]
Integrating this inequality from $\tau^{-1}(\sigma(t))$ to $t$, we get
\[z(\tau^{-1}(\sigma(t))) - z(t) \leq \frac{1}{p_0} z(\tau^{-1}(\sigma(t))) \int_{\tau^{-1}(\sigma(t))}^t \left( \frac{1}{r(s)} \int_s^t q(u) \, du \right)^{\frac{1}{\alpha}} \, ds.
\]
Since $z(t) < 0$, we have
\[\int_{\tau^{-1}(\sigma(t))}^t \left( \frac{1}{r(s)} \int_s^t q(u) \, du \right)^{\frac{1}{\alpha}} \, ds \leq p_0.
\]
This contradicts (3). Thus the impossibility for $z(t)$ satisfies case (ii) of Lemma 2.1. \qed

3 Oscillation results

**Theorem 3.1** Assume that (2) and (3) be satisfied. If there exists a positive function $\rho \in C^1([t_0, \infty), (0, \infty))$, such that, for all sufficiently large $T \geq t_0$,
\[\int_{t_0}^{\infty} \left[ \rho(t) \bar{Q}(t) - \frac{r(t)(\rho'(t))^{\sigma+1}}{(\sigma+1)^{\sigma+1}\rho^\sigma(t)} \right] \, dt = \infty,
\]
where $\bar{Q}(t) = Q(t)\frac{R(t, \sigma(t))}{R(t)}$, $Q(t) = q(t)[1 + \bar{p}(\sigma(t))]^\sigma$, $\bar{p}(t) = \rho(t)\frac{R(t, \sigma(t))}{R(t)}$, then Eq. (1) is oscillatory.

**Proof** Assume that $x(t) > 0$. From Lemma 2.1, $z(t)$ satisfies one of the cases (i) and (ii).

Case (i). Suppose that case (i) holds, from Lemma 2.2, we have
\[\frac{z(t)}{R(t)} \leq \frac{z(\tau(t))}{R(\tau(t))}.
\]
That is,
\[z(\tau(t)) \geq R(\tau(t)) \frac{z(t)}{R(t)}.
\]
We get
\[z(t) = x(t) - p(t)x(\tau(t)) \leq x(t) - p(t)z(\tau(t)) \leq x(t) - p(t)R(\tau(t)) \frac{z(t)}{R(t)}.
\]
That is,
\[x(t) \geq \left[ 1 + p(t)\frac{R(\tau(t))}{R(t)} \right] z(t) = \left[ 1 + \bar{p}(t) \right] z(t),
\]
where $\bar{p}(t) = \rho(t)\frac{R(t, \sigma(t))}{R(t)}$. 

From (1), we conclude that
\[
(r(t)(z'(t))^\alpha)' + q(t)x^\alpha (\sigma(t)) \leq 0.
\]
Then we have
\[
(r(t)(z'(t))^\alpha)' + q(t)[1 + \tilde{p}(\sigma(t))]z^\alpha (\sigma(t)) \leq 0.
\]
That is,
\[
(r(t)(z'(t))^\alpha)' \leq -Q(t)z^\alpha (\sigma(t)),
\]
where \(Q(t) = q(t)[1 + \tilde{p}(\sigma(t))]^\alpha\).

We define a function \(w(t)\) of the generalized Riccati transformation by
\[
w(t) = \frac{\rho(t)r(t)(z'(t))^\alpha}{z^\alpha(t)}.
\]
Then \(w(t) > 0\), from Lemma 2.2, we have \(\frac{\rho(t)}{R(t)} \geq \frac{\rho(t)}{R(t)}\), that is, \(\frac{\rho(t)}{R(t)} \geq \frac{\rho(t)}{R(t)}\).

Using the inequality [2]
\[
Bu - Au^{\alpha+1} \leq \frac{\theta^\alpha}{(\theta + 1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^\alpha}, \quad \theta > 0, A > 0, B \in R,
\]
we have
\[
w'(t) = \rho(t)r(t)(z'(t))^\alpha + \rho(t) \frac{(r(t)(z'(t))^\alpha)'}{z^\alpha(t)} = \rho(t) \frac{\alpha r(t)(z'(t))^\alpha}{z^{\alpha+1}(t)}
\]
\[
\leq \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t)Q(t) \frac{z^\alpha (\sigma(t))}{z^\alpha(t)} - \frac{\alpha}{(\rho(t)r(t))^{1/\alpha}} w^{\alpha+1}(t)
\]
\[
\leq \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t)Q(t) \frac{R^\alpha (\sigma(t))}{R^\alpha(t)} - \frac{\alpha}{(\rho(t)r(t))^{1/\alpha}} w^{\alpha+1}(t)
\]
\[
\leq -\rho(t)\tilde{Q}(t) + \frac{\rho'(t)}{\rho(t)} w(t) - \frac{\alpha}{(\rho(t)r(t))^{1/\alpha}} w^{\alpha+1}(t)
\]
\[
= -\rho(t)\tilde{Q}(t) + \frac{r(t)(\rho(t))^\alpha}{(\alpha + 1)^{\alpha+1} \rho^\alpha(t)},
\]
where \(\tilde{Q}(t) = Q(t) \frac{R^\alpha (\sigma(t))}{R^\alpha(t)}\).

Integrating this inequality from \(T\) to \(t\), we have
\[
w(t) \leq w(T) - \int_T^t \left( \rho(s)\tilde{Q}(s) - \frac{r(s)(\rho(s))^\alpha}{(\alpha + 1)^{\alpha+1} \rho^\alpha(s)} \right) ds.
\]
From (4), we get \(\lim_{t \to \infty} w(t) = -\infty\), this contradicts \(w(t) > 0\).

Case (ii). If \(z(t)\) satisfies (ii), then due to Lemma 2.3, Eq. (1) is oscillatory. \(\square\)
Theorem 3.2 Assume that (2) and (3) are satisfied. If there exists a positive function \( \varphi \in C^1([t_0, \infty), (0, \infty)) \) such that, for all sufficiently large \( T \geq t_0 \),

\[
\int_{t_0}^{\infty} \frac{\mathcal{Q}(t) - \varphi^{\alpha+1}(t)}{r^{1/\alpha}(t)} \exp\left((\alpha + 1) \int_{T}^{t} \frac{\varphi(s)}{r^{1/\alpha}(s)} \, ds\right) = \infty,
\]

then Eq. (1) is oscillatory.

Proof We use the counter-evidence method, suppose we have a non-oscillatory solution \( x(t) \) of Eq. (1), as above, suppose that \( x(t) \) is a positive solution of (1), by using Lemma 2.1, \( z(t) \) satisfies one of (i) and (ii), we discuss each of the two cases separately.

Case (i). Assume that \( z(t) \) has property (i), we obtain (5). We define a function \( V(t) \) of a generalized Riccati transformation by

\[
V(t) = \frac{r(t)(z'(t))^{\alpha}}{z^{\alpha}(t)}.
\]

Then \( V(t) > 0 \), using the Yang inequality \( \frac{1}{p} a^p + \frac{1}{q} b^q \geq ab, \frac{1}{p} + \frac{1}{q} = 1 \), similar to (6), we have

\[
V'(t) = \frac{(r(t)(z'(t))^{\alpha})'}{z^{\alpha}(t)} - \frac{\alpha r(t)(z'(t))^{\alpha+1}}{z^{\alpha+1}(t)}
\]

\[
\leq -\mathcal{Q}(t) - \frac{\alpha}{r^{1/\alpha}(t)} V^{\alpha+1}(t)
\]

\[
= -\left[\mathcal{Q}(t) - r^{\frac{1}{\alpha}}(t)\varphi^{\alpha+1}(t)\right] - (\alpha + 1) r^{\frac{1}{\alpha}}(t) \left[\frac{1}{\alpha + 1} \varphi^{\alpha+1}(t) + \frac{\alpha}{\alpha + 1} V^{\alpha+1}(t)\right]
\]

That is,

\[
V'(t) + (\alpha + 1) r^{-\frac{1}{\alpha}}(t) \varphi(t) V(t) \leq -\left[\mathcal{Q}(t) - r^{-\frac{1}{\alpha}}(t)\varphi^{\alpha+1}(t)\right].
\]

We get

\[
\left[V'(t) + (\alpha + 1) r^{-\frac{1}{\alpha}}(t) \varphi(t) V(t)\right] \exp[(\alpha + 1) \int_{t}^{\infty} \frac{\varphi(s)}{r^{1/\alpha}(s)} \, ds]
\]

\[
\leq -\left[\mathcal{Q}(t) - r^{-\frac{1}{\alpha}}(t)\varphi^{\alpha+1}(t)\right] \exp[(\alpha + 1) \int_{T}^{t} \frac{\varphi(s)}{r^{1/\alpha}(s)} \, ds].
\]

That is,

\[
\left(V(t) \cdot \exp\left[(\alpha + 1) \int_{T}^{t} r^{-\frac{1}{\alpha}}(s) \varphi(s) \, ds\right]\right)'
\]

\[
\leq -\left[\mathcal{Q}(t) - r^{-\frac{1}{\alpha}}(t)\varphi^{\alpha+1}(t)\right] \exp[(\alpha + 1) \int_{T}^{t} \frac{\varphi(s)}{r^{1/\alpha}(s)} \, ds].
\]

Integrating this inequality from \( T \) to \( t \), we get

\[
0 \leq V(t) \cdot \exp\left[(\alpha + 1) \int_{T}^{t} r^{-\frac{1}{\alpha}}(s) \varphi(s) \, ds\right]
\]
\[ \leq V(T) - \int_T^T \left[ \left( \hat{Q}(t) - r^{-\frac{1}{\alpha}}(t) \psi^{\alpha+1}(t) \right) \exp\left[ (\alpha + 1) \int_T^t \frac{\psi(s)}{r^{1/\alpha}(s)} ds \right] \right] dt. \]

This contradicts (7).

**Case (ii).** If \( z(t) \) satisfies (ii), then due to Lemma 2.3, Eq. (1) is oscillatory.

\[ \square \]

**Example** Consider the following equation:

\[ \left( \left( x(t) - px(t - 1) \right) \right)^{\frac{1}{\alpha}} + q_0 x^{\frac{1}{3}}(t - 2) = 0. \] (8)

Comparing Eq. (8) with Eq. (1), let \( r(t) = 1, \alpha = \frac{1}{3}, \tau(t) = t - 1, \sigma(t) = t - 2, q(t) = q_0 > 0, p(t) = p < 1 \) is a positive constant. Choose \( \rho(t) = t, \varphi(t) = 1 \), we now verify (3):

\[ \lim_{t \to \infty} \sup_{t-t^{-1}(\sigma(t))} \int_{t-t^{-1}(\sigma(t))}^{t} \frac{1}{r(s)} \left( \int_{t-t^{-1}(\sigma(t))}^{t} q(u) du \right)^{\frac{1}{\alpha}} ds = \lim_{t \to \infty} \int_{t-t^{-1}}^{t} q_0(t-s)^{3} ds = \frac{q_0}{4} > p_0. \]

Therefore, if \( \frac{q_0}{4} > p_0 \), obviously, the conditions of Theorem 3.1 and Theorem 3.2 are satisfied, then Eq. (8) is oscillatory.

Then the conditions of Theorem 3.1 and Theorem 3.2 are satisfied.

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**Authors’ contributions**

All three authors contributed equally to this work. All authors read and approved the final manuscript.

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