Are we witnessing non-classical phenomena when noncontextuality inequalities are violated?

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Abstract

The view that non-classical phenomena are witnessed when noncontextuality inequalities are violated by quantum correlations has become widespread over the last decades, and many authors describe contextuality as a counter-intuitive and non-classical feature of quantum systems. In this paper we show that a classical input/output device generating contextual data (i.e., data violating noncontextuality inequalities) can easily be found if the following conditions are satisfied: (1) We only have access to a specific collection of “epistemic” measurements, namely measurements which do not necessarily reveal properties possessed by the system to the agent, and (2) not all these measurements can be jointly performed. The conclusion we draw from this example is twofold. Firstly, regarding “model-independent” contextuality, it suggests that, unlike Bell inequalities (whose violation allegedly implies non-locality), noncontextuality inequalities have no phenomenological significance per se: we cannot treat experiments as “black-boxes” and draw conclusions about the phenomenological status of their underlying structure based only on contextuality analysis of the data they generate. Secondly, regarding quantum theory, it indicates that one who accepts, as many interpretations suggest, that quantum theory satisfies their underlying structure based only on contextuality analysis of the data they generate.

1 Introduction

Simon Kochen and Ernst Specker famously showed that, under suitable conditions, one cannot understand self-adjoint operators on a Hilbert space $H$ ($\dim(H) > 2$) as representing properties simultaneously possessed by a physical system [1]. The “suitable conditions” are basically that (1) each bounded selfadjoint operator represents one, and only one, property of the system, (2) the value of a property represented by a selfadjoint operator $A$ lies in its spectrum $\sigma(A)$ and (3) if a selfadjoint operator $B$ is a function of an operator $A$ by means of the Borel functional calculus, i.e., if $B = g(A)$ for some Borel function $g : \sigma(A) \to \mathbb{R}$, then the value $V(B)$ possessed by $B$ has to be equal to $f(V(A))$, where $V(A)$ denotes the value possessed by $A$. A function $V : \mathcal{B}(H)_{sa} \to \mathbb{R}$, where $\mathcal{B}(H)_{sa}$ denotes the set of all bounded selfadjoint operators on $H$, satisfying conditions (2) and (3) listed above is said to be a valuation on $H$, and the Kochen-Specker theorem consists in the statement “there is no valuation on $H$ if $\dim(H) > 2$” [2, 3, 4]. As Kochen and Specker themselves pointed out [1], a corollary of this result is that, if $\dim(H) > 2$, no noncontextual hidden variable (NCHV) model (see definition 10) for $H$ exists. The reason is simple [3]: in such hypothetical model, each selfadjoint operator $A$ would be associated to a random variable $f_A : \Lambda \to \sigma(A)$ on the set $\Lambda$ of “hidden states”, whereas the mapping $A \mapsto f_A$ would “commute with the Borel functional calculus”, that is to say, if $B = g(A)$, then $f_B = g \circ f_A$. As a consequence, a hidden state $\lambda \in \Lambda$ would define a valuation $V_\lambda$ on $\mathcal{B}(H)_{sa}$ by $V_\lambda(A) = f_A(\lambda)$, contradicting the Kochen-Specker theorem. Nowadays, the non-existence of NCHV models for quantum systems is what is usually called “quantum contextuality” (or, more specifically, “Kochen-Specker contextuality”) [5].

All things considered, Kochen-Specker theorem breaks down the analogy between quantum observables (selfadjoint operations) and classical observables (i.e., observables in classical mechanics). In a NCHV model for a quantum system one tries to represent all quantum observables as measurable functions on a sample space, in complete analogy with the theoretical representation of classical observables, so the construction fails. Furthermore, the non-existence of valuations can be proved even in very “small” subsets of $\mathcal{B}(H)_{sa}$ (sets containing only nine operators, for example [6]), thus, even if one neglects the strong requirement (1), the analogy between
quantum and classical observables remains problematic - although not impossible [7]. Despite that, the non-existence of valuations does not preclude the possibility of representing quantum systems (or quantum behaviors - see appendix A) in terms of probability theory; we just need to give up defining quantum observables as classical observables, which, for all practical purposes, means that we cannot associate quantum observables one-to-one with random variables in such a way that conditions (2) and (3) are satisfied. An alternative, based on Spekkens’ ontological model [8], would be to associate selfadjoint operators to Markov Kernels. At any rate, even though we cannot consistently interpret all selfadjoint operators as representing properties simultaneously possessed by a system, we can, up to practical limitations, interpret them as representing measurement procedures which can be performed over the system. Kochen-Specker theorem, thus, endorses – or at least is compatible with – an “operational” view, shared by Bohr [9], on selfadjoint operators: they represent measurement procedures with no necessary connection with properties. As we know, this view is widely accepted among physicists, especially in the field of quantum information, and basically any nonrealist interpretation of quantum theory is compatible with it.

Following this reasoning, it seems plausible to conclude that experimental violation of noncontextuality inequalities [10, 11, 12] only reinforces the distinction between quantum and classical observables - or, according to our new terminology, between quantum measurements and observables. In fact, such tests show that experimental data obtained from quantum systems cannot be described by “NCHV-like models”, i.e., models where measurements are random variables and states are probability measures. Thus, if we insist – despite the Kochen-Specker theorem – in representing measurements in this way, we are forced to say that we cannot represent the state we are working with as a probability measure in the measurable space where such variables are defined. However, we have no decisive evidence in favor of this view; on the contrary, the existence of similarities between quantum states and states of incomplete knowledge, such as probability measures, is well known [13, 14]. Therefore, if on the one hand there is theoretical and conceptual evidence against the way in which measurements are represented in such models, and on the other hand there is no decisive evidence against the way states are represented, we should not blame the state for this failure.

The way one interprets quantum measurements has a strong impact on how one understands the violation of noncontextuality inequalities. As we see it, one who accepts the operational view on operators described above should not be surprised by this violation, given that it only proves that quantum predictions cannot be described by models where the representation of measurements conflicts with the conclusions we draw from the Kochen-Specker theorem. Furthermore, we believe that any phenomenologically meaningful criterion of classicality should tell us when, in some sense, “nature fails to respect classical physics”, as Michael D. Mazurek et al. put it [15]. We see no evidence that this is the case with noncontextuality inequalities, thus we believe one should not carelessly assume that contextuality is a signature of non-classicality. In order to defend this viewpoint, we argue that an “operational” or “epistemic” view on measurements, together with a restriction on which of these measurements can be jointly performed (a condition satisfied by quantum theory), might be capable, at least in some situations, of turning contextuality (i.e., the violation of noncontextuality inequalities) into an expected consequence of experimental procedures. We do it by showing that, under these conditions, a simple classical device can give rise to contextual data. The idea is that, if measurements do not reveal properties possessed by the system to the agent, and if there is a restriction on what can be jointly measured, then we have no ground for expecting noncontextuality in the data, whatever is the system upon which the measurements are performed. Besides that, the classical device we consider constitutes a black-box, i.e., an input/output device, which is in accordance with many “model-independent” or “operational” frameworks to contextuality. Our device is a particular case of the device considered in Refs. [16, 17], for example. Therefore, regardless of what we say about quantum contextuality, a weaker conclusion is still possible, namely that noncontextuality inequalities have no phenomenological significance per se. The way we see it, contextuality is not a physical phenomenon, let alone a non-classical phenomenon, thus noncontextuality inequalities are not on the same footing as Bell inequalities, as Christian de Ronde points out [4].

Many model-independent or operational frameworks intending to make contextuality analysis of experimental data possible have been proposed in recent years [18, 19, 20, 21, 22], and many authors motivate these constructions by saying that contextuality is a non-classical or counter-intuitive feature of quantum systems [18, 22, 5, 12]. That is the reason why we believe this discussion is important.

2 Black-boxes, noncontextuality inequalities and and the CH-approach

As we mentioned, the example we propose is a “black-box” or “input/output device” operating along the lines set out in Refs. [16, 17]. In this section we motivate
this model-independent view on experimental setups and introduce the compatibility-hypergraph (CH) approach to contextuality [5], which is the framework we will work with.

Model-independent or operational views on physics have been dominating many contemporary discussions in quantum foundations, and contextuality is no exception. As Sandu Popescu puts it, such views are grounded on the realization that, allegedly, physics can be represented “in a way that is largely independent of the details of the specific underlying theories” [23]. For all practical purposes, it means that experiments can be seen as “input/output devices” or “black boxes”. Popescu exemplifies this perspective as follows [23].

“Suppose Alice has a box that accepts inputs $x$ and yields outputs $a$. One can imagine that inside the box there is an automated laboratory, containing particles, measuring devices, and so on. The laboratory is prearranged to perform some specific experiments; the input $x$ simply indicates which experiment is to be performed. Suppose also that for every measurement we know in advance the set of the possible outcomes; the output $a$ is simply a label that indicates which of the results has been obtained. In this framework, the entire physics is encapsulated in $P(a|x)$, the probability that output $a$ occurs given that measurement $x$ was made.”

(emphasis added). Note that an entire laboratory counts as a black-box, so, according to this view, basically any experimental setup is a legitimate device.

Popescu emphasizes the model-independent view to illustrate how nonlocality “beyond quantum mechanics” is possible: it does not matter what Alice and Bob have inside their boxes; if the data they obtain violates Bell’s inequalities, we can say that non-locality has been witnessed. As he says, “the entire physics is encapsulated” in the data. Similarly, when contextuality is defined in terms of noncontextuality inequalities or some similar concept like global sections, contextuality “beyond quantum mechanics” becomes mathematically – though not necessarily conceptually – meaningful: if a experimental setup generates contextual data (namely data violating noncontextuality inequalities or having no global section), we can say that contextuality has been witnessed. From this viewpoint, contextuality is a property of experimental data, as those who believe that the entire physics is encapsulated in the data should expect. As we mentioned, lots of model-independent frameworks to contextuality have been proposed in recent years [18, 19, 20, 5, 21, 22]. These frameworks allow us to characterize and quantify contextuality regardless of the reason why it occurs. In this paper we will work with the so-called compatibility-hypergraph (CH) approach to contextuality, which is closely related to the so-called Sheaf-theoretic approach [18]. We will briefly introduce it now. A more precise description of this framework can be found in appendix A.

The starting point of the CH approach is the idea of measurement scenario (definition 2). It consists in a finite set $A$ representing measurements which can be performed over some physical system, together with a collection $C$ of maximal contexts. A context is a set $C \subseteq A$ of compatible measurements, i.e., measurements which can be jointly performed, and we assume that the collection $C$ satisfies the following conditions: (1) Any $C \subseteq C$ is maximal, i.e., if $C', C'' \subseteq C$ satisfy $C'' \subseteq C'$ then $C' = C''$, and (2) each measurement belongs to at least one context, that is, $\cup C = A$. Denoting by $O$ the set of possible outcomes for all measurements, we say that the triple of sets $S \equiv (A, C, O)$ is a measurement scenario [5, 18].

If the system is prepared in a certain way and all measurements of a context $C$ are performed, a joint outcome, which we represent as a function $C \rightarrow O$, will be obtained. If we prepare the system in exactly the same way – by “exactly the same” we mean that the agent cannot distinguish both preparation procedures – and run the experiment a second time, we will not necessarily obtain the same joint outcome. It means that a preparation procedure does not determine an outcome for each measurement of $C$ but actually defines a probability distribution $p(\cdot|C)$ on the set $O^C$ of all functions $C \rightarrow O$; this probability distribution is experimentally accessible by means of successive runs of the experiment. This is true for any context $C \subseteq C$, therefore a preparation procedure defines a mapping $C \xrightarrow{p} p(\cdot|C)$ associating to each context $C$ a probability distribution $p(\cdot|C)$ on $O^C$. This mapping $p$ is said to be a behavior (or empirical model [18]) in the measurement scenario $S$ (see definition 3).

A behavior $p$ in a scenario $S$ is said to be noncontextual if a probability distribution $p$ on $O^A$ exists such that, for any context $C$, its marginalization on $O^C$ coincides with $p(\cdot|C)$ (see definition 7); otherwise $p$ is said to be contextual. The set of all noncontextual behaviors in a scenario $S$ defines a polytope, so it can be characterized by finitely many linear inequalities; a noncontextuality inequality is a linear inequality characterizing the polytope of noncontextual behaviors on $S$ which is satisfied by all noncontextual behaviors but violated by some contextual ones – we suggest Ref. [5] for an explicit definition. The important point is that a noncontextuality inequality is violated by a behavior $p$ only if $p$ is contextual.

To conclude this section, let’s see how this definition of contextuality relates with random variables. Let $p$ be a
behavior in a scenario $S \equiv (\mathcal{A}, C, O)$. We say that a probability space $\Lambda \equiv (\Lambda, \Sigma, \mu)$ is a classical realization for $p$ if a mapping $A \rightarrow f_A$ associating measurements $A \in \mathcal{A}$ to random variables $f_A : \Lambda \rightarrow O$ exists such that, for any context $C$, $p(\cdot|C)$ is the joint distribution of the collection $f_A, A \in C$ (see definition 5). Note that this definition is strongly based on definition 10, i.e., a classical realization is basically a noncontextual hidden variable model for $p$, and a precise connection between these definitions is established by the idea of quantum behavior (see definition 6). In any case, the fact is that a classical realization for $p$ exists if and only if $p$ is noncontextual, as one can easily show [5]. Therefore, an experimental test showing that a quantum behavior (definition 6) violates a noncontextuality inequality tells us that we cannot describe this behavior using a “NCHV-like model”, as we said in the introduction.

3 Classical observables, coin toss and epistemic measurements

We now motivate the choices we make in our classical device. The measurement procedures which take place inside the device are strongly based on coin tossing, which in turn is frequently presented as an example of (or at least as operationally equivalent to) a valid measurement procedure in operational theories – see, for example, Refs. [15, 8]. We also explain what we mean by “epistemic” measurement here.

For the sake of clarity, let’s recall how to describe a system of $N$-particle in classical mechanics. To begin with, we fix a phase space - which we assume to be $\Lambda \equiv \mathbb{R}^{3N} \times \mathbb{R}^{3N}$ here - encoding the collection of all possible pure states for this system w.r.t. a inertial reference frame previously chosen. A pure state is a pair $\lambda \equiv (q, p)$, where $q \in \mathbb{R}^{3N}$ encodes the positions of all particles and $p \in \mathbb{R}^{3N}$ encodes their momenta. A mechanical property (we also say mechanical observable or simply observable) of a classical system can always be represented as a (measurable) function $f : \Lambda \rightarrow \mathbb{R}$, and the reason is that, according to the realist foundation of classical mechanics, the state of affairs of a system (encoded in the theory by a pure state) determines ‘the way things are’ [24]: if $A$ is a mechanical property, by knowing the state of affairs $\lambda$ of the system we must be able to know the value of $A$, given that this state allows us to know everything that, from a mechanical perspective, can be known about the system; thus, the theoretical representation of $A$ is a function $f_A$ associating each state of affairs $\lambda$ to the value $A$ assumes in this state. Note that, by construction, the value of any (mechanical) observable can be inferred from the values of only two classical properties: position and momentum.

A pure state $\lambda$ in a classical system ascribes values for all (mechanical) observables, which means that it defines a “valuation” on $\mathcal{M}(\Lambda)$, the set off all measurable functions on $\Lambda$, by $V_{\lambda}(f) \equiv f(\lambda)$ – note that, if $g = h \circ f$, then $V_{\lambda}(g) = h(f(\lambda)) = h(V_{\lambda}(f))$. A standard measurement in classical mechanics (which we call ontic measurement here) is a procedure which reveals to the agent the value of an observable. When such measurement is performed, only one outcome is obtained, namely the value determined by the pure state $\lambda$ describing the state of affairs of the system at that time. If the agent knows this state, she can predict this value with certainty; if she does not have complete knowledge about the system, then all she can do is to determine a probability measure $\mu$ (or, equivalently, a probability density) on $\Lambda$, which will allow her to infer, given any Borel set $\Delta \subset \mathbb{R}$, the probability $\mu(f^{-1}(\Delta))$ of obtaining an outcome $a \in \Delta$ in this measurement, being $f$ (the theoretical representation of) the observable she is measuring [24].

As we know, a coin toss is totally different from an ontic measurement in classical mechanics, and this distinction is far from being explained by the fact that we usually do not have complete knowledge about the state of a coin when it is tossed. In fact, even though a coin is an object that can be described by classical mechanics – i.e., a coin is a classical system –, the outcomes of a coin toss, namely heads or tails, are not values of an observable of this system. There is no mechanical observable being measured when a coin is tossed; there is nothing like a “flipness” property for such a system. Outcomes of a coin toss only make sense when the measurement event happens: asking someone if the coin I have in my pocket is heads or tails is totally meaningless – not because he does not have complete knowledge about the state of the system but because the measurement has not been performed (or, to put it differently, because there is no such thing as “flipness”). To sum up, a coin toss is a measurement procedure based on a classical system (a coin) which satisfies the following conditions.

- There is no property of the classical system associated with the measurement event.
- Outcomes are intrinsically dependent on the measurement procedure and are meaningless unless the measurement is performed. In particular, they cannot be “possessed” by the system.

From a theoretical perspective, we can say that, if there is no mechanical property associated with a given measurement (like a coin toss), there is no reason to require that the formalism of classical mechanics must be able to represent this measurement as a function on the phase space $\Lambda$ of the system (in the case of a coin toss, we can represent a coin as a system of 3-particles, for the sake of simplicity). This is true even if pure states...
allow us to predict with certainty the outcome of the measurement - again, the coin toss is an example. The formalism accounts for properties only, and it does not have to provide a theoretical representation of a process just because the system participates in it.

A coin toss is an example of what we call an epistemic measurement in classical mechanics. Following Bohr’s well known view on measurements [9, 25], by an epistemic measurement we mean a reproducible interaction between a classical system and an apparatus (which can range from a photon to something way more complex) satisfying the following conditions.

(a) Every time it happens, the interaction gives rise to a measurement event (which we call the outcome of the measurement), depending on the state of the classical system right before the beginning of the interaction.

(b) Each state (pure or not) of the system determines a probability distribution (possibly deterministic) on the set of outcomes of the epistemic measurement. This distribution is experimentally accessible by means of successive runs of the experiment.

(c) The measurement event does not necessarily reveal a property of the classical system to the agent. Consequently, assertions about outcomes are in principle meaningless if no interaction occurs.

Note that the distinction we draw between ontic and more general epistemic measurements is only possible because we have at hand the conceptual framework of classical mechanics. We can only say that a specific measurement “does not reveal a property of the classical system” because we know what a property in classical mechanics is.

4 A contextual but non-disturbing behavior

As we said in section 2, we will use the compatibility-hypergraph (CH) approach to contextuality to analyze data. The measurement scenario (definition 2) we work with is the well known n-cycle scenario. A n-cycle is a scenario $S_n$ containing $n$ dichotomic measurements $A_0, \ldots, A_n$ and $n$ maximal contexts $C_i \equiv \{A_i, A_{i+1}\}$, $i = 0, \ldots, n$ (if $i = n$, then $i + 1 \equiv 0$). Given a behavior $p$ on $S_n$ (definition 3), we will sometimes denote its component $C_i$ by $p(\cdot|A_i, A_{i+1})$ instead of $p(\cdot|C_i)$. We have only four possible joint outcomes in context $C_i$, namely $(\perp, \perp), (\perp, T), (T, \perp)$ and $(T, T)$. We are interested in the following well-known behavior [5].

Definition 1 (generalized coin toss) Let $S_n$ be the n-cycle. A “generalized coin toss” on $S_n$ is the behavior $p$ defined as follows. For any context $C_i \equiv \{A_i, A_{i+1}\}$, $p(\cdot|C_i) \equiv p(\cdot|A_i, A_{i+1})$ is given by

\begin{align}
    p(\perp, \perp|A_i, A_{i+1}) &= 0 = p(T, T|A_j, A_{j+1}) \\
    p(\perp, T|A_i, A_{i+1}) &= \frac{1}{2} = p(T, \perp|A_i, A_{i+1})
\end{align}

Straightforward calculations show that this behavior is nondisturbing (definition 4). As a consequence, $p$ associates, for any measurement $A_i$, a unique (i.e., context-independent) probability distribution $p(\cdot|A_i)$ on $\{\perp, T\}$ (just take any component of $p$ containing $A_i$ and marginalize it). All measurements $A_i, i = 1, \ldots, n$, have the same distribution, given by

$$p(\perp|A_i) = \frac{1}{2} = p(T|A_i).$$

In the CH approach, two measurement events (definition 8) are said to be exclusive if they ascribe distinct outcomes to a common measurement (definition 9). In a n-cycle scenario we can say, without loss of generality, that two events $(u_i, u_{i+1}|C_i)$, $(v_j, v_{j+1}|C_j)$ are exclusive if and only if one of the following conditions are satisfied: (1) $i = j$ and $(u_i, u_{i+1}) \neq (v_i, v_{i+1})$ or (2) $j = i + 1$ and $u_{i+1} \neq v_{i+1}$. Now take $n = 5$. For any $i = 0, \ldots, 4$, denote by $E_i$ the measurement event $(T, \perp|A_i, A_{i+1})$. The exclusivity graph determined by these events consists in a graph $G$ having $\{E_0, \ldots, E_4\}$ as set of vertices and connecting a pair of vertices by an edge if and only if they are exclusive events. This graph $G$ is cyclic and can be represented as follows.

By definition, the generalized coin toss ascribe probability $\frac{1}{2}$ for each of these events (see definition 8), which implies that $\sum_{i=0}^{4} p(E_i) = \frac{5}{2}$, which is greater than the Lovász number of this graph, which is known to be $\sqrt{5}$. As proved in [19], the fact that $\sum_{i=0}^{4} p(E_i)$ is strictly greater than the Lovász number of $G$ implies that this probability assignment on $G$ cannot be reproduced by quantum theory, in the sense that we cannot associate each vertex $E_i$ to a projection $P_i$ on a finite-dimensional Hilbert space $H$ is such a way that the following conditions are satisfied: (1) $P_i P_j = 0 = P_j P_i$ whenever $E_i$ and
$E_i$ are exclusive and (2) There is a density operator $\rho$ on $\mathcal{H}$ satisfying, for any $i = 0, \ldots, 4$, $p(E_i) = \text{tr} (\rho P_i)$ [19]. It proves the well known result [5] that, if $n = 5$, the generalized coin toss has no quantum realization (definition 6); in particular, it is contextual (definition 7).

5 Inside the device: a coin of many sides

Now let’s show how a classical input/output device can give rise to the behavior we named “generalized coin toss” (definition 1).

As we have discussed in section 3, a coin toss is a measurement procedure which, by means of a convention, return one out two previously fixed characteristics (opposite sides) of a coin. Given that, for practical purposes, a coin has only one pair of opposite sides, we usually understand a coin toss as a measurement procedure where only one dichotomic measurement is performed. In a similar fashion, by conceiving a classical object having $n$ pairs of opposite sides, we can propose $n$ distinct epistemic measurements - one for each pair of opposite sides - to be performed with such an object. We represent this object as a regular polygon of $2n$ sides, and figure 1 depicts it for $n = 5$. That is what we mean by a “coin of many sides”.

A coin toss can be idealized as a rotation around a fixed axis passing through the center of the coin, and so will be our measurements. In order to perform measurement $A_0$, for example (see figure 1), we rotate the object around the axis represented by an horizontal dashed line in figure 1; parallel to this axis there is a detector, depicted by a small dashed line, which is capable of identifying colors and distinguishing between light and dark color; it return $(\top | A_0)$ in case it detects a dark red at the end of the movement and $(\perp | A_0)$ in case it detects a light red. Figure 2 represents five axes, and their respective detectors, which allows us to perform single measurements upon this object.

To sum up, each single measurement in this device is associated with a color (a characteristic of the object), and a measurement procedure consists in rotating the object around the axis determined by the measurement; we have two opposite sides for each color, and one, and only one, of these sides is regarded as the outcome of a single measurement.

We also have axes and detectors allowing us to jointly perform pairs of consecutive measurements. The vertical line in figure 1 represents an axis which, together with an appropriate detector (also depicted in figure 1), allows us to jointly measure $A_2, A_3$ (associated, respectively, with yellow and red). Figure 3 represents a set of axes which, together with appropriate detectors, allows us to measure any pair of consecutive measurements.

In our thought experiment, this trivial classical object (a “coin of many sides”) lies inside a black-box, which means that the agent knows nothing about the mechanism giving rise to the data. Outside the black-box there are buttons, one for each color, which we also denote by $A_0, \ldots, A_4$; obviously, button $A_i$ is associated to measurement $A_i$. The axes (i.e., contexts) determine which buttons can be pressed at the same time, therefore only consecutive buttons can be simultaneously pressed. Note that, as we said, it operates along the lines set out
in Refs. [16, 17]. If all we have access to is the data provided by these epistemic measurements, we should expect that, by “flipping this coin” (i.e., rotating the object around an axis with an “uncontrolled kick”) several times in all possible contexts, the resulting data would match behavior 1. In fact, due to a mere geometrical property of the object, in a joint measurement of a pair $A_i, A_{i+1}$ only events $(\top, \bot | A_i, A_{i+1})$ and $(\bot, \top | A_i, A_{i+1})$ can happen, and, for the same reason why the probability of obtaining heads and tails in a coin toss is half, the probability of obtaining each one of these joint outcomes is half. For analogous reasons, equation 3 describes data obtained when single measurements are performed, also in accordance with definition 1. Thus, if all we are able to do is to perform this limited collection of epistemic measurements upon this simple classical system, the data we obtain is contextual.

6 Conclusion

We have described an input/output device working along the lines set out in Refs. [16, 17] which is capable of generating contextual – and even non-quantum – data, despite the fact that it is based on a simple classical object. The reason why contextual data arises from this classical device is twofold. Firstly, we deal with “epistemic” measurements, that is to say, measurements which do not reveal to the agent classical properties possessed by the system inside the box. Secondly, we impose a restriction on which measurements can be jointly performed. According to many interpretations of quantum theory, including the orthodox one, these conditions are satisfied by measurement procedures in quantum systems. Therefore, we believe one should not carelessly assume that quantum contextuality is a signature of non-classicality: as far as we can tell, there is no evidence that, in any sense, “nature fails to respect classical physics” [15] when non-contextuality inequalities are violated by quantum systems. Beyond quantum theory, i.e., considering model-independent contextuality, our example suggests that non-contextuality inequalities have no ontological significance per se: contextuality analysis of data generated by a black-box provides no ground for assertions about the ontological status of its underlying structure. In particular, at least in the case of black-boxes operating along the lines set out in Ref. [16, 17], contextuality does not require non-classical phenomena. The way we see it, contextuality is not, like non-locality, a physical phenomenon, let alone a non-classical phenomenon, so, as Christian de Ronde points out, noncontextuality and Bell inequalities are not on the same footing [4].

To conclude, it is worth emphasizing that it is clear for us that our thought experiment is far from being a decisive step towards a proper understanding of contextuality in quantum theory. The way we interpret measurements and contexts can be criticized in many ways, and the fact that our device generates non-quantum data makes it clear that quantum measurements are not so simple. However, it is important to recall that many model-independent frameworks to contextuality [18, 21, 26, 19] and many operational theories [27] require basically nothing from devices and measurements; coin flips, which motivate our example, are frequently presented as valid measurement procedures in operational theories [8, 15]. Also, as we said, our device follows the rules imposed in Refs. [16, 17]. Thus, even if all we have said about quantum theory is misleading, we believe our example at least shed some light on the dangers of approaching contextuality – and physics in general – from a purely operational perspective. We should not forget that model-independent frameworks to contextuality were originally proposed only as tools for characterizing and quantifying contextuality, not as a means for phenomenological speculation.

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A Compatibility-hypergraph approach to contextuality

The main reference of this appendix is Ref. [5].

Definition 2 (Scenario) A scenario is a triple \( S \equiv (A, C, O) \) where \( A, O \) are finite sets (whose elements represent measurements and outcomes respectively) and \( C \) is a collection of subsets of \( A \) (representing maximal sets of compatible measurements) satisfying the following conditions.

(a) \( A = \cup C \)

(b) For \( C, C' \in C \), \( C' \subseteq C \) implies \( C' = C \)

The approach is named "hypergraph-approach" because a scenario \( (A, C, O) \) can be associated with a hypergraph whose vertices are the elements of \( A \) and whose hyper-edges are the elements of \( C \) [5].

The result of a joint measurement over a context \( C \) can be represented as a function \( C \rightarrow O \). Therefore, the set \( O^C \) of all functions \( C \rightarrow O \) can be understood as the set of all possible outcomes of a joint measurement on \( C \). Behaviors allows us to encode experimental data obtained from joint measurements on a measurement scenario[6].

Definition 3 (behavior) Let \( S \) be a scenario. A behavior on \( S \) is a function \( p \) which associates to each context \( C \) a probability distribution \( p(\cdot|C) \) on \( O^C \), that is to say, for each context \( C \), \( p(\cdot|C) \) is a function \( O^C \rightarrow [0,1] \) satisfying \( \sum_{s \in O^C} p(s|C) = 1 \).

A behavior whose components match in intersections of contexts is said to be non-disturbing [5]:

Definition 4 (Non-disturbance) A behavior \( p \) in a scenario \( S \) is said to be non-disturbing if, for any pair of intersecting contexts \( C, D \), equality

\[
 p(\cdot|C \cap D, C) = p(\cdot|C \cap D, D) \tag{4}
\]

holds true, where, for any \( E \in C \) and \( E' \subset E \), \( p(\cdot|E', E) \) denotes the marginal of \( p(\cdot|E) \) on \( O^{E'} \), namely

\[
 \forall t \in O^{E'}: \quad p(t|E', E)(t) = \sum_{s \in O^{E'}: s|E' = t} p(s|E). \tag{5}
\]

Classical and quantum behaviors arise when ideal measurements are performed upon classical and quantum systems respectively. They are defined as follows [5].

Definition 5 (Classical behavior) Let \( p \) be a behavior in a scenario \( S \equiv (A, C, O) \). A probability space \( \Lambda \equiv (\Lambda, \Sigma, \mu) \) is said to be a classical realization for \( p \) if we can associate each measurement \( A \) of \( S \) to a random variable \( f_A: \Lambda \rightarrow O \) on \( \Lambda \) is such a way that, for any context \( C \), \( p(\cdot|C) \) is the joint distribution of \( f_A: A \subseteq C \), which means that, for any \( s \in O^C \),

\[
 p(s|C) = \mu \left( \bigcap_{A \in C} f_A^{-1}(\{s_A\}) \right), \tag{6}
\]

where \( s_A \equiv s(A) \). We say that a behavior \( p \) is classical if a classical realization for \( p \) exists.

Note that classical realizations are closely related to NCHV models (definition 10). The definitions of quantum realization and quantum behavior goes as follows [5].

Definition 6 (Quantum behavior) Let \( p \) be a behavior in a scenario \( S \equiv (A, C, O) \). A pair \((H, \rho)\), where \( H \) is a finite dimensional Hilbert space and \( \rho \) is a density operator on \( H \), is said to be a quantum realization for \( p \) if we can associate each measurement \( A \in A \) to a selfadjoint operator \( T_A \in B(H) \) is such a way that the following conditions are satisfied

(a) for any measurement \( A \), \( O \) is isomorphic to the spectrum of \( T_A \)

(b) The image of each context lies inside a commutative algebra, i.e., if \( A, B \in C \) for some context \( C \), then \( [T_A, T_B] = 0 \)

- For each context \( C \), \( p(\cdot|C) \) can be reproducted by the Born rule, i.e., for any \( s \in O^C \),

\[
 p(s|C) = \text{tr} \left( \rho \prod_{A \in C} P_{s_A}^{(A)} \right), \tag{7}
\]

where \( P_{s_A}^{(A)} \) denotes the projection associated with the subspace of \( H \) spanned by the eigenvalue \( s_A \) of \( A \).

We say that a behavior \( p \) is quantum if a quantum realization for \( p \) exists.

The definition of non-contextuality in the CH approach goes as follows [5].

Definition 7 (Non-contextuality) A behavior \( p \) in a scenario \( S \) is said to be non-contextual if there is a probability distribution \( \overline{p}: O^A \rightarrow [0,1] \) satisfying, for any context \( C \),

\[
 p(\cdot|C) = \overline{p}_C, \tag{8}
\]

where \( \overline{p}_C \) denotes the marginal of \( \overline{p} \) in \( O^C \), i.e., for any \( s \in O^C \),

\[
 \overline{p}_C(s) = \sum_{t \in O^A: \{t\} = s} \overline{p}(t). \tag{9}
\]
One can easily prove that non-contextuality and classicality (i.e., being classical) are equivalent concepts in the compatibility-hypergraph approach [3].

It is easy to show that any classical behavior has a quantum realization, which means that any classical behavior is quantum [5]. It is also easy to show that any quantum behavior is non-disturbing [5]. Therefore, if we denote by $\mathcal{N}C(S)$, $\mathcal{D}(S)$ and $\mathcal{N}F(S)$ the sets of noncontextual (or, equivalently, classical), quantum and non-disturbing behaviors, respectively, on a scenario $S$, we obtain the well known chain of inclusions

$$\mathcal{N}C(S) \subset \mathcal{D}(S) \subset \mathcal{N}F(S).$$

The behavior we are interested in (definition ??) is disturbing but lies outside the quantum set.

To conclude, we introduce measurement events and the exclusivity relation between them.

**Definition 8 (Measurement event)** Let $S$ be a measurement scenario. A measurement event on $S$ is pair $(s|C)$ where $C$ is a context and $s$ is a joint outcome on $C$ (i.e., $s \in O^C$). Given a behavior $p$ on $S$, the probability of $(s|C)$ with respect to $p$ is the number $p(u|C)$ determined according to definition 3.

**Definition 9 (Exclusive events)** Let $S$ be the a scenario and let $E \equiv (u|C)$, $F \equiv (v|D)$ be measurement events on it. Then $E$ and $F$ are said to be exclusive if there is a measurement $A \in C \cap D$ such that $u(A) \neq v(A)$.

Note that, given any context $C$, $(u|C)$ and $(v|C)$ are exclusive if and only if $u \neq v$.

## B Hidden variable models for quantum systems

By a noncontextual hidden variable model for a quantum system we mean the following [1, 3].

**Definition 10 (NCHV model)** Let $H$ be a separable Hilbert space. Let $M \equiv (\Lambda, \Phi, \Omega)$ a triple where $\Lambda \equiv (\Lambda, \Sigma)$ is a measurable space, $\Phi$ is a mapping associating each bounded selfadjoint operator $A \in \mathcal{B}(H)$ to a measurable function $\tilde{f}_A \equiv \Phi(A)$ on $\Lambda$ onto $\sigma(A)$ and, finally, $\Omega$ is a mapping each associating each pure state $P_\psi \equiv \langle \psi, \cdot \rangle \psi$ to a probability measure $\Omega(P_\psi) \equiv \mu_\psi$ on $\Lambda$. Then $M$ is said to be a NCHV model for $H$ if the following conditions are satisfied.

(a) $\Phi$ “commutes with the Borel functional calculus”, i.e., for any bounded selfadjoint operator $A$ and any measurable function $g : \sigma(A) \rightarrow \mathbb{R}$ we have

$$f_g(A) = g \circ f_\Lambda.$$

(b) For any bounded selfadjoint operator $A \in \mathcal{B}(H)$ and any quantum state $P_\psi \equiv \langle \psi, \cdot \rangle \psi$,

$$\langle \psi, A \psi \rangle = \int_\Lambda f_\Lambda \, d\mu_\psi.$$
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