PALAIS-SMALE VALUES AND STABILITY OF GLOBAL HöLDERIAN ERROR BOUNDS FOR POLYNOMIAL FUNCTIONS

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Abstract. Let \( f \) be a polynomial function of \( n \) variables. In this paper, we study stability of global Hölderian error bound for a sublevel set \([f \leq t]\) under a perturbation of \( t \). Namely, we investigate the following questions:

1. Suppose that \([f \leq t]\) has a global Hölderian error bound, when does there exist an open interval \( I(t) \subset \mathbb{R}, t \in I(t) \), such that for any \( t' \in I(t) \), \([f \leq t']\) has also a global Hölderian error bound?

2. Suppose that \([f \leq t]\) does not have global Hölderian error bound, when does there exist an open interval \( I(t) \subset \mathbb{R}, t \in I(t) \), such that for any \( t' \in I(t) \), \([f \leq t']\) does also not have global Hölderian error bound?

3. Are there other types of stability which are different from types in questions 1 and 2?

To answer these questions, we compute the set \( H(f) \) of all values \( t \in \mathbb{R} \) such that \([f \leq t]\) has a global Hölderian error bound and study the relationship between \( H(f) \) and the set of Palais-Smale values of \( f \). Our main results are the following:

- The list of all possible types of stability is given;
- For an arbitrary \( t \in \mathbb{R} \), we can determine the type of stability of global Hölderian error bound for \([f \leq t]\).

1. Introduction

Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a polynomial function. For \( t \in \mathbb{R} \), put

\[
[f \leq t] := \{ x \in \mathbb{R}^n | f(x) \leq t \}
\]

and \([a]_+ := \max\{0, a\}\).

Definition 1.1. [Hà] We say that the set \([f \leq t]\) has a global Hölderian error bound (GHEB for short) if there exist \( \alpha, \beta, c > 0 \) such that

\[
[f(x) - t]_+^\alpha + [f(x) - t]_+^\beta \geq c \text{dist}(x, [f \leq t]) \quad \text{for all } x \in \mathbb{R}^n.
\]

The existence of error bounds have many important applications, including convergence analysis in optimization problems (see [Luo, P, BNPS, DL]), variational inequalities and identifying active constraints ([LTW]).
The study of error bounds has received rising awareness in many papers of mathematical programming in recent years. We mention some of them [LL, LS, Y, LiG1, LiG2, Ha, Ng, LMP, DHP] (for the case of polynomial functions) and [Hoff, Ro, M, AC, LiW, K, KL, P, LP, Luo, NZ, CM, LTW, I, BNPS, DL] (for non-polynomial cases). The reader is referred to survey papers [LP, P, Az, I] and the references therein for the theory and applications of error bounds.

In this paper, we study stability of a global Hölderian error bound for the set \([f \leq t]\) under a perturbation of \(t\). The following natural questions arise

1. Suppose that \([f \leq t]\) has a GHEB, when does there exist an open interval \(I(t) \subset \mathbb{R}, t \in I(t)\), such that for any \(t' \in I(t)\), \([f \leq t']\) also has a GHEB?

2. Suppose that \([f \leq t]\) does not have GHEB, when does there exist an open interval \(I(t) \subset \mathbb{R}, t \in I(t)\), such that for any \(t' \in I(t)\), \([f \leq t']\) also does not have GHEB?

3. Are there other types of stability which are different from types in questions 1 and 2?

To answer these questions, our idea is computing the set

\[
H(f) := \{t \in \mathbb{R} : [f \leq t] \text{ has a global Hölderian error bound}\}
\]

and investigating the relationship between \(H(f)\) and the set

\[
PS(f) := \{t \in \mathbb{R} : \exists \{x^k\} \subset \mathbb{R}^n, \|x^k\| \to \infty, \|\nabla f(x^k)\| \to 0, f(x^k) \to t\}
\]

of Palais-Smale values of \(f\).

We will show that there exists a value \(h(f) \in PS(f) \cup \{\pm \infty\}\) and a subset \(PS^1(f)\) of \(PS(f)\), such that

Either \(H(f) = [h(f), +\infty) \setminus PS^1(f)\) or \(H(f) = (h(f), +\infty) \setminus PS^1(f)\).

Next, we use the above formulas for studying the stability of a GHEB and answer the questions 1 and 2. Moreover, we discover some other types of stability which are different from the types in questions 1-2 and give the list of all possible types of stability.

The paper is organized as follows. In Section 2, we give two different formulas for computing the set \(H(f)\). The first formula is based on criterion for the existence of GHEB for \([f \leq t]\), given in [Ha]. The second formula follows from a new criterion for the existence of global Hölderian error bounds. In Section 3, the relationship between \(H(f)\) and the set of Palais-Smale values of \(f\) will be established. In Section 4, we use the formulas of \(H(f)\) and relationship between \(H(f)\) and \(PS(f)\) to study our problems. It is well-known that \(PS(f)\) is either empty set, or \(PS(f)\) is non-empty finite subset of \(\mathbb{R}\), or \(PS(f) = \mathbb{R}\). Hence, it is convenient to consider each of these cases separately.
In Subsection 4.1, we consider the case $PS(f) = \emptyset$. In this case, $H(f) = \mathbb{R}$ (Theorem 4.1). Therefore, there is only one type of stability of GHEB’s. Namely, any point $t$ of $\mathbb{R}$ is $y$-stable, by this we mean that $t \in H(f)$ and there exists an open interval $I(t)$ such that $t \in I(t) \subset H(f)$. Note that, for almost every polynomial $f$, $PS(f) = \emptyset$. Hence, $H(f) = \mathbb{R}$ if $f$ is generic (Remark 4.1).

In Subsection 4.2, we consider the case when $PS(f)$ is a non-empty finite set. In this case

- $H(f) \neq \emptyset$ (Proposition 4.1);
- Beside of $y$-stable type, there are at most 4 other types of stability of GHEB’s.

Any value $t$ of $\mathbb{R}$ belongs to one of the following types

**Case A:** If $h(f) = -\infty$, then there are 2 types

1. $t$ is $y$-stable;
2. $t$ is a $n$-isolated point: $t \in \mathbb{R} \setminus H(f)$ and for $\epsilon > 0$ sufficiently small, $(t - \epsilon, t) \cup (t, t + \epsilon) \subset H(f)$.

**Case B:** If $h(f)$ is a finite value, then there are 5 types

1. $t$ is $y$-stable;
2. $t$ is $y$-right stable: $t \in H(f)$ and there exists $\epsilon > 0$ such that $[t, t + \epsilon) \subset H(f)$ and $(t - \epsilon, t) \cap H(f) = \emptyset$;
3. $t$ is $n$-stable: $t \in \mathbb{R} \setminus H(f)$ and there exists an open interval $I(t)$ such that $t \in I(t) \subset \mathbb{R} \setminus H(f)$;
4. $t$ is $n$-left stable: $t \in \mathbb{R} \setminus H(f)$ and there exists $\epsilon > 0$ such that $(t - \epsilon, t] \subset \mathbb{R} \setminus H(f)$ and $(t, t + \epsilon) \cap (\mathbb{R} \setminus H(f)) = \emptyset$;
5. $t$ is an $y$-isolated point;

Note that, if $t$ is $y$-right stable or $t$ is $n$-left stable, then it is necessarily that $t = h(f)$.

- We can determine the type of stability of any $t \in \mathbb{R}$ (Theorem 4.3);
- We give a criterion for $\#PS(f) < +\infty$ (Proposition 4.2);
- We give an estimation of the number of connected components of $H(f)$ (Theorem 4.4);

In Subsection 4.3, we consider the case when $PS(f) = \mathbb{R}$. In this case

- Any value $t$ of $\mathbb{R}$ belongs to one of the following types
  1. $t$ is $y$-stable;
  2. $t$ is $y$-right stable;
  3. $t$ is $y$-left stable: $t \in H(f)$ and there exists $\epsilon > 0$ such that $(t - \epsilon, t] \subset H(f)$ and $(t, t + \epsilon) \cap H(f) = \emptyset$;
  4. $t$ is an $y$-isolated point: $t \in H(f)$ and for $\epsilon > 0$ sufficiently small, $(t - \epsilon, t) \cup (t, t + \epsilon) \subset \mathbb{R} \setminus H(f)$;
  1'. $t$ is $n$-stable;
2’. $t$ is $n$-right stable: $t \in \mathbb{R} \setminus H(f)$ and there exists $\epsilon > 0$ such that 
$[t, t + \epsilon) \subset \mathbb{R} \setminus H(f)$ and $(t - \epsilon, t) \cap \mathbb{R} \setminus H(f) = \emptyset$;

3’. $t$ is $n$-left stable;

4’. $t$ is an $n$-isolated point.

- We can determine the type of stability of any $t \in \mathbb{R}$ (Theorem 4.5).

In Section 5, we consider the case $n = 2$. We show that

- For any polynomial $f$ in two variables, $H(f) \neq \emptyset$ (Theorem 5.1 and Remark 5.1);
- Using the Newton-Puiseux expansions at infinity of affine algebraic curves, we give an algorithm for computing the set $H(f)$ where $f$ is an arbitrary polynomial. This allows us to understand the problem of stability of GHEB’s in the case of two variables completely.

Finally, in Appendix, we recall briefly how to reduce the problem of computation of Newton-Puiseux expansions at infinity to the well-known procedure of computation of Newton-Puiseux expansions in a neighbourhood of a singular point of an algebraic curve.

2. The set $H(f)$

2.1. The first formula of $H(f)$.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial function and $t \in \mathbb{R}$.

**Definition 2.1 ([DHN][Ha]).** We say that

(i) A sequence $\{x^k\} \subset \mathbb{R}^n$ is the first type of $[f \leq t]$ if

$$
\|x^k\| \to \infty,
$$

$$
f(x^k) > t, f(x^k) \to t,
$$

$$
\exists \delta > 0 \text{ s.t. } \text{dist}(x^k, [f \leq t]) \geq \delta.
$$

(ii) A sequence $\{x^k\} \subset \mathbb{R}^n$ is the second type of $[f \leq t]$ if

$$
\|x^k\| \to \infty,
$$

$$
\exists M \in \mathbb{R} : t < f(x^k) \leq M < +\infty,
$$

$$
\text{dist}(x^k, [f \leq t]) \to +\infty.
$$

**Theorem 2.1 ([Ha]).** The following statements are equivalent:

(i) There are no sequences of the first or second types of $[f \leq t]$.

(ii) $[f \leq t]$ has a GHEB, i.e. there exist $\alpha, \beta, c > 0$ such that

$$
[f(x) - t]^\alpha_+ + [f(x) - t]^\beta_+ \geq c \text{dist}(x, [f \leq t]) \text{ for all } x \in \mathbb{R}^n.
$$
Put

\[ PS^1(f) = \{ t \in \mathbb{R} : \exists \{x^k\} \subset \mathbb{R}^n, \{x^k\} \text{ is a sequence of the first type of } [f \leq t] \}, \]

\[ PS^2(f) = \{ t \in \mathbb{R} : \exists \{x^k\} \subset \mathbb{R}^n, \{x^k\} \text{ is a sequence of the second type of } [f \leq t] \}. \]

**Definition 2.2.** Put

\[ h(f) = \begin{cases} 
-\infty & \text{if } PS^2(f) = \emptyset, \\
+\infty & \text{if } PS^2(f) = \mathbb{R}, \\
\sup\{t \in \mathbb{R} : t \in PS^2(f)\} & \text{if } PS^2(f) \neq \emptyset \text{ and } PS^2(f) \neq \mathbb{R}.
\end{cases} \]

We call \( h(f) \) the *threshold* of global Hölderian error bounds of \([f \leq t]\).

**Theorem 2.2** (The first formula of \( H(f) \)). We have

(i) If \( h(f) = -\infty \), then \( H(f) = \mathbb{R} \setminus PS^1(f) \);

(ii) If \( h(f) = +\infty \), then \( H(f) = \emptyset \);

(iii) If \( h(f) \in PS^2(f) \), then \( H(f) = (h(f), +\infty) \setminus PS^1(f) \);

(iv) If \( h(f) \notin PS^2(f) \), then \( H(f) = [h(f), +\infty) \setminus PS^1(f) \).

**Proof.** Clearly, if \( t \in PS^2(f) \) and \( t' \leq t \), then \( t' \in PS^2(f) \). Hence,

either \( PS^2(f) = \emptyset \),

or \( PS^2(f) = \mathbb{R} \),

or \( PS^2(f) = (-\infty, h(f)] \) if \( h(f) \in PS^2(f) \),

or \( PS^2(f) = (-\infty, h(f)) \) if \( h(f) \notin PS^2(f) \).

Therefore, Theorem 2.2 follows from Theorem 2.1.

\[ \square \]

2.2. A new criterion of the existence of a GHEB of \([f \leq t]\) and the second formula of \( H(f) \).

Let \( d \) be the degree of a polynomial \( f \). By a linear change of coordinates, we can put \( f \) in the form

\[ f(x_1, \ldots, x_n) = a_0x_n^d + a_1(x_1, \ldots, x_{n-1})x_n^{d-1} + \cdots + a_d(x_1, \ldots, x_{n-1}) \ (\ast), \]

where \( a_0 \neq 0 \) and \( a_i(x_1, \ldots, x_{n-1}) \) are polynomials in \( (x_1, \ldots, x_{n-1}) \), where degrees \( \deg a_i \leq i, i = 1, \ldots, d \).

Put \( V_1 = \{ x \in \mathbb{R} : \frac{\partial f}{\partial x_n} (x) = 0 \} \).

**Definition 2.3.** We say that
(i) A sequence \( \{x^k\} \) is of the first type of \([f \leq t]\) w.r.t \( V_1 \) if
\[
\|x^k\| \to \infty, \\
f(x^k) > t, f(x^k) \to t, \\
dist(x^k, [f \leq t]) \geq \delta > 0, \\
\text{and } \{x^k\} \subset V_1.
\]
(ii) A sequence \( \{x^k\} \) is of the second type of \([f \leq t]\) w.r.t \( V_1 \) if
\[
\|x^k\| \to \infty, \\
t < f(x^k) \leq M < +\infty, \\
dist(x^k, [f \leq t]) \to \infty, \\
\text{and } \{x^k\} \subset V_1.
\]

The following result, as well as its proof, is inspired by Theorem 2.3 of [HD]

**Theorem 2.3.** Let \( f \) be of the form (\( \ast \)). Then the following statements are equivalent

(i) There are no sequences of the first or second types of \([f \leq t]\) w.r.t \( V_1 \);
(ii) \( \exists \alpha_1, \beta_1, c > 0 \) such that
\[
[f(x) - t]_{+}^{\alpha_1} + [f(x) - t]_{+}^{\beta_1} \geq c_1 \text{dist}(x, [f \leq t]),
\]
for all \( x \in V_1 \);
(iii) \( \exists \alpha_1, \beta_1, c > 0 \) such that
\[
[f(x) - t]_{+}^{\alpha_1} + [f(x) - t]_{+}^{\beta_1} + [f(x) - t]_{+}^{\frac{1}{2}} \geq c \text{dist}(x, [f \leq t]),
\]
for all \( x \in \mathbb{R}^n \);
(iv) \([f \leq t]\) has a global Hölderian error bound.

**Proof.**
We will prove that (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (iv) \( \Rightarrow \) (i).

Proof of (i) \( \Rightarrow \) (ii) :
For \( \tau > 0 \), put
\[
\psi(\tau) := \begin{cases} 
0 & \text{if } [f(x) - t]_{+} = t \text{ is empty} \\
\sup_{[f(x) - t]_{+} = \tau, x \in V_1} \text{dist}(x, [f \leq t]) & \text{if } [f(x) - t]_{+} = t \text{ is not empty}.
\end{cases}
\]
By (i), \( \psi(\tau) \) is well defined on \([0, +\infty)\). Moreover, it follows from Tarski-Seidenberg theorem (see, for example, [BCR, C, HP]), \( \psi(\tau) \) is a semialgebraic function.

To prove (ii), it is important to know the behavior of \( \psi(\tau) \), as \( \tau \to 0 \) or \( \tau \to +\infty \).

We distinguish 4 possibilities

(a) \( \psi(\tau) \equiv 0 \) for \( \tau \) sufficiently small and \( \psi(\tau) \equiv 0 \) for \( \tau \) sufficiently large;
(b) \( \psi(\tau) \equiv 0 \) for \( \tau \) sufficiently small and \( \psi(\tau) \neq 0 \) for \( \tau \) sufficiently large;
(c) \( \psi(\tau) \neq 0 \) for \( \tau \) sufficiently small and \( \psi(\tau) \equiv 0 \) for \( \tau \) sufficiently large;
(d) \( \psi(\tau) \neq 0 \) both for \( \tau \) sufficiently small and \( \tau \) sufficiently large.

We will prove (i) \( \Rightarrow \) (ii) for the case (d) because the proofs of other cases are similar.

In this case, since \( \psi(\tau) \) is semialgebraic and \( \psi(\tau) \neq 0 \) for any \( \tau \in [0, +\infty) \), we have

\[
(2) \quad \psi(\tau) = a_0 \tau^{\tilde{\alpha}} + o(\tau^{\tilde{\alpha}}) \text{ as } \tau \to 0, \text{ where } a_0 > 0.
\]

and

\[
(3) \quad \psi(\tau) = b_0 \tau^{\tilde{\beta}} + o(\tau^{\tilde{\beta}}) \text{ as } \tau \to +\infty, \text{ where } b_0 > 0.
\]

Clearly, \( \tilde{\alpha} > 0 \). It follows from (2) that there exists \( \delta > 0 \) such that

\[
(4) \quad [f(x) - t]_+^{\frac{1}{\tilde{\alpha}}} \geq \frac{a_0}{2} \text{dist}(x, [f \leq t]),
\]

for \( x \in \{x \in V_1 : [f(x) - t]_+ \leq \delta\} \).

It follows from (3) that there exists \( \Delta > 0 \) sufficiently large, such that for any \( x \in \{x \in V_1 : [f(x) - t]_+ \geq \Delta\} \). We have

\[
(5) \quad [f(x) - t]_+ \geq \frac{b_0}{2} \text{dist}(x, [f \leq t])
\]

if \( \tilde{\beta} \leq 0 \) and

\[
(6) \quad [f(x) - t]_+^{\frac{1}{\tilde{\beta}}} \geq \frac{b_0}{2} \text{dist}(x, [f \leq t]),
\]

if \( \tilde{\beta} > 0 \).

Since, by (i), there are no sequences of the second type, the function \( \text{dist}(x, [f \leq t]) \) is bounded on the set

\[
\{x \in V_1 : \delta \leq [f(x) - t]_+ \leq \Delta\}.
\]

This fact, together with (4), (5) and (6), give the proof of (i) \( \Rightarrow \) (ii).

Proof of (ii) \( \Rightarrow \) (iii):
The proof is based on the following classical result

Lemma (van der Corput, [G]). Let \( u(\tau) \) be a real valued \( C^d \)-function, \( d \in \mathbb{N} \), that satisfies \( |u^{(d)}(\tau)| \geq 1 \) for all \( \tau \in \mathbb{R} \). Then the following estimate is valid for all \( \epsilon > 0 \):

\[
\text{mes}\{\tau \in \mathbb{R} : |u(\tau)| \leq \epsilon\} \leq (2\epsilon)((d + 1)!)^{1/d}\epsilon^{1/d}.
\]

Suppose that we have (ii). Then

- If \( x \in [f \leq t] \), then \( \text{dist}(x, [f \leq t]) = 0 \) and (iii) holds automatically.
- If \( x \in V_1 \), then (iii) follows from (ii).
Assume that $x \notin [f \leq t] \cup V_1$.

Clearly

- (ii) holds if and only if there exists $c > 0$ such that

$$
[f(x) - t]_+ \geq c \min \{ \text{dist}(x, [f \leq t])^{\frac{1}{d}}, \text{dist}(x, [f \leq t])^{\frac{1}{n}} \}
$$

for all $x \in V_1$.

- (iii) holds if and only if there exists $c > 0$

$$
[f(x) - t]_+ \geq c \min \{ \text{dist}(x, [f \leq t])^{\frac{1}{d}}, \text{dist}(x, [f \leq t])^{\frac{1}{n}}, \text{dist}(x, [f \leq t])^d \}
$$

for all $x \in \mathbb{R}^n$.

Let $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}, x' = (x_1, \ldots, x_{n-1})$. We put

$$
u_{x'}(\tau) = \frac{f(x', \tau) - t}{a_0 d!}, \tau \in \mathbb{R}
$$

and

$$
\Sigma(x') = \{ \tau \in \mathbb{R} : |\nu_{x'}(\tau)| \leq \frac{f(x) - t}{|a_0| d!} \}.
$$

Since $\nu_{x'}^{(d)}(\tau) = 1$, it follows from the van der Corput Lemma that there exists a constant $c > 0$, independent of $x$ such that

$$
\text{mes} \Sigma(x') \leq c (f(x) - t)^{1/d}.
$$

Clearly, $\Sigma(x') \neq \emptyset$ and $\Sigma(x') \neq \mathbb{R}$. Since $\Sigma(x')$ is a closed semi-algebraic subset of $\mathbb{R}$, we have

$$
\Sigma(x') = \bigcup_{i=1}^m [a_i, b_i] \bigcup \bigcup_{j=1}^s \{c_j\},
$$

where $a_i, b_i, c_j \in \mathbb{R}, i = 1, \ldots, m; j = 1, \ldots, s$, and

$$
|\nu(a_i)| = |\nu(b_i)| = |\nu(c_j)| = \frac{f(x) - t}{|a_0| d!}.
$$

Firstly, we see that $x_n \neq c_j, \forall j = 1, \ldots, s$. In fact, since $c_j$ is an isolated point of $\Sigma(x')$, $c_j$ is a local extremum of $\nu_{x'}(\tau)$. Hence,

$$
\frac{d \nu_{x'}(c_j)}{d \tau} = 0
$$

or $\frac{\partial f}{\partial x_n}(x', c_j) = 0$ i.e. $(x', c_j) \in V_1$, while by assumption, $x = (x', x_n) \notin V_1$. Thus, $x_n \in \{a_i, b_i ; i = 1, \ldots, m\}$.

Without loss of generality, we may assume that $x_n = a_1$. Since $|\nu_{x'}(a_1)| = |\nu_{x'}(b_1)|$, we distinguish two cases

- If $\nu_{x'}(a_1) = -\nu_{x'}(b_1)$, then there exists $\tau_1 \in [a_1, b_1]$ such that $\nu_{x'}(\tau_1) = 0$, which means that $f(x', \tau_1) = t$ or $(x', \tau_1) \in f^{-1}(t) \subset [f \leq t]$. Hence

$$
\text{dist}(x, [f \leq t]) \leq \text{dist}(x, (x', \tau_1)) = |x_n - \tau_1| \leq |a_1 - \tau_1| \leq \text{mes} \Sigma(x').
$$
Then, by (9), (iii) holds.

• If $u_{x'}(a_1) = u_{x'}(b_1)$, then, by Rolle’s Theorem, there exists $\tau_2 \in [a_1, b_1]$ such that
  
  \[ \frac{du_{x'}}{d\tau}(\tau_2) = 0, \]

  which means that $(x', \tau_2) \in V_1$. Applying (7), there exists $c_1 > 0$ such that
  \[ [f(x', \tau_2) - t]_+ \geq c_1 \min \{ \dist((x', \tau_2), [f \leq t])^{1/\alpha_1}, \dist((x', \tau_2), [f \leq t])^{1/\beta_1} \}. \]

  Moreover, since $\tau_2 \in \Sigma(x')$, we have
  \[ f(x) - t \geq [f(x', \tau_2) - t]_+ \geq c_1 \min \{ \dist((x', \tau_2), [f \leq t])^{1/\alpha_1}, \dist((x', \tau_2), [f \leq t])^{1/\beta_1} \}. \]  

  Let $P(x', \tau_2)$ be the point of $[f \leq t]$ such that
  \[ \dist((x', \tau_2), [f \leq t]) = \dist((x', \tau_2), P(x', \tau_2)). \]

  We have
  \[
  \dist(x, [f \leq t]) \leq \dist(x, P(x', \tau_2)) \\
  \leq \dist(x, (x', \tau_2)) + \dist((x', \tau_2), P(x', \tau_2)) \\
  \leq 2 \max \{ \dist(x, (x', \tau_2)), \dist((x', \tau_2), P(x', \tau_2)) \}. 
  \]

  Now:

  - If $\max \{ \dist(x, (x', \tau_2)), \dist((x', \tau_2), P(x', \tau_2)) \} = \dist(x, (x', \tau_2))$, then
    \[ \dist(x, [f \leq t]) \leq 2 \dist((x', \tau_2), x) \leq 2 \mes(\Sigma) \leq 2 \mes(f(x) - t)^{1/d}. \]

  - If $\max \{ \dist(x, (x', \tau_2)), \dist((x', \tau_2), P(x', \tau_2)) \} = \dist((x', \tau_2), P(x', \tau_2))$, then
    \[ \dist(x, [f \leq t]) \leq 2 \dist((x', \tau_2), P(x', \tau_2)) \leq 2 \dist((x', \tau_2), [f \leq t]). \]

  Then (iii) follows from (10).

  Hence, the implication (ii) $\Rightarrow$ (iii) is proved.

Proof of (iii) $\Rightarrow$ (iv):

Clearly, if (iii) holds, then there are no sequences of the first or second types of $[f \leq t]$. Hence, by Theorem 2.1 (iv) holds.

The proof of (iv) $\Rightarrow$ (i) is straightforward. $\square$

**Proposition 2.1.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial function and $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear isomorphism. Then we have

\[ H(f \circ A) = H(f). \]
Proof. Let $y = Ax$ and put $g = f \circ A$.

Firstly, we prove that $t_0 \in H(g) \Rightarrow t_0 \in H(f)$.
We have $f(y) = f(A \circ A^{-1}(y)) = g(A^{-1}(y))$. This implies that

$$[f(y) - t_0]^\alpha_+ + [f(y) - t_0]^\beta_+ = [g(A^{-1}(y)) - t_0]^\alpha_+ + [g(A^{-1}(y)) - t_0]^\beta_+.$$  \hfill (11)

Since $t_0 \in H(g)$, then there exists $\alpha, \beta, c > 0$ such that

$$[g(A^{-1}(y)) - t_0]^\alpha_+ + [g(A^{-1}(y)) - t_0]^\beta_+ \geq c \text{dist}(A^{-1}(y), [g \leq t_0]).$$  \hfill (12)

Suppose that $\text{dist}(A^{-1}(y), [g \leq t_0]) = \|A^{-1}(y) - x_0\|$, where $g(x_0) = t_0$ or $f(A(x_0)) = t_0$. Since $y_0 = Ax_0$ and $A$ is a linear isomorphism, we have $f(y_0) = t_0$ and there exists $c' > 0$ such that

$$c'\|y - y_0\| \geq \|A^{-1}(y) - A^{-1}(y_0)\| \geq \frac{1}{c'}\|y - y_0\|.$$

It follows that

$$\text{dist}(A^{-1}(y), [g \leq t_0]) = \|A^{-1}(y) - A^{-1}(y_0)\| \geq \frac{1}{c'}\|y - y_0\| \geq \frac{1}{c} \text{dist}(y, [f \leq t_0]).$$

Combining (11), (12) and above fact, we have

$$[f(y) - t_0]^\alpha_+ + [f(y) - t_0]^\beta_+ \geq \frac{c}{c'} \text{dist}(y, [f \leq t_0]), \forall y \in \mathbb{R}^n,$$

i.e., $t_0 \in H(f)$. The claim $t_0 \in H(f) \Rightarrow t_0 \in H(g)$ is proved similarly. \hfill \Box

Corollary 2.1. For computing the set $H(f)$, we can always assume that $f$ is of the form $(\ast)$, i.e.

$$f(x_1, \ldots, x_n) = a_0x_n^d + a_1(x_1, \ldots, x_{n-1})x_n^{d-1} + \cdots + a_d(x_1, \ldots, x_{n-1}),$$

where $d$ is the degree of $f$.

Let $f$ be of the form $(\ast)$. Put

$$P^1(f) = \{t \in \mathbb{R} : [f \leq t] \text{ has a sequence of the first type w.r.t. } V_1\};$$

$$P^2(f) = \{t \in \mathbb{R} : [f \leq t] \text{ has a sequence of the second type w.r.t. } V_1\}.$$

Theorem 2.4 (The second formula of $H(f)$). Let $f$ be a polynomial of the form $(\ast)$. Then we have

(i) $h(f) = \sup\{t \in \mathbb{R} : t \in P^2(f)\};$
(ii) If $h(f) = -\infty$, then $H(f) = \mathbb{R} \setminus P^1(f);$  
(iii) If $h(f) = +\infty$, then $H(f) = \emptyset;$
(iv) If $h(f) \in \mathbb{R}$ and $h(f) \in P^2(f)$, then $H(f) = (h(f), +\infty) \setminus P^1(f);$  
(vv) If $h(f) \in \mathbb{R}$ and $h(f) \notin P^2(f)$, then $H(f) = [h(f), +\infty) \setminus P^1(f).$
3. The relationship between $H(f)$ and Palais-Smale values

The relationship between Palais-Smale values and the existence of global Hölderian error bounds is well-known and has been explored in many previous works, see, for example, [Az, CM, LP, Ha, I]. In this section, we will establish this relationship by proving that $h(f) \in PS(f) \cup \{\pm \infty\}$ and $PS^1(f) \subset PS(f)$. We recall

**Definition 3.1.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial function. The set of Palais-Smale values of $f$ is defined by

$$PS(f) := \{ t \in \mathbb{R} : \exists \{x^k\} \subset \mathbb{R}^n, \|x^k\| \to \infty, \|\nabla f(x^k)\| \to 0, f(x^k) \to t \}.$$ 

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial function. Put

$$PS_C(f) := \{ t \in \mathbb{C} : \exists \{x^k\} \subset \mathbb{C}^n, \|x^k\| \to \infty, \|\nabla f(x^k)\| \to 0, f(x^k) \to t \}.$$ 

**Theorem 3.1 ([Sp]).** $PS_C(f)$ is an algebraic subset of $\mathbb{C}$.

According to this result, $PS_C(f)$ is either empty, or $PS_C(f)$ is a finite subset of $\mathbb{C}$, or $PS_C(f) = \mathbb{C}$.

**Corollary 3.1.** If $f : \mathbb{R}^n \to \mathbb{R}$ is a polynomial function, then $PS(f)$ is an algebraic subset of $\mathbb{R}$. Therefore, either $PS(f) = \emptyset$ or $PS(f)$ is a finite subset of $\mathbb{R}$, or $PS(f) = \mathbb{R}$.

**Remark 3.1.** In general, $PS(f)$ can be infinite. For example, if $f(x, y, z) = x + x^2y + x^4yz$, then $PS(f) = \mathbb{R}$ (see [KOS]).

The following proposition is contained implicitly in the proof of Theorem B of [Ha].

**Proposition 3.1.** $PS^1(f) \subset PS(f)$.

**Proof.** Put $X = \{ x \in \mathbb{R}^n : f(x) \geq t \}$. By the metric induced from that of $\mathbb{R}^n$, $X$ is a complete metric space and the function $f : X \to \mathbb{R}$ is bounded from below. Let $t \in PS^1(f)$ and $\{ x^k \}$ be a sequence of the first type of $[f \leq t]$:

$$\|x^k\| \to \infty,$n

$$f(x^k) > t,$n

$$f(x^k) \to t,$n

$$\exists \delta > 0 \text{ s.t. } \text{dist}(x^k, [f \leq t]) \geq \delta.$$n

Let $\epsilon_k = f(x^k) - t$. Then $\epsilon_k > 0$ and $\epsilon_k \to 0$ as $k \to +\infty$. Set $\lambda_k = \sqrt{\epsilon_k}$. By the Ekeland’s Variational Principle ([E]), there exists a sequence $\{y^k\} \subset X$ such that

$$f(y^k) \leq t + \epsilon_k = f(x^k),$$

$$\text{dist}(y^k, x^k) \leq \lambda_k.$$
and for any $x \in X, x \neq y^k$, we have

$$f(x) \geq f(y^k) - \frac{\epsilon_k}{\lambda_k} d(x, y^k), \forall x \in X.$$  \hspace{1cm} (13)

Since $\text{dist}(y^k, x^k) \leq \lambda_k = \sqrt{\epsilon_k} \to 0$ and $\text{dist}(x^k, [f \leq t]) \geq \delta > 0$, the ball $B(y^k, \delta/2) = \{ x \in \mathbb{R}^n : \text{dist}(y^k, x) \leq \delta/2 \}$ is contained in $X$. Then, inequality (13) implies that

$$\frac{f(y^k + \tau u) - f(y^k)}{\tau} \geq -\sqrt{\epsilon_k}$$

holds true for every $u \in \mathbb{R}^n, \|u\| = 1$ and $\tau \in [0, \delta/2)$. This gives us

$$\langle \nabla f(y^k), u \rangle \geq -\sqrt{\epsilon_k}.$$  \hspace{1cm} \Box

Proposition 3.2. If there is a sequence of the second type of $[f \leq t]$:

$$\|x^k\| \to \infty,$$

$$t < f(x^k) \leq M < +\infty,$$

$$\text{dist}(x^k, [f \leq t]) \to +\infty.$$  

then there exists a sequence $\{y^k\}$ of the second type of $[f \leq t]$:

$$\|y^k\| \to \infty,$$

$$t \leq f(y^k) \leq M < +\infty,$$

$$\text{dist}(y^k, [f \leq t]) \to +\infty.$$  

with additional conditions

$$\|\nabla f(y^k)\| \to 0,$$

and $\lim_{k \to \infty} f(y^k) \in PS(f)$.

In particular, the segment $[t, M]$ contains at least one point of $PS(f)$.

Proof. Put $X = \{ x \in \mathbb{R}^n : f(x) \geq t \}, \epsilon_k = f(x^k) - t$ and $\lambda_k = \frac{1}{2} \text{dist}(x^k, [f \leq t])$.

As in the proof of Proposition 3.1, we can find a sequence $\{y^k\} \subset X$ such that

$$\|y^k\| \to \infty,$$

$$t \leq f(y^k) \leq t + \epsilon_k = f(x^k) \leq M < +\infty,$$

$$\lim_{k \to \infty} f(y^k) \in PS(f),$$

$$\|\nabla f(y^k)\| \to 0,$$

$$\text{dist}(y^k, x^k) \leq \lambda_k.$$  \hspace{12} 12
Proposition 3.3. If \( h(f) \notin \{\pm \infty\} \), then \( h(f) \in PS(f) \).

Proof. Assume that \( h(f) \notin \{\pm \infty\} \). By contradiction, suppose that \( h(f) \notin PS(f) \). Firstly, it means that \( PS(f) \neq \mathbb{R} \). Hence, either \( PS(f) = \emptyset \) or \( PS(f) \) is a non-empty finite set.

By definition of \( h(f) \), \([f \leq h(f) - \epsilon]\) has a sequence of second type. Hence, it follows from Proposition 3.2, \( PS(f) \neq \emptyset \). Thus, \( PS(f) \) is a non-empty finite set. Then, for any \( \epsilon > 0 \) sufficiently small, we have \([h(f) - \epsilon, h(f)] \cap PS(f) = \emptyset\) and \( h(f) - \epsilon \in PS^2(f) \).

Let \( \{x^k\} \) be a sequence of the second type of \([f \leq h(f) - \epsilon] \):

\[
h(f) - \epsilon \leq f(x^k) \leq M, \|x^k\| \to \infty \text{ and } \text{dist}(x^k, [f \leq h(f) - \epsilon]) \to \infty.
\]

By Proposition 3.2, we may assume that \( \|\nabla f(x^k)\| \to 0 \) and there exists \( t_1 \in PS(f) \cap [h(f) - \epsilon, M] \) and \( t_1 = \lim_{k \to \infty} f(x^k) \).

Let \( \delta_1 > 0 \) such that \( t_1 - \delta_1 \notin PS(f) \) and \( t_1 - \delta_1 > h(f) \). Since \( f(x^k) \to t_1 \), we can assume that \( f(x^k) > t_1 - \delta_1 \) for all \( k \). Let \( y^k \) be the point of \([f \leq t_1 - \delta_1] \) such that \( \text{dist}(x^k, [f \leq t_1 - \delta_1]) = \|x^k - y^k\| \). Clearly, \( y^k \in f^{-1}(t_1 - \delta_1) \).

Claim: \( \{y^k\} \) is a sequence of second type of \([f \leq h(f) - \epsilon] \).

Proof of Claim. Since \( t_1 - \delta_1 > h(f), t_1 - \delta_1 \notin PS^2(f) \). Hence, for some \( A > 0 \), we have \( \|x^k - y^k\| \leq A < +\infty \) for all \( k \).

Let \( z^k \) be the point of \([f \leq h(f) - \epsilon] \) such that \( \text{dist}(y^k, [f \leq h(f) - \epsilon]) = \|y^k - z^k\| \).

We have

\[
\text{dist}(y^k, [f \leq h(f) - \epsilon]) \geq \text{dist}(x^k, [f \leq h(f) - \epsilon]) - \|x^k - y^k\| \geq \text{dist}(x^k, [f \leq h(f) - \epsilon]) - A.
\]

This shows that \( \text{dist}(y^k, [f \leq h(f) - \epsilon]) \to +\infty \) and the claim is proved.

Since \( \{y^k\} \) is a sequence of the second type of \([f \leq h(f) - \epsilon] \) and \( f(y^k) = t_1 - \delta_1 \notin PS(f) \), by Proposition 3.2, there exists \( t_2 \in [h(f) - \epsilon, t_1 - \delta_1] \cap PS(f) \). Choose \( \delta_2 \) such that \( t_1 - \delta_2 > h(f) \) and \( t_2 - \delta_2 \notin PS(f) \). Similarly as in the proof of Claim, we can find a sequence of the second type \( \{y^{k_j}\} \) of \([f \leq h(f) - \epsilon] \) such that \( f(y^{k_j}) = t_2 - \delta_2 \) and \( t_3 \in PS(f) \) such that \( h(f) - \epsilon < t_3 < t_2 \).

Making this process iteratively, we see that the interval \([h(f) - \epsilon, M] \) contains a infinite number of points in \( PS(f) \), which is a contradiction.
4. Types of stability of global Hölderian error bounds

We will distinguish 3 cases.

4.1. Case 1 - $PS(f) = \emptyset$.

**Theorem 4.1.** If $PS(f) = \emptyset$ then $H(f) = \mathbb{R}$.

**Proof.** Assume that $PS(f) = \emptyset$. Then by Proposition 3.1, $PS^1(f) = \emptyset$. Moreover, it follows from Proposition 3.2 that $PS^2(f)$ is also empty.

Hence, by Theorem 2.1, $H(f) = \mathbb{R} \setminus (PS^1(f) \cup PS^2(f)) = \mathbb{R}$. □

**Definition 4.1.** Let $t \in \mathbb{R}$, $t$ is called $y$-stable if $t \in H(f)$ and there exists an open interval $I(t)$ such that $t \in I(t) \subset H(f)$.

**Corollary 4.1.** If $PS(f) = \emptyset$, then there is only one type of stability of GHEB. Namely, for all $t \in \mathbb{R}$, $t$ is $y$-stable.

**Remark 4.1.** We recall here results of [Ha] about the role that Newton polyhedron plays in studying GHEB’s.

Let $f(x) = \sum a_\alpha x^\alpha$ be a polynomial in $n$ variables. Put $supp(f) = \{\alpha \in (\mathbb{N} \cup \{0\})^n : a_\alpha \neq 0\}$ and denote $\Gamma_f$ the convex hull in $\mathbb{R}^n$ of the set $\{(0,0,\ldots,0)\} \cup supp(f)$. Following [Kou] we call $\Gamma_f$ the Newton polyhedron at infinity of $f$.

Let $\Delta$ be a face (of any dimension) of $\Gamma_f$, set:

$$f_\Delta(x) := \sum_{\alpha \in \Delta} a_\alpha x^\alpha.$$

**Definition ([Kou]).** We say that a polynomial $f$ is nondegenerate with respect to its Newton boundary at infinity (nondegenerate for short), if for every face $\Delta$ of $\Gamma_f$ not containing the origin, the system

$$x_i \frac{\partial f_\Delta}{\partial x_i} = 0, i = 1, \ldots, n.$$

has no solution in $(\mathbb{R} \setminus \{0\})^n$.

**Definition.** A polynomial $f(x) = \sum a_\alpha x^\alpha$ in $n$ variables is said to be convenient if for every $i$, there exists a monomial of $f$ of the form $x_i^{\alpha_i}, \alpha_i > 0$, with a non-zero coefficient.

**Theorem 4.2 ([Ha]).** If $f$ is convenient and nondegenerate w.r.t. its Newton polyhedron at infinity, then there exist $r, \delta > 0$ such that

$$\|\nabla f(x)\| \geq \delta \text{ for } \|x\| \geq r \gg 1.$$

In particular, $PS(f) = \emptyset$. 14
Let $\mathbb{R}[x_1, \ldots, x_n]$ denote the ring of polynomials in $n$ variables over $\mathbb{R}$.

For $g \in \mathbb{R}[x_1, \ldots, x_n]$, as before, $\Gamma_g$ denotes the Newton polyhedron at infinity of $g$. Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a convenient polynomial.

Put $\Gamma := \Gamma_f$ and

$$\mathcal{A}_\Gamma = \{ g \in \mathbb{R}[x_1, \ldots, x_n] : \Gamma_g \subset \Gamma \}.$$ 

The set $\mathcal{A}_\Gamma$ can be identified to the space $\mathbb{R}^m$, where $m$ is the number of integer points of $\Gamma$.

Put $\mathcal{B}_\Gamma = \{ h \in \mathcal{A}_\Gamma : \Gamma_h = \Gamma$ and $h$ is nondegenerate $\}$. According to [Kou], $\mathcal{B}_\Gamma$ is an open and dense subset of $\mathcal{A}_\Gamma$. Hence, Theorem 4.1 and 4.2 show that if $f$ is a generic polynomial, then $H(f) = \mathbb{R}$ and any value $t \in \mathbb{R}$ is $y$-stable.

### 4.2. Case 2 - $PS(f)$ is non-empty finite set.

**Proposition 4.1.** If $\#PS(f) < +\infty$, then $H(f) \neq \emptyset$.

**Proof.** By contradiction, assume that $H(f) = \emptyset$. Since $\#PS(f) < +\infty$, we have $\#PS^1(f) < +\infty$ (Proposition 3.1). Then, it follows from the first formula that $H(f) = \emptyset$ if and only if $h(f) = +\infty$ but the later is impossible, since we have

**Claim:** If $h(f) = +\infty$, then $PS(f) = \mathbb{R}$.

**Proof.** Take $t_1 \in \mathbb{R}$, since $h(f) = +\infty$, $[f \leq t_1]$ has a sequence of the second type. By Proposition 3.2, there exists $M_1 > t_1$ and $a_1 \in [t_1, M_1] \cap PS(f)$. Take $t_2$ such that $M_1 < t_2$, then $[f \leq t_2]$ has a sequence of the second type. Hence, there exists $M_2 > t_2$ and $a_2$ such that $a_2 \in [t_2, M_2] \cap PS(f)$. Continuing this way, we find an infinite sequence $a_1, a_2, a_3, \ldots$ of $PS(f)$. Therefore, $PS(f) = \mathbb{R}$. □

Now, we study the stability of GHEB in the case when $PS(f)$ is a non-empty finite set.

**Definition 4.2.** Let $t \in \mathbb{R}$.

1. Recall that $t$ is called $y$-stable if $t \in H(f)$ and there exists an open interval $I(t)$ such that $t \in I(t) \subset H(f)$;
2. $t$ is called $y$-right stable if $t \in H(f)$ and there exists $\epsilon > 0$ such that $[t, t+\epsilon) \subset H(f)$ and $(t-\epsilon, t) \cap H(f) = \emptyset$;
1'. $t$ is called $n$-stable if $t \in \mathbb{R} \setminus H(f)$ and there exists an open interval $I(t)$ such that $t \in I(t) \subset \mathbb{R} \setminus H(f)$;
2'. $t$ is called $n$-left stable if $t \in \mathbb{R} \setminus H(f)$ and there exists $\epsilon > 0$ such that $(t-\epsilon, t] \subset \mathbb{R} \setminus H(f)$ and $(t, t+\epsilon) \cap (\mathbb{R} \setminus H(f)) = \emptyset$;
3'. $t$ is called $n$-isolated if $t \in \mathbb{R} \setminus H(f)$ and for $\epsilon > 0$ sufficiently small, $(t-\epsilon, t) \cup (t, t+\epsilon) \subset H(f)$. 


We have the following results

**Theorem 4.3.** Let $PS(f)$ be a non-empty finite set and $t \in \mathbb{R}$. Then, $t$ is one of types

**Case A:** If $h(f) = -\infty$, then
(i) $t$ is y-stable if and only if $t \notin PS^1(f)$.
(ii) $t$ is a n-isolated point if and only if $t \in PS^1(f)$.

**Case B:** If $h(f)$ is a finite value, then
1. $t$ is y-stable if and only if $t > h(f)$ and $t / \notin PS^1(f)$;
2. $t$ is y-right stable if and only if $t = h(f)$ and $h(f) \in H(f)$;
1'. $t$ is n-stable if and only if $t < h(f)$;
2'. $t$ is n-left stable if and only if $t = h(f)$ and $h(f) / \notin H(f)$;
3'. $t$ is a n-isolated point if and only if $t > h(f)$ and $t \in PS^1(f)$.

**Remark 4.2.** Here, if we have item 2, then we does not have item 2' and vice versa.

Now, for completeness, we add two following results:

- Necessary and sufficient condition for $\#PS(f) < +\infty$;
- An estimation of the number of connected components of $H(f)$ for the case $\#PS(f) < +\infty$.

We have the following real version of Theorem 2 in [Sp]:

**Proposition 4.2.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial. Then the following statements are equivalent

(i) There exists $\rho \in \mathbb{R}, c > 0$ and $r \gg 1$ such that

\[
\|\nabla f(x)\| \geq c|f(x)|^\rho \text{ for all } x \in \{ x \in \mathbb{R}^n : |f(x)| \geq r \};
\]

(ii) $PS(f)$ is either empty or a finite set.

**Proof.** (i) $\Rightarrow$ (ii): Assume that (i) holds. Then it follows that $PS(f)$ is either empty or a bounded subset of $\mathbb{R}$. Therefore, by Corollary 3.1 $\#PS(f) < +\infty$.

(ii) $\Rightarrow$ (i): Assume that $\#PS(f) < +\infty$. Choose $r_0 > 0$ such that $\forall t \in PS(f), |t| < r_0$. We define

\[
\varphi : (r_0, +\infty) \to \mathbb{R},
\]

\[
r \mapsto \inf_{\{x \in \mathbb{R}^n : |f(x)| = r\}} \|\nabla f(x)\|.
\]

Clearly, $\varphi(r) \neq 0, \forall r > r_0$. By Dichotomy Lemma (see, for example, [HP]), for $r \gg 1$, we have $\varphi(r) = c_0r^n + o(r^n)$, where $\rho \in \mathbb{R}$ (even $\rho \in \mathbb{Q}$). Hence, there exists $r_1 \gg r_0$ such that $\forall x \in \{ x \in \mathbb{R}^n : |f(x)| \geq r_1 \}$:

\[
\|\nabla f(x)\| \geq \frac{c_0}{2}|f(x)|^\rho.
\]

\[ \square \]
Let us denote $C(S)$ the number of connected components of $S \subset \mathbb{R}^n$, we have the following result

**Theorem 4.4.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be an any polynomial of degree $d$. Then, if \( \#PS(f) < +\infty \), we have

$$C(H(f)) \leq (d - 1)^{n-1} + 1.$$ 

**Proof.** Since \( \#PS(f) < +\infty \), we have \( \#PS_C(f) < +\infty \). Then, according to Theorem 1.1 of [Je], we have

$$\#PS(f) \leq \#PS_C(f) \leq (d - 1)^{n-1}.$$ 

Hence, it follows from the first formula that

$$C(H(f)) \leq (d - 1)^{n-1} + 1.$$ 

\[\square\]

4.3. **Case 3 - PS(f) = \mathbb{R}**.

In this case, the following lemma tells us that the set $H(f)$ has still very simple structure

**Lemma 4.1.** $H(f)$ is a semialgebraic subset of $\mathbb{R}$.

Using the first formula for $H(f)$, it is enough to show that $PS^1(f)$ is semialgebraic. To do that, it is more convenient to use the logical formulation of the Tarski-Seidenberg Theorem. Let us to recall it.

A first-order formula is obtained as follows recursively (see, for example, [BCR, C, HP])

1. If $f \in \mathbb{R}[X_1, \ldots, X_n]$, then $f = 0$ and $f > 0$ are first-order formulas (with free variables $X = (X_1, \ldots, X_n)$ and \{ $x \in \mathbb{R}^n | f(x) = 0$ \} and \{ $x \in \mathbb{R}^n | f(x) > 0$ \} are respectively the subsets of $\mathbb{R}^n$ such that the formulas $f = 0$ and $f > 0$ hold.

2. If $\Phi$ and $\Psi$ are first-order formulas, then $\Phi \lor \Psi$ (conjunction), $\Phi \land \Psi$ (disjunction) and $\neg \Phi$ (negation) are also first-order formulas.

3. If $\Phi$ is a formula and $X$ is a variable ranging over $\mathbb{R}$, then $\exists X \Phi$ and $\forall X \Phi$ are first-order formulas.

**Theorem** (Logical formulation of the Tarski–Seidenberg Theorem [BCR, C, HP]). If $\Phi(X_1, \ldots, X_n)$ is a first-order formula, then the set

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n : \Phi(x_1, \ldots, x_n) \text{ holds}\}$$

is semialgebraic.
Proof of Lemma 4.1. We have

\[ PS^1(f) = \{ t \in \mathbb{R} | \exists \delta > 0, \forall R > 0 : \forall \epsilon > 0, \exists x \in \mathbb{R}^n : \|x\|^2 \geq R^2, 0 < f(x) - t < \epsilon, \text{dist}(x, [f \leq t]) \geq \delta \}, \] (a)

\[ \{ x \in \mathbb{R}^n : \text{dist}(x, [f < t]) \geq \delta \} = \{ x \in \mathbb{R}^n : \exists \delta \forall x_0 \in [f \leq t], \|x - x_0\|^2 \geq \delta^2 \}. \] (b)

It follows from (a) and (b) that the set \( PS^1(f) \) can be determined by a first-order formula, hence it is a semialgebraic subset of \( \mathbb{R} \).

\[ \square \]\n
Since \( H(f) \) is a semialgebraic subset of \( \mathbb{R} \), we have

**Corollary 4.2.** If \( H(f) \neq \emptyset \) and \( H(f) \neq \mathbb{R} \), then it is the unions of finitely many points and intervals.

By Corollary 4.2 we have to consider three cases

- **(a)** \( H(f) = \mathbb{R} \);
- **(b)** \( H(f) = \emptyset \);
- **(c)** \( H(f) \) is a non-empty proper semialgebraic subset of \( \mathbb{R} \).

- In the case (a), we have only one stable type: \( t \) is \( y \)-stable for all \( t \in \mathbb{R} \);
- In the case (b), we have only one stable type: \( t \) is \( n \)-stable for all \( t \in \mathbb{R} \);
- In the case (c), \( H(f) \) is a disjoint union of the sets of the following types:

\[ I(a_i^1, a_i^2), I(b_j^1, b_j^2), I(c_k^1, c_k^2], I[d_l^1, d_l^2], A(m), I_{-\infty}, I_{+\infty}. \]

Where

1. \( I(a_i^1, a_i^2) = \emptyset \) or \( I(a_i^1, a_i^2) = (a_i^1, a_i^2), i = 1, \ldots, p \);
2. \( I(b_j^1, b_j^2) = \emptyset \) or \( I(b_j^1, b_j^2) = [b_j^1, b_j^2], j = 1, \ldots, q \);
3. \( I(c_k^1, c_k^2] = \emptyset \) or \( I(c_k^1, c_k^2] = (c_k^1, c_k^2], k = 1, \ldots, r \);
4. \( I[d_l^1, d_l^2] = \emptyset \) or \( I[d_l^1, d_l^2] = [d_l^1, d_l^2], l = 1, \ldots, s \);
5. \( A(m) = \emptyset \) or \( A(m) = \{ e_1, \ldots, e_m \} \), where \( e_1, \ldots, e_m \) are isolated points;
6. \( I_{-\infty} = \emptyset \) or \( I_{-\infty} = (-\infty, a] \) or \( I_{-\infty} = (-\infty, a) \), where \( a \in \mathbb{R} \);
7. \( I_{+\infty} = \emptyset \) or \( I_{+\infty} = [b, +\infty) \) or \( I_{+\infty} = (b, +\infty) \), where \( b \in \mathbb{R} \).

Similarly, \( \mathbb{R} \setminus H(f) \) is a disjoint union of the sets of the following types:

\[ I(a_i^1, a_i^2), I(b_j^1, b_j^2), I(c_k^1, c_k^2], I[d_l^1, d_l^2], A'(m'), I'_{-\infty}, I'_{+\infty}. \]

We have the following definition

**Definition 4.3.** Let \( t \in \mathbb{R} \).

1. Recall that \( t \) is said to be \( y \)-stable if \( t \in H(f) \) and there exists an open interval \( I(t) \) such that \( t \in I(t) \subset H(f) \);
2. Recall that \( t \) is said to be \( y \)-right stable if \( t \in H(f) \) and there exists \( \epsilon \) such that \([t, t + \epsilon) \subset H(f) \) and \((t - \epsilon, t) \cap H(f) = \emptyset \);
3. \( t \) is said to be \( y \)-left stable if \( t \in H(f) \) and there exists \( \epsilon > 0 \) such that \((t - \epsilon, t] \subset H(f) \) and \((t, t + \epsilon) \cap H(f) = \emptyset\);
4. \( t \) is said to be \( y \)-isolated if \( t \in H(f) \) and for \( \epsilon > 0 \) sufficiently small, \((t - \epsilon, t) \cup (t, t + \epsilon) \subset \mathbb{R} \setminus H(f)\);

1'. Recall that \( t \) is called \( n \)-stable if \( t \in \mathbb{R} \setminus H(f) \) and there exists an open interval \( I(t) \) such that \( t \in I(t) \subset \mathbb{R} \setminus H(f)\);
2'. \( t \) is said to be \( n \)-right stable if \( t \in \mathbb{R} \setminus H(f) \) and there exists \( \epsilon > 0 \) such that \([t, t + \epsilon) \subset \mathbb{R} \setminus H(f) \) and \((t - \epsilon, t) \cap (\mathbb{R} \setminus H(f)) = \emptyset\);
3'. Recall that \( t \) is called \( n \)-left stable if \( t \in \mathbb{R} \setminus H(f) \) and there exists \( \epsilon > 0 \) such that \((t - \epsilon, t] \subset \mathbb{R} \setminus H(f) \) and \((t, t + \epsilon) \cap (\mathbb{R} \setminus H(f)) = \emptyset\);
4'. Recall that \( t \) is called \( n \)-isolated if \( t \in \mathbb{R} \setminus H(f) \) and for \( \epsilon > 0 \) sufficiently small, \((t - \epsilon, t) \cup (t, t + \epsilon) \subset H(f)\).

**Theorem 4.5.** Let \( H(f) \) be of the form (c) and \( t \in \mathbb{R} \). Then we have

1. \( t \) is \( y \)-stable if and only if \( t \) is an interior point of the sets

\[
I_{-\infty} \bigcup_{i=1}^{p} I(a_i, a_i^*) \bigcup_{j=1}^{q} I(b_j, b_j^*) \bigcup_{k=1}^{r} I(c_k, c_k^*) \bigcup_{l=1}^{s} I(d_l, d_l^*) \bigcup I_{+\infty};
\]

2. \( t \) is \( y \)-right stable if and only if we have \( t = b_j \) or \( t = d_l^1 \) or \( t = b \) (where \( I_{+\infty} = [b, +\infty) \));
3. \( t \) is \( y \)-left stable if and only if we have \( t = c_k^2 \) or \( t = d_l^2 \) or \( t = a \) (where \( I_{-\infty} = (-\infty, a) \));
4. \( t \) is an \( y \)-isolated point if and only if \( t \in A(m) \).

1'. \( t \) is \( n \)-stable if and only if \( t \) is an interior point of the set:

\[
I'_{-\infty} \bigcup_{i=1}^{p} I'(a_i, a_i^*) \bigcup_{j=1}^{q} I'(b_j, b_j^*) \bigcup_{k=1}^{r} I'(c_k, c_k^*) \bigcup_{l=1}^{s} I'(d_l, d_l^*) \bigcup I'_{+\infty};
\]

2'. \( t \) is \( n \)-right stable if and only if we have \( t = b_j^1 \) or \( t = d_l^1 \) or \( t = b' \) (where \( I'_{+\infty} = [b', +\infty) \));
3'. \( t \) is \( n \)-left stable if and only if we have \( t = c_k^2 \) or \( t = d_l^2 \) or \( t = a' \) (where \( I'_{-\infty} = (-\infty, a') \));
4'. \( t \) is an \( n \)-isolated point if and only if \( t \in A'(m') \).

**Remark 4.3.** In the above list, we collect all types of stability that could theoretically exist. The problem of deciding when this or that type really appears, seems to be very difficult.

5. **Computation of \( H(f) \) for polynomials in two variables**

This section consists of two parts. Firstly, we show that for any polynomial \( f \) in two variables, \( H(f) \) is non-empty. Then, we propose a procedure of computing the set \( H(f) \) explicitly.
5.1. Non-emptiness of $H(f)$.

**Theorem 5.1.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be an arbitrary polynomial in two variables, then $H(f) \neq \emptyset$.

**Proof.** By Corollary 2.1, we can assume that $f$ is of the form $(\ast)$, i.e.

$$f(x, y) = a_0 y^d + a_1(x) y^{d-1} + \cdots + a_d(x),$$

where $a_0 \neq 0$ and $a_i(x)$ are polynomials in $x$, where $\deg a_i(x) \leq i; i = 1, \ldots, d$. Put $V_1 = \{(x, y) \in \mathbb{R}^2 : \frac{\partial f}{\partial y}(x, y) = 0\}$.

Using the second formula, it is enough to prove that

$$\# P^1(f) < +\infty \text{ and } h(f) \neq +\infty.$$ 

Put

$$P(f) := \{t \in \mathbb{R} : \exists \{(x, y^k)\} \subset \mathbb{R}^2, \frac{\partial f}{\partial y}(x, y_k) = 0, \|(x, y_k)\| \to \infty, f(x, y_k) \to t\}.$$

Clearly, $P^1(f) \subset P(f)$.

**Proof of $\# P(f) < +\infty$:**

Let $(x^k, y_k^k) \in V_1$ and $\|(x^k, y^k_k)\| \to \infty$ and $\lim f(x^k, y^k) = a \in P(f)$.

Since $f$ is of the form $(\ast)$, it is easy to see that the curve $V_1$ has no vertical asymptotic. Hence, $x^k \to \infty$. Without loss of generality, we can assume that $0 < r < x^k < x^{k+1} < \ldots$ and $(x^k, y^k) \in \Gamma_a$, where $\Gamma_a$ be a connected component of $V_1 \setminus \{x < r\}$.

$\Gamma_a$ can be parametrized as follows

$$\Gamma_a = \{(x, y) \in \mathbb{R}^2 : x > r, y = y(x)\},$$

where $y(x)$ is a semialgebraic function.

The function $f(x, y(x)), x > r$, is also semialgebraic. By Monotonicity Theorem (see, for example, [HP, Theorem 1.8]), $f(x, y(x))$ is monotone for $x$ sufficiently close to $+\infty$. Since $\lim f(x^k, y^k(x)) = \lim f(x^k, y^k) = a$ and $\{x^k\}$ is monotone, we have

$$\lim f(x, y(x)) = a.$$

Hence, the correspondence $a \in P(f) \mapsto \Gamma_a$ defines an injection of $P(f)$ in the set of connected components of $V_1 \setminus \{x < r\}$. Since $V_1 \setminus \{x < r\}$ is a semialgebraic subset, it has only a finite number of connected components. Hence, the set $P(f)$ is finite. \hfill \Box

**Proof of $h(f) \neq +\infty$:**

By contradiction, assume that $h(f) = +\infty$, which means that $P^2(f) = \mathbb{R}$.
Take \( t_1 \in \mathbb{R} \), then \([f \leq t_1]\) has a sequence of the second type w.r.t \( V_1 \). By Theorem 2.3, there exists \( \{(x^k, y^k)\} \subset V_1 \) such that
\[
\|(x^k, y^k)\| \to \infty,
\]
\[
t_1 < f(x^k, y^k) < M_1 < +\infty,
\]
\[
dist(x^k, [f \leq t_1]) \to +\infty.
\]
Taking a subsequence if necessary, we can assume that \( \lim_{k \to \infty} f(x^k, y^k) = a, t_1 \leq a \leq M_1 \). Hence, \([t_1, M_1] \cap P(f) \neq \emptyset \).

Next, take \( t_2 > M_1 \), by repeating above arguments, we can find \( M_2 > t_2 \) such that \([t_2, M_2] \cap P(f) \neq \emptyset \). Continuing this process, we obtain that \( P(f) \) contains infinitely many points, which is, as we have seen before, impossible.

\[\square\]

Remark 5.1. Let us point out a different proof of Theorem 5.1. Put
\[ P_C(f) = \{ t \in \mathbb{C} : \exists \{(x^k, y^k)\} \subset \mathbb{C}^2, \|(x^k, y^k)\| \to \infty, \frac{\partial f}{\partial y}(x^k, y^k) = 0, f(x^k, y^k) \to t \}. \]

According to [Ha2] (see also [KP]), if \( n = 2 \), then \( P_C(f) = PS_C(f) \). Hence, \( \#PS(f) < +\infty \) which implies, by Proposition 4.1, that \( H(f) \neq \emptyset \).

5.2. Computation of \( H(f) \).

5.2.1. Newton-Puiseux expansions at infinity of affine algebraic curves.

To compute \( H(f) \) explicitly, we need a classical fact of theory of algebraic functions.

Let \( g : \mathbb{R}^2 \to \mathbb{R} \) be a polynomial of the form
\[ g(x, y) = a_0 y^d + a_1(x) y^{d-1} + \cdots + a_d(x), \]
where \( a_0 \neq 0 \) and \( a_i(x), i = 1, \ldots, d \), are polynomials in \( x \) and \( \deg a_i(x) \leq i, i = 1, \ldots, d \).

Proposition 5.1 (see [Hor], [GV] or [W]). Suppose that \( g \) is a polynomial of the above form. Then \( g(x, y) \) is decomposed as
\[ g(x, y) = a_0 \prod_{i=1}^{d} (y - y_i(x)), \]
where each function \( y_i(x) \) is of the form
\[ y_i(x) = \sum_{k=-\infty}^{k_i} c_{ik}(x^{1/p})^k, \tag{15} \]
here, each \( k_i \in \mathbb{Z} \) and \( p \in \mathbb{N} \). Moreover, the series
\[ \hat{y}_i(t) = y(t^p) = \sum_{k=-\infty}^{k_i} c_{ik} t^k \]
converges in the domain \( \{ t \in \mathbb{C} : |t| > A \} \), with \( A \) sufficiently large.

The series \( y_i(x), i = 1, \ldots, d \) of the form (13) are called the Newton-Puiseux expansions at infinity of the curve \( V_C(g) = \{(x, y) \in \mathbb{C}^2 : g(x, y) = 0\} \) (or Newton-Puiseux roots at infinity of the equation \( g(x, y) = 0 \)). Note that they are not necessarily different from each other.

In order to describe the real locus of \( g(x, y) = 0 \), we use the so called real Newton-Puiseux roots at infinity of \( g(x, y) = 0 \).

**Definition 5.1.** Let \( y(x) = \sum_{k=-\infty}^{m} c_k x^k \) be a Newton-Puiseux root at infinity of the equation \( g(x, y) = 0 \). We say that \( y(x) \) is a real Newton-Puiseux roots at infinity of \( g(x, y) = 0 \) if \( c_k \in \mathbb{R} \) for all \( k = m, m-1, \ldots \).

Note that if \( x > 0 \) then \( c_k x^{k/p} \in \mathbb{R} \) if \( c_k \in \mathbb{R} \). Hence the real locus of \( g(x, y) = 0 \), for \( x > r \gg 1 \) is described by the real Newton-Puiseux roots at infinity of \( g(x, y) = 0 \). For describing the real locus of \( g(x, y) = 0 \) in the semi-line \((-\infty, -r), r \gg 1 \). We use the real Newton-Puiseux roots at infinity of \( \overline{g}(x, y) = g(-x, y) \) where \( x > r \).

For a polynomial \( g \), let denote
- \( \mathcal{RP}_+(g) \) the set of all real Newton-Puiseux roots at infinity of \( g \);
- \( \mathcal{RP}_-(g) \) the set of all real Newton-Puiseux roots at infinity of \( \overline{g}(x, y) \).

Put \( \mathcal{RP}(g) = \mathcal{RP}_+(g) \cup \mathcal{RP}_-(g) \). It follows from the definition of \( \mathcal{RP}(g) \) that
\[
V(g) = \{(x, y) \in \mathbb{R}^2 : g(x, y) = 0\} \setminus \{(x, y) \in \mathbb{R}^2 : |x| \leq r\}
= \bigcup_{y(x) \in \mathcal{RP}(g)} \{(x, y) \in \mathbb{R}^2 : x > r, y = y_i(x)\},
\]
where \( r > 0 \) sufficiently large.

Let \( \varphi \) be a function, defined by a fractional series
\[
\varphi(x) = \sum_{k=-\infty}^{m} c_k x^{k/p} = c_m x^{m/p} + \text{terms of lower degrees}.
\]
If \( c_m \neq 0 \), then we denote \( m/p \) by \( v(\varphi) \) and \( v(\varphi) = -\infty \) when \( \varphi \equiv 0 \). \( v(\varphi) \) is called order at infinity of the series \( \varphi \). We have the following lemma

**Lemma 5.1.** [HN, Proposition 2.3] Let \( \varphi(x) \) be any fractional power series as above. Let \( g : \mathbb{R}^2 \to \mathbb{R} \) be a polynomial in two variables of the form
\[
g(x, y) = a_0 y^d + a_1(x) y^{d-1} + \cdots + a_d(x),
\]
where \( a_0 \neq 0 \) and \( \deg a_i(x), i = 0, \ldots, d \) are polynomials of degree \( \leq i \).

Then there exists \( c > 0 \) and \( r \gg 1 \) such that
\[
\frac{1}{c} |x|^{v(\varphi, V(g))} \leq \text{dist}((x, \varphi(x)), V(g)) \leq c |x|^{v(\varphi, V(g))},
\]
for all $|x| \geq r$, where $v(\varphi, V(g)) = \min_{y(x) \in \mathcal{R}(g)} \{v(\varphi(x) - y(x))\}$.

5.2.2. The procedure for computation of $H(f)$.

By Corollary 2.1, we can assume that $f$ is of the form

$$f(x, y) = a_0 y^d + a_1(x) y^{d-1} + \cdots + a_d(x),$$

where $d$ is degree of $f$ and $a_0 \neq 0$.

We will use the second formula to compute $H(f)$.

Firstly, we note that, using the classical Newton-Puiseux algorithm ([BK, Section 8.3] or [W, Chapter 4]), we can compute the set $\mathcal{R}(g)$ for any polynomial $g$ in two variables (see Appendix).

Let $c$ be an arbitrary point of $\mathbb{R}$. It follows from definitions of the sets $P^1(f)$ and $P^2(f)$ and Lemma 5.1 that

- $c \in P^1(f)$ if and only if the following condition, says $(i_c)$, is satisfied

$$\exists \tilde{y}(x) \in \mathcal{R}(\frac{\partial f}{\partial y}) \text{ such that } f(x, \tilde{y}(x)) \geq c,$$

$$\lim_{x \to +\infty} f(x, \tilde{y}(x)) = c,$$

$$\min_{y \in \mathcal{R}(f-c)} v(\tilde{y} - y) \geq 0;$$

- $c \in P^2(f)$ if and only if the following condition, says $(ii_c)$, is satisfied

$$\exists \tilde{y}(x) \in \mathcal{R}(\frac{\partial f}{\partial y}) \text{ such that } f(x, \tilde{y}(x)) \geq c,$$

$$\lim_{x \to +\infty} f(x, \tilde{y}(x)) \neq \infty,$$

$$\min_{y \in \mathcal{R}(f-c)} v(\tilde{y} - y) > 0.$$

By $(i_c)$, we have

$$P^1(f) = \{a \in \mathbb{R} : \exists \tilde{y}(x) \in \mathcal{R}(\frac{\partial f}{\partial y}) \text{ such that } f(x, \tilde{y}(x)) \geq a,$$

$$\lim_{x \to +\infty} f(x, \tilde{y}(x)) = a, \min_{y \in \mathcal{R}(f-c)} v(\tilde{y} - y) \geq 0\};$$

Thus, by the second formula, in order to compute $H(f)$, it remains to compute $h(f)$.

Put

$$P^+(f) = \{a \in \mathbb{R} : \exists \tilde{y} \in \mathcal{R}(\frac{\partial f}{\partial y}) : f(x, \tilde{y}(x)) \geq a, \lim_{x \to +\infty} f(x, \tilde{y}(x)) = a\};$$

Let $P^+(f) = \{a_1 < a_2 < \cdots < a_s\}$. It follows from definition of $P^2(f)$ and the second formula that

$$h(f) \in P^+(f) \cup \{-\infty\}.$$

**Computation of $h(f)$**.

**Step 1.** Check if $h(f) = a_s$ or not:

- If $(ii_c)$ holds for $c = a_s$, then $h(f) = a_s$ and the computation is finished;
• If \((ii_c)\) does not hold for \(c = a_s\), then we take an arbitrary \(b \in (a_{s-1}, a_s)\).

There are two possibilities:

Firstly, if \((ii_c)\) holds for \(c = b\), then \(b \in P^2(f)\) and \(h(f) \in [b, a_s]\). Since \(h(f) \in P^+(f) \cup \{-\infty\}\) and \([b, a_s] \cap P^+(f) = \{a_s\}\), hence, \(h(f) = a_s\) and the computation is finished;

Secondly, if \((ii_c)\) does not hold for \(c = b\), then \(h(f) \leq a_{s-1}\) and we go to the next step.

\textbf{Step 2.} Check if \(h(f) = a_{s-1}\) or not:

Similarly as in Step 1, we can decide whether \(h(f) = a_{s-1}\) or \(h(f) \leq a_{s-2}\). If \(h(f) = a_{s-1}\), then the computation is finished. If \(h(f) \leq a_{s-2}\), then we go to the next step.

In this way, either there exists \(a_{i_0}, i_0 \in \{1, \ldots, s\}\) such that \(h(f) = a_{i_0}\) or \(h(f) < a_1\). In the later case, we have \(h(f) = -\infty\).

\textbf{Example 5.1.} Let \(f(x, y) = y^4 + 2x y^3 + x^2 y^2 - y^2 - xy\).

We have \(\frac{\partial f}{\partial y} = 4y^3 + 6xy^2 + 2x^2 y - 2y - x\). Hence, real Newton-Puiseux expansions at infinity of \(\frac{\partial f}{\partial y}\):

\[
\mathcal{RP}^+\left(\frac{\partial f}{\partial y}\right) = \{\tilde{y}_1(x) = \frac{1}{2x} - \frac{1}{4x^3} + \ldots; \tilde{y}_2(x) = -x - \frac{1}{2x} + \frac{1}{4x^3} \ldots; \tilde{y}_3(x) = -\frac{x}{2}\},
\]

\[
\mathcal{RP}^-\left(\frac{\partial f}{\partial y}\right) = \{\tilde{y}'_1(x) = -\frac{1}{2x} + \frac{1}{4x^3} + \ldots; \tilde{y}'_2(x) = x + \frac{1}{2x} - \frac{1}{4x^3} \ldots; \tilde{y}'_3(x) = \frac{x}{2}\}.
\]

• Computation of \(P^+(f)\).

We have

\[
f(x, \tilde{y}_1(x)) \geq -\frac{1}{4} \quad \text{and} \quad \lim_{x \to \infty} f(x, \tilde{y}_1(x)) = -\frac{1}{4};
\]

\[
f(x, \tilde{y}_2(x)) \geq -\frac{1}{4} \quad \text{and} \quad \lim_{x \to \infty} f(x, \tilde{y}_2(x)) = -\frac{1}{4};
\]

\[
\lim_{x \to \infty} f(x, \tilde{y}_3(x)) = \infty,
\]

\[
\lim_{x \to \infty} f(x, \tilde{y}'_1(x)) \leq 3/4,
\]

\[
\lim_{x \to \infty} f(x, \tilde{y}'_2(x)) = \infty,
\]

\[
\lim_{x \to \infty} f(x, \tilde{y}'_3(x)) = \infty.
\]

This implies \(P^+(f) = \{-\frac{1}{4}\}\).

• Computation of \(h(f)\).

Check if \(h(f) = -\frac{1}{4}\) or not.
We compute Newton-Puiseux roots at infinity of \( f(x, y) + \frac{1}{4} = 0 \). Since \( \tilde{y}_1, \tilde{y}_2 \in \mathbb{RP}_+(\frac{\partial f}{\partial y}) \), we compute

\[
\mathbb{RP}_+(f + \frac{1}{4}) = \{ y_1(x) = \frac{1}{2x} - \frac{1}{4x^3} + \ldots; y_2(x) = -x - \frac{1}{2x} - \frac{1}{2x^2} + \ldots; y_3(x) = -x - \frac{1}{2x} + \frac{1}{2x^2} + \ldots \}.
\]

This implies that

\[
\begin{align*}
\tilde{y}_1(x) - y_1(x) &= 0; \\
\tilde{y}_1(x) - y_2(x) &= x + \frac{1}{x} + \frac{1}{2x^2} + \ldots; \\
\tilde{y}_1(x) - y_3(x) &= x + \frac{1}{x} - \frac{1}{2x^2} + \ldots; \\
\tilde{y}_2(x) - y_1(x) &= -x - \frac{1}{x} + \frac{1}{2x^3} + \ldots; \\
\tilde{y}_2(x) - y_2(x) &= \frac{1}{2x^2} + \frac{1}{4x^3} + \ldots; \\
\tilde{y}_2(x) - y_3(x) &= -\frac{1}{2x^2} + \frac{1}{4x^3} + \ldots.
\end{align*}
\]

Hence, there does not exist \( \tilde{y} \in \mathbb{RP}(\frac{\partial f}{\partial y}) \) satisfies \((ii_c)\), i.e.

\[
\min_{y \in \mathbb{RP}(f + 1/4)} v(\tilde{y} - y) > 0
\]

and it follows that \( -\frac{1}{4} \notin P^2(f) \). Take any \( c < -\frac{1}{4} \), we can check that \( c \in P^2(f) \), thus, \( h(f) = -\frac{1}{4} \).

- **Computation of \( P^1(f) \).**
  
  Since there does not exist \( \tilde{y} \in \mathbb{RP}(\frac{\partial f}{\partial y}) \) which satisfies \((i_c)\), i.e.

  \[
  \min_{y \in \mathbb{RP}(f + 1/4)} v(\tilde{y} - y) \geq 0.
  \]

  It follows that \( -\frac{1}{4} \notin P^1(f) \).

Combining above computations, we conclude that \( H(f) = [-\frac{1}{4}, +\infty) \). In this example, for any \( t \in \mathbb{R} \), we have:

- If \( t > -\frac{1}{4} \), then \( t \) is y-stable;
- If \( t = -\frac{1}{4} \), then \( t \) is y-right stable;
- If \( t < -\frac{1}{4} \), then \( t \) is n-stable.

**Appendix. Computation of Newton-Puiseux roots at infinity of the equation \( g(x, y) = 0 \)**

In this Appendix, we recall how one can reduce the problem of computation of Newton-Puiseux roots at infinity of a polynomial equations to the classical problem...
of computation of Newton-Puiseux roots in a neighborhood of the point \((0,0)\in \mathbb{C}^2\) of germs of analytic functions.

Let \(g(x,y)\) be a polynomial of the form \((\ast)\), i.e.
\[
g(x,y) = a_0 y^d + a_1(x) y^{d-1} + \cdots + a_d(x),
\]
where \(d\) is the degree of \(g\).

We want to compute the Newton-Puiseux roots at infinity of \(g(x,y) = 0\). Let
\[
V_\mathbb{C}(g) = \{(x,y)\in \mathbb{C}^2 : g(x,y) = 0\}.
\]

Let \(\hat{g}(x,y,z) = z^d g(\frac{x}{z}, \frac{y}{z})\). Then \(
\overline{V_\mathbb{C}(g)}\), the compactification of \(V_\mathbb{C}(g)\) in projective plane \(\mathbb{CP}^2\), is given by
\[
\overline{V_\mathbb{C}(g)} = \{[x : y : z] \in \mathbb{CP}^2 : \hat{g}(x,y,z) = 0\}.
\]
The set \(\overline{V_\mathbb{C}(g)}\cap \{z = 0\}\), the intersection of \(\overline{V_\mathbb{C}(g)}\) with the line at infinity of \(\mathbb{CP}^2\), is given by
\[
\overline{V_\mathbb{C}(g)}\cap \{z = 0\} = \{[x : y : 0] \in \mathbb{CP}^2 : g_d(x,y) = 0\},
\]
where \(g_d(x,y)\) is the sum of all the monomials of degree \(d\) of \(g\).

Since \(g\) is of the form \((\ast)\), all the points of \(\overline{V_\mathbb{C}(g)}\cap \{z = 0\}\) are contained in the chart \(\{x = 1\}\) of \(\mathbb{CP}^2\). Hence,
\[
\overline{V_\mathbb{C}(g)}\cap \{z = 0\} = \{[1 : c_i : 0] \in \mathbb{CP}^2, i = 1,\ldots,m\},
\]
where \(\{c_i, i = 1,\ldots,m\}\) are the set of roots of \(g_d(1,y) = 0\).

Let \(\overline{\overline{y}}_{i1}(z), \ldots, \overline{\overline{y}}_{im(i)}, i = 1,\ldots,m\) be the set of all Newton-Puiseux expansions of the germ \(\hat{g}(1,y,z)\) at the point \((z,y) = (0,c_i)\). Note that, the series \(\overline{\overline{y}}_{i1}, \ldots, \overline{\overline{y}}_{im(i)}, i = 1,\ldots,m\) can be computed explicitly by the classical Newton-Puiseux algorithm (see, for examples, [BK] Section 8.3 or [W] Chapter 4). We have \(\sum_{i=1}^{m} m(i) = d\) (counted with multiplicities) and
\[
y(x) = x \overline{\overline{y}}_{ij}(\frac{1}{x}), i = 1,\ldots,m; j = 1,\ldots,m(i),
\]
are the Newton-Puiseux expansions at infinity of the equation \(g(x,y) = 0\).

**Example 5.2 (see Example 5.1).** Let \(g(x,y) = 4y^3 + 6xy^2 + 2x^2y - 2y - x\). Then
\[
\hat{g}(x,y,z) = 4y^3 + 6xy^2 + 2x^2y - 2yz^2 - xz^2.
\]

\[
\overline{V_\mathbb{C}(g)}\cap \{z = 0\} = \{[x : y : 0] \in \mathbb{CP}^2 : 4y^3 + 6xy^2 + 2x^2y = 0\}
= \{[1 : y : 0] \in \mathbb{CP}^2 : 4y^3 + 6y^2 + 2y = 0\}
= \{A_1 = [1 : 0 : 0]; A_2 = [1 : -1 : 0]; A_3 = [1 : -\frac{1}{2} : 0]\} \subset \mathbb{CP}^2.
\]
Let us compute Newton-Puiseux roots at infinity of \( g(x, y) = 0 \), corresponding to the point \( A_1 \). We consider

\[
\tilde{g}(z, y) = \hat{g}(1, y, z) = 4y^3 + 6y^2 + 2y - 2yz^2 - z^2 = 0.
\]

We compute Newton-Puiseux roots of \( \tilde{g}(z, y) = 0 \) in the neighborhood of \((z, y) = (0, 0)\).

According to the Newton-Puiseux algorithm (see, for example, [BK, Section 8.3, p377]), the first term of \( \tilde{y}(z) \) is \( \frac{z^2}{2} \). To compute the next term of \( \tilde{y}(z) \), one puts \( \tilde{y}(z) = \frac{1}{2}z^2(1 + \varphi) \). We have

\[
0 = \tilde{g}(z, \tilde{y}(z)) = \tilde{g}(z, \frac{1}{2}z^2(1 + \varphi))
\]

or

\[
\frac{1}{2}z^4(1 + \varphi)^3 + 2z^2\varphi + 3\frac{1}{2}z^2\varphi^2 + \varphi + \frac{1}{2}z^2 = 0.
\]

Again, the Newton-Puiseux algorithm tells us that

\[
\varphi = -\frac{1}{2}z^2 + o(z^2).
\]

Hence, \( \tilde{y}(z) = \frac{1}{2}z^2 - \frac{1}{4}z^4 + \ldots \). Therefore,

\[
y(x) = x\tilde{y}(\frac{1}{x}) = \frac{1}{2x} - \frac{1}{4x^3} + \ldots
\]

is a Newton-Puiseux root at infinity of \( g(x, y) = 0 \). (Moreover, in this example, we have \( \frac{\partial}{\partial \varphi} [\frac{1}{2}z^4(1 + \varphi)^3 + 2z^2\varphi + 3\frac{1}{2}z^2\varphi^2 + \varphi + \frac{1}{2}z^2](0, 0) = 1 \). By Implicit Function Theorem, \( \varphi \) is a real analytic function. Hence, \( y(x) \) is a real Newton-Puiseux root at infinity of \( g(x, y) = 0 \).

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