CLASSICAL DIFFERENTIAL GEOMETRY AND INTEGRABILITY OF SYSTEMS OF HYDRODYNAMIC TYPE

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ABSTRACT. Remarkable parallelism between the theory of integrable systems of first-order quasilinear PDE and some old results in projective and affine differential geometry of conjugate nets, Laplace equations, their Bianchi-Bäcklund transformations is exposed. These results were recently applied by I.M.Krichever and B.A.Dubrovin to prove integrability of some models in topological field theories. Within the geometric framework we derive some new integrable (in a sense to be discussed) generalizations describing N-wave resonant interactions.

Ten years ago [10] a natural hamiltonian formalism was proposed for the class of homogeneous systems of PDE

\[
\begin{pmatrix}
 u^1_i \\
 \vdots \\
 u^n_i
\end{pmatrix} =
\begin{pmatrix}
 v^1_1(u) & \cdots & v^1_n(u) \\
 \vdots & \ddots & \vdots \\
 v^n_1(u) & \cdots & v^n_n(u)
\end{pmatrix}
\begin{pmatrix}
 u^1_x \\
 \vdots \\
 u^n_x
\end{pmatrix},
\]

(1+1,h)

\[u^i = u^i(x,t), \quad i = 1, \ldots, n\]

(called "one-dimensional systems of hydrodynamic type"). Later (see [11]) it was generalized for the class of multidimensional
Let us recall briefly the main results of [10], [32]. A (generally nondiagonal) system

\[
\begin{pmatrix}
  u^1_t \\
  \vdots \\
  u^n_t
\end{pmatrix}
= 
\begin{pmatrix}
  v_1^{(1)}(u) & \cdots & v_n^{(1)}(u) \\
  \vdots & \ddots & \vdots \\
  v_1^{(n)}(u) & \cdots & v_n^{(n)}(u)
\end{pmatrix}
\begin{pmatrix}
  u^1_x \\
  \vdots \\
  u^n_x
\end{pmatrix}
+ 
\cdots
\]

\[u^i = u^i(t, x_1, \ldots, x_N), \quad i = 1, \ldots, n\]

and non-homogeneous

\[
\begin{pmatrix}
  u^1_t \\
  \vdots \\
  u^n_t
\end{pmatrix}
= 
\begin{pmatrix}
  v_1(u) & \cdots & v_n(u) \\
  \vdots & \ddots & \vdots \\
  v_1(u) & \cdots & v_n(u)
\end{pmatrix}
\begin{pmatrix}
  u^1_x \\
  \vdots \\
  u^n_x
\end{pmatrix}
+ 
\begin{pmatrix}
  f_1(u) \\
  \vdots \\
  f_n(u)
\end{pmatrix},
\]

\[u^i = u^i(x, t), \quad i = 1, \ldots, n\]

systems.

Some systems (1+1,h) of physical importance such as Whitham equations (the averaged 1-phase KdV equation) and Benney equations have the notable property of being diagonalizable: under a suitable choice of field variables \(u^i\) (Riemann invariants) the equations become

\[(1)\]

\[u^i_t(x) = v_i(u)u^i_x\]

(there is no summation over \(i\)). As we have proved in [32], these properties (hamiltonian property and diagonalizability) imply integrability. Deeper insight into this type of integrability is given by the theory of orthogonal curvilinear coordinate systems. This chapter of classical differential geometry was being intensively developed at the beginning of the XX century ([6], [8], [20]). In fact this theory gives the geometric background for integrability of systems (1+1,h), (N+1,h), (1+1,nh). These forgotten corners of differential geometry seem to be worth revisiting.

An example. The well-known Bullough-Dodd-Jiber-Shabat equation \(u_{xx} - u_{tt} = e^u - e^{-2u}\) (in the form \((\ln h)_{uv} = h - 1/h^2\)) was introduced for the first time in [35] where the respective linear problem was given as well as a proper Bäcklund transformation for it! It is much simpler and "geometric" than Bäcklund transformations discussed recently [1], [31] in the context of integrable systems.

In this paper we will sketch some applications of methods originating from classical differential geometry to equations of types (1+1,h), (N+1,h), (1+1,nh).

1. Diagonal systems of hydrodynamic type and orthogonal curvilinear coordinate systems in \(\mathbb{R}^n\)

Let us recall briefly the main results of [10], [32]. A (generally nondiagonal) system \(u^i_t = \sum_{j=1}^n v^i_j(u)u^j_x\) is hamiltonian if there exist a hamiltonian \(H = \int h(u) \, dx\) and a
hamiltonian operator

\[ \hat{A}^{ij} = g^{ij}(u) \frac{d}{dx} + b^{ij}_k(u) u^k_x \]

which define a skew-symmetric Poisson bracket on functionals

\[ \{I, J\} = \int \frac{\delta I}{\delta u^i(x)} \hat{A}_{ij} \frac{\delta J}{\delta u^j(x)} dx \]

satisfying the Jacobi identity and generate the system

\[ u^i_t(x) = \{u^i(x), H\} = \hat{A}_{ij} \frac{\partial H}{\partial u^j(x)} = (g^{ij} \partial_k \partial_j h + b^{ij}_k \partial_j h) u^k_x = v^i_k(u) u^k_x \]

where \( \partial_s = \partial/\partial u^s \). B.A.Dubrovin and S.P.Novikov [10] proved that the necessary and sufficient conditions for \( \hat{A}^{ij} \) to be a hamiltonian operator in the case of non-degeneracy of the matrix \( g^{ij} \) are:

a) \( g^{ij} = g^{ji} \), i.e. the inverse matrix \( g^{-1} \) defines a Riemannian metric.

b) \( b^{ij}_k = -g^{is} \Gamma^j_{sk} \) for the standard Christoffel symbols \( \Gamma^j_{sk} \) generated by \( g_{ij} \).

c) the metric \( g_{ij} \) has identically vanishing curvature tensor.

In such case we have \( v^i_j(u) = \nabla^i \nabla_j h = g^{is} \nabla_s \nabla_j h \) with the covariant derivatives defined by \( g_{ij} \).

Lemma [32], [33]. In order that a matrix \( v^i_j(u) \) be a matrix of a hamiltonian system (1+1,h) with a nondegenerate metric in \( \hat{A}^{ij} \) it is necessary and sufficient that there exists a nondegenerate zero curvature metric \( g_{ij} \) such that

a) \( g_{ik} v^k_j = g_{jk} v^k_i \) and

b) \( \nabla_j v^i_k = \nabla_k v^i_j \), where \( \nabla \) is the covariant differentiation generated by the metric \( g_{ij} \).

For a diagonal matrix \( v^i_j(u) = v_j(u) \delta^i_j \) this implies that (see [32], [33]) \( g_{ij} \) is also diagonal and

\[ \frac{\partial_i v_k}{v_i - v_k} = \Gamma^k_{ki} = \frac{1}{2} \partial_i \ln g_{kk}, \quad \partial_i = \partial/\partial u^i \]

(hereafter we do not imply the summation on repeated indices!). From (3) we deduce

\[ \partial_j \frac{\partial_i v_k}{v_i - v_k} = \partial_i \frac{\partial_j v_k}{v_j - v_k}, \quad i \neq j \neq k. \]

From a differential geometric point of view, to give a zero curvature nondegenerate diagonal metric is equivalent to giving an orthogonal curvilinear coordinate system on a flat (possibly pseudo-Euclidean) space (see [6]). Locally these coordinate systems are determined by \( n(n - 1)/2 \) functions of two variables (L.Bianchi). A striking fact
can be discovered: formula (3) was found in [6] (p. 353)! This formula is crucial for the integrability property of diagonal hamiltonian systems (1): if we interpret it as an overdetermined (compatible in view of zero curvature property of \( g \)) system on \( n \) unknown functions \( v_j(u) (g_{ii} \text{ given}) \) we can generate from every its solution \( \bar{v}_j(u) \) a symmetry (commuting flow)

\[
 u_t^i = \bar{v}_i(u) u_x^i, \quad i = 1, ..., n,
\]

of (1) and a solution of (1) (the generalized hodograph method, see [33] for the details). One can prove ([33]) the completeness property for this class of symmetries and solutions parameterized by \( n \) functions of 1 variable - the generic Cauchy data for our diagonal system (1).

The corresponding geometric notion used in the theory of orthogonal curvilinear coordinate systems corresponding to (3) is the so called Combescure transformation (see [6]).

**Definition.** Two orthogonal curvilinear coordinate systems \( x^i = x^i(u^1, ..., u^n) \) and \( \hat{x}^i = \hat{x}^i(u^1, ..., u^n) \) in the same flat (pseudo)Euclidean space \( \mathbb{R}^n = \{(x^1, ..., x^n)\} \) are said to be related by a Combescure transformation (or simply parallel) iff their tangent frames \( \vec{e}_i = \partial \vec{x}/\partial u^i \) and \( \hat{\vec{e}}_i = \partial \hat{\vec{x}}/\partial u^i \) are parallel in points corresponding to the same values of curvilinear coordinates \( u^i \).

Let us take the quantities \( H_i(u) = |\vec{e}_i| = \sqrt{g_{ii}}, \hat{H}_i(u) = |\hat{\vec{e}}_i| \) (Lamé coefficients).

**Proposition** The quantities \( \bar{v}_i(u) = \hat{H}_i(u)/H_i(u) \) satisfy (3) with \( \Gamma^k_{ki} = \partial_i H_k/H_k \), the connection coefficients for the metric \( g_{ii} = H_i^2 \). Conversely, for any solution \( \bar{v}_i \) of (3) \( \hat{g}_{ii} = (\bar{v}_i H_i)^2 \) will give an orthogonal curvilinear coordinate system related to the coordinate system with the metric \( g_{ii} = H_i^2 \) by a Combescure transformation.

The theory of Combescure transformations coincides with the theory of integrable diagonal systems of hydrodynamic type.

Physical examples of such systems (Whitham equations, Benney equations) have hamiltonian structures (2) with diagonal metrics \( g_{ii} \) possessing the so called Egorov property: \( \partial_i g_{kk} = \partial_k g_{ii} \). As we have demonstrated earlier ([34]) this is a consequence of Galilei invariance of the original systems. See also [12] for the algebro-geometric background of this property for averaged integrable systems. Using this property and homogeneity of coefficients one can find explicit formulas for solutions of (3) for the systems in question [33], [34].

The class of Egorov orthogonal curvilinear coordinate systems is interesting in itself and merits our special attention.

2. Egorov coordinate systems, the \( N \)-wave problem and its generalizations

Introducing \( \beta_{ik}(u) = \partial_i H_k/H_i, \quad i \neq k, \beta_{ii}(u) = 0 \) (rotation coefficients of a given
orthogonal curvilinear coordinate system with \( g_{ii} = H^2_i \), see [6]) one can easily check the following:

a) vanishing of the curvature tensor is equivalent to
\[
\partial_j \beta_{ik} = \beta_{ij} \beta_{jk}, \quad i \neq j \neq k, \tag{5}
\]
\[
\partial_i \beta_{ik} + \partial_k \beta_{ki} + \sum_{s \neq i,k} \beta_{si} \beta_{sk} = 0, \quad i \neq k. \tag{6}
\]
b) the Egorov property \( \partial_i g_{kk} = \partial_k g_{ii} \) reduces to
\[
\beta_{ik} = \beta_{ki}. \tag{7}
\]

In the Egorov case condition (6) is equivalent to \( \hat{T} \beta_{ik} = 0 \), \( \hat{T} \) is an operator defined by
\[
\hat{T} = \partial_1 + \ldots + \partial_n.
\]
Consequently the problem of classification of Egorov coordinate systems is reduced to description of all off-diagonal symmetric matrices \( (\beta_{ik}) \) satisfying (5) and \( \hat{T} \beta_{ik} = 0 \).

B.A.Dubrovin [12] have recently observed that this problem coincides with the purely imaginary reduction of the well-known integrable system describing resonant \( N \)-wave interactions. Namely, restriction of \( \beta_{ik} \) on any \((x,t)\) plane \( u^i = a^i x + b^i t \) gives
\[
[A, \Gamma_t] - [B, \Gamma_x] = [[A, \Gamma], [B, \Gamma]],
\]
\[
A = diag(a^1, \ldots, a^n), B = diag(b^1, \ldots, b^n), \Gamma = (\beta_{ik}) \text{ with additional reduction}
\]
\[
\text{Im} \Gamma = 0, \Gamma^T = \Gamma.
\]
For the case \( N = 3 \) this reduces to
\[
\begin{align*}
\begin{cases}
 b_1^1 + c_1 b_2^1 &= \kappa b^2 b^3, \\
 b_2^1 + c_2 b_3^1 &= \kappa b^1 b^3, \\
 b_3^1 + c_3 b_2^1 &= \kappa b^1 b^2.
\end{cases}
\end{align*} \tag{8}
\]

This is a system of type \((1+1,nh)\), integrable by the IST method ([28]).

Now we can compare the progress achieved in the modern integrability theory for (8) and the results obtained more than 70 years ago in the theory of Egorov coordinate systems initiated by G.Darboux in 1866 and continued by D.Th.Egorov in 1901 in his thesis (see [13]). It was Darboux [6] who proposed to call this special type of coordinate systems Egorov type systems. From the point of view of integrability properties remarkable progress was achieved by L.Bianchi in 1915 (see [2]). He found a Bäcklund transformation for this problem and established the permutability property as well as the superposition formula for it. We shall remark here that the pioneering results on Bäcklund transformations and their permutability in the well-known theory of constant curvature surfaces in \( R^3 \) are due to Bianchi also.

Let us take an orthogonal curvilinear coordinate system (not necessary of Egorov type) with Lamé coefficients \( H_i(u) \) and rotation coefficients \( \beta_{ik}(u) \). Bianchi applied to it a generalization of Ribaucour transformations known in the theory of transformations of surfaces. We recall that two surfaces \( \tilde{x}(u,v) \) and \( \hat{x}(u,v) \) in \( R^3 \) are related
by a Ribaucour transformation iff there exists a two-parametric family of spheres $S(u, v)$ such that each sphere $S(u_0, v_0)$ is tangent to both surfaces in corresponding points $\tilde{x}(u_0, v_0)$, $\hat{x}(u_0, v_0)$ and this correspondence preserves the curvature lines on the surfaces. For the case of a pair of orthogonal curvilinear coordinate systems in $R^3$ we need a three-parametric family of spheres (or an $n$-parametric family for the $n$-dimensional case) tangent in the corresponding points to one of three families of coordinate surfaces as well as to a coordinate surface of the other curvilinear coordinate system. Since due to the classical Dupin theorem coordinate lines in any orthogonal curvilinear coordinate system are curvature lines their correspondence is guaranteed. In terms of the rotation coefficients $\beta_{ik}$ one shall find a solution $\gamma_i(u)$ of

$$\partial_i \gamma_k = \beta_{ki} \gamma_i, \quad i \neq k,$$

(9)

to define the corresponding Ribaucour transformation ([2])

$$\dot{\beta}_{ik} = \beta_{ik} - \frac{2\gamma_i}{A} (\partial_k \gamma_k + \sum_{s \neq k} \beta_{sk} \gamma_s), \quad A = \sum_p (\gamma_p)^2.$$

(10)

For the case of Egorov systems we shall complete (9) and restrict $\gamma_i$ to satisfy

$$\partial_i \gamma_i = c \gamma_i - \sum_{s \neq i} \beta_{si} \gamma_s, \quad (c = \text{const})$$

or

$$\hat{T} \gamma_i = c \gamma_i, \quad \hat{T} = \partial_1 + \ldots + \partial_n$$

(11)

Then the Bäcklund transformation in question is

$$\dot{\beta}_{ik} = \beta_{ik} - \frac{2c \gamma_i \gamma_k}{A}. $$

(12)

The permutability property for any orthogonal coordinate system requires a quadrature, but for Egorov systems it may be found explicitly and provides the following formulas for the fourth Egorov system $\tilde{\beta}_{ik}$ related to $\beta'_{ik}, \beta''_{ik}$ obtained from a given Egorov system $\beta_{ik}$ with constants $c', c''$ ($c' + c'' \neq 0$) and potentials $\gamma'_i, \gamma''_i$ in (10):

$$\gamma'_i = \gamma''_i = \frac{2c' \gamma_i \sum_s (\gamma'_s \gamma''_s)}{(c' + c'') \sum_s (\gamma'_s)^2}, \quad \gamma''_i = \frac{2c'' \gamma_i \sum_s (\gamma'_s \gamma''_s)}{(c' + c'') \sum_s (\gamma''_s)^2}.$$  

(13)

One can enjoy reading [4], [23] where these formulas were rediscovered in the context of 3-wave system. So the basic integrability results for (8) were established long ago by Darboux, Egorov and Bianchi certainly with the exception of the IST transformation.

An unexpected result (hidden in [6]) consists in existence of a homogeneous system $(1+1,h)$ of three equations related to (8) by a nonlocal transformation. Geometrically this is trivial. Given an orthogonal curvilinear coordinate system in $R^3$ we have in each its point $P(x_0, y_0, z_0)$ the orthogonal 3-frame of tangent planes

$$z = p_k (x - x_0) + q_k (y - y_0) + z_0, \quad k = 1, 2, 3.$$
Let us parameterize it by 3 functions \( A(x, y, z), B(x, y, z), C(x, y, z) \), coefficients \( p_k, q_k \) of tangent planes being three solutions of

\[
\begin{align*}
pq + Ap + Bq &= 0, \\
p^2 - q^2 + 2(Cp + Hq) &= 0, \\
2(BC - AH) + 1 &= 0,
\end{align*}
\]

different from the trivial solution \( p = q = 0 \) ([3]). Then the Frobenius compatibility conditions for these three families of distributions (13) give a system of three homogeneous first-order equations of type (2+1,h):

\[
\begin{align*}
2(AC_z - CA_z) &= 2Cy + By - Ax \\
2(BH_z - HB_z) &= 2Hx + By - Ax \\
AH_z - HA_z + BC_z - CB_z &= Ay - Bx
\end{align*}
\]

where \( H = (2BC - 1)/2A \).

For this system one can reformulate the Bäcklund-like transformation (10) given in terms of \( \beta_{ik}(u) \). A number of different transformations producing (with quadratures) solutions of (14) parameterized by arbitrary many functions of one variable may be found in [6]. Thus (14) is integrable in a sense to be discussed elsewhere.

If we will search for solutions of (14) which do not depend on \( z \) then a remarkable integrable (1+1,h) system of three equation appears. Since one can easily prove the equivalence of \( z \)-independence in (14) and the Egorov property (7) we have received a homogeneous system related to (8) by a nonlocal change of variables. In Euler \( \varphi, \psi, \theta \) parameterization of orthogonal 3-frames it reads

\[
\begin{pmatrix}
\psi_t \\
\theta_t \\
\varphi_t
\end{pmatrix} = \begin{pmatrix}
-\cos^2 \varphi & -\sin \varphi \cos \varphi / \sin \theta & 0 \\
-\sin \theta \sin \varphi \cos \varphi & -\sin^2 \varphi & 0 \\
-\cos \theta (1 + \cos^2 \varphi) & -\sin \varphi \cos \varphi \cos \theta / \sin \theta & 1
\end{pmatrix} \begin{pmatrix}
\psi_x \\
\theta_x \\
\varphi_x
\end{pmatrix}
\]

This nonlocal change does not affect the existence of higher order conserved densities. Recently Ferapontov [17] have proved the uniqueness result for such 3×3 homogeneous systems possessing higher-order conserved densities: they may be transformed to (15) by reciprocal and point transformations. Also another nonlocal transition from (8) to (15) as given there.

The matrix of (15) has constant eigenvalues \(-1, 0, +1\) but its eigenvector fields (properly normalized) form \( so(3) \) Lie algebra, consequently (15) is a non-diagonalizable (1+1,h) integrable system.

The complete system (14) certainly may be called a (2+1)-dimensional generalization of the (1+1)-dimensional 3-wave system (8). Orthogonal curvilinear coordinate systems in \( R^n \) provide also only a (2+1)-dimensional generalization of the (1+1)-dimensional \( N \)-wave system since they are parameterized by \( n(n - 1)/2 \) functions of \( two \) variables (L.Bianchi).

3. Semihamiltonian diagonal systems and coordinate systems with conjugate lines
The class of integrable diagonal systems (1) is wider than the class of *hamiltonian* systems of this type. Namely, the property (4) which is a weaker consequence of the hamiltonian property is sufficient ([33]). Let us call a diagonal system *semi-hamiltonian* if \( n = 2 \) or if \( n > 2 \) and \( v_i(u) \) satisfy (4). As a physical example of a semihamiltonian (but non-hamiltonian for \( n > 3 \)) system one can mention the ideal Langmuir chromatography and electrophoresis systems ([33]).

To every semihamiltonian system we can relate a diagonal metric \( g_{ii} \) via

\[
\frac{\partial_i \ln g_{kk}}{2} = \frac{\partial_i v_k}{(v_i - v_k)}.
\]

This metric is *not flat* in general though some coefficients of the curvature tensor vanish as a consequence of (4). Namely introducing \( H_i = \sqrt{|g|} \), \( \beta_{ik}(u) = \frac{\partial_i H_k}{H_i} \), we can find out that (4) is equivalent to the set (5) of equations on \( \beta_{ik} \). Solutions of (5) may be parameterized by \( n(n-1) \) functions of 2 variables. This system coincides with the compatibility conditions for a linear system

\[
\partial_i \psi_k = \beta_{ik} \psi_i, \quad i \neq k.
\]

Restricting (5) on 3-dimensional planes \( u^i = a^i x + b^i y + c^i z \) in \( \mathbb{R}^n \) we obtain (for general nonvanishing constants \( a^i, b^i, c^i \)) a \((2+1, nh)\) system on \( n(n-1) \) quantities \( \beta_{ik}(x, y, z) \).

As we have seen earlier the theory of hamiltonian diagonal systems (1) is closely related to the theory of orthogonal curvilinear coordinate systems in \( \mathbb{R}^n \). The geometric background for the theory of semihamiltonian systems is given by the theory of coordinate systems with conjugate coordinate lines (see [6] and [7], t. 4, ch. 12). A general (non-orthogonal) coordinate system \( \bar{x}(u_1, u_2, u_3) \) in \( \mathbb{R}^3 \) is called a system with conjugate coordinate lines (or simply a conjugate coordinate system) if on every coordinate surface \( S_{u_0} = \{ u_0 = \text{const} \} \) in every point \( P(x_0, y_0, z_0) \) the lines of intersection of this surface with two other coordinate surfaces belonging to one-parametric families of coordinate surfaces and containing \( P(x_0, y_0, z_0) \) are conjugate on \( S_{u_0} \) (with respect to its second fundamental form). Every orthogonal curvilinear coordinate system is conjugate due to Dupin theorem mentioned above. The theory of conjugate coordinate systems was developed by Darboux and others and borrowed a lot of results from the classical theory of conjugate coordinate nets on surfaces in \( \mathbb{R}^3 \) (known as "nets" or "réseaux", see [14], [21], [22], [36]). A number of Bäcklund-like transformations for these coordinate systems was given with permutability properties (though the superposition formulas therein require quadratures).

Every conjugate coordinate system \( x^i = x^i(u^1, \ldots, u^n) \) in \( \mathbb{R}^n = \{(x^1, \ldots, x^n)\} \) is characterized by the conditions of conjugacy of coordinate lines:

\[
\partial_i \bar{x}^k = \Gamma_{ki}^k(u) \partial_k \bar{x} + \Gamma_{ik}^i(u) \partial_i \bar{x}, \quad i \neq k.
\]

This system of equations coincides with the system describing hydrodynamic type conserved quantities of a semihamiltonian system with ([33]). Quantities \( \Gamma_{ki}^k \) in (16)
satisfy its compatibility conditions
\[ \partial_j \Gamma^k_{ki} = \partial_i \Gamma^k_{kj}, \quad \partial_j \Gamma^k_{ki} = \Gamma^k_{kj} \partial^j_i + \Gamma^k_{ki} \Gamma^j_i - \Gamma^k_{ki} \Gamma^j_i, \quad i \neq j \neq k. \]
equivalent to the semihamiltonian property (4). Introducing \( H_i(u) \) as solutions of \( \partial_i H_k(u) = \Gamma^k_{ki} H_k(u) \) and \( \beta_{ik} = \partial_i H_k / H_i, \quad i \neq k \), we receive a set of \( \beta_{ik} \) satisfying (5). The converse is also true: given a solution \( \beta_{ik} \) of (5) one can find (a number of) semihamiltonian systems related to it. Any semihamiltonian system also may be related to a Combesecure transformation of conjugate coordinate systems ([6]).

This geometric interpretation provides another example of integrable \((2+1,h)\) system. Namely, given a conjugate coordinate system in \( R^3 \) one can take the field of its (non-orthogonal) tangent 3-frames \( (\vec{e}_1, \vec{e}_2, \vec{e}_3) \) and parameterize it by 6 independent functions \( e^i_k(x,y,z), i = 1,2, k = 1,2,3 \) (the coefficients \( e^3_k \) may be set to 1 due to normalization). Then the Frobenius compatibility conditions give 3 homogeneous first-order PDE on \( e^i_k(x,y,z), i = 1,2, k = 1,2,3 \). Another 3 equations are given by the conjugacy condition \( \det((\vec{e}_i \cdot \nabla)\vec{e}_k, \vec{e}_i, \vec{e}_k) = 0, i < k \). This system of 6 equations is a homogeneous \((2+1,h)\) system in question. Its \( z \)-independent solutions satisfy a \((1+1,h)\) system enjoying properties analogous to those of (15): it has constant eigenvalues \(-1,0,+1\) (all doubly degenerate) and 6 linearly independent fields of eigenvectors forming (if properly normalized) a nontrivial Lie algebra. This remarkable system will be studied in subsequent publications.

4. Additional topics

Recently O.I.Mokhov and E.V.Ferapontov [27], [16] found a nonlocal generalization of the hamiltonian formalism of hydrodynamic type (2). V.E.Ferapontov communicated to the author about the further generalization resulting in the following beautiful theorem: any semihamiltonian system (1) has a nonlocal hamiltonian structure with a hydrodynamic hamiltonian and a hamiltonian operator with (possibly infinitely many) nonlocal terms similar to those in [16]. Grinevich [19] derived a series of nonlocal symmetries for Whitham equations as well as original KdV equations.

Weakly nonlinear semihamiltonian systems (i.e. systems (6) with \( \partial_i v_i = 0 \) without summation on \( i \), such systems are also called "linearly degenerate") were studied in [15], [30]. The theory of such systems is connected to the theory of \( n \)-webs on Euclidean plane, Dupin cyclids and Stäckel metrics (E.V.Ferapontov). Among the results are: quasiperiodic behavior of their solutions ([30]), complete description of such systems and complete sets of their hydrodynamic symmetries ([15]). E.V.Ferapontov communicated to the author the following fact: any \( n \)-phase (\( n \)-zone) quasiperiodic (or a \( n \)-soliton) solution of the KdV equation can be represented with a solution of a weakly nonlinear semihamiltonian system \( R^i_t = (\sum_{k \neq i} R^k) R^i_x, i = 1, \ldots, n \). These results shall be compared with Curro and Fusco’s results [5] in the soliton-like interactions of Riemann simple waves for some \( 2 \times 2 \) systems.
IST-like methods were developed in [18], [24], [25] for some diagonal hamiltonian systems of physical importance. Certainly this approach shall be related to our geometric methods.

In a series of preprints (see [9],[26] and references therein) I.M.Krichever and B.A.Dubrovin exposed a remarkable link between the theory of Egorov coordinate systems and Witten-Dijgraagh-Verlinder-Verlinder equations for the correlation functions of topological conformal field theories proving their integrability.

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