Entanglement and spin squeezing in non-Hermitian phase transitions

Tony E. Lee,1,2 Florentin Reiter,3 and Nimrod Moiseyev4

1ITAMP, Harvard-Smithsonian Center for Astrophysics, Cambridge, Massachusetts 02138, USA
2Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA
3Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen, Denmark
4Schulich Faculty of Chemistry and Faculty of Physics, Technion - Israel Institute of Technology, Haifa, 32000, Israel

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We show that non-Hermitian dynamics generate substantial entanglement in many-body systems. We consider the non-Hermitian Lipkin-Meshkov-Glick model and show that the non-Hermitian phase transition occurs with maximum multi-particle entanglement: there is full N-particle entanglement at the transition, in contrast to the Hermitian case. The non-Hermitian model also exhibits more spin squeezing than the Hermitian model, showing that non-Hermitian dynamics are useful for quantum metrology. Experimental implementations with cavity-QED and trapped ions are discussed.

Introduction.— Condensed matter and quantum information, two seemingly disparate fields, have found much cross-fertilization in recent years. Many-body physics traditionally falls under condensed matter, but it has been shown that entanglement (a staple of quantum information) provides new insight into many-body systems [4]. Aside from fundamental interest, understanding the entanglement in condensed-matter systems allows one to use such systems for applications like quantum computing and quantum metrology [5–7].

In an interacting many-body system, a quantum phase transition usually changes how the particles are entangled with each other [4]. The Lipkin-Meshkov-Glick model is the prototypical model of interacting spins that undergo a quantum phase transition, so it serves as an important example: two-particle entanglement peaks at the phase transition [8, 9], while multi-particle entanglement becomes macroscopic after the transition [10, 11].

At the same time, the field of non-Hermitian quantum mechanics has drawn significant interest, especially with recent experimental results in cavities [12–15], waveguides [14], and ultracold atoms [15]. The motivation is that non-Hermitian systems behave quite differently from Hermitian ones and can exhibit novel phenomena [16–20]. Non-Hermitian dynamics often arise in systems with dissipation or loss.

In this paper, we view non-Hermitian quantum mechanics from a quantum-information perspective: we see what kind of entanglement it generates. We study the non-Hermitian Lipkin-Meshkov-Glick model and show that the non-Hermitian phase transition coincides with a maximum in multi-particle entanglement [Fig. 1(a)]. In fact, all particles are entangled at the transition, in contrast to the Hermitian transition. The presence of substantial multi-particle entanglement can be seen in the Wigner function, which exhibits fringes of negative value [Fig. 1(b)]. Thus, non-Hermiticity amplifies the entanglement at the phase transition.

We further show that the entanglement is useful for quantum metrology: the non-Hermitian model generates spin squeezing with phase sensitivity near the Heisenberg limit and exhibits more squeezing than the Hermitian model [20, 21]. Thus, non-Hermitian dynamics may be a resource for quantum metrological applications like magnetometry [22] and atomic clocks [23].

We also discuss experimental implementation with cavity-QED and trapped ions. Although the scheme is probabilistic, one can implement the non-Hermitian model for thousands of atoms with a high probability, because the gap increases linearly with system size.

Model.— The (Hermitian) Lipkin-Meshkov-Glick model is the prototypical model of interacting spins [24]. Here, we consider the non-Hermitian version,

\[ H = \frac{V}{N}(J_x^2 - J_y^2) + \frac{i\gamma}{2} J_z - \frac{i\gamma N}{4}, \]

where \( \vec{J} = \frac{1}{2} \sum_n \vec{\sigma}_n \) are collective spin operators, and \( N \) is the number of spins. We focus on theDicke manifold with maximum angular momentum, so the Hilbert space has dimension \( N + 1 \).

The Hermitian terms of Eq. (1) can be experimentally implemented using atoms in a cavity [25] or trapped ions.
In practice, one would optically pump $\vert \uparrow \rangle$ to its finite lifetime given by linewidth $\gamma$; in the absence of a decay event, the atoms evolve according to Eq. (1). In practice, one would optically pump $\vert \uparrow \rangle$ into an auxiliary state instead of $\vert \downarrow \rangle$, and then measure the population in the auxiliary state to determine whether a decay event occurred.

Consider the eigenvalues and eigenstates of the Hamiltonian [Eq. (1)]. Due to the non-Hermitian terms, all the eigenvalues have negative imaginary parts, as shown in Fig. 2(a). Suppose one evolves a wavefunction using $\exp(-iHt)$. The wavefunction can be written as a superposition of the eigenstates of $H$. Under the non-Hermitian evolution, the weight in each eigenstate decreases over time due to the imaginary parts of the eigenvalues. After a sufficient amount of time, the wavefunction consists mostly of the eigenstate whose eigenvalue has the largest imaginary part. We are interested in this surviving eigenstate because it is the one that would be observed experimentally. We call this eigenstate the steady state since the system eventually settles into it.

**Sharp transition.** — We are interested in whether the steady state exhibits a phase transition. We define the spectral gap $\Delta$ as the difference in imaginary parts of the two eigenvalues with largest imaginary parts. The gap indicates how quickly the system reaches steady state. If the gap closes ($\Delta \rightarrow 0$), eigenvalues become degenerate, and the corresponding eigenstates change non-analytically. We define $V_c$ as the value of $V$ at which the gap closes. For later usage, we define $V^*$ as the value of $V$ at which the gap is maximum.

As seen in Fig. 2, the gap closes already for finite $N$, leading to non-analytic behavior of $\langle \sigma_z \rangle$ at $V_c$. Non-Hermitian models are unique in their ability to have singularities for finite $N$, known as “exceptional points” [10–12]. However, Fig. 3(a) shows that $V_c$ increases linearly with $N$, which implies that a singularity does not occur in an infinite system. Thus, the non-Hermitian steady state has sharp transitions for finite $N$ but not for infinite $N$; in contrast, Hermitian models have sharp transitions for infinite $N$ but not for finite $N$.

Figure 2(a) shows that there is actually a sequence of degeneracies as $V$ increases, and the degeneracy of the steady state is the last one to occur. The degeneracies can be understood by noting that the mapping $J_x, J_y, J_z \rightarrow J_y, J_x, -J_z$ leads to

$$H + \frac{i\gamma N}{4} \rightarrow -\left(H + \frac{i\gamma N}{4}\right).$$

This implies that the eigenvalues of $H$ are symmetric around $-i\gamma N/4$ and degenerate in pairs. The Supplemental Material discusses this further.

Given the collective nature of the model, it is natural to use a mean-field approach [22–25]. Mean-field theory predicts that a degeneracy occurs at $V = \gamma/2$; this is actually where the first degeneracy occurs and is unrelated to the steady state. Thus, the transition of the steady state (for finite $N$) is not predicted by mean-field theory.

**Entanglement.** — Having established that there is a sharp transition, we now characterize its entanglement [17]. To quantify two-particle entanglement, we use rescaled concurrence $C_R = (N-1)C$, where $C$ is the concurrence; if $C_R > 0$, there is two-particle entanglement [15]. To quantify multi-particle entanglement, we use the averaged quantum Fisher information (QFI) [6, 7],

$$\bar{F} = \frac{4}{3}[(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2].$$

The magnitude of $\bar{F}$ gives an indication of how much multi-particle entanglement there is; if $\bar{F}/N^2$ is on the order of 1, there is macroscopic multi-particle entanglement. The entanglement properties of the Hermitian Lipkin-Meshkov-Glick model are well-known; the rescaled concurrence peaks at the phase transition [8].
while the QFI becomes macroscopic after the phase transition \cite{10,11}.

Figure \(\text{(a)}\) shows the entanglement for the non-Hermitian model for \(N = 20\). (Other \(N\) behave similarly.) The rescaled concurrence peaks before the transition, while the QFI reaches a plateau at the transition. In fact, the QFI takes the maximum possible value, \(F = (N^2 + 2N)/3\), when \(V \geq V_c\), meaning that the steady state is fully \(N\)-particle entangled \cite{6,7}. Thus, the non-Hermitian transition is associated with multi-particle entanglement, in contrast to the two-particle entanglement of the Hermitian transition.

To understand this behavior, we recall that if a pure state has \(\langle \vec{J} \rangle = 0\), it is \(N\)-particle entangled \cite{7}. Since \(H\) has spin-flip symmetry \((J_x, J_y \rightarrow -J_x, -J_y), \langle J_z \rangle = \langle J_y \rangle = 0\). Also, due to Eq. (2), when the steady state degenerates, \(\langle J_z \rangle = 0\). Hence, at the transition, the steady state is \(N\)-particle entangled. (The other eigenstates are also \(N\)-particle entangled when they degenerate.)

The presence of substantial multi-particle entanglement is reflected in the Wigner function \cite{49}, which exhibits interference fringes with negative values [Fig. \(\text{(a)}\)]. Thus, the steady state is a highly non-classical state \cite{60}. The entanglement here is not of GHZ-type, which is why we use the averaged QFI instead of the non-averaged QFI \cite{6,7,61}. In the Supplemental Material, we show that the entanglement is of Dicke-type by comparing the steady state with a rotated Dicke state.

Figure \(\text{(a)}\) shows that even when \(V < V_c\), the QFI remains large, meaning that there is still a lot of multi-particle entanglement. For example, when \(V = V^*\), there is still 13-particle entanglement.

Spin squeezing.— Now we show that the entanglement is useful for quantum metrology by calculating the spin squeezing of the steady state. When an ensemble of atoms is spin-squeezed, one can measure rotations on the Bloch sphere better than the shot-noise limit, which is important for precision measurements. We use the spin-squeezing parameter as defined by Wineland et al. \cite{30},

\[
\xi^2 = \min_{\vec{n}_z} \frac{N(\Delta \langle \hat{n}_z \rangle)^2}{|\langle \vec{J} \rangle|^2}, \tag{4}
\]

where \(\vec{n}_z\) is a unit vector normal to \(\langle \vec{J} \rangle\). There is squeezing when \(\xi^2 < 1\); the smaller \(\xi^2\) is, the better the phase sensitivity.

Figure \(\text{(a)}\) shows that \(\xi^2\) reaches a minimum at \(V^*\), which is also where the gap is maximum [Fig. \(\text{(b)}\)]. Figure \(\text{(b)}\) shows the squeezing for different \(N\) and indicates \(\xi^2 \approx 3/N\), meaning that the phase sensitivity is near the Heisenberg limit \(\xi^2 = 1/N\).

For comparison, squeezing of the Hermitian ground state scales as \(\xi^2 \sim N^{-1/3}\) \cite{9}. Time evolution with the Hermitian Hamiltonian (two-axis countewristing model \cite{31}) leads to squeezing with \(\xi^2 \approx 4/N\). Thus, the non-Hermitian model has more squeezing than the Hermitian model.

We note that there are other measurement-based spin-squeezing protocols, starting from Kuzmich et al. \cite{52, 53}. Our scheme uses a different type of measurement (absence of a decay event), which leads to the explicit non-Hermitian Hamiltonian in Eq. (1). This non-Hermitian scheme may be advantageous in situations where the decay of \(|\uparrow\rangle\) is nonnegligible. Also, since the scheme is based on a steady state, it is robust to initial conditions.

Probabilities.— The non-Hermitian scheme is a probabilistic one, since it is conditioned on the absence of a decay event among \(N\) atoms. An important question is how scalable the scheme is: for large \(N\), what is the probability that an experimental trial reaches steady state before a decay event? One expects that as \(N\) increases, the probability should decrease exponentially. This turns out to be wrong due to two fortunate coincidences.

The time scale to reach steady state is on the order of \(1/\Delta\). The average number of decay events during this time is \cite{38}

\[
\mu = \gamma N(\langle \sigma_z \rangle + 1)/2\Delta. \tag{5}
\]

The probability of no decay event is \(e^{-\mu}\).

It is advantageous to set \(V = V^*\), since \(\Delta\) is maximum and \(\xi^2\) is minimum there. Now, it turns out that \(\Delta(\nu^*)\) increases linearly with \(N\), as seen in Fig. \(\text{(b)}\). To estimate \(\langle \sigma_z \rangle\), we use its steady-state value, which is independent of \(N\) when \(V = V^*\), as seen in Fig. \(\text{(d)}\). Thus, this rough estimate says that the probability of success is independent of \(N\).

For a more accurate estimate, Fig. \(\text{(e)}\) shows the non-Hermitian evolution of \(N = 1000\) spins starting with all spins in \(|\downarrow\rangle\). As time increases, \(\xi^2\) decreases towards the steady-state value, and the probability of no decay event decreases. The squeezing reaches steady state at a time of about 0.025/\(\gamma\), which corresponds to a probability of 0.4. This clearly shows that the non-Hermitian scheme is feasible for a large number of spins. This is due to two fortunate coincidences: \(\xi^2\) is minimum when \(\Delta\) is maximum, and \(\Delta\) increases linearly with \(N\).

Bosonic approximation.— The above results were obtained numerically using exact diagonalization. It is
possible to obtain many of the results analytically using the Holstein-Primakoff transformation. We expand around $J_z = -N/2$ by mapping $J_x \rightarrow -N/2 + a^\dagger a$ and $J_y \rightarrow \sqrt{N}a$, where $a, a^\dagger$ are bosonic creation and annihilation operators that satisfy $[a, a^\dagger] = 1$. This mapping is accurate when $a^\dagger a \ll N$. Equation (1) becomes

$$H = \frac{V}{2} (a^\dagger a + a^2) - \frac{i\gamma}{2} a^\dagger a,$$

which can be diagonalized using a complex Bogoliubov transformation[20–28]:

$$H = -\frac{i}{2} \sqrt{4V^2 + \gamma^2} \bar{b}\bar{b} - \frac{i}{4} (\sqrt{4V^2 + \gamma^2} - \gamma),$$

where $\bar{b}, \bar{b}$ are bosonic creation and annihilation operators that satisfy $[\bar{b}, \bar{b}^\dagger] = 1$. It is important to realize that $\bar{b} \neq \bar{b}^\dagger$ because $\bar{b}$ is complex, as seen from Eq. (10). The vacuum state of the $b$-bosons is defined via $b|0\rangle = 0$. We identify $|0\rangle$ as the steady state because its eigenvalue has the largest imaginary part.

The eigenvalues are given by Eq. (7), and it is easy to see that the bosonic model never has a degeneracy. We recall that the original model does have degeneracies because its eigenvalues are symmetric around $-i\gamma N/4$ due to Eq. (2). Equation (7) only predicts the eigenvalues above $-i\gamma N/4$. To get the other eigenvalues, we have to expand around $J_z = N/2$. The symmetry implies that a degeneracy occurs when an eigenvalue reaches $-i\gamma N/4$.

This allows us to predict, for large $N$, $V^* = \Delta(V^*) = \frac{\gamma N}{6}, \langle \sigma_z \rangle (V^*) = -\frac{2}{3},$ (11) $V_c = \frac{\gamma N}{2}, \xi^2(V^*) = \frac{27}{8N}, \bar{F}(V^*) = \frac{8}{N^2}.$ (12) Equations (11) are surprisingly accurate, while Eqs. (12) have the right scaling with $N$ but not the right prefactor. See the Supplemental Material for more details.

Experimental considerations.— The Hermitian terms of Eq. (1) can be implemented using atoms in cavities[34] or trapped ions[35–37]: the spin-spin interaction is mediated by the cavity or collective ion motion. To get the non-Hermitian terms, one would optically pump from $|\uparrow\rangle$ into an auxiliary state so that $|\uparrow\rangle$ has linewidth $\gamma$. By measuring the population in the auxiliary state, one can determine with near perfect efficiency whether a decay event occurred[43, 44]. One would do multiple experimental runs, and the runs without decay events are the ones that simulate the non-Hermitian model. The Supplemental Material provides suitable optical-pumping schemes for $^{87}$Rb and $^{43}$Ca$^+$. To see the sharp transition, one would look for the singularity of $\langle \sigma_z \rangle$ as a function of $V$, as shown in Fig. 2c. When $V < V_c$, there is a unique steady state, and each experimental run should last for a time of at least $1/\Delta$ to ensure that the system reaches steady state. When $V > V_c$, there is not a unique steady state, but all eigenstates have $\langle \sigma_z \rangle = 0$, which can be observed by averaging over time. Note that the relevant parameter is $V/\gamma$, which can be made large by setting $\gamma$ small.

Conclusion.— We have shown that quantum information sheds new light on non-Hermitian many-body systems. Non-Hermitian dynamics can amplify the entanglement and spin squeezing near quantum phase transitions. It would be interesting to consider other types of non-Hermitian models to see how general this is. In particular, it would be interesting to see the effect of non-Hermiticity on topological entanglement entropy[53–56]. Finally, one should study how non-Hermitian terms affect the entanglement scaling in one-dimensional spin chains[57, 61].

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**SUPPLEMENTAL MATERIAL**

**Non-Hermitian degeneracies**

Figure 6(a) plots both the real and imaginary parts of the eigenvalues of $H$ as a function of $V$ for $N = 20$. There is a sequence of $N/2$ degeneracies as $V$ increases. Before a degeneracy of an eigenvalue pair, both eigenvalues are purely imaginary; after the degeneracy, they are complex with imaginary part $-i\gamma N/4$.

Figure 6(b) demonstrates the phenomenon of self-orthogonality. Let the right eigenvectors of $H$ be denoted $u_n$. Since $H$ is non-Hermitian, the eigenvectors are normalized and orthogonal according to the c-product, $u_n^T \cdot u_m = \delta_{mn}$, which is different from the usual scalar product, $u_n^\dagger \cdot u_m = \delta_{mn}$. It turns out that at an exceptional point, the eigenvector becomes self-orthogonal, $u_n^T \cdot u_n = 0$. Numerically, this is seen as a divergence in the scalar product. Indeed, Fig. 6(b) shows that the scalar product of the steady state diverges at the exceptional point. For more information, see Chapter 9 of Ref. [16].

Interestingly, at a non-Hermitian degeneracy, the two eigenvectors become parallel. Also, the survival probability develops a linear dependence on time [62].

**Comparison with rotated Dicke state**

Here, we show that the multi-particle entanglement of the non-Hermitian steady state is of Dicke-type by showing that the steady state is similar to a rotated Dicke state. Let $|\psi\rangle$ be the steady state as a function of $V$. (Strictly speaking, when $V \geq V_c$, there is not a unique steady state but we continuously follow the eigenstate that is the unique steady state for $V < V_c$.) Let $|\psi'\rangle$ be the Dicke state $|j = N/2, m = 0\rangle$ rotated by angle $\pi/2$ around the axis $(\hat{x} + \hat{y})/\sqrt{2}$:

$$|\psi'\rangle = \exp\left(-i\pi(J_x + J_y)\right)|j = N/2, m = 0\rangle. \quad (13)$$

Figure 7(a) shows the overlap $|\langle \psi' | \psi \rangle|^2$ as a function of $V$ for $N = 20$. The overlap is maximum (0.97) at $V_c$. Figure 7(b) shows the population in each $m$ component for $|\psi\rangle$ at $V_c$ and $|\psi'\rangle$. They are clearly similar. Thus, at the transition, the steady state is similar to (but not exactly) a rotated Dicke state.

Note that $|\psi'\rangle$ has maximum averaged quantum Fisher information, $\bar{F} = (N^2 + 2N)/3$, so it is $N$-particle entangled [6, 7].
Here, we provide details on calculating expectation values in the bosonic model. It is more convenient to express $a, a^\dagger$ in terms of $b, b^\dagger$ instead of $\bar{b}, \bar{b}^\dagger$:

$$a^\dagger = \frac{b^\dagger \cosh \frac{\theta}{2} + b (\sinh \frac{\theta}{2})^*}{\sqrt{\cosh^2 \frac{\theta}{2} - \sinh^2 \frac{\theta}{2}}}, \quad a = \frac{b^\dagger \sinh \frac{\theta}{2} + b (\cosh \frac{\theta}{2})^*}{\sqrt{\cosh^2 \frac{\theta}{2} - \sinh^2 \frac{\theta}{2}}}.$$  \hspace{1cm} (14)

We also express $b^\dagger$ in terms of $b, \bar{b}$:

$$b^\dagger = \left(\sqrt{\cosh^2 \frac{\theta}{2} - \sinh^2 \frac{\theta}{2}}\right) \bar{b} + \left(\sinh \frac{\theta}{2}\right)^* \sinh \frac{\theta}{2} - \cosh \frac{\theta}{2} \left(\sinh \frac{\theta}{2}\right)^* \bar{b},$$  \hspace{1cm} (15)

whereby we find

$$\langle 0|bb^\dagger|0\rangle = \left|\cosh \frac{\theta}{2}\right|^2 - \left|\sinh \frac{\theta}{2}\right|^2.$$  \hspace{1cm} (16)

We take expectation values with respect to the vacuum of the $b$-bosons, since it is the steady state. First,

$$\langle a^\dagger a \rangle = \frac{\cosh \frac{\theta}{2} (\sinh \frac{\theta}{2})^*}{\left|\cosh \frac{\theta}{2} - \sinh \frac{\theta}{2}\right|^2} \langle 0|bb^\dagger|0\rangle = \frac{\cosh \frac{\theta}{2} (\sinh \frac{\theta}{2})^*}{\left|\cosh \frac{\theta}{2} - \sinh \frac{\theta}{2}\right|^2}.$$  \hspace{1cm} (17)

$$\langle a^2 \rangle = \frac{\sinh \frac{\theta}{2} (\cosh \frac{\theta}{2})^*}{\left|\cosh \frac{\theta}{2} - \sinh \frac{\theta}{2}\right|^2} \langle 0|bb^\dagger|0\rangle = \frac{\sinh \frac{\theta}{2} (\cosh \frac{\theta}{2})^*}{\left|\cosh \frac{\theta}{2} - \sinh \frac{\theta}{2}\right|^2};$$  \hspace{1cm} (18)

$$\langle a^\dagger a \rangle = \frac{|\sinh \frac{\theta}{2}|^2}{\left|\cosh \frac{\theta}{2} - \sinh \frac{\theta}{2}\right|^2} \langle 0|bb^\dagger|0\rangle = \frac{|\sinh \frac{\theta}{2}|^2}{\left|\cosh \frac{\theta}{2} - \sinh \frac{\theta}{2}\right|^2}.$$  \hspace{1cm} (19)

We use hyperbolic identities to obtain

$$\sinh \frac{\theta}{2} = \sqrt{\frac{1}{2} (\cosh \theta - 1)} = \sqrt{\frac{1}{2} \left(\frac{\gamma}{\sqrt{4V^2 + \gamma^2}} - 1\right)} = -i \sqrt{\frac{1}{2} \left(1 - \frac{\gamma}{\sqrt{4V^2 + \gamma^2}}\right)},$$  \hspace{1cm} (20)

$$\cosh \frac{\theta}{2} = \sqrt{\frac{1}{2} (\cosh \theta + 1)} = \sqrt{\frac{1}{2} \left(\frac{\gamma}{\sqrt{4V^2 + \gamma^2}} + 1\right)}.$$  \hspace{1cm} (21)
Since \( \text{Im tanh } \theta < 0 \), we have to let \( \sinh \frac{\theta}{2} \) be negative in the last step of Eq. (20). This gives

\[
\langle 0|b b^\dagger|0\rangle = \left| \cosh \frac{\theta}{2} \right|^2 - \left| \sinh \frac{\theta}{2} \right|^2 = \cosh \theta = \sqrt{4V^2 + \gamma^2},
\]

(22)

\[
\langle a^2 \rangle = + \frac{i}{2} \tan \theta | = \frac{iV}{\gamma},
\]

(23)

\[
\langle a^2 \rangle = - \frac{i}{2} \tan \theta | = - \frac{iV}{\gamma},
\]

(24)

\[
\langle a^\dagger a \rangle = \frac{1 - \cosh \theta}{2 \cosh \theta} = \frac{1}{2} \left( \sqrt{4V^2 + \gamma^2} - 1 \right).
\]

(25)

From these results, we find:

\[
\langle \sigma_z \rangle = -1 + \frac{-1 + \sqrt{4(V/\gamma)^2 + 1}}{N},
\]

(26)

\[
\xi^2 = \frac{N^2 \left[ -2V + \sqrt{4(V/\gamma)^2 + 1} \right]}{\left[ N + 1 - \sqrt{4(V/\gamma)^2 + 1} \right]^2},
\]

(27)

\[
F = \frac{2}{3} \left[ N \sqrt{4(V/\gamma)^2 + 1} + 1 + (V/\gamma)^2 \right].
\]

(28)

Experimental level schemes

As discussed in Ref. [43], the optical pumping should be such that \( |\uparrow\rangle \) decays mostly into an auxiliary state \( |a\rangle \) instead of \( |\downarrow\rangle \). It is advantageous to use atoms with hyperfine structure since they have many ground states. Figure 8 shows suitable level schemes for \(^{43}\text{Ca}^+\) and \(^{87}\text{Rb}^+\).

![Experimental level schemes](image)

FIG. 8. Optical-pumping schemes for (a) \(^{43}\text{Ca}^+\) and (b) \(^{87}\text{Rb}^+\).