Benchmarking non-simulable quantum processes via symmetry conservation

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As quantum devices scale up, many-body quantum gates and algorithms begin to surpass what is possible to simulate classically. Validation methods which rely on such classical simulation, such as process tomography and randomized benchmarking, cannot efficiently check correctness of most of the processes involved. In particular non-Clifford gates are a requirement for not only universal quantum computation but for any algorithm or quantum simulation that yields fundamental speedup in comparison with its classical counterpart. We show that it is in fact still possible to validate such non-simulable processes by taking advantage of expected or engineered conservations laws in the system, combined with a unitary one-design strategy to randomize errors over the computational Hilbert space. We show that in the context of (fault-tolerant) quantum error correction, we can construct a one-design using the logically encoded Clifford group over the engineered error-free stabilizer subspace to obtain average error for arbitrary logically-encoded gates and algorithms. In the case of benchmarking simulation of physical systems, these can have various exotic symmetries over which one-design strategies can nonetheless be constructed. We give examples for fermionic systems which conserve particle number, as well as for the Fermi-Hubbard model. The symmetry benchmarking method preserves robustness to state preparation and measurement imperfections.

A requirement for the successful development of quantum technologies, in addition to designing performance enhancing protocols, and building hardware on which they can run, is being able to ascertain their high-fidelity operation.

This subfield of quantum information broadly groups together the verification, benchmarking, or characterization of underlying black-box physical processes. Which metric can be evaluated typically depends on its complexity. For example, verification of cryptographic black-box security can often be demonstrated via a Bell test [1, 2], while potential solutions to NP problems can be verified classically in polynomial time [3]. On the other hand, full characterization of an unknown process through process tomography scales exponentially with system size and thus is tractable only for small dimension or sparse Liouvlillians [4–7]. A third branch of validation methods has been developed known as Randomized Benchmarking (RB) [8, 9], which scales polynomially with system size and amplifies deviations from the ideal process with respect to other sources of error such as preparation and measurement.

However, complex dynamical protocols such as digital quantum computation [10], adiabatic quantum computation [11], and quantum simulation [12–22] are typically useful because classical emulation of the same tasks can require significantly more time. Yet most processes that can be validated to date involve only classically simulable ideal outcomes. As such, earlier proposals to expand the purview of RB, to include benchmarking individual operations [23], to remove assumptions about leakage and gate-dependent errors [24], and to test certain non-Clifford gates using a different basis [25–27], are nonetheless restricted to processes that have efficient equivalent classical circuits. The benchmarking of arbitrary evolution on the other hand, has shown to result in exponential scaling [28, 29].

In this work, we present a method to efficiently verify symmetry conservation laws in sequences of arbitrary quantum operations. We achieve this by drawing a distinction between randomizing input states to the unknown noise afflicting a finite group of operations (which allows diffusion of errors into a single average error channel), and the random output state of a sequence of quantum operations due to the inherent complexity of a large random Hilbert space. The latter can nonetheless exhibit structure in the form of either inherent stabilized subgroups (due to the form of the dynamics) or engineered redundancies (due to large Hamming distances in encoded subspaces). These conservation laws can of course vary between applications, but typically be found anywhere from the algorithmic level down to the hardware implementation level. Mathematically, to this end, we generalize the benchmarking requirements to eliminate the use of a so-called ‘2-design’, for which the randomization over the finite group must be classically tracked and subsequently inverted (hence the process is ‘squared’). Instead, by constructing a ‘1-design’ strategy, we randomize not only the process itself, but also the desired final outcome within given boundaries. We derive a suitable generalized metric to assess error propagation outside of any problem-specific conserved subspace.

We conclude with some examples of typical conserved computational spaces. As the simplest application of the algorithm, at the lowest level of hardware architecture, we often see computational spaces whose population is preserved (e.g. avoiding auxiliary or leakage subspaces [24]). More generally, local quantum gate operations will have some number of invariants, which they conserve by virtue of being typically generated by low rank
operators. Many-body quantum simulations of more exotic physical system also offer many examples of conserved symmetries. In this work we give in particular prescriptions for how to use number conservation and parity conservation to benchmark population degradation. Finally, at the highest algorithmic level, quantum annealing, cryptography, and fault-tolerant computation all use enforced redundant information to improve system performance. We detail how to use stabilizer-code conservation to benchmark average error propagation in such encoded systems.

**Symmetry Benchmarking Protocol** – We wish to assess a given (set of) error channel(s) that take us out of a restricted subspace \( \mathcal{H}_0 \). The ideal dynamics of the system preserve the eigenstates of a conserved operator \( \hat{C} \), i.e. a stabilizer of the system. Let \( \lambda_i \) be the degenerate eigenvalues of \( \hat{C} \). To conserve the symmetry, all operations in the algorithm (gates) must be block–diagonal in the \( \hat{C} \)--eigensystem, with the blocks corresponding to the eigenspaces.

The approach of the proposed protocol is to find an error channel \( \Lambda \) as a unitary one–design on any of the eigenspaces of \( \Lambda \) over \( \mathcal{H} \). We make use of the following definition: the product ensures the correct succession of the quantum matrices (superoperators) and the inverse order of \( H \).

Here \( \text{Tr} \) is a completely positive, trace preserving map, the entries on its matrix representation are real and positive and hence the absolute values of the \( \lambda_i \) are smaller than or equal to one due to the Perron–Frobenius theorem [31–33]. This implies that the population decay can be fitted with just a few exponential decays despite the maximum number of different eigenvalues scaling as \( d^2 \equiv 2^{2n} \) [24]. Finally, we can extract the averaged error \( \mu \) per time step as

\[
\mu = 1 - \int_{SU(d_0)} \text{Tr}_{\gamma_0} \left[ \Lambda \left( \hat{U} \rho_0 \hat{U}^\dagger \right) \right] \, d\hat{U}
\]

\[
= 1 - \frac{1}{|\gamma\rangle} \sum_{D \in \gamma} \text{Tr}_{\gamma_0} \left[ (\Lambda D) \left( \rho_0 \right) \right] = 1 - \Gamma_1,
\]

namely, the symmetry breaking of \( \hat{C} \). The protocol inherits robustness against state preperation and measurement (SPAM) errors, similarly to Clifford benchmarking protocols [8, 9, 24]. This stabilizer leakage quantifies the error accumulation for any error channel that causes decays out of it. When all error channels are predominantly manifest via decay out of the conserved subspace (i.e. the Hamming distance of the stabilized symmetry is large), this gives a metric for the cumulative average Haar-measure error. We will give examples for both cases.

**Benchmarking arbitrary operations** – The error randomization over the one-design allows us to also benchmark operations outside of the set \( \mathcal{D} \). Thus, we introduce a second set of operations that we want benchmark with respect to the error channel, which we call \( \mathcal{I} \), containing one, several or all possible gates of the algorithm. Inspired by Interleaved Randomized Benchmarking (IRB) [23] we interleave the random \( \mathcal{D} \)-sequence with random elements of \( \mathcal{I} \) to assess a combined stabilizer decay \( \mu_{\mathcal{ID}} \). The symmetry preservation for that combined sequence of length \( 2y \) gives

\[
\Gamma_y = \frac{1}{y \sharp \mathcal{D}} \sum_{\{D_j\} \in \mathcal{D}^y} \text{Tr}_{\gamma_0} \left[ \prod_{j=y}^1 (\Lambda D_j) \left( \rho_0 \right) \right].
\]

Here \( \text{Tr}_{\gamma_0} [\cdot] \) denotes the trace over the preserved subspace \( \mathcal{H}_{\gamma_0} \), the unhatted gates describe the effect on density matrices (superoperators) and the inverse order of the product ensures the correct succession of the quantum gates. We make use of the following definition: the half twirl of \( A \) over \( \mathcal{D} \) is \( \Lambda_{\text{twirl}} = \frac{1}{\sharp \mathcal{D}} \sum_{D \in \mathcal{D}} AD \), in contrast to the usual twirl \( \Lambda_{\text{twirl}} = \frac{1}{\sharp \mathcal{C}} \sum_{C \in \mathcal{C}} CAC^{-1} \) over a group \( \mathcal{C} \) [30]. Similarly to the arguments in [24], \( \Lambda_{\text{ht}} \) to the power of \( y \) is acted on by a linear functional, hence it can be simplified to

\[
\Gamma_y = \text{Tr}_{\gamma_0} \left[ \Lambda_{\text{ht}}^y \left( \rho_0 \right) \right] = \sum_i \alpha_i \lambda_i^y
\]

As \( \Lambda_{\text{ht}} \) is a completely positive, trace preserving map, the entries on its matrix representation are real and positive and hence the absolute values of the \( \lambda_i \) are smaller than or equal to one due to the Perron–Frobenius theorem [31–33]. This implies that the population decay can be fitted with just a few exponential decays despite the maximum number of different eigenvalues scaling as \( d^2 \equiv 2^{2n} \) [24]. Finally, we can extract the averaged error \( \mu \) per time step as

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Quantum chemistry: number conservation — A prominent symmetry in quantum simulation of physical systems, as well as many gate architectures for quantum computing (e.g., iSWAP interactions) is the conservation of excitation or particle number. Most commonly, this symmetry arises when mapping from the para-fermionic to the fermionic basis [34–37]. The electron number operator \( \hat{n} \equiv \sum_i \hat{\sigma}_i^z \) divides the Hilbert space into \( n+1 \) eigenspaces \( \mathcal{H}_n \) with \( 0 \leq \gamma \leq n \) excited qubits and dimension \( \gamma \) itself.

To properly define the conditions for \( \mathcal{D} \) being a one-design we have to review and define a basis for the Hilbert space \( \mathcal{R}_{\gamma_0} \) of density operators of states in \( \mathcal{H}_{\gamma_0} \). \( \mathcal{R}_{\gamma_0} \) can be seen as the union of \( \{ \vert i\rangle \langle i \vert \}_{i \in \mathcal{H}_{\gamma_0}} \), \( \{ \vert i\rangle \langle j \vert + \vert j\rangle \langle i \vert \}_{i,j \in \mathcal{H}_{\gamma_0}, i < j} \) and \( \{ -i \vert i\rangle \langle j \vert + i \vert j\rangle \langle i \vert \}_{i,j \in \mathcal{H}_{\gamma_0}, i < j} \) which we denote \( \{ B_i \} \), \( \{ X_{ij} \} \) and \( \{ Y_{ij} \} \) respectively. Let \( \mathcal{D} \) be the one-design acting on \( \mathcal{R}_{\gamma_0} \), then \( \sum_{D \in \mathcal{D}} D \) maps any density matrix onto the completely mixed state. The action of the one-design in this basis can then be simply rewritten as

\[
\frac{1}{\vert \mathcal{D} \vert} \sum_{D \in \mathcal{D}} D \vert B_i \rangle = \frac{1}{d_0} \sum_k \vert B_k \rangle
\]

(§1)

\[
\sum_{D \in \mathcal{D}} D \vert X_{ij} \rangle = 0
\]

(§2)

\[
\sum_{D \in \mathcal{D}} D \vert Y_{ij} \rangle = 0,
\]

(§3)

with \( \dim(\mathcal{H}_{\gamma_0}) = d_0 \). Because of the linearity of Eq. 1, these are the only nontrivial conditions needed for constructing a unitary one-design. Focusing on (§1), we want to ensure that by sampling over \( \mathcal{D} \) the transition between each two basis states is realized with equal probability. We implement this using arbitrary qubit permutations to randomly redistribute the excited qubits’ sites. We populate each basis state with equal probability. Averaging over the one-design, this yields the completely mixed state of \( \mathcal{H}_{\gamma_0} \) regardless of the initial state, thus satisfying (§1). Note that although qubit permutations implement all transitions between two states it is not equivalent to state permutation. This is in fact crucial for the scalability of this solution and shows importance in the following section.

To examine the remaining conditions (§2) and (§3), we examine the effect of the qubit permutations via iSWAPs on the \( X_{ij} \) and \( Y_{ij} \) matrix elements, where each element again is mapped onto a density operator \( X_{ij} \), or \( Y_{ij} \) on the space \( \mathcal{R}_{\gamma_0} \) corresponding to the preserved number of excited qubits. Introducing a uniformly random \( \pm 1 \) phase between every two states ensures that those occur with opposite signs equally likely, hence sum up to zero, satisfying (§2) and (§3). This random phase is not inherently given by the phases included in the iSWAPs but easily achieved by a probability \( 1/2 \) \( \delta_z \) gate on every qubit. This matches our intuition that \( X_{ij} \) and \( Y_{ij} \) represent coherent phases between states so that randomizing all phases should eliminate them. The above protocol using the derived unitary one-design is simulated in Fig. 1 on a five qubit system which is initialized in a state which has three excited qubits. Every permutation of qubits consists of iSWAPs which, in this example, contain pre-defined errors on the pair of qubits. The error channel is derived as a unitary operator close to the identity acting on a four-qubit Hilbert space, then tracing out two qubits. The \( \gamma = 3 \) subspace is benchmarked via a fit of the population decay to have an average population leakage of \( \mu = 88\% \) and \( \Gamma_1 = 99.12 \).

The dynamics of the random sequence can be viewed even more simply. Since applying \( \sum_{D \in \mathcal{D}} D \) twice is equivalent to a single application, reviewing the average symmetry preservation of equation (2) gives

\[
\Gamma_y = \frac{1}{\vert \mathcal{D} \vert^2 2^y} \sum_{\{C_j\} \in \mathcal{D}^y} \sum_{\{D_j\} \in \mathcal{D}^y} \text{Tr}_{\gamma_0} \left[ \left( \prod_{j=y}^{\gamma} (C_j A D_j) \right) \left( \hat{\rho}_0 \right) \right] 
\]

(8)

The updated \( \Lambda'_n = \frac{1}{\mathcal{D}^y} \sum_{C,D} C A D \) commutes with any unitary evolution within subspaces \( \mathcal{H}_\gamma \), and can therefore be reduced to simple transition rates between those subspaces. This not only provides a more easily approachable concept but gives the intuition for the Ansätze in the following sections.

Fermi–Hubbard model and parity conservation — A symmetry that is often encountered in quantum technologies, such as for measurement-based entanglement generation and for error correction is parity-preserving operations. In the context of quantum simulation, it appears in the Fermi–Hubbard model, used to study strongly correlated electrons in condensed matter physics including basic atomic structure and second quantization [38]. The computationally most costly parts of its simulations can in principle be resolved by medium sized quantum computers [39]. While some of the Hamiltonians employed in that scheme are number conserving and can be treated using the set \( \mathcal{D} \) derived previously, others are not, namely, the terms which induce superconductivity to the model. However, these terms always change the electron number by two (a Cooper pair), preserving parity.
As the Fermi–Hubbard model involves an even number \( n \) of electron sites/qubits, the subspaces \( \mathcal{H}_{\text{even}} \) and \( \mathcal{H}_{\text{odd}} \) are of equal dimension \( 2^{n-1} \); it is in principle possible to map the \( n - 1 \) qubit Clifford group onto those subspaces, but such a protocol would map single qubit gates into multiqubit ones and visa versa, yielding a potentially exponential increase in gate complexity. Instead, we refrain from finding a new unitary one-designs for \( \mathcal{H}_{\text{even}} \) and \( \mathcal{H}_{\text{odd}} \) but rely on the transition rates derived previously. The symmetry preservation on the even subspace for only one individual gate \( I \) is

\[
\Gamma_1 = \frac{1}{2^{\mathcal{D}}} \sum_{D \in \mathcal{D}_{\text{even}}} \text{Tr}_{\text{even}}[(I \Lambda_I \Lambda_D)(\hat{\rho}_0)]
\]

\[
= \frac{1}{2^{n-1}} \text{Tr}_{\text{even}}[(I \Lambda_I \Lambda_D)(\hat{1}_{\text{even}})]
\]

as a unitary one-design on the even subspace would map to the identity thereon. Writing it as sum of the identities of different subspaces gives

\[
= \sum_{\gamma_{\text{even}}} \frac{d_{\gamma}}{2^{n-1}} \sum_{D \in \mathcal{D}} \text{Tr}_{\text{even}}[(I \Lambda_I \Lambda_D)(\hat{\rho}_\gamma)]
\]

\[
\equiv \sum_{\gamma_{\text{even}}} \frac{d_{\gamma}}{2^{n-1}} \Gamma_1^\gamma,
\]

where \( \hat{\rho}_\gamma \) is an initial state in the respective subspace. Each of those \( \Gamma_1^\gamma \) can be derived by sequences of the usual form; an estimation for \( \mu_Z \) can be obtained for each subspace via interleaved symmetry benchmarking and subsequently an overall estimation can be found. As there are only \( n/2 \) or \( n/2 + 1 \) different subspaces, this scales linearly in \( n \) and is therefore efficiently scalable in the number of qubits. The protocol also translates easily to a set \( \mathcal{I} \) of gates allowing for an efficient symmetry benchmarking; Figure 2 shows the data of a simulation for six qubits where we benchmark the symmetry preservation on the even subspace for a interleaved gate \( I = \sigma_2 \sigma_3 + \sigma_2^\dagger \sigma_3^\dagger \) with an exact symmetry breaking of \( \mu_1 = .30\% \). Extracting \( \mu_{\text{ISB}} = .86\% \) and \( \mu_D = .51\% \) provides an estimate of \( \mu_1 = .35\% / \Gamma_1 = 0.9965 \) which is remarkably close to the actual values as \( \Lambda_I \) is not the dominating error term. This better-than-expected performance is similar to observations from Interleaved Randomized Benchmarking [23].

Benchmarking logically encoded processes – This protocol allows benchmarking not only inherent symmetries in (simulated) physical systems, but engineered symmetries for which the vast majority (asymptotically speaking) of error leaks through particular ‘syndrome’ states, that is, avoiding direct transitions between logical stabilizer eigenstates. In this case, we can expect to not only quantify important sources of error, but all sources of error present. There are many examples of engineered redundancy for the purposes of error suppression or monitoring, most notably in (fault-tolerant) quantum error correcting codes (ECC), but also in quantum cryptography, simulation, and adiabatic quantum computation.

Here we are interested in showing that we can obtain error metrics that are amplified and immune to SPAM, as in standard RB, but also that it allows us to do so for entire algorithms and arbitrary logical gates, such as non-Clifford gates which are fundamental to the logically-encoded quantum computation.

We consider here a Hilbert space encoded in the form \( \mathcal{H} = \{0,1\}^n \otimes \{0,1\}^m \) where the latter space is used for syndrome measurements. It can be written as \( \mathcal{H} = \mathcal{H}_{\text{comp}} \oplus \mathcal{H}_{\text{err}} \), with \( \mathcal{H}_{\text{comp}} \equiv \{0,1\}^n \otimes \{|\Psi_+\rangle\} \). A logical operation \( L_j \) at time step \( j \) in the ECC consists of a subsequence \( G_j \) of (faulty) local gates, followed by syndrome measurement \( M_j \) in \( \mathcal{H}_{\text{err}} \) and correction feedback \( F_j \) towards \( \mathcal{H}_{\text{comp}} \). As before, to do randomized benchmarking of one component we require that no phase relationship is built up with the second, for which we use a generalized operation \( R_j \). The one-design sequence can then be written as

\[
\Gamma_y = \frac{1}{2^{|\mathcal{C}_y|}} \sum_{(c_j) \in \mathcal{C}_y} \text{Tr}_{\text{comp}}\left[\left(\prod_{j=y}^{1}(\Lambda_j R_j C_j)\right) \hat{\rho}_0\right]
\]

where \( C_j \) are logically-encoded Clifford gates prior to error correction. As with the general protocol and the examples discussed above, we again ensure we take the correct group average that reduces our error channel to a depolarizing one; in the present case we do so using the half-twirl using the Clifford group amended with phase randomization between the \( \mathcal{H}_{\text{err}} \) and \( \mathcal{H}_{\text{comp}} \) subspaces. There are actually different ways to do this. As with the number conservation, we can do it with single-qubit phase gates on the physical qubits, i.e. \( R_j = Z_j^m \). Another way to randomize the phase is with syndrome measurements, i.e. \( R_j = M_j \). Yet another (more subtle) way is to apply both a syndrome measurement (throwing away the result) and a randomly chosen correction operation \( F_j \) from amongst the possible correction operations \( \{F_j\} \), so that \( R_j = F_j M_j \). At the end, one obtains an average error per gate estimate of the compound operation \( RC \), i.e. \( \mu_{RC} \). Since we can obtain the single-qubit error
rates ($\mu_2$) from standard randomized benchmarking of the logical Clifford operations [40], we can simply derive estimates for the other components as $\mu_C \approx \mu_{RC} - \mu_R$, as with standard interleaved benchmarking.

Using the above protocol, we now have a means to obtain the decay rate into the $\mathcal{H}_{err}$ subspace. Note that this does not correspond to a logical error since the vast majority of these events are suppressed by the error correction feedback. Nonetheless, we can use the protocol with interleaved gates to benchmark arbitrary logical operations $G$ with respect to this error channel, obtaining $\mu_G$. If we further make the standard assumption from ECC that correlated errors longer than the distance $d$ of the code are negligible, we can then simply upperbound the logical error rate corresponding to computational faults as $\mu_G^d < (\mu_F + \mu_M + \mu_G)^d$. Note also that the above considerations could be particularly insightful for concatenated error codes, where the logical error rate at one layer corresponds to the physical error rate at the layer above. For concatenated codes as well as for other ECC protocols, benchmarking non-Clifford gates is pivotal to fault-tolerant universal quantum computation.

Conclusion — Both quantum simulation and universal quantum computation involve complex processes that cannot be efficiently predicted classically. We show that this restriction does not prohibit their validation, provided the implementation being benchmarked can be found or engineered to conserve symmetries in the system. Such is the case for many quantum simulation tasks, such as fermionic systems and the Fermi-Hubbard model, as well as for fault-tolerant quantum computation, where stabilizers of error correcting codes are preserved by logical operations. We present a symmetry benchmarking protocol relying on randomization via unitary one-designs on conserved subspaces, that allows extraction of average channel error while maintaining robustness to state preparation and measurement imperfections.

Acknowledgements — We acknowledge funding through SCALEQIT and a Google Faculty Research Award.

[1] J. S. Bell, (1964).
[2] A. Aspect, J. Dalibard, and G. Roger, Physical review letters 49, 1804 (1982).
[3] O. Goldreich, P. NP, and NP-Completeness: The basics of computational complexity (Cambridge University Press, 2010).
[4] D. Gross, Y.-K. Liu, S. T. Flammia, S. Becker, and J. Eisert, Physical review letters 105, 150401 (2010).
[5] M. Cramer, M. B. Plenio, S. T. Flammia, R. Somma, D. Gross, S. D. Bartlett, O. Landon-Cardinal, D. Poulin, and Y.-K. Liu, Nature communications 1, 149 (2010).
[6] D. Gross, IEEE Transactions on Information Theory 57, 1548 (2011).
[7] A. Shabani, R. Kosut, M. Mohseni, H. Rabitz, M. Broome, M. Almeida, A. Fedrizzi, and A. White, Physical review letters 106, 100401 (2011).
[8] E. Knill, D. Leibfried, R. Reichle, J. Britton, R. Blakestad, J. Jost, C. Langer, R. Ozeri, S. Seidelin, and D. Wineland, Phys. Rev. A 77, 012307 (2008).
[9] E. Magesan, J. Gambetta, and J. Emerson, Phys. Rev. Lett. 106, 180504 (2011).
[10] M. Nielsen and I. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, 2000).
[11] E. Farhi, J. Goldstone, S. Gutmann, J. Lapan, A. Lundgren, and D. Preda, Science 292, 472 (2001).
[12] S. Lloyd, Science 273, 1073 (1996).
[13] A. Aspuru-Guzik, A. D. Dutoi, P. J. Love, and M. Head-Gordon, Science 309, 1704 (2005).
[14] I. Kassal, J. D. Whitfield, A. Perdomo-Ortiz, M.-H. Yung, and A. Aspuru-Guzik, Annual Review of Physical Chemistry 62, 185 (2011).
[15] J. R. McClean, J. Romero, R. Babbush, and A. Aspuru-Guzik, New Journal of Physics 18, 023023 (2016).
[16] R. Barends, L. Lamata, J. Kelly, L. Garcia-Alvarez, A. G. Fowler, A. Megrant, E. Jeffrey, T. C. White, D. Sank, J. Y. Mutus, B. Campbell, Y. Chen, Z. Chen, B. Chiaro, A. Dunsuito, I.-C. Ioi, C. Neill, P. J. J. O’Malley, C. Quintana, P. Roushan, A. Vainsencher, J. Wenner, E. Solano, and J. M. Martinis, Nature Communications 6, 7654 (2015).
[17] D. Wecker, M. B. Hastings, N. Wiebe, B. K. Clark, C. Nayak, and M. Troyer, Phys. Rev. A 92, 062318 (2015).
[18] B. Bauer, D. Wecker, A. J. Millis, M. B. Hastings, and M. Troyer, Physical Review X 6, 10.1103/physrevx.6.031045.
[19] P.-L. Dallaire-Demers and F. K. Wilhelm, Phys. Rev. A 94, 062304 (2016).
[20] Y. Salathe, M. Mondal, M. Oppliger, J. Heinsoo, P. Kurpier, A. Potocnik, A. Mezzacapo, U. L. Heras, L. Lamata, E. Solano, S. Filipp, and A. Wallraff, Physical Rev. X 5, 021027 (2015).
[21] E. A. Martinez, C. A. Muschik, P. Schindler, A. Erhard, M. Heyl, P. Hauke, M. Dalmore, T. Monz, P. Zoller, and R. Blatt, Nature 534, 516 (2016).
[22] E. Zohar, A. Farace, B. Reznik, and J. I. Cirac, Physical Review A 95 (2017), http://dx.doi.org/10.1103/PhysRevA.95.023604.
[23] E. Magesan, J. Gambetta, B. R. Johnson, C. Ryan, J. Chow, S. Merkel, M. da Silva, G. Keefe, M. Rothwell, T. Ohki, M. Ketchen, and M. Steffen, Phys. Rev. Lett. 109, 080505 (2012).
[24] T. Chasseur and F. Wilhelm, Phys. Rev. A 92, 042333 (2015).
[25] J. Wallman, M. Barnhill, and J. Emerson, “Characterization of leakage errors via randomized benchmarking,” (2014), arXiv:1412.4126.
[26] A. Carignan-Dugas, J. Wallman, and J. Emerson, Phys. Rev. A 92, 060302(R) (2015).
[27] A. Cross, E. Magesan, L. Bishop, J. Smolin, and J. Gambetta, npj Quantum Inf. 2, 16012 (2016).
[28] T. Chasseur, D. Reich, F. Wilhelm, and C. Koch, “Hybrid benchmarking of arbitrary quantum gates,” (2016), arXiv:1606.03927.
[29] D. Reich, G. Gualdi, and C. Koch, Phys. Rev. Lett. 111, 030502 (2018).
200401 (2013).
[30] C. Dankert, R. Cleve, J. Emerson, and E. Livine, Phys. Rev. A 80, 012304 (2009).
[31] O. Perron, Math. Ann. 64, 248 (1907).
[32] G. Frobenius, Sitzungsber. Königl. Preuss. Akad. Wiss. 456 (1912).
[33] C. Meyer, Matrix analysis and applied linear algebra (SIAM, 2000) p. 655.
[34] P. Jordan and E. P. Wigner, in The Collected Works of Eugene Paul Wigner (Springer, 1993) pp. 109–129.
[35] J. D. Whitfield, J. Biamonte, and A. Aspuru-Guzik, Molecular Physics 109, 735 (2011).
[36] S. B. Bravyi and A. Y. Kitaev, Annals of Physics 298, 210 (2002).
[37] A. Tranter, S. Sofia, J. Seeley, M. Kaicher, J. McClean, R. Babbush, P. Coveney, F. Mintert, and F. W. P. Love, Int. J. Quant. Chem. 115, 1431 (2015).
[38] P. Lafarge, P. Joyez, D. Esteve, C. Urbina, and M. Devoret, Nature 365, 422 (1993).
[39] P.-L. Dallaire-Demers and F. Wilhelm, Phys. Rev. A 93, 032303 (2016).
[40] J. Combes, C. Granade, C. Ferrie, and S. Flamia, “Logical randomized benchmarking,” (2017), arXiv:1702.03688.