Polynomials Associated with Equilibria of Affine Toda-Sutherland Systems

S. Odake\textsuperscript{a} and R. Sasaki\textsuperscript{b}

\textsuperscript{a} Department of Physics, Shinshu University, Matsumoto 390-8621, Japan
\textsuperscript{b} Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan

Abstract

An affine Toda-Sutherland system is a quasi-exactly solvable multi-particle dynamics based on an affine simple root system. It is a ‘cross’ between two well-known integrable multi-particle dynamics, an affine Toda molecule and a Sutherland system. Polynomials describing the equilibrium positions of affine Toda-Sutherland systems are determined for all affine simple root systems.

1 Introduction

Given a multi-particle dynamical system, to find and describe its equilibrium position has practical as well as theoretical significance. As is well-known, near the equilibrium the system is reduced to a collection of harmonic oscillators and that their spectra give the exact order $\hbar$ part of the full quantum spectra \cite{1}. Naively, one could describe the equilibrium position by zeros of a certain polynomial. In this way one obtains the celebrated classical orthogonal polynomials for exactly solvable multi-particle dynamics. For the Calogero systems \cite{2} based on the $A$ and $B$ ($C$, $BC$ and $D$) root systems, the equilibrium positions correspond to the
zeros of the Hermite and Laguerre polynomials \[3, 4, 5, 6\]. For the Sutherland systems \[7\] based on the \(A\) and \(B\) (\(C, BC\) and \(D\)) root systems, the equilibrium positions correspond to the zeros of the Chebyshev and Jacobi polynomials \[6\]. Polynomials describing the equilibria of the Calogero and Sutherland systems based on the exceptional root systems are also determined \[8\]. In all these cases the frequencies of small oscillations at the equilibrium are “quantised” \[6, 9\]. For another family of multi-particle dynamics based on root systems, the Ruijsenaars-Schneider systems \[10\], which are deformation of the Calogero and Sutherland systems, the corresponding polynomials are determined \[11, 12\]. They turn out to be deformation of the Hermite, Laguerre and Jacobi polynomials which inherit the orthogonality \[12\]. The frequencies of small oscillations at the equilibrium are also “quantised” \[11\]. Another interesting feature is that the equations determining the equilibrium look like Bethe ansatz equations.

One is naturally led to a similar investigation for partially solvable or quasi-exactly solvable \[13\] multi-particle dynamics. From a not-so-long list of known quasi-exactly solvable multi-particle dynamical systems \[14\], we pick up the so-called affine Toda-Sutherland systems \[15\] and determine polynomials describing the equilibrium positions. These polynomials, as well as all the polynomials mentioned above, are characterised as having integer coefficients only.

2 affine Toda-Sutherland systems

The affine Toda-Sutherland systems are quasi-exactly solvable \[13\] multi-particle dynamics based on any crystallographic root system. Roughly speaking, they are obtained by ‘crossing’ two well-known integrable dynamics, the affine-Toda molecule and the Sutherland system. Given a set of affine simple roots \(\Pi_0 = \{\alpha_0, \alpha_1, \ldots, \alpha_r\}, \alpha_j \in \mathbb{R}^r\), let us introduce a prepotential \(W\) \[16\]

\[
W(q) = g \sum_{j=0}^{r} n_j \log |\sin(\alpha_j \cdot q)|, \quad q = t(q_1, \ldots, q_r) \in \mathbb{R}^r,
\]

in which \(g\) is a positive coupling constant and \(\{n_j\}\) are the Dynkin-Kac labels for \(\Pi_0\). That is, they are the integer coefficients of the affine simple root \(\alpha_0\); \(-\alpha_0 = \sum_{j=1}^{r} n_j \alpha_j, n_0 \equiv 1\).

For simply-laced and un-twisted non-simply laced affine root systems \(\alpha_0\) is the lowest long root, whereas for twisted non-simply laced affine root systems, \(\alpha_0\) is the lowest short root.
In either case $h \equiv \sum_{j=0}^{r} n_j$ is the Coxeter number. This leads to the classical Hamiltonian
\[ H_C = \frac{1}{2} \sum_{j=1}^{r} p_j^2 + \frac{1}{2} \sum_{j=1}^{r} \left( \frac{\partial W(q)}{\partial q_j} \right)^2. \] (2)

It is shown [15] that the equilibrium position $\bar{q}$ is given by a universal formula in terms of the dual Weyl vector $\varrho^\vee$:
\[ \frac{\partial W(\bar{q})}{\partial q_j} = 0 \iff \bar{q} = \frac{\pi}{h} \varrho^\vee, \quad \varrho^\vee \equiv \sum_{j=1}^{r} \lambda_j^\vee. \] (3)

The dual fundamental weight $\lambda_j^\vee$ is defined in terms of the fundamental weight $\lambda_j$ by $\lambda_j^\vee \equiv (2/\alpha_j^2) \lambda_j$, which satisfies $\alpha_j \cdot \lambda_k^\vee = \delta_{jk}$. At the equilibrium, the classical multi-particle dynamical system is reduced to a set of harmonic oscillators. The frequencies (not frequencies squared) of small oscillations at the equilibrium of the affine Toda-Sutherland model are given up to the coupling constant $g$ by [15]
\[ \frac{1}{\sin^2 \frac{\pi}{h}} \{m_1^2, m_2^2, \ldots, m_r^2\}, \]
in which $m_j^2$ are the so-called affine Toda masses [17]. Namely, they are the eigenvalues of a symmetric $r \times r$ matrix $M$, $M_{kl} = \sum_{j=0}^{r} n_j (\alpha_j)_k (\alpha_j)_l$, or $M = \sum_{j=0}^{r} n_j \alpha_j \otimes \alpha_j$, which encode the integrability of affine Toda field theory. In [17] it is shown for the non-twisted cases that the vector $\mathbf{m} = ^t(m_1, \ldots, m_r)$, if ordered properly, is the Perron-Frobenius eigenvector of the incidence matrix (the Cartan matrix) of the corresponding root system.

The corresponding quantum Hamiltonian [16] is
\[ H_Q = \frac{1}{2} \sum_{j=1}^{r} p_j^2 + \frac{1}{2} \sum_{j=1}^{r} \left[ \left( \frac{\partial W(q)}{\partial q_j} \right)^2 + \frac{\partial^2 W(q)}{\partial q_j^2} \right], \] (4)
which is partially solvable or quasi-exactly solvable for some affine simple root systems. Namely for $A_{r-1}^{(1)}$, $D_3^{(1)}$, $D_{r+1}^{(2)}$, $C_r^{(1)}$ and $A_{2r}^{(2)}$, the above Hamiltonian is known to have a few exact eigenvalues and corresponding exact eigenfunctions [15].

The polynomials related to the equilibrium position $\bar{q}$ are easy to define for the classical root systems, $A$, $B$, $C$ and $D$. As in the Sutherland cases, we introduce a polynomial having zeros at $\{\sin \bar{q}_j\}$ or $\{\cos 2\bar{q}_j\}$:
\[ P_r(q) \propto \prod_{j=1}^{r} (x - \sin \bar{q}_j), \quad \prod_{j=1}^{r} (x - \cos 2\bar{q}_j). \] (5)
For the exceptional root systems, let us choose a set of $D$ vectors $\mathcal{R}$
\[
\mathcal{R} = \{\mu^{(1)}, \ldots, \mu^{(D)} | \mu^{(a)} \in \mathbb{R}^r\},
\]
which form a single orbit of the corresponding Weyl group. For example, they are the set of roots $\Delta$ itself for simply laced root systems, the set of long (short, middle) roots $\Delta_L$ ($\Delta_S$, $\Delta_M$) for non-simply laced root systems and the so-called sets of \textit{minimal weights}. The latter is better specified by the corresponding fundamental representations, which are all the fundamental representations of $A_r$, the vector ($V$), spinor ($S$) and conjugate spinor ($\bar{S}$) representations of $D_r$ and 27 ($\overline{27}$) of $E_6$ and 56 of $E_7$. By generalising the above examples (5), we define polynomials
\[
P^{\mathcal{R}}_\Delta(x) \propto \prod_{\mu \in \mathcal{R}} (x - \sin(\mu \cdot \bar{q})), \quad \prod_{\mu \in \mathcal{R}} (x - \cos(2\mu \cdot \bar{q})).
\]
(6)

For more general treatment we refer to our previous article [8].

The resulting polynomials for various affine root systems $\Pi_0$ are (we follow the affine Lie algebra notation used in [15, 17]):

$A_{r-1}^{(1)}$ : In this case the equilibrium position is exactly the same as that of the $A_{r-1}$ Sutherland [17] and $A_{r-1}$ Ruijsenaars-Sutherland system [12], $\bar{q} = (\pi/2h)^{(r-1, r-3, \ldots, -(r-1))}$ with $h = r$. Thus the polynomial is also the same, the Chebyshev polynomial of the first kind: $2^{r-1} \prod_{j=1}^{r} (x - \sin \bar{q}_j) = T_r(x) = \cos r\varphi$, $x = \cos \varphi$.

$B_r^{(1)} \& D_{r+1}^{(2)} \& A_{2r}^{(2)}$ : The Coxeter number is $h = 2r$ for $B_r^{(1)}$, $h = r + 1$ for $D_{r+1}^{(2)}$ and $h = 2r + 1$ for $A_{2r}^{(2)}$. The equilibrium position is equally spaced $\bar{q} = (\pi/h)^{(r, r-1, \ldots, 1)}$. We obtain the Chebyshev polynomial of the second kind, $U_n(x) = \sin(n+1)\varphi/\sin \varphi$, $x = \cos \varphi$, for $B_r^{(1)}$ and a product of them for $D_{r+1}^{(2)}$ and a sum of them for $A_{2r}^{(2)}$,

\[
2^{r-1} \prod_{j=1}^{r} (x - \cos 2\bar{q}_j) = \begin{cases} 
(x + 1)U_{r-1}(x), & B_r^{(1)}, \\
(x + 1)U_{r/2}(x)U_{(r-2)/2}(x) + 1/2, & D_{r+1}^{(2)}, \quad r : \text{even}, \\
(x + 1)U_{(r-1)/2}(x), & D_{r+1}^{(2)}, \quad r : \text{odd}, \\
(U_r(x) + U_{r-1}(x))/2, & A_{2r}^{(2)}.
\end{cases}
\]
(7)

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The Coxeter number is $h = 2r$ for $C_r^{(1)}$ and $h = 2r - 1$ for $A_{2r-1}^{(2)}$. The equilibrium position is equally spaced $\bar{q} = (\pi/2h)^i(2r - 1, 2r - 3, \ldots, 3, 1)$. We obtain the Chebyshev polynomial of the first kind $T_r(x)$ for $C_r^{(1)}$ and a sum of them for $A_{2r-1}^{(2)}$.

$$2^{r-1} \prod_{j=1}^{r}(x - \cos 2\bar{q}_j) = \left\{ \begin{array}{ll} T_r(x), & C_r^{(1)}, \\ T_r(x) + T_{r-1}(x), & A_{2r-1}^{(2)} \end{array} \right.$$  \hspace{1cm} (8)

The Coxeter number is $h = 2(r - 1)$ and the equilibrium position is equally spaced $\bar{q} = (\pi/h)^i(r - 1, r - 2, \ldots, 1, 0)$. We obtain the Chebyshev polynomial of the second kind

$$2^{r-2} \prod_{j=1}^{r}(x - \cos 2\bar{q}_j) = (x^2 - 1)U_{r-2}(x).$$  \hspace{1cm} (9)

The Coxeter number is $h = 12$ and the equilibrium position is not equally spaced $\bar{q} = (\pi/h)^i(4\sqrt{3}, 4, 3, 2, 1, 0)$. We consider the set of minimal weights $27$ and the set of positive roots $\Delta_+$, which consists of 36 roots. The polynomials are

$$2^{20} \prod_{\mu \in 27} \left( x - \sin(\mu \cdot \bar{q}) \right) = (-1 + x)x^3(1 + x)(-1 + 2x)^2(1 + 2x)^2(-1 + 2x)^2 \times (-3 + 4x^2)^3(1 - 16x^2 + 16x^4)^2,$$  \hspace{1cm} (10)

$$2^{27} \prod_{\mu \in \Delta_+} \left( x - \cos(2\mu \cdot \bar{q}) \right) = x^6(1 + x)^3(-1 + 2x)^6(1 + 2x)^7(-3 + 4x^2)^7.$$  \hspace{1cm} (11)

The Coxeter number is $h = 18$ and the equilibrium position is not equally spaced $\bar{q} = (\pi/2h)^i(17\sqrt{2}, 10, 8, 6, 4, 2, 0)$. We consider the set of minimal weights $56$ and the set of positive roots $\Delta_+$, which consists of 63 roots. The $56$ is even, i.e if $\mu \in 56$ then $-\mu \in 56$. The positive part of $56$ is denoted as $56_+$. The polynomials are

$$2^{24} \prod_{\mu \in 56_+} \left( x - \cos(2\mu \cdot \bar{q}) \right) = x^4(-3 + 4x^2)^3(-3 + 36x^2 - 96x^4 + 64x^6)^3,$$  \hspace{1cm} (12)

$$2^{59} \prod_{\mu \in \Delta_+} \left( x - \cos(2\mu \cdot \bar{q}) \right) = (1 + x)^4(-1 + 2x)^7(1 + 2x)^7 \times (-1 + 6x + 8x^3)^8(1 - 6x + 8x^3)^7.$$  \hspace{1cm} (13)

The Coxeter number is $h = 30$ and the equilibrium position is not equally spaced $\bar{q} = (\pi/h)^i(23, 6, 5, 4, 3, 2, 1, 0)$. We consider the set of positive roots $\Delta_+$, which consists of
120 roots. The polynomial is
\[
2^{116} \prod_{\mu \in \Delta_+} \left( x - \cos(2 \mu \cdot \bar{q}) \right) = (1 + x)^4 (-1 + 2x)^8 (1 + 2x)^8 (-1 - 2x + 4x^2)^8 (-1 + 2x + 4x^2)^8 \times (1 + 8x - 16x^2 - 8x^3 + 16x^4)^8 (1 - 8x - 16x^2 + 8x^3 + 16x^4)^9. \tag{14}
\]

\(F_4^{(1)} \& E_6^{(2)}\) : The Coxeter number is \(h = 12\) for \(F_4^{(1)}\) and \(h = 9\) for \(E_6^{(2)}\) and the equilibrium position is not equally spaced \(\bar{q} = (\pi/h)^t(8, 3, 2, 1)\). We consider the set of long positive roots \(\Delta_{L+}\) and short positive roots \(\Delta_{S+}\), both of which consist of 12 roots reflecting the self-duality of \(F_4\) Dynkin diagram. The polynomials for \(F_4^{(1)}\) are
\[
2^9 \prod_{\mu \in \Delta_{S+}} \left( x - \cos(2 \mu \cdot \bar{q}) \right) = x^2 (1 + x) (-1 + 2x)^2 (1 + 2x)^3 (-3 + 4x^2)^2, \tag{15}
\]
\[
2^9 \prod_{\mu \in \Delta_{L+}} \left( x - \cos(2 \mu \cdot \bar{q}) \right) = x^2 (1 + x) (-1 + 2x)^2 (1 + 2x) (-3 + 4x^2)^3. \tag{16}
\]

The polynomials associated with the twisted affine root system \(E_6^{(2)}\) are
\[
2^{12} \prod_{\mu \in \Delta_{S+}} \left( x - \cos(2 \mu \cdot \bar{q}) \right) = (1 + 2x)^3 (1 - 6x + 8x^3)^3, \tag{17}
\]
\[
2^{12} \prod_{\mu \in \Delta_{L+}} \left( x - \cos(2 \mu \cdot \bar{q}) \right) = 2 (-1 + x) (1 + 2x)^2 (1 - 6x + 8x^3)^3. \tag{18}
\]

\(G_2^{(1)} \& D_4^{(3)}\) : The Coxeter number is \(h = 6\) for \(G_2^{(1)}\) and \(h = 4\) for \(D_4^{(3)}\) and the equilibrium position is \(\bar{q} = (\pi/2h)^t(3\sqrt{6}, \sqrt{2})\). We consider the set of long positive roots \(\Delta_{L+}\) and short positive roots \(\Delta_{S+}\), both of which consists of 3 roots reflecting the self-duality of \(G_2\) Dynkin diagram. The polynomials for the untwisted \(G_2^{(1)}\) are
\[
2^3 \prod_{\mu \in \Delta_{S+}} \left( x - \cos(2 \mu \cdot \bar{q}) \right) = 2 (1 + x) (-1 + 2x) (1 + 2x), \tag{19}
\]
\[
2^3 \prod_{\mu \in \Delta_{L+}} \left( x - \cos(2 \mu \cdot \bar{q}) \right) = (-1 + 2x)^2 (1 + 2x). \tag{20}
\]

The polynomials for the twisted \(D_4^{(3)}\) are
\[
\prod_{\mu \in \Delta_{S+}} \left( x - \cos(2 \mu \cdot \bar{q}) \right) = x^2 (1 + x), \tag{21}
\]
\[
\prod_{\mu \in \Delta_{L+}} \left( x - \cos(2 \mu \cdot \bar{q}) \right) = x^2 (-1 + x). \tag{22}
\]
Before closing this paper, let us briefly remark on the identities arising from *foldings* of root systems. Among them those relating two un-twisted root systems, *ie* with superscript (1) are quite simple.

**Folding** $A^{(1)}_{2r-1} \rightarrow C^{(1)}_r$ : The vector weights of $A_{2r-1}$ ($2r$ dim.) become those of $C_r$ ($2r$ dim.). This relates $T_{2r}$ to $T_r$ in (8) as

$$A_{2r-1} : \quad T_{2r}(x) = (-1)^r T_r(1 - 2x^2), \quad C^{(1)}_r.$$  \hfill (23)

**Folding** $D^{(1)}_{r+1} \rightarrow B^{(1)}_r$ : This gives a quite obvious relation as seen from (9) and (7).

**Folding** $E^{(1)}_6 \rightarrow F^{(1)}_4$ : In this folding the minimal weights $27$ of $E_6$ become $\Delta_S$ (24 dim.) of $F_4$ plus three zero weights. Thus we obtain

$$E^{(1)}_6 : \quad 2 \frac{[10]}{x^3} = \frac{[15]}{x-1-2x^2}, \quad F^{(1)}_4.$$  \hfill (24)

We also obtain

$$E^{(1)}_6 : \quad [11] = [15]^2 \times [10], \quad F^{(1)}_4,$$  \hfill (25)

since the 72 roots of $E_6$ are decomposed into $2\Delta_S + \Delta_L$ (24 dim.) of $F_4$.

**Folding** $D^{(1)}_4 \rightarrow G^{(1)}_2$ : The vector weights of $D_4$ (8 dim.) decompose into $\Delta_S$ (6 dim.) plus two zero weights of $G_2$ leading to the identity

$$D^{(1)}_4 : \quad 2 \frac{[8]}{r=1/(x-1)} = [19], \quad G^{(1)}_2.$$  \hfill (26)

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