Equivariant cd-structures and descent theory

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Abstract

We construct the equivariant version of cd-structures, and we develop descent theory for topologies coming from equivariant cd-structures. In particular, we reprove several results of Cisinski-Déglies on the étale descent, qfh-descent, and h-descent. Since the étale topos, qfh-topos, and h-topos do not come from usual cd-structures, such results cannot be produced by usual cd-structures. We also apply equivariant cd-structures to study several topologies on the category of noetherian fs log schemes.

1. Introduction

1.1. In [7] and [8], analogous results of the Brown-Gersten theorem ([2]) for the Nisnevich topology and cdh-topology are studied by introducing cd-structures. For instance, if we take $P$ as the collection of Nisnevich distinguished squares, then we recover the Nisnevich cd-structure. In [3, §3.3], it is applied to study descents in triangulated categories of motives over schemes.

However, there are topoi like the étale topos, qfh-topos, and h-topos that cannot be obtained by any cd-structures. In [loc. cit], descent theory for the étale topology, qfh-topology, and h-topology is discussed with equivariant versions of distinguished squares but without cd-structures. The reason why we introduce equivariant cd-structures here is to study descent theory for such a topology more systematically. Here is the definition.

Definition 1.2. An equivariant cd-structure (or ecd-structure for abbreviation) $P$ on a small category $\mathcal{S}$ with an initial object is a collection of pairs $(G, C)$ where $G$ is a group and $C$ is a commutative diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow \phi' & & \downarrow \phi \\
S' & \xrightarrow{g} & S
\end{array}
$$

(1.2.1)

in $\mathcal{S}$ with $G$-actions on $X$ over $S$ and on $X'$ over $S'$ such that

(i) $g'$ is $G$-equivariant over $g$,

(ii) if $(G, C) \cong (G', C')$, then $(G, C) \in P$ if and only if $(G', C') \in P$.

For a pair $(G, C) \in P$, $C$ is called a $P$-distinguished square of group $G$. The $t_P$-topology is the topology generated by the $t_\emptyset$-topology (see (2.2) for the definition) and the families of morphisms of the form

$$\{f, g\}
$$

(1.2.2)

for $(G, C) \in P$. If $G$ is trivial for any element of $P$, then $P$ is a usual cd-structure defined in [7 2.1].
1.3. We can define bounded, complete, and regular ecd-structures as in [7]. Then we obtain the following two theorems generalizing several results in [7] and [3 §3].

**Theorem 1.4.** (See (4.4)) Let $P$ be a complete and regular ecd-structure bounded with respect to a density structure $D$ (see (2.5) for the definition), and let $F$ be a $t_P$-sheaf of $\mathbb{Q}$-modules on a small category $\mathcal{I}$ with an initial object. Then

$$H^n_{t_P}(S, F) = 0$$

for any $S \in \text{ob} \mathcal{I}$ and $n > \text{dim}_P S$.

**Theorem 1.5.** (See (5.8)) Let $\mathcal{I}$ be a small category with an initial object, let $\mathcal{I}^\text{dia}$ denote the 2-category of functors from small categories to $\mathcal{I}$, let $\mathcal{I} : \mathcal{I}^\text{dia} \to \text{Tri}$ be a contravariant pseudofunctor satisfying the conditions (i)–(iv) in (5.3), let $K$ be an object of $\mathcal{I}(T)$ where $T \in \text{ob} \mathcal{I}$, and let $P$ be a bounded, complete, and regular ecd-structure on $\mathcal{I}$. Then the following are equivalent.

(i) For any distinguished square in $\mathcal{I}/T$ of group $G$ of the form (1.2.1), the commutative diagram

$$
\begin{array}{cc}
p_*p^*K & \xrightarrow{ad} & p_*g_*g^*p^*K \\
\downarrow{ad} & & \downarrow{ad} \\
(p_*f_*f^*p^*K)^G & \xrightarrow{ad} & (p_*h_*h^*p^*K)^G
\end{array}
$$

in $\mathcal{I}(T)$ is homotopy Cartesian where $p : S \to T$ is the structural morphism and $h = fg'$.

(ii) $K$ satisfies $t_P$-descent.

**Remark 1.6.** Note that we only work for coefficients that are $\mathbb{Q}$-algebra. The $\mathbb{Z}$-coefficient is not suitable for the above theorems.

1.7. We show that that the étale topos, qfh-topos, and h-topos come from ecd-structures. This reproves several results in [6] and [3 §3] more systematically. The eh-topos also comes from an ecd-structure, so we can apply the above theorems to the eh-topos also, which seems to be new. We also construct several ecd-structures on the category of noetherian fs log schemes, and define strict closed topology, dividing topology, pw-topology, and qw-topology. Then we prove the following theorem.

**Theorem 1.8.** (See (7.14)) Let $\mathcal{I}$ be a category of finite dimensional noetherian fs log schemes, let $\mathcal{I} : \mathcal{I}^\text{dia} \to \text{Tri}$ be a contravariant pseudofunctor satisfying the conditions (i)–(iv) in (5.3), and let $K$ be an object of $\mathcal{I}(T)$ satisfying the dividing descent and strict closed descent where $T$ is an object of $\mathcal{I}$. Then $K$ satisfies the pw-descent if and only if $K$ satisfies the qw-descent.

1.9. This theorem has an application in the theory of motives over fs log schemes. The original motivation to define ecd-structures was to prove the above theorem since ecd-structures make us be able to prove it.

1.10. **Organization of the paper.** In Section 2, we give several definitions. In Section 3, we study the notions of bounded, complete, and regular ecd-structures. In Section 4, we study the descent theory for complex of preseaves of $\mathbb{Q}$-modules, and this is generalized for derivators in Section 5 using techniques in [3 §3]. In Section 6, we apply our theorems to several topologies on the category of noetherian schemes, which reproves several results on the étale topology, qfh-topology,
and h-topology in \[6\] and \[3, \S3\]. In Section 7, we apply our results to several \(\mathcal{E}\-\)structures on the category of noetherian \(fs\) log schemes, which will be useful in the theory of motives over \(fs\) log schemes.

1.11. Acknowledgement. The author’s dissertation contains many of the results of this paper. The author is grateful to Martin Olsson for helpful communications.

2. Definitions

2.1. Throughout this paper, \(\mathcal{S}\) is a small category with an initial object \(\emptyset\). In Section 6, \(\mathcal{S}\) is the category of finite dimensional noetherian schemes, and in Section 7, \(\mathcal{S}\) is the category of finite dimensional noetherian \(fs\) log schemes.

Definition 2.2. Recall from [1, \S4.5.3] that the \(t\emptyset\)-topology on \(\mathcal{S}\) is the minimal topology such that the empty sieve is a covering sieve for the initial object \(\emptyset\). Note that a presheaf \(F\) on \(\mathcal{S}\) is a \(t\emptyset\)-sheaf if and only if \(F(\emptyset) = \ast\).

Definition 2.3. Here are some definitions we will frequently use.

1. Let \(\Lambda\) be a ring. We denote by \(\text{Mod}\_\Lambda\) the category of \(\Lambda\)-modules.
2. Let \(t\) be a topology on \(\mathcal{S}\). For any object \(S\) of \(\mathcal{S}\), we denote by \(\rho(S)\) (resp. \(\rho_t(S)\)) the presheaf represented by \(S\) (resp. \(t\)-sheaf associated with the presheaf represented by \(S\)).
3. Let \(t\) be a topology on \(\mathcal{S}\), and let \(\Lambda\) be a ring. For any object \(S\) of \(\mathcal{S}\), we denote by \(\Lambda(S)\) (resp. \(\Lambda_t(S)\)) the presheaf (resp. \(t\)-sheaf) of \(\Lambda\)-module freely generated by \(\rho(S)\) (resp. \(\rho_t(S)\)).
4. Let \(t\) be a topology on \(\mathcal{S}\), and let \(\Lambda\) be a ring. We denote by \(\text{PSh}(\mathcal{S}, \Lambda)\) (resp. \(\text{Sh}_t(\mathcal{S}, \Lambda)\)) the category of presheaves (resp. \(t\)-sheaves) of \(\Lambda\)-modules on \(\mathcal{S}\). In \(\text{PSh}(\mathcal{S}, \Lambda)\) and \(\text{Sh}_t(\mathcal{S}, \Lambda)\), we denote by \(1\) the constant presheaf (\(t\)-sheaf associated to) \(\Lambda\).
5. An \(\mathcal{S}\)-diagram is a functor from a small category to \(\mathcal{S}\).
6. Let \(\mathcal{X}: I \rightarrow \mathcal{S}\) be an \(\mathcal{S}\)-diagram. We often write it as \((\mathcal{X}, I)\). For \(i \in \text{ob} I\), let \(\mathcal{X}_i\) denote the image of \(i\) in \(\mathcal{S}\).
7. Let \(f: (\mathcal{X}, I) \rightarrow (\mathcal{Y}, J)\) be a morphism of \(\mathcal{S}\)-diagrams, and let \(i\) be an object of \(I\). We denote by \(f_i\) the induced morphism \(\mathcal{X}_i \rightarrow \mathcal{Y}_{f(i)}\) where \(f(i)\) denotes the image of \(i\) in \(J\).
8. A morphism \(f: (\mathcal{X}, I) \rightarrow (\mathcal{Y}, J)\) is called reduced if the induced functor \(I \rightarrow J\) is an equivalence.
9. We denote by \(\mathcal{S}\text{-dia}\) the 2-category of \(\mathcal{S}\)-diagrams.
10. We denote by \(e\) the category with only one object and only one morphism that is the identity morphism.
11. Let \(F: C \rightleftharpoons D: G\) be a pair of adjoint functors between categories. We denote by

\[
\text{ad} : \text{id} \rightarrow GF, \quad \text{ad}' : FG \rightarrow \text{id}
\]

the unit and counit respectively.

Definition 2.4. Here are some definitions about group actions.
(1) Let $G$ be a group. We denote by $e_G$ the category with only one object $\ast$ and $\text{Hom}(\ast, \ast) = G$.

(2) Let $A$ be a set or an abelian group with an action of a group $G$. Then we denote by $A^G$ the limit of the functor $e_G \to \text{Set}$ given by

$$\ast \mapsto A, \quad g \in \text{Hom}(\ast, \ast) \mapsto g : A \to A.$$  

Here, Set denotes the category of sets. Note that $A^G$ is equal to the subset of $A$ fixed by $G$.

(3) Let $t$ be a topology on $\mathcal{S}$, and let $S$ be an object of $\mathcal{S}$ with an action of a group $G$. We denote by $\rho(S)_G$ (resp. $\rho_t(S)_G$) the colimit of the functor $e_G \to \text{PSh}(\mathcal{S})$ (resp. $e_G \to \text{Sh}_t(\mathcal{S})$) induced by the $G$-action. Note that for any $F \in \text{PSh}(\mathcal{S})$,

$$\text{Hom}_{\text{PSh}(\mathcal{S})}(\rho(S)_G, F) = F(S)^G.$$  

Thus the induced morphism $\rho(S) \to \rho(S)_G$ is an epimorphism since $F(S)^G \to F(S)$ is injective.

(4) Let $f : \mathcal{S}' \to \mathcal{S}$ be a functor of categories, and let $\Lambda$ be a ring. We denote by $f^\#$, $f^*$ the left adjoint and right adjoint of $f^* : \text{D}(\text{PSh}(\mathcal{S}', \Lambda)) \to \text{D}(\text{PSh}(\mathcal{S}, \Lambda))$ respectively.

(5) Let $\Lambda$ be a ring, and let $K$ be an object of $\text{C}(\text{Mod}_\Lambda)$ with an action of a group $G$. Consider it as an object of $\text{C}(\text{PSh}(e_G, \Lambda))$. We denote by

$$K^G$$

the object $f_*K$ where $f : e_G \to e$ is the trivial functor.

When $G$ is a finite group and $\Lambda$ is a $\mathbf{Q}$-algebra, we have the following formula for $K^G$ in [3, 3.3.21]. Consider the morphism $p_K : K \to K$ given by the formula

$$p_K = \frac{1}{|G|} \sum_{g \in G} \sigma^g$$  \hspace{1cm} (2.4.1)$$

where $\sigma^g$ denotes the action of $g$ on $K$. The morphism $p_K$ is a projector, and $K^G$ is the image of $p_K$. Note that $K^G$ is a direct summand of $K$.

(6) Let $K$ be an object of $\text{D}(\text{PSh}(\mathcal{S}, \Lambda))$, and let $\mathcal{X} : I \to \mathcal{S}$ be an $\mathcal{S}$-diagram. We denote by $K(\mathcal{X}, I)$ the object of $\text{Ab}$ given by

$$p_*\mathcal{X}^*K$$

where $p : I \to e$ is the functor to the trivial category. Note that we have the formula

$$K(\mathcal{X}, I) \cong R\lim_{i \in I} \mathcal{X}_i.$$  

**Definition 2.5.** Let $P$ be an ecd-structure on $\mathcal{S}$. As in [7], we have the following definition.

(1) A $P$-simple cover is a cover that can be obtained by iterating covers of the form $[1.2.2]$.  

\[4\]
(2) A density structure on $\mathcal{S}$ is a function which assigns to any object $S$ of $\mathcal{S}$ a sequence $D_0(S), D_1(S) \ldots$ of family of morphisms to $S$ with the following conditions:

(i) $(\emptyset \to S) \in D_0(S)$ for all $S$,
(ii) isomorphisms belong to $D_i$ for all $i$,
(iii) $D_{i+1} \subset D_i$,
(iv) if $g : Y \to X$ is in $D_i(X)$ and $f : X \to S$ is in $D_i(S)$, then $gf : Y \to S$ is in $D_i(S)$.

(3) The dimension of $S \in \text{ob} \mathcal{S}$ (with respect to a density structure $D$) is the smallest number $n$ such that every morphism in $D_n(S)$ is an isomorphism. It is denoted by $\dim_D S$.

(4) Let $D$ be a density structure on $\mathcal{S}$. Then $(G, C) \in P$ is called reducing (with respect to $D$) if for any $i \geq 0$, $X_0 \in D_{i+1}(X)$, $S'_0 \in D_{i+1}(S')$, and $X'_0 \in D_i(X')$, there exist a distinguished square

$$
\begin{array}{ccc}
X'_1 & \longrightarrow & X_1 \\
\downarrow & & \downarrow \\
S'_1 & \longrightarrow & S_1
\end{array}
$$

of group $G$ and a morphism $(G, C_1) \to (G, C)$ of distinguished squares of group $G$ such that $S_1 \in D_{i+1}(S)$ and that $X_1 \to X$, $S'_1 \to S'$, $X'_1 \to X'$ factors through $X_0$, $S_0$, and $X'_0$ respectively.

(5) A morphism of $P$-distinguished squares $C \to C'$ of group $G$ is called a refinement if the morphism of the commutative diagrams is an isomorphism on the right corner.

Definition 2.6. Let $P$ be an ecd-structure on $\mathcal{S}$. As in [7], we introduce the notions of bounded, complete, and regular ecd-structures as follows.

(1) $P$ is called bounded (with respect to a density structure $D$) if every element of $P$ has a refinement that is reducing with respect to $D$ and that for any object $S$ of $\mathcal{S}$, $\dim_D S$ is finite.

(2) $P$ is called complete if any covering sieve of an object $X \neq \emptyset$ of $\mathcal{S}$ contains a sieve generated by a $P$-simple cover.

(3) $P$ is called regular if for any $(G, C) \in P$, $C$ is Cartesian, $S' \to S$ is a monomorphism, and the induced morphism

$$
(\rho_{tP}(X') \times_{\rho_{tP}(S')} \rho_{tP}(X')_G) \coprod \rho_{tP}(X) \to \rho_{tP}(X) \times_{\rho_{tP}(S)} \rho_{tP}(X)_G
$$

of $t_P$-sheaves is an epimorphism where $\rho(S)$ denotes the representable $t_P$-sheaf of sets of $S$.

3. Properties of ecd-structures

3.1. From (3.2) to (3.4), we state some results in [7] that can be trivially generalized to ecd-structures. The proofs are identical to those in [loc. cit.].
Proposition 3.2. An ecd-structure $P$ on $\mathcal{S}$ is complete if the following two conditions hold.

(i) Any morphism with values in $\emptyset$ is an isomorphism.

(ii) For any distinguished square $(G, C)$ of the form (1.2.1) and any morphism $Y \to X$, $(G, C \times_X Y)$ is distinguished.

Proposition 3.3. Let $P_1$ and $P_2$ be ecd-structures on $\mathcal{S}$. If $P_1$ and $P_2$ are complete (resp. regular, resp. bounded with respect to the same density structure $D$), then $P_1 \cup P_2$ is complete (resp. regular, resp. bounded with respect to $D$).

Proposition 3.4. Let $P$ be an ecd-structure on $\mathcal{S}$. For any object $S$ of $\mathcal{S}$, if $P$ is complete (resp. regular, resp. bounded with respect to a density structure $D$), then $P/S$ is complete (resp. regular, resp. bounded with respect to $D/S$).

Proposition 3.5. Let $P$ be an ecd-structure on $\mathcal{S}$ such that for any $(G, C) \in P$ of the form (1.2.1),

(i) $C$ is Cartesian,

(ii) $S' \to S$ is a monomorphism,

(iii) in the Cartesian diagram

$$
\begin{array}{ccc}
G \times X' & \longrightarrow & G \times X \\
\downarrow & & \downarrow \\
X' \times_{S'} X' & \longrightarrow & X \times S X
\end{array}
$$

(3.5.1)

in $\mathcal{S}$, the family

$$
\{X' \times_{S'} X' \to X \times_{S} X, G \times X \to X \times S X\}
$$

of morphisms is a $t_P$-cover. Here, the right vertical arrow is induced by the projection and the group action $G \times X \rightrightarrows X$.

Then $P$ is regular.

Proof. We put $t = t_P$. Consider the commutative diagram

$$
\begin{array}{ccc}
p_t(X' \times_{S'} X') \coprod p_t(G \times X) & \longrightarrow & p_t(X \times_{S} X) \\
\alpha' \coprod \beta & & \downarrow \alpha \\
(p_t(X') \times_{p_t(s')} p_t(X')_G) \coprod p_t(X) & \longrightarrow & p_t(X) \times p_t(S) p_t(X)_G
\end{array}
$$

of $t_P$-sheaves where $\alpha$, $\alpha'$, and $\beta$ are induced by the $G$-action, and $\gamma$ and $\gamma'$ are induced by (3.5.1). The morphism $p_t(G \times X) \to p_t(X)$ is an epimorphism since it has a section. Thus morphisms $p_t(X) \to p_t(X)_G$ and $p_t(X') \to p_t(X')_G$ are also epimorphisms by (2.4). Thus the vertical arrows are epimorphisms since

$$p_t(X \times_{S} X) = p_t(X) \times p_t(S) p_t(X), \quad p_t(X' \times_{S'} X') = p_t(X') \times p_t(S') p_t(X').$$

Now the lower horizontal arrow is surjective since the upper horizontal arrow is surjective by (iii).
**Proposition 3.6.** Let $P$ be a regular ecd-structure on $S$, and let $F$ be a $t_P$-sheaf. Then for a distinguished square of the form (1.2.1), the induced diagram

$$
\begin{array}{ccc}
F(S) & \rightarrow & F(S') \\
\downarrow & & \downarrow \\
F(X)^G & \rightarrow & F(X')^G
\end{array}
$$

of sets is Cartesian.

**Proof.** We put $t = t_P$. Since $F$ is a $t$-sheaf, the induced function $F(S) \rightarrow F(S') \times F(X)$ is injective. Thus the induced function

$$
F(S) \rightarrow F(S') \times F(X)^G
$$

is also injective. The remaining is to show that the equalizer of the induced functions

$$
F(S') \times F(X)^G \Rightarrow F(X')^G
$$

(3.6.1)
is $F(S)$. Since $F$ is a $t$-sheaf, $F(S)$ is the equalizer of the induced functions

$$
F(S') \times F(X) = F(S' \times_S S') \times F(X') \times F(X) \times F(X \times_S X),
$$

which is equal to the equalizer of the induced functions

$$
\text{Hom}(\rho(S') \prod \rho_t(X), F) \Rightarrow \text{Hom}(\rho(S' \times_S S') \prod \rho_t(X') \prod \rho_t(X' \times_X X) \times_X X, F).
$$

Note that $S' \times_S S' \cong S'$ since $g$ is a monomorphism. The image of $F(S)$ in $F(S') \times F(X)$ is in $F(S') \times F(X)^G$, so we see that $F(S)$ is the equalizer of the induced functions

$$
\text{Hom}(\rho(S') \prod \rho_t(X)^G, F) \Rightarrow \text{Hom}(\rho(S') \prod \rho_t(X)^G \prod \rho_t(X)^G \prod \rho_t(X) \times_{\rho(S)} \rho_t(X)^G, F).
$$

Thus $F(S)$ is the equalizer of the induced functions

$$
\text{Hom}(\rho(S') \prod \rho_t(X)^G, F) \Rightarrow \text{Hom}(\rho(S') \prod \rho_t(X)^G \prod (\rho_t(X) \times_{\rho(S)} \rho_t(X)^G), F).
$$

Using the fact that (2.6.1) is surjective, we see that $F(S)$ is the equalizer of the induced functions

$$
\text{Hom}(\rho(S') \prod \rho_t(X)^G, F) \Rightarrow \text{Hom}(\rho(S') \prod (\rho_t(X') \times_{\rho_t(S')} \rho_t(X') \prod \rho_t(X)), F). \quad (3.6.2)
$$

Now, let $(a, b) \in F(S') \times F(X)^G$ be an element of the equalizer of the functions in (3.6.1). Then the images of $b$ for the two induced functions

$$
\text{Hom}(\rho_t(X)^G, F) \Rightarrow \text{Hom}(\rho_t(X') \times_{\rho_t(S')} \rho_t(X')^G, F) \quad (3.6.3)
$$

are equal to the images of $a$ for the two equal induced functions

$$
\text{Hom}(\rho_t(S'), F) \Rightarrow \text{Hom}(\rho_t(X') \times_{\rho_t(S')} \rho_t(X')^G, F)
$$

respectively. Thus $b$ is in the equalizer of the functions in (3.6.2). This implies that $(a, b)$ is in the equalizer of the functions in (3.6.3). This completes the proof. 

**Proposition 3.7.** Let \( P \) be a regular ecd-structure on \( \mathcal{S} \). Then for a distinguished square of the form \([1.2.1]\), the induced diagram

\[
\begin{array}{c}
\rho_{tP}(X')_G \\
\downarrow \\
\rho_{tP}(S')
\end{array} \quad \begin{array}{c}
\rho_{tP}(X)_G \\
\downarrow \\
\rho_{tP}(S)
\end{array}
\]

of \( t_P \)-sheaves of sets on \( \mathcal{S} \) is coCartesian.

*Proof.* It suffices to show that for any \( t_P \)-sheaf, the induced diagram

\[
\begin{array}{ccc}
\text{Hom}(\rho_{tP}(S), F) & \longrightarrow & \text{Hom}(\rho_{tP}(S'), F) \\
\downarrow & & \downarrow \\
\text{Hom}(\rho_{tP}(X)_G, F) & \longrightarrow & \text{Hom}(\rho_{tP}(X')_G, F)
\end{array}
\]

of sets is Cartesian. It follows from \([3.6]\). \qed

**Proposition 3.8.** Let \( P \) be a regular ecd-structure on \( \mathcal{S} \), and let \( \Lambda \) be a ring. Then for a distinguished square of the form \([1.2.1]\), the sequence

\[
0 \longrightarrow \Lambda_t(X') \longrightarrow \Lambda_t(X) \oplus \Lambda_t(S') \longrightarrow \Lambda_t(S) \longrightarrow 0
\]

in \( \text{Sh}_{tP}(\mathcal{S}, \Lambda) \) is exact.

*Proof.* The functor \( F \mapsto \Lambda(F) \) preserves colimits and monomorphisms. Since \( P \) is regular, \( X' \rightarrow X \) is a monomorphism, so the second arrow is a monomorphism. The other part follows from \([3.7]\) using the fact that the functor preserves colimits. \qed

**Definition 3.9.** Let \( \Lambda \) be a ring. For a topology \( t \) on \( \mathcal{S} \), we have the sheafification functor \( \text{PSh}(\mathcal{S}, \Lambda) \rightarrow \text{Sh}(\mathcal{S}, \Lambda) \) and the inclusion functor \( \text{Sh}(\mathcal{S}, \Lambda) : \rightarrow \text{PSh}(\mathcal{S}, \Lambda) \). We denote by

\[
a_t^*: \text{D}(\text{PSh}(\mathcal{S}, \Lambda)) \overset{\cong}{\rightarrow} \text{D}(\text{Sh}(\mathcal{S}, \Lambda)): a_t^*
\]

their derived functors.

Let \( K \) be an object of \( \text{D}(\text{PSh}(\mathcal{S}, \Lambda)) \). We say that \( K \) is \( t \)-local if the morphism

\[
K \overset{\text{ad}}{\longrightarrow} a_t^* a_t^* K
\]

is an isomorphism. Note that if \( t' \) is another topology on \( \mathcal{S} \) such that the \( t \)-topos is equivalent to the \( t' \)-topos, then \( K \) is \( t \)-local if and only if it is \( t' \)-local.

**3.10.** Here, under the notations and hypotheses as above, we give describe \( t \)-local objects using \( t \)-hypercovers. By the abelian version of \([4, 1.2]\), we see that an object \( K \) of \( \text{D}(\text{PSh}(\mathcal{S}, \Lambda)) \) is \( t \)-local if and only if the morphism

\[
K(X) \overset{\text{ad}}{\longrightarrow} \underset{i \in I}{\text{Rlim}} K(\mathcal{S}_i) \cong K(\mathcal{S}, I)
\]

is an isomorphism for any object \( X \) of \( \mathcal{S} \) and \( t \)-hypercover \( (\mathcal{S}, I) \) of \( X \).
Lemma 3.11. Let \( t \) be a topology on \( \mathcal{S} \), and let \( K \in D(\text{PSh}(\mathcal{S}, \Lambda)) \) be an element. If \( K \) is \( t \)-local, then
\[
K(S) \cong R\text{Hom}_{D(\text{PSh}(\mathcal{S}, \Lambda))}(\Lambda_t(S), K)
\]
for any \( S \in \text{ob} \mathcal{S} \).

Proof. Since \( K(S) \cong R\text{Hom}_{D(\text{PSh}(\mathcal{S}, \Lambda))}(\Lambda(S), K) \), it follows from the fact that \( a^*_t \Lambda(S) \cong a^*_t \Lambda_t(S) \).

Lemma 3.12. Let \( \Lambda \) be a ring, let \( t \) be a topology on \( \mathcal{S} \), and let \( K \in D(\text{PSh}(\mathcal{S}, \Lambda)) \) be an element. If \( K \) is \( t \)-local, then
\[
K(S)^G \cong R\text{Hom}_{D(\text{PSh}(\mathcal{S}, \Lambda))}(\Lambda(X)_G, K)
\]
for any \( S \in \text{ob} \mathcal{S} \) with an action of a group \( G \).

Proof. Since \( a^*_t(\Lambda(S)_G) \cong a^*_t(\Lambda_t(S)_G) \), we have that
\[
R\text{Hom}_{D(\text{PSh}(\mathcal{S}, \Lambda))}(\Lambda(X)_G, K) \cong R\text{Hom}_{D(\text{PSh}(\mathcal{S}, \Lambda))}(\Lambda(X)_G, K).
\]
Consider the functor \( f : e_G \to \mathcal{S} \) induced by the \( G \)-action. Then
\[
R\text{Hom}_{D(\text{PSh}(\mathcal{S}, \Lambda))}(\Lambda(X)_G, K) \cong R\text{Hom}_{D(\text{PSh}(\mathcal{S}, \Lambda))}(f_t f^* 1, K)
\]
\[
\cong R\text{Hom}_{D(\text{PSh}(e_G, \Lambda))}(1, K(S))
\]
\[
\cong K(S)^G.
\]
The conclusion follows from these.

Proposition 3.13. Let \( P \) be a regular ecd-structure on \( \mathcal{S} \), and let \( K \in D(\text{PSh}(\mathcal{S}, \mathbb{Q})) \) be an element. Then for each distinguished square of group \( G \) of the form (1.2.1), there is a homomorphism
\[
\partial_{(G,C)} : H^n_{t_p}(X', K)^G \to H^{n+1}_{t_p}(S, K)
\]
of \( \mathbb{Q} \)-modules for any \( n \) such that

(i) for any morphism \((G, C_1) \to (G, C)\) of distinguished squares of group \( G \), the diagram
\[
\begin{array}{ccc}
H^n_{t_p}(X', K)^G & \xrightarrow{\partial_{(G,C_1)}} & H^n_{t_p}(S, K) \\
\downarrow & & \downarrow \\
H^n_{t_p}(X'_1, K)^G & \xrightarrow{\partial_{(G,C)}} & H^n_{t_p}(S, K)
\end{array}
\]
commutes,

(ii) the sequence of \( \mathbb{Q} \)-modules
\[
H^n(S, K) \to H^n(X, K)^G \oplus H^n(S', K) \to H^n(X', K)^G \xrightarrow{\partial_{(G,C)}} H^{n+1}(S, K)
\]
is exact.
Proof. We put $K' = a_\ast a_\ast^t F$. Then by \textbf{(3.11)},

$$H_i^n(Y, K) \cong H^n(K'(Y)) \cong H^n(R\text{Hom}(Q_t(Y), K'))$$

for any object $Y$ of $\mathcal{S}$. When $Y$ has a $G$-action, by \textbf{(3.12)},

$$H_i^n(Y, K)^G \cong H^n(K'(Y))^G \cong H^n(K'(Y))^G \cong H^n(R\text{Hom}(Q_t(Y)_G, K')).$$

Here, the second isomorphism comes from the fact that $K'(Y)^G$ is given by the projector \textbf{(2.4.1)}. The conclusion follows from these and \textbf{(3.8)}. \qed

4. Descent theory for complexes of presheaves

4.1. In \textbf{(1.2.4)} and \textbf{(1.3.3)}, we will have the following notations and hypotheses. Let $P$ be a complete ecd-structure on $\mathcal{S}$ bounded with respect to a density structure $D$, and we put $t = t_P$. For $n \geq 0$, let $T^n$ be presheaves of abelian groups on $\mathcal{S}$, and for $(G, C) \in P$ of the form \textbf{(2.2.1)}, let $\partial_{(G, C)} : T^n(X') \to T^n(S)$ be functions. Consider the following conditions.

(i) $\partial_{(G, C)}$ is natural with respect to morphisms of distinguished squares of group $G$.

(ii) The sequence

$$T^{n-1}(X')^G \to T^n(S) \to T^n(S') \times T^n(X)^G$$

is exact for any $n \geq 0$ (put $T^{-1} = 0$).

(iii) $T^0(\emptyset) = 0$.

(iii)' $T^n(\emptyset) = 0$ for any $n \geq 0$.

(iv) $a_\ast T^n = 0$ for any $n \geq 1$.

(iv)' $a_\ast T^n = 0$ for any $n \geq 0$.

Theorem 4.2. Under the notations and hypotheses of \textbf{(4.1)}, assume (i), (ii), (iii), and (iv). Then for any object $S$ of $\mathcal{S}$,

$$T^n(S) = 0$$

for any $n > \dim_D S$.

Proof. We repeat the proof of \textbf{[7, 2.27]} with minor changes. Refining $P$, we may assume that $P$ contains only reducing distinguished squares. For $n \geq 0$, consider the following statement.

$(A_n)$ For any $S \in \text{ob } \mathcal{S}$ and $a \in T^n(S)$, there exists $(j : U \to S) \in D_n(S)$ such that $j^*(a) = 0$.

We only need to prove $(A_n)$ for all $n \geq 0$. Let us prove it by induction.

Since $(\emptyset \to X) \in D_0(S)$ and $T^0(\emptyset) = 0$, $(A_0)$ holds. When $n > 0$, assume $(A_{n-1})$. Let $S$ be an object of $\mathcal{S}$, and let $a \in T^n(S)$ be an element. By (iv), there exists a $t_P$-cover $\{j_i : U_i \to S\}_{i \in I}$ such that $T^n(j_i)(a) = 0$ for all $i$. Since $P$ is complete, we may assume that the $t_P$-cover is a simple cover. Using the induction on the number of distinguished squares used in the cover, we may assume that for some $(G, C) \in P$ of the form \textbf{(2.2.1)}, there exist $X_0 \in D_n(X)$ and $S_0' \in D_n(S')$ such that the images of $a$ in $T^n(X_0)$ and $T^n(S_0')$ are 0.
Since $X' \in D_{n-1}(X')$ and $(G,C)$ is reducing, there exist a distinguished square

$$C_1 = \begin{array}{ccc}
X'_1 & \rightarrow & X_1 \\
\downarrow & & \downarrow \\
S'_1 & \rightarrow & S_1
\end{array}$$

of group $G$ and a morphism $(G,C_1) \rightarrow (G,C)$ of distinguished squares of group $G$ such that $S_1 \in D_n(S)$ and that the images of $a$ in $T^n(X_1)$ and $T^n(S'_1)$ are 0. Let $a_1$ be the image of $a$ in $T^n(S_1)$. Then by (ii), $a_1 = \partial_{(G,C_1)}b_1$ for some $b_1 \in T^{n-1}(X'_1)^G$. By $(A_{n-1})$, there exist $X'_2 \in D_{n-1}(X'_1)$ such that the image of $b_1$ in $H^{n-1}(X'_2)^G$ is 0.

Since $X_1 \in D_n(X_1)$, $S'_1 \in D_n(S'_1)$, and $(G,C)$ is reducing, there exists a distinguished square

$$C_3 = \begin{array}{ccc}
X'_3 & \rightarrow & X_3 \\
\downarrow & & \downarrow \\
S'_3 & \rightarrow & S_3
\end{array}$$

of group $G$ and a morphism $(G,C_3) \rightarrow (G,C_1)$ of distinguished squares of group $G$ such that $S_3 \in D_n(S_1)$ and that the image of $b_1$ in $T^{n-1}(X'_3)^G$ is 0. Note that $S_3 \in D_n(S)$ since $S_3 \in D_n(S_1)$ and $S_1 \in D_n(S)$. Then by (i), the image of $a$ in $T^n(S_3)$ is $\partial_{(G,C_1)}0 = 0$, which proves $(A_n)$ since $S_3 \in D_n(S)$.

**Theorem 4.3.** Under the notations and hypotheses of (11), assume (i), (ii), (iii)', and (iv)'. Then $T^n = 0$ for any $n \geq 0$.

**Proof.** We repeat the proof of [7, 3.2] with minor changes. Refining $P$, we may assume that $P$ contains only reducing distinguished squares. For $d \geq 0$, consider the following statement.

(11) For any $S \in \text{ob} \mathcal{S}$, $n \geq 0$, and $a \in T^n(S)$, there exists $(j : U \rightarrow S) \in D_d(S)$ such that $T^n(j)(a) = 0$.

Let us prove (11) by induction on $d$.

Since $(\emptyset \rightarrow X) \in D_0(S)$ and $T^n(\emptyset) = 0$, (11) holds. When $d > 0$, assume (11). Let $S$ be an object of $\mathcal{S}$, and let $a \in T^n(S)$ be an element. By (iv)', there exists a $t_P$-cover $\{j_i : U_i \rightarrow S\}_{i \in I}$ such that $T^n(j_i)(a) = 0$ for all $i$. Since $P$ is complete, we may assume that the $t_P$-cover is a simple cover. Using the induction on the number of distinguished squares used in the cover, we may assume that for some $(G,C) \in P$ of the form (12.23), there exist $S'_0 \in D_d(S')$ and $X_0 \in D_d(X)$ such that the images of $a$ in $T^n(X_0)$ and $T^n(S'_0)$ are 0.

Since $X' \in D_{d-1}(X')$ and $(G,C)$ is reducing, there exist a distinguished square

$$C_1 = \begin{array}{ccc}
X'_1 & \rightarrow & X_1 \\
\downarrow & & \downarrow \\
S'_1 & \rightarrow & S_1
\end{array}$$

of group $G$ and a morphism $(G,C_1) \rightarrow (G,C)$ of distinguished squares of group $G$ such that $S_1 \in D_d(S)$ and that the images of $a$ in $T^n(X_1)$ and $T^n(S'_1)$ are 0. Let $a_1$ be the image of $a$
in $T^n(S_1)$. Then by (ii), $a_1 = \partial_{(G,C_1)} b_1$ for some $b_1 \in T^{n-1}(X'_1)^G$. By $(B_{d-1})$, there exists $X'_2 \in D_{d-1}(X'_1)$ such that the image of $b_1$ in $T^{n-1}(X'_2)^G$ is 0.

Since $X_1 \in D_d(X_1)$, $S'_1 \in D_d(S'_1)$, and $(G,C)$ is reducing, there exists a distinguished square

$$C_3 = \begin{array}{ccc} X'_3 & \rightarrow & X_3 \\ \downarrow & & \downarrow \\ S'_3 & \rightarrow & S_3 \end{array}$$

of group $G$ and a morphism $(G,C_3) \rightarrow (G,C_1)$ of distinguished squares of group $G$ such that $S_3 \in D_d(S)$ and that the image of $b_1$ in $T^{n-1}(X'_3)^G$ is 0. Note that $S_3 \in D_d(S)$ since $S_3 \in D_d(S'_1)$ and $S_1 \in D_d(S)$. Then by (i), the image of $a$ in $T^n(S_3)$ is $\partial_{(G,C_3)} 0 = 0$, which proves $(B_d)$ since $S_3 \in D_d(S)$.

For large $d$, $D_d(S)$ contains only isomorphisms, so this shows that $a = 0$. Thus $T^n = 0$ for any $n \geq 0$.

**Theorem 4.4.** Let $P$ be a complete and regular ecd-structure bounded with respect to a density structure $D$, and let $F$ be a $t_P$-sheaf of $\mathbb{Q}$-modules on $\mathcal{S}$. Then

$$H^n_{t_P}(S,F) = 0$$

for any $S \in \text{ob} \mathcal{S}$ and $n > \text{dim}_D S$.

**Proof.** We put $T^n = H^n(-,F)$. Then $T^n$ trivially satisfies the condition (iii) of (4.1), and $T^n$ satisfies the conditions (i) and (ii) of (loc. cit) by (3.13). Since higher cohomology $t_P$-sheaf is locally trivial for the $t_P$-topology, $T^n$ satisfies the condition (iv) of (4.1). Then the conclusion follows from (4.2).

**Theorem 4.5.** Let $K \in D(\text{PSh}(\mathcal{S},\mathbb{Q}))$ be an element, and let $P$ be an ecd-structure on $\mathcal{S}$. Consider the following conditions.

(i) $K$ is $t_P$-local.

(ii) For any distinguished square of group $G$ of the form (1.2.1), the commutative diagram

$$\begin{array}{ccc}
K(S) & \rightarrow & K(S') \\
\downarrow & & \downarrow \\
K(X)^G & \rightarrow & K(X')^G
\end{array}$$

in $D(\text{Mod}_Q)$ is homotopy Cartesian.

If $P$ is regular, then (i) implies (ii). If $P$ is bounded, complete, and regular, then (ii) implies (i).

**Proof.** We put $t = t_P$. Assume that $P$ is regular. If $K$ is $t$-local, then to prove (ii), by (3.11) and (3.12), we only need to show that the diagram

$$\begin{array}{ccc}
R\text{Hom}_{D(\text{PSh}(\mathcal{S},\mathbb{Q}))}(Q_t(S),K) & \rightarrow & R\text{Hom}_{D(\text{PSh}(\mathcal{S},\mathbb{Q}))}(Q_t(S'),K) \\
\downarrow & & \downarrow \\
R\text{Hom}_{D(\text{PSh}(\mathcal{S},\mathbb{Q}))}(Q_t(X)^G,K) & \rightarrow & R\text{Hom}_{D(\text{PSh}(\mathcal{S},\mathbb{Q}))}(Q_t(X')^G,K)
\end{array}$$
in $\text{D}(\text{Mod}_Q)$ is homotopy Cartesian. It follows from (3.8).

Assume that $P$ is bounded, complete, and regular. If $K$ satisfies the condition (ii), let $K'$ be a cone of the morphism

$$K \xrightarrow{a_{t+}a_!} K.$$ 

Since both $K$ and $a_{t+}a_!K$ satisfy the condition (ii), $K'$ also satisfies the condition (ii). We put $T^n(S) = H^n(K(S))$ for $S \in \text{ob} \mathcal{S}$. Since $K'$ is acyclic for the topology $t$, $T^n$ satisfies the conditions (iii)' and (iv)' of (4.1). Thus by (4.1), $K'$ is isomorphic to 0 in $\text{D}(\text{PSh}(\mathcal{S},Q))$. Then $K$ is isomorphic to $a_{t+}a_!K$, so $K$ is $t$-local.

4.6. Let $P$ be a bounded and complete but not necessarily a regular $\mathcal{S}$-structure on $S$, and assume that for any $P$-distinguished square of the form (1.2.1), $g : S' \to S$ is a monomorphism. Consider the $\mathcal{S}$-structure $P'$ that consists of distinguished squares of trivial groups

$$G \times X' \xrightarrow{u'} G \times X \xrightarrow{u} X' \times_{\text{ob} \mathcal{S}} X' \xrightarrow{u} X \times_{\text{ob} \mathcal{S}} X$$

in (3.5.1) for any $P$-distinguished squares of group $G$ of the form (1.2.1). Note that $P \cup P'$ is a bounded, complete, and regular $\mathcal{S}$-structure by (3.3) and (3.5). We have a descent theorem about $P \cup P'$ as follows.

**Theorem 4.7.** Under the notations and hypotheses of (4.6), let $K$ be an object of $\text{D}(\text{PSh}(\mathcal{S},Q))$. Then the following are equivalent.

(i) $K$ is $t_P$-local, and for any $P'$-distinguished square in $\mathcal{S}$ of group $G$ of the form (4.6.1), the commutative diagram

$$
\begin{array}{ccc}
K(X \times_{\text{ob} \mathcal{S}} X) & \longrightarrow & K(X' \times_{\text{ob} \mathcal{S}} X') \\
\downarrow & & \downarrow \\
K(G \times X) & \longrightarrow & K(G \times X')
\end{array}
$$

in $\text{D}(\text{Mod}_Q)$ is homotopy Cartesian.

(ii) $K$ is $t_{P \cup P'}$-local.

**Proof.** The statement (ii)⇒(i) is done by (4.5), so the remaining is the inverse direction. Assume (i). By (4.3), it suffices to show that for any $P$-distinguished square of group $G$ of the form (1.2.1), the commutative diagram

$$
\begin{array}{ccc}
K(S) & \longrightarrow & K(S') \\
\downarrow & & \downarrow \\
K(X)^G & \longrightarrow & K(X')^G
\end{array}
$$

in $\text{D}(\text{Mod}_Q)$ is homotopy Cartesian. Let $(\mathcal{S}, \Delta)$ be the Čech hypercover associated to $X \to S$ where $\Delta$ denotes the simplicial category. Consider the Čech hypercover $(\mathcal{Y}, \mathcal{J})$ associated to the family of morphisms $\{f : \mathcal{S} \to S, g : S' \to S\}$. Then for any object $j$ of $\mathcal{J}$, $\mathcal{Y}_j$ is one of

$$\mathcal{S}_i, S' \times_{\mathcal{S}_i} \mathcal{S}_i, S'$$

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where \( i \in \text{ob } \Delta. \)

Then since \( K \) is \( t_p \)-local, by (3.10), it suffices to show that the commutative diagrams

\[
\begin{array}{c}
K(X \times_S S)^G \rightarrow K(X' \times_S S)^G \\
K(X \times_S S)^G \rightarrow K(X' \times_S S)^G
\end{array}
\]

in \( D(\text{Mod}_Q) \) are homotopy Cartesian for any \( i \) where \( G \) acts on left \( X \) and \( X' \). The second and third ones are homotopy Cartesian since \( g : S' \rightarrow S \) is a monomorphism, so it suffices to show that the first one is homotopy Cartesian. The first one is the upper square in the diagram

\[
\begin{array}{c}
K(X \times_S S)^G \rightarrow K(X' \times_S S)^G \\
K(X \times_S S)^G \rightarrow K(X' \times_S S)^G
\end{array}
\]

in \( D(\text{Mod}_Q) \) where the lower square comes from (4.6.1). The big square is homotopy Cartesian trivially, and the lower square is homotopy Cartesian by the condition (i). Thus the upper square is homotopy Cartesian.

5. Descent theory for derivators

5.1. In this section, we use techniques in \([3, \S 3]\) to transform (4.5) and (4.7) into statements for contravariant pseudofunctors \( \mathcal{T} : \mathcal{J}^{\text{dia}} \rightarrow \text{Tri} \) under some conditions.

**Definition 5.2.** Let \( C \) be an additive category. An object \( C \) of \( C \) is called uniquely divisible if for any integer \( n \geq 2 \), there exists a unique morphism \( f_n : C \rightarrow C \) such that \( nf_n = \text{id} \).

5.3. Let \( \mathcal{T} : \mathcal{J}^{\text{dia}} \rightarrow \text{Tri} \) be a contravariant pseudofunctor. For any 1-morphism \( f \) in \( \mathcal{J}^{\text{dia}} \), we put

\[ f^* := \mathcal{T}(f). \]

We will often assume the following conditions.
(i) For any object $S$ of $\mathcal{S}$, the prederivator
\[ \mathcal{F}(S \times (-)) : \mathcal{S} \to \text{Tri} \]
is a triangulated derivator.

(ii) For any 1-morphism $f$ of $\mathcal{S}$-diagrams, the functor $f^*$ has a right adjoint denoted by $f_*$.

(iii) For any Cartesian diagram
\[ (\mathcal{Y}', J) \xrightarrow{g'} (\mathcal{X}', I) \]
\[ \downarrow^f \quad \downarrow^g \]
\[ (\mathcal{Y}, I) \xrightarrow{g} (\mathcal{X}, I) \]
of $\mathcal{S}$-diagrams where $g$ is reduced and $f_j$ is an isomorphism for any $j \in \text{ob} J$, the natural transformation
\[ f^*g_* \xrightarrow{\text{Ex}} g'_*f'^* \]
given by
\[ f^*g_* \xrightarrow{\text{ad}} f^*g'_*f'^* \xrightarrow{\sim} f^*f'_*f'^* \xrightarrow{\text{ad}} g'_*f'^* \]
is an isomorphism.

(iv) For any $\mathcal{S}$-diagram $(\mathcal{X}, I)$, every object of $\mathcal{F}(\mathcal{X}, I)$ is uniquely divisible.

Note that algebraic derivators defined in [1, 2.4.13] satisfy the conditions (i)–(iii).

5.4. Let $\mathcal{F} : \mathcal{S}^{\text{dia}} \to \text{Tri}$ be a contravariant pseudofunctor satisfying the conditions (i)–(iv) in (5.3), and let $X$ be an object of $\mathcal{S}$. For any object $E$ of $\mathcal{F}(X)$, by [3, 3.2.15, 3.2.16] and the conditions (i) and (iv) in (5.3), we have the morphism
\[ \mathcal{R} \text{Hom}(E, -) : \mathcal{F}(X \times (-)) \to \text{D}(\text{PSh}(\mathcal{S}/X, \mathbb{Q})) \]
of derivators (the meaning is that it commutes with $u^*$ for any 1-morphism $u$ in dia where dia denotes the 2-category of small categories) commuting with $u_*$ also.

Consider the $\mathcal{S}$-diagram $(\mathcal{X}, \mathcal{F}/X)$ where for any morphism $U \to V$ in $\mathcal{S}/X$, the morphism $\mathcal{X}_U \to \mathcal{X}_V$ is equal to the morphism $U \to V$. Then consider the 1-morphism
\[ p : (\mathcal{X}, \mathcal{F}/X) \to (X, \mathcal{F}/X) \]
of $\mathcal{S}$-diagrams mapping $\mathcal{X}_U \to \mathcal{X}_V$ to id : $X \to X$ for any morphism $U \to V$ in $\mathcal{S}/X$. Then we denote by
\[ \Phi_E : \mathcal{F}(X) \to \text{D}(\text{PSh}(\mathcal{S}/X, \mathbb{Q})) \]
the functor of triangulated categories given by
\[ \mathcal{R} \text{Hom}(E, p_*f^*(-)) \].

Now, let $g : (\mathcal{Y}, J) \to X$ be a 1-morphism of $\mathcal{S}$-diagrams. Then we have the commutative diagram
\[ \begin{array}{ccc}
(\mathcal{Y}, J) & \xrightarrow{f'} & (X, J) \\
\downarrow^g & \searrow^h & \downarrow^q \\
X & \xrightarrow{f} & (\mathcal{X}, \mathcal{F}/X) \\
\end{array} \]
\[ \xrightarrow{p} (X, \mathcal{F}/X) \]
of \( \mathcal{S} \)-diagrams where \( h \) denotes the 1-morphism mapping \( J \) to the trivial category, and \( q \) and \( q' \) denote the 1-morphisms induced by \( g \). By abuse of notation, we also denote by \( h : J \to e \) and \( q : \mathcal{S}/X \to J \) the corresponding 1-morphisms of diagrams.

Consider the diagram

\[
\begin{array}{ccc}
\mathcal{T}(X) & \xrightarrow{f^*} & \mathcal{T}(\mathcal{Y}, \mathcal{S}/X) \\
\downarrow{g^*} & & \downarrow{q^*} \\
\mathcal{T}(\mathcal{Y}, J) & \xrightarrow{p^*} & \mathcal{T}(X, \mathcal{S}/X) \\
\downarrow{h_*} & & \downarrow{g_*} \\
\mathcal{T}(X) & \xrightarrow{R\text{Hom}(E,-)} & \mathcal{D}(\text{PSh}(\mathcal{S}/X, \Lambda)) \\
\end{array}
\]

of triangulated categories. By the condition (ii) in [3.2.5] and the fact that \( R\text{Hom}(E, -) \) is a morphism of derivators commuting with \( h_* \), the diagram commutes. Then by the commutativity, for any object \( K \) of \( \mathcal{T}(X) \), we have the isomorphism

\[
h_*q^*\Phi_E(K) \sim \to R\text{Hom}(E, g_*g^*K).
\]

We also have that

\[
\Phi_E(K)(\mathcal{Y}, J) = h_*q^*\Phi_E(K).
\]

Thus we have the isomorphism

\[
\Phi_E(K)(\mathcal{Y}, J) \sim \to R\text{Hom}(E, g_*g^*K).
\]

This is functorial in the following sense. Let \( g' : (\mathcal{Y}', J') \to (\mathcal{Y}, J) \) be another 1-morphism of \( \mathcal{S} \)-diagrams. Then the diagram

\[
\begin{array}{ccc}
\Phi_E(K)(\mathcal{Y}, J) & \xrightarrow{\sim} & R\text{Hom}(E, g_*g^*K) \\
\uparrow & & \downarrow{ad} \\
\Phi_E(K)(\mathcal{Y}', J') & \xrightarrow{\sim} & R\text{Hom}(E, g_*g'_*g^*K)
\end{array}
\]

commutes where the left vertical arrow is induced by \( g' \).

**Definition 5.5.** Let \( \mathcal{T} : \mathcal{S}^{\text{dia}} \to \text{Tri} \) be a contravariant pseudofunctor satisfying the conditions (ii) in [3.2.5], let \( S \) be an object of \( \mathcal{S} \), and let \( t \) be a topology on \( \mathcal{S} \). Following [3, 3.2.5], we say that an object \( K \) of \( \mathcal{T}(S) \) satisfies \( t \)-descent if for any morphism \( f : X \to S \) in \( \mathcal{S} \) and any \( t \)-hypercover \( g : (\mathcal{X}, I) \to X \), the morphism

\[
f_*f^*K \xrightarrow{ad} f_*g_*g^*f^*K
\]

in \( \mathcal{T}(S) \) is an isomorphism.

**5.6.** Consider the contravariant pseudofunctor \( \text{D}(\text{PSh}(-, \Lambda)) : \mathcal{S}^{\text{dia}} \to \text{Tri} \) where \( \Lambda \) is a \( \mathbb{Q} \)-algebra. Let \( S \) be an object of \( \mathcal{S} \), let \( t \) be a topology on \( \mathcal{S} \), and let \( K \) be an object of \( \mathcal{T}(S) \). Then \( K \) satisfies \( t \)-descent if and only if for any morphisms \( f : X \to S \) and \( h : Y \to S \) in \( \mathcal{S} \) and any \( t \)-hypercover \( g : (\mathcal{X}, I) \to X \), the homomorphism

\[
(f_*f^*K)(Y) \xrightarrow{ad} (f_*g_*g^*f^*K)(Y)
\]

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in $\text{D}(\text{Mod}_\Lambda)$ is an isomorphism. This is equivalent to the statement that the induced homomorphism
$$K(X \times_S Y) \rightarrow K((\mathcal{X}, I) \times_S Y)$$
in $\text{D}(\text{Mod}_\Lambda)$ is an isomorphism for any $f$, $h$, and $g$. Thus $K$ satisfies $t$-descent if and only if $K$ is $t$-local by (3.10).

5.7. Let $\mathcal{T} : \mathcal{T}^{\text{dia}} \rightarrow \text{Tri}$ be a contravariant pseudofunctor satisfying the conditions (iii) in (5.3), and let $f : X \rightarrow S$ be a morphism in $\mathcal{T}$. Assume that we have a $G$-action on $X$ equivariant over $S$. Then this gives a morphism $u : (\mathcal{X}, e_G) \rightarrow S$ where

(i) $e_G$ denotes the category with single object $*$ and $\text{Hom}(*, *) = G$,
(ii) $\mathcal{X} : e_G \rightarrow \mathcal{S}$ is an $\mathcal{S}$-diagram that gives the $G$-action on $X$.

Let $K$ be an object of $\mathcal{T}(S)$. Then we have defined $(f, f^*K)^G$ in (2.4. Note that by [3, 3.3.31],
$$u^*K \cong (f, f^*K)^G.$$
in \( \text{D}(\text{Mod}_Q) \) is an isomorphism. Now by (5.4) again, this is equivalent to the statement that for any \( E, p, \) and \( q \), the morphism
\[
p_*p^*K \xrightarrow{ad} p_*q_*p^*K
\]
is an isomorphism, which is the statement (ii).

**Theorem 5.9.** Let \( \mathcal{F} : \mathcal{F}^{\text{dia}} \to \text{Tri} \) be a contravariant pseudofunctor satisfying the conditions (i)–(iv) in (5.3), let \( K \) be an object of \( \mathcal{F}(T) \) where \( T \in \text{ob} \mathcal{F} \), and let \( P \) be a bounded and complete ccd-structure on \( \mathcal{F} \). Consider the bounded, complete, and regular ccd-structure \( P' \) on \( \mathcal{F} \) defined in (4.6). Then the following are equivalent.

(i) \( K \) satisfies \( t_P \)-descent, and for any distinguished square in \( \mathcal{F}/T \) of group \( G \) of the form (1.2.1), the commutative diagram
\[
\begin{array}{ccc}
q_*q^*K & \xrightarrow{ad} & q_*v_*v^*q^*K \\
\downarrow{ad} & & \downarrow{ad} \\
qu_*u_*u^*q^*K & \xrightarrow{ad} & qu_*w_*w^*q^*K^G
\end{array}
\]
in \( \mathcal{F}(T) \) is homotopy Cartesian where \( q : X \times_S X \to T \) is the structural morphism, \( u \) and \( v \) are the morphisms given in (4.6), and \( w = uv' \) where \( v' \) is the morphism given in (loc. cit).

1. \( K \) satisfies \( t_{P',P'} \)-descent.

**Proof.** We can transform the statement to that of (1.7) as in the proof of (5.8). \( \square \)

**Example 5.10.** Let \( \Lambda \) be a \( Q \)-algebra, and let \( t \) be a topology on \( \mathcal{F} \). Assume that \( \mathcal{F} \) is the category of noetherian schemes. Consider the contravariant pseudofunctor \( D_{\mathcal{A}}(X, \Lambda) : \mathcal{F}^{\text{dia}} \to \text{Tri} \) in [3, 5.3.31] (it is written as \( D_{\mathcal{A}^1,t}(-, \Lambda) \) in [loc. cit]). Then it satisfies the conditions (i)–(iii) in (5.3) by [1, 4.5.30]. It also satisfies (iv) in (5.3) since \( \Lambda \) is a \( Q \)-algebra. Thus we can apply (5.8) and (5.9) to this case.

**6. Topologies on the category of schemes**

6.1. In this section, we will introduce several ccd-structures that will produce topoi like étale topos, qfh-topos, and h-topos. Throughout this section, \( \mathcal{F} \) is the category of finite dimensional noetherian schemes. We denote by \( \mathcal{F}^{\text{ex}} \) the category of finite dimensional quasi-excellent noetherian schemes.

**Definition 6.2.** Let \( G \) be a finite group. Recall from [3, 3.3.14] that a morphism \( f : X \to S \) of schemes is called a **pseudo-Galois** cover of group \( G \) if it is finite surjective and it has a factorization
\[
X \xrightarrow{g} Y \xrightarrow{h} S
\]
such that \( g \) is a Galois cover of group \( G \) and \( h \) is radical.

**Definition 6.3.** Consider a Cartesian diagram
\[
\begin{array}{ccc}
C & \xrightarrow{g'} & X \\
\downarrow{f'} & & \downarrow{f} \\
S' & \xrightarrow{g} & S
\end{array}
\]
Proof. The étale topology is finer than the Nisnevich topology and Galois topology, so we only need to prove the other inclusion. Let \( f : X \to S \) be an étale cover. Then by \([8] 3.3.26\), \( f \) has

1. Following \([8]\), \( C \) is called an additive distinguished square (of trivial \( G \)) if \( X' = \emptyset \) and \( S = X \amalg S' \).
2. Following \([8]\), \( C \) is called a closed distinguished square (of trivial \( G \)) if \( f \) and \( g \) are closed immersions and \( S = f(X) \cup g(S') \).
3. Following \([8]\), \( C \) is called a Zariski distinguished square (of trivial \( G \)) if \( f \) and \( g \) are open immersions and \( S = f(X) \cup g(S') \).
4. Following \([8]\), \( C \) is called a Nisnevich distinguished square (of trivial \( G \)) if \( f \) is étale, \( g \) is an open immersion, and the morphism \( f^{-1}(S - g(S')) \to S - g(S') \) is an isomorphism. Here, \( S - g(S') \) is considered with the reduced scheme structure.
5. Following \([8]\), \( C \) is called a proper cdh-distinguished square (of trivial \( G \)) if \( f \) is proper, \( g \) is a closed immersion, and the morphism \( f^{-1}(S - g(S')) \to S - g(S') \) is an isomorphism.
6. Following \([8]\), \( C \) is called a cdh-distinguished square (of trivial \( G \)) if it is a Nisnevich or cdh-distinguished square.
7. \( C \) is called a Galois distinguished square of group \( G \) if \( X' = S' = \emptyset \), \( f \) is a Galois cover of group \( G \).
8. Following \([8] 3.3.15\], \( C \) is called a pseudo-Galois qfh-distinguished square of group \( G \) if \( f \) is finite and surjective, \( g \) is a nowhere dense closed immersion, and the induced morphism

\[ f^{-1}(S - g(S')) \to S - g(S') \]

is a pseudo-Galois cover of group \( G \).

Then we obtain the additive, closed, Zariski, Nisnevich, cdh, Galois, and pseudo-Galois qfh ecd-structures and topologies using the definitions in \([1,2]\).

**Definition 6.4.** Recall from \([8] 2.1\] that the eh-topology is the topology generated by the cdh-topology and the étale topology.

6.5. In \([8]\), the notion of the standard density structure \( D_d \) is defined on \( \mathcal{F} \). Let \( X \) be an object of \( \mathcal{F} \). Then \( \dim X = \dim_{D_d} X \). Here, we also recall several lemmas in \([8]\).

1. If \( U, V \in D_d(X) \), then \( U \cap V, \in D_d(X) \).
2. If \( U \in D_d(X) \) and \( V \) is an open subscheme of \( X \), then \( U \cap V \in D_d(V) \).
3. Let \( f : X \to Y \) be a morphism of finite type in \( \mathcal{F} \), and assume that there exists an open subscheme \( U \) of \( Y \) such that \( f^{-1}(U) \) is dense in \( X \) and \( f^{-1}(U) \to U \) has fibers of dimension zero. Then for any \( d \geq 0 \) and \( V \in D_d(X) \), there exists \( W \in D_d(Y) \) such that \( f^{-1}(W) \subset V \).

6.6. From \((6.7)\) to \((6.11)\), we relate some topoi obtained by ecd-structures with étale topos, qfh-topos, or h-topos.

**Proposition 6.7.** Let \( P \) be the union of the Nisnevich and Galois ecd-structures. Then the \( t_P \)-topology on \( \mathcal{F} \) is equal to the étale topology on \( \mathcal{F} \).

**Proof.** The étale topology is finer than the Nisnevich topology and Galois topology, so we only need to prove the other inclusion. Let \( f : X \to S \) be an étale cover. Then by \([8] 3.3.26\], \( f \) has
a refinement that has a factorization $X \xrightarrow{\varphi} Y \xrightarrow{h} S$ such that $g$ is a finite étale cover and $h$ is a Nisnevich cover. Refining $g$ further, we may assume that $g$ is a Galois cover. Then $f$ becomes a $t_p$-cover.

**Proposition 6.8.** Let $P$ be the union of the cdh and Galois ecd-structures. Then the $t_p$-topology on $\mathcal{S}$ is equal to the eh-topology on $\mathcal{S}$.

*Proof.* This is a direct consequence of (6.7) since the cdh ecd-structure contains the Nisnevich ecd-structure.

**Proposition 6.9.** Let $P$ be the union of the closed and pseudo-Galois qfh ecd-structures. Then the $t_p$-topology on $\mathcal{S}^{\text{ex}}$ is generated by the families of finite morphisms in $\mathcal{S}^{\text{ex}}$ that are jointly surjective.

*Proof.* In every $t_p$-covering, morphisms are finite and jointly surjective. Thus we only need to show that every family of finite morphisms that are jointly surjective is a $t_p$-covering. Let $f : X \to S$ be a finite surjective morphism. Since $P$ contains the closed ecd-structure, we only need to show that $f$ is a $t_p$-cover.

(I) Reduction to the case when $X$ is reduced and $S$ is integral and normal. Let $\{S_i\}_{i \in I}$ be the set of irreducible components of $S$. Then the morphism $\amalg S_i, \text{red} \to S$ is a $t_p$-cover since $P$ contains the closed ecd-structure, so to show that $f$ is a $t_p$-cover, it suffices to show that its pullback to each $S_i, \text{red}$ is a $t_p$-cover. Thus we reduce to the case when $S$ is integral.

To show that $f$ is a $t_p$-cover, it suffices to show that the composition $f : X, \text{red} \to S$ is a $t_p$-cover.

Thus we reduce to the case when $X$ is integral.

Let $g : S' \to S$ be the normalization of $S$, and let $h : Z \to S$ be a nowhere dense closed immersion such that $g$ is an isomorphism on $S - h(Z)$. Then by definition, the induced morphism $Z \amalg S' \to S$ is a $t_p$-cover. Thus to show that $f$ is a $t_p$-cover, it suffices to show that the projections $Z \times_S X \to Z$, $S' \times_S X \to S'$ are $t_p$-covers. By noetherian induction, the first one is a $t_p$-cover, so it suffices to show that the second one is a $t_p$-cover. Thus we reduce to the case when $S$ is normal.

(II) Reduction to the case when $X$ is integral. Let $\{X_i\}_{i \in I}$ be the irreducible component of $X$, and consider its images $\{Y_i\}_{i \in I}$ in $Y$. Then the induced morphism $\amalg Y_i \to Y$ is a $t_p$-cover by definition, so to show that $f$ is a $t_p$-cover, it suffices to show that for each $j \in I$, the family $\{X_i \times_Y Y_j \to Y_j\}_{i \in I}$ is a $t_p$-cover. To show this, it suffices to show that $X_i \to Y_i$ is a $t_p$-cover. Thus we reduce to the case when $X$ is integral.

(III) Final step of the proof. Let $K$ (resp. $L$) be the field of functions of $S$ (resp. $X$), and let $X'$ be the normalization of $X$ in $L'$ where $L'$ is a Galois extension of the inseparable closure of $K$ in $L$. Then it suffices to show that $X' \to S$ is a $t_p$-cover. Then by [3, 3.3.16], there is a nowhere dense closed subscheme $Z$ of $S$ such that $Z \amalg X' \to S$ is a $t_p$-cover. Thus it suffices to show that the projections $X' \times_S Z \to Z$ and $X' \times_S X' \to X'$ are $t_p$-cover. The first one is a $t_p$-cover by noetherian induction, so it suffices to show that the second one is a $t_p$-cover. In the proof [3, 3.3.19], it is shown that $X' \times_S X' = \bigcup_{g \in G} X'$ in our case. Thus $X' \times_S X' \to X'$ is a $t_p$-cover.

**Proposition 6.10.** Let $P$ be the union of closed, pseudo-Galois qfh, and Nisnevich ecd-structures. Then the $t_p$-topology on $\mathcal{S}^{\text{ex}}$ is equal to the qfh-topology on $\mathcal{S}^{\text{ex}}$. 20
Proof. It follows from [3, 3.3.27] and (6.1).

**Proposition 6.11.** Let $S$ be an object of $\mathcal{S}^e_x$, and let $t$ be the topology on $\mathcal{S}^e_x$ generated by the cdh-topology and qfh-topology. Then a presheaf $F$ of sets on $\mathcal{S}^e_x/S$ is a $t$-sheaf if and only if $F$ is a $h$-sheaf.

**Proof.** Since $h$-topology is finer than cdh-topology and qfh-topology, the statement is a necessary condition. For sufficiency, by [6, 3.1.9], we only need to show the sheaf condition for $f : X \to S$ that can be factorized into $p : X \to Y$ and $q : Y \to S$ such that $q$ is proper, birational, and surjective.

Choose a nowhere dense closed immersion $g : S' \to S$ such that $q$ is an isomorphism on $S - g(S')$. Then the induced morphism $X \coprod_s S' \to S$ is a $t$-cover. Consider the sequence

$$F(S) \to F(X) \rightrightarrows F(X \times_S S')$$

of sets, and let $a \in F(X)$ be an element whose images in $F(X \times_S S')$ are equal. Let $b$ be the image of $a$ in $F(X \times_S S')$. Then the images of $b$ in $F(X \times_S S' \times_S S')$ are equal. Thus there is an element $c \in F(S')$ whose image in $F(X \times_S S')$ is $b$ since by noetherian induction, the sheaf condition is satisfied for the projection $X \times_S S' \to S'$. The images of $c$ in $F(S' \times_S S')$ are also equal since $S' \to S$ is a closed immersion, so the images of $(b, c)$ in

$$F((X \coprod_s S') \times_s (X \coprod_s S'))$$

are equal. We can use the sheaf condition for $X \coprod_s S' \to S$ since it is a $t$-cover, and then we see that $b$ is in the image of $F(S) \to F(X)$.

**Remark 6.12.** It seems that $h$-topology does not come from an ecd-structure even though $h$-topos comes from an ecd-structure.

6.13. From (6.14) to (6.19), we prove that some ecd-structures are bounded, complete, or regular.

**Proposition 6.14.** The additive, cdh, closed, Nisnevich, and Zariski ecd-structures are complete, regular, and bounded with respect to the standard density structure.

**Proof.** It is [8, 2.2].

**Proposition 6.15.** The Galois and pseudo-Galois qfh ecd-structure are complete.

**Proof.** It follows from (5.2).

**Proposition 6.16.** The pseudo-Galois qfh ecd-structure is bounded with respect to the standard density structure.

**Proof.** Consider a pseudo-Galois qfh-distinguished square $(G, C)$ of the form (1,2,1). Let $Y$ be the scheme theoretic closure of the open subscheme $f^{-1}(S - g(S'))$ in $X$. Refining $C$, we can replace $X$ by $\cap_{g \in G} Y$, so we may assume that $f^{-1}(S - g(S'))$ is dense in $X$.

Assume that we have $X_0 \in D_{i+1}(X)$, $S'_0 \in D_{i+1}(S')$, and $X'_0 \in D_i(X')$. Then by (6.5, 2)), we can replace $X_0$ (resp. $X'_0$) by $\cap_{g \in G}(X_0)$ (resp. $\cap_{g \in G}(X'_0)$), so we may assume that the open immersions $X_0 \to X$ and $X'_0 \to X'$ are $G$-equivariant over $S$ and $S'$ respectively.
By (6.5(3)), we can find $S_0 \in D_{i+1}(S)$ and $S_1 \in D_{i+1}(S)$ such that $f^{-1}(S_0) \subset X_0$ and $g^{-1}(S_1) \subset S'_0$. Then by (6.5(1)), $S_0 \cap S_1 \in D_{i+1}(S)$. Thus by (6.5(2)), replacing $S$ by $S_0 \cap S_1$, we may assume that $X = X_0$ and $S' = S'_0$. We put

$$S_2 = S - fg'(X' - X'_0).$$

Consider the pullback square of $C$ along $S_2 \to S$. To show that this satisfies the condition, the remaining is to show that $S_2 \in D_{i+1}(S)$.

Applying (6.5(3)) to $f$, it suffices to show that $(X - g'(X' - X'_0)) \in D_{i+1}(X)$. Since $X'_0 \in D_i(X_0)$ and $g'$ is a closed immersion, it suffices to show that $X - g'(X')$ is dense in $X$. This follows from the assumption that $X - g'(X') = f^{-1}(S - g(S'))$ is dense in $X$.

**Proposition 6.17.** The union of the additive and Galois ecd-structures is regular.

**Proof.** Let $f : X \to S$ be a Galois cover of group $G$. Then $X \times_S X \cong G \times X$. Thus the conclusion follows from this and (3.5).

**Proposition 6.18.** The union of the closed and pseudo-Galois qfh ecd-structures is regular.

**Proof.** The closed ecd-structure is regular by (6.14), so it suffices to show that a pseudo-Galois qfh-distinguished square of the form (1.2.1) satisfies the condition in (3.5). By [3, 3.3.18], $X \times_S X$ has the closed cover

$$(S' \times_S S') \cup \bigcup_{g \in G} X.$$  

The conclusion follows from this and (3.5).

**Theorem 6.19.** The union of any combination of the additive, closed, Zariski, Nisnevich, cdh, additive + Galois, closed + pseudo-Galois qfh ecd-structures is complete, regular, and bounded with respect to the standard density structure.

**Proof.** It follows from (3.3), (6.14), (6.17), and (6.18).

**6.20.** In (6.21) and (6.22), we collect several our works.

**Theorem 6.21.** Let $t$ be the étale topology on $\mathcal{S}$ (resp. one of the eh-topology, qfh-topology, and $h$-topology on $\mathcal{S}^{\text{ex}}$), and let $F$ be a $t$-sheaf of $\mathbb{Q}$-modules. Then

$$H^n_t(S, F) = 0$$

for any $S \in \text{ob} \mathcal{S}$ (resp. $S \in \text{ob} \mathcal{S}^{\text{ex}}$) and $n > \dim S$.

**Proof.** By (6.19), (6.7), (6.8), (6.10), and (6.11), the $t$-topos is equivalent to the $t_P$-topos for some complete and regular ecd-structure $P$ bounded with respect to the standard density structure. Then the conclusion follows from (4.2).

**Theorem 6.22.** Let $\mathcal{T} : (\mathcal{S}^{\text{ex}})^{\text{dia}} \to \text{Tri}$ be a contravariant pseudofunctor satisfying the conditions (i)–(iv) in (5.5), let $K$ be an object of $\mathcal{T}(T)$ where $T \in \text{ob} \mathcal{S}^{\text{ex}}$. Then the following are equivalent.
(i) For any Nisnevich or Galois (resp. cdh or Galois, resp. Nisnevich or pseudo-Galois qfh, resp. cdh or pseudo-Galois qfh) distinguished squares of group $G$ in $\mathcal{S}^{ex}/U$ of the form (1.2.1), the diagram

$$
p_*p^*K \xrightarrow{ad} p_*g_*g^*p^*K \xrightarrow{ad} (p_*f_*f^*p^*K)^G \xrightarrow{ad} (p_*h_*h^*p^*K)^G
$$

is homotopy Cartesian where $p : S \to T$ is the structural morphism and $h = fg'$.

(ii) $K$ satisfies étale (resp. eh, resp. qfh, resp. h) descent.

Proof. It follows from (6.19), (6.7), (6.8), (6.10), (6.11), and (5.8).

Remark 6.23. The above statement still holds if we replace $\mathcal{S}^{ex}$ by $\mathcal{S}$ in the étale case. In (6.21) and (6.22), the results for the étale topology, qfh-topology, and $h$-topology are already known in [6, 3.4.7], [6, 3.4.8], [6, 3.4.6], [3, 3.3.32], and [3, 3.3.37]. We unified the various proofs.

7. Topologies on the category of fs log schemes

7.1. Throughout this section, $\mathcal{S}$ is a category of finite dimensional noetherian fs log schemes.

Definition 7.2. Consider a Cartesian diagram

$$
C = \begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow f' & & \downarrow f \\
S' & \xrightarrow{g} & S
\end{array}
$$

in $\mathcal{S}$ and a group $G$ acting on $X$ over $S$. We have several ec-d-structures as follows.

1. $C$ is called an additive distinguished square (of trivial $G$) if $X' = \emptyset$ and $S = X \amalg S'$.

2. $C$ is called a strict closed distinguished square (of trivial $G$) if $f$ and $g$ are strict closed immersions and $S = f(X) \cup g(S')$.

3. $C$ is called a Zariski distinguished square (of trivial $G$) if $f$ and $g$ are open immersions and $S = f(X) \cup g(S')$.

4. $C$ is called a strict Nisnevich distinguished square (of trivial $G$) if $f$ is strict étale, $g$ is an open immersion, and the morphism $f^{-1}(S - g(S')) \to S - g(S')$ is an isomorphism. Here, $S - g(S')$ is considered with the reduced scheme structure.

5. $C$ is called a Galois distinguished square of group $G$ if $X' = S' = \emptyset$, $f$ is Galois, and $G$ is the Galois group of $f$.

6. $C$ is called a dividing distinguished square (of trivial $G$) if $X' = S' = \emptyset$ and $f$ is a surjective proper log étale monomorphism.

7. $C$ is called a piercing distinguished square (of trivial $G$) if $C$ is a pullback of the Cartesian diagram

$$
\begin{array}{ccc}
\text{pt}_N & \longrightarrow & \mathbb{A}_N \\
\downarrow & & \downarrow \\
\text{Spec Z} & \longrightarrow & \mathbb{A}^1
\end{array}
$$

(7.2.1)
of $S$-schemes where the lower horizontal arrow is the 0-section and the right vertical arrow is the morphism removing the log structure.

(8) $C$ is called a **quasi-piercing** distinguished square (of trivial $G$) if $C$ is a strict closed distinguished square $C$ is a piercing distinguished square, or $C$ is a pullback of the Cartesian diagram

$$
\begin{array}{ccc}
\text{pt}_N & \to & \mathbf{A}_N \\
\downarrow & & \downarrow \\
\text{pt}_{N^2} & \to & \mathbf{A}_N \times_{\mathbf{A}^1} \mathbf{A}_N
\end{array}
$$

(7.2.2)

where the lower horizontal arrow is the 0-section and the right vertical arrow is the diagonal morphism of $\mathbf{A}_N \to \mathbf{A}^1$ removing the log structure.

(9) For $n \in \mathbb{N}^+$, let $\mu_n$ be an $n$-th root of unity. Then $C$ is called a **winding** distinguished square of group $G$ if $X' = S' = \emptyset$, $f$ is a pullback of the composition

$$
\mathbf{A}_Q \times \text{Spec } \mathbb{Z}[\mu_n] \to \mathbf{A}_Q \xrightarrow{\theta} \mathbf{A}_P
$$

where the first arrow is the projection, $n \in \mathbb{N}^+$, and $\theta : P \to Q$ is a Kummer homomorphism of fs monoids such that the Galois group of $\mathbf{A}_Q \times \text{Spec } \mathbb{Q}[\mu_n]$ over $\mathbf{A}_P \times \text{Spec } \mathbb{Q}$ exists, and $G$ is the Galois group.

Then we obtain the additive, strict closed, Zariski, strict Nisnevich, dividing, piercing, quasi-piercing, Galois, and winding ecd-structures and topologies using the definition in (1.2).

**Definition 7.3.** If $D$ denotes the standard density structure, for any object $X$, we denote by $D_d(X)$ the family $D_d(X)$. It is again called the **standard density structure**.

**Proposition 7.4.** The additive, strict closed, Zariski, and strict Nisnevich, and piercing ecd-structures are complete, regular, and bounded with respect to the standard density structure.

*Proof.* The proof is identical to that of (6.14).

**Proposition 7.5.** The piercing, quasi-piercing, Galois, and winding ecd-structures are complete.

*Proof.* It follows from [8, 2.5].

**Proposition 7.6.** The quasi-piercing ecd-structure is regular.

*Proof.* Consider a commutative diagram

$$
\begin{array}{ccc}
C = & \xrightarrow{g'} & X \\
\xrightarrow{f'} & \downarrow f & \downarrow f \\
S' & \xrightarrow{g} & S
\end{array}
$$

of $S$-schemes. The diagram

$$
\begin{array}{ccc}
\overline{X}' & \xrightarrow{\overline{g}'} & \overline{X} \\
\downarrow \overline{f}' & & \downarrow \overline{f} \\
\overline{S}' & \xrightarrow{\overline{g}} & \overline{S}
\end{array}
$$

is a closed distinguished square, so we are done by (6.14).
Proposition 7.7. The quasi-piercing ecd-structures is bounded with respect to the standard density structure.

Proof. Consider a quasi-piercing distinguished square

$$C = \begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow f' & & \downarrow f \\
S' & \xrightarrow{g} & S
\end{array}$$

of $\mathcal{S}$-schemes. As in the proof of [8, 2.11], if we replace $X$ by the scheme-theoretic closure of the open subscheme $f^{-1}(S' - S)$, we get another quasi-piercing distinguished square which is a refinement of the original one. Then the same proof of [8, 2.12] can be applied to our situation. □

Proposition 7.8. The union of the strict closed and winding ecd-structures is regular.

Proof. Let $f : X \to S$ be a winding cover, which is a pullback of the composition

$$A_Q \times \text{Spec } Z[\mu_n] \to A_Q \xrightarrow{\theta} A_P$$

where the first arrow is the projection, $n \in \mathbb{N}^+$, and $\theta : P \to Q$ is a Kummer homomorphism of fs monoids such that the Galois group $G$ of $A_Q \times \text{Spec } Q[\mu_n]$ over $A_P \times \text{Spec } Q$ exists. We denote by

$$\varphi(g) : Q \oplus \mathbb{Z}/(n) \to \varphi : Q \oplus \mathbb{Z}/(n)$$

the homomorphism induced by $g$. We have

$$X \times_S X = \bigcup_{g \in G} X_g$$

where $X_g$ denotes the graph of the automorphism $X \to X$ induced by $g \in G$.

We will show that $X_g$ is a closed subscheme of $X \times_S X$. We put $Q' = Q \oplus \mathbb{Z}/(n)$. It suffices to show that for any $g \in G$, the homomorphism

$$Q' \oplus_P Q' \to Q', \quad (a, b) \mapsto a + \varphi(g)(b)$$

is strict. Composing with the isomorphism

$$Q' \oplus_P Q' \to Q' \oplus Q', \quad (a, b) \mapsto (a, \varphi(g^{-1})(b)),$$

it suffices to show that the summation homomorphism

$$Q' \oplus_P Q' \to Q'$$

is strict. It follows from [7.9] below. Thus $X_g$ is a closed subscheme of $X \times_S X$. Now we can use [6.9] to get the conclusion. □

Lemma 7.9. Let $\theta : P \to Q$ be a Kummer homomorphism of fs monoids. Then the summation homomorphism $\eta : Q \oplus_P Q \to Q$ is strict.
Proof. The homomorphism \( \eta : Q \oplus_P Q \to Q \) is surjective, so the remaining is to show that \( \eta \) is injective. Let \( q \in Q \) be an element. Since \( \theta \) is Kummer, we can choose \( n \in \mathbb{N}^+ \) such that \( nq \subset \theta(P) \) for any \( q \in Q \). Then \( n(q, -q) = (nq, 0) + (0, -nq) = 0 \) because \( nq \in \theta(P) \). Thus \( (q, -q) \in (Q \oplus_P Q)^* \) since \( Q \oplus_P Q \) is saturated. Let \( Q' \) be the submonoid of \( Q \oplus_P Q \) generated by elements of the form \( (q, -q) \) for \( q \in Q \). Then \( (q, -q) \) belongs to \( (Q \oplus_P Q)^* \) since \( Q \oplus_P Q \) is saturated. Let \( Q' \) be the submonoid of \( Q \oplus_P Q \) generated by elements of the form \( (q, -q) \) for \( q \in Q \). Then \( (q, -q) \) belongs to \( (Q \oplus_P Q)^* \) since \( Q \oplus_P Q \) is saturated. Let \( Q' \) be the submonoid of \( Q \oplus_P Q \) generated by elements of the form \( (q, -q) \) for \( q \in Q \). Then \( (q, -q) \) belongs to \( (Q \oplus_P Q)^* \) since \( Q \oplus_P Q \) is saturated. Let \( Q' \) be the submonoid of \( Q \oplus_P Q \) generated by elements of the form \( (q, -q) \) for \( q \in Q \). Then \( (q, -q) \) belongs to \( (Q \oplus_P Q)^* \) since \( Q \oplus_P Q \) is saturated.

Proposition 7.10. The Galois and winding ecd-structures are bounded with respect to the standard density structure.

Proof. If follows from [8, 2.9].

Theorem 7.11. The union of any combination of the additive, strict closed, Zariski, strict Nisnevich, quasi-piercing, additive + Galois, and strict closed + winding ecd-structures is complete, regular, and bounded with respect to the standard density structure.

Proof. It follows from (6.19), (7.4), (7.5), (7.6), (7.7), (7.8), (7.10), and (3.3).

Definition 7.12. The topology on \( S \) generated by the strict étale, piercing, and winding topologies is called the \( pw \)-topology, and the topology on \( S \) generated by the strict étale, quasi-piercing, and winding topologies is called \( qw \)-topology.

Theorem 7.13. Let \( \mathcal{F} : S^{\text{dia}} \to \text{Tri} \) be a contravariant pseudofunctor satisfying the conditions (i)–(iv) in (5.3), and let \( K \) be an object of \( \mathcal{F}(T) \) satisfying the dividing descent and strict closed descent where \( T \) is an object of \( \mathcal{F} \). Then \( K \) satisfies the piercing descent if and only if \( K \) satisfies the quasi-piercing descent.

Proof. Since the quasi-piercing topology is finer than the piercing topology, the sufficient condition holds. Let us prove the necessary condition. Consider a piercing distinguished square

\[
\begin{array}{ccc}
C &=& X' \xrightarrow{g'} X \\
S' & \xrightarrow{g} & S
\end{array}
\]

in \( \mathcal{F}/T \), and consider the Cartesian diagram

\[
\begin{array}{ccc}
C' &=& X' \xrightarrow{u'} X \\
X' \times_S X' & \xrightarrow{v} & X \times_S X
\end{array}
\]

in \( \mathcal{F}/T \) where \( u \) is the diagonal morphism and \( v = g' \times \eta \). Let \( q : X \times_S X \to T \) be the structural morphism, and we put \( w = uv' \). Then by (5.9), it suffices to show that the commutative diagram

\[
\begin{array}{ccc}
q_*q^*K & \xrightarrow{ad} & q_*v_*q^*K \\
\downarrow{ad} & & \downarrow{ad} \\
q_*u_*w_*q^*K & \xrightarrow{ad} & q_*w_*w^*q^*K
\end{array}
\]
in \( \mathcal{F}(T) \) is homotopypp Cartesian.

The diagram \( C' \) is a pullback of \((7.2.2)\), which has the decomposition

\[
\begin{array}{ccc}
\text{pt}_{N^2} & \to & A_N \\
\downarrow & & \downarrow \text{id} \\
V' & \to & V \\
\downarrow & & \downarrow \alpha \\
\text{pt}_{N^2} & \to & A_N \times _{A^1} A_N \\
\end{array}
\]

(7.13.2)

\[
\begin{array}{ccc}
\to & \to & \to \\
\downarrow & \downarrow & \downarrow \\
\to & \to & \to \\
\end{array}
\]

\[
\begin{array}{ccc}
\to & \to & \to \\
\downarrow & \downarrow & \downarrow \\
\to & \to & \to \\
\end{array}
\]

where

(i) each square is Cartesian,

(ii) \( t \) is the zero section,

(iii) \( t' \) is the morphism induced by the morphism \( A^1 \to Z \),

(iv) \( W \) is the fs log scheme that is the gluing of

\[
A_{N^2 \oplus N(x^{-1}y)}; \quad A_{N^2 \oplus N(y^{-1}x)}
\]

along \( A_{N^2 \oplus Z(x^{-1}y)} \).

(v) \( \alpha \) is the morphism \( A_N \to V \) given by the homomorphisms

\[
N^x \oplus N(x^{-1}y) \to N, \quad N^y \oplus N(y^{-1}x) \to N
\]

mapping \( x \) to 1 and \( y \) to 1,

(vi) \( \beta \) is the proper log étale monomorphism \( V \to A_{N^2 \oplus N^2} \) given by the homomorphism

\[
N^x \oplus N^y \to N^x \oplus N(x^{-1}y), \quad N^x \oplus N^y \to N^y \oplus N(y^{-1}x)
\]

mapping \( x \) to \( x \) and \( y \) to \( y \).

Then \( C' \) has the decomposition

\[
\begin{array}{ccc}
X' & \to & X \\
\downarrow \text{b} & & \downarrow \text{b} \\
Y' & \to & Y \\
\downarrow \text{a} & & \downarrow \text{a} \\
X' \times_S Y' & \to & X \times_S X
\end{array}
\]

(7.13.3)

that is the pullback of the left part of \((7.13.2)\). The upper square is a strict closed distinguished square and \( a \) and \( a' \) are dividing covers. Since \( K \) satisfies the dividing descent, the adjunctions

\[
id \to a_s a^* \quad \text{and} \quad \text{id} \to a'_s a'^* \]

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are isomorphisms. Thus to show that (7.13.1) is homotopy Cartesian, it suffices to show that the diagram

\[
\begin{array}{c}
(qa)_*(qa)^* K \\
\downarrow^{ad} \quad \downarrow^{ad}
\end{array}
\begin{array}{c}
(qa)_* v''v'^*(qa)^* K \\
\end{array}
\]

in \( \mathcal{T}(T) \) is homotopy Cartesian where \( w' = bv' \). It follows from applying (5.8) to strict closed ecd-structure.

**Theorem 7.14.** Let \( \mathcal{T} : \mathcal{D} \to \text{Tri} \) be a contravariant pseudofunctor satisfying the conditions (i)–(iv) in (5.3), and let \( K \) be an object of \( \mathcal{T}(T) \) satisfying the dividing descent and strict closed descent where \( T \) is an object of \( \mathcal{T} \). Then \( K \) satisfies the pw-descent if and only if \( K \) satisfies the qw-descent.

**Proof.** The sufficient condition holds since the qw-topology is finer than the pw-topology, so let us prove the necessary condition. Assume that \( K \) satisfies the pw-descent. Then \( K \) satisfies the piercing descent, so by (7.13), \( K \) satisfies the quasi-piercing descent. Now by (5.8) and (7.14), \( K \) satisfies the qw-descent. \( \square \)

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