Topological phase transition to Abelian anyon phases due to off-diagonal exchange interaction in the Kitaev spin liquid state

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(Dated: March 13, 2020)

We investigate how the Kitaev spin liquid state described by Majorana fermions coupled with $Z_2$ gauge fields is affected by non-Kitaev interactions which exist in real candidate materials of the Kitaev magnet. It is found that the off-diagonal exchange interaction referred to as the $\Gamma'$ term dramatically changes the Majorana band structure in the case of the antiferromagnetic Kitaev interaction, and gives rise to a topological phase transition from a non-Abelian topological phase with the Chern number equal to $\pm 1$ to an Abelian phase with the Chern number equal to $\pm 2$, in which the $Z_2$ vortices behave as Abelian anyons. On the other hand, other non-Kitaev interactions such as the Heisenberg interaction and the $\Gamma$ term, only affect the bandwidth of the Majorana band as long as the spin liquid state is not destabilized.

PACS numbers:

I. INTRODUCTION

Recently, the realization of the Kitaev spin liquid state in real magnetic materials has been extensively explored both theoretically and experimentally. The low-energy excitations in the Kitaev spin liquid state are itinerant Majorana fermions and $Z_2$ vortices (visons), which interact with each other. In the case with an applied magnetic field, the Majorana fermions acquire a mass gap, and the system is changed into a chiral spin liquid state with chiral Majorana edge states, which is also a non-Abelian topological phase. In this phase, visons with zero-energy Majorana bound states behave as non-Abelian anyons which have implications for the application to quantum computation. The Kitaev spin liquid is exactly realized in the Kitaev honeycomb-lattice model which is characterized by the Ising-type interactions between nearest-neighbor spins with three different easy-axis directions depending on three types of bonds on the honeycomb lattice. In real candidate materials of the Kitaev magnet such as $\alpha$-RuCl$_3$ and Na$_2$IrO$_3$, as clarified by ab initio studies, there are non-Kitaev interactions which do not exist in the ideal Kitaev honeycomb-lattice model, e.g. the Heisenberg exchange interaction, and symmetric off-diagonal exchange interactions referred to as the $\Gamma$ term and the $\Gamma'$ term. In the case that the non-Kitaev interactions are sufficiently strong compared to the Kitaev interaction, the spin liquid state is destabilized, and conventional magnetic orders occur, as extensively discussed in many previous studies. However, if the non-Kitaev interactions are not so strong enough to destroy the spin liquid state, they may add novel unique features to the Kitaev spin liquid state, which are not expected for the ideal Kitaev model. In fact, in our previous study it is found that the off-diagonal exchange interaction $\Gamma'$ term significantly increases the magnitude of the mass gap of itinerant Majorana fermions induced by a magnetic field, and enhances the stability of the topological phase rather than destroy it. According to ab initio calculations, the magnitude of $\Gamma'$ in real candidate materials, $\alpha$-RuCl$_3$ and Na$_2$IrO$_3$, is rather small compared to that of the Heisenberg interaction and the $\Gamma$ term. However, this never implies that $\Gamma'$ is negligible. As a matter of fact, the magnitude of $\Gamma'$ is strongly sensitive to trigonal distortion of the edge-shared tetrahedra structure of these materials. Thus, for the clarification of sample-dependence of the spin liquid phase of the candidate materials, it is important to understand effects of the $\Gamma'$ term on the Kitaev spin liquid state.

In this paper, we explore for effects of the non-Kitaev interactions on the spin liquid state more extensively. The starting point of our argument is the Kitaev spin liquid state perturbed by the above-mentioned non-Kitaev interactions. We mainly focus on vortex-free states, where the system is described only by itinerant Majorana fermions. Our argument is restricted to the parameter regions where the vortex-free phases are not destroyed by the non-Kitaev interactions. This assumption is valid as long as the strength of the non-Kitaev interactions are not too large. We derive an effective Hamiltonian for the itinerant Majorana fermions taking into account the non-Kitaev interactions as perturbations, which modify the Majorana band structure. In general, non-Kitaev interactions may give rise to many-body interactions between Majorana fermions. However, unless the coupling strength is large enough, these many-body interactions are irrelevant perturbations to itinerant Majorana fermions, since the density of states vanishes in the low-energy limit. Thus, we do not consider any possibility of spontaneous symmetry breaking due to these interactions, and focus on effects on the band structure of non-interacting Majorana fermions. One of our main results is that, in the chiral spin liquid phase induced by a magnetic field, the $\Gamma'$-term can give rise to a topological phase transition accompanying the change of the first Chern number from $\pm 1$ to $\pm 2$ in the case of the antiferromagnetic Kitaev interaction. As elucidated by Kitaev the phase with the Chern number $\nu = \pm 2$ is the Abelian topological phase with Abelian anyons $a, \bar{a}$, the braiding of which results in the phase change $e^{i\pi}$; i.e. they behave as a quarter of a fermion. On the other hand, it is found that other non-Kitaev interactions such as Heisenberg term and the $\Gamma$-term only change the Majorana band width of the vortex free spin liquid, and does not qualitatively affect the Majorana band structure.

The organization of this paper is as follows: In Sec. II, we consider effects of the $\Gamma'$-term, and derive an effective Hamiltonian by using perturbative expansions up to the second order in $\Gamma'$ around the Kitaev spin liquid state. It is found that the...
perturbed term change drastically the band structure of itinerant Majorana fermions. In Sec. III, from numerical analysis of the effective Hamiltonian, we demonstrate that topological phase transitions with the change of the Chern number occurs, as the magnitude of $\Gamma'$ increases. In Sec. IV, we clarify that the Heisenberg interaction and the $\Gamma'$ term do not affect qualitatively the Majorana fermion band.

II. EFFECTS OF THE $\Gamma'$ TERM ON THE MAJORANA BAND STRUCTURE

In this section, we investigate effects of the $\Gamma'$ term on the Kitaev spin liquid phase on the basis of perturbative expansions with respect to $\Gamma'$. We start with the following Hamiltonian for candidate materials of the Kitaev magnet on a honeycomb lattice such as $\alpha$-RuCl$_3$ and Na$_2$IrO$_3$,

$$\mathcal{H}_f = J \sum_{\langle ij \rangle} S_i \cdot S_j,$$

(1)

$$\mathcal{H}_K = -K \sum_{\langle i \rangle} S_i^\alpha S_i^\alpha,$$

(2)

$$\mathcal{H}_f = \Gamma \sum_{\langle i \rangle} \{ S_i^\beta S_i^\gamma + S_i^\gamma S_i^\beta \},$$

(3)

$$\mathcal{H}_f = \Gamma' \sum_{\langle i \rangle \neq \langle j \rangle} \{ S_i^\beta S_j^\gamma + S_j^\beta S_i^\gamma \},$$

(4)

where $S_i^\alpha$ is an $\alpha = x, y, z$ component of an $s = 1/2$ spin operator at a site $i$. $\mathcal{H}_f$ is the Heisenberg exchange interaction between the nearest neighbor sites, and $\mathcal{H}_K$ is the Kitaev interaction. Here, $\langle ij \rangle$ denotes that the $i$-site and the $j$-site are the nearest-neighbor sites connected via an $\alpha$-bond on the honeycomb lattice (see FIG.1). $H_f$ and $H_f$ are symmetric off-diagonal exchange interactions arising from spin-orbit couplings and oxygen-mediated exchange interactions in the edge-shared octahedra structure.

The ideal Kitaev Hamiltonian $\mathcal{H}_K$ is exactly solvable in terms of the Majorana fermion representation:

$$S_i^x = \frac{1}{2} b_i^\dagger c_j, \quad S_i^y = \frac{1}{2} b_i^\dagger c_j, \quad S_i^z = \frac{1}{2} b_i^\dagger c_j,$$

(5)

where $b_i^\dagger (\alpha = x, y, z)$ and $c_j$ are Majorana fermion operators, and the Hilbert space where these operators act on is restricted to satisfy $D_i |\phi\rangle = |\phi\rangle$ with $D_i = b_i^\dagger b_j^\dagger b_j c_i$, and $|\phi\rangle$ the eigen state of the Kitaev spin liquid. In terms of the Majorana fields, $\mathcal{H}_K$ is expressed as,

$$\mathcal{H}_K = \frac{i}{4} \sum_{i,j} \hat{A}_{ij} c_i c_j,$$

(6)

where $\hat{A}_{ij} = \frac{i}{2} K \hat{u}_{ij}$ and $\hat{u}_{ij} = ib_i^\dagger b_j^\dagger$ with $i, j \in \alpha$-bond. The $Z_2$ gauge fields $\hat{u}_{ij}$ commute with $\mathcal{H}_K$, and can be replaced by the eigenvalues $\pm 1$. For $K > 0 (K < 0)$, in the ground state, we can put $\hat{u}_{ij} \rightarrow 1 (-1)$, and hence, $\hat{A}_{ij} \rightarrow \frac{i}{2} K (-\frac{1}{2} K)$. Then, eq. (6) is reduced to the Hamiltonian of free massless Majorana fermions which can be diagonalized in the momentum representation. When the sign of a $Z_2$ gauge field is flipped, a $Z_2$ vortex (vison), the excitation energy of which is $\sim K$, is created.

In this paper, we consider effects of non-Kitaev interactions, $\mathcal{H}_f$, $\mathcal{H}_f$, and $\mathcal{H}_f$, on the vortex-free spin liquid phase which is the ground state of $\mathcal{H}_K$. As a first step, in this section, we focus on the $\Gamma'$-term, which, as shown below, drastically affects the band structure of Majorana fermions. To see effects on the Kitaev spin liquid state, following the spirit of the original Kitaev’s paper, we carry out perturbative expansions with respect to $\Gamma'$ around the vortex-free Kitaev spin liquid state, which is separated from excited states with visons by a finite energy gap $\sim K$. The corrections to the ground state of the ideal Kitaev Hamiltonian $\mathcal{H}_K$ are expressed in terms of perturbative expansions of the self-energy due to $\mathcal{H}_f$,

$$\Sigma(E) = \Pi_0(\mathcal{H}_f + \mathcal{H}_f G_0(E) \mathcal{H}_f + ...) \Pi_0$$

(7)

$$G_0(E_0) \sim -\frac{1}{\Pi_0 \mathcal{H}_f \mathcal{H}_f \Pi_0}$$

(8)

where, $\Pi_0$ is a projection to the vortex-free spin liquid state. Up to the second order in $\mathcal{H}_f$, we obtain the perturbative corrections to the effective Hamiltonian,

$$\mathcal{H}_{1, eff} = 0,$$

(9)

$$\mathcal{H}_{2, eff} = \Pi_0(\mathcal{H}_f G_0(E) \mathcal{H}_f) \Pi_0$$

$$\sim -\frac{1}{|[K]|} \Pi_0 \mathcal{H}_f \mathcal{H}_f \Pi_0$$

(10)

where $\alpha, \chi, \xi, \nu = x, y, z$. In eq. (10), the second order term arises from the off-diagonal exchange interactions acting on the $\alpha$-bond, and the $\xi$-bond. To analyze this term more precisely, we use the fact that in the Majorana fermion representation of the Kitaev spin liquid state, gauge Majorana fields $b^\alpha$ should be paired on the $\alpha$-bond connecting two sites $i$ and $j$ to form $Z_2$ gauge fields $\hat{u}_{ij} = ib_i^\dagger b_j^\dagger$, since the Kitaev spin liquid state is expressed by the eigen state of the $Z_2$ gauge fields. Then, eq. (10) is recast into,

$$\mathcal{H}_{2, eff} = -\frac{\Gamma^2}{16 |K|} \sum_{\langle \langle ij \rangle \rangle} \sum_{\langle kl \rangle} \sum_{\chi, \alpha} \sum_{\xi, \alpha} \sum_{\nu, \alpha} i c_{\langle \langle \langle \langle ij \rangle \rangle \rangle} \hat{u}_{ij} \hat{u}_{kl} \hat{u}_{kl}$$

$$- \sum_{\langle \langle ij \rangle \rangle} \sum_{\langle \langle \langle \langle kl \rangle \rangle \rangle} \sum_{\chi, \alpha} \sum_{\xi, \alpha} \sum_{\nu, \alpha} i c_{\langle \langle \langle \langle kl \rangle \rangle \rangle} \hat{u}_{ij} \hat{u}_{kl} \hat{u}_{kl}$$

(11)
Therefore, for the vortex-free spin liquid state, we have,

\[
\mathcal{H}_{\tau,\text{eff}}^{(2)} = -\frac{d^2}{4\pi i} \sum_i \left[ \sum_{p=1}^6 (-1)^m c_i N_p(-1)^m c_i + 2 \sum_{p=7,8,9} (-1)^{m-1} c_i N_p(-1)^{m-1} c_i \right].
\]  

(12)

where \( N_p \) is a vector defined in FIG.1 and \( m = 0 \) if the site \( i \) is on the \( A \) sub-lattice of the honeycomb lattice, and \( m = 1 \) if the site \( i \) is on the \( B \) sub-lattice. The factor 2 in front of the second term arises from two shortest paths connecting sites \( i \) and \( i \pm N_p \) with \( p = 7, 8, 9 \). This second-order perturbation term generates the third nearest-neighbor hopping of itinerant Majorana fermions as shown in FIG.1 and changes the Majorana band structure drastically. The total effective Hamiltonian, \( \mathcal{H}_{\text{eff}} = \mathcal{H}_K + \mathcal{H}_{\tau,\text{eff}}^{(2)} \), is,

\[
\mathcal{H}_{\text{eff}} = \sum_k (c_A(-k), c_B(-k)) \begin{pmatrix} 0 & iF(k) \\ -iF^*(k) & 0 \end{pmatrix} \begin{pmatrix} c_A(k) \\ c_B(k) \end{pmatrix},
\]

(13)

with \( F(k) = f(k) + g(k) \), and

\[
f(k) = \frac{\Gamma}{2} (e^{i k \cdot n_1} + e^{-i k \cdot n_2} + 1),
\]

(14)

\[
g(k) = \frac{\Gamma}{4\pi K} \sum_{p=1}^6 e^{-i k \cdot N_p} - \frac{\Gamma}{2\pi K} \sum_{p=7,8,9} e^{-i k \cdot N_p}
\]

\[+ \frac{\Gamma}{2} [1 + e^{i k \cdot n_1} + e^{-i k \cdot n_2}],
\]

(15)

where \( c_{A/B}(k) \) is a Majorana field on the \( A \) (\( B \)) sub-lattice of the honeycomb lattice, and \( n_1 = \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) \), \( n_2 = \left( \frac{1}{2}, -\frac{\sqrt{3}}{2} \right) \). The term \( g(k) \) arises from \( \mathcal{H}_{\tau,\text{eff}}^{(2)} \). In FIG.2 we plot zero-energy Dirac points of the Majorana band of eq.4(13), i.e., \( |F(k)| = 0 \), for several values of \( \Gamma / |K| \). We see that the second-order perturbation term changes the location and the number of Dirac points in the Brillouin zone, which implies the change of the Chern number in the case with a mass gap of Majorana fermions induced by an applied magnetic field. We explore for this possibility in the next section.

### III. Topological Phase Transition With the Change of the Chern Number and Abelian Anyon Phase

In this section, on the basis of the results obtained in Sec. II, we discuss topological phase transitions induced by the second-order perturbation term \( \mathcal{H}_{\tau,\text{eff}}^{(2)} \) in the case with a mass gap of itinerant Majorana fermions which is generated by an applied magnetic field. For the ideal Kitaev model, eq.2, when time-reversal symmetry is broken by an applied magnetic field, itinerant Majorana fermions acquire a mass gap and a non-Abelian topological phase with the Chern number equal to ±1 is realized. This topological feature is changed by \( \mathcal{H}_{\text{eff}}^{(2)} \) as seen below. In the case with a magnetic field

\[
\mathbf{h} = (h_x, h_y, h_z), \quad \text{we obtain the effective Hamiltonian } \mathcal{H}_{\text{eff},\mathbf{h}} \text{ from perturbative calculations up to the third order in } \mathbf{h},
\]

\[
\mathcal{H}_{\text{eff},\mathbf{h}} = \mathcal{H}_K + \mathcal{H}_{\tau,\text{eff}}^{(2)} + \mathcal{H}_{\tau,\text{eff}}^{(2)} + \mathcal{H}_h^{(3)},
\]

(16)

where the third and fourth terms in the right-hand side are,

\[
\mathcal{H}_{\tau,\text{eff}}^{(2)} = -\frac{\Gamma}{4|K|} \sum_{p,m} \sum_{i,j,k} |(i,j)| |(i,j,k)| \sum_{i,j,k} (h_x + h_y)(-1)^m c_i c_k,
\]

(17)

\[
\mathcal{H}_h^{(3)} = \frac{i h_x}{K^2} \sum_{i,m} \sum_{p=1,2,3} (-1)^m c_{i+n_p} c_i.
\]

(18)

Here, \( n_3 = -n_1 - n_2 \), and \( m = 0 \) if the site \( i \) is on the \( A \) sub-lattice of the honeycomb lattice, and \( m = 1 \) if the site \( i \) is on the \( B \) sub-lattice. These term generates the Majorana mass term,

\[
\Delta(k) = \Delta_0(k) + \Delta_1(k),
\]

\[
\Delta_0(k) = \frac{4h_x h_z}{|K|^2} \left[ \sin(k \cdot n_1) + \sin(k \cdot n_2) + \sin(k \cdot n_3) \right],
\]

(19)

\[
\Delta_1(k) = -\frac{\Gamma}{|K|} \left[ (h_x + h_z) \sin(k \cdot n_1) + (h_x + h_z) \sin(k \cdot n_2) + (h_x + h_z) \sin(k \cdot n_3) \right].
\]

(20)

The mass gap terms change the system into topological phases with the nonzero Chern number. Note that the mass term eq.17\( \text{linear in } \mathbf{h} \) was obtained before in ref.39.

To investigate topological characters of this gapped phase,
we calculate the first Chern number $\nu$ which is given by,

$$
\nu = \frac{1}{2\pi} \int_{BZ} dk_x dk_y \Omega_{k_x,k_y},
$$

where

$$
\Omega_{k_x,k_y} = \frac{1}{2} \left\langle \mathbf{\hat{H}} \cdot (\partial_{k_x} \mathbf{\hat{H}} \times \partial_{k_y} \mathbf{\hat{H}}) \right\rangle, \quad \mathbf{\hat{H}} := \frac{\mathcal{H}}{|\mathcal{H}|},
$$

with $\Omega_{k_x,k_y}$ the Berry curvature of the Majorana band. We consider the case with a magnetic field applied in the direction shown in FIG.3(A). The calculated results of the Chern number as a function of $h = |h|$ and $\Gamma'$ are plotted in FIGs.4 and 5. We see that, in the case of the antiferromagnetic Kitaev interaction $K < 0$, as shown in FIG.4, topological phase transitions with the change of the Chern number from $\pm 1$ to $\pm 2$ occur, as the magnitude of $\Gamma'$ increases. This topological phase transition may be experimentally observed by the measurement of the quantized thermal Hall effect. The changes of the Chern number are in accordance with the change of the number of Dirac cones shown in FIG.2. On the other hand, in the case of the ferromagnetic Kitaev interaction $K > 0$, the phase with the Chern number equal to $\pm 2$ is not realized for any values of $\Gamma'$, as shown in FIG.5.

Another interesting feature found in FIGs.4 and 5 is that the sign of the Chern number, i.e. the sign of the thermal Hall conductivity, is flipped from $+1$ to $-1$ (or $-1$ to $+1$) depending on the sign of $\Gamma'$. This phenomenon occurs for both the antiferromagnetic and ferromagnetic Kitaev interactions.

We also examine the bulk-edge correspondence by calculating edge states in the case with open boundaries. The calculated energy spectra in the antiferromagnetic Kitaev interaction are shown in FIG.6. Here, we consider the system with armchair open edges to avoid extrinsic complexity caused by zigzag boundaries which lead to non-topological flat bands of the edge states unrelated to the bulk Chern number. In this calculation, the total number of the unit cell along the $a$-axis is 100, a magnetic field is set to $h = 0.5|K|$, and the field direction is $\theta = \pi/6$. The results are shown in FIG.4. We see that there are two chiral edge states in the case of the Chern number $\nu = \pm 2$ (see FIG.4(d)).

The emergence of the topological phase with the even Chern number is quite intriguing, because, in this case, Abelian topological phases with Abelian anyons are realized. As elucidated by Kitaev, when $\nu = 2 \mod 4$, there are two types of the $Z_2$ vortices which behave as Abelian anyons $a$, $\bar{a}$. They obey the fusion rule $a \times a = \bar{a} \times \bar{a} = e$, $a \times \bar{a} = 1$, $a \times e = \bar{a}$, and $\bar{a} \times e = a$, where $e$ is a fermion field. The braiding of $a$ and $\bar{a}$ accompanies the phase change $e^{i\pi/4}$, which implies that $a$ and $\bar{a}$ are neither bosons nor fermions, but behave as a quarter of a fermion. It is an interesting future issue to explore for novel phenomena associated with these Abelian...
anyons in Kitaev magnets.

IV. EFFECT OF OTHER NON-KITAEV INTERACTIONS

In this section, we discuss effects of other non-Kitaev interactions, i.e. the Heisenberg term $J_{ij}^\alpha \langle i j \rangle^\alpha$ and the off-diagonal exchange interaction $\Gamma$ term $\Gamma$ on the vortex-free spin liquid state. We do not discuss the instability of the Kitaev spin liquid state toward conventional magnetic ordered states induced by these interactions, which has been discussed in many previous studies.\cite{36,37,44,45} However, instead, we consider the issue how properties of the spin liquid state are affected by these interactions provided that the ground state is described by Majorana fermions coupled with $Z_2$ gauge fields. We deal with these non-Kitaev interactions as perturbations to the Kitaev spin liquid state. The main result of this section is that both the $\Gamma$ term and the Heisenberg term generate nearest-neighbor hopping terms of itinerant Majorana fermions, which only affect the Majorana band width, and do not give rise to qualitative changes of the Majorana band structure.

Since the Heisenberg term $J_{ij}^\alpha \langle i j \rangle^\alpha$ acting on $(ij)_\alpha$ sites trivially normalizes the Kitaev interaction, we focus on other terms. Putting $V' = H' + H_F$, where $H_F'$ is the Heisenberg
transition point at $\Gamma$ points (a), (b), (c), and (d) denoted in FIG. 4(a). (a) $\Gamma_{\nu}$ is set to 0, $|K| = 0.5|\nu|$, (b) $\nu = -1$ with $\Gamma' = 0.4|K|$, (c) the topological phase transition point at $\Gamma' = 0.5|K|$, (d) $\nu = 2$ with $\Gamma' = 0.6|K|$. There is one edge state for (a) and (b), while there are two for (d).

FIG. 6: The energy spectra in the case with armchair open boundaries and the antiferromagnetic Kitaev interaction. A magnetic field is set to $h = 0.5|K|$ and $\theta = \pi/6$. The parameters correspond to the points (a), (b), (c), and (d) denoted in FIG. 4(a). (a) $\nu = -1$ with $\Gamma' = 0|K|$, (b) $\nu = -1$ with $\Gamma' = 0.4|K|$, (c) the topological phase transition point at $\Gamma' = 0.5|K|$, (d) $\nu = 2$ with $\Gamma' = 0.6|K|$. There is one edge state for (a) and (b), while there are two for (d).

FIG. 7: The nearest neighbour hopping term between sites $i$ and $j$ generated from the off-diagonal exchange interaction $\Gamma$ term and the Heisenberg interaction term. The path (i) corresponds to the lowest order perturbation term which includes $\Sigma_{\nu}^b S_{ij}^\nu$. The path (ii) corresponds to the lowest order term which includes $JS_{ij}^\nu$.

As mentioned above, we focus on perturbation terms which are expressed in the quadratic form of itinerant Majorana fields $c_i$ in the vortex-free ground state. Let us consider the case that one of Majorana fields in a quadratic term, $c_i$, arises from $\Gamma S_{ij}^\nu = -\frac{\Gamma}{4}b_i^\nu c_j^\nu$ acting on $(ij)_{\alpha}$ where $\alpha, \beta, \gamma$ are given by the cyclic permutation of $x, y, z$. Then, since

$$-\frac{\Gamma}{4}b_i^\nu c_j^\nu|\phi\rangle = \frac{\Gamma}{4}b_i^\nu c_j^\nu|\phi\rangle,$$

there are two cases; (i) another itinerant Majorana field in the quadratic term is $c_j$, and $b_i^\nu$ and $b_j^\nu$ are formed into $Z_2$ gauge fields on the $\beta$-bond and the $\gamma$-bond, respectively. (ii) $b_i^\nu$ and $b_j^\nu$ are formed into $Z_2$ gauge fields on the $\alpha$-bond and $\beta$-bond, respectively. However, the second case (ii) is not possible, because the $Z_2$ gauge field on the $\alpha$-bond, $ib_i^\nu b_j^\nu$, can not be
formed with the lack of $b_i^\sigma$. Thus, only the nearest-neighbor hopping term of itinerant Majorana fermions $c_i\sigma_j$ can be realized. This argument is also applicable to the case that $c_i$ arises from the Heisenberg interaction $J S_i^\beta S_j^\beta$ or $J S_i^\gamma S_j^\gamma$. On the basis of this insight, it is found that all perturbation terms which lead to nearest-neighbor hopping between sites $i$ and $j$ are expressed in terms of paths connecting $i$ and $j$, as depicted in Fig.7 On every bond in these paths, a $Z_2$ gauge field must be formed. For this reason, in each perturbation terms of eq. (24), spin operators on all sites on the paths except the sites $i$ and $j$ should be expressed in terms of gauge Majorana fields only, by using the relations, $S_i^\sigma = \frac{1}{2} b_i^\sigma c_i$, $S_j^\sigma = \frac{1}{2} b_j^\sigma c_j$, and $S_i^\gamma = \frac{1}{2} b_i^\gamma c_i$, $S_j^\gamma = \frac{1}{2} b_j^\gamma c_j$. Then, the lowest order terms which do not vanish in the vortex-free state are the third order perturbation terms, the explicit form of which is given by,

\[ \mathcal{F}^{(3)}(\sigma_i, \sigma_j, \cdots) := \Pi_0 \left[ \sum_{\beta, \gamma} c_\beta^i c_\gamma^i \right] \Pi_0 \]

\[ = 2 \sum_{\alpha} \sum_{(l)} (-\bar{c}_l c_j b_i^\beta b_j^\beta) \times (-b_l^\alpha c_l^\alpha b_i^\alpha b_j^\alpha) \]

\[ \times (-b_l^\beta b_i^\beta b_j^\beta b_l^\beta) \]

\[ = 2 \sum_{\alpha} \sum_{(l)} i \bar{c}_l c_j \left( \bar{u}_{\alpha l}^\beta u_{l \alpha}^\beta \bar{u}_{\alpha l}^\alpha u_{l \alpha}^\alpha \right) \]

\[ = 2 \sum_{\alpha} \sum_{(l)} i \bar{c}_l c_j, \]

As a result, we obtain,

\[ \mathcal{H}_{\text{eff}}^{(3)} = -\frac{2\Gamma^3}{4} \sum_{\alpha} \sum_{(l)} i \bar{c}_l c_j, \]  

which is a nearest neighbor hopping term. This term gives only the change of the band width of itinerant Majorana fermions of the original Kitaev model as $K \rightarrow K + \frac{\Gamma^3}{4}$. The above analysis for the non-vanishing lowest order term can be straightforwardly generalized to higher order terms. It is found that, in all orders, any non-vanishing terms of eq. (24) give only nearest-neighbor hopping terms of itinerant Majorana fermions in the vortex-free state. Therefore, we can conclude that the $\Gamma$ term and the Heisenberg term do not alter qualitative features of the Majorana band structure, as long as the Kitaev spin liquid state is not destabilized.

V. SUMMARY

In this paper, we investigated effects of non-Kitaev interactions, i.e. the Heisenberg exchange interaction, and the symmetric off-diagonal exchange interactions, the $\Gamma$ term and the $\Gamma'$ term, on the Kitaev spin liquid state by exploiting perturbative expansions around the vortex-free spin liquid state. We demonstrated that the Heisenberg term and the $\Gamma$ term change the band width of itinerant Majorana fermions only, and do not alter qualitative features of the spin liquid, provided that the Kitaev spin liquid state is not destabilized. On the other hand, it is found that the $\Gamma'$ term drastically affects the band structure of Majorana fermions changing the number of the Dirac points, and inducing topological phase transitions with the change of the Chern number from $\pm 1$ to $\pm 2$ under an applied magnetic field. The phase with the Chern number equal to $\pm 2$ is the Abelian topological phase, and can be detected by the measurement of the quantized thermal Hall conductivity. Since the magnitude of $\Gamma'$ is quite sensitive to trigonal distortion of the crystal structure, it may be possible to realize the topological phase transition by applying strain on a sample. It is an interesting future issue to explore for the Abelian topological phase in candidate materials of Kitaev magnets.

Finally, we comment on the relation between our study and the recent related paper ref. In ref. the realization of the Abelian topological phases in a Kitaev model with four-spin interaction terms are considered. Some of the four-spin interaction terms considered in ref. which induce the transition to the Abelian topological phases, have the same form as the second-order perturbation term $\mathcal{H}_{\text{eff}}^{(2)}$ obtained in Sec.II in our paper. Thus, our results provide a microscopic origin of the model considered in ref.

Acknowledgments

The authors are grateful to Y. Kasahara and Y. Matsuda for valuable discussions. This work was supported by the Grant-in-Aids for Scientific Research from MEXT of Japan [Grants No. 17K05517, and KAKENHI on Innovative Areas “Topological Materials Science” [No. JP15H05852] and “J-Physics” [No. JP18H04318], and JST CREST Grant Number JPMJCR19T5, Japan.

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