ON ASYMPTOTICALLY AUTONOMOUS DYNAMICS FOR
MULTIVALUED EVOLUTION PROBLEMS

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Dedicated to Peter E. Kloeden on occasion of his 70th birthday

Abstract. In this work we improve the result presented by Kloeden-Simsen-
Stefanello Simsen in [8] by reducing uniform conditions. We prove theoretical
results in order to establish convergence in the Hausdorff semi-distance of the
component subsets of the pullback attractor of a non-autonomous multivalued
problem to the global attractor of the corresponding autonomous multivalued
problem.

1. Introduction. The study of nonautonomous problems which are asymptoti-
cally autonomous has attracted much attention in the recent years and have been
intensively studied.

In [7], the authors provided abstract results which gave sufficient conditions en-
suring that the components of the pullback attractor converges in time to the global
attractor of the semigroup of the corresponding autonomous problem. Moreover,
they gave an application on evolution equations with spatially variable exponents
and a time-dependent operator of subdifferential type.

More recently, the authors in [8], extended the results in [7] to the multival-
ued context and gave an application on evolution inclusions with spatially variable
exponents and a time-dependent operator of subdifferential type.

The authors in [9] relaxed, in the single-valued context, the two uniformity con-
ditions needed in [7] and gave an application to a parabolic equation with discon-
tinuous nonlinearity.

In this work we deal with multivalued abstract problems. We improve the result
given in [8] by reducing uniform conditions. We also compare the limit set of the
pullback attractor with the corresponding global attractor.

The paper is organized as follows. In Section 2 we collect some definitions and
results on multivalued processes and multivalued semigroups theory. In Section 3
we revise the abstract result presented in [8]. The new results are presented in

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Section 4 and an application is given in Section 5 for an evolution inclusion with variable exponents.

2. Preliminaries. In this section we present some definitions on multivalued semigroups and multivalued processes, see e.g., [2, 3, 4, 10, 11, 14] for more details.

Definition 2.1. Let $X$ be a complete metric space. The map $G : \mathbb{R}^+ \times X \to P(X)$ is called a multivalued semigroup (or $m$-semiflow) if

1. $G(0, \cdot) = \text{Id}$ is the identity map;
2. $G(t_1 + t_2, x) \subset G(t_1, G(t_2, x))$, for all $x \in X$ and $t_1, t_2 \in \mathbb{R}^+$.

Definition 2.2. Let $G$ be a multivalued semigroup on $X$. The set $A \subset X$ attracts the subset $B$ of $X$ if $\lim_{t \to \infty} \text{dist}(G(t, B), A) = 0$. The set $M$ is said to be a global attractor for $G$ if $M$ attracts any nonempty bounded subset $B \subset X$ and it is negatively invariant, i.e., $M \subset G(t, M)$, $\forall t \geq 0$.

We refer the reader to [10] and [14] for results that guarantee properties on $\omega$-limits and existence of the global attractor for $m$-semiflows.

Definition 2.3. Let $X$ be a complete metric space, $P(X)$ the set of all nonempty subsets of $X$ and $\mathbb{R}_d := \{(t, s) \in \mathbb{R}^2 : t \geq s\}$. The map $U : \mathbb{R}_d \times X \to P(X)$ is called a multivalued evolution process on $X$ if

1. $U(t, t, \cdot) = \text{Id}$ is the identity map;
2. $U(t, s, x) \subset U(t, \tau, U(\tau, s, x))$, for all $x \in X$, $s \leq \tau \leq t$, where
   
   
   $$U(t, \tau, U(\tau, s, x)) = \bigcup_{y \in U(\tau, s, x)} U(t, \tau, y).$$

The multivalued evolution process $U$ is called strict if

$$U(t, s, x) = U(t, \tau, U(\tau, s, x)),$$

for all $x \in X$, $s \leq \tau \leq t$.

Definition 2.4. Let $U$ be a multivalued evolution process on $X$ and $t \in \mathbb{R}$. The set $D(t) \subset X$ pullback attracts the nonempty bounded subset $B$ of $X$ at time $t$ if

$$\lim_{\tau \to -\infty} \text{dist}(U(t, \tau)B, D(t)) = 0. \tag{1}$$

The set $D(t)$ is said to be pullback attracting at time $t$ if (1) is satisfied for any nonempty bounded subset $B \subset X$.

For a nonempty and bounded subset $B \subset X$ and $t \in \mathbb{R}$, consider $\gamma^s(t, B) := \bigcup_{\tau \leq s} U(t, \tau, B)$ and $\omega(t, B) := \bigcap \gamma^s(t, B)$. The set $\omega(t, B)$ is called the pullback $\omega$-limit set of $B$ at time $t$ with respect to the multivalued evolution process $U$.

Theorem 2.5. [Theorem 6 in [3]] Suppose that for $t \in \mathbb{R}$ and $B$ a nonempty and bounded subset of $X$ there exists a nonempty compact subset $D(t, B)$ of $X$ such that

$$\lim_{s \to -\infty} \text{dist}(U(t, s)B, D(t, B)) = 0.$$

Then $\omega(t, B)$ is nonempty, compact and the minimal closed set attracting $B$ at time $t$.

Definition 2.6. A family of sets $\{A(t) : t \in \mathbb{R}\}$ of $X$ is called a pullback attractor for the multivalued evolution process $U$ if

1. $A(t)$ is pullback attracting at time $t$ for all $t \in \mathbb{R}$;
2. it is semi-invariant (or negatively invariant), that is,

$$A(t) \subset U(t, s, A(s)), \quad \text{for any } (t, s) \in \mathbb{R}_d;$$
it is minimal, that is, for any closed attracting set $Y$ at time $t$, we have $A(t) \subset Y$.

**Theorem 2.7.** [Theorem 18 in [3]] Let us suppose that for all $(t, s) \in \mathbb{R}_d$ the map $x \mapsto U(t, s, x) \in P(X)$ is closed. If, moreover, for any $t \in \mathbb{R}$ there exists a nonempty compact set $D(t)$ which is attracting, then the set $A = \{ A(t) \}_{t \in \mathbb{R}}$, with

$$A(t) = \bigcup_{B \in B(X)} \omega(t, B)$$

where $B(X) = \{ B \in P(X) : B \text{ is bounded} \}$, is the pullback attractor of $U$. Moreover, the sets $A(t)$ are compact.

3. **Asymptotic upper semicontinuity.** Consider the following non-autonomous multivalued problem in a Banach space $X$ of the form

$$\frac{\partial u}{\partial t}(t) + A(t)u(t) + F(u(t)) \ni 0, \quad u(\tau) = \psi_\tau,$$

compared with that of an autonomous multivalued problem of the form

$$\frac{\partial v}{\partial t}(t) + A_\infty v(t) + F(v(t)) \ni 0, \quad v(0) = \psi_0,$$

where $A(t), A_\infty$ are univalued operators in $X$ and $F : X \to P(X)$ is a multivalued map.

The multivalued evolution process associated with (2) is defined by the map $U : \mathbb{R}_d \times X \to P(X)$ defined by

$$U(t, \tau)\xi = \{ z : \text{there exists } u(\cdot) \in \mathcal{D}_\tau(\xi) \text{ such that } u(t) = z \}$$

where $\mathcal{D}_\tau(\xi)$ denotes the set of all solutions of (2) corresponding to the initial condition $u(\tau) = \xi$.

We will suppose that the multivalued evolution process $\{ U(t, \tau) : t \geq \tau \}$ in $X$ associated with problem (2) has a pullback attractor $\mathcal{A} = \{ A(t) : t \in \mathbb{R} \}$, with $A(t)$ compact for each $t \in \mathbb{R}$, and that the multivalued semigroup $G : \mathbb{R}_+ \times X \to P(X)$ associated with problem (3) has a compact global autonomous attractor $A_\infty$ in the Banach space $X$.

The following result was proved in [8] in order to establish sufficient conditions for the convergence in the Hausdorff semi-distance of the component subsets $A(t)$ of the pullback attractor $\mathcal{A}$ to $A_\infty$ as $t \to \infty$.

**Theorem 3.1.** Suppose that $X = H$ is a Hilbert space and that $\mathcal{C} := \bigcup_{\tau \in \mathbb{R}} \overline{A(\tau)}$ is a compact subset of $H$. In addition, suppose that for each solution $u$ of problem (2) there exists a solution $v$ of problem (3) such that $u(t + \tau) \to v(t)$ in $H$ as $\tau \to +\infty$ uniformly in $t \geq 0$ whenever $\psi_\tau \in \mathcal{A}(\tau)$ and $\psi_\tau \to \psi_0$ in $H$ as $\tau \to +\infty$. Then

$$\lim_{t \to +\infty} \text{dist}(A(t), A_\infty) = 0.$$

4. **New results.** In this section $(X, \| \cdot \|)$ will be a Banach space.

4.1. **Improvement of Theorem 3.1.** In this section, we will improve Theorem 3.1 by reducing two uniform conditions.
\textbf{Theorem 4.1.} Suppose that for each solution \( u \) of problem (2) there exists a solution \( v \) of problem (3) such that \( u(t + \tau) \to v(t) \) in \( X \) as \( \tau \to +\infty \) for each \( t \geq 0 \) whenever \( \psi_\tau \in \mathcal{A}(\tau) \) and \( \psi_\tau \to \psi_0 \) in \( X \) as \( \tau \to +\infty \). If \( \bigcup_{s \geq \tau} \mathcal{A}(s) \) is a compact subset of \( X \) for each \( \tau \in \mathbb{R} \), then

\[
\lim_{t \to +\infty} \text{dist}(\mathcal{A}(t), \mathcal{A}_\infty) = 0. \tag{4}
\]

\textit{Proof.} We have that \( K(\tau) := \bigcup_{s \geq \tau} \mathcal{A}(s) \) is a compact subset of \( X \) for each \( \tau \in \mathbb{R} \). In particular, \( K := \bigcup_{s \geq 0} \mathcal{A}(s) \) is a compact subset of \( X \). Suppose that (4) is not true. Then there would exist an \( \epsilon_0 > 0 \) and a real sequence \( \{\tau_n\}_{n \in \mathbb{N}} \) with \( 0 < \tau_n \nearrow +\infty \) such that \( \text{dist}(\mathcal{A}(\tau_n), \mathcal{A}_\infty) \geq 3\epsilon_0 \) for all \( n \in \mathbb{N} \). Since the sets \( \mathcal{A}(\tau_n) \) are compact, there exists \( a_n \in \mathcal{A}(\tau_n) \) such that

\[
\text{dist}(a_n, \mathcal{A}_\infty) = \text{dist}(\mathcal{A}(\tau_n), \mathcal{A}_\infty) \geq 3\epsilon_0, \tag{5}
\]

for each \( n \in \mathbb{N} \). By the attraction property for the multivalued semigroup we have \( \text{dist}(G(\tau_{n_0}, K), \mathcal{A}_\infty) \leq \epsilon_0 \) for \( n_0 > 0 \) large enough. Moreover, by the semi-invariance of the pullback attractor there exist \( b_n \in \mathcal{A}(\tau_n - \tau_{n_0}) \subset K \) for \( n > n_0 \) such that \( a_n \in U(\tau_n, \tau_n - \tau_{n_0})b_n \) for each \( n > n_0 \). Since \( K \) is compact, there is a convergent subsequence \( b_{n'} \to b \in K \). Since \( a_{n'} \in U(\tau_{n'}, \tau_{n'} - \tau_{n_0})b_{n'} \) there exists a solution \( u_{n'} \) of

\[
\frac{\partial u_{n'}(t)}{\partial t} + A(t)u_{n'}(t) + F(u_{n'}(t)) \ni 0, \quad u_{n'}(\tau_{n'} - \tau_{n_0}) = b_{n'},
\]

such that \( a_{n'} = u_{n'}(\tau_{n'}) \). Writing \( \tau_{n'} = \tau_{n_0} + (\tau_{n'} - \tau_{n_0}) \) and using the hypotheses with \( t = \tau_{n_0} \) and \( \tau = \tau_{n'} - \tau_{n_0} \to +\infty \) (as \( n' \to +\infty \)), there exists a solution \( v_{n'} \) of

\[
\frac{\partial v_{n'}(t)}{\partial t} + A_{\infty}v_{n'}(t) + F(v_{n'}(t)) \ni 0, \quad v_{n'}(0) = b,
\]

such that

\[
\|u_{n'}(\tau_{n'}) - v_{n'}(\tau_{n_0})\|_X < \epsilon_0
\]

for \( n' \) large enough. Hence,

\[
\text{dist} (a_{n'}, \mathcal{A}_\infty) = \text{dist} (u_{n'}(\tau_{n'}), \mathcal{A}_\infty) \\
\leq \|u_{n'}(\tau_{n'}) - v_{n'}(\tau_{n_0})\|_X + \text{dist} (v_{n'}(\tau_{n_0}), \mathcal{A}_\infty) \\
\leq \|u_{n'}(\tau_{n'}) - v_{n'}(\tau_{n_0})\|_X + \text{dist} (G(\tau_{n_0}, K), \mathcal{A}_\infty) \\
\leq 2\epsilon_0,
\]

which contradicts (5). \( \Box \)

\textbf{Remark 1.} The condition of precompactness, i.e., that \( \bigcup_{s \geq \tau} \mathcal{A}(s) \) is a compact subset of \( X \) for each \( \tau \in \mathbb{R} \) can be verified in applied problems when the main operators are of the subdifferential type (see [6]). Since sometimes we also have to deal with problems where the main operators are not of the subdifferential type it can be hard to check this precompactness condition. So, improvements without this precompactness condition are welcome.

Under the following extra condition (USC) a kind of reciprocal of the previous theorem holds.

\textit{(USC):} For each \( t \in \mathbb{R}, \, \sigma : [t, +\infty) \times X \to P(X) \), defined by \( \sigma(r, x) := U(r, t)x \), is an upper semicontinuous map and has compact values.

It is a well-known result that for each fixed pair \((t, \tau)\) with \( t \geq \tau \), the multivalued evolution process associated with problem (2) \( U(t, \tau) : X \to P(X) \) is an upper
semicontinuous map if for example the multivalued process is defined by an Exact Generalized Process (see Theorem 12.3 in [13]).

**Theorem 4.2.** Suppose that (USC) is satisfied and that for each solution $u$ of problem (2) there exists a solution $v$ of problem (3) such that $u(t + \tau) \to v(t)$ in $X$ as $\tau \to +\infty$ for each $t \geq 0$ whenever $\psi_\tau \in \mathcal{A}(\tau)$ and $\psi_\tau \to \psi_0$ in $X$ as $\tau \to +\infty$. If

$$\lim_{t \to +\infty} \text{dist}(\mathcal{A}(t), \mathcal{A}_\infty) = 0$$

then $\bigcup_{s \geq \tau} \mathcal{A}(s)$ is a compact subset of $X$ for each $\tau \in \mathbb{R}$.

**Proof.** Suppose that (6) holds true. We will prove the precompactness of $\bigcup_{t \geq r} \mathcal{A}(r)$ for each fixed $t \in \mathbb{R}$. Taking a sequence $\{x_n\}$ from this union, we then choose $r_n \geq t$ such that $x_n \in \mathcal{A}(r_n)$. We will prove that the sequence $\{x_n\}$ has a convergent subsequence in two cases.

**Case 1.** $r_0 := \sup_{n \in \mathbb{N}} r_n < +\infty$.

In this case, $\{r_n\} \subset [t, r_0]$ and so $\{x_n\} \subset \bigcup_{r \leq r \leq r_0} \mathcal{A}(r)$. By assumption (USC), $\sigma : [t, +\infty) \times X \to P(X)$ is upper semicontinuous. By the invariance of the pullback attractor $\mathcal{U}$, we have $\bigcup_{r \leq r \leq r_0} \mathcal{A}(r) \subset \sigma([t, r_0] \times \mathcal{A}(t))$.

Once an upper semicontinuous multivalued map with compact values take compact sets into compact sets (see Proposition 3, p.42 in [1]) we have that $\sigma([t, r_0] \times \mathcal{A}(t))$ is a compact set. So, $\bigcup_{t \geq r \leq r_n} \mathcal{A}(r)$ is a compact set since it is a closed set into a compact set.

Hence $\{x_n\}$ is precompact.

**Case 2.** $\sup_{n \in \mathbb{N}} r_n = +\infty$.

In this case, passing to a subsequence, we may assume $r_n \nearrow +\infty$. By assumption (6), we have

$$\text{dist}(x_n, \mathcal{A}_\infty) \leq \text{dist}(\mathcal{A}(r_n), \mathcal{A}_\infty) \to 0$$

as $n \to +\infty$. So, for each $n \in \mathbb{N}$ we can choose $y_n \in \mathcal{A}_\infty$ such that

$$d(x_n, y_n) \leq \text{dist}(x_n, \mathcal{A}_\infty) + \frac{1}{n}.$$ 

Since $\mathcal{A}_\infty$ is a compact set, it follows that $y_n$ has a convergent subsequence such that $y_{nk} \to y \in \mathcal{A}_\infty$ as $k \to +\infty$. Therefore,

$$d(x_{nk}, y) \leq d(x_{nk}, y_{nk}) + d(y_{nk}, y) \leq \text{dist}(x_{nk}, \mathcal{A}_\infty) + \frac{1}{nk} + d(y_{nk}, y).$$

Using (7), we conclude that $x_{nk} \to y$ as $k \to +\infty$. \hfill \Box

**Remark 2.** A similar result for the univalued context can be found in [9].

4.2. **Construction from the limit-set of a pullback attractor.** In this subsection we will compare the global attractor $\mathcal{A}_\infty$ with the limit-set $\mathcal{A}(\infty)$ defined by $\mathcal{A}(\infty) := \bigcap_{t \in \mathbb{R}} \bigcup_{s \geq \tau} \mathcal{A}(s)$ and can be characterized by

$$\bigcup_{r_n \nearrow \infty} \{x \in X : \exists x_n \in \mathcal{A}(r_n) \text{ s. t. } x_n \to x\}.$$

This kind of comparison was done in [9] for the univalued context. By following a completely analogous procedure as in the proof of Proposition 2.6 in [9] but using our Theorem 4.1 instead of Theorem 2.4 in [9] we obtain the following result.
Theorem 4.3. Suppose the pullback attractor $\mathcal{U}$ is forward compact, i.e., $\bigcup_{\tau \geq t} \mathcal{A}(\tau)$ is precompact for each $t \in \mathbb{R}$. Moreover, suppose that for each solution $u$ of problem (2) there exists a solution $v$ of problem (3) such that $u(t + \tau) \to v(t)$ in $X$ as $\tau \to +\infty$ for each $t \geq 0$ whenever $\psi_\tau \in \mathcal{A}(\tau)$ and $\psi_\tau \to \psi_0$ in $X$ as $\tau \to +\infty$. Then $\mathcal{A}_\infty \supset \mathcal{A}(\infty)$.

To obtain the equality $\mathcal{A}_\infty = \mathcal{A}(\infty)$ we need to assume stronger conditions as in the next result.

Theorem 4.4. Under the same assumptions of Theorem 4.3, we have $\mathcal{A}_\infty = \mathcal{A}(\infty)$ if we further assume the following conditions:

(a) $\mathcal{A}(\infty)$ forward attracts $\mathcal{A}_\infty$ by $U(\cdot, 0)$, i.e.,

$$\lim_{t \to +\infty} \text{dist}(U(t, 0)\mathcal{A}_\infty, \mathcal{A}(\infty)) = 0;$$

(b) $\lim_{t \to +\infty} \sup_{x \in \mathcal{A}_\infty} \text{dist}(G(t)x, U(t, 0)x) = 0$.

Proof. By Theorem 4.3 we have $\mathcal{A}_\infty \supset \mathcal{A}(\infty)$. It remains to show that $\mathcal{A}_\infty \subset \mathcal{A}(\infty)$. Since $\mathcal{U}$ is assumed to be forward compact, it follows from the nested compact theorem that $\mathcal{A}(\infty)$ is a nonempty compact set. Let $t \geq 0$. Then for each $x \in \mathcal{A}_\infty$,

$$\text{dist}(G(t)x, \mathcal{A}(\infty)) \leq \text{dist}(G(t)x, U(t, 0)x) + \text{dist}(U(t, 0)x, \mathcal{A}(\infty))$$

$$\leq \sup_{x \in \mathcal{A}_\infty} \text{dist}(G(t)x, U(t, 0)x) + \text{dist}(U(t, 0)x, \mathcal{A}(\infty)).$$

So, letting $t \to +\infty$ and using both hypotheses (a) and (b) we obtain

$$\lim_{t \to +\infty} \text{dist}(G(t)\mathcal{A}_\infty, \mathcal{A}(\infty)) = 0.$$ 

Since $\mathcal{A}_\infty$ is negatively invariant under $G$, it follows that

$$\text{dist}(\mathcal{A}_\infty, \mathcal{A}(\infty)) \leq \text{dist}(G(t)\mathcal{A}_\infty, \mathcal{A}(\infty)) \to 0$$

as $t \to +\infty$. So, $\mathcal{A}_\infty \subset \overline{\mathcal{A}(\infty)} = \mathcal{A}(\infty)$. \hfill $\square$

Remark 3. Note that in the multivalued context $\text{dist}(G(t)x, U(t, 0)x)$ can be different from $\text{dist}(U(t, 0)x, G(t)x)$ whereas in the univalued case they are the same.

5. Application. In this section we consider an evolution inclusion where the diffusion and the absorption coefficients both depend on time, more precisely, we study the asymptotic behavior of the following non-autonomous multivalued problem, with a homogeneous Neumann boundary condition, in the Hilbert space $H := L^2(\Omega)$

$$\frac{\partial u}{\partial t} - \text{div} \left(D_1(t)|\nabla u|^{p(x)-2}\nabla u\right) + D_2(t)|u|^{p(x)-2}u + F(u(t)) \ni 0, \quad u(\tau) = \psi_\tau, \quad (8)$$

where $\Omega$ is a bounded smooth domain in $\mathbb{R}^n$, $n \geq 1$, the exponent $p(\cdot) \in C(\overline{\Omega})$ satisfies

$$p^+ := \max_{x \in \overline{\Omega}} p(x) \geq p^- := \min_{x \in \overline{\Omega}} p(x) > 2$$

and the initial condition $u(\tau) \in H := L^2(\Omega)$. The terms $D_1$, $D_2$ and $F$ are assumed to satisfy:

Assumption D. $D_1, D_2 : \mathbb{R} \times \Omega \to \mathbb{R}$ are functions in $L^\infty$ such that

(D1) there is a positive constant $\beta$ such that $0 < \beta \leq D_1(t, x), D_2(t, x)$ for almost all $(t, x) \in \mathbb{R} \times \Omega$. 

\( (D2) \) \( D_1(t, s) \geq D_1(s, x) \) and \( D_2(t, x) \geq D_2(s, x) \) for each \( x \in \Omega \) and \( t \leq s \in \mathbb{R} \).

\( (D3) \) \( D_1(t + \tau, \cdot) \to D_1^\tau(\cdot) \) and \( D_2(t + \tau, \cdot) \to D_2^\tau(\cdot) \) in \( L^\infty(\Omega) \) as \( \tau \to \infty \).

**Assumption F.** \( F : H \to P(H) \) is a multifunction with compact values and such that there exists a constant \( K > 0 \) such that \( h(F(x), F(y)) \leq K \| x - y \| \) for all \( x, y \in H \). Here \( h \) denotes the Hausdorff metric given by:

\[
h(A, B) = \max \{ \sup \{ d(a, B) : a \in A \}, \sup \{ d(b, A) : b \in B \} \}
\]

(where \( d(a, B) = \inf \{ \| a - b \| : b \in B \} \); similarly for \( d(b, A) \).

Let us consider the univalued operator in \( H \) defined by

\[
A(t)u := -\text{div} \left( D_1(t) \nabla u |(p(x)-2)\nabla u \right) + D_2(t) |u|^{p(x)-2} u. \quad (9)
\]

We have that \( A(t) \) is a maximal monotone operator in \( H \) for each \( t \in [\tau, T] \). Moreover, the operator \( A(t) \) is the subdifferential \( \partial \varphi^t_{\rho(t)} \) of the convex, proper and lower semicontinuous map \( \varphi^t_{\rho(t)} : L^2(\Omega) \to \mathbb{R} \cup \{ +\infty \} \) given by

\[
\varphi^t_{\rho(t)}(u) := \left\{ \begin{array}{ll}
\int_\Omega \frac{D_1(t, x)}{\rho(x)} |\nabla u|^{p(x)} dx + \int_\Omega \frac{D_2(t, x)}{\rho(x)} |u|^{p(x)} dx & \text{if } u \in W^{1,p(t)}(\Omega) \\
+\infty, & \text{otherwise.}
\end{array} \right. \quad (10)
\]

As our operator is of subdifferential type, we can obtain existence of global solution for problem (8) using the paper of Papageorgiou and Papalini [12] (see also [8]).

Assumptions (D1)-(D2) imply that the pointwise limits \( D_1^\tau(x) \), \( D_2^\tau(x) \) as \( \tau \to \infty \) exists and satisfies \( 0 < \beta \leq D_1^\tau(x), D_2^\tau(x) \) for almost all \( x \in \Omega \). Then the problem

\[
\frac{\partial v}{\partial t}(t) - \text{div} \left( D_1^\tau |\nabla v(t)|^{p(x)-2} \nabla v(t) \right) + D_2^\tau |v(t)|^{p(x)-2} v(t) + F(v(t)) \geq 0, \quad v(0) = \psi_0,
\]

with \( D_1^\tau(x), D_2^\tau(x) \), is autonomous.

We have the following estimates on the solutions:

**Lemma 5.1.** Let \( u \) be a solution of problem (8) and \( v \) a solution of problem (11). There exist positive constants \( T_2, K_2 \) such that

\[
\| u(t) \|_{W^{1,p(t)}(\Omega)} \leq K_2, \quad \forall t \geq T_2 + \tau,
\]

and also \( \| v(t) \|_{W^{1,p(t)}(\Omega)} \leq K_2 \) for all \( t \geq T_2 + \tau \).

Moreover, following the same lines as the proof of Theorem 8 of [8], we have that

**Theorem 5.2.** i) The multivalued evolution process associated with problem (8) has a pullback attractor \( \mathcal{A} = \{ A(t) : t \in \mathbb{R} \} \), and the sets \( A(t) \) are compact. Moreover \( \cup_{\tau \in \mathbb{R}} A(\tau) \) is a compact subset of \( H \).

ii) The multivalued semigroup associated with problem (11) has a compact global autonomous attractor \( A_{\infty} \) in the Hilbert space \( H \).

It will be shown that the dynamics of the non-autonomous problem (8) is asymptotically autonomous and its pullback attractor converges upper-semi continuously to the autonomous global attractor \( A_{\infty} \) of the problem (11).

In particular, we consider the operator

\[
A_{\infty} v := -\text{div} \left( D_1^\tau |\nabla v|^{p(x)-2} \nabla v \right) + D_2^\tau |v|^{p(x)-2} v.
\]
Theorem 5.3. If \( \{ \psi_\tau : \tau \in \mathbb{R} \} \) is a bounded set in \( W^{1,p(x)}(\Omega) \) and \( \psi_\tau \to \psi_0 \) in \( H \) as \( \tau \to +\infty \), then for each \( \tau \in \mathbb{R} \) there exists a function \( g_\tau : [0, \infty) \to [0, \infty) \) given by

\[
g_\tau(t) = K \left( \| D_1(t + \tau, \cdot) - D_1^*(\cdot) \|_{L^\infty(\Omega)} + \| D_2(t + \tau, \cdot) - D_2^*(\cdot) \|_{L^\infty(\Omega)} \right),
\]

where \( K \) is a positive constant, such that

\[
\langle A(t + \tau)u(t + \tau) - A_\infty v(t), u(t + \tau) - v(t) \rangle \geq -g_\tau(t), \quad \text{for all } t \in \mathbb{R}^+,
\]

for any solution \( u \) of (8) and any uniformly bounded function \( v \) with \( v(t) \in D(A_\infty) \) for all \( t \geq 0 \).

Proof. We have that

\[
\langle A(t + \tau)u(t + \tau) - A_\infty v(t), u(t + \tau) - v(t) \rangle \geq
\]

\[
\int_\Omega \left[ D_1(t + \tau, x) - D_1^*(x) \right] \left[ \nabla v(t, x) |^p(x) - 2 \nabla v(t, x) \right] \left[ \nabla u(t + \tau, x) - \nabla v(t, x) \right] dx
\]

\[+ \int_\Omega \left[ D_2(t + \tau, x) - D_2^*(x) \right] \left[ v(t, x) |^p(x) - 2 v(t, x) \right] \left[ u(t + \tau, x) - v(t, x) \right] dx.
\]

Note that

\[
- \int_\Omega \left[ D_1(t + \tau, x) - D_1^*(x) \right] \left[ \nabla v(t, x) |^p(x) - 2 \nabla v(t, x) \right] \left[ \nabla u(t + \tau, x) - \nabla v(t, x) \right] dx
\]

\[
\leq \| D_1(t + \tau) - D_1^*(\cdot) \|_{L^\infty(\Omega)} \int_\Omega \left| \nabla v(t, x) |^p(x) - 2 \nabla v(t, x) \right| \left| \nabla u(t + \tau, x) - \nabla v(t, x) \right| dx,
\]

and

\[
- \int_\Omega \left[ D_2(t + \tau, x) - D_2^*(x) \right] \left[ v(t, x) |^p(x) - 2 v(t, x) \right] \left[ u(t + \tau, x) - v(t, x) \right] dx
\]

\[
\leq \| D_2(t + \tau, \cdot) - D_2^*(\cdot) \|_{L^\infty(\Omega)} \int_\Omega \left| v(t, x) |^p(x) - 2 v(t, x) \right| \left| u(t + \tau, x) - v(t, x) \right| dx.
\]

The fact that \( \{ \psi_\tau : \tau \in \mathbb{R} \} \) is a bounded set in \( W^{1,p(x)}(\Omega) \) and the estimates on Lemma 5.1 guarantee that there exists a positive constant \( K \) such that

\[
\int_\Omega \left| \nabla v(t, x) |^p(x) - 1 \right| \left| \nabla u(t + \tau, x) \right| dx + \int_\Omega \left| \nabla v(t, x) |^p(x) \right| dx \leq K,
\]

and

\[
\int_\Omega \left| v(t, x) |^p(x) - 1 \right| \left| u(t + \tau, x) \right| dx + \int_\Omega \left| v(t, x) |^p(x) \right| dx \leq K.
\]

Thus,

\[
- \int_\Omega \left[ D_1(t + \tau, x) - D_1^*(x) \right] \left[ \nabla v(t, x) |^p(x) - 2 \nabla v(t, x) \right] \left[ \nabla u(t + \tau, x) - \nabla v(t, x) \right] dx
\]

\[
\leq K \| D_1(t + \tau, \cdot) - D_1^*(\cdot) \|_{L^\infty(\Omega)},
\]

and

\[
- \int_\Omega \left[ D_2(t + \tau, x) - D_2^*(x) \right] \left[ v(t, x) |^p(x) - 2 v(t, x) \right] \left[ u(t + \tau, x) - v(t, x) \right] dx
\]

\[
\leq K \| D_2(t + \tau, \cdot) - D_2^*(\cdot) \|_{L^\infty(\Omega)}.
\]

It follows that

\[
\langle A(t + \tau)u(t + \tau) - A_\infty v(t), u(t + \tau) - v(t) \rangle \geq -g_\tau(t).
\]
Observe that by Assumption D3 the function $g_\tau : [0, +\infty) \to [0, +\infty)$ given in Theorem 5.3 satisfies $g_\tau(t) \to 0$ as $\tau \to +\infty$.

In the next result we check the hypothesis of asymptotic continuity of the non-autonomous flow in the Theorem 4.1 for problems like (8).

**Theorem 5.4.** If $\psi_\tau \in \mathcal{A}(\tau)$ and $\psi_\tau \to \psi_0$ in $H$ as $\tau \to +\infty$, then for each solution $u$ of (8) there exists a solution $v$ of (11) such that $u(t + \tau) \to v(t)$ in $H$ as $\tau \to +\infty$ for each $t \geq 0$.

**Proof.** Let $u$ be a solution of (8) then there exists $f \in L^2([\tau, T]; H)$ such that $f(t) \in F(u(t))$ a.e. and

$$
\frac{\partial u}{\partial t}(t) + A(t)u(t) + f(t) = 0, \quad \text{a.e in } [\tau, T],
$$

Using the semi-invariance of the pullback attractor and the estimate (12) it follows that $\{\psi_\tau : \tau \in \mathbb{R}\}$ is a bounded set in $W^{1,p}([\tau, T]; \Omega)$. From the semi-invariance of the pullback attractor, $\mathcal{A}(\tau) \subset U(\tau, s)\mathcal{A}(s)$, $\forall (\tau, s) \in \mathbb{R}^2$. So, there exists a solution $w$ of (8) with $\psi_\tau = w(\tau)$ and $w(s) \in \mathcal{A}(s)$. Consider the concatenate solution

$$
\theta_\tau(\ell) := \begin{cases} u(\ell), & \ell \geq \tau, \\ w(\ell), & \ell \leq \tau. \end{cases}
$$

Using the pullback attracting property, we have that for each given $\epsilon > 0$ there exists $s_\epsilon \in \mathbb{R}$ such that

$$
\operatorname{dist} \left( U(t + \tau, s_\epsilon) \cup_{\tau \in \mathbb{R}} A(\tau), A(t + \tau) \right) < \epsilon.
$$

In particular,

$$
u(t + \tau) = \theta_{s_\epsilon}(t + \tau) \in O_\epsilon(A(t + \tau)) \subset O_\epsilon \left( \cup_{\tau \in \mathbb{R}} A(\tau) \right), \forall \epsilon > 0.
$$

Then,

$$
u(t + \tau) \in \bigcap_{\epsilon > 0} O_\epsilon \left( \cup_{\tau \in \mathbb{R}} A(\tau) \right) = \overline{\bigcup_{\tau \in \mathbb{R}} A(\tau)}.
$$

Considering $z_\tau(t) := f(t + \tau)$, we have

$$
z_\tau(t) \in F(u(t + \tau)) \subset K := F \left( \overline{\bigcup_{\tau \in \mathbb{R}} A(\tau)} \right).
$$

Using that $\overline{\bigcup_{\tau \in \mathbb{R}} A(\tau)}$ is a compact subset of $H$ (see Theorem 5.2) and that $F$ is Lipschitz continuous with compact values we have from Proposition 3, p. 42 in [1] that $K$ is a compact set in $H$. So, for each $t \geq 0$ there exist $z(t) \in K$ and a subnet of $\{z_\tau(t) \}_{\tau \in \mathbb{R}}$, which we do not relabel, such that $z_\tau(t) \to z(t)$ as $\tau \to +\infty$. Let $v$ be the unique solution of the problem

$$
\frac{\partial v}{\partial t}(t) + A_\infty v(t) + z(t) = 0,
$$

We have that $v$ is uniformly bounded. Subtracting the equation in (13) from the equation in (15) gives

$$
\frac{d}{dt}(u(t + \tau) - v(t)) + A(t + \tau)u(t + \tau) - A_\infty v(t) + f(t + \tau) - z(t) = 0
$$

for a.e. $t \in [0, T]$. Multiplying by $u(t + \tau) - v(t)$ and using Theorem 5.3, we obtain

$$
\frac{1}{2} \frac{d}{dt} \|u(t + \tau) - v(t)\|^2_H \leq g_\tau(t) + \|z_\tau(t) - z(t)\|_H \|u(t + \tau) - v(t)\|_H.
$$
Integrating this last inequality from 0 to $t$, $t \leq T$, it gives
\[
\frac{1}{2} \|u(t + \tau) - v(t)\|_H^2 \leq \frac{1}{2} \|\psi_\tau - \psi_0\|_H^2 + T g_\tau(t) + \int_0^t \|z_\tau(s) - z(s)\|_H \|u(s + \tau) - v(s)\|_H ds.
\]

Hence, by the Gronwall inequality
\[
\|u(t + \tau) - v(t)\|_H \leq (\|\psi_\tau - \psi_0\|_H^2 + 2T g_\tau(t))^{1/2} + \int_0^T \|z_\tau(s) - z(s)\|_H ds.
\]

Using (14) and the Dominated Convergence Theorem, we have
\[
\int_0^T \|z_\tau(s) - z(s)\|_H ds \to 0
\]
as $\tau \to +\infty$.

Since $\psi_\tau \to \psi_0$ in $H$ and $g_\tau(t) \to 0$ as $\tau \to +\infty$ for each $t \geq 0$, we obtain $u(t + \tau) \to v(t)$ in $H$ as $\tau \to +\infty$ for each $t \geq 0$.

From Theorem 3.3 in [5], $z \in SelF(v)$ and the result follows.

The next result gives the desired asymptotic upper semi-continuous convergence.

**Theorem 5.5.** \(\lim_{\tau \to +\infty} \text{dist}(A(t), A_\infty) = 0.\)

**Proof.** Suppose that $\psi_\tau \in A(\tau)$ and $\psi_\tau \to \psi_0$ in $H$. From Theorem 5.4, for each solution $u$ of (8) there exists a solution $v$ of (11) such that $u(t + \tau) \to v(t)$ in $H$ as $\tau \to +\infty$ for each $t \geq 0$. Theorem 5.2 and Theorem 4.1 then yield \(\lim_{\tau \to +\infty} \text{dist}(A(t), A_\infty) = 0.\)

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