Remarks on invariants of Hamiltonian loops

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Abstract

In this note the interrelations between several natural morphisms on the $\pi_1$ of groups of Hamiltonian diffeomorphisms are investigated. As an application, the equality of the (non-linear) Maslov index of loops of quantomorphisms of prequantizations of $\mathbb{C}P^n$ and the Calabi-Weinstein invariant is shown, settling affirmatively a conjecture by A. Givental. We also prove the proportionality of the mixed action-Maslov morphism and the Futaki invariant on loops of Hamiltonian biholomorphisms of Fano Kahler manifolds, as suggested by C. Woodward. Finally, a family of generalized action-Maslov invariants is computed for toric manifolds via barycenters of their moment polytopes, with an application to mass-linear functions recently introduced by D. McDuff and S. Tolman.

1 Introduction and main results:

The topology of groups of Hamiltonian diffeomorphisms of symplectic manifolds and of the closely related groups of quantomorphisms of their prequantizations is an intriguing area of modern symplectic geometry. In the present note we study the $\pi_1$ of these groups by investigating the interrelations between several natural morphisms $\pi_1 \to \mathbb{R}$. Our main results are the following:

1. We show that for the complex projective space $\mathbb{C}P^n$ the (non-linear) Maslov index on loops of quantomorphisms \cite{13} of its prequantizations is proportional to the Calabi-Weinstein homomorphism (Theorem 1 below). This settles in positive a conjecture by A. Givental \cite{13}.

2. Following a suggestion by C. Woodward \cite{36}, we prove that the mixed action-Maslov invariant on Fano Kahler manifolds is proportional to the Futaki invariant \cite{8}.

3. Finally, we compute a family of generalized action-Maslov invariants \cite{17,15,19} on toric manifolds via barycenters of their moment polytopes, and present applications to the notion of mass-linear functions recently introduced by D. McDuff and S. Tolman in \cite{22}.

1.1 Mixed action-Maslov invariants.

Definition 1.1.1. (Mixed Action-Maslov homomorphism)

Let $(M, \omega)$ be a compact spherically monotone symplectic manifold. That means that $[\omega] = \kappa c_1(M)$ on $\pi_2(M)$ for some positive $\kappa \in \mathbb{R}$. And denote by $Ham = Ham(M, \omega)$ the group of Hamiltonian diffeomorphisms of $(M, \omega)$.

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Then the Mixed Action-Maslov homomorphism $I : \pi_1(Ham) \to \mathbb{R}$ introduced by L.Polterovich in [27] is defined as follows:

Given a class $a \in \pi_1(Ham, Id)$
a) Choose a representative $\gamma := \{\phi_t\}_{t \in \mathbb{R}/\mathbb{Z}}$ of $a$.
b) Choose a smooth time dependent Hamiltonian for $\gamma : \{F_t\}_{t \in \mathbb{R}/\mathbb{Z}}$, that is normalized by the condition that for all $t$, $\int_M F_t \omega^n = 0$.
c) Choose a point $p$ in $M$. Then the path $\alpha_t := \{\phi_t(p)\}_{t \in \mathbb{R}/\mathbb{Z}}$ is contractible by Floer Theory.
d) Choose a disk $D$ filling the path $\alpha$.

Then define $I(\gamma) := \int_D \omega - \int_0^1 F_t(\alpha_t)dt - \kappa \cdot \text{Maslov}/2(\gamma^D_p)$.

Here, $\gamma^D_p$ is the path of symplectic linear operators obtained when considering the linear maps $\phi_{1,p} : T_pM \to T_{\phi_t(p)}M$ and trivializing the bundle $(TM, \omega)$ symplectically along the disk $D$.

**Convention:** hereafter, our normalization of the Maslov index is such that the Hopf flow $\{R_{2\pi t} : z \mapsto e^{2\pi t}z\}_{0 \leq t \leq 1}$ on $(\mathbb{C}, \omega_{std})$ has Maslov index 2.

This value does not depend on the choices a)-d), and defines a homomorphism $I : \pi_1(Ham) \to \mathbb{R}$. The most essential part is the independence on choice d) and it follows from the assumption that our manifold is spherically monotone. (see [27])

**Remark 1.1.1.** This invariant provides a lower bound on the asymptotic Hofer norm ([27]), and its vanishing is necessary and sufficient for asymptotic spectral invariants to descend to the group $Ham$ from its universal cover ([20]). It is also related to hamiltonianly non-displaceable fibers of moment maps of Hamiltonian torus actions ([6]).

The homomorphism $I$ is known to vanish identically in several cases:

**Fact 1.** Since $I$ is a homomorphism to $\mathbb{R}$, it vanishes whenever $\pi_1(Ham)$ contains only elements of finite order. This holds for 1. all compact surfaces and 2. the product of two spheres with equal areas.

**Fact 2.** For $\mathbb{C}P^n$, $I$ is also known to vanish, by a non-trivial argument involving the Seidel representation (view [6] Theorem 1.11, [7] section 4.3). The link between the two is established by the reinterpretation of $I$ as the homogeneization of the valuation of the Seidel representation.

To fact 2 the following corollary holds:

Consider $(M, \omega) = (\mathbb{C}P^n, \omega_{FS})$, where the Fubini-Study form is normalized to represent the generator of the integer cohomology. Let $(P, \alpha) := (S^{2n+1}, \frac{\theta + \alpha_{std}}{2\pi})$ be a prequantization space of $M$. Denote $H := Ham(M, \omega)$ and $Q := \text{Quant}(P, \alpha)$. In this case two homomorphisms are defined:

The first:

**Definition 1.1.2.** (*Calabi-Weinstein homomorphism* [3 [25])

Given a path $\{\phi_t\}_{0 \leq t \leq 1}$: $\phi_0 = Id$ representing a class in $\widetilde{Q}$, let $h_t$ be its contact Hamiltonian, considered as a function on the base $\mathbb{C}P^n$.

Then

$$cw(\{\phi_t\}) := \int_0^1 dt \int_M h_t \omega^n$$

does not depend on homotopy with fixed endpoints and determines a homomorphism $\widetilde{Q} \to \mathbb{R}$.
Its restriction to $\pi_1(Q)$ is considered.

**Remark 1.1.2.** This definition works for the general situation of prequantization spaces.
And the second:

**Definition 1.1.3.** (Nonlinear Maslov index for loops [13])

Consider a loop \( \{ \tilde{\phi}_t \}_{t \in S^1} \) that represents a class \( b \in \pi_1(Q) \).

Denote by \( \{ \Phi_t \}_{t \in S^1} \) its lifting to a loop of homogenous Hamiltonian diffeomorphisms of \( \mathbb{C}^{n+1} \setminus \{0\} \). (Note that \( \mathbb{C}^{n+1} \setminus \{0\} \) is the symplectization of \( (P, \alpha) \))

For a point \( y \in CP^n \) denote by \( P_y \cong S^1 \) the fiber of \( P \) over \( y \), and by \( SP_y \cong \mathbb{C}^* \) the fiber of \( SP \) over \( y \).

Let \( m(\{ \Phi_{t+} \}_{t \in S^1}) \) be the Maslov index of the linearization of the lift \( \{ \Phi_t \} \) at a point \( 1 \in P_y \subset SP_y \cong \mathbb{C}^* \) in the fiber over \( y \), when \( T(\mathbb{C}^{n+1} \setminus \{0\}) \) is naturally identified with \( \mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C}^{n+1} \) by the linear structure.

Then,

\[
\mu(\{ \tilde{\phi}_t \}) := m(\{ \Phi_{t+} \})
\]

depends only on the class \( b \in \pi_1(Q) \) and is a homomorphism \( \pi_1(Q) \to \mathbb{R} \).

**Remark 1.1.3.** Although this invariant does not a-priori extend to a homomorphism \( \tilde{Q} \to \mathbb{R} \), it is known to extend to a homogenous quasimorphism \( \tilde{Q} \to \mathbb{R} \) ([13] [1]).

The two invariants of loops we’ve defined happen to be equal:

**Theorem 1.** On \( \pi_1(Q) \),

\[
\frac{1}{\text{Vol}(M, \omega^n)^{cw}} = \frac{1}{2(n+1)^\mu}
\]

**Example.** By taking a loop with Hamiltonian \( h_t \equiv 1 \), one obtains the standard Reeb rotations of \( \mathbb{C}^{n+1} \setminus \{0\} \). The Maslov index the linearization at any point is \( 2(n+1) \). Hence, in this case \( \frac{1}{2(n+1)^\mu} \mu = 1 \), and \( \frac{1}{\text{Vol}(M, \omega^n)^{cw}} = 1 \). So the equality holds in this case.

**Remark 1.1.4.** The given equality enables the extension of \( \mu \) to a homomorphism \( \tilde{Q} \to \mathbb{R} \).

**Remark 1.1.5.** This theorem answers a conjecture posed by A.Givental in [13]. The above formula transforms to the one in [13] as follows:

Note that while the Nonlinear Maslov index on loops does not vary when the symplectic form is positively scaled, the Calabi-Weinstein homomorphism does. In fact the right hand side of the equality proved varies homogeneously of degree 1 in the scale parameter. So that after our formula is established, we can scale the form so that \( \text{Vol}(M, \omega^n) = 1 \), to obtain the required formula.

The factor \( \frac{1}{2(n+1)} \) which we omitted comes purely from the definition of the Calabi-Weinstein homomorphism (and relates to the eternal question of whether the length of \( S^1 \) is 1 or \( 2\pi \)).

**Remark 1.1.6.** The main formula in the proof is:

\[
\frac{1}{2(n+1)} m(\hat{\gamma}) - \frac{1}{\text{Vol}(M, \omega^n)^{cw}(\hat{\gamma})} = -I(\gamma)
\]

for all loops \( \hat{\gamma} \) in \( Q \), and their corresponding loops \( \gamma \) in \( H \).

This formula works for prequantizations of arbitrary (spherically) monotone integral symplectic manifolds, with appropriate modifications, since for their symplectizations \( c_1 = 0 \) on spheres.

**Remark 1.1.7.** Another extension of \( \mu \) to a quasimorphism \( \mathcal{G} : \tilde{Q} \to \mathbb{R} \), is obtained by changing \( m \) to the linear Maslov quasimorphism on the universal cover of the symplectic group (refer to [29]), and averaging over the sphere with the natural measure \( \alpha \wedge (\alpha \wedge \alpha)^n \) w.r.t. the point of linearization.

This construction works in the general case of prequantization spaces, with simple modifications, and the quasimorphism thus obtained is equal to the one \( (\mathcal{G} : H \to \mathbb{R}) \) from [29] as follows:

\[
\frac{1}{2(n+1)} \mathcal{G} - \frac{1}{\text{Vol}(M, \omega^n)^{cw}} = -\frac{1}{\text{Vol}(M, \omega^n)^{cw}} \mathcal{G}
\]

And this links between the quasimorphism from [29] and the one from from [5].
Remark 1.1.8. An equality analogous to Theorem [11] works for other prequantizations \( P \) of \((\mathbb{C}P^n, \omega_{FS})\). The modification for \( c_1(P) = p[\omega_{FS}] \) is as follows: one considers the subgroup \( Q_p \subset \text{Quant}(S^{2n+1}) \) of quantomorphisms that commute with the action of the group of roots of unity of order \( p \). This group is a \( p \)-to-1 cover of \( \text{Quant}(P) \). Therefore, one can lift every \( p \)-th power \( \gamma^p \) of a loop \( \gamma \) in \( \text{Quant}(P) \) to a loop \( \tilde{\gamma}^p \) in \( Q_p \subset \text{Quant}(S^{2n+1}) \). One then defines \( \mu(\gamma) := \frac{1}{p} \mu(\tilde{\gamma}^p) \).

Theorem [11] has the following consequence:

**Corollary 1.** For \( (M, \omega) = (\mathbb{C}P^n, \omega_{FS}) \),

\[
\pi_1(H) = \mathbb{Z}/(n+1)\mathbb{Z} \oplus \text{Ker}(\text{cw}|_{\pi_1(Q)}),
\]

where \( \mathbb{Z}/(n+1)\mathbb{Z} = \pi_1(PU(n+1)) \).

This reproves the well-known fact that for \( \mathbb{C}P^n \), \( \mathbb{Z}/(n+1)\mathbb{Z} \) embeds into \( \pi_1(\text{Ham}) \).

Several proofs are already known - e.g. [30, 31, 22, 23].

**Remark 1.1.9.** Also, from this follows a curious fact that the \((n+1)\)-st power of any loop in \( H \) based at identity lifts to a loop in \( Q \) (based at identity).

**Remark 1.1.10.** The splitting works because the groups are abelian. A proof of this corollary can be found in section 2.2.

The mixed action-Maslov invariant has a share of generalizations, which are defined for all, not necessarily spherically monotone, symplectic manifolds.

**Definition 1.1.4.** (Generalized mixed action-Maslov invariants, along [17, 15, 19])

Let \( M := (M, \omega) \) be a symplectic manifold of dimension \( 2n \). \( \text{Ham} := \text{Ham}(M) \) - the group of Hamiltonian diffeomorphisms.

The generalized mixed Action-Maslov Invariants are a family of homomorphisms \( \pi_1(\text{Ham}, \text{Id}) \to \mathbb{R} \), that are defined similarly to the usual Chern numbers of complex vector bundles:

To a Hamiltonian loop based at identity, \( \phi = \{\phi_t\}_{t \in S^1} \subset \text{Ham} \) one can associate by the clutching construction a Hamiltonian fiber bundle \( \pi : P \to S^2 \) over \( S^2 = \mathbb{C}P^1 \) with fiber \( M \). This gives a one-to-one correspondence between isomorphism classes of such Hamiltonian bundles over the 2-sphere and \( \pi_1(\text{Ham}, \text{Id}) \). The group operation in \( \pi_1 \) corresponds to the "fibered"-connected sum operation of bundles up to isomorphism. ([17]).

For \( \alpha \in \pi_1(\text{Ham}, \text{Id}) \) we’ll denote by \( P_\alpha \) the corresponding isomorphism class of bundles (or rather it’s representative).

Let \( V := \text{Ker}(\pi_* : TP \to TS^2) \subset TP \) be the vertical (sub)bundle. It is a symplectic vector bundle of rank \( 2n \), and so, by the canonical-up-to-homotopy choice of a compatible almost complex structure, it has well defined ("vertical") Chern classes

\[
\{c_l := c_l(V) \in H^{2l}(P, \mathbb{Z})\}_{0 \leq l \leq n}.
\]

Let \( u \in H^2(P, \mathbb{R}) \) denote the coupling class of the Hamiltonian vector bundle \( P \), defined by the conditions

\[
1) u|_{\text{fiber}} = \omega; \quad 2) \int_{\text{fiber}} u^{n+1} = 0 \Leftrightarrow u^{n+1} = 0.
\]

**Remark 1.1.11.** The equivalence in 2) is specific to the case where the base is 2-dimensional (consult e.g. [15]).

**Remark 1.1.12.** More generally, these characteristic classes can also be defined as elements in the cohomology of corresponding classifying spaces. ([15])

Now choosing, as in the definition of Chern numbers, a monomial \( \alpha = (c_1)^{i_1} \cdots (c_n)^{i_n} \omega^j \) of degree \( 2n + 2 \), define \( I_\alpha : \pi_1(\text{Ham}, \text{Id}) \to \mathbb{R} \) by
\[ I_\alpha(a) := \int_{P_a} (c_1)^{i_1} \cdots (c_n)^{i_n} u^j. \]

\( I_\alpha \) is a homomorphism, as is seen from the compatibility of the clutching construction with the group operations (turn to \[17\]).

A simple computation shows that in the monotone case, \( I(c_1) \cdot L^{n+1} - L^n \) are all proportional to \( I \).

**Computation 1.** For a monotone symplectic manifold \((M, \omega)\), with \([\omega] = \kappa c_1(M)\),

\[ I(c_1) \cdot L^{n+1} - L^n = -\frac{L}{\kappa^E} \cdot \text{Vol}(M, \omega^n) \cdot I. \]

In particular, \( I(c_1) u^n = -\frac{1}{\kappa} \cdot \text{Vol}(M, \omega^n) \cdot I \), and in the case \( \kappa = 1 \), \( I(c_1) u^{n+1} = -(n+1) \cdot \text{Vol}(M, \omega^n) \cdot I \).

**Proof.** For monotone symplectic manifolds, \( u = \kappa c_1 + I \cdot \pi^*(a) \), where \( a \) is the generator of \( H^2(S^2, \mathbb{R}) \), talking in the language of symplectic fibrations [27]. So that \( c_1 = \frac{1}{\kappa}(u - I \cdot \pi^*(a)) \). Plugging this in, we readily obtain the formula. \( \square \)

### 1.2 Futaki invariants

The mixed action-Maslov invariant and its generalizations happen to be related to Futaki invariants.

**Definition 1.2.1. (Futaki Invariant)**

Let \((M, \omega, J)\) be a compact Fano manifold. That is to say Kahler and \([\omega] = c_1(M)\). Denote by \( \frak{h}(M) \) the complex Lie algebra of holomorphic sections of \( T^{(1,0)} M \). Then the Futaki Invariant \( F : \frak{h}(M) \to \mathbb{C} \) introduced by A. Futaki in [8] is defined as follows:

By the Kahler property, \( \omega \) is a closed, real, \((1,1)\)-form in the class \( c_1(M) \). And the Ricci form \( \text{Ric}(\omega) \) of \( \omega \) is also a closed, real, \((1,1)\)-form in the class \( c_1(M) \) (by the symmetry of the curvature tensor, and by Chern-Weil theory of characteristic classes). Therefore by the \( \text{dd}^c \)-Lemma, there exists a function \( f_{\omega} \in C^\infty(M, i\mathbb{R}) \) unique up to constant, such that,

\[ \text{Ric}(\omega) - \omega = \partial\bar{\partial} f_{\omega}. \]

Given \( Z \in \frak{h}(M) \), define:

\[ F(Z) := \int_M Z(f_{\omega}) \omega^n \]

In [8] it is proven that the integral on the right hand side does not depend on the Kahler form \( \omega \) in the class \( c_1(M) \), and that

\[ F : \frak{h}(M) \to \mathbb{C} \]

is a homomorphism of Lie algebras.

**Remark 1.2.1.** The vanishing of this invariant is a necessary condition for the existence of a Kahler-Einstein metric ([8]). In the case of smooth toric manifolds this is also sufficient ([37]).

Even though \( F \) is defined for vector fields and \( I \) - for loops, when restricted to the group

\[ K := \text{Iso}_0(M) := (\text{Ham}(M, \omega) \cap \text{Aut}(M, J))_0 \]

on which both are defined, such vector fields and loops are essentially equivalent, since

\[ \pi_1(K) \otimes \mathbb{R} \cong \text{Lie}(K)/[\text{Lie}(K), \text{Lie}(K)] \]
So that, by taking duals, one gets an equivalence between homomorphisms of abelian groups $\pi_1(K) \to \mathbb{R}$ and homomorphisms of Lie algebras $\text{Lie}(K) \to \mathbb{R}$.

One then has the following equality:

**Theorem 2.** Let $(M, J, \omega)$ be a compact Fano manifold.

Then

$$F = -\operatorname{Vol}(M, \omega^n) I$$

when restricted to $K = \text{Isom}(M, J, \omega)$.

The equality is understood via the isomorphism $\operatorname{Hom}(\pi_1(K), \mathbb{R}) \cong \operatorname{Hom}(\text{Lie}(K), \mathbb{R})$.

There are lots of generalizations of the Futaki invariants, with perhaps the most inclusive family being the one introduced in [9]. We present the original definition in a slightly rewritten form, for the case of the frame bundle of the holomorphic tangent bundle of a compact Kahler manifold.

**Definition 1.2.2. (Generalized Futaki Invariants)**

Let $(M, \omega, J)$ be a compact Kahler manifold. Denote by $\mathfrak{h}_0(M)$ the space of holomorphic vector fields in $\mathfrak{h}(M)$ that have a zero. It’s known that

$$\mathfrak{h}_0(M) = \{ Z \in \mathfrak{h}(M) \mid \exists ! f_Z \in C^\infty(M, \mathbb{C}) \text{ s.t. } i_Z \omega = -\partial f_Z \land \int_M f_Z \omega^n = 0 \}$$

(e.g. [18])

Consider the frame bundle $Fr$ of $T^{(1,0)}(M)$. This is a principal holomorphic $\text{GL}(n, \mathbb{C})$ bundle over $M$. The automorphism group $\text{Aut}(M, J)$ acts on $Fr$ commuting with the action of $\text{GL}(n, \mathbb{C})$.

The invariant polynomials on $\text{Lie}(\text{GL}(n, \mathbb{C}))$ are generated by elementary symmetric polynomials in the eigenvalues, which are taken with suitable coefficients to correspond to Chern classes. Denote them by $c_1, ..., c_n$.

Take $Z \in \mathfrak{h}_0$. Since $\text{Aut}(M, J)$ acts on $Fr$ by bundle maps, $Z$ lifts to a holomorphic vector field $\hat{Z}$ on $Fr$.

Set $w_Z := \omega + f_Z$. This is a form of mixed degree.

Choose a $(1, 0)$ connection form $\theta$ on $Fr$. Let $\Theta$ be the corresponding curvature form.

Take a polynomial $(c_1)^{i_1} \cdots (c_n)^{i_n}$ of degree $k \leq n$, and take $w_Z^l$, s.t. $k + l = n$.

Denote $\alpha := (c_1)^{i_1} \cdots (c_n)^{i_n} w^l$.

Then

$$F_\alpha(Z) := \int_M (c_1)^{i_1} \cdots (c_n)^{i_n}(\theta(\hat{Z}) + \Theta) w_Z^l.$$  

It is shown in [9] that $F_\alpha(Z)$ is independent of the choice of the $(1, 0)$ connection form $\theta$, and defines a homomorphism of Lie algebras:

$$F_\alpha : \mathfrak{h}_0(M) \to \mathbb{C}$$

**Remark 1.2.2.** $F_\alpha$ also depends only on the cohomology class of the Kahler form $\omega$.

**Remark 1.2.3.** The invariant $F_{\alpha, \omega^n}$, also known as the Bando-Calabi-Futaki character, is an obstruction to the existence of a constant scalar curvature Kahler metric with the Kahler form in the given cohomology class. It also has a definition similar to Definition 1.2.1 (peruse [9] and the references therein).

**Remark 1.2.4.** Following [12, 11], one can relate the invariants $F_\alpha$ to a type of equivariant cohomology.

Theorem 2 then generalizes to the following:

**Theorem 3.** Let $(M, J, \omega)$ be a compact Kahler manifold.

Then

$$F_\alpha = I_\alpha$$

when restricted to $K = \text{Isom}(M, J, \omega)$.

The equality is understood via the isomorphism $\operatorname{Hom}(\pi_1(K), \mathbb{R}) \cong \operatorname{Hom}(\text{Lie}(K), \mathbb{R})$. 

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Remark 1.2.5. Theorem 3 follows from Theorem 2 for the following reason: in [12] (Proposition 2.3) it is proven that for Fano Kahler manifolds, $F_{(c_1)^n+1} = (n + 1)F$. Also from Computation 1 one has $I_{(c_1)^n+1} = -(n + 1)\text{Vol}(M, \omega^n) \cdot I$. So $F_{(c_1)^n+1} = I_{(c_1)^n+1}$ means $F = -\text{Vol}(M, \omega^n)I$.

Remark 1.2.6. Theorem 3 yields the equality $-I_{c_1, u^n}/\text{Vol}(M, \omega^n) = I = I_{c_1, u}/\text{Euler}(M)$ on $K$ for Fano manifolds.

Indeed, talking in the language of symplectic fibrations, $u = c_1 + I \cdot \pi^*(a)$ for Fano manifolds, where $a$ is the generator of $H^2(S^2, \mathbb{R})$, by [24]. So $I_{c_1, u} = I_{c_1} + I < c_n \pi^*(a), [P_\gamma] > = I_{c_n, c_1} + I \cdot \text{Euler}(M)$. By Theorem 3 $I_{c_n, c_1} = F_{c_n, c_1}$ on $K$. And the latter vanishes on its domain of definition ([12] [11] [10]). So $I_{c_n, u} = I \cdot \text{Euler}(M)$, and the rightmost equality is proven. The leftmost equality follows from Computation 1.

Remark 1.2.7. Theorem 3 also lets one recover the toric computations of $I_{c_1, u^n}$ in [33] from the computations of $F_{c_1, u^n}$ in [24] [25].

1.3 Example: the case of toric loops

Although the computation of $I$ on all loops on toric manifolds yet remains an open issue, one can compute its restriction to those Hamiltonian loops that come from the torus action. We’ll address these as “toric loops”.

A homomorphism on $\pi_1(T)$ can be considered as a vector in $\text{Lie}(T)^*$. The corresponding vectors are most conveniently expressed through the following notion of barycenters $\{B_k\}_{0 \leq k \leq n}$ of Delzant polytopes:

Definition 1.3.1. (k-dimensional measure on $\Delta$) As the faces of the polytope are rational, there is a lattice on each induced from the integer lattice of the ambient space. This lattice defines up to a fundamental parallelotope has measure 1. The constant can be normalized in such a way that any multiplicative constant a measure on the face.

The k-dimensional measure is the sum of such measures over all $k$-dimensional faces.

Definition 1.3.2. (The k-th barycenter)

$B_k :=$ barycenter of the k-dimensional measure.

Remark 1.3.1. Note that the k-dimensional measure is the push forward by the moment map of $(\omega^k)/k!$ restricted to the corresponding $2k$ dimensional symplectic invariant subspaces of $M$.

Consider a Delzant Polytope with vertices $\{P_k\}_{1 \leq k \leq V}$, and faces $\{F_j\}_{1 \leq j \leq F}$ given by primitive “normals” $\{l_j\}_{1 \leq j \leq F}$ in the integer lattice of the dual space. So that

$$\Delta = \bigcap_{j=0}^F \{l_j \leq \kappa_j\}$$

for some support numbers $\kappa = (\kappa_1, ..., \kappa_F)$.

For monotone toric manifolds, $I$ was computed in [6]. The result is expressed as follows in our notations:

$$I = -B_0 + B_n$$

For general toric manifolds, $I_{c_1, u^n}$ was computed in [33]. The result is expressed as follows in our notations:

$$I_{c_1, u^n} = -n!\text{Vol}(n-1)(\Delta)(B_{n-1} - B_n)$$

Remark 1.3.2. Comparing the formulas and using Computation 1 one obtains

$$B_0 - B_n = -C_\Delta(B_{n-1} - B_n)$$

where $C_\Delta = \frac{\text{Vol}_{n-1}(\Delta)}{\text{Vol}(\Delta)}$ is a positive constant. Which means, curiously enough, that the three barycenters $B_0, B_{n-1}$ and $B_n$ are collinear for monotone toric manifolds, and are either all distinct or all equal.
Remark 1.3.3. Another way to get the collinearity of $B_0, B_{n-1}, B_n$ in the toric Fano case is to compare the two toric computations of $F$ in [21], and [1].

What follows is a uniform computation of all $\{I_{n,l}u^{n+1-L}\}_{0 \leq L \leq n}$.

**Theorem 4.** $I_{n,l}u^{n+1-L}/(n-L)!Vol(n-L)(\Delta) = -(n+1-L)(B_{n-L} - B_n)$.

**Remark 1.3.4.** In particular, $I_{n,n}/Vol(n)(\Delta) = -B_0 + B_n$, on the torus.

Remark 1.3.5. An explanation of the collinearity phenomenon in the toric Fano case follows from this equality and Remark 12.6 via $Vol(n)(\Delta) =$ number of the vertices in the polytope $= \text{Euler}(M)$.

This computation has a corollary related to a result in [22].

Consider a symplectic toric manifold, and a Hamiltonian loop $\gamma$ coming from the toric action. This loop corresponds to a point $l$ in the integer lattice in $\text{Lie}(T)$.

Let the moment polytope be $\Delta = \Delta(\overline{n}(0)) = \bigcap_{j=0}^{\text{F}} \{ \xi_j \leq \kappa_j(0) \}$ for some support numbers $\overline{n}(0) = (\kappa_1(0), ..., \kappa_{\text{F}}(0))$.

Denote by $C$ the chamber of all support numbers $\overline{n}$ for which the polytope $\Delta(\overline{n})$ is analogous to the original one ([22]). One can think of continuous deformations of $\Delta$ in the space of Delzant polytopes with the given conormals.

In [22] it is proven that if $\gamma$ is contractible in $\text{Ham}$, then the function $f(\overline{n}) := \langle B_0(\overline{n}), l \rangle$ is a linear function of $\overline{n}$ with integer coefficients (that is, the restriction of such a function to $C$).

In other words, $l$ considered as a function on $\text{Lie}(T)^*$ is mass-linear with integer coefficients.

A lemma based on Moser’s homotopy method is used, saying that if such a toric loop is contractible in $\text{Ham}$ for the original polytope, then it will be contractible for all $\overline{n}$ in an open neighbourhood $U$ of the original $\overline{n}(0)$.

Note that as $\langle B_0(\overline{n}), l \rangle$ is a priori a rational function of $\overline{n}$, it’s enough to show linearity on such an open set.

**Theorem 4** lets one prove a related result in the following manner:

**Corollary 2.** Assume a loop $\gamma$ in $\text{Ham}$ coming from the toric action is contractible. Let $l$ be the corresponding point in the integer lattice in $\text{Lie}(T)$.

Then

$$\langle B_0(\overline{n}), l \rangle = \langle B_0(\overline{n}), l \rangle$$

(which is a linear function of $\overline{n}$, with coefficients in $\frac{1}{\text{Vol}_0} \mathbb{Z}$)

and

$$\langle B_0(\overline{n}), l \rangle = \langle B_k(\overline{n}), l \rangle$$

for all other $k$, as well.

**Proof.** If $\gamma$ is contractible then $I_{l,n}(\gamma) = 0$ (for all $\overline{n}$ in $U$). Which, by Theorem 4 means that $\langle B_0(\overline{n}) - B_0(\overline{n}), l \rangle = 0$ for all $\overline{n} \in U$. Hence, as both are rational functions of $\overline{n}$, the equality holds for all $\overline{n}$ in $C$. The second part is obtained similarly, by applying Theorem 4 to $I_{l,n-k}u^{k+1}$.

**Remark 1.3.6.** An alternative way to state this corollary is that if $\gamma$ in $\text{Ham}$ coming from $l \in \pi_1(T) \subset \text{Lie}(T)$ is contractible, then $l$ is perpendicular to the affine span of $\{B_k(\overline{n})\}_{0 \leq k \leq n}$ for all $\overline{n}$ in $C$.

Also, since under the contractibility condition $\langle B_0(\overline{n}), l \rangle$ is linear with integer coefficients ([22]), it follows that all $\langle B_k(\overline{n}), l \rangle$ are linear with integer coefficients. This could be applied to $\langle B_0(\overline{n}), l \rangle$, which has coefficients in $\frac{1}{\text{Vol}_0} \mathbb{Z}$, a-priori, to give lower bounds on the orders of torsion elements of $\pi_1(\text{Ham})$ represented by toric loops.

**Remark 1.3.7.** In [34], for $\mathbb{C}P^n$ bundles over $\mathbb{C}P^l$ and for the blowup $Bl_1(\mathbb{C}P^n)$ of $\mathbb{C}P^n$ at one point, a similar result was obtained by using the invariant $I_{c_1, u^n}$. 
1.4 A map of the rest of the article

In section 2 the theorems are proven in the order of their appearance. A stand-alone differential geometric proof of Theorem 2 is provided, as it is interestingly similar to the proof of Theorem 1.

In section 3 several related questions are posed.

2 Proofs

2.1 Proof of Theorem 1

Let \( \{ \hat{\phi}_t \}_{t \in S^1} \) be a loop in \( Q \), and \( \{ \phi_t \}_{t \in S^1} \) be the corresponding ”downstairs” loop in \( H \). Let \( F_t \) be the normalized Hamiltonian function for \( \{ \phi_t \}_{t \in S^1} \).

Note that \(-h_t\) is a (non-normalized) Hamiltonian for \( \{ \phi_t \}_{t \in S^1} \). (the minus sign follows from the fact that the tautological line bundle has Chern class \(-\alpha\), where \( \alpha \) is the generator of the cohomology of \( CP^n \))

As both \(-h_t\) and \( F_t \) are both hamiltonians for the same loop in \( H \), they differ by a constant dependent on time.

\[ h_t = F_t + c(t) \]

or, for future reference,\[ -F_t = -h_t + c(t) \]

Then by integrating over \( M \), we get:

\[ Vol(M, \omega^n) \cdot \int_0^1 c(t) dt = \int_0^1 dt \int_M h_t \omega^n \]

But the right hand side equals \( cw(\{ \hat{\phi}_t \}) \), therefore

\[ \int_0^1 c(t) dt = \frac{1}{Vol(M, \omega^n)} \cdot cw(\{ \hat{\phi}_t \}) \]

For a point \( 1 \in P_y \) for \( y \in CP^n \) consider the path \( \{ \hat{\phi}_t 1 \}_{t \in S^1} \) in \( P = S^{2n+1} \). It has a unique, up to homotopy with fixed boundary, filling disk \( \hat{D}_0 \), since \( \pi_1(S^{2n+1}) = 0 \) and \( \pi_2(S^{2n+1}) = 0 \). And to this disk there corresponds a canonical filling disk \( D_0 := p \circ \hat{D}_0 \) of the ”downstairs” path \( \{ \phi_t y \}_{t \in S^1} \).

From this point, the theorem follows from the following two lemmas:

**Lemma 1.** Consider a trajectory \( m(\{ \Phi_t(1) \}_{t \in S^1}) \) for a point \( 1 \in P_y \subset SP_y \cong \mathbb{C}^* \) in the fiber over \( y \). Then, for the canonical filling disk \( D_0 \),

\[ m(\{ \Phi_{t+1} \}_{t \in S^1}) = m(\{ \phi_t \}) \]

**Lemma 2.** For the canonical filling disk \( D_0 \) it is true that:

\[ \frac{1}{2(n+1)} m_D(\{ \phi_{t+1} \}) = I(\{ \phi_t \}) + \int_0^1 c(\tau) d\tau \]

These lemmas yield,\[ \frac{1}{2(n+1)} m(\{ \Phi_{t+1} \}) = I(\{ \phi_t \}) + \int_0^1 c(\tau) d\tau \]

But since, \( I \equiv 0 \), by Seidel’s argument, we’ll have
\[
\frac{1}{2(n+1)}m(\{\Phi_{t*}\}) = \int_0^1 c(t)dt = \frac{1}{Vol(M,\omega^n)} \cdot cw(\{\hat{\phi}_t\}),
\]

So that
\[
\frac{1}{2(n+1)}\mu(\{\hat{\phi}_t\}) = \frac{1}{Vol(M,\omega^n)} \cdot cw(\{\hat{\phi}_t\})
\]

and we’re done.

We proceed to prove the lemmas:

**Proof of Lemma 1**

Since \(\hat{\phi}_t\) is a quasimorphism, it preserves the vertical and the horizontal subbundles of \(TP\). Therefore \(\Phi_t\) preserves the corresponding vertical and horizontal subbundles of \(TL^\times\).

So, \(m(\{\Phi_{t*}\}) = m_{D_0}(\{\Phi_{t*}\}) = m_{D_0}(\{\Phi_{t*}|_{Hor}\}) + m_{D_0}(\{\Phi_{t*}|_{Vert}\})\)

But \(m_{D_0}(\{\Phi_{t*}|_{Hor}\}) = m_{D_0}(\{\hat{\phi}_t\})\), since \(Hor \cong p^*TM\) as symplectic vector bundles;

And \(m_{D_0}(\{\Phi_{t*}|_{Vert}\}) = 0\), since \(\Phi_{t*} : Hor_1 \to Hor_{\theta,1}\) is just equal to the parallel translation map:

\(\Gamma_{\{\phi_{t*}\}} : Hor_1 \to Hor_{\theta,1}\) since both preserve the connection 1-form, and the fiber is 1 dimensional.

And since the connection on \(\mathbb{C}^{n+1} \setminus \{0\}\) is trivial, \(m_{D_0}(\{\Phi_{t*}|_{Vert}\}) = m(\{I\}t \in S^1)\).

**Proof of Lemma 2**

By definition, \(I(\{\phi_t\}) = \int_{D_0} \omega - \int_0^1 F_1(\phi_t(y)dt - \frac{n+1}{n-1} \cdot m_{D_0}(\{\phi_{t*}\})\)

But \(-\int_0^1 F_1(\phi_t(y)dt = -\int_0^1 h_t(\hat{\phi}_1) + \int_0^1 c(t)dt\)

Substituting, and noting that: \(\int_{D_0} \omega - \int_0^1 h_t(\hat{\phi}_1) = 0\), by the Stokes formula (indeed, \(\int_{D_0} \omega - \int_0^1 h_t(\hat{\phi}_1) = \int_{D_0} p^*\omega - \int_{\hat{\phi}_1(1)} p^*\omega = 0\) since \(p^*\omega = d\alpha\), and \(\{\hat{\phi}_1\}t \in S^1 = \partial D_0\), we get

\(I(\{\phi_t\}) - \int_0^1 c(t)dt = -\frac{1}{2(n+1)} \cdot m_{D_0}(\{\phi_{t*}\})\)

So \(\frac{1}{2(n+1)}m_{D_0}(\{\phi_{t*}\}) = -I(\{\phi_t\}) + \int_0^1 c(t)dt\)

**2.2 Proof of Corollary**

**Claim:** There exists a homomorphism \(\alpha : \pi_1(H) \to \mathbb{Z}/(n+1)\mathbb{Z}\), such that

1. \(Ker(\alpha) = Ker(cw|_{\pi_1(Q)})\) and
2. on the element \(\theta\) represented by the \(S^1\) action of the rotation of the first homogenous coordinate, it takes value \(1 \in \mathbb{Z}/(n+1)\mathbb{Z}\)

Note that property 2 means that \(\pi_1(H) \ni \theta \mapsto \mathbb{Z}/(n+1)\mathbb{Z}\) and \(\alpha|_{<\theta>} : <\theta> \to \mathbb{Z}/(n+1)\mathbb{Z}\) is an isomorphism.

The theorem now follows from the claim, as \(\pi_1(H)\) is abelian.

**Proof of Claim:**

Consider a loop \(\gamma = \{\phi_t\}t \in S^1\) based at \(Id\) in \(H\), and lift it to a path \(\hat{\gamma} = \{\hat{\phi}_t\}\) in \(Q\) (by the "canonical lifting") using the mean-normalized Hamiltonian. Obviously, this path represents an element of \(Ker(cw)\). Note that if it closes up to a loop, then it represents one in \(Ker(cw|_{\pi_1(Q)})\).

Since it’s a lift of a loop downstairs, we can close it up by a path \(\delta = \{\hat{R}_t\}0 \leq t \leq \alpha, \alpha \in \mathbb{R}/\mathbb{Z}\) of ("Reeb") rotations of the fibers, to get a loop \(\hat{\gamma} \ast \delta\) in \(Q\).

Note that \(\gamma \mapsto \alpha\) gives a homomorphism \(\pi_1(H) \to \mathbb{R}/\mathbb{Z}\). We contend that this homomorphism actually takes values in \(\frac{1}{n+1}\mathbb{Z}/\mathbb{Z} \cong \mathbb{Z}/(n+1)\mathbb{Z}\).

Indeed, choose a representative \(0 \leq \alpha < 1\) for \(\alpha\). Then, \(\frac{1}{2(n+1)}m(\gamma) = \frac{1}{Vol(M,\omega^n)} cw(\hat{\gamma}) = \alpha\). So, \(\alpha \in \frac{1}{n+1}\mathbb{Z}\) (since the Maslov index is always even).
Property 1 now follows from this computation.
Property 2 is a straightforward computation. One obtains that the representative $0 \leq \alpha < 1$ equals $\frac{1}{n+1}$, that is $1 \in \mathbb{Z}/(n + 1)\mathbb{Z}$.

2.3 Proof of Theorem 2

First, note that it is enough to prove the equality on $S^1$ subgroups $\Phi_t$ of $K$, since $K$ is a compact Lie group.

The proof is based on the fact, that the Calabi-Yau theorem gives $(M, \omega)$ a canonical prequantization.

The prequantization is built as follows: Let $L = \bigwedge^n T^{(1,0)} M^*$ be the canonical line bundle on $M$. Let $\eta$ be the Calabi-Yau form with $\text{Ric}(\eta) = \omega$.

It gives us a Hermitian metric on $T^{(1,0)} M$. To the pair $(L, \rho)$ there corresponds a unique complex connection $\nabla$Ch which is compatible with the metric and with the structure of a holomorphic bundle (it is called the Chern connection of $(L, \rho)$). Denote by $\bar{\alpha}$ the corresponding connection one-form $\alpha$ with values in $C$.

Now the prequantization $(P, \alpha)$ is the principal $S^1$-bundle $P$ of $\rho$-unit vectors in $L$, together with the $\mathbb{R}$-valued connection one-form $\alpha = -i\bar{\alpha}|_P$.

As $\Phi_t$ are automorphisms of all the structures involved, they lift canonically to automorphisms $\hat{\Phi}_t$ of $(P, \alpha)$.

In detail the lift acts as follows:

$$
(x, p) \mapsto (\Phi_t(x), ((\Phi_t)^{-1})^* p)
$$

$x \in M, p \in P_x$

To the loop of automorphisms $\hat{\Phi}_t$ there corresponds a contact Hamiltonian $\hat{h} = \alpha(\frac{d}{dt}|_{t=0} \hat{\Phi}_t)$. This is an $S^1$ invariant function, and therefore can be considered as a function $h$ on $M$.

Definition 2.3.1. (divergence of $Z \in \mathfrak{h}(M)$ w. r. t. a Kahler form $\eta$)

For a holomorphic vector field $Z \in \mathfrak{h}(M)$ define $\text{div}_{\eta}(Z)$ to be the unique function $\psi \in C^\infty(M, \mathbb{C})$, such that $d(iZ\eta^n) = \psi \eta^n$.

The proof is composed of a proposition from [12], and two lemmas:

Lemma 3. (A. Futaki, S. Morita [12]) $F(Z) = \frac{1}{i} \int_M \text{div}_\eta(Z) \omega^n$ where $\eta$ is the Calabi-Yau form of $\omega$ that is defined uniquely by the condition $\text{Ric}(\eta) = \omega$.

Lemma 4. Let $h$ be the contact Hamiltonian introduced earlier. Then $ih = -\text{div}(Z)$.

Lemma 5. For the same contact Hamiltonian $h$, 

$$
\int_M h \omega^n = \text{Vol}(M, \omega^n) \cdot I(\gamma)
$$

Where $\gamma$ is the forementioned loop of automorphisms $\{\Phi_t\}, t \in \mathbb{R}/\mathbb{Z}$.

Indeed, given these lemmas one has:

$$
F(Z) = \frac{1}{i} \int_M \text{div}_\eta(Z) \omega^n = \frac{1}{i} \int_M -ih \omega^n = -\text{Vol}(M, \omega^n) \cdot I(\gamma).
$$

We now go on to prove the lemmas:
Proof of Lemma 5. Let $h$ be the contact Hamiltonian introduced earlier. Denote by $H$ the normalized Hamiltonian of the flow $\Phi_t$.

Then $h = H + c$, where $c \in \mathbb{R}$ is a constant.

Therefore

$$\int_M h \omega^n = \text{Vol}(M, \omega^n) \cdot c.$$ 

Claim: $c = I(\gamma)$. 

The lemma follows from the claim by substituting into the last formula. Indeed, we obtain

$$\int_M h \omega^n = \text{Vol}(M, \omega^n) \cdot I(\gamma).$$ 

Proof of Claim:

Indeed, $c = h(p) - H(p)$ for any $p \in M$.

As $M$ is compact, one can choose $p$ to be a critical point of $H$. This point is a fixed point of the flow. Therefore computing $I$ w.r.t. this point and the trivial disk,

$$I(\gamma) = -H(p) - \text{Maslov}/2(\gamma_s)$$

where $\gamma_s(t) = \Phi_{t*}$ is the linearized loop of the flow at this point.

Yet $-h(p)$ = (speed of rotation of $\det(\gamma_s(t))$ in the fibre over $p$). And this, as the speed is constant, in turn equals to (# of full turns of $\det(\gamma_s(t))$) which, by definition of the Maslov index, is $\text{Maslov}/2(\gamma_s(t))$.

Therefore,

$$I(\gamma) = -H(p) + h(p).$$ 

Proof of Lemma 4. We use a formula proved in [14] (Proposition 2.1.2, p. 237):

Let $s$ be a section of the canonical line bundle $L$, and $X$ - a (real) holomorphic vector field on $M$. Then,

$$\mathcal{L}_X s = \nabla_X s + \text{div}(X) \cdot s$$

where $\text{div}(X)$ is defined as follows:

$$\text{div}(X)_p := \text{trace}_C(V \to \nabla_V X).$$

(note that $V \to \nabla_V X$ is a $J$-linear operator $T_pM \to T_pM$.)

Locally, we can choose a flat section $s$ of unit length. This is possible, as the connection preserves the Hermitian metric.

For a flat section, the equation reduces to:

$$\mathcal{L}_X s = \text{div}(X) \cdot s$$

On the other hand:

$$\mathcal{L}_X s = \frac{d}{dt}|_{t=0} \Phi^*_t s$$

and as $\Phi^*_t s$ is also a flat section of unit length,
\[ \Phi_t^* s = e^{ia(t)} \cdot s \]

where \( a(t) \) is a function of the time only (because of the flatness condition). Moreover \( -a'(0) = h \) is the contact Hamiltonian.

Therefore,

\[ L_X s = \frac{d}{dt}|_{t=0} e^{ia(t)} \cdot s = \]
\[ = ia'(0) \cdot s = -i h \cdot s \]

Comparing the two computations of \( L_X s \), we obtain

\[ \text{div}(X) = -i h \]

The last remark is that for \( Z := (X - iJX)/2 \in \mathfrak{h}(M) \)

\[ \text{div}_\eta(Z) = \text{div}(X). \]

(by the same J-linearity; see e.g. [16])

\( \square \)

### 2.4 Proof of Theorem 3

The key lemma of the proof is the following simple fact:

**Lemma 6.** Let \( (M, \omega, J) \) be a compact Kähler manifold. Let \( f \in C^\infty(M, \mathbb{R}) \) be a real valued smooth function on \( M \). Then

\[ i_X \omega = -df \iff i_Z \omega = -\bar{\partial} f \]

where \( Z = (X - iJX)/2 \).

As in the proof of Theorem 2 it’s enough to prove the given equality on \( S^1 \) subgroups of \( K \).

Given such an \( S^1 \) subgroup \( \gamma \) of \( K \) one has the corresponding Hamiltonian fibration \( P \) over \( \mathbb{C}P^1 \).

According to [17] (Remark 3.C), this bundle is just the restriction to \( \mathbb{C}P^1 \) of the universal bundle \( M_{S^1} := M \times_{S^1} S^\infty \to \mathbb{C}P^\infty \). And so, the cohomology \( H^*(P, \mathbb{R}) \) is just \( H_{S^1}(M, \mathbb{R}) \otimes_{\mathbb{R}[z]} \mathbb{R}[z]/z^2 \mathbb{R}[z] \).

Also, the vertical Chern classes of \( P \) are nothing but the restrictions equivariant Chern classes of the tangent bundle.

Denote by \( f \) the normalized Hamiltonian function of the \( S^1 \) action, and by \( X \) the corresponding vector field.

Consider now the Cartan model for equivariant cohomology:

The coupling class is represented by \( u = \omega + zf \).

Given an \( S^1 \)-equivariant connection form \( \theta \) on the frame bundle of the complex vector bundle \( TM \) and its curvature \( \Theta \), the equivariant Chern classes are represented by \( c_r(z\theta(\hat{X}) + \Theta) \), where \( c_r \) is the elementary symmetric polynomial of degree \( r \), and \( \hat{X} \) is the lift of \( X \) to the frame bundle (proven e.g. in [2]).

Therefore the value of the generalized mixed action-Maslov invariant \( I_\alpha \) on \( \gamma \) is given by

\[ I_\alpha(\gamma) = \int_M (c_1)^i_1 \cdots (c_n)^i_n (\theta(\hat{X}) + \Theta)(\omega + f)^j. \]

By invoking Lemma 6 and choosing the connection to be of type \( (1,0) \) this now equals to \( F_\alpha(Z) \) with \( Z = (X - iJX)/2 \).
2.5 Proof of Theorem 4

Take a toric $S^1$ action $\gamma$ with mean normalized Hamiltonian $f$.

Using the language of Hamiltonian fibrations, one has, by definition

$$I_{c_L}u^{n+1-L}(\gamma) = \int_{P_r} c_L u^{n+1-L} = <PD_P(c_L), u^{n+1-L}>$$

Denote by $\Delta_{n-L}$ the formal sum of the faces of $\Delta$ of $dim = n - L$. And let $N = m^{-1}(\Delta_{n-L})$ be the formal sum of the preimages of these faces by the moment map (note that these preimages are all $T$-invariant).

Let $F_{n-L}$ be the chain obtained from $N$ in the same way as $P_\gamma$ is obtained from $M$. Then $PD_P(c_L)$ is represented by $F_{n-L}$. This follows from [15], the fact that the first Chern class of a holomorphic line bundle is the Poincare dual of the corresponding divisor, and choosing an equivariant meromorphic section for each of the relevant holomorphic line bundles.

According to [27], the coupling class is represented by the form:

$$\{ \omega | M \times D_+; \omega + d(\psi(r)f(x)dt) \}$$

where $D_+$ and $D_-$ are two disks (from which the sphere is glued), $r$ and $t$ are the radial and the angular coordinates on the disk $D_-$, $x$ denotes a point on $M$, and $\psi$ is a function that vanishes near 0 and equals to 1 near 1.

So that $u^{n+1-L}$ restricted to $F_{n-L}$, equals to

$$\{ 0 | N \times D_+; (n + 1 - L)\omega^{n-L}\psi(r)f(x)drdt \}$$

Hence $\int_{F_{n-L}} u^{n+1-L} = (n + 1 - L) \int_N f^{(n-L)}$, so that:

$$<PD_P(c_L), u^{n+1-L}> = (n + 1 - L) \int_N f^{(n-L)}$$

Yet, the right hand side is just equal to $-(n + 1 - L)!Vol_{(n-L)}(\Delta)(B_{n-L} - B_n)$ evaluated on the point $l$ in the integer lattice corresponding to $\gamma$ (the "-" sign stems from the difference in sign conventions for hamiltonians and moment maps).

And therefore,

$$I_{c_L}u^{n+1-L}/Vol_{(n-L)}(\Delta) = -(n + 1 - L)!(B_{n-L} - B_n)$$

as required.

3 Discussion and Questions

The following are questions related to the subject of this note.

1. It would be interesting to make further comparisons of the computations in subsection 1.3 to the results of [22]. In particular, Theorem 4 raises the following inverse question:

   **Question 1.** Assume that a loop $\gamma$ in $Ham$ coming from the toric action is such that all $I_\alpha(\gamma) = 0$ for all $\alpha$ in the chamber. Does it follow that $\gamma$ is contractible?

2. It was shown in [29] that for monotone symplectic manifolds the mixed Action-Maslov invariant extends to a homogenous quasimorphism on the universal cover of the Hamiltonian group. Then it was asked by L. Polterovich ([28]) whether one could extend $I_{c_L}$ to such a quasimorphism in the non-monotone case. As a first step, it would be interesting to check this for Kahler manifolds of constant scalar curvature.
3. It would be interesting to investigate the collinearity phenomenon of barycenters of toric Fano polytopes, beyond the triple $B_0, B_n, B_{n-1}$. Are all the barycenters collinear? Are they collinear in triples $B_{L}, B_n, B_{n-L-1}$? Is this related to a certain duality? The first nontrivial case for investigation is in complex dimension 4. For example, taking the Ostrover-Tyomkin polytope (§20, section 5), it would be interesting to check whether $B_1, B_2, B_4$ are collinear. In general, following Remarks 1.2.6 and 1.3.5 further vanishings of Futaki invariants ([10]) might be of use in addressing these questions.

4. For toric Fano manifolds $(M, \omega, J)$, whenever $I = 0$ on toric loops, the Futaki invariant $F$ vanishes. This is equivalent, by [37], to the existence of a Kahler-Einstein metric. However not all Hamiltonian loops are necessarily toric, so it’s interesting to answer

**Question 2.** Do Kahler-Einstein toric Fano manifolds $(M, \omega, J)$ have $I = 0$ identically?

Here an extension of Seidel’s argument from [7], section 4.3 could be of essence. It may also be interesting to investigate the same question for general Kahler-Einstein manifolds.

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