Similarity solutions for two-phase fluids models

Andronikos Paliathanasis

1Institute of Systems Science, Durban University of Technology, Durban, Republic of South Africa
2Instituto de Ciencias Físicas y Matemáticas, Universidad Austral de Chile, Valdivia, Chile

The algebraic properties of drift–flux two-phase fluids models without gravitational and wall friction forces are studied. More precisely, for the two fluids we consider an equation of state of polytropic gases. We perform a classification scheme of the unknown parameters of the model to determine all the possible admitted Lie symmetries. We find that in the most general case the dynamical system of hyperbolic equations is invariant under the action of a four-dimensional Lie algebra, while the larger number of admitted Lie symmetries is 6. For each admitted Lie algebra the one-dimensional optimal system is derived which is applied for the determination of all the unique similarity transformations which lead to similarity solutions. Our results are compared with that of previous studies from where we see that most of the solutions presented in this study have not been found before in the literature.

KEYWORDS
fluid dynamics, hyperbolic equations, Lie symmetries, similarity solutions, two-phase fluids

MSC CLASSIFICATION
35Q35; 58J45; 58J70

1 INTRODUCTION

A systematic approach for the determination of exact and analytical solutions of nonlinear differential equations is Lie’s theory.1–3 The novelty of Lie’s consideration is that someone investigate the invariance properties of a given differential equation under the action of infinitesimal transformations in order to write the differential equation into an equivalent form through algebraic representation.4–7 The generator of the infinitesimal transformation which keeps a differential equation invariant is called Lie symmetry of the right type. A main property for the admitted Lie symmetries for a differential equation is that they form an algebra known as Lie algebra.

The existence of a large-dimensional Lie algebra of Lie symmetries for a differential equation enables one to solve the differential equation by means of repeated reduction of order with the use of similarity transformations, or by means of the determination of a sufficient number of first integrals.8,9 In particular if an nth-order differential equation admit at least n Lie point symmetries with a solvable algebra, one knows that the system is reducible to quadratures and the system is integrable.10 Thus, the absence of the latter property does not immediately obviate the possibility of integrability. For instance, the application of a similarity transformation reduces the differential equation into a new differential equation which may have different algebraic properties and additional (new) symmetries to follow.11 On the other hand, there is a zoology on the nature of symmetries which means that because an equation does not admit Lie point symmetries, it does not mean that it is not invariant under the action of other kind transformations which follow from generalized forms of the definition of symmetries; for instance, see the literature12–14 and references therein. Another important application of the symmetries is that they are applied for the classification of equations and establish classes and families of equations which has as common feature the common admitted Lie algebra.15–20
There are various applications in the literature on the application of Lie symmetries for the study of nonlinear differential equations. Some results on ordinary differential equations are presented in Leach \cite{21} and Mahomed \cite{22} while there are various studies and for elliptic partial differential equations; for instance, see other works\cite{23,24,25} and references therein. As far as the application of Lie symmetries on hyperbolic partial differential equations is concerned, there are various of studies in all areas of applied mathematics,\cite{26,27,28,29} In theory of fluid dynamics Lie symmetries have played an important role on the determination of exact and analytic solutions. The complete symmetry classification of the shallow-water equations with or without a gravitational field is performed in Chesnokov,\cite{30,31} while the case with a Coriolis force is performed in Paliathanasis.\cite{32,33} The case with various class of bottoms investigated in other works,\cite{34,35,36} while for other studies on shallow-water equations we refer the reader to the literature\cite{37,38,39} and references therein. A detailed discussion of the symmetry approach on mechanical theories of continuous media is presented in Pucci et al.\cite{40} The authors by studying some systems of physical interests demonstrate the application of Lie invariants for the derivation of similarity solutions.

Another important system of hyperbolic equations where Lie point symmetries have been used for the determination of new solutions is the two-phase flow model.\cite{41,42} The two-phase fluids model describes the evolution of two fluids with different phases, such as liquid and gas, in a tube. The two-phase models have many physical applications from oil extraction, underground water, nuclear reactor and many others.\cite{43} For some important results on two-phase flow models we refer the reader to the literature\cite{44,45,46,47,48,49,50} and references therein. The exact solution of the Riemann problem for the drift–flux equations for a two-phase flow system was studied in Kuila et al,\cite{51} while numerical simulations of wave propagation in compressible two-phase flow were presented in Zeidan et al.\cite{52}

In Bira and Sekhar\cite{41} and Bira et al,\cite{42} Lie's theory was applied for the simplest two-phase flow model where there is not any mass transfer from the one fluid to the other while the pressure and the energy density of the two fluids is given by a polytropic equation of state as it is given by Lane–Emden equation. In this work we revise the results of Bira and Sekhar\cite{41} and Bira et al,\cite{42} for a non-flip drift–flux model of multi-phase flow defined.\cite{53} More specifically, we find new solutions which have not been presented before, for linear and nonlinear equation of state parameters for the two fluids. More precisely, we derive the admitted Lie point symmetries of the two-phase fluids model and for the admitted Lie symmetries we calculate the commutators and the Adjoint representation. By using these results we are able to determine the one-dimensional optimal system such that to perform all the independent similarity transformations. We find that the similarity solutions are expressed either by closed-form expressions or by quadratures. There are not many known exact solutions in two-phase models; see for instance, other works.\cite{54,55,56} Hence, the analysis presented here is important for the determination of exact solutions. The plan of the paper is as follows.

The main mathematical properties and definitions on the symmetries of differential equations are presented in Section 2. The main results of this study are presented in Section 3. The drift–flux two-phase fluids model is given where we determine the admitted Lie point symmetries. The unknown parameters of the problem are the two polytropic exponents $\gamma_1, \gamma_2$ of the two fluids. For arbitrary values of the polytropic exponents such that $\gamma_1 \geq 1$, the set of three hyperbolic equations admit four Lie point symmetries, while two additional Lie symmetries exist when $\gamma_1 = \gamma_2 = \gamma$ which means that in the latter case the dynamical system is invariant under a sixth dimensional Lie algebra. For each different admitted Lie algebra we calculate the commutators and the Adjoint representation. By using these results we are able to derive the one-dimensional optimal system, necessary for the derivation of all the independent similarity solutions. On the derivation of the similarity solutions emphasis is given in the two cases in which $\gamma_1 = \gamma_2 = 2$ and $\gamma_1 = \gamma_2 = 1$. Finally, in Section 4 we discuss our results and we draw our conclusions.

## 2 | LIE SYMMETRIES OF DIFFERENTIAL EQUATIONS

In this Section we present the basic properties and definitions of the Lie point symmetries. Consider the system of differential equations $H^A \left( x^i, \Phi^A, \Phi^A_j, \ldots \right) = 0$ where $x^i$ denotes the independent variables and $\Phi^A$ are the independent variables while $\Phi^A_j$ indicates derivative with respect to the variable $x^i$, that is, $\Phi^A_j = \frac{\partial}{\partial x^i} \Phi^A$.

We proceed by assuming the infinitesimal one-parameter point transformation of the form

$$
\tilde{x}^i = x^i \left( x^i', \Phi^B; \epsilon \right), \Phi^A = \Phi^A \left( x^i, \Phi^B; \epsilon \right),
$$

where $\epsilon$ is the infinitesimal parameter. Point transformation (1) connects two different points $P \left( x^i, \Phi^B \right) \rightarrow Q \left( \tilde{x}^i, \Phi^B, \epsilon \right)$, while at these two points the system of differential equations is defined $H^A(P)$ and $\tilde{H}^A(Q)$. 


We shall say that the system $H^A$ remains invariant under the action of the one-parameter transformation if and only if
\[
\lim_{\varepsilon \to 0} \frac{\mathcal{H}^A (\tilde{y}, \tilde{u}^A, \ldots ; \varepsilon) - \mathcal{H}^A (y^i, u^A, \ldots)}{\varepsilon} = 0.
\]
(2)

Which means that the solutions $\Phi^A$ of the system $H^A$ at the two different points, that is, $\Phi^A(P)$ and $\Phi^A(Q)$, are related through the point transformation (1).

By definition, expression (2) is the Lie derivative of $H^A$ along the vector field $X$ of the one-parameter point transformation (1), in which $X$ is defined as
\[
X = \frac{\partial \tilde{x}^i}{\partial \varepsilon} \partial_i + \frac{\partial \tilde{U}^A}{\partial \varepsilon} \partial_A.
\]
Hence, an equivalent form of the symmetry condition is
\[
\mathcal{L}_X (H^A) = 0,
\]
(3)
where $\mathcal{L}$ denotes the Lie derivative with respect to the vector field $X^{[n]}$. Vector field $X^{[n]}$ the $n$th-extension of generator $X$ of the transformation (1) in the jet space $\{x^i, \Phi^A, \Phi_{ij}^A, \ldots\}$ where $H^A$ is defined. The vector field $X^{[n]}$ is calculated by the generator of the point transformation $X$ as follows
\[
X^{[n]} = X + \eta^{[1]} \partial_{\Phi^A_i} + \ldots + \eta^{[n]} \partial_{\Phi^A_{ij} \ldots^n},
\]
(4)
where the new terms are
\[
\eta^{[n]} = D_i \eta^{[n-1]} - u_{i,ij} \ldots_{i_n} D_i \left( \frac{\partial \tilde{x}^j}{\partial \varepsilon} \right), i \geq 1, \eta^{[0]} = \left( \frac{\partial \tilde{U}^A}{\partial \varepsilon} \right).
\]
(5)

The Lie symmetries for a given differential equation form a Lie algebra. Lie symmetries can be used in different ways in order to study a differential equation. However, their direct application is on the determination of the so-called similarity solutions. The steps which we follow to determine a similarity solution is based on the determination and application of the Lie invariant functions.

In order to determine all the possible independent solutions of a given dynamical system we should derive the one-dimensional optimal system. Let the $n$-dimensional Lie algebra $G_n$ with elements $X_1, X_2, \ldots X_n$ admitted by the system $H^A$. The two vector fields
\[
Z = \sum_{i=1}^{n} a_i X_i, W = \sum_{i=1}^{n} b_i X_i, a_i, b_i \text{ are constants.}
\]
(6)
are equivalent if and only if
\[
W = Ad (\exp(\varepsilon X_i)) Z
\]
(7)
or
\[
W = cZ, c = \text{const.}
\]
(8)
where the operator $Ad (\exp(\varepsilon X_i)) X_j = X_j - \varepsilon [X_i, X_j] + \frac{1}{2} \varepsilon^2 [X_i, [X_i, X_j]] + \ldots$ is called the adjoint representation.

3 | TWO-PHASE FLOW MODEL

Let us assume a mixture fluid stratified flow in a pipe, where the mixture is consisting by two phases of the same fluid, that is, liquid ($\rho_l$) and gas ($\rho_g$). Furthermore, each phase moves on the pipe at a local section with average velocity $u_l$ and $u_g$. Hence, the continuous equations for the two fluids in the one-dimensional read
\[
(\rho_l a_l)_t + (\rho_l a_l u_l)_x = \Gamma_l a_l,
\]
(9)
\[ (\rho_2 a_g)_x + (\rho_2 a_g u_g)_{xx} = \Gamma_g a_g, \]  

(10)

where \( \Gamma_g \) are the particle creation terms, where for a closed system hold \( \Gamma_l = -\Gamma_g \). Furthermore, \( a_l, a_g \) are the volume fractions for the two fluids with \( a_l + a_g = 1 \). We continue our analysis by assume the simplest model without particle creation term, or any interaction between the two fluids, while we omit any gravitational effect.

Hence, the momentum equations for the fluids read\(^{57}\)

\[ (\rho(a_l u_l)_x + (\rho a_l u_l^2 + a_l p_l)_x - p_l^i(a_l)_x = 0, \]  

(11)

\[ (\rho_2 a_g u_g)_x + (\rho_2 a_g u_g^2 + a_g p_g)_x - p_g^i(a_g)_x = 0. \]  

(12)

in which \( p_l, p_g \) are the pressure terms for the two fluids and \( p_l^i, p_g^i \) describe the interfacial pressures on the gas–liquid interface on the side of the liquid and of the gas respectively.

Hence, from the momentum Equations (11) and (12) it follows

\[ 0 = (\rho a_l u_l + \rho a_g u_g)_x + (\rho a_l u_l^2 + \rho a_g u_g^2)_{xx} + a_l p_{lx} + a_g p_{gx} + (p_l - p_l^i)(a_l)_x + (p_g - p_g^i)(a_g)_x \]  

(13)

Therefore, with the use of the new variables \( u_t = u_g - u_l \) and \( u = \frac{\rho_l a_l u_l + \rho_g a_g u_g}{\rho_g a_g + \rho_l a_l} + u_t \), and by considering that for the drift model that \( u_t = 0 \), we end with the final system\(^{42}\)

\[ \rho_{1,t} + (\rho_1 u)_x = 0, \]  

(14)

\[ \rho_{2,t} + (\rho_2 u)_x = 0, \]  

(15)

\[ ((\rho_1 + \rho_2) u)_x + ((\rho_1 + \rho_2) u^2 + p_1 + p_2)_{xx} = 0. \]  

(16)

where \( \rho_{1,2}(t,x) = a_{(l,g)}(t,x) \) are the energy density of the fluid of phase and \( p_{1,2}(t,x) = a_{(l,g)}(t,x) \) denote the pressure of the fluids. Recall that for the drift–flux model, by definition, the pressure in the gas and the liquid surfaces are equal for the same cross-sectional area.\(^{57}\)

Finally, for the fluids we assume polytropic equation of state parameters of the form \( p_{1,2} = \kappa_{(1,2)} \rho_{(1,2)}^{\gamma_{(1,2)}} \) with \( \gamma_{(1,2)} \geq 1 \).

### 3.1 Arbitrary polytropic exponents

For the system (14)–(16) we apply Lie’s theory and we determine the Lie point symmetries which are\(^{42}\)

\[ X_1 = \partial_t, X_2 = \partial_x, X_3 = t \partial_t + x \partial_x \quad \text{and} \quad X_4 = t \partial_x + \partial_u \]

for arbitrary values of the polytropic exponents \( \gamma_1, \gamma_2 \). The commutators of the admitted Lie symmetries by the system (14)–(16) and the Adjoint representation of the Lie algebra consisted by the elements \( \{X_1, X_2, X_3, X_4\} \) are presented in Tables 1 and 2. The admitted Lie algebra is a solvable four-dimensional Lie algebra classified as \( A_{4,6} \) in the Patera and Winternitz classification scheme.\(^{18}\)

### TABLE 1

| Commutator table for the Lie point symmetries of the two-phase flow system (14)–(16) |
|---------------------|-----|-----|-----|-----|
| \([X_1, X_2]\)       | \(X_1\) | \(X_2\) | \(X_3\) | \(X_4\) |
| \(X_1\)             | 0    | 0    | \(X_1\) | \(X_2\) |
| \(X_2\)             | 0    | 0    | 0    | 0    |
| \(X_3\)             | \(-X_1\) | \(-X_2\) | 0    | 0    |
| \(X_4\)             | \(-X_2\) | 0    | 0    | 0    |
From Tables 1 and 2 we infer that the one-dimensional optimal system is consisted by the elements
\[
\{X_1\}, \{X_2\}, \{X_3\}, \{X_4\}, \\
\{X_1 + \beta X_2\}, \{X_3 + \beta X_4\}, \{X_1 + \beta X_4\}.
\]

We proceed with the application of the Lie symmetries such that to find the invariant transformations where the solution of the system (14)–(16) is expressed by the solution of an ordinary differential equation.

### 3.1.1 Reduction with \(X_1\)

Application of the Lie point symmetry \(X_1\) gives stationary solutions, that is, \(\rho_1 = \rho_1(x), u = u(x)\) where the reduced system is
\[
(r_1 u)_x = 0, (r_2 u)_x = 0, \left((r_1 + r_2) u^2 + \kappa_1 r_1^2 + \kappa_2 r_2^2\right)_x = 0,
\]
which means \(r_1 u = r_{10}, r_2 u = r_{20}\) and
\[
(r_{10} + r_{20}) u + \kappa_1 (r_{10})^{\gamma_1} u^{-\gamma_1} + \kappa_2 (r_{20})^{\gamma_2} u^{-\gamma_2} = u_{10}.
\]

Thus, it follows, \(u(x) = \text{const}\).

### 3.1.2 Reduction with \(X_2\)

From the Lie point symmetry \(X_2\) the stationary similarity transformation are \(\rho_1 = \rho_1(t), u = u(t)\), where
\[
\rho_{1,t} = 0, \rho_{2,t} = 0, \left((\rho_1 + \rho_2) u\right)_t = 0,
\]
that is, \(\rho_1 = \rho_{10}\) and \(u = u_0\).

### 3.1.3 Reduction with \(X_3\)

From the scaling symmetry \(X_3\) it follows the similarity transformation \(\rho_1 = \rho_1(w), u = u(w)\) where the new independent variable is defined as \(w = \frac{x}{t}\). The reduced system is
\[
w r_{1,w} - (r_1 u)_w = 0,
\]
\[
w r_{2,w} - (r_2 u)_w = 0,
\]
\[
w ((r_1 + r_2) u)_w + ((r_1 + r_2) u^2 + p_1 + p_2)_w = 0.
\]

From (20) and (21) it follows
\[
\rho_1(w) = \rho_{10} \exp \left( - \int \frac{u_w}{u - w} dw \right),
\]
where by replacing in (22) we end with the equation
\[
U_{,w} \left( (U_w)^{-1} (r_{01} + r_{02}) (U_0 - U)^2 - \kappa_1 (r_{01})^{\gamma_1} (U_w)^{\gamma_1} - \kappa_1 (r_{02})^{\gamma_2} (U_w)^{\gamma_2}\right) = 0,
\]
in which we have defined \(u(w) = w - (U_0 - U(w)) (U_w)^{-1}\).
Hence, function $U(w)$ is given by the non-static solution of the first-order differential equation

$$\left(U_w\right)^{-1} (\rho_{01} + \rho_{02}) (U_0 - U)^2 - \kappa_1 (\rho_{01})^{\gamma_1} (U_w)^{\gamma_1} - \kappa_1 (\rho_{02})^{\gamma_1} (U_w)^{\gamma_2} = 0. \tag{25}$$

The latter equation is solved by quadratures, however there are some closed-form solutions for specific values of the polytropic exponents $\gamma_i$. Indeed, for $\gamma_i = 1$ the closed-form solution is

$$U_{\pm}(w) = U_0 - U_1 e^{\pm \lambda w} \tag{26}$$

where now $\lambda = \sqrt{\frac{\rho_{01} + \rho_{02}}{\kappa_1 \rho_{01} + \kappa_2 \rho_{02}}}$; thus, the similarity solution is

$$u(w) = w - 1, \rho_i(w) = \rho_i(t)e^{\lambda w}. \tag{27}$$

This is a new close-form solution which has not found before.

### 3.1.4 Reduction with $X_4$

The similarity transformation given by the symmetry vector $X_4$ is $\rho_i = \rho_i(t), u = \xi + v(t)$ where

$$t \rho_{1,t} + \rho_1 = 0, \tag{28}$$

$$t \rho_{2,t} + \rho_2 = 0, \tag{29}$$

$$(t (\rho_{1,t} + \rho_{2,t}) + (\rho_1 + \rho_2)) x + (\rho_1 v + \rho_2 v)_t + 2t (\rho_1 + \rho_2) v = 0, \tag{30}$$

from where it follows the exact solution

$$\rho_i(t) = \rho_i(0) t^{-1} \text{ and } v(t, x) = \frac{(x + u_0)}{t}. \tag{31}$$

### 3.1.5 Travelling waves $X_1 + \beta X_2$

From the Lie symmetry $X_1 + \beta X_2$ it follows $\rho_i = \rho_i(w), u = u(w)$ where $w = x - \beta t$ while the reduced system is

$$\beta \rho_{1,w} - (\rho_1 u)_w = 0, \tag{32}$$

$$\beta \rho_{2,w} - (\rho_2 u)_w = 0, \tag{33}$$

$$((\rho_1 + \rho_2) u)_w - \beta ((\rho_1 + \rho_2) u^2 + p_1 + p_2)_w = 0, \tag{34}$$

which provides the constant solution $\rho_i = \rho_i(0)$ and $u = u_0$, or for $u = \beta$ it follows

$$\rho_i(w) = \left(\rho_{10} - \frac{\kappa_2}{\kappa_1} \rho_2(w)\right)^{\frac{1}{\gamma_1}} \tag{35}$$

where $\rho_2(w)$ is an arbitrary function.

Recall that in order the latter solution to be physically accepted $\rho_2(w)$ should be defined such that $\rho_1, \rho_2$ to be real positive functions. That is a more general solution that the one found before in Bira and Sekhar.\(^{42}\) The only solution which was found in Bira and Sekhar\(^{42}\) provides that one of the $\rho_i(w)$ will be negative, or one of the pressure $p_i(\rho_i(w))$ will be negative. That is not physically accepted, since there are not known physically fluids with negative energy density or negative pressure. Although that kind of fluids are used in theoretical astrophysics as toy models, they have not observed yet.
3.1.6 Reduction with $X_3 + \beta X_4$

Application of the symmetry vector $X_3 + \beta X_4$ gives the similarity transformation $\rho_i = \rho_i(w)$, $u = \beta \ln t + v(w)$ with $w = x - \beta \ln t$. The reduced system provides

$$\rho_i(w) = \rho_{i0}V_w, \gamma(w) = (w + \beta) + \frac{V_0 - V(w)}{V_w}$$  \hspace{1cm} (36)

where $V(w)$ satisfies the second-order ordinary differential equation

$$0 = (\rho_{10} + \beta_{20}) \left( \beta (V_w)^3 - V_{w0}(V - V_0)^2 \right) + V_{w0}V_{w0} \left( \kappa_1(\rho_{10}V_w)^{\gamma_1} + \kappa_2(\rho_{20}V_w)^{\gamma_2} \right),$$

Equation (37) is an autonomous equation; hence, we can define the new dependent variable $\phi = V_w$, and the new independent variable $z = V$.

Therefore, it follows

$$0 = (\rho_{10} + \beta_{20}) \left( \beta(\phi(z))^3 - \phi(z)\phi(z)(z - V_0)^2 \right) + \phi(z)(\phi(z))^2 \left( \kappa_1(\rho_{10}\phi(z))^{\gamma_1} + \kappa_2(\rho_{20}\phi(z))^{\gamma_2} \right),$$

which can be solved by quadratures.

3.1.7 Reduction with $X_1 + \beta X_4$

From the symmetry vector $X_1 + \beta X_4$ it follows $\rho_i = \rho_i(w)$, $u = \beta t + V(w)$ where now $w = x - \frac{\beta}{2} t^2$. The reduced system is

$$\rho_1 (v)_w = 0, \rho_2 (v)_w = 0, \hspace{1cm} (38)$$

$$\beta (\rho_1 + \rho_2) + (\rho_1 + \rho_2) u^2 + p_1 + p_2 = 0 \hspace{1cm} (39)$$

that is, $\rho_1(w) = \rho_{10}v^{-1}$, $\rho_2(w) = \rho_{20}v^{-1}$. Hence, $v(w)$ is given by the equation

$$(\rho_{10} + \beta_{20}) w + v_0 + \frac{1}{2} (\rho_{10} + \rho_{20}) v^2 - \frac{\kappa_1(\rho_{10})^{\gamma_1}}{1 - \gamma_1} v^{1 - \gamma_1} - \frac{\kappa_2(\rho_{20})^{\gamma_2}}{1 - \gamma_2} v^{1 - \gamma_2} = 0 \hspace{1cm} (40)$$

for $\gamma_i \neq 1$, or

$$(\rho_{10} + \beta_{20}) w + v_0 + \frac{1}{2} (\rho_{10} + \rho_{20}) v^2 - \kappa_1 \rho_{10} \ln v - \frac{\kappa_2(\rho_{20})^{\gamma_2}}{1 - \gamma_2} v^{1 - \gamma_2} = 0 \hspace{1cm} (41)$$

for $\gamma_1 = 1$, $\gamma_2 \neq 1$ or

$$(\rho_{10} + \beta_{20}) w + v_0 + \frac{1}{2} (\rho_{10} + \rho_{20}) v^2 - \kappa_1 \rho_{10} \ln v^{\frac{\kappa_1}{\kappa_1 + \kappa_2} + \frac{\kappa_2}{\kappa_1 + \kappa_2}} = 0 \hspace{1cm} (42)$$

for $\gamma_1 = 1$. This is a new solutions which has not found before in the literature.

Nevertheless for specific values of the polytropic exponents the dynamical system (14)–(16) admits additional symmetries.

3.2 Polytropic exponents $\gamma_1 = \gamma_2 = \gamma, \gamma \neq 1$

Consider now the case where the polytropic exponents $\gamma_i$ are equal, that is, $\gamma_1 = \gamma_2 = \gamma$. In such scenario the Lie point symmetries of the two-phase model (14)–(16) are the vector fields $X_1, X_2, X_3, X_4$, plus the additional symmetries $X_5 = \frac{2}{\gamma - 1} (\rho_1 \partial_{\rho_1} + \rho_2 \partial_{\rho_2} + u \partial u - t \partial t)$, $X_6 = f_1 (\rho_1, \rho_2; \kappa_1, \kappa_2, \gamma) \partial_{\rho_1} + f_2 (\rho_1, \rho_2; \kappa_1, \kappa_2, \gamma) \partial_{\rho_2}$.

Vector field $X_5$ is a scaling symmetry, while $X_6$ is a rotation symmetry which indicates the invariance of the two-phase model if $(\rho_1, \rho_2) \rightarrow (\rho_1, \rho_2)$. However, functions $f_1 (\rho_1, \rho_2; \kappa_1, \kappa_2, \gamma)$ are not expressed always into closed-form expressions. Thus, for $\gamma = 2$ it follows

$$f_1 (\rho_1, \rho_2; \kappa_1, \kappa_2, \gamma) = \frac{\kappa_1 \rho_1^2 - 2 \kappa_2 \rho_1 \rho_2 - \kappa_2 \rho_2^2}{\kappa_1 \rho_1 - \kappa_2 \rho_2}. \hspace{1cm} (43)$$
Reduction with $X_5$

Application of the symmetry vector $X_5$ provides the similarity transformation $\rho_1 = \rho_i(x) t^{-2}$ and $u = u(x) t^{-2}$, where

$$\rho_i(x) = \rho_{i0} \exp \left( \int \frac{2 - u_x}{u} \, dx \right)$$

(45)

where $u = (2v(x) + v_0) (v_x)^{-1}$ and $v(x)$ is given by the differential equation

$$v_{xx} \left( (\kappa_1 \rho_{10}^2 + \kappa_2 \rho_{20}^2) (v_x)^3 -(2v + v_0)^2 (\rho_{10} + \rho_{20}) \right) + (v_x)^2 (\rho_{10} + \rho_{20}) (2v + v_0) = 0,$$

(46)

the later equation can be integrated by quadratures.

Reduction with $X_2 + \beta X_5$

From the Lie symmetry vector $X_2 + \beta X_5$ we find the similarity transformation $\rho_1 = \rho_i(w) t^{-2}, u = \tilde{u}(w) t^{-1}$ where $w = x + \frac{1}{\beta} \ln t$. The reduced system provides

$$\rho_1 = \rho_{i0} \exp \left( \beta \int \frac{2 - \tilde{u}_w}{1 + \beta \tilde{u}_w} \, dw \right)$$

(47)
where
\[
\bar{u} = -\frac{1}{\beta} + (2v(w) + v_0) \left( v_w \right)^{-1}
\] (48)

where now \( v(x) \) is given by the second-order differential equation
\[
0 = \beta v_{ww} \left( 2 \left( \kappa_1 \rho_{10}^2 + \kappa_2 \rho_{20}^2 \right) \left( v_w \right)^3 - (2v + v_0)^2 (\rho_{10} + \rho_{20}) \right) + \left( v_w \right)^2 (\rho_{10} + \rho_{20}) \left( \beta (2v + v_0) + v _w \right).
\] (49)

which again can be solved by quadratures.

### 3.2.3 Reduction with \( X_3 + \beta X_5 \)

The Lie symmetry vector \( \{X_3 + \beta X_5\} \) provides the similarity transformation \( \rho_i = \rho_i(w) t^{-\frac{2i}{\beta}}, u = \bar{u}(w) t^{-\frac{2}{\beta}} \) with new independent variable \( w = xt^{\frac{1}{\beta}} \), while the reduced system provides
\[
\rho_i(w) = \rho_{i0} v_x, \bar{u} = -\frac{1}{\beta} + (2v(w) + v_0) \left( v_w \right)^{-1}
\]

where \( v(w) \) satisfies the second-order differential equation
\[
0 = v_{ww} \left( (\rho_{10} + \rho_{20}) (v(2\beta + 1) - v_0 (1 - \beta)) - 2 \left( \kappa_1 \rho_{10}^2 + \kappa_2 \rho_{20}^2 \right) \left( v_w \right)^3 (1 - \beta)^2 \right) + w \left( v_w \right)^3 \beta (\rho_{10} + \rho_{20}) - (\rho_{10} + \rho_{20}) \beta \left( v_w \right)^2 (v(2\beta + 1) - v_0 (1 - \beta))
\] (50)

which can be solved by quadratures.

### 3.2.4 Reduction with \( X_1 + \beta X_6 \)

In order to proceed with the reduction with the use of the rest of the symmetry vectors we define the new variables
\[
\rho(t, x) = \rho_1(t, x) + \rho_2(t, x),
\] (51)
\[
p(t, x) = \kappa_1 \rho_1^2(t, x) + \kappa_2 \rho_2^2(t, x).
\] (52)

where now system (14)–(16) becomes
\[
\rho, \ (\rho u)_x = 0,
\] (53)
\[
p, \ (pu)_x + pu_x = 0,
\] (54)
\[
(\rho u)_x + (p u^2 + p)_x = 0.
\] (55)

In the new coordinates, symmetries \( X_5, X_6 \) are written \( X_5 = -2\rho \partial_{\rho} + u \partial_u - t \partial_t \) and \( X_6 = \rho \partial_{\rho} + p \partial_p \).

Thus, application of the Lie symmetry vector \( X_1 + \beta X_6 \) gives
\[
\rho = e^\rho \rho(x), \ p = e^p p(x), \ u = u(x)
\] (56)

while the reduced system is
\[
\rho, \ (\rho u)_x = 0,
\] (57)
\[
p, \ (pu)_x + pu_x = 0,
\] (58)
\[
(\rho u + (p u^2 + p))_x = 0.
\] (59)
which gives the solution
\[
\rho(x) = \frac{2pu_x + \beta p}{u^2u_x}, p = p_0 \exp \left( -\int \frac{\beta + 2u_x}{u} dx \right)
\]  
(60)

where \(u(x)\) is given by the polynomial
\[
u_0 \beta u^4 - 2\beta u - x - \beta x_0 = 0.
\]  
(61)

This is also a new solution for the two-phase fluids model. As also all the following similarity solutions are new and they have not been calculated before.

### 3.2.5 Reduction with \(X_2 + \beta X_6\)

The similarity transformation which follows from \(X_2 + \beta X_6\) is \(\rho = e^{\beta \psi} \tilde{\rho}(t), p = e^{\beta \psi} \tilde{p}(t)\) and \(u = u(t)\) while system (53)–(55) becomes
\[
\begin{align*}
\tilde{p}_t + \beta \tilde{p}u &= 0, \\
\tilde{p}_t + \beta \tilde{p}u &= 0,
\end{align*}
\]  
(62)
\(63\)

from where we infer the physically accepted solution
\[
u(t) = u_0 t + u_1,
\]  
(65)
\[
\tilde{p}(t) = p_0 \exp \left( -\frac{\beta}{2} \lambda_0 t^2 - \beta \lambda_1 t \right),
\]  
(66)
\[
\tilde{\rho}(t) = -\frac{\beta p_0}{\lambda_0} \exp \left( -\frac{\beta}{2} \lambda_0 t^2 - \beta \lambda_1 t \right).
\]  
(67)

### 3.2.6 Reduction with \(X_3 + \beta X_6\)

From the symmetry vector \(X_3 + \beta X_6\) we find \(\rho = t^\psi \tilde{\rho}(w), p = t^\psi \tilde{p}(w)\) and \(u = u(w)\) with \(w = \frac{x}{t}\). Hence, from (53)–(55) we find the solution
\[
\tilde{\rho}(w) = \rho_0 \exp \left( \frac{\beta + u_w}{w - u} \right), \tilde{p} = p_1 \exp \left( \int \frac{\beta - 2u_w}{w - u} \right),
\]  
(68)

where \(u(w) = w + v(w)\), while \(v(w)\) is given by the integral \(\int v(w) f(s)^{-1} ds = (w - w_0) = 0\), where \(f(s)\) is a solution of the polynomial equation
\[
-s^{-2-\frac{1}{2}} \beta (f(s) + 1)^{-1} - \frac{1}{2} \beta (3f(s) + 1)^{\frac{1}{2}} (2f(s) + \beta + 2)^{\frac{1}{2} + 1} + p_0 = 0.
\]  
(69)

### 3.2.7 Reduction with \(X_4 + \beta X_6\)

Application of the symmetry vector \(X_4 + \beta X_6\) gives the transformation \(\rho = e^{\beta z} \tilde{\rho}(t), p = e^{\beta z} \tilde{p}(t)\) and \(u = \frac{x}{t} + v(t)\), where now the reduced system is derived
\[
\begin{align*}
t \tilde{p}_t + \rho (1 + \beta v) &= 0, \\
t \tilde{p}_t + p (2 + \beta v) &= 0,
\end{align*}
\]  
(70)
(71)

that is,
\[
v(t) = \frac{u_0}{t} + \frac{u_1}{t} \ln t,
\]  
(73)
\[
\begin{align*}
\tilde{p}(t) &= p_0 t^{\frac{\beta}{t} - 2} \exp \left( \frac{\beta}{t} (u_0 + u_1) \right), \\
\tilde{\rho}(t) &= -\beta p_0 t^{\frac{\beta}{u_1} - 1} \exp \left( \frac{\beta}{t} (u_0 + u_1) \right).
\end{align*}
\] (74)

Thus, in order the solution to be physically accepted, \(p_0 > 0\) and \(\beta u_1 < 0\).

### 3.2.8 Reduction with \(X_5 + \beta X_6\)

The Lie symmetry vector \(X_5 + \beta X_6\) provides \(\rho = t^{2-\beta} \bar{\rho}(x), p = t^{-\beta} \bar{p}(x)\) and \(u = t^{-1} \bar{u}(x)\), while the reduced system is

\[
(2 - \beta) \bar{\rho} + (\bar{p} \bar{u})_x = 0,
\] (76)

\[
-\beta \bar{p} \bar{p} + (\bar{p} u)_x + \bar{p} \bar{u} = 0,
\] (77)

\[
(1 - \beta) \bar{p} \bar{u} + (\bar{p} \bar{u}^2 + \bar{p})_x = 0.
\] (78)

That is,

\[
\bar{\rho}(x) = \rho_0 \exp \left( - \int \frac{2 - \beta + \bar{u}_x}{\bar{u}} \, dx \right),
\] (79)

\[
\bar{p}(x) = p_0 \exp \left( - \int \frac{2\bar{u}_x - \beta}{\bar{u}} \, dx \right),
\] (80)

where \(\bar{u}(w)\) is given by the expression

\[
\int \bar{u}(x) (g(s))^{-1} \, ds = (x - x_0),
\] (81)

where

\[-2(sg(s) - 1)^{\frac{s-M}{T}} (3g(s) - 2)^{\frac{2-s}{T}} (2g(s) - \beta) (2g(s) - \beta) + g_0 = 0.\]

### 3.2.9 Reduction with \(X_2 + \beta X_5 + \delta X_6\)

From \(X_2 + \beta X_5 + \delta X_6\) it follows \(\rho = t^{\frac{1}{T} + \delta} \bar{\rho}(w), p = t^{\frac{1}{T} + \delta} \bar{p}(w), u = t^{-1} \bar{u}(w)\), where \(w = x + \frac{1}{\beta} \ln t\). Thus, system (53)–(55) is reduced

\[
\bar{p},_w + (2\beta - \delta) \bar{p} + \beta (\bar{p} \bar{u})_w = 0,
\] (82)

\[
\bar{p},_w - \delta \bar{p} + \beta ((\bar{p} \bar{u}),_w + \bar{p} \bar{u},_w) = 0,
\] (83)

\[
(\bar{p} \bar{u})_w + (\beta - \delta) \bar{p} \bar{u} + \beta (\bar{p} \bar{u}^2 + \bar{p})_w = 0.
\] (84)

The latter system can be integrated further and its solution is expressed in terms of quadratures.

### 3.2.10 Reduction with \(X_3 + \beta X_5 + \delta X_6\)

Similarly, from the symmetry vector \(X_3 + \beta X_5 + \delta X_6\) it follows \(\rho = t^{\frac{2X_3}{2X_3 - 1}} \bar{\rho}(w), p = t^{\frac{2X_3}{2X_3 - 1}} \bar{p}(w), u = t^{-\frac{2X_3}{2X_3 - 1}} \bar{u}(w)\) with \(w = x t^{\frac{1}{2X_3 - 1}}\). Therefore, the reduced system is

\[
\bar{p},_w + (2\beta - \delta) \bar{p} + (\beta - 1) (\bar{p} \bar{u})_w = 0,
\] (85)

\[
\bar{p},_w - \delta \bar{p} + (\beta - 1) ((\bar{p} \bar{u})_w + p \bar{u},_w) = 0,
\] (86)

\[
(\bar{p} \bar{u})_w + (\beta - \delta) \bar{p} \bar{u} + (\beta - 1) (\bar{p} \bar{u}^2 + \bar{p})_w = 0.
\] (87)

which again can be integrated by quadratures.

We continue our analysis with the special case of the polytropic exponents \(\gamma_1 = \gamma_2 = 1\).
3.3 Polytropic exponents $\gamma_1 = \gamma_2 = 1$

In the special case the two polytropic exponents are equal to 1; that is, $\gamma_1 = \gamma_2 = 1$. Similarly with the previous case we define the new dependent variables

$$\rho(t, x) = \rho_1(t, x) + \rho_2(t, x),$$  \hspace{1cm} (88)

$$p(t, x) = \kappa_1 \rho_1(t, x) + \kappa_2 \rho_2(t, x),$$  \hspace{1cm} (89)

where the dynamical system (14)-(16) takes the following form

$$\rho_x + (\rho u)_x = 0,$$  \hspace{1cm} (90)

$$p_x + (pu)_x = 0,$$  \hspace{1cm} (91)

$$(\rho u)_x + (pu^2 + p)_x = 0.$$  \hspace{1cm} (92)

The set of Equations (90)-(92) admits a six dimensional Lie algebra consisted by the symmetry vectors $\{X_1, X_2, X_3, X_4, X_5, X_6\}$. Thus, there is not any different between the symmetry classification of the two cases $\gamma = 1$ and $\gamma \neq 1$, as it was found before in Bira and Sekhar.\textsuperscript{42} We proceed with the application of the symmetry vectors for the derivation of similarity solutions.

3.3.1 Reduction with $X_5$

Application of the symmetry vector $X_5$ gives $\rho = t^{-2} \bar{\rho}(x)$, $p = p(x)$ and $u = t^{-1} \bar{u}(x)$ where the reduced system is

$$(\bar{\rho} u)_x + 2 \bar{\rho} = 0,$$  \hspace{1cm} (93)

$$(p \bar{u})_x = 0,$$  \hspace{1cm} (94)

$$\bar{\rho} \bar{u} + (\bar{\rho} u^2 + p)_x = 0,$$  \hspace{1cm} (95)

from where we find $\bar{\rho} = v_x$, $p = p_0 u^{-1}$ and $\bar{u} = \frac{2(v_0 - v)}{v_x}$, where $v(x)$ satisfies the differential equation

$$6\rho_0 (v - v_0) + \frac{p_0}{2(v - v_0)^2} \left( (v_0 - v) v_{xx} + v_x^2 \right) - \frac{4\rho_0}{v_x^2} (v - v_0)^2 v_{xx} = 0,$$  \hspace{1cm} (96)

which can be solved by quadratures.

3.3.2 Reduction with $X_2 + \beta X_5$

The vector field $X_2 + \beta X_5$ provides $\rho = t^2 \rho(w)$, $p = p(w)$ and $u = t^{-1} u(w)$ where now the independent variable is defined as $w = x + \frac{1}{\beta} \ln t$. Hence, by replacing in (90)-(92) we find

$$\rho(w) = \rho_0 v_{ww}, \ p(w) = p_0(1 + \beta u)^{-1}, \ u(w) = -\frac{1}{\beta} + \frac{2(v_0 - v)}{v_x}$$  \hspace{1cm} (97)

where $v(x)$ satisfies the second-order ode

$$0 = v_{xx} (v_0 - v) \left( 8 \beta \rho_0 (v - v_0)^3 + p_0 (v_x)^2 \right) + p_0 (v_x)^4 +$$

$$+ 2 \rho_0 (v - v_0)^2 (v_x)^2 + 12 \beta \rho_0 (v_x)^2 (v - v_0)^3$$  \hspace{1cm} (98)

which can be solved by quadratures.
3.3.3 Reduction with $X_3 + \beta X_5$

From the vector field $X_3 + \beta X_5$ it follows $\rho = t^\frac{\beta}{\alpha} \bar{p}(w), p = p(w)$ and $u = t^\frac{\beta}{\alpha} \bar{u}(w)$ where $w = x t^\frac{1}{\alpha}$. The reduced system is

\begin{align}
(x + (\beta - 1) \bar{u}) \bar{p}_x + 2 \beta \bar{p} + (\beta - 1) \bar{p} \bar{u}_x &= 0, \\
(x + (\beta - 1) \bar{u}) p_x + (\beta - 1) p \bar{u}_x &= 0, \\
x \bar{u} \bar{p}_x + x \bar{p} \bar{u}_x + \beta \bar{u} \bar{p} + (\beta - 1) (\bar{p} \bar{u}_x^2 + p)_x &= 0.
\end{align}

(99) (100) (101)

For $\beta = 1$, the closed-form solution of the later system is $p(w) = p_0, \bar{p} = \rho_0 w^{-2} \bar{u} = u_0 w$.

3.3.4 Reduction with $X_1 + \beta X_6$

The similarity transformation which follows from the symmetry vector $X_1 + \beta X_6$ is $\rho = e^{\beta x} \bar{p}(x), p = e^{\beta x} \bar{p}(x), u = \bar{u}(x)$, where by replacing in (90)–(92) we find

\[ \bar{p} = \rho_0 v_x, \quad \check{p} = p_0 v_x, \quad \check{u} = \frac{v_0 - \beta v}{v_x} \]

(102)

where $v(x)$ is a solution of the second-order differential equation

\[ v_{xx} \left( p_0 (v_x)^2 - \rho_0 (\beta v - v_0)^2 \right) + \beta \rho_0 v_x^2 (\beta v - v_0) = 0. \]

(103)

which can be solved by quadratures.

3.3.5 Reduction with $X_2 + \beta X_6$

From the symmetry vector $X_2 + \beta X_6$ it follows $\rho = e^{\beta x} \bar{p}(t), p = e^{\beta x} \bar{p}(t)$ and $u = u(t)$, where by replacing in the system (90)–(92) it follows

\[ u(x) = u_1 x + u_0, \]

(104)

\[ \bar{p}(x) = p_0 \exp \left( - \beta \left( \frac{1}{2} u_1 x^2 + u_0 x \right) \right), \]

(105)

\[ \bar{p} = -\beta p_0 \frac{u_1}{u} \exp \left( - \beta \left( \frac{1}{2} u_1 x^2 + u_0 x \right) \right). \]

(106)

3.3.6 Reduction with $X_3 + \beta X_6$

The Lie symmetry vector $X_3 + \beta X_6$ provides $\rho = t^\beta \bar{p}(w), p = t^\beta \bar{p}(w), u = u(w)$ where $w = \frac{x}{t}$. Hence, from the system (90)–(92) we find

\[ \bar{p} = \rho_0 \exp \left( - \int \frac{\beta + u}{u - w} dw \right), \quad \bar{u} = p_0 \exp \left( - \int \frac{\beta + u}{u - w} dw \right). \]

(107)

and $u(w)$ is given by the algebraic equation

\[ \arctan \left( \frac{e^{\frac{\beta}{\alpha}} (u(w) - w)}{\sqrt{\beta + 1}} \right) \beta + \sqrt{(\beta + 1) e (u_0 - u(w))} = 0. \]

(108)

In the special case where $\beta = -1$, function $u(w)$ is expressed as follows

\[ u_\pm(w) = \frac{1}{2} \left( (w - w_0) \pm \sqrt{(w - w_0)^2 - 4p_0} \right). \]

(109)
3.3.7 Reduction with $X_4 + \beta X_6$

The similarity transformation which correspond to the symmetry vector $X_4 + \beta X_6$ is $\rho = \exp^\beta \bar{\rho}(t), p = \exp^\beta \bar{p}(t), u = \frac{x}{t} + v(t)$. Therefore by replacing in the original system (90)–(92) the closed-form solution it follows

\begin{equation}
v(t) = \frac{u_1}{t} + u_0, \tag{110}\end{equation}

\begin{equation}\bar{p}(t) = t^{-1-u_0\beta} \exp\left(\frac{u_1\beta}{t}\right), \tag{111}\end{equation}

\begin{equation}\tilde{p}(t) = -\frac{\beta p_0}{u_0} t^{-1-u_0\beta} \exp\left(\frac{u_1\beta}{t}\right). \tag{112}\end{equation}

3.3.8 Reduction with $X_5 + \beta X_6$

From the Lie symmetry $X_5 + \beta X_6$ we calculate $\rho = t^{-1} \tilde{\rho}(x), p = t^{-\beta} \tilde{p}(x)$ and $u = t^{-1} \tilde{u}(x)$. Thus, from (90)–(92) we calculate

\begin{equation}\tilde{p}(x) = p_0 \exp\left(-\int \frac{\bar{u}_x - \beta}{\bar{u}} dx\right), \tilde{\rho}(x) = \frac{\bar{u}_x \tilde{p} - \beta \tilde{p}}{\bar{u}_x \bar{u}^2 - \bar{u}^2}, \tag{113}\end{equation}

while $\tilde{u}(x)$ is expressed in quadratures, that is

\begin{equation}\int_0^{\tilde{u}(x)} (\rho(s) + \beta)^{-1} ds = x - x_0, \tag{114}\end{equation}

where $r(s)$ is a solution of the algebraic equation

\begin{equation}0 = \rho(s) [Z + p_0] + (\beta - 1) (r(s)\beta + p_0\beta + 1) - \beta (r(s) + \beta - 1) \ln (\rho(s) + \beta - 1) - 2 (\beta - 1) (r(s) + \beta - 1) \ln s. \tag{115}\end{equation}

In the special case where $\beta = 1$ the closed-form solution follow

\begin{equation}\tilde{\rho} = \frac{\rho_0}{(x - x_0)^2}, \tilde{u} = x - x_0, \tilde{p} = \rho_0. \tag{116}\end{equation}

3.3.9 Reduction with $X_2 + \beta X_5 + \delta X_6$

The similarity transformation which correspond to the vector field $X_2 + \beta X_5 + \delta X_6$ is $\rho = t^{1-\delta} \bar{\rho}(w), p = t^{1-\delta} \bar{p}(w), u = t^{-1} \tilde{u}(w)$, where $w = x + \frac{1}{\beta} \ln t$. The reduced system is

\begin{equation}\tilde{\rho}_w + (2\beta - \delta) \tilde{\rho} + \beta (\tilde{\rho} \tilde{u})_w = 0, \tag{117}\end{equation}

\begin{equation}\tilde{p}_w - \delta \tilde{p} + \beta (\tilde{p} \tilde{u})_w = 0, \tag{118}\end{equation}

\begin{equation}(\tilde{\rho} \tilde{u})_w + (\beta - \delta) \tilde{p} \tilde{u} + \beta (\tilde{\rho} \tilde{u}^2 + \tilde{p})_w = 0, \tag{119}\end{equation}

which can be solved by quadratures.

In the limit where $\beta = \frac{\delta}{2}$, the solution is

\begin{equation}\tilde{\rho} = \frac{\rho_0}{2 + \delta \tilde{u}}, \tilde{p}(x) = \rho_0 \frac{(\delta \tilde{u} + 2) \tilde{u}_w - \delta \tilde{u}}{\delta^2 (\tilde{u} - 2)}. \tag{120}\end{equation}
where \( \tilde{u}(w) \) is given by the first-order differential equation

\[
\frac{df(\tilde{u})}{d\tilde{u}} - \frac{2(f(\tilde{u}) - 2)((\tilde{u} \delta + 2)f(\tilde{u}) - (2f(\tilde{u}) - 1)\delta \tilde{u} - 3f(\tilde{u}))}{(\tilde{u} \delta + 4)(\tilde{u} \delta + 2)}, f(\tilde{u}) = \tilde{u},
\]

(120)

### 3.3.10 | Reduction with \( X_3 + \beta X_5 + \delta X_6 \)

The symmetry vector \( X_3 + \beta X_5 + \delta X_6 \) provides the similarity transformation \( \rho = t^{\frac{2\beta - \delta}{\delta}} \tilde{\rho}(w), p = t^{\frac{\delta}{\delta - 1}} \tilde{p}(w), u = t^{\frac{\delta}{\delta - 1}} \tilde{u}(w) \) with \( w = xt^{\frac{\delta}{\delta - 1}} \). Therefore, the reduced system is

\[
\tilde{\rho}_w + (2\beta - \delta) \tilde{\rho} + (\beta - 1)(\tilde{\rho} \tilde{u})_w = 0,
\]

(121)

\[
\tilde{p}_w - \delta \tilde{p} + (\beta - 1)(\tilde{p} \tilde{u})_w = 0,
\]

(122)

\[
(\tilde{\rho} \tilde{u})_w + (\beta - \delta) \tilde{\rho} \tilde{u} + (\beta - 1)(\tilde{\rho} \tilde{u}^2 + \tilde{p})_w = 0.
\]

(123)

which can be integrated by quadratures.

When \( \beta = 1 \) and \( \delta = -1 \) the similarity transformation is \( \rho = x^{-3} \tilde{\rho}(t), p = \tilde{p}(t) \) and \( u = x\tilde{u}(t) \), where now the exact solution is

\[
\tilde{\rho}(x) = -\frac{\delta \tilde{u}(t)}{\tilde{p}^2 + \tilde{p}_t}, \tilde{u} = u_0e^{-(\delta+1)\int \tilde{p}_t dt},
\]

where

\[
\int_{t_0}^{P(t)} \left( e^{W\left(-\frac{s}{W(x+t_0p_0)}\right)} - s^2 \right) ds = t - t_0
\]

(124)

in which \( W(t) \) is the Lambert function.

### 4 | CONCLUSIONS

In this work we considered a drift–flux two-phase fluids model without gravitational and wall friction forces where the equation of state parameter for the fluids is that of a polytropic gas. The system of three hyperbolic differential equations studied with the use of Lie’s theory. In particular we investigate the algebraic properties of the two-phase fluids model by calculate the one-parameter point transformations in which the dynamical system is invariant.

We found that in the general scenario where the polytropic exponents \( \gamma_1, \gamma_2 \) of the two fluids are arbitrary the admitted Lie symmetries from a algebra of dimension fourth, while when in the special case where the polytropic exponents are equal, that is, \( \gamma_1 = \gamma_2 \) the admitted Lie symmetries form a sixth dimensional Lie algebra. That result is different from that previously found in the literature where it was found that a sixth dimensional Lie algebra is admitted only when \( \gamma_1, \gamma_2 \) are equal to 1, that is, \( \gamma_1 = \gamma_2 = 1 \).41

Thus, the missed symmetry vector by the previous study where \( \gamma_1, \gamma_2 \) are equal it is important for the determination of new similarity transformations. Indeed, because the symmetry vector is defined only in the two-dimensional space of the dependent variables and it commutes with the rest symmetries it provides a large number of independent similarity transformations which lead to similarity solutions which can not connect through an Adjoint transformation.

Furthermore, the one-dimensional optimal systems were determined for all the cases which followed by the classification of the Lie symmetries, the knowledge of the one-dimensional optimal system is essential because all the unique similarity solutions can be classified. The results of this work includes and new similarity solutions for the two-phase fluids model of our study.

In this work we have not studied the initial value problem for the two-phase fluids model. Indeed, physical problems are defined with a set of boundary and initial conditions. Consequently, not all similarity solutions found in this work will satisfy the initial value problems for all the physical states. However, we were able to find all the possible similarity
solutions, where these can be constrained according to the initial conditions. Such analysis extends the scope of this work and will be investigated in a future study.

This study contributes on the subject of the study of the algebraic properties of hyperbolic equations in fluid dynamics. From the result of this work it is clear that Lie symmetries play an important role on the determination of exact solutions in the two-phase fluids models. Although in this work we studied the simplest drift–flux two-phase fluids model, in the future we plan to extend our analysis in more general models.

CONFLICT OF INTEREST
This work does not have any conflicts of interest.

ORCID
Andronikos Paliathanasis https://orcid.org/0000-0002-9966-5517

REFERENCES
1. Lie S. Theorie der Transformationsgruppen, Vol. I. New York: Chelsea; 1970.
2. Lie S. Theorie der Transformationsgruppen, Vol. II. New York: Chelsea; 1970.
3. Lie S. Theorie der Transformationsgruppen, Vol. III. New York: Chelsea; 1970.
4. Ibragimov NH. CRC Handbook of Lie Group Analysis of Differential Equations. Volume I: Symmetries, Exact Solutions, and Conservation Laws. Florida: CRS Press LLC; 2000.
5. Bluman GW, Kumei S. Symmetries of Differential Equations. New York: Springer-Verlag; 1989.
6. Stephani H. Differential Equations: Their Solutions Using Symmetry. New York: Cambridge University Press; 1989.
7. Olver PJ. Applications of Lie Groups to Differential Equations. New York: Springer-Verlag; 1993.
8. Sarlet W, Crampin M. A characterisation of higher-order Noether symmetries. J Phys A: Math Gen. 1985;18:L563.
9. Leach PGL. Lie symmetries and Noether symmetries. Appl Anal Discret Math. 2012;6:238-246.
10. Paliathanasis A, Leach PGL. Nonlinear ordinary differential equations: A discussion on symmetries and singularities. Int J Geom Meth Mod Phys. 2016;13:1630009.
11. Abraham-Shrauner B. Hidden symmetries and linearization of the modified Painlevé–Ince equation. J Math Phys. 1993;34:4809-4816.
12. Anco S, Bluman G. Derivation of conservation laws from nonlocal symmetries of differential equations. J Math Phys. 1996;37:2361-2375.
13. Bluman G. Use and construction of potential symmetries. Math Comput Model. 1993;8:1-14.
14. Olver PJ, Sanders JA, Wang JP, Nonl J. Ghost symmetries. Math Phys. 2002;9:164-172.
15. Mubarakzyanov GM. On solvable Lie algebras. Izvestia Vysshikh Uchebn Zavendeniı, Matematika. 1963;32:114.
16. Mubarakzyanov GM. Classification of real structures of five-dimensional Lie algebras. Izvestia Vysshikh Uchebn Zavendeniı, Matematika. 1963;3:34.
17. Mubarakzyanov GM. Classification of solvable six-dimensional Lie algebras with one nilpotent base element. Izvestia Vysshikh Uchebn Zavendeniı, Matematika. 1963;35:104.
18. Patera J, Winternitz P, Zassenhaus H. Continuous subgroups of the fundamental groups of physics. I. General method and the Poincaré group. Math J Phys. 1975;16:1597-1614.
19. Patera J, Winternitz P. Invariants of real low dimension Lie algebras. J Math Phys. 1976;17:986-994.
20. Patera J, Winternitz P. Subalgebras of real three- and four-dimensional Lie algebras. J Math Phys. 1977;18:1449-1455.
21. Leach PGL. On a direct method for the determination of an exact invariant for the time-dependent harmonic oscillator. Soc J Aust Math B. 1977;20:97-105.
22. Mahomed FM. Symmetry group classification of ordinary differential equations: survey of some results. Appl Math Meth Sci. 2007;30:1995-2012.
23. Cherniha R, Davydovych V. Lie symmetries of nonlinear parabolic-elliptic systems and their application to a tumour growth model. J R King. Symmetry. 2018;10:171.
24. Craddock M. Symmetry groups of linear partial differential equations and representation theory: the Laplace and axially symmetric wave equations. J Differential Equations. 2000;166:107.
25. Paliathanasis A. Similarity inner solutions for the Pulsar equation. Appl Math Meth Sci. 2020;43:716.
26. Webb GM. Lie symmetries of a coupled nonlinear Burgers-heat equation system. J Phys A: MathGen. 1990;23:3885.
27. Kontogiorgis S, Popovych RO, Sophocleous C. Enhanced symmetry analysis of two-dimensional Burgers system. Acta Appl Math. 2019;163:91-128.
28. Moyo S, Leach PGL. Symmetry methods applied to a mathematical model of a tumour of the brain. Proc Inst Math NAS of Ukraine. 2004;50:204-210.
29. Paliathanasis A, Krishnakumar K, Tamizhmani KM, Leach PGL. Lie symmetry analysis of the Black–Scholes–Merton Model for European options with stochastic volatility. Mathematics. 2016;4:28.
30. Chesnokov AA. Symmetries and exact solutions of the shallow water equations for a two-dimensional shear flow. *Techn J Appl Mech Phys*. 2008;49:737-748.
31. Chesnokov AA. Symmetries and exact solutions of the rotating shallow-water equations. *Eur J Appl Math*. 2009;20:461-477.
32. Paliathanasis A. One-dimensional optimal system for 2D rotating ideal gas. *Symmetry*. 2019;11:1115.
33. Paliathanasis A. Lie symmetries and singularity analysis for generalized shallow-water equation. *Int J Nonlinear Sci Numer Simul*. 2020;20:739.
34. Meleshko SV. Invariant solutions of the two-dimensional shallow water equations with a particular class of bottoms. *N F Samatova, AIP Conf Proc*. 2019;2164:050003.
35. Bihlo A, Poltavets N, Popoych RO. Lie symmetries of two-dimensional shallow water equations with variable bottom topography. *Chaos*. 2020;30:073132.
36. Aksenov AV, Druzkhov KP. Conservation laws and symmetries of the shallow water system above rough bottom. *Phys J Conf Ser*. 2016;722:012001.
37. Dorodnitsyn VA, Kaptsov EI. Shallow water equations in Lagrangian coordinates: symmetries, conservation laws and its preservation in difference models. *Commun Nonl Sci Num Sim*. 2020;21:105343.
38. Liu J-G, Zeng Z-F, He Y, Ai G-P. A class of exact solution of (3+1)-dimensional generalized shallow water equation system. *Int J Nonl Sci Num Sim*. 2013;16:114.
39. Hematulin A, Meleshko SV, Gavrilyuk SG. Group classification of one-dimensional equations of fluids with internal inertia. *Math Methods Appl Sci*. 2007;30:2101.
40. Pucci E, Saccomandi G, Vitolo R. Bogus transformations in mechanics of continua. *Int J Eng Sci*. 2016;99:13-21.
41. Bira B, Sekhar TR. Exact solutions to drift–flux multiphase flow models through Lie group symmetry analysis. *Appl Math Mech-Engl Ed*. 2015;36:1105-1112.
42. Bira B, Sekhar TR, Zeidan D. Application of Lie groups to compressible model of two-phase flows. *Comput Math Appl*. 2016;71:46-56.
43. Hewitt GF. Two-phase flow and its applications: past, present, and future. *Heat Transfer Eng*. 1983;4:67-79.
44. Zeidan D, Bähr P, Farber P, Gräbel J, Ueberholz P. Numerical investigation of a mixture two-phase flow model in two-dimensional space. *Comput Fluids*. 2019;181:90-106.
45. Goncalves E, Zeidan D. Simulation of compressible two-phase flows using a void ratio transport equation. *Commun Comput Phys*. 2018;24:167-203.
46. Zhai J, Liu W, Yuan L. Solving two-phase shallow granular flow equations with a well-balanced NOC scheme on multiple GPUs. *Comput Fluids*. 2016;134:90-110.
47. Zeidan D, Zhang LT, Goncalves E. High-resolution simulations for aerogel using two-phase flow equations and Godunov methods. *Int J Appl Mech*. 2020;12:2050049.
48. Zeidan D, Bira B. Weak shock waves and its interaction with characteristic shocks in polyatomic gas. *Math Meth Appl Sci*. 2019;42:4679-4687.
49. Kozakevicius A, Zeidan D, Schmidt AA, Jakobsson S. Solving a mixture model of two-phase flow with velocity non-equilibrium using WENO wavelet methods. *Int J Numer Methods Heat Fluid Flow*. 2018;28:2052.
50. Goncalves E, Zeidan D. Numerical study of turbulent cavitating flows in thermal regime. *Int J Numer Methods Heat Fluid Flow*. 2017;27:1487.
51. Kuila S, Sekhar TR, Zeidan D. On the Riemann problem simulation for the drift–flux equations of two-phase flows. *Int J Comput Methods*. 2016;13:1650009.
52. Zeidan D, Romenski E, Slaouti A, Toro EF. Numerical study of wave propagation in compressible two? phase flow. *Int J Numer Methods Fluids*. 2007;54:393-417.
53. Banda MK, Herty M, N Gottschau J-MT. Toward a mathematical analysis for drift-flux multiphase flow models in networks. *SIAM J Sci Comput*. 2010;31:4633.
54. Pan L, Webb SW, Oldenburg CM. Analytical solution for two-phase flow in a wellbore using the drift–flux model. *Adv Water Res*. 2011;34:1656-1665.
55. Sander GC, Paralange J-Y, Lisle IG, Weeks SW. Exact solutions to radially symmetric two-phase flow for an arbitrary diffusivity. *Adv Water Res*. 2005;28:1112-1121.
56. Thanh MD. Exact solutions of a two-fluid model of two-phase compressible flows with gravity. *Anal Nonlinear Real World Appl*. 2012;13:987-998.