A Dirac-type Characterization of $k$-chordal Graphs

R. Krithika$^1$, Rogers Mathew$^2$, N. S. Narayanaswamy$^1$, and N. Sadagopan$^1$

1 Department of Computer Science and Engineering, Indian Institute of Technology Madras, India. {krithika, swamy, sadagopu}@cse.iitm.ac.in
2 Department of Computer Science and Automation, Indian Institute of Science, India. rogers@csa.iisc.ernet.in

Abstract

Characterization of $k$-chordal graphs based on the existence of a simplicial path was shown in [Chvátal et al. Note: Dirac-type characterizations of graphs without long chordless cycles. Discrete Mathematics, 256, 445-448, 2002]. We give a characterization of $k$-chordal graphs which is a generalization of the known characterization of chordal graphs due to [G. A. Dirac. On rigid circuit graphs. Hamburg, 25, 71-76, 1961] that use notions of a simplicial vertex and a simplicial ordering.

1 Introduction

Notations and definitions are as per [1]. The chordality of a graph is the size of its longest induced (chordless) cycle. Acyclic graphs are assumed to be of chordality 0. Low chordality graphs are known to admit rich combinatorial structure and efficient algorithms. In particular, chordal graphs [7], which are graphs of chordality at most 3, have a structural characterization due to Dirac [3] that says: a graph is chordal if and only if every minimal vertex separator is a clique. A vertex is at most 3, have a structural characterization due to Dirac [3] that says: a graph is chordal if and only if every minimal vertex separator is a clique. A vertex is simplicial if it induces a clique in its neighbourhood, has at most 2 edges.

For a graph $G$, the graph $G^{k}$ defined on the vertex set $V(G)$ with $E(G^{k}) = E(G) \cup \{\{u, v\} \mid u, v \in V(G) \text{ and } d_{G}(u, v) \leq k\}$ is referred to as the $k^{th}$ power of $G$. For a vertex $v$ in $G$, $(G - v)^{k}$ refers to the $k^{th}$ power of $G$ induced by the vertex set $V(G) \setminus \{v\}$. A set $A$ of vertices is referred to as a connected non-dominating set if $G[A]$ is connected and $N_{G}[A] \subseteq V(G)$. For every integer $k \geq 3$, a vertex $v$ is $k$-simplicial in $G$ if

(C1) $N_{G}(v)$ induces a clique in $(G - v)^{(k-2)}$

(C2) for every non-adjacent pair $x, y$ in the neighbourhood of $v$, every chordless path between $x$ and $y$, with internal vertices excluding $v$ and its neighbourhood, has at most $k - 2$ edges.

Note that neither of the conditions (C1) or (C2) always implies the other. For a graph $G$ on $n$ vertices, a vertex ordering $\pi = [v_{1}, v_{2}, \ldots, v_{n}]$ is a $k$-simplicial ordering if, for each $i$, $v_{i}$ is $k$-simplicial in $G'[\{v_{1}, \ldots, v_{n}\}]$. We show that a $k$-chordal graph has a $k$-simplicial vertex and as a consequence, we obtain a characterization of $k$-chordal graphs based on $k$-simplicial ordering. In particular, we show the following theorem:
Theorem 1. For a graph $G$ and an integer $k \geq 3$, the following statements are equivalent.

(i) $G$ is $k$-chordal.

(ii) There exists a $k$-simplicial ordering of vertices in $G$.

(iii) Every minimal vertex separator $S$ in $G$ is such that, for all non-adjacent $x, y \in S$, there are two distinct connected components $S_i$ and $S_j$ in $G \setminus S$, each pair $P_{S_i}^S$ and $P_{S_j}^S$ of induced paths between $x$ and $y$ with internal vertices from $S_i$ and $S_j$, respectively, satisfies $||P_{S_i}^S|| + ||P_{S_j}^S|| \leq k$.

Proof. (i) if and only if (ii) follows from Theorem 8 and (ii) if and only if (iii) follows from Theorem 9.

We note that our study generalizes Dirac’s structural results on chordal graphs, in particular, Lemma 4.2 and Theorem 4.1 in the book on perfect graphs by Golumbic [7].

2 Characterization of $k$-chordal graphs

Observation 2. A $k$-simplicial vertex is also $l$-simplicial, for every integer $l > k$. Also, for every integer $k \geq 3$, every vertex in a complete graph is $k$-simplicial.

Observation 3. For an integer $k \geq 3$, every vertex in a $k$-chordal graph $G$ satisfies (C2).

Observation 4. In a non-complete graph $G$, for every vertex $x$ which is non-adjacent to at least one other vertex, there exists a connected non-dominating set containing $x$.

Proof. As there exists a vertex $y$ that is not adjacent to $x$, the set $A = \{x\}$ induces a connected subgraph with $y \notin N_G[A]$.

Lemma 5. Let $A$ be a maximal connected non-dominating set in a non-complete graph $G$. Every vertex in $V(G) \setminus N_G[A]$ is adjacent to every vertex in $N_G(A)$.

Proof. If there exists non-adjacent vertices $x' \in N_G(A)$ and $y' \in V(G) \setminus N_G[A]$, then the set $A' = A \cup \{x'\}$ is a connected non-dominating set in $G$ contradicting the maximality of $A$.

Lemma 6. Let $A$ be a maximal connected non-dominating set in a non-complete graph $k$-chordal graph $G$, where $k \geq 3$ is an integer. There exists a vertex in $V(G) \setminus N_G[A]$ that is $k$-simplicial in $G$.

Proof. We prove the lemma by induction on $|V(G)|$. It is easy to verify that the statement of the lemma is true when $|V(G)| \leq 3$. We assume the statement to be true for all $k$-chordal graphs having less than $n$ vertices, where $n \geq 4$. Consider a non-complete $k$-chordal graph $G$ on $n$ vertices. Let $A$ be a maximal connected non-dominating set in $G$ and $B$ denote $V(G) \setminus N_G[A]$. If $B$ is a clique, then by Observation 2 every vertex in $B$ in $k$-simplicial in $G[B]$. Otherwise, by induction hypothesis, there exists a vertex that is $k$-simplicial in $G[B]$. Let $b \in B$ be $k$-simplicial in $G[B]$ and we now show that $b$ is $k$-simplicial in $G$ too. Since $b \in B$, every vertex in $N_G(b)$ is either in $B$ or in $N_G(A)$. Let $x, y \in N_G(b)$.

Case $(x, y \in B)$: Since $b$ is $k$-simplicial in $G[B]$, there exists a path $P_{xy}$ in $G[B \setminus \{b\}]$ and thereby in $G[V(G) \setminus \{b\}]$ such that $||P_{xy}|| \leq (k - 2)$.

Case $(x \in N_G(A), y \in B)$ or $(x \in B, y \in N_G(A))$: Since every vertex in $N_G(A)$ is adjacent with every vertex in $B$ by Lemma 5 we can take the path $P_{xy}$ to be the edge $\{x, y\}$.

Case $(x, y \in N_G(A))$: If $\{x, y\} \notin E(G)$, then the edge $\{x, y\}$ itself can be thought of as the path $P_{xy}$. Suppose $\{x, y\} \notin E(G)$. Since $A$ is connected and $x, y \in N_G(A)$, there exists a path between $x$ and $y$ whose every internal vertex is from $A$. Let $P_{xy}$ be the shortest of all such paths. Clearly, $P_{xy}$ is present in $G[V(G) \setminus \{b\}]$. We claim that $||P_{xy}|| \leq k - 2$. Suppose $||P_{xy}|| > k - 2$. Then, $C = bP_{xy}b$ is an induced cycle of length at least $k + 1$. This contradicts the fact that $G$ is $k$-chordal.

Lemma 7. For an integer $k \geq 3$, every $k$-chordal graph $G$ has a $k$-simplicial vertex. Moreover, if $G$ is not a complete graph, then it has two non-adjacent $k$-simplicial vertices.

Proof. If $G$ is a complete graph, then by Observation 2 every vertex in $G$ is $k$-simplicial. Suppose $G$ is not a complete graph. Let $A$ be a maximal connected non-dominating set in $G$ and $B$ denote $V(G) \setminus N_G[A]$. The sets $A$ and $B$ are well-defined by Observation 4. By Lemma 6 there exists a vertex $u \in B$ that is $k$-simplicial in $G$. Now, let $A'$ be a maximal connected non-dominating set in $G$ containing $u$ and $B'$ denote $V(G) \setminus N_{G[A']}$. The sets $A'$ and $B'$ are well-defined by Observation 4 as there exists a vertex $w \in A$ that is not adjacent to
u. By Lemma 8, there exists a vertex v ∈ B′ that is k-simplicial in G. Thus, u and v are two non-adjacent k-simplicial vertices in G.

As k-chordality is a hereditary property, we obtain the following characterization of k-chordal graphs.

**Theorem 8.** For an integer k ≥ 3, a graph G is k-chordal if and only if G has a k-simplicial ordering.

**Proof.** (⇒) As every induced subgraph of a k-chordal graph is k-chordal, the proof follows from Lemma 7.

(⇐) Consider a k-simplicial ordering π = [v1, v2, ... , vn] of G. If G is not k-chordal then there exists an induced cycle C = (ui, vij, ... , ui+1) of length at least (k + 1) with vij being the minimum labelled vertex in C as per π. Clearly, the 2 neighbours vj and vi of vij in C satisfy j, l > i by the choice of vij. As vij is k-simplicial, there are no chordless paths between vj and vi of length greater than (k − 2) with internal vertices from V(G) \ {v1, v2, ..., vn−1} and vij \ N(G)[vij] in G[\{v1, v2, ..., vn\}]. However, the path P = vj, ..., vi in C is one such path of length at least (k − 1), contradicting that vij is k-simplicial in G[v1, ..., vn].

It is well-known that a graph is chordal if and only if every minimal vertex separator induces a clique. In the subsequent discussion, we generalize this result and characterize k-chordal graphs based on their minimal vertex separators. We use terms minimal vertex separators and minimal (a, b)-vertex separators interchangeably and the pair (a, b) under consideration will be clear from the context.

**Theorem 9.** Let k ≥ 3 be an integer. A graph G is k-chordal if and only if for all minimal vertex separators S in G, for all non-adjacent x, y ∈ S and for any two distinct connected components S and S′ in G \ S, every pair Pxy and Pxy of induced paths between x and y with internal vertices from S and S′, respectively, satisfies ||Pxy|| + ||Pxy|| ≤ k.

**Proof.** (⇒) Assume on the contrary that there exists a minimal vertex separator S in G containing two non-adjacent vertices x and y such that there exists chordless paths Pxy and Pxy with ||Pxy|| + ||Pxy|| > k. Clearly, V(Pxy) \ U(Pxy) induces a cycle of length at least k + 1 contradicting the fact that G is k-chordal.

(⇐) Suppose there exists an induced cycle C of length at least k + 1 in G. Let C = (x, a, y1, ..., y(r−2) = b, x), l ≥ 3. We observe that any minimal (a, b)-vertex separator S in G must contain x and some yr = z, 1 ≤ r ≤ (l − 3). This shows that there exists chordless paths Pxz and Pxz such that ||Pxz|| + ||Pxz|| ≥ k + 1 contradicting our hypothesis. Therefore, as claimed, G is k-chordal.

**References**

[1] D.B.West. Introduction to graph theory. Prentice Hall of India, 2003.

[2] D.R.Fulkerson and O.A.Gross. Incidence matrices and interval graphs. Pacific J. Math, 15,835-855, 1965.

[3] G.A.Dirac. On rigid circuit graphs. Abh. Math. Sem. Univ. Hamburg, 25, 71-76, 1961.

[4] H.L.Bodlaender and D.M.Thilikos. Treewidth for graphs with small chordality. Discrete Applied Mathematics, 79, 456, 1997.

[5] J.P.Spinrad. Finding large holes. Information Processing Letters, 39, 227-229, 1991.

[6] L.S.Chandran, C.R.Subramanian, and V.V.Lozin. Graphs of low chordality. Discrete Mathematics and Theoretical Computer Science, 7, 25-36, 2005.

[7] M.C.Golumbic. Algorithmic graph theory and perfect graphs, Second edition. Elsevier Science B.V., 2004.

[8] R.B.Hayward. Two classes of perfect graphs. Ph.D Thesis, School of Computer Science, McGill University, 1987.

[9] R.Uehara. Tractable and intractable problems on generalized chordal graphs. Technical Report, COMP98-83, IEICE, Faculty of Natural Sciences, Komazawa University, 1999.

[10] V.Chvátal, I.Rusu, and R.Sritharan. Note: Dirac-type characterizations of graphs without long chordless cycles. Discrete Mathematics, 256, 445-448, 2002.