Abstract

We consider graphs parameterized on a portion \( X \subset \mathbb{Z}^d \times \{1, \ldots, M\}^k \) of a cylindrical subset of the lattice \( \mathbb{Z}^d \times \mathbb{Z}^k \), and perform a discrete-to-continuum dimension-reduction process for energies defined on \( X \) of quadratic type. Our only assumptions are that \( X \) be connected as a graph and periodic in the first \( d \)-directions. We show that, upon scaling of the domain and of the energies by a small parameter \( \varepsilon \), the scaled energies converge to a \( d \)-dimensional limit energy. The main technical points are a dimension-lowering coarse-graining process and a discrete version of the \( p \)-connectedness approach by Zhikov.

1 Introduction

The object of the investigation in this paper is the analysis of discrete thin objects through, at the same time, a discrete-to-continuum and dimension-reduction process. The main focus of our work is the great generality of the geometry of our discrete systems, which we essentially require to be a connected graph periodic in the dimensions that are maintained after a discrete-to-continuum passage.

![Figure 1](image.png)

Figure 1: A discrete thin object in three dimensions with a one-dimensional behaviour

An example of the structure that we have in mind is pictured in Fig.1. The thicker black lines (both the solid ones and the ones dashed for graphic purpose) represent connections
between nodes of a cubic lattice in $\mathbb{R}^3$.  Equivalently, we may think of the same structure as a network of conducting rods.  Note that this object is not trivially a ‘subgraph’ of a function depending of the vertical variable, as it consists of a double helix connected through horizontal bonds. Nevertheless, it can be included in a ‘regular’ thin object; in this case, the cylindrical part of $\mathbb{Z}^3$ whose projection on the two-dimensional horizontal plane are the four vertices of a square. Even if no connection is purely vertical, the overall behaviour of such a structure is expected to be that of a vertical one-dimensional object.

With this example in mind, we are going to look at graphs whose nodes are a subset $X$ of $\mathbb{Z}^{d+k}$ periodic of period $T$ in the first $d$ directions (in the example $d = 1$, corresponding to the vertical direction, with period $T = 2$), bounded in the last $k$ directions (in the example, $k = 2$, corresponding to the horizontal directions), so that we may always think that it is contained in $\mathbb{Z}^d \times \{0, \ldots, M - 1\}^k$ for some $M \in \mathbb{N}$. This graph is equipped with a set of edges $\mathcal{E} \subset X \times X$ which make it connected. This set of edges is supposed to be invariant by the same translations as $X$.

We are going to show that we may define a continuous $d$-dimensional approximation of this set. In order to maintain technicalities to a minimum, we consider only quadratic interactions. The Dirichlet energy of such a set is defined as

$$F(u) = \sum_{(i,j) \in \mathcal{E}} (u(i) - u(j))^2$$

on functions $u : X \to \mathbb{R}$. A discrete-to-continuum and dimensionally reduced limit is then obtained by considering a scaled version of the energy

$$F_\varepsilon(u) = \sum_{(i,j) \in \mathcal{E}} \varepsilon^{d-2} (u_i - u_j)^2$$

defined on functions $u : \varepsilon X \to \mathbb{R}$, where we use the notation $u_i = u(\varepsilon i)$, and taking their limit in a suitable sense as $\varepsilon \to 0$. Note that we may interpret

$$\varepsilon^{-1}(u_i - u_j) = |i - j| \left( \frac{u_i - u_j}{\varepsilon|i - j|} \right)$$

as an inhomogeneous difference quotient, so that $F_\varepsilon$ represent discrete versions of an (inhomogeneous) Dirichlet integral, whose general continuous counterpart is of the form

$$\frac{1}{\varepsilon^k} \int_X f(\nabla u) \, dx,$$

(1)

with $E$ a subset of $\mathbb{R}^d \times \mathbb{R}^k$ uniformly bounded in the last $k$ variables. Energies of the form (1) are the prototype of thin-structure energies on the continuum (see e.g. [6, 7]), which have been treated extensively in the last thirty years. Among the many contribution to the subject we recall the seminal paper by Le Dret and Raoult [20] which gives a general dimension-reduction formula when $E = \mathbb{R}^d \times [0, 1]$ through a lower-dimensional quasiconvexification process. Moreover, a general compactness and integral-representation theorem has been proved by Braides, Fonseca and Francfort [12], which interpret lower-dimensional quasiconvexification through a homogenization formula, and extend the analysis to general thin films with varying profiles. In their approach they deal with $E$ that can be seen as a subgraph of a function defined on $\mathbb{R}^d$. In our case, even if a continuum set $E$ corresponding to $X$ can be constructed, it may not be a subgraph, as it might have holes or even possess a more complex topology. Note that the assumption that the integration be performed on the scaled $\varepsilon E$ cannot be extended to arbitrary $E_\varepsilon$ since in that case the limit might not be simply $d$-dimensional if the complexity of the topology increased as $\varepsilon \to 0$ (see the example by Braides and Bhattacharya [5]). Finally, we note that asymptotic analysis of thin objects can be interpreted as an intermediate step in the study of structures with very fast oscillating profile [9] (see also [4], and e.g. [13] for an example of application in a continuum geometry).

Discrete-to-continuum analyses for lattice energies are usually performed after identification of functions defined on (portions of) lattices with their piecewise-constant interpolations. This identification allows to embed families of energies in a common Lebesgue-space
environment (see the seminal paper by Alicandro and Cicalèse [3]). Using this approach, a discrete-to-continuum analog for thin films of the Braides, Fonseca and Francfort theory, has been studied by Alicandro, Braides and Cicalèse [2] (see also [21], and the work [15] for a connection with aperiodic lattices). Due to the great generality of our discrete set $X$, we will not directly extend functions defined on $\varepsilon X$ but follow a dimension-lowering coarse-graining approach: to each function $u_\varepsilon : \varepsilon X \to \mathbb{R}$ we associate the function $\overline{u}_\varepsilon : \varepsilon T \mathbb{Z}^d \to \mathbb{R}$ where $\overline{u}_\varepsilon(t) = \text{average of } u_\varepsilon$ on $X \cap ([t, T) \times \{0, \ldots, M-1\}^d)$. We then prove that energy bounds on $u_\varepsilon$ imply that the piecewise-constant interpolations of the corresponding $\overline{u}_\varepsilon$ are precompact in $L^2_{\text{loc}}(\mathbb{R}^d)$ and their limit is in $H^1_{\text{loc}}(\mathbb{R}^d)$. In this way a dimensionally reduced continuum parameter can be defined. In order to relate the original $u_\varepsilon$ to this limit, a Poincaré inequality must be used at scale $\varepsilon$, which shows that the original $u_\varepsilon$ converge to $u$ in a ‘perforated domain’ fashion (see e.g. [11]). Both the coarse-graining and the Poincaré-type inequality are very reminiscent of the $\mu$-connectedness approach by Zhikov [22], and of its use in the homogenization of singular structures by Braides and Chiadò Piat [8], even though in those papers $\mu$-connectedness is stated fo local functionals depending on the gradient. Here we deal with non-local interactions, even though the non-locality weakens as $\varepsilon \to 0$, and some additional care has to be taken, similarly to the case of the homogenization of convolution-type energies (see [1] [10] [11]).

The paper is organized as follows. In Section 2 we introduce the notation for the environment $X \subset \mathbb{R}^d \times \mathbb{R}^k$ and for the energies that we consider, which are a little more general than those described above in that a more general inhomogeneity is allowed (introducing interactions coefficients $a_{ij}$) and the energies are localized by considering interactions parameterized on a set $\Omega \subset \mathbb{R}^d$. Section 3 is devoted to the definition of coarse-grained functions, and to the statement and proof of the two-connectedness property and of a Poincaré-Wirtinger’s inequality. Section 4 contains a compactness result for coarse-grained functions, and to the statement and proof of the two-connectedness property and of a Poincaré-type inequality. Section 5 contains a result that allows to consider boundary-values on ‘lateral boundaries’ of thin films. A homogenization theorem for quadratic energies defined on $\varepsilon X$ is stated in Section 6. Its proof is subdivided into a lower bound by blow-up and an upper bound by a direct construction. Moreover, an application to the description of the asymptotic behaviour of boundary-value problems is also described. Finally, Section 7 contains some simple examples illustrating some possible non-trivial shapes of the thin structures we consider.

Notation

- The letter $C$ denotes a generic strictly positive constant not depending on the parameters of the problem considered, whose value may be different at every its appearance.
- If $x, y \in \mathbb{R}^d$ then $x \cdot y$ denotes their scalar product. If $t \in \mathbb{R}$ then $\lfloor t \rfloor$ is its integer part.
- For $T \in \mathbb{N}$, we denote by $Q_{T,d}$ the $d$-dimensional semi-open cube of side length $T$; i.e., $Q_{T,d} := [0,T)^d$. If $T = 1$, we simply write $Q_d = Q_{1,d}$. For $l \in \mathbb{Z}^d$, $Q_{l,T,d} := lT + [0,T)^d$ and for $T = 1$, we write $Q_l = Q_{1,d}$.
- $Q_{T,k}$ denotes the $k$-dimensional semi-open cube of side length $T$; i.e., $Q_{T,k} := [0,T)^k$. For $T = 1$, we set $Q_k = Q_{1,k}$ and $Q_n = n + Q_k$ if $n \in \mathbb{Z}^k$.
- For any measurable set $\Omega$ and $u \in L^1(\Omega)$, $\frac{1}{|\Omega|} \int_\Omega u(x) dx$ denotes the average of $u$ on $\Omega$; i.e., $\frac{1}{|\Omega|} \int_\Omega u(x) dx := \frac{1}{|\Omega|} \int_\Omega u(x) dx$,
  where $| \cdot |$ stands for the Lebesgue measure.
- For any open set $\Omega \subset \mathbb{R}^d$ and for any $\delta > 0$, we let $\Omega(\delta) := \{ x^d \in \Omega : \text{dist}(x, \partial \Omega) > \delta \}$.
2 Setting of the problem

In the following $X$ will be a fixed subset of $\mathbb{Z}^d \times \{0, \ldots, T-1\}^k$, with $d, k \geq 1$ and $T \in \mathbb{N}$. We assume that

(i) $X$ is $T$-periodic in $e_1, \ldots, e_d$;

(ii) $X$ is connected in the following sense: there exists $\mathcal{E} \subset X \times X$ such that for all $i, j \in X$ there exists a sequence $\{i_n\}_{n \geq 0}$ of points of $X$, with $i_0 = i$ and $i_N = j$, such that the segment $(i_n, i_{n+1}) \in \mathcal{E}$. Moreover, the set $\mathcal{E}$ is $T$-periodic: i.e. if the segment $(i, j)$ belongs to $\mathcal{E}$, then, for any $m = 1, \ldots, d$, the segment $(i + Te_m, j + Te_m)$ belongs to $\mathcal{E}$;

(iii) the set $\mathcal{E}$ is equi-bounded; i.e. there exists $R > 0$ such that

$$\max \{|i - j| : (i, j) \in \mathcal{E}\} \leq R.$$ 

Note that it is not restrictive to assume that $R \leq T$, upon taking a larger period.

Let $a_{ij}$ be $T$-periodic coefficients in $e_1, \ldots, e_d$; i.e.,

$$a_{i+Te_m,j+Te_m} = a_{ij} \text{ for all } i, j \in X, m \in \{1, \ldots, d\},$$

such that $a_{ij} > 0$ if $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$ if $(i, j) \notin \mathcal{E}$. For $\varepsilon > 0$, we introduce the family of functionals $F_{\varepsilon}$ defined on functions $u : \varepsilon X \to \mathbb{R}$ by

$$F_{\varepsilon}(u) := \sum_{i,j \in X} \varepsilon^d a_{ij} \left( \frac{u_i - u_j}{\varepsilon} \right)^2,$$

where $u_i := u(\varepsilon i)$. Note that also the case $a_{ij} = 1$ if $(i, j) \in \mathcal{E}$ is non trivial. Note moreover that, by the periodicity of $a_{ij}$, there exists a positive constant $C$ such that $C \leq a_{ij} \leq 1/C$ if $(i, j) \in \mathcal{E}$, so that $F_{\varepsilon}$ is estimated from above and below by the energy corresponding to $a_{ij} = 1$ if $(i, j) \in \mathcal{E}$.

More in general, we will consider ‘localized’ versions of energies $F_{\varepsilon}$, limiting interactions to $i, j \in X$ such that $\varepsilon i, \varepsilon j \in \Omega \times \varepsilon Q_{T,h}$ for some Lipschitz open subset $\Omega$ of $\mathbb{R}^d$.

Remark 2.1. In the notation above, we can include also the case of

$$X \subset \mathbb{Z}^d \times \prod_{n=1}^k \{0, \ldots, M_n - 1\},$$

with $T_m \geq 1$, $m = 1, \ldots, d$, and $M_n \geq 1$, $n = 1, \ldots, k$, and $X$ $T_m$-periodic in $e_m$, for any $m = 1, \ldots, d$. In this case, we take $T = \text{l.c.m.}\{T_1, \ldots, T_d, M_1, \ldots, M_k\}$.

3 Two-connectedness and Poincaré-Wirtinger’s inequality

In this section we prove two technical lemmas, which will allow to use some compactness results for systems of nearest-neighbour interactions. To that end, we define a coarse-grained, lower-dimensional variable as follows. Let $u$ be a real-valued function defined on $X$. For $l \in \mathbb{Z}^d$, set

$$\tilde{u}^l := \frac{1}{\#[(Q_{T,d} \times Q_{T,k}) \cap X]} \sum_{i \in (Q_{T,d}^l \times Q_{T,k}) \cap X} u_i,$$

with $u_i = u(i)$.

The first result of this section states that a nearest-neighbour interaction energy on the coarse-grained variable is (locally) dominated by the energy on $X$.

Proposition 3.1. There exist $C = C(X) > 0$ and $M > 0$ such that

$$|\tilde{u}^l - \tilde{u}^{l'}|^2 \leq C \sum_{i,j \in (Q_{T,d}^l \cup Q_{T,d}^{l'} + (-M,M)^d) \times Q_{T,k}) \cap X} |u_i - u_j|^2,$$

for any $l, l' \in \mathbb{Z}^d$ such that $|l - l'| = 1$.
The connectedness of $u$ substituting $T$ on the original functions contains the path joining $u$ and $u_{m}$ through the points $\gamma \in X$. For any $m \in T,d$ with $u_{m}$, for all $i \in T,d \times Q_{T,k}$, we write that
\[
|u_{i} - u_{i+T_{em}}| = \sum_{n=1}^{N_{i}} (u_{jn-1} - u_{jn}),
\]
so that, due to (4) combined with the Hölder inequality, we have that
\[
|\tilde{u}^{i} - \tilde{u}^{i'}|^{2} \leq \frac{1}{\#(Q_{T,d} \times Q_{T,k}) \cap X} \sum_{i \in (Q_{T,d} \times Q_{T,k}) \cap X} \left| \sum_{n=1}^{N_{i}} (u_{jn-1} - u_{jn}) \right|^{2}
\]
\[
\leq \frac{1}{\#(Q_{T,d} \times Q_{T,k}) \cap X} \sum_{i \in (Q_{T,d} \times Q_{T,k}) \cap X} \sum_{n=1}^{N_{i}} \sum_{n=1}^{N_{i}} |u_{jn-1} - u_{jn}|^{2}
\]
\[
\leq \max\{N_{i} : i \in (Q_{T,d} \times Q_{T,k}) \cap X\} \sum_{i,j} \sum_{n=1}^{N_{i}} \sum_{n=1}^{N_{j}} |u_{i} - u_{j}|^{2},
\]
where in the last inequality we have used the fact that $((Q_{T,d} \cup Q_{T,d}^{j} + (-M,M)^{d}) \times Q_{T,k}) \cap X$ contains the path $\gamma$ joining $i$ and $i + T_{em}$ for all $i \in (Q_{T,d} \times Q_{T,k}) \cap X$. This proves the desired inequality.

**Remark 3.2.** In order to reduce the number of parameters, we can choose $M = T$, up to substituting $T$ with a multiple and taking a slightly larger $M$.

We point out that in the following (2) and (3) will be applied to functions $u : \mathbb{Z} \times X \to \mathbb{R}$, where $u_{i}$ stands for $u(\varepsilon i)$ as in the notation introduced above.

Now, we show a Poincaré-Wirtinger inequality. This will be used to recover information on the original functions $u$ from their coarse-grained versions.

**Proposition 3.3.** (i) There exists $C = C(X) > 0$ such that, for any $l \in \mathbb{Z}^{d}$,
\[
\sum_{i \in (Q_{T,d} \times Q_{T,k}) \cap X} |u_{i} - \tilde{u}^{i}|^{2} \leq C \sum_{i,j} |u_{i} - u_{j}|^{2},
\]
(ii) there exist positive constants $C$ and $M$ such that, for any $l \in \mathbb{Z}^{d}$,
\[
\sum_{i \in (Q_{T,d} \times Q_{T,k}) \cap X} |u_{i} - \tilde{u}^{i}|^{2} \leq C \sum_{i,j} a_{ij} |u_{i} - u_{j}|^{2}. \tag{5}
\]
Proof. (i) Using definition 2 of $\tilde{u}^j$ and thanks to the Hölder inequality, we deduce that

$$\sum_{i\in(Q_T^i\times Q_T,k)\cap X} |u_i - \tilde{u}^j|^2 \leq \frac{1}{\#((Q_T^i\times Q_T,k)\cap X)^2} \sum_{i\in(Q_T^i\times Q_T,k)\cap X} \sum_{j\in(Q_T^i\times Q_T,k)\cap X} (u_i - u_j)^2$$

which concludes the proof.

(ii) Since $X$ is connected and due to the boundedness and periodicity properties of the coefficient $a_{ij}$, there exists $M > 0$ large enough such that if $i, j \in (Q_T^i\times Q_T,k) \cap X$, then there exists a path $\gamma$ joining $i$ and $j$ which is contained in $[(Q_T^i + (-M, M))^d \times Q_T,k] \cap X$. From (6), we deduce (5) as desired. \qed

4 A compactness result

In this section, we are going to show that sequences with equi-bounded energy are compact in $L^2$ with limit in $H^1_{\text{loc}}(\mathbb{R}^d)$.

Let $\Omega$ be an open set of $\mathbb{R}^d$ with Lipschitz boundary. For $\varepsilon > 0$, let $u_\varepsilon$ be a family of functions $u_\varepsilon : (\Omega \times \varepsilon Q_T,k) \cap \varepsilon X \rightarrow \mathbb{R}$. Setting $I_\varepsilon = I_\varepsilon(\Omega) := \{l \in \mathbb{Z}^d : \varepsilon Q_{T,k} \subset \Omega\}$, we define a piecewise-constant function $\bar{u}_\varepsilon$ in $L^2(\Omega)$ by

$$\bar{u}_\varepsilon(x^d) := \sum_{l \in I_\varepsilon} \tilde{u}_\varepsilon^l \chi_{\varepsilon Q_{T,k}^l}(x^d),$$

where $\tilde{u}_\varepsilon^l$ is given by (3), with $u_i = u_i^\varepsilon := u_\varepsilon(\varepsilon i)$, and $\chi_{\varepsilon Q_{T,k}^l}$ is the characteristic function of the cube $\varepsilon Q_{T,k}^l$.

In order to introduce the convergence of $u_\varepsilon$, we need to identify real-valued functions $u$ defined on $\varepsilon X$ with piecewise-constant interpolations as follows. We introduce the set

$$C_\varepsilon(\Omega) := \left\{ u : \mathbb{R}^d \times Q_{T,k} \rightarrow \mathbb{R} : u \text{ is constant on } \varepsilon Q_{T,k}^l \times \varepsilon Q_k \right\},$$

(8)

for $(l, n) \in \left(\mathbb{Z}^d \cap \frac{1}{\varepsilon} \Omega\right) \times \{0, \ldots, T-1\}^k$, so that a function $u : (\varepsilon \mathbb{Z}^d \cap \Omega) \times \varepsilon \{0, \ldots, T-1\}^k \cap \varepsilon X \rightarrow \mathbb{R}$ can be identified with its extension belonging to $C_\varepsilon(\Omega)$. We say that the family of function $u_\varepsilon$ in $C_\varepsilon(\Omega)$ converges to $u \in H^1(\Omega)$ if

$$\bar{u}_\varepsilon \rightarrow u \quad \text{in } L^2_{\text{loc}}(\Omega).$$

(9)

From this convergence, we will obtain that

$$\int_{\bar{\Omega}_\varepsilon} |u_\varepsilon - u|^2 \chi_{(\cup_{i \in \varepsilon X} \varepsilon Q_{T,k}^i)} \rightarrow 0,$$

(10)

where we have set $i = (i^d, i^k) \in \mathbb{Z}^d \times \{0, \ldots, T-1\}^k$ and $\bar{\Omega}_\varepsilon$ is given by

$$\bar{\Omega}_\varepsilon := \bigcup_{i \in \varepsilon X} \varepsilon Q_{T,k}^i \times Q_{T,k},$$

(11)

and $L_\varepsilon := \{l \in \mathbb{Z}^d : \text{dist}(\varepsilon l, \partial \Omega) > 2\varepsilon \sqrt{d} T\}$. The next proposition provides a compactness result for $\bar{u}_\varepsilon$ using the analysis of nearest-neighbour interactions in [3].

Proposition 4.1 (Compactness). Let $\Omega$ be an open set of $\mathbb{R}^d$ with Lipschitz boundary. Let $u_\varepsilon$ be a family of functions defined on $(\Omega \times \varepsilon Q_T,k) \cap \varepsilon X$ such that

$$\sum_{l \in \mathbb{Z}^d} \sum_{i \in (Q_T^i \times Q_T,k) \cap X} \varepsilon^d |u_i^\varepsilon|^2 \leq C, \quad \forall \varepsilon > 0,$$

(12)
where \( u_i^* = 0 \) if \( i \notin [\frac{1}{2} \Omega \cap Q_{T,d}^I] \times Q_{T,h} \) and
\[
\sum_{l,l' \in \mathcal{L}, |l-l'|=1} \sum_{i,j \in [(Q_{T,d}^I) \times Q_{T,d}^I] + (-M,M)^d} \varepsilon^{d-2} |u_i^* - u_j^*|^2 \leq C, \quad \forall \varepsilon > 0. \tag{13}
\]

Then, up to a subsequence, the family \( \overline{\pi}_e \), given by \( \Omega \), strongly converges in \( L^2_{\text{loc}}(\Omega) \) to some \( u \in H^1(\Omega) \).

**Proof.** First, we show that \( \overline{\pi}_e \) weakly converges in \( L^2_{\text{loc}} \) to some \( u \). Indeed, from (12), we deduce that the norm \( \|\overline{\pi}_e\|_\Omega \) is bounded which implies the weak convergence of \( \overline{\pi}_e \). Moreover, thanks to assumption (13), an application of [3] Proposition 3.4 provides us that \( u \in H^1(\Omega) \).

Now, we prove the strong convergence in \( L^2_{\text{loc}}(\Omega) \) of \( \overline{\pi}_e \). To this end, the key tool is the Compactness Criterion by Fréchet and Kolmogorov (see, e.g. [16, Theorem 4.26]). In other words, we have to prove that, for any \( \Omega' \subset \subset \Omega \) and for any \( \eta > 0 \), there exists \( \delta > 0 \), with \( \text{dist}(\Omega', \mathbb{R}^d \setminus \Omega') > \delta \), such that for every \( h \in \mathbb{R}^d \), with \( |h| < \delta \), then
\[
\| \tau_h \overline{\pi}_e - \overline{\pi}_e \|_{L^2(\Omega')} < \eta, \tag{14}
\]
where \( \tau_h \overline{\pi}_e(x) := \overline{\pi}_e(x + h) \). Assume that \( h = \lambda e_m \), for some \( m = 1, \ldots, d \). The inequality (14) for every \( h \in \mathbb{R}^d \) is obtained by triangle inequality. Fix \( \Omega' \subset \subset \Omega \) and set\[
\mathcal{I}_e := \{ l \in \mathcal{L}_e : \Omega' \subset \cup_l \varepsilon Q_{T,d}^l \subset \Omega \}.
\]

Take \( x \in \varepsilon Q_{T,d}^l \). Hence, we have that \( x \in \varepsilon Q_{T,d}^l \) and \( (x + h) \in \varepsilon Q_{T,d}^{l'} \), for some \( l, l' \in \mathcal{I}_e \). By definition of \( \overline{\pi}_e \) given by (7), we deduce that
\[
|\tau_h \overline{\pi}_e(x) - \overline{\pi}_e(x)|^2 = |\overline{\pi}_e(x + h) - \overline{\pi}_e(x)|^2 = |\tilde{u}_e^l - \tilde{u}_e^{l'}|^2 \tag{15}
\]
Since \( l \) and \( l' \) are not necessarily such that \( |l - l'| = 1 \), we need to re-write the two-connectedness inequality in terms of non-neighbouring cubes. In order to show this, let \( S_{l'} \) be union of neighbouring cubes joining \( \varepsilon Q_{T,d}^l \) and \( \varepsilon Q_{T,d}^{l'} \) such that each two consecutive cubes have one face in common; i.e., \( S_{l'} = \bigcup_{n=0}^{N_{l'}} \varepsilon Q_{T,d}^{l_0} \) with \( |l_n - l_{n-1}| = 1, l_0 = l \) and \( l_{N_e} = l' \). Note that the number \( N_e \) of the cubes \( \varepsilon Q_{T,d}^l \) contained in stripes of cubes joining \( \varepsilon Q_{T,d}^l \) and \( \varepsilon Q_{T,d}^{l'} \) is of order \( |h|/T^{-1} \varepsilon^{-1} \). Hence, thanks to inequality (3), we deduce that
\[
|\tilde{u}_e^l - \tilde{u}_e^{l'}|^2 \leq \sum_{n=1}^{N_e} \left( |\tilde{u}_e^{l_n} - \tilde{u}_e^{l_{n-1}}|^2 \right) \leq |h|T^{-1} \varepsilon^{-1} \sum_{n=1}^{N_e} |\tilde{u}_e^{l_n} - \tilde{u}_e^{l_{n-1}}|^2 \\
\leq C|h|T^{-1} \varepsilon^{-1} \sum_{n=1}^{N_e} \sum_{i,j \in [(S_{l_0} + (-M,M)^d) \times Q_{T,h} \setminus X]} |u_i^* - u_j^*|^2 \\
\leq C|h|T^{-1} \varepsilon^{-1} \sum_{i,j \in [(S_{l'}) + (-M,M)^d) \times Q_{T,h} \setminus X]} |u_i^* - u_j^*|^2.
\]
Plugging the above inequality in (15), we obtain that
\[
|\tau_h \overline{\pi}_e(x) - \overline{\pi}_e(x)|^2 \leq C|h|T^{-1} \varepsilon^{-1} \sum_{i,j \in [(S_{l'}) + (-M,M)^d) \times Q_{T,h} \setminus X]} |u_i^* - u_j^*|^2 \\
= C|h|T^{-1} \varepsilon^{-1} \sum_{i,j \in [(S(x,h) + (-M,M)^d) \times Q_{T,h} \setminus X]} |u_i^* - u_j^*|^2,
\]
where we have set $S(x, h) := S_{0V} + (-M, M)^d$ which depends on $x$ and $h$. Now, an integration with respect to $x \in \varepsilon Q_{T,d}$ yields

$$
\int_{\varepsilon Q_{T,d}} |r_n \tilde{\pi}_e(x) - \tilde{\pi}_e(x)|^2 dx \leq C|h|T^{-1} \varepsilon^{-1} \int_{\varepsilon Q_{T,d}} \left( \sum_{i,j \in \{S(x, h) \times Q_{T,k}\} \cap X} |u_i^e - u_j^e|^2 \right) dx
$$

$$
\leq C|h|T^{-1} \varepsilon^{-1} \sum_{i,j \in \{S(\varepsilon Q_{T,d}^1, h) \times Q_{T,k}\} \cap X} |u_i^e - u_j^e|^2,
$$

where we have used the fact that $S(x, h) \subset S(\varepsilon Q_{T,d}^1, h) := \bigcup \{S(x, h) : x \in \varepsilon Q_{T,d}^1\}$.

Summing over $I_x$, we have that

$$
\sum_{i \in I_x} \int_{\varepsilon Q_{T,d}} |r_n \tilde{\pi}_e(x) - \tilde{\pi}_e(x)|^2 dx \leq C|h|T^{-1} \varepsilon^{-1} \sum_{i \in I_x} \sum_{i,j \in \{S(\varepsilon Q_{T,d}^1, h) \times Q_{T,k}\} \cap X} |u_i^e - u_j^e|^2,
$$

which implies that

$$
\int_{\varepsilon Q_{T,d}} |r_n \tilde{\pi}_e(x) - \tilde{\pi}_e(x)|^2 dx \leq \sum_{i \in I_x} \int_{\varepsilon Q_{T,d}} |r_n \tilde{\pi}_e(x) - \tilde{\pi}_e(x)|^2 dx \leq C|h|,
$$

where we have used assumption (13) and the fact that the number of indices $l$ and $l'$ such that $S(\varepsilon Q_{T,d}^1, h) \cap S(\varepsilon Q_{T,d}^{l'1}, h) \neq \emptyset$ is of the order $|h|T^{-1} \varepsilon^{-1}$ (the ratio between the size of $S(\varepsilon Q_{T,d}^1, h)$ and the size of $\varepsilon Q_{T,d}^1$). This concludes the proof of (14).

Finally, applying the compactness criterion, it follows that, up to a subsequence, $\tilde{\pi}_e \rightharpoonup v$. Since we already know that $\tilde{\pi}_e \rightharpoonup u$, we conclude that $v = u$, which is the desired claim.

The next proposition provides a convergence result in the sense of (10).

**Proposition 4.2.** Let $\Omega$ be an open set of $\mathbb{R}^d$ with Lipschitz boundary. Let $u_\varepsilon$ be a sequence of functions defined on $\varepsilon X$ such that

$$
\sup_{\varepsilon > 0} \left( \sum_{i \in I_x} \sum_{i,j \in \{x \times Q_{T,k}\} \cap X} \varepsilon^d |u_i^\varepsilon|^2 + F_i(u_\varepsilon) \right) \leq C. 
$$

Then, up to a subsequence, we have that

$$
\int_{\Omega_x} |u_\varepsilon - u|^2 \chi_{\cup_i \in X \times Q_{T,d}^1 \times Q_{T,k}} \to 0
$$

where $u \in H^1(\Omega)$ is the strong limit in $L^2(\Omega)$ of the sequence $\tilde{\pi}_e$ and $\tilde{\tilde{\pi}}_e$ is given by (11).

**Proof.** Set $x = (x^d, x^k) \in \varepsilon Q_{T,d}^1 \times Q_{T,k}$ and recall that $u_\varepsilon$ is defined on $\varepsilon Q_{T,d}^1 \times \varepsilon Q_{T,k}$. Hence,

$$
\int_{\tilde{\Omega}_x} |(u_\varepsilon - u)(x)\chi_{\cup_i \in X \times Q_{T,d}^1 \times Q_{T,k}}(x)|^2 dx \leq \int_{\tilde{\Omega}_x} |(u_\varepsilon - \tilde{\pi}_e)(x)\chi_{\cup_i \in X \times Q_{T,d}^1 \times Q_{T,k}}(x)|^2 dx + \int_{\tilde{\Omega}_x} |(\tilde{\pi}_e - u)(x)\chi_{\cup_i \in X \times Q_{T,d}^1 \times Q_{T,k}}(x)|^2 dx.
$$

From Proposition 4.1, we know that $\tilde{\pi}_e$ strongly converges to $u$ in $L^2(\Omega)$, so that the second integral in (17) vanishes as $\varepsilon \to 0$. 


In order to estimate the first integral of (17), the key tool is the Poincaré-Wirtinger inequality given by (5). Indeed, due to the fact that $u_\varepsilon$ is constant on $\varepsilon Q_d^d \times Q_k$ and $\pi_\varepsilon$ is constant on $\varepsilon Q_d^d \times Q_k$, we deduce that

$$
\int_{\tilde{\Omega}_\varepsilon} |(u_\varepsilon - \pi_\varepsilon)(x)|^2 + \sum_{i \in J} \sum_{j \in J} |u_i^{\varepsilon} - \tilde{u}_j^{\varepsilon}|^2 \leq \varepsilon^d \sum_{i \in J} \sum_{j \in J} a_{ij} |u_i^{\varepsilon} - u_j^{\varepsilon}|^2 \leq \varepsilon^2 F_\varepsilon(u_\varepsilon).
$$

From this, combined with assumption (10), we have that also the first integral of (17) goes to 0 as $\varepsilon \to 0$, which concludes the proof.

5 Treatment of boundary data

In this section we prove a classical lemma which allows to match boundary conditions. For future reference we prove it in a general form.

For any $u \in H^1(\Omega)$, we define the sequence $v_\varepsilon$ on $\varepsilon X$ by

$$
v_i^{\varepsilon} = v_\varepsilon(x) := \int_{\varepsilon X} u(x) dx.
$$

We have that $v_\varepsilon$ converges to $u$ with respect to convergence (9). For any bounded open set $A$ and for $\delta > 0$, we define $A(\delta) := \{ x \in A : \text{dist}(x, \partial A) > \delta \}$.

**Lemma 5.1.** Let $A$ be a bounded and open set of $\Omega$ with Lipschitz boundary. Let $u_\varepsilon$ be a sequence converging to $u \in H^1(\Omega)$ with respect to convergence (9). For any $\delta > 0$, there exists a sequence $w_\varepsilon$ converging to $u$ with respect convergence (9) such that

$$
w_\varepsilon = u_\varepsilon, \quad \text{if } i \in (A(2\delta) \times Q_{T,k}) \cap X,
$$

$$
w_\varepsilon = v_\varepsilon, \quad \text{if } i \in (A \setminus A(\delta) \times Q_{T,k}) \cap X,
$$

and

$$
\limsup_{\varepsilon \to 0} (F_\varepsilon(w_\varepsilon) - F_\varepsilon(u_\varepsilon)) \leq o(1)
$$

as $\delta \to 0$.

**Proof.** Fixed $N \in \mathbb{N}$ and $\delta \in (0, 1/4)$. For $h \in \{0, \cdots, N\}$, we set

$$
A_h := \left\{ x \in A : \text{dist}(x, A(\delta)) < \frac{\delta h}{N} \right\}.
$$

For $h \in \{0, \cdots, N - 1\}$, let $\phi_h^0$ be a cut-off function between $A_h$ and $A_{h+1}$ with $|\nabla \phi_h^0| \leq 2N/\delta$ and let $w_i^\varepsilon$ be a function defined by

$$
w_i^{\varepsilon} = w_i^\varepsilon(x) := \phi_h^0(\varepsilon x^d) u_i^{\varepsilon} + (1 - \phi_h^0(\varepsilon x^d)) v_i^{\varepsilon}.
$$

Since both $u_\varepsilon$ and $v_\varepsilon$ converge to $u$ with respect to convergence given by (9), we also deduce that $w_\varepsilon$ converges to $u$ with respect to (9). By adding and subtracting the term $\phi_h^0(\varepsilon x^d) u_i^{\varepsilon} + (1 - \phi_h^0(\varepsilon x^d)) v_i^{\varepsilon}$, we get that

$$
w_i^{\varepsilon} - w_i^{\varepsilon} = \phi_h^0(\varepsilon x^d)(u_i^{\varepsilon} - u_i^\varepsilon) + (1 - \phi_h^0(\varepsilon x^d))(v_i^{\varepsilon} - v_i^\varepsilon) + (\phi_h^0(\varepsilon x^d) - \phi_h^0(\varepsilon x^d))(u_i^{\varepsilon} - u_i^\varepsilon).
$$

For $h \in \{1, \cdots, N - 2\}$, we set

$$
S_h^d := A_{h+1} \setminus A_h,
$$

so that $A = A_h \cup A \setminus A_{h+1} \cup S_h^d$. In order to estimate the energy

$$
\sum_{i,j \in \{A \times Q_{T,k}\} \cap X} \varepsilon^{d-2} a_{ij}(w_i^{\varepsilon} - w_j^{\varepsilon})^2,
$$

we separately evaluate the following cases.
i) \( i, j \in (\frac{1}{2}A_h \times Q_{T,h}) \cap X \);

ii) \( i, j \in (\frac{1}{2}(A \setminus A_{h+1}) \times Q_{T,h}) \cap X \);

iii) \( i \in (\frac{1}{2}S^h_{ik} \times Q_{T,h}) \cap X \) and \( j \in (\frac{1}{2}A \times Q_{T,h}) \cap X \);

iv) \( i \in (\frac{1}{2}(A_h \cup (A \setminus A_{h+1})) \times Q_{T,k}) \cap X \) and \( j \in (\frac{1}{2}S^h_{ik} \times Q_{T,h}) \cap X \);

v) \( i \in (\frac{1}{2}A_h \times Q_{T,k}) \cap X \) and \( j \in (\frac{1}{2}(A \setminus A_{h+1}) \times Q_{T,h}) \cap X \);

vi) \( i \in (\frac{1}{2}(A \setminus A_{h+1}) \times Q_{T,k}) \cap X \) and \( j \in (\frac{1}{2}A_h \times Q_{T,h}) \cap X \) as follows

i) In view of definition \([20]\), we deduce that

\[
\sum_{i,j \in (\frac{1}{2}A_h \times Q_{T,h}) \cap X} \varepsilon^{d-2} a_{ij}(u_i^e - u_j^e)^2 = \sum_{i,j \in (\frac{1}{2}A_h \times Q_{T,h}) \cap X} \varepsilon^{d-2} a_{ij}(u_i^e - u_j^e)^2 \\
\leq \sum_{i,j \in (\frac{1}{2}A \times Q_{T,h}) \cap X} \varepsilon^{d-2} a_{ij}(u_i^e - u_j^e)^2. \tag{22}
\]

ii) We have that

\[
\sum_{i,j \in (\frac{1}{2}(A \setminus A_{h+1}) \times Q_{T,h}) \cap X} \varepsilon^{d-2} a_{ij}(w_i^e - w_j^e)^2 = \sum_{i,j \in (\frac{1}{2}(A \setminus A_{h+1}) \times Q_{T,h}) \cap X} \varepsilon^{d-2} a_{ij}(w_i^e - w_j^e)^2 \\
\leq \sum_{i,j \in (\frac{1}{2}(A \setminus A_d) \times Q_{T,h}) \cap X} \varepsilon^{d} a_{ij}(w_i^e - w_j^e)^2. \tag{23}
\]

In view of definition of \( v_i^e \) given by \([18]\) and since \( \varepsilon v^d + \varepsilon Q_d = \varepsilon (v^d - j^d) + \varepsilon j^d + \varepsilon Q_d \), we deduce that

\[
|v_i^e - v_j^e|^2 = \left| \int_{\varepsilon^d + \varepsilon Q_d} u(x)dx - \int_{\varepsilon^d + \varepsilon Q_d} u(x)dx \right|^2 \\
= \int_{\varepsilon^d + \varepsilon Q_d} (u(x + \varepsilon(i^d - j^d)) - u(x))dx \right|^2. \tag{24}
\]

Since \( u \in H^1(\Omega) \), we have that

\[
u(x + \varepsilon(i^d - j^d)) - u(x) = \int_0^1 \frac{\partial u}{\partial t}(x + \varepsilon(i^d - j^d))dt \\
= \int_0^1 \nabla u(x + \varepsilon(i^d - j^d)) \cdot (i^d - j^d)dt.
\]

This, combined with \([24]\) and the Fubini theorem, implies that

\[
|v_i^e - v_j^e|^2 = \left| \int_{\varepsilon^d + \varepsilon Q_d} \int_0^1 \nabla u(x + \varepsilon(i^d - j^d)) \cdot (i^d - j^d)dx \right|^2 \\
= \frac{1}{\varepsilon^d} \left| \int_0^1 \int_{\varepsilon^d + \varepsilon Q_d} \nabla u(x + \varepsilon(i^d - j^d)) \cdot (i^d - j^d)dx \right|^2 \\
\leq \varepsilon^{2-d}|i^d - j^d|^2 \int_0^1 \int_{\varepsilon^d + \varepsilon Q_d} |\nabla u(x + \varepsilon(i^d - j^d))|^2 dx dt \\
\leq \varepsilon^{2-d}T^2 \int_0^1 \int_{\varepsilon^d + \varepsilon Q_d} |\nabla u(x)|^2 dx dt, \tag{25}
\]

where in the last inequality we have used the fact that the nodes \( i \) and \( j \) interact at most at distance \( T \). In view of the assumption of finite range along with estimate above, from \([23]\), it follows that

\[
\sum_{i,j \in (\frac{1}{2}(A \setminus A_{h+1}) \times Q_{T,h}) \cap X} \varepsilon^{d-2} a_{ij}(w_i^e - w_j^e)^2 \leq C \sum_{i \in (\frac{1}{2}(A \setminus A_d) \times Q_{T,h}) \cap X} \int_{\varepsilon^d + \varepsilon Q_d} |\nabla u(x)|^2 dx \\
\leq C \int_{A \setminus A(2d)} |\nabla u(x)|^2 dx, \tag{26}
\]

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where the constant $C$ is due to the fact that a fixed node $i \in \frac{1}{\varepsilon}(A \setminus A(\delta)) \times Q_{T,k}) \cap X$ interacts with a finite number of nodes $j \in \frac{1}{\varepsilon}(A \setminus A(\delta)) \times Q_{T,k}) \cap X$.

iii) First note that due to the assumption of finite range, if $\varepsilon^d \in S^d_k$, then $\varepsilon^d \in \tilde{S}^d_k := S^d_{k-1} \cup S^d_k \cup S^d_{k+1}$. This combined with (21) and the Jensen inequality implies that

$$
\sum_{i \in \frac{1}{\varepsilon}(S^d_k \cap A(\delta)) \cap X} \varepsilon^{-2} a_{ij}(u_i^\varepsilon - w_i^\varepsilon)^2 \leq C \sum_{i \in \frac{1}{\varepsilon}(S^d_k \cap A(\delta)) \cap X} \varepsilon^{-2} a_{ij}(u_i^\varepsilon - w_i^\varepsilon)^2 + C \sum_{i \in \frac{1}{\varepsilon}(S^d_k \cap A(\delta)) \cap X} \varepsilon^{-2} a_{ij}(u_i^\varepsilon - v_i^\varepsilon)^2
$$

$$
+ C \sum_{i \in \frac{1}{\varepsilon}(S^d_k \cap A(\delta)) \cap X} \varepsilon^{-2} a_{ij}(\phi_d(\varepsilon^d u_i^\varepsilon) - \phi_d(\varepsilon^d v_i^\varepsilon))^2 (u_i^\varepsilon - v_i^\varepsilon)^2.
$$

Due to the fact that $|\nabla \phi_d| \leq 2N/\varepsilon$, the last integral in (27) can be estimated as follows

$$
\sum_{i \in \frac{1}{\varepsilon}(S^d_k \cap A(\delta)) \cap X} \varepsilon^{-2} a_{ij}(\phi_d^h(\varepsilon^d u_i^\varepsilon) - \phi_d^h(\varepsilon^d v_i^\varepsilon))^2 (u_i^\varepsilon - v_i^\varepsilon)^2
$$

$$
\leq \frac{N^2}{\varepsilon^d} \sum_{i \in \frac{1}{\varepsilon}(S^d_k \cap A(\delta)) \cap X} \varepsilon^{-2} a_{ij}(u_i^\varepsilon - v_i^\varepsilon)^2 \leq 3 \frac{N^2}{\varepsilon^d} \sum_{i \in \frac{1}{\varepsilon}(S^d_k \cap A(\delta)) \cap X} \varepsilon^{-2} a_{ij}(u_i^\varepsilon - v_i^\varepsilon)^2.
$$

In order to estimate the first two integrals in (27), we may choose $h \in \{1, \ldots, N - 2\}$ such that

$$
\sum_{i \in \frac{1}{\varepsilon}(S^d_k \cap A(\delta)) \cap X} \varepsilon^{-2} a_{ij} \left[ (u_i^\varepsilon - u_i^\varepsilon)^2 + (v_i^\varepsilon - v_i^\varepsilon)^2 \right]
$$

$$
\leq \frac{1}{N - 2} \sum_{i,j \in \frac{1}{\varepsilon}A \cap A(\delta)} \varepsilon^{-2} a_{ij}(u_i^\varepsilon - u_i^\varepsilon)^2 + \frac{1}{N - 2} \sum_{i,j \in \frac{1}{\varepsilon}A \cap A(\delta)} \varepsilon^{-2} a_{ij}(v_i^\varepsilon - v_i^\varepsilon)^2
$$

$$
\leq \frac{1}{N - 2} \sum_{i,j \in \frac{1}{\varepsilon}A \cap A(\delta)} \varepsilon^{-2} a_{ij}(u_i^\varepsilon - u_i^\varepsilon)^2 + \frac{C}{N - 2} \int_{A \setminus A(\delta)} |\nabla u(x)|^2 dx,
$$

where we have used (25) and the assumption of finite range. This, combined with (27), leads us to

$$
\sum_{i \in \frac{1}{\varepsilon}(S^d_k \cap A(\delta)) \cap X} \varepsilon^{-2} a_{ij}(u_i^\varepsilon - w_i^\varepsilon)^2 \leq C \frac{1}{N - 2} \sum_{i,j \in \frac{1}{\varepsilon}A \cap A(\delta)} \varepsilon^{-2} a_{ij}(u_i^\varepsilon - u_i^\varepsilon)^2
$$

$$
+ \frac{C}{N - 2} \int_{A \setminus A(\delta)} |\nabla u(x)|^2 dx + C \frac{N^2}{\varepsilon^d} \sum_{i \in \frac{1}{\varepsilon}(S^d_k \cap A(\delta)) \cap X} \varepsilon^{-2} a_{ij}(u_i^\varepsilon - v_i^\varepsilon)^2.
$$

iv) Note that

$$
\sum_{i \in \frac{1}{\varepsilon}(A \setminus A_{k+1}) \cap X} \varepsilon^{-2} a_{ij}(u_i^\varepsilon - w_i^\varepsilon)^2 \leq \sum_{i \in \frac{1}{\varepsilon}A \cap A(\delta)} \varepsilon^{-2} a_{ij}(u_i^\varepsilon - w_i^\varepsilon)^2,
$$

so that, the same argument as for iii) can be performed, obtaining estimate (28).
In view of the finite-range assumption, the points belonging to sets of items (v) and (vi) do not have any interaction since $\delta/N \gg \varepsilon T$.

Gathering estimates (22), (26) and (28), we obtain that, for $h \in \{1, \ldots, N-2\}$,

$$
\sum_{i,j \in (\frac{1}{T} A \times Q_{T,k}) \cap X} \varepsilon^{d-2} a_{ij}(u_i^\varepsilon - u_j^\varepsilon)^2 \leq
\sum_{i,j \in (\frac{1}{T} A \times Q_{T,k}) \cap X} \varepsilon^{d-2} a_{ij}(u_i - u_j)^2
+ C \int_{A \setminus A(2\delta)} |\nabla u(x)|^2 dx + C \frac{1}{N-2} \sum_{i,j \in (\frac{1}{T} A \times Q_{T,k}) \cap X} \varepsilon^{d-2} a_{ij}(u_i^\varepsilon - u_j^\varepsilon)^2
+ \frac{C}{N-2} \int_{A \setminus A(2\delta)} |\nabla u(x)|^2 dx.
$$

Note that the last sum vanishes as $\varepsilon \to 0$ since both $u_\varepsilon$ and $v_\varepsilon$ converge to $u$ with respect to convergence $[\varrho]$. Hence, taking the limit as $\varepsilon \to 0$ of (29), we obtain that

$$
\limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) - F_\varepsilon(u) \leq C \frac{1}{N-2} \liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon)
+ \frac{C}{N-2} \int_{A \setminus A(2\delta)} |\nabla u(x)|^2 dx.
$$

Letting first $N \to \infty$ and then $\delta \to 0$, we get inequality $[10]$ as desired. \hfill \Box

6 Homogenization

This section is devoted to the limit analysis as $\varepsilon \to 0$ of the family of functionals $F_\varepsilon : C_c(\Omega) \to [0, \infty)$ defined as

$$
F_\varepsilon(u) := \sum_{i,j \in (\frac{1}{T} A \times Q_{T,k}) \cap X} \varepsilon^{d-2} a_{ij}(u_i - u_j)^2 ,
$$

where $\Omega$ is a bounded open set of $\mathbb{R}^d$ with Lipschitz boundary and $C_c(\Omega)$ is given by $[8]$. This is done through the computation of the corresponding $\Gamma$-limit with respect to convergence $[10]$.

**Theorem 6.1.** The family of functionals $[30]$ $\Gamma$-converges to $F_{\text{hom}} : H^1(\Omega) \to \mathbb{R}$ defined by

$$
F_{\text{hom}}(u) := \int_{\Omega} A_{\text{hom}} \nabla u \cdot \nabla u dx,
$$

where

$$
A_{\text{hom}} z \cdot z := \frac{1}{T^d} \min \left\{ \sum_{i \in (Q_T \times Q_{T,k}) \cap X} \sum_{j \in (\mathbb{R}^d \times Q_{T,k}) \cap X} a_{ij}(u_i - u_j)^2 : u_i - z \cdot i^d \text{ is } T\text{-periodic in } e_1, \ldots, e_d \right\} .
$$

In this formula we interpret $u_i = z \cdot i^d$ as the discrete interpolation of the affine function $z \cdot x^d$, with $x^d \in \mathbb{R}^d$.

6.1 Proof of the lower bound

We prove the lower-bound inequality for the family $F_\varepsilon$ using the blow-up method introduced by Fonseca and Müller [18] (see also [13]).

Let $u_\varepsilon$ be a sequence with equi-bounded energy $F_\varepsilon(u_\varepsilon)$ and such that $u_\varepsilon$ converge to $u \in H^1(\Omega)$. Let the sequence of positive measures $\lambda_\varepsilon$ be defined as

$$
\lambda_\varepsilon := \sum_{i \in (\frac{1}{T} \Omega \times Q_{T,k}) \cap X} \left( \sum_{j \in (\frac{1}{T} \Omega \times Q_{T,k}) \cap X} \varepsilon^{d-2} a_{ij}(u_i^\varepsilon - u_j^\varepsilon)^2 \right) \delta_{e_i}.
$$

[12]
where $\delta_x$ is the Dirac measure concentrated at $x$. The $d$-dimensional measure $\mu_\varepsilon$ is defined by
\[
\mu_\varepsilon(B) := \lambda_d(B \times \varepsilon Q_{T,k}) = \sum_{i \in (B \times \varepsilon Q_{T,k}) \cap X} \sum_{j \in (\Omega \times \varepsilon Q_{T,k}) \cap X} \varepsilon^{d-2} a_{ij}(u_i^\varepsilon - u_j^\varepsilon)^2
\]
for Borel sets $B$ of $\mathbb{R}^d$. Note that $\mu_\varepsilon(B)$ takes into account interactions between the nodes with projection in $B$ and the ones in all $\varepsilon$-neighbourhood, which is a finite-range assumption, we can limit the interactions between the nodes with projection in $B$ and those with projection in an $\varepsilon R$-neighbourhood of $B$.

Since $\mu_\varepsilon(\Omega) = F_\varepsilon(u_\varepsilon)$ and thanks to the equi-boundedness of $F_\varepsilon$, the measures $\mu_\varepsilon$ are also equi-bounded, so that, up to subsequences, we deduce that
\[
\mu_\varepsilon \overset{*}{\rightharpoonup} \mu,
\]
where $\mu$ is a $d$-dimensional positive measure on $\Omega$. The Radon-Nikodym decomposition of the limit measure $\mu$ with respect to the $d$-dimensional Lebesgue measure $\mathcal{L}^d$ enables us to write that
\[
\mu = \frac{d\mu}{dx} \mathcal{L}^d + \mu^*,
\]
with $\mu^* \perp \mathcal{L}^d$. Note that the positiveness of $\mu$ ensures that its singular part $\mu^*$ is positive as well.

Now, we perform a local analysis. Let $x_0 \in \Omega$ be a Lebesgue point for $\mu$ with respect to $\mathcal{L}^d$, i.e.,
\[
\frac{d\mu}{dx}(x_0) = \lim_{\rho \to 0} \frac{\mu(Q_{\rho,d}(x_0))}{\rho^d} = \lim_{\rho \to 0} \frac{\mu(Q_{\rho,d}(x_0))}{\rho^d},
\]
with $Q_{\rho,d}(x_0) := x_0 + (0,\rho)^d$. Thanks to the Besicovitch Derivation Theorem, $\mathcal{L}^d$-almost every $x_0 \in \Omega$ is a Lebesgue point for $\mu$ with respect to $\mathcal{L}^d$. Moreover, in view of [23, Theorem 3.4.2], we have that, up to a set of zero Lebesgue measure, $x_0$ is a point such that
\[
\lim_{\rho \to 0} \frac{1}{\rho^d} \left( \frac{1}{\rho^d} \int_{Q_{\rho,d}(x_0)} |u(x) - u(x_0) - \nabla u(x_0) \cdot (x - x_0)|^2 \, dx \right)^{1/2} = 0. \tag{33}
\]
In other words, performing the change of variables $x = \rho y + x_0$ in the above integral, we have that
\[
\frac{u(\rho y + x_0) - u(x_0)}{\rho} \to \nabla u(x_0) \cdot y \quad \text{in} \ L^2(\Omega). \tag{33}
\]
For all $\rho \to 0$ but a countable set, we have that $\mu(Q_{\rho,d}(x_0)) = 0$ and hence for such $\rho$ we have that
\[
\mu(Q_{\rho,d}(x_0)) = \lim_{\varepsilon \to 0} \mu_\varepsilon(Q_{\rho,d}(x_0)). \tag{34}
\]
Therefore, from (32), it follows that
\[
\frac{d\mu}{dx}(x_0) = \lim_{\rho \to 0} \frac{\mu_\varepsilon(Q_{\rho,d}(x_0))}{\rho^d}.
\]
Now, we perform the blow-up argument. Since $x_0 \in \Omega$ is a Lebesgue point and due to a diagonalization argument on [72] and [74], there exists a sequence $\rho_\varepsilon \to 0$ as $\varepsilon \to 0$ such that $\rho_\varepsilon >> \varepsilon$ and the following equalities
\[
\frac{d\mu}{dx}(x_0) = \lim_{\varepsilon \to 0} \frac{\mu_\varepsilon(Q_{\rho_\varepsilon,d}(x_0))}{\rho_\varepsilon^d}, \tag{35}
\]
and
\[
\lim_{\varepsilon \to 0} \frac{1}{\rho_\varepsilon} \int_{Q_{\rho_\varepsilon,d}(x_0) \times Q_{T,k}} |u_\varepsilon - u(x)\chi_{\cup_{i \in X}^\varepsilon Q_{\rho_\varepsilon}^\varepsilon} |^2 \, dx = 0, \tag{36}
\]
hold. Thanks to the link between the measure $\mu_\varepsilon$ and the energy $F_\varepsilon$, equality (35) can be re-written as
\[
\frac{d\mu}{dx}(x_0) = \lim_{\varepsilon \to 0} \frac{1}{\rho_\varepsilon^d} \left( \int_{Q_{\rho_\varepsilon,d}(x_0) \times Q_{T,k}} \sum_{i \in (Q_{\rho_\varepsilon,d}(x_0) \times Q_{T,k}) \cap X} \sum_{j \in (\Omega \times Q_{T,k}) \cap X} \varepsilon^{d-2} a_{ij} |u_i^\varepsilon - u_j^\varepsilon|^2 \right). \tag{35}
\]
Now, the aim is to estimate the limit above. First, note that since the coefficients $a_{ij}$ are positive, we can consider only interactions taking place between nodes inside the cube $Q_{x_0} \times (x_0/ε) \times Q_{T-k}$, so that

$$\frac{\mu_0(Q_{x_0} \times (x_0/ε) \times Q_{T-k})}{ρ^d} \geq \frac{1}{ρ^d_{k}} \sum_{i,j \in (Q_{x_0} \times (x_0/ε) \times Q_{T-k}) \cap X} ε^{d-2} a_{ij}(u_i^* - u_j^*)^2. \quad (37)$$

We need to modify $u_ε$ in order to define a function $v_ε$ converging to the affine function $\nabla u(x_0) \cdot x^d$ in $L^2(Q_d)$. To that end, let $η_ε = \frac{ε}{ρ}$, and let $X_{n,ε}$ be the set $X$ rescaled to $η_ε Z^d \times ε \{0, \ldots, T - 1\}^k$. We define $v_ε$ on $(Q_d \times Q_{T-k}) \cap X_{n,ε}$ by

$$v_ε(η_ε x^d, ε x^k) := \frac{u_ε(ε x^d + x_0, ε x^k) - u(x_0)}{ρ_k}, \quad (38)$$

where $u_ε$ is defined on $(Q_{x_0} \times (x_0/ε) \times Q_{T-k}) \cap εX$. Note that since $u_ε$ is a function in $C_ε(Ω)$, $v_ε$ can be identified with a piecewise-constant function on $η_ε Q_{x_0}^d \times ε Q_k^d$ if $(x^d, x^k) \in X$. For $x = (x^d, x^k) \in \mathbb{R}^d \times Q_{T-k}$, we set $u_0(x) := \nabla u(x_0) \cdot x^d$ and we show that

$$\lim_{ε \to 0} \int_{Q_d \times Q_{T-k}} |v_ε(x) - u_0(x)|^2 \chi_{\cup_{i \in X_{n,ε}} Q_{d}^d \times Q_k^k}(x) dx = 0. \quad (39)$$

To this end, we introduce the function $u_0$ given by $u_0(x^d) := u(x_0) + w_0(x)$. Hence,

$$u(x_0) = u_0(ρ x^d) - ρ \nabla u(x_0) \cdot x^d = u_0(ρ x^d) - ρ w_0(x).$$

This, combined with (38), implies that

$$\int_{Q_d \times Q_{T-k}} |v_ε(x) - u_0(x)|^2 \chi_{\cup_{i \in X_{n,ε}} Q_{d}^d \times Q_k^k}(x) dx$$

$$= \int_{Q_d \times Q_{T-k}} \frac{u_ε(ρ x^d + x_0, ε x^k) - u(x_0)}{ρ_k} - u_0(x) \chi_{\cup_{i \in X_{n,ε}} Q_{d}^d \times Q_k^k}(x) dx$$

$$= \int_{Q_d \times Q_{T-k}} \frac{u_ε(ρ x^d + x_0, ε x^k) - u_0(ρ x^d)}{ρ_k} \chi_{\cup_{i \in X_{n,ε}} Q_{d}^d \times Q_k^k}(x) dx$$

$$\leq \int_{Q_d \times Q_{T-k}} \frac{u_ε(ρ x^d + x_0, ε x^k) - u(ρ x^d + x_0)}{ρ_k} \chi_{\cup_{i \in X_{n,ε}} Q_{d}^d \times Q_k^k}(x) dx$$

$$+ \int_{Q_d \times Q_{T-k}} u_0(ρ x^d + x_0) - u_0(ρ x^d) \chi_{\cup_{i \in X_{n,ε}} Q_{d}^d \times Q_k^k}(x) dx. \quad (40)$$

The first integral in (40) goes to $0$ as $ε \to 0$. Indeed, due to the change of variables $y^d = ρ x^d + x_0$, we deduce that

$$\int_{Q_d \times Q_{T-k}} \frac{u_ε(ρ x^d + x_0) - u_0(ρ x^d)}{ρ_k} \chi_{\cup_{i \in X_{n,ε}} Q_{d}^d \times Q_k^k}(x) dx$$

$$= \frac{1}{ρ_k^{d+2}} \int_{Q_{x_0} \times (x_0/ε) \times Q_{T-k}} |u_ε(y^d, ε x^k) - u(y^d)|^2 \chi_{\cup_{i \in X_{n,ε}} Q_{d}^d \times Q_k^k}(x) dy^d dx^k,$$

which vanishes as $ε \to 0$ thanks to (36). We evaluate the second integral in (40). Using again the change of variables $y^d = ρ x^d + x_0$ and the definition of $u_0$, we have that

$$\int_{Q_{x_0} \times (x_0/ε) \times Q_{T-k}} |u(y^d, ε x^k) - u(y^d)|^2 \chi_{\cup_{i \in X_{n,ε}} Q_{d}^d \times Q_k^k}(x) dy^d dx^k$$

$$\leq T^k \frac{1}{ρ_k^{d+2}} \int_{Q_{x_0} \times (x_0/ε) \times Q_{T-k}} |u(y^d) - u_0(y - x_0)|^2 dy^d$$

$$= T^k \frac{1}{ρ_k^{d+2}} \int_{Q_{x_0} \times (x_0/ε) \times Q_{T-k}} |u(y^d) - u(x_0) - \nabla(x_0) \cdot (y - x_0)|^2 dy^d.$$
Thanks to (33), it follows that also the integral above vanishes as \( \varepsilon \to 0 \) so that we can conclude that (39) holds. Set

\[
v_\varepsilon(\eta_0^d, \varepsilon^k) := u_\varepsilon(\eta_0^d, \varepsilon^k).
\]

Now, using Lemma 5.1, we may modify the sequence \( v_\varepsilon \) to get a new sequence \( \bar{v}_\varepsilon \) which is equal to \( \nabla u(x_0) \cdot \eta_0^d \) near the boundary \( (\partial Q_d \times Q_{T,k}) \cap X_{\eta_0} \), where \( X_{\eta_0} \) is the set \( X \) rescaled to \( \eta_0^d \times \{ 0, \ldots, T - 1 \} \), and

\[
\limsup_{\varepsilon \to 0} \sum_{(\eta_0^d, \varepsilon^k), (\eta_0^d, \varepsilon^k) \in (Q_d \times Q_{T,k}) \cap X_{\eta_0}} \eta_0^{d-2} a_{ij} (\bar{v}_\varepsilon(\eta_0^d, \varepsilon^k) - (\bar{v}_\varepsilon(\eta_0^d, \varepsilon^k))) \leq \limsup_{\varepsilon \to 0} \sum_{(\eta_0^d, \varepsilon^k), (\eta_0^d, \varepsilon^k) \in (Q_d \times Q_{T,k}) \cap X_{\eta_0}} \eta_0^{d-2} a_{ij}(v_\varepsilon(\eta_0^d, \varepsilon^k) - v_\varepsilon(\eta_0^d, \varepsilon^k)) + o(1).
\]

(41)

In order to simplify the notation, we may assume that \( x_0 \in \varepsilon T \mathbb{Z}^d \) so that we avoid the translation of the coefficients \( a_{ij} \). In view of (37) and thanks estimate (41), we have that

\[
\frac{d\mu}{dx}(x_0) \geq \limsup_{\varepsilon \to 0} \sum_{(\eta_0^d, \varepsilon^k), (\eta_0^d, \varepsilon^k) \in (Q_d \times Q_{T,k}) \cap X_{\eta_0}} \eta_0^{d-2} a_{ij}(\frac{u_\varepsilon - u_\varepsilon}{\mu_\varepsilon})^2 \geq \liminf_{\varepsilon \to 0} \sum_{(\eta_0^d, \varepsilon^k), (\eta_0^d, \varepsilon^k) \in (Q_d \times Q_{T,k}) \cap X_{\eta_0}} \eta_0^{d-2} a_{ij}(\bar{w}_\varepsilon(\eta_0^d, \varepsilon^k) - w_\varepsilon(\eta_0^d, \varepsilon^k))^2:
\]

\[
\bar{w}_\varepsilon(\eta_0^d, \varepsilon^k) = \nabla u(x_0) \cdot \eta_0^d, \quad \text{if dist}(\eta_0^d, \partial Q_d) < 2\varepsilon \sqrt{dT}.
\]

Setting \( K_\varepsilon = [1/(\eta_0^d)] \), we have that

\[
\frac{d\mu}{dx}(x_0) \geq \liminf_{\varepsilon \to 0} \frac{1}{(K_\varepsilon)^d} \inf_{i,j \in (\mathbb{Z}^d \times X) \cap X} \frac{1}{\eta_0^d} a_{ij}(w_\varepsilon(\eta_0^d, \varepsilon^k) - w_\varepsilon(\eta_0^d, \varepsilon^k))^2:
\]

\[
\begin{aligned}
&= \liminf_{\varepsilon \to 0} \frac{1}{(K_\varepsilon)^d} \inf_{i,j \in (\mathbb{Z}^d \times X) \cap X} a_{ij} (\bar{w}_\varepsilon - \bar{w}_\varepsilon)^2:
&\quad \bar{w}_\varepsilon = \nabla u(x_0) \cdot i^d, \quad \text{if dist}(i^d, \partial Q_{K_\varepsilon \times d}) < 2\varepsilon \sqrt{dT}
\end{aligned}
\]

\[
= f_0(\nabla u(x_0)),
\]

where we have set \( \bar{w}_\varepsilon := w_\varepsilon(\eta_0^d, \varepsilon^k)/\eta_0^d \). Therefore, for \( \mathcal{L}^d \)-almost every \( x_0 \in \Omega \), we have

\[
\frac{d\mu}{dx}(x_0) \geq A_{\text{hom}} \nabla u(x_0) \cdot \nabla u(x_0).
\]

Integrating on \( \Omega \), we conclude that

\[
\mu(\Omega) \geq \int_\Omega \frac{d\mu}{dx}(x_0) dx \geq \int_\Omega A_{\text{hom}} \nabla u(x) \cdot \nabla u(x) dx.
\]

Since \( \mu_\varepsilon \rightarrow \mu \), we have that \( \liminf_{\varepsilon \to 0} \mu_\varepsilon(\Omega) \geq \mu(\Omega) \). This implies that

\[
\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) = \liminf_{\varepsilon \to 0} \mu_\varepsilon(\Omega) \geq \mu(\Omega) \geq \int_\Omega A_{\text{hom}} \nabla u(x) \cdot \nabla u(x) dx = F_{\text{hom}}(u),
\]

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Due to the assumption of finite range, the interactions between nodes in the first sum in (43) may be estimated as

\[ a_{ij}(u_i - u_j)^2 : u_i = z \cdot t^d \text{ if } \text{dist}(i^d, \partial Q_{K,T,d}) < 2\sqrt[d]{dT}, \]

which concludes the proof of the lower bound.

It remains to prove that \( f_0 \) satisfies formula [31]. First, we prove the existence of the limit.

**Proposition 6.2.** There exists the limit

\[
\lim_{K \to \infty} \frac{1}{(KT)^d} \inf \left\{ \sum_{i,j \in (Q_{K,T,d} \times Q_{T,k}) \cap X} a_{ij}(u_i - u_j)^2 : u_i = z \cdot t^d \text{ if } \text{dist}(i^d, \partial Q_{K,T,d}) < 2\sqrt[d]{dT} \right\},
\]

for \( z \in \mathbb{R} \).

**Proof.** For fixed \( K \in \mathbb{N} \) and \( z \in \mathbb{R}^d \), we set

\[
f^K_0(z) := \frac{1}{(KT)^d} \inf \left\{ \sum_{i,j \in (Q_{K,T,d} \times Q_{T,k}) \cap X} a_{ij}(u_i^K - u_j^K)^2 \right\},
\]

Let \( u^K \) be a function such that

\[
\frac{1}{(KT)^d} \sum_{i,j \in (Q_{K,T,d} \times Q_{T,k}) \cap X} a_{ij}(u_i^K - u_j^K)^2 \leq f^K_0(z) + \frac{1}{K},
\]

and \( u^K = z \cdot t^d \), if \( \text{dist}(i^d, \partial Q_{K,T,d}) < 2\sqrt[d]{dT} \). For \( H > K \), we introduce the set of indices \( I := \{ l \in \mathbb{Z}^d : 0 \leq (K+1)l_m < H, m = 1, \ldots, d \}. \) We define

\[
u^K_i := \begin{cases} u^K(i^d - l, i^d) + z \cdot l, & (i^d, i^d) \in Q^K_{K,T,d} \times Q_{T,k}, l \in I, \\ z \cdot t^d, & \text{otherwise}. \end{cases}
\]

We have that

\[
f^K_0(z) \leq \frac{1}{(HT)^d} \sum_{i,j \in (Q_{H,T,d} \times Q_{T,k}) \cap X} a_{ij}(u_i^K - u_j^K)^2
\]

\[
= \frac{1}{(HT)^d} \sum_{i \in (Q_{H,T,d} \times Q_{T,k}) \cap X} \sum_{j \in (Q_{H,T,d} \times Q_{T,k}) \cap X} a_{ij}(u_i^K - u_j^K)^2
\]

\[
+ \frac{1}{(HT)^d} \sum_{i \in (Q_{H,T,d} \setminus \bigcup_{l \in I} Q^K_{K,T,d}) \times Q_{T,k}} \sum_{j \in (Q_{H,T,d} \times Q_{T,k}) \cap X} a_{ij}(z \cdot t^d - u_j^K)^2.
\]

Due to the assumption of finite range, the interactions between nodes in \((Q_{H,T,d} \setminus \bigcup_{l \in I} Q^K_{K,T,d}) \times Q_{T,k}) \cap X\) do not take place. This implies that the first sum in (43) may be estimated as

\[
\frac{1}{(HT)^d} \sum_{i \in (Q_{H,T,d} \times Q_{T,k}) \cap X} \sum_{j \in (Q_{H,T,d} \times Q_{T,k}) \cap X} a_{ij}(u_i^K - u_j^K)^2
\]

\[
= \frac{1}{(HT)^d} \sum_{i,j \in (Q_{H,T,d} \times Q_{T,k}) \cap X} a_{ij}(u_i^K - u_j^K)^2 \leq \frac{K^d}{H^d} \sum_{l \in I} (f^K_0(z) + K^{-1})
\]

\[
\leq K^d \frac{H}{K+1} \left( f^K_0(z) + K^{-1} \right) \leq \frac{K^d}{(K+1)^d} (f^K_0(z) + K^{-1}).
\]
Taking first the limsup as reduced to a single periodic minimization problem. which concludes the proof.

Using the assumption of finite range, the second sum in (43) may be estimated as

\[
\sum_{i \in \{Q_{HT,d} \cap \Omega \} \times X} \sum_{j \in \{Q_{HT,d} \cap \Omega \} \times X} a_{ij} (z \cdot i^d - u_j^H)^2
\]

\[
\leq \frac{1}{(HT)^d} \sum_{i \in \{Q_{HT,d} \cap \Omega \} \times X} \sum_{j \in \{Q_{HT,d} \cap \Omega \} \times X} a_{ij} |z \cdot i^d - z \cdot j^d|^2
\]

\[
\leq \frac{C}{(HT)^d} \sum_{i \in \{Q_{HT,d} \cap \Omega \} \times X} \sum_{j \in \{Q_{HT,d} \cap \Omega \} \times X} a_{ij} |z \cdot i^d - z \cdot j^d|^2
\]

\[
\leq \frac{C}{(HT)^d} \left(\frac{H}{K+1}\right)^d.
\]

(45)

Combining (44) and (45), from (43) it follows that

\[
\limsup_{H \to \infty} f_0^H(z) \leq \frac{K^d}{(K+1)^d} (f_0^K(z) + K^{-1}) + \frac{1}{Td(K+1)^d}.
\]

Taking first the limsup as \(H \to \infty\) and then the lower limit as \(K \to \infty\), we obtain

\[
\limsup_{H \to \infty} f_0^H(z) \leq \liminf_{K \to \infty} f_0^K(z),
\]

which concludes the proof.

Since we deal with convex energies, asymptotic homogenization formula (42) can be reduced to a single periodic minimization problem.

**Proposition 6.3.** We have that \(f_\delta(z)\) defined by (42) coincides with \(f_{\text{hom}}(z)\) defined as

\[
f_{\text{hom}}(z) := \frac{1}{T^d} \inf_{i \in \{Q_{T,d} \times Q_{T,k}\} \cap X} \sum_{j \in \{\mathbb{R}^d \times Q_{T,k}\} \cap X} a_{ij} (u_i - u_j)^2 \quad \text{is } T\text{-periodic in } e_1, \ldots, e_d,
\]

for \(z \in \mathbb{R}^d\).

**Proof.** Fix \(z \in \mathbb{R}^d\). First, we prove that \(f_\delta(z) \leq f_{\text{hom}}(z)\). To this end, for \(\delta > 0\), let \(u^\delta\) be a function satisfying

\[
\frac{1}{T^d} \sum_{i \in \{Q_{T,d} \times Q_{T,k}\} \cap X} \sum_{j \in \{\mathbb{R}^d \times Q_{T,k}\} \cap X} a_{ij} (u_i^\delta - u_j^\delta)^2 \leq f_{\text{hom}}(z) + \delta
\]

and \(u^\delta \cdot \cdot i^d\) is \(T\)-periodic in \(e_1, \ldots, e_d\). We define \(u_i^\varepsilon := u_i(\varepsilon z) := \varepsilon u_i^\delta(i)\). Note that \(u_i^\varepsilon\) converges to \(z \cdot \cdot i^d\) with respect to the convergence given by (6). Set \(I^\varepsilon := \{l \in \mathbb{Z}^d : \varepsilon lT + \varepsilon Q_{T,d} \cap \Omega \neq \emptyset\}\). In view of Theorem 6.1 and the periodicity of \(a_{ij}\), we deduce that

\[
|\Omega| f_\delta(z) \leq \liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon)
\]

\[
\leq \limsup_{\varepsilon \to 0} \sum_{l \in I^\varepsilon} \sum_{i,j \in \{Q_{T,d} \times Q_{T,k}\} \cap X \cap \{\mathbb{R}^d \times \Omega\}} \varepsilon^{d} a_{ij} (u_i^\varepsilon - u_j^\varepsilon)^2
\]

\[
\leq \limsup_{\varepsilon \to 0} \sum_{l \in I^\varepsilon} \sum_{i,j \in \{Q_{T,d} \times Q_{T,k}\} \cap X \cap \{\mathbb{R}^d \times \Omega\}} \varepsilon^{d} a_{ij} (u_i^\varepsilon - u_j^\varepsilon)^2
\]

\[
\leq |\Omega| (f_{\text{hom}}(z) + \delta).
\]

From the arbitrariness of \(\delta\), it follows the conclusion.

It remains to show that \(f_{\text{hom}}(z) \leq f_\delta(z)\). Let \(v\) be a function defined on \((Q_{K,T,d} \times Q_{T,k}) \cap X\) such that \(v_i = z \cdot i^d\) if \(\text{dist}(i^d, \partial Q_{K,T,d}) \leq 2\sqrt{d}T\). We define a function \(u\) on \((Q_{T,d} \times Q_{T,k}) \cap X\) by

\[
u(i) : = \frac{1}{K^d} \sum_{l \in \{0, \ldots, K-1\}^d} v(i^d + lT, i^d).
\]

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With the help of Jensen’s inequality combined with the assumption of finite range and the periodicity of $a_{ij}$, we deduce that
\[
 f_{\text{hom}}(z) \leq \frac{1}{(KT)^d} \sum_{i \in (Q_{T,d} \times Q_{T,k}) \cap X} \sum_{j \in (Q_{T,d} \times Q_{T,k}) \cap X} a_{ij}(u_i - u_j)^2
 \]
\[
 \leq \frac{1}{(KT)^d} \sum_{i \in (Q_{T,d} \times Q_{T,k}) \cap X} \sum_{j \in (Q_{T,d} \times Q_{T,k}) \cap X} \sum_{i \in \{0, \ldots, K-1\}^d} a_{ij}(v(i^d + iT, i^d) - v(j^d + iT, j^d))^2
 \]
\[
 = \frac{1}{(KT)^d} \sum_{i,j \in (Q_{T,d} \times Q_{T,k}) \cap X} a_{ij}(v_i - v_j)^2.
\]
Taking the infimum, we get
\[
 f_{\text{hom}}(z) \leq \frac{1}{(KT)^d} \inf \left\{ \sum_{i,j \in (Q_{T,d} \times Q_{T,k}) \cap X} a_{ij}(v_i - v_j)^2 : v_i = z \cdot i^d \text{ if } \text{dist}(i^d, \partial Q_{T,d}) < 2\sqrt{d}T \right\}.
\]
Then, passing to the limit as $K \to \infty$, we have the desired inequality which concludes the proof.

6.2 Proof of the upper bound

We are going to prove the $\Gamma$-lim sup inequality. The proof is independent of the blow-up result and it relies on a standard density argument by piecewise-affine functions (see [6, Remark 1.29]). We consider the case when the target function $u$ is piecewise-affine and we assume that the gradient of $u$ takes $\lambda$ values, for some $\lambda$ positive integer. For fixed $z_1, \ldots, z_\lambda \in \mathbb{R}$, we define
\[
 \Omega_q := \{ x^d \in \Omega : u(x^d) = z_q \cdot x^d + c_q \},
\]
for $q = 1, \ldots, \lambda$ (with $c_q$ some constant).

We fix one such $q$. We choose $w^q \in \mathcal{C}_b(Q_{T,d})$ such that $w^q - z \cdot i^d$ is $T$-periodic in $e_1, \ldots, e_d$ and
\[
 \sum_{i,j \in (Q_{T,d} \times Q_{T,k}) \cap X} a_{ij}(w_i^q - w_j^q)^2 = f_{\text{hom}}(z_q).
\]
For any $q = 1, \ldots, \lambda$, we define $u^q_{\varepsilon,i} := u^q(\varepsilon i) = \varepsilon w^q(i) + c_q$. In view of Lemma 5.1, we may modify the sequence $u^q_{\varepsilon,i}$ to obtain a new sequence $v^q_{\varepsilon,i}$ converging to $z_q \cdot x^d + c_q$ with respect to convergence $[\theta]$ such that
\[
 v^q_{\varepsilon,i} = z_q \cdot i^d + c_q, \quad \text{if } i \in (\Omega_q \setminus \Omega_q(\theta)) \times Q_{T,k} \cap X,
\]
\[
 v^q_{\varepsilon,i} = u^q_{\varepsilon,i}, \quad \text{if } i \in (\Omega_q(2\delta) \times Q_{T,k}) \cap X,
\]
and
\[
 \limsup_{\varepsilon \to 0} F^\varepsilon_T(v^q_{\varepsilon,i}) \leq \limsup_{\varepsilon \to 0} F^\varepsilon_T(u^q_{\varepsilon,i}) + o(1) \tag{46}
\]
as $\varepsilon \to 0$, where $F^\varepsilon_T$ is the functional defined as in [30] with $\Omega_q$ in the place of $\Omega$.

Now, we estimate $F^\varepsilon_T(u^q_{\varepsilon})$. To that end, for $q = 1, \ldots, \lambda$, we introduce the set of indices $\mathcal{I}_{\varepsilon,q} := \{ l \in \mathbb{Z}^d : \varepsilon Q_{T,d} \cap \Omega_q \neq \emptyset \}$, and we deduce that
\[
 F^\varepsilon_T(u^q_{\varepsilon,i}) = \sum_{i,j \in (\mathcal{I}_{\varepsilon,q} \times Q_{T,k}) \cap X} \varepsilon^{d-2} a_{ij}(u^q_{\varepsilon,i} - u^q_{\varepsilon,j})^2
 \]
\[
 \leq \sum_{i,j \in (\mathcal{I}_{\varepsilon,q} \times Q_{T,k}) \cap X} \varepsilon^d a_{ij}(w^q(i) - w^q(j))^2
 \]
\[
 \leq \sum_{l \in \mathbb{Z}^d} \varepsilon^d \sum_{i,j \in (Q_{T,d} \times Q_{T,k}) \cap X} a_{ij}(w^q(i) - w^q(j))^2 \leq |\Omega_q| f_{\text{hom}}(z_q) + o(1),
\]
as \( \varepsilon \to 0 \). This combined with (46) implies that
\[
\limsup_{\varepsilon \to 0} F^{\phi}_{\varepsilon}(v_{\varepsilon}^{q,\delta}) \leq |\Omega_q| f_{\text{hom}}(z_q) + o(1) \tag{47}
\]
as \( \delta \to 0 \).

Now, we define the recovery sequence \( v^\varepsilon \) by
\[
v_i^\varepsilon = v_i^{\varepsilon,\eta,\delta} \quad \text{if} \quad i \in \Omega_q,
\]
for \( q = 1, \ldots, \lambda \). To conclude the proof, it remains to show that, given \( q_1, q_2 \in \{1, \ldots, \lambda\} \),
\[
\limsup_{\varepsilon \to 0} \sum_{i \in (\frac{1}{2} \Omega_{q_1} \times Q_{T,k}) \cap X} \varepsilon^d a_{ij}(v_i^\varepsilon - v_j^\varepsilon)^2 = o(1) \tag{48}
\]
as \( \delta \to 0 \).

Since \( \delta \gg \varepsilon T \), the interactions between nodes in \( (\frac{1}{2} \Omega_{q_1} (2\delta) \times Q_{T,k}) \cap X \) and \( (\frac{1}{2} \Omega_{q_2} (2\delta) \times Q_{T,k}) \cap X \) do not take place. This allows to reduce (48) to the following estimate
\[
\limsup_{\varepsilon \to 0} \sum_{i \in (\frac{1}{2} \Omega_{q_1} \{\delta\} \times Q_{T,k}) \cap X} \varepsilon^d a_{ij}(u_i^\varepsilon - u_j^\varepsilon)^2 = o(1) \tag{49}
\]
as \( \delta \to 0 \), where we have set \( u_i^\varepsilon = u(\varepsilon d_i) \), and used the fact that \( v_i^{\varepsilon,\eta,\delta} = z_q \cdot \varepsilon d_i + c_d = u_i^\varepsilon \)
if \( i \in (\Omega_q \setminus \frac{1}{2} \Omega_{q_1}(\delta)) \times Q_{T,k}) \cap X \). By the Lipschitz continuity of \( u \) we deduce that
\[
\sum_{i \in (\frac{1}{2} \Omega_{q_1} \{\delta\} \times Q_{T,k}) \cap X} \varepsilon^{d-2} a_{ij}(u_i^\varepsilon - u_j^\varepsilon)^2 \leq C \sum_{i \in (\frac{1}{2} \Omega_{q_1} \{\delta\} \times Q_{T,k}) \cap X} \varepsilon^d a_{ij}|d_i - d_j|^2
\]
\[
\leq C \max\{a_{ij}\} T^d \sum_{q=1}^{\lambda} \left| \bigcup_{i \in (\frac{1}{2} \Omega_q \{\delta\})} \varepsilon Q_i^d \right| \leq C \delta
\]
(the final \( C \) taking into account the bound for \( a_{ij} \), their range, \( T \) and the \( H^{d-1} \) measure of the union of \( \partial \Omega_q \), which proves (49).

Gathering estimates (47) and (48), we deduce that
\[
\limsup_{\varepsilon \to 0} \sum_{i,j \in (\frac{1}{2} \Omega \times Q_{T,k}) \cap X} \varepsilon^{d-2} a_{ij}(v_i^\varepsilon - v_j^\varepsilon)^2 \leq \sum_{q=1}^{\lambda} |\Omega_q| f_{\text{hom}}(z_q) + o(1).
\]
as \( \delta \to 0 \), which concludes the proof of the upper bound.

Remark 6.4. Recall that the \( \Gamma \)-limit of a family of non-negative quadratic forms is still a non-negative quadratic form (see e.g. [17] Theorem 11.10). Applying this property in our setting, we deduce that the \( \Gamma \)-limit \( F_{\text{hom}} \) of \( F_{\varepsilon} \) is a non-negative quadratic form. In other words, there exists a symmetric matrix \( A_{\text{hom}} \) such that \( f_{\text{hom}}(z) = A_{\text{hom}} z \cdot z \), which finally gives (31).

6.3 Convergence of minimum problems

In this section, we deal with minimum problems with boundary data. To this end, we derive compactness result in the case that the functionals \( F_{\varepsilon} \) are subject to Dirichlet boundary conditions. In the discrete setting, such conditions are imposed by introducing a parameter \( r \in \mathbb{N} \) and fixing the value of \( u \) in a neighbourhood of the ‘lateral boundary’ of \( \Omega \times Q_{T,k} \), corresponding to \( \varepsilon d \) in a neighbourhood of the boundary of \( \Omega \subset \mathbb{R}^d \), of size \( \varepsilon r \).

For any \( r > 0 \) and given \( \varphi \in H^1(\mathbb{R}^d) \), we introduce the set
\[
C_{\varepsilon, r}^\varphi(\Omega) := \left\{ u \in C_{w}(\Omega) : u(\varepsilon t) = \int_{\varepsilon d + t \times Q_d} \varphi(x) dx, \quad \text{if} \quad (\varepsilon d + (-\varepsilon r, r)^d) \cap \mathbb{R}^d \setminus \Omega \neq \emptyset \right\}
\]
We define the functional $F_{\epsilon}^{x,r}$ by

$$F_{\epsilon}^{x,r}(u) := F_{\epsilon}(u), \quad u \in C_0^{x,r}(\Omega).$$

**Theorem 6.5.** For any $\varphi \in H^1(\mathbb{R}^d)$, let $F^\varphi$ be the functional defined by

$$F^\varphi(u) := \begin{cases} \int_{\Omega} A_{\text{hom}} \nabla u \cdot \nabla u \, dx, & u - \varphi \in H_0^1(\Omega), \\ \infty, & \text{otherwise}, \end{cases}$$

where $A_{\text{hom}}$ is given by (11). Then, for any $r > 0$, $F_{\epsilon}^{x,r} \Gamma$-converges to the functional $F^{\varphi}$ with respect to convergence (9).

**Proof.** We prove the $\Gamma$-liminf inequality. To that end, we prove that if $u_\epsilon$ converges to $u$ with respect to convergence (9) and $F_{\epsilon}^{x,r}(u_\epsilon)$ is equibounded, then $u - \varphi \in H_0^1(\Omega)$. First, note that if $\sup_{\epsilon > 0} F_{\epsilon}^{x,r}(u_\epsilon) < \infty$, then, thanks to the coerciveness of the coefficient $a_{ij}$, we deduce that

$$\sup_{\epsilon > 0} \sum_{i,j \in (\frac{1}{\epsilon} \Omega \times Q_T,k) \cap X} \epsilon^{d-2}(u_\epsilon^i - u_\epsilon^j)^2 < \infty.$$

We denote by $\tilde{u}_\epsilon$ the extension of $u_\epsilon$ on the whole $X$ defined by $\tilde{u}_\epsilon^i = \varphi(\epsilon^d)$, for any $\epsilon > 0$ and outside $\Omega$. Analogously, $\tilde{u}$ is the extension of $u$ on $\mathbb{R}^d$ obtained by setting $\tilde{u}(x^d) = \varphi(x^d)$. Let $\Omega'$ be an open set such that $\Omega \subset \subset \Omega'$. Hence, we have that

$$\sum_{i,j \in (\frac{1}{\epsilon} \Omega' \times Q_T,k) \cap X} \epsilon^{d-2} a_{ij}(\tilde{u}_\epsilon^i - \tilde{u}_\epsilon^j)^2 \leq \sum_{i,j \in (\frac{1}{\epsilon} \Omega \times Q_T,k) \cap X} \epsilon^{d-2} a_{ij}(u_\epsilon^i - u_\epsilon^j)^2 + \sum_{i,j \in (\frac{1}{\epsilon} (\Omega \setminus \Omega') \times Q_T,k) \cap X} \epsilon^{d-2} a_{ij}(\varphi(\epsilon^d) - \varphi(\epsilon^d))^2 \leq C.$$

Repeating similar arguments as the proof of $\Gamma$-liminf inequality of Theorem 6.1 and since $\tilde{u}_\epsilon$ converges to $\tilde{u}$, we deduce that $\tilde{u} \in \tilde{u} \in H^1(\Omega')$ and hence $u - \varphi \in H_0^1(\Omega)$. Then, invoking again Theorem 6.1 we have that

$$\liminf_{\epsilon \to 0} F_{\epsilon}^{x,r}(u_\epsilon) = \liminf_{\epsilon \to 0} F_{\epsilon}(u_\epsilon) \geq F^\varphi(u),$$

as desired.

Now, we show the $\Gamma$-limsup inequality. First, consider the case where $u \in H^1(\Omega)$ such that $\text{supp}(u - \varphi) \subset \subset \Omega$. The general case is obtained by a density argument.

Consider a target function $u$ such that $\text{supp}(u - \varphi) \subset \subset \Omega$. In view of Theorem 6.1 we know that there exists a recovery sequence $u_\epsilon$ converging to $u$ such that

$$\lim_{\epsilon \to 0} F_{\epsilon}(u_\epsilon) = \int_{\Omega} A_{\text{hom}} \nabla u \cdot \nabla u \, dx.$$ 

In order to modify the sequence $u_\epsilon$ near the boundary of $\Omega$, we apply Lemma 5.1 with $v_\epsilon = u$. Hence, there exists a sequence $w_\epsilon$ such that $w_\epsilon$ still converges to $u$ with respect to convergence (9). $w_\epsilon = u$ is a neighbourhood of $\Omega$ and

$$\limsup_{\epsilon \to 0} F_{\epsilon}(w_\epsilon) \leq \limsup_{\epsilon \to 0} F_{\epsilon}(u_\epsilon) + o(1).$$

Since $\text{supp}(u - \varphi) \subset \subset \Omega$, it follows that $w_\epsilon$ is equal to $\varphi$ is a neighbourhood of $\Omega$, so that $F_{\epsilon}^{x,r}(w_\epsilon) = F_{\epsilon}(w_\epsilon)$. We may conclude that

$$\limsup_{\epsilon \to 0} F_{\epsilon}^{x,r}(w_\epsilon) \leq \limsup_{\epsilon \to 0} F_{\epsilon}(u_\epsilon) + o(1) = F^\varphi(u) + o(1),$$

which concludes the proof. 

Now, we state the following result which deals with convergence of minimum problems with Dirichlet boundary data.
Proposition 6.6. We have that
\[ \liminf_{\varepsilon \to 0} \{ F_{\varepsilon}(u) : u \in C^{2r}_{\omega}(\Omega) \} = \min \{ F_{\text{hom}}(u) : u - \varphi \in H^1_0(\Omega) \}. \]
Moreover, if \( u_\varepsilon \in C^{2r}_{\omega}(\Omega) \) converges to \( \bar{u} \) with respect to convergence \( (9) \) and it is such that
\[ \liminf_{\varepsilon \to 0} F_{\varepsilon}(u_\varepsilon) = \liminf_{\varepsilon \to 0} \{ F_{\varepsilon}(u) : u \in C^{2r}_{\omega}(\Omega) \}, \]
Then, \( u \) is a minimizer for \( \min \{ F_{\text{hom}}(u) : u - \varphi \in H^1_0(\Omega) \} \).

Proof. We have to show the equi-coerciveness of \( H^1_0(\Omega) \) applied to \( \bar{u} \) by \( (9) \). To that end, consider \( \{ u_\varepsilon \} \subset C^{2r}_{\omega}(\Omega) \) such that \( \sup_{\varepsilon > 0} F_{\varepsilon}(u_\varepsilon) < \infty \). In view of inequality \( (50) \) applied to \( u - \varphi \), we deduce that
\[ \sum_{i \in \mathbb{Z}^d} \sum_{i,j \in (Q_{T,d}^1 \cap X)} \varepsilon d|u_i^1 - \varphi_i| \leq C \sum_{i,j \in (\frac{1}{2} \Omega \times Q_{T,k}) \cap X} \varepsilon d a_{ij} |(u_i^1 - \varphi_i) - (u_j^1 - \varphi_j)|^2 \leq CF_{\varepsilon}(u_\varepsilon) + C \sum_{i,j \in (\frac{1}{2} \Omega \times Q_{T,k}) \cap X} \varepsilon d a_{ij} |\varphi_i - \varphi_j|^2 \leq C. \]
Hence, we may apply Proposition \( 4.3 \) to deduce that there exists a subsequence \( u_{\varepsilon_k} \) such that \( u_{\varepsilon_k} \) is converging. This concludes the proof. \( \square \)

The next proposition shows the Poincaré inequality for functions \( u \in C^{2r}_{\omega}(\Omega) \). We prove it assuming that \( \varphi = 0 \).

Proposition 6.7. Let \( \Omega \) be a bounded open set of \( \mathbb{R}^d \) and let \( u \) be a function in \( C_0(\Omega) \) such that \( u_i = u(\varepsilon i) = 0 \) if \( \text{dist}(\varepsilon i, \partial \Omega) \leq 2\varepsilon T \). Then, there exists a constant \( C > 0 \) such that
\[ \sum_{i,j \in (\frac{1}{2} \Omega \times Q_{T,k}) \cap X} \varepsilon d a_{ij} |u_i - u_j|^2 \leq C \sum_{i,j \in (\frac{1}{2} \Omega \times Q_{T,k}) \cap X} \varepsilon d a_{ij} |u_i - u_j|, \]
where \( C \) is of order of \( |\varepsilon| \).

Proof. We identify \( u \) with its extension to \( (\mathbb{R}^d \times Q_{T,k}) \cap X \) which is equal to 0 outside \( (\Omega \times Q_{T,k}) \cap X \). Due to the boundedness of \( \Omega \), there exists \( M > 0 \) such that \( \Omega \subset [0, M]^d \) and \( (\frac{1}{2} [0, M]^d) \times Q_{T,k} \cap X \) contains a path joining two arbitrary nodes \( i, j \in (\frac{1}{2} \Omega \times Q_{T,k}) \cap X \).

Fix \( i \in (\frac{1}{2} \Omega \times Q_{T,k}) \cap X \) and let \( j \) be a node such that \( \text{dist}(\varepsilon j, \partial \Omega) \leq 2\varepsilon T \) and \( i^d - j^d = \lambda T e_1 \), where, without loss of generality, we may assume that \( \lambda \) is a positive integer. Note that \( \lambda \) depends on the fixed node \( i \) and it is of order \( T^{-1} \varepsilon \). Let \( l_i \) and \( l_j \) be two indices in \( \mathbb{Z}^d \) such that \( i \in Q_{T,d}^l \) and \( j \in Q_{T,d}^j \). Let \( S^\lambda_{T,d} \) be the union of \( (\lambda + 1) \) neighbour cubes joining \( Q_{T,d}^l \) and \( Q_{T,d}^j \) such that each two consecutive cubes having one face in common. In other words,
\[ S^\lambda_{T,d} := \bigcup_{q=0}^\lambda Q_{T,d}^q, \]
where \( l_q = l_i + qTe_1 \), for \( q = 1, \ldots, \lambda \) and \( l_0 = l_i \). Since \( X \) is connected, there exists a path of nodes \( \{ j_q \}_{q=0}^\lambda \) joining \( j_0 = j \) and \( j_\lambda = i \) such that it is contained in \( S^\lambda_{T,d} \). Such a path can be built repeating periodically the path joining \( j \in (Q_{T,d}^0 \times Q_{T,k}) \cap X \) and \( j + T e_1 \in (Q_{T,d}^{\lambda} \times Q_{T,k}) \cap X \). Since \( u_j = 0 \), we have that
\[ u_i = \sum_{q=1}^{\lambda} (u_{j_q} - u_{j_{q-1}}). \]
Hence, an application of the Jensen inequality leads us to
\[ |u_i|^2 \leq \lambda \sum_{q=1}^{\lambda} |u_{j_q} - u_{j_{q-1}}|^2. \]
Summing over \( i \in (\frac{1}{\varepsilon} \Omega \times Q_{T,k}) \cap X \), we get
\[
\sum_{i \in (\frac{1}{\varepsilon} \Omega \times Q_{T,k}) \cap X} \varepsilon^d |u_i|^2 \leq A \sum_{q=1}^{\lambda} \sum_{i,j \in (\frac{1}{\varepsilon} \Omega \times Q_{T,k}) \cap X} \varepsilon^d |u_{jq} - u_{jq-1}|^2 \\
\leq C \lambda \sum_{i,j \in (\frac{1}{\varepsilon} \Omega \times Q_{T,k}) \cap X} \varepsilon^d a_{ij} |u_{jq} - u_{jq-1}|^2,
\]
where the constant \( C \) takes into account the fact that the possible multiplicity of the paths containing the connection joining \( j_{q-1} \) and \( j_q \), which is anyhow uniformly bounded. Recalling that \( \lambda \) is of order \( MT^{-1} \varepsilon^{-1} \), we get the inequality (50), as desired.

7 Examples

In this section, we exhibit some examples of the possible geometries of the set \( X \). We also compute the homogenized matrix \( A_{\text{hom}} \) given by formula (31). In the examples below, we think of \( X \) as a subset of \( \mathbb{Z}^{d+k} \) where \( d = 1 \) is identified with the horizontal direction and \( k = 1 \) or 2. Since \( d = 1 \) the homogenized matrix actually reduces to a single coefficient giving the homogenized energy density \( A_{\text{hom}} \).

In all the following examples the value of the non-zero coefficients \( a_{ij} \) is always 1, and the corresponding connections are represented by solid lines in the figures.

Figure 2: Figure (a) shows \( X \) and Figure (b) shows the periodicity cell \( (Q_{2,0} \times Q_{2,1}) \cap X \).

Example 7.1. Let \( X \) be the set pictured in Figure 2(a). Here, we have that \( d = k = 1 \) and the period \( T \) is equal to 2. Figure 2(b) shows a periodicity cell. The geometry of the set \( X \) can be thought as the discrete version of a perforated domain. Indeed, note that nodes \( (0,1) \) and \( (2,1) \) in Figure 2(b) are missing. A minimizer \( \tilde{u} \) for (31) is given by \( \tilde{u}(0,0) = \tilde{u}(0,2) = 0 \), \( \tilde{u}(1,0) = \tilde{u}(1,1) = \tilde{u}(1,2) = z \) and \( \tilde{u}(2,0) = \tilde{u}(2,2) = 2z \), so that \( A_{\text{hom}} = 4 \).

In the next four examples \( d = k = 1 \), the set \( X \) is always simply \( \mathbb{Z} \times \{0,1\} \) and the period \( T \) is 1, but the set \( \mathcal{E} \) is such that the graph cannot be directly seen as a discretization of a thin film in the continuum parameterized as a subgraph of a function of one real variable.

Example 7.2. Let \( X \) be as drawn in Figure 3(a). In this case \( \mathcal{E} \) contains all ‘cross-connections’ between points of \( X \). The minimizer \( \tilde{u} \) for \( A_{\text{hom}} z^2 \) is \( \tilde{u}(0,0) = \tilde{u}(1,1) = 0 \) and \( \tilde{u}(1,0) = \tilde{u}(0,1) = z \), so that \( A_{\text{hom}} = 4 \).

Example 7.3. Consider \( X \) as drawn in Figure 4(a). Here, the graph is analogous to a nearest-neighbour thin film, but with a translation of a unit of one of the two copies of \( \mathbb{Z} \), which again makes this geometry not immediately seen as a discretization of a continuum thin film. The 1-periodic minimizer \( \tilde{u} \) for \( A_{\text{hom}} z^2 \) is given by \( \tilde{u}(0,0) = \tilde{u}(1,1) = 0 \), \( \tilde{u}(1,0) = z \) and \( \tilde{u}(0,1) = -z \) and the homogenized coefficient is \( A_{\text{hom}} = 4 \).
Example 7.4. Consider $X$ as drawn in Figure 5(a). Here the set of connections has the structure of a triangular lattice. The minimizer $\tilde{u}$ for $A_{\text{hom}}z^2$ is given by $\tilde{u}(0,0) = 0$, $\tilde{u}(1,0) = z$, $\tilde{u}(0,1) = -1/2z$ and $\tilde{u}(1,1) = 1/2z$. The homogenized coefficient is $A_{\text{hom}} = 5/2$.

Example 7.5. Let $X$ be the set pictured in Figure 6(a), where $d = k = 1$ and the period $T$ is equal to 4. The set $X$ is a subset of $\mathbb{Z} \times \{0,1,2\}$. Such a set $X$ can be though as a discrete layered media, whose conductivity is equal to 1 along the straight lines, while in the part corresponding to the rhombus structure the effective conductivity is 2.

The minimizer $\tilde{u}$ for $A_{\text{hom}}z^2$ is given by $\tilde{u}(0,0) = z$, $\tilde{u}(1,1) = 4z/3$, $\tilde{u}(2,1) = 8z/3$, $\tilde{u}(3,0) = \tilde{u}(3,2) = 10z/3$ and $\tilde{u}(4,1) = 4z$ and $A_{\text{hom}} = 8/3$.

Example 7.6. We consider the set $X$ drawn in Figure 1. To uniform the notation introduced in this section, we rotate $X$, obtaining the structure pictured in Figure 7(a). Here $d = 1$ and $k = 2$. The period $T$ is equal to 2 and the periodicity cell is drawn in Figure 5(b). The structure is actually the same as that in Example 7.2 but transposed to a three-dimensional setting, and $A_{\text{hom}} = 4$. Note that in this case the solid lines representing the connections do not intersect and they have all the same length, so that they can also be interpreted as a system of homogeneous conducting rods.

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Figure 5: Figure (a) shows $X$ and Figure (b) shows the periodicity cell $(Q_{1,d} \times Q_{1,k}) \cap X$.

Figure 6: Figure (a) shows $X$ and Figure (b) shows the periodicity cell $(Q_{4,d} \times Q_{4,k}) \cap X$.

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Figure 7: Figure (a) shows $X$ and Figure (b) shows the periodicity cell $(Q_{2,d} \times Q_{2,k}) \cap X$.

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