NECESSARY CONDITIONS FOR POINT EQUIVALENCE OF SECOND-ORDER ODES TO THE SIXTH PAINLEVÉ EQUATION

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We consider the equivalence problem for projective-type scalar second-order ordinary differential equations with respect to invertible point changes of variables. Invariants of the equivalence transformation group of this family of equations are used to find necessary conditions for equivalence to the sixth Painlevé equation. Bibliography: 27 titles.

Dedicated to M. A. Semenov–Tian-Shansky on the occasion of his 70th birthday

1. Introduction

The interest to the equivalence problem for second-order ordinary differential equations (ODEs) of the form

$$\frac{d^2 y}{dx^2} = S(x, y) \left(\frac{dy}{dx}\right)^3 + 3R(x, y) \left(\frac{dy}{dx}\right)^2 + 3Q(x, y) \frac{dy}{dx} + P(x, y),$$

(1.1)

where \(P, Q, R, S\) are arbitrary analytic functions, goes back to Liouville, Lie, Tresse, Cartan \[1–3\]. Lie \[4\] established that the family of equations (1.1) is closed with respect to the point transformations (with analytic functions \(\xi, \eta\))

$$z = \xi(x, y), \quad w = \eta(x, y), \quad \det \frac{\partial(\xi, \eta)}{\partial(x, y)} \neq 0.$$  

(1.2)

This means that any change of variables (1.2) turns an ODE of the form (1.1) into an equation

$$\frac{d^2 w}{dz^2} = \tilde{S}(z, w) \left(\frac{dw}{dz}\right)^3 + 3\tilde{R}(z, w) \left(\frac{dw}{dz}\right)^2 + 3\tilde{Q}(z, w) \frac{dw}{dz} + \tilde{P}(z, w)$$

with the same dependence of the right-hand side on the first derivative, but with some new coefficients \(\tilde{P}, \tilde{Q}, \tilde{R}, \tilde{S}\).

All second-order ODEs that possess the Painlevé property, i.e., have no movable singularities other than poles, belong to the family (1.1). They were classified by Painlevé and Gambier and are listed, e.g., in \[5\], the coefficient \(S(x, y)\) vanishing in all fifty equations. At present, the Painlevé equations are studied in sufficient detail, so their solutions are regarded as special functions. It is important to find as wide a class of equations as possible that can be integrated using Painlevé equations. In the simplest case, these are scalar second-order ODEs related to the Painlevé equations PI–PVI by point transformations.

The point equivalence problem was solved for the first four Painlevé equations \[6–15\]. For very special cases of the fifth and sixth Painlevé equations with one or two nonvanishing parameters, only some necessary equivalence conditions were found in \[15\]. Equivalence conditions are stated in terms of invariants of the family (1.1), because equivalent equations have coinciding sets of invariants. As usual, by invariants of a family of equations we mean invariants of the group \(E\) of equivalence transformations of this family. For the family (1.1), the group

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The third and sixth Painlevé equations, as well as their invariants, have a similar structure. Equations equivalent to the ODEs involved in the study of the self-dual Yang–Mills equations [23]. The ODE (1.3) is regarded as the master equation, the equations as a similarity reduction of the three-wave resonant system [21,22], in the study of the self-dual Yang–Mills equations [23]. The ODE (1.3) is regarded as the master equation, the equations PI–PV being obtained by degenerations of PVI through a certain limiting procedure [24].

The third and sixth Painlevé equations, as well as their invariants, have a similar structure. Both ODEs involve four constant parameters $\alpha$, $\beta$, $\gamma$, $\delta$. By a dilation of $z$, $w$, two parameters of the third Painlevé equation

$$\frac{d^2 w}{dz^2} = \frac{1}{w} \left( \frac{dw}{dz} \right)^2 - \frac{1}{z} \frac{dw}{dz} + \frac{\alpha w + \beta}{z} + \gamma w^3 + \delta w$$

can be normalized. Hence, PIII has two essential parameters. In PVI, all parameters are essential. As shown in [14, Sec. V], in order to obtain criteria of equivalence to PIII, it suffices to use seven invariants $I_1$, $I_2$, $D_{\theta} I_1$, $J_1$, $J_2$, $J_3$, $J_4$ (defined in Sec. 2 below). PVI has two more essential parameters than PIII. So, in Sec. 3 we use the same set of invariants and two more invariants $J_5$ and $J_6$ for PVI. This suggests that the necessary conditions of equivalence to PVI will be also sufficient, even if the proof is not given.

In [14], necessary conditions of equivalence to PIII, stated in terms of the above-mentioned invariants, are imposed on an ODE of the form

$$\frac{d^2 y}{dx^2} = f(x,y).$$

The resulting equation coincides (up to a transformation preserving the form of equations (1.4)) with the special form of PIII from [25],

$$\frac{d^2 z}{dx^2} = \alpha e^{x+y} + \beta e^{x-y} + \gamma e^{2(x+y)} + \delta e^{2(x-y)},$$

which indicates that these conditions are also sufficient. The same approach was implemented in [15] for the fourth Painlevé equation. A special form of PVI defined by an equation of the form (1.4) exists as well. Here we do not apply the method described above to (1.3), since in theory one can obtain necessary conditions for equivalence in terms of invariants only, but they will be useless because of their great size. In actual practice, we prefer to retain the dependence on $z$, $w$, and an auxiliary variable $H$ in Theorem 2 from Sec. 3 below. In Sec. 4, it is shown how to successively eliminate $H$, $z$, and $w$ from the equivalence conditions given by Theorem 2. This procedure leads either to a transformation of the form (1.2) relating an ODE of the form (1.1) to (1.3) if they are equivalent, or to a contradiction, which means that the equation of the form (1.1) under study is not equivalent to PVI.

2. INVARIANTS OF EQUATIONS (1.1) OF THE FOURTH TYPE

A basis of invariants is obtained in [11] for the generic case ($J_0 \neq 0$) and for five degenerate types of equations (1.1). Equations equivalent to the ODE $y'' = cy^{-3}$ fall into the seventh type, and linearizable ODEs belong to the last type. The classification in [11] contained one
Theorem 1. ODEs (1.1) of the fourth type satisfy the conditions
\[ \beta_1 \neq 0, \quad J_0 = 0, \quad j_0 = 0, \quad j_1 \neq 0 \] (2.1)
and have two algebraic basic invariants
\[ I_1 = \frac{\Gamma_0}{\beta_1 \sqrt{j_1}}, \quad I_2 = \frac{5(2j_1j_3 + j_2^2)}{\sqrt{j_1}}. \] (2.2)

Any invariant of Eq. (1.1) can be obtained by applying to (2.2) functional and algebraic operations and the operators of invariant differentiation
\[ \mathcal{D}_1 = \frac{1}{\sqrt{j_1}}(\beta_2 \partial_x - \beta_1 \partial_y), \quad \mathcal{D}_2 = j_1^{1/4} \left( \frac{5j_2}{2j_1} (\beta_2 \partial_x - \beta_1 \partial_y) - \frac{3}{\beta_1} \partial_x \right). \]

Remark. The hodograph transformation \( z = y, \ w = x \) turns (1.1) into the equation
\[ \frac{d^2 w}{dz^2} = -S(\omega, z) - 3R(\omega, z) \frac{dw}{dz} - 3Q(\omega, z) \left( \frac{dw}{dz} \right)^2 - P(\omega, z) \left( \frac{dw}{dz} \right)^3. \] (2.3)

It is readily seen that the values \( \beta_1, \beta_2 \) calculated for Eq. (2.3) after the substitution \( z = y, \ w = x \) coincide with the values \( -\beta_2, -\beta_1 \) calculated for Eq. (1.1). Thus, the hodograph transformation reduces Eq. (1.1) with \( \beta_1 = 0, \beta_2 \neq 0 \) to that with \( \beta_1 \neq 0 \). Hence, equations (1.1) with \( \beta_1 = 0 \) need not be considered separately.
Conditions of equivalence of an equation of the form (1.1) to the sixth Painlevé equation can be stated in terms of the basic invariants (2.2) and invariant derivatives of $I_1$, namely, $D_2I_1$ and $D^n_1I_1$. Instead of the latter invariants, it is convenient to use the series of universal absolute invariants introduced in [15] (for equations of the fourth type only):

$$J_1 = \frac{6}{5} \left( \frac{1}{I_1} - 1 \right), \quad J_{m+1} = (m + (m + 1)J_1)J_m + \frac{1}{I_1}D_1J_m, \quad m \in \mathbb{N}. \quad (2.4)$$

3. **Equivalence to the Sixth Painlevé Equation**

Here we find necessary conditions for equivalence of a second-order ODE to the sixth Painlevé equation.

**Theorem 2.** If an ODE (1.1) of the fourth type is equivalent to the sixth Painlevé equation, then its invariants (2.2) satisfy the conditions

$$\det \frac{\partial (I_1, I_2)}{\partial (x, y)} \neq 0, \quad I_{21} = D_2I_1 \neq 0 \quad (3.1)$$

and there exists a solution

$$z = \xi(x, y), \quad w = \eta(x, y), \quad H = \zeta(x, y) \quad (3.2)$$

of the system

$$3\upsilon_0J_3H^4 + 450\upsilon_2\upsilon_5J_2H^3 - 15(256\upsilon_1\upsilon_2\upsilon_3^2 + 504\upsilon_1^2\upsilon_3\upsilon_4^3 + 45\upsilon_2\upsilon_4^2)J_1H^2$$
$$+ 45(1512\upsilon_2^2\upsilon_4 + 125\upsilon_2\upsilon_4)\upsilon_5H + 128(625\upsilon_2^3 - 3969\upsilon_1\upsilon_3)\upsilon_3^3$$
$$- 5(3600\upsilon_1\upsilon_2\upsilon_3^2 - 1512\upsilon_1\upsilon_3\upsilon_4 - 125\upsilon_2\upsilon_4^2)\upsilon_4 = 0,$$
$$\upsilon_0J_4H^5 - 5(96\upsilon_1\upsilon_2\upsilon_3^2 + 504\upsilon_1^2\upsilon_3\upsilon_4 + 55\upsilon_2\upsilon_4^2)J_2H^3 + 75(504\upsilon_2^2\upsilon_3$$
$$+ 61\upsilon_2\upsilon_4)\upsilon_5J_1H^2 + 2(1008\upsilon_1^2\upsilon_3 + 125\upsilon_2\upsilon_4)(144\upsilon_1\upsilon_3 - 5\upsilon_4)H$$
$$+ 30(1200\upsilon_1\upsilon_2\upsilon_3^2 - 1008\upsilon_1\upsilon_3\upsilon_4 - 125\upsilon_2\upsilon_4^2)\upsilon_5 = 0, \quad (3.3)$$

$$9J_5H^6 - 105\upsilon_4J_3H^4 + 2205\upsilon_5J_4H^3 + 21(976\upsilon_1^2\upsilon_3^2 - 45\upsilon_4^2)J_1H^2$$
$$+ 6300\upsilon_1\upsilon_5H - 20(1280\upsilon_2\upsilon_3^3 + 1008\upsilon_1\upsilon_3^2\upsilon_4 - 35\upsilon_3^3) = 0, \quad$$

$$3J_6H^7 - 35\upsilon_4J_4H^5 + 945\upsilon_5J_3H^4 + 112(96\upsilon_1^2\upsilon_3^2 - 5\upsilon_4^2)J_2H^3$$
$$+ 5880\upsilon_4\upsilon_5J_1H^2 + 40(960\upsilon_2\upsilon_3^3 + 1008\upsilon_1\upsilon_3^2\upsilon_4 - 35\upsilon_3^3)H$$
$$+ 840(48\upsilon_1\upsilon_3^2 - 5\upsilon_4^2)\upsilon_5 = 0$$

such that its substitution into the relation

$$I_1 I_2 - \frac{HN_1}{678\upsilon_1\upsilon_3^2} + \frac{1728I_{21}^2 H^2\upsilon_1\upsilon_3^2 N_2}{25I_1^2 N_0^2} = 0 \quad (3.4)$$
turns it into an identity. And the substitution of (3.2) into the equations

\[
\frac{25l_0^0N_0^2}{21596I_z^2\nu_6^2\nu_4^2H^3} + \alpha\nu_3^3(z + 1 - 3w) + \beta z(w - 1)^3(w - z)^3(3z - (z + 1)w) \\
+ \gamma(z - 1)w_3^3(w - z)^3((z - 2)w + 2z - 1) \\
+ \delta z(z - 1)w^3(w - 1)^3((2z - 1)w + z(z - 2)) = 0,
\]

\[
\frac{25I_0^0N_0^2}{21639I_z^2\nu_6^2\nu_4^2\nu_3^2H^3} + \alpha w^3(w - 1)^4(w - z)^4(6w - z - 1) \\
+ \beta z(w - 1)^4(w - z)^4(6z - (z + 1)w) \\
+ \gamma(z - 1)w^3(w - z)^4((z - 2)w^2 + 6(z - 1)w + 2z - 1) \\
+ \delta z(z - 1)w^3(w - 1)^4((2z - 1)w^2 + 6z(z - 1)w + z^2(z - 2)) = 0,
\]

\[
\frac{25I_0^0N_0^2}{21639I_z^2\nu_6^2\nu_4^2\nu_3^2H^3} [J_2H^3 + (w^2 - z)(J_1H^2 + 4(z + 1)\nu_3) \\
+ 2(w - 1)(w - z)(3(w^2 + z) - 2(z + 1)w)(z - w^2 - H)] \\
+ \alpha \nu_5^5(7(z + 1) - 18w) + \beta z^2(w - 1)^5(w - z)^5(7(z + 1)w - 18z) \\
+ 3\gamma(z - 1)^2w^5(w - z)^5((3z - 7)w + 7z - 3) \\
+ 3\delta z^2(z - 1)^2w^5(w - 1)^5((3 - 7z)w + z(7 - 3z)) = 0,
\]

leads to algebraic relations from which the constant parameters of PVI can be found. The functions \(\xi(x, y), \eta(x, y)\) in (3.2) define the change of variables (1.2) that relates Eqs. (1.1) and (1.3).

In (3.3)–(3.5), we use the notation

\[
\nu_0 = 216\nu_1^2\nu_3 + 25\nu_2\nu_4, \quad \nu_1 = z^2 - z + 1, \quad \nu_2 = (z + 1)(2z - 1)(z - 2), \\
\nu_3 = w(w - 1)(w - z), \quad \nu_5 = (w - z)(w^2 - 2w + z)(w^2 - 2zw + z), \\
\nu_4 = 3(w^2 + z)^2 - 4(z + 1)w(w^2 + z) + 4(z^2 - z + 1)w^2, \\
\nu_6 = 27\nu_1\nu_5J_2H^3 - 6(168\nu_1^2\nu_3^2 + 15\nu_2\nu_3\nu_4 - 2\nu_1^2\nu_4)^2J_1H^2 - 135\nu_1\nu_4\nu_5H \\
+ 4800\nu_2\nu_3^2\nu_4^2 + 2232\nu_1^2\nu_3^2\nu_4 - 25\nu_2\nu_3\nu_4^2 - 15\nu_1^2\nu_4^3, \\
\nu_7 = (288\nu_1^2\nu_3^2 + 30\nu_2\nu_3\nu_4 - \nu_1\nu_4^2)J_2H^3 - 12\nu_1\nu_4\nu_5J_1H^2 - (576\nu_1^2\nu_3^2 \\
+ 50\nu_2\nu_3\nu_4 - 5\nu_1\nu_4^2)\nu_5H - 3(1584\nu_1^2\nu_3^2 + 100\nu_2\nu_3\nu_4 - 5\nu_1\nu_4^2)\nu_5,
\]

\[
N_0 = (6J_2 - J_1^2)\nu_1\nu_5H^2 - 3(5J_2 + 7J_1)\nu_1\nu_7H - 20(5J_1 + 6)\nu_2\nu_3\nu_6 \\
+ (5J_1 + 6)\nu_0(31\nu_1\nu_4 - 72\nu_2\nu_3)J_1H^2 - 270\nu_1\nu_5H + 2784\nu_1^2\nu_3^2 - 20\nu_2\nu_3\nu_4 - 30\nu_1\nu_4^2, \\
N_1 = 5(\nu_1H + 15\nu_7) + \nu_0(18(J_1 - 5J_2)H^2 + 155\nu_4H + 900\nu_5),
\]

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\[ N_2 = (15\nu_1^2 - 12\nu_0\nu_3 H - 2\nu_0^2 J_1 H^2) I_1^2 + 12\nu_0 \nu_3^2 H^2 [540\nu_2 \nu_3 J_2 H^3 \\
- 6(768 \nu_1 \nu_2 \nu_3^2 + 648 \nu_1^2 \nu_2 \nu_4 + 35 \nu_2 \nu_3^2) J_1 H^2 + 23328 \nu_1 \nu_2 \nu_3 H \\
+ 5(768(25\nu_2^2 - 108\nu_1^2)\nu_3^3 + 192 \nu_1 \nu_2 \nu_3^2 \nu_4 + 216 \nu_1^2 \nu_2 \nu_4^2 - 35 \nu_2 \nu_3^3)]]. \]

**Proof.** Let us find relations on the invariants of PVI. The same relations hold for the invariants of an equation (1.1) equivalent to PVI. The invariants (2.2), (2.3), and \( I_{21} \) of Eq. (1.3) are given by

\[ J_1 = \frac{2K_0 K_2}{K_1^2}, \quad J_2 = \frac{2K_2^2 K_3}{K_1^3}, \quad J_3 = \frac{-16K_0^3 K_4}{K_1^4}, \]

\[ J_4 = \frac{16K_0^4 K_5}{K_1^5}, \quad J_5 = \frac{-32K_0^5 K_6}{K_1^6}, \quad J_6 = \frac{32K_0^5 K_7}{K_1^7}, \] \( \tag{3.6} \)

\[ \frac{I_{21}^2}{I_1^2} = \frac{25\bar{N}_0^2}{2^{13} 3^4 \nu_0^2 \nu_1^2 \nu_3^2 K_1^7}, \] \( \tag{3.7} \)

\[ I_1 I_2 = \frac{K_1 \bar{N}_1}{768 \nu_0 \nu_1 \nu_3 K_0^3} + \frac{K_1 \bar{N}_2}{4608 \nu_0 \nu_3 K_0^3}, \] \( \tag{3.8} \)

\[ \bar{N}_0 = 2(3K_1 K_3 - 7K_2^2)\nu_1 K_8 + 3(7K_1 K_2 - 5K_0 K_3)\nu_1 K_9 \\
- 20(3K_1^2 - 5K_0 K_2)\nu_2 \nu_3 K_8 + 2(3K_1^2 - 5K_0 K_2)\nu_0 \nu_3[-135\nu_1 \nu_5 K_1 \\
+ (72\nu_1 \nu_3 - 31 \nu_1 \nu_4) K_2 + (1392 \nu_1^2 \nu_3^2 - 10 \nu_2 \nu_3 \nu_4 - 15 \nu_1 \nu_2^2) K_0], \]

\[ \bar{N}_1 = 5K_1 K_8 + 75K_0 K_9 + \nu_0 \nu_3(900 \nu_5 K_0^2 + 155 \nu_4 K_0 K_1 - 36(K_1 K_2 + 5K_0 K_3)), \]

\[ \bar{N}_2 = \frac{4K_2 K_3^2 - 12K_1 K_8 K_9 + 15K_0 K_3^2}{\nu_0 (3K_1^2 - 5K_0 K_2)} + 4\nu_3^2[23328 \nu_1^2 \nu_3 \nu_5 K_1 \\
+ 1080 \nu_2 \nu_5 K_3 + 5(768(25\nu_2^2 - 108\nu_1^2)\nu_3^3 + 216 \nu_1 \nu_2 \nu_3^2 \nu_4 - 35 \nu_2 \nu_3^3 \\
+ 192 \nu_1 \nu_2 \nu_3^2 \nu_4) K_0 + 12(768 \nu_1 \nu_2 \nu_3^2 + 648 \nu_1^2 \nu_3 \nu_4 + 35 \nu_2 \nu_3^2) K_2]], \]

where we set

\[ K_0 = A(z + 1 - 3w) + B(3z - (z + 1)w) \\
+ \Gamma((z - 2)w + 2z - 1) + \Delta((2z - 1)w + z(z - 2)), \] \( \tag{3.9} \)

\[ B = \beta z(w - 1)^3(w - z)^3, \]

\[ A = \alpha \nu_3^3, \quad \Gamma = \gamma(z - 1)w^3(w - z)^3, \quad \Delta = \delta(z - 1)w^3(w - 1)^3, \]

\[ K_8 = 54\nu_1 \nu_5 K_3 + 12(168 \nu_1^2 \nu_3 K_2 - 135 \nu_1 \nu_3 K_8) \\
+ (4800 \nu_1 \nu_3 K_2^2 + 23328 \nu_1 \nu_3^2 \nu_4 - 25 \nu_2 \nu_3 \nu_4 - 15 \nu_1 \nu_2^2) K_0, \]

\[ K_9 = 2(288 \nu_1^2 \nu_3^2 + 30 \nu_2 \nu_2 \nu_4 - \nu_1 \nu_2^2) K_3 + 24 \nu_1 \nu_4 \nu_5 K_2 - (576 \nu_1^2 \nu_3^3 \\
+ 50 \nu_2 \nu_3 \nu_4 - 5 \nu_1 \nu_2^2) \nu_4 K_1 - 3(10 \nu_4 \nu_3^2 + 100 \nu_2 \nu_3 \nu_4 - 5 \nu_1 \nu_2^2) \nu_5 K_0. \]

Instead of the cumbersome expressions for \( K_1, K_2, K_3, \) it is convenient to deal with the following combinations:

\[ \text{600} \]
\((w^2 - z)K_0 - K_1 = 2A(w - 1)(w - z)(6w - z - 1) + 2B(w - 1)(w - z)(6z - (z + 1)w) + 2\Gamma(w - z)((z - 2)w^2 + 6(z - 1)w + 2z - 1) + 2\Delta(w - 1)((2z - 1)w^2 + 6z(z - 1)w + z^2(z - 2)),
\]
\(K_2 + 2(z + 1)\nu_3K_0 = -3A\nu_3(15w^2 - 8(z + 1)w + 3z) + 3Bz(w - 1)(w - z)(3w^2 - 8(z + 1)w + 15z) - 3\Gamma w(w - z)((7 - 5z)w^2 + 4(2z - 1)(z - 2)w + z(7z - 5)) - 3\Delta zw(w - 1)((5 - 7z)w^2 + 4(2z - 1)(2 - z)w + z(5z - 7)),
\]
\((w - 1)(w - z)(3(w^2 + z) - 2(z + 1)w)((z - w^2)K_0 - K_1)
K_3 + (w^2 - z)(2(z + 1)\nu_3K_0 - K_2) = 12A\nu_3^2(7z + 1) - 18w + 12Bz(w - 1)^2(w - z)^2(7z + 1)w - 18z) + 36\Gamma(z - 1)w^2(w - z)^2((3z - 7)w + 7z - 3) + 36\Delta(z - 1)w^2(w - 1)^2((3z - 7)w + 7z - 3).
\]

Furthermore, for the values \(K_4, K_5, K_6, K_7\), the relations
\[
48\nu_0K_4 - 900\nu_2\nu_5K_3 - 30(256\nu_1\nu_2\nu_3^2 + 504\nu_1^2\nu_2\nu_4^2 + 45\nu_2\nu_4^2)K_2
- 45(1512\nu_1^2\nu_3 + 125\nu_2\nu_4)\nu_5K_1 + (128(3969\nu_1^3 - 625\nu_2^3)\nu_3^2
+ 18000\nu_1\nu_2\nu_3\nu_4 - 7560\nu_1^2\nu_2\nu_4^2 - 625\nu_2\nu_4^2)K_0 = 0,
8\nu_0K_5 - 5(96\nu_1\nu_2\nu_3^2 + 504\nu_1^2\nu_2\nu_4^2 + 55\nu_2\nu_4^2)K_3 - 75(504\nu_1^2\nu_3
+ 61\nu_2\nu_4)\nu_5K_2 + (1008\nu_1^2\nu_3 + 125\nu_2\nu_4)(144\nu_1\nu_3^2 - 5\nu_4^2)K_1
+ 15(1200\nu_1\nu_2\nu_3 - 1008\nu_2^2\nu_3\nu_4 - 125\nu_2\nu_4^2)\nu_5K_0 = 0,
144K_6 - 840\nu_1\nu_4K_4 - 2205\nu_5K_3 + 21(976\nu_1\nu_2\nu_3^2 - 45\nu_4^2)K_2
- 3150\nu_1\nu_2\nu_4K_1 + 10(1280\nu_2^2\nu_3^2 + 1008\nu_1^2\nu_2\nu_4^2 - 35\nu_3^2)K_0 = 0,
12K_7 - 70\nu_4K_5 - 1890\nu_5K_4 + 28(96\nu_1\nu_2\nu_3^2 - 5\nu_4^2)K_3 - 1470\nu_2\nu_5K_2
+ 5(960\nu_2\nu_3^3 + 1008\nu_1\nu_2\nu_3\nu_4 - 35\nu_3^3)K_1 + 105(48\nu_1\nu_2^3 - 5\nu_4^2)\nu_5K_0 = 0
\]
hold. Equations (3.6) can be solved for the values
\[
K_2 = -\frac{K_4^2J_1}{2K_0}, \quad K_3 = \frac{K_4^3J_2}{2K_0^2}, \quad K_4 = -\frac{K_4^4J_3}{16K_0^3}, \quad K_5 = \frac{K_4^5J_4}{16K_0^4}, \quad K_6 = -\frac{K_4^6J_5}{32K_0^5}, \quad K_7 = \frac{K_4^7J_6}{32K_0^6}.
\]

After substituting them, relations (3.11) take the form (3.3), while (3.7) becomes
\[
\frac{I_2^2}{I_1^3} = -\frac{25N_4^2}{21536\nu_2^2\nu_4^2J^2K_1}, \quad H = \frac{K_4}{K_0},
\]
allowing us to express \(K_1\). Substituting the obtained value into (3.8) together with (3.12) leads to (3.4), and substituting it into (3.9), (3.10) leads to (3.5). This completes the proof. \(\square\)
4. Application of Theorem 2

If we need to check whether a given ODE (1.1) is equivalent to PVI, first we should check the conditions (2.1), (3.1) for this ODE and then find the solution (3.2) of the system (3.3). Relations (3.3) can be rewritten in the form of polynomials of degree 4 in $H$:

\begin{align*}
3P_4J_4H^4 &+ 450v_5P_3J_2H^3 - 15P_2J_1H^2 + 45v_5P_1H + P_0 = 0, \\
Q_4J_4H^4 &- 90v_5Q_3J_3H^3 + 5Q_2J_2H^2 + 75v_5Q_1J_1H + Q_0 = 0, \\
3R_4J_4H^4 &- 900v_5R_3J_3H^3 + 15R_2J_2H^2 + 315v_5R_1J_1H + R_0J_1 = 0, \\
S_4J_6H^4 &+ 120v_5S_3J_5H^3 + 15S_2J_4H^2 + 225v_5S_1J_3H + S_0J_2 = 0, \\
\end{align*}

\[ \text{(4.1)} \]

where

\begin{align*}
P_0 &= 128(625\nu_2^2 - 3969\nu_1^3\nu_3^3 - 5(3600\nu_1\nu_2\nu_3^2 - 1512\nu_1^2\nu_3\nu_4 - 125\nu_2\nu_4^2)\nu_4, \\
P_1 &= 1512\nu_1^2\nu_3 + 125\nu_2\nu_4, \\
P_2 &= 256\nu_1\nu_2\nu_3^2 + 504\nu_1^2\nu_3\nu_4 + 45\nu_2^2, \\
P_3 &= \nu_2, \\
P_4 &= 216\nu_1^2\nu_3 + 25\nu_2\nu_4, \\
Q_0 &= 384\nu_3^2R_4, \\
Q_1 &= 128(1525\nu_2^2 - 9261\nu_1^3\nu_3^3 - 7(5520\nu_1\nu_2\nu_3^2 - 504\nu_1^2\nu_3\nu_4 - 25\nu_2\nu_4^2)\nu_4, \\
Q_2 &= 128(9261\nu_1^3 - 1625\nu_2^2\nu_3^3\nu_4 + 7(32256\nu_1^2\nu_2\nu_3^4 + 1200\nu_1\nu_2\nu_3^2\nu_4^2 \\
&- 2520\nu_1\nu_2\nu_3\nu_4^2 - 125\nu_2\nu_4^4), \\
Q_3 &= 1200\nu_1\nu_2\nu_3^2 - 1008\nu_1^2\nu_3\nu_4 - 125\nu_2\nu_4^2, \\
R_0 &= 5376(540125\nu_2^2 - 2259684\nu_1^3\nu_3^3 + 4116000\nu_1^2\nu_2\nu_3\nu_4^2 \\
&+ 6720(28224\nu_1^3 - 6875\nu_2^2)\nu_1\nu_2\nu_3^4 + 875(625\nu_2^2 - 1764\nu_1^3)\nu_4, \\
&+ 6400(38125\nu_2^2 - 168462\nu_1^3)\nu_2\nu_3^3\nu_4, \\
R_1 &= -117600\nu_1^2\nu_2\nu_3\nu_4 \\
&+ 48(18125\nu_2^2 - 86436\nu_1^3)\nu_1\nu_2\nu_3^3 + 25(1764\nu_1^3 - 625\nu_2^2)\nu_4^3, \\
R_2 &= 1290240\nu_1^3\nu_2\nu_3^3 + 48(107604\nu_1^3 - 15625\nu_2^2)\nu_1\nu_2\nu_3^2\nu_4 + 285600\nu_1^2\nu_2\nu_3\nu_4^2 \\
&+ 65(625\nu_2^2 - 1764\nu_1^3)\nu_4^3, \\
R_3 &= 1680\nu_1^2\nu_2\nu_3 + (625\nu_2^2 - 1764\nu_1^3)\nu_4, \\
R_4 &= 112(3125\nu_2^2 - 15876\nu_1^3)\nu_1\nu_2\nu_3^3 - 84000\nu_1^2\nu_2\nu_3\nu_4 + 25(1764\nu_1^3 - 625\nu_2^2)\nu_4^2, \\
S_0 &= 512\nu_1^2(6443747200\nu_1^3\nu_2^2 - 1328694192\nu_1^3 - 27640625\nu_2^2)\nu_3^3 \\
&+ 403200(3625\nu_2^2 - 15582\nu_1^3)\nu_1\nu_2\nu_3^2\nu_4 + 88200(19404\nu_1^3 \\
&- 5375\nu_2^2)\nu_1^2\nu_2\nu_3^3 + 30625(5292\nu_1^3 - 1375\nu_2^2)\nu_2\nu_3^4, \\
S_1 &= 16128(267725\nu_2^2 - 1111908\nu_1^3)\nu_1^2\nu_2^3\nu_4^4 + 823200\nu_1^2\nu_2\nu_3\nu_4^3 \\
&+ 6400(44225\nu_2^2 - 196686\nu_1^3)\nu_2\nu_3^3\nu_4 + 105(625\nu_2^2 - 1764\nu_1^3)\nu_4^4 \\
&+ 20160(7056\nu_1^3 - 1975\nu_2^2)\nu_1\nu_2\nu_3^3\nu_4^2, \\
S_2 &= 5376(2905308\nu_1^3 - 689875\nu_2^2)\nu_1^2\nu_2\nu_3\nu_4 + 4116000\nu_1^2\nu_2\nu_3\nu_4^2 \\
&+ 57600(21266\nu_1^3 - 5125\nu_2^2)\nu_1\nu_2\nu_3^2\nu_4 + 11200(3025\nu_2^2 - 15288\nu_1^3)\nu_1\nu_2\nu_3^3\nu_4^2 \\
&+ 40960(125538\nu_1^3 - 27875\nu_2^2)\nu_1\nu_2\nu_3^5 + 525(1764\nu_1^3 - 625\nu_2^2)\nu_4^5, \\
S_3 &= 320(38125\nu_2^2 - 168462\nu_1^3)\nu_2\nu_3^3\nu_4 + 672(28224\nu_1^3 - 6875\nu_2^2)\nu_1\nu_2\nu_3^3\nu_4 \\
&+ 617400\nu_1^2\nu_2\nu_3\nu_4^2 + 175(625\nu_2^2 - 1764\nu_1^3)\nu_4^3, \\
S_4 &= R_0.
\end{align*}

From (4.1) we obtain that the following quadratic polynomials in $H$ vanish:

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Eliminating $H$ these expressions satisfy the relations \begin{align*}
15U_6H^2 + 45\nu_5U_5H + U_4 &= 0, \\
6\nu_5U_4H^2 + U_3H + 3\nu_2U_2 &= 0, \\
U_2H^2 + 30\nu_5U_1H + 5U_0 &= 0,
\end{align*}
where we set
\begin{align*}
U_0 &= T_0P_2J_1 + T_1P_1 - T_1P_0, \\
U_2 &= -T_0P_4J_3 + T_3P_1 - T_0P_0, \\
U_4 &= 3T_2P_4J_3 + 5T_3P_3J_2 + T_3P_0, \\
U_5 &= T_5P_4J_3 + 5T_6P_3J_2 + T_0P_1, \\
U_6 &= T_7P_4J_3 + 5T_8P_3J_2 - T_9P_2J_1, \\
V_0 &= 3T_6S_2J_4 - 15T_1S_1J_3 + 3T_4S_0J_2, \\
V_1 &= 4T_6S_3J_5 + 45T_2S_1J_3 - 3T_5S_0J_2, \\
V_2 &= 5T_6S_4J_6 - 15T_1S_1J_3 + 3T_6S_0J_2, \\
V_3 &= T_1S_4J_6 - 3T_3S_2J_4 + 3T_8S_0J_2, \\
V_4 &= -3T_2S_3J_5 - 4T_3S_2J_4 - 3T_5S_0J_2, \\
V_5 &= -5T_5S_4J_6 - 4T_6S_3J_5 - 15T_9S_1J_3, \\
V_6 &= -T_7S_4J_6 - 4T_8S_3J_5 - 3T_9S_2J_4,
\end{align*}
thus obtained into relations (3.3), (3.4) turns them into identities. After the substitution (3.2), ODE (1.1) under consideration is equivalent to PVI, then the substitution of the functions (3.2) otherwise, if this system is inconsistent or it is not algebraic, then the given ODE (1.1) is not.

\begin{align*}
U_0U_4 - U_1U_3 &= U_2 \tilde{U}_2, \\
U_2U_6 - U_3U_5 &= U_4 \tilde{U}_4, \\
V_0V_4 - V_1V_3 &= V_2 \tilde{V}_2, \\
V_2V_6 - V_3V_5 &= V_4 \tilde{V}_4,
\end{align*}

Eliminating $H$ from Eqs. (4.2) leads to the relations
\begin{align*}
U_4[3\nu_5^2(4U_4^3 - 30U_3U_4U_5 - 60U_2U_4U_6 + 225\overline{U}_2 \overline{U}_4U_6) + 1215\nu_5^2U_2U_4^2 + 5U_3^2U_6] &= 0, \\
V_4[3\nu_5^2(4V_4^3 - 30V_3V_4V_5 - 60V_2V_4V_6 + 225V_2V_4V_6) + 1215\nu_5^2V_2V_4^2 + 5V_3^2V_6] &= 0, \\
U_2[9\nu_5^2(U_2^3 - 10U_1U_2U_3 - 20U_0U_2U_4 + 100U_0\overline{U}_2U_4) + 16200\nu_5^2U_2^4U_4 + 5U_0U_3^2] &= 0, \\
V_2[9\nu_5^2(V_2^3 - 10V_1V_2V_3 - 20V_0V_2V_4 + 100V_0\overline{V}_2V_4) + 16200\nu_5^2V_2^4V_4 + 5V_0V_3^2] &= 0.
\end{align*}

They are defined by polynomials in $z$, $w$ with coefficients given by functions of the invariants $J_1$, …, $J_6$ of Eq. (1.1) depending on $x$, $y$. Assume that we can find functions $\xi(x, y)$, $\eta(x, y)$ such that the expressions (1.2) solve all equations (4.3). After substituting (1.2), we try to factorize the left-hand sides of (4.2) with a common factor, whose vanishing allows one to obtain an expression for $H = \zeta(x, y)$. If the ODE (1.1) under consideration is equivalent to PVI, then the substitution of the functions (3.2) thus obtained into relations (3.3), (3.4) turns them into identities. After the substitution (3.2), relations (3.5) must reduce to a consistent system of algebraic equations on $\alpha$, $\beta$, $\gamma$, $\delta$. Otherwise, if this system is inconsistent or it is not algebraic, then the given ODE (1.1) is not equivalent to PVI.
5. Other necessary equivalence conditions and autotransformations of PVI

Conditions for equivalence of a second-order ODE to the sixth Painlevé equation with one nonzero essential parameter were found in [15]. If we modify the series (2.4), in whose terms they are stated, and introduce the invariants

\[ E_1 = \frac{6}{5}(\frac{1}{n^2} - 1), \quad E_{m+1} = (2m + (m + 1)E_1)E_m + \frac{1}{n^2}D_1E_m, \quad m \in \mathbb{N}, \quad (5.1) \]

then these conditions take the following form.

**Theorem 3.** The invariants (5.1) of an equation (1.1) equivalent to the sixth Painlevé equation satisfy the relations

\[ 225E_3^2(E_3 - 2E_2 - E_1) - 120(E_1E_3 + E_2^2)^2 \]
\[ - 30E_1E_2(3E_2E_3 + 5E_1E_3 + 2E_2^2) + E_1^3(16E_1E_3 + 7E_2^2) = 0, \quad (5.2) \]
\[ E_1E_5 + 20E_2E_4 - 45E_3^2 = 0 \]

in the following cases: (1) only one of the parameters \( \alpha, \beta, \gamma, \) or \( \delta \) of Eq. (1.3) is nonzero; (2) two parameters have the same nonzero value, and the other two parameters are zero; (3) \( \alpha = \beta = \gamma = \delta \neq 0. \)

Conditions (5.2) are only necessary, because they do not involve the invariants \( I_2 \) and \( D_2I_1. \) But they are easier to verify than the conditions of Theorem 2, and it makes sense to check them before applying Theorem 2.

Theorem 3 suggests that the three special cases of PVI mentioned in the statement are related by point transformations. These transformations can be found using the invariants (5.1) and Theorem 2. For Eq. (1.3) with \( \alpha \neq 0, \beta = \gamma = \delta = 0, \)

\[ \frac{d^2w}{dz^2} = \frac{1}{2}(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z})\left(\frac{dw}{dz}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)\frac{dw}{dz} \]
\[ + \frac{\alpha w(w-1)(w-z)}{2(z-1)^2} + \frac{w(w-1)}{2z(z-1)(w-z)}, \quad (5.3) \]

the invariants (5.1) are given by

\[ E_1 = -\frac{k_0\tau}{3k_1^3}(\rho + \sigma), \quad E_2 = \frac{k_0\tau^2}{3k_1^3}(4\rho + \sigma), \quad E_3 = \frac{k_0\tau^3}{3k_1^4}(7\rho + \sigma), \quad (5.4) \]

and so on, where

\[ k_1 = 15(w - z - 1)w^2 + (2z^2 + 13z + 2)w - z(z + 1), \quad k_0 = 3w - z - 1, \]
\[ \tau = 6w(w-1)(w-z), \quad \rho = 2(z^2 - z + 1), \quad \sigma = 15(3w^2 - 2(z + 1)w + z). \]

For the sixth Painlevé equation with \( \alpha = \beta = \gamma = \delta = A \neq 0, \)

\[ \frac{d^2y}{dx^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)\left(\frac{dy}{dx}\right)^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)\frac{dy}{dx} \]
\[ + \frac{A(y^2 - x)(y^2 - 2y + x)(y^2 - 2xy + x)}{x^2(x-1)^2y(y-1)(y-x)} + \frac{y(y-1)}{2x(x-1)(y-x)}, \quad (5.5) \]
the invariants (5.1) are of the same form (5.4), but with the coefficients
\[ k_1 = 60y(y - 1)(x - y)(y^2 - x)^4(x + 1) - 64x(x + 1)y^3(y - 1)^3(y - x)^3 \]
\[ + 15(y^2 - x)^6 + 16y^2(y - 1)^2(y - x)^2(2x^2 + 13x + 2), \]
\[ k_0 = 3(y^2 - x)^2 + 4y(y - 1)(x - y)(x + 1), \]
\[ \tau = 6(y^2 - x)^2(2y^2 - 2y + x)^2(2y^2 - 2xy + x)^2, \]
\[ \rho = 32y^2(y - 1)^2(y - x)^2(x^2 - x + 1), \]
\[ \sigma = 15(3(y^2 - x)^4 + 8y(y - 1)(y^2 - x)^2(x + 1) + 16xy^2(y - 1)^2(y - x)^2). \]
Comparing the coefficients \( \rho \) allows us to assume that \( z = x \). Then, equating the invariants (5.4) of Eqs. (5.3), (5.5), we infer that
\[ z = x, \quad w = \frac{(y^2 - x)^2}{4y(y - 1)(y - x)}. \]
(5.6)

Let us apply Theorem 2 to Eq. (5.5). After the substitution (5.6), relations (4.3) hold identically and relations (4.2) factorize with common factor \((y^2 - x)(y^2 - 2y + x)(y^2 - 2xy + x)k_0H - k_1\).
Equating it to zero, we can find an expression for \( H \). Substituting it together with (5.6) into (3.3), (3.4) leads to identities, while relations (3.5) reduce to \( \alpha - 16A = 0, \beta = 0, \gamma = 0, \delta = 0 \). Thus, the change of variables (5.6) relates the ODE (5.5) to Eq. (5.3) with \( \alpha = 16A \).
Transformations relating the ODE (5.5) to PVI with one nonzero parameter \( \beta, \gamma, \) or \( \delta \) can be obtained by taking the product of (5.6) with known point autotransformations of PVI [24].
For PVI with \( \alpha = \beta = B \neq 0, \gamma = \delta = 0 \),
\[ \frac{d^2u}{dt^2} = \frac{1}{2} \left( \frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-t} \right) \left( \frac{du}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t-1} + \frac{1}{u-t} \right) \frac{du}{dt} \]
\[ + \frac{B(u-1)(u-t)(u^2-t)}{t^2(t-1)^2u} + \frac{u(u-1)}{2t(t-1)(u-t)}, \]
(5.7)
the invariants (5.1) have the form (5.4) with the coefficients
\[ k_1 = 2u^2(u^2+t)(t^2-16t+1) - u(t+1)(15u^4+2tu^2+15t^2) \]
\[ + 15(u^2+t)^3, \quad k_0 = 3(u^2+t) - (t+1)u, \]
\[ \tau = 6(u-1)(u-t)(u^2-t)^2, \]
\[ \rho = 2u^2(t^2+14t+1), \]
\[ \sigma = 15(3(u^2+t)^2 - 2u(t+1)(u^2+t) - 4tu^2). \]
Equating the correspondent invariants (5.4) of Eqs. (5.3), (5.7) and solving the resulting equations for \( z, w \), we obtain the change of variables
\[ z = \frac{(\sqrt{7} + 1)^2}{4\sqrt{7}}, \quad w = \frac{(u + \sqrt{7})^2}{4\sqrt{7}u}. \]
(5.8)
It transforms the ODE (5.7) into the ODE (5.3) with \( \alpha = 4B \). Equations (5.3), (5.5), (5.7) are discussed in [26], including the case \( A = B = 1/8 \).

For PVI with either two pairs of equal parameters or two nonzero nonequal parameters, relations similar to (5.2) hold. We omit them for reasons of space. It is readily verified that (5.8) defines also an autotransformation of PVI for \( u(t) \) with \( \alpha = \beta = B \neq 0, \gamma = \delta = \Gamma \neq 0, B \neq \Gamma \) into Eq. (1.3) with \( \alpha = 4B, \beta = \gamma = 0, \delta = 4\Gamma \).

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