SOME SPECTRAL PROPERTIES OF THE CANONICAL SOLUTION OPERATOR TO ∂ ON WEIGHTED FOCK SPACES

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Abstract. We characterize the Schatten class membership of the canonical solution operator to ∂ acting on $L^2(e^{-2φ})$, where φ is a subharmonic function with $∆φ$ a doubling measure. The obtained characterization is in terms of $∆φ$. As part of our approach, we study Hankel operators with anti-analytic symbols acting on the corresponding Fock space of entire functions in $L^2(e^{-2φ})$.

Keywords: Schatten classes, canonical solution operator to ∂

1. Introduction

For a (nonharmonic) subharmonic function φ on $\mathbb{C}$ having the property that $∆φ$ is a doubling measure, the generalized Fock space $\mathcal{F}^2_φ$ is defined by

$$\mathcal{F}^2_φ = \{ f \in \mathcal{H}(\mathbb{C}) : \|f\|_{\mathcal{F}^2_φ}^2 = \int_{\mathbb{C}} |f(z)|^2 e^{-2φ(z)} dm(z) < \infty \},$$

where $dm(z)$ denotes the Lebesgue measure on $\mathbb{C}$. We let $μ = ∆φ$ and denote by $ρ(z)$ the positive radius for which we have $μ(D(z, ρ(z))) = 1$, $z ∈ \mathbb{C}$. The function $ρ^{-2}$ can be regarded as a regularized version of $∆φ$ (see [5, 17]). We consider the canonical solution operator $N$ to ∂ given by

$$\bar{∂}Nf = f \quad \text{and} \quad Nf \text{ is of minimal norm in } L^2(e^{-2φ}),$$

or, equivalently

$$\bar{∂}Nf = f \quad \text{and} \quad Nf \perp \mathcal{F}^2_φ.$$

The boundedness and the compactness of $N$ acting on various weighted $L^2$-spaces have been extensively studied in one or several variables (see [6, 8, 9, 10]). Concerning the Schatten class membership of this operator, it was first shown in [8] that for the particular choice $φ(z) = |z|^m$, $N$ fails to be Hilbert-Schmidt, and a more involved study was pursued in [10] in the context of several complex variables, where the authors obtain necessary and sufficient conditions for the canonical solution operator to ∂ to belong to the Schatten class $S^p$, $p > 0$, when restricted to $(0,1)$-forms with holomorphic coefficients in $L^2(μ)$, for measures $μ$ with the property that the monomials form an orthogonal family in $L^2(μ)$. Some particular cases of these results were previously obtained in [16].

Date: July 8, 2010.

The second author is supported by the project MTM2008-05561-C02-01 and the CIRIT grant 2005SGR00611.
In the present paper we are interested in the setting of subharmonic functions $\phi$ with $\Delta \phi$ a doubling measure. For this type of weights, it was proven in [18] that $N$ is compact from $L^2(e^{-2\phi})$ to itself if and only if $\rho(z) \to 0$ as $|z| \to \infty$. We continue the investigation in [18] by characterizing the Schatten class membership of $N$. We find that $N$ fails to be Hilbert-Schmidt and that $N$ belongs to the Schatten class $S^p$ with $p > 2$ if and only if the following holds

$$\int_C \rho^{p-2}(z) \, dA(z) < \infty. \tag{1}$$

We start our approach by noticing that the restriction of $N$ to $F^2_{\phi}$ is actually a (big) Hankel operator with symbol $\bar{z}$. This observation leads us to a study of these properties for Hankel operators on $F^2_{\phi}$ with anti-analytic symbols. We would like to point out that Lin and Rochberg [14, 15] considered these problems for Hankel operators with symbols in $L^2(C)$ for a certain class of subharmonic functions $\phi$. The case of anti-analytic symbols was investigated in [13, 4, 20] for $\phi(z) = |z|^m, m > 0$, and it was shown that a Hankel operator $H_g$ belongs to $S^p$ if and only if the symbol $g$ is a polynomial of degree smaller than $m(p - 2)/(2p)$. For subharmonic functions $\phi$ with $\Delta \phi$ a doubling measure, we find that $H_g$ fails to be Hilbert-Schmidt unless $g$ is constant, and $H_g \in S^p$ for $p > 2$, if and only if its symbol satisfies

$$\int_C |g'(z)|^p \rho^{p-2}(z) \, dA(z) < \infty,$$

that is, $g$ is a polynomial whose degree depends on the order of decay of $\rho$.

Finally, using a result by Russo [19] together with the pointwise estimates obtained in [18] for the kernel of the canonical solution operator $N$, we show that the condition (1) is actually sufficient for $N$ to belong to $S^p$ with $p > 2$, even when defined on the whole of $L^2(e^{-2\phi})$.

2. Preliminaries

In this section we gather a few definitions and some known estimates that will be used in our further considerations. We start with some facts about doubling measures. A nonnegative Borel measure $\mu$ is called doubling if there exists $C > 0$ such that

$$\mu(D(z, 2r)) \leq C \mu(D(z, r)),$$

for all $z \in \mathbb{C}$ and $r > 0$. The smallest constant in the previous inequality is called the doubling constant for $\mu$.

**Lemma 1.** ([5, Lemma 2.1]) Let $\mu$ be a doubling measure on $\mathbb{C}$. There exists a constant $\gamma > 0$ such that for any discs $D, D'$ with respective radius $r > r'$ and with $D \cap D' \neq \emptyset$ the following holds

$$\left( \frac{\mu(D)}{\mu(D')} \right)^\gamma \lesssim \frac{r}{r'} \lesssim \left( \frac{\mu(D)}{\mu(D')} \right)^{1/\gamma}.$$
From now on we shall assume that \( \phi \) is a subharmonic function on \( \mathbb{C} \) such that \( \Delta \phi \) is a doubling measure. We denote \( D^r(z) = D(z, r\rho(z)) \) and for \( r = 1 \) we simply write \( D(z) \) instead of \( D^1(z) \). The function \( \rho \) has at most polynomial growth/decay (see \[17, \text{Remark 1}]\): there exist constants \( C, \beta, \gamma > 0 \) such that

\[
C^{-1} \frac{1}{|z|^{\gamma}} \leq \rho(z) \leq C|z|^\beta, \quad \text{for } |z| > 1.
\]

As an immediate consequence of Lemma 1 one obtains

Lemma 2. \[18] For any \( r > 0 \) there exists \( c > 0 \) depending only on \( r \) and the doubling constant for \( \Delta \phi \) such that

\[
c^{-1} \rho(\zeta) \leq \rho(z) \leq c \rho(\zeta) \quad \text{for } \zeta \in D^r(z).
\]

We also have

Lemma 3. \[5, \text{p. 205}] If \( \zeta \notin D(z) \), then

\[
\frac{\rho(z)}{\rho(\zeta)} \lesssim \left( \frac{|z - \zeta|}{\rho(\zeta)} \right)^{1 - \delta}
\]

for some \( \delta \in (0, 1) \) depending only on the doubling constant for \( \Delta \phi \).

For \( z, \zeta \in \mathbb{C} \), the distance \( d_\phi \) induced by the metric \( \rho^{-2}(z)dz \otimes d\bar{z} \) is given by

\[
d_\phi(z, \zeta) = \inf_{\gamma} \int_0^1 |\gamma'(t)| \frac{dt}{\rho(\gamma(t))},
\]

where \( \gamma \) runs over the piecewise \( C^1 \) curves \( \gamma : [0, 1] \rightarrow \mathbb{C} \) with \( \gamma(0) = z \) and \( \gamma(1) = \zeta \). We observe now that the metric \( \rho^{-2}(z)dz \otimes d\bar{z} \) is comparable to the Bergman metric: it is well known, see \[2\] that the Bergman metric \( B\left( \frac{\partial}{\partial z}, z \right) \) at the point \( z \) is given by the solution to the extremal problem

\[
B\left( \frac{\partial}{\partial z}, z \right) = \sup\{|f'(z)| : f \in \mathcal{F}_\phi^2, f(z) = 0; \|f\|_{\mathcal{F}_\phi^2} = 1\}.
\]

where \( K(z, \zeta) \) is the Bergman kernel for \( \mathcal{F}_\phi^2 \). In \[17, \text{Lemma 20}]\ it is proved that for all \( f \in \mathcal{F}_\phi^2 \) with \( f(z) = 0 \) we have \( |f'(z)| \lesssim \frac{\rho(z)}{\rho^2(z)} \|f\|_{\mathcal{F}_\phi^2} \), thus \( B\left( \frac{\partial}{\partial z}, z \right) \lesssim 1/\rho(z) \). The other inequality follows taking as \( f(\zeta) = C_z(\zeta - z)K(\zeta, z) \) where \( C_z \) is taken in such a way that \( \|f\|_{\mathcal{F}_\phi^2} = 1 \). In view of the estimates of the Bergman kernel stated below it follows that \( B\left( \frac{\partial}{\partial z}, z \right) \sim 1/\rho(z) \).

The following estimates for the Bergman distance \( d_\phi \) hold:

Lemma 4. \[17, \text{Lemma 4}] There exists \( \delta \in (0, 1) \) such that for every \( r > 0 \) there exists \( C_r > 0 \) such that

\[
C_r^{-1} \frac{|z - \zeta|}{\rho(z)} \leq d_\phi(z, \zeta) \leq C_r \frac{|z - \zeta|}{\rho(\zeta)}, \quad \text{for } \zeta \in D^r(z),
\]
and
\[ C_{r}^{-1} \left( \frac{|z - \zeta|}{\rho(z)} \right)^{\delta} \leq d_{\phi}(z, \zeta) \leq C_{r} \left( \frac{|z - \zeta|}{\rho(z)} \right)^{2 - \delta}, \quad \text{for } \zeta \in D^{r}(z). \]

The next result shows that we can replace the weight \( \phi \) by a regular weight \( \tilde{\phi} \) equivalent to it.

**Proposition 1.** [17, Theorem 14] Let \( \phi \) be a subharmonic function with \( \mu = \Delta \phi \) doubling. There exists \( \tilde{\phi} \in C^\infty(\mathbb{C}) \) subharmonic such that
\[ |\phi - \tilde{\phi}| \leq c \] with \( \Delta \tilde{\phi} \) doubling and
\[ \Delta \tilde{\phi} \sim \frac{1}{\rho^2_{\tilde{\phi}}} \sim \frac{1}{\rho^2_{\phi}}. \]

We also need the estimates

**Lemma 5.** [18] Let \( \phi \) be a subharmonic function with \( \mu = \Delta \phi \) doubling. Then for any \( \varepsilon > 0 \) and \( k \geq 0 \)
\[ \int_{\mathbb{C}} \frac{|z - \zeta|^k}{\exp d_{\phi}(z, \zeta)^{\varepsilon}} d\mu(z) \leq c \rho^k(\zeta), \]
where \( c > 0 \) is a constant depending only on \( k, \varepsilon \) and on the doubling constant for \( \mu \).

**Theorem 1.** [18] Let \( K(z, \zeta) \) be the Bergman kernel for \( F^2_{\phi} \). There exist positive constants \( c \) and \( \varepsilon \) (depending only on the doubling constant for \( \Delta \phi \)) such that for any \( z, \zeta \in \mathbb{C} \)
\[ |K(z, \zeta)| \leq c \frac{1}{\rho(z)\rho(\zeta)} \exp d_{\phi}^{\varepsilon}(z, \zeta). \]

**Lemma 6.** [18] There exists \( \alpha > 0 \) such that
\[ |K(z, \zeta)| \sim K(z, z)^{1/2} K(\zeta, \zeta)^{1/2} \sim \frac{e^{\phi(z) + \phi(\zeta)}}{\rho(z)\rho(\zeta)}, \quad \text{if } |z - \zeta| < \alpha \rho(z). \]

On the diagonal we have
\[ K(z, z) \sim \frac{e^{2\phi(z)}}{\rho^2(z)}, \quad z \in \mathbb{C}. \]

For \( \lambda \in \mathbb{D} \), we denote by \( k_{\lambda} \) the normalized reproducing kernel of \( F^2_{\phi} \), i.e.
\[ k_{\lambda}(z) = \frac{K(z, \lambda)}{K(\lambda, \lambda)^{1/2}}, \quad z, \lambda \in \mathbb{C}. \]

Finally, let us recall that a compact operator \( T \) acting on a Hilbert space belongs to the Schatten class \( \mathcal{S}^p \) if the sequence of eigenvalues of \((T^*T)^{1/2}) \) belongs to \( l^p \).
3. Hankel operators on $F_\phi^2$

As already mentioned in the introduction, the canonical solution operator $N$ to $\bar{\partial}$ is defined on $L^2(e^{-2\phi})$ by

$$\bar{\partial}Nf = f \quad \text{and} \quad Nf \perp F_\phi^2.$$  

Let us now consider the restriction of $N$ to $F_\phi^2$. Notice that if $f \in F_\phi^2$ and $\bar{z}f \in L^2(e^{-2\phi})$, then

$$Nf = (I - P)(\bar{z}f),$$

where $P$ is the orthogonal projection of $L^2(e^{-2\phi})$ onto $F_\phi^2$. In general, $\bar{z}f \in L^2(e^{-2\phi})$ does not hold for all $f \in F_\phi^2$ (see e.g. [13]), but it follows from Theorem 1 that $\bar{z}k_\lambda \in L^2(e^{-2\phi})$ for all $\lambda \in \mathbb{C}$. Since the subset $\text{Span}\{k_\lambda : \lambda \in \mathbb{C}\}$ is dense in $F_\phi^2$, we deduce from (4) that $N$ coincides with the big Hankel operator acting on $F_\phi^2$ with symbol $\bar{z}$. Motivated by this last fact, we now aim to study Hankel operators with anti-analytic symbols on $F_\phi^2$. Given an entire function $g$ so that there exists a dense subset $A$ of $F_\phi^2$ with $\bar{g}f \in L^2(e^{-2\phi})$ for $f \in A$, the big Hankel operator with symbol $\bar{g}$ is densely defined by

$$H_{\bar{g}}f = \bar{g}f - P(\bar{g}f) = (I - P)(\bar{g}f), \quad f \in A,$$

where $P$ is the orthogonal projection of $L^2(e^{-2\phi})$ onto $F_\phi^2$. We consider symbols $g$ such that

$$\bar{g}k_\lambda \in L^2(e^{-2\phi})$$

for all $\lambda \in \mathbb{C}$.

It follows from Theorem 1 that, for example, polynomial symbols satisfy this assumption. By the reproducing formula in $F_\phi^2$ we get

$$H_{\bar{g}}k_\lambda(z) = (g(z) - g(\bar{\lambda}))k_\lambda(z), \quad z, \lambda \in \mathbb{C}.$$  

For the sake of completeness we shall first characterize the boundedness and compactness of $H_{\bar{g}}$. Let us state the following theorem due to Hörmander which is essential to our approach.

**Theorem 2.** [11] Let $\Omega \subseteq \mathbb{C}$ be a domain and $\phi \in C^2(\Omega)$ be such that $\Delta \phi \geq 0$. For any $f \in L^2_{loc}(\Omega)$ there exists a solution $u$ to $\bar{\partial}u = f$ such that

$$\int |u|^2 e^{-2\phi} dm \leq \int \frac{|f|^2}{\Delta \phi} e^{-2\phi} dm.$$  

**Theorem 3.** $H_{\bar{g}}$ extends to a bounded linear operator on $F_\phi^2$ if and only if $|g'|\rho$ is bounded.

**Proof.** Assume first that $|g'|\rho$ is bounded. Then notice that for $f \in \text{Span}\{k_\lambda : \lambda \in \mathbb{C}\}$, $H_{\bar{g}}f$ is the solution to $\bar{\partial}u = \bar{g}'f$ of minimal $L^2(e^{-2\phi})$-norm. By Theorem 2 and Proposition 11 we have

$$\int |H_{\bar{g}}f|^2 e^{-2\phi} dm \lesssim \int |f|^2 |g'|^2 \rho^2 dm \leq \langle \sup |g'|\rho \rangle^2 \|f\|^2,$$

which shows that $H_{\bar{g}}$ can be extended to a bounded linear operator on $F_\phi^2$. 

Conversely, assume that $H_{\bar{g}}$ is bounded. Then we have $\|H_{\bar{g}}k\lambda\| < M$ for $\lambda \in \mathbb{C}$, and using relation (5) together with Lemmas 6 and 2 we obtain

$$M > \|H_{\bar{g}}k\lambda\|^2 = \int_{\mathbb{C}} |g(z) - g(\lambda)|^2 |k_\lambda(z)|^2 e^{-2\phi(z)} dm(z)$$

$$\geq \int_{|z - \lambda| < \alpha \rho(\lambda)} |g(z) - g(\lambda)|^2 |k_\lambda(z)|^2 e^{-2\phi(z)} dm(z)$$

$$\geq \frac{1}{\rho^2(\lambda)} \int_{|z - \lambda| < \alpha \rho(\lambda)} |g(z) - g(\lambda)|^2 dm(z),$$

for $\alpha$ small enough. By the subharmonicity of $|g|$ and the Cauchy formula applied to $g_\lambda(z) = g(z) - g(\lambda)$ we can now conclude

$$|g'(\lambda)\rho(\lambda)| \lesssim \frac{1}{\rho^2(\lambda)} \int_{|z - \lambda| < \alpha \rho(\lambda)} |g(z) - g(\lambda)|^2 dm(z) < M, \quad \lambda \in \mathbb{C}.$$ 

\[ \square \]

**Remark.** The fact that $\rho$ can have at most polynomial decay (see relation (2)) implies that $H_{\bar{g}}$ is bounded only for polynomial symbols of degree smaller than the order of decay of $\rho$. Notice also that if $H_{\bar{g}}$ is bounded, then $\rho$ has to be bounded, since $g$ is a polynomial.

**Theorem 4.** $H_{\bar{g}}$ is compact if and only if $|g'(\lambda)|\rho(\lambda) \to 0$ as $|\lambda| \to \infty$.

**Proof.** Assume first that $|g'(\lambda)|\rho(\lambda) \to 0$ as $|\lambda| \to \infty$. As in relation (6) we have

$$\|H_{\bar{g}}f\|^2 \leq \int_{\mathbb{C}} |g'|^2 \rho^2 |f|^2 e^{-2\phi} dm = \|M_{g'\rho}f\|^2,$$

where $M_{g'\rho} : \mathcal{F}_0^2 \to L^2(e^{-2\phi})$ is given by $M_{g'\rho}f = g'\rho f$. Hence, if $M_{g'\rho}$ is compact, then $H_{\bar{g}}$ is compact. We first show that, for $R > 0$, the truncation of $M_{g'\rho}$ given by

$$M_{g'\rho}^R f = \chi_{\{|z|<R\} \} g'\rho f$$

is compact. To this end, let $\{f_n\}$ be a bounded sequence in $\mathcal{F}_0^2$, i.e. $\|f_n\| < M$. Since pointwise evaluation is bounded, we deduce that $\{f_n\}$ is a normal family and it therefore contains a subsequence $\{f_{n_k}\}$ uniformly convergent on compacts to an entire function $f$. By Fatou’s lemma we obtain $f \in \mathcal{F}_0^2$. Then $f_{n_k} - f \to 0$ uniformly on compacts and $\|f_{n_k} - f\| < 2M$. Hence in order to show that $M_{g'\rho}^R$ is compact, it is enough to show that for any sequence $f_n$ (by abuse of notation) that is bounded in the norm and converges uniformly to zero on compact sets, we have $\|M_{g'\rho}^R f_n\| \to 0$ as $n \to \infty$.

But this is quite easy to see, as

$$\|M_{g'\rho}^R f_n\|^2 \leq \sup_{|z|<R} |f_n|^2 \int_{|z|<R} |g'|^2 \rho^2 e^{-2\phi} dm \to 0,$$
as \( n \to \infty \). Now
\[
\|(M'_{g\rho} - M^R_{g\rho})f\|^2 = \int_{|z|>R} |g'|^2 \rho^2 |f|^2 e^{-2\phi} dm \leq \sup_{|z|>R} |g'|^2 \rho^2 \int_C |f|^2 e^{-2\phi} dm, \quad f \in F^2_\phi,
\]
which shows that \( \|M'_{g\rho} - M^R_{g\rho}\| \to 0 \) as \( R \to \infty \), and therefore \( M'_{g\rho} \) is compact, and consequently \( H_g \) is compact.

Suppose now \( H_g \) is compact. The set \( \{k_\lambda\}_{\lambda \in \mathbb{C}} \) is bounded in \( F^2_\phi \). By compactness it follows that the set \( \{H_gk_\lambda\}_{\lambda \in \mathbb{C}} \) is relatively compact in \( L^2(e^{-2\phi}) \). Then by the Riesz-Tamarkin compactness theorem (see [3]) we have
\[
\lim_{R \to \infty} \int_{|z|>R} |H_gk_\lambda|^2 e^{-2\phi} dm = 0,
\]
uniformly in \( \lambda \). Since \( H_g \) is bounded, we have \( B := \sup_{\lambda} \rho(\zeta) < \infty \). For \( |\lambda| > R + B \), the inclusion \( \{|z-\lambda| \leq \rho(\lambda)\} \subset \{|z| > R\} \) holds, and then for \( \alpha > 0 \) sufficiently small we have by Lemma 6
\[
\int_{|z|>R} |H_gk_\lambda|^2 e^{-2\phi} dm \lesssim \int_{|z|<\alpha \rho(\lambda)} |g(z) - g(\lambda)|^2 |k_\lambda(z)|^2 e^{-2\phi(z)} dm(z)
\]
\[
\lesssim \frac{1}{\rho^2(\lambda)} \int_{|z-\lambda|<\alpha \rho(\lambda)} |g(z) - g(\lambda)|^2 dm(z)
\]
\[
\lesssim \frac{\rho^2(\lambda)}{\rho^2(\lambda)} |g'(\lambda)|^2,
\]
where the last step above follows again by the Cauchy formula and the subharmonicity of \( |g| \). This shows that
\[
\lim_{|\lambda| \to \infty} |g'(\lambda)| \rho(\lambda) = 0.
\]
From this we deduce
\[ \int \|TK(\cdot, \lambda)\|^2 e^{-2\phi(\lambda)} dm(\lambda) = \int \sum_n \lambda_n^2 |e_n(\lambda)|^2 e^{-2\phi(\lambda)} dm(\lambda) = \sum_n \lambda_n^2. \]

Hence
\[ \int \|Tk_\lambda\|^2 \frac{dm(\lambda)}{\rho^2(\lambda)} \sim \int \|TK(\cdot, \lambda)\|^2 e^{-2\phi(\lambda)} dm(\lambda) = \|T\|^2_{S^2}. \]

For \( p = \infty \), we have
\[ \sup_\lambda \|Tk_\lambda\| \leq \|T\|_{S^\infty}. \]

Then (8) follows by interpolation.

**Theorem 5.** Suppose \( H_\delta \) is bounded. Then \( H_\delta \in S^p \) with \( p > 2 \) if and only if \( g'\rho \in L^p(1/\rho^2) \). Moreover, \( H_\delta \) fails to be Hilbert-Schmidt, unless \( g \) is constant.

**Proof.** Suppose \( H_\delta \in S^p \) with \( p \geq 2 \). Then by (8) and using arguments similar to those above we have
\[ \infty > \int \|H_\delta k_\lambda\|^2 \frac{dm(\lambda)}{\rho^2(\lambda)} = \int \left( \int |g(z) - g(\lambda)|^2 |k_\lambda(z)|^2 e^{-2\phi(z)} dm(z) \right)^{p/2} \frac{dm(\lambda)}{\rho^2(\lambda)} \]
\[ \gtrsim \int \left( \int_{|z - \lambda| < \alpha(\lambda)} |g(z) - g(\lambda)|^2 |k_\lambda(z)|^2 e^{-2\phi(z)} dm(z) \right)^{p/2} \frac{dm(\lambda)}{\rho^2(\lambda)} \]
\[ \gtrsim \int |g'(\lambda)| \rho(\lambda) |^p \frac{dm(\lambda)}{\rho^2(\lambda)}, \]

for \( \alpha \) small enough. With this the necessity is proven. In particular, the above relation shows that \( H_\delta \) cannot be Hilbert-Schmidt for nonconstant anti-analytic symbols.

To prove the sufficiency, assume \( g'\rho \in L^p(1/\rho^2) \). Then a subharmonicity argument shows that \( |g'(\lambda)| \rho(\lambda) \to 0 \) as \( |\lambda| \to \infty \). As in the proof of Theorem 4 we have
\[ \|H_\delta f\| \lesssim \|M_{g'\rho} f\|, \quad f \in F_\delta^2. \]

Therefore \( M_{g'\rho} \in S^p \) for some \( p > 2 \), implies \( H_\delta \in S^p \). Indeed, this follows from the criterion (see [7]): A linear operator \( S : H_1 \to H_2 \), where \( H_1, H_2 \) are separable Hilbert spaces, belongs to \( S^p, p \geq 2 \), if and only if \( \sum ||S e_n||^p < \infty \), for any orthonormal basis \( \{e_n\} \) of \( H_1 \). We notice that for \( f, h \in F_\delta^2 \) we have
\[ \langle M_{g'\rho}^* M_{g'\rho} f, h \rangle = \langle M_{g'\rho} f, M_{g'\rho} h \rangle = \int f \bar{h} |g'|^2 \rho^2 e^{-2\phi} dm = \langle T_{|g'|^2 \rho^2} f, h \rangle, \]

where \( T_{|g'|^2 \rho^2} \) is the Toeplitz operator on \( F_\delta^2 \) with symbol \( |g'|^2 \rho^2 \). In order to show that \( M_{g'\rho} \in S^p \), we are going to prove that \( T_{|g'|^2 \rho^2} = M_{g'\rho}^* M_{g'\rho} \in S^{p/2} \). Since \( |g'(\lambda)| \rho(\lambda) \to 0 \) as \( |\lambda| \to \infty \), the proof of the sufficiency in Theorem 4 shows that \( M_{g'\rho} \) is compact, and hence \( T_{|g'|^2 \rho^2} \) is compact. Denote \( G = |g'|^2 \rho^2 \) for convenience. The operator \( T_G \) is also positive and self-adjoint, and it is then given by
\[ T_G = \sum_n \lambda_n \langle \cdot, e_n \rangle e_n, \]
where \( \lambda_n \) are the singular numbers of \( T_G \), and \( e_n \) is an orthonormal basis in \( \mathcal{F}_\phi^2 \). Then

\[
\lambda_n = \langle T_G e_n, e_n \rangle = \int_{\mathbb{C}} |e_n|^2 Ge^{-2\phi} \, dm,
\]

and by Jensen’s inequality we get

\[
\lambda_n^{p/2} \leq \int_{\mathbb{C}} G^{p/2} |e_n|^2 e^{-2\phi} \, dm
\]

using the fact that \( |e_n|^2 e^{-2\phi} \, dm \) is a probability measure on \( \mathbb{C} \). Taking into account the fact that \( K(z, \zeta) = \sum e_n(z)e_n(\zeta) \), we can sum up over \( n \) in the previous relation to deduce

\[
\sum_n \lambda_n^{p/2} \leq \sum_n \int_{\mathbb{C}} G^{p/2} |e_n|^2 e^{-2\phi} \, dm
\]

\[
= \int_{\mathbb{C}} G(z)^{p/2} K(z, z)e^{-2\phi(z)} \, dm(z)
\]

\[
\lesssim \int_{\mathbb{C}} G(z)^{p/2} \frac{1}{\rho^2(z)} \, dm(z) < \infty,
\]

by our assumption. Thus \( T_G \in S^{p/2} \), and consequently \( H_\bar{\theta} \in S^p \).

\[\square\]

4. The canonical solution to \( \bar{\partial} \) on \( L^2(e^{-2\phi}) \)

For \( g = z \) in Theorem \[\text{5}\] we obtain that the restriction of the canonical solution operator \( N \) to \( \bar{\partial} \) to the generalized Fock space \( \mathcal{F}_\phi^2 \) is never Hilbert-Schmidt and it belongs to \( S^p \) for \( p > 2 \) if and only if

\[
\int_{\mathbb{C}} \rho^{p-2}(z) \, dm(z) < \infty.
\]

The aim of this section is to show that the condition above is sufficient for \( N \) to belong to \( S^p \), even when defined on the whole of \( L^2(e^{-2\phi}) \).

For the integral kernel \( C(z, \zeta) \) of \( N \), i.e.

\[
Nf(z) = \int_{\mathbb{C}} e^{\phi(z)-\phi(\zeta)} C(z, \zeta) f(\zeta) \, dm(\zeta), \quad f \in L^2(e^{-2\phi}),
\]

the following estimates were obtained in \[\text{18}\]

**Theorem 6.** \[\text{18}\] There exists \( \varepsilon > 0 \) such that

\[
|C(z, \zeta)| \lesssim \begin{cases} |z - \zeta|^{-1}, & |z - \zeta| \leq \rho(z), \\ \rho^{-1}(z) \exp(-d_\phi(z, \zeta)^\varepsilon), & |z - \zeta| \geq \rho(z). \end{cases}
\]

To prove our main result we use these estimates together with a criterion for an integral operator to belong to Schatten classes for \( p \geq 2 \) obtained in \[\text{19}\]. Given a measure
space \((X, \mu)\), let \(G(x, y)\) be a complex-valued measurable function on \(X \times X\) and denote \(G^*(x, y) = G(y, x)\). Consider the mixed normed space 

\[
L^p(L^q) = \left\{ G : \int \left( \int |G(x, y)|^q d\mu(y) \right)^{p/q} d\mu(x) < \infty \right\}
\]

**Theorem 7.**\(^{[19]}\) Let \(p \geq 2\) and let \((X, \mu)\) be as above. If \(G, G^* \in L^p(L^{p'})\), where \(1/p + 1/p' = 1\), then the integral operator with kernel \(G(x, y)\) given by 

\[
T f(x) = \int G(x, y) f(y) d\mu(y), \quad f \in L^2(d\mu),
\]

belongs to \(S^p\).

A first version of the above theorem was proven in \([19]\) (see also \([12]\)) and subsequently improved in \([1]\), where sharper conditions on the kernel \(G\) were given.

**Theorem 8.** The operator \(N\) is never Hilbert-Schmidt. For \(p > 2\), \(N\) belongs to the Schatten class \(S^p\) if and only if \((9)\) holds.

**Proof.** The necessity follows from Theorem 5. It remains to prove the sufficiency. Assume \(\rho\) satisfies \((9)\) for some \(p > 2\). In order to prove that \(N \in S^p\), we want apply Theorem 7. To this end consider the unitary operator \(U : L^2 \to L^2(e^{-2\phi})\) given by 

\[
Uf = f e^{\phi}.
\]

Then \(N \in S^p\) if and only if \(U^*NU \in S^p\). Notice that 

\[
U^*NU f(z) = \int_C C(z, \zeta) f(\zeta) dm(\zeta), \quad f \in L^2.
\]

Now it is enough to show that the kernel \(C(z, \zeta)\) of \(U^*NU\) satisfies the conditions in Theorem 7 and then the conclusion will easily follow. We shall first estimate

\[
\|C\|^p_{L^p(L^{p'})} = \int_C \left( \int_C |C(z, \zeta)|^{p'} dm(\zeta) \right)^{p/p'} dm(z).
\]

**Theorem 6** implies

\[
\int_C |C(z, \zeta)|^{p'} dm(\zeta) \lesssim \int_{|z-\zeta| \leq \rho(z)} \frac{dm(\zeta)}{|z - \zeta|^{p'}} + \int_{|z-\zeta| > \rho(z)} \frac{dm(\zeta)}{\rho(z)^{p'} \exp(p'd_{\phi}(z, \zeta))}
\]

\[
\lesssim \rho(z)^{2-p'} + \int_{|z-\zeta| > \rho(z)} \frac{dm(\zeta)}{\rho(z)^{p'} \exp(d_{\phi}^2(z, \zeta))},
\]

for \(0 < \varepsilon_1 < \varepsilon\). Now for \(|z - \zeta| \leq \rho(z)| or \(|z - \zeta| \leq \rho(\zeta)\) we have \(\rho(z) \sim \rho(\zeta)\) by Lemma 2. On the other hand, for \((z, \zeta) \in \{|z - \zeta| > \rho(z)\} \cap \{|z - \zeta| > \rho(\zeta)\}\), Lemmas 3-4 imply

\[
\frac{\rho(\zeta)^2}{\exp d_{\phi}^2(z, \zeta)} \lesssim \frac{\rho(z)^2}{\exp d_{\phi}^2(z, \zeta)},
\]
for some $\varepsilon_2 > 0$. Using this in (11) we get
\[ \int_{\mathbb{C}} |C(z, \zeta)|^{p'} \, dm(\zeta) \lesssim \rho(z)^{2-p'} + \int_{|z-\zeta|>\rho(z)} \frac{1}{\exp d_{\phi}^{z_0}(z, \zeta)} \, dm(\zeta) \lesssim \rho(z)^{2-p'}, \]
where the last step above follows by Proposition 1 and Lemma 5. Returning to (10) we obtain
\[ \|C\|_{L^p(L^{p'})}^p = \int_{\mathbb{C}} \rho(z)^{(2-p')/p'} \, dm(z) = \int_{\mathbb{C}} \rho(z)^{p-2} \, dm(z) < \infty, \]
by our assumption. It remains to show that $\|C^*\|_{L^p(L^{p'})} < \infty$. Although the estimates are analogous in this case, we include them for the sake of completeness. We have
\[ \|C^*\|_{L^p(L^{p'})}^p = \int_{\mathbb{C}} \left( \int_{\mathbb{C}} |C(z, \zeta)|^{p'} \, dm(z) \right)^{p/p'} \, dm(\zeta). \]
As before, by Theorem 6 and Lemma 2 we get
\[ \int_{\mathbb{C}} |C(z, \zeta)|^{p'} \, dm(z) \lesssim \int_{|z-\zeta|\leq\rho(z)} \frac{dm(z)}{|z-\zeta|^{p'}} + \int_{|z-\zeta|>\rho(z)} \frac{dm(z)}{\rho(z)^{p'} \exp(p'd_{\phi}(z, \zeta))} \lesssim \rho(z)^{2-p'} \left(1 + \int_{|z-\zeta|>\rho(z)} \frac{1}{\exp d_{\phi}^{z_0}(z, \zeta)} \, dm(z) \right), \]
where $c > 0$, and the last step above follows by Lemmas 3-4. By Proposition 1 and Lemma 5 we obtain
\[ \int_{\mathbb{C}} |C(z, \zeta)|^{p'} \, dm(z) \lesssim \rho(z)^{2-p'}, \]
and hence by (12) we get
\[ \|C^*\|_{L^p(L^{p'})}^p \lesssim \int_{\mathbb{C}} \rho(z)^{p-2} \, dm(z) < \infty. \]
With this the proof is complete. \hfill \Box

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