PROCESI BUNDLES AND SYMPLECTIC REFLECTION ALGEBRAS

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Abstract. In this survey we describe an interplay between Procesi bundles on symplectic resolutions of quotient singularities and Symplectic reflection algebras. Procesi bundles were constructed by Haiman and, in a greater generality, by Bezrukavnikov and Kaledin. Symplectic reflection algebras are deformations of skew-group algebras defined in complete generality by Etingof and Ginzburg. We construct and classify Procesi bundles, prove an isomorphism between spherical Symplectic reflection algebras, give a proof of wreath Macdonald positivity and of localization theorems for cyclotomic Rational Cherednik algebras.

1. Introduction

1.1. Procesi bundles: Hilbert scheme case. A Procesi bundle is a vector bundle of rank $n!$ on the Hilbert scheme $\operatorname{Hilb}_n(\mathbb{C}^2)$ whose existence was predicted by Procesi and proved by Haiman, [Hai1]. This bundle was used by Haiman to prove a famous $n!$ conjecture in Combinatorics that, in turn, settles another famous conjecture: Schur positivity of Macdonald polynomials.

1.1.1. $n!$ theorem. Consider the Vandermond determinant $\Delta(x)$, where we write $x$ for $(x_1, \ldots, x_n)$, it is given by $\Delta(x) = \det(x_i^{j-1})_{i,j=1}^n$. Consider the space $\partial \Delta$ spanned by all partial derivatives of $\Delta$. This space is graded and carries an action of the symmetric group $S_n$ (by permuting the variables $x_1, \ldots, x_n$). A deeper fact is that $\dim \partial \Delta = n!$ (and $\partial \Delta \cong \mathbb{C}S_n$ as an $S_n$-module), in fact, $\partial \Delta$ coincides with the space of the $S_n$-harmonic polynomials, i.e., all polynomials annihilated by all elements of $\mathbb{C}[\partial^{\mathbb{S}_n}]$ without constant term.

One can ask if there is a two-variable generalization of that fact. We have several two-variable versions of $\Delta$, one for each Young diagram $\lambda$ with $n$ boxes. Namely, let $(a_1, b_1), \ldots, (a_n, b_n)$ be the coordinates of the boxes in $\lambda$, e.g., $\lambda = (3, 2)$ gives pairs $(0, 0), (1, 0), (2, 0), (0, 1), (1, 1)$.

$$
\begin{array}{cccc}
(0, 0) & (0, 1) & (1, 0) & (2, 0) \\
(1, 1) & & & \\
\end{array}
$$

Then set $\Delta_\lambda(x, y) := \det(x_i^{a_j} y_i^{b_j})_{i,j=1}^n$ so that, for $\lambda = (n)$, we get $\Delta_\lambda(x, y) = \Delta(x)$, for $\lambda = (1^n)$, we get $\Delta_\lambda(x, y) = \Delta(y)$, while, for $\lambda = (2, 1)$, we get $\Delta_\lambda(x, y) = x_1 y_2 + x_2 y_3 + x_3 y_1 - x_2 y_1 - x_3 y_2 - x_1 y_3$.

Theorem 1.1 (Haiman’s $n!$ theorem). The space $\partial \Delta_\lambda$ spanned by the partial derivatives of $\Delta_\lambda$ is isomorphic to $\mathbb{C}S_n$ as an $S_n$-module (where $S_n$ acts by permuting the pairs $(x_1, y_1), \ldots, (x_n, y_n)$) and, in particular, has dimension $n!$.

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This is a beautiful result with an elementary statement and an extremely involved proof, [Hai1].

1.1.2. Macdonald positivity. Before describing some ideas of the proof that are relevant to the present survey, let us describe an application to Macdonald polynomials, particularly important and interesting symmetric polynomials with coefficients in \( \mathbb{Q}(q,t) \). It will be more convenient for us to speak about representations of \( \mathfrak{S}_n \) rather than about symmetric polynomials (they are related via taking the Frobenius character) and to deal with Haiman’s modified Macdonald polynomials.

**Definition 1.2.** The modified Macdonald polynomial \( \tilde{H}_\lambda \) is the Frobenius character of a bigraded \( \mathfrak{S}_n \)-module \( P_\lambda := \bigoplus_{i,j \in \mathbb{Z}} P_\lambda[i,j] \) subject to the following three conditions

(a) The class of \( P_\lambda \otimes \sum_{i=0}^n (-1)^i \bigwedge^i \mathbb{C}^n[1,0] \) is expressed via the irreducibles \( V_\mu \) with \( \mu \geq \lambda \) (in the \( K_0 \) of the bigraded \( \mathfrak{S}_n \)-modules).

(b) \( P_\lambda \otimes \sum_{i=0}^n (-1)^i \bigwedge^i \mathbb{C}^n[0,1] \) is expressed via the irreducibles \( V_\mu \) with \( \mu \geq \lambda' \).

(c) The trivial module \( \mathbb{C}^{n} \) occurs in \( P_\lambda \) once and in degree \( (0,0) \).

Here \( \mu \geq \lambda \) is the usual dominance order on the set of Young diagrams (meaning that \( \sum_{i=1}^k \mu_i \geq \sum_{i=1}^k \lambda_i \) and \( \lambda' \) denotes the transpose of \( \lambda \).

It is not clear from this definition that the representations \( P_\lambda \) exist (the statement on the level of \( K_0 \) is easier but also non-trivial, this was known before Haiman’s work).

**Theorem 1.3** (Haiman’s Macdonald positivity theorem). A bigraded \( \mathfrak{S}_n \)-module \( P_\lambda \) exists (and is unique) for any \( \lambda \). Moreover, \( P_\lambda \) coincides with \( \partial \Delta_\lambda \), where \( \partial \Delta_\lambda \) is is given the structure of a bigraded \( \mathfrak{S}_n \)-module as the quotient of \( \mathbb{C}[\partial_x, \partial_y] \) (via \( f \mapsto f \Delta_\lambda \)).

1.1.3. Hilbert schemes and Procesi bundles. The proofs of the two theorems above given in [Hai1] are based on the geometry of the Hilbert schemes \( \text{Hilb}_n(\mathbb{C}^2) \) of points on \( \mathbb{C}^2 \). A basic reference for Hilbert schemes of points on smooth surfaces is [Nak4]. As a set, \( \text{Hilb}_n(\mathbb{C}^2) \) consists of the ideals \( J \subset \mathbb{C}[x,y] \) of codimension \( n \). It turns out that \( \text{Hilb}_n(\mathbb{C}^2) \) is a smooth algebraic variety of dimension \( 2n \). It admits a morphism (called the Hilbert-Chow map) to the variety \( \text{Sym}_n(\mathbb{C}^2) \) of the unordered \( n \)-tuples of points in \( \mathbb{C}^2 \) to an ideal \( J \) one assigns its support, where points are counted with multiplicities. Of course, \( \text{Sym}_n(\mathbb{C}^2) \) is nothing else but the quotient space \( (\mathbb{C}^2)^{\otimes n}/\mathfrak{S}_n \), the affine algebraic variety whose algebra of functions is the invariant algebra \( \mathbb{C}[x,y]^{\otimes n} \). The Hilbert-Chow map is a resolution of singularities.

Note that the two-dimensional torus \( (\mathbb{C}^*)^2 \) acts on \( \text{Hilb}_n(\mathbb{C}^2) \) and on \( \text{Sym}_n(\mathbb{C}^2) \), the action is induced from the following action on \( \mathbb{C}^2 \): \( (t,s),(a,b) := (t^{-1}a, s^{-1}b) \). The fixed points of this action on \( \text{Hilb}_n(\mathbb{C}^2) \) correspond to the monomial ideals (=ideals generated by monomials) in \( \mathbb{C}[x,y] \), they are in a natural one-to-one correspondence with Young diagrams (as before we fill a Young diagram with monomials and take the ideal spanned by all monomials that do not appear in the diagram). Let \( z_\lambda \) denote the fixed point corresponding to a Young seminar \( \lambda \).

Following Haiman, consider the isospectral Hilbert scheme \( I_n \), the reduced Cartesian product \( \mathbb{C}^{2n} \times_{\text{Sym}_n(\mathbb{C}^2)} \text{Hilb}_n(\mathbb{C}^2) \), let \( \eta : I_n \to \text{Hilb}_n(\mathbb{C}^2) \) be the natural morphism. It is finite of generic degree \( n! \). The main technical result of Haiman, [Hai1], is that \( I_n \) is Cohen-Macaulay and Gorenstein. So \( \mathcal{P} := \eta_* \mathcal{O}_{I_n} \) is a rank \( n! \) vector bundle on \( \text{Hilb}_n(\mathbb{C}^2) \) (the Procesi bundle). By the construction, each fiber of this bundle carries an algebra structure that is a quotient of \( \mathbb{C}[x,y] \). Let us write \( \mathcal{P}_\lambda \) for the fiber of \( \mathcal{P} \) in \( z_\lambda \), this is an
algebra that carries a natural bi-grading because the bundle $\mathcal{P}$ is $(\mathbb{C}^\times)^2$-equivariant by the construction. On the other hand, $\partial \Delta_\lambda$ is a quotient of $\mathbb{C}[\partial x, \partial y]$ by an ideal and so is also an algebra. The latter algebra is bigraded. Haiman has shown that $\mathcal{P}_\lambda \cong \partial \Delta_\lambda$, an isomorphism of bigraded algebras. This finishes the proof of Theorem 1.1.

Let us proceed to Theorem 1.3. The class in (a) of Definition 1.2 is that of the fiber at $z_\lambda$ of the Koszul complex

$$\mathcal{P} \leftarrow \mathfrak{h}_x \otimes \mathcal{P} \leftarrow \Lambda^2 \mathfrak{h}_x \otimes \mathcal{P} \leftarrow \ldots,$$

where $\mathfrak{h}_x$ is the span of $x_1, \ldots, x_n$ viewed as endomorphisms of $\mathcal{P}$. Haiman has shown that $I_n$ is flat over $\text{Spec}(\mathbb{C}[x])$ (with morphism $I_n \rightarrow \text{Spec}(\mathbb{C}[x, y]) \rightarrow \text{Spec}(\mathbb{C}[x])$). It follows that (1) is a resolution of $\mathcal{P}/\mathfrak{h}_x \mathcal{P}$. Now (a) follows from the claim that, for any Young diagram $\lambda$, the support of the isotypic $V_\lambda$-component in $\mathcal{P}/\mathfrak{h}_x \mathcal{P}$ contains only points $z_\mu$ with $\mu \leq \lambda$. This was checked by Haiman. Part (b) is analogous, while (c) follows directly from the construction.

There are several other proofs of Theorem 1.3 available. Two of them use the geometry of Hilbert schemes and Procesi bundle, [G4, BF]. We will discuss (a somewhat modified) approach from [BF] in detail in Section 5.

1.2. Quotient singularities and symplectic resolutions.

1.2.1. Setting. Let $V$ be a finite dimensional vector space over $\mathbb{C}$ equipped with a symplectic form $\Omega \in \bigwedge^2 V^\ast$. Let $\Gamma$ be a finite subgroup of $\text{Sp}(V)$. The invariant algebra $\mathbb{C}[V]^\Gamma$ is a graded Poisson algebra (as a subalgebra of $\mathbb{C}[V]$) and the corresponding quotient $V/\Gamma = \text{Spec}(\mathbb{C}[V]^\Gamma)$ is a singular Poisson affine variety that comes with a $\mathbb{C}^\times$-action induced from the action on $V$ by dilations: $t.v := t^{-1}v$.

1.2.2. Symplectic resolutions. One can ask if there is a resolution of singularities of $V/\Gamma$ that is nicely compatible with the Poisson structure (and with the $\mathbb{C}^\times$-action). This compatibility is formalized in the notion of a (conical) symplectic resolution.

Definition 1.4. Let $X_0$ be a singular normal affine Poisson variety such that the regular locus $X_0^{\text{reg}}$ is symplectic. We say that a variety $X$ equipped with a morphism $\rho : X \rightarrow X_0$ is a symplectic resolution of $X_0$ if $X$ is symplectic (with form $\omega$), $\rho$ is a resolution of singularities and $\rho : \rho^{-1}(X_0^{\text{reg}}) \rightarrow X_0^{\text{reg}}$ is a symplectomorphism.

Definition 1.5. Further, suppose that $X_0$ is equipped with a $\mathbb{C}^\times$-action such that

- the corresponding grading $\mathbb{C}[X_0] = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}[X_0]_i$ is positive, meaning that $\mathbb{C}[X_0]_i = \{0\}$ for $i < 0$ and $\mathbb{C}[X_0]_0 = \mathbb{C}$,
- and the Poisson bracket on $\mathbb{C}[X_0]$ has degree $-d$ for some fixed $d \in \mathbb{Z}_{>0}$: $\{\mathbb{C}[X_0]_i, \mathbb{C}[X_0]_j\} \subset \mathbb{C}[X_0]_{i+j-d}$ for all $i, j$.

We say that a symplectic resolution $X$ is conical if it is equipped with a $\mathbb{C}^\times$-action making $\rho$ equivariant.

The variety $X_0 = V/\Gamma$ is normal and carries a natural $\mathbb{C}^\times$-action (by dilations) as in Definition 1.3 with $d = 2$. Also note that the $\mathbb{C}^\times$-action on $X$ automatically satisfies $t.\omega = t^{-d}\omega$. Finally, note that, under assumptions of Definition 1.4, we have $\mathbb{C}[X] = \mathbb{C}[X_0]$. 

1.2.3. Symplectic resolutions for quotient singularities. In the previous subsection, we have already seen an example of \((V, \Gamma)\) such that \(V/\Gamma\) admits a conical symplectic resolution: \(V = (\mathbb{C}^2)^{\oplus n}, \Gamma = \mathfrak{S}_n,\) in this case one can take \(X = \text{Hilb}_n(\mathbb{C}^2)\) together with the Hilbert-Chow morphism, see [Nak4, Section 1].

There are other examples as well. Let \(\Gamma_1\) be a finite subgroup of \(\text{SL}_2(\mathbb{C})\), such subgroups are classified (up to conjugacy) by Dynkin diagrams of ADE types. Say, the cyclic subgroup \(\mathbb{Z}/(\ell + 1)\mathbb{Z}\) (embedded into \(\text{SL}_2(\mathbb{C})\) via \(n \mapsto \text{diag}(\eta^n, \eta^{-n})\) with \(\eta := \exp(2\pi \sqrt{-1}/(\ell + 1))\)) corresponds to the diagram \(A_\ell\). The quotient singularity \(\mathbb{C}^2/\Gamma_1\) admits a distinguished minimal resolution to be denoted by \(\widehat{\mathbb{C}^2}/\Gamma_1\). This resolution is conical symplectic, see, e.g., [Nak4, Section 4.1].

The examples of \(\mathfrak{S}_n\) and \(\Gamma_1\) can be “joined” together. Consider the group \(\Gamma_n := \mathfrak{S}_n \times \Gamma_1^n\). It acts on \(V_n := (\mathbb{C}^2)^{\oplus n}\): the symmetric group permutes the summands, while each copy of \(\Gamma_1\) acts on its own summand. The quotient singularities \(V_n/\Gamma_n\) admit symplectic resolutions. For example, one can take \(X := \text{Hilb}_n(\widehat{\mathbb{C}^2}/\Gamma_1)\). But there are other conical symplectic resolutions of \(V_n/\Gamma_n\), conjecturally, they are all constructed as Nakajima quiver varieties, we will recall the construction of these varieties in 3.1.4.

To finish this section, let us point out that, presently, two more pairs \((V, \Gamma)\) such that \(V/\Gamma\) admits a symplectic resolutions are known, see [Be, BS]. In this paper, we are not interested in these cases.

1.3. Procesi bundles: general case.

1.3.1. Smash-product algebra. One nice feature of quotient singularities \(V/\Gamma\) is that they always have a nice resolution of singularities which is, however, noncommutative algebraic rather than algebro-geometric: the smash-product algebra \(\mathbb{C}[V] \# \Gamma\). As a vector space, \(\mathbb{C}[V] \# \Gamma\) is the tensor product \(\mathbb{C}[V] \otimes \mathbb{C}[\Gamma]\), and the product on \(\mathbb{C}[V] \# \Gamma\) is given by

\[(f_1 \otimes \gamma_1) \cdot (f_2 \otimes \gamma_2) = f_1 \gamma_1(f_2) \otimes \gamma_1 \gamma_2, f_1, f_2 \in \mathbb{C}[V], \gamma_1, \gamma_2 \in \Gamma,\]

where \(\gamma_1(f_2)\) denotes the image of \(f_2\) under the action of \(\gamma_1\). The definition is arranged in such a way that a \(\mathbb{C}[V] \# \Gamma\)-module is the same thing as a \(\Gamma\)-equivariant \(\mathbb{C}[V]\)-module. Note that the algebra \(\mathbb{C}[V] \# \Gamma\) is graded, for a homogeneous element \(f\) of degree \(n\), the degree of \(f \otimes \gamma\) is \(n\).

Let us explain what we mean when we say that \(\mathbb{C}[V] \# \Gamma\) is a resolution of singularities of \(V/\Gamma\). Note that \(\mathbb{C}[V]^\Gamma\) can be recovered from \(\mathbb{C}[V] \# \Gamma\) in two different but related ways. First, we have an embedding \(\mathbb{C}[V]^\Gamma \hookrightarrow \mathbb{C}[V] \# \Gamma\) given by \(f \mapsto f \otimes 1\). The image lies in the center (this is easy) and actually coincides with the center (a bit harder). Second, consider the element \(e \in \mathbb{C}[\Gamma], e := |\Gamma|^{-1} \sum_{\gamma \in \Gamma} \gamma\), the averaging idempotent. Consider the subspace \(e(\mathbb{C}[V] \# \Gamma)e \subset \mathbb{C}[V] \# \Gamma\). It is obviously closed under multiplication, and \(e\) is a unit with respect to the multiplication there. So \(e(\mathbb{C}[V] \# \Gamma)e\) is an algebra, to be called the spherical subalgebra of \(\mathbb{C}[V] \# \Gamma\). It is isomorphic to \(\mathbb{C}[V]^\Gamma\), an isomorphism is given by \(f \mapsto ef\).

Thanks to the realization of \(\mathbb{C}[V]^\Gamma\) as a spherical subalgebra, we can consider the functor \(M \mapsto eM : \mathbb{C}[V] \# \Gamma\)-mod \(\rightarrow \mathbb{C}[V]^\Gamma\)-mod (an analog of the morphism \(\rho\)). Note that the algebra \(\mathbb{C}[V] \# \Gamma\) has finite homological dimension (because \(\mathbb{C}[V]\) does) and so is “smooth”. The algebra \(\mathbb{C}[V] \# \Gamma\) is finite over \(\mathbb{C}[V]^\Gamma\), which can be thought as an analog of \(\rho\) being proper. Also, after replacing \(\mathbb{C}[V] \# \Gamma, \mathbb{C}[V]^\Gamma\) with sheaves \(\mathcal{O}_{V^{reg}} \# \Gamma, \mathcal{O}_{V^{reg}/\Gamma}\) on \(V^{reg}/\Gamma\),...
where
\( \text{V}^{\text{reg}} := \{ v \in V \mid \Gamma_v = \{1\} \}, \)
the functor \( M \mapsto eM \) becomes a category equivalence. This is an analog of \( \rho \) being birational.

1.3.2. **Procesi bundle: an axiomatic description.** Now let \( X \) be a conical symplectic resolution of \( V/\Gamma \). We want to relate \( X \) to \( \mathbb{C}[V]#\Gamma \).

**Definition 1.6.** A Procesi bundle \( P \) on \( X \) is a \( \mathbb{C}^* \)-equivariant vector bundle on \( X \) together with an isomorphism \( \text{End}_{\mathcal{O}_X}(P) \overset{\sim}{\rightarrow} \mathbb{C}[V]#\Gamma \) of graded algebras over \( \mathbb{C}[X] = \mathbb{C}[V]^\Gamma \) such that \( \text{Ext}^i(P, P) = 0 \) for \( i > 0 \).

Note that the isomorphism \( \text{End}_{\mathcal{O}_X}(P) \overset{\sim}{\rightarrow} \mathbb{C}[V]#\Gamma \) gives a fiberwise \( \Gamma \)-action on \( P \). The invariant sheaf \( eP \) is a vector bundle of rank 1. We say that \( P \) is normalized if \( eP = \mathcal{O}_X \) (as a \( \mathbb{C}^* \)-equivariant vector bundle). We can normalize an arbitrary Procesi bundle by tensoring it with \( (eP)^* \). Below we only consider normalized Procesi bundles.

In particular, Haiman’s Procesi bundle on \( X = \text{Hilb}_n(\mathbb{C}^2) \) fits the definition, this is essentially a part of [Hai2, Theorem 5.3.2] (and is normalized). The existence of a Procesi bundle on a general \( X \) was proved by Bezrukavnikov and Kaledin in [BK2]. We will see that the number of different Procesi bundles on a symplectic resolution of \( \mathbb{C}^{2n}/\Gamma_n \) equals \( 2|W| \) if \( n > 1 \), where \( W \) is the Weyl group of the Dynkin diagram corresponding to \( \Gamma_1 \). For example, when \( \Gamma_1 = \mathbb{Z}/\ell\mathbb{Z} \), we get \( W = \mathcal{S}_\ell \) and so the number of different Procesi bundles is \( 2\ell! \).

1.4. **Symplectic reflection algebras.**

1.4.1. **Definition.** Symplectic reflection algebras were introduced by Etingof and Ginzburg in [EG]. Those are filtered deformations of \( \mathbb{C}[V]#\Gamma \).

By a symplectic reflection in \( \Gamma \) one means an element \( \gamma \) with \( \text{rk}(\gamma - 1_V) = 2 \). Note that the rank has to be even: the image of \( \gamma - 1_V \) is a symplectic subspace of \( V \). By \( S \) we denote the set of all symplectic reflections in \( \Gamma \), it is a union of conjugacy classes, \( S = \sqcup_{i=1}^r S_i \). Now pick \( t \in \mathbb{C} \) and \( c = (c_1, \ldots, c_r) \in \mathbb{C}^r \). We define the algebra \( H_{t,c} \) as the quotient of \( T(V)/\Gamma \) by the relations
\[ u \otimes v - v \otimes u = t\Omega(u, v) + \sum_{i=1}^r c_i \sum_{s \in S_i} \Omega(\pi_s u, \pi_s v) s, u, v \in V. \]

Here we write \( \pi_s \) for the projection \( V \twoheadrightarrow \text{im}(s - 1_V) \) corresponding to the decomposition \( V = \text{im}(s - 1_V) \oplus \ker(s - 1_V) \).

As Etingof and Ginzburg checked in [EG], the algebra \( H_{t,c} \) satisfies the PBW property: if we filter \( H_{t,c} \) by setting \( \deg \Gamma = 0 \), \( \deg V = 1 \), then \( \text{gr} H_{t,c} = \mathbb{C}[V]#\Gamma \) (here we identify \( V \) with \( V^* \) by means of \( \Omega \) so that \( \mathbb{C}[V] \cong S(V) \)). Moreover, we will see that \( H_{t,c} \) satisfies a certain universality property so this deformation of \( \mathbb{C}[V]#\Gamma \) is forced on us, in a way.

1.4.2. **Connection to Procesi bundles.** It may seem that Symplectic reflection algebras and Procesi bundles are not related. This is not so. It turns out that the algebra \( H_{t,c} \) is the endomorphism algebra of a suitable understood deformation of a Procesi bundle \( P \). This connection is beneficial for studying both. On the Procesi side, it allows to classify Procesi bundles, [L4], and prove the Macdonald positivity in the case of groups \( \Gamma_n \) with \( \Gamma_1 = \mathbb{Z}/\ell\mathbb{Z} \), [BF]. On the symplectic reflection side, it allows to relate the algebras \( H_{t,c} \) to
quantized Nakajima quiver varieties, see [EGGO, L3] and references therein, which then allows to study the representation theory of $H_{t,c}(BL)$ and to prove versions of Beilinson-Bernstein localization theorems, [GL, L5]. Connections between Procesi bundles and Symplectic reflection algebras is a subject of this survey.

1.5. **Notation and conventions.** Let us list some notation used in the paper.

**Quantizations and deformations.** We use the following conventions for quantizations. For a Poisson algebra $A$, we write $A$ for its formal quantization. When $A$ is graded, we write $A$ for its filtered quantization. The notation $D$ is usually used for a formal quantization of a variety, while $\mathcal{D}$ usually denotes a filtered quantization.

When $X$ is a conical symplectic resolution of singularities, we write $\tilde{X}$ for its universal conical deformation (over $H_{DR}^2(X)$) and $\tilde{D}$ stands for the canonical quantization of $\tilde{X}$.

**Symplectic reflection groups and algebras.** We write $\Gamma_1$ for a finite subgroup of $\text{SL}_2(\mathbb{C})$ and $\Gamma_n$ for the semidirect product $S_n \ltimes \Gamma_1$. This semi-direct product acts on $V_n := \mathbb{C}^{2n}$. In the case when $\Gamma_1 = \{1\}$, we usually write $V_n$ for $T^*\mathbb{C}^{n-1}$, where $\mathbb{C}^{n-1}$ is the reflection representation of $S_n$.

For a group $\Gamma$ acting on a space $V$ by linear symplectomorphisms, by $S$ we denote the set of symplectic reflections in $\Gamma$. By $e$ we denote the averaging idempotent of $\Gamma$. By $H$ we denote the universal symplectic reflection algebra of $(V, \Gamma)$. Its specializations are denoted by $H_{t,c}$.

**Quotients and reductions.** Let $G$ be a group acting on a variety $X$. If $G$ is finite and $X$ is quasi-projective, then the quotient is denoted by $X/G$ (note that this quotient may fail to exist when $X$ is not quasi-projective). If $G$ is reductive and $X$ is affine, then $X//G$ stands for the categorical quotient. A GIT quotient of $X$ under the $G$-action with stability condition $\theta$ is denoted by $X//\theta G$.

When $X$ is Poisson, and the $G$-action is Hamiltonian, we write $X///\lambda G$ for $\mu^{-1}(\lambda)//G$ and $X///\theta G$ for $\mu^{-1}(\lambda)//\theta G$.

**Miscellaneous notation.**

- $\hat{\otimes}$ the completed tensor product of complete topological vector spaces/ modules.
- $(a_1, \ldots, a_k)$ the two-sided ideal in an associative algebra generated by elements $a_1, \ldots, a_k$.
- $A^\wedge$ the completion of a commutative (or “almost commutative”) algebra $A$ with respect to the maximal ideal of a point $\chi \in \text{Spec}(A)$.
- $A(V)$ the Weyl algebra of a symplectic vector space $V$.
- $D(X)$ the algebra of differential operators on a smooth variety $X$.
- $\mathbb{F}_q$ the finite field with $q$ elements.
- $\text{gr} A$ the associated graded vector space of a filtered vector space $A$.
- $H_{DR}^i(X)$ the $i$th De Rham cohomology of $X$ with coefficients in $\mathbb{C}$.
- $\mathcal{O}_X$ the structure sheaf of a scheme $X$.
- $R_{\mathbb{C}}(A) := \bigoplus_{\ell \in \mathbb{Z}} \mathbb{C}[h^\ell] A_{\ell i}$ the Rees $\mathbb{C}[h]$-module of a filtered vector space $A$.
- $\mathcal{S}_n$ the symmetric group in $n$ letters.
- $S(V)$ the symmetric algebra of a vector space $V$.
- $\text{Sp}(V)$ the symplectic linear group of a symplectic vector space $V$.
- $\Gamma(S)$ global sections of a sheaf $S$. 
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2. Quantizations

In this section we review the quantization formalism. In Section 2.1 we discuss quantizations of Poisson algebras. There are two formalisms here: filtered quantizations and formal quantizations. We introduce both of them, discuss a relation between them and then give examples.

Then, in Section 2.2, we proceed to quantizations of non-necessarily affine Poisson algebraic varieties. Here we quantize the structure sheaf. We explain that to quantize an affine variety is the same thing as to quantize its algebra of functions. Then we mention a theorem of Bezrukavnikov and Kaledin classifying quantizations of symplectic varieties under certain cohomology vanishing conditions.

After that we proceed to modules over quantizations. We define coherent and quasicoherent sheaves of modules and outline their basic properties. For a coherent sheaf of modules, we define its support. Then we discuss global section and localization functors and their derived analogs.

We finish this system by discussing Frobenius constant quantizations in positive characteristic.

2.1. Algebra level. Here we will review formalisms of quantizations of Poisson algebras. Let \( A \) be a Poisson algebra (commutative, associative and with a unit).

2.1.1. Formal quantizations. First, let us discuss formal quantizations. By a formal quantization of \( A \) we mean an associative \( \mathbb{C}[\hbar] \)-algebra \( A_\hbar \) equipped with an algebra isomorphism \( \pi : A_\hbar / (\hbar) \xrightarrow{\sim} A \) such that

(i) \( A_\hbar \cong A[[\hbar]] \) as a \( \mathbb{C}[\hbar] \)-module and this isomorphism intertwines \( \pi \) and the natural projection \( A[[\hbar]] \to A \).

(ii) We have \( \pi(\frac{1}{\hbar}[a,b]) \equiv \{\pi(a),\pi(b)\} \) (note that \( \pi([a,b]) = \pi(a),\pi(b) = 0 \) and so \( \frac{1}{\hbar}[a,b] \) makes sense).

Condition (i) can be stated equivalently as follows: \( A_\hbar \) is flat over \( \mathbb{C}[\hbar] \) and is complete and separated in the \( \hbar \)-adic topology.

2.1.2. Filtered quantizations. Second, we will need the formalism of filtered quantizations. Suppose that \( A \) is equipped with an algebra grading, \( A = \bigoplus_{i \in \mathbb{Z}} A_i \), that is compatible with \( \{\cdot,\cdot\} \) in the following way: \( \{A_i, A_j\} \subseteq A_{i+j-1} \).

First, we consider the case when the grading on \( A \) is non-negative: \( A_i = \{0\} \) for \( i < 0 \). Then, by a filtered quantization of \( A \) one means a \( \mathbb{Z}_{\geq 0} \)-filtered algebra \( A = \bigcup_{i \geq 0} A_{\leq i} \) together with a graded algebra isomorphism \( \pi : \text{gr} A \xrightarrow{\sim} A \) such that, for \( a \in A_{\leq i}, b \in A_{\leq j} \), one has \( \{\pi(a + A_{\leq i-1}),\pi(b + A_{\leq j-1})\} = \pi([a,b] + A_{\leq i+j-2}) \) (note that \( [a,b] \in A_{\leq i+j-1} \) because \( \text{gr} A \) is commutative).

2.1.3. Relation between the two formalisms. Let us explain a connection between the two formalisms (that will also motivate the definition of a filtered quantization in the case when the grading on \( A \) has negative components). Take a filtered quantization \( A_\hbar \) of \( A \). Form the Rees algebra \( R_\hbar(A) := \bigoplus_{i \geq 0} A_{\leq i} \hbar^i \) that is equipped with a graded algebra structure as
a subalgebra in $\mathcal{A}[\hbar]$. We have natural identifications $R_\hbar(\mathcal{A})/(\hbar) \cong \mathcal{A}$, $R_\hbar(\mathcal{A})/(\hbar-1) \cong \mathcal{A}$. The $\hbar$-adic completion $R_\hbar(\mathcal{A})^{\hbar} := \lim_{\leftarrow n} R_\hbar(\mathcal{A})/(\hbar^n)$ satisfies (i) and (ii) and so is a formal quantization of $\mathcal{A}$. Moreover, it comes with a $\mathbb{C}^\times$-action by algebra automorphisms such that $t.\hbar = \hbar t, t \in \mathbb{C}^\times$: the action is given by $t. \sum_{i=0}^{+\infty} a_i \hbar^i := \sum_{i=0}^{+\infty} t^i a_i \hbar^i$. Clearly, the induced action on $\mathcal{A}$ coincides with the action coming from the grading. Conversely, suppose we have a formal quantization $\mathcal{A}_h$ of $\mathcal{A}$ equipped with a $\mathbb{C}^\times$-action by algebra automorphisms such that $t.\hbar = \hbar t$ and the epimorphism $\pi$ is $\mathbb{C}^\times$-equivariant. Assume, further, that the action is pro-rational meaning that it is rational on all quotients $\mathcal{A}_h/(\hbar^n)$. Consider the subspace $\mathcal{A}_{h,fin} \subset \mathcal{A}_h$ consisting of all $\mathbb{C}^\times$-finite elements, i.e., those elements that are contained in some finite dimensional $\mathbb{C}^\times$-stable subspace. This is a $\mathbb{C}^\times$-stable $\mathbb{C}[\hbar]$-subalgebra of $\mathcal{A}_h$. It is easy to see that $\pi$ induces an isomorphism $\mathcal{A}_{h,fin}/(\hbar) \cong \mathcal{A}$. Then $\mathcal{A} := \mathcal{A}_{h,fin}/(\hbar - 1)$ is a filtered quantization.

2.1.4. Filtered quantizations, general case. Let us proceed to the case when the grading on $\mathcal{A}$ is not necessarily non-negative. We can still consider a formal quantization $\mathcal{A}_h$ with a $\mathbb{C}^\times$-action as above, the subalgebra $\mathcal{A}_{h,fin} \subset \mathcal{A}_h$ and the quotient $\mathcal{A} := \mathcal{A}_{h,fin}/(\hbar - 1)$. It is still a filtered quantization in the sense explained above (with the difference that now we have a $\mathbb{Z}$-filtration rather than a $\mathbb{Z}_{\geq 0}$-filtration) but, moreover, the filtration on $\mathcal{A}$ has a special property: it is complete and separated meaning that a natural homomorphism $\mathcal{A} \to \varprojlim_{\leftarrow n} \mathcal{A} / \mathcal{A}_{\leq n}$ is an isomorphism. By a filtered quantization of $\mathcal{A}$ we now mean a $\mathbb{Z}$-filtered algebra $\mathcal{A}$, where the filtration is complete and separated, together with an isomorphism $\pi : \text{gr} \mathcal{A} \xrightarrow{\sim} \mathcal{A}$ of graded algebras such that $\{ \pi(a + \mathcal{A}_{\leq i-1}), \pi(b + \mathcal{A}_{\leq j-1}) \} = \pi(\{ [a, b] + \mathcal{A}_{\leq i+j-2} \}).$

Our conclusion is that the following two formalisms are equivalent: filtered quantizations and formal quantizations with a $\mathbb{C}^\times$-action. To get from a filtered quantization $\mathcal{A}$ to a formal one, one takes $R_\hbar(\mathcal{A})^{\hbar}$. To get from a formal quantization $\mathcal{A}_h$ to a filtered one, one takes $\mathcal{A}_{h,fin}/(\hbar - 1)$.

2.1.5. Examples. Let us proceed to examples. In examples, one usually gets $\mathbb{Z}_{\geq 0}$-filtered quantizations, more general $\mathbb{Z}$-filtered or formal quantizations arise in various constructions (such as (micro)localization or completion).

Example 2.1. Let $\mathfrak{g}$ be a Lie algebra. Then, by the PBW theorem, the universal enveloping algebra $U(\mathfrak{g})$ is a filtered quantization of $S(\mathfrak{g})$.

Example 2.2. Let $Y$ be an affine algebraic variety. The algebra $D(Y)$ of linear differential operators on $Y$ (together with the filtration by the order of differential operators) is a filtered quantization of $\mathbb{C}[T^*Y]$.

Remark 2.3. Often one needs to deal with a more general compatibility condition between the grading and the bracket: $\{ A_i, A_j \} \subset A_{i+j-d}$ for some fixed $d > 0$. In this case, one can modify the definitions of formal and filtered quantizations. Namely, in the definition of a formal quantization one can require that $[\mathcal{A}_a, \mathcal{A}_b] \subset \hbar^d \mathcal{A}_h$ and $\pi(\frac{1}{\hbar^d}[a, b]) = \{ \pi(a), \pi(b) \}$. The definition of a filtered quantization can be modified similarly.

Example 2.4. Let $V$ be a symplectic vector space and $\Gamma \in \text{Sp}(V)$ be a finite group. Consider $A = S(V)^{\Gamma}$ with Poisson bracket $\{ \cdot, \cdot \}$ restricted from $S(V)$. In the notation of Remark 2.3 $d = 2$. As was essentially checked in [EG], the spherical subalgebra $eH_{1, e}$
(with a filtration restricted from $H_{1,c}$) is a quantization of $S(V)^\Gamma$ for any parameter $c$. When $\Gamma = \{1_V\}$, we recover the usual Weyl algebra, $A(V)$, of $V$.

To check that $eH_{1,c}e$ is a quantization carefully we note that the proof of Theorem 1.6 in loc.cit. shows that the bracket on $S(V)^\Gamma$ coming from the filtered deformation $eH_{1,c}e$ coincides with $a\{,\}$, where $a$ is a nonzero number independent of $c$. Then we notice that for $c = 0$ we get $eH_{1,c}e = A(V)^\Gamma$ and so $a = 1$.

In fact, in the previous example we often can also achieve $d = 1$. Namely, if $-1_V \in \Gamma$, then all degrees in $S(V)^\Gamma$ are even and so we can consider the grading $A = \bigoplus_{i \geq 0} A_i$ with $A_i$ consisting of all homogeneous elements with usual degree $2i$. We introduce a filtration on $eH_{1,c}e$ in a similar way (this filtration is not restricted from $H_{1,c}$). Then we get a filtered quantization according to our original definition. When $\Gamma = \Gamma_n$, we only have $-1_V \notin \Gamma$ if $\Gamma = \mathbb{Z}/(2\mathbb{Z})$ for odd $\ell$. For $\Gamma = \mathbb{Z}/(\ell\mathbb{Z})$ (and any $\ell$), $V$ splits as $\mathfrak{h} \oplus \mathfrak{h}^*$, where $\mathfrak{h} = \mathbb{C}^n$. We can grade $S(V)$ by setting $\deg \mathfrak{h}^* = 0$, $\deg \mathfrak{h} = 1$ and take the induced grading on $S(V)^\Gamma$ and the induced filtration on $H_{1,c}$.

2.2. Sheaf level. Above, we were dealing with Poisson algebras or, basically equivalently, with affine Poisson algebraic varieties. Now we are going to consider general Poisson varieties (or schemes). Recall that by a Poisson variety one means a variety $X$ such that the structure sheaf $O_X$ is equipped with a Poisson bracket (meaning that all algebras of sections are Poisson and the restriction homomorphisms respect the Poisson brackets). In this case a quantization of $X$ will be a (formal or filtered) quantization of $O_X$ in the sense explained below in this section.

2.2.1. Formal quantizations. We start with a formal setting. A quantization $D_h$ of $X$ is a sheaf of $\mathbb{C}[[\hbar]]$-algebras on $X$ together with an isomorphism $\pi : D_h/(\hbar) \xrightarrow{\sim} O_X$ such that

(a) $D_h$ is flat over $\mathbb{C}[[\hbar]]$ (equivalently, there are no nonzero local sections annihilated by $\hbar$) and complete and separated in the $\hbar$-adic topology (meaning that $D_h \xrightarrow{\sim} \varprojlim_{n \to +\infty} D_h/(\hbar^n)$).

(b) $\pi(\frac{1}{\hbar}(a,b)) = \{\pi(a), \pi(b)\}$ for any local sections $a, b$ of $D_h$.

2.2.2. Motivation: star-products. The origins of this definition are in the deformation quantization introduced in [BFFLS]. Let us adopt this definition to our situation. Let $A$ be a Poisson algebra. By a star-product on $A$ one means a bilinear map $*: A \otimes A \to A[[\hbar]]$ subject to the following conditions:

1. The $\mathbb{C}[[\hbar]]$-bilinear extension of $*$ to $A[[\hbar]]$ is associative and $1 \in A$ is a unit.

2. $a * b \equiv ab \mod \hbar A[[\hbar]], a * b - b * a \equiv \hbar \{a, b\} \mod \hbar^2 A[[\hbar]]$.

Of course, $A[[\hbar]]$ together with $*$ is a formal quantization of $A$ in the sense of the previous section. Conversely, any formal quantization $A_h$ is isomorphic to $(A[[\hbar]], *)$.

Traditionally, one imposes an additional restriction on $*$: the locality axiom that requires that the coefficients $D_i$ in the $\hbar$-adic expansion of $* (a * b = \sum_{i=0}^\infty D_i(a, b)\hbar^i)$ are bidifferential operators. If $*$ is local, then it naturally extends to any localization $A[a^{-1}]$. So, if $A = \mathbb{C}[X]$ for $X$ affine, then a local star-product defines a quantization of $O_X$.

Let us provide an example of a local star-product. Consider $A = \mathbb{C}[x, y]$ with standard Poisson bracket: $\{x_i, x_j\} = \{y_i, y_j\}, \{y_i, x_j\} = \delta_{ij}$. Then set

$$f * g = m \circ \exp(\hbar \sum_{i=1}^n \partial_{y_i} \otimes \partial_{x_i}) f \otimes g,$$
where $\mu : A \otimes A \to A$ is the usual commutative product. For example, we have $x_i \ast x_j = x_i x_j, y_i \ast y_j = y_i y_j, x_i \ast y_j = x_i y_j, y_j \ast x_i = x_i y_j + \hbar \delta_{ij}$. In this case, $A[h]$ is closed with respect to $\ast$ and is identified with $R_h(D(\mathbb{C}^n))$.

2.2.3. Algebra vs sheaf setting in the affine case. It turns out that any formal quantization $A_h$ of $\mathbb{C}[X]$ for an affine variety $X$ defines a quantization of $X$. The reason is that we can localize elements of $\mathbb{C}[X]$ in $A_h$. The construction is as follows. Pick $f \in \mathbb{C}[X]$ and lift it to $\hat{f} \in A_h$. The operator $\text{ad} \hat{f}$ is nilpotent in $A_h/(\hbar^n)$ for any $n$ and so the set $\{\hat{f}^n\} \subset A_h/(\hbar^n)$ satisfies the Ore conditions, hence the localization $A_h/(\hbar^n)[\hat{f}^{-1}]$ makes sense. It is easy to see that these localizations do not depend on the choice of the lift $\hat{f}$ and form an inverse system. We set $A_h[f^{-1}] := \lim_{\leftarrow n \to +\infty} A_h/(\hbar^n)[\hat{f}^{-1}]$.

**Exercise 2.5.** Check that there is a unique sheaf $D_h$ in the Zariski topology on $X$ such that $D_h(X_f) = A_h[f^{-1}]$ for any $f \in \mathbb{C}[X]$ and that this sheaf is a quantization of $X$.

So we see that there is a natural bijection between the quantizations of $X$ and of $\mathbb{C}[X]$ (to get from a quantization of $X$ to that of $\mathbb{C}[X]$ we just take the global sections). Thanks to this, we can view a quantization of a general variety $X$ as glued from affine pieces.

2.2.4. Filtered quantizations. Let us proceed to the filtered setting. Suppose that $X$ is equipped with a $\mathbb{C}^\times$-action such that the Poisson bracket has degree $-1$. Obviously, for an arbitrary open $U \subset X$, the algebra $\mathbb{C}[U]$ does not need to be graded. However, it is graded when $U$ is $\mathbb{C}^\times$-stable. By a conical topology on $X$ we mean the topology, where “open” means Zariski open and $\mathbb{C}^\times$-stable. One can ask whether this topology is sufficiently rich, for example, whether any point has an open affine neighborhood.

**Theorem 2.6** (Sumihiro). Suppose $X$ is normal. Then any point in $X$ has an open affine neighborhood in the conical topology.

Below we always assume that $X$ is normal. Note that $\mathcal{O}_X$ is a sheaf of graded algebras in the conical topology. By a filtered quantization of $X$ we mean a sheaf $D$ of filtered algebras (in the conical topology on $X$) equipped with an isomorphism $\pi : \text{gr} D \cong \mathcal{O}_X$ of graded algebras such that the filtration on $D$ is complete and separated and $\pi$ is compatible with the Poisson brackets as in [2.1.2].

We still have a one-to-one correspondence between filtered quantizations and formal quantizations with $\mathbb{C}^\times$-actions. This works just as in [2.1.3] (note that $D_{h,\text{fin}}$ makes sense as a sheaf in conical topology).

2.2.5. Quantization in families. Let $X$ be a smooth scheme over a scheme $S$. It still makes sense to speak about closed and non-degenerate forms in $\Omega^2(X/S)$. By a symplectic $S$-scheme we mean a smooth $S$-scheme $X$ together with a closed non-degenerate form $\omega_S \in \Omega^2(X/S)$. Note that from $\omega$ one can recover an $\mathcal{O}_S$-linear Poisson bracket on $X$.

By a formal quantization $D_h$ of $X$ we mean a sheaf of $\mathcal{O}_S$-algebras on $X$ satisfying conditions (a),(b) in [2.2.1].

Note that the definition above still makes sense when $S$ is a formal scheme and $X$ is a formal $S$-scheme.

2.2.6. Classification theorem. Let us finish this section with a classification theorem due to Bezrukavnikov and Kaledin, [BK1] (with a ramification given in [L3]).
Theorem 2.7. Let $X$ be a smooth symplectic variety. Suppose $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$ (this holds when $X$ is affine, for example). Then the formal quantizations of $X$ are parameterized by $H^2_{DR}(X, \mathbb{C})[[\hbar]]$. If $X$ has a $\mathbb{C}^\times$-action compatible with the bracket (where we have $d = 1$), then the filtered quantizations are in one-to-one correspondence with $H^2_{DR}(X, \mathbb{C})$.

Even without the cohomology vanishing assumption, there is a so called period map $\text{Per}$ from the set $\text{Quant}(X)$ of formal quantizations of $X$ (considered up to an isomorphism) to $H^2_{DR}(X) [[\hbar]]$. When the vanishing condition holds, this map is a bijection. The classification of filtered quantizations follows from the observation that once a quantization admits a $\mathbb{C}^\times$-action by automorphisms, its period lies in $H^2_{DR}(X) \subset H^2_{DR}(X) [[\hbar]]$ (and if the vanishing holds, the converse is also true), see [L3, 2.3].

Assume until the end of the section that the vanishing condition holds.

A formal quantization $\mathcal{D}_h$ having a $\mathbb{C}^\times$-action by automorphisms and satisfying $\text{Per}(\mathcal{D}_h) = 0$ has a nice property: it is even. When $X$ is affine this means that the quantization can be realized by a star-product $f \ast g = \sum_{i=0}^{\infty} D_i(f, g) \hbar^i$ with $\deg D_i = -i$ and $D_i(f, g) = (-1)^i D_i(g, f)$. For general $X$, being even means that there is an anti-automorphism $\rho$ of $\mathcal{D}_h$ that commutes with the $\mathbb{C}^\times$-action, is the identity modulo $\hbar$, and maps $\hbar$ to $-\hbar$.

Let us finish this subsection with the discussion of the universal quantization. The variety $X$ has a universal symplectic deformation $\hat{X}$ over the formal disc $\mathcal{S}$ that is the formal neighborhood of 0 in $H^2_{DR}(X)$ (provided $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, 2$), see [KaVe]. The universality means that any other formal symplectic deformation of $X$ is obtained from $\hat{X}$ by pull-back. Further, there is a canonical quantization $\hat{\mathcal{D}}_h$ of $\hat{X}/\mathcal{S}$. All quantizations of $X$ are obtained by pulling back $\hat{\mathcal{D}}_h$. More precisely, we can view $\hat{\mathcal{D}}_h$ as a sheaf of $\mathbb{C}[[H^2_{DR}(X), \hbar]]$-algebras on $X$ (via the sheaf-theoretic pull-back) and then we can obtain quantizations of $X$ by base change to $\mathbb{C}[[\hbar]]$.

In the case when $X$, in addition, has a $\mathbb{C}^\times$-action rescaling the symplectic form, we can consider the universal $\mathbb{C}^\times$-equivariant deformation $\hat{X}$ over $H^2_{DR}(X)$ as well as its canonical quantization $\hat{\mathcal{D}}_h$.

2.3. Modules over quantizations. Let $X$ be a Poisson variety (or scheme). We are going to define coherent and quasi-coherent modules over filtered and formal quantizations of $X$ (to be denoted by $\mathcal{D}$ and $\mathcal{D}_h$, respectively).

2.3.1. Coherent modules over formal quantizations. By definition, a sheaf $\mathcal{M}_h$ of $\mathcal{D}_h$-modules on $X$ is called coherent if $\mathcal{M}_h/\hbar \mathcal{M}_h$ is a coherent $\mathcal{O}_X$-module and $\mathcal{M}_h$ is complete and separated in $\hbar$-adic topology. Note that the condition of being complete and separated in the $\hbar$-adic topology is local. So being coherent is a local condition (as in Algebraic geometry).

Let $X$ be affine and let $\mathcal{A}_h := \Gamma(\mathcal{D}_h)$. Let $\mathcal{N}_h$ be a finitely generated $\mathcal{A}_h$-module. Then it is easy to see that $\mathcal{N}_h$ is complete and separated in the $\hbar$-adic topology. It follows that $\mathcal{D}_h \otimes_{\mathcal{A}_h} \mathcal{N}_h$ is a coherent $\mathcal{D}_h$-module. Conversely, for a coherent $\mathcal{D}_h$-module $\mathcal{M}_h$, the global sections $\Gamma(\mathcal{M}_h)$ is a finitely generated $\mathcal{A}_h$-module.

Lemma 2.8. Let $X$ be affine. Then the functors $\mathcal{D}_h \otimes_{\mathcal{A}_h} \bullet$ and $\Gamma(\bullet)$ are mutually quasi-inverse equivalences between the categories of coherent $\mathcal{D}_h$-modules and finitely generated $\mathcal{A}_h$-modules.
Proof. Note that these functors define compatible equivalences between the categories of coherent \( \mathcal{D}_h/(\hbar^n) \)-modules and of finitely generated \( \mathcal{A}_h/(\hbar^n) \)-modules for any \( n \) (which is proved in the same way as the classical statement for \( n = 1 \)). Then we use that all objects we consider are complete and separated in the \( \hbar \)-adic topology.

From this lemma we easily see that a subsheaf and a quotient sheaf of a coherent \( \mathcal{D}_h \)-module is coherent itself. So the category \( \text{Coh}(\mathcal{D}_h) \) of coherent \( \mathcal{D}_h \)-modules is an abelian category.

2.3.2. Quasi-coherent modules over formal quantizations. By a quasi-coherent \( \mathcal{D}_h \)-module we mean a direct limit of coherent \( \mathcal{D}_h \)-modules. Lemma 2.8 implies that, when \( X \) is affine, the category of quasi-coherent \( \mathcal{D}_h \)-modules is equivalent to the category of \( \Gamma(\mathcal{D}_h) \)-modules.

Analogously to the classical algebro-geometric result, the category \( \text{QCoh}(\mathcal{D}_h) \) of quasi-coherent \( \mathcal{D}_h \)-modules has enough injective objects. Note that the natural functor from \( D^b(\text{Coh}(\mathcal{D}_h)) \) to the full subcategory in \( D^b(\text{QCoh}(\mathcal{D}_h)) \) of all complexes with coherent homology is a category equivalence. This is because any quasi-coherent complex is a union of coherent subcomplexes, as in the usual Algebro-geometric situation.

2.3.3. Modules over filtered quantizations. Let us proceed to modules over filtered quantizations. Let \( \mathcal{M} \) be a sheaf of \( \mathcal{D} \)-modules. We say that \( \mathcal{M} \) is coherent if it can be equipped with a global complete and separated filtration compatible with that on \( \mathcal{D} \) and such that \( \text{gr} \mathcal{M} \) is a coherent sheaf on \( X \) (such a filtration is usually called good). The \( \hbar \)-adic completion of the Rees sheaf \( R_h(\mathcal{M}) \) is then a \( \mathbb{C}^\times \)-equivariant coherent \( \mathcal{D}_h \)-module. Conversely, if we take \( \mathcal{C}^\times \)-equivariant coherent \( \mathcal{D}_h \)-module \( \mathcal{M}_h \), take the \( \mathbb{C}^\times \)-finite part \( \mathcal{M}_{h,\text{fin}} \), then \( \mathcal{M}_{h,\text{fin}}/(\hbar - 1) \) is a coherent \( \mathcal{D} \)-modules.

Lemma 2.9. Consider the full subcategory \( \text{Coh}^\mathbb{C}^\times(\mathcal{D}_h)_{\text{tor}} \) consisting of all modules that are torsion over \( \mathbb{C}[[\hbar]] \). Then taking quotient by \( \hbar - 1 \) gives rise to an equivalence \( \text{Coh}^\mathbb{C}^\times(\mathcal{D}_h)/\text{Coh}^\mathbb{C}^\times(\mathcal{D}_h)_{\text{tor}} \xrightarrow{\sim} \text{Coh}(\mathcal{D}) \).

Proof. Let us produce a quasi-inverse functor. Of course, the \( R_h(\mathcal{D}) \)-module \( R_h(\mathcal{M}) \) depends on the choice of a good filtration. Let \( F, F' \) be two good filtrations. Then one can find positive integers \( d_1, d_2 \) such that \( F_i - d_1 \mathcal{M} \subseteq F'_i \mathcal{M} \subseteq F_i + d_2 \mathcal{M} \) the inclusion of subsheaves (of vector spaces) in \( \mathcal{M} \) (it is enough to check this claim for local sections over open subsets from an affine cover, where it is easy). It follows that modulo \( \hbar \)-torsion the sheaf \( R_h(\mathcal{M}) \) is independent of the choice of a good filtration. Our quasi-inverse functor sends \( \mathcal{M} \) to the \( \hbar \)-adic completion of \( R_h(\mathcal{M}) \). To check that this is indeed a quasi-inverse functor is standard.

2.3.4. Supports. For a coherent \( \mathcal{D}_h \)-module \( \mathcal{M}_h \) we have the notion of support. By definition, \( \text{Supp}(\mathcal{M}_h) := \text{Supp}(\mathcal{M}_h/\hbar \mathcal{M}_h) \), this is a closed subvariety in \( X \).

Now let \( \mathcal{M} \in \text{Coh}(\mathcal{D}) \). Then we can take a good filtration on \( \mathcal{M} \) and set \( \text{Supp}(\mathcal{M}) := \text{Supp}(\text{gr} \mathcal{M}) \). By the argument in the proof of Lemma 2.9 the support of \( \mathcal{M} \) is well-defined, i.e., it does not depend on the choice of a good filtration.

2.3.5. Global sections and localization. Let \( \mathcal{D} \) be a filtered quantization of \( X \). We have natural functors \( \text{Coh}(\mathcal{D}) \to \Gamma(\mathcal{D})\text{-mod} \) of taking global sections (to be denoted by \( \Gamma \)) as well as a functor in the opposite direction \( \text{Loc} : \Gamma(\mathcal{D})\text{-mod} \to \text{Coh}(\mathcal{D}), \mathcal{M} \mapsto \mathcal{D} \otimes_{\Gamma(\mathcal{D})} \mathcal{M} \).

Let us discuss a situation when these functors behave particularly nicely. Namely, let \( X \) be a conical symplectic resolution of singularities of an affine variety \( \mathcal{X}_0 \). Note that,
by the Grauert-Riemenschneider theorem, the higher cohomology of $\mathcal{O}_X$ vanish. This has the following corollary (the proof is left to the reader).

**Lemma 2.10.** We have $H^i(\mathcal{D}) = 0$ for $i > 0$. Moreover, $\Gamma(\mathcal{D})$ is a quantization of $X_0$.

Thanks to this lemma, it makes sense to consider derived functors $R\Gamma : D(\text{Coh}(\mathcal{D})) \to D(\Gamma(\mathcal{D})\text{-mod})$ and $L\text{Loc} : D(\Gamma(\mathcal{D})\text{-mod}) \to D(\text{Coh}(\mathcal{D}))$. In fact, $R\Gamma$ is given by the Čech complex and so restricts to bounded (to the left and to the right) derived categories. The functor $L\text{Loc}$ restricts to $D^{-}$'s. Lemma 2.10 implies that $R\Gamma \circ L\text{Loc}$ is the identity on $D^{-}(\Gamma(\mathcal{D})\text{-mod})$. Furthermore, if $\Gamma(\mathcal{D})$ has finite homological dimension, then $L\text{Loc}$ maps $D^b(\Gamma(\mathcal{D})\text{-mod})$ to $D^b(\text{Coh}(\mathcal{D}))$ and is left inverse to $R\Gamma$. It is likely (and is proved in many cases, see, e.g., [MN]) that $R\Gamma$ and $L\text{Loc}$ are mutually quasi-inverse equivalences in this case.

2.4. **Frobenius constant quantizations.** Above, we were dealing with the case when the ground field is $\mathbb{C}$. Everything works the same for any algebraically closed field of characteristic 0. In this section we are going to work over an algebraically closed field $\mathbb{F}$ of positive characteristic.

The notions of filtered and formal quantizations still make sense, both for algebras and for varieties. But in positive characteristic we have an important special class of quantizations: Frobenius constant ones.

2.4.1. **Basic example.** Let us start our discussion with an example of a quantization: the Weyl algebra $\mathbf{A}(V)$, where $V$ is a symplectic $\mathbb{F}$-vector space. A new feature is that this algebra is finite over its center. Namely, for $v \in V \subset \mathbf{A}(V)$, the element $v^p \in \mathbf{A}(V)$ lies in the center. We have a semi-linear map $\iota : V \to \mathbf{A}(V)$ given by $v \mapsto v^p$ on $v \in V$ with central image that extends to a ring homomorphism $S(V) \to \mathbf{A}(V)$. The semi-linearity condition is $\iota(av) = \text{Fr}(a)\iota(v)$, where $\text{Fr} : \mathbb{F} \to \mathbb{F}$ is the Frobenius automorphism. Let $V^{(1)}$ denote the $\mathbb{F}$-vector space identified with $V$ as an abelian group but with new multiplication by scalars: $a \cdot v = \text{Fr}^{-1}(a)v$. So $\iota$ becomes an algebra homomorphism when viewed as a map $S(V^{(1)}) \to \mathbf{A}(V)$, its image is usually called the $p$-center, in our case it coincides with the whole center. Another important feature of this example is that $\mathbf{A}(V)$ is an Azumaya algebra over $V^{(1)}$, i.e., $\mathbf{A}(V)$ is a vector bundle over $\text{Spec}(S(V^{(1)}))$ and all (geometric) fibers are matrix algebras (of rank $p^{\dim V/2}$).

2.4.2. **Definition.** The notion of a Frobenius constant quantization generalizes the example in 2.4.1. We will give the definition in the filtered setting and only for symplectic varieties – we will only need it in this case. Let $X$ be a smooth symplectic $\mathbb{F}$-variety equipped with an $\mathbb{F}^\times$-action rescaling the symplectic form (by the character $t \mapsto t^d$). Let $X^{(1)}$ be the $\mathbb{F}$-variety that is identified with $X$ as a scheme over $\text{Spec}(\mathbb{Z})$ but with twisted multiplication by scalars in the structure sheaf just as in 2.4.1. We have a natural morphism $\text{Fr} : X \to X^{(1)}$ of $\mathbb{F}$-varieties and hence we have a sheaf $\text{Fr}_*(\mathcal{O}_X)$ on $X^{(1)}$. This is a coherent sheaf of algebras and a vector bundle of rank $p^{\dim X}$.

**Definition 2.11.** A Frobenius constant quantization is a filtered sheaf $\mathcal{D}$ of Azumaya algebras on $X^{(1)}$ together with an isomorphism $\text{gr}\mathcal{D} \cong \text{Fr}_*\mathcal{O}_X$ of graded algebras (in conical topology) that satisfies our usual compatibility condition on Poisson brackets.

It is not difficult to show that a Frobenius constant quantization gives rise to a filtered quantization of $X$. But, as we will see in 3.3.3, not every filtered quantization arises this way.
2.4.3. **Differential operators.** Let us give another example that should be thought as a global analog of 2.4.1. Let $Y$ be a smooth $\mathbb{F}$-variety. Consider the sheaf $D_Y$ of differential operators on $Y$. Let $\xi$ be a vector field on an open subset $Y' \subset Y$. Define a vector field $\xi[p]$ as follows. For every open affine subvariety $Y^0 \subset Y'$, we can regard $\xi$ as a derivation of $\mathbb{F}[Y^0]$. The map $\xi^p : \mathbb{F}[Y^0] \to \mathbb{F}[Y^0]$ is again a derivation. The corresponding vector field on $Y'$ (that is easily seen to be well-defined) is what we denote by $\xi[p]$. It is easy to see that $f^p$, for a function $f$ on $Y$, and $\xi^p - \xi[p]$, for a vector field $\xi$ (here $\xi^p$ is taken with respect to the product on $D_Y$), are central. The maps $f \mapsto f^p, \xi \mapsto \xi^p - \xi[p]$ give rise to a sheaf of algebras homomorphism $\pi^* \mathcal{O}(T^*Y)(1) \to \text{Fr}_* D_Y$, where we write $\pi$ for the projection $(T^*Y)(1) = T^*(Y(1)) \to Y(1)$. The sheaf $D_Y$ then becomes a Frobenius constant quantization of $T^*Y$.

To finish this section, let us mention that, under some restrictions on $X$, there is a classification of Frobenius constant quantizations, see [BK3].

### 3. Hamiltonian reductions

In this section we recall the notions of the classical and quantum Hamiltonian reduction. The classical Hamiltonian reduction produces a new Poisson variety from an existing Poisson variety with suitable symmetries. The quantum Hamiltonian reduction does the same on the level of quantizations.

We start by discussing classical Hamiltonian reductions, Section 3.1. First, we recall Hamiltonian actions and moment maps. Then we define classical Hamiltonian reductions in the settings of categorical quotients and of GIT quotients. We then proceed to the construction and basic properties of Nakajima quiver varieties that are our main examples of Hamiltonian reductions. Next, we explain how quotient singularities $V_n/\Gamma_n$ are realized as quiver varieties. Finally, we construct symplectic resolutions of singularities for $V_n/\Gamma_n$ and establish, following Namikawa, some isomorphisms between some of these resolutions.

In Section 3.2 we proceed to quantum Hamiltonian reductions. We define them on the level of algebras and on the level of sheaves and compare the two levels. After that we state one of the main results of this survey: an isomorphism between spherical SRA for wreath-product groups and quantum Hamiltonian reductions. We finish this section by discussing a quantum version of Namikawa’s Weyl group action.

Section 3.3 deals with Hamiltonian reductions for Frobenius constant quantization. We first recall some basic results on GIT in positive characteristic. Then we discuss Nakajima quiver varieties in sufficiently large positive characteristic. Finally, we prove, following Bezrukavnikov, Finkelberg and Ginzburg, that the quantum Hamiltonian reduction of a Frobenius constant quantization at an integral value of the quantum comoment map is Frobenius constant.

#### 3.1. Classical Hamiltonian reduction.

3.1.1. **Hamiltonian group actions.** Let $X$ be a Poisson variety (over an algebraically closed field) and let $G$ be an algebraic group acting on $X$. The action induces a Lie algebra homomorphism $\mathfrak{g} \to \text{Vect}(X)$, the image of $\xi \in \mathfrak{g}$ under this homomorphism will be denoted by $\xi_X$. We say that the $G$-action on $X$ is Hamiltonian, if there is a $G$-equivariant linear map $\mathfrak{g} \to \mathbb{C}[X], \xi \mapsto H_\xi$, such that $\{H_\xi, \cdot\} = \xi_X$. Note that this map is automatically a Lie algebra homomorphism. This map is called the comoment map, the dual map $\mu : X \to \mathfrak{g}^*$ is the moment map.

Let us provide two examples of Hamiltonian actions.
Example 3.1. Let $Y$ be a smooth variety, $G$ act on $Y$. Then $X := T^*Y$ carries a natural $G$-action. This action is Hamiltonian with $H_\xi = \xi_Y$ (viewed as a function on $X$).

Example 3.2. Let $V$ be a vector space (with symplectic form $\Omega$) and let $G$ act on $V$ by linear symplectomorphisms. The action is Hamiltonian with $H_\xi(v) = \frac{1}{2}\Omega(\xi v, v)$.

Below we will need a standard property of Hamiltonian actions.

Lemma 3.3. Let $x \in X$. Then $\text{im} \, d_x \mu \subset g^*$ coincides with the annihilator of $g_x := \text{Lie}(G_x)$. In particular, $\mu$ is a submersion at $x$ if and only if $G_x$ is finite.

3.1.2. Hamiltonian reduction. Let $A$ be a Poisson algebra and $g$ be a Lie algebra equipped with a Lie algebra homomorphism $g \to A, \xi \mapsto H_\xi$. Consider the ideal $I := A\{H_\xi, \xi \in g\}$. The adjoint action of $g$ on $A$ preserves this ideal so we can take the invariants $A///_g := (A/I)^\theta$. This algebra comes with a natural Poisson bracket: $\{a + I, b + I\} := \{a, b\} + I$ (but $A/I$ has no Poisson bracket!).

This construction has several ramifications. First, let $\lambda : g \to \mathbb{C}$ be a character (i.e., a function vanishing on $[g, g]$). Then we can set $A///_\lambda := (A/A\{H_\xi - \langle \lambda, \xi \rangle\})^\theta$. Also we can set $A///_g := (A/A\{H_\xi, \xi \in [g, g]\})^\theta$. The latter is a Poisson $S(g/[g, g])$-algebra whose specialization at $\lambda \in (g/[g, g])^*$ coincides with $A///_\lambda$ provided that the $g$-action on $A/A\{H_\xi, \xi \in [g, g]\}$ is completely reducible.

Let us proceed to a geometric incarnation of this construction. Suppose the base field is of characteristic 0. To ensure a good behavior of quotients assume that $G$ is a reductive group. Let $X$ be an affine Poisson variety equipped with a Hamiltonian $G$-action. Then we can take $A := \mathbb{C}[X]$ together with the comoment map $\xi \mapsto H_\xi$. We set $A///_0 := (A/I)^G$, this algebra coincides with $A///_g$ when $G$ is connected. It is finitely generated by the Hilbert theorem, here we use that $G$ is reductive. The variety (or scheme) $\text{Spec}(A///_0G)$ is nothing else but the categorical quotient $X///_0G := \mu^{-1}(0)//G$.

Here is a corollary of Lemma 3.3.

Corollary 3.4. Suppose that $X$ is smooth and symplectic and that the $G$-action on $\mu^{-1}(0)$ is free. Then $X///_0G$ is smooth and symplectic of dimension $\dim X - 2 \dim G$.

Proof. The variety $\mu^{-1}(0)$ is smooth by Lemma 3.3. That the quotient is smooth of required dimension is a straightforward corollary of the Luna slice theorem, see, e.g., [PV, Section 6.3].

The form on $X///_0G$ can be recovered as follows. Let $\Omega$ denote the form on $X$, $\iota : \mu^{-1}(0) \hookrightarrow X$ denote the inclusion map and $\pi : \mu^{-1}(0) \to X///_0G$ be the projection. Then there is a unique 2-form $\Omega_{\text{red}}$ on $X///_0G$ such that $\pi^*\Omega_{\text{red}} = \iota^*\Omega$ and this is the form we need.

3.1.3. GIT Hamiltonian reduction. We will be mostly interested in Hamiltonian reductions for linear actions $G \curvearrowright V$. The assumptions of Corollary 3.4 are not satisfied in this case. However, if one uses GIT quotients instead of the usual categorical quotients, one can often get a smooth symplectic variety that will be a resolution of the usual reduction $V///_0G$.

Let us recall the construction of a GIT quotient. Let $G$ be a reductive algebraic group acting on an affine algebraic variety $X$. Fix a character $\theta : G \to \mathbb{C}^\times$. We use the additive notation for the multiplication of characters. Then consider the space $\mathbb{C}[X]^{G, n\theta}$ of $n\theta$-semiinvariants: $\mathbb{C}[X]^{G, n\theta} := \{f \in \mathbb{C}[X] | g.f := \theta(g)^n f\}$ (recall that $g.f(x) := f(g^{-1}x)$). Consider the graded algebra $\bigoplus_{n \geq 0} \mathbb{C}[X]^{G, n\theta}$, where $\deg \mathbb{C}[X]^{G, n\theta} := n$. Then we set
$X//^0 G := \text{Proj}(\bigoplus_{n>0} \mathbb{C}[X]^{G,n\theta})$, this is a projective variety over $X//G$. Note that we no longer have a morphism $X \to X//^0 G$. Instead, consider the open subset of $\theta$-semistable points $X^{\theta-ss}$, a point $x \in X$ is called semistable if there is $f \in \mathbb{C}[X]^{G,n\theta}$ for $n > 0$ with $f(x) \neq 0$. We clearly have a natural morphism $X^{\theta-ss} \to X//^0 G$ that makes the following diagram commutative

$$
\begin{array}{ccc}
X^{\theta-ss} & \to & X//^0 G \\
\subseteq & \searrow & \\
X \quad & X//G
\end{array}
$$

The variety $X//^0 G$ is glued from the varieties of the form $X_f//G$, where $f \in \mathbb{C}[X]^{G,n\theta}$ with some $n > 0$. The intersection of $X_f//G, X_g//G$ inside $X//^0 G$ is identified with $X_{fg}//G$, where the inclusions $X_{fg}//G \hookrightarrow X_f//G, X_g//G$ are induced from the inclusions $X_{fg} \hookrightarrow X_f, X_g$ by passing to the quotients.

In the setting of 3.1.2, we set $X//^0_0 G := \mu^{-1}(0)^{\theta-ss}//G$. This is a Poisson variety (the bracket comes from gluing together the brackets on the open subvarieties $X_f///_0 G$) equipped with a projective morphism $X//^0_0 G \to X//_0 G$ of Poisson varieties. If $X$ is smooth and symplectic, and the $G$-action on $\mu^{-1}(0)^{\theta-ss}$ is free, then $X//^0_0 G$ is smooth and symplectic of dimension $\dim X - 2 \dim G$. The symplectic form on $X//^0_0 G$ is recovered similarly to the case of $X//_0 G$ considered above.

### 3.1.4. Nakajima quiver varieties: construction

Now we are going to introduce an important special class of varieties constructed by means of Hamiltonian reduction: Nakajima quiver varieties, introduced in [Nak1], see also [Nak3].

By a quiver, we mean an oriented graph. Formally, it can be presented as a quadruple $Q = (Q_0, Q_1, t, h)$, where $Q_0, Q_1$ are finite sets of vertices and arrows, respectively, and $t, h : Q_1 \to Q_0$ are maps that to an arrow $a$ assign its tale and head.

Let us proceed to (framed) representations of $Q$. Fix two elements $v, w \in \mathbb{Z}^{Q_0}$ and set $V_i := \mathbb{C}^{v_i}, W_i := \mathbb{C}^{w_i}, i \in Q_0$. Consider the space

$$R(= R(Q, v, w)) := \bigoplus_{a \in Q_1} \text{Hom}_\mathbb{C}(V_{t(a)}, V_{h(a)}) \oplus \bigoplus_{i \in Q_0} \text{Hom}_\mathbb{C}(W_i, V_i).$$

An element of $R$ can be thought as a collection of linear maps, one for each arrow, between the corresponding vector spaces, together with collections of vectors in each $V_i$. This description suggests a group of symmetry of $R$: we set $G := \prod_{i \in Q_0} \text{GL}(V_i)$, this group acts by changing bases in the spaces $V_i$.

A character of $G$ is of the form $\chi = (\chi_i)_{i \in Q_0} \mapsto \prod_{i \in Q_0} \det(\chi_i)^{\theta_i}$, where $\theta = (\theta_i)_{i \in Q_0} \in \mathbb{C}^{Q_0}$. We will identify the character group of $G$ with $\mathbb{C}^{Q_0}$.

A Nakajima quiver variety $\mathcal{M}_Q(\chi, v, w)$ is, by definition, the reduction $T^*R//^\chi G$. Here $\lambda$ is a character of $g$, it can be thought as an element of $\mathbb{C}^{Q_0}$ via $\lambda(x) := \sum_{i \in Q_0} \lambda_i \text{tr}(x_i)$. The moment map $\mu : T^*R \to \bigoplus_{i \in Q_0} \text{End}(V_i) = g(= g^*)$ is explicitly given as follows:

$$(x_a, x_a^*, i, j)_{a \in Q_1, k \in Q_0} \mapsto \sum_{a \in Q_1} (x_a x_a^* - x_a^* x_a) - \sum_{k \in Q_0} j_k i_k,$$

where $x_a \in \text{Hom}(V_{t(a)}, V_{h(a)}), x_a^* \in \text{Hom}(V_{h(a)}, V_{t(a)}), i_k \in \text{Hom}(V_k, W_k), j_k \in \text{Hom}(W_k, V_k)$. [Nak1] [Nak3]
We also would like to remark that the quiver variety is independent of the choice of an orientation of $Q$. Indeed, let $Q'$ be a quiver obtained from $Q$ by changing the orientation of a single arrow $a$ and let $R'$ be the corresponding representation space. Then we have an isomorphism $T^*R \cong T^*R'$ that sends $x_a$ to $x_{a'}$, $x_{a'}$ to $-x_a$ and does not change the other components. This is a $G$-equivariant symplectomorphism that intertwines the moment maps and hence inducing a symplectomorphism of the corresponding Nakajima quiver varieties.

When $\lambda = 0$, we have a $\mathbb{C}^\times$-action on $M_0^\gamma(v, w)$ that rescales the Poisson structure. For example, one can take the action induced by the dilation action on $T^* R$, that is, $t.v := t^{-1}v, t \in \mathbb{C}^\times, v \in T^* R$ to be called the dilation action as well. Then the Poisson bracket on $M_0^\gamma(v, w)$ has degree $-2$. We can also have an action such that the Poisson bracket has degree $-1$ coming from $t. (r, r^*) := (r, t^{-1}r^*), r \in R, r^* \in R$.

3.1.5. Nakajima quiver varieties: structural results. Let us explain some structural results regarding the quiver varieties and the corresponding moment maps. We will need algebro-geometric properties of $\mu^{-1}(\lambda)$ and of $M_0^\gamma(v, w)$ due to Crawley-Boevey and also a criterium for the freeness of the $G$-action on $\mu^{-1}(\lambda)^{\theta - ss}$ due to Nakajima.

**Theorem 3.5** (Crawley-Boevey, [CB1]). The scheme $M_0^\gamma(v, w)$ is reduced and normal.

We now want to provide a criterium for $\mu : T^* R \to \mathfrak{g}^*$ to be flat proved in [CB1]. Define the symmetrized Tits form $\mathbb{C}^{Q_0} \times \mathbb{C}^{Q_0} \to \mathbb{C}$:

$$(v^1, v^2) := \sum_{a \in Q_1} (v^1_{l(a)} v^2_{h(a)} + v^1_{h(a)} v^2_{l(a)}) - 2 \sum_{i \in Q_0} v^1_i v^2_i$$

and quadratic maps $p, p_w : \mathbb{C}^{Q_0} \to \mathbb{C}$ by

$$p(v) := 1 - \frac{1}{2} (v, v), \quad p_w(v) := w \cdot v - \frac{1}{2} (v, v).$$

**Theorem 3.6** (Crawley-Boevey, [CB1]). The following two conditions are equivalent:

(i) $\mu$ is flat.
(ii) $p_w(v) \geq p_w(v^0) + \sum_{i=1}^k p(v^i)$ for any decomposition $v = v^0 + v^1 + \ldots + v^k$ with $v^i \in \mathbb{Z}^{Q_0}_{\geq 0}$ for $i = 1, \ldots, k$.

**Theorem 3.7** (Crawley-Boevey, [CB1]). Suppose that, for a proper decomposition $v = v^0 + v^1 + \ldots + v^k$, we have $p_w(v) > p_w(v^0) + \sum_{i=1}^k p(v^i)$. Then $\mu^{-1}(0)$ is irreducible and a generic $G$-orbit there is closed and free.

Let us proceed to a criterium for the action of $G$ on $\mu^{-1}(\lambda)^{\theta - ss}$ to be free. We can view $Q$ as a Dynkin diagram and form the corresponding Kac-Moody algebra $\mathfrak{g}(Q)$. Then $\mathbb{C}^{Q_0}$ gets identified with the dual of the Cartan of $\mathfrak{g}(Q)$ in such a way that the coordinate vector $e_i, i \in Q_0$, becomes a simple root. Then, [Nak1], the action of $G$ on $\mu^{-1}(\lambda)^{\theta - ss}$ is free if and only if there are no roots $\nu'$ of $\mathfrak{g}(Q)$ such that $\nu' \leq v$ (component-wise) and $\nu' \cdot \theta = \nu' \cdot \lambda = 0$.

The equations $\nu' \cdot \theta = 0$, where $\nu'$ is a root satisfying $\nu' \leq v, \nu' \cdot \lambda = 0$ split the character lattice into the union of cones. It is a classical fact from GIT, that when $\theta, \theta'$ are generic and inside one cone, we have $\mu^{-1}(\lambda)^{\theta - ss} = \mu^{-1}(\lambda)^{\theta' - ss}$. So $M_0^\gamma(v, w) = M_0^\nu(v, w)$. 
3.1.6. Hilb_n(C^2) and C^{2n}\!/\!S_n as quiver varieties. Let Q be a quiver with a single vertex and a single loop (a.k.a. the Jordan quiver). We are going to show that Hilb_n(C^2) is identified with M_0^1(n, 1) and C^{2n}\!/\!S_n is identified with M_0^0(n, 1) (and the Hilbert-Chow map from 1.1.3 becomes the natural morphism M_0^1(n, 1) \rightarrow M_0^0(n, 1) from 3.1.3).

An identification M_0^1(n, 1) \cong Hilb_n(C^2) is an easier part. We have R = End(C^n) \oplus C^n. Using the trace pairing, we identify R* with End(C^n) \oplus C^{n*} so that T^*R = End(C^n) \oplus C^n \oplus C^{n*}. We write (A, B, i, j) for a typical point of T^*R. Identifying g with g* again using the trace pairing, we can write the moment map \mu : T^*R \rightarrow g as \mu(A, B, i, j) = [A, B] + ij.

Using the Hilbert-Mumford theorem from Invariant theory, see, e.g., [PV] 5.3, one shows that (T^*R)^{qss} = \{(A, B, i, j) | C\langle A, B \rangle i = C^n\}. Then it is a nice Linear Algebra exercise to show that if [A, B] + ij = 0 and C\langle A, B \rangle i = C^n, then j = 0. So \mu^{-1}(0)^{qss}/G = \{(A, B, i) | [A, B] = 0, C\langle A, B \rangle i = C^n\}/G that recovers the classical description of Hilb_n(C^2), see [Nak] Theorem 1.14.

An identification M_0^0(n, 1) \cong C^{2n}\!/\!S_n is more subtle. An easy part is to construct a morphism \iota : C^{2n}\!/\!S_n \rightarrow M_0^0(n, 1): we send ((z, y) \in C^{2n}) to (\text{diag}(z), \text{diag}(y), 0, 0) \in \mu^{-1}(0) and this induces a morphism of quotients. Then one checks that \iota is a closed embedding. For this, one uses a classical result of Weyl to see that polynomials of the map from 1.1.3 becomes the natural morphism M_0^1(n, 1) \rightarrow M_0^0(n, 1) from 3.1.3.

**Lemma 3.8.** The isomorphism M_0^0(n, 1) \cong C^{2n}\!/\!S_n intertwines the Poisson brackets.

**Proof.** Consider the principal open subsets

R^{reg} = \{(A, i) | A has distinct e-values\}, C^{n, reg} := \{(x_1, \ldots, x_n) | x_i \neq x_j \text{ for } i \neq j\}.

Note that under the above embedding C^{2n} \hookrightarrow T^*R, we have T^*C^{n, reg} \hookrightarrow T^*R^{reg}. Moreover, the pull-back of the symplectic form from T^*R^{reg} to T^*C^{n, reg} coincides with the natural symplectic form on the latter. Using the description of the symplectic form on the reduction, we conclude that the induced morphism of quotients T^*C^{n, reg}/S_n \rightarrow T^*R^{reg}/G is a symplectomorphism. But T^*R^{reg}/G embeds as an open subset into M_0^0(n, 1) and the symplectomorphism above is the restriction of the isomorphism C^{2n}\!/\!S_n \sim M_0^0(n, 1) to T^*C^{n, reg}/S_n. The claim of the lemma follows.

3.1.7. McKay correspondence. Let \Gamma_1 be a finite subgroup of SL_2(C). It turns out that the singular Poisson variety V_n/\Gamma_n (where recall V_n = C^{2n}) and its symplectic resolutions also can be realized as Nakajima quiver varieties.

The first step in this isomorphism is the McKay correspondence: a way to label the finite subgroups of SL_2(C) by Dynkin diagrams. Let \Gamma_1 be a finite subgroup of SL_2(C) and let N_0, \ldots, N_r be the irreducible representations of \Gamma_1, where N_0 is the trivial representation. Let us define the McKay graph of \Gamma_1: its vertices are 0, 1, \ldots, r and the number of edges (we consider a non-oriented graph) between i and j is dim Hom_T(C^2 \otimes N_i, N_j), note that this is well-defined because C^2 is a self-dual representation of \Gamma and so the number of edges between i and j is the same as between j and i. McKay proved the following facts:

(i) The resulting graph is an extended Dynkin graph of types A, D, E and 0 is the extending vertex.

(ii) The vector (dim N_i)_{i=0} is the indecomposable imaginary root \delta of the corresponding affine Kac-Moody algebra.
3.1.8. $\mathbb{C}^2/\Gamma_1$ as a quiver variety. Let $Q$ be the McKay graph of $\Gamma_1$ with an arbitrary orientation. Then there is an isomorphism $\mathcal{M}_0^0(\delta, 0) \cong \mathbb{C}^2/\Gamma_1$.

Let us explain how this is established following [CBH, Section 8]. For this, we will need the representation varieties.

Let $A$ be a finitely generated associative algebra and $V$ be a vector space. Then the set $X := \text{Hom}(A, \text{End}(V))$ of algebra homomorphisms is an algebraic variety. More precisely, if $A$ is the quotient of $\mathbb{C}(x_1, \ldots, x_n)$ by relations $F_\alpha(x_1, \ldots, x_n)$, where $\alpha$ runs over an indexing set $\mathcal{I}$, then $X = \{(A_1, \ldots, A_n) \in \text{End}(V) | F_\alpha(A_1, \ldots, A_n) = 0, \alpha \in \mathcal{I}\}$. The group $G := \text{GL}(V)$ naturally acts on $X$ and so we can form the quotient $X//G$ (called the representation variety). Recall that, in general, the points of $X//G$ correspond to the closed $G$-orbits on $X$, in our case an orbit is closed if its element is a semisimple representation.

This construction has various ramifications. For example, we can consider a semisimple finite dimensional subalgebra $A_0 \subset A$ and an $A_0$-module $V$. This leads to the variety $X$ of $A_0$-linear homomorphisms $A \to \text{End}(V)$ acted on by the group $G$ of $A_0$-linear automorphisms of $V$. In this situation we still can speak about representation varieties. We will realize $\mathcal{M}_0^0(\delta, 0), \mathbb{C}^2/\Gamma_1$ as the representation varieties of this kind and then show that the algebras involved are Morita equivalent, this will yield an isomorphism of interest.

Let us start with $\mathbb{C}^2/\Gamma_1$. Set $A := \mathbb{C}[x, y] \# \Gamma_1, A_0 := \mathbb{C}\Gamma_1 \subset A$ and $V := \mathbb{C}\Gamma_1$, a regular representation. Then one can show that $\mathbb{C}^2/\Gamma_1$ is the representation variety for this triple.

Let us proceed to $\mathcal{M}_0^0(\delta, 0)$. Let $\bar{Q}$ be the double quiver of $Q$. It is obtained from $Q$ by adding the inverse arrow to each arrow in $Q$. Formally, $\bar{Q}_0 = Q_0, \bar{Q}_1 = Q_1 \sqcup Q_1^\circ$, where $Q_1^\circ$ is in bijection with $Q_1, a \mapsto a^\ast$, in such a way that $t(a^\ast) = h(a), h(a^\ast) = t(a)$. Then form the path algebra $\mathbb{C}\bar{Q}$ of $\bar{Q}$, it has a basis consisting of the paths in $\bar{Q}$, the multiplication is given by concatenation (if two paths cannot be concatenated, the product is zero). This algebra is graded by the length of a path, where, by convention, the degree 0 paths are just vertices so the corresponding graded component $\mathbb{C}Q_0$ is $\mathbb{C}Q_0$.

Let us consider the quotient $\Pi^0(Q)$ of $\mathbb{C}Q$ called the preprojective algebra. It is given by the following relation:

$$\sum_{a \in Q_1} [a, a^\ast] = 0.$$

Note that $\mathbb{C}Q_0$ naturally embeds into $\Pi^0(Q)$. It is easy to see that $\mathcal{M}_0^0(\delta, 0)$ is the representation variety for the triple $(\Pi^0(Q), \mathbb{C}Q_0, \bigoplus_{i \in Q_0} \mathbb{C}^\delta_i)$.

It turns out that there is an idempotent $f \in \mathbb{C}\Gamma_1$ such that $f(\mathbb{C}[x, y] \# \Gamma_1) f \cong \Pi^0(Q)$. Namely, take primitive idempotents $f_i, i = 0, \ldots, r$, in the matrix summands of $\mathbb{C}\Gamma_1$. Set $f := \sum_{i \in Q_0} f_i$. Obviously, $f(\mathbb{C}\Gamma_1) f \cong \mathbb{C}Q_0$. Further, the construction of $Q$ implies that $f(\text{Span}(x, y) \otimes \mathbb{C}\Gamma_1) f \cong \mathbb{C}Q_1$. These identifications induce an isomorphism $f(\mathbb{C}[x, y] \# \Gamma_1) f \cong \mathbb{C}Q$. Under this isomorphism, the ideal $f(xy-yx)f$ becomes $(\sum_{a \in Q_1} [a, a^\ast]),$ see [CBH, Section 2]. Also note that the $\mathbb{C}Q_0$-module $\bigoplus_{i \in Q_0} \mathbb{C}^\delta_i$ is nothing else but $f \mathbb{C}\Gamma_1$. Finally, note that $f$ defines a Morita equivalence between $\mathbb{C}[x, y] \# \Gamma_1, \Pi^0(Q)$. An isomorphism $\mathbb{C}^2/\Gamma_1 \cong \mathcal{M}_0^0(\delta, 0)$ now follows from the next lemma, whose proof is left to the reader.

**Lemma 3.9.** Let $A_0 \subset A$ and $V$ be as above and let $f \in A_0$ be an idempotent giving a Morita equivalence. Then the representation varieties for $(A, A_0, V)$ and $(fAf, fA_0f, fV)$ are naturally isomorphic.
Note that the algebras $\mathbb{C}[x, y] \# \Gamma_1$ and $\Pi^0(Q)$ are graded and an isomorphism $\Pi^0(Q) \cong \mathbb{C}[x, y] \# \Gamma_1$ preserves the grading. From here one easily deduces that the isomorphism $\mathbb{C}^2/\Gamma_1 \cong \mathcal{M}_0^0(\delta, 0)$ is equivariant with respect to the dilation $\mathbb{C}^\times$-actions.

3.1.9. $V_n/\Gamma_n$ as a quiver variety. Let us proceed now to the case of an arbitrary $n$. Let $\epsilon_0 \in \mathbb{C}^{2n}$ be the coordinate vector at the extending vertex.

**Proposition 3.10.** We have a $\mathbb{C}^\times$-equivariant isomorphism $\mathcal{M}_0^0(n\delta, \epsilon_0) \cong V_n/\Gamma_n$ (of Poisson schemes).

**Proof.** We have a diagonal embedding $T^*R(Q, \delta, 0)^{\oplus n} \to T^*R(Q, n\delta, \epsilon_0)$, compare to 3.1.6 that restricts to $\mu_1^{-1}(0)^n \hookrightarrow \mu_1^{-1}(0)$, where $\mu_1$ stands for the moment map $T^*R(Q, \delta, 0) \to \mathfrak{gl}(\delta)^\times$. This gives rise to a $\mathfrak{S}_n$-invariant morphism $\mathcal{M}_0^0(\delta, 0) \to \mathcal{M}_0^0(n\delta, \epsilon_0)$ and hence to a morphism $\iota : \mathbb{C}^{2n}/\Gamma_n = (\mathbb{C}^2/\Gamma_1)^n/\mathfrak{S}_n \to \mathcal{M}_0^0(n\delta, \epsilon_0)$. One can show that this morphism is bijective. Also it is $\mathbb{C}^\times$-equivariant, where the $\mathbb{C}^\times$-actions on $\mathbb{C}^{2n}/\Gamma_n, \mathcal{M}_0^0(n\delta, \epsilon_0)$ are induced from the dilation actions on $\mathbb{C}^{2n}, T^*R(Q, n\delta, \epsilon_0)$. It follows that $\iota$ is finite. By Theorem $3.5$, $\mathcal{M}_0^0(n\delta, \epsilon_0)$ is normal and this implies that $\iota$ is an isomorphism.

We can make the isomorphism $\iota$ Poisson if we rescale it using the $\mathbb{C}^\times$-actions. This is a consequence of the following lemma. \[\Box\]

**Lemma 3.11.** [EG] Lemma 2.23 Let $V$ be a symplectic vector space and $\Gamma \subset \text{Sp}(V)$ be a finite subgroup such that $V$ is symplectically irreducible, i.e., there are no proper symplectic $\Gamma$-stable subspace in $V$. Then there are no nonzero brackets (= skew-symmetric bi-derivations) of degree $< -2$ on $\mathbb{C}[V]^\Gamma$. Further, the space of brackets of degree $-2$ is one-dimensional.

One can ask why we use $\mathcal{M}_0^0(n\delta, \epsilon_0)$ instead of $\mathcal{M}_0^0(n\delta, 0)$ in the proposition. The reason is that the moment map for $T^*R(n\delta, \epsilon_0)$ is flat, this can be checked using Theorem 3.6.

3.1.10. Symplectic resolutions of $V_n/\Gamma_n$. Here we will study symplectic resolutions of $V_n/\Gamma_n$ constructed as non-affine Nakajima quiver varieties for generic stability conditions $\theta$.

Let us consider the case $n = 1$. First let $\tilde{G}$ denote the quotient of $G = \text{GL}(\delta)$ modulo the one-dimensional torus $T_{\text{const}} := \{(x \text{id}_{\mathbb{C}^k})_x \mid x \in \mathbb{C}^\times\}$. Note that the $G$-action on $R := R(Q, \delta, 0)$ factors through $\tilde{G}$. Analogously to Nakajima’s result explained in 3.1.5 the group $\tilde{G}$ acts freely on $\mu_1^{-1}(0)^{\oplus ss}$ if and only if $\theta \cdot \alpha \neq 0$ for every Dynkin root of $Q$ (these are the roots $\alpha \in \mathbb{C}^{2n}$ with $\alpha_0 = 0$). For such $\theta$, we get a conical symplectic resolution $\mathcal{M}_0^0(\delta, 0) \to \mathcal{M}_0^0(\delta, 0)$, this can be deduced, for example, from Theorem 3.7. Of course, all these resolutions are isomorphic to the minimal resolution $\mathbb{C}^2/\Gamma_1$: there are just no other symplectic resolutions.

Let us proceed to the case $n > 1$. We get a projective morphism $\mathcal{M}_0^0(n\delta, \epsilon_0) \to \mathcal{M}_0^0(n\delta, \epsilon_0)$. Theorem 3.7 no longer holds, in fact, $\mu_1^{-1}(0)$ has $n+1$ irreducible components by [GG1] Section 3.2. Still, $\mathcal{M}_0^0(n\delta, \epsilon_0) \to \mathcal{M}_0^0(n\delta, \epsilon_0)$ is a resolution of singularities. One just needs to check that the fiber over a generic point in $\mathcal{M}_0^0(n\delta, \epsilon_0)$ consists of a single point. A generic closed $G$-orbit in $\mu_1^{-1}(0)$ has a point of the form $r^1 \oplus \ldots \oplus r^n$, where $r^1, \ldots, r^n$ are pair-wise non-isomorphic simple representations of $\Pi^0(Q)$ of dimension $\delta$. Then one can analyze the structure of the $G$-action near that orbit using a symplectic slice theorem, see, for example, [CB2] Section 4 or 4.3.3 below. This analysis shows that there is a unique semistable $G$-orbit containing $Gr$ in its closure. So we see that $\mathcal{M}^\theta(n\delta, \epsilon_0) \to \mathcal{M}_0^0(n\delta, \epsilon_0)$ is a conical symplectic resolution.
3.1.11. **Isomorphic resolutions.** Now let us discuss how many resolutions we get. The stability condition $\theta$ is generic if $\theta \cdot \delta \neq 0$ and $\theta \cdot v \neq 0$ for $v$ of the form $v = \alpha + m\delta$, where $\alpha$ is a Dynkin root and $|m| < n$. So we get resolutions labeled by the open cones in the complement to these hyperplanes in $\mathbb{R}^n$. However, some of these resolutions are isomorphic: there is an action of $W \times \mathbb{Z}/2\mathbb{Z}$ on $\mathbb{Z}^{\mathbb{Q}_0}$ such that, for $\theta, \theta'$ lying in one orbit, the resolutions $\mathcal{M}_0^\theta(n\delta, \epsilon_0) \to \mathcal{M}_0^\theta(n\delta, \epsilon_0)$, $\mathcal{M}_0^{\theta'}(n\delta, \epsilon_0) \to \mathcal{M}_0^{\theta'}(n\delta, \epsilon_0)$ are isomorphic (here $W$ denotes the Weyl group of the Dynkin diagram obtained from $Q$ by removing the vertex 0). This is a special case of a construction due to Namikawa, $[\text{Nam}]$, that we are going to explain now.

Let $X \to X_0$ be an arbitrary symplectic resolution. The variety $X_0$ has finitely many symplectic leaves, $[\mathcal{R}]$. Let $L_1, \ldots, L_k$ be the leaves of codimension 2. Take formal slices $S_1^\delta, \ldots, S_k^\delta$ through $L_1, \ldots, L_k$. The slices are formal neighborhoods of 0 in Kleinian singularities $S_1, \ldots, S_k$. From these Kleinian singularities one produces Weyl groups $\tilde{W}_1, \ldots, \tilde{W}_k$ (of the same types as the singularities) acting on the spaces $H^2(S_k, \mathbb{C})$ identified with their reflection representations $\tilde{h}_i$. The fundamental group $\pi_1(L_i)$ acts on the irreducible components of the exceptional divisor in $S_i$. Hence it also acts on $\tilde{W}_i$ (by diagram automorphisms) and on $\tilde{h}_i$. Set $W_i := \tilde{W}_i^{\pi_1(L_i)}$, $h_i := \tilde{h}_i^{\pi_1(L_i)}$ so that $W_i$ is a crystallographic reflection group and $h_i$ is its reflection representation. There is a natural restriction map $H^2(X) \to h := \bigoplus_i h_i$. Namikawa proved that this map is surjective. Furthermore, he has constructed a $W := \prod_i W_i$-action on $H^2_{DR}(X)$ that makes the map equivariant and is trivial on the kernel.

Let us return to our situation. The symplectic leaves in $V/\Gamma$ are in one-to-one correspondence with conjugacy classes of stabilizers of points in $V$. The leaf corresponding to $\Gamma' \subset \Gamma$ is the image of $V^{\Gamma', reg} := \{v \in V|\Gamma_v = \Gamma'\}$ under the quotient morphism $\pi : V \to V/\Gamma$. The leaf is identified with $V^{\Gamma', reg}/N_\Gamma(\Gamma')$. So, in the case when $V = V_n$ and $\Gamma = \Gamma_n$, we get two leaves of codimension 2 (provided $\Gamma_1 \neq \{1\}$, in that case we get just one leaf of codimension 2). One of them, say $L_1$, corresponds to $\Gamma_1 \subset \Gamma_n$ (the stabilizer of a point of the form $(0, p_1, \ldots, p_{n-1})$, where $p_1, \ldots, p_{n-1}$ are pairwise different points of $\mathbb{C}^2$). The other, say $L_2$, corresponds to $\Sigma_2$ (the stabilizer of $(p_1, p_1, p_2, \ldots, p_{n-1})$). The fundamental group actions from the previous paragraph are easily seen to be trivial. So we get $W_1 = W, W_2 = \mathbb{Z}/2\mathbb{Z}$. Further, $H^2(X) = \mathbb{C}^{\mathbb{Q}_0}$ and $h_1 = \{(x_i)_{i \in \mathbb{Q}_0}|x \cdot \delta = 0\}, h_2 = \mathbb{C}\delta$. The group $W_2$ acts on $\mathbb{C}\delta$ by $1$, while $h_1$ is identified with the Cartan space for $W_1$ via $(x_i)_{i \in \mathbb{Q}_0} \mapsto \sum_{i=1}^{\mathbb{Q}_0} x_i \omega_i^{\gamma}$. We write $\omega_i^{\gamma}$ for the fundamental coweights.

Let us remark that the $W$-action can be recovered by using the quiver variety setting as well, see $[\mathbb{M}]$ and $[\mathbb{L}3]$ for more detail.

### 3.2. Quantum Hamiltonian reduction

Here we will explain a quantum counterpart of the constructions of the previous section.

#### 3.2.1. Quantum Hamiltonian reduction: algebra level. Let $A$ be an associative algebra, $g$ a Lie algebra and $\Phi : g \to A$ be a Lie algebra homomorphism. Then, for a character $\lambda$ of $g$, set $\mathcal{I}_\lambda := A\{x - \langle \lambda, x \rangle, x \in g\}$; this is a left ideal in $A$ that is stable under the adjoint action of $g$. We set $A//\lambda g := (A/\mathcal{I}_\lambda)^\theta$. This space has a natural associative product given by $(a + \mathcal{I}_\lambda)(b + \mathcal{I}_\lambda) := ab + \mathcal{I}_\lambda$. With this product, $A//\lambda g$ becomes naturally isomorphic to $\text{End}_A(A/\mathcal{I}_\lambda)^{opp}$, an element $a + \mathcal{I}_\lambda$ gets mapped to the unique endomorphism sending $1 + \mathcal{I}_\lambda$ to $a + \mathcal{I}_\lambda$. We also have a universal variant of quantum Hamiltonian reduction: $A//g := (A/A\Phi([g, g]))^\theta$. 
Now suppose $\mathcal{A}$ is a filtered quantization of $\mathbb{C}[X]$, where $X$ is an an affine Poisson variety (we assume that the bracket on $\mathbb{C}[X]$ has degree $-1$). Suppose that $G$ acts on $X$ in a Hamiltonian way and the functions $\mu^*(\xi)$ have degree $1$ for all $\xi \in \mathfrak{g}$. By a quantization of the Hamiltonian $G$-action on $\mathbb{C}[X]$ we mean a rational $G$-action on $\mathcal{A}$ together with a $G$-equivariant map $\Phi: \mathfrak{g} \to \mathcal{A}$ such that

(i) the filtration on $\mathcal{A}$ is $G$-stable and the isomorphism $\text{Gr} \mathcal{A} \cong \mathbb{C}[X]$ is $G$-equivariant,

(ii) $\Phi(\xi)$ lies in $\mathcal{A}_{\ell_1}$ and coincides with $\mu^*_h(\xi)$ modulo $\mathcal{A}_{\ell_0}$,

(iii) and $[\Phi(\xi), \cdot] = \xi \cdot A$, where $\xi \cdot A$ is the derivation of $\mathcal{A}$ coming from the $G$-action.

Note that $\text{Gr} \mathcal{I}_\lambda \supset I := \mathbb{C}[X] \mu^*(\mathfrak{g})$ and so we have a surjective homomorphism $\mathbb{C}[X//\theta_0 \mathcal{G}] \twoheadrightarrow \text{Gr} \mathcal{A}//\lambda \mathcal{G}$. We want to get a sufficient condition for $\text{Gr} \mathcal{I}_\lambda = I$ for all $\lambda$.

**Lemma 3.12.** Let $\xi_1, \ldots, \xi_n$ be a basis in $\mathfrak{g}$. Suppose $\mu^*(\xi_1), \ldots, \mu^*(\xi_n)$ form a regular sequence. Then $\text{Gr} \mathcal{I}_\lambda = I$ for any $\lambda$.

**Proof.** The proof is based on the observation that the 1st homology in the Koszul complex associated to $\mu^*(\xi_1), \ldots, \mu^*(\xi_n)$ is zero. In other words, if $f_1, \ldots, f_n \in \mathbb{C}[X]$ are such that $\sum_{i=1}^n f_i \mu^*(\xi_i)$, then there are $f_{ij} \in \mathbb{C}[X]$ with $f_{ij} = -f_{ji}$ and $f_i = \sum_{j=1}^n f_{ij} \mu^*(\xi_j)$. Details of the proof are left to the reader. \hfill $\square$

So if $G$ is reductive and the assumptions of Lemma 3.12 hold, then $\mathcal{A}//\lambda \mathfrak{g}$ is a filtered quantization of $\mathbb{C}[X//\theta_0 \mathcal{G}]$.

We can also give the definition of a quantization of a Hamiltonian action in the setting of formal quantizations. One should modify (i)-(iii) as follows. In (i) one requires the $G$-action to be $\mathbb{C}[[h]]$-linear and the isomorphism $\mathcal{A}_0/h \mathcal{A}_0 \cong \mathbb{C}[X]$ has to be $G$-equivariant. In (ii), one requires that $\Phi(\xi)$ coincides with $\mu^*(\xi)$ modulo $h$. In (iii) one requires $\xi_i [\Phi(\xi), \cdot] = \xi_i A$. We then can consider reductions of the form $\mathcal{A}_0/h \mathcal{A}_0 \mathcal{G}$, where $\lambda(h)$ is an element in $(\mathfrak{g}^* \mathcal{G})[[h]]$. If $G$ is reductive, and the elements $\mu^*(\xi_i) - \langle \lambda(0), \xi_i \rangle, i = 1, \ldots, n$, form a regular sequence in $\mathbb{C}[X]$, then $\mathcal{A}_0/h \mathcal{A}_0 \mathcal{G}$ is a formal quantization of $\mathbb{C}[X//\lambda(0) \mathcal{G}]$.

### 3.2.2. Quantum Hamiltonian reduction: sheaf level

Let $X$ be a smooth affine symplectic algebraic variety equipped with a Hamiltonian action of $G$ and let $\theta$ be a character of $G$. Assume that, for a basis $\xi_1, \ldots, \xi_n$ of $\mathfrak{g}$, the elements $\mu^*(\xi_1), \ldots, \mu^*(\xi_n)$ form a regular sequence at points of $\mu^{-1}(0)^{g-ss}$. Let $\mathcal{D}_h$ be a formal quantization of $\mathcal{O}_X$. Our goal is to define a (formal) quantization $\mathcal{D}_h//\theta \mathcal{L}_\lambda \mathcal{G}$ of $X//\theta \mathcal{G}$ (so $\lambda(0) = 0$).

Recall that it is enough to define the following data:

1. For an open affine covering $X//\theta \mathcal{G} := \bigcup Y_i$, the algebras of sections $\Gamma(Y_i, \mathcal{D}_h//\theta \mathcal{L}_\lambda \mathcal{G})$ that quantize $Y_i$,
2. and identifications $\Gamma(Y_i, \mathcal{D}_h//\theta \mathcal{L}_\lambda \mathcal{G})|_{Y_i \cap Y_j} \cong \Gamma(Y_j, \mathcal{D}_h//\theta \mathcal{L}_\lambda \mathcal{G})|_{Y_i \cap Y_j}$ satisfying cocycle conditions.

Recall that we can choose an open covering by setting $Y_i := X_{f_i} // \theta \mathcal{G}$, where polynomials $f_i \in \mathbb{C}[X]^G$ are such that $\mathbb{C}^{\theta-ss} = \bigcup Y_i$. Then we set $\Gamma(Y_i, \mathcal{D}_h//\theta \mathcal{L}_\lambda \mathcal{G}) := \Gamma(X_{f_i} \cap \mathcal{D}_h//\theta \mathcal{L}_\lambda \mathcal{G})$. The sections of the corresponding sheaf on $Y_i \cap Y_j$ are easily seen to be $\Gamma(Y_i \cap Y_j, \mathcal{D}_h//\theta \mathcal{L}_\lambda \mathcal{G})$ and this yields the gluing maps.

Now let us discuss the period map mentioned in [2.2.6]. Suppose that the $G$-action on $\mu^{-1}(0)^{g-ss}$ is free so that $X///\theta \mathcal{G}$ is smooth and symplectic. In this case we have a period map associated to the quantization of $\mathcal{D}_h//\theta \mathcal{L}_\lambda \mathcal{G}$. Assume, for simplicity, that $\lambda(h) := \lambda h$ for $\lambda \in \mathfrak{g}^* \mathcal{G}$ – this is the most interesting case, for example, it is the only case that appears when we work with the filtered setting. Further, assume that $\mathcal{D}_h$ is canonical,
induced from the Bernstein filtration on $D$ identity on the level of associated graded algebras (we consider the filtration on where $c$).

3.2.3. Algebra vs sheaf level. We need to relate the sheaf $D_h//_\lambda G$ to the algebra $D_h//_\lambda G$.

3.2.4. Isomorphism theorem. Recall a $\mathbb{C}^\times$-equivariant isomorphism $\mathcal{C}^{2n}/\Gamma_n \cong \mathcal{M}^\theta(n\delta, \epsilon_0)$ of Poisson varieties. The left hand side admits a family of quantizations, $eH_{1,c,e}$, and so does the right hand side, there quantizations are the quantum Hamiltonian reductions $D(R)//_\lambda G$, where we use the symmetrized quantum comoment map $\Phi(\xi) = \frac{1}{2}(\xi_R + \xi_{R^*})$.

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$$\frac{1}{|\Gamma_1|} \left( 1 + \sum_{\gamma \in \Gamma_1 \setminus \{1\}} c(\gamma) \gamma \right),$$

where $c(\gamma) := c_i$ for $\gamma \in S_i$ (recall that $S_0$ is the conjugacy class of a reflection in $\mathfrak{S}_n \subset \Gamma_n$ and $S_1, \ldots, S_r$ are conjugacy classes of elements of $\Gamma_1 \subset \Gamma_n$).

Theorem 3.14. We have a filtered algebra isomorphism $eH_{1,c,e} \cong D(R)//_\lambda G$ that is the identity on the level of associated graded algebras (we consider the filtration on $D(R)//_\lambda G$ induced from the Bernstein filtration on $D(R)$, where $\deg R = \deg R^* = 1$). Here $\lambda := \sum_{i=0}^r \lambda_i \text{tr}_i$ is recovered from $c$ by the following formulas:

$$\lambda_i := \text{tr}_{N_i} c, \quad i = 1, \ldots, r, \quad \lambda_0 := \text{tr}_{N_0} c - \frac{1}{2}(c_0 + 1),$$

where in the $n = 1$ case one needs to put $c_0 = 1$. 
For $n = 1$, this theorem was proved by Holland in [Ho]. The case of $\Gamma_1 = \{1\}$ was handled in [EG] [GG1] (EG proved a weaker statement and then in [GG1] the proof was completed). The case of cyclic $\Gamma_1$ was done in [O, G2]. In [EGGO] the proof was completed: they considered the case when $Q$ is a bi-partite graph. Let us note that in these papers formulas look different from (5): they use the quantum comoment map $\Phi(\xi) = \xi_R$. A uniform and more conceptual proof was given in [L3] using Procesi bundles.

Theorem 3.14 is of crucial importance for the representation theory of the algebras $H_{1,c}$. It turns out that the representation theory of the algebras $D(R)///_\lambda G$ (actually, of sheaves $D(R)///_\lambda G$) is easier to study. The main ingredient here is the geometry of the quiver varieties $\mathcal{M}^\theta(v, \epsilon_0)$. Using this, in [BL], the author and Bezrukavnikov have proved a conjecture of Etingof [E], on the number of the finite dimensional irreducible representations of $H_{1,c}$.

3.2.5. Automorphisms. Here we are going to explain a quantum version of Namikawa’s construction recalled in [3.11]. In the complete generality this construction was given in [BPW, Section 3.3].

Let $X$ be a conical symplectic resolution of $X_0$. Let $\tilde{X}$ be its universal deformation over $H^2_{DR}(X)$ and let $\tilde{D}_h$ be the canonical quantization of $\tilde{X}$. Let $\tilde{A}_h$ denote the $\mathbb{C}^\times$-finite part of $\Gamma(\tilde{D}_h)$. Then Namikawa’s Weyl group $W$ acts on $\tilde{A}_h$ by graded $\mathbb{C}[h]$-algebra automorphisms preserving $H^2_{DR}(X)^\ast$. Moreover, the action on $H^2_{DR}(X)^\ast$ is as explained in [3.11].

3.3. Quantum Hamiltonian reduction for Frobenius constant quantizations. In this section, we will consider the situation in characteristic $p$. Our main result is that a quantum GIT Hamiltonian reduction under a free Hamiltonian action is again Frobenius constant.

3.3.1. GIT in characteristic $p$. The definition of a reductive group (one with trivial unipotent radical) makes sense in all characteristics. A crucial difficulty of dealing with reductive groups in positive characteristic is that their rational representations are no longer completely reducible, in general. The groups for which the complete reducibility holds are called linearly reductive. Tori are still linearly reductive independently of the characteristic. We need to deal with GIT for reductive groups (such as products of $GL$’s) and so we need to explain how this works in positive characteristic.

It turns out that reductive groups satisfy a weaker condition than being linearly reductive, they are geometrically reductive. This was conjectured by Mumford and proved by Haboush, [Hab]. To state the condition of being geometrically reductive, let us re-formulate the linear reductivity first: a group $G$ is called linearly reductive, if, for any linear $G$-action on a vector space $V$ and any fixed point $v \in V$, there is $f \in (V^\ast)^G$ with $f(v) \neq 0$. A group $G$ is called geometrically reductive if instead of $f \in (V^\ast)^G$, one can find $f \in S^r(V^\ast)^G$ (for some $r > 0$) with $f(v) \neq 0$.

This condition is enough for many applications. For example, if $X$ is an affine algebraic variety acted on by a reductive (and hence geometrically reductive) group $G$, then $\mathbb{F}[X]^G$ is finitely generated. So we can consider the quotient morphism $X \rightarrow X//G$. This morphism is surjective and separates the closed orbits. Moreover, if $X' \subset X$ is a $G$-stable subvariety, then the natural morphism $X'//G \rightarrow X//G$ is injective with closed image.

The claim about the properties of the quotient morphism in the previous paragraph can be deduced from the following lemma, [MFK, Lemma A.1.2].
Lemma 3.15. Let $G$ be a geometrically reductive group acting on a finitely generated commutative $\mathbb{F}$-algebra $R$ rationally and by algebra automorphisms. Let $I \subset R$ be a $G$-stable ideal and $f \in (R/I)^G$. Then there is $n$ such that $fr^n$ lies in the image of $R^G$ in $(R/I)^G$.

In characteristic $p$, we can still speak about unstable and semistable points for reductive group actions on vector spaces, about GIT quotients, etc.

Another very useful and powerful result of Invariant theory in characteristic 0 is Luna’s étale slice theorem, see, e.g., [PV, 6.3]. There is a version of this theorem in characteristic $p$ due to Bardsley and Richardson, see [BR]. We will need a consequence of this theorem dealing with free actions.

Recall that, in characteristic 0, an action of an algebraic group $G$ on a variety $X$ is called free if the stabilizers of all points are trivial. In characteristic $p$ one should give this definition more carefully: the stabilizer may be a nontrivial finite group scheme with a single point. An example is provided by the left action of $G$ on $G^{(1)}$, we will discuss a closely related question in the next subsection. We have the following three equivalent definitions of a free action.

- For every $x \in X$, the stabilizer $G_x$ equals $\{1\}$ as a group scheme.
- For every $x \in X$, the orbit map $G \to X$ corresponding to $x$ is an isomorphism of algebraic varieties.
- For every $x \in X$, $G_x$ coincides with $\{1\}$ as a set and the stabilizer of $x$ in $g$ is trivial.

The following is a weak version of the slice theorem that we need.

Lemma 3.16. Let $X$ be a smooth affine variety equipped with a free action of a reductive algebraic group $G$. Then the quotient morphism $X \to X/G$ is a principal $G$-bundle in étale topology.

3.3.2. Quiver varieties. Let us now discuss Nakajima quiver varieties in characteristic $p \gg 0$. We have a finite localization $\mathfrak{R}$ of $\mathbb{Z}$ with the following properties:

(1) $R$ together with the $G$-action and $\mu$ are defined over $\mathfrak{R}$.
(2) $\mu^{-1}(0)^{\theta-ss}$ and the $G$-bundle $\mu^{-1}(0)^{\theta-ss}/G$ are defined over $\mathfrak{R}$.

For an $\mathfrak{R}$-algebra $\mathfrak{R}'$, let $R_{\mathfrak{R}'}, G_{\mathfrak{R}'}, \mu_{\mathfrak{R}'}$ etc. denote the $\mathfrak{R}'$-forms of the corresponding objects. Let us write $X_{\mathfrak{R}}$ for an $\mathfrak{R}$-form of $\mu^{-1}(0)^{\theta-ss}/G$. After a finite localization of $\mathfrak{R}$, we can achieve that $X_{\mathfrak{R}}$ is a symplectic scheme over $\text{Spec}(\mathfrak{R})$ with $\mathbb{C} \otimes_{\mathfrak{R}} \Gamma(X_{\mathfrak{R}}, \mathcal{O}_{X_{\mathfrak{R}}}) \cong \mathbb{C}[X_C]$ and $H^i(X_{\mathfrak{R}}, \mathcal{O}_{X_{\mathfrak{R}}}) = 0$ for $i > 0$.

For $\mathfrak{R}'$, we can take $\mathbb{F} := \overline{\mathbb{F}}_p$ when $p$ is large enough. So we get a symplectic $\mathbb{F}$-variety $M^\theta_0(n, 1)$ that is naturally identified with $T^*R_{\mathbb{F}'}/\mu_0^G_{\mathbb{F}'}$ as well as with $\text{Spec}(\mathbb{F}) \times_{\text{Spec}(\mathfrak{R})} X_{\mathfrak{R}}$. For $p \gg 0$, we get $\mathbb{F}[X_{\mathfrak{R}}] = \mathbb{F} \otimes_{\mathfrak{R}} \Gamma(X_{\mathfrak{R}}, \mathcal{O}_{X_{\mathfrak{R}}})$ and $H^i(X_{\mathfrak{R}}, \mathcal{O}_{X_{\mathfrak{R}}}) = 0$.

We can take a finite algebraic extension of $\mathfrak{R}$ and assume that the $\Gamma_n$-module $\mathbb{C}^{2n}$ is defined over $\mathfrak{R}$. Now we claim that (again for $p \gg 0$) $M^\theta_0(n, 1)$ is a symplectic resolution of $\mathbb{F}^{2n}/\Gamma_n$. This follows from the claim that both $\Gamma(X_{\mathfrak{R}}, \mathcal{O}_{X_{\mathfrak{R}}}), \mathfrak{R}[x, y]^{T_{\mathfrak{R}}} = \mathfrak{R}$-forms of $\mathbb{C}[x, y]^{T_{\mathfrak{R}}}$ so they coincide after some finite localization of $\mathfrak{R}$.

3.3.3. Quantum Hamiltonian reduction. Now suppose that $R$ is a symplectic vector space over $\mathbb{F}$, $G$ is a reductive group over $\mathbb{F}$ acting on $R$ and $\theta$ is a character of $G$. We suppose that $G$ acts freely on $\mu^{-1}(0)^{\theta-ss}$. We are going to define a Frobenius constant quantization $D_R/\mu^G$ of $T^*R/\mu^G$, where $\lambda \in \text{Hom}(G, \mathbb{F}^\times) \otimes_{\mathbb{Z}} \mathbb{F}_p \subset g^G$. The associated filtered
quantization of \( T^*R/\mathfrak{g}_0 G \) will be a quantization obtained by quantum Hamiltonian reduction, see \([3.2.2]\). We note that for \( \lambda \notin \text{Hom}(G, \mathbb{F}^\times) \) we do not get a Frobenius constant quantization of \( T^*R/\mathfrak{g}_0 G \).

Consider the Frobenius twist \( G^{(1)} \). It is a group and the morphism \( \text{Fr} : G \to G^{(1)} \) is a group epimorphism. Its kernel (a.k.a. the Frobenius kernel) \( G_1 \) is a finite group scheme whose Lie algebra coincides with \( \mathfrak{g} \).

The action of \( G \) on \( R \) induces an action of \( G^{(1)} \) on \( R^{(1)} \). The \( G^{(1)} \)-action on \( T^*R^{(1)} \) is Hamiltonian with moment map \( \mu^{(1)} : T^*R^{(1)} \to \mathfrak{g}^{(1)*} \) induced by \( \mu \). Consider the sheaf \( D_{R/\mathfrak{g}} G_1 \) (a subquotient of \( D_R \)) on \( T^*R^{(1)\theta=ss} \). One can show, see \([BFC]\), that it is supported on \((\mu^{(1)})^{-1}(0)\), here we use that \( \lambda \in \text{Hom}(G, \mathbb{F}^\times) \otimes \mathbb{F}_p \). Moreover, it is a \( G^{(1)} \)-equivariant Azumaya algebra on \((\mu^{(1)})^{-1}(0)\). The descent of this algebra to \( (T^*R/\mathfrak{g}_0 G)^{(1)} = T^*R^{(1)}/\mathfrak{g}_0 G^{(1)} \) is an Azumaya algebra with a filtration induced from that on \( D_R \). We have a natural homomorphism \( \text{gr}(D_{R/\mathfrak{g}} G_1) \to \text{Fr}_* \mathcal{O}_{T^*R/\mathfrak{g}_0 G_1} \). To show that it is an isomorphism one uses that the action of \( G_1 \) is free (that yields the required cohomology vanishing). This isomorphism implies \( \text{gr}(D_{R/\mathfrak{g}} G) \cong \text{Fr}_* \mathcal{O}_{T^*R/\mathfrak{g}_0 G} \). So \( D_{R/\mathfrak{g}} G_1 \) is indeed a Frobenius constant quantization.

Note that if \( \lambda \in \text{Hom}(G, \mathbb{F}^\times) \otimes \mathbb{F}_p \), then \( D_{R/\mathfrak{g}} G_1 \) is supported on a nonzero fiber of \( \mu^{(1)} \), see \([BFC]\) for details, and so \( D_{R/\mathfrak{g}} G \) is no longer a Frobenius constant quantization of \( X/\mathfrak{g}_0 G \).

### 4. Existence and classification of Procesi bundles

In this section we construct and classify Procesi bundles on \( X = M^\theta(n\delta, \epsilon_0) \) and also prove Theorem 3.14.

In Section 4.1 we construct a Procesi bundle on \( X \). The case \( n = 1 \) is relatively easy, it was done in \([KaVa]\). For \( n > 1 \), we follow \([BK2]\). A key step here is to construct a special Frobenius constant quantization of \( X_\mathbb{F} \), where \( \mathbb{F} \) is an algebraically closed field of large enough positive characteristic. This quantization provides a suitable version of derived McKay equivalence and using this equivalence we can produce a Procesi bundle over \( \mathbb{F} \). Then we lift it to characteristic 0.

In Section 4.2 we prove that Symplectic reflection algebras satisfy PBW property and, in some sense, the family of SRA \( H_{t,\epsilon} \) is universal with this property. The proof is based on computing relevant graded components in the Hochschild cohomology of \( SV \# \Gamma \).

Theorem 3.14 is proved in Section 4.3. Using the Procesi bundle, we show that each algebra \( D(R)/\mathfrak{g}_0 G \) is isomorphic to some \( eH_{t,\epsilon} \). Then the task is to show that the correspondence between the parameters \( \lambda \) and the parameters \( c \) is as in Theorem 3.14. We first do this for \( n = 1 \). Then we reduce the case of \( n > 1 \) to \( n = 1 \) by studying completions of the algebras involved. This allows to show that the map between the parameters \( \lambda \) and the parameters \( c \) is conjugate to that in Theorem 3.14 up to a conjugation under an action of the group \( W \times \mathbb{Z}/2\mathbb{Z} \), where \( W \) is the Weyl group of the finite part of the quiver \( Q \). But from \([3.2.3]\) we know that this action lifts to an action on the universal reduction \( D(R)/\mathfrak{g}_0 G \) by automorphisms. This completes the proof of Theorem 3.14.

Then, in Section 4.4 we classify Procesi bundles. Namely, we show that, when \( n > 1 \), there are \( 2|W| \) different Procesi bundles on \( X \). For this, we use Theorem 3.14 to produce this number of bundles. And then we use techniques used in the proof to show that the number cannot exceed \( 2|W| \). Further, we show that each \( X \) carries a distinguished Procesi bundle.
4.1. Construction of Procesi bundles.

4.1.1. Baby case: $n = 1$. In this case it is easy to construct a vector bundle of required rank on $X$. Namely, for $i = 0, \ldots, r$, let $U_i$ be the $G$-module $\mathbb{C}^b_i$ and let $U_i$ be the corresponding vector bundle on $X$. We set $\mathcal{P} := \bigoplus_{i=0}^r U_i^b$. It follows from results of Kapranov and Vasserot, [KaVa], that this bundle satisfies the axioms of a Procesi bundle.

4.1.2. Procesi bundles and derived McKay equivalence. Before we proceed to constructing Procesi bundles in general, let us explain their connection to derived McKay equivalences, i.e., equivalences $\mathcal{D}(\text{Coh } X) \sim \mathcal{D}(\mathbb{K}[V_n]#\Gamma_n)$, here $\mathbb{K}$ stands for the base field.

**Proposition 4.1.** Let $\mathcal{P}$ be a Procesi bundle on $X$. Then the functor $\mathcal{R} \text{Hom}_{\text{Coh } X}(\mathcal{P}, \bullet)$ is a derived equivalence $\mathcal{D}(\text{Coh } X) \rightarrow \mathcal{D}(\mathbb{K}[V_n]#\Gamma_n)$-mod).

The proof is based on the following more general result (Calabi-Yau trick) of (in this form) Bezrukavnikov and Kaledin.

**Proposition 4.2.** [BK2 Proposition 2.2] Let $X$ be a smooth variety, projective over an affine variety, with trivial canonical class. Furthermore, let $\mathcal{A}$ be an Azumaya algebra over $X$ such that $\Gamma(\mathcal{A})$ has finite homological dimension and $H^i(X, \mathcal{A}) = 0$ for $i > 0$. Then the functor $\mathcal{R} \Gamma : \mathcal{D}(\text{Coh }(X, \mathcal{A})) \rightarrow \mathcal{D}(\mathcal{A})$-mod) is an equivalence.

Proposition 4.1 follows from Proposition 4.2 with $\mathcal{A} = \mathcal{E}nd(\mathcal{P})$.

Now suppose that we have a derived equivalence $\iota : \mathcal{D}(\text{Coh } X) \sim \mathcal{D}(\mathbb{K}[V_n]#\Gamma_n)$-mod).

Assume $\mathcal{P}' := \iota^{-1}(\mathbb{K}[V_n]#\Gamma_n)$ is a vector bundle. Then $\mathcal{E}nd_{\text{Coh } X}(\mathcal{P}') = \mathbb{K}[V]#\Gamma$ and $\text{Ext}^i(\mathcal{P}', \mathcal{P}) = 0$ for $i > 0$. So $\mathcal{P}'$ is, basically, a Procesi bundle (it also needs to be $\mathbb{K}^b$-equivariant, but we will see below that this always can be achieved). In fact, this is roughly, how the construction of a Procesi bundle will work, although it is more involved and technical.

4.1.3. Quantization of $X$. Here and in 4.1.4 everything is going to be over an algebraically closed field $\mathbb{F}$ of characteristic $p \gg 0$. The first step in the construction of a Procesi bundle is to produce a Frobenius constant quantization of $X$ with special properties.

**Proposition 4.3.** There is a Frobenius constant quantization $\mathcal{D}$ of $X$ such that $\Gamma(\mathcal{D}) = \mathcal{A}(V_n)^G_n$ (an isomorphism of filtered algebras over $\mathbb{F}[X^{(1)}] = \mathbb{F}[V_n^{(1)}]^G_n$).

Note that this proposition can be thought as a special case of the characteristic $p$ version of Theorem 3.14. Here $\Gamma(\mathcal{D})$ is an analog of $D(R)\lambda/\lambda G$ (indeed, the latter is the algebra of global sections of some filtered quantization of $X_{\mathbb{C}}$, see Proposition 3.13), while $\mathcal{A}(V_n)^G_n$ is the characteristic $p$ analog of $eH_{1,0}e$.

In fact, the following is true.

**Lemma 4.4.** Theorem 3.14 (for $c = 0$) implies Proposition 4.3.

**Proof.** First, let us see that we get an isomorphism $\Gamma(\mathcal{D}) \cong \mathcal{A}(V_n)^G_n$ of filtered algebras that is the identity on the associated graded algebra. Set $\mathcal{D} := D(R)\lambda/\lambda G$, where $\lambda$ is the parameter corresponding to $c = 0$.

The algebra $\mathcal{A}(V_n, \mathbb{C})^G_n$ is finitely generated and so an isomorphism in Theorem 3.14 is defined over some finitely generated subring $\mathfrak{R}$ of $\mathbb{C}$. We can enlarge $\mathfrak{R}$ and assume that we are in the situation described in 3.3.2. We can form filtered quantizations $\mathcal{D}'_{\mathbb{C}}, \mathcal{D}'_{\mathfrak{R}}, \mathcal{D}'_F$ of $X_{\mathbb{C}}, X_{\mathfrak{R}}, X$. Both $\mathcal{D}'_{\mathbb{C}}, \mathcal{D}'_F$ are obtained as suitable completions of base changes of $\mathcal{D}'_{\mathfrak{R}}$. 
(completions are necessary because of our condition on the filtration in the definition of a filtered quantization, see \[2.2.4\]. In particular, \(D(R_\mathcal{C})///_\lambda G_\mathcal{C} = (\Gamma(D) = )\mathbb{C} \otimes_{\mathcal{G}} \Gamma(D_{\mathcal{G}})\), while \(\Gamma(D) = (\Gamma(D_{\mathcal{G}}) = )\mathbb{F} \otimes_{\mathbb{Q}} \Gamma(D_{\mathcal{G}})\).

So we can reduce an isomorphism from Theorem \[3.14\] (for \(c = 0\)) \(\mod p \gg 0\) and get an isomorphism \(\Gamma(D) \cong \mathbb{A}(V_n)^{\Gamma_n}\). What remains to show is that this isomorphism is \(\mathbb{F}[V_n^{(1)}]^{\Gamma_n}\)-linear. The first step here is to show that \(\mathbb{F}[V_n^{(1)}]^{\Gamma_n}\) is the center of \(\mathbb{A}(V_n)^{\Gamma_n}\). It is enough to check that \(\mathbb{F}[V_n^{(1)}]^{\Gamma_n}\) coincides with the center of the Poisson algebra \(\mathbb{F}[V_n]\). Here we just note that the Poisson center of \(\mathbb{F}[V_n]\) is finite and birational over \(\mathbb{F}[V_n^{(1)}]^{\Gamma_n}\) and use that the latter algebra is normal. So the isomorphism \(\Gamma(D) \cong \mathbb{A}(V_n)^{\Gamma_n}\) induces an automorphism of \(\mathbb{F}[V_n^{(1)}]^{\Gamma_n}\). This isomorphism preserves the filtration and is trivial on the level of associated graded algebras.

The second step is to show that the algebra \(\mathbb{F}[V_n^{(1)}]^{\Gamma_n}\) has no nontrivial automorphisms \(\varphi\) with such properties. Let us define a derivation \(\psi\) of \(\mathbb{F}[V_n^{(1)}]^{\Gamma_n}\) that should be thought as \(\ln \varphi\). The degrees of generators of \(\mathbb{F}[V_n^{(1)}]^{\Gamma_n}\) are bounded from above for all \(p \gg 0\) and so are degrees of relations between them. Observe that it is only enough to define a derivation on generators and it will be well-defined as long as it sends all relations to 0. Now to construct \(\psi\) we note that \(\varphi - 1\) decreases degrees, and hence \(\psi := \ln \varphi\) makes sense as long as \(p\) is sufficiently large. The derivation \(\psi\) lifts to \(\mathbb{F}[V_n^{(1)}]\) because the quotient morphism \(V_n^{(1)} \to V_n^{(1)}/\Gamma_n\) is ramified in codimension bigger than 1. Since it decreases degrees, we see that \(\psi\) has the form \(\partial_v\) for some \(v \in \mathbb{F}^{2n(1)}\). But, if \(\Gamma_1 \neq \{1\}\), the vector \(v\) cannot be \(\Gamma_n\)-equivariant and so \(\partial_v\) does not preserve \(\mathbb{F}[V_n^{(1)}]^{\Gamma_n}\). When \(\Gamma_1 = \{1\}\), there is a \(\Gamma_n\)-invariant vector. However, in this case we can modify our construction: consider the reflection representation \(\mathfrak{h}\) of \(\mathfrak{S}_n\) instead of the permutation representation \(\mathbb{C}^n\). We need to replace \(R\) with \(\mathfrak{sl}_n \oplus \mathbb{C}^n\). Theorem \[3.14\] gets modified accordingly. \(\square\)

However, the easiest way to prove Theorem \[3.14\] is by using Procesi bundles (at least for non-cyclic \(\Gamma_1\) or general \(c\), the case \(c = 0\) may be easier). So we need some roundabout way to construct \(\mathcal{D}\). In \[BK2\] the question of existence of \(\mathcal{D}\) was reduced to \(n = 1\). More precisely, let \(V^{sr}\) denote the set of all \(v \in V_n\) such that \(\dim V^{\Gamma_n} > 2\). Let us write \(X_1 := \rho^{-1}(V^{sr}/\Gamma_n)\). This is an open subset in \(X\) with \(\text{codim}_X X \setminus X_1 > 1\). First, Bezrukavnikov and Kaledin produce a Frobenius constant quantization \(\mathcal{D}_1\) of \(X_1\) with \(\Gamma(\mathcal{D}_1) = \mathbb{A}(V_n)^{\Gamma_n}\). This requires the existence of such a quantization in the case when \(n = 1\). The latter case can be handled using Theorem \[3.14\] proved in this case by Holland (that can be alternatively proved using the existence of a Procesi bundle in the case \(n = 1\)). When \(\mathcal{D}_1\) is constructed, Bezrukavnikov and Kaledin use the inequality \(\text{codim}_X X \setminus X_1 > 1\) to show that \(\mathcal{D}_1\) uniquely extends to a Frobenius constant quantization \(\mathcal{D}\) of \(X\), automatically with \(\Gamma(\mathcal{D}) = \mathbb{A}(V_n)^{\Gamma_n}\).

4.1.4. Construction of a Procesi bundle: characteristic \(p\). Let \(\mathcal{D}\) be as in the previous subsection. We will produce a Procesi bundle on \(X^{(1)}\) starting from \(\mathcal{D}\). Since \(X^{(1)} \cong X\) (an isomorphism of \(\mathbb{F}\)-varieties), this will automatically establish a Procesi bundle on \(X\). The isomorphism \(X^{(1)} \cong X\) follows from the observation that \(X\) is defined over \(\mathbb{F}_p\) and \(\text{Fr}\) is an isomorphism of \(\mathbb{F}\) fixing \(\mathbb{F}_p\).

By Proposition \[4.2\] we have a derived equivalence \(D^b(\text{Coh}(X^{(1)}, \mathcal{D})) \sim \mathbb{D}(\mathbb{A}(V_n)^{\Gamma_n}\text{-mod})\). Also we have an abelian equivalence \(\mathbb{A}(V_n)^{\Gamma_n}\text{-mod} \sim \mathbb{A}(V_n)^{\# \Gamma_n}\text{-mod} = \mathbb{A}(V_n)^{\text{-mod}}\).
Composing the two equivalences, we get
\[
D^b(\text{Coh}(X, D)) \sim D^b(\text{A}(V_n)\text{-mod}^\Gamma_n),
\]
while what we need is a derived McKay equivalence
\[
D^b(\text{Coh}(X^{(1)})) \sim D^b(\mathbb{F}[V_n^{(1)}]\text{-mod}^\Gamma_n).
\]
Recall that $D$ is an Azumaya algebra on $X$, while $\text{A}(V_n)$ is a $\Gamma_n$-equivariant Azumaya algebra on $V_n^{(1)}$. If we had a splitting and a $\Gamma_n$-equivariant splitting, respectively, we would get (7) from (6). However, this is obviously not the case: $\text{A}(V_n)$ admits no splitting at all.

This can be fixed by passing to completions at 0. Namely, let $X^{(1)\#0}$ denote the formal neighborhood of $(\rho^{(1)})^{-1}(0)$ in $X^{(1)}$. It was checked in [BK2, Section 6.3] that the restriction of $D$ to $X^{(1)\#0}$ splits. Also it was checked that the restriction of $\text{A}(V_n)$ to the formal neighborhood of 0 in $\mathbb{F}^{2n^{(1)\#0}}$ admits a $\Gamma_n$-equivariant splitting. So, we get an equivalence
\[
i : D^b(\text{Coh}(X^{(1)\#0})) \sim D^b(\mathbb{F}[V_n^{(1)\#0}]\text{-mod}_\Gamma_n)
\]
that makes the following diagram commutative (all arrows are equivalences of triangulated categories and all arrows but $R\Gamma$ come from abelian equivalences):

\[
\begin{array}{cccc}
D^b(\text{Coh}(X^{(1)\#0}, D)) & \xrightarrow{R\Gamma} & D^b(\text{A}(V_n)^{\#0}\text{-mod}^\Gamma_n) & \xrightarrow{\bullet} & D^b(\text{A}(V_n)^{\#0}\text{-mod}^\Gamma_n) \\
| & & | & & | \\
D^b(\text{Coh}(X^{(1)\#0})) & \xrightarrow{\mathcal{B}^{*}\otimes\bullet} & D^b(\mathbb{F}[V_n^{(1)\#0}]\text{-mod}) & \xrightarrow{\mathcal{B}^{*}\otimes\bullet} & D^b(\mathbb{F}[V_n^{(1)\#0}]\text{-mod})
\end{array}
\]

Here $\mathcal{B}$ denotes a splitting bundle for the restriction of $D$ to $X^{(1)\#0}$.

Set $\mathcal{P}': = i^{-1}(\mathbb{F}[V_n^{(1)\#0}]\text{-mod}^\Gamma_n)$. We claim that $\mathcal{P}'$ is a vector bundle on $X^{(1)\#0}$. Indeed, the image of $\mathbb{F}[V_n^{(1)\#0}]\text{-mod}^\Gamma_n$ in $\text{A}(V_n)^{\#0}\text{-mod}^\Gamma_n$ is a projective generator and so is a direct summand in the sum of several copies of $\text{A}(V_n)^{\#0}\text{-mod}^\Gamma_n$. But $R\Gamma^{-1}(\text{A}(V_n)^{\#0}\text{-mod}) = \mathcal{B}^{*}$. So $\mathcal{P}'$ is a direct summand in a vector bundle (the sum of several copies of $\mathcal{B}^{*}$) and hence is a vector bundle itself.

So we get a vector bundle $\mathcal{P}'$ on $X^{(1)\#0}$ that satisfies $\text{End}(\mathcal{P}') \cong \mathbb{F}[V_n^{(1)\#0}]\text{-mod}^\Gamma_n$, $\text{Ext}^i(\mathcal{P}', \mathcal{P}) = 0$ for $i > 0$. The latter vanishing implies that $\mathcal{P}'$ is equivariant with respect to the $\mathbb{F}_X$-action on $X^{(1)\#0}$, see [V]. From here it follows that $\mathcal{P}'$ can be extended to $X^{(1)}$ (this is because $\mathbb{F}_X$ contracts $X^{(1)}$ to the zero fiber, see [BK2, Section 2.3]). Moreover, we can modify the equivariant structure on $\mathcal{P}'$ and achieve that the isomorphism $\text{End}(\mathcal{P}') \cong \mathbb{F}[V_n^{(1)\#0}]\text{-mod}^\Gamma_n$ is $\mathbb{F}_X$-equivariant, see [L4, Section 3.1]. It follows that $\mathcal{P}$ is a Procesi bundle.

4.1.5. Construction of a Procesi bundle: lifting to characteristic 0. Recall the $\mathcal{R}$-scheme $X_{\mathcal{R}}$ from [BK2]. We may assume $\mathcal{R}$ is regular. Taking an algebraic extension of $\mathcal{R}$, we get a maximal ideal $\mathfrak{m}$ such that there is a Procesi bundle $\mathcal{P}_{\mathcal{R}}$ on $X_{\mathcal{R}}$, where $\mathcal{R}$ is an algebraic closure of $\mathbb{F}_0 := \mathcal{R}/\mathfrak{m}$. We may assume that $\mathcal{P}_{\mathcal{R}}$ is defined over $\mathbb{F}_0$, let $\mathcal{P}_{\mathcal{R}_0}$ be the corresponding form. Let $\mathcal{R}_{\mathcal{m}}$ be the $\mathcal{m}$-adic completion of $\mathcal{R}$. Since $\text{Ext}^i(\mathcal{P}_{\mathcal{R}_0}, \mathcal{P}_{\mathcal{R}_0}) = 0$ for $i = 1, 2$, we see that $\mathcal{P}_{\mathcal{R}_0}$ uniquely deforms to a $\mathcal{G}_{\mathcal{m}}$-equivariant vector bundle on the formal neighborhood of $X_{\mathcal{R}_0}$ in $X_{\mathcal{R}}$ (see [BK2, Section 2.3]).

Let us show that the $\mathcal{G}_{\mathcal{m}}$-finite part of $\text{End}(\mathcal{P}_{\mathcal{R}_0})$ is $\mathcal{R}_{\mathcal{m}}[V_n^{\#0}]\text{-mod}^\Gamma_n$. Consider the formal neighborhood $Z$ of $X_{\mathcal{R}_0}^{\text{reg}}$ in $X_{\mathcal{R}}^{\text{reg}}$. Note that $\text{Ext}^1(\mathcal{P}_{\mathcal{R}_0}|_{X_{\mathcal{R}_0}^{\text{reg}}}, \mathcal{P}_{\mathcal{R}_0}|_{X_{\mathcal{R}_0}^{\text{reg}}}) = 0$, see, for example,
So the restriction of $P_{\mathfrak{g}^\Lambda}$ to $Z$ coincides with $\eta_*\mathcal{O}_{(\mathfrak{g}^\Lambda)^{reg}}$, where $\eta$ denotes the quotient morphism $\mathfrak{g}^{\Lambda n} \to \mathfrak{g}^{\Lambda n}/\Gamma$. This implies the claim about endomorphisms.

Since $P_{\mathfrak{g}^\Lambda}$ is $\mathbb{G}_m$-equivariant and the $\mathbb{G}_m$-action is contracting, it extends from a formal neighborhood of $X_{\mathfrak{g}}$ in $X_{\mathfrak{g}^\Lambda}$. So we get a Procesi bundle on $X_K$, where $K = \text{Frac}(\mathfrak{g}^\Lambda)$. But being a finite extension of the $p$-adic field, $K$ embeds into $\mathbb{C}$ and so we get a Procesi bundle on $X$.

### 4.2. Symplectic reflection algebras

#### 4.2.1. Flatness and universality

Let $V$ be a symplectic vector space with form $\Omega$ and $\Gamma \subset \text{Sp}(V)$ be a finite group of symplectomorphisms. We write $S$ for the set of symplectic reflections in $\Gamma$, it is a union of conjugacy classes: $S = S_0 \sqcup S_1 \sqcup \ldots \sqcup S_r$. We pick independent variables $t, c_0, \ldots, c_r$.

Recall the universal Symplectic reflection algebra $H$, the quotient of $T(V)^\#\Gamma[t, c_0, \ldots, c_r]$ by the relations [3]. Let us write $c_{\text{univ}}$ for the vector space with basis $t, c_0, \ldots, c_r$ so that $H$ is a graded $S(c_{\text{univ}})$-algebra.

**Theorem 4.5.** The algebra $H$ is a free graded $S(c_{\text{univ}})$-module. Moreover, assume that $\Gamma$ is symplectically irreducible. Then $H$ is universal with this property in the following sense. Let $c'$ be a vector space and $H'$ be a graded $S(c')$-algebra (with $\deg c' = 2$) that is a free graded $S(c')$-module and $H'/\langle c' \rangle = S(V)^\#\Gamma$. Then there is a unique linear map $\nu : c_{\text{univ}} \to c'$ and unique isomorphism $S(c') \otimes_{S(c_{\text{univ}})} H \cong H'$ of graded $S(c')$-algebras that induces the identity isomorphism of $S(V)^\#\Gamma$.

When $\Gamma_1 \neq \{1\}$, then the action of the group $\Gamma_n$ on $V_n = \mathbb{C}^{2n}$ is symplectically irreducible. When $\Gamma_1 = \{1\}$, the module $\mathbb{C}^{2n}$ over $\Gamma_n$ is not symplectically irreducible, so we replace $\mathbb{C}^{2n}$ with $V_n = \mathfrak{h} \oplus \mathfrak{h}^*$, where $\mathfrak{h}$ is the reflection representation of $\mathbb{G}_n$. Note that we did the same in [4.1.3]

#### 4.2.2. Hochschild cohomology

Before we prove this theorem we will need to get some information about Hochschild cohomology of $S(V)^\#\Gamma$. We need this because the Hochschild cohomology controls deformations of $S(V)^\#\Gamma$.

Let $A$ be a graded algebra. We want to describe graded deformations of $A$. The Hochschild cohomology group $HH^i(A)$ inherits the grading from $A$, let $HH^i(A)^j$ denote the $j$th graded component. The general deformation theory implies the following.

**Lemma 4.6.** Assume that $\dim HH^2(A)^{-2} < \infty$ and $HH^i(A)^j = 0$ for $i + j < 0$. Set $P_{\text{univ}} := (HH^2(A)^{-2})^*$. Then there is a free graded $S(P_{\text{univ}})$-algebra $A_{\text{univ}}$ (with $\deg P_{\text{univ}} = 2$) such that $A_{\text{univ}}/\langle P_{\text{univ}} \rangle = A$ that is a universal graded deformation of $A$ in the same sense as in Theorem 4.5.

What we are going to do is to compute the relevant graded components of $HH^*(SV^\#\Gamma)$. The vanishing result is easy and the computation of $P_{\text{univ}}$ is more subtle.

First, we use the fact that $HH^i(A, M) = \text{Ext}^i_{A \otimes A^{opp}}(A, M)$ (where $M$ is an $A$-bimodule) to see that

$$HH^i(S(V)^\#\Gamma, SV^\#\Gamma) = HH^i(S(V), S(V)^\#\Gamma)^\Gamma.$$  

We have a $\Gamma$-action on $HH^i(S(V), S(V)^\#\Gamma)$ because both $S(V)$-bimodules $S(V), S(V)^\#\Gamma$ are $\Gamma$-equivariant. We have $S(V)^\#\Gamma = \bigoplus_{\gamma \in \Gamma} S(V)_\gamma$ of $S(V)$-bimodules, where $S(V)_\gamma$ is identified with $S(V)$ as a left $S(V)$-module and the right action is given by $f \cdot a := f\gamma(a)$.  

Let us compute $\text{HH}^i(S(V), S(V)\gamma)$ in degrees we are interested in: $j < -i$ and also $j = -2$ for $i = 2$. We have $\gamma = \text{diag}(\gamma_1, \ldots, \gamma_n)$, where we view $\gamma_i$ as elements of cyclic groups acting on $\mathbb{C}$. Then we have an isomorphism of bigraded spaces

\[
\bigoplus_i \text{HH}^i(S(V), S(V)\gamma) \cong \bigotimes_{\ell=1}^n \text{HH}^i(\mathbb{C}[x], \mathbb{C}[x]\gamma_\ell).
\]

For an arbitrary $\gamma_\ell$, we have $\text{HH}^i(\mathbb{C}[x], \mathbb{C}[x]\gamma_\ell) = 0$ when $i > 1$. When $\gamma_\ell = 1$, we have $\text{HH}^0(\mathbb{C}[x], \mathbb{C}[x]) = \mathbb{C}[x]$ and $\text{HH}^1(\mathbb{C}[x], \mathbb{C}[x]) = \mathbb{C}[x]\{1\}$, where $\{1\}$ indicates the grading shift by 1 so that $\text{HH}^1(\mathbb{C}[x], \mathbb{C}[x])$ is a free module generated in degree $-1$. When $\gamma_\ell \neq 1$, then $\text{HH}^0(\mathbb{C}[x], \mathbb{C}[x]\gamma_\ell) = 0$ and $\text{HH}^1(\mathbb{C}[x], \mathbb{C}[x]) = \mathbb{C}\{1\}$.

This computation easily implies that $\text{HH}^i(S(V), S(V)\#\Gamma)^j = 0$ when $i + j < 0$. Now let us explain how to compute $(\text{HH}^2(S(V), S(V)\#\Gamma)^{-2})^\Gamma$. If $\text{HH}^2(S(V), S(V)^i\gamma)^{-2} \neq 0$, then either $\gamma = 1$ or $\gamma$ is a symplectic reflection. When $\gamma = 1$, then $\text{HH}^2(S(V), S(V)^i\gamma)^{-2} = \wedge^2 V$. When $\gamma$ is a symplectic reflection, then $\text{HH}^2(S(V), S(V)^i\gamma)^{-2} = \mathbb{C}$. An element $\gamma_1 \in \Gamma$ maps $S(V)^i\gamma$ to $S(V)(\gamma_1\gamma_1^{-1})$. The action of $\Gamma$ on $\text{HH}^2(S(V), S(V)^i\gamma)^{-2} = \wedge^2 V$ is a natural one. When $\gamma$ is a symplectic reflection, then the action of $Z_\Gamma(\gamma)$ on $\text{HH}^2(S(V), S(V)^i\gamma)^{-2} = \mathbb{C}$ is trivial. From here we deduce that

\[
\dim \text{HH}^2(S(V)\#\Gamma, S(V)\#\Gamma)^{-2} = r + 2,
\]

as claimed.

4.2.3. Proof of Theorem 4.5 Let us write $H_{\text{univ}}$ for the universal deformation, we need to prove that $H_{\text{univ}} \tilde{\to} H$.

First of all, note that degree 0 and 1 components of $H_{\text{univ}}$ are the same as in $S(V)\#\Gamma$. So we have natural embeddings $\Gamma, V \hookrightarrow H_{\text{univ}}$. It is easy to see that $c_{\text{univ}}, V, \Gamma$ generate $H_{\text{univ}}$. This gives rise to an epimorphism $S(c_{\text{univ}}) \otimes T(V)\#\Gamma \twoheadrightarrow H_{\text{univ}}$. Further, for $u, v \in V \subset H_{\text{univ}}$, we have $[u, v] \in (c_{\text{univ}})$. The degree 2 of $(c_{\text{univ}})$ is $c_{\text{univ}} \otimes \mathbb{C}\Gamma$.

So we get $[u, v] = \kappa(u, v)$ in $H_{\text{univ}}$, where $\kappa$ is a map $\wedge^2 V \to c_{\text{univ}} \otimes \mathbb{C}\Gamma$. A computation done in [EG, Section 2] shows that, since $H_{\text{univ}}$ is free over $S(c_{\text{univ}})$, we get

\[
\kappa = t\Omega + \sum_{i=0}^r c_i \sum_{s \in S_i} \Omega_s(u, v)s.
\]

This completes the proof of Theorem 4.5.

4.3. Proof of the isomorphism theorem. We will prove an isomorphism of $e\text{He}$ and the universal Hamiltonian reduction $A := A_h(T^*R)\!/\!/G$, where $A_h(T^*R)$ is the Rees algebra of $D(R)$ (with modified grading so that $\text{deg}T^*R = 1$, $\text{deg}h = 2$). Here we take $R := R(Q, n\delta_0, \epsilon_0)$ for $n > 1$ and $R := R(Q, \delta, 0)$ for $n = 1$. In the case when $n > 1$, we take $G := \text{GL}(n\delta)$. For $n = 1$, for $G$, we take the quotient of $\text{GL}(\delta)$ by the one-dimensional central subgroup of constant elements.

Both $e\text{He}, A$ are graded algebras. The algebra $e\text{He}$ is over $S(c_{\text{univ}})$ with deg $c_{\text{univ}} = 2$. The algebra $A$ is over $S(c_{\text{red}})$, where $c_{\text{red}} := \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \oplus \mathbb{C}h$. We will prove that there is a graded algebra isomorphism $e\text{He} \tilde{\to} A$ that maps $c_{\text{univ}}$ to $c_{\text{red}}$ and induces the identity automorphism $e\text{He}/(c_{\text{univ}}) = \mathbb{C}[V_{\mathfrak{g}}]^H = A/(c_{\text{red}})$. Further, we will explain why the corresponding isomorphism $c_{\text{univ}} \cong c_{\text{red}}$ maps $h$ to $t$ and gives (5) on the hyperplanes.
t = 1 and \( \hbar = 1 \). In other words, the isomorphism \( \nu : c_{\text{univ}} \to c_{\text{red}} \) is the inverse of the following map

\[
\begin{align*}
\hbar \mapsto t, & \quad \epsilon_i \mapsto \frac{1}{1!} \text{tr}_{\mathcal{N}_i} \tilde{c}, \ i \neq 0, \\
& \quad \epsilon_0 \mapsto \frac{1}{1!} \text{tr}_{\mathcal{N}_0} \tilde{c} - \frac{1}{2} (\epsilon_0 + t), \quad (n > 1) \\
\hbar \mapsto t, & \quad \epsilon_i \mapsto \frac{1}{1!} \text{tr}_{\mathcal{N}_i} \tilde{c}, \ i \neq 0, \\
& \quad \epsilon_0 \mapsto \frac{1}{1!} \text{tr}_{\mathcal{N}_0} \tilde{c} - t, \quad (n = 1)
\end{align*}
\]

Here the notation is as follows. We write \( \tilde{c} := 1 \). In other words, the isomorphism \( \text{tr}_i, \epsilon_j := \delta_{ij} \).

### 4.3.1. Application of a Procesi bundle

An isomorphism \( e\mathcal{H} \mathcal{e} \cong \mathcal{A} \) is produced as follows. The algebra \( e\mathcal{H} \mathcal{e} \) does not have good universality properties (although it is expected to be semi-universal), it is \( \mathcal{H} \) that does. We will produce a graded \( S(c_{\text{red}}) \)-algebra \( \tilde{\mathcal{A}} \) deforming \( \mathbb{C}[V_0] \# \Gamma_n \) with \( e\tilde{\mathcal{A}} \mathcal{e} = \mathcal{A} \). This will give rise to a linear map \( \nu : c_{\text{univ}} \to c_{\text{red}} \) and to an isomorphism \( S(c_{\text{red}}) \otimes S(c_{\text{univ}}) \mathcal{A} \cong \tilde{\mathcal{A}} \) and hence also to an isomorphism \( S(c_{\text{red}}) \otimes S(c_{\text{univ}}) \mathcal{H} \cong \mathcal{A} \). The algebra \( \tilde{\mathcal{A}} \) will be constructed from a Procesi bundle \( \mathcal{P} \) on \( X = \mathcal{M}^\theta (n\delta, \epsilon_0) \).

First, let us produce a sheaf version of \( \mathcal{A} \). Consider the variety \( \tilde{X} := T^* R/\mathbb{G}^\theta \), this is a deformation of \( X \) over \( \mathfrak{g}^* \mathbb{G} \). Then we consider its deformation quantization obtained by Hamiltonian reduction, the sheaf \( \tilde{\mathcal{D}}_h := \mathcal{A}_h (T^* R)^{\mathfrak{g}^* \mathbb{G}} / \mathbb{G}^\theta \). The algebra \( \tilde{\mathcal{A}} \) coincides with the \( \mathbb{C}^\times \)-finite part of \( \Gamma(\tilde{\mathcal{D}}_h) \). Now let us take a Procesi bundle \( \mathcal{P} \) on \( X \). Since \( \text{Ext}^i (\mathcal{P}, \mathcal{P}) = 0 \), the bundle \( \mathcal{P} \) deforms to a unique \( \mathbb{C}^\times \)-equivariant vector bundle on the formal neighborhood of \( X \) in \( \tilde{X} \). But the \( \mathbb{C}^\times \)-action contracts \( \tilde{X} \) to \( X \). So \( \mathcal{P} \) extends to a unique \( \mathbb{C}^\times \)-equivariant bundle \( \tilde{\mathcal{P}} \) on \( \tilde{X} \). The extension \( \tilde{\mathcal{P}} \) again satisfies the Ext-vanishing conditions and so further extends to a unique \( \mathbb{C}^\times \)-equivariant right \( \tilde{\mathcal{D}}_h \)-module \( \tilde{\mathcal{P}}_h \).

Consider the endomorphism algebra \( \text{End}_{\tilde{\mathcal{D}}_h^{\text{gen}}} (\tilde{\mathcal{P}}_h) \). Modulo \( (c_{\text{red}}) \), this algebra coincides with \( \text{End}_{\mathcal{O}_X} (\mathcal{P}) = \mathbb{C}[V_0] \# \Gamma_n \). Let \( \tilde{\mathcal{A}} \) be the \( \mathbb{C}^\times \)-finite part of \( \text{End}_{\tilde{\mathcal{D}}_h^{\text{gen}}} (\tilde{\mathcal{P}}_h) \). It is the endomorphism algebra of the right \( \tilde{\mathcal{D}}_{h, \text{fin}} \)-module \( \tilde{\mathcal{P}}_{h, \text{fin}} \). The algebra \( \tilde{\mathcal{A}} \) is a graded \( S(c_{\text{red}}) \)-algebra with \( \tilde{\mathcal{A}}/(c_{\text{red}}) = \mathbb{C}[V_0] \# \Gamma_n \), where \( c_{\text{red}} \) lives in degree 2. We conclude that there is a unique map \( \nu : c_{\text{univ}} \to c_{\text{red}} \) with \( \tilde{\mathcal{A}} \cong S(c_{\text{red}}) \otimes S(c_{\text{univ}}) \mathcal{H} \). Then, automatically, we have

\[
A(\mathcal{A} \otimes \mathcal{H}) \cong S(c_{\text{red}}) \otimes S(c_{\text{univ}}) e\mathcal{H} e.
\]

We will study the linear maps \( \nu : c_{\text{univ}} \to c_{\text{red}} \) such that \((11)\) holds. We will see that

(a) any such \( \nu \) is an isomorphism,

(b) that there are \( |W| \) options for \( \nu \) when \( n = 1 \) and \( 2|W| \) options else,

(c) and that one can choose \( \nu \) as in \((10)\).

(c) will complete the proof of Theorem \(3.14\) while (b) will be used to classify the Procesi bundles.

First of all, let us point out that \( \nu(t) = h \). Indeed, the Poisson bracket on \( \mathbb{C}[\mathcal{M}^\theta (n\delta, \epsilon_0)] \) induced by the deformation \( \mathcal{A} \) equals \( h \{ \cdot, \cdot \} \), where \( \{ \cdot, \cdot \} \) is the standard bracket given by the Hamiltonian reduction (more precisely, if we specialize to \( (h', \lambda) \in \mathbb{C} \oplus \mathfrak{g}^* \mathbb{G} \), then the bracket induced by the corresponding filtered deformation is \( h' \{ \cdot, \cdot \} \)). Similarly, the bracket on \( \mathbb{C}[V_0] \mathcal{P} \), induced by \( e\mathcal{H} e \) coincides with \( t \{ \cdot, \cdot \} \), see Example \(2.24\). Since the isomorphism \( \mathcal{M}^\theta (n\delta, \epsilon_0) \cong V_0 / \Gamma_n \) is Poisson, the equality \( \nu(t) = h \) follows.
4.3.2. Case $n = 1$. We start by proving (a)-(c) for $n = 1$.

Let us prove (c). First of all, recall that $X$ can be constructed as the moduli space of the $\mathbb{C}[x, y]/\mathbb{C}G$-modules isomorphic to $\mathbb{C}G$ as $\mathbb{C}G$-modules that admit a cyclic vector. The universal bundle on $X$ is a Procesi bundle. Moreover, from [CBH] Section 8, it follows that $\tilde{X}$ is the moduli space of the $H/(t)$-modules isomorphic to $\mathbb{C}G$ and admitting a cyclic vector. The corresponding isomorphism $c_{\text{red}}/\mathbb{C}h \cong c_{\text{univ}}/\mathbb{C}t$ is induced from $\nu$.

To show that $\nu$ then is given by [10] we consider the loci of parameters $\lambda$ and $c$ where the homological dimensions of $A_{1,\lambda} := A(T^*R)/\mathbb{C}G$, $eH_{1,\lambda}e$ are infinite. Both are given by the union of hyperplanes of the form $\lambda \cdot \beta = 0$, where $\beta$ runs over the set of the roots of $Q \setminus \{0\}$ (when we speak of the parameter $\lambda$ for the algebra $eH_{1,\lambda}e$ we mean the parameter computed in Theorem 3.11). The claim for $eH_{1,\lambda}e$ follows from [CBH] Theorem 0.4, and that on $A_{1,\lambda}$ then follows from [L3, Section 5] (from an isomorphism of $A_{1,\lambda}$ with a central reduction of a suitable W-algebra) or from [B3].

The same considerations as in the previous paragraph imply (a). To prove (b) one now needs to describe the group $\mathfrak{A}$ of the automorphisms of $A(\cong eHe)$ satisfying the following:

- they preserve the grading,
- they preserve $c_{\text{red}}$ as a subset of $A$,
- they are the identity modulo $c_{\text{red}}$.

We have a natural homomorphism $\mathfrak{A} \to \text{GL}(c_{\text{red}})$ that is easily seen to be injective. From the isomorphism with a W-algebra mentioned above, one sees that $W \subset \mathfrak{A}$ (recall that the $W$-action on $g^{*G}$ was described in [3.1.11]). With some more work, see [L3, Proposition 6.4.5], one shows that actually $W = \mathfrak{A}$. This implies (b).

4.3.3. Completions. The case of a general $n$ is reduced to $n = 1$ using suitable completions of the algebras $A, H$. Let us explain what completions we use as well as general results on their structure.

First, let us describe completions of algebras of the form $A := A_h(V)//G$, where $V$ is a symplectic vector space and $G$ is a reductive group acting on $V$ by symplectomorphisms. Let $b \in V//_0G$. The point $b$ defines a maximal ideal $m \subset A$. So we can form the $b$-adic completion $A^{\wedge_b} := \lim_{n \to +\infty} A/m^n$. Let $v \in V$ be a point with closed $G$-orbit mapping to $b$. Let us write $A_h(V)^{\wedge_G}$ for the completion of $A_h(V)$ with respect to the ideal of $Gv$. Then it is easy to see that $A^{\wedge_b} \cong A_h(V)^{\wedge_G}//G$.

The algebra $A_h(V)^{\wedge_G}$ can be described using a suitable version of the slice theorem. More precisely, it follows, for example, from [CB2, Section 4] that the formal neighborhood $V^{\wedge_G}$ is equivariantly symplectomorphic to the neighborhood of the base $G/K$ in $(T^*G \times U)//_0K$, where $K := G_v, U := (T_vGv)^+//T_vGv$. This statement quantizes: $A_h(V)^{\wedge_G} \cong (D_h(G) \otimes_{\mathbb{C}[G]} A_h(U))//_0K$, this can be proved similarly to [L1, Theorem 2.3.1]. From here one deduces that

$$A^{\wedge_b} \cong C[[\mathfrak{g}^{*G}]] \otimes_{C[[\mathfrak{t}^{*K}]]} (A_h(U)^{\wedge_c} //K),$$

where a homomorphism $C[[\mathfrak{t}^{*K}]] \to C[[\mathfrak{g}^{*G}]]$ is induced from the restriction map $\mathfrak{g}^{*G} \to \mathfrak{t}^{*K}$.

On the other hand, take a symplectic vector space $V'$ and a finite subgroup $\Gamma \subset \text{Sp}(V')$. From these data we can form the symplectic reflection algebra $H$. Pick $b \in V'/\Gamma$. We can produce the completion $H^{\wedge_b}$: the point $b$ defines a natural maximal ideal in $C[V']//\Gamma$, we take its preimage in $H$ and complete with respect to that preimage. The algebra $H^{\wedge_b}$ can also be described in terms of a “smaller” algebra of the same type, [L2, Theorem 1.2.1].
More precisely, let \( \Gamma \) be the stabilizer corresponding to \( b \) and let \( \mathbf{H} \) stand for the SRA corresponding to the pair \((\mathbf{L}, V')\), an algebra over \( S(\mathbf{c}_\text{univ}) \). Then \( \mathbf{H}^{\chi} \cong \mathbb{Z}(\Gamma, \mathbf{H}^{\chi_0}) \), where \( Z(\Gamma, \mathbf{L}, \bullet) \) is the centralizer algebra from [BE 3.2], it is isomorphic to \( \text{Mat}_{\chi}(\bullet) \). A consequence we need is that \( e\mathbf{H}^{\chi_0}e \cong e\mathbf{H}^{\chi_0}e \). The algebra \( \mathbf{H} \) can be described as follows. Let us write \( V^+ \) for a unique \( \mathbf{L} \)-stable complement to \( V^E \) in \( V' \). Consider the SRA \( \mathbf{H}^+ \) over \( S(\mathbf{c}_\text{univ}) \), where \( \mathbf{c}_\text{univ} \) is the parameter space for \( \mathbf{L} \). The inclusion \( \mathbf{L} \hookrightarrow \mathbf{L} \) gives rise to a natural map \( \mathbf{c}_\text{univ} \rightarrow \mathbf{c}_\text{univ} \). Then \( \mathbf{H} = A_t(V^+) \otimes_{C[t]} (S(\mathbf{c}_\text{univ}) \otimes S(\mathbf{c}_\text{univ}) \mathbf{H}) \).

4.3.4. Completions at leaves of codimension 2. We are going to use the completions of \( A \) and \( e\mathbf{H}e \) at points lying in the codimension 2 symplectic leaves. Recall from [3.11] that when \( n > 1 \) and \( \Gamma_1 \neq \{1\} \), we have two such leaves. One corresponds to \( \mathbf{L} = \Gamma_1 \subset \mathbf{G}_n \), the other to \( \mathbf{G}_2 \subset \mathbf{G}_n \). Let \( \mathbf{H}^{1+}, \mathbf{H}^{2+} \) be the corresponding SRA’s. The corresponding parameter spaces are \( \mathbf{c}^{1}_{\text{univ}} = \text{Span}(c_1, \ldots, c_r, t) \) and \( \mathbf{c}^{2}_{\text{univ}} = \text{Span}(c_0, t) \). When \( \Gamma_1 = \{1\} \), we have just one leaf of codimension 2, it corresponds to \( \mathbf{G}_2 \).

Now let us describe the completions on the Hamiltonian reduction side. Let \( v^1, v^2 \) be elements from closed \( G \)-orbits in \( \mu^{-1}(0) \in T^*R \) whose images \( b^1, b^2 \) in \( \mathcal{M}_0((n\delta, \epsilon_0), V_\nu/\Gamma_n \) lie in the two leaves. We can take the points \( v^1, v^2 \) as follows. We have a natural embedding \( \mu^{-1}(0)^n \hookrightarrow \mu^{-1}(0) \) from the proof of Proposition [3.10] Take pairwise different elements \( v_1, \ldots, v_n \in \mu^{-1}(0) \) with closed \( GL(\delta) \)-orbits. Then we can take \( v^1 = (v_1, \ldots, v_{n-1}, 0) \in T^*R(n\delta, 0) \subset T^*R \) and \( v^2 = (v_1, \ldots, v_{n-2}, v_{n-1}, v_{n-1}) \).

Let us describe the completion \( A^{\wedge_1} \). We have \( K_1(= G_{v^1}) = (\mathbb{C}^n)^{n-1} \times GL(\delta) \). So the space \( \mathfrak{t}_{1}K_1 \) coincides \( \mathbb{C}^n \oplus \mathbb{C}^q_0 \). The restriction map \( \mathbb{C}^q_0 = \mathfrak{g}^G \rightarrow \mathfrak{t}_{1}K_1 = (\mathbb{C}^n \oplus \mathbb{C}^q_0) \) sends \( \lambda \) to \( (\lambda, \delta, \ldots, \lambda, \delta, \lambda) \). The symplectic part \( U \) of the normal space \( T^*R/T_{v^1}Gv^1 \) splits into the direct sum of the trivial module \( \mathbb{C}^{(n-1)} \), of the \( (\mathbb{C}^n)^{n-1} \)-module \( (T^*\mathbb{C})^{\oplus n-1} \), and of the \( GL(\delta) \)-module \( T^*R(\delta, \epsilon_0) \). So

\[
A_{h}(U)\bigwedge K_1 \cong C[z_1, \ldots, z_{n-1}] \otimes A_{h}(\mathbb{C}^{(n-1)}) \otimes_{C[q_0]} A_{h}(T^*R(\delta, \epsilon_0)) \bigwedge GL(\delta),
\]

where \( z_1, \ldots, z_{n-1} \) are homogeneous elements of degree 2, the images of the natural basis in \( \text{Lie}(\mathbb{C}^{(n-1)}) \) under the comoment map.

Let us write \( \overline{\text{GL}(\delta)} \) for the quotient of \( GL(\delta) \) by the one-dimensional torus of constant elements. Set \( g^G_0 := g^G/(\mathbb{C}^{n-1}) \), clearly, \( g^G_0 = \overline{\text{GL}(\delta)}^\times \). Set \( A^1 := A_{h}(T^*R(\delta, 0)) \bigwedge GL(\delta). \) It is easy to see that \( A_{h}(T^*R(\delta, \epsilon_0)) \bigwedge GL(\delta) = C[g^G] \otimes_{C[\mathfrak{g}_0]} A^1 \). From here and the description of the map \( \mathfrak{t}_{1}K_1 \rightarrow \mathfrak{g}^G \) given above, we deduce that

\[
C[g^G] \otimes_{C[\mathfrak{t}_{1}K_1]} A_{h}(U) \bigwedge K_1 \cong A_{h}(\mathbb{C}^{2n-2}) \otimes_{C[q_0]} (C[g^G] \otimes_{C[\mathfrak{g}_0]} A^1).
\]

It follows that

\[
A^{\wedge_1} \cong A^{\wedge_0}(\mathbb{C}^{2n-2}) \otimes_{C[q_0]} (C[\mathfrak{c}^*_\text{red}] \otimes_{C[\mathfrak{c}^*_\text{red}]} A^{\wedge_0}),
\]

where we write \( \mathfrak{c}^*_\text{red} \) for \( \{\lambda \in \mathbb{C}[\mathfrak{g}_0] | \lambda \cdot \delta = 0 \} \oplus \mathbb{C}h. \)

Let us now deal with \( A^{\wedge_2} \). We have \( K_2(= G_{v^2}) = (\mathbb{C}^n)^{n-2} \times GL(2). \) The map \( g^G \rightarrow \mathfrak{t}_{2}K_2 \) sends \( \lambda \) to the \( n-1 \)-tuple with equal coordinates \( \lambda \cdot \delta \). The symplectic part \( U^2 \) of the normal space \( T^*R/T_{v^2}Gv^2 \) is the sum of the trivial module \( \mathbb{C}^{(n-1)} \), the \( (\mathbb{C}^n)^{n-2} \)-module \( (T^*\mathbb{C})^{\oplus n-2} \), the \( GL(2) \)-module \( T^*(\mathfrak{sl}_3 \oplus \mathbb{C}^2) \). Let \( \mathfrak{c}^*_\text{red} \) denote the span of \( \sum_{i \in \mathfrak{g}_0} \delta_i \epsilon_i \) and \( h \). Set \( A^2 := A_{h}((T^*(\mathfrak{sl}_3 \oplus \mathbb{C}^2)) \bigwedge GL(2), \) we can view it as an algebra over \( S(\mathfrak{c}^*_\text{red}) \) (where a natural generator of \( \mathfrak{g}_2/[\mathfrak{g}_2, \mathfrak{g}_2] \) corresponds to \( \sum_{i \in \mathfrak{g}_0} \delta_i \epsilon_i \)). As above, we have

\[
C[g^G] \otimes_{C[\mathfrak{t}_{2}K_2]} A_{h}(U^2) \bigwedge K_2 \cong S(\mathfrak{c}^*_\text{red}) \otimes_{S(\mathfrak{c}^*_\text{red})} (A_{h}(\mathbb{C}^{2n-2}) \otimes_{C[q_0]} A^2).
\]
and we get the following description of $A^{\Lambda^2}$:

$$A^{\Lambda^2} \cong A^{a_0}_h(C^{2n-2}) \otimes C[[\hbar]](C[[\epsilon_{\text{red}}]] \otimes_{C[[\epsilon^{*}_{\text{red}}]]} A^{2a_0}).$$

4.3.5. **Reduction to $n = 1$.** Using (12) we see that (11) yields an isomorphism of completions $A^{\Lambda^1} \cong e^1H^{\Lambda^1}e^1$ and hence an isomorphism

$$A^{a_0}_h(C^{2(n-1)}) \otimes C[[\hbar]](C[[\epsilon_{\text{red}}]] \otimes_{C[[\epsilon^{*}_{\text{red}}]]} A^{1a_0}) \cong$$

$$A^{a_0}_h(C^{2(n-1)}) \otimes C[[\hbar]](C[[\epsilon_{\text{red}}]] \otimes_{C[[\epsilon^{*}_{\text{red}}]]} C[[\epsilon^{*}_{\text{univ}}]] \otimes_{C[[\epsilon^{*}_{\text{univ}}]]} e^1H^{1a_0}e^1).$$

It was checked in [L3, Section 6.5] that this isomorphism restricts to

$$S(c_{\text{red}}) \otimes S(c^{*}_{\text{red}}) A^1 \cong S(c_{\text{red}}) \otimes S(c^{*}_{\text{univ}}) e^1H^1e^1$$

that preserves the grading and is the identity modulo $(\epsilon_{\text{red}})$. From here it is easy to deduce that $\nu$ maps $c_{\text{univ}}$ to $c^{*}_{\text{red}}$ and restricts to one of $W$-conjugates of the map in (10) for $n = 1$.

Let us proceed to the second leaf. Similarly to (4.3.4), one can show that $A_h(T^*(sl_2 \oplus C^2))/\!/\! GL(2) \cong e^1H^2C^2$, where the isomorphism sends the element $\sum_{i \in \mathbb{Q}_0} \delta_i \epsilon_i \mapsto \pm (c_0 + t)/2$. It follows that $\nu$ maps $c^*_{\text{univ}}$ to $c^*_{\text{red}}$ and induces one of two maps in the previous sentence. It follows that $\nu$ is an isomorphism that is $W \times \mathbb{Z}/2\mathbb{Z}$-conjugate to the map given by (10) for $n > 1$. Since $W \times \mathbb{Z}/2\mathbb{Z}$-action comes from automorphisms, that preserve the grading, map $c_{\text{red}}$ to $c_{\text{red}}$, and are the identity modulo $(\epsilon_{\text{red}})$ (see 3.2.5), claims (b) and (c) follow. This completes the proof of Theorem 3.14.

4.4. **Classification of Procesi bundles.** Here we are going to prove that the number of different Procesi bundles on $X$ equals $2|W|$ for $n > 1$ and $|W|$ for $n = 1$.

4.4.1. **Upper bound.** Recall that a Procesi bundle $\mathcal{P}$ on $X$ defines a linear isomorphism $\nu_{\mathcal{P}} : c_{\text{univ}} \to c_{\text{red}}$. We claim that if $\nu_{\mathcal{P}_1} = \nu_{\mathcal{P}_2}$, then $\mathcal{P}^1 \cong \mathcal{P}^2$. Indeed, we have

$$(\nu_{\mathcal{P}^1})/(\mathcal{P}^1) = \Gamma(\mathcal{P}_{h,fim}^1) = \text{End}(\mathcal{P}_{h,fim}^1)c = \text{End}(\mathcal{P}_{h,fim}^2)c = \Gamma(\mathcal{P}_{h,fim}^2).$$

(an isomorphism of graded right $H$-modules). Note that $H^1(\tilde{X}, \tilde{\mathcal{P}}^1) = 0$ because $\tilde{\mathcal{P}}^1$ is a direct summand of $\mathcal{E}nd(\mathcal{P}^1)$ and the latter sheaf has no higher cohomology. It follows that

$$\Gamma(\mathcal{P}_{h,fim}^1)/\mathcal{H}(\mathcal{P}_{h,fim}^1) \cong \Gamma(\mathcal{P}^1).$$

Taking the quotient of (13) by $\hbar$, we get an isomorphism $\Gamma(\mathcal{P}^1) \cong \Gamma(\mathcal{P}^2)$ of graded $\mathcal{C}[\tilde{X}]$-modules. We claim that this implies that the vector bundles $\tilde{\mathcal{P}}^1, \tilde{\mathcal{P}}^2$ are $\mathcal{C}^{*}$-equivariantly isomorphic. Indeed, consider the resolution of singularities morphism $\tilde{\rho} : \tilde{X} \to \tilde{X}_0$. This morphism is birational over any $p \in c^{*}_{\text{red}}$. Moreover, for a Zariski generic $p$, the morphism $\rho_p$ is an isomorphism, indeed, $\mu^{-1}(p)^{\theta_{ss}} = \mu^{-1}(p)$. It follows that the restrictions of bundles $\tilde{\mathcal{P}}^1, \tilde{\mathcal{P}}^2$ to some Zariski open subset in $\tilde{X}$ with codimension of complement bigger than 1 are isomorphic. It follows that $\tilde{\mathcal{P}}^1 \cong \tilde{\mathcal{P}}^2$ and hence $\mathcal{P}^1 \cong \mathcal{P}^2$.

We have seen above that $\nu_{\mathcal{P}}$ can only be one of $2|W|$ (for $n > 1$) or $|W|$ (for $n = 1$) maps. This implies the upper bound on the number of Procesi bundles.

4.4.2. **Lower bound.** Let us show that there are $2|W|$ different Procesi bundles in the case of $n > 1$. Recall that one can construct a Procesi bundle $\mathcal{P}_\mathcal{D}$ once one has a Frobenius constant quantization $\mathcal{D}$ of $X_\mathcal{D}$ with $\Gamma(\mathcal{D}) = A(V_{n,p})^{F_n}$. Note that the action of $W \times \mathbb{Z}/2\mathbb{Z}$ on $A$ is defined over some algebraic extension of $\mathbb{Z}$. So, as before, it can be reduced modulo $q$ for $q = p^f, p \gg 0$. Let $\mathcal{D}_\lambda$ be the Frobenius constant quantization obtained by Hamiltonian reduction with parameter $\lambda \in \mathbb{F}_p^{Q_0}$. The parameter $\lambda$ constructed from $c = 0$ belongs to $\mathbb{F}_p^{Q_0}$. Above, we have remarked that $\Gamma(\mathcal{D}_\lambda) \cong A(V_{n,p})^{F_n}$. Moreover, for $q \gg 0,$
Remark 4.4. are different as well, as was checked in [L4, Section 3.3]. By the \( G \in c \nu \) subbundle is as in (10). According to [L4, Section 4.2], this bundle has the following property: the \( W \) of the canonical Procesi bundle \( P \) of a complex reflection group and the corresponding algebra \( \Gamma \) restricts to categories with weight structure. We can also define the category \( \text{Coh}(\mathcal{D}) \), for this bundle. Recall that for \( w \in W \times \mathbb{Z}/2\mathbb{Z} \) we get an isomorphism \( \mathcal{M}(n\delta,\epsilon_0) \cong \mathcal{M}(n\delta,\epsilon_0) \) that yields the map \( c_{\text{red}} = H^2(\mathcal{M}(n\delta,\epsilon_0)) \oplus \mathbb{C} \rightarrow H^2(\mathcal{M}(n\delta,\epsilon_0)) \oplus \mathbb{C} = c_{\text{red}} \) equal to \( w \). It follows that \( \nu_{w,\mathcal{P}} = uwv \). So every other Procesi bundle on \( \mathcal{M}(n\delta,\epsilon_0) \) is obtained as a push-forward of the canonical Procesi bundle \( \mathcal{P} \) on \( \mathcal{M}(n\delta,\epsilon_0) \).

Note that when \( \mathcal{P} \) is a Procesi bundle, then so is \( \mathcal{P}^* \). Indeed, \( \text{End}_{\mathcal{O}_X}(\mathcal{P},\mathcal{P}) \cong \text{End}_{\mathcal{O}_X}(\mathcal{P},\mathcal{P})^{opp} \). The algebra \( \mathbb{C}[V_n]\#\Gamma_n \) is identified with its opposite via \( v \mapsto v, \gamma \mapsto \gamma^{-1}, v \in V_n^*, \gamma \in \Gamma_n \) and this gives a Procesi bundle structure on \( \mathcal{P}^* \). We have \( \nu_{\mathcal{P}^*} = w_0\nu_{\mathcal{P}} \), where \( w_0 \) is the longest element in \( W \) and \( \sigma \) is the image of \( 1 \) in \( \mathbb{Z}/2\mathbb{Z} \), see [L4, Remark 4.4].

5. Macdonald positivity and categories \( \mathcal{O} \)

In this section we provide some applications of results of Section 4. In Section 5.1, we will produce equivalences between categories \( D^b(H_{1,c}) \) and \( D^b(\text{Coh}(\mathcal{D}_\lambda)) \).

Starting from Section 5.2, we will only consider the groups \( \Gamma_n \) with cyclic \( \Gamma_1 \). Here \( \Gamma_n \) is a complex reflection group and the corresponding algebra \( H_{1,c} \) (called a Rational Cherednik algebra) in this case admits a triangular decomposition. This decomposition allows to define Verma modules and, for \( t = 1 \), category \( \mathcal{O} \) for \( H_{1,c} \) that has a so called highest weight structure. We can also define the category \( \mathcal{O} \) for \( \mathcal{D}_\lambda \), this will be a subcategory in \( \text{Coh}(\mathcal{D}_\lambda) \). We will show that the derived equivalence \( D^b(H_{1,c}-\text{mod}) \cong D^b(\text{Coh}(\mathcal{D}_\lambda)) \) restricts to categories \( \mathcal{O} \). This was used in [GL] to establish [R] Conjecture 5.6 for the groups \( \Gamma_n \).

In Section 5.3, we prove Theorem 1.3 and also its generalization to the groups \( \Gamma_n \) due to Bezrukavnikov and Finkelberg. The proof is based on studying the algebras \( H_{0,c} \) and their Verma modules.

Finally, in Section 5.4, we prove an analog of the Beilinson-Bernstein localization theorem, [BB], for the Rational Cherednik algebras associated to the groups \( \Gamma_n \). More precisely, we answer the question when the derived equivalence \( D^b(\text{Coh}(\mathcal{D}_\lambda)) \rightarrow D^b(H_{1,c}-\text{mod}) \) restricts to an equivalence \( \text{Coh}(\mathcal{D}_\lambda) \rightarrow H_{1,c}-\text{mod} \).

5.1. Derived equivalence.

5.1.1. Deformed derived McKay correspondence. Similarly to 4.1.2, the functor \( R\Gamma(\mathcal{P} \otimes_{\mathcal{O}_X} \bullet) \) defines an equivalence \( D^b(\text{Coh} X) \cong D^b(\mathcal{C}[V_n]\#\Gamma_n - \text{mod}) \) with quasi-inverse \( \mathcal{P}^* \otimes^{L}_{\mathcal{C}[V_n]\#\Gamma_n} \bullet \). These equivalence automatically upgrade to the categories of \( \mathbb{C}^* \)-equivariant objects: \( D^b(\text{Coh}^\mathbb{C}^* X) \cong D^b(\mathcal{C}[V_n]\#\Gamma_n - \text{mod}^{\mathbb{C}^*}) \) defined in the same way.

Now let us consider the deformation \( \mathcal{P}_h \) of \( \mathcal{P} \) to a right \( \mathbb{C}^* \)-equivariant \( \mathcal{D}_h \)-module. It gives a functor \( \mathcal{F} := R\Gamma(\mathcal{P}_{h,\text{fin}} \otimes^{L}_{\mathcal{D}_{h,\text{fin}}} \bullet) : D^b(\text{Coh}^\mathbb{C}^* (\mathcal{D}_{h,\text{fin}})) \rightarrow D^b(\mathbb{H} - \text{mod}^{\mathbb{C}^*}) \). This functor has left adjoint and right inverse \( \mathcal{G} = \mathcal{P}_{h,\text{fin}} \otimes^{L}_{\mathbb{H}} \bullet \). So we get the adjunction...
morphism $\tilde{g} \circ \tilde{F} \to \text{id}$. One can show (see [GL] Section 5) for details) that since this morphism is an isomorphism modulo $c_{\text{univ}}$, it is an isomorphism itself.

5.1.2. Specialization. The equivalence $\tilde{F}$ can be specialized to a numerical parameter. In particular, we get equivalences $D^b(\text{Coh}(\mathcal{D}_\lambda)) \to D^b(H_{1,c}-\text{mod})$, where $\lambda$ is recovered from $c$ as in Theorem 3.14. This is done in two steps. First, one gets a derived equivalence between $\text{Coh}^{\times}(R_{h_{1/2}}(\mathcal{D}_\lambda))$ and $R_{h_{1/2}}(H_{1,c})-\text{mod}^{\times}$, the corresponding sheaf and algebra are obtained from $\tilde{D}_{h_{\text{fin}}}, H$ by base change (and the equivalence we need comes from the corresponding base change of $\tilde{P}_{h_{\text{fin}}}$). To do the second step we recall that $H_{1,c}-\text{mod}$ is the quotient $R_{h_{1/2}}(H_{1,c})-\text{mod}^{\times}$ by the full subcategory of the $\mathbb{C}[h]$-torsion modules and the similar claim holds for $\text{Coh}(\mathcal{D}_\lambda)$, see Lemma 2.9. It follows that $D^b(H_{1,c}-\text{mod})$ is the quotient of $D^b(R_{h_{1/2}}(H_{1,c})-\text{mod}^{\times})$ by the category of all complexes whose homology are $\mathbb{C}[h]$-torsion and a similar claim holds for $\mathcal{D}_\lambda$. Since the equivalence $D^b(R_{h_{1/2}}(H_{1,c})-\text{mod}^{\times}) \cong D^b(\text{Coh}^{\times}(R_{h_{1/2}}(\mathcal{D}_\lambda)))$ is $\mathbb{C}[h]$-linear by the construction, they induce

\begin{equation}
D^b(H_{1,c}-\text{mod}) \cong D^b(\text{Coh}(\mathcal{D}_\lambda)).
\end{equation}

5.1.3. Application: shift equivalences. The equivalences (15) can be applied to producing a result that only concerns the symplectic reflection algebras. Namely, we say that parameters $c, c'$ for $H_{1,c}$ have integral difference if $\lambda - \lambda' \in \mathbb{Z}^{Q_0}$ for the corresponding parameters $\lambda$. Recall that we can view $\chi \in \mathbb{Z}^{Q_0}$ as a character of $G$. So $\chi$ defines a line bundle on $X$, explicitly, $\mathcal{O}_\chi = \pi_*(\mathcal{O}_{\mu^{-1}(0) - s_\chi})^G$. This line bundle can be quantized to a $\mathcal{D}_{\lambda + \chi} - \mathcal{D}_{\lambda}$-bimodule to be denoted by $\mathcal{D}_{\lambda, \chi}$. Explicitly,

$$
\mathcal{D}_{\lambda, \chi} := \pi_*(\mathcal{D}_{ss}/\mathcal{D}_{ss}^s\{\Phi(x) - \langle \lambda, x \rangle\})^G.
$$

This bundle carries a natural filtration and an isomorphism $\text{gr}\mathcal{D}_{\lambda, \chi} \cong \mathcal{O}_\chi$ follows from the flatness of the moment map.

Note that there is a natural (multiplication) homomorphism $\mathcal{D}_{\lambda + \chi, \chi'} \otimes \mathcal{D}_{\lambda + \chi} \to \mathcal{D}_{\lambda + \chi'}$ that becomes the isomorphism $\mathcal{O}_{\chi'} \otimes \mathcal{O}_\chi \to \mathcal{O}_{\chi + \chi'}$ after passing to the associated graded. So the multiplication homomorphism itself is an isomorphism. It follows that a functor $\mathcal{D}_{\lambda, \chi} \otimes \mathcal{D}_{\lambda} : \text{Coh}(\mathcal{D}_{\lambda}) \to \text{Coh}(\mathcal{D}_{\lambda + \chi})$ is a category equivalence. We conclude that categories $D^b(H_{1,c}-\text{mod})$ and $D^b(H_{1,c'}-\text{mod})$ are equivalent provided $c, c'$ have integral difference.

5.2. Category $\mathcal{O}$. Starting from now on, we assume that $\Gamma_1$ is a cyclic group $\mathbb{Z}/\ell\mathbb{Z}$. Recall that in this case the space $V_n$ (equal to $\mathbb{C}^{2\ell}$ when $\ell > 1$ and $\mathbb{C}^{2(n-1)}$ when $\ell = 1$) splits as $\mathfrak{h} \oplus \mathfrak{h}^*$, where $\mathfrak{h}$ is a standard reflection representation of the group $\Gamma_n$. The embeddings $\mathfrak{h}, \mathfrak{h}^* \hookrightarrow H$ extend to algebra embeddings $S(\mathfrak{h}), S(\mathfrak{h}^*) \hookrightarrow H$. These embeddings give rise to the triangular decomposition $H = S(\mathfrak{h}^*) \otimes S(c_{\text{univ}})\Gamma_n \otimes S(\mathfrak{h})$. We can also consider the specialization $H_{1,c} = S(\mathfrak{h}^*) \otimes \mathbb{C}\Gamma_n \otimes S(\mathfrak{h})$ (here and below $c$ is a numerical parameter) of this decomposition.

5.2.1. Category $\mathcal{O}$ for $H_{1,c}$. By definition, the category $\mathcal{O}$ for $H_{1,c}$ consists of all $H_{1,c}$- modules $M$ such that

(i) $\mathfrak{h}$ acts locally nilpotently on $M$.

(ii) $M$ is finitely generated over $H_{1,c}$.

Note that, modulo (i), the condition (ii) is equivalent to
(ii') $M$ is finitely generated over $S(\mathfrak{h}^*)$.

An example of an object in the category $\mathcal{O}$ is a Verma module constructed as follows. Pick an irreducible representation $\tau$ of $\Gamma_n$ and view it as a $S(\mathfrak{h})\#\Gamma_n$-module by making $\mathfrak{h}$ act by $0$. Then set $\Delta_{1,c}(\tau) := H_{1,c} \otimes_{S(\mathfrak{h})\#\Gamma_n} \tau$. As a $S(\mathfrak{h}^*)\#\mathcal{W}$-module, $\Delta_{1,c}(\tau)$ is naturally identified with $S(\mathfrak{h}^*) \otimes \tau$ (the algebra $S(\mathfrak{h}^*)$ acts by multiplications from the left, and $\mathcal{W}$ acts diagonally).

The algebra $H_{1,c}$ carries an Euler grading given by $\deg \mathfrak{h} = -1, \deg \mathfrak{h}^* = 1, \deg \mathcal{W} = 0$. This grading is internal: we have an element $h \in H_{1,c}$ with $[h,a] = da$ for $a \in H_{1,c}$ of degree $d$. Explicitly, the element $h$ is given by

$$\sum_{i=1}^m x_i y_i + \sum_{s \in S} \frac{c(s)}{1 - \lambda_s s}.$$

Here the notation is as follows. We write $y_1, \ldots, y_m$ for a basis in $\mathfrak{h}$ (of course, $m = n$ for $\ell > 1$ and $m = n - 1$ for $\ell = 1$) and $x_1, \ldots, x_m$ for the dual basis in $\mathfrak{h}^*$. By $S$ we, as usual, denote the set of reflections in $\Gamma_n$ and $c(s)$ stands for $c_i$ if $s \in S_i$ (note that the formula for $h$ is different from the usual formula for the Euler element, see, e.g., [BE, Section 2.1], because our $c(s)$ is rescaled). Finally, $\lambda_s$ is the eigenvalue of $s$ in $\mathfrak{h}^*$ different from $1$.

Using the element $h$, we can show that every Verma module $\Delta_{1,c}(\tau)$ has a unique simple quotient. These quotients form a complete collection of the simple objects in $\mathcal{O}$. Also one can show that every object in $\mathcal{O}$ has finite length. These claims are left as exercises to the reader.

5.2.2. Category $\mathcal{O}$ for $\mathcal{D}_\lambda$. We have a $\mathbb{C}^\times$-action on $D(R)$ induced by the $\mathbb{C}^\times$-action on $R$ given by $t \cdot r := t^{-1} r$. This action is Hamiltonian, the corresponding quantum comoment map $\Phi : \mathbb{C} \to D(R)$ sends $1$ to the Euler vector field. The action descends to a Hamiltonian $\mathbb{C}^\times$-action on $\mathcal{D}_\lambda$ for any $\lambda$.

Consider the corresponding Hamiltonian $\mathbb{C}^\times$-action on $X = \mathcal{M}_p^0(n\delta, \epsilon_0)$. Recall that the resolution of singularities morphism $X \to (\mathfrak{h} \oplus \mathfrak{h}^*)/\Gamma_n$ becomes $\mathbb{C}^\times$-equivariant if we equip the target variety with the $\mathbb{C}^\times$-action induced by $t.(a,b) = (t^{-1}a, tb), a \in \mathfrak{h}, b \in \mathfrak{h}^*$. This action has finitely many fixed points that are in a natural bijection with the irreducible representations of $\Gamma_n$, see [G3, 5.1]. Namely, $X^{\mathbb{C}^\times}$ is in a natural bijection with $\mathcal{M}_p^0(n\delta, \epsilon_0)^{\mathbb{C}^\times}$, where $p \in \mathfrak{g}^* G$ is generic. Indeed, $\mathcal{M}_p^0(n\delta, \epsilon_0) = \mathcal{M}_p^0(n\delta, \epsilon_0)$ and the sets $\mathcal{M}_p^0(n\delta, \epsilon_0)^{\mathbb{C}^\times}$ are identified for all $p$ by continuity. Let $c$ be a parameter corresponding to $p$ (meaning that $\nu(0,c) = (0, p)$). Then we can consider the Verma module $\Delta_{0,c}(\tau) := H_{0,c} \otimes_{S(\mathfrak{h})\#\Gamma_n} \tau$. The subalgebra $S(\mathfrak{h}^*)^{\Gamma_n}$ is easily seen to be central. Let us write $S(\mathfrak{h}^*)^{\Gamma_n}$ for the augmentation ideal in $S(\mathfrak{h}^*)^{\Gamma_n}$. Following [G1], consider the baby Verma module $\Delta_{0,c}(\tau) := \Delta_{0,c}(\tau)/S(\mathfrak{h}^*)^{\Gamma_n}\Delta_{0,c}(\tau) \cong S(\mathfrak{h}^*)/(S(\mathfrak{h}^*)^{\Gamma_n}) \otimes \tau$ (the last isomorphism is that of $S(\mathfrak{h}^*)\#\Gamma_n$-modules). This module is easily seen to be indecomposable so it has a central character that is a point of $\mathrm{Spec}(Z(H_{0,c})) = \mathcal{M}_p^0(n\delta, \epsilon_0)$. Clearly, this point is fixed by $\mathbb{C}^\times$ and this defines a map $\mathrm{Irr}(\Gamma_n) \to \mathcal{M}_p^0(n\delta, \epsilon_0)^{\mathbb{C}^\times}, \tau \mapsto z_\tau$, that was shown to be a bijection in [G3].

Fix some $p \in \mathfrak{g}^* G$. Consider the attracting locus $Y_p \subset \mathcal{M}_p^0(n\delta, \epsilon_0)$ for the $\mathbb{C}^\times$-action. Since this action has finitely many fixed points, we see that $Y_p$ is a lagrangian subvariety with irreducible components indexed by $\mathrm{Irr}(\Gamma_n)$. Namely, to $\tau \in \mathrm{Irr}(\Gamma_n)$ we assign the attracting locus $Y_p(\tau) := \{ z \in \mathcal{M}_p^0(n\delta, \epsilon_0) | \lim_{t \to 0} t z = z_\tau \}$. The irreducible components
Let us determine the lowest graded component in $H$. Denote an indexing set of the simple objects in this simple module by $L$.

Identification of $\text{Irr}(\Gamma)$ indexed by the lowest graded component in $\Delta$. Namely, consider the category $\tau$. 

5.2.3. Choice of identification $X^{C^\times} \cong \text{Irr}(\Gamma_n)$. We note that despite our identification of $X^{C^\times}$ with $\text{Irr}(\Gamma_n)$ is natural, there are other natural choices as well. The choice we have made is good for working with the category $O$. We could also consider the category $O^*$, where the modules are locally nilpotent for $\mathfrak{h}^*$, not for $\mathfrak{h}$ (and are still finitely generated over $H_{1,c}$). Consequently, we need to use the opposite Hamiltonian $C^\times$-action on $X$, $\mathcal{M}_p(n\delta, \epsilon_0)$ and Verma modules $\Delta_{0,c}(\tau) := H_{0,c} \otimes S(\mathfrak{h}^*)^{\#\Gamma_n} \tau$. Let us explain how the bijection $X^{C^\times} \cong \text{Irr}(\Gamma_n)$ changes.

All simple constituents of $\Delta_{0,c}(\tau)$ are isomorphic modules of dimension $|\Gamma_n|$ (indeed, $H_{0,c}$ is the endomorphism algebra of the rank $|\Gamma_n|$ bundle $\tilde{P}_p$ on $\mathcal{M}_p(n\delta, \epsilon_0)$). Let us denote this simple module by $L_{0,c}(\tau)$. This module is graded, the highest graded component is $\tau$. Let us determine the lowest graded component in $L_{0,c}(\tau)$. This component coincides with the lowest graded component in $\Delta_{0,c}(\tau)$ that is the tensor product of $\tau$ with the lowest degree component in $\mathbb{C}[\mathfrak{h}]/(\mathbb{C}[\mathfrak{h}]^{\Gamma_n})_{+}$. It is easy to see that the latter is $\Lambda^\text{top}\mathfrak{h}$. Abusing the notation, we will denote $\tau \otimes \Lambda^\text{top}\mathfrak{h}$ by $\tau^t$. When $\Gamma_1 = \{1\}$ we can use the standard identification of $\text{Irr}(\mathfrak{S}_n)$ with the set of Young diagrams of $n$ boxes. In this case, $\Lambda^\text{top}\mathfrak{h}$ is the sign representation of $\mathfrak{S}_n$ and $\tau^t$ indeed corresponds to the transposed Young diagram of $\tau$.

The previous paragraph shows that there is an epimorphism $\Delta^*_0(\tau^t) \to L_p(\tau)$. So our new bijection sends the point $z \tau \in X^{C^\times}$ to $\tau^t$.

We also note that the identification $X^{C^\times} \cong \text{Irr}(\Gamma_n)$, $\tau \mapsto z_\tau$, depends on the choice of a Procesi bundle $\mathcal{P}$ but we are not going to use this.

5.2.4. Highest weight structures. Let us recall the definition of a highest weight category. Let $\mathcal{C}$ be an abelian category that is equivalent to the category of modules over a finite dimensional algebra, equivalently, the category $\mathcal{C}$ has finitely many simples, enough projectives and finite dimensional Hom’s (and hence every object has finite length). Let $\mathcal{T}$ denote an indexing set of the simple objects in $\mathcal{C}$, we write $L(\tau)$ for the simple object indexed by $\tau \in \mathcal{T}$ and $P(\tau)$ for its projective cover. The additional structure of a highest weight category is a partial order $\leqslant$ on $\mathcal{T}$ and a collection of so called standard objects $\Delta(\tau)$, $\tau \in \mathcal{T}$, satisfying the following axioms:

1. $\text{Hom}_\mathcal{C}(\Delta(\tau), \Delta(\tau')) \neq 0$ implies $\tau \leqslant \tau'$;
2. $\text{End}_\mathcal{C}(\Delta(\tau)) = \mathbb{C}$;
3. $P(\tau) \to \Delta(\tau)$ and the kernel admits a filtration with quotients $\Delta(\tau')$ for $\tau' > \tau$.

Remark 5.1. Let us point out that the standard objects are uniquely recovered from the partial order. Namely, consider the category $\mathcal{C}_{\leqslant \tau}$ that is the Serre span of the simples $L(\tau')$ with $\tau' \leqslant \tau$. Then $\Delta(\tau)$ is the projective cover of $L(\tau)$ in $\mathcal{C}_{\leqslant \tau}$.

Both categories $\mathcal{O}, \mathcal{O}^{\text{loc}}$ that were described above are highest weight, see [GGOR, Sections 2.6.3.2] for $\mathcal{O}$ and [BLPW] Section 5.3 for $\mathcal{O}^{\text{loc}}$. The standard objects $\Delta(\lambda)$ are
the Verma modules. The order can be introduced as follows. Recall the element \( h \in H_{1,c} \) introduced in [5.2.1]. It acts on \( \tau \subset \Delta(\tau) \) by \( \sum_{s \in S} \frac{c(s)}{1 - \lambda_s} s \). The latter element in \( \mathbb{C} \Gamma_n \) is central and so acts on \( \tau \) by a scalar, denote that scalar by \( c_\tau \). Then we set \( \tau \leq \tau' \) if \( c_\tau - c_{\tau'} \in \mathbb{Z}_{\geq 0} \).

Let us provide a formula for \( c_\tau \). We start with \( \ell = 1 \). Then a classical computation shows that \( c_\tau = c_0 \text{cont}(\tau)/2 \), where the integer \( \text{cont}(\tau) \) is defined as follows. For the box \( b \in \tau \) lying in \( x \)th column and \( y \)th row, we set \( \text{cont}(b) := x - y \). Then \( \text{cont}(\tau) := \sum_{b \in \tau} \text{cont}(b) \). Now let us proceed to \( \ell > 1 \). In this case, the irreducible representations of \( \Gamma_n \) are parameterized by the \( \ell \)-multipartitions \( (\tau^{(1)}, \ldots, \tau^{(\ell)}) \) of \( n \). Define elements \( \lambda_1, \ldots, \lambda_\ell \) by requiring that \( \lambda_i, i = 1, \ldots, \ell - 1 \), is recovered from \( c \) as in Theorem 3.14 and \( \sum_{i=1}^\ell \lambda_i = 0 \). For a box \( b \in \tau^{(j)} \) set \( d_\ell(b) := c_0 \ell \text{cont}(b)/2 + \ell \lambda_j \). Then, up to a summand independent of \( \tau \), we have \( c_\tau = \sum_{b \in \tau} d_\ell(b) \), see [R] Proposition 6.2 or [GL, 2.3.5] (in both papers the notation is different from what we use).

In fact, one can take a weaker ordering on \( \text{Irr}(\Gamma_n) \) making \( \mathcal{O} \) into a highest weight category. Namely, according to [GL], for two boxes \( b, b' \) in \( j \)th and \( j' \)th diagrams respectively we say that \( b \leq b' \) if \( d_\ell(b) - d_\ell(b') \) is congruent to \( j - j' \) modulo \( \ell \) and is in \( \mathbb{Z}_{\geq 0} \). Then \( \lambda \leq \lambda' \) if one can order boxes \( b_1, \ldots, b_n \) of \( \lambda \) and \( b'_1, \ldots, b'_n \) of \( \lambda' \) in such a way that \( b_i \leq b'_i \) for all \( i \).

Let us proceed to the categories \( \mathcal{O}^{\text{loc}} \). They are highest weight with respect to the order \( \leq \) (we will often write \( \leq^\theta \) to indicate the dependence on \( \theta \)) defined as follows. We first define a pre-order \( \leq' \) by setting \( \tau \leq' \tau' \) if \( z_\tau \in \tau_{\tau'} \) and then define \( \leq \) as the transitive closure of \( \leq' \).  

**Example 5.2.** When \( \ell = 1 \) and \( \theta < 0 \), the bijection between the \( \mathbb{C} h^\times \)-fixed points and partitions is the standard one. A combinatorial description of \( \leq^\theta \) follows from [Nak2, Section 4]: we have \( \tau \leq^\theta \tau' \) if \( \tau \leq \tau' \) as Young diagrams.

In the case when \( \ell > 1 \) an a priori stronger order (that automatically also makes \( \mathcal{O}^{\text{loc}} \) into a highest weight category) was described by Gordon in [G3, Section 7] in combinatorial terms. The standard modules are recovered from \( \leq^\theta \) as before. Below we will see that they can be described using the deformations of the Procesi bundle.

### 5.2.5. Derived equivalence.

Here we are going to produce a derived equivalence \( D^b(\mathcal{O}) \cong D^b(\mathcal{O}^{\text{loc}}) \).

Inside \( D^b(\text{H}_{1,c}^{-}\text{mod}) \) we can consider the full subcategory \( D^b(\mathcal{O}^{\text{loc}}) \) consisting of all complexes whose homology lie in the category \( \mathcal{O} \). We then have a natural functor \( D^b(\mathcal{O}) \to D^b(\mathcal{O}^{\text{loc}}) \). This functor is an equivalence by [E, Proposition 4.4]. We can also consider the category \( D^b_{\text{loc}}(\text{Coh}(\mathcal{D}_\lambda)) \), the functor \( D^b(\mathcal{O}^{\text{loc}}) \to D^b(\text{Coh}(\mathcal{D}_\lambda)) \) is an equivalence as well, this follows from [BLPW, Corollary 5.13] and [BPW, Corollary 5.12].

The equivalence \( D^b(\text{H}_{1,c}^{-}\text{mod}) \) is compatible with the supports in the following sense. Recall that we have two commuting \( \mathbb{C}^\times \)-actions. The Hamiltonian torus will be denoted by \( \mathbb{C}^\times \), while, for the contracting torus (which is present even when \( \Gamma_1 \) is not cyclic), we will write \( \mathbb{C}_c^\times \). Pick a closed subvariety \( Y_0 \subset (\mathfrak{h} \oplus \mathfrak{h}^*)/\Gamma_n \) that is stable under the \( \mathbb{C}_c^\times \)-action. Consider the full subcategory \( D^b_{Y_0}(\text{H}_{1,c}^{-}\text{mod}) \) of all complexes with homology supported on \( Y_0 \). Set \( Y := \rho^{-1}(Y_0) \), where, recall, \( \rho \) stands for the resolution of singularities morphism \( \rho : X \to V_n/\Gamma_n \) and consider the subcategory
\[ D_1^b(\text{Coh}(\mathcal{D}_\lambda)) \subset D^b(\text{Coh}(\mathcal{D}_\lambda)). \] Then the equivalence \( D^b(\text{Coh}(\mathcal{D}_\lambda)) \cong D^b(H_{1,c}\text{-mod}) \) restricts to \( D_1^b(\text{Coh}(\mathcal{D}_\lambda)) \cong D_1^b(H_{1,c}\text{-mod}) \).

Note that the bundle \( \mathcal{P} \) on \( X \) is \((\mathbb{C}^\times)^2\)-equivariant. Therefore the deformation \( \bar{\mathcal{P}}_h \) is \((\mathbb{C}^\times)^2\)-equivariant as well. It follows that the equivalence \( D^b(\text{Coh}(\mathcal{D}_\lambda)) \cong D^b(H_{1,c}\text{-mod}) \) preserves complexes whose homology admit \( \mathbb{C}^\times\)-equivariant liftings. Combined with the previous paragraph, this means that we get an equivalence \( D^b_{\text{O}}(H_{1,c}\text{-mod}) \cong D^b_{\text{O}}(\text{Coh}(\mathcal{D}_\lambda)) \) and hence an equivalence \( D^b(\mathcal{O}) \cong D^b(\mathcal{O}^{\text{loc}}) \).

This was used in [GL] Section 5 to prove a conjecture of Rouquier, [R, Conjecture 5.6]. Namely, suppose that we have parameters \( c, c' \) such that the corresponding parameters \( \lambda, \lambda' \) have integral difference. Then we have an abelian equivalence \( \text{Coh}(\mathcal{D}_\lambda) \cong \text{Coh}(\mathcal{D}_{\lambda'}) \), given by tensoring with the bimodule \( \mathcal{D}_{\lambda,\lambda'-\lambda} \). This bimodule is \( \mathbb{C}^\times_n\)-equivariant, this follows from the construction. Also it is clear that tensoring with \( \mathcal{D}_{\lambda,\lambda'-\lambda} \) preserves the supports. So we conclude that \( \mathcal{O}^{\text{loc}}_\lambda \cong \mathcal{O}^{\text{loc}}_{\lambda'} \). It follows that the categories \( \mathcal{O}_c \) and \( \mathcal{O}_{c'} \) are derived equivalent that was conjectured by Rouquier (in the generality of all Cherednik algebras).

5.3. Macdonald positivity. Consider the \( \text{H}\)-module \( \Delta(\lambda) := \text{H} \otimes S(h)\# \Gamma_n \lambda \). Recall the derived equivalence \( D^b(\text{Coh}(\bar{\mathcal{D}}_{h,\text{fin}})) \cong D^b(\text{H}\text{-mod}) \) given by

\[
\mathcal{F} := \Gamma(\bar{\mathcal{P}}_{h,\text{fin}} \otimes \bar{\mathcal{D}}_{h,\text{fin}} \bullet)
\]

and its inverse \( \mathcal{G} \). It turns out that the study of the objects \( \mathcal{G}(\Delta(\lambda)) \) leads to the proof of the Macdonald positivity. The proof that we provide below is morally similar to but different from the original proof in [BF].

5.3.1. Flatness. A key step in the proof is to establish the flatness over \( \mathbb{C}[\mathfrak{h}] \) of an arbitrary Procesi bundle \( \mathcal{P} \), where we view \( \mathcal{P} \) (\( \mathbb{C}[\mathfrak{h}] \) acts on \( \mathcal{P} \) via the inclusion \( \mathbb{C}[\mathfrak{h}] \hookrightarrow S(\mathfrak{h} \oplus \mathfrak{h}^*)\# \Gamma_n = \text{End}_{\Omega}(\mathcal{P})) \). This will imply that the Koszul complex

\[
\mathcal{P} \leftarrow \mathfrak{h}^* \otimes \mathcal{P} \leftarrow \Lambda^2 \mathfrak{h}^* \otimes \mathcal{P} \leftarrow \ldots \leftarrow \Lambda^n \mathfrak{h}^* \otimes \mathcal{P}
\]

is a resolution of \( \mathcal{P}/\mathfrak{h}^* \mathcal{P} \). The proof of the flatness is taken from the proof of [BF] Lemma 3.7.

Note that, since \( \Gamma_n \) is a complex reflection group, \( \mathbb{C}[\mathfrak{h}] \) is free over \( \mathbb{C}[\mathfrak{h}]^{\mathfrak{r}}_{\Gamma_n} \). So it is enough to show that \( \mathcal{P} \) is flat over \( \mathbb{C}[\mathfrak{h}]^{\mathfrak{r}}_{\Gamma_n} \).

Let us recall how \( \mathcal{P} \) was constructed, see [4.1.4] (construction of one Procesi bundle in characteristic \( p \gg 0 \)), [4.1.5] (construction of one Procesi bundle in characteristic 0), [4.4.2] (construction of all Procesi bundles).

1. We start with a suitable Frobenius constant quantization \( \mathcal{D} \) of \( X_{\mathbb{F}} \), where \( \mathbb{F} \) is an algebraically closed field of characteristic 0.
2. Then we take a splitting bundle \( \mathcal{B} \) of \( \mathcal{D}|_{X_{\mathbb{F}}^{(1)} \setminus \varnothing} \).
3. We form a bundle \( \mathcal{P}' \) on \( X^{(1)}_{\mathbb{F}} \setminus \varnothing \) that is the sum of indecomposable summands of \( \mathcal{S}^* \) with suitable multiplicities. Then we extend this bundle to \( X^{(1)}_{\mathbb{F}} \) and get a Procesi bundle \( \mathcal{P}_{\mathbb{F}}^{(1)} \) on \( X^{(1)}_{\mathbb{F}} \).
4. Since \( X_{\mathbb{F}}^{(1)} \cong X_{\mathbb{F}} \) as \( \mathbb{F} \)-varieties, we can view \( \mathcal{P}_{\mathbb{F}}^{(1)} \) as a bundle \( \mathcal{P}_{\mathbb{F}} \) on \( X \).
5. Then we lift \( \mathcal{P}_{\mathbb{F}} \) to characteristic 0.

The procedure in (5) implies that if \( \mathcal{P}_{\mathbb{F}} \) is flat over \( \mathbb{F}[\mathfrak{h}]^{\mathfrak{r}}_{\Gamma_n} \), then \( \mathcal{P} \) is flat over \( \mathbb{C}[\mathfrak{h}]^{\mathfrak{r}}_{\Gamma_n} \) (the reader is welcome to verify the technical details). Obviously, \( \mathcal{P}_{\mathbb{F}} \) is flat over \( \mathbb{F}[\mathfrak{h}]^{\mathfrak{r}}_{\Gamma_n} \).
if and only if $\mathcal{P}_\mathcal{F}^{(1)}$ is flat over $\mathbb{F}[\mathfrak{h}]^{\Gamma_n}$. The latter is equivalent to $\mathcal{B}^*$ being flat over $\mathbb{F}[[\mathfrak{h}]]^{\Gamma_n}$, which, in turn, is equivalent to the claim that $\mathcal{D}$ is a flat $\mathbb{F}[\mathfrak{h}]^{\Gamma_n}$-module. But $\text{gr}\mathcal{D} \cong \text{gr}_X \mathcal{O}_{X_{\mathcal{F}}}$. So it is enough to verify that $\mathcal{O}_{X_{\mathcal{F}}}$ is flat over $\mathbb{F}[\mathfrak{h}]^{\Gamma_n}$. Since $\mathbb{F}[\mathfrak{h}]^{\Gamma_n}$ is flat over $\mathbb{F}[\mathfrak{h}]^{\Gamma_n}$, we reduce to proving that $X_{\mathcal{F}}$ is flat over $h_{\mathcal{F}}/\Gamma_n$, equivalently, all fibers of $X_{\mathcal{F}} \to h_{\mathcal{F}}/\Gamma_n$ have the same dimension, equivalently, the zero fiber has dimension dim $\mathfrak{h}$. But the zero fiber of this map is precisely the contracting variety for the Hamiltonian $\mathbb{F}^\times$-action and so is lagrangian. This completes the proof.

Similarly, $\mathcal{P}$ is flat over $\mathbb{C}[\mathfrak{h}]$. Also let us recall, see [4.4.3], that $\mathcal{P}^*$ can be equipped with a structure of the Procesi bundle, for which we need to convert the right $S(\mathfrak{h} \oplus \mathfrak{h}^*)\# \Gamma_n$-module into a left $S(\mathfrak{h} \oplus \mathfrak{h}^*)\# \Gamma_n$ using a natural anti-automorphism of $S(\mathfrak{h} \oplus \mathfrak{h}^*)\# \Gamma_n$. This shows that $\mathcal{P}^*$ is a flat right module over both $\mathbb{C}[\mathfrak{h}]$ and $\mathbb{C}[\mathfrak{h}]$. This is what we are going to use below.

5.3.2. Upper triangularity. Let $\theta$ be a generic stability condition and take $X = X^\theta$. This gives rise to the partial order $\leq \theta$ on the set $\text{Irr}(\Gamma_n)$ described in [5.2.2]. Recall that we write $z_{\tau}$ for the $\mathfrak{C}_h^\times$-fixed point in $X$ corresponding to $\tau$ as explained in [5.2.2]. We write $Y_{\tau}$ for the $\mathfrak{C}_h^\times$-contracting component of $z_{\tau}$, a lagrangian subvariety in $X^\theta$. Further, write $e_{\tau}$ for a primitive idempotent in $\mathbb{C}\Gamma_n$ corresponding to $\tau$ so that $\tau \cong (\mathbb{C}\Gamma_n)e_{\tau}$.

Proposition 5.3. Let $\mathcal{P}$ be the canonical Procesi bundle on $X^\theta$. Then the sheaf $(\mathcal{P}^*/\mathcal{P}^*\mathfrak{h})e_{\tau}$ is supported on $\bigcup_{\tau' \leq \theta \tau} Y_{\tau'}$.

Proof. Consider the deformation $\tilde{\mathcal{P}}^*$ of $\mathcal{P}^*$ to $\tilde{X}$. It is flat over $\mathbb{C}[\mathfrak{g}^G, \mathfrak{h}]^*$. Therefore $\tilde{\mathcal{P}}^*/\mathcal{P}^*\mathfrak{h}$ is flat over $\mathbb{C}[\mathfrak{g}^G]$. It follows that $\text{Supp}((\mathcal{P}^*/\mathcal{P}^*\mathfrak{h})e_{\tau}) \subset \mathbb{C}_C^\times \text{Supp}(\mathcal{P}^*/\mathcal{P}^*\mathfrak{h})e_{\tau}$ for a generic $\mathfrak{p} \in \mathfrak{g}^G$. But $(\mathcal{P}^*/\mathcal{P}^*\mathfrak{h})e_{\tau}$ is nothing else but $e\Delta_0,\mathfrak{c}(\tau)$. We claim that $\text{Supp} \Delta_0,\mathfrak{c}(\tau) \subset Y_{\mathfrak{p},\tau}$. Indeed, we have shown in [5.2.2] that $\Delta_0,\mathfrak{c}(\tau)/S(\mathfrak{h})\bigwedge^n \Delta_0,\mathfrak{c}(\tau)$ is supported in $z_{\mathfrak{p},\tau}$, the point in $M_{\mathfrak{p}}(n\delta, \epsilon_0)\mathbb{C}_C^\times$ indexed by $\tau$. If $\text{Supp} \Delta_0,\mathfrak{c}(\tau) \not\subset Y_{\mathfrak{p},\tau}$, then there is $\tau' \neq \tau$ with $z_{\mathfrak{p},\tau'} \in \text{Supp} \Delta_0,\mathfrak{c}(\tau)$ (because the latter is closed and contained in $Y_{\mathfrak{p}}$). The support of $\Delta_0,\mathfrak{c}(\tau)$ is disconnected and so the module $\Delta_0,\mathfrak{c}(\tau)$ is indecomposable. From here one deduces that $z_{\mathfrak{p},\tau'}$ lies in the support of $\Delta_0,\mathfrak{c}(\tau)/S(\mathfrak{h})\bigwedge^n \Delta_0,\mathfrak{c}(\tau)$, contradiction.

Now the inclusion

$$\text{Supp}((\mathcal{P}^*/\mathcal{P}^*\mathfrak{h})e_{\tau}) \subset \bigcup_{\tau' \leq \theta \tau} Y_{\tau'}$$

follows from

$$\mathbb{C}_C^\times Y_{\mathfrak{p},\tau} \cap X^\theta \subset \bigcup_{\tau' \leq \theta \tau} Y_{\tau'},$$

see [BF, Lemma 3.8].

In fact, $e\Delta_0,\mathfrak{c}(\tau) = \mathbb{C}[Y_{\mathfrak{p},\tau}]$ but we do not need this fact.

5.3.3. Wreath-Macdonald positivity. Now we are ready to prove the Macdonald positivity theorem, Theorem 1.3, and its “wreath-generalization” due to Bezrukavnikov and Finkelberg.

First of all, Proposition 5.3 implies that if the fiber of $[\mathcal{P}^*/\mathcal{P}^*\mathfrak{h}]e_{\tau}$ in $z_{\tau'}$ is nonzero, then $\tau' \leq \theta \tau$. It follows that if $\tau'$ is a constituent of the fiber $(\mathcal{P}^*/\mathcal{P}^*\mathfrak{h})_{z_{\tau'}}$, then $\tau \geq \theta \tau'$. But since $\mathcal{P}^*$ is a flat right $\mathbb{C}[\mathfrak{h}]$-module, we see that the class of $[\mathcal{P}^*/\mathcal{P}^*\mathfrak{h}^*]_{z_{\tau'}}$ in the $K_0$ of bigraded $\Gamma_n$-modules coincides with that of the Koszul resolution

$$\mathcal{P}^*_{z_{\tau'}} \leftarrow \mathcal{P}^*_{z_{\tau'}} \otimes \mathfrak{h} \leftarrow \ldots$$
Taking the duals, we see that if $\tau$ occurs in the class

$$\mathcal{P}_{z_\tau} \otimes \sum_{i=0}^{\text{dim } \mathfrak{h}} (-1)^i \Lambda^i \mathfrak{h}^*, \quad \text{then } \tau' \leq^\theta \tau.$$

When $\Gamma_1 = \{1\}$, this yields (a) from Definition 1.2.

To get (b) in that definition (and its wreath-generalization), we consider $[\mathcal{P}^*/\mathcal{P}^* \mathfrak{h}^*]_{e_\tau}$. This sheaf is supported on the union of repelling components for $\mathfrak{C}_h^\times$ and can have nonzero fibers only in the fixed points $z_{\tau'}$ with $z_{\tau'} \geq^\theta z_\tau$ meaning $\tau^t \leq^\theta \tau'$. In other words, if $\tau$ appears in

$$\mathcal{P}_{z_\tau} \otimes \sum_{i=0}^{\text{dim } \mathfrak{h}} (-1)^i \Lambda^i \mathfrak{h}, \quad \text{then } \tau^t \leq^\theta \tau.'$$

When $\Gamma_1 = \{1\}$, this yields (b) in Definition 1.2 (c) there follows because $\mathcal{P}$ is normalized.

5.4. Localization theorem. Let $\mathcal{P}_{1,\lambda}$ denote the the right $\mathcal{D}_\lambda$-module obtained by specializing $\tilde{\mathcal{P}}_{\mathfrak{h}}$. One can ask when (i.e., for which $\lambda$) the functor $\Gamma(\mathcal{P}_{1,\lambda} \otimes_{\mathcal{D}_\lambda} \bullet) : \mathcal{O}_{loc} \rightarrow \mathcal{O}_c$ is a category equivalence. The following result answers this question.

Theorem 5.4. Suppose that there is an order $\leq$ on $\text{Irr}(\Gamma_n)$ refining $\leq^\theta$ and making both $\mathcal{O}_{loc}^{\lambda}, \mathcal{O}_c$ into highest weight categories. Then $\Gamma : \text{Coh}(\mathcal{D}_\lambda) \rightarrow H_{1,c}\text{-mod}, \mathcal{O}_{loc}^{\lambda} \rightarrow \mathcal{O}_c$ are equivalences of categories.

This theorem can be viewed as an analog of the Beilinson-Bernstein localization theorem, [BB], from the Lie representation theory.

Sketch of proof. It is enough to prove that $\Gamma$ gives an equivalence between the categories $\mathcal{O}$, see [L5, Section 3.3]. So below in the proof we only deal with the categories $\mathcal{O}$.

Set $\Delta^{loc}(\lambda) := [\mathcal{P}_{1,\lambda}^{\mathfrak{h}}/\mathcal{P}_{1,\lambda}^{\mathfrak{h}}]_{e_\lambda}$. Further, let $\mathcal{F}$ stand for $R\Gamma(\mathcal{P}_{1,\lambda} \otimes_{\mathcal{D}_\lambda} \bullet)$. The flatness of $\mathcal{P}$ over $S(\mathfrak{h})$ from the previous subsection implies that

$$\mathcal{F}\Delta^{loc}(\tau) = \Delta_c(\tau). \quad (16)$$

We have $\Delta^{loc}(\lambda) \in \mathcal{O}_{loc}^{\lambda\lambda}$. The condition on the orders implies that $\Delta^{loc}(\tau)$ is the standard object in $\mathcal{O}_{loc}^{\lambda}$. Now the claim of Theorem 5.4 follows from the next general claim. \hfill $\square$

Lemma 5.5. Let $\mathcal{C}^1, \mathcal{C}^2$ be two abelian categories with the same indexing poset $\mathcal{T}$. Suppose that $\mathcal{F} : D^b(\mathcal{C}^1) \rightarrow D^b(\mathcal{C}^2)$ is a derived equivalence mapping $\Delta^1(\tau)$ to $\Delta^2(\tau)$ for any $\tau \in \mathcal{T}$. Then $\mathcal{F}$ is induced from an abelian equivalence $\mathcal{C}^1 \rightarrow \mathcal{C}^2$.

Theorem 5.4 generalizes results of [GS, KR] for $\Gamma_1 = \{1\}$ to the case of general cyclic $\Gamma_1$.

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