WEYL-HEISENBERG INTEGRAL QUANTIZATION(S):
A COMPENDIUM

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Abstract. We present a list of formulae useful for Weyl-Heisenberg integral quantizations, with arbitrary weight, of functions or distributions on the plane. Most of these formulae are known, others are original. The list encompasses particular cases like Weyl-Wigner quantization (constant weight) and coherent states (CS) or Berezin quantization (Gaussian weight). The formulae are given with implicit assumptions on their validity on appropriate space(s) of functions (or distributions). One of the aims of the document is to accompany a work in progress on Weyl-Heisenberg integral quantization of dynamics for the motion of a point particle on the line.

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I. Introduction

While preparing a paper on the Weyl-Heisenberg (WH) integral quantization of one dimensional dynamics for the motion of a point particle on the real line [1], we have been establishing a compendium of useful formulae. We now think suitable to share these results with the community of researchers interested in similar topics. Most of the presented material is not original. It can be found in many places, and with different notations, since the emergence of Quantum Mechanics [2], specially in References [3] (see also [4]), [5], [6], [7], [8] (see also [9]), [10], [11], [12], [13], [14], [15], [16], [17], in the mathematically oriented [18] and [19], in [20], in the books [21] and [22], in [23], [24]... For most recent, mathematically oriented, works, see for instance [25], [26] and [27], and references therein. However, we do not pretend to have exhausted here the complete relevant bibliography!

Despite that lack of originality of many formulas presented in our paper, the fact to gather all of them in a single document, under an itemized form, might be valuable for people working on the subject. Beyond their computational justifications, one might be concerned by the mathematical validity of our expressions in terms of functional analysis, since most of them should be justified on a mathematical level with regard to involved (generalized) functions. Nevertheless, they are given with implicit assumptions on their validity on appropriate space(s) of functions (or distributions).

The object we start with, namely the Euclidean plane \( \mathbb{R}^2 \), or the complex plane \( \mathbb{C} \), is briefly presented in Section 2 together with its physical content. Weyl-Heisenberg group and algebra together with their representations as operators in Hilbert spaces are described in Section 3, Section 4, and Section 5 respectively. Section 6 is devoted to the so-called WH displacement operator, \( D(z) \) (complex notation), \( D(q,p) \) (real notations), of central importance for
the content of this paper. The short section 7 is devoted to the construction of standard coherent states through the action of the $D(z)$ on the “vacuum” state in Hilbert space. Section 8 deals with discrete symmetries and in Section 9 are considered rotations in the plane. Integral formulas involving the displacement operator $D$ or $\mathcal{D}$ are given in Section 10. Section 11 is devoted to symplectic Fourier analysis of (possibly operator valued) functions on the plane. Trace formulae are presented in Section 12. The heart of the paper lies in Section 14 where is presented what we call Weyl–Heisenberg integral quantization (WHIQ), expressed in either complex or real variables, and the ingredients are just the WH displacement operator and a weight (apodization) function. In the succeeding sections are given technical issues of this quantization procedure, in relation with derivatives in Section 15, with products in Section 16, expansion coefficients in Sections 17, 18, Section 19, and Section 20. Common characteristics and outcomes of WHIQ are presented in Section 21. Section 22 is devoted to WHIQ results obtained from particular weight or measures on the phase space. We examine in Section 23 the quantization of separable functions $f(q, p) = u(q)v(p)$. In Section 24 we give a list of commutation relations which are relevant to examples encountered in previous Sections, and in Section 25 the case of weights which are separable gaussians with different widths. Section 26 is devoted to the WHIQ versions of the harmonic oscillator and their common outcomes. Finally, semiclassical portraits, generalizing Wigner or Husimi functions for operators issued from WHIQ are considered in Section 27, with an interesting outcome in terms of probabilistic interpretation of the weight function.

2. Phase space

- Phase space for motion on the line is $\mathbb{R}^2 \sim \mathbb{C}$

\begin{equation}
\mathbb{C} \sim \mathbb{R}^2 = \left\{ z = \frac{q + ip}{\sqrt{2}} , \ q, p \in \mathbb{R} \right\}.
\end{equation}
\[
\frac{d^2 z}{dq dp} = \frac{1}{2}
\]

- Physical dimensions are restored, particularly in view of classical limit, through the re-scalings

\[
\mathbb{C} \ni z = \frac{q + ip}{\sqrt{2}} \mapsto z = \frac{1}{\sqrt{2}} \left( \frac{q}{\ell} + i \frac{p}{\ell^*} \right),
\]

so that the complex \( z \) remains dimensionless.

- Here \( \ell \) (resp. \( \wp \)) is some length (resp. momentum) appropriate to the scale of the model. Thinking to quantum systems, we can also introduce the Planck constant \( \hbar \) such that \( \ell \wp = \hbar \)

- Useful change of variables formulae
  - With physical constants \( \hbar \) and \( \ell \)

\[
\begin{align*}
\partial_z &= \frac{1}{\sqrt{2}} \left( \ell \partial_q - \frac{i}{\ell} \partial_p \right), & \partial_{\bar{z}} &= \frac{1}{\sqrt{2}} \left( \ell \partial_q + \frac{i}{\ell} \partial_p \right), \\
\partial_q &= \frac{1}{\sqrt{2} \ell} (\partial_z + \partial_{\bar{z}}), & \partial_p &= \frac{i}{\sqrt{2} \hbar} \left( \partial_z - \partial_{\bar{z}} \right).
\end{align*}
\]

  - Without physical constants

\[
\begin{align*}
\partial_z &= \frac{1}{\sqrt{2}} (\partial_q - i \partial_p), & \partial_{\bar{z}} &= \frac{1}{\sqrt{2}} (\partial_q + i \partial_p), \\
\partial_q &= \frac{1}{\sqrt{2}} (\partial_z + \partial_{\bar{z}}), & \partial_p &= \frac{i}{\sqrt{2}} (\partial_z - \partial_{\bar{z}}).
\end{align*}
\]

3. **Weyl-Heisenberg group**

- Forgetting about physical dimensions and \( \hbar = 1 \), an arbitrary element \( g \) of \( G_{WH} \) is of the form

\[
G_{WH} = \{ g = (\zeta, q, p), \zeta \in \mathbb{R}, (q, p) \in \mathbb{R}^2 \}.
\]

- In complex notations

\[
G_{WH} = \{ (\zeta, z), \zeta \in \mathbb{R}, z \in \mathbb{C} \}.
\]
Multiplication law

\[ g_1 g_2 = (\varsigma_1 + \varsigma_2 + \xi((q_1, p_1), (q_2, p_2)), q_1 + q_2, p_1 + p_2) = (\varsigma_1 + \varsigma_2 + \xi(z_1, z_2), z_1 + z_2), \]

where \( \xi \) is the multiplier (or two-cocycle) function (\( \sim \) symplectic form on \( \mathbb{R}^2 \))

\[ \xi((q_1, p_1), (q_2, p_2)) = \frac{1}{2}(p_1 q_2 - p_2 q_1) \equiv \xi(z_1, z_2) = \text{Im} \, z_1 \bar{z}_2 \equiv -z_1 \wedge z_2. \]

Besides this multiplicity of notations, we introduce another the further one, more suitable for many expressions

\[ z_1 \circ z_2 := z_1 \bar{z}_2 - \bar{z}_1 z_2 = 2\xi(z_1, z_2) = 2\text{Im} \, z_1 \bar{z}_2. \]

The two-cocycle condition results from associativity of the WH group

\[ \xi(z_1, z_3) + \xi(z_1 + z_2, z_3) = \xi(z_2, z_3) + \xi(z_1, z_2 + z_3) \]

together with \( \xi(z, 0) = 0 = \xi(0, z) \) and \( \xi(z, -z) = 0 \), resulting from group identity and inverse respectively.

4. Weyl–Heisenberg algebra and its Fock or number representation

Notational convention: set of nonnegative integers is \( \mathbb{N} = 0, 1, 2, \ldots \)

Let \( \mathcal{H} \) be a separable (complex) Hilbert space with orthonormal basis \( e_0, e_1, \ldots, e_n \equiv |e_n\rangle, \ldots , \) (e.g. the Fock space with \( |e_n\rangle \equiv |n\rangle \)).

Define the lowering and raising operators \( a \) and \( a^\dagger \) as

\[ a |e_n\rangle = \sqrt{n} |e_{n-1}\rangle, \quad a |e_0\rangle = 0 \quad \text{(lowering or annihilation operator)} \]

\[ a^\dagger |e_n\rangle = \sqrt{n+1} |e_{n+1}\rangle \quad \text{(raising or creation operator)}. \]

Equivalently

\[ a = \sum_{n=0}^{\infty} \sqrt{n+1} |e_n\rangle \langle e_{n+1}| \quad \text{(weak sense)} \]

\[ a^\dagger = \sum_{n=0}^{\infty} \sqrt{n+1} |e_{n+1}\rangle \langle e_n| \quad \text{(weak sense)}. \]
Operator algebra \( \{a, a^\dagger, I\} \) is defined by the Canonical Commutation Rule (CCR)

\[[a, a^\dagger] = I.\]

- Number operator: \( N = a^\dagger a \), spectrum \( \mathbb{N}, N|e_n\rangle = n|e_n\rangle \).
- Operators \( Q \) and \( P \)

\[ a = \frac{Q + iP}{\sqrt{2}}, \quad a^\dagger = \frac{Q - iP}{\sqrt{2}} \iff Q = \frac{a + a^\dagger}{\sqrt{2}}, \quad P = \frac{1}{i} \frac{a - a^\dagger}{\sqrt{2}}. \]

- Both are essentially self-adjoint in \( \mathcal{H} \), with absolutely spectrum \( \sigma(Q) = \mathbb{R} = \sigma(P) \).
- Familiar form of the CCR

\[[Q, P] = iI.\]

- Consistently to (4.7), if \( Q \) is realised on a multiplication operator on the Hilbert space \( L^2(\mathbb{R}, dx) \) of square-integrable complex valued functions (“wave functions”) on its spectrum \( \sigma(Q) = \mathbb{R}, Q\phi(x) = x\phi(x) \), then \( P \) is realised as \( \frac{-1}{dx} \)

- Alternatively, if \( P \) is realised on a multiplication operator on the Hilbert space \( L^2(\mathbb{R}, dk) \) of square-integrable complex valued functions on its spectrum \( \sigma(P) = \mathbb{R}, P\hat{\phi}(k) = k\hat{\phi}(k) \) then \( Q \) is realised as \( \frac{d}{dk} \)

5. Unitary Weyl-Heisenberg group representation(s)

- Any infinite-dimensional UIR, \( U^\lambda \), of \( G_{WH} \) is characterized by a real number \( \lambda \neq 0 \) (in addition, there are also degenerate, one-dimensional, UIRs corresponding to \( \lambda = 0 \)) and can be identified with an irreducible representation of the CCR

\[ (\zeta, q, p) \mapsto U^\lambda(\zeta, q, p) := e^{i\lambda(\zeta - pq/2)} e^{i\lambda pQ} e^{-i\lambda qP}. \]

- If the Hilbert space carrying the UIR is \( \mathcal{H} = L^2(\mathbb{R}, dx) \), \( \mathbb{R} \) being viewed here as the spectrum \( \sigma(Q) \) of the (essentially) self-adjoint position operator
Q, the unitary operator $U^\lambda$ acts as

$$\tag{5.2} (U^\lambda(\varsigma, q, p)\phi)(x) = e^{i\lambda\varsigma x} e^{i\lambda p(x - q/2)} \phi(x - q), \quad \phi \in L^2(\mathbb{R}, dx).$$

- Alternatively, if the Hilbert space carrying the UIR is $\mathcal{H} = L^2(\mathbb{R}, dx)$, $\mathbb{R}$ being viewed here as the spectrum $\sigma(P)$ of the (essentially) self-adjoint position operator $P$, the unitary operator $U^\lambda$ acts as

$$\tag{5.3} (U^\lambda(\varsigma, q, p)\hat{\phi})(k) = e^{i\lambda\varsigma} e^{-i\lambda q (k - p/2)} \hat{\phi}(k - p), \quad \hat{\phi} \in L^2(\mathbb{R}, dk).$$

- Thus, the infinitesimal generators of $U^\lambda$ read as

$$\tag{5.4} D^\lambda_\varsigma = i\lambda I, \quad (D^\lambda_\varsigma \phi)(x) = i\lambda x \phi(x) \equiv i\lambda Q \phi(x),$$

$$\tag{5.4} (D^\lambda_p \phi)(x) = -\frac{\partial \phi}{\partial x}(x) = -i\lambda P \phi(x), \quad [D^\lambda_p, D^\lambda_\varsigma] = i\lambda = \lambda^2 [Q, P].$$

- Usually, one takes for $\lambda$ the specific value, $\lambda = 1/\hbar$ (in QM) or simply $\lambda = 1$, and write $U$ for $U^1$

- Hence, the representation reads as the action of exponential operators

$$\tag{5.5} (U(\varsigma, q, p) = e^{i\lambda \varsigma} e^{-i\lambda p q / 2} e^{i\lambda p} e^{-i\lambda p} \equiv e^{i\lambda \mathcal{D}(q, p)}$$

where $\mathcal{D}(q, p)$ is the unitary “displacement operator” in representation $(q, p)$

- Now consider the Fock–Bargmann Hilbert space $\mathcal{F}_B$ (resp. $\mathcal{A}_B$) of entire analytical (resp. antianalytical) functions that are square integrable with respect to the Gaussian measure on the complex plane

$$\tag{5.6} \frac{1}{\pi} e^{-|z|^2} d^2 z = \frac{1}{2\pi} e^{-\alpha^2/2} dq dp,$$

$\alpha(z)$ (resp. $\alpha(\bar{z}) = \sum_{n=0}^{+\infty} \alpha_n z^n$ (resp. $\bar{z}^n$) converges absolutely for all $z \in \mathbb{C}$, i.e., its convergence radius is infinite, and

$$\tag{5.7} ||\alpha||^2_{\mathcal{F}_B} \overset{\text{def}}{=} \int_\mathbb{C} |\alpha(z)|^2 e^{-|z|^2} \frac{d^2 z}{\pi} < \infty.$$

- Space $\mathcal{F}_B$ is equipped with the scalar product:

$$\tag{5.8} \langle \alpha_1 | \alpha_2 \rangle = \int_\mathbb{C} \overline{\alpha_1(z)} \alpha_2(z) = \sum_{n=0}^{+\infty} n! \overline{\alpha_{1n}} \alpha_{2n}.$$
The unitary operator \( U \) acts on space \( \mathcal{F}\mathcal{B} \) as
\[
(U(\varsigma, z')\alpha)(\bar{z}) = e^{i\varsigma} e^{-\frac{1}{2}|z'|^2} e^{\varsigma \bar{z}} \alpha(\bar{z} - z'),
\]
where \( z' \circ z = -z \circ z' \) is defined in (3.4)

Equivalently, and more simply, on the space of “Gaussian weighted \( \mathcal{F}\mathcal{B} \) as
\[
(U(\varsigma, z')\alpha)(\bar{z}) = e^{i\lambda \varsigma} e^{-\frac{1}{2}|z - z'|^2} e^{\varsigma \bar{z}} \alpha(\bar{z} - z'),
\]
\[\alpha_g(z) = e^{-\frac{|z|^2}{2}} \alpha(z) \in L^2(C, \frac{d^2z}{\pi}) .\]

On Space \( \mathcal{F}\mathcal{B} \) the annihilation operator \( a \) is represented as a derivation whereas its adjoint is a multiplication operator:

\[
a \alpha(z) = \frac{d}{dz} \alpha(z), \quad a^\dagger \alpha(z) = z \alpha(z).
\]

6. Displacement operator

From the previous section, to each complex number \( z \) is associated the (unitary) displacement operator or “function \( D(z) \)”
\[
C \ni z \mapsto D(z) = e^{z a^\dagger - \bar{z} a}, \quad D(-z) = (D(z))^{-1} = D(z)^\dagger.
\]

In variables \((q, p)\) and operators \( Q \) and \( P \)
\[
D(z) \equiv \mathcal{D}(q, p) = e^{i(pQ - qP)}.
\]

Unitary representation \( U \) with complex notations
\[
(\varsigma, z) \mapsto e^{i\varsigma} D(z),
\]
\[
(\varsigma, z)(\varsigma', z') \mapsto e^{i\varsigma} D(z) e^{i\varsigma'} D(z') = e^{i(\varsigma + \varsigma' + \text{Im} z\bar{z}')} D(z + z').
\]

The Weyl formula
\[
e^A e^B = e^{\frac{1}{2}[A,B]} e^{(A + B)},
\]
(arising from the Baker–Campbell–Hausdorff relation) which is formally valid for any pair of operators that commute with their commutator, \([A, [A, B]] = 0\).\footnote{Actually we have to be seriously aware of domains of involved operators when we apply the “algebraic” formula, see [28].}
\[ 0 = [B, [A, B]], \] yields
\[
D(z) = e^{-i|z|^2} e^{za^\dagger} e^{-\bar{z}a} = e^{\frac{i}{2}|z|^2} e^{-\bar{z}a} e^{za^\dagger},
\]
- In variables \((q, p)\) and operators \(Q\) and \(P\), consistently with \((5.5)\)
\[
D(z) \equiv \mathcal{D}(q, p) = e^{i(pQ-qP)} = e^{-\frac{iqP}{2}} e^{ipQ} e^{-iqp} = e^{\frac{iqP}{2}} e^{-iqp} e^{ipQ}.
\]
- It follows the formulae
\[
\frac{\partial}{\partial z} D(z) = \left( a^\dagger - \frac{1}{2} \bar{z} \right) D(z) = D(z) \left( a^\dagger + \frac{1}{2} \bar{z} \right).
\]
\[
\frac{\partial}{\partial \bar{z}} D(z) = - \left( a - \frac{1}{2} z \right) D(z) = -D(z) \left( a + \frac{1}{2} z \right).
\]
- equivalently
\[
z D(z) = [a, D(z)], \quad \bar{z} D(z) = [a^\dagger, D(z)],
\]
\[
\frac{\partial}{\partial z} D(z) = \frac{1}{2} \{a^\dagger, D(z)\}, \quad \frac{\partial}{\partial \bar{z}} D(z) = -\frac{1}{2} \{a, D(z)\}.
\]
- With variables \(q, p\)
\[
\frac{\partial}{\partial q} \mathcal{D}(q, p) = \left( -iP + \frac{i}{2} q \right) \mathcal{D}(q, p) = -\mathcal{D}(q, p) \left( iP + \frac{i}{2} q \right),
\]
\[
\frac{\partial}{\partial p} \mathcal{D}(q, p) = \left( iQ - \frac{i}{2} q \right) \mathcal{D}(q, p) = \mathcal{D}(q, p) \left( iQ + \frac{i}{2} q \right),
\]
- equivalently
\[
q \mathcal{D}(q, p) = [Q, \mathcal{D}(q, p)], \quad p \mathcal{D}(q, p) = [P, \mathcal{D}(q, p)],
\]
\[
\frac{\partial}{\partial q} \mathcal{D}(q, p) = -i \{P, \mathcal{D}(q, p)\}, \quad \frac{\partial}{\partial p} \mathcal{D}(q, p) = i \{Q, \mathcal{D}(q, p)\}.
\]
- Addition formula
\[
D(z)D(z') = e^{\frac{i}{2}z\bar{z'}} D(z + z'),
\]
\[
\mathcal{D}(q, p) \mathcal{D}(q', p') = e^{-\frac{1}{2}(qp' - pq')^2} \mathcal{D}(q + q', p + p').
\]
• It follows the covariance formula on a global level

\[ D(z)D(z')D(z) = e^{z z'} D(z') , \]

\[ \mathfrak{D}(q, p) \mathfrak{D}(q', p') \mathfrak{D}^\dagger(q, p) = e^{i p q' - q p'} \mathfrak{D}(q', p') . \]

• and on a Lie algebra level

\[ D(z) a D(z)^\dagger = a - z I, \quad D(z) a^\dagger D(z)^\dagger = a^\dagger - z I . \]

• Matrix elements of operator \( D(z) \) in the basis \( \{|e_n\}\) involve associated Laguerre polynomials \( L_n^{(m)}(t) \) \([29, 30]\\):

\[ \langle e_n|D(z)|e_n \rangle = D_{mn}(z) = \sqrt{\frac{n!}{m!}} e^{-|z|^2/2} z^m z^{m-n} L_n^{(m-n)}(|z|^2) , \quad \text{for } m \geq n , \]

with \( L_n^{(m-n)}(t) = \frac{n!}{m!} (-t)^{n-m} L_m^{(n-m)}(t) \) for \( n \geq m \)

• Orthonormality properties straightforwardly derive from unitarity:

\[ \int C \frac{d^2 z}{\pi} D_{mn}(z) \overline{D_{m'n'}(z)} = \delta_{mm'} \delta_{nn'} . \]

• One derives from unitarity the infinite sums:

\[ \sum_{n=0}^{\infty} D_{mn}(z) \overline{D_{m'n}(z)} = \delta_{mm'} = \sum_{n=0}^{\infty} D_{nm'}(z) \overline{D_{nm}(z)} , \]

and particularly

\[ \sum_{n=0}^{\infty} |D_{mn}(z)|^2 = 1 = \sum_{n=0}^{\infty} |D_{nm}(z)|^2 , \quad m \in \mathbb{N} . \]

7. Coherent states

• Standard (Schrödinger \([31], \) Iwata \([32], \) Klauder \([33, 34, 35], \) Glauber \([36, 37, 38], \) Sudarshan \([39], \) see also \([40, 41, 42, 43, 44, 45, 22, 28]\) for reviews and further developments), coherent states are defined as vectors in \( \mathcal{H} \)

\[ |z\rangle = D(z) |e_0\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |e_n\rangle , \]

• They have unit norm and solve the identity in \( \mathcal{H} \)

\[ \langle z|z\rangle = 1 , \quad \int C \frac{d^2 z}{\pi} |z\rangle \langle z| = I . \]
• Their overlap is a Gaussian on $\mathbb{C}$ up to the symplectic phase factor

$$\langle z | z' \rangle = e^{\frac{i}{2} z'^* z} e^{\frac{i}{2} |z - z'|^2}.$$  

(7.3)

• CS is reproducing kernel for functions $\psi(z) := \langle z | \psi \rangle$ built from vectors $|\psi\rangle \in \mathcal{H}$

$$\psi(z) = \int_C \frac{dz'}{\pi} \langle z | z' \rangle \psi(z').$$  

(7.4)

• In representation position or momentum, there are Gaussian states, expressed with $z = \frac{q + ip}{\sqrt{2}}$,

$$\langle x | e_0 \rangle = \frac{1}{\sqrt{\pi}} e^{-\frac{x^2}{2}} \quad \langle x | z \rangle = \frac{1}{\sqrt{\pi}} e^{-i p(x - q/2)} e^{-\frac{(x - q)^2}{2}},$$  

(7.5)

$$\langle k | e_0 \rangle = \frac{1}{\sqrt{\pi}} e^{-\frac{k^2}{2}} \quad \langle k | z \rangle = \frac{1}{\sqrt{\pi}} e^{i q(k - p/2)} e^{-\frac{(k - p)^2}{2}}.$$  

(7.6)

• WH translation covariance of coherent states result from their definition (7.1) as “displaced lowest state”

$$D(z) | z_0 \rangle = e^{\frac{i}{2} z_0 z} | z + z_0 \rangle.$$  

(7.7)

8. Parity and time reversal

• The parity $\mathcal{P}$ acts on $\mathcal{H}$ as a linear operator through

$$\mathcal{P} | e_n \rangle = (-1)^n | e_n \rangle \quad \text{or} \quad \mathcal{P} = e^{i \pi a^\dagger a}.$$  

(8.1)

• The time reversal $\mathcal{T}$ acts on $\mathcal{H}$ as a conjugation, that is an antilinear operator such that

$$\mathcal{T} \sum_n \xi_n | e_n \rangle = \sum_n \overline{\xi_n} | e_n \rangle.$$  

(8.2)
• These discrete symmetries verify

\begin{align*}
(8.3) & \quad P^2 = T^2 = I, \\
(8.4) & \quad PaP = -a, \quad Pa^\dagger P = -a^\dagger, \\
(8.5) & \quad TaT = a, \quad Ta^\dagger T = a^\dagger, \\
(8.6) & \quad PD(z)P = D(-z), \quad TD(z)T = D(\bar{z}).
\end{align*}

9. Rotation in the plane

• A unitary representation \( \theta \mapsto U_T(\theta) \) of the torus \( S^1 \) on the Hilbert space \( \mathcal{H} \) is defined as

\begin{equation}
(9.1) \quad U_T(\theta)|e_n\rangle = e^{i(n+\nu)\theta} |e_n\rangle, \quad \nu \in \mathbb{R}.
\end{equation}

• Note that \( P = U_T(\pi) \) with \( \nu = 0 \)

• Rotational covariance of the displacement operator

\begin{equation}
(9.2) \quad U_T(\theta)D(z)U_T(\theta)^\dagger = D(e^{i\theta} z).
\end{equation}

• Rotational covariance of coherent states

\begin{equation}
(9.3) \quad U_T(\theta)|z\rangle = e^{i\nu\theta} |e^{i\theta} z\rangle.
\end{equation}

10. Integral formulae for \( D(z) \)

• First fundamental integral: from [20]

\begin{equation}
(10.1) \quad \int_0^\infty e^{-\frac{t}{2}} L_n(t) \, dt = (-2)^n \Rightarrow \int_{\mathbb{C}} D_{mn}(z) \frac{d^2z}{\pi} = \delta_{mn}(-2)^m,
\end{equation}

it follows

\begin{equation}
(10.2) \quad \int_{\mathbb{C}} D(z) \frac{d^2z}{\pi} = 2P.
\end{equation}

• Resolution of the identity follows from (10.2)

\begin{equation}
(10.3) \quad \int_{\mathbb{C}} D(z) 2P D(-z) \frac{d^2z}{\pi} = I.
\end{equation}

(At the basis of the Weyl–Wigner quantization (in complex notations), see below)
Second fundamental integral: from (6.21) and the orthogonality of the associated Laguerre polynomials we obtain the “ground state” projector as the Gaussian average of $D(z)$

$$
\int_C e^{-\frac{1}{2}|z|^2} D(z) \frac{d^2 z}{\pi} = |e_0 \rangle \langle e_0 |
$$

More generally for $\text{Re}(s) < 1$

$$
\int_C e^{s|z|^2} D(z) \frac{d^2 z}{\pi} = \frac{2}{1-s} \exp \left( \ln \frac{s+1}{s-1} a^+ a \right),
$$

where the convergence holds in norm for $\text{Re}(s) < 0$ and weakly for $0 \leq \text{Re}(s) < 1$.

II. Harmonic analysis on $\mathbb{C}$ or $\mathbb{R}^2$ and symbol calculus

II.1. In terms of $z$ and $\bar{z}$.

- Symplectic Fourier transform on $\mathbb{C}$ (for a sake of simplicity, we write $f(z, \bar{z}) \equiv f(z)$)

$$
\mathfrak{f}_s[f](z) = \int_C e^{z\xi - \bar{z}\bar{\xi}} f(\xi) \frac{d^2 \xi}{\pi} = \int_C e^{2 \text{Im}(z\bar{\xi})} f(\xi) \frac{d^2 \xi}{\pi} = \int_C e^{z \xi} f(\xi) \frac{d^2 \xi}{\pi} \quad \text{(notation most used in the paper)}.
$$

- Dirac-Fourier formula

$$
\mathfrak{f}_s[1](z) = \int_C e^{z \xi} \frac{d^2 \xi}{\pi} = \int_{\mathbb{R}^2} e^{-i(q y - p x)} \frac{dx dy}{2\pi} = 2\pi \delta(q) \delta(p) = \pi \delta^2(z).
$$

- The symplectic Fourier transform is its inverse: it is an involution

$$
\mathfrak{f}_s[\mathfrak{f}_s[f]](z) = f(z) \iff \mathfrak{f}_s^2 = I.
$$

- The symplectic Fourier transform commutes with the parity operator

$$
\mathfrak{f}_s = P \mathfrak{f}_s P, \quad (P f)(z) = f(-z) = \tilde{f}(z), \quad \tilde{f}(z) := f(-z).
$$

- Reflected symplectic Fourier transform

$$
\mathfrak{f}_{s}[f](z) = \int_C e^{-z \xi - \bar{z}\bar{\xi}} f(\xi) \frac{d^2 \xi}{\pi} = \mathfrak{f}_s[f](-z) = \mathfrak{f}_s[\tilde{f}](z) = \mathfrak{f}_s[\bar{f}](z).
$$
• The reflected symplectic Fourier transform is its inverse

\[(11.7) \quad \overline{\mathcal{T}_s f_s} = I.\]

• Factorization of the parity operator

\[(11.8) \quad \overline{\mathcal{T}_s f_s} = f_s \mathcal{P}.\]

• Symplectic Fourier transform and translation with

\[(11.9) \quad (t_z f)(z') := f(z' - z),\]

\[(11.10) \quad f_s [t_z f](z') = e^{z'x} f_s [f](z'), \quad \overline{\mathcal{T}_s [t_z f]}(z') = e^{z'x} \overline{\mathcal{T}_s [f]}(z').\]

• Symplectic Fourier transform and derivation

\[(11.11) \quad \frac{\partial^k}{\partial z^k} f_s [f](z) = f_s [\xi^k f](z), \quad \frac{\partial^k}{\partial z^k} \overline{\mathcal{T}_s [f]}(z) = \overline{\mathcal{T}_s \left[ (-\xi)^k f \right]}(z),\]

\[(11.12) \quad \frac{\partial^k}{\partial \bar{z}^k} f_s [f](z) = f_s \left[ (-\xi)^k f \right](z), \quad \frac{\partial^k}{\partial \bar{z}^k} \overline{\mathcal{T}_s [f]}(z) = \overline{\mathcal{T}_s \left[ \xi^k f \right]}(z),\]

\[(11.13) \quad f_s \left[ \frac{\partial^k}{\partial \xi^k} f \right](z) = z^k \overline{f_s \left[ f \right]}(z), \quad \overline{\mathcal{T}_s \left[ \frac{\partial^k}{\partial \xi^k} f \right]}(z) = (-z)^k \overline{\mathcal{T}_s \left[ f \right]}(z),\]

\[(11.14) \quad f_s \left[ \frac{\partial^k}{\partial \bar{\xi}^k} f \right](z) = (-z)^k f_s \left[ f \right](z), \quad \overline{\mathcal{T}_s \left[ \frac{\partial^k}{\partial \bar{\xi}^k} f \right]}(z) = z^k \overline{\mathcal{T}_s \left[ f \right]}(z).\]

• Convolution product with complex variables

\[(11.15) \quad (f \ast g)(z) := \int_C d^2 z' f(z - z') g(z') = (g \ast f)(z).\]

• Symplectic Fourier transform of convolution products

\[(11.16) \quad f_s [f \ast g](z) = \pi f_s [f](z) f_s [g](z),\]

\[(11.17) \quad f_s [f \ast g](z) = \frac{1}{\pi} (f_s [f] \ast f_s [g])(z).\]

• Symplectic Fourier transform of Gaussian

\[(11.18) \quad f_s \left[ e^{\nu |\xi|^2} \right](z) = \frac{1}{(\nu^2)} e^{\nu |z|^2} = \overline{\mathcal{T}_s \left[ e^{\nu |\xi|^2} \right]}(z), \quad \text{Re}(\nu) < 0.\]
• Symplectic Fourier transform of $D$

$$\int_C e^{x^\xi} D(z') \frac{d^2z'}{\pi} = 2 D(2z) P = 2P D(-2z).$$

11.2. In terms of $q$ and $p$.

• In terms of coordinates $z = (q + ip)/\sqrt{2}$, $\xi = (x + iy)/\sqrt{2}$,

$$f[s](z) \equiv \mathcal{F}_s[F](q, p) = \int_{\mathbb{R}^2} e^{-i(qy - px)} F(x, y) \frac{dx \, dy}{2\pi} = \mathcal{F}[F](-p, q),$$

where $\mathcal{F}$ denotes the standard two-dimensional Fourier transform,

$$\mathcal{F}[F](k_x, k_y) = \int_{\mathbb{R}^2} e^{-i(k_x x + k_y y)} F(x, y) \frac{dx \, dy}{2\pi},$$

with inverse

$$\mathcal{F}[F](k_x, k_y) = \int_{\mathbb{R}^2} e^{i(k_x x + k_y y)} F(x, y) \frac{dx \, dy}{2\pi} = \mathcal{F}[F](-k_x, -k_y).$$

• $\mathcal{F}_s$ is involutive, $\mathcal{F}_s[\mathcal{F}_s[F]] = \mathcal{F}_s^2[F] = F$ like its “dual” defined as

$$\mathcal{F}_s[F](q, p) = \mathcal{F}_s[F](-q, -p) = \int_{\mathbb{R}^2} e^{i(qy - px)} F(x, y) \frac{dx \, dy}{2\pi} = \mathcal{F}[F](p, -q),$$

• Symplectic Fourier transform and derivation

$$\frac{\partial^k}{\partial q^k} \mathcal{F}_s[F](q, p) = (-i)^k \mathcal{F}_s \left[ y^k \ F \right](q, p), \quad \frac{\partial^k}{\partial p^k} \mathcal{F}_s[F](q, p) = i^k \mathcal{F}_s \left[ x^k \ F \right](q, p),$$

$$\frac{\partial^k}{\partial p^k} \mathcal{F}_s[F](q, p) = i^k \mathcal{F}_s \left[ x^k \ F \right](q, p), \quad \frac{\partial^k}{\partial q^k} \mathcal{F}_s[F](q, p) = (-i)^k \mathcal{F}_s \left[ x^k \ F \right](q, p),$$

$$i^k \mathcal{F}_s \left[ \frac{\partial^k}{\partial x^k} \ F \right](q, p) = p^k \mathcal{F}_s [F] (q, p), \quad (-i)^k \mathcal{F}_s \left[ \frac{\partial^k}{\partial y^k} \ f \right] (q, p) = p^k \mathcal{F}_s [F] (q, p),$$

$$(-i)^k \mathcal{F}_s \left[ \frac{\partial^k}{\partial y^k} \ F \right](q, p) = q^k \mathcal{F}_s [F] (q, p), \quad i^k \mathcal{F}_s \left[ \frac{\partial^k}{\partial y^k} \ F \right](q, p) = q^k \mathcal{F}_s [F] (q, p).$$

• Convolution product

$$(F \ast G)(q, p) := \int_C dq' \, dp' \, F(q - q', p - p') \, G(q - q', p - p') = (G \ast F)(z).$$
• Symplectic Fourier transform of convolution products

\[ \begin{align*}
\mathcal{F}_s[F * G](q, p) &= 2\pi \mathcal{F}_s[F](q, p) \mathcal{F}_s[G](q, p), \\
\mathcal{F}_s[F G](q, p) &= \frac{1}{2\pi} (\mathcal{F}_s[F] * \mathcal{F}_s[G])(q, p).
\end{align*} \tag{11.29} \tag{11.30} \]

• Same formulae for \( \mathcal{F}_s \)

11.3. **In terms of \( r \) and \( \phi \), polar coordinates of \( z \).**

• Consider a function (or distribution) \( f(z) \) with \( z = r e^{i\phi} \) Fourier expandable as

\[ f\left(r e^{i\phi}\right) = \sum_{n=-\infty}^{+\infty} c_n(r) e^{in\phi}. \tag{11.31} \]

• Its symplectic Fourier transform is expressed as the Fourier series

\[ \mathcal{F}_s[f]\left(r e^{i\phi}\right) = \sum_{n=-\infty}^{+\infty} c_{s n}(r) e^{in\phi}, \tag{11.32} \]

where the coefficients are Hankel transform of order \( n \) \([29]\), up to factors 2, of the \( c_n \)'s

\[ c_{s n}(r) = 2 \int_{0}^{\infty} c_n(\rho) J_n(2r\rho) \rho \, d\rho, \tag{11.33} \]

\( J_n \): Bessel function of the first kind, with \( J_{-n} = (-1)^n J_n \)

12. **Trace formulae for \( D(z) \)**

• From the resolution of the identity of the standard coherent states, from Eq. (6.18) and Eq. (11.3), it follows

\[ \text{tr} D(z) = \int_{C} \langle \xi | D(z) | \xi \rangle \frac{d^2\xi}{\pi} = \pi \delta^2(z). \tag{12.1} \]

• Using Eq.(6.16), we have

\[ \text{tr} \left( D(z) \dagger D(z') \right) = \pi \delta^2(z - z'). \tag{12.2} \]
13. Noncommutative Weyl–Heisenberg Harmonic Analysis

- Weyl–Heisenberg transform of a function as the operator in \( \mathcal{H} \)

\[
(13.1) \quad f(z) \mapsto \mathfrak{T}[f] = \int_C D(z) f(z) \frac{d^2 z}{\pi}.
\]

- Adjoint

\[
(13.2) \quad \mathfrak{T}[f]^\dagger = \mathfrak{T}[\tilde{f}].
\]

- Inversion formula is direct consequence of (12.2)

\[
(13.3) \quad f(z) = \text{tr} \left( D(-z) \mathfrak{T}[f] \right).
\]

- Translation covariance

\[
(13.4) \quad D(z_0) \mathfrak{T}[f] D(z_0)^\dagger = \mathfrak{T}[t_z f]
\]

14. Weyl–Heisenberg Integral Quantization

- Pick a weight function \( \omega(z) = \Pi(q, p) \) obeying

\[
(14.1) \quad \omega(0) = 1 = \Pi(0, 0).
\]

- Suppose that

\[
(14.2) \quad \mathcal{M}^{\omega} = \int_C D(z) \omega(z) \frac{d^2 z}{\pi} = \int_{\mathbb{R}^2} \mathfrak{D}(q, p) \Pi(q, p) \frac{dq dp}{2\pi} = \mathcal{M}^{\Pi}.
\]

is bounded on \( \mathcal{H} \)

- Then, the family

\[
(14.3) \quad \mathcal{M}^{\omega}(z) := D(z) \mathcal{M}^{\omega} D(z)^\dagger = \mathfrak{D}(q, p) \mathcal{M}^{\Pi} \mathfrak{D}(q, p)^\dagger = \mathcal{M}^{\Pi}(q, p)
\]

resolves the identity on \( \mathcal{H} \)

\[
(14.4) \quad \int_C \mathcal{M}^{\omega}(z) \frac{d^2 z}{\pi} = I = \int_{\mathbb{R}^2} \mathcal{M}^{\Pi}(q, p) \frac{dq dp}{2\pi}.
\]

- The operator \( \mathcal{M}^{\omega} \) as an integral operator on Gaussian weighted anti-analytic Fock–Bargmann space

\[
(14.5) \quad (\mathcal{M}^{\omega} \varphi_g)(\bar{z}) = \int_C \frac{d^2 z'}{\pi} \mathfrak{M}^{\omega}(z, z') \varphi_g(\bar{z}') \quad \varphi_g \in L^2 \left( \mathbb{C}, \frac{d^2 z}{\pi} \right),
\]
where \( \alpha_g(\bar{z}) := e^{-|\xi|^2/2} \alpha(\bar{z}) \), \( \alpha \in \mathcal{A}\mathcal{F}\mathcal{B} \). The kernel \( \omega^\alpha \) is given by
\[
\mathcal{M}^\omega(z, z') = \omega(z - z') e^{\alpha z'}. 
\]

- In terms of symplectic Fourier transforms,
\[
(M^\omega \alpha_g)(\bar{z}) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dp e^{-ipy} \hat{\Pi}_p \left( x - x', -\frac{x + x'}{2} \right). 
\]
- The operator \( M^\omega \) as an integral operator in representation position
\[
(\mathcal{M}^\omega \phi)(x) = \int_{-\infty}^{+\infty} dx' \mathcal{M}^\omega(x, x') \phi(x'), \quad \phi \in L^2(\mathbb{R}, dx), 
\]
with kernel
\[
\mathcal{M}^\omega(x, x') = \frac{1}{\sqrt{2\pi}} \hat{\Pi}_p \left( x - x', -\frac{x + x'}{2} \right). 
\]
- In the above expression, \( \hat{\Pi}_p \) stands for the partial Fourier transform of \( \Pi(q, p) \) with respect to the \( p \) variable
\[
\hat{\Pi}_p(q, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dp e^{-ipy} \Pi(q, p). 
\]
- A trace formula, issued, if applicable, from (12.1) and (14.1)
\[
\text{tr} (M^\omega(z)) = \text{tr} (\mathcal{M}^\omega) = \omega(0) = 1. 
\]
- Necessary condition on \( \omega(z) \) for that \( M^\omega(z) \) define a normalized Positive Operator Valued Measure (POVM)
\[
\forall z, \ 0 < \langle z|\mathcal{M}^\omega|z \rangle = \int \left| e^{-|\xi|^2/2} \omega(\bar{\xi}) \right|^2 (\xi) = \frac{2}{\pi} \int \left| e^{-|\xi|^2} \right|^2 \hat{\Pi}_p[\omega(\xi)](\xi). 
\]
- Weyl-Heisenberg integral quantization is the linear map
\[
f \mapsto A^\omega_f = \iint_C M^\omega(z) f(z) d^2z, 
\]
such that the constant function \( f = 1 \) is mapped to the identity \( I \).
- Alternatively with the symplectic Fourier transform (11.2)
\[
A^\omega_f = \iint_C D(z) \hat{\Pi}_s[f](-z) \omega(z) d^2z = \iint_C D(z) \hat{\Pi}_s[f](z) \omega(z) d^2z. 
\]
• In terms of variables \( q, p \) and Fourier transform, with \( \varphi(z) \equiv \Pi(q, p) \),

\[
(14.15) \quad A^0_F = A^\Pi_F = \int_{\mathbb{R}^2} \mathcal{D}(q, p) \overline{\Pi}[F][(-q, -p)] \Pi(q, p) \frac{dq \, dp}{2\pi} \\
= \int_{\mathbb{R}^2} \mathcal{D}(q, p) \overline{\Pi}[F](q, p) \Pi(q, p) \frac{dq \, dp}{2\pi} \\
= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-\frac{i}{2} p q} e^{i p \varphi} e^{-i q p} e^{i(qy - px)} F(x, y) \Pi(q, p) \frac{dq \, dp \, dx \, dy}{2\pi} \\
= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-\frac{i}{2} p q} e^{i p \varphi} e^{i(qy - px)} F(x, y) \Pi(q, p) \frac{dq \, dp \, dx \, dy}{2\pi}
\]

• The operator \( A^0_F \) as an integral operator on Gaussian weighted anti-analytic Fock–Bargmann space

\[
(14.19) \quad (A^0_f \alpha_g)(\tilde{z}) = \int_{\mathbb{C}} \frac{d^2 z'}{\pi} \mathcal{A}^0_f(z, z') \alpha_g(z'),
\]

with kernel

\[
(14.20) \quad \mathcal{A}^0_f(z, z') = \varphi(z - z') e^{\frac{i}{2} z' \varphi} \bar{\Pi}[f](z - z') = \mathcal{F}^0_f(z, z') \bar{\Pi}[f](z - z')
\]

• In terms of symplectic Fourier transforms,

\[
(14.21) \quad (A^0_f \phi)(\tilde{z}) = \frac{1}{\pi} e^{-\varphi} f \ast e^{-\varphi} \bar{\Pi}[\varphi] \ast \Pi_{\varphi}(\tilde{z} \frac{\bar{z}}{2}).
\]

• The operator \( A^0_f = A^\Pi_f \) as an integral operator in representation position

\[
(14.22) \quad (A^\Pi_f \phi)(x) = \int_{-\infty}^{+\infty} dx' \mathcal{A}^\Pi_f(x, x') \phi(x'), \quad \phi \in L^2(\mathbb{R}, dx),
\]

with kernel

\[
(14.23) \quad \mathcal{A}^\Pi_f(x, x') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dq \hat{F}_p(q, x' - x) \hat{\Pi}_p \left(x - x', q - \frac{x + x'}{2}\right).
\]

• In the above expression, \( \hat{F}_p \) and \( \hat{\Pi}_p \) stand respectively for the partial Fourier transforms of \( F(q, p) \) and \( \Pi(q, p) \) with respect to the \( p \) variable

• Historical Weyl–Wigner (\( \mathcal{W} = \mathcal{W} \)) case holds for \( \varphi(z) = \Pi(q, p) = 1 \). Then

\[
(14.24) \quad \mathcal{M}^1 \equiv \mathcal{M}^{\mathcal{W} = \mathcal{W}} = 2 \mathcal{P} \quad \mathcal{A}^{\mathcal{W} = \mathcal{W}}_f(x, x') = \frac{1}{\sqrt{2\pi}} \hat{F}_p \left( x + \frac{x'}{2}, x' - x \right).
\]
• Another historical case: coherent state, or Berezin, or anti-Wick quantization. From a normalised fiducial vector $\eta(x) \in L^2(\mathbb{R}, dx)$, e.g. $\eta(x) = e_0(x)$,

\begin{align}
M^{(\eta)} &= |\eta\rangle \langle \eta|, \\
A^{(\eta)^\dagger}(x, x') &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dq \hat{F}_p(q, x' - x) \overline{\eta(x' - q)} \eta(x - q).
\end{align}

• If applicable from (14.11), one has the trace formula

\begin{equation}
\text{tr} \left( A^\omega_f \right) = \int_C f(z) \frac{d^2 z}{\pi}.
\end{equation}

• Complex conjugaison covariance. From

\begin{equation}
\left( A^\omega_f \right)^\dagger = A^{\omega^\dagger}_f,
\end{equation}

we have

\begin{equation}
A^\omega_f = \left( A^\omega_f \right)^\dagger, \forall f \iff \omega(-z) = \omega(z) \forall z.
\end{equation}

• Translational covariance

\begin{equation}
A^\omega_{f(z-z_0)} = D(z_0) A^\omega_f D(z_0)^\dagger.
\end{equation}

• Parity covariance

\begin{equation}
A^\omega_{f(-z)} = P A^\omega_f P, \forall f \iff \omega(z) = \omega(-z), \forall z.
\end{equation}

• Rotational covariance

\begin{equation}
U_T(\theta) A^\omega_f U_T(-\theta) = A^\omega_{U_T(\theta) f} \iff \omega(e^{i\theta} z) = \omega(z), \forall z, \theta \iff M^\omega \text{ diagonal},
\end{equation}

where $T(\theta) f(z) := f(e^{-i\theta} z)$.

• Fundamental separation formulae

\begin{itemize}
  \item If $F(q, p)$ is a function of $q$ only, $F(q, p) \equiv u(q)$, then $A^\Pi_u$ depends on $Q$ only

\begin{equation}
A^\Pi_u = \frac{1}{\sqrt{2\pi}} u * \overline{\hat{F}[\Pi(0, \cdot)]}(Q),
\end{equation}
\end{itemize}
where $\mathcal{F}$ is the inverse 1-D Fourier transform

\begin{equation}
\mathcal{F}[h](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} h(x) \, dx, \quad \mathcal{F}[u](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} u(x) \, dx.
\end{equation}

- If $F(q, p)$ is a function of $p$ only, $F(q, p) \equiv v(p)$, then $A_{\Pi}^v$ depends on $P$ only

\begin{equation}
A_{\Pi}^v = \frac{1}{\sqrt{2\pi}} v \ast \mathcal{F}[\Pi(\cdot, 0)](P).
\end{equation}

- Similar formulae exist for holomorphic and anti-holomorphic functions $f(z)$

### 15. Quantization of derivatives

- From (6.10), (11.13) and (11.14) we derive easily

\begin{equation}
A^\omega_{\partial_x f} = -A^\xi_{\partial_x f} = -[a^+, A^\omega_f], \quad A^\phi_{\partial_x f} = A^\varphi_{\partial_x f} = [a, A^\phi_f].
\end{equation}

\begin{equation}
A^\Pi_{\partial_q F} = -A^i_{\partial_q F} = i[P, A^\Pi_F], \quad A^\Pi_{\partial_p F} = A^{-iQ}_{\partial_p F} = -i[Q, A^\Pi_F].
\end{equation}

- With Lie algebra adjoint action notation

\begin{equation}
\text{ad}_X(Y) = [X, Y],
\end{equation}

we have more generally

\begin{equation}
A^\omega_{\partial^i_\alpha \partial^j_\beta f} = (-1)^i \left( \text{ad}_a \text{ad}^{i}\right) (A^\omega_f),
\end{equation}

\begin{equation}
A^\Pi_{\partial^i_\alpha \partial^j_\beta F} = (i)^{k-l} \left( \text{ad}_p \text{ad}^i_p \right) (A^\Pi_F).
\end{equation}

- Here we recall the Jacobi identity

\begin{equation}
[X, [Y, Z]] + [Z, [X, Y]] + [Z, [X, Y]] = 0
\end{equation}

from which it results

\begin{equation}
[\text{ad}_X, \text{ad}_Y] (Z) = \text{ad}_{[X,Y]}(Z),
\end{equation}
and so in our above particular case

\[(15.8) \quad [\text{ad}_a, \text{ad}_a^\dagger](Z) = \text{ad}_f(Z) = 0, \quad [\text{ad}_Q, \text{ad}_P^\dagger](Z) = \text{ad}_f(Z) = 0.\]

16. Quantization and product(s)

16.1. With $z, \bar{z}$ variables.

- Quantization of product

\[(16.1) \quad A^0_{fg} = \int \int C \, d^2 z \, d^2 z' \frac{1}{\pi^2} e^{-\frac{1}{2} z z'} \omega(z + z') D(z) \bar{\pi}_f(z) D(z') \bar{\pi}_g(z') \]

\[(16.2) \quad = \int \int C \, d^2 z \, d^2 z' \frac{1}{\pi^2} e^{-\frac{1}{2} z z'} \omega(z + z') D(z) \bar{\pi}_f(z) D(z') \bar{\pi}_g(z') \]

\[(16.3) \quad = \sum_{i,i',j,j'} (-1)^{i+j} a_{iijj} A^0_{ii} \partial_i^a \partial_{i'} g^0 \partial_{i'}^a \partial_j \bar{\pi}_f = A^0_{ff} A^0_{gg} + \cdots \]

\[(16.4) \quad = \sum_{i,i',j,j'} (-1)^{i+j} a_{iijj} A^0_{ii} \partial_i^a \partial_{i'} g^0 \partial_{i'}^a \partial_j \bar{\pi}_f = A^0_{ff} A^0_{gg} + \cdots , \]

where coefficients are defined by the (if making sense) development

\[(16.5) \quad e^{-\frac{1}{2} z z'} \frac{\bar{\omega}(z + z')}{\omega(z) \omega(z')} = \sum_{i,i',j,j'} a_{iijj} z^i z^{i'} z^j \bar{z}^{j'} \bar{z}^{j'}, \quad a_{0000} = 1.\]

- Note that the expression (16.3) provides a sort of inverse Moyal product on the level of operators

- Poisson bracket

From

\[(16.6) \quad \{f, g\} := i \left( \partial_z f \partial_{\bar{z}} g - \partial_{\bar{z}} f \partial_{\bar{z}} g \right) = \{F, G\} = \partial_q F \partial_{\bar{p}} G - \partial_{\bar{p}} F \partial_q G, \]

or, after restoring physical dimensions along (2.3) and (2.5),

\[(16.7) \quad \{f, g\} := -\frac{1}{i\hbar} \left( \partial_{\bar{z}} f \partial_{\bar{z}} g - \partial_{\bar{z}} f \partial_{\bar{z}} g \right) = \{F, G\} = \partial_q F \partial_{\bar{p}} G - \partial_{\bar{p}} F \partial_q G, \]
\[
A_{(f,g)}^{\emptyset} = \frac{i}{\hbar} \sum_{i,j} (-1)^{\bar{i} + \bar{j}} a_{\bar{i} \bar{j}} \left[ A_{\bar{i} \bar{j}}^{\emptyset} \partial_{\bar{i}} f \partial_{\bar{j}} g - A_{\bar{i} \bar{j}}^{\emptyset} \partial_{\bar{i}} g \partial_{\bar{j}} f \right]
\]

\[
= \frac{i}{\hbar} \sum_{i,j} a_{i j} \times \left[ \left( \text{ad}_{\bar{a}^j} \text{ad}_{\bar{a}^i} \right) \left( A_{\bar{a}^j}^{\emptyset} \right) \left( \partial_{\bar{a}^j} f \partial_{\bar{a}^i} g \right) \right]
\]

\[
= \frac{i}{\hbar} \left[ A_{\bar{i} \bar{j}}^{\emptyset} \partial_{\bar{i}} f A_{\bar{j}}^{\emptyset} g - A_{\bar{i} \bar{j}}^{\emptyset} \partial_{\bar{i}} g A_{\bar{j}}^{\emptyset} f \right] + \cdots
\]

- Product of quantizations

\[
A_{f}^{\emptyset} A_{g}^{\emptyset} = A_{f * g}^{\emptyset},
\]

where the “\( \otimes \)-Moyal product” is defined by

\[
(f \star g)(z) = \sum_{i,j} (-1)^{\bar{i} + \bar{j}} a_{\bar{i} \bar{j}} \partial_{\bar{i}} f \partial_{\bar{j}} g,
\]

with coefficients defined by the (if making sense) expansion

\[
\varepsilon_{z z^\prime} \frac{\partial(z) \partial(z^\prime)}{\partial(z + z^\prime)} = \sum_{i,j} \bar{a}_{i j} z^i z^j,
\]

Since coefficients \( \bar{a}_{i j} \) result from the inverse of (16.3), we have the convolution relations with coefficients \( a_{\bar{i} \bar{j}} \)

\[
\sum_{i,j} a_{i j} \bar{a}_{k-i \bar{k} \bar{l} - j \bar{l} - j} = 0
\]

for all \( k, \bar{k}, l, \bar{l} \), such that \( k + \bar{k} + l + \bar{l} \geq 1 \).

- Commutator of quantizations

\[
\left[ A_{f}^{\emptyset}, A_{g}^{\emptyset} \right] = A_{f * g - g * f}^{\emptyset}
\]

\[
= \sum_{i,j} (-1)^{\bar{i} + \bar{j}} a_{\bar{i} \bar{j}} \left[ \left( A_{\bar{i} \bar{j}}^{\emptyset} \partial_{\bar{i}} f \right) \left( A_{\bar{j}}^{\emptyset} \partial_{\bar{j}} g \right) - \left( A_{\bar{i} \bar{j}}^{\emptyset} \partial_{\bar{i}} g \right) \left( A_{\bar{j}}^{\emptyset} \partial_{\bar{j}} f \right) \right]
\]

\[
= i A_{(f,g)}^{\emptyset} + \cdots
\]
16.2. **With \( q, p \) variables.**

- Quantization of product

\[
A^\Pi_{FG} = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{dq \, dp \, dq' \, dp'}{2\pi} e^{i(qp' - p'q')} \Pi(q + q', p + p') \times \\
\times \mathcal{D}(q, p) \mathcal{D}(q', p') \mathcal{D}(q', p') \mathcal{D}(q', p')
\]

(16.17)

\[
= \sum_{k, l, m, n} i^{k-l+m-n} a_{klmn} A^\Pi_{q^k \partial_p^l} A^\Pi_{q^m \partial_p^n} A^\Pi_{G} = A^\Pi_F A^\Pi_G + \cdots,
\]

where coefficients are defined by the (if making sense) development

(16.19) \[e^{i(qp' - p'q')} \Pi(q + q', p + p') = \sum_{k, l, k', l'} a_{klk'l'} q^k p^l q^{k'} p^{l'}, \quad a_{0000} = 1.\]

- Poisson bracket

From

(16.20) \[\{ F, G \} = \partial_q F \partial_p G - \partial_p F \partial_q G,\]

(16.21) \[
A^\Pi_{\{ F, G \}} = \sum_{k, l, m, n} i^{k-l+m-n} a_{klmn} \left[ A^\Pi_{q^k \partial_p^l} A^\Pi_{q^m \partial_p^n} - A^\Pi_{\partial_q^k \partial_p^l} A^\Pi_{\partial_q^m \partial_p^n} \right] = \sum_{k, l, l, m, n} a_{klmn} \times
\]

(16.22)

\[
\times \left[ (ad_Q^k \partial_p^{l+1}) (A^\Pi_{F}) (ad_Q^m \partial_p^n) (A^\Pi_{G}) - (ad_Q^k \partial_p^l) (A^\Pi_{F}) (ad_Q^m \partial_p^{n+1}) (A^\Pi_{G}) \right]
\]

(16.23) \[
= [A^\Pi_{\partial_q F \partial_p G} - A^\Pi_{\partial_p F \partial_q G}] + \cdots
\]

- Product of quantizations

(16.24) \[A^\Pi_F A^\Pi_G = A^\Pi_{F \oplus G},\]

where the “\( \Pi \)-Moyal product” is defined by

(16.25) \[(F \oplus_G G)(q, p) = \sum_{k, l, m, n} i^{k+m-l-n} \bar{a}_{klmn} (\partial_q^l \partial_p^k F) (\partial_q^m \partial_p^n G),\]
with coefficients defined by the (if making sense) expansion

\[
 e^{-\frac{i}{2}(q p' - p q')} \frac{\Pi(q,p) \Pi(q',p')}{\Pi(q + q', p + p')} = \sum_{k,l,k',l'} \tilde{a}_{kll'} q^k p'^l q'^k p^l, \quad \tilde{a}_{0000} = 1.
\]

Since coefficients \(\tilde{a}_{kll'}\) result from the inverse of \((16.19)\), we have the convolution relations with coefficients \(a_{kll'}\)

\[
 \sum_{m,n,m',n'} a_{m,n,m',n'} \tilde{a}_{k-m,l-n,k'-m',l'-n'} = 0
\]
for all \(k, l, k', l'\), such that \(k + l + k' + l' \geq 1\).

- Commutator of quantizations

\[
 [A^\Pi_F, A^\Pi_G] = A^\Pi_{F*G*G*F}
\]

\[
 = \sum_{k,l,m,n} i^{k+m-l-n} \tilde{a}_{kllm} \left[ A^\Pi_{(\partial^k q\partial^l p F)}(\partial^m q\partial^n p G) - A^\Pi_{(\partial^k q\partial^l p G)}(\partial^m q\partial^n p F) \right]
\]

\[
 = i A^\Pi_{(F,G)} + \cdots
\]

17. Interlude I: expressions for expansion coefficients, variables \(z, \bar{z}\), arbitrary \(\varnothing\)

- Supposing real analyticity for weight function \(\varnothing\), we define expansion coefficients for it by

\[
 \varnothing(z) = \sum_{l,i} c_{li} z^l \bar{z}^i, \quad c_{00} = 1, \quad c_{ii} = \frac{1}{i!} \frac{\partial^i z}{\partial z} \varnothing(z) \bigg|_{z=0},
\]

and for its inverse

\[
 \frac{1}{\varnothing(z)} = \sum_{l,i} \tilde{c}_{li} z^l \bar{z}^i, \quad \tilde{c}_{00} = 1, \quad \tilde{c}_{ii} = \frac{1}{i!} \frac{\partial^i z}{\partial z} \frac{1}{\varnothing(z)} \bigg|_{z=0},
\]

with the discrete convolution equation holding for all \(n, \tilde{n}\) such that \(n + \tilde{n} \geq 1\),

\[
 \sum_{l,i} c_{n-l,\tilde{n}-i} \tilde{c}_{li} = 0.
\]
• For instance

$${c}_{10} = -c_{10}, \quad c_{01} = -c_{01},$$

$${c}_{11} = -c_{11} + 2c_{10}c_{01},$$

$${c}_{20} = -c_{20} + c_{10}^2, \quad c_{02} = -c_{02} + c_{01}^2.$$

• There results expansion coefficients for the ratios

$${d}_{i,j} = \frac{\partial (z + z')}{\partial (z) \partial (z')} = \sum_{i,j,k,l} d_{i,j,k,l} z^i z^j z'^k z'^l, \quad d_{0000} = 1,$$

$${d}_{i,j,k,l} = \frac{1}{i!j!k!l!} \partial_z^i \partial_{z'}^j \partial_{z'}^k \partial_z^l \frac{\partial (z + z')}{\partial (z) \partial (z')} \bigg|_{z,z' = 0}.$$

$${d}_{i,j,k,l} = \sum_{i,j,k,l} \bar{d}_{i,j,k,l} z^i z^j z'^k z'^l, \quad \bar{d}_{0000} = 1,$$

$${\bar{d}}_{i,j,k,l} = \frac{1}{i!j!k!l!} \partial_z^i \partial_{z'}^j \partial_{z'}^k \partial_z^l \frac{\partial (z + z')}{\partial (z) \partial (z')} \bigg|_{z,z' = 0}.$$

with the discrete convolution equation holding for all $k, \bar{k}, l, \bar{l}$ such that $k + \bar{k} + l + \bar{l} \geq 1,$

$${d}_{i,j,k,l} = \sum_{i,j,k,l} d_{k-l,\bar{k}-\bar{l},i-j,\bar{i}-\bar{j}} \bar{d}_{i,j,k,l} = 0.$$

• Expressions of $d_{i,j,k,l}$ and $\bar{d}_{i,j,k,l}$ in terms of $c_{i,l}$ and $\bar{c}_{i,l}$

$${d}_{i,j,k,l} = \sum_{s,\bar{s},r,\bar{r}} \left( \begin{array}{c} i + j - s \\ i - r \\ \bar{i} + j - \bar{s} \\ \bar{i} - \bar{r} \end{array} \right) c_{i+j-s,i+j-\bar{s}} \bar{c}_{s-r,\bar{s}-\bar{r}} c_{r,\bar{r}},$$

$${\bar{d}}_{i,j,k,l} = \sum_{s,\bar{s},r,\bar{r}} \left( \begin{array}{c} i + j - s \\ i - r \\ \bar{i} + j - \bar{s} \\ \bar{i} - \bar{r} \end{array} \right) c_{i+j-s,i+j-\bar{s}} \bar{c}_{s-r,\bar{s}-\bar{r}} c_{r,\bar{r}}.$$

• Properties of coefficients $d_{i,j,k,l}$ and $\bar{d}_{i,j,k,l}$

$${d}_{0000} = 1, \quad d_{i,i} = 0 = d_{0,j,j} \forall i, \bar{i}, j, \bar{j}, i + \bar{i} \geq 1, j + \bar{j} \geq 1.$$

$${\bar{d}}_{0000} = 1, \quad {\bar{d}}_{i,i} = 0 = {\bar{d}}_{0,j,j} \forall i, \bar{i}, j, \bar{j}, i + \bar{i} \geq 1, j + \bar{j} \geq 1.$$
• Other particular expressions

\[ d_{00j} = \sum_{s, \bar{s}} c_{-s, j - \bar{s}} \bar{e}_{s \bar{s}}, \quad d_{0j0} = \sum_{s, \bar{s}} c_{j - s, \bar{s}} e_{s \bar{s}}. \]  

• Coefficients \( d_{i\bar{i}j} \) for \( i + \bar{i} + j + \bar{j} \leq 2 \) are given by

\[ d_{0001} = 1, \quad d_{1001} = d_{0100} = d_{0010} = d_{0001} = 0, \]

\[ d_{1001} = c_{11} - c_{10} c_{01} = d_{0110}, \]

\[ d_{1010} = 2c_{20} - c_{10}^2, \quad d_{0101} = 2c_{02} - c_{01}^2, \]

\[ d_{2000} = d_{0200} = d_{0020} = d_{0002} = 0, \]

\[ d_{1100} = d_{0011} = 0. \]

• Expressions of \( a_{i\bar{i}j} \) and \( \tilde{a}_{i\bar{i}j} \) in terms of \( d_{i\bar{i}j} \) and \( \tilde{d}_{i\bar{i}j} \) respectively

\[ a_{i\bar{i}j} = \sum_{k=0}^{\min(i, \bar{i})} \sum_{l=0}^{\min(j)} (-1)^k \frac{1}{k! l!} d_{-k, \bar{i} - l, j - l, \bar{j} - k}, \]

\[ \tilde{a}_{i\bar{i}j} = \sum_{k=0}^{\min(i, \bar{i})} \sum_{l=0}^{\min(j)} (-1)^l \frac{1}{k! l!} \tilde{d}_{-k, \bar{i} - l, j - l, \bar{j} - k}. \]

• In particular

\[ a_{00j0} = d_{00j0}, \quad a_{00j0} = \tilde{d}_{00j0}. \]

• Properties of coefficients \( a_{i\bar{i}j} \) and \( \tilde{a}_{i\bar{i}j} \)

\[ a_{0000} = 1, \quad a_{i\bar{i}00} = 0 = a_{00j} \forall i, \bar{i}, j, \bar{j}, i + \bar{i} + j + \bar{j} \geq 1, j + \bar{j} \geq 1, \]

\[ \tilde{a}_{0000} = 1, \quad \tilde{a}_{i\bar{i}00} = 0 = \tilde{a}_{00j} \forall i, \bar{i}, j, \bar{j}, i + \bar{i} + j + \bar{j} \geq 1, j + \bar{j} \geq 1. \]

• Coefficients \( a_{i\bar{i}j} \) for \( i + \bar{i} + j + \bar{j} \leq 2 \) are given by

\[ a_{0000} = 1, \quad a_{1000} = a_{0100} = a_{0010} = a_{0001} = 0, \]

\[ a_{1001} = c_{11} - c_{10} c_{01} - \frac{1}{2}, \quad a_{0110} = c_{11} - c_{10} c_{01} + \frac{1}{2}, \]

\[ a_{1010} = 2c_{20} - c_{10}^2, \quad a_{0101} = 2c_{02} - c_{01}^2, \]

\[ a_{2000} = a_{0200} = a_{0020} = a_{0002} = 0. \]
• Coefficients $\tilde{a}_{i\bar{j}j}$ for $i + \bar{i} + j + \bar{j} \leq 2$ are given by

\begin{equation}
\tilde{a}_{0000} = 1, \quad \tilde{a}_{1000} = \tilde{a}_{0100} = \tilde{a}_{0010} = \tilde{a}_{0001} = 0,
\end{equation}

\begin{equation}
\tilde{a}_{1001} = -c_{11} + c_{10}c_{01} + \frac{1}{2} = -a_{1001}, \quad \tilde{a}_{0110} = -c_{11} + c_{10}c_{01} - \frac{1}{2} = -a_{0110},
\end{equation}

\begin{equation}
\tilde{a}_{1010} = -2c_{20} = -a_{1010} - c_{10}^2, \quad \tilde{a}_{0101} = -2c_{02} = -a_{0101} - c_{01}^2.
\end{equation}

\begin{equation}
\tilde{a}_{2000} = \tilde{a}_{0200} = \tilde{a}_{0020} = \tilde{a}_{0002} = 0.
\end{equation}

18. Interlude II: Expressions for Expansion Coefficients, Variables $q, p$, Arbitrary II

• Supposing analyticity for weight function $\Pi$, we define expansion coefficients for it by

\begin{equation}
\Pi(q, p) = \sum_{k,l} \bar{c}_{kl} q^k p^l, \quad \bar{c}_{00} = 1, \quad \bar{c}_{kl} = \frac{1}{k!l!} \partial_q^k \partial_p^l \Pi(q, p) \bigg|_{q=0=p},
\end{equation}

and for its inverse

\begin{equation}
\frac{1}{\Pi(q, p)} = \sum_{k,l} \bar{c}_{kl} q^k p^l, \quad \bar{c}_{00} = 1, \quad \bar{c}_{kl} = \frac{1}{k!l!} \partial_q^k \partial_p^l \frac{1}{\Pi(q, p)} \bigg|_{q=0=p},
\end{equation}

with the discrete convolution equation holding for all $m, n$ such that $m + n \geq 1$,

\begin{equation}
\sum_{k,l} \bar{c}_{m-k,n-l} \bar{c}_{kl} = 0.
\end{equation}

• For instance

\begin{equation}
\bar{c}_{10} = -c_{10}, \quad \bar{c}_{01} = -c_{01},
\end{equation}

\begin{equation}
\bar{c}_{11} = -c_{11} + 2c_{10}c_{01},
\end{equation}

\begin{equation}
\bar{c}_{20} = -c_{20} + c_{10}^2, \quad \bar{c}_{02} = -c_{02} + c_{01}^2.
\end{equation}
There results expansion coefficients for the ratios

\[
\frac{\Pi(q + q', p + p')}{\Pi(q, p) \Pi(q', p')} = \sum_{k, l, k', l'} d_{k,l,k',l'} q^k p^l q^{k'} p^{l'}, \quad d_{0000} = 1,
\]

\[
d_{k,l,k',l'} = \frac{1}{k! l! k'! l'!} \partial_q^k \partial_p^l \partial_q^{k'} \partial_p^{l'} \left| \frac{\Pi(q + q', p + p')}{\Pi(q, p) \Pi(q', p')} \right|_{q,q',p,p'=0}.
\]

\[
\frac{\Pi(q, p) \Pi(q', p')}{\Pi(q + q', p + p')} = \sum_{k, l, k', l'} \tilde{d}_{k,l,k',l'} q^k p^l q^{k'} p^{l'}, \quad \tilde{d}_{0000} = 1,
\]

\[
\tilde{d}_{k,l,k',l'} = \frac{1}{k! l! k'! l'!} \partial_q^k \partial_p^l \partial_q^{k'} \partial_p^{l'} \left| \frac{\Pi(q, p) \Pi(q', p')}{\Pi(q + q', p + p')} \right|_{q,q',p,p'=0}.
\]

with the discrete convolution equation holding for all \( k, l, k', l' \) such that \( k + l + k' + l' \geq 1 \).

\[
\sum_{m,n,m',n'} d_{k-m,l-i,l-n,k'-m',l'-n'} \tilde{d}_{mm'n'} = 0.
\]

Expressions of \( d_{k,l,k',l'} \) and \( \tilde{d}_{k,l,k',l'} \) in terms of \( c_{kl} \) and \( \bar{c}_{kl} \)

\[
d_{k,l,k',l'} = \sum_{r,s,m,n} \binom{r}{m} \binom{s}{n} c_{rs} \bar{c}_{k-r+m,l-s+n,k'-m',l'-n},
\]

\[
\tilde{d}_{k,l,k',l'} = \sum_{r,s,m,n} \binom{r}{m} \binom{s}{n} \bar{c}_{rs} c_{k-r+m,l-s+n,k'-m',l'-n}.
\]

Properties of coefficients \( d_{k,l,k',l'} \) and \( \tilde{d}_{k,l,k',l'} \)

\[
d_{0000} = 1, \quad d_{k,00} = 0 = d_{00,k'} \forall k, l, k', l', k + l \geq 1, k' + l' \geq 1.
\]

\[
\tilde{d}_{0000} = 1, \quad \tilde{d}_{k,00} = 0 = \tilde{d}_{00,k'} \forall k, l, k', l', k + l \geq 1, k' + l' \geq 1.
\]

Other particular expressions

\[
d_{k,00} = \sum_{r,s} c_{rs} \bar{c}_{k-r,0} \bar{c}_{0,l-s}, \quad d_{00,l} = \sum_{r,s} c_{rs} \bar{c}_{0,l-s} \bar{c}_{k'-r,0}.
\]

\[
d_{00,l'} = \sum_{r,m} \binom{r}{m} c_{r0} \bar{c}_{k-r+m,0} \bar{c}_{k'-m,0}, \quad d_{00,l'} = \sum_{s,n} \binom{s}{n} c_{0s} \bar{c}_{0,l-s+n} \bar{c}_{0,l'-n}.
\]
Coefficients $d_{k\ell k'}$ for $k + l + k' + l' \leq 2$ are given by

\begin{align}
(18.18) & \quad d_{0000} = 1, \quad d_{1000} = d_{0100} = d_{0010} = d_{0001} = 0, \\
(18.19) & \quad d_{1001} = c_{11} - c_{10} c_{01} = d_{0110}, \\
(18.20) & \quad d_{1010} = 2c_{20} - c_{10}^2, \quad d_{0101} = 2c_{02} - c_{01}^2, \\
(18.21) & \quad d_{2000} = d_{0200} = d_{0020} = d_{0002} = 0, \\
(18.22) & \quad d_{1100} = d_{0011} = 0.
\end{align}

Expressions of $a_{k\ell k'}$ and $\tilde{a}_{k\ell k'}$ in terms of $d_{k\ell k'}$ and $\tilde{d}_{k\ell k'}$ respectively

\begin{align}
(18.23) & \quad a_{k\ell k'} = \sum_{m=0}^{\min(k,l')} \sum_{n=0}^{\min(l,k')} \frac{i^{n-m}}{m! n!} \frac{1}{2^{m+n}} d_{k-m,l-n,k'-n,l'-m}, \\
(18.24) & \quad \tilde{a}_{k\ell k'} = \sum_{m=0}^{\min(k,l')} \sum_{n=0}^{\min(l,k')} \frac{i^{n-m}}{m! n!} \frac{1}{2^{m+n}} \tilde{d}_{k-m,l-n,k'-n,l'-m}.
\end{align}

In particular

\begin{align}
(18.25) & \quad a_{k0k0} = d_{k0k0}, \quad a_{010l} = d_{010l}.
\end{align}

Properties of coefficients $a_{k\ell k'}$ and $\tilde{a}_{k\ell k'}$

\begin{align}
(18.26) & \quad a_{0000} = 1, \quad a_{k000} = 0 = a_{00k'} \forall k, l, k', l', k + l \geq 1, k' + l' \geq 1.
\end{align}

\begin{align}
(18.27) & \quad \tilde{a}_{0000} = 1, \quad \tilde{a}_{k000} = 0 = \tilde{a}_{00k'} \forall k, l, k', l', k + l \geq 1, k' + l' \geq 1.
\end{align}

Coefficients $a_{k\ell k'}$ for $k + l + k' + l' \leq 2$ are given by

\begin{align}
(18.28) & \quad a_{0000} = 1, \quad a_{1000} = a_{0100} = a_{0010} = a_{0001} = 0, \\
(18.29) & \quad a_{1001} = c_{11} - c_{10} c_{01} + \frac{i}{2}, \quad a_{0101} = c_{11} - c_{10} c_{01} - \frac{i}{2}, \\
(18.30) & \quad a_{1010} = 2c_{20} - c_{10}^2, \quad a_{0101} = 2c_{02} - c_{01}^2, \\
(18.31) & \quad a_{2000} = a_{0200} = a_{0020} = a_{0002} = 0.
\end{align}
19. Interlude III: quantization results with Cahill-Glauber weight

- Cahill-Glauber weight: \( \omega(z) = e^{\frac{z}{2}|z|^2} \)
- Coefficients are

\[
(19.1) \quad a_{ijij} = \frac{1}{i!j!} \delta_{ij} \delta_{ij} \left( \frac{s-1}{2} \right)^i \left( \frac{s+1}{2} \right)^j.
\]

\[
(19.2) \quad \tilde{a}_{ijij} = (-1)^{i+j} a_{ijij}.
\]

- Quantization of Poisson bracket

\[
(19.3) \quad A_{ij}^\omega = \frac{i}{\hbar} \sum_{l,j} \frac{(-1)^{l+j}}{l!j!} \left( \frac{s-1}{2} \right)^i \left( \frac{s+1}{2} \right)^j \left[ A_{ij}^{\omega l} \delta_{ij} f + A_{ij}^{\omega l} \delta_{ij} g - A_{i+1,j}^{\omega l} \delta_{ij} f - A_{i+1,j}^{\omega l} \delta_{ij} g \right]
\]

\[
= \frac{i}{\hbar} \sum_{l,j} \frac{1}{l!j!} \left( \frac{s-1}{2} \right)^i \left( \frac{s+1}{2} \right)^j \times
\]

\[
\times \left[ (\text{ad}_{a^i} \text{ad}_{a^j}) (A_f^\omega) (\text{ad}_{a^i} \text{ad}_{a^j}) (A_g^\omega) - (\text{ad}_{a^i} \text{ad}_{a^j}) (A_f^\omega) (\text{ad}_{a^i} \text{ad}_{a^j}) (A_g^\omega) \right]
\]

- \( \omega \)-Moyal product

\[
(19.4) \quad (f \star_{\omega} g)(z) = \sum_{l,j} \frac{i}{l!j!} \left( \frac{s-1}{2} \right)^i \left( \frac{s+1}{2} \right)^j \left( \partial_{\hat{z}} \partial_{\hat{z}} f \right) \left( \partial_{\hat{z}} \partial_{\hat{z}} g \right),
\]

- \( \omega \)-Moyal commutator

\[
(19.5) \quad (f \star_{\omega} g)(z) - (g \star_{\omega} f)(z) = \sum_{l,j} \frac{1}{l!j!} \left( \frac{s-1}{2} \right)^i \left( \frac{s+1}{2} \right)^j \times
\]

\[
\times \left[ \left( \partial_{\hat{z}} \partial_{\hat{z}} f \right) \left( \partial_{\hat{z}} \partial_{\hat{z}} g \right) - \left( \partial_{\hat{z}} \partial_{\hat{z}} f \right) \left( \partial_{\hat{z}} \partial_{\hat{z}} g \right) \right]
\]

\[
= \sum_{l,j} \frac{\text{sgn}(j-i)}{2^{l+j} l! j!} \left( s^2 - 1 \right)^{\min(l,j)} \left( s+1 \right)^{|j-i|} \left( s-1 \right)^{|i-j|} \times
\]

\[
\times \left( \partial_{\hat{z}} \partial_{\hat{z}} f \right) \left( \partial_{\hat{z}} \partial_{\hat{z}} g \right) ,
\]

20. Interlude IV: quantization results with Born-Jordan weight

- Born-Jordan weight:
\[
\Pi(q, p) = \frac{\sin qp}{qp} = \text{sinc}(qp), \quad \frac{1}{\Pi(q, p)} = qp \csc(qp).
\]

- Coefficients \(c\) are

\[
C_{kl} = \begin{cases} 
\delta_{kl} \frac{(-1)^r}{(2r+1)!} & \text{if } k = 2r \\
0 & \text{otherwise}
\end{cases}
\]

- Coefficients \(\tilde{c}\) are

\[
\tilde{C}_{kl} = \begin{cases} 
\delta_{kl} \frac{(-1)^{r+1} 2 (2^{2r-1} - 1)}{(2r)!} B_{2r} & \text{if } k = 2r \\
0 & \text{otherwise}
\end{cases}
\]

where \(B_n\) is a Bell number,

\[
B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k, \quad B_0 = 1.
\]

- Coefficients \(d\) are

\[
d_{kk'} = \delta_{k+k'+l+l'} \sum_{u,v} \binom{2u}{k' - 2 v} \binom{2u}{kl' - 2 v} C_{2u,2u} \tilde{C}_{2u,2u} \tilde{C}_{k+k'-2(u+v),k+k'-2(u+v)},
\]

if \(k + k' = l + l'\) is even, otherwise is zero.

- Etc.

21. **Permanent issues of Weyl–Heisenberg integral quantizations with arbitrary weights**

21.1. **Quantization of linear and quadratic expressions in \(z\) and \(\bar{z}\) (or in \(q\) and \(p\)).**

- CCR is a permanent outcome of the above quantization, whatever the chosen complex function \(\varphi(z)\), such that \(\varphi(0) = 1\), provided integrability and derivability at the origin is ensured.
• Quantization of canonical variables

\[(21.1)\]
\[A_q^O = a - \partial_z \varphi|_{z=0} = a - c_{01}\]

\[(21.2)\]
\[A_p^O = a^\dagger + \partial_z \varphi|_{z=0} = a^\dagger + c_{10},\]

\[(21.3)\]
\[A_q^O = \frac{1}{\sqrt{2}} \left[ (a + a^\dagger) - \partial_z \varphi|_{z=0} + \partial_z \varphi|_{z=0} \right] = Q - \frac{1}{\sqrt{2}} [c_{01} - c_{10}],\]

\[(21.4)\]
\[A_p^O = \frac{1}{\sqrt{2i}} \left[ (a - a^\dagger) - \partial_z \varphi|_{z=0} - \partial_z \varphi|_{z=0} \right] = P - \frac{1}{\sqrt{2i}} [c_{01} + c_{10}],\]

• Hence the ccr,

\[(21.5)\]
\[A_q^O A_p^O - A_p^O A_q^O = [Q, P] = i [a, a^\dagger] = iI.\]

• Quadratic terms

\[(21.6)\]
\[A_{z^2}^O = a^2 - 2a \partial_z \varphi|_{z=0} + \partial_z^2 \varphi|_{z=0} = a^2 - 2c_{01} a + 2c_{02},\]

\[(21.7)\]
\[A_{z^2} = (a^\dagger)^2 + 2a^\dagger \partial_z \varphi|_{z=0} + \partial_z^2 \varphi|_{z=0} = (a^\dagger)^2 + 2c_{10} a^\dagger + 2c_{20},\]

\[(21.8)\]
\[A_{|z|^2}^O = a^\dagger a + c_{10} a - c_{01} a^\dagger - c_{11} + \frac{1}{2} = a a^\dagger + c_{10} a - c_{01} a^\dagger - c_{11} - \frac{1}{2},\]

\[(21.9)\]
\[A_{q^2}^O = Q^2 + \sqrt{2} (c_{10} - c_{01}) Q + c_{20} + c_{02} - c_{11},\]

\[(21.10)\]
\[A_{p^2} = P^2 + \sqrt{2} i ((c_{10} + c_{01}) P - c_{20} - c_{02} - c_{11})\]

\[(21.11)\]
\[A_{q^2} = A_q^O A_q^O - \frac{i}{2} - i(c_{02} - c_{20}) + \frac{i}{2} (c_{01} - c_{10}^2)\]

\[(21.12)\]
\[= Q P + \frac{i}{\sqrt{2}} (c_{01} + c_{10}) Q - \frac{1}{\sqrt{2}} (c_{01} - c_{10}) P - i(c_{02} - c_{20}) - \frac{i}{2}\]

\[(21.13)\]
\[= A_q^O A_q^O + \text{(constant} \in \mathbb{C}).\]

where the constant can take any value we wish depending on our choice of \(\varphi\) (think to the so-called \(kp\)-quantization!)

• For more general formulae with variables \(q, p\), and weight \(\Pi(q, p)\), see Subsection 21.3 below
21.2. Quantization of arbitrary monomials in $z$ and $\bar{z}$.

- Recurrence formula for $A_{zn}^\omega$

\[
A_{zn}^\omega = (a - c_{01}) A_{zn-1}^\omega - \sum_{j=1}^{n-1} (-1)^j a_{010j} \frac{(n-1)!}{(n-1-j)!} A_{zn-1-j}^\omega ,
\]

with for $j \geq 1$

\[
a_{010j} = d_{010j} = \sum_{k=0}^{j} (j + 1 - \bar{k}) c_{0,j+1-k} \bar{c}_{0,k} .
\]

- Recurrence formula for $A_{\bar{z}n}^\omega$

\[
A_{\bar{z}n}^\omega = (a^\dagger + c_{10}) A_{\bar{z}n-1}^\omega + \sum_{j=1}^{n-1} (-1)^j a_{10j0} \frac{(n-1)!}{(n-1-j)!} A_{\bar{z}n-1-j}^\omega ,
\]

with for $j \geq 1$

\[
a_{10j0} = d_{10j0} = \sum_{k=0}^{j} (j + 1 - k) c_{j+1-k,0} \bar{c}_{k,0} .
\]

- Separation formula for $A_{zn \bar{z}n}^\omega$

\[
A_{zn \bar{z}n}^\omega = A_{zn}^\omega A_{\bar{z}n}^\omega + \sum_{l+j \geq 1} (-1)^j a_{0lj0} \frac{n! \bar{n}!}{(n-l)! (\bar{n}-j)!} A_{zn-l}^\omega A_{\bar{z}n-j}^\omega ,
\]

with

\[
a_{0lj0} = \sum_{l=0}^{\min(l,j)} \frac{1}{l!} \frac{1}{2^l} d_{0,l-l,j-l,0} .
\]

- It is easily inferred from these formula that
  - $A_{zn}^\omega$ is polynomial of degree $n$ in lowering operator $a$

\[
A_{zn}^\omega = \sum_{m=0}^{n} a_m^\omega a^m .
\]

  - $A_{zn}^\omega$ is polynomial of degree $n$ in raising operator $a^\dagger$

\[
A_{zn}^\omega = \sum_{m=0}^{n} \beta^\omega_m (a^\dagger)^m .
\]

  - $A_{zn \bar{z}n}^\omega$ is polynomial in $a$ and $a^\dagger$, separately of degree $n$ in $a$ and of degree $\bar{n}$ in $a^\dagger$
21.3. **Quantization of arbitrary monomials in** $q$ **and** $p$ **with weight** $\Pi(q,p)$.

- Quantizations of $q$ and $p$

\begin{equation}
A_q^\Pi = Q - iC_{01}, \quad A_p^\Pi = P + iC_{10}.
\end{equation}

- Recurrence formula for $A_q^\Pi$

\begin{equation}
A_q^\Pi n = (Q - iC_{01}) A_q^\Pi (n-1) + \sum_{l'=1}^{n-1} (-i)^{l'+1} a_{010}^{l+1} \frac{(n-1)!}{(n-1-l')!} A_q^\Pi (n-1-l'),
\end{equation}

with for $l' \geq 1$

\begin{equation}
a_{010}^{l+1} = d_{010}^{l+1} = \sum_{s=1}^{l+1} s C_{0s} \bar{C}_{0,l+1-s}.
\end{equation}

- Recurrence formula for $A_p^\Pi$

\begin{equation}
A_p^\Pi n = (P + iC_{10}) A_p^\Pi (n-1) + \sum_{k'=1}^{n-1} (i)^{k'+1} a_{100}^{k+1} \frac{(n-1)!}{(n-1-k')!} A_p^\Pi (n-1-k'),
\end{equation}

with for $k' \geq 1$

\begin{equation}
a_{100}^{k+1} = d_{100}^{k+1} = \sum_{r=1}^{k+1} r C_{r0} \bar{C}_{k+1-r,0}.
\end{equation}

- Separation formula for $A_{q^m p^n}$

\begin{equation}
A_{q^m p^n} = A_{q^m}^\Pi A_{p^n}^\Pi + \sum_{l+k' \geq 1} i^{k' - l} a_{0k'l0} \frac{m! n!}{(m-l)(n-k')!} A_{q^{m-l}}^\Pi A_{p^{n-k'}}^\Pi,
\end{equation}

with

\begin{equation}
a_{0k'l0} = \sum_{s=0}^{\min(l,k')} (-i)^s \frac{1}{s!} \frac{1}{2s} d_{0,l-s,k'-s,0}.
\end{equation}

- It is easily inferred from these formula that

  - $A_{q^n}$ is polynomial of degree $n$ in position operator $Q$

\begin{equation}
A_{q^n}^\Pi = \sum_{m=0}^{n} u_m^\Pi Q^m.
\end{equation}

  - $A_{p^n}$ is polynomial of degree $n$ in momentum operator $P$

\begin{equation}
A_{p^n}^\Pi = \sum_{m=0}^{n} v_m^\Pi P^m.
\end{equation}
22. Quantizations with particular weight functions
(superscript \(\omega\) is omitted)

- **Regular quantizations**
  The weight function \(\omega\) is even and real, \(\omega(-z) = \omega(z), \overline{\omega(z)} = \omega(z)\)

\[
\begin{align*}
A_z &= a, \\
A_{f(z)} &= A_{f(z)}^*.
\end{align*}
\]

- **Elliptic regular quantizations**
  The weight function \(\omega\) is isotropic \(\omega(z) \equiv w(|z|^2) \text{ with } w: \mathbb{R} \to \mathbb{R}\)

Example: Cahill-Glauber choice

\[
\omega_s(z) = e^{s|z|^2/2}, \quad \text{Re } s < 1.
\]

From

\[
\int_0^\infty e^{-\nu x} x^\lambda L_n(x) dx = \frac{\Gamma(\lambda + 1)\Gamma(\alpha + n + 1)}{n!\Gamma(\alpha + 1)} \nu^{-\lambda-1} \frac{1}{2} F_1(-n, \lambda + 1; \alpha + 1; \nu^{-1}),
\]
we get the diagonal elements of \(M_s\):

\[
\langle e_n | M_s | e_n \rangle = \frac{2}{1 - s} \left( \frac{s + 1}{s - 1} \right)^n,
\]

and so

\[
M_s = \int_C \omega_s(z) D(z) \frac{d^2z}{\pi} = \frac{2}{1 - s} \exp \left( \frac{1}{\log s} \frac{s + 1}{s - 1} \right) a^\dagger a.
\]

\(s = -1\) corresponds to the CS (anti-normal) quantization, since

\[
M_{-1} = \lim_{s \to -1+} \frac{2}{1 - s} \exp \left( \frac{1}{\log s} \frac{s + 1}{s - 1} \right) a^\dagger a = |e_0\rangle \langle e_0|,
\]

and so

\[
A_f = \int_C D(z)M_{-1} D(z)\frac{d^2z}{\pi} = \int_C |z\rangle \langle z| f(z) \frac{d^2z}{\pi}.
\]
\( s = 0 \) corresponds to the Weyl-Wigner quantization since, from Eq. (22.5), \( M_0 = 2 \mathcal{P} \), and so
\[
(22.7) \quad A_f = \int_C D(z) \, 2 \mathcal{P} \, D(z)^\dagger \, f(z) \, \frac{d^2z}{\pi}.
\]
\( s = 1 \) is the normal quantization in an asymptotic sense.

The operator \( M_s \) is positive unit trace class for \( s \leq -1 \) (it is just trace class if \( \text{Re} \, s < 0 \))

• Hyperbolic regular quantizations

The weight function \( \wp \) verifies \( \wp(z) \equiv m(\text{Im}(z^2)) \) with \( m : \mathbb{R} \mapsto \mathbb{R} \) (i.e. yields a regular quantization), like the Born-Jordan weight.

It is easily inferred from (14.33) and (14.35) that this case preserves the classical functions
\[
(22.8) \quad A_{f(q)} = f(Q), \quad A_{f(p)} = f(P).
\]

Because \( \text{Im} \, (z^2) \equiv qp \), we only need the Planck constant \( \hbar \) and a mathematical function \( m \) to build the physical expression \( m(\text{Im}(z^2))/\hbar \)

• The common elliptic regular and hyperbolic regular cases must verify \( \wp(z) = w(|z|^2) = m(\text{Im}(z^2)) \) with \( \wp(0) = 1 \). The unique solution is \( \wp(z) = 1 \) corresponds to the Weyl-Wigner quantization

• Isometric quantizations

If we have
\[
(22.9) \quad \text{tr}(A^\dagger_f A_f) = \int_C |f(z)|^2 \frac{d^2z}{\pi}.
\]

\( f \mapsto A_f \) is then invertible (the inverse is given by a trace formula)

From (12.2) we have the trace formula
\[
(22.10) \quad \text{tr}(A^\dagger_f A_f) = \int_C \frac{d^2z}{\pi} |\wp(z)|^2 |\wp_\ell(z)|^2.
\]

From the invariance of the \( L^2 \)-norm under symplectic transform, we find that \( f \mapsto A_f \) is isometric if and only if \( |\wp(z)| = 1 \) for all \( z \)
• Elliptic regular quantizations that are isometric
  
  It is the case $\omega(z) = w(|z|^2) \in \{-1, +1\}$. Simple example

(22.11) $\omega_\alpha(z) = 2\theta(1 - \alpha|z|^2) - 1$

where $\theta$ is the Heaviside function. The Weyl-Wigner quantization is a special case ($\alpha = 0$)

• Hyperbolic regular quantizations that are isometric
  
  It is the case $m(u) \in \{-1, +1\}$.

Simple example

(22.12) $\omega_\alpha(z) = 2\theta(1 - \alpha \text{Im}(z^2)) - 1$.

The Weyl-Wigner quantization is a special case ($\alpha = 0$)

### 23. Separable quantizations

• It is the case $\Pi(q, p) = \lambda(q) \mu(p)$ and $F(q, p) = L(q) M(p)$. From (14.17)

(23.1) $A_{\Pi(q)M(p)} = \int_{\mathbb{R}^2} \frac{dq}{2\pi} \frac{dp}{2\pi} e^{-i \frac{qp}{2}} e^{i pQ} e^{-i qP} \lambda(q) \mu(p) \mathfrak{F}[L](p) \mathfrak{F}[M](-q)$.

• Useful case $M(p) = p^m$ whereas $L(q)$ is a function of class $C^m$ (actually all this may be extended to distributions with appropriate weight functions)

(23.2) $A_{L(q)M(p)} = \sum_{r+s+t=m} 2^{-s} \binom{m}{r,s,t} \lambda(r)(0) (-i)^s \frac{1}{\sqrt{2\pi}} \mathfrak{F}[\mu] * L^{(s)}(Q) P^t$.

where “*” is the convolution product

(23.3) $f * g(x) = \int_{-\infty}^{+\infty} f(x - y) g(y) \, dy = g * f(x)$,

and $\mathfrak{F}$ is the inverse Fourier transform (see (23.10) and (23.11))

• Weyl-Wigner case ($\lambda(q) = 1 = \mu(p)$) for $M(p) = p^m$ and $L(q)$ is a function of class $C^m$

(23.4) $A_{L(q)M(p)} = \sum_t 2^{t-m} \binom{m}{t} (-i)^{m-t} L^{(m-t)}(Q) P^t$.

and this corresponds to a certain type of quantization (“Weyl calculus”)
• Lowest $m$'s

\begin{equation}
A_{L(q)}^{W-W} = L(Q),
\end{equation}

\begin{equation}
A_{L(q) P}^{W-W} = \frac{1}{2} (-i) L'(Q) + L(Q) P = \frac{1}{2} [L(Q) P + P L(Q)],
\end{equation}

\begin{equation}
A_{L(q) P^2}^{W-W} = -\frac{1}{4} L'''(Q) - i L'(Q) P + L(Q) P^2
\end{equation}

\begin{equation}
= \frac{1}{12} L'''(Q) + \frac{1}{3} \left( P^2 L(Q) + P L(Q) P + L(Q) P^2 \right) = \frac{1}{12} L'''(Q) + \text{Sym}_W \left( L(Q) P^2 \right)
\end{equation}

\begin{equation}
A_{L(q) P^3}^{W-W} = \frac{-i}{8} L''' + \frac{7}{4} L'' P + \text{Sym}_W \left( L P^3 \right),
\end{equation}

etc,

where $\text{Sym}_W$ stands for the so-called Weyl ordering. The latter is defined for monomials with powers of two operators $A$ and $B$ as

\begin{equation}
\text{Sym}_W (A^m B^n) = \frac{1}{\binom{m+n}{m}} \sum_{C_1 \cdots C_{m+n}} C_1 \cdots C_{m+n}
\end{equation}

where the $C_1 \cdots C_{m+n}$ ranges over all uples which contain $m$ copies of $A$ and $n$ copies of $B$.
• Ingredients of the proof of (23.1)-(23.4) from basic Fourier analysis and distribution theory

\[(23.10)\]
\[\hat{f}[f](k) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx, \quad \hat{g}[f](k) := \hat{g}[f](-k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ikx} \, dx,\]

\[(23.11)\]
\[\hat{\delta} \hat{\delta} = \hat{\delta} \hat{\delta} = I\]

\[(23.12)\]
\[\langle \delta^{(m)}, \varphi \rangle \equiv \int_{-\infty}^{+\infty} (-1)^m \delta^{(m)}(x) \varphi(x) \, dx = \varphi^{(m)}(0),\]

\[(23.13)\]
\[\hat{\delta} [x^m] = \sqrt{2\pi} i^m \delta^{(m)}\]

\[(23.14)\]
\[\hat{\delta}[f * g](k) = \sqrt{2\pi} \hat{\delta}[f](k) \hat{\delta}[g](k), \quad \hat{\delta}[F(k) G(k)] = \frac{1}{\sqrt{2\pi}} \hat{\delta}[F] * \hat{\delta}[G].\]

24. Usefull commutation relations

• Restoring the presence of \(\hbar\) and starting from \([Q, P] = i\hbar I,\)

• Commutator of \(P\) with powers of \(Q\)

\[(24.1)\]
\[[P, Q^m] = -m\hbar Q^{m-1}\]

• Commutator of \(P^2\) with powers of \(Q\)

\[(24.2)\]
\[[P^2, Q^m] = (-i\hbar)^2 m(m - 1)Q^{m-2} - 2m\hbar Q^{m-1}P\]

• Commutator of powers of \(P\) with \(Q\)

\[(24.3)\]
\[[P^n, Q] = -n\hbar P^{n-1}\]

• Commutator of powers of \(P\) with \(Q^2\)

\[(24.4)\]
\[[P^n, Q^2] = (-i\hbar)^2 n(n - 1)P^{n-2} - 2n\hbar Q P^{n-1}\]

• Commutator of \(P^m\) with \(U(Q), U\) of class \(C^m\) (at least)

\[(24.5)\]
\[[P, U(Q)] = -i\hbar U'(Q),\]
If we consider powers of $P$ and $Q$ we have:

(i) For $n \leq m$

\[
[P^n, U(Q)] = \sum_{k=0}^{m-1} (-i\hbar)^{m-k} U^{(m-k)}(Q) P^k,
\]

where $A_{m,n} = m!/n!$ is the arrangement of $m$ and $n$

(ii) For $n \geq m$

\[
[P^n, Q^m] = \sum_{k=0}^{m-1} (-i\hbar)^{m-k} A_{m,n-k} Q^{m-n+k} P^k,
\]

25. **Gaussian separable quantizations**

- It is the case when the weight is

\[
\varphi(z) = \Pi(q, p) = \lambda(p) \mu(p) = e^{-\frac{q^2}{2\sigma_r^2} - \frac{p^2}{2\sigma_\ell^2}}.
\]

- This case includes all Cahill-Glauber weights $\varphi(z) = e^{\frac{z^2}{2\sigma^2}}$, and in particular the “limit” Weyl-Wigner as both the widths $\sigma_r$ and $\sigma_\ell$ are infinite (Weyl-Wigner is singular in this respect!)

- Specifying (23.2) and taking into account

\[
\lambda^{(r)}(0) = (-1)^r (2)^{-r/2} (\sigma_r)^{-r} H_r(0) = \frac{1}{\sqrt{\pi}} \frac{(-2)^u}{\sigma_r^2} \Gamma \left( u + \frac{1}{2} \right)
\]

\[
= 0 \quad \text{if} \quad r = 2u + 1
\]

and

\[
\overline{\delta} [\mu](x) = \sigma_\ell e^{-\sigma_\ell^2 x^2},
\]
we have
\begin{equation}
A_{t}^{G.o.u35} = \sum_{u,s,t} 2^{-u-s} \left( \frac{m}{2u} \right) \sigma_{t}^{-2u} \frac{\Gamma(u+1/2)}{\sqrt{\pi}} (-i)^{s} \mathbf{6}_{1/\sigma_{0}} \left[ L^{(s)} \right] (Q) P^{t}
\end{equation}
\begin{equation}
= \sum_{u,s,t=0} 2^{-u-s} \left( \frac{m}{2u} \right) \sigma_{t}^{-2u} \frac{\Gamma(u+1/2)}{\sqrt{\pi}} (-i)^{s} \frac{d^{s}}{dQ^{s}} \mathbf{6}_{1/\sigma_{0}} \left[ L \right] (Q) P^{t},
\end{equation}
where \( \mathbf{6}_{\sigma}[F] \) stands for the Gaussian convolution of \( F \),
\begin{equation}
\mathbf{6}_{\sigma}[F](x) := \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{\infty} dy e^{-\frac{(x-y)^{2}}{2\sigma^{2}}} F(y).
\end{equation}

**Particular cases of Gaussian separable quantizations.**

- Kinetic energy with “variable mass”: \( m = m(q) = 1/(2L(q)) \)
\begin{equation}
A_{t}^{G.o.u35} = \mathbf{6}_{1/\sigma_{0}} \left[ L \right] (Q) P^{2} - i \mathbf{6}_{1/\sigma_{0}} \left[ L' \right] (Q) P + \mathbf{6}_{1/\sigma_{0}} \left[ \frac{L}{\sigma_{0}^{2}} - \frac{L''}{4} \right] (Q)
\end{equation}
\begin{equation}
= P \mathbf{6}_{1/\sigma_{0}} \left[ L \right] (Q) P + \mathbf{6}_{1/\sigma_{0}} \left[ \frac{L}{\sigma_{0}^{2}} - \frac{L''}{4} \right] (Q) .
\end{equation}
- \( L(q) = 1, M(p) = p^{m} \)
\begin{equation}
A_{p}^{G.o.u35} = P
\end{equation}
\begin{equation}
A_{p^{2}}^{G.o.u35} = P^{2} + \frac{1}{\sigma_{t}^{2}} .
\end{equation}
where \( H_{m}(x) \) is a Hermite polynomial
- \( L(q), M(p) = 1 \)
\begin{equation}
A_{t}^{G.o.u35} = \mathbf{6}_{1/\sigma_{0}} \left[ L \right] (Q) = \frac{\sigma_{0}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy e^{-\frac{y^{2}}{2\sigma_{0}^{2}}} L(Q - y),
\end{equation}
\begin{equation}
A_{q}^{G.o.u35} = \sum_{u=0} \frac{m!}{2u \sigma_{0}^{2u} u!(m-2u)!} Q^{m-2u} = 2^{-m/2} \sigma_{0}^{-m} (-i)^{m} H_{m} \left( \frac{i\sigma_{0} Q}{\sqrt{2}} \right) ,
\end{equation}
\begin{equation}
A_{p}^{G.o.u35} = P
\end{equation}
\begin{equation}
A_{p^{2}}^{G.o.u35} = P^{2} + \frac{1}{\sigma_{t}^{2}} .
\end{equation}
(25.13) \[ A_{\psi}^{Q.e.u.55} = Q. \]
(25.14) \[ A_{\psi}^{Q.e.u.55} = Q^2 + \frac{1}{\sigma_0^2}. \]

- Harmonic oscillator

(25.15) \[ A_{\psi}^{Q.e.u.55} = \frac{P^2 + Q^2}{2} + \frac{1}{\frac{1}{\sigma_\ell^2} + \frac{1}{\sigma_0^2}} = N + \frac{1}{2} \left( \frac{1}{\sigma_\ell^2} + \frac{1}{\sigma_0^2} + 1 \right). \]

- Comparing with the CS quantization (Cahill-Glauber \( s = -1 \)) of the harmonic oscillator (26.5) below \( (\sigma_\ell = \sigma_0 = \sqrt{2}) \), we infer that the CS case ("zero temperature") and separable Gaussian case yield same HO eigenenergies when

(25.16) \[ \frac{1}{\sigma_\ell^2} + \frac{1}{\sigma_0^2} = 1, \]

- and more generally the "s" Cahill-Glauber quantization \( (\sigma_\ell = \sigma_0 = \sqrt{2-s}) \) and separable Gaussian quantizations yield same HO eigenenergies when

(25.17) \[ \frac{1}{\sigma_\ell^2} + \frac{1}{\sigma_0^2} = -s. \]

26. Quantum harmonic oscillator according to regular \( \omega \)

Given a general weight function \( \omega \), we saw in the above that the quantization of the classical harmonic oscillator energy \( |z|^2 = (p^2 + q^2)/2 \) yields the operator

(26.1) \[ A_{|z|^2}^{\omega} = a^\dagger a + \partial_z \omega|_{z=0} a - \partial_{\overline{z}} \omega|_{\overline{z}=0} a^\dagger + \frac{1}{2} - \partial_z \partial_{\overline{z}} \omega|_{z=0}. \]

In the case of a regular quantization, i.e., \( \omega(-z) = \omega(z) \), \( \overline{\omega(z)} = \omega(z) \), we obtain the operator

(26.2) \[ A_{|z|^2}^{\omega} = a^\dagger a + \frac{1}{2} - \partial_z \partial_{\overline{z}} \omega|_{z=0}. \]

and for \( q^2 \) and \( p^2 \),

(26.3) \[ A_{q^2}^{\omega} = Q^2 - \partial_z \partial_{\overline{z}} \omega|_{z=0} + \frac{1}{2} \left( \partial_z^2 \omega|_{z=0} + \partial_{\overline{z}}^2 \omega|_{\overline{z}=0} \right) \]
(26.4) \[ A_{p^2}^{\omega} = P^2 - \partial_z \partial_{\overline{z}} \omega|_{z=0} - \frac{1}{2} \left( \partial_z^2 \omega|_{z=0} + \partial_{\overline{z}}^2 \omega|_{\overline{z}=0} \right). \]
One observes that the difference between the ground state energy of the quantum harmonic oscillator, namely \( E_0 = \frac{1}{2} - \partial_z \partial_{\bar{z}} |_{z=0} \), and the minimum of the quantum potential energy, namely \( E_m = \frac{1}{2} \left[ \min(A_{q^2}^\omega) + \min(A_{p^2}^\omega) \right] = - \partial_z \partial_{\bar{z}} |_{z=0} \), is \( E_0 - E_m = 1/2 \). It is independent of the particular (regular) quantization chosen.

In the exponential Cahill-Glauber case \( \overline{\omega}_s(z) = e^{i|z|^2/2} \) the above operators reduce to

\[
A_{q^2}^{\overline{\omega}_s} = Q^2 - \frac{s}{2}, \quad A_{p^2}^{\overline{\omega}_s} = P^2 - \frac{s}{2}, \quad A_{|z|^2}^{\overline{\omega}_s} = a^\dagger a + \frac{1-s}{2}.
\]

27. Variations on the Wigner function and lower symbols

- The Wigner function is (up to a constant factor) the Weyl transform of a quantum density operator. For a particle in one dimension it takes the form (in units \( \hbar = 1 \))

\[
W(q,p) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left( q - \frac{y}{2} \right) \left( q + \frac{y}{2} \right) e^{iyp} dy.
\]

- Adapting this definition to the present context, and given an operator \( A \), the corresponding Wigner function is defined as the linear map

\[
A \mapsto W^W - W_A(z) = \text{tr} \left( D(z) 2PD(z)^\dagger A \right),
\]

- In the case of the quantization map \( f \mapsto A_f \) based on a weight function \( \omega \), we get the “lower symbol” \( \check{f} \) of \( A_f \)

\[
W_{A_f}^{\omega}(z) := \text{tr} \left( M^\omega(z) A_f \right) = \check{f}(z) = \int_C \left[ \omega \overline{\omega} \right](\xi - z) f(\xi) \frac{d^2\xi}{\pi}
\]
Hence the map $f \mapsto \tilde{f}$ is an (Berezin-like) integral transform with kernel
\[ f_s[\omega \tilde{\omega}](\xi - z) \]
From (27.5) it is also viewed as the convolution of $f$ with the correlation of the (quasi-) distributions $z \mapsto f_s[\omega \tilde{\omega}](z)$ and $z \mapsto f_s[\omega \tilde{\omega}](z) = f_s[\omega \tilde{\omega}](z) \tilde{f}(z)$ on the phase space $\mathbb{C}$.
If the latter are non-negative, i.e. are both probability distribution on $\mathbb{C}$ equipped with the uniform measure $d^2z$, their convolution product is the probability distribution for the random variable $z_1 - z_2$.
Then the map $f \mapsto \tilde{f}$ is interpreted as an averaging.
Hence we reach here the most sensitive aspect of Weyl-Heisenberg integral quantization through the combination of integrals, symmetry and probabilities, see [45].
In the case of Weyl-Wigner quantization
\[
\mathcal{W}_{A_f}^W \mathcal{W} = \tilde{f} = f
\]
(this one-to-one correspondence of the Weyl quantization is related to the isometry property)
In the case of the anti-normal quantization, the above convolution corresponds to the Husimi transform (when $f$ is the Wigner transform of a quantum pure state).
If the quantization map $f \mapsto A_f$ is regular (i.e. $\omega$ real even) and isometric (i.e. $|\omega| = 1$), which means that $\omega(z)$ is even and $\in \{-1, 1\}$, the corresponding inverse map $A \mapsto \mathcal{W}_A$ is given by
\[
\mathcal{W}_A(z) = \text{tr} \left( D(z)\mathcal{M}(z)^*A \right), \quad \text{where } \mathcal{M} = \mathcal{M}^* = \int_{\mathbb{C}} \omega(z)D(z) \frac{d^2z}{\pi}.
\]
In general this map $A \mapsto \mathcal{W}_A^\omega$ is only the dual of the quantization map $f \mapsto A_f$ in the sense that
\[
\int_{\mathbb{C}} f(z) \mathcal{W}_A^\omega(z) \frac{d^2z}{\pi} = \text{tr}(A A_f).
\]
This dual map becomes the inverse of the quantization map only in the case of a Hilbertian isometry.
It is clear that no compendium is perfect and complete. The hope of this one is to live its own life, being collectively improved with corrections, additions, suggestions. We invite interested researchers to be part of the work in proposing apposite remarks or contributions.

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