CONTRAMODULES OVER PRO-PERFECT TOPOLOGICAL RINGS

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Abstract. For four wide classes of topological rings $\mathcal{R}$, we show that all flat left $\mathcal{R}$-contramodules have projective covers if and only if all flat left $\mathcal{R}$-contramodules are projective if and only if all left $\mathcal{R}$-contramodules have projective covers if and only if all descending chains of cyclic discrete right $\mathcal{R}$-modules terminate if and only if all the discrete quotient rings of $\mathcal{R}$ are left perfect. Three classes of topological rings for which this holds are the complete, separated topological associative rings with a base of neighborhoods of zero formed by open two-sided ideals such that either the ring is commutative, or it has a countable base of neighborhoods of zero, or it has only a finite number of semisimple discrete quotient rings. The fourth class consists of all the topological rings with a base of neighborhoods of zero formed by open right ideals which have a closed two-sided ideal with certain properties such that the quotient ring is a topological product of rings from the previous three classes. The key technique on which the proofs are based is the contramodule Nakayama lemma for topologically T-nilpotent ideals.

Contents

1. Introduction 2
2. Preliminaries on Topological Rings 4
3. Flat Contramodules 17
4. Projective Covers of Flat Contramodules 21
5. Bass Flat Contramodules 22
6. Topologically T-Nilpotent Ideals 26
7. Topological Jacobson Radical 30
8. Products of Topological Rings 35
9. Projectivity of Flat Contramodules 37
10. Existence of Projective Covers 39
11. Proof of Main Theorem 41
12. Examples 45
13. Generalization of Main Theorem 48
References 51
1. Introduction

1.1. In a classical paper of Bass [4] (based on his Ph. D. dissertation), it was shown that every left module over an associative ring $R$ has a projective cover if and only if every flat left $R$-module is projective. Such rings were called left perfect, and a number of further equivalent characterizations of them were provided in the paper (in particular, one of the conditions equivalent to left perfectness is that all descending chains of principal right ideals in $R$ terminate). Many years later, Enochs conjectured [12], and subsequently Bican, El Bashir, and Enochs proved [9] (see also [3]) that, over any associative ring, all modules have flat covers. This assertion became known as the “flat cover conjecture theorem.”

In our recent paper [29], it is shown that, over any complete, separated topological associative ring with a countable base of neighborhoods of zero formed by open right ideals, all left contramodules have flat covers. This provides a contramodule analogue of the result of Bican, El Bashir, and Enochs. In fact, there are two proofs of the existence of flat covers in their paper [9], one following the approach of Bican and El Bashir, and the other one based on the results of Eklof and Trlifaj [11]. Both of these lines of argumentation are extended to contramodules over topological rings with a countable base of neighborhoods of zero in the paper [29].

1.2. The aim of the present paper is to extend the results of Bass’ paper [4] to the contramodule realm. One reason why this is interesting is because perfect rings are relatively rare, while pro-perfect topological rings (for many of which we prove that all contramodules have projective covers and all flat contramodules are projective) are more numerous. In particular, our results apply to the contramodules over all commutative pro-perfect topological rings. This class of topological rings includes complete Noetherian local commutative rings and the $S$-completions of $S$-almost perfect commutative rings (as defined in our previous paper [5]).

Here a pro-perfect topological ring is a complete, separated topological ring with a base of neighborhoods of zero formed by open two-sided ideals such that all its discrete quotient rings are perfect. In fact, it is only because of possible problems with non-well-behaved uncountable projective limits that we do not claim applicability of our results to all pro-perfect topological rings. Such problems do not exist for topological rings with a countable base of neighborhoods of zero, and we find them manageable for commutative topological rings and topological rings having only a finite number of semisimple discrete quotient rings.

Thus three classes of topological rings for which our main results hold are the complete, separated topological associative rings with a base of neighborhoods of zero formed by open two-sided ideals such that either (a) the ring is commutative, or (b) it has a countable base of neighborhoods of zero, or (c) it has only a finite number of semisimple discrete quotient rings.

1.3. The assumption of existence of a base of neighborhoods of zero formed by open two-sided ideals is somewhat restrictive, given that the definition of a left contramodule over a topological ring only requires a base of neighborhoods of zero consisting
of open right ideals. Some, though not all, of our main results are applicable to topological rings with a base of neighborhoods of zero formed by open right ideals. In particular, we prove the existence of projective covers and projectivity of flat contramodules over many topological rings $R$ having a topologically T-nilpotent closed two-sided ideal $\mathfrak{H}$ such that the quotient ring $R/\mathfrak{H}$ is a product of simple Artinian discrete rings (endowed with the product topology).

Here a closed ideal $\mathfrak{H}$ in a topological ring $R$ is said to be topologically left T-nilpotent if, for any sequence of elements $a_1, a_2, a_3, \ldots$ in $\mathfrak{H}$, the sequence of products $a_1, a_1a_2, a_1a_2a_3, \ldots$ converges to zero in $R$. Among our main technical tools, we present two versions of the Nakayama lemma for topologically T-nilpotent ideals. Firstly, a closed left ideal $\mathfrak{H} \subset R$ is topologically left T-nilpotent if and only if any nonzero discrete right $R$-module has a nonzero submodule annihilated by $\mathfrak{H}$. Secondly, a closed left ideal $\mathfrak{H}$ is topologically left T-nilpotent if and only if for any nonzero left $R$-contramodule $C$ the contraction map $\mathfrak{H}[C] \rightarrow C$ is not surjective.

Moreover, there is a wider fourth class of topological rings $R$, containing the three classes (a), (b), and (c), for which we show that all flat left $R$-contramodules have projective covers if and only if all flat left $R$-contramodules are projective if and only if all left $R$-contramodules have projective covers if and only if all descending chains of cyclic discrete right $R$-modules terminate if and only if all the discrete quotient rings of $R$ are left perfect. This class (d) consists of complete, separated topological rings with a base of neighborhoods of zero formed by open right ideals having a topologically left T-nilpotent closed ideal $K \subset R$ such that the quotient ring $R/K$ is a topological product of topological rings satisfying (a), (b), or (c). Topological rings satisfying the condition (d) do not need to have a base of neighborhoods of zero formed by open two-sided ideals.

We also discuss the notion of the topological Jacobson radical of a topological ring $R$ with a base of neighborhoods of zero formed by open right ideals (cf. [14]). The topological Jacobson radical of $R$ is defined as the intersection of all the open maximal right ideals in $R$. Generally speaking, the topological Jacobson radical of $R$ is a closed two-sided ideal containing the usual Jacobson radical of the ring $R$ viewed as an abstract ring. For a topological ring $R$ with a two-sided ideal $\mathfrak{H}$ of the kind mentioned above in Section 1.3, both the topological and the abstract nontopological Jacobson radicals of $R$ coincide with $\mathfrak{H}$.

This paper was originally motivated by the idea to apply contramodules in the study of covers and direct limits in module categories, and more generally, in additive and abelian categories. Such applications to covers and direct limits in the categorical tilting context and beyond, and in particular, for injective homological ring epimorphisms, are discussed in a companion paper [6]. Furthermore, the main results of this paper provide supporting evidence for the discussion of conjectures about topologically perfect topological rings in another companion paper [32]. Finally, a general proof of the assertion that a direct limit of projective contramodules is projective if it has a projective cover is given in the paper [7]. Based on this
result, we prove in [32] that, for a complete separated topological ring $\mathcal{R}$ with a base of neighborhoods of zero formed by open right ideals, all flat left $\mathcal{R}$-contramodules are projective if and only if all flat left $\mathcal{R}$-contramodules have projective covers if and only if all left $\mathcal{R}$-contramodules have projective covers.

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2. **Preliminaries on Topological Rings**

Throughout this paper, by “direct limits” in a category we mean inductive limits indexed by directed posets. Otherwise, these are known as the directed or filtered colimits.

The material in Sections 2.1–2.4 below is well-known (some relevant references are [35], Chapter VI], the books [33, 2], and the paper [8]). So is the example in Section 2.13 (where the book [13, Section 107] can be used as a reference). Sections 2.5–2.10 go back to [20, Remark A.3] and [21, Section 1.2]; for later expositions, see [23, Sections 2.1 and 2.3], [29, Sections 1.2 and 5], [30, Sections 6–7], and [27, Section 2]. The counterexample in Section 2.11 comes from [33] or [2], while the rest of the material of Sections 2.11–2.12 may be somewhat new (it is implicit in [21, Sections 1.3 and B.4]).

For a basic background discussion of topological abelian groups and vector spaces with linear topology, going into many details and correcting some errors in [8], see the very recent preprint [28].

2.1. **Abelian groups with linear topology.** A “topological abelian group” in this paper will always mean a topological abelian group with a base of neighborhoods of zero formed by open subgroups. Any nonempty collection of subgroups $B$ in an abelian group $A$ such that for any two subgroups $U' \in B$ and $U'' \in B$ there exists a subgroup $U \in B$, $U \subseteq U' \cap U''$, forms a base of neighborhoods of zero in a (uniquely defined) topology on $A$ compatible with the abelian group structure.

The completion $\hat{A} = A^\wedge_B$ of a topological abelian group $A$ is the abelian group $\hat{A} = \varprojlim_{U \in B} A/U$ endowed with the projective limit topology, in which a base of neighborhoods of zero in $\hat{A}$ is formed by the kernels $\mathfrak{U} = U^\wedge$ of the projection maps $\mathfrak{A} \twoheadrightarrow A/U$. The topological group $\hat{A} = A^\wedge_B$ does not depend on the choice of a particular base of neighborhoods of zero $B$ in a topological group $A$.

A topological abelian group $A$ is said to be separated if the completion map $A \twoheadrightarrow \hat{A}$ is injective, and complete if it is surjective. The projection maps $\mathfrak{A} \twoheadrightarrow A/U$ induce isomorphisms $\mathfrak{A}/\mathfrak{U} \cong A/U$, so the completion $\hat{A}$ of a topological abelian group
2.2. Subgroup and quotient group topologies. Let $A$ be a topological abelian group and $A' \subset A$ be a subgroup. Then the group $A'$ can be endowed with the topology induced from $A$. If $B$ is a collection of subgroups forming a base of neighborhoods of zero in $A$, then the collection of subgroups $B' = \{A' \cap U \mid U \in B\}$ forms a base of neighborhoods of zero in $A'$.

Lemma 2.1. (a) If a topological abelian group $A$ is separated, then any subgroup $A' \subset A$ is separated in the induced topology.

(b) Let $\mathfrak{A}$ be a complete, separated topological abelian group. Then a subgroup $\mathfrak{A}' \subset \mathfrak{A}$ is complete in the induced topology if and only if it is closed in $\mathfrak{A}$.

(c) More generally, if $\mathfrak{A}$ is separated and complete, then for any subgroup $A' \subset \mathfrak{A}$ the induced map between the completions $A'^{\sim} \to \mathfrak{A}^{\sim} = \mathfrak{A}$ provides a topological group isomorphism of the completion $A'^{\sim}$ of the group $A'$ with the closure $\mathfrak{A}' \subset \mathfrak{A}$ of the subgroup $A' \subset \mathfrak{A}$ (where the topology on $A'$ and $\mathfrak{A}'$ is induced from $\mathfrak{A}$).

Proof. See, e. g., [28, Lemma 1.1].

Let $A$ be a topological abelian group, $A' \subset A$ be a subgroup, and $A'' = A/A'$ be the quotient group. Then the group $A''$ can be endowed with the quotient topology, in which a subset of $A''$ is open if and only if its preimage in $A$ is. If $B$ is a collection of subgroups forming a base of neighborhoods of zero in $A$, then the collection of subgroups $B'' = \{(A' + U)/A' \mid U \in B\}$ forms a base of neighborhoods of zero in $A''$.

Lemma 2.2. (a) If a topological abelian group $A$ is separated, then the quotient group $A/A'$ is separated in the quotient topology if and only if the subgroup $A'$ is closed in $A$.

(b) Let $\mathfrak{A}$ be a complete, separated topological abelian group with a countable base of neighborhoods of zero formed by open subgroups, and let $\mathfrak{A}' \subset \mathfrak{A}$ be a closed subgroup. Then the quotient group $\mathfrak{A}'' = \mathfrak{A}/\mathfrak{A}'$ is complete in the quotient topology.

Proof. We will only prove part (b). Let $\mathfrak{B}$ be a countable base of neighborhoods of zero consisting of open subgroups in $\mathfrak{A}$. Then for any $U \in \mathfrak{B}$ we have a short exact sequence of abelian groups

$$0 \to \mathfrak{A}'/\mathfrak{U}' \to \mathfrak{A}/\mathfrak{U} \to \mathfrak{A}''/\mathfrak{U}'' \to 0,$$

where $\mathfrak{U}' = \mathfrak{A}' \cap \mathfrak{U}$ and $\mathfrak{U}'' = (\mathfrak{A}' + \mathfrak{U})/\mathfrak{A}'$. Now $(\mathfrak{A}'/\mathfrak{U}')_{\mathfrak{U} \in \mathfrak{B}}$ is a projective system of abelian groups and surjective morphisms between them, indexed by a countable directed poset $\mathfrak{B}$. Hence the sequence remains exact after the passage to the projective limits,

$$0 \to \mathfrak{A}'_{\mathfrak{B}'}, \to \mathfrak{A}_{\mathfrak{B}} \to \mathfrak{A}''_{\mathfrak{B}'}, \to 0,$$

where $\mathfrak{B}' = \{\mathfrak{U}' \mid \mathfrak{U} \in \mathfrak{B}\}$ and $\mathfrak{B}'' = \{\mathfrak{U}'' \mid \mathfrak{U} \in \mathfrak{B}\}$. By assumption, we have $\mathfrak{A} = \mathfrak{A}_{\mathfrak{B}}$; by Lemma 2.1(b), $\mathfrak{A}' = \mathfrak{A}'_{\mathfrak{B}}$. Thus the completion map $\mathfrak{A}'' \to \mathfrak{A}''_{\mathfrak{B}''}$ is bijective, that is, the topological abelian group $\mathfrak{A}''$ is (separated and) complete.

For a generalization, see [28, Proposition 1.4].
Without the countability assumption, the assertion of Lemma 2.2(b) is not true. See Section 2.11 for further discussion and counterexamples.

### 2.3. Topological rings

In this paper, the word “ring” means “an associative ring with unit” by default. Unless otherwise mentioned, all ring homomorphisms are supposed to preserve units, and all modules are presumed to be unital. When considering rings without unit or subrings without unit, as we will at some point in Section 6, we will always explicitly refer to them as being “without unit”.

Given an (associative and unital) ring \( R \), we denote the abelian category of (arbitrary unital) left \( R \)-modules by \( R \text{-mod} \) and the abelian category of right \( R \)-modules by \( \text{mod-R} \). The ring with the opposite multiplication to a ring \( R \) is denoted by \( R^{\text{op}} \).

In this paper we are interested in topological associative rings \( R \) such that open right ideals \( I \subset R \) form a base of neighborhoods of zero in \( R \). A nonempty collection of right ideals \( B \) in a ring \( R \) forms a base of neighborhoods of zero in a topology compatible with the ring structure on \( R \) if and only if it satisfies the following conditions (cf. [35, Section VI.4] and [8, Remark 1.1(ii), Claim 1.4, and Lemma 1.4]):

(i) for any two right ideals \( I' \) and \( I'' \in B \), there exists a right ideal \( J \in B \) such that \( J \subset I' \cap I'' \); and

(ii) for any right ideal \( I \in B \) and any element \( r \in R \), there exists a right ideal \( J \in B \) such that \( rJ \subset I \).

The completion \( \mathfrak{R} = \hat{R} \) of a topological ring \( R \) is the abelian group \( \mathfrak{R} = \lim \leftarrow_{I \in B} R/I \) endowed with the projective limit topology (in which a base of neighborhoods of zero \( B \) in \( \mathfrak{R} \) is formed by the kernels \( I = \cap \subset \mathfrak{R} \) of the projection maps \( \mathfrak{R} \to R/I \)). One readily checks, using the conditions (i) and (ii), that there exists a unique associative ring structure on \( \mathfrak{R} \) that is continuous with respect to the projective limit topology and such that the natural map \( R \to \mathfrak{R} \) is a ring homomorphism. Given two elements \( r' = (r'_I)_{I \in B} \) and \( r'' = (r''_I)_{I \in B} \in \mathfrak{R} \), in order to compute the \( I \)-component \( r_I \) of the product \( r = (r_I)_{I \in B} \in \mathfrak{R} \), one has to find an open right ideal \( J \in B \) such that \( J \subset I \) and \( r'_I J \subset I \); then one can set \( r_I = r'_I r''_I + I \).

The open subsets \( I = \cap \subset \mathfrak{R} \) are right ideals in \( \mathfrak{R} \), so the topological ring \( \mathfrak{R} \) has a base of neighborhoods of zero consisting of open right ideals. When the base of neighborhoods of zero \( B \) in \( R \) consists of open two-sided ideals, the topological ring \( \mathfrak{R} \) can be simply defined as the projective limit of (discrete) rings \( R/I \). Then the open subsets \( I = \cap \subset \mathfrak{R} \) are two-sided ideals.

A topological ring \( R \) is said to be separated (resp., complete) if it is separated (resp., complete) as a topological abelian group.

### 2.4. Discrete modules

Let \( R \) be a topological ring with a base of neighborhoods of zero formed by open right ideals. A right \( R \)-module \( N \) is said to be discrete if for every element \( b \in N \) the annihilator \( \text{Ann}_R(b) = \{ r \in R \mid br = 0 \} \) is an open right ideal in \( \mathfrak{R} \). The annihilator of an element in a right \( R \)-module is always a right ideal in \( R \), so topological rings with a base of neighborhoods of zero formed by open right ideals are a natural setting for considering discrete right modules.
The full subcategory of discrete right $R$-modules $\text{discr--} R \subset \text{mod--} R$ is a hereditary pretorsion class in the abelian category of right $R$-modules $\text{mod--} R$, and all hereditary pretorsion classes in $\text{mod--} R$ appear in this way [35, Lemma VI.4.1 and Proposition VI.4.2]. Viewed as an abstract category, the category $\text{discr--} R$ is a Grothendieck abelian category. So, in particular, the abelian category $\text{discr--} R$ is complete and cocomplete, has exact direct limits, and an injective cogenerator.

One readily checks that any discrete right $R$-module has a unique discrete right $\mathcal{R}$-module structure compatible with its $R$-module structure (where $\mathcal{R} = R^+$ denotes the topological ring $R$). Thus the abelian categories of discrete right $R$-modules and discrete right $\mathcal{R}$-modules are naturally equivalent (in fact, isomorphic), $\text{discr--} R \cong \text{discr--} \mathcal{R}$.

2.5. **Convergent formal linear combinations.** Given an abelian group $A$ and a set $X$, we denote by $A[[X]] = A^X$ the direct sum of $X$ copies of the abelian group $A$, viewed as the group of all finite formal linear combinations $\sum_{x \in X} a_x x$ of elements of $X$ with the coefficients in $A$. A formal linear combination $\sum_{x \in X} a_x x$ belongs to $A[[X]]$ if and only if the set of all indices $x \in X$ for which $a_x \neq 0$ is finite.

Given a separated and complete topological abelian group $\mathfrak{A}$ with a base of neighborhoods of zero $\mathfrak{B}$ consisting of open subgroups, and a set $X$, we denote by $\mathfrak{A}[[X]]$ the abelian group $\varprojlim_{U \in \mathfrak{B}} (\mathfrak{A}/U)[X]$. Clearly, the group $\mathfrak{A}[[X]]$ does not depend on the choice of a particular base of neighborhoods of zero $\mathfrak{B}$ in $\mathfrak{A}$. We interpret $\mathfrak{A}[[X]]$ as the group of all infinite formal linear combinations $\sum_{x \in X} a_x x$ of elements of $X$ with the coefficients in $\mathfrak{A}$ forming an $X$-indexed family of elements in $\mathfrak{A}$ converging to zero in the topology of $\mathfrak{A}$. This means that the subgroup $\mathfrak{A}[[X]] \subset \mathfrak{A}^X$ consists of all the infinite formal linear combinations $\sum_{x \in X} a_x x$ such that, for every open subgroup $U \subset \mathfrak{A}$, one has $a_x \in U$ for all but a finite subset of indices $x \in X$.

The map assigning to a set $X$ the abelian group $\mathfrak{A}[[X]]$ extends naturally to a covariant functor from the category of sets to the category of abelian groups. Given a map of sets $f : X \rightarrow Y$, one defines the induced map $\mathfrak{A}[[f]] : \mathfrak{A}[[X]] \rightarrow \mathfrak{A}[[Y]]$ by the rule $\sum_{x \in X} a_x x \mapsto \sum_{y \in Y} (\sum_{f(x) = y} a_x) y$, where the sum of elements $a_x$ in the parentheses is understood as the limit of finite partial sums in the topology of $\mathfrak{A}$. Such a limit is unique and exists because the topological abelian group $\mathfrak{A}$ is separated and complete, while the family of elements $(a_x)_{x \in X}$, and consequently its subfamily indexed by all $x \in X$ with $f(x) = y$ for a fixed $y \in Y$, converges to zero in $\mathfrak{A}$.

2.6. **The monad structure.** Let $\mathcal{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Let us consider the functor $X \mapsto \mathcal{R}[[X]]$ as taking values in the category of sets; so it becomes an endofunctor $\mathcal{T}_\mathcal{R} = \mathcal{R}[[-]] : \text{Sets} \rightarrow \text{Sets}$. The key observation is that the functor $\mathcal{T}_\mathcal{R}$ has a natural structure of a monad on the category of sets [20] Remark A.3], [21, Section 1.2], [23, Section 2.1], [29 Sections 1.1–1.2 and 5], [26, Section 1], [20, Section 6].
This means that the functor $T_R$ is endowed with natural transformations $\epsilon : \text{Id} \to T_R$ and $\phi : T_R \circ T_R \to T_R$ satisfying the monad equations (of unitality and associativity). The monad unit $\epsilon_X : X \to \mathcal{R}[[X]]$ is the “point measure” map, assigning to an element $x_0 \in X$ the (finite) formal linear combination $\sum_{x \in X} r_x x \in \mathcal{R}[[X]]$, where $r_{x_0} = 1$ and $r_x = 0$ for all $x \neq x_0$. The monad multiplication $\phi_X : \mathcal{R}[[\mathcal{R}[[X]]]] \to \mathcal{R}[[X]]$ is the “opening of parentheses” map, assigning a formal linear combination to a formal linear combination of formal linear combinations.

Given a set $X$ and an element $r \in \mathcal{R}[[\mathcal{R}[[X]]]]$, computing the element $\phi_X(r) \in \mathcal{R}[[X]]$ involves opening the parentheses, computing the products of pairs of elements in the ring $\mathcal{R}$, and then computing the infinite sums. The coefficient $t_x$ of $\phi(r) = \sum_{x \in X} t_x x$ at an element $x \in X$ is an infinite sum of products of pairs of elements in $\mathcal{R}$, understood as the limit of finite partial sums in the topology of $\mathcal{R}$. Thus it is crucial for the definition of $\phi_X$ that $\mathcal{R}$ is separated and complete, and that all the infinite sums involved converge. The latter is guaranteed by the assumption that open right ideals form a base of neighborhoods of zero in $\mathcal{R}$.

Indeed, let $Y$ denote the set $\mathcal{R}[[X]]$; then we have $r = \sum_{y \in Y} r_y y$ for some $r_y \in \mathcal{R}$, and $y = \sum_{x \in X} s_{y,x} x$ for all $y \in Y$ and some $s_{y,x} \in \mathcal{R}$. For any $x \in X$, the coefficient $t_x$ is to be computed as $t_x = \sum_{y \in Y} r_y s_{y,x} \in \mathcal{R}$. In order to show that this sum converges in the topology of $\mathcal{R}$, we have to check that, for every open right ideal $\mathcal{I} \subset \mathcal{R}$, the product $r_y s_{y,x}$ belongs to $\mathcal{I}$ for all but a finite set of indices $y \in Y$. Now, one has $r_y s_{y,x} \in \mathcal{I}$ whenever $r_y \in \mathcal{I}$; and there is only a finite set of indices $y$ with $r_y \notin \mathcal{I}$, because $r \in \mathcal{R}[[Y]]$. So the coefficient $t_x$ is well-defined for every $x \in X$. In order to check that $\sum_{x \in X} t_x x \in \mathcal{R}[[X]]$, that is $t_x \in \mathcal{I}$ for all but a finite set of indices $x \in X$, one has to use the condition (ii) from Section 2.3.

2.7. Contramodules. Let $\mathcal{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. By the definition, a left $\mathcal{R}$-contramodule is an algebra or module (depending on the terminology) over the monad $T_R : X \mapsto \mathcal{R}[[X]]$ on the category of sets.

This means that a left $\mathcal{R}$-contramodule $C$ is a set endowed with a map of sets $\pi_C : \mathcal{R}[[C]] \to C$, called the left conraction map. The map $\pi_C$ must satisfy the equations of contramunitality, telling that the composition $\pi_C \epsilon_C$ with the monad unit map $\epsilon_C$ is the identity map $\text{id}_C$,

$$C \to \mathcal{R}[[C]] \to C,$$

and contraassocitativity, asserting that the two maps $\mathcal{R}[[\mathcal{R}[[C]]]] \to \mathcal{R}[[C]]$, one of which is the monad multiplication map $\phi_C$ and the other one is the map $\mathcal{R}[[\pi_C]]$ induced by $\pi_C$, should have equal compositions with the conraction map $\pi_C$,

$$\mathcal{R}[[\mathcal{R}[[C]]]] \to \mathcal{R}[[C]] \to C.$$  

We denote the category of left $\mathcal{R}$-contramodules by $\mathcal{R}$-contra.

In particular, any associative ring $R$ can be considered as a topological ring with the discrete topology. In this case, we have $R[[X]] = R[X]$, and a left $R$-contramodule is the same thing as a left $R$-module [30, Section 6.1]. So the above definition of an
\(R\)-contramodule, restricted to the particular case when the topological ring \(R = R\) is discrete, provides a fancy way to define the familiar notion of a module over an associative ring.

Now, for any complete, separated topological associative ring \(R\) with a base of neighborhoods of zero formed by open right ideals, and for any left \(R\)-contramodule \(C\), one can compose the contraaction map \(\pi_C : R[[C]] \to C\) with the identity embedding \(R[C] \to R[[C]]\) of the set of all finite formal linear combinations into the set of all convergent infinite ones. This defines a natural structure of an algebra/module over the monad \(X \mapsto R[X]\) on the set \(C\), which means a left \(R\)-module structure. Thus all left \(R\)-contramodules have underlying structures of left modules over the ring \(R\), viewed as an abstract (nontopological) ring. We have constructed the forgetful functor \(R\textnormal{-contra} \to R\textnormal{-mod}\). In particular, it means that all left \(R\)-contramodules, which were originally defined as only sets endowed with a contraaction map, are actually abelian groups.

The category \(R\textnormal{-contra}\) is abelian [26, Lemma 1.1], and the forgetful functor \(R\textnormal{-contra} \to R\textnormal{-mod}\) is exact. The category \(R\textnormal{-contra}\) is also complete and cocomplete, with the forgetful functor \(R\textnormal{-contra} \to R\textnormal{-mod}\) preserving infinite products (but not coproducts) are exact in \(R\textnormal{-contra}\). Given two left \(R\)-contramodules \(C\) and \(D\), we denote by \(\text{Hom}^R(C, D)\) the abelian group of morphisms \(C \to D\) in \(R\textnormal{-contra}\). For any set \(X\), the map \(\pi_{R[[X]]} = \phi_X\) endows the set/abelian group \(R[[X]]\) with the structure of a left \(R\)-contramodule. It is called the free left \(R\)-contramodule generated by a set \(X\). For any left \(R\)-contramodule \(C\), morphisms \(R[[X]] \to C\) in the category \(R\textnormal{-contra}\) correspond bijectively to maps of sets \(X \to C\).

\[\text{Hom}^R(R[[X]], C) \cong \text{Hom}_{\text{sets}}(X, C)\]

Hence free left \(R\)-contramodules are projective objects of the category \(R\textnormal{-contra}\). There are also enough of them: for any left \(R\)-contramodule \(C\), the contraaction map \(\pi_C : R[[C]] \to C\) is an \(R\)-contramodule morphism presenting \(C\) as a quotient contramodule of the free left \(R\)-contramodule \(R[[C]]\). So the abelian category \(R\textnormal{-contra}\) has enough projectives, and a left \(R\)-contramodule is projective if and only if it is a direct summand of a free one.

2.8. **Contratensor product.** As above, we denote by \(R\) a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Some of the simplest examples of non-free left \(R\)-contramodules are obtained by dualizing discrete right \(R\)-modules.

Let \(A\) be an associative ring, and let \(N\) be an \(A\)-\(R\)-bimodule whose right \(A\)-module structure is that of a discrete right \(R\)-module. Let \(V\) be a left \(A\)-module. Then the induced left \(R\)-module structure of the abelian group \(\text{Hom}_A(N, V)\) extends naturally to a left \(R\)-contramodule structure. Indeed, to construct a left \(R\)-contraaction map for the set \(D = \text{Hom}_A(N, V)\), consider an element \(r = \sum_{d \in D} r_d d \in R[[D]]\). Set \(f = \pi_D(r)\) to be the left \(A\)-module map \(f : N \to V\) taking any element \(b \in N\) to the
element $\sum_{d \in D} d(br_d) \in V$,
\[
\pi_D \left( \sum_{d \in D} r_d d \right)(b) = \sum_{d \in D} d(br_d),
\]
where the sum in the right-hand side is finite, because one has $br_d = 0$ for all but a finite number of elements $d \in D$. Indeed, the annihilator ideal $\text{Ann}_R(b) \subset R$ is open by assumption, and consequently, it has to contain the coefficient $r_d$ for all but a finite number of elements $d \in D$.

Let $N$ be a discrete right $R$-module and $C$ be a left $R$-contramodule. The contratensor product $N \otimes_R C$ is an abelian group defined as the cokernel of (the difference of) a natural pair of abelian group homomorphisms
\[
N \otimes_Z R[[C]] \to N \otimes_Z C.
\]
Here one of the maps $N \otimes_Z R[[C]] \to N \otimes_Z C$ is just the map $N \otimes \pi C$ induced by the contraaction map $\pi C : R[[C]] \to C$, while the other map is the composition $N \otimes_Z R[[C]] \to N[C] \to N \otimes_Z C$, where (following our general notation system) $N[C]$ denotes the group of all finite formal linear combinations of elements of $C$ with the coefficients in $N$. The map $N \otimes_Z R[[C]] \to N[C]$, induced by the right action map $N \otimes_Z R \to N$, is well-defined due to the assumption that $N$ is a discrete right $R$-module. The map $N[C] \to N \otimes_Z C$ is just the obvious one, taking a finite formal linear combination $\sum_{c \in C} b_c c$ to the tensor $\sum_{c \in C} b_c \otimes c$.

Essentially by the definition, the contratensor product is a quotient group of the tensor product: for any discrete right $R$-module $N$ and any left $R$-contramodule $C$, there is a natural surjective map of abelian groups
\[
N \otimes_R C \twoheadrightarrow N \otimes_R C.
\]
For any $A$-$R$-bimodule $N$ whose right $R$-module structure is that of a discrete right $R$-module, any left $R$-contramodule $C$, and any abelian group $V$, there is a natural adjunction isomorphism of abelian groups \cite[Section 5]{29} \[ \text{Hom}_A(N \otimes_R C, V) \cong \text{Hom}_R(C, \text{Hom}_A(N, V)). \] The functor of contratensor product $\otimes_R : \text{discr} \times R\text{-contra} \to Z\text{-mod}$ preserves colimits (i. e., is right exact and preserves coproducts) in both its arguments. For any discrete right $R$-module $N$ and any set $X$, there is a natural isomorphism of abelian groups
\[
N \otimes_R R[[X]] \cong N[X].
\]

2.9. Change of scalars. Let us start with a discussion of change of scalars for homomorphisms of abstract associative rings (without any topology) before passing to contramodules and discrete modules over topological rings.

Let $g : R \to S$ be a homomorphism of associative rings. Then any $S$-module can be endowed with an $R$-module structure via $g$, so we have the functor of restriction of scalars $g_* : S\text{-mod} \to R\text{-mod}$. The functor $g_*$ is exact and faithful, and it preserves both the infinite coproducts and products.
The functor $g_\ast: S\mod\rightarrow R\mod$ has adjoints on both sides. The functor of extension of scalars $g^*: R\mod\rightarrow S\mod$ given by the rule $g^*(M) = S \otimes_R M$ is left adjoint to $g_\ast$, while the functor of coextension of scalars $g^! : R\mod \rightarrow S\mod$ given by the formula $g^!(M) = \text{Hom}_R(S, M)$ is right adjoint to $g_\ast$.

When one passes to contramodules and discrete modules over topological rings, this picture of three functors splits in two halves. Let $f: \mathcal{R} \rightarrow \mathcal{G}$ be a continuous homomorphism of complete, separated topological rings, each of them having a base of neighborhoods of zero formed by open right ideals. Then for any set $X$ there is the induced map of sets/abelian groups $f[[X]]: \mathcal{R}[[X]] \rightarrow \mathcal{G}[[X]]$.

Let $\mathcal{C}$ be a left $\mathcal{G}$-contramodule. Composing the map $f[[\mathcal{C}]]: \mathcal{R}[[\mathcal{C}]] \rightarrow \mathcal{G}[[\mathcal{C}]]$ with the contraaction map $\pi_\mathcal{C} : \mathcal{G}[[\mathcal{C}]] \rightarrow \mathcal{C}$, we obtain a map $\mathcal{R}[[\mathcal{C}]] \rightarrow \mathcal{C}$ defining a left $\mathcal{R}$-contramodule structure on the set $\mathcal{C}$. We have constructed an exact, faithful functor of contrarestiction of scalars $f_! : \mathcal{G}\text{-contra} \rightarrow \mathcal{R}\text{-contra}$ forming a commutative square diagram with the forgetful functors $\mathcal{R}\text{-contra} \rightarrow \mathcal{R}\mod$, $\mathcal{G}\text{-contra} \rightarrow \mathcal{G}\mod$ and the restriction-of-scalars functor $\mathcal{G}\mod \rightarrow \mathcal{R}\mod$. The functor $f_!$ also preserves infinite products.

The functor $f_!$ has a left adjoint functor of contraextension of scalars $f^! : \mathcal{R}\text{-contra} \rightarrow \mathcal{G}\text{-contra}$. To construct the functor $f^!$, one can first define it on free left $\mathcal{R}$-contramodules by the rule $f^!(\mathcal{R}[[X]]) = \mathcal{G}[[X]]$ for all sets $X$, and then extend to a right exact functor on the whole category $\mathcal{R}\text{-contra}$. As any left adjoint functor, the functor $f^!$ preserves coproducts.

Notice that the forgetful functor $\mathcal{G}\text{-contra} \rightarrow \mathcal{G}\mod$ does not preserve coproducts, generally speaking; and accordingly the functor of contrarestiction of scalars $f_! : \mathcal{G}\text{-contra} \rightarrow \mathcal{R}\text{-contra}$ does not preserve coproducts. (For a counterexample, take $\mathcal{R} = \mathcal{G}$ with the given topology on $\mathcal{G}$ and the discrete topology on $\mathcal{R}$, and let $f : \mathcal{R} \rightarrow \mathcal{G}$ be the identity map; or alternatively, take $\mathcal{R} = \mathbb{Z}$ with the discrete topology and let $f : \mathcal{R} \rightarrow \mathcal{G}$ be the unique ring homomorphism.) Hence the functor $f_!$ does not have a right adjoint, in general.

Similarly, the map $f$ endows any discrete right $\mathcal{G}$-module with a discrete right $\mathcal{R}$-module structure. In other words, the conventional functor of restriction of scalars $\mod\mathcal{G} \rightarrow \mod\mathcal{R}$ takes discrete right $\mathcal{G}$-modules to discrete right $\mathcal{R}$-modules. So we have an exact, faithful functor of restriction of scalars $f_* : \text{discr}\mathcal{G} \rightarrow \text{discr}\mathcal{R}$. The functor $f_*$ also preserves infinite coproducts.

As any colimit-preserving functor between Grothendieck abelian categories, the functor $f_*$ has a right adjoint functor of coextension of scalars $f^* : \text{discr}\mathcal{R} \rightarrow \text{discr}\mathcal{G}$. The functor $f^*$ is left exact and preserves products.

Notice that the inclusion functor $\text{discr}\mathcal{G} \rightarrow \mod\mathcal{G}$ does not preserve products, generally speaking. In fact, the product of a family of objects $\mathcal{M}_\alpha$ in $\text{discr}\mathcal{G}$ can be constructed as the maximal discrete $\mathcal{G}$-submodule of the product of the $\mathcal{G}$-modules $\mathcal{M}_\alpha$ taken in $\mod\mathcal{G}$. Accordingly, the functor of restriction of scalars $f_* : \text{discr}\mathcal{G} \rightarrow \text{discr}\mathcal{R}$ does not preserve products, in general. (Once again, for a counterexample it suffices to take $\mathcal{R} = \mathcal{G}$ with the given topology on $\mathcal{G}$ and the discrete topology on $\mathcal{R}$.) Hence the functor $f_*$ usually does not have a left adjoint.
For any discrete right $\mathfrak{S}$-module $\mathcal{M}$ and any left $\mathfrak{R}$-contramodule $\mathcal{C}$ there is a natural isomorphism of abelian groups

$$f_0(\mathcal{M}) \otimes_{\mathfrak{R}} \mathcal{C} \cong \mathcal{M} \otimes_{\mathfrak{S}} f_0^j(\mathcal{C}).$$

2.10. **Reductions modulo ideals.** Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals.

Let $\mathfrak{N}$ be a discrete right $\mathfrak{R}$-module, and $A \subset \mathfrak{R}$ be an additive subgroup in $\mathfrak{R}$. Then we denote by $N_A \subset \mathfrak{N}$ the additive subgroup in $\mathfrak{N}$ consisting of all the elements $b \in \mathfrak{N}$ such that $br = 0$ for all $r \in A$. If $J$ is a left ideal in $\mathfrak{R}$, then the subgroup $N_J$ is an $\mathfrak{R}$-submodule in $\mathfrak{N}$.

Let $R$ be an associative ring, $C$ be a left $R$-module, and $A \subset R$ be an additive subgroup in $R$. As usually, we will denote by $AC \subset C$ the subgroup in $C$ spanned by all the elements $ac$ with $a \in A$ and $c \in C$. Clearly, one has $AC = A(RC) = (AR)C$, where $AR \subset R$ is the right ideal generated by $A$. When $J \subset R$ is a left ideal, the subgroup $JC$ is an $R$-submodule in $C$.

Let $\mathcal{C}$ be a left $\mathfrak{R}$-contramodule, and $A \subset \mathfrak{R}$ be a closed additive subgroup in $\mathfrak{R}$. Then $A$ is a complete, separated topological abelian group in the topology induced from $\mathfrak{R}$, and for any set $X$ the group $A[[X]]$ is a subgroup in $\mathfrak{R}[[X]]$. Following [20, Remark A.3], [21, Section 1.3], [22, Section D.1], [29, Section 5], we will denote by $A \triangleleft \mathfrak{C} \subset \mathfrak{C}$ the image of the map $A[[\mathfrak{C}]] \rightarrow \mathfrak{C}$ obtained by restricting the contraaction map $\mathfrak{R}[[\mathfrak{C}]] \rightarrow \mathfrak{C}$ to the subgroup $A[[\mathfrak{C}]] \subset \mathfrak{R}[[\mathfrak{C}]]$. Clearly, one has $AC \subset A \triangleleft \mathfrak{C}$.

Let $J \subset \mathfrak{R}$ be a closed left ideal. Then the composition of the identity embedding $\mathfrak{R}[[J[[X]]]] \rightarrow \mathfrak{R}[[\mathfrak{R}[[X]]]]$ with the map $\phi_X: \mathfrak{R}[[\mathfrak{R}[[X]]]] \rightarrow \mathfrak{R}[[X]]$ takes values inside the subset $J[[X]] \subset \mathfrak{R}[[X]]$, so $J[[X]]$ is a left $\mathfrak{R}$-subcontramodule of $\mathfrak{R}[[X]]$. It follows that, for any left $\mathfrak{R}$-contramodule $\mathfrak{C}$, the subgroup $J \triangleleft \mathfrak{C} \subset \mathfrak{C}$ is the image of the composition of left $\mathfrak{R}$-contramodule morphisms $J[[\mathfrak{C}]] \rightarrow \mathfrak{R}[[\mathfrak{C}]] \rightarrow \mathfrak{C}$. Hence $J \triangleleft \mathfrak{C}$ is an $\mathfrak{R}$-subcontramodule in $\mathfrak{C}$.

For any closed right ideal $J \subset \mathfrak{R}$ and any set $X$, one has

$$J \times (\mathfrak{R}[[X]]) = J[[X]] \subset \mathfrak{R}[[X]].$$

Let $I \subset \mathfrak{R}$ be an open right ideal. Then the right $\mathfrak{R}$-module $\mathfrak{R}/I$ is discrete, and for any left $\mathfrak{R}$-contramodule $\mathfrak{C}$ there is a natural isomorphism of abelian groups

$$\mathfrak{R}/I \otimes_{\mathfrak{R}} \mathfrak{C} \cong \mathfrak{C}/I\mathfrak{C}. \quad (1)$$

The isomorphism (1) is a part of the commutative square diagram

$$\begin{array}{ccc}
(\mathfrak{R}/I) \otimes_{\mathfrak{R}} \mathfrak{C} & \cong & \mathfrak{C}/I\mathfrak{C} \\
\downarrow & & \downarrow \\
(\mathfrak{R}/I) \otimes_{\mathfrak{R}} \mathfrak{C} & \cong & \mathfrak{C}/(J \triangleleft \mathfrak{C})
\end{array} \quad (2)$$

with the familiar natural isomorphism in the upper horizontal line and the natural surjective maps of abelian groups shown by the vertical arrows. To convince oneself of the existence of the isomorphism (1), one observes that the two kernels of the surjective maps in the vertical lines of (2) are identified with each other by the
isomorphism in the upper horizontal line. Essentially, this holds because one has $R[\mathcal{C}] = R[\mathcal{C}] + J[\mathcal{C}]$ for any open right ideal $J \subset R$, and specifically, for the open right ideal $J$ provided by the condition (ii) from Section 2.3.

In particular, for any set $X$ one has $R[[X]]/(J \triangleleft R[[X]]) \cong (R/J)[X]$ and therefore $R[[X]] \cong \lim_{\leftarrow} R[[X]]/(J \triangleleft R[[X]])$, where the projective limit is taken over all the open right ideals $J \subset R$. It follows that the natural map

$$\mathfrak{P} \longrightarrow \lim_{\leftarrow} \mathfrak{P} / (J \triangleleft \mathfrak{P})$$

is an isomorphism for every projective left $R$-contramodule $\mathfrak{P}$.

2.11. Strongly closed subgroups. Let $\mathfrak{A}$ be a complete, separated topological abelian group (with a base of neighborhoods of zero formed by open subgroups). Let $\mathfrak{H} \subset \mathfrak{A}$ be a closed subgroup. Then the quotient group $\mathfrak{Q} = \mathfrak{A}/\mathfrak{H}$ is separated by Lemma 2.2(a), but it does not have to be complete. Let $\mathfrak{Q} = \hat{\mathfrak{Q}}$ be the completion of the topological group $\mathfrak{Q}$. Then the natural morphism $p: \mathfrak{A} \longrightarrow \mathfrak{Q}$ is the cokernel of the morphism $\mathfrak{H} \longrightarrow \mathfrak{A}$ in the category of complete, separated topological abelian groups. The group $\mathfrak{Q}$ is a dense subgroup in $\mathfrak{Q}$, and $\mathfrak{H}$ is the kernel of $p$; but $p$ does not need to be surjective.

A counterexample, going back to the book [33, Proposition 11.1], can be also found in the book [2, Theorem 4.1.48]. It is explained in these books how to construct, for any separated topological abelian group $\mathfrak{Q}$, a complete, separated topological group structure on the direct sum $\mathfrak{A} = \bigoplus_{i=1}^{\infty} Q$ of a countable set of copies of $Q$ in such a way that the original topology on $Q$ is the quotient topology of the topology on $\mathfrak{A}$ with respect to the summation map $\mathfrak{A} \longrightarrow Q$. So $\mathfrak{A}$ is complete while $Q$ may be not. (See also [18, Problem 20D].) In a different context of topological vector spaces over the field of real numbers, in its real topology, a counterexample to completeness of quotients was suggested in [10, Exercise IV.4.10(b)]; see also [19, Proposition 11.2].) We refer to [28, Section 2] for a detailed discussion of this construction of counterexamples.

Furthermore, given a set $X$, we have the induced map of sets/abelian groups $p[[X]]: \mathfrak{A}[[X]] \longrightarrow \mathfrak{Q}[[X]]$. The subgroup $\mathfrak{H}[[X]] \subset \mathfrak{A}[[X]]$ is the kernel of $p[[X]]$. But even if the map $p$ is surjective (i.e., the topological group $Q$ is complete), the map $p[[X]]$ does not have to be surjective. Essentially, the problem consists in the following: given an $X$-indexed family of elements in the group $\mathfrak{Q}$ converging to zero in the topology of $\mathfrak{Q}$, how to lift it to an $X$-indexed family of elements in the group $\mathfrak{A}$ converging to zero in the topology of $\mathfrak{A}$?

Once again, the same construction from the books [33, 2] provides a counterexample. One can check that, for any separated topological abelian group $Q$, the topology on the group $\mathfrak{A} = \bigoplus_{i=1}^{\infty} Q$ constructed in [33, Proposition 11.1] and [2, Theorem 4.1.48], has the property that no infinite family of nonzero elements in $\mathfrak{A}$ converges to zero in $\mathfrak{A}$. This is essentially mentioned in the formulation of [33, Proposition 11.1]. So, generally speaking, the above-described lifting problem may be unsolvable. See [28, Example 11.2] for the details.

We will say that a closed subgroup $\mathfrak{H}$ in a complete, separated topological abelian group $\mathfrak{A}$ is strongly closed if the quotient group $\mathfrak{A}/\mathfrak{H}$ is complete and, for every set
\(X\), the induced map \(\mathbb{A}[[X]] \to (\mathbb{A}/\mathfrak{H})[[X]]\) is surjective. Clearly, any open subgroup in a complete, separated topological abelian group is strongly closed.

**Lemma 2.3.** Let \(\mathbb{A}\) be a complete, separated topological abelian group with a countable base of neighborhoods of zero consisting of open subgroups. Then any closed subgroup in \(\mathbb{A}\) is strongly closed.

**Proof.** This is a straightforward extension of Lemma 2.2(b). For a generalization (and some details), see [28, Proposition 11.6]. □

Notice that countability of a base of neighborhoods of zero in the topological quotient group \(Q = \mathbb{A}/\mathfrak{H}\) is not sufficient for the validity of Lemma 2.3, as the very same counterexample shows.

**Lemma 2.4.** Let \(\mathfrak{K} \subset \mathfrak{H} \subset \mathbb{A}\) be two embedded closed subgroups in a complete, separated topological abelian group \(\mathbb{A}\). In this situation,

1. if \(\mathfrak{K}\) is strongly closed in \(\mathbb{A}\), then \(\mathfrak{K}\) is strongly closed in \(\mathfrak{H}\);
2. if \(\mathfrak{K}\) is strongly closed in \(\mathbb{A}\) and \(\mathfrak{H}/\mathfrak{K}\) is strongly closed in \(\mathbb{A}/\mathfrak{K}\), then \(\mathfrak{H}\) is strongly closed in \(\mathbb{A}/\mathfrak{K}\);
3. if \(\mathfrak{H}\) is strongly closed in \(\mathbb{A}\) and \(\mathbb{A}/\mathfrak{K}\) is complete, then \(\mathfrak{H}/\mathfrak{K}\) is strongly closed in \(\mathbb{A}/\mathfrak{K}\);
4. if \(\mathfrak{H}\) is strongly closed in \(\mathbb{A}\) and \(\mathfrak{K}\) is strongly closed in \(\mathfrak{H}\), then \(\mathfrak{K}\) is strongly closed in \(\mathbb{A}\).

**Proof.** Part (a): one observes that for any closed subgroups \(\mathfrak{K} \subset \mathfrak{H} \subset \mathbb{A}\), the subgroup \(\mathfrak{H}/\mathfrak{K}\) is closed in \(\mathbb{A}/\mathfrak{K}\). Hence \(\mathfrak{H}/\mathfrak{K}\) is complete whenever \(\mathbb{A}/\mathfrak{K}\) is (see Lemma 2.1(b)). Furthermore, the square diagram of abelian groups \(\mathbb{A}[[X]] \to (\mathfrak{H}/\mathfrak{K})[[X]] \to (\mathfrak{H}/\mathfrak{K})[[X]] \to \mathbb{A}[[X]] \to (\mathbb{A}/\mathfrak{K})[[X]]\) is Cartesian for any set \(X\). Hence the morphism \(\mathbb{A}[[X]] \to (\mathbb{A}/\mathfrak{K})[[X]]\) is surjective whenever the morphism \(\mathbb{A}[[X]] \to (\mathfrak{H}/\mathfrak{K})[[X]]\) is.

In parts (b) and (c) one uses the isomorphism of topological groups \(\mathbb{A}/\mathfrak{H} \cong (\mathbb{A}/\mathfrak{K})/(\mathfrak{H}/\mathfrak{K})\) and commutativity of the triangle diagram \(\mathbb{A}[[X]] \to (\mathbb{A}/\mathfrak{K})[[X]] \to (\mathbb{A}/\mathfrak{H})[[X]]\).

Part (d): to prove that the topological group \(\mathbb{A}/\mathfrak{K}\) is complete, consider the commutative diagram of abelian groups

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathfrak{H}/\mathfrak{K} & \longrightarrow & \mathbb{A}/\mathfrak{K} & \longrightarrow & \mathbb{A}/\mathfrak{H} & \longrightarrow & 0 \\
& & \downarrow\cong & & \downarrow & & \downarrow\cong & & \\
0 & \longrightarrow & \lim\inf\mathfrak{H}/\mathfrak{K} \cap (\mathfrak{K} + \mathfrak{U}) & \longrightarrow & \lim\inf\mathbb{A}/(\mathfrak{K} + \mathfrak{U}) & \longrightarrow & \lim\inf\mathbb{A}/(\mathfrak{H} + \mathfrak{U}) & \longrightarrow & 0
\end{array}
\]

Here the projective limits in the lower row are taken over all the open subgroups \(\mathfrak{U}\) in \(\mathbb{A}\). The upper row is a short exact sequence. The lower row is left exact as the projective limit of short exact sequences \(0 \to \mathfrak{H}/\mathfrak{K} \cap (\mathfrak{K} + \mathfrak{U}) \to \mathbb{A}/(\mathfrak{K} + \mathfrak{U}) \to \mathbb{A}/(\mathfrak{H} + \mathfrak{U}) \to 0\). The leftmost and rightmost vertical arrows are isomorphisms, since the topological groups \(\mathfrak{H}/\mathfrak{K}\) and \(\mathbb{A}/\mathfrak{H}\) are complete by assumption. It follows
that the lower row is also a short exact sequence and the middle vertical arrow is an isomorphism.

To prove that the map \( A[[X]] \to (A/H)[[X]] \) is surjective for any set \( X \), consider the commutative diagram of abelian groups

\[
\begin{array}{cccccc}
0 & \to & H[[X]] & \to & A[[X]] & \to (A/H)[[X]] & \to 0 \\
& & \downarrow & & \downarrow & & \\
0 & \to & (H/S)[[X]] & \to (A/S)[[X]] & \to (A/S)[[X]] & \to 0 \\
\end{array}
\]

The upper row is obviously left exact; and in fact it is exact since the map \( A[[X]] \to (A/H)[[X]] \) is surjective by assumption. The lower row is also obviously left exact. The leftmost vertical arrow is surjective by assumption. It follows that the lower row is also a short exact sequence and the middle vertical arrow is surjective.

Alternatively, all the assertions of lemma follow from the existence of the strong exact structure on the additive category of complete, separated topological abelian groups; see [28, Theorem 11.5]. □

Example 2.5. A topological vector space over a field \( k \) is said to be pro-finite-dimensional (or pseudo-compact, or linearly compact) if it is isomorphic to the projective limit of a directed diagram of discrete finite-dimensional \( k \)-vector spaces, endowed with the projective limit topology. All pro-finite-dimensional topological vector spaces are complete and separated; a complete, separated topological vector space \( W \) is pro-finite-dimensional if and only if open vector subspaces of finite codimension form a base of neighborhoods of zero in \( W \). Any closed vector subspace of a pro-finite-dimensional vector space is pro-finite-dimensional in the induced topology, and the related quotient space is pro-finite-dimensional in the quotient topology.

The category of pro-finite-dimensional topological vector spaces (and continuous linear maps between them) is abelian; in fact, it is anti-equivalent to the category of discrete \( k \)-vector spaces. The anti-equivalence assigns to a discrete vector space \( V \) the pro-finite-dimensional vector space \( V^* = \text{Hom}_k(V, k) \) with the topology where the annihilators of finite-dimensional vector subspaces of \( V \) form a base of neighborhoods of zero. Conversely, to a pro-finite-dimensional vector space \( W \), the discrete vector space of all continuous linear maps \( W \to k \) is assigned.

It follows that any short exact sequence of pro-finite-dimensional vector spaces splits. In other words, any closed vector subspace in a pro-finite-dimensional vector space is a direct summand (in the category of topological vector spaces). Consequently, all closed vector subspaces in a pro-finite-dimensional topological vector space are strongly closed. Moreover, in any topological vector space with linear topology (i.e., with a base of neighborhoods of zero consisting of vector subspaces), any vector subspace which is pro-finite-dimensional in the induced topology is a direct summand in the category of topological vector spaces, and consequently, is strongly closed (see [28, end of Section 3]).

2.12. **Strongly closed two-sided ideals.** Let \( \mathfrak{R} \) be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Let \( \mathfrak{H} \subset \mathfrak{R} \)
be a closed two-sided ideal. Then the quotient ring $S = \mathcal{R}/\mathcal{H}$ in its quotient topology is a separated topological ring with a base of neighborhoods of zero formed by open right ideals. Hence the completion $\mathcal{G} = S^\sim$ is a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals.

The natural morphism $p: \mathcal{R} \rightarrow \mathcal{G}$ is a continuous homomorphism of complete, separated topological rings with the kernel $\mathcal{H}$, and the universal one with this property; but it needs not be surjective. If $\mathcal{R}$ has a base of neighborhoods of zero consisting of open two-sided ideals, then so do $S$ and $\mathcal{G}$.

The abelian category of discrete right $S$-modules or, equivalently, discrete right $\mathcal{G}$-modules is a full subcategory of the abelian category of discrete right $\mathcal{R}$-modules. In other words, the exact functor $p_*: \text{discr}-\mathcal{G} \rightarrow \text{discr}-\mathcal{R}$ is fully faithful. The full subcategory $\text{discr}-\mathcal{G} \subset \text{discr}-\mathcal{R}$ is closed under arbitrary subobjects, quotient objects, and coproducts.

For any discrete right $\mathcal{R}$-module $\mathcal{N}$, the discrete right $\mathcal{R}$-module structure on the submodule $\mathcal{N}_\mathcal{H} \subset \mathcal{N}$ comes from a discrete right $\mathcal{G}$-module structure. In other words, the discrete right $\mathcal{R}$-module $\mathcal{N}_\mathcal{H}$ belongs to the essential image of the functor $p_*: \text{discr}-\mathcal{G} \rightarrow \text{discr}-\mathcal{R}$. This is the maximal $\mathcal{R}$-submodule in $\mathcal{N}$ with this property. The functor $\mathcal{N} \mapsto \mathcal{N}_\mathcal{H}$ is the coextension-of-scalars functor with respect to the morphism $p: \mathcal{R} \rightarrow \mathcal{G}$, that is

$$p^*(\mathcal{N}) \cong \mathcal{N}_\mathcal{H} \quad \text{for all } \mathcal{N} \in \text{discr}-\mathcal{R}.$$ 

Now let us assume that $\mathcal{H} \subset \mathcal{R}$ is a strongly closed two-sided ideal. Then surjectivity of the maps $p[[X]]: \mathcal{R}[[X]] \rightarrow \mathcal{G}[[X]]$ for all sets $X$ implies that the exact functor of contrarestriction of scalars $p_\sharp: \mathcal{G}\text{-contra} \rightarrow \mathcal{R}\text{-contra}$ is fully faithful. So the abelian category $\mathcal{G}\text{-contra}$ is a full subcategory in the abelian category $\mathcal{R}\text{-contra}$. One easily observes that $\mathcal{G}\text{-contra}$ is closed under arbitrary subobjects, quotient objects, and products in $\mathcal{R}\text{-contra}$.

For any left $\mathcal{R}$-contramodule $\mathcal{C}$, the left $\mathcal{R}$-contramodule structure of the quotient contramodule $\mathcal{C}/(\mathcal{H} \times \mathcal{C})$ comes from a left $\mathcal{G}$-contramodule structure. In other words, the left $\mathcal{R}$-contramodule $\mathcal{C}/(\mathcal{H} \times \mathcal{C})$ belongs to the essential image of the functor $p_\sharp: \mathcal{G}\text{-contra} \rightarrow \mathcal{R}\text{-contra}$. This is the maximal quotient $\mathcal{R}$-contramodule of $\mathcal{C}$ with this property. The functor $\mathcal{C} \mapsto \mathcal{C}/(\mathcal{H} \times \mathcal{C})$ is the contraextension-of-scalars functor with respect to the morphism $p: \mathcal{R} \rightarrow \mathcal{G}$, that is

$$p^\circ(\mathcal{C}) \cong \mathcal{C}/(\mathcal{H} \times \mathcal{C}) \quad \text{for all } \mathcal{C} \in \mathcal{R}\text{-contra}.$$ 

When open two-sided ideals form a base of neighborhoods of zero in $\mathcal{R}$, one can compute the contratensor product $\mathcal{N} \circ_{\mathcal{R}} \mathcal{C}$ as

$$\mathcal{N} \circ_{\mathcal{R}} \mathcal{C} \cong \lim_{\mathcal{I}} \mathcal{N}_\mathcal{I} \circ_{\mathcal{R}} \mathcal{C} \cong \lim_{\mathcal{I}} \mathcal{N}_\mathcal{I} \circ_{\mathcal{R}/\mathcal{I}} (\mathcal{C}/\mathcal{J} \times \mathcal{C})$$

for any discrete right $\mathcal{R}$-module $\mathcal{N}$ and left $\mathcal{R}$-contramodule $\mathcal{C}$, where the inductive limits are taken over all the open two-sided ideals $\mathcal{I} \subset \mathcal{R}$ (cf. [22 Section D.2]).

2.13. **Example.** Let $A$ be an associative ring and $M$ be a left $A$-module. Consider the associative ring $\mathcal{R} = \text{Hom}_A(M,M)^{op}$ opposite to the ring of endomorphisms of the $A$-module $M$. Then the ring $A$ acts in $M$ on the left and the ring $\mathcal{R}$ acts in $M$.
on the right; so \( M \) is an \( A \mathcal{R} \)-bimodule. The following topology on the ring \( \mathcal{R} \) is known as the finite topology in the literature [16, Section IX.6], [13, Section 107] (it is implicit in Jacobson’s density theorem [15], [16, Section IX.13]).

For every finitely generated \( A \)-submodule \( E \subset M \), consider the subgroup \( \text{Ann}(E) = \text{Hom}_A(M/E, M) \subset \text{Hom}_A(M, M) \) consisting of all the endomorphisms of the \( A \)-module \( M \) which annihilate the submodule \( E \). Then \( \text{Ann}(E) \) is a left ideal in the ring \( \text{Hom}_A(M, M) \) and a right ideal in the ring \( \mathcal{R} \). Let \( \mathfrak{B} \) denote the set of all right ideals in \( \mathcal{R} \) of the form \( \text{Ann}(E) \), where \( E \) ranges over all the finitely generated submodules in \( M \). Then \( \mathfrak{B} \) is a base of a complete, separated topology compatible with the associative ring structure on \( \mathcal{R} \) [13, Theorem 107.1], [30, Theorem 7.1]. The right action of \( \mathcal{R} \) in \( M \) makes \( M \) a discrete right \( \mathcal{R} \)-module [30, Proposition 7.3].

This example plays a key role in the categorical tilting theory [30, 31], and it is also our intended example of a topological ring in the companion paper [6]. Besides the categories of left modules over an associative rings, there are also other/wider classes of additive categories \( A \) such that for any object \( M \in A \) there is a natural structure of a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals on the ring \( \mathcal{R} = \text{Hom}_A(M, M)\text{op} \). A detailed discussion of these can be found in [30, Sections 9–10] and [32, Section 3].

### 3. Flat Contramodules

The interactions of flatness with adic completion were studied by Yekutieli for ideals in Noetherian commutative rings [38] and in the greater generality of weakly proregular finitely generated ideals in commutative rings [39]. In the work of the present author, the theory of flat contramodules was developed for ideals in Noetherian commutative rings [21, Sections B.8–B.9], for centrally generated ideals in noncommutative Noetherian rings [22, Section C.5], for topological associative rings with a countable base of neighborhoods of zero formed by open two-sided ideals [22, Section D.1], and for topological rings with a countable base of neighborhoods of zero formed by open right ideals [29, Sections 5–7].

In this section, we obtain some very partial results for topological rings with an uncountable base of neighborhoods of zero.

Let \( \mathcal{R} \) be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. A left \( \mathcal{R} \)-contramodule \( \mathfrak{F} \) is called flat [29, Section 5] if the functor of contratensor product with \( \mathfrak{F} \)

\[
- \otimes_{\mathcal{R}} \mathfrak{F} : \text{discr–} \mathcal{R} \longrightarrow \mathbb{Z}–\text{mod}
\]

is exact as a functor from the abelian category of discrete right \( \mathcal{R} \)-modules to the category of abelian groups. The class of flat left \( \mathcal{R} \)-contramodules is closed under coproducts and direct limits in the category \( \mathcal{R}–\text{contra} \) [29, Lemma 5.6]. All projective left \( \mathcal{R} \)-contramodules are flat.

If \( \mathcal{I} \subset \mathcal{R} \) is an open two-sided ideal, then the left \( \mathcal{R}/\mathcal{I} \)-module \( \mathfrak{F}/\mathcal{I} \times \mathfrak{F} \) is flat for any flat left \( \mathcal{R} \)-contramodule \( \mathfrak{F} \). Indeed, the functor \(- \otimes_{\mathcal{R}/\mathcal{I}} (\mathfrak{F}/\mathcal{I} \times \mathfrak{F}) : \text{mod–} \mathcal{R}/\mathcal{I} \longrightarrow \text{mod–} \mathcal{R}/\mathcal{I} \times \mathcal{R}/\mathcal{I} \)
\( \mathbb{Z} \text{-mod} \) is exact, because there is a natural isomorphism
\[
N \otimes_{\mathcal{R}/\mathfrak{I}} (\mathfrak{J} \otimes \mathfrak{F}) \cong N \otimes_{\mathfrak{I}} \mathfrak{F}
\]
for all right \( \mathcal{R}/\mathfrak{I} \)-modules \( N \).
If open two-sided ideals form a base of neighborhoods of zero in \( \mathcal{R} \), then the converse assertion also holds: a left \( \mathcal{R} \)-contramodule \( \mathfrak{F} \) is flat if and only if the left \( \mathcal{R}/\mathfrak{I} \)-module \( \mathfrak{J}/\mathfrak{I} \otimes \mathfrak{F} \) is flat for every open two-sided ideal \( \mathfrak{J} \subset \mathcal{R} \). (Cf. Sections 2.10 and 2.12.)

The left derived functor
\[
\text{Ctrtor}_i: \text{discr-\mathcal{R} \times \mathcal{R}-contra} \longrightarrow \mathbb{Z} \text{-mod}
\]
is constructed using projective resolutions of the second (contramodule) argument. So, if \( \cdots \rightarrow \mathfrak{P}_2 \rightarrow \mathfrak{P}_1 \rightarrow \mathfrak{P}_0 \rightarrow \mathfrak{C} \rightarrow 0 \) is an exact complex in the abelian category \( \mathcal{R} \text{-contra} \) and \( \mathfrak{P}_i \) are projective left \( \mathcal{R} \)-contramodules for all \( i \geq 0 \), then
\[
\text{Ctrtor}_i^\mathcal{R}(N, \mathfrak{C}) = H_i(N \otimes_{\mathcal{R}} \mathfrak{P}_*) \quad \text{for all } N \in \text{discr-\mathcal{R} \text{ and } } i \geq 0.
\]

As always with derived functors of one argument, for any short exact sequence of left \( \mathcal{R} \)-contramodules \( 0 \rightarrow \mathfrak{A} \rightarrow \mathfrak{B} \rightarrow \mathfrak{C} \rightarrow 0 \) and any discrete right \( \mathcal{R} \)-module \( N \) there is a natural long exact sequence of abelian groups
\[
\cdots \rightarrow \text{Ctrtor}^\mathcal{R}_{i+1}(N, \mathfrak{C}) \rightarrow \text{Ctrtor}^\mathcal{R}_i(N, \mathfrak{A}) \rightarrow \text{Ctrtor}^\mathcal{R}_i(N, \mathfrak{B}) \rightarrow \text{Ctrtor}^\mathcal{R}_i(N, \mathfrak{C}) \rightarrow \cdots
\]

Since the functor of contratensor product \( N \otimes_{\mathcal{R}} - : \mathcal{R} \text{-contra} \rightarrow \mathbb{Z} \text{-mod} \) is right exact on the abelian category \( \mathcal{R} \text{-contra} \) for every \( N \in \text{discr-\mathcal{R} \text{, one has} \quad \text{Ctrtor}_0^\mathcal{R}(N, \mathfrak{C}) = N \otimes_{\mathcal{R}} \mathfrak{C}. \}
\]

Furthermore, for any short exact sequence of discrete right \( \mathcal{R} \)-modules \( 0 \rightarrow \mathcal{L} \rightarrow \mathcal{M} \rightarrow \mathcal{N} \rightarrow 0 \) and any complex of projective left \( \mathcal{R} \)-contramodules \( \mathfrak{P}_* \), the short sequence of complexes of abelian groups \( 0 \rightarrow \mathcal{L} \otimes_{\mathcal{R}} \mathfrak{P}_* \rightarrow \mathcal{M} \otimes_{\mathcal{R}} \mathfrak{P}_* \rightarrow \mathcal{N} \otimes_{\mathcal{R}} \mathfrak{P}_* \rightarrow 0 \) is exact (because projective left \( \mathcal{R} \)-contramodules are flat). Therefore, for any left \( \mathcal{R} \)-contramodule \( \mathfrak{C} \) there is a long exact sequence of abelian groups
\[
\cdots \rightarrow \text{Ctrtor}^\mathcal{R}_{i+1}(N, \mathfrak{C}) \rightarrow \text{Ctrtor}^\mathcal{R}_i(\mathcal{L}, \mathfrak{C}) \rightarrow \text{Ctrtor}^\mathcal{R}_i(\mathcal{M}, \mathfrak{C}) \rightarrow \text{Ctrtor}^\mathcal{R}_i(N, \mathfrak{C}) \rightarrow \cdots
\]

We will say that a left \( \mathcal{R} \)-contramodule \( \mathfrak{F} \) is 1-strictly flat if \( \text{Ctrtor}_1^\mathcal{R}(N, \mathfrak{F}) = 0 \) for all discrete right \( \mathcal{R} \)-modules \( N \). More generally, a left \( \mathcal{R} \)-contramodule \( \mathfrak{F} \) is \( n \)-strictly flat if \( \text{Ctrtor}_n^\mathcal{R}(N, \mathfrak{F}) = 0 \) for all discrete right \( \mathcal{R} \)-modules \( N \) and all \( 1 \leq i \leq n \). A left \( \mathcal{R} \)-contramodule \( \mathfrak{F} \) is \( \infty \)-strictly flat if it is \( n \)-strictly flat for all \( n > 0 \).

Clearly, all projective left \( \mathcal{R} \)-contramodules are \( \infty \)-strictly flat. It follows from the exact sequence [3] that the class of all \( n \)-strictly flat left \( \mathcal{R} \)-contramodules is closed under extensions in \( \mathcal{R} \text{-contra} \) for every \( n \geq 1 \), and that the class of all \( \infty \)-strictly flat left \( \mathcal{R} \)-contramodules is closed under (extensions and) the passage to the kernels of surjective morphisms. Given some \( n \geq 1 \), the class of all \( n \)-strictly flat left \( \mathcal{R} \)-contramodules is closed under the kernels of surjective morphisms if and only if it coincides with the class of all \( \infty \)-strictly flat left \( \mathcal{R} \)-contramodules.
From the exact sequence (1) one can conclude that every 1-strictly flat left \( R \)-contramodule is flat. According to [20] proof of Lemma 6.10, Remark 6.11 and Corollary 6.15, when the topological ring \( R \) has a countable base of neighborhoods of zero (formed by open right ideals), the classes of flat, 1-strictly flat, and \( \infty \)-strictly flat left \( R \)-contramodules coincide.

Let us say that a short exact sequence of left \( R \)-contramodules \( 0 \to A \to B \to C \to 0 \) is contratensor pure if the induced sequence \( 0 \to N \circ_\mathcal{R} A \to N \circ_\mathcal{R} B \to N \circ_\mathcal{R} C \to 0 \) is exact (i.e., the map \( N \circ_\mathcal{R} A \to N \circ_\mathcal{R} B \) is injective) for every discrete right \( R \)-module \( N \). If the left \( R \)-contramodule \( B \) is 1-strictly flat, then the sequence \( 0 \to A \to B \to C \to 0 \) is contratensor pure if and only if the left \( R \)-contramodule \( C \) is 1-strictly flat.

**Lemma 3.1.** The class of all 1-strictly flat left \( R \)-contramodules is closed under infinite coproducts in \( \mathcal{R} \)-contra.

**Proof.** Let \((\mathcal{F}_\alpha)_\alpha\) be a family of 1-strictly flat left \( R \)-contramodules. Choose short exact sequences of left \( R \)-contramodules \( 0 \to K_\alpha \to \mathcal{F}_\alpha \to \mathcal{F}_\alpha \to 0 \), where \( \mathcal{F}_\alpha \) are projective left \( R \)-contramodules. Then the sequence \( \coprod_\alpha K_\alpha \to \coprod_\alpha \mathcal{F}_\alpha \to 0 \) (where the coproducts are taken in \( \mathcal{R} \)-contramodules) is exact, as the functors of coproduct are right exact in any abelian category. Therefore, there is a natural surjective \( R \)-contramodule morphism from \( \coprod_\alpha K_\alpha \) onto the kernel \( K \) of the morphism \( \coprod_\alpha \mathcal{F}_\alpha \to 0 \). The contratensor product functor \( \circ_\mathcal{R} \) preserves colimits, hence for any discrete right \( R \)-module \( N \) the morphism \( N \circ_\mathcal{R} \coprod_\alpha K_\alpha \to N \circ_\mathcal{R} \coprod_\alpha \mathcal{F}_\alpha \) is injective (being isomorphic to the morphism \( \coprod_\alpha N \circ_\mathcal{R} K_\alpha \to \coprod_\alpha N \circ_\mathcal{R} \mathcal{F}_\alpha \), where the coproducts are taken in the category of abelian groups). At the same time, the morphism \( N \circ_\mathcal{R} \coprod_\alpha K_\alpha \to N \circ_\mathcal{R} K \) is surjective. It follows that the morphism \( N \circ_\mathcal{R} \coprod_\alpha \mathcal{F}_\alpha \to N \circ_\mathcal{R} K \) is an isomorphism and the morphism \( N \circ_\mathcal{R} K \to N \circ_\mathcal{R} \coprod_\alpha \mathcal{F}_\alpha \) is injective, that is, the short exact sequence \( 0 \to K \to \coprod_\alpha \mathcal{F}_\alpha \to 0 \) is contratensor pure. Since the left \( R \)-contramodule \( \coprod_\alpha \mathcal{F}_\alpha \) is projective, it follows that the left \( R \)-contramodule \( \coprod_\alpha \mathcal{F}_\alpha \) is 1-strictly flat. \( \square \)

**Lemma 3.2.** The class of all 1-strictly flat left \( R \)-contramodules is closed under countable direct limits in \( \mathcal{R} \)-contra.

**Proof.** Let \( \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to \cdots \) be a sequence of left \( R \)-contramodules and \( R \)-contramodule morphisms between them. Then the direct limit \( \varinjlim_n \mathcal{F}_n \) is the cokernel of the morphism \( \text{id} - \text{shift}: \coprod_{n=1}^\infty \mathcal{F}_n \to \coprod_{n=1}^\infty \mathcal{F}_n \). Denote the image of this morphism by \( \mathcal{L} \). Arguing as in the proof of Lemma 3.1, we have a surjective morphism \( \coprod_{n=1}^\infty \mathcal{F}_n \to \mathcal{L} \) and an exact sequence \( 0 \to \mathcal{L} \to \coprod_{n=1}^\infty \mathcal{F}_n \to \varinjlim_n \mathcal{F}_n \to 0 \). The morphism \( N \circ_\mathcal{R} \coprod_{n=1}^\infty \mathcal{F}_n \to N \circ_\mathcal{R} \coprod_{n=1}^\infty \mathcal{F}_n \) is injective (being isomorphic to the morphism \( \coprod_{n=1}^\infty N \circ_\mathcal{R} \mathcal{F}_n \to \coprod_{n=1}^\infty N \circ_\mathcal{R} \mathcal{F}_n \)) for every discrete right \( R \)-module \( N \). At the same time, the morphism \( N \circ_\mathcal{R} \coprod_{n=1}^\infty \mathcal{F}_n \to N \circ_\mathcal{R} \mathcal{L} \) is surjective. It follows that the morphism \( N \circ_\mathcal{R} \coprod_{n=1}^\infty \mathcal{F}_n \to N \circ_\mathcal{R} \mathcal{L} \) is an isomorphism and the morphism \( N \circ_\mathcal{R} \mathcal{L} \to N \circ_\mathcal{R} \coprod_{n=1}^\infty \mathcal{F}_n \) is injective, that is, the short exact sequence \( 0 \to \mathcal{L} \to \coprod_{n=1}^\infty \mathcal{F}_n \to \varinjlim_n \mathcal{F}_n \to 0 \) is contratensor pure. Now, if the left \( R \)-contramodules \( \mathcal{F}_n \) are 1-strictly flat for all \( n \), then...
n ≥ 1, then the left $\mathcal{R}$-contramodule $\coprod_n f_n$ is 1-strictly flat by Lemma 3.1, and it follows that the left $\mathcal{R}$-contramodule $\lim_{\to n} f_n$ is also 1-strictly flat.

In fact, the class of all 1-strictly flat left $\mathcal{R}$-contramodules is closed under all direct limits in $\mathcal{R}$–contra. This is the result of [7, Corollary 7.1].

A left $\mathcal{R}$-contramodule $C$ is said to be separated if the intersection of its subgroups $I ∖ C$, with $I ⊂ R$ ranging over all the open right ideals in $\mathcal{R}$, vanishes: $\bigcap_{I ⊂ R} I ∖ C = 0$. Following the discussion at the end of Section 2.10, all projective left $\mathcal{R}$-contramodules are separated. Over a topological ring with a countable base of neighborhoods of zero, all flat contramodules are separated [29, Corollary 6.15].

**Lemma 3.3.** Let $\mathfrak{F}_1 → \mathfrak{F}_2 → \mathfrak{F}_3 → \cdots$ be a sequence of left $\mathcal{R}$-contramodules and contramodule morphisms between them. Assume that the left $\mathcal{R}$-contramodule $\coprod_{n=1}^{\infty} \mathfrak{F}_n$ is separated. Then the natural short sequence of left $\mathcal{R}$-contramodules

\[
0 \to \coprod_{n=1}^{\infty} \mathfrak{F}_n \to \coprod_{n=1}^{\infty} \mathfrak{F}_n \to \lim_{\to n \geq 1} \mathfrak{F}_n \to 0
\]

is exact.

**Proof.** In any additive category with countable coproducts, the short sequence (5) is always right exact (cf. the previous proof). The nontrivial assertion is that the left $\mathcal{R}$-contramodule morphism $\text{id} - \text{shift} : \mathfrak{F} → \mathfrak{F}$, where we put $\mathfrak{F} = \coprod_{n=1}^{\infty} \mathfrak{F}_n$, is injective in the assumptions of the lemma. Indeed, let $N$ be a discrete right $\mathcal{R}$-module. Applying to (5) the contratensor product functor $N \otimes_{\mathcal{R}} -$ , we obtain the short sequence of abelian groups

\[
0 \to \coprod_{n=1}^{\infty} N \otimes_{\mathcal{R}} \mathfrak{F}_n \to \coprod_{n=1}^{\infty} N \otimes_{\mathcal{R}} \mathfrak{F}_n \to \lim_{\to n \geq 1} N \otimes_{\mathcal{R}} \mathfrak{F}_n \to 0,
\]

which is exact since direct limits are exact functors in the category of abelian groups. So the abelian group homomorphism $N \otimes_{\mathcal{R}} (\text{id} - \text{shift})$ is injective for any discrete right $\mathcal{R}$-module $N$. In particular, let $I ⊂ \mathcal{R}$ be an open right ideal. Then the morphism $\mathcal{R}/I \otimes_{\mathcal{R}} (\text{id} - \text{shift}) : \mathfrak{F}/I ∖ \mathfrak{F} \to \mathfrak{F}/I ∖ \mathfrak{F}$ is injective. It follows that the kernel of the morphism $\text{id} - \text{shift}$ is contained in the subgroup (in fact, always a left $\mathcal{R}$-subcontramodule) $\bigcap_{I ⊂ \mathcal{R}} I ∖ \mathfrak{F} ⊂ \mathfrak{F}$.

Before formulating the next corollary, we notice that, if a left $\mathcal{R}$-contramodule $\mathfrak{F}$ has projective dimension not exceeding $n$ (as an object of the abelian category $\mathcal{R}–contra$) and $\mathfrak{F}$ is $n$-strictly flat, then $\mathfrak{F}$ is also $\infty$-strictly flat.

**Corollary 3.4.** Any countable direct limit of projective left $\mathcal{R}$-contramodules has projective dimension not exceeding 1 in $\mathcal{R}$–contra. In particular, any such $\mathcal{R}$-contramodule is $\infty$-strictly flat.

**Proof.** The first assertion is a corollary of Lemma 3.3. Indeed, a coproduct of projective contramodules is projective, and any projective contramodule is separated; so the exact sequence (5) is a desired projective resolution. The second assertion follows immediately from the first one together with Lemma 3.2. □
4. Projective Covers of Flat Contramodules

Let $\mathcal{B}$ be an abelian category with enough projective objects. An epimorphism $p: P \rightarrow C$ in $\mathcal{B}$ is called a projective cover (of the object $C$) if the object $P$ is projective and, for any endomorphism $e: P \rightarrow P$, the equation $pe = p$ implies that $e$ is an automorphism of $P$ (i.e., $e$ is invertible).

A subobject $K$ of an object $Q \in \mathcal{B}$ is said to be superfluous if, for any other subobject $G \subset Q$, the equation $K + G = Q$ implies that $G = Q$. If a subobject $K \subset Q$ is superfluous, then, for any subobject $E \subset Q$, the quotient $(K + E)/E$ is a superfluous subobject of the quotient $Q/E$.

**Lemma 4.1.** Let $P \in \mathcal{B}$ be a projective object. Then an epimorphism $p: P \rightarrow C$ in $\mathcal{B}$ is a projective cover if and only if its kernel $K$ is a superfluous subobject in $P$.

**Proof.** Let $p: P \rightarrow C$ be a projective cover with the kernel $K$, and let $G \subset P$ be a subobject such that $K + G = P$. Then the restriction of $p$ onto $G$ is an epimorphism $s: G \rightarrow C$. Since $P$ is projective, there exists a morphism $f: P \rightarrow G$ making the triangle diagram $P \rightarrow G \rightarrow C$ commutative. Let $e: P \rightarrow P$ be the composition of the morphism $f$ with the embedding $G \rightarrow P$. Then $pe = p$, and by assumption it follows that $e$ is invertible. Hence $G = P$.

Conversely, let $p: P \rightarrow C$ be an epimorphism with a superfluous kernel $K \subset P$, and let $e: P \rightarrow P$ be an endomorphism satisfying $pe = p$. Let $G \subset P$ be the image of $e$; then $K + G = P$. By assumption, it follows that $G = P$, so $e$ is an epimorphism. Then, since $P$ is projective, the kernel $L$ of $e$ must be a direct summand of $P$. Denote by $E \subset P$ a complementary direct summand. The equation $pe = p$ implies that $L \subset K$, hence $K + E = P$. Again by assumption, it follows that $E = P$, so $L = 0$ and $e$ is an automorphism of $P$. □

**Proposition 4.2.** Let $R$ be an associative ring. Then any flat $R$-module that has a projective cover is projective.

**Proof.** A proof of this result can be found in [37, Section 36.3]. □

We refer to Sections 2.10 and 2.12 for the discussion of reductions of contramodules modulo strongly closed ideals.

**Lemma 4.3.** Let $\mathcal{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals, $\mathfrak{J}$ be a strongly closed two-sided ideal in $\mathcal{R}$, and $\mathfrak{T} = \mathcal{R}/\mathfrak{J}$ be the topological quotient ring. Assume that a left $\mathcal{R}$-contramodule $\mathfrak{C}$ has a projective cover $p: \mathfrak{P} \rightarrow \mathfrak{C}$ in $\mathcal{R}$-contra. Then the induced map $\overline{p}: \mathfrak{P}/\mathfrak{J} \otimes \mathfrak{P} \rightarrow \mathfrak{C}/\mathfrak{J} \times \mathfrak{C}$ is a projective cover of the left $\mathfrak{T}$-contramodule $\mathfrak{C}/\mathfrak{J} \times \mathfrak{C}$.

**Proof.** Set $\mathfrak{K} = \ker(p)$. Then $0 \rightarrow \mathfrak{K} \rightarrow \mathfrak{P} \rightarrow \mathfrak{C} \rightarrow 0$ is a short exact sequence in $\mathcal{R}$-contra and $0 \rightarrow \mathfrak{K}/(\mathfrak{J} \otimes \mathfrak{P}) \cap \mathfrak{K} \rightarrow \mathfrak{P}/\mathfrak{J} \otimes \mathfrak{P} \rightarrow \mathfrak{C}/\mathfrak{J} \times \mathfrak{C} \rightarrow 0$ is a short exact sequence in $\mathfrak{T}$-contra. The left $\mathfrak{T}$-contramodule $\mathfrak{P}/\mathfrak{J} \otimes \mathfrak{P}$ is projective, since the left $\mathcal{R}$-contramodule $\mathfrak{P}$ is; and the $\mathfrak{T}$-subcontramodule $\mathfrak{K}/(\mathfrak{J} \otimes \mathfrak{P}) \cap \mathfrak{K} \subset \mathfrak{P}/\mathfrak{J} \otimes \mathfrak{P}$ is superfluous, since the $\mathcal{R}$-subcontramodule $\mathfrak{K} \subset \mathfrak{P}$ is. □
Proposition 4.4. Let $\mathcal{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open two-sided ideals, and let $\mathfrak{F}$ be a 1-strictly flat left $\mathcal{R}$-contramodule. Assume that $\mathfrak{F}$ has a projective cover $p : \mathfrak{P} \to \mathfrak{F}$ in $\mathcal{R}$-contra.

Then the map $p$ is an isomorphism and the $\mathcal{R}$-contramodule $\mathfrak{F} \cong \mathfrak{P}$ is projective.

Proof. Let $\mathcal{L} \subset \mathfrak{F}$ be the kernel of the epimorphism $p$. For any open two-sided ideal $\mathfrak{I} \subset \mathcal{R}$, we have a short sequence of left $\mathcal{R}/\mathfrak{I}$-modules $0 \to \mathcal{L}/\mathfrak{I} \to \mathfrak{P}/\mathfrak{I} \to \mathfrak{F}/\mathfrak{I} \to 0$, which is exact since $\text{Cttr}^{\mathcal{R}}_{1}(\mathcal{R}/\mathfrak{I}, \mathfrak{F}) = 0$. The left $\mathcal{R}/\mathfrak{I}$-module $\mathfrak{L}/\mathfrak{I}$ is flat, since the left $\mathcal{R}$-contramodule $\mathfrak{F}$ is. Furthermore, the morphism $\mathfrak{P}/\mathfrak{I} \to \mathfrak{F}/\mathfrak{I}$ is a projective cover in the category of left $\mathcal{R}/\mathfrak{I}$-modules by Lemma 4.3 (applied in the particular case of an open two-sided ideal $\mathfrak{I}$ and a discrete quotient ring $T = \mathcal{R}/\mathfrak{I}$). Using Proposition 4.2, we conclude that the $\mathcal{R}/\mathfrak{I}$-module $\mathfrak{F}/\mathfrak{I}$ is projective and $\mathfrak{L}/\mathfrak{I} = 0$ for every open two-sided ideal $\mathfrak{I} \subset \mathcal{R}$. Since $\mathfrak{P}$ is a projective left $\mathcal{R}$-contramodule, one has $\mathfrak{P} = \lim \leftarrow \mathfrak{P}/\mathfrak{I} \to \mathfrak{F}/\mathfrak{I}$ (see Section 2.10), hence $\mathcal{L} = \bigcap \mathfrak{I} \to \mathcal{L} \subset \bigcap \mathfrak{I} \to \mathfrak{P} = 0$. □

With the additional assumption that the projective dimension of the left $\mathcal{R}$-contramodule $\mathfrak{F}$ does not exceed 1, the result of Proposition 4.4 can be extended to complete, separated topological rings with a base of neighborhoods of zero formed by open right ideals. This is the assertion of [7, Theorem 3.1].

5. Bass Flat Contramodules

Let $a_1, a_2, a_3, \ldots$ be a sequence of elements in the topological ring $\mathcal{R}$. For every $n \geq 1$, the multiplication by $a_n$ on the right is a left $\mathcal{R}$-contramodule morphism $\mathcal{R} \to \mathcal{R}$ (where $\mathcal{R}$ is viewed as a free left $\mathcal{R}$-contramodule with one generator). The direct limit

$$
\mathfrak{B} = \lim \leftarrow (\mathcal{R} \xrightarrow{a_1} \mathcal{R} \xrightarrow{a_2} \mathcal{R} \xrightarrow{a_3} \ldots)
$$

is called the Bass flat left $\mathcal{R}$-contramodule associated with the sequence of elements $(a_n \in \mathcal{R})_{n \geq 1}$. According to Corollary 3.4, the Bass flat left $\mathcal{R}$-contramodules are $\infty$-strictly flat and have projective dimension not exceeding 1.

In particular, when the topological ring $\mathcal{R} = \mathcal{R}$ is discrete, the above construction specializes to the classical definition of a Bass flat left $\mathcal{R}$-module.

Lemma 5.1. If all Bass flat left modules over an associative ring $\mathcal{R}$ are projective, then all flat left $\mathcal{R}$-modules are projective (i. e., the ring $\mathcal{R}$ is left perfect).

Proof. Clear from the proof of the implication (5) $\Rightarrow$ (6) in [4, Theorem P], which only uses projectivity of the Bass flat modules. (For a more recent exposition, see [1, proof of Theorem 28.4 (d) $\Rightarrow$ (e)].) □

Lemma 5.2. Let $\mathcal{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Let $a_1, a_2, a_3, \ldots$ be a sequence of elements in $\mathcal{R}$, and let $\mathfrak{B}$ be the related Bass flat left $\mathcal{R}$-contramodule. Then one has $\mathfrak{B} = 0$ if and only if for every $m \geq 1$ the sequence of elements $a_m, a_m a_{m+1}, \ldots, a_m a_{m+1} \cdots a_n, \ldots$ converges to zero in the topology of $\mathcal{R}$ as $n \to \infty$. 22
Proof. The left $\mathcal{R}$-contramodule $\mathcal{B}$ can be constructed as the cokernel of the left $\mathcal{R}$-contramodule morphism $id - shift: \coprod_{n=1}^{\infty} \mathcal{R} \to \coprod_{n=1}^{\infty} \mathcal{R}$. In other words, $\mathcal{B}$ is the cokernel of the morphism of free left $\mathcal{R}$-contramodules
\[
f = f_{a_1, a_2, a_3, \ldots} : \mathcal{R}[y_1, y_2, y_3, \ldots] \to \mathcal{R}[x_1, x_2, x_3, \ldots]
\]
defined on the generators by the rule $f(y_n) = x_n - a_n x_{n+1}$. According to Lemma \[\text{3.3}\], the morphism $f$ is injective (but we will not need to use this fact).

One has $\mathcal{B} = 0$ if and only if the image of $x_m$ in $\mathcal{B}$ vanishes for every $m \geq 1$. We will show that the image of $x_m$ in $\mathcal{B}$ vanishes if and only if the sequence of elements $a_m, a_m a_{m+1}, \ldots, a_m a_{m+1} \cdots a_n, \ldots$ converges to zero in $\mathcal{R}$. More generally, given an element $r \in \mathcal{R}$, the image of the element $r x_m$ under the map $\mathcal{R}[[x_1, x_2, x_3, \ldots]] \to \mathcal{B}$ vanishes if and only if the sequence of elements $r a_m, r a_m a_{m+1}, \ldots, r a_m a_{m+1} \cdots a_n, \ldots$ converges to zero in $\mathcal{R}$ as $n \to \infty$ (cf. \[\text{3.1}\] and \[\text{3.2}\]).

“If”: assuming that the sequence $r a_m, r a_m a_{m+1}, \ldots$ converges to zero, we have to show that the element $r x_m$ belongs to the image of the morphism $f$. Indeed, one has
\[
\sum_{n=m}^{\infty} r a_m \cdots a_{n-1} y_n = r y_m + r a_m y_{m+1} + r a_m a_{m+1} y_{m+2} + \cdots \in \mathcal{R}[[y_1, y_2, y_3, \ldots]]
\]
and
\[
f \left( \sum_{n=m}^{\infty} r a_m \cdots a_{n-1} y_n \right) = \sum_{n=m}^{\infty} r a_m \cdots a_{n-1} (x_n - a_n x_{n+1})
= \sum_{n=m}^{\infty} r a_m \cdots a_{n-1} x_n - \sum_{n=m}^{\infty} r a_m \cdots a_n x_{n+1} = r x_m.
\]

“Only if”: assume that $r x_m \in \text{im } f$; so there exists an element $z = \sum_{n=1}^{\infty} b_n y_n \in \mathcal{R}[[y_1, y_2, y_3, \ldots]]$ such that $f(z) = r x_m$. Since $z \in \mathcal{R}[[y_1, y_2, y_3, \ldots]]$, the sequence of elements $b_n \in \mathcal{R}$ converges to zero as $n \to \infty$. On the other hand, the equation $f(z) = r x_m$ means that $b_1 = \delta_{m,1} r$ and $b_{n+1} - b_n a_n = \delta_{m,n+1} r$ for $n \geq 1$, where $\delta_{i,j}$ is the Kronecker delta symbol. Hence $b_1 = \cdots = b_{m-1} = 0$ and $b_m = r$, $b_{m+1} = r a_m$, $b_{m+2} = r a_m a_{m+1}$, $\ldots$, $b_n = r a_m \cdots a_{n-1}$ for $n \geq m$.

The following proposition, which is the main result of this section, is the contramodule version of \[\text{3.1}\] (see also \[\text{3.2}\]).

Proposition 5.3. Let $\mathcal{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals, and let $a_1, a_2, a_3, \ldots$ be a sequence of elements in $\mathcal{R}$. Assume that the related Bass flat left $\mathcal{R}$-contramodule $\mathcal{B}$ is a projective left $\mathcal{R}$-contramodule. Then for any open right ideal $\mathcal{I} \subset \mathcal{R}$ the chain of right ideals $\mathcal{R} \supset a_1 \mathcal{R} \supset a_1 a_2 \mathcal{R} \supset a_1 a_2 a_3 \mathcal{R} \supset \cdots$ terminates.

Proof. In the notation of the previous proof, set $\mathcal{P} = \mathcal{R}[[x_1, x_2, x_3, \ldots]]$. Then we have a surjective left $\mathcal{R}$-contramodule morphism $\rho: \mathcal{P} \to \mathcal{B}$. Assuming that $\mathcal{B}$ is a projective left $\mathcal{R}$-contramodule, $\rho$ has a section, i. e., there exists a left $\mathcal{R}$-contramodule morphism $\sigma: \mathcal{B} \to \mathcal{P}$ such that $\rho \sigma = \text{id}_{\mathcal{P}}$.

Following \[\text{27}\] Proposition 4.2(b)], we assign to every left $\mathcal{R}$-contramodule $\mathcal{C}$ the functor of contratempered product $\text{CT}(\mathcal{C}) = - \otimes_{\mathcal{R}} \mathcal{C}$ acting from the category of cyclic discrete right $\mathcal{R}$-modules to the category of abelian groups. So we have $\text{CT}(\mathcal{C}) (\mathcal{R}/\mathcal{I}) = (\mathcal{R}/\mathcal{I}) \otimes_{\mathcal{R}} \mathcal{C} = \mathcal{C} / \mathcal{I} \wedge \mathcal{C}$ for any open right ideal $\mathcal{I} \subset \mathcal{R}$. In particular, $\text{CT}(\mathcal{P}) (\mathcal{R}/\mathcal{I}) = (\mathcal{R}/\mathcal{I})[x_1, x_2, x_3, \ldots]$ is naturally the direct sum of a countable set of
copies of the abelian group \( R/I \). We will denote formally the elements of the group \( \text{CT}(\mathfrak{B})(R/I) \) by expressions like \( \sum_{m=1}^{n} r_m x_m \), where \( r_m \in R/I \). Here the sum is a direct sum, of course.

Similarly, the abelian group \( \text{CT}(\mathfrak{B})(R/I) \) can be computed as the direct limit

\[
R/I \circ \mathfrak{B} = \lim_{\longrightarrow} \left( \frac{R/I}{s_{a_1}} \xrightarrow{R/I \xrightarrow{s_{a_2}}} \frac{R/I}{s_{a_3}} \xrightarrow{\cdots} \right)
\]

of the sequence of the right \( R \)-modules \( R/I \) and the maps of right multiplication by the elements \( a_n \) acting between them. We will denote the elements of the groups \( R/I \) in this sequence by \( r z_n \), where \( r \in R/I \) and \( z_n \) is a formal symbol; so the transition map takes \( r z_n \) to \( r a_n z_{n+1} \). Let \( r \hat{z}_n \in \text{CT}(\mathfrak{B})(R/I) \) be the notation for the images of the elements \( r z_n \) in the direct limit. Then the equations \( r \hat{z}_n = r a_n \hat{z}_{n+1} \) hold in the abelian group \( \text{CT}(\mathfrak{B})(R/I) \) for all \( n \geq 1 \) and \( r \in R/I \).

Applying the functor \( \text{CT} \) to the morphisms \( \rho \) and \( \sigma \), we obtain the abelian group homomorphism \( \rho_\sigma : \text{CT}(\mathfrak{P})(R/I) \to \text{CT}(\mathfrak{B})(R/I) \) taking \( r_n x_n \) to \( r_n \hat{z}_n \), and its section \( \sigma_\mathfrak{P} : \text{CT}(\mathfrak{B})(R/I) \to \text{CT}(\mathfrak{P})(R/I) \) (so \( \rho_\mathfrak{P} \sigma_\mathfrak{P} = \text{id} \)). Furthermore, for any two open right ideals \( I \) and \( J \subset R \) and an element \( s \in R \) such that \( s I \subset J \), there is a morphism of discrete right \( R \)-modules \( R/I \xrightarrow{s} R/I \) taking a coset \( t+J \in R/I \), \( t \in R \) to the coset \( st+J \in R/I \). Applying the functor \( \text{CT}(\mathfrak{P}) \) to the morphism \( s \) produces the abelian group homomorphism \( \text{CT}(\mathfrak{P})(s) : \text{CT}(\mathfrak{P})(R/I) \to \text{CT}(\mathfrak{P})(R/I) \) taking an element \( \sum_{m=1}^{n} t_m x_m \) to the element \( \sum_{m=1}^{n} s_m t_m x_m \), where \( t_m \in R/I \) and \( s_m \in R/I \). Similarly, applying the functor \( \text{CT}(\mathfrak{B}) \) to the morphism \( s \) we get the abelian group homomorphism \( \text{CT}(\mathfrak{B})(s) : \text{CT}(\mathfrak{B})(R/I) \to \text{CT}(\mathfrak{B})(R/I) \) taking \( t \hat{z}_n \) to \( s \hat{z}_n \) for all \( t \in R/I \). Since \( \text{CT}(\rho) \) and \( \text{CT}(\sigma) \) are morphisms of functors, the maps \( \text{CT}(\mathfrak{P})(s) \) and \( \text{CT}(\mathfrak{B})(s) \) form a commutative square diagram with the maps \( \rho_\sigma \) and \( \rho_\mathfrak{P} \sigma_\mathfrak{P} \) and another commutative square diagram with the maps \( \sigma_\mathfrak{P} \) and \( \sigma_\mathfrak{B} \).

Consider the element \( z_1 \in \text{CT}(\mathfrak{B})(R/I) \), where \( z_1 = 1z_1 \in R/I \) is the coset of the element \( 1 \in R \) and \( z_1 \in \text{CT}(\mathfrak{B})(R/I) \) is the image of \( z_1 \) in the direct limit. Then the element \( \sigma_\mathfrak{P}(z_1) \in \text{CT}(\mathfrak{P})(R/I) \) can be presented in the form \( \sigma_\mathfrak{P}(z_1) = \sum_{i=1}^{k} r_i x_i \) with some integer \( k \geq 1 \) and elements \( r_i \in R/I \). So we have

\[
a_1 \cdots a_{k-1} \hat{z}_k = z_1 = \rho_\mathfrak{P} \sigma_\mathfrak{P}(z_1) = r_1 \hat{z}_1 + \cdots + r_k \hat{z}_k = \sum_{i=1}^{k} r_i a_{i+1} \hat{z}_k \in \text{CT}(\mathfrak{B})(R/I).
\]

It follows that there exists an integer \( m \geq k \) such that the equation

\[
a_1 \cdots a_{m-1} z_m = (r_1 a_2 \cdots a_{m-k} + r_2 a_3 \cdots a_{m-1} + \cdots + r_k a_{k+1} \cdots a_{m-1}) z_m
\]

holds in the group \( R/I = (R/I)z_m \). Hence for every \( n \geq m \) we have

\[
a_1 \cdots a_{m-1} \equiv r_1 a_2 \cdots a_{m-1} + r_2 a_3 \cdots a_{m-1} + \cdots + r_k a_{k+1} \cdots a_{m-1} \mod J.
\]

Set \( s = a_1 \cdots a_n \in R \), and choose an open right ideal \( J \subset R \) such that \( s J \subset J \). Consider the element \( \hat{z}_n+1 \in \text{CT}(\mathfrak{B})(R/I) \), where \( z_{n+1} = 1z_{n+1} \in R/I \) is the coset of the element \( 1 \in R \). Then the element \( \sigma_\mathfrak{B}(\hat{z}_{n+1}) \in \text{CT}(\mathfrak{B})(R/I) \) can be presented in the form \( \sigma_\mathfrak{B}(\hat{z}_{n+1}) = \sum_{i=1}^{l} t_i x_i \) with some integer \( l \geq k \) and elements \( t_i \in R/I \). By
commutativity of the square diagram formed by the maps $\text{CT}(\mathcal{B})(s^*)$ and $\text{CT}(\mathcal{P})(s^*)$ together with the maps $\sigma_3$ and $\sigma_5$, we have

$$\sigma_3(s\bar{z}_{n+1}) = \sum_{i=1}^{l} s_i x_i \in \text{CT}(\mathcal{P})(\mathcal{R}/\mathcal{I}).$$

On the other hand, $s\bar{z}_{n+1} = a_1 \cdots a_n \bar{z}_{n+1} = \bar{z}_1 \in \text{CT}(\mathcal{B})(\mathcal{R}/\mathcal{I})$, hence

$$\sigma_3(s\bar{z}_{n+1}) = \sigma_3(\bar{z}_1) = \sum_{i=1}^{k} r_i x_i \in \text{CT}(\mathcal{P})(\mathcal{R}/\mathcal{I}).$$

Since the summation signs in these expressions stand for a direct sum, we can conclude that $r_i \equiv s_i \mod \mathcal{I}$ for every $1 \leq i \leq k$.

Finally, we have

$$a_1 \cdots a_{n-1} \equiv r_1 a_1 \cdots a_{n-1} + r_2 a_2 \cdots a_{n-1} + \cdots + r_k a_k \cdots a_{n-1} \equiv sl_1 a_1 \cdots a_{n-1} + st_2 a_2 \cdots a_{n-1} + \cdots + st_k a_k \cdots a_{n-1} \in s\mathcal{R} = a_1 \cdots a_n \mathcal{R} \mod \mathcal{I}.$$

Thus $a_1 \cdots a_{n-1} \in a_1 \cdots a_n \mathcal{R} + \mathcal{I}$, and it follows that $a_1 \cdots a_{n-1} \mathcal{R} + \mathcal{I} = a_1 \cdots a_n \mathcal{R} + \mathcal{I}$ for all $n \geq m$, as desired.

**Lemma 5.4.** Let $\mathcal{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals, $\mathcal{J}$ be a strongly closed two-sided ideal in $\mathcal{R}$, and $\mathcal{I} = \mathcal{R}/\mathcal{J}$ be the topological quotient ring. Assume that all Bass flat $\mathcal{R}$-contramodules are projective. Then all Bass flat left $\mathcal{I}$-contramodules are projective, too.

**Proof.** Let $\bar{a}_1, \bar{a}_2, \bar{a}_3, \ldots$ be a sequence of elements in $\mathcal{I}$ and $\mathcal{B} = \lim_{\rightarrow} (\mathcal{I} \xleftarrow{s\bar{a}_1} \mathcal{I} \xrightarrow{s\bar{a}_2} \mathcal{I} \xrightarrow{s\bar{a}_3} \cdots)$ be the related Bass flat left $\mathcal{I}$-contramodule. Lift the elements $\bar{a}_n \in \mathcal{I}$ to some elements $a_n \in \mathcal{R}$, and consider the related Bass flat left $\mathcal{R}$-contramodule $\mathcal{B}$. Then we have $\mathcal{B}/\mathcal{J} \otimes \mathcal{B} \cong \mathcal{B}$, since the reduction functors preserve colimits (see Sections 2.9, 2.12). The left $\mathcal{R}$-contramodule $\mathcal{B}$ is projective by assumption, hence the left $\mathcal{I}$-contramodule $\mathcal{B}/\mathcal{J} \otimes \mathcal{B}$ is projective, too.

**Corollary 5.5.** Let $\mathcal{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Assume that all Bass flat left $\mathcal{R}$-contramodules are projective. Then the discrete ring $\mathcal{R} = \mathcal{R}/\mathcal{J}$ is left perfect for every open two-sided ideal $\mathcal{I} \subset \mathcal{R}$.

**Proof.** In view of Lemma 5.1 it suffices to show that all Bass flat left $\mathcal{R}$-modules are projective. In our assumptions, this is a particular case of Lemma 5.4.

**Lemma 5.6.** Let $\mathcal{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals, $\mathcal{J}$ be a strongly closed two-sided ideal in $\mathcal{R}$, and $\mathcal{I} = \mathcal{R}/\mathcal{J}$ be the topological quotient ring. Assume that all Bass flat left $\mathcal{R}$-contramodules have projective covers in $\mathcal{R}$–contra. Then all Bass flat left $\mathcal{I}$-contramodules have projective covers in $\mathcal{I}$–contra.
Proof. As in the proof of Lemma 5.4, let \( \bar{a}_1, \bar{a}_2, \bar{a}_3, \ldots \) be a sequence of elements in \( T \) and \( \mathfrak{B} \) be the related Bass flat left \( T \)-contramodule. Lift the elements \( \bar{a}_n \in T \) to some elements \( a_n \in R \), and consider the related Bass flat left \( R \)-contramodule \( B \). Then we have \( B/J \cong \mathfrak{B} \). By assumption, we know that the left \( R \)-contramodule \( B \) has a projective cover in \( R \)-contra; and by Lemma 4.3 it follows that the left \( T \)-contramodule \( B \) has a projective cover in \( T \)-contra. \( \square \)

Corollary 5.7. Let \( R \) be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Assume that all Bass flat left \( R \)-contramodules have projective covers. Then the discrete ring \( R = R/J \) is left perfect for every open two-sided ideal \( J \subset R \).

Proof. In view of Lemma 5.1 and Proposition 4.2, it suffices to show that all Bass flat left \( R \)-modules have projective covers. In our assumptions, this is a particular case of Lemma 5.6. (In the more special case of a topological ring \( R \) with a base of neighborhoods of zero consisting of open two-sided ideals, one could, alternatively, use Corollary 5.4, Proposition 4.4, and Corollary 5.5.) \( \square \)

6. Topologically T-Nilpotent Ideals

Let \( H \) be a separated topological ring without unit. We will say that \( H \) is topologically nil if for any element \( a \in H \) the sequence of elements \( a, a^2, a^3, \ldots \) converges to zero in the topology of \( H \). Furthermore, we will say that \( H \) is topologically left \( T \)-nilpotent if for any sequence of elements \( a_1, a_2, a_3, \ldots \) in \( H \) the sequence of elements \( a_1, a_1a_2, a_1a_2a_3, \ldots, a_1a_2\cdots a_n, \ldots \) converges to zero in the topology of \( H \).

For examples of topologically left \( T \)-nilpotent two-sided ideals in topological rings we refer to Section 12 and Examples 13.1–13.2.

The following discrete module version of Nakayama lemma is also a topological version of the condition (7) and the implication (7) \( \Rightarrow \) (1) in [1, Theorem P].

Lemma 6.1. Let \( R \) be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals, and let \( H \subset R \) be a left ideal. Then \( H \) is topologically left \( T \)-nilpotent (as a topological ring without unit in the topology induced from \( R \)) if and only if for any nonzero discrete right \( R \)-module \( N \) the submodule \( N_H \subset N \) of all the elements annihilated by \( H \) in \( N \) is also nonzero.

Proof. “Only if”: suppose that \( H \) is topologically left \( T \)-nilpotent, and let \( N \) be a nonzero discrete right \( R \)-module. We have to show that \( N \) contains a nonzero element annihilated by \( H \). This implication does not depend on the assumption that \( H \) is a left ideal in \( R \); so \( H \subset R \) can be an arbitrary subring without unit. Choose an arbitrary nonzero element \( x \in N \). Suppose \( x \) is not annihilated by \( H \); then there exists an element \( a_1 \in H \) such that \( xa_1 \neq 0 \) in \( N \). Suppose \( xa_1 \) is not annihilated by \( H \); then there exists an element \( a_2 \in H \) such that \( xa_1a_2 \neq 0 \) in \( N \), etc. Assuming that \( N_H = 0 \), we can proceed indefinitely in this way and construct a sequence of elements \( (a_n \in H)_{n \geq 1} \) such that \( xa_1 \cdots a_n \neq 0 \) in \( N \) for all \( n \geq 1 \).
Let $\mathfrak{I} \subset \mathfrak{R}$ be the annihilator of $x$; then $\mathfrak{I}$ is an open right ideal in $\mathfrak{R}$, so $\mathfrak{I} \cap H$ is a neighborhood of zero in $H$. Since $H$ is topologically left $T$-nilpotent, there exists $n \geq 1$ such that $a_1a_2 \cdots a_n \in \mathfrak{I} \cap H$. Hence $xa_1 \cdots a_n = 0$ in $\mathfrak{N}$. The contradiction proves that $\mathfrak{N}_H \neq 0$.

“If”: suppose that $\mathfrak{N}_H \neq 0$ for every nonzero discrete right $\mathfrak{R}$-module $\mathfrak{N}$. Since $H \subset \mathfrak{R}$ is a left ideal, $\mathfrak{N}_H$ is an $\mathfrak{R}$-submodule in $\mathfrak{N}$. Whenever $\mathfrak{N}_H \neq \mathfrak{N}$, one then also has $(\mathfrak{N}/\mathfrak{N}_H)_H \neq 0$. Proceeding in a transfinite induction, one constructs a filtration $0 = F_0 \mathfrak{N} \subset F_1 \mathfrak{N} \subset F_2 \mathfrak{N} \subset \cdots \subset F_\alpha \mathfrak{N} = \mathfrak{N}$ of the $\mathfrak{R}$-module $\mathfrak{N}$ by its $\mathfrak{R}$-submodules, indexed by some ordinal $\alpha$, such that $F_i \mathfrak{N}/F_j \mathfrak{N} = (\mathfrak{N}/\mathfrak{N}_H)_H \neq 0$ for all ordinals $i < \alpha$ and $F_j \mathfrak{N} = \bigcup_{i < j} F_i \mathfrak{N}$ for all limit ordinals $j \leq \alpha$.

Now let $(a_n \in H)_{n \geq 1}$ be a sequence of elements. In order to prove that $H$ is topologically left $T$-nilpotent, we have to show that, for every open right ideal $\mathfrak{I} \subset \mathfrak{R}$, there exists $n \geq 1$ such that $a_1 \cdots a_n \in \mathfrak{I} \cap H$. Consider the discrete right $\mathfrak{R}$-module $\mathfrak{N} = \mathfrak{R}/\mathfrak{I}$ and its filtration $(F_i \mathfrak{N})_{i=0}^\alpha$, as constructed above. Let $x \in \mathfrak{N}$ denote the image of the element $1 \in \mathfrak{R}$.

We follow the argument in the proof of (7) $\implies$ (1) in [4, Theorem P]. Let $i_0 \leq \alpha$ be the minimal ordinal such that $x \in F_{i_0} \mathfrak{N}$. Then $i_0$ cannot be a limit ordinal; so either $i_0 = 0$, or $i_0 = i'_0 + 1$ for some ordinal $i'_0$. Since the $\mathfrak{R}$-module $F_{i_0} \mathfrak{N}/F_{i'_0} \mathfrak{N}$ is annihilated by $H$, we have $xa_1 \in F_{i'_0} \mathfrak{N}$. Let $i_1$ be the minimal ordinal such that $xa_1 \in F_{i_1} \mathfrak{N}$; then $i_1 < i_0$. Once again, $i_1$ cannot be a limit ordinal; so either $i_1 = 0$, or $i_1 = i'_1 + 1$ for some ordinal $i'_1$, and then $xa_1a_2 \in F_{i'_1} \mathfrak{N}$. Proceeding in this way, we construct a decreasing chain of ordinals $i_0 > i_1 > i_2 > \cdots$, which must terminate. Thus there exists $n \geq 1$ such that $xa_1 \cdots a_n \in F_0 \mathfrak{N} = 0$, hence $a_1 \cdots a_n \in \mathfrak{I}$. \hfill \Box

The following version of contramodule Nakayama lemma is a generalization of [21, Lemma 1.3.1] (which is, in turn, a generalization of [20, Lemma A.2.1]). For other versions of contramodule Nakayama lemma, see [22, Lemma D.1.2] and [29, Lemma 6.14]. For a module version, see [1, Lemma 28.3 (a) $\iff$ (b)].

**Lemma 6.2.** Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals, and let $\mathfrak{H} \subset \mathfrak{R}$ be a closed left ideal. Then $\mathfrak{H}$ is topologically left $T$-nilpotent (in the topology induced from $\mathfrak{R}$) if and only if for any nonzero left $\mathfrak{R}$-contramodule $\mathfrak{C}$ one has $\mathfrak{H} \triangleleft \mathfrak{C} \neq \mathfrak{C}$.

**Proof.** “Only if”: the argument follows the proof of [21, Lemma 1.3.1] with an additional consideration based on the König lemma (in the spirit of a paragraph from [4, proof of Theorem 2.1]). We will suppose that $\mathfrak{H} \triangleleft \mathfrak{C} = \mathfrak{C}$ and prove that $\mathfrak{C} = 0$ in this case. This implication does not depend on the assumption that $\mathfrak{H}$ is a left ideal in $\mathfrak{R}$; so $\mathfrak{H} \subset \mathfrak{R}$ can be an arbitrary closed subring without unit.

Indeed, let $b \in \mathfrak{C}$ be an element. By assumption, the contraction map $\pi: \mathfrak{H}[[\mathfrak{C}]] \rightarrow \mathfrak{C}$ is surjective. Let $h: \mathfrak{C} \rightarrow \mathfrak{H}[[\mathfrak{C}]]$ be a section of the map $\pi$ (so $\pi \circ h = \operatorname{id}_\mathfrak{C}$). Introduce the notation $h(d) = \sum_{c \in \mathfrak{C}} h_{d,c} \in \mathfrak{H}[[\mathfrak{C}]]$ for all $d \in \mathfrak{C}$, where $h_{d,c} \in \mathfrak{H}$ and the $\mathfrak{C}$-indexed family of elements $c \mapsto h_{d,c}$ converges to zero in the topology of $\mathfrak{H}$ for every $d \in \mathfrak{C}$.

27
For any set \( X \), define inductively \( \mathcal{H}^{(0)}[X] = X \) and \( \mathcal{H}^{(n)}[X] = \mathcal{H}[(\mathcal{H}^{(n-1)}[X])][X] \) for \( n \geq 1 \). Let \( \phi^{(n)}_X : \mathcal{H}^{(n)}[X] \rightarrow \mathcal{H}^n[X] \) denote the iterated monad multiplication (“opening of parentheses”) map. Set \( b_1 = h(b) \in \mathcal{H}[[\mathcal{C}]] \), and define inductively \( b_n = \mathcal{H}^{(n-1)}[[h]](b_{n-1}) \in \mathcal{H}^{(n)}[[\mathcal{C}]] \) for each \( n \geq 2 \), where \( \mathcal{H}^{(n-1)}[[h]] : \mathcal{H}^{(n-1)}[[\mathcal{C}]] \rightarrow \mathcal{H}^{(n)}[[\mathcal{C}]] \) is the map induced by \( h \). Put \( a_n = \phi^{(n)}_X(b_n) \in \mathcal{H}[[\mathcal{C}]] \) for all \( n \geq 1 \).

Furthermore, set \( q_n = \phi^{(n-1)}_X(b_n) = \mathcal{H}[[h]](a_{n-1}) \in \mathcal{H}[[\mathcal{H}[[\mathcal{C}]]]] \) for all \( n \geq 2 \). Then \( \mathcal{H}[[\pi]](q_n) = a_{n-1} \) and \( \phi^X(q_n) = a_n \).

The abelian group \( \mathcal{H}[[X]] \) is separated and complete in its natural topology with a base of neighborhoods of zero formed by the subgroups \( (\mathcal{I} \cap \mathcal{H})[[X]] \subset \mathcal{H}[[X]] \), where \( \mathcal{I} \subset \mathcal{R} \) are open right ideals. For any map of sets \( f : X \rightarrow Y \), the map \( \mathcal{H}[[f]] \) is continuous with respect to such topologies on \( \mathcal{H}[[X]] \) and \( \mathcal{H}[[Y]] \). Besides, the map \( \phi^X : \mathcal{H}[[\mathcal{H}[[X]]]] \rightarrow \mathcal{H}[[X]] \) is continuous, too, with respect to the above-described topology on \( \mathcal{H}[[X]] \) and the similar topology of \( \mathcal{H}[[\mathcal{H}[[X]]]] = \mathcal{H}[[Y]] \), where \( Y = \mathcal{H}[[X]] \) is viewed as an abstract set.

The key observation is that the sequence of elements \( a_n \) converges to zero in the topology of \( \mathcal{H}[[\mathcal{C}]] \) as \( n \rightarrow \infty \). In order to prove this convergence, we will represent the sequence of elements \( b_n \in \mathcal{H}^{(n)}[[\mathcal{C}]] \) by an infinite rooted tree \( B \) in the following way. The root vertex (that is, the only vertex of depth 0) is marked by the element \( b \in \mathcal{C} \). Its children (i.e., the vertices of depth 1) are marked by all the elements \( c \in \mathcal{C} \), one such child for every element \( c \). The edge leading from the root vertex \( b \) to its child \( c \) is marked by the coefficient \( h_{b,c} \in \mathcal{H} \) in the formal linear combination \( b_1 = h(b) = \sum_{c \in \mathcal{C}} h_{b,c} c \in \mathcal{H}[[\mathcal{C}]] \).

The element \( b_2 = \mathcal{H}[[h]](b_1) \in \mathcal{H}[[\mathcal{H}[[\mathcal{C}]]]] \) has the form \( b_2 = \sum_{c \in \mathcal{C}} h_{b,c} h(c) \), where \( h(c) = \sum_{c_2 \in \mathcal{C}} h_{c,c_2} c_2 \) for every \( c \in \mathcal{C} \). The children of a vertex of depth 1 marked by \( c \) in the tree \( B \) are marked by all the elements \( c_2 \in \mathcal{C} \); and the edge leading from \( c \) to \( c_2 \) is marked by the element \( h_{c,c_2} \in \mathcal{H} \).

Generally, the children of any vertex in \( B \) are marked by all the elements of \( \mathcal{C} \); we will write that the children of any fixed vertex of depth \( n - 1 \) are marked by all the elements \( c_n \in \mathcal{C} \), one such child for every element \( c_n \in \mathcal{C} \). Thus, a vertex \( v \) of depth \( n \) in \( B \) is characterized by its root path, which passes from the root vertex \( b \) through vertices marked by \( c_1 = c, c_2, \ldots, \), and comes to the vertex \( v \) marked by \( c(v) = c_n \). So the set of all vertices of depth \( n \) in \( B \) is bijective to \( \mathcal{C}^n = \{ (c_1, \ldots, c_n) \mid c_i \in \mathcal{C} \} \).

The edge going from a vertex marked by \( c_{n-1} \) to its child marked by \( c_n \) is marked by the element \( h_{c_{n-1},c_n} \in \mathcal{H} \).

In addition to marking all the vertices and edges of \( B \), let us also mark all the root paths. A root path going from the root vertex \( b \) to a vertex marked by \( c_1 \), to a vertex marked by \( c_2 \), etc., and coming to a vertex \( v \) marked by \( c_n \), goes along the edges marked by the elements \( h_{b,c_1}, h_{c_1,c_2}, \ldots, h_{c_{n-1},c_n} \in \mathcal{H} \). We mark such a root path by the product \( r(v) = h_{b,c_1} \cdots h_{c_{n-1},c_n} \in \mathcal{H} \) of the elements marking its edges.

The purpose of this construction is to observe that the element \( a_n \in \mathcal{H}[[\mathcal{C}]] \) can be expressed as the infinite sum \( a_n = \sum_{v \in B_n} r(v)c(v) \) over the set \( B_n \) of all vertices of depth \( n \) in the tree \( B \). This sum converges in the topology of \( \mathcal{H}[[\mathcal{C}]] \).
In order to show that the sequence of elements $a_n$ converges to zero in $\mathcal{H}[[\mathcal{C}]]$, choose a proper open right ideal $\mathfrak{I} \subset \mathcal{R}$. Denote by $B^3$ the subtree of $B$ formed by all the vertices $v \in B$ with $r(v) \notin \mathfrak{I}$. The root vertex belongs to $B^3$, since $1 \notin \mathfrak{I}$; and whenever $r(v) \in \mathfrak{I}$ for some $v \in B$, one also has $r(w) \in B$ for all the descendants $w \in B$ of the vertex $v$; so $B^3$ is indeed a tree.

Furthermore, the tree $B^3$ is locally finite, because for every vertex $v \in B$ with $c(v) = c_{n-1}$ there exists an open right ideal $\mathfrak{I} \subset \mathcal{R}$ such that $r(v) \mathfrak{I} \subset \mathfrak{I}$, and the marking element $h_{c_{n-1},c_n}$ of all but a finite subset of the edges going down from $v$ belongs to $\mathfrak{I} \cap \mathcal{H}$ (as $h(c_{n-1}) = \sum_{c_n \in \mathcal{C}} h_{c_{n-1},c_n}$ is an element of $\mathcal{H}[[\mathcal{C}]]$). So, denoting by $vc_n \in B$ the child of $v$ marked by $c_n$, we have $r(vc_n) = r(v)h_{c_{n-1},c_n} \in \mathfrak{I}$ for all but a finite subset of $c_n \in \mathcal{C}$.

Finally, the tree $B^3$ has no infinite branches, since the ring $\mathcal{H}$ is topologically left $T$-nilpotent. Indeed, the sequence of the marking elements $r(v_n)$ of the root paths of the vertices $v_n$ along any infinite branch in $B$ converges to zero in $\mathcal{H}$, hence $r(v_n) \in \mathfrak{I} \cap \mathcal{H}$ for $n \gg 0$. By the König lemma, it follows that the tree $B^3$ is finite, so it has a finite depth $m$. Thus $a_n \in (\mathfrak{I} \cap \mathcal{H})[[\mathcal{C}]]$ for all $n > m$.

Now we can finish the proof of the “only if” assertion of the lemma. Since the sequence $a_n$ converges to zero in $\mathcal{H}[[\mathcal{C}]]$ as $n \to \infty$, the sequence $q_n = \mathcal{H}[[h]](a_{n-1}) \in \mathcal{H}[[\mathcal{H}[[\mathcal{C}]]]]$ converges to zero in the topology of $\mathcal{H}[[\mathcal{Y}]]$, where $\mathcal{Y} = \mathcal{H}[[\mathcal{C}]]$. So the sum $\sum_{n=2}^{\infty} q_n \in \mathcal{H}[[\mathcal{H}[[\mathcal{C}]]]]$ is well-defined as the limit of finite partial sums. Furthermore, we have $\mathcal{H}[[\pi]](q_{n+1}) = a_n = \phi_\mathcal{C}(q_n)$ for all $n \geq 2$ and $\mathcal{H}[[\pi]](q_1) = a_1 = b_1$. Hence
\[
\mathcal{H}[[\pi]] \left( \sum_{n=2}^{\infty} q_n \right) - \phi_\mathcal{C} \left( \sum_{n=2}^{\infty} q_n \right) = b_1
\]
(we recall that the maps $\mathcal{H}[[\pi]]$ and $\phi_\mathcal{C}$ are continuous, as mentioned in the above discussion). Therefore, $b = \pi(b_1) = 0$ by the contraassociativity equation,
\[
\pi \circ (\mathcal{H}[[\pi]] - \phi_\mathcal{C}) = 0.
\]

“If”: we suppose that $\mathcal{H} \times \mathcal{C} = \mathcal{C}$ implies $\mathcal{C} = 0$ for any left $\mathcal{R}$-contramodule $\mathcal{C}$, and prove that $\mathcal{H}$ is topologically left $T$-nilpotent. Given a sequence of elements $(a_n \in \mathcal{H})_{n \geq 1}$, consider the related Bass flat left $\mathcal{R}$-contramodule $\mathcal{B}$. Denote by $z_n \in \mathcal{B}$ the images of the free generators $x_n$, $n \geq 1$, under the surjective left $\mathcal{R}$-contramodule morphism $\mathcal{R}[[x_1, x_2, x_3, \ldots]] \twoheadrightarrow \mathcal{B}$ (in the notation of the proof of Lemma 5.2). Then we have $z_n = a_n z_{n+1} \in \mathcal{H} \times \mathcal{B}$ for all $n \geq 1$. Since $\mathcal{H} \subset \mathcal{R}$ is a closed left ideal, $\mathcal{H} \times \mathcal{B}$ is an $\mathcal{R}$-subcontramodule in $\mathcal{B}$. Since the left $\mathcal{R}$-contramodule $\mathcal{B}$ is generated by its elements $z_n$, $n \geq 1$, it follows that $\mathcal{H} \times \mathcal{B} = \mathcal{B}$. Thus (by assumption) we have $\mathcal{B} = 0$. Applying Lemma 5.2, we can conclude that the sequence of elements $a_1 \cdots a_n \in \mathcal{H}$ converges to zero in $\mathcal{R}$ as $n \to \infty$. 

\[\square\]

**Lemma 6.3.** Let $H$ be a separated topological ring without unit, with a base of neighborhoods of zero formed by open right ideals, and let $K \subset H$ be a closed two-sided ideal. Then $H$ is topologically left $T$-nilpotent if and only if both $K$ and $H/K$ are.

**Proof.** The “only if” assertion is obvious; let us prove the “if”. Let $a_1, a_2, a_3, \ldots$ be a sequence of elements in $H$, and let $I \subset H$ be an open right ideal. Denote by $\tilde{a}_i$ the
images of the elements $a_i$ in $H/K$. For any open right ideal $J \subset H$, we will denote by $\overline{J} \subset H/K$ the image of the ideal $J$. Then $\overline{J}$ is an open right ideal in $H/K$.

Since $H/K$ is topologically left T-nilpotent, there exists an integer $n_1 \geq 1$ such that the product $\bar{a}_1 \cdots \bar{a}_{n_1}$ belongs to $\overline{J}$. Let $J_1 \subset H$ be an open right ideal such that $\bar{a}_1 \cdots \bar{a}_{n_1} J_1 \subset I$. Then there exists an integer $n_2 > n_1$ such that the product $\bar{a}_{n_1+1} \cdots \bar{a}_{n_2}$ belongs to $\overline{J}_1$. Let $J_2 \subset H$ be an open right ideal such that $\bar{a}_{n_1+1} \cdots \bar{a}_{n_2} J_2 \subset J_1$, etc. Proceeding in this way, we construct a sequence of integers $0 = n_0 < n_1 < n_1 < n_2 < \cdots$ and open right ideals $I = J_0, J_1, J_2, \ldots \subset H$ such that $\bar{a}_{n_{m-1}+1} \cdots \bar{a}_{n_m} \in \overline{J}_{m-1}$ and $a_{n_{m-1}+1} \cdots a_{n_m} J_m \subset J_{m-1}$ for all $m \geq 1$.

For every $m \geq 1$, we have $a_{n_{m-1}+1} \cdots a_{n_m} \in J_{m-1} + K$. Choose $b_m \in J_{m-1}$ and $c_m \in K$ such that $a_{n_{m-1}+1} \cdots a_{n_m} = b_m + c_m$. Since $K$ is topologically left T-nilpotent, there exists $m \geq 1$ such that the product $c_1 c_2 \cdots c_m$ belongs to $I \cap K$. Now we have

$$a_1 \cdots a_m = (b_1 + c_1) \cdots (b_m + c_m) = c_1 c_2 \cdots c_m + b_1 c_2 c_3 \cdots c_m + (b_1 + c_1) b_2 c_3 \cdots c_m + \cdots + (b_1 + c_1)(b_2 + c_2) \cdots (b_{m-2} + c_{m-2}) b_{m-1} c_m + (b_1 + c_1)(b_2 + c_2) \cdots (b_{m-1} + c_{m-1}) b_m$$

$$\in I \cap K + J_0 + (b_1 + c_1) J_1 + \cdots + (b_1 + c_1) \cdots (b_{m-1} + c_{m-1}) J_m = I.$$

\[\square\]

7. **Topological Jacobson Radical**

Let $\mathcal{R}$ be a complete, separated topological associative ring with a base of neighborhoods of zero formed by open right ideals. By an open maximal right ideal $\mathcal{M} \subset \mathcal{R}$ we mean a maximal right ideal that is simultaneously an open right ideal. Notice that this is the same thing as a maximal element in the set of all proper open right ideals, as any right ideal containing an open right ideal is open. It follows that every proper open right ideal $\mathcal{I} \subset \mathcal{R}$ is contained in an open maximal right ideal $\mathcal{I} \subset \mathcal{M} \subset \mathcal{R}$.

We define the topological Jacobson radical $\mathcal{J}$ of the topological ring $\mathcal{R}$ as the intersection of its open maximal right ideals. This definition was previously discussed in the paper [14, Section 3.B].

**Lemma 7.1.** For any complete, separated topological associative ring $\mathcal{R}$ with a base of neighborhoods of zero formed by open right ideals, the topological Jacobson radical $\mathcal{J} \subset \mathcal{R}$ is a closed two-sided ideal in $\mathcal{R}$.

**Proof.** The assertion that $\mathcal{J}$ is a closed right ideal in $\mathcal{R}$ is obvious (as any open subgroup is closed, and any intersection of closed right ideals is a closed right ideal). To prove that $\mathcal{J}$ is a two-sided ideal, consider two elements $h \in \mathcal{J}$ and $r \in \mathcal{R}$. Given an open maximal right ideal $\mathcal{M} \subset \mathcal{R}$, we have to show that $rh \in \mathcal{M}$. Let $\mathcal{R} \subset \mathcal{R}$ be the set of all elements $q \in \mathcal{R}$ such that $rq \in \mathcal{M}$. Then $\mathcal{R}$ is an open right ideal in $\mathcal{R}$. The left multiplication with $r$ is an injective right $\mathcal{R}$-module morphism $r* : \mathcal{R}/\mathcal{M} \rightarrow \mathcal{R}/\mathcal{M}$ (taking a coset $s + \mathcal{M}$, $s \in \mathcal{R}$, to the coset $rs + \mathcal{M}$). Since $\mathcal{R}/\mathcal{M}$ is a simple right $\mathcal{R}$-module, it follows that either $\mathcal{R} = \mathcal{M}$, or $\mathcal{R}/\mathcal{M}$ is a simple right
An $\mathcal{R}$-module, too. In the former case, we have $r\mathcal{R} \subset \mathcal{M}$, hence $rh \in \mathcal{M}$. In the latter case, $\mathcal{N}$ is an open maximal right ideal in $\mathcal{R}$, hence $h \in \mathcal{R}$ and $rh \in \mathcal{M}$.

Alternatively, it suffices to observe that $\mathcal{H}$ is the intersection of the annihilators of all the simple discrete right $\mathcal{R}$-modules (see Lemma 7.2(ii) below). These are closed two-sided ideals in $\mathcal{R}$.

Denote by $H \subset \mathcal{R}$ the Jacobson radical of the ring $\mathcal{R}$ viewed as an abstract ring (without the topology). So $H$ is the intersection of all maximal right ideals in $\mathcal{R}$, and as the set of all open maximal right ideals is a subset of the set of all maximal right ideals, it follows that the topological Jacobson radical of the ring $\mathcal{R}$ contains the nontopological one, that is $H \subset \mathcal{H} \subset \mathcal{R}$.

Lemma 7.2. Let $\mathcal{R}$ be a complete, separated topological associative ring with a base of neighborhoods of zero formed by open right ideals, let $\mathcal{H} \subset \mathcal{R}$ be the topological Jacobson radical of $\mathcal{R}$, and let $h \in \mathcal{R}$ be an element. Then the following conditions are equivalent:

(i) $h \in \mathcal{H}$;
(ii) for every (semi)simple discrete right $\mathcal{R}$-module $\mathcal{N}$ one has $\mathcal{N}h = 0$;
(iii) for every element $r \in \mathcal{R}$ and every open right ideal $\mathcal{I} \subset \mathcal{R}$ one has $(1 - hr)\mathcal{R} + \mathcal{I} = \mathcal{R}$;
(iv) for every pair of elements $r, s \in \mathcal{R}$ and every open right ideal $\mathcal{I} \subset \mathcal{R}$ one has $(1 - shr)\mathcal{R} + \mathcal{I} = \mathcal{R}$.

Proof. (i) $\iff$ (ii) holds, because the open maximal right ideals in $\mathcal{R}$ are precisely the annihilators of nonzero elements in simple discrete right $\mathcal{R}$-modules (and the semisimple discrete right $\mathcal{R}$-modules are the direct sums of simple ones).

(i) $\implies$ (iii) Observe that $(1 - hr)\mathcal{R} + \mathcal{I} = \mathcal{R}$ is an open right ideal in $\mathcal{R}$. If different from $\mathcal{R}$, it must be contained in an open maximal right ideal $\mathcal{M}$. Now $1 - hr \in \mathcal{M}$ and $h \in \mathcal{R}$ imply $1 \in \mathcal{M}$, a contradiction.

(iii) $\implies$ (ii) Suppose $\mathcal{N}$ is a simple discrete left $\mathcal{R}$-module and $\mathcal{N}h \neq 0$. Choose an element $b \in \mathcal{N}$ for which $bh \neq 0$ in $\mathcal{N}$. Since $\mathcal{N}$ is simple, one has $bh\mathcal{R} = \mathcal{N}$, so there exists an element $r \in \mathcal{R}$ such that $bhr = b$. Thus we have $b(1 - hr) = 0$. Let $\mathcal{I} = \mathcal{M} \subset \mathcal{R}$ be the annihilator of the element $b \in \mathcal{N}$. Then $\mathcal{I}$ is an open (maximal) right ideal in $\mathcal{R}$ and $(1 - hr) \in \mathcal{I}$, hence $(1 - hr)\mathcal{R} + \mathcal{I} = \mathcal{I} \neq \mathcal{R}$.

The implication (iv) $\implies$ (iii) is obvious, and to prove (iii) $\implies$ (iv) it suffices to observe that the set of all elements $h \in \mathcal{R}$ satisfying (iii) is a two-sided ideal (since (i) $\iff$ (iii) and by Lemma 7.1).

Lemma 7.3. Let $\mathcal{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Then the following two conditions are equivalent:

(i) for every sequence of elements $a_1, a_2, a_3, \ldots \in \mathcal{R}$ and every open right ideal $\mathcal{I} \subset \mathcal{R}$, the chain of right ideals $\mathcal{R} \supset a_1\mathcal{R} + \mathcal{I} \supset a_1a_2\mathcal{R} + \mathcal{I} \supset a_1a_2a_3\mathcal{R} + \mathcal{I} \supset \cdots$ terminates;
(ii) any descending chain of cyclic discrete right $\mathcal{R}$-modules terminates.
Proof. One observes that the right $\mathcal{R}$-module $\mathcal{R}/\mathcal{I}$ is cyclic and discrete for any open right ideal $\mathcal{I} \subset \mathcal{R}$, and any cyclic discrete right $\mathcal{R}$-module has this form. □

**Lemma 7.4.** Let $\mathcal{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Assume that every descending chain of cyclic discrete right $\mathcal{R}$-modules terminates. Then any nonzero discrete right $\mathcal{R}$-module has nonzero socle.

Proof. One easily shows that, in the assumptions of the lemma, any nonzero discrete right $\mathcal{R}$-module has a simple submodule. □

**Lemma 7.5.** Let $\mathcal{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Assume that every nonzero discrete right $\mathcal{R}$-module has nonzero socle. Then the topological Jacobson radical $\mathcal{H} \subset \mathcal{R}$ is topologically left $T$-nilpotent.

Proof. According to Lemma 6.1 it suffices to show that every nonzero discrete right $\mathcal{R}$-module $\mathcal{N}$ contains a nonzero element annihilated by $\mathcal{H}$. Indeed, let $\mathcal{L} \subset \mathcal{N}$ be the socle of $\mathcal{N}$. Then $\mathcal{L} \neq 0$ by assumption and $\mathcal{L}\mathcal{H} = 0$ by Lemma 7.2(ii). □

**Lemma 7.6.** Let $\mathcal{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals, let $\mathcal{H} \subset \mathcal{R}$ be the topological Jacobson radical of $\mathcal{R}$, and let $\mathcal{H} \subset \mathcal{R}$ be the Jacobson radical of the ring $\mathcal{R}$ viewed as an abstract ring (without the topology).

(a) Let $J \subset \mathcal{R}$ be a topologically nil left or right ideal. Then $J \subset H \subset \mathcal{H}$.

(b) In particular, if $\mathcal{H}$ is topologically nil, then $\mathcal{H} = H$.

Proof. In part (a), the inclusion $H \subset \mathcal{H}$ was explained above in this section, and the inclusion $J \subset H$ is true because the element $1 - x$ is invertible in $\mathcal{R}$ for every $x \in J$, as the series $1 + x + x^2 + x^3 + \cdots$ converges in $\mathcal{R}$. Part (b) follows from part (a). □

**Corollary 7.7.** Let $\mathcal{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Assume that every descending chain of cyclic discrete right $\mathcal{R}$-modules terminates. Then the topological Jacobson radical $\mathcal{H}$ of the topological ring $\mathcal{R}$ is topologically left $T$-nilpotent, the Jacobson radical $H$ of the ring $\mathcal{R}$ viewed as an abstract ring is closed in $\mathcal{R}$, and $\mathcal{H} = H$.

Proof. Follows from Lemmas 7.4, 7.5, and 7.6(b). □

**Lemma 7.8.** Let $\mathcal{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals, $\mathcal{A} \subset \mathcal{R}$ be a closed two-sided ideal such that the quotient ring $\mathcal{S} = \mathcal{R}/\mathcal{A}$ is complete in its quotient topology, and $p : \mathcal{R} \to \mathcal{S}$ be the natural surjective map. Let $\mathcal{H} \subset \mathcal{R}$ and $\mathcal{J} \subset \mathcal{S}$ be the topological Jacobson radicals of the topological rings $\mathcal{R}$ and $\mathcal{S}$. Then

(a) $p(\mathcal{H}) \subset \mathcal{J}$;
(b) if $\mathcal{A} \subset \mathcal{H}$, then $\mathcal{H} = p^{-1}(\mathcal{J})$.

Proof. If $\mathcal{A}$ is an open maximal right ideal in $\mathcal{S}$, then $p^{-1}(\mathcal{A})$ is an open maximal right ideal in $\mathcal{R}$ (since $p$ is a surjective continuous ring homomorphism). The set of
all right ideals \( f^{-1}(\mathfrak{M}) \subset \mathfrak{R} \) is contained in the set of all open maximal right ideals \( \mathfrak{M} \subset \mathfrak{R} \), hence \( \mathfrak{H} = \bigcap_{\mathfrak{M} \subset \mathfrak{R}} \mathfrak{M} \) is contained in \( p^{-1}(\mathfrak{J}) = \bigcap_{\mathfrak{R}} f^{-1}(\mathfrak{M}) \). Moreover, if \( \mathfrak{R} \subset \mathfrak{H} \), then one has \( \mathfrak{R} \subset \mathfrak{M} \) for every \( \mathfrak{M} \). It follows that \( \mathfrak{N} = p(\mathfrak{M}) \) is an open maximal right ideal in \( \mathfrak{S} \) (since \( p \) is an open map) and \( \mathfrak{M} = p^{-1}(\mathfrak{M}) \). So the sets of all right ideals \( \mathfrak{M} \) and \( p^{-1}(\mathfrak{M}) \) in \( \mathfrak{R} \) coincide in this case.

The following lemma is a part of [14, Theorem 3.8].

**Lemma 7.9.** Let \( \mathfrak{R} \) be a complete, separated topological ring with a base of neighborhoods of zero formed by open two-sided ideals. For every discrete quotient ring \( R = \mathfrak{R}/\mathfrak{J} \) of the topological ring \( \mathfrak{R} \), consider the Jacobson radical \( H(R) \subset R \) of the ring \( R \). Then the topological Jacobson radical \( \mathfrak{H} \subset \mathfrak{R} \) is the projective limit \( \mathfrak{H} = \lim_{\mathfrak{M} \subset \mathfrak{R}} H(\mathfrak{R}/\mathfrak{J}) \subset \lim_{\mathfrak{M} \subset \mathfrak{R}} \mathfrak{R}/\mathfrak{J} = \mathfrak{R} \) of the Jacobson radicals \( H(\mathfrak{R}/\mathfrak{J}) = H(R) \) taken over all the open two-sided ideals \( \mathfrak{J} \subset \mathfrak{R} \). Furthermore, the Jacobson radical \( H \) of the ring \( \mathfrak{R} \) viewed as an abstract ring coincides with the topological Jacobson radical \( \mathfrak{H} \) (so, in particular, \( H \) is a closed ideal in \( \mathfrak{R} \)).

**Proof.** The projective limit of Jacobson radicals of the rings \( R \) is well-defined, because for any surjective morphism of discrete associative rings \( f : R' \to R'' \) one has \( f(H(R')) \subset H(R'') \). To see that this projective limit coincides with the topological Jacobson radical \( \mathfrak{H} \subset \mathfrak{R} \), one can use the characterization provided by Lemma [7.2(iii)] or (iv), observing that it suffices to consider open two-sided ideals \( \mathfrak{J} \subset \mathfrak{R} \) in this characterization whenever such ideals form a base of neighborhoods of zero in \( \mathfrak{R} \). For the proof of the second assertion of the lemma we refer to [14, Theorem 3.8 (2–3)].

**Remark 7.10.** The reader should be warned that the assertion of [14, Theorem 3.8 (4)] is erroneous. While it is correctly pointed out in [14] proof of Theorem 3.8 (4)] that \( \mathfrak{H} \) is the kernel of the natural ring homomorphism

\[
(6) \quad \mathfrak{R} = \lim_{\mathfrak{M} \subset \mathfrak{R}} \mathfrak{R}/\mathfrak{J} \longrightarrow \lim_{\mathfrak{M} \subset \mathfrak{R}} (\mathfrak{R}/\mathfrak{J})/(H(\mathfrak{R}/\mathfrak{J})),
\]

the map (6) need not be surjective; not even for complete, separated commutative rings \( \mathfrak{R} \) with a countable base of neighborhoods of zero consisting of open ideals. Simply put, the problem is that the passage to projective limit does not preserve surjectivity of homomorphisms of abelian groups.

Here is a specific counterexample. Let \( k \) be a field and \( L \) be a commutative local \( k \)-algebra with a maximal ideal \( m \subset L \) such that \( L = k \oplus m \). Assume further that \( L \) has no nonzero nilpotent elements. Notice that, for any commutative ring \( K \) without nilpotent elements, the Jacobson radical of the polynomial ring \( K[x] \) vanishes. Consider the ring \( E = L[x_1, x_2, x_3, \ldots] \) of polynomials in a countable set of variables \( x_i, \ i \geq 1 \), with coefficients in the ring \( L \). Denote by \( E_n = L[x_n, x_{n+1}, x_{n+2}, \ldots] = E/(x_1, \ldots, x_{n-1}) \) the ring of polynomials in the variables \( x_{n+i}, \ i \geq 0 \), which we view as the quotient ring of the ring \( E \) by the ideal generated by \( x_1, \ldots, x_{n-1} \).

Let \( R \) be the following ring of sequences of polynomials \( (f_1, f_2, f_3, \ldots) \). For every \( n \geq 1 \), we have \( f_n \in E_n \). The elements of the ring \( R \) are all the eventually \( k \)-sequences of polynomials \( (f_1, f_2, f_3, \ldots) \), i.e., all such sequences for which \( f_n \in k \) for \( n \) large
enough. The addition and multiplication operations are performed on the sequences of polynomials \((f_1, f_2, f_3, \ldots)\) termwise; so \(R\) is a subring in \(\prod_{n \geq 1} E_n\). Furthermore, let \(R' \supset R\) be the ring of \textit{eventually \(L\)-sequences}\ of polynomials \((f_n \in E_n)_{n \geq 1}\), i.e., all such sequences for which \(f_n \in L\) for \(n\) large enough.

Let \(I_j \subset R\) be the ideal consisting of all sequences of polynomials \((f_1, f_2, f_3, \ldots)\) such that, for every \(n \geq 1\), the polynomial \(f_n \in E_n\) belongs to the ideal generated by all the elements \(x_i\) with \(i \geq j\) (or equivalently, \(i \geq \max(n, j)\)). In other words, one can say that \(f_n\) belongs to the image of the ideal \((x_j, x_{j+1}, x_{j+2}, \ldots) \subset E\) under the natural surjective ring homomorphism \(E \twoheadrightarrow E_n\). This condition implies that for every sequence \((f_1, f_2, f_3, \ldots) \in I_j\) one has \(f_n = 0\) for \(n\) large enough. Obviously, we have \(R' \supset R' \supset I_1 \supset I_2 \supset I_3 \supset \cdots\); and \(I_j\) is an ideal in \(R'\) as well as in \(R\). Denote by \(\mathfrak{R}\) and \(\mathfrak{R}'\), respectively, the completions of the rings \(R\) and \(R'\) with respect to their topologies in which the ideals \(I_j, j \geq 1\), form a base of neighborhoods of zero. Clearly, \(\mathfrak{R}\) is an open subring in \(\mathfrak{R}'\).

The quotient ring \(R/I_j\) is the ring of all eventually \(k\)-sequences \((f_1, f_2, f_3, \ldots)\), where \(f_n \in L[x_n, \ldots, x_{j-1}]\) for \(n < j\), \(f_n \in L\) for \(n \geq j\), and \(f_n \in k\) for \(n\) large enough. Therefore, the Jacobson radical \(H(R/I_j)\) consists of all sequences \((\bar{h}_1, \bar{h}_2, \bar{h}_3, \ldots)\), where \(\bar{h}_n = 0\) for \(n < j\), \(\bar{h}_n \in m\) for \(n \geq j\), and \(\bar{h}_n = 0\) for \(n\) large enough. So one has \(H(\mathfrak{R}) = \mathfrak{h}(\mathfrak{R}) = \lim_{j \geq 1} H(R/I_j) = 0\). The key observation is that the derived projective limit \(\lim_{j \geq 1} H(R/I_j) = \lim_{j \geq 1} \bigoplus_{n \geq j} m \cong \prod_{n \geq 1} m/\bigoplus_{n \geq 1} m\) does not vanish. Furthermore, \(R'/I_j\) is the ring of all sequences \((f_1, f_2, f_3, \ldots)\), where \(f_n \in L[x_n, \ldots, x_{j-1}]\) for \(n < j\) and \(f_n \in L\) for \(n \geq j\). The Jacobson radical \(H(R'/I_j)\) consists of all sequences \((\bar{h}_1, \bar{h}_2, \bar{h}_3, \ldots)\) with \(\bar{h}_n = 0\) for \(n < j\) and \(\bar{h}_n \in m\) for \(n \geq j\). So \(H(\mathfrak{R}') = \mathfrak{h}(\mathfrak{R}') = \lim_{j \geq 1} H(R'/I_j) = 0\) and also \(\lim_{j \geq 1} H(R'/I_j) = \lim_{j \geq 1} \prod_{n \geq j} m = 0\). Notice the isomorphism of \(k\)-vector spaces \(\lim_{j \geq 1} H(R/I_j) \cong R'/R = \mathfrak{R}'/\mathfrak{R}\).

It is straightforward to compute that the right-hand side of (6), viewed as an abstract ring, is isomorphic to \(\mathfrak{R}'\). The map (6) is the ring monomorphism \(\mathfrak{R} \rightarrow \mathfrak{R}'\), and it is not an isomorphism. The projective limit topology on the right-hand side of (6), however, is different from the topology on the ring \(\mathfrak{R}'\) arising from the above construction (indeed, the image of (6) is dense in the projective limit topology on the right-hand side).

Now we return to the general case of a complete, separated topological ring \(\mathfrak{R}\) with a base of neighborhoods of zero formed by open two-sided ideals. The map (6) has to be distinguished from the ring homomorphism

\[
(7) \quad \mathfrak{R} = \lim_{j \subset \mathfrak{R}} \mathfrak{R}/\mathfrak{I} \longrightarrow \lim_{j \subset \mathfrak{R}} (\mathfrak{R}/\mathfrak{I})/(\mathfrak{I} + \mathfrak{h}).
\]

In fact, there is a natural surjective homomorphism of discrete rings \(\mathfrak{R}/(\mathfrak{I} + \mathfrak{h}) \rightarrow (\mathfrak{R}/\mathfrak{I})/(H(\mathfrak{R}/\mathfrak{I}))\), but it is not always an isomorphism, because the map of Jacobson radicals \(\mathfrak{h}(R) \rightarrow H(R/\mathfrak{I})\) need not be surjective. So we have a commutative triangle diagram of continuous homomorphisms of topological rings

\[
\mathfrak{R} \longrightarrow \lim_{j \subset \mathfrak{R}} \mathfrak{R}/(\mathfrak{I} + \mathfrak{h}) \longrightarrow \lim_{j \subset \mathfrak{R}} (\mathfrak{R}/\mathfrak{I})/(H(\mathfrak{R}/\mathfrak{I})).
\]
The right-hand side of (7) is the completion of the quotient ring \( R/\mathcal{H} \) in its quotient topology. The kernel of the map (7) is the Jacobson radical \( \mathcal{H} \subset R \). The map (7) is surjective when the topological ring \( R \) has a countable base of neighborhoods of zero (see the discussion in Sections 2.2 and 2.11–2.12). We do not know whether the ring homomorphism (7) is surjective in general.

8. Products of Topological Rings

Let \( \Gamma \) be a set and \((A_\gamma)_{\gamma \in \Gamma}\) be a family of topological abelian groups, each of them with a base of neighborhoods of zero \( B_\gamma \) consisting of open subgroups. The product topology on the Cartesian product \( A = \prod_{\gamma \in \Gamma} A_\gamma \) has a base of neighborhoods of zero formed by the subgroups \( \prod_{\delta \in \Delta} U_\delta \times \prod_{\gamma \in \Gamma \setminus \Delta} A_\gamma \), where \( \Delta \subset \Gamma \) are finite subsets and \( U_\delta \in B_\delta \). The topological group \( A = \prod_{\gamma \in \Gamma} A_\gamma \) does not depend on the choice of bases of neighborhoods of zero \( B_\gamma \) in topological groups \( A_\gamma \).

If \( A'_\gamma \subset A_\gamma \) are subgroups in topological abelian groups \( A_\gamma \) and \( A'_\gamma \) are viewed as topological abelian groups in the induced topology, then the product topology on \( \prod_{\gamma \in \Gamma} A'_\gamma \) coincides with the induced topology on \( \prod_{\gamma \in \Gamma} A'_\gamma \subset \prod_{\gamma \in \Gamma} A_\gamma \). When the subgroups \( A'_\gamma \) are closed in \( A_\gamma \), so is the subgroup \( \prod_{\gamma \in \Gamma} A'_\gamma \subset \prod_{\gamma \in \Gamma} A_\gamma \). If the quotient groups \( A''_\gamma = A_\gamma / A'_\gamma \) are viewed as topological abelian groups in the quotient topology, then the product topology on \( A'' = \prod_{\gamma \in \Gamma} A''_\gamma \) coincides with the quotient topology on \( A'' = \prod_{\gamma \in \Gamma} A''_\gamma / \prod_{\gamma \in \Gamma} A'_\gamma \).

Let \((\mathcal{A}_\gamma)_{\gamma \in \Gamma}\) be a family of complete, separated topological abelian groups. Then the group \( \mathcal{A} = \prod_{\gamma \in \Gamma} \mathcal{A}_\gamma \) is complete and separated in the product topology. Moreover, for any set \( X \) there is a natural isomorphism of abelian groups

\[
\mathcal{A}[[X]] \cong \prod_{\gamma \in \Gamma} \mathcal{A}_\gamma[[X]].
\]

If \( \mathcal{H}_\gamma \subset \mathcal{A}_\gamma \) are strongly closed subgroups then \( \mathcal{H} = \prod_{\gamma \in \Gamma} \mathcal{H}_\gamma \) is a strongly closed subgroup in \( \mathcal{A} = \prod_{\gamma \in \Gamma} \mathcal{A}_\gamma \) (in the sense of Section 2.11).

If \((R_\gamma)_{\gamma \in \Gamma}\) is a family of topological rings (with or without unit), then \( R = \prod_{\gamma \in \Gamma} R_\gamma \) is a topological ring (with or without unit, respectively) in the product topology. If each of the topological rings \( R_\gamma \) has a base of neighborhoods of zero formed by open right (resp., two-sided) ideals, then the ring \( R \) also has a base of neighborhoods of zero formed by open right (resp., two-sided) ideals. If \( H_\gamma \) are topologically nil (resp., topologically left T-nilpotent) separated topological rings without unit, then their product \( H = \prod_{\gamma \in \Gamma} H_\gamma \) in its product topology is also a topologically nil (resp., topologically left T-nilpotent) topological ring without unit.

The following lemma is the main result of this section. Part (a) is an easy version of part (b), which is a generalization of [20 Lemma A.2.2] (see also [34 Theorem 4.5]).
Lemma 8.1. Let \((\mathcal{R}_\gamma)_{\gamma \in \Gamma}\) be a family of complete, separated topological rings, each of them having a base of neighborhoods of zero formed by open right ideals; and let \(\mathcal{R} = \prod_{\gamma \in \Gamma} \mathcal{R}_\gamma\) be their product. Then

(a) the coproduct functor \((N_\gamma)_{\gamma \in \Gamma} \mapsto \bigoplus_{\gamma \in \Gamma} N_\gamma\) establishes an equivalence between the Cartesian product of the abelian categories of discrete right \(\mathcal{R}_\gamma\)-modules over all \(\gamma \in \Gamma\) and the abelian category of discrete right \(\mathcal{R}\)-modules;

(b) the product functor \((\mathcal{C}_\gamma)_{\gamma \in \Gamma} \mapsto \prod_{\gamma \in \Gamma} \mathcal{C}_\gamma\) establishes an equivalence between the Cartesian product of the abelian categories of left \(\mathcal{R}_\gamma\)-contramodules over all \(\gamma \in \Gamma\) and the abelian category of left \(\mathcal{R}\)-contramodules.

Proof. Part (a): for every \(\gamma \in \Gamma\), denote by \(e_\gamma = (e_{\gamma, \gamma'})_{\gamma' \in \Gamma} \in \mathcal{R}\) the central idempotent element whose \(\gamma'\)-component \(e_{\gamma, \gamma'}\) is equal to \(0 \in \mathcal{R}_{\gamma'}\) for all \(\gamma' \in \Gamma\), \(\gamma' \neq \gamma\), and whose \(\gamma\)-component \(e_{\gamma, \gamma}\) is equal to \(1 \in \mathcal{R}_\gamma\). For any discrete right \(\mathcal{R}\)-module \(N\), the subgroup \(N e_\gamma \subseteq N\) is the maximal \(\mathcal{R}\)-submodule in \(N\) whose right \(\mathcal{R}\)-module structure comes from a (discrete) right \(\mathcal{R}_\gamma\)-module structure via the natural continuous ring homomorphism \(p_\gamma : \mathcal{R} \rightarrow \mathcal{R}_\gamma\). So, in the notation of Sections 2.9 and 2.12, we have \(p_\gamma^* (N) = N e_\gamma\). We claim that the functor \(N \mapsto (N, N e_\gamma)_{\gamma \in \Gamma}\) is quasi-inverse to the functor \((N_\gamma)_{\gamma \in \Gamma} \mapsto N = \bigoplus_{\gamma \in \Gamma} p_\gamma N_\gamma\), where \(N \in \text{discr-} \mathcal{R}\) and \(N_\gamma \in \text{discr-} \mathcal{R}_\gamma\). In other words, this simply means that any discrete right \(\mathcal{R}\)-module \(N\) is the direct sum of its submodules \(N e_\gamma \subseteq N\).

Indeed, the idempotents \(e_\gamma \in \mathcal{R}\), \(\gamma \in \Gamma\) are orthogonal to each other, which easily implies injectivity of the map \(\bigoplus_{\gamma \in \Gamma} N e_\gamma \rightarrow N\). To prove surjectivity, consider an element \(b \in N\). Since \(N\) is a discrete right \(\mathcal{R}\)-module by assumption, there exists a neighborhood of zero \(\mathcal{U} \subseteq \mathcal{R}\) such that \(b \mathcal{U} = 0\). By the definition of the product topology, there exists a finite subset \(\Delta \subseteq \Gamma\) such that \(\mathcal{J} = \prod_{\gamma \in \Gamma \setminus \Delta} \mathcal{R}_\gamma \subseteq \mathcal{U} \subseteq \mathcal{R}\). Consider the submodule \(N_3 \subseteq N\) of all elements annihilated by the closed two-sided ideal \(\mathcal{J} \subseteq \mathcal{R}\); then we have \(b \in N_3\). Now we have \(1 - \sum_{\delta \in \Delta} e_\delta \in \mathcal{J}\), hence \(b = \sum_{\delta \in \Delta} b e_\delta\) is a decomposition of the element \(b\) into the sum of elements \(b e_\delta \in N e_\delta\).

Part (b): we keep our notation for the central idempotent elements \(e_\gamma \in \mathcal{R}\). For any left \(\mathcal{R}\)-contramodule \(\mathcal{C}\), the map \(e_\gamma : \mathcal{C} \rightarrow e_\gamma \mathcal{C}\) represents \(e_\gamma \mathcal{C}\) as a quotient group of \(\mathcal{C}\). This is the maximal quotient \(\mathcal{R}\)-contramodule of \(\mathcal{C}\) whose left \(\mathcal{R}\)-contramodule structure comes from a left \(\mathcal{R}_\gamma\)-contramodule structure via the homomorphism \(p_\gamma\). So, in the notation of Sections 2.9 and 2.12, we have \(p_\gamma^* (\mathcal{C}) = e_\gamma \mathcal{C}\). We claim that the functor \(\mathcal{C} \mapsto (e_\gamma \mathcal{C})_{\gamma \in \Gamma}\) is quasi-inverse to the functor \((\mathcal{C}_\gamma)_{\gamma \in \Gamma} \mapsto \mathcal{C} = \prod_{\gamma \in \Gamma} p_\gamma \mathcal{C}_\gamma\), where \(\mathcal{C} \in \mathcal{R}\text{-contra}\) and \(\mathcal{C}_\gamma \in \mathcal{R}_\gamma\text{-contra}\). In other words, this simply means that the natural map

\[
eq (e_\gamma)_{\gamma \in \Gamma} : \mathcal{C} \rightarrow \prod_{\gamma \in \Gamma} e_\gamma \mathcal{C}
\]

is an isomorphism for any left \(\mathcal{R}\)-contramodule \(\mathcal{C}\).

Indeed, let us construct an inverse map to \(e\). Given a family of elements \(c_\gamma \in e_\gamma \mathcal{C}\), we consider them as elements of \(\mathcal{C}\) and assign to them the element

\[
f((c_\gamma)_{\gamma \in \Gamma}) = \pi e \left(\sum_{\gamma \in \Gamma} e_\gamma c_\gamma\right).
\]
Here it is important that the family of central idempotent elements $e_\gamma \in R$ converges to zero in the topology of $R$, so the expression $\sum_{\gamma \in \Gamma} e_\gamma c_\gamma$ defines an element of the set $R[[C]]$ of all convergent infinite formal linear combinations of elements of $C$ with the coefficients in $R$ (to which the contraaction map $\pi_\varepsilon: R[[C]] \rightarrow C$ can be applied). To check that $e \circ f = id$, it suffices to compute, for any family of elements $(c_\gamma \in C)_{\gamma \in \Gamma}$ and any fixed element $\gamma' \in \Gamma$,

$$e_{\gamma'} \pi_\varepsilon \left( \sum_{\gamma \in \Gamma} e_\gamma c_\gamma \right) = \pi_\varepsilon \left( \sum_{\gamma \in \Gamma} e_{\gamma'} e_\gamma c_\gamma \right) = e_{\gamma'} c_{\gamma'}$$

using the contraassociativity equation. To check that $f \circ e = id$, one computes, for any element $c \in C$,

$$\pi_\varepsilon \left( \sum_{\gamma \in \Gamma} e_\gamma (e_\gamma c) \right) = \left( \sum_{\gamma \in \Gamma} e_\gamma \right) c = c$$

by the contraassociativity equation and because the infinite sum $\sum_{\gamma \in \Gamma} e_\gamma$ converges to 1 in the topology of $R$.

\[\square\]

9. Projectivity of Flat Contramodules

In this section and in the next one, we consider the following setting. Let $R$ be a complete, separated topological associative ring with a base of neighborhoods of zero formed by open right ideals. Let $H \subset R$ be a strongly closed two-sided ideal in $R$ (see Sections 2.11–2.12). Assume that the quotient ring $S = R/\mathcal{H}$ is isomorphic, as a topological ring, to the product $\prod_{\gamma \in \Gamma} S_\gamma$ of a family of discrete rings $S_\gamma$ (viewed as a topological ring in the product topology), and that every ring $S_\gamma$ is a classically simple (i.e., simple Artinian) ring. In other words, $S_\gamma$ is the matrix ring of some finite order over a division ring (for every $\gamma$). Finally, we will also assume that the ideal $H$ is topologically left $T$-nilpotent.

**Lemma 9.1.** In the above assumptions, $H$ is the topological Jacobson radical of the topological ring $R$, and $H$ coincides with the Jacobson radical $H$ of the ring $R$ viewed as an abstract ring (without the topology).

**Proof.** One observes that any nonzero element of $S$ acts nontrivially in a certain simple discrete right $S$-module, so by Lemma 7.2(ii) the topological Jacobson radical of the ring $S$ vanishes. It remains to use Lemmas 7.6(a) and 7.8(b).

Denote the natural continuous ring homomorphisms by $p: R \rightarrow S$, $q_\gamma: S \rightarrow S_\gamma$, and $p_\gamma = q_\gamma p: R \rightarrow S_\gamma$. Set $J_\gamma = \ker(p_\gamma) \subset R$. Recall that, according to the discussion in Sections 2.9 and 2.12, the fully faithful functor of contrarestriction of scalars $p_\varepsilon: \mathcal{G}^\text{contra} \rightarrow R^\text{contra}$ has a left adjoint functor of contraextension of scalars $p_\varepsilon^*: R^\text{contra} \rightarrow \mathcal{G}^\text{contra}$ computable as $p_\varepsilon^*(C) = C/\mathcal{H} \times C$. Similarly, the fully faithful functor $p_{\gamma\varepsilon}: S_\gamma^\text{mod} = S_\gamma^\text{contra} \rightarrow R^\text{contra}$ has a left adjoint functor $p_{\gamma\varepsilon}^*: R^\text{contra} \rightarrow S_\gamma^\text{mod}$ computable as $p_{\gamma\varepsilon}^*(C) = C/J_\gamma \times C$. The fully faithful functor $q_{\gamma\varepsilon}: S_\gamma^\text{mod} \rightarrow \mathcal{G}^\text{contra}$ has a left adjoint functor $q_{\gamma\varepsilon}^*: \mathcal{G}^\text{contra} \rightarrow S_\gamma^\text{mod}$, which can be computed in the same fashion.
Finally, according to Lemma \[8.1\)(b), for any left \( \mathcal{S} \)-contramodule \( \mathcal{D} \) we have a natural direct product decomposition \( \mathcal{D} \cong \prod_{i \in I} q_i^* q_i^\circ \mathcal{D} \). So, in particular, for any left \( \mathcal{R} \)-contramodule \( \mathcal{C} \) one has \( \mathcal{C} / \mathcal{J} \triangleleft \mathcal{C} \cong \prod_{i \in I} \mathcal{C} / \mathcal{J}_i \triangleleft \mathcal{C} \).

The analogous assertions hold for discrete right modules. The fully faithful functor of restriction of scalars \( p_\circ : \text{disc} - \mathcal{S} \rightarrow \text{disc} - \mathcal{R} \) has a right adjoint functor of coextension of scalars \( p^\circ : \text{disc} - \mathcal{R} \rightarrow \text{disc} - \mathcal{S} \) computable as \( p^\circ (\mathcal{N}) = \mathcal{N}_R \). The fully faithful functor \( p_\circ : \text{mod} - \mathcal{S}_\gamma \rightarrow \text{disc} - \mathcal{R} \) has a right adjoint functor \( p_\circ^\circ : \text{disc} - \mathcal{S} \rightarrow \text{mod} - \mathcal{S}_\gamma \) computable as \( p_\circ^\circ (\mathcal{N}) = \mathcal{N}_R \). The fully faithful functor \( q_\circ : \text{mod} - \mathcal{S}_\gamma \rightarrow \text{disc} - \mathcal{S} \) has a right adjoint functor \( q_\circ^\circ : \text{disc} - \mathcal{S} \rightarrow \text{mod} - \mathcal{S}_\gamma \), which can be computed similarly. Finally, by Lemma \[8.1\](a), for any discrete right \( \mathcal{S} \)-module \( \mathcal{M} \) we have a natural direct sum decomposition \( \mathcal{M} \cong \bigoplus_{\gamma \in \Gamma} q_\circ^\circ \mathcal{M} \); so, in particular, for any discrete right \( \mathcal{R} \)-module \( \mathcal{N} \) one has \( \mathcal{N}_R \cong \bigoplus_{\gamma \in \Gamma} \mathcal{N}_\gamma \).

Lemma 9.2. Let \( \mathcal{R} \) be a complete, separated topological associative ring with a base of neighborhoods of zero formed by open right ideals, and let \( \mathcal{H} \subset \mathcal{R} \) be a topologically left \( T \)-nilpotent strongly closed two-sided ideal. Let \( f: \mathcal{F}' \rightarrow \mathcal{F}'' \) be a morphism of flat left \( \mathcal{R} \)-contramodules such that the induced morphism of left \( \mathcal{S} \)-contramodules \( \mathcal{F}' / \mathcal{H} \triangleleft \mathcal{F}' \rightarrow \mathcal{F}'' / \mathcal{H} \triangleleft \mathcal{F}'' \) is an isomorphism. Then the morphism \( f \) is surjective and its kernel is contained in \( \bigcap_{\mathcal{F} \subset \mathcal{R} \mathcal{I} \triangleleft \mathcal{F} \subset \mathcal{F}'} \), where the intersection is taken over all the open right ideals \( \mathcal{I} \subset \mathcal{R} \).

Proof. The conclusion that \( f \) is surjective does not depend on the flatness assumption on \( \mathcal{F}' \) and \( \mathcal{F}'' \), and only requires surjectivity of the map \( \mathcal{F}' / \mathcal{H} \triangleleft \mathcal{F}' \rightarrow \mathcal{F}'' / \mathcal{H} \triangleleft \mathcal{F}'' \). It suffices to set \( \mathcal{C} = \text{coker} (f) \), observe that \( \mathcal{C} / \mathcal{H} \triangleleft \mathcal{C} = 0 \), and apply the contramodule Nakayama Lemma \[6.2\] in order to conclude that \( \mathcal{C} = 0 \).

In order to prove the assertion about \( \text{ker} (f) \), we will show that the map of abelian groups \( N \circ \circ f: N \circ \circ \mathcal{F}' \rightarrow N \circ \circ \mathcal{F}'' \) is an isomorphism for any discrete right \( \mathcal{R} \)-module \( N \). In particular, it will follow that the map \( \mathcal{F}' / \mathcal{I} \triangleleft \mathcal{F} \rightarrow \mathcal{F}'' / \mathcal{I} \triangleleft \mathcal{F}'' \) is an isomorphism for any open right ideal \( \mathcal{I} \subset \mathcal{R} \), hence \( \text{ker} (f) \subset \mathcal{I} \triangleleft \mathcal{F} \).

Indeed, for any discrete right \( \mathcal{S} \)-module \( \mathcal{M} \), one has \( p_\circ \mathcal{M} \circ \circ \mathcal{F}' = \mathcal{M} \circ \circ p_\circ \mathcal{F}' = \mathcal{M} \circ \circ p_\circ \mathcal{F}'' = p_\circ \mathcal{M} \circ \circ \mathcal{F}'' \) (see Section \[2.2\]), so the map \( \mathcal{M} \circ \circ f \) is an isomorphism for any discrete right \( \mathcal{R} \)-module \( \mathcal{M} \) annihilated by \( \mathcal{H} \). Now, according to the discrete module Nakayama Lemma \[6.1\], any discrete right \( \mathcal{R} \)-module \( N \) has an increasing filtration \( 0 = F_0 N \subset F_1 N \subset F_2 N \subset \cdots \subset F_{\alpha} N = N \), indexed by some ordinal \( \alpha \), such that the quotient module \( F_{i+1} N / F_i N \) is annihilated by \( \mathcal{H} \) for all ordinals \( i < \alpha \) and \( F_j N = \bigcup_{i < j} F_i N \) for all limit ordinals \( j \leq \alpha \). Since the functors of contratensor product with \( \mathcal{F}' \) and \( \mathcal{F}'' \) are exact on the abelian category \( \text{disc} - \mathcal{R} \) by assumption, and since they also preserve colimits, it follows by induction on \( i \) that \( F_i N \circ \circ f \) is an isomorphism for all \( 0 \leq i \leq \alpha \).

Theorem 9.3. Let \( \mathcal{R} \) be a complete, separated topological associative ring with a base of neighborhoods of zero formed by open right ideals, let \( \mathcal{H} \subset \mathcal{R} \) be a topologically left \( T \)-nilpotent strongly closed two-sided ideal, and let \( \mathcal{S} = \mathcal{R} / \mathcal{H} \) be the quotient ring. Let \( \mathcal{F} \) be a flat left \( \mathcal{R} \)-contramodule. Then the left \( \mathcal{R} \)-contramodule \( \mathcal{F} / \mathcal{H} \triangleleft \mathcal{F} \) is projective if and only if the left \( \mathcal{S} \)-contramodule \( \mathcal{F} / \mathcal{H} \triangleleft \mathcal{F} \) is projective.
Proof. The functor of contraextension of scalars $f^\sharp$ with respect to a continuous homomorphism of topological rings $f$ always takes projective contramodules to projective contramodules, since it is left adjoint to an exact functor of contrarestriction of scalars $f_\flat$ (cf. Sections 2.9 and 2.12). So the “only if” assertion is obvious.

To prove the “if”, choose a set $X_0$ such that the projective left $S$-contramodule $Q = F/H \rightleftharpoons F$ is a direct summand of the free left $S$-contramodule $S[[X]]$. Setting $X = \mathbb{Z}_{\geq 0} \times X_0$, so that $S[[X]]$ is the coproduct of a countable family of copies of $S[[X_0]]$ in $S$–contra, and using the cancellation trick, one can see that the left $S$-contramodule $Q \oplus S[[X]]$ is isomorphic to $S[[X]]$.

Consider the left $R$-contramodule $F'' = F \oplus R[[X]]$ and put $Q'' = F''/H \rightleftharpoons F'' \cong Q \oplus S[[X]]$ (where $Y$ is a subset in $Q''$ bijective to $X$). Set $F' = R[[Y]]$ to be the free left $R$-contramodule with $Y$ generators. Then we have natural surjective left $R$-contramodule morphisms $F'' \twoheadrightarrow p_2Q'' = S[[Y]]$ and $F' \twoheadrightarrow S[[Y]]$. Since $F'$ is a projective left $R$-contramodule, the latter morphism lifts to a left $R$-contramodule morphism $f : F' \twoheadrightarrow F''$ satisfying the assumption of Lemma 9.2. (Notice that both the left $R$-contramodules $F'$ and $F''$ are flat.) Thus the morphism $f$ is surjective with $\ker(f) \subseteq \bigcap_{I \in \mathcal{I}} I \subseteq F'$.

Since the left $R$-contramodule $F'$ is projective (and even free) by construction, the natural map $F' \twoheadrightarrow \lim_{\leftarrow I \in \mathcal{I}} F'/I \rightleftharpoons F'$ is an isomorphism (see Section 2.10). So one has $\bigcap_{I \in \mathcal{I}} F' \subseteq F'$. Hence the morphism $f$ is an isomorphism. We have shown that $F''$ is a free left $R$-contramodule. Finally, we can conclude the left $R$-contramodule $F$ is projective as a direct summand of $F''$. \[\Box\]

The following corollary is a generalization of [20, Lemma A.3].

Corollary 9.4. In the assumptions formulated in the beginning of this section, all flat left $R$-contramodules are projective.

Proof. The abelian category $\mathcal{G}$–contra $\cong \prod_{\gamma \in \Gamma} S_\gamma$–mod (see Lemma 8.1(b)) is semisimple in these assumptions. So all left $\mathcal{G}$-contramodules are projective, and the assertion of the corollary follows from Theorem 9.3. \[\Box\]

10. Existence of Projective Covers

This section contains two proofs of its main result, which is Theorem 10.4. The first one is very short, consisting only of two references: one of them to the main result of the previous section, and the other one to a general theorem from category theory. The second proof is longer and more explicit.

Theorem 10.1. Let $\mathcal{B}$ be a locally presentable abelian category with enough projective objects. Assume that the class of all projective objects is closed under direct limits in $\mathcal{B}$. Then every object of $\mathcal{B}$ has a projective cover.
Proof. This is a particular case of \cite{29} Theorem 2.7, or Corollary 3.7, or Corollary 4.17. (This result goes back to \cite{12} Theorems 2.1 and 3.1 and \cite{3} Theorem 1.2.)

Corollary 10.2. Let $\mathcal{R}$ be a complete, separated topological associative ring with a base of neighborhoods of zero consisting of open right ideals. Assume that all flat left $\mathcal{R}$-contramodules are projective. Then every left $\mathcal{R}$-contramodule has a projective cover.

Proof. The abelian category $\mathcal{R}$–contra is locally presentable \cite{29} Section 5]. All the projective left $\mathcal{R}$-contramodules are flat, and the class of all flat left $\mathcal{R}$-contramodules is closed under direct limits (see Section 3]. Thus the assertion of the corollary follows from Theorem 10.1.

This is essentially all we need for our first proof of Theorem 10.4. To prepare ground for the second one, we have to address the question of lifting of idempotents.

It is a classical fact in the associative ring theory that idempotents can be lifted modulo any nil ideal. The following lemma provides a topological generalization. We refer to Section 3 for the definition of a topologically nil topological ring without unit.

Lemma 10.3. Let $\mathcal{R}$ be complete, separated topological associative ring with a base of neighborhoods of zero formed by open right ideals. Let $\mathcal{H} \subset \mathcal{R}$ be a topologically nil closed two-sided ideal, and let $S = \mathcal{R}/\mathcal{H}$ be the quotient ring. Then any idempotent element in $S$ can be lifted to an idempotent element in $\mathcal{R}$.

Proof. We adopt the argument from \cite{17} Tag 00J9 to the situation at hand. Let $\bar{e} \in S$ be an idempotent element. Choose any preimage $f \in \mathcal{R}$ of the element $\bar{e} \in S$. Proceeding by induction, we construct a sequence of elements $e_k \in \mathcal{R}$, $k \geq 0$, starting from $e_0 = f$ and passing from $e_k$ to $e_{k+1}$ by the rule

$$e_{k+1} = e_k - (2e_k - 1)(e_k^2 - e_k) = 3e_k^2 - 2e_k^3,$$

$$k \geq 0.$$

A straightforward computation yields

$$e_{k+1}^2 - e_k = (4e_k^2 - 4e_k - 3)(e_k^2 - e_k)^2.$$

Now let us show that the sequence of elements $e_k \in \mathcal{R}$ converges in the topology of $\mathcal{R}$ as $k \to \infty$, and that its limit $e$ is an idempotent element in $\mathcal{R}$ whose image in $S$ is equal to $\bar{e}$. Indeed, set $h = f^2 - f$; then we have $h \in \mathcal{H}$, since $e^2 - e = 0$ in $S$. Notice that all the elements $f$, $h$, and $e_k$ belong to the subring generated by $f$ in $\mathcal{R}$ over $\mathbb{Z}$; so they commute with each other. It follows from the above formulas by a simple induction on $k$ that $e_k^2 - e_k \in h^2 \mathcal{R}$ for all $k \geq 0$.

Let $\mathcal{I} \subset \mathcal{R}$ be an open right ideal. Since the ideal $\mathcal{H} \subset \mathcal{R}$ is topologically nil, there exists $n \geq 1$ such that $h^n \in \mathcal{I} \cap \mathcal{H}$. Choosing $m$ such that $2^m \geq n$, we find that $e_{k+1} - e_k = -(2e_k - 1)(e_k^2 - e_k) \in \mathcal{I}$ for all $k \geq m$. Thus the sequence of elements $e_k$ converges in $\mathcal{R}$ as $k \to \infty$, and we can consider its limit $e \in \mathcal{R}$. We also have $e_k^2 - e_k \in \mathcal{I}$ for all $k \geq m$, hence $e^2 - e \in \mathcal{I}$, and, as this holds for all the open right ideals $\mathcal{I} \subset \mathcal{R}$, it follows that $e^2 - e = 0$ in $\mathcal{R}$. Finally, $e_{k+1} - e_k \in h\mathcal{R} \subset \mathcal{H}$ for all $k \geq 0$, hence $e - f \in \mathcal{H}$, and therefore the image of $e$ in $S$ is equal to $\bar{e}$. □
Theorem 10.4. In the assumptions formulated in the beginning of Section 9, every left $R$-contramodule has a projective cover.

First proof. The assertion follows from Corollaries 9.4 and 10.2. □

Second proof. Let us first show that for every left $S$-contramodule $D$ there exists a projective left $R$-contramodule $P$ such that the left $S$-contramodule $P/\mathcal{H} \times P$ is isomorphic to $D$. Indeed, Lemma 8.1(b) applied to the ring $S = \prod_{\gamma \in \Gamma} S_{\gamma}$ implies that any left $S$-contramodule, viewed as an object of $S$-contra, is a coproduct of irreducible left $S$-contramodules. The irreducible left $S$-contramodules are indexed by the elements $\gamma \in \Gamma$ and have the form $q_{\gamma} I_{\gamma}$, where $I_{\gamma}$ is the (unique) irreducible left $S_{\gamma}$-module.

One easily finds a (noncentral) idempotent element $i_{\gamma} \in S$ such that $q_{\gamma} I_{\gamma} \cong S i_{\gamma}$. Lifting $i_{\gamma}$ to an idempotent element $i_{\gamma} \in R$ using Lemma 10.3 one can produce a projective left $R$-contramodule $P_{\gamma} = Ri_{\gamma}$ such that $P_{\gamma}/\mathcal{H} \times P_{\gamma} \cong S i_{\gamma}$. Finally, coproducts of projective objects are projective, and the reduction functor $\mathcal{E} \rightarrow P^{\sharp}(\mathcal{E}) = \mathcal{E}/\mathcal{H} \times \mathcal{E}$ preserves coproducts, which allows to construct a projective left $R$-contramodule $P$ such that $P/\mathcal{H} \times P \cong D$.

Now let $\mathcal{C}$ be a left $R$-contramodule. Consider the left $S$-contramodule $D = \mathcal{C}/\mathcal{H} \times \mathcal{C}$ and find a projective left $R$-contramodule $P$ such that $P/\mathcal{H} \times P \cong D$. Then we have two surjective left $R$-contramodule morphisms $P \rightarrow p_{*} D$ and $\mathcal{C} \rightarrow P_{*} D$. Since $P \in R$-contra is a projective object, we can lift the former morphism to a left $R$-contramodule morphism $f: P \rightarrow \mathcal{C}$ such that the induced morphism $P/\mathcal{H} \times P \rightarrow \mathcal{C}/\mathcal{H} \times \mathcal{C}$ is an isomorphism.

Arguing as in Lemma 9.2 and using the contramodule Nakayama Lemma 6.2, one shows that the map $f$ is surjective. We claim that the morphism $f$ is a projective cover of a left $R$-contramodule $\mathcal{C}$. Indeed, in view of Lemma 4.1 it suffices to check that $R = \ker(f)$ is a superfluous $R$-subcontramodule in $P$.

Let $\mathcal{G} \subset P$ be an $R$-subcontramodule such that $R + \mathcal{G} = P$. The morphism of left $S$-contramodules $p^{\sharp}(R) \rightarrow p^{\sharp}(P/\mathcal{G})$ is surjective, because the morphism of left $R$-contramodules $R \rightarrow P/\mathcal{G}$ is. On the other hand, the morphism $p^{\sharp}(R) \rightarrow p^{\sharp}(P)$ is zero, since the composition $R \rightarrow P \rightarrow \mathcal{C}$ vanishes and the morphism $p^{\sharp}(P) \rightarrow p^{\sharp}(\mathcal{C})$ is an isomorphism. Therefore, the composition $p^{\sharp}(R) \rightarrow p^{\sharp}(P) \rightarrow p^{\sharp}(P/\mathcal{G})$ also vanishes. It follows that $p^{\sharp}(P/\mathcal{G}) = 0$, that is $P/\mathcal{G} = \mathcal{H} \times (P/\mathcal{G})$. Applying Lemma 6.2 again, we conclude that $P/\mathcal{G} = 0$, as desired. □

11. Proof of Main Theorem

Let $R$ be a complete, separated topological associative ring with a base of neighborhoods of zero formed by open two-sided ideals. We will need to assume that one of the following three conditions holds:

(a) the ring $R$ is commutative; or
(b) $R$ has a countable base of neighborhoods of zero; or
(c) $\mathcal{R}$ has only a finite number of classically semisimple (semisimple Artinian) discrete quotient rings.

The following theorem is the main result of this paper.

**Theorem 11.1.** Let $\mathcal{R}$ be a complete, separated topological associative ring with a base of neighborhoods of zero formed by open two-sided ideals. Assume that one of the conditions (a), (b), or (c) is satisfied. Then the following conditions are equivalent:

(i) all flat left $\mathcal{R}$-contramodules have projective covers;
(ii) all left $\mathcal{R}$-contramodules have projective covers;
(iii) all flat left $\mathcal{R}$-contramodules are projective;
(iv) all Bass flat left $\mathcal{R}$-contramodules are projective;
(v) $\mathcal{R}$ has a topologically left $T$-nilpotent strongly closed two-sided ideal $\mathcal{H}$ such that the quotient ring $\mathcal{R}/\mathcal{H}$ is topologically isomorphic to a product of simple Artinian discrete rings endowed with the product topology;
(vi) all the discrete quotient rings of $\mathcal{R}$ are left perfect.

**Proof.** The implications (ii) $\implies$ (i) $\implies$ (i) and (iii) $\implies$ (iii) are obvious. So are the implications (iii) $\implies$ (i) and (iii) $\implies$ (i).

For any complete, separated topological ring $\mathcal{R}$ with a countable base of neighborhoods of zero formed by open right ideals, any left $\mathcal{R}$-contramodule has a flat cover [29, Corollary 7.9]. Hence the condition (iii) implies (ii) under the assumption of (b). Moreover, Corollary 10.2 provides the implication (iii) $\implies$ (ii) for any complete, separated topological ring $\mathcal{R}$ with a base of neighborhoods of zero formed by open right ideals. (But we do not need to use either of these arguments.)

The condition (iv) was already formulated in the beginning of Section 9. In particular, by Lemma 9.1 for any complete, separated topological ring $\mathcal{R}$ with a base of neighborhoods of zero formed by open right ideals the condition (iv) implies that $\mathcal{H}$ is the topological Jacobson radical of the ring $\mathcal{R}$, and that $\mathcal{H}$ also coincides with the Jacobson radical of the ring $\mathcal{R}$ viewed as an abstract ring.

The implications (iv) $\implies$ (iii) and (iv) $\implies$ (ii) are provided by Corollary 9.4 and Theorem 10.4, respectively, and hold for any complete, separated topological ring $\mathcal{R}$ with a base of neighborhoods of zero formed by open right ideals.

The implication (iii) $\implies$ (vi) is provided by Corollary 5.5 and the implication (i) $\implies$ (vi) by Corollary 5.7. Using the assumption of open two-sided ideals forming a base of neighborhoods of zero in $\mathcal{R}$, one can obtain the implication (i) $\implies$ (iii) from Corollary 8.4 and Proposition 4.4.

The implication (ii) $\implies$ (v) is provided by Proposition 5.3 with Lemma 7.3. The implication (v) $\implies$ (vi) follows straightforwardly from the characterization of left perfect rings in terms of the descending chain condition on principal right ideals (as in [41 Theorem P(6)] and [11 Theorem 28.4(e)]), as any right module over a discrete quotient ring of $\mathcal{R}$ is a discrete right module over $\mathcal{R}$. The converse implication (vi) $\implies$ (v) holds under the assumption of two-sided ideals forming a base of neighborhoods.
of zero in \( R \), as any finitely generated discrete right \( R \)-module is a module over a discrete quotient ring of \( R \) in this case.

It is the implication \((vi) \implies (iv)\) that needs both the assumption that \( R \) has a base of neighborhoods of zero formed by open two-sided ideals and one of the conditions \((a), (b),\) or \((c)\). Assuming \((vi)\) and denoting by \( H(R) \) the Jacobson radical (= nilradical) of a left perfect discrete ring \( R \), we set

\[
\mathfrak{J} = \lim_{\leftarrow} \mathfrak{J}(R/I) \subset \mathfrak{R},
\]

where the projective limit is taken over all the open two-sided ideals \( I \) in \( \mathfrak{R} \), to be the (topological) Jacobson radical of the ring \( \mathfrak{R} \) (see Lemma 7.9). Notice that for any surjective morphism of left perfect rings \( f : R' \to R'' \) one has \( f(H(R')) = H(R'') \).

In order to finish the proof of the theorem, it remains to apply the next proposition.

\[ \square \]

**Proposition 11.2.** Let \( \mathfrak{R} \) be a complete, separated topological ring with a base of neighborhoods of zero formed by open two-sided ideals such that all the discrete quotient rings of \( \mathfrak{R} \) are left perfect. Assume that one of the conditions \((a), (b),\) or \((c)\) is satisfied. Then \( \mathfrak{J} = \lim_{\leftarrow} \mathfrak{J}(R/I) \) is a topologically left T-nilpotent strongly closed two-sided ideal in \( \mathfrak{R} \), and the quotient ring \( \mathfrak{S} = \mathfrak{R}/\mathfrak{J} \) is topologically isomorphic to a product of simple Artinian discrete rings endowed with the product topology.

**Proof.** The two-sided ideal \( \mathfrak{J} \subset \mathfrak{R} \) is closed by construction and, viewed as a topological ring without unit, it is topologically left T-nilpotent as the projective limit of T-nilpotent discrete rings without unit. Alternatively, one can use the discussion in Section 7 and deduce the topological left T-nilpotency of the topological Jacobson radical \( \mathfrak{J} \) from the condition \((v)\) and Corollary 7.7.

In order to prove the remaining assertions, let us consider the three cases separately.

(b) First of all, any closed subgroup in a topological abelian group with a countable base of neighborhoods of zero is strongly closed (see Lemma 2.3).

Furthermore, for any discrete quotient ring \( R = \mathfrak{R}/\mathfrak{J} \) of the topological ring \( \mathfrak{R} \), we have a short exact sequence

\[
0 \longrightarrow H(R) \longrightarrow R \longrightarrow R/H(R) \longrightarrow 0.
\]

The transition maps in the projective system \((H(R/I))_{I \subset \mathfrak{R}} \) are surjective, so passing to the (countable filtered) projective limit we get a short exact sequence

\[
0 \longrightarrow \mathfrak{J} \longrightarrow \mathfrak{R} \longrightarrow \mathfrak{S} = \lim_{\leftarrow} \mathfrak{R}/H(R) \longrightarrow 0.
\]

This proves that the topological ring \( \mathfrak{S} \) is the topological projective limit of the countable filtered projective system of semisimple Artinian discrete rings \( R/H(R) \) and surjective morphisms between them. All such ring homomorphisms are projections onto direct factors, and it follows that \( \mathfrak{S} \) is a topological product of simple Artinian discrete rings.

(c) Let \( \mathfrak{J}_1 \) and \( \mathfrak{J}_2 \subset \mathfrak{R} \) be two open two-sided ideals such that the quotient rings \( \mathfrak{R}/\mathfrak{J}_1 \) and \( \mathfrak{R}/\mathfrak{J}_2 \) are semisimple Artinian. Since \( R = \mathfrak{R}/(\mathfrak{J}_1 \cap \mathfrak{J}_2) \) is a left perfect discrete ring by assumption, we have \( H(R) \subset \mathfrak{J}_1/(\mathfrak{J}_1 \cap \mathfrak{J}_2) \) and \( H(R) \subset \mathfrak{J}_2/(\mathfrak{J}_1 \cap \mathfrak{J}_2) \), so \( H(R) = 0 \) and \( R \) is a semisimple Artinian ring, too. Since \( \mathfrak{R} \) only has a finite
number of semisimple Artinian discrete quotient rings, it follows that there exists a unique minimal open two-sided ideal \( \mathcal{J} \subset \mathcal{R} \) such that \( \mathcal{R}/\mathcal{J} \) is semisimple Artinian.

Now if \( \mathcal{J} \subset \mathcal{J}' \subset \mathcal{R} \) is an open two-sided ideal, then \( H(\mathcal{R}/\mathcal{J}) = \mathcal{J}/\mathcal{J}' \). Thus we have \( \mathcal{J} = \mathcal{J}' \), so \( \mathcal{J} \) is an open (hence strongly closed) two-sided ideal in \( \mathcal{R} \) and the quotient ring \( \mathcal{R}/\mathcal{J} \) is a finite product of simple Artinian rings.

(a) Any perfect commutative ring uniquely decomposes as a finite product of perfect commutative local rings, while any semisimple commutative ring uniquely decomposes as a finite product of fields.

Let \( \Gamma \) be set of all open ideals \( \mathcal{G} \subset \mathcal{R} \) such that the discrete quotient ring \( \mathcal{R}/\mathcal{G} \) is a field. Furthermore, consider the set \( \Xi \) of all open ideals \( \mathcal{J} \subset \mathcal{R} \) such that the discrete quotient ring \( \mathcal{R}/\mathcal{J} \) is local. For any such ideal \( \mathcal{J} \), there exists a unique open ideal \( \mathcal{G} \in \Gamma \) for which \( \mathcal{J} \subset \mathcal{G} \). Denote by \( \Xi_\mathcal{G} \) the disjoint union of the set \( \{ \mathcal{J} \in \Xi \mid \mathcal{J} \subset \mathcal{G} \} \) with the one-point set \( \{ \mathcal{J} \} \) consisting of the unit ideal of \( \mathcal{R} \). We will say that the element \( \mathcal{R} \) is the marked point of the set \( \Xi_\mathcal{G} \). So the set \( \Xi \) is the disjoint union of the sets \( \Xi_\mathcal{G} \) over all open maximal ideals \( \mathcal{G} \in \Gamma \).

Now for any open ideal \( \mathcal{J} \subset \mathcal{R} \) the subset \( \Delta_3 \subset \Gamma \) of all \( \mathcal{G} \in \Gamma \) such that \( \mathcal{J} \subset \mathcal{G} \) is finite (and bijective to the spectrum of \( \mathcal{R}/\mathcal{J} \)). Furthermore, there exists a unique collection of open ideals \( \mathcal{I}_\mathcal{G} \in \Xi_\mathcal{G} \setminus \{ \mathcal{R} \} \), \( \mathcal{G} \in \Delta_3 \), such that \( \mathcal{J} \subset \mathcal{I}_\mathcal{G} \) and the natural ring homomorphism

\[
\mathcal{R}/\mathcal{J} \longrightarrow \prod_{\mathcal{G} \in \Delta_3} \mathcal{R}/\mathcal{I}_\mathcal{G}
\]

is an isomorphism. Conversely, for any finite subset \( \Delta \subset \Gamma \) and any collection of open ideals \( \mathcal{I}_\mathcal{G} \in \Xi_\mathcal{G} \setminus \{ \mathcal{R} \} \), \( \mathcal{G} \in \Delta \), the intersection \( \mathcal{J} = \bigcap_{\mathcal{G} \in \Delta} \mathcal{I}_\mathcal{G} \) is an open ideal in \( \mathcal{R} \) and the map \( \mathcal{R}/\mathcal{J} \longrightarrow \prod_{\mathcal{G} \in \Delta} \mathcal{R}/\mathcal{I}_\mathcal{G} \) is an isomorphism.

Given an open ideal \( \mathcal{J} \subset \mathcal{R} \) and an element \( \mathcal{G} \in \Gamma \setminus \Delta_3 \), we put \( \mathcal{I}_\mathcal{G} = \mathcal{R} \in \Xi_\mathcal{G} \). We have constructed a bijective correspondence between the set of all open ideals \( \mathcal{J} \subset \mathcal{R} \) and the set of all finitely supported elements of the product \( \prod_{\mathcal{G} \in \Gamma} \Xi_\mathcal{G} \). Here an element \( (\mathcal{I}_\mathcal{G})_{\mathcal{G} \in \Gamma} \in \prod_{\mathcal{G} \in \Gamma} \Xi_\mathcal{G} \) is said to be finitely supported if one has \( \mathcal{I}_\mathcal{G} = \mathcal{R} \) for all but a finite subset of indices \( \mathcal{G} \in \Gamma \).

For any two open ideals \( \mathcal{J}', \mathcal{J}'' \in \Xi_\mathcal{G} \), the intersection \( \mathcal{J}' \cap \mathcal{J}'' \) belongs to \( \Xi_\mathcal{G} \) again. Hence \( \Xi_\mathcal{G} \) is a directed poset with respect to inverse inclusion, with a minimal element which is the marked point \( \mathcal{R} \in \Xi_\mathcal{G} \). The local rings \( \mathcal{R}/\mathcal{J}, \mathcal{J} \in \Xi_\mathcal{G} \), form a projective system indexed by the directed poset \( \Xi_\mathcal{G} \), and we can form their projective limit

\[
\mathcal{R}_\mathcal{G} = \lim_{\leftarrow \mathcal{J} \in \Xi_\mathcal{G}} \mathcal{R}/\mathcal{J},
\]

endowing it with the projective limit topology. There is a natural ring homomorphism \( \mathcal{R} \longrightarrow \mathcal{R}_\mathcal{G} \) whose compositions with the projections \( \mathcal{R}_\mathcal{G} \longrightarrow \mathcal{R}/\mathcal{J} \) are surjective, so these projections are surjective, too. In particular, there is a natural surjective ring homomorphism \( \mathcal{R}_\mathcal{G} \longrightarrow \mathcal{R}/\mathcal{G} \), whose open kernel we denote by \( \mathcal{H}_\mathcal{G} \subset \mathcal{R}_\mathcal{G} \).

The poset of all open ideals \( \mathcal{J} \subset \mathcal{R} \) is isomorphic to the subposet of the product \( \prod_{\mathcal{G} \in \Gamma} \Xi_\mathcal{G} \) consisting of all the finitely supported elements. Moreover, the projective system of quotient rings \( \mathcal{R}/\mathcal{J} \) can be recovered as the finitely supported product of the projective systems \( \mathcal{R}/\mathcal{J}, \mathcal{J} \in \Xi_\mathcal{G} \). The latter words mean that we have a discrete ring isomorphism \( \mathcal{R}/\mathcal{J} \cong \prod_{\mathcal{G} \in \Gamma} \mathcal{R}/\mathcal{I}_\mathcal{G} \) for all open ideals \( \mathcal{J} \subset \mathcal{R} \), where \( \mathcal{I}_\mathcal{G} = \mathcal{R} \) and
so $R/J_\mathfrak{G} = 0$ for all $\mathfrak{G}$ outside of the finite subset $\Delta_\mathfrak{G} \subset \Gamma$; and such isomorphisms agree with the transition maps in the projective systems.

It follows from these considerations that the topological ring $R$ decomposes as the product of topological rings $R_\mathfrak{G}$,

$$R \cong \prod_{\mathfrak{G} \in \Gamma} R_\mathfrak{G},$$

and the topology on $R$ coincides with the product topology. Furthermore, under this isomorphism one has

$$\mathfrak{H} = \prod_{\mathfrak{G} \in \Gamma} \mathfrak{H}_\mathfrak{G}.$$ 

Now the ideal $\mathfrak{H} \subset R$ is strongly closed as a product of open ideals $\mathfrak{H}_\mathfrak{G} \subset R_\mathfrak{G}$ (cf. the discussion in the beginning of Section 8), and the quotient ring

$$\mathcal{S} = R/\mathfrak{H} \cong \prod_{\mathfrak{G} \in \Gamma} R_\mathfrak{G}/\mathfrak{H}_\mathfrak{G} = \prod_{\mathfrak{G} \in \Gamma} R/\mathfrak{G}$$

is the topological product of discrete fields. \[\Box\]

We will say that a topological ring $R$ is left pro-perfect if it is separated and complete, has a base of neighborhoods of zero consisting of open two-sided ideals, and all the discrete quotient rings of $R$ are left perfect. According to Theorem 11.1 over a left pro-perfect topological ring satisfying one of the conditions (a), (b), or (c) all left contramodules have projective covers and all flat left contramodules are projective. Conversely, any complete, separated topological ring with a base of neighborhoods of zero consisting of open two-sided ideals over which all Bass flat left contramodules have projective covers is pro-perfect.

12. Examples

Three (classes of) examples of pro-perfect topological rings are discussed below. The former two are commutative topological rings, while the third one is not.

**Example 12.1.** Let $R$ be a complete Noetherian commutative local ring with the maximal ideal $m \subset R$. We view $R$ as a topological ring in the $m$-adic topology. Then $R = \lim \leftarrow_{n \geq 1} R/m^n$ is a separated and complete topological ring with a base of neighborhoods of zero formed by the ideals $m^n \subset R$. Furthermore, $R$ is pro-perfect, as all of its discrete quotient rings are Artinian and consequently perfect. The maximal ideal $m \subset R$ is strongly closed and topologically T-nilpotent.

By [21, Theorem B.1.1] or [26, Example 2.2(4)], the forgetful functor $R\text{--contra} \rightarrow R\text{--mod}$ is fully faithful, so the abelian category of $R$-contramodules is a full subcategory in the category of arbitrary $R$-modules. This full subcategory consists of all the so-called $m$-contramodule $R$-modules, which means the $R$-modules $C$ such that $\text{Ext}_R^i(R[s^{-1}], C) = 0$ for all $i = 0, 1$ and all $s \in m$. It suffices to check this condition for any chosen set of generators $s_1, \ldots, s_m \in m$ of the ideal $m$ (or of any ideal in $R$ whose radical is equal to $m$) by [24, Theorem 5.1].
According to Theorem 11.1 all $\mathcal{R}$-contramodules have projective covers and all flat left $\mathcal{R}$-contramodules are projective. Let us explain how to obtain these results from the previously existing literature. An $\mathcal{R}$-contramodule is flat if and only if it is flat as an $\mathcal{R}$-module [21, Lemma B.9.2], [24, Corollary 10.3(a)]. All flat $\mathcal{R}$-contramodules are projective by [21, Corollary B.8.2] or [24, Theorem 10.5]. Moreover, the projective objects of the category $\mathcal{R}$–contra are precisely the free $\mathcal{R}$-contramodules $\mathcal{R}[[X]] = \lim_{\leftarrow n \geq 1} (\mathcal{R}/m^n)[[X]]$ (see [21, Lemma 1.3.2] or [24, Corollary 10.7]).

Concerning the projective covers, one observes that all $\mathcal{R}$-contramodules are Enochs cotorsion $\mathcal{R}$-modules [21, Proposition B.10.1], [24, Theorem 9.3]. Let $\mathcal{C}$ be an $\mathcal{R}$-contramodule, and let $f: F \to \mathcal{C}$ be a flat cover of the $\mathcal{R}$-module $\mathcal{C}$. Let $p: \mathcal{P} \to \mathcal{C}$ be a surjective morphism onto $\mathcal{C}$ from a projective $\mathcal{R}$-contramodule $\mathcal{P}$ with the kernel $\mathcal{R}$. Then $\mathcal{P}$ is also a flat $\mathcal{R}$-module, while $\mathcal{R}$ is a cotorsion $\mathcal{R}$-module; so $p$ is a special flat precover of the $\mathcal{R}$-module $\mathcal{C}$. It follows that the $\mathcal{R}$-module $F$ is a direct summand of $\mathcal{P}$; hence $F$ is also an $\mathcal{R}$-contramodule. Thus the morphism $f$ is a projective cover of $\mathcal{C}$ in the category $\mathcal{R}$–contra.

**Example 12.2.** Let $R$ be a commutative ring and $S \subset R$ be a multiplicative subset. The $S$-topology on an $R$-module $M$ has a base of neighborhoods of zero formed by the $R$-submodules $sM \subset M$, where $s \in S$. In particular, the ring $R$ itself is a topological ring in the $S$-topology. Let $\mathcal{R} = \lim_{\leftarrow s \in S} R/sR$ be its completion, endowed with the projective limit topology [25, Section 2]. Then $\mathcal{R}$ is a complete, separated topological commutative ring with a base of neighborhoods of zero formed by open ideals.

Assume that the quotient ring $R/sR$ is perfect for all $s \in S$. Then $\mathcal{R}$ is a proper perfect commutative topological ring, so Theorem 11.1 tells that all $\mathcal{R}$-contramodules have projective covers and all flat $\mathcal{R}$-contramodules are projective. These results do not seem to follow easily from the previously existing literature.

Let us discuss the category of $\mathcal{R}$-contramodules $\mathcal{R}$–contra in some more detail. Following the proof of Proposition 11.2(a), the topological ring $\mathcal{R}$ decomposes as the topological product $\mathcal{R} \cong \prod_{\mathfrak{S} \in \mathcal{R}} \mathcal{R}_\mathfrak{S}$ over the open ideals $\mathfrak{S} \subset \mathcal{R}$ such that the quotient ring $\mathcal{R}/\mathfrak{S}$ is a field. (The same argument allows to obtain such a decomposition in the slightly more general case of an $S$-h-nil ring $R$ [5, Section 6].) Such open ideals $\mathfrak{S} \subset \mathcal{R}$ correspond bijectively to the maximal ideals $\mathfrak{m} \subset R$ for which the intersection $\mathfrak{m} \cap S$ is nonempty, and the topological ring $\mathcal{R}_\mathfrak{S}$ can be described as the $S$-completion of the localization $R_\mathfrak{m}$ of the ring $R$. By Lemma 8.1(b), the abelian category $\mathcal{R}$–contra decomposes as the Cartesian product of the abelian categories $\mathcal{R}_\mathfrak{S}$–contra.

By [5, Theorem 6.13], the localization $R_S$ of the ring $R$ at the multiplicative subset $S$, viewed as an $R$-module, has projective dimension at most 1. Thus the full subcategory of $S$-contramodule $R$-modules $R$–mod$_{S, \text{contra}} \subset R$–mod, consisting of all the $R$-modules $C$ such that $\text{Ext}^i_R(R_S, C) = 0$ for $i = 0$ and 1, is an abelian category, and the identity embedding $R$–mod$_{S, \text{contra}} \to R$–mod is an exact functor [25, Theorem 3.4(a)]. The forgetful functor $\mathcal{R}$–contra $\to R$–mod factorizes as $\mathcal{R}$–contra $\to R$–mod$_{S, \text{contra}} \to R$–mod [26, Example 2.4(2)]. We studied the category $R$–mod$_{S, \text{contra}}$ in [5, Sections 4 and 6].
Still, the functor \( \mathcal{R} \text{- contra} \rightarrow R \text{- mod}_{S, \text{contra}} \) is not an equivalence of categories, generally speaking [26, Example 1.3(6)]. A sufficient condition for it to be an equivalence is that the \( S \)-torsion of \( R \) be bounded [26, Example 2.4(3)]. More generally, using the decomposition of the category \( \mathcal{R} \text{- contra} \) into the Cartesian product over the maximal ideals \( m \) of the ring \( R \) with \( m \cap S \neq \emptyset \) and the similar decomposition of the category \( R \text{- mod}_{S, \text{contra}} \) [5, Corollary 6.15], one shows that the functor \( \mathcal{R} \text{- contra} \rightarrow R \text{- mod}_{S, \text{contra}} \) is an equivalence whenever the \( S \)-torsion in \( R_m \) is bounded for every \( m \). When \( S \) is countable, the functor \( \mathcal{R} \text{- contra} \rightarrow R \text{- mod}_{S, \text{contra}} \) is an equivalence of categories if and only if the \( S \)-torsion in \( \mathcal{R} = \lim \leftarrow_{s \in S} R/sR \) is bounded [26, Example 5.4(2)].

Furthermore, [26, Example 3.7(1)] lists two conditions which, taken together, are sufficient for the functor \( \mathcal{R} \text{- contra} \rightarrow R \text{- mod}_{S, \text{contra}} \) to be fully faithful. By [5, Proposition 2.1(2)], every \( S \)-divisible \( R \)-module is \( S \)-injective, so one of the two conditions always holds in our case. Hence the functor \( \mathcal{R} \text{- contra} \rightarrow R \text{- mod}_{S, \text{contra}} \) is fully faithful whenever the other condition holds, that is, whenever for every set \( X \) the free \( \mathcal{R} \)-contramodule \( \mathcal{R}[\{X\}] = \lim \leftarrow_{s \in S} \{R/sR\}[X] \) is complete in its \( S \)-topology (or in other words, its \( S \)-topology coincides with its projective limit topology [25, Theorem 2.3]).

In particular, the functor \( \mathcal{R} \text{- contra} \rightarrow R \text{- mod}_{S, \text{contra}} \) is fully faithful whenever \( S \) is countable [26, Example 3.7(2)]. In this case, every object of \( R \text{- mod}_{S, \text{contra}} \) is an extension of two objects from \( \mathcal{R} \text{- contra} \) [26, Example 5.4(2)].

**Example 12.3.** Let \( \mathcal{C} \) be a coassociative, counital coalgebra over a field \( k \). Then the dual vector space \( \mathcal{C}^* = \text{Hom}_k(\mathcal{C}, k) \) to the coalgebra \( \mathcal{C} \) has a natural structure of complete, separated topological ring (in fact, topological algebra over \( k \)) with a base of neighborhoods of zero formed by open two-sided ideals. Moreover, any coassociative coalgebra over a field is the union of its finite-dimensional subcoalgebras; so \( \mathcal{C}^* \) is the projective limit of a directed diagram of finite-dimensional algebras (and surjective morphisms between them). The discrete quotient rings of \( \mathcal{C}^* \) are the finite-dimensional algebras dual to the finite-dimensional subcoalgebras of \( \mathcal{C} \) [36, Sections 1.1 and 2.2], [23, Section 1.3].

Since all finite-dimensional algebras are perfect, the topological ring \( \mathcal{C}^* \) is pro-perfect. All closed \( k \)-vector subspaces in the pro-finite-dimensional vector space \( \mathcal{C}^* \) are strongly closed (see Example 2.5). Moreover, denoting by \( \mathcal{D} \subset \mathcal{C} \) the maximal cosemisimple subcoalgebra in \( \mathcal{C} \), one observes that the coalgebra \( \mathcal{D} \) is a direct sum of cosemisimple coalgebras, while the quotient coalgebra without counit \( \mathcal{C}/\mathcal{D} \) is conilpotent. In other words, this means that the closed two-sided ideal \( \mathfrak{N} = (\mathcal{C}/\mathcal{D})^* \subset \mathcal{C}^* \) is not only topologically left T-nilpotent, but even topologically nilpotent, that is, for any neighborhood of zero \( \mathfrak{U} \subset \mathcal{C}^* \) there exists an integer \( n \geq 1 \) such that \( \mathfrak{N}^n \subset \mathfrak{U} \). The quotient ring \( \mathcal{C}^*/\mathfrak{N} \simeq \mathcal{D}^* \) is a product of simple finite-dimensional \( k \)-algebras with the product topology [36, Sections 8.0 and 9.0–9.1], [20, Section A.2].

Thus the condition (iv) of Theorem 11.4 is satisfied for \( \mathcal{C}^* \). As we will see below in Theorem 13.7, it follows that the topological ring \( \mathcal{C}^* \) satisfies the condition (d) (though it does not need to satisfy (a), (b), or (c)).

A discrete right module over \( \mathcal{C}^* \) is the same thing as a right \( \mathcal{C} \)-comodule [36, Section 2.1], [23, Section 1.4]. All \( \mathcal{C} \)-comodules are unions of their finite-dimensional
subcomodules. Left contramodules over the topological ring $C^*$ are otherwise known as left contramodules over the coalgebra $C$. According to the proof of Theorem 11.1, it follows that all flat $C$-contramodules are projective (cf. [20, Sections 0.2.9 and A.3]) and all $C$-contramodules have projective covers.

13. Generalization of Main Theorem

The aim of this section is to generalize the result of Theorem 11.1 so that the class of topological rings covered by its equivalent conditions includes all the rings satisfying the assumptions formulated in the beginning of Section 9.

Let $R$ be a complete, separated topological ring $R$ with a base of neighborhoods of zero formed by open right ideals. We are interested in the following condition on the topological ring $R$, generalizing the conditions (a-c) of Section 11:

(d) there is a topologically left $T$-nilpotent strongly closed two-sided ideal $K \subset R$ such that the quotient ring $R/K$ is isomorphic, as a topological ring, to the product $\prod_{\delta \in \Delta} T_{\delta}$ of a family of topological rings $T_{\delta}$, each of which has a base of neighborhoods of zero consisting of open two-sided ideals and satisfies one of the conditions (a), (b), or (c) of Section 11.

Here the quotient ring $R/K$ is endowed with the quotient topology and the product of topological rings $\prod_{\delta \in \Delta} T_{\delta}$ is endowed with the product topology. The following example shows that a topological ring satisfying (d) does not need to have a base of neighborhoods of zero consisting of open two-sided ideals.

Example 13.1. Let $R = \text{Hom}_{\mathbb{Z}}(Q \oplus Q/\mathbb{Z}, Q \oplus Q/\mathbb{Z})^{\text{op}}$ be the opposite ring to the ring of endomorphisms of the abelian group $Q \oplus Q/\mathbb{Z}$, endowed with the topology defined in Section 2.13. This topological ring, occurring in tilting theory, was described in [30, Example 8.4] as the matrix ring

$$R = \begin{pmatrix} Q & \mathbb{A}^f_Q \hat{\mathbb{Z}} \\ 0 & \hat{\mathbb{Z}} \end{pmatrix},$$

where $\hat{\mathbb{Z}} = \prod_{p} \mathbb{Z}_p$ is the product over the prime numbers $p$ of the topological rings of $p$-adic integers $\mathbb{Z}_p$ endowed with the $p$-adic topology, $\mathbb{A}^f_Q = Q \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ is the ring of finite adeles of the field of rational numbers $Q$ endowed with the adele topology, and the field $Q$ itself is endowed with the discrete topology.

Consider the two-sided ideal $K = \mathbb{A}^f_Q \subset R$. Then one has $K^2 = 0$, so this ideal is even (finitely) nilpotent. It is also clearly strongly closed in $R$. The quotient ring $R/K$ is commutative, so it satisfies the condition (a). The topological ring $R$ has a base of neighborhoods of zero formed by the open right ideals

$$\begin{pmatrix} 0 & r\hat{\mathbb{Z}} \\ 0 & n\hat{\mathbb{Z}} \end{pmatrix} \subset R,$$
where \( r \in \mathbb{Q}_{>0} \) and \( n \in \mathbb{Z}_{>0} \) are an arbitrary positive rational number and a positive integer. But every open two-sided (or even left) ideal in \( \mathcal{R} \) contains \( \mathcal{R} \), so such ideals do not form a base of neighborhoods of zero.

**Example 13.2.** Let \( \alpha \) be an ordinal, and let \((M_i)\) be an \( \alpha \)-indexed sequence of left modules over an associative ring \( R \). Assume that all the morphisms between the \( R \)-modules \( M_i \) go backwards, that is, \( \text{Hom}_R(M_i, M_j) = 0 \) for all \( 0 \leq i < j < \alpha \). Let \( \Sigma_i = \text{Hom}_R(M_i, M_i)^{\text{op}} \) be topological rings opposite to the endomorphism rings of the \( R \)-modules \( M_i \), and let \( \Sigma = \text{Hom}_R(M, M)^{\text{op}} \) be the topological ring opposite to the ring of endomorphisms of the \( R \)-module \( M = \bigoplus_{i<\alpha} M_i \).

Then there is a natural surjective morphism of topological rings \( p: \Sigma \twoheadrightarrow \prod_{i \in \alpha} \Sigma_i \). Set \( \mathcal{K} = \ker(p) \subset \Sigma \); then \( \mathcal{K} \) is a strongly closed two-sided ideal in \( \Sigma \) and the topological quotient ring \( \Sigma/\mathcal{K} \) is isomorphic to the topological product \( \prod_{i \in \alpha} \Sigma_i \). Moreover, the ideal \( \mathcal{K} \) is topologically left \( T \)-nilpotent, because for every element \( b \in M \) and any sequence of endomorphisms \( a_1, a_2, a_3, \ldots \in \mathcal{K} \) one has \( ba_a a_2 \cdots a_n = 0 \) in \( M \) for \( n \) large enough (as one easily shows using König’s lemma).

Thus if for every \( i < \alpha \) the topological ring \( \Sigma_i \) has a base of neighborhoods of zero consisting of open two-sided ideals and satisfies one of the conditions (a), (b), or (c) of Section 11, then the topological ring \( \Sigma \) satisfies the condition (d). Moreover, if for every index \( i \) the topological ring \( \Sigma_i \) satisfies the condition (d), then so does the topological ring \( \Sigma \) (as we will see below in Lemma 13.6(b)).

**Lemma 13.3.** (a) Let \((R_{\gamma})_{\gamma \in \Gamma}\) be a family of topological rings. Then all the discrete quotient rings of the topological ring \( R = \prod_{\gamma \in \Gamma} R_{\gamma} \) are left perfect if and only if all the discrete quotient rings of the topological rings \( R_{\gamma}, \gamma \in \Gamma \), are left perfect.

(b) Let \( R \) be a separated topological ring, and let \( K \subset R \) be a topologically left \( T \)-nilpotent closed two-sided ideal. Then all the discrete quotient rings of the ring \( R \) are left perfect if and only if all the discrete quotient rings of the topological ring \( R/K \) are left perfect.

**Proof.** Part (a) holds, because the discrete quotient rings of \( R \) are the finite products of discrete quotient rings of \( R_{\gamma} \), taken over the finite subsets of the set \( \Gamma \). Furthermore, a finite product of left perfect rings is left perfect. Part (b) follows from its discrete version: if \( \overline{K} \) is a left \( T \)-nilpotent two-sided ideal in an associative ring \( \overline{R} \) and the quotient ring \( \overline{R}/\overline{K} \) is left perfect, then the ring \( R \) is left perfect. The latter is obtainable from the characterization of left perfect rings in [4, Theorem P (1)] and the discrete version of Lemma 6.3. \( \square \)

The following theorem is our generalization of Theorem 11.1.

**Theorem 13.4.** Let \( \mathcal{R} \) be a complete, separated topological associative ring with a base of neighborhoods of zero formed by open right ideals. Assume that the condition (d) is satisfied. Then the following conditions are equivalent:

(i) all flat left \( \mathcal{R} \)-contramodules have projective covers;

(ii) all Bass flat left \( \mathcal{R} \)-contramodules have projective covers;

(iii) all left \( \mathcal{R} \)-contramodules have projective covers;
(iii) all flat left $R$-contramodules are projective;

(iii') all Bass flat left $R$-contramodules are projective;

(iv) $R$ has a topologically left $T$-nilpotent strongly closed two-sided ideal $\mathcal{H}$ such that the quotient ring $R/\mathcal{H}$ is topologically isomorphic to a product of simple Artinian discrete rings endowed with the product topology;

(v) all descending chains of cyclic discrete right $R$-modules terminate;

(vi) all discrete quotient rings of $R$ are left perfect;

(vi') all discrete quotient rings of $R/\mathcal{K}$ are left perfect.

Conversely, if a complete, separated topological associative ring with a base of neighborhoods of zero formed by open right ideals satisfies (iv), then it also satisfies (d).

Proof. The first four paragraphs of the proof of Theorem 11.1 apply in our present context as well. Furthermore, as in Theorem 11.1, the implication (iii') $\Rightarrow$ (vi) is provided by Corollary 5.5, the implication (i') $\Rightarrow$ (vi) by Corollary 5.7, the implication (iii') $\Rightarrow$ (v) by Proposition 5.3 with Lemma 7.3, and the implication (v) $\Rightarrow$ (vi) is easy. The implication (vi) $\Rightarrow$ (vi') is obvious, and the inverse implication (vi') $\Rightarrow$ (vi) is provided by Lemma 13.3(b).

The final implication (vi') $\Rightarrow$ (iv) holds in the assumption of the condition (d). This one, as well as the converse implication (iv) $\Rightarrow$ (d), are provided by the following proposition. In other words, the proposition below shows that (iv) is equivalent to the combination of (vi') and (d).

Proposition 13.5. Let $R$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Suppose that there exists an ideal $\mathcal{K} \subset R$ such that the condition (d) is satisfied and all the discrete quotient rings of the ring $R/\mathcal{K}$ are left perfect. Then there exists an ideal $\mathcal{H} \subset R$ satisfying (iv). Conversely, if an ideal $\mathcal{H} \subset R$ satisfies (iv), then the same ideal $\mathcal{K} = \mathcal{H}$ also satisfies (d).

Proof. The converse assertion is obvious: any simple Artinian ring endowed with the discrete topology satisfies both (b) and (c), so a product of such rings is a product of topological rings satisfying (b) and (c). To prove the direct implication, suppose that $\mathcal{K} \subset R$ is an ideal satisfying (d) such that all the discrete quotient rings of $R/\mathcal{K}$ are left perfect. Then we have $R/\mathcal{K} \cong \prod_{\delta \in \Delta} \mathcal{S}_\delta$, so any discrete quotient ring of $\mathcal{S}_\delta$ is at the same time a discrete quotient ring of $R/\mathcal{K}$.

Applying Proposition 11.2 to the topological ring $\mathcal{S}_\delta$, we conclude that there exists a topologically left $T$-nilpotent strongly closed two-sided ideal $\mathcal{J}_\delta \subset \mathcal{S}_\delta$ such that the quotient ring $\mathcal{S}_\delta / \mathcal{J}_\delta$ is topologically isomorphic to a product of discrete simple Artinian rings, $\mathcal{S}_\delta \cong \prod_{\gamma \in \Gamma_\delta} S_\gamma$. According to the discussion in the beginning of Section 8, it follows that $\mathcal{J} = \prod_{\delta} \mathcal{J}_\delta$ is a topologically left $T$-nilpotent strongly closed two-sided ideal in $\mathcal{T} = \prod_{\delta} \mathcal{S}_\delta$. Furthermore, the topological quotient ring $\mathcal{T}/\mathcal{J} \cong \prod_{\delta \in \Delta} \mathcal{S}_\delta$ is isomorphic to the topological product $\prod_{\gamma \in \Gamma} S_\gamma$ of the discrete simple Artinian rings $S_\gamma$ over the disjoint union $\Gamma = \bigsqcup_{\delta \in \Delta} \Gamma_\delta$ of the sets of indices $\Gamma_\delta$.

Now we have a surjective continuous ring homomorphism $R \rightarrow R/\mathcal{K} \cong \mathcal{T}$. Let $\mathcal{H} \subset R$ be the full preimage of the closed ideal $\mathcal{J} \subset \mathcal{T}$ under this homomorphism. Then the ideal $\mathcal{H}$ is strongly closed in $R$ by Lemma 2.3(b), $\mathcal{H}$ is topologically left
T-nilpotent by Lemma 6.3, and the topological ring $\mathcal{R}/\mathfrak{J} \cong \mathcal{S}/\mathfrak{J}$ is the topological product of discrete simple Artinian rings $S_\gamma$. □

The following lemma shows the class of all topological rings satisfying (d) is closed under the operations that were used to define it.

**Lemma 13.6.** (a) Let $(\mathcal{R}_\gamma)_{\gamma \in \Gamma}$ be a family of topological rings satisfying the condition (d). Then the topological ring $\mathcal{R} = \prod_{\gamma \in \Gamma} \mathcal{R}_\gamma$ also satisfies (d).

(b) Let $\mathcal{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals, and let $\mathfrak{J} \subset \mathcal{R}$ be a topologically left T-nilpotent strongly closed two-sided ideal. Assume that the topological quotient ring $\mathcal{R}/\mathfrak{J}$ satisfies (d). Then the topological ring $\mathcal{R}$ satisfies (d).

**Proof.** Similar to the proof of Proposition 13.5. The proof of part (a) is based on the discussion in the beginning of Section 8, while the proof of part (b) uses Lemmas 2.4(b) and 6.3. □

**Remark 13.7.** More generally, for any complete, separated topological ring $\mathcal{R}$ with a base of neighborhoods of zero formed by open right ideals, the four conditions (i), (ii), (iii), and (iv) are equivalent to each other. This is a result of the paper [32, Theorem 14.1]. Furthermore, in the same (full) generality, the two conditions (i*) and (iii*) are equivalent to each other by [7, Corollary 3.10].

A conjecture claiming all the conditions (i), (i*), (ii), (iii), (iii*), (iv), and (v) to be equivalent to each other is formulated in the paper [32] as [32, Conjecture 14.3]. This conjecture is true for all complete, separated topological rings $\mathcal{R}$ with a countable base of neighborhoods of zero consisting of open right ideals [32, Theorem 14.8].

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