STRONG COUPLING EXPANSION AND
SEIBERG–WITTEN–WHITHAM EQUATIONS

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Abstract

We study the Seiberg-Witten-Whitham equations in the strong coupling regime of the $\mathcal{N} = 2$ super Yang-Mills theory in the vicinity of the maximal singularities. In the case of $SU(2)$ the Seiberg-Witten-Whitham equations fix completely the strong coupling expansion. For higher rank $SU(N)$ they provide a set of non-trivial constraints on the form of this expansion. As an example, we study the off-diagonal couplings at the maximal point for which we propose an ansatz that fulfills all the equations.

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The low energy dynamics of $\mathcal{N} = 2$ supersymmetric gauge theories is governed by a single holomorphic function $F$ known as the effective prepotential. A self-consistent proposal for this function has been done a few years ago by Seiberg and Witten \cite{Seiberg:1994rs} in the $SU(2)$ case and generalized to $SU(N)$ by many authors \cite{Nekrasov:2002qd}. The Seiberg-Witten solution for the effective theory of $\mathcal{N} = 2$ super Yang-Mills can be embedded into the Whitham hierarchy associated to the periodic Toda lattice \cite{Harnad:1996zw}. The link between both constructions is summarized in the statement that the prepotential of the $\mathcal{N} = 2$ Yang-Mills theory corresponds to the logarithm of the Toda’s quasiclassical tau function.

On the Whitham hierarchy side, there is a family of new variables entering into the prepotential known as slow times. They can be promoted to spurion superfields that softly break $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 0$ with higher Casimir perturbations \cite{Intriligator:1995ne}, something that might drive the theory to a vacuum placed in the neighborhood of the so-called Argyres-Douglas singularities \cite{Argyres:1994xu}. Aside from this physical motivation and its mathematical interest, the Seiberg-Witten-Whitham (SWW) correspondence also provides us with new techniques to explore the structure of the effective action of $\mathcal{N} = 2$ theories. In Ref.\cite{Nekrasov:2002qd}, for example, the second derivatives of the prepotential with respect to the Whitham times were computed and shown to be given in terms of Riemann Theta functions associated to the root lattice of the gauge group. When evaluated in the Seiberg-Witten theory, these formulae were shown to provide a powerful tool to compute all instanton corrections to the effective prepotential of $SU(N) \mathcal{N} = 2$ super Yang-Mills theory from the one-loop contribution through a recursive relation \cite{Gaiotto:2008cd}.

It is the aim of this letter to apply the Seiberg-Witten-Whitham equations to the strong coupling regime of the $\mathcal{N} = 2$ super Yang-Mills theory with gauge group $SU(N)$. We will start by considering $\mathcal{N} = 2$ $SU(2)$ super Yang-Mills theory and show that, starting from the Whitham hierarchy side, the exact effective prepotential near the monopole singularity is obtained in a remarkably simple way up to arbitrary order in the dual variable. We will then consider the generic $SU(N)$ case and show that the SWW equations do not give a closed recursive procedure to perform the same computation. Nevertheless, as we will see, these equations can be used to study the couplings between different magnetic $U(1)$ factors at the maximal singularities of the moduli space.

Let us start with a brief description of our general framework. The low-energy dynamics of $SU(N) \mathcal{N} = 2$ super Yang-Mills theory is described in terms of the hyperelliptic curve \cite{Nekrasov:2002qd}

$$y^2 = P^2(\lambda, u_k) - 4\Lambda^{2N},$$  

(1)

where $P(\lambda, u_k) = \lambda^N - \sum_{k=2}^{N} u_k \lambda^{N-k}$ is the characteristic polynomial of $SU(N)$ and
$u_k$, $k = 2, \ldots, N$ are the Casimirs of the gauge group. This curve can be identified with the spectral curve of the $N$ site periodic Toda lattice and, moreover, the prepotential of the effective theory is essentially the logarithm of the corresponding quasiclassical tau function, hence depending on the slow times $T_n$, $1 \leq n \leq N - 1$, of the corresponding Whitham hierarchy\footnote{It was shown in Refs.\cite{ref1, ref2} that the relevant quantities are not $T_n$ but the rescaled Whitham slow times $\hat{T}_n = T_n/T_1$ (in fact, the SWW correspondence also requires a rescaling of the scalar variables $\hat{a}^i = T_1 a^i$ and, consequently, of the Casimirs $\hat{u}_k = T_1^k u_k$). We will always work in what follows with the rescaled variables and, therefore, hats will be omitted everywhere.}. The derivatives of the prepotential with respect to the Whitham slow times have been obtained in Ref.\cite{ref3}. When restricted to the submanifold $T_{n > 1} = 0$, where the Seiberg-Witten solution lives (we may identify $T_1$ with $\Lambda$ in this submanifold\footnote{It was shown in Refs.\cite{ref1, ref2} that the relevant quantities are not $T_n$ but the rescaled Whitham slow times $\hat{T}_n = T_n/T_1$ (in fact, the SWW correspondence also requires a rescaling of the scalar variables $\hat{a}^i = T_1 a^i$ and, consequently, of the Casimirs $\hat{u}_k = T_1^k u_k$). We will always work in what follows with the rescaled variables and, therefore, hats will be omitted everywhere.}), the first derivatives are simply

$$\frac{\partial F}{\partial \log \Lambda} = \frac{\beta}{2\pi i} \mathcal{H}_2 \quad \frac{\partial F}{\partial T_n} = \frac{\beta}{2\pi i n} \mathcal{H}_{n+1} \quad , \tag{2}$$

while the second order derivatives with respect to the slow times result in\footnote{It was shown in Refs.\cite{ref1, ref2} that the relevant quantities are not $T_n$ but the rescaled Whitham slow times $\hat{T}_n = T_n/T_1$ (in fact, the SWW correspondence also requires a rescaling of the scalar variables $\hat{a}^i = T_1 a^i$ and, consequently, of the Casimirs $\hat{u}_k = T_1^k u_k$). We will always work in what follows with the rescaled variables and, therefore, hats will be omitted everywhere.}

$$\frac{\partial^2 F}{\partial (\log \Lambda)^2} = -\frac{\beta^2}{2\pi i} \frac{\partial \mathcal{H}_2}{\partial a^i} \frac{\partial \mathcal{H}_2}{\partial \omega^j} \frac{1}{i\pi} \partial_{r_{ij}} \log \Theta_E(0|\tau) \quad ,$$
$$\frac{\partial^2 F}{\partial \log \Lambda \partial T_n} = -\frac{\beta^2}{2\pi i n} \frac{\partial \mathcal{H}_2}{\partial a^i} \frac{\partial \mathcal{H}_{n+1}}{\partial \omega^j} \frac{1}{i\pi} \partial_{r_{ij}} \log \Theta_E(0|\tau) \quad , \quad \frac{\partial^2 F}{\partial T_m \partial T_n} = -\frac{\beta}{2\pi i} \left( \mathcal{H}_{m+1,n+1} + \frac{\beta n}{\beta m n} \frac{\partial \mathcal{H}_{m+1}}{\partial a^i} \frac{\partial \mathcal{H}_{n+1}}{\partial \omega^j} \frac{1}{i\pi} \partial_{r_{ij}} \log \Theta_E(0|\tau) \right) \quad , \tag{3}$$

with $m, n \geq 2$ and $\beta = 2N$. The functions $\mathcal{H}_{m,n}$ are certain homogeneous combinations of the Casimirs $u_k$, given by

$$\mathcal{H}_{m+1,n+1} = \frac{N}{m n} \text{res}_{\infty} \left( P^{m/N}(\lambda) dP^{n/N}_+(\lambda) \right) \quad \text{and} \quad \mathcal{H}_{m+1} \equiv \mathcal{H}_{m+1,2} = u_{m+1} + \mathcal{O}(u_m) \quad .$$

Here $\left( \sum_{k=-\infty}^{\infty} c_k \lambda^k \right)_+ = \sum_{k=0}^{\infty} c_k \lambda^k$ and $\text{res}_{\infty}$ stands for the usual Cauchy residue at infinity. Notice that, for instance, $\mathcal{H}_{2,2} = \mathcal{H}_2 = u_2$, $\mathcal{H}_{3,2} = \mathcal{H}_3 = u_3$ and $\mathcal{H}_{3,3} = u_4 + \frac{N-2}{2N} u_2^2$. The characteristic $E$ of the Theta function entering the previous expressions can be read, for example, from the blow-up formula derived in Ref.\cite{ref4} for twisted $\mathcal{N} = 2$ supersymmetric gauge theory to be $\vec{\alpha} = (0, \ldots, 0)$ and $\vec{\beta} = (1/2, \ldots, 1/2)$. This is the –even and half-integer– characteristic of the Theta function when we express it in terms of electric variables. If we are to consider the physics near the $\mathcal{N} = 1$ (maximal) singularities, where generically $N - 1$ magnetic monopoles become massless, we should use as local variables the dual $a_{D,k}$ reached from the former $a^k$ after an S duality transformation. In
Ref. [4], a careful study of the $Sp(2r, \mathbb{Z})$ covariance of Eqs.(3) was performed. In general, if we transform under an element $\Gamma$ of the duality group $Sp(2r, \mathbb{Z})$, the variables $a_i, a_{D,i}$ transform as a vector $v' = (a_{D,i}, a_i)$, $v \rightarrow \Gamma v$, whereas the arguments $\xi, \tau$ of the Theta function change as follows:

$$\tau \rightarrow \tau' = (A\tau + B)(C\tau + D)^{-1},$$

$$\xi \rightarrow \xi' = [(C\tau + D)^{-1}]^t \xi,$$

with $A'D - C'B = 1_r$, $A'C = C'A$ and $B'D = D'B$. The characteristics (understood as row vectors) transform as

$$\alpha \rightarrow \alpha' = D\alpha - C\beta + \frac{1}{2} \text{diag}(CD^t),$$

$$\beta \rightarrow \beta' = -B\alpha + A\beta + \frac{1}{2} \text{diag}(AB^t).$$

From these equations, the characteristic of the Theta function near the $\mathcal{N} = 1$ points, which we will call $D$, can be computed to be $\vec{\alpha} = (1/2, \ldots, 1/2)$ and $\vec{\beta} = (0, \ldots, 0)$. The second derivatives of the prepotential with respect to the Whitham slow times can be written in this patch of the moduli space as in Eqs.(3) by replacing $\tau \rightarrow \tau_D$ and using the dual characteristic $D$ in Riemann’s Theta function. These formulae should be suitable to study the physics of $\mathcal{N} = 2$ super Yang-Mills theory near the $\mathcal{N} = 1$ singularities. Let us first consider, for simplicity, the SU(2) case.

**SU(2)**

There are two $\mathcal{N} = 1$ singularities related by the unbroken $\mathbb{Z}_2$ subgroup of the gauge group, occurring at each of the two points of the vanishing locus $\mathcal{C}_\Lambda : \{ u = \pm 2\Lambda^2 \}$ (here $u \equiv u_2 = \mathcal{H}_2$ is the quadratic Casimir of $SU(2)$). They correspond to a massless monopole and a massless dyon, in a given symplectic basis of cycles on the torus. We will study the point $u = 2\Lambda^2$: all quantities at the other point can be obtained through a $\mathbb{Z}_2$ transformation. The generic form of the strong coupling expansion of the prepotential at such singularity is given by:

$$\mathcal{F} = \frac{1}{4\pi i} a_D^2 \log \frac{a_D}{\Lambda} + \frac{i\Lambda^2}{2\pi} \sum_{s=0}^{\infty} \mathcal{F}_s \left( \frac{ia_D}{\Lambda} \right)^s,$$

where the logarithmic term, coming from the one-loop diagram that involves the light monopole, has the appropriate sign and factor making manifest that the theory is non-asymptotically free and that there is a monopole hypermultiplet weakly coupled to the
dual photon for \( a_D \to 0 \). The remaining power series expansion comes from the integration of infinitely many massive BPS states. We shall use in what follows the SWW correspondence to predict the strong coupling expansion of the prepotential \((i.e., the values of the constants \( F_s \) in the expression above). There is just one SWW equation in the \( SU(2) \) case, namely the first one in \( (3) \), and it reads

\[
\frac{\partial^2 \mathcal{F}}{\partial (\log \Lambda)^2} = \frac{8i}{\pi} \left( \frac{\partial u}{\partial a_D} \right)^2 \frac{1}{i\pi} \frac{\partial}{\partial \tau^D} \log \vartheta_2(0|\tau^D) ,
\]

where \( \vartheta_2(0|\tau^D) \) is Jacobi’s Theta function,

\[
\vartheta_2(0|\tau^D) = \sum_{n=-\infty}^{\infty} e^{i\pi (n+1/2)^2 \tau^D} .
\]

Using the ansatz \( (8) \), one can compute the derivative of the Theta function and accommodate the result as follows

\[
\frac{1}{i\pi} \frac{\partial}{\partial \tau^D} \log \vartheta_2(0|\tau^D) = \left[ \sum_{n=0}^{\infty} \Xi(a_D)^{n(n+1)/2} \right]^{-1} \left[ \sum_{n=0}^{\infty} \left( \frac{n+1}{2} \right)^2 \Xi(a_D)^{n(n+1)/2} \right] ,
\]

where \( \Xi(a_D) \) is given by

\[
\Xi(a_D) = e^{3/2} \left( \frac{a_D}{\Lambda} \right)^{\frac{3}{2}} \prod_{s=2}^{\infty} \exp \left\{ s(s-1)F_s \left( \frac{ia_D}{\Lambda} \right)^{s-2} \right\} \approx e^{3/2} \left( \frac{a_D}{\Lambda} \right) + O \left( \frac{a_D^2}{\Lambda^2} \right) .
\]

The expression above in terms of \( \Xi(a_D) \) makes it simple to expand \( (11) \) up to arbitrary order in \( a_D \). It is interesting to mention that Eqs.\( (9) \)–\( (11) \) give a non-trivial consistency check of the dual characteristic previously derived. Indeed, being \( a_D \) a local coordinate of the moduli space, its corresponding characteristic must be even and half–integer. This fact, together with the extra \(-1/2\) factor in the logarithmic term of the prepotential \( (8) \)–related to the fact that the massless magnetic monopole comes in an \( \mathcal{N} = 2 \) hypermultiplet–, leads to a unique possibility that makes both members of Eq.\( (9) \) to fit when an expansion in the dual variable is considered, this being \((\alpha = 1/2, \beta = 0)\). Still, in order to compute the RHS of \( (3) \), we must use the first equation in \( (2) \) (which is nothing but the RG equation derived in Ref.\( (8) \)) to obtain

\[
\frac{\partial u}{\partial a_D} = - \frac{1}{4} a_D + \frac{\Lambda^2}{4a_D} \sum_{s=1}^{\infty} s(s-2)F_s \left( \frac{ia_D}{\Lambda} \right)^s ,
\]

while the LHS of the SWW equation is simply

\[
\frac{\partial^2 \mathcal{F}}{\partial (\log \Lambda)^2} = \frac{i\Lambda^2}{2\pi} \sum_{s=0}^{\infty} (s-2)^2 F_s \left( \frac{ia_D}{\Lambda} \right)^s .
\]
These expressions can be inserted into Eq.(9) and expanded around the vanishing value of the dual variable. At the end, one can recursively compute all the $F_s$ coefficients of the strong coupling expansion, the first few terms being

$$F = \frac{1}{4\pi i} a_D^2 \log \frac{a_D}{\Lambda} + \frac{i a^2}{2\pi} \left\{ -1 + 4 \left( \frac{ia_D}{\Lambda} \right) - \left( \frac{3}{4} + \frac{1}{2} \log(16i) \right) \left( \frac{ia_D}{\Lambda} \right)^2 + \frac{1}{16} \left( \frac{ia_D}{\Lambda} \right)^3 \right. $$

$$+ \frac{5}{2^9} \left( \frac{ia_D}{\Lambda} \right)^4 + \frac{11}{2^4} \left( \frac{ia_D}{\Lambda} \right)^5 + \frac{63}{2^5 \pi} \left( \frac{ia_D}{\Lambda} \right)^6 + \frac{527}{5 \cdot 2^{18}} \left( \frac{ia_D}{\Lambda} \right)^7 + \mathcal{O}(a^8_D) \right\},$$

in precise agreement with the results of Ref.[9]. Notice that the solvability or completeness of the recursive relation that results from this procedure is not granted. It is also interesting to remark that the SWW correspondence allowed us to obtain the exact prepotential near the strong coupling singularity without an explicit knowledge of the actual solution $(a(u), a_D(u))$.

**SU(N)**

The quantum moduli space of $SU(N)$ $\mathcal{N} = 2$ super Yang–Mills theory has $N$ maximal singularities where $N - 1$ monopoles become massless simultaneously. Physical quantities in the neighborhood of any of these singularities can be translated to a patch in the vicinity of any other by the action of the unbroken discrete subgroup $\mathbb{Z}_N$. We will consider in what follows the point where $H_2$ is real and positive. The strong coupling expansion of the prepotential at such singular point can be written in terms of appropriate $a_{D,i}$ variables as

$$F = \frac{N^2}{2\pi i} \Lambda^2 + \frac{2N\Lambda}{\pi} \sum_{k=1}^{N-1} a_{D,k} \sin(\pi k/N) + \frac{1}{4\pi i} \sum_{k=1}^{N-1} a_{D,k}^2 \log \frac{a_{D,k}}{\tilde{\Lambda}_k}$$

$$+ \frac{1}{2} \sum_{k \neq l=1}^{N-1} \tau_{kl}^\text{off} a_{D,k} a_{D,l} + \frac{1}{2\pi i} \sum_{s=1}^{\infty} F_s(a_D) \Lambda^{-s},$$

where $F_s(a_D)$ are polynomials of degree $s + 2$ in dual variables, $\tilde{\Lambda}_k = e^{3/2}\Lambda \sin \hat{\theta}_k$ (with $\hat{\theta}_k = \pi k/N$) and $\tau_{ij}^\text{off}$, $i \neq j$ are the values of the off-diagonal entries of the coupling constant at the $\mathcal{N} = 1$ point, $a_{D,i} = 0$.

Again, we shall introduce this prepotential expansion into the SWW equations (3). These equations allow us to relate the strong coupling expansion of homogeneous combinations of higher Casimir operators $H_m$ with that of the prepotential. Indeed, inserting (4) into (3), we have

$$\frac{\partial H_m}{\partial \log \Lambda} = -\beta \frac{\partial H_2}{\partial a_{D,i}} \frac{\partial H_m}{\partial a_{D,j}} \frac{1}{i\pi} \partial_{\tau_{ij}} \log \Theta_D(0|\tau).$$

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2We follow here the conventions of Ref.[10] to fix the first three terms of the expansion.
In fact, from Eq. (17), both the full expansion of the prepotential around the semiclassical patch for $SU(N) \mathcal{N} = 2$ super Yang–Mills theory \[4\] as well as the strong coupling expansion near the maximal singularities in the $SU(2)$ case were obtained. The generalization of this last result to $SU(N)$, however, faces some difficulties. First, the dual characteristic does not lead to the vanishing of some higher contributions as it happens in the semiclassical expansion with the 'electric' characteristic $E$, a feature that allowed in this last case to explicitly show recursivity \[4\]. Second, the polynomials $\mathcal{F}_s(a_D)$ are not constrained by classical symmetries –as the Weyl group in the semiclassical expansion– being the amount of unknowns, then, considerably higher. Third, the prepotential (16) has powers of $\Lambda$ of both signs such that, in spite of the fact that a grading still exists, many higher terms of the expansion appear in the lowest equations spoiling recursivity\[4\].

The SWW equations do not seem to be instrumental to study the full strong coupling expansion of $SU(N) \mathcal{N} = 2$ super Yang–Mills theory. Other methods have been derived in the literature to tackle the problem of computing the higher threshold corrections $\mathcal{F}_s(a_D)$. For example, in Ref.\[11\] this has been accomplished by parametrizing the neighborhood of the maximal singularities with a family of deformations of the corresponding auxiliary singular Riemann manifold. However, this formalism is not sensitive to quadratic terms in the prepotential. Thus, it does not give an answer for the couplings between different magnetic $U(1)$ factors at the maximal singularities of the moduli space, encoded in the coefficients $\tau_{ij}^{\text{off}}$. The existence and importance of such terms has been first pointed out in Ref.\[10\] by using a scaling trajectory that smoothly connects the maximal singularities with the semiclassical region. These terms also appear in the expression of the Donaldson–Witten functional for gauge group $SU(N)$ \[7\]. To our knowledge, a closed formula for these off-diagonal couplings has not been obtained so far, except for the gauge group $SU(3)$ \[3\]. In the rest of this letter, we will apply the SWW equations to the solution of this problem.

Let us first remark that Eq. (17) is also valid for the higher Casimirs $h_n = 1/n \text{Tr} \phi^n$, $n = 2, ..., N$ \[6\]. They, as well as their particular combinations encoded in $\mathcal{H}_n$, are homogeneous functions of $a_D$ and $\Lambda$ of degree $n$. Thus, at the $\mathcal{N} = 1$ singularities, the LHS of eq. (17) is simply \[10\]

$$\frac{\partial h_n}{\partial \log \Lambda} = nh_n = \sum_{k=1}^{N} (2 \cos \theta_k)^n,$$

(18)

\[3\]Notice that the same difficulties should also appear for $SU(2)$ though, as we have explicitly shown above, this is not the case. The reason for this singular character of the gauge group $SU(2)$ is nothing but the one-dimensional character of its Cartan subalgebra as a detailed analysis makes manifest.
where we used the fact that the eigenvalues of $\phi$ are $\phi_k = 2 \cos \theta_k$ with $\theta_k = (k - 1/2)\pi/N$. The derivative of the Casimir operators with respect to the dual variables can be computed at the same point of the moduli space, by using the explicit representation of the curve in terms of the Chebyshev polynomials $[I]$, resulting in

$$
\frac{\partial h_n}{\partial a_{D,j}} = -2i \sum_{l=0}^{\lfloor n/2-1 \rfloor} \binom{n-1}{l} \sin(n-2l-1)\hat{\theta}_j .
$$

Finally, the leading couplings at the maximal singularity can be easily obtained from the expansion (16) to be

$$
\tau_{Dij} = \frac{1}{2\pi i} \log \left( \frac{a_{D,i}}{\Lambda_i} \right) \delta_{ij} + \tau_{\text{off}}^{ij} ,
$$

where the logarithmic divergence is nothing but the expected running of the coupling constant to the point where $N-1$ monopoles become massless.

The derivative of the Theta function $\Theta_D$ with respect to the period matrix has the following expression when evaluated at the $N = 1$ singularity

$$
\frac{1}{i\pi} \partial_{\tau_{ij}} \log \Theta_D(0, \tau_D) = \frac{1}{4} \left( \sum_{\xi^k=\pm 1} \exp \left( i \frac{\pi}{4} \xi^l \tau_{\text{off}}^{lm} \xi^m \right) \right)^{-1} \sum_{\xi^k=\pm 1} \xi^i \xi^j \exp \left( i \frac{\pi}{4} \xi^l \tau_{\text{off}}^{lm} \xi^m \right) .
$$

Now, we can insert the results (18), (19) and (21) in equation (17):

$$
\frac{1}{2N} \sum_{k=1}^{N} (2 \cos \theta_k)^n = \sum_{l=0}^{\lfloor n/2-1 \rfloor} \binom{n-1}{l} \sin \hat{\theta}_i \sin(n-2l-1)\hat{\theta}_j \left( \sum_{\xi^k=\pm 1} \exp \left( i \frac{\pi}{4} \xi^l \tau_{\text{off}}^{lm} \xi^m \right) \right)^{-1} \times \sum_{\xi^k=\pm 1} \xi^i \xi^j \exp \left( i \frac{\pi}{4} \xi^l \tau_{\text{off}}^{lm} \xi^m \right) .
$$

We have $N-1$ equations and $(N-1)(N-2)/2$ unknowns (the components of the symmetric matrix $\tau_{\text{off}}^{ij}$). Thus, Eq.(22) has predictive power in its own for $SU(3)$ and $SU(4)$. Indeed, we obtain for these two cases the following values:

$$
SU(3) : \quad \tau_{\text{off}}^{12} = i/\pi \log 2 ,
$$

$$
SU(4) : \quad \begin{cases} 
\tau_{\text{off}}^{12} = \tau_{\text{off}}^{23} = -i/\pi \log(\sqrt{2}-1) \\
\tau_{\text{off}}^{13} = i/\pi \log \sqrt{2} .
\end{cases}
$$

Notice that our result for $SU(3)$ coincides with that of Ref.[9] while the ones for $SU(4)$ have not been found previously. For higher $SU(N)$, further ingredients would be necessary in order to obtain the off-diagonal couplings at the $N = 1$ singularity. Instead, inspired by the findings in the last section of Ref.[10], we propose the following ansatz for $\tau_{\text{off}}^{mn}$:

$$
\tau_{\text{off}}^{mn} = \frac{i}{\pi} \frac{2}{N^2} \sum_{k=1}^{N-1} \sin k\hat{\theta}_m \sin k\hat{\theta}_n \sum_{i,j=1}^{N} \tau_{ij}^{(0)} \cos k\theta_i \cos k\theta_j ,
$$

(25)
with \( \tau^{(0)}_{ij} \) being given by
\[
\tau^{(0)}_{ij} = \delta_{ij} \sum_{k \neq i} \log(2 \cos \theta_i - 2 \cos \theta_k)^2 - (1 - \delta_{ij}) \log(2 \cos \theta_i - 2 \cos \theta_j)^2.
\] (26)

Unfortunately, we are not aware of the existence of an equivalent expression in the literature to compare with. We have checked numerically that, with our ansatz (25)–(26) for the off-diagonal couplings, the SWW equations are satisfied up to SU(11). There is a second check that we can do using results that do not rely on Whitham equations at all. Douglas and Shenker showed that the matrix \( \tau^D_{mn} \) at any point of the scaling trajectory diagonalizes in the basis \( \{ \sin k\hat{\theta}_n \} \) with certain particular eigenvalues (see Eqs.(5.9)–(5.12) of Ref.[10]). The couplings (25)–(26) satisfy this restrictive condition in the limit of the scaling trajectory ending at the maximal singularity. As long as our solution (25)–(26) matches two very stringent conditions coming from different places, we believe that it provides a faithful answer for \( \tau^{\text{off}}_{mn} \) as well as a highly non-trivial test of the Seiberg-Witten-Whitham correspondence proposed in Refs.[4, 6].

Although we have focused on \( SU(N) \), what we have presented should be generalizable to other cases. In this respect we point out that relations such as (17) hold for all the simply laced algebras [12]. Many interesting problems are raised by our approach. In particular, it would be of great interest to implement the SWW equations all along the scaling trajectory introduced in Ref.[10], to be able to compute with them at intermediate patches between the maximal singularities and the semiclassical region of the quantum moduli space. We hope to report on this problem elsewhere.

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