On the Twisted (2, 0) and Little-String Theories

Yeuk-Kwan E. Cheung\(^1\), Ori J. Ganor\(^2\) and Morten Krogh\(^3\)

Department of Physics
Jadwin Hall
Princeton University
Princeton, NJ 08544, USA

We study the compactification of the (2, 0) and type-II little-string theories on \(S^1\), \(T^2\) and \(T^3\) with an R-symmetry twist that preserves half the supersymmetry. We argue that it produces the same moduli spaces of vacua as compactification of the (1, 0) theory with \(E_8\) Wilson lines given by a maximal embedding of \(SU(2)\). In certain limits, this reproduces the moduli space of \(SU(2)\) with a massive adjoint hyper-multiplet. In the type-II little-string theory case, we observe a peculiar phase transition where the strings condense. We conjecture a generalization to more than two 5-branes which involves instantons on non-commutative \(T^4\). We conclude with open questions.

May, 1998

\(^1\) cheung@viper.princeton.edu
\(^2\) origa@puhep1.princeton.edu
\(^3\) krogh@phoenix.princeton.edu
1. Introduction

In this paper we will study the compactification of the $(2, 0)$ theory and the little-string theory on $S^1$, $T^2$ and $T^3$. The $(2, 0)$-theory describes the low-energy modes coming from type-IIB on an $A_{k-1}$ singularity [1] or, equivalently, $k$ 5-branes of M-theory [2]. The little-string theory is the theory of $k$ type-II NS5-branes decoupled from gravity [3]. In order to get an interesting low-energy question we will twist the boundary conditions along $T^d$ by elements of the $Spin(5)$ (or $Spin(4)$ for the little-string theory) R-symmetry. In this way we obtain new kinds of theories with 8 supersymmetries. The aim of this paper is to find the low-energy description of these theories. We will present an explicit construction in the case $k = 2$ and suggest a conjecture for higher $k$. The construction for $k = 2$ involves the moduli space of the heterotic 5-brane wrapped on tori. The conjecture for any $k$ involves moduli spaces of instantons on non-commutative tori. In certain limits we recover the known moduli spaces of Super-Yang-Mills theories with a massive adjoint hypermultiplet. In the compactified little-string theories, examination of the moduli space shows that for certain values of the external parameters there is a phase transition to a phase where little-strings condense.

The paper is organized as follows. In section (2) we explain our notation, present the problem and discuss the parameters on which the compactifications depend. In section (3) we present the general solution for $k = 2$. In section (4) we study in more detail various limiting cases of the solution. In particular, we study the limits where Super-Yang-Mills is obtained. In subsection (4.2) we observe the phase transition. In section (5) we explain the relation between the twist and the mass of the adjoint hypermultiplets in the effective low-energy description of Super-Yang-Mills. In section (6) we discuss in more detail what it means to twist the little-string theories. We study what happens to the twists after T-duality and suggest that the R-symmetry twists are a special case of a more general twist. In section (7) we present the conjecture for higher $k$ and the relation to moduli spaces of instantons on non-commutative tori. In section (8) we briefly discuss the questions raised in the M(atrix)-approach to these twists. We end with a discussion and open problems.

2. The problem

The problem that we are going to study is to find the Seiberg-Witten curves of certain $\mathcal{N} = 2$ theories in 3+1D and to find the hyperk"ahler moduli space of certain $\mathcal{N} = 4$ theories in 2+1D. The $\mathcal{N} = 2$ theories will be obtained by compactifying the $(2, 0)$ theory or, slightly generalizing, the little-string theory, on $T^2$ with twisted R-symmetry boundary conditions along the sides of the torus. The $\mathcal{N} = 4$ theories in 2+1D are similarly obtained by compactification on $T^3$. In this section we will describe the setting and the notation.
2.1. Definitions

Let us denote by $T(k)$ the $(2,0)$ low-energy theory of $k$ 5-branes of M-theory \[1,2\]. We denote by $S_A(k)$ ($S_B(k)$) the theory of $k$ type-IIA (type-IIB) NS5-branes in the limit when the string coupling goes to zero keeping the string tension fixed \[3\]. Compactified on a circle, these two theories are T-dual. $T(k)$ is often called “the $(2,0)$ theory” and $S(k)$ is referred to as “the little-string theory”.

2.2. The $(2,0)$ theory and the little-string theories

When $T(k)$ is compactified on $T^2$ we get a 3+1D theory which at low-energy becomes $k$ free vector multiplets (at generic points in the moduli space). The vector-multiplet moduli space is $(S^1 \times \mathbb{R}^5)^k/S_k$ where the size of $S^1$ is $A^{-1/2}$ and $A$ is the area of $T^2$. When we compactify $T(k)$ on $T^3$ the low-energy is (generically on the moduli space) given by a $\sigma$-model on the hyperkähler manifold $(T^3 \times \mathbb{R}^5)^k/S_k$. The $T^3$ in the moduli space has the same shape as the physical $T^3$ but its volume is $V^{-1/2}$, where $V$ is the volume of the physical $T^3$. (See \[4\] for review.) $S_A(k)$ has a low-energy description given by 5+1D SYM and has a scale $M_s$. The scale is related to the SYM coupling constant $M_s^{-1}$. The parameters of the compactification are now the metric on $T^3$ and also the NSNS 2-form on $T^3$. The 2-form couples as a $\theta$-angle in the effective 5+1D low-energy SYM, i.e. as $\int B \wedge \text{tr}\{F \wedge F\}$. Together they parameterize

$$SO(3, 3, \mathbb{Z}) \backslash SO(3, 3, \mathbb{R})/(SO(3) \times SO(3)) = SL(4, \mathbb{Z}) \backslash SL(4, \mathbb{R})/SO(4).$$

(2.1)

The moduli space is given by $(T^4 \times \mathbb{R}^4)^k/S_k$ where $T^4$ has the shape given by the point in $SL(4, \mathbb{Z}) \backslash SL(4, \mathbb{R})/SO(4)$ and has a fixed volume $M_s^2$.

We have to mention that the arguments of \[5\] (see also \[6\]) show that the theories $S(k)$ are far more complicated than the $T(k)$ theories, in the sense that they have a continuous spectrum starting at energy around $M_s$ and this spectrum describes graviton states propagating in a weakly coupled throat. Below the scale $M_s$ there is a discrete spectrum (up to the effect of the $4k$ non-compact scalars). Since there is a mass gap, one can still ask low-energy questions, as we are doing.

2.3. R-symmetry Wilson lines

The compactifications discussed above have 16 supersymmetries and therefore the moduli spaces obtained in 2+1D are flat and only their global structure is interesting. To get interesting metrics on the moduli space we need to break the supersymmetry down by $\frac{1}{2}$. This can be done as follows. Suppose we identify a global symmetry of the $(2,0)$
theory. When we compactify on $S^1$ of radius $R$ and coordinate $0 \leq x \leq 2\pi R$, we can glue the points $x = 0$ and $x = 2\pi R$ by adding a twist of the global symmetry. When we compactify on $T^3$ we can twist along all 3 directions so long as the twists commute. The global symmetry of $T(k)$ is the $Spin(5)$ R-symmetry. Such a twist has been recently used in [7] to break the supersymmetry of the $(2,0)$ theory in compactifications.

When we compactify the little-string theory $S_A(k)$ ($S_B(k)$) it is not immediately obvious that we can use such a twist because the space-time interpretation is not unique. However, since we can embed the twist as a geometrical twist in type-IIA, the question is well defined. We will elaborate more on that point in section (6).

Let us now take the $(2,0)$ theory $T(k)$ on $T^3$ with three commuting twists $g_1, g_2, g_3 \in Spin(5)$ along $T^3$. The 16 super-charges of $T(k)$ transform as a space-time spinor which also has indices in the 4 of $Spin(5)$. The condition that 8 supersymmetries will be preserved is the condition that $g_1, g_2, g_3$ preserve a two-dimensional subspace of the representation 4 of $Spin(5)$. This becomes the following condition. Take $SU(2)_B \times SU(2)_U = Spin(4) \subset Spin(5)$ and let $g_1, g_2, g_3$ be 3 commuting elements in the first $SU(2)_B$ factor. This is the generic twist which preserves $N = 4$ in 2+1D. Similarly, for the little-string theory $S(k)$ the R-symmetry is $Spin(4)$ and we need,

$$g_1, g_2, g_3 \in SU(2)_B \subset SU(2)_B \times SU(2)_U = Spin(4).$$

Since the $g_i$’s are commuting they can be taken inside a $U(1)$ subgroup of $SU(2)_B$. Then $g_i = e^{i\theta_i} \in U(1) \subset SU(2)_B$. The subscripts $B$ and $U$ are short for “broken” and “unbroken” respectively. We can now ask what is the low-energy description of $T(k)$, $S_A(k)$ and $S_B(k)$ compactified, in turn, on $S^1$, $T^2$ and $T^3$ with twists $\theta_i$. The most general question is about $S(k)$ on $T^3$ since all others can be obtained by taking appropriate limits. The low-energy description in 2+1D is a $\sigma$-model on a 4($k - 1$)-dimensional hyperkähler manifold. We will always ignore the decoupled “center of mass”. Furthermore, as will be elaborated in section (4), in appropriate limits we obtain 3+1D or 2+1D $SU(k)$ SYM with a massive adjoint hypermultiplet.

What is the external parameter space? The parameter space for the metric and $B$ fields on $T^3$ is given by $[2.1]$. The parameter space for conjugacy classes of three commuting $SU(2)$ R-symmetry twists along $T^3$ is given by $\tilde{T}^3/\mathbb{Z}_2$ where $\tilde{T}^3$ is the torus dual to $T^3$ and $\mathbb{Z}_2$ is the Weyl group of $SU(2)$. However, with R-symmetry twists, we can no longer divide by the full T-duality group $SO(3,3,\mathbb{Z})$ (see the discussion in section (6)). This means that the parameter space is a fibration of $(\tilde{T}^3/\mathbb{Z}_2)$ over

$$SL(3,\mathbb{Z})\backslash SO(3,3,\mathbb{R})/(SO(3) \times SO(3)).$$

3
2.4. Why is the problem not trivially solved by M-theory?

Let us explain why we cannot just read off the SW-curves and moduli spaces from M-theory. To be specific, let us take the 6-dimensional non-compact space defined as an $\mathbb{R}^4$-fibration over $T^2$ with $Spin(4)$ twists along the cycles of the $T^2$. This is the geometric realization of the R-symmetry twist, that we mentioned above (see section (6) for a more detailed discussion). M-theory compactified on this space preserves 16 supersymmetries if the two twists $\theta_1, \theta_2$ are taken inside $SU(2)_B \subset SU(2)_B \times SU(2)_U = Spin(4)$. Let us wrap $k$ 5-branes on $T^2$. Given the success of the method described in [8] one may at first sight wonder whether the classical moduli space of the $k$ 5-branes immediately gives the right answer. The answer is negative. There is, in fact, a big difference between the situation in [8] and ours. The construction of [8] was used to solve certain QCD questions. As explained there, QCD is not the low-energy description of 5-branes in M-theory. It is not even an approximate one. QCD is only a good approximation in the region of moduli space where the 5-branes are close together and the 11$^{th}$ dimension is very small. When this parameter was increased the dynamics of the system is completely changed except for the vacuum states (i.e. the moduli of the vector-multiplets). This relied on the fact that the parameter that deforms the system from close NS5-branes and D4-branes in type-IIA to M5-branes decoupled from the vector-multiplet moduli space (similarly to the decoupling in [9] and [10,11]). The classical result was correct for the M5-brane limit because all the relevant sizes were much larger than $M_{Pl}$ (the Planck scale).

In our case, not all the relevant sizes of the M5-brane configuration are large. Let $A$ be the area of $T^2$ and let $\Phi$ be the modulus of the tensor multiplet in 5+1D. $\Phi$ is related to the separation $y$ between the 5-branes as $\Phi \sim M_{Pl}^3 y$. The interesting region in moduli space is $\Phi A \sim 1$. This region is $M_{Pl}^3Ay \sim 1$ and at least one of $y$ or $A$ cannot be made large.

3. Solution

In this section we will consider the theory $S_A(2)$ compactified on $T^3$. We recall that $S_A(2)$ is the theory living on 2 coincident NS 5-branes in type IIA in the limit of vanishing string coupling with string scale, $M_s$, kept fixed. The compactified theory has a moduli space of vacua which is a hyperkähler manifold. The purpose of this section is to find this hyperkähler manifold as a function of the parameters of the compactification. These parameters are described above. There is the IIA string scale, $M_s$ (which is already a parameter in 6 dimensions). There is the metric, $G_{ij}^A$ and NS-NS 2-form, $B_{ij}^A$, on the $T^3$. Here $A$ denotes the underlying type IIA theory. Finally, there are the holonomies
of the Spin(4) R-symmetry around the 3 circles. The holonomies are taken inside an
$SU(2)_B$ subgroup of Spin(4) to preserve half of the supersymmetries. The 3 holonomies
must commute and can thus be taken inside $U(1) \subset SU(2)_B$. We denoted the holonomies $e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}$, where $\theta_i$ is periodic with period $2\pi$. Furthermore the Weyl group of $SU(2)_B$ relates $\theta_i$ to $-\theta_i$. These are the parameters of the theory.

The moduli space of vacua has real dimension 4, since we are dealing with 2 5-branes
and we throw away the center of mass motion. We want to find the metric on this as a
function of $M_s, G^A_{ij}, B^A_{ij}$ and $\theta_i$. Our strategy will be to start at the special point $\theta_i = 0$
and then later understand how to do the general case. At $\theta_i = 0$ the theory actually
has $\mathcal{N} = 8$ supersymmetry in 3 dimensions (like $\mathcal{N} = 4$ in 4 dimensions). Here the moduli
space is just the classical one. At the origin of the moduli space the low energy theory is an
$SU(2), \mathcal{N} = 8$ theory. There are also heavy Kaluza-Klein modes with masses that go like
multiples of $\frac{1}{R_i}$, where $R_i$ are the radii of the circles. In $\mathcal{N} = 4$ language the multiplet is a
vector-multiplet and an adjoint hypermultiplet. On the the moduli space of vacua $SU(2)$
is broken to $U(1)$. Dualizing the photon gives an extra scalar, so the vector-multiplet has
4 scalars. In the $\mathcal{N} = 8$ theory the moduli space of vacua is 8 dimensional. Four of the
directions come from scalars in the hypermultiplet. These are lifted as soon as $\theta_i \neq 0$,
because $\theta_i$ supply a mass to the hypermultiplet. We are really only interested in the 4
directions coming from scalars in the vector-multiplet. These 4 scalars are all compact.
From the 5-brane point of view these scalars come about as follows. One of them is the
relative position of the 5-branes on the $11^{th}$ circle. The other 3 are the 2-form living on the
5-brane with indices along the $T^3$. These 4 scalars are obviously compact. The Weyl
group of the SU(2) gauge group changes the sign of all these. We thus see that the moduli space
of vacua is $T^4/\mathbb{Z}_2$. When we deform to $\theta_i \neq 0$, the moduli space remains compact.
The only compact 4 dimensional hyperkähler manifolds are K3 and $T^4$. $T^4/\mathbb{Z}_2$ is topologically
a K3 manifold with a singular metric. We thus conclude that for all parameters $G^A_{ij}, B^A_{ij}, \theta_i$
the moduli space is topologically K3. We just need to find the hyperkähler metric as a
function of these parameters.

Let us first recall the moduli space of hyperkähler metrics on K3. It is $[12],$

$$O(3, 19, \mathbb{Z}) \setminus O(3, 19, \mathbb{R}) / ((O(3) \times O(19)) \times \mathbb{R}^+).$$

$\mathbb{R}^+$ parameterizes the volume. This moduli space nicely coincides with the moduli space
for Heterotic string theory on $T^3$. This is a well-known consequence of the duality of M-theory on K3 with heterotic on $T^3$. On the heterotic side the $\mathbb{R}^+$ denotes the dilaton. The
space $O(3, 19, \mathbb{R}) / O(3) \times O(19)$ can be parameterized by the metric and NS-NS 2-form
on the $T^3$, $G^H_{ij}, B^H_{ij}$ and the Wilson lines around the 3 circles $V_1, V_2, V_3$. We will work
with the $E_8 \times E_8$ Heterotic theory. The reason for that will become clear in a moment. There is a very nice way of obtaining the K3 on the M-theory side as a moduli space of vacua for a 3-dimensional $N = 4$ theory. This is the membrane of M-theory imbedded in $R^{1,6} \times K3$ with the world-volume along $R^{1,6}$ and at a point in K3. On the dual Heterotic side it corresponds to the 5-brane wrapped on $T^3$. This is thus the moduli space of the $(1,0)$ little-string theory obtained from an NS5-brane in the heterotic string by taking the coupling constant to zero [3].

Our aim can now be formulated as finding $G_{ij}^H, B_{ij}^H, V_1, V_2, V_3$ for given $G_{ij}^A, B_{ij}^A, \theta_i$. According to the arguments of [13], the external parameters can be combined into scalar components of auxiliary vector-multiplets which are non-dynamical. Supersymmetry then requires that the periods of the three 2-forms which determine the hyperkähler metric on the moduli space are linear in these combinations of external parameters [14]. To find the map subject to this restriction, we first examine $\theta_i = 0$. We saw earlier that this was the $N = 8$ theory and the moduli space is $T^4/Z_2$. Therefore, we can find the data of the $T^4$ by classical analysis, starting from the $(1,0)$ tensor-multiplet living on the IIA 5-brane. (We have ignored the VEVs along the $(1,0)$ hypermultiplet direction.) The $(1,0)$ tensor-multiplet is also the low-energy description of the $E_8 \times E_8$ Heterotic 5-brane and the scalar is compact since it corresponds to motion in the 11th direction, which is an interval. Let us compactify this theory on $T^3$ with data $G_{ij}^H, B_{ij}^H, V_1, V_2, V_3$. To obtain the same moduli space of vacua as in the $S_A(2)$ case we need to set $G^H = G^A, B^H = B^A$. What about $V_1, V_2, V_3$? The $S_A(2)$ theory had a $T^4/Z_2$ as moduli space. $T^4/Z_2$ has 16 $A_1$ singularities. This means that M-theory on this K3 has $SU(2)^{16}$ gauge symmetry. To achieve this we need very special Wilson lines. We can take $V_1$ to break $E_8 \times E_8$ to $SO(16) \times SO(16)$ and $V_2$ to break each $SO(16)$ to $SO(8) \times SO(8)$ and $V_3$ to break each $SO(8)$ to $SO(4) \times SO(4)$. The unbroken symmetry group is thus $SO(4)^8 = SU(2)^{16}$ as desired. These Wilson lines are unique up to $E_8 \times E_8$ conjugation. We can write down $V_1, V_2, V_3$ explicitly. The two $E_8$’s are treated symmetrically, so we restrict to one of them. Consider $\Gamma^8 \subset R^8$ where $\Gamma^8$ is the weight lattice of $E_8$. Recall that $\Gamma^8$ can be characterized as all sets $(a_1, \ldots, a_8)$ such that either all $a_i$ are half-integers or all $a_i$ are integers. Furthermore $\sum a_i$ is even. A Wilson line around a circle can be specified by an element $V \in \mathbb{R}^{16}$ such that a “state” given by a weight vector $a$ is transformed as $e^{ia \cdot V}$ on traversing the circle. In this notation

\[
V_1 = (0, 0, 0, 0, 0, 0, 0, 1) \\
V_2 = (0, 0, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \\
V_3 = (0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}).
\]
Now we have the map in the case $\theta_i = 0$. We will make a proposal for the general case presently. The 16 singularities in $\mathbb{T}^4/\mathbb{Z}_2$ are due to an adjoint hypermultiplet becoming massless. When $\theta_i \neq 0$ the hypermultiplet is massive and we expect the singularities to disappear. Near the original singularities the theory now looks like pure $SU(2)$ SYM. This does not have any singularities [11,14]. We thus see that the Wilson lines must change when we turn on $\theta_i$. We now make the following proposal. For nonzero $\theta_i$ we still have $G^H_{ij} = G^A_{ij}$, $B^H_{ij} = B^A_{ij}$. The Wilson lines $W_1, W_2, W_3$ are taken to be,

$$W_i = V_i + \frac{\theta_i}{\pi} \left( \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0 \right),$$

in the notation from above. This is the same as embedding $e^{i\theta_i}$ in the diagonal $SU(2) \subset SU(2)^{16} \subset E_8 \times E_8$. The coefficients of $\theta_i$ are chosen such that the period is $\theta_i \rightarrow \theta_i + 2\pi$.

We do not have a proof of this proposal, but this certainly satisfies the requirements of linearity in external parameters, because this is also the moduli space of the compactified $(1,0)$ little-string theory. In the coming sections we will show that our proposal is consistent with string theory and field theory expectation.

There is another very similar theory. This is the theory on 2 coincident type IIB NS 5-branes in the limit of vanishing string coupling and fixed string mass. We call this theory $S_B(2)$. As soon as we compactify it on a circle it is T-dual to the theory studied above. When we compactify it on a $\mathbb{T}^3$ with R-symmetry twists we get a 3-dimensional theory with a K3 as the moduli space of vacua. Arguing exactly as in the IIA case we propose that this K3 is given in the same way as in the IIA case, except that we replace Heterotic $E_8 \times E_8$ with Heterotic SO(32). This is because the low energy description of the theory living on a IIB 5-brane is a gauge theory. The Heterotic SO(32) 5-brane is also described, at low energy, by a gauge theory. When we do the comparison at the point without an R-symmetry twist, the $\mathcal{N} = 8$ point, the moduli spaces will automatically agree. This is analogous to the $\mathcal{N} = 8$ point in the IIA case where we compared two tensor-multiplets. The T-duality between the IIA and IIB 5-brane theories on $\mathbb{T}^3$ fits very nicely with the T-duality between Heterotic SO(32) and Heterotic $E_8 \times E_8$ on $\mathbb{T}^3$ at the point $\theta_i = 0$. For $\theta_i \neq 0$, the R-symmetry twists do not remain R-symmetry twists after T-duality.

4. Limits

Now that we have identified the moduli space of vacua for $S_A(2)$ compactified on $\mathbb{T}^3$ with arbitrary R-symmetry twists, we can decompactify one or two of the circles to obtain the moduli space of vacua for $S_A(2)$ compactified to 4 and 5 dimensions. Another limit is to take $M_s \rightarrow \infty$ in the $S_A(2)$ theory. This takes us to the $(2,0)$ theory, which we call $T(2)$. In this section we will consider these limits.
4.1. Decompactification limits

Let us first recall the correspondence between M-theory on K3 and Heterotic \( E_8 \times E_8 \) on \( T^3 \). M-theory on K3 has a Planck mass, \( M_{Pl} \), and a moduli space

\[
O(3, 19, \mathbb{Z}) \setminus O(3, 19, \mathbb{R})/(\langle O(3) \times O(19) \rangle \times \mathbb{R}^+) 
\]

\( \mathbb{R}^+ \) denotes the volume of K3, \( \text{Vol}(K3) \). In Heterotic \( E_8 \times E_8 \) on \( T^3 \) there is a string mass, \( M_s \), and a moduli space, which is the same as for M-theory on K3. There is a 10-dimensional string coupling, \( \lambda \). The \( T^3 \) has a volume, \( \text{Vol}(T^3) \), which is part of \( O(3, 19, \mathbb{R})/(\langle O(3) \times O(19) \rangle) \). Under the duality an M5-brane wrapped on K3 is mapped to the Heterotic string. Equating the tensions gives,

\[
M_{Pl}^6 \text{Vol}(K3) = M_s^2. \tag{4.1}
\]

Equating the 7-dimensional gravitational couplings gives,

\[
M_{Pl}^9 \text{Vol}(K3) = \frac{M_s^8 \text{Vol}(T^3)}{\lambda^2}. \tag{4.2}
\]

We thus see, that the \( \mathbb{R}^+ \) on the Heterotic side is \( \frac{\text{Vol}(T^3)}{\lambda^2} \), which of course is T-duality invariant. Eq.(4.1) agrees with the fact that the volume of the moduli space of vacua of the Heterotic 5-brane is \( M_s^2 \) and \( M_{Pl}^6 \text{Vol}(K3) \) is the volume of the moduli space of the M 2-brane probe. We remember that scalar fields have dimension \( \frac{1}{2} \) in 3 dimensions. A concrete way of tracing the duality between these two theories is to use T-duality from Heterotic \( E_8 \times E_8 \) on \( T^3 \) to Heterotic SO(32) on \( T^3 \), and then S-duality to type-I on \( T^3 \), then T-duality to type-IA on \( T^3 \) which can be viewed as M-theory on K3.

Let us now consider the decompactification to 4 dimensions. This can be done by taking \( \theta_3 = 0 \) and \( R_3 \to \infty \). In this limit the K3 becomes elliptically fibered with the fiber shrinking. The area of the fiber \( A \) is

\[
M_{Pl}^3 A = \frac{1}{R_3}
\]

This can be seen by noting that a membrane wrapped on the fiber corresponds to momentum around the circle \( R_3 \) in the Heterotic theory. This limit of M-theory on an elliptically fibered K3 is exactly what gives F-theory on this K3. The M2-brane probe becomes the D3-brane probe in F-theory on K3 [15,10]. Since the volume of K3 stays fixed and the fiber shrinks this means that the base grows. One might thus think that the moduli space seen by the D3-brane probe is infinite. However we should remember that a scalar field
in 4 dimensions has dimension one, so we need a factor of the type-IIB string mass in the area of the moduli space. Inserting this makes the area up to a constant, $M_s^2$. This agrees with the expectation from $S_A(2)$ compactified on $T^2$. We can thus summarize our result for the 4-dimensional case. Take the theory $S_A(2)$ with mass scale $M_s$. Compactify it on a $T^2$ with R-symmetry twists given by $\theta_1, \theta_2$. The $T^2$ is specified by $G_{i\bar{j}}, B^A_{i\bar{j}}$. The moduli space of vacua for this $N = 2$ theory in $D = 4$ is the same as the moduli space of vacua for the $E_8 \times E_8$ Heterotic 5-brane wrapped on $T^2$ with string mass, $M_s$, and a point in $O(18, 2)/O(18) \times O(2)$ given as follows. The metric and 2-form on $T^2$ is $G^A_{i\bar{j}}, B^A_{i\bar{j}}$. The Wilson lines on $T^2$ depend on $\theta_1, \theta_2$. In the case $\theta_i = 0$ they are the essentially unique Wilson lines that break $E_8 \times E_8$ to $SO(8)^4$. For non-zero $\theta_i$ the Wilson lines are constructed as in the last section by embedding in a diagonal $SU(2)^{16} \subset SO(8)^4$.

This wrapped 5-brane in the Heterotic theory is dual to the 3-brane probe in F-theory on the corresponding elliptic-fibered K3. This K3 is the Seiberg-Witten curve for the moduli space. As in the 3 dimensional case, we are not saying that the compactified $S_A(2)$ theory is equal to the little-string theory on the Heterotic 5-brane, but just that the low-energy description is the same. It is obvious that they are not equal since the $S_A(2)$ theory has enhanced supersymmetry when $\theta_i = 0$.

Decompactifying to 5 dimensions is now easy. The correspondence becomes the following. Consider the theory $S_A(2)$ compactified on $S^1$ of radius $R$, string scale $M_s$ and R-symmetry twist $\theta$. This is a 5 dimensional theory with $N = 1$ supersymmetry. The coulomb branch is 1-dimensional. Topologically it is $S^1/Z_2$. This moduli space is the same as the moduli space of the Heterotic $E_8 \times E_8$ 5-brane compactified on a circle with an $E_8 \times E_8$ Wilson line. The Wilson line for one $E_8$ is,

$$W = (0, 0, 0, 0, 0, 0, 0, 1) + \frac{\theta}{2\pi} \left( \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0 \right),$$

and the same for the other $E_8$.

Completely analogous statements can be made for the type-IIB 5-brane theory, $S_B(2)$. Here the moduli space is given by the 5-brane in the Heterotic SO(32) theory. Let us describe this in detail for the case of 5 dimensions. Consider $S_B(2)$ on a circle of radius $R$, with R-symmetry twist $\theta$ and string scale $M_s$. The moduli space of vacua of this is the same as the 5-brane of SO(32) Heterotic string theory on a circle with radius $R$, string scale $M_s$ and SO(32) Wilson line

$$W = \left( \frac{1}{2}, \cdots, \frac{1}{2}, 0, \cdots, 0 \right) + \frac{\theta}{\pi} \left( \frac{1}{2}, 0, \frac{1}{2}, \cdots, \frac{1}{2}, 0 \right).$$

9
The string coupling \( \lambda \) goes to zero to give a decoupled theory on the 5-brane.

There is a dual type-IA picture of the Heterotic theory. The 5-brane becomes a D4-brane living on an interval with 8-branes. The parameters of the type-IA theory are

\[
M'_s = \frac{M_s}{\sqrt{\lambda}} \\
R' = \frac{\lambda}{M_s^2 R} \\
\lambda' = \frac{1}{\sqrt{\lambda M_s R}}
\]

All quantities of interest to the D4-brane theory have a limit as \( \lambda \to 0 \). The positions of the D8-branes are given by the Wilson line.

**Fig. 1:** The dual type-IA picture.

At each end there are 4 D8-branes. There are two more stacks of 4 D8-branes at distance \( \frac{\theta}{2} R' \) from each end. When \( \theta = 2\pi \) the 8-branes reach the other end. This has to be the case since \( \theta \) is periodic with period \( 2\pi \). We also remark that something interesting happens when \( \theta = \pi \). Here 8 D8-branes are on top of each other. We will return to a discussion of this point later. There behavior in \( D = 3, 4 \) is similar.

\( S_3(2) \) compactified to 5 dimensions was described above by mapping the R-symmetry twists to \( E_8 \times E_8 \) Wilson lines. For later purposes it will be more convenient to use the type-IA dual description as the theory living on a D4-brane. The chain of dualities going from Heterotic \( E_8 \times E_8 \) on \( S^1 \) to type-IA is to first invoke T-duality from Heterotic \( E_8 \times E_8 \)}
to Heterotic SO(32), and then proceed as above to reach type-IA. The parameters of the type-IA theory in terms of the parameters of the $S_A(2)$ theory become

$$M'_s = \frac{M_s}{\sqrt{\lambda}} \left(2\left(\frac{\theta^2}{\pi^2} + (M_s R)^2\right)\right)^{\frac{1}{4}}$$

$$R' = \frac{\lambda}{M'_s R} \left(2\left(\frac{\theta^2}{\pi^2} + (M_s R)^2\right)\right)^{\frac{1}{2}}$$

$$\lambda' = \left(\frac{2\left(\frac{\theta^2}{\pi^2} + (M_s R)^2\right)}{\sqrt{\lambda M_s R}}\right)^{\frac{1}{4}}$$

(4.4)

The configuration of D8-branes is as in the $S_B(2)$ case. At each end there are 4 D8-branes. At distance $\frac{\theta}{2} R'$ from the ends are 4 D8-branes. Here

$$\theta' = \frac{\theta}{2\left(\frac{\theta^2}{\pi^2} + (M_s R)^2\right)}.$$  

(4.5)

We see an interesting effect here. When $\theta = 0$, $\theta'$ is also zero. For small $\theta, \theta'$ is an increasing function. At $\theta = \pi M_s R$, $\theta'$ reaches its maximum and start to decrease.

The moduli spaces of all the $S_A(2)$ theories with all possible $\theta$-twists occupy some subspace

$$\mathcal{M}' \subset SO(1, 17, \mathbb{Z}) \backslash SO(1, 17, \mathbb{R}) / SO(17).$$

We wish to know what is the locus $\mathcal{M}''$ which the $S_A(2)$ theories with the T-dual $\eta$-twists span. In the above chain of dualities we started with the heterotic $E_8 \times E_8$ 5-brane wrapped on $S^1$. This represented $S_A(2)$ on a circle with a $\theta$-twist. By definition, $S_A(2)$ with a $\eta$-twist is T-dual to $S_B(2)$ with some $\theta$-twist and therefore corresponds to a point in the moduli space of the $SO(32)$ 5-brane. We have seen that the points on the $SO(32)$ which correspond to our $S_B(2)$ theories map under heterotic T-duality to the points on the $E_8 \times E_8$ moduli space which correspond to the $S_A(2)$ theories. Thus $\mathcal{M}'$ and $\mathcal{M}''$ are the same locus. Nevertheless, $S_A(2)$ with a $\theta$-twist is not equivalent to $S_A(2)$ with a $\eta$-twist.

The $S_A(2)$ theory with a $\eta$-twist is T-dual, by definition, to the $S_B(2)$ theory with an $\theta$-twist. The latter is After T-duality to the heterotic $SO(32)$ we would expect

4.2. A peculiar phase transition

As we have explained above, when we compactify $S_B(2)$ with a twist $\theta$ on $S^1$ of radius $R$ we get a 4+1D theory whose low-energy is the same as that of a D4-brane probe in a configuration of D8-branes on an interval. In this configuration there are 2 stacks of four coincident D8-branes. Whenever the D4-brane crosses the stack, a particle of $U(1)$ charge 2 (coming from the adjoint of $SU(2) \supset U(1)$) becomes massless. When the two stacks of
D8-branes coincide we get two massless hypermultiplets. Since the low-energy description of a $U(1)$ with two massless particles is weakly coupled, we can trust field-theory and the conclusion is that there exists a Higgs phase where the massless hyper-multiplets get a VEV and break the $U(1)$.

In the $S_B(2)$ case, this phase transition occurs for $\theta = \pi$ and for all values of $R$. In contrast, for $S_A(2)$ this only happens for small enough $R$. We can see this from eq.(4.3). The phase transition occurs when $\theta' = \pi$ since this is where two stacks of D8-branes coincide in the type-IA picture. This has a real solution $\theta$, only if $M_sR \leq \frac{1}{4}$. Such a bound is certainly to be expected for $S_A(2)$. The reason is that this phase transition happens when two NS 5-branes are on opposite points of the 11th circle. The 2 hypermultiplets that become massless originate from membranes stretched between the two 5-branes and wrapped on the compactified circle (the circle which takes us from a 6-dimensional theory to a 5-dimensional theory). The tension of the membrane gives a mass to these states. However there is also a contribution to the mass from the zero-point energy of the fields on the membrane. This contribution depends on $\theta$. For certain values of the parameters the zero-point energy can cancel the mass from the tension. This is how the hypermultiplet can become massless. Obviously the mass from the tension can not be canceled if the 11th circle is too big, or equivalently the compactified circle is too large. This is the reason for the above inequality.

4.3. The (2, 0) limit

Let us briefly consider another limit, namely the limit where $S_A(2)$ on $T^3$ becomes $T(2)$ on $T^3$. This happens when the 11th circle opens up. We see from eq.(4.1) and eq.(4.2) that $\text{Vol}(K3) \to \infty$, i.e. the moduli space becomes non-compact. This is as expected. Basically we just get half of the K3. The other half goes to infinity. In the $E_8 \times E_8$ Heterotic 5-brane picture it means that the distance between the ends of the world go to infinity and we only look at one end.

4.4. Field theory limits

In this section we will compare the moduli spaces of vacua found in the other sections with field theory results. At each point of the moduli spaces for the $T(k)$ and $S(k)$ theories, we can find a field theory description for the light modes. We are fortunate that such field theories in $D = 3, 4, 5$ are known. The metric on the moduli space around the chosen point will be determined by the light matter. We are going to compare our exact metric with this field theory expectation.
Let us start with $S_B(2)$ compactified on $S^1$. The effective field theory is that of the $D4$-brane probe in type-IA. From $S_B(2)$ we have $SU(2)$ gauge theory with (1,1) supersymmetry. (The field content of the (1,1) vector multiplet are a (1,0) vector multiplet and a (0,1) adjoint hypermultiplet.) Upon compactification on $S^1$ of radius $R$ with a R-symmetry twist $\theta$, the moduli space is parameterized by the sixth-component of the gauge field, $A_6$, in $U(1) \subset SU(2)$. The full R-symmetry of $S_B(2)$ theory is 

$$SO(4) = SU(2)_U \otimes SU(2)_B$$

which is broken down to $SU(2)_U$ by $e^{i\theta} \in U(1) \subset SU(2)_B$. We get in 5D an $SU(2)$ vector-multiplet and massive adjoint vector-multiplets (with masses $\frac{n}{R}$ with $n \in \mathbb{Z}_{\neq 0}$) transforming non-trivially under $SU(2)_U$ R-symmetry. The boundary conditions on the two complex scalars in the adjoint hypermultiplet are shifted by $\theta$:

$$\phi_1(2\pi R) = e^{i\theta} \phi_1(0), \quad \phi_2(2\pi R) = e^{-i\theta} \phi_2(0).$$

This shifts the periodicity of the fields around the circle. The reduction also gives a tower of adjoint hypermultiplets in 5D with masses,

$$m^2 = \left(\frac{n + \frac{\theta}{2\pi R}}{R^2}\right)^2, \quad n \in \mathbb{Z}.$$

For small $\theta > 0$, we get one light adjoint hypermultiplet of mass $\frac{\theta}{2\pi R}$. Now let us look at the moduli space around $A_6 = 0$. From field theory it looks like $SU(2)$ theory with an adjoint hypermultiplet of mass $\frac{\theta}{2\pi R}$. The gauge coupling is then given by \[16\],

$$\frac{1}{g^2} = b + cA_6,$$

where $b$ and $c$ are constants. The slope, $c$, changes when charged matter becomes massless. The change in the slope is proportional to the cube of the charge of the multiplet becoming massless. In $U(1) \subset SU(2)$ an adjoint field has components of charge $-2, 0, +2$ under the $U(1)$ in units where the 2 of $SU(2)$ has charge $\pm 1$. This means that the change in the slope, $c$, is 8 times bigger for an adjoint hypermultiplet than for a fundamental. Let us calculate at what value of $A_6$ the charge 2 component of the adjoint hypermultiplet becomes massless. The holonomy around the circle is

$$\phi \rightarrow e^{-4\pi i RA_6} \phi.$$

To cancel $e^{i\theta}$ we thus need

$$A_6 = \frac{\theta}{4\pi R}.$$
Let us now compare to the solution from the previous section. Here $\theta$ parameterizes the position of 4 D8-branes. For $\theta = 2\pi$ they reach the other end of the interval. In terms of $A_6$ the other end of the interval is at $\frac{1}{2R}$, so the position of the 4 D8-branes is,

$$A_6 = \frac{\theta}{2\pi} \times \frac{1}{2R} = \frac{\theta}{4\pi R}$$

in exact agreement. However the number of D8-branes is 4 and not 8 as naively expected from the discussion above. It seems like the change in slope is half of what should be expected from field theory. There is no discrepancy for a subtle reason. We compare, on one hand, the U(1) low energy effective action for a D4-brane moving in an orientifold setting, with, on the other hand, a U(1) from a $SU(2)$ gauge theory. The U(1) on the D4-brane probe corresponds not to the $U(1) \subset SU(2)$ but to one of the U(1) factors in $U(1) \times U(1) \subset U(2)$. The action for the diagonal $U(1) \subset U(2)$ is twice the action for a single $U(1)$ factor. The normalization would contain an extra $\sqrt{2}$ factor. Taking this factor of 2 into account the change in the slope becomes 8 instead of 4.

Let us now consider the case of $S_A(2)$ on $T^3$ with twists $\theta_1, \theta_2, \theta_3$. For simplicity the torus is taken to be rectangular with radii $R_1, R_2, R_3$ with $B_{ij} = 0$. We will also take $\theta_i$ to be small. We want to find the light fields. Finding the light fields in this case is not as easy as in the previous case, because $S_A(2)$ does not have a Lagrangian description. However we can figure out the result by first compactifying on a small $R_1$, with $\theta_1 = 0$. Then the low energy description is a 5-dimensional $N = 2$ $SU(2)$ gauge theory. In $N = 1$ language it comprises a vector-multiplet and a hypermultiplet. Now we can compactify this on a second circle of radius $R_2 \gg R_1$. At scale $R_2$ the $SU(2)$ gauge theory is weakly coupled and we can perform a classical analysis to include the twists $\theta_2$. We get an $SU(2)$ gauge theory in $D = 4$ with a hypermultiplet of mass $\frac{\theta_2}{2\pi R_2}$. In $D = 4$, $N = 2$ a hypermultiplet mass is complex. Since there is no distinction between direction 1 and 2 we expect a contribution $\frac{\theta_1}{2\pi R_1}$ from direction 1. They have to combine into a complex mass

$$m = \frac{\theta_1}{2\pi R_1} + i \frac{\theta_2}{2\pi R_2}.$$ 

On compactifying down to 3 dimensions on $R_3$ (we assume that $R_3 > R_2$) there will similarly be a contribution $\frac{\theta_3}{2\pi R_3}$. In $D = 3$ a hypermultiplet mass consists of 3 real numbers that transform in the 3 of $SO(3)$ [14]. This $SO(3)$ is part of the R-symmetry group. We thus conclude that the 3 real numbers are $\frac{\theta_i}{2\pi R_i}$. There is a region in moduli space where the theory looks like $N = 4$, $SU(2)$ gauge theory with an adjoint hypermultiplet with mass $m_i = \frac{\theta_i}{2\pi R_i}$. As we have seen, this region is when $|\theta_i| \ll \pi$ and when the mass scale set by

14
the 2+1D SYM coupling constant (the smallest of $R_1 R_2 R_3$, $R_2 R_3 R_1$, and $R_3 R_1 R_2$) is much smaller than the smallest compactification scale (the smallest of $R_1^{-1}$, $R_2^{-1}$ and $R_3^{-1}$). In our setting, $R_1 \ll R_2 < R_3$, this condition is met. Note that if $|\theta| \ll \pi$ but $R_1 \sim R_2 \sim R_3$ are of the same order of magnitude, the correct approximation is to start with the 2+1D CFT to which $\mathcal{N} = 8$ 2+1D SYM flows \[17,18\] and deform it by the relevant operator to which the mass deformation flows. When $m_i = 0$ we obtain a $\mathcal{N} = 8$ theory and the moduli space is $(\mathbb{R}^3 \times S^1)/\mathbb{Z}_2$. This has two $A_1$ singularities. When $m_i \neq 0$ these are blown up. From our solution in the previous section the sizes of the blow up can be read off as a function of $\theta_i$. This means that we have derived a formula for the size of the blow-up of the singularities in $D = 3$, $\mathcal{N} = 4$ $SU(2)$ gauge theory with a massive adjoint hypermultiplet.

We can do the same analysis in $D = 4$. For $\theta_1 = \theta_2 = 0$ there are 4 singularities. Close to any one of them the system should be describable as an $\mathcal{N} = 2$, $SU(2)$ gauge theory with an adjoint hypermultiplet. For small $\theta_i$ the mass of the hypermultiplet is

$$m = \frac{\theta_1}{2\pi R_1} + i \frac{\theta_2}{2\pi R_2}.$$ 

We expect this to change the Seiberg-Witten curve. Our result also predicts how this goes. Our picture is that the Seiberg-Witten curve is the same as the D3-brane probe in F-theory on the K3 as described earlier. For $\theta_i = 0$ this has a description as a type-IIB orientifold 8 plane with 4 D7-branes on top making a $D_4$ singularity \[13,10\]. For non-zero $\theta_i$ two of the 7-branes move away, giving a $U(2) \times SO(4) = U(2) \times SU(2) \times SU(2)$ singularity. In a field theory setting this corresponds to the $SU(2)$ Seiberg-Witten theory with 4 fundamental hypermultiplets, 2 of them massless and 2 of them massive with equal mass. Our analysis thus predicts that this situation should have the same curve as the massive adjoint hypermultiplet. In the second Seiberg-Witten paper \[19\] this was indeed found to be the case. In comparing the curves with the low energy effective action there is again a factor of 2 in the coupling constant $\tau$ because of a difference in conventions between the adjoint and fundamental case. This is the same factor of 2 as explained in the 5-dimensional case above.

5. Reduction of the twisted $(2,0)$ theory to $4+1$D

In this section we will study $T(2)$ on $S^1$ with a twist $\theta$. Neglecting the overall center-of-mass, the moduli space is 1-dimensional. The low-energy physics is a $U(1)$ vector-multiplet. Let $\phi$ be the scalar partner of the vector field. In this section we will study the BPS states in the theory. There are two different regions in moduli space to consider. Let $R$ be the
radius of $S^1$. When $\phi R \ll 1$ we can use the effective 4+1D SYM Lagrangian. We will show that for small $\theta$, the BPS states come from the $W^\pm$ bosons and the charged states of a massive adjoint hyper-multiplet. When $\phi R \gg 1$ we can identify the charged BPS states with strings wound around $S^1$.

The BPS masses in 4+1D are $[16]$,

$$2\phi, \quad m_0 + 2\phi, \quad m_0 - 2\phi.$$ (5.1)

In the D4-brane and D8-brane picture, these come from strings connecting the D4-brane to its image, and to the two mirror D8-brane stacks. Here,

$$m_0 = \frac{\theta}{2\pi R}.$$

This can be seen from eq.(1.4) and eq.(4.5). The states with mass $2\phi$ are vectors while those with masses $2\phi \pm m_0$ are hyper-multiplets.

5.1. Yang-Mills limit

When $\theta = 0$, the low-energy description of $T(2)$ on $S^1$ is $SU(2)$ SYM with a coupling constant $g^2$ which is proportional to $R$. As long as our energy scale is below the compactification scale $R^{-1}$, the coupling constant is weak and the effective description is good. When $|\theta| \ll 1$ it can be incorporated as a small perturbation in the effective Lagrangian. It corresponds to giving a bare mass of $m_0$ to the hyper-multiplet in the Lagrangian. After spontaneous breaking of $SU(2)$ down to $U(1)$, the masses in (5.1) are easily calculated in field theory. $2\phi$ is the mass of the $W^\pm$ bosons while $2\phi \pm m_0$ come from the hypermultiplet. The adjoint hypermultiplet also gives rise to a neutral multiplet with a mass $m_0$.

5.2. The large-tension limit

Let us assume that $\phi R \gg 1$. In this case, we can first reduce to the 5+1D low-energy of a single $\mathcal{N} = (2,0)$ tensor multiplet and then reduce this tensor multiplet to 4+1D since the scale of the VEV $\phi$ is much higher than the compactification scale. In 4+1D, the neutral states come from the hypermultiplet in 5+1D with twists along $S^1$ as in (4.6). The mass of these states is therefore (for small $\theta$),

$$m = \frac{\theta}{2\pi R}.$$

The charged states come from quantization of the strings wrapped on $S^1$. Up to a correction proportional to $\frac{\alpha^2}{R}$ (see [16,20,21]), the tension of the string in 5+1D is $\Phi = \phi/2R$. In the
limit we are considering, $\Phi R^2 \gg 1$, it is enough to quantize only the low-energy excitations of the strings. This is just as well, since the low-energy excitations are the only things we know about these strings! This means that our results are correct up to $O(1/\Phi R^2)$. The low-energy description is given by a 1+1D $\mathcal{N} = (4, 4)$ theory. The VEV of the tensor multiplet of the 5+1D bulk breaks the $Spin(5)$ R-symmetry down to $Spin(4)$. The 1+1D low-energy description of a string contains 4 left-moving bosons and 4 right-moving bosons, 4 left-moving fermions and 4 right-moving fermions. The bosons are not-charged under the $Spin(5)$ R-symmetry. The 8 fermions can be decomposed into representations, of

$$(SU(2)_B \times SU(2)_U \times SU(2)'_1 \times SU(2)'_2)_{SO(1,1)}$$

Here $Spin(4) = SU(2)_B \times SU(2)_U$ is the unbroken R-symmetry of the 5+1D theory, $Spin(4) = SU(2)'_1 \times SU(2)'_2$ is the subgroup of $Spin(5, 1)$ of rotations transverse to the string and $SO(1, 1)$ is the world-sheet rotation group. The fermions are in the

$$(2, 1, 2, 1)_{+\frac{1}{2}} + (1, 2, 1, 2)_{-\frac{1}{2}}$$

with an added reality condition. Under the embedding

$$U(1) \subset SU(2)_B \subset SU(2)_B \times SU(2)_U = Spin(4) \subset Spin(5),$$

we find 2 left-moving fermions with charge +1 under $U(1)$, 2 left-moving fermions with charge −1 under $U(1)$, and 4 right-moving fermions with charge 0 under $U(1)$. The boundary conditions on the charged fermions are twisted. Quantization of this system gives low-lying vector-multiplets and hyper-multiplets with masses,

$$\Phi R, \quad \frac{\theta}{2\pi R} \pm \Phi R.$$

Recall that the derivation assumed that $\Phi R^2 \gg 1$ and $|\theta| \ll \pi$. This agrees with eq.(5.1).

6. R-symmetry twists in the little-string theories

For the $(2, 0)$ theories, which are believed to have a local description, a twist by a global symmetry along a circle makes perfect sense. For the little-string theories, the issue of locality is more complicated and the meaning of an R-symmetry twist has to be elaborated. In this section we will describe the construction in more detail. We will then see explicitly that T-duality of $S(k)$ does not preserve the $\theta$-twists. Instead it maps them to T-dual “$\eta$-twists”. This raises the intriguing possibility to combine both kinds of twists simultaneously.
6.1. Geometrical realization

One way to define an R-symmetry twist is to realize it geometrically as follows. We can start with $\mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R}^4$ and mod out by a discrete $\mathbb{Z}^3$ symmetry which is generated by elements which act as a shift in $\mathbb{R}^3$ and rotations in $\mathbb{R}^4$. We obtain $Z \times \mathbb{R}^2$ where $Z$ is an $\mathbb{R}^4$-fibration over $T^3$. Explicitly, we define the 7-dimensional space

$$Z_{\theta_1, \theta_2, \theta_3} = (\mathbb{R}^3 \times \mathbb{C}^2)/\mathbb{Z}^3,$$

where $\mathbb{Z}^3$ is the freely acting group generated by,

\begin{align*}
  s_1 :& (x_1, x_2, x_3, z_1, z_2) \mapsto (x_1 + 2\pi R_1, x_2, x_3, e^{i\theta_1}z_1, e^{-i\theta_1}z_2), \\
  s_2 :& (x_1, x_2, x_3, z_1, z_2) \mapsto (x_1, x_2 + 2\pi R_2, x_3, e^{i\theta_2}z_1, e^{-i\theta_2}z_2), \\
  s_3 :& (x_1, x_2, x_3, z_1, z_2) \mapsto (x_1, x_2, x_3 + 2\pi R_3, e^{i\theta_3}z_1, e^{-i\theta_3}z_2),
\end{align*}

(6.1)

Here $(x_1, x_2, x_3)$ are coordinates on $\mathbb{R}^3$. We can similarly define

$$Y_{\theta_1, \theta_2} = (\mathbb{R}^2 \times \mathbb{C}^2)/\mathbb{Z}^2, \quad X_{\theta} = (\mathbb{R} \times \mathbb{C}^2)/\mathbb{Z}. \quad (6.2)$$

The theory that we study in this paper, $S_A(k)$ on $T^3$ with a twist, can be obtained if we compactify type-IIA on $Z_{\theta_1, \theta_2, \theta_3}$, wrap $k$ NS5-branes on $T^3$ and take $\lambda_s \to 0$ as in [3]. This shows that it makes sense to include an R-symmetry twist in $S(k)$.

What is the meaning of these twists in terms of the theory $S(k)$ itself, without appealing to the underlying string-theory? Let us first refine our terminology. Let $p$ be a generic point in the parameter space

$$\mathcal{M}_A \equiv SO(3, 3, \mathbb{Z}) \backslash O(3, 3, \mathbb{R})/(O(3) \times O(3)).$$

We will denote the theory derived from $k$ type-IIA NS5-branes at the type-IIA moduli space point $p \in \mathcal{M}_A$ by $S_A(k; p)$. Similarly there is an identical moduli space $\mathcal{M}_B$ for type-IIB NS5-branes. We will denote the theory derived from $k$ type-IIB NS5-branes at the type-IIB moduli space point $p \in \mathcal{M}_B$ by $S_B(k; p)$. T-duality implies that there is a map,

$$T : \mathcal{M}_A \to \mathcal{M}_B,$$

with $T^2 = I$ such that $S_A(k, p) = S_B(k; T(p))$. This map can be defined as follows. Pick an element $v \in O(3, 3, \mathbb{Z})$ with $\det v = -1$ (all such elements are $SO(3, 3, \mathbb{Z})$ conjugate to each other). For $g \in O(3, 3, \mathbb{R})$ which is a representative of a point in $p \in \mathcal{M}_A$ take $v \circ g$ to be a representative of $T(p) \in \mathcal{M}_B$.}

18
A generic point \( p' \) in the cover, 
\[
SL(3, \mathbb{Z}) \backslash O(3, 3, \mathbb{R})/(O(3) \times O(3)),
\]
of the parameter space (note that we divided by \( SL(3, \mathbb{Z}) \) instead of \( SO(3, 3, \mathbb{Z}) \)) will be called a \textit{locality-frame}. A generic point \( p'' \) in the cover, 
\[
O(3, 3, \mathbb{R})/(O(3) \times O(3))
\]
will be called a \textit{coordinate-frame}. There are the obvious maps, 
\[
p'' \to p' \to p.
\]
Now suppose that we are in a specific point \( p \in \mathcal{M}_A \), say, and we fix a locality-frame \( p' \) for \( p \) and a coordinate-frame \( p'' \) for \( p' \). For a given \( p'' \) we can contemplate whether it makes sense to define R-symmetry twists along the cycles of \( T^3 \). If they commute with each other, an \( SL(3, \mathbb{Z}) \) transformation will permute the cycles and will act on the twists in an obvious way. However, a full \( SO(3, 3, \mathbb{Z}) \) transformation takes one locality-frame to another and an R-symmetry twist is not mapped back to an R-symmetry twist.

6.2. The T-dual of an R-symmetry twist

What does become of an R-symmetry twist after T-duality? The effect of the R-symmetry twist is to make a state which is R-charged have a fractional momentum, because its boundary conditions are not periodic. The momentum modulo \( \mathbb{Z} \) is related to the R-charge and the twist in a linear way. Since T-duality replaces the momentum charge with another \( U(1) \) charge – the winding number of little-strings, one would deduce that after T-duality, R-charged states should have fractional winding number.

To be more precise, let us take weakly coupled type-IIA on \( X_\theta \) from (6.2) and perform T-duality. Recall that, 
\[
X_\theta = (\mathbb{R} \times \mathbb{C}^2)/\mathbb{Z},
\]
with \( \mathbb{Z} \) generated by, 
\[
s : (x, z_1, z_2) \to (x + 2\pi R, e^{i\theta} z_1, e^{-i\theta} z_2).
\]
The world-sheet theory is the free type-IIA theory. Let
\[
X = x + w\sigma + p\tau + \sum_{n \in \mathbb{Z} \neq 0} \frac{\alpha_n}{n} e^{in(\tau - \sigma)} + \sum_{n \in \mathbb{Z} \neq 0} \frac{\tilde{\alpha}_n}{n} e^{in(\tau + \sigma)},
\]
\[
Z_1 = \sum_{s \in \mathbb{Z} + \gamma_1} \frac{\zeta^{(1)}_s}{s} e^{is(\tau - \sigma)} + \sum_{s \in \mathbb{Z} + \gamma_1} \frac{\tilde{\zeta}^{(1)}_s}{s} e^{is(\tau + \sigma)},
\]
\[
Z_2 = \sum_{s \in \mathbb{Z} + \gamma_2} \frac{\zeta^{(2)}_s}{s} e^{is(\tau - \sigma)} + \sum_{s \in \mathbb{Z} + \gamma_2} \frac{\tilde{\zeta}^{(2)}_s}{s} e^{is(\tau + \sigma)},
\]
\( \gamma_{1,2} \) are real numbers which depends on the sector in a manner that we will write down below. When \( \gamma_i = 0 \), we need to add a piece \( z_i + p_i \tau \) to \( Z_i \). \( p_1, p_2 \) are complex while \( w, p \) are real. Also \( \alpha_{-n}^\dagger = \alpha_n \) and \( \tilde{\alpha}_{-n}^\dagger = \tilde{\alpha}_n \). Let \( L \) be the total number of \( \zeta^{(1)} \) creation operators minus the total number of \( \zeta^{(2)} \) creation operators in a state. If some \( \gamma_i = 0 \) we also need to add the rotation generator \( i(z_i p_i^\dagger - z_i^\dagger p_i) \).

\[
L \equiv \sum_{s \in \mathbb{Z}+\gamma_1} \frac{1}{s} (\zeta^{(1)}_{-s})^\dagger \zeta^{(1)}_{-s} - \sum_{s \in \mathbb{Z}+\gamma_2} \frac{1}{s} (\zeta^{(2)}_{-s})^\dagger \zeta^{(2)}_{-s} + (\zeta \leftrightarrow \tilde{\zeta}). \tag{6.5}
\]

Now we can determine which sectors are allowed. First we require invariance under \( s \) in (6.3). This is the world-sheet operator \( e^{2\pi ipR - i\theta L} \), so we require,

\[
pR - \frac{\theta}{2\pi} L \in \mathbb{Z}.
\]

The sector twisted by \( s^k \) has

\[
\frac{w}{R} = k, \quad \gamma = k \frac{\theta}{2\pi}.
\]

What happens after T-duality? In a world-sheet formulation, T-duality replaces \( p \) with \( w \) and replaces \( R \) with \( R' = 1/R \). We now have the conditions

\[
\frac{w'}{R'} - \frac{\theta}{2\pi} L \in \mathbb{Z}, \quad p'R' \in \mathbb{Z}, \quad \gamma = p'R' \frac{\theta}{2\pi}.
\]

This suggests a more general twist, which can no longer be described as modding out by a discrete symmetry. This time we keep the sectors with

\[
pR - \frac{\theta}{2\pi} L \in \mathbb{Z}, \quad \frac{w}{R} - \frac{\eta}{2\pi} L \in \mathbb{Z}, \quad 2\pi \gamma = \frac{\theta w}{R} + \eta pR. \tag{6.6}
\]

We admit to not having checked that this is consistent with modular invariance. The following argument suggests that turning on both \( \theta \) and \( \eta \) twists is consistent. For small \( \theta \), turning on a \( \theta \)-twists corresponds to making a small perturbation with a certain operator to the Hamiltonian of \( S(k) \). An infinitesimal \( \eta \)-twist also corresponds to a perturbation but with another operator. Now we can make a small perturbation with both a \( \theta \)-twist as well as a \( \eta \)-twist. They preserve exactly the same supersymmetry. It could, however, happen that after we turn on both \( \theta \)-twists and \( \eta \)-twists there is no longer any super-symmetric vacuum. We do not know of any way to settle this question.

We will check in subsequent sections what becomes of the hyper-Kähler moduli spaces after a T-duality.
7. Relation with instantons on non-commutative tori

So far we have discussed only $T(k)$ and $S(k)$ for $k = 2$, which reduces to $SU(2)$ SYM in appropriate limits. In this section we are tempted to present conjectures for higher $k$. To motivate the conjecture let us first look at the following table, which lists the limiting moduli spaces when all the R-symmetry twists are zero, for $S(k)$ on $T^3$, $T(k)$ on $T^3$ and the masses are zero for 3+1D $SU(k)$ SYM on $S^1$ and 2+1D $SU(k)$ SYM:

| Theory                          | $S(k)/T^3$                   | $T(k)/T^3$                  | 3+1D $SU(k)$ on $S^1$ | 2+1D $SU(k)$ |
|---------------------------------|------------------------------|----------------------------|------------------------|--------------|
| Moduli space                    | $(T^4)^k/S_k$                | $(T^3 \times \mathbb{R})^k/S_k$ | $(T^2 \times \mathbb{R}^2)^k/S_k$ | $(S^1 \times \mathbb{R}^3)^k/S_k$ |

As we discussed above each column is an appropriate limit of the one left to it. Now let us turn on $\theta$-twists. As we have argued, in the 3+1D SYM on $S^1$ and 2+1D SYM case, the $\theta$-twists correspond to turning on a mass to the adjoint hyper-multiplet. The moduli space of 2+1D $SU(k)$ SYM with massive adjoint hypermultiplets has been recently constructed by Kapustin and Sethi \[22\]. It was shown there that this moduli space is identical to the moduli space of $k$ $U(1)$ instantons on a non-commutative $S^1 \times \mathbb{R}^3$. The moduli space of instantons on non-commutative $\mathbb{R}^4$ has been recently discussed in \[23,24\]. The non-commutativity is characterized by 6 parameters which transform as a tensor of $SO(4)$. For the moduli space of instantons one only turns on 3 parameters which transform as a self-dual 2-form on $\mathbb{R}^4$. In \[22\], the moduli spaces of instantons on the non-commutative $S^1 \times \mathbb{R}^3$ was identified with the moduli space of the gauge theories by setting the 3 non-commutativity parameters to be proportional to the 3 mass parameters of the gauge theory. In fact, Kapustin and Sethi considered a more general question, namely the moduli space of 3+1D $SU(k)$ SYM with massive adjoint hypermultiplets compactified on $S^1$. This was mapped to the moduli space of non-commutative instantons on $T^2 \times \mathbb{R}^2$.

It is now tempting to conjecture that $S_A(k)$ on $T^3$ with 3 R-symmetry $\theta$-twists has the moduli space of $k$ $U(1)$ instantons on the non-commutative $T^4$ (obtained from the external parameters as in \(2.1\)). The non-commutativity is determined by the 3 $\theta_i$’s.

In section (6), we suggested that there is a more general perturbation of the $S(k)$ on $T^3$ where we turn on both $\theta$-twists and $\eta$-twists. As we mentioned, it could be that there is no super-symmetric vacuum after we turn on both twists. However, the theory probably still makes sense. Perhaps there is a deeper relation between non-commutative $T^4$ and the twisted $S(k)$ theories such that all 6 twists are mapped to all 6 non-commutativity parameters on $T^4$. The instanton moduli space only depends on 3 out of the 6 parameters which form the self-dual combination. Note that in Super-Yang-Mills, when we turn on a generic configuration of all 6 non-commutativity parameters, (say in the setting of \[25,26\]}
with a a D0-brane and $k$ D4-branes on a small $T^4$ with some NSNS 2-form flux) the instanton charge breaks supersymmetry completely. In our setting this might suggest that for the theories with a generic configuration of all 6 twists, there is no super-symmetric vacuum.

8. A M(atrix) approach

We will now study the M(atrix) description of the R-symmetry twists. In the process we will find non-local field theories which depend on a continuous parameter (the R-symmetry twist) and which can be mapped to local theories compactified on a smaller space for rational values of the parameter. This phenomenon is similar in spirit to Yang-Mills theories on non-commutative spaces as described in [25, 26, 27] and also reminiscent of the continuous limit of $(p, q)$ 5-branes theories suggested in [28, 29].

The M-theory setting that we study is somewhat similar to the vacua of [28] which were of the form $(S^1 \times C^2)/\Gamma$ where $\Gamma$ is a discrete subgroup of $SO(4)$ and shifts.

8.1. The model

We start with the following geometrical vacuum of M-theory. Take the 5-dimensional space

$$X_\theta = (\mathbb{R} \times C^2)/Z$$

where $Z$ is the freely acting group generated by,

$$s : x \rightarrow x + 2\pi R, \quad z_1 \rightarrow e^{i\theta} z_1, \quad z_2 \rightarrow e^{-i\theta} z_2. \quad (8.1)$$

Here $-\infty < x < \infty$ is a real coordinate on $\mathbb{R}$ and $(z_1, z_2)$ are complex coordinates on $C^2$.

For simplicity, let us start with the M(atrix)-model of M-theory on $X_\theta$ without any 5-branes. The division by $\Gamma$ is performed according to the rules of going to the covering space and imposing the following constraints [30, 31]. We pick $U \in U(N)$ and solve,

$$UXU^{-1} = X + 2\pi R, \quad UZ_1U^{-1} = e^{i\theta} Z_1, \quad UZ_2U^{-1} = e^{-i\theta} Z_2. \quad (8.2)$$

Here $X$ is the $N \times N$ matrix field corresponding to $x$ in (8.1) and $Z_1, Z_2$ are the complex matrices corresponding to $z_1$ and $z_2$. Generically, we can assume that $U$ is diagonal and let $e^{2\pi i \sigma R}$ be its eigenvalues, with $0 \leq \sigma \leq \frac{1}{R}$. We assume that each eigenvalue appears

---

1 We are grateful to M. Berkooz for pointing this out.
\( M \) times. The solution is then, \( X = i\partial_\sigma + A(\sigma) \) with \( A(\sigma) \) and \( M \times M \) gauge field. For \( Z_1 \) and \( Z_2 \) we find,

\[
(Z_1)_{\sigma,\sigma'} = \Phi_1(\frac{\sigma + \sigma'}{2})\delta(\sigma - \sigma' - \frac{\theta}{2\pi R}), \quad (Z_2)_{\sigma,\sigma'} = \Phi_2(\frac{\sigma + \sigma'}{2})\delta(\sigma - \sigma' + \frac{\theta}{2\pi R}).
\]

Let us define

\[
\xi = \frac{\theta}{4\pi R}.
\]

The fields \( A(\sigma, \tau) \) and \( X(\sigma, \tau) \) are in the adjoint of \( U(N) \) and live on the dual circle of radius \( 1/2\pi R \). The fields \( \Phi_1(\sigma, \tau) \) are not in the adjoint of \( U(N) \) but rather in the product \( N \otimes \bar{N} \) of the gauge group

\[
U(N)_{(\sigma - \xi)} \otimes U(N)_{(\sigma + \xi)}.
\]

Here \( U(N)_{(\sigma)} \) is the group at the point \( \sigma \). In the Lagrangian, to preserve gauge invariance, the field \( \Phi_1(\sigma) \) can be multiplied on the left by local fields at \( \sigma - \xi \) \( (X(\sigma - \xi) \text{ or } A(\sigma - \xi)) \) but on the right, it will have to be multiplied by local fields as \( \sigma + \xi \). Similarly, \( \Phi_1^\dagger \) is in \( N \otimes \bar{N} \) of \( U(N)_{(\sigma + \xi)} \otimes U(N)_{(\sigma - \xi)} \). Similar statements hold for \( \Phi_2 \) and \( \Phi_2^\dagger \) with \( \sigma + \xi \) replaced by \( \sigma - \xi \). Thus, we will see expressions like

\[
\text{tr}\{X(\sigma)\Phi(\sigma + \xi)X(\sigma + 2\xi)\Phi(\sigma + \xi)^\dagger\}, \ldots
\]

We denote,

\[
D_\mu \Phi_1(\sigma) \equiv \partial_\mu \Phi_1(\sigma) - A_\mu(\sigma - \xi)\Phi_1(\sigma) + \Phi_1(\sigma)A_\mu(\sigma + \xi),
\]

\[
D_\mu \Phi_2(\sigma) \equiv \partial_\mu \Phi_2(\sigma) - A_\mu(\sigma + \xi)\Phi_2(\sigma) + \Phi_2(\sigma)A_\mu(\sigma - \xi),
\]

\[
D_\mu \Phi_1(\sigma)^\dagger \equiv \partial_\mu \Phi_1(\sigma)^\dagger - A_\mu(\sigma + \xi)\Phi_1(\sigma)^\dagger + \Phi_1(\sigma)^\dagger A_\mu(\sigma - \xi),
\]

\[
D_\mu \Phi_2(\sigma)^\dagger \equiv \partial_\mu \Phi_2(\sigma)^\dagger - A_\mu(\sigma - \xi)\Phi_2(\sigma)^\dagger + \Phi_2(\sigma)^\dagger A_\mu(\sigma + \xi),
\]

The Lagrangian is given schematically by:

\[
\mathcal{L} = \frac{1}{\Lambda} \int \text{d}\sigma \text{d}\tau \text{ Tr}\{D_\mu XD^\mu X + F_{\mu\nu}F^{\mu\nu} + D_\mu \Phi_i D^\mu \Phi_i^\dagger + \}
\]

\[
+ (X(\sigma)\Phi(\sigma + \xi) - \Phi(\sigma + \xi)X(\sigma + 2\xi))(X(\sigma + 2\xi)\Phi(\sigma + \xi)^\dagger - \Phi(\sigma + \xi)^\dagger X(\sigma))
\]

\[
+ (\Phi_1(\sigma + \xi)\Phi_2(\sigma + \xi) - \Phi_2(\sigma + \xi)\Phi_1(\sigma + \xi))(\Phi_1(\sigma - \xi)^\dagger \Phi_2(\sigma - \xi) - \Phi_2(\sigma - \xi)^\dagger \Phi_1(\sigma + \xi))
\]

\[
+ (\Phi_1(\sigma + \xi)\Phi_2(\sigma + 3\xi)^\dagger - \Phi_2(\sigma + 3\xi)^\dagger \Phi_1(\sigma + 3\xi))
\]

\[
(\Phi_1(\sigma - \xi)^\dagger \Phi_2(\sigma - 3\xi) - \Phi_2(\sigma - 3\xi)^\dagger \Phi_1(\sigma - 3\xi)^\dagger )\},
\]

(8.4)

where \( \mu = 0, 1 \) is along the 1+1D space. The Lagrangian (8.4) is non-local for irrational \( \theta \). For rational \( \theta = \frac{m}{n} \) we can redefine the theory on a short interval of length \( 1/l \) smaller. We
then get a local theory but with a gauge group $U(N)_l$. The fields $\Phi_i$ become hypermultiplets in the $(N, \bar{N})$ representation. The matrix model then reduces to the one described in [28], as expected since the space $X_\theta$ becomes one of the spaces of [28] in this case.

We can now insert $k$ M5-branes at position $Z_1 = Z_2 = 0$. This amounts to compactifying the model of [32] according to (8.2). The new ingredient is that [32] also has fields $v$ in the fundamental $N$ of $U(N)$. The scalars are not charged under the $SO(5)$ R-symmetry but they satisfy $Uv = v$ and become localized at impurities at $\sigma = 0$ (see [33,34,22]).

How do we see the non-trivial moduli space coming out in the M(atrix) description? In [22] the Higgs branch of the impurity system was studied. In these cases, the impurity system was a M(atrix) model for a system with 16 supersymmetries and the dependence on external parameters is not expected to be quantum corrected. In our case, the system is a M(atrix) model for a vacuum with 8 supersymmetries. One can define a metric on parameter space as,

$$g_{\alpha\beta} \equiv -\langle 0|\partial_\alpha \partial_\beta |0\rangle - A_\alpha A_\beta, \quad A_\alpha = \langle 0|\partial_\alpha |0\rangle.$$ 

This is very similar to the Zamolodchikov metric in conformal field-theory. The relation between this metric and the metric on the moduli space of the theories will be explored in a future work [35].

9. Discussion

We have argued that the moduli space of vacua of $S_A(2)$ ($S_B(2)$) compactified on $\mathbf{T}^3$ with 3 R-symmetry twists, $\theta_1, \theta_2, \theta_3$, is the same as the moduli space of vacua of the heterotic $E_8 \times E_8$ ($SO(32)$) (1,0) NS5-brane theory compactified on the same $\mathbf{T}^3$ with Wilson lines given by an embedding of the twists in the gauge group. We have presented a conjecture for higher $k$ involving instantons on non-commutative tori. We have also studied how T-duality of the little-string theory acts on the R-symmetry $\theta$-twists. We have seen that they get mapped to other types of twists ($\eta$-twists). We have suggested that there exist theories with both kinds of twists simultaneously.

Let us suggest a few questions for further research:
1. Confirm or disprove the conjecture of section (7) about the relation between the moduli spaces of $S(k)$ on $\mathbf{T}^3$ and instantons on a non-commutative $\mathbf{T}^4$.
2. Find an M-theoretic derivation of the moduli spaces, or perhaps using compactification on a Calabi-Yau manifold.
3. Study the BPS spectrum of the theories in 3+1D and 4+1D. We have identified the moduli spaces of the twisted $(2,0)$ theory with the moduli space of the compactified
\(E_8 (1, 0)\) theory. However, these two theories are not identical. It would be interesting to see how this distinction is manifested in the multiplicities of BPS states [36,37,38,39,40].

4. Study the M(atrix) models of these compactifications. In M(atrix)-theory the moduli space of vacua of the theory should be manifested as the space of external parameters of the M(atrix) Hamiltonian. It would be interesting to see how the non-flat metric and the non-trivial topology of the moduli space arises.

5. Study the theories with combined \(\theta\)-twists and \(\eta\)-twists. In particular, do they have a super-symmetric vacuum?

6. Study the other phase where little-strings condense (see section (4.2)).

Acknowledgments

We wish to thank M. Berkooz, S. Ramgoolam and S. Sethi for discussions. The research of OJG was supported by a Robert H. Dicke fellowship and by DOE grant DE-FG02-91ER40671 and the research of MK was supported by the Danish Research Academy.
References

[1] E. Witten, “Some Comments on String Dynamics,” *hep-th/9507121*, published in “Future perspectives in string theory,” 501-523.
[2] A. Strominger, “Open p-Branes,” Phys. Lett. **383B** (1996) 44-47, *hep-th/9512059*
[3] N. Seiberg, “New Theories in Six-Dimensions and Matrix Description of M-theory on \( T^5 \) and \( T^5/\mathbb{Z}_2 \),” *hep-th/9705221*, Phys. Lett. **408B** (97) 98
[4] N. Seiberg, “Notes on Theories with 16 Supercharges,” *hep-th/9705117*
[5] J. Maldacena and A. Strominger, “Semiclassical Decay of Near Extremal 5-Branes,” *JHEP* **12**(97)008, *hep-th/9710014*
[6] S.S. Gubser and I.R. Klebanov, “Absorption by Branes and Schwinger Terms in the World Volume Theory,” Phys. Lett. **413B** (1997) 41–48, *hep-th/9708005*
[7] E. Witten, “Anti-de Sitter Space, Thermal Phase Transition, and Confinement in Gauge Theories,” *hep-th/9803131*
[8] E. Witten, “Solutions Of Four-Dimensional Field Theories Via M Theory,” Nucl. Phys. **B500** (1997) 3–42, *hep-th/9703166*
[9] S. Kachru and C. Vafa, “Exact Results For \( N = 2 \) Compactifications Of Heterotic Strings,” Nucl. Phys. **B450** (95) 69, *hep-th/9505105*
[10] T. Banks, M.R. Douglas and N. Seiberg, “Probing F-theory With Branes,” Phys. Lett. **387B** (1996) 278–281, *hep-th/9605199*
[11] N. Seiberg, “IR Dynamics on Branes and Space-Time Geometry,” Phys. Lett. **384B** (1996) 81–85, *hep-th/9606017*
[12] P. Aspinwall, “K3 Surfaces and String Duality,” *hep-th/9611137*
[13] N. Seiberg, “Naturalness Versus Supersymmetric Non-Renormalization Theorems,” Phys. Lett. **318B** (1993) 469–475, *hep-ph/9309335*
[14] N. Seiberg and E. Witten, “Gauge Dynamics And Compactification To Three Dimensions,” *hep-th/9607163*
[15] A. Sen, “F-theory and Orientifolds,” Nucl. Phys. **B475** (1996) 562-578, *hep-th/9605150*
[16] N. Seiberg, “Five Dimensional SUSY Field Theories, Non-trivial Fixed Points and String Dynamics,” Phys. Lett. **388B** (1996) 753-760, *hep-th/9608111*
[17] S. Sethi and L. Susskind, “Rotational Invariance in the M(atrix) Formulation of Type IIB Theory,” Phys. Lett. **400B** (1997) 265–268, *hep-th/9702101*
[18] T. Banks and N. Seiberg, “Strings from Matrices,” Nucl. Phys. **B497** (1997) 41–55, *hep-th/9702187*
[19] N. Seiberg and E. Witten, “Monopoles, Duality and Chiral Symmetry Breaking in \( N = 2 \) Supersymmetric QCD,” Nucl. Phys. **B431** (1994) 484–550, *hep-th/9408099*
[20] O.J. Ganor, D.R. Morrison and N. Seiberg, “Branes, Calabi-Yau Spaces, and Toroidal Compactification of the \( N = 1 \) Six-Dimensional \( E_8 \) Theory,” Nucl. Phys. **B487** (1997) 93, *hep-th/9610251*
[21] K. Intriligator, D.R. Morrison and N. Seiberg, “Five-Dimensional Supersymmetric
Gauge Theories and Degenerations of Calabi-Yau Spaces,” Nucl. Phys. B497 (97)
56-100, hep-th/9702198
[22] A. Kapustin and S. Sethi, “The Higgs Branch of Impurity Theories,” hep-th/9804027
[23] N. Nekrasov, A. Schwarz, “Instantons on noncommutative $R^4$ and (2,0) superconformal
six dimensional theory,” hep-th/9802068
[24] M. Berkooz, “Non-local Field Theories and the Non-commutative Torus,” hep-th/9802069
[25] A. Connes, M.R. Douglas and A. Schwarz, “Noncommutative Geometry and Matrix
Theory: Compactification on Tori,” hep-th/9711162
[26] M.R. Douglas and C. Hull, “D-branes and the Noncommutative Torus,” hep-th/9711165
[27] Y.-K. E. Cheung and M. Krogh, “Noncommutative Geometry from 0-branes in a
Background B-field,” hep-th/9803031
[28] E. Witten, “New “Gauge” Theories In Six Dimensions,” hep-th/9710065
[29] B. Kol, “On 6d “Gauge” Theories with Irrational Theta Angle”, hep-th/9711017
[30] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, “M Theory As A Matrix Model:
A Conjecture,” hep-th/9610043, Phys. Rev. D55 (1997) 5112-5128
[31] W. Taylor, “D-brane field theory on compact spaces,” hep-th/9611042, Phys. Lett.
394B (1997) 283.
[32] M. Berkooz and M.R. Douglas, “Five-branes in M(atrix) Theory,” hep-th/9610236
[33] S. Sethi, “The Matrix Formulation of Type IIB Five-Branes,” hep-th/9710005
[34] O.J. Ganor and S. Sethi, “New Perspectives on Yang-Mills Theories With Sixteen
Supersymmetries,” hep-th/9712071
[35] Work in progress.
[36] O.J. Ganor, “A Test Of The Chiral E8 Current Algebra On A 6D Non-Critical String,”
Nucl. Phys. B479 (1996) 197–217, hep-th/9607020
[37] A. Klemm, P. Mayr and C. Vafa, “BPS States of Exceptional Non-Critical Strings,”
hep-th/9607139
[38] J.A. Minahan, D. Nemeschansky and N.P. Warner, “Investigating the BPS Spectrum
of Non-Critical $E_n$ Strings,” Nucl. Phys. B508 (1997) 64–106, hep-th/9705237
[39] J. A. Minahan, D. Nemeschansky and N. P. Warner, “Partition Functions for BPS
States of the Non-Critical $E_8$ String,” Adv.Theor.Math.Phys. 1 (1998) 167-183, hep-th/9707149
[40] J. A. Minahan, D. Nemeschansky, C. Vafa and N. P. Warner, “E-Strings and $N = 4$
Topological Yang-Mills Theories,” hep-th/9802168

27