What Can Be Done with Consensus Number One:
Relaxed Queues and Stacks

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Abstract
Sequentially specified linearizable concurrent data structures must be relaxed in order to
support scalability. In this work, we identify and formally define relaxations of queues and
stacks that can be non-blocking or wait-free implemented using only read/write operations. We
use set-linearizability to specify the relaxations formally, and precisely identify the subset of
executions which preserve the original sequential behavior. The relaxations allow for an item to
be extracted more than once by different operations, but only in case of concurrency; we call such
a property multiplicity. The stack implementation is wait-free, while the queue implementation
is non-blocking. We also use interval-linearizability to describe a queue with multiplicity, with
the additional and new relaxation that a dequeue operation can return weak-empty, which
means that the queue might be empty. We present a read/write wait-free implementation of
the interval-sequential queue. As far as we know, this work is the first that provides simple and
clear formalizations of the notions of multiplicity and weak-emptiness, which can be implemented
from read/write registers only.

1 Introduction
Linearizable implementations of concurrent counters, queues, stacks, pools, and other concurrent
data structures [32] need extensive synchronization among processes, which in turn jeopardize
performance and scalability. Moreover, it has been formally shown that this cost is sometimes
unavoidable, under various specific assumptions [10, 11, 17]. However, often applications do not
require all guarantees offered by a linearizable sequential specification [38]. Thus, much research
has focused on improving performance of concurrent data structures by relaxing their semantics.
Furthermore, several works have focused on relaxations for queues and stacks, achieving significant
performance improvements.

It is impossible however to implement queues and stacks with only Read/Write operations,
without relaxing the specification [21]. Thus, atomic Read-Modify-Write operations, such as Com-
pare&Swap or Test&Set, are required in any queue or stack implementation. To the best of our
knowledge, even relaxed versions of queues or stacks have not been designed that avoid the use of
Read-Modify-Write operations.

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Another challenge in designing relaxed data structures is the difficulty of formally specifying what is meant by “relaxed specification”. One approach is to stay with the original sequential specification, and demand a consistency notion weaker than linearizability. Alternatively, one may define a weaker specification, and demand linearizability only with respect to the relaxed specification. A combination of both approaches is even more confusing. In the literature, some works simply do not provide a formal specification of the relaxation.

1.1 Contributions

Our goal is to identify and formally define relaxations of queues and stacks that can be implemented using only simple Read/Write operations. As far as we know, our work is the first theoretical advancement in implementing relaxed versions of queues and stacks with consensus number one operations.

To provide a formal specification of our relaxations, we use set-linearizability [33] and interval-linearizability [15], specification methods that are useful to specify the behavior of a data structure in concurrent patterns of operation invocations, instead of only in sequential patterns. Using these specification methods, we are able to precisely state in which executions the relaxed behavior of the data structure should take place, and demand a strict behavior (not relaxed), in other executions, especially when operation invocations are sequential.

We consider queues and stacks relaxations with multiplicity, where an item can be extracted by more than one dequeue or pop operation, instead of exactly once. We use set-linearizability to specify that this may happen only in the presence of concurrent operations. As already argued [30], this type of relaxation could be useful in a wide range of applications, such as parallel garbage collection, fixed point computations in program analysis, constraint solvers (e.g. SAT solvers), state space search exploration in model checking, as well as integer and mixed programming solvers.

First contribution.

We define a set-sequential stack with multiplicity, in which no items is lost, all items are pushed/popped in LIFO order but an item can be popped by multiple operations, which are then concurrent. We define a set-sequential queue with multiplicity similarly. In both cases we present set-linearizable implementations based only on Read/Write operations. The stack implementation is wait-free [21], while the queue implementation is non-blocking [23].

We have thus shown that Read/Write operations are strong enough to guarantee FIFO/LIFO order (at the cost of getting an item more than once in well-defined concurrency patterns). Also, that queues and stack with multiplicity allows us to overcome the impossibility results of [11], by exhibiting that it is possible to avoid atomic Read-Modify-Write operations and Read-After-Write synchronization patterns. As far as we know, all previous relaxed queue or stack implementations use more powerful (in the sense of the consensus number hierarchy) Read-Modify-Write operations. Our set-sequential implementations imply Read/Write solutions for idempotent work-stealing [30] and $k$-FIFO [26] queues and stacks.

Second contribution.

We define an interval-sequential queue with a weak-emptiness check, which behaves like a classical sequential queue with the exception that a dequeue operation can return a control value denoted
weak-empty. Intuitively, this value means that the operation was concurrent with dequeue operations that took the items that were in the queue when it started, thus the queue might be empty. First, we describe a wait-free interval-sequential implementation based on Fetch&Inc and Swap operations. Then, using the techniques in our set-linearizable stack and queue implementations, we obtain a wait-free interval-sequential implementation using only Read/Write operations.

Our interval-sequential queue with weak-emptiness check is motivated by a question that has been open for more than two decades [4]: it is unknown if there is a wait-free linearizable queue implementation based on objects with consensus number two (e.g. Fetch&Inc or Swap), i.e. objects that allow consensus to be solved among two processes but not three. There are only such non-blocking implementations in the literature, or wait-free implementations for restricted cases (e.g. [13, 16, 27, 28, 29]). Interestingly, our interval-sequential queue allows to go from non-blocking to wait-freedom. Our interval-sequential queue models the tail-chasing problem that one faces when trying to obtain a wait-free queue implementation from objects with consensus number two.

Since we are interested in the computability power of Read/Write operations to implement relaxed concurrent objects (that otherwise are impossible), our algorithms are presented in an idealized shared-memory computational model. We hope these algorithms can help to find solutions for real multicore architectures, with good performance and scalability.

1.2 Related Work

It has been frequently pointed out that classic concurrent data structures have to be relaxed in order to support scalability, and examples are known showing how natural relaxations on the ordering guarantees of queues or stacks can result in higher performance and greater scalability [38]. Thus, for the past ten years there has been a surge of interest in relaxed concurrent data structures, from practitioners (e.g. [34]). Also, theoreticians have identified inherent limitations in achieving high scalability in the implementation of linearizable objects [10, 11, 17].

Some articles relax the sequential specification of traditional data structures, while others relax their correctness condition requirements. As an example of relaxing the requirement of a sequential data structure, [25, 26, 35] present a k-FIFO queue in which elements may be dequeued out of FIFO order up to a constant $k \geq 0$. A family of relaxed queues and stacks is introduced in [39], and studied from a computability point of view (consensus numbers). The previous works use relaxed specifications, but still sequential, while we relax the specification to make it concurrent (using set-linearizability and interval-linearizability).

Relaxed priority queues (in the flavor of [39]) and associated performance experiments are presented in [6, 42].

Other works design a weakening of the consistency condition. For instance, quasi-linearizability [3], which allows to model relaxed data structures through a distance function from valid sequential executions. This work provides examples of quasi-linearizable concurrent implementations that outperform state of the art standard implementations. A quantitative relaxation framework to formally specify relaxed objects is introduced in [19, 20] where relaxed queues, stacks and priority queues are studied. This framework is more powerful than quasi-linearizability. It is shown in [40] that linearizability and three data type relaxations studied in [20], k-Out-of-Order, k-Lateness, and k-Stuttering, can also be defined as consistency conditions. The notion of local linearizability is introduced in [18]. It is a relaxed consistency condition that is applicable to container-type concurrent data structures like pools, queues, and stacks. The notion of distributional linearizability [5]
allows to capture *randomized* relaxations. This formalism is applied to MultiQueues [37], a family of concurrent data structures implementing relaxed concurrent priority queues.

The notion of *idempotent work stealing* is introduced in [30], where are described LIFO, FIFO and double-ended queue algorithms that exploit the relaxed semantics to deliver better performance than usual work stealing algorithms. Similarly to our queues and stacks with multiplicity, the *idempotent* relaxation means that each inserted item is eventually extracted at least once, instead of exactly once. In contrast to our work, the algorithms presented in [30] use Compare&Swap (in the *Steal* operation). Being a practical-oriented work, formal specifications of the implemented data structures are not given.

In a few words:

To summarize, the novelty and the main contributions of the paper are the following.

1. We identify relaxations for queues and stacks that can be implemented from consensus number one (implementations of previous relaxations are based on Read-Modify-Write operations).

2. We provide simple and formally-defined specifications of the relaxations.

3. We provide simple algorithms and formal correctness proofs. Interestingly the proposed algorithms are based on a similar skeleton, which could allow for the design of a generic family of algorithms.

1.3 Organization

The article is organized as follows. Section 2 presents the model of computations and the correctness conditions, namely, linearizability, set-linearizability and interval-linearizability. Section 3 introduces the notion of set-sequential stack with multiplicity and presents a read/wait wait-free solution of it, while Section 4 defines the set-sequential queue with multiplicity and shows a non-blocking wait/free implementation. Some consequences of the set-sequential queue and stack implementations are discussed in Section 5. The new interval-sequential queue with weak-emptiness check and its implementation are presented in Section 6. Section 7 concludes the paper with a final discussion.

2 Preliminaries

2.1 Model of Computation

We consider the standard concurrent system model with $n$ asynchronous processes, $p_1, \ldots, p_n$, which may crash at any time during an execution, namely, a process that crashes stops taking steps. The index of process $p_i$ is $i$. Processes communicate with each other by invoking atomic operations on shared *base objects*. A base object can provide atomic Read/Write operations (such an object is henceforth called a *register*), or more powerful atomic Read-Modify-Write operations, such as Fetch&Inc, Swap or Compare&Swap.

The operation $R.\text{Swap}(x)$ atomically reads the current value of $R$, sets its value to $x$ and returns $R$’s old value. The operation $R.\text{Fetch}\&\text{Inc}()$ atomically adds 1 to the current value of $R$ and returns the previous value. The operation $R.\text{Compare}\&\text{Swap}(\text{new},\text{old})$ is a conditional replacement
operation that atomically checks if the current value of R is equal to old, and if so, replaces it with new and returns true; otherwise, R remains unchanged and the operation returns false.

A (high-level) concurrent object, or data type, is, roughly speaking, defined by a state machine consisting of a set of states, a finite set of operations, and a set of transitions between states. The specification does not necessarily have to be sequential, namely, (1) a state might have pending operations and (2) state transitions might involve several invocations. The following subsections formalize this notion and the different types of objects.

An implementation of a concurrent object T is a distributed algorithm A consisting of local state machines A_1, ..., A_n. Local machine A_i specifies which operations on base objects p_i executes in order to return a response when it invokes a high-level operation of T. Each of these base objects operation invocations is a step.

An execution of A is a possibly infinite sequence of steps, namely, executions of base objects operations, plus invocations and responses for each operation op() of the high-level concurrent object T, with the following properties:

1. Each process first invokes a high-level operation, and only when it has a corresponding response, it can invoke another high-level operation, i.e., executions are well-formed.

2. For any invocation to an operation op, denoted inv(op), of a process p_i, the steps of p_i between that invocation and its corresponding response (if there is one), denoted res(op), are steps that are specified by A when p_i invokes op.

An operation in an execution is complete if both its invocation and response appear in the execution. An operation is pending if only its invocation appears in the execution. A process is correct in an execution if it takes infinitely many steps. For sake of simplicity, and without loss of generality, we identify the invocation of an operation with its first step, and its response with its last step.

In subsequent sections, we will formally define and implement relaxed versions the classical queues and stacks. For sake of simplicity, and without loss of generality, we will suppose that in every execution an item can be enqueued/pushed at most once.

An implementation is wait-free if every process completes each operation in a finite number of its steps. Formally, if a process executes infinitely many steps in an execution, all its invocations are completed. An implementation is non-blocking if whenever processes take steps, at least one of the operations terminates. Formally, in every infinite execution, infinitely many invocations are completed. Thus, a wait-free implementation is non-blocking but not necessarily vice versa.

The consensus number of a shared object O is the maximum number of processes that can solve the well-known consensus problem, using any number of instances of O in addition to any number of Read/Write registers. Consensus numbers induce the consensus hierarchy where objects are classified according their consensus numbers. The simple Read/Write operations stand at the bottom of the hierarchy, with consensus number one; these operations are the least expensive ones in real multicore architectures. At the top of the hierarchy we find operations with infinite consensus number, like Compare&Swap, that provide the maximum possible coordination.

2.2 The Linearizability Correctness Condition

Linearizability is the standard notion used to identify a correct implementation. Intuitively, an execution is linearizable if operations can be ordered sequentially, without reordering non-overlapping operations, so that their responses satisfy the specification of the implemented object.
A *sequential specification* of a concurrent object $T$ is a state machine specified through a transition function $\delta$. Given a state $q$ and an invocation $\text{inv}(\text{op})$, $\delta(q, \text{inv}(\text{op}))$ returns the tuple $(q', \text{res}(\text{op}))$ (or a set of tuples if the machine is non-deterministic) indicating that the machine moves to state $q'$ and the response to $\text{op}$ is $\text{res}(\text{op})$. The sequences of invocation-response tuples, $\langle \text{inv}(\text{op}) : \text{res}(\text{op}) \rangle$, produced by the state machine are its *sequential executions*.

To formalize linearizability we define a partial order $<_\alpha$ on the completed operations of an execution $\alpha$: $\text{op} <_\alpha \text{op}'$ if and only if $\text{res}(\text{op})$ precedes $\text{inv}(\text{op}')$ in $\alpha$. Two operations are concurrent, denoted $\text{op}||_\alpha \text{op}'$, if they are incomparable by $<_\alpha$. The execution is sequential if $<_\alpha$ is a total order.

**Definition 1 (Linearizability).** Let $\mathcal{A}$ be an implementation of a concurrent object $T$. An execution $\alpha$ of $\mathcal{A}$ is linearizable if there is a sequential execution $S$ of $T$ such that

1. $S$ contains every completed operation of $\alpha$ and might contain some pending operations. Inputs and outputs of invocations and responses in $S$ agree with inputs and outputs in $\alpha$.

2. For every two completed operations $\text{op}$ and $\text{op}'$ in $\alpha$, if $\text{op} <_\alpha \text{op}'$, then $\text{op}$ appears before $\text{op}'$ in $S$.

We say that $\mathcal{A}$ is linearizable if each of its executions is linearizable.

### 2.3 The Set-Linearizability Correctness Condition

To formally specify our relaxed queues and stacks, we use the formalism provided by the set-linearizability and interval-linearizability consistency conditions [15, 33]. Roughly speaking, set-linearizability allows to linearize several operations in the same point, namely, all these operations are executed concurrently, while interval-linearizability allows operations to be linearized concurrently with several non-concurrent operations. Figure 1 schematizes the differences between the three consistency conditions where each double-end arrow represents an operation execution. It is known that set-linearizability has strictly more expressiveness power than linearizability, and interval-linearizability is strictly more powerful than set-linearizability. Moreover, as linearizability, both set-linearizability and interval-linearizability are composable (also called local) [15].

A *set-sequential specification* of a concurrent object differs from a sequential execution in that $\delta$ receives as input the current state $q$ of the machine and a set $\text{Inv} = \{\text{inv}(\text{op}_1), \ldots, \text{inv}(\text{op}_t)\}$ of operation invocations that happen concurrently. Thus $\delta(q, \text{Inv})$ returns $(q', \text{Res})$ where $q'$ is the next state and $\text{Res} = \{\text{res}(\text{op}_1), \ldots, \text{res}(\text{op}_t)\}$ are the responses to the invocations in $\text{Inv}$. The sets $\text{Inv}$ and $\text{Res}$ are called *concurrency classes*. This, in a sequential specification all concurrency classes have a single element.

**Definition 2 (Set-linearizability).** Let $\mathcal{A}$ be an implementation of a concurrent object $T$. An execution $\alpha$ of $\mathcal{A}$ is set-linearizable if there is a set-sequential execution $S$ of $T$ such that

1. $S$ contains every completed operation of $\alpha$ and might contain some pending operations. Inputs and outputs of invocations and responses in $S$ agree with inputs and outputs in $\alpha$.

2. For every two completed operations $\text{op}$ and $\text{op}'$ in $\alpha$, if $\text{op} <_\alpha \text{op}'$, then $\text{op}$ appears before $\text{op}'$ in $S$.

We say that $\mathcal{A}$ is set-linearizable if each of its executions is set-linearizable.
Figure 1: Linearizability requires a total order on the operations, set-linearizability allows several operations to be linearized at the same linearization point, while interval-linearizability allows an operation to be decomposed into several linearization points.

2.4 The Interval-Linearizability Correctness Condition

In an interval-sequential specification, some operations might be pending in a given state \( q \). Thus in \((q',\text{Res}) = \delta(q,\text{Inv})\), some of the operations that are pending in \( q \) might still be pending in \( q' \) and operations invoked in Inv may be pending in \( q' \), therefore Res contains the responses to the operations that are completed when moving from \( q \) to \( q' \).

Definition 3 (Interval-linearizability). Let \( A \) be an implementation of a concurrent object \( T \). An execution \( \alpha \) of \( A \) is interval-linearizable if there is an interval-sequential execution \( S \) of \( T \) such that

1. \( S \) contains every completed operation of \( \alpha \) and might contain some pending operations. Inputs and outputs of invocations and responses in \( S \) agree with inputs and outputs in \( \alpha \).

2. For every two completed operations \( \text{op} \) and \( \text{op}' \) in \( \alpha \), if \( \text{op} <_\alpha \text{op}' \), then \( \text{op} \) appears before \( \text{op}' \) in \( S \).

We say that \( A \) is interval-linearizable if each of its executions is interval-linearizable.

3 Set-Sequential Stacks with Multiplicity

By the universality of consensus [21], we know that, for every sequential object there is a linearizable wait-free implementation of it, for any number of processes, using Read/Write registers and base objects with consensus number \( \infty \), e.g. Compare&Swap [22, 36, 41]. However, the resulting implementation might not be efficient because first, as it is universal, the construction does not exploit the semantics of the particular object, and second, Compare&Swap are expensive operations in multicore architectures. Thus, it is desirable to have efficient implementations using “cheaper” operations, i.e., operations whose consensus number is a small constant. In this direction, an interesting example is the Snapshot object for which there are several linearizable wait-free Read/Write implementations, e.g. [1, 8, 12, 24].
3.1 A Wait-free Linearizable Stack from Consensus Number Two

Afek, Gafni and Morisson proposed in [2] a simple linearizable wait-free stack implementation for \( n \geq 2 \) processes, using Fetch&Inc and Test&Set base objects, whose consensus number is 2. Figure 2 contains a slight variant of this algorithm that uses Swap and readable Fetch&Inc objects, both with consensus number 2. A Push operation reserves a slot in Item by atomically reading and incrementing Top (Line 01) and then places its item in the corresponding position (Line 02). A Pop operation simply reads the Top of the stack (Line 04) and scans down Items from that position (Line 05), trying to obtain an item with the help of a Swap operation (Lines 06 and 07); if the operation cannot get an item (a non-⊥ value), it returns empty (Line 09). In what follows, we call this implementation Seq-Stack. It is worth mentioning that, although Seq-Stack has a simple structure, its linearizability proof is far from trivial, being the difficult part proving that items are taken in LIFO order.

In a formal sense, Seq-Stack is the best we can do, from the perspective of the consensus hierarchy: if there were a wait-free (or non-blocking) linearizable implementation based only on Read/Write registers, we could solve consensus among two processes in the standard way, by popping a value from the stack initialized to a single item containing a predefined value winner; this is a contradiction as consensus cannot be solved from Read/Write registers [22, 36, 41]. Therefore, there is no exact wait-free linearizable stack implementation from Read/Write registers only. However, we could search for approximate solutions.

Below, we show a formal definition of the notion of a relaxed set-sequential stack and prove that it can be wait-free implemented from Read/Write registers. Informally, our solution consists in implementing relaxed versions of Fetch&Inc and Swap with Read/Write registers, and plug these implementations in Seq-Stack.

![Shared Variables:
Top: Fetch&Inc base object initialized to 1
Items[1,:]: array of Swap base objects initialized to ⊥

Operation Push \((x_i)\) is
(01) \(top_i \leftarrow Top.Fetch&Inc()\)
(02) \(Items[top_i].Write(x_i)\)
(03) return true
end Push

Operation Pop is
(04) \(top_i \leftarrow Top.Read() - 1\)
(05) for \(r_i \leftarrow top_i\) down to 1 do
(06) \(x_i \leftarrow Items[r_i].Swap(\perp)\)
(07) if \(x_i \neq \perp\) then return \(x_i\) end if
(08) end for
(09) return \(\epsilon\)
end Pop

Figure 2: Stack implementation Seq-Stack of Afek, Gafni and Morisson [2] (code for process \(p_i\)).

3.2 A Set-linearizable Read/Write Stack with Multiplicity

Roughly speaking, our relaxed stack allows concurrent Pop operations to obtain the same item, but all items are returned in LIFO order, and no pushed item is lost. Formally, our set-sequential stack

1The authors themselves explain in [2] how to replace Test&Set with Swap.
is specified as follows:

**Definition 4** (Set-Sequential Stack with Multiplicity). The universe of items that can be pushed is \( N = \{1, 2, \ldots\} \), and the set of states \( Q \) is the infinite set of strings \( N^* \). The initial state is the empty string, denoted \( \epsilon \). In state \( q \), the first element in \( q \) represents the top of the stack, which might be empty if \( q \) is the empty string. The transitions are the following:

1. For \( q \in Q \), \( \delta(q, \text{Push}(x)) = (x \cdot q, (\text{Push}(x) : \text{true})) \).

2. For \( q \in Q \), \( 1 \leq t \leq n \) and \( x \in N \): \( \delta(x \cdot q, \{\text{Pop}_1(), \ldots, \text{Pop}_t()\}) = (q, \{\text{Pop}_1() : x, \ldots, (\text{Pop}_t() : x)\}) \).

3. \( \delta(\epsilon, \text{Pop}()) = (\epsilon, (\text{Pop}() : \epsilon)) \).

**Remark 1.** Every execution of the set-sequential stack with all its concurrency classes containing a single operation, is an execution of the sequential stack.

The following lemma shows that any algorithm implementing the set-sequential stack keeps the behavior of a sequential stack in several cases. In fact, the only reason the implementation does not provide linearizability is due only to the \( \text{Pop} \) operations that are concurrent.

**Lemma 1.** Let \( A \) be any set-linearizable implementation of the set-sequential stack with multiplicity. Then,

1. All sequential executions of \( A \) are executions of the sequential stack.

2. All executions with no concurrent \( \text{Pop} \) operations are linearizable with respect to the sequential stack.

3. All executions with \( \text{Pop} \) operations returning distinct values are linearizable with respect to the sequential stack.

4. If \( \text{Pop} \) operations return the same value in an execution, then they are concurrent.

**Proof.** Consider any execution \( E \) of \( A \) and a set-linearization \( \text{SetLin}(E) \) of it. The definition of the set-sequential stack implies that if all concurrency classes in \( \text{SetLin}(E) \) have a single operation, then \( \text{SetLin}(E) \) is an execution of the sequential stack. We prove each item separately:

1. Since \( E \) is sequential, all concurrency classes in \( \text{SetLin}(E) \) have a single operation.

2. If there are no concurrent \( \text{Pop} \) operations in \( E \), then every concurrency class of \( \text{SetLin}(E) \) contains at most one \( \text{Pop} \) operation. By the specification of the set-sequential stack, every \( \text{Push} \) operation appears alone in its concurrency class. Thus, every concurrency class of \( \text{SetLin}(E) \) contains a single operation.

3. A similar reasoning implies that every concurrency class of \( \text{SetLin}(E) \) contains a single operation.

4. If any pair of \( \text{Pop} \) return distinct values, then, the definition of the set-sequential stack implies that every \( \text{Pop} \) operation appears alone in its concurrency class. As observed before, the definition of the objects also implies that the same happens with \( \text{Push} \) operations. Thus, every concurrency class of \( \text{SetLin}(E) \) contains a single operation.
Shared Variables:

- **Top**: Read/Write wait-free linearizable Counter base object initialized to 1
- **Items**[1, ...,][1, ..., n]: array of Read/Write registers initialized to ⊥

Operation **Push(x)** is

1. top_i ← Top.Read()
2. Top.Increment()
3. Items[top_i, i].Write(x)
4. return true

end Push

Operation **Pop()** is

1. top_i ← Top.Read() – 1
2. for r_i ← top_i down to 1 do
3. for s_i ← n down to 1 do
4. x_i ← Items[r_i][s_i].Read()
5. if x_i ≠ ⊥ then
6. Items[r_i][s_i].Write(⊥)
7. return x_i
8. end if
9. end for
10. end for
11. return ϵ

end Pop

Figure 3: Read/Write wait-free set-sequential stack **Set-Seq-Stack** with multiplicity (code for process p_i).

The algorithm in Figure 3 is a set-linearizable Read/Write wait-free implementation of the stack with multiplicity, which we call **Set-Seq-Stack**. This implementation is a modification of **Seq-Stack**. The Fetch&Inc operation in Line 01 in **Seq-Stack** is replaced by a Read and Increment operations of a Read/Write wait-free linearizable Counter, in Lines 01 and 02 in **Set-Seq-Stack**. This causes a problem as two Push operations can set the same value in their top_i local variables. This problem is resolved with the help of a two-dimensional array **Items** in Line 03, which guarantees that no pushed item is lost. Similarly, the Swap operation in Line 06 in **Seq-Stack** is replaced by Read and Write operations in Lines 08 and 10 in **Set-Seq-Stack**, together with the test in Line 09 which ensures that a Pop operation modifies an entry in **Items** only if an item has been written in it. Thus, it is now possible that two distinct Pop operations get the same non-⊥ value, which is fine because this can only happen if the operations are concurrent. Object **Top** in **Set-Seq-Stack** can be any of the known Read/Write wait-free linearizable Counter implementations.

**Theorem 1.** The algorithm **Set-Seq-Stack** (Figure 3) is a Read/Write wait-free set-linearizable implementation of the stack with multiplicity.

**Proof.** Since all base objects are wait-free, it follows directly from the code that the implementation is wait-free. Thus, we focus on proving that the implementation is set-linearizable. Let E be any execution of **Set-Seq-Stack**. Since the algorithm is wait-free, there is an extension of E in which all its pending operations are completed, and no new operation is started. Any set-linearization of such extension is a set-linearization of E. Thus, without loss of generality, we can assume all operations in E are completed.

2To the best of our knowledge, the best implementation is in [7] with polylogarithmic step complexity, on the number of processes, provided that the number of increments is polynomial.
The rest of the proof is a "reduction" that proceeds as follows. First, we modify $E$ and remove some of its operations to obtain another execution $G$ of the algorithm. Then, from $G$, we obtain an execution $H$ of Seq-Stack, and show that we can obtain a set-linearization $SetLin(G)$ of $G$ from any linearization $Lin(H)$ of $H$. Finally, we add to $SetLin(G)$ the operations of $E$ that were removed to obtain a set-linearization $SetLin(E)$ of $E$.

We start with the following simple remarks:

**Remark 2.** Every Push operation gets a unique pair $(top_i, i)$, hence no pushed value is lost.

**Remark 3.** If two Push operations store their values at entries in the same row of Items (both set the same value in top$_i$ in Line 02), they are concurrent.

**Remark 4.** If two Pop operations return the same value, they are concurrent.

To obtain the execution $G$ mentioned above, we first obtain intermediate executions $F$ and then $F'$, from which we derive $G$.

For any value $y \neq \epsilon$ that is returned by more than one Pop operation in $E$, we remove from $E$ all these operations (invocations, responses and steps) except for the first one that executes Line 10, i.e., the first among these operations that marks $y$ as taken in Items. Let $F$ be the resulting sequence. We claim that $F$ is an execution of the algorithm: (1) any Pop operation reads Top in Line 05 and Items$[r_i][t_i]$ in Line 08, thus no step of any other operation depends on such a step of a removed Pop operation, and (2) $F$ keeps the Pop operation that marks first $y$ as taken in Line 10, hence the subsequent Write steps of the removed Pop operations are superfluous. As we only removed some operations from $E$, the remaining operations in $F$ respect the partial order $<_E$, namely, $<_F \subseteq <_E$. Since there are no two Pop operations in $F$ popping the same item $y \neq \epsilon$, then for every Pop operation we can safely move backward each of its steps in Line 10 next to its previous step in Line 08 (which corresponds to the same iteration of the for loop in Line 07). Thus, for every Pop operation, Lines 08 to 10 correspond to a Swap operation. Let $F'$ denote the resulting equivalent execution.

We now permute the order of some steps in $F'$ to obtain an execution $G$ with the same operations and $<_G = <_{F'}$. For each integer $b \geq 0$, let $t(b) \in [0, \ldots, n]$ be the number of Pop operations in $F'$ that store their items in row Items$[b]$. Namely, each these operations obtains $b$ in its Read steps in Line 01. Let Push$_1^b, \ldots, \text{Push}_{t(b)}^b$ denote all these operations. For each Push$_j^b$, let $x_j^b$ denote the item the operation pushes, let $e_j^b$ denote its Read step in Line 01 and let ind$_j^b$ be the index of the process that performs operation Push$_j^b$. Hence, Push$_j^b$ stores its item $x_j^b$ in Items$[b][\text{ind}_j^b]$ when performs Line 03. Without loss of generality, let us suppose that ind$_1^b < \text{ind}_2^b < \ldots < \text{ind}_{t(b)}^b$. Observe the following:

- Push$_1^b, \ldots, \text{Push}_{t(b)}^b$ are concurrent.

- By linearizability of Top, all Read operations in Lines 02 and 05 read monotonically increasing values. Therefore, for $a < b$, the step $e_k^a$ of any Push$_k^a$ appears before the step $e_j^b$ of any Push$_j^b$.

- Since all $e_1^b, \ldots, e_{t(b)}^b$ read the same value from Top, there are no Top.Increment() steps in the shortest sub-string of $F'$ containing $e_1^b, \ldots, e_{t(b)}^b$.

- Let $f^b$ be the step among $e_1^b, \ldots, e_{t(b)}^b$ that appears first in $F'$. Then, the step in Line 03 of any Push$_j^b$ appears after $f^b$ in $F'$. 


The last two items imply that moving forward each $e_b$ right after $f_b$ produces another execution equivalent to $F'$. Thus, we obtain $G$ by moving forward all steps $e_1^b, \ldots, e_{t(b)}^b$ up to the position of $f^b$, and place them in that order, $e_1^b, \ldots, e_{t(b)}^b$, for every $b \geq 0$. Figure 4 shows a graphical description of the transformation. Observe that $<G = <F'$.

The main observation now is that $G$ already corresponds to an execution of Seq-Stack, if we consider the entries in Items in their usual order (first row, then column). We say that Items $[r][s]$ is touched in $G$ if there is a Push operation that writes its item in that entry; otherwise, Items $[r][s]$ is untouched. Now, for every $b \geq 0$, in $G$ all Push$_b^1, \ldots, $ Push$_b$ execute Line 01 one right after the other, in order $e_1^b, \ldots, e_{t(b)}^b$. Also, the items they push appear in row Items$[b]$ from left to right in order Push$_b^1, \ldots, $ Push$_b$. Thus, we can think of the touched entries in row Items$[b]$ as a column with the left most element at the bottom, and pile all rows of Items with Items$[0]$ at the bottom. Figure 5 depicts an example of the transformation. In this way, each $e_j^b$ corresponds to a Fetch&Inc operation and every Pop operations scans the touched entries of Items in the order Seq-Stack does (note that it does not matter if the operation start scanning in a row of Items with no touched entries, since untouched entries are immaterial). Following this idea, we do the following to obtain an execution $H$ of Seq-Stack from $G$:

- For each Push$_j$:
  1. Replace its step in Line 01 (denoted $e_j^b$ above) with $top_i \leftarrow Top.Fetch&Inc()$, corresponding to Line 01 of Seq-Stack (Figure 2).
  2. Remove its step in Line 02
3. Replace its step in Line 03 with $\text{Items}[\text{top}_i].\text{Write}(x_i)$, corresponding to Line 02 of Seq-Stack (Figure 2).

- For each Pop operation:
  1. Note that the step in Line 05 directly corresponds to Line 04 of Seq-Stack (Figure 2). Thus, this step remains unchanged.
  2. Let $e$ be any of its steps corresponding to Line 08. If $e$ reads a touched entry, then replace $e$ with $x_i \leftarrow \text{Items}[\text{top}_i].\text{Swap}(\bot)$, corresponding to Line 06 of Seq-Stack (Figure 2); otherwise, remove $e$.
  3. Remove all its steps corresponding to Line 10.

As already argued, $H$ is an execution of Seq-Stack; furthermore, $G$ and $H$ have the same operations and $<_H = <_G$. Let $\text{Lin}(H)$ be any linearization of Seq-Stack. To conclude the proof of the theorem, we obtain a set-linearization of $E$ from $\text{Lin}(H)$. We have that $H$, $G$ and $F$ have the same operations and $<_H = <_G = <_F$, and then $\text{Lin}(H)$ is indeed a set-linearization of $F$ and $G$ with each concurrency class having a single operation. To obtain a set-linearization $\text{SetLin}(E)$ of $E$, we put every Pop operation of $E$ that is removed to obtain $F$, in the concurrency class of $\text{Lin}(H)$ with the Pop operation that returns the same item. The resulting set-sequential execution, $\text{SetLin}(E)$, respects $<_E$ because any two operations returning the same item are concurrent, as observed at the beginning of the proof. Therefore, $\text{SetLin}(E)$ is a set-linearization of $E$, and hence Seq-Stack is set-linearizable.

It is worth observing that indeed it is very simple to prove that Seq-Stack is a set-linearizable implementation the set-sequential pool with multiplicity, namely, Definition 4 without LIFO order (i.e. $q$ is a set instead of a string). The hard part in the previous proof is the LIFO order, which is proved through a simulation argument.

4 Set-Sequential Queues with Multiplicity

4.1 A Non-Blocking Linearizable Queue from Consensus Number Two

We now consider the linearizable queue implementation in Figure 6, which uses objects with consensus number two. The idea of the implementation, which we call Seq-Queue, is similar to that of Seq-Stack in the previous section. Differently from Seq-Stack, whose operations are wait-free, Seq-Queue has a wait-free Enqueue and a non-blocking Dequeue.

Seq-Queue is a slight modification of the non-blocking queue implementation of Li [27], which in turn is a variation of the blocking queue implementation of Herlihy and Wing [23]. Each Enqueue operation simply reserves a slot for its item by performing Fetch&Inc to the tail of the queue, Line 01, and then stores it in Items, Line 02. A Dequeue operation repeatedly tries to obtain an item scanning Items from position 1 to the tail of the queue (from its perspective), Line 07; every time it sees an item has been stored in an entry of Items, Lines 09 and 10, it tries to obtain the item by atomically replacing it with $\bot$, which signals that the item stored in that entry has been taken, Line 11. While scanning, the operation records the number of items that has been taken (from its perspective), Line 13, and if this number is equal to the number of items that were taken in the previous scan, it declares the queue is empty, Line 16. For completeness, the correctness
proof of \textbf{Seq-Queue} is in Appendix A. Despite its simplicity, \textbf{Seq-Queue}'s linearizability proof is far from trivial.

\begin{center}
\begin{figure}[h]
\begin{mdframed}
\textbf{Shared Variables:}
- \textit{Tail}: Fetch\&Inc base object initialized to 1
- \textit{Items}[1,\ldots] : array of Swap base objects initialized to ⊥

\textbf{Operation Enqueue}(x_i) is
\begin{enumerate}
  \item \textit{tail}_i \leftarrow \textit{Tail}.Fetch\&Inc()
  \item \textit{Items}[\textit{tail}_i].Write(x_i)
  \item return true
\end{enumerate}
\textbf{Operation Dequeue}()
\begin{enumerate}
  \item \textit{taken}'_i \leftarrow 0
  \item while true do
  \begin{enumerate}
    \item \textit{taken}_i \leftarrow 0
    \item \textit{tail}_i \leftarrow \textit{Tail}.Read() - 1
    \item for \textit{r}_i \leftarrow 1 \text{ up to } \textit{tail}_i do
      \begin{enumerate}
        \item \textit{x}_i \leftarrow \textit{Items}[\textit{r}_i].Read()
        \item if \textit{x}_i \neq ⊥ then
          \begin{enumerate}
            \item \textit{x}_i \leftarrow \textit{Items}[\textit{r}_i].Swap(⊤)
            \item if \textit{x}_i \neq ⊤ then return \textit{x}_i end if
            \item \textit{taken}_i \leftarrow \textit{taken}_i + 1
          \end{enumerate}
        end if
      \end{enumerate}
    end for
  \end{enumerate}
  \item if \textit{taken}_i = \textit{taken}'_i then return \epsilon
  \item \textit{taken}'_i \leftarrow \textit{taken}_i
  \item end while
\end{enumerate}
\end{mdframed}
\end{figure}
\end{center}

Figure 6: Non-blocking linearizable queue \textbf{Seq-Queue} from base objects with consensus number 2 (code for \textit{p}_i).

Similarly to the case of the stack, \textbf{Seq-Queue} is optimal from the perspective of the the consensus hierarchy as there is no non-blocking linearizable queue implementation from \textbf{Read}/\textbf{Write} operations only. However, as we will show below, we can obtain a \textbf{Read}/\textbf{Write} non-blocking implementation of a set-sequential queue with multiplicity.

\subsection{A Set-linearizable Read/Write Queue with Multiplicity}

Our relaxed queue follows a similar idea of that of the set-sequential stack in Definition 4: concurrent \textbf{Dequeue} operations might obtain the same item, but all items are returned in FIFO order, and no enqueued item is lost.

\textbf{Definition 5 (Set-Sequential Queue with Multiplicity).} The universe of items that can be enqueued is \( \mathbb{N} = \{1, 2, \ldots\} \), and the set of states \( \mathcal{Q} \) is the infinite set of strings \( \mathbb{N}^* \). The initial state is the empty string, denoted \( \epsilon \). In state \( q \), the first element in \( q \) represents the head of the queue, which might be empty if \( q \) is the empty string. The transitions are the following:

1. For \( q \in \mathcal{Q} \), \( \delta(q, \text{Enqueue}(x)) = (q \cdot x, (\text{Enqueue}(x) : \text{true})) \).

2. For \( q \in \mathcal{Q}, 1 \leq t \leq n, x \in \mathbb{N} : \delta(x \cdot q, \{\text{Dequeue}_1(), \ldots, \text{Dequeue}_t()\}) = (q, \{\text{Dequeue}_1() : x\}, \ldots, \{\text{Dequeue}_t() : x\}) \).

3. \( \delta(\epsilon, \text{Dequeue}()) = (\epsilon, \{\text{Dequeue}() : \epsilon\}) \).
Remark 5. Every execution of the set-sequential queue with all its concurrency classes containing a single operation, is an execution of the sequential queue.

The proof of the following lemma is similar to the proof of Lemma 1.

Lemma 2. Let $A$ be any set-linearizable implementation of the set-sequential queue with multiplicity. Then,

1. All sequential executions of $A$ are executions of the sequential queue.

2. All executions with no concurrent Dequeue operations are linearizable with respect to the sequential queue.

3. All executions with Dequeue operations returning distinct values are linearizable with respect to the sequential queue.

4. If two Dequeue operations return the same value in an execution, then they are concurrent.

The algorithm in Figure 7 is a set-linearizable Read/Write non-blocking implementation of a queue with multiplicity, which we call Set-Seq-Queue. As for the case of the stack before, we obtain Set-Seq-Queue from Seq-Queue by: (1) replacing the Fetch&Inc object in Seq-Queue with a Read/Write wait-free Counter, (2) extending Items to a matrix to handle collisions, and (3) simulating the Swap operation with a Read followed by a Write. The correctness proof of Set-Seq-Queue is similar to the correctness proof of Set-Seq-Stack in Theorem 1.

Shared Variables:
Tail : Read/Write wait-free linearizable Counter base object initialized to 1
Items[1,...][1,...,n] : array of Read/Write registers initialized to ⊥

Operation Enqueue($x$) is
(01) $\text{tail}_i \leftarrow \text{Tail.Read}()$
(02) $\text{Tail.Increment}()$
(03) Items[$\text{tail}_i,i$].Write($x$)
(04) return true
end Enqueue

Operation Dequeue() is
(05) $\text{taken}_i' \leftarrow 0$
(06) while true do
(07) $\text{taken}_i \leftarrow 0$
(08) $\text{tail}_i \leftarrow \text{Tail.Read}() - 1$
(09) for $r_i \leftarrow 1$ up to $\text{tail}_i$ do
(10) for $s_i \leftarrow 1$ up to $\text{n}$ do
(11) $x_i \leftarrow \text{Items}[r_i][s_i].\text{Read}()$
(12) if $x_i \neq \bot$ then
(13) Items[$r_i][s_i$].Write($T$)
(14) if $x_i \neq T$ then return $x_i$ end if
(15) $\text{taken}_i \leftarrow \text{taken}_i + 1$
(16) end if
(17) end for
(18) end for
(19) if $\text{taken}_i = \text{taken}_i'$ then return $\epsilon$ end if
(20) $\text{taken}_i' \leftarrow \text{taken}_i$
(21) end while
end Dequeue

Figure 7: Read/Write non-blocking set-sequential queue Set-Seq-Queue with multiplicity (code for $p_i$).
**Theorem 2.** The algorithm Set-Seq-Queue (Figure 7) is a Read/Write non-blocking set-linearizable implementation of the queue with multiplicity.

**Proof.** First, observe that the Enqueue method is wait-free. To prove that Dequeue is lock-free, it is enough to observe that the only way a Dequeue operation never terminates is because it sets larger values in taken at the end of each iteration of the while loop, which can only happen if there are new Enqueue operations and Dequeue operations, implying that infinitely many operations are completed.

The set-linearizability proof of the implementation is nearly the same as in the proof of Theorem 1. Given any execution $E$ without pending operations, we obtain in the same way $F'$ and then $F''$ to obtain an “equivalent” execution $G$ of Set-Seq-Queue. Again, $G$ naturally corresponds to an execution of $H$ of Seq-Stack, and hence we consider any linearization Lin$(H)$ of $H$, which is a set-linearization of $G$ in which all concurrency classes have a single element. Finally, from Lin$(H)$ we obtain a set-linearization SetLin$(E)$ of $E$ by adding the Dequeue operations that were removed from $E$. 

In fact, proving Set-Seq-Queue implements the set-sequential pool with multiplicity is simple, the difficulty comes from the FIFO order requirement of the queue, which is shown through a simulation argument.

## 5 Implications

### 5.1 Avoiding Costly Synchronization Operations/Patterns

It is worth observing that Set-Seq-Stack and Set-Seq-Queue allow us to circumvent the linearization-related impossibility results in [11], where it is shown that every linearizable implementation of a queue or a stack, as well as other concurrent operation executions as encountered for example in work-stealing, must use either expensive Read-Modify-Write operations (e.g. Fetch&Inc and Compare&Swap) or Read-After-Write patterns [11] (i.e. a process writing in a shared variable and then reading another shared variable, maybe performing operation on other variables in between). It is well known that such synchronization mechanisms are slow in real multicore architectures.

In the simplest Read/Write Counter implementation we are aware of, the object is represented via a shared array $M$ with an entry per process; process $p_i$ performs Increment by incrementing its entry, $M[i]$, and Read by reading, one by one, the entries of $M$ and returning the sum. Using this simple Counter implementation, we obtain from Set-Seq-Stack a set-sequential stack implementation with multiplicity, devoided of (1) Read-Modify-Write operations, as only Read/Write operations are used, and (2) Read-After-Write patterns, as in both operations, Push and Pop, a process first reads and then writes. It similarly happens with Set-Seq-Queue.

### 5.2 Work-stealing with multiplicity

Our implementations also provide relaxed work-stealing solutions without expensive synchronization operation/patterns. Work-stealing is a popular technique to implement load balancing in a distributed manner, in which each process maintains its own pool of tasks and occasionally steals tasks from the pool of another process. In more detail, a process can Put and Take tasks in its own pool and Steal tasks from another pool. To improve performance, it is introduced in [30] the notion of idempotent work-stealing which allows a task to be extracted at least once instead of exactly
once as in previous work. Using this relaxed notion, three different solutions are presented in that paper where the Put and Take operations avoid Read-Modify-Write operations and Read-After-Write patterns; however, the Steal operation still uses costly Compare&Swap operations.

Our set-sequential queue and stack implementations provide idempotent work-stealing solutions in which no operation uses Read-Modify-Write operations and Read-After-Write patterns. Moreover, in our solutions both Take and Steal are implemented by Pop (or Dequeue), hence any process can invoke those operations, allowing more concurrency. If we insist that Take and Steal can be invoked only by the owner, Items can be a 1-dimensional array. Additionally, differently from [30], whose approach is practical, our queues and stacks with multiplicity are formally defined, with a clear and simple semantics.

5.3 Out-of-order queues and stacks with multiplicity

The notion of a $k$-FIFO queue is introduced in [26], in which items can be dequeued out of FIFO order up to an integer $k \geq 0$. More precisely, dequeuing the oldest item may require up to $k + 1$ dequeue operations, which may return elements not younger than the $k + 1$ oldest elements in the queue, or nothing even if the queue is not empty. It is also presented in [26] a simple way to implement a $k$-FIFO queue, through $p$ independent FIFO queue linearizable implementations. When a process want to perform an operation, it first uses a load balancer to pick one of the $p$ queues and then performs its operation. The value of $k$ depends on $p$ and the load balancer. Examples of load balancers are round-robin load balancing, which requires the use of Read-Modify-Write operations, and randomized load balancing, which does not require coordination but can be computational locally expensive. As explained in [26], the notion of a $k$-FIFO stack can be defined and implemented similarly.

We can relax the $k$-FIFO queues and stacks to include multiplicity, namely, an item can be taken by several concurrent operations. Using $p$ instances of our set-sequential stack or queue Read/Write implementations, we can easily obtain set-sequential implementations of $k$-FIFO queues and stacks with multiplicity, where the use of Read-Modify-Write operations or Read-After-Write patterns are in the load balancer.

6 Interval-Sequential Queues with Weak-Emptiness Check

A natural question is if in Section 4 we could start with a wait-free linearizable queue implementation instead of Seq-Queue, which is only non-blocking, and hence derive a wait-free set-linearizable queue implementation with multiplicity. It turns out that it is an open question if there is a wait-free linearizable queue implementation from objects with consensus number two. (Concretely, such an algorithm would show that the queue belongs to the Common2 family of operations [4].) This question has been open for more than two decades [4] and there has been several papers proposing wait-free implementations of restricted queues [13, 16, 27, 28, 29], e.g., limiting the number of processes that can perform a type of operations.

6.1 The Tail-Chasing Problem

One of the main difficulties to solve when trying to design such an implementations using objects with consensus number two is known as the tail-chasing problem. This problem can be easily
Shared Variables:

tail : Fetch&Inc base object initialized to 1
items[1,...] : array of Swap base objects initialized to ⊥

Operation Enqueue(x) is
(01) tail_i ← Tail.Fetch&Inc()
(02) items[tail_i].Write(x)
(03) return true
end Enqueue

Operation Dequeue() is
(04) tail_i ← Tail.Read() - 1
(05) for r_i ← 1 up to tail_i do
(06) x_i ← items[r_i].Swap(⊥)
(07) if x_i ≠ ⊥ then return x_i end if
(08) end for
(09) return ⊥
end Dequeue

Figure 8: A non-linearizable queue implementation (code for process p_i).

Exemplified with the help of the non-linearizable queue implementation in Figure 8. The implementation is similar to Seq-Stack with the difference that Dequeue operations scan Items in the opposite order, i.e. from the head to the tail.

The problem with this implementation is that once a Dequeue has scanned unsuccessfully Item (i.e., the items that were in the queue were taken by “faster” operations), it returns ⊥; however, while the operation was scanning, more items could have been enqueued, and indeed it is not safe to return ⊥ as the queue might not be empty. Figure 9 describes an execution of the implementation that cannot be linearized because there is no moment in time during the execution of the Dequeue operation returning ⊥ in which the queue is empty. Certainly, this problem can be solved as in Seq-Queue: read the tail and scan again; thus, in order to complete, a Dequeue operation is forced to chase the current position of the tail until it is sure there are no new items.

Figure 9: An example of the tail-chasing problem.

Inspired by this problem, below we introduce a relaxed interval-sequential queue that allows a Dequeue operation to return a weak-empty value, with the meaning that the operation was not able take any of the items that were in the queue when it started but it was concurrent with all the Dequeue operation that took those items, i.e., it has a sort of certificate that the items were taken, and the queue might be empty. Then, we show that such a relaxed queue can be wait-free implemented from objects with consensus number two.
6.2 A Wait-Free Interval-Sequential Queue with Weak-Emptiness

Roughly speaking, in our relaxed interval-sequential queue, the state is a tuple \((q, P)\), where \(q\) denotes the state of the queue and \(P\) denotes the pending \texttt{Dequeue} operations that eventually return \texttt{weak-empty}, denoted \(\epsilon^w\). More precisely, \(P[i] \neq \perp\) means that process \(p_i\) has a pending \texttt{Dequeue} operation. \(P[i]\) is a prefix of \(q\) and represents the remaining items that have to be dequeued so that the current \texttt{Dequeue} operation of \(p_i\) can return \(\epsilon^w\). As \texttt{Dequeue} operations take items from the queue, these items are removed from \(P[i]\), and the operation can return \(\epsilon^w\) only if \(P[i]\) is \(\epsilon\). Intuitively, the semantics of \(\epsilon^w\) is that the queue could be empty as all items that were in the queue when the operations started, have been taken. So this \texttt{Dequeue} operation virtually occurs after all the items have been dequeued.

\[
\text{Dequeue()} : \epsilon^w
\]

\begin{figure}[h]
\centering
\begin{tikzpicture}
  \node (q) at (0,0) {Enqueue(x)}; \node (r) at (2,0) {Enqueue(y)};
  \node (s) at (4,0) {Enqueue(z)}; \node (t) at (6,0) {Dequeue():x}; \node (u) at (8,0) {Dequeue():y}; \node (v) at (10,0) {Dequeue():\epsilon w};
  \draw[->] (q) -- (r); \draw[->] (r) -- (s); \draw[->] (s) -- (t); \draw[->] (t) -- (u); \draw[->] (u) -- (v);
\end{tikzpicture}
\caption{An interval-sequential execution with a \texttt{Dequeue} operations returning weak-empty.}
\end{figure}

Figure 10 shows an example of an interval-sequential execution of our relaxed queue where the \texttt{Dequeue} operation returning \(\epsilon^w\) is allowed to return only when \(x\) and \(y\) have been dequeued, as the queue contains those values when the operations starts. Observe this execution is an interval-linearization of the execution obtained from Figure 9 by replacing \(\epsilon\) with \(\epsilon^w\).

\textbf{Definition 6} (Interval-Sequential Queue with Weak-Empty). The universe of items that can be enqueued is \(N = \{1, 2, \ldots\}\) and the set of states is \(Q = N^* \times (N^* \cup \{\perp\})^n\), with the initial state being \((\epsilon, \perp, \ldots, \perp)\). Below, a subscript denotes the ID of the process invoking an operation. The transitions are the following:

1. For \((q, P) \in Q, 0 \leq t, \ell \leq n - 1, \delta(q, P, \text{Enqueue}(x), \text{Dequeue}_{i(1)}(), \ldots, \text{Dequeue}_{i(\ell)}())\) contains the transition \((q \cdot x, S, (\text{Enqueue}(x) : \text{true}), (\text{Dequeue}_{j(1)}() : \epsilon^w), \ldots, (\text{Dequeue}_{j(\ell)}() : \epsilon^w))\), satisfying that
   \begin{enumerate}
   \item \(i(k) \neq i(k')\), \(i(k) \neq \text{id of the process invoking Enqueue}(x)\), and \(j(k) \neq j(k')\),
   \item for each \(i(k)\), \(P[i(k)] = \perp\),
   \item for each \(j(k)\), either \(P[j(k)] = \epsilon\), or \(P[j(k)] = \perp\) and \(q = \epsilon\) and \(j(k) = i(k')\) for some \(k'\),
   \item for each \(1 \leq s \leq n\), if there is a \(k\) with \(s = j(k)\), then \(S[s] = \perp\); otherwise, if there is \(k'\) with \(s = i(k')\), \(S[s] = q\), else \(S[s] = P[s]\).
   \end{enumerate}

2. For \((x \cdot q, P) \in Q, 0 \leq t, \ell \leq n - 1, \delta(x \cdot q, P, \text{Dequeue}(), \text{Dequeue}_{i(1)}(), \ldots, \text{Dequeue}_{i(\ell)}())\) contains the transition \((q, S, (\text{Dequeue}() : x), (\text{Dequeue}_{j(1)}() : \epsilon^w), \ldots, (\text{Dequeue}_{j(\ell)}() : \epsilon^w))\), satisfying that
   \begin{enumerate}
   \item \(i(k) \neq i(k')\), \(i(k) \neq \text{id of the process invoking Dequeue}()\), and \(j(k) \neq j(k')\),
   \item for each \(i(k)\), \(P[i(k)] = \perp\),
   \item for each \(j(k)\), either \(P[j(k)] = x\), or \(P[j(k)] = \perp\) and \(q = \epsilon\) and \(j(k) = i(k')\) for some \(k'\),
   \end{enumerate}
(d) for each $1 \leq s \leq n$, if there is a $k$ with $s = j(k)$, then $S[s] = \perp$; otherwise, if there is $k'$ with $s = i(k')$, $S[s] = q$, else $S[s]$ is the string obtained by removing the first symbol of $P[s]$ (which must be $x$).

(e) if $x \cdot q = \epsilon$ and $t, \ell = 0$, then $x \in \{\epsilon, \epsilon^w\}$.

Remark 6. Every execution of the interval-sequential queue with no dequeue operation returning $\epsilon^w$ is an execution of the sequential queue.

The proof of the following lemma is similar to the proof of Lemmas 1 and 2.

Lemma 3. Let $A$ be any interval-linearizable implementation of the interval-sequential queue with weak-empty. Then,

1. All sequential executions of $A$ are executions of the sequential queue.

2. All executions in which no Dequeue operation is concurrent with any other operation are linearizable with respect to the sequential queue.

The algorithm in Figure 11, which we call Int-Seq-Queue, is an interval-linearizable wait-free implementation of a queue with weak-emptiness, which uses base objects with consensus number two. Int-Seq-Queue is a simple modification Seq-Queue in which an Enqueue operation proceed as in Seq-Queue, while a Dequeue operation scans at most two times Items to obtain an item, in both cases recording the number of taken items. If the two numbers are the same (cf. double clean scan), then the operations returns $\epsilon$, otherwise it returns $\epsilon^w$.

Shared Variables:
- $\text{Tail} : \text{Fetch\&Inc}$ base object initialized to 1
- $\text{Items}[1, \ldots] : \text{array of Swap}$ base objects initialized to ⊤

Operation Enqueue\$(x_i)\$

(01) $\text{tail} \leftarrow \text{Tail}.\text{Fetch\&Inc}()$
(02) $\text{Items}[\text{tail}].\text{Write}(x_i)$
(03) return true
end Enqueue

Operation Dequeue\$(\cdot)\$

(04) for $k \leftarrow 1$ up to 2 do
(05) $\text{taken}_i[k] \leftarrow 0$
(06) $\text{tail} \leftarrow \text{Tail}.\text{Read()} - 1$
(07) for $r_i \leftarrow 1$ up to $\text{tail}$, do
(08) $x_i \leftarrow \text{Items}[r_i].\text{Read}()$
(09) if $x_i \neq \perp$ then
(10) $x_i \leftarrow \text{Items}[r_i].\text{Swap}(\top)$
(11) if $x_i \neq \top$ then return $x_i$ end if
(12) $\text{taken}_i[k] \leftarrow \text{taken}_i[k] + 1$
(13) end if
(14) end for
(15) end for
(16) if $\text{taken}_i[1] = \text{taken}_i[2]$ then return $\epsilon$
(17) else return $\epsilon^w$
(18) end if
end Dequeue

Figure 11: Wait-free interval-sequential queue from consensus number 2 (code for $p_i$).
**Theorem 3.** The algorithm Int-Seq-Queue (Figure 11) is a wait-free interval-linearizable implementation of the queue with weak-empty, using objects with consensus number two.

**Proof.** It is clear that Int-Seq-Queue is wait-free as all its base objects are wait-free, thus we focus on showing it is interval-linearizable. Let $E$ be any execution of Int-Seq-Queue. Since the algorithm is wait-free, there is an extension of $E$ in which all its pending operations are completed, and no new operation is started. Observe that any interval-linearization of such an extension is an interval-linearization of $E$. Thus, without loss of generality, we can assume all operations in $E$ are completed.

Consider the sequence $F$ obtained by removing from $E$ every Dequeue operation (invocation, response and steps) that returns $e^w$. Observe that none of these operations changes the state of Tail and Items (as they return $e^w$), hence $F$ is indeed an execution of Int-Seq-Queue. Furthermore, $F$ is an execution of Seq-Queue: Enqueue operations behave the same in Seq-Queue and Int-Seq-Queue, and every Dequeue operation scans Items at most twice and either returns an item or finds the queue empty and hence returns $e$. Since Seq-Queue is linearizable, consider a linearization Lin($F$) of it.

We will obtain an interval-linearization IntLin($E$) of $E$ from Lin($F$) by adding to it the Dequeue operations of $E$ that return $e^w$. The idea is the following. When any such operation, say DEQ$^\alpha$, starts, the queue is in a state $q$. Since DEQ$^\alpha$ returns $e^w$, it is the case that DEQ$^\alpha$ runs concurrently to Dequeue operations that take the items in $q$, and other Dequeue operations that take the new items that are concurrently enqueued while DEQ$^\alpha$ runs. Roughly speaking, if $q \neq e$, DEQ$^\alpha$ will be interval-linearized with all the Dequeue operations that take the items in $q$; otherwise, DEQ$^\alpha$ will be interval-linearized alone, since the queue is empty in this case.

To obtain IntLin($E$), below we suppose, without loss of generality, that Lin($F$) has the following property, which, roughly speaking, say that Enqueue operations are linearized in Lin($F$) “as late as possible”.

**Assumption 1.** If Lin($F$) has two consecutive operations $(\text{Enqueue}(x) : \text{true})$ and $(\text{Dequeue}() : y)$ that are concurrent in $F$ with $x \neq y$, then the order of the operations can be exchanged to obtain another sequential execution of the queue which is a linearization of $F$ too. Thus, we assume that Lin($F$) does not have such a pair of operations.

Let DEQ$^\alpha$ be any Dequeue operation of $E$ that returns $e^w$. Let $\alpha$ be the shortest prefix of Lin($F$) with every operation $\text{op}$ of $F$ with $\text{op} <_E \text{DEQ}^\alpha$; in words, $\alpha$ is the first moment in which all operations that happen before DEQ$^\alpha$ in $E$ are linearized in Lin($F$).

In what follows, let OP$^{\text{last}}$ denote the operation at the end of $\alpha$ (if there is one). First, we observe that OP$^{\text{last}} <_E \text{DEQ}^\alpha$: if not, the prefix of Lin($F$) obtained by removing $\text{op}$ from $\alpha$ has all operations in $F$ that happen before DEQ$^\alpha$ in $E$, which contradicts minimality of the length of $\alpha$. Then:

**Claim 1.** If $\alpha \neq e$, OP$^{\text{last}} <_E \text{DEQ}^\alpha$.

We also note that for every $\text{op} \neq \text{OP}^{\text{last}}$ in $\alpha$ (if there is one), either $\text{op} <_E \text{DEQ}^\alpha$ or $\text{op}||_E \text{DEQ}^\alpha$: if not, then DEQ$^\alpha <_E \text{op}$ and then OP$^{\text{last}} <_E \text{op}$, which leads to a contradiction as OP$^{\text{last}}$ appears before OP$^{\text{last}}$ in Lin($F$) and thus the partial order $<_E$ is not respected in Lin($F$), implying that Lin($F$) is not a linearizability of $F$. Thus, we have:

**Claim 2.** If $\alpha \neq e$, for every $\text{op} \neq \text{OP}^{\text{last}}$ in $\alpha$, either $\text{op} <_E \text{DEQ}^\alpha$ or $\text{op}||_E \text{DEQ}^\alpha$. 

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Let $q$ be the state of the queue at the end of $\alpha$. We split the rest of proof in the following two cases:

**Case** $q = \epsilon$. This case is simple. Recall that if the queue is empty, a $\text{Dequeue}$ operation can be linearized at a single point and non-deterministically return either $\epsilon$ or $\epsilon^w$ (case 2.e in the definition of the interval-sequential queue with weak-empty). Thus, $\text{DEQ}^w$ is linearized in $\text{IntLin}(E)$ alone between the last operation in $\alpha$ and the next one in $\text{Lin}(F)$. Since $q = \epsilon$ at the end of $\alpha$, $\text{DEQ}^w$ returning $\epsilon$ follows the specification of the interval-sequential queue with weak-empty. Moreover, the linearization of $\text{DEQ}^w$ in $\text{IntLin}(E)$ respects the partial order $<_E$, by Claim 2 and since $\alpha$ contains every operations that happens before $F$. If there is another $\text{DEQ}^w$ having the same prefix $\alpha$, then $\text{DEQ}^w$ and $\text{DEQ}^w$ are linearized respecting $<_E$, namely, if they are concurrent, their order does not matter, otherwise, we follows the order imposed by $<_E$.

**Case** $q = x_1x_2 \cdots x_t \neq \epsilon$. This is the hard case. Since $q \neq \epsilon$, $\alpha$ must contain the $\text{Enqueue}$ operations enqueuing the items in $q$, and thus $\alpha \neq \epsilon$.

**Claim 3.** For every item $x_i$ in $q$, there is a $\text{Dequeue}$ operation in $F$ that dequeues $x_i$.

We observe first that it is enough to prove that there is a $\text{Dequeue}$ operation that dequeues $x_i$.

As a corollary, we get that the same happens for every item $x_i$ in $q$, namely, there is a $\text{Dequeue}$ operation in $F$ that dequeues $x_i$: $\text{Lin}(F)$ is a sequential execution of the queue and thus $x_t$ can be dequeued only if $x_i$ (which is enqueued before $x_t$ as it appears before $x_t$ in $q$) is dequeued first.

Consider the operation $\langle \text{Enqueue}(x_i) : \text{true} \rangle$ in $\alpha$. We identify the following subcases:

- **Op$_{\text{last}} = \langle \text{Enqueue}(x_i) : \text{true} \rangle$ or $\langle \text{Enqueue}(x_i) : \text{true} \rangle <_E \text{DEQ}^w$.** In any case, we have $\langle \text{Enqueue}(x_i) : \text{true} \rangle <_E \text{DEQ}^w$ and thus $x_t$ is stored in some entry in $\text{Items}$ before $\text{DEQ}^w$ starts scanning $\text{Items}$, and thus there must exists a $\text{Dequeue}$ operations that takes $x_t$ before $\text{DEQ}^w$ reads the entry where $x_t$ is written.

- **Op$_{\text{last}} \neq \langle \text{Enqueue}(x_i) : \text{true} \rangle$ and $\text{DEQ}^w <_E \langle \text{Enqueue}(x_i) : \text{true} \rangle$.** We already saw that $\text{Op}_{\text{last}} <_E \text{DEQ}^w$, from which follows that $\text{Op}_{\text{last}} <_E \langle \text{Enqueue}(x_i) : \text{true} \rangle$. Since $\langle \text{Enqueue}(x_i) : \text{true} \rangle$ appears before $\text{Op}_{\text{last}}$ in $\text{Lin}(F)$, we have that the partial order $<_E$ is not respected in $\text{Lin}(F)$, which is a contradiction as $\text{Lin}(F)$ is a linearizability of $F$.

- **Op$_{\text{last}} \neq \langle \text{Enqueue}(x_i) : \text{true} \rangle$ and $\langle \text{Enqueue}(x_i) : \text{true} \rangle | E \text{DEQ}^w$ (see Figure 12).** Let $\text{Items}[\text{pos}]$ be the entry where $x_t$ is written by $\langle \text{Enqueue}(x_i) : \text{true} \rangle$. By contradiction, suppose that there is no $\text{Dequeue}$ operation in $F$ that dequeues $x_t$. Observe that the only way this can happen is that $\langle \text{Enqueue}(x_i) : \text{true} \rangle$ writes $x_t$ (Line 02) after $\text{DEQ}^w$ has scanned two times that entry (i.e. it executes Line 08 with $\text{Items}[\text{pos}]$ in the two iterations of the for loop); Figure 12 schematizes this (and the rest of the argument for this case).

Observe the following:

- all operations of $\alpha$ after $\langle \text{Enqueue}(x_i) : \text{true} \rangle$ are $\text{Dequeue}$ operations (if not, then $x_t$ would not be the last item in $q$), and

- each of these $\text{Dequeue}$ operations dequeues an item that is not in $q$ (if not, that item would not appear in $q$).
Thus, we have \( \text{Op}_\text{last} = \langle \text{Dequeue}() : z \rangle \). Let \( \langle \text{Dequeue}() : y \rangle \) be the operation of \( \alpha \) right after \( \langle \text{Enqueue}(x_t) : \text{true} \rangle \). We argue that \( \text{Op}_\text{last} = \langle \text{Dequeue}() : z \rangle <_E \langle \text{Dequeue}() : y \rangle \), which leads to a contradiction as \( \langle \text{Dequeue}() : z \rangle \) appears before \( \langle \text{Dequeue}() : y \rangle \) in \( \text{Lin}(F) \) and thus the partial order \( <_E \) is not respected in \( \text{Lin}(F) \), implying that \( \text{Lin}(F) \) is not a linearizability of \( F \).

We already saw that \( \langle \text{Dequeue}() : z \rangle <_E \text{DEQ}^w \), and thus the response of \( \langle \text{Dequeue}() : z \rangle \) in \( F \) appears before the first read of \( \text{DEQ}^w \) to entry \( \text{Items}[\text{pos}] \) (where \( x_t \) is written). We also observe that \( \langle \text{Enqueue}(x_t) : \text{true} \rangle \) and \( \langle \text{Dequeue}() : y \rangle \) are not concurrent in \( F \), by Assumption 1; thus it must be that \( \langle \text{Enqueue}(x_t) : \text{true} \rangle <_E \langle \text{Dequeue}() : y \rangle \), from which follows that the invocation of \( \langle \text{Dequeue}() : y \rangle \) appears in \( F \) after the second read of \( \text{DEQ}^w \) to entry \( \text{Items}[\text{pos}] \). Therefore, \( \text{Op}_\text{last} = \langle \text{Dequeue}() : z \rangle <_E \langle \text{Dequeue}() : y \rangle \) (see Figure 12), which concludes the argument for this subcase, and completes the proof of Claim 3.

We now define the interval-linearization of \( \text{DEQ}^w \) in \( \text{IntLin}(E) \) (see Figure 13). Let \( \beta \) be the sequence of \( \text{Lin}(F) \) containing all its operations after the operation right after \( \text{Op}_\text{last} \) to \( \langle \text{Dequeue}() : x_t \rangle \) (whose existence is shown above). The interval-linearization of \( \text{DEQ}^w \) spans the interval of \( \text{Lin}(F) \) defined by \( \beta \); formally, the invocation of \( \text{DEQ}^w \) is added to the first concurrency class of \( \beta \) and the response of it to the last concurrency class of \( \beta \). By construction, at the beginning of the interval-linearization of \( \text{DEQ}^w \), the state of the queue is \( q \) and at the end of it, all items in \( q \) are dequeued, which follows the specification of the interval-sequential queue with weak-empty.

To conclude this case, we show that for every operation \( \text{op} \) of \( \beta \), \( \text{op} ||_E \text{DEQ}^w \) in Claim 4, which together with Claim 2 implies that the interval-linearization of \( \text{DEQ}^w \) respects the partial order \( <_E \).

![Figure 13: Interval-linearization of DEQ^w when q ≠ ε.](image-url)

**Claim 4.** For every operation \( \text{op} \) of \( \beta \), \( \text{op} ||_E \text{DEQ}^w \).

There are two subcases to be proven:

- It is not the case that \( \text{op} <_E \text{DEQ}^w \). If so, then \( \alpha \) is not the shortest prefix of \( \text{Lin}(F) \) containing all operations that happen before \( \text{DEQ}^w \) in \( F \), which is a contradiction.
It is not the case that $\text{DEQ}^w <_E \text{op}$. For contradiction, suppose that $\text{DEQ}^w <_E \text{op}$. Let $\text{Item}[^{\text{pos}}]$ the entry where $x_1$ is written by $\langle \text{Enqueue}(x_1) : \text{true} \rangle$, and consider the operation $\langle \text{Dequeue}() : x_1 \rangle$.

- First, suppose that $\langle \text{Enqueue}(x_1) : \text{true} \rangle$ writes $x_1$ in $\text{Items}[^{\text{pos}}]$ (hence, $\text{Items}[^{\text{pos}}]$ is $\text{Items}[^{\text{tail}}]$ if the invoking process is $p_i$, Line 02 before $\text{DEQ}^w$ performs its second read of $\text{Items}[^{\text{pos}}]$ (Line 07, corresponding to the second iteration of the for loop). Note that $\text{DEQ}^w$ might or might not obtain $x_1$ from that read operation, however, certainly it must happen that $\langle \text{Dequeue}() : x_1 \rangle$ takes $x_1$ (i.e. it successfully obtains $x_1$ from $\text{Items}[^{\text{pos}}]$ in Line 09) before $\text{DEQ}^w$ completes. Thus, in $F$ the response of $\langle \text{Dequeue}() : x_1 \rangle$ appears before the response of $\text{DEQ}^w$. Observe that either $\langle \text{Dequeue}() : x_1 \rangle <_E \text{DEQ}^w$ or $\langle \text{Dequeue}() : x_1 \rangle ||_E \text{DEQ}^w$. Thus $\text{op} \neq \langle \text{Dequeue}() : x_1 \rangle$ as $\text{DEQ}^w <_E \text{op}$. Moreover, we have that $\langle \text{Dequeue}() : x_1 \rangle <_E \text{op}$ This is a contradiction as $\text{op}$ appears before $\langle \text{Dequeue}() : x_1 \rangle$, which contradicts that $\text{Lin}(F)$ respects $<_E$.

- Otherwise, note that either $\langle \text{Enqueue}(x_1) : \text{true} \rangle ||_E \text{DEQ}^w$ or $\text{DEQ}^w <_E \langle \text{Enqueue}(x_1) : \text{true} \rangle$. Thus, $\text{OP}^{\text{last}} 
eq \langle \text{Enqueue}(x_1) : \text{true} \rangle$ because $\text{OP}^{\text{last}} <_E \text{DEQ}^w$, by Claim $\text{[1]}$. We first observe that it cannot happen $\text{DEQ}^w <_E \langle \text{Enqueue}(x_1) : \text{true} \rangle$, because if so, $\text{OP}^{\text{last}} <_E \langle \text{Enqueue}(x_1) : \text{true} \rangle$ but in $\text{Lin}(F)$ $\langle \text{Enqueue}(x_1) : \text{true} \rangle$ appears first and then $\text{OP}^{\text{last}}$, which contradicts that $\text{Lin}(F)$ is a linearization of $F$. Thus, we have $\langle \text{Enqueue}(x_1) : \text{true} \rangle ||_E \text{DEQ}^w$. As observed above (third subcase in the proof of Claim $\text{[3]}$), all operations of $\alpha$ after $\langle \text{Enqueue}(x_1) : \text{true} \rangle$ are $\text{Dequeue}$ operations, and each of these operations dequeues an item that is not in $q$. Let $\text{OP}^{\text{last}} = \langle \text{Dequeue}() : z \rangle$ and $\langle \text{Dequeue}() : y \rangle$ be the operation in $\alpha$ right after $\langle \text{Enqueue}(x_1) : \text{true} \rangle$. By Assumption $\text{[4]}$ we must have $\langle \text{Enqueue}(x_1) : \text{true} \rangle <_E \langle \text{Dequeue}() : y \rangle$. We also have $\langle \text{Dequeue}() : z \rangle <_E \text{DEQ}^w$, by Claim $\text{[4]}$. Therefore, we obtain that $\langle \text{Dequeue}() : z \rangle <_E \langle \text{Dequeue}() : y \rangle$, which leads to a contradiction because $\langle \text{Dequeue}() : y \rangle$ appears before $\langle \text{Dequeue}() : z \rangle$ in $\text{Lin}(F)$, implying that $\text{Lin}(F)$ is not a linearization of $F$.

The second subcase follows and hence Claim $\text{[4]}$ follows too.

To obtain $\text{IntLin}(E)$, we repeat the construction above for any such operation $\text{DEQ}^w$ of $E$. To conclude the proof of the Theorem, we prove that $\text{IntLin}(E)$ respects the partial order $<_E$. Since the order of operations in $\text{Lin}(F)$ is not modified to built $\text{IntLin}(E)$, we only need to check that any two $\text{Dequeue}$ operations returning $e^w$ respect the partial order $<_E$. Namely, given any two $\text{DEQ}^w_1 <_E \text{DEQ}^w_2$ such operations, it holds that the interval-linearizations of the operations in $\text{IntLin}(E)$ do not overlap and the interval-linearization of $\text{DEQ}^w_1$ appears before the interval-linearization of $\text{DEQ}^w_2$.

Let $\alpha_i$ be the prefix of $\text{Lin}(F)$ used to define the interval-linearizability of $\text{DEQ}^w_i$. Recall that $\alpha_i$ is the shortest prefix of $\text{Lin}(F)$ containing all operations in $F$ that happens before $\text{DEQ}^w_i$ in $E$. First note that $\alpha_2$ cannot be a proper prefix of $\alpha_1$ because $\text{DEQ}^w_1 <_E \text{DEQ}^w_2$.

Consider first the case where the interval-linearization of $\text{DEQ}^w_1$ boils down to a single linearization point. This can happen only if the state of the queue at the end of $\alpha_1$ is empty. Thus, if $\alpha_1 = \alpha_2$, the interval-linearization of $\text{DEQ}^w_2$ is a point too and, by construction, $\text{DEQ}^w_2$ is linearized after $\text{DEQ}^w_1$. And if $\alpha_1 \neq \alpha_2$, then the interval-linearization of $\text{DEQ}^w_2$, which might be a point or an interval, necessarily appears before the linearization of $\text{DEQ}^w_1$, by construction.
Consider now the case that the interval-linearization of $\text{DEQ}_1^w$ is an interval (a sequence of linearization points). Then, the state of the queue at the end of $\alpha_1$ is $q_1 = x_1x_2\cdots x_t \neq \epsilon$. The interval-linearization of $\text{DEQ}_1^w$ is the interval $\beta_1$ of $\text{Lin}(F)$ starting at the next operation after $\alpha_1$ and ending at the $\text{Dequeue}$ operation that returns the item $x_t$. Observe that there is no moment in $\beta_1$ in which the queue is empty, and thus, if the interval-linearization of $\text{DEQ}_1^w$ is a point, it cannot belong to $\beta_1$.

The only case that remains to be analyzed is when both interval-linearizations of $\text{DEQ}_1^w$ and $\text{DEQ}_2^w$ are intervals. Let $\text{OP}^{\text{last}}_1$ be the last operation $\alpha_1$ and consider $(\text{Enqueue}(x_t) : \text{true})$ that appears in $\alpha_1$ too. For the sake of contradiction, suppose that the interval-linearizations overlap. Thus, $(\text{Dequeue}() : x_t)$ belong to both intervals. This implies that in $E$, $(\text{Dequeue}() : x_t)$ takes $x_t$ (i.e., it successfully executes Lines 10 and 11) when $\text{DEQ}_2^w$ is running. Thus, $(\text{Dequeue}() : x_t)$ starts before of concurrently with $\text{DEQ}_1^w$ and ends concurrently with $\text{DEQ}_2^w$. We analyze what happens with $(\text{Enqueue}(x_t) : \text{true})$:

- $(\text{Enqueue}(x_t) : \text{true}) = \text{OP}^{\text{last}}_1$. This case cannot happen because if so, $(\text{Enqueue}(x_t) : \text{true}) <_E \text{DEQ}_1^w$, by Claim [1] and when $\text{DEQ}_1^w$ starts, $x_t$ would be already in $\text{Items}$ and $\text{DEQ}_1^w$ would take it, as $(\text{Dequeue}() : x_t)$ takes its item only after $\text{DEQ}_2^w$ starts.

- $(\text{Enqueue}(x_t) : \text{true}) \neq \text{OP}^{\text{last}}_1$. In this case, consider the operation of $(\text{Dequeue}() : y)$ right after $(\text{Enqueue}(x_t) : \text{true})$ (it was already observed above that operation must be a $\text{Dequeue}$ with $y$ not appearing in $q_1$). By Assumption [1] $(\text{Enqueue}(x_t) : \text{true}) <_E (\text{Dequeue}() : y)$. If $(\text{Dequeue}() : y) = \text{OP}^{\text{last}}_1$, then we have $(\text{Enqueue}(x_t) : \text{true}) <_E \text{DEQ}_1^w$, because $(\text{Dequeue}() : y) <_E \text{DEQ}_1^w$, by Claim [1]. As observed in the previous case, this cannot happen.

We only remain to consider that $(\text{Dequeue}() : y) \neq \text{OP}^{\text{last}}_1$. Since $(\text{Enqueue}(x_t) : \text{true})$ appears before $\text{OP}^{\text{last}}_1$ in $\text{Lin}(F)$, we must have either $(\text{Enqueue}(x_t) : \text{true}) <_E \text{OP}^{\text{last}}_1$ or $(\text{Enqueue}(x_t) : \text{true}) ||_E \text{OP}^{\text{last}}_1$. If $(\text{Enqueue}(x_t) : \text{true}) <_E \text{OP}^{\text{last}}_1$, then $(\text{Enqueue}(x_t) : \text{true}) <_E \text{DEQ}_1^w$, and we already saw that leads to a contradiction; thus, that cannot happen. Finally, if $(\text{Enqueue}(x_t) : \text{true}) ||_E \text{OP}^{\text{last}}_1$, then the first step of $(\text{Enqueue}(x_t) : \text{true})$ (Line 02) occurs in $E$ before $\text{OP}^{\text{last}}_1$ terminates but its last step (Line 03) can occur only after $\text{DEQ}_1^w$ terminates because, as already explained, $(\text{Enqueue}(x_t) : \text{true})$ takes $x_t$ after $\text{DEQ}_2^w$ starts. Since $(\text{Enqueue}(x_t) : \text{true}) <_E (\text{Dequeue}() : y)$, by Assumption [1], we thus have $\text{OP}^{\text{last}}_1 <_E (\text{Dequeue}() : y)$. We have reached a contradiction because $(\text{Dequeue}() : y)$ appears before $\text{OP}^{\text{last}}_1$, and then $\text{Lin}(F)$ does not respect the partial order $<_F$.

We conclude that $\text{Int-Seq-Queue}$ is interval-linearizable and wait-free. This concludes the proof of the theorem.

Observe that the definition of the interval-sequential queue with weak-empty allows a $\text{Dequeue}$ operation to be linearized at a single point and non-deterministically return either $\epsilon$ or $\epsilon^w$, when the queue is empty (case 2.e in Definition [3]). The proof of Theorem [3] indeed interval-linearizes some $\text{Dequeue}$ operations returning $\epsilon^w$ at single points, although the operations return that value due to concurrency. Certainly, we can remove case 2.e in Definition [3] and modify the proof of Theorem [3] so that every interval-linearization of a $\text{Dequeue}$ operation returning $\epsilon^w$ is an interval, at the cost of making the proof more complex and longer.
6.3 Interval-Sequential Queue with Weak-emptiness and Multiplicity

Using the techniques in Sections 3 and 4, we can obtain a Read/Write wait-free implementation of a even more relaxed interval-sequential queue in which an item can be taken by several dequeue operations, i.e., with multiplicity. In more detail, the interval-sequential queue with weak-emptiness is modified such that concurrent Dequeue operations can return the same item and are set-linearized in the same concurrency class, as in Definitions 4 and 5.

We obtain a Read/Write wait-free interval-sequential implementation of the queue with weak-emptiness and multiplicity by doing the following: (1) replace the Fetch&Inc object in Int-Seq-Queue with a Read/Write wait-free Counter, (2) extend Items to a matrix to handle collisions, and (3) simulate the Swap operation with a Read followed by a Write. Thus, we have:

**Theorem 4.** There is a Read/Write wait-free interval-linearizable implementation of the queue with weak-emptiness and multiplicity.

7 Final Discussion

This work has introduced relaxation of classical data structures to adapt them to concurrency (with precisely defined behavioral properties), and implementations of these relaxed data structures. First, it introduced the notion of set-sequential queues and stacks with multiplicity, a relaxed version of queues and tasks in which an item can be dequeued more than once by concurrent operations. Non-blocking and wait-free implementations were presented, both based only on the simplest Read/Write operations. These are the first implementations of relaxed queues and stacks using only these operations. The implementations imply algorithms for idempotent work-stealing and out-of-order stacks and queues.

The paper also introduced the relaxed interval-linearizable queue with weak-emptiness check, which allows a dequeue operation to return a “weak-empty certificate” reporting that the queue might be empty. A wait-free algorithm using objects with consensus number two was presented. There are only non-blocking queue implementations using objects with consensus number two, and it is an open question if there is such a wait-free implementation. The proposed queue relaxation allowed us to go from non-blocking to wait-free.

This work also can be seen as a work prolonging the results described in [15] where the notion of interval-linearizability is introduced, and the set-linearizability [33] is studied. It has shown that linearizability, set-linearizability and interval-linearizability constitutes a (strict) hierarchy of consistency conditions that allow us to formally express the behavior of non-trivially relaxed queues and stacks on top of simple base objects such as Read/Write registers.

An interesting extension to this work is to explore if the proposed algorithms can lead to practical efficient implementations. Another interesting extension is to explore if set-sequential or interval-sequential relaxations of other concurrent data structures allow to avoid atomic Read-Modify-Write operations.

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A  Correctness proof of Seq-Queue (Fig. 3, Section 4.1)

As already explained, Seq-Queue is a slight modification of Li’s non-blocking queue implementation \[27\]. The only difference between the two implementations is that Seq-Queue relaxes the condition for returning $\epsilon$: while Li’s implementations requires $\text{tail}_i = \text{tail}'_i \land \text{taken}_i = \text{taken}'_i$ in the condition in Line 16, Seq-Queue only requires $\text{taken}_i = \text{taken}'_i$. Roughly speaking, we show below that any execution of Seq-Queue can be modified so that it corresponds to an execution of Li’s queue implementation, and hence any linearization of it is also a linearization of the execution of Seq-Queue.

Let $E$ be any execution of Seq-Queue. Since Seq-Queue is non-blocking, there is an extension of $E$ in which all operations are completed and no new operations started. Thus, we can assume that all operations in $E$ are completed. For every Dequeue operation in $E$, let $\text{tail}'_i$ be the value of $\text{tail}_i$ in the second-to-last while iteration, or zero if there is only one iteration. Let $k$ be the number of Dequeue operations returning $\epsilon$ with $\text{tail}'_i$ being distinct to $\text{tail}_i$. By induction on $k$, we show that $E$ is linearizable.

For the base case, $k = 0$, for every Dequeue operation returning $\epsilon$, it holds $\text{tail}_i = \text{tail}'_i$ (additionally to $\text{taken}_i = \text{taken}'_i$), namely, it satisfies the condition in Li’s queue, and thus $E$ is indeed an execution of that algorithm, which implies that $E$ is linearizable. Assuming the claim holds when $E$ has $k - 1$ such operations, let us show it holds for $k$. We will modify $E$ such that it has $k + 1$ such operations. Figure 14 exemplifies the transformation.

Among those $k$ Dequeue operations in $E$ (all of them returning $\epsilon$), let $\text{deq}^*$ denote the one that executes first its last step in Line 07 (or to Line 04 if $\text{deq}^*$ executes only one iteration of the while loop). We have $\text{taken}_i = \text{taken}'_i$ and $\text{tail}_i \neq \text{tail}'_i$ when $\text{deq}^*$ returns $\epsilon$ in Line 16. Thus, $\text{tail}_i = \text{tail}'_i + \Delta$, for some integer $\Delta \geq 1$, which can only happen if there is at least one Enqueue operation that executes its step in Line 01 in the interval $I$ of $E$ from step $e'$ to step $e$. Let $\text{enq}^*$ be any of such Enqueue operations. Note that $\text{enq}^*$ writes its item, Line 02, after $\text{deq}^*$ has scanned all entries in $\text{Items}[1,\ldots,\text{tail}'_i]$ in its last while iteration (if not, $\text{deq}^*$ finds $\text{taken}_i > \text{taken}'_i$ and
it does not decides $\epsilon$ in that iteration). For any such operation $\text{enq}^*$, we move forward its step in Line [01] right after step $e$ of $\text{deq}^*$, respecting the relative order among all the steps. Thus, in the resulting execution $E'$, the step $e$ of $\text{deq}^*$ reads the same value from $\text{Tail}$, name, it finds $\text{tail}_i = \text{tail}_i'$ (additionally to $\text{taken}_i = \text{taken}_i'$), hence it holds Li’s queue condition for $\text{deq}^*$ and returns $\epsilon$ in $E'$ too. Therefore, $E'$ has $k - 1$ $\text{Dequeue}$ operations that return $\epsilon$ with $\text{tail}_i'$ being distinct to $\text{tail}_i$. Observe that operations in $E$ and $E'$ have the same real-time order. By induction hypothesis, $E'$ is linearizable, however, a linearization of $E'$ might not be a linearization of $E$ because it could be that, after the modifications, a $\text{Dequeue}$ operation returns distinct values in $E$ and $E'$. Thus, to conclude, we modify $E'$ to be sure that all $\text{Dequeue}$ operations in $E'$ return the same value.

Let $f$ be a step of a $\text{Dequeue}$ operation (distinct from $\text{deq}^*$) corresponding to Line [07] and appearing in the interval $I$ in $E$. If $f$ reads the value $\text{tail}_i'$, then that step obtains the same value in $E'$ and there is nothing to do; intuitively, $f$ happens before all $\text{Enqueue}$ operations that execute Line [01] in $I$, and hence all modifications to $E$ happen after $f$. If $f$ reads a value greater than $\text{tail}_i'$, then there are two sub-cases (see Figure [14]). If the $\text{Dequeue}$ operation returns in the while iteration $f$ belongs to, an item that is stored in an entry of $\text{Items}[1, \ldots, \text{tail}_i']$, then we do not move $f$, because in $E'$ that step reads the value $\text{tail}_i'$, which is fine since it just needs to scan up to that

Figure 14: An example of the transformation from $\text{Seq-Queue}$ to Li’s queue.
entry in Items to obtain its item. Otherwise, we move forward f right after step e of deq∗ (and all its scan steps that appear in I), respecting the relative order among all the steps in E. Thus, f reads the same value in E and in the resulting execution. Note that we can do this because deq∗ finds takeni = taken′ i when it decides, and hence the state of Items[1, . . . , taili] does not change between e and the first read step (Line 09) of the last scan of deq∗. We do the same for each such Dequeue operation. Let E′′ be the resulting execution. Therefore, the operations E and E′′ have the same real-time order and return the same values. By induction hypothesis, E′′ is linearizable, and any linearization of E′′ is a linearization of E too. The claim follows.

B  A computability view

Considering the consensus numbers 1 and 2 of the base objects on top of which are built the stack and the queue presented in the present article, this appendix gives a global view of the tradeoff between the liveness and safety properties of the constructed stacks and queues.

| Base object | Liveness       | Safety             | Algorithm                  |
|-------------|----------------|--------------------|----------------------------|
| CN = 1      | push(): wait-freedom | push(): linearizability | Figures 3 and 15           |
|             | pop(): wait-freedom   | pop(): set-linearizability |                           |
| CN = 2      | push(): wait-freedom   | push(): linearizability | Figure 2 [2]              |
|             | pop(): wait-freedom   | pop(): linearizability |                           |

Table 1: Stack in the consensus number (CN) 1 and 2 worlds.

| Base object | Liveness       | Safety             | Algorithm                  |
|-------------|----------------|--------------------|----------------------------|
| CN = 1      | enqueue(): wait-freedom | enqueue(): linearizability | Figure 7                  |
|             | dequeue(): non-blocking | dequeue(): set-linearizability |                           |
| CN = 2      | enqueue(): wait-freedom | enqueue(): linearizability | Figure 6 (modified version of [27]) |
|             | dequeue(): non-blocking | dequeue(): linearizability |                           |
| CN = 2      | enqueue(): wait-freedom | enqueue(): linearizability | Figure 11                  |
|             | dequeue(): wait-freedom | dequeue(): interval-linearizability |                           |

Table 2: Queue in the consensus number (CN) 1 and 2 worlds.

C  A Renaming-based Performance-related Improvement (when contention is small)

When the contention on the shared memory accesses is small, a Pop operation in Set-Seq-Stack might perform several “useless” Read and Write operations in Lines 08 and 10 as it scans all entries of Items in every row while trying to get a non-⊥ value, and some of these entries might never store an item in the execution (called untouched in the proof of Theorem 1). The algorithm in Figure 15 solves this issue with the help of an array Ren with instances of any Read/Write f(n)-adaptive renaming. In f(n)-adaptive renaming [9], each process starts with its index as input
and obtains a unique name in the space \(\{1, \ldots, f(p)\}\), where \(p\) denotes the number of processes participating in the execution. Several adaptive renaming algorithms have been proposed (see e.g. [14]); a good candidate is the simple \(p^2/2\)-adaptive renaming algorithm of Moir and Anderson with \(O(p)\) individual step complexity [31].

Shared Variables:
- \(Top: \text{Read}/\text{Write} \) wait-free linearizable Counter base object initialized to 1
- \(NOPS[1, \ldots]: \) array of \(\text{Read}/\text{Write} \) wait-free linearizable Counter base objects initialized to 0
- \(Ren[1, \ldots]: \) array of instances of \(\text{Read}/\text{Write} \) \(f(n)\)-adaptive renaming
- \(Items[1, \ldots][1, \ldots, f(n)]: \) array of \(\text{Read}/\text{Write} \) registers initialized to \(\perp\)

Operation Push\((x)\) is
1. \(top_i \leftarrow \text{Top. Read()}\)
2. \(\text{tiebreaker}_i \leftarrow Ren[top_i]\). Rename\((i)\)
3. \(\text{Top. Increment()}\)
4. \(NOPS[top_i]\). Increment\((i)\)
5. \(\text{Items}[top_i, \text{tiebreaker}_i]\). Write\((x)\)
6. return true

end Push

Operation Pop\((i)\) is
7. \(top_i \leftarrow \text{Top. Read()} + 1\)
8. for \(r_i \leftarrow top_i\) down to 1 do
9. \(\text{nops}_i \leftarrow NOPS[r_i]\). Read\((i)\)
10. \(\text{max}_i \leftarrow f(\text{nops}_i)\)
11. for \(s_i \leftarrow \text{max}_i\) down to 1 do
12. \(x_i \leftarrow Items[r_i][s_i]\). Read\((i)\)
13. if \(x_i \neq \perp\) then
14. \(\text{Items}[r_i][s_i]\). Write\((\perp)\)
15. return \(x_i\)
16. end if
17. end for
18. end for
19. return \(\epsilon\)

end Pop

Figure 15: An improved \(\text{Read}/\text{Write} \) wait-free set-sequential stack with multiplicity (code for process \(p_i\)).

Push operations storing their items in the same row \(Items[b]\), now dynamically decide where in the row they store their items, with the help of \(Ren[b]\). Rename\((\cdot)\) in Line 02. Additionally, these operations announce in \(NOPS[b]\) the number of processes that store values in row \(Items[b]\), in Line 04 hence helping Pop operations to scan only the segment of \(Items[b]\) where there might be items, in Line 09. The correctness proof of this implementation is very similar to the correctness proof of Set-Seq-Stack. Note that if the contention is small, say \(O(\log^2 n)\), every Pop operation scans only the first entries \(O(\log^{2x} n)\) of row \(Items[b]\) as the processes storing items in that row rename in the space \(\{1, \ldots, (\log^{2x} n)/2\}\).

The same analysis and improvement can be applied to the set-sequential queue implementation Set-Seq-Queue.