On the difference between a D. H. Lehmer number and its inverse over short interval

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Abstract

Let $q > 2$ be an odd integer. For each integer $x$ with $0 < x < q$ and $(q, x) = 1$, we know that there exists one and only one $\overline{x}$ with $0 < \overline{x} < q$ such that $x\overline{x} \equiv 1 \pmod{q}$. A Lehmer number is defined to be any integer $a$ with $2 \nmid (a + \overline{a})$. For any nonnegative integer $k$, Let

$$ M(x, q, k) = \sum_{a=1}^{q} \sum_{b \leq x}^{'} (a - b)^{2k}. $$

The main purpose of this paper is to study the properties of $M(x, q, k)$, and give a sharp asymptotic formula, by using estimates of Kloosterman’s sums and properties of trigonometric sums.

Key words: D. H. Lehmer problem; Dirichlet Character; short interval; inverse of integers; estimate.

1. Introduction

Let $q > 2$ be an odd integer. For each integer $x$ with $0 < x < q$ and $(q, x) = 1$, we know that there exists one and only one $\overline{x}$ with $0 < \overline{x} < q$ such that $x\overline{x} \equiv 1 \pmod{q}$. Let $r(q)$ be the number of cases in which $x$ and $\overline{x}$ are of opposite parity. For $q = p$ a prime, D. H. Lehmer [1] asks us to find $r(p)$ or at least to say something nontrivial about it. About this problem, a lot of scholars [2,3] have studied it. For the sake of simplicity, we call such a number $x$ as a D. H. Lehmer number.

W. Zhang [4] has given an asymptotic estimate:

$$ r(p) = \frac{1}{2}p + O(p^{\frac{3}{4}} \ln^2 p). $$

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Later, W. Zhang [5, 6] also proved that for every odd integer \( q \geq 3 \),
\[
r(q) = \frac{1}{2} \phi(q) + O(q^{\frac{3}{2}} d^2(q) \ln^2 q),
\]
where \( \phi(q) \) is the Euler function and \( d(q) \) is the divisor function.

For any nonnegative integer \( k \), let
\[
M(q, k) = \sum_{a=1}^{q} (a - \bar{a})^{2k},
\]
W. Zhang [7] gave a sharp asymptotic formula for \( M(q, k) \) as following:
\[
M(q, k) = \frac{1}{(2k + 1)(2k + 2)} \phi(q) q^{2k} + O(4^k q^{2k+\frac{1}{2}} d^2(q) \ln^2 q).
\]
where \( \sum'_{a} \) denotes the summation over all \( a \) such that \( (a, q) = 1 \). Moreover, he [8] also proved
\[
\sum_{a=1}^{q} (a - \bar{a})^{2k} = \frac{1}{(2k + 1)(k + 1)} \phi(q) q^{2k} + O(4^k q^{2k+\frac{1}{2}} d^2(q) \ln^2 q).
\]

The main purpose of this paper is to study the distribution properties of D. H. Lehmer numbers and the asymptotic properties of the 2kth power mean
\[
M(x, q, k) = \sum_{a=1}^{q} \sum_{b \leq x} (a - b)^{2k}.
\]
It seems that no one has studied this problem yet. The problem is interesting because it can help us to find how large is the difference between a D. H. Lehmer number and its inverse modulo \( q \). In this paper, we use estimates of Kloostermans sums and properties of trigonometric sums to give a sharper asymptotic formula for \( M(x, q, k) \) for any fixed positive integer \( k \). That is, we shall prove the following:

**Theorem.** For any odd number \( q \) and integer \( k \), we have the asymptotic formula
\[
M(x, q, k) = \frac{1}{(2k + 1)(2k + 2)} x \phi(q) q^{2k} + O(q^{2k+\frac{1}{2}} \ln^2 q)
\]
where \( \phi(q) \) is the Euler function.

2. Some lemmas

In this section, we prove some elementary lemmas which are necessary in the proof of the theorems.

**Lemma 1.** Let \( q \) be an odd number. For any integer \( n \) and nonnegative integer \( r \), define
\[
K(n, r) = \sum_{a=1}^{\frac{q}{2}} a^r e \left( \frac{an}{q} \right), \quad H(n, r) = \sum_{a=1}^{\frac{q}{2}} (-1)^a a^r e \left( \frac{an}{q} \right),
\]
where \( e(y) = e^{2\pi iy} \). We have the estimates

\[
K(n, r) \begin{cases}
= \frac{q^{r+1}}{\sin(\pi n/q)} + O(q^r), & q|n \\
\ll \frac{q^r}{|\sin(\pi n/q)|}, & q \nmid n
\end{cases}
\]

(1)

\[
H(n, r) \ll \frac{q^r}{|\cos(\pi n/q)|}.
\]

(2)

**Proof.** See Ref. [7].

**Lemma 2.** For any integer \( K \geq 1 \) and \( 0 < \alpha < 1 \), we have

\[
\left| \sum_{n=1}^{K} e(\alpha n) \right| \leq \min \left( K, \frac{1}{2\langle \alpha \rangle} \right),
\]

where \( \langle \alpha \rangle = \min(\{\alpha\}, 1 - \{\alpha\}) \), \( \{\alpha\} \) is the decimal part of \( \alpha \).

**Proof.** See Ref. [9].

**Lemma 3.** For any integer \( q \geq 3 \), \( x > \frac{1}{2} \), and any positive integer \( n \geq 1 \), we have

\[
\sum_{l=1}^{q-1} \left| \sum_{b \leq xq} \left( \frac{b(n - l)}{q} \right) \right| \ll q^{1+\epsilon}.
\]

**Proof.** From Lemma 2, when \( n \not\equiv 0 \pmod{q} \), for \( 1 \leq l \leq q - 1 \), there must be one and only one \( l \) such that \( n - l \equiv 0 \pmod{q} \), so we get

\[
\sum_{l=1}^{q-1} \left| \sum_{b \leq xq} \left( \frac{b(n - l)}{q} \right) \right| = \sum_{l=1}^{q-1} \left| \sum_{b \leq xq/d} \mu(d) \sum_{b \equiv l \pmod{d}} e \left( \frac{b(n - l)}{q} \right) \right| = \sum_{l=1}^{q-1} \left| \sum_{d|q} \mu(d) \sum_{b \leq xq/d} e \left( \frac{b(n - l)}{q} \right) \right|
\]

\[
\leq \sum_{l=1}^{q-1} \left| \sum_{d|q} \mu(d) \sum_{b \leq xq/d} e \left( \frac{b(n - l)}{q} \right) \right| + xq \sum_{d|q} \frac{\mu(d)}{d} | \ll q^{1+\epsilon}.
\]

\[
= \sum_{l=1}^{q-1} \left| \frac{1}{2\langle \frac{n}{q} \rangle} \right| - \sum_{d|q} \left| \frac{1}{2\langle \frac{n}{q} \rangle} \right| + x\phi(q)
\]

\[
= q - \sum_{d|q} \frac{1}{d} + x\phi(q).
\]

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\[ \leq 2 \sum_{1 \leq l \leq \lfloor \frac{q}{2} \rfloor} \frac{1}{d(q)} - \sum_{d(q)} \left| \frac{1}{2\left\langle \frac{n}{q} \right\rangle} \right| + x\phi(q) \]
\[ = q \sum_{1 \leq l \leq \lfloor \frac{q}{2} \rfloor} \frac{1}{l} - \sum_{d(q)} \left| \frac{1}{2\left\langle \frac{n}{q} \right\rangle} \right| + x\phi(q) \]
\[ = qd(q) \sum_{1 \leq l \leq \lfloor \frac{q}{2} \rfloor} \frac{1}{l} - d(q) \left| \frac{1}{2\left\langle \frac{n}{q} \right\rangle} \right| + x\phi(q) \]
\[ \ll q^{1+\epsilon}. \quad (3) \]

When \( n \equiv 0(\text{mod} q) \), from Lemma 2, we also get
\[ \sum_{l=1}^{q-1} \left| \sum_{b \leq xq} e\left( \frac{b(n-l)}{q} \right) \right| \]
\[ = \sum_{l=1}^{q-1} \left| \sum_{b \leq xq} e\left( \frac{b(n-l)}{q} \right) \right| \sum_{d|(b,q)} \mu(d) \]
\[ = \sum_{l=1}^{q-1} \sum_{d|q} \mu(d) \sum_{b \leq xq/d} e\left( \frac{bl}{q} \right) \]
\[ \leq \sum_{l=1}^{q-1} \sum_{d|q} \left| \sum_{b \leq xq/d} e\left( \frac{bl}{q} \right) \right| \]
\[ \leq \sum_{l=1}^{q-1} \sum_{d|q} \left| \frac{1}{2\left\langle \frac{l}{q} \right\rangle} \right| \]
\[ \leq 2 \sum_{1 \leq l \leq \lfloor \frac{q}{2} \rfloor} \sum_{d|q} \frac{1}{2\left\langle \frac{l}{q} \right\rangle} \]
\[ = q \sum_{1 \leq l \leq \lfloor \frac{q}{2} \rfloor} \frac{1}{l} \]
\[ = qd(q) \sum_{1 \leq l \leq \lfloor \frac{q}{2} \rfloor} \frac{1}{l} \]
\[ \ll q^{1+\epsilon}. \quad (4) \]

Therefore, combining (3) and (4), for any positive integer \( n \geq 1 \), we have the estimate
\[ \sum_{l=1}^{q-1} \left| \sum_{b \leq xq} e\left( \frac{b(n-l)}{q} \right) \right| \ll q^{1+\epsilon}. \]

This proves Lemma 3.

**Lemma 4.** Let \( q \geq 3 \) be an integer, and \( \chi \) denote the Dirichlet character modulo \( q \). The Gauss sum is defined by
\[ G(n, \chi) = \sum_{a=1}^{q} \chi(a)e\left( \frac{na}{q} \right). \]
Then by the principal Dirichlet character \( \chi_0 \) modulo \( q \), we have the identity

\[
G(n, \chi_0) = \mu \left( \frac{q}{(n, q)} \right) \phi(q) \phi^{-1} \left( \frac{q}{(n, q)} \right),
\]

where \( \mu(n) \) is the M"obius function, \( \phi(q) \) is the Euler function, and \((n, q)\) is the greatest common divisor of \( n \) and \( q \).

**Proof.** See Ref. [10].

**Lemma 5.** For \( q > 2 \) an integer, any non-principal Dirichlet character \( \chi \) modulo \( q \) and any positive integers \( m \) and \( l \), we have the estimate

\[
\sum_{\chi \mod q} G(m, \chi) G(l, \chi) \ll (m, l, q)^{\frac{1}{2}} q^{\frac{3}{2} + \epsilon}, \tag{5}
\]

where \((m, l, q)\) is the greatest common divisor of \( m, l \) and \( q \), \( \epsilon \) is any fixed positive real number.

On the other hand, for the principal Dirichlet character \( \chi_0 \), any positive integers \( m \) and \( l \), we also have

\[
G(m, \chi_0) G(l, \chi_0) \ll (m, q)(l, q) q^\epsilon. \tag{6}
\]

**Proof.** First, we prove (5). According to the orthogonality of character sums, we have

\[
\sum_{\chi \mod q} \chi(n) = \begin{cases} 
\phi(q), & n \equiv l \pmod{q} \\
0, & n \not\equiv l \pmod{q}
\end{cases}
\]

and hence we have

\[
\sum_{\chi \mod q} G(m, \chi) G(l, \chi)
\]

\[
= \sum_{\chi \mod q} \sum_{s=1}^{q} \chi(s) e \left( \frac{ms}{q} \right) \sum_{t=1}^{q} \chi(t) e \left( \frac{lt}{q} \right)
\]

\[
= \sum_{s=1}^{q'} \sum_{t=1}^{q'} e \left( \frac{ms + lt}{q} \right) \sum_{\chi \mod q} \chi(s) \chi(t)
\]

\[
= \phi(q) \sum_{s=1}^{q-1} \sum_{t=1}^{q-1} e \left( \frac{ms + lt}{q} \right)
\]

\[
= \phi(q) \sum_{t=1}^{q-1} e \left( \frac{m \bar{t} + lt}{q} \right)
\]

\[
\ll \phi(q)(m, l, q)^{\frac{1}{2}} q^{\frac{3}{2}}
\]

\[
\ll (m, l, q)^{\frac{1}{2}} q^{\frac{3}{2} + \epsilon}.
\]

Now we show (6). From Lemma 4 and the definition of Gauss sum, we have

\[
G(m, \chi_0) G(l, \chi_0)
\]
\[
\mu \left( \frac{q}{(m, q)} \right) \phi(q) \phi^{-1} \left( \frac{q}{(m, q)} \right) \mu \left( \frac{q}{(l, q)} \right) \phi(q) \phi^{-1} \left( \frac{q}{(l, q)} \right) \ll \phi^2(q) \frac{(m, q)(l, q) d^2(q)}{q^2} \ll (m, q)(l, q) q^\epsilon,
\]
where we have used \( \phi(q) \gg \frac{q}{d(q)} \) (see Ref. [11]), and \( d(q) \) is the divisor function.

**Lemma 6.** Let \( m, n \) and \( q \) be integers, and \( q > 2 \). Then we have the estimates

\[
S(x, m, n; q) = \sum_{a=1}^{q} \sum_{b \leq xq \atop ab \equiv 1 (\mod q)} e \left( \frac{am + bn}{q} \right) \ll q^{1/2 + \epsilon}
\]

**Proof.** From the orthogonality of character sums, we have

\[
S(x, m, n; q) = \sum_{a=1}^{q} \sum_{b \leq xq \atop ab \equiv 1 (\mod q)} e \left( \frac{am + bn}{q} \right)
\]

\[
= \frac{1}{\phi(q)} \sum_{a=1}^{q} \sum_{b \leq xq} e \left( \frac{am + bn}{q} \right) \sum_{\chi \mod q} \chi(a) \chi(b)
\]

\[
= \frac{1}{\phi(q)} \sum_{\chi \mod q} \left( \sum_{a=1}^{q} \chi(a) e \left( \frac{am}{q} \right) \right) \left( \sum_{b \leq xq} \chi(b) e \left( \frac{bn}{q} \right) \right)
\]

\[
= \frac{1}{\phi(q)} \sum_{\chi \equiv \chi_0} \left( \sum_{a=1}^{q} \chi(a) e \left( \frac{am}{q} \right) \right) \left( \sum_{b \leq xq} \chi(b) e \left( \frac{bn}{q} \right) \right)
\]

\[
+ \frac{1}{\phi(q)} \left( \sum_{a=1}^{q-1} e \left( \frac{am}{q} \right) \right) \left( \sum_{b \leq xq} e \left( \frac{bn}{q} \right) \right)
\]

\[
= S_1 + S_2.
\]

(7)

Now we will estimate both \( S_1 \) and \( S_2 \) respectively. Firstly, we shall estimate \( S_1 \). From the identity, for any Dirichlet character \( \chi \neq \chi_0 \) modulo \( q \),

\[
\chi(a) = \frac{1}{q} \sum_{k=1}^{q-1} G(k, \chi) e \left( -\frac{ak}{q} \right).
\]

Hence according to Lemma 3 and Lemma 5, we have

\[
S_1 = \frac{1}{\phi(q)} \sum_{\chi \equiv \chi_0} \left( \sum_{a=1}^{q} \frac{1}{q} \sum_{k=1}^{q-1} G(k, \chi) e \left( -\frac{ak}{q} \right) e \left( \frac{am}{q} \right) \right) \times
\]

\[
\times \left( \sum_{b \leq xq} \frac{1}{q} \sum_{l=1}^{q-1} G(l, \chi) e \left( -\frac{bl}{q} \right) e \left( \frac{bn}{q} \right) \right)
\]

\[
= \frac{1}{q^2 \phi(q)} \sum_{\chi \neq \chi_0} \left( \sum_{k=1}^{q-1} G(k, \chi) \sum_{a=1}^{q} e \left( \frac{a(m - k)}{q} \right) \right) \times
\]
\[
\times \left( \sum_{l=1}^{q-1} G(l, \chi) \sum_{b \leq xq} e\left( \frac{b(n-l)}{q} \right) \right) \\
= \frac{1}{q^2 \phi(q)} \sum_{\chi \neq \chi_0} \left( \sum_{k=1}^{q-1} G(k, \chi) \sum_{a=1}^{q} e\left( \frac{a(m-k)}{q} \right) \sum_{d|a} \mu(d) \right) \times \\
\times \left( \sum_{l=1}^{q-1} G(l, \chi) \sum_{b \leq xq} e\left( \frac{b(n-l)}{q} \right) \right) \\
= \frac{1}{q^2 \phi(q)} \sum_{\chi \neq \chi_0} \left( \sum_{k=1}^{q-1} G(k, \chi) \sum_{d|q} \mu(d) \sum_{a=1}^{q} e\left( \frac{a(m-k)}{q} \right) \right) \times \\
\times \left( \sum_{l=1}^{q-1} G(l, \chi) \sum_{b \leq xq} e\left( \frac{b(n-l)}{q} \right) \right) \\
= \frac{q}{q^2 \phi(q)} \sum_{\chi \neq \chi_0} \left( \sum_{k=1}^{q-1} G(k, \chi) \sum_{d|q} \frac{\mu(d)}{d} \right) \times \\
\times \left( \sum_{l=1}^{q-1} G(l, \chi) \sum_{b \leq xq} e\left( \frac{b(n-l)}{q} \right) \right) \\
= \frac{1}{q^2} \sum_{\chi \neq \chi_0} G(m, \chi) \left( \sum_{l=1}^{q-1} G(l, \chi) \sum_{b \leq xq} e\left( \frac{b(n-l)}{q} \right) \right) \\
= \frac{1}{q^2} \sum_{l=1}^{q-1} \sum_{b \leq xq} e\left( \frac{b(n-l)}{q} \right) \sum_{\chi \neq \chi_0} G(m, \chi) G(l, \chi) + \\
+ \frac{1}{q^2} \sum_{l=1}^{q-1} \sum_{b \leq xq} e\left( \frac{b(n-l)}{q} \right) G(m, \chi_0) G(l, \chi_0) \\
\ll \frac{1}{q^2} \sum_{l=1}^{q-1} \sum_{b \leq xq} e\left( \frac{b(n-l)}{q} \right) \sum_{\chi \mod q} G(m, \chi) G(l, \chi) + \\
+ \frac{1}{q^2} \sum_{l=1}^{q-1} \sum_{b \leq xq} e\left( \frac{b(n-l)}{q} \right) G(m, \chi_0) G(l, \chi_0) \\
\ll \frac{1}{q^2} \sum_{l=1}^{q-1} \sum_{b \leq xq} e\left( \frac{b(n-l)}{q} \right) (m, l, q)^{\frac{1}{2}} q^{\frac{d}{2}} + \\
+ \frac{1}{q^2 \phi(q)} \sum_{l=1}^{q-1} \sum_{b \leq xq} e\left( \frac{b(n-l)}{q} \right) (m, q)(l, q)^{\epsilon} \\
\ll q^{\frac{1}{2} + \epsilon} \sum_{l=1}^{q-1} \sum_{b \leq xq} e\left( \frac{b(n-l)}{q} \right) \\
\ll q^{\frac{1}{2} + \epsilon}.
\] (8)
Now we estimate $S_2$, we have

$$|S_2| = \frac{1}{\phi(q)} \left| \sum_{a=1}^{q-1} e\left(\frac{am}{q}\right) \left( \sum_{b \leq xq} e\left(\frac{bn}{q}\right) \right) \right|$$

$$= \frac{1}{\phi(q)} \left| \sum_{a=1}^{q-1} e\left(\frac{am}{q}\right) \sum_{b \leq xq} e\left(\frac{bn}{q}\right) \right|$$

$$\leq \frac{1}{\phi(q)} \left| \sum_{b \leq xq} e\left(\frac{bn}{q}\right) \right|$$

$$\leq \frac{xq}{\phi(q)}$$

$$\leq xq^\epsilon$$

$$\ll q^{\frac{3}{2}+\epsilon}, \quad (9)$$

where $x < q^{\frac{1}{2}+\epsilon}$.

Therefore, from (7)-(9), we have

$$S(x, m, n; q) = \sum_{a=1}^{q} \sum_{b \leq xq} e\left(\frac{am + bn}{q}\right) \ll q^{\frac{1}{2}+\epsilon},$$

where $\epsilon$ is any positive real number.

**Lemma 7.** Let $r$, $s$ and $q$ be positive integers and $q > 2$. Then

$$\sum_{a=1}^{q} \sum_{b \leq xq} a^rb^s = \frac{x\phi(q)q^{r+s}}{(r+1)(s+1)} + O(q^{r+s+\frac{3}{2}} \ln^2 q),$$

where $\phi(q)$ is the Euler function.

**Proof.** From the trigonometric identity

$$\sum_{a=1}^{q} e\left(\frac{an}{q}\right) = \begin{cases} q, & q \mid n \\ 0, & q \nmid n \end{cases}$$

we get the identity

$$\sum_{a=1}^{q} \sum_{b \leq xq} a^rb^s$$

$$= \frac{1}{q^2} \sum_{a=1}^{q} \sum_{b \leq xq} \sum_{c,d=1}^{q} \sum_{m,n=1}^{q} e\left(\frac{m(a-c) + n(b-d)}{q}\right)$$

$$= \frac{1}{q^2} \sum_{m,n=1}^{q} \left( \sum_{a=1}^{q} \sum_{b \leq xq} e\left(\frac{am + bn}{q}\right) \right) \left( \sum_{c=1}^{q} c^r e\left(\frac{-mc}{q}\right) \right) \left( \sum_{d=1}^{q} d^{s-r} e\left(\frac{-nd}{q}\right) \right)$$
\[
\frac{1}{q^2} S(x, q; q) K(-q, r) K(-q, s) = \frac{1}{q^2} \sum_{m=1}^{q} \sum_{n=1}^{q} S(x, m, n; q) K(-m, r) K(-n, s) + \frac{1}{q^2} \sum_{m=1}^{q-1} S(x, m, q; q) K(-m, r) K(-q, s) + \frac{1}{q^2} S(x, m, q; q) K(-m, r) K(-q, s),
\]

where \( K(-m, r) \) is defined in Lemma 1. From (2) of Lemma 1, Lemma 6 and noting that \( 2/\pi \leq (\sin x/x) \) for \( |x| \leq \pi/2 \), we get

\[
\frac{1}{q^2} S(x, q; q) K(-q, r) K(-q, s) = \frac{1}{q^2} \left( \sum_{a=1}^{q} \sum_{b \leq x_q \atop ab \equiv 1 \pmod{q}} \left( \frac{aq + bq}{q} \right) \right) \left( \frac{q^{r+1}}{r+1} + O(q^r) \right) \left( \frac{q^{s+1}}{s+1} + O(q^s) \right)
\]

\[
= \frac{1}{q^2} \left( \sum_{a=1}^{q} \sum_{b \leq x_q \atop ab \equiv 1 \pmod{q}} 1 \right) \left( \frac{q^{r+1}}{r+1} + O(q^r) \right) \left( \frac{q^{s+1}}{s+1} + O(q^s) \right)
\]

\[
= \frac{1}{q^2} \left( \sum_{b \leq x_q} \mu(d) \right) \left( \frac{q^{r+1}}{r+1} + O(q^r) \right) \left( \frac{q^{s+1}}{s+1} + O(q^s) \right)
\]

\[
= \frac{1}{q^2} \left( xq \sum_{d | q} \frac{\mu(d)}{d} \right) \left( \frac{q^{r+1}}{r+1} + O(q^r) \right) \left( \frac{q^{s+1}}{s+1} + O(q^s) \right)
\]

\[
= \frac{x \phi(q)}{q^2} \left( \frac{q^{r+1}}{r+1} + O(q^r) \right) \left( \frac{q^{s+1}}{s+1} + O(q^s) \right)
\]

\[
= \frac{x \phi(q) q^{r+s}}{(r+1)(s+1)} + O(q^{r+s}),
\]

(11)
\[
\ll \sum_{m=1}^{q-1} q^{2} q^{s+1} q^{r} \frac{q}{2m} \\
\ll q^{r+s+\frac{5}{2}} \sum_{m=1}^{q-1} \frac{1}{m} \\
\ll q^{r+s+\frac{5}{2}} \ln q. \quad (12)
\]

Similarly, we can get the estimate
\[
\sum_{n=1}^{q-1} S(x, q, n; q) K(-q, r) K(-n, s) \ll q^{r+s+\frac{5}{2}} \ln q. \quad (13)
\]
\[
\sum_{m=1}^{q-1} S(x, m, n; q) K(-m, r) K(-n, s) \ll q^{r+s+\frac{5}{2}} \sum_{m=1}^{q-1} \sum_{n=1}^{q-1} \frac{1}{mn} \\
\ll q^{r+s+\frac{5}{2}} \ln^{2} q. \quad (14)
\]

Combining (10)-(14) we immediately deduce that
\[
\ll \sum_{a=1}^{q} \sum_{b \leq xq} \alpha^{a} \beta^{b} = \frac{x \phi(q) q^{r+s}}{(r+1)(s+1)} + O(q^{r+s+\frac{5}{2}} \ln^{2} q).
\]

This is the conclusion of Lemma 7.

**Lemma 8.** Let \( r, s \) and \( q \) be positive integers and \( q > 2 \). Then
\[
\ll \sum_{a=1}^{q} \sum_{b \leq xq} (-1)^{a+b} q^{r} b^{s} = O(q^{r+s+\frac{5}{2}} \ln^{2} q).
\]

**Proof.** Similarly, we get
\[
\ll \sum_{a=1}^{q} \sum_{b \leq xq} (-1)^{a+b} q^{r} b^{s} \\
= \frac{1}{q^{2}} \sum_{a=1}^{q} \sum_{b \leq xq} (-1)^{c+d} d^{s} \sum_{m,n=1}^{q} \frac{1}{q} \left( m(a-c) + n(b-d) \right) \\
= \frac{1}{q^{2}} \sum_{m,n=1}^{q} \left( \sum_{a=1}^{q} \sum_{b \leq xq} (-1)^{c+d} e^{\left( \frac{am+bn}{q} \right)} \right) \left( \sum_{c=1}^{q} (-1)^{c} e^{\left( \frac{-mc}{q} \right)} \right) \left( \sum_{d=1}^{q} (-1)^{d} e^{\left( \frac{-nd}{q} \right)} \right)
\]
Lemma 7 and Lemma 8 we get used (2) of Lemma 1 in the proof above. 

\[ H \approx q^{r+s} q^2 \]

\[ H \approx q^{r+s} + \frac{1}{2} \ln^2 q. \]

where \( H(-m, r) \) is defined in Lemma 1.

Noting that \( |\cos(\pi m/q)| = |\sin(\pi(q-2m)/(2q))| \) and \( q - 2m \neq 0 \), and we have used (2) of Lemma 1 in the proof above.

### 3. Proof of The Theorem

In this section, we shall complete the proof of the theorem. By the binomial formula, Lemma 7 and Lemma 8 we get

\[ M(x, q, k) = \sum_{a=1}^{q} \sum_{1 \leq b \leq qx} (a-b)^{2k} \]

\[ = \frac{1}{2} \sum_{a=1}^{q} \sum_{1 \leq b \leq qx} (1 + (-1)^{a+b})(a-b)^{2k} \]

\[ = \frac{1}{2} \sum_{a=1}^{q} \sum_{1 \leq b \leq qx} (-1)^{a+b}(a-b)^{2k} \]

\[ = \frac{1}{2} \sum_{i=0}^{2k} C_{2k}^{i} (-1)^{i} \left( \sum_{a=1}^{q} \sum_{1 \leq b \leq qx \mod q} (a^{2k-i}b^{i}) - \sum_{a=1}^{q} \sum_{b \equiv 1 \mod (q-2n)} (-1)^{a+b} a^{2k-i}b^{i} \right) \]

\[ = \frac{1}{2} \sum_{i=0}^{2k} C_{2k}^{i} (-1)^{i} \left( \frac{x\phi(q)q^{2k}}{(i+1)(2k-i+1)} + O(q^{2k+\frac{1}{2}} \ln^2 q) \right) + \]

\[ + O \left( \sum_{i=0}^{2k} C_{2k}^{i} q^{2k+\frac{1}{2}} \ln^2 q \right) \]

\[ = \frac{x\phi(q)q^{2k}}{2} \sum_{i=0}^{2k} \frac{C_{2k}^{i} (-1)^{i}}{(i+1)(2k-i+1)} + O \left( q^{2k+\frac{1}{2}} \ln^2 q \right) \]

\[ = \frac{x\phi(q)q^{2k}}{2(2k+1)(2k+2)} \sum_{i=0}^{2k} (-1)^{i} C_{2k+2}^{i+1} + O \left( q^{2k+\frac{1}{2}} \ln^2 q \right) \]

\[ = \frac{x\phi(q)q^{2k}}{2(2k+1)(2k+2)} \left( - \sum_{i=0}^{2k+2} (-1)^{i} C_{2k+2}^{i+2} + 2 \right) + O \left( q^{2k+\frac{1}{2}} \ln^2 q \right) \]
\[
\frac{x \phi(q) q^{2k}}{2(2k + 1)(2k + 2)} \left((-1 - 1)^{2k + 2} + 2\right) + O\left(q^{2k + \frac{1}{2} \ln^2 q}\right) = \frac{x \phi(q) q^{2k}}{2(2k + 1)(2k + 2)} + O\left(q^{2k + \frac{1}{2} \ln^2 q}\right).
\]

This is the conclusion of Theorem.

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