MatchKAT: An Algebraic Foundation For Match-Action

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Abstract
We present MatchKAT, an algebraic language for modeling match-action packet processing in network switches. Although the match-action paradigm has remained a popular low-level programming model for specifying packet forwarding behavior, little has been done towards giving it formal semantics. With MatchKAT, we hope to embark on the first steps in exploring how network programs compiled to match-action rules can be reasoned about formally in a reliable, algebraic way. In this paper, we give details of MatchKAT and its metatheory, as well as a formal treatment of match expressions on binary strings that form the basis of “match” in match-action. Through a correspondence with NetKAT, we show that MatchKAT’s equational theory is sound and complete with regards to a similar packet filtering semantics. We also demonstrate the complexity of deciding equivalence in MatchKAT is PSPACE-complete.

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1 Introduction

The match-action paradigm has remained a popular low-level programming model for specifying packet forwarding behavior in network switches. In this model, a switch is organized as one or more match tables in sequence, each containing rules with patterns and actions. The pattern is some match specification on the binary data fields in a packet header, such as a ternary expression containing 0, 1 or don’t-care. The action is some modification on the packet header. When a packet arrives at a match table, a rule is selected among those with matching patterns and the associated action is executed. The selection criterion could be some pre-configured priority ordering on the rules, or based on some property of the pattern such as selecting the one with the fewest don’t-cares (longest prefix matching) [13].

There are efficient hardware implementations of match-action [12], and it is a simple model accepted by network programmers. Nevertheless, high-level domain specific languages (DSLs) such as NetKAT [1] and P4 [2] are available to provide abstractions for network policies that can then be compiled down to match-action tables in the target switch [3, 14, 15]. Despite much theoretic work surrounding these DSLs, there has been comparatively little investigation towards putting match-action itself on a firm theoretical foundation.

Towards the goal of formalizing match-action, we present MatchKAT, a Kleene algebra with tests (KAT) that employs match expressions on binary strings as tests. It is able to encode match and action while having a metatheory closely related to NetKAT. Leveraging results from NetKAT, we are able to show MatchKAT is sound and complete with respect to its own packet filtering semantics. Through a translation to NetKAT, decision procedures such as those in [5] can also be adapted to MatchKAT. Although this paper will mainly introduce the basics of MatchKAT and its metatheory, the application-level motivation is that in the future we may be able to give a formal semantics for match-action as used in network switches. It is hoped that MatchKAT will eventually allow for algebraic reasoning.
on local switch configurations similar to NetKAT for global network policies, which could allow applications such as proving the equivalence of match-action switch configurations and decompiling match-action rules to higher-level policies. Previous attempts at reasoning with match expressions on binary strings in the context of packet classification, such as in [8, 9], have been more ad hoc and without a formal metatheory.

Our contributions can be summarized as follows:

- We give an algebraic formalization of ternary (0, 1, don’t-care) match expressions on binary strings (Section 2.2). Although others have studied aspects of the theory of match expressions, for example [9], we present it here in a formal algebraic language as match expressions will be integral to the formalization of MatchKAT.
- We give the syntax and a packet filtering semantics for MatchKAT (Section 3), and show how it is able to encode match-action (Section 3.4). Despite being related to NetKAT, MatchKAT is able to encode operations that would require much longer expressions in NetKAT.
- We show that MatchKAT has a sound and complete equational theory with respect to its semantics by leveraging a correspondence with the dup-free fragment of NetKAT (Sections 4 and 5). The problem of deciding equivalence between MatchKAT terms is shown to be PSPACE-complete (Section 5.1).

2 Preliminaries

In this section we give some background on KATs, as well as a formal presentation of match expressions on binary strings. We will defer discussion on NetKAT to Section 4 when we clarify its connection with MatchKAT.

2.1 Kleene Algebras with Tests

A Kleene algebra with tests (KAT) [10] has a signature \((P, B, +, \cdot, *, 0, 1, \bar{)}\) such that

- \((P, +, *, 0, 1)\) is a Kleene algebra.
- \((B, +, \cdot, 0, 1)\) is a Boolean algebra.
- \((B, +, *, 0, 1)\) is a subalgebra of \((P, +, *, 0, 1)\).

\(P\) is usually called the set of primitive actions while members of \(B\) are primitive tests. Note that 0 and 1 are the identities of \(+\) and \(\cdot\) respectively and 0 is an annihilator for \(\cdot\). Terms of the KAT are then freely generated by \(P\) and \(B\) with the operators. We omit most of the algebraic theories here as they are well-covered in literature [4, 10, 11]. We will however highlight that KATs can possess interesting equational theories, as we will be studying later. It is possible to axiomatically derive equivalences between KAT terms, as well as assign some denotational semantics to them. We say that the equational theory is sound with respect to those semantics if all provably equal terms have equal semantics, and complete if proofs of equivalence exist for any two terms that are semantically equal. The decision problem of whether two terms are equal can also be studied and its complexity classified. Since KATs can often be used to encode programs, the equational theory is important for studying program equivalence.

2.2 Match Expressions

We give a formalization of match expressions on binary strings as found in match-action tables implemented in network switches. These expressions will form the tests within MatchKAT.
\[ E_0 ::= \bullet \mid \bot \mid E_0 + E_0 \mid E_0 \cap E_0 \mid \bar{E}_0 \]

\[ E_{n+1} ::= E_n @ 1 \mid E_n @ 0 \mid E_n @ x \mid \bot \mid E_{n+1} + E_{n+1} \mid E_{n+1} \cap E_{n+1} \mid \bar{E}_{n+1} \]

\[ \top_0 \triangleq \bullet \quad \top_n \triangleq \underbrace{x \cdots x}_n \text{ for all } n > 0 \]

**Figure 1** Syntax of match expressions.

\[
\mathbf{0} = 1 \quad 1 + 0 = 0 + 1 = x \quad 1 \cap 0 = 0 \cap 1 = \bot
\]

\[ \bullet e = e \bullet = e \quad \bot e = e \bot = \bot \quad \bar{e} \top = e \top n_2 + \top n_1 \bar{e} \top
\]

\[
e_1 (e_2 + e_3) = e_1 e_2 + e_1 e_3 \quad e_1 (e_2 \cap e_3) = e_1 e_2 \cap e_1 e_3
\]

\[
(e_1 + e_2) e_3 = e_1 e_3 + e_2 e_3 \quad (e_1 \cap e_2) e_3 = e_1 e_3 \cap e_2 e_3
\]

**Figure 2** Axioms for match expressions for \( e_1, e_2, e_3 \in E \), \( e \in E_{n_1} \), and \( e' \in E_{n_2} \). These are in addition to axioms that enforce \( \langle E_n, +, \cap, \bot, \top_n \rangle \) for each \( n \) as Boolean algebras.

The set \( E \) will be the set of all match expressions that we will define. The syntax of expressions is found in Figure 1. \( E \) is equipped with a concatenation operation \( @ \) and is stratified into subsets \( E_n \) for all \( n \geq 0 \), such that \( E \triangleq \bigcup_n E_n \). Each \( E_n \) is said to be the set of match expressions with width \( n \), and has an algebraic signature \( \langle E_n, +, \cap, \bot, \top_n \rangle \). + is union, \( \cap \) is intersection, \( \bot \) is complementation, and \( \bot \) and \( \top_n \) are identities of + and \( \cap \) respectively. Terminology-wise, we will refer to the size of binary strings to be matched on as the width, reserving the word length for later quantifying the size of match expressions themselves.

The actual members of the set \( E_n \) are defined inductively on the width \( n \). In the base case, there are the empty \( \bullet \) and bottom \( \bot \) expressions. Note that we distinguish between the empty expression and the empty binary string \( \epsilon \). \( E_{n+1} \) is then built from members of \( E_n \) with concatenation \( @ \). Notationally we will usually elide this operator.

Intuitively, \( 1, 0 \) and \( x \) will correspond to matching 1, 0 or anything (don’t-care) at a given position in the binary string, \( \bullet \) is for matching \( \epsilon \), and \( \bot \) matches nothing. We use \( x \) to avoid confusion with the * operator of KATs. This intuition of an expression matching bits will be made formal shortly. Since \( E_n \) at each width has the signature \( \langle E_n, +, \cap, \bot, \top_n \rangle \), it is also extended freely with expressions built from \( +, \cap \) and \( \bot \).

Axiomatically, for every \( n \) we require \( \langle E_n, +, \cap, \bot, \top_n \rangle \) to be a Boolean algebra. The Boolean algebra axioms determine the behavior of \( \bot \) and \( \top_n \) when combined with the Boolean operators. However, additionally we also need axioms that relate 1, 0, \( x \) and concatenation. They are found in Figure 2. Note that axiomatically all expressions concatenated with \( \bot \) collapse to just \( \bot \). It is therefore unnecessary to distinguish \( \bot \)s of different widths. On the other hand \( \top_n \) is syntactic sugar for the wildcard expression matching any string of width \( n \), and there is a distinct such expression for each \( n \).

To formalize the semantics of match expressions, we come back to the notion of width and length as mentioned at the start of this section. Let \( 2^n \) be the set of binary strings of width \( n \), and in particular let \( 2^0 = \{ \epsilon \} \) be the set only containing the empty string \( \epsilon \). An expression \( e \in E_n \) is said to have width \( n \) and matches strings in \( 2^n \).
We can model what it means for a match expression to match a binary string by interpreting an expression as the set of all strings that match it. For some expression \( e \in E_n \), its interpretation \( \{e\} \) is the set of all strings in \( 2^n \) that matches \( e \). The definition of \( \{e\} \subseteq 2^n \) is made inductively on \( e \):

\[
\begin{align*}
\{\bullet\} &\triangleq \{e\} \\
\{0\} &\triangleq \{0\} \\
\{\bot\} &\triangleq \emptyset \\
\{1\} &\triangleq \{1\}
\end{align*}
\]

\[
\begin{align*}
\{e + e'\} &\triangleq \{e\} \cup \{e'\} \\
\{e \cap e'\} &\triangleq \{e\} \cap \{e'\}
\end{align*}
\]

**Example 1.** Since \( \top_n \triangleq \underbrace{x \ldots x}_n \) for \( n > 0 \), \( \{\top_n\} = 2^n \) by derivation from the definitions \( \{x\} \) and \( \{ee'\} \).

For any \( S \subseteq 2^n \), we say that \( e \) captures \( S \) if and only if \( \{e\} = S \). When reasoning with binary strings, we often wish to refer to individual bits within the string. For \( b \in 2^n \), we write \( b[i] \) for the \( i \)-th bit of \( b \), and \( b[i \leftarrow x] \) for the string that is \( b \) but with \( x \) as the \( i \)-th bit. Conventionally, we will use base-1 for bit indices, strings are read left-to-right, and the most significant bit is to the left whenever a string is interpreted as a binary number.

Therefore it can be seen that a match expression has the same width as the strings it matches. On the other hand, its length could be arbitrary in size, and it is a measure of the complexity of the expression.

**Example 2.** Suppose we are interested only in counting occurrences of \( \cap \) and \( + \). For some even width \( 2n \), consider the expression

\[
\prod_{i=1}^{n} (\top_{i-1} 0 \top_{n-i-1} 1 \top_{n-1-i} + \top_{i-1} 1 \top_{n-1-i} 1 \top_{n-i}).
\]

It captures exactly the set \( \{bb \mid b \in 2^n\} \) and its length is \( O(n) \) since that many \( \cap \) and + operators were used. An equivalent expression that captures the same set is

\[
\sum_{i=1}^{n} (\top_{i-1} 1 \top_{n-i-1} + \top_{i-1} 1 \top_{n-1-i} 1 \top_{n-i}),
\]

which is also length \( O(n) \). However, if we are only allowed to use +, but not \( \cap \) and complementation, an expression capturing this set must have length at least exponential in \( n \). This is because each string of the form \( bb \) must occur in the match expression explicitly.

The same match expression could have different lengths depending on which operators we are interested in counting. This is useful for the application of relating match expression length to the complexity of a match program in a network switch. Some operations may be expensive, such as \( \cap \), while concatenation can be “free” and do not need to be counted as it is simply multiple hardware units placed in parallel.

We end the discussion on match expressions by speaking briefly on the soundness and completeness of the equational theory of match expressions with respect to the binary strings model. Proving soundness is a straightforward albeit tedious task. We simply go through each axiom and show that the expressions on both sides of the equality capture the same set. Completeness is also fairly easy. We can decide whether two match expressions \( e \) and \( e' \) are equivalent by expanding both to their disjunctive normal forms and then eliminate all occurrences of \( \cap \). Equality can then be checked if the expressions are identical up to commutativity of \( + \). Unfortunately, this axiomatic proof of equivalent introduces an exponential blowup. A more tractable, co-NP decision procedure is to non-deterministically guess a string in the symmetric difference of \( \{e\} \) and \( \{e'\} \), which succeeds if and only if \( e \) and \( e' \) are not equivalent.
3 MatchKAT

Our discussion of MatchKAT starts with the intuition that each width-$n$ space of match expressions $E_n$ can be seen as a Boolean algebra over $n$ variables. A binary string corresponds to an assignment of truth values and a match expression is a propositional formula that is satisfied by exactly the assignments of matching binary strings. This is an alternative way to think of the underlying model that we are working with as our definition of MatchKAT evolves.

3.1 Definitions

Let $n$ be a constant positive integer, which as before was used to denote the widths of binary strings, but now we will refer to it as the packet size. Intuitively, MatchKAT is a KAT whose terms operate on the finite state space created by $n$ bits of random access memory occupied by a packet header. It is defined by:

- Primitive tests are match expressions in $E_n$, matching the whole memory at once. For $1 \leq i \leq n$ and $k \in \{0, 1\}$, we will adopt the shorthand $i \simeq k$ for the match expression $\top_{i-1} k \top_{n-i}$, which solely tests whether the $i$-th bit is $k$.

- Primitive actions are in the form $i \leftarrow k$, for $1 \leq i \leq n$ and $k \in \{0, 1\}$, intended to mean assigning 0 or 1 to bit $i$.

- The operations are plus $+$, composition $\cdot$, complementation $\bar{p}$, and Kleene star $p^*$. For tests, $+$ and $\cdot$ correspond respectively to $\top$ and $\cap$ within match expressions $E_n$ (not concatenation within $\mathbb{E}$). Sometimes we may write composition as $\cap$ between terms that are known to be tests.

- The identity of $+$ is $\bot$, and for $\cdot$ it is $\top_n$, or just $\top$ for short.

We admit all the axioms required of a KAT, and those of match expressions presented previously. This is already a sufficient definition for a valid KAT. However, we require additional packet algebra axioms in order to allow commutation of actions and tests on unrelated memory locations, and absorption of related ones. For $i \neq j$:

$$ i \leftarrow k \cdot j \leftarrow k' \equiv j \leftarrow k' \cdot i \leftarrow k \quad i \leftarrow k \cdot i \simeq k \equiv i \simeq k $$

We use $\equiv$ to denote the equivalence of terms in order to avoid ambiguity with $\simeq$ and $\equiv$. Readers familiar with NetKAT may wonder why we do not require axioms of the forms $i \simeq k \cdot i \simeq k \equiv i \simeq k$, $k \neq k' \implies i \simeq k \cdot i \simeq k' \equiv \bot$, and $\sum_k i \simeq k \equiv \top$. These are derivable theorems within the algebra of match expressions.

Example 3. If $k \neq k'$, then

$$ i \simeq k \cdot i \simeq k' \equiv (\top_{i-1} k \top_{n-i}) \cap (\top_{i-1} k' \top_{n-i}) \equiv \top_{i-1} (k \cap k') \top_{n-i} \equiv \bot $$

as $k \cap k' \equiv \bot$.

3.2 Packet Filtering Semantics

We now discuss the semantics of MatchKAT as applied to packet forwarding. Naturally, the $n$ bits of state we have in mind will be modeled by packet headers, which we will just refer to as packets. The following semantics operate on sets of packets at both input and output, intending to model the packets that arrive at a switch and what packets will be forwarded after filtering by the MatchKAT term. We denote the set of packets as $Pk$, which we will
represent as strings in $2^n$ (so really $P_k = 2^n$). The semantics of a MatchKAT term $e$ is a function $\llbracket e \rrbracket : \mathcal{P}(P_k) \to \mathcal{P}(P_k)$:

- $\llbracket \bot \rrbracket (P) \triangleq \emptyset$
- $\llbracket \top \rrbracket (P) \triangleq P$
- $\llbracket a \rrbracket (P) \triangleq P \cap \{a\}$, $a \in E_n$
- $\llbracket i \leftarrow k \rrbracket (P) \triangleq \{\pi | i \leftarrow k | \pi \in P\}$
- $\llbracket p + q \rrbracket (P) \triangleq \llbracket p \rrbracket (P) \cup \llbracket q \rrbracket (P)$
- $\llbracket p \cdot q \rrbracket (P) \triangleq (\llbracket q \rrbracket \circ \llbracket p \rrbracket) (P)$
- $\llbracket \tilde{p} \rrbracket (P) \triangleq P_k - \llbracket p \rrbracket (P)$
- $\llbracket p^* \rrbracket (P) \triangleq \bigcup_{k \geq 0} \llbracket p \rrbracket^k (P)$

We call this the packet filtering semantics as the semantic functions are transformers on sets of packets. Suppose a network switch is modeled by a MatchKAT term, the output denotes the set of packets that is produced given some set of input packets. The next sections will give examples of how MatchKAT terms can be used in practice, while later in Section 5 we will show the equational theory of MatchKAT is sound and complete with respect to this semantics, and deciding equivalence is PSPACE-complete.

### 3.3 Encoding Actions on Packets

In match-action, “match” refers to matching of binary data in packet headers, which we have covered so far. On the other hand, “actions” in this context refer to simple modifications of the packet header. Once a rule is matched, its action is performed and the switch then forwards (or keeps on processing) the packet based on the updated header fields. For example, there may be a port field specifying the egress port the packet should be moved. It is possible to encode modifications on fields in MatchKAT.

#### 3.3.1 Direct and indirect assignment/test.

Assignment/test of a constant value over a range of bits can be performed by assigning/testing the value’s binary representation.

▶ **Example 4.** Assigning the value 6 (binary 110) to bits 2 through 4 can be written as $2 \leftarrow 1 \cdot 3 \leftarrow 1 \cdot 4 \leftarrow 0$.

Test and assignment of a range of bits against another range can be done in a single match expression bitwise.

▶ **Example 5.** To assign the values contained in bits 1 through 3 to bits 4 through 6, we can write

$$
(1 \simeq 0 \cdot 4 \leftarrow 0 + 1 \simeq 1 \cdot 4 \leftarrow 1) \\
\cdot (2 \simeq 0 \cdot 5 \leftarrow 0 + 2 \simeq 1 \cdot 5 \leftarrow 1) \\
\cdot (3 \simeq 0 \cdot 6 \leftarrow 0 + 3 \simeq 1 \cdot 6 \leftarrow 1).
$$

We can simply replace $\leftarrow$ with $\simeq$ above instead to test for equality.

#### 3.3.2 Arithmetic.

Since we know in advance the packet size $n$, and the range of bits to operate on, we can encode arithmetic on sets of bits in MatchKAT through simple fixed-width algorithms. We give incrementation just as an example.
Example 6. Suppose a range of bits contains a binary value we wish to increment. We write \([i \ldots j]^{++}\) for the term that increments the value contained in bits \(i\) through \(j\). It can be defined inductively as:

\[
[i \ldots j]^{++} \triangleq \begin{cases} 
\top & j < i \\
0 \cdot j \leftarrow 1 + j \cdot 0 \leftarrow 0 \cdot (i \ldots j - 1)^{++} & \text{otherwise}
\end{cases}
\]

3.4 Encoding Match-Action Tables

In real match-action tables, match patterns and actions are paired in rules. A single rule can be easily encoded in MatchKAT as the composition of a test with actions. Less straightforward is capturing the rule selection mechanism of the table. For example, let match expressions be \(b_1 \ldots b_k\) and actions \(p_1 \ldots p_k\). In a table with rules \((b_1 p_1) \ldots (b_k p_k)\), we may have multiple \(b\) expressions matching an incoming packet. In a priority-ordered table, the rule that is actually selected and has its action executed is based on some pre-assigned priority ordering on the rules. Here suppose \(1\) is the highest priority and \(k\) the lowest. A naive MatchKAT encoding of the table as \(b_1 p_1 + \cdots + b_k p_k\) does not work, since in a KAT + is commutative.

To impose an order, the simplest way is to negate all higher-priority tests:

\[b_1 p_1 + b_1 b_2 p_2 + \cdots + b_1 \ldots b_{k-1} b_k p_k.\]

This term contains \(O(k)\) sums and \(O(k^2)\) compositions. Albeit inefficient, in this case indeed a rule’s action will only be executed if no higher-priority rule matched.

An alternative encoding is to set aside some metadata bits as a counter to record the current rule being matched. Suppose this counter resides in bits \(i\) through \(j\), then using incrementation from the previous section, we can write:

\[
([i \ldots j] \leftarrow 1) \left[ \sum_{r=1}^{k} ([i \ldots j] = r \cdot (b_r p_r \cdot [i \ldots j] \leftarrow (k + 1) + b_{1} [i \ldots j]^{++})) \right]^*. 
\]

Here we write \([i \ldots j] = r\) as shorthand for testing the range bitwise for the binary number \(r\). The encoded term works by only testing rule \(r\) if \([i \ldots j]\) has value \(r\). If \(b_r\) succeeds then action \(p_r\) is executed, and the rule counter is set to the end value \(k + 1\). If \(b_r\) fails then the rule counter is incremented. Kleene star is used to iterate through all the rules.

The above examples are not the only possible ways to encode match-action tables in MatchKAT. However, since we will prove that the equational theory of MatchKAT is sound and complete with respect to its packet filtering semantics, in principle we should be able to prove equivalence between all possible valid encodings. Even though different encodings have equivalent semantics, they may have different implementation qualities such as the length of match expressions, the depth of nesting, and the use of additional bits to store metadata such as \([i \ldots j]\) in the example above. Nevertheless we can establish a notion of program equivalence between these two ways of representing a table of match-action rules.

In some network switches there exist more than one match-action table organized in a pipeline [2, 13]. The tables can be sequentially composed, or possibly be in parallel with branching and loops. These can all be handled by MatchKAT’s \(\cdot\) for sequencing, \(+\) for parallelism or branching, and \(*\) for loops.

4 Connection with NetKAT

NetKAT is an algebraic language based on Kleene algebra with tests that is able to specify packet forwarding policies in a network [1]. Before we study the equational theory of
MatchKAT, we will precisely define a connection between MatchKAT and NetKAT in both a syntactic and also semantic sense. This will allow us to leverage known results about NetKAT in the MatchKAT setting. Syntactically, there is a correspondence between MatchKAT and the dup-free fragment of NetKAT, and we will elaborate on this shortly. Semantically, NetKAT is mainly concerned with the possible progressions of a packet through the network, whereas we are more interested in the behavior of a single, local switch on packets. The syntactic and semantic relationships are entirely consistent.

NetKAT can be used in NetKAT to record the states of a packet at different hops, so it is natural that without dup, we instead reason about what happens on the local hop. This is referred to in [15] as the “local program”, where the switch configuration is still in NetKAT but agnostic about the network topology. However, we emphasize that MatchKAT is not intended to serve the same purpose as NetKAT. The language instead focuses on lower-level match expressions and manipulation of bits as this is closer to what is implemented in hardware.

We give a short description of NetKAT’s syntax and its axioms, but since NetKAT is well-presented elsewhere, we will not discuss too many details here. What we will see at by the end of this section, however, are mutual translations between MatchKAT and NetKAT that will come in useful when we study MatchKAT’s equational theory.

4.1 Syntax and Axioms of NetKAT

Let $F = \{f_1, f_2, \ldots, f_n\}$ be some fixed, finite set of fields. NetKAT is a KAT again with signature $(P, B, +, \cdot, \ast, 0, 1, \bar{\cdot})$ whose primitive tests $B$ and actions $P$ are defined with respect to $F$:

- In addition to 0 and 1, primitive tests are of the form $f_i = k$, for some natural number $k$ and $f_i \in F$.
- There is a special primitive action named dup. Other primitive actions are in “assignment” form $f_i \leftarrow k$.

We assume for each field $f_i$ there exists a finite set of natural numbers that could be associated with the field. Hence a NetKAT term is not well-formed if it contains $f_i = k$ or $f_i \leftarrow k$ for $k$ not in that set. Just like MatchKAT, in addition to the standard KAT axioms, NetKAT requires packet algebra axioms governing mainly when tests and actions can commute. They can be found in [1] and it suffices for us to say that they are similar to those in MatchKAT, except for one additional axiom involving dup.

We highlight the fact that tests and actions in NetKAT involve constant values. At first glance this may appear more limited than MatchKAT’s ability to perform indirect assignment and computation on fields as demonstrated previously. We point out that this is only possible in MatchKAT’s case since the size $n$ of the state space is known and we are performing fixed-width arithmetic. Although we will see later that there is a close connection between the two, this difference in focus between NetKAT and MatchKAT means they are still separate languages dealing with different levels of abstraction of network programs.

4.2 Semantics

We will talk briefly about the semantics of NetKAT, while readers interested in a formal detailed treatment are invited to read [1]. In NetKAT, a packet is a record of field-value pairs \( \{f_1 = k_1, \ldots, f_n = k_n\} \) where each field has a valid assignment of values. This represents the header of a real-life packet that is of interest when we are deciding on its forwarding behavior. A packet history is simply a list of packets with the head being the most recent.
Definition 7. Let $H$ be the set of packet histories. For $\pi \in Pk$, we write $\pi :: ()$ for the packet history with $\pi$ at its head and nothing else, and $hd$ to be the function that takes packet history to their head packets. When we conflate notation and write $hd H$ for $H \subseteq H$, we mean the set $\{hd h \mid h \in H\}$.

In NetKAT’s packet filtering semantics, the interpretation of a term $e$ is a function $[e] : H \to \mathcal{P}(H)$. Composition of these functions is done through Kleisli composition in the powerset monad. The semantics can be thought of as the behavior of a switch when it is presented with the head packet in a packet history. Each packet in the history represents a previous state of the head packet, possibly at a previous switch in the network. Using packet histories, as opposed to simply packets, allows us to distinguish packets that have taken different paths in the network. However, the input history beyond the head packet cannot be accessed directly by NetKAT terms, consistent with a switch not being able to see the operations that previous switches have done to the packet.

The semantics of NetKAT can be explained intuitively. $\bot$ lets a packet through unchanged, while $\top$ drops the packet. $f = k$ and $f \leftarrow k$ tests and assigns the field $f$ with the value $k$ respectively, in the head packet of the input packet history. $\text{dup}$ duplicates the current head packet and places a copy of it at the head of the history, i.e.

$$\text{dup}(\pi :: h) \triangleq \{\pi :: \pi :: h\}.$$ 

Note also that the codomain of the semantic function is sets of packet histories. This accommodates the fact that it is possible for a switch to egress multiple packets in response to a packet at ingress, possibly different in content and to different destinations. Composition $\cdot$ of interpretations having type $H \to \mathcal{P}(H)$ is done through Kleisli composition in the powerset monad, in contrast to function composition in MatchKAT. $+$ and $\bar{}$ becomes union and complementation in the result sets respectively, and $\ast$ takes the usual meaning of iterated composition.

Example 8. Suppose the set of fields is $\{\text{pt}, \text{proto}, \text{ttl}\}$ and $\text{pt}$ is understood to be the switch port where the packet is located. The NetKAT term $\text{pt} = 1 \cdot \text{proto} = 6 \cdot \text{dup} \cdot \text{ttl} \leftarrow 40 \cdot \text{pt} = 3$ is the policy “If the packet is at port 1 and has $\text{proto}$ value 6, take a snapshot of its current state, change the $\text{ttl}$ value to 40 and move the packet to port 3. Otherwise drop the packet.”

4.3 MatchKAT to NetKAT

We will now formally define a translation from MatchKAT to NetKAT. For a MatchKAT with packet size $n$, the corresponding NetKAT will be over $n$ fields, $f_1$ through $f_n$, each taking 0 or 1 in value. We define a homomorphism $[\cdot]$ that takes terms in this MatchKAT to the corresponding NetKAT terms as follows:

$$[\bot] \triangleq 0 \quad [\top] \triangleq 1 \quad [i \leftarrow k] \triangleq f_i \leftarrow k$$

The definitions for $+, \cdot, \ast$ and $\bar{}$ terms extend homomorphically, i.e.

$$[e + e'] \triangleq [e] + [e'] \quad [e \cdot e'] \triangleq [e] \cdot [e'] \quad [e^\ast] \triangleq [e]^\ast \quad [\bar{e}] \triangleq \overline{[e]}.$$ 

We complete the definition for primitive tests by giving the translation in terms of match expressions on single bits and then concatenation. Translations of more complex match expressions extend naturally from the definitions for $+$ and $\cdot$.

$$[0] \triangleq f_i = 0 \quad [1] \triangleq f_i = 1 \quad [x] \triangleq 1 \quad [e @ e'] \triangleq [e] \cdot [e']$$
Here \( i \) refers to the bit position that the single-bit expression 0 or 1 is matching. We can pre-compute these position values for every 0 or 1 that appears in the expression before carrying out the translation. Notice that the translation does not introduce any \( \text{dup} \), and is a straightforward syntactic embedding into NetKAT. More importantly, this translation is semantic preserving in the following way.

\[ \text{Theorem 9.} \quad \text{For any MatchKAT term } e, \ [e] (P) = \bigcup_{e \in E} \text{hd} \, [\,[e]\,] (\pi :: \langle \rangle). \]

The proof is a standard induction on \( e \). We will simply observe that since the translation introduces no \( \text{dup} \), \( \text{hd} (\pi :: \langle \rangle) = \pi \), and it is clear that \( [e] \) performs the same operations in NetKAT as \( e \) does in MatchKAT.

### 4.4 NetKAT to MatchKAT

Similarly, there is a translation from NetKAT to MatchKAT. Since the latter is \( \text{dup} \)-free, such a translation is forgetful in the sense that we lose the packet history structure entirely and only track the state of the head packet.

Suppose the particular NetKAT we wish to translate from has fields \( f_1 \) through \( f_m \). We assume it is possible to represent the values in each field in binary, and let \( |f_i| \) denote the number of bits required to store \( f_i \). We set the target MatchKAT packet size to be \( n = \sum |f_i| \).

The translation from NetKAT terms to MatchKAT terms is again a homomorphic function \([\cdot]\), and it is only necessary for us to specify its action on the on primitives:

\[ [0] = \bot \quad [1] = \top \quad [\text{dup}] = \top \]

\[ [f_i = k] = \bigcap_{j=1}^{\sum |f_i|} \text{pos}_{j} (f_i) \simeq \text{bin}_{j} (k) \quad [f_i \leftarrow k] = \prod_{j=1}^{\sum |f_i|} \text{pos}_{j} (f_i) \leftarrow \text{bin}_{j} (k) \]

The function \( \text{pos}_{j} (f_i) \) gives bit position for the \( j \)-th bit in the header space allocated for \( f_i \), i.e. it is \( \text{pos}_{j} (f_i) = j + \sum_{i < i} |f_v| \).

On the other hand, \( \text{bin}_{j} (k) \) is the \( j \)-th bit of the binary representation of \( k \). The translation for assignment \( f_i \leftarrow k \) just sets each bit in the space allocated for \( f_i \) in the target MatchKAT bitwise. This is the same method for test \( f_i = k \), with the resulting bitwise tests composed by \( \cap \), and we can always equivalently combine the tests into a match expression without \( \cap \) by using match expression axioms like in Example 3. We forget the existence of \( \text{dup} \) by translating it as \( \top \). Just like the translation to NetKAT, \([\cdot]\) implies a semantic correspondence.

\[ \text{Theorem 10.} \quad \text{For any NetKAT term } e, \ \text{hd} ([e] (h)) = [\,[e]\,] ((\text{hd} h)) . \]

Again the proof proceeds by induction on \( e \), but we will elaborate slightly this time. The base cases are all straightforward by the following reasoning. Both \( 0 \) and \( \bot \) filter out all packet (histories), while \( 1 \) and \( \top \) let through everything. Assignments and tests in both worlds perform the same operations on the (head) packet. \( \text{dup} \) does not change the head packet, and on both sides we only consider the head packets. The inductive cases then rely on \( \text{hd} \) commuting with the semantics of the NetKAT operators \( +, \cdot \) and \( \ast \), which it does since NetKAT terms do not examine or modify packets in the packet history beyond the head.

### 5 Equational Theories of MatchKAT and \text{dup}-Free NetKAT

As promised, we show that the equational theory of MatchKAT is sound and complete with respect to the packet filtering semantics, through borrowing soundness and completeness results of NetKAT’s equational theory from [1].
Consider two NetKAT terms $e$ and $e'$.

Suppose \( \forall h \in H. hd([e](h)) = hd([e'](h)) \),

it is not necessarily the case that \( [e] = [e'] \). Although NetKAT terms cannot access packets beyond the head in the input packet history, \( [e] \) may still produce different output packet histories compared to \( [e'] \) by using \( \text{dup} \). If $e$ and $e'$ are \( \text{dup} \)-free however, we are then able to deduce \( [e] = [e'] \). The equational theory of \( \text{dup} \)-free NetKAT is therefore determined entirely by the operations on the head packet. This idea can be developed into a proof for the soundness and completeness for the equational theory of MatchKAT. First we require two lemmas.

**Lemma 11.** For any MatchKAT expression $e$, $[e] = [[e]].$

**Lemma 12.** For all MatchKAT expressions $e$ and $e'$, $e \equiv e' \iff [e] \equiv [e']$.  

Proofs of these results can be found in the Appendix, with the insight in both being that translations to/from NetKAT preserve equations syntactically and semantically.

**Theorem 13.** (Soundness and completeness.) For all MatchKAT expressions $e$ and $e'$, 

\[ e \equiv e' \iff [e] \equiv [e'] \]  

This follows from the implications

\[
\begin{align*}
\quad & e \equiv e' \
\iff & [e] \equiv [e'] \
\iff & [[e]] = [[e']] \
\iff & [[e]] = [[e']] \
\iff & [e] = [e'] \
\iff & [e] = [e'] \
\end{align*}
\]

This third step follows from Theorem 10 since \( [e] \) and \( [e'] \), being translations from MatchKAT and therefore \( \text{dup} \)-free, have interpretations determined entirely by modifications on the head packet.

### 5.1 Complexity of Deciding Equivalence

In this section, we discuss the complexity of deciding equivalence in MatchKAT, and how the result relates to NetKAT.

**Theorem 14.** Deciding equivalence in MatchKAT is PSPACE-complete.

Membership of PSPACE is argued by translating the MatchKAT terms to the \( \text{dup} \)-free fragment of NetKAT as shown previously. The equational theory of this fragment is in PSPACE since that of NetKAT is in PSPACE [1].

For hardness, we can encode a word problem for a linear-bounded automaton as a MatchKAT term $e$, such that the automaton accepts the given word if and only if $e \not\equiv \bot$.  

The proof is given in the Appendix. The word problem for a linear-bounded automaton is known to be PSPACE-hard [6].

The hardness result in [1] of deciding equivalence in NetKAT relies on a simple translation of regular expressions to NetKAT expressions containing many \( \text{dup} \)s. Our result improves this slightly:

**Corollary 15.** Deciding equivalence of \( \text{dup} \)-free NetKAT terms is PSPACE-complete.

This can be seen through a similar encoding of the linear-bounded automaton.
6 Discussion and Conclusion

We will end by discussing the potential applications and decision procedures of MatchKAT, as the latter will be crucial in any real-world application in reasoning with match-action tables. Efficient procedures for NetKAT have already been discovered, such as in [5, 15], that work well on many real-life cases. Much of the difficult work is in reasoning with \texttt{dup}, which MatchKAT does without. We conjecture that it should be possible to adapt these previous decision procedures to MatchKAT with much simplification. A coalgebraic treatment of MatchKAT directly is also conjectured to be possible.

Application-wise, it is envisaged that MatchKAT could be used to reason about local switch behavior, in contrast to NetKAT on global network policies, when the switch has already been configured by match-action rules. This could be useful for various reasons:

- MatchKAT has a sound and complete equational theory. Equivalence of terms can be decided and is guaranteed to be sound. This helps in the verification of correctness as well as potential configuration optimizations in reducing the number of rules. We have previously talked about the notion of length for MatchKAT terms, and so equivalence of terms of different lengths is potentially proof of equivalence between optimized and unoptimized configurations.

- MatchKAT is equivalent to \texttt{dup}-free NetKAT, and there is a well-defined translation between the two. This could help in decompiling match-action tables to NetKAT in order to make sense of the global policies they are implementing.

- MatchKAT’s match expressions is closer to how bits in packet headers are matched on switches at low-level. MatchKAT could potentially help with efficient implementations of hardware that performs matching.

These all distinguish our work from previous attempts such as [8, 9] that also reasoned with binary data in packet headers theoretically. We also note with interest that other authors have also created new algebraic systems with a strong relationship to NetKAT, such as [7].

In the future, we intend to further develop concrete applications of MatchKAT in the setting of match-action tables, and demonstrate the usefulness of its algebraic theory.

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For all $h$, we have

$$hd[[e]](h) = [[e]]([hdh])$$

by Theorem 10. Hence for all $H \subseteq H$,

$$\bigcup_{h \in H} \left[ e \right](hdH) = \left[ e \right](hdH)$$

On the other hand, Theorem 9 gives us

$$[e](hdH) = \bigcup_{\pi \in hdH} \left[ e \right](\pi :: \langle \rangle)$$
for all $H \subseteq H$. Since $h \approx (hdh) :: \langle \rangle$ for all $h \in H$, it is safe to rewrite the latter equation to 
$$[e] (hdH) = \bigcup_{h \in H} hd [[e]] (h)].$$

Combining this with the second equation gives the required result.

Proof of Lemma 12

$$e \equiv e' \iff [e] \equiv [e']$$

On the left we have MatchKAT terms operating on $n$ bits. On the right are NetKAT terms operating on $n$ fields each containing 1 bit. Through exhaustion we can prove that every axiom in the MatchKAT world gives rise to a corresponding axiom (or derivable theorem) in the NetKAT world, or vice versa, and hence a proof of equality in one produces automatically a proof of equality in the other. Instead of going through the full proof for every axiom, we give some reasons for why it works.

- KAT axioms are clearly present in both worlds, and $[\cdot]$ is a homomorphism.
- The packet algebra axioms are present in both as mentioned in Section 3.1.
- The axioms for manipulating match expressions are present in MatchKAT, but they are not in NetKAT. However NetKAT has extra axioms of the forms $i = k \cdot i = k \equiv i = k$, $k \neq k' \Rightarrow i = k \cdot i = k' \equiv 0$, and $\sum_k (i = k) \equiv 1$. These, along with the axioms of the Boolean algebra, are sufficient to derive equivalents of match expression axioms as theorems.

Proof of Theorem 14

A linear-bounded automaton $M = (Q, \Sigma, \vdash, \dashv, \delta, s, t, r)$ is composed of:

- Finite set of states $Q$.
- Tape alphabet $\Sigma$.
- Left $\vdash$ and right $\dashv$ tape-end markers.
- Transition relation $\delta \subseteq [Q \times (\Sigma \cup \{\vdash, \dashv\})] \times [Q \times (\Sigma \cup \{\vdash, \dashv\})] \times \{L, R\}$.
- Start $s$, accept $t$, and reject $r$ states.

$M$ can be seen as a non-deterministic Turing machine where the tape is finite and marked on both ends by $\vdash$ and $\dashv$. At the start, an input word is present on the tape, while the tape is bound to a linear size $n$ with respect to the length of the input. $\delta$ is restricted such that the end markers are unmodified and the tape head does not move off the ends of the tape. The automaton never transitions out of the accept or reject states once it enters them. We will further restrict $\Sigma$ to two symbols $\{0, 1\}$. This is without loss of generality with a linear increase in the amount of tape required.

A packet in the MatchKAT encoding of $M$ contains the following bits:

- $n$ bits that we will refer to by convenience as $tape_1, \ldots, tape_n$, representing the tape.
- Bits $state_1, \ldots, state_{\log Q}$ to record the binary encoding of the current state.
- Bits $head_1, \ldots, head_{\log n}$ to record the binary encoding of the head position.

Instead of referring to each state and head bit individually, we will assign and test for them collectively for a particular $Q$ state or tape head position. We then construct expressions in the MatchKAT as follows:

- The setup expression $\alpha$, which is an assignment of $state$ with $s$, $head$ with $1$, and the $tape$ fields as appropriate for the initial tape contents for a given input word. .
The transition expression $\beta$, consisting of sums guarded by $\text{state} \times \text{head} \times \text{tape}_\text{head}$ conditions. For a packet in a given configuration, it rewrites it as per one action of the transition relation.

The decision expression $\gamma$, which is just a test for $\text{state} = t$.

Consider the expression $\alpha(\beta^*)\gamma$, which is not equivalent to $\bot$ if and only $M$ accepts the given word. For any non-empty set of input packets, $\alpha(\beta^*)$ constructs the set of all reachable configurations of $M$, while $\gamma$ filters this set to include only the packets that contain the accept state. On the other hand, $\bot$ drops all packets. The size of the expressions are polynomial in the size of the automaton specification.