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Discrete spectrum of the magnetic Laplacian on perturbed half-planes

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Abstract
The existence of bound states for the magnetic Laplacian in unbounded domains can be quite challenging in the case of a homogeneous magnetic field. We provide an affirmative answer for almost flat corners and slightly curved half-planes when the total curvature of the boundary is positive.

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1 INTRODUCTION
We consider the Neumann magnetic Laplacian with constant magnetic field $B = 1$ on an open set $\Omega_\delta$, which is defined through the following (closed) quadratic form:

$$\forall \psi \in H^1_A(\Omega_\delta), \quad Q_\delta(\psi) := \int_{\Omega_\delta} |(-iV + A)\psi|^2 \, dx.$$
FIGURE 1 (a) The corner $C_\delta$ with opening $\pi - \delta$. (b) Its smoothed version $\Omega_\delta$ with curvature $\kappa_\delta = \delta \kappa$, where $\kappa$ is a fixed compactly supported function.

Here, $A = (-x_2, 0)$ is a vector potential associated with $B = 1$ in the sense that

$$\partial_1 A_2 - \partial_2 A_1 = 1.$$ 

Then, we consider the magnetic Laplacian as the self-adjoint operator $\mathcal{L}_\delta$ associated with $Q_\delta$. We denote by $\lambda(\delta)$ the bottom of its spectrum

$$\lambda(\delta) = \inf_{\psi \in H^1_\Lambda(\Omega_\delta), \psi \neq 0} \frac{\|(-i \nabla + A)\psi\|^2_{L^2(\Omega_\delta)}}{\|\psi\|^2_{L^2(\Omega_\delta)}}.$$ 

The study of the spectrum of such operators in different geometries has been the focus of much interest in recent decades. Below, we will give a short overview of results relevant for this article. For more in depth coverage, we refer to [7, 9].

In general, when $\Omega_\delta$ is unbounded, we do not know if $\lambda(\delta)$ is an eigenvalue. For instance, in the case when $\Omega_\delta = \Omega_0 := \mathbb{R} \times \mathbb{R}_+$, it is well known that the spectrum is absolutely continuous and given by the half-line $[\Theta_0, +\infty)$, where $\Theta_0 \approx 0.59$ is a positive universal constant (see (2.2) below). In this article, $\Omega_\delta$ will be a perturbation of the half-plane $\Omega_0$. Let us describe the two types of perturbations that we consider. The first type is a singular perturbation when $\Omega_\delta$ is of the (corner) form

$$\Omega_\delta = C_\delta := \{x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}_+ : \frac{x_2}{\tan \delta} > -x_1\}, \quad \delta \in (0, \pi), \quad (1.1)$$

with the convention $C_{\pi/2} := \mathbb{R}_+ \times \mathbb{R}_+$.

The second type of perturbation under consideration is regular (see Figure 1b). Consider a bounded continuous compactly supported function $\kappa : \mathbb{R} \to \mathbb{R}$ and, for all $\delta \geq 0$, a simple $\mathcal{C}^2$ curve $\gamma_\delta : \mathbb{R} \to \mathbb{R}^2$, parameterized by arc-length and such that its algebraic curvature $\kappa_\delta$ satisfies

$$\kappa_\delta(s) = \delta \kappa(s).$$

Note that $\gamma''(s) = \delta \kappa(s) \mathbf{n}_\delta(s)$, where $\mathbf{n}_\delta(s)$ is the unit normal such that $\det(\gamma''(s), \mathbf{n}_\delta(s)) = 1$. We also assume $\gamma_0(s) = (s, 0), \gamma_0(0) = (0, 0)$, and that $\delta \mapsto \gamma'(s)$ is continuous for the uniform topology at $\delta = 0$. Let us now define our perturbed half-space. We write $\mathbb{R}^2 \setminus \gamma_\delta = \Gamma^+_\delta \sqcup \Gamma^-_\delta$ in such a way that $\mathbf{n}_\delta$ is the inward pointing normal to $\partial \Gamma^+_\delta$. We let $\Omega_\delta = \Gamma^+_\delta$. A typical example of such a configuration is given by $\gamma_\delta(s) = (s, f_\delta(s))$, where $f_\delta(s) = \delta \int_0^s g(\sigma)(1 - \delta^2 g(\sigma)^2)^{-1/2}d\sigma$ and $g(\sigma) = \int_0^\sigma \kappa(x)dx$, with $\delta$ chosen in $(0, \delta_0)$ for a sufficiently small $\delta_0$ to ensure that $\delta g(\sigma) \in (-1, 1)$, in which case, $\Gamma^+_\delta = \{(x, y) \in \mathbb{R}^2 | y > f_\delta(x)\}$.
By the Persson’s theorem (see, for instance, [1]), it is well known in both cases that the essential spectrum of this operator $\mathcal{L}_\delta$ is the same as the spectrum of $\mathcal{L}_0$, that is, $[\Theta_0, +\infty)$. Let us now state our main two results, both establishing the existence of a bound state.

**Theorem 1.1** (Almost flat corner). We assume $\Omega_\delta = C_\delta$. There exists $\delta_0 \in (0, \pi)$ such that, for all $\delta \in (0, \delta_0)$,

$$
\lambda(\delta) \leq \Theta_0 - \frac{C_1^2}{4} \delta^2 + o(\delta^2) < \Theta_0,
$$

where $C_1 > 0$ is a universal constant (defined below in (2.4)). In particular, the bottom of the spectrum of $\mathcal{L}_\delta$ belongs to the discrete spectrum.

In the corner setting, $\lambda(\delta)$ is only known to be an eigenvalue for the nonflat situation when $\delta \in (\frac{\pi}{2} - \varepsilon, \pi)$, where $\varepsilon > 0$ is a constant, see [1]. We even know that $\varepsilon \approx 0.083\pi$ after [5]. Numerical simulations are also given in [1] and suggest that $\varepsilon$ can be taken all the way up to $\pi/2$ but this is far from being rigorously proved. The regime $\delta \to 0$ considered in this article tackles the subtle situation when we know that the discrete spectrum tends to disappear. Let us also underline that it is still an open question to know if $\delta \mapsto \lambda(\delta)$ is monotone nonincreasing (as suggested by the numerical simulations in [1]). If one were able to establish this monotonicity, Theorem 1.1 would imply $\lambda(\delta) < \Theta_0$ for all $\delta \in (0, \pi)$, and thus the existence of a bound state for all (nonflat) convex corners.

We next state the result for the slightly curved half-plane.

**Theorem 1.2** (Slightly curved half-plane). We assume $\Omega_\delta = \Gamma^+ + \delta$ and $\int_R \kappa(s)ds > 0$. There exists $\delta_0 > 0$ such that, for all $\delta \in (0, \delta_0)$, the discrete spectrum of $\mathcal{L}_\delta$ is nonempty. Moreover,

$$
\lambda(\delta) \leq \Theta_0 - \left( \frac{C_1}{2} \int_R \kappa(s)ds \right)^2 \delta^2 + o(\delta^2),
$$

where $C_1$ is the same universal constant as in Theorem 1.1.

Moreover, if $\int_R \kappa(s)ds = 0$ but $\kappa$ does not vanish identically, then

$$
\lambda(\delta) \leq \Theta_0 - (M(\kappa) - \hat{C} \|\kappa\|_2^2) \delta^4 + o(\delta^4),
$$

where $M(\kappa) = -\frac{1}{16} C_1^2 \int_R \kappa(s) |s - s'| \kappa(s') \, ds' > 0$, $(\kappa)_+ = \max\{\kappa, 0\}$ denotes the positive part, and $\hat{C}$ is a constant independent of $\kappa$. In particular, if $M(\kappa) > \hat{C} \|\kappa\|_2^2$, which is easily realizable by scaling, $\lambda(\delta)$ is an element of the discrete spectrum of $\mathcal{L}_\delta$ for small values of $\delta$.

Our analysis suggests that there is only one simple eigenvalue below $\Theta_0 - C \delta^{\frac{1}{2}}$, for some constant $C > 0$. This is likely to follow from a dimensional reduction as in [6, 8] or in the Grushin spirit (see [2]). The question of estimating the number of bound states (below $\Theta_0$) remains open. This requires the derivation of a very precise operator near the threshold of the essential spectrum, that is quite an interesting problem sharing similar features as in [3].

**Remark 1.3.** Actually, Theorem 1.1 can be seen as a *formal* consequence of Theorem 1.2. Indeed, it is possible to exhibit a normal parameterization of $\partial C_\delta$ by considering

$$
\gamma_\delta(s) = (s 1_{R^+}(s) + s \cos \delta 1_{R^-}(s), -s \sin \delta 1_{R^-}(s)).
$$
In the sense of distributions, we have
\[ \gamma''_\delta = (1 - \cos \delta, \sin \delta) b_0 = 2 \sin \left( \frac{\delta}{2} \right) n_\delta b_0, \quad n_\delta = (\sin(\delta/2), \cos(\delta/2)), \]
where \( b_0 \) is the Dirac distribution at 0 and \( n_\delta \) is the direction of the bisector of \( C_\delta \). Formally, the curvature is \( \kappa_\delta = 2 \sin \left( \frac{\delta}{2} \right) b_0 \).

1.1 Organization of the article

In Section 2, we introduce the constants \( \Theta_0 \) and \( C_1 \), related to the de Gennes operator (2.1). Section 3 is devoted to the proof of Theorem 1.1, whereas Section 4 deals with that of Theorem 1.2. In both situations, the proof follows by construction of an appropriate trial state. In the corner case, the phase of the trial state (see (3.5)) is reminiscent of the construction appearing in the nonlinear setting of [4], while the amplitude (see (3.11)) is obtained by minimizing a new energy functional in (3.16). In the regular case, we use a tensorized trial state in curvilinear coordinates (see (4.1)) involving the bound state of a 1D model operator studied in [10] and revisited in Appendix A.

2 THE DE GENNES OPERATOR AND THE CONSTANT \( C_1 \)

The material of this section is standard and only included for convenience and to fix notation. We refer to [7] for more material and reference to earlier works. The constants \( \Theta_0 \) and \( C_1 \) in Theorems 1.1 and 1.2 are defined starting from a family of 1D harmonic oscillators on the half axis. For all \( \xi \in \mathbb{R} \), let us denote by \( \mu(\xi) \) the first eigenvalue of the operator
\[ H_\xi:=-\frac{d^2}{dt^2}+(t-\xi)^2 \text{ in } L^2(\mathbb{R}_+), \quad (2.1) \]
with Neumann boundary condition at 0, \( u'(0) = 0 \).

We introduce the de Gennes constant
\[ \Theta_0 := \inf_{\xi \in \mathbb{R}} \mu(\xi). \quad (2.2) \]
We know that \( \frac{1}{2} < \Theta_0 < 1 \) and there exists a unique \( \xi_0 \) such that
\[ \Theta_0 = \mu(\xi_0). \]
Furthermore, \( \xi_0 = \sqrt{\Theta_0} \) and \( \mu''(\xi_0) > 0 \). Let us denote by \( f_* \) the positive normalized ground state of \( \Theta_0 \), that is,
\[ H_{\xi_0} f_* = \Theta_0 f_* , \quad f_* > 0 , \quad f'_*(0) = 0 , \quad \int_{\mathbb{R}_+} |f_*(t)|^2 dt = 1. \quad (2.3) \]
We introduce the constant \( C_1 \) as follows:
\[ C_1 := \frac{|f_*(0)|^2}{3}. \quad (2.4) \]
The function $f_\star$ belongs to $S(\mathbb{R}_+)$ and decays exponentially at infinity. It satisfies the additional property (Feynman–Hellmann)

$$
\int_{\mathbb{R}_+} (t - \xi_0)|f_\star(t)|^2 \, dt = 0.
$$

(2.5)

Noticing that

$$
\int_0^{+\infty} t|f'_\star(t)|^2 \, dt = -\int_0^{+\infty} (tf'_\star(t))' \, dt = \int_0^{+\infty} (t(-f''_\star(t)) - f'_\star(t)) \, dt
$$

$$
= \int_0^{+\infty} (\Theta_0 t - t(\xi_0 - t)^2)|f_\star(t)|^2 \, dt + \frac{f_\star(0)^2}{2},
$$

we get, using (2.4), the interesting identity

$$
\int_{\mathbb{R}_+} \left(|f'_\star(t)|^2 + (\xi_0 - t)^2|f_\star(t)|^2 - \Theta_0|f_\star(t)|^2\right) \, dt = \frac{3C_1}{2}.
$$

(2.6)

Another interesting identity is

$$
\int_{\mathbb{R}_+} (t - \xi_0)t(t - 2\xi_0)|f_\star(t)|^2 \, dt = \frac{C_1}{2},
$$

(2.7)

which follows by writing

$$(t - \xi_0)(t - 2\xi_0)t = (t - \xi_0)^3 - \xi_0^2(t - \xi_0),$$

using (2.5) and the formula (see [7, Lemma 3.2.7]):

$$
\int_{\mathbb{R}_+} (t - \xi_0)^3|f_\star(t)|^2 \, dt = \frac{C_1}{2}.
$$

For a positive number $\ell$, let us introduce the function

$$
f_{\ell}(t) := \zeta \left( \frac{t}{\ell} \right)f_\star(t),
$$

(2.8)

where $\zeta \in C_0^\infty(\mathbb{R})$ satisfies

$$
0 \leq \zeta \leq 1, \quad \text{supp } \zeta \subset [-1, 1], \quad \zeta = 1 \text{ on } \left[ -\frac{1}{2}, \frac{1}{2} \right].
$$

(2.9)

Consequently, as $\ell \to +\infty$, we have

$$
\int_{\mathbb{R}_+} |f_{\ell}(t)|^2 \, dt = 1 + O(\ell^{-\infty}),
$$

$$
q(f_{\ell}) := \int_{\mathbb{R}_+} \left(|f'_{\ell}(t)|^2 + (t - \xi_0)^2|f_{\ell}(t)|^2\right) \, dt = \Theta_0 + O(\ell^{-\infty}),
$$

(2.10)

$$
\int_{\mathbb{R}_+} (t - \xi_0)^k|f_{\ell}(t)|^2 \, dt = \int_{\mathbb{R}_+} (t - \xi_0)^k|f_\star(t)|^2 \, dt + O(\ell^{-\infty}),
$$

where $O(\ell^{-\infty})$ denotes a quantity equal to $O(\ell^{-N})$ for all $N > 0$. 
Since $f_* \in \mathcal{S}(\mathbb{R}_+)$, we deduce the following two identities from (2.6), which will be useful below in our proof of Theorem 1.1,

\[
\begin{align*}
\int_{\mathbb{R}_+} (|f'_\ell(t)|^2 + (\xi_0 - t)^2 |f_\ell(t)|^2 - \Theta_0 |f_\ell(t)|^2) t dt &= \frac{3C_1}{2} + \mathcal{O}(\varepsilon^{-\infty}), \\
\int_{\mathbb{R}_+} (t - \xi_0)(t - 2\xi_0) |f_\ell(t)|^2 t dt &= \frac{C_1}{2} + \mathcal{O}(\varepsilon^{-\infty}).
\end{align*}
\]

(2.11)

3 \quad ALMOST FLAT SECTORS

This section is devoted to the proof of Theorem 1.1, so $\Omega_\delta = C_\delta$ hereafter. The proof is by construction of a quasi-mode having approximately the form (after truncation)

$$
\psi(t) \approx f_*(t)e^{i\Phi(s,t)},
$$

where $s$ denotes the tangential variable along $\partial C_\delta$ and $t$ denotes the transversal variable. More precisely, $t$ is the distance to $\partial C_\delta$, and the coordinates $(s,t)$ are defined except on the bisector of $C_\delta$. For instance, $(s,t) = (x_1,x_2)$ when $\delta = 0$ (in which case, $C_\delta$ is the half-plane $\mathbb{R} \times \mathbb{R}_+$).

The phase term $\Phi$ is, up to symmetry considerations, a perturbation of $i\xi_0 s$. As already mentioned, the idea of perturbing the phase term in an almost flat sector was first introduced in the nonlinear framework of the Ginzburg–Landau functional [4]. We use the same construction and add to this by proving that the phase term proposed in [4] is rather the optimal choice. Interestingly, we determine the best truncation profile by minimizing a nonlinear functional, which allows us to capture the $\delta^2$-term of Theorem 1.1.

3.1 \quad Geometric framework

We denote by $T^+$ the following trapezoid:

$$
T^+ := \{x \in (0, +\infty) \times (0, \ell) : x_2 \tan(\delta/2) < x_1\}.
$$

Consider the angle bisector

$$
D_\delta = \{x \in \mathbb{R}^2 : x_1 = x_2 \tan(\delta/2)\}.
$$

We denote by $S_\delta$ the reflection in the line $D_\delta$, whose matrix is

$$
S_\delta = \begin{pmatrix}
-\cos \delta & \sin \delta \\
\sin \delta & \cos \delta
\end{pmatrix}.
$$

We denote by $T^-$ the reflection of $T^+$, that is,

$$
T^- := S_\delta T^+.
$$
### 3.2 Toward a test function

Let us try to define a test function compatible with the symmetry and the magnetic field. Let us consider a function $u_+$ such that

$$E_+ := \int_{T^+} |(-i\nabla + A)u_+|^2dx < +\infty .$$

(3.1)

Now, we want to extend $u_+$ by using the symmetry. We wish to do this in such a way that the magnetic energy on $T^-$ coincides with the one on $T^+$.

**Lemma 3.1.** Considering

$$\phi(y) := \frac{\sin(2\delta)}{4}(y_1^2 - y_2^2) - y_1y_2 \sin^2 \delta$$

(3.2)

and

$$u_-(x) := e^{-i\phi(S\delta x)}u_+(S\delta x),$$

(3.3)

we have

$$\int_{T^+} |(-i\nabla + A)u_+|^2dx = \int_{T^-} |(-i\nabla + A)u_-|^2dx .$$

(3.4)

**Proof.** For a given function $u_-$ on $T^-$, we use the change of variable given by $x = S\delta y$ (remember that $S\delta = S\delta^{-1} = S\delta^*$) and notice that

$$\int_{T^-} |(-i\nabla + A)u_-|^2dx = \int_{T^+} |(-i\nabla y + \tilde{A}(y))\circ u_-|^2dy ,$$

with

$$\tilde{A} := S\delta(A\circ S\delta), \quad \circ u_- := u_-\circ S\delta(y) .$$

A straightforward computation gives

$$\tilde{A}(y) = S\delta\begin{pmatrix} -y_1 \sin \delta - y_2 \cos \delta \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{y_1}{2} \sin(2\delta) + y_2 \cos^2 \delta \\ -y_1 \sin^2 \delta - \frac{y_2}{2} \sin(2\delta) \end{pmatrix} .$$

The magnetic field associated with $\tilde{A}$ is

$$\partial_{y_1}\tilde{A}_2 - \partial_{y_2}\tilde{A}_1 = -1 .$$

Let us consider the function $\phi$ defined in (3.2). It satisfies

$$\tilde{A} = \begin{pmatrix} y_2 \\ 0 \end{pmatrix} = \nabla \phi .$$
Therefore,

\[ \int_{T^-} |(-i\nabla + A)u_-|^2 \, dx = \int_{T^+} \left| \left(-i\nabla_y + \left( \begin{array}{c} y_2 \\ 0 \end{array} \right) \right) e^{i\phi(y)} \circ u_- (y) \right|^2 \, dy \]

\[ = \int_{T^+} \left| (-i\nabla_y + A(y)) e^{-i\phi(y)} v_- (y) \right|^2 \, dy, \quad \text{with} \quad v_- := \circ \]

so that (3.4) holds if

\[ e^{-i\phi(y)} \circ u_- (y) = u_+(y), \]

or, equivalently,

\[ u_-(x) = e^{-i\phi(S_\delta x)} \circ u_+(S_\delta x). \] 

Remark 3.2. If we choose

\[ u_+(x_1, x_2) = f(x_2) e^{i\xi_0 x_1}, \]

with a real-valued \( f \) in the Schwartz class, (3.3) gives

\[ u_-(x) = f(x_1 \sin \delta + x_2 \cos \delta) e^{-i\xi_0 (-x_1 \cos \delta + x_2 \sin \delta)} e^{-i\phi(x)}, \]

since \( \phi(S_\delta x) = \phi(x) \). Of course, this choice of \( u_+ \) is not appropriate since (3.1) is not satisfied due to the lack of integrability with respect to \( x_1 \). However, up to using a cut-off function with respect to \( x_1 \), this gives a rather good idea of the shape of our test function in \( T^+ \). Now, if we consider

\[ u(x) = \begin{cases} 
    f(x_2) e^{i\xi_0 x_1}, & \text{if } x \in T^+ \\
    f(x_1 \sin \delta + x_2 \cos \delta) e^{i\xi_0 (x_1 \cos \delta - x_2 \sin \delta)} e^{-i\phi(x)}, & \text{if } x \in T^- 
\end{cases} \]

we see that \( u \) does not belong to \( H^1 \) near the symmetry axis \( D_\delta \) due to the phase shift, which does not vanish on \( D_\delta \). In the next section, we slightly modify this function \( u \) to solve this inconvenience.

### 3.3 Smoothing the transition near \( D_\delta \)

For the given \( \delta \), we let \( 0 < \gamma < \pi - \delta \) and define the following trial state (in polar coordinates):

\[ \Psi(r \cos \theta, r \sin \theta) := \begin{cases} 
    f(r \sin \theta) e^{i\xi_0 r \cos \theta}, & \text{if } (r \cos \theta, r \sin \theta) \in T^+_{\delta, \gamma}, \\
    f(r \sin \theta) e^{i\alpha(r, \theta)}, & \text{if } (r \cos \theta, r \sin \theta) \in V^+_{\delta, \gamma}, \\
    f(r \sin(\theta + \delta)) e^{i\alpha(r, \theta)}, & \text{if } (r \cos \theta, r \sin \theta) \in V^-_{\delta, \gamma}, \\
    f(r \sin(\theta + \delta)) e^{i\xi_0 r \cos(\theta + \delta) - ir^2 \Phi(\cos \theta, \sin \theta)}, & \text{if } (r \cos \theta, r \sin \theta) \in T^-_{\delta, \gamma}, \\
    0, & \text{else}, 
\end{cases} \]

(3.5)

where (cf. Figure 2):
FIGURE 2 Geometric setting: The figure illustrates the sector cut-off at height $\ell$. Here, $T^+_\delta = T^+_{\delta,\gamma} \cup V^+_{\delta,\gamma}$, and similarly for $T^-$. The symmetry axis $D_\delta$ is drawn in blue.

- $V^+_{\delta,\gamma}$ is the sector defined in polar coordinates by $0 < r < r_\ast := \frac{\ell}{\cos \left( \frac{\delta + \gamma}{2} \right)}$ and $\theta \in \left( \theta_\delta - \frac{\gamma}{2}, \theta_\delta \right)$
- $V^-_{\delta,\gamma} := S_\delta V^+_{\delta,\gamma}$, with $\theta_\delta := \frac{\pi - \delta}{2}$;
- the trapezoids $T^+_{\delta,\gamma}$ and $T^-_{\delta,\gamma}$ are given by

$$T^+_{\delta,\gamma} := \left\{ x \in (0, +\infty) \times (0, \ell) : x_2 \tan \left( \frac{\delta + \gamma}{2} \right) < x_1 \right\},$$

and $T^-_{\delta,\gamma} := S_\delta T^+_{\delta,\gamma}$.

The function $f$ is given by $f = f_{\ell}$, see (2.8), and the phase $\alpha$ is chosen so that the function $\Psi$ belongs to $H^1_{\text{loc}}(C_\delta)$. Let us give an explicit choice of $\alpha$. Note that

$$\phi(\cos \theta, \sin \theta) = \frac{\sin \delta}{2} \cos(2\theta + \delta),$$

and consider the two phases

$$\alpha_+(r, \theta) := r \xi_0 \cos(\theta), \quad \alpha_-(r, \theta) := r \xi_0 \cos(\theta + \delta) - \frac{r^2}{2} \sin \delta \cos(2\theta + \delta).$$

Notice that the transition zone is simply given by $\theta \in \left( \theta_\delta - \frac{\gamma}{2}, \theta_\delta + \frac{\gamma}{2} \right)$ and that we have

$$\alpha_+(r, \theta) = r \xi_0 \sin \left( \frac{\delta + \gamma}{2} \right)$$

and

$$\alpha_-(r, \theta) = -r \xi_0 \sin \left( \frac{\delta + \gamma}{2} \right) + \frac{r^2}{2} \sin \delta \cos \gamma.$$ 

This leads to the choice

$$\alpha(r, \theta) = br^2 - \chi_{\delta,\gamma}(\theta)(ar - br^2),$$

(3.6)
with

\[ a = \xi_0 \sin \left( \frac{\delta + \gamma}{2} \right), \quad b = \frac{1}{4} \sin \delta \cos \gamma, \]

and

\[ \chi_{\delta, \gamma}(\theta) = \chi \left( \frac{2(\theta - \theta_{\delta})}{\gamma} \right), \]

where \( \chi \) is a smooth odd function such that \( \chi(t) = 1 \) for \( t \geq 1 \).

**Remark 3.3.** The function \( \Psi \) has no decay in the \( x_1 \)-direction, so it will be complemented by a well-chosen cut-off in that variable.

### 3.4 Estimate of the energy

We consider a function \( \eta_+ \in H^1(\mathbb{R}_+) \) equal to 1 in a fixed neighborhood of 0 (so that \( (x_1, x_2) \mapsto \eta_+(x_1) \) equals 1 on \( V^+_{\delta, \gamma} \) as soon as \( \delta \) and \( \gamma \) are small enough). This function will be chosen later on in Section 3.5.

The aim of this section is to establish the following proposition valid in the regime where \( \ell \to +\infty \) and \( (\delta, \gamma) \to 0 \) provided that \( \delta = o(\gamma) \).

**Proposition 3.4.** For small enough values of \( \delta, \gamma, \ell^{-1} \) and under the assumption that \( \delta \leq \gamma \), we have

\[
\int_{T^+} |\eta_+(x_1)|^2 |(-i\nabla + A)\Psi|^2 \, dx \leq (\Theta_0 + O(\ell^{-\infty})) \|\eta_+\|_{L^2(\mathbb{R}_+)}^2 - J_{\delta/2} \left( \frac{\gamma^3}{2} + O(\delta^2) + O(\gamma^3) + O(\ell^{-\infty}), \right)
\]

with

\[
J := \int_0^{+\infty} \left( |f'|^2 + (t - \xi_0)^2 |f(t)|^2 \right) t \, dt - \int_0^{+\infty} (t - \xi_0)(t - 2\xi_0) t \, |f(t)|^2 \, dt.
\]

The remainder terms \( O(\gamma^{-1}\delta^2), O(\ell^{-\infty}), \) and \( O(\gamma^3) \) are controlled uniformly with respect to the function \( \eta_+ \).

Proposition 3.4 is a consequence of Lemma 3.5, Remark 3.6, and Lemma 3.7.

**Lemma 3.5.** We have the estimate

\[
\int_{V^+_{\delta, \gamma}} |(-i\nabla + A)\Psi|^2 \, dx = \frac{\gamma}{2} J_{\delta, \gamma} + O(\gamma^3 + \gamma \ell^{-\infty}),
\]

where

\[
J_{\delta, \gamma} := \int_0^\infty \left( |f'(t)|^2 + \left(t - \xi_0 \left(1 + \frac{\delta}{\gamma} \right) - \frac{\delta}{2\gamma} \right)^2 |f(t)|^2 \right) t \, dt.
\]
Remark 3.6. Expanding $J_{\delta,\gamma}$, we observe that
\[
\frac{\gamma}{2} J_{\delta,\gamma} = \frac{\gamma}{2} \int_0^{+\infty} (|f'|^2 + (t - \xi_0)^2 |f(t)|^2) \, dt + \frac{\delta}{2} \int_0^{+\infty} (t - \xi_0)(t - 2\xi_0) t |f(t)|^2 \, dt + O(\gamma^{-1}\delta^2),
\]
where we used the exponential decay of $f_*$ to control the remainder.

Proof of Lemma 3.5. We have
\[
\int_{V_+^{\delta,\gamma}} |(-i\nabla + A)\Psi|^2 \, dx = \int_{V_+^{\delta,\gamma}} |(-i\nabla + A_0)\tilde{\Psi}|^2 \, dx, \quad A_0 = \frac{1}{2}(-x_2, x_1),
\]
where $\Psi = e^{-ix_1 x_2/2} \Psi$. We get
\[
\int_{V_+^{\delta,\gamma}} |(-i\nabla + A)\Psi|^2 \, dx = \int_{\Theta_0 - \gamma}^{\Theta_0} \int_0^{r_*} \left( |\partial_r \Psi|^2 + r^{-2} \left( -i \partial_\Theta + \frac{r^2}{2} \right)^2 \right) r \, dr \, d\Theta.
\]
Then, we write
\[
\tilde{\Psi} = e^{i(-r^2 \sin(2\Theta)/4 + \alpha(r, \Theta))} f(r \sin \Theta),
\]
and, using that $f$ is real valued, we find
\[
|\partial_r \tilde{\Psi}|^2 = \sin^2 \Theta |f'(r \sin \Theta)|^2 + (-r \sin(2\Theta)/2 + \partial_r \alpha)^2 |f(r \sin \Theta)|^2,
\]
and
\[
\left| \left( -i \partial_\Theta + \frac{r^2}{2} \right) \tilde{\Psi} \right|^2 = \left( -\frac{r^2}{2} \cos(2\Theta) + \partial_\Theta \alpha + \frac{r^2}{2} \right)^2 |f(r \sin \Theta)|^2 + r^2 \cos^2 \Theta |f'(r \sin \Theta)|^2
\]
\[
= (r^2 \sin^2(\Theta) + \partial_\Theta \alpha)^2 |f(r \sin \Theta)|^2 + r^2 \cos^2 \Theta |f'(r \sin \Theta)|^2.
\]
It follows that
\[
\int_{V_+^{\delta,\gamma}} |(-i\nabla + A)\Psi|^2 \, dx
\]
\[
= \int_{\Theta_0 - \gamma}^{\Theta_0} \int_0^{r_*} |f'(r \sin \Theta)|^2
\]
\[
\quad + \left( (-r \sin(2\Theta)/2 + \partial_\Theta \alpha)^2 + (r \sin^2(\Theta) + r^{-1} \partial_\Theta \alpha)^2 \right) |f(r \sin \Theta)|^2 r \, dr \, d\Theta
\]
\[
= \int_{\Theta_0 - \gamma}^{\Theta_0} \int_0^{r_*} |f'(r \sin \Theta)|^2 + F(r, \Theta, x_{\delta,\gamma}) |f(r \sin \Theta)|^2 r \, dr \, d\Theta,
\]
(3.8)
where
\[ F(r, \theta, \chi_{\delta, \gamma}) = (-r \sin(2\theta)/2 - \chi_{\delta, \gamma}(\theta)(a - 2br) - 2br)^2 + \left( r \sin^2(\theta) - \chi'_{\delta, \gamma}(\theta)(a - br) \right)^2. \]

Notice that the interval of integration in \( \theta \) has length \( \gamma/2 \ll 1 \) and \( \theta \in \left[ \theta_{\delta} - \frac{\gamma}{2}, \theta_{\delta} \right] \) implies
\[ 1 \geq \sin \theta \geq 1 - C(\delta - \gamma)^2, \quad \text{and} \quad 0 \leq \sin(2\theta) \leq \delta + \gamma. \]

Furthermore,
\[ a = \xi_0 \frac{\gamma + \delta}{2} + O(\gamma^3), \quad b = \frac{\delta}{4} + O(\gamma^3) \]
and
\[ \chi'_{\delta, \gamma}(\theta) = \frac{2}{\gamma} \chi' \left( \frac{2(\theta - \theta_{\delta})}{\gamma} \right). \]

Therefore, the first part of \( F(r, \theta, \chi_{\delta, \gamma}) \) is small, and we get, using the decay of \( f \) and \( f' \),
\[
\int_{V^+_{\delta, \gamma}} |(-i\nabla + A)\Psi|^2 \, dx
= \int_{\delta_{\delta} - \frac{\gamma}{2}}^{\delta_{\delta}} \int_0^r \left( |f'(r \sin \theta)|^2 + \left( r \sin^2(\theta) - \chi'_{\delta, \gamma}(\theta)(a - br) \right)^2 |f(r \sin \theta)|^2 \right) r \, dr \, d\theta + O(\gamma^3)
= \int_{\delta_{\delta} - \frac{\gamma}{2}}^{\delta_{\delta}} \int_0^{+\infty} \left( |f'(t)|^2 + \left( t - \chi' \left( \frac{2(\theta - \theta_{\delta})}{\gamma} \right) \left( \xi_0 \left( 1 + \frac{\delta}{\gamma} \right) - \frac{\delta}{2\gamma} t \right) \right)^2 |f(t)|^2 \right) t \, dt \, d\theta
+ O(\gamma^3 + \gamma \epsilon^{\infty})
= \frac{\gamma}{2} \int_0^{+\infty} \left( |f'(t)|^2 + t^2 |f(t)|^2 \right) t \, dt - \gamma \int_0^{+\infty} \left( \xi_0 \left( 1 + \frac{\delta}{\gamma} \right) - \frac{\delta}{2\gamma} t \right) |f(t)|^2 t^2 \, dt
+ \frac{\gamma}{2} \left( \int_{-1}^0 \chi'^2 \, d\theta \right) \int_0^{+\infty} \left( \xi_0 \left( 1 + \frac{\delta}{\gamma} \right) - \frac{\delta}{2\gamma} t \right)^2 |f(t)|^2 t \, dt + O(\gamma^3 + \gamma \epsilon^{\infty}), \quad (3.9)
\]
where we used that \( \chi \) is odd with \( \chi(-1) = -1 \), to get the last equality. Therefore, the optimal choice is that \( \chi(x) = x \) on \([-1, 1]\) and the result follows. \( \square \)

**Lemma 3.7.** We have
\[
\int_{T^+_{\delta, \gamma}} \eta_+ (x_1)^2 |(-i\nabla + A)\Psi|^2 \, dx \leq (\Theta_0 + O(\epsilon^{\infty})) \| \eta_+ \|^2_{L^2(R_+)}
- \frac{\gamma + \delta}{2} \int_0^{+\infty} \left( f'(t)^2 + (t - \xi_0)^2 |f|^2 \right) t \, dt
+ O(\gamma^3) + O(\epsilon^{\infty}).
\]
Proof. We recall that $\Psi(x) = u(x) = f(x_2) e^{i\xi_0 x_1}$, on $T_{\delta,\gamma}^+$, and that $\eta_+ = 1$ on $[0, \varepsilon]$. For small enough $\gamma, \delta$ we can therefore write

$$\int_{T_{\delta,\gamma}^+} \eta_+ (x_1)^2 (-i \nabla + A)\Psi|^2 \, dx = \int_R \eta_+ (x_1)^2 (-i \nabla + A)u|^2 \, dx - E,$$

with $R = (0, +\infty) \times (0, \varepsilon)$ and

$$E = \int_{R \setminus T_{\delta,\gamma}^+} |(-i \nabla + A)u|^2 \, dx.$$

The calculation of $E$ is similar to the beginning of the proof of Lemma 3.5. We have

$$E = \int_{R \setminus T_{\delta,\gamma}^+} |(-i \nabla + A_0)\tilde{u}|^2 \, dx, \quad \tilde{u}(x) = e^{-i x_1 x_2/2} u(x),$$

with $A_0(x) = \frac{1}{2}(-x_2, x_1)$. We let $w(r, \theta) = \tilde{u}(r \cos \theta, r \sin \theta) = f(r \sin \theta) e^{i \left( \xi_0 r \cos \theta - r^2 \frac{\sin(2\theta)}{2} \right)}$ to get

$$E = \int_{\delta - \gamma / 2}^{\gamma / 2} \int_0^{r_*(\theta)} \left( |\partial_r w|^2 + r^{-2} \left| -i \partial_\theta + \frac{r^2}{2} \right|^2 w \right)^2 rdr d\theta, \quad r_*(\theta) = \frac{\ell}{\sin \theta} \geq \ell.$$

Since $f$ is real-valued,

$$|\partial_r w|^2 = \sin^2 \theta |f'(r \sin \theta)|^2 + (\xi_0 \cos \theta - r \sin \theta \cos \theta)^2 |f(r \sin \theta)|^2$$

$$r^{-2} \left| -i \partial_\theta + \frac{r^2}{2} \right|^2 w^2 = \cos^2 \theta |f'(r \sin \theta)|^2 + (-\xi_0 \sin \theta + r \sin^2(\theta))^2 |f(r \sin \theta)|^2.$$

We get

$$E = \int_{\delta - \gamma / 2}^{\gamma / 2} \int_0^{\ell / \sin \theta} \left( |f'(r \sin \theta)|^2 + (\xi_0 - r \sin \theta)^2 |f(r \sin \theta)|^2 \right) rdr d\theta. \quad (3.10)$$

Thus, with the change of variable $r = t / \sin \theta$, we get for all $N > 0$,

$$E \geq \frac{\gamma + \delta}{2} \int_0^{+\infty} (tf'(t)^2 + t(\xi_0 - t)^2 |f|^2) dt - C\gamma^3 - C\gamma \ell^{-N}.$$
Moreover,
\[
\int_R \eta_+(x_1)^2 |(-i \nabla + A)u|^2 \, dx = \int_R \eta_+(x_1)^2 \left( |(-i \partial_1 - x_2)u|^2 + |\partial_{x_2} u|^2 \right) \, dx
\]
\[
= \int_R \eta_+(x_1)^2 (|\xi_0 - x_2)f(x_2)|^2 + |f'(x_2)|^2 \, dx
\]
\[
= ||\eta_+||_{L^2(\mathbb{R}^+)}^2 \int_0^\ell \left( |(\xi_0 - x_2)f(x_2)|^2 + |f'(x_2)|^2 \right) \, dx_2
\]
\[
= (\Theta_0 + \mathcal{O}(\varepsilon^{-\infty})) ||\eta||_{L^2(\mathbb{R}^+)}^2. \quad \square
\]

3.5 | Proof of Theorem 1.1

We have now almost all the elements at hand to establish Theorem 1.1. Let us first truncate the function \( \Psi \) to produce a test function in \( H^1_A(C_\delta) \). We introduce the function

\[ \Psi^{\text{tr}} = \eta \Psi, \]

where

\[ \eta(x) = \begin{cases} 
\eta_+(x_1) & \text{if } x \in T^+ \\
\eta_+(-x_1 \cos \delta + x_2 \sin \delta) & \text{if } x \in T^-, 
\end{cases} \quad (3.11) \]

and \( \eta_+ \in H^1(\mathbb{R}^+) \) is equal to 1 on the interval \((0, \varepsilon)\), for a fixed (and arbitrary) \( \varepsilon > 0 \). The construction of \( \eta \) and \( \Psi^{\text{tr}} \) respects the symmetry considerations in Section 3.2 (since \( \eta \circ S_\delta = \eta \)). Notice that on \( T^+ \),

\[ \Psi^{\text{tr}}(x) = \eta_+(x_1) \Psi(x). \]

Recall that our main (small) parameter is \( \delta \in (0, \pi) \). We have also introduced another small parameter \( \gamma \) with \( 0 < \delta < \gamma \) and the large parameter \( \ell \) in the definition of \( \Psi \). Below we will need the condition that

\[ \ell \delta \to 0. \quad (3.12) \]

We will make the choices

\[ \gamma = \delta^{\frac{1}{2}}, \quad \ell = \delta^{-\frac{1}{2}}, \quad (3.13) \]

the first one ensuring that \( \gamma^{-1} \delta^2 = \gamma^3 \) (see the remainders in Proposition 3.4) and the second one being rather arbitrary. The function \( \eta_+ \) will be chosen to depend on \( \delta \) at the very end of the proof.
3.5.1 Estimates of the $L^2$-norm and of the energy of $\Psi^{tr}$

Let us estimate the $L^2$-norm of our test function $\Psi^{tr}$. For small enough $\delta$, we have by (2.10), and using (3.12) in the second term,

$$
\int_{T^+} |\Psi^{tr}|^2 \, dx = \int_{(0,+\infty)\times(0,\varepsilon)} \eta_+^2 |f_\varepsilon(x_2)|^2 \, dx - \int_{\frac{\pi}{2} - \varepsilon}^{\frac{\pi}{2}} |f(r \sin \theta)|^2 r \, dr \, d\theta
$$

$$
= (1 + \mathcal{O}(\varepsilon^{-\infty})) \|\eta_+\|^2_{L^2(\mathbb{R}^+)} - \int_{\frac{\pi}{2} - \varepsilon}^{\frac{\pi}{2}} \sin^{-2} \theta \int_0^\varepsilon |f(t)|^2 r \, dr \, d\theta.
$$

We deduce that

$$
\int_{T^+} |\Psi^{tr}|^2 \, dx = (1 + \mathcal{O}(\varepsilon^{-\infty})) \|\eta_+\|^2_{L^2(\mathbb{R}^+)} - \frac{\delta}{2} \int_0^{+\infty} t |f(t)|^2 \, dt + \mathcal{O}(\delta^3). \quad (3.14)
$$

Let us now estimate the energy of $\Psi^{tr}$. An integration by parts and a symmetry consideration yield the following identity ($\| \cdot \|$ denotes the norm in $L^2(C_\delta)$):

$$
Q_\delta(\Psi^{tr}) = \int_{C_\delta} |\eta|^2 |(-i\nabla + A)\Psi|^2 \, dx - \int_{C_\delta} \eta \Delta \eta |\Psi|^2 \, dx
$$

$$
= \|\eta(-i\nabla + A)\Psi\|^2 - 2 \int_{T^+} \eta \Delta \eta |\Psi|^2 \, dx
$$

$$
= \|\eta(-i\nabla + A)\Psi\|^2 - 2 \int_{T^+} \eta(x_1)\eta''(x_1) |\Psi|^2 \, dx
$$

$$
= \|\eta(-i\nabla + A)\Psi\|^2 + 2 \|f\|_{L^2(\mathbb{R}^+)}^2 \int_{\mathbb{R}^+} |\eta'_+(x_1)|^2 \, dx_1.
$$

Notice that for the integration by parts we needed $\eta_+ \in H^2(\mathbb{R}^+)$, but a density argument gives the identity for all $\eta_+ \in H^1(\mathbb{R}^+)$, with $\eta_+ = 1$ on the interval $(0, \varepsilon)$.

3.5.2 Upper bound and proof of Theorem 1.1

Using the symmetry of our construction with respect to $D_\delta$, we get (as in Lemma 3.1):

$$
\|\eta(-i\nabla + A)\Psi\|^2 - \Theta_0 \|\Psi^{tr}\|^2 = 2(\|\eta_+(-i\nabla + A)\Psi\|^2_{L^2(T^+)} - \Theta_0 \|\Psi^{tr}\|^2_{L^2(T^+)}).
$$

With (3.13), (3.14), and Proposition 3.4, we get

$$
Q_\delta(\Psi^{tr}) - \Theta_0 \|\Psi^{tr}\|^2 \leq -\delta \left( J - \Theta_0 \int_0^{+\infty} t |f(t)|^2 \, dt \right) + 2 \|\eta'_+\|^2_{L^2(\mathbb{R}^+)}
$$

$$
+ \mathcal{O}(\delta^{\frac{3}{2}}) + \mathcal{O}(\delta^\infty) \|\eta_+\|^2_{H^1(\mathbb{R}^+)}.\]
Recalling (3.7) and (2.11), we get
\[ J - \Theta_0 \int_0^{+\infty} t |f(t)|^2 \, dt = C_1 + O(\delta^{\infty}), \]
so that
\[ Q_\delta(\Psi^{tr}) - \Theta_0 \|\Psi^{tr}\|^2 \leq -C_1 \delta + 2\|\eta_+\|^2_{L^2(\mathbb{R}_+)} + O(\delta^{\frac{3}{2}}) + O(\delta^{\infty})\|\eta_+\|^2_{H^1(\mathbb{R}_+)}. \]

Therefore,
\[ \frac{Q_\delta(\Psi^{tr})}{\|\Psi^{tr}\|^2} \leq \Theta_0 + \frac{1}{\|\Psi^{tr}\|^2} \left( -C_1 \delta + 2\|\eta_+\|^2_{L^2(\mathbb{R}_+)} + O(\delta^{\frac{3}{2}}) + O(\delta^{\infty})\|\eta_+\|^2_{H^1(\mathbb{R}_+)} \right), \]
and by (3.14) and (2.5), we have
\[ \frac{1}{\|\Psi^{tr}\|^2} = \frac{1}{2\|\eta_+\|^2} \left( 1 + \frac{\delta_0 \delta}{2\|\eta_+\|^2} + O(\delta^2) \right), \]
where we used that \( \|\eta_+\|^2 \geq \varepsilon \). Then,
\[ \frac{Q_\delta(\Psi^{tr})}{\|\Psi^{tr}\|^2} \leq \Theta_0 + I_\delta(\eta_+) + O(R_\delta(\eta_+)) + O(\delta^{\infty}\|\eta_+\|^2_{H^1(\mathbb{R}_+)}), \]
where
\[ I_\delta(\eta_+) = \frac{1}{\|\eta_+\|^2} \left( -\frac{C_1 \delta}{2} + \|\eta_+\|^2_{L^2(\mathbb{R}_+)} \right), \]
and
\[ R_\delta(\eta_+) = \frac{\delta^{\frac{3}{2}}}{\|\eta_+\|^2} + \delta \frac{\|\eta_+\|^2_{L^2(\mathbb{R}_+)}^2}{\|\eta_+\|^4} + \delta^{3} \frac{\|\eta_+\|^2_{L^2(\mathbb{R}_+)}^2}{\|\eta_+\|^2}. \]

The estimate (3.15) leads us to minimize the functional \( I_\delta \) over the \( \eta_+ \in H^1(\mathbb{R}_+) \) such that \( \eta_+ = 1 \) on \((0, \varepsilon), \varepsilon > 0 \) being fixed. For our purpose, namely, to finish the proof of Theorem 1.1, it suffices to come up with a sufficiently good trial \( \eta_+ \), so we will be brief. The Euler–Lagrange equation
\[ -\eta_+''' = I_\delta(\eta_+)\eta_+, \quad \text{on} \ [\varepsilon, +\infty), \]
leads us to consider test functions \( \eta_+ = \eta_\alpha \) of the form
\[ \eta_\alpha(x) = \begin{cases} 1 & \text{on} \ (0, \varepsilon) \\ e^{\alpha x} e^{-\alpha x} & x \geq \varepsilon, \end{cases} \]
where \( \alpha > 0 \). We notice that
\[ I_\delta(\eta_\alpha) = \frac{-C_1 \delta}{\varepsilon + \|\eta_\alpha\|^2_{L^2(\varepsilon, +\infty)}} = \frac{\alpha^2 - d\alpha}{1 + 2\alpha}, \quad d = C_1 \delta. \]
This last quantity is minimal for \( \alpha_\delta = \frac{1}{2\varepsilon} \left( -1 + \sqrt{1 + 2\varepsilon d} \right) \), and we have

\[
\alpha_\delta \xrightarrow{\delta \to 0} \frac{C_1 \delta}{2} + O(\delta^2).
\]

Therefore, we choose \( \alpha = \frac{C_1 \delta}{2} \) and we have

\[
I_\delta(\eta_\alpha) = -\frac{C_1^2}{4} \delta^2 + O(\delta^3).
\]

We notice that

\[
\| \eta_\alpha \|_{L^2(\mathbb{R}^+)}^2 = \varepsilon + \frac{1}{C_1 \delta}, \quad \| \eta_\alpha' \|_{L^2(\mathbb{R}^+)}^2 = \frac{C_1 \delta}{4},
\]

so that

\[
R_\delta(\eta_\alpha) = O(\delta^5).
\]

With the upper bound (3.15), this concludes the proof of Theorem 1.1.

### 4. REGULAR PERTURBATION

The purpose of this section is to prove Theorem 1.2, so \( \Omega_\delta = \Gamma^+ \) hereafter. We construct a test function supported in a tubular neighborhood of \( \partial \Omega_\delta \) where we can use the Frenet coordinates.

We choose \( \ell = \delta - \rho \) with \( \rho \in (0, 1) \) and we let \( B_\ell = \mathbb{R} \times (0, \ell) \). For \( \delta \) small enough, the classical tubular coordinates,

\[
\Phi_\delta : B_\ell \ni (s, t) \mapsto \gamma_\delta(s) + tn_\delta(s),
\]

are well defined in the sense that \( \Phi_\delta \) is injective and induces a local (and then global) \( C^1 \)-diffeomorphism (the Jacobian of which being \( 1 - t\delta \cdot \kappa(s) \)). Indeed, by using \( \gamma_\delta' = (1, 0) \), the continuity of \( \delta \mapsto \gamma_\delta' \), and the Taylor formula, we can check that there exist \( c, \delta_0 > 0 \) such that, for all \( \delta \in (0, \delta_0) \) and all \( (s_1, s_2) \in \mathbb{R}^2 \),

\[
|\Phi_\delta(s_2, t_2) - \Phi_\delta(s_1, t_1)| \geq c(|s_2 - s_1| + |t_2 - t_1|).
\]

We let \( \Omega_{\delta, \ell} = \Phi_\delta(B_\ell) \). We consider a function of the form

\[
\psi(s, t) = \zeta(\ell^{-1} t) f_\ast(t) g(s),
\]

where \( \zeta \) is a cut-off function (see (2.9)), \( f_\ast \) was defined in (2.3), and where \( g \) has to be determined and will be chosen real-valued and normalized in \( L^2(\mathbb{R}) \). There exists a suitable phase \( \varphi \) such that (see [7, Lemma F.1.1]) if we let

\[
\Psi = (e^{i\varphi} \psi) o\Phi_\delta^{-1},
\]

which is supported in \( \Omega_{\delta, \ell} \), we have

\[
Q_\delta(\Psi) = Q_\delta(\psi),
\]
where (with $\kappa = \kappa(s)$)
\[
\tilde{Q}_\delta(\psi) := \int_{B_F} (1 - t\delta\kappa)|\delta_t\psi|^2\,ds\,dt \\
+ \int_{B_F} (1 - t\kappa(\delta - 1)|(-i\delta_s + \xi_0 - t + \frac{\delta xt^2}{2})|\psi|^2\,ds\,dt.
\]

**We start by proving in the case where** $\int \kappa > 0$.

By the exponential decay of $f_*$, we have
\[
\int_{B_F} (1 - t\delta\kappa)|\delta_t\psi|^2\,ds\,dt = \int_{B_F} (1 - t\delta\kappa)|g|^2|e^{-t}e^{-t}f_* + \xi(\delta^{-1}t)f_*|^2\,ds\,dt \\
\leq \|f_*\|^2\|g\|^2 - \delta \left(\int_0^\infty t|f_*|^2\,dt\right) \int_{\mathbb{R}} |g|^2\,ds + O(\varepsilon^{-\infty})\|g\|^2. \tag{4.2}
\]

Also,
\[
\int_{B_F} (1 - t\delta\kappa)^{-1}\left(-i\delta_s + \xi_0 - t + \frac{\delta xt^2}{2}\right)|\psi|^2\,ds\,dt \leq I + O(\varepsilon^{-\infty})(\|g\|^2 + \|g'\|^2) \tag{4.3}
\]

where, using first that $g$ is real-valued and then that $(1 - t\delta\kappa)^{-1} = 1 + \delta xt + O(\delta^2t^2\kappa^2)$ as well as (2.5),
\[
I = \int_{\mathbb{R} \times \mathbb{R}^+} (1 - t\delta\kappa)^{-1} - i\delta g f_* + \left(\xi_0 - t + \frac{\delta xt^2}{2}\right) f_* g \, ds\,dt \\
= \int_{B_F} (1 - t\delta\kappa)^{-1}\left(|g|^2f_*^2 + |g|^2\left(\xi_0 - t + \frac{\delta xt^2}{2}\right)^2 f_*^2\right)\,ds\,dt. \\
\leq (1 + \xi_0\|\kappa\|_{\infty}\delta)\|g'\|^2 + \|(t - \xi_0)f_*\|^2\|g\|^2 \\
+ \left(\int_{\mathbb{R}} \delta|g|^2\,ds\right) \int_0^{+\infty} ((\xi_0 - t)t^2 + t(t - \xi_0)^2)f_*^2\,dt + C \int_{\mathbb{R}} \delta^2\kappa^2|g|^2\,ds. \tag{4.4}
\]

Combining this with (4.2) and (4.3), we deduce that
\[
\tilde{Q}_\delta(\psi) \leq \Theta_0\|g\|^2 + (1 + \xi_0\|\kappa\|_{\infty}\delta)\|g'\|^2 \\
+ \left(\int_0^{+\infty} \left[((\xi_0 - t)t^2 + t(t - \xi_0)^2)f_*^2 - tf_*^2\right]\,dt\right) \left(\int_{\mathbb{R}} \delta|g|^2\,ds\right) \\
+ C \int_{\mathbb{R}} \delta^2\kappa^2|g|^2\,ds + O(\delta^\infty)\|g\|^2_{H^1(\mathbb{R})}.
\]
Using the decay of $f_*$ and (2.5), the norm of $\psi$ is given by

$$
\|\psi\|^2 = \int_{B_r} (1 - \delta t\xi)|g(s)|^2|f_*(t)^2|^2 \xi(e^{-1} t)^2 \, dt \, ds
$$

$$
= \|g\|^2 - \delta \xi_0 \int_{\mathbb{R}^+} \kappa |g(s)|^2 \, ds + \mathcal{O}(\delta^\infty)\|g\|^2. \tag{4.5}
$$

It follows that

$$
\bar{Q}_d(\psi) - \Theta_0\|\psi\|^2 \leq (1 + \tilde{C}\|\kappa\|_\infty\delta)\|g'\|^2
$$

$$
+ \delta \int_0^{+\infty} (((\xi_0 - t)t^2 + t(t - \xi_0)^2)f_*^2 - tf_*'^2 + t\Theta_0 f_*^2) \, dt \left(\int_{\mathbb{R}} \kappa |g|^2 \, ds\right)
+ \mathcal{O}(\delta^2) \int_{\mathbb{R}} \kappa^2 |g(s)|^2 \, ds + \mathcal{O}(\delta^\infty)\|g\|^2. \tag{4.6}
$$

We have to investigate the sign of

$$
(1 + \tilde{C}\|\kappa\|_\infty\delta)\|g'\|^2 + A \int_{\mathbb{R}} \delta \kappa |g|^2 \, ds + \mathcal{O}(\delta^2) \int_{\mathbb{R}} \kappa^2 |g(s)|^2 \, ds + \mathcal{O}(\delta^\infty)\|g\|^2, 
$$

with

$$
A := \int_0^{+\infty} ((\xi_0 - t)t^2 + t(t - \xi_0)^2)f_*^2 - tf_*'^2 + t\Theta_0 f_*^2) \, dt.
$$

By using (2.6) and (2.7), we have

$$
A = -\frac{3C_1}{2} + \int_0^{+\infty} t(t - \xi_0)(t - 2\xi_0)f_*^2(t) \, dt = -\frac{3C_1}{2} + \frac{C_1}{2} = -C_1.
$$

We choose $g$ such that $\|g\|^2 = 1$ and observe that (4.5) yields for $\delta$ small enough,

$$
\|\psi\|^{-2} = 1 + \mathcal{O}(\delta\|\kappa\|_\infty). 
$$

By the min–max principle, we deduce from (4.6)

$$
\lambda(\delta) \leq \Theta_0 + F_\delta(g) + \mathcal{O}(\delta^\infty),
$$

where

$$
F_\delta(g) = (1 + \tilde{C}\|\kappa\|_\infty\delta)^2\|g'\|^2 + \delta \int_{\mathbb{R}} (-C_1 \kappa + \tilde{C}\delta(\|\kappa\|_\infty|\kappa| + \kappa^2)) |g(s)|^2 \, ds. \tag{4.7}
$$

Minimizing over $g$ and using the analysis in Appendix A, we get if $\int_{\mathbb{R}} \kappa(s) \, ds > 0$,

$$
\lambda(\delta) \leq \Theta_0 - \frac{C_1^2}{4} \left(\int_{\mathbb{R}} \kappa(s) \, ds\right)^2 \delta^2 + \mathcal{O}(\delta^3).
$$
More precisely, it is enough to consider the trial function

\[ g(s) = \sqrt{|V|} \exp \left( \frac{\delta V}{2} |s| \right), \quad V(s) = -C_1 \kappa(s), \quad \langle V \rangle = \int_\mathbb{R} V(s) ds < 0. \]  

(4.8)

Notice that, by dominated convergence,

\[ \int_\mathbb{R} (\|\kappa\|_\infty |\kappa| + \kappa^2) |g(s)|^2 ds = \mathcal{O}(\delta) \int_\mathbb{R} (\|\kappa\|_\infty |\kappa| + \kappa^2) ds. \]

The case where \( \int_\mathbb{R} \kappa = 0. \)

In the case \( \int_\mathbb{R} \kappa = 0, \) we need to make the calculation to precision \( \delta^4, \) but the approach is the same and the trial state still has the form from (4.1).

Calculating as in (4.2), (4.3), but expanding \((1 - \delta \kappa t)^{-1}\) to fourth order, we get

\[ \tilde{Q}_\delta(\psi) \leq \Theta_0 \|g\|^2 + 2\|g'\|^2 + \sum_{j=1}^4 c_j \delta^j \int \kappa^j |g|^2 ds + \mathcal{O}(\delta^5) \|g\|^2_{H^1(\mathbb{R})}, \]

(4.9)

where the constants come from calculating moments of \( f_\star, \) in particular, \( c_1 \) is the same as before,

\[ c_1 = \int_0^{+\infty} \left[ \left( (\xi_0 - t)^2 + t (t - \xi_0)^2 \right) f_\star^2 - tf_\star' \right] dt, \]

while \( c_2 \) is\(^\dagger\)

\[ c_2 = \int_0^{+\infty} \left[ \frac{1}{4} t^4 + (t - \xi_0)^2 t^2 - (t - \xi_0) t^3 \right] |f_\star(t)|^2 dt \]

\[ = \frac{1}{4} \int_0^{+\infty} (t - \xi_0)^4 |f_\star(t)|^2 dt > 0. \]

We now insert (4.5) and the calculation that the constant \( A = -C_1, \) to get (similarly to (4.6)) that (for \( \delta \) sufficiently small, depending only on \( \|\kappa\|_\infty \))

\[ \tilde{Q}_\delta(\psi) - \Theta_0 \|\psi\|^2 \leq 2\|g'\|^2 + \int 2\delta V_\delta |g|^2 ds + \mathcal{O}(\delta^5) \|g\|^2_{H^1(\mathbb{R})}, \]

(4.10)

where

\[ V(\delta) = -\frac{C_1}{2} \kappa(s) + \sum_{j=2}^4 c_j \delta^{j-1} \kappa(s)^j. \]

(4.11)

The question of a bound state now becomes the question of a negative eigenvalue for

\[ g \mapsto \int |g'|^2 + V_\delta |g|^2 ds. \]

(4.12)

\(^\dagger\)The moments \( M_k := \int_0^{+\infty} (t - \xi_0)^k |f_\star(t)|^2 dt \) are calculated for \( k = 0, 1, 2, 3, 4, \) see, for example, [7, Lemma 3.2.7].
By [10, Theorem 2.5], this operator has a negative eigenvalue $-\alpha(\delta)^2$, if $\alpha(\delta) > 0$, where

$$\alpha(\delta) = -\frac{\delta}{2} \int V_\delta \, ds - \frac{\delta^2}{4} \iint V_\delta(s)|s - t|V_\delta(t) \, ds \, dt + o(\delta^2). \quad (4.13)$$

Using $\int_\mathbb{R} \kappa = 0$, we find

$$\alpha(\delta) = \delta^2 \left\{ \frac{-c_2}{2} \int \kappa^2 + M(\kappa) \right\} + o(\delta^2), \quad (4.14)$$

with (where the inequality was proved in [10])

$$M(\kappa) = -\frac{C_1^2}{16} \iint \kappa(s)|s - s'|\kappa(s') \, ds \, ds' > 0. \quad (4.15)$$

Under the substitution $\kappa(s) \mapsto \kappa(s/\mu)$, we see that $M(\kappa)$ scales as $\mu^3$, whereas $\int \kappa^2$ scales as $\mu$, so it is clearly possible to make the leading coefficient in the expression for $\alpha(\delta)$ positive. This assures the existence of a negative eigenvalue of order $\delta^4$.

**APPENDIX A: WEAK PERTURBATION LIMIT**

In this appendix, we explain where the choice (4.8) is coming from.

Let $V \in C_0^\infty(\mathbb{R})$ such that $\langle V \rangle = \int_\mathbb{R} V(x)dx < 0$ and $\delta \in \mathbb{R}$. Let us consider the self-adjoint operator $L_\delta$ associated with the following quadratic form:

$$q_\delta(\psi) := \|\psi'|^2 + \delta \int_\mathbb{R} V(x)|\psi(x)|^2 \, dx, \quad \forall \psi \in H^1(\mathbb{R}),$$

which is the main term in (4.7), with $V = -C_1 \kappa$.

By the rescaling $x = \delta^{-1} y$, we see that $L_\delta$ is unitarily equivalent to $\delta^2 M_\delta$ where

$$M_\delta := -\frac{\delta^2}{2} + \delta^{-1} V(\delta^{-1} y).$$

At a formal level, we see that $M_\delta$ converges to $M^{\text{eff}} := -\frac{\delta^2}{2} + \langle V \rangle \delta_0$, whose spectrum is

$$\left\{ -\frac{\langle V \rangle^2}{4} \right\} \cup [0, +\infty),$$

and with groundstate $\exp(\frac{\langle V \rangle}{2}|y|)$.

Let us make this heuristics rigorous by comparing the quadratic forms. The quadratic form associated with $M_\delta$ is

$$p_\delta(\psi) = \|\psi'|^2 + \int_\mathbb{R} \delta^{-1} V(\delta^{-1} y)|\psi|^2 \, dy, \quad \forall \psi \in H^1(\mathbb{R}).$$

We denote by $p^{\text{eff}}$ the quadratic form associated with $M^{\text{eff}}$:

$$p^{\text{eff}}(\psi) = \|\psi'|^2 + \langle V \rangle |\psi(0)|^2, \quad \forall \psi \in H^1(\mathbb{R}).$$
Let us estimate the difference $p_\delta - p_{\text{eff}}$. For all $\psi \in H^1(\mathbb{R})$, we have

$$|p_\delta(\psi) - p_{\text{eff}}(\psi)| \leq \left| \int_\mathbb{R} \delta^{-1} V(\delta^{-1}y)|\psi(y)|^2 dy - \langle V \rangle |\psi(0)|^2 \right|$$

$$\leq \left| \int_\mathbb{R} V(y)|\psi(\delta y)|^2 dy - \langle V \rangle |\psi(0)|^2 \right|$$

$$\leq \int_\mathbb{R} |V(y)||\psi(\delta y)|^2 - |\psi(0)|^2| dy.$$ 

It is well known that

$$\left| |\psi(\delta y)|^2 - |\psi(0)|^2 \right| \leq \delta^{\frac{1}{2}} \| |\psi|^2 \|^2 \sqrt{|y|} \leq 2 \delta^{\frac{1}{2}} \| \psi \psi' \| \sqrt{|y|},$$

and also that, by the usual Sobolev embedding,

$$\| \psi \|_{L^\infty(\mathbb{R})} \leq C \| \psi \|_{H^1(\mathbb{R})}.$$ 

Thus,

$$\left| |\psi(\delta y)|^2 - |\psi(0)|^2 \right| \leq C \delta^{\frac{1}{2}} \| \psi \|_{H^1(\mathbb{R})}^2 \sqrt{|y|},$$

so that

$$|p_\delta(\psi) - p_{\text{eff}}(\psi)| \leq C \delta^{\frac{1}{2}} \| \psi \|_{H^1(\mathbb{R})}^2.$$ 

Therefore,

$$p_\delta^-(\psi) \leq p_\delta(\psi) \leq p_\delta^+(\psi),$$

where

$$p_\delta^-(\psi) := (1 - C \delta^{\frac{1}{2}}) \| \psi' \|^2 + \langle V \rangle |\psi(0)|^2 - C \delta^{\frac{1}{2}} \| \psi \|^2,$$

$$p_\delta^+(\psi) := (1 + C \delta^{\frac{1}{2}}) \| \psi' \|^2 + \langle V \rangle |\psi(0)|^2 + C \delta^{\frac{1}{2}} \| \psi \|^2.$$ 

If $\nu(\delta)$ is the bottom of the spectrum of $M_\delta$, we have

$$\nu(\delta) = -\frac{\langle V \rangle^2}{4} + O(\delta^{\frac{1}{2}}) < 0.$$ 

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