Desirable Companion for Vertical Federated Learning: New Zeroth-Order Gradient Based Algorithm

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Abstract

Vertical federated learning (VFL) attracts increasing attention due to the emerging demands of multi-party collaborative modeling and concerns of privacy leakage. A complete list of metrics to evaluate VFL algorithms should include model applicability, privacy security, communication cost, and computation efficiency, where privacy security is especially important to VFL. However, to the best of our knowledge, there does not exist a VFL algorithm satisfying all these criteria very well. To address this challenging problem, in this paper, we reveal that zeroth-order optimization (ZOO) is a desirable companion for VFL. Specifically, ZOO can 1) improve the model applicability of VFL framework, 2) prevent VFL framework from privacy leakage under curious, colluding, and malicious threat models, 3) support inexpensive communication and efficient computation. Based on that, we propose a novel and practical VFL framework with black-box models, which is inseparably interconnected to the promising properties of ZOO. We believe that it takes one stride towards designing a practical VFL framework matching all the criteria. Under this framework, we raise two novel asynchronous zeroth-order algorithms for vertical federated learning (AsyREVEL) with different smoothing techniques. We theoretically drive the convergence rates of AsyREVEL algorithms under nonconvex condition. More importantly, we prove the privacy security of our proposed framework under existing VFL attacks on different levels. Extensive experiments on benchmark datasets demonstrate the favorable model applicability, satisfied privacy security, inexpensive communication, efficient computation, scalability and losslessness of our framework.

1. Introduction

Federated learning McMahan et al. (2016); Smith et al. (2017); Kairouz et al. (2019); Gascón et al. (2016) is a prevailing distributed machine learning paradigm for collaboratively training a machine learning model with privacy-preserving. A line of recent works McMahan et al. (2016); Smith et al. (2017) focus on the horizontal federated learning, where different parties have different samples IDs but they all share the same complete features. The other line of works Hardy et al. (2017); Yang et al. (2019b); Zhang et al. (2021a,b) studying the vertical federated learning (VFL), where data owned by different parties have the same sample IDs but disjoint subsets of features. Such scenario is common in the industry applications of emerging cross-organizational collaborative learning, including but not limited to medical study, financial risk, and targeted marketing Gong et al. (2016); Yang et al. (2019b); Cheng et al. (2019); Hu et al. (2019). For example, E-commerce companies owning the online shopping information could collaboratively train joint-models with banks and digital finance companies that own other information of the same people such as the average monthly deposit and online consumption, respectively,
to achieve a precise customer profiling. In this paper, we focus on VFL due to its wide applications for emerging multi-organization collaborative modeling with privacy-preserving.

There have been extensive works studying VFL systems from several perspectives. For example, some works focus on developing fast and scalable optimization methods for training VFL models based on stochastic gradient descent (SGD) methods Wan et al. (2007); Hu et al. (2019); Liu et al. (2019a); Gu et al. (2020) and stochastic quasi-Newton methods Yang et al. (2019a). Some works study attack models under different threat models, such as inference attack under the honest-but-curious Gu et al. (2020), inference attacks under the honest-but-colluding Cheng et al. (2019); Weng et al. (2020) and backdoor attack under the malicious Liu et al. (2020). And some works study different (auxiliary) defense strategies such as the scalar product protocol for defense Du et al. (2004); Hu et al. (2019); Liu et al. (2019a); Gu et al. (2020) and the auxiliary strategies, including the differential privacy (DP) Liu et al. (2019b); Xu et al. (2019); Chen et al. (2020) and the gradient sparsification Liu et al. (2020), for alleviating different attacks. Besides, there are several works focusing on reducing the communication cost (number of communication rounds) Liu et al. (2019a); Yang et al. (2019a), and also some works concerning different computation manners such as the synchronous Gong et al. (2016); Zhang et al. (2018); Liu et al. (2019a) and asynchronous ones Hu et al. (2019); Gu et al. (2020).

In fact, the above perspectives can be summarized into a complete list of criteria, i.e., model applicability, privacy security, communication cost and computation efficiency, which can be used to comprehensively evaluate the performance of a VFL algorithm. Specifically, 1) model applicability means the ability to solve different problems, Wan et al. (2007); Hu et al. (2019); Liu et al. (2019a); Gu et al. (2020); Yang et al. (2019a), 2) privacy security depends on the ability to defense different attacks, which is especially important to VFL Gu et al. (2020); Cheng et al. (2019); Weng et al. (2020); Liu et al. (2020), 3) communication cost depends on the number of communication rounds and the per-round communication overhead (PRCO) Liu et al. (2019a); Yang et al. (2019a), and 4) computation efficiency is mainly dominated by the computation manner, i.e., the asynchronous or synchronous Hu et al. (2019); Gu et al. (2020); Chen et al. (2020); Liu et al. (2019a).

However, to the best of our knowledge, there does not exist a VFL algorithm that is well designed to satisfy all these metrics together. Specifically,

1. Most existing VFL frameworks adopt SGD methods Wan et al. (2007); Hu et al. (2019); Liu et al. (2019a); Gu et al. (2020). However, these optimization methods will fail when applied to the widely-existing problems whose explicit expressions of gradients are difficult or infeasible to obtain, such as the structure prediction Sokolov et al. (2018), bandit learning Shamir (2017) and black-box learning Liu et al. (2018) problems. Thus, these VFL frameworks have the poor model applicability when applied to these problems.

2. Privacy security is especially important for VFL, thus there have many attack manners Gu et al. (2020); Cheng et al. (2019); Weng et al. (2020); Liu et al. (2020) and defense (or auxiliary defense) strategies Liu et al. (2019b); Xu et al. (2019); Chen et al. (2020); Liu et al. (2020) been proposed. However, they still can not totally defense some existing VFL attacks, especially, the latest proposed inference attacks in Luo et al. (2020); Weng et al. (2020); Liu et al. (2020) and backdoor attack in Liu et al. (2020) due to transmitting the informative knowledge, e.g., the model parameters and (intermediate) gradients. Thus, existing VFL frameworks have the unsatisfied privacy security.

3. Meanwhile, in the real VFL system, different parties always represent different companies or organizations across different networks. In this case, most existing VFL frameworks directly transmitting the model parameters Gong et al. (2016); Yang et al. (2019a); Liu et al. (2019b); Xu et al. (2019) or gradients Weng et al. (2020); Chen et al. (2020); Liu et al. (2020) between parties are much communication-expensive due to the large PRCO.

4. Moreover, it is common in the real-world applications that both large and small companies collaboratively learn the model, where the former have better computational capacity while the later have the poorer. In this case, algorithms using synchronous computation Gong et al. (2016); Zhang et al. (2018) are inefficient.
Table 1: A summary of evaluating existing VFL frameworks following these four metrics, where ERCR denotes “exchanging the raw computation results”, TIG denotes “transmitting intermediate gradients”, TG denotes “transmitting gradients”, MA denotes “Model Applicability”, PS denotes “Privacy Security”, IC denotes “Inexpensive Communication”, CE denotes “Computational Efficiency”. VFL framework adopting AsyREVEL is proposed in Section 3, and the results of privacy security means which attack these methods cannot defense (“1–feature inference attack Gu et al. (2020)”, “2–label inference attack in Liu et al. (2020)”, “3–feature inference attack in Luo et al. (2020)”, “4–reverse multiplication and reverse sum attacks in Weng et al. (2020)”, “5–backdoor attack in Liu et al. (2020)”, “–” means that can prevent attacks proposed in Gu et al. (2020); Luo et al. (2020); Weng et al. (2020); Liu et al. (2020)).

| Methods                                      | MA | PS | IC | CE |
|----------------------------------------------|----|----|----|----|
| Asynchronous ERCR-based methods Hu et al. (2019); Gu et al. (2020) | ×  | 2  | ✓  | ✓  |
| Communication-efficient TIG-based method Liu et al. (2019a)         | ×  | 2.5| ✓  | ✗  |
| Asynchronous TG-based methods Vepakomma et al. (2018); Chen et al. (2020) | ×  | 3  | ✗  | ✓  |
| Communication-efficient HE-based method Yang et al. (2019a)          | ×  | 4  | ✓  | ✗  |
| Synchronous HE-based methods Gong et al. (2016); Hardy et al. (2017) | ×  | 4  | ✗  | ✗  |
| VFL framework adopting AsyREVEL (ours)                               | ✓  | –  | ✓  | ✓  |

1 When not all parties have the labels, these methods can not prevent the label inference attack Liu et al. (2020).

Because, parties possessing better computational capacity have to waste the computational capacity to wait the stragglers for synchronization.

As discussed above, although there have been extensive works towards studying better VFL frameworks following these criteria, existing VFL frameworks still can not satisfy all criteria well because of the poor model applicability, the unsatisfied privacy security, expensive communication, or the inefficient computation. Thus, it is challenging to design a practical VFL framework that not only supports inexpensive communication and efficient computation but also has favorable model applicability and satisfied privacy security.

In this paper, we address this challenging problem by revealing the promising properties of ZOO and, be inseparably interconnected, proposing a novel practical VFL framework with black-box models, under which the asynchronous zeroth-order optimization algorithms (AsyREVEL) are proposed. Specifically, 1) ZOO only needs the function values for updating rather than the gradients with explicit expressions and thus can improve the model applicability of VFL to more ML problems. 2) Only black-box information i.e., function values, is necessary to be transmitted for ZOO, which can prevent existing VFL attacks under three levels of threat models, i.e., the curious, colluding, and malicious. 3) Only function values are transmitted for ZOO (have low PRCO) and the asynchronous computation is adopted for AsyREVEL, thus, ZOO-VFL can also support inexpensive communication and efficient computation. We summarize the contributions of this paper as follows.

- We are the first to reveal that ZOO is a desirable companion for VFL, which not only support inexpensive communication and efficient computation but also has favorable model applicability and satisfied privacy security. Moreover, we also propose a novel practical VFL framework with black-box models, which inherits the promising properties of ZOO.

- We propose two AsyREVEL algorithms with different smoothing techniques, i.e. AsyREVEL-Gau and -Uni, under our practical VFL framework. Moreover, we theoretically prove their convergence rates for the nonconvex problems.
2. A Desirable Companion for VFL

In this section, we first give a brief review to VFL and then, importantly, we reveal that ZOO is a desirable companion for VFL. Moreover, we give the thorough privacy security analyses of ZOO for VFL (named ZOO-VFL) under existing VFL attacks.

2.1 Vertical Federated Learning

Vertical federated learning Gascón et al. (2016); Yang et al. (2019b); Hu et al. (2019); Liu et al. (2019a); Gu et al. (2020) is a paradigm for multi-party collaborative learning with privacy preserving. In the VFL system, each party holds different features for one sample. Specifically, for a VFL system with \( q \) parties and training data \( \{x_i, y_i\}_{i=1}^n \), the \( x_i \in \mathbb{R}^d \) can be represented as a concatenation of all feature blocks, i.e., \( x_i = [x_{i,1}; x_{i,2}; \cdots; x_{i,q}] \), where \( x_{i,m} \in \mathbb{R}^{d_m} \) is stored privately on party \( m \), and \( \sum_{m=1}^{q} \tilde{d}_m = \tilde{d} \). Moreover, each party in the VFL system privately maintains and learns a local model, and all parties collaboratively learn the joint model.

Currently, much efforts have been made towards designing better VFL frameworks for real-world applications from various aspects. In this paper, we summarize these aspects into four metrics, i.e., model applicability, privacy security, communication cost and computation efficiency, which can be used to comprehensively evaluate the performance of VFL frameworks. Although there have been many works studying VFL following those metrics, to the best of our knowledge, existing VFL frameworks are still not well designed to match those criteria simultaneously. In the following, we reveal that ZOO is a promising choice for designing VFL framework matching these metrics.

2.2 A Desirable Companion for VFL

Zeroth-Order Optimization: ZOO methods Huang et al. (2020, 2019c) have been developed to effectively solve many ML problems, whose explicit gradient expressions are difficult or infeasible to obtain, such as the structure prediction problems whose explicit gradients are difficult to obtain Sokolov et al. (2018), the bandit and black-box learning problems Shamir (2017); Liu et al. (2018), whose explicit gradients are infeasible to obtain. Specifically, ZOO only uses the function values for optimizing instead of gradients with explicit expressions. Although there have been many works focusing on ZOO, it is still vacant to explore the application of ZOO to VFL, especially, reveal its promising properties for VFL.

In the following we present the promising properties of ZOO-VFL concerning those four practical metrics and reveal that ZOO is a desirable optimization methods for VFL.

Model Applicability: Model applicability is a basic property for the VFL frameworks. Currently, most existing VFL frameworks adopt the gradient-based optimization methods for training. However, frameworks adopting gradient-based optimization methods have the poor model applicability to ML problems whose explicit expressions of gradients are difficult or infeasible to obtain. ZOO only needs the function values for optimizing, which thus is a promising choice to improve the model applicability of VFL to these problem.
Privacy Security: Privacy security is the most important character distinguishing FL from the distributed learning. Currently, there have many attack models and defense strategies been proposed Weng et al. (2020); Luo et al. (2020); Liu et al. (2020). Especially, the latest proposed inference attacks in Weng et al. (2020); Luo et al. (2020); Liu et al. (2020) and backdoor attack in Liu et al. (2020) are difficult for existing VFL frameworks to totally defense. Two data inference attacks are proposed in Weng et al. (2020), which, however, require the adversary to access the gradient of the local model and then utilize it for attack. To perform the label inference attack in Liu et al. (2020), the adversary must be able to access the intermediate gradient. Similarly, the gradient-replacement backdoor attack proposed in Liu et al. (2020) has to access the intermediate gradient and then replace it with the targeted one. In fact, existing attacks that are difficult to defense have to access the informative knowledge such as the model parameters and the gradients. Thus, to prevent these attacks, one can design a VFL system with the model unknown and without transmitting the informative knowledge between the parties. A natural and promising idea to achieve this is letting the model a black box and only transmitting the black-box knowledge, such as the function values (the outputs of local and global models).

However, it is impossible to leverage existing optimization methods for VFL to optimize these black-box models when only function values are transmitted. Currently, there have been many optimization methods for black-box learning, such as the Bayesian optimization Karro et al. (2017), heuristic algorithms Yoo and Han (2014), and ZOO Liu et al. (2018). Among them the ZOO is the optimal choice due to its superiority of theoretical guarantee to heuristic algorithms and less computation complexity than Bayesian optimization. Thus, ZOO is a desired optimization method for improving the privacy security of VFL framework. Especially, since privacy security is considerably important for FL, in the next subsection, we give the detailed privacy security analyses.

Communication Cost and Computation Efficiency: Note that, in terms of ZOO-VFL, only the function values are necessary to be transmitted. Thus, ZOO-VFL is communication-inexpensive because the PRCO of only transmitting the function values is considerably low. Moreover, we can also design the corresponding asynchronous ZOO algorithm, i.e., AsyREVEL proposed in Section 3.3, that keeps the computation resource being utilized all the time during training for better computation efficiency. Thus, ZOO-VFL is communication-inexpensive and computation-efficient.

In above analyses, we reveal that ZOO is naturally a desirable optimization method for VFL. Specifically, ZOO-VFL has favourable model applicability (ability to optimize black-box models), provides satisfied privacy security (ability to defense existing attacks for VFL), support inexpensive communication (low PRCO), and efficient computation (adopting asynchronous computation). For a strong support to our claim, we compare a VFL framework that adopts ZOO (proposed in Section 3) with existing VFL frameworks following these four metrics and show the results in Table 1.

2.3 Privacy Security of ZOO-VFL

In this section, we detailedly analyze the privacy security of ZOO-VFL under following three types of threat model, which capsule existing attacks for VFL. We introduce them as follows, whose illustrations are shown in Fig. 1.

Honest-but-Curious: All parties perform operations following the FL protocol but they may try to learn the private information of the other parties based on the accessed knowledge.

Honest-but-Colluding: All parties perform operations following the FL protocol but they may collude by sharing the accessed knowledge and use it to learn the private information of the other parties.

Malicious: Some (adversarial) parties may perform operations deviating arbitrarily from the FL protocol, and to learn the private information of other honest parties or inject a backdoor task by modifying, re-playing, or even removing transmitted messages.

Importantly, we have the theorem for the privacy security of VFL.

Theorem 1 ZOO for vertical federated learning can defense existing VFL attacks under honest-but-curious, honest-but-colluding, and malicious threat models.

Proof Honest-but-curious: Under this setting, only inference attacks can be performed by leveraging the intermediate computational results. Specifically, the feature inference attack is considered in Yang et al. (2019b); Gu et al.
(2020), where the adversary maintains the intermediate computational results of \(w^T x_i = z_i\) and uses them to infer \(w^T\) and \(x_i\). While, this attack will fail in ZOO-VFL because of the inability of solving \(n\) equations in more than \(n\) unknowns Du et al. (2004); Yang et al. (2019b); Gu et al. (2020). The label inference attack is proposed in Liu et al. (2020), which need access the intermediate gradient \(g_i = \frac{\partial L}{\partial H_i}\). The adversary uses the element values of \(g_i\) and formula \(g_i = \frac{\partial L}{\partial H_i}\) to refer the label of sample \(i\). As for ZOO-VFL, no knowledge about the intermediate gradients is exposed, thus it can prevent such attack totally.

**Honest-but-colluding:** Under this setting, the feature inference attacks (FIA) and the reverse multiplication attack (RMA) are proposed in Luo et al. (2020) and Weng et al. (2020), respectively. In the FIA proposed in Luo et al. (2020) is performed the adversary party is supposed to have its own input \(x_{\text{adv}}\), its local model \(\theta_{\text{adv}}\), local model of the target party \(\theta_{\text{target}}\), and the final prediction \(z\). And then it uses the formula \(x_{\text{adv}} \cdot \theta_{\text{adv}} + x_{\text{target}} \cdot \theta_{\text{target}} = z\) to infer the feature of the target party \(x_{\text{target}}\) during the model prediction stage. Moreover, the generative regression network is also designed in Luo et al. (2020) for such inference attack, which uses a generative regression network to iteratively approximate the original sample based on multiple model predictions. This attack seems very suitable for the ZOO because it also only uses the model outputs (the predictions). However, the strong primary assumption of both inference attacks that the adversary knows the local model of the target party does not hold in ZOO-VFL, where the local models are private and black-box. Thus, ZOO-VFL can prevent both FIAS totally. In the RMA, the adversary party accesses the intermediate computational results of successive training epoches, i.e., \(w^T_{t-1} x_i\) and \(w^T_t x_i\), and the gradient \(g_t\), and then uses the iterative gradient-based update rule \(w^T_t x_i - w^T_{t-1} x_i = -\eta g_t x_i\) to infer \(x_i\) (\(\eta\) is the learning rate). ZOO-VFL can prevent such RMA totally due to not transmitting the gradients necessary for such attack.

**Malicious:** Under this setting, the reverse sum attack and backdoor attack are proposed in Weng et al. (2020) and Liu et al. (2020), respectively. In the former, the adversary party encodes a magic number\(^1\) into the ciphertext of the first and second gradients (this operation revolves re-playing the gradient), which is used as the global unique identifier to infer the partial orders of training data. The targeted backdoor task is to assign an attacker-chosen label to input data with a specific pattern (i.e., a trigger) Liu et al. (2020). Specifically, the adversary party records the received intermediate gradient of the target sample (denoted as \(g_{\text{rec}}\) and replaces the intermediate gradient of the poisoned sample with \(g_{\text{rec}}\). As introduced, both reverse sum and backdoor attacks require the adversary to access the intermediate gradient. While, ZOO-VFL does not transmit the intermediate gradients necessary for these attacks, thus can prevent the reverse sum and backdoor attacks totally.

Thus, we have that ZOO-VFL can defense existing VFL attacks and protect the privacy security. This completes the proof.

In fact, all existing VFL attacks Luo et al. (2020); Liu et al. (2020); Weng et al. (2020) that are difficult to defense have to access the informative knowledge, i.e., the model parameters or the (intermediate) gradients. While, for ZOO-VFL, only the black-box knowledge (function values) are exposed. Thus, it can defense these attacks. Moreover, it can also prevent the potential VFL attacks that have to access such informative knowledge.

### 3. Practical VFL Framework and the AsyREVEL Algorithms

In this section, we propose a novel practical VFL framework with black-box models and asynchronous ZOO algorithms, which inherits the promising properties of ZOO, and is inseparably interconnected by above analyses of ZOO-VFL.

#### 3.1 Generalized Form of VFL

This paper considers a generalized VFL system with \(q\) parties and a server, where each party owns the vertically partitioned feature data and the server (maybe a party or trusty third-party) owns the labels. In this VFL system, all

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\(^1\) https://en.wikipedia.org/wiki/Magic_number\(^{\text{programming}}\).
parties and the server want to solve a finite-sum problem in the following composite form

\[
f(w_0, w) := \frac{1}{n} \sum_{i=1}^{n} F_0(w_0, c_{i,1}, \cdots, c_{i,q}; y_i) + \lambda \sum_{m=1}^{q} g(w_m) \quad \text{with} \quad c_{i,m} = F_m(w_m; x_{i,m}) \quad \forall m \in [q] \quad (P)
\]

where \( f_i(w_0, w) : \mathbb{R}^d \to \mathbb{R} \) is the cost function of the \( i \)-th sample, \( w = \{w_1, \cdots, w_q\} \in \mathbb{R}^d \), \( w_m \in \mathbb{R}^{d_m} \) for \( m \in [q] \) (given a positive integer \( q \), \( [q] \) denotes a set \( \{1, \cdots, q\} \)) defines a local model \( F_m \) on party \( m \), which maps input \( x_{i,m} \) to output \( c_{i,m} \), \( w_0 \in \mathbb{R}^d \) defines a global model \( F_0 \) learned and maintained by the server, \( d = \sum_{m=0}^{q} d_m \), and \( q \) is the regularized function. Especially, problem (P) is a generalized form that encapsulates a wide range of machine learning models. Two examples are shown as follows.

**Generalized Linear Model:** For \( m \in [q] \), \( F_m \) can be a linear model, such as \( F_m(w_m; x_{i,m}) = w_m^T x_{i,m} \). In this case, if we choose \( F_0(c_i; y_i) = \log(1 + e^{-y_i \sum_{m=1}^{q} c_{i,m}}) \) for binary classification tasks, Problem (P) will reduce to the classical logistic regression model. We can also choose suitable \( F_0 \) to obtain other linear models such as linear regression and support vector machine.

**Neural Network Model:** For \( m \in [q] \), \( F_m \) can also be a nonlinear model such as neural networks. In this case, \( c_{i,m} \) is presented in the following composite form

\[
\text{input layer: } u_0 = x_{i,m}, \quad \text{and output layer: } c_{i,m} = u_K \\
\text{intermediate layers: } u_l = \sigma_l(h_l u_{l-1} + b_l), \quad l = 1, \cdots, K
\]

Figure 2: A diagram of the proposed framework with black-box models, where only black-box knowledge (function values) is transmitted between parties and exposed to the adversary.
Algorithm 1 AsyREVEL SGD

0: initialize variables for workers $m \in [q]$
1: while not convergent do
2:     when client $m$ is activated, do:
3:         Sample an index $i \sim \text{Unif} [n]$
4:         Compute $c_{i,m}, \hat{c}_{i,m}$ and upload them to the server
5:         Receive $h_{i,m}$ and $\bar{h}_{i,m}$ from the server (in a listen manner)
6:         Compute $\hat{v}_m = \nabla_m f_i(w_0, \bar{w})$
7:     Update $w_m \leftarrow w_m - \eta_m \hat{v}_m$
8:     when server receives $c_{i,m}$ and $\hat{c}_{i,m}$, do:
9:         Compute $h_{i,m}, \bar{h}_{i,m}, \hat{h}_{i,m}$, and sent $h_{i,m}, \bar{h}_{i,m}$ to client $m$
10:        Compute $\hat{v}_0 = \nabla_0 f_i(w_0, \bar{w})$
11:        Update $w_0 \leftarrow w_0 - \eta_0 \hat{v}_0$
12: end while

where $\sigma_l$ is an active function with linear or nonlinear form, $h_l$ and $b_l$ for $l \in [K]$ correspond to the parameter $w_m$, $K$ is the number of layer. In this case, $F_0$ can be either a simple network, e.g., the fully connection networks or other complicated deep neural networks.

3.2 Practical Vertical Federated Learning Framework with Black-Box Models

Aiming at the generalized VFL problem in the form of (P), we propose a novel VFL framework with black-box models, whose diagram is presented in Fig. 2. As illustrated, the whole data are vertically stored on each party locally and privately. Especially, the local models and the global model are black-box models, which are privately maintained and learned by the parties and server, respectively. Moreover, each local model cascades to the global model and all local models are connected by this global model. Information such as model parameter and data sharing between parties is prohibited, which thus can prevent the data and model from directly leaking. Importantly, the function values transmitted between all parties and the server is black-box knowledge, which is useful to defense existing attacks for VFL (refer to Section 2.3). In the following, we present how to propose the AsyREVEL algorithms.

3.3 AsyREVEL Algorithms

Given a function $F_i(\bar{x})$, a typical two-point stochastic gradient estimator for ZOO is defined as

$$\hat{\nabla} F_i(\bar{x}) = \frac{d}{\mu} [F_i(\bar{x} + \mu u_i) - F_i(\bar{x})] u_i$$  \hspace{1cm} (14)$$

where $\bar{x} \in \mathbb{R}^d \mu > 0$ is the smoothing parameter, and random directions $\{u_i\}$ are i.i.d. drawn from a specific distribution.

However, it is difficult to apply this zeroth-order estimation (ZOE) technique to the VFL due to the much different problem form and application scenario. Specifically, the models to be optimized are distributed over the parties and the server but in a composite form. As shown in Fig. 2, each local model cascades to the global model and all local models are connected by this global model. Information such as model parameter and data sharing between parties is prohibited, which thus can prevent the data and model from directly leaking. Importantly, the function values transmitted between all parties and the server is black-box knowledge, which is useful to defense existing attacks for VFL (refer to Section 2.3). In the following, we present how to propose the AsyREVEL algorithms.

In this paper, we apply the ZOE technique to each model separately, i.e., to estimate $\frac{\partial F_m}{\partial w_m}$, $m = 0, 1, \cdots, q$. Because if we take all black-box models (both local and global) as a whole and then apply ZOE technique to estimate $\frac{\partial F_0}{\partial w_0}$, we can not leverage the feature-distributed character of VFL and can only design the synchronous algorithms. Moreover, we use the cascade relation between each $F_m$ ($m \in [q]$) and $F_0$ to compute the function value of $F_0$, and then use it to compute the zeroth-order estimation of $\frac{\partial F_0}{\partial w_m}$ directly. Note that we do not apply ZOE
The proposed AsyREVEL algorithm under our VFL framework is shown in Algorithm 1.

Motivated by the above analyses and Eq. (14), we defined the ZOE of $f_i$ with respect to (w.r.t.) $w_m$, $m = 1, \ldots, q$, as

$$\hat{\nabla}_m f_i(w_0, w) = \frac{d_m}{\mu_m} [f_i(w_m + \mu_m u_{i,m}) - f_i(w_m)] u_{i,m}$$

where $f_i(w_m + \mu_m u_{i,m}) = f_i(w_0, \ldots, w_m + \mu_m u_{i,m}, \ldots)$ denotes function $f_i(w_0, w)$ with the other parameters fixed and only $w_m$ as the variable, $d_m$ is the dimension of $w_m$, $\mu_m > 0$ is the smoothing parameter, and $\{u_{i,m}\}$ are i.i.d. random directions drawn from different distributions. For notation brevity, we define that $c_i = \{c_{i,m}\}_{m=1}^q$ contains function values of sample $i$ from all parties. And $f_i(w_m)$ and $f_i(w_m + \mu_m u_{i,m})$ are computed as follows.

$$f_i(w_m) = F_0(w_0, c_i) + \lambda g(w_m) = h_{i,m} + \lambda g(w_m),$$
$$f_i(w_m + \mu_m u_{i,m}) = F_0(w_0, c_{i,-m}) + \lambda g(w_m + \mu_m u_{i,m}) = \bar{h}_{i,m} + \lambda g(w_m + \mu_m u_{i,m})$$

where $c_{i,-m} = \{c_{i,j}\}_{j \neq m} \hat{c}_{i,m}$ means $c_i$ with $c_{i,m}$ being replaced by $\hat{c}_{i,m} = F_m(w_m + \mu_m u_{i,m}; x_{i,m})$. For $w_0$, there is

$$\hat{\nabla}_0 f_i = \frac{d_{w_0}}{\mu_m} (\bar{h}_{i,m} - h_{i,m}) u_{i,m},$$

where $\bar{h}_{i,m} = F_0(w_0 + \mu_m u_{i,m}, c_i)$.

**AsyREVEL algorithm:** The proposed AsyREVEL algorithm under our VFL framework is shown in Algorithm 1. At step 4, the activated party $m$ computes $c_{i,m}$ and $\hat{c}_{i,m}$ using its private data and local model and then sent them to the server. When the server receives $c_{i,m}$ and $\hat{c}_{i,m}$ from party $m$, it uses them together with the other parties’ function values received previously (stored in the server) to compute $h_{i,m}, \bar{h}_{i,m}$ and $\bar{h}_{i,m}$. Note that those function values of the other $q-1$ parties are steal due to the asynchronously updating. At step 9, the server then uses $h_{i,m}$ and $\bar{h}_{i,m}$ to compute the ZOE of $\nabla_0 f_i$ following Eq. (17). For client $m$, it needs to query the server for the values of $h_{i,m}$ and $\bar{h}_{i,m}$ and then uses them to compute the ZOE of local gradient at step 6. Note that, $w$ used at step 6 is the steal state of $w$ because of both the asynchronous updates and communication delay. An auxiliary illustration of these steps is shown in Fig. 2.

Moreover, we consider two different AsyREVEL algorithms, i.e., AsyREVEL-Gau and -Uni. Specifically, the algorithmic steps of them are the same as those of Algorithm 1, while the random directions used in Eqs. (15) and (17) are i.i.d. drawn from a zero-mean isotropic multivariate Gaussian distribution for AsyREVEL-Gau and a uniform distribution over a unit sphere for AsyREVEL-Uni.

### 4. Convergence Analysis and Complexity Analysis

In this section, we provide the convergence and complexity analyses of our proposed AsyREVEL algorithms. Note that we only give the sketch of convergence analysis and one can refer to the arXiv version of this paper for the details. First we present some preliminaries necessary for the convergence analysis.

**Assumption 1** Function $f$ is bounded below that is,

$$f^* := \inf_{[w_0, w] \in \mathbb{R}^d} f(w_0, w) > -\infty.$$  \hspace{1cm} (18)

**Assumption 2** For $f_i$, $i = 1, \ldots, n$ in problem (P), we assume the following conditions hold:

Lipschitz Gradient: $\nabla f_i$ is $L$-Lipschitz continuous, i.e., there exists a constant $L$ for $\forall [w_0, w], [w'_0, w']$ such that

$$\|\nabla f_i(w_0, w) - \nabla f_i(w'_0, w')\| \leq L \|[w_0, w] - [w'_0, w']\|.$$
and there exists an $L_m > 0$ for $m = 0, \ldots, q$ such that $\nabla_m f_i$ is $L_m$-Lipschitz continuous.

**Bounded Block-Coordinate Gradient:** For $m = 0, \ldots, q$, there exists a constant $\sigma_m$ such that $\|\nabla_m f_i(w_0, w)\|^2 \leq \sigma_m^2$.

Above assumptions are standard in previous optimization works Zhang et al. (2021c); Huang et al. (2019a,c,b), where Assumption 1 guarantees the feasibility of problem (P), Assumption 2 imposes (block-coordinate) smoothness on the individual functions and introduces bounded block-coordinate gradients. We also introduced Assumption 3 to handle the asynchronous updates, which is helpful for tracking the behavior of the global model.

**Assumption 3** The activated client $m_t$ is independent of $m_0, \ldots, m_{t-1}$ and satisfies $\mathbb{P}(m_t = m) := p_m$

Moreover, the function values of the other $q-1$ parties used to compute $h_{i,m}$ (or $\hat{h}_{i,m}, \tilde{h}_{i,m}$) are stolen due to the asynchronously updating manner and possible communication delay. To handle this case, we introduce the following assumption to bound the delay.

**Assumption 4** Bounded Delay: For $\bar{w}_t$ that is the $w$ used for computing at current iteration $t$, there is

$$
\bar{w}_t = w^{t-\tau_t^{n,m}} = w^t + \eta_{m_t} \sum_{t' \in D'(t)} \tilde{v}_{m_t,t'}^{t'}. 
$$

where $D'(t) = \{t-1, \ldots, t - \tau_t^{n,m}\}$ is a subset of previous iterations and $\tau_t^{n,m} \leq \tau$.

### 4.1 Convergence Analyses

**Theorem 2** Under Assumptions 1-4, to solve problem $P$ with AsyREVEL-Gau, let $\eta = \min\left\{\frac{1}{4(\tau + 1) L^*}, \frac{\mu_0}{\sqrt{T}}\right\}$ with constant $m_0 > 0$ and $\mu_0 = O\left(\frac{1}{\sqrt{T}}\right)$ such as $\mu_0 = \frac{1}{\sqrt{T} \bar{d}_L d}$, then we have

$$
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\|\nabla f(w_0, w)\|^2 \leq \frac{4 p_* (f^0 - f^*)}{\sqrt{T} m_0} + \frac{8 p_* m_0 (L + \tau L) \sigma_*^2}{\sqrt{T}} + \frac{(q + 1) + (q + 1) + 3 p_*}{2T} \frac{1}{L^* \bar{d}_L^2} 
$$

Figure 3: Results for solving black-box federated learning problem on different datasets.
Table 2: Dataset Descriptions.

|          | For logistic regression task | For deep learning task |
|----------|-----------------------------|------------------------|
| #Samples | D1 24,000                    | D7 60,000              |
|          | D2 96,257                    | D8 60,000              |
|          | D3 677,399                   |                        |
|          | D4 32,561                    |                        |
|          | D5 45,749                    |                        |
|          | D6 400,000                   |                        |
|          | D7 60,000                    |                        |
| #Features| 90                          | 784                    |
|          | 92                          | 784                    |
|          | 47,236                      |                        |
|          | 127                         |                        |
|          | 300                         |                        |
|          | 2,000                       |                        |

where $d^* = \max_m d_m + 3$, $p^* = \min_m p_m$, $\tau$ is independent of $T$.

**Theorem 3** Under Assumptions 1-4, to solve problem $P$ with AsyREVEL-Uni, let $\eta = \min\{\frac{1}{4(\tau+1)L}, \frac{m_0}{\sqrt{T}}\}$ with constant $m_0 > 0$ and $\mu_m = O(\frac{1}{\sqrt{TM^*d^*}})$, then we have

$$
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \| \nabla f(w_0, w) \|^2 \leq \frac{4p^*(f^0 - f^*)}{\sqrt{Tm_0}} + \frac{8p^*m_0(L + \tau L)d^*_s}{\sqrt{T}}
\frac{(q + 1)}{2T^2L_s d^*_s} + \frac{(q + 1) + 3p^*}{2T} \tag{21}
$$

where $d^* = \max_m d_m$, $p^* = \min_m p_m$, $\tau$ is independent of $T$.

**Remark 1** Under Assumptions 1-4, given the parameters in corresponding theorems, the convergence rates of both AsyREVEL-Gau and -Uni are $O(\frac{1}{\sqrt{T}})$.

### 4.2 Complexity Analyses

The total computation complexity at steps 4, 6 and 7 is $O(d_m)$, and that at steps 10, 11 and 12 is $O(d_0)$. Thus, the whole computation complexity of Algorithm 1 is $O(d_m + d_0)$. Importantly, only the local outputs and global outputs are transmitted between the party $m$ and the server, and the total communication complexity of Algorithm 1 is $O(1)$. Thus, our framework is communication-inexpensive compared with those transmitting the (intermediate) gradients.

### 5. Experiments

In this section, we implement extensive experiments to demonstrate the model applicability, privacy security, inexpensive communication and efficient computation of our proposed algorithms. Moreover, we also show that AsyREVEL is scalable and lossless.

**Experiment Settings:** All experiments are performed on a machine with four sockets, and each socket has 12 cores. The MPI is used for communication. Following previous works, we vertically partition the data into $q$ non-overlapped parts with nearly equal number of features. An optimal $\eta$ for all clients is chosen from $\{5e^{-1}, 1e^{-1}, \ldots\}$, and the learning rate for server is $\eta/q$.

**Datasets:** We use eight datasets for evaluation, which are summarized in Table 2, among which $D_1$ (UCICreditCard), $D_2$ (GiveMeSomeCredit), $D_3$ (Rcv1), $D_4$ (a9a), $D_5$ (w8a) and $D_6$ (Epsilon) are used for logistic regression problem, $D_7$ (MNIST) and $D_8$ (Fashion MNIST) are used for the deep learning tasks.

**Framework for Comparison:** We introduce a framework that has the same structure of our framework but directly transmits the intermediate gradient (called TIG-based framework, refer to Liu et al. (2020); Vepakomma et al. (2018) for details) instead of the function values. Specifically, in TIG-based framework, intermediate gradient $\frac{\partial F_0}{\partial F_m}$ is computed by the server and transmitted to party $m$, and then party $m$ uses the chain rule, i.e., $\frac{\partial F_0}{\partial w_m} = \frac{\partial F_0}{\partial F_m} \frac{\partial F_m}{\partial w_m}$ to compute the local gradient.
In these experiments, we set

\[
\eta_{AsyREVEL-Gau}\quad \text{and}\quad \lambda
\]

where \(q\) (ReLU) and the global model is a 1-layer (\(\text{FCN}\)).

To demonstrate the efficiency of asynchronous computation, we compare AsyREVEL algorithm with its synchronous counterpart, i.e. SynREVEL. When implementing the synchronous algorithms, there is a synthetic straggler party which maybe 20% to 60% slower than the faster one to simulate the industry application scenario.

Asynchronous Efficiency: In these experiments, we set \(q = 8\). As shown in Fig. 3, the loss v.s. runtime curves demonstrate that our algorithms are more computation-efficient than the synchronous ones.

| Algorithm        | \(D_1(\%)\) | \(D_2(\%)\) | \(D_3(\%)\) | \(D_4(\%)\) | \(D_5(\%)\) | \(D_6(\%)\) | \(D_7(\%)\) | \(D_8(\%)\) |
|------------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| NonF             | 81.93±0.36  | 93.50±0.28  | 95.24±0.06  | 85.16±0.08  | 89.85±0.08  | 87.79±0.09  | 91.89±0.25  | 81.32±0.11  |
| AsyREVEL-Gau     | 81.93±0.24  | 93.50±0.31  | 95.24±0.14  | 85.16±0.08  | 89.85±0.10  | 87.79±0.11  | 91.89±0.29  | 81.32±0.15  |
| NonF             | 81.88±0.10  | 93.48±0.09  | 95.14±0.12  | 85.14±0.12  | 89.88±0.07  | 87.89±0.12  | 91.84±0.32  | 81.45±0.11  |
| AsyREVEL-Uni     | 81.88±0.14  | 93.48±0.11  | 95.14±0.09  | 85.14±0.09  | 89.88±0.12  | 87.89±0.07  | 91.84±0.38  | 81.45±0.09  |

5.1 Evaluation of Favorable Model Applicability

5.2 Evaluation of Inexpensive Communication

5.3 Evaluation of Computation Efficiency
Asynchronous Scalability: We also consider the asynchronous speedup scalability in terms of $q$. Given $q$ parties, there is

$$q\text{-parties speedup} = \frac{\text{training time of using 1 party}}{\text{training time of using } q \text{ parties}}, \quad (23)$$

where training time is the time spending on reaching a certain precision of sub-optimality, i.e., $5e^{-4}$ for $D_4$. The results are shown in Fig. 4, which demonstrate that our asynchronous algorithms has much better $q$-parties speedup scalability than the synchronous ones and can achieve near linear speedup.

5.4 Evaluation of Losslessness

To demonstrate the losslessness of our algorithms, we compare AsyREVEL with its non-federated (NonF) counterpart whose only difference to AstREVEL is that all data are integrated together for modeling. For datasets without testing data, we split the data set into 10 parts, and use one of them for testing. Each comparison is repeated 10 times with $q = 8$, and a same stop criterion, e.g., $5e^{-4}$ for $D_4$. As shown in Table 4, the accuracies of our algorithms are the same with those of NonF algorithms.

6. Conclusion

In this paper, we revealed that ZOO is a desirable companion for VFL. Specifically, ZOO can 1) improve the model applicability of VFL framework. 2) prevent VFL framework from attacks under three levels of threat models, i.e., the curious, colluding, and malicious. 3) support inexpensive communication and efficient computation. We proposed a novel practical VFL framework with black-box models, which inherits the promising properties of ZOO. Under this framework, we raised the novel AsyREVEL algorithms with two smoothing techniques. Moreover, we prove the privacy security of ZOO-VFL under different attacks and theoretically drive the convergence rates of AsyREVEL algorithms under nonconvex condition.
Appendix

Lemma 1 Suppose that Assumption 2 holds, then we have
1) $f_{\mu_m}$ is $L_m$-smooth and $f_\mu$ is $L$-smooth

$$\nabla_m f_{\mu_m} = E_{u_m} [\nabla_m f(w_0, \mathbf{w})], \quad \nabla f_\mu = E_u [\nabla f(w_0, \mathbf{w})]$$  \hspace{1cm} (24)

where $\nabla_m f(w_0, \mathbf{w})$ is given by Eq. (6).
2) For any $w_m \in \mathbb{R}^{d_m}$,

$$|f_{\mu_m}(w_m) - f(w_m)| \leq \frac{L_m d_m \mu_m^2}{2}$$  \hspace{1cm} (25)

$$\|\nabla_m f_{\mu_m}(w_0, \mathbf{w}) - \nabla_m f(w_0, \mathbf{w})\|^2 \leq \frac{\mu_m^2 L_m^2 (d_m + 3)^3}{4},$$  \hspace{1cm} (26)

$$E_u \left[\|\nabla_m f(w_0, \mathbf{w})\|^2\right] \leq 2(d_m + 4) \|\nabla_m f(w_0, \mathbf{w})\|^2 + \frac{\mu_m^2 L_m^2 (d_m + 6)^3}{2}.$$  \hspace{1cm} (27)

3) For any $w_m \in \mathbb{R}^{d_m}$,

$$E_u \left[\|\nabla_m f(w_0, \mathbf{w}) - \nabla_m f_{\mu_m}(w_0, \mathbf{w})\|^2\right] \leq 2(2d_m + 9) \|\nabla_m f(w_0, \mathbf{w})\|^2 + \mu_m^2 L_m^2 (d_m + 6)^3.$$  \hspace{1cm} (28)

Lemma 2 Under Assumptions 1 to 4, for $m = 0, 1, \ldots, q$ there is

$$E\|\tilde{v}_{mt}^t - v_{mt}^t\|^2 = E\|\tilde{v}_{mt}^t - \tilde{v}_{mt}^t + \tilde{v}_{mt}^t - v_{mt}^t\|^2$$

$$\leq 2E\|\tilde{v}_{mt}^t - \tilde{v}_{mt}^t\|^2 + 2E\|\tilde{v}_{mt}^t - v_{mt}^t\|^2$$

$$\leq \frac{\mu_m^2 L_m^2 (d_m + 3)^3}{2} + 2L^2 \|\tilde{w}^t - w^t\|^2$$  \hspace{1cm} (29)

and

$$E\|\tilde{v}_{mt}^t\|^2 = E\|\tilde{v}_{mt}^t - v_{mt}^t + v_{mt}^t\|^2$$

$$\leq 2E\|\tilde{v}_{mt}^t - v_{mt}^t\|^2 + 2E\|v_{mt}^t\|^2$$

$$\leq \mu_m^2 L_m^2 d_m^2 + 4L^2 \|\tilde{w}^t - w^t\|^2 + 2E\|v_{mt}^t\|^2$$

$$\leq 3\mu_m^2 L_m^2 (d_m + 3)^3 + 4L^2 \|\tilde{w}^t - w^t\|^2 + 4\sigma_{mt}^2$$  \hspace{1cm} (30)

Proof

Under Assumption 2, and taking expectation w.r.t. the sample index $i_t$, we have

$$E f_{\mu_m}(w_0^{t+1}, w^{t+1})$$

$$= E \left(f_{\mu_m}(w_0 - \eta_0 \tilde{v}_0^t, \cdots, w_{mt} - \eta_{mt} \tilde{v}_{mt}^t, \cdots)\right)$$

$$\leq E \left(f_{\mu_m}(w_0^t, w^t) - \eta_0 \langle V_0^t, \tilde{v}_0^t \rangle - \eta_{mt} \langle V_{mt}^t, \tilde{v}_{mt}^t \rangle + \frac{L_n \eta_0^2}{2} \|\tilde{v}_0^t\|^2 + \frac{L \eta_{mt}^2}{2} \|\tilde{v}_{mt}^t\|^2\right)$$

$$= E \left(f_{\mu_m}(w_0^t, w^t) - \eta_0 \langle V_0^t, \tilde{v}_0^t - v_0^t + v_0^t \rangle - \eta_{mt} \langle V_{mt}^t, \tilde{v}_{mt}^t - v_{mt}^t + v_{mt}^t \rangle + \frac{L_n \eta_0^2}{2} \|\tilde{v}_0^t\|^2 + \frac{L \eta_{mt}^2}{2} \|\tilde{v}_{mt}^t\|^2\right)$$

$$\leq E \left(f_{\mu_m}(w_0^t, w^t) - \eta_0 \|V_0^t\|^2 - \eta_0 \langle V_0^t, \tilde{v}_0^t - v_0^t \rangle - \eta_{mt} \|V_{mt}^t\|^2 - \eta_{mt} \langle V_{mt}^t, \tilde{v}_{mt}^t - v_{mt}^t \rangle + \frac{L_n \eta_0^2}{2} \|\tilde{v}_0^t\|^2 + \frac{L \eta_{mt}^2}{2} \|\tilde{v}_{mt}^t\|^2\right)$$

$$\leq E \left(f_{\mu_m}(w_0^t, w^t) - \eta_0 \|V_0^t\|^2 + \eta_0 \|\tilde{v}_0^t\|^2 - \eta_{mt} \|V_{mt}^t\|^2 + \eta_{mt} \|\tilde{v}_{mt}^t\|^2 + \frac{L_n \eta_0^2}{2} \|\tilde{v}_0^t\|^2 + \frac{L \eta_{mt}^2}{2} \|\tilde{v}_{mt}^t\|^2\right)$$

$$\leq E \left(f_{\mu_m}(w_0^t, w^t) - \eta_0 \|V_0^t\|^2 - \eta_0 \|\tilde{v}_0^t\|^2 + (\eta_0 + \eta_{mt} + 2L_n \eta_0^2 + 2L \eta_{mt}^2) L^2 \|\tilde{w}^t - w^t\|^2\right)$$
We than bound the term

\[
+ \left( \frac{\eta_m^2}{4} + \frac{3L\eta_m^2}{2} \right) \mu_m^2 L_m^2 (d_m + 3)^3 + \left( \frac{\eta_0^2}{4} + \frac{3L\eta_0^2}{2} \right) \mu_m^2 L_0^2 (d_0 + 3)^3 + 2L\eta_m^2 \sigma_m^2 + 2L\eta_0^2 \sigma_0^2
\]  

(31)

Taking expectation w.r.t. \( m_t \), and using Assumption 3, there is

\[
\mathbb{E} f_{\mu_m}(w_0^{t+1}, w^{t+1}) 
\leq \mathbb{E} f_{\mu_m}(w_0^t, w^t) - \frac{\eta_m}{2} \mathbb{E} \| V_0^t \|^2 - \sum_{m=1}^q p_m \frac{\eta_m}{2} \mathbb{E} \| V_m^t \|^2 + \left( \frac{\eta_0}{2} + 2L\eta_0^2 + \max_m (2L\eta_m^2 + \eta_m) \right) L^2 \mathbb{E} \| \hat{w}^t - w^t \|^2
\]

\[
+ \sum_{m=1}^q p_m \left( \frac{\eta_m}{4} + \frac{3L\eta_m^2}{2} \right) \mu_m^2 L_m^2 (d_m + 3)^3 + \left( \frac{\eta_0}{4} + \frac{3L\eta_0^2}{2} \right) \mu_m^2 L_0^2 (d_0 + 3)^3 + \sum_{m=1}^q p_m 2L\eta_m^2 \sigma_m^2 + 2L\eta_0^2 \sigma_0^2
\]  

(32)

According to Assumption 4, there is

\[
\| \hat{w}^t - w^t \|^2 = \| \sum_{i \in D(t)} w^{i+1} - w^i \|^2 \leq \tau \sum_{i=1}^\tau \| w_t^{t+1-i} - w_t^{t-i} \|^2
\]  

(33)

We then bound the term \( \| \hat{w}^t - w^t \|^2 \). First, for \( \mathbb{E} \| w^{t+1} - w^t \|^2 \)

\[
\mathbb{E} \| w^{t+1} - w^t \|^2 = \mathbb{E} \eta_m^2 \| \hat{w}_m \|^2 \leq \sum_{m=1}^q p_m \eta_m^2 (3\mu_m^2 L_m^2 (d_m + 3)^3 + 4\sigma_m^2) + \max_m \eta_m^2 4L^2 \| \hat{w}^t - w^t \|^2
\]  

(34)

Define a Lyapunov function as

\[
M^t = f_{\mu_m}(w_0^t, w^t) + \sum_{i=1}^\tau \theta_i \| w^{i+1} - w^i \|^2
\]  

(35)

Following Lemma 3 and Eq. 57, there is

\[
\mathbb{E} (M^{t+1} - M^t)
\]

\[
= \mathbb{E} \left( f_{\mu_m}(w_0^{t+1}, w^{t+1}) + \sum_{i=1}^\tau \theta_i \| w^{t+1+i} - w^{t+1-i} \|^2 - f_{\mu_m}(w_0^t, w^t) - \sum_{i=1}^\tau \theta_i \| w^{t+1-i} - w^{t-i} \|^2 \right)
\]

\[
= -\frac{\eta_0}{2} \mathbb{E} \| V_0^t \|^2 - \sum_{m=1}^q p_m \frac{\eta_m}{2} \mathbb{E} \| V_m^t \|^2 + \beta^t \mathbb{E} \| \hat{w}^t - w^t \|^2 + \sum_{m=1}^q p_m 2L\eta_m^2 \sigma_m^2 + 2L\eta_0^2 \sigma_0^2
\]

\[
+ \sum_{m=1}^q p_m \left( \frac{\eta_m}{4} + \frac{3L\eta_m^2}{2} \right) \mu_m^2 L_m^2 (d_m + 3)^3 + \left( \frac{\eta_0}{4} + \frac{3L\eta_0^2}{2} \right) \mu_m^2 L_0^2 (d_0 + 3)^3
\]

\[
+ \theta_1 \mathbb{E} \| w^{t+1} - w^t \|^2 + \sum_{i=1}^{\tau-1} (\theta_{i+1} - \theta_i) \mathbb{E} \| w^{t+1-i} - w^{t-i} \|^2 - \theta_\tau \mathbb{E} \| w^{t+1-\tau} - w^{t-\tau} \|^2
\]

\[
\leq -\frac{\eta_0}{2} \mathbb{E} \| V_0^t \|^2 - \sum_{m=1}^q p_m \frac{\eta_m}{2} \mathbb{E} \| V_m^t \|^2 + \beta^t \tau \sum_{i=1}^\tau \| w^{t+1-i} - w^{t-i} \|^2 + \sum_{m=1}^q p_m 2L\eta_m^2 \sigma_m^2 + 2L\eta_0^2 \sigma_0^2
\]

\[
+ \sum_{m=1}^q p_m \left( \frac{\eta_m}{4} + \frac{3L\eta_m^2}{2} \right) \mu_m^2 L_m^2 (d_m + 3)^3 + \left( \frac{\eta_0}{4} + \frac{3L\eta_0^2}{2} \right) \mu_m^2 L_0^2 (d_0 + 3)^3
\]

\[
+ \sum_{i=1}^{\tau-1} (\theta_{i+1} - \theta_i) \mathbb{E} \| w^{t+1-i} - w^{t-i} \|^2 - \theta_\tau \mathbb{E} \| w^{t+1-\tau} - w^{t-\tau} \|^2
\]
+ \theta_1(\sum_{m=1}^{q} p_m \eta_m^2 (3\mu_m^2 L_m^2 d_m^2 + 4\sigma_m^2) + \max_{m} \eta_m^2 4L_m^2 \tau \sum_{i=1}^{\tau} ||w^{i+1-i} - w^{i-i}||^2)
\leq -\frac{-\eta_0}{2} E\|V_0\|^2 - \sum_{m=1}^{q} p_m \eta_m \frac{\eta}{2} E\|V_m\|^2 + \sum_{m=1}^{q} p_m \eta_m^2 (L + 4\theta_1)\sigma_m^2 + 2L\eta_0^2 \sigma_0^2
+ \sum_{m=1}^{q} p_m (\eta_m^2 + 2\frac{3L\eta_m^2}{2} + 3\theta_1 \eta_m^2)\mu_m^2 L_m^2 (d_m + 3)^3 + (\frac{\eta_0^2}{2} + \frac{3L\eta_0^2}{2})\mu_m^2 L_0^2 (d_0 + 3)^3
+ \sum_{i=1}^{\tau-1} (\beta_t \tau + \tau \eta_1 \max_{m} \eta_m^2 4L^2 + \theta_{i+1} - \theta_i) E\|w^{i+1-i} - w^{i-i}||^2 + (\beta_t \tau + \tau \eta_1 \max_{m} \eta_m^2 4L^2 - \theta_{\tau}) E\|w^{i+1-\tau} - w^{i-\tau}||^2
\tag{36}

If we choose \eta_0, \eta_m \leq \eta_0 \leq \frac{1}{4(\tau + 1)\eta L} , then there is \beta_t \leq \frac{3\eta L^2}{2}. Then for Eq. 58 there is
\begin{align*}
\mathbb{E}(M^{t+1} - M^t)
\leq -\frac{1}{2} \min \{\eta_0, p_m \eta_m\} E\|\nabla f_{\mu_m}(w_0, w)\|^2 + \sum_{m=1}^{q} p_m \eta_m^2 (L + 4\theta_1)\sigma_m^2 + 2L\eta_0^2 \sigma_0^2
+ \sum_{m=1}^{q} p_m (\eta_m^2 + 2\frac{3L\eta_m^2}{2} + 3\theta_1 \eta_m^2)\mu_m^2 L_m^2 (d_m + 3)^3 + (\frac{\eta_0^2}{2} + \frac{3L\eta_0^2}{2})\mu_m^2 L_0^2 (d_0 + 3)^3
- \sum_{i=1}^{\tau-1} (\theta_t - \theta_{i+1} - \frac{3}{2} \eta L^2 \tau - 4\tau \theta_1 L^2 \eta^2) E\|w^{i+1-i} - w^{i-i}||^2 - (\theta_t - \frac{3}{2} \eta L^2 \tau - 4\tau \theta_1 L^2 \eta^2) E\|w^{i+1-\tau} - w^{i-\tau}||^2
\tag{37}
\end{align*}

Let \theta_1 = \frac{3/2 \eta L^2}{1- 4\tau \eta^2 L^2} \leq \frac{1}{2} \tau L and \eta_0 = \eta_m = \eta \leq \frac{1}{4(\tau + 1)\eta L} and choose \theta_2, \cdots , \theta_\tau as
\begin{align*}
\theta_{i+1} = \theta_t - \frac{3}{2} \eta L^2 \tau - 4\tau \theta_1 L^2 \eta^2 \quad \text{for } i = 1, \cdots , \tau - 1
\tag{38}
\end{align*}

Following form Eq. xx and the definition of \theta_1, there is \theta_\tau = \theta_t - (\tau - 1) \frac{3\eta L^2}{2} \tau - 4(\tau - 1)\tau \theta_1 L^2 \eta^2 \geq 0. Then Eq. 59 reduces to
\begin{align*}
\mathbb{E}(M^{t+1} - M^t)
\leq -\frac{1}{2} \min \{\eta_0, p_m \eta_m\} E\|\nabla f_{\mu_m}(w_0, w)\|^2 + 2L\eta^2 \sigma_0^2
+ \sum_{m=1}^{q} p_m \eta_m^2 (L + 2\tau L)\sigma_m^2 + \sum_{m=1}^{q} p_m (\eta_m^2 + 2\frac{3L\eta_m^2}{2} + 3\tau \eta_m^2)\mu_m^2 L_m^2 (d_m + 3)^3 + (\frac{\eta_0^2}{2} + \frac{3L\eta_0^2}{2})\mu_m^2 L_0^2 (d_0 + 3)^3
\tag{39}
\end{align*}

Summing Eq. 61 over \( t = 0, \cdots , T - 1 \), there is
\begin{align*}
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\|\nabla f_{\mu_m}(w_0, w)\|^2 \leq \frac{f_0^{\mu_m} - f_{\mu_m}}{2} \min_m p_m T \eta + \frac{\sum_{m=1}^{q} p_m \eta(L + 2\tau L)\sigma_m^2 + 2L\eta\sigma_0^2}{\frac{1}{2} \min_m p_m}
+ \frac{\sum_{m=1}^{q} p_m (\frac{1}{4} + \frac{3L}{2} \tau \eta)\mu_m^2 L_m^2 (d_m + 3)^3 + (\frac{1}{4} + \frac{3L}{2} \tau \eta)\mu_m^2 L_0^2 (d_0 + 3)^3}{\frac{1}{2} \min_m p_m}
\tag{40}
\end{align*}

According to Lemma 3, there is
\begin{align*}
\mathbb{E}\|\nabla_m f(w_0, w)\|^2 \leq 2\mathbb{E}\|\nabla_m f_{\mu_m}(w_0, w)\|^2 + \frac{\mu_m^2 L_m^2 (d_m + 3)^3}{2}.
\tag{41}
\end{align*}
Thus, there is
\[
\mathbb{E}\|\nabla f(w_0, w)\|^2 \leq 2 \sum_{m=0}^{q} \mathbb{E}\|\nabla f_{\mu_m}(w_0, w)\|^2 + \sum_{m=0}^{q} \frac{\mu_m^2 L_m^2 (d_m + 3)^3}{2} \\
\leq 2\mathbb{E}\|\nabla f_{\mu_m}(w_0, w)\|^2 + \sum_{m=0}^{q} \frac{\mu_m^2 L_m^2 (d_m + 3)^3}{2}.
\] (42)

Similarly, according to Lemma 3, there is
\[
f(w_0, w_0) - f^* \leq f_{\mu_m}(w_0, w_0) - f^* + \sum_{m=0}^{q} L_m d_m \mu_m^2 
\] (43)

Applying Eqs. 64 and 65 to Eq. 62, there is
\[
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\|\nabla f(w_0, w)\|^2 \leq \frac{f^0 - f^*}{T} + \sum_{m=1}^{q} \frac{p_m \eta (L + 2\tau L) \sigma_m^2 + 2L \eta \sigma_m^2}{\frac{1}{4} \min_m p_m T} + \sum_{m=0}^{q} \frac{L_m d_m \mu_m^2}{2T} \\
+ \sum_{m=0}^{q} \frac{\mu_m^2 L_m^2 (d_m + 3)^3}{2} + \sum_{m=0}^{q} \frac{p_m (1 + \frac{3}{2} \tau L) \mu_m^2 L_m^2 (d_m + 3)^3}{\frac{1}{4} \min_m p_m ^2 T}
\] (44)

Let \( L_* = \max\{\{L_m\}_{m=0}^{q}, L\}, d_* = \max\{d_m + 3\}_{m=0}^{q}, \sigma_*^2 = \max_m \sigma_m^2, \frac{1}{p_*} = \min_m p_m \), then Eq. xx reduces to
\[
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\|\nabla f(w_0, w)\|^2 \leq \frac{4p_* (f^0 - f^*)}{T \eta} + 8p_* (L + \tau L) \eta \sigma_*^2 + \frac{(q + 1) L_* d_* \mu_*^2}{2T} + \frac{(q + 1) \mu_*^2 L_*^2 d_*^3}{2} \\
+ p_* (2 + 3L_* \eta + \frac{3}{2} \tau L_* \eta) \mu_*^2 L_*^2 d_*^3
\] (45)

Choosing \( \eta = \min\{\frac{1}{4(\tau + 1) L}, \frac{m_0}{\sqrt{T}}\} \) with constant \( m_0 > 0 \) and \( \mu_m = \mathcal{O}\left(\frac{1}{\sqrt{T}}\right) \) such as \( \mu_m = \frac{1}{\sqrt{\tau L_* d_*^2}} \), there is
\[
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\|\nabla f(w_0, w)\|^2 \leq \frac{4p_* (f^0 - f^*)}{\sqrt{T} m_0} + \frac{8p_* m_0 (L + \tau L) \sigma_*^2}{\sqrt{T}} + \frac{(q + 1)}{2T^2 L_* d_*^2} + \frac{(q + 1) + 3p_*}{2T}
\] (46)

Thus, if \( \tau \) is a constant independent to \( T \), then there
Lemma 3 Suppose that Assumption 2 holds, then we have
1) \( f_{\mu_m}(w_m) \) is \( L_m \)-smooth and \( f_{\mu_m}(w_0, w) \) is \( L \)-smooth

\[
\nabla_m f_{\mu_m}(w_0, w) = E_u \left[ \nabla f(w_0, w) \right] \tag{47}
\]
\[
\nabla f_{\mu_m}(w_0, w) = E_u \left[ \nabla f(w_0, w) \right] \tag{48}
\]

where \( u \) is drawn from the uniform distribution over the unit Euclidean sphere, and \( \nabla_w f(w_0, w) \) is given by Eq. (6).

2) For any \( w_m \in \mathbb{R}^{d_m} \),

\[
|f_{\mu_m}(w_m) - f(w_m)| \leq \frac{L_m d_m \mu_m^2}{2} \tag{49}
\]
\[
\|\nabla_m f_{\mu_m}(w_0, w) - \nabla_m f(w_0, w)\|^2 \leq \frac{\mu_m^2 L_m^2 d_m^2}{4}, \tag{50}
\]

Lemma 4 Under Assumptions 1 to 4, for \( m = 0, 1, \ldots, q \) there is

\[
E\|\hat{v}_{m+1}^t - v_m^t\|^2 = E\|\hat{v}_{m+1}^t - \hat{v}_{m+1}^t + \hat{v}_{m+1}^t - v_m^t\|^2 \leq 2E\|\hat{v}_{m+1}^t - \hat{v}_{m+1}^t\|^2 + 2E\|\hat{v}_{m+1}^t - v_m^t\|^2 \leq \frac{\mu_m^2 L_m^2 d_m^2}{2} + 2L^2\|\hat{w}^t - w^t\|^2 \tag{51}
\]

and

\[
E\|\hat{v}_{m+1}^t\|^2 = E\|\hat{v}_{m+1}^t - v_m^t + v_m^t\|^2 \leq 2E\|\hat{v}_{m+1}^t - v_m^t\|^2 + 2E\|v_m^t\|^2 \leq \mu_m^2 L_m^2 d_m^2 + 4L^2\|\hat{w}^t - w^t\|^2 + 2E\|v_m^t\|^2 \leq 3\mu_m^2 L_m^2 d_m^2 + 4L^2\|\hat{w}^t - w^t\|^2 + 4\sigma_m \tag{52}
\]

Proof Under Assumption 2, and taking expectation w.r.t. the sample index \( i_t \), we have

\[
E f_{\mu_m}(w_0^{t+1}, w^{t+1}) = E \left( f_{\mu_m}(w_0^{t} - \eta_t \hat{v}_0^t, \ldots, w_0^{t} - \eta_t \hat{v}_m^t, \ldots) \right) \leq E \left( f_{\mu_m}(w_0^{t}, w^{t}) - \eta_t \left( V_0^t, \hat{v}_0^t, \ldots, V_m^t, \hat{v}_m^t \right) \right) + \frac{L\eta_0^2}{2} \|\hat{v}_0^t\|^2 + \frac{L\eta_m^2}{2} \|\hat{v}_m^t\|^2 \leq E \left( f_{\mu_m}(w_0^{t}, w^{t}) - \eta_t \left( V_0^t, \hat{v}_0^t, \ldots, V_m^t, \hat{v}_m^t \right) \right) + \frac{L\eta_0^2}{2} \|\hat{v}_0^t\|^2 + \frac{L\eta_m^2}{2} \|\hat{v}_m^t\|^2 + \frac{L^2}{2} \|\hat{w}^t - w^t\|^2 \tag{53}
\]
Following Lemma 3 and Eq. 57, there is

\[ \begin{align*}
\text{Define a Lyapunov function as}
\end{align*} \]

\[ \begin{align*}
\theta \left\| \tilde{w}^t \right\| + \left( \frac{\eta_0}{4} + \frac{3L_0^2}{2} \right) \mu_m^2 L_m^2 d_m^2 + \left( \frac{\eta_0}{4} + \frac{3L_0^2}{2} \right) \mu_m L_0^2 d_0^2 + \left( \frac{\eta_0}{4} + \frac{3L_0^2}{2} \right) \mu_m^2 L_m^2 d_m^2 + \left( \frac{\eta_0}{4} + \frac{3L_0^2}{2} \right) \mu_m^2 L_m^2 d_m^2 + \left( \frac{\eta_0}{4} + \frac{3L_0^2}{2} \right) \mu_m L_0^2 d_0^2
\end{align*} \]

(54)

According to Assumption 4, there is

\[ \left\| \tilde{w}^t - w^t \right\|^2 \leq \left\| \sum_{i = 1}^{\tau} \tilde{w}^{t+1-i} - w^{t-i} \right\|^2 \]

\[ \left( \frac{\eta_0}{4} + \frac{3L_0^2}{2} \right) \mu_m L_0^2 d_0^2 \]

(55)

We then bound the term \( \left\| \tilde{w}^t - w^t \right\|^2 \). First, for \( \mathbb{E} \| w^{t+1} - w^t \|^2 \)

\[ \begin{align*}
\mathbb{E} \| w^{t+1} - w^t \|^2 &= \mathbb{E} \left( \sum_{i = 1}^{\tau} \tilde{w}^{t+1-i} - w^{t-i} \right) \left( \sum_{i = 1}^{\tau} \tilde{w}^{t+1-i} - w^{t-i} \right)
\end{align*} \]

(56)

Define a Lyapunov function as

\[ \begin{align*}
M^t \geq f_{\mu_m}(w_0^t, w^t) \sum_{i = 1}^{\tau} \left\| \tilde{w}^{i+1} - w^i \right\|^2
\end{align*} \]

(57)

Following Lemma 3 and Eq. 57, there is

\[ \begin{align*}
\mathbb{E} (M^{t+1} - M^t)
\end{align*} \]

(58)

\[ \begin{align*}
\leq \frac{\eta_0}{2} \mathbb{E} \left( \sum_{i = 1}^{\tau} \right) \tilde{w}^{t+1-i} - w^{t-i} \left( \sum_{i = 1}^{\tau} \right) \tilde{w}^{t+1-i} - w^{t-i} \right)
\end{align*} \]

(59)

\[ \begin{align*}
+ \sum_{m = 1}^{q} \left( \frac{\eta_m}{4} + \frac{3L_m^2}{2} \right) \mu_m^2 L_m^2 d_m^2 + \left( \frac{\eta_0}{4} + \frac{3L_m^2}{2} \right) \mu_m L_0^2 d_0^2
\end{align*} \]

(60)

\[ \begin{align*}
+ \sum_{m = 1}^{q} \left( \frac{\eta_m}{4} + \frac{3L_m^2}{2} \right) \mu_m L_0^2 d_0^2
\end{align*} \]

(61)

\[ \begin{align*}
\geq \frac{\eta_0}{2} \mathbb{E} \left( \sum_{i = 1}^{\tau} \right) \tilde{w}^{t+1-i} - w^{t-i} \left( \sum_{i = 1}^{\tau} \right) \tilde{w}^{t+1-i} - w^{t-i} \right)
\end{align*} \]

(62)

\[ \begin{align*}
+ \sum_{m = 1}^{q} \left( \frac{\eta_m}{4} + \frac{3L_m^2}{2} \right) \mu_m^2 L_m^2 d_m^2 + \left( \frac{\eta_0}{4} + \frac{3L_m^2}{2} \right) \mu_m L_0^2 d_0^2
\end{align*} \]

(63)

\[ \begin{align*}
\leq \frac{\eta_0}{2} \mathbb{E} \left( \sum_{i = 1}^{\tau} \right) \tilde{w}^{t+1-i} - w^{t-i} \left( \sum_{i = 1}^{\tau} \right) \tilde{w}^{t+1-i} - w^{t-i} \right)
\end{align*} \]

(64)

\[ \begin{align*}
+ \sum_{m = 1}^{q} \left( \frac{\eta_m}{4} + \frac{3L_m^2}{2} \right) \mu_m L_0^2 d_0^2
\end{align*} \]

(65)

\[ \begin{align*}
\leq \frac{\eta_0}{2} \mathbb{E} \left( \sum_{i = 1}^{\tau} \right) \tilde{w}^{t+1-i} - w^{t-i} \left( \sum_{i = 1}^{\tau} \right) \tilde{w}^{t+1-i} - w^{t-i} \right)
\end{align*} \]

(66)
\[ + \sum_{m=1}^{q} p_m (\eta_m^2 + \frac{3L_\eta_m^2}{2} + 3\theta_1 \eta_m^2) \mu_m^2 L_m^2 d_m^2 + (\frac{\eta_0}{4} + \frac{3L_\eta_0^2}{2}) \mu_0^2 L_0^2 d_0^2 \\
+ \sum_{i=1}^{\tau} (\beta_i \tau + \tau_1 \max_m \eta_m^2 4L_t^2 + \theta_{i+1} - \theta_i) E\|w^{t+1-i} - w^{t-i}\|^2 + (\beta_i \tau + \tau_1 \max_m \eta_m^2 4L_t^2 - \theta_i) E\|w^{t+1-\tau} - w^{t-\tau}\|^2 \]

(58)

If we choose \( \eta_0, \eta_m \leq \bar{\eta} \leq \frac{1}{4(L+2\theta_1)} \), then there is \( \beta^t \leq \frac{3\theta L^2}{2} \). Then for Eq. 58 there is

\[ E(M^{t+1} - M^t) \]

\[ \leq \frac{1}{2} \min_{m} \{\eta_m, p_m \eta_m\} E\|\nabla f_{\mu_m}(w_0, w)\|^2 + \sum_{m=1}^{q} p_m \eta_m^2 (L + 4 \theta_1) \sigma_m^2 + 2L_\eta^2 \sigma_0^2 \]

\[ + \sum_{m=1}^{q} p_m (\eta_m^2 + \frac{3L_\eta_m^2}{2} + 3\theta_1 \eta_m^2) \mu_m^2 L_m^2 d_m^2 + (\frac{\eta_0}{4} + \frac{3L_\eta_0^2}{2}) \mu_0^2 L_0^2 d_0^2 \]

\[ - \sum_{i=1}^{\tau} (\theta_i - \theta_{i+1} - \frac{3}{2} \bar{\eta} L_t^2 \tau - 4\tau_1 L_t^2 \eta^2) E\|w^{t+1-i} - w^{t-i}\|^2 - (\theta_{\tau} - \frac{3}{2} \bar{\eta} L_t^2 \tau - 4\tau_1 L_t^2 \eta^2) E\|w^{t+1-\tau} - w^{t-\tau}\|^2 \]

(59)

Let \( \theta_1 = \frac{3\theta L_t^2}{1 - 4\tau_1 \eta L_t^2} \leq \frac{1}{2} \tau L \) and \( \eta_0 = \eta_m = \eta \leq \frac{1}{4(\tau+1)\eta} \) and choose \( \theta_2, \cdots, \theta_\tau \)

\[ \theta_{i+1} = \theta_i - \frac{3}{2} \eta L_t^2 \tau - 4\tau_1 L_t^2 \eta^2, \quad \text{for } i = 1, \cdots, \tau - 1 \]

(60)

Following form Eq. 57 and the definition of \( \theta_1 \), there is \( \theta_\tau = \theta_1 - (\tau - 1) \frac{3\theta L_t^2}{2} \tau - 4(\tau - 1) \theta_1 L_t^2 \eta^2 \geq 0 \). Then Eq. 59 reduces to

\[ E(M^{t+1} - M^t) \leq \frac{1}{2} \min_{m} \eta \bar{\|\nabla f_{\mu_m}(w_0, w)\|^2 + 2L_\eta^2 \sigma_0^2} \]

\[ + \sum_{m=1}^{q} p_m \eta_m^2 (L + 2\tau L) \sigma_m^2 + \sum_{m=1}^{q} p_m (\eta_m^2 + \frac{3L_\eta_m^2}{2} + \frac{3\tau L_\eta_m^2}{2}) \mu_m^2 L_m^2 d_m^2 + (\frac{\eta_0}{4} + \frac{3L_\eta_0^2}{2}) \mu_0^2 L_0^2 d_0^2 \]

(61)

Summing Eq. over \( t = 0, \cdots, T - 1 \), there is

\[ \frac{1}{T} \sum_{t=0}^{T-1} E\|\nabla f_{\mu_m}(w_0, w)\|^2 \leq \frac{f_{\mu_m}^0 - f_{\mu_m}^*}{2 \min_{m} \eta_T^m T} + \sum_{m=1}^{q} p_m \eta (L + 2\tau L) \sigma_m^2 + 2L_\eta^2 \sigma_0^2 \]

\[ + \sum_{m=1}^{q} p_m (\frac{1}{4} + \frac{3L_\eta_m + \frac{3\tau L_\eta_m}{2}}{2}) \mu_m^2 L_m^2 d_m^2 + (\frac{1}{4} + \frac{3L_\eta_0^2}{2}) \mu_0^2 L_0^2 d_0^2 \]

(62)

According to Lemma 3, there is

\[ E\|\nabla m f(w_0, w)\|^2 \leq 2E\|\nabla m f_{\mu_m}(w_0, w)\|^2 + \frac{\mu_0^2 L_0^2 d_0^2}{2} \]

(63)

Thus, there is

\[ E\|\nabla f(w_0, w)\|^2 \leq 2 \sum_{m=0}^{q} E\|\nabla m f_{\mu_m}(w_0, w)\|^2 + \sum_{m=0}^{q} \frac{\mu_0^2 L_0^2 d_0^2}{2} \]

20
\[
\leq 2\mathbb{E}\|\nabla f_{\mu_m}(w_0, w)\|^2 + \sum_{m=0}^{q} \frac{\mu_m^2 L_m^2 d_m^2}{2}.
\] (64)

Similarly, according to Lemma 3, there is
\[
f(w_0, w^0) - f^* \leq f_{\mu_m}(w_0, w^0) - f^* + \sum_{m=0}^{q} \frac{L_m \mu_m^2}{2}.
\] (65)

Applying Eqs. 64 and 65 to Eq. 62, there is
\[
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\|\nabla f(w_0, w)\|^2 \leq \frac{f^0 - f^*}{\frac{1}{4} \min_m p_m T \eta} + \frac{\sum_{m=1}^{q} p_m \eta (L + 2\tau L) \sigma_m^2 + 2L \eta \sigma_0^2}{\frac{1}{4} \min_m p_m} + \sum_{m=0}^{q} \frac{L_m d_m \mu_m^2}{2T} + \sum_{m=0}^{q} \frac{\mu_m^2 L_m^2 d_m^2}{2}
\]
\[
+ \sum_{m=0}^{q} \frac{p_m (\frac{\frac{1}{4} + \frac{3L \eta}{2} + \frac{3}{2} \tau L \eta) \mu_m^2 L_m^2 d_m^2}{\frac{1}{4} \min_m p_m}}{2}.
\] (66)

Let \( L_s = \max\{\{L_m\}_{m=0}^{q}, L\}, d_s = \max\{d_m\}_{m=0}^{q}, \sigma_s^2 = \max_m \sigma_m^2, \frac{1}{p_s} = \min_m p_m \), then Eq. 62 reduces to
\[
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\|\nabla f(w_0, w)\|^2 \leq \frac{4p_s (f^0 - f^*)}{T \eta} + \frac{8p_s (L + \tau L) \eta \sigma_s^2}{T \eta} + \frac{(q + 1) L_s \mu_m^2}{2T} + \frac{(q + 1) \mu_m^2 L_s^2 d_s^2}{2}
\]
\[
+ p_s (2 + 3L_s \eta + \frac{3}{2} \tau L_s \eta) \mu_m^2 L_s^2 d_s^2.
\] (67)

Choosing \( \eta = \min\{\frac{1}{4(\tau + 1)L}, \frac{m_0}{\sqrt{T}}\} \) with constant \( m_0 > 0 \) and \( \mu_m = \mathcal{O}(\frac{1}{\sqrt{T}}) \) such as \( \mu_m = \frac{1}{\sqrt{T}L_s d_s} \), there is
\[
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\|\nabla f(w_0, w)\|^2 \leq \frac{4p_s (f^0 - f^*)}{\sqrt{T} m_0} + \frac{8p_s m_0 (L + \tau L) \sigma_s^2}{\sqrt{T}} + \frac{(q + 1) L_s^2 d_s^2}{2T} + \frac{(q + 1) + 3p_s}{2T}.
\] (68)

Thus, if \( \tau \) is a constant independent to \( T \), we can drive the corresponding result.
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