ON FRACTIONAL VECTOR OPTIMIZATION OVER CONES WITH SUPPORT FUNCTIONS

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ABSTRACT. In this paper we give necessary and sufficient optimality conditions for a fractional vector optimization problem over cones involving support functions in objective as well as constraints, using cone-convex functions. We also associate Mond-Weir type and Schaible type duals with the primal problem and establish weak and strong duality results under cone-convexity, pseudo-convexity and quasiconvexity assumptions. A number of previously studied problems appear as special cases.

1. Introduction. The optimization problems, in which the function to be maximized or minimized is given as a ratio of functions, are called fractional programming problems. The problems, where we have two or more such fractional functions to be optimized, are called multiobjective fractional programming problems. These problems arise in different areas of modern research such as game theory, goal programming, minimum risk problems, portfolio selection, production, information theory and numerous decision-making problems in management science and heat exchange networking. More specifically, it can be used in engineering and economics to minimize a ratio of physical or economical functions, or both, such as cost/time, cost/volume, cost/benefit, etc., in order to measure the efficiency or productivity of the system. Many economic, non-economic and indirect applications of fractional programming problems have also been given by Schaible and Ibaraki [23].

Those fractional programming problems where the functions concerned are linear in nature were introduced by Charnes and Cooper [3] in their classical paper in 1962. They used a transformation to convert a linear fractional objective function into an ordinary linear program. Then Dinkelbach [7] introduced the most known approach used for solving nonlinear fractional programming problems called the parametric approach. Later in 1976 Schaible [21] developed duality theory for linear and concave-convex fractional programs and in the same year he [22] proposed a revised version of Dinkelbach’s Algorithm using duality theory introduced in [21]. Liang et al. [15] introduced the concept of \((F, \alpha, p, d)\)-convexity and presented optimality and duality results for a class of nonlinear fractional programming problems which were based on the properties of sublinear functionals and generalized convex functions.

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Fractional programming problems have also been studied in nonsmooth settings where either the quotients are taken to be nondifferentiable or a support function of compact convex set is added to numerator or denominator of the objective function or both. Husain and Jabeen [8] worked in this direction and gave optimality conditions and duality results for a nonlinear fractional program in which a support function appears in the numerator and denominator of the objective function as well as in each constraint function.

Later Liang et al. [16] gave efficiency conditions and duality results for a class of multiobjective fractional programming problems using the concept of \((F, \alpha, \rho, d)\)-convexity introduced in [15]. Antczak [1] used a new method for solving nonlinear multiobjective fractional programming problems having \(V\)-invex objective and constraint functions with respect to the same function \(\eta\). Recently, Jayswal et al. [9] established sufficient optimality conditions and duality results for a multiobjective fractional programming problem under the assumption of \((p, r)\)-\((\eta, \theta)\)-invexity.

Also many authors have studied nonsmooth multiobjective fractional programming problems. For example, Kuk et al. [14] considered a class of such problems in which functions are taken to be locally Lipschitz and obtained optimality conditions and derived duality results using \((V, \rho)\)-invex functions. Kim et al. [12] studied a class of nondifferentiable multiobjective fractional programs in which the numerator of each component of the objective function contains a term involving the support function of a compact convex set. Recently Kim and Kim [13] considered a generalized nondifferentiable fractional optimization problem, which consists of a maximum objective function defined by finite differentiable fractional functions involving support functions and they have obtained optimality conditions and duality results by modifying the approach of Kim et al. [12].

In 2005, Kim [10] considered a multiobjective fractional programming problem in which all the concerned functions were taken to be continuously differentiable and the denominator of each objective function consisted of the same scalar function. He gave the necessary and sufficient optimality conditions and saddle point theorems under the generalized invexity assumptions. Then, in 2006, he [11] again considered the same problem with the only difference that all the concerned functions were taken to be locally Lipschitz. He also introduced the property of generalized invexity for fractional functions and presented necessary and sufficient optimality conditions and duality relations under suitable generalized invexity assumptions.

In this direction we keep a step forward and formulate a fractional vector optimization problem over arbitrary cones in which the constraint function, the numerator and denominator of each component of the objective function include support functions of compact convex sets and also each component of the objective function contains the same scalar function in the denominator. We give Fritz-John type necessary optimality conditions by applying the parametric approach and a result given by Craven [6]. Using the constraint qualification given by Suneja et al. [25], we prove KKT type necessary optimality conditions. We establish sufficient optimality conditions under cone-convexity assumptions and illustrate them with examples. We formulate Mond-Weir type and Schaible type dual and give weak and strong duality results. Finally, we relate our primal and dual problems with special cases that often occur in the literature in which a support function is the square root of a positive semi-definite quadratic form or an \(L_p\) norm.
2. Notations and definitions. Let $K \subseteq \mathbb{R}^m$ be a closed convex pointed $(K \cap (-K) = \{0\})$ cone such that int$K \neq \emptyset$, where int$K$ denotes the interior of $K$. The positive dual cone $K^+$ is defined as follows:

$$K^+ = \{ \lambda \in \mathbb{R}^m : \lambda^T x \geq 0, \text{ for all } x \in K \}.$$ 

In this section, we recall some of the basic definitions and results, which are to be used throughout the paper.

Let $h : \mathbb{R}^n \rightarrow \mathbb{R}$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, where $f = (f_1, f_2, \ldots, f_m)^T$.

**Definition 2.1.** The function $h$ is said to be convex at $\overline{x} \in \mathbb{R}^n$, if for all $x \in \mathbb{R}^n$ and $t \in [0, 1]$,

$$th(x) + (1-t)h(\overline{x}) \geq h(tx + (1-t)\overline{x}).$$

The function $h$ is said to be convex, if it is convex at each $\overline{x} \in \mathbb{R}^n$.

**Remark 1.** [2] If $f$ is a differentiable function, then $f$ is $K$-convex if and only if for all $x, \overline{x} \in \mathbb{R}^n$,

$$f(x) - f(\overline{x}) - \nabla f(\overline{x})(x - \overline{x}) \in K$$

where $\nabla f(\overline{x}) = [\nabla f_1(\overline{x}), \nabla f_2(\overline{x}), \ldots, \nabla f_m(\overline{x})]^T$ is the $m \times n$ Jacobian matrix of $f$ at $\overline{x}$ and for each $i = 1, 2, \ldots, m$, $\nabla f_i(\overline{x}) = \left( \frac{\partial f_i}{\partial x_1}, \frac{\partial f_i}{\partial x_2}, \ldots, \frac{\partial f_i}{\partial x_n} \right)^T$ is the $n \times 1$ Gradient vector of $f_i$ at $\overline{x}$.

**Definition 2.2.** [19] The function $f$ is said to be strictly $K$-convex at $\overline{x} \in \mathbb{R}^n$, if for all $x \in \mathbb{R}^n$ and $t \in [0, 1]$,

$$tf(x) + (1-t)f(\overline{x}) - f(tx + (1-t)\overline{x}) \in K.$$ 

The function $f$ is said to be $K$-convex, if it is $K$-convex at each $\overline{x} \in \mathbb{R}^n$.

**Remark 2.** Every function that is strictly $K$-convex is also $K$-convex but the converse is not true.

**Lemma 2.4.** If $f$ is a differentiable function and strictly $K$-convex, then for all $x, \overline{x} \in \mathbb{R}^n$ such that $x \neq \overline{x}$,

$$f(x) - f(\overline{x}) - \nabla f(\overline{x})(x - \overline{x}) \in \text{int}K.$$ 

**Proof.** Since $f$ is strictly $K$-convex, so for all $x, \overline{x} \in \mathbb{R}^n$ such that $x \neq \overline{x}$, and $t \in (0, 1)$,

$$tf(x) + (1-t)f(\overline{x}) - f(tx + (1-t)\overline{x}) \in \text{int}K. \quad (1)$$

By Remark 2 it follows that $f$ is $K$-convex and by Remark 1 we have that for all $x, \overline{x} \in \mathbb{R}^n$

$$f(x) - f(\overline{x}) - \nabla f(\overline{x})(x - \overline{x}) \in K.$$ 

In the above relationship we replace $x$ by $tx + (1-t)\overline{x}$ for any $t \in (0, 1)$ and get

$$f(tx + (1-t)\overline{x}) - f(\overline{x}) - t\nabla f(\overline{x})(x - \overline{x}) \in K. \quad (2)$$
Adding (1) and (2), we get
\[ t [ f(x) - f(\mathbf{x}) - \nabla f(\mathbf{x})(x - \mathbf{x}) ] \in \text{int} K. \]
Since \( t > 0 \), we have
\[ f(x) - f(\mathbf{x}) - \nabla f(\mathbf{x})(x - \mathbf{x}) \in \text{int} K. \]

**Definition 2.5.** The function \( h \) is said to be pseudoconvex at \( x \in \mathbb{R}^n \), if for every \( x \in \mathbb{R}^n \),
\[ \nabla h(\mathbf{x})(x - \mathbf{x}) \geq 0 \implies h(x) \geq h(\mathbf{x}). \]

**Definition 2.6.** The function \( h \) is said to be quasiconvex at \( x \in \mathbb{R}^n \), if for every \( x \in \mathbb{R}^n \),
\[ h(x) \geq h(\mathbf{x}) \implies \nabla h(\mathbf{x})(x - \mathbf{x}) \geq 0. \]

Now we briefly describe the Clarke's [5] notion of generalized directional derivative and subdifferential of a locally Lipschitz function.

The function \( h \) is said to be locally Lipschitz at \( x \in \mathbb{R}^n \) if there exists a nonnegative constant \( L \) and a neighborhood \( N(\mathbf{x}) \) of \( x \) such that for all \( x, y \in N(\mathbf{x}) \), we have
\[ |h(x) - h(y)| \leq L \| x - y \|. \]

The function \( f \) is vector-valued locally Lipschitz at \( x \in \mathbb{R}^n \) if for each \( i = 1, 2, \ldots, m \), \( f_i \) is locally Lipschitz at \( x \in \mathbb{R}^n \).

The Clarke generalized directional derivative of a locally Lipschitz function \( h \) at \( \mathbf{x} \in \mathbb{R}^n \) in the direction \( d \in \mathbb{R}^n \) is given as
\[ h^0(\mathbf{x}; d) = \limsup_{y \to \mathbf{x}, t \to 0^+} \frac{h(y + td) - h(y)}{t}, \]
where \( y \in \mathbb{R}^n \) and \( t > 0 \).

The Clarke generalized gradient or the Clarke subdifferential of a locally Lipschitz function \( h \) at \( \mathbf{x} \in \mathbb{R}^n \) is given as
\[ \partial c h(\mathbf{x}) = \{ \xi \in \mathbb{R}^n : h^0(\mathbf{x}, d) \geq \xi^T d, \text{ for all } d \in \mathbb{R}^n \}. \]

If \( h \) is convex function, then for any \( \mathbf{x} \in \mathbb{R}^n \), \( h \) is locally Lipschitz at \( \mathbf{x} \) and in this case
\[ \partial h(\mathbf{x}) = \partial h(\mathbf{x}) = \{ \xi \in \mathbb{R}^n : h(x) - h(\mathbf{x}) \geq (x - \mathbf{x})^T \xi, \text{ for all } x \in \mathbb{R}^n \}. \]

Also, if \( h \) is continuously differentiable at \( \mathbf{x} \), then \( h \) is locally Lipschitz at \( \mathbf{x} \) and \( \partial h(\mathbf{x}) = \{ \nabla h(\mathbf{x}) \}. \)

We now review some well known facts about support functions. Let \( C \subseteq \mathbb{R}^n \) be a compact convex set. The support function of \( C \) is defined by
\[ s(x|C) = \max_{z \in C} x^T z. \]

The support function of a compact convex set, being convex and finite everywhere, has a subgradient at every \( \mathbf{x} \in \mathbb{R}^n \) and the set of all subgradients at \( \mathbf{x} \), that is, the subdifferential at \( \mathbf{x} \) is given by
\[ \partial s(\mathbf{x}|C) = \{ z \in C : \mathbf{z}^T z = s(\mathbf{x}|C) \}. \]

**3. Optimality conditions.** Consider the following fractional vector optimization problem
(FVP) \[ K \text{-Min} \frac{f(x) + s(x|C)k}{h(x) - s(x|E)} \]

\[ = \left( \frac{f_1(x) + s(x|C)k_1}{h(x) - s(x|E)}, \frac{f_2(x) + s(x|C)k_2}{h(x) - s(x|E)}, \ldots, \frac{f_m(x) + s(x|C)k_m}{h(x) - s(x|E)} \right) \]

subject to

\[-g(x) - s(x|D)q = -(g_1(x) + s(x|D)q_1, g_2(x) + s(x|D)q_2, \ldots, g_p(x) + s(x|D)q_p) \in Q,\]

where \( f : \mathbb{R}^n \to \mathbb{R}^m, h : \mathbb{R}^n \to \mathbb{R} \) and \( g : \mathbb{R}^n \to \mathbb{R}^p \) are continuously differentiable functions, \( C, D \) and \( E \) are non-empty compact convex subsets of \( \mathbb{R}^n \), \( K \) and \( Q \) are closed convex pointed cones in \( \mathbb{R}^m \) and \( \mathbb{R}^p \) respectively with non-empty interiors and \( k = (k_1, k_2, \ldots, k_m) \) \( t \in \mathbb{R}^m \) and \( q = (q_1, q_2, \ldots, q_p) \) \( t \in \mathbb{Q} \) are any arbitrary but fixed vectors.

The feasible set of (FVP) is given by \( S_0 = \{ x \in \mathbb{R}^n : -g(x) - s(x|D)q \in Q \} \).

Also we assume that \( h(x) - s(x|E) > 0 \) for every \( x \in S_0 \).

Remark 3. 1. If we replace \( m \) by \( k \), \( p \) by \( m \), the function \( h \) by \( g : \mathbb{R}^n \to \mathbb{R}^m \) and we take \( K = \mathbb{R}^k_+, Q = \mathbb{R}^m_+ \) and \( C = D = E = \{0\} \), then our problem (FVP) reduces to the problem (MFP) considered by Kim [10] where \( X = \mathbb{R}^n \).

2. If we interchange the roles of \( m \) and \( p \), replace the function \( h \) by \( g : \mathbb{R}^n \to \mathbb{R}^m \) and we take \( K = \mathbb{R}^k_+, Q = \mathbb{R}^m_+ \) and \( C = D = E = \{0\} \), then our problem (FVP) reduces to the problem (NMFP) considered by Kim [11] where \( X_0 = \mathbb{R}^n \).

3. If we replace \( m \) by \( k \), \( p \) by \( m \), the function \( h \) by \( q : \mathbb{R}^n \to \mathbb{R}^m \) and if we take \( K = \mathbb{R}^k_+, Q = S \) and \( C = D = E = \{0\} \), then our problem (FVP) reduces to the problem (MFP) considered by Chen et al. [4].

4. If we replace \( m \) by \( k \), \( p \) by \( m \), \( g_j \) by \( h_j \) and we take \( K = \mathbb{R}^k_+, Q = \mathbb{R}^m_+ \), \( C = D = E = \{0\} \), then our problem (FVP) reduces to the problem (FP) considered by Suneja and Gupta [24] where \( g_i = h \) and \( S = \mathbb{R}^n \) and also to the problem (FP) given by Antczak [1] where \( g_i = h \) and \( X = \mathbb{R}^n \).

5. If we interchange the roles of \( m \) and \( p \) and replace \( g_j \) by the function \( h_j : \mathbb{R}^n \to \mathbb{R}^m \), also if we take \( K = \mathbb{R}^k_+, Q = \mathbb{R}^m_+ \), \( D = \{0\} \), \( E = \{0\} \) and \( k = (1, 1, \ldots, 1) \in \mathbb{R}^p \). Then our problem (FVP) reduces to (MFP) considered by Kim et al. [12] where \( X_0 = \mathbb{R}^n, C_i = C \) and \( g_i = h \) for each \( i = 1, 2, \ldots, p \).

6. If we replace \( m \) by \( k \), \( p \) by \( m \), \( g_i \) by \( h_i \) and we take \( K = \mathbb{R}^k_+, Q = \mathbb{R}^m_+ \), \( C = D = E = \{0\} \), then our problem (FVP) reduces to the problem (FP) given by Jayswal et al. [9] where \( X = \mathbb{R}^n \).

7. If we take \( m = 1 \), interchange \( p \) by \( m \), replace \( g_j \) by the function \( h_j : \mathbb{R}^n \to \mathbb{R}^m \), for each \( j = 1, 2, \ldots, m \) and the function \( h \) by \( g : \mathbb{R}^n \to \mathbb{R} \). Also \( K = \mathbb{R}_+, k = 1, Q = \mathbb{R}^m_+, C = C_1, E = C_2 \), then our problem (FVP) reduces to the problem (FP) considered by Husain et al. [8] where \( D_j = D \) for \( j = 1, 2, \ldots, p \).

Definition 3.1. A point \( \overline{x} \in S_0 \) is called

1. a weak minimum of (FVP), if for every \( x \in S_0 \),

\[ \frac{f(\overline{x}) + s(\overline{x}|C)k}{h(\overline{x}) - s(\overline{x}|E)} - \frac{f(x) + s(x|C)k}{h(x) - s(x|E)} \notin \text{int} K. \]
2. a minimum of (FVP), if for every \( x \in S_0 \),

\[
f(\overline{x}) + s(\overline{x}|C)k - \frac{f(x) + s(x|C)k}{h(x) - s(x|E)} - (h(x) - s(x|E)) \notin K \setminus \{0\}.
\]

Consider the following vector optimization problem.

\[
(FVP)_v \quad K-\text{Min } f(x) + s(x|C)k - v(h(x) - s(x|E)) = (f_1(x) + s(x|C)k_1 - v_1(h(x) - s(x|E)),
\]

\[
f_2(x) + s(x|C)k_2 - v_2(h(x) - s(x|E)),
\]

\[
\ldots, f_m(x) + s(x|C)k_m - v_m(h(x) - s(x|E))
\]

subject to

\[
-g(x) - s(x|D)q \in Q,
\]

Since the constraint in both the problems (FVP) and (FVP)_v is same therefore their feasible sets are also same.

**Lemma 3.2.** \( \overline{x} \in S_0 \) is a weak minimum of (FVP) if and only if \( \overline{x} \) is also a weak minimum of (FVP)_\( \overline{\pi} \), where \( \overline{\pi} = \frac{f(\overline{x}) + s(\overline{x}|C)k}{h(\overline{x}) - s(\overline{x}|E)} \) i.e. \( \pi_i = \frac{f_i(\overline{x}) + s(\overline{x}|C)k_i}{h(\overline{x}) - s(\overline{x}|E)} \).

**Proof.** Suppose \( \overline{x} \) is a weak minimum of (FVP) and if possible not a weak minimum of (FVP)_\( \overline{\pi} \), then there exists \( x \in S_0 \) such that

\[
[f(\overline{x}) + s(\overline{x}|C)k - \overline{\pi}(h(\overline{x}) - s(\overline{x}|E))] - [f(x) + s(x|C)k - \overline{\pi}(h(x) - s(x|E))] \in \text{int}K
\]

which implies that

\[
\left[ f(\overline{x}) + s(\overline{x}|C)k - (h(\overline{x}) - s(\overline{x}|E)) \left( \frac{f(x) + s(x|C)k}{h(x) - s(x|E)} \right) \right]
\]

\[
- \left[ f(x) + s(x|C)k - (h(x) - s(x|E)) \left( \frac{f(\overline{x}) + s(\overline{x}|C)k}{h(\overline{x}) - s(\overline{x}|E)} \right) \right] \in \text{int}K.
\]

Since \( h(x) - s(x|E) > 0 \), therefore multiplying the above relation by \( \frac{1}{h(x) - s(x|E)} \),

we get

\[
\frac{f(\overline{x}) + s(\overline{x}|C)k}{h(\overline{x}) - s(\overline{x}|E)} - \frac{f(x) + s(x|C)k}{h(x) - s(x|E)} \in \text{int}K
\]

which is a contradiction to the fact that \( \overline{x} \) is a weak minimum of (FVP).

Therefore, \( \overline{x} \) is also a weak minimum of (FVP)_\( \overline{\pi} \).

Conversely, suppose \( \overline{x} \) is a weak minimum of (FVP)_\( \overline{\pi} \) and not a weak minimum of (FVP), then there exists \( x \in S_0 \) such that

\[
\frac{f(\overline{x}) + s(\overline{x}|C)k}{h(\overline{x}) - s(\overline{x}|E)} - \frac{f(x) + s(x|C)k}{h(x) - s(x|E)} \in \text{int}K
\]

Since \( h(x) - s(x|E) > 0 \) for every \( x \in S_0 \), therefore multiplying the above relation by \( h(x) - s(x|E) \) and using the fact that

\[
f(\overline{x}) + s(\overline{x}|C)k - (h(\overline{x}) - s(\overline{x}|E)) \overline{\pi} = 0,
\]

we get

\[
[f(\overline{x}) + s(\overline{x}|C)k - \overline{\pi}(h(\overline{x}) - s(\overline{x}|E))] - [f(x) + s(x|C)k - \overline{\pi}(h(x) - s(x|E))] \in \text{int}K
\]

which is a contradiction to the fact that \( \overline{x} \) is a weak minimum of (FVP)_\( \overline{\pi} \).

Therefore, \( \overline{x} \) is also a weak minimum of (FVP).
Lemma 3.3. Consider the problem
\[ \begin{array}{l}
K\text{-Minimize } \Phi(x) \\
subject to \\
- \Psi(x) \in Q,
\end{array} \]
where \( \Phi: \mathbb{R}^n \to \mathbb{R}^m \) and \( \Psi: \mathbb{R}^n \to \mathbb{R}^p \) are locally Lipschitz functions. Let \( \pi \) be a weak minimum of this problem, then there exists \( (0,0) \neq (\lambda,\mu) \in K^+ \times Q^+ \) such that
\[ 0 \in \partial^c \left( \lambda^T \Phi + \mu^T \Psi \right)(\pi) \]
and
\[ (\mu^T \Psi)(\pi) = 0. \]

Proof. This Lemma can be proved by taking \( F = \Phi, G = \Psi, Q = K, S = Q \) and \( T = \emptyset \) (the empty set) in the problem (P) considered by Craven and applying Theorem 2 given by him.

We now establish Fritz-John type necessary optimality conditions for the problem (FVP), based on the above lemma.

Theorem 3.4. Let \( \pi \in S_0 \) be a weak minimum of (FVP). Then there exist \( \lambda \in K^+, \pi \in Q^+ \) with \( (\lambda,\pi) \neq 0 \) and \( \pi \in \partial s(\pi|C), \pi \in \partial s(\pi|E) \) and \( \pi \in \partial s(\pi|D) \) such that
\[ \sum_{i=1}^m \lambda_i \nabla \left( f_i(\pi) + (\pi^T \pi) k_i \right) + \sum_{j=1}^p \pi_j \left[ \nabla g_j(\pi) + \bar{w} q_j \right] = 0 \]
(3)
and
\[ \sum_{j=1}^p \pi_j \left[ g_j(\pi) + (\pi^T \pi) q_j \right] = 0. \]
(4)

Proof. Since \( \pi \) is a weak minimum of (FVP), therefore by Lemma 3.2 \( \pi \) is also a weak minimum of (FVP)|\( \pi \), where \( \pi = f(\pi) + s(\pi|C) k \)
\[ h(\pi) - s(\pi|E). \]

Let \( \Phi: \mathbb{R}^n \to \mathbb{R}^m \) and \( \Psi: \mathbb{R}^n \to \mathbb{R}^p \) be such that
\[ \Phi(x) = f(x) + s(x|C) k - \pi (h(x) - s(x|E)) \quad \text{and} \quad \Psi(x) = g(x) + s(x|D) q. \]

Since \( s(\cdot|C), s(\cdot|E) \) and \( s(\cdot|D) \) are convex functions, therefore they are locally Lipschitz at any \( x \in \mathbb{R}^n \) and hence \( s(\cdot|C) k, s(\cdot|E) \pi \) and \( s(\cdot|D) q \) are vector-valued locally Lipschitz functions at any \( x \in \mathbb{R}^n \).

Also \(-h\) is differentiable, therefore it is locally Lipschitz function at any \( x \in \mathbb{R}^n \) and hence \(-h(\cdot)\pi \) is a vector-valued locally Lipschitz function at any \( x \in \mathbb{R}^n \) and \( f \) and \( g \) are vector-valued continuously differentiable functions, so they are vector-valued locally Lipschitz at any \( x \in \mathbb{R}^n \). Due to the fact that sum of locally Lipschitz functions is locally Lipschitz, therefore \( \Phi \) and \( \Psi \) are vector-valued locally Lipschitz functions.

Now due to Lemma 3.3, there exist \( \lambda, \mu^* \in Q^+ \) with \( (\lambda, \mu^*) \neq 0 \) such that
\[ 0 \in \partial^c \left( \lambda^T \Phi + \mu^* \Psi \right)(\pi) \subseteq \partial^c (\lambda^T \Phi)(\pi) + \partial^c (\mu^* \Psi)(\pi) \]
and
\[ (\mu^* \Psi)(\pi) = 0. \]
That is,
\[ 0 \in \{ \nabla (\lambda^T f)(\pi) \} + \partial^c s(\pi|C)(\lambda^T k) - \{ \nabla h(\pi)(\lambda^T \pi) \} + \partial^c s(\pi|E)(\lambda^T \pi). \]
where 

\[
\frac{1}{h(\pi) - \pi^T \mu} \text{ and substituting the value of } \pi_i \text{ for } i = 1, 2, \ldots, m, \text{ we get}
\]

\[
\sum_{i=1}^{m} \frac{\lambda_i}{(h(\pi) - \pi^T \mu)^2} \left[ (h(\pi) - \pi^T \mu) (\nabla f_i(\pi) + \nabla k_i) \right] - (f_i(\pi) + (\pi^T \pi) k_i) (\nabla h(\pi) - \nabla \pi^T \mu) + \sum_{j=1}^{p} \mu_j^* (g_j(\pi) + \nabla g_j(\pi) + \bar{w}) q_j = 0
\]

where \((\pi_1, \pi_2, \ldots, \pi_p) = \frac{1}{h(\pi) - \pi^T \mu} (\mu_1^*, \mu_2^*, \ldots, \mu_p^*) \in Q^+.

Hence substituting the value \(\mu^* = (h(\pi) - \pi^T \mu) \pi\) in (6) we get that there exists \(\lambda \in K^+, \pi \in Q^+ \text{ with } (\lambda, \pi) \neq 0 \text{ and } \pi \in \partial s(\pi|C), \bar{w} \in \partial s(\pi|E) \text{ and } w \in \partial s(\pi|D) \text{ such that (3) and (4) hold.}

On the lines of Suneja et al. [25], we establish the following Kuhn-Tucker type necessary optimality conditions for the problem (FVP).

**Theorem 3.5.** Let \(\pi \in S_0\) be a weak minimum of (FVP). Then there exist \(\lambda \in K^+, \pi \in Q^+ \text{ with } (\lambda, \pi) \neq 0 \text{ and } \pi \in \partial s(\pi|C), \bar{w} \in \partial s(\pi|E) \text{ and } w \in \partial s(\pi|D) \text{ such that (3) and (4) hold.}

Moreover if \(g\) is Q-convex at \(\pi\) and there exists \(x^* \in \mathbb{R}^n\) such that

\[
(\pi^T g)(x^*) + s(x^*|D)(\pi^T \mu) < 0.
\]

Then \(\lambda \neq 0\).

**Proof.** Let \(\pi\) be a weak minimum of (FVP), then we invoke Theorem 3.4 to deduce that there exist \(\lambda \in K^+ \text{ and } \pi \in Q^+ \text{ with } (\lambda, \pi) \neq 0 \text{ and } \pi \in \partial s(\pi|C), \bar{w} \in \partial s(\pi|E) \text{ and } w \in \partial s(\pi|D) \text{ such that (3) and (4) hold.}

Suppose now that \(g\) is Q-convex at \(\pi\) and there exists \(x^* \in \mathbb{R}^n\) such that

\[
(\pi^T g)(x^*) + s(x^*|D)(\pi^T \mu) < 0.
\]

We have to prove that \(\lambda \neq 0\).

Let, if possible, \(\lambda = 0\), then \(\pi \neq 0\) and (3) reduces to

\[
\nabla (\pi^T g)(\pi) + \bar{w}(\pi^T \mu) = 0.
\]

(8)

Now as \(g\) is Q-convex at \(\pi\), hence

\[
g(x^*) - g(\pi) - \nabla g(\pi)(x^* - \pi) \in Q.
\]

(9)
ON FRACTIONAL VECTOR OPTIMIZATION OVER CONES

Since \( \overline{w} \in \partial s(\overline{\pi}|D) \), therefore
\[
s(x^*|D) - s(\overline{\pi}|D) \geq (x^* - \overline{\pi})^T \overline{w},
\]
which implies that
\[
[s(x^*|D) - s(\overline{\pi}|D) - (x^* - \overline{\pi})^T \overline{w}] q \in Q.
\]
Adding relations (9) and (10) and using the fact that \( \overline{\pi} \in Q^+ \) and \( \overline{w} \in \partial s(\overline{\pi}|D) \),
that is, \( \overline{\pi}^T \overline{w} = s(\overline{\pi}|D) \) we get
\[
(\overline{\pi}^T g)(x^*) + s(x^*|D)(\overline{\pi}^T q) - (\overline{\pi}^T g)(\overline{\pi}) - \overline{\pi}^T \overline{w}(\overline{\pi}^T q)
- (x^* - \overline{\pi})^T [\nabla (\overline{\pi}^T g)(\overline{\pi}) + \overline{w}(\overline{\pi}^T q)] \geq 0.
\]
Using (4) and (8) in the above inequality, we get
\[
(\overline{\pi}^T g)(x^*) + s(x^*|D)(\overline{\pi}^T q) \geq 0,
\]
which is a contradiction to (7). Hence, \( \overline{\lambda} \neq 0 \). □

Next we give sufficient optimality conditions for a point to be weak minimum of the problem (FVP).

**Theorem 3.6.** Let \( f \) be \( K \)-convex, \( -h \) be convex and \( g \) be \( Q \)-convex at \( \overline{\pi} \in S_0 \). Suppose that there exist \( \overline{\lambda} \in K^+ \setminus \{0\} \), \( \overline{\pi} \in Q^+ \), \( \overline{\pi} \in \partial s(\overline{\pi}|C) \), \( \overline{y} \in \partial s(\overline{\pi}|E) \) and \( \overline{w} \in \partial s(\overline{\pi}|D) \) such that (3) and (4) hold and \( f(\overline{\pi}) + (\overline{\pi}^T \overline{\pi})k \in K \). Then \( \overline{\pi} \) is a weak minimum of (FVP).

**Proof.** Let, if possible, \( \overline{\pi} \) be not a weak minimum of (FVP), then there exists \( x \in S_0 \)
such that
\[
\frac{f(\overline{\pi}) + s(\overline{\pi}|C)k}{h(\overline{\pi}) - s(\overline{\pi}|E)} - \frac{f(x) + s(x|C)k}{h(x) - s(x|E)} \in \text{int}K.
\]
Since, \( \overline{\pi} \in \partial s(\overline{\pi}|C) \) and \( \overline{y} \in \partial s(\overline{\pi}|E) \), therefore
\[
\frac{f(\overline{\pi}) + (\overline{\pi}^T \overline{\pi})k}{h(\overline{\pi}) - \overline{\pi}^T \overline{y}} - \frac{f(x) + s(x|C)k}{h(x) - s(x|E)} \in \text{int}K.
\]
Using the fact that \( h(x) - s(x|E) > 0 \), in the above relation, we have
\[
(h(x) - s(x|E)) \frac{f(\overline{\pi}) + (\overline{\pi}^T \overline{\pi})k}{h(\overline{\pi}) - \overline{\pi}^T \overline{y}} - f(x) - s(x|C)k \in \text{int}K.
\]
Again since, \( \overline{\pi} \in C \), we have \( s(x|C) \geq x^T \overline{\pi} \), and hence
\[
(s(x|C) - x^T \overline{\pi})k \in K.
\]
Also since, \( \overline{y} \in E \), therefore \( s(x|E) \geq x^T \overline{y} \) and then using the fact that \( f(\overline{\pi}) + (\overline{\pi}^T \overline{\pi})k \in K \) and \( h(\overline{\pi}) - \overline{\pi}^T \overline{y} = h(\overline{\pi}) - s(\overline{\pi}|E) > 0 \), we get
\[
(s(x|E) - x^T \overline{y}) \frac{f(\overline{\pi}) + (\overline{\pi}^T \overline{\pi})k}{h(\overline{\pi}) - \overline{\pi}^T \overline{y}} \in K
\]
Adding (11), (12) and (13), we get
\[
(h(x) - x^T \overline{y}) \frac{f(\overline{\pi}) + (\overline{\pi}^T \overline{\pi})k}{h(\overline{\pi}) - \overline{\pi}^T \overline{y}} - f(x) - (x^T \overline{\pi})k \in \text{int}K.
\]
Since \( \overline{y} \in E \), so \( s(x|E) \geq x^T \overline{y} \) and hence \( h(x) - x^T \overline{y} \geq h(x) - s(x|E) > 0 \). Therefore the above relation can be written as
\[
\frac{f(\overline{\pi}) + (\overline{\pi}^T \overline{\pi})k}{h(\overline{\pi}) - \overline{\pi}^T \overline{y}} - \frac{f(x) + (x^T \overline{\pi})k}{h(x) - x^T \overline{y}} \in \text{int}K,
\]
or,
\[- \left[ \frac{f(x) + (x^T y)k}{h(x) - x^T y} - \frac{f(x) + (\bar{x}^T y)k}{h(\bar{x}) - \bar{x}^T y} \right] \in \text{int}K\]
which gives that
\[
\frac{f(\bar{x}) + (\bar{x}^T y)k}{h(\bar{x}) - \bar{x}^T y} - \frac{f(x) + (\bar{x}^T y)k}{h(x) - x^T y} + \frac{f(\bar{x}) + (\bar{x}^T y)k}{h(\bar{x}) - \bar{x}^T y} - \frac{f(x) + (\bar{x}^T y)k}{h(x) - x^T y} \in \text{int}K.
\]
That is
\[
\left[ \frac{h(x) - h(\bar{x}) - (x - \bar{x})^T y}{(h(\bar{x}) - \bar{x}^T y)(h(x) - x^T y)} \right] (f(\bar{x}) + (\bar{x}^T y)k)
+ \frac{1}{h(x) - x^T y} [f(\bar{x}) - f(x) - (x - \bar{x})^T y k] \in \text{int}K.
\]
Since \( \bar{y} \in E \), we have \( s(x|E) \geq x^T \bar{y} \) and hence \( h(x) - x^T \bar{y} \geq h(x) - s(x|E) > 0 \).
Therefore, multiplying the above relation by \( h(x) - x^T \bar{y} \), we get
\[
\left[ \frac{h(x) - h(\bar{x}) - (x - \bar{x})^T y}{(h(\bar{x}) - \bar{x}^T y)(h(x) - x^T y)} \right] (f(\bar{x}) + (\bar{x}^T y)k)
+ [f(\bar{x}) - f(x) - (x - \bar{x})^T y k] \in \text{int}K. \tag{14}
\]
Since \( f \) is \( K \)-convex at \( \bar{x} \), therefore
\[
f(x) - f(\bar{x}) - \nabla f(\bar{x})(x - \bar{x}) \in K \tag{15}
\]
and since \( -h \) is convex at \( \bar{x} \), therefore
\[
-h(x) + h(\bar{x}) + (x - \bar{x})^T \nabla h(\bar{x}) \geq 0
\]
Now, since \( f(\bar{x}) + (\bar{x}^T y)k \in K \) and \( h(\bar{x}) - \bar{x}^T \bar{y} = h(\bar{x}) - s(\bar{x}|E) > 0 \), we get
\[
\left[ \frac{-h(x) + h(\bar{x}) + (x - \bar{x})^T \nabla h(\bar{x})}{h(\bar{x}) - \bar{x}^T \bar{y}} \right] (f(\bar{x}) + (\bar{x}^T y)k) \in K \tag{16}
\]
Adding \( \tag{14}, \tag{15} \) and \( \tag{16} \), we get
\[
\left[ \frac{(x - \bar{x})^T \nabla h(\bar{x}) - (x - \bar{x})^T \bar{y}}{h(\bar{x}) - \bar{x}^T \bar{y}} \right] (f(\bar{x}) + (\bar{x}^T y)k)
- [\nabla f(\bar{x})(x - \bar{x}) + (x - \bar{x})^T y k] \in \text{int}K,
\]
Again, since \( h(\bar{x}) - \bar{x}^T \bar{y} \geq 0 \), so multiplying the above relation by \( \frac{1}{h(\bar{x}) - \bar{x}^T \bar{y}} \) gives that
\[
\left( \frac{[h(\bar{x}) - \bar{x}^T \bar{y}] [\nabla f_1(\bar{x}) + z k_1] - [f_1(\bar{x}) + (\bar{x}^T \bar{y})k_1] [\nabla h(\bar{x}) - \bar{y}]}{(h(\bar{x}) - \bar{x}^T \bar{y})^2}, \frac{[h(\bar{x}) - \bar{x}^T \bar{y}] [\nabla f_2(\bar{x}) + z k_2] - [f_2(\bar{x}) + (\bar{x}^T \bar{y})k_2] [\nabla h(\bar{x}) - \bar{y}]}{(h(\bar{x}) - \bar{x}^T \bar{y})^2}, \ldots, \frac{[h(\bar{x}) - \bar{x}^T \bar{y}] [\nabla f_m(\bar{x}) + z k_m] - [f_m(\bar{x}) + (\bar{x}^T \bar{y})k_m] [\nabla h(\bar{x}) - \bar{y}]}{(h(\bar{x}) - \bar{x}^T \bar{y})^2} \right)^T \in \text{int}K
\]
which implies that
\[-\left(\nabla \left( \frac{f_1(x) + (x^T \bar{z})k_1}{h(x) - (x^T \bar{y})} \right), \nabla \left( \frac{f_2(x) + (x^T \bar{z})k_2}{h(x) - (x^T \bar{y})} \right), \ldots, \nabla \left( \frac{f_m(x) + (x^T \bar{z})k_m}{h(x) - (x^T \bar{y})} \right) \right)^T (x - \bar{x}) \in \text{int}K.\]

That is,
\[-\nabla \left( \frac{f(x) + (x^T \bar{z})k}{h(x) - (x^T \bar{y})} \right) (x - \bar{x}) \in \text{int}K.\]

which implies that
\[-(x - \bar{x})^T \sum_{i=1}^{m} \lambda_i \nabla \left( \frac{f_i(x) + (x^T \bar{z})k_i}{h(x) - (x^T \bar{y})} \right) > 0.\]

Using (3) in above inequality, we get
\[(x - \bar{x})^T \left[ \nabla (\lambda^T g)(\bar{x}) + w(\lambda^T q) \right] > 0 \quad (17)\]

Now since \(w \in \partial s(x|D)\), therefore
\[(s(x|D) - s(x|D) - (x - \bar{x})^T w)|q \in Q, \quad (18)\]

and since \(g\) is \(Q\)-convex at \(\bar{x}\), therefore
\[g(x) - g(\bar{x}) - \nabla g(\bar{x})(x - \bar{x}) \in Q. \quad (19)\]

Adding (18) and (19), we get
\[g(x) - g(\bar{x}) + s(x|D)q - s(x|D)q - \nabla g(\bar{x})(x - \bar{x}) - (x - \bar{x})^T wq \in Q.\]

Using \(\lambda \in Q^+\) in above relation, we get
\[(\lambda^T g)(x) - (\lambda^T g)(\bar{x}) + s(x|D)(\lambda^T q) - s(x|D)(\lambda^T q)\]
\[= \left( x - \bar{x} \right)^T \left[ \nabla (\lambda^T g)(\bar{x}) + w(\lambda^T q) \right] \geq 0. \quad (20)\]

Adding (17) and (20), we have
\[(\lambda^T g)(x) + s(x|D)(\lambda^T q) - (\lambda^T g)(\bar{x}) - s(x|D)(\lambda^T q) > 0\]

Using (4) in above inequality, we get
\[(\lambda^T g)(x) + s(x|D)(\lambda^T q) > 0 \quad (21)\]

Since \(x \in S_0\), therefore
\[-g(x) - s(x|D)q \in Q.\]

Using \(\lambda \in Q^+\) in above inequality, gives that
\[(\lambda^T g)(x) + s(x|D)(\lambda^T q) \leq 0,\]

which is contradiction to the inequality (21).
Hence \(\bar{x}\) is a weak minimum of \(FVP\). \(\square\)

Now we give an example to illustrate the above theorem.
Example 1. Consider the problem

\[
\text{(FVP)} \quad \text{K-Min} \quad \frac{f(x) + s(x|C)k}{h(x) - s(x|E)} = \left( \frac{f_1(x) + s(x|C)k_1}{h(x) - s(x|E)}, \frac{f_2(x) + s(x|C)k_2}{h(x) - s(x|E)} \right)^T
\]

subject to

\[-g(x) - s(x|D)q = -(g_1(x) + s(x|D)q_1, g_2(x) + s(x|D)q_2)^T \in Q,\]

where \( f : \mathbb{R} \rightarrow \mathbb{R}^2, h : \mathbb{R} \rightarrow \mathbb{R} \) and \( g : \mathbb{R} \rightarrow \mathbb{R}^2 \)

\[
f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = \begin{pmatrix} -1 - x^2 \\ x \end{pmatrix}, \quad C = [-1, 1], \quad s(x|C) = \begin{cases} -x & x \leq 0 \\ x & x > 0 \end{cases}
\]

\[
K = \left\{ \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \in \mathbb{R}^2 : x_1 \leq x_2, x_2 \geq 0 \right\}, \quad k = \left( \begin{array}{c} 1 \\ 2 \end{array} \right) \in \text{int } K.
\]

\[
h(x) = 1 - x^2, \quad E = [0, 2], \quad s(x|E) = \begin{cases} 0 & x \leq 0 \\ 2x & x > 0 \end{cases}
\]

\[
g(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix} = \begin{pmatrix} 2 - 7x^4 \\ x^4 + x + 2 \end{pmatrix}, \quad D = [0, 1], \quad s(x|D) = \begin{cases} 0 & x \leq 0 \\ x & x > 0 \end{cases}
\]

\[
Q = \left\{ \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \in \mathbb{R}^2 : x_1 \leq x_2 \leq -x_1 \right\}, \quad q = \left( \begin{array}{c} -11/4 \\ 1 \end{array} \right) \in Q.
\]

The feasible set of the problem (NVP) is \( S_0 = \left[ -\frac{1}{2}, 0 \right] \). Let \( \bar{\pi} = 0 \).

Then \( f \) is \( K \)-convex at \( \bar{\pi} \), because for every \( x \in \mathbb{R} \), we have

\[
f(x) - f(\bar{\pi}) - \nabla f(\bar{\pi})(x - \bar{\pi}) = \begin{pmatrix} -x^2 \\ 0 \end{pmatrix} \in K,
\]

g is \( Q \)-convex at \( \bar{\pi} \), because for every \( x \in \mathbb{R} \), we have

\[
g(x) - g(\bar{\pi}) - \nabla g(\bar{\pi})(x - \bar{\pi}) = \begin{pmatrix} -7x^4 \\ x^4 \end{pmatrix} \in Q.
\]

and it is clear that \( -h \) is convex at \( \bar{\pi} \) and \( h(x) - s(x|E) > 0 \) for all \( x \in S_0 \).

\[
K^+ = \left\{ \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \in \mathbb{R}^2 : 0 \leq -x_1 \leq x_2 \right\}, \quad Q^+ = Q,
\]

\[
\partial s(\bar{\pi}|C) = [-1, 1], \quad \partial s(\bar{\pi}|E) = [0, 2] \quad \text{and} \quad \partial s(\bar{\pi}|D) = [0, 1].
\]

Also there exists \( \bar{\lambda} = (-1, 3)^T \in K^+ \setminus \{0\}, \quad \bar{\mu} = (-1, 1)^T \in Q^+, \quad \bar{x} = -1 \in \partial s(\bar{\pi}|C), \quad \bar{w} = \frac{1}{15} \in \partial s(\bar{\pi}|D), \quad \bar{y} = \frac{3}{4} \in \partial s(\bar{\pi}|E) \) such that

\[
\sum_{i=1}^{2} \bar{\lambda}_i \nabla \left( \frac{f_i(\bar{\pi}) + (\bar{\pi}^T \bar{x})k_i}{h(\bar{\pi}) - (\bar{\pi}^T \bar{y})} \right) + \sum_{j=1}^{2} \bar{\mu}_j \left[ \nabla g_j(\bar{\pi}) + \bar{w}q_j \right] = 0,
\]

\[
\sum_{j=1}^{p} \bar{\mu}_j \left[ g_j(\bar{\pi}) + (\bar{\pi}^T \bar{w}) q_j \right] = 0,
\]

and \( f(\bar{\pi}) + (\bar{\pi}^T \bar{x})k = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \in K. \)

Therefore, \( \bar{\pi} \) is a weak minimum of problem (FVP).
Now proceeding on the lines of proof of Theorem 3.6 we give the following sufficient optimality conditions for a point to be a minimum of (FVP).

**Theorem 3.7.** Let \( f \) be strictly \( K \)-convex, \( -h \) be convex and \( g \) be \( Q \)-convex at \( \bar{x} \in S_0 \). Suppose that there exist \( \bar{\lambda} \in K^+ \setminus \{0\} \), \( \bar{\mu} \in Q^+ \), \( \bar{\tau} \in \partial s(\bar{x}|C) \), \( \bar{\eta} \in \partial s(\bar{x}|E) \) and \( \bar{w} \in \partial s(\bar{x}|D) \) such that (3) and (4) hold and \( f(\bar{x}) + (\bar{x}^T \bar{z}) k \in K \). Then \( \bar{x} \) is a minimum of (FVP).

Now we give an example to illustrate the above theorem.

**Example 2.** Consider the problem given in Example 1, where

\[
\begin{align*}
K &= \left\{ \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \in \mathbb{R}^2 : x_1 \leq 0, x_1 \leq x_2 \right\}, \\
k &= \left( \begin{array}{c} -1 \\ 1 \end{array} \right) \in \text{int}K.
\end{align*}
\]

and rest all the functions and sets are same as that in Example 1.

Now, let

\[
K^+ = \left\{ \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \in \mathbb{R}^2 : x_1 \leq -x_2 \leq 0 \right\}, \quad Q^+ = Q,
\]

\[
\partial s(\bar{x}|C) = [-1, 1], \quad \partial s(\bar{x}|E) = [0, 1] \quad \text{and} \quad \partial s(\bar{x}|D) = [0, 1]
\]

The feasible set of the problem (FVP) is \( S_0 = \left[ -\frac{1}{2}, 0 \right] \). Let \( \bar{x} = 0 \).

Then \( f \) is strictly \( K \)-convex at \( \bar{x} \), because for every \( x \in \mathbb{R}^n \setminus \{\bar{x}\} \), we have

\[
f(x) - f(\bar{x}) - \nabla f(\bar{x})(x - \bar{x}) = \left( -\frac{x^2}{x_2} \right) \in \text{int}K.
\]

Also it was shown in Example 1 that \( g \) is \( Q \)-convex at \( \bar{x} \), \( -h \) is convex at \( \bar{x} \) and \( h(x) - s(x|E) > 0 \) for all \( x \in S_0 \) and there exists \( \bar{\lambda} = \left( \begin{array}{c} -5 \\ 3 \end{array} \right)^T \in K^+ \setminus \{0\} \), \( \bar{\mu} = (-1, 1)^T \in Q^+ \), \( \bar{\tau} = -\frac{3}{4} \in \partial s(\bar{x}|C) \), \( \bar{w} = \frac{1}{15} \in \partial s(\bar{x}|D) \), \( \bar{\eta} = \frac{1}{2} \in \partial s(\bar{x}|E) \) such that

\[
\sum_{i=1}^{2} \lambda_i \nabla \left( \frac{f_i(\bar{x}) + (\bar{x}^T \bar{z}) k_i}{h(\bar{x}) - (\bar{x}^T \bar{y})} \right) + \sum_{j=1}^{p} \mu_j \left[ \nabla g_j(\bar{x}) + \bar{w} q_j \right] = 0,
\]

\[
\sum_{j=1}^{p} \bar{\eta} [g_j(\bar{x}) + (\bar{x}^T \bar{w}) q_j] = 0,
\]

and \( f(\bar{x}) + (\bar{x}^T \bar{z}) k = \left( \begin{array}{c} -1 \\ 1 \end{array} \right) \in K \).

Therefore, \( \bar{x} \) is a minimum of problem (FVP).

4. **Mond-Weir type dual.** Consider the following Mond-Weir type dual.

\[
\begin{array}{ll}
\text{(FMD)} & \text{K-Max} \quad \frac{f(u) + (u^T z) k}{h(u) - u^T y} \\
& = \left( \begin{array}{l}
f_1(u) + (u^T z) k_1 \\
f_2(x) + (u^T z) k_2 \\
\vdots \\
f_m(x) + (u^T z) k_m \\
\end{array} \right) \\
& \text{subject to}
\end{array}
\]
Remark 4. If we take $m = 1$, interchange $p$ by $m$, replace $g_j$ by the function $h_j : \mathbb{R}^n \to \mathbb{R}$, for each $j = 1, 2, \ldots, m$ and the function $h$ by $g : \mathbb{R}^n \to \mathbb{R}$. Also $K = \mathbb{R}^+ \times \mathbb{R}^m$, $C = C_1$, $E = C_2$, then our dual (FMD) reduces to the dual (FD) considered by Husain et al. \[8\] where $D_j = D$ for $j = 1, 2, \ldots, p$.

Next we establish the Weak Duality relation between the primal problem (FVP) and its Mond-Weir type dual (FMD).

**Theorem 4.1** (Weak Duality). Let $x$ be feasible for (FVP) and $(u, z, y, w, \lambda, \mu)$ be feasible for (FMD). If $f$ is $K$-convex, $-h$ is convex, $g$ is $Q$-convex at $u$ and $f(u) + (u^T z)k \in K$. Then

$$\frac{f(u) + (u^T z)k}{h(u) - u^Ty} - \frac{f(x) + s(x|C)k}{h(x) - s(x|E)} \notin \text{int}K.$$

**Proof.** Let, if possible,

$$\frac{f(u) + (u^T z)k}{h(u) - u^Ty} - \frac{f(x) + s(x|C)k}{h(x) - s(x|E)} \in \text{int}K.$$

Using the fact that $z \in C$ and $y \in E$ and proceeding as in the proof of Theorem \[3.6\] the above relation can be written as follows

$$\frac{f(u) + (u^T z)k}{h(u) - u^Ty} - \frac{f(x) + (x^T z)k}{h(x) - x^Ty} \in \text{int}K.$$

Since $f$ is $K$-convex, $-h$ is convex at $u$, so again following the lines of Theorem \[3.6\] we arrive at

$$-\nabla \left( \frac{f(u) + (u^T z)k}{h(u) - u^Ty} \right) (x - u) \in \text{int} K$$

which implies that

$$(x - u)^T \sum_{i=1}^{m} \lambda_i \nabla \left( \frac{f_i(u) + (u^T z)k_i}{h(u) - u^Ty} \right) > 0$$

Since $(u, z, y, w, \lambda, \mu)$ is feasible for (FMD), therefore

$$(x - u)^T \left[ \nabla (\mu^T g)(u) + w(\mu^T q) \right] > 0 \quad (22)$$

Now since $g$ is $Q$-convex at $u$ and $\mu \in Q^+$, therefore

$$(\mu^T g)(x) - (\mu^T g)(u) - (x - u)^T \nabla (\mu^T g)(u) \geq 0. \quad (23)$$

Adding \[22\] and \[23\], we get

$$(\mu^T g)(x) - (\mu^T g)(u) + x^T w(\mu^T q) - u^T w(\mu^T q) > 0.$$
Since \( w \in D \), therefore \( s(x|D) \geq x^T w \) and hence
\[
[s(x|D) - x^T w] q \in Q.
\]
Using \( \mu \in Q^+ \), we get
\[
s(x|D)(\mu^T q) - x^T w(\mu^T q) \geq 0 \tag{25}
\]
Adding (24) and (25), we get
\[
(\mu^T g)(x) + s(x|D)(\mu^T q) > 0. \tag{26}
\]
Since \( x \) is feasible for the problem (FVP), we have
\[
-g(x) - s(x|D)q \in Q.
\]
Using \( \mu \in Q^+ \), we get
\[
(\mu^T g)(x) + s(x|D)(\mu^T q) \leq 0,
\]
which is contradiction to the inequality (26).
Hence
\[
\frac{f(u) + (u^T z)k}{h(u) - u^T y} - \frac{f(x) + s(x|C)k}{h(x) - s(x|E)} \notin \text{int} K.
\]

Now we give an example to illustrate the above weak duality result.

**Example 3.** Consider the problem in Example 1. The dual associated with that problem (FVP) is given by
\[
K-\text{Max } \left(\begin{array}{c}
-1 - u^2 + uz \\
1 - u^2 - uy
\end{array}\right)^T
\]
subject to
\[
\frac{\lambda_1 \{u^2 z + u^2 y + z - y - 4u\} + \lambda_2 (u^2 + 1)(1 + 2z)}{(1 - uy - u^2)^2} - \mu_1 \left(28u^3 + \frac{11}{4}w\right) + \mu_2 \left(4u^3 + w + 1\right) = 0,
\]
\[
\mu_1 \left(-7u^4 - \frac{11}{4}uw + 2\right) + \mu_2 \left(u^4 + uw + u + 2\right) \geq 0,
\]
where \( u \in \mathbb{R}, z \in [-1, 1], y \in [0, 2], w \in [0, 1], \lambda \in K^+ \setminus \{0\}, \mu \in Q^+ \).

Take \( x = -\frac{1}{4} \) feasible for (FVP) and a feasible point \((u, z, y, w, \lambda, \mu) = (0, -1, \frac{3}{4}, \frac{1}{15}, (-1, 3)^T, (-1, 1)^T)\) for (FMD). As shown in Example 1, \( f \) is \( K \)-convex, \(-h\) is convex, \( g \) is \( Q \)-convex at \( u \) and \( f(u) + (u^T z)k \in K \).

Here,
\[
\frac{f(u) + (u^T z)k}{h(u) - u^T y} - \frac{f(x) + s(x|C)k}{h(x) - s(x|E)} = \left(-\frac{2}{15}, -\frac{4}{15}\right)^T \notin \text{int} K.
\]
Thus Weak Duality result (Theorem 4.1) holds.

Now we prove the Strong Duality result.
**Theorem 4.2** (Strong Duality). Let \( \pi \in S_0 \) be a weak minimum of (FVP). Then there exist \( \lambda \in K^+ \), \( \mu \in Q^+ \) with \( (\lambda, \mu) \neq 0 \) and \( \pi \in \partial s(\pi|C) \), \( \gamma \in \partial s(\pi|E) \), \( \varpi \in \partial s(\pi|D) \) such that \([3]\) and \([4]\) hold. Moreover if \( g \) is \( Q \)-convex at \( \pi \) and there exists \( x^* \in \mathbb{R}^n \) such that \( (\pi^T g)(x^*) + s(x^*|D)(\pi^T \gamma) < 0 \) then \( (\pi, \lambda, \gamma, \mu, \varpi, \lambda, \mu) \) is feasible for (FMD). Suppose conditions of Weak Duality Theorem 4.1 are satisfied for each feasible solution \( (u, z, y, w, \lambda, \mu) \) of (FMD), then \( (\pi, \lambda, \gamma, \mu, \varpi, \lambda, \mu) \) is weak maximum for (FMD).

**Proof.** Since \( \pi \) is a weak minimum of (FVP), therefore by Theorems \([3.3]\) and \([3.5]\) there exist \( \lambda \in K^+ \setminus \{0\} \), \( \mu \in Q^+ \), \( \gamma \in \partial s(\pi|C) \), \( \varpi \in \partial s(\pi|D) \), \( \varpi \in \partial s(\pi|D) \) such that \([3]\) and \([4]\) hold. Hence, \( (\pi, \pi, \gamma, \varpi, \lambda, \mu) \) is feasible for (FMD).

Now consider the conditions of Weak Duality Theorem 4.1 hold for all feasible points of (FMD) and let if possible \( (\pi, \lambda, \gamma, \varpi, \lambda, \mu) \) be not a weak maximum of (FMD), then there exists a feasible point \( (u, z, y, w, \lambda, \mu) \) of (FMD) such that

\[
\frac{f(u) + (u^T z)k}{h(u) - u^T y} - \frac{f(\pi) + (\pi^T \pi)k}{h(\pi) - \pi^T y} \in \text{int} K.
\]

Since \( \pi \in \partial s(\pi|C) \) and \( \gamma \in \partial s(\pi|E) \), therefore \( \pi^T \varpi = s(\pi|C) \) and \( \pi^T \gamma = s(\pi|E) \) and hence from the above relation we have

\[
\frac{f(u) + (u^T z)k}{h(u) - u^T y} - \frac{f(\pi) + s(\pi|C)k}{h(\pi) - s(\pi|E)} \in \text{int} K.
\]

which is a contradiction to the Weak Duality Result. Hence, \( (\pi, \lambda, \gamma, \varpi, \lambda, \mu) \) is a weak maximum of (FMD).

5. **Schaible type dual.** Consider the following Schaible type dual for our problem (FVP).

(SD) \( K\text{-Max } v = (v_1, v_2, \ldots, v_m) \)

subject to

\[
\begin{align*}
\sum_{i=1}^{m} \lambda_i \left[ \{ \nabla f_i(u) + z k_i \} - v_i \{ \nabla h(u) - y \} \right] + \sum_{j=1}^{p} \mu_j \left[ \nabla g_j(u) + w q_j \right] &= 0, \\
\sum_{i=1}^{m} \lambda_i \left[ \{ f_i(u) + (u^T z) k_i \} - v_i \{ h(u) - u^T y \} \right] &\geq 0, \\
\sum_{j=1}^{p} \mu_j \left[ g_j(u) + (u^T w) q_j \right] &\geq 0,
\end{align*}
\]

where \( u \in \mathbb{R}^n, v \in K, z \in C, y \in E, w \in D, \lambda \in K^+ \setminus \{0\}, \mu \in Q^+ \).

**Remark 5.** If we replace \( m \) by \( k \), \( p \) by \( m \), \( g_j \) by \( h_j \) and we take \( K = \mathbb{R}_+^k, Q = \mathbb{R}_+^m, C = D = E = \{0\} \), then our dual (SD) reduces to the dual (D) considered by Suneja and Gupta \([24]\) where \( g_i = h \) and \( S = \mathbb{R}^n \).

Theorem 5.1 (Weak Duality). Let \( x \) be feasible for (FVP) and \( (u, v, z, y, w, \lambda, \mu) \) be feasible for (SD). If \( f \) is \( K \)-convex, \( -h \) is convex and \( g \) is \( Q \)-convex at \( u \). Then

\[
\frac{v - f(x) + s(x|C)k}{h(x) - s(x|E)} \notin \text{int} K.
\]
Proof. Let, if possible, \( v - f(x) + s(x|C)k \) \( \frac{h(x) - s(x|E)}{h(x) - s(x|E)} \in \text{int} K. \)

Since \( x \) is feasible for (FVP), so \( h(x) - s(x|E) > 0 \) and hence we have

\( v \{ h(x) - s(x|E) \} - \{ f(x) + s(x|C)k \} \in \text{int} K. \)

Using the fact that \( z \in C \) and \( k \in K \), we have

\( (s(x|C) - x^T z) k \in K, \)

and \( y \in E \) and \( v \in K \), gives

\( (s(x|E) - x^T y) v \in K. \)

Adding the above three relations we get

\[ v \{ h(x) - x^T y \} - \{ f(x) + (x^T z) k \} \in \text{int} K. \]

Using \( \lambda \in K^+ \setminus \{0\} \), we get

\[ \sum_{i=1}^{m} \lambda_i \left[ v_i \{ h(x) - x^T y \} - \{ f_i(x) + (x^T z) k_i \} \right] > 0. \]

Since \( f \) is \( K \)-convex and \( \lambda \in K^+ \setminus \{0\} \), we have

\[ \sum_{i=1}^{m} \lambda_i \left[ f_i(x) - f_i(u) - (x - u)^T \nabla f_i(u) \right] \geq 0, \]

and \(-h\) is convex at \( u \), \( v \in K \) and \( \lambda \in K^+ \setminus \{0\} \), implies that

\[ \sum_{i=1}^{m} \lambda_i v_i \left[ -h(x) + h(u) + (x - u)^T \nabla h(u) \right] \geq 0. \]

Now adding the above three inequalities, we get

\[ - \sum_{i=1}^{m} \lambda_i \left[ \{ f_i(u) + (u^T z) k_i \} - v_i \{ h(u) - u^T y \} \right] \]

\[ - (x - u)^T \sum_{i=1}^{m} \lambda_i \left[ \{ \nabla f_i(u) + z k_i \} - v_i \{ \nabla h(u) - y \} \right] > 0. \]

Since \((u, v, z, y, w, \lambda, \mu)\) is feasible for (SD), therefore

\[ (x - u)^T \sum_{j=1}^{p} \mu_j [\nabla g_j(u) + w q_j] > 0 \]

Now since \( g \) is \( Q \)-convex at \( u \) and \( \mu \in Q^+ \), therefore

\[ \sum_{j=1}^{p} \mu_j \left[ g_j(x) - g_j(u) - (x - u)^T \nabla g_j(u) \right] \geq 0. \]

Adding above two inequalities, we get

\[ \sum_{j=1}^{p} \mu_j \left[ g_j(x) + (x^T w) q_j - g_j(u) - (u^T w) q_j \right] > 0. \]
Using the fact that \((u, v, z, y, w, \lambda, \mu)\) is feasible for (SD), we get
\[
\sum_{j=1}^{p} \mu_j \left[ g_j(x) + (x^T w) q_j \right] > 0.
\]
Since \(w \in D\), therefore \(s(x|D) \geq x^T w\) and then using \(\mu \in Q^+\), we get
\[
\sum_{j=1}^{p} \mu_j \left[ s(x|D) - x^T w \right] q_j \geq 0.
\]
Adding above two inequalities, we have
\[
\sum_{j=1}^{p} \mu_j \left[ g_j(x) + s(x|D) q_j \right] > 0.
\]
(27)
Since \(x\) is feasible for the problem (FVP), we have
\[
-g(x) - s(x|D)q \in Q.
\]
Using \(\mu \in Q^+\), we get
\[
\sum_{j=1}^{p} \mu_j \left[ g_j(x) + s(x|D) q_j \right] \leq 0,
\]
which is contradiction to the inequality (27).
Hence
\[
v - f(x) + s(x|C) k - h(x) - s(x|E) \notin \text{int} K.
\]

Now we give an example to illustrate the above weak duality result.

**Example 4.** Consider the problem in Example 1. The dual associated with that problem (FVP) is given by

\[
K-\text{Max} \ (v_1, v_2)^T
\]
subject to

\[
\begin{align*}
\lambda_1 \left[ \{-2u + z\} - v_1 \{-2u - y\} \right] + \lambda_2 \left[ \{1 + 2z\} - v_2 \{-2u - y\} \right] & - \mu_1 \left( 28u^3 + \frac{11}{4} w \right) + \mu_2 \left( 4u^3 + w + 1 \right) = 0, \\
\lambda_1 \left[ \{-1 - u^2 + uz\} - v_1 \{1 - u^2 - uy\} \right] + \lambda_2 \left[ \{u + 2uz\} - v_2 \{1 - u^2 - uy\} \right] & \geq 0, \\
\mu_1 \left( -7u^4 + \frac{11}{4} uw + 2 \right) + \mu_2 \left( u^4 + uw + u + 2 \right) & \geq 0,
\end{align*}
\]
where \(u \in \mathbb{R}, v \in K, z \in [-1, 1], y \in [0, 2], w \in [0, 1], \lambda \in K^+ \setminus \{0\}, \mu \in Q^+\).

Take \(x = -\frac{1}{4}\) feasible for (FVP) and a feasible point \((u, v, z, y, w, \lambda, \mu) = \left(0, \left(-\frac{1}{2}, \frac{1}{6}\right)^T, -1, \frac{1}{4}, \frac{1}{15}, (-1, 3)^T, (-1, 1)^T\right)\) for (SD). As shown in Example 1, \(f\) is \(K\)-convex, \(-h\) is convex and \(g\) is \(Q\)-convex at \(u\).
Here,
\[ v - \frac{f(x) + s(x|C)k}{h(x) - s(x|E)} = \left( \frac{11}{30}, -\frac{1}{10} \right)^T \notin \text{int}K. \]

Thus Weak Duality result (Theorem 5.1) holds.

**Theorem 5.2 (Weak Duality).** Let \( x \) be feasible for (FVP) and \((u, v, z, y, w, \lambda, \mu)\) be feasible for (SD). If \( \sum_{i=1}^{m} \lambda_i \left[ \{ f_i(\cdot) + (\cdot)^T z_i \} - v_i \{ h(\cdot) - (\cdot)^T y \} \right] \) is pseudoconvex and \( \sum_{j=1}^{p} \mu_j \left[ g(\cdot) + ((\cdot)^T w_j \right] \) is quasiconvex at \( u \). Then

\[ v - \frac{f(x) + s(x|C)k}{h(x) - s(x|E)} \notin \text{int}K. \]

**Proof.** Let, if possible,
\[ v - \frac{f(x) + s(x|C)k}{h(x) - s(x|E)} \in \text{int}K. \]

Proceeding as in Theorem 5.1, we get
\[ \sum_{i=1}^{m} \lambda_i \left[ v_i \{ h(x) - x^T y \} - \{ f_i(x) + (x^T z) k_i \} \right] > 0. \]

Using above inequality and the fact that \((u, v, z, y, w, \lambda, \mu)\) is feasible for (SD), we have
\[ \sum_{i=1}^{m} \lambda_i \left[ \{ f_i(x) + (x^T z) k_i \} - v_i \{ h(x) - x^T y \} \right] < \sum_{i=1}^{m} \lambda_i \left[ \{ f_i(u) + (u^T z) k_i \} - v_i \{ h(u) - u^T y \} \right]. \]

Since \( \sum_{i=1}^{m} \lambda_i \left[ \{ f_i(\cdot) + (\cdot)^T z \} k_i \right] - v_i \{ h(\cdot) - (\cdot)^T y \} \right] \) is pseudoconvex at \( u \), therefore
\[ (x - u)^T \sum_{i=1}^{m} \lambda_i \left[ \{ \nabla f_i(u) + z k_i \} - v_i \{ \nabla h(u) - y \} \right] < 0, \]

Since \((u, v, z, y, w, \lambda, \mu)\) is feasible for (SD), therefore
\[ (x - u)^T \sum_{j=1}^{p} \mu_j \left[ \nabla g_j(u) + w q_j \right] > 0 \]

Now since \( \sum_{j=1}^{p} \mu_j \left[ g(\cdot) + ((\cdot)^T w) q_j \right] \) is quasiconvex at \( u \), therefore
\[ \sum_{j=1}^{p} \mu_j \left[ g(x) + ((x)^T w) q_j \right] > \sum_{j=1}^{p} \mu_j \left[ g(u) + ((u)^T w) q_j \right]. \]
Using the fact that \((u, v, z, y, w, \lambda, \mu)\) is feasible for (SD) in the above inequality, we get
\[
\sum_{j=1}^{p} \mu_j \left[ g_j(x) + (x^T w_j) q_j \right] > 0.
\]

Now again proceeding as in Theorem 5.1 we arrive at a contradiction and hence
\[
v - \frac{f(x) + s(x|C)k}{h(x) - s(x|E)} \notin \text{int} K.
\]

\[\square\]

**Theorem 5.3 (Strong Duality).** Let \(\pi \in S_0\) be a weak minimum of (FVP). Then there exist \(v \in K, \lambda \in K^+, \mu^* \in Q^+\) with \((\lambda, \mu^*) \neq 0\) and \(\pi \in \partial s(\pi|C), \gamma \in \partial s(\pi|E), \\nu \in \partial s(\pi|D)\) such that \(g\) and \(h\) hold. Moreover if \(g\) is Q-convex at \(\pi\) and there exists \(x^* \in \mathbb{R}^n\) such that \((\mu^* g)(x^*) + x^*|D|(\mu^* g) < 0\) then \((\pi, v, z, y, w, \lambda, \mu^*)\) is feasible for (SD). Suppose conditions of Weak Duality Theorem 5.1 or 5.2 are satisfied for each feasible solution \((u, v, z, y, w, \lambda, \mu)\) of (SD), then \((\pi, v, z, y, w, \lambda, \mu^*)\) is weak maximum for (SD).

**Proof.** Since \(\pi\) is a weak minimum of (FVP), therefore by Lemma 3.2 \(\pi\) is also a weak minimum of (FVP)\(\pi\), where \(\nu = \frac{f(\pi) + s(\pi|C)}{h(\pi) - s(\pi|E)} \in K\). Now proceeding as in Theorem 3.4 we get \(\lambda \in K^+, \mu^* \in Q^+\) with \((\lambda, \mu^*) \neq 0\) and \(\pi \in \partial s(\pi|C), \gamma \in \partial s(\pi|E), \\nu \in \partial s(\pi|D)\) such that \(g\) and \(h\) hold. Now following the lines of Theorem 3.3 by replacing \(\mu\) with \(\mu^*\), we can prove that \(\lambda \neq 0\). Hence, \((\pi, v, z, y, w, \lambda, \mu^*)\) is feasible for (SD).

Now suppose the conditions of Weak Duality Theorem 5.1 or 5.2 hold for all feasible points of (SD) and let if possible \((\pi, v, z, y, w, \lambda, \mu)\) be not a weak maximum of (SD), then there exists a feasible point \((u, v, z, y, w, \lambda, \mu)\) of (SD) such that
\[
v - \nu \not\in \text{int} K,
\]
or,
\[
v - \frac{f(\pi) + s(\pi|C)k}{h(\pi) - s(\pi|E)} \not\in \text{int} K
\]
which is a contradiction to the Weak Duality Result. Hence, \((\pi, v, z, y, w, \lambda, \mu^*)\) is a weak maximum of (SD). \(\square\)

6. **Special cases.** In this section we specialize our problem.

1. Let the compact convex set \(C, E\) and \(D\) be
\[
C = \{ Mz : z^T Mz \leq 1 \},
\]
\[
E = \{ Ny : y^T Ny \leq 1 \},
\]
\[
D = \{ Pw : w^T Pw \leq 1 \},
\]
where \(M, N\) and \(P\) are positive semidefinite matrices of order \(n\). As discussed by Mond and Schechter [20], we may write
\[
s(x|C) = (x^T Mx)^{1/2}, \quad s(x|E) = (x^T Nx)^{1/2}
\]
and
\[
s(x|D) = (x^T Px)^{1/2}.
\]
Putting these values in our problem (FVP), we get
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\[(FVP)_1\] \quad K\text{-Min} \quad \frac{f(x) + (x^T M x)^{1/2} k}{h(x) - (x^T N x)^{1/2}}

subject to

\[-g(x) - (x^T P x)^{1/2} q \in Q,\]

Similarly substituting these values in dual problems, we get

\[(FMD)_1\] \quad K\text{-Max} \quad \frac{f(u) + (u^T M z) k}{h(u) - u^T N y}

subject to

\[\sum_{i=1}^{m} \lambda_i \nabla \left( \frac{f_i(u) + (u^T M z) k_i}{h(u) - u^T N y} \right) + \sum_{j=1}^{p} \mu_j \left[ \nabla g_j(u) + (P w) q_j \right] = 0,\]

\[\sum_{j=1}^{p} \mu_j \left[ g_j(u) + (u^T P w) q_j \right] \geq 0.\]

where \(u, z, y, w \in \mathbb{R}^n, z^T M z \leq 1, y^T N y \leq 1, w^T P w \leq 1, \lambda \in K^+ \setminus \{0\}, \mu \in \mathcal{Q}^+.\)

\[(SD)_1\] \quad K\text{-Max} \quad v = (v_1, v_2, \ldots, v_m)

subject to

\[\sum_{i=1}^{m} \lambda_i \left[ \nabla f_i(u) + (M z) k_i \right] - v_i \left\{ h(u) - N y \right\} \]
\[+ \sum_{j=1}^{p} \mu_j \left[ \nabla g_j(u) + (P w) q_j \right] = 0,\]

\[\sum_{i=1}^{m} \lambda_i \left[ \{ f_i(u) + (u^T M z) k_i \} - v_i \{ h(u) - u^T N y \} \right] \geq 0,\]

\[\sum_{j=1}^{p} \mu_j \left[ g_j(u) + (u^T P w) q_j \right] \geq 0.\]

where \(u, z, y, w \in \mathbb{R}^n, v \in K, z^T M z \leq 1, y^T N y \leq 1, w^T P w \leq 1, \lambda \in K^+ \setminus \{0\}, \mu \in \mathcal{Q}^+.\)

If we take \(m = p\) and \(p = q\) and replace \(g_i\) by the function \(G_i\), also if we take \(K = \mathbb{R}^p_+, Q = \mathbb{R}^q_+, k = (1, 1, \ldots, 1) \in \mathbb{R}^p\) and \(q = (1, 1, \ldots, 1) \in \mathbb{R}^q.\) Then our problem \((FVP)_1\) reduces to the problem \((P4)\) considered by Zalmai \([26]\) where \(X = \mathbb{R}^n, \) the set \(r\) is empty, \(g_i = h, P_i = M, Q_i = N\) for each \(i = 1, 2, \ldots, p\) and \(R_j = P\) for each \(i = 1, 2, \ldots, q.\)

2. Let the compact convex set \(C, E\) and \(D\) be

\[C = \{ M z : \|z\|_{p_1} \leq 1 \} \]
\[E = \{ N z : \|z\|_{p_1} \leq 1 \}\]
\[ D = \{ Pw : \|w\|_{p_1} \leq 1 \}, \]

where \( M, N \) and \( P \) are \( n \times n \) matrices. As shown by Mond and Schechter [20], we can write

\[ s(x|C) = \|Mx\|_{p_2}, \quad s(x|E) = \|Nx\|_{p_2} \]

and

\[ s(x|D) = \|Px\|_{p_2} \]

where \( p_1 \) and \( p_2 \) are conjugate exponents, i.e., for \( p_2 > 1, \frac{1}{p_1} + \frac{1}{p_2} = 1 \).

Then substituting these values of \( s(x|C) \) and \( s(x|D) \) in the problem \((FVP)\), we get

\[ (FVP)_2 \quad K\text{-Minimize} \quad \frac{f(x) + \|Mx\|_{p_1}k}{h(x) - \|Nx\|_{p_2}} \]

subject to

\[ -g(x) - \|Px\|_{p_2}q \in Q, \]

and its Mond-Weir type dual is given as under

\[ (FMD)_2 \quad K\text{-Maximize} \quad \frac{f(u) + (u^T Mz)k}{h(u) - u^T Ny} \]

subject to

\[ \sum_{i=1}^{m} \lambda_i \left( f_i(u) + (u^T Mz)k_i \right) + \sum_{j=1}^{p} \mu_j \left[ \nabla g_j(u) + (Pw)q_j \right] = 0, \]

\[ \sum_{j=1}^{p} \mu_j \left[ g_j(u) + (u^T Pw)q_j \right] \geq 0, \]

where \( u, z, y, w \in \mathbb{R}^n, \|z\|_{p_1} \leq 1, \|y\|_{p_1} \leq 1, \|w\|_{p_1} \leq 1, \lambda \in K^+ \setminus \{0\}, \mu \in Q^+. \)

\[ (SD)_2 \quad K\text{-Max} \quad v = (v_1, v_2, \ldots, v_m) \]

subject to

\[ \sum_{i=1}^{m} \lambda_i \left[ \{\nabla f_i(u) + (Mz)k_i\} - v_i \{\nabla h(u) - Ny\} \right] + \sum_{j=1}^{p} \mu_j \left[ \nabla g_j(u) + (Pw)q_j \right] = 0, \]

\[ \sum_{i=1}^{m} \lambda_i \left[ \{f_i(u) + (u^T Mz)k_i\} - v_i \{h(u) - u^T Ny\} \right] \geq 0, \]

\[ \sum_{j=1}^{p} \mu_j \left[ g_j(u) + (u^T Pw)q_j \right] \geq 0. \]
where \( u, z, y, w \in \mathbb{R}^n, v \in K, \|z\|_{p_1} \leq 1, \|y\|_{p_1} \leq 1, \|w\|_{p_1} \leq 1, \lambda \in K^+ \setminus \{0\}, \mu \in Q^+ \).

If we take \( m = p \) and \( p = q \) and replace \( g_i \) by the function \( G_i \), also if we take \( K = \mathbb{R}^n_+, Q = \mathbb{R}^n_+ \), \( k = (1, 1, \ldots, 1) \in \mathbb{R}^p \) and \( q = (1, 1, \ldots, 1) \in \mathbb{R}^q \). Then our problem (FVP) reduces to the problem \((P)\) considered by Zalmai [20] where \( X = \mathbb{R}^n \), the set \( r \) is empty, \( g_i = h \), \( A_i = M \), \( B_i = N \), \( a(i) = p_2 \), \( b(i) = p_2 \) for each \( i = 1, 2, \ldots, p \) and \( C_j = P \), \( c_j = p_2 \) for each \( i = 1, 2, \ldots, q \).

If we take \( m = 1, p = m, K = \mathbb{R}^n_+, k = 1, Q = \mathbb{R}^n_+ \) and \( q = (1, 1, \ldots, 1)^T \in Q \) then our problems (FVP)\(_1\) and (FMD)\(_1\) reduce to the problem (FP)\(_1\) and (FD)\(_1\), respectively considered by Husain and Jabeen [8] as special case where \( g = h \), \( A = M \), \( B = N \) and for \( j = 1, 2, \ldots, m \), \( h_j = g_j \) and \( E_j = P \). Further our problems (FVP)\(_2\) and (FMD)\(_2\) reduce to the problem (FP)\(_2\) and (FD)\(_2\), respectively, considered in [8] where \( p = p_2 \), \( q = p_1 \), \( g = h \), \( P_1 = M \), \( P_2 = N \) and for \( j = 1, 2, \ldots, m \), \( h_j = g_j \) and \( Q_j = P \).

7. Conclusions. The advantage of this paper is that the problem (FVP) considered here generalises many existing problems in the literature. Under particular conditions our problem (FVP) reduces to the problems considered by Kim [10], Kim [11], Chen et al. [4], Suneja and Gupta [24], Antczak [1], Kim et al. [12], Jaywal et al. [9] and Husain et al. [8]. The main result of the paper, Theorem 3.4, is proved without using any convexity assumptions. Moreover, two type of duals Mond-Weir type duals (FMD) and Schaible type dual (SD) are formulated and duality results are studied.

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