ON THE BAUM–CONNES CONJECTURE FOR DISCRETE QUANTUM GROUPS WITH TORSION AND THE QUANTUM ROSENBERG CONJECTURE

YUKI ARANO AND ADAM SKALSKI

Abstract. We give a decomposition of the equivariant Kasparov category for discrete quantum group with torsions. As an outcome, we show that the crossed product by a discrete quantum group in a certain class preserves the UCT. We then show that quasidiagonality of a reduced C*-algebra of a countable discrete quantum group Γ implies that Γ is amenable, and deduce from the work of Tikuisis, White and Winter, and the results in the first part of the paper, the converse (i.e. the quantum Rosenberg Conjecture) for a large class of countable discrete unimodular quantum groups. We also note that the unimodularity is a necessary condition.

1. Introduction

In [RoS], Rosenberg and Schochet have introduced a property of C*-algebras called the Universal Coefficient Theorem (UCT in short) for K-theory of C*-algebras and have shown that it holds for all C*-algebras in the so-called bootstrap class. The UCT gives a formula computing the KK-groups only from the K-groups. This property plays an important role in the classification of nuclear C*-algebras (see e.g. [TWW] and the last section of this paper).

The UCT for group C*-algebras is related to (a variation of) the Baum–Connes conjecture of groups. In [Tu], Tu proved that the group C*-algebra of a discrete group with Haagerup property satisfies the UCT using the Higson–Kasparov type argument [HiK] for groupoids.

The Baum–Connes conjecture for quantum groups first appeared in the series of works of Meyer and Nest [MN3, Mey] (after an early paper [GoK]). Even though there is no unified method proving the Baum–Connes conjecture for fairly general quantum groups, it is proven for many known examples of discrete quantum groups: [FrM], [VeV], [Vo1], [Vo2].

In this paper, we study the general theory of the Baum–Connes conjecture for discrete quantum group with possible torsion. In particular we give a decomposition of the equivariant category KK^G (where G is any compact quantum group), which gives the Baum–Connes assembly map. As a byproduct of the general theory, we prove that the group C*-algebra of a discrete quantum group satisfying the Baum–Connes conjecture satisfies also the UCT. This is applied in the last section of the paper to the considerations regarding the quantum version of the Rosenberg Conjecture, connecting amenability of a discrete group to quasidiagonality of its C*-algebra.

The detailed plan of the paper is as follows: in the following section we introduce the notation and some background related to discrete/compact quantum groups and triangulated categories. In Section 3 we present a ‘crossed product type’ construction for two C*-algebras equipped respectively with left and right action of a given compact quantum group, which is then applied in Section 4 to build an adjoint functor between certain KK-categories. In Section 5 we establish as a consequence a relationship between the (Cof)-Baum–Connes property of a discrete quantum group and the Universal Coefficient Theorem for some crossed products. Finally in Section 6 the applications to quantum Rosenberg Conjecture are discussed.

Acknowledgment. This work was initiated in the workshop “The 6th Workshop on Operator Algebras and their Applications” in the School of Mathematics of Institute for Research in Fundamental Sciences (IPM). The authors would like to thank the organizers and IPM for their hospitality. Y.I. is supported by JSPS KAKENHI Grant Number JP18K13424. A.S. was partially

1991 Mathematics Subject Classification. Primary 46L67, Secondary 46L80.
Key words and phrases. Quantum group, triangulated categories, UCT, Rosenberg conjecture.
supported by the National Science Centre (NCN) grant no. 2014/14/E/ST1/00525. He acknowledges discussions with Paweł Józiak, Piotr Sołtan, Stuart White and Joachim Zacharias on the subject of the last section of the paper.

2. Preliminaries

2.1. Quantum groups. Let \( \Gamma \) be a discrete quantum group (so that \( \hat{\Gamma} \) is a compact quantum group in the sense of [Wo1] – we refer to that paper for the details of the facts introduced below, and often write simply \( \mathbb{G} \) for the dual compact quantum group). We study \( \Gamma \) via its algebra of functions, \( c_0(\Gamma) \). Recall that

\[
c_0(\Gamma) = \bigoplus_{\alpha \in \text{Irr}_\mathbb{F}} M_{n_\alpha},
\]

where \( \text{Irr}_\mathbb{F} \) denotes the set of equivalence classes of irreducible unitary representations of \( \hat{\Gamma} \); the span of coefficients of the latter is a Hopf *-algebra denoted \( \mathcal{O}(\hat{\Gamma}) \), admitting a Haar (bi-invariant) state \( h \). Note that we will also write \( c_c(\mathbb{F}) \) for the algebraic direct sum:

\[
c_c(\mathbb{F}) = \bigoplus_{\alpha \in \text{Irr}_\mathbb{F}} M_{n_\alpha},
\]

The \( \text{C}^* \)-algebra \( C^*_r(\mathbb{F}) \), often written as \( C(\hat{\Gamma}) \), is the \( \text{C}^* \)-completion of \( \mathcal{O}(\hat{\Gamma}) \) in the GNS representation with respect to \( h \). For each \( \alpha \in \text{Irr}_\mathbb{F} \) we choose a representative, i.e. a unitary matrix \( U^\alpha = (u^\alpha_{ij})_{i,j=1}^{n_\alpha} \in M_{n_\alpha}(C^*_r(\hat{\Gamma})) \). We may and do assume that \( c_0(\Gamma) \) is represented on the Hilbert space \( \ell^2(\Gamma) \) (viewed here as the GNS space of the Haar state of \( \hat{\Gamma} \), so also the space on which \( C^*_r(\hat{\Gamma}) \) acts); this representation will be later denoted by \( \lambda \). The matrix units in \( M_{n_\alpha} \subset c_0(\Gamma) \) will be denoted by \( e^\alpha_{ij} \).

The multiplicative unitary of \( \Gamma \) is the unitary \( W \in B(\ell^2(\Gamma) \otimes \ell^2(\Gamma)) \) given by the formula:

\[
W = \sum_{\alpha \in \text{Irr}_\mathbb{F}} \sum_{i,j=1}^{n_\alpha} e^\alpha_{ij} \otimes (u^\alpha_{ij})^*.
\]

The von Neumann completion of \( c_0(\Gamma) \) will be denoted by \( \ell^\infty(\Gamma) \). The predual of \( \ell^\infty(\Gamma) \) will be denoted by \( \ell^1(\Gamma) \).

The coproduct of \( \Gamma \), a coassociative normal unital *-homomorphism \( \Delta : \ell^\infty(\Gamma) \to \ell^\infty(\Gamma) \otimes \ell^\infty(\Gamma) \) is implemented by \( W \) via the following formula:

\[
(2.1) \quad \Delta(x) = W^*(1 \otimes x)W, \quad x \in \ell^\infty(\Gamma).
\]

Given a functional \( \phi \in \ell^1(\Gamma) \) we define the (normal, bounded) maps \( L_\phi : \ell^\infty(\Gamma) \to \ell^\infty(\Gamma) \) and \( R_\phi : \ell^\infty(\Gamma) \to \ell^\infty(\Gamma) \) via the formulas

\[
L_\phi = (\phi \otimes \text{id}) \circ \Delta, \quad R_\phi = (\text{id} \otimes \phi) \circ \Delta.
\]

A discrete quantum group \( \Gamma \) is said to be finite, if \( \text{Irr}_\mathbb{F} \) is finite (equivalently, \( c_0(\Gamma) \) is finite-dimensional), and countable, if \( \text{Irr}_\mathbb{F} \) is countable (equivalently, \( c_0(\Gamma) \) is separable); it is unimodular if its left and right Haar weights coincide; equivalently the Haar state \( h \) of \( \hat{\Gamma} \) is tracial.

A discrete quantum group \( \Gamma \) is called amenable if it admits a bi-invariant mean, i.e. a state \( m \in \ell^\infty(\Gamma)^* \), such that for all \( \phi \in \ell^1(\Gamma) \) there is

\[
m \circ L_\phi = m \circ R_\phi = \phi(1)m.
\]

By [DQV] a discrete quantum group \( \Gamma \) is amenable if it admits a left invariant mean \( m \in \ell^\infty(\Gamma)^* \); a state such that for each \( \phi \in \ell^1(\Gamma) \) there is \( m \circ L_\phi = \phi(1)m \). In fact it suffices to check the last formula for the functionals of the form \( e^\alpha_{ij} \), \( \alpha \in \text{Irr}_\mathbb{F} \), \( i,j = 1, \ldots, n_\alpha \), as the latter are linearly dense in \( \ell^1(\Gamma) \), and the map \( \phi \mapsto L_\phi \) is a (complete) isometry. Thus we will need the following
explicit form of the map $L_{\phi}$ for $\phi = \hat{c}_{i,j}^\alpha$:

\begin{equation}
L_{\phi}(x) = \sum_{p=1}^{n_m} u_{i,p}^\alpha x (u_{j,p}^\alpha)^*
\end{equation}

(with $x \in \ell^\infty(\Gamma)$).

Recall that $G$ denotes the dual compact quantum group of $\Gamma$. The left regular representation $\lambda: C(G) \to B(L^2(G))$ is the GNS representation with respect to the Haar state $\varphi$; note that $L^2(G)$ is canonically isomorphic to $\ell^2(\Gamma)$. We also have the right regular representation $\rho(x) = JR(x)^* J$, $x \in C(G)$, where $J$ is the modular conjugation and $R$ is the unitary antipode.

Via the natural pairing

$$\mathcal{O}(G) \times c_c(\Gamma) \to \mathbb{C},$$

we put a (multiplier) Hopf algebra structure on $c_c(\Gamma)$.

For details of quantum group actions and the associated crossed products we refer for example to [DC] and [Vae]; note that we always work with reduced/faithful actions. Given a left action $\alpha: A \to C(G) \otimes A$ we call $A$ a $G$-$C^*$-algebra. Such an action induces a right $c_c(\Gamma)$-comodule algebra structure on $A$:

$$a \triangleright x := (x \otimes \text{id})_\alpha(a)$$

for $a \in A$ and $x \in c_c(\Gamma)$. Similarly a right action $\beta: B \to B \otimes C(G)$ induces a left $c_c(\Gamma)$-comodule algebra structure on $B$:

$$x \triangleright b := (\text{id} \otimes x)_\beta(b)$$

for $b \in B$ and $x \in c_c(\Gamma)$.

For a finite dimensional $C^*$-algebra $D$ and a left action $\alpha$ of $G$ on $D$, there always exists a $G$-invariant state $\varphi_D$ on $D$, which is of the form $\varphi_D = \text{Tr}(\rho)$, where $\text{Tr}$ is the trace taking value 1 at each minimal projection and $\varrho \in D$. Let $(\lambda_D, L^2(D) = L^2(D, \varphi_D), \Omega_D)$ be the GNS representation of $\varphi_D$. Then $L^2(D)$ also carries a $*$-representation $\rho_D$ of the opposite $C^*$-algebra $D^{\text{op}}$, given by

$$\rho_D(x^{\text{op}}) \lambda_D(y) \Omega_D = \lambda_D(y \rho^{1/2} x \rho^{-1/2}) \Omega_D, \quad x, y \in D.$$  

Furthermore the formula

$$U^*(a \otimes x \Omega) = \alpha(x)(a \otimes \Omega), \quad a \in C(G), x \in D,$$

defines a unitary representation $U \in C(G) \otimes B(L^2(D))$.

**Definition 2.1.** [BS] For a $C^*$-algebra $A$ with a $G$-action $\alpha: A \to C(G) \otimes A$ and a Hilbert $A$-module $E$, a $G$-action on $E$ is a linear map $\alpha_E: E \to C(G) \otimes E$ such that

1. $\alpha_E(xa) = \alpha_E(x)\alpha(a)$ for all $x \in E, a \in A$,
2. the linear span $\alpha_E(E)(C(G) \otimes 1)$ is dense in $C(G) \otimes E$ and
3. $(\text{id} \otimes \alpha_E)\alpha_E = (\Delta \otimes \text{id})\alpha_E$.

This is equivalent to say that $\alpha_E$ is the corner of an action of $G$ on the linking algebra $\mathcal{K}(E \oplus A) \simeq \begin{pmatrix} \mathcal{K}(E) & E \\ E^* & A \end{pmatrix}$ which coincides with $\alpha$ on $A \cong \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}$. See [BS] for details.

Finally for the notion of torsion in the context of compact quantum groups we refer for example to [ADC].

### 2.2. Triangulated category

In [MN], Meyer and Nest introduced a framework to work on $KK$-theory in terms of triangulated categories. In this section, we review their work, not going into the full generality of triangulated categories but only restricting ourselves to describe the situation in terms of $G$-equivariant $KK$-theory, where $G$ is a fixed compact quantum group with a countable dual.

To each equivariant $*$-homomorphism $\varphi: A \to B$, one can associate an exact sequence called the mapping cone exact sequence:

$$0 \to C_\varphi \xrightarrow{\iota} M_\varphi \xrightarrow{\text{ev}_1} B \to 0$$
where \( M_\varphi = \{(f, a) \in (C[0, 1] \otimes B) \oplus A) \mid f(1) = \varphi(a) \} \supset C_\varphi = \{(f, a) \in M_\varphi \mid f(0) = 0 \}. \) Notice that \( M_\varphi \) is homotopy equivalent to \( A. \) We may continue this construction for \( \iota \) to get another mapping cone exact sequence:

\[ 0 \to C_\iota \to M_\iota \to M_\varphi \to 0. \]

Then \( C_\iota \) is actually homotopy equivalent to the suspension \( SB = C_0(\mathbb{R}) \otimes B. \) Hence in the category \( KK^G, \) we get a diagram

\[ \cdots \to SC_\varphi \to SA \to B \to C_\varphi \to A \to B \]

or

\[
\begin{array}{c}
A \\
\downarrow^{\varphi}
\end{array} \downarrow^{C_\varphi} \begin{array}{c}
B
\end{array}
\]

which gives the six-term exact sequence after taking the \( K \)-groups. (Here the circle on the arrow represents the change of the degree.) A distinguished triangle is a diagram which is \( KK^G \)-equivalent to some mapping cone triangle as above.

**Definition 2.2.** A localizing subcategory of \( KK^G \) is a full subcategory which is closed under taking countable direct sums, suspensions and mapping cones.

Let \( \mathcal{P}, \mathcal{N} \) be localizing subcategories of \( KK^G. \) We say that the pair \( (\mathcal{P}, \mathcal{N}) \) is complementary if

1. \( KK^G(P, N) = 0 \) for any \( P \in \mathcal{P} \) and \( N \in \mathcal{N}. \)
2. For any \( A \in KK^G, \) there exists a distinguished triangle

\[
\begin{array}{c}
P(A) \\
\downarrow
\end{array} \begin{array}{c}
A
\end{array} \begin{array}{c}
\downarrow
\end{array} \begin{array}{c}
N(A)
\end{array}
\]

where \( P(A) \in \mathcal{P} \) and \( N(A) \in \mathcal{N}. \)

**Remark 2.3.** For arbitrary choice of \( P(A) \) and \( N(A) \) as above, the morphism \( P(A) \to A \) is universal among all morphisms \( P \to A \) for \( P \in \mathcal{P}. \) In fact, we write the six-term exact sequence of \( KK^G: \)

\[
\cdots \to KK^G(P, SN(A)) \to KK^G(P, P(A)) \to KK^G(P, A) \to KK^G(P, N(A)) \to \ldots .
\]

Since \( KK^G(P, SN(A)) = KK^G(P, N(A)) = 0 \) by (1), the map \( KK^G(P, P(A)) \to KK^G(P, A) \) is an isomorphism. This is what we claimed.

This in particular shows that the triangle \( P(A) \to A \to N(A) \) is unique up to isomorphism.

The following result holds in a more general setting, namely, when the adjoint is only partially defined, but we only use it in the following form.

**Theorem 2.4.** [MN3 Theorem 3.31] Let \( \mathbb{H}, \mathbb{G} \) be compact quantum groups with countable duals and let \( F_i \) be a countable family of functors \( KK^G \to KK^\mathbb{H} \) which preserve the distinguished triangles. Assume that there exist left adjoint functors \( F_i^\perp: KK^\mathbb{H} \to KK^G, \) i.e. \( KK^G(F_i^\perp(A), B) \simeq KK^\mathbb{H}(A, F_i(B)) \) for all \( C^* \)-algebras \( A, B \) equipped respectively with a \( \mathbb{G} \) and \( \mathbb{H} \) action. We set \( \mathcal{P} \) to be the smallest thick subcategory containing \( F_i^\perp(A) \) and \( \mathcal{N} \) to be the full subcategory whose object is \( N \in KK^G \) such that \( F_i(N) \) is \( KK^\mathbb{H} \)-contractible (note that \( \mathcal{N} \) is automatically thick). Then \( (\mathcal{P}, \mathcal{N}) \) is localizing.

In this case, an explicit construction of \( P(A) \) is given by the phantom castle construction [MN3 Section 3]. We recall the construction in Section 5.
3. Crossed products

Let $G$ be a compact quantum group. For two C*-algebras with $G$-actions, there is no general way of constructing the “product” action on the tensor product. However it is possible to construct a C*-algebra like a “crossed product” with respect to the product action. The construction works for any locally compact quantum group actions in an obvious manner, but we restrict ourselves to work with the compact case.

Let $A$ (resp. $B$) be a C*-algebra with a right (resp. left) $G$-action:

$$\alpha: A \to A \otimes C(G), \beta: B \to C(G) \otimes B;$$

these will be fixed throughout this section. By a covariant representation of $(A, B, G)$ on a Hilbert space $H$ we understand a triple of representations $\pi_A: A \to B(H), \pi_B: B \to B(H)$ and $U \in M(C(G) \otimes K(H))$, a unitary representation of $G$, which satisfies the following.

- $\pi_A(A)$ and $\pi_B(B)$ commute;
- $U^* (1 \otimes \pi_B(b))U = (\text{id} \otimes \pi_B(\beta(b)))$ for any $b \in B$;
- $\sigma(U)(\pi_A(a) \otimes 1)\sigma(U)^* = (\pi_A \otimes \text{id})(\alpha(a))$ for any $a \in A$ (where $\sigma$ denotes the tensor flip).

Recall that we denote the dual discrete quantum group of $G$ by $\Gamma$. Take the algebraic cores [DC, Definition 3.15] $A_0$ and $B_0$ of $A$ and $B$. We define a *-algebra $\mathcal{A} = A_0 \rtimes_{\text{alg}} G \ltimes_{\text{alg}} B_0$ as follows.

- As a vector space, $\mathcal{A}$ is isomorphic to $A_0 \otimes_{\text{alg}} c_0(\Gamma) \otimes_{\text{alg}} B_0$. The element in $\mathcal{A}$ corresponding to $a \otimes x \otimes b$ is denoted by $axb$ for $a \in A_0, x \in c_0(\Gamma), b \in B_0$.
- The product is given by $(axb)(a'x'b') = a(x_1 \triangleright a')x_2 x'_2(b \triangleleft x_2(x'_1)b')$ for $a, a' \in A_0, x, x' \in c_0(\Gamma), b, b' \in B_0$. Notice that the sum in the right hand side is finite since $a'$ and $b$ are in the respective algebraic cores.

Let $A \rtimes G \rtimes B$ be the universal C*-completion of $\mathcal{A}$.

Remark 3.1. In the von Neumann algebra setting, a similar construction arises from Popa’s symmetric enveloping algebra [Popa] (or the Longo–Rehren inclusion [LoR]) of a subfactor of the form $M^G \subset M$ for a minimal action of $G$ on a factor $M$.

Proposition 3.2. We have the following.

- The C*-algebras $A, B$ and $c_0(\Gamma)$ are nondegenerate C*-subalgebras in the multiplier C*-algebra $M(A \rtimes G \rtimes B)$.
- There is a natural one-to-one correspondence between the covariant representations of $(A, B, G)$ and *-representations of $A \rtimes G \rtimes B$.

Proof. Let $A^m$ (resp. $B^m$) be the universal C*-envelope of $A_0$ (resp. $B_0$) equipped with a right (resp. left) universal action of $G$. We first observe that $A^m$ coincides with the maximalization of $A$ in the sense of [FJS, Definition 6.1] (See [EcQ] for the group case). To this end, we only need to show

$$A \rtimes G \rtimes \Gamma \simeq A^m \otimes K(L^2(\Gamma))$$

(see the proof of [FJS, Theorem 6.4]). Since $\text{id} \otimes \varphi: A^m \otimes C(G) \rightarrow (A^m)^\alpha \otimes 1 = A^\alpha \otimes 1$ is a faithful conditional expectation, the action map $A_0 \rightarrow A_0 \otimes_{\text{alg}} \mathcal{O}(G)$ extends to

$$\alpha^m: A \rightarrow A^m \otimes C(G).$$

Now the triplet $((\text{id} \otimes \rho)\alpha^m, \text{id}_{A^m} \otimes \lambda, \text{id}_{A^m} \otimes \lambda)$ gives a *-homomorphism

$$A \rtimes G \rtimes \Gamma \rightarrow A^m \otimes K(L^2(\Gamma)).$$

Conversely the algebraic crossed product $A_0 \rtimes_{\text{alg}} G \rtimes_{\text{alg}} \Gamma$ is isomorphic to $A_0 \otimes_{\text{alg}} F(L^2(\Gamma))$ where $F(L^2(\Gamma)) = \text{span}\{xy \in x \in \mathcal{O}(G), y \in c_0(\Gamma)\} \subset K(L^2(\Gamma))$ is the *-algebra of all finite rank operators supported on finitely many components in $\text{Irr}_G$. Hence the universal completion of $A_0 \rtimes_{\text{alg}} G \rtimes_{\text{alg}} \Gamma$ is naturally isomorphic to $A^m \otimes K(L^2(\Gamma))$ and hence the *-homomorphism $A_0 \rtimes_{\text{alg}} G \rtimes_{\text{alg}} \Gamma \rightarrow A \rtimes G \rtimes \Gamma$ induces a map $A^m \otimes K(L^2(\Gamma)) \rightarrow A \rtimes G \rtimes \Gamma$. Since the two maps described above are inverse to each other, we get the conclusion. In particular, the natural map $A^m \rtimes G \rightarrow A \times G$ is an isomorphism.
Since $A_0$ and $c_c(\Gamma)$ satisfy the commutation relation as in $A_0 \rtimes_{\text{alg}} G$, we obtain a nondegenerate $*$-homomorphism $A \rtimes G \simeq A^m \rtimes G \to M(A \rtimes G \ltimes B)$. Since there exists a non-degenerate $*$-homomorphism

$$A \rtimes G \times B \to M((A \rtimes G) \otimes (G \rtimes B)): axb \mapsto (a \otimes 1)\Delta(x)(1 \otimes b),$$

the map is injective. Similarly we get a nondegenerate injective $*$-homomorphism $B \to M(A \rtimes G \ltimes B)$. This proves (1).

For the assertion (2), by definition of $A$, we obtain a $*$-representation of $A$ from a covariant representation of $(A, B, G)$. Conversely the covariant representation of $(A, B, G)$ is obtained by (1) from a $*$-representation of $A$. □

**Lemma 3.3.** Let $\varphi_A$ be a $G$-invariant state. Then $\varphi_A$ induces a conditional expectation

$$A \rtimes G \ltimes B \to \mathbb{G} \rtimes B: axb \mapsto \varphi_A(axb),$$

where $a \in A$, $x \in c_c(\Gamma)$, $b \in B$.

**Proof.** Take the GNS construction $(L^2(A, \varphi_A), \Omega)$ for $\varphi_A$. Consider the Hilbert $G \rtimes B$-module $E = L^2(A, \varphi_A) \otimes \mathbb{G} \rtimes B$. Then $A \rtimes G \times B$ admits a representation on $E$ defined by (the continuous extension of) the formula

$$(axb)(a'\Omega \otimes x'b') = a(x_{(1)} \triangleright a')\Omega \otimes x_{(2)}x'_{(1)}(b \triangleleft x'_{(2)})b'$$

for $a, a' \in A_0$, $x, x' \in c_c(\Gamma)$, $b, b' \in B_0$, where again $A_0$ and $B_0$ denote the respective algebraic cores. Take an approximate unit $(e_i)_{i \in I}$ of $G \rtimes B$. Then the desired conditional expectation is given by

$$x \mapsto \lim_{i \in I}(\Omega \otimes e_i, x(\Omega \otimes e_i)),$$

hence it is well-defined. □

Using this lemma, we give an easy structural result on this crossed product for later use.

**Proposition 3.4.** Let $D$ be a finite dimensional C*-algebra with a right $G$-action and $B$ be a separable C*-algebra with a left $G$-action.

1. If $B$ is finite dimensional, then the C*-algebra $D \rtimes G \rtimes B$ is a direct sum of matrix algebras.
2. If $B$ is of type I, then the C*-algebra $D \rtimes G \rtimes B$ is also of type I.

**Proof.** (1) We only need to show that any representation of $D \rtimes G \rtimes B$ decomposes into a direct sum of finite dimensional representations. To this end, we take a representation $\pi$ of $D \rtimes G \rtimes B$ on a Hilbert space $H$ and take the associated covariant representation $(\pi_D, \pi_D, U)$. Since $G$ is compact, $U$ decomposes into a direct sum of finite dimensional irreducible representations: $H = \bigoplus_i H_i$. Then for each $\xi \in H_i$, its orbit $(D \rtimes G \rtimes B)\xi = \pi_D(D)\pi_B(B)H_i$ is finite dimensional. By a simple maximality argument, we get the conclusion.

(2) We fix a faithful $G$-invariant state $\varphi_D$ on $D$. Since $D$ is finite dimensional, there exists $\lambda > 0$ such that

$$\varphi_D(d^*d) \geq \lambda d^*d.$$  

By Lemma 3.3, the map

$$E: D \rtimes G \times B \to \mathbb{G} \rtimes B: dxa \mapsto \varphi_D(d)xa$$

defines a conditional expectation. Then $E(x^*x) \geq \lambda x^*x$ for any $x \in D \rtimes G \times B$. Therefore the conditional expectation $E^{**}$ from $(D \rtimes G \times B)^{**} \rtimes (G \rtimes B)^{**}$ is of finite index, hence $D \rtimes G \times B$ is of type I, as so is $G \rtimes B \subset B \otimes K(L^2(G))$ since in the separable context the type I property passes to any C*-subalgebra, as explained in the proof of [BrG, Corollary 9.4.5]. □

Similarly for a $G$-equivariant Hilbert $B$-module $E$, one can define a Hilbert $A \rtimes G \times B$-module $A \rtimes G \rtimes E$ as a corner of the linking algebra $A \rtimes G \times K(E \otimes B)$. More concretely, the Hilbert module $A \rtimes G \times E$ is the completion of the pre-Hilbert module $\tilde{E}_0$ defined as follows:

- As a vector space, $\tilde{E}_0$ is isomorphic to $A_0 \otimes_{\text{alg}} c_c(\Gamma) \otimes_{\text{alg}} E$, where again $A_0$ denotes the respective algebraic core. Again the element in $\tilde{E}_0$ corresponding to $a \otimes x \otimes b$ is denoted by $axb$ for $a \in A_0, x \in c_c(\Gamma), b \in E$. 

• The right $A \times G \times B$-module structure is given by
  $$(axb)(a'x'b') = a(x_{(1)} \triangleright a')x_{(2)}'(b \triangleleft x_{(2)})b'$$
  for $a, a' \in A_0, x, x' \in c_c(\Gamma), b \in B$ and $b' \in B_0$.

• The inner product is given by
  $$(bxa, b'x'a') = a^* x^*(b, b') x' a'$$
  for $a, a' \in A_0, x, x' \in c_c(\Gamma), b, b' \in E$. Here $bxa$ expresses an element of $\tilde{E}_0$ by the same commutation relation as in $A$.

It is easy to see that $K(A \times G \times E)$ is naturally isomorphic to $A \times G \times K(E)$. In particular we get the following result.

**Lemma 3.5.** For each $G$-equivariant Hilbert $B$-module $E$, $A \times G \times K(E)$ is Morita equivalent to $A \times G \times B$.

Finally we show that the construction preserves the exact sequences in a natural sense.

**Lemma 3.6.** Let $A$ be a $C^*$-algebra with a right $G$-action $\beta$. For a $C^*$-algebra $B$ with a left $G$-action $\alpha$ and a $G$-invariant ideal $I \subseteq B$, the sequence

$$0 \rightarrow A \times G \times I \rightarrow A \times G \times B \rightarrow A \times G \times (B/I) \rightarrow 0$$

is exact.

**Proof.** Clearly the map $A \times G \times B \rightarrow A \times G \times (B/I)$ is surjective. To see the injectivity of $A \times G \times I \rightarrow A \times G \times B$, take a faithful nondegenerate representation $\pi$ of $A \times G \times I$ on a Hilbert space $H$. Consider the associated covariant representation $(\pi_A, \pi_I, U)$ and the unique extension of $\pi_I$ to $B$, say $\pi_B$. Since

$$U^*(1 \otimes \pi_I(b))U = (\text{id} \otimes \pi_I)\beta(b)$$

for any $b \in M(I)$, the triple $(\pi_A, \pi_B, U)$ is a covariant representation. Since the associated representation $A \times G \times B$ is an extension of $\pi$, we conclude that the map $A \times G \times I \rightarrow A \times G \times B$ is injective.

It remains to prove that the sequence in the lemma is exact at the middle term. Since the composition is zero, we can induce a homomorphism

$$(A \times G \times B)/(A \times G \times I) \rightarrow A \times G \times (B/I).$$

On the other hand, the universality of the crossed product induces the inverse of the map. Hence the homomorphism is an isomorphism. □

**Proposition 3.7.** Let $G$ be a compact quantum group with a countable dual. Fix a separable $C^*$-algebra $A$ with a right $G$-action. The crossed product introduced above gives rise to a triangulated functor on the equivariant Kasparov category

$$A \times G \times \cdot : KK^G \rightarrow KK.$$

**Proof.** This is a direct consequence of [NeV], Theorem 4.4] and the last two lemmas. □

4. Adjunction

In this section we develop a construction which will allow us to establish a (natural) isomorphism of certain equivariant and non-equivariant $KK$-groups.

Let $D$ be a finite dimensional $C^*$-algebra with a left $G$-action, where $G$ is again a compact quantum group (with $\Gamma$ its discrete dual). The opposite algebra $D^{\text{op}}$ admits a right $G$-action:

$$\alpha^{\text{op}} : D^{\text{op}} \rightarrow D^{\text{op}} \otimes C(G) : x^{\text{op}} \mapsto (\text{id} \otimes R) \circ \sigma \circ \alpha(x)^{\text{op}}.$$ 

Our goal is to show the functor $D^{\text{op}} \times G \times \cdot : KK^G \rightarrow KK$, developed in the previous section, is the right adjoint functor of $D \otimes \cdot : KK \rightarrow KK^G$. This is done by constructing the counit and unit.

From now on, we fix a faithful $G$-invariant state $\varphi_D = \text{Tr}(\rho)$ on $D$, where $\text{Tr}$ is the trace taking value 1 at each minimal projection and $\rho \in D$ is the ‘density matrix’ of $\varphi_D$. 

4.1. **Unit.** The modular group of \( \varphi_D \) is given by the formula \( \sigma^\varphi_t(a) = \rho^t a \rho^{-t} \), \( a \in D, t \in \mathbb{R} \). By [Eno, Théorème 2.9], we get

\[
(\rho^t a \rho^{-t}) \triangleleft x = \rho^t (a \triangleleft \hat{\tau}_t(x)) \rho^{-t}, \quad a \in D, t \in \mathbb{R},
\]

where \( \hat{\tau} \) is the scaling group on \( c_0(\Gamma) \). Let \( p^0 \) be the support of the counit on \( c_0(\Gamma) \).

**Lemma 4.1.** Let \( D \) be a finite dimensional C*-algebra with a left \( \mathbb{G} \)-action. There exists \( X \in D^{\text{op}} \otimes D \subset M(D^{\text{op}} \ltimes \mathbb{G} \ltimes D) \) such that

1. \( X \) is positive;
2. \( aX = (\rho^{-1/2}a \rho^{1/2})^{\text{op}} X \) and \( Xb = X(\rho^{1/2}b \rho^{-1/2})^{\text{op}} \) for all \( a, b \in D \);
3. \( p^0X = Xp^0 \);
4. \( (\varphi_D^{\text{op}} \otimes \text{id})(X) = 1 \).

**Proof.** Since \( D \) is finite dimensional, we write \( D \) as a direct sum of matrix algebras

\[
D = \bigoplus \pi M_{n(\pi)}
\]

and fix a matrix unit \( (e_{ij}^\pi) \) for each matrix algebra. Let

\[
X = \sum_{\pi, i,j} (\rho^{-1/2} e_{ij}^\pi \rho^{-1/2})^{\text{op}} e_{ji}^\pi \in D^{\text{op}} \otimes D \subset M(D^{\text{op}} \ltimes \mathbb{G} \ltimes D).
\]

The assertions (2) and (4) follow from a straightforward computation.

For (1), recall that \( \sum_{\pi, i,j} e_{ij}^\pi \otimes e_{ji}^\pi \in D \otimes D \) is positive. Now using the component-wise transpose map viewed as the an isomorphism \( t: D \rightarrow D^{\text{op}}: e_{ij}^\pi \mapsto (e_{ji}^\pi)^{\text{op}} \), we conclude

\[
X = ((\rho^{-1/2})^{\text{op}} \otimes 1)(t \otimes \text{id}) \left( \sum_{\pi, i,j} e_{ij}^\pi \otimes e_{ji}^\pi \right) ((\rho^{-1/2})^{\text{op}} \otimes 1)
\]

is still positive.

For (3), we define a vector space isomorphism \( \iota: D \otimes D \rightarrow \text{End}(D) \) by

\[
\iota(a \otimes b)(d) = \varphi_D(db)a.
\]

This is equivariant with respect to the tensor representation of \( \mathbb{G} \) on \( D \otimes D \) and the adjoint representation of \( \mathbb{G} \) on \( \text{End}(D) \). Hence \( \iota^{-1}(1_{\text{End}(D)}) = \sum_{\pi, i,j} \rho^{-1} e_{ij}^\pi \otimes e_{ji}^\pi \) is invariant under the tensor representation.

Now we observe that for all \( x \in c_c(\Gamma) \)

\[
p^0Xx = p^0 \sum_{\pi, i,j} \left( (\rho^{-1/2} e_{ij}^\pi \rho^{-1/2})^{\text{op}} \triangleleft \hat{\tau}_{-i/2}(x_{(2)}) \right)^{\text{op}} (e_{ij}^\pi \triangleleft x_{(1)})
\]

\[
= p^0 \sum_{\pi, i,j} \left( (\rho^{1/2} \rho^{-1/2})^{\text{op}} \triangleleft x_{(2)} \rho^{-1/2} \right)^{\text{op}} (e_{ij}^\pi \triangleleft x_{(1)})
\]

\[
= p^0 \hat{\varepsilon}(x)X.
\]

In particular \( p^0Xp^0 = p^0X \). Since \( X \) is self-adjoint, we get \( Xp^0 = p^0Xp^0 = p^0X \).

Thanks to the lemma above, the vector space \( D \) admits a right Hilbert \( D^{\text{op}} \ltimes \mathbb{G} \ltimes D \)-module structure defined as follows:

- The right module structure is given by \( d \triangleleft (a^{\text{op}}xb) = ((\rho^{-1/2}a \rho^{1/2}d) \triangleleft x)b \) for \( a, b, d \in D \), \( x \in c_c(\Gamma) \).
- The \( D^{\text{op}} \ltimes \mathbb{G} \ltimes D \)-valued inner product is given by

\[
(d, d') = d^* X p^0 d', \quad d, d' \in D.
\]

We denote \( D \) equipped with the right Hilbert \( D^{\text{op}} \ltimes \mathbb{G} \ltimes D \)-module structure above by \( \mathcal{D} \).
4.2. **Counit.** Suppose now that we also have a C*-algebra $B$ equipped with a right action $\beta$ of $G$, with the algebraic core $B_0$. Recall that we have a conditional expectation $D^{op} \times G \times B \to G \times B$ as in Lemma 4.3. Composing $D^{op} \times G \times B \to G \times B$ with the natural operator-valued weight from $G \times B$ to $B$, we get an operator-valued weight $E: D^{op} \times G \times B \to B$. On $D^{op} \times_{alg} G \times_{alg} B_0$ it is given by

$$E: D^{op} \times G \times B \to B; \ axb \mapsto \varphi_B^G(a)\hat{\psi}(b),$$

with $\hat{\psi}$ denoting the right Haar weight of $\Gamma$. The corresponding GNS module is isomorphic to $E_B := L^2(D) \otimes L^2(G) \otimes B$ with the GNS map

$$\Lambda: D^{op} \times_{alg} G \times_{alg} B \to L^2(D) \otimes L^2(G) \otimes B; \ d^{op}bx \mapsto d\Omega \otimes \Lambda(x) \otimes b$$

for $d \in D$, $x \in c_c(\Gamma), b \in B$. Furthermore $E_B$ carries a natural $G$-equivariant $D \otimes (D^{op} \times G \times B)$-$B$-bimodule structure defined as follows:

- The $G$-action $\beta_{E_B}$ on $E_B$ is given by
  $$\beta_{E_B}(x) = U_{12}V_{13}^*(\id \otimes \id \otimes \beta)(x) \in E_B.$$

- The left action of $D \otimes (D^{op} \times G \times B)$ structure is given by
  1. $D \otimes 1 \ni d \otimes 1 \mapsto \lambda_D(d) \otimes 1 \otimes 1$
  2. $1 \otimes D^{op} \ni 1 \otimes d^{op} \mapsto (\rho_D \otimes \rho)\alpha^{op}(d^{op}) \otimes 1,$
  3. $1 \otimes c_c(\Gamma) \ni 1 \otimes x \mapsto 1 \otimes \hat{\lambda}(x) \otimes 1,$
  4. $1 \otimes B_0 \ni 1 \otimes b \mapsto 1 \otimes (\lambda \otimes \id) \beta(b).$

From this presentation, it follows that the image of $D \otimes (D^{op} \times G \times B)$ is in $K(E_B)$.

4.3. **Adjunction.** We begin by stating a general lemma.

**Lemma 4.2.** Let $A,B,C$ be C*-algebras, $E$ a right Hilbert $A$-module, $D$ a Hilbert $A \otimes B$-$C$-bimodule. Then we have a natural isomorphism

$$(E \otimes D) \otimes_{A \otimes B} F \simeq E \otimes_A F.$$

**Proof.** Direct computation. \hfill $\Box$

**Lemma 4.3.** There exists a unitary

$$U: D \otimes_{D^{op} \times G \times D} E_D \to D$$

defined by

$$U(d \otimes_{D^{op} \times G \times D} \Lambda(d)) = \rho(d \triangleleft x)$$

for $d \in D$, $x \in D^{op} \times G \times D$.

**Proof.** We only need to show that $U$ is an isometry. This is done by a straightforward computation. Indeed, take $x, x' \in c_c(\Gamma)$, $a^{op}, a'^{op} \in D^{op}$ and $b, b' \in D$. Then

$$(d \otimes_{D^{op} \times G \times D} \Lambda(xa^{op}b), d' \otimes_{D^{op} \times G \times D} \Lambda(x'a'^{op}b')) = E(b^* (a^{op})^* x^* d^* Xp^0 d' x' a'^{op}b')$$

$$= E(b^* (a^{op})^* (d \triangleleft x)^* Xp^0 (d' \triangleleft x') a^{op} b)$$

$$= E(b^* (d \triangleleft x)^* \rho^{1/2} a^* \rho^{-1/2} Xp^0 \rho^{-1/2} a' \rho^{1/2} (d' \triangleleft x') b)$$

$$= (d \triangleleft xa^{op}b) \rho^{2}(d' \triangleleft x' a'^{op}b').$$

\hfill $\Box$

**Lemma 4.4.** There exists a unitary

$$V: D \otimes_{D^{op} \times G \times D} D^{op} \times G \times E_D \simeq D^{op} \times G \times D$$

defined by

$$V(d \otimes_{D^{op} \times G \times D} a^{op} x \Lambda(b^{op} y)) = (d \triangleleft a^{op} x) b^{op} y c$$
Proof. Since
\[
(d \otimes (a^{op}x \Lambda(b^{op}yc))) = 1 \otimes \Lambda(y)(\rho^{1/2}(d \otimes a^{op}x \rho^{1/2}b \rho^{-1/2})\rho^{-1/2})^{op}c
\]
for \(a, b, c, d \in D\) and \(x, y \in c_c(\Gamma)\), we only need to show
\[
(1 \otimes \Lambda(x), 1 \otimes \Lambda(y)) = x^*y.
\]
To this end, we compute, using the antipode of \(c_c(\Gamma)\) denoted \(\hat{S}\),
\[
(1 \otimes \Lambda(x), 1 \otimes \Lambda(y)) = (\Lambda(x), X \rho^0 \Lambda(y)) = \sum_{\pi, i, j} (\Lambda(x), \Lambda(\rho^{1/2} e_{ij} \rho^{-1/2})^{op} y \hat{S}^{-1}(p_0^0)(\rho^{1/2} e_{ij} \rho^{-1/2})p_0^0) = \sum_{\pi, i, j} \varphi_D(\rho^{1/2} e_{ij} \rho^{-1/2}) \psi(x^* y \hat{S}^{-1}(p_0^0))p_0^0 = x^*y.
\]
Here we have used the identity
\[
p_0^0x \otimes p_0^0 = p_0^0x(1) \otimes p_0^0x(2) \hat{S}(x(3)) = p_0^0 \otimes p_0^0 \hat{S}(x).
\]
\[\square\]

We have two functors
\[
KK \to KK^G: A \mapsto D \otimes A, KK^G \to KK: A \mapsto D^{op} \times G \ltimes A.
\]

**Theorem 4.5.** Let \(G\) be a compact quantum group with a countable dual. For \(A \in KK\) and \(B \in KK^G\), we have natural isomorphisms
\[
KK^G(D \otimes A, B) \simeq KK(A, D^{op} \times G \ltimes B).
\]

Proof. We construct the counit-unit adjunction. The unit is given by
\[
\eta_A = [D] \otimes 1_A \in KK(A, (D^{op} \times G \ltimes D) \otimes A)
\]
and the counit is given by
\[
\varepsilon_B = [E_B] \in KK^G(D \otimes (D^{op} \times G \ltimes B), B).
\]
We need to prove
\[
\varepsilon_{D \otimes A}(\text{id}_D \otimes \eta_A) = \text{id}_{D \otimes A}, (D^{op} \times G \ltimes \varepsilon_B)\eta_{D^{op} \times G \ltimes B} = \text{id}_{D^{op} \times G \ltimes B}.
\]
The first identity is due to Lemma 1.43
\[
(D \otimes D \otimes A) \otimes_{D \otimes (D^{op} \times G \ltimes D) \otimes D} (E_D \otimes A) \simeq D \otimes A.
\]
This follows from \(D \otimes D^{op} \times G \ltimes D \simeq D\). The second identity is due to Lemma 1.44
\[
(D \otimes D^{op} \times G \ltimes B) \otimes_{(D^{op} \times G \ltimes D) \otimes (D^{op} \times G \ltimes B)} (D^{op} \times G \ltimes E_B) \simeq D^{op} \times G \ltimes B.
\]
\[\square\]

5. **Application to UCT**

In this section we will apply the last theorem to questions regarding the Universal Coefficient Theorem. Let \(G\) be again a compact quantum group.

Recall that a \(G\)-C*-algebra is said to be cofibrant if it is of the form \(D \otimes A\) where \(D\) is a finite dimensional C*-algebra with a \(G\)-action \(\alpha\) and the \(G\)-action on \(D \otimes A\) is given by \(\alpha \otimes \text{id}\). Let \(\text{Cof}\) be the full subcategory of cofibrant objects in \(KK^G\) and let \(N\) be the full subcategory of \(A \in KK^G\) such that \(D^{op} \times G \ltimes A\) is \(KK\)-contractible for any finite dimensional \(G\)-C*-algebra \(D\).
Corollary 5.1. Suppose that the dual of \( \mathcal{G} \) is countable. The subcategories \( (\text{Cof}, \mathcal{N}) \) are complementary, i.e., for any \( A \in KK^\mathcal{G} \), there exists a unique triangle

\[
P(A) \\ A \longrightarrow N(A),
\]

where \( P(A) \in \text{Cof} \) and \( N(A) \in \mathcal{N} \).

Proof. By the help of Theorem 2.4 and Theorem 4.5, we only need to show the isomorphism class of finite dimensional \( \mathcal{G} \)-C*-algebras is at most countable. First from [ADC], the isomorphism classes of finite dimensional \( \mathcal{G} \)-C*-algebra are in one-to-one correspondence with the \( Q \)-systems in \( \text{Rep}(\mathcal{G}) \). There exists only countably many objects in \( \text{Rep}(\mathcal{G}) \) and each of them has at most finitely many structures of a \( Q \)-system by [IzK]. \( \Box \)

Recall now the phantom tower construction.

For a separable \( \mathcal{G} \)-C*-algebra \( A \), define \( P_n, N_n \in KK^\mathcal{G} \) inductively as follows:

- Put \( A = N_0 \).
- For \( N_n \), we set \( P_{n+1} = \bigoplus_{D \text{ torsion}} D \otimes (D^{\text{op}} \rtimes \mathcal{G} \ltimes N_n) \). Then we have the counit morphism \( \bigoplus_{D \in \mathcal{N}_n} : P_{n+1} \rightarrow N_n \). We embed this morphism into a triangle \( P_{n+1} \rightarrow N_n \rightarrow N_{n+1} \rightarrow SP_{n+1} \) to define \( N_{n+1} \).

With the construction above, we get a diagram:

\[
P_1 \quad \rightarrow \quad P_2 \quad \leftarrow \quad N_1 \quad \rightarrow \quad \ldots
\]

Now consider the morphism \( A \rightarrow N_n \) to fit in a triangle \( A \rightarrow N_n \rightarrow \tilde{A}_n \). The octahedral axiom shows that \( \tilde{A}_n \) also fits to a triangle

\[
\tilde{A}_n \rightarrow \tilde{A}_{n+1} \rightarrow P_n \rightarrow S\tilde{A}_n.
\]

We take the homotopy limit \( N = \text{ho-lim} N_n \) and \( \tilde{A} = \text{ho-lim} \tilde{A}_n \). Then we have a triangle

\[
\tilde{A} \rightarrow \quad \rightarrow \quad \tilde{A} \quad \rightarrow \quad \ldots
\]

where \( \tilde{A} \in \langle \text{Cof} \rangle \) and \( N \in \mathcal{N} \).

Theorem 5.2. Let \( \mathcal{G} \) be a compact quantum group with a countable dual and let \( A \) be a separable \( \mathcal{G} \)-C*-algebra. Suppose that either

(i) \( D^{\text{op}} \rtimes \mathcal{G} \rtimes A \) satisfies the UCT for any finite dimensional \( \mathcal{G} \)-C*-algebra \( D \), or

(ii) \( \mathcal{G} \) has no torsion and \( \mathcal{G} \rtimes A \) satisfies the UCT.

Then \( P(A) \) satisfies the UCT.

Proof. We prove only (i); (ii) follows similarly, using the construction preceding the theorem.

First, by induction, we show that \( D^{\text{op}} \rtimes \mathcal{G} \rtimes N_n \) satisfies the UCT for any \( n \in \mathbb{N} \cup \{0\} \). This holds for \( n = 0 \) by assumption. Assume \( D^{\text{op}} \rtimes \mathcal{G} \rtimes N_n \) satisfies the UCT. Since \( D^{\text{op}} \rtimes \mathcal{G} \rtimes D \) is a direct sum of matrix algebras, \( D^{\text{op}} \rtimes \mathcal{G} \rtimes P_{n+1} \simeq (D^{\text{op}} \rtimes \mathcal{G} \rtimes D) \otimes (D^{\text{op}} \rtimes \mathcal{G} \rtimes N_n) \) satisfies the UCT. Hence the mapping cone \( D^{\text{op}} \rtimes \mathcal{G} \rtimes N_{n+1} \) also satisfies the UCT.

In particular, \( P_n \) satisfies the UCT. Again by induction we see that \( \tilde{A}_n \) satisfies the UCT for any \( n \in \mathbb{N} \) by \( \tilde{A}_1 = P_1 \) and the triangle \( (5.1) \). Passing to the homotopy limit, we get that \( P(A) \) satisfies the UCT. \( \Box \)
Fix a discrete quantum group $\Gamma$ with the dual compact quantum group $\mathbb{G}$, and consider (Cof) and $\mathcal{N}$, the subcategories of $KK^\mathbb{G}$ introduced above. We say that $\Gamma$ satisfies the (Cof)-Baum–Connes property if $N$ is $KK$-contractible for any $N \in \mathcal{N}$, or equivalently, $P(A) \to A$ is a $KK$-equivalence.

We say that $\Gamma$ satisfies the (Cof)-strong Baum–Connes property if $N$ is $KK^\mathbb{G}$-contractible for any $N \in \mathcal{N}$, or equivalently, $KK^\mathbb{G} = (\text{Cof})$.

Remark 5.4. When $\Gamma$ is a classical discrete group, then the (Cof)-Baum–Connes property is equivalent to the fact that the strong Baum–Connes conjecture as introduced in [MN], Definition 9.1, holds, by [MN, Theorem 9.3]. We do not know whether $KK^\mathbb{G} = (\text{Cof})$ even when $\Gamma$ is a finite group.

In spite of that, many discrete quantum groups actually satisfy the strong (Cof)-Baum–Connes property. In particular, it holds for compact connected groups [MN], free orthogonal quantum groups $\{\text{Vo}\}$, free unitary quantum groups $\{\text{Ve}\}$ and free permutation groups $\{\text{Vp}\}$. It passes through the monoidal equivalence and is closed under taking free products $\{\text{Ve}\}$, subgroups and free wreath product $\{\text{Fw}\}$.

Corollary 5.5. Let $A$ be a separable $\Gamma$-$C^*$-algebra. We have the following.

1. For any torsion-free countable discrete quantum group $\Gamma$ with the (Cof)-Baum–Connes property, the $C^*$-algebra $\Gamma \ltimes A$ satisfies the UCT if $A$ does.
2. For any countable discrete quantum group $\Gamma$ with the (Cof)-Baum–Connes property, the $C^*$-algebra $\Gamma \ltimes A$ satisfies the UCT if $A$ is of type I. In particular the reduced group $C^*$-algebra $C(\mathbb{G})$ satisfies the UCT.

Proof. Recall that we denote the dual of $\Gamma$ by $\hat{\Gamma}$.

1. Consider the $\mathbb{G}$-$C^*$-algebra $\Gamma \ltimes A$. Then by the Baaj–Skandalis duality (see [Vac] and recall that discrete/compact quantum groups are automatically regular), $\mathbb{G} \ltimes \Gamma \ltimes A \simeq K(L^2(\mathbb{G})) \otimes A$, so that $\mathbb{G} \ltimes \Gamma \ltimes A$ satisfies the UCT. Now apply Theorem 5.2 to show $P(\Gamma \ltimes A)$ satisfies the UCT. By (Cof)-Baum–Connes property, the $C^*$-algebra $\Gamma \ltimes A$ also satisfies the UCT.

2. We denote the $\mathbb{G}$-$C^*$-algebra $\Gamma \ltimes A$ by $B$. First we will show that $D^{op} \ltimes \mathbb{G} \ltimes B$ is of type I. Again by the Baaj–Skandalis duality, the $C^*$-algebra $\mathbb{G} \ltimes B \simeq K(L^2(\mathbb{G})) \otimes A$ is of type I. Hence by Proposition 5.3, $D^{op} \ltimes \mathbb{G} \ltimes B$ is of type I. In particular $D^{op} \ltimes \mathbb{G} \ltimes B$ satisfies the UCT. The rest of the proof is the same as (1).

6. Quantum Rosenberg Conjecture

One of very well-known conjectures regarding group $C^*$-algebras, the Rosenberg Conjecture, stating that reduced $C^*$-algebras of a countable amenable discrete groups are quasidiagonal, was established in [TWW] (with the converse implication proved much earlier by Rosenberg in [HaR]). Here we show how as a corollary of that result and the progress on the UCT conjecture for quantum group algebras made in the last section one can obtain a similar statement for a large class of (unimodular) discrete quantum groups. A key observation is that the proof of the original result of Rosenberg found by Davidson ([Dav]) we refer to this book also for the definition of quasidiagonality) passes to the quantum case in a straightforward manner.

Theorem 6.1. Assume that $\Gamma$ is a countable discrete quantum group. Then quasidiagonality of $C^*_r(\Gamma)$ implies amenability of $\Gamma$ and if $\Gamma$ is unimodular, amenable and satisfies the (Cof)-Baum–Connes property then $C^*_r(\Gamma)$ is quasidiagonal. Finally there exist amenable nonunimodular countable discrete quantum groups $\Gamma$ (e.g. $SU_q(2), \alpha \in (0,1)$) such that $C^*_r(\Gamma)$ is not quasidiagonal.

Proof. Assume first that $C^*_r(\Gamma)$ is a quasidiagonal $C^*$-algebra.

If $\Gamma$ is finite, then it is obviously amenable (in that case it is compact and the Haar state yields an invariant mean). If $\Gamma$ is infinite, then the left regular representation of $C^*_r(\Gamma)$ is essential, i.e. contains no compact operators [Kal]. This means (via one of the versions of the Voiculescu Theorem) that $C^*_r(\Gamma)$ is quasidiagonal as the set of operators in $B(\ell^2(\Gamma))$. Let then $(P_n)_{n=1}^\infty$ be a sequence of finite rank projections in $B(\ell^2(\Gamma))$ increasing to $I$ and such that for each $\alpha \in \text{Irr}_{\hat{\Gamma}}$, $
Let then \( \tau_n \) be the normalised trace on the matrix algebra \( B(P_n \ell^2(\Gamma)) \), fix an ultrafilter \( \mathcal{U} \) on \( \mathbb{N} \) and define the state \( \omega \) on \( B(\ell^2(\Gamma)) \) via the prescription

\[
\omega(a) = \lim_{\mathcal{U}} \tau_n(P_n a P_n), \quad a \in B(\ell^2(\Gamma)),
\]

and let \( m = \omega|_{l^\infty(\Gamma)} \). Fix \( \alpha \in \text{Irr}_r \), \( i, j \in \{1, \ldots, n_\alpha \} \) and a non-zero \( x \in l^\infty(\Gamma) \). Put \( \phi = e_{i,j}^\alpha \in l^\infty(\Gamma)^* \). Note that \( \phi(1) = \delta_{i,j} \).

For each \( \epsilon > 0 \) we can find \( N \in \mathbb{N} \) such that for all \( n \geq N \) and \( p = 1, \ldots, n_\alpha \) we have

\[
\|P_n u_{i,p}^\alpha - u_{i,p}^\alpha P_n\| \leq \epsilon(n \|x\|)^{-1}
\]

and

\[
\|P_n u_{j,p}^\alpha - u_{j,p}^\alpha P_n\| \leq \epsilon(n \|x\|)^{-1}
\]

(note that the latter estimate is valid if one replaces \( u_{j,p}^\alpha \) by \( (u_{j,p}^\alpha)^* \)). Thus for such \( n \geq N \) (see (2.2))

\[
|\tau_n(P_n L_\phi(x) P_n) - \delta_{i,j} \tau_n(P_n x P_n)| = \left| \tau_n \left( P_n \left( \sum_{p=1}^{n_\alpha} u_{i,p}^\alpha x (u_{j,p}^\alpha)^* P_n \right) \right) - \delta_{i,j} \tau_n(P_n x P_n) \right|
\]

\[
\leq \sum_{p=1}^{n_\alpha} \tau_n(P_n u_{i,p}^\alpha P_n x (u_{j,p}^\alpha)^* P_n) - \delta_{i,j} \tau_n(P_n x P_n) + \frac{2\epsilon}{3}
\]

\[
= \sum_{p=1}^{n_\alpha} \tau_n(P_n x P_n (u_{j,p}^\alpha)^* P_n u_{i,p}^\alpha P_n) - \delta_{i,j} \tau_n(P_n x P_n) + \frac{2\epsilon}{3}
\]

\[
\leq \sum_{p=1}^{n_\alpha} \tau_n(P_n x P_n (u_{j,p}^\alpha)^* u_{i,p}^\alpha P_n) - \delta_{i,j} \tau_n(P_n x P_n) + \epsilon = \epsilon,
\]

where in the third equality we used the fact that \( \tau_n \) is a trace and in the last one the unitarity of the matrix \( (u_{i,j}^\alpha)_{i,j=1}^{n_\alpha} \). This implies that in the limit we obtain

\[
m(L_\phi(x)) = \lim_{\mathcal{U}} \tau_n(P_n L_\phi(x) P_n) = \delta_{i,j} m(x) = \phi(1) m(x)
\]

and the proof of the forward implication is finished.

Assume then that \( \Gamma \) is amenable and unimodular. By Theorem 1.1 of [BMT] \( C_r^*(\Gamma) \) is nuclear; by the unimodularity of \( \Gamma \), \( C_r^*(\Gamma) \) admits a faithful trace. Then the main result of [LWW] shows that \( C_r^*(\Gamma) \) is quasidiagonal if only it satisfies the UCT. This however follows from the assumption that \( \Gamma \) satisfies the (Cof)-Baum-Connes property by Corollary 5.5 (2).

It remains to note that Woronowicz’s compact quantum group \( SU_q(2) \) (with \( q \in (0,1) \)) is amenable, as first noted in [Ban]. Thus \( SU_q(2) \) is amenable. On the other hand the \( C^* \)-algebra \( C_r^*(SU_q(2)) \), which is in fact independent of \( q \), as observed in Theorem A.2 in [Wor], contains a proper isometry, so in particular cannot be quasidiagonal.

Note that if we knew that all group \( C^* \)-algebras of discrete amenable unimodular quantum groups satisfy UCT we could drop the (Cof)-Baum-Connes property assumption in the second part of the theorem above.

**References**

[ADC] Y. Arano and K. De Commer, Torsion-freeness for fusion rings and tensor \( C^* \)-categories, *J. Noncommut. Geom.* 13 (2019), no. 1, 35–58.

[BS] S. Baaj and G. Skandalis, \( C^* \)-algèbres de Hopf et théorie de Kasparov équivariante, *K-Theory,* 2 (1989), no. 6, 683–721.

[Ban] T. Banica, Representations of compact quantum groups and subfactors, *J. Reine Angew. Math.* 509 (1999), no. 1, 167–198.

[BMT] E. Bédos, G. J. Murphy, and L. Tuset, Amenability and co-amenability of algebraic quantum groups. II. *J. Funct. Anal.* 201 (2003), no. 2, 303–340.
N. Brown and N. Ozawa, “C*-algebras and finite-dimensional approximations”. Graduate Studies in Mathematics, 88. American Mathematical Society, Providence, RI, 2008.

K. Davidson, “C*-algebras by example”, Fields Institute Monographs, 6. American Mathematical Society, Providence, RI, 1996.

K. De Commer, Actions of compact quantum groups, Banach Center Publ. 111 (2017), 33–100.

P. Desmedt, J. Quaegebeur and S. Vaes, Amenability and the bicrossed product construction, Illinois J. Math. 46 (2002), no. 4, 1259–1277.

S. Echterhoff and J. Quigg, Full duality for coactions of discrete groups, Math. Scand. 90 (2002), no. 2, 267–288.

K. Davidson, "C*-algebras by example", Fields Institute Monographs, 6. American Mathematical Society, Providence, RI, 1996.

K. De Commer, Actions of compact quantum groups, Banach Center Publ. 111 (2017), 33–100.

P. Desmedt, J. Quaegebeur and S. Vaes, Amenability and the bicrossed product construction, Illinois J. Math. 46 (2002), no. 4, 1259–1277.

S. Echterhoff and J. Quigg, Full duality for coactions of discrete groups, Math. Scand. 90 (2002), no. 2, 267–288.

M. Enock, Sous-facteurs intermédiaires et groupes quantiques mesurés, J. Operator Theory 18 (1987), no. 1, 3–18.

D. Hadwin, Strongly quasidiagonal C*-algebras, (with an appendix by J. Rosenberg), J. Operator Theory 18 (1987), no. 1, 3–18.

N. Higson and G. Kasparov, E-theory and KK-theory for groups which act properly and isometrically on Hilbert space, Invent. Math. 144 (2001), no. 1, 23–74.

M. Izumi and H. Kosaki, On a subfactor analogue of the second cohomology, Rev. Math. Phys. 14 (2002), no. 7-8, 733–757.

R. Nest and C. Voigt, Equivariant Poincaré duality for quantum group actions, J. Funct. Anal. 229 (2005), no. 4, 409–425.

J. Rosenberg and C. Schochet, The Künneth theorem and the universal coefficient theorem for Kasparov’s generalized K-functor, Duke Math. J. 55 (1987), no. 2, 431–474.

A. Tikuisis, S. White and W. Winter, Quasidiagonality of nuclear C*-algebras, Ann. of Math. (2) 185 (2017), no. 1, 229–284.

J.-L. Tu, The Baum-Connes conjecture for groupoids, in “C*-algebras (Münster, 1999)”, Springer, 2000, pp.227–242.