Split Attractor Flow
in $\mathcal{N} = 2$ Minimally Coupled Supergravity

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ABSTRACT

We classify the stability region, marginal stability walls (MS) and split attractor flows for two-center extremal black holes in four-dimensional $\mathcal{N} = 2$ supergravity minimally coupled to $n$ vector multiplets.

It is found that two-center (continuous) charge orbits, classified by four duality invariants, either support a stability region ending on a MS wall or on an anti-marginal stability (AMS) wall, but not both. Therefore, the scalar manifold never contains both walls. Moreover, the BPS mass of the black hole composite (in its stability region) never vanishes in the scalar manifold. For these reasons, the “bound state transformation walls” phenomenon does not necessarily occur in these theories.

The entropy of the flow trees also satisfies an inequality which forbids “entropy enigma” decays in these models.

Finally, the non-BPS case, due to the existence of a “fake” superpotential satisfying a triangle inequality, can be treated as well, and it can be shown to exhibit a split attractor flow dynamics which, at least in the $n = 1$ case, is analogous to the BPS one.
1 Introduction

The present paper is devoted to the study of the two-center extremal black hole (BH) solution and split attractor flow \cite{1} in $\mathcal{N} = 2$, $d = 4$ supergravity minimally coupled to $n$ Abelian vector multiplets \cite{2}. Within such a theory, the entropy of a single-center extremal BH with dyonic charge vector $(p^0, p^i, q_0, q_i)$ is given by

$$S = \frac{1}{2} |I_2(Q)| = \frac{1}{2} |p_0^2 + q_0^2 - p_i^2 - q_i^2|,$$

(1.1)

with $I_2 \geq 0$ for BPS and non-BPS solutions, respectively. Note that Eq. (1.1) reduces to the Reissner-Nordström BPS BH entropy if one sets $p^i = q_i = 0$. However, the ADM mass \cite{3} depends on scalars, due to the presence of the $e^{\mathcal{K}/2}$ Kähler factor in the $\mathcal{N} = 2$ central charge function:

$$Z \left( |t^i|^2 ; p_0, q_0 \right) = \frac{(q_0 + i p_0)}{\sqrt{2 \left( 1 - |t^i|^2 \right)}}.$$

(1.2)

Two-center solutions exist as well, with general different properties with respect to the single-center cases. As we will show in the subsequent treatment, a peculiar feature of the $\mathcal{N} = 2$ minimally coupled is that AMS walls, when they exist, are not supported by charge configurations which admit a split

\footnote{Note that the “physically sensible” charges are actually given by $Q/\sqrt{2}$ and $X/\sqrt{2}$ where $X$ and $Q$ are real and complex parameterizations of the charges, respectively defined in (2.9) and (2.12) further below.}
attractor flow. Moreover, within such configurations, the single-center entropy with charge \( Q_1 + Q_2 \) is always larger than the corresponding two-center entropy, namely:

\[
S(Q_1 + Q_2) > S(Q_1) + S(Q_2).
\]

The inequality (1.3) implies that the ADM masses of the constituents, as well as the one of the composite solution, are always bounded from below in the scalar manifold. As a consequence, “entropy enigma decays” [4, 5, 6, 7] do not occur, and the bound states do not necessarily have “recombination walls” [8].

The non-BPS branch can be investigated as well, exhibiting a split dynamics analogous to the BPS case. However, an important difference with respect to the BPS case is the presence of a “moduli space” of non-BPS solutions [9]. This is ultimately due to the fact that non-BPS attractor equations (given by (2.19) further below) define hyper-planes, and not points [9, 10].

The plan of the paper is as follows.

Sec. 2 presents some basic facts on the geometric structure and on the duality symmetries of \( \mathcal{N} = 2, d = 4 \) supergravity minimally coupled to \( n \) vector multiplets, which will then be exploited in the subsequent treatment of split flow in this theory.

In Sec. 3 we analyse the one-modulus case, namely the model which is electric-magnetic dual to the axion-dilaton model obtained as a truncation (to two different \( U(1) \)'s) of “pure” \( \mathcal{N} = 4, d = 4 \) supergravity (for a review and a list of Refs., see e.g. [17]). The corresponding non-BPS branch is studied in Sec. 3.2.

Sec. 4 extends the analysis of the BPS two-center split flow to an arbitrary number of Abelian vector multiplets.

In Sec. 5 a comparison with the so-called \( \mathcal{N} = 2, d = 4 \) t-model is worked out.

The paper ends with some comments and remarks in Sec. 6, along with a couple of Appendices, providing some technical details on the MS and AMS conditions for the split scalar flows.

### 2 Basics

The scalar manifold of the \( \mathcal{N} = 2, d = 4 \) minimally coupled Maxwell-Einstein supergravity theory provides the simplest example of symmetric special Kähler space, which is locally a (non-compact version of the) \( \mathbb{CP}^n \) space:

\[
\frac{SU(1,n)}{SU(n) \times U(1)}.
\]

The main feature of the corresponding special geometry is the vanishing of the tensor \( C_{ijk} \), yielding the following Riemann and Ricci tensors (see e.g. [11], and Refs. therein)

\[
R_{i\bar{j}k\bar{l}} = -g_{\bar{j}k}g_{i\bar{l}} - g_{i\bar{l}}g_{\bar{j}k} \Rightarrow R_{i\bar{j}} = - (n + 1) g_{i\bar{j}}. \tag{2.2}
\]

The special coordinates preserving the \( SU(1,n) \) symmetry are based on the holomorphic prepotential function

\[
F(X) = -\frac{i}{2} \left( (X^0)^2 - (X^i)^2 \right) \equiv (X^0)^2 F(t),
\]

such that the holomorphic symplectic sections read \( (F_\Lambda (X) \equiv \frac{\partial F}{\partial X^\Lambda}, \Lambda = 0, 1, ..., n \) throughout

\[
\mathbf{V} = (X^\Lambda, F_\Lambda (X))^T = (X^0, X^i, -iX^0, iX^i)^T = (1, t^i, -i, it^i), \tag{2.4}
\]

where in (2.3) and in the second line of (2.4) projective coordinates \( t^i \equiv X^i / X^0 \) have been introduced, with \( X^0 \equiv 1 \) eventually fixed by choosing a suitable Kähler gauge. Correspondingly, the covariantly
holomorphic symplectic sections read
\[ V \equiv (L^A, M_A)^T \equiv e^{K/2} V, \]  
where the Kähler potential \( K \) is then given by (see e.g. [11], and Refs. therein)
\[ K = - \ln \left[ i \left( X^{\Lambda} F_\Lambda - X^A \tilde{F}_A \right) \right] = - \ln \left[ 2 \left( 1 - |t|^2 \right) \right]. \]  
Note that, as a consequence of \( C_{ijk} = 0 \), the special geometry relations are very simple:
\[ D_i V = 0, \quad D_i D_j V = 0, \quad D_j D_i V = g_{ij} V, \]  
where \( D_i \) and \( D_i \) respectively denote the Kähler-covariant differential operators, whose action on \( V \) reads
\[ D_i V = (\partial_i + \partial_i K) V, \quad D_i V = 0. \]  
The scalar-dependent central extension \( Z \) (central charge) of the \( \mathcal{N} = 2 \) local supersymmetry algebra is built from the symplectic product of the dyonic vector of (magnetic \( p \) and electric \( q \)) charges of the two-form field strengths
\[ Q \equiv \left( p^0, p^i, q_0, q_i \right)^T \]  
and of the vector of covariantly holomorphic symplectic sections \( V \) (2.5) as follows:
\[ Z \equiv \langle Q, V \rangle = Q^T \Omega V = q_0 L^0 + q_i L^i - p^0 M_0 - p^j M_i = e^{K/2} (q_\Lambda X^{\Lambda} - p^\Lambda F_\Lambda) \]  
\[ = \frac{\left[ q_0 + i p^0 + (q_i - i p^i) t^i \right]}{\sqrt{2} \sqrt{1 - |t|^2}}. \]  
where \( \Omega \) is the \( Sp(2n+2, \mathbb{R}) \)-metric. The corresponding Kähler covariant derivatives (also named matter charges) read as follows:
\[ D_i Z \equiv Z_i = 2 e^{3K/2} \left( \frac{1}{2} e^{-K} X^i + X^0 \tilde{F}^i + X^j t^j \tilde{F}^i \right). \]  
It is also convenient to switch to a complex parametrization of the charge vector (in the fundamental irrepr. \( 1 + n \) of \( U(1, n) \)):
\[ \mathcal{X} \equiv \left( q_0 + i p^0, q_i - i p^i \right)^T, \]  
such that (2.10) and (2.11) can be recast in the following simple form:
\[ Z \equiv e^{K/2} \left( X^0 + X^i t^i \right); \]  
\[ Z_i = 2 e^{3K/2} \left( \frac{1}{2} e^{-K} X^i + X^0 \tilde{F}^i + X^j t^j \tilde{F}^i \right). \]  
In the basis in which the charges \( Q \) or \( \mathcal{X} \) are dressed by the scalar fields into the central charge \( Z \) and its Kähler covariant derivatives \( Z_i \), the quadratic invariant \( I_2 \) of the symplectic representation \( 1 + n \) of the electric-magnetic duality group \( U(1, n) \) reads [12]
\[ I_2 = 2 \left( Z \tilde{Z} - g_{ij} Z_i Z_j \right), \]  
\[ ^2 \text{We will henceforth simply refer to electric-magnetic duality as to duality. In string theory, electric-magnetic duality can be seen as the “continuous” version, valid for large values of the charges, of the } U\text{-duality [13].} \)
where $g_{ij} \equiv \partial_j \partial_i \mathcal{K}$ is the metric of the scalar manifold.

The BH effective potential and its criticality equations (alias Attractor Eqs. [14]) respectively read

\[ V = \mathcal{Z} \mathcal{Z} + g^{ij} \mathcal{Z}_i \mathcal{Z}_j \]  \quad (2.16)
\[ \partial_i V = 2 \mathcal{Z} \mathcal{Z}_i = 0. \]  \quad (2.17)

The solutions to (2.17), such that $V|_{\partial_i V = 0} \neq 0$ and its Hessian is positive definite, correspond to the various classes of attractor solutions (namely the manifold $\mathbb{C}P^{n-1}$) in the BH near-horizon geometry.

\[ \begin{align*}
BPS & : Z_H \neq 0, Z_{i,H} = 0 \forall i; \quad \mathcal{I}_2 (\mathcal{Q}) > 0; \\
non-BPS & : Z_H = 0, \quad Z_{i,H} \neq 0 \text{ for some } i; \quad \mathcal{I}_2 (\mathcal{Q}) < 0.
\end{align*} \]  \quad (2.18) \quad (2.19)

The attractor configurations are usually named “large”, because they correspond, through the Bekenstein-Hawking entropy-area formula [16], to a non-vanishing (semi-)classical BH entropy given by (1.1).

In the minimally coupled models under consideration, there is also a class of charge configurations supporting “small” single-center BHs (which are BPS) with $\mathcal{I}_2 = 0$. Note that in Eqs. (2.18) and (2.19) $\mathcal{Q}$ is assumed to support single-center solutions. Within the same assumption, note that

\[ \begin{align*}
\mathcal{I}_2 (\mathcal{Q}) & > 0 \Rightarrow Z (\mathcal{Q}) \neq 0; \\
\mathcal{I}_2 (\mathcal{Q}) & < 0 \Rightarrow D_i Z (\mathcal{Q}) \neq 0 \text{ for some } i.
\end{align*} \]  \quad (2.20) \quad (2.21)

Thus, as mentioned above, the minimally coupled models have the remarkable feature that the BPS (non-BPS) scalar flow trees never cross points at which $Z = 0$ ($D_i Z = 0 \forall i$), due to the very constraints on the supporting charge vectors.

Considering two different symplectic charge vectors

\[ \mathcal{Q}_1 \equiv (p^0, p^i, q_0, q_i)^T; \quad \mathcal{Q}_2 \equiv (p^0, p^i, Q_0, Q_i)^T, \]  \quad (2.22)

all the quadratic $U(1,n)$-invariants built out with $\mathcal{Q}_1$ and $\mathcal{Q}_2$ read as follows:

\[ \begin{align*}
\mathcal{I}_1 & \equiv \mathcal{I}_2 (\mathcal{Q}_1) = (p^0)^2 - (p^i)^2 + q_0^2 - q_i^2; \\
\mathcal{I}_2 & \equiv \mathcal{I}_2 (\mathcal{Q}_2) = (P^0)^2 - (P^i)^2 + Q_0^2 - Q_i^2; \\
\mathcal{I}_s & \equiv p^0 P^0 - p^i P^i + q_0 Q_0 - q_i Q_i; \\
\mathcal{I}_a & \equiv p^0 Q_0 + p^i Q_i - q_0 P^0 - q_i P^i = - \langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle.
\end{align*} \]  \quad (2.23) \quad (2.24) \quad (2.25) \quad (2.26)

In the complex parametrization of the charge vectors, (2.22) amounts to considering

\[ \mathcal{X}_1 \equiv (q_0 + ip^0, q_i - ip^i)^T; \quad \mathcal{X}_2 \equiv (Q_0 + iP^0, Q_i - iP^i)^T, \]  \quad (2.27)

and thus the four quadratic $U(1,n)$-invariants (2.23)-(2.26) can be re-written as follows:

\[ \begin{align*}
\mathcal{I}_1 & = \mathcal{X}_1 \cdot \overline{\mathcal{X}_1}; \\
\mathcal{I}_2 & = \mathcal{X}_2 \cdot \overline{\mathcal{X}_2}; \\
\mathcal{I}_s & = \text{Re} (\mathcal{X}_1 \cdot \overline{\mathcal{X}_2}); \\
\mathcal{I}_a & = \text{Im} (\mathcal{X}_1 \cdot \overline{\mathcal{X}_2}).
\end{align*} \]  \quad (2.28) \quad (2.29) \quad (2.30) \quad (2.31)

\textsuperscript{3}In the non-BPS case $\partial_i V = 0$ corresponds to only one complex equation ($Z = 0$). Thus, a complex ($n-1$)-dimensional “moduli space” of attractor solutions (namely the manifold $\mathbb{C}P^{n-1}$) exists in this case [9].

\textsuperscript{4}The subscript “H” denotes evaluation at the BH horizon throughout.

\textsuperscript{5}Note that we adopt a different normalization of $\mathcal{I}_2$ with respect to [10].

Moreover, the subscripts “s” and “a” respectively stand for “symmetric” and “antisymmetric” with respect to the exchange $\mathcal{Q}_1 \leftrightarrow \mathcal{Q}_2$.\[ \text{4} \]}
where "·" is the bilinear Hermitian form defined by the Lorentzian metric
\[ \eta_{\Lambda\Sigma} = \text{diag} \left( 1, -1, \ldots, -1 \right), \quad (2.32) \]

namely:
\[ \mathcal{X}_1 \cdot \mathcal{X}_1 \equiv \mathcal{X}_1^\Lambda \mathcal{X}_1^\Sigma \eta_{\Lambda\Sigma}; \quad (2.33) \]
\[ \mathcal{X}_1 \cdot \mathcal{X}_2 \equiv \mathcal{X}_1^\Lambda \mathcal{X}_2^\Sigma \eta_{\Lambda\Sigma}. \quad (2.34) \]

From the expression (2.13), it is easy to see that \( Z \) transforms as
\[ Z \rightarrow Z e^{i\alpha} \quad (2.35) \]
under
\[ \begin{cases} \mathcal{X} \rightarrow \mathcal{X} e^{i\alpha}, \\ t^i \rightarrow t^i, \end{cases} \quad (2.36) \]

namely a finite transformation of the **global** (inactive on scalar fields) \( U (1) \) factor of the duality group \( U (1, n) = U (1) \times SU (1, n) \). Such a \( U (1) \) is a global electric-magnetic duality, which enlarges the actual duality group from the numerator group \( SU (1, n) \) of the (non-compact) \( \mathbb{C}P^n \) scalar manifold to \( U (1, n) \). Note that this is consistent also with the fact that in the \( n = 0 \) case of minimal coupling sequence (corresponding to "pure" \( \mathcal{N} = 2 \) supergravity), the resulting duality group is \( U (1) \).

In the one-modulus case, the presence of the global \( U (1) \) factor in the duality group can also be understood by noticing that a consistent truncation of "pure" \( \mathcal{N} = 4 \) supergravity produces the \( n = 1 \) **minimally coupled** \( \mathcal{N} = 2 \) model in the so-called axion-dilaton symplectic basis (which is not the one considered in Sec. 3; see e.g. [17] for a recent review and a list of Refs.). At the level of duality group, the aforementioned truncation amounts to the following group embedding:
\[ \text{SL} (2, \mathbb{R}) \times SO (6) \supset \text{SL} (2, \mathbb{R}) \times SO (2), \quad (2.37) \]

Notice that the axion-dilaton of the resulting **minimally coupled** \( \mathcal{N} = 2 \) theory is nothing but the axio-dilatonic scalar of the \( \mathcal{N} = 4 \) supergravity multiplet. Moreover, four of the six \( \mathcal{N} = 2 \) graviphotons are truncated away, and the remaining two ones split into the \( \mathcal{N} = 2 \) graviphoton and in the Maxwell field of the axio-dilatonic \( \mathcal{N} = 2 \) multiplet. At fermionic level, two out of the four \( \mathcal{N} = 4 \) gravitinos are truncated away, consistent with the lower local supersymmetry. The supersymmetry uplift of \( \mathcal{N} = 2 \) axion-dilaton model into extended supergravities has been recently discussed e.g. in [18].

It is worth remarking that without the extra global \( U (1) \) factor in the duality group, the analysis that we are going to perform in Sec. 3 would have been incomplete. Indeed, from their very definitions, \( \mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_s \) and \( \mathbf{I}_a \) are two-center invariants of both \( SU (1, 1) \) and \( U (1, 1) \). However, \( SU (1, 1) \) has an extra two-center quadratic invariant, defined as:
\[ \mathfrak{I} \equiv \mathcal{X}_1 \wedge \mathcal{X}_2 \equiv \mathcal{X}_1^\Lambda \mathcal{X}_2^\Sigma \epsilon_{\Lambda\Sigma}, \quad (2.38) \]

where \( \epsilon \) denotes the antisymmetric Levi-Civita symbol. Note that \( \mathfrak{I} \) is the unique quadratic two-center \( SU (1, 1) \)-invariant which is complex, and thus which is not an \( U (1, 1) \)-invariant. Its squared absolute value is related to \( \mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_s \) and \( \mathbf{I}_a \) as follows:
\[ |\mathfrak{I}|^2 = -\mathbf{I}_1^2 + \mathbf{I}_2^2 + \mathbf{I}_a^2. \quad (2.39) \]

\[ \text{Note that, from its very definition (2.38), } \mathfrak{I} \text{ exists for the duality group } SU (1, n) \text{ only when } n + 1 \text{ centers are considered (this statement holds irrespective of the non-compact nature of the duality group itself).} \]
Since (2.39) holds, I would only have introduced a further real degrees of freedom (charge) in the discussion of Sec. 3.

The following relation will prove to be useful in the treatment given below:

\[ I_2 (Q_1 + Q_2) = I_1 + I_2 + 2I_s; \]  
(2.40)

\[ \downarrow \]

\[ I_2 (Q_1 + Q_2) \geq 0 \Leftrightarrow I_s \geq -\frac{1}{2} (I_1 + I_2); \]  
(2.41)

\[ I_2 (Q_1 + Q_2) \geq I_1 + I_2 \Leftrightarrow I_s \geq 0. \]  
(2.42)

In particular, it holds that

\[ I_s > 0 \Rightarrow I_2 (Q_1 + Q_2) > I_1 + I_2. \]  
(2.43)

In the case of two-center BH solutions with both BPS centers (i.e. \( I_1 \geq 0 \) and \( I_2 \geq 0 \)), in Secs. 3.1 and 4.1 we will obtain, in terms of the aforementioned duality-invariants, the stability region \( S \) of the composite solution, and the MS region (if any). In fact, depending on the sign of some invariants, we have found that MS or AMS walls can occur in the scalar manifold, but not both. Therefore, the scalar flow supported by the physical charge orbit, whose stability region ends when crossing the MS wall, never encounters the AMS wall, which instead pertains to another (un-physical) charge orbit which does not support a MS wall.

More interestingly, under the assumption of existence of a MS wall (and of a stability region of the two-center solution), we have found that \( I_2 (Q_1 + Q_2) > 0 \), and in particular that (2.43) necessarily holds. This latter, through (1.1), leads to the following fundamental relation (1.3) anticipated above:

\[ S_{1-ctr,BPS} (Q_1 + Q_2) > S_{2-ctr,BPS} (Q_1, Q_2) = S_{1-ctr,BPS} (Q_1) + S_{1-ctr,BPS} (Q_2), \]  
(2.44)

namely that the entropy of the single-center solution with charge \( Q_1 + Q_2 \) is always larger than the entropy of the two-center solution with charges \( Q_1 \) and \( Q_2 \) for the centers 1 and 2, respectively. This can ultimately be traced back to the fact that the BH entropy (1.1) can be written as

\[ S(Q) = \frac{\pi}{2} \chi \cdot \chi, \]  
(2.45)

and that the MS condition requires

\[ I_s > 0, \]  
(2.46)

from which (2.43) and (2.44) follow.

Mutatis mutandis, the same holds for two-center BH solutions with both non-BPS centers (i.e. \( I_1 < 0 \) and \( I_2 < 0 \)), at least in the minimally coupled model with \( n = 1 \) complex scalar. In such a framework, it holds

\[ I_s < 0 \Rightarrow I_2 (Q_1 + Q_2) < I_1 + I_2 = -|I_1 + I_2|. \]  
(2.47)

In Sec. 3.2 we will show that the assumption of existence of a MS wall (and of a stability region of the two-center solution) necessarily implies \( I_2 (Q_1 + Q_2) < 0 \), and in particular (2.47). This latter, through the formula (1.1), implies

\[ S_{1-ctr,nBPS} (Q_1 + Q_2) > S_{2-ctr,nBPS} (Q_1, Q_2) = S_{1-ctr,nBPS} (Q_1) + S_{1-ctr,nBPS} (Q_2). \]  
(2.48)

The treatment of the non-BPS case is possible in virtue of the observation [19] that the “fake” superpotential [20, 21], which gives the non-BPS ADM mass, also satisfies a Cauchy-Schwarz triangle inequality:

\[ W(Q_1 + Q_2) \leq W(Q_1) + W(Q_2), \]  
(2.49)

as it holds for the central charge \( Z \) in the BPS case.
Eqs. (1.3), (2.44) and (2.48) express an interesting feature of the minimally coupled models of \( \mathcal{N} = 2, d = 4 \) Maxwell-Einstein supergravity: the constituents always have an entropy which is smaller than the entropy of the original composite (if considered as a single-center solution). Thus, the corresponding split dynamics of the scalar flows exhibits a different behavior with respect to the \( \mathcal{N} = 2 \) models with special Kähler geometry based on cubic prepotentials. Indeed, in these latter models, MS and AMS walls are known to co-exist, for a suitable choice of the charge vectors \( Q_1 \) and \( Q_2 \), in different zones of the scalar manifold itself (see e.g. [8], and the analysis in Sec. 5 [4]). Furthermore, in cubic \( \mathcal{N} = 2 \) models, also by assuming \( \mathcal{I}_4 (Q_1) \geq 0 \) and \( \mathcal{I}_4 (Q_2) \geq 0 \), \( \mathcal{I}_4 (Q_1 + Q_2) \) is not necessarily positive (see e.g. [4] and [22]).

Eqs. (2.44) and (2.48) also imply that “entropy enigma” decays [5, 6, 7] never occur in these models, and thus that in the corresponding regime of large charges the microscopic state counting is still dominated by the single-center configurations (see e.g. the discussion in [3, 6]).

As mentioned in Sec. 1, Eqs. (2.44) and (2.48) also imply that the BPS (non-BPS) mass is bounded from below by the single-center entropy

\[
\begin{align*}
|Z (t^i (r), \bar{t} (r) ; Q_1 + Q_2) | & \geq \sqrt{S_{1-ctr,BPS} (Q_1 + Q_2) \pi} = \sqrt{\frac{\mathcal{I}_2 (Q_1 + Q_2)}{2}}, \quad (2.50) \\
W (t^i (r), \bar{t} (r) ; Q_1 + Q_2) & \geq \sqrt{S_{1-ctr,nBPS} (Q_1 + Q_2) \pi} = \sqrt{\frac{-\mathcal{I}_2 (Q_1 + Q_2)}{2}}. \quad (2.51)
\end{align*}
\]

As a consequence of (2.44) - (2.48), the inequality (2.50) (and its non-BPS counterpart (2.51)) implies that \( Z (W) \) never vanishes in the scalar manifold, neither for single-center nor for two-center solutions. For this reason, and for the fact that MS and AMS walls cannot co-exist in the scalar manifold, the “paradox” which led to the introduction of “bound state transformation walls” [8] does not occur in the class of theories under consideration.

It is worth recalling that the so-called \( t^2 \) and \( t^4 \) models are the only one-modulus \( \mathcal{N} = 2, d = 4 \) Maxwell-Einstein supergravity models with homogeneous scalar manifolds [26]. As mentioned above, in cubic special geometries “bound state recombination walls” and “entropy enigma” decays are possible, respectively because (2.50) (with \( \mathcal{I}_2 \) replaced by \( \mathcal{I}_4 \)) and (2.44) do not necessarily apply.

### 3 One Modulus

We start and consider the simplest model, namely the one with \( n = 1 \) minimally coupled vector multiplet, duality-related to the so-called axion-dilaton model (see e.g. [17] for a review and a list of Refs.). The metric function in this case reads:

\[
\begin{align*}
g^{\bar{t} t} &= (1 - \bar{t} t)^2 = |e^l_\bar{t} e^l_t|^2; \\
e^l_\bar{t} &= i (1 - \bar{t} t),
\end{align*}
\]

where the phase of the Vielbein \( e^l_\bar{t} \) is chosen for later convenience.

The domain of definition of the Kähler potential \( K \) and of the metric \( g^{\bar{t} t} \) is the open unit disc centered in the origin of the Argand-Gauss plane (we use the notation \( b \equiv \text{Re}(t) \) and \( a \equiv \text{Im}(t) \)):

\[
b^2 + a^2 < 1. \quad (3.3)
\]

The expressions of the central charge and of the matter charge are given by the \( n = 1 \) case of Eqs. (2.10) - (2.11), whereas the BPS and non-BPS attractor values of the complex scalar \( t \) respectively read as follows [10]:

\[
\begin{align*}
t_{\text{BPS}} &= -\frac{(q_1 + i p^1)}{(q_0 - i p^0)}; \\
t_{\text{nBPS}} &= -\frac{(q_0 + i p^0)}{(q_1 - i p^1)}.
\end{align*}
\]
3.1 BPS MS or AMS Wall

Within this Subsection, we assume

\[ (Q_1, Q_2) : \begin{cases} \Re (Z_1 Z_2) = 0; \\ \Im (Z_1 Z_2) = 0; \\ \Re (Z_1 Z_2) > 0; \\ \Re (Z_1 Z_2) < 0; \end{cases} \]

(3.6)

as well as \( \mathbb{CP}^1 \) to be the spatially asymptotical \((r \to \infty)\) scalar manifold.

Depending on the various cases, the \textit{a priori} possible BPS “large” two-center configurations are\footnote{Throughout the present paper, we consider only “large” initial states. From the reasonings done in Sec. 4 and the main results of the present investigation, when requiring the existence of a stability region and of a MS wall, this assumption does not imply any loss of generality.}

1. BPS “large” \( \to \) BPS “large” + BPS “large”; \hspace{1cm} (3.7)

2. BPS “large” \( \to \) BPS “large” + BPS “small”; \hspace{1cm} (3.8)

3. BPS “large” \( \to \) BPS “small” + BPS “small”. \hspace{1cm} (3.9)

The BPS MS and AMS walls are defined as (within \( \mathbb{CP}^1 \); we use the notation \( Z_a \equiv Z (b, a; Q_a) \), \( a = 1, 2 \) throughout):

\[
MS_{BPS} \equiv \begin{cases} b + ia : \left[ \Im (Z_1 Z_2) = 0; \right. \\ \Re (Z_1 Z_2) > 0; \end{cases} \\
AMS_{BPS} \equiv \begin{cases} b + ia : \left[ \Im (Z_1 Z_2) = 0; \right. \\ \Re (Z_1 Z_2) < 0; \end{cases},
\]

(3.10)

(3.11)

where

\[
2 (1 - b^2 - a^2) \Re (Z_1 Z_2) = \begin{bmatrix} q_0 Q_0 + p^0 P^0 \\ + (q_1 Q_1 + q^1 P^1) b^2 + (q_0 Q_1 + q_1 Q_0 - p^0 P^1 - p^1 P^0) b \\ + (q_1 Q_1 + q^1 P^1) a^2 + (q_0 P^1 + q^1 Q_1 + p^1 Q_0) a \end{bmatrix};
\]

(3.12)

\[
2 (1 - b^2 - a^2) \Im (Z_1 Z_2) = \begin{bmatrix} p^0 Q_0 - q^0 P^0 \\ + (q_1 P^1 - q^1 Q_1) b^2 + (q_0 P^1 - q_1 P^0 + q^0 Q_1 - p^1 Q_0) b \\ + (q_1 P^1 - q^1 Q_1) a^2 + (-q_0 Q_1 + q^1 Q_0 + p^1 P^1 - p^0 P^0) a \end{bmatrix};
\]

(3.13)

The region of stability \( S_{BPS} \) of the two-center BPS solution is defined as

\[
S_{BPS} \equiv \{ b + ia \in \mathbb{CP}^1 : \langle Q_1, Q_2 \rangle \Im (Z_1 Z_2) > 0 \}.
\]

(3.14)

The distance between the centers 1 and 2 can be \( SU (1, 1) \)-invariantly written as \[4\]

\[
| \vec{x}_1 - \vec{x}_2 | = \frac{\langle Q_1, Q_2 \rangle |Z_1 + Z_2|}{2 \Im (Z_1 Z_2)}.
\]

(3.15)

and the corresponding configurational angular momentum reads \[1, 4\]

\[
\vec{J} = \frac{\langle Q_1, Q_2 \rangle (\vec{x}_1 - \vec{x}_2)}{2 | \vec{x}_1 - \vec{x}_2 |} = \frac{\Im (Z_1 Z_2)}{|Z_1 + Z_2|} (\vec{x}_1 - \vec{x}_2).
\]

(3.16)

It is also here worth observing that the “large” BPS single-center solution with charge \( Q = Q_1 + Q_2 \) would exist \textit{iff}

\[
\mathcal{I}_2 (Q_1 + Q_2) > 0 \iff 2 \mathcal{I}_s > - (\mathbf{I}_1 + \mathbf{I}_2),
\]

(3.17)

where \( \mathbf{I}_2 (Q_1 + Q_2) > 0 \) has been used, and the condition (3.6) must be taken into account.
### 3.1.1 Case 1

The most general charge configuration supporting the two-center solution (3.7) is duality-related to

\[ Q_1 \equiv (0, 0, q_0, 0) \Rightarrow \begin{cases} 
I_1 = q_0^2 > 0; \\
 t_{H,BPS}(Q_1) = 0; 
\end{cases} \tag{3.18} \]

\[ Q_2 \equiv (P^0, P^1, Q_0, 0) \Rightarrow \begin{cases} 
I_2 = (P^0)^2 + Q_0^2 - (P^1)^2 > 0; \\
 t_{H,BPS}(Q_2) = -i \frac{P^1}{(Q_0 - iP^0)}. \tag{3.19} 
\end{cases} \]

which can thus be considered without any loss in generality. Indeed, for the charge configuration (3.18)-(3.19), the four quadratic \( U(1, 1) \)-invariants (2.23)-(2.26) are all generally non-coinciding and non-vanishing:

\[ (Q_1, Q_2) : \begin{cases} 
I_1 = q_0^2; \\
I_2 = (P^0)^2 - (P^1)^2 + Q_0^2 > 0; \\
I_s = q_0 Q_0; \\
I_a = -q_0 P^0. 
\end{cases} \tag{3.20} \]

It is worth noting that the charge vector \( Q_1 \) given by Eq. (3.18), in which only \( q_0 \) is non-vanishing, is nothing but the Reissner-Nordström black hole embedded in \( \mathbb{C}P^1 \), with attractor value at the origin of such a space.

A manifestly \( U(1, 1) \)-invariant characterization of the four non-vanishing charges of the general BPS two-center configuration (3.18)-(3.19) reads as follows:

\[ \begin{align*}
q_0^2 &= I_1; \\
(P^0)^2 &= \frac{I_2}{I_1}; \\
(P^1)^2 &= \frac{(I_s^2 + I_a^2 - I_1 I_2)}{I_1}; \\
Q_0^2 &= \frac{I_2}{I_1}. 
\end{align*} \tag{3.21} \]

where

\[ \begin{cases} 
I_1 > 0 \\
I_2 > 0 \end{cases} \Rightarrow I_s^2 + I_a^2 - I_1 I_2 > 0. \tag{3.25} \]

Within the configuration (3.18)-(3.19), the real and imaginary part of \( Z_1 \bar{Z}_2 \) respectively read (recall (3.12) and (3.13)):

\[ \begin{align*}
\text{Re} (Z_1 \bar{Z}_2) &= \frac{q_0 (Q_0 + P^1 a)}{2 (1 - b^2 - a^2)}; \\
\text{Im} (Z_1 \bar{Z}_2) &= \frac{q_0 (-P^0 + P^1 b)}{2 (1 - b^2 - a^2)}. \tag{3.27} 
\end{align*} \]

---

\^We remind that at the attractor points, \( 2\text{Im}(Z_1 \bar{Z}_2) = -\langle Q_1, Q_2 \rangle \) as pointed out in [19]. It turns out that this relation still holds in our case at the single center attractor point with charge \( Q_1 + Q_2 \).

Furthermore, by using the fundamental identities of special Kähler geometry in presence of two symplectic charge vectors \( Q_1 \) and \( Q_2 \) (see e.g. [23], [24]), one can compute that at BPS attractor points for the centers 1 or 2:

\[ \text{Re} (Z_1 \bar{Z}_2) = -\frac{1}{2} \bar{Q}_1^T \mathcal{M} Q_2, \]

where \( \mathcal{M} \) is the symplectic, symmetric, negative definite \( 2(n_V + 1) \times 2(n_V + 1) \) matrix with entries depending on the real and imaginary part of the vector kinetic matrix \( \mathcal{N} \) (see e.g. [25], [11], and Refs. therein). Notice that \( \bar{Q}_1^T \mathcal{M} Q_2 \) does not have a definite sign.
Figure 1: Stability region $S_{BPS}$ and MS wall $MS_{BPS}$ for the BPS two-center extremal BH solution of 1-modulus minimally coupled $\mathcal{N} = 2$ model, represented as functions of $b$ and $a$, respectively the real and imaginary part of the scalar $t$. The charges has been chosen all positive. Here $b_0 = P^0/P^1$ and $a_0 = Q_0/P^1$ are respectively the values at which $\text{Im}(Z_1\bar{Z}_2)$ and $\text{Re}(Z_1\bar{Z}_2)$ vanish. The attractor points associated to the centers with charges $Q_1$, $Q_2$ and $Q_1 + Q_1$ are respectively labeled by AP1, AP2 and AP12.

Let us start by computing the region of stability $S_{BPS}$ defined in (3.14):

$$S_{BPS}: \left\{ \begin{array}{l} P^0 P^1 > 0 : \frac{P^0}{P^1} < b < \sqrt{1 - a^2}; \\
 P^0 P^1 < 0 : -\sqrt{1 - a^2} < b < \frac{P^0}{P^1} 
\end{array} \right. \quad (3.28)$$

Note that $a$ enters Eq. (3.28) only through the constraint to belong to the domain of definition of the metric of the scalar manifold, defined by (3.3):

$$b^2 + a^2 < 1, \quad (3.29)$$

implying that

$$\left| \frac{P^0}{P^1} \right| < 1 \iff (P^1)^2 - (P^0)^2 > 0 \iff I_1^2 - I_1 I_2 > 0. \quad (3.30)$$

By using (3.22)-(3.23), the region of stability $S_{BPS}$ (3.28)-(3.30) can be re-expressed as follows:

$$S_{BPS}: \left\{ \begin{array}{l} \pm \frac{\sqrt{I_1^2 + I_2^2 - I_1 I_2}}{|I_a|} b > 1; \\
 b^2 + a^2 < 1. 
\end{array} \right. \quad (3.31)$$

Then, one can study the existence of the BPS MS and AMS walls, defined by (3.10)-(3.12).
Within the condition (3.30), it is convenient to define (see Eqs. (A.3)-(A.4))
\[ a_{BPS} = \sqrt{(P_1)^2 - (P_0)^2} = \sqrt{\frac{I_2^2 - I_1I_2}{I_1^2 + I_2^2 - I_1I_2}} > 0; \] (3.32)
\[ \mathcal{A} = \begin{cases} b, a \in \mathbb{C} \mathbb{P}^1 : & \begin{cases} b = \pm \frac{|I_2|}{\sqrt{I_1^2 + I_2^2 - I_1I_2}}; \\ - \frac{|Q_0|}{P_0^1} < -a_{BPS} < a < a_{BPS} < \frac{|Q_0|}{P_0^1}. \end{cases} \\ \end{cases}; \] (3.33)
\[ \frac{|Q_0|}{P_0^1} = \frac{|I_2|}{\sqrt{I_1^2 + I_2^2 - I_1I_2}}. \] (3.34)

Then, through some straightforward computations (detailed in App. A), one obtains that within the two-center charge configuration (3.18)-(3.19) the existence of BPS MS or AMS walls depends on the sign of \( I_s \):
\[ I_s > 0 : \begin{cases} MS_{BPS} = \mathcal{A}; \\ \#AMS_{BPS}; \end{cases} \] (3.35)
\[ I_s < 0 : \begin{cases} \#MS_{BPS}; \\ AMS_{BPS} = \mathcal{A}. \end{cases} \] (3.36)

\( S_{BPS} \) and \( MS_{BPS} \) are graphically depicted in Fig. 1 for an all positive charge configuration.

**Single-Center Solution and MS Wall** By recalling (2.40), it follows that
\[ I_1 + I_2 + 2I_s > 0 \] (3.37)
is the general condition of existence of the “large” BPS single-center solution with charge \( Q_1 + Q_2 \).

By denoting the entropy of the BH solution with \( S \), one then obtains that
\[ S_{1-ctr,BPS} (Q_1 + Q_2) \geq S_{2-ctr,BPS} (Q_1, Q_2) \iff \begin{cases} I_s > 0; \\ I_s = 0; \\ -\frac{1}{I_1 + I_2} < I_s < 0. \end{cases} \] (3.38)

As anticipated in Sec. 1 within the general conditions (3.19) and (3.30) on \( Q_2 \) (corresponding to assuming the existence of a stability region for the two-center configuration “large” BPS + “large” BPS (3.7)), the existence of a BPS MS wall \( MS_{BPS} \) (see (3.35)) implies the existence of the “large” BPS single-center solution with charge \( Q_1 + Q_2 \), with entropy strictly larger than the entropy of the two-center solution, as given by Eq. (2.44).

### 3.1.2 Case 2

The most general charge configuration supporting the two-center solution (3.8) is duality-related to
\[ Q_1 \equiv (0, 0, q_0, 0) \Rightarrow \begin{cases} I_1 = q_0^2 > 0; \\ \#t_{H,BPS} (Q_1) = 0; \end{cases} \] (3.39)
\[ Q_2 \equiv (P_0^0, P_1^1, Q_0, 0) \Rightarrow \begin{cases} I_2 = (P_0^0)^2 + Q_0^2 - (P_1^1)^2 = 0; \\ \#t_{H} (Q_2) = 0. \end{cases} \] (3.40)
which can thus be considered without any loss in generality.

This case can be consistently obtained as the limit \( I_2 \to 0^+ \) of the treatment given in Sec. 3.1.1 and in App. 3 enforcing the addition restriction

\[
a_{BPS} = \frac{|Q_0|}{P^1} = \frac{|I_s|}{\sqrt{T_s + T_a}}. \tag{3.41}
\]

**Single-Center Solution and MS Wall** Clearly, in this case the limit \( I_2 \to 0^+ \) of Eqs. (3.37) and (3.38), and related comments, hold, as well.

Within the general condition (3.40) on \( Q_2 \) within \( \mathbb{CP}^1 \) (namely, by assuming the existence of a stability region for the two-center configuration “large” BPS + “small” BPS (3.8)), the existence of a BPS MS wall \( MS_{BPS} \) (cfr. the limit \( I_2 \to 0^+ \) of (3.35)) implies the existence of the “large” BPS single-center solution with charge \( Q_1 + Q_2 \), with entropy strictly larger than the entropy of the two-center solution, as given by the limit \( I_2 \to 0^+ \) of Eq. (2.44).

Since one of the two centers is “small”, this case is similar to the one treated e.g. in Sec. 5 of [4], with the important difference that for the \( \mathbb{CP}^1 \) model under consideration the corresponding existing single-center solution is necessarily BPS with entropy larger than the corresponding two-center solution (see the discussion in Sec. 1, as well as the comment below Eq. (5.4)).

### 3.1.3 Case 3

This case cannot be consistently obtained by performing the \( I_1 \to 0^+ \) limit of the treatment of case 2 given in Sec. 3.1.3 due to the 1-charge nature of the charge vector \( Q_1 \) given by (3.18).

On the other hand, it is immediate to realize that the most general charge configuration supporting the two-center solution (3.9) is duality-related to

\[
Q_1 = (0, 0, q_0, q_1) \Rightarrow \begin{cases} I_1 = q_0^2 - q_1^2 = 0 \iff |q_0| = |q_1|; \\ \#t_H(Q_1) \end{cases}; \tag{3.42}
\]

\[
Q_2 = (P^0, 0, 0, Q_1) \Rightarrow \begin{cases} I_2 = (P^0)^2 - Q_1^2 = 0 \iff |P^0| = |Q_1|; \\ \#t_H(Q_2) \end{cases}, \tag{3.43}
\]

which can thus be considered without any loss in generality. Indeed, for the charge configuration (3.42) - (3.43), besides \( I_1 = I_2 = 0 \), it holds that:

\[
(Q_1, Q_2) : \begin{cases} I_s = -q_1 Q_1; \\ I_a = -q_0 P^0. \end{cases} \tag{3.44}
\]

Note that

\[
\begin{cases} I_1 = 0; \\ I_2 = 0; \end{cases} \Rightarrow I_s^2 = I_a^2. \tag{3.45}
\]

Within the configuration (3.42) - (3.43), the real and imaginary part of \( Z \bar{Z} \) respectively read (recall (3.12) and (3.13)):

\[
\text{Re}\left( Z \bar{Z} \right) = q_1 Q_1 \frac{\left( b^2 + \frac{q_0 b}{q_1} + a^2 + \frac{P^0}{Q_1} a \right)}{(1 - b^2 - a^2)}; \tag{3.46}
\]

\[
\text{Im}\left( Z \bar{Z} \right) = -q_0 P^0 \frac{\left( 1 + \frac{q_0 b}{q_1} + \frac{Q_1}{P^0} a \right)}{(1 - b^2 - a^2)}. \tag{3.47}
\]
The region of stability $S_{BPS}$ defined in (3.14) corresponds to the region of $\mathbb{C}P^1$ in which the inequality
\[ S_{BPS} : 1 \pm b \pm a < 0 \] (3.48)
is satisfied. Note that in the second step we used $I_1 = I_2 = 0$, and the two “±” are reciprocally independent, depending on the signs of $q_0q_1$ and $P^0Q_1$, respectively. By solving (3.48) in a consistent way with the metric constraint (3.29), one achieves the following manifestly $U(1,1)$-invariant result:
\[
S_{BPS} : \begin{cases} 
I_a I_a > 0 : & \left\{ \begin{array}{l}
-\sqrt{1 - b^2} < a < -(1 + b) ; \\
b \in (-1,0) ; \\
1 - b < a < \sqrt{1 - b^2} ; \\
b \in (0,1) ; 
\end{array} \right. \\
I_a I_a < 0 : & \left\{ \begin{array}{l}
1 + b < a < \sqrt{1 - b^2} ; \\
b \in (-1,0) ; \\
-\sqrt{1 - b^2} < a < -(1 + b) ; \\
b \in (0,1) ; 
\end{array} \right.
\end{cases}
\] (3.49)

Then, one can study the existence of the BPS MS and AMS walls, defined by (3.10)-(3.12). By solving the condition $\text{Im}(Z_1 \overline{Z_2}) = 0$ consistently with the metric constraint (3.29) yields to the following manifestly $U(1,1)$-invariant result:
\[
\text{Re} \left( Z_1 \overline{Z_2} \right) |_{\text{Im}(Z_1 \overline{Z_2}) = 0} = \frac{I_s}{2}.
\] (3.50)

Therefore, one can formulate the conditions of existence of the BPS MS or AMS wall in the manifestly $U(1,1)$-invariant following way:
\[
I_a > 0 : ~ MS_{BPS} = \begin{cases} 
I_a > 0 : & \left\{ \begin{array}{l}
a = -1 - b \\
b \in (-1,0) ; \\
a = 1 - b \\
b \in (0,1) ; 
\end{array} \right. \\
I_a < 0 : & \left\{ \begin{array}{l}
a = 1 + b \\
b \in (-1,0) ; \\
a = -1 + b \\
b \in (0,1) ; 
\end{array} \right.
\end{cases}
\] (3.51)

$\# MS_{BPS};$
\[
I_a < 0 : ~ AMS_{BPS} = \begin{cases} 
I_a > 0 : & \left\{ \begin{array}{l}
a = 1 + b \\
b \in (-1,0) ; \\
a = -1 + b \\
b \in (0,1) ; 
\end{array} \right. \\
I_a < 0 : & \left\{ \begin{array}{l}
a = -1 - b \\
b \in (-1,0) ; \\
a = 1 - b \\
b \in (0,1) ; 
\end{array} \right.
\end{cases}
\] (3.52)

Single-Center Solution and MS Wall  By recalling (2.40), in this case it follows that
\[ I_a > 0 \] (3.53)
is also the general condition of existence of the “large” BPS single-center solution with charge $Q_1 + Q_2$. Thus, the unique possibility is
\[
\frac{1}{\pi} S_{1-ctr,BPS} (Q_1 + Q_2) = I_a > S_{2-ctr,BPS} (Q_1, Q_2) = 0.
\] (3.54)

The results holding for cases 1 and 2 respectively treated in Secs. 3.1.1 and 3.1.2 still hold for this case: within the general conditions (3.42) and (3.43) on $Q_1$ and $Q_2$ within $\mathbb{C}P^1$ (namely, by assuming the existence of a stability region for the two-center configuration “small” BPS $+$ “small” BPS (3.29),

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the condition of existence of a BPS MS wall $MS_{BPS}$ (see (3.51)) matches the condition (3.53) of existence of the “large” BPS single-center solution with charge $Q_1 + Q_2$, with entropy $I_1 > 0$ strictly larger than the entropy of the two-center solution. Indeed, the limit $I_1, I_2 \to 0^+$ of Eq. (2.44) trivially yields that the entropy of the two-center solution vanishes.

This case is similar to the ones treated e.g. in [27, 22], with the important difference that for the $\mathbb{C}P^1$ model under consideration the corresponding existing single-center solution is necessarily BPS (see the discussion in Sec. 3.1).

3.2 Non-BPS MS or AMS Wall

Within this Subsection, we assume

$$
(Q_1, Q_2) : \begin{cases}
    I_1 < 0; \\
    I_2 < 0,
\end{cases} \quad (3.55)
$$

as well as $\mathbb{C}P^1$ to be the spatially asymptotical scalar manifold (as in Sec. 3.1). Only one possibility a priori exists, namely:

non-BPS “large” $\rightarrow$ non-BPS “large” $+$ non-BPS “large”.  \quad (3.56)

A crucial observations (not holding for the minimally coupled models with $n \geq 2$ complex scalars, treated in Sec. 4) is that one can switch from $I_2(Q) > 0$ (“large” BPS BH states) to $I_2(Q) < 0$ (“large” non-BPS BH states) e.g. by performing the following simple transformation on the charge vector:

$$
Q \equiv (p^0, p^1, q_0, q_1)^T \rightarrow (\pm p^1, \pm p^0, \pm q_0, \pm q_1)^T, \quad (3.57)
$$

where all “±”’s in the r.h.s. are reciprocally independent. The relevant transformation for the following treatment is the one with all “+” or all “−” in the r.h.s. of (3.57). Without any loss of generality, we will consider the one with all “+”s:

$$
Q \equiv (p^0, p^1, q_0, q_1)^T \rightarrow (p^1, p^0, q_0, q_1)^T, \quad (3.58)
$$

which is ultimately equivalent to the interchanging the $N = 2$ graviphoton with the Maxwell field of the minimally coupled vector multiplet.

By performing transformation (3.58) on both $Q_1$ and $Q_2$, the symplectic product $\langle Q_1, Q_2 \rangle$ gets unchanged, $t_{BPS} \rightarrow t_{nBPS}$, and

$$
Z \rightarrow i D_t Z, \quad (3.59)
$$

where $D_t Z$ is the “flat” matter charge:

$$
D_t Z \equiv \epsilon^b_t D_t Z = i (1 - t\tilde{t}) D_t Z. \quad (3.60)
$$

As a consequence, the known BPS formulæ (3.15) and (3.16) [1, 4] get mapped into their corresponding non-BPS counterparts, namely (we use the notation $D_t Z_a \equiv D_t Z(b, a; Q_a)$, $a = 1, 2$ throughout):

$$
|\vec{x}_1 - \vec{x}_2| = -\frac{\langle Q_1, Q_2 \rangle}{2} \frac{|D_t Z_1 + D_t Z_2|}{\text{Im} (D_t Z_1 D_t Z_2)}; \quad (3.61)
$$

$$
\vec{J} = -\frac{\text{Im} (D_t Z_1 D_t Z_2)}{|D_t Z_1 + D_t Z_2|} (\vec{x}_1 - \vec{x}_2). \quad (3.62)
$$

9Note that (3.59) is consistent with the treatment given e.g. in [13] (see for instance Eq. (5.5) therein).
By applying (3.58) to (3.10) and (3.11), also the definitions of non-BPS MS and AMS walls can thus be given (within $\mathbb{CP}^1$):

$$MS_{nBPS} \equiv \begin{cases} 
  b + ia : & \text{Im} \left( D_{\tau}Z_1 D_{\tau}Z_2 \right) = 0; \\
  & \text{Re} \left( D_{\tau}Z_1 D_{\tau}Z_2 \right) > 0; 
\end{cases} \tag{3.63}$$

$$AMS_{nBPS} \equiv \begin{cases} 
  b + ia : & \text{Im} \left( D_{\tau}Z_1 D_{\tau}Z_2 \right) = 0; \\
  & \text{Re} \left( D_{\tau}Z_1 D_{\tau}Z_2 \right) < 0; 
\end{cases} \tag{3.64}$$

where

$$2 \left( 1 - b^2 - a^2 \right) \text{Re} \left( D_{\tau}Z_1 D_{\tau}Z_2 \right) = \left[ q_1 Q_1 + p^1 P^1 \right. \\
+ \left( q_0 Q_0 + p^0 P^0 \right) b_2 + \left( q_1 Q_0 + q_0 Q_1 - p^1 P^0 - p^0 P^1 \right) b \\
+ \left( q_0 Q_0 + p^0 P^0 \right) a^2 + \left( q_1 P^0 + q_0 P^1 + p^1 Q_0 + p^0 Q_1 \right) a \right]; \tag{3.65}$$

$$-2 \left( 1 - b^2 - a^2 \right) \text{Im} \left( D_{\tau}Z_1 D_{\tau}Z_2 \right) = \left[ p^1 Q_1 - q_1 P^1 \right. \\
+ \left( q_0 P^0 - p^0 Q_0 \right) b_2 + \left( q_1 P^0 - q_0 P^1 + p^1 Q_0 - p^0 Q_1 \right) b \\
+ \left( q_0 P^0 - p^0 Q_0 \right) a^2 + \left( -q_1 Q_0 + q_0 Q_1 + p^1 P^0 - p^0 P^1 \right) a \right]. \tag{3.66}$$

Analogously, by applying (3.58) to (3.14), the region of stability $S_{nBPS} \left( b; a; Q_1, Q_2 \right)$ of the two-center non-BPS solution can be defined as

$$S_{nBPS} \equiv \left\{ b + ia \in \mathbb{CP}^1 : \langle Q_1, Q_2 \rangle \text{Im} \left( D_{\tau}Z_1 D_{\tau}Z_2 \right) < 0 \right\}. \tag{3.67}$$

It is also worth observing that the “large” non-BPS single-center solution with charge $Q = Q_1 + Q_2$ would exist iff (recall (2.40))

$$I_2 \left( Q_1 + Q_2 \right) < 0 \iff 2I_s < - \left( I_1 + I_2 \right), \tag{3.68}$$

where the condition (3.55) must be taken into account.

3.2.1 Analysis

The most general charge configuration supporting the two-center solution (3.59) is duality-related to

$$Q_1 \equiv (0, 0, 0, q_1) \Rightarrow \left\{ \begin{array}{l}
  I_1 = -q_1^2 < 0; \\
  t_{H,nBPS} (Q_1) = 0; 
\end{array} \right. \tag{3.69}$$

$$Q_2 \equiv (p^0, p^1, 0, Q_1) \Rightarrow \left\{ \begin{array}{l}
  I_2 = \left( P^0 \right)^2 - \left( P^1 \right)^2 - Q_1^2 < 0; \\
  t_{H,nBPS} (Q_2) = -i \frac{p^0}{Q_1 - i P^1}; 
\end{array} \right. \tag{3.70}$$

which can thus be considered without any loss in generality. Indeed, for the charge configuration (3.69) - (3.70), the four quadratic $U(1,1)$-invariants (2.23) - (2.26) are all generally non-coinciding and non-vanishing:

$$\langle Q_1, Q_2 \rangle : \left\{ \begin{array}{l}
  I_1 = -q_1^2; \\
  I_2 = \left( P^0 \right)^2 - \left( P^1 \right)^2 - Q_1^2 < 0; \\
  I_s = -q_1 Q_1; \\
  I_a = -q_1 P^1. 
\end{array} \right. \tag{3.71}$$
A manifestly $U(1,1)$-invariant characterization of the four non-vanishing charges of the general non-BPS two-center configuration (3.69)-(3.70) reads as follows:

$$q_1^2 = -I_1;$$

$$P^0 = \frac{(I_1 I_2 - I_2^2 - I_1^2)}{I_1};$$

$$P^1 = \frac{I_2}{I_1};$$

$$Q_1^2 = \frac{I_2}{I_1},$$

where

$$I_1 < 0 \quad I_2 < 0 \implies I_1 I_2 - I_2^2 - I_1^2 < 0.$$

Note that the configuration (3.69)-(3.70) (and in general all the treatment of non-BPS case given below) can be obtained from (3.18)-(3.19) (and in general all the treatment of BPS case given in Sec. 3.1.1) by performing the transformation (3.58) on both $Q_1$ and $Q_2$.

Within the configuration (3.69)-(3.70), the real and imaginary part of $D_t Z_1 D_t Z_2$ respectively read (recall (3.65) and (3.66)):

$$\text{Re} \left( D_t Z_1 D_t Z_2 \right) = q_1 \frac{Q_1 + P^0 a}{2(1 - b^2 - a^2)};$$

$$\text{Im} \left( D_t Z_1 D_t Z_2 \right) = q_1 \frac{P^1 - P^0 b}{2(1 - b^2 - a^2)}.$$
Within the condition \((3.80)\), it is convenient to introduce

\[
a_{nBPS} \equiv \sqrt{\frac{I_1 I_2 - I_s^2}{I_1 I_2 - I_s^2 - I_2}} > 0; \quad (3.82)
\]

\[
\mathcal{B} \equiv \left\{ b, a \in \mathbb{C}P^1 : \begin{array}{c}
b = \pm \frac{|I_s|}{\sqrt{I_1^2 + I_2^2 - 11 I_2}}; \\
-\frac{|Q_1|}{p^0} < -a_{nBPS} < a < a_{nBPS} < \frac{|Q_1|}{p^0}.
\end{array} \right\} = A|_{p^0 \leftrightarrow p^1}; \quad (3.83)
\]

\[
\left| \frac{Q_1}{p^0} \right| = \frac{|I_s|}{\sqrt{I_s^2 + I_1^2 - I_1 I_2}}. \quad (3.84)
\]

Then, through some straightforward computations (detailed in App. B), one obtains that within the two-center charge configuration \((3.69)-(3.70)\) the existence of non-BPS MS or AMS walls depends on the sign of \(I_s\):

\[
I_s < 0 : \begin{cases} 
MS_{nBPS} = \mathcal{B}; \\
\#AMS_{nBPS}; \\
\end{cases} \quad (3.85)
\]

\[
I_s > 0 : \begin{cases} 
\#MS_{nBPS}; \\
AMS_{nBPS} = \mathcal{B}, \\
\end{cases} \quad (3.86)
\]

It is interesting to compare Eqs. \((3.82)-(3.86)\) with their BPS counterparts, respectively given by \((3.32)-(3.36)\).

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\[
I_1 + I_2 + 2I_s < 0 \quad (3.87)
\]

is the general condition \((3.68)\) of existence of the “large” non-BPS single-center solution with charge \(Q_1 + Q_2\). Notice that

\[
S_{1-ctr,nBPS} (Q_1 + Q_2) \geq S_{2-ctr,nBPS} (Q_1, Q_2) \iff \begin{cases} I_s < 0; \\
I_s = 0; \\
0 < I_s < -\frac{1}{2} (I_1 + I_2).
\end{cases} \quad (3.88)
\]

*Mutatis mutandis*, the story goes as in the BPS case treated in Sec. 3.1.1. Indeed, as anticipated in Sec. 1 within the general conditions \((3.70)\) and \((3.80)\) on \(Q_2\) (corresponding to assuming the existence of a stability region for the two-center configuration “large” non-BPS + “large” non-BPS \((3.56)\)), the existence of a non-BPS MS wall \(MS_{nBPS}\) (see \((3.83)\)) implies the existence of the “large” non-BPS single-center solution with charge \(Q_1 + Q_2\), with entropy strictly larger than the entropy of the two-center solution, as given by Eq. \((2.48)\).

Note that the analysis of Sec. 3.2 provides the an example worked out in full generality of non-BPS two-center BH solution with constrained positions of the centers \(i.e.\) mutually non-local charges \(Q_1\) and \(Q_2\). In fact, it is worth pointing out that this case does not belong to the class of non-BPS multi-center solutions studied \(e.g.\) in [28] and [29], nor to the \(I_4 < 0\) two-center solution of [30].

This is also due to the fact that the \(t^2\) model is the unique known model in which the non-BPS fake superpotential is the absolute value of a complex quantity linear in the charges, namely [21] \(W_{nBPS} = |D_t Z|\).

This remarkably form of \(W_{nBPS}\) allowed for an especially simple treatment of non-BPS two-center solution in full generality in Sec. 3.2.
4 Many Moduli

We now turn to consider the \( \mathcal{N} = 2, d = 4 \) supergravity models with \( n \geq 2 \) Abelian vector multiplets minimally coupled to the gravity multiplet [2]. The metric of the scalar manifold can be computed to read [2, 10] (Einstein summation convention on repeated indices is used, and \( i = 1, \ldots, n \), throughout):

\[
g^{ij} \equiv \partial_i \partial_j K = \left(1 - \left| t^i \right|^2 \right)^2 \left(1 - \left| t^k \right|^2 \right)^2 \delta^{ij} + 2e^{2K} t^i t^j; \tag{4.1}
\]

\[
g^{\overline{i}\overline{j}} = \left(1 - \left| t^k \right|^2 \right)^2 \delta^{\overline{i}\overline{j}} - 2e^{-K} \left| t^i \right|^2 \delta^{\overline{i}\overline{j}}; \tag{4.2}
\]

\[
g^{ij}g_{ij} = \delta^{k\overline{j}}. \tag{4.3}
\]

The domain of definition of the Kähler potential \( K \) and of the metric \( g^{ij} \) is the interior of the \( 2n \)-hypersphere of unitary radius centered in the origin:

\[
\sum_{i=1}^{n} \left| t^i \right|^2 < 1. \tag{4.4}
\]

The expressions of the central charge and of the matter charges are given by Eqs. (2.10)-(2.11), whereas the BPS and non-BPS attractor values of scalar fields, respectively read as follows [10]:

\[
t^1_{\text{BPS}} = -\left( \frac{q_i + ip^i}{q_0 - ip^0} \right), \forall i; \tag{4.5}
\]

\[
t^1_{\text{nBPS}} = \frac{q_a - ip^a}{q_1 - ip^1} \frac{t^a_{\text{nBPS}}}{q_1 - ip^1}. \tag{4.6}
\]

Without any loss of generality (up to re-labelling), in (4.6) the complex scalar field \( t^1 \) is stabilized in terms of the non-BPS values \( t^a_{\text{nBPS}} \) of the remaining \( n - 1 \) scalars. Notice that \( t^a_{\text{nBPS}} \) are not fixed by the Attractor Mechanism; indeed, such scalars are known to coordinatise the “moduli space” of non-BPS attractor solutions in minimally coupled sequence, which is nothing but \( \mathbb{C}P^{n-1} \) [9]

\[
\{t^a_{\text{nBPS}}\}_{a=2,\ldots,n} \in \mathcal{M}_{\text{nBPS}} = \mathbb{C}P^{n-1}. \tag{4.7}
\]

As evident from the treatment of Sec. 3 in the 1-modulus case there is no non-BPS \( Z_H \) = 0 “moduli space” at all.

It is worth remarking that the number of quadratic \( U(1,n) \)-invariants does not depend on the number \( n \) of minimally coupled vector multiplets, and it is then always equal to four. Thus, the \( n \geq 2 \) generalization of the most general charge configuration (3.18)-(3.19) supporting the two-center “large” BPS + “large” BPS BH solution (3.7) is duality-related to (\( a = 2, \ldots, n \) throughout)

\[
Q_1 \equiv (0, p^i = 0, q_0, q_i = 0) \Rightarrow I_1 = q_0^2 > 0; \tag{4.8}
\]

\[
Q_2 \equiv (P^0, P^1, P^a = 0, Q_0, Q_i = 0) \Rightarrow I_2 = (P^0)^2 + Q_0^2 - (P^1)^2 > 0, \tag{4.9}
\]

implying

\[
\begin{cases}
t^i_{\text{BPS}} (Q_1) = 0, \forall i; \\
t^1_{\text{BPS}} (Q_2) = -\frac{iP^1}{(q_0 - iP^0)}; \\
t^a_{\text{BPS}} (Q_2) = 0, \forall a. \tag{4.10}
\end{cases}
\]
On the same respect, the \( n \geq 2 \) generalization of the most general charge configuration (3.69)-(3.70) supporting the two-center non-BPS BH solution (3.56) is duality-related to

\[
\begin{align*}
Q_1 &\equiv (0, p^i = 0, 0, q_1, q_a = 0) \Rightarrow I_1 = -q_1^2 < 0; \\
Q_2 &\equiv (P^0, P^1, P^a = 0, 0, Q_1, Q_a = 0) \Rightarrow I_1 = (P^0)^2 - (P^1)^2 - Q_1^2 < 0,
\end{align*}
\] (4.11)

implying (recall (4.7))

\[
\begin{align*}
t^{i^a}_{nBPS} (Q_1) & = 0; \\
t^{i^a}_{nBPS} (Q_2) & = i\frac{(P^{n^a}_{nBPS} - P^0)}{(q_1 - iP^1)}; \\
t^{n}_{nBPS} & \in \mathbb{CP}^{n-1}, \forall a.
\end{align*}
\] (4.13)

Thus, both (4.8)-(4.9) and (4.11)-(4.12) can be considered without any loss in generality. As a consequence, the treatment of BPS MS/AMS in the case \( n \geq 2 \) (see Sec. 4.1 further below) is very similar to the treatment done in the case \( n = 1 \) in Sec. 3.1, the main difference consisting in the change of the metric constraint, which is now given by (4.4).

On the other hand, the treatment of non-BPS MS/AMS walls in the case \( n \geq 2 \) (see Sec. 4.2 further below) is different from the treatment done in the case \( n = 1 \) in Sec. 3.1. Indeed, for \( n \geq 2 \) the accidental \( n = 1 \) symmetry between the \( N = 2 \) central charge \( Z \) and the “flat” matter charge \( iD_t Z \) is spoiled.

We will briefly consider the treatment of non-BPS two-center configuration in the case \( n \geq 2 \), based on the results of [19], in Sec. 4.2.

### 4.1 BPS MS or AMS Wall

Within this Subsection, we assume \( Q_1 \) and \( Q_2 \) to satisfy (3.6), as well as \( \mathbb{CP}^n \) to be the spatially asymptotical scalar manifold. The \textit{a priori} possible BPS “large” two-center configurations are given by (3.7)-(3.9), with the BPS MS and AMS walls defined by (3.10) and (3.11) (clearly, with \( \mathbb{CP}^1 \) replaced by \( \mathbb{CP}^n \)). The \( n \geq 2 \) generalizations of the explicit \( n = 1 \) expressions (3.12) and (3.13) are cumbersome and, within the choice of charges (4.8)-(4.9), useless; thus, we will refrain from reporting them here.

The BPS stability region, the distance between the centers 1 and 2, and the corresponding configurational angular momentum are still given by the formulæ (3.14)-(3.16). Moreover, the condition of existence of the “large” BPS single-center solution with charge \( Q = Q_1 + Q_2 \) is given by (3.17).

#### 4.1.1 Case 1

Without any loss of generality, we consider the two-charge configuration (4.8)-(4.9). Within such a configuration, the four quadratic \( U(1, n) \)-invariants (2.23)-(2.26) are all generally non-coinciding and non-vanishing, and they do match the expressions (3.20) holding for the case \( n = 1 \) itself. Consequently, a manifestly \( U(1, n) \)-invariant characterization of the four non-vanishing charges of the general BPS two-center configuration (4.8)-(4.9) is the very same as the \( n = 1 \) one given by (3.21)-(3.25).

Within the general configuration (4.8)-(4.9), one obtains that

\[
\begin{align*}
Z_1 & = \frac{q_0}{\sqrt{2}\sqrt{1 - a^2 - b^2 - |t_a|^2}}; \\
Z_2 & = \frac{[Q_0 + iP^0 - iP^1 (b + ia)]}{\sqrt{2}\sqrt{1 - a^2 - b^2 - |t_a|^2}}.
\end{align*}
\] (4.14)

\[
\begin{align*}
\text{[Footnote 10]} \quad \text{Recall Eq. (3.59). This can be interpreted as exchange of the two skew-eigenvalues of the central charge matrix in the \( \mathcal{N} = 4 \) supersymmetry uplift; see e.g. Eq. (5.5) of [18], as well as [17] and Refs. therein.}
\end{align*}
\]
which are thus coinciding, up to the different Kähler overall factor, with their $n = 1$ counterparts. As a consequence, it holds that the real and imaginary part of $Z_1 Z_2$ respectively read:

\[
\text{Re} \left( Z_1 Z_2 \right) = \frac{q_0 \left( Q_0 + P^1 a \right)}{2 \left( 1 - a^2 - b^2 - |t|^2 \right)}; \\
\text{Im} \left( Z_1 Z_2 \right) = \frac{q_0 \left( -P^0 + P^1 b \right)}{2 \left( 1 - a^2 - b^2 - |t|^2 \right)},
\]

which still match, up to the different Kähler overall factor, their $n = 1$ counterparts, respectively given by (3.26) and (3.27).

By recalling (3.28), the region of stability $S_{\text{BP} S} (Q_1, Q_2)$ defined in (3.14) can be easily computed to be

\[
S_{\text{BP} S} \equiv \frac{P^1}{P^0} b > 1 \iff \begin{cases} 
 p^0 P^1 > 0 : & P^0 < b < \sqrt{1 - a^2 - |t|^2}; \\
 p^0 P^1 < 0 : & -\sqrt{1 - a^2 - |t|^2} < b < P^0. 
\end{cases}
\]

(4.18)

Note that $a$ and the remaining $n - 1$ complex fields $t^a$'s enter Eq. (4.18) only through the constrain to belong to the domain of definition of the metric of the scalar manifold, defined by (4.4). By using (3.22)-(3.23), the region of stability $S_{\text{BP} S}$ (4.18) can thus be re-expressed through the $n \geq 2$ generalization of Eq. (3.31):

\[
S_{\text{BP} S} \equiv \begin{cases} 
 \pm \frac{\sqrt{I_1^2 + I_2^2 - I_1 I_2}}{I_a} b > 1; \\
 b^2 + a^2 + |t|^2 < 1. 
\end{cases}
\]

(4.19)

Then, one can study the existence of the BPS MS and AMS walls, which are defined by (3.30) and (3.11) (with $\mathbb{C}P^1$ replaced by $\mathbb{C}P^n$).

By assuming the condition ($n \geq 2$ generalization of the (3.30))

\[
(I^2_a + I^2_a - I_1 I_2) \left( 1 - |t|^2 \right) - I^2_a > 0,
\]

(4.20)

one can recall (3.32) and define

\[
A_n \equiv \left\{ t^i \right\}_{i=1, \ldots, n} \in \mathbb{C}P^n : \\
\begin{cases} 
 b = \pm \frac{|I_a|}{\sqrt{I_1^2 + I_2^2 - I_1 I_2}}; \\
 a^2 - |I_a|^2 < a < \sqrt{a^2_{\text{BPS}} - |t|^2}. 
\end{cases}
\]

(4.21)

which is the $n \geq 2$ generalization of the region $A \subseteq \mathbb{C}P^1$ defined in (3.33). Thus, through some straightforward computations (the $n \geq 2$ analogues of the ones detailed in App. A), one obtains that within the two-center charge configuration (4.8)-(4.9) the existence of BPS MS or AMS walls depends on the sign of $I_a$:

\[
I_a > 0 : \begin{cases} 
 MS_{\text{BPS}} = A_n; \\
 \n MS_{\text{BPS}}; \\
 AMS_{\text{BPS}} = A_n.
\end{cases}
\]

(4.22)

\[
I_a < 0 : \begin{cases} 
 MS_{\text{BPS}}; \\
 AMS_{\text{BPS}} = A_n.
\end{cases}
\]

(4.23)
Single-Center Solution and MS Wall  Let us also remark that, within the configuration (4.8)-(4.9), Eq. (3.38) keeps holding true.

Within the general conditions (4.9) and (4.20) on $Q_2$ (corresponding to assuming the existence of a stability region for the two-center configuration “large” BPS + “large” BPS (3.7) in the case $n \geq 2$), the existence of a BPS MS wall $MS_{BPS}$ (see (4.22)) implies the existence of the “large” BPS single-center solution with charge $Q_1 + Q_2$, with entropy strictly larger than the entropy of the two-center solution, as given by Eq. (2.44).

4.2 Non-BPS

As mentioned above, for the study of two-center non-BPS solutions in presence of $n \geq 2$ Abelian vector multiplets coupled to $N = 2$, $d = 4$ supergravity multiplet, a different approach with respect to the case $n = 1$ (treated in Sec. 3.2) must be adopted.

This approach relies on the general formulæ of the MS wall, AMS wall, distance between centers 1 and 2, and configurational angular momentum for two-center non-BPS “large” + non-BPS “large” solutions in minimally coupled $N = 2$, $d = 4$ supergravity. By using the notation

$$W \left( \{ t^i, t^j \}_{i=1,...,n}, Q_a \right) \equiv W_a, \ a = 1, 2,$$

such formulæ respectively read [19]:

$$MS_{nBPS} : \quad W_{1+2} = W_1 + W_2;$$

$$AMS_{nBPS} : \quad W_{1+2} = |W_1 - W_2|;$$

$$|\vec{x}_1 - \vec{x}_2| = \pm \frac{\langle Q_1, Q_2 \rangle W_{1+2}}{\sqrt{4W_1^2W_2^2 - (W_{1+2}^2 - W_1^2 - W_2^2)^2}};$$

$$\mathcal{J} = \frac{\langle Q_1, Q_2 \rangle (\vec{x}_1 - \vec{x}_2)}{2 |\vec{x}_1 - \vec{x}_2|} = \pm \frac{(\vec{x}_1 - \vec{x}_2) \sqrt{4W_1^2W_2^2 - (W_{1+2}^2 - W_1^2 - W_2^2)^2}}{W_{1+2}},$$

where the branch “±” must be chosen for $\langle Q_1, Q_2 \rangle \gtrless 0$, respectively. In these formulæ, $W \equiv W_{nBPS}$ is nothing but the Euclidean norm of the complex vector of matter charges $D_i Z$ in local “flat” indices of $\mathbb{CP}^n$ [21] [19]:

$$W = \sqrt{g^{\overline{j}} D_\overline{j} Z D_j Z} = \sqrt{\sum_{i=1}^{n} |D_i Z|^2},$$

and it has been explicitly computed in [10]:

$$W = \frac{1}{\sqrt{2(1 - |t^m|^2)}} \left( g^{\overline{j}} - t^i t^j \right) \cdot \left[ (q_i - ip^i) \left( 1 - |t^m|^2 \right) + (q_0 + ip^0) t^i + (q_r - ip^r) t^i t^j \right] \cdot \left[ (q_j + ip^j) \left( 1 - |t^m|^2 \right) + (q_0 - ip^0) t^j + (q_n + ip^n) t^i t^j \right].$$

In the present paper, we are not going to deal with a general analysis of Eqs. (4.25)-(4.30), which will be given elsewhere.
4.2.1 “Moduli Spaces” of Multi-Center Flows

We now briefly discuss the “moduli spaces” of p-center non-BPS solutions in minimally coupled $\mathbb{CP}^n$ $\mathcal{N} = 2, d = 4$ models. It is known that for $p = 1$ the “moduli space” is

$$\mathcal{M}_{n\text{BPS, }\mathbb{CP}^p, p=1} = \frac{U(1, n-1)}{U(1) \times U(n-1)}. \quad (4.31)$$

Its generalization to the case of $2 \leq p \leq n$ centers is

$$\mathcal{M}_{n\text{BPS, }\mathbb{CP}^n, p} = \frac{U(1, n-p)}{U(1) \times U(n-p)}. \quad (4.32)$$

In order to prove this, we notice that the generic orbit of $p$ $(n+1)$-dimensional complex vectors $\{X\}_a = X_a \cdot \overline{X}_a < 0 \forall a$ (see e.g. (2.28)) is

$$\mathcal{O}_{n\text{BPS, }\mathbb{CP}^n, p} = \frac{U(1, n)}{U(1, n-p)}. \quad (4.33)$$

With $p$ complex vectors $\{X\}_a$, one can build $p^2 U(1, n)$-invariants $X_a \cdot \overline{X}_b (a, b = 1, ..., p)$; recall definition (2.31), corresponding to $p^2$ real degrees of freedom. Thus, the following consistent counting holds:

$$p^2 + \dim_{\mathbb{R}}(\mathcal{O}_{n\text{BPS, }\mathbb{CP}^n, p}) = 2p(n+1), \quad (4.34)$$

where $2p(n+1)$ is the number of real charge degrees of freedom pertaining to $p$ $(n+1)$-dimensional vectors $\{X\}_a$ of complexified charges (recall definition (2.12)).

Therefore, “flat directions” (and thus “moduli spaces”) for non-BPS $p$-center flows in $\mathcal{N} = 2$ $\mathbb{CP}^n$ models arise only for $p < n$. In particular, for $p = 2$ centers, one needs at least $n = 3$. Incidentally, this model is “dual” to $\mathcal{N} = 3$ supergravity with one matter multiplet (for a discussion of split flows and marginal stability in extended $d = 4$ supergravities, see [9]).

5 A Comparison: BPS MS and AMS Walls in the $t^3$ Model

We refer to the treatment of the BPS two-center solutions in $\mathcal{N} = 2, d = 4$ $t^3$ model, given in Sec. 5 of [5]. The symplectic charge vectors $\mathcal{Q}_1$ and $\mathcal{Q}_2$ of the two centers are chosen as follows ($u, q, v \in \mathbb{R}_0^+$):

$$\mathcal{Q}_1 \equiv (-P^0, P^1, q_0, q_1/3)^T \equiv (v, 0, 0, q) \Rightarrow \mathcal{I}_4(\mathcal{Q}_1) > 0; \quad (5.1)$$
$$\mathcal{Q}_2 \equiv (-P^0, P^1, Q_0, Q_1/3)^T \equiv (0, 0, u, 0) \Rightarrow \mathcal{I}_4(\mathcal{Q}_2) = 0, \quad (5.2)$$

yielding a mutual non-locality:

$$\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle = -uv < 0. \quad (5.3)$$

Thus, this is a case with “large” (and thus attractive) BPS center 1, and “small” (1-charge) center 2. Note that, as also observed in [5], iff $\mathcal{I}_4(\mathcal{Q}_1 + \mathcal{Q}_2) > 0$ the “large” BPS single-center solution with charge vector $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$ would exist, as well. If this occurs, it should be pointed out that

$$\sqrt{\mathcal{I}_4(\mathcal{Q}_1 + \mathcal{Q}_2)} < \sqrt{\mathcal{I}_4(\mathcal{Q}_1)} + \sqrt{\mathcal{I}_4(\mathcal{Q}_2)} = \sqrt{\mathcal{I}_4(\mathcal{Q}_1)}, \quad (5.4)$$

namely that the two-center BPS solutions with charges $\mathcal{Q}_1$ and $\mathcal{Q}_2$, if it exists, has more entropy than the corresponding BPS single-center solution with charge $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$. As discussed in Sec. 5 this is the opposite of what holds for the BPS split flows in $\mathcal{N} = 2, d = 4$ minimally coupled models.

\[\text{11}\text{We are grateful to R. Stora for an enlightening discussion on this point.}\]
The corresponding holomorphic central charges read \( t = b + ia \) \footnote{1} 
\[
\begin{align*}
Z(Q_1) & \equiv Z_1 = 3qt - vt^3 = (3q - vb^2 + 3a^2v) b + i (3q - 3b^2v + a^2v) a; \\
Z(Q_2) & \equiv Z_2 = u.
\end{align*}
\]  
(5.5)  
(5.6)

Note that, within the conventions of \cite{4}, the domain of definition of the metric of the scalar manifold is \( a \in \mathbb{R}_0^+ \).

(5.5) implies that
\[
Z(Q_1) = 0 \iff (b, a) = \begin{cases} 
\pm \left( \sqrt{\frac{3q}{v}}, 0 \right); \\
(0, 0),
\end{cases}
\]  
(5.7)

which are both outside the domain of definition of the metric of the scalar manifold itself. On the other hand, (5.6) implies that \( Z(Q_2) \) never vanishes (because \( u > 0 \)).

From (5.5) and (5.6), one can compute that
\[
\begin{align*}
\text{Re} \left( Z_1 \overline{Z_2} \right) & = \text{Re} \left( Z_1 \right) \text{Re} \left( Z_2 \right) + \text{Im} \left( Z_1 \right) \text{Im} \left( Z_2 \right) = [3q - v (b^2 - 3a^2)] ub; \\
\text{Im} \left( Z_1 \overline{Z_2} \right) & = \text{Im} \left( Z_1 \right) \text{Re} \left( Z_2 \right) - \text{Re} \left( Z_1 \right) \text{Im} \left( Z_2 \right) = [3q - v (3b^2 - a^2)] ua.
\end{align*}
\]  
(5.8)  
(5.9)

Exploiting (5.9) one can compute
\[
\text{Im} \left( Z_1 \overline{Z_2} \right) = 0 \iff b^2 = \frac{a^2}{3} + \frac{q}{v} \iff b = \pm \sqrt{\frac{a^2}{3} + \frac{q}{v}},
\]  
(5.10)

where the argument of the root is always positive. Notice that (5.10) automatically implies \( b^2 > q/v \) which is a required condition to have a well-defined axion \( b \). Moreover, by combining Eqs. (5.10) and (5.8), one obtains
\[
\text{Re} \left( Z_1 \overline{Z_2} \right) = 2ub \left( q + 4a^2v \right),
\]  
(5.11)

and thus it is immediate to realize that the sign of \( b \) determines the very nature of the wall itself. Indeed, by recalling the definitions (5.10) and (5.11), the BPS MS and AMS walls \( MS_{BPS} \) and \( AMS_{BPS} \) can be computed to be given by:
\[
\begin{align*}
MS_{BPS}(Q_1, Q_2) : & \quad b = \sqrt{\frac{a^2}{3} + \frac{q}{v}}; \\
AMS_{BPS}(Q_1, Q_2) : & \quad b = -\sqrt{\frac{a^2}{3} + \frac{q}{v}}.
\end{align*}
\]  
(5.12)  
(5.13)

Notice that by solving \( \text{Im} \left( Z_1 \overline{Z_2} \right) = 0 \) with respect to the axion \( b \) and plugging the solution into (5.8), leads to the following expression for \( \text{Re} \left( Z_1 \overline{Z_2} \right) \) at the points at which \( \text{Im} \left( Z_1 \overline{Z_2} \right) = 0 \):
\[
\text{Re} \left( Z_1 \overline{Z_2} \right) \big|_{\text{Im} \left( Z_1 \overline{Z_2} \right) = 0} = [3q - v (b^2 - 3a^2)] ub \big|_{b^2 = 3(b^2 - \frac{a^2}{3})} = 8uvb \left( b^2 - \frac{3q}{4v} \right),
\]  
(5.14)

which matches the expression given by Eq. (5.8) of \cite{1}, but does not have a definite sign.

In general, the flow is directed from stability to instability. Assuming the flowing dynamics from the BPS MS wall \( MS_{BPS}(Q_1, Q_2) \) towards the BPS AMS wall \( AMS_{BPS}(Q_1, Q_2) \)
\[
\begin{align*}
MS_{BPS}(Q_1, Q_2) : & \quad \begin{cases} 
\text{Im} \left( Z_1 \overline{Z_2} \right) = 0; \\
\text{Re} \left( Z_1 \overline{Z_2} \right) > 0.
\end{cases} \\
\implies AMS_{BPS}(Q_1, Q_2) : & \quad \begin{cases} 
\text{Im} \left( Z_1 \overline{Z_2} \right) = 0; \\
\text{Re} \left( Z_1 \overline{Z_2} \right) < 0.
\end{cases}
\end{align*}
\]  
(5.15)
Figure 2: Plot of $\text{Im} \left( Z_1 Z_2 \right)$ (red curve) and $\text{Re} \left( Z_1 Z_2 \right)$ (black curve) as a function of $b$. MS and AMS walls (blue lines) are identified by $b_{MS} = \sqrt{a^2/3 + q/v}$ and $b_{AMS} = -\sqrt{a^2/3 + q/v}$. It is manifest that the (physically sensible) flow connecting MS to AMS wall must cross the instability region.

to be continuous, then surely the flow itself will crash into a point in which $\text{Re} \left( Z_1 Z_2 \right) = 0$. The locations at which this occurs can be identified by

$$\text{Re} \left( Z_1 Z_2 \right) = 0 \iff \left[ 3q - v \left( b^2 - 3a^2 \right) \right] ub = 0 \iff \left[ 3q - v \left( b^2 - 3a^2 \right) \right] b = 0$$

$$\implies \begin{cases} \text{i) } b = 0; \\ \text{or} \\ \text{ii) } b^2 = 3 \left( a^2 + \frac{2}{v} \right) \iff b = \pm \sqrt{3} \sqrt{a^2 + \frac{2}{v}}. \end{cases}$$

(5.16)

In order to understand if the flow connecting $MS_{BPS} (Q_1, Q_2)$ to $AMS_{BPS} (Q_1, Q_2)$ belongs to the BPS stability region

$$S_{BPS} (Q_1, Q_2) : \langle Q_1, Q_2 \rangle \text{ Im} \left( Z_1 Z_2 \right) > 0,$$

(5.17)

one has to check if the condition

$$\text{Im} \left( Z_1 Z_2 \right) = \left[ 3q - v \left( 3b^2 - a^2 \right) \right] ua < 0$$

(5.18)

holds, since (5.17) reduces to (5.17) by using (5.3). By plugging the solutions i and ii of (5.16) into (5.9), one finds:

at solution $i$ ($\notin S_{BPS} (Q_1, Q_2)$) : $\text{Im} \left( Z_1 Z_2 \right) \big|_i = (3q + va^2) ua > 0 \rightarrow$ unstable ;

(5.19)

at solution $ii$ ($\in S_{BPS} (Q_1, Q_2)$) : $\text{Im} \left( Z_1 Z_2 \right) \big|_ii = -2 \left( 4va^2 + 3q \right) ua < 0 \rightarrow$ stable .

(5.20)

On top of that, the last equations clearly prove that

$$\text{Re} \left( Z_1 Z_2 \right) = 0 \not\Rightarrow \text{Im} \left( Z_1 Z_2 \right) = 0,$$

(5.21)

and thus $\text{Re} \left( Z_1 Z_2 \right) = 0$ does not imply $Z (Q_1) = 0$ nor $Z (Q_2) = 0$. One can reach the same conclusion by recalling (5.5), from which we gain (from (5.6), $Z_2 = u \in \mathbb{R}_0^+$)

at solution $i$ ($\notin S_{BPS} (Q_1, Q_2)$) : $Z_1 \big|_i = i \left( 3q + va^2 \right) a \neq 0$;

(5.22)

at solution $ii$ ($\in S_{BPS} (Q_1, Q_2)$) : $Z_1 \big|_ii = -i \left( 8va^2 + 6q \right) a \neq 0$.

(5.23)
Thus, the flow encounters points at which $\text{Re}(Z_1Z_2) = 0$, but at which $\text{Im}(Z_1Z_2) \neq 0$, and also both $Z_1$ and $Z_2$ are non-vanishing.

Note that, in order to go from the BPS MS wall $MS_{BPS}(Q_1,Q_2)$ to the BPS AMS wall $AMS_{BPS}(Q_1,Q_2)$, the flow necessarily cross the instability region $-\sqrt{\frac{q}{v}} \leq b \leq \sqrt{\frac{q}{v}}$. In particular, the flow crosses the axis $a$ (at which $b = 0$, and thus $\text{Re}(Z_1Z_2) = 0$; see solution $i$ of (5.16)).

The situation is depicted in Fig. 2.

6 Conclusion

The analysis carried out for minimally coupled Maxwell-Einstein supergravity is rather different from the one holding for $N = 2$ special Kähler geometries based on cubic prepotential [7], even though it exhibits many general properties of the split attractor flow for multi-center BHs. The properties of the latter have been considered to a large extent in the literature, as they are related to Calabi-Yau compactifications.

On the other hand, $N = 3$ [31] supergravity is expected to have a split flow analysis analogous to the one studied for minimally coupled $N = 2$ models in the present paper. Indeed, such a theory also has a duality quadratic invariant $I_2$, with the charges sitting in the fundamental representation of the duality group $U(3,n)$ [12, 10]. Furthermore, the $N = 3$ 1-modulus supergravity is “dual” to the minimally coupled 3-moduli ($\mathbb{C}P^3$) $N = 2$ theory, with the BPS and non-BPS supersymmetry features interchanged [10].

For theories with a duality invariant $I_4$ which is quartic in the charges, the analysis is more involved, because the charge orbits have a more intricate structure. These latter theories are expected to exhibit various phenomena, such as “recombination walls” [8] and “entropy enigmas” [4, 6, 7], which are not present in the class of theories analyzed in this work.

It is worth of notice that (non-compact forms of) $\mathbb{C}P^n$ spaces as moduli spaces of string compactifications have appeared in the literature, either as particular subspaces of complex structure deformations of certain Calabi-Yau manifold [32, 33] or as moduli spaces of some asymmetric orbifolds of Type II superstrings [34–37], or of orientifolds [38].

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A $I_s > 0$ as Condition of Existence for the BPS MS Wall in $\mathbb{C}P^n$ Models (Cases 1 and 2)

In this Appendix, we detail the derivation of the results (3.35)-(3.36), relating the sign of the $U(1,1)$-invariant $I_s$ to the existence of the BPS MS wall or of the BPS AMS wall.

Through the conditions (3.19) and (3.30) and the definition (3.32), the general solution to the
condition of compatibility of the metric constraint (3.3) with the condition \( \text{Im}(Z_1 Z_2) = 0 \):

\[
\begin{cases}
  a^2 + b^2 < 1; \\
  \text{Im}(Z_1 Z_2) = 0
\end{cases} \iff a^2 + \frac{I_1 I_2 - I_s^2}{(I_s^2 + I_a^2 - I_1 I_2)} < 0 \quad (A.1)
\]

reads (see also (3.33))

\[-a_{BPS} < a < a_{BPS}. \quad (A.2)\]

Therefore, due to (3.19), the following ordering on the \( a \)-axis holds:

\[-\mid \frac{Q_0}{P^1} \mid < -a_{BPS} < a < a_{BPS} < \mid \frac{Q_0}{P^1} \mid; \quad (A.3)\]

\[\mid \frac{Q_0}{P^1} \mid = \frac{|I_s|}{\sqrt{I_s^2 + I_a^2 - I_1 I_2}}. \quad (A.4)\]

After defining \( \mathcal{A} \) through (3.33), let us now analyse all sign possibilities for the relevant quantities

\[Q_0 P^1 = \pm \frac{I_s}{I_1} \sqrt{I_a^2 + I_s^2 - I_1 I_2}; \quad (A.5)\]

\[q_0 P^1 = \pm \sqrt{I_a^2 + I_s^2 - I_1 I_2}, \quad (A.6)\]

where the “±” branches in (A.5) and (A.6) are clearly independent.

- Let us start by choosing the branch “+” in (A.5). If one chooses the branch “+” also in (A.6), then the exploitation of (3.10)-(3.11) yields

\[\text{MS}_{BPS} = \mathcal{A}; \quad \#\text{AMS}_{BPS}. \quad (A.7)\]

On the other hand, if one chooses the branch “−” in (A.5), then (3.10)-(3.11) imply

\[\#\text{MS}_{BPS}; \quad \text{AMS}_{BPS} = \mathcal{A}. \quad (A.8)\]

- Let us now consider the branch “−” in (A.5). If one chooses the branch “−” in (A.6), then (3.10)-(3.11) imply (A.8). On the other hand, if the branch “−” is chosen also in (A.6), (3.10)-(3.11) yield to the result (A.7).

By summarizing the various results, it is immediate to realize that only the sign of \( q_0 Q_0 = I_s \) (recall (3.20)) is relevant: when this quantity is positive, the BPS MS wall exists, but not the AMS wall, and vice versa when such a quantity is negative, as given by Eqs. (3.35)-(3.36). 

Mutatis mutandis, an analogous treatment holds for the case \( n \geq 2 \), leading to the results (4.22)-(4.23).

B \( I_s < 0 \) as Condition of Existence for the non-BPS MS Wall in the \( \mathbb{CP}^1 \) Model

In this Appendix, we detail the derivation of the results (3.85)-(3.86), relating the sign of the \( U(1,1) \)-invariant \( I_s \) to the existence of the non-BPS MS wall or of the BPS AMS wall. Essentially, all the treatment of this Appendix can be obtained from the treatment given in App. A (for \( n = 1 \)) by applying the transformation (3.38) to both \( Q_1 \) and \( Q_2 \).
Through the conditions (3.70) and (3.80) and the definition (3.82), the general solution to the condition of compatibility of the metric constraint (3.3) with the condition \( \text{Im}(D_t \bar{Z}_1 D_t \bar{Z}_2) = 0 \) reads (see also (3.82))

\[
- a_{nBPS} < a < a_{nBPS}.
\]

Therefore, due to (3.70), the following ordering on the \( a \)-axis holds:

\[
- |Q_1| \frac{P_0}{P_0} < -a_{nBPS} < a < a_{nBPS} \left\{ \begin{array}{l}
|Q_1| \frac{P_0}{P_0} = \\
\sqrt{I_s^2 + I^2_{a} - I_1 I_2}.
\end{array} \right.
\]

After defining \( \mathcal{B} \) through (3.83), let us now analyse all sign possibilities for the relevant quantities

\[
Q_1 P_0 = \pm \frac{I_s}{I_1} \sqrt{I_s^2 + I^2_{a} - I_1 I_2};
\]

\[
q_1 P_0 = \pm \sqrt{I_s^2 + I^2_{a} - I_1 I_2},
\]

where the “\( \pm \)” branches in (B.5) and (B.6) are clearly independent.

- Let us start by choosing the branch “+” in (B.5). If one chooses the branch “+” also in (B.6), then the exploitation of (3.63)-(3.64) yields

\[
MS_{nBPS} = \mathcal{B};
\]

\[\not\exists \text{AMS}_{nBPS}.\]

(B.7)

On the other hand, if one chooses the branch “-” in (B.6), then (3.63)-(3.64) imply

\[\not\exists MS_{nBPS};\]

\[\text{AMS}_{nBPS} = \mathcal{B}.\]

(B.8)

- Let us now consider the branch “-” in (B.5). If one chooses the branch “-” in (B.6), then (3.63)-(3.64) yield the result (B.7). On the other hand, if the branch “-” is chosen also in (B.6), (3.63)-(3.64) yield to the result (B.7).

By summarizing the various results, it is immediate to realize that only the sign of \( q_1 Q_1 = -I_s \) (recall (3.71)) is relevant: when \( I_s < 0 \), the non-BPS MS wall exists, but not the AMS wall, and vice versa when \( I_s > 0 \), as given by Eqs. (3.85)-(3.86).

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