Equilibrium States and SRB-like measures of $C^1$ Expanding Maps of the Circle.

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Abstract

For any $C^1$ expanding map $f$ of the circle we study the equilibrium states for the potential $\psi = -\log |f'|$. We formulate a $C^1$ generalization of the Pesin’s Entropy Formula that holds for all the SRB measures if they exist, and for all the (necessarily existing) SRB-like measures. In the $C^1$-generic case the Pesin’s Entropy Formula holds for a unique SRB measure which is not absolutely continuous with respect to Lebesgue. The result also stands in the non generic case for which no SRB measure exists.

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1 Introduction

For any map $f$ on a compact manifold, if no invariant measure is equivalent to Lebesgue, or if $f$ is non ergodic with respect to the invariant measures that are equivalent to Lebesgue, many substitutive concepts of natural invariant measures have been defined. They translate the statistical asymptotic behavior of Lebesgue-positive sets of orbits, into spacial probabilities. Nevertheless, except under specific conditions in the $C^{1+\alpha}$ scenario, those statistically good

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measures do not necessarily coincide, and moreover, they do not necessarily exist. For instance in [5] and [14] the natural measures are defined as the weak* limit (if it exists) of the averages \( (1/n) \sum_{j=0}^{n-1} (f^*)^j \nu \) for any probability \( \nu \ll m \), where \( m \) is the Lebesgue measure and \( f^* \) denotes the pull back operator in the space of Borel-probabilities. Similarly, SRB or physical measures are defined as the weak* limit (if it exists) of the averages \( \sigma_n(x) := (1/n) \sum_{j=0}^{n-1} (f^*)^j \delta_x \) for any probability \( \nu \ll m \), where \( m \) is the Lebesgue measure and \( f^* \) denotes the pull back operator in the space of Borel-probabilities. Similarly, SRB or physical measures are defined as the weak* limit (if it exists) of the averages \( \sigma_n(x) := (1/n) \sum_{j=0}^{n-1} (f^*)^j \delta_x \) for a Lebesgue-positive set of initial states \( x \). In [14] a method is exhibited to construct \( C^0 \) non singular expanding maps of the circle \( S^1 \), for which there exists a unique natural measure with respect to \( m \), and nevertheless, for \( m \)-almost all the points \( x \in S^1 \) the averages \( \sigma_n(x) \) are non convergent. Thus, even in the case that topological expansion is exhibited, the notions of SRB and natural measures differ widely. In [12], [22], [10] diverse maps are constructed without any natural limit measure, but with other good ergodic properties (for instance, the existing of a mixing probability). In a general context, neither the existence of natural measures nor of SRB measures is required for a map \( f \) exhibit statistically good properties with respect to the Lebesgue measure (8).

A third notion of good measure from the statistical viewpoint raises from the thermodynamic formalism when considering, if it exists, a probability \( \mu \) for which the Pesin’s Entropy Formula holds [15]. If the hypothesis of \( C^2 \) (or \( C^{1+\alpha} \)) regularity is added, plenty of tight relations were proved among the SRB measures, the absolute continuity with respect to Lebesgue (of the conditional measures along the unstable manifolds), and the Pesin’s Entropy Formula. See for instance [13], [18], [20], [1], [3]. But to prove those results, the \( C^1 \)-plus Hölder regularity is essential. In the \( C^1 \) scenario, generic volume preserving diffeomorphisms still have an invariant measure satisfying the Pesin’s Entropy Formula [21], [7]. But contrarily to the situation of the \( C^{1+\alpha} \) maps, the \( C^1 \)-generic dynamical systems, under some hyperbolic-like assumptions, have no invariant measure \( \mu \) being (either \( \mu \) or its conditional unstable measures) absolutely continuous with respect to Lebesgue [2], [6]. Nevertheless, those \( C^1 \)-generic systems still have a unique SRB measure [9], [16].

Along this paper we consider the family \( \mathcal{E}^1 \) of all the \( C^1 \) expanding maps of the circle \( S^1 \). We recall that a \( C^1 \) map \( f \colon S^1 \mapsto S^1 \) is expanding if \( |f'(x)| > 1 \) for all \( x \in S^1 \). We denote \( \mathcal{E}^{1+\alpha} = \mathcal{E}^1 \cap C^{1+\alpha} \). Namely \( f \in \mathcal{E}^{1+\alpha} \) if and only if \( f \in \mathcal{E}^1 \) and besides \( f' \) is \( \alpha \)-Hölder continuous. We will focus with particular emphases on the systems in \( \mathcal{E}^1 \setminus \mathcal{E}^{1+\alpha} \). Our purpose is to state and prove a reformulation of the Pesin’s Entropy Formula for all those systems, including the non generic ones for which no SRB exists.

For \( C^1 \) systems Ruelle’s Inequality [19] states that for any \( f \)-invariant prob-
ability measure \( \mu \) on the Borel \( \sigma \)-algebra of \( S^1 \), the corresponding measure entropy \( h_\mu(f) \) satisfies:

\[
h_\mu(f) \leq \int \log |f'| \, d\mu.
\] (1)

Therefore, \( h_\mu(f) - \int \log |f'| \, d\mu \leq 0 \). By definition, the Pesin’s Entropy Formula holds if the latter difference is equal to zero:

\[
h_\mu(f) = \int \log |f'| \, d\mu
\] (2)

For any map \( f \in \mathcal{E}^{1+\alpha} \), [14] and [13] prove that Formula (2) holds if and only if \( \mu \ll m \), where \( m \) is the Lebesgue measure. On the contrary, as said above, if \( f \) is only \( C^1 \) then \( \mathcal{E}^1 \)-generically \( f \) has no invariant measure \( \mu \) such that \( \mu \ll m \) [9]. Thus, two questions raise: First, do there exist, for any \( f \in \mathcal{E}^1 \), invariant probability measures satisfying Formula (2)? From the thermodynamic formalism, the answer to this question is known to be affirmative, since \( f \) is topologically expansive. Second, what statistical properties do those probabilities exhibit with respect to the (non invariant) Lebesgue measure? In Theorem 2.3 of this paper we give a statistical simple description of a nonempty subset of invariant measures that satisfy Formula (2). We call that description the SRB-like property [8]. As a Corollary, if the measure that satisfies Formula (2) is unique, then it is SRB (i.e. physical), also when \( \mathcal{E}^1 \)-generically it is mutually singular with respect to the Lebesgue measure. Besides, if there exist physical measures, all of them satisfy Formula (2). Finally, if SRB measures do not exist, there still exist (uncountably many) probabilities that are distinguished from the general invariant measures by a weak physical condition, which is similar to the statistical property of SRB measures, and besides, satisfy the Pesin’s Entropy Formula.

Even if we conjecture that the results are also true for \( C^1 \) expanding maps in any dimension, the proofs along this paper work only on one-dimensional compact manifolds. In fact, in Lemma 4.1 we use that there exists a partition of the ambient manifold with arbitrarily small norm, such that the measure of the union of the boundaries of its pieces is zero, for all the invariant probability measures. This property is trivially satisfied by any one-dimensional map whose set of periodic orbits is, at most, countable.
2 Definitions and Statement of the Result.

The classic thermodynamic formalism defines the pressure $P_f$ respect to the potential $\psi := -\log |f'|$ by

$$P_f = \sup_{\mu \in \mathcal{M}_f} \{ h_\mu(f) - \int \log |f'| \, d\mu \},$$

where $\mathcal{M}_f$ is the set of all $f$-invariant Borel probabilities in $S^1$. For any $f \in \mathcal{C}^1$ the pressure $P_f$ is equal to zero (see [17]). Let us denote with $ES_f$ the (a priori maybe empty) set of all the $f$-invariant probability measures $\mu$ that realize the pressure $P_f$ as a maximum equal to zero. Precisely:

$$\mu \in ES_f \quad \text{if and only if} \quad h_\mu(f) = \int \log |f'| \, d\mu. \tag{3}$$

Namely, the set $ES_f$ is the set of invariant measures that satisfy the Pesin’s Entropy Formula (2) of the entropy. The thermodynamic formalism (see for instance [11]) for expansive maps states that $ES_f$ is weak$^*$ compact and convex in the space $\mathcal{M}$ of all the Borel probabilities in $S^1$, and, if non empty, its extremal points are ergodic measures. The measures in $ES_f$ are called equilibrium states of $f$ for the $C^0$ real function $\psi = -\log |f'|$.

Let us recall some definitions from the statistical viewpoint. Consider for each initial point $x \in S^1$, the following sequence of probabilities $\{\sigma_n(x)\}_{n \geq 1}$, that are called empirical probabilities.

$$\sigma_n(x) := \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}. \tag{4}$$

In the Equality above $\delta_y$ denotes the Dirac-delta measure supported on $y$.

**Definition 2.1** We call a Borel probability measure $\mu$ physical or SRB if

$$B(\mu) := \{x \in S^1 : \lim_{n \to +\infty} \sigma_n(x) = \mu\} \tag{5}$$

has positive Lebesgue measure. (In the definition of the set $B(\mu)$ the limit of the measures is taken in the space $\mathcal{M}$ endowed with the weak$^*$ topology.)

We call the set $B(\mu)$ the basin of attraction of the physical measure $\mu$.

It is standard to check that any physical measure is $f$-invariant. After the definition above, if there exist physical measures, then they describe the spatial probabilistical distribution in $S^1$ of the asymptotic behavior of the
empirical distributions in Equality (4), for a Lebesgue-positive set $B(\mu) \subset S^1$ of initial states. This is the physical role of the SRB measures from the statistical viewpoint. As said in the introduction, the existence and uniqueness of an SRB measure $\mu$ are generic for $f \in \mathcal{E}^1$, but $\mu$ is mutually singular with respect to the Lebesgue measure \[9\]. On the other hand, for any $f \in \mathcal{E}^{1+\alpha}$, the existence and uniqueness of the SRB measure $\mu$ are necessary (Ruelle’s Theorem). Besides, in this case $\mu$ is equivalent to Lebesgue and is the unique equilibrium state for the potential $\psi = -\log |f'|$. Namely, it is the unique probability that satisfies the Pesin’s Entropy Formula (2). In Theorem 2.3 we prove a generalization of Ruelle’s Theorem and of Pesin’s Entropy Formula (2) for any $C^1$ expanding map of the circle. We apply the definition of SRB-like (weakly physical) measure, instead of considering only physical or SRB measures. The gain in this generalization is that the SRB-like measures always exist, they still have a physical-like meaning (see Proposition 2.2) and they are equilibrium states for $-\log |f'|$ of any $C^1$ expanding map in $S^1$, regardless weather physical measures exist and weather such equilibrium state is unique.

Before stating the precise result, we need to revisit the definition of SRB-like measure. In brief, the nonempty set $\mathcal{O}_f$ of the SRB-like probability measures (defined for any continuous map acting on a compact manifold) is the minimal weak*-compact nonempty set of $\mathcal{M}$ that contains all the limits of the convergent subsequences of $\mu$ for Lebesgue-almost all the initial states $x \in S^1$ (see Definition 3.2). Immediately, if there exist SRB measures, they are SRB-like; and there exists a unique SRB-like measure if and only if there exists a unique SRB probability $\mu$ and its basin $B(\mu)$ has full-Lebesgue measure. But besides, in the cases that no SRB measure exists, the SRB-like measures still exist and preserve the statistical role that the nonexisting SRB measures would exhibit. Although the construction above is global, each SRB-like measure $\mu \in \mathcal{O}_f$ preserves an individual weakly physical meaning, independently of the other measures in the set $\mathcal{O}_f$. This is stated in the following Proposition 2.2. It gives a characterization of the SRB-like measures. To state Proposition 3.3 and to argue along the paper, the space $\mathcal{M}$ of all the Borel probabilities on $S^1$ is endowed with the weak* topology. For each point $x \in S^1$ we denote:

$$p_\omega(x) = \{\mu \in \mathcal{M} : \exists n_i \to +\infty \text{ such that } \lim_{i \to +\infty} \sigma_{n_i}(x) = \mu\}$$

(6)

where $\sigma_n(x)$ is the empirical probability defined in Equality (4). The set $p_\omega(x)$ is the limit set in $\mathcal{M}$ of the empirical sequence with initial state $x$. We call $p_\omega(x)$ the p-limit set of $x$. We fix any weak*-metric in $\mathcal{M}$. We denote this metric with $\text{dist}^*$. 

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Proposition 2.2 A probability measure $\mu$ is SRB-like if and only if for all $\epsilon > 0$ the following set $A_\epsilon(\mu) \subset S^1$ (called basin of $\epsilon$-weak attraction of $\mu$) has positive Lebesgue measure:

$$A_\epsilon(\mu) := \{ x \in S^1 : \text{dist}^* (p_\omega(x), \mu) < \epsilon \}. \quad (7)$$

For a seek of completeness, and although Proposition 2.2 can be easily obtained from the results in [8], we give an independent proof in the paragraph 3.3 of this paper. Let us state now our main result:

**Theorem 2.3** For any $C^1$-expanding map $f : S^1 \mapsto S^1$ there exist SRB-like measures and all of them are equilibrium states for the potential $-\log |f'|$.

The following assertions are immediate consequences or restatements of Theorem 2.3 for all the $C^1$ expanding maps that are non necessarily $C^{1+\alpha}$:

**2.3.1** The set $ES_f$ of equilibrium states for $-\log |f'|$ contains the weak*-compact convex hull of the never empty set $O_f$ of SRB-like measures.

**2.3.2** If $ES_f$ contains a single measure $\mu$, then $\mu$ is ergodic and $O_f = \{ \mu \}$. Besides, $O_f = \{ \mu \}$ if and only if $\mu$ is SRB and its basin $B(\mu)$ has full Lebesgue measure.

**2.3.3** Any SRB-like measure $\mu$ (and in particular any SRB measure if it exists) satisfies the Pesin’s Entropy Formula $\mathcal{P}$ of the entropy.

**2.3.4** There exist $f$-invariant probability measures such that $m(A_\epsilon(\mu)) > 0$ for all $\epsilon > 0$, where $m$ denotes the Lebesgue measure and $A_\epsilon(\mu)$ denotes the basin of $\epsilon$-weak attraction of $\mu$ defined by (7). All those measures satisfy the Pesin’s Entropy Formula.

We prove Theorem 2.3 in Section 5. It is a stronger version of Theorem 6.1.8 of the book of Keller [11], that states that observable measures belong to $ES_f$. In fact, the definition of SRB-like measures in Section 2 of this paper is non trivially weaker than the definition of observable measures in [11]. While SRB-like measures do exist for any $f \in \mathcal{E}^{1+\alpha}$, the stronger observable measures according to [11] may not exist. Nevertheless, some of the arguments that we use to prove Theorem 2.3 are taken from the proof of Theorem 6.1.8 in [11]. The different idea when using SRB-like measures instead of the observable measures defined in [11], resides in the proof of Lemma 4.3. We have to manage with sets of probabilities (neighborhoods of the SRB-like measures), instead of fixed probabilities (the observable measures according with [11]).
The statement 2.3.2 can be equivalently reformulated, substituting the assumption \( \#ES_f = 1 \) by the following condition (see Lemma 2.4 [16]):

\[
\lim_{t \to 0^+} \frac{1}{t} \sup_{\nu \in \mathcal{M}_f} \left( h_\nu + \int (t \varphi - \log |f'|) \, d\nu \right) = \int \varphi \, d\mu \quad \forall \varphi \in C^0(S^1) \quad \forall \mu \in ES_f.
\]

For any expansive map (in any finite-dimensional manifold) the above condition is \( C^1 \) generic (see Corollary 2.5 and Proposition 3.1 of [16]). Thus, the statement 2.3.2 provides a new proof of a remarkable result in [9]: \( C^1 \)-generically the expanding maps of the circle have a unique ergodic SRB measure whose basin covers Lebesgue-almost all the orbits.

Let us state three Corollaries of Theorem 2.3. We say that a probability measure is atomic if it is supported on a finite set.

**Corollary 2.4.**

There is no atomic SRB-like measure of a \( C^1 \) expanding map in \( S^1 \).

We prove this Corollary in the paragraph 5.1.

**Corollary 2.5** Denote by \( m \) the Lebesgue measure in \( S^1 \). For any \( C^1 \) expanding map \( f \) in \( S^1 \) the following assertions are equivalent:

(a) There exists some SRB-like measure \( \mu \) such that \( m \ll \mu \)

(b) There exists a unique SRB-like measure \( \mu \), it is equivalent to Lebesgue and ergodic.

Besides, if the conditions above hold, then \( \mu \) is SRB and its basin \( B(\mu) \) has full Lebesgue measure.

We prove this Corollary in the paragraph 5.1 at the end of this paper. This corollary has a similar version for natural measures, when they exist, instead of SRB-like measures (Theorem 2.4, Part (3) of [10]). From the definition of SRB-like measure, it is immediate that if there exists some ergodic SRB-like measure \( \mu \) such that \( \mu \ll m \), then it is SRB. Nevertheless it may exist non ergodic invariant measures \( \mu \ll m \) that are neither SRB nor SRB-like (see [17]). In this latter case the weak* closure of the ergodic components of \( \mu \) is composed by SRB-like measures, but they are not necessarily absolutely continuous with respect to \( m \). In spite of the considerations above, we have
Corollary 2.6 Let $f$ be a $C^1$ expanding map of $S^1$. Let $\mu$ be a non ergodic $f$-invariant probability such that $\mu \ll m$, where $m$ is the Lebesgue measure. Then $\mu$ satisfies the Pesin’s Entropy Formula.

The proof of Corollary 2.6 is in the paragraph 5.5. This corollary has a similar formulation for $C^1$-diffeomorphisms in any dimension with a dominated splitting (see [20]).

3 SRB-like measures.

We revisit the definition and properties of the SRB-like (weakly physical) measures. The content of this section is a reformulation of a part of [8].

Proposition 3.1 There exists a unique minimal nonempty and weak* compact set $O_f \subset M$ such that $p\omega(x) \subset O_f$ for a full-Lebesgue set of initial states $x \in S^1$.

Proof: Consider the family $\Upsilon$ of all the non empty and weak* compact sets $A \subset M$ such that $p\omega(x) \subset A$ for a full Lebesgue set of initial states $x \in S^1$. The family $\Upsilon$ is not empty, since trivially $M \subset \Upsilon$. Define in $\Upsilon$ the partial order $A_1 \leq A_2$ if and only if $A_1 \subset A_2$. Each countable chain $\{A_n\}_{n \geq 0} \subset \Upsilon$ (namely, $A_{n+1} \leq A_n$ for all $n \geq 0$) has a minimal element $A = \bigcap_{n=0}^{+\infty} A_n \in \Upsilon$. In fact, if $m$ denotes the Lebesgue measure in $S^1$ and if for each fixed $n \geq 0$ the Borel set $B_n$ contains all the points $x$ such that $p\omega(x) \subset A_n$, then $m(B_n) = 1$, $m(\bigcap_{n=0}^{+\infty} B_n) = 1$, $p\omega(x) \in A$ for all $x \in \bigcap_{n=0}^{+\infty} B_n$, and thus $A \subset \Upsilon$. Therefore, after Zorn Lemma there exist minimal elements in $\Upsilon$, namely, minimal non empty and weak* compact sets $O \subset M$ such that $p\omega(x) \subset O$ for Lebesgue almost all $x \in S^1$. Finally, this minimal $O \subset \Upsilon$ is unique since the intersection of two of them is also in $\subset \Upsilon$. □

Definition 3.2 (SRB-like probability measures.) A probability measure $\mu \in M$ is SRB-like or weakly physical if $\mu \in O_f$, where $O_f$ is the set of Proposition 3.1.

It is immediate that any SRB-like measure is $f$-invariant. In fact, the set of $f$-invariant Borel probabilities is non empty, weak*–compact and contains $p\omega(x)$ for all $x \in S^1$. It is also immediate that all the physical or SRB measures (according with Definition 2.1), if they exist, are SRB-like measures. In fact, if $\mu \notin O_f$, then since $p\omega(x) \subset O_f$ for Lebesgue-almost all $x \in S^1$, the set $B(\mu) = \{x \in S^1 : p\omega(x) = \{\mu\}\}$ has zero Lebesgue-measure, and thus $\mu$ is non SRB.
3.3 Proof of Proposition 2.2

As said in Section 2, this Proposition gives an individual (weakly) physical meaning to each of the SRB-like measures.

Proof: Let us denote with $m$ the Lebesgue measure. For any $\epsilon > 0$ and any $\mu \in M$ let us denote $B_\epsilon(\mu)$ to the ball of center $\mu$ and radius $\epsilon$ in $M$, defined with the metric $\text{dist}^*$. If $\mu \in O_f$ then $m(A_\epsilon(\mu)) > 0$ for all $\epsilon > 0$, because if not, the compact set $K := O_f \setminus B_\epsilon(\mu)$ would be strictly contained in $O_f$ and such that $p_\omega(x) \subset K$ for Lebesgue almost all $x \in S^1$. (Therefore $K$ is not empty.) This last contradicts the minimality condition of $O_f$ in Proposition 3.1. Thus any SRB-like measure satisfies the statement $m(A_\epsilon(\mu)) > 0$ for all $\epsilon > 0$. On the other hand, if a Borel probability measure $\mu$ satisfies the inequality $m(A_\epsilon(\mu)) > 0$ for all $\epsilon > 0$, and since $p_\omega(x) \subset O_f$ for $m$ a.e. $x \in S^1$, we obtain $B_\epsilon(\mu) \cap O_f \neq \emptyset$ for all $\epsilon > 0$. Namely, $\mu$ is in the weak$^*$-closure of $O_f$. Since $O_f$ is weak$^*$-compact (see Proposition 3.1) we conclude that $\mu \in O_f$ as wanted. □

4 Proof of Theorem 2.3

Denote by $M_f \subset M$ the set of all the $f$-invariant Borel probability measures on $S^1$. Let us recall the definition of the measure theoretic entropy. For any Borel measurable finite partition $P$ of $S^1$, and for any (non necessarily invariant) probability $\mu$ it is defined

$$H(P,\mu) := \sum_{X_i \in P} \mu(X_i) \log \mu(X_i) \quad \text{and} \quad h(P,\mu) := \lim_{q \to +\infty} \frac{H(P^q,\mu)}{q}.$$ 

In the equality above $P^q := \bigvee_{j=0}^{q-1} f^{-j}(P)$, where for any pair of finite partitions $P$ and $Q$ it is defined $P \sqcup Q := \{X \cap Y \neq \emptyset : X \in P, Y \in Q\}$. It is a classic result that the limit defining $h(P,\mu)$ exists. Finally, the measure theoretic entropy $h_\mu$ of an $f$-invariant measure $\mu$ is defined by $h_\mu := \sup_P h(\mu,P)$, where the sup is taken on all the Borel measurable finite partitions $P$ of the space.

A well known result (see Proposition 2.5 of [4]) states that if $f$ is expansive (in particular if $f \in \mathcal{E}^1$), and if $P$ is a partition with norm smaller than the expansivity constant, then $h_\mu = h(\mu,P)$ for all $\mu \in M_f$. Applying this result, in the sequel we will consider only finite partitions with norm smaller than the expansivity constant $\alpha$ of $f \in \mathcal{E}^1$. So, we will compute the measurable theoretic entropy by
For any (non necessarily \( f \)-invariant) Borel probability \( \nu \), denote \( f^* \nu \) to the probability defined by \( f^* \nu(B) := \nu(f^{-1}(B)) \) for any Borel-measurable set \( B \).

For a given finite partition \( \mathcal{P} \) denote \( \partial \mathcal{P} := \bigcup_{X \in \mathcal{P}} \partial X \), where \( \partial X \) denotes the topological boundary of the piece \( X \). The only step along the proof of Theorem 2.3 (which is one of the key-points in its proof) for which we use the dimension one of the space, resides in the application of the following lemma, in particular in its statements (ii) and (iii):

\[ h_\mu = \lim_{q \to +\infty} \frac{H(\mathcal{P}^q, \mu)}{q} \text{ if } \|\mathcal{P}\| < \alpha. \quad (8) \]

\textbf{Lemma 4.1} Let \( f \) be a \( C^1 \) expanding map on \( S^1 \). Let \( \alpha > 0 \) be an expansivity constant. For all \( 0 < \delta \leq \alpha \) there exists a finite partition \( \mathcal{P} \) of \( S^1 \) such that:

(i) \( \|\mathcal{P}\| < \delta \leq \alpha \),
(ii) \( \mu(\partial \mathcal{P}) = 0 \) for all \( \mu \in \mathcal{M}_f \),
(iii) For any sequence of non necessarily invariant probabilities \( \nu_n \), for any \( \mu \in \mathcal{M}_f \) equal to the weak* limit of a convergent subsequence \( \{\mu_n\}_{i \geq 1} \) of \( \mu_n := \frac{1}{n} \sum_{j=0}^{n-1} (f^j)^* \nu_n \), and for any \( \epsilon > 0 \), there exists \( i_0 \) such that

\[ \frac{1}{n_i} H(\mathcal{P}^{n_i}, \nu_{n_i}) \leq h_\mu + \epsilon \quad \forall \ i \geq i_0. \]

\textbf{Proof:} Take any finite covering \( \mathcal{U} \) of \( S^1 \) with open intervals with length smaller than \( \delta \). Denote \( \partial \mathcal{U} := \bigcup_{X \in \mathcal{U}} \partial X \). It is a finite set. Therefore \( \mu(\partial \mathcal{U}) = 0 \) for all \( \mu \in \mathcal{M}_f \) if and only if \( \partial \mathcal{U} \) does not contain periodic points of \( f \). Since \( f \in \mathcal{E}^1 \), the set of periodic points is countable. Then, changing if necessary the open intervals \( X \in \mathcal{U} \) to slightly smaller ones such that they still cover \( S^1 \) and their boundary points are non periodic, we get a new covering \( \mathcal{U}' = \{Y_i\}_{1 \leq i \leq p} \) such that \( \mu(\partial \mathcal{U}') = 0 \) for all \( \mu \in \mathcal{M}_f \). Therefore, the partition \( \mathcal{P} = \{X_i\}_{1 \leq i \leq p} \) defined by \( X_1 := Y_1 \in \mathcal{U}', \ X_{i+1} := Y_{i+1} \setminus (\bigcup_{j=1}^{i} X_i) \), satisfies the assertions (i) and (ii). Let us prove that (i) and (ii) imply (iii). Fix the integer numbers \( q \geq 1 \), and \( n \geq q \). Write \( n = Nq + j \) where \( N, j \) are integer numbers such that \( 0 \leq j \leq q - 1 \). Fix a (non necessarily invariant) probability \( \nu \). From the properties of the entropy function \( H \) of \( \nu \) with respect to the partition \( \mathcal{P} \), we obtain

\[ H(\mathcal{P}^n, \nu) = H(\mathcal{P}^{Nq+j}, \nu) \leq H(\bigvee_{i=0}^{j-1} f^{-i} \mathcal{P}, \nu) + H(\bigvee_{i=1}^{N} f^{-iq} \mathcal{P}^q, \nu) \]

\textsuperscript{1}The condition \( \mu \in \mathcal{M}_f \) is redundant: if \( \mu \) is the weak* limit of a convergent subsequence of \( \{\mu_n\}_{n \geq 1} \), then \( \mu \in \mathcal{M}_f \).
\begin{align*}
\sum_{i=0}^{j-1} H(f^{-i}\mathcal{P}, \nu) + \sum_{i=1}^{N} H(f^{-iq}\mathcal{P}^q, \nu) &= \sum_{i=0}^{j-1} H(\mathcal{P}, (f^i)^*\nu) + \sum_{i=1}^{N} H(\mathcal{P}^q, (f^{iq})^*\nu) \\
\Rightarrow H(\mathcal{P}^n, \nu) &\leq q \log p + \sum_{i=1}^{N} H(\mathcal{P}^q, (f^{iq})^*\nu) \forall q \geq 1, \ n \geq q.
\end{align*}

To obtain the inequality above recall that $H(\mathcal{P}, \nu) \leq \log p \ \forall \nu \in \mathcal{M}$, where $p$ is the number of pieces of the partition $\mathcal{P}$. The inequality above holds also for $f^{-l}\mathcal{P}$ instead of $\mathcal{P}$, for any $l \geq 0$, because it holds for any partition with exactly $p$ pieces. Thus:

\begin{align*}
H(f^{-l}\mathcal{P}^n, \nu) &\leq q \log p + \sum_{i=1}^{N} H(f^{-l}\mathcal{P}^q, (f^{iq})^*\nu) = \\
q \log p + \sum_{i=1}^{N} H(\mathcal{P}^q, (f^{iq+l})^*\nu).
\end{align*}

Adding the above inequalities for $0 \leq l \leq q - 1$, we obtain:

\begin{align*}
\sum_{l=0}^{q-1} H(f^{-l}\mathcal{P}^n, \nu) &\leq q^2 \log p + \sum_{l=0}^{q-1} \sum_{i=1}^{N} H(\mathcal{P}^q, (f^{iq+l})^*\nu) \\
\Rightarrow \sum_{l=0}^{q-1} H(f^{-l}\mathcal{P}^n, \nu) &\leq q^2 \log p + \sum_{l=0}^{Nq+q-1} H(\mathcal{P}^q, (f^l)^*\nu). \quad (9)
\end{align*}

On the other hand, for all $0 \leq l \leq q - 1$

\begin{align*}
H(\mathcal{P}^n, \nu) &\leq H(\mathcal{P}^n+l, \nu) \leq \left( \sum_{i=0}^{l-1} H(f^{-i}\mathcal{P}, \nu) \right) + H(f^{-l}\mathcal{P}^n, \nu) \\
&\leq q \log p + H(f^{-l}\mathcal{P}^n, \nu).
\end{align*}

Therefore, adding the above inequalities for $0 \leq j \leq q - 1$ and joining with the inequality (9), we obtain:

\begin{align*}
qH(\mathcal{P}^n, \nu) &\leq 2q^2 \log p + \sum_{l=0}^{Nq+q-1} H(\mathcal{P}^q, (f^l)^*\nu).
\end{align*}
Recall that \( n = Nq + j \) with \( 0 \leq j \leq q - 1 \). So \( Nq + q \leq n + q \) and then

\[
qH(\mathcal{P}^n, \nu) \leq 2q^2 \log p + \sum_{l=0}^{n-1} H(\mathcal{P}^q, (f^l)^*\nu) + \sum_{l=n}^{Nq+q-1} H(\mathcal{P}^q, (f^l)^*\nu)
\]

\[
\Rightarrow qH(\mathcal{P}^n, \nu) \leq 3q^2 \log p + \sum_{l=0}^{n-1} H(\mathcal{P}^q, (f^l)^*\nu).
\]

In the last inequality we have used that the number of nonempty pieces of \( \mathcal{P}^q \) is at most \( p^q \). Now we put \( \nu = \nu_n \) and divide by \( n \). Recall that the convex combination of the function \( H \) for a finite set of probability measures is not larger than the function \( H \) for the convex combination of the measures. We deduce:

\[
\frac{qH(\mathcal{P}^n, \nu_n)}{n} \leq \frac{3q^2 \log p}{n} + \frac{1}{n} \sum_{l=0}^{n-1} H(\mathcal{P}^q, (f^l)^*\nu_n)
\]

\[
\Rightarrow \frac{qH(\mathcal{P}^n, \nu_n)}{n} \leq \frac{3q^2 \log p}{n} + H(\mathcal{P}^q, \mu_n).
\]

For any fixed \( \epsilon > 0 \) (and the natural number \( q \geq 1 \) still fixed), take \( n \geq n(q) := \max\{q, 9q \log p/\epsilon\} \) in the inequality above. We deduce:

\[
\frac{q}{n}H(\mathcal{P}^n, \nu_n) \leq \frac{q\epsilon}{3} + H(\mathcal{P}^q, \mu_n) \quad \forall \ n \geq n(q) \quad \forall \ q \geq 1.
\]

\[
\Rightarrow \frac{1}{n}H(\mathcal{P}^n, \nu_n) \leq \frac{\epsilon}{3} + \frac{H(\mathcal{P}^q, \mu_n)}{q} \quad \forall \ n \geq n(q) \quad \forall \ q \geq 1. \quad (10)
\]

The inequality above holds for any fixed \( q \geq 1 \) and for any \( n \) large enough, depending on \( q \).

By hypothesis, \( \mu \) is \( f \)-invariant equal to the weak*-limit of a convergent subsequence of \( \mu_n \). After Equality (8) there exists \( q \geq 1 \) such that

\[
\frac{H(\mathcal{P}^q, \mu)}{q} \leq h_\mu + \frac{\epsilon}{3}.
\]

(11)

Fix such a value of \( q \). Since \( \mu(\partial(\mathcal{P})) = 0 \) for all \( \mu \in \mathcal{M}_f \):

\[
\lim_{i \to +\infty} H(\mathcal{P}^q, \mu_{n_i}) = H(\mathcal{P}^q, \mu) \text{ if } \lim_{i \to +\infty} (\text{weak}^*) \mu_{n_i} = \mu, \quad \text{because } \mu(\partial\mathcal{P}^q) = 0.
\]
Therefore, there exists $i_0$ such that for all $i \geq i_0$:

$$n_i \geq n(q) \quad \text{and} \quad \frac{H(P^q, \mu_{n_i})}{q} \leq \frac{H(P^q, \mu)}{q} + \frac{\epsilon}{3}.$$ 

Joining the last assertion with Inequalities (10) and (11) we deduce (iii), as wanted. \(\square\)

4.2 Notation: For any $f \in E^1$ denote $\psi := -\log |f'| < 0$, and for all $r \geq 0$ construct

$$K_r := \{ \nu \in \mathcal{M}_f : \int \psi \, d\nu + h_\nu \geq -r \}. \quad (12)$$

The notation above is taken from the book [11]. The set $K_r$ is non empty, weak* compact and convex. In fact, join the proof of Theorem 4.2.3 of the book in [11] with Theorem 4.2.4 and Remark 6.1.10 of the same book. Joining the assertions (1), (3) and (12) we deduce that $E_{S_f} = K_0$ is weak* compact and convex.

For all integer $n \geq 1$ and all $x \in S^1$ recall the definition of the empirical probability $\sigma_n(x)$ in Equality (4), and the definition of the p-limit set $p_\omega(x)$ in the set $\mathcal{M}$ of Borel probabilities, according with Equality (6). In $\mathcal{M}$ fix the following weak* metric:

$$\text{dist}(\mu, \nu) := \sum_{i=0}^{+\infty} \frac{1}{2^i} \left| \int \psi_i \, d\mu - \int \psi_i \, d\nu \right|,$$ 

where $\psi_0 := -\log |f'|$ and $\{\psi_i\}_{i \geq 1}$ is countable family of continuous functions, dense in the space $C^0(S^1, [0, 1])$. Trivially with this distance, for any $\mu_0 \in \mathcal{M}$ and any $\epsilon > 0$ the ball $B := \{ \nu \in \mathcal{M} : \text{dist}(\mu_0, \nu) < \epsilon \}$ is convex.

Lemma 4.3 Let $f$ be a $C^1$ expanding map on $S^1$. Let $m$ be the Lebesgue measure on $S^1$. Fix $r > 0$ and let $K_r$ be defined by Equality (12). Consider the weak* distance defined in (13). Then, for all $0 < \epsilon < r/2$ there exists $n_0 \geq 1$ such that

$$m(\{ x \in S^1 : \text{dist}(\sigma_n(x), K_r) \geq \epsilon \}) \leq e^{n(\epsilon-r)} < e^{-nr/2} \quad \forall \ n \geq n_0. \quad (14)$$

Proof: Fix $0 < \epsilon < r/2$. Observe that the set $\{ \mu \in \mathcal{M} : \text{dist}(\mu, K_r) \geq \epsilon \}$ is weak* compact, so it has a finite covering $\{ B_i \}_{1 \leq i \leq k}$, for a minimal cardinal $k \geq 1$, with open balls $B_i \subset \mathcal{M}$ of radius $\epsilon/3$. For any fixed $n \geq 1$ denote
\[ C_{n,i} := \{ x \in S^1 : \sigma_n(x) \in B_i \}, \quad C_n := \bigcup_{i=1}^k C_{n,i}. \]

Then:

\[ \{ x \in S^1 : \text{dist}(\sigma_n(x), \mathcal{K}_r) \geq \epsilon \} \subset C_n. \]

Therefore, to prove the lemma it is enough to find \( n_0 \) such that \( m(C_n) \leq e^{n(\epsilon-r)} \) for all \( n \geq n_0 \). Fix \( 1 \leq i \leq k \). We claim:

\[ \exists n_i \text{ such that } m(C_{n,i}) \leq e^{n(\epsilon-r+\epsilon/2)} \quad \forall \ n \geq n_i \quad \text{(to be proved)} \quad (15) \]

First, let us see that it is enough to prove Assertion (15) to end the proof of the lemma. In fact, if Assertion (15) holds, define:

\[ n_0 := \max \{ 2(\log k)/\epsilon, \ max_{1 \leq i \leq k} n_i \}. \]

Then, we deduce the following inequalities for all \( n \geq n_0 \), as wanted:

\[ m(C_n) \leq \sum_{i=1}^k m(C_{n,i}) \leq k \ e^{n(\epsilon-r+\epsilon/2)} = e^{n(\epsilon-r+(\log k/n))} \leq e^{n(\epsilon-r)}. \]

Second and last, let us prove Assertion (15). Consider an expansivity constant \( \alpha > 0 \) of \( f \). Take \( \epsilon/10 \) and for such value, fix a continuity modulus \( 0 < \delta < \alpha \) of the function \( \psi = -\log |f'| \). Namely \( |\psi(x) - \psi(y)| < \epsilon/6 \) if \( \text{dist}(x,y) < \delta \). Take a finite partition \( \mathcal{P} = \{ X_i \}_{1 \leq i \leq p} \) of \( S^1 \) with norm smaller than \( \delta \) and satisfying also the conditions (ii) and (iii) of Lemma 4.1. The map \( f \) is conjugated to a linear expanding map. Therefore, if the norm of the partition \( \mathcal{P} \) is chosen small enough, the restricted map \( f^n|_X : X \mapsto f^n(X) \) is a diffeomorphism for all \( X \in \mathcal{P}^n \) and for all \( n \geq 1 \). Thus, for all \( X \in \mathcal{P}^n \):

\[ m(X \cap C_{n,i}) = \int_{f^n(X \cap C_{n,i})} |(f^n)'| \ dm = \int_{f^n(X \cap C_{n,i})} e^{\sum_{j=0}^{n-1} \psi \circ f^j} \ dm \]

\[ \Rightarrow m(C_n) = \sum_{X \in \mathcal{P}^n} \int_{f^n(X \cap C_{n,i})} e^{\sum_{j=0}^{n-1} \psi \circ f^j} \ dm. \]

Either \( C_{n,i} = \emptyset \), and Assertion (15) becomes trivially proved, or the finite family of pieces \( \{ X \in \mathcal{P}^n : X \cap C_{n,i} \neq \emptyset \} = \{ X_1, \ldots, X_N \} \) has \( N = N(n,i) \) pieces for some \( N \geq 1 \). In this latter case, take a unique point \( y_k \in X_k \cap C_{n,i} \) for each \( k = 1, \ldots, N \). Denote by \( Y(n,i) = \{ y_1, \ldots, y_N \} \) the collection of such points. Due to the construction of \( \delta > 0 \), and since the partition \( \mathcal{P} \) has norm smaller than \( \delta \), we deduce:
\[
\sum_{j=0}^{n-1} \psi(f^j(y)) \leq \sum_{j=0}^{n-1} (\psi(f^j(y_k)) + \epsilon/6) \quad \forall \; y, y_k \in X_k, \; \forall \; k = 1, \ldots, N.
\]

Therefore \( m(C_n) \leq e^{n\epsilon/6} \sum_{k=1}^{N} e^{\sum_{j=0}^{n-1} \psi(f^j(y_k))} m(f^n(X_k \cap C_{n,i})) \), and thus:

\[
m(C_n) \leq e^{n\epsilon/6} \sum_{k=1}^{N} e^{\sum_{j=0}^{n-1} \psi(f^j(y_k))}.
\]

Define

\[
L := \sum_{k=1}^{N} e^{\sum_{j=0}^{n-1} \psi(f^j(y_k))}, \quad \lambda_k := \frac{1}{L} e^{\sum_{j=0}^{n-1} \psi(f^j(y_k))} > 0.
\]

Then, \( \sum_{k=1}^{N} \lambda_k = 1 \) and

\[
m(C_n) \leq e^{(n\epsilon/6) + \log L}, \quad \log L = \left( \sum_{k=1}^{N} \lambda_k \sum_{j=0}^{n-1} \psi(f^j(y_k)) \right) - \left( \sum_{k=1}^{N} \lambda_k \log \lambda_k \right).
\]

(To check the last equality substitute \( \lambda_k \) by the quotient which defines it.) Define the probability measures

\[
\nu_n := \sum_{k=1}^{N} \lambda_k \delta_{y_k}, \quad \mu_n := \frac{1}{n} \sum_{j=0}^{n-1} (f^j)^* \nu_n = \sum_{k=1}^{N} \lambda_k \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(y_k)} = \sum_{k=1}^{N} \lambda_k \sigma_n(y_k).
\]

It is standard to check that

\[
\sum_{k=1}^{N} \lambda_k \sum_{j=0}^{n-1} \psi(f^j(y_k)) = n \int \psi \, d\mu_n, \quad \sum_{k=1}^{N} \lambda_k \log \lambda_k = H(\mathcal{P}^n, \nu_n), \quad \text{and then}
\]

\[
m(C_n) \leq \exp \left( \frac{n\epsilon}{6} + \log L \right) = \exp \left( n \left( \frac{\epsilon}{6} + \int \psi \, d\mu_n + \frac{H(\mathcal{P}^n, \nu_n)}{n} \right) \right).
\]

Take a subsequence \( n_t \to +\infty \) such that

- \( \lim_{t \to +\infty} m(C_{n_t}) = \limsup_{n \to +\infty} m(C_n) \) and
• the sequence \( \{\mu_n\}_{n \geq 1} \) is weak*-convergent.

Denote \( \mu = \lim_{i \to +\infty} \mu_{n_i} \). After Assertion (iii) of Lemma 4.1 and the definition of the weak* topology, there exists \( n_i \geq 1 \) such that

\[
m(C_n) \leq \exp \left( n \left( \frac{\epsilon}{2} + \int \psi \, d\mu + h_\mu \right) \right) \quad \forall \ n \geq n_i.
\] (16)

By construction \( y_k \in C_{n,i} \) for all \( k = 1, \ldots, N \). Thus \( \sigma_n(y_k) \in B_i \). Since the ball \( B_i \) is convex and \( \mu_n \) is a convex combination of the measures \( \sigma_n(y_k) \) (recall that \( \sum_{k=1}^N \lambda_k = 1 \)), we deduce that \( \mu_n \in B_i \). Therefore, the weak* limit \( \mu \) of any convergent subsequence of \( \{\mu_n\}_{n \geq 1} \) belongs to \( \overline{B}_i \). Since the ball \( B_i \) has radius \( \epsilon/3 \) and intersects \( \{\mu \in M : \text{dist}(\mu, K_r) \geq \epsilon\} \), we have \( \mu \in \overline{B}_i \subset M \setminus K_r \). Therefore, by the definition of the set \( K_r \), we have: \( h_\mu + \int \psi \, d\mu < -r \). Substituting this last inequality in (16) we conclude (15) ending the proof. \( \square \)

4.4 End of the proof of Theorem 2.3

For any \( r > 0 \) consider the compact set \( K_r \subset M \) defined by Equality (12).

Since \( \{K_r\}_r \) is decreasing with \( r \):

\[
K_0 = \bigcap_{r > 0} K_r
\]

From Equalities (1) and (3) and from the definition of \( K_0 \) in Equality (12), we have

\[
K_0 = ES_f.
\]

So, to prove Theorem 2.3 we must prove that the set \( O_f \) of SRB-like measures satisfy: \( O_f \subset K_r \) for all \( r > 0 \). Since \( K_r \) is weak* compact, we have

\[
K_r = \bigcap_{\epsilon > 0} B(r, \epsilon), \quad \text{where } B(r, \epsilon) := \{\mu \in M : \text{dist}(\mu, K_r) \leq \epsilon\},
\]

with the weak* distance defined in (13). Therefore, it is enough to prove that \( O_f \subset B(r, \epsilon) \) for all \( 0 < \epsilon < r/2 \) and for all \( r > 0 \). After Proposition 3.1 and since \( B(r, \epsilon) \) is weak* compact, it is enough to prove that the following set \( B(r, \epsilon) \) (called basin of attraction of \( B(r, \epsilon) \)) has full Lebesgue measure:

\[
B(r, \epsilon) := \{x \in S_1 : \text{pw}(x) \subset B(r, \epsilon)\}.
\]

From Lemma 4.3 there exists \( n_0 \) such that, for any \( n > n_0 \):

\[
m\{x : \sigma_n(x) \notin B(r, \epsilon)\} \leq e^{n(\epsilon-r)} \leq e^{-nr/2},
\]
where \( m \) denotes the Lebesgue measure. This implies that
\[
\sum_{n=1}^{\infty} m(x : \sigma_n(x) \not\in B(r, \epsilon)) < +\infty.
\]

After Borel-Cantelli Lemma it follows that
\[
m\left(\bigcap_{n_0=1}^{\infty} \bigcup_{n=n_0}^{\infty} \{x : \sigma_n(x) \not\in B(r, \epsilon)\}\right) = 0.
\]

In other words for \( m \)-a.e. \( x \in S^1 \), there exists \( n_0 \geq 1 \) such that \( \sigma_n(x) \in B(r, \epsilon) \) for all \( n \geq n_0 \). Hence, \( pw(x) \subset B(r, \epsilon) \) for \( m \)-almost all the points \( x \in S^1 \), as wanted. \( \square \)

5 Proofs of the Corollaries

5.1 Proof of Corollary 2.4

If \( \mu \) is an atomic invariant measure for an expanding map \( f \), then \( h_\mu(f) = 0 \). Since \( \psi = -\log f' < 0 \), we have \( h_\mu(f) + \int \psi \, d\mu < 0 \). Therefore \( \mu \) does not satisfy Pesin’s formula (3). After Theorem 2.3.3 the measure \( \mu \) is not SRB-like. \( \square \)

To prove Corollaries 2.5 and 2.6 we will use the following definition:

**Definition 5.2** For any \( f \)-invariant probability measure \( \mu \) the weak*-closure \( \mathcal{K}(\mu) \) of the ergodic components of \( \mu \), is the minimal nonempty and weak*-
compact set of probabilities such that
\[
\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} \in \mathcal{K}(\mu) \quad \mu\text{-a.e. } x \in S^1.
\]

After Birkhoff’s Ergodic Theorem and the Ergodic Decomposition Theorem, the limit above exists (in the weak* topology) and is an ergodic invariant measure for \( \mu \)-almost all the points \( x \in S^1 \). It is immediate, after applying Zorn Lemma, the existence of \( \mathcal{K}(\mu) \).

**Lemma 5.3** For any \( f \)-invariant measure \( \mu \), consider the weak*-closure \( \mathcal{K}(\mu) \) of its ergodic components, as defined in 5.2. Then, \( \mu \) is ergodic if and only if \( \mu \in \mathcal{K}(\mu) \), and if this latter inclusion occurs, then \( \mathcal{K}(\mu) = \{\mu\} \). Thus, \( \mu \) is non ergodic if and only if \( \text{dist}^*(\mu, \mathcal{K}(\mu)) > 0 \).
Proof: After Definition 5.2 and the definition of ergodicity, we have $\mathcal{K} = \{\mu\}$ if and only if $\mu$ is ergodic. Now, it is enough to prove that if $\mu \in \mathcal{K}(\mu)$ then $\mu$ is ergodic. Consider the weak* distance defined by Equality (13). For any $\epsilon > 0$ consider the ball $B_\epsilon = \{\nu \in \mathcal{M} : \text{dist}^*(\nu, \mu) < \epsilon\}$ and the set

$$A_\epsilon = \{x \in S^1 : p_\omega(x) \subset B_\epsilon < \epsilon\}. \quad (17)$$

We claim that $\mu(A_\epsilon) > 0$ for all $\epsilon > 0$. In fact, arguing by contradiction if $\mu(A_\epsilon) = 0$, and since $p_\omega(x)$ is a single measure for $\mu$-almost all the points $x \in S^1$, then $\mu(\{x \in S^1 : p_\omega(x) \in \mathcal{K}(\mu) \setminus B_\epsilon\}) = 1$. This contradicts the minimality of $\mathcal{K}(\mu)$ in Definition 5.2. Consider the sequence of continuous functions $\psi_i$ in the equality (13) which defines the distance $\text{dist}^*$. Applying the Ergodic Decomposition Theorem:

$$\int_{A_\epsilon} \psi_i d\mu = \int d\mu \int_{x \in A_\epsilon} \psi_i(x) d\mu_x,$$

where $\mu_x$ is an ergodic measure. Since $A_\epsilon$ satisfies the inequality (17), and $\mu_x$ is ergodic, we deduce $p_\omega(x) = \{\mu_x\}$, $\mu_x \in B_\epsilon$ for $\mu$-a.e. $x \in A_\epsilon$. Therefore

$$\sum_{i=0}^{+\infty} \frac{1}{2^i} \left| \int_{A_\epsilon} \psi_i d\mu - \int \psi_i d\mu \right| \leq \sum_{i=0}^{+\infty} \frac{1}{2^i} \left| \int_{x \in A_\epsilon} \psi_i(x) d\mu_x \right| \mu_x \leq \epsilon.$$

Thus, the bounded linear operator $\varphi \in C^0(S^1) \mapsto \int_{A_\epsilon} \varphi \ d\mu$ (which is by Riesz Representation Theorem the finite measure $\mu_x(B) := \mu(A_\epsilon \cap B)$ for all Borel-set $B \subset S^1$) is at weak* distance smaller than $\epsilon$ of $\mu$, in the space of all the finite Borel-measures $\nu$ such that $\nu(S^1) \leq 1$. Therefore, applying in particular this operator to the continuous constant function $\phi = 1$, we obtain $\lim_{\epsilon \to 0^+} \mu(A_\epsilon) = 1$. Consider the basin $B(\mu)$ of attraction of $\mu$ defined by Equality (15). By construction, the sets $A_\epsilon$ decrease to $B(\mu)$ when $\epsilon > 0$ decrease going to zero. Thus, $B(\mu) = \bigcup_{\epsilon > 0} A_\epsilon$ and $\mu(B(\mu)) = \lim_{\epsilon \to 0^+} \mu(A_\epsilon) = 1$. We conclude that $\lim_{n \to +\infty} (1/n) \sum_{j=0}^{n-1} \delta_{f^j(x)} = P$ for $\mu$-a.e. $x \in S^1$, and thus, $\mu$ is ergodic as wanted. \qed

5.4 Proof of Corollary 2.5

Trivially (b) implies (a) and also implies that $\mu$ is SRB and that its basin of attraction has full-Lebesgue measure. So, it is only left to prove that (a)
implies (b). Assume (a). Since $\mu$ is SRB-like, it is invariant. Since $m \ll \mu$, applying Birkhoff Theorem and the Ergodic Decomposition Theorem, for $m$ a.e. $x \in S^1$: $p\omega(x) = \{\mu(x)\}$, where $\mu(x)$ is an ergodic component of $\mu$. Applying Proposition 2.2 to the SRB-like measure $\mu$, for all $\epsilon > 0$ there exists a $m$-positive set $A_{\epsilon}(\mu)$ such that $\text{dist}(p\omega(x), \mu) < \epsilon$. Joining with the assertion above, we deduce that $\mu$ is in the weak$^*$-closure of the set of its ergodic components. As proved in Lemma 5.3 any non ergodic invariant measure is isolated from the weak$^*$ closure set of its ergodic components. So, we deduce that $\mu$ is ergodic. Therefore for $m$-a.e. $x \in S^1$ $p\omega(x) = \{\mu\}$. This implies that \( \mu \) is the unique SRB-like measure, and so it is SRB and its basin of attraction covers Lebesgue almost all the points. To end the proof of (b) it is left to check that $\mu \ll m$. Take any Borel set $B \subset S^1$ such that $\mu(B) > 0$ and construct the set $C = \bigcup_{j=0}^{n}(f^{-n})(B)$, which satisfies $f^{-1}(C) \subset C$. Since $\mu$ is ergodic and $\mu(C) \geq \mu(B) > 0$, we have $\mu(C) = 1$. As $m \ll \mu$ we deduce $m(C) > 0$. Therefore $m(f^{-n}(B)) > 0$ for some $n \geq 0$. Recall that $f$ is absolutely continuous with respect to Lebesgue (because $f$ is $C^1$). We conclude the $m(B) > 0$. This shows that $m(B) > 0$ if $\mu(B) > 0$, or in other words $\mu \ll m$, ending the proof.

5.5 Proof of Corollary 2.6

Proof: After the Birkhoff ergodic theorem and the ergodic decomposition theorem we have $\lim_{n}(1/n)\sum_{j=0}^{n-1}\delta_{f^{j}(x)} = \mu_{x}$ (where $\mu_{x}$ is an ergodic component of $\mu$) for $\mu$-almost all the points $x$. Denote by $K$ to the weak$^*$-closure of the ergodic components of $\mu$, as defined in 5.2. We claim that any $\nu \in K$ is SRB-like. In fact, arguing by contradiction and applying Proposition 2.2 if there existed a weak$^*$ neighborhood $B$ of $\nu$ such that $m(\{x \in S^1 : p\omega(x) \cap B\}) = 0$, since $\mu \ll m$, we would have $\mu(\{x \in S^1 : p\omega(x) \subset K \setminus B\}) = 1$, contradicting the minimality of $K$. Applying theorem 2.3 and recalling Assertion (3): $K \subset ES_f$. From the ergodic decomposition theorem $\mu$ is in the weak$^*$-compact convex hull of $K \subset ES_f$. Since $ES_f$ is weak$^*$-compact and convex (because $f$ is expansive), then $\mu \in ES_f$, as wanted. □

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