Guaranteed Lower Bounds for the Elastic Eigenvalues by Using the Nonconforming Crouzeix–Raviart Finite Element

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Abstract: This paper uses a locking-free nonconforming Crouzeix–Raviart finite element to solve the planar linear elastic eigenvalue problem with homogeneous pure displacement boundary condition. Making full use of the Poincaré inequality, we obtain the guaranteed lower bounds for eigenvalues. Besides, we also use the nonconforming Crouzeix–Raviart finite element to the planar linear elastic eigenvalue problem with the pure traction boundary condition, and obtain the guaranteed lower eigenvalue bounds. Finally, numerical experiments with MATLAB program are reported.

Keywords: elastic eigenvalue problem; lower eigenvalue bounds; the Poincaré inequality; nonconforming Crouzeix–Raviart finite element

1. Introduction

The linear elasticity discusses how solid objects deform and become internally stressed under prescribed loading conditions, and is widely used in structural analysis and engineering design. It has been well-known that using the finite element methods for the elasticity equations/eigenvalue problems, the displacement field can be determined numerically. There have been many studies on the finite element methods for the elastic equations/eigenvalue problems (e.g., see [1–25]). For the elastic equations/eigenvalue problems with pure displacement boundary condition, [1–3] studied the conforming finite element methods. As we all know, when $\lambda \to \infty$ (or Poisson ratio $\nu \to \frac{1}{2}$), i.e., the elastic material is nearly incompressible, the locking phenomenon will occur by using the conforming finite elements to solve the equations/eigenvalue problems. In order to overcome the phenomenon of locking, various numerical methods for the linear elasticity equations have been developed. For instance, Brenner et al. [4,5] applied nonconforming Crouzeix–Raviart finite element (CR element). Based on the nonconforming CR element Hansbo [6] proposed a discontinuous Galerkin method, the discontinuous Galerkin method is closely related to the classical nonconforming CR element, which is obtained when one of the stabilizing parameters tends to infinity. Lee et al. [7] proposed a nonconforming Galerkin method based on triangular and quadrilateral elements. Botti et al. [8] constructed a low-order nonconforming approximation method. Rui [9] constructed a finite difference method on staggered grids. Zhang et al. [10] developed the nonconforming virtual element method. Chen et al. [11] presented a nonconforming triangular element and a nonconforming rectangular element. For the elasticity equations with interfaces, Lee [12] adopted the immersed finite element method based on the nonconforming CR element. Jo [13] introduced a low order finite element method for three dimensional elasticity problems. In recent
years, mixed nonconforming finite element methods seem to be much more attractive (see [14–17]), among them, [14,15] studied the linear elasticity equations, [16,17] discussed the elastic eigenvalue problems. Besides, for the pure traction problem, the classical finite element methods can be found in the literature (e.g., see [15,18–25]). In this paper, we apply the nonconforming CR element to the planar linear elastic eigenvalue problem with the pure displacement and the pure traction boundary conditions, and obtain the guaranteed lower bounds for the eigenvalues.

In 1973, Crouzeix and Raviart in [26] first introduced the triangular CR element to solve the stationary Stokes equations. Armentano and Durán in [27] first used the CR element to get the asymptotic lower bounds for the eigenvalues of the Laplace operator. On the basis of their work, finding the asymptotic lower bounds of eigenvalues was further developed by many researchers (e.g., see [28–38]). In recent years, the guaranteed lower bounds for eigenvalues have attracted academic attention. Carstensen in [39] first used the CR element to get the guaranteed lower bounds for eigenvalues of the Laplacian operator. In [40], Carstensen provided a guaranteed lower bounds for the biharmonic operator by nonconforming elements. Li in [41] discussed the guaranteed lower bounds for eigenvalues of the Stokes operator in any dimension. In [42] Liu further developed the work of [39], and gave a framework that provides guaranteed lower eigenvalue bounds for the self-adjoint eigenvalue problems. Later, in [43] Xie et al. presented an guaranteed lower bounds of Stokes eigenvalues by nonconforming elements. You et al. in [44] studied the guaranteed lower bounds for the Steklov eigenvalue problem. Hu et al. in [45] discussed the guaranteed lower bounds for eigenvalues of elliptic operators.

As far as we all know, there is no report on the lower eigenvalue bounds for the elastic eigenvalue problem. Based on the above work, we verify all conditions of the framework of Liu in [42] and apply the framework to the elastic eigenvalue problem, and obtain lower eigenvalue bounds. For the planar linear elastic eigenvalue problem with the pure displacement boundary, we make full use of the Poincaré inequality with an explicit bound in [42] (see (23)) to prove (22) and (26), which is important to lower eigenvalue bounds. We further develop the work of [44] to obtain the guaranteed lower bounds for eigenvalues by using the nonconforming CR element, and prove that is locking-free (see Theorem 1 for details). Besides, we also apply the work of Carstensen in [40] and Liu in [42] to the planar linear elastic eigenvalue problem with the pure traction boundary. We prove that using the nonconforming CR element also can obtain the guaranteed lower bounds for eigenvalues (see Theorem 2 for details), and it can be seen from the numerical experiments that it is locking-free. Further, numerical experiments show that using the linear conforming finite element to solve the planar linear elastic eigenvalue problem with the pure traction boundary is locking-free, this is an interesting phenomenon.

Throughout the paper, C denotes a positive constant independent of the mesh size h and Lamé parameters u and λ, which may not be the same in different occurrences. The bold letters represent vector-valued operators, functions and associated spaces.

### 2. The Pure Displacement Problem

Let \( \mathbf{x} = (x, y)^T \in \Omega, \Omega \subset \mathbb{R}^2 \) be a bounded polygonal domain. The standard notation \( L^2(\Omega) \) and \( W^{k,p} \) are used to denote Lebesgue function space and Sobolev spaces, respectively. For \( p = 2 \), \( H^p(\Omega) \) and their associated norms \( \| \cdot \|_{H^p(\Omega)} \) and seminorms \( |\cdot|_{H^p(\Omega)} \) are used. We denote \( H^1_0(\Omega) = \{ v \in H^1(\Omega) : v|_{\partial\Omega} = 0 \} \), the space \( L^2(\Omega) = L^2(\Omega)^n \times L^2(\Omega)^2 \) and Hilbert space \( H^1_0(\Omega) := H^1_0(\Omega)^n \times H^1_0(\Omega)^2 \). We also define the norms and seminorms on the space \( H^1(\Omega) \), for any \( \mathbf{v}(\mathbf{x}) = (v_1(\mathbf{x}), v_2(\mathbf{x}))^T \),

\[
\| \mathbf{v} \|_{H^1(\Omega)} := \left( \int_{\Omega} \sum_{\alpha\leq 2} |\partial^\alpha \mathbf{v}|^2 d\mathbf{x} \right)^{\frac{1}{2}} = \sqrt{\| v_1 \|_{H^1(\Omega)}^2 + \| v_2 \|_{H^1(\Omega)}^2}.
\]
\[ |v|_{H^s(\Omega)} := \left( \int_\Omega \sum_{k=s}^2 |\partial^k v|^2 \, dx \right)^{\frac{1}{2}} = \sqrt{|v_1|^2_{H^s(\Omega)} + |v_2|^2_{H^s(\Omega)}}. \]

Let the displacement vector \( \mathbf{u}(x) = (u_1(x), u_2(x))^T \), and the displacement gradient tensor \( \nabla \mathbf{u} \) be defined by
\[
\nabla \mathbf{u} = \begin{bmatrix}
\partial_x u_1 & \partial_y u_1 \\
\partial_x u_2 & \partial_y u_2
\end{bmatrix}.
\]

The strain tensor \( \varepsilon(\mathbf{u}) \) is defined by
\[
\varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T),
\]
and the stress tensor \( \sigma(\mathbf{u}) \) is defined by
\[
\sigma(\mathbf{u}) = 2\mu \varepsilon(\mathbf{u}) + \lambda \text{tr}(\varepsilon(\mathbf{u})) \mathbf{I},
\]
\[
= \begin{bmatrix}
(2\mu + \lambda)\partial_x u_1 + \lambda \partial_y u_2 & \mu(\partial_x u_1 + \partial_y u_2) \\
\mu(\partial_x u_1 + \partial_y u_2) & (2\mu + \lambda)\partial_x u_2 + \lambda \partial_y u_1
\end{bmatrix},
\]
where \( \mathbf{I} \in \mathbb{R}^{2 \times 2} \) is the identity matrix, and the positive constants \( \mu, \lambda \) are Lamé parameters given by
\[
\mu = \frac{E}{2(1 + \nu)}, \quad \lambda = \frac{E\nu}{(1 + \nu)(1 - 2\nu)},
\]
here the parameter \( \nu \in (0, \frac{1}{2}) \) is the Poisson ratio and \( E \) denotes Young’s modulus. We note that the coefficients \( \mu \) and \( \lambda \) are \( 0 < \mu_1 < \mu < \mu_2 < \infty \) and \( 0 < \lambda < \infty \).

Consider the elastic eigenvalue problem with homogeneous pure displacement boundary condition:
\[
\begin{cases}
-\nabla \cdot \sigma(\mathbf{u}) = \gamma \rho \mathbf{u}, & \text{in } \Omega, \\
\mathbf{u} = \mathbf{0}, & \text{on } \partial \Omega,
\end{cases}
\]
(1)

where \( \rho(\mathbf{x}) \) is the mass density. Without loss of generality, we assume that \( \rho \equiv 1 \) throughout this paper.

The weak formulation of (1) is: Find \( (\gamma, \mathbf{u}) \in \mathbb{R} \times H_0^1(\Omega), \mathbf{u} \neq \mathbf{0} \), such that
\[
a(\mathbf{u}, \mathbf{v}) = \gamma b(\mathbf{u}, \mathbf{v}), \quad \forall \mathbf{v} \in H_0^1(\Omega),
\]
(2)

where
\[
a(\mathbf{u}, \mathbf{v}) = \int_\Omega \sigma(\mathbf{u}) : \nabla \mathbf{v} \, dx
\]
\[
= 2\mu \int_\Omega \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) \, dx + \lambda \int_\Omega (\text{div}\mathbf{u})(\text{div}\mathbf{v}) \, dx
\]
\[
= \mu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} \, dx + (\mu + \lambda) \int_\Omega (\text{div}\mathbf{u})(\text{div}\mathbf{v}) \, dx,
\]
(3)
\[
b(\mathbf{u}, \mathbf{v}) = \int_\Omega \rho \mathbf{u} \cdot \mathbf{v} \, dx,
\]
and $A : B = \text{tr}(AB^T)$ is the Frobenius inner product of matrices $A$ and $B$, and we use $\|A\|_{L^2(\Omega)}^2 = \int_\Omega A : A \, dx$ to denote the $L^2$-norm of matrix $A$. From the following Korn’s inequality (see [4], Corollary 11.2.25)

$$\|e(v)\|_{L^2(\Omega)} \geq C\|v\|_{H^1(\Omega)}^2 \quad \forall v \in H^1_0(\Omega),$$

we know that the bilinear form $a(\cdot, \cdot)$ is $H^1_0(\Omega)$-coercive. We use $a(\cdot, \cdot)$ and $\cdot \cdot = \sqrt{a(\cdot, \cdot)}$ as an inner product and norm on $H^1_0(\Omega)$, respectively, use $b(\cdot, \cdot)$ and $\cdot \cdot = \sqrt{b(\cdot, \cdot)}$ as an inner product and norm on $L^2(\Omega)$, respectively.

Let $\pi_h := \{\kappa\}$ be a regular triangular mesh of $\Omega$, and the mesh diameter $h = \max_{\kappa \in \pi_h} h_\kappa$ where $h_\kappa$ is the diameter of element $\kappa$. Let the set of all interior edges of $\pi_h$ as $\mathcal{e}^i_h$, the set of the edges on the boundary as $\mathcal{e}^b_h$ and $\mathcal{e}_h = \mathcal{e}^i_h \cup \mathcal{e}^b_h$. The nonconforming CR element $V_h(\Omega)$ is defined by

$$V_h(\Omega) := V_h^i(\Omega) \times V_h^b(\Omega),$$

where

$$V_h^i(\Omega) = \{v \in L^2(\Omega) : v|_\kappa \in \text{span}\{1, x, y\}, \kappa \in \pi_h; \int_e v|_{\kappa_1} ds = \int_e v|_{\kappa_2} ds, \quad \forall e \in \mathcal{e}^i_h, \partial \kappa_1 \cap \partial \kappa_2 = e; \int_e \nu ds = 0, \forall e \in \mathcal{e}^b_h\}.$$ 

For any $v \in V_h(\Omega)$ we define $(\nabla_h v)|_\kappa = \nabla (v|_\kappa)$, $(\text{div}_h v)|_\kappa = \text{div}(v|_\kappa)$ and $(\Delta_h v)|_\kappa = \Delta(v|_\kappa)$. For the nonconforming CR element, the interpolation operator $I_h : H^1_0(\Omega) \to V_h(\Omega)$ is defined by

$$\int_e (v - I_h v) ds = 0, \quad \forall e \in \mathcal{e}_h, v \in H^1_0(\Omega).$$

Define

$$I_h v := (I_h v_1, I_h v_2) \in V_h(\Omega), \quad \forall v = (v_1, v_2).$$

Then operator $I_h : H^1_0(\Omega) \to V_h(\Omega)$, and

$$\int_e (v - I_h v) ds = 0, \quad \forall e \in \mathcal{e}_h, v \in H^1_0(\Omega).$$

(6)

Denote $H(h) = V_h(\Omega) + H^1_0(\Omega) = \{v_h + v : v_h \in V_h(\Omega), v \in H^1_0(\Omega)\}$.

The nonconforming CR element approximation of (2) is: Find $(\gamma_h, u_h) \in \mathbb{R} \times V_h(\Omega)$, $\|u_h\|_{L^2(\Omega)} = 1$, satisfying

$$a_h(u_h, v_h) = \gamma_h b(u_h, v_h), \quad \forall v_h \in V_h(\Omega),$$

(7)

where

$$a_h(u_h, v_h) = \mu \int_\Omega \nabla u_h : \nabla v_h dx + (\mu + \lambda) \int_\Omega (\text{div}_h u_h)(\text{div}_h v_h) dx.$$ 

Korn’s inequality for piecewise $H^1$-vector fields (see [20]) plays an important role in the existence and uniqueness of the solution for the linear elasticity problem discreted by the discontinuous Galerkin method. For the nonconforming CR element, discrete Equation (7) has a unique solution because $a_h(\cdot, \cdot)$ is positive definite (see page 325 in [4]). In fact, the nonconforming bilinear form $a_h(\cdot, \cdot)$ is
symmetric and positive definite on $H(h)$. Because $a_h(v_h, v_h) = 0$ implies that $v_h$ is piecewise constant, and the zero boundary condition together with continuity at the midpoints imply $v_h \equiv 0$.

Define the nonconforming energy norm $\| \cdot \|_h$ and the norm $| \cdot |_{1,h}$ on $H(h)$, respectively:

$$\|v\|_h = \sqrt{a_h(v, v)},$$

$$|v|_{1,h} = \sqrt{\sum_{k \in \mathcal{T}_h} |v|_{H^1(x)}^2},$$

and

$$|v|_{1,h} = \sqrt{\sum_{k \in \mathcal{T}_h} |v|_{H^1(x)}^2} = \sqrt{\int_{\Omega} \nabla v : \nabla v dx \leq C\|v\|_h}. \tag{8}$$

Consider the associated boundary value problem of (2) and discrete form:

$$w \in H^1_0(\Omega), \ a(w, v) = b(f, v), \ \forall v \in H^1_0(\Omega). \tag{9}$$

$$w_h \in V_h(\Omega), \ a_h(w_h, v) = b(f, v), \ \forall v \in V_h(\Omega). \tag{10}$$

where $f = (f_1(x), f_2(x))^T \in L^2(\Omega)$ is the body force.

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygonal domain, from (11.4.4) in [4] (or Lemma 2.2 in [5]) we have the elliptic regularity estimate:

For any $f \in L^2(\Omega)$, (9) exists a unique solution $w \in H^2(\Omega) \cap H^1_0(\Omega)$ satisfying

$$\|w\|_{H^2(\Omega)} + \lambda \|\text{div}w\|_{H^1(\Omega)} \leq C_\Omega \|f\|_{L^2(\Omega)}, \tag{11}$$

where the positive constant $C_\Omega$ is independent of Lamé parameters $\mu$ and $\lambda$.

The above inequality (11) plays an essential role in showing the robustness of our numerical approximation to (9).

Brenner et al. in [4,5] studied and proved the following estimates:

**Proposition 1.** Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygonal domain, $w$ and $w_h$ be the $k$th eigenvalues of (9) and (10), respectively. There exists a positive constant $C$ such that

$$\|w_h - w\|_h \leq Ch\|f\|_{L^2(\Omega)}', \tag{12}$$

$$\|w_h - w\|_{L^2(\Omega)} \leq Ch^2\|f\|_{L^2(\Omega)}', \tag{13}$$

where $C$ is independent of $h$ and $(\mu, \lambda)$, which indicates that the nonconforming CR element method is locking-free.

**Proof of Proposition 1.** See the proof method of Theorem 3.1 on page 331 and Theorem 3.2 on page 332 in [5] or the proof method of Theorem 11.4.15 on page 325 in [4] to prove (12) and (13). \(\square\)

Define the operator $T : L^2(\Omega) \rightarrow H^1_0(\Omega) \rightarrow L^2(\Omega)$ such that

$$a(Tf, v) = b(f, v), \ \forall v \in H^1_0(\Omega), \tag{14}$$

and $T_h : L^2(\Omega) \rightarrow V_h(\Omega) \not\subset H^1_0(\Omega)$ such that

$$a_h(T_h f, v) = b(f, v), \ \forall v \in V_h(\Omega). \tag{15}$$
Then both $T$ and $T_h$ are self-adjoint and completely continuous operators. (2) and (7) have the following equivalent operator forms, respectively:

$$
Tu = \gamma^{-1}u, \\
T_h u_h = \gamma_h^{-1}u_h.
$$

**Proposition 2.** Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygonal domain, $(\gamma_h, u_h)$ be the $k$th eigenpair of (7), and $\|u_h\|_{L^2(\Omega)} = 1$, $\gamma$ be the $k$th eigenvalue of (2). Then $\gamma_h \to \gamma (h \to 0)$, and there exists an eigenfunction $u$ corresponding to $\gamma$, $\|u\|_{L^2(\Omega)} = 1$, such that

$$
\|u_h - u\|_h \leq C h, \\
\|u_h - u\|_{L^2(\Omega)} \leq C h^2,
$$

where the positive constant $C$ is independent of $h$ and $(\mu, \lambda)$, i.e., the nonconforming CR element method is locking-free.

**Proof of Proposition 2.** From (13) we know

$$
\|T_h - T\|_{L^2(\Omega)} = \sup_{f \in L^2(\Omega) \setminus \{0\}} \frac{\|T_h f - Tf\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}} \leq Ch^2 \to 0 (h \to 0).
$$

From the spectral approximation theory, we have $\gamma_h \to \gamma (h \to 0)$. According to Lemma 2.3 in [32] we get

$$
\|u_h - u\|_{L^2(\Omega)} \leq C \| (T_h - T)u \|_{L^2(\Omega)},
$$

$$
\|u_h - u\|_h = \gamma \| (T_h - T)u \|_h + R,
$$

where $R \leq C \| (T_h - T)u \|_{L^2(\Omega)}$.

Let $f = u$ in (9) and (10), then $w = Tu, w_h = T_h u$. Since

$$
\|u_h - u\|_{L^2(\Omega)} \leq C \|w_h - w\|_{L^2(\Omega)},
$$

$$
\|u_h - u\|_h \leq \gamma \|w_h - w\|_h + C \|w_h - w\|_{L^2(\Omega)},
$$

which, together with (12) and (13), yields (16) and (17). □

**The Lower Bounds for the Eigenvalues of the Pure Displacement Problem**

In [42] Liu established a framework that provides guaranteed lower eigenvalue bounds for the self-adjoint eigenvalue problems. We verify all conditions of the framework and apply it to obtain lower bounds for the eigenvalues of the planar linear elastic eigenvalue problem with the pure displacement boundary.

**A1** $V$ is a Hilbert space of real function on $\Omega$ with the inner product $M(\cdot, \cdot)$ and the corresponding norm $\| \cdot \|_M := \sqrt{M(\cdot, \cdot)}$.

**A2** $N(\cdot, \cdot)$ is another inner product of $V$. The corresponding norm $\| \cdot \|_N := \sqrt{N(\cdot, \cdot)}$ is compact for $V$ with respect to $\| \cdot \|_M$, i.e., every sequence in $V$ which is bounded in $\| \cdot \|_M$ has a subsequence, which is Cauchy in $\| \cdot \|_N$.

**A3** $V^h$ is a finite dimensional space of real function over $\Omega$, $\dim(V^h) = n$. Define $V(h) := V + V^h = \{ v + v_h \mid v \in V, v_h \in V^h \}$.

**A4** Bilinear forms $M_h(\cdot, \cdot)$ and $N_h(\cdot, \cdot)$ on $V(h)$ are extension of $M(\cdot, \cdot)$ and $N(\cdot, \cdot)$ to $V(h)$, such that
(1) $M_h(u,v) = M(u,v), N_h(u,v) = N(u,v), \forall u,v \in V$.

(2) $M_h(\cdot,\cdot)$ and $N_h(\cdot,\cdot)$ are symmetric and positive definite on $V(h)$.

The assumption A4 means that $M_h(\cdot,\cdot)$ and $N_h(\cdot,\cdot)$ are inner products of $V(h)$. In order to simplicity, the extended bilinear forms $M_h(\cdot,\cdot)$ and $N_h(\cdot,\cdot)$ are still denoted by $M(\cdot,\cdot)$ and $N(\cdot,\cdot)$, and the corresponding norms are denoted by $\| \cdot \|_M$ and $\| \cdot \|_N$, respectively.

Under the above assumptions, in [42] Liu gave the following Lemma.

**Lemma 1** (the Theorem 2.1 in [42]). Suppose that $\gamma$ and $\gamma_h$ are the $k$th eigenvalues of (2) and (7), respectively, $P_h : V(h) \to V^h$ is the projection, $\forall u \in V(h)$

$$M(u - P_h u, v_h) = 0, \ \forall v_h \in V^h. \quad (18)$$

Suppose the following error estimation holds, $\forall u \in V$

$$\|u - P_h u\|_N \leq C_h \|u - P_h u\|_M. \quad (19)$$

Then, there holds

$$\frac{\gamma_h}{1 + \gamma_h C_h} \leq \gamma_i \ (i = 1, 2, \cdots, n). \quad (20)$$

For the pure displacement problem, we take the following settings:

$$V := H^1_0(\Omega), V^h := V_h(\Omega),$$

$$M(\cdot,\cdot) := a_h(\cdot,\cdot), N(\cdot,\cdot) := b(\cdot,\cdot), \| \cdot \|_h, \| \cdot \|_r, \| \cdot \|_N := \| \cdot \|_{L^2(\Omega)}.$$ 

It is easy to verify that the above settings satisfy the assumption A1 - A4. In the following discussion, for consistency of notations, we use $H^1_0(\Omega), V_h(\Omega), H(h), a_h(\cdot,\cdot), b(\cdot,\cdot), \| \cdot \|_h, \| \cdot \|_r, \| \cdot \|_{L^2(\Omega)}$ to denote $V, V^h, V(h), M(\cdot,\cdot), N(\cdot,\cdot), \| \cdot \|_M, \| \cdot \|_N$, respectively.

**Theorem 1** (guaranteed lower bounds for eigenvalues). Let $\gamma_k$ and $\gamma_{h,k}$ be the $k$th eigenvalue of (2) and (7), respectively, then the following guaranteed lower bounds holds

$$\frac{\gamma_{h,k}}{1 + \gamma_{h,k} C_h} \leq \gamma_k \ (k = 1, 2, \cdots, n), \quad (21)$$

where $\gamma_{h,k} = \frac{\gamma_k}{1 + \gamma_k C_h}, C_h = 0.346 h^{-\frac{1}{\sqrt{r}}}$.

**Proof of Theorem 1.** For any $u \in H(h), v_h \in V_h(\Omega)$, using integration by parts and noticing that $\frac{\partial v_h}{\partial n}$ is a constant function on edges ($n$ is the unit outward normal vector), $v_h$ is a linear function on element $k$ and $\Delta_h v_h = 0$, which together with (6) we have

$$a_h(u - I_h u, v_h)$$
$$= \mu \int_\Omega \nabla_h (u - I_h u) : \nabla_h v_h dx + (\mu + \lambda) \int_\Omega \text{div}_h (u - I_h u)(\text{div}_h v_h) dx$$
$$= \mu \left( \int_\Omega -\Delta_h v_h (u - I_h u) dx + \sum_{e \in \partial k} \int_e (u - I_h u) \frac{\partial v_h}{\partial n} ds \right)$$
$$+ (\mu + \lambda) \left( \int_\Omega -\nabla_h (\text{div}_h v_h) (u - I_h u) dx + \sum_{e \in \partial k} \int_e (u - I_h u) \frac{\partial v_h}{\partial n} ds \right)$$
$$= 0,$$ 

$$\quad (22)$$
thus the interpolation operator $I_h$ is the orthogonal projection.

For $\kappa \in \pi_h$, the edges of which are denoted by $e_1, e_2, e_3$, let

$$V_e(\kappa) = \{ v \in H^1(\kappa) : \int_{e_i} v ds = 0, \ i = 1, 2, 3 \}.$$  

The following the Poincaré inequality with an explicit bound (see Theorem 3.2 in [42]) plays an crucial role in studying guaranteed lower eigenvalue bounds.

$$\| v \|_{L^2(\kappa)} \leq 0.346 h_\kappa |v|_{H^1(\kappa)}, \ \forall v \in V_e(\kappa). \ (23)$$  

Since $u - I_h u \in V_e(\kappa)$, by (23) we get

$$\| u - I_h u \|_{L^2(\kappa)} \leq 0.346 h_\kappa |u - I_h u|_{H^1(\kappa)}, \ (24)$$  

then

$$\| u - I_h u \|_{L^2(\Omega)}^2 \leq \frac{1}{\mu} \sum_{\kappa \in \pi_h} (0.346 h_\kappa)^2 \mu |u - I_h u|_{H^1(\kappa)}^2$$

$$\leq \frac{1}{\mu} (0.346 h)^2 \left\{ \sum_{\kappa \in \pi_h} \mu |u - I_h u|_{H^1(\kappa)}^2 + (\mu + \lambda) \| \text{div}(u - I_h u) \|_{L^2(\Omega)}^2 \right\}, \ (25)$$

thus we have

$$\| u - I_h u \|_{L^2(\Omega)} \leq 0.346 h \frac{1}{\sqrt{\mu}} \| u - I_h u \|.$$  

Taking $P_h = I_h, C_h = 0.346 h \frac{1}{\sqrt{\mu}}$, and combining (22), (26) and Lemma 1 we deduce (21).  

**Remark 1.** Actually, when the angles of meshes meet contain condition (see Theorem 4.2 in [42] for details), such as uniform meshes used in our numerical experiments, the value of $C_h$ can be $0.1893 h \frac{1}{\sqrt{\mu}}$.  

3. The Pure Traction Problem

In this section, we present the nonconforming CR finite element for the planar linear elastic eigenvalue problem with the pure traction boundary condition. Unless explicitly noted in this section, we use the same notation as in Section 2.

We consider the planar linear elastic eigenvalue problem with the pure traction boundary:

$$\begin{cases}
- \nabla \cdot \sigma(u) = \gamma \rho u, & \text{in } \Omega, \\
\sigma(u) n = 0, & \text{on } \partial \Omega,
\end{cases} \quad (27)$$

where $n$ is the unit outward normal vector with respect to the domain $\Omega$.

The weak formulation of (27) can be described as to find $(\omega, u) \in \mathbb{R} \times H^1(\Omega)$ such that

$$a(u, v) = \omega b(u, v), \ \forall v \in H^1(\Omega), \ (28)$$
where $\omega = \gamma + 1$ and
\[
a(u, v) = \int_{\Omega} \sigma(u) : \nabla v dx + \int_{\Omega} \rho u \cdot v dx
\]
\[
= 2\mu \int_{\Omega} \varepsilon(u) : \varepsilon(v) dx + \lambda \int_{\Omega} (\text{div} u)(\text{div} v) dx + \int_{\Omega} \rho u \cdot v dx
\]
\[
= \mu \int_{\Omega} \nabla u : \nabla v dx + (\mu + \lambda) \int_{\Omega} (\text{div} u)(\text{div} v) dx + \int_{\Omega} \rho u \cdot v dx,
\]
\[
b(u, v) = \int_{\Omega} \rho u \cdot v dx.
\]

By the Korn’s inequality (see [4], Theorem 11.2.16)
\[
\|e(v)\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} \geq C\|v\|_{H^1(\Omega)}' \quad \forall v \in H^1(\Omega),
\]
we can know that the bilinear form $a(\cdot, \cdot)$ is $H^1(\Omega)$-coercive. We use $a(\cdot, \cdot)$ and $\|\cdot\|_a = \sqrt{a(\cdot, \cdot)}$ as an inner product and norm on $H^1(\Omega)$, respectively.

Let $V_h(\Omega)$ be the nonconforming CR element space:
\[
V_h(\Omega) = \{ v \in L^2(\Omega) : v|_\kappa \in \text{span}\{1, x, y\}, \kappa \in \pi_h; \int_{\kappa} v|_{\kappa_1} ds = \int_{\kappa} v|_{\kappa_2} ds, \forall \kappa \in e_h, \partial \kappa_1 \cap \partial \kappa_2 = e \}.
\]

Denote $V_h(\Omega) = V_h(\Omega) \times V_h(\Omega)$, and
\[
H(h) = V_h(\Omega) + H^1(\Omega) = \{ v_h + v : v_h \in V_h(\Omega), v \in H^1(\Omega) \}.
\]

The nonconforming CR element approximation of (28) is: Find $(\omega_h, u_h) \in \mathbb{R} \times V_h(\Omega)$, satisfying
\[
a_h(u_h, v_h) = \omega_h b(u_h, v_h), \quad \forall v_h \in V_h(\Omega),
\]
where $\omega_h = \gamma_h + 1$ and
\[
a_h(u_h, v_h) = \mu \int_{\Omega} \nabla u_h \cdot \nabla v_h dx + (\mu + \lambda) \int_{\Omega} (\text{div} u_h)(\text{div} v_h) dx + \int_{\Omega} \rho u_h \cdot v_h dx.
\]

It is easy to know that the nonconforming bilinear form $a_h(\cdot, \cdot)$ is symmetric and positive definite on $H(h)$.

Define the nonconforming energy norm $\|\cdot\|_h$ and the norm $|\cdot|_{1,h}$ on $H(h)$, respectively:
\[
\|v\|_h = \sqrt{a_h(v, v)}, \quad |v|_{1,h} = \sqrt{\sum_{\kappa \in \pi_h} |v|^2_{H^1(\kappa)'}}
\]
and
\[
|v|_{1,h} = \sqrt{\sum_{\kappa \in \pi_h} \int_{\kappa} \nabla v : \nabla v dx} \leq C\|v\|_h.
\]

Consider the associated boundary value problem of (28) and discrete form:
\[
w \in H^1(\Omega), \quad a(w, v) = b(f, v), \quad \forall v \in H^1(\Omega).
\]
\[
w_h \in V_h(\Omega), \quad a_h(w_h, v) = b(f, v), \quad \forall v \in V_h(\Omega).
\]
Define the operator $K : H^1(\Omega) \rightarrow H^1(\Omega) \hookrightarrow L^2(\Omega)$ such that
\[
a(Kf, v) = b(f, v), \quad \forall v \in H^1(\Omega),
\] (33)
and $K_h : V_h(\Omega) \rightarrow V_h(\Omega) \not\subset H^1(\Omega)$ such that
\[
a_h(K_hf, v) = b(f, v), \quad \forall v \in V_h(\Omega).
\] (34)

Then both $K$ and $K_h$ are self-adjoint and completely continuous operators.

**The Lower Bounds for Eigenvalues of the Pure Traction Problem**

For the elastic eigenvalue problem with the pure traction boundary condition, we also use $H^1(\Omega), V_h(\Omega), H(h) a_h(\cdot, \cdot), b(\cdot, \cdot), \| \cdot \|_{H^1(\Omega)}$ to denote $V, V^h, V(h), M(\cdot, \cdot), N(\cdot, \cdot), \| \cdot \|_{M^*}, \| \cdot \|_N$, respectively, which also satisfy the assumption A1–A4.

Let $P_h : H(h) \rightarrow V_h(\Omega)$ be the projection operator. For $u \in H(h), P_h u$ satisfies
\[
a_h(u - P_h u, v_h) = 0, \quad \forall v_h \in V_h(\Omega). \tag{35}
\]

Using the argument in [44] we give the estimate between the interpolation and projection operators.

**Lemma 2.** Let $I_h : H^1(\Omega) \rightarrow V_h(\Omega)$ be the interpolation operator of the nonconforming CR element, for all $u \in H(h)$, we have
\[
\|I_h u - P_h u\|_{L^2(\Omega)} \leq \frac{1}{\sqrt{\omega_{h,1}}} \|I_h u - u\|_{L^2(\Omega)}. \tag{36}
\]

**Proof of Lemma 2.** For the convenience of readers, we refer to the Lemma 3.6 in [44] to give a detailed proof. From (34), for any $\Phi_h \in V_h(\Omega)$, we have $K_h \Phi_h \in V_h(\Omega)$ and
\[
a_h(K_h \Phi_h, v_h) = b(\Phi_h, v_h), \quad \forall v_h \in V_h(\Omega). \tag{37}
\]

Let $v_h := K_h \Phi_h$ in the above formula, then
\[
\|K_h \Phi_h\|_{h}^2 = b(\Phi_h, K_h \Phi_h) \leq \|\Phi_h\|_{L^2(\Omega)} \|K_h \Phi_h\|_{L^2(\Omega)}. \tag{38}
\]

From the definition of $\omega_{h,1}$ in (30) and the min–max principle, we get
\[
\omega_{h,1} \leq \frac{\|K_h \Phi_h\|_{h}^2}{\|\Phi_h\|_{L^2(\Omega)}}, \tag{39}
\]
which together with (38) yields the estimate
\[
\|K_h \Phi_h\|_{h} \leq \frac{1}{\sqrt{\omega_{h,1}}} \|\Phi_h\|_{L^2(\Omega)}. \tag{40}
\]

For any $u \in H(h)$, using the same argument as (22) we get
\[
a_h(I_h u - u, v_h) = \mu \int_\Omega \nabla_h (I_h u - u) : \nabla_h v_h dx
+ (\mu + \lambda) \int_\Omega \nabla_h (I_h u - u) (\nabla_h v_h) dx + \int_\Omega \rho(I_h u - u) \cdot v_h dx
= \int_\Omega \rho(I_h u - u) \cdot v_h dx. \tag{41}
\]
Let \( v_h := I_h u - P_h u \) and \( \Psi_h := K_h v_h \in V_h(\Omega) \). Combining (35), (37) and \( \rho \equiv 1 \) we have

\[
\|v_h\|^2_{L^2(\Omega)} = b(v_h, v_h) = a_h(\Psi_h, I_h u - u + u - P_h u) = a_h(\Psi_h, I_h u - u).
\]

Let \( v_h = \Psi_h \) in (41), together with (40) and (42) we obtain

\[
\|v_h\|^2_{L^2(\Omega)} = \int_{\Omega} \rho(I_h u - u) \cdot \Psi_h \, dx \leq \|\Psi_h\|_{L^2(\Omega)} \|I_h u - u\|_{L^2(\Omega)}
\]

\[
\leq \|\Psi_h\|_{L^2(\Omega)} \|I_h u - u\|_{L^2(\Omega)} \leq \frac{1}{\sqrt{\omega_{h,1}}} \|v_h\|_{L^2(\Omega)} \|I_h u - u\|_{L^2(\Omega)}.
\]

From (43) we can immediately obtain (36). \( \square \)

**Theorem 2** (guaranteed lower bounds for eigenvalues). Let \( \omega_h \) and \( \omega_{h,k} \) be as defined in (28) and (30), respectively. Then, there holds

\[
\omega_{h,k} \leq \omega_k \quad (k = 1, 2, \ldots, n),
\]

where \( \omega_{h,k} = \frac{\omega_{h,k}}{1 + \omega_{h,k} C_h} \), \( C_h = \left( \frac{1}{\sqrt{\omega_{h,1}}} + 1 \right) 0.346 h^{1/\sqrt{p}} \).

**Proof of Theorem 2.** For any \( u \in H(h) \), from (23) we get the following estimates:

\[
\|u - I_h u\|_{L^2(\Omega)} \leq \left( \sum_{k \in \mathcal{N}_h} (0.346 h_k)^2 \|u - I_h u\|^2_{H^1(\kappa)} \right)^{1/2}
\]

\[
\leq 0.346 \max_{k \in \mathcal{N}_h} h_k |u - I_h u|_{1,h}.
\]

From the triangle inequality, (36) and (45) we obtain

\[
\|u - P_h u\|_{L^2(\Omega)} \leq \|u - I_h u\|_{L^2(\Omega)} + \|I_h u - P_h u\|_{L^2(\Omega)}
\]

\[
\leq (1 + \frac{1}{\sqrt{\omega_{h,1}}}) \|u - I_h u\|_{L^2(\Omega)}
\]

\[
\leq (1 + \frac{1}{\sqrt{\omega_{h,1}}}) 0.346 \max_{k \in \mathcal{N}_h} h_k |u - I_h u|_{1,h}
\]

\[
\leq C_h |u - P_h u|_{1,h}
\]

\[
\leq C_h \|u - P_h u\|_{h}, \quad \forall u \in V_h(\Omega),
\]

here \( C_h = \left( \frac{1}{\sqrt{\omega_{h,1}}} + 1 \right) 0.346 h^{1/\sqrt{p}} \), \( h = \max_{k \in \mathcal{N}_h} h_k \).

From Lemma 1 we can immediately obtain (44). \( \square \)

**Remark 2.** Same as Remark 1, when the angles of meshes meet contain condition the value of \( C_h \) can be

\[
\frac{1}{\sqrt{\omega_{h,1}}} + 1 \right) 0.1893 h^{1/\sqrt{p}}.
\]

4. Numerical Experiments

In this section, we report some numerical experiments. In our computation, our program was completed under the package of iFEM [46], and the discrete eigenvalue problems were solved in MATLAB 2012a on a DELL inspiron 5480 PC with 8G memory, and the MATLAB codes are in Appendix A. The following notations are adopted in tables.

\( h \): the meshes \( h = \max_{k \in \mathcal{N}_h} h_k \).
Combining with the analysis of the Morley finite element in [40], we know that the corrected eigenvalues are the (1) or (2) obtained by correcting $\gamma_{h,k}$, i.e., the lower bounds of the $k$th eigenvalue of (1) or (27).

$\gamma_{h,k}^c$: the $k$th eigenvalue obtained by the linear conforming finite element.

$\gamma_{h,k}^c$: the $k$th eigenvalue obtained by the linear conforming finite element.

$\gamma_{h,k}^c$: guaranteed lower bounds for the elastic eigenvalues.

$\gamma_{glob}$: the condition number of the stiffness matrix of (30) obtained by the nonconforming CR element.

$\gamma_{glob}$: the condition number of the stiffness matrix obtained by the linear conforming element.

Example 1. Consider the planar linear elastic eigenvalue problem with the pure displacement boundary condition (1), we take the mass density $\rho = 1$, Lamé parameter $\mu = 1$ and take $\lambda = 1, \lambda = 100, \lambda = 10^4, \lambda = 10^8$, respectively.

We use the nonconforming CR element to solve (1) on the unit square $\Omega_S = [0,1]^2$ and L-shaped domain $\Omega_L = [0,1]^2 \setminus [\frac{1}{2}, 1]^2$, respectively. On each domain, we select two eigenvalues to execute correction. One is the minimum eigenvalue, and the other is selected because it is an upper bound of the exact value on the coarsest mesh.

For the uniform meshes, $C_h$ methods of obtaining the guaranteed lower eigenvalue bounds are locking-free.

Table 1. The first selected eigenvalue on the uniform meshes for $\Omega_S$.

| $h$   | $\lambda = 1$ | $\lambda = 100$ | $\lambda = 10^4$ | $\lambda = 10^8$ |
|-------|---------------|------------------|------------------|------------------|
|       | $\gamma_{h,1}$ | $\gamma_{h,1}$   | $\gamma_{h,1}$   | $\gamma_{h,1}$   |
| $1/16$ | 17.886861     | 26.322914        | 19.651300        | 30.330628        |
| $1/32$ | 29.113233     | 33.479158        | 38.248713        | 46.156631        |
| $1/64$ | 34.755856     | 36.163354        | 47.885721        | 51.107816        |
| $1/128$| 36.589358     | 36.968038        | 51.0599032       | 51.107816        |
| $1/256$| 37.222052     | 37.246310        | 52.283033        | 52.283033        |
| $1/512$| 37.255010     | 37.261082        | 52.305604        | 52.305604        |
| $1/1024$| 37.263300    | 37.264818        | 52.339468        | 52.339468        |
| $1/2048$| 37.265378    | 37.265758        | 52.319313        | 52.319313        |

$\gamma_{glob}$: the condition number of the stiffness matrix obtained by the linear conforming element.

From Tables 1, 3 and 5, we see that $\gamma_{h,k}$ approximate the exact ones from below, and from Tables 2, 4 and 6, we see that on the coarsest triangulation of the square and the L-shaped domain, the discrete eigenvalues $\gamma_{h,k}$ computed by the nonconforming CR element cannot be a lower bound for the exact ones. But after correction, it must be the lower bound for eigenvalue. The corrected eigenvalues all converge to the exact ones from below. Combining with the analysis of the Morley finite element in [40], we know that the corrected eigenvalues are the guaranteed lower bounds, which coincide with our theoretical result of Theorem 1. Furthermore, from Figure 2, we can see that the approximations for the first eigenvalue of (1) obtained by the linear finite element become large as $\lambda$ increases, but the approximations for the first eigenvalue of (1) obtained by the nonconforming CR element and the corrected eigenvalues become stable as $\lambda$ increases, which indicates the nonconforming CR element method and the method of obtaining the guaranteed lower eigenvalue bounds are locking-free.
Figure 1. (Left panel) uniform meshes for $\Omega_S$. (Middle panel) uniform meshes for $\Omega_L$. (Right panel) nonuniform meshes for $\Omega_L$.

Table 2. The second selected eigenvalue on the uniform meshes for $\Omega_S$.

| $h$ | $\lambda = 1$ | $\gamma_{h,1}$ | $\lambda = 100$ | $\gamma_{h,10}$ | $\lambda = 10^4$ | $\gamma_{h,10}$ | $\lambda = 10^8$ | $\gamma_{h,10}$ |
|-----|---------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
|     | $\lambda$     | $\lambda$       | $\lambda$       | $\lambda$       | $\lambda$       | $\lambda$       | $\lambda$       | $\lambda$       |
| 46.751983 | 288.000000 | 53.879884 | 1556.281308 | 55.791490 | 150701.95 | 55.812150 | 1506524739 |
| 74.70195  | 112.267444 | 75.756482 | 114.667336 | 75.851660 | 114.885537 | 75.852618 | 114.887735 |
| 134.73582 | 158.676489 | 158.167030 | 192.211501 | 194.928781 | 160.018233 | 194.952294 |
| 162.768463 | 217.346150 | 231.427979 | 219.753392 | 234.159216 | 219.773060 | 234.181547 |
| 171.36978 | 236.789056 | 240.779418 | 239.244782 | 243.319052 | 239.264608 | 243.339595 |
| 173.64950 | 242.042378 | 243.071809 | 244.508667 | 245.559229 | 244.528375 | 245.579106 |
| 174.22872 | 236.31658 | 243.383107 | 245.852024 | 246.116707 | 245.870846 | 246.135810 |
| 174.34229 | 243.72067 | 243.785043 | 246.189635 | 246.259394 | 246.206029 | 246.273337 |
| 174.41061 | 243.80422 | 243.820673 | 246.274148 | 246.290731 | 246.280228 | 246.298612 |

Table 3. The first selected eigenvalue on the uniform meshes for $\Omega_L$.

| $h$ | $\lambda = 1$ | $\gamma_{h,1}$ | $\lambda = 100$ | $\gamma_{h,10}$ | $\lambda = 10^4$ | $\gamma_{h,10}$ | $\lambda = 10^8$ | $\gamma_{h,10}$ |
|-----|---------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
|     | $\lambda$     | $\lambda$       | $\lambda$       | $\lambda$       | $\lambda$       | $\lambda$       | $\lambda$       | $\lambda$       |
| 20.494339 | 32.386862 | 21.134242 | 34.014378 | 21.148868 | 34.052280 | 21.149018 | 34.052668 |
| 38.771461 | 46.920038 | 55.671045 | 74.165556 | 55.710938 | 74.236376 | 55.711332 | 74.237075 |
| 48.852331 | 51.679521 | 94.220010 | 105.333799 | 94.429370 | 105.595537 | 94.431453 | 105.598135 |
| 52.534607 | 53.318789 | 115.066100 | 118.896168 | 115.432059 | 119.286937 | 115.435625 | 119.290745 |
| 53.737136 | 53.940006 | 123.207359 | 124.279040 | 123.656871 | 124.736421 | 123.661998 | 124.740824 |
| 54.137166 | 54.188497 | 126.144324 | 126.423364 | 126.635772 | 126.916993 | 126.640338 | 126.921579 |
| 54.278441 | 54.291332 | 127.234951 | 127.305805 | 127.747230 | 127.818656 | 127.751345 | 127.822776 |
| 54.331611 | 54.334832 | 127.66037 | 127.683864 | 128.188741 | 128.206713 | 128.190307 | 128.208281 |
| 54.352676 | 54.353484 | 127.846819 | 127.851288 | 128.374768 | 128.397273 | 128.365844 | 128.370349 |
| 54.352676 | 128.374768 | 128.397273 | 128.365844 | 128.370349 |

Trend $\gamma_{h,1}$ $\gamma_{h,10}$ $\gamma_{h,1}$ $\gamma_{h,10}$ $\gamma_{h,1}$ $\gamma_{h,10}$ $\gamma_{h,1}$ $\gamma_{h,10}$
Table 4. The second selected eigenvalue on the uniform meshes for $\Omega_L$.

| $h$ | $\lambda = 1$ | $\lambda = 100$ | $\lambda = 10^4$ | $\lambda = 10^8$ |
|-----|---------------|----------------|-----------------|----------------|
|     | $\gamma_{h,8}$ | $\gamma_{h,8}$ | $\gamma_{h,6}$ | $\gamma_{h,6}$ |
| 46.751983 | 288.000000 | 52.539466 | 894.981566 | 55.775556 |
| 76.421607 | 116.198092 | 77.875935 | 119.593964 | 78.440738 |
| 129.040140 | 150.836415 | 199.851499 | 166.991337 | 167.027271 |
| 154.609650 | 161.604549 | 247.202534 | 239.805738 | 239.855617 |
| 162.809875 | 164.684628 | 257.154280 | 266.073733 | 266.121735 |
| 165.032625 | 165.505557 | 267.748904 | 275.590903 | 275.648410 |
| 165.650350 | 167.253565 | 276.434666 | 274.268484 | 274.284106 |
| 165.751008 | 165.781058 | 276.874900 | 277.067763 | 277.084258 |
| 165.788047 | 165.795562 | 277.813925 | 277.835028 | 277.869366 |

Trend: $\uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \uparrow \downarrow$

$\gamma_{glb}$ = 165.788047, 277.813925, 277.835028, 277.869366

Table 5. The first selected eigenvalue on the nonuniform meshes for $\Omega_L$.

| $h$ | $\lambda = 1$ | $\lambda = 100$ | $\lambda = 10^4$ | $\lambda = 10^8$ |
|-----|---------------|----------------|-----------------|----------------|
|     | $\gamma_{h,8}$ | $\gamma_{h,8}$ | $\gamma_{h,6}$ | $\gamma_{h,6}$ |
| 0.6115 | 14.323135 | 39.913359 | 15.160216 | 54.279847 |
| 0.3057 | 32.193412 | 61.835112 | 94.978070 | 15.173447 |
| 0.1529 | 46.126594 | 111.841249 | 123.698974 | 15.173447 |
| 0.0764 | 51.903333 | 123.594495 | 124.168450 | 14.218245 |
| 0.0382 | 54.672221 | 127.154900 | 128.648988 | 128.169263 |
| 0.0191 | 54.13439 | 127.307725 | 127.110189 | 127.154900 |
| 0.0096 | 54.304519 | 127.519231 | 128.040936 | 128.074000 |
| 0.0048 | 54.366670 | 127.816022 | 128.343900 | 128.394842 |
| $\gamma_{glb}$ | 54.366670 | 127.816022 | 128.343900 | 128.394842 |

Table 6. The second selected eigenvalue on the nonuniform meshes for $\Omega_L$.

| $h$ | $\lambda = 1$ | $\lambda = 100$ | $\lambda = 10^4$ | $\lambda = 10^8$ |
|-----|---------------|----------------|-----------------|----------------|
|     | $\gamma_{h,19}$ | $\gamma_{h,19}$ | $\gamma_{h,14}$ | $\gamma_{h,14}$ |
| 0.6115 | 21.133209 | 21.484746 | 54.145284 | 54.272636 |
| 0.3057 | 61.835112 | 200.749622 | 61.978314 | 61.980750 |
| 0.1529 | 169.546563 | 179.023183 | 179.023183 | 179.023183 |
| 0.0764 | 287.588211 | 360.009427 | 360.009427 | 360.009427 |
| 0.0382 | 346.264391 | 404.968887 | 404.968887 | 404.968887 |
| 0.0191 | 369.758806 | 371.259020 | 371.259020 | 371.259020 |
| 0.0096 | 371.019253 | 371.395723 | 371.395723 | 371.395723 |
| $\gamma_{glb}$ | 371.019253 | 371.395723 | 371.395723 | 371.395723 |
We also use the nonconforming CR element and the linear conforming finite element to solve (27) by taking correction. One is the first nonzero eigenvalue, and the other is selected because it is an upper bound of \( \lambda \) on the unit square \( \Omega \) and \( \mu \). We also use the nonconforming CR element and the linear conforming finite element to solve (27) by taking \( \lambda = 1, \lambda = 10^3, \lambda = 10^5, \lambda = 10^8, \lambda = 10^{10} \) while \( \mu = 1 \) and \( h = \frac{1}{256} \), and depict the curves of the corrected eigenvalues and approximations for the first nonzero eigenvalue of (27) in Figure 3.

Table 7. The first selected eigenvalue on the uniform meshes for \( \Omega_T \).

| \( h \) | \( \lambda = 1 \) | \( \lambda = 10^3 \) | \( \lambda = 10^5 \) | \( \lambda = 10^8 \) | \( \lambda = 10^{10} \) |
|---------|----------------|----------------|----------------|----------------|----------------|
| 4.679529 | 4.684617 | 8.592845 | 8.684738 | 8.593190 | 8.684740 | 8.593194 |
| 8.054279 | 9.807919 | 8.102456 | 9.876278 | 9.878011 | 9.103683 | 9.878029 |
| 9.625991 | 10.157304 | 9.723257 | 10.263230 | 9.724902 | 10.266126 | 9.729419 |
| 10.107923 | 10.247834 | 10.223430 | 10.366285 | 10.226510 | 10.369444 | 10.226525 |
| 10.35374 | 10.278262 | 10.356354 | 10.392575 | 10.359589 | 10.395830 | 10.395869 |
| 10.267705 | 10.276598 | 10.390111 | 10.399198 | 10.393386 | 10.402479 | 10.394353 |
| 10.275819 | 10.278044 | 10.400859 | 10.400859 | 10.401871 | 10.404146 | 10.405085 |
| 10.277649 | 10.278405 | 10.400706 | 10.401274 | 10.403995 | 10.404564 | 10.396304 |
| 10.278357 | 10.278496 | 10.401236 | 10.401378 | 10.404525 | 10.404667 | 10.428847 |

Figure 2. (Left panel) the curves of eigenvalues on \( \Omega_S \). (Right panel) the curves of eigenvalues on the uniform meshes for \( \Omega_L \).

Figure 3. (Left panel) the curves of eigenvalues on \( \Omega_T \). (Right panel) the curves of eigenvalues on \( \Omega_S \).
Table 8. The second selected eigenvalue on the uniform meshes for $\Omega_T$.

| $h$ | $\lambda = 1$ | $\lambda = 10^2$ | $\lambda = 10^4$ | $\lambda = 10^8$ |
|-----|---------------|-----------------|-----------------|-----------------|
|     | $\gamma_{h,12}$ | $\gamma_{h,12}$ | $\gamma_{h,12}$ | $\gamma_{h,12}$ |
| 11.570888 | 125.905191 | 11.570972 | 125.913754 | 11.570972 | 125.913931 |
| 32.080588 | 80.221812 | 33.033058 | 86.214751 | 33.044201 | 86.287965 |
| 66.683469 | 96.131275 | 69.383473 | 101.790033 | 69.427480 | 101.871605 |
| 89.594800 | 100.151320 | 94.525205 | 105.989428 | 94.605273 | 106.068143 |
| 98.316148 | 101.156335 | 103.875452 | 107.476262 | 107.484310 | 107.497167 |
| 100.679017 | 101.470690 | 107.179404 | 107.380042 | 107.456039 | 107.479767 |
| 101.297213 | 101.486351 | 107.396686 | 107.421208 | 107.472653 | 107.464208 |
| 101.478753 | 101.478753 | 101.478753 | 101.478753 | 101.478753 | 101.478753 |

Table 9. The first selected eigenvalue on the uniform meshes for $\Omega_S$.

| $h$ | $\lambda = 1$ | $\lambda = 10^2$ | $\lambda = 10^4$ | $\lambda = 10^8$ |
|-----|---------------|-----------------|-----------------|-----------------|
|     | $\gamma_{h,13}$ | $\gamma_{h,13}$ | $\gamma_{h,13}$ | $\gamma_{h,13}$ |
| 4.652581 | 8.501969 | 4.652581 | 8.501969 | 4.652581 | 8.501969 |
| 7.891788 | 9.576853 | 7.900475 | 9.589148 | 7.906965 | 9.584616 |
| 9.302245 | 9.806662 | 9.305798 | 9.804567 | 9.305887 | 9.804666 |
| 9.723205 | 9.852614 | 9.723299 | 9.853635 | 9.723324 | 9.853658 |
| 9.834241 | 9.865372 | 9.832677 | 9.865629 | 9.832683 | 9.865635 |
| 9.860286 | 9.868547 | 9.860345 | 9.868612 | 9.860352 | 9.868613 |
| 9.867273 | 9.869340 | 9.867289 | 9.869356 | 9.867290 | 9.869357 |
| 9.869022 | 9.869538 | 9.869026 | 9.869542 | 9.869026 | 9.869543 |
| 9.869459 | 9.869588 | 9.869460 | 9.869589 | 9.869459 | 9.869588 |

Table 10. The second selected eigenvalue on the uniform meshes for $\Omega_S$.

| $h$ | $\lambda = 1$ | $\lambda = 10^2$ | $\lambda = 10^4$ | $\lambda = 10^8$ |
|-----|---------------|-----------------|-----------------|-----------------|
|     | $\gamma_{h,9}$ | $\gamma_{h,9}$ | $\gamma_{h,9}$ | $\gamma_{h,9}$ |
| 9.860459 | 48.000000 | 9.860459 | 48.000000 | 9.860459 | 48.000000 |
| 20.827963 | 34.848012 | 20.925542 | 35.111955 | 20.927973 | 35.118549 |
| 32.472581 | 38.380191 | 32.522585 | 38.445672 | 32.523776 | 38.447321 |
| 37.474579 | 39.206889 | 37.489648 | 39.223346 | 37.490026 | 39.223758 |
| 38.958651 | 39.410714 | 38.962681 | 39.414835 | 38.962782 | 39.414938 |
| 39.347245 | 39.461503 | 39.348269 | 39.462533 | 39.44938 | 39.462559 |
| 39.445546 | 39.474190 | 39.445804 | 39.474447 | 39.445810 | 39.474454 |
| 39.470195 | 39.477361 | 39.470259 | 39.477425 | 39.470261 | 39.477427 |
| 39.476362 | 39.478153 | 39.476378 | 39.478169 | 39.476377 | 39.478169 |

| Trend | $\gamma_{q,10}$ | $\gamma_{q,11}$ | $\gamma_{q,12}$ | $\gamma_{q,13}$ | $\gamma_{q,19}$ | $\gamma_{q,20}$ | $\gamma_{q,29}$ | $\gamma_{q,30}$ | $\gamma_{q,31}$ |
|-----|---------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 39.476362 | 39.476378 | 39.476377 | 39.476316 | 39.476316 | 39.476316 | 39.476316 | 39.476316 | 39.476316 | 39.476316 |
From Tables 7–10, we see that $\gamma_{h,k}$ approximate the exact ones from below. From Tables 8 and 10, we know that the eigenvalue obtained on the coarsest grid cannot be a lower bound for the exact ones, but after correction, it must be the lower bound for eigenvalue. The corrected eigenvalues converge to the exact ones from below, which coincide with our theoretical results. From Figure 3, we can see that curves of the corrected eigenvalues are parallel to that of the uncorrected eigenvalues, and the corrected eigenvalues are locking-free. As $\lambda$ increases, the approximations for eigenvalue of (27) obtained by the nonconforming CR element and the linear conforming finite element are locking-free. Furthermore, from Figure 3, we find that the eigenvalues are abnormal as $\lambda$ increases, so we compute the condition number of the stiffness matrix, and the results are shown in Tables 11 and 12. From Tables 11 and 12, we can see that the condition number increases as $\lambda$ increases, we think it may be because the influence of the condition number and the rounding error.

| $\lambda$ | 1  | $10^3$ | $10^5$ | $10^8$ | $10^{10}$ |
|-----------|----|--------|--------|--------|----------|
| $\text{cond}_2(\text{CR})$ | $9.817 \times 10^6$ | $3.312 \times 10^9$ | $3.305 \times 10^{11}$ | $3.321 \times 10^{14}$ | $3.060 \times 10^{16}$ |
| $\text{cond}_2(\text{P1})$ | $1.650 \times 10^6$ | $5.557 \times 10^8$ | $5.498 \times 10^{10}$ | $5.496 \times 10^{13}$ | $5.085 \times 10^{15}$ |

Table 11. The condition number of the stiffness matrix on $\Omega_T$.

| $\lambda$ | 1  | $10^3$ | $10^5$ | $10^8$ | $10^{10}$ |
|-----------|----|--------|--------|--------|----------|
| $\text{cond}_2(\text{CR})$ | $1.004 \times 10^7$ | $3.408 \times 10^9$ | $3.395 \times 10^{11}$ | $3.416 \times 10^{14}$ | $3.179 \times 10^{16}$ |
| $\text{cond}_2(\text{P1})$ | $1.681 \times 10^6$ | $5.708 \times 10^8$ | $5.636 \times 10^{10}$ | $5.618 \times 10^{13}$ | $5.191 \times 10^{15}$ |

Table 12. The condition number of the stiffness matrix on $\Omega_S$.

**Remark 3.** In Tables 1–10, we can see that the eigenvalues selected are the guaranteed lower bounds after correction. We take the corrected eigenvalues on the smallest mesh size as the guaranteed lower bounds.

5. Conclusions

Generally, we know that the exact eigenvalues of the planar linear elastic eigenvalue problem are unknown, according to the min–max principle we can obtain upper bounds for eigenvalues by using conforming finite element. But it is generally more difficult to obtain lower bounds for the numerical eigenvalues. Noticing that the lower and upper bounds can produce intervals to which exact eigenvalue belongs. This is important for the design of the coefficient of safety in practical engineering. Therefore, we apply a locking-free nonconforming CR element to the planar linear elastic eigenvalue problem, and obtain the guaranteed lower bounds for eigenvalues.

In this paper, for the planar linear elastic eigenvalue problem with the pure displacement and the pure traction boundary conditions in two spatial dimensions, we prove that using the nonconforming CR element can obtain the guaranteed lower bounds for exact eigenvalues (see Theorems 1 and 2), and that is locking-free when the elliptic regularity estimate (11) holds. Besides, the discussion of the guaranteed lower eigenvalue bounds in this paper can be extended to mixed boundary-value problem and three spacial dimensions, and the schemes are locking-free when the elliptic regularity estimate holds, i.e., the constant $C_{\Omega}$ is independent of $\mu$ and $\lambda$. Furthermore, it is a meaningful and difficult work to extend to higher order elements.

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Appendix A. The MATLAB Codes for the Planar Linear Elastic Eigenvalue Problem

In our computation, our program is completed under the package of iFEM [46], and the discrete eigenvalue problems are solved in MATLAB 2012a on a DELL inspiron 5480 PC with 8 G memory. The following MATLAB codes are to obtain the guaranteed lower eigenvalue bounds on $\Omega$. For the case of $\Omega_L, \Omega_T$, we need change the codes of elems and nodes.

Listing A1. MATLAB codes for the pure displacement problem.

```matlab
function [eigv,eigvlower]=Elasticuniforminterioreig(nodeH,elemH)
nodeH=[0 0; 1 0; 1 1; 0 1];
elemH=[2 3 1; 4 1 3];
mu=1;Lbd=10^8;rou=1;
for i=1:8
[nodeH,elemH]=uniformrefine(nodeH,elemH);
end
[nodeH,elemH]=uniformrefine(nodeH,elemH);
figure(1);showmesh(node,elem)
T=auxstructure(elem);
elem2edge=T.elem2edge;edge=T.edge;
N=size(node,1);NE=size(edge,1);NT=size(elem,1);
Ndof=2*NE;elem2dof=[elem2edge elem2edge+NE];
ve1 = node(elem(:,3),:)-node(elem(:,2),:);
ve2 = node(elem(:,1),:)-node(elem(:,3),:);
ve3 = node(elem(:,2),:)-node(elem(:,1),:);

area = 0.5*abs(-ve3(:,1).*ve2(:,2) + ve3(:,2).*ve2(:,1));
Dlambda(1:NT,:,1) = [-ve1(:,2)./(2*area), ve1(:,1)./(2*area)];
Dlambda(1:NT,:,2) = [-ve2(:,2)./(2*area), ve2(:,1)./(2*area)];
Dlambda(1:NT,:,3) = [-ve3(:,2)./(2*area), ve3(:,1)./(2*area)];
[lambda,weight] = quadpts(2); nQuad=length(weight);
phi(:,1)=1-2*lambda(:,1);
phi(:,2)=1-2*lambda(:,2);
phi(:,3)=1-2*lambda(:,3);
Dphi(:,:,1)=-2*Dlambda(:,:,1);
Dphi(:,:,2)=-2*Dlambda(:,:,2);
Dphi(:,:,3)=-2*Dlambda(:,:,3);
A=sparse(Ndof,Ndof);M=sparse(Ndof,Ndof);
uphi1=zeros(nQuad,6);uphi2=uphi1;
Dphi1x=zeros(NT,2,6);Dphi1y=Dphi1x;Dphi2x=Dphi1x;Dphi2y=Dphi1x;
for i=1:6
if i<=3;
uphi1(:,i)=phi(:,i);uphi2(:,i)=0;
Dphi1x(:,1,i)=Dphi(:,1,i);Dphi1y(:,2,i)=Dphi(:,2,i);
Dphi2x(:,1,i)=0;Dphi2y(:,2,i)=0;
else
uphi1(:,i)=0;uphi2(:,i)=phi(:,i-3);
Dphi1x(:,1,i)=0;Dphi1y(:,2,i)=0;
Dphi2x(:,1,i)=Dphi(:,1,i-3);Dphi2y(:,2,i)=Dphi(:,2,i-3);
end
end
```
for i=1:6
for j=i:6
Aij=mu*(Dphi1x(:,1,i).*Dphi1x(:,1,j)+Dphi1y(:,2,i).*Dphi1y(:,2,j)+... 
Dphi2x(:,1,i).*Dphi2x(:,1,j)+Dphi2y(:,2,i).*Dphi2y(:,2,j)).*area;
Bij=(mu+Lbd)*((Dphi1x(:,1,i)+... 
Dphi2y(:,2,i)).*(Dphi1x(:,1,j)+Dphi2y(:,2,j))).*area;
Kij=Aij+Bij;
if (i==j)
A=A+sparse(double(elem2dof(:,i)),double(elem2dof(:,j)),Kij,Ndof,Ndof);
else
A=A+sparse([double(elem2dof(:,i));... 
double(elem2dof(:,j))],[double(elem2dof(:,j));... 
double(elem2dof(:,i))],Kij,Kij,Ndof,Ndof);
end
end
end
for i=1:6
for j=i:6
Mij=0;
for p=1:nQuad
Mij=Mij+weight(p)*(dot(uphi1(p,i),uphi1(p,j),2)+... 
dot(uphi2(p,i),uphi2(p,j),2));
end
Mij=rou*Mij.*area;
if (i==j)
M=M+sparse(double(elem2dof(:,i)),double(elem2dof(:,j)),Mij,Ndof,Ndof);
else
M=M+sparse([double(elem2dof(:,i));... 
double(elem2dof(:,j))],[Mij,Mij],Ndof,Ndof);
end
end
end
bdEage=setboundary(node,elem,'Dirichlet');
bdedgejudge=false(NE,1);
bdedgejudge(elem2edge(bdEage==1))=1;
inedge=find(bdedgejudge~=1);
bdedge=find(bdedgejudge==1);
bdedge=[bdedge;bdedge+NE];
A(bdedge,:)=[];A(:,bdedge)=[];
M(bdedge,:)=[];M(:,bdedge)=[];
[eigf,eigv]=eigs(A,M,8,'sm');
eigv=sort(diag(eigv))
i=6;
eigvi=eigv(i)
hc=sqrt(ve1(:,1).^2+ve1(:,2).^2);
ha=sqrt(ve2(:,1).^2+ve2(:,2).^2);
hb=sqrt(ve3(:,1).^2+ve3(:,2).^2);
hh=max([hc,ha,hb]); h=max(hh);
Ch=0.1893*h;
eigvlower=eigvi/(1+eigvi*(1/mu)*(Ch^2));
Listing A2. MATLAB® codes for the pure traction problem.

function [eigv1,eigvlower]=Elastictractionuniformeig(nodeH,elemH)
nodeH=[0 0;1 0;1 1;0 1];
elemH=[2 3 1;4 1 3];
mu=1;Lbd=10^8;rou=1;
node=nodeH;elem=elemH;
for i=1:9
[node,elem]=uniformrefine(node,elem);
end
figure(1);showmesh(node,elem)
T=auxstructure(elem);
elem2edge=T.elem2edge;
edge=T.edge;
N=size(node,1);
NE=size(edge,1);NT=size(elem,1);
Ndof=2*NE;
elem2dof=[elem2edge edge+NE];
ve1 = node(elem(:,3),:)-node(elem(:,2),:);
ve2 = node(elem(:,1),:)-node(elem(:,3),:);
ve3 = node(elem(:,2),:)-node(elem(:,1),:);
area = 0.5*abs(-ve3(:,1).*ve2(:,2) + ve3(:,2).*ve2(:,1));
Dlambda(1:NT,:,1) = [-ve1(:,2)./(2*area), ve1(:,1)./(2*area)];
Dlambda(1:NT,:,2) = [-ve2(:,2)./(2*area), ve2(:,1)./(2*area)];
Dlambda(1:NT,:,3) = [-ve3(:,2)./(2*area), ve3(:,1)./(2*area)];
[lambda,weight] = quadpts(2); nQuad=length(weight);
phi(:,1)=1-2*lambda(:,1);
phi(:,2)=1-2*lambda(:,2);
phi(:,3)=1-2*lambda(:,3);
Dphi(:,1)=(-2)*Dlambda(:,1);
Dphi(:,2)=(-2)*Dlambda(:,2);
Dphi(:,3)=(-2)*Dlambda(:,3);
A=sparse(Ndof,Ndof);M=sparse(Ndof,Ndof);
uphi1=zeros(nQuad,6);uphi2=uphi1;
Dphi1x=zeros(NT,2,6);Dphi1y=Dphi1x;Dphi2x=Dphi1x;Dphi2y=Dphi1x;
for i=1:6
if i<=3;
uphi1(:,i)=phi(:,i);uphi2(:,i)=0;
Dphi1x(:,1,i)=Dphi(:,1,i);Dphi1y(:,2,i)=Dphi(:,2,i);
Dphi2x(:,i,1)=0;Dphi2y(:,i,2)=0;
else
uphi1(:,i)=0;uphi2(:,i)=phi(:,i-3);
Dphi1x(:,1,i)=0;Dphi1y(:,2,i)=0;
Dphi2x(:,1,i)=Dphi(:,1,i-3);Dphi2y(:,2,i)=Dphi(:,2,i-3);
end
end
for i=1:6
for j=i:6
Aij=mu*(Dphi1x(:,1,i).*Dphi1x(:,1,j)+Dphi1y(:,2,i).*Dphi1y(:,2,j)+
Dphi2x(:,1,i).*Dphi2x(:,1,j)+Dphi2y(:,2,i).*Dphi2y(:,2,j)).*area;
Bij=(mu+Lbd)*((Dphi1x(:,1,i)+Dphi2y(:,2,i)).*(Dphi1x(:,1,j)+Dphi2y(:,2,j))).*area;
Kij=Aij+Bij;
if (i==j)
A=A+sparse(double(elem2dof(:,i)),double(elem2dof(:,j)),Kij,Ndof,Ndof);
else
end
\[
A = A + \text{sparse}([\text{double}(\text{elem2dof}(:,i));...
\text{double}(\text{elem2dof}(:,j))], [\text{double}(\text{elem2dof}(:,j));...
\text{double}(\text{elem2dof}(:,i))], [\text{Kij}, \text{Kij}], \text{Ndof}, \text{Ndof});
\]

end

end

end

for i = 1:6
    for j = i:6
        \[M_{ij} = 0;\]
        for p = 1:nQuad
            \[M_{ij} = M_{ij} + \text{weight}(p) * (\text{dot}(\text{uphi1}(p,i), \text{uphi1}(p,j), 2) + ...
            \text{dot}(\text{uphi2}(p,i), \text{uphi2}(p,j), 2));\]
        end
        \[M_{ij} = \text{rou} * M_{ij} * \text{area};\]
        if (i == j)
            \[M = M + \text{sparse}((\text{double}(\text{elem2dof}(:,i)));
            \text{double}(\text{elem2dof}(:,j)), [\text{Mij}, \text{Mij}], \text{Ndof}, \text{Ndof});\]
        else
            \[M = M + \text{sparse}([\text{double}(\text{elem2dof}(:,i));...
            \text{double}(\text{elem2dof}(:,j))], [\text{double}(\text{elem2dof}(:,j));...
            \text{double}(\text{elem2dof}(:,i))], [\text{Mij}, \text{Mij}], \text{Ndof}, \text{Ndof});\]
        end
    end
end

A = A + M;
\[cA = \text{condest}(A, 2);\]
\[cB = \text{condest}(M, 2);\]
\[[\text{eigf}, \text{eigv}] = \text{eigs}(A, M, 12, 'sm');\]
\[\text{eigv} = \text{sort}(\text{diag}(\text{eigv})\]
\[i = 12;\]
\[\text{eigvi} = \text{eigv}(i);\]
\[\text{eigv3} = \text{eigvi} - 1; \text{eigv1} = \text{eigv}(1);\]
\[\text{eigf} = \text{eigf}(:, 3);\]
\[\text{hc} = \text{sqrt}((\text{ve1}(:, 1).^2 + \text{ve1}(:, 2).^2));\]
\[\text{ha} = \text{sqrt}((\text{ve2}(:, 1).^2 + \text{ve2}(:, 2).^2));\]
\[\text{hb} = \text{sqrt}((\text{ve3}(:, 1).^2 + \text{ve3}(:, 2).^2));\]
\[\text{hh} = \text{max}([\text{hc}, \text{ha}, \text{hb}]); \text{h} = \text{max}(\text{hh});\]
\[\text{Ch} = (\text{1/sqrt(\text{eigvi})} + 1) \times 0.1893 \times \text{h} \times (\text{1/sqrt(\text{mu})});\]
\[\text{eigvlower} = \text{eigvi} / (1 + \text{eigvi} * (1/\text{mu}) * (\text{Ch}^{-2}));\]
\[\text{eigvlower} = \text{eigvlower} - 1;\]

In Listings A1 and A2, those subprograms \([\text{nodeH, elemH}] = \text{uniformrefine} (\text{nodeH, elemH})\) (It divides each triangle into four small similar triangles); \(\text{T = auxstructure (elem)}\) (It constructs the indices map between elements, edges and nodes, and the boundary information); \([\text{lambda, weight}] = \text{quadpts} (2)\) (It returns quadrature points with given order (up to nine) in the barycentric coordinates); \(\text{bdEage = setboundary (node, elem,'Dirichlet')}\) (It sets the type of boundary edges) comes from the package of iFEM [46].
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