A precise definition of the Standard Model

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Abstract

To write the $SU_c(3) \times SU_L(2) \times U(1) \times SU_f(3)$ Standard Model in Minkowski space-time in a precise way, we assume that a special space-time such as the $SU_c(3) \times SU_L(2) \times U(1) \times SU_f(3)$ Minkowski space-time exists to begin with. Thus, the scalar fields $\Phi(1, 2)$ (the Standard-Model Higgs), $\Phi(3, 1)$ (the purely family Higgs), and $\Phi(3, 2)$ (the mixed family Higgs), with the first family label and the second $SU_L(2)$ label, all pre-exist with all gauge bosons, each with well-defined group assignment and the "purpose" (of making a certain gauge boson massive). Moreover, this space-time turns out to support the lepton world, and it also supports the quark world. In this language, all the various gauge bosons are born with the special gauge-group Minkowski space-time. (Or, the Minkowski space-time is further characterized by the force fields, or the gauge group.) This may be the most efficient and clearest way to spell out the Standard Model.

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1 Prelude

The entries in the "$SU_c(3) \times SU_L(2) \times U(1) \times SU_f(3)$ Minkowski space-time" mean the objects that have definitive properties both under the force group $SU_c(3) \times SU_L(2) \times U(1) \times SU_f(3)$ and under the Lorentz group defining Minkowski space-time. The Standard Model defined in this way would be called "the Standard Model in the $SU_c(3) \times SU_L(2) \times U(1) \times SU_f(3)$ Minkowski space-time". In this paper, we caution the mathematical rigor in the terminology.

The adjective of "the gauge group" means "the force fields"; it means that the force fields, or the gauge fields, exist simultaneously with the Minkowski space-time.

What precisely is the difference between "the $SU_c(3) \times SU_L(2) \times U(1) \times SU_f(3)$ Minkowski space-time" and just "Minkowski space-time"? In the former, every object must have the designated-group assignments while, in the latter, it could be any 4-dimensional object in the Minkowski space-time. Is the adjective necessary in this context? We propose "yes" in the strictly mathematical sense.

We have to do all these with extra care, since "the Standard Model" is the beginning of everything. There exist basically no rules in doing things.

Besides the force fields or gauge fields, most of the gauge fields, upon spontaneous symmetry breaking (SSB), become massive, including weak bosons and family gauge bosons,
if the standard wisdom is assumed. That calls for the Standard-Model (SM) Higgs $\Phi(1, 2)$, the purely family Higgs $\Phi(3, 1)$, and the mixed family Higgs $\Phi(3, 2)$ - they interact "strongly" whenever possible, i.e., between $\Phi(1, 2)$ and $\Phi(3, 2)$, and between $\Phi(3, 1)$ and $\Phi(3, 2)$. (See below about the two numbers/labels.) Thus, these things provide the "background" of everything else. That may be what we mean by the $SU_c(3) \times SU_L(2) \times U(1) \times SU_f(3)$ Minkowski space-time.

The next question associated with the Higgs fields is to understand "the origin of mass" - a question that we have recently gained some understanding [1]. In that [1], we may set all the mass terms of the various Higgs to identically zero, except one spontaneous-symmetry-breaking (SSB) igniting term. All the mass terms are the results of this SSB, when switched on. Therefore, the "mass" is the result of SSB - a generalized Higgs mechanism. Thus, when the temperature is higher than a certain critical temperature, notion of "mass" does not exist.

The set of the "various" Higgs includes the Standard-Model (SM) Higgs $\Phi(1, 2)$, the mixed family Higgs $\Phi(3, 2)$, and the pure family Higgs $\Phi(3, 1)$, where the first label refers to the group $SU_f(3)$ while the second the group $SU_L(2)$. The ignition could be on the pure family Higgs $\Phi(3, 1)$ [1], and it is clear that that may not be not necessarily on the SM Higgs $\Phi(1, 2)$.

These related Higgs, being the scalar fields, act as the systems of energies, self-interacting via dimensionless $\lambda(\phi^\dagger \phi)^2$ and interacting equivalently with other Higgs. When the temperature is low enough, it becomes the "mass" phase, or the phase in which the particles have masses.

## 2 The Lepton World and the Quark World

The $SU_c(3) \times SU_L(2) \times U(1) \times SU_f(3)$ Minkowski space-time supports the lepton world, and also supports the quark world, to the least.

It is interesting to propose [2] $((\nu_\tau, \tau)_L, (\nu_\mu, \mu)_L, (\nu_e, e)_L) \ (columns) \ (\equiv \Psi(3, 2))$ as the $SU_f(3)$ triplet and $SU_L(2)$ doublet. It is also essential to complete the (extended) Standard Model [3] by working out the Higgs dynamics in detail [1]. Here it is important to realize the role of neutrino oscillations - it is the change of a neutrino in one generation (flavor) into that in another generation; or, we need to have the coupling $i\eta \bar{\Psi}_L(3, 2) \times \Psi_R(3, 1) \cdot \Phi(3, 2)$, exactly the coupling introduced by Hwang and Yan [2]. Then, it is clear that the mixed family Higgs $\Phi(3, 2)$ must be there. The remaining purely family Higgs $\Phi(3, 1)$ helps to complete the picture, so that the eight gauge bosons are massive in the $SU_f(3)$ family gauge theory [4].

With a complete Standard Model such as [3], we could address a few basic questions. After all, all "building blocks of matter" seem to be point-like particles (point-like Dirac particles if fermions), and vice versa. And nothing more. If quantum field theory (QFT) can describe the Nature, it should mean more - such as the various ultraviolet divergences, do they cancel out in some way? See below on tests for a "complete" theory.

Usually in an old textbook [5], the QCD chapter precedes the one on Glashow-Weinberg-Salam (GWS) electroweak theory, but we are talking about the $SU_c(3) \times SU_L(2) \times U(1) \times SU_f(3)$ Minkowski space-time and what happens in it. The so-called "basic units" are made
up from quarks (of six flavors, of three colors, and of the two helicities) and leptons (of three generations and of the two helicities).

If we look at the basic units as compared to the original particle, i.e. the electron, the starting basic units are all "point-like" Dirac particles. Dirac invented Dirac electrons eighty years ago and surprisingly enough these "point-like" Dirac particles are the basic units of the Standard Model. Thus, we call it "Dirac Similarity Principle" - a salute to Dirac; a triumph to mathematics. Our world could indeed be described by the proper mathematics. The proper mathematical language may be the renormalizable quantum field theory, as advocated in this paper.

For the lepton world or the quark world, the story is fixed if the so-called "gauge-invariant derivative", i.e. $D^\mu$ in the kinetic-energy term $-\bar{\Psi} \gamma^\mu D_\mu \Psi$, is given for a given basic unit, one on one [5].

For the lepton world, we introduce the family triplet, $(\nu^R_\tau, \nu^R_\mu, \nu^R_e)$ (column), under $SU_f(3)$. Since the minimal Standard Model does not see the right-handed neutrinos, it would be a natural way to make an extension of the minimal Standard Model. Or, we have, for $(\nu^R_\tau, \nu^R_\mu, \nu^R_e)$,

$$D_\mu = \partial_\mu - i\kappa \lambda^a \frac{1}{2} F^a_\mu. \quad (1)$$

and, for the left-handed $SU_f(3)$-triplet and $SU_L(2)$-doublet ($(\nu^L_\tau, \tau^L), (\nu^L_\mu, \mu^L), (\nu^L_e, e^L)$) (all columns),

$$D_\mu = \partial_\mu - i\kappa \lambda^a \frac{1}{2} F^a_\mu - ig \tilde{A}_\mu + i\frac{g'}{2} B_\mu. \quad (2)$$

The right-handed charged leptons form the triplet $\Psi_R^C(3,1)$ under $SU_f(3)$, since it were singlets their common factor $\bar{\Psi}_L(3,2) \Psi_R(1,1) \Phi(3,2)$ for the mass terms would involve the cross terms such as $\mu \rightarrow e$.

The neutrino mass term assumes a new form [2]:

$$\frac{i \eta}{2} \bar{\Psi}_L(3,2) \times \Psi_R(3,1) \cdot \Phi(3,2) + h.c., \quad (3)$$

where $\Psi(3,i)$ are the neutrino triplet just mentioned above (with the first label for $SU_f(3)$ and the second for $SU_L(2)$). The cross (curl) product is somewhat new [4], referring to the singlet combination of three triplets in $SU(3)$. The Higgs field $\Phi(3,2)$ is new in this effort, because it carries some nontrivial $SU_L(2)$ charge.

Note that, for charged leptons, the Standard-Model choice is $\Psi^\dagger_R(3,2) \Psi_C^R(3,1) \Phi(1,2) + c.c.$, which gives three leptons an equal mass. But, in view of that if $(\phi_1, \phi_2)$ is an $SU(2)$ doublet then $(\phi_2^\dagger, -\phi_1^\dagger)$ is another doublet, we could form $\bar{\Phi}^\dagger(3,2)$ from the doublet-triplet $\Phi(3,2)$.

So, we have [3]

$$\frac{i \eta^C}{2} \bar{\Psi}_L(3,2) \times \Psi_R^C(3,1) \cdot \bar{\Phi}^\dagger(3,2) + h.c., \quad (4)$$

which gives rise to the imaginary off-diagonal (hermitian) elements in the $3 \times 3$ mass matrix, so removing the equal masses of the charged leptons. Note that $\eta \sim \eta^C \sim \kappa$, up to the sign or a factor of two.
The fact that an $SU(2)$ doublet out of two complex scalar fields could form another doublet has been used to generate the upper-type quark - thus, we assume that it is legitimate to do the same for a complicated object $\Phi(3,2)$; we note that this may be illegal for the basic unit such as $\Psi_L(3,2)$ since certain entities associated with the basic units are supposed to appear once and only once. This is tricky in mathematics.

The expressions for neutrino oscillations and the off-diagonal mass term are in $i\epsilon_{abc}$, or curl-dot, product - it is allowed for $SU(3)$. Note that such coupling has nothing to do with the kinetic-energy term of the particle, though the coupling $h$ (and $h^c$) is related to the gauge coupling $\kappa$.

We now turn our attention to the quark world, which our special gauge-group Minkowski space-time supports. Thus, we have, for the up-type right-handed quarks $u_R$, $c_R$, and $t_R$,

$$\mathcal{D}_\mu = \partial_\mu - ig_c \frac{\lambda^a}{2} G^a_\mu - i \frac{2}{3} g' B_\mu,$$

(5)

and, for the rotated down-type right-handed quarks $d'_R$, $s'_R$, and $b'_R$,

$$\mathcal{D}_\mu = \partial_\mu - ig_c \frac{\lambda^a}{2} G^a_\mu - i (\frac{1}{3}) g' B_\mu.$$

(6)

On the other hand, we have, for the $SU_L(2)$ quark doublets,

$$\mathcal{D}_\mu = \partial_\mu - ig_c \frac{\lambda^a}{2} G^a_\mu - i g \frac{\tau}{2} \cdot A_\mu - i \frac{1}{6} g' B_\mu.$$

(7)

These are the standard way to generate mass for quarks and for leptons. For quarks, we use the "old-fashion" way as in the minimal Standard Model. For charged leptons, the old-fashion way gets modified in view of $SU_f(3)$. For neutrinos, the family group takes over the role of generating the masses, in accord with the observed neutrino oscillations.

The $SU_c(3) \times SU_L(2) \times U(1) \times SU_f(3)$ Minkowski space-time supports the lepton world and also supports the quark world. Under $SU_c(3) \times SU_L(2) \times U(1) \times SU_f(3)$, each object has its own group assignments. If the way of "supporting" turns out to be a little "ugly" (like in the quark world), our suggestion is to look deeper to find another one, since, after all, there are many elegant stories associated with the Standard Model.

### 3 The Higgs Sector

We should present our reasonings which lead to the formulation of "The Origin of Mass" \[\text{[1]}\]. It stresses that, before the spontaneous symmetry breaking (SSB), the Standard Model does not contain any parameter that is pertaining to "mass", but, after the SSB, all particles in the Standard Model acquire the mass terms as it should - a way to explain "the origin of mass". In this way, we sort of tie "the origin of mass" to the effects of the SSB, or the generalized Higgs mechanism.

It is amusing to observe that it is so easy, by construction, to have the scalar fields but, among the building blocks of matter, the scalar fields are so rare. It is much harder for two scalar fields in co-existence, as though they are mutually "repulsive". Only if they belong to the "same" family, they would be mutually attractive.
In the 4-dimensional Minkowski space-time, it is an amusing fact for the complex scalar field that the dimensionless interaction $\lambda(\phi^\dagger\phi)^2$ exists - we don’t know how to determine the dimensionless $\lambda$; this might have to do with the 4-dimensional nature and maybe more. The determination of $\lambda$, that should be done a priori in the Standard Model, poses an important conceptional question.

The reason that we try to write together a force-field Minkowski space-time is that when put together the $SU_c(3) \times SU_L(2) \times U(1) \times SU_f(3)$ Minkowski space-time (that is already specific enough) the complex scalar field $\phi$ in this space should have a specific $\lambda$ in the dimensionless interaction $\lambda(\phi^\dagger\phi)^2$. If we agree that a specific $\lambda$ is needed, then there is the universal $\lambda$ for the various complex scalar fields allowed in this force-field Minkowski space-time. The complex scalar field(s) should have the existence a priori. [These arguments sound fairly philosophical and logical, but they are needed for clarification.]

A complex scalar field in our space-time has the dimensionless coupling:

$$L(x) = \lambda(\phi^\dagger(x)\phi(x))^2.$$  

(8)

The space-time integral of it gives the action. In our 4-dimensional Minkowski space-time, we find that $\lambda = \frac{1}{8}$ numerically. This number should come out topologically (after the normalizations of the various fields in a given space [5]), although, at this point, we don’t know why this is the case.

If there are more than a complex scalar field, we should have

$$L(x) = \lambda\{(\phi_a^\dagger\phi_a)^2 + (\phi_b^\dagger\phi_b)^2 + \ldots\}.$$  

(9)

There should be only one $\lambda$.

For the unrelated complex fields, we propose to write

$$L(x) = \lambda(\phi_a^\dagger\phi_a + \phi_b^\dagger\phi_b + \ldots)^2,$$  

(10)

to signify the universal repulsive interactions among them, since we do not see much of them together ("minimum Higgs hypothesis").

Now we return to our generalized Standard Model [3]. We have the Standard-Model Higgs $\Phi(1,2)$, the purely family Higgs $\Phi(3,1)$, and the mixed family Higgs $\Phi(3,2)$, with the first label for $SU_f(3)$ and the second for $SU_L(2)$. We need another triplet $\Phi(3,1)$ since all eight family gauge bosons are massive [4]. Note that, under $SU_c(3) \times SU_L(2) \times U(1) \times SU_f(3)$ Minkowski space-time, the Higgs fields all have their own group assignments.

It is clear that $\Phi(1,2)$ would interact with $\Phi(3,2)$ while $\Phi(3,1)$ would also interact with $\Phi(3,2)$. These interactions are presumably attractive to explain why they are showing up together. Thus, we have [4]

$$V_{Higgs} = \mu_2^2\Phi^\dagger(\bar{\Phi})(\Phi(3,1) + \lambda(\Phi^\dagger(1,2)\Phi(1,2) + \cos\theta_P\Phi^\dagger(3,2)\Phi(3,2))^2
$$
$$+ \lambda(-4\cos\theta_P)(\Phi^\dagger(3,2)(\Phi^\dagger(1,2)))(\Phi^\dagger(1,2)\Phi(3,2))
$$
$$+ \lambda(\Phi^\dagger(3,1)\Phi(3,1) + \sin\theta_P\Phi^\dagger(\bar{\Phi})(\Phi(3,2))^2 + \lambda(-4\sin\theta_P)(\Phi^\dagger(3,2)(\Phi(3,1))(\Phi^\dagger(3,1)\Phi(3,2))
$$
$$+ \lambda_2^2\Phi^\dagger(3,1)\Phi(3,1)\Phi^\dagger(1,2)\Phi(1,2) + (terms \ in \ i\delta's \ and \ in \ decay).$$  

(11)
Here \( \lambda'_2 \) (describing the interaction between \( \Phi(1, 2) \) and \( \Phi(3, 1) \)) would be vanishingly small. As said earlier [1], the positive definitiveness serves naturally as the guideline in fixing up the lagrangian.

On the other hand, we may write down the terms for potentials among the three Higgs fields, subject to (1) that they are renormalizable, and (2) that symmetries are only broken spontaneously (the Higgs or induced Higgs mechanism). We have [3, 5]

\[
V = V_{SM} + V_1 + V_2 + V_3, \quad (12)
\]

\[
V_{SM} = \mu^2 \Phi^\dagger(1, 2) \Phi(1, 2) + \lambda (\Phi^\dagger(1, 2) \Phi(1, 2))^2 \quad (13)
\]

\[
V_1 = M^2 \Phi^\dagger(3, 2) \Phi(3, 2) + \lambda_1 (\Phi^\dagger(3, 2) \Phi(3, 2))^2
+ \epsilon_1 (\Phi^\dagger(3, 2) \Phi(3, 2))(\Phi^\dagger(1, 2) \Phi(1, 2)) + \eta_1 (\Phi^\dagger(3, 2) \Phi(1, 2))(\Phi^\dagger(1, 2) \Phi(3, 2))
+ \epsilon_2 (\Phi^\dagger(3, 2) \Phi(3, 2))(\Phi^\dagger(3, 1) \Phi(3, 1)) + \eta_2 (\Phi^\dagger(3, 2) \Phi(3, 1))(\Phi^\dagger(3, 1) \Phi(3, 2))
+ (\delta_1 \Phi^\dagger(3, 2) \times \Phi(3, 2) \cdot \Phi^\dagger(3, 1) + h.c.), \quad (14)
\]

\[
V_2 = \mu_2^2 \Phi^\dagger(3, 1) \Phi(3, 1) + \lambda_2 (\Phi^\dagger(3, 1) \Phi(3, 1))^2
+ (\delta_2 \Phi^\dagger(3, 1) \cdot \Phi(3, 1) \times \Phi^\dagger(3, 1) + h.c.)
+ \lambda'_2 \Phi^\dagger(3, 1) \Phi(3, 1) \Phi^\dagger(1, 2) \Phi(1, 2), \quad (15)
\]

\[
V_3 = (\delta_3 \Phi^\dagger(3, 2) \cdot \Phi(3, 2) \times (\Phi^\dagger(1, 2) \Phi(3, 2)) + h.c.)
+ (\delta_4 \Phi^\dagger(3, 2) \Phi(1, 2) \cdot \Phi^\dagger(3, 1) \times \Phi(3, 1) + h.c.)
+ \eta_3 (\Phi^\dagger(3, 2) \Phi(1, 2) \Phi(3, 1) + c.c.). \quad (16)
\]

In doing the renormalization analysis of the three Higgs fields, we realize that even if we start from the well-motivated lagrangian such as Eq. (11), it will spill over to the more generalized lagrangian such as Eqs. (12)-(16).

On the other hand, we should pay special attention to the so-called "U-gauge" (unitary gauge). In the U-gauge, every particle is a real particle (not a ghost). We find it to be useful in the analysis of the situation with the spontaneous symmetry breaking (SSB) with dimensional regularization. For the \( SU_f(3) \times SU_L(2) \times U(1) \times SU_f(3) \) Minkowski space-time in the U-gauge, we have \( W^\pm, Z^0 \), and eight massive family gauge bosons, one Standard-Model Higgs and four neutral family Higgs (three mixed plus one pure).

Thus, we choose to have, in the U-gauge, as in [1],

\[
\Phi(1, 2) = (0, \begin{pmatrix} v + \eta \end{pmatrix}), \quad \Phi^0(3, 2) = \begin{pmatrix} u_1 + \eta'_1, u_2 + \eta'_2, u_3 + \eta'_3 \end{pmatrix}, \quad \Phi(3, 1) = \begin{pmatrix} \frac{1}{\sqrt{2}} (w + \eta', 0, 0) \end{pmatrix}, \quad (17)
\]

all in columns. The five components of the complex triplet \( \Phi(3, 1) \) get absorbed by the \( SU_f(3) \) family gauge bosons and the neutral part of \( \Phi(3, 2) \) has three real parts left - together making all eight family gauge bosons massive.

Before the mixing, the masses of the various Higgs are given by, for Eqs. (12)-(16),

\[
\eta : \quad (\mu^2/\lambda) + \frac{1}{4}(\epsilon_1 + \eta_1)u_1u_4 + \frac{\lambda'}{4}v^2,
\]

\[
\eta' : \quad (\mu'_2/\lambda_2) + \frac{\lambda'}{4}u_4u_1 + \frac{2\lambda'}{4}v_1^2 + \frac{\lambda'}{4}v^2,
\]

\[
\eta'_1 : \quad M^2 + \frac{1}{4}(\epsilon_1 + \eta_1)v_2^2 + \frac{1}{4}(\epsilon_2 + \eta_2)w^2 + (\lambda_1 - \text{term}),
\]

\[
\eta'_2 : \quad M^2 + \frac{1}{4}(\epsilon_1 + \eta_1)v_2^2 + \frac{1}{4}(\epsilon_2 + \eta_2)w^2 + (\lambda_1 - \text{term}),
\]

\[
\eta'_3 : \quad M^2 + \frac{1}{4}(\epsilon_1 + \eta_1)v_2^2 + \frac{1}{4}(\epsilon_2 + \eta_2)w^2 + (\lambda_1 - \text{term}),
\]

\[
\eta'_4 : \quad M^2 + \frac{1}{4}(\epsilon_1 + \eta_1)v_2^2 + \frac{1}{4}(\epsilon_2 + \eta_2)w^2 + (\lambda_1 - \text{term}).
\]
\( \eta_{1,2,3}' : \quad M^2 + \frac{1}{2}(\epsilon_1 + \eta_1)v^2 + \frac{4}{9}w^2 + (\lambda_1 - \text{term}), \)
\( \phi_1 : \quad M^2 + \frac{1}{2}\epsilon_1 v^2 + \frac{1}{2}\epsilon_2 w^2 + \frac{1}{2}\eta_2 w^2 + \frac{\lambda}{2}u_1 u_i, \)
\( \phi_{2,3} : \quad M^2 + \frac{1}{2}\epsilon_1 v^2 + \frac{1}{2}\epsilon_2 w^2 + \frac{\lambda}{2}u_1 u_i. \)  
(18)

The mixing term looks like, apart from some common factor:
\[
2(\epsilon_1 + \eta_1)u_{1i}' v \eta + 2\epsilon_2 u_{1i} w \eta' + 2\eta_2 u_{1i} w \eta' + 2\lambda_2 w \eta' v \eta. \tag{19}
\]

And we also neglect the mixing (and the mixing inside \( \eta_{1,2,3}' \)). To understand the origin of mass, we would drop out all “mass” terms to begin with.

To understand the origin of mass, we find that the ignition term would better be in the purely family sector, i.e., the \( \eta \) term. When \( \mu^2 = 0 \), the \( \Phi(3, 2) \) is equally partitioned between \( \Phi(1, 2) \) and \( \Phi(3, 1) \) (i.e., \( \theta_P = 45^\circ \) in Eq. (11)).

It is easy to see that only one SSB-driving term is enough for all the three Higgs fields—there may be several SSBs for the neutral fields—in our case, it works for all of them. SSB for one Higgs but is driven by other Higgs—a unique feature for the complex scalar fields.

As for the SSB-driving term, we decide to keep the purely family term, \( \mu^2 \Phi^\dagger(3, 1)\Phi(3, 1) \). We don’t know if the symmetry breaking would occur much earlier (in the history of the early Universe), or at a much higher temperature, mainly because the channel for \( \eta' \) is rather elusive.

Further simplification associated with Eq. (11) leads to [1]:
\[
v^2(3\cos^2 \theta_P - 1) = \sin \theta_P \cos \theta_P w^2. \tag{20}
\]

And the SSB-driven \( \eta' \) yields
\[
w^2(1 - 2\sin^2 \theta_P) = -\frac{\mu^2}{\lambda} + (\sin 2\theta_P - \tan \theta_P)v^2. \tag{21}
\]

These two equations show that it is necessary to have the driving term, since \( \mu^2 = 0 \) implies that everything is zero. Also, \( \theta = 45^\circ \) is the (lower) limit.

The mass squared of the SM Higgs \( \eta \) is \( 2\lambda \cos \theta_P u_{1i} u_i \) (noting the factor of two), as known to be (125 \( \text{GeV} \))^2. The famous \( v^2 \) is the number divided by \( 2\lambda \), or (125 \( \text{GeV} \))^2/(2\lambda). Using PDG’s for \( e, \sin^2 \theta_W \), and the W-mass [10], we find \( v^2 = 255 \text{ GeV} \). So, we set \( \lambda = \frac{1}{8} \), a simple model indeed.

The mass squared of \( \eta' \) is \(-2(\mu^2 - \sin \theta_P u_{1i}^2 + \sin \theta_P (u_{2i}^2 + u_{3i}^2)) \). The other condensates are \( u_{1i}^2 = \cos \theta_P v^2 + \sin \theta_P w^2 \) and \( u_{2,3}^2 = \cos \theta_P v^2 - \sin \theta_P w^2 \) while the mass squared of \( \eta_1' \) is \( u_{2,3}^2 \lambda \), those of \( \eta_{2,3}' \) be \( u_{2,3}^2 \lambda \). The mixings among \( \eta_1' \) themselves are neglected in the paper.

There is no SSB for the charged Higgs \( \Phi^\dagger(3, 2) \). The mass squared of \( \phi_1 \) is \( \lambda(\cos \theta_P v^2 - \sin \theta_P w^2) + \frac{\lambda}{2} u_{1i} u_i \) while \( \phi_{2,3} \) be \( \lambda(\cos \theta_P v^2 + \sin \theta_P w^2) + \frac{\lambda}{2} u_{1i} u_i \).

A further look of these equations tells that \( 3\cos^2 \theta_P - 1 > 0 \) and \( 2\sin^2 \theta_P - 1 > 0 \). A narrow range of \( \theta_P \) is allowed (greater than \( 45^\circ \) while less than \( 57.4^\circ \), which is determined by the group structure). For illustration, let us choose \( \cos \theta_0 = 0.6 \) and work out the numbers as follows: (Note that \( \lambda = \frac{1}{8} \) is used.)
\[
6w^2 = v^2, \quad -\frac{\mu^2}{\lambda} = 0.32v^2; \\
\eta : \quad m^2(\eta) = (125 \text{ GeV})^2, \quad v^2 = (250 \text{ GeV})^2; \\
\]
\begin{align*}
\eta' : & \quad m^2(\eta') = (51.03 \text{ GeV})^2, \quad w^2 = v^2/6; \\
\eta'_1 : & \quad m^2(\eta'_1) = (107 \text{ GeV})^2, \quad u_1^2 = 0.7333v^2; \\
\eta'_{2,3} : & \quad m^2(\eta'_{2,3}) = (85.4 \text{ GeV})^2, \quad u_{2,3} = 0.4667v^2; \\
\phi_1 : & \quad \text{mass} = 100.8 \text{ GeV}; \\
\phi_{2,3} : & \quad \text{mass} = 110.6 \text{ GeV}.
\end{align*}

(22)

All numbers appear to be reasonable. Since the new objects need to be accessed in the lepton world, it would be a challenge for our experimental colleagues.

As for the range of validity, \( \frac{1}{3} \leq \cos^2 \theta_P \leq \frac{1}{2} \). The first limit refers to \( w^2 = 0 \) while the second for \( \mu_2^2 = 0 \).

We may fix up the various couplings, using our common senses. The cross-dot products would be similar to \( \kappa \), the basic coupling of the family gauge bosons. The electroweak coupling \( g \) is 0.6300 while the strong QCD coupling \( g_s = 3.545 \) (order of unity); my first guess for \( \kappa \) would be about 0.1 (which is rather small). The masses of the family gauge bosons would be estimated by using \( \frac{1}{2} \kappa \cdot w \), so slightly less than 10 GeV. (In the numerical example with \( \cos \theta_P = 0.6 \), we have \( 6w^2 = v^2 \) or \( w = 102 \text{ GeV} \). This gives \( m = 5 \text{ GeV} \) as the estimate.) So, the range of the family forces, existing in the lepton world, would be 0.04 fermi.

In [1], the term that ignites the SSB is chosen to be with \( \eta' \), the purely family Higgs. This in turn ignites EW SSB and others. It explains the origin of all the masses, in terms of the spontaneous symmetry breaking (SSB). SSB in \( \Phi(3,2) \) is driven by \( \Phi(3,1) \), while SSB in \( \Phi(1,2) \) from the driven SSB by \( \Phi(3,2) \), as well. The different, but related, scalar fields can accomplish so much, to our surprise.

We note that, at the Lagrangian level, the \( SU_c(3) \times SU_L(2) \times U(1) \times SU_f(3) \) gauge symmetry is protected but the symmetry is violated via spontaneous symmetry breaking (via the Higgs mechanisms).

We iterate that the mathematics of the three neutral Higgs, \( \Phi(1,2) \) (standard Higgs), \( \Phi(3,1) \) (purely family Higgs), and \( \Phi^0(3,2) \), subject to the renormalizabilty (up to the fourth power), turns to be rather rich. In our earlier work regarding the "colored Higgs mechanism" [6], we show how the eight gauge bosons in the \( SU(3) \) gauge theory become massive using two complex scalar triplet fields (with the resultant four real Higgs fields), with a lot of choices. We suspect that, even within QCD, there might be some elegant choice of "colored" Higgs, or there must be a good reason for massless gluons.

## 4 Tests for a complete theory

With a theory in the \( SU_c(3) \times SU_L(2) \times U(1) \times SU_f(3) \) Minkowski space-time, we could ask, for example, whether ultraviolet divergences of the quadratic order of the self-energy diagrams of, e.g., the SM Higgs could be summed up to zero (or some finite number). A lot of questions like this deserve the answers since it would determine if quantum field theory is the proper language or not. The burden of being the proper language is rather heavy, indeed.

Here we could only begin the analysis of "one" such question while leaving the heavy burdens to the others. In the textbook, e.g., Ch. 10 of [5] on the ultraviolet divergences in...
QED, the divergences are there, but QED is only part of the theory, including the leading-order calculation related to \( g - 2 \). We propose that many issues could be examined in a theory in the \( SU_c(3) \times SU_L(2) \times U(1) \times SU_f(3) \) Minkowski space-time. It makes sense to examine whether the theory is complete or not.

Maybe we should study the ”completeness” question in light of the Standard Model. First of all, the ”building blocks of matter” seems to be complete. Next, all the fermions are ”point-like” Dirac particles. The Standard Model provides a consist and complete ”mathematical” system, that we can pursue after.

For example, although one electron self-energy diagram shown in Ch. 10 of [5] is infinite, this is true in QED but QED is part of the Standard Model. Maybe all of the self-energy diagrams of similar type may add up to a finite number. The fact that the QED-part is infinite is the symptom of that QED is an incomplete theory. In a complete theory, we have everything and nothing more - if there is still some infinities, then there is something more; this is a way, the hard way, to find something new.

In fact, we would like to show that the U-gauge and the dimensional regularization offer us the machinery to handle these infinities. Staying in the U-gauge means that we are treating ”physical particles”. Dimensional regularization means that the results in all dimensions are included, including finite results in fractional dimensions.

We follow the details of [1] in discussing the ultraviolet divergences of the quadratic order of the self-energy of the SM Higgs:

![Figure 1: The within-Higgs diagrams for the Standard-Model Higgs Φ(1, 2).](image)

In Fig. 1, the wave-function renormalization of the Standard-Model Higgs Φ(1, 2) is shown, for simplicity, in the U-gauge in the absence of Dirac fermions. The lowest-order loop diagrams, from the above interaction lagrangian, are shown from 1(b) [in \( \lambda \)] to 1(g) [in \( \eta_3 \)], where the first five are of quadratic divergence while the last one of logarithmic divergence. The higher-order connected loop diagrams, many of them and also of quadratic divergence, are also troublesome and should be dealt with at the same time. We will discuss the worst divergent, i.e., the quadratic divergent, cases in what follows.
Using dimensional regularization (i.e. the appendix of Ch. 10, the Wu-Hwang book, Ref. 3), we could write down the one-loop results:

We try to use one explicit example to illustrate our point related to the infinities - the quadratic divergences of the wave function of the SM Higgs \( \eta \). We may draw the diagrams for the wave-function renormalizations for the Standard-Model Higgs, the mixed family Higgs, and purely family-triplet Higgs, respectively. The result for the Standard-Model Higgs is shown by Fig. 1. Here, in Fig. 1, we try to show only the Higgs sectors themselves; in a complete Standard Model, the (Dirac) fermion loop diagrams, and those with gauge bosons, also present divergences of quadratic order and should be dealt with simultaneously.

As said earlier, we know that the formulae in dimensional regularization work and that it works in the U-gauge. For example, the \( Z^0 - \text{boson} \) loop for Fig. 1 would give us the vanishing result - so, it does not bother us.

In details, the coupling of the SM Higgs is (\[5\], e.g., Wu/Hwang, Ch. 13)

\[
- \frac{1}{8} (v^2 + 2v \eta + \eta^2) \{ 2g_\mu^+ W_\mu^- + |g|^2 + (g')^2 \} Z_\mu^0 Z_\mu^0, \tag{23}
\]

which gives rise to, to the first order, the one-loop \( W^\pm \) or \( Z^0 \) diagram. To evaluate them, we use the propagator in the U-gauge (see the appendix of Ch. 13) and the formulae in the dimensional regularization (see the appendix of Ch. 10). They cancel between two terms for each diagram.

We proceed to examine those diagrams in Fig. 1 which are "simple" quadratically divergent - those at the one-loop order. These are among the various Higgs.

In Fig. 1, we show the wave-function renormalization of the Standard-Model Higgs \( \Phi(1,2) \), among the Higgs, in the U-gauge. The lowest-order loop diagrams, from the above interaction lagrangian, are shown from 1(b) [in \( \lambda \)] to 1(g) [in \( \eta_3 \)], where the first five are of quadratic divergence while the last one of logarithmic divergence. The higher-order connected loop diagrams, many of them and of quadratic divergence multiplied by logarithmic divergences, are also troublesome.

The one-loop diagrams involving the quark (or charged lepton), when simplified, are sums of quadratic and logarithmic divergences.

Using dimensional regularization (i.e. the appendix of Ch. 10, the Wu-Hwang book, \[5\]), we obtain the one-loop and quadratic-divergence results as follows. In the dimensional regularzation, the factor \( \Gamma(1 - \frac{n}{2}) \) stands for where the quadratic divergence appears. Maybe the fractional dimensions, which are represented as finite numbers, could get some meaning, but we have to remember that, as a drawback, we bypass the \(-ie\) in the propagators.

\[
-4 \cdot \frac{n}{2} \cdot (S_q + S_{\text{c.l.}}) \Gamma(1 - \frac{n}{2}) \\
+ \{ 3\lambda m^2(\eta) + \frac{\eta}{2} \sum_i m^2(\eta_i) + c_1 \sum_i m^2(\phi_i) \\
+ \frac{\eta'}{2} m^2(\eta') + \eta \sum_i m^2(\eta_i) \} \Gamma(1 - \frac{n}{2}) \equiv 0; \tag{24}
\]

\[
S_q = \sum_{\text{quarks}} 3 \cdot G^2_1 \cdot (m^2_i - \frac{1}{3} m^2(\eta)), \\
S_{\text{c.l.}} = \sum_{\text{c.l.}} G^2_1 \cdot (m^2_i - \frac{1}{3} m^2(\eta)). \tag{25}
\]

Or, using the Standard Model, we have

\[
-4 \cdot \frac{n}{2} \cdot (S_q + S_{\text{c.l.}}) \Gamma(1 - \frac{n}{2})
\]

10
\[ + \{ \lambda (3m^2(\eta) - \cos \theta P \sum_i m_i^2(\eta_i') + 2 \cos \theta P \sum_i m_i^2(\phi_i)) + \frac{\lambda'}{2} m^2(\eta') \} \Gamma(1 - \frac{n}{2}) \equiv 0. \] (26)

Here we "reinforce" "\(\equiv 0\)" for the sake of a "complete" theory.

There are a few general characteristics: (1) In the contributions from quarks and from charged leptons, the mass of the SM Higgs enters (as external momentum squared). This makes all contributions equivalent in some sense. (2) We assume that the quarks enter in the theory in a SM way - if we examine the theory closely, there might still be some colored-Higgs sector \[6\]. In other words, we are not sure that the "identity" in this SM Higgs would hold out; instead, the identity in the case of pure Higgs \(\Phi(3,1)\), or that for \(\Phi(3,2)\), has better reasons to hold out.

Here the coefficients of \(\Gamma(1 - \frac{n}{2})\) are the coefficients of quadratic divergences while those of \(\Gamma(2 - \frac{n}{2})\) are the coefficients of logarithmic divergences - for the latter, divergence is less severe and the contributions could be everywhere; the treatment is far more complicated.

As mentioned in \[1\], the diagrams which are of multiple quadratic divergence are troublesome since the series could be blown up, of \(2n\)-th divergence with \(n \to \infty\). Mathematically, we should avoid such terms by all means. This is the requirement beyond the naive renormalizability.

We are hinting that we have to study the mathematics of divergences; they are there, because of the uncountably infinite degrees of freedom and other reasons, and there are regularities to be discovered \[7\]. So, is the Standard Model a complete theory?

The result for \(\Phi(1,2)\) in the one-loop result certainly gives rise to an "approximate" formula for the masses and the couplings - they are "approximate" because the cancelation could be modified by terms in the higher-loop orders. But it should be approximate - that is why it is worthwhile looking for them.

According to dimensional-regularization results, the three-loop diagram gives rise to (quadratic divergence) \(\times\) (logarithmic divergence), and so on. They have to organized differently. One simple way out is that they cancel completely in their own group, such as all the four-loop diagrams. In any event, these divergences could be systematically analyzed to display what is going on.

If we believe that all particles are there, then the cancelation conjecture should hold, first for the simple quadratic divergences of a certain Higgs, and other divergences. And for other Higgs, and other particles. The game would be "completed" in some way.

5 Other theoretical physics issues

We think that it is the most important to get a clear picture of infinities. In the previous section, we only begin the discussions by considering the ultraviolet divergences of quadratic order in the one-loop renormalization of the SM Higgs - the real game is far more complicated and far beyond the author’s personal ability to handle. The Standard Model as a "complete" theory, if verified, means that at least quantum field theory is the language for describing the physics, especially about the Standard Model of particle physics. That is one of the dreams that the author and others have as theorists.

Why do we need the "ignition" channel? Why is the ignition channel not the SM Higgs \(\eta\)? We set out to use the SM Higgs as the "ignition" channel, but soon realized that the
"ignition" channel with the purely family $\eta'$ works out "perfectly". In fact, the first version of $^1$ was based on that the ignition channel was the SM Higgs $\eta$ - the standard wisdom. As for why we need the "ignition" channel, we still need a good answer.

Note that $\theta_P = 45^\circ$ corresponds to the situation that $\Phi(3_2, 2)$ be equally divided by $\Phi(1_2, 2)$ (SM Higgs) and $\Phi(3_1, 1)$ (purely family). The situation corresponds to that it is not yet "ignited" ($\mu_2^2 = 0$). How do these translate into the temperature situations (via the Big Bang or via big inflation)? The elusive purely family Higgs $\eta'$ as the "ignition" channel gives us a few interesting questions.

6 Some footnote remarks

To close this paper, we append a few remarks just to remind ourselves the leading physical issues that we may pursue after - even though most of which have been said elsewhere. For a new theory, we might have to emphasize things again when necessary.

To verify this Standard Model is the experimental search for the family Higgs $\eta'_1$, or $\eta'_{2,3}$, or charged family Higgs $\phi^+_1$ and $\phi^+_2$, or pure family Higgs $\eta'$, in a proposed 120 GeV $\mu^+e^-$ collider $^3$.

The major implication of the family gauge theory is in fact a multi-GeV or sub-sub-fermi gauge theory (a new force field of a few $10^{-15}$ cm in the range), assuming that the ordering in the coupling constants, $g_W/g_c \sim \kappa/g_W$, is reasonable. Note that the lepton world are shielded from this $SU_f(3)$ theory against the QED Landau’s ghost, and similarly the quark world from strong-interaction $SU_c(3)$. The $g - 2$ anomaly certainly deserves another serious look in this context $^9$.

In the Standard Model, the masses of quarks are diagonal, or the singlets in the $SU_f(3)$ space, those of the three charged leptons are $m_0 + a\lambda_2 + b\lambda_5 + c\lambda_7$ (before diagonalization) and the masses of neutrinos are purely off-diagonal, i.e. $a'\lambda_2 + b'\lambda_5 + c'\lambda_7$. This result is very interesting and very intriguing. This is opposite to our intuition that the quark masses be off-diagonal while the lepton masses be diagonal (i.e. they can be observed).

This result follows from the above curl-dot product, or, the $\epsilon^{abc}\bar{\Psi}_{L,a}\Psi_{R,b}\Phi_c$ product, i.e. the $SU_f(3)$ operation, in writing the coupling(s) to the right-handed lepton triplets. In fact, we have $a'/a^* = b'/b^* = c'/c^*$ for the coupling strengths.

In addition, neutrinos oscillate among themselves, giving rise to a lepton-flavor-violating interaction (LFV). There are other oscillation stories, such as the oscillation in the $K^0 - \bar{K}^0$ system, but there is a fundamental "intrinsic" difference here - the $K^0 - \bar{K}^0$ system is composite while neutrinos are "point-like" Dirac particles. We have standard Feymann diagrams for the kaon oscillations but similar diagrams do not exist for point-like neutrino oscillations - our Standard Model solves the problem, maybe in a unique way.

Thinking it through, it is true that neutrino masses and neutrino oscillations may be regarded as one of the most important experimental facts over the last thirty years $^10$.

In fact, certain LFV processes such as $\mu \rightarrow e + \gamma$ $^10$, $\mu + A \rightarrow A^* + e$, etc., are closely related to the most cited picture of neutrino oscillations $^10$. In our previous publications $^11$, it was pointed out that the cross-generation or off-diagonal neutrino-Higgs interaction may serve as the detailed mechanism of neutrino oscillations, with some vacuum expectation values of the family Higgs $\Phi^0(3, 2)$. So, even though we haven’t seen, directly, the family
gauge bosons and family Higgs particles, we already see the manifestations of their vacuum expectation values.

Under the Standard Model in the $SU_c(3) \times SU_L(2) \times U(1) \times SU_f(3)$ Minkowski spacetime, we can close the Universe; that is, all the dark-matter particles and all the ordinary-matter particles are accounted for. Our Standard Model provides a description of the entire matter world - the 25% dark-matter world and the 5% ordinary-matter world.

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