BIVARIANT HERMITIAN K-THEORY AND KAROUBI’S FUNDAMENTAL THEOREM

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Abstract. Let \( \ell \) be a commutative ring with involution \( * \) containing an element \( \lambda \) such that \( \lambda + \lambda^* = 1 \) and let \( \text{Alg}_\ell \) be the category of \( \ell \)-algebras equipped with a semilinear involution and involution preserving homomorphisms. We construct a triangulated category \( \text{kk}^h \) and a functor \( j^h: \text{Alg}_\ell^* \rightarrow \text{kk}^h \) that is homotopy invariant, matricially and hermitian stable and excisive and is universal initial with these properties. We prove that a version of Karoubi’s fundamental theorem holds in \( \text{kk}^h \). By the universal property of the latter, this implies that any functor \( H: \text{Alg}_\ell^* \rightarrow T \) with values in a triangulated category which is homotopy invariant, matricially and hermitian stable and excisive satisfies the fundamental theorem. We also prove a bivariant version of Karoubi’s 12-term exact sequence.

1. Introduction

Let \( \ell \) be a commutative ring with involution \( * \). Assume that \( \ell \) contains an element \( \lambda \) such that
\[
\lambda + \lambda^* = 1.
\]
A \(*\)-algebra over \( \ell \) is an algebra \( A \) equipped with a semilinear involution \( *: A \rightarrow A^{\text{op}} \). Write \( \text{Alg}_\ell \) for the category of \( \ell \)-algebras and \( \text{Alg}_\ell^* \) for the subcategory of \(*\)-algebras and involution preserving homomorphisms. We construct a triangulated category \( \text{kk}^h \), with the same objects as \( \text{Alg}_\ell^* \), and a functor \( j^h: \text{Alg}_\ell^* \rightarrow \text{kk}^h \) which is the identity on objects and is homotopy invariant, matricially and hermitian stable and excisive (all these terms are defined below), and is universal initial with these properties (see Proposition 6.2.7). We write \([-1]\) for the suspension functor, and consider the bivariant hermitian \( K \)-theory groups
\[
\text{kk}^h_n(A,B) := \text{hom}_{\text{kk}^h}(j^h(A),j^h(B)[n]), \quad \text{kk}^h(A,B) = \text{kk}^h_0(A,B).
\]
Setting \( A = \ell \) we recover a Weibel style \([16]\), homotopy invariant version of \( K^h \), the \( K \)-theory of hermitian forms of \([9]\). We prove in Proposition 8.1 that
\[
\text{kk}^h_n(\ell,A) = \text{KH}^h_n(A) \quad (n \in \mathbb{Z}).
\]
The triangulated category \( \text{kk}^h \) is related to the bivariant \( K \)-theory category \( \text{kk} \) of \([5]\) by means of a pair of functors
\[
\text{res}: \text{kk}^h \rightleftarrows \text{kk}: \text{Ind}
\]
which are both left and right adjoint to each other (Proposition 9.2). There is a \(*\)-algebra \( \Lambda \) such that for \( \Lambda A = \Lambda \otimes_\ell A \) we have \( j^h \circ \Lambda = \text{Ind} \circ \text{res} \circ j^h \). Usual homotopy \( K \)-theory is recovered from \( KH^h \) via
\[
K\Lambda_n(A) = KH^h_n(\Lambda A).
\]
Under the isomorphisms (1.2) and (1.4), the unit and counit maps
\[
\eta_A \in \text{kk}^h_1(A,\Lambda A) \quad \text{and} \quad \phi_A \in \text{kk}^h_0(\Lambda A, A)
\]
correspond, respectively, to the forgetful and the hyperbolic maps. There are \(*\)-algebras \( U,V \in \text{Alg}_\ell^* \) such that for \( UA = U \otimes_\ell A \) and \( VA = V \otimes_\ell A \) we have natural
transformations $UA \to \Lambda A$ and $j^b(VA) \to j^b(A)$ which fit into distinguished triangles

$$j^b(UA) \to j^b(\Lambda A) \to j^b(A) \to j^b(UA)[-1]$$

$$j^b(VA) \to j^b(A) \to j^b(\Lambda A) \to j^b(VA)[-1].$$

In particular, for $\mathcal{V}_n(A) := \ker_h^b(\ell, VA)$ and $\mathcal{V}_n(A) := \ker_h^b(\ell, UA)$, we have long exact sequences

$$\mathcal{V}_n(A) \to KH_n(A) \to \ker_h^b(A) \to \mathcal{V}_{n-1}(A)$$

$$\mathcal{V}_n(A) \to KH_n(A) \to \ker_h^b(A) \to \mathcal{V}_{n-1}(A).$$

An invertible element $u$ in a unital *-algebra is unitary if $u^* = u^{-1}$. For unitary $\epsilon \in \ell$ let $_{\epsilon}M_2$ be the algebra of $2 \times 2$-matrices with coefficients in $\ell$ equipped with the involution

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} d^* & eb^* \\ c^* & a^* \end{bmatrix}$$

For $A \in \operatorname{Alg}_\ell^*$, we write $_{\epsilon}M_2A = _{\epsilon}M_2 \otimes _{\ell} A$. A main result of this article is Corollary 11.3, which establishes the following.

**Theorem 1.5.** Assume that $\ell$ contains an element $\lambda$ satisfying (1.1). Then there is a natural isomorphism

$$j^b(_{\epsilon}M_2V(A)) \cong j^b(-_{\epsilon}M_2U(A))[1].$$

Applying $\ker_h^b(\ell, -)$ to the isomorphism of Theorem 1.5 and writing $\mathcal{V}_* \mathcal{H}$ and $\mathcal{V}_* \mathcal{H}$ for $KH_*$ of $_{\epsilon}M_2U$ and $_{\epsilon}M_2V$, we obtain

$$\mathcal{V}_* \mathcal{H}(A) = -_{\epsilon}V_\mathcal{H}_{*+1}(A).$$

We remark that because $j^b$ is hermitian stable and we are assuming that $\ell$ has an element as in (1.1), we have a natural isomorphism

$$j^b(A) \cong j^b(_{1}M_2A).$$

Hence $\mathcal{V}_* \mathcal{H} = _1\mathcal{V}_* \mathcal{H}$. We regard Theorem 1.5 as a bivariant, homotopy invariant version of Karoubi’s fundamental theorem [9]. The latter establishes an equivalence

$$(1.7) \quad _{\epsilon}V(R) \xrightarrow{\sim} \Omega_{-\kappa} \mathcal{V}(R)$$

between the fiber of the forgetful map $_{\epsilon}K^b(R) \to K(R)$ from $\epsilon$-hermitian forms of a unital *-algebra $R$, and the loops of the fiber of the hyperbolic map $K(R) \to -_{\epsilon}K^b(R)$. By Lemma 3.10, we have

$$K^b_{\epsilon}(R) = K^b_{\epsilon}(_{1}M_2R).$$

Hence we may regard (1.6) as a version of (1.7) for homotopy hermitian $K$-theory and Theorem 11.2 as a bivariant version. Karoubi uses (1.7) to establish a 12 term exact sequence connecting Witt and co-Witt groups with the $\mathbb{Z}/2\mathbb{Z}$ Tate cohomology of $K$-theory under the action of the involution. We also prove a bivariant version of Karoubi’s sequence. The algebra $\Lambda$ is equipped with an involutive automorphism
t. The latter induces an action of $\mathbb{Z}/2\mathbb{Z}$ on $kk^h(A, \Lambda B)$. Put
\[
\langle kk^h_n(A, B) = kk^h_n(A, \Lambda_2 B) \rangle
\]
\[
\langle W_n(A, B) = \text{Coker}(\langle kk^h_n(A, \Lambda B) \rangle \to \langle kk^h_n(A, B) \rangle) \rangle
\]
\[
\langle W'_n(A, B) = \text{Ker}(\langle kk^h_n(A, \Lambda B) \rangle \to \langle kk^h_n(A, B) \rangle) \rangle
\]
\[
k_n(A, B) = \{x \in kk^h_n(A, \Lambda B) : x = t_\cdot x\}/\{x = y + t_\cdot y\}
\]
\[
k'_n(A, B) = \{x \in kk^h_n(A, \Lambda B) : x = -t_\cdot x\}/\{x = y - t_\cdot y\}.
\]

**Theorem 1.8** (cf. [9, Théorème 4.3]). Assume that $\ell$ contains an element $\lambda$ as in (1.1). Let $A, B \in \text{Alg}_\ell^*$ and let $n \in \mathbb{Z}$. There is an exact sequence
\[
k_n(A, B) \to W_{n+1}(A, B) \to W'_{n-1}(A, B) \to k'_n(A, B) \to W'_n(A, B) \to W_n(A, B)
\]
\[
\text{By Remark 12.4, Theorem 1.8 follows from Theorem 12.2.}
\]

The rest of this article is organized as follows. In Section 2 we recall some basic facts and fix notations and vocabulary about involutions and (semi-split) $*$-algebra extensions. We recall how an invertible hermitian element $\Phi$ in a unital $*$-algebra $R$ induces a new involution on $R$ and on any $*$-ideal $A \subset R$, given by the adjoint with respect to the form $(x, y) \mapsto x^*\Phi y$. We write $A^\Phi$ for $A$ equipped with this new involution. For example, $\mathcal{M}_2$ is $\mathcal{M}_2$ with the involution associated to the $\epsilon$-hermitian hyperbolic form. We write $\Lambda_\ell$ for $\mathcal{M}_2$ equipped with the involution associated to the $1$-hermitian element diag$(1, -1)$. We give the name “$\lambda$-assumption” (2.1.14) to the hypothesis that $\ell$ contains an element satisfying (1.1). Under this assumption, $\Lambda_\ell \cong \mathcal{M}_2$. We also introduce the concept of semi-split extension as a $*$-algebra extension which admits a splitting in a fixed underlying category such as sets, sets with involution, $\ell$-modules (with involution). We call a $*$-algebra extension
\[
A \longrightarrow B \longrightarrow C
\]
semi-split if $F(p)$ admits a right inverse. In Subsection 2.4 we prove several technical lemmas (Lemmas 2.4.1, 2.4.2 and 2.4.3) concerning stability with respect to a corner inclusion in the $*$-algebra $M_X$ of finitely supported matrices indexed by a set $X$, which we call $M_X$-stability. We also define the concept of hermitian stability for a functor $H : \text{Alg}_\ell^* \to \mathfrak{C}$. We say that $H$ is hermitian stable if whenever $R$ is a unital $*$-algebra and $\Phi, \Psi \in R$ are invertible hermitian elements of the same parity $\epsilon$ then for every $*$-ideal $A \subset R$, $H$ maps the upper left hand corner inclusion $A^\Phi \to (M_2 A)^{\Phi \oplus \Psi}$ to an isomorphism. We show in Proposition 2.4.4 that if $\ell$ satisfies the $\lambda$-assumption and $H : \text{Alg}_\ell^* \to \mathfrak{C}$ is a matricially stable functor, then $F \circ M_\lambda$ is hermitian stable. Section 3 is concerned with hermitian $K$-theory. We write $K^h$ for the hermitian $K$-theory used by Karoubi in [9] and $K^h_V$ for the variant due to Karoubi and Villamayor [12]; we also introduce homotopy hermitian $K$-theory $KH^h$. We present $KH^h$ both as the homotopy groups of the total spectrum of a simplicial spectrum, and by means of an algebraic construction, as done in [16] and [3] for usual homotopy algebraic $K$-theory. We explain that these two versions are equivalent. Lemma 3.8 establishes the basic properties of $KH^h$, including excision. We also consider the comparison maps $K^h \to KV^h \to KH^h$. In particular we prove in Lemma 3.9 that if 2 is invertible in $\ell$ and $A$ is a $K$-regular $*$-algebra which satisfies excision in $K$-theory, then $K^h(A) \to KH^h(A)$ is an isomorphism. In Section 4 we use our algebraic description in $KH^h$ to equip it with a cup product and show that
the latter is compatible with that of $K^h$ via the comparison map (Lemma 4.5). Beginning in Section 5 and for the rest of the paper we assume that $\ell$ satisfies the $\lambda$-assumption 2.1.14. This section introduces, for an infinite set $X$—which will be fixed for the rest of the article—a $\mathbb{Z}$-linear category $\{\text{Alg}_k^*\} = \{\text{Alg}_k^*\}_X$ and a functor $\text{Alg}_k^* \to \{\text{Alg}_k^*\}$ which is homotopy invariant, $M_X$-stable and hermitian stable and is universal initial for these properties (Lemmas 5.11 and 5.12). This section also contains a useful technical lemma (Lemma 5.4) which says that a not necessarily involution preserving homotopy between inner $*$-endomorphisms can be made to preserve involutions upon stabilization. Section 6 introduces bivariant hermitian $K$-theory. For a fixed choice of infinite set $X$ and of forgetful functor $F : \text{Alg}_k^* \to \mathcal{U}$, we introduce a triangulated category $kk^h$ and a functor $j^h : \text{Alg}_k^* \to kk^h$ which is universal among functors to a triangulated category which are homotopy invariant, $M_X$-stable, hermitian stable, and satisfy excision for those $*$-algebra extensions which are semi-split with respect to $F$ in the sense explained above (Proposition 6.2.7). The category $kk^h$ is also equipped with a tensor product of homomorphisms

\begin{equation}
\otimes : kk^h(A_1, A_2) \times kk^h(B_1, B_2) \to kk^h(A_1 \otimes_\ell B_1, A_2 \otimes_\ell B_2),
\end{equation}

defined whenever $A_1 \otimes_\ell$ and $A_2 \otimes_\ell$ (or $\otimes_\ell B_1$ and $\otimes_\ell B_2$) preserve semi-split extensions (Lemma 6.2.10). In Section 7 we consider, for a pair of split $*$-algebra monomorphisms $C \to A$ and $C \to B$, the coproduct $A \coprod C B$ and the amalgamated sum $A \oplus C B$. There is an obvious quotient map $\pi : A \coprod C B \to A \oplus C B$ and we show in Proposition 7.1 that $j^h(\pi)$ is an isomorphism. The proof follows the lines of the analogue result for $j : \text{Alg}_k^* \to kk^h$ proved in [5, Theorem 7.1.1], except that there are certain homotopies that need to be adapted to preserve the involutions; for this we use the technical Lemma 5.4 mentioned above. Section 8 uses the results of the previous sections and the argument of [5, Theorem 8.2.1] to establish the isomorphism (1.2). Under this isomorphism, the product defined in Section 4 for $KH^h$ is the particular case of (1.9) when $A_1 = B_1 = \ell$. The functors in (1.3) are the subject of Section 9. We prove in Proposition 9.2 that they are both right and left adjoint to each other. The $*$-algebras $U$ and $V$ are introduced in Section 10. They are $kk^h$-isomorphic to a desuspension of similar algebras $U^\ell$ and $V^\ell$ introduced by Karoubi in [9] (see Remark 10.1). The main result of this section is Lemma 10.7, which, using the adjunction established in the previous section, establishes an isomorphism

\begin{equation}
j^h(UV)[-1] \cong j^h(\ell).
\end{equation}

Here $UV = U \otimes_\ell V$. Using (1.9) we obtain that the functors $U \otimes_\ell -$ and $V \otimes_\ell -$ : $kk^h \to kk^h$ are category equivalences and that the suspension of each is inverse to the other. Theorem 1.5 is proved in Section 11, as Corollary 11.3. The section starts by recalling an equivalent form of Karoubi’s theorem that says that the cup product with a certain element $\theta_0 \in K^h_b((-1M_2(U^2)))$ induces an isomorphism

\begin{equation}
\theta_0 \ast - : K^h_b(R) \xrightarrow{\cong} K^h_{b+2}((-1M_2U^2R))
\end{equation}

for every unital $*$-algebra $R$. We use Karoubi’s result together with the compatibility between products in $K^h$ and $KH^h$ established in Section 4 to show that the composite $\theta \in kk^h(\ell, -1M_2U^2)$ of the image of $\theta_0$ under the comparison map with the isomorphism $j^h((-1M_2U^2)) \cong j^h((-1M_2U^2))$ induces an isomorphism

\begin{equation}
\theta \ast - : KH^h_b(A) \to KH^h_b((-1M_2U^2A))
\end{equation}

for any $*$-algebra $A$. We use this together with (1.2), (1.9) and (1.10) to show that $\theta$ is an isomorphism (Theorem 11.2). Again applying (1.10), we use the latter theorem to deduce (1.5) in Corollary 11.3. The 12-term exact sequence of Theorem 1.8 is established in Section 12, (Remark 12.4) as a consequence of Theorem 12.2.
The latter combines Karoubi’s argument for the proof of his 12-term exact sequence [9, Théorème 4.3] with properties of the functors Λ, U and V established in Sections 9 and 10.

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2. Preliminaries

2.1. Rings, algebras and involutions. Fix a commutative, unital ring ℓ and an involutive automorphism

\[ * : ℓ \to ℓ, \ x \mapsto x^* \]

A unitary ℓ-bimodule M is called symmetric if \( λx = xλ \) holds for every \( λ \in ℓ \) and \( x \in M \). By an algebra over ℓ we understand a symmetric bimodule A together with an associative, ℓ-linear multiplication \( A \otimes ℓ A \to A \). An involution in an algebra A is an involutive ring homomorphism \( A \to A^{\text{op}} \) that is semilinear with respect to the ℓ-module action; \( (\lambda a)^* = λ^* a^* \) for \( λ \in ℓ \) and \( a \in A \). A *-algebra is an ℓ-algebra equipped with an involution; for \( ℓ = ℤ \) we use the term *-ring. A *-ideal in an *-algebra A is a two-sided ideal closed under the involution. We write \( \text{Alg}_ℓ \) for the category of ℓ-algebras and ℓ-algebra homomorphisms and \( \text{Alg}_ℓ^* \) for the subcategory of *-algebras and involution preserving homomorphisms; we set \( \text{Ring} = \text{Alg}_{ℓ, \text{op}} \) and \( \text{Ring}^* = \text{Alg}_{ℓ, \text{op}}^* \). If A, B are *-algebras then \( A \otimes ℓ B \) is again a *-algebra with involution \( (a \otimes b)^* = a^* \otimes b^* \). If \( L \) is a *-algebra and \( A \in \text{Alg}_ℓ^* \), we shall often write \( LA \) for \( L \otimes ℓ A \).

Example 2.1.1. Let A be a ring; write \( \text{inv}(A) \) for \( A \oplus A^{\text{op}} \) equipped with the involution \( (a, b)^* = (b, a) \). If \( ℓ \) is a commutative ring, then \( \text{inv} : \text{Alg}_ℓ \to \text{Alg}_ℓ^{\text{inv}(ℓ)}, \ A \mapsto \text{inv}(A) \) is a category equivalence, with inverse \( B \mapsto (1,0)B \).

Examples 2.1.2 (Polynomial and matrix rings). The identity map of any commutative ring is an involution, the trivial involution. We shall always regard the polynomial ring \( ℤ[t] \) as a ring with trivial involution and \( ℓ[t] \) with the tensor product involution. For \( λ \in ℓ \), write

\[ \text{ev}_λ : ℓ[t] \to ℓ, \ f \mapsto f(λ) \]

for the evaluation map. Put

\[ P = \text{Ker} \text{ev}_0, \ \Ω = \text{Ker} \text{ev}_0 \cap \text{Ker} \text{ev}_1. \]

We equip the ring \( ℤ[t, t^{-1}] \) of Laurent polynomials with the involution that changes \( t \) and \( t^{-1} \), and \( ℓ[t, t^{-1}] \) with the tensor product involution. For any set \( X \) we put

\[ Γ_X = \{ a : X \times X \to ℓ : |\text{Im}(a)| < ∞ \text{ and } (\exists N)(∀x \in X) \max\{|\text{supp}(a(x, -))|, |\text{supp}(a(-, x))|\} < N\}. \]

Pointwise addition, the convolution or matricial product and transpose involution, given by \( (ab)(x, y) = \sum_z a(x, z)b(z, y) \) and \( a^*(x, y) = a(y, x)^* \) respectively, together make \( Γ_X \) into a ring with involution. This structure is inherited by the *-ideal \( M_X < Γ_X \) of finitely supported functions and by the quotient \( Σ_X = Γ_X/M_X \). If \( |X| = n < ∞ \), then \( M_X = Γ_X \) is *-isomorphic to \( M_n \) with the transpose involution. When \( |X| = ω \), \( Γ_X \) is isomorphic to Karoubi’s cone [12]. We write \( M_∞, Γ \) and Σ for \( M_∞, Γ_∞ \) and \( Σ_∞ \). If \( ℓ \) and X are as above and \( A \in \text{Alg}_ℓ^* \), we write \( A[t], PA, ΩA, A[t, t^{-1}], Γ_X A, M_X A \) and \( Σ_X A \), for the tensor products of \( A \) with \( ℓ[t], P, Ω, ℓ[t, t^{-1}], Γ_X, M_X \) and \( Σ_X \), equipped with the tensor product involution.
Example 2.1.3 (Functions on a simplicial set). For $A \in \text{Alg}_F^*$ and $n \geq 0$, we regard the polynomial algebra in $n$-variables with coefficients in $A$ as $*$-algebra via the isomorphism $A[t_1, \ldots, t_n] = A \otimes \mathbb{Z}[t_1] \otimes \cdots \otimes \mathbb{Z}[t_n]$. In particular, for $n \geq 0$ \( \mathbb{Z}^{\Delta^n} = \mathbb{Z}[t_0, \ldots, t_n]/(t_0 + \cdots + t_n - 1) \) is a $*$-ring and $A^{\Delta^n} = A \otimes \mathbb{Z}^{\Delta^n}$ is a $*$-algebra. Thus we have a simplicial $*$-algebra
(2.1.4)
\[ A^{\Delta^n} : \Delta^n \rightarrow \text{Alg}_F^*, \quad [n] \mapsto A^{\Delta^n}. \]
Write $\mathcal{S}$ for the category of simplicial sets, and $\text{map}_{\mathcal{S}}$ for hom_{\mathcal{S}}. If $X$ is a simplicial set and $B$ is a simplicial $*$-algebra, then $\text{map}_{\mathcal{S}}(X, B)$ is a $*$-algebra. Following [5], if $X \in \mathcal{S}$ and $A \in \text{Alg}_F^*$ we put
(2.1.5)
\[ X^A = \text{map}_{\mathcal{S}}(X, A^\Delta). \]
Let $(X, x)$ be a pointed simplicial set, and let $x : pt = \Delta^0 \rightarrow X$ be the inclusion mapping $0 \mapsto x$; write $e\nu_x$ for the induced $*$-homomorphism $X^A \rightarrow A^{pt} = A$. Put
(2.1.6)
\[ A^{(X, x)} = \text{Ker}(X^A \xrightarrow{e\nu_x} A). \]
Example 2.1.7 (Unitalization). The unitalization $\tilde{A} = \tilde{A}_\ell$ of a $*$-algebra $A$ is the $\ell$-module $A \oplus \ell$ with the following multiplication and involution:
\[ (a, \lambda)(b, \mu) = (ab + \lambda b + a\mu, \lambda \mu), \quad (a, \lambda)^* = (a^*, \lambda^*). \]
Examples 2.1.8 (Hermitian elements and involutions). Let $R$ be a unital $*$-algebra. An invertible element $\epsilon \in R$ is unitary if $\epsilon \epsilon^* = 1$. Let $\epsilon \in R$ be a central unitary; an element $\phi \in R$ is $\epsilon$-hermitian if $\phi^* = \epsilon \phi$. If $\phi$ is $\epsilon$-hermitian and invertible then
(2.1.9)
\[ R \rightarrow R, \quad a \mapsto a^\phi := (\phi)^{-1} a^* \phi \]
is an involution. We write $R^\phi$ for $R$ with the involution (2.1.9). If $u \in R$ is invertible and $\psi = u^* \phi u$, then $\psi$ is $\epsilon$-hermitian again and the inner automorphism $ad(u) : R^\psi \rightarrow R^\phi, x \mapsto uxu^{-1}$ is a $*$-isomorphism. If $S$ is another $*$-algebra and $\psi$ is $\mu$-hermitian, then $\phi \otimes \psi$ is $\epsilon \otimes \mu$-hermitian and
(2.1.10)
\[ (R \otimes \ell S)^{\phi \otimes \psi} = R^\phi \otimes \ell S^\psi. \]
Let $\epsilon \in \ell$ be a central unitary. Consider the following element of $M_2$
\[ h_\epsilon = \begin{bmatrix} 0 & \epsilon \\ 1 & 0 \end{bmatrix}. \]
To abbreviate notation, we write $\epsilon M_2$ for $M_2^{h_\epsilon \ell}$. It follows from (2.1.10) that we have isomorphisms
(2.1.11)
\[ \epsilon M_2 \cong_{\ell} \epsilon M_2 \cong \epsilon M_2 M_2 \]
Let $X$ be a set; put $\epsilon M_2 X = \epsilon M_2 M_2$. Assume that $X$ is infinite. Using (2.1.11) and a bijection $\{1, 2\} \times X \cong X$, we obtain $*$-isomorphisms
(2.1.12)
\[ \epsilon M_2 X, M_2 \cong \epsilon M_2, M_2 M_2 \cong \epsilon M_2 M_2 M_2 \cong \epsilon M_2 M_2 X \cong \epsilon M_2 X. \]
Example 2.1.13. Let $\ell_0$ be a commutative ring and $R_0$ a unital $\ell_0$-algebra. As in Example 2.1.1, consider the $\ell = \text{inv}(\ell_0)$-$*$-algebra $R = \text{inv}(R_0)$. An element $\epsilon \in R$ is a central unitary if and only if $\epsilon = (\epsilon_0, \epsilon_0^{-1})$ for some central invertible element $\epsilon_0 \in R_0$. Any $\epsilon$-hermitian element is of the form $\phi = (\phi_0, \epsilon_0 \phi_0) = (\phi_0, 1)(1, \epsilon_0)(\phi_0, 1)^*$. It follows that $R^\phi \cong R$.

In the article we will often assume that $\ell$ satisfies the following.
\[ \lambda \text{-assumption 2.1.14.} \quad \ell \text{ contains an element } \lambda \text{ such that} \]
(2.1.15)
\[ \lambda + \lambda^* = 1. \]
Example 2.1.16. Hypothesis 2.1.15 is satisfied, for example, if 2 is invertible in \( \ell \), and also if \( \ell = \inv(\ell_0) \) for some commutative ring \( \ell_0 \).

Consider the following element of \( M_2 \)

\[
h_\pm = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\]

For brevity, we write \( M_\pm = M_2^h \pm \) and \( M_\pm^\phi = M_2^\phi \). Let \( \epsilon \in \ell \) be a unitary and \( \phi \in R \) an invertible \( \epsilon \)-hermitian element. If \( \lambda \) is as in (2.1.15), then the element

\[
u = u_\lambda = \begin{bmatrix} 1 & 1 \\ \lambda \phi^* & -\lambda^* \phi^* \end{bmatrix} \in M_2R
\]

satisfies \( u^*(\epsilon \otimes 1)u = h_\pm \otimes \phi \). Hence we have a \( * \)-isomorphism

\[
ad(u) : M_\pm R^\phi \overset{\cong}{\longrightarrow} \epsilon M_2R.
\]

In particular,

\[
M_\pm \cong 1 M_2.
\]

If \( A \triangleright R \) is a \( * \)-ideal for the involution \( x \mapsto x^* \), then it is also an ideal for the involution (2.1.9); we write \( A^\phi \) for \( A \) equipped with the latter involution. The isomorphism (2.1.18) holds with \( A \) substituted for \( R \).

2.2. Extensions. An extension of \( * \)-algebras is a sequence

\[
A \rightarrowtail i \xrightarrow{p} B \rightarrow C
\]

such that \( p \) is surjective and \( i \) is a kernel of \( p \). The extension (2.2.1) is split if there is a \( * \)-homomorphism \( C \rightarrow B \) which is a section of \( p \).

Examples 2.2.2. Let \( A \in \Alg_\ell^* \) and \( X \) a set. Then

\[
PA \xrightarrow{ev_\beta} A \xrightarrow{ev_a} A
\]

\[
\Omega A \xrightarrow{ev_1} PA \xrightarrow{ev_a} A
\]

\[
M_X A \xrightarrow{\Gamma_X A} \Sigma_X A
\]

are extensions. Remark that (2.2.3) is split by the inclusion \( A \subset A[\ell] \), which is a \( * \)-homomorphism. The map \( A \rightarrow PA, a \mapsto at \) is an involution-preserving, \( \ell \)-linear splitting of (2.2.4). By [5, first paragraph of page 92], (2.2.5) admits an \( \ell \)-linear splitting.

2.3. Underlying data and semisplit extensions. A \( * \)-algebra may be viewed as an \( \ell \)-module, an abelian group or a set with involution plus some added structure; one can also put the involution as part of the added structure. Each of these choices of underlying data leads to an underlying category \( \mathfrak{U} \) and a forgetful functor \( F : \Alg_\ell^* \rightarrow \mathfrak{U} \) which admits a left adjoint \( T' : \mathfrak{U} \rightarrow \Alg_\ell^* \). In what follows, we shall fix a choice of \( \mathfrak{U}, F \) and \( T' \). We say that a surjective homomorphism \( f \) of \( * \)-algebras is semi-split if \( F(f) \) admits a right inverse in \( \mathfrak{U} \). An extension (2.2.1) is semi-split if \( p \) is. If \( \ell \) satisfies the \( \lambda \)-assumption 2.1.14, every map in \( \mathfrak{U} \) which is a section of a \( * \)-algebra homomorphism can be averaged to a involution preserving section. Thus a linearly split extension is semisplit for any of the aforementioned choices of underlying data.
2.4. Stability. Let $G_1,G_2 : \text{Alg}_\ell^* \to \text{Alg}_\ell^*$ be functors and $\iota : G_1 \to G_2$ a natural transformation. Let $\mathcal{C}$ be a category, $H : \text{Alg}_\ell^* \to \mathcal{C}$ a functor and $A \in \text{Alg}_\ell^*$. We say that $H$ is $\iota$-stable on $A$ if the natural map $H(\iota_A) : H(G_1(A)) \to H(G_2(A))$ is an isomorphism. We say that $H$ is $\iota$-stable if it is so on every $A \in \text{Alg}_\ell^*$.

We say that $H$ is homotopy invariant if it is stable with respect to the canonical inclusion $\text{inc} : A \to A[1]$. We say that $A \in \text{Alg}_\ell^*$ is $H$-regular if $H$ is inc-stable on $A[[t_1, \ldots, t_n]]$ for every $n \geq 0$.

Let $X$ be a set; if $x, y \in X$, write $e_{x,y}$ for the matrix unit. Consider the natural map $\text{id}_{\text{Alg}_\ell^*} \to M_X$,

$$t_x : A \to M_X A, \quad a \mapsto e_{x,x} \otimes a.$$  

**Lemma 2.4.1.** Let $X$ be a set, $x, y \in X$, $\mathcal{C}$ a category, and $H : \text{Alg}_\ell^* \to \mathcal{C}$ a functor. If $H$ is $i_x$-stable on $A$ and on $M_X A$, then $H(i_x) = H(i_y)$. In particular, $H$ is $i_y$-stable on $A$.

**Proof.** The proof is essentially the same as that of [3, Lemma 2.2.4]. One shows that there are permutation matrices $\sigma_2$ and $\sigma_3$ in $M_{X \times X} = M_X \otimes M_X$ of orders 2 and 3, respectively, both of which conjugate $(i_x \otimes_i M_X) i_x$ into $(i_y \otimes_i M_X) i_y$. Observe that permutation matrices are unitary, so they induce $\ast$-automorphisms. \hfill $\Box$

In view of Lemma 2.4.1, one says that a functor is $M_X$-stable to mean it is $i_x$-stable for some (and then any) element $x \in X$.

**Lemma 2.4.2.** Let $X$ be a set with at least two elements. Let $H : \text{Alg}_\ell^* \to \mathcal{C}$ be an $M_X$-stable functor, $A \subset B \in \text{Alg}_\ell^*$ and $a \in B$ such that $uA, Au^* \subset A, \quad au^* u' = a' u' (a,a' \in A)$.

Assume that $H$ is $M_X$-stable on both $A$ and $M_X A$. Then $\text{ad}(u) : A \to A$, $\text{ad}(u)(a) = u^* u a$ is a $\ast$-homomorphism, and $H(\text{ad}(u)) = \text{id}_{H(A)}$.

**Proof.** Immediate from Lemma 2.4.1 and the argument of [3, Proposition 2.2.6] \hfill $\Box$

**Lemma 2.4.3.** Let $X$ be as in Lemma 2.4.2 and let $Y$ be a set with $|Y| > |X|$. Then any $M_Y$-stable functor $H : \text{Alg}_\ell^* \to \mathcal{C}$ is also $M_X$-stable.

**Proof.** We may assume that $X \subset Y$; write inc : $M_X \subset M_Y$ for the canonical inclusion. Let $x \in X$ and and $i = i_x$. Because $H$ is $M_Y$-stable, $H(\text{inc} \circ i)$ is an isomorphism. Hence $H(i)$ is a split monomorphism and $H(\text{inc})$ is a split epimorphism. Let $\tau : M_X \otimes M_Y \rightarrowtail M_Y \otimes M_X$, $\tau(a \otimes b) = b \otimes a$. We have $\tau \circ (i \otimes M_Y) \circ \text{inc} = \text{inc} \circ i$. Let $\sigma : Y \times X \to Y \times Y$ be any bijection which restricts to coordinate permutation on $X \times \{x\}$; also write $\sigma$ for the corresponding permutation matrix in $M_{Y \times Y} = M_Y \otimes M_Y$. Then $\text{ad}(\sigma) \circ (\text{inc} \circ i) = i \otimes M_X$. Hence $H(\text{inc} \circ i)$ is an isomorphism, and thus $H(\text{inc})$ is a split monomorphism. This finishes the proof. \hfill $\Box$

Let $F : \text{Alg}_\ell^* \to \mathcal{C}$ be a functor and $A \in \text{Alg}_\ell^*$. We say that $F$ is hermitian stable on $A$ if for every $\ast$-ideal embedding $A \subset R$ into a unital $\ast$-algebra $R$, every central unitary $\epsilon \in R$, and any two $\epsilon$-hermitian invertible elements $\phi, \psi \in R$, $F$ sends the upper left hand corner inclusion

$$\iota^\phi : A^\phi \to (M_2 A)^{\phi \oplus \psi}$$  

to an isomorphism.

**Proposition 2.4.4.** Assume that $\ell$ satisfies the $\lambda$-assumption 2.1.14. Let $F : \text{Alg}_\ell^* \to \mathcal{C}$ be a functor and let $A \in \text{Alg}_\ell^*$. If $F$ is $M_2$-stable on $\iota M_2 A$ for every $\ast$-ideal embedding $A \subset R$ into a $\ast$-unital algebra $R$ and central unitary element $\epsilon \in R$, then $F \circ M_\ell$ is hermitian stable on $A$. 


Proof. The isomorphisms $M_{\pm}A^\phi \cong {}_1\epsilon M_2A$ and $M_{\pm}(M_2A)^{\phi \oplus \psi} \cong {}_1\epsilon M_2M_2A$ of (2.1.18) and the permutation isomorphism ${}_1\epsilon M_2M_2 \cong M_2(\epsilon M_2)$ fit into a commutative diagram

\[
\begin{array}{c}
M_{\pm}A^\phi \cong {}_1\epsilon M_2A \\
M_{\pm}(M_2A)^{\phi \oplus \psi} \cong {}_1\epsilon M_2M_2A \\
\end{array}
\]

The proposition is now immediate. \qed

3. HERMITIAN $K$-THEORIES

Let $A$ be a $*$-ring; write

\[ \mathcal{U}(A) = \{ x \in A \mid xx^* = x^*x, \; x + x^* + xx^* = 0 \}. \]

Observe $\mathcal{U}(A)$ is a group under the operation

\[ x \cdot y = x + y + xy. \]

If $A$ is unital then $\mathcal{U}(A)$ is isomorphic to the group of unitary elements of $A$ via the map $a \mapsto 1 + a$.

Let $A \subset R$ be a $*$-ideal embedding into a unital $*$-algebra $R$ and let $\epsilon \in R$ be a central unitary. Put

\[ \epsilon \mathcal{O}(A) = \mathcal{U}(\epsilon M_2M_\infty A). \]

By (2.1.19) and (2.1.12) we have a group isomorphism

\[ \epsilon \mathcal{O}(A) \cong _1\epsilon \mathcal{O}(M_2A). \]

The $\epsilon$-hermitian $K$-theory groups of a unital $*$-ring $A$ are the stable homotopy groups of a spectrum $\epsilon K^hA = \{ \epsilon K^hA_n \}$ whose $n$-th space is $\epsilon K^hA_n = \Omega B(\mathcal{O}(\Sigma^{n+1}A))^+$, the loopspace of the $+$-construction ([13, Section 3.1.6]). As usual we also write

\[ \epsilon K^h_n(A) = \pi_n(\epsilon K^hA), \; (n \in \mathbb{Z}) \]

for the $n$-th stable homotopy group. When $\epsilon = 1$ we drop it from the notation. For a nonunital $*$-ring $A$, we put

\[ \pm_1K^h_n(A) = \text{Ker}(\pm_1K^h_n(\hat{A}_\mathbb{Z}) \rightarrow \pm K^h_n(\mathbb{Z})). \]

If $A$ is unital, these groups agree with those defined above since hermitian $K$-theory of unital $*$-rings is additive. Recall that a ring $A$ is $K$-excisive if for any embedding $A \subset R$ as an ideal of a unital ring $R$ and every unital homomorphism $R \rightarrow S$ mapping $A$ isomorphically onto an ideal of $S$, the map of relative $K$-theory spectra $K(R : A) \rightarrow K(S : A)$ is an equivalence. Now assume that $A$ is a $K$-excisive ring that is a $*$-algebra over some commutative ring $\ell$ with involution, and let $A \subset R$ be a $*$-ideal embedding into a unital $*$-algebra. Let $f : R \rightarrow S$ be a unital $*$-algebra homomorphism mapping $A$ isomorphically onto a $*$-ideal of $S$. Let $\epsilon \in \ell$ be a central unitary. Then by [2, Corollary 3.5.1], the map $\epsilon K^h(R : A) \rightarrow \epsilon K^h(S : A)$ is an equivalence. Taking this into account, we set, for any $K$-excisive $A \in \text{Alg}_K^e$, any unitary $\epsilon \in \ell$ and $n \in \mathbb{Z}$,

\[ \epsilon K^h_n(A) = \text{Ker}(\epsilon K^h_n(\hat{A}) \rightarrow \epsilon K^h_n(\ell)). \]

For $n \leq 0$ and not necessarily $K$-excisive $A$, we take (3.2) as a definition. The nonpositive hermitian $K$-groups agree with Bass’ quadratic $K$-groups [1] for the maximum form parameter. In particular, by [1, Chapter III, Theorem 1.1], hermitian $K$-theory as defined above satisfies excision in nonpositive dimensions.
Remark 3.3 \((K_0^h \text{ and quasi-homomorphisms})\). For unital \(R \in \text{Ring}^*\), \(K_0^h\) is the group completion of the monoid \(Y^h_\infty(R)\) of unitary conjugacy classes of self-adjoint idempotents in \(\mathcal{M}_2 M_\infty R\). We may thus regard elements of \(K_0^h(R)\) as differences of classes of \(*\)-homomorphisms \(Z \to \mathcal{M}_2 M_\infty R\); this can be formalized in terms of quasi-homomorphisms. Let \(A, B, C\) be \(*\)-rings such that \(B \subseteq C\). A quasi-homomorphism from \(A \to B\) through \(C\), noted as \((f_+, f_-) : A \cong C \supseteq B\), consists of two \(*\)-homomorphisms \(f_+, f_- : A \to C\) such that \(f_+(a) - f_-(a) \in B\) for all \(a \in A\). Thus an element of \(K_0^h(R)\) is a class of quasi-homomorphism \(Z \cong \mathcal{M}_2 M_\infty R \supseteq \mathcal{M}_2 M_\infty R\). Moreover it follows from (3.2) that if \(A\) is any \(*\)-ring then any element of \(K_0^h(A)\) is represented by a quasi-homomorphism \(Z \cong \mathcal{M}_2 M_\infty A \supseteq \mathcal{M}_2 M_\infty A\). If \(A \in \text{Alg}_\ell^*\) then the same holds with \(\ell\) substituted for \(Z\).

For any, not necessarily unital \(*\)-ring, and \(\epsilon = \pm 1\), Karoubi and Villamayor also introduce hermitian \(K\)-groups for \(n \geq 1\). They agree with the homotopy groups of the simplicial group \(\mathcal{O}(A^\Delta)\) up to a degree shift

\[\epsilon K^h_{n+1}(A) = \pi_{n-1,\epsilon} \mathcal{O}(A^\Delta) \quad (n \geq 1)\]

The argument of [3, Proposition 10.2.1] shows that the definition above is equivalent to that given in [12]; we have

\[\epsilon K^h_{n+1}(A) = \epsilon K^h_{n}(\Omega^n A) \quad (n \geq 1)\]

Similarly, if \(A\) is unital, for all \(n \geq 1\) we have

\[\epsilon K^h_{n+1}(A) = \pi_n B, \mathcal{O}(A^\Delta) = \pi_n B, \mathcal{O}(A^\Delta)^+ = \pi_n B, \mathcal{O}(\Sigma A^\Delta)^+\]

Applying \(\epsilon K^h\) to the path extension \((2.2.4)\) an using excision, we obtain a natural map

\[\epsilon K^h_n(A) \to \epsilon K^h_{n-1}(\Omega A) \quad (n \leq 0)\]

For \(n \in \mathbb{Z}\), the \(n^{th}\) homotopy \(\epsilon\)-hermitian \(K\)-theory group of \(A\) is

\[\epsilon K^h_n(A) = \text{colim}_m \epsilon K^h_{m}(\Omega^m A)\]

One can also describe \(\epsilon K^h\) in terms of \(\epsilon K^h\); by [12, Théorème 4.1], \(\epsilon K^h\) satisfies excision for the cone extension \((2.2.5)\). Hence we have a map

\[\epsilon K^h_{n+1}(A) \to \epsilon K^h_n(\Sigma A)\]

The argument of [5, Proposition 8.1.1] shows that

\[\epsilon K^h_{n+1}(A) = \text{colim}_m \epsilon K^h_{n+m}(\Sigma^m A)\]

Now assume that \(A\) is unital; let \(\epsilon K^h(A)\) be the total spectrum of the simplicial spectrum \(\epsilon K^h(A^\Delta)\). We have

\[\pi_n(\epsilon K^h(A)) = \text{colim}_m \pi_{n+m} \Omega B, \mathcal{O}(\Sigma^m A^\Delta)^+ = \text{colim}_n \epsilon K^h_{n+m}(\Sigma^m A) = \epsilon K^h_n(A)\]

Lemma 3.8. Homotopy hermitian \(K\)-theory is homotopy invariant, matricially stable and satisfies excision.

Proof. The same argument as in [3, Theorem 5.1.1] shows this. \(\square\)

In the proof of the following lemma we use an argument communicated to the first author by Max Karoubi.

Lemma 3.9. Assume that \(2\) is invertible in \(\ell\). Let \(n \in \mathbb{Z}\) and let \(A \in \text{Alg}_\ell^*\) be \(K_n\)-regular.

i) If either \(n \leq 0\) or \(A\) is \(K\)-excisive, then \(A\) is \(K_n^h\) regular and \(K^h_n(A) \to KH^h_{rn}(A)\) is an isomorphism for \(m \leq n\).

ii) If \(n = 0\) then \(KV^h_0(A) \to KH^h_{rn}(A)\) is an isomorphism for all \(m \geq 1\).

iii) If \(n \geq 1\) and \(A\) is \(K\)-excisive, then \(KV^h_{n+1}(A) \to KH^h_{rn}(A)\) is an isomorphism.
Proof. First assume that $A$ is unital. Consider Karoubi’s 12-term exact sequence [9, Théorème 4.3]. Because 2 is invertible in $\ell$, the groups $k_*(A)$ and $k'_*(A)$ appearing therein vanish by [15, Corollary 3.3]. Using this and the fact that by [11, Corollaire 3.10] and [10, Théorème 1.1.1], the Witt and co-Witt groups $\mathcal{W}_0$ and $\mathcal{W}'_0$ are homotopy invariant for all $\epsilon$, it follows that $W_m$ is homotopy invariant for all $m$. Hence by diagram chasing $A$ is $K^h_n$-regular whenever $A$ is, at least for unital $A$. By [2, Corollary 3.5.1], if $A$ is $K$-excisive then it is also $K^h$-excisive. By [4, Proposition A.5.3], $A[t_1, \ldots, t_n]$ is $K$-excisive whenever $A$ is. Using excision, we see that if $A$ is $K_n$-regular and either $n \leq 0$ or $A$ is $K$-excisive then it is also $K^h_n$-regular. If $n \leq 0$ and $A$ is $K^h_n$-regular, then $K^h_m(A) \to K^h_{m-1}(\Omega A)$ is an isomorphism for all $m \leq n$, so $K^h_m(A) \to KH^h_m(A)$ is an isomorphism. The argument of [3, Proposition 5.2.3] shows that if $A$ is $K_0$-regular then $KV^h_m(A) \to KH^h_m(A)$ is an isomorphism for $m \geq 1$. If $A$ is unital, we reach the same conclusions also for $n > 0$, using the argument of [16, Proposition 1.5]. The case when $A$ is excisive and $n > 0$ follows from the unital case using excision.

□

Lemma 3.10. Let $\epsilon \in \ell$ be unitary. If either $n \leq 0$ or $A$ is $K$-excisive, then there is a canonical isomorphism

$$\epsilon K^h_n(A) \cong K^h_n(\epsilon M_2 A).$$

Moreover for all $A \in \text{Alg}^+_\ell$ we have canonical isomorphisms

$$\epsilon KV^h_n(A) \cong KV^h_n(\epsilon M_2 A) \quad (n \geq 1) \quad \text{and} \quad \epsilon KH^h_n(A) \cong KH^h_n(\epsilon M_2 A) \quad (n \in \mathbb{Z}).$$

Proof. The isomorphism (3.1) comes from (2.1.12), which is canonical up to the choices of an element $\lambda \in \ell$ in the $\lambda$-assumption 2.1.14 and a bijection $[1, 2] \times X \to X$. By [13, Lemme 1.2.7], if $A$ is unital, then varying those choices has no effect on the homotopy type of the induced isomorphism $B(\epsilon O(A))^+ \cong B(\epsilon O(M_2 A))^+$. Applying this to $\Sigma^m A^m \quad (r \in \mathbb{Z}, m \geq 0)$, we obtain the statement of the lemma for unital $A$. The nonunital case follows from the unital one using split-exactness. □

4. Homotopy hermitian $K$-theory and cup products

Hermitian $K$-theory of unital $\ast$-rings is equipped with products [13, Chapitre III]. In fact these are defined in much greater generality and at the spectrum level [14, Section 5]. In this section we construct a product on the homotopy $K$-theory groups $KH^h_n$, and show it is compatible with that in $KH^h$, via the comparison map. Our construction is purely algebraic, and based on our elementary algebraic definition of $KH^h$ (3.5). Using that $KH^h$ satisfies excision in nonpositive dimensions we obtain, for $R, A \in \text{Alg}^+_\ell$ with $R$ unital, $m \in \mathbb{Z}$ and $n \leq 0$, a natural product

$$K^h_m(R) \otimes K^h_n(A) \xrightarrow{\cup} K^h_{m+n}(R \otimes_\ell A).$$

If moreover $m \leq 0$, we also obtain the product above for not necessarily unital $R$.

Lemma 4.2. Let $R, S \in \text{Alg}^+_\ell$ be unital and let $I \triangleleft S$ be a $\ast$-ideal. Assume that the sequence

$$0 \to R \otimes_\ell I \to R \otimes_\ell S \to R \otimes_\ell (S/I) \to 0$$

is exact. Let $\partial$ be the index map. Then the following diagram commutes

$$\begin{array}{ccc}
K^h_0(R) \otimes K^h_0(S/I) & \xrightarrow{\cdot} & K^h_0(R \otimes_\ell (S/I)) \\
\downarrow \otimes \partial & & \downarrow \partial \\
K^h_0(R) \otimes K^h_0(I) & \xrightarrow{\cdot} & K^h_0(R \otimes_\ell I).
\end{array}$$
Let $ii)$ There is an associative product

$$\partial \star$$

Part $i)$ is immediate from Lemma 4.3.

Proof. Let $j_1: \ell \to \ell \oplus \ell$ be the inclusion in the second summand. The extensions (2.2.4) and (2.2.3) are connected by a map of extensions

$$\begin{array}{ccc}
\Omega & \xrightarrow{P} & \ell \\
\Omega & \xrightarrow{inc} & \ell \oplus \ell
\end{array}$$

Let $i \leq 0$. Applying Lemma 4.2 with $S = \Sigma^{-1}i[\ell]$, $I = \Sigma^{-1}i\Omega$ and $R = \hat{A}$ and using naturality and excision, we obtain that the boundary map $\partial: K^h_n(A) \to K^h_n(\Omega A)$ is the cup product with $\partial([1]) \in K^h_1(\Omega)$. The proof now follows from associativity of $\star$.

Corollary 4.4. Let $R, A \in \text{Alg}^*_\ell$ with $R$ unital and let $m, n \in \mathbb{Z}$.

i) There is an associative product

$$\star: K^h_n(R) \otimes KH^h_n(A) \to KH^h_{m+n}(R \otimes \ell A).$$

ii) Let $c_\star: K^h_n(R) \to KH^h_n(R)$ be the comparison map. Then for all $m \in \mathbb{Z}$ and $\xi \in K^h_n(R)$, $c_m(\xi) = \xi \star c_0([1])$.

Proof. Part i) is immediate from Lemma 4.3 upon taking colimits. For $m \leq 0$, part ii) is clear from the construction of $\star$ and the definition of $KH^h$. For $m \geq 1$, $c_m$ factors as the comparison map $c'_m: K^h_m(R) \to KV^h_m(R)$ followed by the iterated boundary map $KV^h_m(R) \to KV^h_1(\Omega^{m-1}R) \to K^h_0(\Omega^mR) \to KH^h_m(R)$, which is the transfinite composition of $\partial([1])\star$. So part ii) follows from the fact that $c'_m(x \star y) = c'_m(x) \star y$ for all $x \in K^h_m(R)$ and $y \in K^h_1(\Omega)$.

Lemma 4.5. Let $A, B \in \text{Alg}^*_\ell$ and $m, n \in \mathbb{Z}$. Then (4.1) induces an associative product

$$KH^h_m(A) \otimes KH^h_n(B) \xrightarrow{\star} KH^h_{m+n}(A \otimes \ell B).$$

If $m \leq 0$ or $A$ is unital, then the following diagram commutes

$$\begin{array}{ccc}
K^h_m(A) \otimes KH^h_n(B) & \xrightarrow{c_m \otimes 1} & KH^h_m(A) \otimes KH^h_n(B) \\
& \xrightarrow{\star} &
\end{array}$$
Proof. As observed above, the boundary map \( \partial : K_* \to K_0 \) is cup product with \( \partial[1] \in K_{-1}(\Omega) \). It follows that the following diagram commutes for all \( r \leq 0 \)

\[
\begin{array}{ccc}
K_r(A) \otimes K_r(B) & \overset{*}{\longrightarrow} & K_{2r}(A \otimes \ell B) \\
\downarrow{\partial \otimes \partial} & & \downarrow{\partial^2} \\
K_{r-1}(A) \otimes K_{r-1}(B) & \overset{*}{\longrightarrow} & K_{2r-2}(A \otimes \ell B).
\end{array}
\]

Taking colimit along the columns we get the desired product map for \( r = s = 0 \). The general case is obtained from the latter applying the suspension and loop functors as many times as appropriate. Commutativity of the diagram of the Lemma follows from Corollary 4.4.

\[ \square \]

Corollary 4.6. Let \( A \in \text{Alg}_{\ell}^f \) and \( n \in \mathbb{Z} \), then \( _\ell KH_n(A) \) is a \( KH_0(\ell) \)-module.

5. STABILIZATION AND HOMOTOpy

Standing assumption 5.1. From now on we shall assume that \( \ell \) satisfies the \( \lambda \)-assumption 2.1.14.

An Ind-object in a category \( \mathcal{C} \) is a pair \((C, I)\) consisting of an upward filtered poset \( I \) and a functor \( C : I \to \mathcal{C} \). The Ind-objects of \( \mathcal{C} \) form a category \( \text{Ind} - \mathcal{C} \) where homomorphisms are given by

\[ \hom_{\text{Ind} - \mathcal{C}}((C, I), (D, J)) = \lim_{i} \colim_{j} \hom_{\mathcal{C}}(C_i, D_j). \]

Any functor \( F : \mathcal{C} \to \mathcal{D} \) extends to \( \text{Ind} - \mathcal{C} \to \text{Ind} - \mathcal{D} \) by applying it indexwise, \( F(C)_i = F(C_i) \). We shall also use the dual concept of a pro-object in a category. In Examples 2.1.2 we have already introduced, for a simplicial set \( X \) and a \( * \)-algebra \( A \), the \( * \)-algebra \( A^X \). By iteration of the simplicial subdivision functor \( \text{sd} : \mathcal{S} \to \mathcal{S} \) and the natural last vertex map \( \text{sd} \to \text{id}_{\mathcal{S}} \) \cite{[8]}, one obtains a pro-simplicial set \( \text{sd}^n X = \{\text{sd}^n X\} \), and thus an Ind-*algebra \( A^\text{sd}^n X \). For \( S^1 = \Delta^1 / \partial \Delta^1 \) we write

\[ (5.2) \quad \mathcal{P}A = A^\text{sd}^0, \quad A^1 = A^\text{sd}^1(S^1, pt), \quad A^{n+1} = A^\text{sd}^n \otimes \mathbb{Z}^{2n} \quad (n \geq 0). \]

Observe that the two endpoint inclusions \( \Delta^0 \to \Delta^1 \) induce inclusions \( \Delta^0 \to \text{sd}^n(\Delta^1) \) and evaluation maps \( \text{ev}_i : A^\text{sd}^n \Delta^1 \to A, i = 0, 1 \). Two \(*\)-algebra homomorphisms \( f_0, f_1 : A \to B = * \)-homotopic if there is a \(*\)-homomorphism \( H : A \to B^\text{sd}^n \Delta^1 \) in \( \text{Ind} - \text{Alg}^* \) such that \( \text{ev}_i H = f_i \) for \( i = 0, 1 \); \(*\)-homotopy is an equivalence relation, compatible with composition. We write \([A, B]^*\) for the set of homotopy classes of \(*\)-homomorphisms and \([\text{Alg}^*]^{-}\) for the category with the same objects as \( \text{Alg}^* \) with \( \hom_{[\text{Alg}^*]}(A, B) = [A, B] \). If \( A, B \in \text{Ind} - \text{Alg}_{\ell}^* \), we write \([A, B]^* = \hom_{\text{Ind} - [\text{Alg}_{\ell}^*]}(A, B)\).

Let \( C \in \text{Alg}_{\ell}^* \) and let \( A, B \subset C \) be subalgebras. If \( u, v \in C \) satisfy \( uAv \subset B \) and \( avu = aa' \) for all \( a, a' \in A \), then

\[ (5.3) \quad \text{ad}(u, v) : A \to B, \quad a \mapsto uav \]

is an algebra homomorphism. In this situation, we say that the pair \((u, v)\) multiplies \( A \) into \( B \). Let \((u_i, v_i) \in C^2 \) \((i = 0, 1)\) be pairs multiplying \( A \) into \( B \). A homotopy between \((u_0, v_0)\) and \((u_1, v_1)\) is a pair \((u(t), v(t)) \in C[t]^2\) which multiplies \( A \) into \( B[t] \), and such that \( u(0), v(0) = u_0, v_0 \). In this case \( \text{ad}(u(t), v(t)) : A \to B[t] \) is a homomorphism and a homotopy between \( \text{ad}(u_0, v_0) \) and \( \text{ad}(u_1, v_1) \). Now suppose that \( C \) is a \(*\)-algebra and that \( A, B \) are \(*\)-subalgebras; if \( u \in C \) and \((u, u^*)\) multiplies \( A \) into \( B \), then \( \text{ad}(u, u^*) \) is a \(*\)-homomorphism, and we say that \( u \) \(*\)-multiplies \( A \) into \( B \). If \( v \in C \) is another element with the same property, a \(*\)-homotopy from \( u \) to \( v \) is an element \( w(t) \in C[t] \) \(*\)-multiplying \( A \) to \( B[t] \) such that \( w(0) = u \) and \( w(1) = v \).

\[ \square \]
$w(1) = v$. We shall often encounter examples of elements $u_0, u_1 \in C$ which multiply $A$ into $B$ and which are homotopic via a pair $(u, v)$ with $v \neq u^*$ so that the homotopy $\text{ad}(u, v)$ is not a $*$-homomorphism. We shall see in Lemma 5.4 below that this can be fixed upon stabilization.

Write $i_+$ and $i_-$ for the upper left and lower right corner inclusions $\ell \to M_k$; both are $*$-homomorphisms.

**Lemma 5.4.** Let $C \in \text{Alg}_+^*$, let $A, B \subset C$ be $*$-subalgebras and let $u_0, u_1 \in C$ $*$-multiply $A$ into $B$. Let $(v, w) \in C[t]^2$ be a homotopy of multipliers from $(u_0, u_0^*)$ to $(u_1, u_1^*)$. Assume that

$$w^*Aw \subset B[t] \supset vAv^*. \tag{5.5}$$

Let $\lambda \in \ell$ be as in (2.1.15). Then the element

$$c(v, w) = \begin{bmatrix} \lambda^*v + \lambda w^* & \lambda^*(v-w^*) \\ \lambda(v-w^*) & \lambda v + \lambda^*w^* \end{bmatrix}$$

$*$-multiplies $i_+(A)$ into $M_kB[t]$, and $\text{ad}(c(v, w), c(v, w)^*) \circ i_+$ is a $*$-homomorphism from $i_+ \circ \text{ad}(u_0, u_0^*)$ to $i_+ \circ \text{ad}(u_1, u_1^*)$.

**Proof.** One checks that $c(v, w)^*c(v, w) = c(wv, wv)$. Hence if $a, a' \in A$, then

$$\begin{align*}
i_+(a)c(v, w)^*c(v, w)i_+(a') &= i_+(a(\lambda^*wv + \lambda(wv)^*)a') \\
&= i_+(aa').
\end{align*}$$

Similarly one checks, using (5.5), that $c(v, w)i_+(a)c(v, w)^* \subset M_kB[t]$. Thus $H = \text{ad}(c(v, w)) \circ i_+ : A \to M_kB[t]$ is a $*$-homomorphism. Moreover for $i = 0, 1$, $ev_i(c(v, w)) = c(u_i, u_i) = \text{diag}(u_i, u_i)$. Hence $ev_i \circ H = i_+ \circ \text{ad}(u_i, u_i^*)$. □

Let $p, q \geq 0$ and $n = p + q$. Set

$$i_+^{p,q} := M^p_+ \otimes i_+ \otimes M^q_+ : M^n_+ \to M^{n+1}_+.$$  

**Lemma 5.6.** Let $p, q$ and $n$ be as above, and let $p', q' \geq 0$ be such that $p' + q' = n+1$. Then $i_+^{p', q', p, q}$ is $*$-homotopic to $i_+^{0, n+1}$.

**Proof.** We have $i_+^{0,0} = i_+$ and $i_+^{1,0} = i_+^{0,1}$. Next observe that, under the identification $\ell^2 \otimes \ell^2 = \ell^{(1,2)}$, $i_+^{1,0}(e_{i,j}) = e_{(i,1),(j,1)}$ and $i_+^{0,1}(e_{i,j}) = e_{(1,i),(j,1)}$. Also, observe that the matrix

$$u = e_{(1,1),(1,1)} - e_{(1,2),(2,1)} + e_{(2,1),(1,2)} + e_{(2,2),(2,2)}$$

is a unitary element of $M^2_+$ and satisfies $u(1), u(1) = e_{1,1}$. Moreover by [5, Section 6.4], there exists an invertible element $u(t) \in M^2_+$ such that $u(0) = 1$ and $u(1) = u$. Hence the composites of $i_+^{0,2}$ with $i_+^{1,0}$ and $i_+^{0,1}$ are $*$-homotopic by Lemma 5.4. Tensoring on both sides with identity maps, we get that $i_+^{p,q+1} \simeq i_+^{p+1,q}$.

Let $p', q' \geq 0$ such that $p' + q' = n + 1$. Permuting factors in the tensor product $M^{p+1}_+ \otimes M^{q+1}_+$ we obtain a $*$-isomorphism $\sigma : M^{p+1}_+ \to M^{q+1}_+$ such that $\sigma i_+^{p+1,q} = i_+^{p', q'}$. Hence we have

$$i_+^{p', q', p+1,q} \simeq i_+^{p+1,q} i_+^{p,q+1} \tag{5.7}$$

for all $p, q, p', q'$ as above. The lemma follows from (5.7) using the identity

$$i_+^{r,s+1} i_+^{r,s} = i_+^{r+1,s} i_+^{r,s}.$$ □
Consider the Ind-$\ast$-algebra
\[ M^\ast_X = \{ \iota^n_{0,+} : M^n_+ \to M^{n+1}_+ \}. \]
Let $X$ be an infinite set. Put
\[ \mathcal{M}_X = M^\ast_X \mathcal{M}_X. \]
Any bijective map $f : X \to Y$ induces an isomorphism
\[ f_* : \mathcal{M}_X \cong \mathcal{M}_Y. \]

**Lemma 5.8.** Let $f, g : X \to Y$ be bijections. Then $[f_*] = [g_*] \in [\mathcal{M}_X, \mathcal{M}_Y]^\ast$.

**Proof.** It suffices to consider the case when $X = Y$ and $g = \text{id}_X$. The matrix $u = \sum_{x \in X} E_{f(x), x}$ is a unitary element of $\Gamma_X$, and $f_*$ is the restriction to $\mathcal{M}_X$ of the automorphism $\text{ad}(u) : \Gamma_X \to \Gamma_X$. Hence $[f_*] = [\text{id}]$ by Lemma 5.4. \qed

**Lemma 5.9.** Let $x, y \in X$. Then $\iota_{x+y} \cong \iota_{x+y}$.

**Proof.** If $x = y$, the maps of the lemma are equal, hence homotopic. Assume $x \neq y$; let $X' = X \setminus \{x, y\}$. Let $u = E_{y,y} - E_{x,y} + \sum_{z \in X'} E_{z,z}$; $u$ is a unitary element of $\Gamma_X$, and satisfies $\text{ad}(u) \circ \iota_x = \iota_y$. Moreover, by [5, Section 3.4], there is an invertible element $u(t) \in \Gamma_X[t]$ such that $u(0) = 1$ and $u(1) = u$. Hence $\iota_{x+y} \cong \iota_{x+y}$. \qed

In view of Lemma 5.9, we shall pick any $x \in X$ and write $\iota_X$, or simply $\iota$, for the homotopy class of $\iota_x$, as well as for that of $\iota_X \otimes \iota \text{id}_A$ ($A \in \text{Alg}_f^\ast$). Because we are assuming that $X$ is infinite, there is a bijection $X \coprod X \cong X$; this induces an $\ast$-homomorphism $\mathcal{M}_X \oplus \mathcal{M}_X \to \mathcal{M}_X$; write $\boxtimes$ for its Ind-$\ast$-homotopy class. By Lemma 5.8, $\boxtimes$ is independent of the choice of bijection above. As in [5, Section 4.1] one checks, using the lemmas above, that $\mathcal{M}_X$ equipped with $\boxtimes$ and with the class of the zero map, is an abelian monoid in $[\text{Ind} \ast X \ast X, \boxtimes]$. Similarly, any choice of bijection $X \times X \to X$ gives rise to the same $\text{Ind} \ast$-homotopy class $\mu$ of $\ast$-homomorphism $\mathcal{M}_X \otimes \mathcal{M}_X \to \mathcal{M}_X$. Again, one checks that $\mu$ is associative with unit the $\text{Ind} \ast$-homotopy class of $\iota : \ell \to \mathcal{M}_X$ and that it distributes over $\cdot$.

Hence $(\mathcal{M}_X, \boxtimes, \mu, 0, [\iota_{+1}])$ is a semi-ring in $[\text{Alg}_f^\ast]$. Let $A, B \in \text{Ind} \ast \text{Alg}_f^\ast$; put
\[ (A, B)_X^\ast = [A, \mathcal{M}_X B]^\ast. \]
The monoid operation $\boxtimes$ on $\mathcal{M}_X$ induces one on $(A, B)_X^\ast$. One checks that the product $\mu$ induces a bilinear, associative composition
\[ \ast : (B, C)_X^\ast \times (A, B)_X^\ast \to (A, C)_X^\ast \]
\[ [f] \ast [g] = [\mu \mathcal{M}_X f \circ g]. \]
Thus we have a category $\{(\text{Ind} \ast \text{Alg}_f^\ast)_X^\ast \}$ with the same objects as $\text{Ind} \ast \text{Alg}_f^\ast$, where homomorphisms are given by (5.10) and which is enriched over abelian monoids. Moreover, for $n \geq 1$, $(A, B^{\otimes n})^\ast$ is an abelian group by [5, Theorem 3.3.2] and the Hilton-Eckmann argument.

There is a canonical functor $[\text{Alg}_f^\ast] \to (\text{Alg}_f^\ast)_X^\ast$, which is the identity on objects and sends the class of a map $f$ to that of $\iota f$.

**Lemma 5.11.** The composite functor $\text{can} : \text{Alg}_f^\ast \to (\text{Alg}_f^\ast)_X^\ast$ is homotopy invariant, $\mathcal{M}_X$-stable and $\iota_{+1}$-stable. Moreover any functor $H : \text{Alg}_f^\ast \to \mathcal{C}$ which is homotopy invariant $\mathcal{M}_X$-stable and $\iota_{+1}$-stable, factors uniquely through $\text{can}$. \[ \text{Proof.} \text{ Straightforward.} \]

**Lemma 5.12.** Let $\epsilon \in \ell$ be a unitary element. Let $R \in \text{Alg}_f^\ast$ be a unital $\ast$-algebra, let $\phi, \psi \in R$ be $\epsilon$-hermitian elements, and let $A < R$ be a $\ast$-ideal. Then for every infinite set $X$, the canonical functor $\text{can} : \text{Alg}_f^\ast \to (\text{Alg}_f^\ast)_X^\ast$ maps the canonical inclusion $A^\circ \to M_2 A^{\otimes \psi}$ to an isomorphism.
Proof. The proof follows from Lemma 5.11 and Proposition 2.4.4.

Example 5.13. Lemma 5.12 applied to $R = \hat{A}$ and $\phi = 1$ says that the upper left hand corner inclusion $i : A \to M_2A$ is an isomorphism in $\{\text{Ind} - \text{Alg}_R^e\}$.

Lemma 5.14. Let $A \in R$ be as in Lemma 5.12 and let $\lambda_1, \lambda_2 \in R$ be central elements satisfying (2.1.15). Let

$$p_i = p_{\lambda_i} = \begin{bmatrix} \lambda_i^* & 1 \\ \lambda_2 \lambda_1^* & \lambda_2 \end{bmatrix}$$

and let $\iota_i : A \to 1M_2A$, $\iota_i(a) = p_i a$. Then $\text{can}(\iota_1) = \text{can}(\iota_2)$ is an isomorphism in $\{\text{Alg}_R^e\}_X$.

Proof. Let $u_i = u_{\lambda_i}$ be as in (2.1.17). Under the isomorphism (2.1.18), $\iota_i$ corresponds to $\iota_+$. Thus $\text{can}(\iota_i)$ is an isomorphism. Moreover, since $u = u_2u_1^{-1} \in 1M_2R$ is unitary, $\text{can}(\text{ad}(u)) = \text{id}_{1M_2A}$ by Lemma 2.4.2, so

$$\text{can}(\iota_2) = \text{can}(\text{ad}(u_2u_1^{-1})) \text{can}(\iota_1) = \text{can}(\iota_1).$$

6. Bivariant hermitian $K$-theory

6.1. Extensions and classifying maps. As in Subsection 2.3, let $\mathfrak{U}$ be our fixed underlying category and $\mathfrak{F} : \text{Alg}_R^e \to \mathfrak{U}$ and $T' : \mathfrak{U} \to \text{Alg}_R^e$ the forgetful functor and its left adjoint. Put $T = T' F : \text{Alg}_R^e \to \text{Alg}_R^e$. If $A \in \text{Alg}_R^e$, write $JA = \text{Ker}(TA \to A)$ for the kernel of the counit of the adjunction. A sequence of Ind-$*$ algebras (2.2.1) is an extension if $p$ is a cokernel of $i$ and $i$ is a kernel of $p$; it is semi-split if $F(p)$ is split. Any semi-split extension (2.2.1) gives rise to a $*$-homomorphism $JC \to A$, which is unique up to homotopy [5, Proposition 4.4.1]; its homotopy class, and, by abuse of notation, any map in it, is called the classifying map of the extension. We write $\gamma_A$ and $\rho_A$ for the classifying maps of the extensions

$$\begin{align*}
(JA)^{S^1} & \xrightarrow{(TA)^{S^1}} A^{S^1} \\
A^{S^1} & \xrightarrow{\mathfrak{F}A} A
\end{align*}$$

(6.1.1)

(6.1.2)

For $m, n \geq 0$, put also

$$\begin{align*}
\gamma_A^{1,n} & = \sigma_{i=0}^{n-1} \iota_{AF_i} : J(A^{S^n}) \to (JA)^{S^n} \\
\gamma_A^{m,n} & = \sigma_{i=0}^{m-1} J^i(\gamma_{J_m-1} A) : J^m(A^{S^n}) \to J^m(B)^{S^n}.
\end{align*}$$

Lemma 6.1.5. Let $A \in \text{Ind} - \text{Alg}_R^e$ and let $X$ be an infinite set. Then the following diagram commutes in $\{\text{Ind} - \text{Alg}_R^e\}_X$

$$\begin{align*}
J^2(A) \xrightarrow{\rho_{JA}} JA^{S^1} \\
\downarrow_{J(\rho_A)} & \\
J(A^{S^1}) \xrightarrow{\gamma_A}
\end{align*}$$

Proof. The analogue of the lemma in the setting of bornological algebras is proved in [7, Lemma 6.30]. With some obvious modifications, the same argument works in the present case.

Remark 6.1.6. The analogue of Lemma 6.1.5 for algebras without involution also holds as stated, and can be deduced from the equivalence of Example 2.1.1. This corrects a mistake in [5, Lemma 6.2.2], where the sign is missing. A sign is also missing in the definition of composition in the category $kk$ [5, Theorem 6.2.3]; this is fixed as in (6.2.3) below.
6.2. The category $kk^h$. Fix an infinite set $X$ and let $A, B \in \text{Alg}_X^*$. As in [5, Section 6.1], there is a map $[A, B]^*_X \to [JA, BS^\infty]^*_X$ which sends the class of $f$ to that of $\rho_B J(f)$. Thus one can form the colimit
\[(6.2.1) \quad kk^h(A, B) = kk^h(A, B)_X = \text{colim}_n \{J^nA, BS^n\}^*_X.\]

Since $X$ is fixed, we drop it from our notation. Define a composition law
\[(6.2.2) \quad \circ : kk^h(B, C) \otimes kk^h(A, B) \to kk^h(A, C),\]
\[(\xi, \eta) \mapsto \xi \circ \eta\]
as follows. Let
\[\gamma_{B}^{m,n} = \circ_{i=0}^{m} J^i J^{m-i} : J^m(B^{S^n}) \to J^m(B^{S^n}).\]
If $\xi$ is represented by a class $[f] \in \{J^nB, C^{S^n}\}^*$ and $\eta$ is represented by $[g] \in \{J^mA, BS^m\}^*$, put
\[(6.2.3) \quad \xi \circ \eta = [g^{S^m} \circ (-1)^{mn}\gamma_{B}^{m,n} \star J^m(f)].\]

As in [7, Section 6.3], one checks that the composition above is well-defined and associative. Hence we have a category $kk^h$ with the same objects as $\text{Alg}_X^*$, where the identity map of $A \in \text{Alg}_X^*$ is represented by the class of $\iota_A : A \to M_X A$. Define a functor $\{\text{Alg}_X^*\} \to kk^h$ as the identity on objects and as the canonical map to the colimit $\{A, B\}_X \to kk^h(A, B)$ on arrows. Composing the latter with the functor $\text{Alg}_X^* \to \{\text{Alg}_X^*\}_X$ we obtain
\[(6.2.4) \quad j^h : \text{Alg}_X^* \to kk^h.\]

Let $\Sigma$ be a triangulated category; write $[-n]$ for the $n$-fold suspension in $\Sigma$. Let $E$ be the class of all semi-split extensions
\[(E) \quad A \xrightarrow{i} B \xrightarrow{p} C\]
An excisive homology theory on $\text{Alg}_X^*$ is a functor $H : \text{Alg}_X^* \to \Sigma$ together with a family of maps $\{\partial_E : H(C)[1] \to H(A)[E \in E]\}$ such that for every $E \in E$,
\[H(C)[1] \xrightarrow{\partial_E} H(A) \xrightarrow{} H(B) \xrightarrow{} H(C)\]
is a triangle in $\Sigma$, and such that $\partial$ is compatible with maps of extensions ([5, Section 6.6]).

Let $f : A \to B \in \text{Alg}_X^*$ be a $*$-homomorphism. The homotopy fiber of $f$ is the pullback $P_f$ of $f$ and $\text{ev}_1 : PB \to B$. Thus the following diagram is cartesian
\[(6.2.5) \quad \begin{array}{ccc} P_f & \rightarrow & PB \\ \downarrow & & \downarrow \text{ev}_1 \\ A & \xrightarrow{f} & B \end{array}\]

Observe that $\Omega B$ embeds naturally as a $*$-ideal in $P_f$. The sequence
\[(6.2.6) \quad \Omega B \to P_f \to A \xrightarrow{f} B\]
is the homotopy fibration associated to $f$.

**Proposition 6.2.7.** The category $kk^h$ has a triangulation which makes the functor $j^h : \text{Alg}_X^* \to kk^h$ into an excisive homology theory which is homotopy invariant, $i_+\text{-stable}$ and $M_X\text{-stable}$. Moreover any other excisive, homotopy invariant, $i_+\text{-stable}$ and $M_X\text{-stable}$ homology theory $\hat{H} : \text{Alg}_X^* \to \Sigma$ factors uniquely through a triangulated functor $\hat{H} : kk^h \to \Sigma$. 
Proof. As in [5, Corollary 6.4.2] one proves that the endofunctor of $kk^h$ induced by $\Omega_S X$ is naturally isomorphic to the identity. For $A \in \text{Alg}_k^*$, put $j^h(A)[-n] = j^h(\Omega^n X A)$. Thus $j^h(A) \mapsto j^h(A)[-1]$ is an equivalence with inverse $j^h(A) \mapsto j^h(A)[1] := j^h(\Omega A)$. As in [5, Definition 6.5.1] we declare a triangle in $kk^h$

$$j^h(C')[1] \rightarrow j^h(A') \rightarrow j^h(B') \rightarrow j^h(C')$$

distinguished if it is isomorphic to the image under $j^h$ of a homotopy fibration sequence (6.2.6). The proof that the triangles above make $kk^h$ triangulated is the same as in [5, Theorem 6.5.2]. The universal property is proved as in [5, Theorem 6.6.2].

Example 6.2.8. Let $\ell_0$ be any commutative ring and let $\ell = \text{inv}(\ell_0)$ and $\text{inv} : \text{Alg}_k \rightarrow \text{Alg}_k^*$ be as in Example 2.1.1. Then $\text{inv}$ is excisive, homotopy invariant and matricially stable, so it induces a triangulated functor from $kk$ to $kk^h(\ell_0)$ to $kk^h(\ell)$. Similarly, its inverse, $B \mapsto (1,0)B = (\ell/(0,1) \ell) \otimes B$ is excisive, homotopy invariant, and matricially stable; by 2.1.13 it is also hermitian stable. Hence it induces a functor $kk^h(\ell) \rightarrow kk^h(\ell_0)$ which is inverse to $\text{inv}$. This shows that $kk$ is a particular case of $kk^h$.

Example 6.2.9. Let $L \in \text{Alg}_k^*$: then $L \otimes_{\ell} -$ preserves all $F$-linearly split extensions if either $L$ is flat an $\ell$-module or every $F$-split extension is $\ell$-linearly split. In either case, $j^h(L \otimes_{\ell} -) : \text{Alg}_k^* \rightarrow kk^h$ is homotopy invariant, matricially stable, hermitian stable and excisive, and therefore induces a triangulated functor $L \otimes_{\ell} - : kk \rightarrow kk^h$.

Lemma 6.2.10. Let $A_1, A_2 \in \text{Alg}_k^*$ such that $A_1 \otimes_{\ell} -$ preserves $F$-split extensions, $i = 1, 2$. Then we have a natural bilinear, associative product

$$kk^h(A_1, A_2) \times kk^h(B_1, B_2) \rightarrow kk^h(A_1 \otimes_{\ell} B_1, A_2 \otimes_{\ell} B_2), \text{ } (\xi, \eta) \mapsto \xi \otimes \eta$$

that is compatible with composition in all variables.

Proof. If $F$-split extensions are linearly split, this is standard [5, Example 6.6.5]. For the other case, if $C \in \text{Alg}_k^*$ is flat then $J^nC$ is flat for all $n \geq 0$. Hence from the universal property of $T(C \otimes_{\ell} D)$ one obtains maps $\gamma_{C,D} : J(C \otimes_{\ell} D) \rightarrow C \otimes_{\ell} J D$, and $\gamma^C_{D} : J(C \otimes_{\ell} D) \rightarrow J(C) \otimes_{\ell} D$, both of which $j^h$ maps to isomorphisms. Define inductively $\gamma_{C,D}^n = \gamma_{C,J^n-1 D} \circ J(\gamma_{C,D}^{n-1})$ and $\gamma_{C,D}^n = \gamma_{J^n-1 C,J} \circ J(\gamma_{C,D}^n)$. Now, let $f \in \{ J^n A_1 \rightarrow A_2^m \}$ and $g \in \{ J^m B_1, B_2^m \}$ be representatives of $\xi$ and $\eta$. Define $\xi \otimes \eta \in kk^h(A_1 \otimes_{\ell} B_1, A_2 \otimes_{\ell} B_2)$ as the element represented by the composite $(f \otimes g) \circ \gamma_{J^n A_1, B_1} \circ J^m(\gamma_{A_1, B_1})$. It is tedious but straightforward to check that this gives a well-defined product with all the required properties.

Proposition 6.2.11. Let $\mathcal{E}$ be an abelian category and $H : \text{Alg}_k^* \rightarrow \mathcal{E}$ a functor. Assume that $H$ is split-exact, homotopy invariant, $\ell_+-$stable and $M_\mathcal{E}$-stable. Then there is a unique homological functor $H : kk^h \rightarrow \mathcal{E}$ such that $H \circ j^h = H$.

Proof. The proof is the same as in [5, Theorem 6.6].

Let $\ell \in \ell$ be a unitary, $A, B \in \text{Alg}_k^*$ and $n \in Z$. Put

$$kk^h_\ell(A, B) := \text{hom}_{kk}(j^h(A), j^h(B)[n]), \text{ } kk^h_\ell(A, B) = kk^h_\ell(A, B, M_2 B)$$

$$kk^h(A, B) := kk^h_0(A, B), \text{ } kk^h(A, B) = kk^h_0(A, B).$$

Remark 6.2.13. In view of (2.1.18), there is a $*$-isomorphism $1 M_2 \cong M_4$. It follows from this and from Proposition 6.2.7 that for all $A, B \in \text{Alg}_k^*$, $\ell_+$ induces a canonical isomorphism

$$\ell_+ kk^h(A, B) \cong kk^h(A, B).$$
Example 6.2.14. The functor $KH^h_0 : \text{Alg}_\ell^* \to KH^h_0(\ell) - \text{Mod}$ satisfies the hypothesis of Proposition 6.2.11. Hence the functor $KH^h_0$ of the proposition induces a natural homomorphism

$$kk^h(A, B) \to \text{hom}_{KH^h_0(\ell)}(KH^h_0(A), KH^h_0(B))$$

Setting $A = \ell$ we obtain a natural map

$$(6.2.15) \quad kk^h(\ell, B) \to KH^h_0(B).$$

We shall show in Proposition 8.1 below that $(6.2.15)$ is an isomorphism.

Example 6.2.16. Let $f : A \to B$ be a $*$-homomorphism. Then $j^h$ maps $(6.2.6)$ to a distinguished triangle, by definition of the latter. One can also fit $f$ into an equivalent triangle by other natural constructions. For example, let $T_f$ be the pullback of $TB \to B$ and $f$; we have a commutative diagram

$$(6.2.17) \quad \\
\xymatrix{ JB \ar[r] \ar[d] & T_f \ar[r] \ar[d] & A \ar[r]^f \ar[d] & B \ar[d] \ar[r] \ar[d] & \\
\Omega B \ar[r] & P_f \ar[r] & A \ar[r]_f & B }$$

By the argument of [5, Lemma 6.3.10], the vertical map $JB \to \Omega B$ is a $kk^h$-equivalence. Because the first three terms of the top row above form an extension, it follows that the vertical map $T_f \to P_f$ is a $kk^h$-equivalence. Summing up, the top row is $kk^h$-isomorphic to the bottom row, and is thus a triangle in $kk^h$. For another example, let $\Gamma_f$ be the pullback of $\Sigma f$ and the canonical surjection $\Gamma B \to B$. By the argument of [5, Theorem 6.4.1], $j^h(\Gamma B) = 0$. Hence the classifying map of the cone extension $J\Sigma B \to M_\infty B$ is a $kk^h$-equivalence. Hence $T_{\Sigma f} \to \Gamma_f$ is a $kk^h$-equivalence. Thus the vertical maps in the commutative diagram below form an isomorphism of triangles

$$(6.2.18) \quad \\
\xymatrix{ J\Sigma B \ar[r] \ar[d] & T_{\Sigma f} \ar[r] \ar[d] & \Sigma A \ar[r]^{\Sigma f} \ar[d] & \Sigma B \ar[d] \ar[r] \ar[d] & \\
M_\infty B \ar[r] & \Gamma_f \ar[r] & \Sigma A \ar[r]_{\Sigma f} & \Sigma B. }$$

It follows that the bottom line of $(6.2.18)$ is a distinguished triangle in $kk^h$. The map $(6.2.18)$ together with that of $(6.2.17)$ with $\Sigma(f)$ substituted for $f$ is a zig-zag of $kk^h$-equivalences. In particular $j^h(\Gamma_f) \cong j^h(P_{\Sigma f}) = j^h(\Sigma P_f)$ and the bottom line of $(6.2.18)$ is isomorphic in $kk^h$ to the homotopy fibration associated to $\Sigma f$ which in turn is the suspension of $(6.2.6)$:

$$(6.2.19) \quad \\
\xymatrix{ j^h(M_\infty B) \ar[r] \ar[d]_{\cong} & j^h(\Gamma_f) \ar[r] \ar[d]_{\cong} & j^h(\Sigma A) \ar[r]^{j^h(\Sigma f)} \ar[d]_{\cong} & j^h(\Sigma B) \ar[d]_{\cong} \ar[r] & \\
j^h(B) \ar[r] & j^h(P_f)[-1] \ar[r] & j^h(A)[-1] \ar[r] & j^h(B)[-1]. }$$

7. Coproducts

A retract of a $*$-algebra $A \in \text{Alg}_\ell^*$ consists of a $*$-algebra $C$ and $*$-homomorphisms $i : C \to A$ and $\alpha : A \to C$ such that $ai = \text{id}_C$. Let $j : C \leftrightarrow B : \beta$ be another retract and consider the coproduct $A \coprod_C B$. Let $I = \text{Ker} \alpha$, $J = \text{Ker} \beta$. Then $A \coprod_C B$ is the $\ell$-module

$$C \oplus I \oplus J \oplus I \oplus C \oplus J \oplus I \oplus \ldots$$
equipped with the obvious product and involution. The projection onto \( C \) is a \(*\)-homomorphism, and its kernel is the coproduct \( I \coprod J \). The direct sum of all tensors with 2 factors or more is an ideal in \( K \lhd A \coprod_C B \), and the quotient

\[
A \oplus_C B := (A \coprod_C B)/K
\]
is the \( \ell \)-module \( C \oplus I \oplus J \) equipped with the summand-wise involution and the product

\[
(c_1, i_1, j_1)(c_2, i_2, j_2) = (c_1c_2, c_1i_2 + i_1c_2 + i_1i_2, c_1j_2 + j_1c_2 + j_1j_2).
\]

**Proposition 7.1.** The projection \( \pi : A \coprod_C B \to A \oplus_C B \) is a \( kk^h \)-equivalence.

**Proof.** As in [5, Theorem 7.1.1] one reduces to proving the case \( C = 0 \). Let \( \mathrm{diag} : A \oplus B \to M_2(A \coprod B) \), \( \mathrm{diag}(a, b) = aE_{1,1} + bE_{2,2} \). Let \( i : A \oplus B \to M_2(A \oplus B) \) be as in Example 5.13. The matrix \( u = E_{1,1}(1,0) + E_{1,2}(0,1) + E_{2,1}(1,0) + E_{2,2}(1,0) \in M_2\mathbb{A} \) is unitary, satisfies \( \mathrm{ad}(u) \pi(\mathrm{diag}(a, b)) = i \) and is connected to the identity by a path \( u(t) \in \mathrm{GL}(A[t]) \). Thus \( \ell^h \circ \mathrm{diag} \sim^\ast \ell^h \circ i \) by Lemma 5.4; by Example 5.13, this implies that \( M_2(\pi) \circ \mathrm{diag} \) is an isomorphism in \( \{\text{Alg}^*_\ell\} \) and therefore also in \( kk^h \). A similar argument shows that \( \mathrm{diag} \circ \pi \) is an isomorphism in \( kk^h \); this concludes the proof. \( \square \)

In the next corollary and elsewhere, for \( A \in \text{Alg}^*_\ell \), we write \( QA = A \coprod A \) and \( QA \lhd qA = \text{Ker} (\mathrm{id}_A \coprod \mathrm{id}_A : QA \to A) \).

**Corollary 7.2.** Let \( \delta : qA \to A \) be the restriction of the map \( \mathrm{id}_A \coprod 0 : QA \to A \). Then \( j^h(\delta) \) is an isomorphism in \( kk^h \).

**Proof.** This is a formal consequence of Proposition (7.1); see [6, Proposition 3.1 (b)]. \( \square \)

**8. Recovering \( KH^h \) from \( kk^h \)**

**Proposition 8.1.** The natural map (6.2.15) is an isomorphism for every \( B \in \text{Alg}^*_\ell \).

**Proof.** Write \( \beta : kk^h(\ell, B) \to KH^h_0(B) \) for (6.2.15). Our proof proceeds much in the same way as [5, Theorem 8.2.1]. By Remark 3.3, the set \( \{q \in \ell(\ell, B) \} \) of quasi-homomorphisms \( \ell \rightrightarrows M_2M_\infty B \geq M_2M_\infty B \) maps onto \( KH^h_0(B) \). As in loc. cit. we use this together with Corollary 7.2, to obtain a natural map \( \alpha_A : KH^h_0(A) \to kk^h(\ell, A) \). Moreover, \( \alpha_B : KH^h_0(\ell) \to kk^h(\ell, \ell) \) is a ring homomorphism and for every \( B \), \( \alpha_B \) is a \( KH^h_0(\ell) \)-module homomorphism. Taking this into account, the argument of [5, Theorem 8.2.1] applies verbatim to show that that \( \alpha \) and \( \beta \) are inverse isomorphisms. \( \square \)

**Corollary 8.2.** For every unitary element \( \epsilon \in \ell \), every \( A \in \text{Alg}^*_\ell \) and every \( n \in \mathbb{Z} \), there is a natural isomorphism

\[
\epsilon kk^h_n(\ell, A) \cong \epsilon KH^h_n(A).
\]

**Lemma 8.3.** If either \( A \) or \( B \) are flat or if \( F \)-split extensions are \( \ell \)-linearly split, then under the isomorphism of Proposition 8.1 the cup-product of Lemma 4.5 corresponds to the tensor product of Lemma 6.2.10.

**Proof.** Straightforward. \( \square \)
Let \( \text{inv}(\ell) \) be as in Example 2.1.1; the map \( \ell \to \text{inv}(\ell) \), \( a \mapsto (a, a^*) \) is a homomorphism of \( \ast \)-rings. Composing \( \text{inv} : \text{Alg}_\ell \to \text{Alg}_{\text{inv}(\ell)} \) with restriction of scalars along the map we have just defined, we obtain a functor \( \text{ind} : \text{Alg}_\ell \to \text{Alg}_\ell^* \). The forgetful functor \( \text{res} : \text{Alg}_\ell^* \to \text{Alg}_\ell \) is left adjoint to \( \text{ind} \). The unit and counit of this adjunction are the maps

\[ \eta_A : A \to A \oplus A^{\text{op}}, \quad \eta(a) = (a, a^*), \quad \text{pr}_1 : B \oplus B^{\text{op}} \to B, \quad \text{pr}_1(x, y) = x. \]

By the universal properties of \( j : \text{Alg}_\ell \to kk \) and \( j^h : \text{Alg}_\ell^* \to kk^h \), \( \text{res} \) and \( \text{ind} \) induce functors \( kk^h \leftrightarrow kk \) which by abuse of notation we shall still call \( \text{res} \) and \( \text{ind} \).

**Proposition 9.2.** The functors \( \text{res} : kk^h \leftrightarrow kk \) are both right and left adjoint to one another; in other words, for every \( A \in \text{Alg}_\ell^* \) and \( B \in \text{Alg}_\ell \) there are natural isomorphisms

\[ kk(\text{res}(A), B) \cong kk^h(A, \text{ind}(B)) \quad \text{and} \quad kk^h(\text{ind}(B), A) \cong kk(B, \text{res}(A)). \]

**Proof.** The first isomorphism is immediate from the isomorphism \( \text{inv}(B^{\text{op}}) \cong \text{inv}(B)^{\text{op}} \) and the fact that \( \text{res} \) and \( \text{ind} \) are the left and right part of an adjunction \( \text{Alg}_\ell \leftrightarrow \text{Alg}_\ell^* \). To prove the second isomorphism, it suffices to give, for each \( B \in \text{Alg}_\ell \) and each \( A \in \text{Alg}_\ell^* \), natural maps \( \alpha_B \in kk(B, \text{res}\ ind B) \) and \( \beta_A \in kk^h(\text{ind}\ res A, A) \) satisfying the unit and counit conditions. Let \( \phi_B : B \to B \oplus B^{\text{op}}, \quad b \mapsto (b, 0), \quad \alpha_B = j(\phi_B). \)

Define \( \psi_A : A \oplus A^{\text{op}} \to 1M_2A, \quad \psi_A(x, y) = \text{diag}(x, y^*). \)

The composite \( \psi_A \circ \text{res}(A) \circ \phi_{\text{res}(A)} : A \to 1M_2A \) becomes the scalar \( \text{can}(1M_2A) \cong \text{can}(A) \) of Lemma 5.14. The matrix

\[
\begin{pmatrix}
\lambda^* \\
\lambda \\
-1
\end{pmatrix}
\]

is invertible and conjugates the composite \( \psi_A \circ \phi_{\text{res}(A)} : A \to 1M_2A \) to the upper left hand corner inclusion, so \( \text{res}(\beta_A) \circ \text{can}(1M_2A) \) is the identity. The composite \( \psi_{\text{ind}(B)} \circ \phi_B \) is the map

\[ B \oplus B^{\text{op}} \to 1M_2(B \oplus B^{\text{op}}), \quad (x, y) \mapsto \begin{pmatrix}
(x, 0) \\
(0, 0)
\end{pmatrix}. \]

Let \( \lambda = (0, 1) \in \text{ind}(\hat{B}) \), let \( p = p_\lambda \) be as in Lemma 5.14 and let \( \epsilon_p : \text{ind}(B) \to 1M_2\text{ind}(B), \quad \epsilon_p(x, y) = (x, y)p. \) The matrix

\[ u = \begin{pmatrix}
(1, -1) \\
(0, 0) \\
(-1, 1)
\end{pmatrix} \in 1M_2(\text{ind}(\hat{B})) \]

is unitary and conjugates \( \epsilon_p \) to the map (9.4). It follows that \( \beta_{\text{ind}(B)} \circ \text{ind}(\alpha_B) \) is the identity map. \( \square \)

Let \( \Lambda = \ell \oplus \ell \) equipped with involution

\[ (\lambda, \mu)^* = (\mu^*, \lambda^*). \]

If \( B \in \text{Alg}_\ell^* \) we shall identify \( \text{ind}(\text{res}(B)) \) with \( \Lambda B = \Lambda \otimes \ell B \) via the isomorphism

\[ \Lambda B \to \text{ind}(\text{res}(B)), \quad (x, y) \mapsto (x, y^*). \]

Under this identification, the maps \( \eta_A \) of (9.1) and \( \phi_B \) of (9.3) become the scalar extensions of the diagonal embeddings

\[ \eta : \ell \to \Lambda, \quad \eta(x) = (x, x), \]

\[ \phi : \Lambda \to 1M_2, \quad \phi(x, y) = E_{1,1}x + E_{2,2}y. \]
Remark 9.8. The functor induced by tensoring with Λ is left and right adjoint to itself since Λ ≃ ind res ℓ. Also, Proposition 9.2 shows that $kk^h(\cdot, \Lambda(\cdot)) = kk(res(\cdot), res(\cdot))$. In other words, Λ represents $kk$. Note in particular, that

$$kk^h(\cdot, \Lambda(\cdot)) = kk^h(\cdot, \Lambda(\cdot))$$

for any unitary $\epsilon \in \ell$. Moreover by Example 2.1.13, if $R \in \text{Alg}_{\ell}^*$ is unital, $\epsilon \in R$ a central unitary and $\psi \in R$ an invertible $\epsilon$-hermitian element, then

$$(9.9) \quad \text{ad}(1, \psi^{-1}) : \Lambda R \to \Lambda R^\psi$$

is an isomorphism in $\text{Alg}_\ell^*$. In particular, we have $*$-isomorphisms

$$(9.10) \quad \Lambda M_\ell \cong \Lambda(M_2) \cong \Lambda M_2.$$

10. The functors $U$ and $V$

In 6.2.5 we have defined the homotopy fiber $P_f$ of any $*$-homomorphism $f$. Now consider the homotopy fibers of the maps $(9.7)$ and $(9.6)$

$$U = \Lambda_0, \quad V = \Lambda_1.$$

For $A \in \text{Alg}_{\ell}^*$, put $UA = U \otimes \ell A$ and $VA = V \otimes \ell A$; these are, respectively, the homotopy fibers of $\phi \otimes \text{id}_A : \Lambda A \to \Lambda M_2 A$ and $\eta \otimes \text{id}_A : A \to \Lambda A$. Because $U$ and $V$ are flat $\ell$-modules, they define functors $U, V : kk^h \to kk^h$.

Remark 10.1. In Example 6.2.16 we have defined, for every $*$-homomorphism $f$, a $*$-algebra $\Gamma_f$, and shown that $j^h(\Gamma_f) = j^h(P_f)[-1]$. In his paper [9], Karoubi uses, in the case $\ell = \mathbb{Z}$, the letters $U$ and $V$ for $U' := \Gamma_0$ and $V' := \Gamma_1$. Thus $j^h(U)[-1] = j^f(U')$ and $j^h(V)[-1] = j^f(V')$. Karoubi shows that for every $*$-ring $R$, there are weak equivalences $\Omega K(\Lambda U'R) \xrightarrow{\sim} K^h(\Lambda R)$, $K^h(\Lambda U'R) \xrightarrow{\sim} K^h(\Lambda R)$ and $\Omega K^h(U'V'R) \xrightarrow{\sim} K^h(R)$ (see [9, Sections 1.3 and 1.4]. Lemmas 10.2 and 10.7 recast the latter equivalences into the framework of $kk^h$.

Lemma 10.2. There are isomorphisms

$$j^h(U \Lambda) \cong j^h(\Lambda) \quad \text{and} \quad j^h(V \Lambda) \cong j^h(\Omega \Lambda).$$

Proof. Let us prove the first equivalence. To ease the notation we omit the functor $j^h$. Let

$$U \Lambda \to \Lambda^2 \xrightarrow{\phi \otimes \text{id}_1} 1 \Lambda M_2 \Lambda$$

be the triangle in $kk^h$ defining $U \Lambda$. We have an isomorphism

$$\tau : \Lambda^2 \xrightarrow{\cong} \Lambda \oplus \Lambda$$

$$(x_1, x_2) \otimes (x_3, x_4) \mapsto (x_1 x_3, x_2 x_4, x_1 x_4, x_2 x_3).$$

Put $\lambda_1 = (0, 1)$, $\lambda_2 = (1, 0)$ and $\iota \in : \Lambda \to 1 \Lambda M_2 \Lambda$ as in Lemma 5.14. Let $j_1 : \Lambda \to \Lambda \oplus \Lambda$ be the inclusion into the $j^h$ summand. Observe that

$$((\phi \otimes \text{id}_\Lambda) \circ \tau^{-1} \circ j_1)(x, y) = \begin{bmatrix} (x, 0) \\ (0, 0) \\ (0, y) \end{bmatrix}.$$  

Hence as in the proof of Proposition 9.2, the matrix $u$ in (9.5) satisfies

$$\text{ad}(u) \circ \iota_1 = (\phi \otimes \text{id}_\Lambda) \tau^{-1} j_1 : \Lambda \to 1 \Lambda M_2 \Lambda.$$

So the following diagram commutes in $kk^h$

$$\Lambda \xrightarrow{j_1} \Lambda \oplus \Lambda \xrightarrow{\iota_1} 1 \Lambda M_2 \Lambda.$$
Similarly, the diagram

\[
\Lambda \xrightarrow{j_2} \Lambda \oplus \Lambda \xrightarrow{\phi \otimes id} 1M_2\Lambda
\]

which commutes in \(kk^h\). Next consider the following split distinguished triangle in \(kk^h\):

\[
\Lambda \xrightarrow{id \oplus -id} \Lambda \oplus \Lambda \xrightarrow{\pi_1 + \pi_2} \Lambda
\]

Since the diagrams (10.3) and (10.4) commute and since by Lemma 5.14 we have \(j^h(\iota_1) = j^h(\iota_2)\), the following solid arrow diagram commutes in \(kk^h\):

\[
UL \xrightarrow{\phi \otimes id} \Lambda^2 \xrightarrow{\tau^{-1}} 1M_2\Lambda \xrightarrow{\eta \otimes id} \Lambda
\]

Because the middle and right vertical arrows are isomorphisms in \(kk^h\), we get that the dashed map is an isomorphism in \(kk^h\). Next we prove the second isomorphism of the lemma. Let

\[
VA \xrightarrow{\eta \otimes id} \Lambda^2
\]

be the triangle in \(kk^h\) defining \(VA\). Let

\[
t : \Lambda \to \Lambda, \ t(x, y) = (y, x).
\]

Then \(t\) is a \(*\)-homomorphism, \(t^2 = id_{\Lambda}\), and one checks that the following square commutes

\[
\Lambda \xrightarrow{\eta \otimes id} \Lambda^2 \xrightarrow{\tau} \Lambda
\]

Let \(\pi_i : \Lambda \oplus \Lambda \to \Lambda\) be the \(i\)-th coordinate projection. The map \(id \oplus t\) completes to a split distinguished triangle in \(kk^h\):

\[
\Lambda \xrightarrow{id \oplus t} \Lambda \oplus \Lambda \xrightarrow{\pi_1 + t\pi_2} \Lambda.
\]

Rotating the split triangle above we get the triangle

\[
\Omega\Lambda \xrightarrow{0} \Lambda \xrightarrow{id \oplus t} \Lambda \oplus \Lambda.
\]

Then (10.6) extends to a map of triangles in \(kk^h\):

\[
VA \xrightarrow{\eta \otimes id} \Lambda^2 \xrightarrow{\tau} \Lambda
\]

It follows that the dashed map is an isomorphism.

\[\square\]

**Lemma 10.7.** There is an isomorphism

\[j^h(\Sigma VU) \cong j^h(\ell)\]

In particular, \(j^h(VU) \cong j^h(\Omega)\).
Proof. As before, we omit $j^h$ from the notation. In view of (2.1.19) and because $\iota_+ : \ell \to M_\pm$ is an isomorphism in $kk^h$, it suffices to show that $\Sigma U$ is $kk^h$-isomorphic to $1M_2$. Let

$$VU \to U \xrightarrow{\eta \otimes id_U} \Lambda U$$

be the triangle in $kk^h$ that defines $VU$. The $kk^h$ isomorphism between $\Lambda U = \Lambda A$ and $\Lambda$ established in Lemma 10.2 is induced by mapping $A^2$ to $\Lambda \oplus \Lambda$ and then retracting onto the first coordinate. Using this fact we get that there is a map of triangles in $kk^h$

$$U \xrightarrow{\eta \otimes id_U} \Lambda U \xrightarrow{i} \Sigma VU \xrightarrow{\phi} 1M_2.$$ 

It follows that the dashed $kk^h$-map is an isomorphism.

Remark 10.8. By Lemma 6.2.10, the isomorphisms of Lemmas 10.2 and 10.7 induce isomorphisms $j^h(UA) \cong j^h(AA)$, $j^h(VAA) \cong j^h(\Omega AA)$ and $j^h(VUA) = j^h(A)[1]$ for every $A \in Alg^*_\ell$.

11. Karoubi’s Fundamental Theorem

Theorem 11.1 (Karoubi, [9]). Assume that $\ell$ satisfies the $\lambda$-assumption 2.1.14. There is an element $\theta_0 \in -1K^h_2(U^2)$ such that

i) The composite $-1K^h_2((U')^2) \to -1K^h_2(\Sigma inv(U')) \cong -1K^h_2(\Sigma \Omega inv(U')) \cong K^h_1(U') \cong K_0(U) = \mathbb{Z}$ maps $\theta_0$ to 1.

ii) For every unital $*$-$\ell$-algebra $R$, the product with $\theta_0$ induces an isomorphism

$$\theta_0 \star - : K^h_*(R) \to K^h_{*-2}((-1M_2(U')^2)R)$$

Proof. By Lemma 3.10, for any unital $*$-ring $S$, we have $-1K^h_2(S) = K^h_1((-1M_2S)$. The element $\theta_0$ of the present theorem appears under the name of $\sigma$ in the first line of [9, Section 3.1]. Thus as mentioned by Karoubi in [9, Paragraph 1.3 of page 263 through line 3 of page 264], the current theorem is just another way of phrasing Karoubi’s fundamental theorem. Indeed, as said in Remark 10.1, Karoubi showed that $\Omega K^h_0(V'U'R) \xrightarrow{\cong} K^h_0(R)$, so the theorem as stated here is equivalent to that proved in [9, Section 3.5], which says that cup-product with $\theta_0$ induces an isomorphism $K^h_0(V'R) \xrightarrow{\cong} -1K^h_{-1}(U'R)$.

We shall use Theorem 11.1 to prove the following.

Theorem 11.2. Assume that $\ell$ satisfies the $\lambda$-assumption 2.1.14. Then the image $\theta$ of $c_2(\theta_0)$ under the isomorphism $KH^h_2((-1M_2(U')^2) \cong KH^h_0((-1M_2U^2) = kkh((\ell, (-1M_2U^2))$ induces a natural isomorphism

$$\theta_A := \theta \otimes id^{j^h}_{A} : j^h(A) \xrightarrow{\cong} j^h((-1M_2U^2)A) \forall A \in Alg^*_\ell.$$

Corollary 11.3. Let $\ell \in \ell$ be unitary. For every $A \in Alg^*_\ell$, $j^h(M_2VA)[1] \cong j^h((-1M_2UA).$

Proof. It is immediate from Theorem 11.2, Lemma 10.7 and Remark 10.8 that $j^h(VA)[1] \cong j^h((-1M_2UA)$. The corollary follows from this applied to $M_2A$ using the isomorphism

(11.4) $-1M_2(M_2) \cong M_\pm - M_2$

and hermitian stability.

□
Proof of Theorem 11.2. By Lemma 6.2.10 it suffices to show that $\theta = \theta_\ell$ is an isomorphism, or what is the same, that $kk^h(\ell, \theta)$ and $kk^h(-1M_2U^2, \theta)$ are isomorphisms. Taking into account that $-1M_2(-1M_2) \cong M_2(1M_2) \cong M_2^2$ and that the latter is $kk^h$-equivalent to $\ell$ and using Lemma 10.7, we see that $kk^h(-1M_2U^2, \theta)$ is an isomorphism if and only if $kk^h(\ell, \theta, M_2(\Sigma)_2^2)$ is one. Hence the theorem will follow if we prove that $(\theta_A)_*: kkh(\ell, \theta_A)$ is an isomorphism for all $A$. By Lemma 8.3, under the isomorphism of Proposition 8.1, $(\theta_A)_*$ corresponds to the cup-product with $\theta$. By Lemma 4.5 the latter is induced by the product of Corollary 4.4 i) with $\theta_m$, up to a degree shift. By Theorem 11.1, $\theta_0 * - : \Lambda_m^h(A) \to \Lambda_{m+2}(-1M_2(\Sigma)^2)A$ is an isomorphism for all $m \in \mathbb{Z}$ and all unital $A \in \text{Alg}_\ell$. Using excision, we obtain that the same is true also for not necessarily unital $A$ if $m \leq -2$. It follows that $\theta * - : KH_\ell(A) \to KH_\ell(-1M_2U^2A)$ is an isomorphism for all $A$, concluding the proof.  

We finish the section with a lemma that will be used in Section 12.

Lemma 11.5. Consider the isomorphisms $j^h(UA) \cong j^h(A)$ of Lemma 10.2 and $M_2A \cong -1M_2A$ of (9.10). Then the following diagram commutes

$$
\begin{array}{ccc}
j^h(A) & \xrightarrow{\theta_h} & j^h(-1M_2U^2A) \\
\downarrow{\sim} & & \downarrow{\sim} \\
j^h(M_2A) & \cong & j^h(-1M_2A)
\end{array}
$$

Proof. By Theorem 11.1, $\theta_0$ goes to $[1] \in K_0(\mathbb{Z})$; it is straightforward from this that $\theta$ goes to $[1] \in KH_0^h(A) \cong KH_0(\ell)$. This proves the lemma.

12. The Bivariant 12 term exact sequence

Definition 12.1. Let $A, B \in \text{Alg}_\ell^\epsilon, \epsilon \in \ell$ a unitary, $\epsilon kk^h(A, B)$ as in (6.2.12) and $t$ as in (10.5). Put

$$
\begin{align*}
\epsilon W(A, B) &= \text{Coker}(kk^h(A, \Lambda B) \xrightarrow{\phi_{\text{proj}}} \epsilon kk^h(A, B)) \\
\epsilon W'(A, B) &= \text{Ker}(kk^h(A, B) \xrightarrow{\eta_m} \epsilon kk^h(A, \Lambda B)) \\
k(A, B) &= \{x \in kk^h(A, \Lambda B) : x = t_x, \{x = y + t_x y\} \\
k'(A, B) &= \{x \in kk^h(A, \Lambda B) : x = -t_x, \{x = y - t_x y\}
\end{align*}
$$

Where if $\epsilon = 1$ we omit it from the notation.

Theorem 12.2 (cf. [9, Théorème 4.3]). There is an exact sequence

$$
\begin{array}{ccc}
k(A, \Omega B) & \xrightarrow{\sim} & -1W(A, \Omega^2B) \\
\epsilon W(A, B) & \xrightarrow{\beta} & W'(A, B) \\
k'(A, \Omega B) & \xrightarrow{\phi} & -1W'(A, \Omega B) \\
W(A, \Omega B) & \leftrightarrow & W'(A, \Omega B) \\
k(A, \Omega B) & \leftrightarrow & \epsilon W'(A, \Omega B)
\end{array}
$$

Proof. As above, we omit $j^h$ in our notation. Consider the following distinguished triangles in $kk^h$

$$
\begin{align*}
\Omega AB & \xrightarrow{\partial} VB \xrightarrow{\rho} B \xrightarrow{\eta_m} \Lambda B \\
\Omega^2 - 1M_2B & \xrightarrow{\delta} \Omega - 1M_2UB \xrightarrow{\psi} \Omega - 1M_2AB \xrightarrow{\phi m} \Omega - 1M_2B.
\end{align*}
$$
Recall the isomorphism \( \tau: \Lambda^2 \to \Lambda \oplus \Lambda \) used in Lemma 10.2. Let \( \eta_B = \eta \oplus \id_B : B \to \Lambda B \). Using Lemma 11.5 we get the following commutative diagram in \( \kk^h \):

\[
\begin{array}{cccc}
\Omega AB & \xrightarrow{\partial} & VB & \xrightarrow{\theta} \Omega_{-1}M_2 UB & \xrightarrow{\psi} \Omega_{-1}M_2 \Lambda B \\
\downarrow \id_{\Omega AB} \otimes \eta_B & & \downarrow \id_V \otimes \eta_B & & \downarrow \id_{\Omega_{-1}M_2 UB} \otimes \eta_B \\
\Omega^2 B & \xrightarrow{\partial} & VAB & \xrightarrow{\theta} \Omega_{-1}M_2 UAB & \xrightarrow{\psi} \Omega_{-1}M_2 \Lambda^2 B \\
\tau \oplus \id_B & & \iota & & \iota \\
\Omega AB \oplus \Omega AB & \xrightarrow{\pi_1 - t \pi_2} & \Lambda B & \xrightarrow{\id \oplus - \id} & \Lambda B \oplus \Omega AB.
\end{array}
\]

Here \( \tilde{\tau} \) is the composition of the isomorphism \( 9.10 \), the inverse of the corner inclusion and the map \( \tau \otimes \id_{\Omega_{-1}M_2 B} \). From the diagram we get following equality

\[
(\tau \otimes \id_B)(\id_{\Omega_{-1}M_2 B} \otimes \eta_B) = \id_{\Omega_{-1}M_2 B} \otimes \id_{\Omega AB}.
\]

Similarly, for \( h_{-1} \) as in Examples 2.1.8 and \( \iota_1 \) the upper left-hand corner inclusion, we have

\[
(\tilde{\tau} \otimes \id_{\Omega_{-1}M_2 B})(\id_{\Omega_{-1}M_2 B} \otimes \eta_B) = (\id_{\Omega_{-1}M_2 B} \otimes \id_{\Omega AB})(\iota_1)^{-1} \mathrm{ad}(1, h_{-1}^{-1}).
\]

Therefore, composing both sides of the equality \((12.3)\) on the left with the projection onto the first coordinate, we get

\[
(\iota_1)^{-1} \mathrm{ad}(1, h_{-1}^{-1}) \psi \theta \partial = (\pi_1 - t \pi_2)(\tau \otimes \id_B)(\id \oplus \id) = \id - t.
\]

After applying \( \kk^h(A, \cdot) \) and using the identification \( \_1 \kk^h(A, \Lambda B) \cong k\kk^h(A, \Lambda B) \), we get

\[
(\psi \theta \partial)_* = \id - t_*.
\]

The rest of the proof follows exactly as in [9, Théorème 4.3].

**Remark 12.4.** Theorem 1.8 follows from Theorem 12.2 applied with \( B \) replaced by \( \Omega^n\Sigma B \) if \( n \geq 0 \) and by \( \Sigma^{-n-1}B \) otherwise, using the isomorphism \((11.4)\) and hermitian stability.

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