Positive Robinson theories and h-maximal models

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Introduction

In this paper we continue the exploration of the classes of positively closed and h-maximal model of an h-inductive theory in the context of positive logic. In the section 2 we give a concrete description of the class of h-maximal models of an h-inductive theory and theirs companion theories. The section 3 is concerned to the study of the positive Robinson and locally positive Robinson theories and their connexion with the properties of the class of h-maximal models of the companion theories, and their connexion with the property of elimination of quantifiers. Before dealing with the topics mentioned above we give in section 1 a brief introduction to the positive model theory.

1 Positive model theory

The positive logic is a continuation of the line of research on universal theories initiated by Abraham Robinson, based on the study of the notions of inductive theories, existentially closed models, model-complete theories through the notions of embedding, existential formula. The systematic treatment of the positive model theory has been undertaken by Ben Yaacov and Poizat in [2]. In short consists of non-use of negation in building of formulas.

Let $L$ be a first order language. The positive formulas are expressed as: $\exists \bar{x} \psi(\bar{x}, \bar{y})$, where $\psi$ is a formula, the variables $\bar{y}$ are said to be free.

A sentence is a formula without free variables. A sentence is said to be h-inductive (resp. f-inductive) if it is a finite conjunction of sentences of the form:

$$\forall \bar{x} \exists \bar{y} \psi(\bar{x}, \bar{y}) \rightarrow \exists \bar{z} \varphi(\bar{x}, \bar{z})$$

(resp. $\forall \bar{x} \alpha(\bar{x}) \rightarrow \beta(\bar{x})$) where $\psi, \varphi$ (resp. $\alpha, \beta$) are quantifier-free positive formulas.

The h-universal sentences represent a special case of h-inductive sentences, they are the sentences that can be written as negation of a positive sentence.

Given two $L$-structures $A$ and $B$ be over an arbitrary language $L$. A mapping $f$ from $A$ into $B$ is a homomorphism if for every $\bar{a} \in A$ and for every positive
atomic formula $\phi$;
$$A \models \phi(\bar{a}) \Rightarrow B \models \phi(f(a)).$$

A structure $B$ is said to be a continuation of a structure $A$ if and only if there exists a homomorphism from $A$ into $B$.

A homomorphism $f$ is an embedding if and only if for every $\bar{a} \in A$ the tuples $\bar{a}$ and $f(\bar{a})$ satisfy the same atomic formulas.

A homomorphism $f$ is an immersion if and only if for every $\bar{a} \in A$ and for every positive formula $\varphi$: $A \models \varphi(\bar{a})$ if and only if $B \models \varphi(f(\bar{a}))$. We say that $A$ is immersed in $B$ if there exist an immersion from $A$ into $B$.

A class of $L$-structures is said to be $h$-inductive if it is closed with respect to inductive limits of homomorphisms. In [2] it is shown that the class of models of an $h$-inductive theory is $h$-inductive and the class of models of an arbitrary theory $T$ is $h$-inductive if $T$ is axiomatized by an $h$-inductive theory.

1.1 positively closed structures

**Definition 1** A member $M$ of a class $\Gamma$ of $L$-structures is said to be positively closed (pc from now on) in $\Gamma$, if every homomorphism from $M$ into a member of $\Gamma$ is an immersion.

**Fact 1 ([2] Theorem 1)** Every member of an $h$-inductive class of $L$-structures has a positively closed continuation in the same class.

The $h$-inductivity of the class $\Gamma$ is a necessary condition of the existence of pc structure. In this case the class of pc members of $\Gamma$ forms an $h$-inductive and $h$-cofinal subclass of $\Gamma$.

Let $\Gamma$ be an $h$-inductive class of $L$-structure. We denote by $\Pi(\Gamma)$ the class of positively closed member of $\Gamma$. If $\Gamma$ is the class of models of an $h$-inductive theory $T$, we use the notation $\Pi(T)$.

**Definition 2** Two $h$-inductive theories over a language $L$ are said to be companions if they have the same pc models.

Note that every $h$-inductive theory $T$ admits:

- A maximal companion theory denoted $T_k(T)$, called the Kaiser’s hull theory of $T$. By definition $T_k(T)$ is the set of $h$-inductive sentences satisfied by the pc models of $T$.
- A minimal companion theory denoted $T_u(T)$, it is the set of $h$-universal sentences true in the pc models of $T$.

Let $L$ be a first order language and $M$ be a $L$-structure.

- we denote by $T_i(M)$ (resp. $T_u(M)$) the set of $h$-inductive (resp. of $h$-universal) sentences satisfied by $M$ in the language obtained from $L$ by adding the elements of $M$ as constants.
• we denote by $T^*_i(M)$ (resp. $T^*_v(M)$) the set of $h$-inductive (resp. of $h$-universal) $L$-sentences satisfied by $M$.

Note that for every $L$-structure $M$ we have;

$$T^*_i(M) \subset T_k(T^*_i(M)), \quad T^*_v(M) \subset T_u(T^*_v(M)).$$

In the language obtained from $L$ by adding the elements of $M$ as constants. We have;

$$T_k(T^*_i(M)) = T^*_i(M), \quad T_u(T^*_v(M)) = T^*_v(M).$$

If $A$ a pc model of an $h$-inductive theory $T$. We obtain;

$$T^*_i(A) = T_k(T^*_i(A)), \quad T^*_v(A) = T_u(T^*_v(A)).$$

In this case we use the notation $T^*_k(A)$ instead of $T^*_i(A)$.

**Definition 3** Let $T$ be an $h$-inductive theory.

• $T$ is said to be model-complete if every model of $T$ is a pc model of $T$.

• We say that $T$ has a model-companion whenever $T_k(T)$ is model-complete.

• An $n$-type is a maximal set of positive formulas in $n$ variables that is consistent with $T$. We denote by $S_n(T)$ the space of $n$-types of a theory $T$.

Let $M$ be a $L$-structure and $\bar{m}$ a tuple of $M$. We denote by $tp_M(\bar{m})$ the set of positive formulas satisfied by $\bar{m}$ in $M$.

**Fact 2** $M$ is pc model of $T$ if and only if, for every $\bar{a} \in A$, the set of positive formulas satisfied by $\bar{a}$ is a type of $T$.

For every positive formula $\phi$, we denote by $Ctr_T(\phi)$ the set of positive formulas $\psi$ such that $T \vdash \neg \exists x(\phi(x) \land \psi(x))$.

Let $A$ be a pc model of $T$. Let $\bar{a} \in A$ such that $A \not \models \phi(\bar{a})$ where $\phi$ is a positive formula. By the maximality of $tp_A(\bar{a})$, there is a positive formula $\psi \in Ctr_T(\phi)$ such that $A \models \psi(\bar{a})$.

This property is in fact the inner characteristic of these subclass of models of $T$. We have the following fact.

**Fact 3** $A$ is pc model of $T$ if and only if for every $\bar{a} \in A$, and for every positive formula $\varphi$; if $A \not \models \varphi(\bar{a})$ there exists a positive formula $\psi$ such that $A \models \psi(\bar{a})$ and $\psi \in Ctr_T(\phi)$.

Consider a pc model $A$ of $T$ and $\bar{a} \in A$. We denote by $tp(\bar{a})$ (resp. $tpqf(\bar{a})$) the type of $\bar{a}$ in $A$ (resp. the set of quantifier-free positive formulas satisfied by $\bar{a}$ in $A$).

One defines on $S_n(T)$ the topology generated by the following basis of closed sets:

$$F_\varphi = \{ p \in S_n(T) \mid p \vdash \varphi \}.$$

where $\varphi$ ranges over the set of positive formulas.

Note that for every $n$, The space of positive types $S_n(T)$ is compact but generally is not Hausdorff.
Definition 4 Let $T$ be an h-inductive theory and $\varphi$ a positive formula;

- $\varphi$ is said to be $T$-complemented if and only if there is a positive formula $\psi \in \text{Ctr}_T(\varphi)$ such that:
  \[ T \vdash \forall \bar{x} \ (\varphi(\bar{x}) \lor \psi(\bar{x})). \]

The formula $\psi$ is called the $T$-complement of $\varphi$.

- Let $\Gamma$ be a subset of $\text{Ctr}_T(\varphi)$. We say that $\text{Ctr}_T(\varphi)$ is logically equivalent to $\Gamma$ modulo $T$ and we write $\text{Ctr}_T(\varphi) \approx_T \Gamma$; if and only if for every $\psi \in \text{Ctr}_T(\varphi)$ there is $\phi \in \Gamma$ such that
  \[ T \vdash \forall \bar{x} \ (\psi(\bar{x}) \rightarrow \phi(\bar{x})). \]

Remark 1 Let $T$ be an h-inductive theory.

- A formula $\varphi$ is $T_k(T)$-complemented if and only if there is a positive formula $\psi$ such that $\text{Ctr}_T(\varphi) \approx_{T_k(T)} \psi$.

- The class of pc models of $T$ is elementary if and only if, for every positive formula $\psi$, $\text{Ctr}_T(\psi)$ is logically equivalent modulo $T_k(T)$ to a positive formula.

Examples 1

1. Let $L$ be the relational language formed a binary relation $S$. Consider the following h-inductive theory:
   \[ T = \{ \neg \exists xy \ (S(x,y) \land S(y,x)), \forall xyz \ ((S(x,z) \land S(y,z)) \rightarrow x = y) \}. \]

   The model of $T$ formed by the p-cycles where $p = 4$ or $p$ is a prime number greater-than or equal to 3 is the unique pc model of $T$.

   Let $T'$ be the theory obtained from $T$ by adding the h-universal sentence
   \[ \neg \exists x_1x_2x_3x_4 ((\bigwedge_{i=1}^{3} S(x_i, x_{i+1})) \land S(x_4, x_1)). \]

   The structure formed by the p-cycles where $p$ ranges over the set of prime numbers greater-than or equal to 3 is the unique pc model of $T'$.

2. Let $L$ and $T'$ be the language and the theory given in the example above. Let $n$ be an integer greater-than 3. Consider $T_n$ the h-inductive theory obtained from $T'$ by adding the following set of h-inductive sentences
   \[ \{ \forall x_1 \cdots x_m ((\bigwedge_{i=1}^{m} S(x_i, x_{i+1})) \land S(x_m, x_1)) \rightarrow \bigvee_{i \neq j} x_i = x_j \mid m > n \}. \]

   The structure formed by the p-cycles where $p$ is a prime number less than $n$ is the unique pc model of $T_n$. Thereby $T_n$ has a model-companion.
3. Let $T_{ag}$ be the $h$-inductive theory of abelian groups in the language $L = \{, -1, e\}$. In the positive logic $T_{ag}$ has a model-companion. The trivial group $\{e\}$ is the unique pc model of $T_{ag}$. However, in the context of first order logic the class of existentially closed abelian groups is the class of divisible abelian groups which contain for each prime $p$ an infinite number elements of order $p$ (Theorem 2.4 [3]).

To extend the discussion began on the last example. Consider the language $L^*$ obtained from the language $L$ of the theory $T_{ag}$ by adding a constant $a$. Let $T_{ag}^+$ be the $h$-inductive theory $T_{ag}, \{\neg a = e\}$.

Let $(G, g)$ be a pc model of $T_{ag}^+$ where $g$ is the interpretation of the constant $a$ in $G$, we have the following properties

1. The constant $g$ belongs to every non trivial subgroup of $G$. Indeed let $N$ be a non trivial subgroup of $G$ and $\pi$ the $L$-homomorphism $G \to G/N$. Suppose that $\pi$ is a $L^*$-homomorphism. Then $\pi$ is an immersion. Consequently $N = \{e\}$. Thereby $\pi$ can not be a $L^*$-homomorphism, so $\pi(g) = e$. The constant $g$ belongs to the intersection of all subgroups of $G$. Thereby for every $x \in G$ there is $k \in \mathbb{Z}$ such that $g = x^k$.

2. $G$ cannot admit distinct subgroups of order $p$ and $q$ respectively, where $p$ and $q$ are prime to each other. Because if not, the order of $g$ will be a common divisor of $p$ and $q$.

3. $G$ cannot be the direct sum of some of its subgroups; because the constant must belong to the intersection of all subgroups.

**Lemma 1** The pc models of $T_{ag}^+$ are the groups $\{(\mathbb{Z}(p), z_p)\}$, where $p$ is a prime number, $z_p \in \{\mathbb{Z}(p) - 1\}$, and $\mathbb{Z}(p)$ is the group of all complex $p^n$-th roots of unity.

**Proof.** Let $(G, g)$ be a pc model of $T_{ag}^+$. We distinguish two cases:

- $o(g)$ (the order of $g$ in $G$) is finite. In this case $o(g)$ is a prime number. Indeed, if not we can find a subgroup of $H = \langle g \rangle$ that does not contain the constant $g$.

- $o(g)$ is infinite. This case cannot take place because we can find a subgroup of $H = \langle g \rangle$ which does not contain the constant $g$.

Therefore, if $(G, g)$ is a pc model of $T_{ag}^+$ there is $p$ a prime number such that $(G, g)$ is the group of all complex $p^n$-th roots of unity.

### 1.2 Amalgamation property

The notion of amalgamation in positive logic provides a useful means for intuition and motivation. One of these facts is the characterization of the Hausdorff property by the amalgamation property given in [2]. For more expositions of these facts see [1, 2].
Definition 5 Let $\Gamma$ be a class of $L$-structures. An element $A$ of $\Gamma$ is said to be an amalgamation basis of $\Gamma$ if and only if, for every $B, C$ in $\Gamma$, and $f, g$ homomorphisms from $A$ respectively into $B$ and $C$, there exist $D \in \Gamma$ and $f', g'$ homomorphisms such that the following diagram commutes:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{g} & & \downarrow{g'} \\
C & \xrightarrow{f'} & D
\end{array}
\]

We say that $\Gamma$ has the amalgamation property if every element of $\Gamma$ is an amalgamation basis of $\Gamma$.

Note that under certain conditions, each structure can benefit of the property of being an amalgamation basis. On other words in every class of $L$-structures, we can always find universal amalgamations. The useful following fact provides an example of these universal amalgamations.

Fact 4 ([1, lemma 4]) Let $A, B, C$ be $L$-structures such that; $A$ is immersed in $B$ and continued in $C$ by a homomorphism $f$. Then there is $D$ a model of $T_k(C)$ such that the following diagram commutes.

\[
\begin{array}{ccc}
A & \xrightarrow{i_m} & B \\
\downarrow{f} & & \downarrow{g} \\
C & \xrightarrow{i_m} & D
\end{array}
\]

Where $i_m$ in the diagram are immersions and $g$ a homomorphism.

One of the most important property of the class of pc models of an h-inductive theory is the amalgamation property (theorem 9[2]). As a simple application of the amalgamation property we have the following lemma.

Lemma 2 Let $T$ be an h-inductive theory such that the class of pc model (resp. of amalgamation bases) is closed under product. Then $T_k(T)$ has only one pc model, this pc model has only one point.

Proof. Let $A, B$ be two pc models of $T$. Since $A \times B$ is a pc model (resp. an amalgamation basis), we obtain the following commutative diagram:

\[
\begin{array}{ccc}
A \times B & \xrightarrow{\text{Pr}_A} & A \\
\downarrow{\text{Pr}_B} & & \downarrow{f} \\
B & \xrightarrow{g} & C
\end{array}
\]

where $C$ is a pc model of $T$, $f$ and $g$ are immersions. Thus for all $a \in A$ and $b \in B$ we have $f(a) = g(b)$, and so $\text{tp}(a) = \text{tp}(b)$. Consequently every constant mapping from $A$ into $A$ is an immersion. Thereby $A = \{a\}$. 
1.3 Complete theories

**Definition 6** An $h$-inductive theory $T$ is said to be complete or has joint continuation property if any two of its models can be simultaneous continued into a third one.

**Remark 2**
- Let $A, B$ be two pc models of $T$. $T^*_k(A) = T^*_k(B)$ if and only if $A$ and $B$ have the same continuation.
- An $h$-inductive theory is complete if and only if its pc models have the same $h$-inductive theory.

**Lemma 3** Let $A$ be a pc model of $T$ then $T^*_k(A)$ is a complete theory.

**Proof.** Let $B, C$ be models of $T^*_k(A)$. Firstly we show that 
\{ $T^*_k(A), \text{Diag}^+(A), \text{Diag}^+(B)$ \} is consistent, then we conclude that 
\{ $T^*_k(A), \text{Diag}^+(B), \text{Diag}^+(C)$ \} is consistent.

Since for every $\forall \varphi(\bar{a}) \in \text{Diag}^+(A)$ we have $\exists \bar{x} \varphi(\bar{x}) \in T^*_k(A)$, and $B \vdash T^*_k(A)$, then 
by compactness we obtain the consistency of \{ $T^*_k(A), \text{Diag}^+(A), \text{Diag}^+(B)$ \}.

Now, let $B^*$ be a model of \{ $T^*_k(A), \text{Diag}^+(A), \text{Diag}^+(B)$ \} and $C^*$ a model of \{ $T^*_k(A), \text{Diag}^+(A), \text{Diag}^+(C)$ \}. Since the class of pc models of $T$ has the amalgamation property, $B^*$ and $C^*$ are models of $T$. We obtain the following commutative diagram:

\[
\begin{array}{ccc}
B & \xrightarrow{f_1} & B^* \\
\downarrow{i_1} & & \downarrow{f_2} \\
A & \xrightarrow{i_2} & C \\
\end{array}
\]

where $i_1, i_2$ are immersions, $f_1, f_2$ are homomorphisms and $D$ is a model of $T$ that can be assumed a pc model of $T$. Thus, $f_1 \circ i_1$ is an immersion, and $D$ is a model of $T^*_k(A)$ in which $B$ and $C$ are immersed. Consequently \{ $T^*_k(A), \text{Diag}^+(B), \text{Diag}^+(C)$ \} is consistent, and $T^*_k(A)$ is complete theory.

**Corollary 1** Let $A$ be a pc model of $T$. Every pc model of $T^*_k(A)$ is a pc model of $T$.

**Proof.** Let $C$ be a pc model of $T^*_k(A)$ and $B$ a pc model of $T$ such that $C$ is continued in $B$ by a homomorphism $f$. By the lemma[3] and the fact[4] we obtain...
the following commutative diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{i'} & B' \\
\downarrow{g} & & \downarrow{g} \\
C & \xrightarrow{f} & D \\
\uparrow{f} & & \uparrow{f} \\
B & \xrightarrow{i_m} & D' \\
\end{array}
\]

where \(B'\) is a model of \(T_k^*(A)\) in which \(A\) and \(C\) are immersed (lemma \(3\)). \(D \models T_1(B)\) (fact \(4\)). \(D'\) is a pc model of \(T\) in which \(D\) is continued by a homomorphism \(h\).

Now, since \(D'\) and \(A\) are pc models of \(T\), then \(h \circ g \circ i'\) is an immersion and \(D' \models T_k^*(A)\). By the fact that \(C\) is a pc model of \(T_k^*(A)\) we deduce that \(h \circ i_m \circ f\) is an immersion. Consequently \(f\) is an immersion and \(C\) is a pc model of \(T\).

**Lemma 4** Let \(T_1\) and \(T_2\) be two h-inductive theories such that \(T_1\) is a complete theory and there exists a common pc model \(A\) of \(T_1\) and \(T_2\), then every pc model of \(T_1\) is a pc model of \(T_2\).

**Proof.** Let \(B\) be a pc model of \(T_1\). Since \(A\) is a common pc model of \(T_1\) and \(T_2\), and \(T\) is a complete theory, then

\[T_k(T_2) \subset T_k^*(A) = T_k^*(B)\]

This implies that \(B\) is a model of \(T_2\).

On the other hand, since \(T_1\) is complete there exist \(D\) a pc model of \(T_1\) in which \(A\) and \(B\) are immersed, and so \(D\) is a model of \(T_2\). Let \(C\) be a pc model of \(T_2\) in which \(D\) is continued by an homomorphism \(f\) as shown in the following diagram

\[
\begin{array}{ccc}
B & \xrightarrow{i_m} & C \\
\downarrow{f} & & \\
A & \xrightarrow{i_m} & D \\
\end{array}
\]

where \(i_m\) denotes immersions. Given that \(A\) and \(C\) are pc models of \(T_2\) we obtain \(T_k^*(A) = T_k^*(C)\). Consequently \(C \models T_1\) and \(f\) is an immersion, so \(D\) and \(B\) are pc models of \(T_2\).

2 **H-maximal models**

In \(5\) Kungozhin introduced the notion of h-maximal model in the context of studying the elementarity of the classes of pc modelees and h-maximal models of finitely universal theories. In this section ???
Definition 7 Let $T$ be an $h$-inductive theory. A model $A$ of $T$ is said to be $h$-maximal if every homomorphism from $A$ into a model of $T$ is an embedding.

Note that the class of $h$-maximal models of an $h$-inductive theory $T$ forms an inductive class and every model of $T$ is continued in a $h$-maximal model of $T$.

Examples 2

1. Let $T$ and $T'$ be the theories defined in [1, example 1]. The class of $h$-maximal models of $T$ (resp. $T'$) is the class of substructures of the $pc$ model of $T$ (resp. $T'$).

2. The class of $h$-maximal models of the theory $T_n$ given in [2, example 1] is the class of substructures of the $pc$ model of $T_n$. This implies that the class of $h$-maximal models of $T_n$ is not elementary.

3. The class of $h$-maximal models of $T^+_a$ is the class of $pc$ models of $T^+_a$.

4. Consider $T^+_g = T^+_a - \{ \forall x y x y = yx \}$ the theory of groups in the language $L^*$. The $h$-maximal models of $T^+_g$ are the groups whose non trivial normal subgroups contain the constant of the language $L^*$. Indeed, since the $L^*$-homomorphisms are the homomorphisms $f$ of groups such that $f(a) \neq 0$ where $a$ is the interpretation constant of $L^*$. Thus if $(G, a)$ is a $h$-maximal model of $T^+_g$ and $N$ a non trivial normal subgroup of $G$ such that $a \notin N$. The canonical mapping from $G$ into $G/N$ is a $L^*$-homomorphism but not an embedding. Thereby $(G, a)$ can not be $h$-maximal model.

Remark 3 If $\Gamma_1$ and $\Gamma_2$ are $h$-inductive classes that have the same $h$-maximal models. Then they are companion theories.

Let $T$ be an $h$-inductive theory and $T_m(T)$ the set of $h$-inductive sentences satisfied in each $h$-maximal model of $T$. Given that the class of $pc$ models of $T$ is a subclass of the class of $h$-maximal models of $T$, then $T \subseteq T_m(T) \subseteq T_k(T)$. So $T$ and $T_m(T)$ are companion theories.

Definition 8 Let $T$ be an $h$-inductive theory and $\Gamma$ the set of sentences of the form $\forall x \phi(x) \rightarrow \psi(x)$ satisfied in every $h$-maximal model of $T$ and such that $\phi$ and $\psi$ are quantifier-free positive formulas. We denote by $T_f(T)$ the $h$-inductive theory $T, \Gamma$. we have $T \subseteq T_f(T) \subseteq T_m(T)$.

Lemma 5 The $h$-inductive theories $T, T_f(T)$ and $T_m(T)$ have the same class of $h$-maximal models.

Proof. Let $A$ be a $h$-maximal model of $T$, $B$ a model of $T_f(T)$ and $C$ a model of $T_m(T)$. Let $f$ (resp. $g$) be a homomorphism from $A$ into $B$ (resp. into $C$). Since $B$ and $C$ are models of $T$, and $A$ is a model of both theories $T_m(T)$ and $T_f(T)$ then $f$ and $g$ are embeddings, and $A$ is a $h$-maximal of $T_f(T)$ and $T_m(T)$.
Let $A$ be a h-maximal model of $T_m(T)$. Given that $A$ is a model of the theories $T_f(T)$ and $T$, there are $B$ a model of $T$ and $C$ a model $T_f(T)$ such that $A$ is continued in $B$ by a homomorphism $g$ and continued in $C$ by a homomorphism $f$. Since $B$ and $C$ are also models of $T$, there exist $B'$ and $C'$ h-maximal models of $T$ such that $B$ is continued in $B'$ by a homomorphism $f'$, $C$ is continued in $C'$ by a homomorphism $g'$. Now given that $B'$ and $C'$ are models of $T_m(T)$ then $f' \circ f$ and $g' \circ g$ are embeddings. Thereby $g$ and $f$ are embeddings. Consequently $A$ is a h-maximal of $T$ and $T_f(T)$.

By the same way we show that every h-maximal of $T_f(T)$ is a h-maximal of $T$. Therefore the theories $T, T_f(T)$ and $T_m(T)$ have the same class of h-maximal models.

**Remark 4** Consider $T$ an h-inductive theory. We denote by $\Sigma_T$ the class of h-maximal model of $T$. We have $\Sigma_{T_f(T)} \subseteq \Sigma_T = \Sigma_{T_f(T)} \subseteq \Sigma_{T_m(T)} \subseteq \Sigma_T$.

**Lemma 6** A model $A$ of $T$ is h-maximal model if and only if for every quantifier-free positive formula $\varphi$ and a tuple $\bar{a} \in A$ such that $A \not\models \varphi(\bar{a})$, there is $\psi(\bar{x})$ a positive formula $\psi \in Ctr_T(\varphi)$ such that $A \models \psi(\bar{a})$.

**Proof.** Let $A$ be a h-maximal model of $T$, $\bar{a} \in A$ and $\varphi$ a quantifier-free positive formula such that $A \not\models \varphi(\bar{a})$. Since every homomorphism from $A$ into a model of $T$ is an embedding then the set of h-inductive sentences $\{T, Diag^+(A), \varphi(\bar{a})\}$ is inconsistent. Thus by compactness there exists $\phi(\bar{a}, \bar{b}) \in Diag^+(A)$ such that $T \vdash \neg \exists \bar{x}(\varphi(\bar{x}) \land \psi(\bar{x}))$ where $\psi$ is the positive formula $\exists \bar{y} \varphi(\bar{x}, \bar{y})$.

Conversely, let $A$ be a model of $T$ such that for every quantifier-free positive formula $\varphi$ and $\bar{a} \in A$, if $A \not\models \varphi(\bar{a})$ then there is $\psi(\bar{x})$ a positive formula such that $A \models \psi(\bar{a})$ and $T \vdash \neg \exists \bar{x}(\varphi(\bar{x}) \land \psi(\bar{x}))$. It is obvious that every homomorphism from $A$ into a model of $T$ is an embedding, then $A$ is a h-maximal model of $T$.

**Corollary 2** If $A$ is immersed in a h-maximal model of $T$ then $A \in \Sigma_T$.

**Proof.** Since $A$ is immersed in a model of $T$ then $A \models T$. The fact that $A$ is h-maximal results of the lemma.

**Theorem 1** $\Sigma_T$ is elementary class if and only if for every quantifier-free positive formula $\varphi$, there is a positive formula $\psi$ such that

$$Ctr_{T_m(T)}(\varphi) \approx_{T_m(T)} \{\psi\}.$$ 

**Proof.** Suppose that $\Sigma_T$ is elementary and axiomatized by $T_m(T)$. Assume the existence of a quantifier-free positive formula $\varphi$ such that $Ctr_{T_m(T)}(\varphi)$ is not equivalent modulo $T_m(T)$ to any positive formula. By compactness, there is $B$ a model of $T_m(T)$ and $\bar{b} \in B$ such that $B \not\models \varphi(\bar{b})$, and for every positive formula $\psi \in Ctr_{T_m(T)}(\varphi)$ we have $B \not\models \psi(\bar{b})$, which contradicts the lemma.

For the reverse direction, suppose that for every quantifier-free positive formula $\varphi$, there is a positive formula $\psi \in Ctr_{T_m(T)}(\varphi)$ such that $T_m(T) \vdash \forall x \varphi(x) \lor \psi(x)$. Let $A$ be a model of $T_m(T)$. By the lemma it is clear that $A$ is a h-maximal model of $T$. 

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**2 H-MAXIMAL MODELS**

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**10**
3 POSITIVE ROBINSON AND LOCALLY POSITIVE ROBINSON THEORIES

**Corollary 3** If $\Sigma_T$ is elementary then $\Sigma_{T_k(T)}$ is elementary and axiomatized by $T_k(T)$.

*Proof.* Suppose that $\Sigma_T$ is axiomatised by $T_m(T)$. By the theorem (1), for every quantifier-free positive formula $\varphi$ there is a positive formula $\psi \in Ctr_{T_m(T)}(\varphi)$ such that $T_m(T) \vdash \forall x \varphi(x) \lor \psi(x)$. Given that $T_k(T) \supseteq T_m(T)$, then every model of $T_k(T)$ is a h-maximal model of $T_k(T)$. □

**Lemma 7** If $\Sigma_{T_k(T)}$ is elementary then it is axiomatized by $T_k(T)$.

*Proof.* Suppose that $\Sigma_{T_k(T)}$ is axiomatized by an h-inductive theory $T^*$. Then $T^*$ and $T_k(T)$ are companion theories. Given that $T_k(T)$ is the maximal companion of $T$ we obtain $T_k(T) \sim T^*$.

### 3 Positive Robinson and locally positive Robinson theories

In [4] Hrushovski defined Robinson theories to be the universal theory that admits the quantifier separation. The quantifier-free types are the main object of the study of Robinson theories. In our context we adopt this property to define the notion of positive Robinson theories and locally positive Robinson theories.

**Definition 9** An h-inductive theory $T$ is said to be positive Robinson theory (in short. pR theory) if it satisfies the following condition:

For any pc models $A$ and $B$ of $T$, $\bar{a} \in A$ and $\bar{b} \in B$. If $tpqf(\bar{a}) \subseteq tpqf(\bar{b})$ then $tp(\bar{a}) = tp(\bar{b})$. (where $tpqf(\bar{a})$ is the set of quantifier-free positive formulas satisfied by $\bar{a}$ in $A$).

An h-inductive theory $T$ is said to be a locally positive Robinson theory (in short. lpR theory) if the following conditions is satisfied for any pc model $A$ of $T$:

$$\forall \bar{a}, \bar{b} \in A; \; tpqf(\bar{a}) \subseteq tpqf(\bar{b}) \Rightarrow tp(\bar{a}) = tp(\bar{b})$$

Given that the property of being a pR theory or a lpR theory concerns the class of pc models. we have the following remarks.

**Remark 5**

- $T$ is a pR theory (resp. lpR theory) provided that each companion theory of $T$ is a pR theory (resp. lpR theory).
- If $T$ is a pR theory then $T$ is a lpR theory.

**Fact 5** [Lemma 8, [1]] An h-inductive theory $T$ is a pR theory if and only if for every positive formula $\varphi$, $Ctr_T(\varphi)$ is equivalent modulo $T_k(T)$ to a set of quantifier-free positive formulas.

**Remark 6** Let $T$ be a pR theory. If $A \in \Sigma_T$ and $B$ a model of $T$ which is embedded in $A$. Then $B \in \Sigma_T$. 

Theorem 2 \( T \) is lpR h-inductive theory if and only if for every pc model \( A \) of \( T \), and \( \varphi \) a positive formula we have the following property:

for every tuple \( \bar{a} \in A \), if \( A \not\models \varphi(\bar{a}) \) then there exists \( \psi \) a free positive formula such that, \( A \models \psi(\bar{a}) \) and \( T^*_k(A) \models \neg \exists \bar{x} \ (\varphi(\bar{x}) \land \psi(\bar{x})) \).

Proof. Suppose that \( T \) is a lpR theory. Let \( A \) be a pc model of \( T \), \( \bar{a} \in A \) and \( \varphi \) a positive formula such that, \( A \not\models \varphi(\bar{a}) \). We will show that \( T^* = \{ T^*_k(A), tpqf_A(\bar{a}), \varphi(\bar{a}) \} \) is inconsistent, where \( tpqf_A(\bar{a}) \) is the set of quantifier-free positive formulas satisfied by \( \bar{a} \) in the pc model \( A \).

Suppose that \( T^* \) is consistent. Let \( B \) a model of \( T^* \) in the language \( L^* = \{ L, \{ \bar{a} \} \} \). We claim that \( \{ T, Diag^+(A), Diag^+(B) \} \) is consistent. Indeed if not, by compactness there exist \( \psi(\bar{a}, \bar{x}) \in Diag^+(A) \) a quantifier-free positive formula such that \( \{ T, Diag^+(B), \psi(\bar{a}, \bar{x}) \} \) is inconsistent. Given that \( B \models \{ T, Diag^+(B) \} \) then \( B \models \neg \exists \bar{x} \ \psi(\bar{a}, \bar{x}) \). On the other hand, since \( B \models T^*_k(A) \) and \( A \models \exists \bar{x} \ \psi(\bar{a}, \bar{x}) \) we obtain \( B \models \exists \bar{x} \ \psi(\bar{a}, \bar{x}) \), contradiction. Thereby \( \{ T, Diag^+(A), Diag^+(B) \} \) is consistent. Let \( C \) be a model of \( \{ T, Diag^+(A), Diag^+(B) \} \), so \( A \) and \( B \) are consistent in \( C \). Let \( \bar{b} \) the interpretation of \( \bar{a} \in B \) in \( C \), and \( \bar{a} \) the interpretation of \( \bar{a} \in A \) in \( C \). Given that every model of \( T \) is continued in some pc model of \( T \), we can take \( C \) a pc model of \( T \).

Considering that \( A \) is a pc model of \( T \), it is immersed in \( C \), thereby we obtain \( tpqf_A(\bar{a}) = tpqf_C(\bar{a}) \). Since \( B \models T^* \), we have

\[
 tpqf_A(\bar{a}) \subseteq tpqf_B(\bar{a}) \subseteq tpqf_C(\bar{b}).
\]

Since \( T \) is lpR theory we have \( tp(\bar{a}) = tp(\bar{b}) \).

Given that \( A \not\models \varphi(\bar{a}) \) and \( B \models \varphi(\bar{b}) \) then \( C \not\models \varphi(\bar{a}) \) and \( C \models \varphi(\bar{b}) \). Contradiction. Therefore \( T^* \) is inconsistent, by compactness there exists \( \psi(\bar{x}) \in tpqf_A(\bar{a}) \) such that \( T^*_k(A) \models \neg \exists \bar{x} \ (\varphi(\bar{x}) \land \psi(\bar{x})) \).

For the reverse direction, suppose that for every pc model \( A \) of \( T \), \( \bar{a} \in A \), and \( \varphi \) a positive formula we have the following property:

if \( A \not\models \varphi(\bar{a}) \) then there exists a quantifier-free positive formula \( \psi \) such that \( A \models \psi(\bar{a}) \) and \( T^*_k(A) \models \neg \exists \bar{x} \ (\varphi(\bar{x}) \land \psi(\bar{x})) \). Let \( \bar{a} \) and \( \bar{b} \) be tuples of \( A \) such that \( tpqf_A(\bar{a}) = tpqf_A(\bar{b}) \). Assume the existence of a positive formula \( \varphi \) such that \( A \models \varphi(\bar{a}) \) and \( A \not\models \varphi(\bar{b}) \). By hypothesis there is a quantifier-free positive formula \( \psi \) such that \( A \models \psi(\bar{b}) \) and \( T^*_k(A) \models \neg \exists \bar{x} \ (\varphi(\bar{x}) \land \psi(\bar{x})) \). Thus \( \psi \in tpqf(\bar{b}) \), but we have \( \psi \not\in tpqf(\bar{a}) \). Contradiction. Thereby \( T \) is lpR theory.

Lemma 8 An h-inductive theory \( T \) is lpR if and only if for every pc model \( A \) of \( T \), \( T^*_k(A) \) is a pR theory.

Proof. Suppose that \( T \) is a lpR theory. Let \( A \) be a pc model of \( T \). Let \( B \) and \( C \) be two pc models of \( T^*_k(A) \). Consider \( \bar{b} \in B \) and \( \bar{c} \in C \) such that \( tpqf_B(\bar{b}) \subseteq tpqf_C(\bar{c}) \). By the lemma 5 the sets \( S_1 = \{ T^*_k(A), Diag^+(B), Diag^+(A) \} \) and \( S_2 = \{ T^*_k(A), Diag^+(C), Diag^+(A) \} \) are consistent. From the fact 4 we obtain
the following diagram:

\[
\begin{array}{c}
B \xrightarrow{e_1} B^* \\
| \downarrow e_1 \downarrow f_1 \\
A \\
| \downarrow i_2 \downarrow f_2 \\
C \xrightarrow{e_2} C^*
\end{array}
\rightarrow
\begin{array}{c}
\xrightarrow{j_1} D_1 \\
\xrightarrow{j_2} D_2
\end{array}
\]

where \( B^* \models S_1 \) and \( C^* \models S_2 \) that can be taken pc models of \( T_k^*(A) \). \( e_1, e_2, i_1, i_2, f_1, f_2 \) are immersions, \( D \) a model of \( \{ T_k^*(A), \text{Diag}^+(B), \text{Diag}^+(C) \} \) and \( D_1 \) a pc model of \( T \) in which \( D \) is continued. Given that \( A \) and \( D_1 \) are pc models of \( T \) then \( j_1 \circ f_1 \circ i_1 \) is an immersion, which implies that \( D_1 \) is a model of \( T_k^*(A) \). Let \( D_2 \) be a pc model of \( T_k^*(A) \) in which \( D_1 \) is continued.

Since \( j_1 \circ f_1 \circ e_1 \) and \( j_1 \circ f_2 \circ e_2 \) are immersions, we have;

\[
\text{tpqf}_{D_1}(\bar{b}) = \text{tpqf}_{B}(\bar{b}) \subseteq \text{tpqf}_{C}(\bar{c}) = \text{tpqf}_{D_1}(\bar{c})
\]

Given that \( T \) is a lpR theory and \( D_1 \) is a pc model of \( T \) we obtain \( \text{tp}_{D_1}(\bar{b}) = \text{tp}_{D_1}(\bar{c}) \). On the other hand, as \( D_2 \) is a pc model of \( T_k^*(A) \) and \( D_1 \) is immersed in \( D_2 \), then \( D_1 \) is a pc model of \( T_k^*(A) \), thereby \( \text{tp}(\bar{b}) = \text{tp}(\bar{c}) \), wich implies that \( T_k^*(A) \) is a pR theory.

The other direction of the proof is obvious.

**Fact 6** ([Lemma, [7]]) Let \( T \) be a pR theory. We have the following properties:
- Every model of \( T \) that embeds in a pc model of \( T \) is h-maximal model of \( T \).
- The h-maximal models of \( T \) have the amalgamation property.
Moreover, if \( T \) is h-universal the two conditions above imply that \( T \) is a pR theory.

From the fact 6 and the definition of pR theories we obtain the following slightly modified version of the fact 6.

**Lemma 9** An h-inductive theory \( T \) is pR if and only if the class of substructures of the h-maximal models of \( T \) has the amalgamation property.

**Proof.** Suppose that \( T \) is a pR theory, then \( T_u(T) \) the h-universal companion of \( T \) is also a pR theory. Given that the class of substructures of the h-maximal models of \( T \) is \( \Sigma_{T_u(T)} \) the class of h-maximal model of \( T_u(T) \), by the fact 6, \( \Sigma_{T_u(T)} \) has the amalgamation property.

For the other direction of the proof, assume that the class \( \Sigma_{T_u(T)} \) has the amalgamation property. By the fact 6, \( T_u(T) \) and thus \( T \) are pR theories.

**Corollary 4** If the class of pc model of \( T \) is closed under substructures then \( T \) is a pR theory.

**Proof.** Since every pc model is a h-maximal model and the class of pc models has the amalgamation property, the proof follows from the lemma 9.
Corollary 5 $T_{a_2}^*$ is a pR theory.

Examples 3

1. Let $T$ be an $h$-inductive theory that satisfies the following property:

For every pc model $A$ of $T$ and for every positive formula $\varphi$, there exists a family of free-positive formulas $\{\phi_i \mid i \in I\}$ such that:

- for every $i \in I$ we have $T^*_k(A) \vdash \forall \bar{x}. \phi_i(\bar{x}) \rightarrow \varphi(\bar{x})$.
- $\forall \bar{a} \in A$, if $A \models \varphi(\bar{a})$ then there is $i \in I$ such that $A \models \phi_i(\bar{a})$.

We claim that $T$ is an $lpR$ theory, and for every pc model $A$ of $T$ the class of $h$-maximal models of $T_k^* (A)$ is the class of pc model of $T_k^* (A)$.

Indeed, let $A$ be a pc model of $T$, $\bar{a}$ and $\bar{b}$ be tuples from $A$ such that $tpqf(\bar{a}) \subseteq tpqf(\bar{b})$. Let $\varphi$ be a positive formula such that $A \models \varphi(\bar{a})$, by hypothesis there is $\phi$ a quantifier-free positive formula such that $T_k^* (A) \vdash \forall \bar{x}. \phi(\bar{x}) \rightarrow \varphi(\bar{x})$, and $A \models \phi(\bar{a})$. Given that $tpqf(\bar{a}) \subseteq tpqf(\bar{b})$ and $\phi \in tpqf(\bar{a})$, then $A \models \phi(\bar{b})$ and thereby $A \models \varphi(\bar{b})$. Consequently $tpqf(\bar{a}) \subseteq tpqf(\bar{b})$, as $A$ is a pc, by maximality of types we obtain $tpqf(\bar{a}) = tpqf(\bar{b})$.

Now, let $B$ be a $h$-maximal of $T_k^* (A)$ and $C$ a pc model of $T_k^* (A)$ in which $B$ is embedded. Let $\varphi$ a positive formula such that $C \models \varphi(\bar{b})$ where $\bar{b} \in B$, then there is $\phi$ a quantifier-free positive formula such that:

$$T_k^*(A) \equiv T_k^*(C) \vdash \forall \bar{x}. \phi(\bar{x}) \rightarrow \varphi(\bar{x}) \quad \text{and} \quad C \models \phi(\bar{b}).$$

Since $B$ is embedded in $C$ and $\phi$ is a quantifier-free formula then $B \models \phi(\bar{b})$, so $B \models \varphi(\bar{b})$. Consequently $B$ is immersed in $C$ which implies that $B$ is a pc model of $T_k^* (A)$.

Theories whose pc are finite provide a concrete example of theories with the above property.

2. Let $T$ be an $h$-inductive theory such that for every positive formula $\varphi$ there is $\{\phi_i \mid i \in I\}$ a family of quantifier-free positive formula such that

- $T_k(T) \vdash \forall \bar{x}. \phi_i(\bar{x}) \rightarrow \varphi(\bar{x})$.
- For every pc model $A$ of $T$ and $\bar{a}$ a tuple of $A$. If $A \models \varphi(\bar{a})$ then $A \models \phi_i(\bar{a})$ for same $i \in I$.

Then $T$ is a pR theory, and every $h$-maximal model of $T_k(T)$ is a pc model of $T$.

3. Let $L$ be the language formed by the function symbol $f$ or arity 1. Let $T$ be the $h$-universal theory $\{-\exists x \ f(x) = x\}$. In [13], example 3 it is shown that the unique pc model of $T$ is the model formed by the $p$-cycle (cycle of length $p$) where $p$ runs through the set of prime numbers. The $h$-maximal models of $T$ are the substructures of the pc model of $T$.

It is obvious that $T$ is a pR theory.
Theorem 3 Let $T$ be a pR theory with a model-companion. Then every positive formula is equivalent modulo $T_k(T)$ to a quantifier-free positive formula.

Proof. Since $T$ is pR theory with a model-companion then for each positive formula $\varphi$ there is $\phi$ a quantifier-free positive formula such that

$$T_k(T) \vdash \neg \exists \phi(\bar{x}) \land \phi(\bar{x}), \quad T_k(T) \vdash \forall \bar{x} \varphi(\bar{x}) \lor \phi(\bar{x}).$$

We repeat the same reasoning for the quantifier-free positive formula $\phi$ and we obtain a quantifier-free positive formula $\psi$ such that,

$$T_k(T) \vdash \neg \exists \phi(\bar{x}) \land \psi(\bar{x}), T_k(T) \vdash \forall \bar{x} \phi(\bar{x}) \lor \psi(\bar{x})$$

which implies that $T_k(T) \vdash \forall \bar{x} \varphi(\bar{x}) \leftrightarrow \psi(\bar{x})$.

Corollary 6 Let $T$ be a lpR theory such that for every pc model $A$ of $T$, the theory $T^*_k(A)$ has a model-companion. Then every positive formula is equivalent modulo $T^*_k(A)$ to a quantifier-free positive formula.

Exemple 1 For every pc model $Z_p$ of $T^+_a$. Every positive formula in the language of the theory $T^*_a$ is equivalent modulo $T_k(Z_p)$ to a quantifier-free positive formula.

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