Two-qubit geometric discord. The solution

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Abstract. We report the analytical solution of the problem of description of quantum geometric discord for a two-qubit system.

1. Introduction
Contrary to the common belief not all important questions for compound multi qubit systems can be answered explicitly. Important characterization of apparently one of the simplest compound systems i.e. two-qubit system, known as the geometric discord, has been not known for a long time. Only recently has been found strict answer \cite{1}. Explicit construction turns out to be rather involved, but can be brought to the compact form. Here we want to report the final result in a generic form. Partial answers for selected families of states were known before including: the Bell diagonal states (or states with maximally mixed marginals) (2013) \cite{2}, X-shaped states (2014) \cite{3}, states with the correlation matrix having only one nonzero singular value and arbitrary Bell vectors of the marginals and so-called quantum-classical states (2014) \cite{3}. In the present contribution we provide brief description of the general solution for arbitrary type of mixed or pure state of two-qubit compound system.

2. Notation and tools
The set $\mathcal{E}_2$ of all states of 2-level system is conventionally parametrized in the following way

$$
\rho = \frac{1}{2} \left( I_2 + \langle n, \sigma \rangle \right), \quad n \in \mathbb{R}^3
$$

where $||n|| \leq 1$. The $\mathcal{E}_2$ forms a unit ball in $\mathbb{R}^3$ with the pure states laying on the unit sphere $||n|| = 1$.

The set of states of two-qubit compound system can be described by

$$
\rho = \frac{1}{4} \left( I_2 \otimes I_2 + \langle x, \sigma \rangle \otimes I_2 + I_2 \otimes \langle y, \sigma \rangle + \sum_{j,k=1}^{3} K_{jk} \sigma_j \otimes \sigma_k \right)
$$

where $x, y \in \mathbb{R}^3$ and $K = (K_{jk})$ is the correlation matrix. In the case of qubits we can use the group of orthogonal transformations to diagonalize correlation matrix, hence any two-qubit state is locally equivalent to the state with diagonal $K$.

Typical examples:

\textsuperscript{1} Talk given by A.F.
• States with maximally mixed marginals (MMM)
  \[ \text{tr}_A \rho = \frac{1}{2} \mathbb{I}_2 \quad \text{and} \quad \text{tr}_B \rho = \frac{1}{2} \mathbb{I}_2 \]

• X-states
  \[ \rho_X = \begin{pmatrix}
  \rho_{11} & 0 & 0 & \rho_{14} \\
  0 & \rho_{22} & \rho_{23} & 0 \\
  0 & \rho_{32} & \rho_{33} & 0 \\
  \rho_{41} & 0 & 0 & \rho_{44}
\end{pmatrix} \]

• Bell diagonal states (special case of X-states)
  \[ \rho = \frac{1}{4} \left( \mathbb{I}_2 \otimes \mathbb{I}_2 + \sum_{j=1}^{3} c_j \sigma_j \otimes \sigma_j \right) \]

**Geometric discord**

Let us consider a state \( \rho \) of bipartite system \( AB \). When we perform local measurement on the subsystem \( A \), the state \( \rho \) may be disturbed due to such measurement. The trace-norm (one-sided) measurement induced geometric discord is defined as the minimal disturbance induced by projective measurement \( P_A \) on subsystem \( A \), computed using the trace distance in the set of states. It can be compared with the standard geometric discord equal to the distance from a given state to the set of classical - quantum states [4, 5, 6, 7, 8]. As it was already stated in the Introduction, in the case of qubits these two notions coincide [7]. It is more convenient to use the quantity based on the disturbance induced by the measurement which will be simply called trace - norm geometric discord. The formal definition is as follows [2]

\[
D_1(\rho) = \min_{P_A} ||\rho - P_A(\rho)||_1
\]  

where \( ||A||_1 = \text{tr} |A| \).

In the case of qubits, the local projective measurement \( P_A \) is given by the one - dimensional projectors \( P_1, P_2 \) on \( \mathbb{C}^2 \), such that

\[ P_1 + P_2 = \mathbb{I}_2, \quad P_j P_k = \delta_{jk} P_k \]

and \( P_A = P \otimes \text{id} \), where

\[ P(A) = P_1 A P_1 + P_2 A P_2 \]

One - dimensional projectors \( P_k \) can be always chosen as

\[ P_k = u P_k^0 u^* \quad \text{for some} \quad u \in SU(2) \]

where

\[ P_1^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2^0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \]

A real orthogonal projector \( P \) on \( \mathbb{R}^3 \) can be defined by the following relation

\[ \langle P m, \sigma \rangle = P(\langle m, \sigma \rangle), \quad m \in \mathbb{R}^3 \]

If \( P_0 \) denotes such projector given by (4), where we take \( P_1^0 \) and \( P_2^0 \), then

\[ P_0 = \text{diag}(0, 0, 1) \]

and

\[ P = V P_0 V^T, \quad V \in SO(3) \]
It is convenient to use orthogonal complements to $P_0$ and $P$:

$$M_0 = 1_3 - P_0, \quad M = 1_3 - P$$

(7)

Obviously $M_0 = \text{diag}(1,1,0)$,

$$M = V M_0 V^T, \quad V \in SO(3)$$

and

$$\text{dim Ran } M_0 = \text{dim Ran } M = 2$$

Such projectors $M$ cover the whole set of projectors with dimension 2.

**Disturbance**

As a disturbance of the state (2) caused by measurement $P_A$ we shall understand

$$S(M) = \rho - P_A(\rho) = \frac{1}{4} \left( \langle Mx, \sigma \rangle \otimes I_2 + \sum_{k=1}^3 \langle M Ke_k, \sigma \rangle \otimes \langle e_k, \sigma \rangle \right)$$

(8)

where $e_k, k = 1, 2, 3$ are the vector of the canonical basis of $\mathbb{R}^3$. So

$$D_1(\rho) = \min_{M} |S(M)| = \min_{M} \sqrt{Q(M)}$$

(9)

where $Q(M) = S(M)S(M)^*$ and the minimum is taken over all projectors $M$ on two dimensional subspaces of $\mathbb{R}^3$.

3. **Characterization of the disturbance**

The first step in obtaining the analytical solution consists on a derivation of suitable expression of the trace norm of the disturbance allowing convenient study of behaviour of matrix dependent real function. One gets the following elegant result [1].

**Theorem 1.** The trace norm of the disturbance $S(M)$ of the state (2) is given by the formula

$$||S(M)||_1 = \frac{1}{\sqrt{2}} \sqrt{||Mx||^2 + \text{tr} (MKKT) + \sqrt{[||Mx||^2 + \text{tr} (MKKT)]^2 - 4 \left[ ||Mx||^2 + \text{tr} (E^T(E - ME^T)) \right]}}$$

where $|| \cdot ||$ denotes the Euclidean norm in $\mathbb{R}^3$ and $E = \text{adj} K$ is the adjunct matrix of the correlation matrix $K$ (i.e. the transpose of its cofactor matrix).

Now, the question of finding a minimum of the trace norm of the $S(M)$ can be addressed. To find critical points of the mapping $\mathcal{M} \rightarrow ||S(M)||_1$. We use a unit vector $v \in \mathbb{R}^3$ to characterize $P = P_v$, given by the third column of the matrix $V \in SO(3)$ relating $P$ and $P_0$. Introducing auxiliary function $g$ on the unit sphere $S^2 \subset \mathbb{R}^3$ with values in $\mathbb{R}^2$

$$g(v) = (g_1(v), g_2(v))$$

(10)

where

$$g_1(v) = ||(1_3 - P_v)x||^2 + \text{tr} ((1_3 - P_v)KK^T)$$

(11)

and

$$g_2(v) = 4 \left( ||K^T(1_3 - P_v)x||^2 + \text{tr} (E^T(E P_v)) \right)$$

(12)
Using the functions (11) and (12) we get
\[ ||S(\mathcal{M})||_1 = ||S(\mathbb{1}_3 - P_v)||_1 = \frac{1}{\sqrt{2}} \sqrt{g_1(v) + g_1(v)^2 - g_2(v)} \]

The mapping \( g \) can be viewed as a function of two-dimensional projectors \( \mathcal{M} \) or vectors \( v \in S^2 \).

Introducing the following operators
\[ W_yz = \langle z, y \rangle y, \quad y, z \in \mathbb{R}^3 \]
and
\[ L_+ = KK^T + W_y \]
where \( x \) is a Bloch vector, we can obtain relations
\[ g_1(v) = \text{tr} L_+ - \langle v, L_+ v \rangle \]
and
\[ g_2(v) = 4 \left( ||K^T (x - \langle x, v \rangle x)||^2 + \langle v, E^T Ev \rangle \right) \]

It turns out that there are basically two types of situations emerging in the search of a minimum, depending of the type of set of points on which it is achieved. We shall name them as: singular critical points, smooth critical points.

**Singular critical points**
When the minimum of the mapping \( \mathcal{M} \to ||S(\mathcal{M})||_1 \) is achieved on the set
\[ D = \{ v \in S^2 : g_1^2(v) = g_2(v) \} \]
a critical point \( v_0 \in D \) we call the singular critical point. In such a case the minimal value of the \( ||S(\mathcal{M})||_1 \) is given by the minimum of the function \( g_1 \). New representations of the functions \( g_1 \) and \( g_2 \)
\[ g_1(v) = \text{tr} L_- - \langle v, L_- v \rangle + 2 (||x||^2 - \langle x, v \rangle^2) \]
and
\[ g_2(v) = 4 \left( ||x||^2 - \langle x, v \rangle^2 \right) \left[ \text{tr} L_- - \langle v, L_- v \rangle + ||x||^2 - \langle x, v \rangle^2 \right] + 4 \langle v, \text{adj} L_- v \rangle \]

where \( L_- = KK^T - W_y \), allow simplification of analysis of properties of the set \( D \). One can show that the set (17) is discrete or is equal to the whole sphere \( S^2 \). The characterization of the function \( g_1(v) \) for such a case is following: for all vectors \( v \in D \)
\[ g_1(v) = 2 \left( \text{int} \{ \tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3 \} + ||x||^2 - \langle x, v \rangle^2 \right) \]

where \( \text{int} \) denotes the intermediate value and \( \tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3 \) are the eigenvalues of the matrix \( L_- \).

**Smooth critical points**
By a smooth critical point we understand the critical point belonging to the set \( S^2 \setminus D \), in such a case for \( v_0 \in S^2 \setminus D \) we have
\[ g_1^2(v_0) \neq g_2(v_0) \]

Any smooth critical point of the mapping \( \mathcal{M} \to ||S(\mathcal{M})||_1 \) is a critical point of the function \( g \) and satisfies equation
\[ \left[ W_y P_v KK^T + KK^T P_v W_y - KK^T W_y - W_y KK^T + E^T E + \mu L_+ \right] v = \omega_0 v \]
where a vector \( v \in S^2 \) is a solution of (21) if there exists a real number \( \mu \) such that \( v \) is a solution of the above eigenvector problem.
4. The solution
In this section we describe the solution of the main problem, namely we find the value of geometric discord in general case. It is convenient to consider the matrix

\[ G_\mu = -W_\mu \mathcal{M} K K^T - K K^T \mathcal{M} W_\mu + E^T E + \mu L_+ \]  

represented in the ordered basis of orthonormal eigenvectors of the operator \( L_- \) assuming that \( \tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \tilde{\lambda}_3 \), and neglecting parts which commute with the projectors \( P_\mu \), we obtain the matrix

\[
\tilde{G}_\mu = W_\mu P_\mu \Lambda + \Lambda P_\mu W_\mu + 2 P_\mu W_\mu (\tilde{\lambda}_1 - \tilde{\lambda}_2) W_\mu + \tilde{\mu}(\Lambda + 2 W_\mu) + (\tilde{\lambda}_1 - \tilde{\lambda}_2)(\tilde{\lambda}_2 - \tilde{\lambda}_3) P_{\mu_0}
\]

where

\[
\Lambda = \begin{pmatrix}
\tilde{\lambda}_1 - \tilde{\lambda}_3 & 0 & 0 \\
0 & \tilde{\lambda}_2 - \tilde{\lambda}_3 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

and

\[
\tilde{\mu} = \mu - \tilde{\lambda}_2 - ||x||^2
\]

Let us observe that if \( L_- \) has a degenerate spectrum, the form of the operator \( \tilde{G}_\mu \) is particularly simple and the desired minimal value of \( \mathcal{M} \rightarrow ||S(\mathcal{M})||_1 \) can be found in a straightforward way.

Nondegenerate spectrum
This is a harder part of the problem under consideration. The proof is rather subtle, but final result can be formulated in the form of the following theorem

**Theorem 2.** In the case of non-degenerate spectrum of the matrix \( L_- = KK^T - W_\mu \), the value of trace-norm geometric discord is given by

\[ D_1(\rho) = \sqrt{\min\{d_1, d_2, d_3\}} \]

where

\[ d_1 = \tilde{\lambda}_1 + x_1^2, \]

\[ d_2 = \frac{1}{2} \left( \tilde{\lambda}_1 + \tilde{\lambda}_2 + ||x||^2 + x_3^2 - \sqrt{\tilde{\lambda}_1 - \tilde{\lambda}_2 + ||x||^2 - x_3^2} \right)^2 - 4(\tilde{\lambda}_1 - \tilde{\lambda}_2)x_3^2 \]

\[ d_3 = \min\{\mu_\ast, \mu_{\ast\ast}\} \]

with

\[ \mu_\ast = \frac{1}{2} \left( \tilde{\lambda}_1 + \tilde{\lambda}_3 + ||x||^2 + x_3^2 + N_\ast \left(1 - 2p(\theta_\ast)\right) \right) \]

\[ \mu_{\ast\ast} = \tilde{\lambda}_2 + ||x||^2 - (||x||^2 - x_3^2) r(\theta_{\ast\ast}) \]

and

\[
\cos \theta_\ast = \frac{\sqrt{2|x_1 x_3|}}{\sqrt{N_\ast (N_\ast - \tilde{\lambda}_1 + \tilde{\lambda}_3 - x_1^2 + x_3^2)}}, \quad \sin \theta_\ast = \frac{-\sqrt{2|x_1 x_3|}}{\sqrt{N_\ast (N_\ast - \tilde{\lambda}_1 - \tilde{\lambda}_3 + x_1^2 - x_3^2)}}
\]

where

\[ N_\ast = \sqrt{(\tilde{\lambda}_1 - \tilde{\lambda}_3 + ||x||^2 - x_3^2)^2 - 4(\tilde{\lambda}_1 - \tilde{\lambda}_3)x_3^2} \]

and

\[
\cos \theta_{\ast\ast} = \frac{\sqrt{2|x_1 x_3|}}{\sqrt{N_{\ast\ast} (N_{\ast\ast} - x_1^2 + x_3^2)}}, \quad \sin \theta_{\ast\ast} = \frac{-\sqrt{2|x_1 x_3|}}{\sqrt{N_{\ast\ast} (N_{\ast\ast} + x_1^2 - x_3^2)}}
\]

with

\[ N_{\ast\ast} = ||x||^2 - x_3^2. \]
Degenerate spectrum

In the case of degenerate spectrum the analysis definitely simplifies and final answer takes the form

**Theorem 3.** 1. Let \( \tilde{\lambda}_1 = \tilde{\lambda}_2 \geq \tilde{\lambda}_3 \), then

\[
D_1(\rho) = \sqrt{\tilde{\lambda}_2}.
\]

2. Let \( \tilde{\lambda}_1 > \tilde{\lambda}_2 = \tilde{\lambda}_3 \), then

\[
D_1(\rho) = \frac{1}{\sqrt{2}} \sqrt{\lambda_1 + \lambda_2 + ||x||^2 - \sqrt{(\lambda_1 - \lambda_2 + ||x||^2)^2 + 4(\tilde{\lambda}_1 - \tilde{\lambda}_2)x_1^2}}.
\]

5. Example: new class of states \( \rho_{\tilde{\gamma},a} \)

Knowing the generic formulas which determine the explicit values of the geometric discord we can distinguish completely new two-parameter family of states

\[
\rho_{\tilde{\gamma},a} = \frac{1}{4} \begin{pmatrix}
1 + a(1 + w_2) & -iaz & -iaz & a(2 - w_1) \\
iaz & 1 + a(1 - w_2) & a(2 + w_1) & ia(z) \\
a(2 - w_1) & -iaz & 1 - a(1 + w_2) & ia(z) \\
\end{pmatrix},
\]

where for \( \tilde{\gamma} = \sqrt{1 + 16\gamma^2} \) we have

\[
w_1 = \frac{\sqrt{7 - \tilde{\gamma}(\tilde{\gamma} - 1) + \sqrt{7 + \tilde{\gamma}(\tilde{\gamma} + 1)}}}{2\sqrt{2}\tilde{\gamma}}
\]

\[
w_2 = \frac{\sqrt{7 - \tilde{\gamma}(\tilde{\gamma} + 1) + \sqrt{7 + \tilde{\gamma}(\tilde{\gamma} - 1)}}}{2\sqrt{2}\tilde{\gamma}}
\]

\[
z = \frac{\sqrt{7 + \tilde{\gamma} - \sqrt{7 - \tilde{\gamma}}}}{\tilde{\gamma}} \sqrt{2}\tilde{\gamma}.
\]

The (25) is a state if \( |\gamma| \leq \sqrt{3} \), where \( a \) is the function of \( \gamma \). The correlation matrix and Bloch vector are the following

\[
K = a \begin{pmatrix}
2 & 0 & 0 \\
0 & w_1 & z \\
0 & z & w_2
\end{pmatrix}, \quad x = \begin{pmatrix}
0 \\
0 \\
a
\end{pmatrix}
\]

therefore \( a \) is equal to the norm \( ||x|| \) of the Bloch vector. The correlation matrix of the \( \rho_{\tilde{\gamma},a} \) yields the relation

\[
KK^T = ||x||^2 \begin{pmatrix}
4 & 0 & 0 \\
0 & 4 & 2\gamma \\
0 & 2\gamma & 3
\end{pmatrix}
\]

providing that the \( L_- \) matrix has a simple form

\[
L_- = ||x||^2 \begin{pmatrix}
4 & 0 & 0 \\
0 & 4 & 2\gamma \\
0 & 2\gamma & 2
\end{pmatrix}
\]

and the ordered eigenvalues are the following

\[
\tilde{\lambda}_1 = ||x||^2 \left( 3 + \sqrt{1 + 4\gamma^2} \right), \quad \tilde{\lambda}_2 = 4||x||^2, \quad \tilde{\lambda}_3 = ||x||^2 \left( 3 - \sqrt{1 + 4\gamma^2} \right).
\]

The spectrum of \( L_- \) is non-degenerate. The value of \( D_1(\rho_{\tilde{\gamma},a}) \) can be shown to be fully determined by the norm of the Bloch vector and to be equal

\[
D_1(\rho_{\tilde{\gamma},a}) = 2||x||^2.
\]
Conclusions

Strict general formulas characterising quantum correlations valid for all class of quantum states of compound systems in general were not known. In the present report we have sketched how the problem of finding values of quantum correlation measure - the geometric quantum discord - for two-qubit compound system can be solved completely. We have given the new compact formula for the trace norm of the disturbance $S(M)$ for the general two-qubit state.

Let us observe a remarkable effect, that the information about critical points is encoded in the spectrum of the matrix $L_\gamma = KK^T - W_\gamma$. Moreover, for non-degenerate spectrum critical points are located on big circles in $S^2$. Therefore the problem of finding geometric discord could be reduced to minimization of the quadratic form restricted to big circles. The complete solution of this problem was rather involved and it simplified significantly, when any degeneracy of the spectrum of $L_\gamma$ was present. The full analysis can be found in the work [1] (see also Ref. [9]).

Our solution agrees with all known particular answers for selected families of states. Moreover, as a non-trivial illustration, we have discussed the totally new family of mixed states $\rho_{\gamma}^{\alpha}$ and have found the value of its geometric discord.

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