Low rank solutions to differentiable systems over matrices and applications*

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Abstract

Differentiable systems in this paper means systems of equations that are described by differentiable real functions in real matrix variables. This paper proposes algorithms for finding minimal rank solutions to such systems over (arbitrary and/or several structured) matrices by using the Levenberg-Marquardt method (LM-method) for solving least squares problems. We then apply these algorithms to solve several engineering problems such as the low-rank matrix completion problem and the low-dimensional Euclidean embedding one. Some numerical experiments illustrate the validity of the approach.

On the other hand, we provide some further properties of low rank solutions to systems linear matrix equations. This is useful when the differentiable function is linear or quadratic.

Keywords: rank minimization problem, generalized Levenberg-Marquardt method, positive semidefinite matrix, low-rank matrix completion problem, Euclidean distance matrix

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1 Motivation and preliminaries

Several problems in either engineering or computational mathematics can be reformulated as rank minimization problems (shortly, RM-problems) in the form

\[
\begin{align*}
\text{minimize} & \quad \text{rank}(X) \\
\text{subject to} & \quad X \in \mathcal{C},
\end{align*}
\]  

where \( \mathcal{C} \) is a subset of \( \mathbb{R}^{m \times n} \), the set of all \( m \) by \( n \) matrices with real entries.

RM-problem (1) is computationally NP-hard in general, even when \( \mathcal{C} \) is an affine subset of \( \mathbb{R}^{m \times n} \). There hence is a number of algorithms for solving this problem with respect to special cases of \( \mathcal{C} \), see, e.g., [9, 16, 24] and the references there in. When the constraints are defined by linear matrix equations, i.e., \( \mathcal{C} \) is the solution set of a linear system of equations \( \ell(X) = b \in \mathbb{R}^k \), the present problem is called affine rank minimization problem (shortly, ARM-problem) and is in the form [24]

\[
\begin{align*}
\text{minimize} & \quad \text{rank}(X) \\
\text{subject to} & \quad X \in \mathbb{R}^{m \times n}, \\
& \quad \ell(X) = b.
\end{align*}
\]  

When the constraint region is considered on the cone of positive semidefinite matrices, the authors in [20] relaxed the non-convex rank objective function in problem (2) into the nuclear norm that is a convex function. The whole problem is then a semidefinite program [30] and can be efficiently solved by SDP solvers. In our point of view, by using the Cholesky decomposition, each positive semidefinite matrix \( X \) can be written as \( X = YY^T \), \( Y \in \mathbb{R}^{n \times n} \). The linear map in the later problem (2) now becomes a quadratic map in \( Y \).

In this paper, we focus on the problem over a more general set \( \mathcal{C} \), in comparison with the sets we have discussed above. Such a set is determined by a differentiable map. That is, we focus on the problem

\[
\begin{align*}
\text{minimize} & \quad \text{rank}(X) \\
\text{subject to} & \quad X \in \mathcal{C}, \\
& \quad \phi(X) = b,
\end{align*}
\]  

where \( \mathcal{C} \subseteq \mathbb{R}^{m \times n} \) and \( \phi : \mathbb{R}^{m \times n} \to \mathbb{R}^k \) is a differentiable map. This function is clearly non-convex in general. Our method applies the generalized
Levenberg-Marquardt method [27] for checking whether there exists a solution of rank $r$, step by step, for $r = 1, 2, \ldots$. The differentiability of $\phi$ guarantees for the existence of its Jacobian in the Levenberg-Marquardt steps. It turns out that the problem of finding a matrix of rank $r = 1, 2, \ldots, \min\{m, n\}$, solving the equation $\phi(X) = b$ is the most important in our method.

We now recall some important results on matrix factorization in linear algebra that are used in the paper.

By $^T$ we denote the transpose of matrices. For a real symmetric matrix $A$, i.e., $A^T = A$, by $A \succeq 0$ we mean $A$ is positive semidefinite, i.e., $x^T A x \geq 0$ for all $x \in \mathbb{R}^n$. This, equivalently, means its eigenvalues are all non-negative.

For any two real symmetric matrices $A$ and $B$, we write $A \succeq B$ if $A - B \succeq 0$.

Let $S_n$ denote the set of $n \times n$ real symmetric matrices, and $S_n^+$ denote the cone of positive semidefinite matrices in $S_n$.

**Proposition 1.** (see, e.g, [4] or [12, Section 2.6, Observation 7.1.6]) Any positive semidefinite matrix (PSD matrix) $A \in S_n^+$ has a Cholesky decomposition $A = LL^T$, where $L \in \mathbb{R}^{n \times r}$ is a lower triangular matrix which is called a Cholesky factor of $A$. In particular, if $r = \text{rank}(A)$ then one can find $L \in \mathbb{R}^{n \times r}$.

Another fact is that for two matrices $A, B \in S_n$, then $A \succeq B$ if and only if $P^T A P \succeq P^T B P$ for any nonsingular matrix $P \in \mathbb{R}^{n \times n}$.

**Proposition 2.** (see e.g, [12, Section 0.4.6] Let $A$ be an $m \times n$ real matrix. Then

i) $\text{rank}(A) = \text{rank}(A^T) = \text{rank}(AA^T) = \text{rank}(A^T A)$.

ii) $A \in \mathbb{R}^{m \times n}$ has rank $r$ if and only if there exist matrices $X \in \mathbb{R}^{r \times m}$, $Y \in \mathbb{R}^{r \times n}$ and $B \in \mathbb{R}^{r \times r}$ nonsingular with $\text{rank}(X) = \text{rank}(Y) = r$ such that $A = X^TBY$.

A consequence, $A$ can be written as $A = X^T Z$ with $Z = BY \in \mathbb{R}^{r \times n}$ and $\text{rank}(X) = \text{rank}(Z) = r$.

**Proposition 3.** For any linear map $\ell : \mathbb{R}^{m \times n} \to \mathbb{R}$ one can find a matrix $A \in \mathbb{R}^{m \times n}$ such that

$$\ell(X) = \text{Tr}(A^T X) = \text{Tr}(AX^T), \quad \forall X \in \mathbb{R}^{m \times n}.$$ 

Specially, if $\ell : \mathbb{S}^n \to \mathbb{R}$ then $A$ can be found in $\mathbb{S}^n$, i.e., $A^T = A$. 

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Proof. Suppose \( \ell : \mathbb{R}^{m \times n} \to \mathbb{R} \) is a linear map. Consider \( \mathbb{R}^{m \times n} \) as a real vector space endowed with the basis \( \{ E_{ij} | i = 1, \ldots, m; j = 1, \ldots, n \} \), where \( E_{ij} \) is the \( m \times n \) matrix whose entries are zeros except for the \((i, j)\)th one being 1. Let \( A = [\ell(E_{ij})]_{j=1}^{j=1} \in \mathbb{R}^{m \times n} \). Then for every \( X = [x_{ij}] \in \mathbb{R}^{m \times n} \),

\[
X = \sum_{i,j} x_{ij} E_{ij},
\]

we have

\[
\ell(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \ell(E_{ij}) = \text{Tr}(AX^T) = \text{Tr}(A^TX).
\]

If \( \ell : S^n \to \mathbb{R} \), then it follows that

\[
\ell(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \ell(E_{ij}) = \sum_{i=1}^{n} x_{ii} \ell(E_{ii}) + \sum_{1 \leq i < j \leq n} x_{ij} [\ell(E_{ij}) + \ell(E_{ji})]
\]

\[
= \text{Tr}[(\frac{A^T + A}{2})X],
\]

and the proof is done. \( \square \)

We now recall some notation and results from matrix calculus. Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) be a \( m \times 1 \) vector function of a \( n \times 1 \) vector \( x \). The derivative (or Jacobian matrix) of \( f \) is the \( m \times n \) matrix defined by

\[
\text{Jac} f(x) \triangleq \frac{\partial f(x)}{\partial x} = \begin{bmatrix}
\frac{\partial f_1(x)}{\partial x_1} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_m(x)}{\partial x_1} & \cdots & \frac{\partial f_m(x)}{\partial x_n}
\end{bmatrix} \in \mathbb{R}^{m \times n}.
\]

We now recall a general definition for the derivative of a matrix valued function. Suppose \( F : \mathbb{R}^{m \times n} \to \mathbb{R}^{p \times q} \) is a \((p \times q)\)-matrix valued function of an \((m \times n)\)-matrix variable \( X \). Suppose that \( F = [F_{rs}] \in \mathbb{R}^{p \times q} \) and we define the derivative of this function as the \( pq \times mn \) matrix

\[
\text{Jac} F(X) \triangleq \frac{\partial \text{vec} F(X)}{\partial \text{vec} X} = \begin{bmatrix}
\frac{\partial F_{11}(X)}{\partial x_{11}} & \frac{\partial F_{11}(X)}{\partial x_{12}} & \cdots & \frac{\partial F_{11}(X)}{\partial x_{1n}} \\
\frac{\partial F_{21}(X)}{\partial x_{11}} & \frac{\partial F_{21}(X)}{\partial x_{12}} & \cdots & \frac{\partial F_{21}(X)}{\partial x_{1n}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial F_{pq}(X)}{\partial x_{11}} & \frac{\partial F_{pq}(X)}{\partial x_{12}} & \cdots & \frac{\partial F_{pq}(X)}{\partial x_{1n}}
\end{bmatrix} \in \mathbb{R}^{pq \times mn},
\]

where \( \text{vec} X \in \mathbb{R}^{mn \times 1} \) denotes the vector obtained by stacking its columns one underneath the other, i.e., if \( X \in \mathbb{R}^{m \times n} \) and \( X_j, j = 1, \ldots, n, \) are the columns of \( X \) then

\[
\text{vec} X = [X_1^T \ldots X_n^T]^T.
\]

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We list below the important properties of the derivative of trace functions that will be used in either paper or MATLAB codes (see, eg., [22]).

- Let $A$ be a given matrix in $\mathbb{R}^{m \times n}$. If $F : \mathbb{R}^{m \times n} \to \mathbb{R}$ is defined by $F(X) = \text{Tr}(A^T X)$, $\forall X \in \mathbb{R}^{m \times n}$, then
  \[ \text{Jac} F(X) = \text{vec}(A^T). \]  
  (4)

- Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{m \times m}$ be given. If $F : \mathbb{R}^{m \times n} \to \mathbb{R}$ is defined by $F(X) = \text{Tr}(XAX^T B)$ then
  \[ \text{Jac} F(X) = \text{vec}(B^T XA^T +BXA)^T. \]  
  (5)

- Let $A, B$ be two given matrices in $\mathbb{R}^{m \times m}$. Then for all $X \in \mathbb{R}^{m \times m}$,
  \[ \text{JacTr}(AXB) = \text{vec}(A^T B^T)^T \quad \text{and} \quad \text{JacTr}(AX^T B) = \text{vec}(BA)^T. \]  
  (6)

This paper is organized as follows. Section 2 presents the main algorithm for solving problem (3). This algorithm will be applied to particular problems with respect to several types of constraint sets. The affine rank minimization problem over arbitrary matrices is presented in Section 3 and a similar method applied for positive semidefinite matrices is handled in Section 4. Section 5 summarizes some applications of our solution method to several problems in engineering. The corresponding numerical experiments are exhibited in Section 6. The last section presents the conclusion and discussion for the future works.

2 The idea for solving problem (3)

In this work, with the help of Proposition 2, we solve problem (3) by using the generalized Levenberg-Marquardt method [27] to find a matrix $X \in \mathcal{C} \subset \mathbb{R}^{m \times n}$ step by step for $\text{rank}(X) = 1, 2, \ldots, \min\{m, n\}$ such that $\phi(X) = b$. In this situation, we consider the least square problem with respect to the function $F : \mathbb{R}^\mu \to \mathbb{R}^k$, with appropriate integer number $\mu$, whose coordinate functions are defined by

\[ F_j(X) = \phi_j(X) - b_j, \quad \forall j = 1, \ldots, k, \quad \forall X \in \mathcal{C}. \]  
(7)

We can summary this algorithm as follows.
Algorithm 1. Find minimal-rank matrix solving problem (3).

Input: Scalars \( b_1, \ldots, b_k \) and function \( \phi \).

Output: a solution \( X \in \mathcal{C} \subset \mathbb{R}^{m \times n} \) to (3).

1. Set \( r = 1 \).
2. Solve system (7) by applying the Levenberg-Marquardt method [27].
3. If (7) has a numerical solution then stop.
   Else, set \( r = r + 1 \) and go to Step 2.

In fact, to perform the experiments, the variable matrices are vectorized. Namely, the functions \( F_j \) in (7) is \( \text{vec}(X) \). This suggests us to study the rank one solutions to systems of equations.

3 Affine rank minimization problem over arbitrary matrices

In this section we are concentrating on numerically solving ARM-problem [2]. Using Proposition 3 each of the \( k \) linear equations \( \ell_i(X) = b_i, \ i = 1, 2, \ldots, k \) is written as

\[
\ell_i(X) = \text{Tr}(A_i^TX) = \text{Tr}(A_iX^T) = b_i.
\]

The function of the least square problem in this case is determined as:

\[
F_j(X) = \ell_j(X) - b_j, \quad j = 1, \ldots, k.
\]

It is clear that such a matrix \( X \) of rank \( r \) can be found only first \( r \) columns \( X(:,1:r) \) and its \( n-r \) last ones are identified to zero vectors. However, we will see later this may not applicable in some particular cases, for example, the matrix completion problem below. A modification is to apply Proposition 2 to find \( X \) as \( X = Y^TZ \) for two matrix variables \( Y \in \mathbb{R}^{r \times m} \) and \( Z \in \mathbb{R}^{r \times n} \). Namely we now focus on the following problem

minimize \( \text{rank}(Y^TZ) \)

subject to

\[
(Y, Z) \in \mathbb{R}^{r \times m} \times \mathbb{R}^{r \times n},
\]

\[
\ell(Y^TZ) = b. \tag{8}
\]
Problem (8) is then a special case of problem (3) with $\phi(Y, Z) = \ell(Y^T Z)$.

To perform this modification, we need the following auxiliary results. Set $W = [Y \ Z] \in \mathbb{R}^{r \times (m+n)}$ for each $(Y, Z) \in \mathbb{R}^{r \times m} \times \mathbb{R}^{r \times n}$. Recall that the linear map $\ell : \mathbb{R}^{m \times n} \to \mathbb{R}^k$ is defined by $k$ matrices $A_1, \ldots, A_k \in \mathbb{R}^{m \times n}$. That is

$$\ell(U) = [\text{Tr}(A_1^T U) \ldots \text{Tr}(A_k^T U)]^T, \quad \forall U \in \mathbb{R}^{m \times n}.$$ 

For each $r = 1, 2, \ldots, p = \min\{m, n\}$, the least squares problem in this situation is then defined by the function $F : \mathbb{R}^{r \times (m+n)} \equiv \mathbb{R}^{r \times m} \times \mathbb{R}^{r \times n} \to \mathbb{R}^k$,

$$F(W) = F(Y, Z) = \ell(Y^T Z) - b, \quad \forall W = (Y, Z) \in \mathbb{R}^{r \times m} \times \mathbb{R}^{r \times n}.$$ 

It is clear that each coordinate function $F_i : \mathbb{R}^{r \times (m+n)} \equiv \mathbb{R}^{r \times m} \times \mathbb{R}^{r \times n} \to \mathbb{R}$ is determined by

$$F_i(W) = F_i(Y, Z) = \text{Tr}(A_i^T Y^T Z) - b_i, \quad \forall W = (Y, Z) \in \mathbb{R}^{r \times m} \times \mathbb{R}^{r \times n}.$$ 

The Jacobian matrix of $F$ can hence be calculated as

$$\text{Jac}(F) = \frac{\partial F}{\partial W} = \frac{\partial \text{vec} F}{\partial \text{vec} W} = \begin{bmatrix} \frac{\partial F_1}{\partial W} \\ \vdots \\ \frac{\partial F_k}{\partial W} \end{bmatrix} = \begin{bmatrix} \frac{\partial F_1}{\partial Y} & \frac{\partial F_1}{\partial Z} \\ \vdots & \vdots \\ \frac{\partial F_k}{\partial Y} & \frac{\partial F_k}{\partial Z} \end{bmatrix} \in \mathbb{R}^{k \times r(m+n)},$$

where

$$\text{vec}(W) = [\text{vec}(Y)^T \text{vec}(Z)^T]^T,$$

$$\frac{\partial F_i}{\partial Y} = \frac{\partial \text{Tr}(A_i^T Y^T Z)}{\partial Y} = \text{vec}(ZA_i^T)^T,$$

$$\frac{\partial F_i}{\partial Z} = \frac{\partial \text{Tr}(A_i^T Y^T Z)}{\partial Z} = \text{vec}(YA_i)^T.$$ 

Algorithm 1 will find a solution $W = [Y \ Z]$ and then a solution to problem (8) can be defined as $X = Y^T Z$. Some corresponding numerical results are presented in Section 6.

## 4 Rank minimization problem over positive semidefinite matrices

In this section we focus on the ARM-problem for semidefinite matrices (3). By Proposition 3, we can characterize the linear map $\ell$ by $k$ symmetric matrices $A_1, \ldots, A_k \in \mathbb{S}^n$, with respect to $b_1, \ldots, b_k$. In the subsection below, we
develop some more properties of the solutions to a system of linear equations. This might be not for our algorithm but it could be useful information in literature.

4.1 Solutions to systems of linear equations

Consider the system of linear equations as follows:

\[ \text{Tr}(A_iX) = b_i, \quad i = 1, \ldots, m, \]  

(9)

where \( A_i, X \) are real symmetric of order \( n \) and \( b = [b_1 \ldots b_m]^T \in \mathbb{R}^m \). The corresponding homogeneous of system (9) is defined by

\[ \text{Tr}(A_iX) = 0, \quad i = 1, \ldots, m. \]  

(10)

On the other hand, for such a nonhomogeneous system (9), we call the system

\[ \text{Tr}(\tilde{A}_i\tilde{X}) = 0, \quad i = 1, \ldots, m, \]  

(11)

with \( \tilde{A}_i = \begin{bmatrix} A_i & 0 \\ 0 & -b_i \end{bmatrix} \), its “dominating system”.

We also note that system (9) can be written in the classical form:

\[ \text{svec}(A_i)^T\text{svec}(X) = b_i, \quad \text{or} \quad \mathcal{A}\text{svec}(X) = b, \]  

(12)

where

\[ \mathcal{A} = \begin{bmatrix} \text{svec}(A_1)^T \\ \vdots \\ \text{svec}(A_m)^T \end{bmatrix} \in \mathbb{R}^{m\times\tau(n)}. \]

Such a system has \( \tau(n) \) variables. It is well known by Kronecker-Capelli theorem [15] that the system \( \mathcal{A}\text{svec}(X) = b \) has a solution if and only if \( \text{rank}(\mathcal{A}) = \text{rank}(\tilde{A}) \), where \( \tilde{A} := [A \mid b] \). Moreover, if \( \text{rank}(\mathcal{A}) = \text{rank}(\tilde{A}) = r \) then such a system has only one solution when \( r = \tau(n) \). In the case \( r < \tau(n) \), such a system has many solutions in which \( r \) variables linearly dependent on \( \tau(n) - r \) other variables. We also note that the system \( \{A_i\} \) is linearly (in)dependent in \( \mathbb{S}^n \) if and only if so is the system \( \{\text{svec}(A_i)\} \) in \( \mathbb{R}^{\tau(n)} \).

The following result gives us an equivalence of the non-homogeneous linear system and a homogeneous linear system in one more variable, in the case that the matrices \( A_i \) are linearly independent.
Proposition 4. System (9), with linearly independent matrices $A_i$'s and $b \neq 0$, $m \leq \tau(n)$, has a solution (must be nonzero) if and only if system (11) has a nontrivial solution.

Moreover, if the positive semidefiniteness of a solution to one of these two systems is valuable then so is a solution to the other system.

Proof. If $0 \neq X = [x_{ij}] \in S^n$ is a solution to (9) then one can check

$$0 \neq \tilde{X} = \begin{bmatrix} X & 0_{n\times 1} \\ 0_{1 \times n} & 1 \end{bmatrix} \in S^{n+1}$$

is a solution to (11) since

$$\text{Tr}(\tilde{A}_i \tilde{X}) = \text{Tr}(A_i X) - b_i = 0, \quad \forall i = 1, \ldots, m.$$

For the opposite direction, we first note that if the homogeneous dominating system (11) has nonzero solutions then there exists one whose $(n + 1, n + 1)$st entry is nonzero. Indeed, since $\{A_i\}_{i=1}^m$ is linearly independent, so is $\{\tilde{A}_i\}_{i=1}^m$. But the homogeneous dominating system (11) has $m$ equations and $\tau(n + 1)$ variables. Its solution vector space is hence of $\tau(n + 1) - m > 0$ dimensional since it has a nonzero positive semidefinite solution, provided by the hypothesis. A basis vector can be chosen with $t := \tilde{x}_{(n+1)(n+1)} \neq 0$. Indeed, if every solution $t$ was zeros then there would exist a nonsingular matrix $P$ (exists from the Gaussian elimination) such that

$$P\tilde{A} = \begin{bmatrix} \bullet & \bullet & \cdots & \bullet \\ 0 & \bullet & \cdots & \bullet \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$ 

This implies $\text{rank}(A) < m$. This contradicts to the fact that $\{A_i\}_{i=1}^m$ is linearly independent.

With a solution $\tilde{X}$ satisfying the above discussion, let $X$ be the $n \times n$ leading principle submatrix of $\tilde{X}$, we have

$$0 = \text{Tr}(\tilde{A}_i \tilde{X}) = \text{Tr}(A_i X) - b_i t, \quad i = 1, \ldots, m.$$ 

This $\frac{1}{t}X$ is a solution of (9).

The rest of the proposition is an immediate consequence of what have shown above. \hfill $\square$
Remark 1. Even though some nonzero solutions of two systems \(9\) and \(11\) stated in Proposition 4 simultaneously exist, they do not need have the same rank. To see this, let us consider the linear system

\[
\begin{align*}
\text{Tr}(A_1X) &= \text{Tr}(A_3X) = 0, \\
\text{Tr}(A_2X) &= -1,
\end{align*}
\]

where

\[
A_1 = \text{diag}(1, -1, 0), \quad A_2 = \text{diag}(1, 0, -1) \quad \text{and} \quad A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

and they are linearly independent. The matrices \(\tilde{A}_i\) are then defined as

\[
\tilde{A}_1 = \text{diag}(1, -1, 0, 0), \quad \tilde{A}_2 = \text{diag}(1, 0, -1, -1) \quad \text{and} \quad \tilde{A}_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

It is shown in \([32]\) that the dominating homogeneous system \((11)\) has no rank-one solution but a rank-three solution \(\tilde{X} = \text{diag}(1, 1, 0, 0)\). In our situation, we can find a rank-two solution, for example, \(\tilde{Y} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}\). However, the initial non-homogeneous system defined by the matrices \(A_1, A_2, A_3\) has a rank-one solution \(X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\).

The following proposition tells us the relationship between the existence of a positive definite element in \(\text{Span}(A_1, \ldots, A_m) := \{ \sum_{i=1}^m t_i A_i : t_i \in \mathbb{R} \} \) and that of trivial solution of the system \(\text{Tr}(A_iX) = 0, \forall i = 1, m\) over \(S^n_+\). This is due to Bohnenblust \([5]\) and is restated in some equivalent versions in \([1,13,32]\). In their works, the proofs are mainly based on either the separation theorem for two nonempty convex sets (see, eg., \([2, \text{Theorem III.1.2}]\)) or the SDP duality theory (see, eg., \([31]\)). In our situation, we use only knowledge on linear algebra, in particular, the theory of orthogonal complement in an inner-product vector space. This also gives us a stronger result, compared with the existence one.

**Proposition 5.** \([13,32]\) With the notation above, we have

\[
\{ X \in S^n_+ | \text{Tr}(A_iX) = 0, \forall i = 1, m \} = \{0\} \iff S^n_+ \cap \text{Span}(A_1, \ldots, A_m) \neq \emptyset.
\]
We have already known by Proposition 1 i) that any positive semidefinite matrix $X \in S^n_+$ with $\text{rank}(X) = r$ can be expressed as $X = \sum_{i=1}^{r} x_i x_i^T$ for some $x_i \in \mathbb{R}^n$, $i = 1, \ldots, r$. Based on the fact

$$\text{Tr}(A_i^T X) = \text{Tr}(A_i \sum_{j=1}^{r} x_j x_j^T) = \sum_{j=1}^{r} x_j^T A_i x_j, \quad \forall i = 1, \ldots, m,$$

the problem of finding a low-rank solution to (9) is of the form

$$\hat{x}^T \hat{A}_i \hat{x} = b_i, \quad i = 1, \ldots, m, \quad (13)$$

where the coefficient matrices now are $\hat{A}_i := \oplus_{j=1}^{r} A_j$ and $\hat{x} = [x_1^T \ldots x_r^T]^T$. It is clear that a nonzero solution to (13) gives a solution to system (9) with rank less than or equal to $r$. This is because of that $x_1, \ldots, x_r$ might be linearly dependent. We thus have the following.

**Proposition 6.** If system (9) has a solution of rank $r$ then system (13) has a nonzero solution. Conversely, if system (13) has a nonzero solution then system (9) has a solution of rank less than or equal to $r$.

**Remark 2.** i) When system (9), with $b \neq 0$, has a nonzero positive semidefinite solution then by the work of Barvinok [3], there is another positive solution with the rank at most $\frac{\sqrt{8m+1} - 1}{2}$. This upper bound is smaller than $n$ since $m \leq \tau(n)$. For us this bound is sharpest by now.

For homogeneous system (10), this bound does not necessary hold [32].

ii) According to the works [7, 18], one obtains

$$\max \{\text{rank}X : X \in \text{Span}(A_1, \ldots, A_m)\} \geq \frac{2n+1 - \sqrt{(2n+1)^2 - 8m}}{2}.$$ 

This, indeed, follows from the proofs for lower bound in [7, 18].

### 4.2 Algorithm

Even though this is a particular case of the ARM-problem over arbitrary matrices, Proposition 1 allows us to find a Cholesky factor $Y \in \mathbb{R}^{n \times r}$ instead of a positive semidefinite matrix, and this leads to a reduction in number of
variables for the ARM-problem. So problem (3) can be cast in the following form

\[
\begin{align*}
\text{minimize} & \quad \text{rank}(Y) \\
\text{subject to} & \quad [\text{Tr}(A_1^TYY^T) \ldots \text{Tr}(A_k^TYY^T)]^T = \ell(YY^T) = b.
\end{align*}
\]

The idea for solving this problem is similar to the previous case, where one checks whether there exists a matrix with lowest possible rank satisfying the requirements. In this situation, at the step corresponding to \( r \), the following function is applied: \( F : \mathbb{R}^{n \times r} \rightarrow \mathbb{R}^k \) defined by

\[
F(Y) = \ell(YY^T) - b, \quad \forall Y \in \mathbb{R}^{n \times r}.
\]

The coordinate functions \( F_i : \mathbb{R}^{n \times r} \rightarrow \mathbb{R}^k \) are obviously defined by

\[
F_i(Y) = \text{Tr}(A_i^TYY^T) - b_i, \quad \forall Y \in \mathbb{R}^{n \times r}.
\]

The Jacobian matrix of \( F \) in this case follows from (5):

\[
\text{Jac}(F) = \frac{\partial F}{\partial Y} = \frac{\partial \text{vec}F}{\partial \text{vec}Y} = \begin{bmatrix} \frac{\partial F_1}{\partial Y} \\ \vdots \\ \frac{\partial F_k}{\partial Y} \end{bmatrix} \in \mathbb{R}^{k \times nr},
\]

where

\[
\frac{\partial F_i}{\partial Y} = \frac{\partial \text{Tr}(A_i^TYY^T)}{\partial Y} = \text{vec}[(A_i^T + A_i)Y]^T.
\]

5 Applications

In this section, we consider three applications of problem (3).

5.1 Low-rank matrix completion

In machine learning scenarios, e.g., in factor analysis, collaborative filtering, and latent semantic analysis [24,25,28], there are several problems that can be reformulated as the low-rank matrix completion problem. Given the values of some entries of a matrix, this problem fills the missing entries of the matrix
such that its rank is small as possible. This problem is summarized and reformulated as follows. Given a set of triples

\[(R, C, S) \in \{1, \ldots, m\}^k \times \{1, \ldots, n\}^k \times \mathbb{R}^k,\]

and we wish to construct a small-as-possible rank matrix \(X = [X_{rs}] \in \mathbb{R}^{m \times n}\), such that \(X_{R(i), C(i)} = S(i)\) for all \(i = 1, \ldots, p\). This can be reformulated as

\[
\begin{align*}
\text{minimize} & \quad \text{rank}(X) \\
\text{subject to} & \quad X_{R(i), C(i)} = S(i), \quad \forall i = 1, \ldots, p.
\end{align*}
\]

This problem can then be solved by using Algorithm 1.

5.2 Low-dimensional Euclidean embedding problems

Euclidean distance matrices, shortly EDMs, have received increased attention because of its many applications which can be found in eg., [6, 8, 21, 24] and references there in.

We first recall this problem. Let \(D = [d_{ij}] \in \mathbb{S}^n\) be a Euclidean distance matrix (EDM) associated to the points \(x_1, \ldots, x_n \in \mathbb{R}^r\), i.e.,

\[
d_{ij} = \|x_i - x_j\|^2 = x_i^T x_i + x_j^T x_j - 2x_i^T x_j, \quad i, j = 1, \ldots, n. \tag{16}
\]

The smallest positive integer number \(r\) is said to be the embedding dimension of \(D\).

Following [24], let \(1 \in \mathbb{R}^{n \times 1}\) be the column vector of ones. Define \(V \triangleq I_n - \frac{1}{n} 11^T\). Note that \(V\) is the orthogonal projection matrix onto the hyperplane \(\{v \in \mathbb{R}^{n \times 1} : 1^Tv = 0\}\). In particular,

\[
\text{rank}(V) = n - 1
\]

and \(V\) has an eigenvector \(1\) with respect to the eigenvalue zero. It follows from the work in [26] that \(D\) is an EDM of \(n\) points in \(\mathbb{R}^r\) if and only if three following conditions hold:

\[
\begin{align*}
d_{ii} &= 0, \quad \forall i = 1, \ldots, n; \\
-VDV &\succeq 0; \\
\text{rank}(VDV) &\leq r.
\end{align*}
\]
Given a positive integer number $n$ and partial Euclidean matrix $D_0$, i.e., every entry of $D_0$ is either “specified” or “unspecified”, $\text{diag}(D_0) = 0$, and every fully specified principal sub-matrix of $D_0$ is also a Euclidean distance matrix. The low-dimensional Euclidean embedding problem finds a Euclidean matrix $D$ consistent with the known pairwise distances described by $D_0$ and associated to a number of points in the smallest dimensional space $\mathbb{R}^r$. Such a problem can be reformulated as the ARM-problem 

$$\min \quad \text{rank}(VDV)$$

$$\text{subject to}$$

$$-VDV \succeq 0,$$

$$\ell(D) = b,$$

where $\ell: \mathbb{S}^n \rightarrow \mathbb{R}^p$ is an appropriate linear map, corresponding to the specified entries in $D_0$, including the condition that makes the diagonal of $D$ to be zero. If one sets $X = [x_1 \ldots x_n] \in \mathbb{R}^{r \times n}$ then $D$ can be found in form

$$D = \mathcal{D}(X) := \text{diag}(X^T X)1^T + 1\text{diag}(X^T X)^T - 2X^T X$$

because of (16). Since $V1 = 0$,

$$-VD(X)V = 2VX^T XV.$$ (19)

Substituting this fact into problem (17) we get the equivalent one:

$$\min \quad \text{rank}(XV)$$

$$\text{subject to}$$

$$X \in \mathbb{R}^{r \times n},$$

$$\ell(\mathcal{D}(X)) = b.$$ (20)

It is clear that the above problem is of the form of problem (3) with $\phi(X) = \ell(\mathcal{D}(X))$.

We make the linear map $\ell$ more explicit as in [14]. Let $H$ be the 1-0 adjacency matrix, i.e.,

$$h_{ij} = \begin{cases} 
1 & \text{if } (i, j) \in E, \\
0 & \text{if } (i, j) \notin E,
\end{cases}$$

for the set of subscripts $E$ corresponding to the specified entries of $D_0$. The main problem is to find an as-small-as-possible rank completion $D$ of $D_0$. 

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Namely, one needs to find $D$ in the form

$$D = \text{diag}(Z)1^T + 1\text{diag}(Z)^T - 2Z,$$

$$Z = X^TX,$$

$$H \odot D = H \odot D_0,$$  \hspace{1cm} (21)

where $\odot$ denotes the component-wise (or Hadamard) matrix product. With the help of the fact $\text{rank}(XV) \leq \text{rank}(X)$, problem (21) is then reduced to the rank minimization problem in the form

$$\begin{align*}
\text{minimize} & \quad \text{rank}(X) \\
\text{subject to} & \quad X \in \mathbb{R}^{r \times n}, \\
& \quad H \odot D(X) = H \odot D_0.
\end{align*}$$  \hspace{1cm} (22)

In many applications one seeks the embedding dimension being two or three.

Note that when $D$ is determined as $D = D(X)$, the following formula is used to compute the Jacobian matrices used in the Levenberg-Marquardt algorithm:

$$\frac{\partial d_{ij}}{\partial X} = \begin{bmatrix} 0 \ldots 0 & 0 \ldots 0 & 2(x_i^T - x_j^T) & 0 \ldots 0 & 2(x_j^T - x_i^T) & 0 \ldots 0 \end{bmatrix} \in \mathbb{R}^{1 \times nr}.$$

Here we assume $i \leq j$.

### 6 Numerical experiments

#### 6.1 General RM-problem and quadratic systems

From the theoretical point of view, the rank minimization problem is NP-hard so that there has not been any method directly solve this one in the literature. A good way for solving the RM-problem over positive semidefinite matrices is to solve the corresponding problem that minimize the nuclear norm (see, e.g., [19, 24]). The nuclear norm minimization problem (NNM-problem) is a really good one to give suitable lower and upper bounds for the original RM-problem. Additionally, in [24] it is proved that the NNM-problem over general matrices is tractable to solve since it can be reformulated as a semidefinite program [10]. Another method for solving the NNM-problem
over positive semidefinite matrices was proposed in [19] by using Modified Fixed Point Continuation Method.

We now illustrate the RM-problem over generic matrices. Table 1 shows the results obtained by Algorithms 1 for this case. The matrices $A_1, \ldots, A_k$ are randomly chosen with entries in $(0,1)$. The backward errors are determined by

$$\text{err} = \frac{\|\ell(X) - b\|_2}{\|b\|_2}.$$ 

The result for each case is averagely taken per three experiments. The numerical results show that Algorithm 1 gives better solutions if the factorization in Proposition 2 is applied. More precisely, we see in Table 1 the results when Proposition 2 is applied have smaller rank. Table 1 also exhibits a comparison between our method and the one described in [24].

The NNM-problem approximating problem (2) is followed in [24] and can be summarize as follows

$$\text{minimize} \quad \|X\|_*$$
subject to
$$\ell(X) = b,$$

where $\|\cdot\|_*$ denotes the nuclear norm of $X$, which is the sum of all its singular values. If $X$ has a singular value decomposition $X = U\Sigma V^T$ then one can solve problem (23) by solving the semidefinite program:

$$\text{minimize} \quad \frac{1}{2}(\text{Tr}(W_1) + \text{Tr}(W_2))$$
subject to
$$\begin{bmatrix} W_1 & X \\ X^T & W_2 \end{bmatrix} \succeq 0,$$
$$\ell(X) = b.$$ 

(24)

This is nice formulation in theoretical point of view but in practice the resulting matrices may have “high-rank” by SDP solvers. One can see in Table 1 where problem (24) is implemented in CVX toolbox of MATLAB calling Sedumi [29], that the semidefinite program seems to give resulting matrices with full rank and less accuracy.

For the RM-problem over positive semidefinite matrices, the experiments perform with randomly chosen symmetric matrices $A_1, \ldots, A_k$. The result for each case is also averagely taken per three experiments.
Table 1: Comparison between LM-method and SDP solving the RM-problem over \( m \times n \) matrices.

Table 2 shows a comparison between our method and the one described in [19]. The errors in this table are computed as

\[
\text{err} = \frac{\| \ell(X) - b \|_2}{\| b \|_2}.
\]

What we see in Table 2 that the values of the rank of resulting matrices obtained by our method are smaller the ones obtained by solving the corresponding NNM-problem.

Table 3 shows the results for several values of \( m, n \). We take \( R, C \in \mathbb{N}^k \) with the entries are random in \( \{1, \ldots, m\}, \{1, \ldots, n\} \), respectively, and so is

\begin{table}[h]
\centering
\begin{tabular}{ccc|cc|cc|cc}
\hline
\( m \) & \( n \) & \( k \) & \( \text{rank}(X) \) & \( \text{err} \) & \( \text{rank}(X) \) & \( \text{err} \) & \( \text{SDP} \) & \( \text{err} \) \\
\hline
5 & 6 & 4 & 1 & 5.31e-16 & 1 & 2.44e-16 & 5 & 1.02e-09 \\
51 & 50 & 51 & 1 & 7.13e-15 & 1 & 4.49e-15 & 50 & 8.42e-10 \\
50 & 100 & 81 & 2 & 5.56e-16 & 1 & 9.42e-16 & 50 & 3.38e-09 \\
50 & 200 & 100 & 3 & 5.46e-16 & 1 & 7.31e-16 & 50 & 4.08e-09 \\
100 & 200 & 300 & 3 & 2.63e-14 & 2 & 3.83e-15 & out of memory & \\
500 & 550 & 300 & 1 & 1.66e-15 & 1 & 1.93e-14 & out of memory & \\
500 & 500 & 450 & 1 & 2.04e-15 & 1 & 1.10e-14 & out of memory & \\
\hline
\end{tabular}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{ccc|cc|cc|cc}
\hline
\( n \) & \( k \) & \( \text{rank}(X) \) & \( \text{err} \) & \( \text{rank}(X) \) & \( \text{err} \) & \( \text{SDP} \) & \( \text{err} \) \\
\hline
100 & 579 & 6 & 10 & 1.89e-16 & 9.46e-4 \\
200 & 1221 & 7 & 10 & 1.87e-15 & 9.84e-4 \\
500 & 5124 & 11 & 10 & 2.52e-15 & 4.90e-3 \\
500 & 3309 & 7 & 27 & 3.00e-15 & NA \\
\hline
\end{tabular}
\end{table}
$S \in (0,1)^k$. For the cases $m = n$, it turns out the results for the systems of quadratic equations. More precisely, the system has solution if the solutions’ have rank one.

\[
\begin{array}{cccc|cccc}
 m & n & k & \text{rank}(X) & m & n & k & \text{rank}(X) \\
 5 & 6 & 4 & 4 & 50 & 50 & 51 & 2 \\
 51 & 50 & 51 & 3 & 100 & 100 & 50 & 1 \\
 50 & 100 & 81 & 1 & 150 & 150 & 100 & 1 \\
 50 & 200 & 100 & 1 & 200 & 200 & 200 & 2 \\
 100 & 200 & 300 & 3 & 400 & 400 & 350 & 1 \\
 500 & 550 & 300 & 1 & 500 & 500 & 450 & 1 \\
\end{array}
\]

Table 3: Solution to the low-rank matrix completion using LM-method.

### 6.2 Euclidean distance matrix problem

This section shows the numerical results for problem (22). All tests are dealt with partial Euclidean matrices $D_0$ with entries randomly taken in the interval $[0,1]$.

Table 4 shows the Euclidean embedding dimensions for all cases that $D_0$ are dense, i.e., the entries of the corresponding matrix $H$ are all one.

\[
\begin{array}{ccc|ccc}
 n & \text{rank}(X) & \text{err} & n & \text{rank}(X) & \text{err} \\
 4 & 2 & 5.09e-16 & 100 & 2 & 1.33e-14 \\
 10 & 2 & 9.38e-16 & 150 & 2 & 1.99e-14 \\
 20 & 2 & 2.09e-15 & 200 & 2 & 2.54e-14 \\
 30 & 2 & 4.04e-15 & 300 & 2 & 3.84e-14 \\
 40 & 2 & 6.10e-15 & 400 & 2 & 4.20e-14 \\
 50 & 2 & 5.98e-15 & 500 & 2 & 4.32e-14 \\
\end{array}
\]

Table 4: Solution to EDM problems with respect to dense partial EDM matrices.

Table 5 shows the results for sparse matrices $D_0$, i.e., the entries of the corresponding matrix $H$ are either zero or one.
Table 5: Solutions to EDM problems with respect to randomly-chosen sparse partial EDM matrices.

| n   | rank(X) | err      | n   | rank(X) | err      |
|-----|---------|----------|-----|---------|----------|
| 4   | 2       | 1.95e-16 | 100 | 2       | 4.03e-16 |
| 10  | 2       | 5.16e-16 | 150 | 2       | 3.97e-16 |
| 20  | 2       | 4.56e-16 | 200 | 2       | 4.23e-16 |
| 30  | 2       | 4.85e-16 | 300 | 2       | 3.89e-16 |
| 40  | 2       | 4.15e-16 | 400 | 2       | 4.42e-16 |
| 50  | 2       | 4.00e-16 | 500 | 2       | 4.54e-16 |

The backward errors of all tests in both cases of $D_0$ are determined as

$$
\text{err} = \frac{\|D - D_0\|_2}{\|D_0\|_2}.
$$

It turns out that the configurations of our experiments are all in two dimensional spaces.

7 Conclusion and discussion

We have proposed an algorithm for solving the rank minimization problem over a subset of $\mathbb{R}^{m \times n}$ determined by a differentiable function. As a consequence, the affine rank minimization problems over either arbitrary or positive semidefinite matrices have been numerically tested. This algorithm was then applied to solve the low-rank matrix completion problem and the low-dimensional Euclidean embedding problem. Some numerical experiments have been performed to illustrate our algorithms as well as the applications.

We have also developed some useful properties for low rank solutions to systems of linear matrix equations. This suggests us a reformulation of the IIR and FIR low-pass filter problems described in [17] as optimization problems over rank-one positive semidefinite matrices. In the future we will deal with this method to solve such filter design problem. This might be suitable because the resulting positive semidefinite matrices derived by SDP solvers in [17] are usually full rank. Obviously, this requires a much more amount of memory and complexity in comparison with rank-one setting.
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