We consider a Brownian particle which, in addition to being in contact with a thermal bath, is driven by active fluctuations. These active fluctuations do not fulfill a fluctuation-dissipation relation and therefore play the role of a non-equilibrium environment. Using an Ornstein-Uhlenbeck process as a model for the active fluctuations, we derive the path probability of the Brownian particle subject to both thermal and active noise. From the case of passive Brownian motion, it is well-known that the log-ratio of path probabilities for observing a certain particle trajectory forward in time versus observing its time-reversed twin trajectory quantifies the entropy production. We calculate this path probability ratio for active Brownian motion and derive a generalized “entropy production”, which fulfills an integral fluctuation theorem. We show that those parts of this “entropy production” which are different from the usual dissipation of heat in the thermal environment, can be associated with the mutual information between the particle trajectory and the history of the non-equilibrium environment. When deriving and discussing these results we keep in mind that the active fluctuations can occur due to either a suspension of active particles pushing around a passive colloid or due to active self-propulsion of the particle itself; we point out the similarities and differences between these two situations. Finally, we illustrate our general results by analyzing a Brownian particle which is trapped in a static or moving harmonic potential.

I. INTRODUCTION

Active particle systems consist of individual entities (“particles”) which have the ability to perform motion by consuming energy from the environment and converting it into a self-propulsion drive [1–7]. Prototypical examples are collections of macro-organisms, such as animal herds, schools of fish, flocks of birds, or ant colonies [8–10], and suspensions of biological microorganisms or artificial microswimmers, such as bacteria and colloidal particles with catalytic surfaces [2, 3, 5, 11, 12]. Systems of this kind exhibit a variety of intriguing properties, e.g. clustering and swarming [13–18], bacterial turbulence [19], or motility-induced phase separation [20, 21], to name but a few.

Microorganisms and microswimmers are usually dispersed in an aqueous solution at room temperature and therefore experience thermal fluctuations which give rise to a diffusive component in their self-propelled swimming motion. In addition, the self-propulsion mechanism is typically noisy in itself [2], for instance, due to environmental factors or intrinsic stochasticity of the mechanisms creating self-propulsion. These “active fluctuations” exhibit two essential features. First, a certain persistence in the direction of driving over length and time scales comparable to observational scales. Second, an inherent non-equilibrium character as a consequence of permanently converting and dissipating energy in order to fuel self-propulsion. Interestingly, similar active fluctuations with the same characteristics can be observed in a complementary class of active matter systems, namely a passive colloidal “tracer” particle which is suspended in an aqueous solution of active swimmers. The collisions with the active particles in the environment entail directional persistence and non-equilibrium features in the motion of the passive tracer particle [22–24].

Despite the inherent non-equilibrium properties of active matter systems they appear to bear striking similarities to equilibrium systems [21, 25–27], for instance, the dynamics of individual active particles at large scales often looks like passive Brownian diffusion [3] (i.e., at scales beyond at least the “persistence length” of the particle motion). In connection with the ongoing attempts to describe active matter systems by thermodynamic (-like) theories this observation raises the important question “How far from equilibrium is active matter?” [28, 29]. More specifically, the questions are: in which respects do emergent properties of active systems resemble the thermodynamics of thermal equilibrium systems, in which respects do they not, and how do these deviations manifest themselves in observables describing the thermodynamic character of the active matter system, like, e.g., the entropy production?

We here address these questions from a fundamental non-equilibrium statistical mechanics viewpoint by quantifying the probability density of particle trajectories generated under the combined influence of thermal and active fluctuations, the latter stemming either from an active bath the (passive) particle is dispersed in or from active self-propulsion. In order to account for the active fluctuations without resolving the microscopic processes that drive self-propulsion or govern the interactions between active and passive particles, we follow the common approach to include stochastic “active forces”...
in the equations of motion of the individual particle of interest (see, e.g., [2, 5] and references therein). When calculating the path probabilities, we treat these active non-equilibrium forces in the same way as thermal equilibrium noise, namely as a “bath” the particle is exposed to with unknown microscopic details, but known statistical properties (which are correlated in time and break detailed balance [30]). Adopting the viewpoint of stochastic thermodynamics [31–35], we study the breakdown of time-reversal symmetry in the motion of the particle and the associated irreversibility of particle trajectories as a measure for non-equilibrium. We do so by comparing the probability of a specific particle trajectory to occur forward in time to that of its time-reversed counterpart, and by linking the corresponding probability ratios to extensive quantities produced along the forward trajectory.

In the case of passive particles in contact with a single, purely thermal bath this procedure is well established, and is known to provide relations between thermodynamic quantities, such as entropy production, work, or heat, and dynamical properties encoded in path probability ratios. Many such relations have been found over the last two decades [32, 33, 36–38], which, in their integrated forms, typically yield refinements of the second law of thermodynamics [32, 37, 39]. In particular, the fluctuation theorem for the total entropy production of a passive particle in a thermal environment reinforces the fundamental interpretation of entropy as a measure of irreversibility (we give a brief summary of the results most relevant to our present work in Sec. III). In the presence of active fluctuations, however, the identification of dissipated heat and entropy production is less straightforward, and is connected to the problem of how to calculate and interpret the path probability ratio in a physically meaningful way [40, 41].

The results we present in this paper contribute to resolving these questions. Our main findings are the following: (i) Using a path-integral approach, we calculate the probability density of particle trajectories as a result of the statistical properties of thermal and active fluctuations simultaneously affecting the dynamics of the particle [Sec. IV B, Eqs. (34) and (36)]. When calculating these path weights, we consider the general situation in which the particle is also subject to external (or interaction) forces. We summarize the mathematical details of the derivation as well as various known limiting cases in Appendices A–C. (ii) We then find that the log-ratio of the probabilities for particle trajectories to occur forward in time versus backward in time can be expressed as a functional along the forward path with a non-local “memory kernel” [Sec. IV C, Eqs. (38)]. This functional (we denote it \( \Delta \Sigma \)) therefore quantifies irreversibility in our active matter systems. Combined with the change in system entropy, it fulfills an integral fluctuation theorem [Sec. IV C, Eq. (40)], valid for any duration of the particle motion, and consequently obeys a second-law like relation [Eq. (41)]. (iii) By keeping track of the specific realization of the active fluctuations which participated in generating the particle trajectory, we can identify the two individual contributions to \( \Delta \Sigma \) stemming from the thermal environment and the active bath. The thermal part is the usual entropy produced in the thermal environment along the particle trajectory (for the given active noise realization). The part associated with the active bath is given by the difference in the amount of correlations (measured in terms of path-wise mutual information), that are built up between the particle trajectory and the active noise in the time-forward direction as compared to the time-backward direction (Sec. V). This splitting is valid no matter if we assume the active fluctuations to be even [Eq. (54)] or odd [Eq. (60)] under time-reversal. From the fluctuation theorem for \( \Delta \Sigma \) we then obtain an integral fluctuation theorem and a second-law relation for the mutual information difference [Eqs. (55), (61) and (56), (62)].

We illustrate all these findings in Sec. VI by discussing the example of a colloidal particle subject to thermal and active fluctuations, which is trapped in a (static or moving) harmonic potential. For this simple linear system all relevant quantities can be calculated explicitly. We give the most important mathematical details in Appendix D.

Our results (i) and (ii) rely on the specific model of a Gaussian Ornstein-Uhlenbeck process [42, 43] for the active fluctuations, while result (iii) is valid for general types of active fluctuations. The Ornstein-Uhlenbeck process has become quite popular and successful in describing active fluctuations [7, 15–18, 21–24, 28, 41, 44–54], because it constitutes a minimal model for persistency in the active forcings due its exponentially decaying correlations with finite correlation time, and it can easily be set up to break detailed balance by a “mismatched” damping term which does not validate the fluctuation-dissipation relation [55, 56]. Moreover, it is able to describe both situations of interest mentioned above, namely passive motion in a bath of active swimmers [22–24] and active motion driven by self-propulsion [7, 15–18, 21, 28, 41, 44–54]. The details of our model are given in the next section.

II. MODEL

We consider a colloidal particle at position \( \mathbf{x} \) in \( d = 1, 2, \) or 3 dimensions, which is suspended in an aqueous solution at thermal equilibrium with temperature \( T \). The particle diffuses under the influence of deterministic external forces, in general consisting of potential forces \( -\nabla U(\mathbf{x}, t) \) (with the potential \( U(\mathbf{x}, t) \)) and non-conservative force components \( \mathbf{F}(\mathbf{x}, t) \). In addition, it experiences fluctuating driving forces due to permanent energy conversion from active processes in the environment or the particle itself. The specific examples we have in mind are a passive tracer in an active non-thermal bath (composed, e.g., of bacteria in aqueous solution) [22–24], or a self-propelled particle (e.g., a bacterium or colloidal microswimmer) [7, 15–18, 21, 28, 41, 44–54].
Neglecting inertia effects [57], we model the overdamped Brownian motion of the colloidal particle by the Langevin equation [42, 43, 58, 59]

\[ \dot{x}(t) = v(x(t), t) + \sqrt{2D_\alpha} \eta(t) + \sqrt{2D} \xi(t). \]  

(1)

The deterministic forces are collected in

\[ v(x, t) = \frac{1}{\gamma} f(x, t), \]  

(2a)

where \( \gamma \) is the hydrodynamic friction coefficient of the particle and

\[ f(x, t) = -\nabla U(x, t) + F(x, t). \]  

(2b)

Thermal fluctuations are described by unbiased Gaussian white noise sources \( \xi(t) \) with mutually independent, delta-correlated components \( \xi_i(t) \), i.e. \( \langle \xi_i(t) \xi_j(t') \rangle = \delta_{ij}\delta(t-t') \), where the angular brackets denote the average over many realizations of the noise and \( i, j \in \{1, \ldots, d\} \).

The strength of the thermal fluctuations is given by the particle’s diffusion coefficient \( D \), which is connected to the temperature \( T \) and the friction \( \gamma \) by the fluctuation-dissipation relation \( D = k_B T / \gamma \) (\( k_B \) is Boltzmann’s constant), as a consequence of the equilibrium properties of the thermal bath.

The term \( \sqrt{2D_\alpha} \eta(t) \) in (1) represents the active force components, with \( \eta(t) = (\eta_1(t), \ldots, \eta_d(t)) \) being unbiased, mutually independent noise processes, and \( D_\alpha \) being an effective “active diffusion” characterizing the strength of the active fluctuations. The model (1) does not contain “active friction”, i.e. an integral term over \( \dot{x}(t) \) with a friction kernel modelling the damping effects associated with the active forcing. It thus represents the limiting case in which such friction effects are negligibly small. In spite of this approximation, the model (1) has been applied successfully to describe various active particle systems [15, 17, 21–24, 54], and has become quite popular in an even more simplified variant which neglects thermal fluctuations \( (D = 0) \) [7, 16, 18, 28, 41, 44–52]. However, for thermodynamic consistency it is necessary to consider both noise sources [41], especially when assessing the flow of heat and entropy, which is in general produced in the thermal environment as well as in the processes fuelling the active fluctuations [40, 63].

As active fluctuations are the result of perpetual energy conversion (e.g., by the bacteria in the bath, or by the propulsion mechanism of the particle), their salient feature is that they do not fulfill a fluctuation-dissipation relation. In our model (1), this is particularly obvious as active friction effects are absent by the assumption of being negligibly small; in general, the active friction kernel would not match the active noise correlations [64, 65]. In that sense, the active fluctuations \( \eta(t) \) characterize a non-thermal environment or bath for the Brownian particle with intrinsic non-equilibrium properties. We emphasize that \( \eta(t) \) is not a quantity directly measurable, but rather embodies the net effects of the active components in the environment or the self-propulsion mechanism of the particle, similar to the white noise \( \xi(t) \) representing an effective description of the innumerable collisions with the fluid molecules in the aqueous solution.

Although several results in the present paper will be generally valid for any noise process \( \eta(t) \) (with finite moments), our main aim is to study the specific situation of exponentially correlated, Gaussian non-equilibrium fluctuations [28, 41, 44–47, 49, 66], generated from an explicitly solvable Ornstein-Uhlenbeck process,

\[ \dot{\eta}(t) = -\frac{1}{\tau_\alpha} \eta(t) + \frac{1}{\tau_\alpha} \zeta(t), \]  

(3)

i.e., so-called Gaussian colored noise [67]. Here, \( \zeta(t) = (\zeta_1(t), \ldots, \zeta_d(t)) \) are mutually independent, unbiased, delta-correlated Gaussian noise sources, just like \( \xi(t) \), but completely unrelated to them. The characteristic time \( \tau_\alpha \) quantifies the correlation time of the process \( (i, j \in \{1, \ldots, d\}) \),

\[ \langle \eta_i(t) \eta_j(t') \rangle = \frac{\delta_{ij}}{2\tau_\alpha} e^{-|t-t'|/\tau_\alpha}, \]  

(4)

as can be verified easily from the explicit steady-state solution of (3). It is thus a measure for the persistence of the active fluctuations.

The model (1), (2) describes a single Brownian particle in simultaneous contact with a thermal bath and an active environment as a source of non-equilibrium fluctuations \( \eta(t) \). Our main aim in this paper is to investigate in detail the role these non-equilibrium fluctuations play for the stochastic energetics [68] and thermodynamics [33] of the Brownian particle. While we adopt the single particle picture for simplicity, all our results hold for multiple, interacting Brownian particles as well. In this case, the symbol \( \mathbf{x} \) in (1), (2) denotes a super-vector collecting the positions \( x_i \) (\( i = 1, 2, \ldots, N \)) of all \( N \) particles, i.e. \( \mathbf{x} = (x_1, x_2, \ldots, x_N) \), and similarly for the forces, velocities, and so on. The only requirement is that the particles are identical in the sense that they have the same coupling coefficients \( D \) and \( D_\alpha \), and that the active fluctuations \( \eta_i(t) \) of the individual constituents are independent, but share identical statistical properties.

**III. THE IDEAL THERMAL BATH: \( D_\alpha = 0 \)**

In order to set up the framework and further establish notation, we start with briefly recalling the well-known case of a Brownian particle in sole contact with a thermal equilibrium reservoir,

\[ \dot{x}(t) = v(x(t), t) + \sqrt{2D} \xi(t), \]  

(5)

where the deterministic driving \( v(x(t), t) \) is defined as in (2). The Brownian particle can be prevented from equilibrating with the thermal bath by time-variations in the potential \( U(x, t) \) or by non-conservative external forces \( \mathbf{F}(x, t) \).
A. Energetics

Following Sekimoto [68, 69], the heat \( \delta Q \) that the particle exchanges with the thermal bath while moving over an infinitesimal distance \( d\mathbf{x}(t) \) during a time-step \( dt \) from \( t \) to \( t+dt \) is quantified as the energy the thermal bath transfers to the particle along this displacement due to friction \(-\dot{\mathbf{x}}(t)\) and fluctuations \( \sqrt{2k_B T \gamma} \xi(t) \), i.e.

\[
\delta Q(t) = \left( -\dot{\mathbf{x}}(t) + \sqrt{2k_B T \gamma} \xi(t) \right) \cdot d\mathbf{x}(t), \tag{6}
\]

where the product needs to be interpreted in Stratonovich sense [70]. With the definition (6), heat is counted as positive if received by the particle and as negative when dumped into the environment (this sign convention is thus the same as Sekimoto’s original one [68, 69]). From the equation of motion (5), we immediately see that the heat exchange can be equivalently written as

\[
\delta Q(t) = -\left( -\nabla U(\mathbf{x}(t), t) + \mathbf{F}(\mathbf{x}(t)) \right) \cdot d\mathbf{x}(t). \tag{7}
\]

The change of the particle’s “internal” energy \( dU(t) \) over the same displacement \( d\mathbf{x}(t) \) is given by the total differential

\[
dU(t) = \nabla U(\mathbf{x}(t), t) \cdot d\mathbf{x}(t) + \frac{\partial U(\mathbf{x}(t), t)}{\partial t} dt. \tag{8}
\]

Hence, if the potential does not vary over time, the only contribution to \( dU(t) \) comes from the change in the potential energy associated with the displacement \( d\mathbf{x} \).

On the other hand, even if the particle does not move within \( dt \), its “internal” energy can still change due to variations of the potential landscape by an externally applied, time-dependent protocol. Prototype examples are intensity- or position-variations of optical tweezers, which are introduced the short-hand notation \( \mathbf{F}(\mathbf{x}, t) \). The second source of external forces which may contribute to such work are the non-conservative components \( \mathbf{F}(\mathbf{x}, t) \) in (5) (originally not considered by Sekimoto [69], but systematically analyzed later by Speck et al. [75]). The total work applied on the particle by external forces is therefore given by

\[
\delta W(t) = \frac{\partial U(\mathbf{x}(t), t)}{\partial t} dt + \mathbf{F}(\mathbf{x}(t), t) \cdot d\mathbf{x}(t). \tag{9}
\]

Combining Eqs. (7), (8), and (9) we obtain the first law

\[
dU = \delta Q + \delta W \tag{10}
\]

for the energy balance over infinitesimal displacements \( d\mathbf{x}(t) \), valid at any point in time \( t \). After integration along a specific trajectory \( \{\mathbf{x}(t)\}_{t=0}^{\tau} \) of duration \( \tau \), which starts at \( \mathbf{x}(0) = \mathbf{x}_0 \) and ends at some \( \mathbf{x}(\tau) = \mathbf{x}_\tau \) (which is different for every realization of the thermal noise \( \xi(t) \) in (5), even if \( \mathbf{x}(0) = \mathbf{x}_0 \) is kept fixed), we find the first law at the trajectory level [34]

\[
\Delta U[\mathbf{x}] = U(\mathbf{x}_\tau, \tau) - U(\mathbf{x}_0, 0) = Q[\mathbf{x}] + W[\mathbf{x}]. \tag{11}
\]

B. Path probability and entropy

The next step towards a thermodynamics characterization of the Brownian motion (5) is to introduce an entropy change or entropy production associated with individual trajectories [37]. From the viewpoint of irreversibility, such an entropy concept has been defined as the log-ratio of the probability densities for observing a certain particle trajectory and its time-reversed twin [76]. We express the probability density of a particle trajectory by the standard Onsager-Machlup path integral [77–80],

\[
\mathbf{p} [\mathbf{x} | \mathbf{x}_0] \propto \exp \left\{ - \int_0^\tau dt \left[ (\dot{x}_i - v_i)^2 / 4D + \nabla \cdot \mathbf{v}_i / 2 \right] \right\}, \tag{12}
\]

where we condition on a fixed starting position \( \mathbf{x}_0 \), and, accordingly, introduce the notation \( \mathbf{p} = \{\mathbf{x}(t)\}_{t>0} \) to denote a trajectory which starts at fixed position \( \mathbf{x}(0) = \mathbf{x}_0 \). In contrast to \( \mathbf{x} \) from above the set of points \( \mathbf{p} \) does therefore not contain the initial point \( \mathbf{x}_0 \) (i.e. \( \mathbf{p} = \mathbf{x}_0 \cup \mathbf{x} \)). The path probability (12) is to be understood as a product of transition probabilities in the limit of infinitesimal time-step size, using a mid-point discretization rule. The divergence of \( \mathbf{v} \) in the second term represents the path-dependent part of normalization [79, 80], while all remaining normalization factors are path-independent constants and thus omitted. For convenience, we have introduced the short-hand notation \( \mathbf{x}_t = \mathbf{x}(t) \) and \( \mathbf{v}_t = \mathbf{v}(\mathbf{x}(t), t) \).

We now consider the time-reversed version of this trajectory, which is traced out backward from the final point \( \mathbf{x}(\tau) = \mathbf{x}_\tau \) to the initial point \( \mathbf{x}_0 \) when advancing time,

\[
\tilde{x}(t) := x(\tau - t), \tag{13}
\]

and ask how likely it is that \( \tilde{x} = \{\tilde{x}(t)\}_{t>0} \) is generated by the same Langevin equation (5) as \( \mathbf{x} \), with the same deterministic forces \( \mathbf{v} \) acting at identical positions along the path. The latter requirement implies that in case of explicitly time-dependent external forces the force protocol has to be time-inverted, i.e. \( \mathbf{v}(\mathbf{x}, t) \) is replaced by \( \mathbf{v}(\mathbf{x}, \tau - t) \) in (5) to construct the Langevin equation for the time-reserved path [81]. From that Langevin equation we can deduce the probability \( \mathbf{p}[\tilde{x} | \mathbf{x}_0] \) for observing the backward trajectory \( \tilde{x} \), conditioned on its initial position \( \tilde{x}_0 = \mathbf{x}(\tau) = \mathbf{x}_\tau \), in analogy to (12). Using (13),
we can then express $\tilde{p}[\mathbf{x}|\mathbf{x}_0]$ in terms of the forward path $\mathbf{x}$, so that we find for the path probability ratio
\begin{equation}
\frac{\tilde{p}[\mathbf{x}|\mathbf{x}_0]}{p[\mathbf{x}|\mathbf{x}_0]} = e^{-\Delta S[\mathbf{x}]/k_B},
\end{equation}
with the quantity $\Delta S[\mathbf{x}]$ being a functional of the forward path only,
\begin{align}
\Delta S[\mathbf{x}] &= \frac{1}{T} \int_0^T dt \dot{x}(t) \cdot f(x(t), t) \\
&= \frac{1}{T} \int_0^T f(x(t), t) \cdot d\mathbf{x}(t).
\end{align}
As a quantitative measure of irreversibility, $\Delta S[\mathbf{x}]$ is identified with the entropy production along the path $\mathbf{x}$ with given initial position $\mathbf{x}_0$.

For an infinitesimal displacement $d\mathbf{x}(t)$ the corresponding entropy change reads
\begin{equation}
\delta S(t) = \frac{1}{T} f(x(t), t) \cdot d\mathbf{x}(t) = -\frac{\delta Q(t)}{T}.
\end{equation}

The last equality follows from comparison with (7) [see also (2)] and states that the entropy production along $d\mathbf{x}(t)$ is given by the heat $-\delta Q(t)$ dissipated into the environment during that step divided by the bath temperature. For that reason, $\delta S(t)$, and $\Delta S[\mathbf{x}]$, are more accurately called entropy production in the environment.

The entropy of the Brownian particle itself (i.e. the entropy associated with the system degrees of freedom $\mathbf{x}$) is defined as the state function [33, 37]
\begin{equation}
S_{\text{sys}}(\mathbf{x}, t) = -k_B \ln p(\mathbf{x}, t),
\end{equation}
where $p(\mathbf{x}, t)$ is the time-dependent solution of the Fokker-Planck equation [42, 43, 59] associated with (5), for the same initial distribution $p_0(\mathbf{x}_0) = p(\mathbf{x}_0, 0)$ from which the initial value $\mathbf{x}_0$ for the path $\mathbf{x}$ is drawn. The change in system entropy along the trajectory is therefore given by
\begin{equation}
\Delta S_{\text{sys}}(\mathbf{x}_0, \mathbf{x}_r) = -k_B \ln p(\mathbf{x}_r, \tau) + k_B \ln p(\mathbf{x}_0, 0).
\end{equation}

Combining $\Delta S[\mathbf{x}]$ and $\Delta S_{\text{sys}}(\mathbf{x}_0, \mathbf{x}_r)$, we obtain the total entropy production along a trajectory $\mathbf{x}$,
\begin{equation}
\Delta S_{\text{tot}}[\mathbf{x}] = \Delta S[\mathbf{x}] + \Delta S_{\text{sys}}(\mathbf{x}_0, \mathbf{x}_r).
\end{equation}
It fulfills the integral fluctuation theorem [31, 33]
\begin{equation}
\left\langle e^{-\Delta S_{\text{tot}}/k_B} \right\rangle = 1
\end{equation}
as a direct consequence of (14) [see also (18)]. The average in (20) is over all trajectories with a given, but arbitrary distribution $p_0(\mathbf{x}_0)$ of initial values $\mathbf{x}_0$ [37].

\section{IV. The Non-Equilibrium Environment}

We now focus on the full model (1), (2), (3) for Brownian motion subject to active fluctuations $\eta(t)$. Our main goal is to develop a trajectory-wise thermodynamic description as a natural generalization of stochastic energetics and thermodynamics in a purely thermal environment (see previous section). We thus treat the active forces $\eta(t)$ in the same way as the thermal noise $\xi(t)$, namely as a source of fluctuations whose specific realizations are not accessible but whose statistical properties determine the probability $p[\mathbf{x}|\mathbf{x}_0]$ for observing a certain particle trajectory $\mathbf{x}$. We are interested in how the non-equilibrium characteristics of the active fluctuations affect the irreversibility measure encoded in the ratio between forward and backward path probabilities, and how this measure is connected to the energetics of the active Brownian motion.

\subsection{A. Energetics and entropy}

Comparing (1) with (5), we may conclude that the stochastic energetics associated with (1) can be obtained from the energetics for (5) by adjusting the total forces acting on the Brownian particle, i.e. by replacing $\mathbf{v}(\mathbf{x}, t)$ with $\mathbf{v}(\mathbf{x}, t) + \sqrt{2D_a} \eta(t)$. However, there is another, maybe less obvious way of turning (5) into the model (1) with active fluctuations, namely by substituting $\dot{x}$ with $\dot{x} - \sqrt{2D_a} \eta(t)$. In the following we argue that these two approaches correspond to two different physical situations, with different trajectory-wise energy balances.

\subsubsection{1. Active bath}

In case the $\eta(t)$ model non-equilibrium fluctuations from an active environment (consisting, e.g., of swimming bacteria), they indeed can be interpreted as additional time-dependent forces from sources external to the Brownian particle. Hence, the total external force acting on the Brownian particle at time $t$ is given by $\gamma (\mathbf{v}(\mathbf{x}(t), t) + \sqrt{2D_a} \eta(t))$. This modification of the external force does obviously not affect the basic definition (6) of heat exchanged with the thermal bath. Using the force balance expressed in the Langevin equation (1) to replace $-\gamma \dot{x}(t) + \sqrt{2D_a} \xi(t)$ we obtain
\begin{equation}
\delta Q_+(t) = \left( f(\mathbf{x}(t), t) + \sqrt{2D_a} \gamma \eta(t) \right) \cdot d\mathbf{x}(t).
\end{equation}

The active fluctuations $\eta(t)$ formally play the role of additional non-conservative force components, and thus affect the heat which is dissipated into the thermal environment in order to balance all acting external forces. However, they cannot be controlled to perform work on the particle due to their inherent fluctuating character as an active bath, such that the definition (9) of the
work remains unchanged. Finally, the change of “internal” energy (8) over a time-interval \( dt \) is determined by the potential \( U(x, t) \) only, and thus is not altered by the presence of the active fluctuations \( \eta(t) \) either.

Combining (8), (9) and (21) we find the energy balance (first law)

\[
dU = \delta Q_+ + \delta W + \delta A_+ \tag{22}
\]

for Brownian motion in an active bath. Here, we have introduced the energy exchanged with the active bath,

\[
\delta A_+ = \sqrt{2D_a} \eta(t) \cdot dx(t).
\tag{23}
\]

It might be best interpreted as “heat” in the sense of Sekimoto’s general definition, that any energy exchange with unknown or inaccessible degrees of freedom may be identified as “heat” [68]. In our setup, the active fluctuations \( \eta(t) \) represent an effective description of the forces from the active environment, and are thus in general not directly measurable in an experiment.

Based on the heat exchanged with the thermal environment, we can identify the entropy production in the thermal environment as

\[
\delta S_+(t) = -\frac{\delta Q_+(t)}{T} \equiv \frac{1}{T} \left( f(x(t), t) + \sqrt{2D_a} \eta(t) \right) \cdot dx(t),
\tag{24}
\]

in analogy to (16). We refrain, however, from defining an entropy production in the active bath. Such an environment, itself being in a non-equilibrium state due to continuous dissipation of energy, does not possess a well-defined entropy, so that the “heat” dissipated into the active bath cannot be associated with a change of bath entropy.

2. Self-propulsion

In case the active fluctuations \( \eta(t) \) represent self-propulsion of the Brownian particle, systematic particle motion can occur already without any external forces being applied. For \( f(x, t) \equiv 0 \), and without thermal fluctuations, \( \sqrt{2D} \xi(t) \equiv 0 \), the momentary particle velocity \( \dot{x}(t) \) is exactly equal to the active velocity \( \sqrt{2D_a} \eta(t) \).

In that sense, self-propulsion is force-free. More precisely, the driving force, which is created locally by the particle for self-propulsion, is compensated according to actio est reactio, such that the total force acting on a fluid volume comprising the particle and its active self-propulsion mechanism is zero. The corresponding dissipation (and entropy production) inside such a fluid volume, which results from the conversion of energy or fuel to generate the self-propulsion drive, can not be quantified by our effective description of the active propulsion as fluctuating forces \( \eta \), because it does not contain any information on the underlying microscopic processes. In order to quantify such entropy production, a specific model for the self-propulsion mechanism is required [40, 63]. In other words, our “coarse-grained” description (3) does not allow us to assess how much energy the conversion process behind the self-propulsion drive dissipates. We can only measure the dissipation associated with deviations of the particle trajectory from the self-propulsion path, which occurs if the particle velocity differs from \( \sqrt{2D_a} \eta(t) \) due to the action of external forces or thermal fluctuations. Accordingly, the heat exchange with the thermal bath for a displacement \( dx(t) \) taking place over a time-interval \( dt \) at time \( t \) is given by

\[
\delta Q_-(t) = -f(x(t), t) \cdot (dx(t) - \sqrt{2D_a} \eta(t) dt),
\tag{25}
\]

The work \( \delta W(t) \), on the other hand, performed on or by the particle during the time-step \( dt \), which can be controlled or harvested by an external agent, is exactly the same as without active propulsion, given in (9), because from the operational viewpoint of the external agent it is irrelevant what kind of mechanisms propel the particle. Likewise, the definition (8) for the change in “internal” energy \( dU(t) \) is independent of how particle motion is driven, and thus remains unaffected by our interpretation of \( \sqrt{2D_a} \eta(t) \) as active propulsion.

Combining (8), (9) and (25), we find the first law-like relation

\[
\delta U = \delta Q_- + \delta W + \delta A_- \tag{26}
\]

for active, self-propelled Brownian motion described by the Langevin equation (1). Here, we balance the different energetic contributions by introducing

\[
\delta A_-(t) = -f(x(t), t) \cdot \sqrt{2D_a} \eta(t) dt.
\tag{27}
\]

This quantity represents the contribution to the heat exchange with the thermal bath, which is contained only in the active component \( \sqrt{2D_a} \eta(t) dt \) of the full particle displacement \( dx(t) \), see (25). We can therefore interpret it as the “heat” transferred from the active fluctuations to the thermal bath via the Brownian particle. Again, this is a quantity which can in general not be measured in an experiment, since it is produced by the inaccessible active fluctuations \( \eta(t) \).

Finally, we can relate the heat exchange \( \delta Q_- \) with the thermal bath to dissipation and define a corresponding entropy production in the environment,

\[
\delta S_-(t) = -\frac{\delta Q_-(t)}{T} = \frac{1}{T} f(x(t), t) \cdot (dx(t) - \sqrt{2D_a} \eta(t) dt).
\tag{28}
\]

B. Path probability

We now calculate the probability \( p(\mathbf{x} | \mathbf{x}_0) \) for observing a certain path \( \mathbf{x} = \{x(t)\}_{t=0}^T \) starting at \( \mathbf{x}_0 \), generated under the combined influence of thermal and
active fluctuations. We treat these two noise sources on equal terms, namely as fluctuating forces with unknown specific realizations but known statistical properties. Due to the memory of the active noise \( \eta(t) \), the system (1) is non-Markovian. Therefore, the standard Onsager-Machlup path integral [77–79] cannot be applied directly to obtain \( p[\mathbf{x}]|\mathbf{x}_0 \). However, for the Ornstein-Uhlenbeck model (3) of \( \eta(t) \) the combined set of variables \( (x, \eta) \) is Markovian and we can easily write down the path probability \( p[\mathbf{x}, \eta]|\mathbf{x}_0, \eta_0 \) for the joint trajectory \( (\mathbf{x}, \eta) = \{(x(t), \eta(t))\}_{t > 0} \), conditioned on an initial configuration \( (x_0, \eta_0) \):

\[
p[\mathbf{x}, \eta]|\mathbf{x}_0, \eta_0 \propto \exp\left\{ -\int_0^\tau dt \left[ \frac{(\dot{x}_t - v_t - \sqrt{2D_\alpha} \eta_t)^2}{4D} + \frac{(\tau_a \dot{\eta}_t + \eta_t)^2}{2} + \nabla \cdot v_t \right] \right\}. \tag{29}
\]

To calculate the path weight for the particle trajectories \( p[\mathbf{x}]|\mathbf{x}_0 \), we have to integrate out the active noise history,

\[
p[\mathbf{x}]|\mathbf{x}_0 = \int \mathcal{D}\eta \, d\eta_0 \, p[\mathbf{x}, \eta]|\mathbf{x}_0, \eta_0 \, p_0(\eta_0|\mathbf{x}_0), \tag{30}
\]

where \( p_0(\eta_0|\mathbf{x}_0) \) characterizes the initial distribution of the active fluctuations at \( t = 0 \).

Since the variables \( \eta(t) \) represent an effective description of the active fluctuations which are not experimentally accessible, it is in general not possible to set up a specific initial state for \( \eta(t) \). As a consequence there are basically two physically reasonable choices for \( p_0(\eta_0|\mathbf{x}_0) \).

On the one hand, we can assume that the active bath has reached its stationary state, before we immerse the Brownian particle, such that the bath’s initial distribution is independent of \( \mathbf{x}_0 \) and given by the stationary distribution of \( \eta(t) \), i.e. \( p_0(\eta_0|\mathbf{x}_0) = p_0(\eta_0) \). For the Ornstein-Uhlenbeck process (3), the stationary distribution reads [42, 82]

\[
p_0(\eta_0) = \sqrt{\frac{\tau_a}{\pi}} e^{-\frac{\tau_a \eta_0^2}{2}}. \tag{31}
\]

At \( t = 0 \) we place the Brownian particle into the fluid with an initial distribution \( p_0(\mathbf{x}_0) \) of particle positions which can be prepared arbitrarily, and start measuring immediately.

On the other hand, we may let the Brownian particle adapt to the active and thermal environments before performing measurements, and assume that the system \( (\mathbf{x}_0, \eta_0) \) is in a joint steady state \( p_0(\mathbf{x}_0, \eta_0) = p_0(\eta_0|\mathbf{x}_0)p_0(\mathbf{x}_0) \) at \( t = 0 \). In that case, control over the distribution \( p_0(\mathbf{x}_0) \) of initial particle positions is limited, as it is influenced by the active fluctuations. The form of \( p_0(\mathbf{x}_0, \eta_0) \) depends on the particular set-up, i.e. the specific choices for \( U(\mathbf{x}, t) \) and \( F(\mathbf{x}, t) \) in (2).

In the following, we will perform the path integration (30) for the first option, starting from independent initial conditions \( p_0(\eta_0|\mathbf{x}_0) = p_0(\eta_0) \); the calculation for the second option can be carried out along the same lines, if the joint steady state \( p_0(\mathbf{x}_0, \eta_0) \) is Gaussian in the active noise \( \eta_0 \) [83]. Plugging (29) and (31) into (30), and performing a partial integration of the term proportional to \( \dot{\eta}_t \) in the exponent, we obtain

\[
p[\mathbf{x}]|\mathbf{x}_0 \propto \exp\left\{ \int_0^\tau dt \left[ \frac{(\dot{x}_t - v_t)^2}{4D} - \nabla \cdot v_t \right] \right\} \times \int \mathcal{D}\eta \, \exp\left\{ \int_0^\tau dt \frac{\sqrt{2D_\alpha}}{2D} \eta_t^2 (\dot{x}_t - v_t) - \frac{1}{2} \int_0^\tau dt \int_0^\tau dt' \eta_t \Gamma_r(t,t') \eta_{t'} \right\}, \tag{32}
\]

where we have used the abbreviation \( \eta = \eta_0 \cup \eta \) for the full path including the initial point \( \eta_0 \), in order to write \( \mathcal{D}\eta \, d\eta_0 = \mathcal{D}_1 \). The differential operator

\[
\dot{V}_r(t,t') := \delta(t - t') \left[ \dot{V}(t) + \dot{V}_0(t) + \dot{V}_r(t) \right] \tag{33a}
\]

consists of an ordinary component,

\[
\dot{V}(t) := -\tau_a^2 \partial_t^2 + (1 + D_\alpha/D), \tag{33b}
\]

and two boundary components

\[
\dot{V}_0(t) := \delta(t) \left( \tau_a - \tau_a^2 \partial_t \right) , \tag{33c}
\]
\[
\dot{V}_r(t) := \delta(t - \tau) \left( \tau_a + \tau_a^2 \partial_t \right) , \tag{33d}
\]

which include the boundary terms picked up by the partial integration of \( \dot{\eta}_t^2 \) and from the initial distribution (31) of \( \eta_0 \). The subscript \( \tau \) in (33a) indicates that the operator \( \dot{V}_r \) is acting on trajectories of duration \( \tau \).

Since the path integral over the active noise histories \( \eta(t) \) is Gaussian, we can perform it exactly [84]. We find

\[
\int_0^\tau dt' \dot{V}_r(t,t') \Gamma_r(t',t'') = \delta(t - t''). \tag{35}
\]

where \( \Gamma_r(t,t') \) denotes the operator inverse or Green’s function of \( \dot{V}_r(t,t') \) in the sense that

\[
\int_0^\tau dt' \dot{V}_r(t,t') \Gamma_r(t',t'') = \delta(t - t''). \tag{35}
\]
Roughly speaking, this Gaussian integration can be understood by thinking of $V_\tau$ and $\Gamma_\tau$ as matrices with continuous indices $t, t'$. The path integral (32) is then a continuum generalization of an ordinary Gaussian integral for finite-dimensional matrices, and can be performed by “completing the square”. We provide a rigorous derivation of (34) and (35) in Appendix A.

In order to obtain the explicit form of the Green’s function $\Gamma_\tau(t, t')$, we need to solve the integro-differential equation (35). In our case, the operator $\tilde{V}_\tau(t, t')$ is proportional to $\delta(t-t')$ [see (33a)] and thus has a “diagonal” structure, such that (35) turns into an ordinary linear differential equation. We can solve it by following standard methods [85, 86], details are given in Appendix B. We obtain

$$\Gamma_\tau(t, t') = \left( \frac{1}{2\tau_0^2 \lambda} \right) \kappa^2_+ e^{-\lambda|t-t'|} + \kappa^2_- e^{-\lambda(2\tau-t-t')} - \kappa_\pm \left\{ e^{-\lambda(t+t')} + e^{-\lambda(2\tau-t-t')} \right\} \frac{\kappa^2_+ - \kappa^2_- e^{-2\lambda\tau}}{\kappa^2_+ - \kappa^2_- e^{-2\lambda\tau}},$$

(36a)

with

$$\lambda = \frac{1}{\tau_0} \sqrt{1 + D_a / D},$$

(36b)

and

$$\kappa_\pm = 1 \pm \frac{\lambda \tau_0}{1 + D_a / D}.$$  

(36c)

With this expression for $\Gamma_\tau(t, t')$, (34) represents the exact path probability density for the dynamics of the Brownian particle (1), under the influence of active Ornstein-Uhlenbeck fluctuations (3). This is our first main result. We see that the active fluctuations with their colored noise character lead to correlations in the path weight via the memory kernel $\Gamma_\tau(t, t')$. They relate trajectory points at different times by an exponential weight factor similar to the active noise correlation function (4), but with a correlation time which is a factor of $(1 + D_a / D)^{-1/2}$ smaller. We emphasize again that we assumed independent initial conditions for the particle’s position $x_0$ and the active bath variables $\eta_0$, $p_\eta(\eta_0|x_0) = p_\eta(\eta_0)$ [see also the discussion around Eq. (31)]. A different choice for the initial distribution, for instance the joint stationary state for $x_0$ and $\eta_0$, would result in a modified $\Gamma_\tau(t, t')$, whose precise form in general depends on the specific implementation of the deterministic forces $f$. The correlations in the path weight we measure via the memory kernel $\Gamma_\tau(t, t')$ are thus influenced by our choice of the time instant at which we start observing the particle trajectory. In that sense, the system “remembers its past” even prior to the initial time point $t = 0$, because of the finite correlation time in the active fluctuations.

We finally remark that our general expression (34) with (36a) reduces to known results in the three limiting cases $D_a \rightarrow 0$ (passive particle), $\tau_a \rightarrow 0$ (white active noise), and $D \rightarrow 0$ (no thermal bath); details of the calculations can be found in Appendix C. Without active fluctuations ($D_a \rightarrow 0$), we trivially recover the standard Onsager-Machlup expression (12) for passive Brownian motion. In the white noise limit for $\eta(t)$ ($\tau_a \rightarrow 0$), the equation of motion (1) involves two independent Gaussian white noise processes with vanishing means and variances $2D$ and $2D_a$, respectively. Their sum is itself a white noise source with zero mean, but variance $2(D + D_a)$. Accordingly, as $\tau_a \rightarrow 0$, we obtain from (34) an Onsager-Machlup path weight of the form (12), but with the diffusion coefficient $D$ being replaced by $D + D_a$. In the third limiting case of vanishing thermal fluctuations, we are left with a pure colored noise path weight [87, 88],

$$p[x|x_0]_{D \rightarrow 0} \propto \exp \left\{ - \int_0^\tau dt \frac{\left( \dot{x}_t - \nu_t \right)^2}{2D_a} - \frac{\nabla \cdot \nu_t}{2} \right\} p_\eta \left( \frac{\dot{x}_0 - \nu_0}{\sqrt{2D_a}} \right),$$

(37)

where $p_\eta$ denotes the steady-state distribution (31) of the colored noise.

C. Fluctuation theorem

With the explicit form (34) of the path probability, we can now derive an exact fluctuation theorem which relates forward and backward paths, following exactly the same line of reasoning as described in Sec. III B for the case of a passive Brownian particle. We consider the ratio of probabilities for observing a specific trajectory $x$ and its time-reversed twin $\tilde{x}$. created under a time-reversed protocol $v(x, \tau - t)$ [30]. With the definition (13) of the time-reversed trajectory, we can express its probability in terms of the forward path. Using the property $\Gamma_\tau(\tau - t, \tau - t') = \Gamma_\tau(t, t')$ of the memory kernel [see Eq. (36a)], we then obtain the path probability ratio

$$\tilde{p}[\tilde{x}|\tilde{x}_0] = e^{\Delta \Sigma[\tilde{x}] / k_B},$$

(38a)

with

$$\Delta \Sigma[\tilde{x}] = \frac{1}{T} \int_0^\tau dt \int_0^\tau dt' \dot{x}_t^T \int_0^\tau dt' \dot{x}_t^T \left( \delta(t - t') - \frac{D_a}{D} \Gamma_\tau(t, t') \right).$$

(38b)
As a stochastic integral, this is to be understood in the Stratonovich sense. Note that the procedure of time inversion does not involve the active fluctuations $\eta(t)$ in any way, and thus does not require any assumptions on their properties under time reversal. In fact, the probability density $p[\mathbf{x}|x_0]$ for the trajectories of the Brownian particle is a result of integrating over all possible realizations $\eta(t)$ of the active fluctuations, containing any pair of conceivable time-forward and time-backward twins with their natural weight of occurrence [see also Eq. (29)]. Hence, for the probability ratio of particle trajectories (34) the behavior of the active fluctuations $\eta(t)$ under time inversion is irrelevant.

As described in Sec. III B a relation like (38a) based on path probability ratios entails an integral fluctuation theorem, if the entropy production in the system $\Delta S_{\text{sys}}(x_0, x_\tau) = -k_B \ln p(x_\tau, \tau) + k_B \ln p(x_0, 0)$ [see Eq. (18)] is taken into consideration. Explicitly, we find

$$\frac{\dot{p}[\mathbf{x}]}{p[\mathbf{x}]} = -\frac{\dot{\Delta} \Sigma[\mathbf{x}],\mathbf{f}}{\Delta \Sigma[\mathbf{x}],\mathbf{f}} = \left(\begin{array}{c} \Delta \Sigma + \Delta S_{\text{sys}} \end{array}\right) / k_B,$$

and therefore

$$\left\langle e^{-\left(\Delta \Sigma + \Delta S_{\text{sys}}\right)/k_B}\right\rangle = 1.$$  

By Jensen’s inequality we conclude

$$\langle \Delta \Sigma + \Delta S_{\text{sys}} \rangle \geq 0,$$

where equality is achieved if and only if the dynamics is symmetric under time reversal.

The setting considered here, however, is generally not symmetric under time-reversal because of our choice of particle position and active fluctuations being independent initially. The approach to a correlated (stationary) particle position and active fluctuations being independent and conservative.

By Jensen’s inequality we conclude

$$\langle \Sigma \rangle \geq 0,$$

where equality is achieved if and only if the dynamics is symmetric under time reversal.

We will argue that a valid interpretation is provided by the mutual information between the active fluctuations $\eta(t)$ and the particle trajectory $x(t)$.

**V. MUTUAL INFORMATION**

The path-wise mutual information [90] between the particle trajectory $\mathbf{x}$ (starting at $x_0$) and a realization $\mathbf{f}$ of the active fluctuations (with $\mathbf{f}(0) = \eta_0$) is given as

$$J[\mathbf{x}, \mathbf{f}] = \ln \frac{p[\mathbf{x}, \mathbf{f}] / p[\mathbf{x} | \mathbf{f}]}{p[\mathbf{f}] / p[\mathbf{x}]},$$

and therefore

$$\varphi_\tau[\mathbf{x}, t] = f(x(t), t)$$

$$- \frac{D_\alpha}{D_t} \int_0^\tau dt' f(x(t'), t') \Gamma_\tau(t, t'),$$

we can bring (38b) into the form

$$\Delta \Sigma[\mathbf{x}] = \frac{1}{T} \int_0^\tau dt \dot{x}(t) \cdot \varphi_\tau[\mathbf{x}, t]$$

$$= \frac{1}{T} \int_0^\tau dt \varphi_\tau[\mathbf{x}, t],$$

in obvious analogy to (15), but with the essential difference that the “force” $\varphi_\tau[\mathbf{x}, t]$ at time $0 \leq t \leq \tau$ depends not only on $x(t)$ but rather on the full trajectory $\mathbf{x} = \{x(t)\}_{t=0}^\tau$ via the memory term $\int_0^\tau dt' f(x(t'), t') \Gamma_\tau(t', t')$. Similar “memory forces” have already been found to affect irreversibility by contributing to dissipation in Langevin systems with colored noise, which does not obey the fluctuation-dissipation relation [89].

From (43) we can read off a production rate for $\Sigma$,

$$\sigma_\tau(t) = \dot{x}(t) \cdot \frac{1}{T} \varphi_\tau[\mathbf{x}, t],$$

or

$$\delta \Sigma_\tau(t) = dx(t) \cdot \frac{1}{T} \varphi_\tau[\mathbf{x}, t].$$

Even though this expression for the $\Sigma$-production is analogous to the actual entropy productions in the environment as identified in (24) or (28) [see also (16)], and even contains a term $[dx(t) \cdot f(x(t), t)] / T$, which quantifies dissipation due to the external force $f$, its physical meaning beyond formally defined “memory forces” is unclear. In the following, we will argue that a valid interpretation is provided by the mutual information between the active fluctuations $\eta(t)$ and the particle trajectory $x(t)$.
where \( p[\bar{x}, \eta] = p[x, \eta | x_0, \eta_0] p(x_0, \eta_0) \), \( p[x] = p[x | x_0] p(x_0) \), and \( p[\eta] = p[\eta | \eta_0] p(\eta_0) \), include the initial densities \( p(x_0, \eta_0) \), \( p(x_0) = \int d\eta_0 p(x_0, \eta_0) \), and \( p(\eta_0) = \int dx_0 p(x_0, \eta_0) \) of the Brownian particle and the active fluctuations (see also the discussion in Sec. IV B). This path-wise mutual information quantifies the reduction in uncertainty about the path \( \bar{x} \) when we know the realization \( \eta \) and vice versa, and can therefore, loosely speaking, be seen as a measure of the correlations between \( \bar{x} \) and \( \eta \). Note that it can become negative if \( p[\bar{x}, \eta] < p[\bar{x}] \), while its average is always positive.

Likewise, the path-wise mutual information between the time-reversed trajectory \( \bar{x} \) from (13) and a suitably chosen time-reversed realization \( \bar{\eta} \) of the active fluctuations is

\[
\mathcal{J}[\bar{x}, \bar{\eta}] = \ln \frac{p[\bar{x}, \bar{\eta}]}{p[\bar{x}]} .
\]  

(46)

For their difference

\[
\Delta \mathcal{J} = \mathcal{J}[\bar{x}, \bar{\eta}] - \mathcal{J}[\bar{x}, \eta]
\]  

(47a)

we thus find

\[
\Delta \mathcal{J} = \ln \frac{p[\bar{x}, \eta]}{p[\bar{x}]} - \ln \frac{p[\bar{x}, \bar{\eta}]}{p[\bar{x}]} .
\]  

(47b)

This expression represents the path-wise mutual information difference between a combined forward process \( \bar{x} \) and its backward twin. If \( \Delta \mathcal{J} \) is positive, the path-wise mutual information \( \mathcal{J}[\bar{x}, \bar{\eta}] \) along the combined forward path is larger than \( \mathcal{J}[\bar{x}, \eta] \) for the time-reversed path [see (47a)], implying that correlations between the particle trajectory \( \bar{x} \) and the active fluctuation \( \bar{\eta} \) are stronger in the time-forward direction. Intuitively, we may thus say that they are more likely to occur together than their time-reversed twins, making the combined forward process \( \{\bar{x}, \bar{\eta}\} \) the more “natural” one of the two processes in terms of path-wise mutual information.

To make the connection to irreversibility more rigorous, we rewrite in (47b) the path-wise mutual information difference \( \Delta \mathcal{J} \) explicitly as path probability ratios between time-forward and time-backward processes. We now see that \( \Delta \mathcal{J} \), via the term \(-\ln(p[\bar{x}] / p[x])\), is directly related to the irreversibility measure \( \Delta \Sigma \) for the actively driven Brownian particle and its change in system entropy \( \Delta S_{sys} \). The additional log-ratio \( \ln(p[\bar{x}, \bar{\eta}] / p[\bar{x}, \eta]) \) involves the probability of the forward path \( \bar{x} \) being generated by the specific realization \( \bar{\eta} \) of the active fluctuations for which we measure the mutual information content with \( \bar{x} \) in \( \Delta \mathcal{J} \), and, likewise, the path probability of the time-reversed twin being generated by the time-reversed fluctuation. We can rewrite this term by splitting off the contributions from the initial densities,

\[
\ln \frac{p[\bar{x}, \bar{\eta}]}{p[\bar{x}, \eta]} = \ln \frac{p[\bar{x}, \bar{\eta}]}{p[x_0, \bar{\eta}]} + \ln \frac{p(x_0 | \bar{\eta})}{p(\bar{x}_0 | \eta_0)} .
\]  

(48)

We here keep the possibility that the initial particle position is conditioned on the initial state of the active fluctuations, \( p(x_0 | \eta) = p(x_0 | \eta_0) \), which is more general than the situation \( p(x_0 | \eta) = p(x_0) \) for which we calculated \( \Delta \Sigma \) in Section IV B [see also the discussion around Eq. (31)]. According to (13), the time-reversed initial position is given by the final point of the forward path, \( x_\tau = x_\tau \). It therefore depends on the complete history of the active fluctuations, which is captured equivalently by \( \eta \) or \( \bar{\eta} \), for any reasonable choice of the time-reversed fluctuations in terms of the forward realization [see also Eq. (51) below]. With \( p(x_0 | \eta) = p(x, \eta) \) we can identify the boundary term in (48) as the change in system entropy

\[
\Delta S_{sys}[\bar{x} | \bar{\eta}] = -k_B \ln p(x, \eta) + k_B \ln p(x_0 | \eta_0) .
\]  

(49)

which occurs along the trajectory \( \bar{x} \) under the specific realization \( \bar{\eta} \) of the active fluctuations (we explicitly indicate the conditioning on the realization of the active fluctuation by adding ‘\( \eta \)’ as a superscript).

To quantify the first term in (48), we exploit that a prescribed active fluctuation \( \eta \) is an odd process. Without needing to know any further details about \( \eta(t) \) (e.g., of how it is generated), we can thus derive the conditioned forward path probability \( p[x | x_0, \eta] \) directly from (1) as a standard Onsager-Machlup path integral,

\[
p[x | x_0, \eta] \propto \exp \left\{ - \int_0^\tau d\tau \left( \frac{\dot{x}_\tau - v_\tau - \sqrt{2D_\eta} \eta_\tau}{4D} - \frac{\nabla \cdot v_\tau}{2} \right) \right\} .
\]  

(50)

In order to evaluate the time-reversed counterpart, we have to specify the behavior of the active fluctuation signal \( \eta(t) \) under inversion of the direction of time. There are essentially two choices,

\[
\eta_\pm(t) := \pm \eta(\tau - t) ,
\]  

(51)

corresponding to \( \eta(t) \) being an even or odd process. While both these options a priori appear to be equally valid, they are connected to different interpretations of the active fluctuations, namely as an active bath (\( \eta(t) \) is even) or as active particle propulsion (\( \eta(t) \) is odd).

\[\text{A. Active bath}\]

If we interpret \( \eta(t) \) as fluctuations coming from an active bath, they are external to the particle and as such are similar to externally applied forces. The question of irreversibility under a prescribed realization \( \eta \) is then related to the question of how likely the backward particle trajectory \( \bar{x}(t) \) [see (13)] is to be observed when exactly the same forces act on the particle at identical positions during the forward and backward motion. Hence, this
case is described by active fluctuations which are even under time-reversal, i.e. by the plus-sign in (51).

Using \( \dot{\eta}_+(t) = +\eta(\tau - t) \) as the time-reversed fluctuation, we find

\[
\ln \frac{p[x|x_0,\eta_+]}{p[x|x_0,\eta]} = -\Delta S_+ \mathbb{1} / k_B \tag{52a}
\]

for the path probability ratio, where \( \Delta S_+ \) is given by the entropy production in the thermal environment from (24),

\[
\Delta S_+ \mathbb{1} = \int_0^\tau dt \delta S_+(t) = \frac{1}{T} \int_0^\tau dx(t) \cdot \left( f(x(t),t) + \sqrt{2D_a} \eta(t) \right). \tag{52b}
\]

Together with (49), we thus obtain the total entropy production

\[
\Delta S^\mathbb{1}_{tot} \mathbb{1} = \Delta S_+ \mathbb{1} + \Delta S^\mathbb{1}_{sys} \mathbb{1} \tag{53}
\]

along the particle trajectory which is generated by a given realization \( \eta(t) \) of the active fluctuations, assumed to be even under time-inversion.

Combining this result with (48), (39) and (47), we then infer that \( \Delta \Sigma(x) + \Delta S^\mathbb{1}_{sys}(x_0,x) \) is a combination of the conditioned entropy production and the difference in mutual information between time-forward and time-backward processes,

\[
\Delta \Sigma(x) + \Delta S^\mathbb{1}_{sys}(x_0,x) = \Delta S^\mathbb{1}_{tot}(x) - k_B \Delta J_+ \mathbb{1}, \tag{54}
\]

where \( \Delta J_+ \mathbb{1} \) is given by (47) with \( \eta = \eta_+ = \left( \dot{\eta}_+(t) \right)_{t=0}^\tau \). As an immediate consequence, the total conditioned entropy production together with the mutual information difference fulfill the integral fluctuation theorem

\[
\left\langle e^{-(\Delta S^\mathbb{1}_{tot}/k_B - \Delta J_+)} \right\rangle = 1, \tag{55}
\]

and the second-law like relation

\[
\langle \Delta S^\mathbb{1}_{tot} \rangle - k_B \langle \Delta J_+ \rangle \geq 0. \tag{56}
\]

We note that these results (54), (55) and (56) are completely independent of the specific model behind the active fluctuations. Moreover, they are formally similar to the fluctuation theorem with information exchange derived by Sagawa and Ueda [91] for a completely different physical setup. Sagawa and Ueda considered the difference between initial and final correlations of a system of interest with an “information reservoir”, which provides correlations as a resource of entropy changes.

For the Ornstein-Uhlenbeck representation (3), we can use our result (38b) for \( \Delta \Sigma \mathbb{1} \) to derive an explicit expression for the mutual information difference,

\[
\Delta J_+ \mathbb{1} = \left[ \ln \frac{p(x_\tau|\eta)}{p(x_\tau)} - \ln \frac{p(x_0|\eta_0)}{p(x_0)} \right] = \frac{1}{k_B T}\int_0^\tau \left[ \sqrt{2D_a} \int_0^\tau dt \dot{x}^2 \eta + \frac{D_a}{D} \int_0^\tau dt \int_0^\tau dt' \dot{x}^2 \left( f(t,t') \right) \right] \tag{57}
\]

where in the last equality we have used \( \Delta A_+ \mathbb{1} \) with the “heat” \( \delta A_+ \mathbb{1} \) exchanged with the active bath as identified in (23).

**B. Self-propulsion**

If the active fluctuations represent an effective model for a self-propulsion mechanism of the particle, we choose \( \eta(t) \) to be odd under time-reversal for a proper assessment of irreversibility, i.e. we choose the minus-sign in (51). Otherwise we would relate irreversibility to the likelihood that thermal fluctuations make the particle move a certain path backward against the probability that thermal fluctuations make the particle move a certain path backward against self-propulsion, rather than to the probability that a self-propelled particle traces out a given trajectory backward in time driven by the same propulsion forces [92].

Calculating the path probability ratio \( p[x|x_0,\eta] / p[x|x_0,\eta] \) for odd active fluctuations, \( \eta(t) = -\eta(\tau - t) \), we obtain

\[
\ln \frac{p[x|x_0,\eta]}{p[x|x_0,\eta]} = -\Delta S_- \mathbb{1} / k_B. \tag{58a}
\]

The quantity \( \Delta S_- \mathbb{1} \) is given by

\[
\Delta S_- \mathbb{1} = \int_0^\tau dt \delta S_-(t) = \frac{1}{T} \int_0^\tau \left( dx(t) - \sqrt{2D_a} \eta(t) dt \right) \cdot f(x(t),t), \tag{58b}
\]

and thus represents the entropy production in the environment along the trajectory \( \mathbb{1} \), because \( \delta S_-(t) \) results from the heat \( -\delta Q_-(t) \) dissipated into the thermal bath during a displacement of the active self-propelled particle, see (28). We finally combine \( \Delta S_- \mathbb{1} \) with (49) to identify the total entropy production under a given realization \( \eta \) of the active propulsion as

\[
\Delta S^\mathbb{1}_{tot} \mathbb{1} = \Delta S_- \mathbb{1} + \Delta S^\mathbb{1}_{sys} \mathbb{1}. \tag{59}
\]

Hence, we find again that \( \Delta \Sigma \mathbb{1} \) of the active propulsion as consistent of conditional mutual entropy production and mutual
information difference [compare Eqs. (47), (39) and (59)],
\[ \Delta \Sigma[x] + \Delta S_{\text{sys}}(x_0, x_\tau) = \Delta S_{\text{tot}}[x|\eta] - k_B \Delta J_-[x, \eta]. \]
(60)
This is exactly the same interpretation of \( \Delta \Sigma[x] + \Delta S_{\text{sys}}(x_0, x_\tau) \) as we found earlier for the case of an active bath [c.f. (54)], but with different expressions for the entropy production and the mutual information difference.

The mutual information in the backward process is now measured with respect to the sign-inverted active fluctuation process \( \eta_-(t) = -\eta(\tau - t) \), i.e. \( \Delta J_-[x, \eta] \) in (60) is given by (47) with \( \eta_0 = \eta_- = \{\eta_-(t)\}_{t=0}^\tau \).

Accordingly, we immediately find the integral fluctuation theorem
\[ \left< e^{-\Delta S_{\text{tot}}(x)/k_B - \Delta J_-} \right> = 1, \] (61)
and the second-law like relation
\[ \langle \Delta S_{\text{tot}}^- \rangle - k_B \langle \Delta J_- \rangle \geq 0. \] (62)
for active self-propulsion. Again, these findings (60), (61) and (62) are valid for any model of the active fluctuations, and not only for the Ornstein-Uhlenbeck process (3), because they are a direct consequence of the particle’s equation of motion (1). Moreover, they are formally similar to the Sagawa-Ueda fluctuation theorem with information [91]. If the active propulsion \( \eta(t) \) is modeled by (3), we can write the mutual information difference in the explicit form
\[
\Delta J_-[x, \eta] = \frac{1}{k_B T} \left[ \int_0^\tau dt \int_0^\tau dt' \eta^\dagger_t f_t \right. \\
+ \left. \int_0^\tau dt \int_0^\tau dt' \eta^\dagger_t f_t \Gamma_{t, t'} \right] \\
= \frac{1}{k_B T} \left[ \Delta A_-[x, \eta] \\
+ \frac{D_a}{D} \int_0^\tau dt \int_0^\tau dt' \eta^\dagger_t f_t \Gamma_{t, t'} \right],
\] (63)
by using (38b) and by defining \( \Delta A_-[x, \eta] = \int_0^\tau dt \delta A_-[t, \longrightarrow] \) [see (27)], as the total “heat” transferred from the active propulsion to the thermal bath along the trajectory \( \Gamma \).

C. Relation to information theory

The second-law like relations (56) and (62) involve the averages \( \langle \Delta J^\pm \rangle \) of the mutual information difference. As we will show in the following, these averages are closely related to central concepts in information theory, entailing additional bounds on \( \langle \Delta J^\pm \rangle \) and an interpretation in terms of hypothesis testing.

Due to the unbiased Gaussian character of the active fluctuations, the probabilities for observing a time-forward and its time-reversed realization are the same, i.e. \( \dot{p}[\eta] = p[\eta] \). We can therefore rewrite (47b) as
\[
\Delta J^\pm[x, \eta] = \ln \frac{\dot{p}[x]}{p[x]} - \ln \frac{\dot{p}[x, \eta_\pm]}{p[x, \eta_\pm]},
\] (64)
where we have inserted our two options \( \eta_\pm = \eta_\pm \) from (51) for the time-reversed active fluctuation. Performing the average over \( \eta \) with density \( p[x, \eta] \), we then find
\[
\langle \Delta J^\pm[x, \eta] \rangle = D_{\text{KL}}(p[x, \eta]||p[x, \eta_\pm]) - D_{\text{KL}}(p[x]||p[x]),
\] (65)
Here we use the definition
\[
D_{\text{KL}}(p[\eta]||p[\eta']) = \int \eta \ln \frac{p[\eta]}{p[\eta']} d\eta
\] (66)
of the Kullback-Leibler divergence between a probability density \( p[\eta] \) for a process \( \eta \) and another density \( p[\eta'] \) for a second process \( \eta' \); in our case, the two processes are related by time-reversal. The Kullback-Leibler divergence is a standard concept in information theory to measure how distinct two probability densities are. It is non-negative and equals zero if and only if the two probabilities are identical [93, 94].

The result (65) shows that the average mutual information difference \( \langle \Delta J^\pm[x, \eta] \rangle \) is the difference between the Kullback-Leibler divergence of the particle trajectory \( x \) relative to its time-reversed twin \( \bar{x} \) and the Kullback-Leibler divergence of the combined path \( (x, \eta) \), relative to the time-reversed realization \( (\bar{x}, \eta_{\pm}) \). We can therefore interpret it to measure how much harder it is to discriminate between time-forward and time-backward realizations if only the particle trajectory is known, rather than the full dynamics including the active noise realization \( \eta \). Since \( x \) can be seen as a “coarse-graining projection” of \( (x, \eta) \), we expect the discrimination to become harder, i.e. \( D_{\text{KL}}(p[x]||p[\bar{x}]) \) to become smaller compared to \( D_{\text{KL}}(p[x, \eta]||p[\bar{x}, \eta_{\pm}]) \) (see [95] for a similar discussion). This intuition is corroborated by the so-called data processing inequality [93, 94], which, applied to our situation, proves
\[
D_{\text{KL}}(p[x, \eta]||p[\bar{x}, \eta_{\pm}]) \geq D_{\text{KL}}(p[x]||p[\bar{x}]).
\] (67)
As a direct consequence, we find from (65) the bound
\[
\langle \Delta J^\pm[x, \eta] \rangle \geq 0
\] (68)
on the average mutual information difference. Applying this bound to (56) and (62) we infer that \( \langle \Delta S_{\text{tot}}^\pm \rangle \geq 0 \), consistent with the fact that \( \Delta S_{\text{tot}}^\pm \) are given as log-ratios of path probabilities [see (52) and (58)] and thus each obey an integral fluctuation theorem.

More interestingly, we can also use (68) to equip the second-law like relations for the total irreversibility measure \( \Delta \Sigma + \Delta S_{\text{sys}} \) with an upper bound. Taking the average of (54) and (60), and combining (56) and (62), respectively, with (68), we find
\[
\langle \Delta S_{\text{tot}}^\pm[x, \eta] \rangle \geq \langle \Delta \Sigma[x] + \Delta S_{\text{sys}}(x_0, x_\tau) \rangle \geq 0.
\] (69)
The total average irreversibility of a particle trajectory, measured as \( \langle \Delta \Sigma + \Delta S_{\text{sys}} \rangle \), is thus always smaller than (or equal to) the entropy change which would occur if we treated the active fluctuations as a known external force contributing to the dissipation in the thermal environment, no matter whether these forcings come from an active bath and thus are considered even under time-reversal (\( \pm \) sign), or from active self-propulsion, which is odd under time-reversal (\( \pm \) sign). The difference is compensated by the build-up of mutual information.

In information theory the Kullback-Leibler divergence has an interesting interpretation in the context of hypothesis testing. For our purposes, the null hypothesis to be tested by measuring particle trajectories is: the observed process runs forward in time. We recall [see also the discussion of (13)] that \( p(\mathbf{x}) \) represents the probability of observing a trajectory that indeed evolves forward in time and thus confirms our null hypothesis, whereas \( p(\mathbf{x}, \mathbf{\eta}) \) is the path probability consistent with a backward evolution such that a corresponding observation would lead us to reject our null hypothesis. The Chernoff-Stein lemma [94, 96] identifies the Kullback-Leibler divergence \( D_{\text{KL}}(p(\mathbf{x}) || p(\mathbf{x}, \mathbf{\eta})) \) as the best achievable rate at which the probability of a false negative decision (type II error), i.e., of accepting the null hypothesis that the path was generated by a forward dynamics when, in reality, it was generated by a backward one, exponentially decreases with repeated observations. The Kullback-Leibler divergence can therefore be interpreted as a measure of how simple it is to discriminate between the hypotheses represented by the two probabilities, in this case, forward evolution versus backward evolution in time (corresponding to telling the direction of the arrow of time [32, 97]). An analogous interpretation holds for the probabilities \( p(\mathbf{x}, \mathbf{\eta}) \), \( p(\mathbf{x}, \mathbf{\eta}) \) of the combined set \((\mathbf{x}, \mathbf{\eta})\) of trajectories, and their Kullback-Leibler divergence \( D_{\text{KL}}(p(\mathbf{x}, \mathbf{\eta}) || p(\mathbf{x}, \mathbf{\eta})) \). The inequality (67) thus states that the probability of a false negative decision decreases faster with the number of observations if more detailed information on the dynamics of the system is available by additionally monitoring the realizations \( \mathbf{\eta} \) of the active fluctuations.

D. Discussion

The fluctuation theorems and second-law like relations (55), (56) and (61), (62) for an active bath and for active self-propulsion, respectively, and their interpretation in terms of mutual information differences are our third main result, see also (69). For both cases, we find that the total irreversibility measure \( \Delta \Sigma(\mathbf{x}) + \Delta S_{\text{sys}}(x_0, x_r) \) consists of two contributions: First, the “usual” entropy production \( \Delta S_{\text{tot}} \), which we would measure for the motion of the Brownian particle under a given realization of the active fluctuations (as if the active fluctuations were just some additional known “external” driving force), and, second, the difference in mutual information \( \Delta I_{\pm} \) accumulated between the particle trajectory and the active non-equilibrium environment along the forward versus the backward path. We note that even though the individual contributions in \( \Delta S_{\text{tot}} - k_B \Delta I_{\pm} \) depend on the specific realization \( \mathbf{\eta} \) of the active fluctuations, their sum does not [see (54) and (60)], i.e. entropy production and change in mutual information always compensate their dependency on the realization of the active fluctuations.

The entropy productions \( \Delta S_{\text{tot}} \) for an active bath and for active self-propulsion, respectively, as obtained from irreversibility arguments in (52) and (58) are consistent with the energetics derived in Secs. IV.A.1 and IV.A.2 [compare with (21) and (25)]. It is thus directly related to the heat dissipated into the thermal part of the environment. Hence, the appearance of the path-wise mutual information difference in the fluctuation theorem is a consequence of the active fluctuations being present as a non-equilibrium bath in addition to the usual thermal bath. Indeed, we can easily see from (57) and (63) that \( \Delta I_{\pm} \) vanish identically in the absence of active fluctuations, \( D_a = 0 \). Accordingly, \( \Delta S_{\pm} \) reduce to the standard entropy production in the thermal environment in this limit [see (52), (58), and (15)].

We emphasize that the path-wise mutual information quantifies how the active fluctuations contribute to the irreversibility of the particle trajectory, but does not capture the unavoidable dissipation connected with maintaining the active nature of the non-equilibrium environment itself. In fact, our effective description (3) of the active fluctuations as a time-correlated Gaussian noise source does not have any knowledge about the microscopic details generating this noise, so that it can obviously not assess the associated dissipative processes. On the other hand, we may therefore suspect that the appearance of the path-wise mutual information is a consequence of this “coarse-grained” description of the active environment, and might be replaced by a “finer” measure once all the details are known [40, 63]. On the other hand, we may argue that these microscopic details behind the active fluctuations are irrelevant if we are only interested in characterizing the irreversibility of the particle trajectory. Then, knowledge of the statistical properties of the active fluctuations, as provided by (3), is sufficient, and the path-wise mutual information between particle trajectory and active noise realization emerges as a natural and adequate irreversibility measure. This situation may be comparable to the one for entropy production in a thermal bath: we quantify entropy production solely from the statistical properties of the thermal noise without having to rely on a detailed microscopic description of the bath.

VI. EXAMPLE: HARMONIC POTENTIAL

To illustrate our results, we consider the example of a one-dimensional Brownian particle trapped in a harmonic potential \( U(x, t) = \frac{k}{2}(x - at)^2 \), whose center is either held at fixed position, \( a = 0 \), or else is displaced
compute the averages of the irreversibility measure $\Delta \Sigma$, the mutual information difference $\Delta I_{\perp}$, and the change in system entropy $\Delta S_{\text{sys}}$, for a Gaussian initial distribution $p_0(x_0)$ of particle positions with zero mean and variance $c_0^2$. Although the calculations are straightforward (we present some details in Appendix D), the resulting formulae for $\langle \Delta \Sigma \rangle$, $\langle \Delta S_{\text{sys}} \rangle$ and $\langle \Delta I_{\perp} \rangle$ are lengthy and bulky, so that we discuss them mostly in graphical form (see Figs. 1 and 3) and give explicit expressions only in some limiting cases [see Eqs. (71), (72), (75) below]. For the initial variance $c_0^2$, two specific cases are of particular physical relevance. First, $c_0^2 = k_B T/k$, corresponding to a Gaussian distribution that would be created as an equilibrium state by thermal fluctuations only. Starting from this distribution, the time evolution of our irreversibility measures includes the transient relaxation from the thermal state towards the steady state which develops due to the presence of the active fluctuations. Second, $c_0^2 = \frac{1}{\gamma} \left[ k_B T + \frac{\gamma D_a}{1+\gamma D_a} \right]$, corresponding to a Gaussian distribution with a variance which is exactly the same as the one the particle distribution has when particle and active fluctuations are in their joint steady state. In that case, the particle distributions at the beginning and end of the process are identical, so that any contributions to the irreversibility measure $\Delta \Sigma$ that are not associated with the displacement $u$ of the trap are solely due to the build-up of correlations between particle position and active degrees of freedom. In the following, we will focus on the first alternative, and briefly come back to the second alternative at the end of Sec. VIA and in Sec. VIB.

### A. Irreversibility

In Fig. 1 we compare $\langle \Delta \Sigma \rangle$, $\langle \Delta S_{\text{sys}} \rangle$ and $\langle \Delta \Sigma + \Delta S_{\text{sys}} \rangle$ for $u = 0$ and $u = 1$ as a function of the duration $\tau$ of the trajectories. We find that the average change in system entropy $\langle \Delta S_{\text{sys}} \rangle$ is independent of the driving velocity $u$, because it only depends on the variance of the initial and final Gaussian distributions, but not on their centers. For long trajectories, i.e. large $\tau$, it approaches the constant value

$$\lim_{\tau \to \infty} \langle \Delta S_{\text{sys}} \rangle = \frac{k_B}{2} \ln \left[ 1 + \frac{D_a}{D} \frac{1}{1 + \frac{k_B T}{D_a \gamma}} \right]. \quad (71)$$

In contrast, the irreversibility measures $\langle \Delta \Sigma \rangle$ for the static trap $u = 0$ and the moving trap $u = 1$ are similar only during a short transient phase, but then become qualitatively different. In the static trap, $\langle \Delta \Sigma \rangle$ becomes constant at large times $\tau$,

---

**FIG. 1.** Average contributions to irreversibility $\langle \Delta \Sigma \rangle$ and average change in system entropy $\langle \Delta S_{\text{sys}} \rangle$ as a function of the observation time $\tau$. We compare the cases of a particle trapped in a static ($u = 0$, orange lines) or moving ($u = 1$, blue lines) harmonic potential having an initial distribution which is Gaussian with variance (a) $c_0^2 = \frac{k_B T}{k} = \gamma D_a / k$ and (b) $c_0^2 = \frac{1}{\gamma} \left[ k_B T + \frac{\gamma D_a}{1+\gamma D_a} \right]$. Parameter values are $k = 0.1$, $\gamma = 1$, $D_a = 0.2$, $D = D_a = 0.2$. Note that the blue and orange lines for the change in system entropy are on top of each other in both figures, as $\langle \Delta S_{\text{sys}} \rangle$ does not depend on $u$. At constant velocity $u \neq 0$. Such a setup can be readily implemented in experiment with state-of-the-art optical tweezers, and in fact has been used to study various aspects of stochastic thermodynamics and active matter, for instance in [23, 72, 98–100].

The Langevin equation of motion (1) for this specific setup is linear,

$$\dot{x}(t) = -\frac{k_B}{\gamma} x(t) - ut + \sqrt{2D \xi(t)} + \sqrt{2D_a \eta(t)}. \quad (70)$$

We can therefore solve the associated Fokker-Planck equation analytically [42, 82, 101] to obtain the propagator in closed form. From that we can explicitly compute
of the trajectories. For large $\tau$, the growth rate is given by the ensemble average of the time-averaged production with variance $D$ by a thermal white-noise process with diffusion constant $\gamma/k$. This is exactly the entropy, which would be produced by a thermal white-noise process with diffusion rate (73) is independent of the relevant system time-

"effective temperature". Remarkably, the average growth rate (73) is independent of the relevant system time-scales, i.e. the correlation time $\tau_a$ of active fluctuations and the relaxation time $\gamma/k$ in the harmonic potential.

In the static trap $u = 0$, obviously no such extensive growth occurs for $\langle \Delta \Sigma + \Delta S_{\text{sys}} \rangle$, because the system reaches a dissipation-less steady state. Nevertheless, we observe $\lim_{\tau \to \infty} \langle \Delta \Sigma + \Delta S_{\text{sys}} \rangle > 0$ regardless of the initial distribution $p_0(x_0)$. For the case of a Gaussian with "thermal variance" $c_0^2 = k_B T/k$ the relevant results are given in Fig. 1 and Eqs. (71), (72). Both, $\lim_{\tau \to \infty} \langle \Delta S_{\text{sys}} \rangle$ and $\lim_{\tau \to \infty} \langle \Delta \Sigma \rangle$, depend on just two dimensionless parameters: the ratio of the two noise intensities $D_a$ and the ratio of the two system timescales $\tau_a/\tau$. The limits of large and small correlation time and noise amplitudes can be easily computed and present no difficulties. As an interesting example, we consider the case of vanishing correlation time of the active fluctuations,

$$
\lim_{\tau_a \to 0} \lim_{\tau \to \infty} \langle \Delta \Sigma + \Delta S_{\text{sys}} \rangle|_{u=0} = k_B \frac{k_B}{2} \left[ \frac{D_a}{D} \right] - k_B \frac{D_a}{2} \frac{D}{D + D_a}. 
$$

This is exactly the entropy, which would be produced by a thermal white-noise process with diffusion constant $D + D_a$ relaxing from an initial Gaussian distribution with variance $D \gamma/k = k_B T/k$ to its equilibrium state, a Gaussian with variance $(D + D_a) \gamma/k$.

It is quite obvious that for a variance $c_0^2 = k_B T/k$ of the initial Gaussian distribution, there must be some change in system entropy and a build-up of irreversibility, because the Gaussian particle distribution approached at long times has a larger variance $1/k \left[ k_B T + \frac{\gamma D_a}{1 + (k \tau_a/\gamma)} \right]$ [see Eq. (D2d)]. But even if we start with the initial variance $c_0^2 = 1/k \left[ k_B T + \frac{\gamma D_a}{1 + (k \tau_a/\gamma)} \right]$, we find a positive $\langle \Delta \Sigma \rangle$ while reaching the steady state at large $\tau$.

$$
\lim_{\tau \to \infty} \langle \Delta \Sigma \rangle|_{u=0} = k_B \frac{k_B}{2} \left[ \frac{D_a}{D} \right]^2 \frac{D_a}{D + D_a}. 
$$

The origin of this positive contribution is our choice of independent initial conditions in form of a product density $p_0(x_0, \eta_0) = p_0(x_0)p_0(\eta_0)$ [see also the discussion of Eq. (31)]. Hence, during the initial transient there must be an irreversible build-up of correlations between the particle and the active fluctuations, becoming manifest in a non-zero total $\langle \Delta \Sigma + \Delta S_{\text{sys}} \rangle$. It turns out that the associated change in $\langle \Delta \Sigma + \Delta S_{\text{sys}} \rangle$ is non-monotonic [see Fig. 1(b)], indicating that the variance of the particle distribution departs over some (transient) time period, even though the initial variance $c_0^2 = 1/k \left[ k_B T + \frac{\gamma D_a}{1 + (k \tau_a/\gamma)} \right]$ is identical to the final one. In order to obtain a vanishing average $\langle \Delta \Sigma + \Delta S_{\text{sys}} \rangle$ we would have to start from a joint stationary state $p_s(x_0, \eta_0)$ of particle positions and active fluctuations instead of a factorized one. It can even be shown that forward and backward paths are equally likely in that case (see [83]), implying that $\Delta \Sigma(\tau) + \Delta S_{\text{sys}}(x_0, x_\tau) = 0$ holds already on the level of individual trajectories.

We emphasize again that these correlations are the very reason that the irreversibility measure $\Delta \Sigma$ is non-additive [see also the discussion below Eq. (41)]. The curves in Fig. 1 apply only to trajectories evolving over the complete time-interval $[0, \tau]$. We cannot split a trajectory at an intermediate time, calculate the individual $\Delta \Sigma$ for the two parts of the trajectory from (38b), and then add them up to obtain $\Delta \Sigma$ for the full time interval, because the "initial" state for the second part will inevitably depend on correlations between particle and active fluctuations accumulated during the first part. Such correlations are not taken into account in (38b) which is based on the assumption of an initial product state.

### B. Fluctuation theorem

To illustrate the integral fluctuation theorem (40) satisfied by $\Delta \Sigma + \Delta S_{\text{sys}}$, we show in Fig. 2 probability densities for the path probability ratio (39) for the same two situations of a static and a moving harmonic trapping
potential already analyzed in Fig. 1(b). The probability densities are obtained from simulating $10^5$ trajectories of length $\tau = 1$, well within the transient regime of the system evolution (see Fig. 1). While the distribution for $\exp[-(\Delta \Sigma + \Delta S_{\text{sys}})/k_B]$ is almost symmetric about 1 in the static trap $u = 0$, the most probable value lies visibly below 1 for the moving trap $u = 1$, indicating that trajectories with a positive $\Delta \Sigma + \Delta S_{\text{sys}}$ are more likely than those with a negative value (see insets in Fig. 1). The sample mean for the path probability ratio lies well within one standard deviation of 1 in both cases, in accordance with the exact result (40).

FIG. 2. Probability densities of the path probability ratio (39) for a harmonically trapped particle (a) in a static trap, $u = 0$, and (b) in a moving trap, $u = 1$. The densities are obtained from numerically simulating $10^5$ sample trajectories of duration $\tau = 1$. Parameters are $k = 0.1$, $\gamma = 1$, $\tau_a = 0.2$, $D = D_a = 0.2$, $c_0^2 = \frac{1}{2} \left[ D + \frac{D_a}{1 + k \tau_a / \gamma} \right]$. The insets show the corresponding densities of $\Delta S_{\text{tot}} = \Delta \Sigma + \Delta S_{\text{sys}}$. The average values $\langle \Delta \Sigma \rangle$, $\langle \Delta S_{\text{sys}} \rangle$ and $\langle \Delta \Sigma + \Delta S_{\text{sys}} \rangle$ as obtained from the numerical simulations are consistent with the corresponding theoretical predictions: (a) $u = 0$, simulation: $\langle \Delta \Sigma \rangle = 0.010(4)$, $\langle \Delta S_{\text{sys}} \rangle = -0.0090(13)$, $\langle \Delta \Sigma + \Delta S_{\text{sys}} \rangle = 0.00139(17)$, $u = 0$, theory: $\langle \Delta \Sigma \rangle = 0.00975$, $\langle \Delta S_{\text{sys}} \rangle = -0.00828$, $\langle \Delta \Sigma + \Delta S_{\text{sys}} \rangle = 0.000139(46)$. (b) $u = 1$, simulation: $\langle \Delta \Sigma \rangle = 0.017(4)$, $\langle \Delta S_{\text{sys}} \rangle = -0.0073(13)$, $\langle \Delta \Sigma + \Delta S_{\text{sys}} \rangle = 0.0104(5)$. $u = 1$, theory: $\langle \Delta \Sigma \rangle = 0.01883$, $\langle \Delta S_{\text{sys}} \rangle = -0.00828$, $\langle \Delta \Sigma + \Delta S_{\text{sys}} \rangle = 0.01056$.

C. Mutual information

FIG. 3. Average mutual information and conditional entropy production as a function of the observation time $\tau$ for a particle trapped in a static ($u = 0$, orange lines) or moving ($u = 1$, blue lines) harmonic potential. Parameters are again $k = 0.1$, $\gamma = 1$, $\tau_a = 0.2$, $D = D_a = 0.2$. (a) Interpretation of the active fluctuations as an active bath, i.e. they are considered being even under time-reversal. (b) Interpretation of the active fluctuations as self-propulsion, i.e. they are considered being odd under time-reversal. Note that even for $u = 0$ (orange lines) the curves for $\langle \Delta S^+ \rangle$, $k_B \langle \Delta I^+ \rangle - \langle \Delta S_{\text{sys}}^\tau \rangle$ and $\langle \Delta S^- \rangle$, $k_B \langle \Delta I^- \rangle - \langle \Delta S_{\text{sys}}^\tau \rangle$, respectively, are not identical, but just appear very similar on the shown scale.

The quantities $\Delta \Sigma$ and $\Delta S_{\text{sys}}$ analyzed in the previous two sections as a measure for the irreversibility in the system evolution depend only on the particle trajectories, but not on the realizations of the active fluctuations, and thus are readily accessible in experiments and simulations. The specific role of the active fluctuations, on the other hand, is nicely captured by the splitting of irreversibility, i.e. the log-ratio of path probabilities, into total conditional entropy production $\Delta S_{\text{tot}}^\tau$ and mutual information $\Delta I^\tau$, as described Sec. V A for an active bath (‘+’ sign) and in Sec. V B for self-propulsion (‘−’ sign). Since for the present example of a particle trapped in a harmonic potential we have analytical expressions at hand for the combined propagator of particle position and active fluctuations (see Appendix D), we can calculate the conditional entropy production from (52) and...
(58), and the mutual information from (57) and (63) explicitly. Only the quantity in \(p(x, \overline{r}; \overline{y})\) is not easily accessible because it is conditioned on the full realization \(\overline{y}\), such that we add it to the change in mutual information in form of the change in (conditional) system entropy \(\Delta S_{\text{sys}}^{\overline{y}}\) [see Eq. (49)]. Note that \(\Delta S_{\text{sys}}^{\overline{y}}\) is non-extensive with \(\tau\), and thus only a small correction to \(\Delta I_{\pm}\) which becomes negligible for long times.

In Fig. 3, we show the average conditional entropy production \(\langle \Delta S_{\pm}^{\overline{y}} \rangle\) and the average mutual information \(k_B \langle \Delta J_{\pm} \rangle \langle \Delta S_{\text{sys}}^{\overline{y}} \rangle\) for an active bath (Fig. 3a) and active self-propulsion (Fig. 3b). The system parameters are the same as before in Figs. 1, 2, in particular we again compare the a static trap \(u = 0\) with a moving trap \(u = 1\). We can see that now in all cases, both \(\langle \Delta S_{\pm}^{\overline{y}} \rangle\) and \(\langle \Delta J_{\pm} \rangle\) grow linearly with time for large \(\tau\). This conforms nicely with our previous findings: For the conditional total entropy \(\Delta S_{\text{tot}}^{\overline{y}}\), the active fluctuations are treated like an external, time-dependent forcing. Such a force is then naturally expected to produce entropy extensively. For the ensemble- and time-averaged rate of total entropy production

\[
\langle \sigma_{\text{tot}}^+ \rangle = k_B \frac{1}{\tau_a} \left[ \frac{1 + u_D^2}{D + D_a} + \frac{D_a}{D} \left( \frac{1}{1 + \frac{k_{\text{eff}}}{\gamma}} \right) \frac{1}{\tau_a} \right],
\]

(76a)

\[
\langle \sigma_{\text{tot}}^- \rangle = k_B \frac{1}{\tau_a} \left[ \frac{1 + u_D^2}{D + D_a} + \frac{D_a}{D} \left( \frac{1}{1 + \frac{k_{\text{eff}}}{\gamma}} \right) \frac{k}{\gamma} \right].
\]

(76b)

The first terms in these expressions contain the production rate of \(\Delta \Sigma\) from (73). The second terms quantify the additional contributions from the active fluctuations. They are balanced by the mutual information production rates \(\langle \sigma_{\pm}^\pm \rangle := \lim_{\tau \to \infty} k_B \langle \Delta J_{\pm} \rangle / \tau\) (note that we include a factor of \(k_B\) in this definition of the rates so that they have units of entropy/time), which explicitly read

\[
\langle \sigma_{\pm}^+ \rangle = k_B \frac{D_a}{D} \left[ \frac{1}{D + D_a} + \left( \frac{1}{1 + \frac{k_{\text{eff}}}{\gamma}} \right) \frac{1}{\tau_a} \right],
\]

(77a)

\[
\langle \sigma_{\pm}^- \rangle = k_B \frac{D_a}{D} \left[ \frac{1}{D + D_a} + \left( \frac{1}{1 + \frac{k_{\text{eff}}}{\gamma}} \right) \frac{k}{\gamma} \right].
\]

(77b)

In total, we therefore find in both cases that \(\langle \sigma \rangle = \langle \sigma_{\text{tot}}^+ \rangle - \langle \sigma_{\pm}^\pm \rangle\) holds, where \(\langle \sigma \rangle\) is given in (73).

As discussed in Sec. IV A, the two interpretations of the active fluctuations as active bath or as self-propulsion mechanism correspond to measuring the mutual information with respect to, respectively, even or odd time-reversal of the active forcing, so that the rate of “mutual information production” is different in the two cases, compare (77a) and (77b). Their difference reads

\[
\langle \sigma_{\pm}^+ \rangle - \langle \sigma_{\pm}^- \rangle = k_B \frac{D_a}{D} \left( \frac{1}{1 + \frac{k_{\text{eff}}}{\gamma}} \right) \left( \frac{1}{\tau_a} - \frac{k}{\gamma} \right).
\]

(78)

We conclude that for the active bath the relevant time scale the trajectory duration \(\tau\) is measured against is the correlation time of fluctuations \(\tau_a\), whereas it is the system’s relaxation time in the harmonic potential \(\gamma/k\) for the self-propelled particle.

VII. CONCLUSIONS AND DISCUSSION

Our present work contributes to assessing the out-of-equilibrium character of active matter [28, 29, 41, 44, 45, 54]. Having in mind that in a typical experiment the central observables are particle trajectories, we quantify irreversibility in active matter systems based on particle trajectories alone, without resolving the microscopic mechanisms and associated dissipation underlying the active fluctuations which drive particle motion. In other words, we are interested in how (ir-) reversible a specific particle trajectory is, out of the set of all possible trajectories which can be generated by the combined influence of thermal and active fluctuations, but not in how (ir-) reversible the processes which underly the active fluctuations are. In that spirit, we treat the active fluctuations as an active (non-thermal) “bath” the particle is in contact with in addition to the thermal bath, with properties which are completely independent of the dynamical state of the particle, i.e. we assume that the microscopic processes behind the active fluctuations are not altered by the particle motion.

We focus on a colored noise model for the active fluctuations, more specifically a Gaussian Ornstein-Uhlenbeck process [7, 15–18, 21–24, 28, 41, 44–54]. We calculate the exact expression for the probability density of particle trajectories by integrating over all possible realizations of the active fluctuations [see Eq. (34)]. Since the colored noise renders the particle’s dynamics non-Markovian, the standard Onsager-Machlup path integral [77–80] cannot be applied directly to obtain the path probabilities. While expressions for a single, colored noise source exist in the literature [87, 88], we here derive for the first time the path weight for the superposition of colored Ornstein-Uhlenbeck noise (the active fluctuations) and white thermal noise.

Building on this result, we then relate the probabilities of time-forward and time-backward processes and establish an integral fluctuation theorem [31–34] for their log-ratio \(\Delta \Sigma\), a functional over the forward particle trajectory [see Eqs. (40) and (38b)] which quantifies the time (ir-) reversibility of the particle dynamics. From the integral fluctuation theorem we directly obtain a corresponding second law-like relation for \(\Delta \Sigma\) [see Eq. (41)]. To the best of our knowledge this is the first (exact) fluctuation theorem for a Brownian particle in contact not only with a thermal bath but at the same time also with an active non-equilibrium bath. In particular, it applies to trajectories of arbitrary, finite duration. We expect that it can be tested experimentally with state-of-the-art micro(-fluidic) technology for biological or synthetic active matter systems, as used, e.g., in [5, 22–24, 102–106]. We point out again that our results for the path
weight and the fluctuation theorem are based on the central assumption that the active fluctuations are Gaussian with exponential correlations in time, generated by an Ornstein-Uhlenbeck process.

Like in the case of usual Brownian motion in contact with a thermal bath only, the path probability ratio (38a) is equal to the identity if there are no external forces acting on the particle. Therefore, for \( f = 0 \) any time-backward trajectory is equally likely to occur as its time-forward twin, so that the particle dynamics looks reversible and equilibrium-like, even though the whole system is out of equilibrium due to the active fluctuations. In other words, in the absence of external forces the probability ratio of particle trajectories alone does not reveal the non-equilibrium nature of the system. This should be true for any stationary (and unbiased) non-equilibrium bath, not just the Ornstein-Uhlenbeck implementation considered here. In order to detect the irreversibility connected with the active fluctuations we would have to resolve the corresponding degrees of freedom and analyze their behavior under time-reversal [40, 63].

However, as we can see from comparing (15) and (38b), for non-vanishing forces \( f \) the irreversibility measure \( \Delta \Sigma \) is distinctively different from the entropy production of purely Brownian motion, because it contains the non-local memory kernel \( \Gamma_{\tau}(t, t') \). Driving the particle by an external force thus reveals the non-equilibrium character of the environment. We leave for future exploration how this observation may be used to probe properties of the external force affecting the particle dynamics, but preliminary results indicate that the external force \( f \) has to be non-linear or time-dependent, as a simple linear \( f \) leads to \( \Delta \Sigma = 0 \) already on the level of individual trajectories [83], see also [28].

Our irreversibility measure \( \Delta \Sigma \) from (38) quantifies the combined “dissipation” into the thermal and the active bath, which occurs along a (time-forward) particle trajectory \( \xi \), but cannot be interpreted easily as entropy production or dissipated heat. However, if we keep track of a specific realization \( \eta \) of the active noise as a fluctuating force affecting the particle dynamics, we find that \( \Delta \Sigma \) can be split into two parts which have a direct physical interpretation: the usual entropy production in the thermal environment, and a complementary dissipative component in the active bath, which is expressed as the difference of path-wise mutual information accumulated along the time-forward process \( (\xi, \eta) \) versus its time-backward twin process [see Eqs. (54), (60)]. This partition of \( \Delta \Sigma \) is valid for any particle trajectory \( \xi \) and any realization of the active noise fluctuation \( \eta \), but with process-dependent amounts of dissipation in the two baths.

All these general results and interpretations are independent of how we choose the active fluctuations to behave under time reversal, even [Secs. IV A.1 and V A] or odd [Secs. IV A.2 and V B]. The quantitative details, however, are different; compare, e.g., Eq. (57) with Eq. (63). These quantitative differences are a consequence of the different amounts of heat exchanged with the thermal environment along a displacement \( d\xi \) in the two cases, see (21) and (25). When we interpret the active fluctuations \( \eta(t) \) as an active environment the particle is moving through, the viscous friction forces from the thermal bath which balance these active fluctuations are included in the heat exchange. In contrast, they are not counted as contributing to heat in the case of self-propulsion, because self-propelled motion occurs without external forces, i.e. on the coarse-grained level of description based on particle degrees of freedom self-propulsion appears to be force- and thus dissipation-free. We argue in Sec. IV A that the interpretation of the active fluctuations \( \eta(t) \) as an active environment requires \( \eta(t) \) to be even under time-reversal, while their interpretation as self-propulsion corresponds to \( \eta(t) \) being odd [92].

We here have analyzed the model (1), (2) in great detail from the viewpoint of a Brownian particle moving under the influence of active fluctuations, which are represented by the Ornstein-Uhlenbeck process (3). However, most of our results, in particular the path weight (34) and the associated integral fluctuation theorem (40), as well as its formulation in information-theoretic terms in Sec. V, are a direct consequence of the mathematical structure of the model, and are therefore valid in a much broader physical context. In principle, our results can be applied to any Brownian dynamics that is driven by an additional Gaussian Ornstein-Uhlenbeck process. For instance, in [107] an information-theoretic analysis similar to ours has been conducted for a colored noise driven Brownian model. Other examples are Brownian motion in a harmonic trap with fluctuating location [71, 108–110], and thermodynamic nonequilibrium processes using external (artificial) colored noise sources [111, 112].

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**Appendix A: Integrating over the active fluctuations \( \eta(t) \)**

Here we will show that the path integral (32) evaluates to expression (34). Gaussian path integrals of this form are ubiquitous in field theories [84, 113] with well-established results. Using the abbreviation \( w_t := \sqrt{2D_\tau} \left( \bar{\xi}_t - \bar{v}_t \right) \), the relevant terms in (32) involving the
active noise variable \( \eta_i \) become

\[
\int D\eta \exp \left[-\frac{1}{2} \int_0^\tau dt \int_0^\tau dt' \eta_i^\tau \dot{V}_r(t, t') \eta_i + \int_0^\tau dt \eta_i^t w_i \right].
\]

(\text{A1})

Complementing the square, we wish to write this as

\[
\int D\eta \exp \left\{ \frac{1}{2} \int_0^\tau dt \int_0^\tau dt' \left[ w_i^t \Gamma_r(t, t') w_i \right] - (\eta_i + \varepsilon_i)^{t} V_r(t, t') (\eta_i + \varepsilon_i) \right\} ,
\]

(\text{A2})

where \( \varepsilon_i \) and \( \Gamma_r(t, t') \) are yet unknown functions. We can then shift the active noise histories and integrate over \( \eta_i^t := \eta_i + \varepsilon_i \) instead of \( \eta_i \). Since the path integral effectively integrates over all possible states of \( \eta_i \) from \(-\infty \) to \(+\infty\) for any point in time \( t \), this shift of trajectories does not alter the domain of integration. Moreover, the Jacobian associated with the transformation is the identity. Performing the remaining functional integral over \( \eta_i^t \), expression (\text{A2}) then reduces to

\[
(\text{Det} \dot{V}_r)^{-1/2} \exp \left[ \frac{1}{2} \int_0^\tau dt \int_0^\tau dt' w_i \Gamma_r(t, t') w_i \right] .
\]

(\text{A3})

assuming proper normalization of the integration measure \( \text{Det} \dot{V}_r \). Absorbing the path-independent functional determinant \( (\text{Det} \dot{V}_r)^{-1/2} \) into the normalization, we obtain the path weight stated in Eq. (\text{34}).

However, we still have to show that \( \Gamma_r(t, t') \) is the operator inverse of \( \dot{V}_r(t, t') \), i.e. we have to verify (\text{35}). Requiring equality of (\text{A1}) and (\text{A2}), we find that

\[
\int_0^\tau dt \eta_i^t w_i \frac{1}{2} \int_0^\tau dt \int_0^\tau dt' \left[ w_i^t \Gamma_r(t, t') w_i \right]\nonumber \\
- \varepsilon_i^t \dot{V}_r(t, t') \varepsilon_i' - 2 \eta_i^t \dot{V}_r(t, t') \varepsilon_i'.
\]

(\text{A4})

Note that we have used \( \dot{V}_r(t, t') = \dot{V}_r(t', t) \). We now observe that the term on the left-hand side as well as the last term on the right-hand side are of first order in \( \eta_i \), whereas the remaining terms on the right-hand side do not contain \( \eta_i \). Therefore, these two types of expressions must cancel individually, i.e.

\[
\int_0^\tau dt \int_0^\tau dt' \varepsilon_i^t \dot{V}_r(t, t') \varepsilon_i' = \int_0^\tau dt \int_0^\tau dt' \eta_i^t \Gamma_r(t, t') \eta_i',
\]

(\text{A5a})

\[
\int_0^\tau dt \eta_i^t w_i = - \int_0^\tau dt \eta_i^t \int_0^\tau dt' \dot{V}_r(t, t') \varepsilon_i'.
\]

(\text{A5b})

From (\text{A5b}) we immediately infer

\[
\eta_i = - \int_0^\tau dt' \dot{V}_r(t, t') \varepsilon_i'.
\]

(\text{A6})

Substituting this result into the left-hand side of (\text{A5a}), we obtain

\[
- \int_0^\tau dt \varepsilon_i^t w_i = \int_0^\tau dt \eta_i^t \int_0^\tau dt' \Gamma_r(t, t') \eta_i'.
\]

(\text{A7})

implying

\[
\varepsilon_i = - \int_0^\tau dt' \Gamma_r(t, t') w_i.
\]

(\text{A8})

This gives the shift \( \varepsilon_i \) required to complete the square in (\text{A1}) (see also (\text{A2})) as a function of \( \Gamma_r(t, t') \) and \( w_i \). Moreover, substituting (\text{A8}) back into (\text{A6}), we find

\[
\eta_i = \int_0^\tau dt' \int_0^\tau dt'' \dot{V}_r(t, t') \Gamma_r(t', t'') \eta_i',
\]

(\text{A9})

which implies (\text{35}) and thus establishes that \( \Gamma_r(t, t') \) is indeed the Green’s function of the differential operator \( \dot{V}_r(t, t') \).

**Appendix B: Construction of the Green’s function \( \Gamma_r(t, t') \)**

In this appendix, we will construct the Green’s function (\text{36a}) by solving its defining equation (\text{35}) with the differential operator \( \dot{V}_r(t, t') \) from (\text{33}). Exploiting the “diagonal” structure of \( \dot{V}_r(t, t') \) we can directly evaluate the integral and obtain

\[
\left[ \dot{V}(t) + \dot{V}_0(t) + \dot{V}_r(t) \right] \Gamma_r(t, t') = \delta(t - t').
\]

(\text{B1})

This is a linear, second-order ordinary differential equation with a \( \delta \)-inhomogeneity. We will calculate its solution in two steps. First, we will compute the Green’s function \( \Gamma(t, t') \) of the ordinary component \( \dot{V}(t) \) with vanishing boundary conditions, i.e. \( \Gamma(0, t') = \Gamma(t', 0) = 0 \). Then we will add a solution \( \Gamma_{0, r}(t, t') \) of the corresponding homogeneous problem, \( \dot{V}(t) \Gamma_{0, r}(t, t') = 0 \), that fixes the two boundary terms. Their sum \( \Gamma_r(t, t') = \Gamma(t, t') + \Gamma_{0, r}(t, t') \) satisfies (\text{B1}), and thus gives the desired solution.

We can construct both these parts, \( \Gamma(t, t') \) and \( \Gamma_{0, r}(t, t') \), from the homogeneous problem associated with \( \dot{V}(t, t') \), which reads (see (\text{33a}))

\[
[-\tau_a^2 \partial_t^2 + (1 + D_a/D)] \Gamma(t) = 0.
\]

(\text{B2})

We make an exponential ansatz \( \Gamma(t) \sim e^{\lambda t} \) and easily obtain

\[
\Gamma(t) = \alpha^+ e^{\lambda t} + \alpha^- e^{-\lambda t} , \quad \lambda = \frac{1}{\tau_a} \sqrt{1 + \frac{D_a}{D}} ,
\]

(\text{B3})

with constants \( \alpha^\pm \) to be determined by the boundary conditions or the \( \delta \)-inhomogeneity.

There exists a standard recipe [85, 86] for the construction of Green’s functions of boundary value problems for ordinary differential equations, which we will follow here to calculate \( \Gamma(t, t') \). Splitting the interval \([0, \tau]\) at \( t = t' \), we write

\[
\Gamma(t, t') = \Theta(t' - t) \tilde{\Gamma}_< (t, t') + \Theta(t - t') \tilde{\Gamma}_> (t, t'),
\]

(\text{B4})
where Θ is the Heaviside step function and both, \( \tilde{\Gamma}_c(t,t') \) and \( \Gamma_>(t,t') \), satisfy the homogenous problem, i.e. they are of the form (B3) with constants \( \alpha_c^< \) and \( \alpha_> \), respectively, to be determined. The constants are fixed by the boundary conditions,

\[
\Gamma_<(0,t') = 0 \quad \text{and} \quad \Gamma_>(\tau, t') = 0, \quad (B5a)
\]

and \( \Gamma_<(t,t') = \Gamma_>(t',t) \). With (B4), we can thus write the Green’s function of the ordinary component on the entire interval \([0, \tau]\) in the form

\[
\tilde{\Gamma}(t,t') = \left( \frac{1}{2\tau_a^2} \right) e^{-\lambda|t-t'|} - e^{-\lambda(t+t')} + e^{-\lambda(t-t')} - e^{-\lambda(t+t')}
\]

We can immediately check that indeed \( \tilde{V}(t)\tilde{\Gamma}(t,t') = \delta(t-t') \) as well as \( \tilde{\Gamma}(0,t') = 0 \) and \( \Gamma(\tau, t') = 0 \). Moreover, we note that \( \tilde{\Gamma}(t,t') \) is symmetric in its arguments, \( \tilde{\Gamma}(t,t') = \Gamma(t',t) \).

As announced earlier, we now add a homogeneous solution \( \Gamma_0(t,t') = \alpha^+ e^{\lambda t} + \alpha^- e^{-\lambda t} \) to the ordinary inverse \( \tilde{\Gamma}(t,t') \). Since \( \tilde{V}(t)\Gamma_0(t,t') = 0 \) and \( \Gamma(t,t') \) vanishes at both boundaries of the interval \([0, \tau]\), the differential equation (B1) for the sum \( \Gamma(\tau, t') = \Gamma(t,t') + \Gamma_0(\tau, t') \) becomes

\[
\delta(t-\tau) \left[ \frac{\tau_a^2}{2} \partial_{t'} \Gamma_c, t', t' \right]_{t'=\tau} + \alpha^+ \tau_a \kappa_+ e^{\lambda t} + \alpha^- \tau_a \kappa_- e^{-\lambda t} + \delta(t) \left[ \frac{\tau_a^2}{2} \partial_t \Gamma_<(t,t') \right]_{t=0} + \alpha^+ \tau_a \kappa_- + \alpha^- \tau_a \kappa_+ = 0, \quad (B8)
\]

with \( \kappa_\pm = 1 \pm \lambda \tau_a / 2 \). We require the two terms in square brackets to vanish individually, and thus obtain two linear equations for the two unknowns \( \alpha^+ \) and \( \alpha^- \). Solving them and combining the results for \( \Gamma(t,t') \) and \( \Gamma_0(t,t') \) into \( \Gamma(t, t') = \Gamma(t,t') + \Gamma_0(t, t') \), we finally find the Green’s function (36a).

**Appendix C: Limiting cases for the path weight**

Here, we analyze three relevant limiting cases of the path weight (34), namely \( D_a \to 0 \) (usual Brownian particle without active fluctuations), \( \tau_a \to 0 \) (memory-less active fluctuations), and \( D \to 0 \) (no thermal fluctuations). All three limits reduce the Langevin equation (1) to simpler set-ups, for which the path probabilities are already known in the literature. We recover all these results when performing the respective limits in our general expression (34).

1. **Usual Brownian motion \( D_a \to 0 \)**

Removing the effects of the active fluctuations from the system amounts to setting \( \eta(t) = 0 \) or, equivalently, \( D_a = 0 \) in the equation of motion (1). Performing this limit for the path weight is straightforward: In this case, \( \lambda = 1 / \tau_a \) is well-behaved, so that the term containing the Green’s function simply drops out in (34). The remaining path probability is just the standard Onsager-Machlup expression (12) for a Brownian particle in a thermal bath.

2. **Vanishing correlation time \( \tau_a \to 0 \)**

In the limit \( \tau_a \to 0 \), the correlator (4) of the active noise approaches a \( \delta \)-distribution, i.e. \( \langle \eta_i(t)\eta_j(t') \rangle \to \delta_{ij} \delta(t-t') \). This means that \( \eta(t) \) becomes just another Gaussian white noise with diffusion constant \( D_a \). The equation of motion (1) thus contains two independent Gaussian white noises with zero mean and variance \( 2D \) and \( 2D_a \), respectively. Their sum is itself a Gaussian white noise with zero mean, but variance \( 2(D + D_a) \).
Therefore, we expect (34) to reduce to
\[ p[x|x_0]_{\tau_n \to 0} \propto \exp \int_0^\tau dt \left[ \frac{(\dot{x}_t - v_t)^2}{4(D + D_n)} - \frac{\nabla \cdot v_t}{2} \right] \] (C1)

as \( \tau_n \to 0 \).

We first note that \( \lambda \) diverges in this limit, so that we have to take more care when analyzing the Green's function (36a). Since the limit is expressed more compactly as \( \lambda \to \infty \), we rewrite all occurrences of \( \tau_n \) in (36a) in terms of \( \lambda, D \) and \( D_n \). By definition (see (36b) and (B3)),
\[ \tau_n^2 = (1 + \frac{D_n}{D}) / \lambda^2 \]
Thus the leading order behavior of \( \Gamma_\tau(t, t') \) is
\[ \Gamma_\tau(t, t') \sim \left( \frac{\lambda}{2} \right) e^{-\lambda|t-t'|} \frac{e^{-\lambda(t+t') + e^{-\lambda(2t-t')}}}{1 + \frac{D}{D_n}} \] (C2)

We observe that
\[ \lim_{\lambda \to \infty} \frac{\lambda}{2} e^{-\lambda|t-t'|} = \delta(t-t'), \] (C3a)
\[ \lim_{\lambda \to \infty} \frac{\lambda}{2} e^{-\lambda \Theta(t)} = \frac{1}{2} \delta(t), \] (C3b)

such that the exponentials become \( \delta \)-distributions as \( \lambda \to \infty \):
\[ \Gamma_\tau(t, t') \sim \frac{\delta(t-t') - \frac{\lambda}{2\tau_n} [\delta(t + t') + \delta(2t - t - t')]}{1 + \frac{D}{D_n}} \]

However, the last two \( \delta \)-distributions map either of the two times \( t \) and \( t' \) outside of the interval of integration when integrating over the other. Therefore, they do not contribute when integrating over both \( t \) and \( t' \). We can see this also from partially integrating the corresponding term in (C2) with two test functions \( f \) and \( g \),
\[ \int_0^\tau dt \int_0^\tau dt' f(t)g(t') \frac{\lambda}{2} [e^{-\lambda(t+t')} + e^{-\lambda(2t-t')}} \]
\[ = \int_0^\tau dt \int_0^\tau dt' \left\{ g(t') \frac{1}{2} [-e^{-\lambda(t+t')} + e^{-\lambda(2t-t')}] \right\} \]
\[ - \int_0^\tau dt' \dot{g}(t') \frac{1}{2} [-e^{-\lambda(t+t')} + e^{-\lambda(2t-t')}] \]
\[ \to 0 \quad \text{as} \quad \lambda \to \infty. \]

Therefore, \( \Gamma_\tau(t, t') \sim \delta(t-t')/(1 + \frac{D_n}{D}) \) as \( \tau_n \to 0 \). Substituting this finding into the path weight (34) leads precisely to the expected limit (C1).

3. No thermal fluctuations \( D \to 0 \)

The limit of vanishing thermal white noise is a little more involved. If we let \( D \to 0 \) in the equation of motion (1), we are left with a system driven by a single, colored noise source. The resulting path probability density is known in the literature [87, 88] and given by Eq. (37). However, it is not immediately obvious how we can obtain this result from (34), because both the prefactor \( 1/(4D) \) and the exponent \( \lambda \) diverge as \( D \to 0 \).

We first rewrite (37), so that the action can be expressed in the form of (34). To this end, we introduce the abbreviation \( \dot{w}_t := \dot{x}_t - \dot{v}_t \) in (37), and remember that we chose \( p_n \) to be the stationary distribution (31) of the colored noise. After partial integration of the \( \dot{w}_t^2 \) term, we find
\[ p[x|x_0]_{D \to 0} \propto \exp \left\{ -\frac{1}{4D_n} \left[ \int_0^\tau dt \left( -\tau_n^2 \dot{w}_t^T \dot{w}_t + \dot{w}_t^2 \right) \right. \right. \]
\[ + \tau_n^2 \dot{w}_t^T w_i|_0 \tau_n + \tau_n w_i^2|_0 + 2\tau_n^2 \dot{w}_t^2 \int_0^\tau dt \frac{\nabla \cdot v_t}{2} \left. \right\}. \] (C4)

We will now show that the action
\[ A[x] = \frac{1}{4D} \int_0^\tau dt \int_0^\tau dt' \dot{w}_t^T \left( \delta(t-t') - \frac{D_n}{D} \Gamma_\tau(t, t') \right) w_t' \] (C5)

of the full path weight \( p[x|x_0] \propto e^{-A[x]} - J_0^\tau dt \nabla \cdot v_t/2 \) as given in (34), reduces to the form of the action in (C4) in the limit \( D \to 0 \).

We rewrite the limit \( D \to 0 \) again as \( \lambda \to \infty \), i.e. we express all occurrences of \( D \) in terms of \( \lambda, D_n \) and \( \tau_n \) according to \( \frac{D_n}{D} = \lambda^2 \tau_n^2 - 1 \) (see (36b) and (B3)). For the leading order behavior of the Green’s function \( \Gamma_\tau(t, t') \) we then find
\[ \Gamma_\tau(t, t') \sim \frac{e^{-\lambda|t-t'|} + \left[ 1 - \frac{2}{\tau_n \lambda} + 0(\lambda^{-2}) \right] e^{-\lambda(t+t')} + e^{-\lambda(2t-t')}}{2\tau_n^2 \lambda}, \] (C6)
which implies
\[ A[\mathbf{x}]_{D \to 0} \sim \frac{1}{4D_a} \int_0^T dt \int_0^T dt' \, \mathbf{w}_t^\intercal \mathbf{w}_{t'} \left\{ \left( \lambda^2 \tau_a^2 - 1 \right) \delta(t - t') - \left( \frac{\lambda^2 \tau_a^2}{2} - \lambda \right) \left[ e^{-\lambda(t-t')} + e^{-\lambda(t+t')} + e^{-\lambda(2\tau-t-t')} \right] \right. \]
\[ + \left. \left[ \lambda^2 \tau_a + \mathcal{O}(\lambda) \right] \left[ e^{-\lambda(t-t')} + e^{-\lambda(2\tau-t-t')} \right] \right\}. \]

We divide our further analysis into three parts by writing \( A[\mathbf{x}] \sim \frac{1}{4D_a} (B_1[\mathbf{x}] + B_2[\mathbf{x}] + B_3[\mathbf{x}]) \) with
\[ B_1[\mathbf{x}] = \int_0^T dt \int_0^T dt' \, \mathbf{w}_t^\intercal \mathbf{w}_{t'} \left[ \left( \lambda^2 \tau_a^2 - 1 \right) \delta(t - t') - \left( \frac{\lambda^2 \tau_a^2}{2} - \lambda \right) e^{-\lambda(t-t')} \right], \tag{C7a} \]
\[ B_2[\mathbf{x}] = -\int_0^T dt \int_0^T dt' \, \mathbf{w}_t^\intercal \mathbf{w}_{t'} \left( \frac{\lambda^2 \tau_a^2}{2} - \lambda \right) \left[ e^{-\lambda(t+t')} + e^{-\lambda(2\tau-t-t')} \right], \tag{C7b} \]
\[ B_3[\mathbf{x}] = \int_0^T dt \int_0^T dt' \, \mathbf{w}_t^\intercal \mathbf{w}_{t'} \left[ \lambda^2 \tau_a + \mathcal{O}(\lambda) \right] \left[ e^{-\lambda(t+t')} + e^{-\lambda(2\tau-t-t')} \right]. \tag{C7c} \]

Upon repeated partial integration using \( \lambda e^{-\lambda t} = -\partial_t e^{-\lambda t} \) to remove powers of \( \lambda \), and upon performing the \( \lambda \to \infty \) limiting procedure for well behaved terms, we find for the first part
\[ B_1[\mathbf{x}] \sim \int_0^T dt \left[ -\tau_a^2 \mathbf{w}_t^\intercal \dot{\mathbf{w}}_t + \mathbf{w}_t^2 \right] + (w_0^2 + w_t^2) \frac{\lambda \tau_a^2}{2}. \]

Similarly, the second part reduces to
\[ B_2[\mathbf{x}] \sim (w_0^2 + w_t^2) \frac{\lambda \tau_a^2}{2} + \tau_a^2 (w_t^2 + \mathbf{w}_t^\intercal \mathbf{w}_t - w_0^2 \mathbf{w}_0). \]

For the third part, we first remark that the terms of order \( \lambda \) vanish here again under double time integrals because they become \( \delta \)-distributions that map one of the times outside the interval of integration, similarly to the case we had for the limit of vanishing correlation time. Hence,
\[ B_3[\mathbf{x}] \sim \int_0^T dt \int_0^T dt' \, \mathbf{w}_t^\intercal \mathbf{w}_{t'} \lambda^2 \tau_a \left[ e^{-\lambda(t+t')} + e^{-\lambda(2\tau-t-t')} \right] \]
\[ = \int_0^T dt \int_0^T dt' \, \mathbf{w}_t^\intercal \mathbf{w}_{t'} \lambda \tau_a \partial_{t'} \left[ e^{-\lambda(t+t')} + e^{-\lambda(2\tau-t-t')} \right] \]
\[ = \int_0^T dt \mathbf{w}_t^\intercal \mathbf{w}_t \lambda \tau_a \left[ e^{-\lambda(t+t')} + e^{-\lambda(2\tau-t-t')} \right] \bigg|_{t'=0} \]
\[ - \int_0^T dt \int_0^T dt' \, \mathbf{w}_t^\intercal \mathbf{w}_{t'} \lambda \tau_a \left[ e^{-\lambda(t+t')} + e^{-\lambda(2\tau-t-t')} \right] \]
\[ \sim \int_0^T dt \mathbf{w}_t^\intercal \left\{ \mathbf{w}_t \tau_a \lambda \left[ e^{-\lambda(t+t')} + e^{-\lambda(2\tau-t-t')} \right] \right\} \]
\[ - \int_0^T dt \mathbf{w}_t^\intercal \left\{ \mathbf{w}_0 \tau_a \lambda \left[ e^{-\lambda(t+t')} + e^{-\lambda(2\tau-t-t')} \right] \right\} \]
\[ \sim \tau_a \left( w_t^2 - w_0^2 \right) + 2\tau_a w_0^2. \]

Combining these three results, we finally find
\[ A[\mathbf{x}]_{D \to 0} = \frac{1}{4D_a} \int_0^T dt \left( -\tau_a^2 \mathbf{w}_t^\intercal \dot{\mathbf{w}}_t + \mathbf{w}_t^2 \right) \]
\[ + \tau_a^2 (w_t^2 + \mathbf{w}_t^\intercal \mathbf{w}_t - w_0^2 \mathbf{w}_0) + \tau_a (w_t^2 - w_0^2) + 2\tau_a w_0^2, \tag{C8} \]
which is identical to the path weight (C4).

**Appendix D: Details for the harmonically trapped particle**

In this appendix, we summarize some details and a few key steps behind the calculations for the Brownian particle in a harmonic potential from Sec. VI.

1. **Dynamics**

The equations of motion for the joint Markovian system of particle and active fluctuations, combined from (70) and (3) (for \( d = 1 \)), read
\[ \left( \dot{x}_t \right) = -A \left( \begin{array}{c} x_t \\ \eta_t \end{array} \right) - \left( \begin{array}{c} 0 \\ t \end{array} \right) + B \left( \begin{array}{c} \xi_t \\ \zeta_t \end{array} \right) \tag{D1a} \]
with
\[ A = \left( \begin{array}{cc} k/\gamma & -\sqrt{2D_a} \\ 0 & 1/\tau_a \end{array} \right) \quad \text{and} \quad B = \left( \begin{array}{cc} \sqrt{2D_a} & 0 \\ 0 & 1/\tau_a \end{array} \right). \tag{D1b} \]

Due to its Markovian character, all statistical properties of this joint system are encoded in the propagator \( p(q, t|q_0, t_0) \), where \( q = (x, \eta) \), \( q_0 = (x_0, \eta_0) \) are “generalized coordinates” summarizing particle position and state of the active fluctuations. This propagator represents the probability to be at “position” \( q \) at time \( t \) when
having been at \(q_0\) at an earlier time \(t_0 < t\). We obtain its explicit form by solving the the Fokker-Planck equation associated with (D1),

\[
p(q, t|q_0, t_0) = \frac{e^{-\frac{1}{2}[q-\mu(t|q_0, t_0)]^T C(t|t_0)^{-1}[q-\mu(t|q_0, t_0)]}}{\sqrt{(2\pi)^2 \det C(t|t_0)}}. \tag{D2a}
\]

Here, the expectation vector and covariance matrix conditioned on the initial time point are given by

\[
\mu(t|q_0, t_0) = (t - A^{-1})u + e^{-(t-t_0)A} [q_0 - (t_0 - A^{-1})u], \tag{D2b}
\]

\[
C(t|t_0) = C(\infty) - e^{-(t-t_0)A} C(\infty) e^{-(t-t_0)A^T}. \tag{D2c}
\]

with \(u = (u, 0)\) and the stationary covariance matrix

\[
C(\infty) = \left( \begin{array}{cc} \frac{\sigma_0^2}{2} & 0 \\ 0 & 1/2\tau_a \end{array} \right). \tag{D2d}
\]

As described in the main text, we consider the situation in which initially the distribution of particle positions is independent of the active fluctuations (which are in their stationary state), so that the initial distribution for the joint system factorizes as \(p_0(x_0, \eta_0) = p_0(x_0)p_0(\eta)\). We further assume that \(p_0(x_0)\) is Gaussian with zero mean and variance \(\sigma_0^2\); whereas \(p_0(\eta)\) is given in (31). The initial probability density \(p_0(x_0, \eta_0)\) is thus also Gaussian with mean \(\mu(0) = 0\) and covariance matrix

\[
C(0) = \left( \begin{array}{cc} \sigma_0^2 & 0 \\ 0 & 1/2\tau_a \end{array} \right). \tag{D3}
\]

Due to the linearity of the system (see (D1)), the distribution remains Gaussian for all later times \(t > t_0\), with the expectation values and covariances evolving according to

\[
\mu(t) = [t - (1 - e^{-tA}) A^{-1}] u, \tag{D4a}
\]

\[
C(t) = C(\infty) + e^{-tA} [C(0) - C(\infty)] e^{-tA^T}. \tag{D4b}
\]

In order to calculate averages of, e.g., \(\Delta \Sigma\) or \(\Delta J_{\pm}\), we have to evaluate correlators of particle positions and/or active fluctuations at two different time points (see next section). Using the propagator (D2) and the time-dependent probability density (D4), we find

\[
\left\langle [q(t) - \mu(t)] [q(t') - \mu(t')] \right\rangle = \begin{cases} 
\left[ e^{(t-t')A} C(t') \right]_{ij} & \text{for } t \geq t' \\
\left[ e^{(t-t')A} C(t) \right]_{jj} & \text{for } t < t'. \end{cases} \tag{D5}
\]

2. Evaluation of averages

We are interested in the averages \(\langle \Delta S_{\text{sys}} \rangle\), \(\langle \Delta \Sigma \rangle\) and \(\langle \Delta J_{\pm} \rangle\), where the general, trajectory-wise expressions of all these quantities are given in (18), (38b), and (57), (63), respectively. While calculating \(\langle \Delta S_{\text{sys}} \rangle\) is relatively straightforward (and basically amounts to computing the variances of the Gaussians at initial and final time), the evaluation of \(\langle \Delta \Sigma \rangle\) and \(\langle \Delta J_{\pm} \rangle\) is more complicated. It involves the non-local memory kernel \(\Gamma_x(t, t')\) and the averages \(\langle \dot{x}(t)k[x(t') - ut'] \rangle\) for \(\langle \Delta \Sigma \rangle\) and \(\langle \Delta J_{\pm} \rangle\), see (38b) and (57), (63), \(\langle \dot{x}(t)\eta(t') \rangle\) or \(\langle \eta(t)k[x(t') - ut'] \rangle\) for \(\langle \Delta J_{\pm} \rangle\), see (57), (63)). Here, we used \(f_t = f(x(t'), t') = k[x(t') - ut']\), see (70). It turns out to be convenient to move the time-derivative from \(\dot{x}(t)\) over to \(\Gamma_x(t, t')\) by partial integration. Then all the correlations reduce to \(\langle x(t)x(t') \rangle\) and \(\langle x(t)\eta(t') \rangle\), which we have already calculated in (D5).

We exemplify the procedure in more detail for \(\langle \Delta \Sigma \rangle\), \(\langle \Delta J_{\pm} \rangle\) can be evaluated in an analogous way. For the explicit calculation, it is useful to split (38b) into a white-noise and a colored-noise contribution,

\[
\Delta \Sigma_c = \frac{1}{T} \int_0^T dx_t f_t, \tag{D6a}
\]

\[
\Delta \Sigma_w = -\frac{1}{T} \left( \frac{D \tau_a}{D} \right) \int_0^T dt \int_0^T dt' \dot{x}_t f_t \Gamma_x(t, t'), \tag{D6b}
\]

such that \(\Delta \Sigma = \Delta \Sigma_w + \Delta \Sigma_c\). The white-noise part \(\Delta \Sigma_w\) is an ordinary stochastic integral and its average can be evaluated using standard techniques; we recall that we use the Stratonovich convention throughout this work. For evaluating \(\Delta \Sigma_c\) we observe that \(\Gamma_x(t, t')\) is continuously differentiable except at \(t = t'\) (see (B5c)). Splitting the \(t\)-integral into the intervals \([0, t']\) and \([t', T]\), we can transfer the time derivative from the trajectory \(\dot{x}(t)\) to the memory kernel by partial integration, so that

\[
\Delta \Sigma_c = \frac{1}{T} \left( \frac{D \tau_a}{D} \right) \int_0^T dt' \left[ x_0 f_t \Gamma_x(0, t') - x_\tau f_t \Gamma_x(\tau, t') + \int_0^\tau dt x_t f_t \partial_t \Gamma_x(t, t') + \int_0^T dt x_t f_t \partial_t \Gamma_x(t, t') \right]. \tag{D7}
\]

where \(\Gamma_x(t, t')\) and \(\Gamma_x(\tau, t')\) are defined in analogy to (B4). Now, the average \(\langle \Delta \Sigma_c \rangle\) involves correlations
\[ \langle x_t f_t \rangle = k(x_t | x_{t'} - u t') \] between different time points. Using the autocorrelation functions (D5) and the result (36a) for \( \Gamma_0(t, t') \), it can thus be evaluated as an ordinary integral.

In its full general form the resulting expression for \( \langle \Delta \Sigma \rangle = \langle \Delta \Sigma_w \rangle + \langle \Delta \Sigma_c \rangle \) is rather lengthy, so that we omit it here. A few relevant limiting cases are given in Sec. VI.
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