Abstract

We present here the field equations describing a non-stationary spherically symmetric $n$-dimensional charged black hole with a varying mass $m(v)$ and/or electric charge $q(v)$, described by a generic charged Vaidya metric with a cosmological constant $\Lambda_1$. This formulation of the metric has been shown to be particularly useful for perturbative studies and it was used in some recent works. Here, we also discuss some issues related to the apparent and event horizons of the black hole.

PACS numbers: 04.30.Nk, 04.40.Nr, 04.70.Bw

(Some figures may appear in colour only in the online journal)
The numerical results are obtained with a generalization of the semi-analytical method used in [24, 25] that allows us to construct the structure of the spacetime from the behavior of the outgoing null geodesics.

The structure of this paper is as follows. In section 2, we present the derivation of the most general Vaidya metric in double-null coordinates. In section 3, we present a discussion on the structure of the spacetime and the numerical results obtained with the semi-analytical method. Finally, in section 4, we present our final discussion of the results.

2. The general Vaidya metric in double-null coordinates

The $n$-dimensional Vaidya metric was first discussed in [26]. It can be easily cast in $n$-dimensional radiation coordinates $(v, r, \theta_1, \ldots, \theta_{n-2})$ as done, e.g., in [27]. The $n$-dimensional charged Vaidya metric in radiation coordinates, obtained originally in [8], reads

$$\text{d}s^2 = -\left(1 - \frac{2m(v)}{(n-3)r^{n-3}} + \frac{2q^2}{(n-2)(n-3)r^{2(n-3)}}\right)\text{d}v^2 + 2c\text{d}r \text{d}v + r^2\text{d}\Omega^2_{n-2},$$

(1)

where $n > 3$, $c = \pm 1$ and $\text{d}\Omega^2_{n-2}$ stands for the metric of the unit $(n-2)$-dimensional sphere, assumed here to be spanned by the angular coordinates $(\theta_1, \theta_2, \ldots, \theta_{n-2})$:

$$\text{d}\Omega^2_{n-2} = \sum_{i=1}^{n-2} \left(\prod_{j=1}^{i-1} \sin^2 \theta_j\right) \text{d}\theta_i^2.$$  

(2)

For the case of an ingoing radial flow, $c = 1$ and $m(v)$ is a monotonically increasing mass function in the advanced time $v$, while $c = -1$ corresponds to an outgoing radial flow, with $m(v)$ being in this case a monotonically decreasing mass function in the retarded time $v$. The constant $q$ corresponds to the total electric charge. In principle, one can also consider time-dependent charges $q$ as done, e.g., in [9]. This situation will of course require the presence of charged null fluids and currents, whose realistic nature we do not address here.

It has been known since a long time that the radiation coordinates are defective at the horizon [28], implying that the Vaidya metric (1), with or without the electric charge, is not geodesically complete in any dimension. The radiation coordinates are not enough to cover the entire Vaidya spacetime. (The radiation coordinates are defective at horizons where $v^2 \to \infty$.)

As can be seen in [28], for a four-dimensional Vaidya metric with $\text{d}m/\text{d}u < 0$ and without the electric charge (in the context of [28], a radiating star), the hypersurface $r = 2m(\infty)$ at $v = \infty$ in the Vaidya metric is analogous to the Schwarzschild hypersurface $r = 2m$ at Schwarzschild’s time coordinate $T = +\infty$ in the Kruskal metric. (See [10] and [29] for further discussions about possible analytical extensions and properties of the horizon of the Vaidya metric in the radiation coordinates.)

The cross term $\text{d}r \text{d}v$ introduces extra terms in the hyperbolic equations governing the evolution of physical fields on spacetimes with the metric (1). Typically, the double-null coordinates are far more convenient for the QNM analysis. This was the main motivation of the series of works based on Waugh and Lake’s approach [23], where the problem of casting the four-dimensional Vaidya metric in double-null coordinates was originally addressed. As all previous attempts to construct a general transformation from radiation to double-null coordinates had failed, Waugh and Lake considered the problem of solving Einstein’s equation with spherical symmetry directly in double-null coordinates. The resulting equations, however, are not analytically solvable in general. Waugh and Lake’s work was revisited in [24], where a semi-analytical approach allowing for general mass functions was proposed.

More recently this semi-analytical approach was extended to the case of an $n$-dimensional Vaidya metric with a cosmological constant $\Lambda$ [25]. This approach consists in a qualitative
study of the null-geodesics, allowing the description of light-cones and revealing many features of the underlying causal structure. It can also be used for more quantitative analyses; indeed, it has already enhanced considerably the accuracy of the quasinormal modes analysis of varying mass black holes [4, 5], and it can also be applied to the study of gravitational collapse [24].

In this section, we extend the approach proposed in [25] and derive the double-null formulation for the most general Vaidya metric: \( n \)-dimensional, in the presence of a cosmological constant, and with varying electric charge and mass. Only the main results are presented. The reader can get more details on the employed semi-analytical approach in [25] and the references cited therein. We recall that the \( n \)-dimensional spherically symmetric line element in double-null coordinates \((u, v, \theta_1, \ldots, \theta_{n-2})\) is given by

\[
d s^2 = -2f(u, v) \, du \, dv + r^2(u, v) \, d\Omega_{n-2}^2,
\]

where \( f(u, v) \) and \( r(u, v) \) are non-vanishing smooth functions. The energy–momentum tensor of a unidirectional radial null-fluid in the eikonal approximation in the presence of an electromagnetic field \( F_{ab} \) is given by

\[
T_{ab} = \frac{1}{8\pi} h k_a k_b + \frac{1}{4\pi} \left( F_{ac} F_c^b - \frac{1}{4} g_{ab} F^{cd} F_{cd} \right),
\]

where \( k_a \) is a radial null vector and \( h(u, v) \) is a smooth function characterizing the null-fluid radial flow. In the double-null coordinates \((u, v, \theta_1, \ldots, \theta_{n-2})\), we can choose either \( k_u = (1, 0, 0, \ldots, 0) \) (flow along the \( u \)-direction) or \( k_v = (0, 1, 0, \ldots, 0) \) (flow along the \( v \)-direction). Since the \( u \) and \( v \) directions are unspecified, it is not in fact necessary to consider flows along both directions. We will consider here, without loss of generality, the case of a flow along the \( v \)-direction, as done in [23].

Maxwell equations are given by

\[
\frac{1}{\sqrt{-g}} \partial_a \sqrt{-g} F^{ab} = 4\pi J^b,
\]

\[
F_{ab,c} + F_{ca,b} + F_{bc,a} = 0,
\]

where, for the metric (3),

\[
\sqrt{-g} = f r^{n-2} \prod_{j=1}^{n-3} (\sin \theta_j)^{n-j-2}.
\]

All geometrical quantities relevant to this work are listed in the appendix of [25]. The equations (5a) have the following static spherically symmetric solution:

\[
F^{uv} = -F^{vu} = \frac{q}{f r^{n-2}},
\]

with all other components of the electromagnetic tensor vanishing, where \( q \) is a constant that one identifies as the \( n \)-dimensional electric monopole charge. This case, of course, corresponds to \( J^b = 0 \). In order to allow for a time-dependent charge \( q(v) \), one needs to assume the presence of a current

\[
J^u = \frac{1}{4\pi} \frac{\dot{q}(v)}{f r^{n-2}},
\]

(with all other components vanishing) which is obtained from the continuity equation \( J^a_{;a} = 0 \), as done, e.g., in [9]. Such a current must naturally appear, as we will see, in the \( h(u, v) \) function characterizing the radial flow in the energy–momentum tensor (4). We note here that our solution (7) for \( F^{ab} \) with \( q(v) \) also consistently satisfies the sourceless Maxwell equations (5b).
Einstein’s equations with a cosmological constant $\Lambda$,  
\[ R_{ab} - \frac{1}{2} g_{ab} R = -\Lambda g_{ab} + 8\pi T_{ab}, \tag{9} \]

imply that for the energy–momentum tensor (4),
\[ \frac{f_u}{f} r_{uu} - \frac{r_{uu}}{f_u} = 0, \tag{11} \]
\[ \frac{f_v}{f} r_{vv} = \frac{h}{n-2} \frac{r}{r_v}, \tag{12} \]

where ' , u' and ' , v' denote, respectively, differentiation with respect to $u$ and $v$ as usual. The $uu$ and $vv$ components of Einstein’s equations for this case read
\[ f_u = \frac{2}{f} Br_r + \frac{r_{uv}}{f}, \tag{17} \]

For $n \neq 3$, differentiating equation (14) with respect to $u$ and then inserting equation (11) leads to
\[ \frac{r^{n-2}r_{uv}}{f} = \frac{2\Lambda}{(n-2)(n-1)} r^{n-1} - \frac{2}{n-2} \frac{q^2}{r^{2(n-2)}} \tag{15} \]

Note that (13) and (17) reproduce (15). Einstein’s equations are, therefore, equivalent to equations (17)–(19), generalizing the previous results of [23–25].

As already mentioned, the physical interpretation of the arbitrary integration functions $m(v)$ and $B(v)$ are the same of the $q = 0$ case. Transforming from the double-null coordinates
back to the radiation coordinates by the coordinate change \((u, v) \rightarrow (r(u, v), v)\), the metric (3) will read
\[
ds^2 = 4Br,v \, dv^2 - 4B \, dr \, dv + r^2 d\Omega_{n-2}^2,
\]
where (17) was explicitly used. Comparing (1) and (20) and taking (19) into account, it is clear that with the choice \(B = \pm 1/2\), the function \(m(v)\) indeed represents the mass of the \(n\)-dimensional charged solution. The coordinate transformation leading to (20) also ensures that the Vaidya metric in radiation coordinates (1) and the double-null metric (3) constructed in this paper are (locally) isometric. It is important to stress this fact, given the absence of a Birkhoff theorem for non-vacuum spacetimes.

As in the \(q = 0\) case, the weak energy condition applied for (4) requires, from (18), that
\[
B \left( m,v - \frac{1}{n-2} \frac{(q^2)_v}{r^{n-2}} \right) < 0.
\]
If there are both mass and charge variations, then \(m,v\) and \(q,v\) cannot be chosen arbitrarily (see [9] for a discussion) and must chosen satisfy the energy condition (21).

Taking (as mentioned above) \(B = \pm 1/2\), if we have only mass (or charge) varying with time, the energy condition requires \(m(v)\) (or \(q(v)\)) to be a monotonic function and fixes \(B\) in the following way:
- If \(m,v > 0\) (or \(q,v < 0\)), then \(B = -1/2\),
- If \(m,v < 0\) (or \(q,v > 0\)), then \(B = +1/2\),
where we consider, without loss of generality, \(q(v) > 0\).

3. The spacetime structure

The problem of constructing a double-null formulation for the general Vaidya metric may be stated as follows: given the mass function \(m(v)\), the electric charge function \(q(v)\), the cosmological constant \(\Lambda\) and the constant \(B\), one needs to solve equation (19), obtaining the function \(r(u, v)\). Then, \(f(u, v)\) and \(h(u, v)\) are calculated from (17) and (18). The arbitrary function of \(u\) appearing in the integration of (19) must be chosen properly [23], so that \(f(u, v)\) given in (17) is a non-vanishing function. Unfortunately, as stressed previously by Waugh and Lake [23], such a procedure is not analytically solvable in general. In [25], a semi-analytical procedure is introduced to attack the problem of solving equations (17)–(19) for the \(q = 0\) case, generalizing in this way the results of [24] obtained for \(n = 4\) and \(\Lambda = 0\). The approach, which we will not reproduce here, allows us to qualitatively construct conformal diagrams, identifying horizons and singularities, and also to evaluate specific geometric quantities. The main idea, however, is to solve equation (19) numerically as an initial value in \(v\), for constant \(u\). In other words, we obtain numerically \(r(u, v)\) for \(u\) constant, starting with an initial condition
\[
r(u, v_0) = F(u),
\]
where we must have \(F'(u) \neq 0\), as can be seen from (17). In analogy with the flat spacetime case, we choose here \(F(u) = -\frac{1}{n-2}\). Since the lines of constant \(u\) (or \(v\)) are null geodesics for any metric in double-null coordinates, knowing \(r(u, v)\) for \(u\) constant is enough, for instance, to construct the causal conformal diagrams. Figure 1 depicts a simple example, corresponding to the usual Reissner–Nordström solution.

Figures 2 and 3 present the behavior of the \(u\)-constant null-geodesics in two different time-dependent cases: increasing the charge function \(q(v)\) and increasing the mass function \(m(v)\). Charge and mass variations produce opposite results for the horizons. Also, \(r_-\) and \(r_+\),
Figure 1. Example of $u$-constant null-geodesics (dotted lines) obtained from equation (19) for the usual Reissner–Nordström case, corresponding to $B = -1/2$, $m = 0.5$, $q = 0.4$, $n = 4$, $\Lambda_1 = 0$, and taking $r(u, v = 0) = -u/2$. (See [24]). Here, $r_+$ and $r_-$ (solid lines) are defined as usual as $r_{\pm} = m \pm \sqrt{m^2 - q^2}$. In the exterior region ($r > r_+$), the constant $u$ null geodesics reach $I^+$, while in the interior region they are confined, giving origin to the typical black hole causal structure.

Figure 2. Example of $u$-constant null-geodesics (dotted lines) obtained from equation (19), with the same parameter values used in figure 1 but now with a time-dependent charge function given by $2q(v) = (q_f + q_i) + (q_f - q_i) \tanh \rho(v - v_m)$, with $q_i = 0.4$, $q_f = 0.48$, $\rho = 4.0$ and $v_m = 1.5$. We can see in this case that $r_{\pm}$ (solid lines) are no longer constant, and the event horizon $r_h$ (dashed line) no longer coincides with $r_+$.

are now time dependent, and the event horizon $r_h$ is no longer coincident with the apparent horizon $r_+$. The case presented in figure 2 deserves a more detailed analysis. The charge increase is analogous to the mass evaporation studied in [24, 25].
Following our discussion in section 2, when \( q,v > 0 \), we must have \( B = +\frac{1}{2} \), in order to satisfy the weak energy condition (21). Now there is a subtlety regarding the sign of \( B \). We can see from equation (17) that the sign of \( f \) depends on the signs of \( B \) and \( r,u \), which is of negative sign with our choice of \( F(u) \). Therefore, for \( B = -\frac{1}{2} \), we have \( f > 0 \) and \( \partial_u + \partial_v \) is timelike and \( \partial_v - \partial_u \) is spacelike. However, for \( B = +\frac{1}{2} \), \( f \) has the opposite sign and the timelike and spacelike directions are now exchanged.

The transformation \((u,v) \rightarrow (v,-u)\) restores the temporal and spatial directions. Under this transformation, equation (19) (with \( B = +\frac{1}{2} \)) becomes

\[
-r_u = -\frac{1}{2} \left( 1 - \frac{2m(-u)}{(n-3)r^{n-3}} - \frac{2\Lambda}{(n-2)(n-1)}r^2 + \frac{2}{(n-2)(n-3)} q^2(-u) r^{2(n-3)} \right),
\]

which is formally identical to a case with \( B = -\frac{1}{2} \), mass function \( m(-u) \) and charge function \( q(-u) \). Therefore, the transformed equation results in a situation with decreasing charge, i.e. a time reversal of the original situation with increasing charge. Stated in a different way, this means that the case with increasing charge and \( B = +\frac{1}{2} \) must be interpreted as the time reversal of the case with decreasing charge and \( B = -\frac{1}{2} \) (with both cases satisfying the weak energy condition).

However, in order to describe an actual charge increase (equivalent to an evaporation), we need to violate the weak energy condition, since there are no classical processes that can lead to black hole evaporation. In order to do that, we deliberately choose \( B = -\frac{1}{2} \) together with \( q,v > 0 \). The weak energy condition is violated, and the resulting evaporation process is shown in figure 2.

In figure 4, we show an example of a higher dimensional spacetime with \( n = 5 \). The results are qualitatively the same as for the four-dimensional cases we presented in figures 2 and 3. In figure 5, we have a qualitatively different behavior, due to the appearance of a cosmological horizon (\( \Lambda \neq 0 \)).

The calculation of the apparent horizons \( r \pm (v) \) shown in figures 1–5 follows the semi-analytic method described in [24, 25]. We can see that the curves defined by the vanishing of
Figure 4. The same as figure 3, but this time exploring a five-dimensional solution \((n = 5)\), with \(q = 0.2, m_i = 0.5\) and \(m_f = 0.65\). The apparent horizons are now at \(r_{\pm} = \left[ \frac{1}{2} (m \pm \sqrt{m^2 - \frac{4}{3} q^2}) \right]^{1/2}\), and therefore, we must have \(\sqrt{3} m > 2q\). The behavior is qualitatively the same as for the \(n = 4\) case.

Figure 5. The same as figure 3, but this time allowing a non-zero cosmological constant \(\Lambda = 0.3\), with \(q = 0.4, m_i = 0.5, m_f = 0.65, \rho = 1.0\) and \(v_m = 15\). The striking feature in this case is the existence of three time-dependent apparent horizons \(r_-, r_c, r_+\), located at the positive roots of equation (19), which is a fourth-order polynomial in this case.

the rhs of equation (19) always describe the frontier between two regions of the \((v, r)\) plane, where the solutions of equation (19) have distinct behaviors. For \(r_v < 0\), the null geodesics approach the singularity, whereas for \(r_v > 0\), the null geodesics tend to escape from the singularity.
The determination of the event horizon \( r_h \) is done numerically by inspection of the initial value \( r(u, v_0) \). The event horizon is found as the last geodesic that escapes toward infinity and does not fall into the singularity. Note that in the case of increasing charge (or, equivalently, decreasing mass) there are null geodesics that escape toward infinity even though they were initially inside the apparent horizon \( r_+ \).

4. Final discussion

We have presented a formulation in double-null coordinates of the most general Vaidya metric: \( n \)-dimensional, with varying mass and/or charge and cosmological constant \( \Lambda \).

By exploring the numerical solutions of equation (19), we were able to highlight some interesting features of the behavior of time-dependent horizons in multiple-horizon spacetimes. The \( n \)-constant geodesics can be used to track the time-dependent event and Cauchy horizons that no longer coincide with \( r_+ \) and \( r_- \).

The formulation presented here for Einstein’s equations (17)–(19) was recently used in a quasinormal mode analysis of the Vaidya metric [22] and provided the framework needed to obtain the quasinormal frequencies with sufficient accuracy to verify their non-stationary behavior.

Acknowledgments

This work was supported by CNPq, FAPESP and the Max Planck Society.

Appendix

Here, we present a short discussion and the results for Einstein’s equations (11)–(14) for the \( n = 3 \) case, generalizing the discussion of [25]. For \( n = 3 \), equation (11) is still valid and can be integrated to give
\[
f = 2Br_u, \tag{A.1}
\]
the same as equation (17).

Taking \( n = 3 \), equation (14) reads
\[
rr_{uv} - (\Lambda r^2 + q^2)f = 0, \tag{A.2}
\]
and we can use equation (A.1) to integrate equation (A.2) and obtain
\[
r_{uv} = -B(-m - \Lambda r^2 - 2q^2 \ln r), \tag{A.3}
\]
which is the \( n = 3 \) version of equation (19). Here, \( m(u) \) is an integration function that has the same interpretation as before, i.e. the mass of the solution. Compare (A.3) with the charged BTZ black hole [30].

From equation (12) with \( n = 3 \), we have
\[
h = \frac{r_u f_w}{r} f - \frac{r_{uw}}{r}, \tag{A.4}
\]
and using equations (A.3) and (A.1) to obtain
\[
f_w = 2B \left( \Lambda r + \frac{q^2}{r} \right) f, \tag{A.5}
\]
we obtain the \( n = 3 \) version of equation (12):
\[
h = -B \left( m_w + \frac{4 \ln r}{r} - qq_w \right). \tag{A.6}
\]
We also note here that the $n = 3$ version of equation (13),
\[
\frac{f_{,u}f_{,v}}{f^2} - \frac{f_{,uv}}{f} - \frac{r_{,uv}}{r} = -2\Lambda f,
\]
(A.7)
together with equation (A.1) still reproduce equation (15) with $n = 3$.

So finally we can conclude that Einstein’s equations (11)–(14), which in the $n \geq 4$ case are equivalent to equations (17)–(19), are equivalent in the $n = 3$ case to equations (A.1), (A.3) and (A.6).

References

[1] Vaidya P C 1951 Proc. Indian Acad. Sci. A 33 264
[2] Xue L H, Shen Z X, Wang B and Su R K 2004 Mod. Phys. Lett. A 19 239
[3] Shao C G, Wang B, Abdalla E and Su R K 2005 Phys. Rev. D 71 044003
[4] Abdalla E, Chirenti C B M H and Saa A 2006 Phys. Rev. D 74 084029
[5] Abdalla E, Chirenti C B M H and Saa A 2007 J. High Energy Phys. JHEP10(2007)086
[6] He X, Wang B, Wu S F and Lin C Y 2009 Phys. Lett. B 673 156
[7] Krori K D and Barua J 1974 J. Phys. A: Math. Nucl. Gen. 7 2125
[8] Chatterjee S, Bhui B and Banerjee A 1990 J. Math. Phys. 31 2208
[9] Ori A 1991 Class. Quantum Grav. 8 1559
[10] Fayos F, Martin-Prats M M and Senovilla J M M 1993 Class. Quantum Grav. 12 2565
[11] Parikh M K and Wizcek F 1999 Phys. Lett. B 449 24
[12] Hong S E, Hwang D, Stewart E D and Yeom D 2010 Class. Quantum Grav. 27 045014
[13] Hwang D and Yeom D 2011 Phys. Rev. D 84 064020
[14] Joshi P S 1993 Global Aspects in Gravitation and Cosmology (Oxford: Oxford University Press)
[15] Lake K 1992 Phys. Rev. Lett. 68 3129
[16] Hiscock W A 1981 Phys. Rev. D 23 2813
[17] Kuroda Y 1984 Prog. Theor. Phys. 71 100
Kuroda Y 1984 Prog. Theor. Phys. 71 1422
[18] Biernacki W 1990 Phys. Rev. D 41 1356
[19] Parentani R 2001 Phys. Rev. D 63 041503
[20] Hu B and Verduguer E 2004 Living Rev. Rel. 7 3
[21] Nielsen A B, Jasiulek M, Krishnan B and Schnetter E 2011 Phys. Rev. D 83 124022
[22] Chirenti C and Saa A 2011 Phys. Rev. D 84 064006
[23] Waugh B and Lake K 1986 Phys. Rev. D 34 2978
[24] Girotto F and Saa A 2004 Phys. Rev. D 70 084014
[25] Saa A 2007 Phys. Rev. D 75 124019
[26] Iyer B R and Vishveshwara C V 1989 Pramana 32 749
[27] Ghosh S G and Dadhich N 2001 Phys. Rev. D 64 047501
Ghosh S G and Dadhich N 2001 Phys. Rev. D 65 127502
[28] Lindquist R, Schwartz R and Misner C 1965 Phys. Rev. 137 1364
[29] BooH B and Martin C 2010 Phys. Rev. D 82 124046
[30] Bañados M, Teitelboim C and Zanelli J 1992 Phys. Rev. Lett. 69 1849
Bañados M, Henneaux M, Teitelboim C and Zanelli J 1993 Phys. Rev. D 48 1506
Kamata M and Kotakawa T 1995 Phys. Lett. B 353 196