LÉVY DIFFERENTIAL OPERATORS AND GAUGE INVARIANT EQUATIONS FOR DIRAC AND HIGGS FIELDS
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Dedicated to the memory of Sergei Starodubov

Abstract: We study the Lévy infinite-dimensional differential operators (differential operators defined by the analogy with the Lévy Laplacian) and their relationship to the Yang-Mills equations. We consider the parallel transport on the space of curves as an infinite-dimensional analogue of chiral fields and show that it is a solution to the system of differential equations if and only if the associated connection is a solution to the Yang-Mills equations. This system is an analogue of the equation of motion of chiral fields and contains the Lévy divergence. The systems of infinite-dimensional equations containing Lévy differential operators, that are equivalent to the Yang-Mills-Higgs equations and the Yang-Mills-Dirac equations (the equations of quantum chromodynamics), are obtained. The equivalence of two ways to define Lévy differential operators is shown.

keywords: Lévy Laplacian, Yang-Mills equations, Yang-Mills-Higgs equations, QCD equations
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Introduction

The current paper has two goals. We provide a classification of infinite-dimensional differential operators defined by the analogy with the Lévy Laplacian and naturally connected to the Yang-Mills equations. And we describe by such differential operators solutions to the Yang-Mills-Higgs equations and the QCD equations (the equations of quantum chromodynamics).

In the current paper we will use the following scheme of the definition of differential operators, and not the form in which they were originally introduced. Let $E$ be a real normed space and $S$ be a linear functional acting on a subspace of $L(E,E^*)$. Then $S$ defines a linear differential operator of

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1If $X$ and $Y$ are normed spaces, the symbol $L(X,Y)$ denotes the space of linear continuous operators from $X$ to $Y$. The symbol $C^k(X,Y)$ denotes the space of $k$-times Fréchet differentiable functions from $X$ to $Y$. 
the second order $D^{2,S}$ acting on $f \in C^2(E, \mathbb{R})$ by the formula

$$D^{2,S}f(x) = S(f''(x)),$$

and the linear differential operator of the first order $D^{1,S}$ acting on $B \in C^1(E, E^*)$ by the formula

$$D^{1,S}B(x) = S(B'(x)).$$

For example, if we choose $E = \mathbb{R}^d$ and $S = tr$ (trace), then $D^{2,tr}$ is the Laplacian $\Delta$ and $D^{1,tr}$ is the divergence $div$. Now let $E$ be a real normed space continuously embedded in a real separable Hilbert space $H$. Let $\{e_n\}$ be an orthonormal basis in $H$ that consists of elements of $E$. Then the value of the Lévy trace $tr^{\{e_n\}}_L$ (generated by the orthonormal basis $\{e_n\}$) on $K \in L(E, E^*)$ is defined by

$$tr^{\{e_n\}}_L K = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} <Ke_k, e_k>.$$(1)

The Lévy Laplacian (generated by the orthonormal basis $\{e_n\}$) $\Delta^{\{e_n\}}_L$ is the differential operator $D^{2,tr^{\{e_n\}}_L}$. This operator was introduced by Paul Lévy for the case $E = H = L^2(0,1)$ (see Ref. [20]).

Another original definition of the Lévy Laplacian from Ref. [20] (the definition of the Lévy Laplacian by means of a second-order derivative of special form) will be given in terms of the Lévy trace as follows. Let $K$ be a bilinear functional on $L^2(0,1)$ such that for all $u, v \in L^2(0,1)$ the following equality holds

$$K(u, v) = \int_0^1 \int_0^1 K_V(t, s)u(t)v(s)dtds + \int_0^1 K_L(t)u(t)v(t)dt,$$(2)

where $K_V \in L^2([0, 1] \times [0, 1])$ and $K_L \in L^\infty[0, 1]$. The value of the Lévy trace $tr_L$ on $K$ is defined by

$$tr_L K = \int_0^1 K_L(t)dt.$$

Then the Lévy Laplacian $\Delta_L$ is the differential operator $D^{2,tr_L}$. The orthonormal bases $\{e_n\}$ in $L^2(0,1)$, such that $tr_L$ is a restriction of $tr^{\{e_n\}}_L$, form a special class of weakly uniformly dense bases. Hence for weakly uniformly dense bases $\Delta_L$ is a restriction of $\Delta^{\{e_n\}}_L$. 

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The article deals with functionals which are generalizations of the traces described above. Using uniformly dense bases it is proved that these functionals coincide under certain conditions. Such functionals will be also called the Lévy traces. We consider differential operators of the first and the second order (Laplacian, d’Alembertian, divergence) generated by these traces arising in the study of gauge fields. Such operators will be called the Lévy differential operators.

The following papers are devoted to the connection between the Lévy differential operators and the Yang-Mills equations. In Ref. [9] by Aref’eva and Volovich the functional divergence was introduced by the analogy with the Lévy Laplacian defined by means of a second-order derivative of special form. The relationship of this divergence with gauge fields were studied. In the current paper it is shown that the functional divergence is a restriction of the Lévy divergence. In Refs. [2,3] Accardi, Gibilisco and Volovich introduced an analogue of the Lévy Laplacian acting on functions on the space of piecewise smooth functions (see also Ref. [1]). This operator was defined by a more complicated form of the second derivative than in [2]. It is also called the Lévy Laplacian. In Refs. [2,3] the following was proved. The connection in the trivial bundle over a Euclidean space is a solution to the Yang-Mills equations if and only if the parallel transport, considered as an operator-valued functional on a space of curves, is a solution to the Laplace equation for such Lévy Laplacian. Moreover in Refs. [2,3] the Lévy d’Alembertian was introduced. The similar theorem about the equivalence of the Yang-Mills equations for a connection on a Minkowski space and the Lévy-d’Alembert equation for the parallel transport holds for this operator. The case of a Riemannian manifold and the stochastic case were considered by Leandre and Volovich in Ref. [19]. In Refs. [25,27] the theorem about the connection between the Lévy Laplacian and the gauge fields was proved in the case of a manifold and the Lévy Laplacian and the Lévy d’Alembertian defined as the Césaro mean of the second directional derivatives (see also Ref. [8] for the definition). Thus, the problem has been solved, set in Ref. [2]. In Refs. [28,29] it was proved that the Lévy Laplacians and Lévy d’Alembertians defined as the Cesàro mean of the second directional derivatives and defined by a special form of the second derivative coincide in the plane case. The instantons were described in terms of the Lévy Laplacian and the parallel transport in Ref. [29]. In Refs. [30,31] the Levy Laplacian defined as the Cesario mean of the directional derivatives in the stochastic case was studied. It was shown that, unlike the deterministic case, the

\footnote{Note that the different approach to the Yang-Mills equations based on the stochastic
equivalence of the Yang-Mills equations and the Levy-Laplace equation is not valid for such Laplacian. It means that this Levy Laplacian does not coincide with the Levy Laplacian introduced in Ref. [19].

The path 2-form is a function on the space of curves in $\mathbb{R}^d$ beginning at the origin such that the value of this function on a curve is a matrix-valued 2-form at the endpoint of the curve. In Ref. [17] by Gross the infinite-dimensional non-commutative Poincaré lemma was proved. It implies that an infinite-dimensional 1-form (a smooth function from the space of curves to the space of linear matrix-valued operators on the space of curves) is generated by the path 2-form and is closed if and only if it is generated by a parallel transport associated with a connection in the trivial bundle over $\mathbb{R}^d$. Using this non-commutative Poincaré lemma in Ref. [17] Gross proved that the differential equations on the path 2-form, that are generalization of Maxwell’s equations, are equivalent to the Yang-Mills equations. Also in Ref. [17] the differential equations on the path 2-forms and the infinite-dimensional fields, that are equivalent to the QCD equations and the Yang-Mills-Higgs equations, were obtained.

In the current paper we show that an infinite-dimensional 1-form generated by the path 2-form is closed and the value of the Lévy divergence on this 1-form is zero if and only if it is associated with a finite-dimensional connection, that is a solution to the Yang-Mills equations.\footnote{The converse statement was proved in Ref. [9] for the functional divergence.} Thus, we obtain the differential equations that are analogue of the equations of motion of the chiral field and are equivalent to the Yang-Mills equations (the parallel transport can be considered as an analogue of the chiral field). In addition, we study the connection between the Lévy infinite-dimensional differential operators and Yang-Mills-Higgs equations and the QCD equations. For this purpose the equations derived in Ref. [17] are used. But in contrast to our approach the Lévy differential operators were not considered in Ref. [17]. We generalize the Accardi-Gibilisco-Volovich theorem and reformulate the Yang-Mills-Higgs equations and the QCD equations in terms of the parallel transport and the Lévy d’Alembertian. Using the non-commutative Poincaré lemma we reformulate the Yang-Mills-Higgs equations and the QCD equations in terms of the infinite-dimensional 1-form and the Lévy divergence. The relationship of the Lévy d’Alembertian and the QCD equations was considered in Ref. [28].

The paper is organized as follows. The first section provides the definition parallel transport but without using the Lévy differential operators was considered in Refs. [13] [10] [11].
of the Lévy traces as the Cesàro mean of diagonal elements and differential operators generated by these traces. The second section provides the definition of the Lévy traces as an integral functional and associated differential operators. Also in the second section the equivalence of two ways to define Lévy differential operators is shown. The connection between the Lévy differential operators and the Yang-Mills equations is discussed in the third section. In the fourth and the fifth sections systems of infinite-dimensional differential equations, containing Lévy differential operators, which are equivalent to the Yang-Mills-Higgs equations and to the QCD equations respectively, are obtained.

1 Lévy trace and Lévy Differential Operators

Let us recall the general scheme of the definition of homogeneous linear differential operators from the paper Ref. [12] (see also Ref. [23]). Let $X,Y,Z$ be real normed vector spaces. Let $C^n(X,Y)$ be the space of $n$ times Fréchet differentiable functions from $X$ to $Y$. Then for any $x \in X$ it is hold that $f^{(n)}(x) \in L_n(X,Y)$, where the space $L_n(X,Y)$ is defined by induction: $L_1(X,Y) = L(X,Y)$ and $L_n(X,Y) = L(X,L_{n-1}(X,Y))$. Let $S$ be a linear operator from $domS$ to $Z$, where $domS \subseteq L_n(X,Y)$. The domain $domD_{n,S}$ of the differential operator $D_{n,S}$ of order $n$ generated by the linear operator $S$ consists of all $f \in C^n(X,Y)$ such that for all $x \in E$ holds $f^{(n)}(x) \in domS$. Then the value of $D_{n,S}$ on $f \in domD_{n,S}$ is defined by the following formula

$$D_{n,S} f(x) = S(f^{(n)}(x)).$$

If we choose $X = \mathbb{R}^d$, $Y = Z = \mathbb{R}$ and $S = tr$, then $D^{2,\text{tr}}$ is the Laplace operator $\Delta$. If we choose $X = Y = \mathbb{R}^d$, $Z = \mathbb{R}$ and $S = tr$, then $D^{1,\text{tr}}$ is the divergence $\text{div}$ acting on the space of $C^1$-smooth 1-forms.

Let $E_1$ be a real normed space. Let $E_1$ be continuously embedded in a real separable infinite-dimensional Hilbert space $H_1$ in such a way that the image of $E_1$ is dense in $H_1$. Set $E = \mathbb{R}^d \otimes E_1$ and $H = \mathbb{R}^d \otimes H_1$.[3] Then $E \subset H \subset E^*$ is a rigged Hilbert space. Let \{e_n\} be an orthonormal basis in $H_1$ that consists of elements of $E_1$. Let \{p_1, \ldots, p_d\} be an orthonormal basis in $\mathbb{R}^d$. Denote the Euclidean metric on $\mathbb{R}^d$ by $\delta = (\delta_{\mu\nu})$. Also let the Minkowski metric $\eta = (\eta_{\mu\nu})$ be defined on $\mathbb{R}^d$. We assume that this metric is diagonal \{\eta_{\mu\nu}\} = diag\{1,-1,\ldots,-1\} in the basis \{p_1, \ldots, p_d\}. Everywhere below $g \in \{\delta, \eta\}$.\footnote{We assume that the topology of the tensor product of $\mathbb{R}^d \otimes E_1$ is defined by some cross-norm (see Ref. [24]). We assume that $H = \mathbb{R}^d \otimes H_1$ is a Hilbert tensor product.}
Definition 1. The Lévy trace $\text{tr}_L^{(e_n)} g$ generated by the orthonormal basis $\{e_n\}$ and the metric $g$ is a linear functional on $\text{dom}\text{tr}_L^{(e_n)} g$ defined by

$$
\text{tr}_L^{(e_n)} g (T) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (T(p_1 \otimes e_k), p_1 \otimes e_k) + \sum_{\mu=2}^{d} \sigma_g^2 (T(p_\mu \otimes e_k), p_\mu \otimes e_k),
$$

where $\sigma_g = 1$ or $\sigma_g = i$, if $g = \delta$ and $g = \eta$ respectively, and $\text{dom}\text{tr}_L^{(e_n)} g$ consists of all $T \in L(E, E^*)$ for which the right-hand side of (3) exists.

Definition 2. Choose $X = E$ and $Y = Z = \mathbb{R}$. The Lévy Laplacian $\Delta_L^{(e_n)}$ generated by the orthonormal basis $\{e_n\}$ is the differential operator $D^2\text{tr}_L^{(e_n)} \delta$. The Lévy d’Alembertian $\square_L^{(e_n)}$ generated by the orthonormal basis $\{e_n\}$ is the differential operator $D^2\text{tr}_L^{(e_n)} \eta$.

Definition 3. Choose $X = E$, $Z = \mathbb{R}$ and $Y = E^*$. In the Euclidean case the Lévy divergence $\text{div}_L^{\delta}$ generated by the orthonormal basis $\{e_n\}$ is the operator $D^1\text{tr}_L^{(e_n)} \delta$. In the Minkowski case the Lévy divergence $\text{div}_L^{\eta}$ generated by the orthonormal basis $\{e_n\}$ is the operator $D^1\text{tr}_L^{(e_n)} \eta$.

Remark 1. The Lévy trace was introduced for $d = 1$ in Ref. [5].

Remark 2. The non-classical Lévy Laplacian is a generalization of the Lévy Laplacian defined in the following way. Let $R$ be a linear mapping from $\text{span}\{e_n: n \in \mathbb{N}\}$ to $E_1$. The non-classical Lévy trace $\text{tr}_L^{(e_n)} g^R$ generated by the orthonormal basis $\{e_n\}$, the metric $g$ and the linear operator $R$ acts on $T \in L(E, E^*)$ by the formula

$$
\text{tr}_L^{(e_n)} g^R (T) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (T(p_1 \otimes Re_k), p_1 \otimes Re_k) + \sum_{\mu=2}^{d} \sigma_g^2 (T(p_\mu \otimes Re_k), p_\mu \otimes Re_k),
$$

where $\sigma_g = 1$ or $\sigma_g = i$, if $g = \delta$ or $g = \eta$ respectively. The non-classical Lévy Laplacian $\Delta_L^{(e_n)}$ and the non-classical Lévy d’Alembertian $\square_L^{(e_n)}$ are the differential operators $D^2\text{tr}_L^{(e_n)} \delta$ and $D^2\text{tr}_L^{(e_n)} \eta$ respectively. The non-classical Lévy divergence $\text{div}_L^{\eta}$ is the differential operator $D^1\text{tr}_L^{(e_n)}$. The non-classical Lévy trace and the associated non-classical Lévy Laplacian were
introduced for \( d = 1 \) in Ref. [6]. As we show below, such operators can be useful in the study of gauge fields (see also [31]). Moreover, the exotic Lévy Laplacians (see Refs. [5, 7] for the definition), which are studied in the white noise analysis (see Ref. [3] and references therein), can be represented as non-classical Lévy Laplacians (see Refs. [26, 28]).

2 Lévy trace as an integral functional

Let

\[ W^{1, p}_0([0, 1], \mathbb{R}^d) = \{ \sigma \text{ is absolutely continuous, } \sigma(0) = 0, \dot{\sigma} \in L_p((0, 1), \mathbb{R}^d) \} \]

and \( C^1([0, 1], \mathbb{R}^d) = \{ \sigma \in C^1([0, 1], \mathbb{R}^d), \sigma(0) = 0 \} \).

Below, unless specifically stated,

\[ E \in \{ C^1([0, 1], \mathbb{R}^d), W^{1,1}_0([0, 1], \mathbb{R}^d), W^{1,2}_0([0, 1], \mathbb{R}^d) \} \]

Denote by \( E_0 \) the space of all \( \sigma \in E \) such that \( \sigma(1) = 0 \).

Denote by \( T^2_E \) the space of continuous bilinear functionals on \( E \times E \), which restriction on \( E_0 \times E_0 \) have the following form

\[
Q(u, v) = \int_0^1 \int_0^1 Q^V_{\mu\nu}(t, s)u^\mu(t)v^\nu(s)dtds + \\
\quad + \int_0^1 Q^L_{\mu\nu}(t)u^\mu(t)v^\nu(t)dt + \frac{1}{2} \int_0^1 Q^S_{\mu\nu}(t)(\dot{u}^\mu(t)v^\nu(t) + v^\mu(t)u^\nu(t))dt,
\]

where \( Q^V_{\mu\nu} \in L_1([0, 1] \times [0, 1], \mathbb{R}), Q^L_{\mu\nu} \in L_1([0, 1], \mathbb{R}), Q^S_{\mu\nu} \in L_\infty([0, 1], \mathbb{R}), Q^L_{\mu\nu} \) is a symmetric tensor: \( Q^L_{\mu\nu} = Q^L_{\nu\mu} \), and \( Q^S_{\mu\nu} \) is an anti-symmetric tensor: \( Q^S_{\mu\nu} = -Q^S_{\nu\mu} \).

**Definition 4.** The Lévy trace \( tr^g_L \) generated by the metric \( g \) is a linear functional on \( T^2_E \) defined by the formula

\[
tr^g_L Q = \int_0^1 Q^L_{\mu\nu}(t)g^{\mu\nu}dt.
\]

**Remark 3.** The first term in (5) is the Volterra part, the second term is the Lévy part, the third term is a singular part.

**Definition 5.** Choose \( X = E, Z = Y = \mathbb{R} \). The Lévy Laplacian \( \Delta_L \) is the differential operator \( D^2_{tr^g_L} \). The Lévy d’Alembertian \( \Box_L \) is the differential operator \( D^2_{\Box_L} \).
Definition 6. Choose $X = E$, $Z = \mathbb{R}$ and $Y = E^*$. In the Euclidean case the Lévy divergence $\text{div}_L^\delta$ is the operator $D_{\text{tr}_L^\delta}^1$. In the Minkowski case the Lévy divergence $\text{div}_L^\eta$ is the differential operator $D_{\text{tr}_L^\eta}^1$.

The definition of the functional divergence by Aref’ieva and Volovich (see Ref. [22]) can be reformulated as follows. Let $E = C^1([0, 1], \mathbb{R}^d)$ and let $T^2_v$ be the space of $Q \in T^2_E$ such that $Q_{\mu\nu} \in C([0, 1] \times [0, 1])$ for any $\mu \in \{1, \ldots, d\}$ and $\sum_{\mu=1}^{d} \int_0^1 Q_{\mu\nu}(t, t)dt = 0$. Let $\text{tr}_v^\delta$ be the restriction of $\text{tr}_L^\delta$ on the space $T^2_v$. The functional divergence is the differential operator $D_{\text{tr}_v^\delta}^1$. Thus, the functional divergence is a restriction of the Lévy divergence.

Recall the following definition from Ref. [20] (see also Refs. [16, 18]).

Definition 7. An orthonormal basis $\{e_n\}$ in $L^2(0, 1)$ is weakly uniformly dense (or equally dense) if

$$\lim_{n \to \infty} \int_0^1 h(t)\left(\frac{1}{n} \sum_{k=1}^{n} e_k^2(t) - 1\right)dt = 0$$

(6)

for any $h \in L_\infty[0, 1]$.

Example 1. The sequence $e_n(t) = \sqrt{2}\sin nt$ is a weakly uniformly dense basis in $L^2(0, 1)$.

Theorem 1. Let $\{e_n\}$ be a weakly uniformly dense orthonormal basis in $L^2(0, 1)$ that consists of functions from $E$ such that $e_n(1) = 0$ for all $n \in \mathbb{N}$. Let the sequence $\{e_n\}$ be uniformly bounded. If $Q \in T^2_E$, then

$$\text{tr}_{L}^g(\{e_n\}) = \text{tr}_{L}^g(Q)$$

(7)

Proof. Since $e_n(0) = e_n(1) = 0$ for all $n \in \mathbb{N}$ and $Q^S_{\mu\nu}$ is anti-symmetric tensor, the following equality holds

$$\frac{1}{n} \sum_{k=1}^{n} Q(p_{\mu} e_k, p_{\mu} e_k) = \frac{1}{n} \sum_{k=1}^{n} \int_0^1 \int_0^1 Q^V_{\mu\nu}(t, s)e_k(t)e_k(s)dtds +$$

$$+ \int_0^1 Q^L_{\mu\nu}(t)\left(\frac{1}{n} \sum_{k=1}^{n} e_k^2(t)\right)dt.$$ 

For a weakly uniformly dense and uniformly bounded orthonormal basis in $L^2(0, 1)$ equality (6) holds for any $h \in L_1[0, 1]$ and also

$$\lim_{n \to \infty} \int_0^1 \int_0^1 K(s, t)e_n(s)e_n(t)dtds = 0.$$
holds for any \( K \in L_1([0, 1] \times [0, 1]) \) (see Ref. [18]). Since \( Q_{\mu\mu}^L \in L_1(0, 1) \) and \( Q_{\mu\nu}^V \in L_1([0, 1] \times [0, 1]) \) we obtain (7).

The following theorem is a direct corollary of Theorem 1.

**Theorem 2.** Let an orthonormal basis \( \{ e_n \} \) in \( L^2(0, 1) \) satisfy the conditions of Theorem 1. If \( f \in \text{dom}\Delta_L \), then
\[
\Delta_L^{\{ e_n \}} f = \Delta_L f \quad \text{and} \quad \Box_L^{\{ e_n \}} f = \Box_L f.
\]
If \( B \in \text{dom} \text{div}_L^g \), then
\[
\text{div}_L^g, \{ e_n \} B = \text{div}_L^g B.
\]

**Remark 4.** The definitions of the Lévy Laplacian and the Lévy d’Alembertian from section 1 and section 2 can be naturally transferred to the space \( C^2(E, M_m(\mathbb{C})) \). Additionally the definitions of the Lévy traces can be transferred to the space \( L(E, E^*_0) \). This fact allows to define the Lévy divergence on the space \( C^1(E, L(E_0, M_m(\mathbb{C}))) \). Such divergence will be used in the following sections.

**Remark 5.** Choose \( E = H = W^{1,2}_0([0, 1], \mathbb{R}^d) \). Let \( \{ f_n \} \) be an orthonormal basis in \( E_1 = W^{1,2}_0([0, 1], \mathbb{R}) \) defined by \( f_1(t) = t \) and \( f_n(t) = \frac{\sqrt{2}}{\pi(n-1)} \sin(\pi(n-1)t) \) for \( n > 1 \). Consider the operator \( N: \text{span}\{ f_n : n \in \mathbb{N} \} \rightarrow E_1 \) defined by \( N f_n = (n-1)f_n \). Then \( \text{tr}_{L_N}^{\{ f_n \}} \) coincides with \( \text{tr}_L^g \) on the space \( T^2_E \). The differential operators generated by the trace \( \text{tr}_L^g \) are restrictions of the corresponding differential operators generated by the trace \( \text{tr}_{L_N}^{\{ f_n \}} \).

3 The Yang-Mills equations

The summation over repeated upper and lower indices is assumed. Let \( G \) be a closed Lie group realized as a subgroup of \( U(N) \), and let \( \text{Lie}(G) \) be a Lie algebra realized as a subalgebra of \( u(N) \). A connection in the trivial vector bundle with base \( \mathbb{R}^d \), fiber \( \mathbb{C}^N \), structure group \( G \) is defined as a \( \text{Lie}(G) \)-valued \( C^\infty \)-smooth 1-form \( A_\mu(x)dx^\mu \) on \( \mathbb{R}^d \). For the function \( \varphi \in C^1(\mathbb{R}^d, \text{Lie}(G)) \) the covariant derivative along the field \( \frac{\partial}{\partial x^\mu} \) is defined by
\[
\nabla_\mu \varphi = \partial_\mu \varphi + [A_\mu, \varphi].
\]
The tensor of curvature is defined by \( \text{Lie}(G) \)-valued 2-form
\[
F(x) = \sum_{\mu < \nu} F_{\mu\nu}(x)dx^\mu \wedge dx^\nu,
\]
where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. The connection $A$ is a solution to the Yang-Mills equations with the current $j(x) = j_\mu(x)dx^\mu$ if

$$g^{\lambda\mu} \nabla_\lambda F_{\mu\nu} = j_\nu. \quad (8)$$

For any curve $\sigma \in E$ an operator $U^A_{t,s}(\sigma)$, where $0 \leq s \leq t \leq 1$, is defined as a solution to the system of differential equations

$$
\begin{align*}
\frac{d}{dt} U^A_{t,s}(\sigma) &= -A_\mu(\sigma(t))\dot{\sigma}^\mu(t)U^A_{t,s}(\sigma), \\
\frac{d}{ds} U^A_{t,s}(\sigma) &= U^A_{t,s}(\sigma)\sigma_\mu(\sigma(s))\dot{\sigma}^\mu(s), \\
U^A_{t,s}(\sigma)|_{t=s} &= I_N.
\end{align*}
$$

Here and below the symbol $I_N$ denotes the unitary matrix in $M_N(\mathbb{C})$. The symbol $U^A_{t,s}(\sigma)$ denotes $U^A_{s,t}(\sigma)^{-1}$, where $0 \leq s \leq t \leq 1$. The operator $U^A_{1,0}(\sigma)$ is a parallel transport along $\sigma$.

**Proposition 1.** It is valid that $U^A_{1,0} \in C^\infty(E, M_N(\mathbb{C}))$. If $u, v \in E$, then

$$(U^A_{1,0})'(u) = -\int_0^1 U^A_{1,t}(\sigma)F_{\mu\nu}(\sigma(t))u^\mu(t)v^\nu(t)dt - A_\mu(\sigma(1))u^\mu(1)U^A_{1,0}(\sigma).$$

For a proof of this proposition we refer the reader to Ref. [17] and also to Ref. [14].

The results of Proposition 2 of Theorem 3 and of Theorem 4 for $j = 0$ belong to Accardi, Gibilisco and Volovich.

**Proposition 2.** If $\sigma \in E$ and $u, v \in E_0$, the following equality holds

$$
(U^A_{1,0})''(u, v) = \int_0^1 \int_0^1 K^V_{\mu\nu}(\sigma)(t, s)u^\mu(t)v^\nu(s)dt ds + \int_0^1 \int_0^1 K^L_{\mu\nu}(\sigma)(t)u^\mu(t)v^\nu(t)dt + \frac{1}{2} \int_0^1 \int_0^1 K^S_{\mu\nu}(\sigma)(t)\dot{u}^\mu(t)v^\nu(t) + \dot{v}^\mu(t)u^\nu(t)dt,
$$

where

$$
K^V_{\mu\nu}(\sigma)(t, s) = \begin{cases} U^A_{1,t}(\sigma)F_{\mu\lambda}(\sigma(t))\dot{\sigma}^\lambda(t)U^A_{1,s}(\sigma)F_{\nu\kappa}(\sigma(s))\dot{\sigma}^\kappa(s)U^A_{s,0}(\sigma), & \text{if } t \geq s \\
U^A_{1,t}(\sigma)F_{\mu\lambda}(\sigma(s))\dot{\sigma}^\lambda(s)U^A_{s,t}(\sigma)F_{\nu\kappa}(\sigma(t))\dot{\sigma}^\kappa(t)U^A_{t,0}(\sigma), & \text{if } t < s,
\end{cases}
$$

$$
K^L_{\mu\nu}(\sigma)(t) = \frac{1}{2} U^A_{1,t}(\sigma)\left(-\nabla_\mu F_{\nu\lambda}(\sigma(t))\dot{\sigma}^\lambda(t) - \nabla_\nu F_{\mu\lambda}(\sigma(t))\dot{\sigma}^\lambda(t)\right)U^A_{t,0}(\sigma),
$$

$$
K^S_{\mu\nu}(\sigma)(t) = U^A_{1,t}(\sigma)F_{\mu\nu}(\sigma(t))U^A_{t,0}(\sigma).
$$

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For a proof of this proposition we refer the reader to Ref. [3] and also to Ref. [29].

The following theorem is a direct corollary of Proposition 2.

**Theorem 3.** The following equality holds

\[
D_{tr}^2 U_{1,0}^A(\sigma) = - \int_0^1 U_{1,t}^A(\sigma) g^{\mu\nu} \nabla_\mu F_{\nu\lambda}(\sigma(t)) \dot{\sigma}^\lambda(t) U_{t,0}^A(\sigma) dt
\]

If \( \sigma \in E \), everywhere below the symbol \( \sigma^r \), where \( 0 \leq r \leq 1 \), denotes the curve from \( E \) defined by \( \sigma^r(t) = \sigma(rt) \).

**Theorem 4.** The following two assertions are equivalent:

1. the connection \( A \) is a solution to the Yang-Mills equations with the current (8);

2. the equality

\[
D_{tr}^2 U_{1,0}^A(\sigma) = - \int_0^1 U_{1,t}^A(\sigma) j_\nu(\sigma(t)) \dot{\sigma}^\nu(t) U_{t,0}^A(\sigma) dt. \tag{11}
\]

holds for the parallel transport \( U_{1,0}^A \).

**Proof.** Let (11) hold. Note that \( \dot{\sigma}^\nu(t) = r \dot{\sigma}^\nu(rt) \) and \( U_{(rt),(rs)}(\sigma) = U_{t,s}^A(\sigma^r) \).

Let \( \sigma \in E \cap C^1([0,1], \mathbb{R}^d) \). Then

\[
\int_0^r U_{1,t}^A(\sigma)(-g^{\lambda\mu} \nabla_\lambda F_{\mu\nu}(\sigma(t)) \dot{\sigma}^\nu(t) U_{t,0}^A(\sigma) dt = \]

\[
= U_{1,r}^A(\sigma) D_{tr}^2 U_{1,0}^A(\sigma^r) = -U_{1,r}^A(\sigma) \int_0^1 U_{1,s}^A(\sigma^s) j_\nu(\sigma^s(t)) \dot{\sigma}^\nu(t) U_{s,0}^A(\sigma^s) dt = \]

\[
= \int_0^r U_{1,t}^A(\sigma)(-g^{\lambda\mu} \nabla_\lambda F_{\mu\nu}(\sigma(t)) \dot{\sigma}^\nu(t) U_{t,0}^A(\sigma) dt. \tag{12}
\]

Differentiating (12) with respect to \( r \), we obtain

\[
U_{1,r}^A(\sigma) g^{\lambda\nu} \nabla_\lambda F_{\mu\nu}(\sigma(r)) \dot{\sigma}^\nu(r) U_{r,0}^A(\sigma) = U_{1,r}^A(\sigma) j_\nu(\sigma^r) \dot{\sigma}^\nu(r) U_{r,0}^A(\sigma). \]

Choose an appropriate \( \sigma \in E \cap C^1([0,1], \mathbb{R}^d) \), we see that the connection \( A \) is a solution to the Yang-Mills equations with current (8). The other side of the theorem is obvious. \qed
Consider the function $B^A: E \to L(E_0, \text{Lie}(G))$ defined by

$$B^A(\sigma)u = U^A_{0,1}(\sigma)\partial_uU^A_{1,0}(\sigma), \quad \sigma \in E, \ u \in E_0.$$ 

It is known that $B^A \in C^\infty(E, L(E_0, \text{Lie}(G)))$ (see Ref. [17]).

**Proposition 3.** The following equalities hold

$$\text{div}^E A^A(\sigma) = -\int_0^1 U^A_{0,t}(\sigma)\partial_uU^A_{1,0}(\sigma)dt = U^A_{0,1}(\sigma)D^2_{tr}U^A_{1,0}(\sigma). \ (13)$$

**Proof.** If $u, v \in E_0$, the following equality holds

$$\partial_u B^A(\sigma)v = \partial_u U^A_{0,1}(\sigma)\partial_uU^A_{1,0}(\sigma) + U^A_{0,1}(\sigma)\partial_u\partial_uU^A_{1,0}(\sigma). \ (14)$$

Then, using (9), we obtain that for all $\sigma \in E$ and $u, v \in E_0$ the following holds

$$\partial_u B^A(\sigma)v = \int_0^1 \int_0^1 R^{V}_{\mu\nu}(t, s)u^\mu(t)v^\nu(s)dtds + \int_0^1 R^{L}_{\mu\nu}(t)u^\mu(t)v^\nu(t)dt + \frac{1}{2}\int_0^1 R^{S}_{\mu\nu}(t)(\dot{u}^\mu(t)v^\nu(t) + \dot{v}^\nu(t)u^\mu(t))dt, \ (15)$$

where

$$R^{V}_{\mu\nu}(t, s) = \begin{cases} [U^A_{0,s}(\sigma)F_{\mu\nu}(\sigma(s))\dot{\sigma}^\nu(s)U^A_{s,0}(\sigma), U^A_{0,t}(\sigma)F_{\mu\lambda}(\sigma(t))\dot{\sigma}^\lambda(t)U^A_{1,0}(\sigma)], & \text{if } t \leq s \\ 0, & \text{if } t > s, \end{cases}$$

$$R^{L}_{\mu\nu}(t) = \frac{1}{2} U^A_{0,t}(\sigma)(-\nabla_\mu F_{\nu\lambda}(\sigma(t))\dot{\sigma}^\lambda(t) - \nabla_\nu F_{\mu\lambda}(\sigma(t))\dot{\sigma}^\lambda(t))U^A_{1,0}(\sigma),$$

$$R^{S}_{\mu\nu}(t) = U^A_{0,t}(\sigma)F_{\mu\nu}(\sigma(t))U^A_{1,0}(\sigma).$$

Thus, we obtain (13). \hfill \Box

The following result belongs to Gross (see Ref. [17]). It is an analogue of the Poincaré lemma.

**Theorem 5.** Let $E \in \{W^{1,1}_0([0,1], \mathbb{R}^d), W^{1,2}_0([0,1], \mathbb{R}^d)\}$. Let a function $B$ from $E$ to $L(E_0, \text{Lie}(G))$ (an infinite-dimensional 1-form) be defined by

$$B(\sigma)u = \int_0^1 h(\sigma^r) < \dot{\sigma}(r), u(r) > dr, \quad \sigma \in E, \ u \in E_0,$$

where $h(\sigma)$ is a Lie(G)-valued anti-symmetric tensor for any $\sigma \in E$, the function $E \ni \sigma \mapsto h(\sigma)$ is continuously Frechét differentiable.\footnote{The function $h$ is called the path 2-form.}
Let 1-form $B$ be closed in the sense that for all $\sigma \in E$ and $u,v \in E_0$ the following equality holds

$$\partial_u B(\sigma)v - \partial_v B(\sigma)u + [B(\sigma)u, B(\sigma)v] = 0$$

Then there is a connection $A$ such that $B = B^A$.

In the Gross’s theorem the space $P$ of piecewise $C^\infty$-smooth functions with the norm $\|\sigma\| = \int_0^1 |\dot{\sigma}(t)| dt$ instead of $E$ was considered. The proof can be transferred unchanged to the case of the spaces $W^{1,1}_0([0,1], \mathbb{R}^d)$ and $W^{1,2}_0([0,1], \mathbb{R}^d)$.

**Remark 6.** It is known (see Ref. [17]) that if an infinite-dimensional 1-form $B$ can be represented as (16) then $B \in C^\infty(E, \text{Lie}(G))$. It was also proved in Ref. [17] that for two connection $A$ and $A'$ such that $B^A = B^{A'}$ there exists a unique function $b \in C^\infty(\mathbb{R}^d, G)$ with $b(0) = I_N$ such that $A(x) = b^{-1}(x)A'(x)b(x) + b^{-1}(x)db(x)$.

**Theorem 6.** Let

$$E \in \{ W^{1,1}_0([0,1], \mathbb{R}^d), W^{1,2}_0([0,1], \mathbb{R}^d) \}. $$

Let $B \in C^\infty(E, \text{Lie}(G))$ can be represented as (16). Then $B$ is a solution to the system

$$\begin{align*}
\partial_u B(\sigma)v - \partial_v B(\sigma)u + [B(\sigma)u, B(\sigma)v] &= 0, \quad \text{if } u,v \in E_0 \\
\text{div}_L^\sigma B(\sigma) &= 0
\end{align*}$$

if and only if there is a connection $A$ which is a solution to the Yang-Mills equations

$$g^{\lambda\mu} \nabla_\lambda F_{\mu\nu} = 0,$$

such that $B = B^A$.

**Proof.** Let $B$ be a solution to system (17). The existence of the $C^\infty$-smooth connection $A$ such that $B = B^A$ follows from Theorem 5. Then, due to Proposition 3 the equality $D^2_{\nu_L^\sigma} U_{1,0}^A(\sigma) = 0$ follows from the equality $\text{div}_L^\sigma B^A(\sigma) = 0$. This implies $A$ is a solution to (18). The other side of the theorem is trivial.

**Remark 7.** The parallel transport can be considered as an infinite-dimensional analogue of the general chiral field. Recall that if $G$ is a matrix Lie group, the
field \( b : \mathbb{R}^d \to G \) is a general chiral field. Its Dirichlet integral (see Ref. [15]) has a form
\[
\frac{1}{2} \int_{\mathbb{R}^d} \text{tr}(\partial_\mu b(x) \partial^\mu b^{-1}(x)) dx = -\frac{1}{2} \int_{\mathbb{R}^d} \text{tr}(Z_\mu(x) Z^\mu(x)) dx,
\]
where \( Z_\mu = b^{-1}(x) \partial_\mu b(x) \). The equations of motion of the chiral field have the form
\[
\begin{align*}
\text{div} Z &= 0 \\
\partial_\mu Z_\nu - \partial_\nu Z_\mu + [Z_\mu, Z_\nu] &= 0,
\end{align*}
\]
where \( Z = (Z_1, \ldots, Z_d) \). System (17) is an infinite-dimensional analogue of system (20).

Conservation laws for the 3-dimensional Yang-Mills theory were considered in Ref. [9] (see also Refs. [21, 22]). In particular, it was shown that \( B^A \) is the first non-trivial conserved current if the connection \( A \) is a solution to the Yang-Mills equations. The conservation of the current was understood in the sense that the value of the functional divergence on this current is zero. Thus, Theorem 6 is a generalization of this result.

4 The Yang-Mills-Higgs equations

Let \( V \) be a finite-dimensional vector space. For any \( h \in \mathbb{R}^d \) consider a \( V \)-valued linear functional \( S_h \) defined in the following way. The domain \( \text{dom} S_h \) consists of all \( T \in L(E, V) \) having the following form
\[
T(u) = \int_0^1 (u(t), \nu_T(dt))_{\mathbb{R}^d}, \quad u \in E,
\]
where \( \nu_T \) is a \( V \times (\mathbb{R}^d) \)-valued Borel measure. The functional \( S_h \) acts on \( T \in \text{dom} S_h \) as
\[
S_h(T) = (h, \nu_T(\{1\}))_{\mathbb{R}^d}.
\]

**Remark 8.** Let \( E = W_{0}^{1,2}([0,1], \mathbb{R}^d) \). In terms of the orthonormal basis \( \{f_n\} \) of \( W_{0}^{1,2}([0,1], \mathbb{R}) \) (see Remark 5) the operator \( S_h \) can be defined in the following way. For any \( h \in \mathbb{R}^d \) consider a linear operator \( S_h^{\{f_n\}} : \text{dom} S_h^{\{f_n\}} \to V \) defined by
\[
S_h^{\{f_n\}}(T) = T(h f_0) + \sum_{n=1}^\infty \sqrt{2} (-1)^n T(h f_n),
\]
where the domain \( \text{dom} S_h^{\{f_n\}} \) consists of those \( T \in L(E, V) \) for which the right side of (21) exists. Then \( S_h^{\{f_n\}} \) coincides with \( S_h \) on \( \text{dom} S_h \).
Definition 8. Choose $X = E$, $Z = Y = V$. The endpoint derivation $D_h$ in the direction $h \in \mathbb{R}^d$ is the differential operator $D_{S_h}^1$. (The symbol $D_\mu$ denotes $D_{p_\mu}$.)

In this and the next sections we assume that the Greek indices run from 0 to 3. We assume the basis $\{p_0, p_1, p_2, p_3\}$ is given in the Minkowski space $\mathbb{R}^{1,3}$ and the Minkowski metric $\eta$ is given by $\text{diag}\{1, -1, -1, -1\}$ in this basis. We will raise and lower indices using the Minkowski metric. We assume that the connection is defined as a $\text{su}(N)$-valued $C^\infty$ smooth 1-form $A_\mu(x)dx^\mu$ on $\mathbb{R}^{1,3}$. Below

$$E \in \{W^{1,1}_0([0, 1], \mathbb{R}^{1,3}), W^{1,2}_0([0, 1], \mathbb{R}^{1,3})\}.$$  

The Yang-Mills-Higgs equations are the following equations on the $C^\infty$-smooth $\text{su}(N)$-valued 1-form $A$ and $C^\infty$-smooth $\text{su}(N)$-valued field $\phi$:

$$\begin{cases}
\nabla^\mu \nabla_\mu \phi(x) - (m^2 - l \text{tr}(\phi^*(x)\phi(x)))\phi(x) = 0 \\
\nabla^\mu F_{\mu\nu}(x) - [\phi(x), \nabla_\nu \phi(x)] = 0,
\end{cases}$$

(22)

where $m, l \geq 0$.

Remark 9. System of the differential equations (22) is the Euler-Lagrange equations for the Lagrangian with the Lagrangian density:

$$L(A_\nu, \partial_\mu A_\nu, \phi, \partial_\mu \phi) = \frac{1}{4} \text{tr}(F_{\mu\nu}^* F^{\mu\nu}) + \frac{1}{2} \frac{1}{4} (\text{tr}(\nabla_\mu \phi)^* \nabla^\mu \phi) + \\
+ m^2 \frac{1}{2} \text{tr}(\phi^* \phi) - l \frac{1}{4} (\text{tr}(\phi^* \phi))^2.$$

In terms of the parallel transport and the Lévy d’Alembertian solutions to the Yang-Mills-Higgs equations can be described as follows.

Let the function $\Phi^{A,\phi}: E \to \text{su}(N)$ be defined by

$$\Phi^{A,\phi}(\sigma) = U^{A}_{\mu,0}(\sigma)\phi(\sigma(1))U^{A}_{1,0}(\sigma), \sigma \in E.$$

Theorem 7. A pair $(A, \phi)$ is a solution to system (22) if and only if for the parallel transport $U^{A}_{\mu,0}$ and the function $\Phi^{A,\phi}$ the following relations hold

$$\begin{cases}
\n\eta^{\mu\nu} D_\mu D_\nu \Phi^{A,\phi}(\sigma) - (m^2 - l \text{tr}(\Phi^{A,\phi})^*(\sigma)\Phi^{A,\phi}(\sigma)))\Phi^{A,\phi}(\sigma) = 0 \\
\n\square L U^{A}_{\mu,0}(\sigma) + U^{A}_{\mu,0}(\sigma) \int_0^1 [\Phi^{A,\phi}(\sigma r), D_\nu \Phi^{A,\phi}(\sigma r)] d\nu(r) dr = 0.
\end{cases}$$

(23)
Proof. Since
\[ d_u \Phi^A,\phi(\sigma) = [B^A(\sigma)(u), \Phi^A,\phi(\sigma)] + U_0^A(\sigma)\nabla_\mu \phi(\sigma(1))u^\mu(1)U_1^A(\sigma), \]
we have
\[ D_\nu \Phi^A,\phi(\sigma) = U_{0,1}^A(\sigma)\nabla_\nu \phi(\sigma(1))U_{1,0}^A(\sigma). \] (24)
Since
\[ d_u D_\nu \Phi^A,\phi(\sigma) = [B^A(\sigma)(u), D_\nu \Phi^A,\phi(\sigma)] + U_{0,1}^A(\sigma)\nabla_\mu \nu \phi(\sigma(1))u^\mu(1)U_{1,0}^A(\sigma), \]
we obtain
\[ D_\mu D_\nu \Phi^A,\phi(\sigma) = U_{0,1}^A(\sigma)\nabla_\mu \nabla_\nu \phi(\sigma(1))U_{1,0}^A(\sigma). \] (25)
Relations (24) and (25) imply that the pair \((A, \phi)\) satisfies the first equation of system (22) if and only if the pair \((U_{1,0}, \Phi)\) satisfies the first equation of system (23). Since
\[ \int_0^1 U_{1,0}^A(\sigma)\phi(\sigma(r)), \nabla_\mu \phi(\sigma(r))|\dot{\sigma}(r)|U_{1,0}^A(\sigma)dr = \]
\[ = U_{1,0}^A(\sigma) \int_0^1 [\Phi^A,\phi(\sigma(r)), D_\nu \Phi^A,\phi(\sigma(r))]|\dot{\sigma}(r)|dr, \]
the pair \((A, \phi)\) satisfies the second equation of system (22) if and only if the pair \((U_{1,0}^A, \Phi^A,\phi)\) satisfies the second equation of system (23). \(\Box\)

Solutions to the Yang-Mills-Higgs equations can be described in terms of the Lévy divergence as follows.

**Theorem 8.** Let \(B \in C^\infty(E, L(E_0, su(N))\) can be represented as (16). Then a pair \((B, \Phi)\), where \(\Phi \in C^\infty(E, su(N))\), is a solution to the system
\[
\begin{cases}
\partial_u B(\sigma)v - \partial_v B(\sigma)u + [B(\sigma)u, B(\sigma)v], \text{if } u, v \in E_0 \\
div_L B(\sigma) + \int_0^1 [\Phi(\sigma(r)), D_\nu \Phi(\sigma(r))]|\dot{\sigma}(r)|dr = 0 \\
\eta^\mu\nu D_\mu D_\nu \Phi(\sigma) - (m^2 - l tr(\Phi^*(\sigma)\Phi(\sigma)))\Phi(\sigma) = 0 \\
\partial_u \Phi(\sigma) + [B(\sigma)u, \Phi(\sigma)] = 0, \text{if } u \in E_0
\end{cases}
\] (26)
where \(m, l \geq 0\), if and only if there is a pair \((A, \phi)\), where \(A\) is a \(su(N)\)-valued \(C^\infty\)-smooth 1-form and \(\phi \in C^\infty(\mathbb{R}^{1,3}, su(N))\), which is a solution of the Yang-Mills-Higgs equations (22) such that \(B = B^A\) and \(\Phi = \Phi^A,\phi\).
Proof. Let the pair \((B, \Phi)\) be a solution to system (26). Due to Theorem 5, the second equation of the system implies that there is a connection \(A\) such that \(B = B^A\). Consider the function \(\Phi_1: E \to su(N)\) defined by the formula

\[ \Phi_1(\sigma) = U^A_{1,0}(\sigma)\Phi(\sigma)U^1_{0,1}(\sigma). \]

Then \(\Phi_1(\sigma) \in C^\infty(E, su(N))\). Due to the third equation of system (26), if \(u \in E_0\), then

\[ \partial_u \Phi_1(\sigma) = U^A_{1,0}(\sigma)(\partial_u \Phi(\sigma) + [B(\sigma)u, \Phi(\sigma)])U^1_{0,1}(\sigma) = 0. \]

This implies that the field \(\phi: \mathbb{R}^{1,3} \to su(N)\) is well defined by \(\phi(x) = \Phi_1(\sigma_x)\), where \(\sigma_x\) is an arbitrary curve from \(E\) with \(\sigma_x(1) = x\). It is valid that \(\Phi = \Phi^A, \phi\). Since \(\Phi, U^A_{1,0}\) and \(U^A_{0,1}\) are \(C^\infty\)-smooth functions, \(\phi \in C^\infty(\mathbb{R}^{1,3}, su(N))\). Note that the equality

\[ \text{div}^\mathbb{H}_x B^A(\sigma) + \int_0^1 [\Phi^A, \phi(\sigma^r), D_\nu \Phi^A, \phi(\sigma^r)]\dot{\sigma}^r(\nu)dr = 0 \]

holds if and only if the equality

\[ \square L U^A_{1,0}(\sigma) + U^A_{1,0}(\sigma) \int_0^1 [\Phi^A, \phi(\sigma^r), D_\nu \Phi^A, \phi(\sigma^r)]\dot{\sigma}^r(\nu)dr = 0 \]

holds. Then we can apply Theorem 7. It follows that \((A, \phi)\) is a solution to the Yang-Mills-Higgs equations.

If the pair \((A, \phi)\) is a solution to the Yang-Mills-Higgs equations, it can be checked by direct calculations that the pair \((B^A, \Phi^A, \phi)\) is a solution to system (26). \(\square\)

Remark 10. Let \((A, \phi)\) and \((A', \phi')\) be such that \(B^A = B^{A'}\) and \(\Phi^A, \phi = \Phi^{A', \phi'}\). Then there exists a unique function \(b \in C^\infty(\mathbb{R}^{1,3}, SU(N))\) with \(b(0) = \text{I}_N\) such that \(A(x) = b^{-1}(x)A'(x)b(x) + b^{-1}(x)db(x)\) and \(\phi(x) = b^{-1}(x)\phi'(x)b(x)\).

5 The Yang-Mills-Dirac equations

Let \(\{g_\alpha\}_{\alpha=1}^4\) be an orthonormal basis in \(\mathbb{C}^4\). If \(\varphi \in \mathbb{C}^N \otimes \mathbb{C}^4\), the symbols \(\varphi_\alpha (\alpha = 1, \ldots, 4)\) denote such vectors from \(\mathbb{C}^N\) that \(\varphi = \sum_{\alpha=1}^4 \varphi_\alpha \otimes g_\alpha\). Let \(\gamma^\mu\) be the Dirac matrices (we assume \(\gamma^\mu \in M_4(\mathbb{C})\) and \(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}I_4\), \(\gamma_0^* = \gamma_0, \gamma_\mu^* = -\gamma_\mu\)). The symbol \(\slashed{\gamma}_\mu \varphi\) denotes the operator acting in \(\mathbb{C}^N\) defined by \(\slashed{\gamma}_\mu \varphi = \sum_{\alpha=1}^4 ((I_N \otimes \gamma_\mu) \varphi_\alpha) \otimes \varphi_\alpha^*\). I. e.,

\[ \left( \sum_{\alpha=1}^4 (I_N \otimes \gamma_\mu) \varphi_\alpha \otimes \varphi_\alpha^* \right) \xi = \sum_{\alpha=1}^4 (\xi, \varphi_\alpha)((I_N \otimes \gamma_\mu) \varphi_\alpha)_{\alpha}, \text{if } \xi \in \mathbb{C}^N. \]
For all $\varphi \in \mathbb{C}^N \otimes \mathbb{C}^4$ it holds that $\nabla^{\gamma}_{\mu} \varphi \in u(N)$. The symbol $pr_{su(N)}$ denotes the orthogonal projection in $u(N)$ on $su(N)$.

The QCD equations (the Yang-Mills-Dirac equations) are the following differential equations on a $C^\infty$-smooth $su(N)$-valued 1-form $A$ and a $C^\infty$-smooth $\mathbb{C}^N \otimes \mathbb{C}^4$-valued function $\psi$ on $\mathbb{R}^{1,3}$:

\[
\begin{cases}
(I_N \otimes \gamma^\mu)(\partial_{\mu} + A_{\mu} \otimes I_4)\psi + i m \psi = 0 \\
\nabla^\mu F^\mu_\nu = -pr_{su(N)}(i(\overline{\psi} \gamma^\nu \psi))
\end{cases}
\]  \(27\)

**Remark 11.** System of differential equations (27) is the Euler-Lagrange equations for the Lagrangian with the Lagrangian density

\[
\mathcal{L}(A_\nu, \partial_{\mu} A_\nu, \psi, \partial_{\mu} \psi) = \frac{1}{4} tr(F^*_{\mu\nu} F^\mu_{\nu}) + \overline{\psi}(i(I_N \otimes \gamma^\mu)((\partial_{\mu} + A_{\mu} \otimes I_4) - m)\psi),
\]

where $\overline{\psi} = \psi^*(I_N \otimes \gamma_0)$.

In terms of the parallel transport and the Lévy d’Alembertian the solutions to the Yang-Mills-Dirac equations can be described in the following way.

The function $\Psi^{A,\psi}: \mathbb{R} \to \mathbb{C}^N \otimes \mathbb{C}^4$ is defined by the formula

\[
\Psi^{A,\psi}(\sigma) = (U^A_{0,1}(\sigma) \otimes I_4)\psi(1), \quad \sigma \in \mathbb{R}.
\]

**Theorem 9.** A pair $(A, \psi)$, where $\psi \in C^\infty(\mathbb{R}^{1,3}, \mathbb{C}^N \otimes \mathbb{C}^4)$, is a solution to the Yang-Mills-Dirac equations \(27\) if and only if for the parallel transport $U^A_{1,0}$ and the function $\Psi^{A,\psi}$ the following system holds

\[
\begin{cases}
(I_N \otimes \gamma^\mu) D_{\mu} \Psi^{A,\psi} + i m \Psi^{A,\psi} = 0 \\
\Box_L U^A_{1,0}(\sigma) = U^A_{1,0}(\sigma)pr_{su(N)}(i \int_0^1 \Psi^{A,\psi}(\sigma^r)\gamma^\nu \Psi^{A,\psi}(\sigma^r)\sigma^\nu(r)dr). \quad (28)
\end{cases}
\]

**Proof.** We obtain by direct calculation

\[
D_{\mu} \Psi^{A,\psi}(\sigma) = (U^A_{0,1}(\sigma) \otimes I_4)\sigma_{\mu} \psi(1) + (A_{\mu}(\sigma(1)) \otimes I_4)\psi(1)).
\]

This implies the pair $(A, \psi)$ satisfies the first equation of system \(27\) if and only if the pair $(U^A_{1,0}, \Psi^{A,\psi})$ satisfies the first equation of system \(28\). Due to Theorem 4 the pair $(A, \psi)$ is a solution to the second equation of system \(27\) if and only if the following equality holds for all $\sigma \in \mathbb{R}$

\[
\Box_L U^A_{1,0}(\sigma) = \int_0^1 U^A_{1,r}(\sigma)(pr_{su(N)}(i\overline{\psi}(\sigma(r))\gamma^\nu \psi(\sigma(r))\sigma^\nu(r)))U^A_{r,0}(\sigma)dr.
\]
For any $b \in SU(N)$ and any $\varphi \in \mathbb{C}^N$ the following equality holds (see for example Ref. [17])

$$ pr_{su(N)}(i(b \otimes I_4)\varphi_\mu (b \otimes I_4)\varphi) = pr_{su(N)}(ib\overline{\gamma}_\mu \varphi b^{-1}). $$

Hence,

$$ U_{1,0}^A(\sigma) \int_0^1 U_{0,r}(\sigma) pr_{su(N)}(i\overline{\psi((r)})\gamma_\nu \psi((r))\sigma'^\nu (r)U_{r,0}^A(\sigma))dr = $$

$$ = U_{1,0}^A(\sigma) pr_{su(N)}(i \int_0^1 \overline{\Psi(A,\psi)}(\sigma')(\gamma_\nu \Psi(A,\psi))\sigma'^\nu (r)dr). $$

This implies that the pair $(A, \psi)$ satisfies the second equation of system (27) if and only if the pair $(U_{1,0}^A, \Psi(A,\psi))$ satisfies the second equation of system (28).

In terms of the Lévy divergence the solutions to the Yang-Mills-Dirac equations can be described in the following way.

**Theorem 10.** Let $B \in C^\infty(E,E_0,su(N))$ can be represented as (10). Then a pair $(B, \Psi)$, where $\Psi \in C^\infty(E,\mathbb{C}^N \otimes \mathbb{C}^4)$, is a solution to the system of the equations:

$$ \left\{ \begin{array}{l}
\operatorname{div}^\mu B(\sigma) = pr_{su(N)}(i \int_0^1 \overline{\Psi(\sigma')} \gamma_\nu \Psi(\sigma')\sigma'^\nu (r)dr) \\
\partial_\mu B(\sigma)v - \partial_v B(\sigma)u + [B(\sigma)u, B(\sigma)v], \text{if } u, v \in E_0 \\
\partial_\mu \Psi(\sigma) + (B(\sigma)u)\Psi(\sigma) = 0, \text{if } u \in E_0 \\
(I_N \otimes \gamma^\mu)D_\mu \Psi + im\Psi = 0,
\end{array} \right. $$

(29)

if and only if there is a pair $(A, \psi)$ which is a solution to the Yang-Mills-Dirac equations (27) such that $B = B^A$ and $\Psi = \Psi(A,\psi)$.

**Proof.** Let a pair $(B, \Psi)$ be a solution to system (29). Due to Theorem 5 from the second equation of system (29) it follows that there is a connection $A$ such that $B = B^A$. Consider the function $\Psi_1: E \to su(N)$ defined by $\Psi_1(\sigma) = (U_{1,0}^A(\sigma) \otimes I_4)\Psi(\sigma)$. Then $\Psi_1(\sigma) \in C^\infty(E, su(N))$. Due to the third equation of system (26), if $u \in E_0$, then

$$ \partial_\mu \Psi_1(\sigma) = U_{1,0}^A(\sigma)(\partial_\mu \Psi(\sigma) + (B^A(\sigma)u)\Psi(\sigma)) = 0. $$

Then the field $\psi: \mathbb{R}^{1,3} \to \mathbb{C}^N \otimes \mathbb{C}^4$ is well defined by $\psi(x) = \Psi_1(\sigma_x)$ for arbitrary $\sigma_x \in E$ with $\sigma_x(1) = x$. It holds that $\Psi = \Psi(A,\psi)$. Since $\Psi$ and $U_{1,0}^A$
are $C^\infty$-smooth functions, $\psi \in C^\infty(\mathbb{R}^{1,3}, \mathbb{C}^N \otimes \mathbb{C}^4)$. Note that the equality

$$\text{div}_L B^A(\sigma) = pr_{su(N)}(i \int_0^1 \Psi_{A,\psi}(\sigma^r) \gamma_\nu \Psi^A_{A,\psi}(\sigma^r) \dot{\sigma}^\nu(r) dr).$$

holds if and only if the following is true

$$\Box_L U^A_{1,0}(\sigma) = U^A_{1,0}(\sigma) pr_{su(N)}(i \int_0^1 \Psi_{A,\psi}(\sigma^r) \gamma_\nu \Psi^A_{A,\psi}(\sigma^r) \dot{\sigma}^\nu(r) dr).$$

Then we can apply Theorem 9 and obtain that $(A, \psi)$ is a solution of the Yang-Mills-Dirac equations.

If a pair $(A, \psi)$ is a solution to the Yang-Mills-Dirac equations, it can be checked by direct calculations that the pair $(B^A, \Psi^{A,\psi})$ is a solution to system (29).

**Remark 12.** In Ref. [17] systems of infinite-dimensional equations equivalent to the Yang-Mills-Higgs equations and to the Yang-Mills-Dirac equations were obtained. The third and the fourth equation of system (29) and system (26) were derived in that work.

**Remark 13.** Let $(A, \psi)$ and $(A', \psi')$ be such that $B^A = B^{A'}$ and $\Psi^{A,\psi} = \Psi^{A',\psi'}$. Then there exists a unique function $g \in C^\infty(\mathbb{R}^{1,3}, SU(N))$ with $b(0) = I_N$ such that $A(x) = b^{-1}(x)A'(x)b(x) + b^{-1}(x)db(x)$ and $\psi(x) = (b^{-1}(x) \otimes I_4)\psi'(x)$.

**Remark 14.** Due to Remarks 5 and 8 all differential operators in systems (29) and (26) can be defined by the basis $\{f_n\}$ in $W^{1,2}_{0}([0, 1], \mathbb{R})$.

**Conclusion**

We provided the classification of the Levy differential operators and obtained the equations contained these operators that are equivalent to the QCD equations and the Yang-Mills equations. It would be interesting to investigate is it possible to find the stochastic analogue of Gross’s noncommutative Poincare lemma. With the help of such analogue it would be possible to develop the results of the paper Ref. [30], where the stochastic parallel transport was considered as a general chiral field. Also it would be interesting to investigate the connection between the stochastic Lévy Laplacians introduced in Ref. [19] and in Ref. [31]. In the next paper we develop some results of the current paper for the case of the Riemannian and the pseudo-Riemannian manifold.
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References

[1] L. Accardi, Yang-Mills equations and Lévy-Laplacians, in: Dirichlet Forms and Stochastic Processes (Beijing, 1993) de Gruyter, Berlin, 1995, pp. 1–24

[2] L. Accardi, P. Gibilisco, I.V. Volovich, The Lévy Laplacian and the Yang-Mills equations, Rendiconti Lincei 4 (1993) 201–206 doi: 10.1007/BF03001574

[3] L. Accardi, P. Gibilisco, I. V. Volovich, Yang-Mills gauge fields as harmonic functions for the Levy-Laplacians, Russ. J. Math. Phys. 2 (1994) 235–250.

[4] L. Accardi, U.C. Ji and K. Saitô, Higher order multi-dimensional extensions of Cesàro theorem, Infinite Dimensional Analysis, Quantum Probability and Related Topics 18 (2015) 1550030 [14 pages]. doi: 10.1142/S0219025715500307

[5] L. Accardi, O.G. Smolyanov, On Laplacians and traces, Conf. Semin. Univ. Bari 250 (1993) 1–25.

[6] L. Accardi, O.G. Smolyanov, Classical and nonclassical Levy Laplacians, Doklady Mathematics 76 (2007) 801–805. doi:10.1134/S1064562407060014

[7] L. Accardi, O. G. Smolyanov, Generalized Lévy Laplacians and Cesaro means, Doklady Mathematics 79 (2009) 90–93. doi:10.1134/S106456240901027X

[8] L. Accardi, O.G. Smolyanov, Feynman formulas for evolution equations with Levy Laplacians on infinite-dimensional manifolds, Doklady Mathematics 73 (2006) 252–257. doi:10.1134/S106456240602027X

[9] I. Ya. Arefieva, I. V. Volovich, Higher order functional conservation laws in gauge theories, in: Proc. Int. Conf. Generalized Functions and their Applications in Mathematical Physics, Academy of Sciences of the USSR, 1981, pp. 43–49 (In Russian).
[10] M. Arnaudon, R. O. Bauer and A. Thalmaier, A probabilistic approach to the Yang-Mills heat equation, J. Math. Pures Appl. 81 (2002) 143–166. doi:10.1016/S0021-7824(02)01254-0

[11] M. Arnaudon and A. Thalmaier, Yang–Mills fields and random holonomy along Brownian bridges, The Annals of Probability 31 (2) (2003) 769–790. doi:10.1214/aop/1048516535

[12] V. I. Averbukh, O. G. Smolyanov and S. V. Fomin. Generalized functions and differential equations in linear spaces. II, Differential operators and their Fourier transform, Trudy Moskov. Mat. Obshch. 27 (1972) 247–262 (in Russian).

[13] R. O. Bauer, Characterizing Yang-Mills Fields by Stochastic Parallel Transport, J. Funct. Anal., 155(1) (1998) 536–549.

[14] B. Driver, Classifications of Bundle Connection Pairs by Parallel Translation and Lassos, Journal of Functional Analysis 83 (1989) 185–231 doi:10.1016/0022-1236(89)90035-9

[15] B. A. Dubrovin, S. P. Novikov and A. T. Fomenko, Sovremennaja geometrija: Metody i prilozhenija. 2-e izd. [Modern Geometry: Methods and Applications. 2nd ed.], Moscow, Nauka publ., 1986 (In Russian).

[16] M. N. Feller, The Lévy Laplacian, Cambridge Tracts in Math 166, Cambridge, Cambridge Univ. Press, 2005.

[17] L. Gross, A Poincarè lemma for connection forms, Journal of Functional Analysis 63 (1985) 1–46. doi:10.1016/0022-1236(85)90096-5

[18] H.-H. Kuo, N. Obata and K. Saitô, Lévy-Laplacian of Generalized Functions on a Nuclear Space, Journal of Functional Analysis 94 (1990) 74–92. doi:10.1016/0022-1236(90)90028-J

[19] R. Leandre, I. V. Volovich, The Stochastic Levy Laplacian and Yang-Mills equation on manifolds. Infinite Dimensional Analysis, Quantum Probability and Related Topics 4 (2001) 151–172. doi:10.1142/S0219025701000449

[20] P. Lévy, Probèmes concrets d’analyse fonctionnelle, Paris, Gauthier-Villars, 1951.

[21] A.M. Polyakov, String representations and hidden symmetries for gauge fields, Phys. Lett. B 82 (1979) 247-250. doi:10.1016/0370-2693(79)90747-0

22
[22] A. M. Polyakov, Gauge fields as rings of glue, Nuclear Physics B. 164 (1980) 171–188. doi:10.1016/0550-3213(80)90507-6

[23] O. G. Smolyanov, Linear differential operators in spaces of measures and functions on Hilbert space, Uspekhi Mat. Nauk 28 (1973) 251–252 (in Russian).

[24] H.H. Schaefer with M.P. Wolff, Topological Vector Spaces Graduate Texts in Mathematics (Book 3) Springer, 2nd edition, 1999.

[25] B. O. Volkov, Lévy-Laplacian and the Gauge Fields, Infinite Dimensional Analysis, Quantum Probability and Related Topics 15 (2012) 1250027-1/19. doi: 10.1142/S0219025712500270

[26] B. O. Volkov, Hierarchy of Lévy-Laplacians and Quantum Stochastic Processes, Infinite Dimensional Analysis, Quantum Probability and Related Topics 16 (2013) 1350027-1/20. doi: 10.1142/S0219025713500276

[27] B. O. Volkov, Lévy Laplacians and related constructions, PhD thesis, 2014 (in Russian).

[28] B. O. Volkov, Lévy d’Alambertians and their application in the quantum theory, Vestn. Samar. Gos. Tekhn. Univ. Ser. Fiz.-Mat. Nauki 19 (2015) 241–258 (in Russian). doi: 10.14498/vsgtu1372

[29] B. O. Volkov, Lévy Laplacians and instantons, Proceedings of the Steklov Institute of Mathematics 290 (2015) 210–222. doi: 10.1134/S037196851503019X

[30] B.O. Volkov, Stochastic Lévy Differential Operators and Yang-Mills Equations, Infin. Dimens. Anal. Quantum Probab. Relat. Top., Vol. 20, No. 2 (2017) 1750008 (23 pages) DOI: 10.1142/S0219025717500084

[31] B.O. Volkov, Lévy Laplacians in Hida calculus and Malliavin calculus, Proceedings of the Steklov Institute of Mathematics 301 (2018) 11–24 doi: 10.1134/S0081543818040028