Propagation of moments for large data and semiclassical limit to the relativistic Vlasov equation

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Abstract

We investigate the semiclassical limit from the semi-relativistic Hartree-Fock equation describing the time evolution of a system of fermions in the mean-field regime with a relativistic dispersion law and interacting through a singular potential of the form

\[ K_p x q \sim \gamma |x|^a, \]

\[ a \in (\max\{d - 2, -1\}, d - 2), \quad d \in \{2, 3\} \quad \text{and} \quad \gamma \in \mathbb{R}, \]

with the convention \( K_p x q \sim \gamma \log(|x|) \) if \( a = 0 \).

For mixed states, we show convergence in Schatten norms with explicit rate towards the Weyl transform of a solution to the relativistic Vlasov equation with singular potentials, thus generalizing [12] where the case of smooth potentials has been treated. Moreover, we provide new results on the well-posedness theory of the relativistic Vlasov equations with singular interactions.

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1 Introduction

We consider the time evolution of a large system of fermions with relativistic dispersion law and its mean-field approximation given by the semi-relativistic Hartree-Fock equation

\[ i\varepsilon \partial_t \omega_{N,t} = \left[ \sqrt{1 - \varepsilon^2 \Delta} + K * \rho_t - X_t, \omega_{N,t} \right] \]

(1)

where \( \omega_{N,t} \) is a sequence of time-dependent self-adjoint operators acting on \( L^2(\mathbb{R}^d) \) with \( \text{Tr} \omega_{N,t} = N \) and \( 0 \leq \omega_{N,t} \leq 1 \). The semiclassical parameter \( \varepsilon \) plays the role of the Planck constant \( \hbar \) and depends on the number of particles \( N \) as \( \varepsilon = N^{-\frac{1}{d}} \). \( \sqrt{1 - \varepsilon^2 \Delta} \) is the pseudodifferential operator defined by the multiplication by the symbol \( \sqrt{1 + \varepsilon^2 |\xi|^2} \), \( K : \mathbb{R}^d \to \mathbb{R} \) is the two-body interaction potential, \( \rho_t(x) = N^{-1} \omega_{N,t}(x;x), \) where \( \omega_{N,t}(\cdot;\cdot) \) denotes the kernel of the operator \( \omega_{N,t} \), and \( X_t \) is referred to as the exchange term, whose integral kernel is given by \( X_t(x;y) = N^{-1} K(x-y) \omega_{N,t}(x;y). \)

Furthermore, we introduce the semi-relativistic Hartree equation

\[ i\varepsilon \partial_t \omega_{N,t} = \left[ \sqrt{1 - \varepsilon^2 \Delta} + K * \rho_t, \omega_{N,t} \right] \]

(2)

obtained from equation (1) by setting the exchange term \( X_t \) to zero.

The aim of the paper is to present a rigorous analysis of the semiclassical limit \( \varepsilon \to 0 \) in the case of singular interactions, including the most physically interesting cases of the Coulomb and gravitational interactions necessary for the purpose of the theoretical description of fermionic systems in the relativistic regime.

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potentials in dimension two and three. More precisely, let \( d \in \{2, 3\} \), \( \gamma \in \mathbb{R} \), \( a \in (-1, d - 2] \). The potential \( K : \mathbb{R}^d \rightarrow \mathbb{R}^d \) is given by

\[
K(x) = \gamma \begin{cases} \ln \left( |x| \right) & \text{if } a = 0 \\ \frac{1}{|x|^a} & \text{else.} \end{cases} \tag{3}
\]

In this setting we prove (see Theorem 1.1) that as \( \varepsilon \rightarrow 0 \) the dynamics described by (1) is well approximated by the relativistic Vlasov equation, the following nonlinear transport equation for the probability density \( f : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \)

\[
\partial_t f + \frac{v}{\sqrt{1 + v^2}} \cdot \nabla_x f + E \cdot \nabla_v f = 0 \tag{4}
\]

where \( E = -\nabla K \ast \rho_f \) and \( \rho_f(t, x) = \int_{\mathbb{R}^d} f(t, x, v) dv \). If \( \gamma = \pm 1 \) and \( (d, a) = \{(2, 0), (3, 1)\} \) the function \( K \) is (modulo a constant) the Green function for the Poisson equation in \( \mathbb{R}^d \). In these cases (4) is called the relativistic Vlasov-Poisson system.

**State of the art and strategy.** The semiclassical limit from the semi-relativistic Hartree equation (2) towards the relativistic Vlasov equation has been tackled in [12] in the case of smooth interactions \( K \in W^{2,\infty}(\mathbb{R}^3) \) and space dimension \( d = 3 \) and in [1] for \( K(x) = \pm \frac{1}{|x|^a} \) and \( d = 3 \). In [12] the authors provide strong convergence with an explicit bound on the convergence rate. In [1] weak convergence as \( \varepsilon \rightarrow 0 \) has been considered for Coulomb and gravitational interactions in the case of mixed states, but the methods of [1] do not allow for an explicit rate of convergence. The question of getting explicit control on the semiclassical approximation is not only of speculative nature, as real physical systems are made of a finite, although big, number of particles. It is therefore important for applications to specify how the semiclassical approximation depends on the number of particles. In this paper we generalise [1] in several directions: we show strong convergence in dimension \( d = 2, 3 \), exhibit an explicit rate, consider a larger class of interaction potentials and include the exchange term estimating it explicitly, thus proving the semiclassical limit from the semi-relativistic Hartree-Fock equation towards the relativistic Vlasov equation with singular interactions in two and three spatial dimensions in strong topology and with explicit rate.

We strongly rely on the method used in [12] Theorem 2.1, generalising it to spatial dimension \( d \in \{2, 3\} \) and, most importantly, to singular interactions. The key idea is to adapt to the relativistic context estimates from [31], where the semiclassical limit of the non-relativistic Hartree equation towards the non-relativistic Vlasov equation with singular potentials has been considered. The main difficulties to deal with come from the well-posedness theory and propagation of regularity for the relativistic Vlasov equation, which is less understood than its non-relativistic counterparts. The semiclassical limit from the non-relativistic Hartree equation towards the Vlasov equation with regular and singular interaction potentials has indeed been extensively studied in [34] [35] [15] [5] [23] [38] [4] [13] [2] [6] [19] [20] [22] [21] [10] [29] [11] [30] and in [37] [12] [9], where convergence was established directly starting from the many-body quantum evolution.

The well-posedness theory for the Vlasov equation is well understood even in the case of singular interactions and general initial data from the works of Lions and Perthame [35] and Pfafflmoser [39]. In the relativistic setting the case of smooth interactions is fully understood (see e.g. [12]), whereas for singular potentials, and in particular for Coulomb and gravitational interactions, the literature is somewhat limited. In the three dimensional attractive case \( (\gamma = -1, a = 1 \text{ and } d = 3) \) global existence has been established in [16] [17] [24] [28] [44] for radially symmetric initial data. We revisit these works and provide a well-posedness theory through propagation of velocity moments (global or local depending on the space dimension \( d \in \{2, 3\} \) and the parameter \( a \)) and propagation of regularity for general initial data in Theorem 1.8 and Propositions 1.10 and 1.11.

As for the semi-relativistic Hartree and Hartree-Fock equations, the case of pure states (namely density matrices which are projection operators) has been studied in [14] [33] [25] [26] and the case of mixed states has been outlined in [1] [26]. In Proposition 1.13 below we generalize these results to singular potentials of the form (3) in spatial dimension \( d = 3 \).
Theorem 1.1. equation (2) towards the relativistic Vlasov equation (4) with potentials of the form given in (3).

Its inverse is referred to as the Weyl quantization and given by

$$\omega_N(x;y) = \varepsilon^{-d} \int_{\mathbb{R}^d} \omega_N \left( x + \frac{\varepsilon y}{2}, x - \frac{\varepsilon y}{2} \right) e^{-iv \cdot x} dv.$$  

For $\sigma \in \mathbb{N}_0$, $k \geq 0$ and $1 \leq p \leq \infty$ let $L_{k,p}^{\sigma} (\mathbb{R}^d)$ be the Sobolev space equipped with the norm

$$\| f \|_{L_{k,p}^{\sigma} (\mathbb{R}^d)} = \left( \sum_{|\alpha| \leq \sigma} \| \langle \cdot \rangle^k D^\alpha f \|_{L^p (\mathbb{R}^d)}^p \right)^{1/p} \quad 1 \leq p < \infty,$$

$$\max_{|\alpha| \leq \sigma} \| \langle \cdot \rangle^k D^\alpha f \|_{L^\infty (\mathbb{R}^d)} = \infty.$$  

where $\langle x \rangle^2 = 1 + |x|^2$. In the cases $\sigma = 0$ or $p = 2$ we use the shorthand notation $L_k^p (\mathbb{R}^d) := L_{k,0}^p (\mathbb{R}^d)$ and $H_k^p (\mathbb{R}^d) := L_{k,2}^p (\mathbb{R}^d)$. Vectors in $\mathbb{R}^{2d}$ are written as $z = (x, v)$ so that $\langle z \rangle^2 = 1 + |v|^2$ and $D_z^\beta = (\partial/\partial x_1)^{\beta_1} (\partial/\partial v_1)^{\gamma_1} \cdots (\partial/\partial x_d)^{\beta_d} (\partial/\partial v_d)^{\gamma_d}$ with $\beta = (\beta_i)_{i=1,d} \in \mathbb{N}_0^d$ and $\gamma = (\gamma_i)_{i=1,d} \in \mathbb{N}_0^d$ such that $|\alpha| = \sum_{i=1}^d \beta_i + \gamma_i$. For two Banach spaces $A$ and $B$ we denote by $A \cap B$ the Banach space of vectors $f \in A \cap B$ with norm $\| f \|_{A \cap B} = \| f \|_A + \| f \|_B$. We use $C$ to denote a generic positive constant that might depend on the dimension $d$, the strength of the singularity $\alpha$ and the parameters $k$ and $\sigma$ appearing in (7).

Let $\mathfrak{S}^p (L^2 (\mathbb{R}^d))$ be the set of all bounded operators on $L^2 (\mathbb{R}^d)$ and $\mathfrak{S}^1 (L^2 (\mathbb{R}^d))$ be the set of trace class operators on $L^2 (\mathbb{R}^d)$. More generally, for $p \in [1, \infty)$, we denote by $\mathfrak{S}^p (\mathbb{R}^d)$ the $p$-Schatten space equipped with the norm $\| A \|_{\mathfrak{S}^p} = (\text{Tr} |A|^p)^{1/p}$, where $A$ is an operator, $A^*$ its adjoint and $|A| = \sqrt{A^* A}$.

We define

$$\mathfrak{S}^{1,\perp} (L^2 (\mathbb{R}^d)) = \left\{ \omega : \omega \in \mathfrak{S}^\infty (L^2 (\mathbb{R}^d), \omega^* = \omega \text{ and } (1 - \Delta)^{1/4} \omega (1 - \Delta)^{1/4} \in \mathfrak{S}^1 (L^2 (\mathbb{R}^d)) \right\} \quad (8)$$

with norm

$$\| \omega \|_{\mathfrak{S}^{1,\perp}} = \left\| (1 - \Delta)^{1/4} \omega (1 - \Delta)^{1/4} \right\|_{\mathfrak{S}^1}.$$  

(9)

The positive cone of this space is defined as

$$\mathfrak{S}^{1,\perp}_+ (L^2 (\mathbb{R}^d)) = \left\{ \omega : \omega \in \mathfrak{S}^\infty (L^2 (\mathbb{R}^d), \omega \geq 0, \text{ and } (1 - \Delta)^{1/4} \omega (1 - \Delta)^{1/4} \in \mathfrak{S}^1 (L^2 (\mathbb{R}^d)) \right\} \quad (10).$$

**Main results.** Our main result concerns the semiclassical limit from the semi-relativistic Hartree equation (2) towards the relativistic Vlasov equation (4) with potentials of the form given in (3).

**Theorem 1.1 (Semiclassical limit towards the relativistic Vlasov equation).** Let $d \in \{2, 3\}$, $\alpha \in (\max \{ \frac{d}{2} - 2, -1 \}, d - 2)$, $\gamma \in \mathbb{R}$ and $K$ be defined as in (5). Let $\omega_N$ be a sequence of reduced density matrices on $L^2 (\mathbb{R}^d)$, $\text{Tr} \omega_N = N$, $0 \leq \omega_N \leq 1$. Let $\omega_{N,\varepsilon}$ be the unique solution of (2) with initial condition $\omega_{N,0} = \omega_N$. In addition, assume that $x \omega_N x \in \mathfrak{S}^1 (L^2 (\mathbb{R}^d))$ if $a \in (\max \{ \frac{d}{2} - 2, -1 \}, 0]$ and $\omega_N \in \mathfrak{S}^{1,\perp}_+ (L^2 (\mathbb{R}^d))$ if $a \in (0, 1]$. Moreover, let $\tilde{W}_{N,\varepsilon}$ be the unique solution of (4) with initial datum $\tilde{W}_{N,0} = \tilde{W}_N \geq 0$ verifying

$$\tilde{W}_{N,\varepsilon} \in L^{\infty} (\mathbb{R}_+, W^{3,\infty} (\mathbb{R}^{2d}) \cap H^{2+}_\sigma (\mathbb{R}^{2d})) \quad (11)$$

and

$$\tilde{p} \in L^{\infty} (\mathbb{R}_+, L^1 (\mathbb{R}^{2d}) \cap H^{\alpha} (\mathbb{R}^{2d})), \quad (12)$$

with...
where \( \nu = 4 + a - d \) and \( \sigma = 4 + n \) with \( n \in 2\mathbb{N} \) such that \( n \geq \frac{d(a+1)}{d-(a+1)} \). For the Weyl quantization \( \tilde{\omega}_{N,t} \) of \( \tilde{W}_{N,t} \) it holds
\[
\| \omega_{N,t} - \tilde{\omega}_{N,t} \|_{\mathcal{E}^1} \leq \left[ \| \omega_N - \tilde{\omega}_N \|_{\mathcal{E}^1} + N \varepsilon \right] C(t) \left[ 1 + \int_0^t ds \, C(s) \lambda(s) e^{\int_0^s \lambda(r) \, dr} \right].
\]

Here,
\[
\lambda(t) = C|\gamma| \left\| \nabla \tilde{W}_{N,s} \right\|_{W^{2,\infty}(\mathbb{R}^{2d}) \cap H^{s}_{\mathcal{E}}(\mathbb{R}^{2d})}
\]
and
\[
C(t) = 1 + C \int_0^t (1 + |\gamma|) \left( 1 + \| \tilde{p}(s) \|_{L^1(\mathbb{R}^d) \cap H^{s}(\mathbb{R}^d)} \right) \left\| \tilde{W}_{N,s} \right\|_{H^{s}_{\mathcal{E}}(\mathbb{R}^{2d})} ds,
\]
where \( C \) is a numerical constant depending only on the dimension \( d \) and the parameters \( n, a \).

Some remarks are in order:

**Remark 1.2.** In \( d = 3 \), if \( \gamma < 0 \) and \( a = 1 \) we assume in addition that the condition \((30)\) stated in Proposition \((1.1)\) for the well-posedness of the semi-relativistic Hartree-Fock equation holds true.

**Remark 1.3.** Theorem \((1.1)\) generalizes \([12] \) Theorem 2.1] which considered the case \( d = 3 \) and regular interaction potentials \( K \in W^{2,\infty}(\mathbb{R}^3) \). Notice that we obtain for singular interactions the same expected optimal rate of convergence as in the case of smooth potentials.

**Remark 1.4.** Observe that thanks to \([31] \) Section 5), we can include the exchange term to exhibit an explicit rate of convergence from the semi-relativistic Hartree-Fock equation \((1)\) towards the relativistic Vlasov equation as \( \varepsilon \to 0 \). More precisely we can bound the exchange term as in \([31] \) Proposition 5.2] (see estimate \((42a)\)) and conclude that the rate of convergence towards the relativistic Vlasov equation does not change whether we consider the semi-relativistic Hartree or the Hartree-Fock equation as starting point.

**Remark 1.5.** The dependence of \( \lambda(t) \) and \( C(t) \) on high regularity norms of the solution to the Vlasov equation restricts our analysis to the case of mixed states, namely one-particle density matrices of the form
\[
\omega_{N,t} := \sum_{j \in \mathbb{N}} \lambda_j |\psi_j(t)\rangle \langle \psi_j(t)|
\]
where \( \{ \psi(t) \}_{j \in \mathbb{N}} \) is an orthonormal system of \( L^2(\mathbb{R}^d) \), \( \lambda_j \in \ell^1 \) and \( \lambda_j \geq 0 \).

**Remark 1.6.** Below it is shown (Theorem \((1.3)\), Corollary \((1.5)\) and Proposition \((1.10)\) that \((11)\) hold globally in time for \( a \in (-1,1/2) \) and locally in time for \( a \in (1/2,1] \) if \( W_N \) is regular enough. By means of \([44] \) Theorem 1.1], Corollary \((1.2)\) and Proposition \((1.10)\) it is, moreover, possible to infer \((11)\) globally in time for \( a = 1 \) if \( W_N \) is spherically symmetric and polynomially decaying as \( (x,v) \to \infty \).

Theorem \((1.1)\) can be extended to general \( p \)-Schatten norm, with \( p \in [1, \infty) \), in the same spirit of \([31] \) Theorem 1.2]. However, in the relativistic setting we have to deal with the kinetic term, that cannot be absorbed by unitary transformations (as it was the case in \([31] \) Theorem 1.2]), to obtain the following result.

**Proposition 1.7.** Under the same assumptions and notations of Theorem \((1.1)\), it holds
\[
\| \omega_{N,t} - \tilde{\omega}_{N,t} \|_{\mathcal{E}^p} \leq \| \omega_N - \tilde{\omega}_N \|_{\mathcal{E}^p} + \left[ \| \omega_N - \tilde{\omega}_N \|_{\mathcal{E}^1} + C(t)N \varepsilon \right] e^{\int_0^t \lambda(r) \, dr}
\]
for \( p \in \left[1, \min\{d/(a+1), 2\}\right) \), and
\[
\| \omega_{N,t} - \tilde{\omega}_{N,t} \|_{\mathcal{E}^p} \leq C(t) \left[ \| \omega_N - \tilde{\omega}_N \|_{\mathcal{E}^1}^q + \| \omega_N - \tilde{\omega}_N \|_{\mathcal{E}^1}^q + N \varepsilon \right] e^{\int_0^t \lambda(r) \, dr}
\]
for \( p \in \left[\min\{d/(a+1), 2\}, \infty\right) \) and \( q \in \left[1, \min\{d/(a+1), 2\}\right) \).
Theorem 1.8 (Velocity moments). Let $d \in \{2, 3\}$, $a \in (-1, d - 2]$, $\gamma \in \mathbb{R}$ and $K$ be defined as in (3). Let $f_0 \geq 0$, $f_0 \in L^1 \cap L^\infty (\mathbb{R}^d \times \mathbb{R}^d)$.

(a) Let $a \in (-1, 1/2]$ and assume that
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} |v|^k f_0(x, v) \, dx \, dv < +\infty \quad \text{for } k \in \mathbb{N}. \tag{16}
\]
Then, for all $1 \leq p < +\infty$ there exists a solution $f \in C (\mathbb{R}_+, L^p (\mathbb{R}^d \times \mathbb{R}^d)) \cap L^\infty (\mathbb{R}_+, L^\infty (\mathbb{R}^d \times \mathbb{R}^d))$ of (1) satisfying
\[
\sup_{t \in [0, T]} \int_{\mathbb{R}^d \times \mathbb{R}^d} |v|^k f(t, x, v) \, dx \, dv < +\infty \quad \text{for all } T < +\infty. \tag{17}
\]

(b) Let $d = 3$, $a \in (1/2, 1]$ and assume
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^k f_0(x, v) \, dx \, dv < +\infty \quad \text{for } k \in \mathbb{N} \text{ such that } k \geq \frac{da}{d - (a + 1)}. \tag{18}
\]
Then, for all $1 \leq p < +\infty$ there exists $T \in \mathbb{R}_+$ and a solution $f \in C ([0, T], L^p (\mathbb{R}^3 \times \mathbb{R}^3)) \cap L^\infty ([0, T], L^\infty (\mathbb{R}^3 \times \mathbb{R}^3))$ of (1) satisfying
\[
\sup_{t \in [0, T]} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^k f(t, x, v) \, dx \, dv < +\infty. \tag{19}
\]

Corollary 1.9. Let $f$ be a solution of (1) satisfying the assumptions of Theorem 1.8. In addition, let $f_0 |v|^n \in L^\infty (\mathbb{R}^d \times \mathbb{R}^d)$ for $n > d$ and let the parameter $k$ of Theorem 1.8 satisfy $k > \frac{d(a+1)}{d-(a+1)}$. Then there exists a time $T \in \mathbb{R}_+$ such that
\[
E \in L^\infty ([0, T], L^\infty (\mathbb{R}^d)) \tag{20},
\]
\[
\rho_f \in L^\infty ([0, T], L^1 \cap L^\infty (\mathbb{R}^d \times \mathbb{R}^d)); \tag{21}
\]

Proposition 1.10 (Propagation of regularity). Let $d \in \{2, 3\}$, $a \in (-1, d - 2]$, $\gamma \in \mathbb{R}$, $K$ be defined as in (3). Let $(n, \sigma) \in \mathbb{N}^2$ be such that $n > 2d$ and $f \geq 0$ be a solution of the relativistic Vlasov equation (1) with initial data $f_0 \in W^{n, \infty}_{\nu} (\mathbb{R}^{2d})$. Moreover, assume that $\rho_f \in L^\infty_{0\nu} (\mathbb{R}_+, L^1 \cap L^\infty (\mathbb{R}^d \times \mathbb{R}^d))$. Then the following regularity estimates hold
\[
f \in L^\infty_{0\nu} (\mathbb{R}_+, W^{n, \infty}_{\nu} (\mathbb{R}^{2d})); \tag{22}
\]
\[
\nabla^\sigma \rho_f \in L^\infty_{0\nu} (\mathbb{R}_+, L^\infty (\mathbb{R}^d)). \tag{23}
\]

If in addition $f_0 \in H^\sigma_k (\mathbb{R}^{2d})$ for some $k \in \mathbb{R}_+$, then
\[
f \in L^\infty_{0\nu} (\mathbb{R}_+, H^\sigma_k (\mathbb{R}^{2d})). \tag{24}
\]

Note that (22) - (24) hold globally and not only locally if $\rho_f \in L^\infty (\mathbb{R}_+, L^1 \cap L^\infty (\mathbb{R}^d \times \mathbb{R}^d))$. 

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**Proposition 1.11** (Uniqueness criterion). Let \(d \in \{2, 3\}\) and \(a \in (-1, d-2]\). Moreover, let \(f_1\) and \(f_2\) be two solutions of the Vlasov equation \(\Phi\) in \(L^\infty ([0, T], L^1(\mathbb{R}^d))\) for some \(T > 0\) such that
\[
\int_{\mathbb{R}^d} |\nabla v f_2| dv \in L^1 \left( [0, T], L^{\frac{d}{a-(a+1)}} \cap L^{\frac{d}{a-(a+1)} + \delta} (\mathbb{R}^d) \right)
\]
for some \(0 < \delta < \frac{d}{a-(a+1)}\). Then there exists \(C > 0\) such that
\[
\| (f_1 - f_2)(t) \|_{L^1(\mathbb{R}^d)} \leq C \| (f_1 - f_2)(0) \|_{L^1(\mathbb{R}^d)} e^{C \int_0^t \| \rho_1 \nabla f_2 \|_{L^{\frac{d}{a-(a+1)} + \delta}} \cap L^{\frac{d}{a-(a+1)} - \delta} (\mathbb{R}^d)} ds,
\]
where \(\rho_1 \nabla f_2(t, x) := \int_{\mathbb{R}^d} |\nabla v f_2(t, x, v)| dv\).

**Remark 1.12.** Note that (25) can be replaced by the stronger condition \(f_2 \in L^1 \left( [0, T], W^{1,\infty}_n(\mathbb{R}^d) \right)\) with \(n > 2d\). Indeed, note that by interpolation it holds
\[
\| \rho_1 \nabla f_2(t) \|_{L^{\frac{d}{a-(a+1)} + \delta}} \cap L^{\frac{d}{a-(a+1)} - \delta} (\mathbb{R}^d) \leq C \left( \| \rho_1 \nabla f_2(t) \|_{L^1(\mathbb{R}^d)} + \| \rho_1 \nabla f_2(t) \|_{L^{\frac{d}{a-(a+1)} + \delta}} \cap L^{\frac{d}{a-(a+1)} - \delta} (\mathbb{R}^d) \right).
\]
Furthermore, we use the assumption \(n/2 > d\) to estimate
\[
\| \rho_1 \nabla f_2(t) \|_{L^1(\mathbb{R}^d)} \leq \| \langle x \rangle^{n/2} \langle v \rangle^{-n/2} \nabla f_2(t) \|_{L^{\infty}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \langle x \rangle^{-n/2} \langle v \rangle^{-n/2} dx dv 
\leq C \| f_2(t) \|_{W^{1,\infty}_n(\mathbb{R}^d)}.
\]
Similarly,
\[
\| \rho_1 \nabla f_2(t) \|_{L^{\frac{d}{a-(a+1)} + \delta}} \cap L^{\frac{d}{a-(a+1)} - \delta} (\mathbb{R}^d) \leq C \| f_2 \|_{W^{1,\infty}_n(\mathbb{R}^d)}
\]
because \(n/2 > d\) and \(\frac{d}{a-(a+1)} + \delta \geq 1\) for \(a \in (-1, d-2]\). This shows that the solutions considered in Proposition [1.10] are unique.

We will rely on the following well-posedness result generalizing to general singular potentials [26, Theorem 1] and [1, Lemma 2.3 and Lemma 2.4] which considered the case \(d = 3\) and \(a = 1\). For the sake of completeness we provide its proof in Appendix A. Note that we deliberately choose to restrict the well-posedness result to \(d = 3, K(x) = \gamma \frac{1}{|x|^a}, 0 < a \leq 1\) and coupling constant \(\gamma \in \mathbb{R}\), in order to keep the presentation shorter. The cases \(d = 2\) or \(a < 0\) would require propagation of spatial moments, making the presentation much longer.

**Proposition 1.13.** Let \(d = 3, a \in (0, 1]\) and \(\omega_{N,0} \in \mathcal{S}_+^{1,\frac{1}{2}} (L^2(\mathbb{R}^3))\) such that \(0 \leq \omega_{N,0} \leq 1\). If \(\gamma < 0\) and \(a = 1\) assume in addition that
\[
|\gamma| < \frac{N}{\gamma_{cr} \left( \text{Tr} \left( \omega_{N,0} \right) \right)^{2/3}},
\]
where \(\gamma_{cr}\) is a universal constant of order 1. Then, the Cauchy problems for (the integral versions of) equation (1) and equation (2) have a unique global solution in \(\mathcal{S}_+^{1,\frac{1}{2}} (L^2(\mathbb{R}^3))\).

**Remark 1.14.** Without condition (30), equation (1) and (2) are still locally well-posed in \(\mathcal{S}_+^{1,\frac{1}{2}} (L^2(\mathbb{R}^3))\) for \(d = 3, a = 1\) and for all \(\gamma \in \mathbb{R}\).

The paper is structured as follows: in Section 2 we collect estimates on the relativistic Vlasov equation and the semi-relativistic Hartree equation; Section 3 is devoted to the detailed proofs of the semiclassical limit (Theorem 1.1), propagation of velocity moments (Theorem 1.2), propagation of regularity (Proposition 1.10) and uniqueness for the relativistic Vlasov equation with singular interactions (Proposition 1.11); Appendices A and B are concerned with the well-posedness theory for the Hartree and Hartree-Fock equations (Proposition 1.12) and address the existence of the two-parameter semigroup used in the Duhamel expansion in the proof of Theorem 1.1.
2 Preliminary estimates

Before we prove our main results, let us state some preliminary facts.

**Proposition 2.1.** Let \( d \in \mathbb{N}, c \geq b > 0 \) and \( f := f(x,v) \geq 0 \). Then for all \( x \in \mathbb{R}^d \),

\[
\int_{\mathbb{R}^d} |v|^{\frac{b}{b}} f(x,v) \, dv \leq C \|f\|_{L^{\infty}(\mathbb{R}^{2d})} \left( \int_{\mathbb{R}^d} |v|^{c} f(x,v) \, dv \right)^{\frac{b}{b+c}},
\]

with \( C \) depending only on \( b, c \) and \( d \). In particular, setting \( p(x) = \int_{\mathbb{R}^d} f(x,v) \, dv \), we have for any \( b > 0 \)

\[
\|p\|_{L^{\frac{b+c}{b}}(\mathbb{R}^d)} \leq C \|f\|_{L^{\infty}(\mathbb{R}^{2d})} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |v|^{b} f(x,v) \, dx \, dv \right)^{\frac{d}{d+b+c}}.
\]

**Proof.** The inequalities are proven in analogy to [11, Proposition 2.1] and the proof of estimate (14) in [33].

**Lemma 2.2.** Let \( d \in \mathbb{N}, l \geq 0, f := f(x,v) \geq 0 \) and \( M_l[f] = \int_{\mathbb{R}^d \times \mathbb{R}^d} |v|^{l} f(x,v) \, dx \, dv \). For \( 0 \leq b \leq a \leq c \) the following holds

\[
M_a[f] \leq M_b^{\frac{c-b}{b}}[f] M_c^{\frac{a-b}{c-a}}[f].
\]

**Proof.** Let \( \mu = \frac{b(a-c)}{(c-b)}, \nu = \frac{c(b-a)}{(c-b)}, p = \frac{c}{c-b} \) and \( q = \frac{a}{b} \). Then \( \mu, \nu \geq 0 \) and

\[
\mu + \nu = \frac{b(c-a) + c(a-b)}{c-b} = \frac{bc - ab + ac - bc}{c-b} = \frac{a(c-b)}{c-b} = a.
\]

Moreover, \( p, q \geq 1 \) and

\[
\frac{1}{p} + \frac{1}{q} = \frac{c-a}{c-b} + \frac{a-b}{c-b} = \frac{c-a + a-b}{c-b} = \frac{c-b}{c-b} = 1.
\]

Since \( \mu p = \frac{b(c-a)}{(c-b)} \frac{c-b}{c-a} = b \) and \( \nu q = \frac{c(b-a)}{(c-b)} \frac{c-b}{c-a} = c \), Hölder’s inequality leads to

\[
M_a[f] = \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |v|^{\mu} f^\frac{1}{\mu}(x,v) |v|^\nu f^\frac{1}{\nu}(x,v) \, dx \, dv \right) \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |v|^{\nu} f(x,v) \, dx \, dv \right) \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |v|^{\mu} f(x,v) \, dx \, dv \right) \leq \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |v|^{p\mu} f(x,v) \, dx \, dv \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |v|^{q\nu} f(x,v) \, dx \, dv \right)^{\frac{1}{q}} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |v|^{\frac{b}{b+c}} f(x,v) \, dx \, dv \right)^{\frac{b}{b+c}}.
\]

**Lemma 2.3.** Let \( d \in \{2,3\}, \gamma \in \mathbb{R}, a \in (-1,d-2) \) and \( K \) be defined as in [8]. Let \( f := f(x,v) \geq 0 \), \( \rho_f = \int_{\mathbb{R}^d} f(x,v) \, dv \), \( E = -\nabla K \ast \rho_f \), \( n \in \mathbb{N} \) be such that \( n > 2d, p > \frac{d}{a-(a+1)} \) and \( 1 \leq q < \frac{d}{a+1} \). Then there exists \( C > 0 \) such that

\[
\|E\|_{L^{\infty}(\mathbb{R}^d)} \leq C |\gamma| \left( \|\rho_f\|_{L^{p}(\mathbb{R}^d)} + \|\rho_f\|_{L^{q}(\mathbb{R}^d)} \right),
\]

\[
\|\nabla E\|_{L^{\infty}(\mathbb{R}^d)} \leq C |\gamma| \left( 1 + \|\rho_f\|_{L^{1}(\mathbb{R}^d)} + \|\rho_f\|_{L^{\infty}(\mathbb{R}^d)} \right) \left( 1 + \ln(1 + \|\nabla \rho_f\|_{L^{\infty}(\mathbb{R}^d)}) \right),
\]

\[
\|E\|_{W^{0,\infty}(\mathbb{R}^d)} \leq C |\gamma| \min \left\{ \left( \|f\|_{W^{0,\infty}(\mathbb{R}^{2d})} \right), \left( 1 + \|f\|_{W^{2,\infty}(\mathbb{R}^{2d})} \right) \left( 1 + \ln \left( 1 + \|f\|_{W^{0,\infty}(\mathbb{R}^{2d})} \right) \right) \right\}. \]

**For** \( \frac{d}{a+1} < q < \infty \) and \( 1 + \frac{1}{q} = \frac{d}{a+1} + \frac{1}{p} \) there exists \( C > 0 \) such that

\[
\|E\|_{L^{q}(\mathbb{R}^d)} \leq C |\gamma| \|\rho_f\|_{L^{p}(\mathbb{R}^d)} \leq C |\gamma| \|f\|_{L^{\infty}(\mathbb{R}^{2d})} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |v|^{(p-1)d} f(x,v) \, dx \, dv \right)^{\frac{1}{p}}.
\]
Proof of Lemma 2.3. The first inequality is obtained by the splitting \( K := K_{|y|<1} + K_{|y|>1} \) and Young’s inequality. Similarly, one derives \( \| \nabla E \|_{L^\infty(\mathbb{R}^d)} \leq C|\gamma| \left( \| \rho_j \|_{L^\infty(\mathbb{R}^d)} + \| \rho_j \|_{L^1(\mathbb{R}^d)} \right) \) for \( a \in (-1, d-2) \) because \( |\partial_x \delta \chi_K(x)| \leq C|\gamma| \left( \frac{1}{|\mathcal{D}|} \right)^\frac{1}{d} \). Inequality (33) with \( a = d - 2 \) is proven in [18, p.83]. Note that \( D_{\xi}^a E = - (\nabla K) * D_{\xi}^a \rho_f \). Together with (33) we get

\[
\| \nabla D_{\xi} E \|_{L^\infty(\mathbb{R}^d)} \leq C|\gamma| \left( 1 + \| D_{\xi}^a \rho_f \|_{L^1(\mathbb{R}^d)} + \| D_{\xi}^a \rho_f \|_{L^\infty(\mathbb{R}^d)} \right) \left( 1 + \ln \left( 1 + \| \nabla D_{\xi}^a \rho_f \|_{L^\infty(\mathbb{R}^d)} \right) \right).
\]  

(41)

Since \( n > 2d \) we have

\[
\| D_{\xi}^a \rho_f \|_{L^\infty(\mathbb{R}^d)} \leq \sup_x \int_{\mathbb{R}^d} |D_{\xi}^a f(t, x, v)| \, dv \leq \| f \|_{W^{n,\infty}_0(\mathbb{R}^{2d})} \int_{\mathbb{R}^d} \langle v \rangle^{-\frac{a}{d}} \, dv \leq C \| f \|_{W^{n,\infty}_0(\mathbb{R}^{2d})}
\]  

(42)

and

\[
\| D_{\xi}^a \rho_f \|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} D_{\xi}^a f(t, x, v) \, dv \, dx \leq \| f \|_{W^{n,\infty}_0(\mathbb{R}^{2d})} \int_{\mathbb{R}^{2d}} \langle z \rangle^{-n} \, dz \leq C \| f \|_{W^{n,\infty}_0(\mathbb{R}^{2d})}.
\]  

(43)

In total, this shows

\[
\| E \|_{W^{n,\infty}_0(\mathbb{R}^{2d})} \leq C|\gamma| \left( 1 + \| f \|_{W^{n-1,\infty}_0(\mathbb{R}^{2d})} \right) \left( 1 + \ln \left( 1 + \| f \|_{W^{n,\infty}_0(\mathbb{R}^{2d})} \right) \right).
\]  

(44)

By similar means and (37) one obtains \( \| E \|_{W^{n,\infty}_0(\mathbb{R}^{2d})} \leq C|\gamma| \| f \|_{W^{n,\infty}_0(\mathbb{R}^{2d})} \). In order to show the first inequality of (40) we use that \( \| \nabla K \| \leq \| \gamma \| \langle x \rangle^{-\frac{a}{d}} \) in \( L^{\frac{d+1}{a}}(\mathbb{R}^d) \), where \( L^{\frac{d+1}{a}}(\mathbb{R}^d) \) denotes the weak-\( L^p \) space of all measurable functions \( f \) such that \( \sup_{\alpha > 0} \left\{ \alpha \left( \int_{\mathbb{R}^d} |f|^{\frac{d+1}{a}} \right)^{\frac{a}{d+1}} \right\} < \infty \) (see, e.g., [32, p. 106]). By means of the weak Young inequality we obtain

\[
\| E \|_{L^p(\mathbb{R}^d)} = \| \nabla K * \rho_f \|_{L^p(\mathbb{R}^d)} \leq C|\gamma| \| \rho_f \|_{L^p(\mathbb{R}^d)}
\]  

(45)

where \( 1 + \frac{1}{q} = \frac{a+1}{d} + \frac{1}{d} \). Note that \( p > 1 \) because \( q > \frac{d}{a+1} \). Writing \( p = \frac{b+d}{d} \) with \( b > 0 \) and applying Proposition 2.1 leads to the second inequality in (40). \( \square \)

Proposition 2.4. For a solution \( f \) of (1) with initial datum \( f(0) = f_0 \geq 0 \) the energy

\[
\mathcal{E}(f(t)) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle v \rangle f(t, x, v) \, dx \, dv + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} K(x - y) \rho_f(t, x) \rho_f(t, y) \, dx \, dy
\]  

(46)

is conserved during the time evolution, i.e., \( \mathcal{E}(f(t)) = \mathcal{E}(f_0) \) for all \( t \in \mathbb{R} \). Moreover, \( \| \rho_f(t) \|_{L^1(\mathbb{R}^d)} = \| \rho_0 \|_{L^1(\mathbb{R}^d)} \) and \( \| f(t) \|_{L^p(\mathbb{R}^d)} = \| f_0 \|_{L^p(\mathbb{R}^d)} \) for all \( p \in [1, \infty) \) and \( f_0 \geq 0 \) implies \( f(t) \geq 0 \) for all \( t \in \mathbb{R} \).

Proof. The conservation of the energy is obtained by a straightforward calculation. The remaining relations hold because \( f \) is constant along a Lebesgue measure preserving flow. \( \square \)

Moreover, we will use the following results stating that the potential energy is dominated by the kinetic energy if \( 0 < a < 1 \) and \( f \in L^1 \cap L^\infty(\mathbb{R}^3) \).

Lemma 2.5. Let \( d = 3, \gamma \in \mathbb{R}, 0 < a < 1 \) and \( K \) be defined as in (31). Moreover, let \( f := f(x, v) \geq 0 \) such that \( f \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3) \) and \( \mathcal{E}(f) \) be defined as in Proposition 2.4. Then there exists a constant \( C \) depending only on \( a, |\gamma| \) and \( \| f \|_{L^1 \cap L^\infty(\mathbb{R}^3)} \) such that

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} \langle v \rangle f(x, v) \, dx \, dv - C \leq 2\mathcal{E}(f) \leq 3 \int_{\mathbb{R}^3 \times \mathbb{R}^3} \langle v \rangle f(x, v) \, dx \, dv + C.
\]  

(47)

Remark 2.6. Note that in the relativistic setting we experience a loss of control in the velocity moments. In fact, it is possible to control only the first moment in velocity by means of the energy of the system, whereas in the non-relativistic setting the kinetic energy controls the second velocity moment.
Proof of Lemma 2.5. By means of the Hardy-Littlewood-Sobolev inequality (see e.g. [32] Chapter 4.3) there exists a constant $C > 0$ (depending on $a$) such that

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} |x-y|^{-a} \rho_f(t,x) \rho_f(t,y) \, dx \, dy \leq C \| \rho_f \|^2_{L^p(\mathbb{R}^3)} \quad \text{with} \quad p = \frac{6}{6-a}.$$  

(48)

Using the second inequality of (40) we obtain

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} |x-y|^{-a} \rho_f(t,x) \rho_f(t,y) \, dx \, dy \leq C \left( \frac{\| f \|^2_{L^{2(p-1)}(\mathbb{R}^3)}}{L^{2(p-1)}(\mathbb{R}^3)} \right)^{\frac{3}{6(p-1)}} \| \rho_f \|^2_{L^p(\mathbb{R}^3)}.$$

(49)

Note that $3(p-1) = \frac{3a}{6-a} < \frac{3}{2}$ for all $0 < a < 1$. By means of Lemma 2.4, we estimate

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} |x-y|^{-a} \rho_f(t,x) \rho_f(t,y) \, dx \, dy \leq C \left( \frac{\| f \|^2_{L^{2(p-1)}(\mathbb{R}^3)}}{L^{2(p-1)}(\mathbb{R}^3)} \right)^{\frac{3}{6(p-1)}} \| \rho_f \|^2_{L^p(\mathbb{R}^3)}.$$

(50)

Since $\frac{6(p-1)}{p} = a < 1$ the result follows from Young's inequality for products.

In the next proposition we collect some estimates we will use in Appendix A.

Proposition 2.7. Let $s \geq 1/2$, $0 < a \leq 1$ and $K(x) = \gamma \frac{1}{|x|^s}$. Then,

$$\| K \ast (fg) \|^2_{L^s(\mathbb{R}^3)} \leq C \| f \|^2_{H^{1/2}(\mathbb{R}^3)} \| g \|^2_{H^{1/2}(\mathbb{R}^3)} \quad \text{for all} \quad f, g \in H^{1/2}(\mathbb{R}^3),$$

(51)

$$\| K \ast (fg) \|^2_{H^s(\mathbb{R}^3)} \leq C \gamma \left[ \| f \|^2_{H^{1/2}(\mathbb{R}^3)} \| g \|^2_{H^{1/2}(\mathbb{R}^3)} \| h \|^2_{H^{1/2}(\mathbb{R}^3)} \right] + \| f \|^2_{H^s(\mathbb{R}^3)} \| g \|^2_{H^s(\mathbb{R}^3)} \| h \|^2_{H^s(\mathbb{R}^3)} \quad \text{for all} \quad f, g, h \in H^s(\mathbb{R}^3).$$

(52)

For $i \in \{1, 2\}$, $\omega_i \in \mathbb{E}^{1/2} (L^2(\mathbb{R}^3))$, $\rho_{\omega_i}(x) = N^{-1} \omega_i(x, x)$ and $X_i(x; y) = N^{-1} K(x - y) \omega_i(x; y)$ we have

$$\left| \int_{\mathbb{R}^3} (K \ast \rho_{\omega_1}) (x) \rho_{\omega_2} (x) \, dx \right| \leq C \gamma \left| \text{Tr} \left( \sqrt{-\Delta} \omega_1 \right) \text{Tr} \left( \sqrt{-\Delta} \omega_2 \right) \right| \left( \| \omega_1 \|^2_{\mathbb{E}^1} \| \omega_2 \|^2_{\mathbb{E}^1} \right)^{\frac{2-a}{2}},$$

(53)

$$| \text{Tr} (K \ast \rho_{\omega_1} \omega_1) | \leq C \gamma \left| \left( \text{Tr} \left( \sqrt{-\Delta} \omega_1 \right) \right) \right| \left( \| \omega_1 \|^2_{\mathbb{E}^1} \right)^{2-a},$$

(54)

$$| \text{Tr} (X_1 \omega_2) | \leq N \left| \int_{\mathbb{R}^3} (K \ast \rho_{\omega_1}) (x) \rho_{\omega_2} (x) \, dx \right| = \left| \text{Tr} \left( K \ast \rho_{\omega_1} \omega_2 \right) \right|.$$ 

(55)

Proof. The first two inequalities are proven by similar estimates as in [33] [27]. If we split the potential into two parts and use Young’s inequality, we obtain

$$\| K \ast (fg) \|^2_{L^s(\mathbb{R}^3)} \leq \| f \|^2_{L^{2*(\mathbb{R}^3)}} \| g \|^2_{L^{2*(\mathbb{R}^3)}} + \| f \|_{L^{2*(\mathbb{R}^3)}} \| g \|_{L^{2*(\mathbb{R}^3)}} + \| f \|_{L^{2*(\mathbb{R}^3)}} \| g \|_{L^{2*(\mathbb{R}^3)}} + \| f \|^2_{L^{2*(\mathbb{R}^3)}} \| g \|^2_{L^{2*(\mathbb{R}^3)}}.$$

(56)

By means of the Cauchy-Schwarz inequality and the estimate $\sup_{y \in \mathbb{R}^3} \left| \int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x-y|} \, dx \right| \leq C \| u \|^2_{H^{1/2}(\mathbb{R}^3)}$ (see [33] inequality (17)) the second term can be bounded by

$$\| f \|^2_{L^{2*(\mathbb{R}^3)}} \| g \|^2_{L^{2*(\mathbb{R}^3)}} \leq \left( \sup_{y \in \mathbb{R}^3} \left| \int_{\mathbb{R}^3} \frac{|f(x)|^2}{|x-y|} \, dx \right| \right)^{1/2} \left( \sup_{y \in \mathbb{R}^3} \left| \int_{\mathbb{R}^3} \frac{|g(x)|^2}{|x-y|} \, dx \right| \right)^{1/2} \leq C \| f \|^2_{H^{1/2}(\mathbb{R}^3)} \| g \|^2_{H^{1/2}(\mathbb{R}^3)}.$$ 

(57)
Now, let \( f, g, h \in H^{s}(\mathbb{R}^3) \) and \( \frac{1}{p} + \frac{1}{q} = \frac{1}{2} \). By the generalized Leibniz rule (see for example Lemma 5) we estimate
\[
\|(K \ast (fg))h\|_{H^{s}(\mathbb{R}^3)} = \|(1 - \Delta)^{s/2} (K \ast (fg))h\|_{L^2(\mathbb{R}^3)} \\
\leq \|(K \ast (fg))h\|_{L^2(\mathbb{R}^3)} + \|(-\Delta)^{s/2} (K \ast (fg))h\|_{L^2(\mathbb{R}^3)} \\
\leq \|K \ast (fg)\|_{L^{\infty}(\mathbb{R}^3)} \|h\|_{L^2(\mathbb{R}^3)} + C \|K \ast (fg)\|_{L^{\infty}(\mathbb{R}^3)} \|(-\Delta)^{s/2} h\|_{L^2(\mathbb{R}^3)} \\
+ C \|(-\Delta)^{s/2} K \ast (fg)\|_{L^p(\mathbb{R}^3)} \|h\|_{L^s(\mathbb{R}^3)} + C \|K \ast (fg)\|_{L^{\infty}(\mathbb{R}^3)} \|(-\Delta)^{s/2} h\|_{L^2(\mathbb{R}^3)} \\
\leq C \|K \ast (fg)\|_{L^{\infty}(\mathbb{R}^3)} \|h\|_{H^s(\mathbb{R}^3)} + C \|(-\Delta)^{s/2} K \ast (fg)\|_{L^p(\mathbb{R}^3)} \|h\|_{L^q(\mathbb{R}^3)}. \quad (58)
\]
The first summand is suitably bounded by means of \((51)\). In order to estimate the remaining terms we distinguish between the cases \( 1/2 \leq s < 3/2 \) and \( s \geq 3/2 \).

**The cases** \( 1/2 \leq s < 3/2 \): We choose \( p = \frac{3}{s}, q = \frac{6}{3-2s} \) and recall the Sobolev inequality \( \|h\|_{L^{\frac{6}{3-2s}}(\mathbb{R}^3)} \leq C \|h\|_{H^s(\mathbb{R}^3)} \). Using the representation of the Riesz potential \((-\Delta)^{(r-3)/2}(fg) = c_r \cdot \cdot \cdot (fg)\) which holds for \(-3 < r < 0\) and some \( c_r \in \mathbb{R} \) the fact that \(| \cdot |^{-(s+a)} \in L^{\frac{6}{3-2s}}(\mathbb{R}^3)\) and the weak Young inequality we estimate
\[
\|(-\Delta)^{s/2} K \ast (fg)\|_{L^\frac{6}{3-2s}(\mathbb{R}^3)} = |\gamma| c_a \|(-\Delta)^{\frac{4s+3}{2}} (fg)\|_{L^\frac{6}{3-2s}(\mathbb{R}^3)} \\
\leq C |\gamma| \| |\cdot|^{-(s+a)} (fg)\|_{L^\frac{6}{3-2s}(\mathbb{R}^3)} \\
\leq C |\gamma| \|fg\|_{L^{\frac{6}{3-2s}}(\mathbb{R}^3)}. \quad (59)
\]
Together with the Cauchy Schwarz and Sobolev’s inequality this shows
\[
\|(-\Delta)^{s/2} K \ast (fg)\|_{L^\frac{6}{3-2s}(\mathbb{R}^3)} \|h\|_{L^{\frac{6}{3-2s}}(\mathbb{R}^3)} \leq C |\gamma| \|f\|_{L^{\frac{6}{3-2s}}(\mathbb{R}^3)} \|g\|_{L^{\frac{6}{3-2s}}(\mathbb{R}^3)} \|h\|_{H^s(\mathbb{R}^3)} \\
\leq C |\gamma| \|f\|_{H^{1/2}(\mathbb{R}^3)} \|g\|_{H^{1/2}(\mathbb{R}^3)} \|h\|_{H^s(\mathbb{R}^3)} \quad (60)
\]
for all \( 1/2 \leq s < 3/2 \).

**The cases** \( s \geq 3/2 \): We choose \( p = 6 \) and \( q = 3 \). Note that \( \|h\|_{L^3(\mathbb{R}^3)} \leq C \|h\|_{H^{1/2}(\mathbb{R}^3)} \) holds because of Sobolev’s inequality. Using again the representation of the Riesz potential and Sobolev’s inequality we obtain
\[
\|(-\Delta)^{s/2} K \ast (fg)\|_{L^6(\mathbb{R}^3)} \leq C |\gamma| \|(-\Delta)^{\frac{3s-3}{2}} fg\|_{L^6(\mathbb{R}^3)} \leq C |\gamma| \|(-\Delta)^{\frac{3s-3}{2}} fg\|_{L^2(\mathbb{R}^3)}. \quad (61)
\]
If \( a \in (0,1] \) such that \( s + 2 \leq 2 \) we can proceed similar as in \((59)\) and obtain
\[
\|(-\Delta)^{s/2} K \ast (fg)\|_{L^6(\mathbb{R}^3)} \leq C \|fg\|_{L^{\frac{6}{2(s+a)}}(\mathbb{R}^3)}. \quad (62)
\]
Since \( 1 \leq \frac{6}{2(s+a)} \leq 2 \) for \( \frac{3}{2} \leq s + a \leq 2 \) we have
\[
\|fg\|_{L^{\frac{6}{2(s+a)}}(\mathbb{R}^3)} \leq C \|fg\|_{L^1 \cap L^2(\mathbb{R}^3)} \leq C \|f\|_{L^2 \cap H^1(\mathbb{R}^3)} \|g\|_{L^2 \cap H^{1/2}(\mathbb{R}^3)} \leq C \|f\|_{H^s(\mathbb{R}^3)} \|g\|_{H^{1/2}(\mathbb{R}^3)} \quad (63)
\]
by interpolation, Hölder’s inequality and Sobolev’s inequality. Using the general Leibniz rule we get for $\mu \geq 0$

$$\left\|(-\Delta)^{\mu/2} (fg)\right\|_{L^2(\mathbb{R}^3)} \leq C \left\|(-\Delta)^{\mu/2} f\right\|_{L^{\infty}(\mathbb{R}^3)} \left\|g\right\|_{L^2(\mathbb{R}^3)} + C \left\|f\right\|_{L^2(\mathbb{R}^3)} \left\|(-\Delta)^{\mu/2} g\right\|_{L^{\infty}(\mathbb{R}^3)}.$$

\[\leq C \left\|(-\Delta)^{\mu/2} f\right\|_{L^2(\mathbb{R}^3)} \left\|g\right\|_{H^{1/2}(\mathbb{R}^3)} + C \left\|f\right\|_{H^{1/2}(\mathbb{R}^3)} \left\|(-\Delta)^{\mu/2} g\right\|_{L^2(\mathbb{R}^3)}. \quad (64)\]

For $s \geq 3/2$ and $0 < a \leq 1$ such that $s + a \geq 2$ we consequently have

$$\left\|(-\Delta)^{s/2} K * (fg)\right\|_{L^6(\mathbb{R}^3)} \leq C\|f\|_{H^{s-s-1}(\mathbb{R}^3)} \|g\|_{H^{1/2}(\mathbb{R}^3)} + C\|f\|_{H^{1/2}(\mathbb{R}^3)} \|g\|_{H^{s-s-1}(\mathbb{R}^3)}$$

\[\leq C\|f\|_{H^{s}(\mathbb{R}^3)} \|g\|_{H^{1/2}(\mathbb{R}^3)} + C\|f\|_{H^{1/2}(\mathbb{R}^3)} \|g\|_{H^{s}(\mathbb{R}^3)}. \quad (65)\]

In total this shows

$$\left\|(-\Delta)^{s/2} K * (fg)\right\|_{L^6(\mathbb{R}^3)} \|h\|_{L^2(\mathbb{R}^3)} \leq C\|f\|_{H^{s}(\mathbb{R}^3)} \|g\|_{H^{1/2}(\mathbb{R}^3)} + \|f\|_{H^{1/2}(\mathbb{R}^3)} \|g\|_{H^{s}(\mathbb{R}^3)} \|h\|_{H^{1/2}(\mathbb{R}^3)} \|h\|_{H^{1/2}(\mathbb{R}^3)}. \quad (66)\]

for all $s \geq 3/2$. Summing up, this shows $\text{(62)}$.

Next, we continue with $\text{(63)}$. Similarly as in the proof of Lemma 2.5 we use the Hardy-Littlewood-Sobolev inequality (see e.g. [32, Chapter 4.3]) to estimate

$$\int_{\mathbb{R}^3} (K * \rho_1)(x) \rho_2(x) \, dx \leq C\|f\|_{L^{\infty}(\mathbb{R}^3)} \|g\|_{L^{\infty}(\mathbb{R}^3)},$$

\[\text{where } C \text{ is a numerical constant depending only on } a. \text{ Let } \{\lambda_j, \varphi_j\}_{j \in \mathbb{N}} \text{ be the spectral set of } \omega_1. \text{ Then}
\]

$$\|\rho_1\|_{L^{6/5}(\mathbb{R}^3)} \leq N^{-1}\sum_{j \in \mathbb{N}} |\lambda_j| \|\varphi_j\|_{L^{6/5}(\mathbb{R}^3)}^2 \|L^{6/5}(\mathbb{R}^3)}, \quad (68)\]

with $1 < \frac{6}{5} - a \leq \frac{6}{5}$. By the interpolation inequality we obtain $\|\varphi_j\|_{L^{6/5}(\mathbb{R}^3)} \leq \|\varphi_j\|_{L^2(\mathbb{R}^3)}^{2-a} \|\varphi_j\|_{L^3(\mathbb{R}^3)}^a$. Using the Sobolev inequality for higher order fractional derivatives ([10, Theorem 1.1])

$$\|f\|_{L^{q}(\mathbb{R}^3)} \leq C \left\|(-\Delta)^{s/2} f\right\|_{L^2(\mathbb{R}^3)} \quad \text{where } s < \frac{3}{2}, q = \frac{6}{3 - 2s} \text{ and } f \in H^{s}(\mathbb{R}^3) \quad (69)\]

together with Hölder’s inequality ($p = \frac{2}{a}$ and $q = \frac{2}{2-a}$) we get

$$\|\rho_1\|_{L^{6/5}(\mathbb{R}^3)} \leq CN^{-1} \left(\sum_{j \in \mathbb{N}} |\lambda_j| \left\|(-\Delta)^{1/4} \varphi_j\right\|_{L^2(\mathbb{R}^3)}^2\right)^{\frac{2}{6}} \left(\sum_{j \in \mathbb{N}} |\lambda_j| \|\varphi_j\|_{L^2(\mathbb{R}^3)}^2\right)^{\frac{2}{5}}$$

\[\leq CN^{-1} \left(\text{Tr} \left(\sqrt{-\Delta} |\omega_1|\right)\right)^{\frac{2}{6}} \left(\|\omega_1\|_{H^1}^2\right)^{\frac{2}{5}}. \quad (70)\]

Plugging this expression into the estimate of the potential shows $\text{(63)}$. Inequality $\text{(63)}$ is an immediate consequence of $\text{(63)}$ and $\text{Tr} (K * \rho_1 \omega_1) = N \int_{\mathbb{R}^3} K * \rho_1(x) \rho_1(x) \, dx$. In order to show $\text{(53)}$ we use again the spectral decomposition $\{\lambda_j, \varphi_j\}_{j \in \mathbb{N}}$ of $\omega_1$ and the Cauchy-Schwarz inequality to estimate

$$|\omega_1(x; y)| \leq \left(\sum_{j \in \mathbb{N}} |\lambda_j| \|\varphi_j(x)\|^2\right)^{1/2} \left(\sum_{j \in \mathbb{N}} |\lambda_j| \|\varphi_j(y)\|^2\right)^{1/2} \leq \sqrt{|\omega_1(x; x)|} \sqrt{|\omega_1(y; y)|}. \quad (71)\]
Together with the analogue estimate for \(\omega_2\) we obtain
\[
|\text{Tr} \, (X_1 \omega_2)| \leqslant N^{-1} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |K(x-y)| \left| \left( \omega_1|(x; x)\right) \omega_1|(y; y) \omega_2|(x; x) \omega_2|(y; y) \right|^{1/2} 
\leqslant N^{-1} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |K(x-y)||\omega_1|(x; x) |\omega_2|(y; y) 
= N^{-1} \int_{\mathbb{R}^3 \times \mathbb{R}^3} K(x-y)||\omega_1|(x; x) |\omega_2|(y; y)|. 
\]
(72)

3 Proofs of the results

3.1 Derivation of the relativistic Vlasov equation

In this subsection we provide the proof of Theorem 1.1 on the accuracy of the use of the relativistic Vlasov equation as effective equation for the semi-relativistic Hartree equation. In order to achieve this, we use some estimates proved in [12] and estimates for singular interaction potentials shown in [31]. For the sake of completeness, we report below the statements of the results we will use.

Note that within this section we will use both notations \(\hat{\rho}\) and \(\bar{\rho}(t)\) to denote the dependence of \(\rho\) on the time variable.

**Lemma 3.1** (Theorem 4 in [31]). Let \(d \in \{2, 3\}\), \(\omega \in \mathfrak{S}^1 \left( L^2(\mathbb{R}^d) \right)\), \(a \in \left( \frac{d}{2} - 2, d - 2 \right)\), \(\gamma \in \mathbb{R}\) and \(K\) be defined as in [3]. Let \(b := \frac{d}{a + 2}\) so that \(\nabla K \in L^{b, \infty}(\mathbb{R}^d)\) and \(b'\) be the conjugated Hölder exponent of \(b\). Then, for any \(\mu \in (0, b' - 1]\), there exists a constant \(C > 0\) such that
\[
\sup_{z \in \mathbb{R}^d} \|[[K(\cdot - z), \omega]](\mathfrak{S}^1) \| \leqslant C |\gamma| \|\text{diag} ([x, \omega])\|_{L^{b' - \mu}} \|\text{diag} ([x, \omega])\|_{L^{b' + \mu}}^\frac{1}{2}, 
\]
for any \(\tilde{\mu} \in (0, \frac{\mu}{2b'})\).

**Lemma 3.2** (Proposition 3.1 in [31]). Let \(d \in \{2, 3\}\), \(n \in 2\mathbb{N}\) and define \(\sigma = 4 + n\). Then, for any \(\hat{W}_N \in W^{3, \infty}(\mathbb{R}^d \times \mathbb{R}^d) \cap H^{\sigma + 1}_\sigma(\mathbb{R}^d \times \mathbb{R}^d)\) with Weyl quantization \(\hat{\omega}_N\), there exists a constant \(C > 0\) such that
\[
\|\text{diag} ([x, \hat{\omega}_N])\|_{L^p(\mathbb{R}^d)} \leqslant C \varepsilon \|
\nabla_v \hat{W}_N \|_{W^{2, \infty}(\mathbb{R}^{2d}) \cap H^{\sigma}_\sigma(\mathbb{R}^{2d})} \| \hat{\varepsilon} \|
\]
for any \(p \in \left[1, 1 + \frac{d}{2}\right]\).

Moreover, we will need to estimate the operator \(C_{N,t}\) with kernel
\[
C_{N,t}(x; y) = \left( (K * \hat{\rho}_t)(x) - (K * \hat{\rho}_t)(y) - \nabla (K * \hat{\rho}_t) \left( \frac{x + y}{2} \right) \cdot (x - y) \right) \hat{\omega}_{N,t}(x; y). 
\]
(75)

To this end we recall

**Lemma 3.3** (Proposition 4.4 in [31]). Let \(a \in \left( \frac{d}{2} - 2, d - 2 \right)\), \(\gamma \in \mathbb{R}\) and \(K\) be defined as in [3]. Moreover, let \(p \in [1, 2]\) and \(\nu = 4 + a - d\). Then, there exists a constant \(C\) independent of \(\varepsilon\) such that
\[
\|C_{N,t}\|_{L^p\sigma} \leqslant C |\gamma| \varepsilon^2 N^{\frac{1}{p}} \|\hat{\rho}(t)\|_{L^1\cap H^\nu(\mathbb{R}^d)} \|\nabla^2_v \hat{W}_N,t\|_{H^\nu_\nu(\mathbb{R}^{2d})}. 
\]
(76)
Proof of Theorem 1.1. In the following we denote the integral kernel of an operator $O : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by $O(x; y)$ and write the Fourier transform of the kernel as
\[
\hat{O}(p; q) = (2\pi)^{-d} \int_{\mathbb{R}^d \times \mathbb{R}^d} O(x; y) e^{-i x \cdot p} e^{i y \cdot q} \, dx \, dy.
\]
The Weyl quantization of the solution to the relativistic Vlasov equation $\tilde{W}_{N,t}$ evolves according to
\[
\hat{\tilde{W}}_{N,t} = A_{N,t} + B_{N,t}
\]
where $A_{N,t}$ and $B_{N,t}$ are defined as
\[
\hat{A}_{N,t}(p; q) = \frac{\varepsilon^2 (p - q) \cdot (p + q)}{2 \sqrt{1 + \frac{\varepsilon^2}{4} (p + q)^2}} \hat{\tilde{W}}_{N,t}(p; q)
\]
and
\[
B_{N,t}(x; y) = (x - y) \cdot \nabla (K * \tilde{\rho}_t) \left( \frac{x + y}{2} \right) \hat{\tilde{W}}_{N,t}(x; y).
\]
The trace norm difference of the solution of the semi-relativistic Hartree equation and the Weyl quantization of the Vlasov solution is bounded by
\[
\|\tilde{\omega}_{N,t} - \tilde{w}_{N,t}\|_{\mathcal{S}} \leq \|\tilde{\omega}_N - \tilde{w}_N\|_{\mathcal{S}}
\]
\[
+ \frac{\varepsilon}{2} \int_0^t \left\| \left[ 1 - \varepsilon^2 \Delta, \tilde{w}_{N,s} \right] - A_{N,s} \right\|_{\mathcal{S}} \, ds \tag{81}
\]
\[
+ \frac{1}{\varepsilon} \int_0^t \left\| [K * (\rho_s - \tilde{\rho}_s), \tilde{w}_{N,s}] \right\|_{\mathcal{S}} \, ds \tag{82}
\]
\[
+ \frac{1}{\varepsilon} \int_0^t \left\| C_{N,s} \right\|_{\mathcal{S}} \, ds \tag{83}
\]
with $C_{N,s}$ being defined as in (75). This estimate is easily derived (see [12, Section 3]) by means of a Duhamel expansion in analogy to [12, Section 3] (where the case $d = 3$ has been treated) one gets
\[
\left\| \left( 1 - \varepsilon^2 \Delta \right) \left( 1 + x^2 \right)^{-1} \left[ \sqrt{1 - \varepsilon^2 \Delta}, \tilde{w}_{N,s} \right] - A_{N,s} \right\|_{\mathcal{S}} \leq C N^\frac{4}{3} \varepsilon^2 \int_0^t \|\tilde{W}_{N,s}\|_{H^3_{\gamma}(\mathbb{R}^d)} \, ds \tag{84}
\]
Hölder’s inequality for Schatten spaces then implies
\[
\leq C N \varepsilon \int_0^t \|\tilde{W}_{N,s}\|_{H^3_{\gamma}(\mathbb{R}^d)} \, ds \tag{85}
\]
since $\left( (1 - \varepsilon^2 \Delta)^{-1} (1 + x^2)^{-1} \right) \|_{\mathcal{S}} \leq C N^\frac{4}{3}$ and $\sigma = 4 + n \geq 6$ holds by assumption. Note that
\[
\left\| \rho_s - \tilde{\rho}_s \right\|_{L^1(\mathbb{R}^d)} = \sup_{O \in L^\infty(\mathbb{R}^d), \|O\|_{L^\infty} \leq 1} \left| \int_{\mathbb{R}^d} O(x) (\rho_s(x) - \tilde{\rho}_s(x)) \, dx \right| \leq \frac{1}{N} \left\| \omega_{N,s} - \tilde{w}_{N,s} \right\|_{\mathcal{S}} \tag{86}
\]
because the space of bounded operators is the dual of the space of trace-class operators and every function $x \mapsto O(x)$ defines a multiplication operator. Hence
\[
\leq \frac{1}{\varepsilon} \int_0^t \left| \rho_s(z) - \tilde{\rho}_s(z) \right| \left\| [K(\cdot - z), \tilde{w}_{N,s}] \right\|_{\mathcal{S}} \, dz \, ds \tag{87}
\]
\[
\leq \frac{1}{\varepsilon} \int_0^t \left\| \omega_{N,s} - \tilde{w}_{N,s} \right\|_{\mathcal{S}} \sup_{z \in \mathbb{R}^d} \left\| [K(\cdot - z), \tilde{w}_{N,s}] \right\|_{\mathcal{S}} \, ds.
\]
Now let \( b' = \frac{d}{d - (a+1)} \), \( n \in 2\mathbb{N} \) such that \( n > \frac{d}{b} = d(b' - 1) \) and \( \mu \in (0, b' - 1) \) such that \( b' + \mu \leq 1 + \frac{n}{d} \).

By means of Lemma 3.1 and Lemma 3.2 we then get

\[
\sup_{z \in \mathbb{R}^3} \| [K(\cdot - z), \tilde{v}_{N,s}] \|_{\mathcal{E}^1} \leq C |\gamma| \| \text{diag} [x, \tilde{v}_{N,s}] \|_{L^{b'+\mu}(\mathbb{R}^d)} \\
\leq C |\gamma| |\varepsilon N| \| \nabla_v \tilde{W}_{N,s} \|_{W^{2,\infty}(\mathbb{R}^d) \cap H^{\frac{4n}{d}+\mu}(\mathbb{R}^d)}
\]

and

\[
\| \omega_{N,s} - \tilde{\omega}_{N,s} \|_{\mathcal{E}^1} \leq C |\gamma| \int_0^t \| \omega_{N,s} - \tilde{\omega}_{N,s} \|_{\mathcal{E}^1} + N \varepsilon C(s) \] ds.
\]

Together with Lemma 3.3 (\( \sigma \geq 6 \)) this leads to

\[
\| \omega_{N,t} - \tilde{\omega}_{N,t} \|_{\mathcal{E}^1} \leq \| \omega_N - \tilde{\omega}_N \|_{\mathcal{E}^1} + \int_0^t \lambda(s) \| \omega_{N,s} - \tilde{\omega}_{N,s} \|_{\mathcal{E}^1} + N \varepsilon C(s) \] ds,
\]

where

\[
\lambda(s) = C |\gamma| \| \nabla_v \tilde{W}_{N,s} \|_{W^{2,\infty}(\mathbb{R}^d) \cap H^{\frac{4n}{d}+\mu}(\mathbb{R}^d)}
\]

and

\[
C(s) = C (1 + |\gamma|) \left( 1 + \| \tilde{\gamma}_s \|_{L^1(\mathbb{R}^d) \cap H^{\frac{4n}{d}+\mu}(\mathbb{R}^d)} \right) \| \tilde{W}_{N,s} \|_{H^{n}(\mathbb{R}^d)}.
\]

Inequality (13) then follows from Gronwall’s Lemma.

By means of (86) and Theorem 1.1 it is possible to control the \( L^1 \)-distance between the semi-relativistic Hartree- and relativistic Vlasov-density. This enables us to extend Theorem 1.1 to arbitrary \( p \)-Schatten norms, as shown below.

**Proof of Proposition 1.7.** We proceed as in the proof of Theorem 1.1 and replace the trace norm by the \( p \)-Schatten norm in (81)–(83):

\[
\| \omega_{N,t} - \tilde{\omega}_{N,t} \|_{\mathcal{E}^p} \leq \| \omega_N - \tilde{\omega}_N \|_{\mathcal{E}^p} + \frac{1}{\varepsilon} \int_0^t \| \left( \sqrt{1 - \varepsilon^2 \Delta} \right) \omega_{N,s} - A_{N,s} \|_{\mathcal{E}^p} \] ds \\
+ \frac{1}{\varepsilon N} \int_0^t \| \omega_{N,s} - \tilde{\omega}_{N,s} \|_{\mathcal{E}^1} \sup_{z \in \mathbb{R}^d} \| [K(\cdot - z), \tilde{v}_{N,s}] \|_{\mathcal{E}^p} \] ds \\
+ \frac{1}{\varepsilon} \int_0^t \| C_{N,s} \|_{\mathcal{E}^p} \] ds,
\]

where we used (81) and the analogue of (81) for \( p \)-Schatten norms. Notice that the existence of the two parameter semi-group is addressed in Appendix 3 for \( p \)-Schatten norms.

We first look at \( p \in \left[ 1, \min \left\{ \frac{d}{a+1}, 2 \right\} \right) \). The third term is bounded by [31 Proposition 4.3], the refinement of the Calderón-Vaillancourt inequality [17] and Theorem 1.1. The fourth term is bounded in Lemma 3.3 Both terms provide a bound of order \( O(N^\frac{1}{2} \varepsilon) \). As for the second term, we proceed by interpolation. By (81) we get

\[
\left\| \left( \sqrt{1 - \varepsilon^2 \Delta} - A_{N,s} \right) \right\|_{\mathcal{E}^1} \leq C(t) N \varepsilon^2 \quad \text{and} \quad \left\| \left( \sqrt{1 - \varepsilon^2 \Delta} - A_{N,s} \right) \right\|_{\mathcal{E}^2} \leq C(t) N^\frac{1}{4} \varepsilon^2.
\]
Hence, for \( p = \min \left\{ \frac{d}{a+1}, 2 \right\} - \eta \), for \( \eta > 0 \) small,
\[
\left\| \sqrt{1 - \varepsilon^2 \Delta} \hat{\omega}_{N,t} - A_{N,t} \right\|_{\mathfrak{L}^p} \leq C(t) N^{\frac{1}{p}} \varepsilon^2.
\]
This is enough to get a bound on \( \| \omega_{N,t} - \hat{\omega}_{N,t} \|_{\mathfrak{L}^p} \) of order \( O(N^{\frac{1}{p}} \varepsilon^2) \), \( p \in \left[ \min \left\{ \frac{d}{a+1}, 2 \right\}, \infty \right] \).

To get a convergence rate for \( p \in \left[ \min \left\{ \frac{d}{a+1}, 2 \right\}, \infty \right] \), we interpolate between \( \| \omega_{N,t} - \hat{\omega}_{N,t} \|_{\mathfrak{L}^p} \) with \( q = \min \left\{ \frac{d}{a+1}, 2 \right\} - \eta \), for \( \eta > 0 \) arbitrarily small, and \( \| \omega_{N,t} - \hat{\omega}_{N,t} \|_{\mathfrak{L}^\infty} \leq C \| \omega_{N,t} \|_{\mathfrak{L}^\infty} + \| \hat{\omega}_{N,t} \|_{\mathfrak{L}^\infty} \leq C \). This yields
\[
\| \omega_{N,t} - \hat{\omega}_{N,t} \|_{\mathfrak{L}^p} \leq \| \omega_{N,t} - \hat{\omega}_{N,t} \|_{\mathfrak{L}^Q} \left\| \omega_{N,t} - \hat{\omega}_{N,t} \right\|_{\mathfrak{L}^2} \leq C \| \omega_{N,t} - \hat{\omega}_{N,t} \|_{\mathfrak{L}^Q} \}
\]
thus providing a bound of order \( O(N^{\frac{1}{p}} \varepsilon^2) \), with \( \frac{2}{p} < 1 \). Keeping track of all the constants and of the dependence on time concludes the proof.

3.2 Results about the relativistic Vlasov equation

Proof of Theorem 3.8 Let \( f \) be a solution of (11) and denote by
\[
M_k(t) = \sup_{0 \leq s \leq t} \int_{\mathbb{R}^d \times \mathbb{R}^d} |v|^k f(s, x, v) \, dx \, dv \quad \text{with } k \in \mathbb{N}
\]
the velocity moments of solutions of the relativistic Vlasov equation. Using integration by parts we get
\[
\frac{d}{dt} M_k(t) \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |v|^k f(t, x, v) \, dx \, dv \leq k \int_{\mathbb{R}^d \times \mathbb{R}^d} |E(t, x)| |v|^{k-1} f(t, x, v) \, dx \, dv.
\]
By means of Hölder’s inequality we obtain
\[
\frac{d}{dt} M_k(t) \leq k \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |v|^k f(t, x, v) \, dx \, dv \right)^{\frac{k-1}{k}} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |E(t, x)|^k f(t, x, v) \, dx \, dv \right)^{\frac{1}{k}},
\]
leading to
\[
\frac{d}{dt} M_k^\frac{1}{k}(t) \leq \left( \int_{\mathbb{R}^d} |E(t, x)|^k \rho_f(t, x) \, dx \right)^{\frac{1}{k}}.
\]
If we again apply Hölder’s inequality with \( p = \frac{(k+1)d}{k(d-(\alpha+1)) + \alpha d} \) and \( q = \frac{(k+1)d}{k(\alpha+1)} \), we get
\[
\frac{d}{dt} M_k^\frac{1}{k}(t) \leq \| \rho_f(t) \|_{L^k(\mathbb{R}^d)}^\frac{k+1}{k} \| E(t) \|_{L^q(\mathbb{R}^d)}^k = \| \rho_f(t) \|_{L^k(\mathbb{R}^d)}^\frac{k}{k} \| E(t) \|_{L^q(\mathbb{R}^d)}^k.
\]
Using (110) we obtain
\[
\frac{d}{dt} M_k^\frac{1}{k}(t) \leq C \| \rho_f(t) \|_{L^k(\mathbb{R}^d)}^\frac{k+1}{k} \| E(t) \|_{L^q(\mathbb{R}^d)}^k = C \| f(t) \|_{L^{\frac{(d-(\alpha+1)) + \alpha d}{d-(\alpha+1)}}(\mathbb{R}^d)}^\frac{k+1}{k} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |v|^{-(p-1)d} f(t, x, v) \, dx \, dv \right)^{\frac{1}{p-1}}.
\]
For \( a \in (-1, d-2) \) such that \( k \geq \frac{da}{d-(\alpha+1)} \) we have \( (p-1)d \leq k \). Thus if we use Lemma 2.2 we get
\[
\frac{d}{dt} M_k^\frac{1}{k}(t) \leq C \| f(t) \|_{L^\frac{a+1}{d-(\alpha+1)}(\mathbb{R}^d)} \| \rho_f(t) \|_{L^1(\mathbb{R}^d)}^\frac{k+1}{k} M_k^{\frac{a+1}{k}}(t).
\]
Together with \( \| \rho_f(t) \|_{L^1(R^d)} = \| \rho_f(0) \|_{L^1(R^d)} \) and \( \| f(t) \|_{L^\infty(R^{2d})} = \| f(0) \|_{L^\infty(R^{2d})} \) (see Proposition 2.4 and Gronwall’s Lemma, this shows the second part of the Theorem and the first part for \( a \in (-1, 0] \).

Next, we consider the case \( d = 3 \) and \( 0 < a \leq \frac{2k}{3 + k} \). For \( k \geq \frac{d}{3d + a + 1} \) and \( p \) defined as previously we have that \( 1 \leq 3(p - 1) \leq k \). By Proposition 2.4 and Lemma 2.5 we have \( M_1(t) \leq 3M_1(0) + C \left( a, \| f(0) \|_{L^{1+\infty}(R^3)} \right) \). If we use Lemma 2.2 to estimate the \((p - 1)\)-th velocity moment on the right-hand side of (99) by the first and \( k \)-th velocity moment, we get

\[
\frac{d}{dt} M_k \left( t \right) \leq C \left( f(0) \right) \left[ \frac{d}{k + 1} M_k \left( t \right) \right]^{\frac{2(3a - 3k + 3k + 3)}{3d + a + 1}} M_k \left( t \right) - \frac{d}{k + 1} M_k \left( t \right)
\]

Now let \( 0 \leq a \leq \frac{1}{2} \). Then \( a \leq \frac{2k}{3 + k} \) for all \( k \in \mathbb{N} \) and \( \frac{d}{3d + a + 1} \leq \frac{1}{2} \). For all \( k \geq \frac{3}{4a + 1} \) we consequently obtain long time estimates of the \( k \)-th velocity moments by means of Gronwall’s Lemma. For \( k \leq \frac{3}{4a + 1} \) we have \( 3(p - 1) \leq 1 \) and the right-hand side of (99) can solely be controlled by \( M_1(t) \leq 3M_1(0) + C \left( a, \| f(0) \|_{L^{1+\infty}(R^3)} \right) \).

**Proof of Corollary 1.9** Using (37), the interpolation inequality and Proposition 2.4 we obtain for all \( k > \frac{d(\alpha + 1)}{\alpha - (\alpha + 1)} \) that

\[
\| E(t) \|_{L^\infty(R^d)} \leq C \left[ \| \rho_f(t) \|_{L^1(R^d)} + \| f(t) \|_{L^\infty(R^{2d})} + \int_{R^d \times R^d} |v|^k f(t, x, v) dxdv \right].
\]

By Theorem 1.8 this shows \( E \in L^\infty \left( [0, T], L^\infty(R^d) \right) \) because \( \| \rho_f(t) \|_{L^1(R^d)} = \| f_0 \|_{L^1(R^d)} \) and \( \| f(t) \|_{L^\infty(R^{2d})} = \| f_0 \|_{L^\infty(R^{2d})} \). For \( n > d \) we obtain analogously to the non-relativistic case (see [24, Corollary 5.1]) the estimate

\[
\| \rho_f(t) \|_{L^\infty(R^d)} \leq C \left( \| f(t) \|_{L^\infty(R^{2d})} + \| f_0 \|_{L^\infty(R^{2d})} + \int_0^t \| f(s) \|_{L^\infty(R^d)} E(s) \|_{L^\infty(R^d)} ds \right)^{\frac{1}{n}},
\]

showing the claim.

**Proof of Proposition 1.10** Proposition 1.10 is proved in analogy to [31, Proposition A.1]. More explicitly, we show the inequalities

\[
\| E(t) \|_{W^0_{n,2} (R^{2d})} \leq CJ(t) \left( n(t) + \ln \left( 1 + \| f_0 \|_{W^0_{n,2} (R^{2d})} \right) \right) e^{C\int_0^t J(s) ds},
\]

\[
\| f(t) \|_{W^0_{\sigma, \infty} (R^{2d})} \leq \left( 1 + \| f_0 \|_{W^0_{\sigma, \infty} (R^{2d})} \right) \frac{Cn(t) + \sqrt{\left\| E(s) \right\|_{W^0_{\sigma, \infty} (R^{2d})}} ds}{e^{Cn(t) + \sqrt{\left\| E(s) \right\|_{W^0_{\sigma, \infty} (R^{2d})}} ds}},
\]

\[
\| f(t) \|_{H^\infty_x (R^d)} \leq \left( 1 + \sup_{s \in [0,t]} \| f(s) \|_{W^\sigma_{n, \infty} (R^{2d})} \right) \| f_0 \|_{H^\infty_x (R^d)} e^{Cn(t) + \sqrt{\left\| E(s) \right\|_{W^0_{\sigma, \infty} (R^{2d})}} ds},
\]

where \( J(t) = 1 + \| \rho_f(t) \|_{L^1(R^d)} + \| \rho_f(t) \|_{L^\infty(R^d)} \). These together with \( \| \nabla^\alpha \rho_f \|_{L^\infty(R^d)} \leq C \| f \|_{W^\alpha_{n, \infty} (R^{2d})} \) (which holds because \( n > d \)) prove the claim.

Define the transport operator \( T = -\frac{\nabla \cdot E}{\sqrt{1 + \nabla^2 \cdot \nabla \cdot E}} \cdot \nabla_x + E \cdot \nabla_v \) and to note that every solution \( f(t) \) of (4) satisfies

\[
\partial_t D^\alpha_x f(t) + T(D^\alpha_x f(t)) = - [D^\alpha_x, T] f(t),
\]

for a certain multi-index \( \alpha \). For sufficient regular functions \( f, g : \mathbb{R}^{2d} \rightarrow \mathbb{R} \), by means of

\[
p \int_{R^d \times R^d} g |f|^{p-2} f T(f) dx dv = \int_{R^d \times R^d} g T(|f|^p) dx dv = - \int_{R^d \times R^d} T(g) |f|^p dx dv
\]
and $T((z)^{np}) \leq np \langle z \rangle^{np-1} (1 + |E(t,x)|)$ with $z = (x,v) \in \mathbb{R}^{2d}$, we estimate

$$
\frac{d}{dt} \| \langle z \rangle^n D_v^\alpha f(t) \|_{L^p(\mathbb{R}^{2d})}^p
= \int_{\mathbb{R}^{2d}} \left| D_v^\alpha f \right|^p T((z)^{np}) \ dz - p \int_{\mathbb{R}^{2d}} \langle z \rangle^{np} \left| D_v^\alpha f \right|^{p-2} \left( D_v^\alpha f \right) [D_v^\alpha, T] f \ dz
\leq np(1 + \|E(t)\|_{L^\infty(\mathbb{R}^d)}) \langle \langle z \rangle^n D_v^\alpha f(t) \rangle_{L^p(\mathbb{R}^{2d})}^p + p \int_{\mathbb{R}^{2d}} \langle z \rangle^{np} \left| D_v^\alpha f \right|^{p-1} \| [D_v^\alpha, T] f \| \ dz,
$$

where we omitted the dependence of $f$ on $t, x, v$.

**Inequalities (103) and (105) for $\sigma = 1$:** Using that

$$
| [D_v^\alpha D_v^\beta, T] f(t, x, v) | \leq C \left( \left| \nabla_x f(t, x, v) \right| + \max_{i \in [1, d]} | \partial_x E(t, x) \| \nabla_v f(t, x, v) | \right)
$$

holds for all $\alpha, \beta \in \mathbb{N}_0^d$ such that $|\alpha| + |\beta| \leq 1$ and the multiplicative Young inequality, $pa^{p-1} \leq a^p + (p-1)b^p$, we get

$$
\frac{d}{dt} \left\{ \sum_{|\alpha| \leq 1} \| \langle z \rangle^n D_v^\alpha f(t) \|_{L^p(\mathbb{R}^{2d})}^p \right\} \leq Cnp(1 + \|E(t)\|_{W^{1,\infty}_0(\mathbb{R}^d)}) \sum_{|\alpha| \leq 1} \| \langle z \rangle^n D_v^\alpha f(t) \|_{L^p(\mathbb{R}^{2d})}^p.
$$

By Gronwall’s Lemma we obtain

$$
\| f(t) \|_{W^{1,p}_0(\mathbb{R}^{2d})} \leq \| f_0 \|_{W^{1,p}_0(\mathbb{R}^{2d})} e^{Cn \int_0^t \| E(s) \|_{W^{1,\infty}_0(\mathbb{R}^d)} ds}
$$

and

$$
\| f(t) \|_{W^{1,\infty}_0(\mathbb{R}^{2d})} \leq \| f_0 \|_{W^{1,\infty}_0(\mathbb{R}^{2d})} e^{Cn \int_0^t \| E(s) \|_{W^{1,\infty}_0(\mathbb{R}^d)} ds}
$$

if we take the limit $p \to \infty$. Note that

$$
\| E(t) \|_{W^{1,\infty}_0(\mathbb{R}^d)} \leq C \left( 1 + \| \rho f(t) \|_{L^1(\mathbb{R}^d)} + \| \rho f(t) \|_{L^\infty(\mathbb{R}^d)} \left( 1 + \ln \left( 1 + \left\| \nabla \rho f(t) \right\|_{L^\infty(\mathbb{R}^d)} \right) \right) \right)
$$

because of Lemma (103) and $\| \nabla \rho f(t) \|_{L^\infty(\mathbb{R}^d)} \leq C \| f(t) \|_{W^{1,\infty}_0(\mathbb{R}^{2d})}$ since $n > d$. Altogether this gives

$$
\| E(t) \|_{W^{1,\infty}_0(\mathbb{R}^d)} \leq C \left( 1 + \| \rho f(t) \|_{L^1(\mathbb{R}^d)} + \| \rho f(t) \|_{L^\infty(\mathbb{R}^d)} \right)
\times \left( n \left\| t \right\| + \ln \left( 1 + \| f_0 \|_{W^{1,\infty}_0(\mathbb{R}^{2d})} \right) + \int_0^t \| E(s) \|_{W^{1,\infty}_0(\mathbb{R}^d)} ds \right).
$$

Applying Gronwall’s Lemma again leads to (104).

**Inequality (105) for $\sigma > 1$:** Next, we show

$$
\sup_{s \in [0, t]} \| f(s) \|_{W^{\sigma,\infty}_0(\mathbb{R}^{2d})} \leq \left( 1 + \| f_0 \|_{W^{\sigma,\infty}_0(\mathbb{R}^{2d})} \right) + \sup_{s \in [0, t]} \| f(s) \|_{W^{\sigma-1,\infty}_0(\mathbb{R}^{2d})}^4 \ e^{Cn \int_0^t \| E(s) \|_{W^{1,\infty}_0(\mathbb{R}^d)} ds},
$$

(115)
which in combination with (112) implies, by induction, inequality (105) for all \( \sigma \in \mathbb{N} \). Using (108), the multiplicative Young inequality and that \( D^\gamma v (t) \leq C \) holds for all \( \gamma \in \mathbb{N}_0^d \), we obtain

\[
\frac{d}{dt} \| f(t) \|_{W^{\sigma,p}(\mathbb{R}^d)}^p \\
\leq np \left( 1 + \| E(t) \|_{L^\infty(\mathbb{R}^d)} \right) \| f(t) \|_{W^{\sigma,p}(\mathbb{R}^d)}^p \\
+ \sum_{|\beta| + |\gamma| \leq \sigma} \int_{\mathbb{R}^d} |D_x^\beta D_v^\gamma f(t)|^p \left( |D_x^\beta, E(t,x)| \cdot \nabla_v D_v^\gamma f(t) \right) dx 
\]

\[
\leq np \left( 1 + \| E(t) \|_{L^\infty(\mathbb{R}^d)} \right) \| f(t) \|_{W^{\sigma,p}(\mathbb{R}^d)}^p \\
+ \sum_{|\beta| + |\gamma| \leq \sigma} \int_{\mathbb{R}^d} |D_x^\beta D_v^\gamma f(t)|^p \left( |D_x^\beta, E(t,x)| \cdot \nabla_v D_v^\gamma f(t) \right) dx 
\]

\[
\leq np \left( 1 + \| E(t) \|_{L^\infty(\mathbb{R}^d)} \right) \| f(t) \|_{W^{\sigma,p}(\mathbb{R}^d)}^p \\
+ \sum_{|\beta| + |\gamma| \leq \sigma} \int_{\mathbb{R}^d} |D_x^\beta D_v^\gamma f(t)|^p \left( |D_x^\beta, E(t,x)| \cdot \nabla_v D_v^\gamma f(t) \right) dx, 
\]

where we omitted the dependence of \( f \) from \( x \) and \( v \) and use the variable \( z = (x,v) \in \mathbb{R}^d \). Note that, for \( \beta, \gamma \) and \( \delta \) multi-indices,

\[
\sum_{|\beta| + |\gamma| \leq \sigma} |D_x^\beta D_v^\gamma f(t)|^p \leq \left| E(t) \right|_{W^{\sigma \cdot, \infty}(\mathbb{R}^d)} \sum_{|\beta| + |\gamma| \leq \sigma} \sum_{|\delta| = |\beta| - 1} |D_x^\beta D_v^\gamma f(t)|^p |D_x^\delta \nabla_v D_v^\gamma f(t)| \\
+ \sum_{|\beta| + |\gamma| \leq \sigma} \sum_{|\delta| \leq |\beta| - 2} |D_x^\beta D_v^\gamma f(t)|^p |E(t)\|_{W^{\sigma \cdot, \infty}(\mathbb{R}^d)} \| D_x^\delta \nabla_v D_v^\gamma f(t) \| \\
\leq C \left( 1 + \| E(t) \|_{W^{\sigma \cdot, \infty}(\mathbb{R}^d)} \right) \sum_{|\beta| + |\gamma| \leq \sigma} |D_x^\beta D_v^\gamma f(t)|^p + \frac{C}{p} \| E(t) \|_{W^{\sigma \cdot, \infty}(\mathbb{R}^d)}^p \sum_{|\beta| + |\gamma| \leq \sigma - 1} |D_x^\beta D_v^\gamma f(t)|^p, 
\]

and therefore

\[
\frac{d}{dt} \| f(t) \|_{W^{\sigma,p}(\mathbb{R}^d)}^p \leq Cnp \left( 1 + \| E(t) \|_{W^{\sigma \cdot, \infty}(\mathbb{R}^d)} \right) \| f(t) \|_{W^{\sigma,p}(\mathbb{R}^d)}^p + C \| E(t) \|_{W^{\sigma \cdot, \infty}(\mathbb{R}^d)} \| f(t) \|_{W^{\sigma - 1,p}(\mathbb{R}^d)}^p. 
\]

Applying Gronwall’s Lemma, taking the \( p \)-th root and using (110) lead to

\[
\| f(t) \|_{W^{\sigma,p}(\mathbb{R}^d)} \leq e^{Cn \left( \frac{t}{\sqrt{\| E(s) \|_{W^{\sigma \cdot, \infty}(\mathbb{R}^d)}} \right) \| f_0 \|_{W^{\sigma,p}(\mathbb{R}^d)}} + \sup_{s \in [0,t]} \| E(s) \|_{W^{\sigma \cdot, \infty}(\mathbb{R}^d)} \| f(s) \|_{W^{\sigma - 1,p}(\mathbb{R}^d)} \\
\leq e^{Cn \left( \frac{t}{\sqrt{\| E(s) \|_{W^{\sigma \cdot, \infty}(\mathbb{R}^d)}} \right) \| f_0 \|_{W^{\sigma,p}(\mathbb{R}^d)}} + \sup_{s \in [0,t]} \| f(s) \|_{W^{\sigma - 1,p}(\mathbb{R}^d)} \left( 1 + \| f(s) \|_{W^{\sigma - 1,\infty}(\mathbb{R}^d)} \right) \left( 1 + \ln \left( 1 + \| f(s) \|_{W^{\sigma \cdot, \infty}(\mathbb{R}^d)} \right) \right). 
\]

Taking the limit \( p \rightarrow \infty \) and estimating the logarithm by \( C(1 + \| f(s) \|_{W^{\sigma \cdot, \infty}(\mathbb{R}^d)}^{1/2} \) give (115).
Inequality (109): Inequality (111) proves (106) for $\sigma = 1$. From (119) we get
\[
\|f(t)\|_{H^\sigma_x(\mathbb{R}^d)} \leq e^{Cn(t) + \int_0^t \|E(s)\|_{L^\infty(\mathbb{R}^d)} ds} \sup_{s \in [0,t]} \left( 1 + \|f(s)\|_{W^{\infty,x}_x(\mathbb{R}^d)} \right) \left( \|f_0\|_{H^\sigma_x(\mathbb{R}^d)} + \|f(s)\|_{H^{\sigma-1}_x(\mathbb{R}^d)} \right)
\]
(120)
which, by induction, enables to infer (110) for all $\sigma \in \mathbb{N}$.

Proof of Proposition 1.11 The statement is proven in analogy to [31, Proposition 2.1]. By means of
\[
\partial_t (f_1 - f_2) = -\langle v \rangle \cdot \nabla_x (f_1 - f_2) + \nabla K * \rho_{f_1} \cdot \nabla_v (f_1 - f_2) + \nabla K * (\rho_{f_2} - \rho_{f_1}) \cdot \nabla_v f_2
\]
we denote $f := f_1 - f_2$ and estimate
\[
\frac{d}{dt} \int_{\mathbb{R}^d \times \mathbb{R}^d} |(f_1 - f_2)(t, x, v)| \, dx \, dv \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \text{sign}[f(t, x, v)] (\nabla K * (\rho_{f_1} - \rho_{f_2}))(t, x) \cdot \nabla_v f_2(t, x, v) \, dx \, dv \\
\leq \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} |\nabla K(x - y)| |(\rho_{f_1} - \rho_{f_2})(t, y)| \nabla_v f_2(t, x, v) \, dx \, dy \\
\leq \|\rho_{f_1} - \rho_{f_2}\|_{L^1(\mathbb{R}^d)} \left\| \nabla K \right\| \left( \int_{\mathbb{R}^d} \|\nabla_v f_2(t, , v)\| \, dv \right) \right\|_{L^1(\mathbb{R}^d)}.
\]
(122)
Using (37) we obtain
\[
\frac{d}{dt} \|f_1 - f_2(0)\|_{L^1_x(\mathbb{R}^d)} \leq C \|f_1 - f_2(0)\|_{L^1_x(\mathbb{R}^d)} \|\rho_{\nabla_v f_2}(t)\|_{L^{\frac{d}{d-(a+1)}}(\mathbb{R}^d)}
\]
(123)
for arbitrary $0 < \delta < \frac{d}{d-(a+1)}$. Together with Gronwall’s Lemma this leads to (20).

A Well-posedness of the Hartree and Hartree-Fock equations

In this section we will prove Proposition 1.13. We restrict our consideration to the Hartree-Fock equation (1). By setting the exchange term to zero in each estimate one obtains the respective result for the Hartree equation (2). We first consider a finite rank version of equation (1) and sketch how one generalizes the global well-posedness result from [14] to general inverse power law potentials. Following the approach of [8] this will allow us to finally prove Proposition 1.13. It should be pointed out that this route was used in [11] to obtain the result for the case $a = 1$. We, nevertheless, decided to present the details for the convenience of the reader. Throughout this section we will use the notations $D^s = (1 - \Delta)^{s/2}$ and $D^s_\varepsilon = (1 - \varepsilon^2 \Delta)^{s/2}$. The norm of the space $\mathfrak{P}^{1,\frac{1}{2}}$ can then be written as
\[
\|\omega\|_{\mathfrak{P}^{1,\frac{1}{2}}} = \left\| D^{1/2} \omega D^{1/2} \right\|_{\mathfrak{P}^{1}}.
\]
(124)

A.1 Finite rank system

In the proof of Proposition 1.13 we will use that the following set of $M$ coupled equations
\[
i\varepsilon \partial_t \psi_k(t) = \sqrt{1 - \varepsilon^2 \Delta} \psi_k(t) + \frac{1}{N} \sum_{l=1}^M \left( K * |\psi_l(t)|^2 \right) \psi_k(t) - \frac{1}{N} \sum_{l=1}^M \left( K * \{\psi_l(t) \psi_k(t)\} \right) \psi_l(t)
\]
(125)
is globally well-posed in $H^{s,M} = (H^s(\mathbb{R}^3))^M$ with the norm $\left\| (\psi_k)_{k=1}^M \right\|_{H^{s,M}} = \left( \sum_{k=1}^M \|\psi_k\|_{H^s(\mathbb{R}^3)}^2 \right)^{1/2}$ and $s = 1/2$. For the case $a = 1$ this has been proven in [14]. We will rely on the following slight generalization of [14, Theorem 2.1 and Theorem 2.2].
Lemma A.1. Let $s \geq 1/2$, $0 < a \leq 1$, $\gamma \in \mathbb{R}$ and $M, N \geq 1$ be integers. Let $\{\psi_{k,0}\}_{k=1}^{M} \subset H^s(\mathbb{R}^3)$ satisfying $0 \leq \langle \psi_{k,0}, \psi_{l,0} \rangle \leq \delta_{k,l}$, where $\delta_{k,l}$ is the Kronecker delta. If $\gamma < 0$ and $a = 1$, in addition, assume that
\[
|\gamma| < \frac{N \varepsilon}{\gamma_{\text{cr}} \left( \sum_{k=1}^{M} \|\psi_{k,0}\|^2_{L^2(\mathbb{R}^3)} \right)^{2/3}},
\]
(126)
where $\gamma_{\text{cr}}$ is a universal constant of order 1. Then, there exists a unique global solution, $\{\psi_k(t)\}_{k=1}^{M} \subset H^s(\mathbb{R}^3)$ solving
\[
\psi_k(t) = \psi_{k,0} \quad \text{and} \quad \psi_k \in C^0(\mathbb{R}_+, H^s(\mathbb{R}^3)) \cap C^1(\mathbb{R}_+, H^{s-1}(\mathbb{R}^3))
\]
holds for all $k = 1, \ldots, N$. The solution continuously depends on the initial data and
\[
\langle \psi_k(t), \psi_l(t) \rangle_{L^2(\mathbb{R}^3)} = \langle \psi_{k,0}, \psi_{l,0} \rangle_{L^2(\mathbb{R}^3)} \quad \text{and} \quad \mathcal{E}_{\text{HF}} \left[ \left\{ \psi_k(t) \right\}_{k=1}^{M} \right] = \mathcal{E}_{\text{HF}} \left[ \left\{ \psi_k(0) \right\}_{k=1}^{M} \right]
\]
holds for all $1 \leq k, l \leq M$ and $t \in \mathbb{R}_+$ with the energy $\mathcal{E}_{\text{HF}}$ being defined as in (131).

Proof of Lemma A.1. Let $s \geq 1/2$, $M \geq 1$ be an integer and $F = (F_1, \ldots, F_M) : H^s, M \rightarrow H^{s, M}$ be given by $F_k \left( \left\{ \psi_{i} \right\}_{i=1}^{M} \right) = \frac{1}{\sqrt{2}} \sum_{i=1}^{M} \left( K * |\psi_i(t)|^2 \right) \psi_i(t) - \frac{1}{\sqrt{2}} \sum_{i=1}^{M} \left( K * |\psi_i(t)\psi_i(t)| \right) \psi_i(t)$. Using (52) and straightforward manipulations we obtain for
\[
\| F(\left\{ \psi_k \right\}_{k=1}^{M}) - F(\left\{ \phi_k \right\}_{k=1}^{M}) \|_{H^{s, M}} \leq C \left( \| \left\{ \psi_k \right\}_{k=1}^{M} \|_{H^s, M}^2 + \| \left\{ \phi_k \right\}_{k=1}^{M} \|_{H^s, M}^2 \right) \| \psi_k - \phi_k \|_{H^{s, M}}^2
\]
(129)
\[
\| F(\left\{ \psi_k \right\}_{k=1}^{M}) \|_{H^{s, M}} \leq C \left( \| \left\{ \psi_k \right\}_{k=1}^{M} \|_{H^s, M}^2 \right) \| \psi_k \|_{H^{s, M}}^2
\]
(130)
for all $\left\{ \psi_k \right\}_{k=1}^{M}, \left\{ \phi_k \right\}_{k=1}^{M} \in H^s, M$. By standard methods we obtain the local-in-time existence and uniqueness of $\left\{ \psi_k(t) \right\}_{k=1}^{M}$ as well as the continuous dependence on the initial data, i.e. the analogue of [14 Theorem 2.1]. If $t$ is smaller than the maximal time of existence we, moreover, have both that
\[
\langle \psi_k(t), \psi_l(t) \rangle_{L^2(\mathbb{R}^3)} = \langle \psi_k(0), \psi_l(0) \rangle_{L^2(\mathbb{R}^3)}
\]
for all $1 \leq k, l \leq M$ and that the energy, defined by
\[
\mathcal{E}_{\text{HF}} \left[ \left\{ \psi_k(t) \right\}_{k=1}^{M} \right] = \text{Tr} \left( \sqrt{1 - \varepsilon^2 \Delta} \omega_t \right) + \frac{1}{2} \text{Tr} \left( (K * \rho_t - X_t) \omega_t \right)
\]
(131)
with $\omega_t = \sum_{i=1}^{M} |\psi_i(t)|^2 \langle \psi_i(t) \rangle_{\mathbb{R}^3}$, $\rho_t = N^{-1} \omega_t \langle \psi_t \rangle_{\mathbb{R}^3}$ and $X_t = N^{-1} K(x-y) \omega_t \langle \psi_t \rangle_{\mathbb{R}^3}$, are conserved quantities.

By means of the integral version of (123), inequality (130) and Gronwall’s Lemma one can show in the same spirit of [33 p. 57] that, for any $s > 1/2$ and all times $T_s$ smaller than the maximal time of existence
\[
\sup_{0 \leq t \leq T_s} \| \left\{ \psi_k(t) \right\}_{k=1}^{M} \|_{H^s, M} \leq C \left( T_s, \| \left\{ \psi_k(0) \right\}_{k=1}^{M} \|_{H^s, M} \right) \Rightarrow \sup_{0 \leq t \leq T_s} \| \left\{ \psi_k(t) \right\}_{k=1}^{M} \|_{H^{s, M}}.
\]
(132)
This implies that the maximal time of existence of any $H^s$-valued solution with $s > 1/2$ is the same as the maximal time of existence of the $H^{1/2}$-valued solution. Hence it suffices to show global well-posedness of the $H^{1/2}$-valued solution. In this regard note that for $0 < a < 1$
\[
\mathcal{E}_{\text{HF}} \left[ \left\{ \psi_k(t) \right\}_{k=1}^{M} \right] = \text{Tr} \left( \sqrt{1 - \varepsilon^2 \Delta} \omega_t \right) \leq \frac{1}{2} \text{Tr} \left( \sqrt{-\varepsilon^2 \Delta} \omega_t \right) + C \left( \varepsilon N^{-1} \right)^{1/2} \left( \varepsilon N^{-1} \right)^{1/2} \left( \varepsilon N^{-1} \right)^{1/2}
\]
(133)
holds because of (54), (55) and Young’s inequality for products. We then get
\[
\| \left\{ \psi_k(t) \right\}_{k=1}^{M} \|_{H^{s, M}}^{2} \leq \varepsilon^{-1} \text{Tr} \left( \sqrt{1 - \varepsilon^2 \Delta} \omega_t \right) \leq 2 \varepsilon^{-1} \mathcal{E}_{\text{HF}} \left[ \left\{ \psi_k(t) \right\}_{k=1}^{M} \right] + C \left( \varepsilon^a N^{-1} \right)^{1/2} \left( \varepsilon N^{-1} \right)^{1/2}
\]
(134)
\[
\leq C \varepsilon^{-1} \text{Tr} \left( \sqrt{-\varepsilon^2 \Delta} \omega_0 \right) + C \left( \varepsilon^a N^{-1} \right)^{1/2} \left( \varepsilon N^{-1} \right)^{1/2}
\]
(135)
for all $0 < a < 1$ and $\gamma \in \mathbb{R}$ by the conservation of the energy and mass. If $\gamma \in \mathbb{R}_+$ the second summand on the right-hand side of (131) is positive because of (55). Together with (54) and the conservation of energy this gives

$$
\left\| \psi_{k}(t) \right\|_{H^{\frac{1}{2},M}}^{2} \leq \varepsilon^{-1} \mathcal{E}_{HF} \left\{ \left\{ \psi_{k}(t) \right\}_{k=1}^{M} \right\} \leq C \varepsilon^{-1} \left( \sqrt{1-\varepsilon^{2} \Delta} \omega_{0}^{\leq M} \right) \left( 1 + \text{Tr} \left( \omega_{0}^{\leq M} \right) \right)
$$

(136)

for $a = 1$ and $\gamma \in \mathbb{R}_+$. For $a = 1$ and $\gamma \in \mathbb{R}_-$ we proceed as in [14]. First, we notice that the exchange term is negative in this case. Second, we use the Hardy–Littlewood–Sobolev inequality and the interpolation inequality to estimate

$$
\text{Tr} \left( K * \rho_{t}^{\leq M} \omega_{t}^{\leq M} \right) \leq C N_{\gamma} \left( \int_{\mathbb{R}^{3}} \rho_{t}^{\leq M}(x) \, dx \right)^{2/3} \int_{\mathbb{R}^{3}} (-\Delta)^{1/2} \rho_{t}^{\leq M}(x) \, dx
$$

(137)

Third, we apply $\int_{\mathbb{R}^{3}} \rho_{t}^{\leq M}(x) \, dx = N^{-1} \text{Tr} \left( \omega_{0}^{\leq M} \right)$ and [14] Lemma A.1 to get

$$
\text{Tr} \left( K * \rho_{t}^{\leq M} \omega_{t}^{\leq M} \right) \leq C N_{\gamma} \left( \text{Tr} \left( \omega_{0}^{\leq M} \right) \right)^{2/3} \left\| \left\{ \psi_{k}(t) \right\}_{k=1}^{M} \right\|_{H^{\frac{1}{2},M}}^{2}.
$$

(138)

This leads to

$$
\left\| \left\{ \psi_{k}(t) \right\}_{k=1}^{M} \right\|_{H^{\frac{1}{2},M}}^{2} \leq \varepsilon^{-1} \mathcal{E}_{HF} \left\{ \left\{ \psi_{k}(t) \right\}_{k=1}^{M} \right\} + (2\varepsilon)^{-1} \left| \text{Tr} \left( K * \rho_{t}^{\leq M} \omega_{t}^{\leq M} \right) \right|
$$

$$
\leq \varepsilon^{-1} \mathcal{E}_{HF} \left\{ \left\{ \psi_{k}(t) \right\}_{k=1}^{M} \right\} + C \varepsilon^{-1} \left( \text{Tr} \left( \omega_{0}^{\leq M} \right) \right)^{2/3} \left\| \left\{ \psi_{k}(t) \right\}_{k=1}^{M} \right\|_{H^{\frac{1}{2},M}}^{2}.
$$

(139)

Assuming $1 > \left( \frac{C \varepsilon}{2\varepsilon} \text{Tr} \left( \omega_{0}^{\leq M} \right)^{2/3} \right)$ we obtain

$$
\left\| \left\{ \psi_{k}(t) \right\}_{k=1}^{M} \right\|_{H^{\frac{1}{2},M}}^{2} \leq C \varepsilon^{-1} \mathcal{E}_{HF} \left\{ \left\{ \psi_{k}(t) \right\}_{k=1}^{M} \right\}.
$$

A.2 Proof of Proposition 1.13

Note that the integrated form of (11) is given by

$$
\omega_{N,t} = e^{-i\sqrt{1-\varepsilon^{2}}\Delta t} \omega_{0} e^{i\sqrt{1-\varepsilon^{2}}\Delta t} - \int_{0}^{t} e^{-i\sqrt{1-\varepsilon^{2}}\Delta (t-s)} i \left[ \left( K * \rho_{s} - X_{s} \right), \omega_{N,s} \right] e^{i\sqrt{1-\varepsilon^{2}}\Delta (t-s)} \, ds.
$$

(140)

Lemma A.2. For $a \in (0,1]$ equation (140) has a unique local solution in $\mathcal{E}^{1,\frac{1}{2}} \left( L^{2}(\mathbb{R}^{3}) \right)$.

Proof of Lemma A.2: Since $\sqrt{1-\varepsilon^{2}}\Delta$ is self-adjoint and commutes with $D^{\frac{1}{2}}$ we have that $\mathbb{R}_{+} \rightarrow \mathcal{E}^{1,\frac{1}{2}} \left( L^{2}(\mathbb{R}^{d}) \right)$, $\omega \rightarrow e^{-i\sqrt{1-\varepsilon^{2}}\Delta t} \omega e^{i\sqrt{1-\varepsilon^{2}}\Delta t}$ defines a strongly continuous semigroup. Lemma A.2 then follows from [13] Theorem 1 and the Lipschitz property of $\mathcal{E}^{1,\frac{1}{2}} \left( L^{2}(\mathbb{R}^{3}) \right) \rightarrow \mathcal{E}^{1,\frac{1}{2}} \left( L^{2}(\mathbb{R}^{3}) \right)$, $\omega \rightarrow i \left[ \left( K * \rho - X \right), \omega \right]$. To prove this fact we let $\omega, \tilde{\omega} \in \mathcal{E}^{1,\frac{1}{2}} \left( L^{2}(\mathbb{R}^{3}) \right)$, $\rho(x) = N^{-1} \omega(x; x)$ and $X(x; y) = N^{-1} (K * \rho - X) \omega(x; y)$. By Hölder’s inequality for Schatten spaces we obtain

$$
\left\| i \left[ K * \rho, \tilde{\omega} \right] \right\|_{\mathcal{E}^{1,\frac{1}{2}}} \leq 2 \left\| D^{\frac{1}{2}} K * \rho \tilde{\omega} \right\|_{\mathcal{E}^{1,\frac{1}{2}}} \left\| \tilde{\omega} \right\|_{\mathcal{E}^{1,\frac{1}{2}}} \left\| D^{\frac{1}{2}} \right\|_{\mathcal{E}^{1,\frac{1}{2}}} \left\| \omega \right\|_{\mathcal{E}^{1,\frac{1}{2}}}
$$

$$
= 2 \left\| D^{\frac{1}{2}} K * \rho \tilde{\omega} K * \rho D^{\frac{1}{2}} \right\|_{\mathcal{E}^{1,\frac{1}{2}}} \left\| \omega \right\|_{\mathcal{E}^{1,\frac{1}{2}}} \left\| D^{\frac{1}{2}} \right\|_{\mathcal{E}^{1,\frac{1}{2}}} \left\| \tilde{\omega} \right\|_{\mathcal{E}^{1,\frac{1}{2}}} \left\| D^{\frac{1}{2}} \right\|_{\mathcal{E}^{1,\frac{1}{2}}} \left\| \omega \right\|_{\mathcal{E}^{1,\frac{1}{2}}}
$$

$$
\leq 2 \left\| D^{\frac{1}{2}} K * \rho \right\|_{\mathcal{E}^{1,\frac{1}{2}}} \left\| D^{\frac{1}{2}} \tilde{\omega} \right\|_{\mathcal{E}^{1,\frac{1}{2}}} \left\| D^{\frac{1}{2}} K * \rho \right\|_{\mathcal{E}^{1,\frac{1}{2}}} \left\| \tilde{\omega} \right\|_{\mathcal{E}^{1,\frac{1}{2}}} \left\| D^{\frac{1}{2}} \right\|_{\mathcal{E}^{1,\frac{1}{2}}} \left\| \omega \right\|_{\mathcal{E}^{1,\frac{1}{2}}}
$$

$$
\leq 2 \left\| D^{\frac{1}{2}} K * \rho \right\|_{\mathcal{E}^{1,\frac{1}{2}}} \left\| D^{\frac{1}{2}} \tilde{\omega} \right\|_{\mathcal{E}^{1,\frac{1}{2}}} \left\| D^{\frac{1}{2}} K * \rho \right\|_{\mathcal{E}^{1,\frac{1}{2}}} \left\| \tilde{\omega} \right\|_{\mathcal{E}^{1,\frac{1}{2}}} \left\| D^{\frac{1}{2}} \right\|_{\mathcal{E}^{1,\frac{1}{2}}} \left\| \omega \right\|_{\mathcal{E}^{1,\frac{1}{2}}}.
$$

(141)
Similarly, \( \| i [ X, \tilde{\omega} ] \|_{L^1} \leq 2 \| D^\frac{1}{2} X D^{-\frac{1}{2}} \|_{L^\infty} \| \tilde{\omega} \|_{L^1} \). By the spectral theorem there exists a spectral set \( \{ \lambda_j, \varphi_j \}_{j \in \mathbb{N}} \) with \( \lambda_j \geq 0 \) for all \( j \in \mathbb{N} \) such that \( \omega = \sum_{j \in \mathbb{N}} \lambda_j | \varphi_j \rangle \langle \varphi_j | \). From \( D^\frac{1}{2} \varphi_j = \lambda_j^{-1} D^\frac{1}{2} \omega \varphi_j \) we conclude \( \| D^\frac{1}{2} \varphi \|_{L^2(\mathbb{R}^3)} \leq \lambda_j^{-1} \| D^\frac{1}{2} \omega D^\frac{1}{2} \|_{L^2(\mathbb{R}^3)} \| \varphi \|_{L^2(\mathbb{R}^3)} \). This implies \( \varphi \in H^\frac{1}{2}(\mathbb{R}^3), D^\frac{1}{2} \omega D^\frac{1}{2} = \sum_{j \in \mathbb{N}} \lambda_j | D^\frac{1}{2} \varphi_j \rangle \langle D^\frac{1}{2} \varphi_j | \),

\[
\| \varphi \|_{H^\frac{1}{2}} = \text{Tr} \left( D^\frac{1}{2} \omega D^\frac{1}{2} \right) = \sum_{j \in \mathbb{N}} \lambda_j \| D^\frac{1}{2} \varphi_j \|_{L^2(\mathbb{R}^3)}^2 \quad \text{and} \quad \rho(x) = N^{-1} \sum_{j \in \mathbb{N}} \lambda_j | \varphi_j (x) |^2. \tag{142}
\]

Using (122) we estimate for \( h \in L^2(\mathbb{R}^3) \)

\[
\| D^\frac{1}{2} K \ast \rho D^{-\frac{1}{2}} h \|_{L^2(\mathbb{R}^3)} \leq N^{-1} \sum_{j \in \mathbb{N}} \lambda_j \| D^\frac{1}{2} K \ast (| \varphi_j |^2) D^{-\frac{1}{2}} h \|_{L^2(\mathbb{R}^3)} \leq CN^{-1} | \gamma | \| \varphi \|_{H^\frac{1}{2}} \| h \|_{L^2(\mathbb{R}^3)}, \tag{143}
\]

\[
\| D^\frac{1}{2} X D^{-\frac{1}{2}} h \|_{L^2(\mathbb{R}^3)} \leq N^{-1} \sum_{j \in \mathbb{N}} \lambda_j \| D^\frac{1}{2} K \ast (| \varphi_j |^2) D^{-\frac{1}{2}} h \|_{L^2(\mathbb{R}^3)} \leq CN^{-1} | \gamma | \| \varphi \|_{H^\frac{1}{2}} \| h \|_{L^2(\mathbb{R}^3)}, \tag{144}
\]

which leads to

\[
\| i \left( [ K \ast \rho - X ], \tilde{\omega} \right) \|_{L^1} \leq CN^{-1} | \gamma | \| \varphi \|_{H^\frac{1}{2}} \| \tilde{\omega} \|_{L^1}. \tag{145}
\]

If \( \omega \in \mathfrak{c}^1(\mathbb{R}^3) \) is not a positive operator we can split the compact and self adjoint operator \( D^\frac{1}{2} \omega D^\frac{1}{2} = \left( D^\frac{1}{2} \omega D^\frac{1}{2} \right)_+ - \left( D^\frac{1}{2} \omega D^\frac{1}{2} \right)_- \) into its positive and negative part. By its spectral decomposition one easily checks the properties \( \left( D^\frac{1}{2} \omega D^\frac{1}{2} \right)_+ \left( D^\frac{1}{2} \omega D^\frac{1}{2} \right)_- = 0 \) and \( \left( D^\frac{1}{2} \omega D^\frac{1}{2} \right)_+ = \left( D^\frac{1}{2} \omega D^\frac{1}{2} \right)_+ + \left( D^\frac{1}{2} \omega D^\frac{1}{2} \right)_- \). The positive operators \( \omega_+ = D^\frac{1}{2} \left( D^\frac{1}{2} \omega D^\frac{1}{2} \right)_+ D^{-\frac{1}{2}} \) and \( \omega_- = D^{-\frac{1}{2}} \left( D^\frac{1}{2} \omega D^\frac{1}{2} \right)_- D^{-\frac{1}{2}} \) satisfy \( \omega_+ - \omega_- = \omega \) and \( \| D^\ast \omega D^\ast \|_{\mathfrak{c}^1} = \| D^\ast \omega_+ D^\ast \|_{\mathfrak{c}^1} + \| D^\ast \omega_- D^\ast \|_{\mathfrak{c}^1} \). By means of the splitting and the triangular inequality it is easily shown that

\[
\| i \left( [ K \ast \rho - X ], \tilde{\omega} \right) \|_{\mathfrak{c}^1} \leq CN^{-1} | \gamma | \| \varphi \|_{\mathfrak{c}^1} \| \tilde{\omega} \|_{\mathfrak{c}^1}. \tag{146}
\]

holds for all \( \omega, \tilde{\omega} \in \mathfrak{c}^1(\mathbb{R}^3) \) and that the mapping \( \mathfrak{c}^1(\mathbb{R}^3) \to \mathfrak{c}^1(\mathbb{R}^3) \), \( \omega \to i \left( [ K \ast \rho - X ], \omega \right) \) is locally Lipschitz.

\[ \square \]

**Lemma A.3.** Suppose the initial data \( \omega_0 \) is a finite rank operator in \( \mathfrak{c}^1_+ \left( L^2(\mathbb{R}^3) \right) \) such that \( 0 \leq \omega_0 \leq 1 \), i.e. \( \omega_0 = \sum_{j=1}^M \lambda_j \psi_{j,0}(x) \psi_{j,0}(y) \) where \( \{ \lambda_j \geq 0, \psi_{j,0} \}_{j=1}^M \) is a spectral set in \( L^2(\mathbb{R}^3) \) with \( \{ \psi_{j,0} \}_{j=1}^M \subset H^{1/2}(\mathbb{R}^3) \). If \( a = 1 \) and \( \gamma < 0 \) assume in addition that \( | \gamma | < \frac{N \epsilon}{\gamma_{ct} (\text{Tr} (\omega_0))^{3/2}} \), where \( \gamma_{ct} \) is the universal constant in Lemma A.1. Denote by \( \{ \psi_{j,t} \}_{j=1}^M \subset H^{1/2}(\mathbb{R}^3) \) the unique (global) solution of (125) with initial data \( \sqrt{\lambda_j} \psi_{j,0} \) given by Lemma A.7. Then \( \omega_t \) with integral kernel

\[
\omega_t(x; y) = \sum_{j=1}^M \psi_{j,t}(x) \psi_{j,t}(y), t = \sum_{j=1}^M \lambda_j \left( \psi_{j,t}(x) / \sqrt{\lambda_j} \right) \left( \psi_{j,t}(y) / \sqrt{\lambda_j} \right) \tag{147}
\]

is the unique global solution of (140) in \( \mathfrak{c}^1_+ \left( L^2(\mathbb{R}^3) \right) \) with initial datum \( \omega_0 \).

**Proof.** The statement is proven in the exact same manner as [8, Proposition 2.4].

\[ \square \]

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Proof of Proposition 1.13. Note that \( \omega_0 \in \mathfrak{S}^{1, \frac{1}{2}} (L^2(\mathbb{R}^3)) \) can be written as \( \omega_0 = \sum_{j=1}^{\infty} \lambda_j |\psi_j, 0 \rangle \langle \psi_j, 0 | \) where \( \{ \lambda_j \geq 0, \psi_j, 0 \}_{j \in \mathbb{N}} \) is a spectral set and

\[
\| \omega_0 \|_{\mathfrak{S}^{1, \frac{1}{2}}} = \sum_{j=1}^{\infty} \lambda_j \left\| D^{1/2} \psi_j, 0 \right\|_{L^2(\mathbb{R}^3)}^2 = \sum_{j=1}^{\infty} \lambda_j \left\| \psi_j, 0 \right\|_{H^{1/2}(\mathbb{R}^3)}^2 < +\infty.
\]

Then \( \left\{ \omega_{t}^{\leq M} = \sum_{j=1}^{M} \lambda_j |\psi_j(t) \rangle \langle \psi_j, 0 | \right\}_{M=1}^{\infty} \) is a sequence of finite rank operators approximating \( \omega_0 \) in \( \mathfrak{S}^{1, \frac{1}{2}} (L^2(\mathbb{R}^3)) \). Let \( \{ \psi_j(t) \}_{j=1}^{M} \subset H^{1/2}(\mathbb{R}^3) \) be the unique (global) solution of (125) with initial data \{ \sqrt{\lambda_j} \psi_j, 0 \}_{j=1}^{M} \subset H^{1/2}(\mathbb{R}^3) \). According to Lemma A.3

\[
\omega_t^{\leq M} = \sum_{j=1}^{M} |\psi_j(t) \rangle \langle \psi_j, 0 | = \sum_{j=1}^{M} \lambda_j \left| \langle \psi_j(t) / \sqrt{\lambda_j} \rangle \langle \psi_j(t) / \sqrt{\lambda_j} \rangle \right|
\]

is the unique global solution of (140) with data \( \omega_0^{\leq M} \) at \( t = 0 \). In the following we show that \( \lim_{M \to \infty} \omega_t^{\leq M} \) converges in \( \mathfrak{S}^{1, \frac{1}{2}} (L^2(\mathbb{R}^3)) \) and that the limiting operator is a solution of (140). Let \( t \in (0, \infty) \) and \( L, M \in \mathbb{N} \) such that \( L \geq M \). To this end note that

\[
\| \omega_t^{\leq L} - \omega_t^{\leq M} \|_{\mathfrak{S}^{1, \frac{1}{2}}} \leq \varepsilon^{-1} \Tr \left( \sqrt{1 - \varepsilon^2 \Delta} \left( \omega_t^{\leq L} - \omega_t^{\leq M} \right) \right).
\]

Using once more the conservation of the energy (131) we write the right-hand side as

\[
\Tr \left( \sqrt{1 - \varepsilon^2 \Delta} \left( \omega_t^{\leq L} - \omega_t^{\leq M} \right) \right) = \Tr \left( \sqrt{1 - \varepsilon^2 \Delta} \left( \omega_0^{\leq L} - \omega_0^{\leq M} \right) \right)
\]

\[
+ \frac{1}{2} \left[ \Tr \left( K * \rho_0^{\leq L} \omega_0^{\leq L} \right) - \Tr \left( K * \rho_0^{\leq M} \omega_0^{\leq M} \right) \right]
\]

\[
- \frac{1}{2} \left[ \Tr \left( X_0^{\leq L} \omega_0^{\leq L} \right) - \Tr \left( X_0^{\leq M} \omega_0^{\leq M} \right) \right]
\]

\[
- \frac{1}{2} \left[ \Tr \left( K * \rho_t^{\leq L} \omega_t^{\leq L} \right) - \Tr \left( K * \rho_t^{\leq M} \omega_t^{\leq M} \right) \right]
\]

\[
+ \frac{1}{2} \left[ \Tr \left( X_t^{\leq L} \omega_t^{\leq L} \right) - \Tr \left( X_t^{\leq M} \omega_t^{\leq M} \right) \right].
\]

In the following, we drop the \( t \)-dependence to simplify the notation. By (153) and the fact that \( \omega^{\leq M} \leq \omega^{\leq L} \) we get

\[
\left| \Tr \left( K * \rho_t^{\leq L} \omega_t^{\leq L} \right) - \Tr \left( K * \rho_t^{\leq M} \omega_t^{\leq M} \right) \right|
\]

\[
\leq N \int_{\mathbb{R}^3} dx K * \rho_t^{\leq L}(x) \left( \rho_t^{\leq L}(x) - \rho_t^{\leq L}(x) \right) + N \int_{\mathbb{R}^3} K * \rho_t^{\leq M}(x) \left( \rho_t^{\leq L}(x) - \rho_t^{\leq M}(x) \right) dx
\]

\[
\leq CN^{-1} \varepsilon^{-a} \left( \Tr \left( \sqrt{-\varepsilon^2 \Delta} \left( \omega_t^{\leq L} - \omega_t^{\leq M} \right) \right) \right)^{\frac{2}{2+a}} \left( \Tr \left( \omega_t^{\leq L} - \omega_t^{\leq M} \right) \right)^{2\alpha}
\]

\[
\times \left[ \left( \Tr \left( \sqrt{-\varepsilon^2 \Delta} \omega_t^{\leq L} \right) \right)^{\frac{2}{2+a}} + \left( \Tr \left( \sqrt{-\varepsilon^2 \Delta} \omega_t^{\leq M} \right) \right)^{\frac{2}{2+a}} \right]
\]

\[
\leq \left( \Tr \left( \sqrt{1 - \varepsilon^2 \Delta} \left( \omega_t^{\leq L} - \omega_t^{\leq M} \right) \right) \right)^{\alpha} + CN^{-2} \varepsilon^{-2a} \left( \Tr \left( \omega_t^{\leq L} - \omega_t^{\leq M} \right) \right)^{2-a} \left( \Tr \left( \sqrt{-\varepsilon^2 \Delta} \omega_t^{\leq L} \right) \right)^{\alpha} \left( \Tr \left( \omega_t^{\leq L} \right) \right)^{2-a}.
\]

Since

\[
\left| \Tr \left( X_t^{\leq L} \omega_t^{\leq L} \right) - \Tr \left( X_t^{\leq M} \omega_t^{\leq M} \right) \right|
\]

\[
\leq \left| \Tr \left( (X_t^{\leq L} - X_t^{\leq M}) \omega_t^{\leq L} \right) \right| + \left| \Tr \left( X_t^{\leq M} \left( \omega_t^{\leq L} - \omega_t^{\leq M} \right) \right) \right|
\]

\[
\leq N \int_{\mathbb{R}^3} K * \rho_t^{\leq L}(x) \left( \rho_t^{\leq L}(x) - \rho_t^{\leq M}(x) \right) dx + N \int_{\mathbb{R}^3} K * \rho_t^{\leq M}(x) \left( \rho_t^{\leq L}(x) - \rho_t^{\leq M}(x) \right) dx
\]
holds because of (155) we can estimate the exchange term by the same means and obtain
\[
\text{Tr} \left( \sqrt{1 - \varepsilon^2 \Delta} \left( \omega_t^{\leq L} - \omega_t^{\leq M} \right) \right) \\
\leq C \text{Tr} \left( \sqrt{1 - \varepsilon^2 \Delta} \left( \omega_0^{\leq L} - \omega_0^{\leq M} \right) \right) \\
+ CN^{-2} \varepsilon^{-2\alpha} \sup_{t \in [0, t]} \left( \text{Tr} \left( \omega_t^{\leq L} - \omega_t^{\leq M} \right) \right)^{2-\alpha} \left( \text{Tr} \left( \sqrt{-\varepsilon^2 \Delta \omega_t^{\leq L}} \right) \right)^{\alpha} \left( \text{Tr} \left( \omega_t^{\leq L} \right) \right)^{2-\alpha}.
\]
(154)

The conservation \(\text{Tr} \left( \omega_t^{\leq L} \right) = \text{Tr} \left( \omega_0^{\leq L} \right)\), and \(\|\omega_0\|_{\mathcal{H}^{1, \frac{1}{6}}} < \infty\) imply
\[
\text{Tr} \left( \sqrt{1 - \varepsilon^2 \Delta} \left( \omega_0^{\leq L} - \omega_0^{\leq M} \right) \right) \leq \varepsilon^{-1} \sum_{j=M}^{L} \lambda_j \|\psi_{j,0}\|_{H^{1/2}(\mathbb{R}^3)}^2 \to 0,
\]
\[
\text{Tr} \left( \omega_t^{\leq L} - \omega_t^{\leq M} \right) = \text{Tr} \left( \omega_0^{\leq L} - \omega_0^{\leq M} \right) \to 0
\]
(155)
as \(M, L \to \infty\). Since (see sketch of proof of Lemma A.1)
\[
\text{Tr} \left( \sqrt{-\varepsilon^2 \Delta \omega_t^{\leq L}} \right) \leq C(N, \varepsilon) \left\|\{\psi_j(t)\}_{j=1}^L\right\|_{H^{1/2}(\mathbb{R}^3)}^2 \leq C(N, \varepsilon) \sum_{j=1}^{L} \lambda_j \|\psi_{j,0}\|_{H^{1/2}(\mathbb{R}^3)}^2
\]
\[
\leq C(N, \varepsilon) \|\omega_0\|_{\mathcal{H}^{1, \frac{1}{6}}} < +\infty
\]
(156)
we obtain
\[
\|\omega_t^{\leq L} - \omega_t^{\leq M}\|_{\mathcal{H}^{1, \frac{1}{6}}} \leq \varepsilon^{-1} \text{Tr} \left( \sqrt{1 - \varepsilon^2 \Delta} \left( \omega_t^{\leq L} - \omega_t^{\leq M} \right) \right) \to 0 \quad \text{as} \quad M, L \to \infty.
\]
(157)

We consequently have that \(\{\omega^{\leq M}\}_{M \in \mathbb{N}}\) is a Cauchy sequence in \(\mathcal{H}^{1, \frac{1}{6}}\) uniformly in \(t \in (0, \infty)\). Hence, it converges to an operator \(\omega_t\). The operator is continuous in \(t\) because of the uniform limit theorem and the fact that \(\omega_t^{\leq M}\) is continuous in \(t\) (see Lemma A.1). From the fact that \(\omega_t^{\leq M}\) is a solution of (140) and
\[
\lim_{M \to \infty} \|\omega_t^{\leq M} - \omega_t\|_{\mathcal{H}^{1, \frac{1}{6}}} = 0,
\]
\[
\lim_{M \to \infty} \|e^{-i\sqrt{1 - \varepsilon^2 \Delta}t} \left( \omega_t^{\leq M} - \omega_0 \right) \|_{\mathcal{H}^{1, \frac{1}{6}}} = \|\omega_t^{\leq M} - \omega_0\|_{\mathcal{H}^{1, \frac{1}{6}}} = 0,
\]
\[
\lim_{M \to \infty} \|\left[ K * \rho_s^{\leq M}, \omega_s^{\leq M} \right] - \left[ K * \rho_s, \omega_s \right]\|_{\mathcal{H}^{1, \frac{1}{6}}} \leq C \lim_{M \to \infty} \left( \|\omega_s^{\leq M}\|_{\mathcal{H}^{1, \frac{1}{6}}} + \|\omega_s\|_{\mathcal{H}^{1, \frac{1}{6}}} \right) \|\omega_t^{\leq M} - \omega_s\|_{\mathcal{H}^{1, \frac{1}{6}}}
\]
(158)
for all \(t, s \in (0, \infty)\) it directly follows that \(\omega_t\) satisfies (140). The uniqueness of the global solution follows from the uniqueness of the local solution (recall Lemma A.2). Since \(\mathcal{H}^{1, \frac{1}{6}} \left( L^2(\mathbb{R}^3) \right) \) is a closed subspace of \(\mathcal{H}^{1, \frac{1}{6}} \left( L^2(\mathbb{R}^3) \right) \) and \(\{\omega_t^{\leq M}\}_{M \in \mathbb{N}}\) is a positive sequence for all \(t \in (0, \infty)\) by construction we have that \(\omega_t\) is positive for all \(t \in (0, \infty)\). \(\square\)

**B Rigorous Duhamel expansion**

In this section, we give the details on how one obtains the Duhamel expansion from Section 3.1.
Derivation of (51)–(53). Let $\Lambda > 1$ and $K_\Lambda : \mathbb{R}^d \to \mathbb{R}$ be a potential (whose explicit form will be chosen later) such that $t \mapsto \| K_\Lambda \rho_0 \|_{\mathcal{E}^\infty}$ is a strongly continuous map of $\mathbb{R}$ into the bounded self-adjoint operators. In analogy to Chapter 3 we define the two parameter group $U_\Lambda(t;s)$ satisfying
\begin{equation}
\begin{aligned}
i\varepsilon \partial_t U_\Lambda(t;s) &= \left( \sqrt{1 - \varepsilon^2 \Delta} + K_\Lambda \right) U_\Lambda(t;s) \quad \text{and} \quad U_\Lambda(s;s) = 1
\end{aligned}
\end{equation}
by means of the interaction picture. Using Duhamel’s formula we obtain
\begin{equation}
U_\Lambda^*(t;0)(\omega_{N,t} - \tilde{\omega}_{N,t})U_\Lambda(t;0) = \omega_{N,0} - \tilde{\omega}_{N,0}
\end{equation}
In the following we show for suitable chosen $K_\Lambda$ that the last term on the right-hand side converges to zero if we take the limit $\Lambda \to \infty$. This shows the claim.

We study separately the following two cases:

Case $d = 3$ and $a \in (0, 1]$: Note that $\omega_t \in \mathcal{E}_+^{1/2} \left( L^2(\mathbb{R}^3) \right)$ holds locally in time, respectively globally if $|\gamma|$ is small enough, because of the initial conditions of Theorem 1.1 and Proposition 1.13. Since
\begin{equation}
| K \ast \rho_s |_{L^\infty(\mathbb{R}^3)} \leq C |\gamma| N^{-1} \| \tilde{\omega}_s \|_{\mathcal{E}_+^{1/2}} < +\infty
\end{equation}
it is possible to choose $K_\Lambda = K$, implying that the last summand on the right hand side of (101) equals zero. In order to show consider the spectral set $\{ \lambda_j, \varphi_j \}_{j \in \mathbb{N}}$ with $\lambda_j \geq 0$ of an operator $\omega \in \mathcal{E}_+^{1/2} \left( L^2(\mathbb{R}^3) \right)$ and recall (112). Together with (101) we obtain
\begin{equation}
| K \ast \rho |_{L^\infty(\mathbb{R}^3)} \leq N^{-1} \sum_{j \in \mathbb{N}} \lambda_j | K \ast |\varphi_j|^2 |_{L^\infty(\mathbb{R}^3)} \leq C |\gamma| N^{-1} \lambda_j \| \varphi_j \|^2_{H^{1/2}(\mathbb{R}^3)} = C |\gamma| N^{-1} \| \omega \|_{\mathcal{E}_+^{1/2}}.
\end{equation}

Case $d \in (2, 3)$ and $a \in \left( \max \left\{ \frac{a}{2} - 2, -1 \right\}, 0 \right]$: Let $K_\Lambda(x) = K(x)1_{|x| \leq \Lambda}$. Then
\begin{equation}
\begin{aligned}
\| (K - K_\Lambda) \ast \rho_s (1 + |x|)^{-1} \|_{L^\infty(\mathbb{R}^d)} &\leq \Lambda^{a - 1} \sup_{x \in \mathbb{R}^d} \left\{ (1 + |x|)^{-1} \int_{|x-y| > \Lambda} |x-y| \rho_s(y) \, dy \right\} \\
&\leq \Lambda^{a - 1} \left( \| \rho_s \|_{L^1(\mathbb{R}^d)} + N^{-1} \text{Tr} \left( |x| \omega_{N,s} \right) \right) \\
&\leq \Lambda^{a - 1} \left( 1 + N^{-1/2} \sqrt{\text{Tr} \left( x^2 \omega_{N,s} \right)} \right).
\end{aligned}
\end{equation}
To obtain the ultimate inequality we have used the estimate \( |\text{Tr} (|x| \omega_{N,s})| \leq \|x| \sqrt{\omega_{N,s}}\|_2 \sqrt{\omega_{N,s}} |\omega|_2 \). Together with the embedding \( \|\omega|_p = 2^\frac{1}{p} \|\omega|_2 \) for \( 1 \leq p < \infty \) and \( \left( 1 + |x| \right) \left( 1 + x^2 \right)^{-1/2} \left( 1 - \varepsilon^2 \Delta \right)^{-1} \|\omega|_2 \leq C \sqrt{N} \) this leads to

\[
\|[(K - K_A) * \rho_s, \omega_{N,s} - \tilde{\omega}_{N,s}]\|_p \leq C \left\| (K - K_A) * \rho_s \left( 1 + |x| \right)^{-1} \right\|_{L^p(S^d)} \left( (1 + |x|) \omega_{N,s} \right) + \left( 1 + |x| \right) \tilde{\omega}_{N,s} \|_2 \\
\leq C \Lambda |\lambda|^{-1} \left( N^{1/2} + \sqrt{\text{Tr} (x^2 \omega_{N,s})} \right) \left( N + \sqrt{\text{Tr} (x^2 \omega_{N,s})} + \left( 1 - \varepsilon^2 \Delta \right)^{1/2} \tilde{\omega}_{N,s} \right). \tag{165}
\]

Using \( \sqrt{1 - \varepsilon^2 \Delta, x} = -\frac{i e^2 \nabla}{\sqrt{1 - \varepsilon^2 \Delta}} \), \( \frac{i e^2 \nabla}{\sqrt{1 - \varepsilon^2 \Delta}} \leq 1 \), \( \text{Tr} (\omega_{N,t}) = N \) and the cyclicity of the trace we get

\[
\frac{d}{dt} \text{Tr} (x^2 \omega_{N,t}) = \text{Tr} \left( \left[ x^2, \sqrt{1 - \varepsilon^2 \Delta} \right] \omega_{N,t} \right) \\
= -\text{Tr} \left( \left\{ \frac{i e^2 \nabla}{\sqrt{1 - \varepsilon^2 \Delta}}, x \right\} \omega_{N,t} \right) \\
\leq 2 \varepsilon \left\| x \sqrt{\omega_{N,t}} \right\|_2 \left\| \sqrt{\omega_{N,t}} \right\|_2 \left\| \frac{i e^2 \nabla}{\sqrt{1 - \varepsilon^2 \Delta}} \right\|_2 \\
\leq 2N^{1/2} \sqrt{\text{Tr} (x^2 \omega_{N,t})}. \tag{166}
\]

For the initial data of Theorem \(1.1\) it consequently holds that \( |\text{Tr} (x^2 \omega_{N,t})| < +\infty \) for all \( t \in \mathbb{R} \). By similar estimates as in \(8\) Chapter 3 one derives

\[
\left\| (1 - \varepsilon^2 \Delta)^{1/2} \left( 1 + x^2 \right)^{3/2} \tilde{\omega}_{N,s} \right\|_2 \leq C N^{1/2} \left\| \tilde{W}_{N,s} \right\|_{H^3_0(S^d)}. \tag{167}
\]

Since the right-hand side is finite for \( \tilde{\omega}_{N,s} \) from Theorem \(1.1\) we obtain

\[
\lim_{\Lambda \to \infty} \left\| [(K - K_A) * \rho_s, \omega_{N,s} - \tilde{\omega}_{N,s}] \right\|_p = 0 \tag{168}
\]

by means of inequality \(165\). \(\square\)

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