Partite Saturation of Complete Graphs

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Abstract

We study the problem of determining $sat(n, k, r)$, the minimum number of edges in a $k$-partite graph $G$ with $n$ vertices in each part such that $G$ is $K_r$-free but the addition of an edge joining any two non-adjacent vertices from different parts creates a $K_r$. Improving recent results of Ferrara, Jacobson, Pfender and Wenger, and generalizing a recent result of Roberts, we define a function $α(k, r)$ such that $sat(n, k, r) = α(k, r)n + o(n)$ as $n → ∞$. Moreover, we prove that

$$k(2r - 4) ≤ α(k, r) ≤ \begin{cases} (k - 1)(4r - k - 6) & \text{for } r ≤ k ≤ 2r - 3, \\ (k - 1)(2r - 3) & \text{for } k ≥ 2r - 3, \end{cases}$$

and show that the lower bound is tight for infinitely many values of $r$ and every $k ≥ 2r - 1$. This allows us to prove that, for these values, $sat(n, k, r) = k(2r - 4)n + O(1)$ as $n → ∞$. Along the way, we disprove a conjecture and answer a question of the first set of authors mentioned above.

1 Introduction

Given a graph $H$, the classical Turán-type extremal problem asks for the maximum number of edges in an $H$-free graph on $n$ vertices. While the corresponding minimization problem is trivial, it is interesting to determine the minimum number of edges in a maximal $H$-free graph on $n$ vertices. We say that a graph is $H$-saturated if it is $H$-free but the addition of an edge joining any two non-adjacent vertices creates a copy of $H$. The minimum number $sat(n, H)$ of edges in an $H$-saturated graph on $n$ vertices was first studied in 1949 by Zykov [17] and independently in 1964 by Erdős, Hajnal, and Moon [2] who proved that $sat(n, K_r) = (r - 2)(n - 1) - \left(\binom{r-2}{2}\right)$. Soon after this, Bollobás [1] determined exactly $sat(n, K_r^{(s)})$ where $K_r^{(s)}$ is the complete $s$-uniform hypergraph on $r$ vertices. Later, in 1986, Kászonyi and Tuza [10] showed that the saturation number $sat(n, H)$ for a graph $H$ on $r$ vertices is maximized

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Theorem 1. Our first result states that for any \( H \) with \( H = K_r \), and consequently, \( sat(n, H) \) is linear in \( n \) for any \( H \). For results on the saturation number, we refer the reader to the survey [3].

This concept can be generalized to the notion of \( H \)-saturated subgraphs which are maximal elements of a family of \( H \)-free subgraphs of a fixed host graph. A subgraph of a graph \( G \) is said to be \( H \)-saturated in \( G \) if it is \( H \)-free but the addition of an edge in \( E(G) \) joining any two non-adjacent vertices creates a copy of \( H \). The problem of determining the minimum number \( sat(G, H) \) of edges in an \( H \)-saturated subgraph of \( G \) was first proposed in the above mentioned paper of Erdős, Hajnal, and Moon. They conjectured a value for the saturation number \( sat(K_{m,n}, K_r) \) which was verified independently by Bollobás [2, 3] and Wessel [15, 16]. Very recently, Sullivan and Weng [14] studied the analogous saturation numbers for tripartite graphs within tripartite graphs and determined \( sat(K_{n_1,n_2,n_3}, K_{l,l,l}) \) for every fixed \( l \geq 1 \) and every \( n_1, n_2 \) and \( n_3 \) sufficiently large. Several other host graphs have been considered, including hypercubes [4, 9, 12] and random graphs [11].

In this paper, we are interested in the saturation number \( sat(n, k, r) = sat(K_{k\times n}, K_r) \) for \( k \geq r \geq 3 \) where \( K_{k\times n} \) is the complete \( k \)-partite graph containing \( n \) vertices in each of its \( k \) parts. This function was first studied recently by Ferrara, Jacobson, Pfender and Wenger [7] who determined \( sat(n, k, 3) \) for \( n \geq 100 \). Later, Roberts [13] showed that \( sat(n, 4, 4) = 18n - 21 \) for sufficiently large \( n \).

For convenience, we say that a \( k \)-partite graph with a fixed \( k \)-partition is \( K_r \)-partite-saturated if it is \( K_r \)-free but the addition of an edge joining any two non-adjacent vertices from different parts creates a \( K_r \). Therefore, \( sat(n, k, r) \) is the minimum number of edges in a \( k \)-partite graph \( G \) with \( n \) vertices in each part which is \( K_r \)-partite-saturated.

Our first result states that \( sat(n, k, r) \) is linear in \( n \) where the constant \( \alpha(k, r) \) in front of \( n \) is defined as follows. Given \( k \geq r \geq 3 \), consider a \( K_r \)-partite-saturated \( k \)-partite graph \( G \) containing an independent set \( X \) of size \( k \) consisting of exactly one vertex from each part of \( G \). We define \( \alpha(k, r) \) to be the minimum number of edges between \( X \) and \( X^c \) taken over all such \( G \) and \( X \).

**Theorem 1.** For \( k \geq r \geq 3 \),

\[
sat(n, k, r) = \alpha(k, r)n + o(n)
\]

as \( n \to \infty \).

Let us shift our focus to the function \( \alpha(k, r) \). The next theorem states what we know about it.

**Theorem 2.** For \( k \geq r \geq 3 \),

\[
(i) \quad k(2r - 4) \leq \alpha(k, r) \leq \begin{cases} (k-1)(4r-k-6) & \text{for } r \leq k \leq 2r-3, \\ (k-1)(2r-3) & \text{for } k \geq 2r-3. \end{cases}
\]

\[
(ii) \quad \alpha(k, r) = k(2r-4) \text{ if } \begin{cases} k = 2r-3, & \text{or} \\ k \geq 2r-2 \text{ and } r \equiv 0 \mod 2, & \text{or} \\ k \geq 2r-1 \text{ and } r \equiv 2 \mod 3. \end{cases}
\]

\[
(iii) \quad \alpha(k, 3) = 3(k-1), \quad \alpha(4, 4) = 18 \quad \text{and} \quad 33 \leq \alpha(5, 5) \leq 36.
\]
(iv) $\alpha(r, r) \geq r(2r - 4) + 1$ for $r \geq 4$.

The bounds in (i), together with Theorem 1, imply that $sat(n, k, r) = O(krn)$, answering a question of Ferrara, Jacobson, Pfender and Wenger [7]. In (ii), we determine exactly $\alpha(k, r)$ for some values of $r$ and every $k$ large enough, allowing us to disprove a conjecture in [7] which states that $sat(n, k, r) = (k - 1)(2r - 3)n - (2r - 3)(r - 1)$ for $k \geq 2r - 3$ and sufficiently large $n$. In (iii), we deal with the cases $r = 3, 4, 5$ which have not been covered by (ii). Finally, (iv) shows that the lower bound in (i), which is attained for certain values of $r$ and $k$ mentioned in (ii), is not tight when $k = r$.

Theorem 1 and Theorem 2 imply that $sat(n, k, r) = k(2r - 4)n + o(n)$ for the values of $k$ and $r$ in (ii). We show that, in this case, the $o(n)$ term can be replaced by a constant.

**Theorem 3.** For $k \geq r \geq 3$,

$$sat(n, k, r) = k(2r - 4)n + O(1) \text{ if } \begin{cases} k = 2r - 3, & \text{or} \\ k \geq 2r - 2 \text{ and } r \equiv 0 \pmod{2}, & \text{or} \\ k \geq 2r - 1 \text{ and } r \equiv 2 \pmod{3}, & \end{cases}$$

as $n \to \infty$.

Now we give a summary of the values of $sat(n, k, r)$ in the case $r = 3, 4, 5$ which are immediate consequences of the first three results.

**Corollary 4.**

(i) $sat(n, k, 3) = 3(k - 1)n + o(n)$ for $k \geq 3$ and as $n \to \infty$.

(ii) $sat(n, k, 4) = \begin{cases} 18n + o(n) & \text{for } k = 4, \text{ as } n \to \infty, \\ 4kn + O(1) & \text{for } k \geq 5, \text{ as } n \to \infty. \end{cases}$

(iii) $sat(n, k, 5) = \begin{cases} 33n + o(n), 36n + o(n) & \text{for } k = 5, \text{ as } n \to \infty, \\ 36n + o(n), 40n + o(n) & \text{for } k = 6, \text{ as } n \to \infty, \\ 48n + o(n), 49n + o(n) & \text{for } k = 8, \text{ as } n \to \infty, \\ 6kn + O(1) & \text{for } k = 7 \text{ or } k \geq 9, \text{ as } n \to \infty. \end{cases}$

We note that (i) and the first half of (ii) are not the best known results. In fact, Ferrara, Jacobson, Pfender and Wenger [7] proved that $sat(n, k, 3) = 3(k - 1)n - 6$ for sufficiently large $n$ and Roberts [13] proved that $sat(n, 4, 4) = 18n - 21$ for sufficiently large $n$.

Let us give some more definitions which will be used throughout the paper. For a $k$-partite $G = V_1 \cup V_2 \cup \cdots \cup V_k$, we refer to each $V_i$ as a part of $G$. We say that an edge (or a non-edge) $uv$ of a $k$-partite graph is *admissible* if $u, v$ lie in different parts. We say that a non-edge $uv$ of a $K_r$-free graph is $K_r$-saturated if adding $uv$ to the graph completes a $K_r$. In other words, a $k$-partite graph is $K_r$-partite-saturated if it is $K_r$-free and every admissible non-edge is $K_r$-saturated.

The rest of this paper is organized as follows. Section 2 is devoted to the proof of Theorem 1. In Section 3 we study the function $\alpha(k, r)$ and prove Theorem 2 (i). In Section 4 we prove Theorem 2 (ii)
by describing constructions matching the lower bound \(\alpha(k, r) \geq k(2r - 4)\) in Theorem 2(i). We prove Theorem 2(iii), Theorem 2(iv) and Theorem 3 in Section 5, Section 6 and Section 7 respectively. Finally, we conclude the paper in Section 8 with some open problems.

\section{Proof of Theorem 1}

First we show that the upper bound follows easily from the definition of \(\alpha(k, r)\).

\textbf{Proposition 5.} For every \(k \geq r \geq 3\) and any integer \(n \geq \alpha(k, r) + 1\), we have \(\text{sat}(n, k, r) \leq \alpha(k, r)n + \alpha(k, r)^2\).

\textit{Proof.} Let \(G\) be a \(K_r\)-partite-saturated \(k\)-partite graph containing an independent set \(X\) of size \(k\) consisting of exactly one vertex from each part of \(G\) with \(e(X, X^c) = \alpha(k, r)\). We may assume that \(|X^c| \leq \alpha(k, r)\). Indeed, since there are \(\alpha(k, r)\) edges between \(X\) and \(X^c\), deleting all the vertices in \(X^c\) with no neighbors in \(X\) leaves at most \(\alpha(k, r)\) vertices in \(X^c\). Note that any admissible non-edge with at least one endpoint in \(X\) is still \(K_r\)-saturated. We finish by keeping adding admissible edges inside \(X^c\) until every admissible non-edge inside \(X^c\) is \(K_r\)-saturated.

Let \(V_1, V_2, \ldots, V_k\) be the parts of \(G\). It follows that \(|V_i| = |V_i \cap X| + |V_i \cap X^c| \leq 1 + \alpha(k, r) \leq n\), and so we can modify \(G\) to have exactly \(n\) vertices in each part by blowing up the vertex of \(X\) in \(V_i\) to a class of size \(n - |V_i \cap X^c|\) for each \(i\). The resulting graph is \(K_r\)-partite-saturated and has exactly \(n\) vertices in each of its \(k\) parts. Moreover, the number of edges is at most \(\alpha(k, r)n + e(G[X^c]) \leq \alpha(k, r)n + \alpha(k, r)^2\). \(\Box\)

Now we prove the lower bound \(\text{sat}(n, k, r) \geq \alpha(k, r)n + o(n)\).

Let \(\varepsilon > 0\) and let \(G = V_1 \cup V_2 \cup \cdots \cup V_k\) be a \(K_r\)-partite-saturated \(k\)-partite graph with \(|V_i| = n\) for all \(i \in [k]\). We shall show that \(e(G) \geq \alpha(k, r)n - \varepsilon n\) for all sufficiently large \(n\). Let \(d\) be a large natural number to be chosen later. For each \(i\), we partition \(V_i\) into \(V_i^+ = \{v \in V_i : d(x) \geq d\}\) and \(V_i^- = \{v \in V_i : d(x) < d\}\). First we show that \(V_i^+\) is small. Since \(e(G) \geq \frac{2\alpha(k, r)}{d} |V_i^+|\), we are done unless \(|V_i^+| \leq 2\alpha(k, r)^2/n\). Now we show that we can delete a constant number of vertices from \(\bigcup_{i=1}^k V_i^-\) to make it independent.

\textbf{Lemma 6.} There exists a subset \(U \subset \bigcup_{i=1}^k V_i^-\) of size \(C_{k,d}\) such that \(\left(\bigcup_{i=1}^k V_i^-\right) \setminus U\) forms an independent set in \(G\) for some constant \(C_{k,d}\).

Let us first show how to finish the proof of Proposition 5 using the lemma. For each \(1 \leq i \leq k\), let \(v_i\) be a vertex of smallest degree in \(V_i^-\). Since \(G\) is a \(K_r\)-partite-saturated \(k\)-partite graph and \(X = \{v_1, v_2, \ldots, v_k\}\) is an independent set with exactly one vertex in each part of \(G\), we have \(\sum_{i=1}^k d(v_i) \geq \alpha(k, r)\) by the definition of \(\alpha(k, r)\). Since \(\left(\bigcup_{i=1}^k V_i^-\right) \setminus U\) forms an independent set,

\[
e(G) \geq \sum_{i=1}^k \sum_{v \in V_i^- \setminus U} d(v) \geq \sum_{i=1}^k |V_i^- \setminus U| d(v_i) \geq (n - |V_i^+| - |U|) \sum_{i=1}^k d(v_i)
\]

\[
\geq \alpha(k, r) \left(n - \frac{2\alpha(k, r)}{d} n - C_{k,d}\right) = \alpha(k, r)n - \left(\frac{2\alpha(k, r)^2}{d} + \frac{\alpha(k, r)C_{k,d}}{n}\right) n \geq \alpha(k, r)n - \varepsilon n
\]
by taking \( d \) and \( n \) sufficiently large. It remains to prove the lemma.

**Proof of Lemma 6.** It is sufficient to show that any matching between \( V_i^- \) and \( V_j^- \) has size less than \( 4d^2 \) for all \( i \neq j \). Indeed, we can take \( U \) to be the endpoints of maximal matchings between \( V_i^- \) and \( V_j^- \) for all \( i \neq j \) and \( |U| < 4d^2 (k) \).

Suppose for contradiction that \( \{ x_1 y_1, x_2 y_2, \ldots, x_d y_d \} \) is a matching of size \( 4d^2 \) where \( X = \{ x_1, x_2, \ldots, x_d \} \subset V_i^- \) and \( Y = \{ y_1, y_2, \ldots, y_d \} \subset V_j^- \). The strategy of the proof is to iteratively find vertices \( x_{t_1}, x_{t_2}, \ldots, x_{t_d} \) of \( X \) such that \( d(x_{t_i}, y_i) \geq i \) for all \( 1 \leq i \leq d \), which would contradict the fact that \( x_{t_d} \in V_i^- \). In fact, we shall find vertices \( x_{t_1}, x_{t_2}, \ldots, x_{t_d} \) of \( X \) such that

(i) there exists a common neighbor of \( x_{t_i} \) and \( y_{t_j} \) which is not a neighbor of \( y_{t_1}, y_{t_2}, \ldots, y_{t_{j-1}} \) for all \( i > j \).

Clearly, this implies that \( d(x_{t_i}) \geq i \) for all \( 1 \leq i \leq d \). To find such vertices, it is sufficient to find vertices \( x_{t_1}, x_{t_2}, \ldots, x_{t_d} \) of \( X \) satisfying

(ii) \( x_{t_i} \) and \( y_{t_j} \) are not neighbors for all \( i > j \), and

(iii) \( N(x_{t_i}) \cap N(y_{t_j}) = N(x_{t_j}) \cap N(y_{t_j}) \) for all \( i > j > l \).

First we show that (ii) and (iii) imply (i). Let \( i > j \). By (ii), \( x_{t_i}, y_{t_j} \) is a non-edge. Since \( G \) is \( K_r \)-partite-saturated, there exists a clique \( W \) of size \( r - 2 \) in the common neighborhood of \( x_{t_i} \) and \( y_{t_j} \). Since \( r \geq 3 \), we are done by picking a required vertex from \( W \) unless each vertex in \( W \) is joined to some \( y_{t_j} \) with \( l < j \). In this case, \( W \cup \{ x_{t_i}, y_{t_j} \} \) forms a clique of size \( r \), contradicting the fact that \( G \) is \( K_r \)-free. Indeed, each \( w \in W \) belongs to some \( N(y_{t_j}) \) with \( l < j \), and since \( w \in N(x_{t_i}) \), we must have \( w \in N(x_{t_j}) \), by (iii).

Now, we find vertices \( x_{t_1}, x_{t_2}, \ldots, x_{t_d} \) of \( X \) satisfying (ii) and (iii). To help us do so, we shall iteratively construct a nested sequence of sets \( X \supset X_1 \supset X_2 \supset \cdots \supset X_d \) with \( x_{t_i} \in X_i \) for all \( 2 \leq i \leq d \), satisfying

(iv) \( x \) and \( y_{t_{i-1}} \) are not neighbors for all \( x \in X_i \), and

(v) \( N(x) \cap N(y_{t_{i-1}}) = N(x') \cap N(y_{t_{i-1}}) \) for all \( x, x' \in X_i \).

Clearly, such vertices \( x_{t_1}, x_{t_2}, \ldots, x_{t_d} \) satisfy (ii) and (iii). Start with \( x_{t_1} = x_1 \) and \( X_1 = X \). Let \( i \leq d \) and suppose that we have found vertices \( x_{t_1}, x_{t_2}, \ldots, x_{t_{i-1}} \) and sets \( X_1 \supset X_2 \supset \cdots \supset X_{i-1} \) with \( x_{t_j} \in X_j \) for all \( j < i \), satisfying (iv) and (v). We delete the neighbors of \( y_{t_{i-1}} \) from \( X_{i-1} \) and partition the remaining vertices into \( 2^{d(y_{t_{i-1}})} \leq 2^d \) subsets according to their common neighborhood with \( y_{t_{i-1}} \). In other words, \( X_{i-1} \setminus N(y_{t_{i-1}}) \) is partitioned into subsets \( \{ x : N(x) \cap N(y_{t_{i-1}}) = S \} \) for \( S \subset N(y_{t_{i-1}}) \). We choose \( X_i \) to be such subset of maximum size, i.e. \( |X_i| \geq \frac{|X_{i-1}| - d}{2^d} \). Clearly, \( X_i \) satisfies (iv) and (v). We then choose \( x_{t_i} \) be any vertex in \( X_i \). It remains to prove that \( |X_i| > 0 \). Recall that \( |X_1| = |X| = 4^d \), and we can see, by induction, that \( |X_i| \geq 4^{d(d-i)} \) for \( i \leq d \). Indeed,

\[
|X_i| \geq \frac{|X_{i-1}| - d}{2^d} \geq \frac{|X_{i-1}|}{4^d} \geq \frac{4^d(d-1)}{4d} \geq 4^d(d-i)
\]

as required. □
3 Bounding $\alpha(k, r)$

In this section, we establish a number of results that will help us prove Theorem 2. We shall deduce Theorem 2(i) at the end of the section.

For $k \geq r \geq 2$ and $1 \leq i \leq k - r + 1$, let $\beta_i(k, r)$ be the minimum number of vertices in a $K_r$-free $k$-partite graph such that the subgraph induced by any $k - i$ parts contains a $K_{r-1}$, i.e. the deletion of any $i$ parts does not destroy all the $K_{r-1}$.

We observe that $\beta_1$ and $\beta_2$ are useful for bounding $\alpha$.

**Proposition 7.** For $k \geq r \geq 3$,

$$k\beta_1(k-1, r-1) \leq \alpha(k, r) \leq (k-1)\beta_2(k, r-1).$$

*Proof.* To prove the lower bound, let $G$ be a $K_r$-partite-saturated $k$-partite graph containing an independent set $X$ of size $k$ consisting of exactly one vertex from each part of $G$. We shall show that $e(X, X^c) \geq k\beta_1(k-1, r-1)$. It is sufficient to show that each vertex in $X$ has degree at least $\beta_1(k-1, r-1)$. Let $x \in X$ and consider the $(k-1)$-partite graph $H = G[N(x)]$. Clearly, it is $K_{r-1}$-free since $G$ is $K_r$-free. It remains to show that, for each part $U$ of $H$, $H \setminus U$ contains a $K_{r-2}$. If $x'$ is a vertex of $X$ in the corresponding part of $U$ in $G$ then, since the non-edge $xx'$ is $K_r$-saturated in $G$, $H \setminus U$ must contain a $K_{r-2}$. Hence, $|N(x)| = |H| \geq \beta_1(k-1, r-1)$.

For the upper bound, let $G_1$ be a $K_{r-1}$-free $k$-partite graph on $\beta_2(k, r-1)$ vertices such that the subgraph induced by any $k - 2$ parts contains a $K_{r-2}$. Let $G_2$ be the graph obtained from $G_1$ by adding one vertex of $X = \{x_1, x_2, \ldots, x_k\}$ to each part of $G_1$ and joining each $x_i$ to every vertex of $G_1$ outside its part. By construction, $X$ forms an independent set and $e(X, X^c) = (k-1)\beta_2(k, r-1)$ edges. Note that $G_2$ is $K_r$-free since a clique in $G_2$ contains at most one vertex from $X$ and $G_1$ is $K_{r-1}$-free. Now, let $G$ be the graph obtained from $G_2$ by adding admissible edges inside $X^c$, until every admissible non-edge inside $X^c$ is $K_r$-saturated. To conclude that $G$ is $K_r$-partite-saturated, we need to show that every admissible non-edge inside $X$ is $K_r$-saturated. Note that, for every pair of distinct vertices $x, x' \in X$, $G_1$ contains a $K_{r-2}$ not using vertices from the parts containing $x$ and $x'$. Since $x$ and $x'$ are joined to every vertex outside their parts, the addition of the edge $xx'$ completes a $K_r$. Hence, $\alpha(k, r) \leq e(X, X^c) = (k-1)\beta_2(k, r-1)$. \qed

In the next sections, the argument above used in the proof of the lower bound will be used several times. Let us state it as a lemma.

**Lemma 8.** Let $G$ be a $k$-partite $K_r$-free graph containing an independent set $X$ of size $k$ consisting of exactly one vertex from each part of $G$ such that the non-edges inside $X$ are $K_r$-saturated. Then, for each $x \in X$, $G[N(x)]$ is a $K_{r-1}$-free $(k-1)$-partite graph such that the subgraph induced by any $k - 2$ parts contains a $K_{r-2}$. In particular, $d(x) \geq \beta_1(k-1, r-1)$ for all $x \in X$.

In the next two subsections, we shall bound $\beta_1$ from below and $\beta_2$ from above.
3.1 Upper bounds for $\beta_i$

We start with an easy observation which helps us bound $\beta_i$ from above.

**Lemma 9.** For $k \geq r \geq 3$ and $1 \leq i \leq k - r + 1$, $\beta_i(k, r) \leq \beta_i(k - 1, r - 1) + i + 1$.

**Proof.** Let $H = U_1 \cup U_2 \cup \cdots \cup U_{k-1}$ be a $K_{r-1}$-free $(k-1)$-partite graph on $\beta_i(k - 1, r - 1)$ vertices such that the subgraph induced by any $k-i-1$ parts contains a $K_{r-2}$. We shall construct a $K_r$-free $k$-partite graph $G = V_1 \cup V_2 \cup \cdots \cup V_k$ from $H$ with $|G| = |H| + (i + 1)$ as follows. First, add new vertices $v_1$ to $U_1$, $v_2$ to $U_2$, ..., $v_i$ to $U_i$ and $v_{i+1}$ to the new part $V_k$. This is possible since $k \geq i + 2$. Now, join $v_{i+1}$ to every vertex in $H$ and, for every $1 \leq j \leq i$, join $v_j$ to every vertex in $H \setminus U_j$. Clearly, $G$ is $K_r$-free since $H$ is $K_{r-1}$-free.

Let $C$ be a collection of $k-i$ parts of $G$. It remains to check that the subgraph of $G$ induced by $C$ contains a $K_{r-1}$. First, suppose that $V_k \in C$. By the induction hypothesis, the other $(k-1)-i$ parts $C \setminus \{V_k\}$ induce a subgraph of $H$ containing a $K_{r-2}$. Together with $v_{i+1} \in V_k$, they form a $K_{r-1}$ in the subgraph of $G$ induced by $C$ as required. Now, let us suppose that $V_k \notin C$. Then $C$ must contain at least one of $V_1, V_2, \ldots, V_i$. Without loss of generality, we may assume that $C$ contains $V_1$. By the induction hypothesis, the other $(k-1)-i$ parts $C \setminus \{V_1\}$ induce a subgraph of $H$ containing a $K_{r-2}$. Together with $v_1 \in V_1$, they form a $K_{r-1}$ in the subgraph of $G$ induced by $C$ as required. \qed

Lemma 9 immediately implies the following upper bound on $\beta_i$.

**Corollary 10.** $\beta_i(k, r) \leq (i + 1)(r - 1)$ for $k \geq r \geq 2$ and $1 \leq i \leq k - r + 1$.

**Proof.** It is clear that $\beta_i(k, 2) = i + 1$ for $k \geq i + 1$ by considering the empty graph on $i + 1$ vertices where each vertex is in a different part and the remaining $k-i-1$ parts are empty.

By induction on $r$ and applying Lemma 9 $\beta_i(k, r) \leq \beta_i(k - 1, r - 1) + i + 1 \leq (i + 1)(r - 2) + i + 1 = (i + 1)(r - 1)$ as required. \qed

We remark that there is a straightforward construction proving Corollary 10 for the case $k \geq (i + 1)(r - 1)$, namely, a disjoint union of $i + 1$ cliques of size $r - 1$ where each vertex is in a different part and the remaining $k - (i + 1)(r - 1)$ parts are empty. Clearly, the deletion of any $i$ parts does not destroy all the $K_{r-1}$.

Now we prove a better upper bound for $\beta_i(k, r)$ in the case when $i \geq 2$ and $k \geq i(r - 1) + 1$ by considering the $(r - 2)$th power of the cycle $C_{i(r-1)+1}$.

**Proposition 11.** $\beta_i(k, r) \leq i(r - 1) + 1$ for $k \geq i(r - 1) + 1$ and $r, i \geq 2$.

**Proof.** Since $\beta_i(k, r)$ is decreasing in $k$ (by adding empty parts), it is enough to show that $\beta_i(k, r) \leq i(r - 1) + 1$ for $k = i(r - 1) + 1$. Let $G$ be the $(r - 2)$th power of the cycle $C_{i(r-1)+1}$, i.e. $G$ is a graph on $Z_{i(r-1)+1}$ where $u, v$ are neighbors if $u - v = 1, 2, \ldots, r - 2$. We view $G$ as a $(i(r - 1) + 1)$-partite graph with one vertex in each part. Clearly, $G$ is $K_r$-free if $i \geq 2$. Note that, after deleting any $i$ vertices of $G$, there are at least $r - 1$ consecutive vertices remaining in $Z_{i(r-1)+1}$, which form a $K_{r-1}$ as required. \qed
Proposition 11 together with Lemma 9 imply a better upper bound than that in Corollary 10 for $\beta_2(k, r)$ in the remaining cases, i.e when $k < 2r - 1$.

**Proposition 12.** $\beta_2(k, r) \leq 4r - k - 2$ for $2 \leq r < k < 2r - 1$.

**Proof.** We proceed by induction on $2r - k$. The base case when $2r - k = 1$ follows from Proposition 11. Now, suppose that $2r - k \geq 2$. Applying Lemma 9,

$$\beta_2(k, r) \leq \beta_2(k - 1, r - 1) + 3 \leq (4(r - 1) - (k - 1) - 2) + 3 = 4r - k - 2,$$

by the induction hypothesis, since $2r - k > 2(r - 1) - (k - 1) \geq 1$.

Let us remark that a similar upper bound for general $\beta_i$ can be obtained by the same method. We believe that the bound in Proposition 12 is, in fact, an equality.

**Conjecture 13.** $\beta_2(k, r) = 4r - k - 2$ for $2 \leq r < k < 2r - 1$.

For the remaining values of $k$, we shall see in the next subsection that $\beta_2(k, r) = 2r - 1$ for $k \geq 2r - 1$.

### 3.2 Determining $\beta_1$

We shall show that the upper bound for $\beta_1$ given by Corollary 10 is an equality. Recall that the clique number of a graph is the order of a maximum clique.

**Proposition 14.** $\beta_1(k, r) = 2(r - 1)$ for $k \geq r \geq 2$.

The lower bound, is a consequence of the following observation.

**Proposition 15.** Let $G$ be a graph on at most $2s - 1$ vertices with clique number $s$. Then there is a vertex which lies in every $K_s$ of $G$.

**Proof of Proposition 14.** The upper bound follows from Corollary 10. To prove the lower bound, suppose for contradiction that $G$ is a $K_r$-free $k$-partite graph on at most $2r - 3$ vertices such that the subgraph induced by any $k - 1$ parts contains a $K_{r-1}$. Applying Proposition 15 with $s = r - 1$, there is a vertex $v$ which lies in every $K_{r-1}$. In particular, the deletion of the part containing $v$ destroys all the $K_{r-1}$. Hence, $\beta_1(k, r) \geq 2r - 2$.

Let us remark that Proposition 15 is a consequence of the clique collection lemma of Hajnal [8] which states that the sum of the number of vertices in the union and the intersection of a collection of maximum cliques is at least twice the clique number. Our argument below can also be used to give a new proof of Hajnal’s clique collection lemma.

**Proof of Proposition 15.** Let $V_1, V_2, \ldots, V_m \subset V(G)$ be the vertex sets of the copies of $K_s$ in $G$. For a vertex $v \in V(G)$, let $I_v = \{i \in [m] : v \in V_i\}$ be the set of $K_s$ containing $v$. For a collection $C \subset \mathcal{P}([m])$ of subsets of $[m]$, let $V_C = \{v \in V(G) : I_v \in C\}$. Observe that if $C \subset \mathcal{P}([m])$ is intersecting then $V_C$ induces a clique in $G$. Indeed, $u, v \in V_C$ are neighbors since $I_u \cap I_v \neq \emptyset$, i.e. there is a clique containing both $u$ and $v$. Therefore, $|V_C| \leq s$ since $G$ is $K_{s+1}$-free. The following lemma implies the result.
Lemma 16. For \( m \geq 3 \), there exist intersecting families \( \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{m-2} \subset \mathcal{P}([m]) \) such that, for \( I \subset [m] \), the number of \( \mathcal{C}_j \) containing \( I \) is 

\[
\begin{cases} 
0 & \text{if } I = \emptyset \\
|I| - 1 & \text{if } I \neq \emptyset, [m] \\
m - 2 & \text{if } I = [m].
\end{cases}
\]

Proof. The proof is by induction on \( m \). For \( m = 3 \), \( \mathcal{C}_1 = \{\{1, 2\}, \{2, 3\}, \{3, 1\}, \{1, 2, 3\}\} \) satisfies the required property. For \( m \geq 4 \), suppose by induction that there exist intersecting families \( \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{m-3} \subset \mathcal{P}([m-1]) \) satisfying the property. We define \( \mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_{m-2} \subset \mathcal{P}([m]) \) as follows. For \( 1 \leq j \leq m - 3 \), let

\[
\mathcal{D}_j = \mathcal{C}_j \cup \{I \cup \{m\} : I \in \mathcal{C}_j\}
\]

and

\[
\mathcal{D}_{m-2} = \{I \subset [m] : m \in I \text{ and } |I| \geq 2 \} \cup \{[m - 1]\}.
\]

It is easy to check that \( \mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_{m-2} \) satisfy the required property. \( \square \)

Let us deduce the result. This is trivial when \( m = 1, 2 \) so we may assume that \( m \geq 3 \). Observe that

\[
\sum_{i=1}^{m} |V_i| = \left| \bigcup_{i=1}^{m} V_i \right| + \sum_{j=1}^{m-2} |V_{C_j}| + \left| \bigcap_{i=1}^{m} V_i \right|.
\]

Indeed, a vertex \( v \) is counted on both sides \( |I_v| \) times by the lemma. Using \( |V_i| = s \), \( \left| \bigcup_{i=1}^{m} V_i \right| \leq 2s - 1 \) and \( |V_{C_j}| \leq s \), we have

\[
ms \leq (2s - 1) + (m - 2)s + \left| \bigcap_{i=1}^{m} V_i \right|
\]

i.e. \( \left| \bigcap_{i=1}^{m} V_i \right| \geq 1 \) as required. \( \square \)

We remark that the fact that \( \beta_1(k, r) = 2(r - 1) \) allows us to show that the upper bound for \( \beta_2(k, r) \) when \( k \geq 2r - 1 \) in Proposition [14] is an equality.

Corollary 17. \( \beta_2(k, r) = 2r - 1 \) for \( k \geq 2r - 1 \) and \( r \geq 2 \).

Proof. Observe that \( \beta_i(k, r) \geq \beta_{i-1}(k - 1, r) + 1 \). Indeed, if \( G \) is a \( K_r \)-free \( k \)-partite graph on \( \beta_i(k, r) \) vertices such that the subgraph induced by any \( k - i \) parts contains a \( K_{r-1} \), then, by deleting a non-empty part of \( G \), we obtain a \( K_r \)-free \( (k - 1) \)-partite graph such that the subgraph induced by any \( (k - 1) - (i - 1) \) parts contains a \( K_{r-1} \). This graph must contains at least \( \beta_{i-1}(k - 1, r) \) vertices and therefore, \( |G| - 1 \geq \beta_{i-1}(k - 1, r) \).

Hence, \( \beta_2(k, r) \geq \beta_1(k - 1, r) + 1 = 2(r - 1) + 1 = 2r - 1 \) by Proposition [14] \( \square \)

3.3 Proof of Theorem [2](i)

The lower bound follows from Proposition [7] and Proposition [14]. The upper bound follows from Proposition [7], Proposition [12] and Corollary [17]. \( \square \)
4 Proof of Theorem 2(ii)

For $k = 2r - 3$, we are done since the lower and upper bounds in Theorem 2(i) match, i.e. $\alpha(k, r) = k(2r - 4) = (k - 1)(2k - 3)$.

Now we shall describe constructions that match the lower bound $\alpha(k, r) \geq k(2r - 4)$ in Theorem 2(i) for the cases when $(k \geq 2r - 2$ and $r$ is even) and $(k \geq 2r - 1$ and $r \equiv 2 \mod 3$), i.e. a $K_r$-partite-saturated $k$-partite graph $G$ containing an independent set $X$ of size $k$ consisting of exactly one vertex from each part of $G$ with $e(X, X^c) = k(2r - 4)$. Lemma 3 tells us that such a graph must satisfy $d(x) = 2r - 4$, for all $x \in X$.

Note that we do not have to worry about making the admissible non-edges inside $X^c$, $K_r$-saturated since we can keep adding admissible edges inside $X^c$ until every admissible non-edge inside $X^c$ is $K_r$-saturated.

Let $p \in \{2, 3\}$ be a divisor of $r - 2$. First we shall construct such $k$-partite graph $G$, for $k = 2r - 4 + p$. We define $X = \{x_1, x_2, \ldots, x_k\}$ and $X^c = \{y_1, y_2, \ldots, y_k\}$, where the parts of $G$ are $\{x_i, y_i\}$, for $i = 1, 2, \ldots, k$. There are no edges inside $X$. Let $y_iy_j$ be an edge iff $i, j$ are not consecutive elements of the circle $\mathbb{Z}_k$, and so $G[X^c]$ is the graph $K_k$ minus a cycle $C_k$. Let $x_iy_j$ be an edge iff $i \neq j \mod \frac{k}{p}$, i.e. $x_i$ is joined to all but $p$ equally spaced $y_j$. We claim that $G$ satisfies the required properties.

Clearly, we have $d(x) = k - p = 2r - 4$ for all $x \in X$ and $e(X, X^c) = k(2r - 4)$. Let us verify that $G$ is $K_r$-free. A clique inside $X^c$ is a set of non-consecutive elements of $\mathbb{Z}_k$, and so a largest clique inside $X^c$ has size $\left\lfloor \frac{2r - 4}{p} \right\rfloor = r - 1$ for $p \in \{2, 3\}$. Since a clique which is not inside $X^c$ can contain at most one vertex of $X$, it remains to check that the neighborhood of each $x_i$ does not contain a clique of size $r - 1$. Viewing $X^c$ as a circle, $N(x_i)$ consists of $p$ segments of the circle, each of size $\frac{2r - 4}{p}$, separated by gaps of size one. Since $\frac{2r - 4}{p}$ is even, a largest clique in $N(x_i)$ has size $\frac{p(2r - 4)}{2p} = r - 2$.

It remains to show that the admissible non-edges inside $X$, and those between $X$ and $X^c$ are $K_r$-saturated. Let $x_iy_j$ be an admissible non-edge, and so $j = i \pm \frac{k}{p}$ in $\mathbb{Z}_k$. Clearly, $N(x_i)$ contains $r - 2$ vertices which form a non-consecutive set of the circle with $y_j$. Therefore, there exists a $K_{r-2}$ in the common neighborhood of $x_i$ and $y_j$ as required. Now let $x_ix_j$ be an admissible non-edge. Then the common neighborhood of $x_i$ and $x_j$ consists of $2p$ segments of the circle separated by gaps of size one such that they form $p$ pairs where the sum of the sizes of each pair is $\frac{2r - 4}{p} - 1$, and so each pair consists of a segment of even size and a segment of odd size. Therefore, a largest non-consecutive set in $N(x_i) \cap N(x_j)$ has size $\frac{p(2r - 4)}{2p} = r - 2$. Hence, there exists a $K_{r-2}$ in $N(x_i) \cap N(x_j)$ as required.

We have constructed such $k$-partite graph $G_k$ for $k = 2r - 4 + p$ with. Let us obtain $G_k$ for $k > 2r - 4 + p$ from $G_{2r - 4 + p}$ by blowing up $x_1$ to a class $\{x_1\} \cup \{x_i : 2r - 3 + p \leq i \leq k\}$ of size $k - (2r - 4 + p) + 1$ where each copy of $x_1$ (not including itself) forms a part of $G_k$ of size one. Clearly, we have $d(x) = 2r - 4$ for all $x \in X = \{x_1, x_2, \ldots, x_k\}$ and $e(X, X^c) = k(2r - 4)$. Since $G_{2r - 4 + p}$ is $K_r$-free, so is $G_k$.

It remains to check that the admissible non-edges inside $X$, and those between $X$ and $X^c$ are $K_r$-saturated. Any admissible non-edge inside $X$ which is not inside the blow up class of $x_1$ is $K_r$-saturated by the same property of $G_{2r - 4 + p}$. Any admissible non-edge inside the blow up class of $x_1$ is $K_r$-saturated since $N(x_1)$ contains a $K_{r-2}$ by the construction of $G_{2r - 4 + p}$. Any admissible non-edge $x_iy_j$
where \( j \neq 1 \) or \((j = 1 \text{ and } i \leq 2r - 4 + p)\), is \(K_r\)-saturated by the same property of \(G_{2r-4+p}\). Any admissible non-edge \(x_iy_j\) where \(j = 1\) and \(2r - 3 + p \leq i \leq k\), is \(K_r\)-saturated since \(N(x_1) \cap N(y_1)\) contains a \(K_{r-2}\) by the construction of \(G_{2r-4+p}\).

\[\square\]

5 Proof of Theorem 2 \((iii)\)

In this section, we study \(\alpha(k,r)\) for \(r = 3,4,5\). The values of \(\alpha(k,3)\) and \(\alpha(k,4)\) are completely determined while the values of \(\alpha(k,5)\) are unknown for \(k = 5,6,8\).

5.1 The function \(\alpha(k,3)\)

We shall prove that \(\alpha(k,3) = 3(k-1)\) for \(k \geq 3\). The upper bound follows from Theorem 2 \((i)\). Let us prove the lower bound.

Let \(G = V_1 \cup V_2 \cup \cdots \cup V_k\) be a \(K_3\)-partite-saturated \(k\)-partite graph \(G\) containing an independent set \(X = \{x_1, x_2, \ldots, x_k\}\) with \(x_i \in V_i\) for all \(i\). By Lemma 8 the deletion of any part of \(G\) does not destroy all vertices of \(N(x_i)\) for all \(i\), i.e. \(x_i\) is joined to at least two parts of \(G\). Suppose for contradiction that \(\epsilon(X,X^c) < 3(k-1)\), i.e. \(X\) contains at least four vertices of degree 2, say \(x_1, x_2, x_3, x_4\). Let \(y_i \in V_i\) and \(y_j \in V_j\) with \(1 < i < j \leq k\) be the neighbors of \(x_1\), and so \(y_i\) and \(y_j\) are not neighbors otherwise \(x_iy_iy_j\) forms a triangle. Since \(\{2,3,4\} \setminus \{i,j\} \neq \emptyset\), we may assume that \(i,j \neq 2\), i.e. \(x_1, x_2, y_i, y_j\) are from different parts of \(G\). Since any pair in \(X\) forms a \(K_3\)-saturated non-edge in \(G\), they have a common neighbor. So \(x_1\) and \(x_2\) have a common neighbor, say \(y_i\).

First we suppose that \(x_2y_j\) is a non-edge. Then \(x_2\) and \(y_j\) have a common neighbor \(y_l \in V_l\). Since \(y_i\) and \(y_j\) are not neighbors, \(l \neq i\). We obtain a contradiction by observing that \(x_iy_iy_j\) forms a triangle. We observe that \(x_iy_j\) are neighbors since \(x_1\) and \(x_i\) have a common neighbor and \(N(x_1) = \{y_i, y_j\}\). Similarly, \(x_iy_l\) are neighbors since \(x_2\) and \(x_i\) have a common neighbor and \(N(x_2) = \{y_i, y_l\}\).

Now, suppose that \(x_2y_j\) is an edge, and so \(N(x_1) = N(x_2) = \{y_i, y_j\}\). Then \(x_iy_j\) are neighbors since \(x_1\) and \(x_i\) have a common neighbor. Similarly, \(x_jy_l\) are neighbors. We know that \(x_i\) and \(x_j\) have a common neighbor \(y_l\) with \(l \neq i,j\). Then either \(l \neq 1\) or \(l \neq 2\), say \(l \neq 1\). Since the non-edge \(x_1y_l\) is \(K_3\)-saturated, \(y_l\) is joined to either \(y_i\) or \(y_j\). This implies a contradiction that either \(x_jy_iy_l\) or \(x_iy_jy_l\) forms a triangle.

\(\square\)

5.2 The function \(\alpha(k,4)\)

As a consequence of Theorem 2 \((ii)\), we obtain that \(\alpha(k,4) = 4k\) for \(k \geq 5\). For the remaining case \(k = 4\), we have the bounds \(16 \leq \alpha(4,4) \leq 18\) from Theorem 2 \((i)\). We shall show that \(\alpha(4,4) = 18\).

Consider the family of graphs appearing in the definition of \(\alpha(r,r)\). Let \(G = V_1 \cup V_2 \cup \cdots \cup V_r\) be an \(K_r\)-partite-saturated \(r\)-partite graph \(G\) containing an independent set \(X = \{x_1, x_2, \ldots, x_r\}\) with \(x_i \in V_i\) for all \(i\). We shall establish some properties of \(G\) which will be useful in this subsection, the next subsection and Section 4.
We say that a vertex \( y \in X_i \) is \( i \)-special if \( y \) is the only neighbor of \( x_i \) in the part of \( G \) containing \( y \). The special degree of a vertex \( y \in X_i \) is the number of \( i \in [r] \) such that \( y \) is \( i \)-special. We say that a vertex \( y \in X_i \) is special if the special degree of \( y \) is at least one. Let us make some easy observations regarding the special vertices.

**Lemma 18.** Let \( G = V_1 \cup V_2 \cup \cdots \cup V_r \) be an \( K_r \)-partite-saturated \( r \)-partite graph \( G \) containing an independent set \( X = \{x_1, x_2, \ldots, x_r\} \) with \( x_i \in V_i \) for all \( i \). The following hold for \( r \geq 4 \).

(i) A special vertex \( y_i \in V_i \) is joined to every vertex of \( X \) except \( x_i \).

(ii) Each \( V_i \) contains at most one special vertex.

(iii) If \( y_i \in V_i \) is \( i' \)-special and \( y_j \in V_j \) is \( j' \)-special with \( i' \neq j \) and \( j' \neq i \) then \( y_i y_j \) is an edge.

(iv) The number of vertices of special degree at least 2 is at most \( r - 2 \).

(v) If \( y_i \in V_i \) is \( i' \)-special and \( y_j \in V_j \) with \( j \neq i, i' \) then \( y_j \) is joined to either \( y_i \) or \( x_{i'} \).

(vi) For a special vertex \( y_i \in V_i \), there exist parts \( V_j \) and \( V_l \) where \( i, j, l \) are distinct such that \( N(x_i) \cap V_j \) and \( N(x_i) \cap V_l \) both contain a non-neighbor of \( y_i \).

**Proof.** (i) Let \( y_i \in V_i \) be \( i' \)-special and let \( j \neq i, i' \). Since the non-edge \( x_{i'} x_j \) is \( K_r \)-saturated, the common neighborhood of \( x_{i'} \) and \( x_j \) contains a \( K_{r-2} \) consisting of one vertex from each part of \( G \setminus (V_{i'} \cup V_j) \). Then \( y_i \) is in this \( K_{r-2} \) since \( y_i \) is the only neighbor of \( x_{i'} \) in \( V_i \), and so \( y_i \) is joined to \( x_j \).

(ii) Suppose for contradiction that \( V_i \) contains two special vertices \( y_i \) and \( z_i \) where \( y_i \) is \( i' \)-special. Then, by (i), \( x_{i'} \) is joined to both \( y_i \) and \( z_i \), contradicting the fact that \( y_i \) is the only neighbor of \( x_{i'} \) in \( V_i \).

(iii) First, suppose that \( i' \neq j' \). Since the non-edge \( x_{i'} x_{j'} \) is \( K_r \)-saturated, the common neighborhood of \( x_{i'} \) and \( x_{j'} \) contains a \( K_{r-2} \) consisting of one vertex from each part of \( G \setminus (V_{i'} \cup V_{j'}) \). Since \( y_i \) is the only neighbor of \( x_{i'} \) in \( V_i \) and \( y_j \) is the only neighbor of \( x_{j'} \) in \( V_j \), both \( y_i \) and \( y_j \) lie in this \( K_{r-2} \). In particular, \( y_i y_j \) is an edge.

Now, suppose that \( i' = j' \). We can pick \( l \neq i, j, i' \) because \( r \geq 4 \). Since the non-edge \( x_{i'} x_l \) is \( K_r \)-saturated, the common neighborhood of \( x_{i'} \) and \( x_l \) contains a \( K_{r-2} \) consisting of one vertex from each part of \( G \setminus (V_{i'} \cup V_l) \). Since \( y_i \) is the only neighbor of \( x_{i'} \) in \( V_i \) and \( y_j \) is the only neighbor of \( x_{i'} \) in \( V_j \), both \( y_i \) and \( y_j \) lie in this \( K_{r-2} \). In particular, \( y_i y_j \) is an edge.

(iv) Suppose for contradiction that there exist vertices \( y_1, y_2, \ldots, y_{r-1} \) of special degree at least 2. By (ii), they lie in different parts of \( G \), say \( y_i \in V_i \) for \( 1 \leq i \leq r - 1 \). We claim that they form a \( K_{r-1} \) which would be a contradiction since, together with \( x_r \), they form a \( K_r \) by (i). Now we show that any \( y_i y_j \) is an edge. Since \( y_i \) and \( y_j \) have special degree at least 2, there exist \( i' \neq j \) and \( j' \neq i \) such that \( y_i \) is \( i' \)-special and \( y_j \) is \( j' \)-special. Therefore, \( y_i y_j \) is an edge by (iii).

(v) Suppose that \( x_{i'} y_j \) is a non-edge. Then the common neighborhood of \( x_{i'} \) and \( y_j \) contains a \( K_{r-2} \) consisting of one vertex from each part of \( G \setminus (V_{i'} \cup V_j) \). Then \( y_i \) is in this \( K_{r-2} \) since \( y_i \) is the only neighbor of \( x_{i'} \) in \( V_i \), and so \( y_i \) is joined to \( y_j \).
(vi) Suppose for contradiction that there exists \( j \in [r] \setminus \{i\} \) such that \( y_j \in V_i \) is joined to every vertex in \( N(x_i) \cap V_i \) for all \( l \neq i, j \). Since the non-edge \( x_i x_j \) is \( K_r \)-saturated, the common neighborhood of \( x_i \) and \( x_j \) contains a \( K_{r-2} \) consisting of one vertex from each part of \((G \setminus X) \setminus (V_i \cup V_j)\). We obtain a contradiction by observing that this \( K_{r-2} \), together with \( x_j \) and \( y_i \), form a \( K_r \). Indeed, by assumption, this \( K_{r-2} \) is also in the neighborhood of \( y_i \) and \( x_j y_i \) is an edge by (i).

Now we are ready to show that \( \alpha(4,4) \geq 18 \). Suppose for contradiction that \( \alpha(4,4) \leq 17 \), i.e. there exists a \( K_4 \)-partite-saturated 4-partite graph \( G = V_1 \cup V_2 \cup V_3 \cup V_4 \) containing an independent set \( X = \{x_1, x_2, x_3, x_4\} \) with \( x_i \in V_i \) for all \( i \) such that \( \sum_{i=1}^{4} d(x_i) \leq 17 \). By Lemma \([8]\), \( d(x_i) \geq \beta_1(3,3) = 4 \) and each \( x_i \) has some neighbor in \( V_j \) for \( j \neq i \). Therefore, there are at least three vertices of degree 4 and possibly one of degree 5. Since a vertex of degree 4 in \( X \) creates at least two special vertices and a vertex of degree 5 in \( X \) creates at least one special vertex, the sum of the special degrees of the vertices in \( X \) is at least \( 2 + 2 + 2 + 1 = 7 \). By Lemma \([18(iv)]\), there is a vertex of special degree 3, say \( y_1 \in V_1 \).

For \( i = 2, 3, 4 \), since \( y_1 \) is \( i \)-special, \( x_i \) has at least three neighbors in \( N(y_1) \cup \{y_1\} \), each in a different part of \( G \), by Lemma \([8]\). On the other hand, \( y_1 \) has at least two non-neighbors, say \( y_2 \in V_2 \) and \( y_3 \in V_3 \), by Lemma \([18(vii)]\). By Lemma \([18(v)]\), \( x_i y_2 \) is an edge for \( i \neq 2 \) and \( x_i y_3 \) is an edge for \( i \neq 3 \). So \( x_4 \) has five neighbors, i.e. \( y_2, y_3 \) and three vertices in \( N(y_1) \cup \{y_1\} \), and \( d(x_1) = d(x_2) = d(x_3) = 4 \). Since \( x_2 \) has four neighbors including \( y_1 \) and it has some neighbor in \( (N(y_1) \cup \{y_1\}) \cap V_j \) for each \( j = 1, 3, 4 \), it has exactly one neighbor in \( V_4 \), say \( y_4 \). Similarly, \( x_3 \) has exactly one neighbor in \( V_4 \) which has to be the same vertex \( y_4 \) by Lemma \([18(ii)]\).

We obtain a contradiction by observing that \( x_1 y_2 y_3 y_4 \) forms a \( K_4 \). First, note that \( x_1 y_4 \) is an edge by Lemma \([18(i)]\). Now \( y_4 \) is not 1-special otherwise \( y_4 \) would have special degree 3 and by repeating the argument above with \( y_1 \) replaced by \( y_4 \), we could deduce that \( x_1, x_2, \) or \( x_3 \) had degree 5. Therefore, the neighbors of \( x_1 \) are \( y_2, y_3, y_4 \) and a vertex in \( V_4 \). Since \( y_2, y_3 \) are both 1-special and \( y_4 \) is 2, 3-special, \( y_2 y_3 y_4 \) forms a triangle by Lemma \([18(iii)]\).

### 5.3 The function \( \alpha(k, 5) \)

As a consequence of Theorem \([2(i)]\) and \((ii)\), we obtain that

\[
\alpha(k, 5) = 6k \quad \text{for } k = 7 \text{ or } k \geq 9,
\]

\[
30 \leq \alpha(5, 5) \leq 36,
\]

\[
36 \leq \alpha(6, 5) \leq 40,
\]

\[
48 \leq \alpha(8, 5) \leq 49.
\]

We shall improve the lower bound for \( \alpha(5, 5) \) to 33.

Suppose for contradiction that \( \alpha(5, 5) \leq 32 \), i.e. there exists a \( K_5 \)-partite-saturated 5-partite graph \( G = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5 \) containing an independent set \( X = \{x_1, x_2, x_3, x_4, x_5\} \) with \( x_i \in V_i \) for all \( i \) such that \( \sum_{i=1}^{5} d(x_i) \leq 32 \). Write \( V_i \) for \( V_i \setminus \{x_i\} \). By Lemma \([8]\), \( d(x_i) \geq \beta_1(4,4) = 6 \) and each \( x_i \) has some neighbor in \( V_j \) for \( j \neq i \). Therefore, there are either four vertices in \( X \) of degree 6 or there
are three vertices of degree 6 and two of degree 7. Since a vertex of degree 6 in $X$ creates at least two special vertices and a vertex of degree 7 in $X$ creates at least one special vertex, the sum of the special degrees of the vertices in $X^c$ is at least 8, and hence, there exists a vertex of special degree at least two. Let $i$ be such that there is a special vertex $y \in Y_i$ with special degree $d_s(y)$ at least two where $(d(y_i), d_s(y))$ is maximum in lexicographical order. Without loss of generality we can assume that $i = 1$. Let $N = N(y) \setminus X$. By Lemma 18(vii) $x_1$ has two neighbours, say $y_2, y_3$, belonging to two distinct parts of $G$, different from $V_1$, which are non-neighbours of $y$. Without loss of generality, we can assume that $y_2 \in Y_2$ and $y_3 \in Y_3$.

For a pair of non-adjacent vertices $u, v \in G$ and $S \subseteq G$, we say that $S$ is $uv$-saturating if adding the edge of $uv$ to $G$ creates a copy $K$ of $K_5$ such that $S \subseteq K$. If $S = \{z\}$ then we simply say that $z$ is $uv$-saturating. Notice that if $S$ is $uv$-saturating then $S$ induces a clique.

In the rest of the proof, we shall repeatedly use the following lemma.

**Lemma 19.** Given $i \in \{2, 3, 4, 5\}$ the following hold.

(i) If $j \in \{2, 3, 4, 5\} \setminus \{i\}$ then $x_i$ has a neighbour in $V_j \cap N$. In particular, $d_N(x_i) \geq 3$.

(ii) If $y$ is $i$-special then $x_i$ is adjacent to $y_j$ for every $j \in \{2, 3\} \setminus \{i\}$.

(iii) If $y$ is $x_ix_j$-saturating, for every $j \in \{2, 3, 4, 5\} \setminus \{i\}$, then $d_N(x_i) \geq 4$.

(iv) If $y$ is $i$-special or $d_s(y) \geq 3$ then $d_N(x_i) \geq 4$.

(v) If $y$ is $2, 3$-special and $i \in \{4, 5\}$, then $d(x_i) \geq 7$.

(vi) If $i \in \{2, 3\}$ and there are $p$ vertices in $X \setminus \{x_1\}$ all of which have neighbours in $Y_i \setminus N$ then there is no vertex in $V_i$ with special degree bigger than $\max \{1, 3 - p\}$.

(vii) $Y_3 \cup Y_4 \subseteq N$.

**Proof.** (i) Observe that we can choose $k \in \{2, 3, 4, 5\} \setminus \{i, j\}$ such that $y$ is either $i$-special or $k$-special. Since there must be a triangle in the common neighbourhood of $x_i$ and $x_j$ which uses $y$, we have that the remaining two vertices belong to $N$. Hence $x_i$ has a neighbour in $N \cap x_j$.

(ii) This follows directly from Lemma 18(vii)

(iii) We shall show that $d_N(x_i) \geq \beta_3(3, 3) = 4$. Take any $j \in \{2, 3, 4, 5\} \setminus \{i\}$. Since $y$ is $x_ix_j$-saturating then there is an edge in the common neighbourhood of $x_i$ and $x_j$ in $N \setminus (V_i \cup V_j)$). Observe that the common neighbourhood of $x_i$ and $y$ cannot contain a $K_3$, hence $d_N(x_i) \geq \beta_3(3, 3) = 4$.

(iv) Take any $j \in \{2, 3, 4, 5\} \setminus \{i\}$. Since $y$ is either $i$- or $j$-special, it follows that $y$ is $x_ix_j$-saturating. Hence, by (ii), $d_N(x_i) \geq 4$.

(v) Without loss of generality we can assume that $i = 4$. If $y$ is also 4-special then it follows from (ii) and (iv) that $d_N(x_4) \geq 4$ and $x_4$ is adjacent to $y, y_2, y_3$, therefore $d(x_4) \geq 7$. Hence we can assume that $y$ is not 4-special. Suppose for contradiction that $d(x_4) = 6$. From (i), we have that $d_N(x_4) \geq 3$.

---

1We say that $(a, b) \preceq (c, d)$ if $a < c$ or $a = c$ and $b \leq d$, where $\preceq$ denotes the lexicographical order relation.
and since $y$ is not 4-special we have that $d_Y(y) \geq 2$. Moreover, $x_4$ has to have at least one neighbour not in $Y_1 \cup N$ as otherwise there would be a copy of $K_5$ in $G$, as seen by considering the non-edge $x_1x_4$. Therefore, $d(x_4) = d_Y(x_4) + d_N(x_4) + |N(x_4) \setminus (Y_1 \cup N)| \geq 3 + 2 + 1 = 6 = d(x_4)$. Hence, $d_Y(x_4) = 3$, $d_N(x_4) = 4$ and $|N(x_4) \setminus (Y_1 \cup N)| = 1$. We shall obtain a contradiction by finding a copy of $K_5$ in the graph $G$.

Suppose $\{z_1, z_2, z_3\}$ is $x_4x_5$ saturating, with $z_i \in V$. We claim that $y \neq z_1$ and $\{z_2, z_3\} \not\subset N$. Suppose for contradiction that it is not the case. If $y$ is $x_4x_5$-saturating then from (iii) we have that $d_N(x_4) \geq 4$ hence we obtain a contradiction. We can therefore assume that $y$ is not $x_4x_5$-saturating and hence $z_1 \neq y$. Hence $z_2, z_3 \in N$. Recall that $\{z_1, z_2, z_3\}$ form a triangle and therefore there is an edge between $z_2, z_3$. By assumption $z_2$ and $z_3$ are neighbours of $y$, hence $y, z_2, z_3$ form a triangle, and therefore $y$ is $x_4x_5$-saturating since $y, z_2, z_3$ belong to the common neighbourhood of $x_4$ and $x_5$, which contradicts the assumption that $y$ is not $x_4x_5$-saturating.

Without loss of generality we can assume that $z_2 \not\in N$. Using (i), we can therefore suppose that $N(x_4) \cap Y_1 = \{y, z_1\}$, $N(x_4) \cap Y_2 = \{w, z_2\}$, $N(x_4) \cap Y_3 = \{z_3\}$ and $N(x_4) \cap Y_5 = \{z_5\}$, for some $w, z_3, z_5 \in N$. We shall obtain a contradiction by observing that $z_1, z_2, z_3, x_4, z_5$ form a copy of $K_5$. First we claim that $\{z_2, z_3, z_5\}$ is $x_1x_4$-saturating. Indeed, there must be a triangle in the common neighbourhood of $x_1$ and $x_4$, with one vertex in each $V_3, V_4, V_5$. There are only two candidates for the triangle: $z_2, z_3, z_5$ or $w, z_3, z_5$. It cannot be $w, z_3, z_5$ since they are all neighbours of $y$, hence $y, w, z_4, z_5$ would form a copy of $K_5$. Hence we must have that the set $\{z_2, z_3, z_5\}$ is $x_1x_4$-saturating. Now, since $x_4$ is not adjacent to $y_3$, and $y_3$ is not adjacent to $y$ we must have an edge between $z_1$ and $z_5$. Indeed, there must be a triangle in the common neighbourhood of $x_4$ and $y_3$ with a vertex in each $V_1, V_2, V_3$. Since $x_4$ has only one neighbour in $V_5$, i.e. $z_5$, and $x_4$ and $y_3$ have only one common neighbour in $V_1$, i.e. $z_1$, we must have an edge between $z_1$ and $z_5$.

Therefore we have that $z_1, z_2, z_3$ form a triangle, $z_2, z_3, z_5$ form a triangle, and $z_1, z_5$ are adjacent. It easy to see now that $z_1, z_2, z_3, x_4, z_5$ form a copy of $K_5$.

(vi) Let $v$ be a special vertex in $V_2 \cup V_4$, say in $V_2$. First observe that if $v$ is 1-special then $x_3, x_4, x_5$ are all adjacent to $y_2 \in Y_2 \setminus N$. On the other hand, it follows from (i) that $x_3, x_4, x_5$ all have neighbours in $N \cap Y_2$ hence they all have degree at least 2 in $Y_2$. It follows that $v$ has special degree 1. If we assume that $v$ is not 1-special then $v$ has special degree at most $3 - p$, since $p$ of the vertices $x_3, x_4, x_5$ have degree 2 in $Y_2$.

(vii) Assume for contradiction that there is $v$, say in $Y_4 \setminus N$. Observe that if $y$ is i-special then it follows from (ii) and (iv) that $d(x_i) \geq 7$, hence if $d_s(y) \geq 3$ we obtain contradiction by finding three vertices in $X$ of degree at least 7. Therefore we can assume that $d_s(y) = 2$.

If $y$ is 5, i-special, then from (ii) and (iv) we have that $d(x_5) \geq 8$ and $d(x_i) \geq 7$ hence again we obtain a contradiction. Therefore we can assume that $y$ is not 5-special. If $y$ is 2, 3-special then $d(x_2), d(x_3) \geq 7$ and from (iv) we have that $d(x_4), d(x_5) \geq 7$. Hence we can assume that $y$ is 2, 4-special or 3, 4-special. Suppose that the former is the case. Then $d(x_2), d(x_4) \geq 7$. It follows that $d(x_1) = 6$. Therefore by maximality $(x_1, y)$ and from (v) we have that every vertex in $Y_2 \cup Y_3 \cup Y_4$ has special degree at most 1 and no vertex in $Y_5$ has special degree bigger than 2. Which gives a contradiction since the sum of special degree is then at most 7.
We are now ready to finish showing that \( \alpha(5,5) \geq 33 \). We consider several cases depending on the special degree of \( y \).

**Case 1.** \( d_s(y) = 4 \)

Consider the 4-partite graph \( H = G[N(y)] \) with an independent set \( X' = \{x_2, x_3, x_4, x_5\} \). Clearly, \( H \) is \( K_4 \)-free since \( G \) is \( K_5 \)-free. We modify \( H \) by keeping adding admissible edges inside \( H \setminus X' \) until every admissible non-edge inside \( H \setminus X' \) is \( K_4 \)-saturated. We claim that \( H \) is \( K_4 \)-partite-saturated, which would imply that \( e(X', H \setminus X') \geq \alpha(4,4) = 18 \) by the previous subsection. It remains to show that the admissible non-edges with at least one endpoint in \( X' \) are \( K_4 \)-saturated.

Consider the non-edge \( x_i y_j \) with \( y_j \in V_j \cap H \) (possibly \( y_j = x_j \)) and distinct \( 2 \leq i, j \leq 5 \). Since the non-edge \( x_i y_j \) is \( K_3 \)-saturated in \( G \), the common neighborhood in \( G \) of \( x_i \) and \( y_j \) contains a \( K_3 \) consisting of one vertex from each part of \( G \setminus (V_i \cup V_j) \). Since \( y \) is \( i \)-special, this \( K_3 \) must contain \( y \), and so the common neighborhood in \( H \) of \( x_i \) and \( y_j \) contains a \( K_2 \), i.e. \( x_i y_j \) is \( K_4 \)-saturated in \( H \) as required.

Recall that \( y \) has two non-neighbors, \( y_2 \in V_2 \) and \( y_5 \in V_3 \). By Lemma \( \ref{lem:18}(v) \), \( x_i y_2 \) is an edge for \( i \neq 2 \) and \( x_i y_5 \) is an edge for \( i \neq 3 \). We shall partition the edges between \( X \) and \( X' \) as follows:

\[
e(X, X') \geq e(X', H \setminus X') + d(x_1) + e(X, y_2) + e(X', y_3) + e(X', y_3) \\
\geq 18 + 6 + 4 + 3 + 3 = 34,
\]

contradicting the assumption.

**Case 2.** \( d_s(y) = 3 \)

If \( y \) is 4, 5-special then from Lemma \( \ref{lem:19}(ii) \) and \( \ref{lem:19}(iii) \) we have that \( d(x_4), d(x_5) \geq 7 \). Otherwise \( y \) is 2, 3-special and hence it follows from Lemma \( \ref{lem:19}(v) \) that \( d(x_4), d(x_5) \geq 7 \). We shall obtain a contradiction by showing that \( d(x_1) \geq 7 \), hence showing that there are three vertices in \( X \) with degrees at least 7, which is against an assumption made in the beginning of the subsection. It follows from Lemma \( \ref{lem:19}(vi) \) with \( p \geq 2 \), that the sum of special degrees in \( Y_2 \cup Y_3 \) is at most 2. Since the sum of special degrees is at least 8, it follows that there is a special vertex in \( Y_4 \cup Y_5 \) with special degree at least 2. Therefore from the maximality of \( d(x_1) \) we have that \( d(x_1) \geq 7 \).

**Case 3.** \( d_s(y) = 2 \)

We split this case into three subcases.

**Case 3.1.** \( y \) is 2, 3-special

It follows from Lemma \( \ref{lem:19}(v) \) that \( d(x_4), d(x_5) \geq 7 \). We shall obtain a contradiction by showing that \( d(x_1) \geq 7 \), hence showing that there are three vertices in \( X \) with degrees at least 7, which is against an assumption made in the beginning of the subsection. It follows from Lemma \( \ref{lem:19}(vi) \) that the sum of special degrees in \( Y_2 \cup Y_3 \) is at most 2. Since the sum of special degrees is at least 8, it follows that there is a special vertex in \( Y_4 \cup Y_5 \) with special degree at least 2. Therefore from the maximality of \( d(x_1) \) we have that \( d(x_1) \geq 7 \).
Case 3.2. \( y \) is 4,5-special

It follows from Lemma 19(ii) and 19(iv) \( d(x_4), d(x_5) \geq 7 \). We shall obtain a contradiction by showing that at least one of \( d(x_1) \geq 7 \), hence showing that there are three vertices in \( X \) with degrees at least 7, which is against an assumption made in the beginning of the subsection. It follows from Lemma 19(vi) that the sum of special degrees in \( Y_2 \cup Y_3 \) is at most 3. Since the sum of special degrees is at least 8, it follows that there is a special vertex in \( Y_4 \cup Y_5 \) with special degree at least 2. Therefore from the maximality of \( d(x_1) \) we have that \( d(x_1) \geq 7 \).

Case 3.3. \( y \) is neither 2,3-special nor 4,5-special

Without loss of generality we can assume that \( y = 2,4 \)-special. It follows from Lemma 19(ii) and 19(iv) \( d(x_4) \geq 7 \) and from Lemma 19(vi) with \( p \geq 2 \) that there is no special vertex in \( Y_2 \cup Y_3 \) with special degree bigger than 1. Hence there is either a vertex in \( Y_4 \) with special degree at least 2 or a vertex in \( Y_5 \) with special degree at least 3. Therefore we can assume that \( d(x_1) = 7 \) as otherwise we obtain a contradiction to the maximality of \( (d(x_1), d_s(y)) \).

We shall obtain a contradiction by showing that at least one of \( x_2, x_3 \) or \( x_5 \) has degree at least 7, thus finding three vertices with degree at least 7. Suppose \( d(x_2) = d(x_3) = d(x_5) = 6 \). Observe that if there is a vertex in \( X_4 \) of special degree bigger than 2 then we obtain a contradiction to the maximality of \( (d(x_1), d_s(y)) \). Therefore there are two vertices in \( X \) with at least two neighbours in \( X_4 \). Suppose that \( i \in \{3, 5\} \) and \( x_i \) has at least two neighbours in \( X_4 \). Then it follows from Lemma 19(i) that \( d_N(x_i) \geq 4 \), and hence \( x_i \) has degree at least 7 as \( x_i \) has at least three neighbours outside \( N \). We can therefore assume that \( x_3 \) and \( x_5 \) have only one neighbour in \( X_4 \). For the same reason we can assume that \( x_3 \) has only one neighbour in \( Y_5 \). If \( x_2 \) has two neighbours in \( Y_5 \) then \( d_N(x_2) \geq 5 \) and therefore \( d(x_2) \geq 7 \). Hence we can assume that there is \( z_5 \in Y_5 \) which is 2,3-special.

Suppose \( \{z_1, z_2, z_4\} \) is \( x_3x_5 \)-saturating, with \( z_i \in V_f \). We claim that \( y \neq z_1 \) and \( z_2 \notin N \). Suppose for contradiction that it is not the case. If \( y \) is \( x_3x_5 \)-saturating then from \( (iii) \) we have that \( d_N(x_3) \geq 4 \) hence we obtain a contradiction. We can therefore assume that \( y \) is not \( x_3x_5 \)-saturating and hence \( z_1 \neq y \). Whence \( z_2 \in N \). Observe that by Lemma 19(vii) we have \( z_4 \in N \). Recall that \( \{z_1, z_2, z_4\} \) form a triangle and therefore there is an edge between \( z_2, z_4 \). By assumption \( z_2 \) and \( z_4 \) are neighbours of \( y \), hence \( y, z_2, z_4 \) form a triangle, and therefore \( y \) is \( x_3x_5 \)-saturating since \( y, z_2, z_4 \) belong to the common neighbourhood of \( x_3 \) and \( x_5 \), which contradicts the assumption that \( y \) is not \( x_3x_5 \)-saturating.

We shall obtain a contradiction by showing that \( z_1, z_2, x_3, z_4, z_5 \) form a copy of \( K_5 \). Indeed, by assumption \( \{z_1, z_2, z_4\} \) is \( x_3x_5 \)-saturating and similar analysis to the one made in the proof of Lemma 19(vi) shows that \( \{z_2, z_4, z_5\} \) is \( x_1x_3 \)-saturating. Since \( y \) is 2-special it follows that \( x_2 \) is not adjacent to \( z_1 \), and moreover \( z_5 \), as the only neighbour of \( x_2 \) in \( Y_5 \), is \( x_2z_1 \)-saturating, and therefore there is an edge between \( x_2 \) and \( z_5 \). Hence we have that \( z_2, z_4, z_5 \) form a triangle, \( z_1, z_2, z_4 \) form a triangle, and \( z_1, z_5 \) are adjacent. It easy to see now that \( z_1, z_2, x_3, z_4, z_5 \) form a copy of \( K_5 \).
6 The diagonal case $\alpha(r, r)$

6.1 Proof of Theorem 2(iv)

We have seen that the lower bound $\alpha(k, r) \geq k(2r - 4)$ in Theorem 2(i) is attained for some $k$. In this subsection, we show that this is not the truth for the diagonal case $k = r \geq 4$, i.e. $\alpha(r, r) \geq r(2r-4)+1$. We shall again use the concept of special vertices introduced in Section 5.

Suppose for contradiction that for some $r \geq 4$, $\alpha(r, r) = r(2r-4)$, i.e. there exists a $K_r$-partite-saturated $r$-partite graph $G = V_1 \cup V_2 \cup \cdots \cup V_r$ containing an independent set $X = \{x_1, x_2, \ldots, x_r\}$ with $x_i \in V_i$ for all $i$ such that $\sum_{i=1}^r d(x_i) = r(2r-4)$. Lemma 5 tells us that we must have $d(x_i) = 2r - 4$ for all $i$ and each $x_i$ has some neighbor in $V_j$ for $j \neq i$. Therefore, each $x_i$ creates at least two special vertices, and so the sum of the special degrees of the vertices in $X^c$ is at least $2r$. By Lemma 18(ii), there is a vertex of special degree at least 3, say $y_1 \in V_1$.

We observe that $y_1$ has at least two non-neighbors, say $y_2 \in V_2$ and $y_3 \in V_3$ by Lemma 18(iv). Since $y_1$ has special degree at least 3, we can pick $i \geq 4$ such that $y_1$ is $i$-special. By Lemma 18(v) $y_2$ and $y_3$ are neighbors of $x_i$. Therefore,

$$|N(x_i) \cap N(y_1)| = d(x_i) - |N(x_i) \setminus N(y_1)| \leq (2r - 4) - 3 = 2r - 7.$$

On the other hand, we shall obtain a contradiction by showing that the graph $H = G[N(x_i) \cap N(y_1)]$ contains at least $\beta_1(r-2, r-2) = 2r-3$ vertices. It is sufficient to prove that $H$ is an $(r-2)$-partite $K_{r-2}$-free graph such that the subgraph induced by any $k-3$ parts contains a $K_{r-3}$. Clearly, $H$ is $K_{r-2}$-free since $G$ is $K_r$-free. The parts of $H$ are $N(x_i) \cap N(y_1) \cap V_j$ for $j \notin [r] \setminus \{1, i\}$. It remains to verify that the deletion of the part $N(x_i) \cap N(y_1) \cap V_j$ does not destroy all the $K_{r-3}$. Since the non-edge $x_ix_j$ is $K_r$-saturated in $G$, the common neighborhood in $G$ of $x_i$ and $x_j$ contains a $K_{r-2}$ consisting of one vertex from each part of $G \setminus (V_1 \cup V_j)$. Since $y_1$ is $i$-special, this $K_{r-2}$ must contain $y_1$, and so the common neighborhood $N(x_i) \cap N(y_1) \cap N(x_j) \subset H$ contains a $K_{r-3}$ not using the vertices of $V_j$ as required.

6.2 Remark on $\beta_2(r, r - 1)$

Recall from Proposition 7 that $\alpha(r, r) \leq (r - 1)\beta_2(r, r - 1)$. Thus, a better estimate on $\beta_2$ would translate to a better understanding of the saturation numbers. While we could not find the exact value of $\beta_2(r, r - 1)$, we suspect that $\beta_2(r, r - 1) = 3r - 6$ as mentioned in Conjecture 13. In this subsection, we make an observation about $\beta_2(r, r - 1)$ which can be viewed as a first step towards determining its exact value. For simplicity of notation, let us write $\beta_2(r) = \beta_2(r, r - 1)$.

**Proposition 20.** Either

- $\beta_2(r) = 3r - 6$ for all $r \geq 3$, or
- $\beta_2(r) \leq (c + o(1))r$ for some constant $c < 3$, as $r \to \infty$. 

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Proof. The result is an immediate consequence of the following lemma.

Lemma 21. $\beta_2(r_1 + r_2) \leq \beta_2(r_1) + \beta_2(r_2) + 6$ for $r_1, r_2 \geq 3$.

Proof. For $i \in \{1, 2\}$, let $G_i = V_i,1 \cup V_i,2 \cup \cdots \cup V_i,r$, be a $K_{r_i-1}$-free $r_i$-partite graph on $\beta_2(r_i)$ vertices such that the subgraph induced by any $r_i - 2$ parts contains a $K_{r_i-2}$. We shall construct a $K_{r_1+r_2-1}$-free $(r_1 + r_2)$-partite graph $G$ from $G_1$ and $G_2$ with $|G| = |G_1| + |G_2| + 6$ by starting with the disjoint union of $G_1$ and $G_2$ and then adding six new vertices $U = \{x_1, x_2, y_1, y_2, z_1, z_2\}$ as follows: add $x_i, y_i$ to $V_i,1$ and add $z_i$ to $V_i,2$ for $i \in \{1, 2\}$. Now, join all admissible pairs between $U$ and $V(G) \setminus U$, and add the edges $x_1z_1, x_2z_2, y_1y_2, z_1z_2, y_1z_2, z_1y_2$ inside $U$.

First, we show that $G$ is $K_{r_1+r_2-1}$-free. Suppose otherwise. Since $G_i$ is $K_{r_i-1}$-free for $i \in \{1, 2\}$, this $K_{r_1+r_2-1}$ must contain at least three vertices forming a triangle in $U$, contradicting the fact that $G[U]$ is triangle-free. It remains to show that the deletion of any two parts does not destroy all the $K_{r_1+r_2-2}$. Suppose first that both deleted parts are from $G_1$. Since $G_1$ contains a $K_{r_1-2}$ not using these two parts and $G_2$ contains a $K_{r_2-2}$ not using $V_2,1$ and $V_2,2$, we obtain a $K_{r_1+r_2-2}$ not using the deleted parts, formed by these two cliques and $x_2, z_2$. Now suppose that one of the deleted parts is from $G_1$ and the other is from $G_2$. For $i \in \{1, 2\}$, let $V_i$ be a part in $\{V_i,1, V_i,2\}$ which was not deleted. By construction, $G[U]$ contains an edge between $V_{i,j}$ and $V_{i,l}$ for all $j, l \in \{1, 2\}$ and so there exists an edge in $G[U]$ between $V_1$ and $V_2$, say $e$. Since $G_1$ contains a $K_{r_1-2}$ not using the deleted part in $G_1$ and $V_1$, and $G_2$ contains a $K_{r_2-2}$ not using the deleted part in $G_2$ and $V_2$, we obtain a $K_{r_1+r_2-2}$ not using the deleted parts, formed by these two cliques and the endpoints of $e$. 

Suppose that $\beta_2(s) < 3s - 6$ for some $s \geq 3$. We shall show that $\beta_2(r) \leq (c+o(1))r$ with $c = \frac{\beta_2(s)+6}{s} < 3$. Applying the lemma and induction on $m$, we deduce that $\beta_2(ms) \leq cms - 6$ for all positive integer $m$. Hence, writing $r = ms + t$ with $3 \leq t \leq s + 2$ and applying the lemma again,

$$\beta_2(r) \leq \beta_2(ms) + \beta_2(t) + 6 \leq cms + d \leq (c + \frac{d}{r}) r = (c + o(1))r$$

where $d = \max\{\beta_2(t) : 3 \leq t \leq s + 2\}$. 

\section{Proof of Theorem 3}

Theorems 1 and Theorem 2(ii) imply that

$$sat(n, k, r) = k(2r - 4)n + o(n)$$

if

$$k = 2r - 3, \text{ or } \begin{cases} k \geq 2r - 2 \text{ and } r \equiv 0 \mod 2, \text{ or } \k \geq 2r - 1 \text{ and } r \equiv 2 \mod 3. \end{cases}$$

In this section, we shall show that the $o(n)$ term can be replaced with $O(1)$. The upper bound follows from Proposition 3 and Theorem 2(ii). We prove that the lower bound holds for any $k \geq r \geq 3$ using the fact that $\beta_1(k - 1, r - 1) = 2r - 4$. 

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Proposition 22. For $k \geq r \geq 3$, there is an integer $C_{k,r}$ such that $\text{sat}(n,k,r) \geq k(2r - 4)n + C_{k,r}$, for every integer $n \geq 0$.

Proof. Suppose, as we may, that $n$ is sufficiently large. Let $G = V_1 \cup V_2 \cup \cdots \cup V_k$ be a $K_r$-partite-saturated $k$-partite graph with $|V_i| = n$ for all $i$. We shall find a subset $U$ of $V(G)$ of constant size such that every vertex in $U^c$ has at least $2r - 4$ neighbors in $U$. Then we would be done since $e(G) \geq e(U,U^c) \geq (2r - 4)(kn - |U|)$. Let $v_1$ be a vertex of smallest degree in $V_1$. Having defined $v_1, v_2, \ldots, v_{i-1}$, let $v_i \in V_i$ be a vertex of smallest degree in $V_i \setminus (N(v_1) \cup N(v_2) \cup \cdots \cup N(v_{i-1}))$. We shall take $U$ to be $N(v_1) \cup N(v_2) \cup \cdots \cup N(v_k)$. Now we may assume that $d(v_i) < 2k(2r - 4)$ for all $1 \leq i \leq k$. Indeed, if $v_i$ is the first vertex in the sequence such that $d(v_i) \geq 2k(2r - 4)$ then we are done since

$$e(G) \geq e(V_i, V_i^c) \geq d(v_i) \left(n - \sum_{j < i} d(v_j)\right) \geq 2k(2r - 4) \left(n - 2k(2r - 4)(i - 1)\right) \geq k(2r - 4)n$$

for sufficiently large $n$. Therefore, $U$ has size bounded by a function of $k$ and $r$. It remains to show that every vertex $v \in U^c$ has at least $2r - 4$ neighbors in $U$. We shall prove that $H = G[N(v) \cap U]$ contains at least $\beta_1(k - 1, r - 1) = 2r - 4$ vertices by showing that $H$ is a $K_{r-1}$-free $(k - 1)$-partite graph such that the subgraph induced by any $k - 2$ parts contains a $K_{r-2}$. Clearly, $H$ is $K_{r-1}$-free since $G$ is $K_r$-free. Without loss of generality, $v \in V_1$. The parts of $H$ are $N(v) \cap U \cap V_i$ for $2 \leq i \leq k$. The deletion of the part $N(v) \cap U \cap V_1$ does not destroy all the $K_{r-2}$ since the non-edge $vv_i$ is $K_{r-1}$-saturated in $G$, i.e. $N(v) \cap N(v_i) \subset H$ contains a $K_{r-2}$ not using the vertices of $V_i$. \hfill \Box

8 Concluding remarks

We have reduced the problem of determining $\text{sat}(n,k,r)$ for large $n$ to that of $\alpha(k,r)$. Although, we have determined $\alpha(k,r)$ for some values of $k$ and $r$, a large number of cases remain unknown. In particular, the seemingly easiest case when $r$ is fixed and $k$ is large, is still open.

Problem 23. Determine $\alpha(k,r)$ for $k \geq 2r - 2$ and $r \equiv 1, 3 \mod 6$.

For $k \geq 2r - 2$ and $r \equiv 0, 2, 4, 5 \mod 6$, we have determined $\alpha(k,r)$ except one missing case when 3 is the smallest divisor of $r - 2$ and $k = 2r - 2$. Theorem 2(i) implies that $\alpha(2r - 2, r) \in \{(2r - 3)^2, (2r - 3)^2 - 1\}$ and we suspect that $\alpha(2r - 2, r) = (2r - 3)^2$. Not only we believe that $\beta_2(k,r) = 4r - k - 2$ for $r < k \leq 2r - 1$ (see Conjecture 13) but we also think that the upper bound $\alpha(k,r) \leq (k - 1)\beta_2(k,r - 1) \leq (k - 1)(4r - k - 6)$ in Theorem 2(i) is the correct value for $\alpha(k,r)$ in this case.

Conjecture 24. $\alpha(k,r) = (k - 1)(4r - k - 6)$ for $5 \leq r \leq k \leq 2r - 4$.

We have shown that $33 \leq \alpha(5,5) \leq 36$. This is the smallest case for which the value of $\alpha$ is not yet known.
Problem 25. Find $\alpha(5, 5)$.

To prove the lower and upper bounds on $\alpha(k, r)$, we extensively used the bounds on $\beta_1(k, r)$ and $\beta_2(k, r)$. We believe that determining the values of $\beta_i(k, r)$ is an interesting problem on its own.

Problem 26. Determine $\beta_i(k, r)$ for $k \geq r \geq 2$ and $2 \leq i \leq k - r + 1$.

We end the paper with a remark on a related problem. Recall that $\text{sat}(n, K_r)$ is the minimum number of edges in a $K_r$-free graph on $n$ vertices but the addition of an edge joining any two non-adjacent vertices creates a $K_r$. In the pioneer paper of Erdős, Hajnal, and Moon [5], they determined $\text{sat}(n, K_r)$ by considering a more general problem where the graphs were not required to be $K_r$-free. Interestingly, the two problems have the same answer since the extremal graph is $K_r$-free. We remark that this phenomenon does not happen for partite saturation. Roberts [13] studied the corresponding more general problem for $\text{sat}(K_{r \times n}, K_r)$ and showed that the minimum number of edges in a $K_r$-saturated subgraph of $K_{r \times n}$, where the subgraph is allowed to contain $K_r$, is $\binom{r}{2}(2n - 1)$ for $r \geq 4$ and sufficiently large $n$. On the other hand, Theorem 1 and Theorem 2 imply that $\text{sat}(K_{r \times n}, K_r) \geq r(2r - 4)n + o(n) > \binom{r}{2}(2n - 1)$ for sufficiently large $n$.

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