QUADRATIC ALGEBRAS WITH EXT ALGEBRAS
GENERATED IN TWO DEGREES

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Abstract. We show that there exist non-Koszul graded algebras that appear to be Koszul up to any given cohomological degree. For any integer $m \geq 3$ we exhibit a non-commutative quadratic algebra for which the corresponding bigraded Yoneda algebra is generated in degrees $(1,1)$ and $(m,m+1)$. The algebra is therefore not Koszul but is $m$-Koszul (in the sense of Backelin). These examples answer a question of Green and Marcos [3].

1. Introduction

A connected graded algebra $A$ over a field $\mathbb{k}$ with generators in degree one is called Koszul [5] if its associated bigraded Yoneda (or Ext) algebra $E(A) = \bigoplus_{m \leq n} Ext_A^{m,n}(\mathbb{k},\mathbb{k})$ is generated as an algebra by $Ext_A^{1,1}(\mathbb{k},\mathbb{k})$. Koszul algebras are always quadratic, i.e. the elements in a minimal collection of defining relations will always be of degree two, however not every quadratic algebra is Koszul. A quadratic algebra $A$ will fail to be Koszul if and only if $Ext_A^{m,n}(\mathbb{k},\mathbb{k}) \neq 0$ for some $m < n$.

The notion of $m$-Koszul described in [4] and credited to Backelin [1] serves as a measure of how close a graded $\mathbb{k}$-algebra comes to being Koszul. The
following definition of $m$-Koszul should not be confused with Berger’s $N$-Koszul [2], which refers to $N$-homogeneous algebras with Yoneda algebras generated in degrees one and two.

**Definition 1.1.** A graded algebra $A$ is called $m$-Koszul if $\text{Ext}^i_A(k,k) = 0$ for all $i < j \leq m$.

While any quadratic algebra is 3-Koszul, only a Koszul algebra will be $m$-Koszul for every $m \geq 1$. Conversely, if $A$ is $m$-Koszul for every $m \geq 1$ then $A$ is Koszul. It is natural to ask whether a quadratic algebra could be $m$-Koszul for large values of $m$ and yet still fail to be Koszul.

The purpose of this paper is to show that there exist non-Koszul graded algebras that appear to be Koszul up to any given cohomological degree. Specifically, we show that for any integer $m \geq 3$ there exists an $m$-Koszul algebra $C$ of global dimension $m$ for which the corresponding Yoneda algebra $E(C)$ is generated as an algebra in cohomology degrees one and $m$. Therefore it is not possible to use a single $m$ to confirm Koszulity by means of $m$-Koszulity.

The algebra $C$ also satisfies the following two conditions:

1. $\text{Ext}^{m,n}_C(k,k) = 0$ unless $m = \delta(n)$ for a function $\delta : \mathbb{N} \rightarrow \mathbb{N}$;
2. $\text{Ext}_C(k,k)$ is finitely generated.

Such algebras are called $\delta$-Koszul by Green and Marcos [3]. In our case the function $\delta$ is given by

$$\delta(n) = \begin{cases} n & \text{if } n < m \\ n + 1 & \text{if } n = m \end{cases}$$

Our examples answer the third question posed Green and Marcos in [3]. They ask if there is a bound $N$ such that if $A$ is a $\delta$-Koszul algebra, then $E(A)$ is generated in degrees 0 to $N$. The algebras $C$ illustrate that no such bound exists. Moreover, the bound does not exist even if we restrict ourselves to quadratic algebras.

A quadratic algebra $A$ is determined by a vector space of generators $V = A_1$ and an arbitrary subspace of quadratic relations $I \subset V \otimes V$. The free algebra $k(V)$ carries a standard grading and $A$ inherits a grading from this free algebra. We denote by $A_n$ the component of $A$ degree $n$. For any graded algebra $A = \oplus_k A_k$, let $A[j]$ be the same vector space with the shifted
grading $A[j]_k = A_{j+k}$. Throughout we assume our graded algebras $A$ are locally finite-dimensional with $A_i = 0$ for $i < 0$ and $A_0 = k$.

2. The algebra $C$

Let $m$ be an greater than 2. If $m = 3$ the algebra $C$ has 10 generators and 8 relations. If $m = 4$ then $C$ has 12 generators and 14 relations. For $n \geq 5$, $C$ has $3m$ generators and $4 + 3m$ relations. The case $m = 3$ is already well known, and the case $m = 4$ will be encompassed in the proof of Lemma 2.6 as the algebra $B$, so we will henceforth assume that $m \geq 5$.

The algebra $C$ is defined as follows. The generating vector space $V$ has the basis $\bigcup_{i=1}^{m+1} S_i$ with sets $S_1 = \{n\}$, $S_2 = \{p, q, r\}$, $S_3 = \{s, t, u\}$, $S_4 = \{v, w, x_1, y_1, z_1\}$, $S_5 = \{x_2, y_2, z_2\}$, $\cdots$, $S_{m-1} = \{x_{m-4}, y_{m-4}, z_{m-4}\}$, $S_m = \{x_{m-3}, y_{m-3}\}$, and $S_{m+1} = \{x_{m-2}\}$. For all $m \geq 5$ the space of relations $I$ contains the generators $\{np - nq, np - nr, ps - pt, qt - qu, rs - ru, sv - sw, tw - tx_1, vw - ux_1, vx_2, wx_2, x_1x_1 + sv - sy_1, tw - ty_1, ux_1 - uy_1, sz_1, tz_1, uz_1, yi - 1 xi + z_i - yi\}$ where $i \leq m - 3$. In addition, if $m \geq 6$ then $I$ also contains $\{z_i z_{i+1}\}$ where $i \leq m - 5$.

Remark 2.1. We have chosen this large set of generators to clarify the exactness of the resolution below. It may be possible to construct examples with fewer generators.

Notice that any basis for $I$ is formed from certain sums of elements of $S_i$ right multiplied by elements of $S_{i+1}$. This ordering on a basis of $I$ makes $C$ highly noncommutative. Indeed the center of $C$ is just the field $k = C_0$. Moreover, this ordering tells us that the left annihilator of $n$ is zero and more generally that the left annihilator of an element of $S_i$ is generated by sums of elements from $\Pi_{k=j}^{i-1} S_k$ for $1 \leq j \leq i - 1$. The structure of $I$ will be exploited in the proofs of Lemmas 2.2, 2.4, 2.5 and 2.6.

Our proof relies on constructing an explicit projective resolution for $k$ as a left $C$-module. Let $(P^*, \lambda)$ be the sequence of projective $C$-modules

$$P^m \xrightarrow{\lambda_m} P^{m-1} \xrightarrow{\lambda_{m-1}} P^{m-2} \cdots \xrightarrow{\lambda_2} P^1 \xrightarrow{\lambda_1} C \to k$$

where $P^m = C[-m - 1]$, $P^{m-1} = (C[1 - m])^7$, $P^{m-2} = (C[2 - m])^{16}$, $P^{2} = (C[-2])^{3m+4}$, $P^{1} = (C[-1])^{3m}$, and for $3 \leq i \leq m-3$, $P^{i} = (C[-i])^{3m+12-3i}$. 


The map from $C$ to $k$ is the usual augmentation. For convenience we will use $\lambda_i$ to denote both the map from $P^i$ to $P^{i-1}$ and the matrix which gives this map via right multiplication. The map $\lambda_1$ is right multiplication by the transpose of the matrix

$$(n \ p \ q \ r \ s \ t \ u \ z_1 \ \ldots \ z_{m-4} \ v \ w \ x_1 \ y_1 \ x_2 \ y_2 \ \ldots \ x_{m-3} \ y_{m-3} \ x_{m-2})$$

and the map $\lambda_m$ is right multiplication by the matrix

$$(0 \ 0 \ 0 \ 0 \ np \ np \ -np \ 0 \ \ldots \ 0) .$$

The remaining maps $\lambda_i$ will be defined as right multiplication by matrices given in block form, for which we will need the following components. Let

$$\alpha = \begin{pmatrix} 0 & 0 & 0 & p & 0 & 0 & -p & p & 0 \\ 0 & 0 & 0 & 0 & q & 0 & 0 & -q & q \\ 0 & 0 & 0 & 0 & 0 & r & -r & 0 & r \end{pmatrix}, \quad \alpha' = \begin{pmatrix} p & 0 & 0 & -p & p & 0 \\ 0 & q & 0 & 0 & -q & q \\ 0 & 0 & r & -r & 0 & r \end{pmatrix},$$

$$\beta = \begin{pmatrix} s & -s & 0 & 0 \\ 0 & t & -t & 0 \\ u & 0 & -u & 0 \\ s & 0 & 0 & -s \\ 0 & t & 0 & -t \\ 0 & 0 & u & -u \end{pmatrix}, \quad \beta' = \begin{pmatrix} s & -s & 0 \\ 0 & t & -t \\ u & 0 & -u \end{pmatrix},$$

$$\gamma = \begin{pmatrix} v \\ w \\ x_1 \end{pmatrix}, \quad \gamma' = \begin{pmatrix} s \\ t \\ u \end{pmatrix}, \quad \chi_j = \begin{pmatrix} x_j & 0 & z_j \\ y_j & 0 & \ldots & 0 \\ z_1 & 0 & 0 & \ldots & 0 \\ z_2 & 0 & 0 & \ldots & 0 \\ z_3 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & z_j \end{pmatrix},$$

$$\delta = \begin{pmatrix} -p & p & 0 \\ 0 & -q & q \\ -r & 0 & r \end{pmatrix}, \quad \epsilon = \begin{pmatrix} v \\ w \\ x_1 \end{pmatrix},$$

$$\zeta_j = \begin{pmatrix} z_1 & 0 & 0 & \ldots & 0 \\ 0 & z_2 & 0 & \ldots & 0 \\ 0 & 0 & z_3 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & z_j \end{pmatrix}, \quad \eta = (0, n, n, -n), \quad \eta' = \begin{pmatrix} 0 & n & -n & 0 \\ 0 & n & 0 & -n \end{pmatrix} .$$

The matrix defining map $\lambda_2$ has this block form

$$\begin{pmatrix} \eta' \\ \delta \\ \epsilon \\ \zeta_{m-5} \\ \beta \\ \gamma \\ \chi_2 \\ \cdots \\ \chi_{m-4} \\ x_{m-3} \end{pmatrix} .$$
The matrix $\lambda_3$ has the form
\[
\begin{pmatrix}
0 & \eta & \delta \\
\epsilon & \zeta_{m-6} & \alpha' \\
\beta & \gamma & \chi_2 \\
& & \ddots \\
& & & \chi_{m-5} & x_{m-4}
\end{pmatrix}.
\]

For $4 \leq j \leq m - 3$ the matrix $\lambda_j$ has the form
\[
\begin{pmatrix}
\eta & \delta \\
\epsilon & \zeta_{m-j-3} & \alpha \\
\beta & \gamma & \chi_2 \\
& & \ddots \\
& & & \chi_{m-j-2} & x_{m-j-1}
\end{pmatrix}.
\]

Note that $\chi_2$ is the only $\chi$ block in $\lambda_{m-4}$ and that the matrix $\lambda_{m-3}$ has no $\zeta$ or $\chi$ blocks.

The matrix $\lambda_{m-2}$ has the form
\[
\begin{pmatrix}
\eta & \delta \\
\alpha & \beta \\
\beta' & \gamma
\end{pmatrix},
\]

and the matrix $\lambda_{m-1}$ has the form
\[
\begin{pmatrix}
\eta & \alpha \\
\alpha' & \beta
\end{pmatrix}.
\]

**Lemma 2.2.** Let $(Q^*, \phi)$ be a minimal projective resolution of $C^k$ where the map $Q^i \to Q^{i-1}$ is given as right multiplication by a matrix $\phi_i$. Then the matrices $\phi$ can be chosen to have block form such that all the entries in $\phi_i$ are elements from the subalgebra generated by the set $\bigcup_{j=1}^{m+2-i} S_j$. 

Proof. We prove this by induction on $i$. $\phi_1$ can be chosen to be $\lambda_1$, which has entries from $V = \bigcup_{j=1}^{m+1} S_j$. Since the set $\phi_2\phi_1$ must span the space $I$, $\phi_2$ can be chosen to be $\lambda_2$, where the blocks have entries of the appropriate form.

Now suppose $\phi_i$ has block form with entries from the subalgebra generated by $\bigcup_{j=m+i}^{m+3-i} S_j$. Since the rows of $\phi_{i+1}$ must annihilate the columns of $\phi_i$, $\phi_{i+1}$ can be chosen to have block form corresponding to the blocks of $\phi_i$.

Recall that any basis for $I$ is ordered so that only elements of $S_j$ appear on the left of elements of $S_{j+1}$. Since the entries in $\phi_i$ contain no elements from $\bigcup_{j=m+i}^{m+3-i} S_j$, no elements from $\bigcup_{j=m+2-i}^{m+3-i} S_j$ will appear in entries of $\phi_{i+1}$. Thus the entries in $\phi_{i+1}$ are from the subalgebra generated by the set $\bigcup_{j=m+1-i}^{m+i} S_j$. \hfill \Box

Remark 2.3. The lemma implies that $\phi_{m+1}$, if it exists, can only contain elements of $S_1$ and that there can be no map $\phi_{m+2}$. Therefore a minimal resolution of $C^k$ would have length no more than $m + 1$ and so the global dimension of $C$ is at most $m + 1$. We will see later that the global dimension of $C$ is exactly $m$.

Lemma 2.4. The left annihilators of $\eta$, $\eta'$, $\lambda_m$ and $\alpha$ are zero.

Proof. The relations for $C$ make it clear that nothing annihilates $n$ from the left, and consequently $\eta$, $\eta'$ and $\lambda_m$ cannot be annihilated from the left. Since the entries in $\alpha$ are all from $S_2$, the annihilator would have to be made from left multiples of $n$. However $p$, $q$ and $r$ each appear alone in the first columns of $\alpha$ and $n$ does not annihilate these individually. \hfill \Box

Lemma 2.5. The rows of $\gamma'$ generate the left annihilator of $x_2$.

Proof. The left annihilator of $x_2$ can have generators made from sums of elements in $S_4$, $S_3S_4$, $S_2S_3S_4$ or $S_1S_2S_3S_4$. It is easy to check that linear combinations of $v$, $w$ and $x_1$ are the only elements in the span of $S_4$ that annihilate $x_2$. The elements of $S_3S_4$ span a six dimensional subspace of $C_2$ with basis $\{sv, sx_1, tv, tw, uv, uw\}$ and these are all left multiples of $v$, $w$ or $x_1$. It follows that in $C$ the elements of $S_2S_3S_4$ and $S_1S_2S_3S_4$ are also all left multiples of $v$, $w$ or $x_1$. \hfill \Box
**Lemma 2.6.** The left annihilator of $\beta$ is generated by the rows of $\alpha$, the left annihilator of $\delta$ is generated by $\eta$, the left annihilator of $\beta'$ is generated by $\lambda_{m+1}$, and the left annihilator of $\gamma'$ is generated by the rows of $\beta'$.

**Proof.** We consider an algebra $B$ closely related to $C$. Let $B$ be the algebra with generators $\{n, p, q, r, s, t, u, v, w, x, y, a, b\}$ and defining relations $\{np-qr, np-nq, ps-pt, qt-qu, rs-ru, sw-sw, tw-tw, vw-vx, wa-vb, wa-wb, x_1a-x_1b\}$. We will resolve $k$ as a $B$-module using maps made from the same blocks as those which appear in the resolution of $k$ as a $C$ module.

Let $(R^*, \psi)$ be this sequence of projective $B$-modules

$$0 \rightarrow B[-5] \xrightarrow{\psi_4} B[-3]^7 \xrightarrow{\psi_3} B[-2]^7 \xrightarrow{\psi_2} B[-1]^5 \xrightarrow{\psi_1} B \rightarrow k$$

The map from $B$ to $k$ is the usual augmentation, the map $\psi_1$ is right multiplication by the transpose of the matrix

$$\begin{pmatrix} n & p & q & r & s & t & u & v & w & x_1 & a & b \end{pmatrix},$$

the map $\psi_2$ is right multiplication by the the matrix

$$\begin{pmatrix} \eta' \\ \delta \\ \beta' \\ \gamma' & -\gamma' \end{pmatrix},$$

the map $\psi_3$ is right multiplication by the the matrix

$$\begin{pmatrix} 0 & \eta \\ \alpha & \beta' \end{pmatrix},$$

and the map $\psi_4$ is right multiplication by the the matrix

$$\begin{pmatrix} 0 & 0 & 0 & np & np & -np \end{pmatrix}.$$

Since the product $\psi_i\psi_{i-1}$ is zero in $B$, $R^*$ is a complex. The list of generators and relations for $B$ ensure that this complex is exact at $R^1$ and $R^0$. Our goal is to show that $(R^*, \psi)$ is a minimal projective resolution of $Bk$.

We observe that $B$ is the associated graded algebra for the splitting algebra (see [7]) corresponding to the layered graph below.
The methods of [7] (see also [6]) show that $B$ has Hilbert series $H_B(g) = (1 - 13g + 14g^2 - 7g^3 + g^5)^{-1}$. Let $P_B(f, g) = \sum_{i,j=0}^{\infty} \dim(\text{Ext}^{ij}_B(\mathbb{k}, \mathbb{k}) f^i g^j)$ be the double Poincaré series for $B$. From the formula $P_B(-1, g) = H_{B}^{-1}(g) = 1 - 13g + 14g^2 - 7g^3 + g^5$ we can deduce something about the shape of a minimal projective resolution of the $B$-module $\mathbb{k}$. Moreover, the entries in the maps of a minimal projective resolution come from certain sets as in Lemma 2.2. It follows that the resolution must have the form:

$$
0 \rightarrow B[-5]^{d_3-d_2} \rightarrow R^4 \oplus B[-5]^{d_3} \oplus B[-4]^{d_1} \rightarrow R^3 \oplus B[-4]^{d_1} \oplus B[-5]^{d_2} \rightarrow R^2 \xrightarrow{\psi_2} R^1 \xrightarrow{\psi_1} R^0 \rightarrow \mathbb{k}
$$

We will show that $d_1 = d_2 = d_3 = 0$ so that $R^\bullet$ is in fact a resolution of $B\mathbb{k}$.

Consider first the possibility that $d_1 > 0$. This can only happen if $\eta$, $\alpha$ or $\beta'$ are annihilated from the left by a linear vector. Clearly nothing annihilates $\eta$ from the left and by Lemma 2.4 nothing annihilates $\alpha$ from
the left. Suppose the vector

\[
\begin{pmatrix}
  e_1 & e_2 & e_3
\end{pmatrix}
\]

left annihilates \( \beta' \) where each \( e_i \) is a sum of elements from \( S_2 \). Then

\[
\vec{e} = \begin{pmatrix}
  e_1 & e_2 & e_3 & 0 & 0 & 0
\end{pmatrix}
\]

would left annihilate the \( \beta \) which appears in \( \psi_2 \). However the Poincaré series for \( B \) assures us that the dimension of \( \text{Ext}^{3,3}_B(\mathbb{k}, \mathbb{k}) \) is seven, which means that \( \vec{e} \) would be in the span of the rows of \( \alpha' \). Now observe that no nonzero combination of the rows of \( \alpha' \) could produce \( \vec{e} \). We conclude that \( d_1 = 0 \).

Since \( d_1 \) is zero, \( \text{Ext}^{3,4}(\mathbb{k}, \mathbb{k}) = \text{Ext}^{4,4}(\mathbb{k}, \mathbb{k}) = 0 \). Now observe that the absence of \( \text{Ext}^{4,4}(\mathbb{k}, \mathbb{k}) \) means that \( \text{Ext}^{5,5}(\mathbb{k}, \mathbb{k}) \) must also be zero, so that \( d_2 = d_3 \). For \( \text{Ext}^{3,5}(\mathbb{k}, \mathbb{k}) \) to be nonzero, the matrix defining the map \( R^3 \oplus B[-5]^{d_2} \rightarrow R^2 \) must contain sums of elements from \( S_1S_2S_3 \) which annihilate \( \gamma' \). The elements of \( S_1S_2S_3 \) span a one dimensional subspace of \( C_3 \) with basis \( \{nps\} \). Suppose \( nps( h_1 \ h_2 \ h_3 ) \gamma' = \) for some \( h_i \in \mathbb{k} \). Since in \( C_4 \)

\[
npsv = npsw = npx_1,
\]

we get \( h_1 + h_2 + h_3 = 0 \), and thus \( nps( h_1 \ h_2 \ h_3 ) \gamma' \) is just a row of \( \beta' \) multiplied on the left by \( np \). Therefore the rows of \( \beta' \) generate the left annihilator of \( \gamma' \) and \( d_3 = d_2 = 0 \), which means \( (R^\bullet, \psi) \) is exact.

In general one would not expect information from resolutions over other algebras to be useful in resolving \( C \), however the relations of \( C \) and \( B \) both follow the pattern that the left annihilator of an element of \( S_i \) is generated by elements of \( S_{i-1} \), so that information from \( B \) is applicable to \( C \). Thus from the fact that \( (R^\bullet, \psi) \) is exact, we see that the left annihilator of \( \beta \) is generated by the rows of \( \alpha \), the left annihilator of \( \delta \) is generated by \( \eta \), the left annihilator of \( \beta' \) is generated by \( \lambda_{m+1} \), and the left annihilator of \( \gamma' \) is generated by the rows of \( \beta' \).

\[\square\]

**Theorem 2.7.** For any integer \( m \geq 3 \) the complex \( P^\bullet \) is a minimal projective resolution of the left \( C \)-module \( \mathbb{k} \). It follows that the algebra \( C \) has global dimension \( m \) and \( \text{Ext}^{ij}_C(\mathbb{k}, \mathbb{k}) = 0 \) for all \( i < j \leq m \). Moreover \( C \) is not a Koszul algebra because \( \text{Ext}^{m,m+1}_C(\mathbb{k}, \mathbb{k}) \neq 0 \).

**Proof.** Direct calculation shows that \( \lambda_i \lambda_{i-1} = 0 \) for all \( i \) so that \( P^\bullet \) is a complex. It is clear from the block form of the matrices \( \lambda_i \) that their rows
are linearly independent. Since nothing annihilates \( n \) from the left, it is clear that \( P^\bullet \) is exact at \( P^m \) and that none of the \( \lambda_i \) needs an additional row to annihilate \( \eta \) or \( \eta' \). The complex is exact at \( P^1 \) since the product \( \lambda_2 \lambda_1 \) gives the defining relations for \( C \). We will show that \( P^\bullet \) is exact elsewhere by examining the component blocks in the matrices \( \lambda_i \).

The block \( \epsilon \) appears in \( \lambda_1 \) and is annihilated on the left by the \( \delta \) which appears in \( \lambda_2 \). For \( i > 1 \), the column of \( \lambda_i \) containing \( \epsilon \) has no other nonzero entries. Suppose that for some \( d_i \in C \) we have \((d_1 \ d_2 \ d_3 \ d_i)\ 0 = 0 \). Since \( \lambda_2 \lambda_1 \) give a basis for \( I \), it follows that

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & d_1 & d_2 & d_3 & \cdots & 0 \\
\end{pmatrix}
\]

is a sum of left multiples of the rows of \( \lambda_2 \). By the block form of \( \lambda_2 \) this means that \((d_1 \ d_2 \ d_3)\) is a sum of left multiples of the rows of \( \delta \). Therefore the rows of \( \delta \) generate the left annihilator of \( \epsilon \). In the same manner, one sees that the rows of \( \epsilon \) generate the left annihilator of \( z_1 \) and that \( z_i \) generates the left annihilator of \( z_{i+1} \).

While \( \gamma \) does not appear in \( \lambda_1 \), the matrix \((v, w, x_1, y_1)^t\) does, and is annihilated by \( \beta \). Any row annihilating \( \gamma \) must also annihilate \((v, w, x_1, y_1)^t\), and hence the rows of \( \beta \) generate the left annihilator of \( \gamma \). Likewise, since \((x_i, y_i)^t\) appears in \( \lambda_1 \) we see that the rows of \( \gamma \) generate the left annihilator of \( \chi_2 \) and that the rows of \( \chi_i \) generate the left annihilator of \( \chi_{i+1} \). For \( i > 2 \), when \( x_i \) appears as the only nonzero entry in a column of \( \lambda_{m-1-i} \), it can be annihilated by at most the rows of \( \chi_{i-1} \) since \( \chi_{i-1} \) annihilates the \((x_i, y_i)^t\) in \( \lambda_1 \). Since the second row of \( \chi_{i-1} \) does not annihilate \( x_i \), we conclude that \( x_{i-1} \) generates the left annihilator of \( x_i \).

Lemmas 2.4, 2.5 and 2.6 complete the proof that \( P^\bullet \) is exact. Since \( C \) is a graded algebra and the maps \( \lambda_i \) are all of degree at least one, this resolution is minimal.

\[ \square \]

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