On the Blandford–Znajek mechanism of the energy loss of a rotating black hole

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Summary. — The Grad–Shafranov approach to the problem of the structure of the black hole magnetosphere is discussed. For the double transonic flow, the number of boundary conditions in the pair creation region is shown to be sufficient to determine not only the longitudinal electric current, but also the angular velocity of a flow as a solution to a problem. As a result, the energy loss is determined by the physical parameters at the particle creation region rather than the "boundary conditions" at the event horizon.

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1. – Introduction

The Blandford–Znajek [1] process of electromagnetic energy extraction from a rotating black hole is considered as the most appreciable source of activity of the central engine in Active Galactic Nuclei (see e.g. [2]). Indeed, the energy loss of a black hole with a mass $m \sim 10^9 m_\odot$ and radius $r_H = 2Gm/c^2 \sim 10^{14}$ cm embedded into an external poloidal magnetic field $B \sim 10^4$ G and rotating with the angular velocity $\Omega_H$ [3]

$$W_{BZ} = k \frac{\Omega_F(\Omega_H - \Omega_F)}{\Omega_H^2} \left( \frac{a}{m} \right)^2 B^2 r_H^2 c$$

$$\approx 10^{45} \frac{\Omega_F(\Omega_H - \Omega_F)}{\Omega_H^2} \left( \frac{a}{m} \right)^2 \left( \frac{m}{10^9 m_\odot} \right)^2 \left( \frac{B}{10^4 G} \right)^2 \text{erg/s}$$

is large enough to explain the energy of jets and their radiation. Here $\Omega_F < \Omega_H$ is the angular velocity of plasma, $B_{Edd} \sim 10^4$ G is the standard estimate of the poloidal magnetic field near a supermassive black hole, and the factor $k \sim 1$ depends on the geometry of the magnetic field. The same process may play the leading role in some

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galactic sources containing black holes of solar mass and even in the sources of gamma–ray bursts.

In what follows we use the standard expressions for the metric in Boyer–Lindquist coordinates (for $c = G = 1$)

$$ds^2 = -\alpha^2 dt^2 + g_{ik}(dx^i + \beta^i dt)(dx^k + \beta^k dt),$$

where

$$\alpha = \frac{\rho}{\Sigma} \sqrt{\Delta}, \quad \beta^r = \beta^\theta = 0, \quad \beta^\phi = -\omega = -\frac{2amr}{\Sigma^2},$$

$$g_{rr} = \frac{\rho^2}{\Delta}, \quad g_{\theta\theta} = \rho^2, \quad g_{\phi\phi} = \varpi^2.$$  

Here $\alpha$ is the lapse function vanishing at the event horizon

$$r_H = m + \sqrt{m^2 - a^2},$$

$\omega$ is the Lense–Thirring angular velocity, and

$$\Delta = r^2 + a^2 - 2mr, \quad \rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Sigma^2 = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta, \quad \varpi = \frac{\Sigma}{\rho} \sin \theta.$$  

As a result, $a/m = 2\Omega_H r_H$, where $\Omega_H = \omega(r_H)$ is the angular velocity of a rotating black hole. In what follows, all three–dimensional vectors correspond to the 3 + 1 split language [3].

The Blandford–Znajek mechanism of energy extraction is analogous in many respects to the one working in the pulsar magnetosphere [4, 5]. Indeed, in the presence of the longitudinal electric current $I$ producing the toroidal magnetic field

$$B_\phi = -\frac{2I}{\alpha \varpi}$$

and the angular rotation of plasma $\Omega_F$ resulting from the electric field

$$E = \frac{(\Omega_F - \omega)}{2\pi \alpha} \nabla \Psi,$$

the flux of the Poynting vector at large distances

$$W_{em} = \frac{c}{4\pi} \int |E \times B_\phi| dS$$

just corresponds to the value $W_{BZ}$ (1). Here $\Psi(r, \theta)$ is the magnetic flux function determining the poloidal magnetic field

$$B_p = \frac{\nabla \Psi \times e_\phi}{2\pi \varpi}.$$
It is necessary to stress the key point – we propose that in the magnetosphere there is enough plasma to screen the longitudinal electric field. Only in this case can the electric field $E$ be presented in a form (7).

As a result, the energy loss for a magnetically dominated flow can be rewritten in the form

$$W_{\text{BZ}} = \int_0^{\Psi_{\text{max}}} \frac{\Omega_P(\Psi) I(\Psi)}{2\pi} d\Psi.$$  

(10)

In other words, we have $W_{\text{BZ}} = IV_{\text{max}}$, where

$$V_{\text{max}} \approx \Omega_F \Psi_{\text{max}}/c$$

(11)

is the potential drop between the central ($\Psi = 0$) and the marginal ($\Psi = \Psi_{\text{max}}$) magnetic surfaces passing through the horizon. The nonrelativistic relations (8) and (11) (in which we restore the dimension, $c \neq 1$) can be used for the black hole magnetosphere as well because at distances $r \gg r_H$ the General Relativity effects are insignificant.

On the other hand, the effects of General Relativity were actually used in expression (1). The point is that the natural regularity condition at the horizon (the absence of infinite electromagnetic fields in the reference frame comoving with freely falling observer) $E'_\theta \to (E_\theta + B_\phi)/\alpha < \infty$, i.e. $E_\theta + B_\phi \to 0$ can be rewritten in the form of a "boundary condition" [6]

$$4\pi I(\Psi) = [\Omega_H - \Omega_F(\Psi)] \frac{r_H^2 + a^2}{r_H^2 + a^2 \cos^2 \theta} \sin \theta \left( \frac{d\Psi}{d\theta} \right)_{r_H}.$$  

(12)

It is the proportionality $I \propto (\Omega_H - \Omega_F)$ that is responsible for the appropriate factor in (1).

Moreover, to forward the analogy, within the membrane paradigm it is possible to introduce a finite "surface charge density" $\sigma_H$ and "surface currents" $J_H$ [3]

$$\sigma_H = \frac{E_H}{4\pi},$$

(13)

$$4\pi J_H \times n = B_H e_\phi = \alpha B_\phi e_\phi.$$  

(14)

As a result, the energy loss can be formally rewritten in the form

$$W_{\text{tot}} = \int [E_H J_H - \beta_H e_\phi (\sigma_H E_H + J_H \times B_H)]dS,$$

(15)

where $\beta_H = \beta(r_H)$ and $E_H = \alpha E_\theta$. This expression is similar to the energy loss of radio pulsars

$$W_{\text{tot}} = \Omega K.$$  

(16)

Here

$$K = \frac{1}{c} \int [r \times [J_s \times B]]dS$$  

(17)
is the braking torque resulting from the Ampère action of the surface currents \( J_s \) that close the electric currents flowing in the pulsar magnetosphere. Hence, one could imagine that it is the surface currents flowing along the horizon that result in the black hole braking. This physically clear picture was used later in many papers devoted to the central engine in Active Galactic Nuclei.

Nevertheless, this approach met with some problems. First of all, in the last several years some critical papers appeared concerning the efficiency of the BZ process. The point is that, as one can see from (1), the necessary energy loss \( 10^{45} \text{ erg/s} \) can only be achieved for extreme values of the parameters. Actually, all parts of this equation were criticised. First of all, the Eddington magnetic field \[ B_{\text{Edd}} = \left( \frac{8 \pi e^4 m_p}{\sigma_T G m} \right)^{1/2} \sim 10^4 \left( \frac{m}{10^9 m_\odot} \right)^{-1/2} \text{ G} \] (whose energy density is equal to that of the accreting matter giving Eddington luminosity) is actually the upper limit of the magnetic field which can be generated by the accreting plasma in the vicinity of the black hole horizon. Up to now, the possibility of the generation of such a large regular magnetic field has not been confirmed by direct calculations [7]. Further, an extreme rotation \( a/m \approx 1 \) as well as very large masses \( m \sim 10^9 m_\odot \) of a black hole are necessary to reach an energy \( W_{\text{tot}} \sim 10^{45} \text{ erg/s} \). It is not clear whether these parameters can be achieved during the evolution [8, 9]. Finally, the numerical coefficient \( k \) is actually smaller than 1 [10]. For example, for a monopole magnetic field \( \Psi = 2 \pi B_n r_H^2 (1 - \cos \theta) \) (and for \( \Omega_F = \text{const} \)) the exact expression (10) results in

\[
W_{\text{tot}} = \frac{1}{6} \frac{\Omega_F (\Omega_H - \Omega_F)}{\Omega_H^2} \left( \frac{a}{m} \right)^2 B_n^2 r_H^2 c
\]

for \( a \ll m \), and

\[
W_{\text{tot}} = (\pi/4 - 1/2) \frac{\Omega_F (\Omega_H - \Omega_F)}{\Omega_H^2} B_n^2 r_H^2 c
\]

for \( a = m \). On the other hand, for a homogeneous magnetic field \( \Psi = \pi B_n r^2 \sin^2 \theta \) one can obtain

\[
W_{\text{tot}} = \frac{1}{30} \frac{\Omega_F (\Omega_H - \Omega_F)}{\Omega_H^2} \left( \frac{a}{m} \right)^2 B_n^2 r_H^2 c
\]

for \( a \ll m \), and

\[
W_{\text{tot}} = (5/6 - \pi/4) \frac{\Omega_F (\Omega_H - \Omega_F)}{\Omega_H^2} B_n^2 r_H^2 c
\]

for \( a = m \). Here we do not include into consideration the disturbance of the homogeneous magnetic field near a rotating black hole. This disturbance is to diminish the magnetic flux passing through the hole i.e. to reduce the energy loss. Thus, in real objects a black hole may play a passive role only, and the main energy release can be connected with magnetic field lines passing through the accretion disk [11].
Another criticism was connected with the validity of the Blandford–Znajek mechanism itself [12]. The point is that, as was demonstrated, during the derivation of expression (1), the condition (12) was actually used. But, clearly, the horizon is not in a casual connection with the external space and, hence, it cannot affect the flow structure or determine the energy flux flowing away from the rotating black hole. For the same reason, the surface currents cannot play any role in the black hole braking. As a result, the conclusion was drawn that a rotating black hole cannot work as a unipolar inductor.

The goal of our paper is to clarify the ground of the Blandford–Znajek mechanism of the black hole energy loss. In particular, we are going to determine the role of the horizon. In our opinion, a self–consistent analysis can be made on the ground of the Grad–Shafranov approach only. For this reason, in Sec. 2 we consider a simple example of the quasi–monopole particle dominated accretion/ejection flow where the exact solution of the stream equation can be obtained. We will show that the natural boundary conditions at the pair creation region are enough to determine not only the longitudinal electric current $I$, but also the angular velocity of a flow $\Omega_F$ as a solution to a problem. As a result, the energy loss of a rotating black hole is to be determined by the physical parameters at the particle creation region rather than the "boundary conditions" at the event horizon. Next, in Sec. 3 we consider some properties of a magnetically dominated flow when the flow structure is to be close to the force–free one. Finally, in Sec. 4 we discuss the general properties of transonic flows. It is demonstrated that such flows can be realised only if there is no restriction of the longitudinal electric current in the source of plasma.

2. – The particle dominated flow. Exact solution

Let us consider the ideal magnetohydrodynamical cold flow in the vicinity of a slowly rotating black hole. For simplicity we shall consider the case when the energy density of the magnetic field is much higher than the plasma energy density

$$
\varepsilon_1 = \frac{\varepsilon_{\text{part}}}{\varepsilon_B} \ll 1.
$$

It does not mean that it is an electromagnetic energy flux that plays the main role in the black hole braking because for a nonrotating black hole the flux of electromagnetic energy vanishes. In this section we consider the particle dominated case when the energy flux of particles is much larger than the energy flux of the electromagnetic field:

$$
\frac{W_{\text{part}}}{W_{\text{em}}} \gg 1.
$$

The opposite case will be considered in Sec. 3.

In the cold limit there are four critical surfaces – two Alfvénic and two fast magnetosonic ones for ingoing and outgoing flows. Indeed, it is known that plasma can pass through the Alfvénic surface in one direction only – outward for an external Alfvénic surface and inward for an internal one (see e.g. [13]). Hence, plasma is to be created between two Alfvénic surfaces. As we shall see, it is the properties of the pair creation region that fully determine the flow structure including the energy loss of a rotating black hole.

Unfortunately, the efficiency of pair creation in the magnetosphere of a black hole is not determined up to now. In particular, this process depends on the density and
energy of photons in a close vicinity of the black hole. As a result, if the density of hard
gamma–quanta with energies $E_\gamma > 1$ MeV is high enough, the particle creation can be
connected with the direct process $\gamma + \gamma \rightarrow e^+ + e^-$ [14]. On the other hand, if the density
of hard photons is not high, the only possibility to create pairs is connected with the
narrow sheet near the surface where the charge density
\[
\rho_{\text{GJ}} = \frac{1}{8\pi^2} \nabla_k \left( \frac{\Omega_F - \omega}{\alpha} \nabla^k \Psi \right),
\]
which is necessary to screen longitudinal electric field, changes the sign (for more details
see [15, 16]). This surface is similar to the outer gap in the pulsar magnetosphere [17].
In what follows we consider the last mechanism of particle creation.

The stream equation describing magnetic surfaces $\Psi(r, \theta)$ in the vicinity of a rotating
black hole was first formulated in [18]. In the $3 + 1$ split language it can be written down
in the compact form [19]
\[
\frac{1}{\alpha} \nabla_k \left\{ \frac{1}{\alpha \omega^2} \left[ \alpha^2 - (\Omega_F - \omega)^2 \omega^2 - M^2 \nabla^2 \Psi \right] \right\} + \frac{\Omega_F - \omega}{\alpha^2} (\nabla \Psi)^2 \frac{d\Omega_F}{d\Psi}
+ \frac{64\pi^4}{\alpha^2 \omega^2} \frac{1}{2M^2} \frac{\partial}{\partial \Psi} \left( \frac{G}{A} \right) - 16\pi^3 \mu n^2 \frac{1}{\eta} \frac{d\eta}{d\Psi} = 0.
\]
Here
\[
A = \alpha^2 - (\Omega_F - \omega)^2 \omega^2 - M^2
\]
is an Alfvénic factor,
\[
M^2 = \frac{4\pi \eta^2 \mu}{n}
\]
is an Alfvénic Mach number ($n$ is the concentration in the comoving reference frame), and
\[
G = \alpha^2 \omega^2 (E - \Omega_F L)^2 + \alpha^2 M^2 L^2 - M^2 \omega^2 (E - \omega L)^2.
\]
In (26), the covariant operator $\nabla_k$ acts in the three–dimensional metric (3), and the
derivative $\partial/\partial \Psi$ acts on the invariants $E(\Psi)$, $L(\Psi)$, and $\Omega_F(\Psi)$ only.

Indeed, for a cold flow equation (26) contains four invariants. Two of them are the
fluxes of the energy and angular momentum
\[
E = E(\Psi) = \frac{\Omega_F I}{2\pi} + \mu \eta (\alpha \gamma + \omega \omega u_\phi),
\]
\[
L = L(\Psi) = \frac{I}{2\pi} + \mu \eta \omega u_\phi,
\]
where $\mu = (\rho_m + P)/n$ is relativistic specific enthalpy. For the cold flow under consider-
ation, $\mu = \text{const}$. The other two invariants are the angular velocity $\Omega_F(\Psi)$ and the
particle to magnetic flux ratio $\eta(\Psi)$
\[
nu_p = \eta(\Psi) B_p.
\]
As we see, in the general case the energy flux $E(\Psi)$ contains not only the electromagnetic part $\Omega F / 2\pi$, but the particle part $\mu \eta (\alpha \gamma + \omega \mp u_\phi)$ as well. The same takes place for the angular momentum $L(\Psi)$. On the other hand, according to (32), $\eta$ has different signs for ingoing and outgoing flows. Equation (26) is to be added by the Bernoulli one

$$\frac{K}{\omega^2 A^2} = \frac{1}{64\pi^4} \frac{M^4(\nabla \Psi)^2}{\omega^2} + \alpha^2 \eta^2 \mu^2,$$

where

$$K = \alpha^2 \omega^2 (E - \Omega F L)^2 (A^2 - M^2) + M^4 [\omega^2 (E - \omega L)^2 - \alpha^2 L^2].$$

This equation gives the implicit expression for the Mach number as a function of the magnetic flux $\Psi$ and four invariants

$$M^2 = M^2[(\nabla \Psi)^2, E(\Psi), L(\Psi), \Omega_F(\Psi), \eta(\Psi)].$$

As a result, knowing the structure of the poloidal magnetic field and the four invariants, one can determine all the characteristics of a flow. In particular [20],

$$\frac{I}{2\pi} = \frac{\alpha^2 L - (\Omega_F - \omega)\omega^2 (E - \omega L)}{\alpha^2 - (\Omega_F - \omega)^2 \omega^2 - M^2},$$

$$\gamma = \frac{1}{\alpha \mu \eta} \frac{\alpha^2 (E - \Omega_F L) - M^2 (E - \omega L)}{\alpha^2 - (\Omega_F - \omega)^2 \omega^2 - M^2},$$

$$u_\phi = \frac{1}{\omega \mu \eta} \frac{(E - \Omega_F L)(\Omega_F - \omega)\omega^2 - L M^2}{\alpha^2 - (\Omega_F - \omega)^2 \omega^2 - M^2}.$$

The general equation (26) is too complicated to be comprehensively analysed. Nevertheless, the solution can be obtained for a flow which is not too far from the known one. As a zero approximation one can take the (split) monopole magnetic field

$$\Psi = \Psi_0 (1 - \cos \theta),$$

which can be realised in the presence of the accreting disk terminating the ingoing and outgoing magnetic fluxes. It is clear that the monopole magnetic field (39) is an exact solution to equation (26) for a nonrotating black hole (and for $E = \text{const}$, $\eta = \text{const}$, and $\Omega_F = \Omega_H = 0$). As was found earlier [21], for a monopole magnetic field and for $\Omega_F = \Omega_H / 2$ the pair creation region (i.e. the surface where the Goldreich–Julian charge density $\rho_{\text{GJ}}$ (25) changes sign) is a sphere with a radius

$$r_{\text{inj}} = 2^{1/3} r_H.$$

On the other hand, using relations (33)–(34) for $\Omega_F = \Omega_H = 0$

$$\frac{1}{64\pi^4} \frac{M^4(\nabla \Psi)^2}{\omega^2} = E^2 - \alpha^2 \eta^2 \mu^2,$$
one can readily obtain the positions of Alfvénic and fast magnetosonic surfaces for an outgoing \((M^2 = 1)\)

\[
\begin{align*}
      r^{(\text{out})}_A &= r^{(\text{out})}_F = \left( \frac{\Psi_0}{8\pi^2 \sqrt{E^2 - \mu^2 \eta^2}} \right)^{1/2}
\end{align*}
\]

and an ingoing \((M^2 = \alpha^2)\)

\[
\begin{align*}
      \alpha^{(\text{in})}_A &= \alpha^{(\text{in})}_F = \left( \frac{8\pi^2 r_H^2 |E|}{\Psi_0} \right)^{1/2}
\end{align*}
\]

flows. Here the condition \(\gamma_{\text{inj}} \gg 1\) and relation (23) resulting in

\[
\begin{align*}
      r^{(\text{out})}_F &\gg r_H, \\
      \alpha^{(\text{in})}_F &\ll 1
\end{align*}
\]

were included into consideration. As we see, Alfvénic and fast magnetosonic surfaces for a nonrotating black hole have a spherical form and coincide each other.

For slow rotation one can seek the solution of the full equation (26) as

\[
\Psi = \Psi_0 [1 - \cos \theta + \varepsilon^2 f(r, \theta)].
\]

Here

\[
\varepsilon = \frac{a}{m},
\]

\(\Psi_0\) is the total magnetic flux in the upper hemisphere, and \(f \approx 1\). As a result, after the linearization of the general equation (26) we have

\[
\begin{align*}
      \varepsilon^2 a_0^2 \frac{\partial}{\partial r} \left[(\alpha_0^2 - M_0^2) \frac{\partial f}{\partial r}\right] + \frac{\varepsilon^2}{r^2} a_0^2 \sin \theta \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial f}{\partial \theta}\right) \\
      + \frac{\alpha_0^2}{(\alpha_0^2 - M_0^2)^2} \frac{E_0^2}{E_0^2 - \alpha_0^2 \mu^2 \eta^2} \left[(2\alpha_0^2 - M_0^2)(\Omega_F - \omega_A)^2 - M_0^2(\Omega_F - \omega)^2\right] \sin^2 \theta \cos \theta \\
      + 2 \left[ \frac{M_0^4}{(\alpha_0^2 - M_0^2)^2} \frac{E_0^2}{E_0^2 - \alpha_0^2 \mu^2 \eta^2} - 1 \right] (\Omega_F - \omega)^2 \sin^2 \theta \cos \theta \\
      - 2 \operatorname{sign} \eta \frac{\alpha_0^2 M_0^2}{(\alpha_0^2 - M_0^2)^2} \frac{E_0^2}{E_0^2 - \alpha_0^2 \mu^2 \eta^2} (\Omega_F - \omega_A) \sin^2 \theta \cos \theta \\
      - \alpha_0^2 \frac{a^2 r_H}{r^5} \left( \frac{\mu^2 \eta^2}{E_0^2 - \alpha_0^2 \mu^2 \eta^2} + 2 \right) \sin^2 \theta \cos \theta + \frac{\alpha_0^2 a^2}{r^3} M_0^2 \sin^2 \theta \cos \theta = 0.
\end{align*}
\]

Here the values \(a(r), M_0(r), \) and \(E_0\) correspond to a nonrotating flow. Hence, in the limit \(\varepsilon \ll 1\) it is possible to neglect the disturbance of the critical surfaces. Next, in (48) we have already used the expressions for the invariants \(L^{(\text{in})}\) and \(L^{(\text{out})}\) which can
be obtained from the critical conditions on the Alfvénic surfaces. Indeed, analysing the nominators of the equation (36), one can obtain

\[ L^{(\text{out})} = \frac{\Omega^{(\text{out})} - \omega^{A}_{\text{inj}}}{8\pi^2} \frac{E_0}{\sqrt{E_0^2 - \mu^2\eta^2}} \Psi_0 \sin^2 \theta, \]  

(49)

\[ L^{(\text{in})} = -\frac{\Omega^{(\text{in})} - \omega^{A}_{\text{inj}}}{8\pi^2} \Psi_0 \sin^2 \theta. \]  

(50)

It is interesting that within our approximation equation (48) does not depend on the disturbance of the energy \( \delta E = E - E_0 \). Finally, as we shall see, the critical conditions result in \( \Omega_F \approx \text{const} \). For this reason in (48) we omit the terms containing \( d\Omega_F/d\theta \).

As equation (48) contains sign\( \eta \) (to say nothing of the fact that the parameters \( \omega^{A}_{\text{out}} \approx 0 \) and \( \omega^{A}_{\text{in}} \approx \Omega_H \) are different for ingoing and outgoing flows), we actually have two different equations for internal and external regions. In other words, the whole equation (48) has a discontinuity at the pair creation region \( r = r_{\text{inj}} \). Equation (48) contains all the information on the disturbance of the monopole magnetic field.

In spite of the simplification, some properties of equation (48) are general and coincide with the properties of the full equation (26). At first, one can see that the region near the horizon \( r_H < r < r_{\text{inj}} \) corresponds to the hyperbolic domain of equation (26). It means that, indeed, the horizon does not affect the magnetic field structure outside the black hole. On the other hand, it is clear that the number of boundary conditions does not depend on the simplification either. Using the relation [19]

\[ b = 2 + i - s = 4 \]  

(51)

for both ingoing and outgoing flows (\( b \) is the number of boundary conditions, \( i = 4 \) is the number of invariants, and \( s = 2 \) is the number of critical surfaces), one can see that to determine fully all the characteristics of a flow it is necessary to specify eight boundary condition in the pair creation region.

First of all, it is necessary to know the injection values of the particle concentrations and the Lorentz–factors of the flow. For simplicity, we shall further consider the case

\[ n_{\text{inj}}^\pm = n_{\text{inj}} = \text{const}, \]  

(52)

\[ \gamma_{\text{inj}}^\pm = \gamma_{\text{inj}} = \text{const}. \]  

(53)

These four values determine the four invariants

\[ E^{(\text{out})} = \alpha_{\text{inj}} \mu n_{\text{inj}} \gamma_{\text{inj}}, \]  

(54)

\[ E^{(\text{in})} = -\alpha_{\text{inj}} \mu n_{\text{inj}} \gamma_{\text{inj}}, \]  

(55)

\[ \eta^{(\text{out})} = \eta_{\text{inj}} = \frac{\alpha_{\text{inj}} n_{\text{inj}} \sqrt{\gamma_{\text{inj}}^2 - 1}}{B_H}, \]  

(56)

\[ \eta^{(\text{in})} = -\eta_{\text{inj}} = -\frac{\alpha_{\text{inj}} n_{\text{inj}} \sqrt{\gamma_{\text{inj}}^2 - 1}}{B_H}. \]  

(57)

Here \( B_H = \Psi_0/(2\pi r_H^2) \). Bellow, we shall sometimes omit the symbol (out). As we can see, in the particle dominated case (24) the energies \( E^{(\text{in})} \) and \( E^{(\text{out})} \) have different signs.
for ingoing and outgoing flows. It means that in this case it is the pair creation region that plays the role of the energy source. On the other hand, different signs of $\eta$s represent the general property of the accretion/ejection magnetosphere. It is necessary to stress the difference from the model presented in [22], where the velocity of plasma was assumed to vanish in the injection domain.

Further, as a boundary condition one can use the absence of discontinuity of the magnetic flux

$$\Psi(r_{\text{inj}} - 0) = \Psi(r_{\text{inj}} + 0)$$

across the pair creation region $r = r_{\text{inj}}$. Then it is necessary to know two components of the surface electric current $J_s$ flowing along the pair creation region:

$$J(r_{\text{inj}}) = J_s.$$

Finally, it is necessary to know the potential drop $V_{cr}$ along the magnetic field lines in this region

$$V(r_{\text{inj}} + 0) - V(r_{\text{inj}} - 0) = V_{cr}.$$

The boundary conditions (52)–(53) and (58)–(60) are sufficient to determine all the properties of the accretion/ejection flow in the vicinity of a black hole.

As has already been stressed, in the general case the solution of equation (26) cannot be obtained. On the other hand, the linearised equation (48) can be solved analytically. Indeed, separating the variables by the substitution

$$f(r, \theta) = g(r) \sin^2 \theta \cos \theta,$$

one can obtain the following ordinary differential equation for the radial function $g(r)$:

$$\frac{\alpha_0^2}{4} \frac{d}{dr} \left( \frac{\alpha_0^2 - M_0^2}{E_0 - \alpha_0^2 \mu^2 \eta^2} \right) \frac{dg(r)}{dr} - 6 \alpha_0^2 \left( \frac{r_H}{r} \right)^2 g(r)$$

$$+ \frac{1}{4} \frac{\alpha_0^2 E_0^2}{(\alpha_0^2 - M_0^2)^2} \left( \frac{2\alpha_0^2 - M_0^2}{\Omega_H^2} \frac{(\Omega_F - \omega_A)^2}{\Omega_F^2} - M_0^2 \frac{(\Omega_F - \omega)^2}{\Omega_H^2} \right)$$

$$+ \frac{1}{2} \left( \frac{M_0^4}{(\alpha_0^2 - M_0^2)^2} \frac{E_0^2}{E_0 - \alpha_0^2 \mu^2 \eta^2} - 1 \right) \frac{(\Omega_F - \omega)^2}{\Omega_H^2}$$

$$- \frac{1}{2} \frac{\mu^2 \eta^2}{(\alpha_0^2 - M_0^2)^2} \frac{E_0^2}{E_0 - \alpha_0^2 \mu^2 \eta^2} \frac{\Omega_F^2 - \omega_L}{\Omega_F^2 - \omega_F}$$

$$- \frac{1}{4} \alpha_0^2 \left( \frac{r_H}{r} \right)^5 \left( \frac{\mu^2 \eta^2}{E_0 - \alpha_0^2 \mu^2 \eta^2} + 2 \right) + \frac{1}{4} \alpha_0^2 \left( \frac{r_H}{r} \right)^4 M_0^2 = 0.$$

The boundary conditions (58)–(60) for the injection radius $r = r_{\text{inj}}$ now have the form

$$g(r_{\text{inj}} - 0) = g(r_{\text{inj}} + 0),$$

$$\frac{dg}{dr}(r_{\text{inj}} - 0) = \frac{dg}{dr}(r_{\text{inj}} + 0) + \Delta j,$$

$$I(r_{\text{inj}} - 0) = I(r_{\text{inj}} + 0) + \Delta I,$$

$$\Omega_F^{(\text{out})} = \Omega_F^{(\text{in})} + \Delta \Omega_F,$$
where the values $\Delta j$, $\Delta I$, and $\Delta \Omega_F$ are to be determined from the mechanism of the pair creation. In particular, $\Delta \Omega_F$ is related to the potential drop in the pair creation region

$$\Delta \Omega_F \approx \Omega_F \frac{V_{cr}}{V_{max}}.$$  \hspace{1cm} (67)

Finally, these boundary conditions are to be supplemented by the regularity conditions on the fast magnetosonic surfaces $\alpha_{0}^2 = M_{0}^2$.

In spite of the fact that the solution of equation (48) can now be easily obtained, we are not going to discuss it in detail. The point is that the main properties of the flow for $\varepsilon \ll 1$ do not actually depend on the radial function $g(r)$. In other words, in the simple monopole geometry (and for a slow rotation) the boundary conditions (65)–(66) together with the regularity conditions on the Alfvénic surfaces resulting in (49) and (50) are sufficient to determine the energy loss of a rotating black hole separately from the solution of equation (62). For this reason, we describe here the main properties of equation (62) only.

First of all, one can see that equation (62) contains no singularity at the horizon. It is not surprising because it results from the main property of the stream equation (26), namely, hyperbolicity near the horizon. Hence, it is not necessary to add any extra boundary condition for $r = r_H$. On the other hand, it means that the disturbance of the monopole magnetic field remains small up to the very horizon

$$\varepsilon^2 f(r_H, \theta) \sim \varepsilon^2 \ll 1.$$  \hspace{1cm} (68)

Secondly, in this approximation the positions of critical surfaces coincide with ones in zero approximation. Nevertheless, as was demonstrated, to obtain the solution it is necessary to use regularity conditions at Alfvénic and fast magnetosonic surfaces separately. As the Alfvénic singularity has already been used in (49) and (50), second, third, and fourth lines in (62) are regular when $\alpha_{0}^2 = M_{0}^2$. Finally, at a large distance $r \gg r_A$ the solution of equation (48) does not depend on the boundary conditions at all and coincides with the Bogovalov [23] solution in a flat space

$$\varepsilon^2 f(r, \theta) = 2 \left( \frac{\Omega_{F\gamma}}{c} \right)^2 \frac{1}{r^2_{inj}} \ln \left( \frac{r}{r_A} \right) \sin^2 \theta \cos \theta.$$  \hspace{1cm} (69)

A much more important thing is that our approach allows the determination of both the electric current $I$ and the angular velocity $\Omega_F$ and, hence, the energy loss of the rotating black hole. Indeed, using the expressions for angular momentum (49) and (50) together with the boundary conditions (65)–(66), one can obtain

$$\Omega_F = \frac{\omega_{\Lambda}^{(in)} + \omega_{\Lambda}^{(out)} + \Delta \Omega_F - \varepsilon^2 \Delta \Omega_{F}}{2(1 - \varepsilon^2)} + \frac{2\pi (\alpha_{inj}^2 - M_{inj}^2) \Delta I}{\alpha_{inj}^2 \Psi_0 (1 - \varepsilon^2) \sin^2 \theta}.$$  \hspace{1cm} (70)

Here

$$\varepsilon^2 = \frac{8\pi^2 r_{inj}^2 E}{\alpha_{inj}^2 \Psi_0},$$  \hspace{1cm} (71)
so that $\varepsilon_2 \sim \varepsilon_1$. In the case under consideration, when the energy density of a secondary plasma is much smaller than the energy density of the poloidal magnetic field $\varepsilon_2 \ll 1$, we have $\omega_A^{(\text{out})} \ll \Omega_H$ and $\omega_A^{(\text{in})} \approx \Omega_H$, so that

\begin{equation}
\Omega_F = \frac{1}{2} \left[ \Omega_H + \Delta \Omega_F + \frac{4\pi(a_{\text{inj}}^3 - M_{\text{inj}}^2)\Delta I}{a_{\text{inj}}^2\Psi_0 \sin^2 \theta} \right],
\end{equation}

\begin{equation}
I = \frac{\Omega_F}{4\pi} \Psi_0 \sin^2 \theta.
\end{equation}

In particular, for $\Delta I \ll I_{GJ} = \Omega_F \Psi_0/4\pi$ and $\Delta \Omega_F \ll \Omega_F$ we obtain

\begin{equation}
\Omega_F = \Omega_H/2.
\end{equation}

Hence, according to (19),

\begin{equation}
W_{\text{em}} = \frac{1}{24} \left( \frac{a}{m} \right)^2 B_r^2 c.
\end{equation}

Thus, one can conclude that the physical conditions at the pair creation region do allow the determination of not only the electric current flowing in the magnetosphere (this property holds in the flat space, see e.g. [23]), but the angular velocity $\Omega_F$ as well. Clearly, this conclusion is general and does not depend on our approximation.

As we see, our results (73)–(74) formally coincide with those obtained by Blandford and Znajek [1] within the force–free approximation. Nevertheless, they actually correspond to absolutely different tasks. Indeed, Blandford and Znajek found the values of current $I = 2\pi L$ and $\Omega_F$ for which the magnetic field structure for a slowly rotating black hole in the force–free approximation does not differ strongly from the monopole one. But within the force–free approximation, when we have only two critical surfaces (Alfvénic surfaces for ingoing and outgoing flows), in general case the current $I$ and the angular velocity $\Omega_F$ can be arbitrary. On the other hand, we have demonstrated that within the full MHD approach two extra critical (fast magnetosonic) surfaces fix the current $I$ and angular velocity $\Omega_F$ values for a double transonic flow.

3. – The magnetically dominated flow

In this section we consider some properties of the double transonic flows when the flux of the electromagnetic energy is much larger than the energy flux of plasma

\begin{equation}
\frac{W_{\text{part}}}{W_{\text{em}}} \ll 1.
\end{equation}

As is well–known, this relation can be rewritten in the form $\sigma \gg 1$, where

\begin{equation}
\sigma = \frac{\Omega e\Psi_0}{8\pi \lambda mc^3}
\end{equation}

is the Michel [24] magnetization parameter and $\lambda = n/n_{GJ} \gg 1$. For simplicity, we again consider the slowly rotating black hole $\Omega_H r_H \ll 1$. In this case the magnetospheric structure is to be close to the one obtained in the force–free approximation. Nevertheless,
there are two important differences. First of all, as previously, for any small but finite
mass of particles the region in the vicinity of the horizon corresponds to the hyperbolic
region of the general equation (26). On the other hand, the force–free equation remains
elliptical to the very horizon. Secondly, in the full MHD approach there are two additional
critical surfaces – fast magnetosonic ones which are absent within the force–free approach.

Unfortunately, no exact analytical solution can be obtained for magnetically dom-
inated accretion/ejection in the vicinity of a rotating black hole. Nevertheless, it is
possible to evaluate the main properties of the transonic flow by analysing the algebraic
relations only. This approach has already been used in many papers (see e.g. [25]). But
in almost all of them the magnetic field structure has been considered as given. It does
not allow a self–consistent analysis of the flow structure. As has recently been demon-
strated in [26], some key properties, e.g. the position of a fast magnetosonic surface,
cannot be determined correctly in a given (monopole) magnetic field. For this reason we
now consider a more general case including the disturbance of the monopole magnetic
field.

So, let us consider again a flow with a monopole magnetic field in the pair creation
region. Then we can rewrite the Bernoulli equation (33) in the form

\begin{equation}
\frac{g^3}{2} - \frac{1}{2} \left[ \xi + 2 \frac{\alpha^2}{(\Omega F - \omega)^2 \omega^2} - \frac{\alpha^2 L^2}{E^2 \omega^2} \right] g^2 \\
+ \frac{1}{2} \alpha^2 \left( \frac{\mu^2 \eta^2}{E^2} \right) + \frac{1}{2} \frac{\alpha^2}{(\Omega F - \omega)^2 \omega^2} \left( \frac{E - \Omega F L}{E} \right)^2 = 0,
\end{equation}

where we omit the term \( g^4 \) resulting in the unphysical root \( g < 0 \). Here, by definition

\begin{equation}
g = \frac{M^2}{(\Omega F - \omega)^2 \omega^2},
\end{equation}

and

\begin{equation}
\xi = \frac{(E - \omega L)^2}{E^2} - \frac{(\Omega F - \omega)^4 \omega^2 (\nabla \Psi)^2}{64 \pi^4 E^2}.
\end{equation}

In particular, far from a black hole [26] we have

\begin{equation}
\xi(r, \theta) \approx - \frac{2\pi^2}{\sin \theta} \frac{\partial f}{\partial \theta} + 4\pi^2 \frac{\cos \theta}{\sin^2 \theta} f,
\end{equation}

so that \( \xi = 0 \) for a monopole magnetic field.

First of all, for an outgoing flow it is possible to use our results in a flat space [26].
Indeed, a fast magnetosonic surface corresponds to the intersection of two roots of equa-
tion (78) at one point. On the other hand, equation (78) has two real positive roots if
\( Q \leq 0 \), where \( Q \) is the discriminant of the third–order algebraic equation (78). Near an
external fast magnetosonic surface \( r \approx r_F \), where the last term in (78) can be neglected,
we have

\begin{equation}
Q = \frac{1}{16} \frac{\mu^4 \eta^4}{E^4} - \frac{1}{432} \frac{\mu^2 \eta^2}{E^2} \left( \xi + \frac{1}{\Omega F^2 r^2 \sin^2 \theta} \right)^3,
\end{equation}
the regularity conditions at the fast magnetosonic surface \( r = r_F \) being

\[
Q = 0, \quad \frac{\partial Q}{\partial r} = 0, \quad \frac{\partial Q}{\partial \theta} = 0.
\]

(83)

As a result, we can rewrite the condition \( Q = 0 \) near the external fast magnetosonic surface as

\[
\xi(r_F, \theta) + \frac{1}{\Omega_F r_F^2 \sin^2 \theta} = 3 \left( \frac{\mu \eta}{E} \right)^{2/3},
\]

(84)

and the condition \( \partial Q / \partial r = 0 \) as

\[
r_F \left( \frac{\partial \xi}{\partial r} \right)_{r_F} - \frac{2}{\Omega_F^2 r_F^2 \sin^2 \theta} = 0.
\]

(85)

As previously, the third condition \( \partial Q / \partial \theta = 0 \) is necessary for the determination of the magnetic disturbance. Using now the estimate \( r(\partial \xi / \partial r) \approx \xi \), one can obtain the position of the fast magnetosonic surface. For \( \gamma_{in} \ll \sigma^{1/3} \)

\[
r_F(\theta) \approx R_L \sigma^{1/3} \sin^{-1/3} \theta,
\]

(86)

when \( \theta > \sigma^{-1/2} \), and

\[
r_F \approx R_L (\sigma / \gamma_{in})^{1/2},
\]

(87)

near the axis. Here \( R_L = c / \Omega_F \) is the radius of the light cylinder.

Further, as the root \( g(r_F) \) for \( r = r_F \) does not depend on the second term in (78), we have exactly

\[
g(r_F, \theta) = \left( \frac{\mu \eta}{E} \right)^{2/3}.
\]

(88)

Hence,

\[
\gamma(r_F, \theta) = \left( \frac{E}{\mu \eta} \right)^{1/3} = \sigma^{1/3} \sin^{2/3} \theta,
\]

(89)

which corresponds to Michel’s [24] result. The only difference is that this energy is achieved at a finite distance \( r_F \) (86). As we see from (85), it takes place because we included the dependence of \( \xi \) on the field disturbance \( \varepsilon f \) into consideration. Indeed, according to (84) and (85), it is the disturbance of the magnetic surfaces \( \varepsilon f \) that plays the main role in (85) at the fast magnetosonic surface. Finally, according to (81), the disturbance \( \varepsilon^2 f \) itself

\[
\varepsilon^2 f(r_F) \approx \sigma^{-2/3}
\]

(90)

is to be small at the fast magnetosonic surface. Moreover, as was found in [22, 26], outside the fast magnetosonic surface \( r \gg r_F \), the particle acceleration and the magnetic
field collimation become ineffective. As a result, the disturbance of magnetic surfaces remains small up to infinity

\[ \varepsilon^2 f(r, \theta) \approx \sigma^{-2/3} \ll 1. \]  

Clearly, these results remain true for our problem as well.

As to an ingoing flow, one can check that, according to (12), \( \xi(rH, \theta) = 0 \). Hence, \( \xi \ll 1 \) for \( \sigma \gg 1 \) for an ingoing flow as well. As a result, using the same procedure as for an outgoing flow, where the discriminant \( Q \) now has a form

\[ Q = \left[ \frac{\alpha^2}{4(\Omega_F - \omega)^2 \omega^2} \frac{e}{E} \right]^2 - \frac{\alpha^2}{432(\Omega_F - \omega)^2 \omega^2} \frac{e^2}{E^2} \left[ \xi + \frac{2\alpha^2}{(\Omega_F - \omega)^2 \omega^2} - \frac{\alpha^2 L^2 \gamma}{E^2 \omega^2} \right]^3, \]

one can obtain for \( \theta \neq 0 \)

\[ g(r_F) \approx \frac{e}{E}, \]
\[ \alpha^2(r_F) \approx (\Omega_H - \Omega_F)^2 \omega^2 H e, \]
\[ \gamma(r_F) = \frac{\gamma_{\text{inj}}}{\alpha(r_F)}, \]
\[ \varepsilon^2 f(r_F) \approx \sigma^{-2/3}. \]

Here we introduce by definition \( e = E - \Omega_F L \). Together with (30)–(31) one can find that

\[ \frac{e}{E} \approx \frac{\gamma_{\text{inj}}}{\sigma}. \]

Thus, \( e/E \ll 1 \) for a magnetically dominated flow.

First of all, we see that an internal fast magnetosonic surface for \( \theta \neq 0 \) is located much closer to the horizon than an Alfvénic one [25]

\[ \alpha^2(r_F) = \alpha^2(r_A) \frac{\gamma_{\text{inj}}}{\sigma}, \]

so that \( \alpha^2(r_F) \ll \alpha^2(r_A) \). Here

\[ \alpha^2(r_A) \approx (\Omega_H - \Omega_F)^2 \omega^2 H \]

corresponds to the position of the internal Alfvénic surface. Then, the Lorentz–factor at an internal fast magnetosonic surface differs only by the factor \( \alpha(r_F) \) from its value in the injection region. But this fact has the coordinate reason only, and the formal increase of \( \gamma \) results from the difference in the ZAMO’s positions in the regions of injection and fast magnetosonic surface. Thus, there is no intrinsic acceleration of ingoing particles, at any rate within the fast magnetosonic surface. Finally, according to (96), the disturbance of the monopole magnetic field remains small at the fast magnetosonic surface. Hence, as for the particle dominated flow, the disturbance of the monopole magnetic field remains small up to the very horizon

\[ \varepsilon^2 f(r_H, \theta) \ll 1. \]
But the most important property, in our opinion, is that the values of the invariants \( L^{(\text{in})} \), \( L^{(\text{out})} \), \( \Omega^{(\text{in})}_F \), and \( \Omega^{(\text{out})}_F \)

\begin{equation}
L^{(\text{in})} \approx L^{(\text{out})} \approx \frac{\Omega_F}{8\pi^2} \Psi_0 \sin^2 \theta, \tag{101}
\end{equation}

\begin{equation}
\Omega^{(\text{in})}_F \approx \Omega^{(\text{out})}_F \approx \frac{\Omega_H}{2} \tag{102}
\end{equation}

(and, hence, the longitudinal electric current flowing in the magnetosphere) for a magnetically dominated MHD flow are close to that considered by Blandford & Znajek for the force–free magnetosphere. It means that the double transonic flow is close to the force–free one. The difference occurs outside the fast magnetosonic surface only. But even in this region the disturbance of the monopole magnetic field remains small.

4. – Discussion

We have demonstrated that for a double transonic flow neither the longitudinal electric current \( I \), nor the angular velocity \( \Omega_F \) is a free parameter, but is determined from the solution. Clearly, this result is general and does not depend on our approximation. On the other hand, exact values of the critical electric current \( I \) and the angular velocity \( \Omega_F \) corresponding to the double transonic flow certainly depend on the geometry of the magnetic field. Nevertheless, relations

\begin{equation}
\Omega_F \sim \frac{\Omega_H}{2}, \quad I \sim I_{\text{GJ}}, \tag{103}
\end{equation}

remain true for an arbitrary source of the external magnetic field.

Unfortunately, for an arbitrary external magnetic field analytical calculations are impossible because, in particular, the position of the critical surfaces are unknown and they themselves are to be determined from the solution. Moreover, it is impossible to say that Alfvénic singularities determine the values of the angular momentum \( L \) and fast magnetosonic surfaces – the structure of the poloidal magnetic field. This was luckily realised for the monopole magnetic field only. In the general case, it is only all the critical surfaces taken together that determine invariants and the magnetic field structure.

It is necessary to clarify the role of the "boundary condition" (12) which is formally used to determine the energy loss (1). This relation is true for any solution of the Grad–Shafranov equation which can be extended up to horizon [19]. The point is that relation (12) can be obtained by direct integration of the stream equation (26) which becomes parabolic at the horizon. Indeed, in the limit \( \alpha \to 0 \) we have

\begin{equation}
\frac{1}{\alpha} \nabla \theta \left[ \frac{1}{\varpi^2 \alpha} A \nabla^\theta \Psi \right] + \frac{\Omega_F - \Omega_H}{\alpha^2} (\nabla \Psi)^2 \frac{d \Omega_F}{d \Psi} - \frac{32\pi^4}{\alpha^2} \frac{\theta}{\partial \Psi} \left[ \frac{(E - \Omega_H L)^2}{A} \right] = 0. \tag{104}
\end{equation}

As a result, multiplying (104) by \( 2A/\left( d \Psi / d \theta \right) \), one can obtain

\begin{equation}
2A \varpi^2 (\Omega_F - \Omega_H) \frac{d \Omega_F}{d \Psi} \left[ \frac{1}{\varpi^2 \rho^2} \left( \frac{d \Psi}{d \theta} \right)^2 - \frac{64\pi^4 (E - \Omega_H L)^2}{A^2} \right] + \frac{d}{d \theta} \left[ \frac{A^2}{\varpi^2 \rho^2} \left( \frac{d \Psi}{d \theta} \right)^2 - 64\pi^4 (E - \Omega_H L)^2 \right] = 0, \tag{105}
\end{equation}

\[ \]
resulting in (12).

Hence, in reality the "boundary condition" (12) contains no additional information. The point is that we do not know the magnetic flux $\Psi(r_h, \theta)$ at the horizon (it is to be found as a solution of a problem), so that in general case relation (12) gives us no connection between the current $I$ and the angular velocity $\Omega_F$. On the other hand, as was demonstrated, the necessary connection between $I$ and $\Omega_F$ results from the extra critical conditions on singular surfaces which are in casual contact with the pair creation region. The critical conditions give the same value of the longitudinal electric current because for a transonic inflow the disturbance of the monopole magnetic field at the horizon both for particle (68) and magnetically dominated (100) flows remains small.

Finally, it is necessary to stress that our results depend sufficiently on the proposal that there is no additional restriction of the longitudinal electric current in the source of particles. In this case it is natural to assume the flow to be transonic and the current to be determined from the critical condition at the singular surfaces. On the other hand, if the electric current $I$ is determined by the particle creation process, the flow structure can be far from the transonic solution.

This property is well-known in the flat space considered in connection with the pulsar magnetosphere. Indeed, if the electric current $I$ is much smaller than the Goldreich–Julian one

$$I_{GJ} = \frac{\Omega_F}{4\pi}\Psi_0\sin^2\theta,$$

(106)

the force–free solution can be extended only up to the light surface $|E| = |B|$ located in the vicinity of the light cylinder $R_L = c/\Omega$ [27]. A shock front must exist here resulting in a very effective particle acceleration. Simultaneously, the longitudinal electric currents $I$ flowing in the magnetosphere are closed in this region (in more details see [5]). As a result, outside the shock almost all the energy is transported by particles.

On the other hand, if the electric current is larger than $I_{GJ}$, there is a strong collimation of the magnetic surfaces along the rotational axis [28], the flow remaining subsonic. In this sense the Michel [29] force–free monopole solution with $I = I_{GJ}$ terminates the flows with different asymptotics. Moreover, as has been demonstrated, a magnetically dominated MHD flow ($\sigma \gg 1$) can pass through the fast magnetosonic surface only if the current $I$ is close to the critical one. It is therefore not surprising that the transonic magnetically dominated inflow differs only slightly from the force–free Blandford–Znajek solution. In all other cases the properties of the magnetically dominated flow will be similar to the force–free one.

Thus, one can conclude that within the MHD approach for a given current $I \neq I_{GJ}$ there is an infinite number of solutions. Indeed, adding one extra boundary condition on the pair creation surface – the value of the electric current $I$ – we simultaneously lose two critical conditions at the internal and external fast magnetosonic surfaces. In this case, the magnetically dominated flow will be similar to the force–free solution. In particular, for an ingoing flow it means that for small enough longitudinal electric currents the light surface $|E| = |B|$ does not coincide with the horizon. Hence, in this case it is impossible to extend the solution up to horizon even within the MHD approach.

In the table we classify different possibilities which can be realised if the longitudinal electric current is determined by the pair creation mechanism. For simplicity, we consider the case $\Delta I = 0$, $\Delta \Omega_F = 0$, and the monopole geometry only. As we see, flows with small electric currents $I$ do contain shocks in the vicinity of the light surface. On the
Table I. – Different possibilities of the flow with electric current fixed by the pair creation region

| $I < I_{GJ}$ | $I = I_{GJ}$ | $I > I_{GJ}$ |
|--------------|--------------|--------------|
| $\Omega_F < \Omega_H/2$ | $\Omega_F < \Omega_H/2$ | $\Omega_F < \Omega_H/2$ |
| outflow with shock | transonic outflow | subsonic outflow |
| inflow with shock | inflow with shock | arbitrary inflow |
| $\Omega_F = \Omega_H/2$ | $\Omega_F = \Omega_H/2$ | $\Omega_F = \Omega_H/2$ |
| outflow with shock | double transonic flow | subsonic outflow |
| inflow with shock | | subsonic inflow |
| $\Omega_F > \Omega_H/2$ | $\Omega_F > \Omega_H/2$ | $\Omega_F > \Omega_H/2$ |
| outflow with shock | transonic outflow | subsonic outflow |
| arbitrary inflow | subsonic inflow | subsonic inflow |

other hand, flows with large enough currents are to be subsonic. Even for the current $I$ coinciding with the critical one, the structure of the flow may be arbitrary.

It is necessary to stress that there are well–known arguments against stability of subsonic flows. Moreover, as has already been pointed out, the subsonic flow cannot pass the event horizon. Hence, either the subsonic inflow does not cross the horizon at all so that the magnetic field lines bend to the equatorial region [30], or this possibility corresponds to an unphysical solution with zero velocity and infinite particle density at the horizon [13]. It is clear that only the solution of the stream equation (26) could determine the real structure of the magnetic field in this case. Unfortunately, up to now such solutions have not been constructed. On the other hand, solutions with shock (i.e., those containing light surfaces $|E| = |B|$) are real and can be realised. Then, outside the light surface the Grad–Shafranov approach itself becomes invalid. These solutions are interesting in connection with the possibility of an effective particle acceleration near the light surface (see e.g. [30, 31]).

Incidentally, the fact that there is a restriction of the magnetic field structure in the vicinity of the horizon has already been discussed by Hirotani at al [25] for a magnetically dominated flow. In their interpretation, the magnetic field cannot have an arbitrary structure (in particular, homogeneous) near the horizon of a black hole. In our opinion, their results actually confirm our point of view – in the presence of the light surface $|E| = |B|$ it is impossible to prolong the MHD solution to the very horizon. But it does not mean that there is a restriction of the magnetic field structure near the horizon.

As to the restriction found in [25], it can be obtained by an even simpler way. As is well–known, on the horizon the electric field is equal to the magnetic one for any geometry of the magnetic field: it results from the membrane paradigm. On the other hand, one can extend the solution up to horizon only if the electric field is lower than the magnetic one. Hence, one can write down this condition as

$$
\frac{d}{dr}(B^2 - E^2)_{rh} > 0.
$$

Using now the definitions (6) and (7), one can rewrite (107) in the form

$$
-2 \frac{\cos \theta}{\sin \theta} P - 4 \alpha^2 \frac{\sin \theta \cos \theta}{\rho^2_H} P + 2 \frac{\partial P}{\partial \theta} - 2 \left(1 - \frac{m}{\rho_H}ight) P^2
$$
\[ + \frac{r_H}{(\Omega_H - \Omega_F)^2 \omega_H^2} \left( \frac{\partial \alpha^2}{\partial r} \right)_{r_H} - 2 \frac{r_H}{\Omega_H - \Omega_F} \left( \frac{\partial \omega}{\partial r} \right)_{r_H} - 2 \frac{r_H}{\omega_H} \left( \frac{\partial \omega}{\partial r} \right)_{r_H} + 2 \frac{r_H^2}{\rho_H^2} > 0, \]

where the parameter
\[ P = -r_H \frac{(\partial \psi / \partial r)_{r_H}}{(\partial \psi / \partial \theta)_{r_H}} \]

depends on the magnetic field structure near horizon. One can easily check that the condition (108) coincides identically with the one obtained by Hirotani et al [25].

5. – Conclusions

Thus, it was demonstrated that:

1. It is impossible to consider relation (12) as a boundary condition. This relation is true for any solution of the Grad–Shafranov equation which can be extended up to horizon. In reality, we do not know the magnetic flux \( \Psi(r_H, \theta) \) at the horizon (it is to be found as a solution of a problem), so that in the general case relation (12) gives us no connection between the current \( I \) and the angular velocity \( \Omega_F \).

2. For the finite mass of particles in the very vicinity of the horizon there is a hyperbolic region of the stream equation which is absent altogether in the force–free approximation. Hence, the stream equation needs no boundary condition at the horizon.

3. It is impossible to say that it is the surface electric current that results in the braking of the rotating black hole. The surface electric currents \( J_H \) (14) as well as the surface charges \( \sigma_H \) (13) are unphysical and cannot act on the surface of a black hole. They only give a convenient way to describe the flux of the negative energy falling onto the black hole horizon.

Hence, the Blandford–Znajek mechanism of the electromagnetic energy extraction from the rotating black hole faces no problem connecting with the causality disconnection between the event horizon and the outer magnetosphere. In particular, for a double transonic flow the values of the longitudinal electric current \( I \) and the angular velocity \( \Omega_F \) (and, hence, the energy loss \( W_{\text{tot}} \)) are to be determined by the physical parameters in the pair creation region and by the critical conditions at the singular surfaces which are in casual contact to each other.

On the other hand, if the longitudinal electric current is determined by the pair creation region, then there is an infinite set of solutions which do not pass one or both fast magnetosonic surfaces. In particular, for small enough longitudinal electric currents the effective particle acceleration can take place in the vicinity of the light surfaces. In this case the ideal MHD solution cannot be extended to the horizon.

As in the pulsar magnetosphere, if the secondary plasma is enough to screen the longitudinal electric field, its charge density and electric currents produce the flux of the electromagnetic energy propagating from the central star to infinity. For the same reason, a rotating black hole embedded into an external magnetic field works as a unipolar inductor extracting its energy of rotation by the flux of the electromagnetic energy.

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