Growth of matrix products and mixing properties of the horocycle flow

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1 Introduction

In this paper we investigate the following problem. Let $H(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and let $\Phi_\ast = \{\Phi_n\}$ be an arbitrary sequence of matrices from $SL(2, \mathbb{R})$. We will consider the sequence of products $P_n(t) = \Phi_n H(t)\Phi_{n-1} H(t) \ldots \Phi_1 H(t)$ and denote by $\mathcal{B}(\Phi_\ast)$ the set of those periods $t \in \mathbb{R}_+$ for which the sequence $\{P_n(t)\}$ is bounded:

$$\mathcal{B}(\Phi_\ast) = \left\{ t \in \mathbb{R}_+ : \sup_{n \geq 1} ||P_n(t)|| < \infty \right\}.$$

The question is: how large the set $\mathcal{B}(\Phi_\ast)$ can be? We present three results on this subject. The first one shows that for every $\{\Phi_n\}$ the set $\mathcal{B}(\Phi_\ast)$ is not “very large”:

**Theorem 1.** For every sequence $\Phi_\ast$, the set $\mathcal{B}(\Phi_\ast)$ has finite measure.

It should be noted that for sequences $\Phi_\ast$ of some special types this was already established in [1]. Our main innovation, which gives us the possibility to handle the general case, is using of potential theory (Lemma 5).

The next two results demonstrate that the conclusion of Theorem 1 cannot be strengthened too much. Namely, Theorem 2 (section 5) shows that the exceptional set $\mathcal{B}(\Phi_\ast)$ can contain an arbitrary given sequence. In Theorem 3 (section 6) we produce an example of a sequence $\Phi_\ast$ for which the set $\mathcal{B}(\Phi_\ast)$ is essentially unbounded, that is $|\mathcal{B}(\Phi_\ast) \cap [a, +\infty)| > 0$ for all $a > 0$ (We denote by $|E|$ the Lebesgue measure of a set $E \subset \mathbb{R}$).

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Motivation: stable quasi-mixing of the horocycle flow. In a recent work of L. Polterovich and Z. Rudnick \cite{1} the authors considered the behavior of one-parameter subgroup of a Lie group under the influence of a sequence of kicks. We remind some basic concepts of their paper.

Let a Lie group $G$ act on a set $X$, and $(h^t)_{t \in \mathbb{R}}$ be a one-parameter subgroup of $G$; we consider it as a dynamical system acting on $X$ with continuous time $t$. We perturb this system by a sequence of kicks \{\phi_i\} $\subset G$. The kicks arrive periodically in time with some positive period $t$. The dynamics of the kicked system is described by a sequence of products $P_i(t) = \phi_i h^t \phi_{i-1} h^t \ldots \phi_1 h^t$ that depend on the period $t$. We treat $t$ as a parameter and $i$ as a discrete time. Then the trajectory of a point $x \in X$ is defined as $x_i = P_i(t)x$.

A dynamical property of a subgroup $(h^t)$ is called kick stable, if for every sequence of kicks \{\phi_i\}, the kicked sequence $P_i(t)$ inherits this property for a “large” set of periods $t$. The property we will concentrate on in this paper, is quasi-mixing.

A sequence \{P_i\} acting on a compact measure space $(X, \mu)$ by measure-preserving automorphisms is called mixing if for any two $L_2$-functions $F_1$ and $F_2$ on $X$
\[
\int_X F_1(P_i x) F_2(x) d\mu \to \int_X F_1(x) d\mu \int_X F_2(x) d\mu
\]
when $i \to \infty$. A sequence \{P_i\} is called quasi-mixing if there exists a subsequence \{i_k\} $\to \infty$ such that for any two $L_2$-functions $F_1$ and $F_2$ on $X$
\[
\int_X F_1(P_{i_k} x) F_2(x) d\mu \to \int_X F_1(x) d\mu \int_X F_2(x) d\mu
\]
when $k \to \infty$.

In what follows, $G = PSL(2, \mathbb{R})$, $\Gamma \subset PSL(2, \mathbb{R})$ is a lattice, that is a discrete subgroup such that the Haar measure of the quotient space $X = PSL(2, \mathbb{R})/\Gamma$ is finite. The group $PSL(2, \mathbb{R})$ acts on $X$ by left multiplication. This action evidently preserves the Haar measure. The principal tool used in \cite{1} for the study of stable mixing in this setting, is the Howe-Moore theorem which gives the geometric description of mixing systems: if the sequence \{P_i\} tends to infinity then it is mixing. It was also shown that the converse is true. In a similar way, the quasi-mixing is equivalent to the unboundedness of the sequence \{P_i\}.

\footnote{i.e., for every compact subset $Q \subset G$ the sequence \{P_i\} eventually leaves $G$}
It follows from the Howe-Moore theorem that the horocycle flow
\[ H(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \]
on $PSL(2,\mathbb{R})/\Gamma$ is mixing. An example given in [1, Remark 3.3.E] shows that this flow is not stably mixing. Our Theorem 1 says that it is stably quasi-mixing. This answers the question raised by Polterovich and Rudnick [1, Question 3.3.B].

Let us mention a corollary to Theorem 1 that pertains to second order difference equations. It was shown in [1] that for a kick sequence of the form \( \begin{pmatrix} 1 & 0 \\ c_n & 1 \end{pmatrix} \), the unboundedness of the evolution is equivalent to the existence of unbounded solutions for the discrete Schrödinger-type equation
\[ q_{k+1} - (2 + tc_k)q_k + q_{k-1} = 0, \quad k \geq 1. \tag{1} \]

So our result implies

**Corollary 1.** For every sequence \( \{c_n\} \), the set of the parameters \( t \in \mathbb{R}_+ \) for which all solutions of the difference equation (1) are bounded, has finite measure.

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## 2 Outline of the proof of Theorem 1

Our proof of Theorem 1 consists of several steps and uses some preliminary results (Lemmas 1-5 below). For convenience of a reader we begin with an outline of this proof.

**Step 1.** First of all we show, that the problem can be reduced to the case of bounded sequences of kicks $\Phi_*$ (Lemma 1).

**Step 2.** For bounded sequences we use the Iwasawa’s decomposition of $2 \times 2$ matrices:
\[ \Phi_n = \begin{pmatrix} 1 & s_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_n & 0 \\ 0 & \frac{1}{\lambda_n} \end{pmatrix} \begin{pmatrix} \cos \alpha_n & -\sin \alpha_n \\ \sin \alpha_n & \cos \alpha_n \end{pmatrix} := H(s_n)D(\lambda_n)R(\alpha_n). \]
with bounded sequences \( \{H(s_n)\} \) and \( \{D(\lambda_n)\} \) and \(-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}\). Denoting by
\[
q = \limsup_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} |\alpha_j|
\]
we then consider two cases separately: \( q = 0 \) and \( q > 0 \).

**Step 3.** In the case of “small” angles (\( q = 0 \)), the sequence \( \{\Phi_n\} \) is “close” (in some sense which we define below) to the bounded sequence \( \{H(s_n)D(\lambda_n)\} \) of upper-triangular matrices. This implies that the set \( \mathfrak{B}(\Phi_n) \) is bounded (see Lemma 3).

**Step 4.** In the case \( q > 0 \) we extend our problem to the complex plane and consider \( SL(2, \mathbb{C}) \)-matrices \( H(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \). Respectively, \( P_n(z) = \Phi_n H(z) \cdot \ldots \cdot \Phi_1 H(z) \). We show that the set
\[
E = \{ z \in \mathbb{C} : \limsup_{n \to \infty} \frac{\log \|P_n(z)\|}{n} = 0 \}
\]
is contained in \( \mathbb{R} \) and has finite length. So we not only prove that the sequence \( \{P_n\} \) is unbounded but prove that it has exponential growth for all \( t \) apart of a set of finite measure.

In order to show that
\[
\limsup_{n \to \infty} \frac{\log \|P_n(z)\|}{n} > 0 \quad z \in \mathbb{C} \setminus \mathbb{R}, 
\tag{2}
\]
we have to estimate \( \|P_n(z)\| \) from below. To this aim we use the quadratic form
\[
Q(x) = \text{Im} \left( x_1 \bar{x}_2 \right), \quad x = (x_1, x_2) \in \mathbb{C}^2,
\]
which has the following properties:

(i) for arbitrary \( y \), \( \|y\|^2 \geq 2Q(y) \),

(ii) for every \( z \in \mathbb{C} \) with \( \text{Im} z > 0 \), one has
\[
Q( H(z) \Phi_n H(z) x ) \geq Q(x) \left( 1 + \frac{|\alpha_n| |\text{Im} z|}{2k(1 + |z|)} \right). \tag{3}
\]

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Due to these properties we get that
\[
\lim_{n \to \infty} \frac{\log \| P_n(z) \|}{n} = \lim_{n \to \infty} \frac{1}{n} \log \left\| \prod_{1 \leq j \leq n} H(z/2) \Phi_j H(z/2) \right\| > 0.
\]

(We denote by \( \prod_{1 \leq j \leq n} A_j \) the matrix product \( A_n A_{n-1} \ldots A_1 \).)

The claim that \( |E| < \infty \) follows now from a potential theory lemma (Lemma 5) applied to the subharmonic functions
\[
u_n(z) = \frac{\log \| P_n(z) \|}{n}.
\]

3 Preliminaries

Lemma 1. If the sequence of kicks \( \Phi_* \) is unbounded then the set \( \mathcal{B}(\Phi_*) \) is empty.

Proof. Aiming at a contradiction, we assume that for some \( t > 0 \) the sequence \( \{ \| P_n(t) \| \} \) is bounded by \( M \). Taking into account that \( \| A^{-1} \| = \| A \| \) for \( A \in SL(2, \mathbb{R}) \), we obtain
\[
\| \Phi_n \| = \| P_n(t) (P_{n-1}(t))^{-1} (H(t))^{-1} \|
\leq \| P_n(t) \| \cdot \| (P_{n-1}(t))^{-1} \| \cdot \| (H(t))^{-1} \| \leq M^2 \| H(t) \|
\]
which contradicts to the unboundedness of \( \{ \Phi_n \} \).\]

Thus, in what follows, we assume that the sequence \( \{ \Phi_n \} \) is bounded.

Lemma 2. Let \( \Psi_* \) be a bounded sequence of upper-triangular matrices:
\[
\Psi_n = \begin{pmatrix} \lambda_n & s_n \\ 0 & \frac{1}{\lambda_n} \end{pmatrix}
\]
and
\[
t_0 = \max\{|s_n/\lambda_n|, n \in \mathbb{N}\}.
\]
Then, for all \( t > t_0 \) and all \( K > 0 \), there exists \( N \in \mathbb{N} \) such that, for each \( j \in \mathbb{N} \), at least one of \( N \) products
\[
\Psi_{j+m} H(t) \cdot \ldots \cdot \Psi_{j+1} H(t), \quad m = 1, 2, \ldots, N
\]
has norm larger than \( K \).
Proof. Let us fix \( t > t_0 \). It follows by induction in \( m \) that

\[
\Psi_{j+m}H(t) \cdot \ldots \cdot \Psi_{j+1}H(t) = \left( \frac{\Pi_{j,m} t \Pi_j H(t)}{\prod_{j,m} S_{j,m}(t)} \right)
\]

where

\[
\Pi_{j,m} = \lambda_{j+1} \cdot \ldots \cdot \lambda_{j+m}, \quad (7)
\]

\[
S_{j,m}(t) = \left( t + \frac{s_{j+1}}{\lambda_{j+1}} \right) + \frac{t + \frac{s_{j+2}}{\lambda_{j+2}}}{\lambda_{j+1}^2} + \ldots + \frac{t + \frac{s_{j+m}}{\lambda_{j+m}}}{\lambda_{j+1}^2 \cdot \ldots \cdot \lambda_{j+m-1}^2}. \quad (8)
\]

Suppose that the assertion of the lemma is wrong. Then there exist \( K > 0 \) such that for any \( N \) one can find \( j \) with the property that all products (6) have norm less than \( K \). It follows that \( K^{-1} |\Pi_{j,m}| < K \) and \( |S_{j,m}(t)| < \frac{K^2}{t} \).

The denominators of the summands in the right hand side of (8) are equal to \( \Pi_{j,k}^2 \), so they do not exceed \( K^2 \). On the other hand, \( t + \frac{s_{j+1}}{\lambda_{j+1}} > t - t_0 \).

Hence

\[
|S_{j,m}(t)| > m \frac{t - t_0}{K^2}.
\]

Thus \( m \frac{t - t_0}{K^2} < \frac{K^2}{t} \). In particular, this is true for \( m = N \). But the inequality \( N \frac{t - t_0}{K^2} < \frac{K^2}{t} \) can not hold for all \( N \). We obtained a contradiction.

Let us say that a sequence \( \{a_n\} \) of complex numbers satisfies the condition (\(^*\)) if

\[
\forall \varepsilon > 0 \quad \forall N \in \mathbb{N} \quad \exists i \in \mathbb{N} \quad \text{such that} \quad \max_{1 \leq j \leq N} |a_{i+j}| < \varepsilon. \quad (\ast)
\]

Lemma 3. Let \( \Psi_* \) be a bounded sequence of upper-triangular matrices. If a sequence of matrices \( \Phi_* \) is so close to \( \Psi_* \) that \( \{\|\Phi_n - \Psi_n\|\} \) satisfies (\( \ast \)), then \( \mathfrak{B}(\Phi_*) \subseteq [0; t_0] \) with \( t_0 \) the same as in (5).

Proof. Assume, to the contrary, that for some \( t > t_0 \) there exists \( M > 0 \) such that \( \|P_n(t)\| \leq M \) for all \( n \in \mathbb{N} \). Applying Lemma 2 to the sequence \( \Psi_* \) with \( K = 2M^2 \), we obtain a positive integer \( N \) such that, for any \( j \), at least one of the products (6) with \( m \leq N \) has norm larger than \( 2M^2 \).

Fix arbitrary \( \delta > 0 \) and \( C > 1 + \sup \|\Phi_n\| \) and choose \( \varepsilon > 0 \) with

\[
\varepsilon < \frac{\delta(C - 1)}{(C^N - 1)\|H(t)\|^N}.
\]
For these $N$ and $\varepsilon$, find $i$ according to condition (*):

$$\| \Phi_{i+j} - \Psi_{i+j} \| < \varepsilon \quad j = 1, 2, \ldots, N.$$  

By our choice of $N$, there exists $m$, $1 \leq m \leq N$, for which

$$\| \Psi_{i+m} H(t) \cdot \ldots \cdot \Psi_{i+1} H(t) \| > K.$$  

Now, we estimate the product $\Phi_{i+m} H(t) \cdot \ldots \cdot \Phi_{i+1} H(t)$. On the one hand, it is close to $\Psi_{i+m} H(t) \cdot \ldots \cdot \Psi_{i+1} H(t)$:

$$\| \Phi_{i+m} H(t) \cdot \ldots \cdot \Phi_{i+1} H(t) - \Psi_{i+m} H(t) \cdot \ldots \cdot \Psi_{i+1} H(t) \| \leq \| \Phi_{i+m} H(t) \cdot \ldots \cdot \Phi_{i+2} H(t) \Phi_{i+1} H(t) - \Psi_{i+m} H(t) \cdot \ldots \cdot \Psi_{i+1} H(t) \|$$

$$+ \| \Phi_{i+m} H(t) \cdot \ldots \cdot \Phi_{i+2} H(t) H(t) - \Psi_{i+m} H(t) \cdot \ldots \cdot \Psi_{i+1} H(t) \|$$

$$\leq \ldots \leq \| \Phi_{i+m} H(t) \cdot \ldots \cdot \Phi_{i+2} H(t) (\Phi_{i+1} - \Psi_{i+1}) H(t) \|$$

$$+ \| \Phi_{i+m} H(t) \cdot \ldots \cdot (\Phi_{i+2} - \Psi_{i+2}) H(t) \Psi_{i+1} H(t) \|$$

$$+ \ldots + \| (\Phi_{i+m} - \Psi_{i+m}) H(t) \Psi_{i+m-1} H(t) \cdot \ldots \cdot \Psi_{i+1} H(t) \|$$

$$\leq \varepsilon (C^{m-1} + C^{m-2} + \ldots + 1)\| H(t) \| \leq \frac{\varepsilon (C^m - 1)}{C - 1} \leq \varepsilon (C^m - 1) H(t) \| ^m < \delta.$$  

Therefore

$$\| \Phi_{i+m} H(t) \cdot \ldots \cdot \Phi_{i+1} H(t) \| \geq \| \Psi_{i+m} H(t) \cdot \ldots \cdot \Psi_{i+1} H(t) \| - \delta > 2 M^2 - \delta. \quad (9)$$

On the other hand,

$$\| \Phi_{i+m} H(t) \cdot \ldots \cdot \Phi_{i+1} H(t) \| = \| P_{i+m}(t) (P_i(t))^{-1} \|$$

$$\leq \| P_{i+m}(t) \| \cdot \| (P_i(t))^{-1} \| = \| P_{i+m}(t) \| \cdot \| P_i(t) \| \leq M^2.$$  

which contradicts (9). \qed
We will also need two auxiliary results from the classical potential theory in the spirit of Wiener’s criterion \cite{2}. Let $\Omega$ be a bounded domain in the complex plane, $z_0 \in \Omega$. Recall that the harmonic measure $\omega$ on the boundary $\partial \Omega$ with respect to a point $z_0$ is defined by the condition
\[ u(z_0) = \int_{\partial \Omega} u(z) d\omega, \tag{10} \]
for all harmonic continuous function on $\overline{\Omega}$, and that for subharmonic $u$ the equality should be changed by the inequality
\[ u(z_0) \leq \int_{\partial \Omega} u(z) d\omega. \tag{11} \]
We will denote by $|E|$ the Lebesgue measure of a set $E \subset \mathbb{R}$.

**Lemma 4.** Let $E$ be a closed subset of $[1, +\infty]$ of infinite length with the property that
\[ |E \cap [a, 4a]| \leq 1 \quad \text{for each } a > 0. \tag{12} \]
Fix $z_0 \in \mathbb{C} \setminus E$. Take $R > |z_0|$ and consider the domain
\[ \Omega_R = \{ z \in \mathbb{C} : |z| < 2R, z \notin E_{1,R} \}, \]
where by $E_{a,b}$ we denote $E \cap [a, b]$. Then the harmonic measure $\omega_R$ on $\partial \Omega_R$, associated with the point $z_0$, satisfies the condition
\[ \lim_{R \to \infty} \omega_R(T_R) \log(1 + 2R) = 0 \]
where $T_R = \partial \Omega_R \cap \{|z| = 2R\}$.

**Proof.** Observe, first of all, that (12) implies that for each $a > 0$,
\[ \int_{E_{a,\infty}} \frac{dt}{t} \leq \sum_{k=0}^{\infty} \frac{1}{4^k a^k} |E_{4^k a, 4^{k+1} a}| \leq \frac{4}{3a}. \]
Consider the auxiliary potential
\[ U(z) = \int_{E_{1,R}} \log \left| 1 - \frac{z}{t} \right| dt. \]
Notice that $U \in C(\overline{\Omega_R})$ and $U$ is harmonic in $\Omega_R$. Since $\partial \Omega_R$ consists of the circumference $T_R$ of radius $2R$ centered at 0 and the set $E_{1,R}$, we have

$$U(z_0) = \int_{T_R} U(z) d\omega_R(z) + \int_{E_{1,R}} U(z) d\omega_R(z). \quad (13)$$

Hence

$$\int_{T_R} U(z) d\omega_R(z) \leq |U(z_0)| - \int_{E_{1,R}} U(z) d\omega_R(z). \quad (14)$$

It follows from the definition of $U$ that

$$|U(z_0)| \leq \int_{E_{1,R}} \log \left\{ 1 + \frac{|z_0|}{t} \right\} dt \leq |z_0| \int_{E_{1,R}} \frac{dt}{t} \leq \frac{4}{3} |z_0|. \quad (15)$$

Choose $b > 0$ so that $|E_{1,b}| = 1$. We may suppose that $R > 2 + b + |z_0|$.

We claim that for every $z \in E_{1,R}$,

$$U(z) \geq \int_{E_{1,b}} \log \left| 1 - \frac{z}{t} \right| dt + \int_{E_{z,2z}} \log \left| 1 - \frac{z}{t} \right| dt + \int_{E_{2z,\infty}} \log \left| 1 - \frac{z}{t} \right| dt = J_1 + J_2 + J_3$$

The reason for this inequality is that $J_2$ and $J_3$ together give exactly the integral of all negative values of the function $t \to \log \left| 1 - \frac{z}{t} \right|$ on $E_{1,\infty}$, so the extension of the upper limit from $R$ to $+\infty$ in $J_3$ and possible overlapping with $J_1$ are not problems: essentially what is said is that the integral of a real-valued function over a set $F$ is not less than its integral over any subset of $F$ plus the integral of all its negative values over any superset of $F$.

Observe that, since $\log |1 - x| \geq -2x$ for $0 < x < 1/2$, we have

$$J_3 \geq -2z \int_{E_{2z,\infty}} \frac{dt}{t} \geq -\frac{4}{3}$$

regardless of $z$. We also have (recall that $|E_{1,b}| = 1$)

$$J_1 - \log z = \int_{E_{1,b}} \log \left| \frac{1}{z} - \frac{1}{t} \right| dt \to -\int_{E_{1,b}} \log t \ dt \quad \text{as} \quad z \to +\infty.$$
Since \( J_1 - \log z \) is a continuous function on \([1, +\infty)\), we get \( J_1 \geq \log z - C_1 \) where \( C_1 \) is some large constant independent on \( R \). Next,

\[
J_2 = \int_{E_{\frac{1}{z}, 2z}} \log |t - z| \, dt - \int_{E_{\frac{1}{z}, 2z}} \log t \, dt \geq \int_{[z-1, z+1]} \log |t - z| \, dt - \log(2) |E_{\frac{1}{z}, 2z}| \geq -2 - \log z - \log 2
\]

(here, we used the inequality \(|E_{\frac{1}{z}, 2z}| \leq 1\)). Therefore, \( U(z) \geq -C_1 - 10/3 - \log 2 \) on \( E_{1,R} \) and, taking into account that \( \omega_R(E_{1,R}) \leq \omega_R(\partial \Omega) = 1 \), we get

\[
\int_{E_{1,R}} U(z) \, d\omega_R(z) \geq -C_2 \omega_R(E_{1,R}) \geq -C_2.
\]  

(16)

Now by (14),

\[
\int_{T_R} U(z) \, d\omega_R(z) \leq C_3,
\]  

(17)

where \( C_3 = C_2 + \frac{4}{3} |z_0| \).

The last observation we need is that for \( z \in T_R \),

\[
U(z) \geq \int_{E_{1,R}} \log \left| 1 - \frac{2R}{t} \right| \, dt.
\]  

(18)

So (17) gives

\[
\omega_R(T_R) \leq C_3 \left\{ \int_{E_{1,R}} \log \left| 1 - \frac{2R}{t} \right| \, dt \right\}^{-1}.
\]

Thereby,

\[
\omega_R(T_R) \log(1 + 2R) \leq C_3 \left\{ \frac{1}{\log(1 + 2R)} \int_{E_{1,R}} \left| 1 - \frac{2R}{t} \right| \, dt \right\}^{-1}
\]

\[
= \left\{ \int_{E_{1,R}} L_R(t) \, dt \right\}^{-1}.
\]

Note that \( 0 \leq L_R(t) \leq 1 \) for each \( t \in E_{1,R} \) and \( L_R(t) \to 1 \) as \( R \to +\infty \) for every fixed \( t \in E \). Therefore, \( \int_{E_{1,R}} L_R(t) \, dt \to |E| = +\infty \), and we are done.
Lemma 5. Let $u_n(z)$ be a sequence of continuous subharmonic functions satisfying the estimate $u_n(z) \leq \log(1 + |z|) + A$ for all $n \geq 1$, $z \in \mathbb{C}$ and some $A > 0$. If $E \subset \mathbb{R}$ has infinite length and $\limsup_{n \to \infty} u_n(z) \leq 0$ for all $z \in E$, then $\limsup_{n \to \infty} u_n(z) \leq 0$ for all $z \in \mathbb{C}$.

Proof. Since every measurable set of infinite length contains a closed subset of infinite length we may assume without loss of generality that $E$ is closed. Also we may assume that $|E \cap [1, +\infty)| = +\infty$. Indeed, otherwise $E \cap [\infty, -1]| = +\infty$ and we may consider the set $-E$ and functions $u_n(-z)$ instead. Thus we can always assume that $E \subset [1, +\infty]$. The last regularization we need is the following. Take any dyadic interval $I_k = [2^{k-1}, 2^k)(k = 1, 2, ...)$.

If $|E \cap I_k| < 1/3$, leave the corresponding piece of $E$ alone. Otherwise replace it by some subset of length exactly 1/3. The resulting set still has infinite length. Indeed if we made finitely many replacements, we dropped only a set of finite length from $E$, and if we made infinitely many replacements, we have infinitely many disjoint pieces of length 1/3 in the resulting set. After such regularization, the set $E$ enjoys the property (12). We will use the notation introduced in the previous Lemma.

Choose $z_0 \in \mathbb{C}$; we have to prove that $\limsup_{n \to \infty} u_n(z_0) \leq 0$. This is evident if $z_0 \in E$, so we assume that $z_0 \in \mathbb{C} \setminus E$.

For $R > |z_0|$, we have, by (11),

$$u_n(z_0) \leq \int_{\partial \Omega_R} u_n(z) d\omega_R(z) = \int_{T_R} u_n(z) d\omega_R(z) + \int_{E_{1,R}} u_n(z) d\omega_R(z).$$

Note that, for fixed $R$, the length of $E_{1,R}$ is finite, $u_n$ are uniformly bounded from above on $E_{1,R}$, and $\limsup_{n \to \infty} u_n(z) \leq 0$ for all $z \in E_{1,R}$. Therefore, the Fatou lemma yields

$$\limsup_{n \to \infty} \int_{E_{1,R}} u_n(z) d\omega_R(z) \leq 0$$

and thereby

$$\limsup_{n \to \infty} u_n(z_0) \leq \sup_{n \geq 1} \int_{T_R} u_n(z) d\omega_R(z) \leq \omega_R(T_R)(\log(1 + 2R) + A)$$

for any fixed $R$. By Lemma 4, the result follows. □
4 The proof of Theorem

Now we can prove Theorem.

Proof. We use Iwasawa’s decomposition of 2 × 2 matrices:

\[ \Phi_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = \begin{pmatrix} 1 & s_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_n & 0 \\ 0 & \frac{1}{\lambda_n} \end{pmatrix} \begin{pmatrix} \cos \alpha_n & -\sin \alpha_n \\ \sin \alpha_n & \cos \alpha_n \end{pmatrix} = H(s_n)D(\lambda_n)R(\alpha_n), \]

where \( \alpha_n = \arcsin \frac{c_n \text{sign}(d_n)}{\sqrt{c_n^2 + d_n^2}} \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \), \( s_n = \frac{a_n c_n + b_n d_n}{c_n^2 + d_n^2} \), \( \lambda_n = \text{sign}(d_n) \sqrt{c_n^2 + d_n^2} \).

By Lemma we may assume that the system \( \Phi_* \) is bounded. It follows that both sequences \( \{H(s_n)\} \) and \( \{D(\lambda_n)\} \) are bounded. Indeed,

\[ s_n = \frac{(a_n, b_n) \cdot (c_n, d_n)}{\| (c_n, d_n) \|^2} \leq \frac{\|(a_n, b_n)\|}{\| (c_n, d_n) \|} \leq \frac{\|(a_n, b_n)\|}{\| (d_n, -c_n) \|} \leq \frac{1}{\| (a_n, b_n) \|} \leq C^2 \]

where \( C > 1 + \sup \{\| \Phi_n \|\} \). Thus \( \{H(s_n)\} \) is a bounded sequence. The boundedness of \( \{D(\lambda_n)\} \) follows from the equality

\[ D(\lambda_n) = H(s_n)^{-1} \Phi_n R(\alpha_n)^{-1}. \]

Now denote

\[ q = \limsup_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} |\alpha_j| \]

and consider two cases separately: \( q = 0 \) and \( q > 0 \). In both cases we will prove stronger statements than the assertion of Theorem.

Case A: \( q = 0 \). We will show that in this case the set \( \mathfrak{B}(\Phi_*) \) is bounded. Note, that in this case the sequences \( \{\alpha_n\} \) and hence \( \{\sin \alpha_n\} \) satisfy condition \((*)\).

Since

\[ \| \Phi_n - H(s_n)D(\lambda_n) \| \leq \| H(s_n) \| \| D(\lambda_n) \| \left\| \begin{pmatrix} \cos(\alpha_n) - 1 & -\sin \alpha_n \\ \sin \alpha_n & \cos(\alpha_n) - 1 \end{pmatrix} \right\| \leq 2|\sin \alpha_n| \| H(s_n) \| \| D(\lambda_n) \|, \]

the sequence \( \Phi_* \) is close in \((*)\)-sense to the sequence of upper-triangular matrices \( \{H(s_n)D(\lambda_n)\} \), i.e, the sequence of norms \( \| \Phi_n - H(s_n)D(\lambda_n) \| \) satisfies
Thus, according to Lemma 3, the sequence of evolutions \( \{P_n(t)\} \) is unbounded for every \( t > t_0 \), where

\[
t_0 = \max\{|s_n/\lambda_n^2|, n \in \mathbb{N}\}.
\]

This completes the proof in the case A. \( \square \)

**Case B: \( q > 0 \).** Extending our problem to the complex plane, we consider \( SL(2, \mathbb{C}) \)-matrices \( H(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \) instead of \( H(t) \). Respectively, \( P_n(z) = \Phi_n H(z) \ldots \Phi_1 H(z) \). We will show that the set

\[
E = \{ z \in \mathbb{C} : \limsup_{n \to \infty} \frac{\log \| P_n(z) \|}{n} = 0 \}
\]

is contained in \( \mathbb{R} \) and has finite length. So we not only prove that the sequence \( P_n \) is unbounded but prove that it has exponential growth for all \( t \) apart of a set of finite measure.

Our first task is to show that

\[
\limsup_{n \to \infty} \frac{\log \| P_n(z) \|}{n} > 0, \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (19)
\]

Assume that \( \text{Im} z > 0 \) (the case \( \text{Im} z < 0 \) can be considered in a similar way). Let us consider the quadratic form

\[
Q(x) = \text{Im}(x_1 \bar{x}_2), \quad x = (x_1, x_2) \in \mathbb{C}^2.
\]

For any matrix \( A \in SL(2, \mathbb{R}) \) and \( x \in \mathbb{C}^2 \), one has \( Q(Ax) = Q(x) \). On the other hand, for every \( z \in \mathbb{C} \) with \( \text{Im} z > 0 \), one has

\[
Q(H(z)x) = Q(x) + \text{Im} z |x_2|^2 \geq Q(x) \left( 1 + \text{Im} \frac{|x_2|}{|x_1|} \right).
\]

Now, we need one more lemma.

**Lemma 6.** Let \( \alpha_n \in [-\pi/2, \pi/2] \), then there exists \( k \geq 1 \) such that for all \( n \in \mathbb{N} \)

\[
Q(H(z)H(s_n)D(\lambda_n)R(\alpha_n)H(z)x) \geq Q(x) \left( 1 + \frac{|\alpha_n| \text{Im} z}{2k(1 + |z|)} \right). \quad (20)
\]
Proof of Lemma. We split the proof into two cases.

Case 1: \[
\frac{|x_2|}{|x_1|} \geq \frac{|\alpha_n|}{2(1 + |z|)}. \]
Then
\[
\begin{align*}
Q(H(z)H(s_n)D(\lambda_n)R(\alpha_n)H(z)x) & \geq Q(H(s_n)D(\lambda_n)R(\alpha_n)H(z)x) = Q(H(z)x) \\
& \geq Q(x) \left(1 + \text{Im}z \cdot \frac{|x_2|}{|x_1|}\right) \geq Q(x) \left(1 + \frac{|\alpha_n|\text{Im}z}{2(1 + |z|)}\right).
\end{align*}
\]

Case 2: Now, we suppose that \[
\frac{|x_2|}{|x_1|} \leq \frac{|\alpha_n|}{2(1 + |z|)}, \]
and split the proof of the estimate \(20\) into 4 steps.

1. Let us estimate how the matrix \(H(z)\) changes the ratio of coordinates. Since \(|\alpha_n| \leq \frac{\pi}{2}\) we have
\[
|[H(z)x]_2| = |x_2| \leq \frac{|\alpha_n|}{2} |x_1| \cdot \frac{1}{1 + |z|}
= \frac{|\alpha_n|}{2} |x_1| \cdot \left(1 - \frac{|z|}{1 + |z|}\right) \leq \frac{|\alpha_n|}{2} |x_1| \cdot \left(1 - \frac{|\alpha_n||z|}{2(1 + |z|)}\right)
\]
where \([ \ ]_2\) means the second coordinate. Next,
\[
|[H(z)x]_1| = |x_1 + z x_2| \geq |x_1| \left|1 - |z| \frac{|x_2|}{|x_1|}\right| > |x_1| \cdot \left(1 - \frac{|\alpha_n||z|}{2(1 + |z|)}\right)
\]
since \(|z| \frac{|x_2|}{|x_1|} \leq \frac{|\alpha_n|}{2} \frac{|z|}{1 + |z|} < \frac{|\alpha_n|}{2} < 1\) Thus
\[
|[H(z)x]_2| \leq \frac{|\alpha_n|}{2} |[H(z)x]_1|.
\]

2. It is easy to check the following property of an orthogonal matrix \(R(\alpha)\): for any \(x \in \mathbb{C}^2\) and \(|\alpha| \leq \frac{\pi}{2}\), the inequality \(|x_2| \leq \frac{|\alpha|}{2} |x_1|\) implies \(|[R(\alpha)x]_2| \geq \frac{|\alpha|}{2} |[R(\alpha)x]_1|\). Therefore
\[
|[R(\alpha_n)H(z)x]_2| \geq \frac{|\alpha_n|}{2} |[R(\alpha_n)H(z)x]_1|.
\]
Denote temporarily $R(\alpha_n)H(z)x = y$, then $|y_2| \geq \frac{1}{2} |\alpha_n| |y_1|$. 

3. We set $k = C^2$ with $C = \sup_i ||\Phi_i||$ and obtain

$$\frac{|[D(\lambda_n)y_2]|}{|D(\lambda_n)y_1|} = \frac{1}{(\lambda_n)} \frac{|y_2|}{|\lambda_n||y_1|} \geq \frac{1}{C^2} \frac{|\alpha_n|}{2} > \frac{|\alpha_n|}{2k}.$$ 

4. At last

$$Q(H(z)H(s_n)D(\lambda_n)R(\alpha_n)H(z)x) = Q(H(z)H(s_n)D(\lambda_n)y)$$

$$= Q(H(s_n)D(\lambda_n)y) + \text{Im} z \cdot [D(\lambda_n)y_2]^2 = Q(D(\lambda_n)y)$$

$$+ \text{Im} z \cdot [D(\lambda_n)y_2]^2 \geq Q(D(\lambda_n)y) \left(1 + \text{Im} z \cdot \frac{|[D(\lambda_n)y_2]|}{|[D(\lambda_n)y_1]|}\right)$$

$$\geq Q(D(\lambda_n)y) \left(1 + \frac{|\alpha_n| |\text{Im} z|}{2k} \right) \geq Q(x) \left(1 + \frac{|\alpha_n| |\text{Im} z|}{2k} \right)$$

$$> Q(x) \left(1 + \frac{|\alpha_n| |\text{Im} z|}{2k(1 + |z|)} \right).$$

\[\square\]

Returning to the proof of the theorem, we have

$$P_n(z) = H(s_n)D(\lambda_n)R(\alpha_n)H(z) \cdots \cdot H(s_1)D(\lambda_1)R(\alpha_1)H(z) = \prod_{1 \leq j \leq n} H(s_j)D(\lambda_j)R(\alpha_j)H(z)$$

(recall that $\prod_{1 \leq j \leq n} A_j$ stands for the matrix product $A_nA_{n-1} \ldots$).

Denote

$$B_n(z) = H(z/2)P_n(z)H(z/2)^{-1} = \prod_{1 \leq j \leq n} H(z/2)H(s_j)D(\lambda_j)R(\alpha_j)H(z).$$

Then

$$\lim_{n \to \infty} \frac{\log ||P_n(z)||}{n} = \lim_{n \to \infty} \frac{\log ||B_n(z)||}{n}.$$
Let us consider the vector $x = \left( \frac{i}{\sqrt{2}} \frac{1}{\sqrt{2}} \right)$, $|x|^2 = 2Q(x) = 1$. Then, due to the fact that for arbitrary $y$

$$\|y\|^2 = y_1^2 + y_2^2 \geq 2|y_1y_2| = 2|y_1\overline{y_2}| \geq 2\text{Im}(y_1\overline{y_2}) = 2Q(y),$$

we obtain

$$\log \|B_n(z)\| \geq \frac{1}{2} \log \|B_n(z)x\|^2 \geq \frac{1}{2} \log (2Q(B_n(z)x))$$

$$\geq \frac{1}{2} \sum_{j=1}^n \log \left( 1 + \frac{|\alpha_j|\text{Im}\frac{z}{2}}{2k(1 + |\frac{z}{2}|)} \right) \geq \frac{1}{4} \sum_{j=1}^n \frac{|\alpha_j|\text{Im}\frac{z}{2}}{2k(1 + |\frac{z}{2}|)} = \frac{\text{Im}\frac{z}{2}}{8k(1 + |\frac{z}{2}|)} \sum_{j=1}^n |\alpha_j|$$

where we used that $\log(1+x) \geq \frac{1}{2}x$ for $x \in (0; 1)$ and $\frac{|\alpha_j|\text{Im}z}{2k(1 + |z|)} \leq \frac{|\alpha_j|}{2k} < 1$.

As a consequence, we obtain:

$$\lim_{n \to \infty} \frac{\log \|P_n(z)\|}{n} \geq \lim_{n \to \infty} \frac{\text{Im}\frac{z}{2}}{8k(1 + |\frac{z}{2}|)} \cdot \frac{1}{n} \sum_{j=1}^n |\alpha_j| = \frac{q \cdot \text{Im}\frac{z}{2}}{8k(1 + |\frac{z}{2}|)} > 0.$$

This proves (19), that is the exponential growth of $\|P_n(z)\|$ for non-real $z$. Thus $E \subset \mathbb{R}$.

The claim that $|E| < \infty$ follows now from Lemma 5 applied to the sub-harmonic functions

$$u_n(z) = \frac{\log \|P_n(z)\|}{n}.$$ 

Indeed, the norm of the matrix $H(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ does not exceed $1 + |z|$. Hence

$$\|P_n(z)\| = \left\| \prod_{1 \leq j \leq n} [\Phi_j H(z)] \right\| \leq \|H(z)\|^n \cdot \prod_{1 \leq j \leq n} \|\Phi_j\| \leq (1 + |z|)^n \cdot k^n$$

Therefore

$$\frac{1}{n} \log \|P_n(z)\| \leq \log (1 + |z|) + \log k.$$

Thus the functions $u_n$ satisfy the majorization condition of Lemma 5. By definition, $\lim \sup u_n(z) = 0$ for $z \in E$. If $|E| = \infty$ then Lemma 5 implies that $\lim \sup u_n(z) \leq 0$ for all $z \in \mathbb{C}$, however, this contradicts (19).
5 Constructing an exceptional set containing a given sequence

Let, as above, \( H(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \), \( \Phi_* \) be a sequence of matrices from \( SL(2, \mathbb{R}) \).

The exceptional set was defined as follows

\[
\mathcal{B}(\Phi_*) = \{ t \geq 0 : \sup_K \left\| \prod_{1 \leq k \leq K} \Phi_k H(t) \right\| < \infty \}
\]

We have proved that this set always has finite measure. Nevertheless it can be unbounded. Moreover, it can contain an arbitrary given sequence:

**Theorem 2.** For every sequence \( \{t_n\} \) of positive numbers there exists a sequence \( \Phi_* \) such that \( \{t_n\} \subset \mathcal{B}(\Phi_*) \).

**Proof.** First let us note the following fact: for every \( SL(2, \mathbb{R}) \)-matrix \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) there exists an orthogonal matrix \( R = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \) for which \( (RA)^2 = -1 \). To prove it is sufficient to choose \( R \) such that \( \text{tr} \ (RA) = 0 \), for example to take \( \alpha = \arctg \frac{c-b}{a+d} \).

Now let us construct two sequences \( \{A_n(t)\} \) and \( \{R_n\} \) in the following way. For \( A_1(t) = H(t) \) we will choose \( R_1 \) such that \( (R_1A_1(t_1))^2 = -1 \). Further for each \( n \in \mathbb{N} \) we define \( A_n(t) = (R_{n-1}A_{n-1}(t))^2 \) and we choose \( R_n \) such that \( (R_nA_n(t_n))^2 = -1 \).

Now we define sequence \( \Phi_* \) as follows: \( \Phi_n = R_{k+1} \ldots R_3 R_1 \) where \( k \) is the the largest \( j \) such that \( 2^j \) divides \( n \). Thus we have: \( \Phi_1 = R_1, \Phi_2 = R_2 R_1, \Phi_3 = R_3, \Phi_4 = R_3 R_2 R_1, \Phi_5 = R_1, \Phi_6 = R_2 R_1, \Phi_7 = R_1, \Phi_8 = R_4 R_3 R_2 R_1, \Phi_9 = R_1, \ldots \)

Then the evolution sequence \( P_n(t) = \Phi_n H(t) \Phi_{n-1} H(t) \ldots \Phi_1 H(t) \) has the form

\[
\ldots R_3 R_2 R_1 H(t) R_1 H(t) R_2 R_1 H(t) R_1 H(t) R_3 R_2 R_1 H(t) R_1 H(t) R_2 R_1 H(t) R_3 R_2 R_1 H(t) R_1 H(t)
\]
\[
= \ldots R_2 A_2(t) R_1 R_3 R_2 A_2(t) R_2 A_2(t) R_3 R_2 A_2(t) R_2 A_2(t)
\]
\[
= \ldots R_4 R_3 A_3(t) R_3 A_3(t) R_4 R_3 A_3(t) R_3 A_3(t)
\]

and so on. Thus, for any \( k \),

\[
P_n(t) = B(n, k) \ldots R_{k+2} R_{k+1} R_k A_k(t) R_k A_k(t) R_{k+1} R_k A_k(t) R_k A_k(t)
\]
where the factor $B(n, k)$ is a product of not more than $N = N(k)$ matrices which are either orthogonal or equal to $H(t)$. It follows that the norm of $B(n, k)$ does not exceed a constant depending only on $k$ and $t$:

$$\|B(n, k)\| \leq C(k, t).$$

Since $(R_k A_k(t))^2 = -1$ for $t = t_k$, one has

$$\|P_n(t_k)\| \leq C(k, t_k).$$

This means that $t_k \in \mathfrak{B}(\Phi^*)$.

6 Constructing an essentially unbounded exceptional set

In this section we will construct an example of a sequence $\Phi^* \subset SL(2, \mathbb{R})$ for which the exceptional set

$$\mathfrak{B}(\Phi^*) = \{t \geq 0 : \sup_{K} \left\| \prod_{1 \leq k \leq K} \Phi_k H(t) \right\| < \infty \}$$

is essentially unbounded, that is $|\mathfrak{B}(\Phi^*) \cap [a, +\infty)| > 0$ for all $a > 0$.

Let us consider a sequence of matrices $M_j = M(c_j) = \begin{pmatrix} 1 & 0 \\ c_j & 1 \end{pmatrix}$ ($j \geq 0$) with $c_j \neq 0$. We define the sequence $\{\Phi_k\}$ ($k \geq 1$) in the following way: $\Phi_k = M(c_{j(k)})$ where $j(k)$ is the largest $j$ such that $2^j$ divides $k$. The first few terms of the sequence $\Phi*$ are

$$M_0, M_1, M_0, M_2, M_0, M_1, M_0, M_3, M_0, M_1, M_0, M_2, M_0, M_1, M_0, \ldots$$

("the abacaba order" [4]).

Theorem 3. There exists a sequence $\{c_j\}$ such that the set $\mathfrak{B}(\Phi^*)$ is essentially unbounded.

The proof of this statement will be given in the next section. Here we only outline its basic ideas.
First of all note that our choice of the sequence \( \Phi_k \) implies that the partial products \( \prod_{1 \leq k \leq K} \Phi_k H(t) \) with diadic numbers \( (K = 2^m) \) are related by a simple recurrent formula. Namely let us define a sequence of matrix-functions \( A_n(t) \), \( n \geq -1 \), as follows:

\[
A_{-1}(t) = H(t), \quad A_{n+1}(t) = A_n(t)M(c_{n+1})A_n(t).
\]

Then it is easy to check that

\[
\prod_{1 \leq k \leq 2^m} \Phi_k H(t) = M(c_m)A_{m-1}(t).
\]

More generally,

\[
\prod_{2^l+1 \leq k \leq 2^{l+m}} \Phi_k H(t) = M(c_m)A_{l-1}(t).
\]

So it is possible to express all partial products via \( A_n(t) \). For example, for \( K = 84 \), we have \( 84 = 2^6 + 2^4 + 2^2 \) and, respectively,

\[
\prod_{1 \leq k \leq 84} \Phi_k H(t) = M(c_2)A_1(t)M(c_4)A_3(t)M(c_6)A_5(t).
\]

The general formula is

\[\prod_{1 \leq k \leq K} \Phi_k H(t) = \prod_{1 \leq l \leq L} M_{j_l}A_{j_l-1}(t) \quad (21)\]

(here \( \{j_l\} \) is the strictly increasing finite sequence of integers such that \( K = 2^{j_1} + 2^{j_2} + \ldots + 2^{j_L} \)). It follows from (21) that for proving the boundedness of the sequence of all partial products for a given \( t \), it will be sufficient to find an upper bound for the norms of partial products \( \prod_{1 \leq k \leq K} \Phi_k H(t) \) for all \( K \) that are multiples of \( 2^m \) for some integer \( m \). Note that only \( A_n \) with \( n \geq m \) can appear in such partial products.

Suppose that for some \( t > 0 \) and for some integer \( m \) the following condition holds:

\[-2 < \text{tr}(A_n(t)) < 2, \text{ for all } n \geq m.\] (22)

Then the eigenvalues of \( A_n(t) \) are complex conjugate and the matrices are similar to diagonal ones:

\[A_n(t) = S_n(t)D_n(t)S_n(t)^{-1}\]

where \( D_n(t) = \begin{pmatrix} \lambda_n(t) & 0 \\ 0 & \lambda_n(t) \end{pmatrix} \).
Suppose also that, for all \( n \geq m \),
\[
\| S_{n+1}(t) - S_n(t) \| < \varepsilon_n
\]  \hfill (23)
where the numbers \( \varepsilon_n \) are such that \( \sum_{n=m}^{\infty} \varepsilon_n < \infty \).

Under these conditions it can be proved, using (21), that all products
\[
\prod_{1 \leq k \leq K} \Phi_k H(t) \text{ with } K \in 2^{m+1}\mathbb{Z}
\]
are bounded by the same constant.

Thus, it remains to construct an essentially unbounded set \( E \) such that the conditions (22) and (23) are satisfied for all \( t \in E \) (with \( m \) depending on \( t \)). We will define \( E \) as
\[
\bigcup_{k=0}^{\infty} \bigcap_{n=k}^{\infty} E_n \text{ where sets } E_n \text{ are constructed inductively.}
\]

The possibility of the induction steps is provided by two auxiliary results (Lemmas 7 and 8) which state that (under some conditions) a set \( F \subset \mathbb{R} \) on which the condition (22) holds, can be slightly reduced and, respectively, can be extended by adding an interval located arbitrarily far away from the origin in such a way that on the new set the inequalities of type (23) hold.

For the beginning of the induction process we take a closed interval \( E_0 \subset (0, +\infty) \) such that \( \text{tr} A_0(t) \in (-2, 2) \) on \( E_0 \), and choose a sequence \( \{ \varepsilon_n \} \) with \( \sum_{n=1}^{\infty} \varepsilon_n < \frac{1}{3} |E_0| \).

Then we choose (using Lemma 7) a closed subset \( \tilde{E}_0 \subset E_0 \) such that \( |E_0 \setminus \tilde{E}_0| < \varepsilon_1 |E_0| \) and the conditions \( \text{tr} A_1(t) \in (-2, 2) \) and \( \| S_1(t) - S_0(t) \| < \varepsilon_1 \) hold on \( \tilde{E}_0 \).

Now using Lemma 8 we find a closed interval \( I_1 \) such that its left endpoint is larger than \( \sup \tilde{E}_0 \) and \( \text{tr} A_1(t) \in (-2, 2) \) on \( I_1 \). We put \( E_1 = \tilde{E}_0 \cup I_1 \).

In this manner we proceed with constructing the intervals \( I_n \) and sets \( E_n \). Namely, on the \( n \)-th step we get a set \( E_n \) such that \( I_n \subset E_n \), and choose its subset \( \tilde{E}_n \), satisfying \( |E_n \setminus \tilde{E}_n| < \varepsilon_n |I_n| \), in such a way that the conditions \( \text{tr} A_j(t) \in (-2, 2) \) and \( \| S_j(t) - S_{j-1}(t) \| < \varepsilon_n \) are fulfilled for \( t \in \tilde{E}_n \) and for all \( j \leq n \). Then we set \( E_{n+1} = E_n \cup I_{n+1} \) where the left endpoint of \( I_{n+1} \) is larger than \( \sup E_n \).

The smallness of the deleted parts of the sets \( E_n \) and the condition \( I_n \subset (n, \infty) \) provide the essential unboundedness of the set \( E \).

## 7 Proof of Theorem 3

We start with two auxiliary results.
Let us call a real polynomial matrix function \( A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix} \) and a compact set \( E \subset (0, +\infty) \) a good pair if the following conditions hold:

(i) \( \det A(t) = 1 \) for all \( t \);

(ii) \( \text{tr} A(t) = a_{11}(t) + a_{22}(t) \) is a non-constant polynomial;

(iii) \( \text{tr} A(t) \in (-2, 2) \) for all \( t \in E \).

If \((A(t), E)\) is a good pair, then, according to the spectral theorem, one can find continuous functions \( \lambda : E \rightarrow \mathbb{T}, \text{Im} \lambda \neq 0 \) and \( S : E \rightarrow \text{SL}(2, \mathbb{C}) \) such that

\[
A(t) = S(t)D(t)S(t)^{-1}
\]

where \( D(t) = \begin{pmatrix} \lambda(t) & 0 \\ 0 & \lambda(t) \end{pmatrix} \).

Choosing \( c \in \mathbb{R} \), we set \( \tilde{A}(t) := A(t)M(c)A(t) \).

**Lemma 7.** Assume that \((A(t), E)\) is a good pair. Let \( \varepsilon > 0 \). Then there exists \( \delta > 0 \) such that, for every real \( c \) with \( |c| < \delta \), there exists a compact set \( \tilde{E} \subset E \) such that

(a) \( |E \setminus \tilde{E}| < \varepsilon \);

(b) the pair \((\tilde{A}(t), \tilde{E})\) is good;

and

(c) A matrix-function \( \tilde{S}(t) \), diagonalizing \( \tilde{A}(t) \):

\[
\tilde{A}(t) = \tilde{S}(t)\tilde{D}(t)\tilde{S}(t)^{-1}
\]

can be chosen in such a way that \( \|\tilde{S}(t) - S(t)\| < \varepsilon \).

**Proof.** Note that if \( |c| \) is small then the function \( \tilde{A}(t) \) is close to \( A(t)^2 \) on \( E \):

\[
\|\tilde{A}(t) - A(t)^2\| = \|A(t)[M(c) - I]A(t)\| \leq |c| \cdot \|A(t)\|^2 \leq C_1\delta
\]

It follows that it is “almost diagonalized” by means of \( S(t) \)

\[
\|S(t)^{-1}\tilde{A}(t)S(t) - D(t)^2\| \leq C_1\delta \cdot \|S(t)\|^2 \leq C_2\delta
\]

on \( E \). Since the matrix function \( B(t) = S(t)^{-1}\tilde{A}(t)S(t) \) is similar to \( \tilde{A}(t) \) (so has the same diagonal part \( \tilde{D}(t) \)), the pair \((\tilde{A}(t), \tilde{E})\) (whatever \( \tilde{E} \) be chosen) is good if and only if \((B(t), \tilde{E})\) is good. So we will deal with \( B(t) \). Let us first of all show that \( B(t) \) can be diagonalized

\[
B(t) = V(t)\tilde{D}(t)V(t)^{-1}
\]
via a matrix $V(t)$ which is close to $I$. It will follow that
\[ \tilde{A}(t) = S(t)V(t)\tilde{D}(t)V(t)^{-1}S(t)^{-1}, \]
and \[ S(t) = S(t)V(t) \]
is close to $S(t)$.

We already have that
\[ \|B(t) - D(t)^2\| \leq C_2\delta \]
on $E$. Now, $D(t)^2 = \begin{pmatrix} \lambda^2(t) & 0 \\ 0 & \lambda^2(t) \end{pmatrix}$ is a continuous diagonal matrix-function with distinct diagonal elements for all $t \in E$ except, maybe, finitely many $t$ satisfying the equation $\text{tr} A(t) = 0$ (in which case $\lambda(t) = \pm i$ and $\lambda^2(t) = \lambda^2(\lambda) = -1$). Let $G$ be any open set containing those exceptional $t$ and such that $|G| < \varepsilon$. Put $\tilde{E} = E \setminus G$. Then $\text{Tr} [D(t)^2] = 2\text{Re} [\lambda^2(t)] \subset [-a,a]$ for all $t \in \tilde{E}$ with some $a < 2$. Let $\delta > 0$ be so small that $2C_2\delta < 2 - a$. Then $\text{tr} B(t) \in (-2,2)$ and, therefore, the eigenvalues of $B(t)$ are $\tilde{\lambda}(t)$ and $\lambda(t)$ with $|\tilde{\lambda}(t)| = 1$. Moreover, $\tilde{\lambda}(t)$ is a continuous function of $t$ and $|\tilde{\lambda}(t) - \lambda^2(t)| \leq C_3\sqrt{\delta}$. Let now $m = \min \{\text{Im} [\lambda^2(t)]\}$ and note that $m > 0$. Also, let
\[ B(t) - D(t)^2 =: \Delta(t) = \begin{pmatrix} \Delta_{11}(t) & \Delta_{12}(t) \\ \Delta_{21}(t) & \Delta_{22}(t) \end{pmatrix}. \]

Then the matrix $V(t)$ whose columns are eigenvectors of $B(t)$ is
\[
V(t) = \begin{pmatrix} \lambda^2(t) - \lambda(t) + \Delta_{22}(t) & \Delta_{12}(t) \\ -\Delta_{21}(t) & \lambda(t) - \lambda^2(t) - \Delta_{11}(t) \end{pmatrix}.
\]
The exact formula for $V(t)$ doesn’t matter but it is important that $V(t)$ can be chosen to be a continuous function of $t$ that is close to a diagonal matrix with equal non-zero elements on the diagonal when the perturbation $\Delta(t)$ is close to 0. Note that $\det V(t) \neq 0$ if $\delta$ is small enough. Let now $\tilde{V}(t) = \frac{1}{\sqrt{\det V(t)}}V(t)$ where the branch of the square root of the determinant is chosen in such a way that it equals $\lambda^2(t) - \lambda^2(t)$ when $\Delta(t) = 0$. Then the norm $\|\tilde{V}(t) - I\|$ can be made arbitrarily small if $\delta$ is small enough. It remains to note that $B(t) = \tilde{V}(t)\tilde{D}(t)\tilde{V}(t)^{-1}$ where $\tilde{D}(t) = \begin{pmatrix} \tilde{\lambda}(t) & 0 \\ 0 & \lambda(t) \end{pmatrix}$, so we can put $\tilde{S}(t) = S(t)\tilde{V}(t)$.
We proved the statement (c) of the lemma. The statement (a) follows from the inequality $|G| < \varepsilon$. To have (b) we must check conditions $(i - iii)$ for the matrix $\tilde{A}(t)$. The condition (i) is obvious because the product of three matrices of determinant 1 is still a matrix of determinant 1. The condition (iii) is proved above: we have shown that $\text{tr} B(t) \subset (-2, 2)$ but $\text{tr} \tilde{A}(t) = \text{tr} B(t)$. It remains to check (ii). A direct computation yields

$$
\text{tr} \tilde{A}(t) = a_{11}(t) + 2a_{12}(t)a_{21}(t) + a_{22}(t) + ca_{12}(t)[a_{11}(t) + a_{22}(t)] = [\text{tr} A(t)] \cdot [\text{tr} A(t) + ca_{12}(t)] - 2.
$$

Since $\text{tr} A(t)$ is a non-constant polynomial, the whole expression is a non-constant polynomial for all sufficiently small $c$ and we are done. 

Note that by the construction, the matrix $V(t)$ is unimodular. Hence $\tilde{S}(t)$ is unimodular if $S(t)$ is such. This shows that in further constructions, based on Lemma 8, we may assume that the obtained matrix functions $S_n(t)$ are unimodular.

In the following lemma, which can be regarded as a modification of Lemma 7 we preserve the notations $E(t)$ and $\tilde{S}(t)$. For brevity, let us call a polynomial matrix function $P(t) = (p_{ij}(t))$ upper right dominating if the degree of the polynomial $p_{12}(t)$ is more than the degrees of others $p_{ij}(t)$.

**Lemma 8.** Assume that $(A(t), E)$ is a good pair and that the polynomial matrix $A(t)^2$ is upper right dominating. Let $\varepsilon > 0$ and $N > 0$ be given. Then there exists $\delta > 0$ and a compact interval $I \subset (N, \infty)$ such that, for every real $c$ with $0 < |c| \leq \delta$, there exists a compact set $\tilde{E} \subset E$ such that $|E \setminus \tilde{E}| < \varepsilon$; $(\tilde{A}(t), \tilde{E} \cup I)$ is a good pair; and, moreover, $\|\tilde{S}(t) - S(t)\| < \varepsilon$ on $\tilde{E}$.

**Proof.** By the proof of Lemma 7 we get $\delta_1$ such that for $|c| < \delta_1$ one can find $\tilde{E} \subset E$ satisfying the conditions: $|E \setminus \tilde{E}| < \varepsilon$, $(\tilde{A}(t), \tilde{E})$ is a good pair and $\|\tilde{S}(t) - S(t)\| < \varepsilon$ on $\tilde{E}$.

Let $A(t)^2 = (b_{ij}(t))$. Since $\tilde{A}(t) = A(t)M(c)A(t)$,

$$
\text{tr} \tilde{A}(t) = \text{tr} [M(c)A(t)^2] = cb_{12}(t) + q(t),
$$

where the degree of $q(t)$ is less than the degree of $b_{12}(t)$. Hence if $|c|$ is less than some $\delta_2$ and has the appropriate sign, then there is $t_0 > N$ for which $\text{tr} \tilde{A}(t_0) = 0$. So the condition (ii) holds on some interval $I$ around $t_0$. This shows that $(\tilde{A}(t), \tilde{E} \cup I)$ is a good pair.

It remains to set $\delta = \min\{\delta_1, \delta_2\}$.

\[ \Box \]
The number $\delta$, constructed in Lemma 8, will be denoted by $\delta(A(t), E, \varepsilon, N)$. To underline that in the construction of the interval $I$ and the set $\Omega = \tilde{E} \cup I$ the number $c$ from $(-\delta, 0)$ or $(0, \delta)$ is used, we will denote them by $I = I(A(t), E, \varepsilon, N, c)$ and $\Omega(A(t), E, \varepsilon, N, c)$ respectively.

Now we can prove the theorem.

**Proof of Theorem 3.** We shall start with the matrix $A_0(t) = \mathcal{H}(t)M_0\mathcal{H}(t)$ and note that if $c_0 < 0$, then there exists a closed interval $E_0 \subset (0, +\infty)$ of positive length such that $(A_0(t), E_0)$ is a good pair. Choose $\varepsilon_0 = |E_0|/3$.

We will construct the sequences of numbers $c_j$, $\varepsilon_j$, matrix functions $A_j(t)$, compact sets $E_j$ and compact intervals $I_j$ inductively.

Suppose that these sequences are constructed for $j < n$. Then set

$$
\varepsilon_n = \frac{1}{3n} \min\{|I_j|\}, \quad \delta = \delta(A_{n-1}, E_{n-1}, \varepsilon_n, n)
$$

and choose $c_n$ with $|c_n| < \delta$ and with appropriate sign. Let

$$
I = I(A_{n-1}, E_{n-1}, \varepsilon_n, n, c_n), \quad E_n = \Omega(A_{n-1}, E_{n-1}, \varepsilon_n, n, c_n).
$$

For these definitions be correct, we have to check that the pairs $(A_n(t), E_n)$ are good and that $A_n(t)^2$ are right upper dominating.

The first property follows by induction from Lemma 8. To prove the second one, note that for each $n$, the function $A_n(t)^2$ is a product $H M H M H \ldots M H$ of matrices in which each $H$ is either $H(t)$ or $H(2t)$ and each $M$ is $M(c)$ with some $c \neq 0$ (possibly different for different $M$’s). Let $p$ be the number of $M$’s in the product. Then $A_n(t)^2 = (b_{ij}(t))$ where $b_{ij}$ are polynomials with degrees of $b_{11}(t)$ and $b_{22}(t)$ equal $p$, degrees of $b_{12}$ and $b_{21}$ equal $(p + 1)$ and $(p - 1)$ respectively. The correctness is proved.

Let us set

$$
E = \bigcup_{k=0}^{\infty} \bigcap_{n=k}^{\infty} E_n.
$$

It follows from the choice of the numbers $\varepsilon_j$ that $|E \cap I_n| \neq 0$ for each $n$. Hence $E$ is essentially unbounded. We have to prove that for each $t \in E$ the sequence $\left\| \prod_{1 \leq k \leq K} \Phi_k H(t) \right\|$ is bounded.

If $t \in E$ then there is $m$ such that $t \in E_n$ for each $n \geq m$. Then for any $K$ which is divided by $2^{m+1}$, the partial product $\prod_{1 \leq k \leq K} \Phi_k H(t)$ can be
written (see [21]) as
\[
\prod_{1 \leq k \leq K} \Phi_k H(t) = \prod_{1 \leq \ell \leq L} M_{j\ell} A_{j\ell-1}(t) = \prod_{1 \leq \ell \leq L} [M_{j\ell} S_{j\ell-1}(t)D_{j\ell-1}(t)S_{j\ell-1}(t)^{-1}]
\]
where \(\{j_\ell\}\) is the strictly increasing finite sequence of integers such that \(K = 2^{j_1} + 2^{j_2} + \ldots + 2^{j_L}\) and all \(j_\ell > m\). To estimate the norm of this product, note that it consists of several diagonal matrices of norm 1, the matrix \(M_{j_\ell} S_{j\ell-1}(t)\) in the beginning, the matrix \(S_{j\ell-1}(t)^{-1}\) in the end and several matrices of the kind \(S_{j\ell-1}(t)^{-1} M_{j_{\ell+1}} S_{j_{\ell+1}-1}(t)\) in the middle. Now, \(\|M_j\|\) are bounded. Also
\[
\|S_j(t)\| \leq \|S_j(t) - S_{j-1}(t)\| + \ldots + \|S_{m+1}(t) - S_m(t)\| + \sum_{j \geq m+1} \varepsilon_j
\]
are bounded for each such \(t\). Since the matrices \(S_j(t)\) are unimodular, their inverse are also bounded:
\[
\|S_j(t)\| \leq C(t), \quad \|S_j(t)^{-1}\| < C(t).
\]
It remains to estimate the norms of \(S_{j\ell-1}(t)^{-1} M_{j_{\ell+1}} S_{j_{\ell+1}-1}(t)\). We have
\[
\|S_{j\ell-1}(t)^{-1} M_{j_{\ell+1}} S_{j_{\ell+1}-1}(t) - I\| =
\|S_{j\ell-1}(t)^{-1} \left((M_{j_{\ell+1}} - I) S_{j_{\ell+1}-1}(t) + (S_{j_{\ell+1}-1}(t) - S_{j\ell-1}(t))\right)\| \leq
\|S_{j\ell-1}(t)^{-1}\| \cdot \left(\|M_{j_{\ell+1}} - I\| \cdot \|S_{j_{\ell+1}-1}(t)\| + \|S_{j_{\ell+1}-1}(t) - S_{j\ell-1}(t)\|\right)
\leq C(t) \left(|c_{j_{\ell+1}}| C(t) + \sum_{j = j_{\ell+1}}^{j_{\ell+1} - 1} \varepsilon_j\right).
\]
Hence
\[
\|S_{j\ell-1}(t)^{-1} M_{j_{\ell+1}} S_{j_{\ell+1}-1}(t)\| \leq \exp\left\{C(t) \left[C(t) |c_{j_{\ell+1}}| + \sum_{j = j_{\ell+1}}^{j_{\ell+1} - 1} \varepsilon_j\right] \right\}.
\]
Multiplying all the above estimates, we see that, for \(K \in 2^{m+1} \mathbb{Z}\), one has
\[
\left\| \prod_{1 \leq k \leq K} \Phi_k H(t) \right\| \leq C(t) \exp\left\{C(t) \sum_{j \geq 1} |c_j| + \sum_{j \geq 1} \varepsilon_j\right\}.
\]
Therefore the partial products corresponding to $K \in 2^{m+1}\mathbb{Z}$ are bounded for each $t \in \bigcap_{n \geq m} E_n$. The products corresponding to other $K$ differ from the products corresponding to $K \in 2^{m+1}\mathbb{Z}$ by just $N$ couples of (uniformly) bounded matrices ($N \in \{1, 2, \ldots, 2^{m+1} - 1\}$). Therefore, all the sequence of partial products is bounded for such $t$.

\[ \square \]

**References**

[1] L. Polterovich, Z. Rudnick *Kick stability in groups and dynamical systems*, Nonlinearity 14 (2001), 1331–1363.

[2] N. S. Landkof, *Foundations of Modern Potential Theory*, Grundlehren der Mathematischen Wissenschaften 180, Springer-Verlag, Berlin, 1972.

[3] T. Ransford *Potential Theory in the Complex Plane*, Cambridge University Press, Cambridge, 1995.

[4] [http://www.ac.wwu.edu/~mnaylor/abacaba/abacaba.html](http://www.ac.wwu.edu/~mnaylor/abacaba/abacaba.html)

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