Some Liouville theorems for the $p$-Laplacian

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1 Introduction

In this paper we present several Liouville type results for the $p$-Laplacian in $\mathbb{R}^N$. Let us present an example of the results here obtained. Suppose that $h$ is a nonnegative regular function such that

$$h(x) = a|x|^{\gamma} \text{ for } |x| \text{ large, } a > 0 \text{ and } \gamma > -p.$$  (1.1)

**Theorem 1.1**  
1) Suppose that $N > p > 1$, and $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ is a nonnegative weak solution of

$$-\text{div}(|\nabla u|^{p-2}\nabla u) \geq h(x)u^q \text{ in } \mathbb{R}^N$$  (1.2)

with $h$ as in (1.1). Suppose that

$$p - 1 < q \leq \frac{(N + \gamma)(p - 1)}{N - p}$$

then $u \equiv 0$.

2) Let $N \leq p$. If $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ is a weak solution bounded below of

$$-\text{div}(|\nabla u|^{p-2}\nabla u) \geq 0 \text{ in } \mathbb{R}^N$$

then $u$ is constant.

We shall denote the $p$-Laplacian by $\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$. 

Remark 1.2 By weak solutions of $-\Delta_p u = f$ in $\mathbb{R}^N$ we mean that $u \in W^{1,p}_{loc}(\mathbb{R}^N)$ and
\[
\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi = \int_{\mathbb{R}^N} f \varphi
\]
for all $\varphi \in \mathcal{D}(\mathbb{R}^N)$. Similarly by weak solutions of $-\Delta_p u \geq f(\leq f)$ we mean that $u \in W^{1,p}_{loc}(\mathbb{R}^N)$ and
\[
\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \geq \int_{\mathbb{R}^N} f \varphi (\leq \int_{\mathbb{R}^N} f \varphi)
\]
for all $\varphi \in \mathcal{D}(\mathbb{R}^N)$, $\varphi \geq 0$.

Non existence results for uniformly elliptic semi-linear equations have been the subject of many papers. When $N > p = 2$, the result in Theorem 1.1 is due to Gidas [11]. It was extended to semi-linear equations in cones by Berestycki, Capuzzo Dolcetta and Nirenberg in [1], [2]. The case where the operator is fully non linear and uniformly elliptic was treated by Cutrì and Leoni in [7]. For other non linear Liouville theorems, see e.g. [3], [16] ....

We would like to remark that the first result of Theorem 1.1 is optimal in the sense that for any $q > \frac{(N+\gamma)(p-1)}{N-p}$ we construct a nonnegative solution of (1.2). A similar example was given in [4] when $p = 2$.

Let us also remark that the condition on $\gamma$ in (1.1) is optimal. Indeed, for $\gamma > -p$, Drábek in [9] has proved the existence of non trivial weak solutions in $\mathbb{R}^N$ (see e.g. Theorem 4.1 of [10]).

When treating the equation instead of the inequality, the values of $q$ for which non existence results hold true are not the same. Precisely for the following equation
\[
-\Delta_p u = r^\gamma u^q, \quad u \geq 0 \text{ in } \mathbb{R}^N, \tag{1.3}
\]
we prove in section 3.3 that for $p-1 < q < \frac{(N+\gamma)(p-1)+p+\gamma}{N-p}$ and $\gamma \geq 0$ any radial solution of (1.3) is identically zero.

We believe that it should be possible to prove that as in the case $p = 2$, any solution of (1.3) is radial (see e.g. the work of Damascelli and Pacella [8], for symmetry results for equations involving the $p$-Laplacian). Hence we expect the non existence result to be true for any solution of (1.3).

When $p = 2$, Gidas and Spruck [13] have proved that for $1 < q < \frac{N+2}{N-2}$ any solution of (1.3) is trivial (see also Chen and Li [6] for a simpler proof of this result).
Gidas and Spruck have used this to obtain a priori estimates for solutions of the following problem:

\[
\begin{aligned}
& L u + f(x, u) = 0 \quad \text{in } \Omega \\
& u = \phi \quad \text{on } \partial \Omega
\end{aligned}
\]  

(1.4)

where \( L \) is a second order uniformly elliptic operator and \( f \) satisfies some growth conditions. This is done through a blow up argument (see also [4]).

Analogously Theorem 1.1 constitutes the first step to obtain a priori estimates for reaction diffusion equations involving \( p \)-Laplacian type operators in bounded domains, this will be the object of future works.

The other Liouville type theorems here enclosed concern the following equation

\[
\Delta_p u \geq h(x) u^q \quad \text{in } \mathbb{R}^N
\]

(1.5)

for \( h \) as in (1.1). We prove that if \( p - 1 < q \) and \( \gamma > -p \) then \( u \equiv 0 \).

For completeness sake, we begin this paper by proving that bounded \( p \)-harmonic functions in \( \mathbb{R}^N \) are constant. This result is probably known, but since we have not found it in the litterature, we include its proof here.

2 \( p \)-harmonic functions

Recall that \( u \) is \( p \)-harmonic in a domain \( \Omega \) if \( u \in W^{1,p}_{\text{loc}}(\Omega) \) and

\[
- \Delta_p u = -\text{div}(\|Du\|^{p-2} Du) = 0 \quad \text{in } \Omega. 
\]

(2.1)

**Theorem 2.1** Suppose that \( u \) is a bounded \( p \)-harmonic function in \( \mathbb{R}^N \), then \( u \) is constant.

Before giving the proof of Theorem 2.1 we we need three lemma:

**Lemma 2.2** Suppose that \( u \) is \( p \)-harmonic and belongs to \( W^{1,p}_{\text{loc}}(\mathbb{R}^N) \). Then, there exists some constant \( C \) such that for all \( R \) and \( \sigma > 0 \)

\[
\int_{B(0,R)} |\nabla u|^p \leq C \frac{1}{\sigma^p} \int_{B(0,R+\sigma)} |u|^p.
\]
Lemma 2.3 Suppose that $u$ is and $p$-harmonic then, there exists some positive constant $C$ such that for all $\lambda \geq 1$, $R$ and $\sigma$

$$\int_{B(0,R)} |\nabla (|\nabla u|^{p+\lambda})|^{2} \leq \frac{C}{\sigma^2} \int_{B(0,R+\sigma)} |\nabla u|^{p+\lambda}.$$  

In particular for all $k$ and $n \in \mathbb{N}$

$$\int_{B(0,R)} |\nabla (|\nabla u|^{\frac{nk-n-1}{2}})|^{2} \leq \frac{C}{\sigma^2} \int_{B(0,R+\sigma)} |\nabla u|^{nk-n-1}.$$  

Lemma 2.4 Suppose that $\phi_n$ is some sequence of positive numbers satisfying for some constants $c > 0$ and $k > 0$

$$\phi_n \leq c^n \phi_{n-1}^k$$

Then

$$\phi_n^{k-n} \leq \frac{c}{(k-1)^2} \phi_0.$$  

Proof of Lemma 2.4

Let us start by noticing that

$$\phi_n \leq c^n \phi_{n-1}^k \leq c^n + (k)(n-1) \phi_{n-2}^{k} \leq \sum_{p=0}^{n-1} (n-p) k^p \phi_0^{k^n} \phi_0$$

One can easily compute

$$\sum_{p=0}^{n-1} (n-p) k^p = (n+1) \frac{k^n - 1}{k - 1} - \frac{d}{dk} \left( \frac{k^n - 1}{(k-1)^2} \right)$$

and then

$$\phi_n^{k-n} \leq \frac{c}{(k-1)^2} \phi_0 \leq c \phi_0^{(k-1)^2} \phi_0.$$  

Proof of Lemma 2.2

First let us remark that by the regularity results of Tolksdorff [15] $u$ is $C^1(\mathbb{R}^N)$ and $u \in W^{2,p}_{\text{loc}}$ for $p < 2$, $u \in W^{2,2}_{\text{loc}}$ for $p \geq 2$. Multiply equation (2.1) by $u\zeta^p$, where $\zeta$ is some smooth function which equals one on $B(0,R)$ and is zero outside.
$B(0, R + \sigma) 0 \leq \zeta \leq 1$. One has then some universal constant such that $|\nabla \zeta|_{\infty} \leq \frac{2}{\sigma}$. One obtains
\[
\int |\nabla u|^{p} \zeta^{p} = -p \int |\nabla u|^{p-2} \nabla u. (\nabla \zeta) \zeta^{p-1} u \leq |(\nabla u)\zeta|_{p-1}^{p-1}(\int u^p |\nabla \zeta|^p)^{\frac{1}{p}}.
\]
Finally
\[
\int_{B(0,R)} |\nabla u|^p \leq |(\nabla u)\zeta|^p \leq \frac{C}{\sigma^p} \int_{B(0,R+\sigma) - B(0,R)} |u|^p.
\]
In particular, if $u$ is bounded
\[
\int_{B(0,R)} |\nabla u|^p \leq C \frac{(R + \sigma)^N - R^N}{\sigma^p}.
\]
For $R = \sigma$ one obtains
\[
\int_{B(0,R)} |\nabla u|^p \leq 2^N C^N R^{N-p}.
\]
**Proof of Lemma 2.3**
We use formal calculations which can be justified, as it is done in [13]. Differentiate equation (2.1) with respect to $x_k$ and multiply by $|u, k|^{\lambda} u, k \zeta^2$, integrating we obtain
\[
\int (|\nabla u|^{p-2} u, j, k |u_k|^{\lambda} u, k \zeta^2) = (\lambda + 1) \int |\nabla u|^{p-2} |u_k|^{\lambda} (u^2, k) \zeta^2 +
\]
\[
+ (p - 2)(\lambda + 1) \int |\nabla u|^{p-4} (\nabla u. \nabla u, k)^2 \zeta^2 +
\]
\[
+ \int |\nabla u|^{p-2} |u_k|^{\lambda} 2 \zeta_j u, kj
\]
\[
+ (p - 2) \int |\nabla u|^{p-4} (\nabla u. \nabla u, k) 2 \zeta_j |u, k|^{\lambda} u, k.
\]
Let us observe that
\[
|\int |\nabla u|^{p-4} (\nabla u. \nabla u, k)^2 \zeta^2| \leq \int |\nabla u|^{p-2} |u_k|^{\lambda} (u^2, k) \zeta^2
\]
by Schwartz inequality. Using $|p - 2| \leq 1$ if $p < 2$ and if not $(p - 2)(\lambda + 1)(\int |\nabla u|^{p-4} (\nabla u. \nabla u, k)^2 \zeta^2) \geq 0$, summing over $k$ and using once more Schwartz inequality one obtains
\[
(\lambda + 1)(p - 1) \int |\nabla u|^{p-2} |\nabla u|^{\lambda} |\nabla^2 u|^2 \zeta^2 \leq 2(p - 1) \int |\nabla u|^{p+\lambda - 1} |\nabla^2 u| \zeta \nabla \zeta|
\]
\[
\leq \frac{\lambda + 1}{2} \int |\nabla u|^{p+\lambda - 2} |\nabla^2 u|^2 +
\]
\[
+ \frac{C}{\lambda + 1} \int |\nabla u|^{p+\lambda} |\nabla \zeta|^2.
\]
Finally
\[
\int_{B(0,R)} |\nabla u|^{p-2+\lambda} |\nabla \nabla u|^2 \leq \frac{C}{(\lambda + 1)^2 \sigma^2} \int_{B(0,R+\sigma)} |\nabla u|^{p+\lambda},
\]
i.e.
\[
\frac{4}{(p + \lambda)^2} \int |\nabla (|\nabla u|^{\frac{p+\lambda}{2}})|^2 \leq \frac{C}{(\lambda + 1)^2 \sigma^2} \int_{B(0,R+\sigma)} |\nabla u|^p
\]
which implies, iterating and beginning with \( \lambda = 0, \)
\[
\int_{B(0,R)} |\nabla (|\nabla u|^{\frac{p+\lambda}{2}})|^2 \leq \frac{C}{\sigma^2} \int_{B(0,R+\sigma)} |\nabla u|^{p^{kn} - 1}.
\]
This concludes the proof of Lemma 2.4

**Proof of Theorem 2.1.** Let us recall that according to some Poincaré’s inequality and Sobolev embedding one has that for all \( k \leq N \), there exists some constant \( C_N \) which depends only on \( N \) and \( p \) such that for all \( R \) and \( w \in H_0^1(R^N) \)
\[
\left( \frac{1}{R^N} \int_{B(0,R)} |w|^2k \right)^{\frac{1}{k}} \leq C_N \left( \frac{1}{R^{N-2}} \int_{B(0,R)} |\nabla w|^2 + \frac{1}{R^N} \int_{B(0,R)} |w|^2 \right).
\]
Let us choose \( w = |\nabla u|^{\frac{p+\lambda}{2}} \) in this inequality in order to obtain
\[
\left( \frac{1}{R^N} \int_{B(0,R)} |\nabla u|^{p^{kn}} \right)^{\frac{1}{k}} \leq C_N \left( \frac{1}{R^{N-2}} \int_{B(0,R)} |\nabla (|\nabla u|^{\frac{p+\lambda}{2}})|^2 + \frac{1}{R^N} \int_{B(0,R)} |\nabla u|^{p^{kn} - 1} \right).
\]
We are in a position to apply Lemma 2.3, and the previous inequality becomes
\[
\left( \frac{1}{R^N} \int_{B(0,R)} |\nabla u|^{p^{kn}} \right)^{\frac{1}{k}} \leq C'_N \left( \frac{1}{R^{N-2}} \int_{B(0,R)} |\nabla u|^{p^{kn} - 1} \right) (1 + \frac{R^2}{\sigma^2}) \int_{B(0,R+\sigma)} |\nabla u|^{p^{kn} - 1} \quad (2.2)
\]
\[
\leq C'_N (1 + \frac{R^2}{\sigma^2})(\frac{R + \sigma}{R})^N \frac{1}{(R + \sigma)^N} \int_{B(0,R+\sigma)} |\nabla u|^{p^{kn} - 1}.
\]
We define \( \phi_n = \left( \frac{1}{\rho_n} \int_{B(0,\rho_n)} |\nabla u|^{p^{kn}} \right)^{\frac{1}{k}} \) with \( \rho_n = r(1 + 2^{-n}) \), which satisfies \( \rho_{n-1} \leq 2 \rho_n \). With \( R = \rho_n \) and \( \sigma = \rho_{n-1} - \rho_n \), (2.2) becomes
\[
\phi_n \leq C(1 + \left( \frac{\rho_n}{\rho_{n-1} - \rho_n} \right)^2) \phi_{n-1}^{k\frac{1}{2}}
\]
Using
\[
1 + \left( \frac{\rho_n}{\rho_{n-1} - \rho_n} \right)^2 = 1 + (2^n + 1)^2 \leq 5^n,
\]
one finally has
\[ \phi_n \leq C \phi_{n-1}^k. \]

Using Lemma 2.2 one gets that
\[ \lim_{n \to \infty} \phi_n \leq C \phi_0 = C \frac{1}{(2r)^N} \int_{B(0,2r)} |\nabla u|^p. \]

The left hand side tends to \( \sup_{B(0,r)} |\nabla u|^p \) and the right hand side is less than a constant multiplied by \( (2r)^{N-p-N} \) by using Lemma 2.2. Taking \( r \) large enough one gets that \( \nabla u = 0 \), which ends the proof.

3 The positive semi-linear case

3.1 The inequation

When \( N > p \) our main non-existence result in this section is the following

**Theorem 3.1** Suppose that \( N > p > 1 \). Let \( u \in W^{1,p}_{\text{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N) \) be a nonnegative weak solution of
\[ -\Delta_p u \geq h(x)u^q \text{ in } \mathbb{R}^N, \]
with \( h \) satisfying (1.4). Suppose that \( p - 1 < q \leq \frac{(N+\gamma)(p-1)}{N-p} \), then \( u \equiv 0 \).

The proof is inspired by the one given in [7], where the authors treat fully nonlinear strictly elliptic equations.

Let us start by one remark and two propositions.

**Remark 3.2** The following comparison result holds true: let \( u \) and \( \phi \) satisfy \( u, \phi \in W^{1,p}(\Omega) \)
\[ \begin{cases} -\Delta_p u \geq -\Delta_p \phi = 0 & \text{in } \Omega \\ u \geq \phi & \text{on } \partial \Omega \end{cases} \]
then \( u \geq \phi \) in \( \Omega \). This is a standard result and it is easy to see for example by multiplying \( -\Delta_p u + \Delta_p \phi \) by \( (\phi - u)^+ \).

**Proposition 3.3** Let \( \Omega \) be an open set in \( \mathbb{R}^N \), and let \( f \in C(\overline{\Omega}) \). Suppose that \( u \in W^{1,p}_{\text{loc}}(\Omega) \cap C(\overline{\Omega}) \) is a weak solution of \( -\Delta_p u \geq f \) in \( \Omega \). Then, if \( x_0 \in \Omega \), and \( \varphi \in C^2(\Omega) \cap C(\overline{\Omega}) \), are such that
\[ \nabla \varphi(x_0) \neq 0, \quad u(x_0) - \varphi(x_0) = \inf_{y \in \Omega} u(y) - \varphi(y) \]
then

\[ -\Delta_p \varphi(x_0) \geq f(x_0) \]

This proof is inspired by Juutinen [14].

**Proof.** Without loss of generality we can suppose that \( u(x_0) = \varphi(x_0) \).

Let us note first that it is sufficient to prove that the property holds for every \( \varphi \) such that \( \varphi(y) < u(y) \) for all \( y \neq x_0 \) in a sufficiently small neighbourhood of \( x_0 \). Indeed, suppose that the property holds for such functions then taking \( \varphi_\varepsilon(y) = \varphi(y) - \varepsilon|y - x_0|^4 \) and letting \( \varepsilon \) go to zero, one obtains the result for every \( \varphi \).

Suppose by contradiction that there exists some \( x_0 \in \Omega \) and some \( C^2 \) function \( \varphi \) such that \( \nabla \varphi(x_0) \neq 0 \), \( \varphi(x_0) = u(x_0) \) and \( \varphi(y) < u(y) \) on some ball \( B(x_0, r) \) and \(-\Delta_p \varphi(x_0) < f(x_0)\). By continuity, one can choose \( r \) sufficiently small such that \( \nabla \varphi(y) \neq 0 \), as well as \(-\Delta_p \varphi(y) < f(y)\), for all \( y \in B(x_0, r) \).

Let \( m = \inf_{|x-x_0|=r} (u(x) - \varphi(x)) > 0 \), and define

\[ \bar{\varphi} = \varphi + \frac{m}{2} \]

One has \(-\Delta_p \bar{\varphi} < f \) in \( B(x_0, r) \) and \( \bar{\varphi} \leq u \) on \( \partial B(x_0, r) \).

Using the comparison principle one gets that \( \bar{\varphi} \leq u \) in the ball which contradicts \( \bar{\varphi}(x_0) = \varphi(x_0) + \frac{m}{2} > u(x_0) \). This ends the proof of Proposition 3.3.

Finally let us recall that if \( v \) is radial i.e. \( v(x) = V(|x|) \equiv V(r) \) for some function \( V \) and \( V \) is \( C^2 \), then if \( x \) is such that \( V'(|x|) \neq 0 \)

\[ \Delta_p v(x) = |V'(r)|^{p-2} \left( (p-1)V''(r) + \frac{N-1}{r} V'(r) \right). \]

Hence for any constants \( C_1 \) and \( C_2 \) if \( N \neq p \) and for \( \lambda = \frac{p-N}{p-1} \) the function \( \phi(x) = C_2|x|^\lambda + C_1 \) satisfies \( \Delta_p \phi = 0 \) for \( x \neq 0 \).

Before giving the proof of Theorem 3.1 let us define \( m(r) = \inf_{x \in B_r} u(x) \) and prove the following Hadamard type inequality

**Proposition 3.4** Let \( N \neq p \). Suppose that \(-\Delta_p u \geq 0\) and \( u \geq 0 \). Let \( \lambda = \frac{p-N}{p-1} \).

For any \( 0 < r_1 < r < r_2 \) :

\[ m(r) \geq \frac{m(r_1)(r^\lambda - r_2^\lambda) + m(r_2)(r_1^\lambda - r^\lambda)}{r_1^\lambda - r_2^\lambda}. \]  

(3.2)
Let $N = p$ then

$$m(r) \geq \frac{m(r_1) \log(\frac{r}{r_2}) + m(r_2) \log(\frac{r}{r_1})}{\log(\frac{r_1}{r_2})}. \quad (3.3)$$

**Proof:** Let $N \neq p$. Let $0 < r_1 < r_2$. Let us consider $\phi(r) = C_2 r^\lambda + C_1$ with $C_2$ and $C_1$ such that $\phi(r_1) = m(r_1)$ and $\phi(r_2) = m(r_2)$. It is easy to see that

$$\phi(r) = \frac{m(r_2)(r^\lambda - r_1^\lambda) + m(r_1)(r_2^\lambda - r_1^\lambda)}{r_2^\lambda - r_1^\lambda}.$$ 

Obviously $\phi > 0$ and for $i = 1$ and $i = 2$, $u(x) \geq m(r_i) = \phi(r_i)$ for $x \in \partial B_{r_i}$, hence $u$ and $\phi$ satisfy the conditions of Remark 3.2, and $u(x) \geq \phi(|x|)$ in $B_{r_2} \setminus B_{r_1}$. Taking the infimum we obtain that $\inf_{|x| = r} u(x) \geq \phi(r)$ for $r \in [r_1, r_2]$. By the minimum principle $m(r) = \inf_{|x| = r} u(x)$. This ends the proof of the first part of proposition 3.4.

For $N = p$ consider

$$\psi(r) = \frac{m(r_1) \log(\frac{r}{r_2}) + m(r_2) \log(\frac{r}{r_1})}{\log(\frac{r_1}{r_2})}.$$ 

Remark that $\Delta_N \psi = 0$ and $\psi(r_1) = m(r_1)$ and $\psi(r_2) = m(r_2)$. Now proceed as above.

**Remark 3.5** Clearly if $\lambda < 0$ i.e. $p < N$, then $g(r) := m(r)r^{-\lambda}$ is an increasing function. Just observe that $r_1^\lambda - r_2^\lambda \geq 0$ and let $r_2$ tend to $+\infty$ in (3.2) and one obtains for $r \geq r_1$:

$$m(r) \geq \frac{m(r_1)r_1^\lambda}{r_1^\lambda}.$$ 

**Proof of Theorem 3.1** We suppose by contradiction that $u \neq 0$ in $\mathbb{R}^n$, but since $u \geq 0$ by the strict maximum principle of Vasquez [17] we get that $u > 0$.

Let $0 < r_1 < R$, define $g(r) = m(r_1) \left\{1 - \frac{(r-r_1)^{k+1}}{(R-r_1)^{k+1}}\right\}$ with $k$ such that

$$k \geq 3 \quad \text{and} \quad \frac{1}{k} < p - 1.$$ 

Let $\zeta(x) = g(|x|)$. Clearly for $|x| < r_1$, $u(x) > m(r_1) = \zeta(x)$ while for $|x| \geq R$, $\zeta(x) \leq 0 < u(x)$. On the other hand there exists $\tilde{x}$ such that $|\tilde{x}| = r_1$ and $u(\tilde{x}) =$
\[ \zeta(\bar{x}). \] Hence the minimum of \( u(x) - \zeta(x) \) occurs for some \( \bar{x} \) such that \( |\bar{x}| = \bar{r} \) with 
\[ r_1 \leq \bar{r} < R. \]

Let \( |x| = r \), it is an easy computation to see that for \( r \geq r_1 \)
\[ -\Delta_p \zeta(x) = \left( \frac{(k+1)m(r_1)}{(R-r_1)^{k+1}} \right)^{(p-1)} \left[ 2(p-1) + (N-1) \left( \frac{r-r_1}{r} \right)^+ \right] \left( (r-r_1)^+ \right)^{kp-(k+1)}. \]

Clearly with our choice of \( k \), \( kp - (k+1) > 0 \) and hence, for \( |x| = r_1 \), \( -\Delta_p \zeta(x) = 0 \) while, of course, \( \nabla \zeta(x) = 0 \).

Now we have two cases.

First case \( \bar{r} = r_1 \). This implies
\[ u(\bar{x}) - m(r_1) = u(\bar{x}) - \zeta(\bar{x}) \leq u(x) - \zeta(x) \]
for all \( x \). In particular choosing \( x = \tilde{x} \), one gets
\[ u(\bar{x}) - m(r_1) \leq u(\tilde{x}) - \zeta(\tilde{x}) = 0. \]

Finally
\[ u(\bar{x}) = m(r_1) \]
and \( \tilde{x} \) is a minimum for \( u \) on \( B(0, r_1) \). Since \( -\Delta_p u \geq 0 \), Hopf’s principle as stated in Vasquez [17] implies that \( \nabla u(\tilde{x}) \neq 0 \). On the other hand \( \nabla u(\bar{x}) = \nabla \zeta(\bar{x}) = 0 \), a contradiction.

Second case : \( r_1 < \bar{r} < R \). Now \( \nabla \zeta(\bar{x}) \neq 0 \), and using Proposition 3.3 one has
\[ h(\bar{x})u^q(\bar{x}) \leq -\Delta_p \zeta(\bar{x}). \]

We choose \( r_1 \) and \( R \) sufficiently large in order that \( h(x) = a|x|^\gamma \) for \( |x| \geq \min(r_1, \frac{R}{2}) \).

Combining this with (3.4), we obtain
\[ a\bar{r}^\gamma m(\bar{r})^q \leq a\bar{r}^\gamma u^q(\bar{x}) \leq (k+1)^{(p-1)}(N+2p-3)m(r_1)^{(p-1)}(R-r_1)^{-p}. \]

Since \( m \) is decreasing we have obtained for some constant \( C > 0 \)
\[ m(R) \leq Cm(r_1)^{\frac{(p-1)}{q}} \left( \frac{R}{2} \right)^{-\frac{\gamma}{q}} (R-r_1)^{-\frac{p}{q}}. \]

Now we choose \( r_1 = \frac{R}{2} \), we use Remark 3.3 and the previous inequality becomes
\[ m(R) \leq Cm(R)^{\frac{(p-1)}{q}} \left( \frac{R}{2} \right)^{-\frac{p-\gamma}{q}}, \]
which implies

\[ m(R)R^{-\lambda} \leq CR^{-\lambda - \frac{p+\gamma}{q-p+1}}. \]  \hspace{1cm} (3.5)

Clearly \(-\lambda - \frac{p+\gamma}{q-p+1} = \frac{N-p}{p-1} - \frac{p+\gamma}{q-p+1} \leq 0\) when \(q \leq \frac{(N+\gamma)(p-1)}{N-p}\).

If \(q < \frac{(N+\gamma)(p-1)}{N-p}\) we have reached a contradiction since the right hand side of (3.5) tends to zero for \(R \to +\infty\) while the left hand side is an increasing positive function as seen in Remark 3.5.

We now treat the case \(q = \frac{(N+\gamma)(p-1)}{N-p}\). Let us remark that for this choice of \(q\) we have that for some \(C_1 > 0\), \(c > 0\) and \(r > r_1 > 0\), with \(r_1\) large enough:

\[-\Delta_p u \geq ar^\gamma u^q \geq C_1 r^{-N} \text{ since } m(r) \leq cr^{\frac{p-n}{p-1}}. \]  \hspace{1cm} (3.6)

We choose \(\psi(x) = g(|x|)\) with

\[ g(r) = \gamma_1 r^{\frac{p-N}{p-1}} \log^\beta r + \gamma_2 \]

where \(\gamma_1\) and \(\gamma_2\) are two positive constants such that for some \(r_1 > 1\) and some \(r_2 > r_1\):

\[ m(r_2) = g(r_2), \]
\[ m(r_1) \geq g(r_1), \]

while \(\beta\) is a positive constant to be chosen later. It is easy to see that

\[ \Delta_p \psi = |\gamma_1|^{p-1} r^{-N} \left| \frac{p-N}{p-1} \log^\beta r + \beta \log^{\beta-1} r \right|^{p-2} \cdot \left[ (p-1)\beta(\beta-1) \log^{\beta-2} r - \beta(3N-2p-2) \log^{\beta-1} r \right] \]

Suppose now that \(p > 2\), and choose \(0 < \beta < \frac{1}{p-1} < 1\), then there exists \(C > 0\) such that

\[ \Delta_p \psi \geq -|\gamma_1|^{p-1} C r^{-N} (\log r)^{\beta(p-1)-1} \geq -|\gamma_1|^{p-1} C r^{-N} (\log r_1)^{\beta(p-1)-1}. \]

On the other hand for \(p \leq 2\) we can choose \(\beta = 1\) and a calculation similar to the one above implies that

\[ \Delta_p \psi \geq -c|\gamma_1|^{p-1} r^{-N} (\log r_1)^{p-2}. \]
In both cases we can choose $\gamma_1$ small enough to get

$$\Delta_p \psi \geq -C_1 r^{-N} \geq \Delta_p u.$$  

Since $u \geq \psi$ on the boundary of $B_{r_2} \setminus B_{r_1}$, one obtains by the comparison principle (Remark 3.2) that $u \geq \psi$ everywhere in $B_{r_2} \setminus B_{r_1}$. When $r_2$ goes to infinity it is easy to see that $\gamma_2 \to 0$, and we obtain

$$u(x) \geq c |x|^{\frac{p-N}{p-1}} \log^\beta |x|,$$

for $|x| \geq r_1$. This implies that

$$m(r) \geq cr^{-N} \log r$$

for $r > r_1$. We have reach a contradiction since

$$m(r) \leq Cr^{-N}.$$  

Hence $u \equiv 0$. This concludes the proof of Theorem 3.1.

We treat now the case $N \leq p$ where the result is much stronger.

**Theorem 3.6** Let $N \leq p$. If $u \in W^{1,p}_{loc}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ is bounded below and is a weak solution of

$$-\Delta_p u \geq 0 \text{ in } \mathbb{R}^N$$

then $u$ is constant.

**Remark 3.7** For $N \leq p$, for any $q > 0$ and for any nonnegative $h$, if $u \in W^{1,p}_{loc}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ is a weak solution of

$$-\Delta_p u \geq h(x)u^q \text{ in } \mathbb{R}^N$$

then $u \equiv 0$.

**Proof of Theorem 3.6.** Without loss of generality we can suppose that $u \geq 0$. First we will consider $N < p$. Let $m(r) = \inf_{x \in B_r(0)} u(x)$. From Proposition 3.4 we know that for $0 < r_1 < r < r_2$

$$m(r) \geq \frac{m(r_1)(r_2^\lambda - r_1^\lambda) + m(r_2)(r_\lambda^\lambda - r_1^\lambda)}{r_2^\lambda - r_1^\lambda}, \quad (3.7)$$
where \( \lambda = \frac{p-N}{p-1} > 0. \)

If we let \( r_2 \to +\infty \) inequality (3.7) becomes

\[
m(r) \geq m(r_1).
\]

But of course \( m(r) \) is decreasing hence (3.8) implies that \( m(r) \) is constant i.e. \( m(r) = m(0) = u(0) \) for any \( r > 0 \). Clearly this can be repeated with balls centered in any point of \( \mathbb{R}^N \). Hence \( u \) is constant.

For the case \( N = p \) just use inequality (3.3) in Proposition 3.4 and proceed as above.

This concludes the proof of Theorem 3.6.

3.2 Counterexample

We are going to show that for \( N > p \), for \( \gamma \geq 0 \) and for any \( q > \frac{(N+\gamma)(p-1)}{N-p} \) there exists a non-negative function \( u \) such that

\[
-\Delta_p u \geq r^\gamma u^q \text{ in } \mathbb{R}^N
\]

hence proving that \( \frac{(N+\gamma)(p-1)}{N-p} \) is an optimal upper bound for \( q \) in Theorem 3.1.

Indeed consider \( g(r) = C(1 + r)^{-\alpha} \) with \( \alpha \) and \( C \) two positive constants to be determined. Clearly \( \Gamma(x) = g(|x|) \) satisfies

\[
-\Delta_p \Gamma = C^{p-1} \alpha^{p-1} (1+r)^{-(\alpha+1)(p-2)} \left[-(\alpha+1)(p-1)(1+r)^{-(\alpha+2)} + \right. \\
+ \frac{(N-1)}{r} (1+r)^{-(\alpha+1)} \\
\geq C^{p-1} \alpha^{p-1} (1+r)^{-\alpha(p-1)-p} \left[N-1-(\alpha+1)(p-1)\right]
\]

with \( r = |x| \).

Now let \( \epsilon > 0 \) such that \( q = \frac{(N+\gamma-\epsilon)(p-1)}{(N-p-\epsilon)} \) and let \( \alpha = \frac{N-p-\epsilon}{p-1} \). Clearly we have \( \alpha(p-1) + p + \gamma = N + \gamma - \epsilon = \alpha q \). Furthermore \( N-1-(\alpha+1)(p-1) = N-p-\alpha(p-1) = \epsilon + \gamma > 0 \). Hence choosing \( C \) such that \( C^{p-1} \alpha^{p-1}(\epsilon + \gamma) = C^q \) we obtain that \( \Gamma(x) \) satisfies

\[
-\Delta_p \Gamma \geq C^q (1+r)^{\gamma(1+r)^{-\alpha(p-1)-p-\gamma}} \geq r^\gamma \Gamma^q \text{ in } \mathbb{R}^N.
\]
3.3 The equation

In this section we are interested in studying non-existence results concerning the equation. Clearly in view of Theorem 3.6, we are only interested in the case \( N > p \):

**Theorem 3.8** Suppose that \( u \in W^{1,p}_{\text{loc}}(\mathbb{R}^N) \) is nonnegative and satisfies

\[
-\Delta_p u = r^\gamma u^q,
\]

for some \( \gamma \geq 0 \). If

\[
p - 1 < q \leq \frac{(N + \gamma)(p - 1) + p + \gamma}{N - p}
\]

and \( u \) is radial then \( u \equiv 0 \).

**Remark 3.9** One can get the same result for \( -\Delta_p u = Cr^\gamma u^q \) by considering \( u \) multiplied by some convenient constant.

The proof given here is similar to the one given by Caffarelli, Gidas and Spruck in [3].

**Proof.**

It is sufficient to consider the case \( q \geq \frac{(N+\gamma)(p-1)}{N-p} \), since the other cases are proved in Theorem 3.1.

If \( u \) is a radial solution and satisfies (3.9) in a weak sense, then it is not difficult to see that it satisfies in the weak sense

\[
-(r^{N-1}|u'|^{p-2}u')' = r^{N-1+\gamma}u^q.
\]

Integrating between 0 and \( r \), one has

\[
r^{N-1}|u'|^{p-2}u' = - \int_0^r s^{N-1+\gamma}u^q(s)ds.
\]

Since \( u' < 0 \), \( u \) is decreasing and then,

\[
r^{N-1}|u'|^{p-2}u' \leq -u(r)^q \frac{r^{N+\gamma}}{N+\gamma}.
\]

Hence

\[
u'u^{\frac{-q}{p-1}} \leq -cr^{\frac{1+\gamma}{p-1}}
\]

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and integrating one gets

\[ u(r) \leq Cr^{\frac{\gamma+p}{p-1-q}}. \]

Coming back to the equation one obtains

\[ r^{N-1}|u'|^{p-1} = \int_0^r s^{N-1+\gamma}u^q(s)\,ds \leq C \int_0^r s^{N-1+\gamma}s^{\frac{(\gamma+p)q}{p-1-q}}\,ds. \]

Clearly \( N + \gamma + \frac{(\gamma+p)q}{p-1-q} \geq 0 \) when \( q \geq \frac{(N+\gamma)(p-1)}{N-p} \) and therefore

\[ |u'(r)|^{p-1} \leq C r^{\gamma+\frac{(\gamma+p)q}{p-1-q}+1} \]

and then

\[ |u'| \leq C r^{\frac{(\gamma+q+1)}{p-1-q}}. \]

In order to conclude, we need to use Pohozaev identity:

\[
(N - p) \int_B |\nabla u|^p + p \int_{\partial B} \sigma . n(\nabla u) \cdot x (\Delta u \cdot x) + \int_{\partial B} |\nabla u|^p \cdot (x, \vec{n})
\]

here \( \sigma = |\nabla u|^{p-2} \nabla u \) and \( B = B(0, R) \). From the equation we know that

\[
\int_B |\nabla u|^p - \int_{\partial B} (\sigma . \vec{n}) = \int_B r^\gamma u^{q+1}
\]

and then

\[
(N - p) \left( \int_B r^\gamma u^{q+1} + \int_{\partial B} \sigma . \vec{n}u \right) + p \int_{\partial B} (\sigma . \vec{n})(\nabla u . x)
\]

\[
= -p \int_B r^\gamma u^q(\nabla u . x) + \int_{\partial B} |\nabla u|^p x . \vec{n}. \quad (3.10)
\]

Using the fact that \( u \) is radial, for \( \omega_n = |B_1| \) one gets

\[
\frac{1}{\omega_n} \int_{B_R} r^\gamma u^q \nabla u \cdot x \,dx = \int_0^R r^{\gamma+N-1} u^q(r) u'(r)dr
\]

\[
= \int_0^R r^{\gamma+N-1} \frac{u^{q+1}(r)}{q+1}dr
\]

\[
= -\frac{\gamma+N}{q+1} \int_0^R r^{\gamma+N-1} u^{q+1} + \frac{1}{q+1} R^{\gamma+N} u^{q+1}(R). \]

We have finally obtained

\[
(N - p - \frac{(\gamma+N)p}{q+1}) \int_0^R r^{\gamma+N-1} u^{q+1}dr = (N - p)|u'|^{p-1}ur^{N-1} + (1 - p)|u'|^p r^N
\]

\[
- \frac{p}{q+1} r^{\gamma+N} u^{q+1}. \]
Let us note that since \( q < \frac{(N+\gamma)(p)+p-N}{N-p} \), one has
\[
\frac{(\gamma + N)p}{q + 1} + p - N > 0.
\]

Moreover the estimates on \( u \) and \( u' \) imply that the terms \( |u'|^{p-1}u(R)R^{N-1} \), \( |u'|^p(R)R^N \) and \( R^{\gamma+N}u^{q+1}(R) \) behave respectively as \( R^{N-1+\frac{2p}{p+1}+\frac{(\gamma+2+1)(p-1)}{p-1}} \), \( R^{\gamma+N+\frac{2p}{p+1}(q+1)} \) and \( R^{N-p\left(\frac{\gamma+2+1}{q-p+1}\right)} \). All the exponents are negative, and then \( \int_0^R r^{\gamma+N-1}u^{q+1}dr \to 0 \) when \( R \to +\infty \), hence \( u \equiv 0 \). This conclude the proof.

4 The negative semi-linear case

In this section we prove analogous results for inequations in which \( \Delta_p \) is replaced
by \( -\Delta_p \), in this case no upper bound for \( q \) is required.

**Theorem 4.1** Suppose that \( u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n) \cap C(\mathbb{R}^N) \) is a weak nonnegative solution
of
\[
-\Delta_p u + h(x)u^q \leq 0,
\]
where \( h \) satisfies (1.2). If \( q > p - 1 \), then \( u \equiv 0 \).

**Proof:** In a first step, we prove the result for \( q < \frac{N(p-1)+(\gamma+1)p}{N-p} \).

Let us multiply (1.1) by \( u\zeta^\alpha \) where \( \zeta \) is some nonnegative cut-off function,
supported in \( B(0,2R) \) and equals 1 on \( B(0,R) \), where \( R \) is large enough to have
\( h(x) = a|x|^{\gamma} \) for \( |x| \geq R \). One may choose such function with in addition \( |\nabla \zeta|_\infty \leq \frac{C}{R} \).

After integrating by parts, (1.1) becomes
\[
\int_{B_{2R}} |\nabla u|^p \zeta^\alpha + \alpha \int_{B_{2R}} |\nabla u|^{p-2} \nabla u.\nabla \zeta u^{\alpha-1} + \int_{B_{2R}} h u^{q+1} \zeta^\alpha \leq 0. \tag{4.2}
\]

Using Holders’ inequality on the second integral one gets
\[
|\int_{B(0,2R)\setminus B(0,R)} |\nabla u|^{p-2} \nabla u.\nabla \zeta u^{\alpha-1}| \leq \alpha \left( \int_{B(0,2R)\setminus B(0,R)} |\nabla u|^p \zeta^\alpha \right)^{\frac{p-1}{p}} \cdot \\
\left( \int_{B(0,2R)\setminus B(0,R)} |\nabla \zeta|^p u^p \zeta^{\alpha-p} \right)^{\frac{1}{p}}. \tag{4.3}
\]
Once more by Holders’ inequality one has:

\[
\int_{B(0,2R)\setminus B(0,R)} |\nabla \zeta|^{p} u^{p} \zeta^{\alpha - p} \leq \left( \int_{B(0,2R)\setminus B(0,R)} u^{q+1} h \zeta^{\alpha} \right)^{\frac{p}{q+1}}.
\]

Choosing \( \alpha > \frac{p(q+1)}{q+1-p} \), it is easy to see that

\[
\int_{B(0,2R)\setminus B(0,R)} |\nabla \zeta|^{\frac{p(q+1)}{q+1-p}} \zeta^{\frac{\alpha-p(q+1)}{q+1-p}} h^{\frac{-p}{q+1-p}} \leq CR^{N - \frac{\gamma p + p(q+1)}{q+1-p}}.
\]

Defining

\[
I_{R} = \int_{B_{2R}} |\nabla u|^{p} \zeta^{\alpha}
\]

and

\[
J_{R} = \int_{B_{2R}} h \zeta^{\alpha} u^{q+1}.
\]

Inserting (4.3), (4.4), (4.5) in (4.2), we have obtained that

\[
I_{R} + J_{R} \leq C \left( R^{N - \frac{\gamma p + p(q+1)}{q+1-p}} \right)^{\frac{1}{q+1}} I_{R}^\frac{p-1}{q+1} J_{R}^\frac{1}{q+1}.
\]

Now choose \( \beta = 1 + \frac{p}{(q+1)(p-1)} \) and \( \delta := \beta^{\left(\frac{p-1}{q+1}\right)} = \beta'(\frac{1}{q+1}) \), then using Young inequality, (4.6) becomes:

\[
I_{R} + J_{R} \leq CR^{\left( N - \frac{\gamma p + p(q+1)}{q+1-p} \right) \frac{q+1-p}{q+1-p}} \left( \frac{1}{\beta} I_{R}^{\delta} + \frac{1}{\beta'} J_{R}^{\delta} \right)
\]

It is easy to see that \( \delta < 1 \) when \( p < q+1 \), and furthermore that \( N - \left( \frac{\gamma p + p(q+1)}{q+1-p} \right) < 0 \) when \( q < \frac{(N+\gamma)(p-1)+p+\gamma}{N-p} \).

This concludes the proof, just let \( R \to +\infty \) and then \( I_{R} + J_{R} \to 0 \) which implies that \( u \equiv 0 \).

In a second step, we observe that if \( u \) satisfies (4.1), then \( u^{\alpha} \) with \( \alpha \geq 1 \), is a solution of

\[
-\Delta_{p}(u^{\alpha}) + \alpha^{p-1} h(u^{\alpha}) \frac{\alpha^{(\alpha-1)(p-1)}}{\alpha} \leq 0.
\]
More precisely

\[-\Delta_p(u^\alpha) + \alpha^{p-1}hu^{q+(\alpha-1)(p-1)} \leq -\alpha^{p-1}(\alpha - 1)(p - 1)u^{(\alpha-1)(p-1)-1}|\nabla u|^p.\]

This can be seen by taking \(\tilde{\varphi} = \alpha^{p-1}u^{(\alpha-1)(p-1)}\varphi, \varphi \in \mathcal{D}(\mathbb{R}^N), \varphi \geq 0\) as test function in the equation

\[-\Delta_p u + hu^q \leq 0.\]

As a consequence one has \(u \equiv 0\) as soon as \(\frac{q+(\alpha-1)(p-1)}{\alpha} \leq \frac{(N+\gamma)(p-1)+p+\gamma}{N-p}\). This will be always possible for \(\alpha\) large since

\[\lim_{\alpha \to +\infty} \frac{q+(\alpha-1)(p-1)}{\alpha} = p - 1 < \frac{(N+\gamma)(p-1)+p+\gamma}{N-p}\]

for \(\gamma > -p\). Finally for any \(q > p - 1\) there exists a power \(\alpha\) such that \(u^\alpha \equiv 0\), hence \(u \equiv 0\).

**Acknowledgements** This work was mainly done while the first author was visiting the Laboratoire d’Analyse, Géométrie et Modélisation of the University of Cergy-Pontoise, she wishes to thank the people of the laboratoire for the kind invitation and their welcome.

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