Error Processing of Sparse Identification of Nonlinear Dynamical Systems via $L_\infty$ Approximation

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Sparse identification of nonlinear dynamical systems (SINDy) is a recently presented framework in the reverse engineering field. It soon gains general interests due to its interpretability and efficiency. Error processing, as an important issue in the SINDy framework, yet remains to be an open problem. To date, literature about error processing focuses on data processing methods which aim to improve the accuracy of data. However, the relationship between data and the identification framework is largely ignored. In this paper, error processing is studied from an optimization perspective. In detail, $L_\infty$ approximation is introduced to the objective function in SINDy framework in place of the former $L_2$ approximation. This is especially appropriate for dealing with the derivative approximation error in SINDy because the derivative approximation error has no exact distribution. To verify the effectiveness of $L_\infty$ approximation, identification scenarios with different types of derivative approximation error are tested. The results indicate that $L_\infty$ approximation could become an alternative of $L_2$ approximation especially when lacking prior knowledge of derivative approximation error. The performances of $L_\infty$ approximation and $L_2$ approximation are evaluated in the cases where the measurement noise of system state is considered. Experimental results show that $L_\infty$ approximation has equal performance compared to $L_2$ approximation under the assumption of Gaussian measurement noise, which is promising in applications.

I. INTRODUCTION

Reverse engineering, referring to extracting certain information from observed data of an unknown system, is an important topic in physics\cite{1} and many other disciplines\cite{2, 3}. With various data, models and tasks, a lot of mathematical methods developed in data science have been employed to reverse engineering such as statistical models\cite{4}, information theory\cite{5} and deep learning\cite{6}. Although these methods largely enrich the knowledge of system modeling, they conclude with black-box models of the system, which has no capability of revealing inner mechanisms in physical level. Towards better interpretability of the identified model, sparse identification of nonlinear dynamical systems (SINDy) was presented\cite{7} which searches for a parsimonious mathematical expression of the unknown system. In detail, at first, a large dictionary containing plenty of possible items of system dynamics is constructed according to professional information or prior knowledge. Then, regression is performed with additional sparsity to select few proper items. Because SINDy model owns interpretability and the algorithm is of high efficiency, it rapidly becomes popular and gains researchers’ attention\cite{8-12}.

Although SINDy successfully discovers many systems governed by different types of differential equations, this framework has an intrinsic defect that the successful discovery relies greatly on the accuracy of data, mainly consists of the system states and the approximated derivatives\cite{7}. Hence, error processing becomes a fairly important task in the SINDy framework. In this field, recent studies mainly focus on advanced numerical difference method\cite{13, 14}, which aims at improving the accuracy of approximating derivative. Despite modification from data perspective, it is still unknown whether the modified data is closer to the true value or not. Obviously, it is far from enough to study error processing only in data aspect.

In this paper, error processing is tackled from an optimization perspective. Specifically, $L_\infty$ approximation is introduced to the objective function in place of the former $L_2$ approximation. The motivation is that $L_\infty$ approximation bounds the fitting error (or the residual) in an interval without considering its distribution, which is intuitively more appropriate to describe derivative approximation error in general conditions. For evaluating the performance of $L_\infty$ approximation, several identification scenarios are considered in this paper with the change of derivative approximation error. The results of $L_2$ approximation and $L_\infty$ approximation are compared and some conclusions are drawn. Moreover, further issues of $L_\infty$ approximation in applications are discussed. The remaining parts of this paper are organized as follows. Section II introduces SINDy and $L_\infty$ approximation. Section III thoroughly shows the comparative experiments of $L_2$ approximation and $L_\infty$ approximation in different derivative approximation error scenarios. Section IV talks about $L_\infty$ approximation in the presence of the measurement noise of system states. Section V concludes this paper.
II. PROBLEM STATEMENT

A. SINDy framework

For convenience, the dynamical system of 1-D ordinary differential equation (ODE) is considered, which is described as follow:

\[ \frac{dx}{dt} = f(x), \]  

(1)

where \( x \in \mathbb{R} \) is the system state and \( f \) is the unknown system dynamics. Assume that only the observation of \( x \) is available and the derivatives of the state, \( \dot{x} \) is derived from \( x \) through numerical difference methods. In the time scale \([t_1, t_n]\), they are written as follow:

\[ x(t) = [x(t_1) \ x(t_2) \ldots x(t_n)]^T, \]  

(2)

and

\[ \dot{x}(t) = [\dot{x}(t_1) \ \dot{x}(t_2) \ldots \dot{x}(t_n)]^T. \]  

(3)

After the data is obtained, a dictionary of possible \( f \) items is constructed according to the prior knowledge of the system. For example, a polynomial dictionary is written as follow:

\[ \Theta(t) = \begin{bmatrix} 1 & x(t_1) & x^2(t_1) & \ldots & x^m(t_1) \\ 1 & x(t_2) & x^2(t_2) & \ldots & x^m(t_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x(t_n) & x^2(t_n) & \ldots & x^m(t_n) \end{bmatrix}. \]  

(4)

Then, the sparse regression is performed. The sparse weight vector \( \xi \) is written as follow:

\[ \xi = [\xi_1 \ \xi_2 \ \ldots \ \xi_m]^T. \]  

(5)

And the optimization problem is formulated as follow:

\[ \xi^* = \arg \min \| \tilde{x}(t) - \Theta(t) \xi \|_2 + \lambda \| \xi \|_0, \]  

(6)

where \( \xi^* \) is the optimal solution and \( \lambda \) is the regularization factor which controls the level of sparsity. Note that, equation (6) could be directly solved by SINDy algorithm [7] or be relaxed to the least absolute shrinkage and selection operator (LASSO) problem [12] and then solved by well-studied algorithms. Finally, according to \( \xi^* \), the relative items in the dictionary are consulted and the governing equation is recovered.

B. \( L_2 \) and \( L_\infty \)

In the following, \( L_2 \) norm and \( L_2 \) approximation are introduced with comparison of \( L_\infty \) norm and \( L_\infty \) approximation. For a vector \( x \in \mathbb{R}^n \), its \( L_2 \) norm and \( L_\infty \) norm are defined as follow:

\[ \|x\|_2 = (x_1^2 + x_2^2 + \ldots + x_n^2)^{\frac{1}{2}}, \]  

(7)

FIG. 1. Distribution of \( r \). The curve in (a) represents the normal distribution and the shadow in (b) means that the distribution is unknown in such area.

\[ \|x\|_\infty = \max(|x_1|, |x_2|, \ldots, |x_n|). \]  

(8)

In equation (6), \( L_2 \) norm is adopted to measure the distance between \( \tilde{x}(t) \) and \( \Theta(t) \xi \). Minimizing \( L_2 \)-norm distance in equation (6) is named \( L_2 \) approximation [? ]. Similarly, using \( L_\infty \) norm to measure the distance between \( \tilde{x}(t) \) and \( \Theta(t) \xi \) in equation (6) is named \( L_\infty \) approximation. Define the distance between \( \tilde{x}(t) \) and \( \Theta(t) \xi \) as the residual \( r \) as follow:

\[ r = \| \tilde{x}(t) - \Theta(t) \xi \| = [r_1 \ r_2 \ \ldots \ r_n]^T. \]  

(9)

Then define the derivative approximation error \( e \) as follow:

\[ e = \tilde{x} - \dot{x}, \]  

(10)

where \( \dot{x} \) is the true derivative of the system state. If \( e \) is known of normal distribution \( \mathcal{N}(0, \sigma^2) \), \( L_2 \) approximation can exactly reveal \( r \) through dividing \( e \) into \( r \) by \( \mathcal{N}(0, \sigma^2) \). However, \( e \) is unknown of distribution in general conditions. In such cases, the revealed \( \xi \) of equation (6) is inaccurate. While in \( L_\infty \) approximation, \( r \) is bounded in the interval of \([-r_{max}, r_{max}]\), in which \( r_{max} \) is the maximum of \(|r|\). Here, the specific form of the distribution of \(|r|\) is unclear. In other words, it can be an arbitrary distribution. This characteristic gives a better description of the derivative approximation error. An intuitive illustration of \( L_2 \) approximation and \( L_\infty \) approximation is shown in FIG. 1.

C. \( L_\infty \) approximation formulation

According to the discussions of \( L_2 \) approximation and \( L_\infty \) approximation, it is natural to introduce \( L_\infty \) approximation to equation (6) in place of \( L_2 \) approximation. Hence, the problem can be reformulated as follow:

\[ \xi^* = \arg \min \| \tilde{x}(t) - \Theta(t) \xi \|_\infty + \lambda \| \xi \|_0. \]  

(11)

Equation (11) is known as NP-hard problem, which has no rigorous mathematical solution. For efficiency, particle swarm optimization (PSO) [16, 17] is employed in
III. DEMONSTRATION EXAMPLE

To verify the effectiveness of \( L_\infty \) approximation, 3 different derivative approximation error scenarios are considered. Note that in this section, the measurement noise of system state \( x \) is not considered. Simulations are based on Lorenz system with the formulation:

\[
\begin{align*}
\dot{x} &= \sigma (y - x) \\
\dot{y} &= x (r - z) - y \\
\dot{z} &= xy - yz
\end{align*}
\] (12)

where \( \sigma = 10 \), \( r = 28 \), and \( \beta = 8/3 \). Initial state \( x(0) \) is \([-8, 8, 27]^T\) in the simulation of identification stage while \([1, 1, 1]^T\) in the simulation of reconstruction stage. Time interval \( \Delta t \) keeps the same in both two stages, and time scale is \([0, 50s]\) in all experiments.

A. Scenario: measurable derivative \( \dot{x} \)

Usually, derivative \( \dot{x} \) could not be obtained directly from observation. This scenario considers measurable derivative \( \dot{x} \) because it provides a clear explanation on the difference between \( L_2 \) approximation and \( L_\infty \) approximation. Assuming derivative \( \dot{x} \) is measurable and the measurement noise \( \varepsilon \) satisfies normal distribution \( \mathcal{N}(0, \sigma^2) \). Different noise levels are tested and the results of the reconstruction error are shown in Table I. In the noise-free case, error indicators of \( L_2 \) approximation are close to 0 and pretty smaller than \( L_\infty \) approximation. In other cases, error indicators of \( L_2 \) approximation and \( L_\infty \) approximation are close to each other and \( L_2 \) approximation performs slightly better than \( L_\infty \) approximation in most cases.

B. Scenario: different sampling interval \( \Delta t \)

Derivative \( \dot{x} \) needs to be approximated from state \( x \). In such a situation, sampling interval \( \Delta t \) plays an important role in the approximation since it controls the accuracy of approximation. For simplicity, the central difference technique is used here as an example which is a basic technique in derivative approximation. Different \( \Delta t \) is tested with reconstruction error as shown in Table II. It is observed that \( L_\infty \) approximation has a better performance in most cases.

C. Scenario: different derivative approximation techniques

In scenario B, the central difference is adopted as the derivative approximation technique. In this section, different derivative approximation techniques are tested with reconstruction error as shown in Table III.

D. Summary

In summary, from Table I, Table II and Table III, the performances of \( L_2 \) approximation and \( L_\infty \) approximation are roughly on the same level because their error indicators are very close. \( L_2 \) approximation performs better than \( L_\infty \) approximation when derivative approximation error is more likely to be normal distribution. \( L_\infty \) approximation performs better than \( L_2 \) approximation when distribution of derivative approximation error is unknown, which is the general scenario in applications. Hence, it can be concluded that \( L_\infty \) approximation could become an alternative of \( L_2 \) approximation. Further, according to the illustration of \( L_2 \) and \( L_\infty \) in section II, \( L_\infty \) approximation is regarded as the baseline formulation of the identification problem. \( L_2 \) approximation is a modified version which additionally uses prior information of the derivative approximation error. The use of such extra information may induce more accurate results but may also induce less accurate results because it is actually unknown of the derivative approximation error. Hence, it is advisable to use \( L_\infty \) approximation in place of \( L_2 \) approximation when there is no prior knowledge of the derivative approximation error.

IV. MEASUREMENT NOISE OF \( x \)

Although the effectiveness of \( L_\infty \) approximation has been verified, there is still a gap towards application. Especially, in applications of the SINDy framework, the measurement noise \( \nu \) of the system state \( x \) always exists. When \( \nu \) is considered, the identification problem becomes more complex in theoretical aspect. Here, simulation experiments are carried out to test the performance of \( L_\infty \) approximation and \( L_2 \) approximation in noisy environment. For simplicity, the measurement noise is assumed to be additive \( \nu \sim \mathcal{N}(0, \sigma^2) \). Lorenz system and Chen system are tested with different levels of \( \sigma \). Lorenz system is formulated as equation (12) and Chen system is formulated as follow:

\[
\begin{align*}
\dot{x} &= a(y - x) \\
\dot{y} &= (c - a)x + cy - xz \\
\dot{z} &= xy - bz
\end{align*}
\] (13)

where \( a = 35 \), \( b = 3 \), and \( c = 28 \). Results are displayed in Table IV and Table V. It shows that \( L_\infty \) approximation and \( L_2 \) approximation almost have the equal perfor-
TABLE I. Reconstruction results in different noise levels in scenario A. The bold number is the better in the comparison between $L_2$ approximation and $L_\infty$ approximation. Note that the sampling interval is 0.01s.

| $\sigma$ | Sub-system 1 | Sub-system 2 | Sub-system 3 |
|----------|--------------|--------------|--------------|
|         | RMSE         | STD          | RMSE         | STD          | RMSE         | STD          |
| 0        | $0.2881$ 8.6895 | $0.2879$ 8.4919 | $0.4577$ 9.8529 | $0.4575$ 9.6780 | $0.5917$ 9.4325 | $0.5917$ 9.4244 |
| 0.001    | $0.5832$ 8.5985 | $0.5772$ 8.5786 | $0.9762$ 9.8542 | $0.9671$ 9.8343 | $0.9415$ 10.6614 | $0.9413$ 10.6622 |
| 0.005    | $0.6649$ 8.9438 | $0.3725$ 8.8966 | $0.9510$ 10.1695 | $0.6912$ 10.1287 | $0.8295$ 7.9664 | $0.9329$ 7.9633 |
| 0.01     | $0.6558$ 8.4965 | $0.9403$ 8.4974 | $10.1527$ 9.7571 | $10.0705$ 9.7581 | $10.8942$ 10.4300 | $10.8938$ 10.4311 |
| 0.05     | $0.9147$ 10.3681 | $0.9311$ 9.7319 | $10.3336$ 11.4948 | $10.1216$ 10.9034 | $10.5555$ 11.0698 | $10.5558$ 11.0670 |
| 0.1      | $0.2816$ 10.1375 | $0.2771$ 9.9941 | $10.4860$ 11.4029 | $10.4831$ 11.2798 | $10.1537$ 11.0649 | $10.1539$ 11.0639 |
| 0.5      | $0.8260$ 9.3277 | $0.8021$ 9.1752 | $10.2290$ 10.5119 | $10.2090$ 10.3786 | $0.9271$ 10.3269 | $0.9271$ 10.3277 |
| 1       | $0.0076$ 10.3535 | $0.0073$ 10.1637 | $0.9418$ 11.6793 | $0.9416$ 11.5084 | $0.9353$ 9.7219 | $0.9352$ 9.7192 |

TABLE II. Reconstruction results with different $\Delta t$ in scenario B. The bold number is the better in the comparison between $L_2$ approximation and $L_\infty$ approximation. Note that the derivative approximation technique is the central difference.

| $\Delta t$ | Sub-system 1 | Sub-system 2 | Sub-system 3 |
|------------|--------------|--------------|--------------|
|            | RMSE         | STD          | RMSE         | STD          | RMSE         | STD          |
| 0.001      | $10.4323$ 9.8680 | $10.4321$ 9.7612 | $11.9036$ 11.0866 | $11.9030$ 11.0963 | $0.1648$ 11.1678 | $0.1648$ 11.1642 |
| 0.0025     | $0.1264$ 9.5385 | $0.0210$ 9.5243 | $10.3349$ 10.9249 | $10.2441$ 10.9135 | $10.1341$ 7.6042 | $10.1361$ 7.5950 |
| 0.005      | $0.9645$ 9.5531 | $0.9573$ 9.5087 | $10.8999$ 10.5686 | $10.8155$ 10.5302 | $0.8923$ 8.9186 | $0.8927$ 8.9179 |
| 0.0075     | $9.4223$ 9.3537 | $9.3978$ 9.2677 | $10.7415$ 10.6525 | $10.7211$ 10.5818 | $0.9629$ 9.3519 | $0.9618$ 9.3514 |
| 0.01       | $10.4789$ 9.7742 | $10.0651$ 9.6871 | $11.6054$ 11.1438 | $11.1824$ 11.0698 | $11.2347$ 10.1744 | $11.2358$ 10.1752 |
| 0.02       | $9.8337$ 9.5330 | $9.5712$ 9.5060 | $11.1490$ 10.8641 | $10.9185$ 10.8417 | $10.6882$ 10.4970 | $10.6855$ 10.4990 |

V. CONCLUSION

This paper deals with the error processing problem in the SINDy framework from an optimization perspective. Derivative approximation error is concerned and $L_\infty$ approximation is introduced to the identification problem in place of the former $L_2$ approximation. The characteristics of $L_\infty$ approximation and $L_2$ approximation are illustrated in both theoretical and experimental aspects. It shows that $L_\infty$ approximation could become an alternative of $L_2$ approximation and it is reasonable to use $L_\infty$ approximation rather than $L_2$ approximation when no prior knowledge of derivative approximation error is available. Furthermore, experiments show that $L_\infty$ approximation performs no worse than $L_2$ approximation even if the measurement noise of system state is under consideration. This shows great potential of $L_\infty$ approximation in applications.

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| Technique                  | Sub-system 1 |                  | Sub-system 2 |                  | Sub-system 3 |                  |
|---------------------------|--------------|-----------------|--------------|-----------------|--------------|-----------------|
|                           | RMSE $L_2$   | STD $L_\infty$  | RMSE $L_2$   | STD $L_\infty$  | RMSE $L_2$   | STD $L_\infty$  |
| Central difference        | 10.4789      | 9.7742          | 10.0051      | 9.6871          | 11.6054      | 11.0698         |
| Polynomial interpolation  | 9.8039       | 9.8283          | 9.7365       | 9.7220          | 11.1710      | 11.1118         |

| TABLE III. Reconstruction results with different derivative approximation techniques in scenario C. The bold number is the better in the comparison between $L_2$ approximation and $L_\infty$ approximation. Note that the sampling interval is 0.01s. |

| TABLE IV. Reconstruction results in Lorenz system considering the measurement noise $\nu$ of the system state $x$. The bold number is the better in the comparison between $L_2$ approximation and $L_\infty$ approximation. Note that the central difference is adopted. |

| $\Delta t$ | $\sigma$ | Sub-system 1 |                  | Sub-system 2 |                  | Sub-system 3 |                  |
|------------|----------|--------------|-----------------|--------------|-----------------|--------------|-----------------|
|            |          | RMSE $L_2$   | STD $L_\infty$  | RMSE $L_2$   | STD $L_\infty$  | RMSE $L_2$   | STD $L_\infty$  |
| 0.01       | 0.01     | 9.2212       | 10.0033         | 9.0940       | 11.2324        | 11.2143      | 10.4182         |
|            | 0.03     | 9.5975       | 9.3425          | 9.7365       | 10.9833        | 10.8940      | 10.7662         |
|            | 0.05     | 10.3548      | 10.2667         | 9.9072       | 11.5361        | 11.4914      | 11.4145         |
| 0.01       | 0.01     | 8.8212       | 9.4977          | 8.7741       | 11.1075        | 10.8457      | 10.0436         |
|            | 0.03     | 8.7960       | 8.5449          | 8.5759       | 10.2307        | 9.8823       | 9.8024          |
|            | 0.05     | 10.1170      | 9.6095          | 9.8105       | 11.3654        | 10.0306      | 10.0259         |
| 0.01       | 0.01     | 9.7939       | 9.4977          | 8.7741       | 11.1075        | 10.8457      | 10.0436         |
|            | 0.03     | 9.9642       | 9.5216          | 10.7341      | 11.3432        | 12.2733      | 10.9511         |
|            | 0.05     | 10.4431      | 9.0049          | 9.1982       | 10.7483        | 9.8403       | 10.4040         |

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TABLE V. Reconstruction results in Chen system considering the measurement noise $\nu$ of the system state $x$. The bold number is the better in the comparison between $L_2$ approximation and $L_\infty$ approximation. Note that the central difference is adopted.

| $\Delta t$ | $\sigma$ | Sub-system 1 | Sub-system 2 | Sub-system 3 |
|-----------|------|--------------|--------------|--------------|
|           |      | $L_2$  $L_\infty$ | $L_2$  $L_\infty$ | $L_2$  $L_\infty$ |
| 0.01      | 0.01 | 11.661 11.5986 | 11.664 11.5944 | 12.6088 12.4503 |
| 0.05      | 0.03 | 12.8201 11.8909 | 12.2007 11.8909 | 13.0935 12.7406 |
| 0.01      | 0.03 | 11.8036 11.4879 | 11.8031 11.4876 | 12.6919 12.4162 |
| 0.05      | 0.03 | 12.1168 11.5378 | 12.1175 11.5359 | 13.0866 12.4665 |
| 0.01      | 0.03 | 12.2153 11.6783 | 12.1989 11.0736 | 13.1231 12.5253 |
| 0.05      | 0.03 | 10.9731 11.6689 | 10.9688 11.6700 | 11.8033 12.5719 |
| 0.01      | 0.03 | 11.4369 11.3026 | 11.4367 11.2973 | 12.3227 12.2262 |
| 0.02      | 0.03 | 11.2958 11.6028 | 11.2831 11.6032 | 12.1547 12.4993 |
| 0.05      | 0.03 | 11.7023 11.8192 | 11.6926 11.8158 | 12.5949 12.8017 |