On Homogenized Conductivity and Fractal Structure in a High Contrast Continuum Percolation Model

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Abstract. In the previous article (S. Matsutani and Y. Shimosako and Y. Wang, Physica A 391 (2012) 5802-5809) we numerically investigated an electric potential problem with high contrast local conductivities ($\gamma_0$ and $\gamma_1$, $0 < \gamma_0 \ll \gamma_1$) for a two-dimensional continuum percolation model (CPM). As numerical results, we showed there that the equipotential curves exhibit the fractal structure around the threshold $p_c$ and gave an approximated curve representing a relation between the homogenized conductivity and the volume fraction $p$ over $[p_c, 1]$. In this article, using the duality of the conductivities and the quasi-harmonic properties, we re-investigate these topics to improve these results. We show that at $\gamma_0 \to 0$, the quasi-harmonic potential problem in CPM is quasiconformally equivalent to a random slit problem, which leads us to an observation between the conformal property and the fractal structure at the threshold. Further we extend the domain $[p_c, 1]$ of the approximated curve to $[0, 1]$ based on these results, which is partially generalized to three dimensional case. These curves represent well the numerical results of the conductivities.

1. Introduction

In the series of articles [17, 18, 19], we numerically investigated the electric potential problem with high contrast local conductivities ($\gamma_0$ and $\gamma_1$, $0 < \gamma_0 \ll \gamma_1$) for continuum percolation models (CPMs) in order to reveal the electrical properties of a real material consisting of conductive nano-particles in an insulator. We solved generalized Laplace equations with mixed boundary conditions using the finite difference method. In Ref. [19], we investigated two-dimensional case and numerically showed that the equipotential curves exhibit the fractal structure around the threshold $p_c$ and gave an approximated curve representing a relation between the homogenized conductivity and the volume fraction $p$ over $p \in [p_c, 1]$, which we call conductivity curve. The fractal structure in Ref. [19] shows the electrical properties of the system, i.e., there appear quasi-equipotential clusters there, in which the potential distribution is constant or nearly constant. On the other hands, due to the ultra high conductivity of an insulator, we have a non-vanishing homogenized conductivity even for a smaller volume-fraction $p$ than the threshold $p_c$. It is also important to determine the dependence of the homogenized conductivity on volume fraction $p$ for every $p \in [0, 1]$.

In this article, we go on to investigate the electric potential distribution on a two-dimensional continuum percolation model (CPM) [20] with a high contrast local conductivity to improve the results in Ref. [19] from more mathematical viewpoints. We consider the simplest Boolean model of CPM, in which disks of unit radius are centered at the points of the Poisson point process with density $\lambda$ [20, 23]. In order to consider the duality between the occupancy state (OS) and the vacancy state (VS), we handle both types in this article; the OS-type is set so that occupied regions have the local conductivity $\gamma_1 = 1$ and the vacant regions have infinitesimal one $\gamma_0$, whereas in the VS-type, the occupied regions have $\gamma_0$ and the vacant regions have $\gamma_1$. VS-type is sometimes called the Swiss cheese model in physics literature if $\gamma_0 = 0$.

Using the duality of two dimensional case, the purposes of this article are to find the geometric properties of the potential distribution, especially the fractal structure of the equipotential curves around the threshold,
and to extend the domain \([p_c, 1]\) of conductivity curve for the homogenized conductivity to \([0, 1]\), which leads us a novel parameterization of the conductivity curve even of three dimensional case.

The homogenized conductivity in CPM has been rigorously studied well and has a long history as recently Kontogiannis \[13\] gave a nice review of these studies and an extension. The studies from more practical viewpoints based on the numerical results are in Refs. \[11\] and \[26\]. In this article, based on these rigorous results and numerical computations, we give an improvement of the previous arguments in Ref. \[19\]. Thus the conventions and notations in this article completely differ from the previous ones \[17\], \[18\], \[19\] since they were written in the framework of physics. In general, the homogenization is assumed a periodicity of its associated system. Instead of the periodicity, we use the ergodicity of CPM to homogenize the conductivity following Refs. \[3\] and \[6\] because the Boolean model has the ergodic properties \[6\], \[13\].

Due to the duality of OS- and VS-types, it is well-known that our second order partial differential equations (PDEs) have the quasiconformal properties of the PDEs as the quasi-harmonic system \[2\]. Using the quasiconformal property and the Keller-Dykhne reciprocity law \[7\], \[12\] of the conductivity, the equipotential curves are represented mathematically. We showed that in Theorem \[4.1\] for the infinitesimal limit of \(\gamma_0\), the OS-type potential distribution in CPM forms a quasiconformal map from the domain to \([0, 1] \times [0, 1]\) with random slits. It implies that at \(\gamma_0 \to 0\), the quasi-harmonic potential problem in CPM is quasiconformally equivalent to a random slit problem. Since one of our purposes is to reveal the fractal structure of the equipotential curves \[19\], we give Theorem \[4.2\] and using it, show an observation in Remark \[4.2\] that the equipotential curves at \(p \nearrow p_c\) are conformally mapped to the family of the lines with infinite length under some assumptions. Further the conformal map gives a relation to a Riemann sphere \(PC^1\).

The other purpose is to find the properties of the conductivity under the threshold. In order to use the duality, we give our numerical computations of the conductivity of VS-type using the Monte-Carlo method and finite difference method as we did for OS-type in Ref. \[19\]. Then based upon these results, we give novel approximation formulae of the conductivity curves. In other words, using the duality, we give approximation formulae of the conductivity curves in Remark \[6.1\] and Figure 8 (a). Figure 8 (a) shows that the approximation formulae represent well the computational results of the homogenized conductivities.

It leads us another approximation formula of the conductivity curve of three-dimensional case as in Remark \[6.2\] and Figure 8 (b). In Refs. \[17\] and \[18\], we have numerically studied the homogenized conductivity on the CPM with OS-type local conductivity mainly in three dimensional case by solving the generalized Laplace equation with a certain Dirichlet-Neumann boundary conditions. The proposed formula also approximates well the conductivity curve of the random spheres in Remark \[6.2\] as in Figure 8 (b). It means that in terms of the novel formula, we can parameterize the conductivity properties discussed in Refs. \[17\] and \[18\].

Contents in this article are as follows. Sec.2 gives our model and mathematical preliminaries. Sec. 2.1 is an review on our Boolean model of CPM based upon the Poisson point process, in which we introduce OS- and VS- types, and Sec. 2.2 gives our PDEs or the generalized Laplace equations on the CPMs. Sec. 3 also shows the well-established results homogenization of the conductivity associated with our PDEs. In Sec. 4, after we review the quasiconformal properties of the PDEs as the quasi-harmonic system, we show that in the infinitesimal limit of \(\gamma_0\), the potential distribution is quasiconformally equivalent to a configuration of a random slit model in Theorem \[4.1\] which is the first our main result. We also consider a conformal structure as a special case of the quasiconformal structures. The conformal structure leads us Theorem \[4.2\] and a novel observation on the fractal structure in Remark \[4.2\] that the equipotential curves which is the second our main result. In Sec. 5 we give our numerical computations using the Monte-Carlo method. Based upon these results, in Sec. 6 we give novel approximation formulae of the conductivity curves in Remark \[6.1\] and their extension to three-dimensional case in Remark \[6.2\] which are the third our main result. Figure 8 shows that they represent the computational results well.
2. Models and Preliminary

2.1. Model, Definitions and Notations. Let \( \hat{Q}(\mathbb{C}) \) be the set of all countable subsets \( \hat{X} \) of the region \( \mathbb{C} \equiv \mathbb{R}^2 \) satisfying \( N_K(\hat{X}) < \infty \) for every compact subset \( K \subset \mathbb{C} \), where \( N_K(\hat{X}) \) is the number of the points of \( \hat{X} \cap K \). Thus \( \hat{Q}(\mathbb{C}) \) has the non-negative valued Radon measure and is equipped with \( \sigma \)-field \( \mathcal{B}(\hat{Q}(\mathbb{C})) \). Let \( \ell(\cdot) \) be the Lebesgue measure of \( \mathbb{R}^2 = \mathbb{C} \) and \( N_0 = \{0, 1, 2, \ldots\} \). Then we consider the Poisson point process \( (\hat{Q}(\mathbb{C}), \mathcal{B}(\hat{Q}(\mathbb{C})), \hat{P}_\lambda) \), i.e., for any disjoint region \( \{A_1, A_2, \ldots, A_m\} \subset \mathcal{B}(\mathbb{C}) \) such that \( \ell(A_i) < \infty i = 1, 2, \ldots, m \), \( N_{A_1}(\hat{X}), \ldots, N_{A_m}(\hat{X}) \) are independent random variables on the probability space \( (\hat{Q}(\mathbb{C}), \mathcal{B}(\hat{Q}(\mathbb{C})), \hat{P}_\lambda) \) and for \( n \in N_0 \),

\[
\hat{P}_\lambda(N_{A_i} = n) = \left( \frac{\lambda(A_i))}{n!} \right)^n \exp(-\lambda(A_i)), \quad i = 1, 2, \ldots, m, n \in \mathbb{N}_0.
\]

The simplest Boolean model of CPM \( (Q_r(\mathbb{C}), \mathcal{B}(Q_r(\mathbb{C})), P_\lambda) \) associated with \( (\hat{Q}(\mathbb{C}), \mathcal{B}(\hat{Q}(\mathbb{C})), \hat{P}_\lambda) \) is given via \( Q_r(\mathbb{C}) := \{U_r(X) \mid \hat{X} \in \hat{Q}(\mathbb{C})\} \), where \( U_r(A) \) is the \( r \)-neighborhood of \( A \) for fixed \( r > 0 \). \( E_\lambda \) denotes the expectation value of \( P_\lambda \). Since for the Borel sets \( A \subset \mathbb{C} \), we have a natural measure \( \ell(A) := \int_{\partial A} \) \( C \) is equipped with \( \mathcal{B}(Q_r(\mathbb{C})) \) induced from \( \mathcal{B}(\hat{Q}(\mathbb{C})) \).

The following result is well-known [20]: By letting the distance of two points in \( \mathbb{C} \) denoted by \( d(x, y) \), let \( d(A) \) be \( \sup\{d(x, y) \mid x, y \in A\} \).

**Proposition 2.1.** In two dimensional case, for

\[
\lambda_c \equiv \inf\{\lambda : P_\lambda[d(X) = \infty] > 0\}, \quad \lambda^*_c \equiv \sup\{\lambda : P_\lambda[d(X^c) = \infty] > 0\},
\]

we have the equality and inequality \( \lambda_c = \lambda^*_c < \infty \).

In this article, we consider a finite domain \( B \equiv B_L := [-L, L]^2 \subset \mathbb{C} = \mathbb{R}^2 \) mainly for \( L \gg r \). As in Chap.5 in Ref. [20], the statistical properties over \( B_{nL} \) is naturally extended to \( \mathbb{C} = \mathbb{R}^2 \) by taking the large \( n \) limit. \( B \) has the natural Borel field \( \mathcal{B}(B) \) induced from that of \( \mathbb{R}^2 \). As our convention, \( z \in \mathbb{C} \) is expressed by the Cartesian coordinate system \( z = x + \sqrt{-1}y \). We let the left, right, upper and bottom edges of the boundary \( \partial B \) of \( B \) denoted by \( \partial_x, \partial_y \) be \( \partial_x B, \partial_y B, \partial_x B \) and \( \partial_y B \) respectively. \( \partial B = \bigcup \partial_i B \) and \( \partial_i B \cap \partial_i B \) is a corner of \( B \) and so on.

For example, as in Ref. [20], we have the covered volume fraction of \( X \in Q_r(\mathbb{C}) \),

\[
(1) \quad p(\lambda, r) := E_\lambda(\nu f(X)), \quad p(X) \equiv \nu f(X) := \lim_{n \to \infty} \nu f(B_{nL}(X),
\]

by letting \( \nu f(A') := \ell(A')/\ell(A) \) for \( A, A' \subset \mathbb{B}(\mathbb{C}) \) \( \ell(A) \neq 0 \). Using the volume fraction and Proposition 2.1 \( \lambda_c = \lambda^*_c \) provides the critical volume fraction \( p_c \), which is called percolation threshold or merely threshold in this article,

\[
p_c := 1 - e^{-\pi \lambda_c r^2} = 1 - e^{-\pi \lambda^*_c r^2}.
\]

Since it is known that our Boolean model \( (Q_r(\mathbb{C}), \mathcal{B}(Q_r(\mathbb{C})), P_\lambda) \) is ergodic [20, 23], the ergodic theorem shows the following proposition as Theorem 6.2 in Ref. [24]:

**Proposition 2.2.** For \( X_0 \in Q_r(\mathbb{C}) \),

\[
p(X_0) = \lim_{n \to \infty} E_\lambda(\nu f(B_{nL}(X))) \quad P_\lambda - a.s.
\]

Since we handle the boundary problem associated with CPM, we mainly considered these models in the finite region \( B_L \). We also define

\[
Q_r(B_L) := \{X \cap B_L \mid X \in Q_r(\mathbb{C})\}
\]
and \((\mathcal{Q}_r(B_L), \mathfrak{B}(\mathcal{Q}_r(B_L)), \mathbf{P}_\lambda)\) by induced by the restriction map \(\varrho_{B,B'} : \mathcal{Q}_r(B') \to \mathcal{Q}_r(B), (\varrho_{B,B'}(X') = (X' \cap B))\) for \(B \subset B' \subset \mathbb{C}\).

As mentioned above, in \((\mathcal{Q}_r(B), \mathfrak{B}(\mathcal{Q}_r(B)), \lambda)\), the ordinary measure is given by \(\ell_U(X) = \ell(X \cap U)\) for a Borel set \(U \subset B\) and \(X \in \mathcal{Q}_r(B)\) as the occupied region \(X\). As its vacancy, \(\ell_U^*(X) = \ell(X^c \cap U)\) is also handled where \(X^c = B \setminus X\) in ordinary CPM. In this article, we also consider the binary measures such that for a Borel set \(U \subset B\),

\[
\begin{align*}
\gamma_+^X(U) &:= \gamma_0 \ell_U^*(X) + \gamma_1 \ell_U(X), \\
\gamma_-^X(U) &:= \gamma_0 \ell_U(X) + \gamma_1 \ell_U^*(X),
\end{align*}
\]

where \(\gamma_1 = 1\) and \(\gamma_1 \gg \gamma_0 > 0\). In the limit \(\gamma_0 \to 0\), \(\gamma_+^X\) (resp. \(\gamma_-^X\)) corresponds to the measure of the occupancy (resp. vacancy). As mentioned in Introduction, we call the systems with \(\gamma_+^X\) \textit{OS-type} and \(\gamma_-^X\) \textit{VS-type} respectively. We regard the measure \(\gamma_\pm\) as an element of the real valued generalized functions \(\mathcal{D}(B,\mathbb{R})\) over \(B\) and we treat it as the binary local conductivity \(\gamma_\pm(z)\), i.e., \(\gamma_\pm(z)\) is defined through the measure \(\mathfrak{B}(\mathbb{C}), \gamma_\pm^X\) for every Borel set \(U \subset \mathbb{C}\),

\[
\int_U \gamma_\pm^X(dz) = \int_U \gamma_\pm^X(z)d^2 z,
\]

where \(d^2 z = dx dy = \frac{1}{2\sqrt{-1}}dz d\bar{z}\) is the Lebesgue measure of \(\mathbb{R}^2 = \mathbb{C}\). Here these \(\gamma_\pm^X\) are weakly first differentiable functions \([8]\).

### 2.2. Dirichlet-Neumann Boundary Problems on \(B_L\).

For a configuration \(X \in \mathcal{Q}_r(B_L)\), using the binary local conductivity \(\gamma_\pm^X(z)\) of the generalized functions \(\mathcal{D}(B_L,\mathbb{R})\), we consider second order PDEs with mixed boundary conditions.

The mixed boundary problem of the elliptic differential equation with measurable coefficients is studied well in Refs. \([8, 24]\) and \([25]\). We consider the symmetric form on the function space \(L^2(B_L)\),

\[
\mathcal{E}_{B_L}[\gamma_\pm^X; v, v'] := \int_{B_L} \gamma_\pm^X(z)(\nabla v(z))(\nabla v'(z))d^2 z
\]

\[
= \int_{B_L} (\nabla v(z))(\nabla v'(z))\gamma_\pm^X(dz).
\]

Let \(\mathcal{E}_{B_L}[\gamma_\pm^X; v] := \mathcal{E}_{B_L}[\gamma_\pm^X; v, v]\). For a non-negative parameter \(\mu\), we define the seminorm:

\[
\|v\|_{\gamma_\pm^X,\mu,B_L} := \left\{\int_U (\gamma_\pm^X(z)(\nabla v(z))^2 + \mu v(z)^2) d^2 z\right\}^{1/2},
\]

and the Sobolev spaces,

\[
\mathcal{W}_2^1(\gamma_\pm^X, \mu, B_L) := \{v \in L^2(B_L) \mid \|v\|_{\gamma_\pm^X,\mu,B_L} < \infty\},
\]

\[
\mathcal{H}(\gamma_\pm^X, \mu, B_L) := \{v \in \mathcal{W}_2^1(B_L) \mid \text{there exists a sequence } \{v_m\} \subset C^1(B_L) \text{ satisfying } \|v_m - v\|_{\gamma_\pm^X,\mu,B_L} \to 0\}.
\]

From Proposition 1 in Ref. \([23]\), \(\mathcal{W}_2^1(\gamma_\pm^X, \mu, B_L)\) and \(\mathcal{H}(\gamma_\pm^X, \mu, B_L) = \mathcal{H}(\gamma_\pm^X,0, B_L)\) are Hilbert spaces. The associated PDE, which we call \textit{the generalized Laplace equation} \([5,8]\), is given by

\[
\nabla \gamma_\pm^X(z)\nabla u^\pm(z) = 0 \quad \text{for } z \in B_L^0.
\]
In this article, we consider the potential problems of $\gamma_+^X$ in [5] with the following boundary conditions respectively,

\begin{align}
\text{BC}^{+}_{B_L} : \quad & \begin{cases}
   u^+(z) = u_0^+, & \text{for } z \in \partial_a B_L, \\
   u^+(z) = 0, & \text{for } z \in \partial_4 B_L, \\
   \frac{\partial}{\partial z} u^+(z) = 0, & \text{for } z \in \partial_1 B_L \cup \partial_3 B_L,
\end{cases} \\
\text{BC}^{-}_{B_L} : \quad & \begin{cases}
   u^-(z) = u_0^-, & \text{for } z \in \partial_a B_L, \\
   u^-(z) = 0, & \text{for } z \in \partial_4 B_L, \\
   \frac{\partial}{\partial z} u^-(z) = 0, & \text{for } z \in \partial_1 B_L \cup \partial_3 B_L.
\end{cases}
\end{align}

(6)

Here $u_0^\pm$ is a real positive constant number, $u_0^+ > 0$. The boundary conditions are listed in Table 1.

| OS | VS |
|----|----|
| $\gamma_+$ | $\gamma_0$ | $u^+ = u_0^+$ | $u^- = u_0^-$ | $\partial_a B_L$ | $\partial_4 B_L$ | $\partial_1 B_L$ | $\partial_3 B_L$ | $\partial_2 B_L$ |
| $\gamma_0$ | $\gamma_1$ | $\partial_2 u^+ = 0$ | $\partial_3 u^- = 0$ | $u^+ = 0$ | $\partial_3 u^+ = 0$ | $u^- = 0$ |

Table 1. The conductivity distribution (CD) and boundary condition (BC):

These problems are equivalent with the energy minimal problems [5], in which we obtain the minimal functional $\mathcal{E}_B^{(0)}[\gamma^X]$,

$$\mathcal{E}_B^{(0)}[\gamma^X] := \min_{v \in \mathcal{H}(B_L|BC^{\pm}, u_0^\pm)} \mathcal{E}_B^{(0)}[\gamma^X; v],$$

where $\mathcal{H}(B_L|BC^{\pm}, u_0^\pm)$ is the function spaces satisfying these boundary conditions,

$$\mathcal{H}(B_L|BC^{\pm}, u_0^\pm) := \{ v \in \mathcal{H}(\gamma^X, \mu, B_L) | v(z) \text{ satisfies BC}^{\pm}_{B_L} \}.$$

This $\mathcal{E}_B^{(0)}[\gamma^X]$ is the same as the capacity, or total conductivity in our problem [8]. In other words, for the solutions of the generalized Laplace equations of (5) with (6) for a configuration $X$, we have

$$\mathcal{E}_B^{(0)}[\gamma^X] = (u_0^\pm)^2 \Gamma^{\pm}_{B_L}(X),$$

(8)

where the total conductivities $\Gamma^{\pm}_{B_L}(X)$ of OS-type and $\Gamma^{-}_{B_L}(X)$ of VS-type are defined by

$$\Gamma^{\pm}_{B_L}(X) := \frac{1}{u_0^\pm} \int_{\partial_a B_L} \gamma^X(z) \frac{\partial}{\partial y} u^+(z) dx,$$

$$\Gamma^{-}_{B_L}(X) := \frac{1}{u_0^\pm} \int_{\partial_a B_L} \gamma^X(z) \frac{\partial}{\partial x} u^- (z) dy,$$

(9)

respectively. [8] is obtained by Gauss-Stokes theorem. The OS-type case is given by integrating the normal current $\gamma^X_+ \partial u^+ / \partial y$ along the line parallel to the $x$-axis and the VS case is given as integral of $\gamma^X_+ \partial u^+ / \partial x$ for the $y$-axis.

Our motivation is illustrated in Figure 1 in which we show the numerical result of the equation (5) with the boundary conditions (6) as mentioned in Sec. 5. It shows that 1) the equipotential curves of OS-type and VS-type intersect orthogonally, 2) for $\lambda < \lambda_c$, the equipotential curves of OS-type don’t penetrate into $X^c$ whereas for $\lambda > \lambda_c$, the equipotential curves of VS-type avoid $X^c$, and 3) the equipotential curves look very complicated. In this article, we characterize these properties as follows.
Figure 1. The equipotential curves of OS-type and VS-type: The black curve represents an equipotential curve of OS-type whereas the white one corresponds to VS-type. (c) and (d) are parts of (a) and (b) respectively. In (a) and (c), the black curves avoid $X$ whereas in (b) and (d), the white curves avoid $X^c$.

3. Homogenized Conductivity

Before we mention the homogenization of the conductivity, we note that our mixed boundary conditions are connected to a Dirichlet problem. For a subset $A$ in $B_L$, we consider the reflection to $\partial_r B_L$,

$$R_{\partial_r B_L} A := \{2L - x + \sqrt{-1}y \mid x + \sqrt{-1}y \in A\}$$

and identify the image $R_{\partial_r B_L}(A \cap \partial_r B_L)$ with itself ($A \cap \partial_r B_L$). The above problem can be extended to the differential equation on $B_L \cup R_{\partial_r B_L} B$ with periodic boundary condition. Since the Neumann conditions in our systems are naturally satisfied in the geometrical setting, our problem can be formulated on an annulus $A_{L_1, L_2} = \{\xi \in \mathbb{C} \mid L_1 \leq |\xi| \leq L_2\} \subset \mathbb{R}^2$ as Dirichlet boundary problem. Hence we can apply the results [8, 24, 25] for the Dirichlet boundary condition to our mixed boundary problems (3) with (6). More precise arguments among the boundary conditions in homogenization is in Ref. [16].

For $\hat{X} \in \hat{Q}(B_L)$, we define the homothety transformation $\hat{D}_a \hat{X} = \{az_i \mid z_i \in \hat{X}\}$ for $a > 0$, and $\hat{D}_a X = U_{ar}(\hat{D}_a \hat{X})$: for every $a > 0$,

$$\mathcal{E}_{B_L}[\gamma^X] = \mathcal{E}_{B_{Lr}}[\gamma^{D_a X}].$$

Further as mentioned before, we note that our Boolean model $(Q_r(\mathbb{C}), \mathcal{B}(Q_r(\mathbb{C})), P_\lambda)$ has ergodic properties [20, 23].
In order to consider a fixed configuration $X$ which stands a typical configuration in $Q_r(B_L)$ of $\lambda < \lambda_c$ and $\lambda > \lambda_c$, let

$$Q_{r,\lambda}(C) := \bigcap_{U \in \mathcal{B}(Q_r(C)), \mathbf{P}_\lambda(U) = 1} U,$$

and $Q_{r,\lambda}(B_L) := \{X \cap B_L \mid X \in Q_{r,\lambda}(C)\}$. For fixed $\lambda$ and $X \in Q_{r,\lambda}(C)$, we are concerned with $\Gamma_{B_{nL}}(X)$ of $n \to \infty$.

In the homogenization problem, the asymptotic expansion of

$$\Gamma^\pm_{B_{nL}}(X) := \inf_{u \in U_{B_{nL}}^0} \frac{1}{(u_0')^2} \int_{B_{nL}} \frac{\gamma_\pm^X(z)}{\varepsilon} \nabla U(z)^2 dz$$

with respect to $\varepsilon$, $\Gamma^\pm_{B_{nL}}(X) = \Gamma^{\pm}_{B^0_{nL}}(X) + \Gamma^{\pm}_{B^0_{nL}}(X)\varepsilon^2 + \cdots$, is investigated. The constant term $\Gamma^\pm(\lambda)$ is the homogenized or the effective conductivity $[14]$, which are the same as the asymptotic behavior of the energy functional $\mathcal{E}_{B_{nL}, 0}[\gamma_\lambda^X]$ and homogenized conductivity $\Gamma^\pm_{B_{nL}}(X)$ for a large $n$ limit of $B_{nL}$.

Homogenization problem is basically for a local conductivity $\gamma_\lambda^X$ with a periodic property. Murat and Tartar proved that in the limit of $\varepsilon \to 0$ in [10], the solution of the generalized Laplace equation weakly converges which is known as $H$-convergence [21, 22]. On the other hand, De Giorgi and Franzoni showed that the energy functional converges as the minimization problem which is known as $\Gamma$-convergence [9]. Recently Ansini, Dal Maso and Ida Zeppieri proved that both are equivalent for such Dirichlet problems [11].

Further based upon $\Gamma$-convergence, if the configuration satisfies the ergodic property, Dal Maso and Modica showed that the energy functional converges for the large $n$ limit [6]. Bourgeat, Mikelić and Wright proved a stronger theorem in a two-scale convergence [4].

Since the Poisson point process has ergodic property, these results guarantee the convergence of the conductivity $\Gamma^\pm_{B_{nL}}(X \cap B_{nL})$ and the energy functional $\mathcal{E}_{B_{nL}, 0}[\gamma_\pm^X \cap B_{nL}]$ for a configuration $X \in Q_{r,\lambda}(C)$ by taking the large $n$ limit of $B_{nL}$. The asymptotic behaviors are defined by

$$\mathcal{E}_0[^X_\lambda] := \lim_{n \to \infty} \mathcal{E}_{B_{nL}, 0}[\gamma_\pm^X \cap B_{nL}], \quad \Gamma^\pm(\lambda) := \lim_{n \to \infty} \Gamma^\pm_{B_{nL}}(X \cap B_{nL}).$$

The ergodic theorem as Theorem 6.2 in Ref. [23] can be applicable to our problem. In our Boolean model $(Q_r(C), \mathcal{B}(Q_r(C)), \mathbf{P}_\lambda)$, Refs. [1] and [6] (Examples in Refs. [6]) mean the following proposition:

**Proposition 3.1.** For $X_0 \in Q_r(C)$,

$$\Gamma^\pm(X_0) = \lim_{n \to \infty} \mathbf{E}_\lambda \left( \Gamma^\pm_{B_{nL}}(X \cap B_{nL}) \right), \quad \mathbf{P}_\lambda - a.s..$$

The right hand side is also denoted by $\lim_{n \to \infty} \mathbf{E}_\lambda \left( \Gamma^\pm_{B_{nL}}(X) \right)$ or $\mathbf{E}_\lambda \left( \Gamma^\pm(X) \right)$ simply, which is called homogenized total conductivity, or homogenized conductivity. Due to the Proposition 3.1 we also call $\Gamma^\pm(X)$ homogenized conductivity $\mathbf{P}_\lambda$ a.s..

**Remark 3.1.** We note the homothety in our system and the energy functional as a function of $Q_r(B_L)$. For a given $X \in Q_{r,\lambda}(B_L)$ and $r' > 0$, there exist $\lambda', X' \in Q_{r,\lambda}(B_{r'L/r'})$ and $r' > 0$ such that $X = D_{r'/r}X'$. Hence for a positive number $r'$, there is an intensity $\lambda'$ such that

$$\lim_{n \to \infty} \mathbf{E}_\lambda \left( \Gamma^\pm_{B_{nL}}(X \in Q_r(C)) \right) = \lim_{n \to \infty} \mathbf{E}_\lambda \left( \Gamma^\pm_{B_{r'L/r}}(X \in Q_{r,r'}(C)) \right).$$

Since the volume fraction is also given by

$$\lim_{n \to \infty} \mathbf{E}_\lambda \left( \text{vf}_{B_{nL}}(X \in Q_r(C)) \right) = \lim_{n \to \infty} \mathbf{E}_\lambda \left( \text{vf}_{B_{r'L/r}}(X \in Q_{r,r'}(C)) \right),$$

$\lambda$ is a function of $r$ and $p$ for our model, which is denoted by $\lambda = \lambda(p, r)$. In other words, for a fixing the radius $r$, $\mathbf{E}_\lambda(p, r)(\Gamma^\pm(X))$ is a function of the volume fraction $p$, as it is well-known.
From Propositions 2.2 and 3.1, $E_\lambda(\Gamma^\pm(X))$ is a function of $p(\lambda, r)$. We have the following proposition:

**Proposition 3.2.**
1. $E_{\lambda(p,r)}(\Gamma^+(X_r))$ is a monotonic increasing function of the volume fraction $p \in [0,1]$.
2. $E_{\lambda(p,r)}(\Gamma^-(X_r))$ is a monotonic decreasing function of the volume fraction $p \in [0,1]$.

Proof. Let us consider (1). For $\hat{X} \in \hat{Q}(C)$, let $X_r := \bigcup_{z \in \hat{X}} U_{z,r}$, $X_{r-\varepsilon} := \bigcup_{z \in \hat{X}} U_{z,r-\varepsilon}$ and $\delta \gamma^{r+}_+: :=\gamma^+_r - \gamma^{r-}_r$. The relation between $E_{B_L}(X_r)$ and $E_{B_L}(X_{r-\varepsilon})$ is computed:

$$
\min_{v \in H(B_L | BC_{B_L}^+, u_0^+)} E[\gamma^+_r; v] = \int_{B_L} \gamma^+_r (\nabla u^+)^2 d^2z,
$$

$$
\geq \int_{B_L} \gamma^{r+}_- (\nabla u^+)^2 d^2z + \int_{B_L} \delta \gamma^+_r (\nabla u^+)^2 d^2z
$$

Since $\int_{B_L} \delta \gamma^+_r (\nabla u^+)^2 d^2z$ must be positive, $\Gamma^+_r \geq \Gamma^+_{r-\varepsilon}$. Since $P_\lambda$-almost surely $\Gamma^\pm(X)$ and $p(X)$ equal to their expectation values from Propositions 2.2 and 3.1 we have the result. (2) is also obtained in the similar computation to the above estimation. \(\square\)

For the volume fraction $p \in [0,1]$, we define the conductivity curves,

$$
\Gamma^+(p) := E_{\lambda(p,r)}(\Gamma^+(X)), \quad \Gamma^-(p) := E_{\lambda(p,r)}(\Gamma^-(X)).
$$

From Proposition 2.1 the following corollary is obtained:

**Corollary 3.1.**

$$
\lim_{\gamma_0 \rightarrow +0} \Gamma^+(p) = \begin{cases} 
0 & \text{for } p < p_c, \\
\neq 0 & \text{for } p > p_c.
\end{cases}
$$

$$
\lim_{\gamma_0 \rightarrow +0} \Gamma^-(p) = \begin{cases} 
\neq 0 & \text{for } p < p_c, \\
0 & \text{for } p > p_c.
\end{cases}
$$

4. Quasiconformal Properties of Binary System

4.1. Quasiconformal Properties and Keller-Dykhne Reciprocity Law. Let us fix a configuration $X \in Q_r(B_L)$, and consider the solutions $u^\pm$ of (3) with (3) from quasi-harmonic map theory [2]. In this section, we consider the complex-valued generalized function,

$$
\psi^X := u^- + \sqrt{-1} u^+ \in D(B_L, C),
$$

and the Beltrami coefficient,

$$
\mu_{\psi^X}(z) := \frac{\overline{\partial} \psi^X(z)}{\partial \psi^X(z)} \in D(B_L, C),
$$

where $\partial = \frac{1}{2}(\partial_x - \sqrt{-1} \partial_y)$, $\overline{\partial} = \frac{1}{2}(\partial_x + \sqrt{-1} \partial_y)$, $\partial_x = \frac{\partial}{\partial x}$, and $\partial_y = \frac{\partial}{\partial y}$.

The theory of the quasiconformal mappings [2] provides the following proposition:

**Proposition 4.1.** For a measurable non-negative valued function $\gamma$ on $B_L$, a weak solution $v$ in $H(B_L | BC_{B_L}^+, u_0^+)$ of the elliptic differential equation,

$$
\nabla \gamma \nabla v = 0,
$$
Remark 4.1. We should note that the families of the equipotential curves, \( \alpha \) and \( \gamma \) which are constant positive numbers, i.e., the correspondence between systems of OS-type and VS-type can be regarded as the \( \gamma \)-harmonic conjugation.

For each \( u_0^+ = u_0^- \) case, we illustrate examples in Figure 1 which are the numerical computational results as mentioned in Section 5 and show that the equipotential curves of OS-types and VS-types cross perpendicularly. We have the following proposition:

**Proposition 4.2.** \( \psi^X = u^- + \sqrt{-1}u^+ \) is a quasiconformal map from \( B_L \) to \( I_{u_0^+}^{u^-} := [0, u_0^-] \times [0, u_0^+] \) for \( \gamma_0 > 0 \).

Let us define the natural maps for \( u \in [0, u_0^-] \) and \( v \in [0, u_0^+] \)

\[ \iota_u : [0, u_0^+] \to I_{u_0^+}^{u^-}; \quad (v \mapsto (u, v)), \quad \iota_v : [0, u_0^-] \to I_{u_0^-}^{u_0^+}; \quad (u \mapsto (u, v)). \]

**Lemma 4.1.** For each \( u \in [0, u_0^-] \) and \( v \in [0, u_0^+] \), the equipotential curves \( C_u^-(X) \) of the potential \( u^- = u \) and \( C_v^+(X) \) of the potential \( u^+ = v \) are given by

\[ C_u^-(X) = \psi^{X^{-1}}\iota_u[0, u_0^+] \quad \text{and} \quad C_v^+(X) = \psi^{X^{-1}}\iota_v[0, u_0^-]. \]

**Remark 4.1.**

1. We should note that the families of the equipotential curves, \( \{ C_u^-(X) = \psi^{X^{-1}}\iota_u[0, u_0^+] \mid u = u_0, u_1, \ldots \} \) and \( \{ C_v^+(X) = \psi^{X^{-1}}\iota_v[0, u_0^-] \mid v = v_0, v_1, \ldots \} \) are given as non-intersection curves in \( B_L \).

2. As mentioned in Sec.5, we numerically computed the equipotential curves \( C_u^-(X) \) and \( C_v^+(X) \) in Figure 1 and Figure 2. In Figure 2 by numerically solving (4) we obtain the potential distributions \( u^+(x, y) \) and \( u^-(x, y) \) for each volume fraction \( p = 0.2, 0.6, 0.9, u_0^+ = u_0^- = 1 \) and a seed \( \iota_s \) of the pseudo-randomness. We display the equipotential curves whose interval \( \delta u \) equals 0.1 there. The curves are very complicated for the \( p = 0.6 \) case which is near the threshold \( p_c \). With Figure 1 these curves of OS-type and VS-type show their duality.

Due to the duality of \( \gamma \)-harmonic property, we have the following well-known results of the Keller-Dykhne reciprocity law of the conductivity [7,12]:

\[ \nabla \frac{1}{\gamma} \nabla u = 0. \]

The function \( f := u + \sqrt{-1}v \) and the Beltrami coefficient \( \mu := \frac{1 - \gamma}{1 + \gamma} \) obey

\[ \overline{\partial}f = \mu \partial f. \]

Noting the relation,

\[ \gamma_X^-(z) = \frac{\gamma_0 \gamma_1}{\gamma_X^+(z)}, \]

we apply Proposition 4.1 to our system. In other words, we regard our system associated with the \( \gamma \)-harmonic conjugation by letting

(14) \[ \gamma := \alpha \gamma_X^+(z), \quad u := \beta u^-, \quad v := u^+, \]

where \( \alpha \) and \( \beta \) are constant positive numbers, i.e., the correspondence between systems of OS-type and VS-type can be regarded as the \( \gamma \)-harmonic conjugation.

It is obvious that above boundary conditions are consistent with \( \gamma \)-harmonic relation because they are orthogonal,

\[ (\nabla u^+, \nabla u^-) = 0 \]

everywhere. For \( u_0^+ = u_0^- \) case, we illustrate examples in Figure 1 which are the numerical computational results as mentioned in Section 5 and show that the equipotential curves of OS-types and VS-types cross perpendicularly. We have the following proposition:
Figure 2. Equipotential curves of the potential distributions of the seed $i_s = 1$ with the volume fractions 0.2, 0.6, 0.8: (a), (b), (c) are for OS-type and (d), (e), (f) for VS-type respectively. The curves correspond to the values $u = 0.1, 0.2, \ldots, 0.8, 0.9$. The interval of the curves $\delta u$ is 0.1.

Proposition 4.3. (1) For every $X \in Q_r(B_L)$, $\Gamma_{B_L}^-(X)\Gamma_{B_L}^+(X) = \gamma_0\gamma_1$.
(2) $E_\lambda(\Gamma^+(X))E_\lambda(\Gamma^-(X)) = \gamma_1\gamma_0$ for every $\lambda$, and
(3) $\Gamma^+(p)\Gamma^-(p) = \gamma_1\gamma_0$ for every $p \in [0, 1]$.

Proof. Since the direct computations show

$$u_0^+\Gamma_{B_L}^+(X) = \int_{\partial_{B_L}} \gamma^X \frac{\partial}{\partial y} u^+(z) dx = \frac{\beta}{\alpha} \int_{\partial_{B_L}} \frac{\partial}{\partial x} u^-(z) dx = \frac{\beta}{\alpha} u_0^-,$$

(1) is obvious. Due to Proposition 5.1 we have (2) and (3).

As Kozlov showed for the random checkerboard model in Theorem 8 of Ref. [14], it is expected that the effective conductivity of the high contrast local conductivity for a more general model also behaves like $\sqrt{\gamma_0\gamma_1}$ at the threshold. Thus we show that $\Gamma^\pm(p)$ is identical to $\sqrt{\gamma_0\gamma_1}$ at $p_c$ by assuming that $\Gamma^\pm(p)$ is a continuous function over $p \in [0, 1]$, which is also numerically shown in Remark 6.1.

We should have the following lemma.

Lemma 4.2. Assume the continuous property of $\Gamma^\pm(p)$. Then there exists a monotonic increasing continuous function $h$ over $[0, 1]$ such that $h(0) = \sqrt{\gamma_0/\gamma_1}$, $h(p_c) = 1$, $h(1) = \sqrt{\gamma_1/\gamma_0}$,

$$\Gamma^+(p) = \sqrt{\gamma_1\gamma_0}h(p) \quad \text{and} \quad \Gamma^-(p) = \sqrt{\gamma_1\gamma_0}/h(p).$$
Proof. The region of \( \Gamma^\pm(p) \) is \([\gamma_0, \gamma_1]\). It is obvious that \( \Gamma^+(0) = \Gamma^-(1) = \gamma_0 \) and \( \Gamma^+(1) = \Gamma^-(0) = \gamma_1 \). They are monotonic continuous functions from the assumption. They must cross at a point \( p_0 \), i.e., \( \Gamma^+(p_0) = \Gamma^-(p_0) \). Corollary 3.1 means that \( p_0 \) must be \( p_c \). From the Keller-Dykhne reciprocity law in Proposition 4.3 we have \( \Gamma^+(p_c) = \Gamma^-(p_c) = \sqrt{\gamma_0 \gamma_1} \). The assumption asserts the existence of the function. \( \square \)

For the case of \( \beta = 1 \) in (14) we have the following conformal relation;

**Proposition 4.4.** For \( X \in Q_c(B_L) \), we have the conformal properties,

\[
\text{supp}(\mu^{B_L}_{\psi^X}) = \begin{cases} 
X & \text{for } \alpha = 1, \\
X^c & \text{for } \alpha = \frac{1}{\gamma_0} \frac{\alpha \gamma_0 \gamma_1}{\gamma_1} u_0^- u_0^+.
\end{cases}
\]

**Proof.** \( \beta = \frac{u_0^+}{u_0^-} \frac{\alpha}{\Gamma^+(X)} = \frac{u_0^+}{u_0^-} \frac{\alpha \gamma_0 \gamma_1}{\gamma_1} = 1. \) \( \square \)

**Figure 3.** The quasiconformal map

4.2. Quasiconformal and Conformal Maps for Infinitesimal \( \gamma_0 \). Let us introduce some geometrical terms [20]. For \( X \in Q_c(B_L) \) and two disjoint regions \( A_1 \) and \( A_2 \in B_L \), we say that a continuous curve \( C \subset B_L \) is an occupied (resp. a vacant) connection of \( A_1 \) and \( A_2 \) with respect to \( X \), if \( C \cap A_1 \neq \emptyset, C \cap A_2 \neq \emptyset \), and \( C \subset X \) (resp. \( C \subset X^c \)).

Further we say that such \( A_1 \) and \( A_2 \) are occupied (resp. vacant) connected with respect to \( X \), if there is a continuous curve \( C \) which is an occupied (resp. vacant) connection of \( A_1 \) and \( A_2 \) with respect to \( X \). We also say that such \( A_1 \) and \( A_2 \) are occupied (resp. vacant) disconnected with respect to \( X \), if there is no continuous curve \( C \) which is an occupied (resp. vacant) connection of \( A_1 \) and \( A_2 \) with respect to \( X \).

**Definition 4.1.** (1) For \( X \in Q_c(B_L) \) we say that \( X^c \) (resp. \( X^c \)) is totally connected with \( \partial B_L \) if every \( \partial_\alpha B_L \) (\( \alpha = u, d, r, \ell \)) are mutually occupied (resp. vacant) connected.
(2) For \( X \in \mathcal{Q}_r(B_L) \) we say that \( X^\circ \) (resp. \( X^c \)) is totally disconnected with \( \partial B_L \) if every \( \partial_\alpha B_L \) \((\alpha = u,d,r,\ell)\) are mutually occupied (resp. vacant) disconnected.

(3) We say that \( X \) dominates \( B_L \) if \( X^\circ \) is totally connected with \( \partial B_L \) and \( X^c \) is totally disconnected with \( \partial B_L \).

(4) We say that \( X^c \) dominates \( B_L \) if \( X^c \) is totally connected with \( \partial B_L \) and \( X \) is totally disconnected with \( \partial B_L \).

Let us define the subset of \( \mathcal{Q}_r(B_L) \) which
\[
\mathcal{Q}_<(B_L) := \{ X \in \mathcal{Q}_r(B_L) \mid X^c \text{ dominates } B_L \} \subset \mathcal{Q}_r(B_L),
\]
\[
\mathcal{Q}_>(B_L) := \{ X \in \mathcal{Q}_r(B_L) \mid X \text{ dominates } B_L \} \subset \mathcal{Q}_r(B_L).
\]

As in Figure 11 we show that the domain of the equipotential curves.

**Proposition 4.5.**

(1) For \( X \in \mathcal{Q}_<(B_L) \), let the decomposition of \( X \) by the connected parts \( X = \bigsqcup X^{(i)} \).

In the limit of \( \gamma_0 \to 0 \), the solution of (2) with \( BC_{B_L}^+ \), \( u^+ \) is constant in each \( X^{(i)} \) and every equipotential curve \( C^+_u \) of \( u \in [0,u_0^+ \) does not penetrate into each \( X^{(i)} \).

(2) For \( X \in \mathcal{Q}_>(B_L) \), let the decomposition of \( X^c \) by the connected parts \( X^c = \bigsqcup X^{c(j)} \).

In the limit of \( \gamma_0 \to 0 \), the solution of (2) with \( BC_{B_L}^- \), \( u^- \) is constant in each \( X^{c(j)} \) every equipotential curve \( C^-_u \) of \( u \in [0,u_0^- \) does not penetrate into each \( X^{c(j)} \).

**Proof.** Let \( X \in \mathcal{Q}_<(B_L) \). Then it is not connected with \( \partial_\alpha B \) and thus \( \lim_{\gamma_0 \to 0} \Gamma_{B_L}^+(X) = 0 \). Hence there is a positive number \( a_0(> 0) \) given by
\[ a_0 := \sup \{ a \in (0,1) \mid \lim_{\gamma_0 \to 0} \frac{\Gamma_{B_L}^+(X)}{\gamma_0^a} = 0 \} \]

Let \( \mathcal{E}_{B_L,0<}^+(X) := \frac{\Gamma_{B_L}^+(X)}{u_0^+} / \gamma_0^{a_0} \), \( \mathcal{E}_{B_L,0<}^+(X) \) is decomposed like
\[
\mathcal{E}_{B_L,0<}^+(X) = \frac{\gamma_1}{\gamma_0^a} \sum_i \int_{X^{(i)}} (\nabla u^+)^2 d^2 z + \gamma_0^{a_0} \int_{X^c} (\nabla u^+) d^2 z.
\]

The energy functional \( \mathcal{E}_{B_L,0<}^+ \) must vanish for \( \gamma_0 \to 0 \) but \( \Gamma_{B_L}^+(X) \) has the non-trivial solution; thus we have assumed \( a_0 < 1 \). Since for the limit of the \( \gamma_0 \to 0 \), the first term must vanish and \( (\nabla u)^2 \) vanish on \( X^{(i)} \).

Hence \( u^+ \) in each \( X^{(i)} \) becomes a constant function \( u_i^+ \) for \( \gamma_0 \to 0 \). (2) is the same as (1). \( \square \)

Let us consider the lateral slit model \( S_<(I_{u_0^+}^{u_0^-}) \) which is given as a point process \( \hat{\mathcal{Q}}(I_{u_0^+}^{u_0^-}) \) with lateral slits.

For \( \hat{\mathcal{X}} \in \hat{\mathcal{Q}}(I_{u_0^+}^{u_0^-}) \), \( \hat{\mathcal{X}} = \{ (x_i, y_i) \mid i = 1, 2, \ldots, m \} \) we consider random variables \( c = (c_i)_{i=1,\ldots,m} \in [0,u_0^+]^m \) and a set of slits
\[ S_{\hat{\mathcal{X}},c} := \{ s(x_i-c_i,y_i),(x_i+c_i,y_i) \cap I_{u_0^+}^{u_0^-} \mid (x_i, y_i) \in \hat{\mathcal{X}} \}, \]

where \( s_{z,z'} \) is the segment between \( z \) and \( z' \). The element \( s \) of \( S_<(I_{u_0^+}^{u_0^-}) \) is given by \( s := I_{u_0^+}^{u_0^-} \setminus S_{\hat{\mathcal{X}},c} \).

As Figure 3 displays the map \( \psi^X \) for \( X \in \mathcal{Q}_<(B_L) \), there is an element \( s = I_{u_0^+}^{u_0^-} \setminus S_{\hat{\mathcal{X}},c} \) of \( S_<(I_{u_0^+}^{u_0^-}) \) and we have the quasiconformal map \( \psi^X|_{X^c} \) from \( X^c \to s \). In other words, \( \psi^X|_{X^c} \) is a map from \( X^c \to I_{u_0^+}^{u_0^-} \equiv \mathfrak{X} \) and each slit \( s_i \in S_{\hat{\mathcal{X}},c} \) corresponds to a connected element \( X^{(i)} \) in \( X \), i.e., \( \partial X^{(i)} = \psi^{X^{-1}}(s_i) ; X^c \) is quasiconformally equivalent to \( s \).

The length of each slit and the configuration depend on the configuration of the disk in each \( X^{(i)} \) and others \( X^{(j)} (j \neq i) \). It implies that the configuration \( X \in \mathcal{Q}_<(B_L) \) is completely parameterized by the element of
\(S_c(I_{\lambda_0}^\pm)\). In other words, there is a random slit model on a certain probability space \((S_c(I_{\lambda_0}^\pm), \mathcal{B}(\hat{Q}(I_{\lambda_0}^\pm))) \times \mathcal{B}([0, u_0]), \hat{\mathbb{P}_{r,\lambda}}\) which corresponds to our Boolean model for \(\lambda\) and \(r\).

Similarly the vertical slits model are defined as \(S_v(I_{\lambda_0}^\pm)\) whose element is given as \(X \in Q(I_{\lambda_0}^\pm)\) with vertical slits with random length.

Then we have the following proposition.

**Proposition 4.6.** In the limit \(\gamma_0 \to 0\), the function \(\psi^X\) induces injections \(\Psi_\leq : Q_\leq(B_L) \to S_c(I_{\lambda_0}^\pm)\) and \(\Psi_\geq : Q_\geq(B_L) \to S_v(I_{\lambda_0}^\pm)\).

**Proof.** Proposition 4.5 show the results. \(\square\)

By considering the \(n\) large limit of \(B_{nL}\), as the behavior of \(Q(C)\), we have the main theorem:

**Theorem 4.1.**

1. For \(\lambda < \lambda_c\) there is an element \(s \in S_c(I_{\lambda_0}^\pm)\) such that by \(\psi^X\), \(X^c \bigcap \bigcup_{n=1}^\infty B_{nL}\) is quasiconformally equivalent to \(s\), \(P_{\lambda}\)-a.s..

2. For \(\lambda > \lambda_c\), there is an element \(s \in S_v(I_{\lambda_0}^\pm)\) such that by \(\psi^X\), \(X^c \bigcap \bigcup_{n=1}^\infty B_{nL}\) is quasiconformally equivalent to \(s\), \(P_{\lambda}\)-a.s..

**Proof.** We show that \(\psi^X(X^c)\) belongs to \(S_c(I_{\lambda_0}^\pm)\). Let us consider

\[
Q_\leq, B_L(C) := \{X \in Q_r(C) \mid X^c \cap B_L\ \text{dominates} B_L\} \subset Q_r(C).
\]

Then \(Q_\leq, B_{nL}(C) \subset Q_\leq, B_{n'L}(C)\) for \(n < n'\). Due to Corollary 3.1 and Proposition 4.6, for \(\lambda < \lambda_c\) we have the relation,

\[
Q_{r,\lambda}(C) \subset \bigcup_{n=1}^\infty Q_\leq, B_{nL}(C),
\]

and the relation (1). The second one is also obtained similarly. \(\square\)

4.3. Conformal Map and Fractal Structure. In this subsection, let us consider the conformal map by letting \(\beta = 1\) of [14] for \(\gamma_0 \to 0\). For an element \(X \in Q_r(B_L)\), we consider

\[
u_0^- = f_0^+(X) := \frac{\Gamma_{f_0^+}(X)}{\gamma_0}u_0^+, \quad \text{or} \quad u_0^- = f_0^-(X) := \frac{\Gamma_{f_0^+}(X)}{\gamma_0}u_0^-.
\]

Then we directly have the following lemma:

**Lemma 4.3.**

1. For \(X \in Q_\leq(B_L)\) and a fixed \(u_0^+ (> 0)\), the map

\[
\psi^X : X^c \to I_{f_0^+}^\pm(X)
\]

is conformal and its image is \(\tau = I_{f_0^+}^\pm(X)\) of a certain \(s \in S_c(I_{f_0^+}^\pm(X))\).

The arclength of \(C_v^+(X)\) for each \(v \in (0, u_0^+)\) is given by

\[
\int_{[0,f_0^+(X)]} \left| \frac{\partial z}{\partial u} \right| (u,v) du.
\]
For $X \in \mathcal{Q}_{>}(B_{L})$ and a fixed $u_{0}^{-} (> 0)$, the map
\[
\psi^{X} : X \rightarrow I^{f_{u_{0}^{-}}(X)}
\]
is conformal and its image is $\mathfrak{F} = I^{f_{u_{0}^{-}}(X)}_{u_{0}^{-}}$ of a certain $s \in \mathcal{S}_{>}(I^{f_{u_{0}^{-}}(X)})$.

The arclength of $C_{u}^{-}$ for each $u \in (0, u_{0}^{-})$ is given by
\[
\int_{[0,f_{u_{0}^{-}}(X)]} \left| \frac{\partial z}{\partial u} \right| (u,v)dv.
\]

We consider the large $n$ limit of the map $\psi^{X} : B_{nL} \rightarrow I^{u_{n}^{-}}_{u_{0}^{-}}$. In the limit, the homogenized conductivity is a function of the volume fraction.

We consider the conformal property of $X \in \mathcal{Q}_{>}(\mathbb{C})$ under the limit by letting $f_{u_{0}^{-}}^{+}(p) := \Gamma_{\gamma_{0}}^{+}(p)u_{0}^{-}/\gamma_{0}$ and $f_{u_{0}^{-}}^{-}(p) := \Gamma_{\gamma_{0}}^{-}(p)u_{0}^{-}/\gamma_{0}$ $\mathbb{P}_{\lambda(p)}$-a.s.

Noting Lemma 4.2, we have the following lemma:

**Lemma 4.4.** Assume the continuous property of $\Gamma_{\gamma}^{\pm}(p)$. Then the following properties hold:

1. $I^{u_{n}^{-}}_{f_{u_{0}^{-}}^{+}(p)}$ approaches to $[0, \infty] \times [0, u_{0}^{-}]$ for $p \not> p_{c}$ and $\gamma_{0} \rightarrow 0$, and
2. $I^{f_{u_{0}^{-}}^{-}}(p)$ approaches to $[0, u_{0}^{-}] \times [0, \infty]$ for $p \not< p_{c}$ and $\gamma_{0} \rightarrow 0$.

**Proof.** It is obvious. \(\square\)

In order to see the meaning of Lemma 4.4, we will consider paths $\xi$ in $\mathcal{Q}_{>}(\mathbb{C})$:
\[
\text{Path}_{\mathcal{Q}_{>}}(\mathbb{C}) := \{ \xi : [0, 1] \rightarrow \mathcal{Q}_{>}(\mathbb{C}) \mid \forall f(\xi(p)) = p, \xi(p) \in \mathcal{Q}_{>}(\mathbb{C}) \}
\]
\[
\xi(p) \in \xi(p') \text{ for every } p \leq p'.
\]

For a given $\xi \in \text{Path}_{\mathcal{Q}_{>}}(\mathbb{C})$, there is an integer $N(\xi)$ such that for every $n > N(\xi)$, $\Gamma_{B_{nL}}^{+}(\xi(p) \cap B_{nL})$ is a monotonic increasing function of $p \in [0, 1]$. We fix $n > N(\xi)$.
\[
p_{c}(\xi, n) := \inf \{ p \in [0, 1] \mid \lim_{\gamma_{0} \rightarrow 0} \Gamma_{B_{nL}}^{+}(\xi(p) \cap B_{nL}) \neq 0 \}.
\]
\[
p_{c}^{-}(\xi, n) := \sup \{ p \in [0, 1] \mid \lim_{\gamma_{0} \rightarrow 0} \Gamma_{B_{nL}}^{-}(\xi(p) \cap B_{nL}) \neq 0 \}.
\]

We also have the conformal map $\psi^{\xi(p)} : \xi(p)^{c} \cap B_{nL} \rightarrow I^{u_{n}^{-}}_{f_{u_{0}^{-}}^{+}(\xi(p))}$ denoted by $\psi^{\xi(p), n}$ for $p \in [0, p_{c}(\xi, n))$, and the conformal map $\psi^{\xi(p)} : \xi(p) \cap B_{nL} \rightarrow I^{f_{u_{0}^{-}}^{-}(\xi(p))}$ denoted by $\psi^{\xi(p), n}$ for $p \in (p_{c}(\xi, n), 1]$.

Recalling Lemma 4.2, the finite version of Lemma 4.4 is given by the following theorem:

**Theorem 4.2.** For $\xi \in \text{Path}_{\mathcal{Q}_{>}}(\mathbb{C})$, $n > N(\xi)$, and $\psi^{\xi(p), n}$, assume that there are a positive number $\varepsilon$ and a monotonic increasing continuous functions $g$ and $g^{*}$ over $(-\varepsilon, \varepsilon)$ such that $g(0) = g^{*}(0) = 0$, $\Gamma_{B_{nL}}^{+}(\xi(p) \cap B_{nL}) = \sqrt{\gamma_{0}}(1 + g(p - p_{c}))$ for $p \in (p_{c}(\xi, n) - \varepsilon, p_{c}(\xi, n) + \varepsilon)$, and $\Gamma_{B_{nL}}^{-}(\xi(p) \cap B_{nL}) = \sqrt{\gamma_{0}}(1 - g^{*}(p - p_{c}))$ for $p \in (p_{c}^{-}(\xi, n) - \varepsilon, p_{c}^{-}(\xi, n) + \varepsilon)$. Then
\begin{enumerate}
\item (1) the length $f_{u_{0}^{-}}^{+}(\xi(p))$ of the image of $C_{u}^{+}(X)$ by $\psi^{\xi(p), n}$ diverges for $p \not> p_{c}(\xi, n)$ and $\gamma_{0} \rightarrow 0$, and
\item (2) the length $f_{u_{0}^{-}}^{-}(\xi(p))$ of the image of $C_{u}^{-}(X)$ by $\psi^{\xi(p), n}$ diverges for $p \not< p_{c}^{-}(\xi, n)$ and $\gamma_{0} \rightarrow 0$.
\end{enumerate}

**Remark 4.2.** Let us give an observation to mention the meaning of Theorem 4.2 by handling the case of $p < p_{c}(\xi, n)$ and $(\alpha, \beta) = (1/\gamma_{0}, 1)$ of [14] under the assumptions. There we have the two important properties:
The expectation values for \( n \to \infty \):

\[
E_{\lambda(p,\lambda)} \left( \frac{\partial u^+}{\partial y} (u,v) \right) = \frac{u^+_0}{nL}, \quad E_{\lambda(p,\lambda)} \left( \frac{\partial u^-}{\partial x} (u,v) \right) = \frac{u^-_0}{nL} \Gamma^+(p). \]

The first one is finite but the second one diverges for the limit \( p \nearrow p_c \).

(2) The conformal property for \( z \in \xi(p)^c \):

\[
\frac{\partial u^-}{\partial z} = \sqrt{\frac{\partial u^+}{\partial z}}, \quad \left| \frac{\partial u^-}{\partial z} \right| = \left| \frac{\partial u^+}{\partial z} \right|.
\]

For \( \xi(p) \) of \( p \to p_c(\xi,n) \), both properties mean that there exists, at least, a narrow slit such that two clusters connected with \( \partial B_{nL} \) and \( \partial B_{nL} \) respectively face each other distance \( d \). (Figure 2 (b) shows that there is such a slit.) Then at the slit, the intensity \( \left| \frac{\partial u^+}{\partial z} \right| \) has the order of \( \frac{u^+_0}{d} \). If \( d \) is proportional to \((p_c(\xi,n) - p)^b \) of \( b > 0 \), the intensity diverges for the limit \( p \nearrow p_c(\xi,n) \) and \( \gamma_0 \to 0 \).

Let us assume that there are such slits in \( B_{nL} \). Then the assumption consists with the fact that the average along \( y \) direction is finite. In other words, we realize the situation that the second property of the conformal structure holds for every \( z \in \xi(p)^c \) and in the first property of the expectation value, \( E_{\lambda(p,\lambda)} \left( \frac{\partial u^+}{\partial y} \right) \) is finite but \( E_{\lambda(p,\lambda)} \left( \frac{\partial u^-}{\partial x} \right) \) diverges for \( p \nearrow p_c(\xi,n) \) and \( n \to \infty \).

It means that the family of \( \{ C^+_v(\xi(p)) \}_v \) gather in the very narrow slit along a path from \( \partial B_{nL} \) to \( \partial B_{nL} \). The density of the curves at the cross section should be parameterized by \( (p_c(\xi,n) - p)^b \) of \( b > 0 \). Due to the ergodic properties, the position of the slits and the cluster are governed by a point process associated with the Poisson point process. (It reminds us of the continuum fractal percolation \([20]\).) Since the equipotential curves \( C^+_v(\xi(p)) \) do not penetrate into \( \xi(p) \), they must go along the front of these clusters, whereas the front is expected to have a fractal structure \([10]\). If the slit \( d \) is bounded from below \( d > \epsilon \), from Lemma \( [32](1) \), \( \left| \frac{\partial z}{\partial u} \right| \) does not vanish and thus the arclength \( C^+_v(\xi(p)) \) has the infinite length for the limit \( p \nearrow p_c(\xi,n) \) and \( \gamma_0 \to 0 \) under the assumption.

When we consider the small distance and the fractal structure, we also consider the limit of the radius \( r \to 0 \) by keeping the volume fraction \( p \), which is relatively the same as large \( n \)-limit of \( B_{nL} \) fixing \( r \). Hence we can go on to consider the finite \( r \) for a while.

We are concerned with the probability of existence of the slits which has the gap \( d < \epsilon \) for given \( \epsilon \) (parameterized by finite \( r \)). More precisely, we are interested in the positions in which the intensity \( |\nabla u^+| \) is greater than \( 1/\epsilon' \) of \( \epsilon' > 0 \). Since \( \Gamma^+(p) \) is finite, we may have the fact \( \lim_{r \to 0} E_{\lambda}(\ell(\{ z \in C \mid |\nabla u^+(z)| > 1/\epsilon' \}) = 0 \) for \( n \to \infty \). Even for the limit \( p \nearrow p_c(\xi,n) \). Since the essentials preserve even for finite \( n \), it means that the length of \( C^+_v(\xi(p)) \) may basically diverge for sufficiently large \( n \) under the limit \( p \nearrow p_c(\xi,n) \). The situation that curves with infinite length are embedded into a finite region such as \( B_{nL} \) might be related to the fractal structure.

As we mentioned numerically in Ref. \([19]\), based on the above arguments, we conjecture the following propositions for \( n \to \infty \):

1. The equipotential curve \( C^+_v(\xi(p)) \subset \xi(p)^c \) \((v \in (0,u^+_0))\) has the fractal structure \( P_{\lambda(p)} \)-a.s. for the limit \( p \nearrow p_c \), \( r \to 0 \), and \( \gamma_0 \to 0 \), and
2. the equipotential curve \( C^-_v(\xi(p)) \subset \xi(p) \) \((u \in (0,u^-_0))\) has the fractal structure \( P_{\lambda(p)} \)-a.s. for the limit \( p \searrow p^*_c \), \( r \to 0 \), and \( \gamma_0 \to 0 \).
Remark 4.3. Instead of the large limit in above arguments, we will consider another limit. For \( X \in Q_<(B_{nL}) \), by extending \( I_{mL}^{u_0^X} \) to \( I_{mL}^{u_0^X} \) we also have a conformal map \( \psi_{m,n}^X : B_{nL} \rightarrow I_{mL}^{u_0^X} \). By taking the limit \( n,m \rightarrow \infty \) and compactification of them appropriately, the limit of the map \( \psi_{m,n}^X \) could be regarded as a map between Riemann spheres \( PC \). Then \( \psi_{m,n}^X \) has the properties that its degree of the cover is 1 and each ramification index at each slit in \( I_{mL}^{u_0^X} \) is also 1. For the case of finite points \( \hat{X} \in \hat{Q}(PC) \), it is related to the cylindrical flow \( \psi_{flow}^{(N)} \) problem to dilute \( N \) cylinders case;

\[
\psi_{flow}^{(N)} = \frac{u_0}{L} \left( z + \sum_{i=0}^{N} \frac{\rho^2}{z - z_i} \right)
\]

for \( \rho \ll \min_{i,j,i \neq j} |z_i - z_j| \) illustrated in Figure 4.

![Figure 4. A cylindrical flow \( \psi_{flow}^{(N)} \)](image)

Since it is stated from the viewpoint of the conformal field theory [3] that two-dimensional turbulence has the fractal dimension of 4/3 whereas the fractal dimension of the equipotential curve in our system is also conjectured [19] to be 4/3, the two-dimensional flow problem and our model might be connected.

Further the fact that the volume fraction \( p \) parameterizes the deformation of \( I_{mL}^{u_0^X} : [0,1]^2 \rightarrow [0,1] \times [0,\infty] \) recalls the fundamental region of the elliptic module \( \tau \) and then the duality might correspond to the Jacobi transformation \( \tau \rightarrow 1/\tau \).

5. Numerical Computations of Conductivity in OS- and VS-types

As we did in Ref. [19], we numerically solved the generalized Laplace equations (5) with (6) for \( X_{p,i,s} \) which is parameterized by the volume fraction \( p \) and pseud random seed \( i,s \). By monitoring the volume fraction \( \text{vf}_{B_L}(X_n) \), we set the particles in \( B_L \) one by one as long as \( \text{vf}_{B_L}(X_n) \leq p \) for the given volume fraction \( p \). We found the step \( n(p) \) such that \( \text{vf}_{B_L}(X_{n(p)-1}) \leq p \) and \( \text{vf}_{B_L}(X_{n(p)}) > p \). Since for sufficiently large size of box, the difference \( \delta \text{vf}_{B_L}(X_{n(p)-1}) := \text{vf}_{B_L}(X_{n(p)}) - \text{vf}_{B_L}(X_{n(p)-1}) \) is sufficiently small, we regard \( \text{vf}_{B_L}(X_{n(p)}) \) as each \( p \) hereafter under this accuracy.
Since we used the pseudo-randomness to simulate the random configuration $X_{n(p)}$ for given $p$, the configuration $X_{n(p)}$ depends upon the seed $i_s$ of the pseudo-random which we choose and thus let it be denoted by $X_{p,i_s}$. For the same seed $i_s$ of the pseudo-random, a configuration $X_{p,i_s}$ of a volume fraction $p$ naturally contains a configuration $X_{p',i_s}$ of $p' < p$ due to our algorithm, i.e.,

$$X_{p',i_s} \subset X_{p,i_s}.$$ 

In other words, we handle a path $\xi_{i_s}$ parameterized $i_s$ in $Q_r(B_L)$ as in Path $Q_r(C)$. $\xi_{i_s} : [0,1] \to Q_r(B_L)$ such that $\text{vf}_{B_L}(\xi(p)) = p$, $\xi(p) \in Q_{r,p}(B_L)$ and $\xi_{i_s}(p) \subset \xi_{i_s}(p')$ for every $p \leq p'$.

Hence the elements in the set of the configurations $\{X_{p,i_s} | p \in [0,1]\}$ with the same seed $i_s$ are relevant. Figure 5 illustrates the configurations of the seed $i_s = 1$ for several $p$'s.

**Figure 5.** The configurations of the seed $i_s = 1$ with the volume fractions 0.5, 0.6, 0.65, 0.7, 0.75, 0.8 for (a), (b), (c), (d), (e), (f) respectively.

5.1. **Numerical Parameters of Computations.** In the finite difference method computations, we used three lattices, $1024 \times 1024$, $2048 \times 2048$, and $4096 \times 4096$, to represent $B_L$, $B_{2L}$ and $B_{3L}$ respectively. Since the radius of the particle $r$ corresponds to 25 meshes, we regard that $L = 40.96r$. Hence the difference $\delta \text{vf}_{B_L}(X_{n(p)-1})$ is determined at most $1.87 \times 10^{-3}$ for $B_L$, $4.68 \times 10^{-4}$ for $B_{2L}$, and $1.17 \times 10^{-4}$ for $B_{3L}$. We set the infinitesimal conductivity $\gamma_0 = 10^{-4}$ and the conductive one $\gamma_1 = 1$ as in Ref. [19].

5.2. **Results of Numerical Computations.**
5.2.1. **Conductivity Curve.** We used the finite difference method to find the solutions of (5) by handling the binary conductivity distribution \( \gamma(x, y) \), as in the previous work [19].

Figure 6(a) exhibits the linear scale behavior of average of the total conductivities \( \Gamma^{\pm}_{B_{3L}}(X) \) in \( B_{3L} \) whereas Figure 6(b) shows its logarithm property. Even though Figure 6(b) illustrates the property of the binary materials, the linear scale behavior of the conductivity curves are described well by

\[
\Gamma^+(p) = \frac{(p - p_c)^{t^+}}{(1 - p_c)^{t^+}} \vartheta(p - p_c),
\]

\[
\Gamma^-(p) = \frac{(p_c^* - p)^{t^-}}{(p_c^*)^{t^-}} \vartheta(p_c^* - p),
\]

where \( p_c \) and \( p_c^* \) are the thresholds, \( t^+ \) and \( t^- \) are the critical exponents, or merely called exponents, and \( \vartheta \) is a Heaviside function, i.e., \( \vartheta(x) = 1 \) if \( x \geq 0 \) and vanishes otherwise.

We computed six paths \( \xi_i \) with different seeds \( i \) of the pseudo-randomness for each \( B_{aL} \), \( a = 1, 2, 3 \). Using the results we determined the thresholds \( p_c \) and the exponents \( t^+ \) for each path using the least mean square method *in the linear scale resolution* respectively. The obtained results are shown in Table 2. Let the fluctuations mean the interval of the minimal and maximal values. It shows that the dependence of the threshold and the exponent on the size of boxes. The larger size of boxes is, the smaller the fluctuations of the threshold and the exponent are.

**Table 2.** The Size dependence of the threshold and the exponent:

|      | Threshold Average | Max-Min | Exponent Average | Max-Min |
|------|-------------------|---------|------------------|---------|
|      | \( B_L \)         |         |                  |         |
| OS-type | \( B_{2L} \)   | 0.664   | 0.059            | 1.297   | 0.312 |
|       | \( B_{3L} \)   | 0.661   | 0.032            | 1.312   | 0.190 |
| VS-type | \( B_{2L} \)   | 0.676   | 0.028            | 1.433   | 0.087 |
|       | \( B_{3L} \)   | 0.669   | 0.042            | 1.390   | 0.125 |
5.2.2. Fractal Dimension of Equipotential Curves. We computed the fractal dimensions of these equipotential curves which are shown in Figure 7 as in Ref. [19]; we computed only OS-type there. On the computation of the fractal dimensions, we used the box-counting method [15]. Since the curves in Figure 2 of the seed \( i_s = 1 \) seem to have the fractal structure, their numerical evaluations of the fractal dimensions in Figure 7 show that both equipotential curves of OS- and VS-types might have non-trivial fractal dimension.

**Figure 7.** The fractal dimension vs. volume fraction \( p \) for six seeds: (a) for OS-type, (b) for VS-type.

6. Approximation Formulae of Conductivity Curves

In this section, we consider the conductivity curves \( \Gamma^\pm(p) \) over all of the volume fraction \([0,1]\). Using the Keller-Dykhne reciprocity law [7,12] in Proposition 4.3, we consider the following formulae \( \Gamma_0^\pm(p) \):

**Lemma 6.1.** For sufficiently small \( \gamma_0 > 0 \) and \( L \to \infty \),

\[
\Gamma_0^+ (p) := \gamma_1 \frac{(p - p_c)^t}{(1 - p_c)^t} \vartheta(p - p_c) + \frac{(p_c)^t \sqrt{\gamma_1 \gamma_0}}{(p_c - p)^t \sqrt{\gamma_1 \gamma_0} \vartheta(p_c - p) + (p_c)^t},
\]

\[
\Gamma_0^- (p) := \gamma_1 \frac{(p_c - p)^t}{(p_c)^t} \vartheta(p_c - p) + \frac{(p_c)^t \sqrt{\gamma_1 \gamma_0}}{(p - p_c)^t \sqrt{\gamma_1 \gamma_0} \vartheta(p - p_c) + (1 - p_c)^t}.
\]

the following relations hold:

1. \( \Gamma_0^+ (p)\Gamma_0^- (p) = \gamma_0 \gamma_1 \).
2. \( \Gamma_0^+ (p) = \sqrt{\gamma_0 \gamma_1} \left( 1 + \sqrt{\frac{\gamma_1}{\gamma_0} \frac{(p - p_c)^t}{p_c} + O((p - p_c)^2)} \right) \) for \( p \in (p_c(1 - 2\sqrt{\gamma_1/\gamma_0}), p_c(1 + 2\sqrt{\gamma_1/\gamma_0})) \).
3. \( \Gamma_0^- (p) = \sqrt{\gamma_0 \gamma_1} \left( 1 - \sqrt{\frac{\gamma_1}{\gamma_0} \frac{(p_c - p)^t}{p_c} + O((p - p_c)^2)} \right) \) for \( p \in (p_c(1 - 2\sqrt{\gamma_1/\gamma_0}), p_c(1 + 2\sqrt{\gamma_1/\gamma_0})) \).
Lemma 6.1 for (p,0) . Figure 8 shows that our formulae (19) for (3) for (p,0), we expand \( \Gamma_0^+ (p) \) by \( \sqrt{\gamma_1} \left( 1 - \sqrt{\gamma_1} \frac{p_c - p}{p_c} + \cdots \right) \). Similarly we also have (3) for \( (p_c, p_c(1 + \sqrt{\gamma_1})), \) The relation (1) gives the (2) and (3).

Remark 6.1. Figure 8 shows that our formulae \( (19) \) for \( (t = 1.451, p_c = 0.656) \) exhibits well the numerical computational results. It implies that the results are expressed by continuous curves. We conjecture that \( \Gamma^\pm (p) \) is approximated well by \( \Gamma_0^+ (p) \), whose behavior around \( p_c \) are given by the formulae in (2) and (3) of Lemma 6.1 for \( (p_c(1 - \sqrt{\gamma_1}/\gamma_0), p_c(1 + \sqrt{\gamma_1}/\gamma_0)) \).

Figure 8. The approximation formulae and computed results: (a) is the two dimensional case and (b) the three dimensional case. The curves are given by the formulae (6.1) and (6.2) and + and × are the average of numerical computations for six seeds.
Remark 6.2. It is expected that the formula (19) might be generalized to three dimensional case. As illustrated in Figure 8, the computational results of three dimensional CPM computed in Ref. [18] may be written by

$$\Gamma_{3d}^{+}(p) = \gamma_1 \frac{(p-p_c)^t}{(1-p_c)^t} \gamma_2 p - p_c + \frac{(p_c)^t}{(p_c-p)^t} \sqrt{\gamma_1} \gamma_0 \gamma_2 (p_c - p) + (p_c)^t t',$$

where $t = 1.7$, $t' = 1.2$, and $p_c = 0.25$. Figure 8 (b) shows this formula also represents well the numerical computational results in Ref. [18].

In Ref. [17], we investigated the shape effects in the continuum percolation conductivity problems and in Ref. [18], we studied the agglomeration effects on the homogenized conductivity. In terms of this novel formula, we can parameterize these conductivity properties in Refs. [17] and [18] more precisely. Further it might be also related to the elastic properties in CPM as in [20].

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REFERENCES

[1] N. Ansini, G. Dal Maso and C. Ida Zeppieri, $\Gamma$-convergence and $H$-convergence of linear elliptic operator, J. Math. Pures Appl. 99 (2013) 321-329.
[2] K. Astala, T. Iwaniec and G. Martin, Elliptic partial differential equations and quasiconformal mappings in the plane, (Princeton, 2008).
[3] D. Bernard, G. Boffetta, A. Celani and G. Falkovich, Conformal invariance in two-dimensional turbulence, Nature Physics, 2 (2006) 124-128.
[4] A. Bourgeat, A. Mikelić and S. Wright, Stochastic two-scale convergence in the mean and applications, J. die reine angew. Math. 456 (1994) 19-51.
[5] H. Brezis, Functional analysis, Sobolev spaces and partial differential equations, (Springer, 2011).
[6] G. Dal Maso and L. Modica, Nonlinear stochastic homogenization and ergodic, J. die reine angew. Math. 326 (1986) 28-42.
[7] A. M. Dykhne, Conductivity of a two-dimensional two-phase system, Sovit Phys. JETP 32 (1971) 63-65.
[8] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, (Springer, 2008).
[9] E. De Giorgi, Ennio De Giorgi: selected papers, (Springer, 2006).
[10] G. Grinstein, Percolation, second ed., (Springer, (Berlin, 1999).
[11] D. Jeulin, Morphology and effective properties of multi-scale random sets: A review, C. R. Mecanique 340 (2012) 219-229.
[12] J. B. Keller, A theorem on the conductivity of a composite medium, J. Math. Phys. 5 (1964) 548-549.
[13] D. Kontogiannis, Homogenization problems in random media, Iowa State University, Thesis (2010).
[14] S. M. Kozlov, Geometric aspect of averaging, Russian. Math. Surveys 44 (1989) 91-144.
[15] B. Mandelbrot, Fractal geometry of nature, (Freeman, 1982).
[16] V. A. Marcher and E. Y. Khruhshchev, Homogenization of partial differential equations, (Birkhäuser, 2004).
[17] S. Matsutani and Y. Shimosako and Y. Wang, Numerical computations of conductivity in continuum percolation for overlapping spheroids, Int. J. Mod. Phys. C 21 (2010) 709-729.
[18] S. Matsutani and Y. Shimosako and Y. Wang, Numerical computations of conductivity over agglomerated continuum percolation modes, Appl. Math. Modeling 37 (2013) 4007-4022.
[19] S. Matsutani and Y. Shimosako and Y. Wang, Fractal structure of equipotential curves on a continuum percolation model, Physica A 391 (2012) 5802-5809.
[20] R. Meester and R Roy, Continuum percolation., Cambridge Tracts in Mathematics 119 (Cambridge, 1996).
[21] F. Murant and L. Tartar, $H$-convergence, 21-44 Topics in the mathematical modelling of composite materials, edited by A. Cherkaev and R. Kohn, (Birkhäuser, 1997).
[22] F. Murant and L. Tartar, Calculus of variations and homogenization, 139-174 Topics in the mathematical modelling of composite materials, edited by A. Cherkaev and R. Kohn, (Birkhäuser, 1997).
[23] D. Stoyan, W. S. Kendall and J. Mecke, *Stochastic geometry and its applications*, second ed., (Wiley, 1995).

[24] N. S. Trudinger, Linear elliptic operators with measurable coefficients, *Annali Scuola Normale Superiore Pisa* 27 (1973) 265-308.

[25] N. S. Trudinger, Maximum principles for linear, non-uniformly elliptic operators with measurable coefficients, *Math. Z.* 156 (1977) 291-301.

[26] F. Willot and D. Jeulin, Elastic and electrical behavior of some random multiscale highly-contrasted composites, *Int. J. Multiscale Comp. Eng.* 9 (2011) 305-326.

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