The Neyman-Pearson lemma for convex expectations

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Abstract. We study the Neyman-Pearson problem for convex expectations on $L^\infty(\mu)$. The existence of the optimal test is given. Without assuming that the level sets of penalty functions are weakly compact, we prove that the optimal tests for convex expectations on $L^\infty(\mu)$ are just the classical Neyman-Pearson tests between a fixed representative pair of simple hypotheses. Then we show that the Neyman-Pearson problem for convex expectations on $L^1(\mu)$ can be solved similarly.

Key words. Convex risk measure, Hypothesis testing, Neyman-Pearson lemma, Minimax theorem

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1 Introduction

The classical Neyman-Pearson lemma gives the most powerful test for discriminating between two probability measures and has important applications in various fields (see [5], [16]).

It is well known that many phenomena need to be explored by nonlinear probabilities or expectations. In 1954, Choquet [3] extended the probability measure to the capacity and gave a nonlinear integral named after him. The coherent risk measure was proposed by Artzner et al. [1] and the $g$-expectation was initiated by Peng [17] in 1999. Föllmer and Schied [11] generalized the coherent risk measure to the convex risk measure in 2002. Divergence risk measures were considered by Ben-Tal and Teboulle [2] under the name of optimized certainty equivalents.

Along with the development of the above concepts, several nonlinear versions of Neyman-Pearson lemma have also been established. In 1973, Huber and Strassen [12] studied the Neyman-Pearson lemma for capacities. Cvitanić and Karatzas [4] extended the classical Neyman-Pearson theory for testing composite hypotheses versus composite alternatives which can also be understood as discriminating between two sublinear expectations in 2001. Later Schied [20] gave a Neyman-Pearson lemma for law-invariant coherent risk measures and robust utility functionals. Ji and Zhou [13] studied hypothesis tests for $g$-probabilities in 2008. Rudloff and Karatzas [18] studied composite hypothesis by using convex duality in 2010. Apart from their own theoretical value, the nonlinear versions of Neyman-Pearson lemma have been found to have many

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applications especially in finance. For instance, Föllmer and Leukert \[7\] and \[8\] studied the quantile hedging and efficient hedging which minimize the shortfall risk in an incomplete financial market. Rudloff \[19\] found a self-financing strategy that minimize the convex risk of the shortfall using convex duality method.

In most literatures, the convex duality method is employed to study the nonlinear Neyman-Pearson lemma or ”Neyman-Pearson type” optimization problem. Without assuming the set of densities which generate the sublinear expectation is weakly compact, Cvitanić and Karatzas \[4\] studied the Neyman-Pearson lemma for sublinear expectations on \(L^\infty(\mu)\). However, due to the additional penalty function terms, the convex duality method in \[4\] is difficult to apply to the case for convex expectations on \(L^\infty(\mu)\). In order to measure the shortfall risk, Föllmer and Leukert \[8\] adopted a specific convex risk measure and Rudloff \[19\] used the convex risk measure on \(L^1(\mu)\). In both cases, the sets of densities which generate the convex risk measure are weakly compact.

As in \[4\], it is natural to study the Neyman-Pearson lemma on \(L^\infty(\mu)\). So in this paper, we investigate the Neyman-Pearson lemma for convex expectations (convex risk measures) on \(L^\infty(\mu)\) and do not assume that the set of densities which generate the convex expectation is weakly compact. In more details, for two given convex expectations \(\rho_1, \rho_2\) and a significance level \(\alpha\), we want to find an optimal test \(X^*\) which minimize the convex expectation of Type II error, among all tests that keep the convex expectation of Type I error below the given acceptable significance level \(\alpha \in (0,1)\). In other words, we study the following problem:

\[
\minimize \rho_2(1 - X),
\]

over the set \(\mathcal{X}_\alpha = \{X \in L^\infty(\mu) : 0 \leq X \leq 1, \rho_1(X) \leq \alpha\}\).

Instead of being weakly compact, we only assume that the level sets of penalty functions are closed under the \(\mu\)-a.e. convergence. Under this assumption, we can’t directly apply the approach in \[4\]. The key to solving this problem is that we find the feasible set \(\mathcal{X}_\alpha\) is compact in the weak* topology \(\sigma(L^\infty, L^1)\). Based on this, we can apply the minimax theorem and find the representative probability measure \(Q^*\) for \(\rho_2\). By solving the dual problem, the representative probability measure \(P^*\) for \(\rho_1\) is also found. Thus, the optimal tests for convex expectations are just the classical Neyman-Pearson tests between a fixed representative pair \((P^*, Q^*)\).

As a by-product, we found that similar ideas can be used to solve the Neyman-Pearson problem for convex expectations on \(L^1(\mu)\). So we put this result in the appendix and gave a brief proof.

This paper is organized as follows: In Section 2, we give some preliminaries and formulate the simple hypothesis testing problem for convex expectations on \(L^\infty(\mu)\). The existence of the optimal tests is derived in section 3. In section 4, we obtain the form of the optimal tests. An application is given to illustrate our main result in section 5. Finally, in the appendix we show that if convex expectations are continuous from above, then Assumption \[4\] holds naturally and give the Neyman-Pearson lemma for convex expectations on \(L^1(\mu)\).

## 2 Preliminaries and Problem Formulation

Let \((\Omega, \mathcal{F}, \mu)\) be a probability space and \(\mathcal{M}\) be the set of probability measures on \((\Omega, \mathcal{F})\) that are absolutely continuous with respect to \(\mu\). \(P\) and \(Q\) are probability measures and their Radon-Nikodym derivatives \(\frac{dP}{d\mu}\)
Definition 2.1 A mapping \( \rho: L^\infty(\mu) \to \mathbb{R} \) is called a convex expectation if for any \( X, Y \in L^\infty(\mu) \), we have

(i) Monotonicity: If \( X \geq Y \), then \( \rho(X) \geq \rho(Y) \);

(ii) Invariance: If \( c \) is a constant, then \( \rho(X + c) = \rho(X) + c \);

(iii) Convexity: If \( \lambda \in [0, 1] \), then \( \rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y) \).

If we take \( \rho'(X) := \rho(-X) \), then \( \rho' \) is a convex risk measure.

Definition 2.2 We call a convex expectation \( \rho \) is continuous from below iff for any sequence \( \{X_n\}_{n \geq 1} \subset L^\infty(\mu) \) increases to some \( X \in L^\infty(\mu) \), then \( \rho(X_n) \to \rho(X) \).

The following theorem comes from Theorem 6 and Proposition 7 in [11].

Theorem 2.3 If a convex expectation \( \rho \) is continuous from below, then

i) For any \( X \in L^\infty(\mu) \),

\[
\rho(X) = \sup_{P \in \mathcal{P}} \left( E_P[X] - \rho^*(P) \right),
\]

where \( \rho^* \) is the penalty function of \( \rho \) and \( \rho^*(P) = \sup_{X \in L^\infty(\mu)} \left( E_P[X] - \rho(X) \right) \).

ii) For any bounded sequence \( \{X_n\}_{n \geq 1} \subset L^\infty(\mu) \), if \( X_n \) converges to some \( X \in L^\infty(\mu) \) in probability, then \( \rho(X) \leq \liminf_{n \to \infty} \rho(X_n) \).

Given two convex expectations \( \rho_1 \) and \( \rho_2 \), for a significance level \( \alpha \) and two random variables \( K_1 \) and \( K_2 \) belonging to \( L^\infty(\mu) \) such that \( 0 \leq K_1 < K_2 \), we are interested in the following problem:

\[
\text{minimize} \quad \rho_2(K_2 - X),
\]

over the set \( \mathcal{X}_\alpha = \{X : K_1 \leq X \leq K_2, \rho_1(X) \leq \alpha, X \in L^\infty(\mu)\} \). Without loss of generality, we assume \( \rho_1(K_1) \leq \alpha \leq \rho_1(K_2) \). Note that if \( K_1 = 0 \) and \( K_2 = 1 \), then the above problem becomes the problem [14].

We call \( X^* \) is the optimal test of (2.2) if \( X^* \in \mathcal{X}_\alpha \) and

\[
\rho_2(K_2 - X^*) = \inf_{X \in \mathcal{X}_\alpha} \rho_2(K_2 - X).
\]

By (i) of Theorem 2.3

\[
\rho_1(X) = \sup_{P \in \mathcal{P}} E_P[X] - \rho_1^*(P) \quad \text{and} \quad \rho_2(X) = \sup_{Q \in \mathcal{Q}} E_Q[X] - \rho_2^*(Q).
\]

If we denote

\[
\mathcal{P} = \{P : P \in \mathcal{M}, \rho_1^*(P) < \infty\} \quad \text{and} \quad \mathcal{Q} = \{Q : Q \in \mathcal{M}, \rho_2^*(Q) < \infty\},
\]

then \( \mathcal{P} \) and \( \mathcal{Q} \) are nonempty convex sets and

\[
\rho_1(X) = \sup_{P \in \mathcal{P}} E_P[X] - \rho_1^*(P) \quad \text{and} \quad \rho_2(X) = \sup_{Q \in \mathcal{Q}} E_Q[X] - \rho_2^*(Q).
\]

Thus, the problem (2.2) can also be considered as discriminating between two convex expectations \( \rho_1 \) and \( \rho_2 \) generated by \( \mathcal{P} \) and \( \mathcal{Q} \).
3 The existence of the optimal test

Set $\beta = \inf_{X \in \mathcal{X}_\alpha} \rho_2(K_2 - X)$. The following result shows that the optimal test exists.

**Theorem 3.1** If $\rho_1$ and $\rho_2$ are convex expectations continuous from below, then the optimal test of (2.2) exists.

**Proof.** Take a sequence $\{X_n\}_{n \geq 1} \subset \mathcal{X}_\alpha$ such that

$$\rho_2(K_2 - X_n) < \beta + \frac{1}{2^n}.$$ 

By the Komlós theorem, there exist a subsequence $\{X_{n_i}\}_{i \geq 1}$ of $\{X_n\}_{n \geq 1}$ and a random variable $X^*$ such that

$$\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^k X_{n_i} = X^*, \quad \mu - a.e.. \quad (3.1)$$

Since for any $n$, $K_1 \leq X_n \leq K_2$, we have $K_1 \leq X^* \leq K_2, \mu$-a.e.. By (ii) of Theorem 2.3,

$$\rho_1(X^*) \leq \lim \inf_{k \to \infty} \rho_1\left(\frac{1}{k} \sum_{i=1}^k X_{n_i}\right) \leq \lim \inf_{k \to \infty} \frac{1}{k} \sum_{i=1}^k \rho_1(X_{n_i}) \leq \alpha$$

which leads to $X^* \in \mathcal{X}_\alpha$. On the other hand,

$$\rho_2(K_2 - X^*) \leq \lim \inf_{k \to \infty} \frac{1}{k} \sum_{i=1}^k \rho_2(K_2 - X_{n_i}) \leq \beta + \lim_{k \to \infty} \frac{1}{k} = \beta.$$ 

Thus,

$$\rho_2(K_2 - X^*) = \inf_{X \in \mathcal{X}_\alpha} \rho_2(K_2 - X).$$

This completes the proof. ■

4 The form of the optimal test

Note that

$$\inf_{X \in \mathcal{X}_\alpha} \rho_2(K_2 - X) = \inf_{X \in \mathcal{X}_\alpha} \sup_{Q \in \mathcal{Q}} \left(E_Q[K_2 - X] - \rho_2^*(Q)\right).$$

Then $X^*$ is the optimal test of (2.2) if and only if it is the optimal test of the problem:

$$\minimize_{Q \in \mathcal{Q}} \left(E_Q[K_2 - X] - \rho_2^*(Q)\right), \quad (4.1)$$

over $\mathcal{X}_\alpha$.

Now we focus on solving the problem (4.1). Denote the level sets of penalty functions $\rho_1^*$ and $\rho_2^*$ as

$$\mathcal{G}_c = \{G_P : P \in \mathcal{P}, \rho_1^*(P) \leq c\} \quad \text{and} \quad \mathcal{H}_c = \{H_Q : Q \in \mathcal{Q}, \rho_2^*(Q) \leq c\},$$

where $c$ is a constant. Since $\rho_1^*$ and $\rho_2^*$ are convex functions on $M$, then both $\mathcal{G}_c$ and $\mathcal{H}_c$ are convex sets.

Since $K_1$ and $K_2$ belong to $L^\infty(\mu)$, we denote the least upper bound of them by $M$. 

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Lemma 4.3 For any sequence $Q_n$, there exists $Q$ such that $G$ leads to $E$ which leads to $Q$.

Since the penalty function of the sublinear expectation takes only the values $0$ and $+\infty$, for sublinear case, Assumption 4.1 is equal to require $\{G_P : P \in P\}$ and $\{H_Q : Q \in Q\}$ are both closed under the $\mu$-a.e. convergence, which is similar as the assumption given by Cvitanić and Karatzas in [4]. In Appendix, we show that if $\rho_1$ and $\rho_2$ are continuous from above, then Assumption 4.1 holds naturally.

4.1 The existence of a representative probability $Q^*$

In this subsection, we want to find a representative probability $Q^*$ such that

$$\inf_{X \in \mathcal{X}_\alpha} \sup_{Q \in Q} (E_Q[K_2 - X] - \rho_2^*(Q)) = \inf_{X \in \mathcal{X}_\alpha} E_Q^*[K_2 - X] - \rho_2^*(Q^*).$$

If such a $Q^*$ exists, then for any optimal test $X^* \in \mathcal{X}$, we have

$$\sup_{Q \in Q} (E_Q[K_2 - X^*] - \rho_2^*(Q)) = \inf_{X \in \mathcal{X}_\alpha} E_Q^*[K_2 - X] - \rho_2^*(Q^*),$$

which leads to $E_Q^*[K_2 - X^*] = \inf_{X \in \mathcal{X}_\alpha} E_Q^*[K_2 - X]$.

Theorem 4.2 If $\rho_1$ and $\rho_2$ are convex expectations continuous from below and Assumption 4.1 holds naturally, then there exists $Q^* \in Q$ such that for any optimal test $X^*$ of (4.2), we have

$$E_Q^*[K_2 - X^*] = \inf_{X \in \mathcal{X}_\alpha} E_Q^*[K_2 - X].$$

Before proving Theorem 4.2, we first give some lemmas.

Lemma 4.3 For any sequence $\{Q_n\}_{n \geq 1} \subset M$, if $H_{Q_n}$ converges to some $H_{Q_\infty}$ under $L^1(\mu)$ norm, then

$$\inf_{X \in \mathcal{X}_\alpha} E_{Q_\infty}[K_2 - X] \geq \limsup_{n \to \infty} \inf_{X \in \mathcal{X}_\alpha} E_{Q_n}[K_2 - X].$$

Proof. For any $X \in \mathcal{X}_\alpha$, we have

$$E_{Q_\infty}[K_2 - X] = \lim_{n \to \infty} E_{Q_n}[K_2 - X] \geq \limsup_{n \to \infty} \inf_{X \in \mathcal{X}_\alpha} E_{Q_n}[K_2 - X].$$

Then

$$\inf_{X \in \mathcal{X}_\alpha} E_{Q_\infty}[K_2 - X] \geq \limsup_{n \to \infty} \inf_{X \in \mathcal{X}_\alpha} E_{Q_n}[K_2 - X].$$

This completes the proof.

Lemma 4.4 If $\rho_1$ is a convex expectation continuous from below, then $\mathcal{X}_\alpha$ is compact in the weak* topology $\sigma(L^\infty(\mu), L^1(\mu))$.

Proof. Define $\phi(Y) = \sup_{X \in \mathcal{X}_\alpha} E_\mu[X \cdot Y]$ where $Y \in L^1(\mu)$. Then $\phi$ is a sublinear function on $L^1(\mu)$ and dominated by $\mathcal{M}||\cdot||_{L^1(\mu)}$. Set

$$\tilde{\mathcal{X}}_\alpha = \{X \in L^\infty(\mu) : E_\mu[X \cdot Y] \leq \phi(Y) \text{ for any } Y \in L^1(\mu)\}.$$
By Theorem 4.2 of chapter I in [21], \( \hat{X}_\alpha \) is compact in the weak* topology \( \sigma(L^\infty(\mu), L^1(\mu)) \). Then we only need to show

\[ \mathcal{X}_\alpha = \hat{X}_\alpha. \]

Since \( \mathcal{X}_\alpha \subset \hat{X}_\alpha \), obviously, in the next, we will show \( \hat{X}_\alpha \subset \mathcal{X}_\alpha \).

Firstly, for any \( \hat{X} \in \hat{X}_\alpha \), we show that \( K_1 \leq \hat{X} \leq K_2, \mu\text{-a.e.} \). If there exists \( \hat{X} \in \hat{X}_\alpha \) such that \( \mu(\omega : \hat{X}(\omega) < K_1) \neq 0 \), then there will exist a constant \( \epsilon > 0 \) such that \( \mu(A) \neq 0 \), where \( A = \{\omega : \hat{X}(\omega) \leq K_1 - \epsilon\} \).

For any \( X \in \mathcal{X}_\alpha \), since \( \hat{X} \leq K_1 - \epsilon \) on \( A \), we have \( \hat{X} \leq X - \epsilon \) on \( A \). Let \( h_A = -\frac{I_A}{\mu(A)} \). Then

\[ E_\mu[\hat{X}h_A] = -\frac{1}{\mu(A)} E_\mu[\hat{X}I_A] \geq -\frac{1}{\mu(A)} E_\mu[(X - \epsilon)I_A] = E_\mu[Xh_A] + \epsilon. \]

Due to \( X \) can be taken in \( \mathcal{X}_\alpha \) arbitrarily, we have

\[ E_\mu[\hat{X}h_A] \geq \sup_{X \in \mathcal{X}_\alpha} E_\mu[Xh_A] + \epsilon > \sup_{X \in \mathcal{X}_\alpha} E_\mu[Xh_A] = \phi(h_A). \]

Since \( h_A \in L^1(\mu) \), it contradicts with \( \hat{X} \in \hat{X}_\alpha \). Thus, \( \hat{X} \geq K_1, \mu\text{-a.e.} \). Similarly, we can prove \( \hat{X} \leq K_2, \mu\text{-a.e.} \).

Next, we show for any \( \hat{X} \in \hat{X}_\alpha \), \( \rho_1(\hat{X}) \leq \alpha \). Since \( \hat{X} \in \hat{X}_\alpha \), for any \( P \in \mathcal{P}, \)

\[ E_P[\hat{X}] = E_\mu[\hat{X}G_P] \leq \sup_{X \in \mathcal{X}_\alpha} E_\mu[XG_P] = \sup_{X \in \mathcal{X}_\alpha} E_P[X]. \]

Then

\[ \rho_1(\hat{X}) = \sup_{P \in \mathcal{P}} (E_P[\hat{X}] - \rho_1^*(P)) \]
\[ \leq \sup_{P \in \mathcal{P}} \sup_{X \in \mathcal{X}_\alpha} (E_P[X] - \rho_1^*(P)) \]
\[ = \sup_{X \in \mathcal{X}_\alpha} \sup_{P \in \mathcal{P}} (E_P[X] - \rho_1^*(P)) \]
\[ = \sup_{X \in \mathcal{X}_\alpha} \rho_1(X) \leq \alpha. \]

Thus, \( \hat{X} \in \mathcal{X}_\alpha \). ■

**Remark 4.5** If \( \rho_1 \) degenerates to be a sublinear expectation, the above result can also be found in [14].

**Lemma 4.6** If \( \rho_1 \) and \( \rho_2 \) are convex expectations continuous from below, then

\[ \inf_{X \in \mathcal{X}_\alpha} \sup_{Q \in \mathcal{Q}} (E_Q[K_2 - X] - \rho_2^*(Q)) = \sup_{Q \in \mathcal{Q}} \inf_{X \in \mathcal{X}_\alpha} (E_Q[K_2 - X] - \rho_2^*(Q)). \quad (4.5) \]

**Proof.** Since for each \( X \in \mathcal{X}_\alpha \), \( E_Q[K_2 - X] - \rho_2^*(Q) \) is a concave function on \( \mathcal{Q} \) and for each \( Q \in \mathcal{Q} \), \( E_Q[K_2 - X] - \rho_2^*(Q) \) is a linear continuous function on \( L^\infty(\mu) \), with \( \mathcal{X}_\alpha \) is compact in the weak* topology \( \sigma(L^\infty(\mu), L^1(\mu)) \), then by the minimax theorem (Refer to Theorem 3.2 of chapter I in [21]), the equation (4.5) holds. ■

The following lemma shows that \( \rho^* \) is lower semi-continuous.
Lemma 4.7 If \( \rho \) is a convex expectation continuous from below, for any sequence \( \{Q_n\}_{n \geq 1} \subset \mathcal{M} \) and \( Q_0 \in \mathcal{M} \) such that \( H_{Q_n} \) converges to \( H_{Q_0} \), \( \mu \)-a.e., then

\[
\rho^*(Q_0) \leq \liminf_{n \to \infty} \rho^*(Q_n).
\]

**Proof.** Set

\[
L_+^\infty(\mu) = \{X \in L^\infty(\mu) : X \geq 0\}.
\]

Then \( \rho^* \) can be redefined as

\[
\rho^*(Q) = \sup_{X \in L_+^\infty(\mu)} (E_Q[X] - \rho(X)),
\]

since \( E_Q[X] - \rho(X) = E_Q[X + m] - \rho(X + m) \) for any \( Q \in \mathcal{M}, X \in L^\infty(\mu) \) and positive real number \( m \).

Take \( J_k = \inf_{n \geq k} H_{Q_n} \). Then \( \{J_k\}_{k \geq 1} \) is an increasing sequence and \( H_{Q_0} = \sup_{k \geq 1} J_k \). We have

\[
\rho^*(Q_0) = \sup_{X \in L_+^\infty(\mu)} (E_{\mu}[X(\sup_{k \geq 1} J_k)] - \rho(X))
\]

\[
= \sup_{k \geq 1} \sup_{X \in L_+^\infty(\mu)} (E_{\mu}[XJ_k] - \rho(X))
\]

\[
= \sup_{k \geq 1} \sup_{X \in L_+^\infty(\mu)} (E_{\mu}[\inf_{n \geq k} (XH_{Q_n})] - \rho(X))
\]

\[
\leq \sup_{k \geq 1} \sup_{X \in L_+^\infty(\mu)} (E_{Q_n}[X] - \rho(X))
\]

\[
\leq \liminf_{n \to \infty} \rho^*(Q_n).
\]

This completes the proof. \( \Box \)

Lemma 4.8 If \( \rho_1 \) and \( \rho_2 \) are convex expectations continuous from below and Assumption 4.1 holds, then there exists \( Q^* \in \mathcal{Q} \) such that

\[
\inf_{X \in \mathcal{A}_0} E_{Q^*}[K_2 - X] - \rho_2^*(Q^*) = \sup_{Q \in \mathcal{Q}} \inf_{X \in \mathcal{A}_0} (E_Q[K_2 - X] - \rho_2^*(Q)).
\]  (4.6)

**Proof.** Take a positive constant \( 0 < \epsilon < 1 \) and a sequence \( \{Q_n\}_{n \geq 1} \subset \mathcal{Q} \) such that

\[
\inf_{X \in \mathcal{A}_0} E_{Q_n}[K_2 - X] - \rho_2^*(Q_n) \geq \gamma - \frac{\epsilon}{2^n},
\]

where \( \gamma = \sup_{Q \in \mathcal{Q}} \inf_{X \in \mathcal{A}_0} (E_Q[K_2 - X] - \rho_2^*(Q)) \). By Lemma 4.7

\[
\gamma = \inf_{X \in \mathcal{A}_0} \sup_{Q \in \mathcal{Q}} (E_Q[K_2 - X] - \rho_2^*(Q)) = \inf_{X \in \mathcal{A}_0} \rho_2(K_2 - X).
\]

Since

\[
\rho_2(0) \leq \inf_{X \in \mathcal{A}_0} \rho_2(K_2 - X),
\]

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then \( \rho_2(0) \leq \gamma \). For any \( n \),

\[
M - \rho_2^*(Q_n) \geq \inf_{X \in \mathcal{X}_n} E_{Q_n}[K_2 - X] - \rho_2^*(Q_n) \geq \gamma - \frac{\epsilon}{2n} \geq \gamma - \epsilon,
\]

which leads to

\[
\rho_2^*(Q_n) \leq M - \gamma + \epsilon \leq M - \rho_2(0) + 1.
\]

For \( v \) defined in Assumption 4.4, we have \( \rho_2^*(Q_n) \leq v \) which implies \( \{H_{Q_n}\}_{n \geq 1} \subseteq \mathcal{H}_v \).

By the Komlós Theorem, there exist a subsequence \( \{Q_{n_i}\}_{i \geq 1} \) of \( \{Q_n\}_{n \geq 1} \) and a random variable \( H^* \in L^1(\mu) \) such that

\[
\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} H_{Q_{n_i}} = H^*, \quad \mu \text{-a.e.}
\]

Since \( \mathcal{H}_v \) is a convex set and closed under the \( \mu \)-a.e. convergence, then \( H^* \in \mathcal{H}_v \). Denote \( Q^* \) as the corresponding probability measure of \( H^* \). Since

\[
1 = E_{\mu}[H^*] = \lim_{k \to \infty} E_{\mu}[\frac{1}{k} \sum_{i=1}^{k} H_{Q_{n_i}}],
\]

we have \( \left\{ \frac{1}{k} \sum_{i=1}^{k} H_{Q_{n_i}} \right\}_{k \geq 1} \) converges to \( H^* \) under \( L^1(\mu) \) norm. By Lemma 4.3 and Lemma 4.7

\[
\inf_{X \in \mathcal{X}_n} E_{Q^*}[K_2 - X] - \rho_2^*(Q^*) \\
\geq \limsup_{k \to \infty} \inf_{X \in \mathcal{X}_n} E_{\mu}[(K_2 - X)(\frac{1}{k} \sum_{i=1}^{k} H_{Q_{n_i}})] - \liminf_{k \to \infty} \rho_2^*(\frac{1}{k} \sum_{i=1}^{k} Q_{n_i}) \\
\geq \limsup_{k \to \infty} \inf_{X \in \mathcal{X}_n} \frac{1}{k} \sum_{i=1}^{k} \left( E_{Q_{n_i}}[(K_2 - X)] - \rho_2^*(Q_{n_i}) \right) \\
\geq \limsup_{k \to \infty} \inf_{X \in \mathcal{X}_n} \frac{1}{k} \sum_{i=1}^{k} E_{Q_{n_i}}[(K_2 - X)] - \rho_2^*(Q_{n_i}) \\
\geq \lim_{k \to \infty} (\gamma - \frac{\epsilon}{k}) = \gamma.
\]

Since \( Q^* \in \mathcal{Q} \), we have

\[
\inf_{X \in \mathcal{X}_n} E_{Q^*}[K_2 - X] - \rho_2^*(Q^*) = \sup_{Q \in \mathcal{Q}} \inf_{X \in \mathcal{X}_n} (E_{Q}[K_2 - X] - \rho_2^*(Q)).
\]

This completes the proof. \( \blacksquare \)

Summarizing all the lemmas above, we obtain the following proof of Theorem 4.2

**Proof.** By Lemma 4.8 there exists \( Q^* \in \mathcal{Q} \) such that

\[
\inf_{X \in \mathcal{X}_n} E_{Q^*}[K_2 - X] - \rho_2^*(Q^*) = \sup_{Q \in \mathcal{Q}} \inf_{X \in \mathcal{X}_n} (E_{Q}[K_2 - X] - \rho_2^*(Q)).
\]

If \( X^* \) is the optimal test of \( (2.2) \), then

\[
sup_{Q \in \mathcal{Q}} \left( E_{Q^*}[K_2 - X^*] - \rho_2^*(Q) \right) = \inf_{X \in \mathcal{X}_n} \sup_{Q \in \mathcal{Q}} (E_{Q}[K_2 - X] - \rho_2^*(Q)).
\]
By Lemma 4.6
\[ \inf_{X \in \mathcal{X}} \sup_{Q \in \mathcal{Q}} (E_Q[K_2 - X] - \rho_2^*(Q)) = \sup_{Q \in \mathcal{Q}} \inf_{X \in \mathcal{X}} (E_Q[K_2 - X] - \rho_2^*(Q)). \]
Thus,
\[ \inf_{X \in \mathcal{X}} E_{Q^*}[K_2 - X] - \rho_2^*(Q^*) = \sup_{Q \in \mathcal{Q}} (E_Q[K_2 - X^*] - \rho_2^*(Q)). \]
Since
\[ \inf_{X \in \mathcal{X}} E_{Q^*}[K_2 - X] - \rho_2^*(Q^*) \leq E_{Q^*}[K_2 - X^*] - \rho_2^*(Q^*) \leq \sup_{Q \in \mathcal{Q}} (E_Q[K_2 - X^*] - \rho_2^*(Q)), \]
then
\[ E_{Q^*}[K_2 - X^*] - \rho_2^*(Q^*) = \inf_{X \in \mathcal{X}} E_{Q^*}[K_2 - X] - \rho_2^*(Q^*), \]
i.e.,
\[ E_{Q^*}[K_2 - X^*] = \inf_{X \in \mathcal{X}} E_{Q^*}[K_2 - X]. \]
This completes the proof. \( \blacksquare \)

**Example 4.9** Consider the probability space \((\Omega, \mathcal{F}, \mu)\), where \(\Omega = \{0, 1\}\), \(\mathcal{F} = \{\emptyset, \{0\}, \{1\}, \Omega\}\). Set
\[
\mu(\omega) = \begin{cases} 
\frac{1}{2}, & \text{if } \omega = 0, \\
\frac{1}{2}, & \text{if } \omega = 1,
\end{cases}
\text{ and } Q_0(\omega) = \begin{cases} 
\frac{1}{2}, & \text{if } \omega = 0, \\
\frac{1}{2}, & \text{if } \omega = 1.
\end{cases}
\]
Let \(K_1 = 0, K_2 = 1, \alpha = \frac{1}{2}\), \(\rho_1(X) = E_\mu[X]\) and \(\rho_2(X) = \ln E_{Q_0}[e^X]\). We solve the problem (2.2). Let \(Q = qI_{\{0\}} + (1 - q)I_{\{1\}}\), where \(0 \leq q \leq 1\). Then
\[ \rho_2^*(Q) = E_{Q_0}\left[ \frac{dQ}{dQ_0} \ln \frac{dQ}{dQ_0} \right] = q \ln q + (1 - q) \ln(1 - q) - q \ln 3 + 2 \ln 2. \]
Let \(X = x_0I_{\{0\}} + x_1I_{\{1\}}\), where \(0 \leq x_0, x_1 \leq 1\). If \(X \in \mathcal{X}_0\), then \(\frac{1}{2}x_0 + \frac{1}{2}x_1 \leq \frac{1}{2}\), i.e., \(x_0 \leq 1 - x_1\). When \(q = \frac{3}{e + 3}\), \(\inf_{X \in \mathcal{X}_0} E_Q[1 - X] - \rho_2^*(Q)\) attains its maximum. Thus,
\[ Q^* = \frac{3}{e + 3}I_{\{0\}} + \frac{e}{e + 3}I_{\{1\}} \text{ and } X^* = I_{\{0\}}. \]

### 4.2 The existence of a representative probability \(P^*\)

In the rest of this paper, \(Q^*\) is always the probability measure found in Theorem 4.2. Define
\[ \gamma_\alpha = \inf_{X \in \mathcal{X}_\alpha} E_{Q^*}[K_2 - X]. \]
If \(\gamma_\alpha = 0\), then it is trivial and the optimal test \(X^* = K_2, Q^*-a.e.\). In the following, we only consider the case \(\gamma_\alpha > 0\).

**Lemma 4.10** If \(\gamma_\alpha > 0\), \(\rho_1\) and \(\rho_2\) are convex expectations continuous from below and Assumption 4.7 holds, then for any optimal test \(X^*\) of (2.2), we have \(X^* \in \mathcal{X}_\gamma\) and
\[ \rho_1(X^*) = \inf_{X \in \mathcal{X}_\gamma} \rho_1(X) = \alpha, \quad (4.7) \]
where \(\mathcal{X}_\gamma = \{X : E_{Q^*}[K_2 - X] \leq \gamma_\alpha, K_1 \leq X \leq K_2, X \in L^\infty(\mu)\}\).
Proof. $X^\ast \in \mathcal{X}\gamma_\alpha$ comes from Theorem 4.1. For any $X \in \mathcal{X}_\alpha$, if $\rho_1(X) < \alpha$, we claim $E_{Q^\ast}[K_2 - X] > \gamma_\alpha$. If not, then there will exist a test $X' \in \mathcal{X}_\alpha$ such that $\rho_1(X') < \alpha$ and

$$E_{Q^\ast}[K_2 - X'] = \gamma_\alpha.$$  

Set

$$\rho_1(X') = \alpha' < \alpha$$

and

$$X'' = (X' + \alpha - \alpha') \wedge K_2.$$  

By the definition of convex expectation,

$$\rho_1(X'') \leq \rho_1(X' + \alpha - \alpha') = \rho_1(X') + \alpha - \alpha' = \alpha,$$

which implies that $X'' \in \mathcal{X}_\alpha$. As $X'' \in \mathcal{X}_\alpha$ and $X'' \geq X'$, we have $E_{Q^\ast}[K_2 - X''] = E_{Q^\ast}[K_2 - X']$, i.e., $E_{Q^\ast}[X''] = E_{Q^\ast}[X']$. Set $A = \{X' \neq K_2\}$. Since

$$X'' - X' \geq 0 \quad \text{and} \quad E_{Q^\ast}[X'' - X'] \geq 0,$$

we have $X'' = X'$, $Q^\ast$-a.e., which implies that $Q^\ast(A) = 0$ and $X' = K_2$, $Q^\ast$-a.e.. Then $\gamma_\alpha = 0$, which contradicts with $\gamma_\alpha > 0$.

Thus, for any $X \in \mathcal{X}\gamma_\alpha$, we have $\rho_1(X) \geq \alpha$. With $\rho_1(X^\ast) = \alpha$, the result holds. ■

Theorem 4.11 Suppose that $\gamma_\alpha > 0$, $\rho_1$ and $\rho_2$ are convex expectations continuous from below and Assumption 4.1 holds. Then there exists $P^* \in \mathcal{P}$ such that for any optimal test $X^\ast$ of (4.8),

$$E_{P^*}[X^\ast] = \inf_{X \in \mathcal{X}\gamma_\alpha} E_{P^*}[X].$$

Proof. Set $Y = K_2 - X$, $Y^\ast = K_2 - X^\ast$ and

$$\mathcal{Y}\gamma_\alpha = \{Y : E_{Q^\ast}[Y] \leq \gamma_\alpha, 0 \leq Y \leq K_2 - K_1, Y \in L^\infty(\mu)\}.$$  

By Lemma 4.10

$$\rho_1(K_2 - Y^\ast) = \inf_{Y \in \mathcal{Y}\gamma_\alpha} \rho_1(K_2 - Y),$$

i.e.,

$$\sup_{P \in \mathcal{P}} (E_P[K_2 - Y^\ast] - \rho_1^*(P)) = \inf_{Y \in \mathcal{Y}\gamma_\alpha} \sup_{P \in \mathcal{P}} (E_P[K_2 - Y] - \rho_1^*(P)). \tag{4.8}$$

Applying similar analysis as in Lemma 4.1 we obtain that $\mathcal{Y}\gamma_\alpha$ is compact in the topology $\sigma(L^\infty(\mu), L^1(\mu))$.

By the minimax theorem,

$$\inf_{Y \in \mathcal{Y}\gamma_\alpha} \sup_{P \in \mathcal{P}} (E_P[K_2 - Y] - \rho_1^*(P)) = \sup_{P \in \mathcal{P}} \inf_{Y \in \mathcal{Y}\gamma_\alpha} (E_P[K_2 - Y] - \rho_1^*(P)). \tag{4.9}$$

Now we prove that there exists a probability measure $P^* \in \mathcal{P}$ such that

$$\inf_{Y \in \mathcal{Y}\gamma_\alpha} (E_{P^*}[K_2 - Y] - \rho_1^*(P^*)) = \sup_{P \in \mathcal{P}} \inf_{Y \in \mathcal{Y}\gamma_\alpha} (E_P[K_2 - Y] - \rho_1^*(P)). \tag{4.10}$$
If we replace \( X \) by \( Y \), \( X_\alpha \) by \( Y_\alpha \), \( P \) by \( Q \) and \( \rho_1^* \) by \( \rho_2^* \) in (4.9), then (4.9) becomes (4.10). Using the same proof method as in Lemma 4.8, we deduce that (4.10) holds.

By (4.8), (4.9) and (4.10),
\[
\inf_{Y \in Y_\alpha} (E_{P^*}[K_2 - Y] - \rho_1^*(P^*)) = \sup_{P \in P} (E_{P^*}[K_2 - Y^*] - \rho_1^*(P)).
\]
Since
\[
\inf_{Y \in Y_\alpha} E_{P^*}[K_2 - Y] - \rho_1^*(P^*) \leq E_{P^*}[K_2 - Y^*] - \rho_1^*(P^*) \leq \sup_{P \in P} (E_{P}[K_2 - Y^*] - \rho_1^*(P)),
\]
we have
\[
E_{P^*}[K_2 - Y^*] - \rho_1^*(P^*) = \inf_{Y \in Y_\alpha} E_{P^*}[K_2 - Y] - \rho_1^*(P^*).
\]
Thus,
\[
E_{P^*}[K_2 - Y^*] = \inf_{Y \in Y_\alpha} E_{P^*}[K_2 - Y],
\]
i.e.,
\[
E_{P^*}[X^*] = \inf_{X \in X_\alpha} E_{P^*}[X].
\]
This completes the proof. \[\blacksquare\]

**Example 4.12** \hspace{1em} Consider the probability space \((\Omega, \mathcal{F}, \mu)\), where \(\Omega\), \(\mathcal{F}\) and \(\mu\) are defined as the same as in Example 4.5. Set \(K_1 = 0\), \(K_2 = 1\), \(\alpha = \ln(e + 3) - 2 \ln 2\), \(\rho_1(X) = \ln E_{P_0}[e^X]\) and \(\rho_2(X) = E_{\mu}[X]\), where

\[
P_0(\omega) = \begin{cases} 
\frac{1}{4}, & \text{if } \omega = 0, \\
\frac{3}{4}, & \text{if } \omega = 1.
\end{cases}
\]

We solve the problem (2.2). It is easy to check that
\[
\inf_{X \in X_\alpha} E_{\mu}(1 - X) = \frac{1}{2},
\]
i.e., \(\gamma_\alpha = \frac{1}{2}\). By Lemma 4.10 to solve the problem (2.2) is equivalent to solve the following problem:

\[
\text{minimize } \rho_1(X),
\]
over the set \(X_\alpha = \{X : E_{\mu}[X] \geq \frac{1}{2}, 0 \leq X \leq 1\}\). Let \(X = x_0I_{\{0\}} + x_1I_{\{1\}}\), where \(0 \leq x_0, x_1 \leq 1\). If \(X \in X_\alpha\), then \(x_0 \geq 1 - x_1\). Let \(P = pI_{\{0\}} + (1 - p)I_{\{1\}}\), where \(0 \leq p \leq 1\). Then
\[
\rho_1^*(P) = E_{P_0}[\frac{dP}{dP_0} \ln \frac{dP}{dP_0}] = 2 \ln 2 + p \ln p + (1 - p) \ln(1 - p) - (1 - p) \ln 3.
\]
When \(p = \frac{e}{e + 3}\), \(\inf_{X \in X_\alpha} E_{P_0}[X] - \rho_1^*(P)\) attains its maximum. Thus,
\[
P^* = \frac{e}{e + 3}I_{\{0\}} + \frac{3}{e + 3}I_{\{1\}} \quad \text{and} \quad X^* = I_{\{0\}}.
\]
4.3 Main result

**Theorem 4.13** If $\rho_1$ and $\rho_2$ are convex expectations continuous from below and Assumption 4.1 holds, then there exist $P^* \in \mathcal{P}$ and $Q^* \in \mathcal{Q}$ such that for any optimal test $X^*$ of (2.2), it can be expressed as

\[ X^* = K_2 I\{H_{Q^*} > z G_{P^*}\} + B I\{H_{Q^*} = z G_{P^*}\} + K_1 I\{H_{Q^*} < z G_{P^*}\}, \quad \mu - a.e., \tag{4.12} \]

where $z \in [0, +\infty) \cup \{+\infty\}$ and $B$ is a random variable taking values in the random interval $[K_1, K_2]$.

**Proof.** We divide our proof into two cases:

i) The case $\gamma_\alpha > 0$. By Theorem 4.11 $X^*$ is the optimal test of the following problem:

minimize $E_{P^*}[X],

over the set $X_{\gamma_\alpha} = \{X : E_{Q^*}[K_2 - X] \leq \gamma_\alpha, K_1 \leq X \leq K_2, X \in L^\infty(\mu)\}$. Set

\[ Z^* = \frac{K_2 - X^*}{K_2 - K_1}, \quad Z = \frac{K_2 - X}{K_2 - K_1} = \frac{\gamma_\alpha}{E_{Q^*}[K_2 - K_1]}, \quad \frac{d\hat{P}}{dP^*} = \frac{K_2 - K_1}{E_{P^*}[K_2 - K_1]} \quad \text{and} \quad \frac{d\hat{Q}}{dQ^*} = \frac{K_2 - K_1}{E_{Q^*}[K_2 - K_1]}, \]

Then $Z^*$ is the optimal test of the problem:

maximize $E_{\hat{P}}[Z], \tag{4.13}$

over the set $Z_{\gamma_\alpha'} = \{Z : E_{Q^*}[Z] \leq \gamma_\alpha', 0 \leq Z \leq 1, Z \in L^\infty(\mu)\}$.

By the classical Neyman-Pearson lemma (see [4] or Theorem A.30 in [9]), any optimal test $Z^*$ of (4.13) has the form

\[ Z^* = I_{\{z' H_{Q^*} < z G_{P^*}\}} + B', \quad I_{\{z' H_{Q^*} = z G_{P^*}\}}, \quad \mu - a.e. \tag{4.14} \]

for some constant $z' \geq 0$ and random variable $B'$ taking values in the interval $[0, 1]$. Since

\[ \frac{d\hat{P}}{dP^*} = \frac{K_2 - K_1}{E_{P^*}[K_2 - K_1]} \quad \text{and} \quad \frac{d\hat{Q}}{dQ^*} = \frac{K_2 - K_1}{E_{Q^*}[K_2 - K_1]}, \]

if we take (with conventions $+\infty = \frac{1}{0}$ and $0 = \frac{1}{0}$)

\[ B = K_2 - (K_2 - K_1)B' \quad \text{and} \quad z = \frac{E_{Q^*}[K_2 - K_1]}{z' E_{P^*}[K_2 - K_1]}, \]

then $z' \in (0, +\infty) \cup \{+\infty\}$ and

\[ X^* = K_2 I_{\{H_{Q^*} > z G_{P^*}\}} + B I_{\{H_{Q^*} = z G_{P^*}\}} + K_1 I_{\{H_{Q^*} < z G_{P^*}\}}, \quad \mu - a.e.. \tag{4.15} \]

ii) The case $\gamma_\alpha = 0$. For this case, $X^* = K_2, Q^* - a.e..$. This is a special case of (4.13) when $z$ equals 0. ■

**Example 4.14** Except $\rho_2(X) = \ln E_{Q_0}[e^X]$ where $Q_0$ is defined as in Example 4.7, all the notations in this example are defined as the same as in Example 4.12. We solve the problem (2.2).

Denote $Z = \{X : 0 \leq X \leq 1, E_{\mu}[X] \leq \frac{1}{2}\}$. By Example 4.12 we have $\sup_{X \in X_\alpha} E_{\mu}[X] = \frac{1}{2}$. Then $X_\alpha \subset Z$ and

\[ \inf_{X \in Z} \rho_2(1 - X) \leq \inf_{X \in X_\alpha} \rho_2(1 - X). \tag{4.16} \]
Take $\hat{X} = I_{\{0\}}$. By Example 4.9,

$$\rho_2(1 - \hat{X}) = \inf_{X \in \mathbb{Z}} \rho_2(1 - X).$$

Since $\hat{X} \in \mathcal{X}$, with (4.16), we have

$$\rho_2(1 - \hat{X}) = \inf_{X \in \mathcal{X}} \rho_2(1 - X),$$

which implies $I_{\{0\}}$ is the optimal test. Furthermore, if we take $Q^* = \frac{1}{\varepsilon + 3}I_{\{0\}} + \frac{1}{\varepsilon + 3}I_{\{1\}}$ as in Examples 4.9 and 4.12, then

$$I_{\{0\}} = I(\frac{1}{\varepsilon + 3}H_{Q^*} > G_{P^*}).$$

5 Application

In a financial market, if an investor does not have enough initial wealth, then he may fail to (super-) hedge an contingent claim and will face some shortfall risk. In this case, we need a criterion expressing the investor’s attitude towards the shortfall risk (see [7, 8, 10, 11]). Föllmer and Leukert [8] use the expectation of the shortfall weighted by the loss function as a shortfall risk measure. In this section, we use a general measure, the convex risk measure, to evaluate the shortfall and consequently minimize such a shortfall risk.

In more details, we adopt the same financial market model as in [8]. The discounted price process of the underlying asset is described as a semimartingale $S = (S_t)_{t \in [0, T]}$ on a complete probability space $(\Omega, \mathcal{F}, \mu)$. The information structure is given by a filtration $\mathcal{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ with $\mathcal{F}_T = \mathcal{F}$. Let $\mathcal{P}$ denote the set of equivalent martingale measures, we assume that $\mathcal{F}_0$ is trivial and $\mathcal{P} \neq \emptyset$. For an initial investment $X_0 \geq 0$ and a portfolio process $\pi$ such that the wealth process

$$X_t = X_0 + \int_0^t \pi_s dS_s \quad \forall t \in [0, T]$$

is well defined. A strategy $(X_0, \pi)$ is called admissible if the corresponding wealth process $X$ is nonnegative. For a given nonnegative contingent claim $H \in L^\infty(\mu)$, we define that

$$U_0 = \sup_{P \in \mathcal{P}} E_P[H].$$

It is well known that if the investor’s initial wealth $\tilde{X}_0 < U_0$, then some shortfall $(H - X_T)^+$ will occur at time $T$.

In this section, we introduce a general convex expectation $\rho$ to measure the shortfall $(H - X_T)^+$.

**Definition 5.1** For a given convex expectation $\rho$, the shortfall risk is defined as

$$\rho((H - X_T)^+).$$

Consequently, the investor wants to find an admissible strategy $(X_0, \pi)$ which minimizes the shortfall risk and control his initial investment $X_0 \leq \tilde{X}_0$. Thus, we will solve the following optimization problem:

$$\min_{(X_0, \pi)} \rho((H - X_T)^+),$$

subject to $X_0 \leq \tilde{X}_0$, (5.2)
where $\tilde{X}_0$ is the initial wealth of the investor.

Now we show that the optimal $X_T^*$ must satisfy $0 \leq X_T^* \leq H$. In fact, if $P(X_T^* > H) > 0$, we can construct a feasible terminal wealth $\tilde{X}_T$ such that $0 \leq \tilde{X}_T \leq H$ and $(H - \tilde{X}_T)^+ < (H - X_T^*)^+$. Thus, $\rho((H - \tilde{X}_T)^+) < \rho((H - X_T^*)^+)$ by the monotonicity property of $\rho$. This leads to a contradiction.

Thus, without loss of generality we assume that $0 \leq X_T \leq H$ and (5.2) becomes

$$\begin{align*}
\min_{0 \leq X_T \leq H} \rho(H - X_T),
\text{subject to } \sup_{P \in P} E_P[X_T] \leq \tilde{X}_0.
\end{align*}$$

(5.3)

By Theorem 4.13 and the classical Neyman-Pearson lemma (Proposition 4.1 in [8]),

$$X_T^* = HI_{\{zH_Q^* > G_P^*\}} + BI_{\{zH_Q^* = G_P^*\}}, \quad \mu - a.e.,$$

where

$$z = \sup\{\tilde{z} \mid \int_{\{zH_Q^* > G_P^*\}} H dP^* \leq \tilde{X}_0\}$$

and

$$B = \begin{cases} \frac{\tilde{X}_0 - \int_{\{zH_Q^* > G_P^*\}} H dP^*}{\int_{\{zH_Q^* = G_P^*\}} H dP^*}, & \text{when } P^*[\{H > 0\} \cap \{zH_Q^* = G_P^*\}] > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Then by the optional decomposition theorem (see [15] and [4]), we obtain the optimal strategy $(\tilde{X}_0, \pi^*)$ corresponding to $X_T^*$.

**Remark 5.2** Instead of minimizing the convex risk measure under the initial investment constraint, we can solve the following essentially equivalent problem: fix a bound on the convex risk measure and minimize the initial investment.

**Remark 5.3** We assume that the given nonnegative contingent claim $H \in L^\infty(\mu)$. If $H \in L^1(\mu)$, then we can use Theorem 6.4 in the appendix.

### 6 Appendix

In this appendix, we first prove that when the convex expectations are continuous from above, Assumption 4.1 holds naturally. Then an example is given to show that Assumption 4.1 is only a sufficient condition for the existence of $Q^*$. Finally, we give the Neyman-Pearson lemma for convex expectations on $L^1(\mu)$.

**Definition 6.1** We call a convex expectation $\rho$ is continuous from above iff for any sequence $\{X_n\}_{n \geq 1} \subset L^\infty(\mu)$ decreases to some $X \in L^\infty(\mu)$, then $\rho(X_n) \to \rho(X)$.

**Proposition 6.2** If $\rho_1$ and $\rho_2$ are continuous from above, then Assumption 4.1 holds.
Proof. We only show the result holds for $\rho_1$.

For any $u > \max\{0, M - \rho_1(0) + 1\}$, we have $u > \max\{0, -\rho_1(0)\}$. By Theorem 3.6 in [14], $G_n$ is uniformly integrable. For any sequence $\{G_{P_n}\}_{n \geq 1} \subset G_n$ that converges to $G_P$, $\mu$-a.e., since $\{G_{P_n}\}_{n \geq 1}$ is uniformly integrable,

$$E_\mu[G_P] = \lim_{n \to \infty} E_\mu[G_{P_n}] = 1,$$

which shows $\hat{P} \in M$. On the other hand, for any $u > \max\{0, M - \rho_1(0) + 1\}$, by Lemma [17] we have

$$\rho^*(\hat{P}) \leq \lim\inf_{n \to \infty} \rho^*(P_n) \leq u.$$

Then $G_P \in G_n$. Thus, $G_n$ is closed under the $\mu$-a.e. convergence.

Now we show that even if Assumption 4.1 does not hold, the probability measure $Q^*$ may still exist.

Example 6.3 Consider the probability space $(\Omega, \mathcal{B}, \mu)$, where $\Omega$ is the interval $[0, 1]$, $\mathcal{B}$ is the collection of all Borel sets in $[0, 1]$ and $\mu$ is the Lebesgue measure defined on $[0, 1]$. Set $K_1 = 0$, $K_2 = 1$, $\alpha = \frac{3}{e-1}$, $\rho_1(X) = E_P[X]$ and $\rho_2(X) = \ln E_\mu[e^X]$, where

$$\frac{dP}{d\mu} = \left\{ \begin{array}{ll}
\frac{e^x}{e-1}, & \omega \in [0, \frac{e-2}{e-1}], \\
\frac{1}{e-1}, & \omega \in (\frac{e-2}{e-1}, 1].
\end{array} \right.$$  

To solve the problem (2.2), one can check that Assumption 4.1 does not hold. Let

$$X^* = I_{(\frac{e-2}{e-1}, 1]} \quad \text{and} \quad \frac{dQ^*}{d\mu} = \left\{ \begin{array}{ll}
\frac{e}{e-1}, & \omega \in [0, \frac{e-2}{e-1}], \\
\frac{1}{e-1}, & \omega \in (\frac{e-2}{e-1}, 1].
\end{array} \right.$$  

We will show that $X^*$ is the optimal test and $Q^*$ satisfies

$$\sup_{X \in \mathcal{X}_n} E_{Q^*}[X] + \rho_2^*(Q^*) = \inf_{Q \in \mathcal{Q}} \sup_{X \in \mathcal{X}_n} \left( E_Q[X] + \rho_2^*(Q) \right).$$  

In fact, through simple calculations, we obtain

$$E_{Q^*}[1 - X^*] - \rho_2^*(Q^*) = \rho_2(1 - X^*).$$

Otherwise, by the classical Neyman-Pearson lemma, we know that $X^*$ is also the optimal test for discriminating between probability measures $P$ and $Q^*$, i.e.,

$$E_{Q^*}[1 - X^*] = \inf_{X \in \mathcal{X}_n} E_{Q^*}[1 - X].$$

Since

$$\inf_{X \in \mathcal{X}_n} \rho_2(1 - X) \geq \inf_{X \in \mathcal{X}_n} E_{Q^*}[1 - X] - \rho_2^*(Q^*) = E_{Q^*}[1 - X^*] - \rho_2^*(Q^*)$$

and

$$E_{Q^*}[1 - X^*] - \rho_2^*(Q^*) = \rho_2(1 - X^*) \geq \inf_{X \in \mathcal{X}_n} \rho_2(1 - X),$$

we have

$$\inf_{X \in \mathcal{X}_n} E_{Q^*}[1 - X] - \rho_2^*(Q^*) = \inf_{X \in \mathcal{X}_n} \rho_2(1 - X),$$

which leads to (6.1).
In the next, we consider the case that $\rho_1$ and $\rho_2$ are two convex expectations defined on $L^1(\mu)$. Then our problem becomes:

$$\text{minimize} \quad \rho_2(K_2 - X), \quad (6.2)$$

over the set $\mathcal{X}_\alpha = \{ X : K_1 \leq X \leq K_2, \rho_1(X) \leq \alpha, X \in L^1(\mu) \}$, where $K_1, K_2 \in L^1(\mu)$.

**Theorem 6.4** If $\rho_1$ and $\rho_2$ are two finite convex expectations defined on $L^1(\mu)$ space, then the optimal test of (6.2) exists and has the same form as in Theorem 4.13.

**Proof.** Since $\rho_1$ and $\rho_2$ are finite, then they are Lebesgue-continuous. Repeating the proof of Theorem 3.1, we will get the optimal test exists. On the other hand, since $\rho_1$ and $\rho_2$ can be represented by some set $\mathcal{P}$ and $\mathcal{Q}$ with their densities sets $\{ G_P \in L^\infty(\mu) : P \in \mathcal{P} \}$ and $\{ H_Q \in L^\infty(\mu) : Q \in \mathcal{Q} \}$ are weakly compact, the form in Theorem 4.13 can also be obtained by using the same method as in section 4. The detailed proof is omitted. ■

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