A brief introduction to multipartite entanglement

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(Dated: December 22, 2016)

A concise introduction to quantum entanglement in multipartite systems is presented. We review entanglement of pure quantum states of three–partite systems analyzing the classes of GHZ and W states and discussing the monogamy relations. Special attention is paid to equivalence with respect to local unitaries and stochastic local operations, invariants along these orbits, momentum map and spectra of partial traces. We discuss absolutely maximally entangled states and their relation to quantum error correction codes. An important case of a large number of parties is also analysed and entanglement in spin systems is briefly reviewed.

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I. INTRODUCTION

These notes are based on a new chapter written to the second edition of our book \textit{Geometry of Quantum States. An introduction to Quantum Entanglement} \cite{bengtsson2006geometry}. The book is written at the graduate level for a reader familiar with the principles of quantum mechanics. It is targeted first of all for readers who do not read the mathematical literature everyday, but we hope that students of mathematics and of the information sciences will find it useful as well, since they also may wish to learn about quantum entanglement.

Individual chapters of the book are to a large extent independent of each other. For instance, we hope that the new chapter presented here might become a source of information on recent developments on quantum entanglement in multipartite systems also for experts working in the field. Therefore we have compiled these notes, which aim to present an introduction to the subject as well as an up to date review on basic features of pure states entanglement of multipartite systems.

All references to equations or the numbers of section refers to the draft of the second edition of the book. To give a reader a better orientation on the topics covered we provide its contents in Appendix A. The second edition of the book includes also a new chapter 12 on discrete structures in the Hilbert space and several other new sections.

II. HOW MUCH IS THREE LARGER THAN TWO?

In the simplest setup one discusses quantum entanglement for bipartite systems, described by states in a composite Hilbert space $\mathcal{H}_{AB}$. Any product state $|\psi_{AB}\rangle = \Gamma_{ij} |i_A\rangle \otimes |j_B\rangle$, (2.1) can always be written in the form

$$|\psi_{AB}\rangle = (U_A \otimes U_B) \sum_{i=1}^{N} \sqrt{\lambda_i} |i_A\rangle \otimes |i_B\rangle ,$$

\texttt{(2.2)}

The answer is ‘yes’. Recall that a general pure state of two subsystems with $N$ levels each,
and its entanglement properties are characterized by its Schmidt vector $\vec{\lambda}$. But a general tripartite pure state

$$|\psi_{ABC}\rangle = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \Gamma_{ijk} |i_A\rangle \otimes |j_B\rangle \otimes |k_C\rangle$$

cannot be written in the form

$$|\psi_{ABC}\rangle = (U_A \otimes U_B \otimes U_C) \sum_{i=1}^{N} \sqrt{\lambda_i} |i_A\rangle \otimes |i_B\rangle \otimes |i_C\rangle .$$

As the dimension counting argument at the end of Section 9.2 reveals, such states are really very rare (although we will see that they have interesting properties). Turning to the mathematical literature in order to standardize the tensor $\Gamma_{ijk}$ in some way, we learn many interesting things \cite{33,52,102}, but there is no magical recipe that solves our problem. Not all algebraic notions developed for matrices work equally fine for tensors.

In short, multipartite entanglement is much more sophisticated than bipartite entanglement, and it has a rich phenomenology already for pure states. If one considers the number of parties in a quantum composite system, then three is much more than two, four is more than three, and so it goes on. The issue of entanglement in multipartite quantum systems deserves therefore a chapter of its own. We will focus our attention on multi–qubit systems and on pure states, otherwise several chapters would be needed!

### III. BOTANY OF STATES

In Section 16.2 we provided rather elaborate pictures showing where separable and entangled states can be found. If we are content with just a sampling of possibilities we can make do with less. Let the states in a product basis be represented by the corners of a square, in the fashion of Figure 1a. We use bitstrings to label the corners, so that $10$ (say) represents the state $|10\rangle \equiv |1_A\rangle \otimes |0_B\rangle$. Occasionally we will also use notation from the multipartite Heisenberg group, so that $Z = \sigma_z$, $X = \sigma_x$, $Y = iXZ = \sigma_y$. The qubit basis is chosen so that

$$Z|0\rangle = |0\rangle , \quad Z|1\rangle = -|1\rangle , \quad X|0\rangle = |1\rangle , \quad X|1\rangle = |0\rangle .$$

Any superposition of two states forming an edge of the square is separable. On the other hand, equal weight superpositions of states represented by two corners on a diagonal give maximally entangled Bell states, $|\phi^\pm\rangle = |00\rangle \pm |11\rangle$ and $|\psi^\pm\rangle = |01\rangle \pm |10\rangle$, where normalization constants got suppressed. Note that the local unitary transformation $I \otimes Z$ interchanges $|\phi^+\rangle$ with $|\phi^-\rangle$ and $|\psi^+\rangle$ with $|\psi^-\rangle$, while the equally local transformation $I \otimes X$ interchanges the two diagonals of the square.

This kind of picture is easily generalized to the case of three qubits. The eight separable basis states will form the corners of a unit cube, see Figure 1b. To describe how far two corners are apart we count the number of edges one must traverse in order to go from the one to the other. The corners have been labelled so that this leads to the Hamming distance between two bitstrings of the same length, equal to the minimal number of bits which need to be flipped to transform one string into the other. Superpositions of two states at Hamming distance one, belonging to the same edge of the cube, are separable. Superpositions of states at Hamming distance two display only bipartite entanglement. For instance,

$$|\psi_{AB|C}\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |110\rangle) = |\phi^+_{AB}\rangle \otimes |0_C\rangle .$$

States that cannot be decomposed in any such way are said to exhibit genuine multipartite entanglement.

It is then thinkable that a balanced superposition of two states corresponding to two maximally distant corners, at Hamming distance three, is highly entangled. This is indeed the case. The GHZ state

$$|GHZ\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle),$$

is named for Greenberger, Horne, and Zeilinger. These authors went beyond Bell’s theorem to obtain a contradiction between quantum mechanics and local hidden variables not relying on statistics \cite{58,59}. The three–qubit GHZ state was created in 1999 \cite{10}. The present experimental record is a fourteen–qubit GHZ state in the form of trapped ions \cite{100}.
Entanglement of the GHZ state is quite fragile: If we trace out any subsystem from the GHZ state we obtain a separable state, which means that all the entanglement is of a global nature. Interestingly, this property holds if and only if the state is Schmidt decomposable \[133\], that is to say if it can be written on the form \[2.4\]. Among such states the GHZ state is distinguished by the property that if one traces out any two subsystems, the maximally mixed state results.

The GHZ state has many curious properties. If we rewrite it by introducing a new basis in Charlie’s factor, \(|+\rangle = |0\rangle + |1\rangle\) and \(|-\rangle = |0\rangle - |1\rangle\), we find that it becomes

\[
|\text{GHZ}\rangle = |\phi^+_{AB}\rangle \otimes |+\rangle_C + |\phi^-_{AB}\rangle \otimes |-\rangle_C,
\]

where again the maximally entangled Bell states appear. At the outset all three parties agree that the state assignment governing the measurements of Alice and Bob is \(\rho_{AB} = \text{Tr}_C |\text{GHZ}\rangle\langle \text{GHZ}|\). However, depending on what he chooses to do and on what the outcomes of his measurements are, Charlie can change his state assignment to one of \(|0_40_B\rangle\), \(|1_A1_B\rangle\), \(|\Phi^+_{AB}\rangle\), or \(|\Phi^-_{AB}\rangle\). When Alice and Bob report the outcome of their measurement it will be consistent with that—as well as with the original state assignment \(\rho_{AB}\). From this point of view the GHZ state describes entangled entanglement \[148\].

The GHZ state is an example of a stabilizer state, defined in Section 12.6 as an eigenstate of a maximal abelian subgroup of the three–partite Heisenberg group. Here this subgroup is generated by the four commuting group elements \(\bar{Z}Z, Z\bar{Z}, ZZ, XXX\). All other equal weight superpositions related to two ‘antipodal’ corners of the cube separated by three Hamming units are locally equivalent to the GHZ state. There are eight of them, two on each diagonal (such as \(|010\rangle \pm |101\rangle\)), and all can be brought to the GHZ form by means of a local transformation (such as \(1X1\)) belonging to the Heisenberg group. And together these eight states form an orthonormal basis composed of states locally equivalent to the GHZ state.

Another genuinely multipartite entangled state is the \(W\) state

\[
|W\rangle = \frac{1}{\sqrt{3}} (|001\rangle + |010\rangle + |100\rangle).
\]

Some experts associate this name to the representation of the state in the space spanned by the energy and labels of the subsystems, which could resemble the letter \(W\). Others tend to believe it is related to the name of one of the authors of the paper \[42\] in which this state was investigated. It could be called ‘ZH’ or ‘anti–GHZ’, as it appeared in a work \[148\] by the authors of the GHZ paper \[58\]. Incidentally, one can make a case for calling GHZ the ‘Svetlichny state’ \[132\].

In Figure 1, it appears as a triangle. Its entanglement is more robust than that of the GHZ state, in the sense that an entangled mixed state results if we trace out any chosen subsystem, and indeed one can argue that in this respect its entanglement is maximally robust \[42\]. Hence the \(W\) state cannot be Schmidt decomposable. Indeed it cannot be written as a superposition of less than three separable states \[42\].

Introducing a fourth qubit into the game we need an extra dimension to construct a hypercube with 16 corners. See Figure 1c. Superpositions of two states at Hamming distance one are fully separable, as the tensor product consists of four factors. Corners distant by two and three Hamming units correspond to states with bi- and tri-partite
entanglement, respectively. Superpositions of two ‘antipodal’ corners of the hypercube distant by four Hamming units form genuinely four-party entangled states, such as

\[ |GHZ_4 \rangle = \frac{1}{\sqrt{2}} (|0000\rangle + |1111\rangle). \] (3.6)

This is called a four-qubit GHZ state, or sometimes a cat state to honour the Schrödinger cat—which existed, or so we are told, in an equal weight superposition of classical states. A four qubit analogue of the W state corresponds to the tetrahedron obtained by four permutations of the bitstring 0001. Again the entanglement of the W state is more robust than that of the GHZ state; we can define the persistency of entanglement \( K \) as the minimum number of local measurements that need to be performed in order to ensure that the state is fully disentangled regardless of the measurement outcomes. For the GHZ\(_K\) state the persistency equals 2, for the W\(_K\) state it equals \( K - 1 \). But this is not to say that the one state is more entangled than the other: they are entangled in different ways.

The list of interesting states can be continued, but Figure 1 is already making it very clear that we are going to need an organizing principle to survey them. Botanists divide the Kingdom of Plants into classes, orders, families, genera, and species. It is reasonably clear that what corresponds to a division into species of the Kingdom of Multipartite States must be a division into orbits of the local unitary group. Hence we would say that two states \( |\psi\rangle \) and \( |\phi\rangle \) are LU equivalent, written \( |\psi\rangle \sim_{LU} |\phi\rangle \), if and only if there exist unitary operators \( U_k \) such that

\[ |\phi\rangle = U_1 \otimes U_2 \otimes \cdots \otimes U_K |\psi\rangle. \] (3.7)

Since the overall phase of each unitary can be fixed, we can choose unitary matrices with determinant set to unity and divide the set of states into orbits of the product group \( G_U = SU(N)^{\otimes K} \), if we are studying \( K \) quNits.

Of more direct relevance to \( K \) parties trying to perform some quantum task would be a division into LOCC equivalent sets of states, consisting of states that can be transformed (with certainty) into each other by means of local operations and classical communication, abbreviated LOCC. Fortunately, for pure states LU equivalence coincides with LOCC equivalence. In the case of two qubits this result was an immediate consequence of Nielsen’s majorization theorem. The argument in the multipartite case is similar but more involved. For many qubits this result is due to Bennett et al. \[14\]. The criterion for SLOCC equivalence—which we are coming to—was presented by Dür, Vidal, and Cirac \[42\].

How much will this buy us? The state space of \( K \) qubits has \( 2 \cdot (2^K - 1) \) real dimensions, while an orbit of the local unitary group has at most \( 3K \) dimensions because \( SU(2) \) is a three dimensional group. Already for three qubits we can look forwards to a five dimensional set of inequivalent orbits.

A coarser classification may therefore be useful, corresponding to the division of plants into genera. This is offered by stochastic local operations and classical communication (SLOCC). The definition of SLOCC operations presented in Section 16.4 for the bipartite case can be extended for multipartite systems in a natural way. For completeness let us write an analogue of Eq. (16.44)presenting a SLOCC transformation acting on a \( K \)-partite state,

\[ |\psi\rangle \rightarrow A_1 \otimes A_2 \otimes \cdots A_K |\psi\rangle. \] (3.8)

To formulate a relation between two SLOCC–equivalent states all matrices have to be invertible,

\[ |\psi\rangle \sim_{SLOCC} |\phi\rangle = L_1 \otimes L_2 \otimes \cdots \otimes L_K |\psi\rangle. \] (3.9)

Since normalization does not play any role here we can freely assume that the matrices \( L_i \) have unit determinant. Thus the group that governs SLOCC equivalence for a system consisting of \( K \) subsystems with \( N \) levels each is the special linear group composed with itself \( K \) times, \( G_L = SL(N, \mathbb{C})^{\otimes K} \). The group \( G_L \) is the complexification of the group \( G_U \), and has twice as many real dimensions as the latter. Only twice, so again the number of inequivalent orbits grows quickly with the number of qubits. Like good botanists we must stand ready to introduce yet coarser classification schemes as we proceed.

**IV. PERMUTATION SYMMETRIC STATES**

What we really need is a way of being able to recognize a given state at a glance (so that we do not have to rely on the way it looks like with respect to some perhaps arbitrary basis). We cannot quite do this, but we can if we restrict ourselves to the symmetric subspace \( \mathcal{H}_K^{\text{sym}} \) of the full Hilbert space \( \mathcal{H}^{\otimes K} \). For \( K \) qubits this subspace is in itself a Hilbert space of dimension \( K + 1 \), and many states of interest such as the GHZ and W states—as well as the ground states of many condensed matter systems, and more—reside in it. We will look at it through the glasses of
the stellar representation from our Chapter 7. Many nice descriptions of this idea can be found [10, 56, 71, 83, 96]. Ours is perhaps closest to that of Ribeiro and Mosseri [118].

The symmetric subspace admits an orthonormal basis consisting of the states

$$|K - k, k\rangle = \binom{K}{k}^{-\frac{1}{2}} \sum_{\text{permutations}} |0\rangle \otimes K^{-k} \otimes |1\rangle \otimes k,$$

where $k = 0, 1, \ldots, K$ and the summation is over all $\binom{K}{k}$ permutations of $K$-letter strings with $K - k$ symbols $|0\rangle$ and $k$ symbols $|1\rangle$. The basis states $|K - k, k\rangle$ can be identified with the Dicke states—which were thought of as angular momentum states in Chapter 7, but above all they were thought of as sets of stars placed at the North and South celestial poles only. The notion of composition of pure states, introduced in (7.27) is useful here. For any two symmetric states $|\psi_1\rangle$ and $|\psi_2\rangle$ of $K_1$ and $K_2$ qubits, respectively, the composite state $|\psi_1\rangle \otimes |\psi_2\rangle$ of $K = K_1 + K_2$ qubits is described by superposition of all $n$ stars. The reason why the stellar representation turns out to be useful is that the group of local unitaries acts on the symmetric subspace through its diagonal subgroup $SU(2)$. In other words, a local unitary transformation corresponds to a rotation of the celestial sphere housing the stars. Constellations of stars that can be brought into coincidence by means of a rotation correspond to states that are equivalent under local unitaries. A SLOCC transformation can also be visualized; it is effected by some $SL(2, \mathbb{C})$ matrix. More precisely the group of Möbius transformations acts effectively on the states, and this group is famously isomorphic to the group of Lorentz transformations $SO(3, 1)$. A general Lorentz transformations will cause the constellations of stars on the sky to change. In particular, if the observer moves with constant velocity towards the North Pole of the celestial sphere all the stars will be seen closer to the North Pole than they would be by an observer standing still. This connection is explained by Penrose and Rindler [112]. The best possible introduction to Möbius transformations is the first chapter of the book by Ford [49]. A detailed description of the action of the group is given in the caption of Figure 2.

Once these points are understood we can address the issue of how the symmetric subspace is divided into orbits under local unitaries and SLOCC transformations. In fact the former question was already discussed, in considerable detail, in Section 7.2. The set of all symmetrized states has real dimension $2K$, and a typical orbit has dimension $3$ (equal to the dimension of the rotation group), so the set of all orbits has dimension $2K - 3$ and will quickly become unmanageable as the number $n$ of qubits grows. For $K = 3$ things are still simple enough. Any set of three stars sits on some circle on the sphere. By means of rotations this can be brought to a circle of constant latitude (one parameter), with one star at the Greenwich meridian, and two stars placed at other longitudes (two parameters). Hence the set of differently entangled three qubit states has three dimensions. By the time we get to four or more qubits a full description becomes at best very unwieldy. Still some special orbits are easily recognized in all cases. Although the physical interpretation is new, mathematically a GHZ state is identical to the noon state that we presented in Section 7.1, and its constellation of stars is a regular $K$-gon on some Great Circle on the sky. The set of all GHZ states form an orbit isomorphic to $SO(3)/Z_K = \mathbb{RP}^3/Z_K$. The $W$ states are Dicke states, and their stars are confined to two antipodal points on the sphere. The set of $W$ states is a 2-sphere.

Things simplify if we consider states equivalent under SLOCC. For symmetric pairs of qubits there are only two orbits: one entangled, and one separable. See Figure 3. Moving on to $K = 3$ qubits, and keeping in mind the picture of the effect a Lorentz boost has on the sphere, we see that every set of three non-coinciding stars can be brought
into coincidence with three stars placed at the corners of a regular triangle on the equator. An equivalent way to see this is to observe that any three distinct complex numbers can be mapped to any other three by means of a Möbius transformations. In fact there are only three orbits in this case. One orbit consists of all states with three coinciding stars. In Chapter 7 these states were called SU(2) coherent states. Now they reappear as (symmetric) separable states. Then there is an orbit consisting of states for which exactly two stars coincide. This is the W class. Finally there is the GHZ class for which all stars are distinct. This gives us three degeneracy types, denoted respectively by \( \{1\}, \{2,1\}, \) and \( \{1,1,1\} \). Throwing randomly three stars on the sphere they will land in different points, hence type \( \{1,1,1\} \) is the generic one. States with two stars merged together can be well approximated with states of the latter type, while the converse statement does not hold.

When \( n = 4 \) the classification remains manageable, but requires some work \[113, 118\]. We define the anharmonic cross ratio of four ordered complex numbers as

\[
(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}.
\]  (4.2)

Recall that we are on the extended complex plane, so that \( \infty \) is a respectable number representing the South Pole of the Poincaré sphere. The point about this definition is that the cross ratio is invariant under Möbius transformations, or in our language that this function of the positions of the stars is invariant under SLOCC. We are not quite done though, because we are interested in unordered constellations of stars, and as we permute the stars the cross ratio will assume six different values

\[
\left\{ \lambda, \frac{1}{\lambda}, 1 - \lambda, \frac{1}{1 - \lambda}, \frac{\lambda}{\lambda - 1}, \frac{\lambda - 1}{\lambda} \right\}.
\]  (4.3)

The set of orbits is given by the set of values that the cross ratio can assume, provided it is understood that values related in this way represent the same orbit. This completely solves the problem of characterizing the set of differently entangled states of 4 symmetrized qubits, in the sense of SLOCC. Figure 3 is a map of the set of all SLOCC orbits in the 4-qubit case. Note that if we pick two states at random they are likely to end up in different places on the map, and then they are inequivalent under SLOCC.

Some interesting special cases deserve mention. First of all there are five different ways in which stars can coincide: \( \{1,1,1,1\}, \{2,1,1\}, \{2,2\}, \{3,1\}, \) and \( \{4\} \). Thus there are five degeneracy types. Type \( \{1,1,1,1\} \), in which all the stars sit at distinct points, is the generic one and the only one to contain a continuous family of SLOCC orbits. Within this family some special cases can be singled out. If the stars are lined up on a circle on the sky—they are then said to be concyclic—the cross ratio is real. If \( \lambda = -1, 2, \) or \( \frac{1}{2} \) the state is SLOCC equivalent to a GHZ state—the four stars form a regular 4-gon on a Great Circle. If \( \lambda = 0, 1, \infty \) two stars coincide. If \( \lambda = e^{\pi i/3}, e^{-\pi i/3} \) the state is SLOCC equivalent to the tetrahedral state, in which the stars form the vertices of a regular tetrahedron.

The set of states for which some stars coincide is closed, but splits into four different orbits under SLOCC. Degeneracy type \( \{2,1,1\} \) is the largest of these, but in itself it is an open set since states with more than two coinciding stars are not included. Its closure contains types \( \{2,2\} \) and \( \{3,1\} \), which are in themselves open orbits. Finally type \( \{4\} \) consists of the separable states and forms a 2-sphere sitting in the closure of all the preceding types. Degeneracy types are called ‘Petrov types’ by Penrose and Rindler \[113\].
FIG. 4: The cross ratio can take any value in the complex plane, but values related according to (4.3) are equivalent. Left: Therefore the complex plane is divided into 6 equivalent regions—the picture would look more symmetric if drawn on the Riemann sphere. Right: Using region I only we show the location of some states of special interest. Only a part of its boundary is included in the set of SLOCC equivalence classes.

As \( K \) increases beyond 4 the story becomes increasingly complicated. It is not only difficult to survey the SLOCC orbits, it even becomes hard to count the degeneracy types—this is the number \( p \) of partitions of the number \( n \) into positive integers. This well known number theoretical problem was studied by Hardy and Ramanujan who obtained the asymptotic expression

\[
p(K) \sim \frac{1}{4\sqrt{3K}} \exp\left(\pi\sqrt{2K/3}\right),
\]

valid when \( K \) is large. The original result [63] goes back to 1918. An elementary proof of this formula was given in 1942 by Erdős [46] and simplified in 1962 by Newman [102].

Distinguishing between the possible degeneracy types by means of suitable invariants is then practically impossible. A typical SLOCC orbit has dimension 6 since this is the number of free parameters in the group \( SL(2, \mathbb{C}) \), which means that the set of orbits representing states with \( K \) non–coinciding stars always has real dimension \( 2K - 6 \). For \( K = 5 \) we need two complex valued invariants—which are known functions of cross ratios, each cross ratio being a function of only four stars—but writing them down explicitly is a lengthy affair. For \( K \) qubits we would need \( K - 3 \) cross ratios, each computed using four stars at a time.

In Section 7.4 we mentioned the classical Thomson-problem: assume the stars are charged electrons, and place them on a sphere in a way that minimizes the electric potential. In a similar vein we can ask for the ‘maximally entangled’ constellation. We can for instance try to maximize the Fubini-Study distance to the closest separable pure state. For \( n \geq 3 \) it is known that the latter lies in the symmetric subspace too (although the proof is not easy [71]). The actual calculation is hard; for \( n = 3 \) one finds that the \( W \)-state is the winner. For \( n = 4 \) numerical results point to the tetrahedral state as being ‘maximally entangled’ in this sense, and for higher \( n \) the resulting constellations tend to be spread out in interesting patterns on the sphere [7, 8, 93, 94].

The stellar representation was used here to describe symmetric states of \( K \) qubits. The same approach works also for symmetric states of a system consisting of \( K \) subsystems with \( N \) levels each [98]. Separable states are then \( SU(N) \) coherent states, see Sections 6.4 and 6.5 so a state can be represented by \( K \) unordered points in \( \mathbb{CP}^{N-1} \). For symmetric states of higher dimensional systems a stellar representation works fine, but the sky in which the stars shine is just larger than ours.

V. INVARIANT THEORY AND QUANTUM STATES

In this section we make a head-on attempt to divide the set of multi-qubit states into species and genera. We would like to have one or several simple functions of the components of their state vectors, such that they take the same values only if the states are locally equivalent. Let us call such functions local unitary or SLOCC invariants as the case may be. We also want enough of them, so that when we have evaluated them all we can say that the states are locally equivalent if and only if the invariants agree.

In the case of bipartite entanglement we know how to do this, because the states \( |\psi\rangle \) and \( |\psi'\rangle \) are locally equivalent if and only if their Schmidt coefficients agree, that is if and only if the eigenvalues of the reduced density matrices \( \rho_A = \text{Tr}_B |\psi\rangle\langle \psi| \) and \( \rho_A' = \text{Tr}_B |\psi'\rangle\langle \psi'| \) agree. When we restrict ourselves to qubits only (with just one independent eigenvalue) this will happen if and only if their determinants agree. Letting the matrix \( \Gamma \) carry the components of
the state vector we see from Eq. (16.19) that
\[ \det \rho_A = \frac{1}{N^2} |\det \Gamma|^2, \]  
(5.1)

For qubits, when \(N = 2\), we are in fact dealing with the tangle
\[ \tau(|\Gamma\rangle) = |\det \Gamma|^2 = \frac{1}{4}|\epsilon_{ij} \epsilon_{i'j'} \Gamma^{ii'} \Gamma^{jj'}|^2. \]  
(5.2)

From now on we will be careful with tensors, just as we were in the earlier chapters. Because the transformation group is \(U(2) \times U(2)\) there are two kinds of indices, unprimed and primed, and in the language of Section 4.4 \(\Gamma^{ij'}\) is a contravariant tensor. In section 4.4 we used \(A, B, \ldots\) for the indices, but in this chapter \(A\) is for Alice, \(B\) is for Bob, and \(C\) is (perhaps—opinions diverge on this point) for Charlie, so a change was called for. Indices always run from 0 to \(N - 1\) (and usually \(N = 2\)).

Indices may be summed over only if they are of the same kind and if one of them is upstairs and the other downstairs. Einstein’s summation convention is in force, so repeated indices are automatically summed over. Thus when a local operation acts on a state it means that
\[ \Gamma^{ii'} \rightarrow \Gamma^{ii'} = L^i_j L^{i'}_{j'} \Gamma^{jj'}, \]  
(5.3)

where \(L^i_j\) and \(L^{i'}_{j'}\) are two independent matrices, unitary or generalinvertible as the case may be. As usual the antisymmetric tensor \(\epsilon_{ij}\) obeys \(\epsilon_{00} = \epsilon_{11} = 0\) and \(\epsilon_{01} = 1 = -\epsilon_{10}\), always. This means that it transforms not like a tensor but like a tensor density,
\[ \epsilon_{ij} \rightarrow \epsilon'_{ij} = \epsilon_{ij} (\det L)^{-1} = \epsilon_{ij}. \]  
(5.4)

Hence \(\det \Gamma\) changes with a determinantal factor under general linear transformations. (See Schrödinger [122] for a short introduction to tensor calculus, and Penrose and Rindler [112] for a long one.) This will become important as we proceed.

For a pair of qubits the tangle is the only invariant we need. Can we do something similar for many qubits? This is again a homogeneous polynomial of the same degree, and its coefficients \(a'_i\) depend linearly on the original coefficients \(a_i\). A small yet satisfying calculation confirms that
\[ \Delta(a) = a_0 a_2 - a_1^2 = (\alpha \delta - \beta \gamma)^2 (a'_0 a'_2 - a'_1^2) = (\det G)^2 \Delta(a'), \]  
(5.9)

where we defined the discriminant of the polynomial,
\[ \Delta \equiv a_0 a_2 - a_1^2. \]  
(5.7)

The discriminant vanishes if and only if the roots coincide. Now suppose we make a Möbius transformation, according to the recipe in Eq. (4.34). We declare that the quadratic form \(Q\) transforms into a new quadratic form \(Q'\), according to the prescription that
\[ Q'(u', v') = Q'(\alpha u + \beta v, \gamma u + \delta v) = Q(u, v). \]  
(5.8)

This is again a homogeneous polynomial of the same degree, and its coefficients \(a'_i\) depend linearly on the original coefficients \(a_i\). A small yet satisfying calculation confirms that
where \( \det G \) is the non-zero determinant of the invertible two-by-two matrix occurring in Eq. (4.34). The calculation shows that the number of distinct roots will be preserved by any linear transformation of the variables, and it also makes the discriminant our—and everybody’s—first example of a relative invariant under the group \( GL(2, \mathbb{C}) \). It is strictly invariant under \( SL(2, \mathbb{C}) \).

This example can be generalized to homogeneous polynomials \( Q(tu, tv) = t^n Q(u, v) \) of arbitrary order \( n \), giving rise to \( n \)-th order polynomials in one variable, and indeed to homogeneous polynomials in any number of variables. Under a linear transformation of the variables Eq. (5.5) will force the coefficients of \( Q' \) to become linear functions of the coefficients of \( Q \). An invariant of weight \( m \) is a polynomial function of the coefficients which changes under invertible linear transformation only with a determinantal factor,

\[
I(a) = (\det G)^m I(a') .
\]

We will also need covariants of weight \( m \), which are polynomial functions of the coefficients and variables of a homogeneous polynomial \( Q(u, v) \) such that

\[
J(a, u, v) = (\det G)^m J(a', u', v') .
\]

Evidently \( Q \) itself is a covariant, but not the only one—fortunately, because we need enough of them, so that they characterize the behaviour of the \( n \)-th order polynomial \( p(z) = Q(u, v)/t^n \).

Now a polynomial \( p(z) \) has a multiple root if and only if it has a root in common with the polynomial \( \partial_z p(z) \). In general two polynomials, one of order \( n \) and the other of order \( m \), have a common root if and only if their resultant vanishes. By definition this is the determinant of a matrix which is constructed by writing the coefficients of the \( n \)-th order polynomial followed by \( m-1 \) zeros in the first row. In the next \( m-1 \) rows one shifts the entries of the previous row cyclically one step to the right. Then comes a row which contains the coefficients of the \( m \)th order polynomial followed by \( n-1 \) zeros, and this is followed by \( n-1 \) rows where the entries are permuted as before. (We present the construction as a piece of magic here, and refer to the literature for proofs.)

We can now check if a cubic polynomial \( p(z) = a_0 z^3 + 3a_1 z^2 + 3a_2 z + a_3 \) has a multiple root by taking the resultant with its derivative. It is convenient to divide by an overall factor, and define

\[
\Delta = \frac{1}{27 a_0} \begin{vmatrix}
  a_0 & 3a_1 & 3a_2 & a_3 & 0 \\
  0 & a_0 & 3a_1 & 3a_2 & a_3 \\
  3a_0 & 6a_1 & 3a_2 & 0 & 0 \\
  0 & 3a_0 & 6a_1 & 3a_2 & 0 \\
  0 & 0 & 3a_0 & 6a_1 & 3a_2
\end{vmatrix} .
\] (5.12)

This invariant is called the discriminant of the cubic, and has weight 6 in the sense that \( \Delta(a) \equiv (\det G)^6 \Delta(a') \). It vanishes if and only if the corresponding quantum state has a pair of coinciding stars (in the sense of Section IV). But it does not allow us to distinguish between the degeneracy classes \( \{2, 1, 1\} \) and \( \{3\} \). To do so we need to bring in a covariant as well. For the cubic form we have the Hessian covariant \( H \), which by definition is the determinant of the matrix of second derivatives of the form \( Q \). In itself this is a quadratic form whose associated discriminant is precisely equal to \( \Delta \). One can show that \( H = 0 \) and \( Q \neq 0 \) if and only the cubic polynomial has a triple root. An additional covariant, denoted \( T \), is often listed. It obeys

\[
T^2 = 2^{4/3} \Delta Q^2 - H^3 .
\]

This is an example of a syzygy between invariants. (In astronomy a syzygy is said to occur when three celestial bodies line up on a straight line. The term is also used in poetry, but we omit the details.)

Note that the order of the discriminant, considered as a polynomial function of the coefficients, increases with the order of the polynomial. However, something interesting happens when we consider quartic polynomials (meaning, for us, symmetric states of four qubits). After calculating the discriminant \( \Delta \) of the quartic according to the recipe, one can show that

\[
\Delta = I_1^3 - 27 I_2^2 ,
\]

where

\[
I_1 = a_0 a_4 - 4a_1 a_3 + 3a_2^2 , \quad I_2 = \begin{vmatrix}
  a_0 & a_1 & a_2 \\
  a_1 & a_2 & a_3 \\
  a_2 & a_3 & a_4
\end{vmatrix} .
\] (5.15)

Both \( I_1 \) and \( I_2 \) are in themselves invariants, of weight 4 and 6 respectively. Hence we have three invariants, but the third is given as a polynomial function of the two others. The interpretation is that the quartic has a multiple root...
if and only if $\Delta = 0$. It has a triple or quadruple root if and only if $I_1 = I_2 = 0$. The corresponding quantum state is SLOCC equivalent to the tetrahedral state if and only if $I_1 = 0$ and $I_2 \neq 0$. There are three covariants in addition to the two independent invariants.

The difficulties increase with the order of the polynomial. Even Sylvester got himself into trouble with the septic polynomial \cite{104}, and a modern mathematician (Dixmier) says that "la situation pour $n \geq 9$ est obscure". We leave that as an exercise in French.

The question, how many independent polynomial invariants may exist occupied numerous eminent mathematicians. An important issue was to check whether there exists a finite collection of invariants, such that any invariant can be written as their polynomial function. A deep theorem was proved by Hilbert himself: Hilbert’s basis theorem. For any system of homogeneous polynomials, every invariant is a polynomial function of a finite number among them.

In highbrow language the invariants belong to a finitely generated ring. In a way the theorem offers more information than we really need, since we may be happy to regard a set of invariants as complete once every other invariant can be given as some not necessarily polynomial function of that set, as happens in Eq. \eqref{5.13}. Hilbert eventually provided a constructive proof of his theorem, but it is a moot question if his procedure can be implemented computationally. Fortunately the result holds also for the subgroups we are interested in, namely the local groups $G_U$ and $G_L$ \cite{104}. It also holds for some finite subgroups: an accessible example of what Hilbert’s theorem is about is the fact that every symmetric polynomial can be written as a polynomial function of the elementary symmetric functions \eqref{2.11}. The 14-th problem on Hilbert’s famous list concerns the issue of finiteness of a polynomial invariant basis for every subgroup of linear invertible transformations. It was settled (in the negative) in 1959 \cite{57}.

Leaving binary forms behind—or, in quantum mechanical terms, stepping out of the symmetric subspace—we can now try to characterize a given orbit of LU equivalent states by looking for the local unitary invariants. Each individual invariant will be a homogeneous polynomial in the components of the state vector, and we start our search with a guarantee that there exists a finite number of independent invariants, in terms of which all other invariants can be obtained by taking sums and products. If we can find such a set we are done: to verify if two pure states are locally equivalent it will be enough to compute a finite number of invariants for both states and check whether the corresponding values coincide.

But we still have to identify a suitable set of independent invariants. In general this task requires lengthy calculations, and here we confine ourselves to describing the results in the three qubit case. We rely on Sudbery \cite{131} and others \cite{88}. In their turn these authors relied on Weyl \cite{144}.

Any invariant under the group $G_U$ of local unitaries can be represented as a homogeneous polynomial in the entries of the tensor \eqref{2.23} and their complex conjugates. There exists only one invariant of order two, $I_1 = \langle \psi | \psi \rangle = \Gamma_{i_1 i_2 i_3} \Gamma_{i_1 i_2 i_3}$, equal to the norm of the state. (We now need three different kinds of indices!) There are three easily identified polynomial invariants of order four,

$$I_2 = \mbox{Tr} \rho_A^2 = \Gamma_{i_1 i_2 i_3} \Gamma_{j_1 j_2 j_3} \Gamma_{j_1 j_2 j_3} \Gamma_{i_1 j_2 j_3}$$

$$I_3 = \mbox{Tr} \rho_B^2 = \Gamma_{i_1 i_2 i_3} \Gamma_{j_1 j_2 j_3} \Gamma_{i_1 j_2 j_3} \Gamma_{j_1 j_2 j_3}$$

$$I_4 = \mbox{Tr} \rho_C^2 = \Gamma_{i_1 i_2 i_3} \Gamma_{j_1 j_2 j_3} \Gamma_{j_1 j_2 j_3} \Gamma_{i_1 j_2 j_3}$$

These are the purities of the three single–party reductions. By construction one has $1/2 \leq I_i \leq 1$ for $i = 2, 3, 4$. Note that the use of Einstein’s summation convention saved us from writing out six sums per invariant.

At order six we have invariants such as $\mbox{Tr} \rho_A^3$. However, because of the characteristic equation (see Section 8.1) applied to two-by-two matrices it happens that we have the syzygy $2 \mbox{Tr} \rho_A^3 = 3 \mbox{Tr} \rho_A^2 \mbox{Tr} \rho_A - (\mbox{Tr} \rho_A)^3$, so this is an example of an invariant dependent on those we introduced already. An independent polynomial invariant of order six, in this context called the Kempe invariant \cite{74}, is

$$I_5 = \Gamma_{i_1 i_2 i_3} \Gamma_{j_1 j_2 j_3} \Gamma_{k_1 k_2 k_3} \Gamma_{j_1 j_2 k_3} \Gamma_{j_1 k_2 j_3} \Gamma_{k_1 k_2 j_3}$$

It is clearly symmetric with respect to exchange of the subsystems, and one can show that $2/9 \leq I_5 \leq 1$. The minimum is attained by the $W$ state. By considering various bipartite splittings of our systems one finds some further possibilities, but they are not independent invariants because

$$I_5 = 3 \mbox{Tr}[(\rho_A \otimes \rho_B) \rho_{AB}] - \mbox{Tr} \rho_A^3 - \mbox{Tr} \rho_B^3 = 3 \mbox{Tr}[(\rho_B \otimes \rho_C) \rho_{BC}] - \mbox{Tr} \rho_B^3 - \mbox{Tr} \rho_C^3.$$

Readers who try to verify these identities should be aware that doing so requires very deft handling of the characteristic equation for two-by-two matrices \cite{131}. 

\[ I_5 = 3 \mbox{Tr}[(\rho_A \otimes \rho_B) \rho_{AB}] - \mbox{Tr} \rho_A^3 - \mbox{Tr} \rho_B^3 = 3 \mbox{Tr}[(\rho_B \otimes \rho_C) \rho_{BC}] - \mbox{Tr} \rho_B^3 - \mbox{Tr} \rho_C^3. \]
A sixth independent invariant is of order eight,

$$I_6 = |2\text{Det}_3(\Gamma)|^2,$$

(5.21)

where the hyperdeterminant of the tensor $$\Gamma_{i_1,i_2,i_3}$$ is, for the moment, defined to be

$$\text{Det}_3(\Gamma) = \frac{1}{2} \epsilon_{i_1,j_1,k_1} \epsilon_{i_2,j_2,k_2} \epsilon_{i_3,j_3,k_3} \Gamma_{i_1,i_2,i_3} \Gamma_{j_1,j_2,j_3} \Gamma_{k_1,k_2,k_3} \Gamma_{\ell_1,\ell_2,\ell_3} =$$

$$= \left[\Gamma_{000}^2 \Gamma_{111}^2 + \Gamma_{001}^2 \Gamma_{110}^2 + \Gamma_{010}^2 \Gamma_{101}^2 + \Gamma_{100}^2 \Gamma_{011}^2 \right]$$

$$- 2 \left[\Gamma_{000}^2 \Gamma_{111} (\Gamma_{011} \Gamma_{100} + \Gamma_{010} \Gamma_{110} \Gamma_{001}) + \Gamma_{101} \Gamma_{010} \Gamma_{110} \Gamma_{001} \right]$$

$$+ 4 \left[\Gamma_{000} \Gamma_{110} \Gamma_{101} \Gamma_{011} + \Gamma_{111} \Gamma_{001} \Gamma_{010} \Gamma_{100} \right].$$

(5.22)

(Hyperdeterminants were introduced by Cayley in 1845 [30], who had finalized the theory of determinants a couple of years earlier. Hyperdeterminants returned in force 150 years later in the book by Gelfand, Kapranov, and Zelevinsky [51], and were brought into this story by Coffman, Kundu, and Wootters [33].)

In the final expression indices are lowered for typographical reasons. To write out all the sums—and in fact for doing tensor algebra too—the graphical technique given in Figures 5-6 is helpful. It is due to Penrose [111, 112], who also provided the mathematical rigour behind it. Graphical notation as such goes back to Clifford [104]. Coecke and his collaborators have devised graphical techniques tailored to quantum mechanics [32].

![FIG. 5: In graphical notation a tensor is represented as a shape with arms (upper indices) and legs (lower indices) attached. Open lines are labelled by the free indices, while lines connecting two tensors represent contracted indices (which need no label). A complication is that our tensors are tensors under a product group, so three different kinds of lines occur. In private calculations clarity is gained (but speed lost) if one uses three different colours to distinguish them. a) Four examples of tensors. b) New tensors from old. The pure state density matrix $$\rho$$ equals an outer product of two tensors; it does not matter how we place the latter on the paper. We also show $$\rho_{BC}$$, obtained from a 3-party state $$\rho_{ABC}$$ by means of a partial trace.](image)

It is clear that the hyperdeterminant is somewhat analogous to the ordinary determinant, and that the invariant $$I_6$$ is analogous to the single invariant $$I_2$$ used in the two-qubit case. We postpone the general definition of hyperdeterminants a little, and just observe that each term is a product of four components such that their ‘barycentres’ coincide with the centre of the cube shown in Figure 1b. Examining the explicit expression for the hyperdeterminant one can convince oneself that $$0 \leq I_6 \leq 1/4$$. See also Table I.

If an invariant takes different values for two states then these states cannot be locally equivalent. Reasoning in the opposite direction is more difficult, since we have to show that we have a complete set of invariants. For a normalized state $$|\psi_{ABC}\rangle$$ the first invariant is fixed, $$I_1 = 1$$, so we have only five remaining invariants to describe a local orbit. Note that one needs at least five such numbers, as the dimensionality of $$CP^7$$ — the space of three-qubit pure states — is 14 and the number of parameters of the local unitary group $$SU(2)^{\otimes 3}$$ is $$3 \times 3 = 9$$. Unfortunately our invariants are still not enough to uniquely single out a local orbit. Clearly $$I_2(|\psi\rangle) = I_2(|\psi^*\rangle)$$ for all six invariants, so they cannot distinguish between a state and its complex conjugate. Grassl has found an additional invariant of order 12 which
FIG. 6: Examples of tensor calculations. a) Here we take the trace of products of two different reduced density matrices. b) As a non-trivial exercise we prove that $I_5 = \text{Tr}(\rho^T_{BC})^3$, where $\rho^T_{BC}$ is the partial transpose with respect to subsystem $C$ of the reduced density matrix $\text{Tr}_A \rho$.

TABLE I: LU invariants for some exemplary (normalized) states.

| State                  | $I_2$ | $I_3$ | $I_4$ | $I_5$ | $I_6$ |
|------------------------|-------|-------|-------|-------|-------|
| separable              | 1     | 1     | 1     | 0     |       |
| $|\phi_A\rangle \otimes |\psi_{BC}\rangle$ | 1/2   | 1/2   | 1/4   | 0     |       |
| $|\phi_B\rangle \otimes |\psi_{AC}\rangle$ | 1/2   | 1/2   | 1/4   | 0     |       |
| $|\phi_C\rangle \otimes |\psi_{AB}\rangle$ | 1/2   | 1/2   | 1     | 1/4   | 0     |
| $|W\rangle$              | 5/9   | 5/9   | 5/9   | 2/9   | 0     |
| $|GHZ\rangle$            | 1/2   | 1/2   | 1/4   | 1/4   | 1/4   |

does complete the set [2]. The somewhat more modest problem of deciding when two given multiqubit states can be connected by local unitaries has been discussed by Kraus [83].

Having described the local orbits in terms of polynomial invariants we can travel along the orbit in order to find some state distinguished by a simple canonical form. In the two-qubit case the Schmidt decomposition provides an obvious choice. In the three-qubit case things are not so simple. By means of three local unitary transformation we can always set 3 out of the 8 components of a given state vector to zero. See Figure 7 (For $K$ quNits we can set $K \cdot N(N-1)/2$ components to zero [29]). An elegant way to achieve this is to observe that there must exist a closest separable state, in the sense of Fubini-Study distance. By changing the bases among the qubits we can assume this to be the state $|111\rangle$. An easy exercise (Problem 17.4) shows that this forces three of the components to be zero. Having come so far we can adjust the phase factors of the basis vectors so that four out of the five remaining components are

FIG. 7: Using the nearest separable state as one element of the basis, basis vectors at Hamming distance 1 from this state do not contribute to the superposition, as shown here with open circles for a) 2, b) 3, and c) 4 qubits.
real and non-negative. Thus we arrive at the canonical form
\[ |\psi\rangle = r_0 e^{i\phi} (000) + r_1 |100\rangle + r_2 |010\rangle + r_3 |001\rangle + r_4 |111\rangle. \] (5.23)

Taking normalization into account only five free parameters remain. (A word of warning: The state |\psi\rangle can be presented in this way if |111\rangle is the closest separable state, but the converse does not hold.) We would like to know, for instance, what values the invariants have to take in order for a further reduction to four, three, or two, non-vanishing components to be possible, but for this purpose other choices of the canonical form are preferable \[2\].

We leave these matters here, and turn to the coarser classification based on stochastic LOCC (SLOCC), that is operations which locally transform the initial state into the target state with a non-zero probability. We will analyze the orbits arising when the group GL acts on given pure states. In contrast to the group GU of local unitaries, the group GL is not compact and the orbits may not be closed—one orbit can sit in the closure of another, and hence the set of orbits will have an intricate topology.

The unitary invariants I1, . . . , I5 will not survive as invariants under the larger group GL—unsurprisingly, since they were constructed using complex conjugation, and GL is the complexification of GU. SLOCC operations can change even the norm I1 of a state. We have one card left to play through, namely the hyperdeterminant Det3(Γ).

From its definition in terms of the tensor Γi1i2i3 and the ε-tensor, in Eq. (5.22), it is clear that it changes with a determinantal factor under SLOCC. Indeed
\[ \text{Det}_3(Γ) = (\text{det} L_1)^2(\text{det} L_2)^2(\text{det} L_3)^2 \text{ Det}_3(Γ'). \] (5.24)

Hence the hyperdeterminant is a relative invariant, and a state for which it vanishes—such as the W state—cannot be transformed into a state for which it is non-zero—such as the GHZ state. Pure states of three qubits can be entangled in two inequivalent ways. There is some additional fine structure left, and indeed a set of six independent covariants exist that together provide a full classification. Rather than listing them all Table II takes the easy way out, and uses the observation that the rank of the reduced density matrices, rA = r(ρA) = r(TRBCρABC), cannot be changed by an invertible SLOCC transformation, so if |ψ⟩ ≡SLOCC |φ⟩ then all three local ranks have to be pairwise equal, e.g. rA(|ψ⟩) = rA(|φ⟩). From the Table it is clear that the local ranks are enough to characterize a state as separable, or as containing bipartite entanglement only, but they cannot distinguish between the GHZ and W states—which, as we know from our tour of the symmetric subspace in Section V, are SLOCC inequivalent. For this we need the hyperdeterminant.

This division into SLOCC equivalence classes is complete. (This was first shown by Dür, Vidal and Cirac \[42\]. Actually, in a way, the classification had already been given by Gelfand et al. \[51\] in their book. But they used very different words.) The GHZ-class is dense in the set of all pure states while the W-class is of measure zero, and any state in the latter can be well approximated by a state in the former \[143\]. This is an important point, and we will return to it in Section VII.

| TABLE II: SLOCC equivalence classes for three-qubit pure states |ψABC⟩: rA, rB and rC denote ranks of single-partite reductions, while Det3(Γ) is the hyperdeterminant of the 3-tensor representing the state. |
|---|---|---|---|---|
| Class | rA | rB | rC | Det3(Γ) | entanglement |
| separable | 1 | 1 | 1 | = 0 | none |
| ψA ⊗ ψB | 1 | 2 | 2 | = 0 | bipartite |
| ψB ⊗ ψAC | 2 | 1 | 2 | = 0 | bipartite |
| ψC ⊗ ψAB | 2 | 2 | 1 | = 0 | bipartite |
| |W⟩ | 2 | 2 | 2 | = 0 | triple bipartite |
| |GHZ⟩ | 2 | 2 | 2 | > 0 | global tripartite |

How do these results generalize to larger systems? For any number of quNits the local unitary group GU = SU(N)⊗K is compact, it has closed orbits when acting on the complex projective space CPN−1, and Hilbert’s theorem (and its extensions) guarantees that these orbits can be characterized by a finite set of invariants. However, it is already clear from our overview of the situation in the symmetric subspace (Section V) that the results would be so complicated that one may perhaps prefer not to know them. If instead we consider equivalence under SLOCC for modest numbers of qubits things do look better. Polynomial invariants are known for four \[80\] and five qubits \[91\], although the results for the latter are partial only. In the 4-qubit Hilbert space there exists infinitely many SLOCC orbits, and their classification is not straightforward \[54\] \[107\]. Still they can be organized into nine families of genuinely four-partite entangled states, six of them depending on a continuous parameter \[31\] \[137\]. We will give a slightly coarser classification of this case (into seven subcases) in Section VII.
To go to higher numbers of qubits we definitely need a yet coarser classification of entangled states. This is where the hyperdeterminant comes into its own—although it will have occurred to the reader that we have not really defined the hyperdeterminant, we simply gave an expression for it in the simplest case. To outline its geometrical meaning, let us return to the bipartite case, but for two $N$-level subsystems. The separable states are then described by the Segre embedding of $\mathbb{CP}^{N-1} \times \mathbb{CP}^{N-1}$ into $\mathbb{CP}^{N^2-1}$. Any state can be described by a tensor $\Gamma^{ij}$, and generic entangled states can be characterized by the fact that its determinant is non-zero. For two quNits there are many intermediate cases for which the determinant vanishes even though the state is non-separable. In fact the rank of $\Gamma$ gives rise to an onion-like structure, with all the non-generic cases characterized by the single equation $\det \Gamma = 0$. To unravel its geometric meaning we have to recall the notion of projective duality from Section 4.1. The idea is that a hyperplane in $\mathbb{CP}^{N^2-1}$ can be regarded as a point in a dual copy of the same space. Now consider a hyperplane corresponding to the point $\bar{\Gamma}$ in the dual space, and tangent to the separable Segre variety at the point $x^iy^j$. This hyperplane is defined by the equation

$$F = \bar{\Gamma}_{ij}x^iy^j = 0 \quad (5.25),$$

together with

$$\frac{\partial}{\partial x^i}F = \frac{\partial}{\partial y^j}F = 0 \quad \Leftrightarrow \quad \bar{\Gamma}_{ij}y^j = x^i\bar{\Gamma}_{ji} = 0 \quad (5.26).$$

The second set of equations ensures that the hyperplane has ‘higher order contact’ with the Segre variety, as befits a tangent hyperplane. But the condition that these equations do have a solution is precisely that $\det \bar{\Gamma} = 0$. This gives a geometrical interpretation of this very equation. It only remains to find the generalization to the set of tangent planes of the generalized Segre variety $\mathbb{CP}^{N-1} \times \mathbb{CP}^{N-1} \times \cdots \times \mathbb{CP}^{N-1}$.

This is not an easy task. For three qubits it turns out to be precisely the condition that the hyperdeterminant, as given above, vanishes. A similar polynomial equation (in the components of the tensor that defines the state) exists for any number of qubits. The polynomial is called the hyperdeterminant, and its degree is known. Unfortunately this is not an easy task. For three qubits it turns out to be precisely the condition that the hyperdeterminant, as given above, vanishes. A similar polynomial equation (in the components of the tensor that defines the state) exists for any number of qubits. The polynomial is called the hyperdeterminant, and its degree is known.

VI. MONOGAMY RELATIONS AND GLOBAL MULTIPARTITE ENTANGLEMENT

Any system consisting of three subsystems can be split into two parts in three different ways. Furthermore one can consider any two parties and investigate their entanglement. How is the entanglement between a single party $A$ and the composite system $BC$, written $A|BC$, related to the pairwise entanglement $A|B$ and $A|C$?

Coffman, Kundu and Wootters analysed this question using tangle as a measure of entanglement. Recall that this quantity equals the square of the concurrence, $\tau(\rho) = C^2(\rho)$, and is known analytically for any mixed state of two qubits $|\psi\rangle$. See Eq. (16.103). They established the monogamy relation

$$\tau_{A|BC} \geq \tau_{A|B} + \tau_{A|C} \quad (6.1).$$

Here $\tau_{A|B}$ denotes the tangle of the two–qubit reduced state, $\rho_{AB} = \text{Tr}_CP_{ABC}$, while $\tau_{A|BC}$ represents the tangle between part $A$ and the composite system $BC$. Although the state of subsystems $BC$ lives in four dimensions, its rank is not larger than two, as it is obtained by the reduction of the pure state $\rho_{ABC} = |\psi_{ABC}\rangle\langle\psi_{ABC}|$. This observation allows one to describe entanglement along the partition $A|BC$ with the two–qubit tangle (16.103) and to establish inequality (6.1).

Even though subsystem $A$ can be simultaneously entangled with the remaining subsystems $B$ and $C$, the sum of these two entanglements cannot exceed the entanglement between $A$ and $BC$. This implies, for instance, that if $\tau_{A|BC} = \tau_{A|B} = 1$ then $\tau_{A|C} = 0$. Hence if the qubit $A$ is maximally entangled with $B$, then it cannot be entangled with $C$.

To characterize entanglement in the three–qubit systems it is natural to consider the tangle averaged over all possible splittings. The average tangle of a pure state $\rho_{ABC} = |\psi_{ABC}\rangle\langle\psi_{ABC}|$ with respect to $1 + 2$ splittings reads

$$\tau_1(|\psi_{ABC}\rangle) \equiv \frac{1}{3} \left( \tau_{A|BC} + \tau_{B|AC} + \tau_{C|AB} \right) \quad (6.2).$$
Another related quantity describes average entanglement contained in two-partite reductions,
\[
\tau_2(\ket{\psi_{ABC}}) \equiv \frac{1}{3}(\tau_{A|B} + \tau_{B|C} + \tau_{C|A}).
\]
(6.3)

By construction, the quantities \(\tau_1, \tau_2\) are non-negative and for any pure state can be computed analytically with help of the Wootters formula (16.103). As entanglement is monotone with respect to partial trace, \(\tau_{A|BC} \geq \tau_{A|B}\) (see Section 16.8) the relation \(\tau_1(\ket{\psi_{ABC}}) \geq \tau_2(\ket{\psi_{ABC}})\) holds for every pure state.

It is easy to check that \(\tau_1\) achieves its maximum for the GHZ state,
\[
0 \leq \tau_1(\ket{\psi_{ABC}}) \leq \tau_1(\ket{GHZ}) = 1.
\]
(6.4)

As discussed in Problem 17.1 after tracing out any single part of \(\ket{GHZ}\), the remaining two subsystems become separable, so \(\tau_2(\ket{GHZ}) = 0\). See a cartoon sketch in Figure 8. The latter quantity, characterizing the mean bipartite entanglement, is maximized by the \(W\) state [42],
\[
0 \leq \tau_2(\ket{\psi_{ABC}}) \leq \tau_2(\ket{W}) = 4/9.
\]
(6.5)

Thus taking into account the quantity \(\tau_1\) the GHZ state is ‘more entangled’ than the \(W\) state, while the opposite is true if we look at the measure \(\tau_2\). The fact that the maximum in (6.5) is smaller than unity is another feature of the monogamy of entanglement: an increase of the entanglement between \(A\) and \(B\) measured by tangle has to be compensated by a corresponding decrease of the entanglement between \(A\) and \(C\) or \(B\) and \(C\).

The last, and the most important, measure, which characterizes the \textit{global entanglement}, is the \textit{3–tangle} [33],
\[
\tau_3(\ket{\psi_{ABC}}) \equiv \tau_{A|BC} - \tau_{A|B} - \tau_{A|C}.
\]
(6.6)

This quantity, invariant with respect to permutation of the parties, is also called \textit{residual entanglement}, as it marks the fraction of entanglement which cannot be described by any two-body measures.

For any three qubit pure state \(\ket{\psi_{ABC}}\) written in terms of the components \(G^{ijk}\) one can use the Wootters formula (16.103) for tangle and find an analytical expression for the 3-tangle. It turns out [33, 42] that the three–tangle is proportional to the modulus of the hyperdeterminant,
\[
\tau_3(\ket{\psi_{ABC}}) = 4|\text{Det}_3(G)|.
\]
(6.7)

which already appeared in (5.22). Thus \(\tau_3\) is indeed invariant with respect to local unitary operations, as necessary for an entanglement measure and forms also an entanglement monotone [42]. Furthermore, \(\tau_3\) is a relative invariant

![Figure 8: Schematic comparison between the distinguished 3–qubit pure states with the help of strings and knots: if one subsystem is traced away from a system in the \(W\) state (panels a and c), the other two subsystems remain entangled. This is not the case for the GHZ state, thus it can be represented by three objects joined with a single thread or three \(Borromean rings\), which enjoy an analogous property. See panels b) and d).](image-url)
under the action of the group $G_L = GL(2, \mathbb{C})^\otimes 3$, so it distinguishes the class of $W$ states and the GHZ states with respect to SLOCC transformations. Due to the monogamy relation (6.1), the 3–tangle is non-negative, and it is equal to zero for any state separable under any cut. Its maximum is achieved for the GHZ state, for which $\tau_{A|BC} = 1$ and $\tau_{A|B} = \tau_{A|C} = 0$, so that $\tau_3(\rho_{\text{GHZ}}) = 1$. With respect to the Fubini–Study measure on the space of pure states of a three–qubit system it is possible to obtain \cite{72} the average value $\langle \tau_3 \rangle = 1/3$.

The monogamy relation (6.1), originally established for three qubits \cite{33}, was later generalized for several qubits by Osborne and Verstraete \cite{104}, while Eltschka and Siewert \cite{45} derived monogamy equalities—exact relations between different kinds of entanglement satisfied by pure states of a system consisting of an arbitrary number of qubits.

A generalization to quNits, with $N > 2$, is even more tricky. Monogamy relation based on tangle does not hold for several subsystems with three or more levels each \cite{104}. Such relations can be formulated for negativity extended to mixed states by convex roof \cite{76} and squared entanglement of formation \cite{39}. Furthermore, for systems of an arbitrary dimension general monogamy relations hold for the squashed entanglement \cite{77}, as this measure is known to be additive.

Observe that the tangle of a pure state of two qubits can be written as $\tau(\psi) = |\langle \psi | \sigma_y \otimes \sigma_y | \psi^* \rangle|^2$, where $|\psi^*\rangle$ denotes the state after complex conjugation in the computational basis. In a similar way for a four–qubit pure state one can construct the four–tangle $\tau_4(\psi) = |\langle \psi | \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y | \psi^* \rangle|^2$. This quantity, introduced by Wong and Christiansen \cite{146}, can be interpreted as 4-party residual entanglement that cannot be shared between two–qubit bipartite cuts \cite{54}. Four–tangle is invariant with respect to permutations and extended for mixed states by convex roof forms an entanglement monotone \cite{146}.

\section{VII. LOCAL SPECTRA AND THE MOMENTUM MAP}

The story of this section begins with an after-dinner-speech by C. A. Coulson, who observed that the most interesting properties of many-electron systems tend to concern observables connecting at most two parties. To calculate them, only the reduced two–party density matrices are needed. (Coulson’s question (1960) \cite{34} inspired important work by A. J. Coleman \cite{34} \cite{55}. Their version of the problem concerned fermionic constituents. We consider distinguishable subsystems only.)

The question he raised was: what conditions on a two–party density matrix ensure that it can have arisen as a partial trace over $K – 2$ subsystems of a pure $K$-partite state? This is a version of the quantum marginal problem. In the analogous classical problem one asks for the conditions on a pair distribution $p_{ij}^2$ (say) ensuring that it can arise as a marginal distribution from the joint distribution $p_{123}$. This is a question about the projection of a simplex in $\mathbb{R}^{N^3}$ to a convex body in $\mathbb{R}^{N^2} \oplus \mathbb{R}^{N^2} \oplus \mathbb{R}^{N^2}$. After finding the pure points of the projected body one has to calculate the facets of their convex hull. But this is a computationally demanding problem. Still the projection of a point is always a point, so the classical problem is trivial for pure states. Not so the quantum problem. Already when one starts from a pure quantum state one needs sophisticated tools from algebraic geometry, and from invariant theory, in order to set up an algorithm for it. Finding the restrictions on the resulting two–party density matrices is really hard. The problem was reduced to that of describing momentum polytopes (see below) by Klyachko \cite{79} and by Daftuar and Hayden \cite{37}, in both cases relying on earlier work by Klyachko \cite{80}.

For bipartite pure states the story is simple. There are two reduced density matrices $\rho_A = Tr_B \rho$ and $\rho_B = Tr_A \rho$, and their spectra are the same (up to extra zeros, if the two subsystems differ in dimensionality). That spectrum can be anything however, which is dramatically different from the classical case where the subsystems are in pure states whenever the composite system is. To figure out what happens for 3-qubit systems we choose a partition, say $A|BC$, and perform a Schmidt decomposition. Thus

$$|\psi_{ABC}\rangle = \sqrt{\lambda_A} \sum_{i,j} a^{ij} |0_A i B J C\rangle + \sqrt{1 - \lambda_A} \sum_{i,j} b^{ij} |1_A i B J C\rangle .$$

(7.1)

By construction

$$\sum_{i,j} a^{ij} \tilde{a}_{ij} = \sum_{i,j} b^{ij} \tilde{b}_{ij} = 1 , \quad \sum_{i,j} a^{ij} \tilde{b}_{ij} = 0 .$$

(7.2)

As always we are free to choose the local bases, and this we do in such a way that $|0_A\rangle$ is the eigenstate of $\rho_A$ with the smallest eigenvalue $\lambda_A$, and similarly for all three subsystems. We can then read off that

$$\lambda_B = \lambda_A \sum_j a^{0j} \tilde{a}_{0j} + (1 - \lambda_A) \sum_j b^{0j} \tilde{b}_{0j} .$$

(7.3)
and similarly for $\lambda_C$. From this we find the inequalities

$$
\lambda_B + \lambda_C = \lambda_A (a^{0j} b_{0j} + a^{0j} b_{0j}) + (1 - \lambda_A)(b^{0j} b_{0j} + b^{0j} b_{0j}) \geq
$$

$$
\geq \lambda_A \left( \sum_{i,j} a^{ij} a_{ij} - |a_{11}|^2 \right) + (1 - \lambda_A) \left( \sum_{i,j} b^{ij} b_{ij} - |b_{11}|^2 \right) \geq
$$

$$
\geq \lambda_A \left( 2 |a_{11}|^2 - |b_{11}|^2 \right),
$$

where we used $\lambda_A \leq 1 - \lambda_A$ and Eqs. (7.2) in the last step. However, $a_{11}$ and $b_{11}$ are the corresponding components of two orthonormal vectors, and hence they obey $|a_{11}|^2 + |b_{11}|^2 \leq 1$. (This is easy to prove using the Cauchy-Schwarz inequality, perhaps avoiding the composite indices for clarity.) Thus, repeating the exercise for all three partitions, we arrive at the triangle inequalities obeyed by the smallest eigenvalues of the three reduced density matrices, namely

$$
\lambda_A \leq \lambda_B + \lambda_C ; \quad \lambda_B \leq \lambda_A + \lambda_C ; \quad \lambda_C \leq \lambda_A + \lambda_B .
$$

Hence there are definite restrictions on the local spectra in the 3-qubit case.

There are no other restrictions. To see this, consider the state

$$
|\psi\rangle = a|001\rangle + b|010\rangle + c|100\rangle + d|111\rangle ,
$$

where all the components are real. Straightforward calculation verifies that

$$
\begin{align*}
\lambda_A &= a^2 + b^2 \\
\lambda_B &= a^2 + c^2 \\
\lambda_C &= b^2 + c^2
\end{align*}
$$

$$
\Rightarrow \begin{cases} 
2a^2 = \lambda_A + \lambda_B - \lambda_C \\
2b^2 = \lambda_A + \lambda_C - \lambda_B \\
2c^2 = \lambda_B + \lambda_C - \lambda_A .
\end{cases}
$$

By choosing $a, b, c$ we can realize any triple $(\lambda_A, \lambda_B, \lambda_C)$ obeying the triangle inequalities.

At the expense of some notational effort the argument can be repeated for $K$-qubit systems, resulting in polygon inequalities of the form

$$
\lambda_k \leq \lambda_1 + \cdots + \lambda_{k-1} + \lambda_{k+1} + \cdots + \lambda_K .
$$

Here $\lambda_k$ denotes the smallest eigenvalue of the reduced density matrix for the $k$-th subsystem. Again the inequalities are sharp: no further restrictions occur. Full details can be found in the paper by Higuchi, Sudbery, and Szulc [70], from which the whole argument is taken. Bravyi [21] provides some further results.

Together with the inequalities $0 \leq \lambda_k \leq 1/2$ the polygon inequalities define a convex polytope known as the entanglement polytope, or sometimes as the Kirwan polytope (for a reason we will come to). It is the convex hull of $2^K - K$ extreme points which are easily found since the whole polytope is inscribed in a hypercube with corners whose $K$ coordinates equal either 0 or 1/2. Let us refer to the corner $(0,0,\ldots,0)$ as the separable corner, since separable states end up there. The inequalities (7.3) imply that all the $K$ corners at Hamming distance 1 from the separable corner are missing, whereas all the other corners of the cube are there. On the long diagonal connecting the separable corner to the GHZ corner $(1/2,1/2,\ldots,1/2)$ — which is where the image of the GHZ state is to be found—we find the images of states in the symmetric subspace. The case $K = 3$ is an easily visualized bipyramid with 5 corners and 6 faces (Figure 9). When $K = 4$ there are 12 corners and 12 facets; the latter are of three different kinds and include 4 copies of the $K = 3$ bipyramid. When $K = 5$ there are 27 corners and 15 facets, and so it goes on.

The story becomes much more interesting once we ask where differently entangled states land in the polytope. For three qubits the story is simple. The $W$ state lands at $(1/3,1/3,1/3)$, and for all states that are SLOCC equivalent to the $W$ state one finds

$$
\lambda_A + \lambda_B + \lambda_C \leq 1 .
$$

In fact the image of this class of states forms a polytope of its own, making up the lower pyramid in Figure 9. In this particular case elementary arguments suffice to prove it [62]. One can also show that states with bi-partite entanglement only form the three edges emanating from the separable corner. These edges are polytopes of their own, which we can call $P_{A|BC}$, $P_{B|AC}$, and $P_{C|AB}$. The image of the separable states is a single point $P_{\text{sep}}$. Images of the generic states (SLOCC equivalent to the GHZ state) form an open set whose closure is the entire polytope. In this way we have a hierarchy of polytopes, $P_{\text{sep}} \subseteq P_{A|BC} \subseteq P_W \subseteq P_{\text{GHZ}}$. A polytope in the hierarchy is a subpolytope.
FIG. 9: Entanglement polytope for three qubits: a) it is formed by three smaller eigenvalues of the single–party reduced states, satisfying the triangle inequalities (7.5); b) Corners $O$ and $G$ represent fully separable and GHZ states, respectively. Three edges of the polytope connected to $O$ denote states with bipartite entanglement only. States equivalent with respect to SLOCC to $W$ belong to the shaded pyramid. Glueing eight such bipyramids together around the GHZ point we obtain a stellated octahedron or ‘stella octangula’, which is not convex. Compare Figure 11.3.

of another if it corresponds to an orbit that lies in the closure of the other orbit, or in physical terms if a state from the former can be approximated by a state from the latter with arbitrary precision.

We have learned that the local spectra carry some information about the kind of entanglement: if $\lambda_A + \lambda_B + \lambda_C > 1$ the state investigated is of the GHZ type. (Perhaps we should note though that the spectra of the set of all mixed states will fill the entire cube, so if we want to use the local spectra as witnesses of GHZ–type entanglement we need a guarantee that the states we measure are close to pure.) For a state of the GHZ class there is a two parameter family of orbits under local unitaries ending up at the same point in the entanglement polytope, while a pure state in the closure of the $W$ class is determined uniquely up to local unitaries by its local spectra [1 20]. Generically this ambiguity can be resolved by considering also the spectra of all bipartite reduced density matrices, but for Schmidt decomposable states some ambiguity remains [89]: For states of the form $a|000\rangle + b|111\rangle$ the spectrum of every reduced density matrix agrees with that obtained from the mixed state $|a|^2|000\rangle\langle 000| + |b|^2|111\rangle\langle 111|$.

The four dimensional entanglement polytope characterizing the four–qubit system was analysed in detail by Walter et al. [142], and by Sawicki et al. [119]. Their arguments rely on invariant theory and symplectic geometry, and are no longer elementary. (The first step in one of these papers is to calculate the 170 independent covariants of the four-qubit system [24] for the nine different families found by Verstraete et al. [137].) The results are easy to describe though, and are summarized in Table III. There are six kinds of four-dimensional subpolytopes, partially ordered by the inclusion relations $P_5 \subseteq P_3 \subseteq P_2 \subseteq P_1$ and $P_5 \subseteq P_4 \subseteq P_6$. In addition to those we find lower dimensional subpolytopes describing states without genuine four-partite entanglement. This includes four facets (called $F_2$ in the Table) that are simply copies of the three-qubit polytope. Anyway the botany has become surveyable.

Our story is not complete since we have said nothing about restrictions on the spectra when the partial trace is taken over two subsystems only. And an explicit generalization to quNits is not easy (although the case of three qutrits is manageable [68]). But for us there is a more burning question to discuss: Why are the results as simple as they are? Why are all the conditions we encountered given by linear inequalities on the spectra of the reduced density matrices?

In the case of pure states of a system consisting of $K$ subsystems of size $N$ the entanglement polytope lives in $\mathbb{R}^{K(N-1)}$, so we are dealing with a map

$$\mathbb{CP}^{N^K-1} \to \mathbb{R}^{K(N-1)}.$$  

At this point we have to take up a thread that we left dangling at the end of Section 13.5, and explain the concepts of momentum maps and momentum polytopes. Their names suggest that they have something to do with momentum, and indeed as our motivating example (forcing us to take a detour through analytical mechanics) we choose angular momentum. On the phase space of a particle, with coordinates $q^i$ and $p_i$, there exist three functions $J_i = J_i(q,p)$ such that

$$\delta q^i = \{q^i, \xi^k J_k\}, \quad \delta p_i = \{p_i, \xi^k J_k\},$$

$$\delta J_i = \{J_i, \xi^k J_k\} = \epsilon_{ik}^j \xi^k J_j \equiv \xi^i J^j.$$  

\[\text{(7.11)}\]

\[\text{(7.12)}\]
Here we should think of $\xi^j$ as a vector in the Lie algebra of the rotation group $SO(3)$, but we also rewrote it as a matrix $\xi^j$ using the natural matrix basis in the Lie algebra. For any given point $x$ in phase space the vector $J_i$ is a linear functional on the Lie algebra, meaning that $J_i\xi^j$ is a real number. Thus $J_i$ sits in a vector space which is dual to the Lie algebra. The rotation group acts on its own Lie algebra by means of conjugation, and it also acts on the dual vector space by means of what we will soon refer to as the coadjoint action. In general, any Lie group $G$ has a Lie algebra $\mathfrak{g}$. This is a vector space, and there exists a dual vector space $\mathfrak{g}^*$ such that for any element $\alpha$ in $\mathfrak{g}^*$ and any element $\xi$ in $\mathfrak{g}$ there exists a real number $\langle \alpha, \xi \rangle$. The Lie group acts on $\mathfrak{g}$ by conjugation, and this gives rise to an action on $\mathfrak{g}^*$ as well. It is known as the coadjoint action, denoted $\text{Ad}_g^*$ and defined by

$$
\langle \text{Ad}_g^*\alpha, \xi \rangle = \langle \alpha, \text{Ad}_g^{-1}\xi \rangle = \langle \alpha, g^{-1}\xi g \rangle .
$$

(7.13)

For a compact group like $SO(3)$ the distinction between $\mathfrak{g}$ and $\mathfrak{g}^*$ is slight, and the bilinear form $\langle \ , \ \rangle$ is simply given by the trace of a product of two matrices. But we have reached a point of view from which Eq. (7.12) is really quite remarkable. On the one hand, the rotation group acts on the functions $J_i$ by means of canonical transformations of phase space. On the other hand, it acts on them through its adjoint action. And the equation says that these two actions are consistent with each other. This key observation motivates the definition of the momentum map, that we are now coming to. (A standard reference for the momentum map is the book by Guillemin and Steinberg [61]. Readers who want a gentle introduction may prefer to begin with the book by Springer [128].)

In general, suppose that we have a manifold with a symplectic form $\Omega$ defined on it (as in Section 3.4) Let a Lie group $G$ act on it in the symplectic way. Thus to each group element $g$ there is a map $x \rightarrow \Phi_g(x)$ from the manifold to itself, preserving the symplectic form. Let there exist a map $\mu$ from the manifold to the vector space $\mathfrak{g}^*$ dual to the Lie algebra $\mathfrak{g}$ of $G$. The group acts there too through the coadjoint action. Then $\mu$ is a momentum map provided it is equivariant,

$$
\mu(\Phi_g(x)) = \text{Ad}_g^*\mu(x) ,
$$

(7.14)

and provided that the vector field

$$
\xi(x) = \frac{d}{dt} \Phi_{e^{t\xi}}(x) ,
$$

(7.15)

is the Hamiltonian vector field for the function $\mu_\xi = \langle \mu(x), \xi \rangle$. In our example the second condition is given (in different notation) by Eq. (7.11), and the first by Eq. (7.12).
Complex projective space—the space of pure quantum states—is a symplectic manifold, and in fact we have come across an important example of a momentum map already, namely

$$\left(Z^0, Z^1, \ldots, Z^n\right) \rightarrow \frac{1}{Z^*Z}(|Z^0|^2, |Z^1|^2, \ldots, |Z^n|^2). \quad (7.16)$$

The map is from \(\mathbb{C}P^n\) to \(\mathbb{R}^n\), and the image is a probability simplex. If we normalize the state, and write \(Z^i = \sqrt{|Z^i|}e^{i\nu^i}\), it looks even simpler. We know from Section 4.7 that \(p_i\) and \(\nu_i\) are action-angle variables. The action variables \(p_i\) generate the action of an abelian Lie group which is the direct product of \(n\) copies of the circle group \(U(1)\). Both the conditions for a momentum map are fulfilled. From the present point of view the measure of the bipartite entanglement can be made.

For the amount of bipartite entanglement averaged over every possible bipartition of the system. Various choices of that is certainly present. We can ask for the average bipartite entanglement of the individual subsystems, or perhaps entanglement. One possible way to proceed is to restrict our attention to the much simpler, bipartite entanglement there are many different kinds of entanglement, we must learn to be pragmatic if we want to talk about ‘maximal entanglement’. That is, maximally entangled states serve as a kind of gold standard. In the multipartite case, where

Third convexity theorem. Let \(G\) be the complexification of a compact group, and let it act on \(\mathbb{C}P^n\). Let \(G \cdot x\) be an orbit through a point \(x\), and let \(\overline{G \cdot x}\) be its closure. Let the map \(\Psi\) be defined as above. Then

a) the set \(\Psi(G \cdot x)\) is a convex polytope,

b) there is an open dense set of points for which \(\Psi(G \cdot x) = \Psi(\mathbb{C}P^n)\),

c) the number of such polytopes is finite.

Concerning the proof we confine ourselves to the remark that invariant theory is present behind the scenes; the finiteness properties of the polytopes are related to the finitely generated ring of covariants. Anyway this proposition is the platform from which the results of this section have been derived, in the references we have cited [119, 142]. From it the problem of classifying multipartite entangled states looks at least more manageable than before.

VIII. AME STATES AND ERROR–CORRECTING CODES

In the bipartite case a maximally entangled state is singled out by the fact that any other state can be reached from it means of LOCC. Moreover, a natural way to quantify bipartite entanglement is to start with a large but fixed number of copies of a state, and ask how many maximally entangled states we can distill from them by means of LOCC. That is, maximally entangled states serve as a kind of gold standard. In the multipartite case, where there are many different kinds of entanglement, we must learn to be pragmatic if we want to talk about ‘maximal entanglement’. One possible way to proceed is to restrict our attention to the much simpler, bipartite entanglement that is certainly present. We can ask for the average bipartite entanglement of the individual subsystems, or perhaps for the amount of bipartite entanglement averaged over every possible bipartition of the system. Various choices of the measure of the bipartite entanglement can be made.
A very pragmatic way to proceed is to average the linear entropy (from Section 2.7) over all the reduced one-partite density matrices. Since we assume that the global state of the $K$ qubits is pure this is an entanglement measure. Under the name of the Mayer–Wallach measure it is defined by

$$ Q_1(|\psi\rangle) = 2\langle S_L \rangle = 2\left(1 - \frac{1}{K} \sum_{k=1}^{K} \text{Tr} \rho_k^2 \right). $$

(8.1)

It is normalized so that $0 \leq Q_1 \leq 1$, with $Q_1 = 0$ if and only if the state is separable and $Q_1 = 1$ if and only if each qubit taken individually is in a maximally mixed state. (It was written in this form by Brennen [23]. Mayer and Wallach wrote it differently [97].)

Now choose a subset $X$ of qubits, with $|X| = k \leq K - k$ members, and trace out the remaining $K - k$ qubits; the assumption that $2k \leq K$ will simplify some statements. There are $K!/(K-k)!$ such bipartitions altogether, and we can define the entanglement measures

$$ Q_k(|\psi\rangle) = \frac{2^k}{2^k - 1} \left(1 - \frac{k!(K-k)!}{K^k} \sum_{|X|=k} \text{Tr} \rho_X^2 \right). $$

(8.2)

Again $0 \leq Q_k \leq 1$, with $Q_k = 0$ if and only if the global state $|\psi\rangle$ is separable, and $Q_k = 1$ if and only if it happens that all the reduced density matrices are maximally mixed. (Scott introduced these measures with the cautionary remark that they ‘provide little intellectual gratification’ [124].) For the $W$ and GHZ states we find

$$ Q_k(|W_K\rangle) = \frac{2^{k+1}}{2^k - 1} \frac{(K-k)k}{K^2}, \quad Q_k(|GHZ_K\rangle) = \frac{2^{k-1}}{2^k - 1}. $$

(8.3)

For the $W_6$ state we note that $Q_1 < Q_3 < Q_2$, which may seem odd. Moreover one can find pairs of states such that $Q_k(|\psi\rangle) > Q_k(|\phi\rangle)$ and $Q_{k'}(|\psi\rangle) < Q_{k'}(|\phi\rangle)$, for some $k' \neq k$. So there is no obvious ordering of the states into more or less entangled. Rather the measures $Q_k$ capture different aspects of multipartite entanglement as $k$ is varied. Moreover, if one changes the linear entropy to, say, the von Neumann entropy in the definition of $Q_k$, one may change the ordering of the states also when $k$ is kept fixed. See Problem 17.7. Maximizing $Q_k$ over the set of all states is a difficult optimization problem, becoming computationally more expensive if we use the von Neumann entropy [18, 47]. In general, we do not obtain a convincing definition of ‘maximally entangled’ in this way.

States for which the upper bound $Q_k = 1$ is reached are called $k$–uniform. The GHZ state, like some other states we know (see Problem 17.3), is 1–uniform for any number of qubits. A $k$–uniform state (with $k \geq 2$) is always $(k-1)$–uniform, since the partial trace of a maximally mixed state is maximally mixed. If all reduced density matrices that result when tracing over at least half of the subsystems is maximally mixed, the state is said to be absolutely maximally entangled, abbreviated AME [67]. Every measure of bipartite entanglement will have to agree that AME states—if they exist—are maximally entangled.

Let us move on from qubits to the general case of $K$ subsystems with $N$ levels each. In the product basis a pure state is described by

$$ |\psi\rangle = \Gamma_{i_1i_2\ldots i_K} |i_1i_2\ldots i_K\rangle, $$

(8.4)

where the indices run from 1 to $N$. If a bipartition is made the tensor can be described by two collective indices, $\mu$ running from 1 to $N^k$ and $\nu$ running from 1 to $N^{K-k}$. Again we assume that $k \leq K - k$, and write the state as

$$ |\psi\rangle = \Gamma_{\mu\nu} |\mu\nu\rangle. $$

(8.5)

Tracing out $K - k$ subsystems we obtain the reduced state

$$ \rho_X = (\Gamma^\dagger)^\nu_\mu |\mu\nu\rangle \langle \mu\nu| \,. $$

(8.6)

This state is maximally entangled (Section 16.3) if and only if the rectangular matrix $\sqrt{N^k} \Gamma_{\mu\nu}$ is a right unitary matrix (also known as an isometry), that is if and only if

$$ N^K \Gamma^\dagger = 1_{N^k}. $$

(8.7)

For a $k$–uniform state this has to be so for every bipartition of the $K$ subsystems into $k + (K - k)$ subsystems. This is clearly putting a constraint on the tensor $\Gamma$ which becomes increasingly severe as $k$ grows. For an AME state, with
an even number of subsystems, such tensors are known as perfect \[110\], while the corresponding reshaped matrices are known as multiunitary \[55\].

AME states do exist for 2, 3, 5, and 6 qubits. For 3 qubits they are the GHZ states. For 5 qubits an AME(5,2) state is
\[
|\Phi_5^2\rangle = |00000\rangle + |11000\rangle + |01100\rangle + |00110\rangle + |00011\rangle + |10001\rangle - |11100\rangle - |01110\rangle - |00111\rangle - |00011\rangle - |10001\rangle - |11110\rangle - |10111\rangle - |10111\rangle - |11101\rangle .
\]

It was given in this form by Bennett et al. [13], and by Laflamme et al. [84] in a locally equivalent form with only eight terms in the superposition.

To see how this state arises, recall the multipartite Heisenberg group from Section 12.4, and in particular the notation used in Eq. (12.50). The cyclic properties of the state make it easy to see that it is left invariant by the operators
\[
G_1 = XXZIZ , \quad G_2 = ZXXZ1 , \quad G_3 = 1ZXXZ , \quad G_4 = Z1ZXX , \quad G_5 = ZZZZZ .
\]

Moreover these five group elements generate a maximal abelian subgroup of the Heisenberg group, having 32 elements. In the terminology of Section 12.6 this means that \(|\Phi_5^2\rangle\) is a stabilizer state.

For the 6 qubit AME state, see Problem 17.8. AME states do not exist for \(K = 4\) \[52, 69\], or for more than 6 \[72, 124\], qubits. AME states built from four qutrits do exist. An example is \[65\]
\[
|\Phi_4^3\rangle = |0000\rangle + |0112\rangle + |0221\rangle + |1011\rangle + |1120\rangle + |1202\rangle + |2022\rangle + |2101\rangle + |2210\rangle .
\]

Glancing at this state we recognize the finite affine plane of order 3, constructed in Eq. (12.58). The labels from the first two factors of the basis vectors are used to define the position in the array, and then the labels from the last two factors are labelling the remaining two sets of parallel lines. There are a number of combinatorial ideas one can use to construct highly entangled states \[55, 56\]. It is not hard to show (Problem 17.9) that this is a stabilizer state.

The same construction, using an affine plane of order \(N = 2\), gives the Svetlichny state discussed in Problem 17.5. For \(N > 2\) the combinatorics of the affine plane ensures that we obtain a projector onto an \(N^2\) dimensional subspace whenever we trace out two of the subsystems, which means that we obtain \(2^\text{uniform}\) states of \(N + 1\) qubits—provided an affine plane of order \(N\) exists, as it will if \(N\) is a prime number, or a power of a prime number. Using results from classical coding theory one can show that AME states exist for any number of subsystems provided that the dimension of the subsystems is high enough \[64\]. For qubits, we can still ask for the highest value of \(k\) such that \(k\)-uniform states exist, and some asymptotic bounds are known for this. It is also known that if we restrict our states to the symmetric subspace, the upper bound for qubits is \(k = 1\) \[3\]. So we cannot ask what AME states look like in the stellar representation, because it contains none (beyond GHZ3).

We now turn to this section’s other strand: error–correcting codes. We begin classically, with the problem of sending four classical bits through a noisy channel. (Perhaps the message is sent by a space probe far out in the solar system.) Assume that there is a certain probability \(p\) for any given bit to be flipped, and that these errors happen independently. In technical language, this is a binary symmetric channel. It may not be an accurate model of the noise. For instance, real noise has a tendency to come in bursts. But we adopt this model, and we want to be able to detect and correct the resulting errors. A simple–minded solution is to repeat the message thrice. Thus, instead of sending the message 0101 we send the message 010101010101. We can then correct any single error by means of a majority vote. But since the length of the message has increased, the probability that two errors occur has increased too. The trade-off is studied in Problem 17.10.

It pays to adopt a geometric point of view. We regard each bitstring of length \(n\) as a vector in the discrete vector space \(Z_2^n\) over the finite field \(Z_2\) (the integers modulo 2; see Section 12.2). Our repetition code is a subspace of dimension four in \(Z_2^{12}\). This information is summarized by the generator matrix of the code, in this case
\[
G_{(12,4)} = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{bmatrix} .
\]
The four row vectors form a basis for the code, i.e. for a linear subspace containing altogether $2^4$ vectors. Their first four entries carry the actual information we want to send. The reason why this code enables us to correct single errors can now be expressed geometrically. Define the *weight* of a vector as the number of its non-zero components. One can convince oneself that the minimal weight of any non-zero vector in this code is 3. Therefore the minimal Hamming distance between any pair of vectors in the code is 3, because by definition the Hamming distance between two vectors $u$ and $v$ is the weight of the vector $u - v$, which is necessarily a vector in the code since the latter is a linear subspace. Thus, what the embedding of $\mathbb{Z}_2^7$ into $\mathbb{Z}_2^{12}$ has achieved is to ensure that the $2^4$ code words are well separated in terms of Hamming distance. Suppose a single error occurs during transmission, somewhere in this string. This means that we receive a vector at Hamming distance 1 from the vector $u$. But the Hamming distance between two code words is never smaller than 3. If we surround each code vector with a ‘ball’ of vectors at Hamming distance ≤ 1 from the centre, we find that the vector received lies in one and only one such ball, and therefore we know with certainty which code word was being sent. This is why we can correct the error.

In general, a linear *classical* $[n, k, d]$ code is a linear subspace of dimension $k$ in a discrete vector space of dimension $n$, with the minimal Hamming distance between the vectors in the subspace equal to $d$. The repetition code is the code $[12, 4, 3]$. Can we improve on it? Indeed, improvements are easily found by consulting the literature. Classical error-correcting codes have been much studied since the time of Shannon’s breakthrough in communication theory [123]. A standard reference is the book by MacWilliams and Sloane [92]. For a briefer account, see Pless [115].

The generator matrix of the $[7, 4, 3]$ *Hamming code* is

$$ G_{(7,4)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \quad (8.12) $$

With this code, the bitstring 0101 is encoded as the message 0101010. One can convince oneself that the minimum Hamming distance between the vectors in the code is again 3 (and indeed that these balls fill all of $\mathbb{Z}_2^7$), so that all single errors can be corrected by this code as well. Moreover, an elegant and efficient decoding procedure can be devised, so that we do not actually have to look through all the balls. To see how this works, write $G = [I | A]$, and introduce the *parity check matrix* $H = [-A^T | I]$. By construction

$$ HG^T = 0 \quad (8.13) $$

In fact $Hu = 0$ for every vector in the code subspace. Now suppose that an error occurs during transmission, so that we receive the vector $u + e$. Then

$$ H(u + e) = He \quad (8.14) $$

It is remarkable, but easy to check, that if the error vector $e$ is assumed to have weight 1 then it can be uniquely reconstructed from the vector $He$. The conclusion is that the Hamming code allows us to correct all errors of weight 1 without, in fact, ever inspecting the message itself. It is enough to inspect the *error syndrome* $He$.

The Hamming code is clearly more efficient than the simple repetition code, which required us to send three times as many bits as the number we want to send. If more than one error occur during the transmission we can no longer correct it using the $[7, 4, 3]$ code, but there is a $[23, 12, 7]$ code, known as the *Golay code*, that can deal with three errors. And so on. In fact, for a binary symmetric channel Shannon proved, with probabilistic methods, that the probability of errors in the transmission can be made smaller than any preassigned $\epsilon$ if we choose the dimension of the code subspace and the dimension of the space in which it is embedded suitably (with an eye on the properties of the channel) [123].

Quantum error correction, of a string of qubits rather than bits, is necessarily a subtle affair, since it has to take place without gaining any information about the quantum state that is to be corrected. The first quantum error-correcting codes took the community by surprise when they were presented by Shor [126] and Steane [129] in the mid 1990s. They were linked to the Heisenberg group soon after that [28, 53], and the theory then grew quickly.

Suppose that the initial state is $|\Psi\rangle$, belonging to some multipartite Hilbert space $\mathcal{H}_N^{\otimes K}$. In the course of time—for concreteness assume that the transmission is through time, in the memory of a quantum computer—the state will suffer decoherence due to the unavoidable interaction with the environment. There is nothing digital about this process. But we do assume that the state is subject to a CP map of the form (10.53). We then introduce a unitary operator basis (Section 12.1), and expand all the Kraus operators in this basis. For our purposes it is best to express this in the environmental representation. Thus what has happened is that

$$ |\Psi\rangle|0\rangle_{\text{env}} \rightarrow \sum_I E_I |\Psi\rangle|\psi_I\rangle_{\text{env}} \quad (8.15) $$
The elements of the unitary operator basis are called $E_I$ here, since this basis is soon going to deserve its alternative name ‘error basis’. The environmental states $|\psi_I\rangle_{env}$ are not assumed to be orthogonal or normalized, and at this stage the introduction of the error basis is a purely formal device. If it were true that

$$\langle \Psi | E_I^+ E_J | \Psi \rangle = \text{Tr} | \Psi \rangle \langle \Psi | E_I^+ E_J = \delta_{IJ},$$

then we would be in good shape. The elements of the error basis would then give rise to mutually exclusive alternatives, and a measurement could be devised so that the state collapses according to

$$\sum_I E_I | \psi_I \rangle_{env} \rightarrow E_I | \psi_I \rangle_{env}.$$

Once we knew the outcome, we could apply the operator $E_I^+$ to the state, and recover the (still unknown) state $| \Psi \rangle$.

If we replace the pure state $| \Psi \rangle \langle \Psi |$ with the maximally mixed state, the second equality in Eq. (8.16) would hold, but the maximally mixed state is not worth correcting. However, this suggests that we should restrict both the noise (the set of allowed error operators $E_I$) and the state $| \Psi \rangle$ in such a way that the state behaves as the maximally mixed state as far as the allowed noise is concerned. Beginning with the noise, we assume that it can be expanded in terms of error operators of the form

$$E_I = E_1 \otimes E_2 \otimes \cdots \otimes E_K,$$

with at most $w$ of the operators on the right hand side not equal to the identity operator. This is an error operator of weight $w$. Physically we are assuming that the individual qubits are subject to independent noise, which is a reasonable assumption, and also that not too many of the qubits are affected at the same time. The operator $E_I^+ E_J$ will therefore have at most $2w$ factors not equal to the identity. Next, we assume that the state is a $2w$-uniform state. Coming back to Eq. (8.16), we can perform the trace over the $K-2w$ factors where the error operators contribute with just the identity, obtaining (for quNits)

$$\langle \Psi | E_I^+ E_J | \Psi \rangle = \text{Tr} | \Psi \rangle \langle \Psi | E_I^+ E_J = \frac{1}{N^{2w}} \text{Tr} E_I^+ E_J,$$

where $E_I^+$ is the non-trivial part of $E_I$. It follows immediately from the definition of the error basis that we obtain the desired conclusion (8.16). Hence the $2w$-uniform state can be safely sent over this noisy channel, and $w$ errors can be corrected afterwards.

This is only a beginning of a long story. In general, a quantum error–correcting code of distance $d$, denoted $[[K,M,d]]_N$, is an $N^M$ dimensional subspace of an $N^K$ dimensional Hilbert space, such that errors affecting only $(d-1)/2$ quNits can be corrected along the lines we have described. The nicest error basis of all, the Heisenberg group (Chapter 12), comes into its own when such codes are designed. We have seen that a $k$–uniform state of $K$ quNits is a $[[K,1,k+1]]_N$ quantum error–correcting code [124], but for more information we refer elsewhere [40, 116, 130]. For us it is enough that we have pointed a moral: beautiful states have a tendency to be useful too.

IX. ENTANGLEMENT IN QUANTUM SPIN SYSTEMS

So far we have had, at the back of our minds, the picture of a bold experimentalist able to explore all of Hilbert space. In many–body physics that picture is to be abandoned. For concreteness, imagine a cubic lattice with ‘atoms’ described as quNits at each lattice site. The Hamiltonian is such that only nearest neighbours interact. There are $K$ lattice sites altogether. We will always assume that $K$ is finite, but we may let $K$ approach Avogadro’s number, say $K = 10^{23}$. Then the total Hilbert space under consideration has $N^{10^{23}}$ dimensions. This is enormous. No experimentalist is going to explore this in full detail. Indeed, Nature itself cannot have done so during the $10^{17}$ seconds that have passed since the creation of the universe. Still multipartite entanglement in such systems will be important. Many good reviews exist [4, 134], and a forthcoming book by Zeng, Chen, Zhou, and Wen [149] describes how quantum information meets quantum matter. In today’s laboratories it is possible to design quantum many–body systems with desirable properties [10]. Indeed the tensor product structure of Hilbert space gives rise to an intricate geography of quantum state space, and we may hope to find the ground states of physically interesting systems in very special places.

The kind of systems we will focus on are called quantum spin systems. Some notation: Subsets of lattice sites will be denoted $X, Y, \ldots$, and the number of sites they contain will be denoted by $|X|, |Y|, \ldots$. Such a number can be regarded as the volume of the subset. The complementary subsets are denoted by $\bar{X}, \bar{Y}, \ldots$, meaning that the
entanglement between any subregion $X$. The number of edges in the lattice passing through the boundary is denoted $|\partial X|$, and is regarded as the area of the boundary. Each lattice site is associated with a Hilbert space of finite dimension $N$, and the total Hilbert space is the tensor product of all them. The Hilbert space associated to a region $X$ is the tensor product of all the Hilbert spaces associated to sites therein.

With our change of perspective in mind, let us first divide the lattice into two parts (somehow). Thus the Hilbert space is

$$H = H_X \otimes H_{\bar{X}} = (\otimes_{x \in X} H_x) \otimes (\otimes_{x \in \bar{X}} H_x) .$$

(9.1)

Now consider the amount of bipartite entanglement that arises if the global state is a pure state $|\psi\rangle$ chosen at random according to the Fubini-Study measure. By tracing out the complementary set $\bar{X}$ we obtain the reduced state $\rho_X = \text{Tr}_{\bar{X}} |\psi\rangle \langle \psi| \bar{X}$ associated to the subsystem formed by the atoms in subset $X$. Assume that $|X|$ is smaller than $|\bar{X}|$.

We can now apply the Page formula (15.73), keeping in mind that the size of the system is $N^{|X|}$ and the size of the environment is $N^{|\bar{X}|}$. The result is

$$\langle E(|\phi\rangle_X = \langle S(\rho_X) \rangle \approx |X| \ln N - \frac{1}{2} N^{|X|-|\bar{X}|} .$$

(9.2)

The second term describes a negative correction which can be neglected if $|X| \ll |\bar{X}|$. The leading term on the other hand grows proportionally to the number $|X|$ of subsystems in the region $X$. In this way we arrive at the following statement:

**Volume law:** For a generic multipartite quantum state the entanglement between any subregion $X$ and a larger environment $\bar{X}$ scales as the volume of the region, as measured by the number $|X|$ of its subsystems.

This is also how the thermodynamical entropy behaves for physical systems with short range interactions—except that the thermodynamical entropy may well vanish at zero temperature, while the entanglement entropy does not, so they are distinct. The hope is that the relevant low energy states in a many–body system are far from generic—and that they are more amenable to computer calculations than the generic states are. Indeed the number of parameters needed to describe a generic state grows exponentially with the number of atoms, and this poses a quite intractable problem in computer simulations.

At this point a sideways glance on a seemingly very different part of physics is useful. In the classical theory of general relativity black holes are assigned an entropy proportional to their area. It is known as the Bekenstein-Hawking entropy, and as it stands it has no microscopic origin. In an attempt to provide one it was noticed that in certain situations the entanglement entropy also grows with area. (This was first seen in 1983 by Sorkin [127]; the concrete calculation performed by him and his coworkers was in the context of a non-interacting quantum field theory [17].)

For a review of black hole thermodynamics see Wald’s book [141]. This suggests that we should replace the volume law with:

**Area law:** For physically important states of many–body quantum systems described by local interactions the entanglement between any subregion $X$ and a larger environment $\bar{X}$ scales as the area of the region, as measured by the number of interactions across its boundary $\partial X$.

We will give a precise statement later, but first we want to understand what kinds of states that can give rise to it.

The spatial dimension of the lattice matters here. We begin with a one dimensional lattice, an open chain with $K$ sites. To find a suitable representation of the states relevant to such a system we are going to apply the Schmidt decomposition (see Section 9.2) stepwise to a tensor carrying $K$ indices. First we recall Eq. (9.14), which gives the singular value decomposition of an $N_1 \times N_2$ matrix $C$ as $C = UDV$, where $D$ is a diagonal matrix with Schmidt rank $r \leq \min(N_1, N_2)$. The (in general) rectangular matrices $U$ and $V$ obey $U^\dagger U = VV^\dagger = 1_r$. We also define the matrix

$$\Lambda \equiv D^2 .$$

(9.3)

If we start from a normalized state then $\Lambda$ is simply the reduced density matrix in diagonal form.

Now any $K$–index tensor with $N^K$ components can be viewed as an $N \times N^{K-1}$ matrix, and the singular value decomposition can be applied to it:

$$\Gamma^{i_1 i_2 \ldots i_K} = \Gamma^{i_1 |i_2 \ldots i_K} = \sum_{a=1}^r A^{i_1}_a (D_1 V_1)_a |i_2 i_3 \ldots i_K .$$

(9.4)

The matrix of left eigenvectors $U_1$ was renamed $A$, because we focus on its column vectors $A^{i_1}$. It will be observed that

$$\sum_{i_1} (A^{i_1})^\dagger A^{i_1} = U_1^\dagger U_1 = 1_r ,$$

(9.5)
\[ \sum_{i_1} A^{i_1} \Lambda(A^{i_1}) = \sum_{i_1} A^{i_1} D V V^\dagger D(A^{i_1}) = \Gamma^{i_1 i_2 \ldots i_K} \Gamma_{i_1 i_2 \ldots i_K} = 1 \cdot \]  

(9.6)

To get the final equality we had to assume that the state is normalized.

In the next step the \( r_1 \times N^{K-1} \) matrix on the right hand side of Eq. (9.4) is reshaped into an \( r_1 N \times N^{K-2} \) matrix, and the singular value decomposition again does its job:

\[
\Gamma^{i_1 i_2 \ldots i_K} = \sum_{a_1=1}^{r_1} \sum_{a_2=1}^{r_2} \ldots \sum_{a_{K-1}=1}^{r_{K-1}} A^{i_1}_{a_1 a_1} A^{i_2}_{a_1 a_2} \ldots A^{i_{K-1}}_{a_{K-2} a_{K-1}} A^{i_K}_{a_{K-1}} \cdot \ 
\]

(9.7)

The matrix \( U_{a_i}^{a_2} \) was renamed \( A^{i_2}_{a_1 a_2} \), and is regarded as a collection of \( N \) matrices of size \( r_1 \times r_2 \). This time we have

\[
\sum_{i_2} (A^{i_2})^\dagger A^{i_2} = U_2^\dagger U_2 = 1_{r_2} 
\]

(9.8)

\[
\sum_{i_2} A^{i_2} \Lambda_2(A^{i_2}) = \sum_{i_2} A^{i_2} D_2 V_2 V_2^\dagger D_2(A^{i_2})^\dagger = \Lambda_1 \cdot 
\]

(9.9)

Clearly this procedure can be iterated. We have arrived at

**Vidal’s theorem.** Every \( K \)-partite state of \( K \) quNits can be expressed in a separable basis as

\[
\Gamma^{i_1 \ldots i_K} = \sum_{a_1=1}^{r_1} \sum_{a_2=1}^{r_2} \ldots \sum_{a_{K-1}=1}^{r_{K-1}} A^{i_1}_{a_1 a_1} A^{i_2}_{a_1 a_2} \ldots A^{i_{K-1}}_{a_{K-2} a_{K-1}} A^{i_K}_{a_{K-1}} , \n\]

(9.10)

where, if the state is normalized,

\[
\sum_{i_{K-1}}^{N-1} (A^{i_{K-1}})^\dagger A^{i_{K-1}} = 1_{r_{K-1}} \cdot 
\]

(9.11)

\[
\sum_{i_{K-1}}^{N-1} A^{i_{K-1}} \Lambda_{K-1}(A^{i_{K-1}})^\dagger = \Lambda_{K-1} \cdot 
\]

(9.12)

\( \Lambda_k \) is a reduced density matrix in diagonal form, and the Schmidt ranks are bounded by \( r_k \leq N^{[K/2]} \).

The largest Schmidt ranks occur in the middle of the chain; \( [N/2] \) denotes the largest integer not larger than \( N/2 \). Conditions (9.11)(9.12) fix the representation uniquely up to orderings of the Schmidt vector and possible degeneracies there.

We have chosen to begin this story with a theorem by Vidal (2003) [139], which gives a canonical form for Matrix Product States. But the story itself is much older. It really began with Affleck, Kennedy, Lieb, and Tasaki (1987) [3], who studied the ground states of isotropic quantum antiferromagnets. This was followed up by Fannes, Nachtergaele and Werner (1992) [48]. Important precursors include works by Baxter (1968) [11] and Accardi (1981) [1].

Of course, once we have a state expressed in the form (9.10) we can relax conditions (9.11)(9.12) without changing the state—they simply offer a canonical form for the representation. An important variation of the theme must be mentioned. Eq. (9.10) is said to use *open boundary conditions*, but one can also use *periodic boundary conditions*, which here means that one uses a collection of matrices such that

\[
\Gamma^{i_1 \ldots i_K} = \text{Tr} A^{i_1} A^{i_2} \ldots A^{i_{K-1}} A^{i_K} \cdot 
\]

(9.13)

This time there are no vectors at the ends, and indeed there are no ends—the chain is closed. Physically this formulation is often preferred, but there is no longer an obvious way to impose a canonical form on the matrices.

The only problem is that we risk getting lost in a clutter of indices. This is where the graphical notation for tensors comes into its own, see Figure 10.

Having organized the \( N^K \) components of the state into products of a collection of \( KN \) matrices we must ask: was this a useful thing to do? Indeed yes, for two reasons. The first reason is that we are not interested in generic states, in fact our aim is to find special states obeying the Area Law. For this reason we now focus on states that can be
written on the form (9.10), but with all the Schmidt ranks obeying $r_k \leq D$, where $D$ is some modest integer called the bond dimension. We call them matrix product states, or MPS for short. Indispensable references, in addition to those mentioned above include papers by Perez-Garcia et al. [114] and by Verstraete, Murg, and Cirac [138]. Splitting the chain in two and tracing out the contribution from one of the regions will result in a mixed state whose entanglement entropy behaves as

$$S(\rho_X) \sim \ln D .$$  

(9.14)

We conclude that the strict Area Law holds for a one dimensional chain if and only if $D$ is independent of the number of subsystems.

There are many interesting states with low bond dimensions. Separable states have bond dimension one. The GHZ$_K$ state has bond dimension two, and moreover the matrices $A^k$ for all sites $k$ between 2 and $K - 1$ are the same, namely

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A^1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} .$$  

(9.15)

With a suitable choice of vectors at the ends we recover the GHZ state. The $W_K$ state also has bond dimension two, but the bond dimension increases for the other Dicke states. Importantly, ground states of interesting Hamiltonians (leading to the Area Law) will have modest bond dimension, in particular the AKLT ground state [3] (which started off the subject [48]) has bond dimension two. See Problem 17.12 for some examples.

The special properties of the singular value decomposition ensure that we have a reasonable approximation scheme on our hands. The Eckart-Young theorem says that if we want to approximate a matrix $M$ with another matrix $\hat{M}$ of lower rank $D$, in such a way that the $L_2$-norm $\text{Tr}(M - \hat{M})^\dagger(M - \hat{M})$ is the smallest possible, then we can do this by performing the singular values decomposition of $M$ and setting all but the $D$ largest Schmidt coefficients to zero. (The Eckart-Young theorem (1936) [143] was later extended to cover all the $L_p$-norms.) This means that we can approximate any state by setting all but the $D$ largest Schmidt coefficients to zero (and renormalizing the remaining Schmidt coefficients so that they again sum to one), at each step of the exact expression (9.10). The number of parameters in the approximation scales as $KND^2$, that is to say linearly in the number of subsystems. For a given $D$ the set of all MPS is a subset of measure zero in the set of all states, but we can make the approximation of any state better by increasing $D$. Truncating at some reasonable integer $D$ in a computer calculation is not so very different from representing real numbers by rational numbers with reasonable denominators. It works fine in everyday calculations. Conversely, if an exponential growth in the bond dimensions is encountered then the calculation cannot be done efficiently on a classical computer [139].

We still have to present at least some evidence that ground states of interesting physical systems can be well approximated by MPS. (Much more can be said on the topic why and how MPS represent ground states faithfully.
We take a short-cut here. The question is if they obey the Area Law. They do, as proved by Hastings [64]. The conditions imposed on the Hamiltonian are that

$$H = \sum_{k=1}^{K} H_{k,k+1} \quad \Rightarrow \quad \|H_{k,k+1}\|_{\infty} \leq J \quad \Delta E > 0 \quad (9.16)$$

for some $J$ and for some energy gap $\Delta E$ between the ground state and the first excited state. The first condition insists that only nearest neighbours are coupled, the second bounds the largest eigenvalue of each individual term. Interactions of finite range can be dealt with by grouping sites together into single sites, so the restriction to interactions between nearest neighbours is not all that severe.

Hastings’ proof is beyond our scope, but one key ingredient must be mentioned because we need it to state the theorem. Consider two disjoint regions $X$ and $Y$ of the lattice. Take operators $A$ and $B$ supported in $X$ and $Y$, respectively. Hence $[A,B] = 0$. Time evolve $A$ with a Hamiltonian obeying conditions (9.16).

$$A(t) = e^{iHt}Ae^{-iHt} \quad (9.17)$$

Then one finds [87]

**Lieb–Robinson theorem.** Under the conditions stated there exist constants $c$ and $a$ and a velocity $v$ such that

$$\|\{A(t),B\}\|_{\infty} \leq c\|A\|_{\infty}\|B\|_{\infty}e^{-a(d(X,Y) - v|t|)} \quad (9.18)$$

The constant $a$ is adjustable and can be chosen to be large provided that $d(X,Y)/v|t|$ is sufficiently large. Again the proof is beyond us, but note the physical meaning: this is a rigorous statement saying that an effective ‘light cone’ appears in the system. Up to an exponentially decaying tail, influences cannot propagate outwards from the region $X$ faster than the Lieb–Robinson velocity $v$, and there will be definite bounds on how fast entanglement can spread through the system under local interactions [22].

Now we can state [64]

**Hastings’ Area Law theorem.** Let $X$ be the region to the left (or right) of any site on a one-dimensional chain. For the ground state of a Hamiltonian obeying the conditions stated there holds the Area Law

$$S(\rho_X) \leq c_0 \xi \ln \xi \ln N\xi \ln N \quad (9.19)$$

where $c_0$ is a numerical constant of order unity and

$$\xi = \max \left( \frac{12v}{\Delta E}, \frac{6a}{\Delta E} \right) \quad (9.20)$$

Here $\Delta E$ is the energy gap, $v$ is the Lieb–Robinson velocity, and the constant $a$ is the one that appears in the Lieb–Robinson theorem.

The point is that the upper bound on the entanglement entropy depends on the dimensionality $N$ of the subsystems, and on the parameters of the model, but not on the number of subsystems within the region $X$. All the vagueness in our statement of the Area Law has disappeared, at the expense of some precise limitations on the Hamiltonians that we admit. In the case of critical systems, for which $\Delta E \to 0$, one observes [27, 44, 85] a logarithmic dependence of entropy on volume, $S(\rho_X) \approx c_1 \ln |X| + c_2$. This is a much milder growth than one expects from a generic state obeying the Volume Law. For one-dimensional chains area laws can also be derived assuming exponential decay of correlations in the system, without any assumptions about the energy gap [20].

Once it is admitted that the Area Law holds for the ground states of physically interesting Hamiltonians, the task of finding these states is greatly simplified. A standard approach is to solve the Rayleigh-Ritz problem: the ground state of the Hamiltonian $H$ is the vector that minimizes the Rayleigh quotient

$$\frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} \quad (9.21)$$

But the set of all states grows exponentially with the number $K$ of subsystems, so the best that can be done in practice is to find the minimum over a suitable set of trial states. The calculations become feasible once we assume that $|\psi\rangle$ belongs to the set of all MPS with a bond dimension growing at most polynomially in $K$. This idea is at the root of the density matrix renormalization group (DMRG) method, which has proven immensely useful in the study of strongly correlated one-dimensional systems. (The DMRG is due to White (1992) [142]. The link to MPS was forged by Östlund and Rommer (1995) [108]. For a review – with useful calculational details of MPS – see Schollwöck [121].)
Note that if the computer is to be able to evaluate the Rayleigh quotient (9.21), it has to be told how. The first aim is to evaluate the single number
\[ \langle \psi | \psi \rangle = \sum_{i_1, i_2, \ldots, i_K} \text{Tr} A_{i_1}^* A_{i_2}^* \cdots A_{i_K}^* . \] (9.22)
It would clearly be a mistake to perform the traces first and the summation over the explicit indices afterwards—doing so means that the computer has to store \(2^{NK}\) numbers at an intermediate stage of the calculation. In this case it is not difficult to propose a better strategy (and we do so in Figure 11) but for spin systems in spatial dimensions larger than one there are difficult issues of computational complexity to be addressed (123). (To learn how to actually perform calculations the review by Orús [105] and the paper by Huckle et al. [73] may be helpful.)

![Diagram](image)

FIG. 11: Normalizing an MPS state: Eventually all the sums have to be done. Doing the contractions in the order indicated ensures that the number of components that have to be stored during intermediate stages of the calculation stays reasonable (as indicated by the arrows).

So what about higher spatial dimension? Then the situation is not quite as clear-cut, but one still expects an area law to hold in suitable circumstances, even though the boundary \(\partial X\) of a region \(X\) in the lattice can have a complicated structure. In particular, with some assumptions about the decay of correlations and on the density of states, Masanes proved (95) that the entanglement entropy for a reduction of the ground state scales as
\[ S(\rho_X) \leq C|\partial X| (\ln |X|)^N + O(\ln |X|^{N-1}) , \] (9.23)

where the constant \(C\) depends on the parameters of the model, but not on the volume \(|X|\). The ratio between the right hand side and the volume tends to zero as the size of the region grows, so although there is a logarithmic correction this again deserves to be called an Area Law.

So it makes sense to look for a way to generalize MPS to higher dimensions. To this end we begin by looking at the one-dimensional construction in a different way. We begin by doubling each site in the chain, so that the total Hilbert space becomes \(H \otimes 2^K\). The dimension of each factor is set to \(D\), which may be larger than the original dimension \(N\) of the physical subsystems. Then we consider a quite special state there, namely a product of maximally entangled bipartite states
\[ |\phi_{k,k+1}^+\rangle = \sum_{c=0}^{D-1} |c\rangle_k |c\rangle_{k+1} \] (9.24)

(We worry about normalization only at the end of the construction.) The factor Hilbert spaces that occur here are the rightmost factor from site \(k\) and the leftmost factor for site \(k+1\). In effect entanglement is being used to link the sites together. The total state of the ‘virtual’ (doubled) chain is taken to be the entangled pair state
\[ |\Psi\rangle = |\phi_{12}^+\rangle |\phi_{23}^+\rangle \cdots |\phi_{K1}^+\rangle . \] (9.25)

Next, once for every site in the original chain, we introduce a linear map from the doubled Hilbert space \(\mathbb{C}^D \otimes \mathbb{C}^D\) back to the \(N\)-dimensional Hilbert space \(\mathbb{C}^N\) we started out with:
\[ A_k = A_{ik}^{k+1} |i_k\rangle \langle a_k b_k| , \] (9.26)
FIG. 12: The PEPS construction. We start from a state consisting of entangled pairs in an auxiliary Hilbert space (forming a ring in this example), apply a linear map, and obtain a Matrix Product State (with periodic boundary conditions).

where summation over repeated indices is understood. It is now a straightforward exercise to show that

\[ A_1 A_2 \ldots A_K |\Psi\rangle = A_{i_1}^{a_1} A_{i_2}^{a_2} \ldots A_{i_K}^{a_K} |a_1 a_2 \ldots a_K i_1 i_2 \ldots i_K\rangle. \]  

(9.27)

We have recovered a Matrix Product State with periodic boundary conditions, as in Eq. (9.13). When arrived at in this way it is called a projected entangled pair state, abbreviated PEPS \[135\]. See Figure 12.

The PEPS construction is easily generalized to any spatial dimension of the lattice. As shown in Figure 13 to describe a two-dimensional lattice we have to expand the Hilbert space \( \mathbb{C}^N \) into a four–partite Hilbert space \( (\mathbb{C}^N)^{\otimes 4} \), and then we introduce a perhaps site-dependent linear map \( (\mathbb{C}^N)^{\otimes 4} \rightarrow \mathbb{C}^N \). Clearly any lattice, in any spatial dimension, can be handled in a similar way. But going beyond one dimension does increase every calculational difficulty, and we break off the story here. More can be found in the references we have cited.

FIG. 13: The PEPS construction works in all spatial dimensions. Applying the map we find that the tensors in the interior of the lattice have 4 + 1 arms. In this way we obtain a tensor network, rather than just a matrix product state.

We have now seen some simple examples of tensor networks. The basic idea is to view a tensor of valence \( m \) as an object with \( m \) free legs, or with both arms and legs if the distinction between upper and lower indices is important, and perhaps with several different kinds of arms and legs. Then we choose an arbitrary undirected graph—this may be a one dimensional chain, a cubic lattice, or something much more general—and assign a tensor to each vertex, in such a way that each edge in the graph corresponds to a pair of contracted indices. Tensor networks have a long history and an active present. From the beginning there was a dream to see the geometry of space emerge from the geometry of quantum states \[111\]. Perhaps the application of tensor networks to quantum spin systems is beginning to substantiate the dream?

X. CONCLUDING REMARKS

The aim of these notes is literally to present an concise introduction to the subject of multipartite entanglement. We believe such a knowledge will contribute to a better understanding of quantum mechanics. We hope also that it will provide a solid foundation for various modern applications of quantum theory including quantum cryptography, quantum error correction, and quantum computing.

There is much more to say. We have said nothing about graph states or toric codes, which is where the multipartite Heisenberg group comes into play. We have said nothing about the vast field of mixed multipartite states, nothing
about multipartite Bell inequalities, and nothing about fermionic systems. The list of omissions can be made longer. But we have to stop somewhere.

* * *

Entanglement plays a crucial role in quantum information processing: it can be considered as our enemy or as our friend [117]. On the negative side, the inevitable interaction systems with their environment induces entanglement between the controlled subsystems and the rest of the world, which influences the state of the qubits and induces errors. On the positive side, entanglement allows us to encode a single qubit in larger quantum systems, so that the entire quantum information will not be destroyed if the environment interacts with a small number of qubits. These issues become specially significant if one considers systems consisting of several parties.

It is tempting to compare quantum entanglement with the snow found high in the mountains during a late spring excursion. A mountaineer equipped with touring skis or crampons and iceaxe typically looks for couloirs and slopes covered by snow. On the other hand, his colleague in light climbing shoes will try to avoid all snowy fields to find safe passages across the rocks. The analogy holds as neither quantum entanglement nor spring snow lasts forever, even though the decay timescales do differ.

We wrap up these notes with the remark that multipartite entanglement offers a lot of space for effects not present in the case of systems consisting of two subsystems only. Entanglement in many body systems is not well understood, so we are pleased to encourage the reader to contribute to this challenging field.

We are indebted to Radoslaw Adamczak, Ole Andersson, Marcus Appleby, Runyao Duan, Shmuel Friedland, Dardo Goyeneche, Marcus Grassl, David Gross, Michal and Pawel Horodeccy, Ted Jacobson, Marek Kuś, Ion Nechita, Zbigniew Puchała, Wojciech Roga, Adam Sawicki, Andreas Winter, Iwona Wintrowicz, and Huangjun Zhu, for reading some fragments of the text and providing us with valuable remarks. We thank Kate Blanchfield, Piotr Gawron, Lia Pugliese, Konrad Szymański, and Maria Życzkowska, for preparing for us two dimensional figures and three dimensional printouts, models and photos for the book.

Financial support by Narodowe Centrum Nauki under the grant number DEC-2015/18/A/ST2/00274 is gratefully acknowledged.

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