Evaporation of Dynamical Horizon with the Hawking Temperature in the K-essence Emergent Vaidya Spacetime

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In the K-essence Vaidya Schwarzschild spacetime, the dynamical horizon equation to measure the mass-loss due to Hawking radiation and the tunneling formalism (Hamilton-Jacobi method) to calculate the hawking temperature is applied. Assuming the Dirac-Born-Infeld kind of non-standard action for the K-essence here, the background physical spacetime is a static spherically symmetric black hole, and the K-essence scalar field to be a function only of either forward or backward time is constrained. The K-essence emergent gravity and the generalizations of Vaidya spacetime have been linked by Manna et al. In this paper, Sawaya’s modified description of the dynamical horizon to show that the obtained findings deviate from the standard Vaidya spacetime geometry are used.

1. Introduction

The type Ia Supernova (SNe Ia), baryon acoustic oscillations (BAO), cosmic microwave background (WMAP7), and Planck findings[1–4] all clearly suggest that the late time universe is speeding and dominated by dark energy.[5,6] which indirectly contradicted traditional gravity theories. Several scientists have already started exploring various aspects of gravitational theories and come up with solutions that show cosmic acceleration. There are several theoretical ideas that have been thoroughly investigated.

Many of the existing models of dark energy struggle because they rely on a very precise tuning of the initial energy density of the highest order. To avoid the necessity for such precise modifications, a non-canonical theory, a new family of scalar field models known as K-essence theory, has been developed,[7–14] in which the negative pressure originates from the scalar field’s non-linear kinetic energy term.

Although the magnitude of the energy decreases by many orders and remains constant in the dust-dominated period, the K-essence theory is unable to remove the dustlike equation of state for a couple of dynamical reasons. But after a long time (about now), the field prevails over the matter density and leads the cosmos towards the cosmic acceleration. According to the K-essence theory, kinetic energy dominates over potential energy of the K-essence scalar field. Lorentz invariance is broken spontaneously by non-trivial dynamical solutions of the K-essence equation of motion with non-canonical kinetic terms. Furthermore, it modifies the metric for the perturbations close to these solutions. In the curved spacetime referred to as emergent or analogue, the perturbation propagates along with the metric. Taking the Dirac-Born-Infeld (DBI) model[15–17] into account, Manna et al.[18–21] have generated the simplest version of the emergent gravity metric $G_\mu\nu$. It’s important to note that this form is not conformally equal to the standard gravitational metric $g_{\mu\nu}$. A form of the Lagrangian for the K-essence model[7–14] is $L = -V(\phi)F(X)$ where $X = \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$.

There exists another form of Lagrangian[22] such as $L = [1 + f(\gamma)]X + (1 + \gamma)\nabla_\phi V_{,\phi}$, where $V_{,\phi} = V_{,\phi}\exp(-\lambda\phi)$, $V_0$ and $\lambda$ are constants, $\gamma = X/V_{,\phi}$, and $f(\gamma)$ and $g(\gamma)$ are arbitrary functions. The authors of this paper use the K-essence model to the study of primordial dark energy. In general, the Lagrangian can depend on arbitrary functions of $\phi$ and $X$. This theory is unique since it only accounts for the radiation epoch when tracing the background energy density. The K-essence theory is more applicable now than ever before since the matter density in the expanding cosmos is falling faster than the energy density. The dark energy component of the K-essence models, in which the speed of sound does not exceed the speed of light, may be able to account for the variations in the cosmic microwave background (CMB) at large angular scales.[23–25] On the other hand, the K-essence theory may be employed just from a gravitational or geometrical aspect[26–28] other than the dark energy model, because the existence of dark energy is still debatable[29] according to current observation.[30]

In addition, we analyze the non-canonical Lagrangian from a different perspective. In general, the canonical or standard form of the Lagrangian is $L = T - V$, where $T$ is the kinetic energy and $V$ is the potential energy of the system. However, according to Goldstein and Rana,[31,32] the general non-canonical form of
Lagrangian leads to the canonical form given a particular condition. Because forces in scleronomic systems cannot be generated from any potential, the canonical Lagrangian has no explicit time dependency. Again, all scleronomic systems are not always conservative for systems exposed to dissipative processes. The canonical Lagrangian is easily derived from the non-canonical one. Fundamentally, \( L \) does not have a unique functional form since the form of the Euler-Lagrange equations of motion may be kept for a variety of Lagrangian options.\(^{[31,32]}\) Furthermore, in special relativistic dynamics, the classical idea of \( L(=T-V) \) is no longer appropriate.\(^{[32,33]}\)

As a result, we may infer that the generic form of the Lagrangian is non-canonical type.

It is widely agreed that a black hole is the result of a gravitational collapse and emits thermal radiation as if it were very hot, with its temperature being directly proportionate to its surface gravity.\(^{[34–43]}\) Assuming the spacetime is to be static or stationary,\(^{[44]}\) the black hole’s mass will slowly diminish due to thermal radiation as long as the radiation emitted is negligible in comparison to the black hole’s mass energy. It may be enhanced by using the Einstein equation for sufficiently large black hole. In this connection, a non-static solution of the Einstein’s field equations for spheres of fluids radiating energy has been derived by Vaidya.\(^{[44,45]}\) He has also established the non-static analogs of Schwarzschild’s interior solution in refs.\(^{[46,47]}\) and solved the problem of gravitational collapse with radiation in ref.\(^{[48]}\). This solution satisfies the physical feature of allowing a positive definite value of the density of collapsing matter and ensures that, from the perspective of a stationary observer at infinity, the total luminosity of the object is zero as it collapses to the Schwarzschild’s singularity. It makes sense to think of the Vaidya spacetime as a non-stationary Schwarzschild spacetime.

Husain\(^{[49]}\) and Wang et al.\(^{[50]}\) proposed the generalized Vaidya spacetime, which is analogous to the gravitational collapse of a null fluid. Recently, Manna et al.\(^{[26,27]}\) has established the \( K \)-essence generalizations of Vaidya spacetime, in which the time dependence of the metric is determined from the kinetic energy \( (\phi')^2 \) of the \( K \)-essence scalar field \( \phi \).

In the following works,\(^{[51–55]}\) authors have discussed about Hawking radiation\(^{[44–43]}\) using tunneling mechanism and this process in semi-classical quantum mechanics can be described by the method of complex path analysis which is proposed in refs.\(^{[56,57]}\). Kernar and Mann\(^{[58]}\) have established that the Hawking temperature is independent of the angular part of the spacetime in general. It is possible to use either the radial-geodesic or Hamilton-Jacobi methods to analyze Hawking radiation. The Hawking radiation of a static and stationary black hole was studied using the radial geodesic approach created by Parikh and Wilczek.\(^{[52]}\) Since its initial popularity in the 1990s, the Hamilton-Jacobi method has been revived as a tool for investigating black hole’s non-thermal radiation. By solving Hamilton-Jacobi equations, one may determine the particle activity of both stationary and non-stationary black holes using this approach. The Hawking radiation in both Vaidya and Vaidya-Bonnor spacetimes is investigated in refs.\(^{[59–65]}\). Considering the Vaidya-Bonnor-de Sitter black hole, Chen and Yang\(^{[63]}\) have fixed the total energy and charge while allowing those of the black hole to vary. According to their theory the particle crosses the horizon radially. Basically their study addresses Hawking radiation when the particle tunnels through the potential barrier and the black hole does not absorb or emit any additional particles at \( \Delta T \). Since the particle penetrates through the barrier instantly, \( \Delta T \) is infinitely tiny. This suggests that the black hole cannot absorb or emit particles at this moment.

The expression of the energy and angular momentum fluxes transported by gravitational waves across the dynamical horizons, as well as the equation describing the variation of the dynamical horizon radius, are obtained in refs.\(^{[66–71]}\). The definition of dynamical horizon is: A smooth, three-dimensional, space-like submanifold \( H \) in a space-time \( M \) is said to be a dynamical horizon if it can be foliated by a family of closed two-surfaces such that, on each leaf \( S \), the expansion \( \Theta \) of one null normal \( l \) vanishes and the expansion \( \Theta \) of the other null normal \( n \) is strictly negative.

However, the modified definition proposed by Sawaya\(^{[72]}\) is as follows: A smooth, three-dimensional, space-like or timelike submanifold \( H \) in a space-time is said to be a dynamical horizon if it is foliated by a preferred family of two-spheres such that, on each leaf \( S \), the expansion \( \Theta \) of a null normal \( l \) vanishes and the expansion \( \Theta \) of the other null normal \( n \) is strictly negative. For an example\(^{[73]}\): in Minkowski spacetime, i) radially outgoing light-rays, \( l = \partial_\nu, v = t + r \), have expansion: \( \Theta_l = \nabla_\nu (\partial_\nu r) = \frac{1}{r} \partial_\nu (r^2 (\partial_\nu + \partial_r)) = \frac{1}{r^2} > 0 \), and ii) radially ingoing light-rays \( n = \partial_\nu, u = t - r \), have expansion: \( \Theta_n = \nabla_\nu (\partial_\nu r) = \frac{1}{r} \partial_\nu (r^2 (\partial_\nu - \partial_r)) = -\frac{1}{r^2} < 0 \), which are indicating that outgoing light-rays expand while ingoing light-rays contract.

Following the work of Ashokkar and Galloway\(^{[68]}\) under the concept of world tubes, if the marginally trapped tube (MTT) is

i) spacelike, then it is called a dynamical horizon (DH) and under some conditions that it provides a quasi-local representation of an evolving black hole.

ii) timelike, then the causal curves can transverse it in both inward and outward directions, where it does not represent the surface of a black hole in any useful sense, it is called a time-like membrane (TLM).

iii) null, then it describes a quasi-local description of a black hole in equilibrium and is called an isolated horizon (IH).

It is now highly promising to determine the behavior of non-static mass under a massive radiation scenario in this non-canonical theory, for the reasons stated above. In this work, we use the dynamical horizon equation based on Sawaya\(^{[72]}\) and the tunneling formalism\(^{[18–20,51–57,59,60]}\) to investigate the Hawking effect in the \( K \)-essence emergent generalized Vaidya spacetime for purely gravitational view point.

The paper is structured as follows: Section 2 provides a short overview of the \( K \)-essence emergent geometry and the corresponding \( K \)-essence Vaidya spacetime. In Section 3, we explain the dynamical horizons for the emergent Vaidya spacetime with the \( K \)-essence using the Schwarzschild black hole as a background. The dynamical horizon equation for the \( K \)-essence Vaidya Schwarzschild spacetime has been studied in depth in the Section 4. We have also presented in Section 5 the related Hawking radiation by discussing the dynamical horizon equation and the tunneling mechanism. The last Section 6 serves as a summary of our findings.
2. Brief Review of K-essence and K-essence-Vaidya Geometry

2.1. K-essence Geometry

The Scalar field $\phi$ of K-essence possesses action$^{[8–12]}$

$$ S_\phi[g_{\mu\nu}] = \int d^4x \sqrt{-g} L(X, \phi), \quad (1) $$

when it is minimally coupled to the background gravitational metric $g_{\mu\nu}$ and where $X = \frac{1}{2}g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi$ and the energy-momentum tensor is

$$ T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta S_\phi}{\delta g^{\mu\nu} \phi} = L_X \nabla_\mu \phi \nabla_\nu \phi - g_{\mu\nu} L, \quad (2) $$

where $L_X = \frac{dL}{dX}$, $L_{XX} = \frac{d^2L}{dX^2}$, $L_\phi = \frac{dL}{d\phi}$ and $\nabla_\mu$ is the covariant derivative defined with respect to the gravitational metric $g_{\mu\nu}$.

The scalar field equation of motion is

$$ -\frac{1}{\sqrt{-g}} \frac{\delta S_\phi}{\delta \phi} = C^{\mu\nu} \nabla_\mu \phi + 2XLXX_\phi - L_\phi = 0, \quad (3) $$

where

$$ C^{\mu\nu} \equiv L_X g^{\mu\nu} + L_{XX} \nabla^\mu \phi \nabla^\nu \phi $\quad (4) $$

and $1 + \frac{2XLXX}{L_X} > 0$.

Using the conformal transformations $G^{\mu\nu} \equiv \frac{L_X}{L_{XX}} C^{\mu\nu}$ and $G_{\mu\nu} \equiv \frac{L_X}{L_{XX}} G_{\mu\nu}^\nu$, with $c_2(X, \phi) \equiv (1 + 2X \frac{L_{XX}}{L_X})^{-1}$ we have$^{[18–20]}$

$$ \tilde{C}_{\mu\nu} = g_{\mu\nu} - \frac{L_{XX}}{L_X + 2XLXX} \nabla_\mu \phi \nabla_\nu \phi. \quad (5) $$

In order to investigate the behavior of minor perturbations as they propagate in the preferred reference frame, when the background is at rest, Babichev et al.$^{[10]}$ has proposed the definition of sound speed ($c_s$), as indicated above. The speed of sound cannot often be higher than the speed of light, however, there are times when field fluctuations may spread at a superluminal rate ($c_s > 1$).

For Equations (1)–(4), to have any physical significance, $L_X \neq 0$ must always hold and $c_2^2$ must be positive definite.

Now the equation of motion (3) simplifies to

$$ -\frac{1}{\sqrt{-g}} \frac{\delta S_\phi}{\delta \phi} = C^{\mu\nu} \nabla_\mu \phi, \quad (6) $$

if and only if $L$ is not an explicit function of $\phi$.

One may take a note that the emergent metric $\tilde{C}_{\mu\nu}$ is not conformally equal to $g_{\mu\nu}$ for non-trivial spacetime configurations of $\phi$. The local causal structure of $\phi$ also differs from that specified by $g_{\mu\nu}$, indicating that $\phi$ possesses properties distinct from canonical scalar fields.

The Lagrangian of the DBI type$^{[15–20]}$ is assumed to be

$$ L(X, \phi) = 1 - V(\phi) \sqrt{1 - 2X}, \quad (7) $$

for $V(\phi) = V = \text{constant and kinetic energy of } \phi \gg V$, i.e., $(\phi^\dagger)^2 \gg V$. However, when it comes to K-essence fields, it is common for kinetic energy to be more dominant over potential. In such case, $c_2^2(X, \phi) = 1 - 2X$. For scalar fields $\nabla_\mu \phi = \partial_\mu \phi$. Then, Equation (5) becomes

$$ \tilde{C}_{\mu\nu} = g_{\mu\nu} - \partial_\mu \phi \partial_\nu \phi. \quad (8) $$

In order to get to the K-essence emergent gravity metric described in Equation (8), we first apply a conformal transformation to determine the identify of the inverse metric $g_{\mu\nu}$, and then perform a second conformal transformation to realize the mapping onto the metric given in Equation (8) for the Lagrangian (Equation 7),$^{[18]}$

The geodesic equation for the K-essence theory in terms of the new Christoffel connections $\Gamma^\nu_{\mu\nu}$

$$ dt^2 = e^{\delta\mu \phi \phi} dx^\mu dx^\nu \delta\nu \phi \phi = 0, \quad (9) $$

where $\lambda$ is an affine parameter and

$$ \Gamma^\nu_{\mu\nu} = \Gamma^\nu_{\mu\nu} - \frac{1}{2(1-2X)} [\delta^\mu_\lambda \phi \phi, \delta^\nu_\lambda \phi \phi]. \quad (10) $$

It is worth noting that the symmetry of $\Gamma$ is preserved by the interchange of $\mu$ and $\nu$ in the second term on the right-hand side of Equation (10). The second term is unique to the K-essence Lagrangian and represents further interaction (forces). Also, note that the Einstein tensor is not the same as the field K-essence emergent gravity metric ($\tilde{C}_{\mu\nu}$) (Equation 8).

It follows that $G_{\mu\nu} \equiv \tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{C}_{\mu\nu} \tilde{R} = k \tilde{T}_{\mu\nu}$ is the corresponding Emergent Einstein Equation, where $G_{\mu\nu}$ is the emergent Einstein tensor, $T_{\mu\nu}$ is the corresponding energy-momentum tensor, $k = 8\pi G$, $R_{\mu\nu}$ is Ricci tensor and $\tilde{R} = \tilde{R}_{\mu\nu} \tilde{C}^{\mu\nu}$ is the Ricci scalar of the emergent spacetime. In this case, any emergent metric $\tilde{C}_{\mu\nu}$ that satisfies the “Emergent Einstein Equation” may be considered as a solution. Components of the emergent Einstein tensor ($G_{\mu\nu}$), energy-momentum tensor ($T_{\mu\nu}$), and energy conditions for a generalized K-essence Vaidya metric are discussed in depth in ref. [26]. Notably, the authors of refs. [75–77] have used the K-essence emergent gravity metric (Equation 8) or (Equation 12)(below) in cosmology, where they have assumed that the underlying metric is of the Friedmann-Lemaître-Robertson-Walker (FLRW) type.

2.2. K-essence-Vaidya Geometry

The Eddington-Finkelstein line element for a general spherically symmetric static black hole$^{[26]}$

$$ ds^2 = f(r) dv^2 - 2vdvdr - r^2 d\Omega^2, \quad (11) $$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$.

It is to note that when $e = +1$, the Eddington advanced time (outgoing) is represented by the null coordinate $v$. Further, the Eddington retarded time (incoming) is represented by the null coordinate $\nu$ when $e = -1$. 
From Equation (8) the emergent spacetime is described by the line element

$$dS^2 = ds^2 - \partial_\nu \phi \partial_\nu dx^\nu dx'. \quad (12)$$

The emergent spacetime line element is

$$dS^2 = \left[ f(r) - \phi_4^2 \right] dv^2 - 2edvdr - r^2d\Omega^2, \quad (13)$$

if we assume the scalar field $\phi(x) = \phi(v)$ where $\phi_4 = \frac{\partial \phi}{\partial v}$.

Note that, in general, spherical symmetry would only require that $\phi(x) = \phi(v, r)$, hence the assumption on $\phi$ contradicts local Lorentz invariance. However, the dynamical solutions break Lorentz invariance spontaneously in the K-essence theory. Therefore, the K-essence scalar field in the form we have chosen is physically acceptable in this context. It should also be noted that the Lorentz invariance of the Minkowski spacetime is preserved by the action (Equation 1), even if the dynamical solutions of the K-essence scalar field violate it.

Now, we derive

$$dS^2_\nu = \left( 1 - \frac{2m(v, r)}{r} \right) dv^2 - 2edvdr - r^2d\Omega^2, \quad (14)$$

by comparing the metric of the generalized Vaidya spacetimes corresponding to the gravitational collapse of a null fluid (choose $\epsilon = +1$) with the metric of the emergent spacetime (Equation 13), where the mass function is

$$m(v, r) = \frac{1}{2}r\left[ 1 + \phi_4^2 - f(r) \right]. \quad (15)$$

For the generalized K-essence emergent spacetime with the Vaidya spacetime provided $\phi, \phi_4 > 0; 1 + \phi_4^2 > f + rf'; 2f, + rf'_r > 0$ which have established by Manna et al., these forms of metrics (Equation 13) or (Equation 14) meet all the necessary energy conditions (weak, strong, dominating).

### 3. Dynamical Horizons

Following the works of some authors, we now discuss how the dynamical horizon of the K-essence emergent Vaidya spacetime behaves.

When the kinetic energy of the K-essence scalar field $\phi_4^2$, is present, the generalized Vaidya spacetime (Equation 13) or (Equation 14) may be expressed

$$dS^2 = F(v, r)dv^2 - 2edvdr - r^2d\Omega^2, \quad (16)$$

where $v^\nu$ is a null vector.

Based on Sawayama’s work, we can define

$$a = \frac{dr}{dr^*}, \quad (17)$$

where $r^*$ is tortoise coordinate defined as $v = t + r^*$. Thus the two null vectors are

$$p^\nu = \begin{bmatrix} \frac{F}{2} f' \\ -a^{-1} \end{bmatrix}, \quad (18)$$

associating to the null vector $v^\nu$, and the other one is

$$n^\nu = \begin{bmatrix} n^1 \\ n^2 \\ 0 \end{bmatrix} = \begin{bmatrix} n^2 \\ a^{-1} \\ 0 \end{bmatrix}. \quad (19)$$

The expansions of the two null vectors $l^\nu$ and $n^\nu$, $\Theta_{(0)}$ and $\Theta_{(1)}$ are

$$\Theta_{(0)} = \frac{1}{r}(2F - a) \quad (20)$$

and

$$\Theta_{(1)} = \frac{1}{ar} \left( -2F^2 + aF - 2a^2 \right). \quad (21)$$

One can note that as $\Theta_{(0)} = 0 \Rightarrow 2F - a = 0$, and the other null expansion $\Theta_{(1)}$ is strictly negative which are the required conditions for the dynamical horizon. As a result, the horizons of our case are dynamical. The following concepts explain the aforementioned necessary requirements:

- The intuition that a black hole emits nothing – not even light – leads to a strictly zero null expansion, and the fact that null matter disappears into a black hole leads to a strictly negative null expansion. Note that the dynamical horizon for the emergent Vaidya spacetime in terms of the K-essence has been developed by Manna et al. on the basis of Ashtekar et al.

### 3.1. Schwarzschild Black Hole as Background

For Schwarzschild black hole as background, we consider $f(r) = 1 - \frac{2M}{r}$, then Equation (16) becomes

$$dS^2 = \left( 1 - \frac{2M}{r} - \phi_4^2 \right) dv^2 - 2edvdr - r^2d\Omega^2, \quad (22)$$

with

$$F = \left( 1 - \frac{2M}{r} - \phi_4^2 \right) \equiv \left( 1 - \frac{2m(v, r)}{r} \right). \quad (23)$$

The mass function can be provided as

$$m(v, r) = M + \frac{r}{2}\phi_4^2, \quad (24)$$

where the tortoise coordinate

$$r^* = \frac{1}{N} \left[ r + B \ln(r - B) \right]. \quad (25)$$
with \( N \equiv N(v) = (1 - \phi_i^2) \), \( B = 2M'(v) = \frac{2M}{N} \) and \( M'(v) = \frac{M}{1 - \phi_i^2} \).

Please note that in the aforementioned spacetime (Equation 22), \( \phi_i < 1 \) is always the case. The signature of this spacetime is ill-defined if \( \phi_i > 1 \). Moreover, the K-essence hypothesis is rendered useless if \( \phi_i = 0 \). It follows that \( \phi_i \neq 0 \) holds good. Again, \( \phi_i \neq 1 \) since if \( \phi_i = 1 \) in Equation (22), it does not have a Newtonian limit,\(^{73}\) which makes it unsuitable for describing astrophysical objects. For these reasons, \( \phi_i \) might be anywhere from 0 and 1. Importantly, this article assumes only that the background gravitational metric is Schwarzschild. Future research into the horizon’s dynamical behavior may also take into consideration the general spherically symmetric background metrics. Additionally, it is noted that Manna et al.\(^{18,19}\) have shown that for a specific solution of the K-essence scalar field, the emergent gravity metrics transfer onto the Barriola-Vilenkin (BV) and Robinson-Trautman (RT) type metric, respectively, when the Schwarzschild metrics transfer onto the Barriola-Vilenkin (BV) and Robinson-Nordström metrics are considered as a backdrop. For instance, these BV and RT metrics are static, spherically symmetric emergent metrics.

In this case, we obtain

\[
a = F\left[1 - \frac{1}{N}\left(\frac{dB}{dv} \ln(r - B) - \frac{B}{r - B} \frac{dB}{dv}\right)\right]
\]

by taking the derivative of the preceding Equation (25) with respect to \( r \).

The dynamical horizon radius

\[
2F - a = 2F - F\left[1 - \frac{1}{N}\left(\frac{dB}{dv} \ln(r - B) - \frac{B}{r - B} \frac{dB}{dv}\right)\right]
\]

is found by solving \( \Theta_{(r)} = 0 \).

As a result, we have two possible solutions for \( r \):

\[
r_D = 2M'(v) = \frac{2M}{N}. \quad (28)
\]

by solving \( F = 0 \) and

\[
(r_D - B)^2\phi_i = e^{\omega N}, \quad (29)
\]

by solving \( 1 + \frac{d}{dv} B \ln(r_D - B) - \frac{B}{r_D - B} \frac{dB}{dv} = 0 \).

The Wright Omega function (\( \omega \))\(^{80}\) may be used to express this value of \( r_D \) as

\[
r_D = B[1 + \omega(Z)] \approx \frac{2M}{N}[1 + \omega(Z)] \quad (30)
\]

with \( Z = -1 + \ln(B) + vC \), \( C = \frac{N^2}{2M} = \frac{(1 - \phi_i^2)^2}{2M} \) where the dynamical radius lies outside \( r = \frac{2M}{1 - \phi_i^2} \).

The Wright Omega function (\( \omega \)) is a single-valued function, defined in terms of the Multi-valued Lambert W function\(^{81}\) as \( \omega(Z) \in W_{1/2}(e^Z) \) where \( K(Z) = \left[ \frac{\ln(2Z - 1)}{2} \right] \) is the unwinding number of \( Z \). The sign of this unwinding number is such that \( \ln(e^Z) = Z + 2\pi i K(Z) \) which is opposite to the sign used in ref. \( [82] \). The algebraic properties\(^{80}\) of the Wright Omega function (\( \omega \)) are

\[
\frac{d\omega}{dZ} = \frac{\omega}{1 + \omega} \quad (31)
\]

having the analytical property \( Z = \omega + l n\omega \).

Here Equation (29) allows us to re-establish Sawayama’s\(^{72}\) finding

\[
r_D = 2M(v)[1 + e^{-2M/v}] \quad (33)
\]

in the usual Vaidya spacetime without the K-essence scalar field \( \phi \) if we consider \( m(v, r) \equiv M(v) \) and \( \phi_i = 0 \).

### 4. Dynamical Horizon Equation

The Ricci scalar (\( \bar{R} \)) and Ricci tensors (\( \bar{R}_{\mu\nu} \)) of the K-essence emergent Vaidya spacetime may be derived from Equation (16) as

\[
\bar{R}_{\mu\nu} = \frac{1}{r} \partial_r F - F \frac{1}{2} \partial_r^2 F - \frac{F}{r} \partial_r F;
\]

\[
\bar{R}_{\mu\nu} = \frac{1}{r} \partial_r F + \frac{1}{r} \partial_r F ; \quad \bar{R}_{\mu\nu} = 0 \quad (34)
\]

The components of the energy momentum tensor for the K-essence emergent Vaidya spacetime may be derived from the “emergent” Einstein’s equation \( \frac{1}{2} \bar{G}_{\mu\nu} \bar{R} = 8\pi \bar{T}_{\mu\nu} \) using these values for the Ricci scalar and the Ricci tensors (Equation 34) as

\[
8\pi \bar{T}_{\mu\nu} = -\frac{1}{r} \partial_r F + \frac{1}{r} F \partial_r F + \frac{F}{r^2} (F - 1), \quad (35)
\]

\[
8\pi \bar{T}_{\mu\nu} = -\frac{1}{r} \partial_r F + \frac{1}{r^2} (F - 1), \quad (36)
\]

\[
8\pi \bar{T}_{\mu\nu} = 0, \quad (37)
\]

\[
8\pi \bar{T}_{\mu\nu} = -\frac{1}{2} \partial_r F + r \partial_r F \quad (38)
\]

where the gravitational constant \( G = 1 \).

Now according to Equation (22), for the Schwarzschild background case, the energy-momentum tensor components are:

\[
8\pi \bar{T}_{\mu\nu} = -\frac{2}{r^2} [\partial_m m + F \partial_m F] = \left[ \frac{2\phi_i \phi_{i\alpha}}{r} + \frac{\phi_i^2}{r^2} (1 - \frac{2M}{r} - \phi_i^2) \right], \quad (39)
\]
for the spherically symmetric K-essence emergent Schwarzschild Vaidya spacetime.

It is worth noting that in our previous work\cite{26} we derived the energy requirements of the emergent Vaidya spacetime with K-essence (Equation 22) for the Schwarzschild background. The energy conditions are: $\gamma \geq 0$, $\rho \geq 0$. $P \geq 0$ ($\gamma \neq 0$), which satisfies the weak and strong energy conditions and $\gamma \geq 0$, $\rho \geq 0$, $P \geq 0$ ($\gamma \neq 0$), which satisfy the dominant energy condition provided $\phi, \phi' > 0$, where $\gamma = \frac{2\phi\phi'}{\phi'^2}$, $\rho = \frac{\phi'}{\phi'^2}$ and $P = 0$ with $\kappa = 8\pi G$. The associated energy-momentum tensor is of the type-II class,\cite{26,78} which is characterized by a double null vector. It is possible to integrate the dynamical horizon equation precisely given in refs.\cite{66,67,72} by deriving the energy-momentum tensor $\tilde{T}_{\mu\nu}$, as shown in ref.\cite{72} as

$$\frac{1}{2}G (\tilde{R}_2 - \tilde{R}_1) = \int_{\mathcal{H}} \tilde{T}_{ab} \tilde{\phi}^a \tilde{\phi}^b \, d^4 V + \frac{1}{16\pi G} \int_{\mathcal{H}} N \left[ \sigma | r^2 + 2 | \xi | r^2 \right] d^4 V,$$

where $R_2, R_1$ are the radii of the dynamical horizon, $T_{ab}$ is the stress-energy tensor, $| \sigma | = \sigma_{\alpha\beta} \sigma^\alpha \sigma^\beta$, $| \xi | = \xi_{\alpha\beta} \xi^\alpha \xi^\beta$, $\sigma_{\alpha\beta}$ is the shear, $\xi_{\alpha\beta} = \tilde{\gamma}_{\alpha\beta} \tilde{\nu} \tilde{\nu}$, with the two-dimensional metric $\tilde{\gamma}_{\alpha\beta}$, and $\tilde{\gamma} = \tilde{\gamma}_{\alpha\beta} \tilde{\nu} \tilde{\nu}$ with $\tilde{\gamma} = \partial \mathcal{R}$, where $\mathcal{R} = \text{radius of the dynamical horizon}$.

This is the dynamical horizon equation (42), and it describes how the horizon radius varies as a function of matter flow, shear, and expansion. The right-hand side of the dynamical equation is simplified since the second component disappears when the system is spherically symmetric.

To begin with, we may express $T_{\mu\nu}$ as the combination of $T_{\nu}$ and $\tilde{T}_{\nu\nu}$, where

$$\tilde{T}_{\nu} = \tilde{T}_{\nu} - \tilde{T}_{\nu\nu} = -\frac{1}{4\pi r^2} \frac{5}{2} \partial_{\nu} m = -\left(\frac{1}{4\pi r^2}\right) \frac{5}{2} \phi \phi_{\nu\nu}.$$

Assuming $\hat{r}$ is the unit vector in the direction of $r$, we get

$$\tilde{T}_{\nu} = -\frac{1}{4\pi r^2} \frac{5}{2} (\partial_{\nu} m) F^{-1},$$

with $F$ is defined in Equation (23).

At the horizon $r = r_0$, the expression of $T_{\nu\nu}$ is $T_{\nu\nu} = \frac{1}{4\pi r_0^2} (\partial_{\nu} m) F^{-1}$ using $m \equiv m(r, v) = M + \frac{1}{2} \psi (1 - N)$. We therefore generate the following terms:

$$\frac{d}{d\nu} \tilde{T}_{\nu} = \left[ -\frac{2M}{N^2} (1 + \omega(Z) + \frac{2M}{N} \frac{\alpha(Z)}{1 + \omega(Z)} \times \left(\frac{1}{N} - \frac{N\nu}{M}\right) \right] \partial_{\nu} N,$$

(45)

Using the above Equation (45), we have

$$\partial_{\nu} N = \frac{N \omega(Z)}{1 + \omega(Z)} \times \left[ \frac{2M}{N^2} (1 + \omega(Z)) + \frac{2M}{N} \frac{\alpha(Z)}{1 + \omega(Z)} \left(\frac{1}{N} - \frac{N\nu}{M}\right) \right]^{-1},$$

at the horizon.

Again, from Equation (30)

$$\frac{d\nu}{dN} = \left(\frac{N}{\partial_{\nu} N}\right) \frac{\omega(Z)}{1 + \omega(Z)}.$$

Now, rewriting Equation (23) in terms of Write $\omega$ function as

$$F = \frac{N \omega(Z)}{1 + \omega(Z)}$$

and also from Equation (24), we have

$$\partial_{\nu} m = -\left(\frac{r_0}{2}\right) \partial_{\nu} N.$$

Now, by changing the order of integration $r_0$ to $N$ in Equation (42), we get

$$\frac{1}{2} \left[ \frac{2M}{N} (1 + \omega(Z)) \right] N_1 = \frac{5}{4} \int_{N_1}^{N_2} \frac{2M}{N} (1 + \omega(Z)) dN,$$

(50)

since the functions $\partial_{\nu} N$ and $F^{-1}$ with fixed $r_0$ are used only in the integration.

By substituting the aforementioned Equation (50) into Equation (42) as well as using Equations (44) to (49) and then considering the limit $N_2 \rightarrow N_1 = N$, one can obtain

$$= \frac{M}{N^2} (1 + \omega(Z)) + \frac{M}{N} \frac{\alpha(Z)}{(1 + \omega(Z))} \left(\frac{1}{N} - \frac{N\nu}{M}\right) - \frac{5M}{2} \left(\frac{1 + \omega(Z)}{2}\right)$$

(51)

This is the dynamical horizon equation (51) for the spherically symmetric K-essence emergent Schwarzschild-Vaidya spacetime. When the kinetic energy of the K-essence scalar field is present, the usual Vaidya dynamical horizon equation in ref.\cite{72} is radically changed.

5. Hawking Radiation in the K-essence Schwarzschild-Vaidya Spacetime

Now we are going to discuss about the Hawking radiation.\cite{14-43} In order to find a solution, we investigate two different approaches: 1) the dynamical horizon equation (42); and 2) the K-essence Schwarzschild-Vaidya metric (Equation 22) using a tunneling mechanism.\cite{51-57,59,60}

5.1. Dynamical Horizon Equation

Taking into account Candela’s conclusion\cite{79} for matters on the dynamical horizon, which is appropriate close to the horizon\cite{72}
and supposes that spacetime is almost static, from Equation (22) and $r \sim 2m(\equiv \frac{2m}{N})$, we get

$$T_{\text{eff}} = -\frac{1}{2\pi^2(1 - \frac{2m}{r})} \int_0^\infty \frac{w^3dw}{\varepsilon - \omega - 1},$$

$$= -\frac{1}{2m^2c^2(1 - \frac{2m}{r})}. \tag{52}$$

with $c = 15 \times 8^3 = 61440$ and $\int_0^\infty \frac{w^3dw}{\varepsilon - \omega - 1} = \frac{\pi^4}{15c^2}$.

In the dynamical horizon equation, the matter-energy of Equation (52) is negative around $r \sim 2m$. Hence the K-essence Schwarzschild-Vaidya black hole absorbs negative energy, causing the black hole’s radius to shrink. At this juncture, one may note that the Schwarzschild mass is less than the K-essence Schwarzschild-Vaidya mass as $\phi^0_v < 1$, whereas larger at the horizon, where the K-essence Schwarzschild mass $m(\equiv \frac{M}{1+\phi^0_v})$. Also, the kinetic energy ($\phi^0_v$) of the K-essence scalar field, as described by Equation (24), carries the dynamic behavior of the mass function $m(\nu, r)$.

Here, we employ the dynamical horizon equation in place of solving the entire Einstein equation with the backreaction, since we are only looking for information about matter close to the horizon. Following,[72] we get

$$T_{\text{eff}} = \frac{1}{2m^2c^2(1 - \frac{2m}{r})}. \tag{53}$$

Again, one may obtain

$$\int_{r_D}^\infty 4\pi r^2_{\text{D}} T_{\text{D}} dr_D \approx b \int_{N_1}^{N_2} \left[ \frac{2M}{N}(1 + \omega(Z)) \right]^2 \frac{(1 + \omega(Z))}{(N\omega(Z))} \left[ M + \frac{M}{N}(1 + \omega(Z))(1 - N) \right]^{-4} \left( \frac{dr_D}{dN} \right) dN,$$

$$= b \int_{N_1}^{N_2} \left[ \frac{2M}{N}(1 + \omega(Z)) \right]^2 \left[ M + \frac{M}{N}(1 + \omega(Z))(1 - N) \right]^{-4} \left( \frac{1}{dN} \right) dN, \tag{54}$$

by changing the order of integration from $r_D$ to $N$ in the right hand side of the dynamical horizon equation (42), where $b = \frac{2}{3} = \text{a constant and } \frac{dr_D}{dN} = \frac{N\omega(Z)}{1+\omega(Z)} \times \frac{1}{0,N}$.

Specifically, the K-essence Schwarzschild-Vaidya metric has a time dependency that carries $\phi^0_v$, i.e., $N(= 1 - \phi^0_v)$.

We derive

$$1\left( \frac{2M}{N}(1 + \omega(Z)) \right)^2 \int_{N_1}^{N_2} \left[ M + \frac{M}{N}(1 + \omega(Z))(1 - N) \right]^{-4} \left( \frac{1}{dN} \right) dN,$$

$$\times \left( \frac{1}{dN} \right) dN + \frac{5}{4} \int_{N_1}^{N_2} \frac{2M}{N}(1 + \omega(Z)) dN, \tag{55}$$

by substituting Equation (54) into the right-hand side of the dynamical horizon equation (42) and where

$$r_D = B[1 + \omega(Z)] = \frac{2M}{N}[1 + \omega(Z)].$$

When we choose the limit $N_2 \rightarrow N_1 = N$ and $\omega(Z) \neq 0$, the dynamical horizon equation (55) simplifies to

$$2\left[ 1 + \left( 1 - \frac{1}{2} \right) (1 + \omega(Z)) \right]^{-4} \times \left[ \frac{4(1 + \omega(Z))}{N\omega(Z)} \left[ \frac{1}{2} - \frac{1}{2}(1 + \omega(Z)) \frac{1 + \omega(Z)}{1 + \omega(Z)} \left( \frac{1}{N} + \frac{N_1}{M} \right) \right]^{-1} \right],$$

$$\text{(56)}$$

as long as both $N$ and $M$ are non-zero.

If $N$ is a function of $\nu$ according to the above Equation (56), then $\phi^0_v$ must also be a function of $\nu$ according to the same logic, because $\phi^0_v = 1 - N$. The Equation (56) contains the Wright omega function $\omega(Z)$ (where $Z = [-1 + \ln(B + vC)]$ in a comprehensive fashion. As such, there is no method to derive $N$ analytically, making it quite distinct from Sawayaama’s[72] transcendental equation (37).

In this subsection, however, we are primarily interested in determining the behavior of the black hole mass $m(\nu, r)$ with $\nu$ for fixed $M$. Here, we must first solve Equation (56) for $N$ for a given value of $M$ assuming $M \neq 0$, while in ref. [72], the behavior of $M(\nu)$ may be found immediately using numerical techniques from Equation (37) of Sawayaama’s work.[72] Then, by substituting into the formula $\phi^0_v = 1 - N$, we can determine what $\phi^0_v$ is when $M$ is a specific value. Last but not least, the relation (Equation 24) allows us to observe the change in mass $m(\nu, r)$ as $\nu$ increases.

As shown in the left column of Figure 1, if we numerically solve Equation (56) for $M = 1, M = 5,$ and $M = 10$, then the solutions of $\phi^0_v$ consistently decrease with increasing $\nu$. Now substituting these values of $\phi^0_v$ for $M = 1$ and $r = 10$ in Equation (24), we find that the black hole’s mass $m(\nu, r)$ decreases with increasing $\nu$ but does not go to zero, as seen from the first panel in the right column of Figure 1. For $M = 5, r = 100$ and $M = 10, r = 50$, we find similar behavior of $m(\nu, r)$ when solving Equations (56) by using Equation (24). These solutions are shown in the final two figures in the right column. In addition, it is said in ref. [72] that the black hole’s mass completely vanished at the moment when time ($\nu$) axis considered as $-\infty$ to zero (shown as $\nu = 0$ in Figure 2). In contrast, our black hole mass $m(\nu, r)$ (Equation 24) is not completely evaporated at $\nu = 0$, even while taking into account the same kind of axis representation. In the K-essence Schwarzschild-Vaidya spacetime, the blackhole mass diminishes but does not completely evaporate as shown by the decreasing behavior of $\phi^0_v$ and $m(\nu, r)$ with increasing $\nu$.

Analytical proof of this situation may be derived from Equation (24): in the limiting case, $m(\nu, r) \rightarrow M$ as $\phi^0_v \rightarrow 0^+$ when $r$ and $M$ are fixed, or $m(\nu, r) \rightarrow M$ as $r \rightarrow 0$ when $\phi^0_v$ and $M$ are fixed since $M \neq 0$. This is in sharp contrast to Sawayaama’s[72] conclusion, according to which the black hole’s mass always vanishes.
Figure 1. (Color online) Figures in the left column show the numerical solutions of $\phi_2^2$ for $M = 1$, $M = 5$, and $M = 10$. Figures in the right column show the numerical solutions of $m(v, r)$ for $(M = 1, r = 10)$, $(M = 5, r = 100)$ and $(M = 10, r = 50)$. 
However, if we write the mass $m(v, r)$ as $\zeta \phi^2$ (using Equation 24) where $M$ is extremely tiny and negligible, then Equation (22) may be represented as (in the usual coordinate system): by rescaling, we may convert $dS^2 = (1 - \phi^2)dt^2 - \frac{dr^2}{(1 - \phi^2)} - r^2d\Omega^2$ to $dS^2 = dt^2 - dr^2 - r^2(1 - \phi^2)d\Omega^2$. Because the rescaled metric reflects a space with a deficit solid angle, the area of a sphere with radius $r$ is less than $4\pi r^2$, specifically, it is $(1 - \phi^2)4\pi r^2$ since $\phi^2$ takes the values in between 0 and 1. As a result, black holes of this type cannot totally evaporate since their associated spaces are not asymptotically flat but asymptotically bound.

5.2. The Tunneling Formalism: Hamilton-Jacobi Method

Using the tunneling approach,[18–20,51–57,59,60] one may get the Hawking temperature for a massless particle in a black hole (Equation 22) whose background is given by the Klein-Gordon equation

$$\tilde{G}^{\nu\mu}\partial_{\nu}S\partial_{\mu}S = 0,$$

where $\Psi$ is in the form

$$\Psi = \exp\left(\frac{i}{\hbar} S + \ldots\right).$$

To find the leading order in $\hbar$ the Hamilton-Jacobi equation is

$$\tilde{G}^{\nu\mu}\partial_{\nu}(\partial_{\mu}S) = 0,$$

where we consider $S$ is independent of $\theta$ and $\Phi$.

Thus

$$2\partial_{\nu}S\partial_{\mu}S + (1 - \frac{2M}{r} - \phi^2)(\partial_{\nu}S)^2 = 0.$$  (60)

In the conventional Hamilton-Jacobi description, the action $S(v, r)$ may be broken down into two parts: the time component, denoted by $E\nu$, and the radius component, denoted by $S_0(r)$, which normally only relies on radius. For black holes of varying masses, however, the energy of the departing particle must change over time since the metric coefficients rely on both radius and time. In view of this, the conventional method obviously fails. Yet, we may extend the procedure[60] if we want to determine the action $S(v, r)$. Hence, following,[60] we can choose the action $S$ to be form

$$S(v, r) = -\int_{v_0}^{v} E(v')dv' + S_0(v, r).$$  (61)

The term $\int_{v_0}^{v} E(v')dv'$ is simpler to understand due to the continuous and time-dependent energy of the discharged particles. So that[60]

$$\partial_{\nu}S = -E(v) + \partial_{\nu}S_0$$

and $\partial_{\mu}S = \partial_{\mu}S_0$.  (62)

Due to the fact that $S_0$ is proportional to $v$ and $r$, we have

$$\frac{dS_0}{dr} = \partial_{\nu}S_0 + \frac{dv}{dr}\partial_{\nu}S_0 = \partial_{\nu}S_0 + \frac{2}{F}\partial_{\nu}S_0.$$  (63)

where we apply the formula $\frac{dv}{dr} = \frac{2}{F}$, $F$ being specified by the given Equation (23).

The solution to (60) is

$$F\frac{dS_0}{dr} = 2E(v),$$

which has been obtained by plugging the results of (62) and (63) into (60) and because $\partial_{\nu}S_0 \neq 0$.

As a result, we may write the solution to $S_0$ as

$$S_0 = 2E(v)\int \frac{dr}{F} = 2E(v)\int \frac{dr}{(1 - \frac{2M}{r} - \phi^2)}$$

$$= \frac{2E(v)}{N} \int \frac{r}{r - 2M/N} = 2\pi i \frac{4ME(v)}{N^2},$$

having $N = 1 - \phi^2$. Since $r$ is analytic within and on any simple closed contour $C$, considered in the positive sense and $\frac{2M}{N}$ is a point inside $C$, we have employed the Cauchy-integral formula. Thus, Equation (61) becomes

$$S(v, r) = -\int_{v_0}^{v} E(v')dv' + 2\pi i \frac{4ME(v)}{N^2}.$$  (66)

As a result, the wave function for the massless particle outgoing (and ingoing) may be written in the following forms:

$$\Psi_{\text{out}}(v, r) = \exp \left[ \frac{i}{\hbar} \left( -\int_{v_0}^{v} E(v')dv' + \pi i \frac{4ME(v)}{N^2} \right) \right],$$

$$\Psi_{\text{in}}(v, r) = \exp \left[ \frac{i}{\hbar} \left( -\int_{v_0}^{v} E(v')dv' - \pi i \frac{4ME(v)}{N^2} \right) \right].$$

For an outgoing particle, the tunneling rate is

$$\Gamma = e^{-2\text{Im}S} \sim e^{-\frac{4\pi ME(0)}{\hbar}} = e^{-\frac{4\pi}{\hbar} K_B T},$$

where $K_B$ is Boltzman Constant.

Consequently

$$T_H = \frac{1}{8\pi K_B} \frac{1}{M} = \frac{1}{8\pi K_B} \left(1 - \frac{\phi^2}{r^2}\right),$$

is the Hawking temperature.

One can note that the usual Hawking temperature for the Vaidya spacetime[61] is retrieved under the assumptions $\phi^2 = 0$ and $m(v, r) = M(v)$.

6. Conclusion

Manna et al.[26] have established a connection between the K-essence geometry and the Vaidya spacetime, which produces a new spacetime called the K-essence emergent Vaidya spacetime. In contrast to the usual Vaidya dynamical horizon equation,
which is based on Sawayama’s finding, we have obtained the dynamical horizon equation (51) for the spherically symmetric K-essence emergent Schwarzschild-Vaidya spacetime.

Some of the salient features of the present works are as follows:

i) We have studied Hawking radiation in the K-essence emergent Vaidya spacetime using the modified definition of the dynamical horizon.\[72\] Using our work on Hawking radiation and applying the dynamical horizon equation, we get a transcendental equation (56) that is far more difficult than the one found in Sawayama’s transcendental equation (37). In the presence of the Wright omega function,\[80\] this transcendental equation (56) can only be solved numerically, not analytically.

ii) Since $M \neq 0$, the numerical solutions of the mass $m(\nu; r)$ always decrease with increasing $\nu$ but do not tend to zero. This is true for different values of $M$ and $r$, $\phi_i^2$. By analyzing the dynamical horizon equation as $\phi_i^2 \to 0^+$, we have demonstrated analytically that the black hole’s mass $m(\nu; r)$ in the K-essence emergent Schwarzschild-Vaidya spacetime constantly diminishes but does not completely evaporate.

iii) One key distinction between Vaidya spacetime and K-essence emergent Schwarzschild-Vaidya spacetime is that the mass always disappears in Vaidya spacetime, but not in K-essence emergent Schwarzschild spacetime. Nonetheless, we can witness the same trend that the black hole’s mass shrinking in all scenarios. Possible explanations for the varied outcomes include the following: in our situation, standard gravity ($g_{\mu\nu}$) is minimally coupled with the scalar filed $\phi$, which causes additional interactions (forces) and the new form of emergent spacetime, as well as changes to the associated equation of motion and geodesic equation.

iv) We have calculated the Hawking temperature using the tunneling process,\[52,60\] which is distinct from the typical Vaidya case and the result is $T_H = \frac{1}{8\pi K_2} \left[1-\phi_i^2\right]^{-1/2}$. In this work, we have used the K-essence model as purely gravitational standpoint as,\[26-28\]

v) Finally, though the cosmic implications of dark energy are important in the current scenario, the K-essence theory has received universal support as an explanation, and the $\phi_i^2$ quantity may be thought of as dark energy density in units of \text{critical density}.\[18-20\]

Acknowledgements

S.R. is thankful to the Inter-University Centre for Astronomy and Astrophysics (IUCAA), Pune, India for providing Visiting Associateship under which a part of this work was carried out who also gratefully acknowledges the facilities under ICARD, Pune at CCASS, GLA University, Mathura.

Data Availability Statement

No data associated in the manuscript.

Keywords

K-essence geometry, Hawking radiation, Vaidya spacetime

Received: May 22, 2023
E. Battaner, R. Battey, K. Benabed, A. Benolt, A. Benoit-Lévy, J.-P. Bernard, M. Bersanelli, P. Bielewicz, J. J. Bock, A. Bonaldi, L. Bonavera, J. R. Bond, J. Borrill, F. R. Bouchet, F. Boulanger, M. Bucher, C. Burigana, R. C. Butler, E. Calabrese, J.-F. Cardoso, et al., (Planck Collaboration), Astron. Astrophys. **2016**, 594, 20.

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