One and two spin–1/2 particles systems under Lorentz transformations

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Lorentz transformation (LT) is used to connect two inertia frames, including the lab and moving frames, and the effect of LT on the states of one spin–1/2 particle system is studied. Moreover, we address the predictions made by Czachor’s and the Pauli spin operators about the spin behavior and compare our results with the behavior of system’s state under Lorentz transformation. This investigation shows that the predictions made by considering the Pauli spin operator about the spin of system are in better agreement with the system’s state in comparison with that of made by considering Czachor’s spin operator. In continue, we focus on two-particle pure entangled systems including two spin–1/2 particles which moves away from each other. Once again, the behavior of this system’s states under Lorentz transformation are investigated. We also point to the behavior of Bell’s inequality, as a witness for non-locality, under Lorentz transformation. Our study shows that the Bell operator made by the Pauli operator has better consistency with the behavior of spin state of system under Lorentz transformation compared with the Bell operator made by Czachor’s operator. Our approach can be used to study the relation between the various spin operators and the effect of LT on system, which provides the basement to predict the outcome of a Stern-Gerlach type experiment in the relativistic situations.

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I. INTRODUCTION

In quantum mechanics, systems may blurt a non-local behavior from themselves [1]. Bohm and Aharanov provided a spin version for exhibiting this behavior [2]. In their setup, non-locality leads to entanglement, i.e. the state of system is not equal to the product of its constituent particles’ states [2]. Firstly, Bell tried to get a criterion for distinguishing the local and non-local phenomena from each other [3]. His work leads to a well-known inequality called the Bell inequality which may be violated by non-local states. In fact, There are various models for this inequality [3, 4]. In the two-particle systems, the Bell operator is defined as [4]

\[ B = a \otimes (b + b') + a' \otimes (b - b'), \quad (1) \]

where \((a, a')\) and \((b, b')\) are yes or no operators applying on the first and second particles, respectively. For every local state, the Bell operator meets the \(\langle B \rangle \leq 2\) condition [4].

Some forehand experimental attempts have been done to detect non-locality can be found in [3, 10]. It is shown that non-locality is not limited to the multi-particle systems and indeed, a one-particle system may also behave non-locally [11, 12]. Non-locality is a source for entropy which has vast implications in current science [3, 13]. It has also been shown that it may be a source for the entropy of horizons in the gravitational and cosmological setups [14].

Spin is a quantum mechanical property of systems which was exhibited in investigating the relativistic quantum mechanical systems. Pauli derived an operator for describing the spin of particles in the low velocity limit. By considering the low velocity limit, Pauli got \(2 \times 2\) matrixes, called the Pauli matrixes or operator \((\sigma_i)\), and the corresponding spin operator \(S_i = \frac{\hbar}{2} \sigma_i, \forall i = x, y, z\) for spin-1/2 particles [15]. Nowadays, it is believed that the predictions made by the Pauli spin operator \((S_i)\) about the spin of systems are in line with the Stern-Gerlach type experiments in the lab frame [16]. But, is it the only candidate for the spin operator which leads to the consistent results with a Stern-Gerlach type experiment in the lab frame? Moreover, what is the result of a Stern-Gerlach type experiment, if it is observed by a moving observer which moves with respect to the lab frame with a constant velocity \((\beta)\)? Indeed, there are various attempts to get a candidate for describing spin and thus the results of applying a Stern-Gerlach type experiment on a system which is in relative motion with respect to observer [17, 22].

Czachor followed the Pryce [23] and Fleming [24] arguments to get

\[ \hat{A} = \frac{(\sqrt{1 - \beta^2} \hat{A}_\perp + \hat{A}_\parallel) \cdot \vec{\sigma}}{\sqrt{1 + \beta^2(\vec{\epsilon} \cdot \hat{A})^2 - 1}}, \quad (2) \]

as the spin operator along the unit vector \(\hat{A}\), which commutes with the Hamiltonian [17]. Based on this result, this operator may be used instead of the Pauli operator along the \(\hat{A}\) vector \((\hat{A}, \vec{\sigma})\) whenever, states with zero
momenum uncertainty are taken into account. Here, \( \sigma \) and \( \hat{\sigma} \) are the Pauli operator and the unit vector along the boost direction, respectively. Additionally, \( \beta \) is the boost velocity, and, independent of \( \vec{A} \), the Pauli spin operator is recovered by substituting \( \beta = 0 \). Moreover, the subscripts \( \perp \) and \( \parallel \) denote the perpendicular and parallel components of the vector \( \vec{A} \) to the boost direction. This operator also covers the Pauli spin operator whenever either \( \vec{A}_{\parallel} = 0 \) or \( \vec{A}_{\perp} = 0 \) meaning that \( \vec{A} = \vec{A}_{\perp} \). It is worth to note that the uncertainty principle leads to \( \Delta \beta \neq 0 \) and therefore, this principle prevents such possibility in a realistic experiment. Its generalization to the wave-packets can be found in Ref. [23]. Some of the shortcomings and strengths of Czachor’s and the Pauli operators are investigated in Refs. [17, 20, 22]. Although, just the same as the Pauli operator, Czachor’s spin operator should indeed be defined as \( C = \frac{\hbar}{2} \hat{A} \) to coverer the spin-1/2 particles, but, we should note that the eigenvalues of Czachor’s spin operator are not always equal to \( \pm \frac{1}{2} \). [20, 21].

Whenever the effects of considering high velocities such as the probability of pair production are ignored, the quantum mechanical interpretations of phenomena are satisfactory and the lab frame is connected to the moving frame, which moves with a constant velocity with respect to the lab frame, by a LT [27]. Therefore, one may apply LT on the system state in the lab frame to get state seen by the moving observer. By this approach, the spin state of system is affected by a rotation with the Wigner angle [21]. The effects of LT on the single-particle entangled states are investigated by Palge et al. [29]. It is shown that such rotations may also affect the spin entropy of one spin 1/2 particle as well as the two spin 1/2 entangled particles [29, 31]. There are also various attempts in which authors investigate the behavior of non-locality under LT [32, 52]. Their results can also be used to get some theoretical predictions about the outcome of a Stern-Gerlach type experiment which may lead to get a more suitable spin operator. The acceleration effects on non-locality are also investigated in [35, 54, 62].

Some authors have used the Pauli spin operator to build the Bell operator and considered bi-partite pure entangled state [36, 37]. Thereinafter, they considered a special set of measurement directions which leads to violate Bell’s inequality to its maximum violation amount (2\( \sqrt{2} \)) in the lab frame. In addition, they have been considered a moving observer connected to the lab frame by a LT, and applied LT on the system state in the lab frame to get the corresponding state in the moving frame. They took into account the same set of measurement directions for the moving frame as the lab frame, and investigate the behavior of Bell’s inequality in the moving frame. In fact, they use Bell’s inequality as a witness for the bi-partite non-locality. Finally, they find that the violation of Bell’s inequality in the moving frame is decreased as a function of the boost velocity and the particles energy in the lab frame [36, 37]. It should be noted that if one applies LT on both of the Bell operator and the system state, Bell’s inequality is violated to the same value as the lab frame [36, 37]. The generalization of this work to three-particle non-local systems can be found in [51, 52].

In a similar approach, Ahn et al. have been considered the Bell states and used Czachor’s operator to construct the Bell operator [53]. Bearing in mind this fact that Czachor’s and the Pauli operators are the same operators in the lab frame (\( \beta = 0 \)), authors have considered the special set of spin measurements which violates Bell’s inequality to its maximum violation amount in the lab frame. They applied LT on the system state in the lab frame to get the corresponding state in the moving frame. They also assumed that the moving frame uses the same set of spin measurements as the lab frame for evaluating Bell’s inequality. Therefore, their setup has some similarity with those of Refs. [36, 37]. There are also some differences between setups investigated in these papers. Their LT differs from each other, and they used different spin operator to build the Bell operator. Finally, Ahn et al. found out that the expectation value of the Bell operator in the moving frame is decreased as a function of the boost velocity and the energy of particles in the lab frame [53]. It should be noted again that Bell’s inequality will be violated in the moving frame to the same value as the lab frame, if one applies LT on both of the Bell operator and the system state [39, 41]. More studies on this subject and its generalization to the three-particle non-local systems can be found in [41, 43, 49].

In fact, both of the mentioned approaches found out that the expectation value of the Bell operator in the moving frame is decreased by increasing the boost velocity and the energy of particles in the lab frame. Although it seems that this conclusion is a common result between the mentioned attempts, but they are completely different from each other. For example, based on the results obtained in [36, 37], Bell’s inequality, in the moving frame and the \( \beta \rightarrow 1 \) limit, is violated to its maximum violation amount for the low energy particles, whilst the results observed by Ahn et al. suggest that this inequality is preserved at this limit independent of the particles energy. However, the question is which view is correct? Here we used the Stern-Gerlach type experiment as an appropriate approach to solve this problem. Is it possible to get more theoretical information about this inconsistency appeared in these studies? Indeed, this inconsistency between the results of considering Czachor’s operator and those of considered the Pauli operator will be more complicated in the three-particle non-local systems [52]. It is reported that the Pauli operator and its corresponding spin operator lead to better agreement with the behavior of spin state of the three-particle non-local systems under LT compared with Czachor’s operator [52]. Moreover, the Pauli-Lubanski spin operator is not suitable to describe the spin interaction with a magnetic field in the moving frame connected to the lab frame by a LT [53]. Indeed, authors took into account the Pauli-Lubanski definition of spin operator and introduced a
Hamiltonian for spin interaction with magnetic field. In continue, they considered the effects of LT on the reduced spin density matrix of one spin-$\frac{1}{2}$ particle and the results of applying a Stern-Gerlach experiment on the system in various frames by focusing on the quantization axes in the various frames. Finally, they concluded that the Pauli-Lubanski spin operator (and similar operators such as Czachor’s operator) is not suitable for describing the system, which includes a one spin-$\frac{1}{2}$ particle interacting with magnetic field in the lab frame, in all inertial frames connected to each other by LT. Hence, what is the origin of these differences between the results of considering the Pauli operator and that of Czachor? Loosely speaking, one of these operators is in better agreement with the spin states of one and two-particle systems, and helps us get more suitable predictions about the results of applying a Stern-Gerlach type experiment on a system which is in relative motion with respect to observer?

In this paper, we study the differences between the results of considering the Pauli operator for describing spin and those of used Czachor’s operator to investigate spin. Unlike Ref. 53, we do not consider any magnetic field. Moreover, in order to avoid any paradoxes due to apply LT on the system, we consider a situation in which the particles momentums are specified with zero uncertainty, in the lab frame. In addition, we focus on the behavior of the system state and spin operator, and show that, even in the absence of the magnetic fields, Czachor’s spin operator is not probably suitable to describe spin. We start from the one particle system and consider a moving observer connected to the lab frame by a LT. By studying the behavior of spin state in the lab and moving frames, we try to establish a theoretical criterion to decide about the validity of spin behavior predicted by either using Czachor’s or the Pauli spin operators. In addition, we generalize our study to the two-particle non-local system. Our results indicate that the Pauli operator is in better agreement with the behavior of spin state in both of the lab and moving frames compared with Czachor’s operator.

The paper is organized as follows. In the next section, we focus on the one particle system and investigate the behavior of expectation values of Czachor’s and the Pauli spin operators under LT. We also point to the behavior of the spin state under LT, and compared the results with the behavior of expectation values of Czachor’s and the Pauli spin operators under LT to get the better spin operator. In section (III), we focus on the two-particle non-local system including two purely entangled spin-$\frac{1}{2}$ particles which moves away from each other along the z direction with the same momentum. In addition, we point to the above mentioned inconsistency and try to eliminate that by considering the behavior of the system state under LT. The last section is devoted to the summary and concluding remarks. Throughout this paper we set $c = 1$ for simplicity.

II. QUANTUM MECHANICS UNDER LT

In the lab frame ($S$) for a spin-$\frac{1}{2}$ particle, with the momentum state $|\vec{p}\rangle$ and the spin state $|\Sigma\rangle$, the state of system is written as

$$|\xi\rangle = |\vec{p}\rangle|\Sigma\rangle. \quad (3)$$

Here, we take into account that $\vec{p} = p_0\hat{z}$. The state of particle is viewed by an observer which moves along the $x$ axis ($\vec{\beta} = \beta\hat{z}$) as

$$|\xi\rangle^\Lambda = |\vec{p}\rangle^\Lambda D(W(\Lambda, p_1))|\Sigma\rangle, \quad (4)$$

where $|\vec{p}\rangle^\Lambda$ denotes the momentum state of particle in the moving frame ($S'$), and $D(W(\Lambda, p))$ is the spin-$\frac{1}{2}$ Wigner representation of the Lorentz group $\mathbb{R}$.

$$D(W(\Lambda, p)) = \cos \frac{\Omega_p}{2} - i\sigma_y \sin \frac{\Omega_p}{2}. \quad (5)$$

In this equation, $\sigma_y$ and $\Omega_p$ are the Pauli matrix and the Wigner angle, respectively, evaluated as

$$\tan \frac{\Omega_p}{2} = \frac{\sinh \alpha \sinh \delta}{\cosh \alpha + \cosh \delta} \quad (6)$$

$cosh \delta = \frac{p_m}{m}$ and $cosh \alpha = \sqrt{1 - \beta^2}$ are related to the particle energy in the lab frame and the boost effects, respectively. Therefore, simple calculations lead to

$$|+\rangle^\Lambda = \cos \frac{\Omega_p}{2} |+\rangle + \sin \frac{\Omega_p}{2} |-\rangle, \quad (7)$$

$$|\rangle^\Lambda = \cos \frac{\Omega_p}{2} |\rangle - \sin \frac{\Omega_p}{2} |+\rangle,$$

where $|+\rangle$ and $|-\rangle$ denote the up and down spin states along the $z$ direction in the lab frame, respectively. Moreover, the superscript $\Lambda$ is used to specify the corresponding spin state in the moving frame. In the $\beta \to 1$ limit, $\sin \frac{\Omega_p}{2} \sim \sqrt{\frac{\Gamma_1 - \beta_1}{2}}$ where $\Gamma = \sqrt{1 - \beta_1^2}$ and $\beta_1$ are the energy factor and the velocity of particle in the lab frame, respectively. Using Eq. (7) to get

$$|+\rangle^\Lambda \approx |+\rangle, \quad \approx |\rangle, \quad (8)$$

for a low energy particle ($\Gamma \to 1$) and

$$|+\rangle^\Lambda \approx \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle), \quad (9)$$

$$|\rangle^\Lambda \approx \frac{1}{\sqrt{2}}(|-\rangle - |+\rangle),$$

for a high energy particle ($\Gamma \to \infty$). Now, consider a situation in which the lab and moving observers apply a Stern-Gerlach experiment in the same direction $\vec{A} = (0, 0, \frac{A}{2})$, while, $|+\rangle$ is the spin state of particle in the lab frame. Therefore, the spin state of system in the
moving frame can be found in Eq. \(7\). Let us focus on the results obtained by taking into account Czachor’s and the Pauli spin operators. If the lab and moving frames use the Pauli spin operator and the same spin measurement direction (\(\vec{A}\)), simple calculations lead to

\[
\langle S \rangle_1 = \frac{\hbar}{2\sqrt{2}},
\]

(10)

and

\[
\langle S \rangle_2 = \frac{\hbar}{2\sqrt{2}}(\cos \Omega_p + \sin \Omega_p),
\]

(11)

for the lab and moving frames, respectively. Here, the subscript 1 and 2 are also used to denote the lab and moving frames, respectively. In addition, \(S = \frac{\hbar}{2}(\vec{A}, \vec{\sigma}) = \frac{\hbar}{2\sqrt{2}}(\sigma_x + \sigma_z)\) is the spin operator along the \(\vec{A}\) direction.

It is easy to check that Eq. (11) is in line with the asymptotic behavior explained in Eqs. \(8\) and \(9\). It is useful to mention here that, in the \(\beta \to 1\) limit, Eq. (11) leads to \(\frac{\hbar}{2\sqrt{2}}\) for both of the low \((\Omega_p \sim 0)\) and high \((\Omega_p \sim \frac{\pi}{2})\) energy particles. This result is in agreement with the asymptotic behaviors addressed in Eqs. \(8\) and \(9\).

If Czachor’s spin operator together with the vector \(\vec{A} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})\) are considered, one can use Eq. 2 in order to evaluate the spin operator in the lab and moving frame as

\[
C_1 = \frac{\hbar}{2\sqrt{2}}(\sigma_x + \sigma_z),
\]

(12)

and

\[
C_2 = \frac{\hbar}{2}(\sigma_x + \sqrt{1 - \beta^2} \sigma_z),
\]

(13)

respectively. In these equations, the subscripts 1 and 2 denote the lab and moving frames, respectively. Since \(|+\rangle\) is the spin state of particle in the lab frame, by using Eq. (12) we get

\[
\langle C_1 \rangle = \frac{\hbar}{2\sqrt{2}},
\]

(14)

which is the same as the previous results obtained by considering the Pauli spin operator. For the moving frame, using Eqs. (7) and (13) to obtain

\[
\langle C_2 \rangle = \frac{\hbar}{2\sqrt{2} - \beta^2}(\sqrt{1 - \beta^2} \cos \Omega_p + \sin \Omega_p),
\]

(15)

which differs from the result obtained by using the Pauli operator, Eq. (11). But which approach is right? In order to check this prediction, we focus on the \(\beta \to 1\) limit. For the low energy particle \(\Omega_p \to 0\) and we get \(\langle C_2 \rangle \to 0\) which is fully inconsistent with Eq. \(8\). Additionally, considering \(\Omega_p \to \frac{\pi}{2}\) leads to \(\langle C_2 \rangle \to \frac{\hbar}{2}\) which is again in contrast with the result predicted by Eq. (11). All in all, using both of these operators lead to the same predictions for the spin in the lab frame but, in the moving frame the results obtained using Czachor’s spin operator differ from that of considered the Pauli spin operator. Our approach shows that the predictions made by considering the Pauli spin operator is in agreement with the behavior of the spin state in the lab and moving frames. The same conclusion is not accessible by considering Czachor’s spin operator which indicates that our approach is in full agreement with previous results obtained in \([53]\).

We should note that the Stern-Gerlach type experiment is required to identify the correct result between Eq. (11) or Eq. (15). Based on Eq. (2), in the moving frame, \(C = \frac{1}{2}A = \frac{1}{2}(\vec{A}, \vec{\sigma}) = \frac{1}{2}\vec{S}\) under one of the following conditions \(\vec{e}.\vec{A} = 0\) \((\vec{A}_y = 0)\) or \(\vec{e}.\vec{A} = 1\) \((\vec{A}_\perp = 0)\). Under this condition Czachor’s operator is the same as the Pauli operator and the results lead to the same predictions in the lab and moving frames.

### III. Pure Bi-Partite Entangled States Under LT

In order to investigate the LT effects on the pure bi-partite non-locality, the above arguments is required to be generalized to the two-particle system. This generalization is as follows. For a system, including two spin-\(\frac{1}{2}\) particles, in the lab frame \((S)\), with the spin state \(|\psi\rangle\) and the momentum state \(|\vec{p}_1\vec{p}_2\rangle\), the state of system is

\[
|\xi\rangle = |\vec{p}_1\vec{p}_2\rangle|\psi\rangle.
\]

(16)

Now, consider a moving frame \((S')\) which moves along the \(x\) axis \((\vec{\beta} = \beta\vec{e}_x)\). In the \(S'\) frame, the state of system is

\[
|\xi\rangle^A = |\vec{p}_1\vec{p}_2\rangle^A \prod_{i=1}^{2} D(W(\Lambda, p_i))|\psi\rangle.
\]

(17)

\(|\vec{p}_1\vec{p}_2\rangle^A\) denotes the momentum state of system in the moving frame, and \(D(W(\Lambda, p_i))\) is the spin-\(\frac{1}{2}\) Wigner representation of the Lorentz group for the \(i\)th particle Eq. 3. Consider a system, including two particles which moves away from each other along the \(z\) direction in the lab frame, with the total state as

\[
|\xi\rangle = |\vec{p}_1\vec{p}_2\rangle \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle),
\]

(18)

where \(\vec{p}_1 = -\vec{p}_2 = p\vec{e}_z\). If the moving frame considers the same basis as the lab frame, and applying LT on the system state, the system state for the moving frame is given by \([38, 40]\)

\[
|\xi\rangle^A = \frac{|\vec{p}_1\vec{p}_2\rangle^A}{\sqrt{2}}(\cos \Omega_p(|++\rangle + |--\rangle) - \sin \Omega_p(|-+\rangle - |+-\rangle)).
\]

(19)
The maximum violation of Bell’s inequality in the lab frame \((\langle B \rangle = 2\sqrt{2})\) is obtainable by choosing the
\[
\vec{a} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \quad \vec{a}' = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right), \quad (20)
\]
and
\[
\vec{b} = (0, 1, 0), \quad \vec{b}' = (1, 0, 0), \quad (21)
\]
for the directions of Pauli’s operators applying on the first and second particles, respectively \([38, 39, 40]\). As noted in the introduction, since \(\beta\) meets the \(\beta = 0\) condition in the lab frame, Czachor’s operator is compatible with the Pauli operator in the lab frame. Therefore, this result is also obtainable in the lab frame if Czachor’s operator is used to build the Bell operator. Let us focus on the moving observer. If the moving observer uses Czachor’s operator and the special set of measurement directions, given in Eqs. (20) and (21), then \([38]\)

\[
\langle B_C \rangle = \frac{2}{\sqrt{2 - \beta^2}}(\sqrt{1 - \beta^2} + \cos \Omega_p), \quad (22)
\]

here \(C\) index indicates the consideration of Czachor’s operator. The result of lab frame is obtainable by inserting \(\beta = 0\) and \(\Omega_p = 0\) simultaneously. It is obvious that \(\langle B_C \rangle \leq 2\) for the \(\beta \to 1\) limit, which means that Bell’s inequality is preserved in the moving frame and is independent from the particles energy in the lab frame. This behavior indicates that non-locality is vanished in this limit if the moving frame uses the same set of measurements as the lab frame violating the Bell inequality to its maximum violation amount in the lab frame \([38]\). From Eq. (19), we see that, in the \(\beta \to 1\) limit and for the low energy particles system \((\Omega_p \sim 0)\),

\[
|\xi\rangle^A \sim \left|\vec{p}_1\vec{p}_2\right|^A \left|\sqrt{2}(|+\rangle - |\rangle - \rangle)\right)\text{ is the same as the system state in the lab frame. Therefore, the bipartite non-locality does not completely disappear at this limit. We believe a true Bell operator should show this behavior. Therefore, we see that, once again, Czachor’s operator does not lead to the results compatible with the behavior of the spin state of system. This weakness of Czachor’s operator was also reported in the multi-particle non-local systems \([52]\). Using the Pauli operators to form the Bell operator in the moving frame, and applying that on the spin state of system, we get

\[
\langle B_l \rangle = 2\sqrt{2}\cos^2 \Omega_p, \quad (23)
\]

here \(l\) index indicates that the Pauli operators are used to build the Bell operator. Moreover, we took into account the special directions explained in Eqs. (20) and (21). \(\langle B_l \rangle\) is displayed in Fig. (1). In the non-relativistic limit \((\Omega_p \to 0)\), the result of lab frame is obtainable. In addition, for the low energy particles in the \(\beta \to 1\) limit, \(\Omega_p \to 0\) and therefore, Bell’s inequality is maximally violated which is in agreement with the asymptotic behavior of the system state. For the high energy particles in the \(\beta \to 1\) limit, \(\Omega_p \to \frac{\pi}{2}\) which leads to

\[
|\xi\rangle^A \sim \left|\vec{p}_1\vec{p}_2\right|^A \left|\sqrt{2}|(+\rangle - |\rangle - \rangle)\right). \quad (24)
\]

where again \(\vec{p}_1 = -\vec{p}_2 = p\vec{e}^\beta\). This state violates Bell’s inequality to its maximum violation amount \((2\sqrt{2})\) in the lab frame by choosing \(\vec{a} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)\), \(\vec{a}' = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)\), \(\vec{b} = (0, 1, 0)\) and \(\vec{b}' = (1, 0, 0)\) \([38, 39, 40]\). In the moving frame, this state is given as \([38]\)

\[
|\xi\rangle^A = \left|\vec{p}_1\vec{p}_2\right|^A \left|\sqrt{2}|(+\rangle + |+\rangle\right), \quad (25)
\]

which means that the considered LT leaves this state unchanged. Therefore, it is crystal clear that when the moving observer uses the Pauli spin operator and the same set of measurements as the lab frame to get the Bell operator and Bell’s inequality, Bell’s inequality is violated to the same value as the lab frame. The latter means that, in the moving frame, this inequality is violated to
its maximum violation amount \( (2\sqrt{2}) \) in this situation. If the moving observer uses Czachor’s operator to construct the Bell operator and investigates the behavior of Bell’s inequality and thus the corresponding non-locality, then

\[
\langle B_C \rangle = \frac{2}{\sqrt{2 - \beta^2}} (\sqrt{1 - \beta^2} + 1),
\]

which claims that non-locality is decreased by increasing the boost velocity and thus, in the \( \beta \to 1 \) limit, Bell’s inequality is marginally satisfied in the moving frame \[38\]. This result is in contrast with the invariant form of this state under LT Eq. \([24]\) and the results made by considering the Pauli operator. Once again, It is figured out that, whenever the lab and moving frames use the same set of measurements which violate Bell’s inequality to its maximum violation amount in the lab frame, the behavior of the bi-partite pure entangled state \([24]\) under LT is fully consistent with the behavior of the Bell operator under LT if the Pauli operator applied to form the Bell operator. Loosely speaking, the same as the results of the previous section and the three-particle non-local systems \([52]\), the predictions of Czachor’s spin operator about the spin differ from those of the Pauli operator and the system state. It is useful to note here that a Stern-Gerlach type experiment is needed to experimentally distinguish these results. Finally, we should note that since LT introduced in this paper is a unitary operator, it should be possible to get the same violation amount as the lab frame for Bell’s inequality in the moving frame. Indeed, if one applies LT on both of the system state and the Bell operator, Bell’s inequality is also maximally violated in the moving frame \([38\), 37, 41, 52].

IV. SUMMARY AND CONCLUSION

Spin is a quantum mechanical property of systems. It has vast implications in the spectroscopy, quantum information theory and etc. Therefore, it is necessary to find a suitable operator to describe this property. Indeed, there are various operators suggested for this aim \([17, 20–22]\). We believe that our approach potentially can be used to study the relation between the various spin operators and the effects of LT on the system state, provides a frame to get some predictions about the outcomes of a Stern-Gerlach type experiment in the relativistic situations. Here, we focused on the two spin operators, including Czachor’s and the Pauli spin operators. Firstly, we saw that Czachor’s operator is in agreement with the Pauli operator in the limit of low velocity, both of them predict the same outcome for a Stern-Gerlach type experiment, applied on a spin–\( \frac{1}{2} \) particle, in the lab frame. In continue, we considered a moving frame which moves along the \( z \) direction with the \( \frac{1}{2} \) particle, in the lab frame. It means that we discard the relativistic effects such as the pair productions, and in fact, one should use relativistic quantum mechanics or quantum field theory to get a more precise results \([52, 53]\). Moreover, the lab and moving frames use the same set of measurement directions in our setup. We found that the Pauli spin operator predictions about the spin of particle are compatible with the behavior of the system state in the moving frame, whiles, Czachor’s spin operator predictions differ from those of the Pauli spin operator and the behavior of system state under LT. In addition, we focused on the two purely bi-partite entangled states, known as the Bell states, which include two spin–\( \frac{1}{2} \) particles moving away from each other along the \( z \) direction with the same momentum in the lab frame. Bearing LT in mind, we evaluated the corresponding system states in the previously mentioned moving frame. Thereinafter, we used the Pauli operator to construct the Bell operator and the special set of measurement directions which violates Bell’s inequality in the lab frame to its maximum violation amount \( (2\sqrt{2}) \). In continue, by taking into account the same directions as the lab frame for the Bell operator in the moving frame, we have investigated the expectation value of the Bell operator in the moving frame. We saw that, for a Bell state introduced in Eq. \([13]\), the expectation value of the Bell operator in the moving frame is decreased as a function of the boost velocity together with the energy of particles in the lab frame. It is also found that, for particles with low energy in the lab frame, Bell’s inequality in the moving frame is violated to the same value as the lab frame (the maximum violation amount) in the \( \beta \to 1 \) limit, which is the same as the lab frame. We also addressed another Bell state Eq. \([24]\) which is invariant under LT, and found out that the expectation of the Bell operator is also invariant under LT which is in agreement with the behavior of the system state. In addition, the same as the one-particle system, our study shows that the predictions of the Pauli spin operator are in line with the behavior of system state in both of the lab and moving frames. We have also pointed to the results obtained by considering Czachor’s operator and compared them with ours \([38]\). Finally, we found out that the Pauli spin operators are in better agreement with the behavior of spin system in these situations. Our study helps us make more clear the origin of differences between the predictions about the spin behavior made by considering Czachor’s and the Pauli spin operators. It is useful to note that although our results are in agreement with those of three-particle non-local systems \([52]\), but a Stern-Gerlach type experiment is needed to get a decision about the quality of validity of these results in nature.

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