Invariance in adelic quantum mechanics

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Abstract

Adelic quantum mechanics is form invariant under an interchange of real and \( p \)-adic number fields as well as rings of \( p \)-adic integers. We also show that in adelic quantum mechanics Feynman’s path integrals for quadratic actions with rational coefficients are invariant under changes of their entries within nonzero rational numbers.

1 Introduction

Invariance of basic theoretical objects under some transformations has played very important role in developments of quantum theory. Quantum models invariance is usually related to transformations in Archimedean spaces characterized by real or complex numbers. However, since 1987 many models have been constructed over \( p \)-adic and adelic spaces. In particular, much attention has been paid to \( p \)-adic and adelic strings (see reviews in [1] and [2]), \( p \)-adic [3] and adelic [4, 5] quantum mechanics (see a review in [6]), \( p \)-adic and adelic quantum cosmology [7], and to some systems which spaces of states exhibit ultrametric hierarchical structures (see reviews [8], [9] and [10]). The present status of application of \( p \)-adic numbers in physics and related branches of sciences is reflected in the recent proceedings [11].

There are many mathematical and physical motivations [6] to employ \( p \)-adic numbers and non-Archimedean geometry in modern mathematical physics. \( p \)-Adic and conventional models are connected by the corresponding adelic models. Adelic quantum mechanics (AQM) contains ordinary and \( p \)-adic quantum mechanics in a natural way. It presents mathematically more complete and theoretically more profound approach to quantum phenomena. Many aspects of AQM have been investigated in detail [6], but a systematic analysis of symmetries was not considered.

This paper contains a brief presentation of results obtained in a recent analysis of some invariance of AQM.
2 p-Adic numbers and adeles

p-Adic numbers are discovered at the end of the 19th century by German mathematician Kurt Hensel. They can be obtained by completion of the field \( \mathbb{Q} \) of rational numbers in the same way as the field \( \mathbb{R} \) of real numbers, but using \( p \)-adic absolute value instead of the ordinary one. \( p \)-Adic absolute value satisfies strong triangle inequality \( |x + y|_p \leq \max\{ |x|_p, |y|_p \} \) and consequently it belongs to non-Archimedean (ultrametric) norm. According to the Ostrowski theorem, \( \mathbb{R} \) and all fields \( \mathbb{Q}_p \) of \( p \)-adic numbers, where \( p \) is a prime number, exhaust all number fields which can be obtained by completions of \( \mathbb{Q} \). In other words, only \( \mathbb{R} \) and \( \mathbb{Q}_p \), for every \( p \), contain \( \mathbb{Q} \) as a dense subfield.

Recall that a real number can be presented in the form

\[
x = \pm 10^n \sum_{k=0}^{+\infty} a_k 10^k, \quad a_k \in \{0, 1, \ldots, 9\}, \quad a_0 \neq 0, \quad n \in \mathbb{Z},
\]

where \( \mathbb{Z} \) is the ring of rational integers. A \( p \)-adic number has a unique expansion

\[
y = p^m \sum_{k=0}^{+\infty} b_k p^k, \quad b_k \in \{0, 1, \ldots, p-1\}, \quad b_0 \neq 0, \quad m \in \mathbb{Z}.
\]

It is evident that (2) is not convergent with respect to the usual absolute value, but it becomes quite convergent applying \( p \)-adic absolute value.

Due to ultrametricity, \( p \)-adic spaces have many properties rather different from the real ones. For an introductory course to \( p \)-adic numbers and \( p \)-adic analysis, one can see [12], [1] and [2].

To consider real and \( p \)-adic numbers simultaneously and on an equal footing one uses space \( \mathbb{A} \) of adeles. An adele \( x \) (see, e.g. [13] and [2]) is an infinite sequence

\[
x = (x_{\infty}, x_2, \ldots, x_p, \ldots),
\]

where \( x_{\infty} \in \mathbb{R} \) and \( x_p \in \mathbb{Q}_p \) with the restriction that for all but a finite set \( \mathcal{P} \) of primes \( p \) one has \( x_p \in \mathbb{Z}_p = \{ y \in \mathbb{Q}_p : |y|_p \leq 1 \} \). Componentwise addition and multiplication endow the ring structure to \( \mathbb{A} \) and it can be presented in the following form:

\[
\mathbb{A} = \bigcup_{\mathcal{P}} A(\mathcal{P}), \quad A(\mathcal{P}) = \mathbb{R} \times \prod_{p \in \mathcal{P}} \mathbb{Q}_p \times \prod_{p \notin \mathcal{P}} \mathbb{Z}_p.
\]

A multiplicative group of ideles \( \mathbb{I} \subset \mathbb{A} \) has elements \( x = (x_{\infty}, x_2, \ldots, x_p, \ldots) \), where \( x_{\infty} \in \mathbb{R}^* = \mathbb{R} \setminus \{0\} \) and \( x_p \in \mathbb{Q}_p^* = \mathbb{Q}_p \setminus \{0\} \) with the restriction that for all but a finite set \( \mathcal{P} \) one has that \( x_p \in U_p = \{ y \in \mathbb{Q}_p : |y|_p = 1 \} \). Thus the whole set of ideles is

\[
\mathbb{I} = \bigcup_{\mathcal{P}} I(\mathcal{P}), \quad I(\mathcal{P}) = \mathbb{R}^* \times \prod_{p \in \mathcal{P}} \mathbb{Q}_p^* \times \prod_{p \notin \mathcal{P}} U_p.
\]
A principal adele (idele) is a sequence \((x, x, \cdots, x, \cdots) \in \mathbb{A}\), where \(x \in \mathbb{Q}\) (\(x \in \mathbb{Q}^* = \mathbb{Q} \setminus \{0\}\)). \(\mathbb{Q}\) and \(\mathbb{Q}^*\) are naturally embedded in \(\mathbb{A}\) and \(I\), respectively.

An additive character on \(\mathbb{A}\) is
\[
\chi(x) = \chi_{\infty}(x_{\infty}) \prod_p \chi_p(x_p) = \exp(-2\pi i x_{\infty}) \prod_p \exp(2\pi i \{x_p\}_p), \quad x \in \mathbb{A}, \tag{6}
\]
where \(\{x_p\}_p\) is the fractional part of \(x_p\). A multiplicative character on \(\mathbb{I}\) is
\[
|x|^s = |x_{\infty}|^s \prod_p |x_p|^s, \quad s \in \mathbb{C}, \quad x \in \mathbb{I}, \tag{7}
\]
where \(\mathbb{C}\) is the field of complex numbers and \(|\cdot|_\infty\) denotes standard absolute value. One can easily see that only finitely many factors in (6) and (7) are different from unity.

### 3 Adelic quantum mechanics

Adelic quantum mechanics (see, e.g. [4], [5], [6]) can be defined as a triple \((L_2(\mathbb{A}), W(z), U(t))\), where \(L_2(\mathbb{A})\) is the Hilbert space of complex-valued square integrable functions with respect to the Haar measure on \(\mathbb{A}\), \(W(z)\) is a unitary representation of the Heisenberg-Weyl group on \(L_2(\mathbb{A})\), and \(U(t)\) is a unitary representation of the evolution operator on \(L_2(\mathbb{A})\).

A basis of \(L_2(\mathbb{A})\) can be the set of orthonormal eigenfunctions in a spectral problem of the evolution operator \(U(t), t \in \mathbb{A}\). Such eigenfunctions have the form
\[
\psi_p(x, t) = \psi_{\infty}(x_{\infty}, t_{\infty}) \prod_{p \not\in P} \psi_p(x_p, t_p) \prod_{p \in P} \Omega(|x_p|_p), \quad x, t \in \mathbb{A}, \tag{8}
\]
where \(\psi_{\infty} \in L_2(\mathbb{R})\) and \(\psi_p \in L_2(\mathbb{Q}_p)\) are eigenfunctions in the ordinary and \(p\)-adic cases, respectively. Eigenfunction \(\Omega(|x_p|_p)\) is defined as a characteristic function of \(\mathbb{Z}_p\), i.e.
\[
\Omega(|x_p|_p) = \begin{cases} 
1, & |x_p|_p \leq 1, \\
0, & |x_p|_p > 1.
\end{cases} \tag{9}
\]
This \(\Omega(|x_p|_p)\) provides convergence of the infinite product in (8) and is invariant under transformation of \(p\)-adic evolution operator \(U(t_p)\).

In AQM quantization performs according to the Weyl procedure. An adelic evolution operator is defined [6] by
\[
U(t'') \psi(x'') = \int_\mathbb{A} \mathcal{K}(x'', t''; x', t') \psi(x', t') dx' = \prod_v \int_{\mathbb{Q}_v} \mathcal{K}_v(x''_v, t''_v; x'_v, t'_v) \psi_v(x'_v, t'_v) dx'_v, \tag{10}
\]
where \(v = \infty, 2, 3, \cdots, p, \cdots\). The eigenvalue problem for \(U(t)\) reads
\[
U(t) \psi_P(x) = \chi(E t) \psi_P(x), \quad x, t, E \in \mathbb{A}, \tag{11}
\]
where $\psi_P(x)$ are adelic eigenfunctions. Since all information on quantum dynamics may be derived from the kernels $K_v(x''; x', t')$ they can be regarded as basic ingredients of AQM. It is natural to evaluate $K_v(x''; x', t')$ by Feynman’s path integral, which for quadratic systems is defined by

$$K_v(x''; x', t') = \int \chi_v\left(-\frac{1}{\hbar} \int_{t'_v}^{t''_v} L(\dot{q}, q, t) dt\right) D_v q.$$  \hfill (12)

As a result of adelic approach and $p$-adic effects in AQM one obtains discreteness of space at the characteristic (Planck) scale. In the limit of large distances, AQM effectively becomes the ordinary one. AQM may be regarded as a starting point for more complete quantum cosmology, quantum field theory and string/M-theory.

3.1 Adelic form invariance

In mathematical models of physical systems number sets which have field structure play a very important role. In 1987 Volovich suggested that a fundamental physical theory should be formulated in such way that it is invariant under change of any number field. So far it has not been constructed such number field invariant quantum model. Especially it seems difficult to have a theory simultaneously invariant on fields which are completions of $\mathbb{Q}$ and finite Galois fields. However if we require invariance only with respect to $\mathbb{R}$ and $\mathbb{Q}_p$ then adelic quantum mechanics satisfies such requirement.

To illustrate this kind of invariance let us consider exact expression for kernel for quantum-mechanical systems with quadratic Lagrangian

$$L(\dot{q}, q, t) = \frac{1}{2} A(t) \dot{q}^2 + B(t) \dot{q} q + \frac{1}{2} C(t) q^2 + D(t) \dot{q} + E(t) q + F(t),$$  \hfill (13)

where $A(t), \ldots, F(t)$ are some analytic functions of the time $t$ with rational coefficients. It is worth noting that (13) may be regarded as real as $p$-adic and it yields quadratic classical action $\bar{S}(x'', t''; x', t')$. The corresponding $v$-adic kernel is

$$K_v(x''; x', t') = \lambda_v \left(-\frac{1}{2\hbar} \frac{\partial^2}{\partial x'' \partial x'} \bar{S}\right) \left[\frac{1}{\hbar} \frac{\partial^2}{\partial x'' \partial x'} \bar{S}\right]^{\frac{1}{2}} v \chi_v \left(-\frac{1}{\hbar} \bar{S}(x'', t''; x', t')\right),$$  \hfill (14)

where we omitted index $v$ in arguments because it is understood. Arithmetic functions $\lambda_v$ are well defined complex-valued functions (see, e.g. [1]).

It is now easy to observe that $K_v(x'', t''; x', t')$ in (14) has the same form for any $v = \infty, 2, 3, \ldots, p, \ldots$, i.e. it is form invariant under interchange of any two values of index $v$, which characterize number fields $\mathbb{R} \equiv \mathbb{Q}_\infty$ and $\mathbb{Q}_p$.

Also the space of adeles $\mathbb{A}$ is defined in $v$-adic number field invariant way. Moreover, we see here that not only fields of numbers are important but also rings $\mathbb{Z}_p$ of $p$-adic integers. Thus adelic space is also form invariant under interchange of any two rings $\mathbb{Z}_p$ and $\mathbb{Z}_p''$, as well as under interchange of $\mathbb{Q}_p$ and $\mathbb{Z}_p''$. It follows that in adelic theory $\mathbb{Z}_p$ is not less important than $\mathbb{Q}_p$.
Eigenfunctions (8) make also an example of adelic form invariance. $\Omega(|x|_p)$ functions are a consequence of $\mathbb{Z}_p$ structure of $\mathbb{A}$ and they are necessary ingredient of AQM.

### 3.2 Adelic rational invariance

Now we are interested in existence of adelic quantities which are invariant under some adelic transformations. If we restrict transformations to principal adeles or principal ideles then there exist such invariant quantities. Since principal adeles and ideles are related to rational numbers we shall call this kind of invariance \textit{adelic rational invariance} (ARI). The simplest case of ARI gives the following example of adelic multiplicative character (7)

$$|x| = |x|_\infty \prod_p |x|_p = 1, \quad x \in \mathbb{Q}^*,$$

which is valued for any rational $x \neq 0$. One can easily show validity of (15) as well as its extension to $|x|^s = 1$ if $s \in \mathbb{C}$ and $x \in \mathbb{Q}^*$.

The next example of ARI is related to additive character (6). Namely

$$\chi(x) = \chi_\infty(x) \prod_p \chi_p(x) = \exp(-2\pi i x) \prod_p \exp(2\pi i \{x\}_p) = 1, \quad x \in \mathbb{Q}.$$  

One can show that adelic kernel

$$K(x'', t''; x', t') = \prod_v K_v(x'', t''; x', t') = 1, \quad x'', x', t'', t' \in \mathbb{Q}^*,$$

where $K_v(x'', t''; x', t')$ is given in (14), if classical action $\bar{S}(x'', t''; x', t')$ is rational function of its arguments. Proof that $\prod_v \lambda_v(x) = 1$ if $x \in \mathbb{Q}^*$ can be found in [1].

It is worth noting that infinite products in (15), (16) and (17) are equal to unity. They connect real and $p$-adic counterparts of the same quantity. As a result, real quantity of rational arguments can be expressed as product of all inverse $p$-adic analogues.

### 4 Conclusion

We presented some results on two kinds of symmetries in adelic quantum mechanics: form invariance and rational invariance. In addition to number fields it is pointed out importance of the rings of $p$-adic integers. These adelic quantum symmetries should stimulate further developments in adelic approach to QFT and string/M-theory.

Acknowledgement. The work on this article was partially supported by the Ministry of Science and Environmental Protection, Serbia, under contract No 144032D.
References

[1] V.S. Vladimirov, I.V. Volovich and E.I. Zelenov: *p-Adic Analysis and Mathematical Physics*, World Scientific, Singapore 1994.

[2] L. Brekke and P.G.O. Freund: Phys. Rep. **233** (1993) 1.

[3] V.S. Vladimirov and I.V. Volovich: Commun. Math. Phys. **123** (1989) 659.

[4] B. Dragovich: Theor. Math Phys. **101** (1994) 1404; hep-th/0402193.

[5] B. Dragovich: Int. J. Mod. Phys. A **10** (1995) 2349; hep-th/0404160.

[6] B. Dragovich: in *Proc. V.A. Steklov Inst. Math.*, Vol. 245 (Eds. I.V. Volovich, M.O. Katanaev, B. Dragovich and S.V. Kozyrev), Nauka Publ., Moscow 2004, p. 72; hep-th/0312046.

[7] G.S. Djordjević, B. Dragovich, Lj. Nešić and I.V. Volovich: Int. J. Mod. Phys. A **17** (2002) 1413; gr-qc/0105050.

[8] R. Rammal, G. Toulouse and M.A. Virasoro: Rev. Mod. Phys. **58** (1986) 765.

[9] A.Yu. Khrennikov and M. Nilsson: *p-Adic Deterministic and Random Dynamics*, Kluwer Acad. Publishers, Dordrecht, 2004.

[10] S.V. Kozyrev: in *p-Adic Mathematical Physics*, AIP Conference Proceedings, Vol. 826 (Eds. A.Yu. Khrennikov, Z. Rakić and I.V. Volovich), New York, 2006, p. 121.

[11] 2nd International Conference on *p-Adic Mathematical Physics* (Belgrade, 15-21.09.2005): *p-Adic Mathematical Physics*, AIP Conference Proceedings, Vol. 826 (Eds. A.Yu. Khrennikov, Z. Rakić and I.V. Volovich), New York, 2006.

[12] W.H. Schikhof: *Ultrametric Calculus: An introduction to p-adic analysis*, Cambridge Univ. Press, Cambridge, 1984.

[13] I.M. Gel’fand, M.I. Graev and I.I. Piatetskii-Shapiro: *Representation Theory and Automorphic Functions*, Nauka, Moscow, 1966 (in Russian).

[14] I.V. Volovich: *Number Theory as the Ultimate Physical Theory*, CERN preprint, CERN-Th. 4781/87.

[15] G.S. Djordjević, B. Dragovich and L. Nešić: Inf. Dim. Anal. Quan. Probab. and Rel. Topics **6** (2003) 179; hep-th/0105030.