General plane wave mode functions for scalar-driven cosmology

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We give a solution for plane wave scalar, vector and tensor mode functions in the presence of any homogeneous, isotropic and spatially flat cosmology which is driven by a single, minimally coupled scalar. The solution is obtained by rescaling the various mode functions so that they reduce, with a suitable scale factor and a suitable time variable, to those of a massless, minimally coupled scalar. We then express the general solution in terms of co-moving time and the original scale factor.

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1. Introduction: On the largest scales the universe is homogeneous and isotropic [1]. When results from the WMAP satellite are combined with other data the energy density of the universe as a fraction of the critical density is found to be $\Omega_{\text{tot}} = 1.02 \pm 0.02$ [2]. This is consistent with the high degree of spatial flatness predicted by most models of inflation [3]. It is therefore safe to assume that the invariant element relevant to cosmology takes the form,

$$ds^2 = -dt^2 + a^2(t)d\vec{x} \cdot d\vec{x}.$$  \hspace{0.5cm} (1)

The scale factor $a(t)$ is not observable but the ratio of its current value to its value at past time $t$ gives the cosmological redshift (plus one) experienced by light emitted at that time and received now,

$$z \equiv \frac{a_0}{a(t)} - 1.$$ \hspace{0.5cm} (2)

It is common to use redshift as a time variable in cosmology, even for epochs from which we detect no radiation. For example, the cosmic microwave radiation was emitted within about 100 redshifts before and after $z_{\text{dec}} = 1089$, and the transition from radiation domination to matter domination occurred at $z_{\text{eq}} \sim 3200$ [2]. Primordial inflation is conjectured to have ended at $z \gtrsim e^{50}$.

The scale factor’s logarithmic derivative defines the Hubble parameter,

$$H(t) \equiv \frac{\dot{a}}{a}.$$ \hspace{0.5cm} (3)

Its physical meaning is the rate at which distant matter is receding due to the expansion of the universe. Combining WMAP with other data sets determines its current value to be, $H_0 = (71^{+4}_{-3}) \text{ km s}^{-1} \text{ Mpc}^{-1}$ [2]. The weak energy condition implies that the Hubble parameter can never increase. One typically assumes it has fallen to $H_0$ from an initial period of inflation during which it was up to 55 orders of magnitude larger.

The scale factor’s second time derivative defines the deceleration parameter,

$$q(t) \equiv \frac{\ddot{a}}{a^2} = -1 - \frac{\dot{H}}{H^2}.$$ \hspace{0.5cm} (4)

It is less well measured than $H(t)$, but Hubble plots of Type Ia supernovae at high redshift are consistent with a current value of $q_0 \sim -0.6$ [4]. This phase of negative deceleration is a relatively late phenomenon, having set in about $z \sim 1$. Before this it was near the value $q = +1$ characteristic of a matter dominated universe. At redshifts much larger than $z_{eq} \sim 3200$ the deceleration parameter was near the value $q = +1$ of a perfectly radiation dominated universe. And during the conjectured epoch of primordial inflation $(z \gtrsim e^{50})$ it would have been near the value $q = -1$.

The fact that the deceleration parameter has changed so much over the course of cosmic history is frustrating because many of the phenomena which we can observe from the epoch of matter domination $(q = +\frac{1}{2})$ are believed to have had their origins in quantum fluctuations during the epoch of primordial inflation $(q = -1)$ [3]. The various effects can be understood using free quantum field theory but the mode functions are known only for the case of constant $q(t)$. Hence one must employ either perturbative or numerical methods to connect the inflationary mode functions — whose normalization is fixed — with their matter dominated descendants whose effects can be observed [4, 2, 5].

The purpose of this paper is to present an exact solution for the scalar, vector and tensor mode functions of gravity plus a single, minimally coupled scalar with any sort of potential. That we can do this derives from previous work [2] in which an exact solution was obtained, for any $a(t)$, for the mode functions of a massless, minimally coupled scalar,

$$\frac{\partial^2 u(t,k)}{\partial t^2} + H^2(t) \left[ a^2(t,k) + \frac{3}{2} q(t) - \frac{3}{4} \right] u(t,k) = 0.$$ \hspace{0.5cm} (5)

The dimensionless parameter $x(t,k) \equiv k/aH$ is the physical wave number expressed in Hubble units. We seek a
solution \( (6) \) which is conventionally normalized,
\[
 u(t, k)u^*(t, k) - \dot{u}(t, k)u^*(t, k) = i \, , \tag{6}
\]
and initially (at \( t = t_i \)) positive frequency. The desired solution takes the form of a row vector containing the two Bessel functions of order \( \nu(t) \equiv \frac{1}{2}q^2(t) \), which comprise (most of) the solution for constant \( q(t) \), multiplied into a \( 2 \times 2 \) matrix that mixes the two solutions whenever \( q(t) \) evolves,
\[
 u(t, k) = \sqrt{\frac{\pi a}{2k}} \left( -\frac{x}{2q} \right)^{\frac{1}{2}} \left( -\text{icsc}(\nu\pi)J_\nu(-\frac{x}{2q}) - J_{\nu}(-\frac{x}{2q}) \right)
\times \mathcal{M}(t, t_i, k) \left( 1 + i\cot[\nu(t_i)\pi] \right). \tag{7}
\]
The transfer matrix \( \mathcal{M}(t, t_i, k) \) is the time-ordered product of the exponential of a line integral,
\[
 \mathcal{M}(t, t_i, k) = P \left\{ \exp \left[ \int_{t_i}^t dt' A(t', k) \right] \right\}, \tag{8}
\]
\[
 \equiv \sum_{n=0}^{\infty} \int_{t_i}^t dt_1 \int_{t_i}^{t_1} dt_2 \cdots \int_{t_i}^{t_{n-1}} dt_n \mathcal{A}(t_1, k) \cdots \mathcal{A}(t_n, k). \tag{9}
\]
The exponent matrix \( \mathcal{A}(t, k) \) vanishes whenever \( q(t) \) is constant. It has the form,
\[
 \mathcal{A}(t, k) = -\frac{\pi}{4} \nu \left( \text{csc}(\nu\pi)c_\nu(-\frac{x}{2q}) - 2id_\nu(-\frac{x}{2q}) \right), \tag{10}
\]
where the various coefficient functions are,
\[
 b_\nu(z) = \frac{1}{2\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{(-1)^n \Gamma(n-\nu - \frac{1}{2}) z^{2n-2\nu(n-\nu)^{-1}}}{\Gamma(n)\Gamma(n-\nu+1)\Gamma(n-2\nu+1)}, \tag{11}
\]
\[
 c_\nu(z) = -\frac{4}{\pi} \sin(\nu\pi) \left[ \psi(\nu) - 1 - \ln(\frac{z}{2}) \right] \left( -\frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{(-1)^n \Gamma(n-\frac{1}{2})}{\Gamma(n+\nu)\Gamma(n+\nu+1)\Gamma(n-\nu+1)} \right), \tag{12}
\]
\[
 d_\nu(z) = \frac{1}{2\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n+\nu - \frac{1}{2}) z^{2n+2\nu(n+\nu)^{-1}}}{\Gamma(n+\nu+2)\Gamma(n+\nu+1)\Gamma(n+1)}, \tag{13}
\]
Here \( \psi(z) \equiv \Gamma'(z)/\Gamma(z) \).

Equation \( (5) \) is of phenomenological interest because it gives the mode functions of linearized gravitons \( [10] \). However, the dominant perturbations imprinted in the cosmic microwave background seem to be from the scalar modes of gravity plus a minimally coupled scalar \( [2] \). The perturbative background and the equations defining the mode functions of this system are reviewed in Section 2. In Section 3 we solve for the mode functions by altering the scale factor and time parameter in \( \psi_0(t,k) \) in Section 4 the general solutions are re-expressed in terms of the actual scale factor and the co-moving time \( t \). We give simple and accurate approximations for the ultraviolet regime \( (x(t,k) \gg 1) \) and the infrared regime \( (x(t,k) \ll 1) \) in Section 5. Section 6 discusses applications.

2. Gravity with a minimally coupled scalar: The Lagrangian in which we are interested is,
\[
 \mathcal{L} = \frac{1}{16\pi G} R \sqrt{-g} - \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi g^{\mu\nu} \sqrt{-g} - V(\varphi) \sqrt{-g} \tag{14}
\]
This system can be solved perturbatively in a variety of different field variables and gauge conditions \( [11] \). We will use a recent formulation \( [11, 12] \) based upon generalizing the de Donder gauge condition typically employed for quantum gravity in flat space.

The full scalar and metric fields are expressed in terms of their background values plus quantum fields,
\[
 \varphi(\eta, x) = \varphi_0(\eta) + \psi(\eta, x), \tag{15}
\]
\[
 g_{\mu\nu}(\eta, x) = a^2 \left( \eta_{\mu\nu} + \kappa \psi_{\mu\nu}(\eta, x) \right). \tag{16}
\]
Here \( \kappa^2 \equiv 16\pi G \) is the loop counting parameter of quantum gravity. As usual, indices on the graviton field \( \psi_{\mu\nu} \) are raised and lowered using the Minkowski metric \( \eta_{\mu\nu} \).

Note that we consider the fields to be functions of conformal time \( \eta \),
\[
 \eta \equiv \eta_i + \int_{t_i}^t dt' \left( \frac{d\eta}{dt} \right)^{-1} = \int_{t_i}^t dt' \left( a(t) \right)^{-1}. \tag{17}
\]
This in no way precludes expressing the relations between the background fields \( \varphi_0(\eta) \) and \( a(t) \) in terms of co-moving time derivatives,
\[
 3H^2 = 8\pi G \left( \frac{1}{2} \dot{\varphi}_0^2 + V(\varphi_0) \right), \tag{18}
\]
\[
 -2H \dot{H} - 3H^2 = 8\pi G \left( \frac{1}{2} \dot{\varphi}_0^2 - V(\varphi_0) \right). \tag{19}
\]
As usual, an overdot denotes differentiation with respect to \( t \) whereas a prime stands for differentiation with respect to \( \eta \).

It is traditional with this system to consider the scalar potential \( V(\varphi) \) to be a known function from which the scale factor is inferred. However, for our purposes it is more convenient to regard \( a(t) \) as the known function. If one desires the scalar and its potential they can be reconstructed using the relations,
\[
 \dot{\varphi}_0 = -\frac{\sqrt{1 + q} H}{\sqrt{4\pi G}} \quad \text{and} \quad V(\varphi_0) = \frac{(2-q)H^2}{8\pi G}. \tag{20}
\]

The generalized de Donder gauge condition is \( [11] \),
\[
 F_\mu \equiv a \left[ \dot{\psi}_{\mu,\nu} - \frac{1}{2} \dot{\psi}_{\nu,\mu} - 2aH \dot{\psi}_{\mu,0} + 2\dot{\phi}_0 \sqrt{1+q} H a \phi \right] = 0. \tag{21}
\]
In this gauge it can be shown that the linearized fields of this system — and the Green’s functions needed to solve at higher orders — are described by three plane wave mode functions \( Q_I(\eta, k) \) \( [12] \).

\[
 Q_I'' + \left( k^2 - \frac{\theta_I'}{\theta_I} \right) Q_I = 0 \quad \text{for} \quad I = A, B, C, \tag{22}
\]
The three parameter functions $\theta_I$ are,

$$\theta_A = a \quad , \quad \theta_B = \frac{1}{a} \quad \text{and} \quad \theta_C = \frac{1}{\sqrt{1 + qa}} .$$

We normalize the various mode functions to obey,

$$Q_I Q_{I'} - Q_1' Q_I = i .$$

The linearized fields in de Donder gauge can be expressed using two mode sums [12],

$$\psi_{00} = -\sqrt{1+q} H \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{1}{k} Q_C e^{ik \cdot x} Y(k) + c.c. \right\},$$

$$\psi_{ij}^{al} = \frac{1}{a} \int \frac{d^3k}{(2\pi)^3} \sum_{\lambda} \left\{ \epsilon_{ij}(k,\lambda) Q_A e^{i k \cdot x} \alpha(k,\lambda) + c.c. \right\}.$$

Note that in these mode sums we have suppressed the arguments of previously defined functions such as $a(t)$, $H(t)$, $q(t)$ and $Q_I(\eta, k)$. The scalar and graviton creation and annihilation operators are canonically normalized,

$$\left[ Y(k) , Y'(k') \right] = (2\pi)^3 \delta^3(k - k'),$$

$$\left[ \alpha(k,\lambda), \alpha'(k',\lambda') \right] = (2\pi)^3 \delta^3(k - k') \delta_{\lambda\lambda'} .$$

As usual, graviton polarization tensors are transverse ($k_i \epsilon_{ij}(k,\lambda) = 0$), traceless ($\delta_{ij} \epsilon_{ij}(k,\lambda) = 0$), and orthonormal ($\epsilon_{ij}(k,\lambda) \epsilon_{ij}(k',\lambda') = \delta_{\lambda\lambda'}$). The other linearized fields are,

$$\phi(\eta, \vec{x}) = \frac{1}{\sqrt{1+q} H a} \frac{\partial}{\partial t} \left( a \psi_{00}(\eta, \vec{x}) \right) ,$$

$$\psi_{ai}(\eta, \vec{x}) = 0 ,$$

$$\psi_{ij}(\eta, \vec{x}) = \delta_{ij} \psi_{00}(\eta, \vec{x}) + \psi_{ij}^{al}(\eta, \vec{x}) .$$

Although the $B$ modes are not required at linearized order it is no trouble to get them, and they would appear in the retarded Green’s functions needed to obtain the next order solutions [12].

3. Reduction to the MMC scalar mode functions: Our procedure is to define a new scale factor and a new time coordinate so that the $Q_I$ mode equation [22] reduces to the mode equation [13] of the massless, minimally coupled scalar. There does not seem to be any way to motivate why this works, but the trick is simple enough that we shall simply state it.

We define the new time variable $t_I$ as follows,

$$dt_I = \pm \theta_I \eta \quad \Rightarrow \quad \frac{d}{d\eta} = \pm \theta_I \frac{d}{dt_I} .$$

The plus sign is used for functions $\theta_I$ which grow with conventional time (hence $I = A$) while the minus is taken when $\theta_I$ falls (hence $I = B, C$). Note that $\theta_I$ plays the same role with respect to $\pm t_I$ as the scale factor $a(t)$ does with respect to the co-moving time $t$. This suggests that we define associated Hubble and deceleration parameters,

$$H_I \equiv \frac{1}{\theta_I} \frac{d\theta_I}{dt_I} \quad \text{and} \quad q_I \equiv -1 - \frac{1}{H_I^2} \frac{dH_I}{dt_I} .$$

A little calculus establishes the identity,

$$\left( \frac{d}{d\eta} \right)^2 = \theta_I^2 \left[ \left( \frac{d}{dt_I} \right)^2 + H_I \frac{d}{dt_I} \right] .$$

Applying this to the function $\theta_I$ itself we can re-express the quantity $-\theta_I''/\theta_I$ which appears in the $Q_I$ mode equation [22],

$$- \frac{\theta''_I}{\theta_I} = \theta_I^2 H_I^2 (-1 + q_I) .$$

In keeping with the geometrical analogy, we define the “physical” wave number in “Hubble” units as,

$$x_I \equiv \frac{k}{\theta_I H_I} .$$

The final step is to rescale $Q_I$ by a factor of $\sqrt{\theta_I}$,

$$u_I \equiv \sqrt{\theta_I} Q_I \quad \Rightarrow \quad \frac{d^2 u_I}{dt_I^2} + H_I^2 \left[ x_I^2 + 3 \frac{3}{2} H_I - \frac{3}{4} \right] u_I = 0 .$$

This is the same equation for a plane wave mode function $u_I$ evolving in the geometry $ds_I^2 = -dt_I^2 + \theta_I^2 d\vec{x} \cdot d\vec{x}$ as [15] is for the mode function $u(t,k)$ evolving in the geometry $ds^2 = -dt^2 + a^2 d\vec{x} \cdot d\vec{x}$. Even the Wronskian agrees up to a sign,

$$u_I \frac{d u_I'}{dt_I} - \frac{d u_I}{dt_I} u_I' = \pm \left[ Q_I Q_{I'} - Q_I' Q_I \right] = \pm i .$$

We can absorb the sign by making the $B$ and $C$ modes scale to $Q^*$, which does not change [24],

$$u_A(t_A, k) \equiv \sqrt{\theta_A} \frac{Q_A(\eta, k)}{H_A} ,$$

$$u_B(t_B, k) \equiv \sqrt{\theta_B} \frac{Q_B(\eta, k)}{H_B} ,$$

$$u_C(t_C, k) \equiv \sqrt{\theta_C} \frac{Q_C(\eta, k)}{H_C} .$$

It follows that $u_I$ must be the same function of $t_I$, with scale factor $\theta_I$, as $u(t, k)$ is of $t$, with the scale factor $a(t)$.

4. Expressing the solution in conventional form: Of course physics is not based on the fictitious geometry $ds_I^2 = -dt_I^2 + \theta_I^2 d\vec{x} \cdot d\vec{x}$, so we must express the solutions $u_I$ as functions of co-moving time $t$ and as functionals of the true scale factor $a(t)$. From their definitions [28] we already know how the functions $\theta_I(t)$ depend upon the co-moving time. To determine $H_I(t)$ and $q_I(t)$ we need the differential relation between $t_I$ and $t$. By comparing the transformation [17] from $\eta$ to $t_I$ with the transformation from $\eta$ to $t$ we infer,
As before, the + sign pertains for A modes and the − sign for B and C modes.

Now apply (42) to the parameters (43) of the $d^2$ geometry. The $A$ and $B$ mode parameters are simple,

$$H_A = H, \quad q_A = q,$$
$$H_B = H a^2, \quad q_B = -q.$$  

(43)
(44)

The $C$ mode parameters are more complicated,

$$H_C = (1+r)\sqrt{1+q} Ha^2, \quad q_C = -\frac{q}{1+r} + \frac{r}{(1+r)^2},$$  

(45)

where we define the additional parameter,

$$r(t) \equiv \frac{1}{H} \frac{d}{dt} \ln \left( \sqrt{1+q} \right).$$  

(46)

However, note that whenever $q(t)$ is constant — which is the case for all familiar cosmologies — the $C$ mode parameters degenerate,

$$\dot{q} = 0 \implies H_C = \sqrt{1+q} Ha^2 \quad \text{and} \quad q_C = -q.$$  

(47)

The rescaled wave number is in all cases simple,

$$x_A = x(t,k), \quad x_B = x(t,k) \quad \text{and} \quad x_C = \frac{x(t,k)}{1+r(t)}.$$  

(48)

In fact, whenever $q(t)$ is constant, $x_C = x(t,k)$.

We have reached the point where the parameters of the $d^2$ geometry can all be considered to be functions of co-moving time $t$: $\theta_I(t), H_I(t), q_I(t)$ and $x_I(t,k)$. We make the additional definition $\nu_I(t) \equiv \frac{1}{2} - q_I^2(t)$. Based on the work of the previous section the $I = A,B,C$ mode functions can be read off from (41) by replacing the parameters of the $d^2$ geometry with the corresponding $d^2$ parameters, considered as functions of $t$,

$$u_I(t,k) \equiv \frac{\pi \theta_I}{2k} \left( \frac{x_I}{2q_I} \right)^{\nu_I} \left( \frac{\nu_I}{2} \right) \left( \frac{\nu_I}{2} \right) J_{\nu_I}(-\frac{i\nu_I}{2}) \left( \frac{x_I}{2q_I} \right), \quad J_{\nu_I}(-\frac{i\nu_I}{2}) \times M_I(t,t,\nu_I(t,k)) \left( \frac{1}{1+i \cot[\nu_I(t,k)]} \right).$$  

(49)

As with $u(t,k)$, the transfer matrix takes the form,

$$M_I(t,t,\nu_I(t,k)) \equiv P \left\{ \exp \left[ \int_{t_i}^t dt' A_I(t',k) \right] \right\}. \quad (50)$$

Its exponent matrix obeys the same scheme,

$$A_I(t,k) = \frac{\pi}{4} \nu_I \left( -\frac{csc(\nu_I \pi) c_{\nu_I}(\frac{x_I}{2q_I})}{2i \nu_I} - 2i c_{\nu_I}(\frac{x_I}{2q_I}) - csc(\nu_I \pi) c_{\nu_I}(\frac{x_I}{2q_I}) \right),$$

(51)

where the coefficient functions (44) are unchanged.

Note that we were able to express the transfer matrix in terms of integrations over the co-moving time owing to the relation,

$$\int_{t_i}^{t_2} dt' \frac{d\nu_I}{dt'} F(x_I,q_I) = \int_{t_i}^{t_2} dt' \nu_I F(x_I,q_I).$$

(52)

5. The ultraviolet and infrared regimes: The ultraviolet regime is defined by $x(t,k) \gg 1$. In this limit we know from previous work (31) that the three mode functions approach the form,

$$u_I(t,k) \bigg|_{x \gg 1} \longrightarrow \frac{\theta_I}{2k} \exp \left[ i \chi_I(t_i,k) - i k \int_{t_i}^{t} dt' \theta_I \right],$$

(53)

where the phase $\chi_I$ is,

$$\chi_I(t,k) \equiv -\frac{x_I(t,k)}{q_I(t)} + \frac{1-q_I(t)}{q_I(t)} \frac{\pi}{2}.$$  

(54)

Recalling $dt_I = \pm \theta_I dt/a$, and the definition (47) of conformal time, we see that the various $Q_I$’s approach,

$$Q_I(\eta,k) \bigg|_{x \gg 1} \longrightarrow \frac{1}{\sqrt{2k}} e^{-ik(\eta-\eta_i) \pm i\chi_I(t_i,k)}.$$  

(55)

Note that the + signs in the relation between $dt_I$ and $dt$ exactly cancels the variations (45) in the relations between $u_I(t,k)$ and $Q_I(\eta,k)$.

The infrared regime ($x \ll 1$) is more subtle and more interesting. We assume that the wave number was initially ultraviolet, i.e., $x(t_i,k) \gg 1$. During inflation $x(t,k)$ falls, typically exponentially. This drives the lower ultraviolet modes through first horizon crossing ($x(t_1,k) = 1$) into the infrared regime of $x(t,k) \ll 1$. In this regime one of the Bessel functions will enormously dominate the other. It is therefore good to re-express the general solution in terms of solutions which are the pure $\pm \nu_I$ Bessel functions at first horizon crossing,

$$u_I(t,k) \equiv \frac{1}{\sqrt{2}} \left( u^-_I(t,t_1,k) + u^+_I(t,t_1,k) \right)$$

$$\times \chi_I(t_1,k) \left( \frac{1}{1+i \cot[\nu_I(t_1,k)]} \right).$$  

(56)

Here $u^+_I(t,t_1,k)$ are the pure $J_{\nu_I}$ solutions at first horizon crossing, evolved to time $t = t_1$.

As long as the condition $x_I(t,k) \ll 1$ is satisfied, the transfer matrix can be worked out up to negligible correction factors of order $x^2$. To the same fractional error we can also use the leading terms in the series expansions of the Bessel functions. The resulting form for the $-$ solution is,

$$u^-_I(t,k) \bigg|_{x \ll 1} \longrightarrow \frac{-i\theta_I^2(t)}{\sqrt{\pi k}} \left[ \frac{\Gamma(\frac{1}{2} - \frac{1}{q_I})}{\Gamma^2(\frac{1}{2} - \frac{1}{q_I})} \right]_{t_i}.$$

(57)

The $+$ solution involves an integral,

$$u^+_I(t,k) \bigg|_{x \ll 1} \longrightarrow \sqrt{\pi} \theta_I^2(t) \left[ \frac{\Gamma(\frac{1}{2} - \frac{1}{q_I})}{\Gamma(\frac{1}{2} - \frac{1}{q_I})} \right]_{t_i} \int_{t_i}^{t} dt' \left[ \frac{1}{2\nu_I q_I} H_I(\nu_I) \right]_{t_i} \times \left[ \frac{1}{2\nu_I q_I} H_I(\nu_I) \right]_{t_i}.$$

(58)
For the $A$ mode it is the $u_A^+$ solution that dominates. We express $u_A(t,k)$ with the standard normalization times two $k$-dependent correction factors,

$$u_A(t,k) \bigg|_{x<1} = \frac{-iH_1}{2k^3} a(t) C_{1A}(k) C_{iA}(k). \quad (59)$$

Here $H_1 = H(t_1)$ is the Hubble parameter at first horizon crossing. Note that it can depend upon $k$ because the time of first horizon crossing depends upon the wave number. Indeed, some modes never experience horizon crossing.

The correction factor in $[59]$ that depends upon the system's state at first horizon crossing is,

$$C_{1A}(k) = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1}{2} - \frac{1}{q_1}\right)}{\left(\frac{2q_1}{q_1}\right)^{\frac{1}{2}}} \equiv M_{1A}(t_1, t_i, k) + M^{12}_{A}(t_1, t_i, k) e^{i\frac{q_1}{q_1} \sec\left(\frac{q_1}{q_1}\right)} \bigg|_{x<1}. \quad (60)$$

Here $q_1 \equiv q(t_1)$ is the deceleration parameter at first horizon crossing. The correction factor depending upon evolution from $t_i$ to $t_1$ is,

$$C_{iA}(k) = M^{11}_{A}(t_1, t_i, k) + M^{12}_{A}(t_1, t_i, k) e^{i\frac{q_1}{q_1} \sec\left(\frac{q_1}{q_1}\right)} \bigg|_{x<1}. \quad (61)$$

Here $q_i \equiv q(t_i)$ is the initial value of the deceleration parameter. Since $\dot{q}(t) = \frac{\dot{q}(t)}{\sqrt{q(t)}}$ is typically small during inflation, it ought to be a very good approximation to simply take the first several terms of the series expansion of the transfer matrix in estimating $C_{iA}(k)$. Combining $[59]$ and $[60]$ we see that the infrared limit of $Q_A$ takes the form,

$$Q_A(\eta, k) \bigg|_{x<1} = \frac{-iH_1}{\sqrt{2k^3}} a(t) C_{1A}(k) C_{iA}(k). \quad (62)$$

For the $B$ mode it is the $u_B^+$ solution that dominates, and only the integral is significant,

$$u_B(t,k) \bigg|_{x<1} = \frac{H_1}{\sqrt{2k^3}} a(t) \int_{t_1}^{t} dt' a(t') C_{1B}(k) C_{iB}(k). \quad (63)$$

The correction factor from horizon crossing is,

$$C_{1B}(k) = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1}{2} - \frac{1}{q_1}\right)}{\left(\frac{2q_1}{q_1}\right)^{\frac{1}{2}}} e^{i\frac{q_1}{q_1} \cos\left(\frac{q_1}{q_1}\right)} \bigg|_{x<1}. \quad (64)$$

The other correction factor depends upon evolution up to this time,

$$C_{iB}(k) = M^{11}_{B}(t_1, t_i, k) + M^{12}_{B}(t_1, t_i, k) e^{-i\frac{q_1}{q_1} \sec\left(\frac{q_1}{q_1}\right)} \bigg|_{x<1}. \quad (65)$$

And we can combine $[63]$ and $[64]$ to extract the infrared limit of $Q_B$,

$$Q_B(\eta, k) \bigg|_{x<1} = \frac{H_1}{\sqrt{2k^3}} a(t) \int_{t_1}^{t} dt' a(t') C_{1B}^*(k) C_{iB}^*(k). \quad (66)$$

It is the $u_C^+$ solution that dominates the infrared $C$ mode,

$$u_C(t,k) \bigg|_{x<1} = \frac{H_1}{\sqrt{2k^3}} \frac{1}{1+q(t)} C_{1C}(k) C_{iC}(k) \bigg|_{x<1} \int_{t_1}^{t} dt' a(t') [1 + q(t')] C_{1C}(k) C_{iC}(k). \quad (67)$$

The correction factor at horizon crossing takes the form,

$$C_{1C}(k) = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1}{2} + \frac{1}{q_1}\right)}{\left(\frac{2q_1}{q_1}\right)^{\frac{1}{2}}} e^{-i\frac{q_1}{q_1} \cos\left(\frac{q_1}{q_1}\right)}(1 + r_1) \bigg|_{x<1}. \quad (68)$$

where $q_1C \equiv q_C(t_1)$ and $r_1 \equiv r(t_1)$. We remind the reader that the $C$ deceleration parameter and $r(t)$ are,

$$q_C(t) = -\frac{q(t)}{1+r(t)} + \frac{\dot{r}(t)}{H(t)} \bigg|_{x<1}, \quad (69)$$

$$r(t) = \frac{1}{H(t)} \frac{d}{dt} \ln \bigg(\sqrt{1+q(t)}\bigg). \quad (70)$$

The other correction factor depends upon $q_{1C} \equiv q_C(t_1)$,

$$C_{iC}(k) = M^{21}_{C}(t_1, t_i, k) + M^{22}_{C}(t_1, t_i, k) e^{-i\frac{q_1}{q_1} \sec\left(\frac{q_1}{q_1}\right)} \bigg|_{x<1}. \quad (71)$$

Taking $[67]$ together with $[61]$ gives the last of our infrared limits,

$$Q_C(\eta, k) \bigg|_{x<1} = \frac{H_1}{\sqrt{2k^3}} \frac{1}{1+q(t)} \bigg|_{x<1} \int_{t_1}^{t} dt' a(t') [1 + q(t')] C_{1C}^*(k) C_{iC}^*(k). \quad (72)$$

6. Applications: The formalism we have developed might seem intimidating, and one must of course approximate the transfer matrix in order to compute anything. However, it is often of great value to possess general expressions. For example, one can search for unexpected effects such as the consequences of the universe having passed through $q = 0$ at the end of inflation. We can also explore cosmologies for which the $C$ mode deceleration parameter $[69]$ differs appreciably from its small $r(t)$ limit of $-q(t)$. Our formalism is also useful. For example, our infrared limiting form $[72]$ allows one to compute the Sachs-Wolfe contribution to the primordial scalar power spectrum for any evolution $a(t)$ which does not compromise the infrared limit. It should be noted, for non-experts, that this primordial power spectrum is almost perfectly flat. The rich structure imaged by WMAP $[2]$ derives from a variety of late-time effects which occur after second horizon crossing, when the stress-energy is no longer dominated by the inflaton and our scalar mode functions are not relevant. (However, note that our tensor mode functions continue to apply. They depend only upon $a(t)$, not what
caused it.) The primordial power spectrum sets one of the initial conditions for this late-time cosmology, whose dynamics are well understood.

Assuming the cosmic microwave radiation radiation was emitted during pure matter domination ($q = +\frac{1}{2}$), our result for the primordial scalar power spectrum is [13].

$$\mathcal{P}_{SW}(k) = \frac{9}{4\pi} \frac{G \Theta_1^2}{1 + q_1} |\mathcal{C}_{1C}(k)|^2 |\mathcal{C}_{1C}(k)|^2 .$$  \hspace{1cm} (73)

The standard result is $\frac{9}{4\pi} G \Theta_1^2 (1 + q_1)$ so the correction factors [8] and (41) represent improvements. Because different conventions exist in the literature we correspond $\mathcal{P}_{SW}(k)$ below to the symbol $\delta(k)$ used by Mukhanov, Feldman and Brandenberger [6], to the variable $\mathcal{P}_{\mathcal{R}}(k)$ used by Liddle and Lyth [7], and to the quantity $A_S^2(k)$ used by Lidsey et al. [8].

$$\mathcal{P}_{SW}(k) = \frac{25}{4} |\delta(k)|^2 = \frac{9}{4} \mathcal{P}_{\mathcal{R}}(k) = \frac{225}{16} A_S^2(k) .$$  \hspace{1cm} (74)

We also specify how it enters the correlation function between temperature fluctuations observed from directions $\hat{e}_1$ and $\hat{e}_2$,

$$\left\langle \Omega \frac{\Delta T_R(\hat{e}_1) \Delta T_R(\hat{e}_2)}{T_R} \right\rangle_{SW} = \int_0^\infty \frac{dk}{k} \mathcal{P}_{SW}(k) \int_0^{\infty} \frac{d^2 k}{4\pi} e^{2ix_0 k \cdot (\hat{e}_1 - \hat{e}_2)} ,$$  \hspace{1cm} (75)

where $x_0 \equiv x(t_0, k)$ is the physical wave number in current Hubble units.

A very interesting problem we now have the analytical power to tackle is the apparent paradox associated with the fact that the scalar power spectrum (73) diverges whenever first horizon crossing occurs during a temporary period of de Sitter inflation (i.e., $q_1 = -1$). It is difficult to understand why the result in this case is not of gravitational strength, and one would think it should not diverge. This has led to controversy in the past [14, 15] which it is now possible to resolve.

Finally, these mode functions can be used to study back-reaction in the gravity plus scalar system. It has been established that no secular back-reaction occurs at one loop for simple models [12, 13]. This formalism allows one to study an arbitrary single-scalar model of inflation.

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