Capacity of a Class of Multi-source Relay Networks

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Abstract

Characterizing the capacity region of multi-source wireless relay networks is one of the fundamental issues in network information theory. The problem is, however, quite challenging because the transmission of other sessions acts as inter-user interference when there exist multiple source-destination (S-D) pairs in the network. By focusing on a special class of networks, we show that the capacity can be found. Namely, we study a linear finite-field network with time-varying channels, which exhibits broadcast, interference, and fading natures of wireless communication. We observe that fading can play an important role in mitigating inter-user interference effectively for both single-hop and multi-hop networks. We propose new block Markov encoding and relaying schemes with randomized channel pairing, which exploit such channel variations, and derive their achievable rates. By comparing them with the general cut-set upper bound, the capacity region of single-hop networks and the sum-capacity of multi-hop networks can be characterized for some classes of channel distributions and networks topologies. For these classes, we show that the capacity of multi-source networks can be interpreted as the max-flow min-cut theorem.

I. INTRODUCTION

Capacity characterization of general wireless relay networks is a fundamental problem in network information theory. However, the capacity is not fully characterized even for the simplest network consisting of single source, single relay, and single destination [1]. In wireless environments, a transmit signal will be heard by multiple nodes, which we call the broadcast nature of wireless communication, and a receiver will receive the superposition of simultaneously transmitted signals from multiple nodes, which we call the interference nature of wireless communication. Furthermore wireless channels may be time-varying due to fading, and there is noise at each receiver. Considering all these makes the problem vary hard.

Hence, one of the promising approaches is to study simplified relay networks, whose results can provide insights towards exact or approximate capacity characterization for more general wireless relay networks. Let us first look at some cases that the capacity is known. For wireline relay networks or the relay networks with no broadcast and no interference, routing is enough to achieve the unicast capacity [2]. On the other hand, routing alone cannot achieve the multicast capacity and network coding has been shown to be optimal in this case [3], [4], [5], [6]. For deterministic relay networks with no interference, the unicast capacity has been characterized in [7] and the extension to the multicast case has been recently studied in [8]. The multicast capacity of erasure networks with no interference has been also characterized in [9]. When there is no broadcast, the unicast capacity of erasure networks has been characterized in [10], which is the dual network studied in [9]. For all these mentioned networks, the unicast or multicast capacity can be interpreted as the max-flow min-cut theorem.

Notice that although such orthogonal transmission or reception is possible in practice by using time, frequency, or code-division techniques, it is suboptimal in general channels. Therefore, simplification of wireless relay networks while preserving both broadcast and interference natures is crucially important to capture the essence of wireless communication. The simplest model that successfully reflects both broadcast and interference natures might be linear finite-field relay networks [11], [12], [13], where a node transmits an element in the finite-field and receives the sum of transmit signals in the same finite-field. Recently, the work in [13] has shown that the max-flow min-cut theorem also holds for deterministic linear finite-field relay networks. After the capacity characterization of linear finite-field relay networks, the approximate capacity of Gaussian relay networks has been characterized within a constant number of bits using the quantize-random-map-and-forward by the same authors [14].

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The material in this paper was presented in part at the Information Theory and Applications Workshop, University of California San Diego, La Jolla, CA, February 2009, and at the IEEE International Symposium on Information Theory (ISIT), Seoul, Korea, June/July 2009.
In spite of the surging importance of multi-source relay networks, capacity characterization is much more challenging if there exist multiple source-destination (S-D) pairs in a network. Even for linear finite-field relay networks, the extension of the results in [13] to the multi-source does not seem to be straightforward. Notice that the main difficulty arises from the fact that the transmission of other sessions acts as inter-user interference and, as a result, the cut-set upper bound is not tight in general. Due to these difficulties, the existing capacity or approximate capacity results are limited in specific network topologies such as two-user interference channel [15], [16], many-to-one interference channel [17], two-way channel [18], [19], and two-user two-hop relay network [20], [21]. Therefore, one of the basic questions is whether we can characterize the capacity or approximate capacity for more general network topologies or other classes of relay networks.

In this paper, we study multi-source linear finite-field relay networks with time-varying channels, which capture three key characteristics of wireless environment, i.e., broadcast, interference, and fading. Note that a random coding strategy, which is still optimal in single-source fading linear finite-field relay networks [22], [23], does not work anymore due to the inter-user interference. As mentioned before, a fundamental issue in multi-source networks is how to manage inter-user interference in a network. We observe that fading can play an important role in mitigating such interference efficiently, that leads to the capacity characterization for certain classes of networks. More specifically, for single-hop networks, inter-user interference can be removed completely at each destination by using two particular channel instances jointly. For multi-hop networks, by using a series of particular channel instances over multiple hops, each destination can also decode its message without interference.

As an example, consider the three-user linear binary-field relay network in Fig. 1 where \( s_k \in \mathbb{F}_2 \) denotes the information bit of the \( k \)-th source and the symbol in each node denotes the transmit signal of that node. For single-hop networks, as shown in Fig. 1 (a), by transmitting the same bit twice at each source through \( H_1^{(1)} \) and \( H_1^{(2)} \) such that \( H_1^{(1)} + H_1^{(2)} = I \), each destination can cancel interference by adding the two received signals, where \( H_1^{(1)} \) and \( H_1^{(2)} \) denote the two channel instances of the first hop and \( I \) denotes the identity matrix. Related works dealing with the inseparability of parallel interference channels can be found in [24], [25], [26], [27] and the references therein, where the idea of opportunistically pairing two channel instances, i.e., \( H_1^{(1)} + H_1^{(2)} = I \), also appeared in [26], [27]. This can be considered as a different and simpler way of doing interference alignment [28], [29]. For two-hop networks, as shown Fig. 1 (b), we notice that each destination can receive the information bit without interference if \( H_2 H_1 = I \), where \( H_1 \) and \( H_2 \) denote the channel instances of the first and second hop, respectively. Similarly the interference-free communication is possible for \( M \)-hop networks by opportunistically pairing the series of channel instances from \( H_1 \) to \( H_M \) such that \( H_M H_{M-1} \cdots H_1 = I \), where \( H_m \) denotes the channel instance of the \( m \)-th hop.

Based on these key observations, we propose block Markov encoding and relaying schemes which make such opportunistic pairing of channel instances possible. By comparing their achievable rate regions with the cut-set upper bound, we characterize the capacity region of single-hop networks and the sum capacity of multi-hop networks for some classes of network topologies and channel distributions.

This paper is organized as follows. In Section II, we define the network model and state the multi-source relay problem and the notations used in the paper. In Section III we derive the general cut-set upper bound, which will be used to prove the converses in Sections IV and V. In Section IV the block Markov encoding scheme is proposed for single-hop networks and its achievable rate region is derived, which characterizes the capacity region for certain classes of networks. In Section V the block Markov encoding and relaying scheme is proposed for multi-hop networks and its achievable rate region is derived, which characterizes the sum capacity for certain classes of networks. We conclude this paper in Section VI and refer the proofs of the lemmas to Appendices I and II.

II. SYSTEM MODEL

In this section, we first explain the underlining network model and then define the achievable rate region and the notations used in the paper. Throughout the paper, \( A \) and \( a \) denote a matrix and a vector, respectively. The symbol \( |A| \) denotes the cardinality of \( A \).

A. Linear Finite-field Relay Networks

We study a layered network in Fig. 2 that consists of \( M + 1 \) layers having \( K_m \) nodes at the \( m \)-th layer, where \( m \in \{1, \cdots, M + 1\} \). Let us denote \( K_{\max} = \max_m \{K_m\} \) and \( K_{\min} = \min_m \{K_m\} \). The \((k,m)\)-th node refers to...
to the $k$-th node at the $m$-th layer. Then the $(k,1)$-th node and the $(k,M+1)$-th node are the source and the destination of the $k$-th S-D pair, respectively. Thus $K = K_1 = K_{M+1}$ is the number of S-D pairs. Notice that if $M = 1$, the network becomes a $K$-user interference channel.

Consider the $m$-th hop transmission. The $(i,m)$-th node and the $(j,m+1)$-th node become the $i$-th transmitter (Tx) and the $j$-th receiver (Rx) of the $m$-th hop, respectively, where $i \in \{1,\cdots,K_m\}$ and $j \in \{1,\cdots,K_{m+1}\}$. Let $x_{i,m}[t] \in \mathbb{F}_2$ denote the transmit signal of the $(i,m)$-th node at time $t$ and let $y_{j,m}[t] \in \mathbb{F}_2$ denote the received signal of the $(j,m+1)$-th node at time $t$. Let $h_{j,i,m}[t] \in \mathbb{F}_2$ be the channel from the $(i,m)$-th node to the $(j,m+1)$-th node at time $t$. The relation between the transmit and received signals is given by

$$
y_{j,m}[t] = \sum_{i=1}^{K_m} h_{j,i,m}[t] x_{i,m}[t],$$

where all operations are performed over $\mathbb{F}_2$. We assume time-varying channels such that

$$\Pr(h_{j,i,m}[t] = 1) = p_{j,i,m}$$

and $h_{j,i,m}[t]$ are independent from each other for different $i$, $j$, $m$, and $t$. Let $x_{m}[t]$ and $y_{m}[t]$ be the $K_m \times 1$ transmit signal vector and $K_{m+1} \times 1$ received signal vector of the $m$-th hop, respectively, where $x_{m}[t] = [x_{1,m}[t], \cdots, x_{K_m,m}[t]]^T$, $y_{m}[t] = [y_{1,m}[t], \cdots, y_{K_{m+1},m}[t]]^T$. Then the transmission of the $m$-th hop can be represented as

$$y_{m}[t] = H_m[t] x_{m}[t],$$

where $H_m[t]$ is the $K_{m+1} \times K_m$ channel matrix of the $m$-th hop having $h_{j,i,m}[t]$ as the $(j,i)$-th element. We assume that at time $t$ both Txs and Rxs of the $m$-th hop know $H_1[t]$ through $H_m[t]$. For notational simplicity, we use $Pr(H_m)$ to denote $Pr(H_m[t] = H_m)$, where $H_m \in \mathbb{F}_2^{K_{m+1} \times K_m}$.

We will study the following class of networks in this paper.

**Definition 1:** Let $m_0 = \arg\min_{m \in \{1,\cdots,M\}} \mathbb{E}(\text{rank}(H_m))$. A linear finite-field relay network is said to have a minimum-dimensional bottleneck-hop $m_0$ if $K_m \geq K_{m_0}$ and $K_{m+1} \geq K_{m_0+1}$ (or $K_m \geq K_{m_0+1}$ and $K_{m+1} \geq K_{m_0}$) for all $m \in \{1,\cdots,M\}$.

Notice that any networks having $K_m = K$ for all $m \in \{1,\cdots,M+1\}$ or any 1-hop or 2-hop networks are included in this class of networks regardless of channel distributions.

**B. Problem Statement**

Based on the previous network model, we define a set of length-$n$ block codes. Let $W_k$ be the message of the $k$-th source uniformly distributed over $\{1,2,\cdots,2^n R_k\}$, where $R_k$ is the rate of the $k$-th source. For simplicity, we assume $n R_k$ is an integer. Then a $(2^n R_k, \cdots, 2^n R_k; n)$ code consists of the following encoding, relaying, and decoding functions.

- **(Encoding)**
  For $k \in \{1,\cdots,K\}$, the set of encoding functions of the $k$-th source is given by $\{f_{k,1,t}\}_{t=1}^n : \{1,\cdots,2^n R_k\} \rightarrow \mathbb{F}_2^n$ such that

$$x_{k,1}[t] = f_{k,1,t}(W_k) \quad \text{for} \quad t \in \{1,\cdots,n\}. \quad (4)$$

- **(Relaying)**
  For $m \in \{2,\cdots,M\}$ and $k \in \{1,\cdots,K_m\}$, the set of relaying functions of the $(k,m)$-th node is given by $\{f_{k,m,t}\}_{t=1}^n : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$ such that

$$x_{k,m}[t] = f_{k,m,t}(y_{k,m-1}[1], \cdots, y_{k,m-1}[t-1]) \quad \text{for} \quad t \in \{1,\cdots,n\}. \quad (5)$$

- **(Decoding)**

1\text{We focus on the binary field } \mathbb{F}_2 \text{ in this paper, but some results can be directly extended to } \mathbb{F}_q \text{ (see Remarks 1 and 2).}

2\text{Since } \mathbb{E}(\text{rank}(H_m[t])) \text{ is the same for all } t, \text{ we use } \mathbb{E}(\text{rank}(H_m)) \text{ to denote } \mathbb{E}(\text{rank}(H_m[t])).
For \( k \in \{1, \cdots, K\} \), the decoding function of the \( k \)-th destination is given by \( g_k : \mathbb{F}_2^n \to \{1, \cdots, 2^{nR_k}\} \) such that
\[
\tilde{W}_k = g_k( y_{k,M}[1], \cdots, y_{k,M}[n] ).
\]

If \( M = 1 \), the sources transmit directly to the destinations without relays. The probability of error at the \( k \)-th destination is given by \( P_{e,k}^{(n)} = P_{e}(\tilde{W}_k \neq W_k) \). A set of rates \((R_1, \cdots, R_K)\) is said to be achievable if there exists a sequence of \((2^{nR_1}, \cdots, 2^{nR_K}; n)\) codes with \( P_{e,k}^{(n)} \to 0 \) as \( n \to \infty \) for all \( k \in \{1, \cdots, K\} \). Then the achievable sum-rate is simply given by \( R_{\text{sum}} = \sum_{k=1}^{K} R_k \). The capacity region is the convex hull of the closure of all achievable \((R_1, \cdots, R_K)\) and the sum capacity is the supremum of all achievable sum-rates.

C. Preliminaries

In this subsection, we introduce the notations for directed graphs and define sets of channel instances and sets of nodes.

1) Notations for directed graphs: The considered network can be represented as a directed graph \( G = (\mathcal{V}, \mathcal{E}) \) consisting of a vertex set \( \mathcal{V} \) and a directed edge set \( \mathcal{E} \). Let \( v_{k,m} \) denote the \((k,m)\)-th node and \( \mathcal{V}_m = \{v_{k,m}\}_{k=1}^{K_m} \) denote the set of nodes in the \( m \)-th layer. Then \( \mathcal{V} \) is given by \( \cup_{m=1}^{M+1} \mathcal{V}_m \). The sets of sources and destinations are given by \( S = \mathcal{V}_1 \) and \( D = \mathcal{V}_{M+1} \), respectively.

There exists a directed edge \((v_{i,m}, v_{j,m+1})\) from \( v_{i,m} \) to \( v_{j,m+1} \) if \( p_{j,i,m} > 0 \). For \( \mathcal{V}_i \subseteq \mathcal{V} \) and \( \mathcal{V}_j \subseteq \mathcal{V} \), define \( \mathcal{E}(\mathcal{V}_i, \mathcal{V}_j) \) as the set of edges going from \( \mathcal{V}_i \) to \( \mathcal{V}_j \) given by \( \{(v,v')|v \in \mathcal{V}_i, v' \in \mathcal{V}_j, (v,v') \in \mathcal{E} \} \). We say node \( v' \) is connected to node \( v'' \) if there exists a series of edges from \( v' \) to \( v'' \), where we assume \( v' \) is always connected to itself \( v' \). We further define \( v' \) is connected to \( v'' \) under \( \mathcal{V}' \) if there exists a series of edges in \( \mathcal{E}(\mathcal{V}', \mathcal{V}') \) from \( v' \) to \( v'' \). We define cut \( \Omega \subseteq \mathcal{V} \) as a subset of nodes such that at least one source is in \( \Omega \) and at least one corresponding destination is in \( \Omega^c \). We define the following sets related to \( \Omega \):
\[
\begin{align*}
\mathcal{K}_\Omega & = \{ k|v_{k,1} \in \Omega, v_{k,M+1} \in \Omega^c, k \in \{1, \cdots, K\} \}, \\
\mathcal{D}_\Omega & = \{ v_{k,M+1}|k \in \mathcal{K}_\Omega \}, \\
\mathcal{S}_\Omega & = \{ v_{k,1}|k \in \mathcal{K}_\Omega \}, \\
\Omega_D & = \{ v|\mathcal{E}(\Omega, \{v\}) \neq \emptyset, v \text{ is connected to at least one of the destinations in } \mathcal{D}_\Omega \text{ under } \Omega^c, v \in \Omega^c \}, \\
\Omega' & = \{ v|\text{At least one of the sources in } \mathcal{S}_\Omega \text{ is connected to } v, v \in \Omega \}, \\
\Omega_S & = \{ v|\mathcal{E}(\{v\}, \Omega_D) \neq \emptyset, v \in \Omega' \}.
\end{align*}
\]

Let \( X_{\mathcal{V}}[t] \) and \( Y_{\mathcal{V}}[t] \) denote the sets of transmit and receive signals of the nodes in \( \mathcal{V} \) at time \( t \), respectively. Let \( H_{\mathcal{V}',\mathcal{V}''}[t] \) be the \(|\mathcal{V}'| \times |\mathcal{V}'| \) channel matrix at time \( t \) from the nodes in \( \mathcal{V}' \) to the nodes in \( \mathcal{V}'' \). Hence \( H_{\mathcal{V}_m,\mathcal{V}_{m+1}}[t] = H_m[t] \). For notational simplicity, we use \( \hat{H}_\Omega[t] \) to denote \( H_{\Omega^c,\Omega_D}[t] \) in this paper.

2) Sets of channel instances and nodes: Suppose \( \mathcal{V}_1 \subseteq \mathcal{V}' \), \( \mathcal{V}_1 \subseteq \mathcal{V}'' \), and \( G \) is a \(|\mathcal{V}'| \times |\mathcal{V}'| \) matrix. We define the following sets of channel instances:
\[
\begin{align*}
\mathcal{H}_{\mathcal{V}',\mathcal{V}''}(G, \tilde{\mathcal{V}}', \tilde{\mathcal{V}}'') & = \{ H_{\mathcal{V}',\mathcal{V}''}|H_{\tilde{\mathcal{V}}',\tilde{\mathcal{V}}''} = G, H_{\mathcal{V}',\mathcal{V}''} \in \mathbb{F}_2^{3|\mathcal{V}'| \times |\mathcal{V}'|} \}, \\
\mathcal{H}_{\mathcal{V}',\mathcal{V}''}^{\phi}(G, \tilde{\mathcal{V}}', \tilde{\mathcal{V}}'') & = \{ H_{\mathcal{V}',\mathcal{V}''}|H_{\tilde{\mathcal{V}}',\tilde{\mathcal{V}}''} = G, \text{rank}(H_{\mathcal{V}',\mathcal{V}''}) = \text{rank}(G), \\
& \quad H_{\mathcal{V}',\mathcal{V}''} \in \mathbb{F}_2^{3|\mathcal{V}'| \times |\mathcal{V}'|} \}.
\end{align*}
\]

Notice that \( \mathcal{H}_{\mathcal{V}',\mathcal{V}''}(G, \tilde{\mathcal{V}}', \tilde{\mathcal{V}}'') \) is the set of all instances of \( H_{\mathcal{V}',\mathcal{V}''} \) that contain \( G \) in \( H_{\mathcal{V}',\mathcal{V}''} \). Similarly, \( \mathcal{H}_{\mathcal{V}',\mathcal{V}''}^{\phi}(G, \tilde{\mathcal{V}}', \tilde{\mathcal{V}}'') \) is the set of all instances of \( H_{\mathcal{V}',\mathcal{V}''} \) that contain \( G \) in \( H_{\mathcal{V}',\mathcal{V}''} \) and have the same rank as \( G \). We further define the following pairs of node subsets:
\[
\begin{align*}
\mathcal{V}(a, b, \mathcal{V}', \mathcal{V}'') & = \{ (\tilde{\mathcal{V}}', \tilde{\mathcal{V}}'')| |\tilde{\mathcal{V}}'| = a, |\tilde{\mathcal{V}}''| = b, (\tilde{\mathcal{V}}', \tilde{\mathcal{V}}'') \subseteq (\mathcal{V}', \mathcal{V}'') \}, \\
\mathcal{V}(H_{\mathcal{V}',\mathcal{V}''}) & = \{ (\tilde{\mathcal{V}}', \tilde{\mathcal{V}}'')| |\tilde{\mathcal{V}}'| = |\tilde{\mathcal{V}}''| = \text{rank}(H_{\mathcal{V}',\mathcal{V}''}), \text{rank}(H_{\tilde{\mathcal{V}}',\tilde{\mathcal{V}}''}) = \text{rank}(H_{\mathcal{V}',\mathcal{V}''}), \\
& \quad (\tilde{\mathcal{V}}', \tilde{\mathcal{V}}'') \subseteq (\mathcal{V}', \mathcal{V}'') \},
\end{align*}
\]
where \( a \) and \( b \) are positive integers satisfying \( a \leq |\mathcal{V}'| \) and \( b \leq |\mathcal{V}''| \) and we define \( \mathcal{V}(H_{\mathcal{V}',\mathcal{V}''}) = \emptyset \) if \( \text{rank}(H_{\mathcal{V}',\mathcal{V}''}) = 0 \). The set \( \mathcal{V}(a, b, \mathcal{V}', \mathcal{V}'') \) consists of all \((\tilde{\mathcal{V}}', \tilde{\mathcal{V}}'')\) such that the number of nodes in \( \mathcal{V}' \) and the number of nodes in

\[\text{(6)}\]
\( \mathcal{V}'' \) are equal to \( a \) and \( b \), respectively. The set \( \mathcal{V}(H_{\mathcal{V}', \mathcal{V}'}) \) consists of all \( (\mathcal{V}', \mathcal{V}'') \) such that \( H_{\mathcal{V}', \mathcal{V}''} \) is a full rank matrix and has the same rank as \( H_{\mathcal{V}', \mathcal{V}''} \).

### III. Upper Bound

In this section, we obtain a general cut-set upper bound then derive some useful rate upper bounds from the general bound, which will be used to prove the converses in Sections IV and V.

#### A. Cut-set Upper Bound

In this subsection, we show that any sequence of \( (2^{nR_1}, \ldots, 2^{nR_k}; n) \) codes with \( P_{e,k}^{(n)} \to 0 \) for all \( k \) satisfies the rate constraints in the following theorem.

**Theorem 1.** Suppose a linear finite-field relay network. For a cut \( \Omega \), the set of achievable rates \( (R_1, \ldots, R_K) \) is upper bounded by

\[
\sum_{k \in \mathcal{K}_a} R_k \leq \mathbb{E}(\text{rank}(H_{\Omega})). 
\]

**Proof:** Let us define \( W_{\mathcal{K}_a} = \{ W_k | k \in \mathcal{K}_a \} \). We further define a length-\( n \) sequence \( a^n \) to denote \( [a[1], \ldots, a[n]] \). Then

\[
\sum_{k \in \mathcal{K}_a} R_k = H(W_{\mathcal{K}_a}) = I(W_{\mathcal{K}_a}; y^n(\mathcal{D}_\Omega), H^n_1, \ldots, H^n_M) + H(W_{\mathcal{K}_a}| y^n(\mathcal{D}_\Omega), H^n_1, \ldots, H^n_M)
\]

\[
\leq I(W_{\mathcal{K}_a}; y^n(\mathcal{D}_\Omega), H^n_1, \ldots, H^n_M) + n\epsilon_n
\]

\[
\leq I(W_{\mathcal{K}_a}; y^n(\mathcal{D}_\Omega), H^n_1, \ldots, H^n_M) + n\epsilon_n
\]

\[
\leq I(W_{\mathcal{K}_a}; y^n(\Omega_D)| x^n(\Omega - \Omega'), H^n_1, \ldots, H^n_M) + n\epsilon_n
\]

\[
\leq I(x^n(\Omega_S); y^n(\Omega_D)| x^n(\Omega - \Omega'), H^n_1, \ldots, H^n_M) + n\epsilon_n
\]

\[
\leq H(y^n(\Omega_D)| x^n(\Omega - \Omega'), H^n_1, \ldots, H^n_M) + n\epsilon_n
\]

\[
\leq \sum_{t=1}^n H(y^n(\Omega_D)[t]| x^n(\Omega - \Omega'), H^n_1[t], \ldots, H^n_M[t]) + n\epsilon_n
\]

\[
\leq n\mathbb{E}(\text{rank}(H_{\Omega})) + n\epsilon_n,
\]

where \( \epsilon_n > 0 \) is an arbitrarily small constant satisfying \( \epsilon_n \to 0 \) as \( n \to \infty \). Notice that (a) holds from Fano’s inequality, (b) holds since the messages are independent with the channel matrices, (c) holds since \( W_{\mathcal{K}_a} \rightarrow (y^n(\Omega_D), H^n_1, \ldots, H^n_M) \) forms a Markov chain, (d) holds since \( x^n(\Omega - \Omega') \) is independent with \( W_{\mathcal{K}_a} \), (e) holds since \( W_{\mathcal{K}_a} \rightarrow (x^n(\Omega_S), x^n(\Omega - \Omega'), H^n_1, \ldots, H^n_M) \) forms a Markov chain, (f) holds since \( y^n(\Omega_D) \) is a deterministic function of \( x^n(\Omega_S), x^n(\Omega - \Omega'), H^n_1, \ldots, H^n_M \), (g) holds since conditioning reduces entropy, and (h) holds with equality if each element of \( x^n(\Omega_S) \) is uniformly distributed over \( \mathbb{F}_2 \). Therefore, we obtain (10), which completes the proof.

Theorem 1 shows that the aggregate rate of the S-D pairs divided by a cut is upper bounded by the average rank of the channel matrix constructed by the cut. As an example, consider the cut in Fig. 3 where \( \Omega = \{v_{1,1}, v_{2,1}, v_{3,1}, v_{2,2}, v_{3,2}, v_{3,3}, v_{3,4}\} \). Then we obtain \( \mathcal{D}_\Omega = \{v_{1,4}, v_{2,4}\}, \mathcal{S}_\Omega = \{v_{1,1}, v_{2,1}\}, \Omega_D = \{v_{2,2}, v_{1,4}\}, \Omega_S = \{v_{2,2}, v_{3,3}\} \), and \( H_{\Omega} = [h_{2,2,2,2,0}^T, [0, h_{1,3,3}^T]^T]^T \). Therefore, \( R_1 + R_2 \) is upper bounded by \( \mathbb{E}(\text{rank}(H_{\Omega})) = p_{2,2,2} + p_{1,3,3} \).

#### B. Rate Bounds for Single-hop and Multi-hop Networks

In this subsection, we obtain useful rate upper bounds from Theorem 1 which will be used to show the converses in Sections IV and V. Let us first consider single-hop networks, that is \( M = 1 \). If we set \( \Omega = \{v_k, 1\} \), then \( \sum_{i \in \mathcal{K}_a} R_i = R_k \) and \( H_{\Omega}[t] = h_{k,k,1}[t] \). Thus, we obtain

\[
R_k \leq p_{k,k,1}
\]
for all \( k \in \{1, \cdots, M\} \), which coincides with the point-to-point rate of the \( k \)-th S-D pair assuming no interference. From (12), the achievable sum-rate of single-hop networks is upper bounded by \( R_{\text{sum}} \leq \sum_{k=1}^{K} p_{k,k,1} \). We will show in Section IV that (12) is achievable if \( p_{k,k,1} = 1/2 \) for all \( k \) regardless of the channel distributions of the interfering links, which characterizes the capacity region of such single-hop networks.

Let us now consider multi-hop networks, that is \( M \geq 2 \). For general multi-hop networks, we set \( \Omega = \cup_{t \leq m} \mathcal{V}_t \), where \( m \in \{1, \cdots, M\} \). Then, from \( \sum_{k \in \mathcal{V}_t} R_k = R_{\text{sum}} \) and \( \mathbf{H}_\Omega[t] = \mathbf{H}_m[t] \), we obtain

\[
R_{\text{sum}} \leq \min_m \mathbb{E}(\text{rank}(\mathbf{H}_m)) \tag{13}
\]

for all \( m \in \{1, \cdots, M\} \), or equivalently \( R_{\text{sum}} \leq \mathbb{E}(\text{rank}(\mathbf{H}_{ma})) \). We will show in Section V that if the network has a minimum-dimensional bottleneck-hop and the channels are uniformly distributed, the above sum-rate bound is achievable, which characterize the sum capacity.

**IV. Achievability for Single-hop Networks**

In this section, we propose a transmission scheme for single-hop linear finite-field networks and derive an achievable rate region when we apply the proposed scheme. As mentioned in Introduction, each source can transmit one bit without interference by using two instances \( \mathbf{H}_1^{(1)} \) and \( \mathbf{H}_1^{(2)} \) jointly if

\[
\mathbf{H}_1^{(1)} + \mathbf{H}_1^{(2)} = \mathbf{I} \tag{14}
\]

The proposed block Markov encoding makes such pairing possible. As a simple example, we first study 2-2 networks and then extend the results of the two-user case to general \( K\)-\( K \) networks.

A. 2-2 Networks

Consider the 2-2 network with \( p_{j,i,1} = 1/2 \) for all \( i \) and \( j \). Fig. 4 (a) shows 16 possible instances of \( \mathbf{H}_1[t] \) that are equally distributed, where the dashed lines and the solid lines denote the corresponding channels are zeros and ones, respectively. The symbols in the figure denote the transmit signals of the sources and the nodes with no symbol transmit zeros, where \( s_k \) denotes the information bit of the \( k \)-th source. If we use each instance separately as shown in Fig. 4 (a), then \( R_{\text{sum}} = \frac{13}{16} \) is achievable. However, one can use some instances jointly as shown in Fig. 4 (b) to improve the achievable sum-rate to 1. More specifically, \( R_1 = R_2 = 1/2 \) are achievable. In the first case, for example, each source can transmit one bit using \( \mathbf{H}_1^{(1)} = \begin{bmatrix} [0,1]^T, [0,0]^T \end{bmatrix}^T \) and \( \mathbf{H}_1^{(2)} = \begin{bmatrix} [1,1]^T, [0,1]^T \end{bmatrix}^T \) jointly. Because the first destination receives the interference bit \( s_2 \) and the interference-added information bit \( s_1 + s_2 \) separately, it can decode the intended bit. Whereas if we use these two instances separately, then only one of the two sources can transmit one bit. Since the cut-set bound in (12) shows \( R_k \leq 1/2 \), this simple scheme indeed achieves the capacity region.

Based on this key observation, we will obtain an achievable rate region for general single-hop networks in the next subsection, which provides the capacity region when \( p_{k,k,1} = 1/2 \) for all \( k \in \{1, \cdots, K\} \).

B. General Single-hop Networks

Now consider the achievability for general single-hop networks. We assume the symmetric rate for all S-D pairs, that is \( R_1 = \cdots = R_K = R \). Similar to 2-2 networks, each source can transmit one bit by transmitting the same bit over two channel instances satisfying \( \mathbf{H}_1^{(1)} + \mathbf{H}_1^{(2)} = \mathbf{I} \). The following block Markov encoding makes such pairing of channel instances possible.

1) Block Markov encoding: Let us first divide a block into two sub-blocks having length \( n/2 \) for each sub-block. Define \( T^{[b]}(\mathbf{H}_1) \) as the set of time indices of the \( b \)-th sub-block whose channel realizations are equal to \( \mathbf{H}_1 \in \mathbb{F}_2^{K \times K} \), where \( b \in \{1,2\} \). We further define

\[
n(\mathbf{H}_1) = nR \min\{\Pr(\mathbf{H}_1), \Pr(\mathbf{H}_1 + \mathbf{I})\}c_1^{-1}, \tag{15}\]

where

\[
c_1 = \sum_{\mathbf{H}_1 \in \mathbb{F}_2^{K \times K}} \min\{\Pr(\mathbf{H}_1), \Pr(\mathbf{H}_1 + \mathbf{I})\}. \tag{16}\]
Notice that \( |T[0](\mathbf{H}_1)| \) is random depending on the channel realizations but \( n(\mathbf{H}_1) \) is a deterministic function of \( R \). Each source will transmit \( \sum_{\mathbf{H}_1} n(\mathbf{H}_1) \) bits during \( n \) channel uses or two sub-blocks. From (15) and (16), we can check that \( R \) is equal to \( \sum_{\mathbf{H}_1} n(\mathbf{H}_1) \).

The detailed encoding is as follows, where for simplicity we assume \( n(\mathbf{H}_1) \) is an integer.

- **(Encoding of the first sub-block)**
  For \( \mathbf{H}_1 \in \mathbb{F}_2^{K \times K} \), if \( |T[1](\mathbf{H}_1)| < n(\mathbf{H}_1) \) declare an error, otherwise each source transmits \( n(\mathbf{H}_1) \) information bits using the time indices in \( T[1](\mathbf{H}_1) \).

- **(Encode of the second sub-block)**
  For \( \mathbf{H}_1 \in \mathbb{F}_2^{K \times K} \), if \( |T[2](\mathbf{H}_1 + \mathbf{I})| < n(\mathbf{H}_1) \) declare an error, otherwise each source retransmits \( n(\mathbf{H}_1) \) information bits that was transmitted during \( T[1](\mathbf{H}_1) \) using the time indices in \( T[2](\mathbf{H}_1 + \mathbf{I}) \).

Let \( s_k(i) \) denote the \( i \)-th information bit of the \( k \)-th source, where \( i = \{1, \ldots, nR\} \). Let \( t_1(i) \) and \( t_2(i) \) denote the time indices over which \( s_k(i) \) was transmitted. Then the detailed decoding is as the follow.

- **(Decoding)**
  For \( i = \{1, \ldots, nR\} \), the \( k \)-th destination sets \( \hat{s}_k(i) = y_{k,1}[t_1(i)] + y_{k,1}[t_2(i)] \). Then estimate \( W_k \) based on \( nR \) estimated bits.

### C. Achievable Rate Region

In this subsection, we derive the achievable rate region of general single-hop networks by applying the proposed scheme.

Let \( E[1] \) denote the event such that \( |T[1](\mathbf{H}_1)| < n(\mathbf{H}_1) \) for any \( \mathbf{H}_1 \) and \( E[2] \) denote the event such that \( |T[2](\mathbf{H}_1 + \mathbf{I})| < n(\mathbf{H}_1) \) for any \( \mathbf{H}_1 \). The following lemma shows that, there is no error if \( (E[1] \cup E[2])^c \) occurs.

**Lemma 1:** Suppose a linear finite-field relay network with \( M = 1 \). The probability of error is upper bounded by

\[
P_{e,k}^{(n)} \leq \Pr(E[1]) + \Pr(E[2])
\]

for all \( k = \{1, \ldots, K\} \).

**Proof:** The proof is in Appendix I. \( \square \)

From the previous lemma, the probability of error can be arbitrarily small if both \( \Pr(E[1]) \) and \( \Pr(E[2]) \) tend to zero as \( n \) increases. In essence, although \( T[0](\mathbf{H}_1) \) is random, by the weak law of large numbers, we can bound \( |T[0](\mathbf{H}_1)| \) and make both \( \Pr(E[1]) \) and \( \Pr(E[2]) \) tend to zero as \( n \to \infty \) by appropriately setting the value of \( n(\mathbf{H}_1) \). From the definition of \( n(\mathbf{H}_1) \), this is equivalent is to determining the symmetric rate \( R \) that guarantees \( P_{e,k}^{(n)} \to 0 \) as \( n \to \infty \). The following theorem characterizes such \( R \).

**Theorem 2:** Suppose a linear finite-field relay network with \( M = 1 \). Then for any \( \delta > 0 \),

\[
R_k = \frac{1}{2} \sum_{\mathbf{H}_1 \in \mathbb{F}_2^{K \times K}} \min \{ \Pr(\mathbf{H}_1), \Pr(\mathbf{H}_1 + \mathbf{I}) \} - \delta
\]

is achievable for all \( k = \{1, \ldots, K\} \).

**Proof:** Let us consider \( |T[0](\mathbf{H}_1)| \). By the weak law of large numbers [30], there exists a sequence \( \epsilon_n \to 0 \) as \( n \to \infty \) such that the probability

\[
|T[1](\mathbf{H}_1)| \geq \frac{n}{2} (\Pr(\mathbf{H}_1) - \delta_n) \quad \text{for all } \mathbf{H}_1
\]

is greater than or equal to \( 1 - \epsilon_n \), where \( \delta_n \to 0 \) as \( n \to \infty \). This indicates that \( \Pr(E[1]) \leq \epsilon_n \) if \( n(\mathbf{H}_1) \leq \frac{n}{2} (\Pr(\mathbf{H}_1) - \delta_n) \) for all \( \mathbf{H}_1 \). Similarly, \( \Pr(E[2]) \leq \epsilon_n \) if \( n(\mathbf{H}_1) \leq \frac{n}{2} (\Pr(\mathbf{H}_1 + \mathbf{I}) - \delta_n) \) for all \( \mathbf{H}_1 \). Hence, from (15) and (17), \( P_{e,k}^{(n)} \leq 2\epsilon_n \) if

\[
R \leq \frac{c_1}{2} - \frac{c_1}{2 \min \{ \Pr(\mathbf{H}_1), \Pr(\mathbf{H}_1 + \mathbf{I}) \}} \delta_n
\]

for all \( \mathbf{H}_1 \). Thus we set \( R = \frac{c_1}{2} - \delta_n^* \), where \( \delta_n^* = \frac{c_1}{2 \min \{ \Pr(\mathbf{H}_1) \}} \delta_n \), which tends to zero as \( n \to \infty \). In conclusion,

\[
R = \frac{1}{2} \sum_{\mathbf{H}_1 \in \mathbb{F}_2^{K \times K}} \min \{ \Pr(\mathbf{H}_1), \Pr(\mathbf{H}_1 + \mathbf{I}) \} - \delta_n^*
\]
is achievable for all $k$. Since $R$ is the symmetric rate for all S-D pairs and $\delta_n^* \to 0$ as $n \to \infty$, Theorem 2 holds.

**Corollary 1:** Suppose a linear finite-field relay network with $M = 1$. If $p_{k,k,1} = 1/2$ for all $k \in \{1, \cdots, K\}$, the capacity region is given by all rate tuples $(R_1, \cdots, R_K)$ satisfying

$$R_k \leq \frac{1}{2}$$

for all $k \in \{1, \cdots, K\}$.

**Proof:** Since $\Pr(H_1) = \Pr(H_1 + I)$ for all $H_1$ if $p_{k,k,1} = 1/2$, from (21),

$$R = \frac{1}{2} \sum_{H_1 \in \mathbb{F}_q^{K \times K}} \Pr(H_1) - \delta_n^* = \frac{1}{2} - \delta_n^*$$

is achievable for all $k$. From the fact that $\delta_n^* \to 0$ as $n \to \infty$, the achievable rate region asymptotically coincides with the upper bound in (12), which provides the capacity region. Thus, Corollary 1 holds.

**Remark 1:** The result of Corollary 1 can be directly extended for $q$-ary case in which inputs, outputs, and channels are in $\mathbb{F}_q$, if the channels are uniformly distributed over $\mathbb{F}_q$. Specifically, the capacity region is given by all rate tuples $(R_1, \cdots, R_K)$ satisfying $R_k \leq \frac{1}{2} \log q$ for all $k \in \{1, \cdots, K\}$.

Corollary 1 shows that the sources can transmit simultaneously to their destinations at a rate of the point-to-point communication assuming no interference if the direct channels are uniformly distributed. This result also shows that, in the case of multi-source one-hop networks, the max-flow min-cut theorem holds for a certain class of channel distributions. Similar to the result of Gaussian one-hop networks in [28] where 1/2 degrees of freedom is achievable for each S-D pair, each source can transmit data to its destination with a non-vanishing rate even as $K$ tends to infinity.

**V. Achievability for Multi-hop Network**

In this section, we propose a transmission scheme for multi-hop linear finite-field relay networks and derive an achievable rate region. As mentioned in Introduction, due to the time-varying nature of wireless channels, each source can transmit one bit to its destination without interference through particular instances from $H_1$ to $H_M$ if

$$H_M H_{M-1} \cdots H_1 = I.$$  

(24)

The block Markov encoding and relaying structure makes a series of pairing from $H_1$ to $H_M$ possible. We first show an example 2-2-2 networks and then extend the idea to general multi-hop networks. The abbreviations used in this section is given by Table 1.

**A. 2-2-2 Networks**

Let us consider 2-2-2 networks with $p_{j,i,m} = 1/2$ for all $i$, $j$, and $m$. There are 16 possible instances for each $H_1[t]$ and $H_2[t]$, which are uniformly distributed. For each time $t$, if information bits are transmitted through $H_1[t]$ and $H_2[t+1]$, there exist 256 possible instances from $H_1[t]$ to $H_2[t+1]$ and

$$R_{\text{sum}} = \mathbb{E}(\text{rank}(H_2 H_1)) = \frac{177}{256}$$

(25)

is achievable. Notice that the achievable sum-rate is less than that of the single-hop case since $\mathbb{E}(\text{rank}(H_2 H_1)) < 1$, which is achievable for 2-2 networks.

However, we can achieve a sum-rate higher than $\frac{177}{256}$ and also higher than the single-hop case by appropriately pairing $H_1$ and $H_2$. Fig. 5 illustrates the deterministic pairing of $H_1$ and $H_2$ and related encoding and relaying. The dashed lines and the solid lines again denote the corresponding channels are ones and zeros, respectively. The symbols in the figure denote the transmit signals of the sources and the nodes with no symbol transmit zeros, where $s_k$ denotes the information bit of the $k$-th source. Then

$$R_{\text{sum}} = \mathbb{E}(\text{rank}(H_1)) = \frac{21}{16}$$

(26)

is achievable, which coincides with the cut-set upper bound in (13). Thus, this simple scheme achieves the sum capacity.
Based on the deterministic pairing in Fig. 5 we can characterize the sum capacity for more general channel distributions as shown in the following theorem. We will explain the detailed block Markov encoding which allows pairing between particular instances possible in the next subsection.

**Theorem 3:** Suppose a linear finite-field relay network with \( M = 2 \) and \( K_1 = K_2 = K_3 = 2 \). Then we can characterize sum capacity for the following cases.

1) For a symmetric channel satisfying \( p_{1,1,1} = p_{2,2,1} = p_{1,1,2} = p_{2,2,2} \) and \( p_{1,2,1} = p_{2,1,2} = p_{1,2,2} \) or a \( Z \) channel satisfying \( p_{2,1,1} = p_{2,1,2} = 0 \), \( p_{1,1,1} = p_{1,1,2} \), \( p_{1,2,1} = p_{2,2,2} \), and \( p_{2,2,1} = p_{2,2,2} \), the sum capacity is given by

\[
C_{\text{sum}} = \mathbb{E}(\text{rank}(H_1)).
\]

(27)

2) For a \( Z \) channel satisfying \( p_{2,1,1} = p_{2,1,2} = 0 \), \( p_{1,1,1} = p_{2,2,2} \), \( p_{1,2,1} = p_{2,2,2} \), and \( p_{2,2,1} = p_{1,1,2} \), the sum capacity is given by

\[
C_{\text{sum}} = \begin{cases} 
\mathbb{E}(h_{2,2,1}) + \mathbb{E}(h_{1,1,2}) & \text{if } p_{1,1,1} \geq p_{2,2,1} \\
\mathbb{E}(h_{1,1,1}) + \mathbb{E}(h_{2,2,2}) & \text{if } p_{1,1,1} < p_{2,2,1},
\end{cases}
\]

(28)

**Proof:** Let us first derive the achievable sum-rate by using the deterministic pairing in Fig. 5. Let \( p_m^{(1)} \) to \( p_m^{(16)} \) denote \( 16 \) possible instances of \( H_m \) as shown in Fig. 6 where \( m \in \{1,2\} \). In general, the probabilities of \( H_1 \) and \( H_2 \) in each pairing in Fig. 5 are not the same. However, we can construct \( n \min\{\Pr(H_1), \Pr(H_2)\} \) pairs for each pairing during a length-\( n \) block. Thus the achievable sum-rate is given by

\[
R_{\text{sum}} = \sum_{i \in \{2,4,6,9,11,13,16\}} \min\{p_1^{(i)}, p_2^{(i)}\} + \min\{p_1^{(3)}, p_2^{(5)}\} + \min\{p_1^{(5)}, p_2^{(3)}\} + 2 \sum_{i \in \{7,10,12,14\}} \min\{p_1^{(1)}, p_2^{(2)}\} + 2 \min\{p_1^{(8)}, p_2^{(15)}\} + 2 \min\{p_1^{(15)}, p_2^{(8)}\}.
\]

(29)

Let us now consider the first case in which the probabilities of \( H_1 \) and \( H_2 \) in each pairing are the same. Then, from (29),

\[
R_{\text{sum}} = \sum_{i \in \{2,3,4,5,6,9,11,13,16\}} p_1^{(i)} + 2 \sum_{i \in \{7,8,10,12,14,15\}} p_1^{(i)} = \mathbb{E}(\text{rank}(H_1)).
\]

(30)

Since the achievable sum-rate coincides with the sum-rate upper bound in (13), it characterizes the sum capacity.

Now consider the second case. Because \( p_{2,1,1,2} = 0 \), there are \( 8 \) possible instances of \( H_m \). Unlike the first case, the probabilities of \( H_1 \) and \( H_2 \) in each pair in Fig. 5 are not the same. Let us denote \( p_a = p_{1,1,1} = p_{2,2,2} \), \( p_b = p_{1,2,1} = p_{2,1,2} \), and \( p_c = p_{2,1,1} = p_{1,1,2} \). For \( p_a \geq p_c \), from (29),

\[
R_{\text{sum}} = p_1^{(2)} + p_1^{(6)} + p_2^{(9)} + p_2^{(13)} + 2p_1^{(10)} + 2p_1^{(14)}
\]

\[
= (1 - p_a)(1 - p_b)p_c + (1 - p_a)p_b p_c + (1 - p_a)(1 - p_b)p_c + (1 - p_a)p_b p_c
\]

\[
+ 2p_a(1 - p_b)p_c + 2p_a p_b p_c
\]

\[
= p_c = \mathbb{E}(h_{2,2,1}) + \mathbb{E}(h_{1,1,2}).
\]

(31)

If we consider \( \Omega_1 \) in Fig. 7, \( R_{\text{sum}} \leq \mathbb{E}(\text{rank}([h_{2,2,1}, 0, 0, h_{1,1,2}]^T)) = \mathbb{E}(h_{2,2,1}) + \mathbb{E}(h_{1,1,2}), \) which coincides with the achievable sum-rate. Similarly, for \( p_a < p_c \), from (29),

\[
R_{\text{sum}} = p_2^{(10)} + p_1^{(13)} + 2p_1^{(10)} + 2p_1^{(14)} = \mathbb{E}(h_{1,1,1}) + \mathbb{E}(h_{2,2,2}).
\]

(32)

If we consider \( \Omega_2 \) and \( \Omega_3 \) in Fig. 7, \( R_1 \leq \mathbb{E}(h_{1,1,1}) \) and \( R_2 \leq \mathbb{E}(h_{2,2,2}) \), respectively. Then \( R_{\text{sum}} \leq \mathbb{E}(h_{1,1,1}) + \mathbb{E}(h_{2,2,2}), \) which coincides with the achievable sum-rate. In conclusion, Theorem 3 holds.

**B. General Multi-hop Networks**

In this subsection, we study the achievability for linear finite field relay networks when \( M \geq 2 \) and \( p_{j,i,m} = p \) for all \( i, j, \) and \( m \). We assume that the considered network has a minimum-dimensional bottleneck-hop and derive a symmetric rate region, that is \( R_1 = \cdots = R_K \).

If a series of pairing from \( H_1 \) to \( H_M \) satisfies the condition (24), each destination can receive one bit without interference. But if some instances are rank-deficient, we cannot find such pairs by using these rank-deficient
instances. Furthermore, the number of possible pairs increases exponentially as the number of nodes in a layer or the number of layers increases. Even for 3-3-3 networks, we should consider $2^3 \times 3^3$ times $2^3 \times 3^3$ possible candidates. Thus, instead of constructing deterministic pairs for all instances, we randomize a series of pairing such that $H_m$ is paired with one instance at random in a subset of $H_{m+1}$’s.

1) Block Markov encoding and relaying: The proposed scheme divides a block into $B + M - 1$ sub-blocks having length $n_B$ for each sub-block, where $n_B = \frac{n}{B + M - 1}$. Since block Markov encoding and relaying are applied over $M$ hops, the number of effective sub-blocks is equal to $B$. Thus, the overall rate is given by $\frac{B}{B + M - 1} R$, where $R$ is the symmetric rate of each sub-block. As $n \rightarrow \infty$, the fractional rate loss $1 - \frac{B}{B + M - 1}$ will be negligible because we can make both $n_B$ and $B$ large enough. For simplicity, we omit the block index in describing the proposed scheme.

2) Balancing the average rank of each hop: Recall that the $m_{th}$ hop becomes a bottleneck for the entire multi-hop transmission, which can be seen from the sum-rate upper bound in (13). As an example, consider 3-2-3 networks in which the second hop becomes a bottleneck. If each source transmits at a rate of $\frac{1}{K} \mathbb{E}(\text{rank}(H_1))$, then it will cause an error at the second hop. To prevent this error event, the rate of each source should be decreased to $\frac{1}{K} \mathbb{E}(\text{rank}(H_{m_0}))$. For this reason, we select $V_{m,tx}[t] \subseteq V_m$ and $V_{m,rx}[t] \subseteq V_{m+1}$ randomly such that

$$(V_{m,tx}[t], V_{m,rx}[t]) \in V(K_{m_0}, K_{m_0+1}, V_m, V_{m+1})$$

with equal probabilities (or in $V(K_{m_0+1}, K_{m_0}, V_m, V_{m+1})$). For each time $t$, only the nodes in $V_{m,tx}[t]$ and $V_{m,rx}[t]$ will become active at the $m$-th hop. Notice that since the considered network has a minimum-dimensional bottleneck-hop, it is possible to construct such $V_{m,tx}[t]$ and $V_{m,rx}[t]$. In the case of 3-2-3 networks, only the nodes in $V_{1,tx}[t]$ and $V_{1,rx}[t]$ satisfying $(|V_{1,tx}[t]|, |V_{1,rx}[t]|) = (2, 2)$ become active at the first hop. The same is true for the last hop. Whereas the whole nodes in $V_2$ and $V_3$ become active at the second hop, that is $V_{2,tx}[t] = V_2$ and $V_{2,rx}[t] = V_3$.

The following lemma shows the probability distribution of $H_{V_{m,tx}[t], V_{m,rx}[t][t]}$, which will be used to derive the achievable rate region of general multi-hop networks.

**Lemma 2:** Suppose a linear finite-field relay with $p_{j,i,m} = p$ for all $i$, $j$, and $m$. If the network has a minimum-dimensional bottleneck-hop, then

$$\text{Pr}(H_{V_{m,tx}[t], V_{m,rx}[t][t]} = H) = p^u(1 - p)^{K_{m_0+1}K_{m_0} - u},$$

where $u$ is the number of zeros in $H$.

**Proof:** The proof is in Appendix II.

The probability distribution of $H_{V_{m,tx}[t], V_{m,rx}[t][t]}$ is the same as that of $H_{m_0}[t]$, which is the channel matrix of the bottleneck-hop. Thus if only the nodes in $V_{m,tx}[t]$ and $V_{m,rx}[t]$ are activated at the $m$-th hop, each hop can deliver information bits that are sustainable at the bottleneck-hop.

3) Construction of sets of transmit and receive nodes: Because the maximum number of bits transmitted at the $m$-th hop is determined by $\text{rank}(H_{V_{m,tx}[t], V_{m,rx}[t][t]})$, we further select $\tilde{V}_{m,tx}[t] \subseteq V_{m,tx}[t]$ and $\tilde{V}_{m,rx}[t] \subseteq V_{m,rx}[t]$ randomly such that

$$(\tilde{V}_{m,tx}[t], \tilde{V}_{m,rx}[t]) \in V(H_{V_{m,tx}[t], V_{m,rx}[t][t]})$$

with equal probabilities. For each time $t$, the nodes in $\tilde{V}_{m,tx}[t]$ transmit and the nodes in $\tilde{V}_{m,rx}[t]$ receive through the channel $H_{V_{m,tx}[t], V_{m,rx}[t][t]}$ at the $m$-th hop. Then, as we will show later, information bits can be transmitted using particular time indices $t_1, \ldots, t_M$ such that

$$H_{\tilde{V}_{m,tx}[t_M], \tilde{V}_{m,rx}[t_M][t_M]} \cdots H_{\tilde{V}_{1,tx}[t_1], \tilde{V}_{1,rx}[t_1][t_1]} = I,$$

which guarantees interference-free reception at the destinations. One of the simplest ways is to set $H_{\tilde{V}_{1,tx}[t_1], \tilde{V}_{1,rx}[t_1][t_1]} = G$ and $H_{\tilde{V}_{m,tx}[t], \tilde{V}_{m,rx}[t][t]} = (G^{M-1})^{-1}$. It is possible to construct those pairs because the resulting $H_{\tilde{V}_{m,tx}[t], \tilde{V}_{m,rx}[t][t]}$ is always invertible. There is no rate loss by using $(\tilde{V}_{m,tx}[t], \tilde{V}_{m,rx}[t][t])$ instead of using $(\tilde{V}_{m,tx}[t], V_{m,rx}[t][t])$ because $\text{rank}(H_{\tilde{V}_{m,tx}[t], V_{m,rx}[t][t]}) = \text{rank}(H_{\tilde{V}_{m,tx}[t], \tilde{V}_{m,rx}[t][t]})$.

The following lemmas show the probability distributions related to $H_{\tilde{V}_{m,tx}[t], \tilde{V}_{m,rx}[t][t]}$, which will be used to derive the achievable rate region of general multi-hop networks.

\footnote{We ignore the instances having all zeros, which give zero rate.}
Lemma 3: Suppose a linear finite-field relay network with \( p_{j,i,m} = p \) for all \( i, j, \) and \( m \). If the network has a minimum-dimensional bottleneck-hop, then for \( \text{rank}(G) = r \neq 0 \), we obtain

\[
\Pr(H) = \sum_{(V',V'') \in \mathcal{H}_{V''}^{G_m,V_{m+1}}(G,V',V'')} \frac{\Pr(H)}{|V(H)|},
\]

where \( \Pr(H) \) is given by (34).

Proof: The proof is in Appendix II.

Lemma 4: Suppose a linear finite-field relay network with \( p_{j,i,m} = p \) for all \( i, j, \) and \( m \). If the network has a minimum-dimensional bottleneck-hop, then for \( \text{rank}(G) = r \neq 0 \), we obtain

\[
\Pr(H) = \sum_{(V',V'') \in \mathcal{H}_{V''}^{G_m,V_{m+1}}(G,V',V'')} \frac{\Pr(G)}{(K_m)(K_{m+1})},
\]

where \( \Pr(G) \) is given by (37).

Proof: The proof is in Appendix II.

Based on Lemma 3, we derive \( \Pr(G) \) when \( p_{j,i,m} = 1/2 \). The important aspect is that the resulting \( \Pr(G) \) is a function of \( \text{rank}(G) \).

Lemma 5: Suppose a linear finite-field relay network with \( p_{j,i,m} = 1/2 \) for all \( i, j, \) and \( m \). If the network has a minimum-dimensional bottleneck-hop, then for \( \text{rank}(G) = r \neq 0 \), we obtain

\[
\Pr(H) = 2^{-K_{m+1}K_{m}}N_{K_{m+1},K_{m}}(r)N_{r,p}(r),
\]

where \( N_{a,b}(i) \) is the number of instances in \( \mathbb{F}_2^{a \times b} \) having rank \( i \).

Proof: The proof is in Appendix II.

4) Encoding, relaying, and decoding functions: In this part, we explain the encoding, relaying, and decoding scheme based on \( V_{m,tx}[t] \) and \( V_{m,rx}[t] \). Let us define \( \mathcal{T}_m(G, V'_m, V'_{m+1}) \) as the set of time indices of the sub-block at the \( m \)-th hop satisfying \( V_{tx,m}[t] = V'_m, V_{mx,m}[t] = V'_{m+1} \), and \( H_{V'_m, V'_{m+1}}[t] = G \), where \( m \in \{ 1, \cdots, M \} \). We further define

\[
n(G) = n_B R \min \left\{ \Pr(G), \Pr\left((G^{M-1})^{-1}\right) \right\} c_2^{-1},
\]

where

\[
c_2 = \frac{1}{K} \sum_{i=1}^{K_{\min}} \sum_{G \in \mathbb{F}_2^{i \times i}, \text{rank}(G) = i} \min \{ \Pr(G), \Pr((G^{M-1})^{-1}) \}.
\]

Each source will transmit \( \frac{1}{K} \sum_G \text{rank}(G)n(G) \) bits during \( n_B \) channel uses. From (40) and (41), we can check that \( R \) is equal to \( \frac{K_{\min}}{K} \sum_G \text{rank}(G)n(G) \).

Let us consider the detailed encoding and relaying procedure. For all full-rank matrices \( G \in \bigcup_{j=1}^{K_{\min}} \mathbb{F}_2^{i \times i} \), the encoding and relaying are as follows, where \( r = \text{rank}(G) \) and for simplicity we assume \( n(G)/(\binom{K_m}{r}\binom{K_{m+1}}{r}) \) is an integer.

- **(Encoding)**
  - For all \( (V'_0, V''_0) \in \mathcal{V}(r, V_0, V_0) \), if \( |T_0(G, V'_0, V''_0)| < n(G)/(\binom{K_0}{r}\binom{K_0}{r}) \) declare an error, otherwise each source in \( V'_0 \) transmits \( n(G)/(\binom{K_0}{r}\binom{K_0}{r}) \) information bits using the time indices in \( T_0(G, V'_0, V''_0) \) to the nodes in \( V''_0 \).
  - **(Relaying for \( m \in \{ 2, \cdots, M-1 \} \))**
    - For all \( (V'_m, V'_{m+1}) \in \mathcal{V}(r, V_m, V_{m+1}) \), if \( |T_m(G, V'_m, V'_{m+1})| < n(G)/(\binom{K_m}{r}\binom{K_{m+1}}{r}) \) declare an error, otherwise each node in \( V'_m \) relays \( n(G)/(\binom{K_m}{r}\binom{K_{m+1}}{r}) \) bits using the time indices in \( T_m(G, V'_m, V'_{m+1}) \) to the nodes in \( V'_{m+1} \), where the transmit bits are constructed by evenly allocating the received bits that arrive from difference paths.
  - **(Relaying for \( m = M \))**
    - For all \( (V'_M, V'_{M+1}) \in \mathcal{V}(r, V_M, V_{M+1}) \), if \( |T_M(G^{M-1})^{-1}, V'_M, V'_{M+1})| < n(G)/(\binom{K_M}{r}\binom{K_{M+1}}{r}) \) declare an error, otherwise each node in \( V'_M \) relays \( n(G)/(\binom{K_M}{r}\binom{K_{M+1}}{r}) \) bits to the destinations in \( V'_{M+1} \) using the time
indices in $T_M((G^{M-1})^{-1}, V'_m, V'_{m+1})$. The nodes in $V'_m$ relay all received bits that originated from $V'_1$ to the nodes in $V'_{m+1}$, where $V'_1$ is the set of the corresponding sources of the destinations in $V'_{m+1}$.

Let $s_k(i)$ denote the $i$-th information bit of the $k$-th source and $t_{k,m}(i)$ denote the time index that the received signal at the $m$-th hop originated from $s_k(i)$, where $i \in \{1, \cdots, 2^nR\}$. That is, $s_k(i)$ is transmitted using the time indices $t_{k,1}(i)$ to $t_{k,M}(i)$ during the multi-hop transmission. The detailed decoding of the $k$-th destination is as follows.

- (Decoding)

For $i \in \{1, \cdots, nBR\}$, set $\hat{s}_k(i) = y_{k,M}[t_{k,M}(i)]$. Then estimate $W_k$ based on $nR_k$ estimated bits.

Fig. 8 illustrates the encoding and relaying of 3-3-3-3 networks when $G = [(1, 1)^T, [0, 1]^T]^T$. The nodes represented as filled circles at the $m$-th hop denote the nodes belonging to $V_{m,tx}[t]$ or $V_{m,rx}[t]$, where the channel between $V_{m,tx}[t]$ and $V_{m,rx}[t]$ are only denoted in the figure. If we assume $|T_1(G, V'_1, V'_2)| \geq n(G)/9$, $|T_2(G, V'_2, V'_3)| \geq n(G)/9$, and $|T_3((G^2)^{-1}, V'_3, V'_4)| \geq n(G)/9$, then we can prove that $\text{rank}(G)n(G)$ bits can be delivered using the time indices in these sets. The formal proof including how to determine the value of $n(G)$, equivalently $R$, that makes $P_e(n_R) \to 0$ as $n_B \to \infty$ will be shown in the next subsection.

Let us focus on the transmission from the sources in $\{v_{1,1}, v_{2,1}\}$ to the corresponding destinations in $\{v_{1,4}, v_{2,4}\}$. At the first hop, each source in $\{v_{1,1}, v_{2,1}\}$ transmits $n(G)/9$ information bits each to the nodes in $\{v_{1,2}, v_{2,2}\}$, $\{v_{1,2}, v_{3,2}\}$, and $\{v_{2,2}, v_{3,2}\}$ using the time indices in $T_1(G, \{v_{1,1}, v_{2,1}\}, \{v_{1,2}, v_{2,2}\})$, $T_1(G, \{v_{1,1}, v_{2,1}\}, \{v_{1,2}, v_{3,2}\})$, and $T_1(G, \{v_{1,1}, v_{2,1}\}, \{v_{2,2}, v_{3,2}\})$, respectively. Then, at the end of the first hop, the nodes in $\{v_{1,2}, v_{2,2}\}$ receive $n(G)/9$ bits each from the sources in $\{v_{1,1}, v_{2,1}\}$, $\{v_{1,1}, v_{3,1}\}$, and $\{v_{2,1}, v_{3,1}\}$. The same is true for $\{v_{1,2}, v_{3,2}\}$ and $\{v_{2,2}, v_{3,2}\}$.

Let us now consider the second hop. Each node in $\{v_{1,2}, v_{2,2}\}$ relays the received bits to the nodes in $\{v_{1,3}, v_{2,3}\}$, $\{v_{1,3}, v_{3,3}\}$, and $\{v_{2,3}, v_{3,3}\}$ using the time indices in $T_2(G, \{v_{1,2}, v_{2,2}\}, \{v_{1,3}, v_{2,3}\})$, $T_2(G, \{v_{1,2}, v_{2,2}\}, \{v_{1,3}, v_{3,3}\})$, and $T_2(G, \{v_{1,2}, v_{2,2}\}, \{v_{2,3}, v_{3,3}\})$, respectively. Note that each node in $\{v_{1,2}, v_{2,2}\}$ construct $n(G)/9$ transmit bits to be delivered to $\{v_{1,3}, v_{2,3}\}$ by evenly choosing the received bits that originated from the sources in $\{v_{1,1}, v_{2,1}\}$, $\{v_{1,2}, v_{3,1}\}$, and $\{v_{1,1}, v_{2,1}\}$, respectively. Similarly, the nodes in $\{v_{1,2}, v_{3,2}\}$ and $\{v_{2,2}, v_{3,2}\}$ transmit their received bits.

Then at the last hop, the destinations in $\{v_{1,4}, v_{2,4}\}$ should collect all received signals that originated from the sources in $\{v_{1,1}, v_{2,1}\}$. This can be done since at the end of the second hop, each node in $\{v_{1,3}, v_{2,3}\}$ receives $n(G)/9$ bits that originated from the sources in $\{v_{1,1}, v_{2,1}\}$, which is less than $|T_3((G^2)^{-1}, V'_3, V'_4)|$ from the assumption. Hence, each source in $\{v_{1,3}, v_{2,3}\}$ can transmit $n(G)/3$ bits to the corresponding destinations in $\{v_{1,4}, v_{2,4}\}$. Because the same number of bits are transmitted from $\{v_{1,4}, v_{3,4}\}$ to $\{v_{1,4}, v_{3,4}\}$ and from $\{v_{2,1}, v_{3,1}\}$ to $\{v_{2,4}, v_{3,4}\}$, we can deliver $n(G)n(G)$ bits by using the time indices such that $H_{V_{m,tx}[t], V_{m,rx}[t]} = G$ for $m = \{1, 2\}$ and $H_{V_{3,tx}[t], V_{3,rx}[t]} = (G^2)^{-1}$.

C. Achievable Rate Region

In this subsection, we derive an achievable rate region by applying the proposed block Markov encoding and relaying. We first show in Lemma 9 that each destination can receive information bits without interference if there is no encoding and relaying error. Then in Theorem 4 we obtain the value of $R$ that makes the probabilities of the encoding and relaying errors arbitrarily small.

For $m \in \{1, \cdots, M - 1\}$, let $E_m$ denote the event such that

$$|T_m(G, V'_m, V'_{m+1})| < \frac{n(G)}{\binom{K_m}{\text{rank}(G)} \binom{K_{m+1}}{\text{rank}(G)}}$$

for any $V'_m$, $V'_{m+1}$, and $G$. Similarly, let $E_M$ denote the event such that

$$|T_M(G, V'_M, V'_{M+1})| < \frac{n(G^{M-1})^{-1}}{\binom{K_M}{\text{rank}(G)} \binom{K_{M+1}}{\text{rank}(G)}}$$

for any $V'_M$, $V'_{M+1}$, and $G$. The following lemma shows that there is no error if $(\cup_{m=1}^M E_m)^c$ occurs.
**Lemma 6:** Suppose a linear finite-field relay network having a minimum-dimensional bottleneck-hop with \( M \geq 2 \) and \( p_{j,i,m} = p \) for all \( i, j, \) and \( m \). The probability of error is upper bounded by

\[
P_{e,k}^{(n_B)} \leq \sum_{m=1}^{M} \Pr(E_m)
\]

for all \( k \in \{1, \cdots, K\} \).

**Proof:** The proof is in Appendix I.

Then the remaining thing is to derive the symmetric rate \( R \) that guarantees \( P_{e,k}^{(n_B)} \to 0 \) as \( n_B \to \infty \). The following theorem characterizes such \( R \).

**Theorem 4:** Suppose a linear finite-field relay network with \( M \geq 2 \). If the network has a minimum-dimensional bottleneck-hop and \( p_{j,i,m} = p \) for all \( i, j, \) and \( m \), then for any \( \delta > 0 \),

\[
R_k = \frac{1}{K} \sum_{i=1}^{K} \sum_{G \in \mathbb{F}_2^{p_{j,i,m}}} \min \{ \Pr(G), \Pr((G^M)^{-1}) \} - \delta
\]

is achievable for all \( k \in \{1, \cdots, K\} \), where \( \Pr(G) \) is given by (37).

**Proof:** Let us first consider \( |T_m(G, V'_m, V'_{m+1})| \), where \( r = \text{rank}(G) \). By the weak law of large numbers [30], there exists a sequence \( \epsilon_{nB} \to 0 \) as \( n_B \to \infty \) such that the probability

\[
|T_m(G, V'_m, V'_{m+1})| \geq n_B (\Pr(G, V'_m, V'_{m+1}) + \delta_{nB})
\]

for all \( G, V'_m, \) and \( V'_{m+1} \) is greater than or equal to \( 1 - \epsilon_{nB} \), where \( \delta_{nB} \to 0 \) as \( n_B \to \infty \). This indicates that, for \( m \in \{1, \cdots, M-1\} \), \( \Pr(E_m) \leq \epsilon_{nB} \) if

\[
n(G) \leq n_B \left( \Pr(G) - \left( \begin{array}{c} K_m \\ r \end{array} \right) \left( \begin{array}{c} K_{m+1} \\ r \end{array} \right) \delta_{nB} \right)
\]

for all \( G \). Note that we use the fact that \( \Pr(G, V'_m, V'_{m+1}) = \Pr(G)/\left( \left( \begin{array}{c} K_m \\ r \end{array} \right) \left( \begin{array}{c} K_{m+1} \\ r \end{array} \right) \right) \). Similarly, \( \Pr(E_M) \leq \epsilon_{nB} \) if

\[
n(G) \leq n_B \left( \Pr \left( (G^M)^{-1} \right) - \left( \begin{array}{c} K_M \\ r \end{array} \right) \left( \begin{array}{c} K_{M+1} \\ r \end{array} \right) \delta_{nB} \right)
\]

for all \( G \). Then, \( P_{e,k}^{(n_B)} \leq M \epsilon_n \) if (47) and (48) hold for all \( G \). This condition can be satisfied if

\[
R \leq c_2 - \frac{c_2 (K_{\max})^2 \delta_{nB}}{\min \{ \Pr(G), \Pr((G^M)^{-1}) \}}
\]

for all \( G \), where we use the definition of \( n(G) \) in (40) and the fact that \( \left( \begin{array}{c} K_m \\ r \end{array} \right) \leq K_{\max}! \). Thus we set \( R = c_2 - \delta_{nB}^* \), where \( \delta_{nB}^* = \frac{c_2 (K_{\max})^2 \delta_{nB}}{\min \{ \Pr(G), \Pr((G^M)^{-1}) \}} \), which tends to zero as \( n_B \to \infty \). In conclusion, we obtain

\[
R = \frac{1}{K} \sum_{i=1}^{K} \sum_{G \in \mathbb{F}_2^{p_{j,i,m}}} \min \{ \Pr(G), \Pr((G^M)^{-1}) \} - \delta_{nB}^*
\]

is achievable. Since \( R \) is the symmetric rate and \( \delta_{nB}^* \to 0 \) as \( n_B \to \infty \), Theorem 4 holds.

Now let us consider the capacity achieving case. If \( \Pr(G) = \Pr((G^M)^{-1}) \) for all possible \( G \), then the achievable sum-rate in Theorem 4 will coincide with the upper bound in (13). When the channel instances are uniformly distributed, the above condition holds and, as a result, the sum capacity can be characterized. The following corollary shows that the sum capacity is given by the average rank of the channel matrix of the bottleneck-hop when \( p = 1/2 \).

**Corollary 2:** Suppose a linear finite-field relay network with \( M \geq 2 \). If the network has a minimum-dimensional bottleneck-hop and \( p_{j,i,m} = 1/2 \) for all \( i, j, \) and \( m \), the sum capacity is given by

\[
C_{\text{sum}} = 2^{-K_{m+1}K_{m}} \sum_{H \in \mathbb{F}_2^{K_{m+1}+K_{m}}} \text{rank}(H).
\]
Proof: Consider the case that $p_{j,i,m} = 1/2$. Then, from (39), $\Pr(G)$ is a function of $\text{rank}(G)$. Since $G$ and $(G^{-1})^{-1}$ have the same rank, $\Pr(G) = \Pr((G^{-1})^{-1})$ for all possible $G$. Hence (50) is given by

$$R = \frac{1}{K} \sum_{i=1}^{K_{\text{min}}} \sum_{G \in \mathbb{F}_q^{K_{m+1} \times K_{m}} \mid \text{rank}(G) = i} \Pr(G) - \delta_{n_B}^*$$

$$= \frac{1}{K} 2^{-K_{m+1}K_{m}} \sum_{i=1}^{K_{\text{min}}} i N K_{m+1}K_{m} (i) - \delta_{n_B}^*$$

$$= \frac{1}{K} 2^{-K_{m+1}K_{m}} \sum_{\text{rank}(H) \leq \delta_{n_B}^*} \text{rank}(H) - \delta_{n_B}^*,$$

(52)

where we use the fact that $\sum_{G \in \mathbb{F}_q^{K_{m+1} \times K_{m}} \mid \text{rank}(G) = i} = N_{i,i}(i)$ and (39) for the second equality. Since $\delta_{n_B}^* \to 0$ as $n_B \to \infty$ the achievable sum-rate $K \bar{R}$ asymptotically coincides with the upper bound in (13), which completes the proof.

Remark 2: The result of Corollary 2 can be directly extended for $q$-ary case in which inputs, outputs, and channels are in $\mathbb{F}_q$. Specifically, the sum capacity is given by $C_{\text{sum}} = q^{-K_{m+1}K_{m}} \sum_{H \in \mathbb{F}_q^{K_{m+1} \times K_{m}}} \text{rank}(H) \log q$.

Notice that Corollary 2 shows that the sum-rate of $\mathbb{E}(\text{rank}(H_{m_0}))$ is achievable, that is the multi-input multi-output (MIMO) capacity of the bottleneck-hop. This result also shows that, in the case of multi-source multi-hop networks, the max-flow min-cut theorem holds for a certain class of channel distributions and network topologies.

Fig. 10 plots sum-rates of two-hop networks, where we assume a linear finite-field relay network with $p_{j,i,m} = p$. For 2-2-2 networks, the sum capacity is given by $C_{\text{sum}} = 4pq^3 + 8p^2q^2 + 8p^3q + p^4$, where $q = 1 - p$. Notice that the considered channel distribution is a special case of the symmetric channel in Theorem 3. Therefore, we can characterize the sum capacity for all $p \in [0,1]$. For 3-3-3 networks, we obtain $C_{\text{sum}} \geq 9pq^3 + 54pq^2q^7 + 168p^3q^6 + 279p^4q^5 + 216p^5q^4 + 72p^6q^3 + 216 \text{min}(p^3q^4, q^6p^4, 90p^6q^2 + 18p^6q + p^9)$ and $C_{\text{sum}} \leq 9pq^3 + 54pq^2q^7 + 168p^3q^6 + 279p^4q^5 + 324p^5q^4 + 198p^6q^3 + 90p^7q^2 + 18p^8q + p^9$. The lower and upper bounds are the same when $p = \frac{1}{2}$, which coincides with the result of Corollary 2 (if $p = 0$ or 1 the lower and upper bounds are trivially the same).

VI. CONCLUSION

In this paper, we studied fading linear finite-field relay networks, which exhibit broadcast, interference, and fading natures of wireless communication. Capacity characterization of such relay networks with multiple S-D pairs is quite challenging because the transmission of other session acts as inter-user interference. One of the main interests will be the possibility of the extension of the max-flow min-cut interpretation to the multi-source problem. We observed that the fading can play an important role in mitigating interference that leads to the capacity characterization for some classes of channel distributions and network topologies. For these classes, we showed that the capacity region of single-hop networks and the sum capacity of multi-hop networks can be interpreted as the max-flow min-cut theorem.

APPENDIX I

UPPER BOUND ON THE PROBABILITY OF ERROR

Proof of Lemma 7: Let us assume that $(E[1] \cup E[2])^c$ occurs. From the assumption, each source can transmit $n(H_1)$ bits using the time indices in $|T[1](H_1)|$ and $|T[2](H_1)|$, respectively.

Then consider the estimated bit $\hat{s}_k(i)$ at the $k$-th destination. Since the effective channel matrix after adding two received signals is given by

$$H_1[t_1(i)] + H_1[t_2(i)] = I,$$

(53)

we obtain $\hat{s}_k(i) = s_k(i)$. Hence there is no error if $(E[1] \cup E[2])^c$ occurs. In conclusion, from the union bound, we obtain $P_{e,k}^{(n)} \leq \Pr(E[1]) + \Pr(E[2])$, which completes the proof.

Proof of Lemma 6: Let us assume that $(\cup_{m=1}^M E_m)^c$ occurs. From the assumption, for $m \in \{1, \cdots, M - 1\}$, each node in $V_m^\prime$ can transmit $n(G)/(\text{rank}(G) \text{rank}(G))$ bits to the nodes in $V_{m+1}^\prime$ using the time indices in
$T_m(G, V'_m, V'_{m+1})$. Similarly each node in $V'_m$ can transmit $n(G)/((K_m^r)_{\text{rank}(G)})$ bits to the nodes in $V'_{m+1}$ using the time indices in $T_m(G^{M-1}, V'_m, V'_{M+1})$.

For given $G$ and $V'_m$, each node in $V'_m$ receives $n(G)/((K_m^r)_{\text{rank}(G)})$ bits and then relays $n(G)/((K_m^r)_{\text{rank}(G)})$ received bits to the nodes in $V'_{m+1}$, where $r = \text{rank}(G)$ and $m \in \{2, \cdots, M - 1\}$. Since there exist $(K_m^r)$ possible $V'_{m+1}$'s, the total number of received bits is the same as the total number of transmit bits at each node in $V'_m$. Thus each node in $V'_m$ can form $n(G)/((K_m^r)_{\text{rank}(G)})$ transmit bits by selecting the same number of received bits that arrive from different paths, and then relays them to the nodes in $V'_{m+1}$.

Let us now consider the last hop. Suppose that the nodes in $V'_{M+1}$ are the corresponding destinations of the sources in $V'_1$. Then, at the last hop, the nodes in $V'_{M+1}$ should receive all bits that originated from the sources in $V'_1$. Note that, for given $G$ and $V'_M$, the number of the received bits of each node in $V'_m$ that originated from $V'_1$ is given by $n(G)/((K_1^r)_{\text{rank}(G)})$. Thus, each node in $V'_M$ can relay these bits to the nodes in $V'_{M+1}$ since $n(G)/((K_1^r)_{\text{rank}(G)})$, the number of bits able to transmit to $V'_{M+1}$ is the same as $n(G)/((K_1^r)_{\text{rank}(G)})$, where we use the fact that $K = K_1 = K_{M+1}$.

Lastly, consider the estimated bit $s_k(i)$ at the $k$-th destination. Since the overall channel matrix from $V_{1x,1}[t_k,1(i)]$ and $V_{tx,M}[t_k,M(i)]$ is given by

$$
H_{V_{tx,M}[t_k,1,i],V_{tx,M}[t_k,M,i],V_{tx,M}[t_k,1,i],V_{tx,M}[t_k,1,i]} = I,
$$

we obtain $s_k(i) = s_k(i)$. Hence there is no error if $(\bigcup_{m=1}^{M} E_m)$ occurs. In conclusion, from the union bound, we obtain $P_{e,k} \leq \sum_{m=1}^{M} Pr(E_m)$, which completes the proof.

## APPENDIX A

### PROBABILITY DISTRIBUTIONS FOR SUB-CHANNEL MATRICES

**Proof of Lemma 2** We assume that $|V_{m,tx}[t]| = K_{m_0}$ and $|V_{m,rx}[t]| = K_{m_0+1}$ for the proof. But the same result holds for the case that $|V_{m,tx}[t]| = K_{m_0+1}$ and $|V_{m,rx}[t]| = K_{m_0}$. For a given $H$, we obtain

$$
Pr(H) = \sum_{(V', V'') \in \mathcal{V}(K_{m_0}, K_{m_0+1}, V_{m}, V_{m+1})} \sum_{H_m \in \mathbb{F}_2^{K_{m_0+1} \times K_m}} Pr(H_m) Pr(V', V''|H_m) Pr(H|H_m, V', V'')
$$

\((a)\) \[ \equiv \sum_{(V', V'') \in \mathcal{V}(K_{m_0}, K_{m_0+1}, V_{m}, V_{m+1})} Pr(V', V'') \sum_{H_m \in \mathbb{F}_2^{K_{m_0+1} \times K_m}} Pr(H_m) Pr(H|H_m, V', V'') \]

\((b)\) \[ \equiv \sum_{(V', V'') \in \mathcal{V}(K_{m_0}, K_{m_0+1}, V_{m}, V_{m+1})} Pr(V', V'') \sum_{H_m \in \mathcal{H}_{V_{m}, V_{m+1}}(V', V'')} Pr(H_m) \]

\((c)\) \[ \equiv p^n (1 - p)^{K_{m_0+1} - K_{m_0} - u}, \]

where \((a)\) holds from the fact that $Pr(V', V''|H_m) = Pr(V', V'')$ because $V_{m,tx}[t]$ and $V_{m,rx}[t]$ are chosen regardless of channel realizations, \((b)\) holds since

$$
Pr(H|H_m, V', V'') = \begin{cases} 1 & \text{if } H_m \in \mathcal{H}_{V_{m}, V_{m+1}}(V', V', V'') \\ 0 & \text{otherwise,} \end{cases}
$$

and \((c)\) holds since $\sum_{H_m \in \mathcal{H}(V', V'')} Pr(H_m) = p^n (1 - p)^{K_{m_0+1} - K_{m_0} - u}$. Therefore, Lemma 2 holds.

**Proof of Lemma 3** We assume that $|V_{m,tx}[t]| = K_{m_0}$ and $|V_{m,rx}[t]| = K_{m_0+1}$ for the proof. But the same
result holds for the case that $|\mathcal{V}_{m,tx}[t]| = K_{m+1}$ and $|\mathcal{V}_{m,rx}[t]| = K_m$. For a given $G$, we obtain

$$Pr(G) = \sum_{(\mathcal{V}', \mathcal{V}'') \in H_{F^{K_m+1 \times K_m}}^{r\times m}} \sum_{H \in F^{r\times m,tx[t],rx[t]}_{V_{m,tx}[t],V_{m,rx}[t]}} Pr(H) Pr(\mathcal{V}', \mathcal{V}'', \mathcal{V}') Pr(G|H, \mathcal{V}', \mathcal{V}'')$$

(a) Equality holds since

$$\sum_{(\mathcal{V}', \mathcal{V}'') \in H_{F^{K_m+1 \times K_m}}^{r\times m}} \sum_{H \in F^{r\times m,tx[t],rx[t]}_{V_{m,tx}[t],V_{m,rx}[t]}} Pr(H) Pr(\mathcal{V}', \mathcal{V}'', H)$$

(b) Equality holds since $Pr(\mathcal{V}', \mathcal{V}'', H) = 1/|H|$ if $H \in H_{F^{K_m+1 \times K_m}}^{r\times m,tx[t],rx[t]}(G, \mathcal{V}', \mathcal{V}'')$, and (c) holds from the facts that

$$Pr(H_{V_{m,tx}[t],V_{m,rx}[t]}[t] = H)$$

is the same for all $m$, which is the result of Lemma 2 and $|H_{V_{m,tx}[t],V_{m,rx}[t]}[t] = H|$ is the same for all $m$. Therefore, Lemma 3 holds.

Proof of Lemma 4 From the definitions of $Pr(G)$ and $Pr(G, \mathcal{V}_m, \mathcal{V}_{m+1})$, we obtain

$$Pr(G) = \sum_{(\mathcal{V}', \mathcal{V}'') \in H_{F^{K_m+1 \times K_m}}^{r\times m}} \sum_{H \in F^{r\times m,tx[t],rx[t]}_{V_{m,tx}[t],V_{m,rx}[t]}} Pr(H) Pr(\mathcal{V}', \mathcal{V}'', \mathcal{V}') = \left( \begin{array}{c} K_m \\ r \end{array} \right) \left( \begin{array}{c} K_{m+1} \\ r \end{array} \right) Pr(G, \mathcal{V}_m, \mathcal{V}_{m+1})$$

(59)

where the second equality holds since $|\mathcal{V}(r, r, V_m, V_{m+1})| = \left( \begin{array}{c} K_m \\ r \end{array} \right) \left( \begin{array}{c} K_{m+1} \\ r \end{array} \right)$ and $Pr(G, \mathcal{V}_m, \mathcal{V}_{m+1})$ is the same for all $\mathcal{V}'$ and $\mathcal{V}''$. Thus, $Pr(G, \mathcal{V}_m, \mathcal{V}_{m+1}) = Pr(G)/\left( \begin{array}{c} K_m \\ r \\ r+1 \end{array} \right)$, which completes the proof.

Proof of Lemma 5 Consider the case $p_{j,i,m} = 1/2$. Since $\text{rank}(H_{V_{m,tx}[t],V_{m,rx}[t]}[t]) = \text{rank}(H_{V_{m,tx}[t],V_{m,tx}[t]}[t]),$ we obtain

$$Pr(G') = \sum_{G' \in E^{r\times r'}_{F^{K_{m+1} \times K_m}}} \sum_{H \in F^{K_{m+1} \times K_m}_{V_{m,rx}[t],V_{m,tx}[t]} \text{rank}(H) = r'} Pr(H).$$

(60)

From Lemma 2 we obtain

$$Pr(H) = 2^{-K_{m+1}K_m},$$

(61)

which is the same for all $H$. From Lemma 3 we obtain

$$Pr(G') = 2^{-K_{m+1}K_m} \sum_{(\mathcal{V}', \mathcal{V}'') \in H_{F^{K_{m+1} \times K_m}}^{r\times m}} \sum_{H \in F^{K_{m+1} \times K_m}_{V_{m,rx}[t],V_{m,tx}[t]}(G', \mathcal{V}', \mathcal{V}'', H)} \frac{1}{|H|}.$$ (62)

Then let us consider $\sum_{H \in H_{F^{K_{m+1} \times K_m}}^{r\times m}} \frac{1}{|H|}$. We will prove the following two properties:

1. $\sum_{H \in H_{F^{K_{m+1} \times K_m}}^{r\times m}} \frac{1}{|H|}$ is the same for all $\mathcal{V}'$ and $\mathcal{V}''$.
2. $\sum_{H \in H_{F^{K_{m+1} \times K_m}}^{r\times m}} \frac{1}{|H|}$ is the same for all $G'$ having the same rank.

To prove the first property, consider two $(\mathcal{V}'_a, \mathcal{V}'_a')$ and $(\mathcal{V}'_b, \mathcal{V}'_b')$. Then we can find a row permutation matrix $E_{row}$ and a column permutation matrix $E_{col}$ such that

$$H_{F^{K_{m+1} \times K_m}}^{r\times m}(G', \mathcal{V}'_a, \mathcal{V}'_a') = \{E_{row}HE_{col} | H \in H_{F^{K_{m+1} \times K_m}}^{r\times m}(G', \mathcal{V}'_b, \mathcal{V}'_b')\}.$$ (63)

Therefore, from the fact that $|\mathcal{V}(H)| = |\mathcal{V}(E_{row}HE_{col})|$, the first property holds.
Now consider the second property. Let us define \( r' = \text{rank}(G') \neq 0 \). We assume that \( \mathcal{V}' = \{ v_{1,m}, \ldots, v_{r',m} \} \) and \( \mathcal{V}'' = \{ v_{1,m+1}, \ldots, v_{r'',m+1} \} \) for the proof, but the same property can be easily derived for arbitrary \( \mathcal{V}' \) and \( \mathcal{V}'' \) by using the first property. Fig. 9 illustrates the construction of \( \mathcal{H}^E_{V_{m+1},V_{m+1}} (G', \mathcal{V}', \mathcal{V}'') \). We obtain \( r' \times (K_m - r') \) matrix \( G_1 = G'A \), where \( A \in \mathbb{F}_2^{r' \times (K_m - r')} \). Then \( (K_m + 1 - r') \times K_m \) matrix \( G_2 \) is obtained by setting \( G_2 = B[G', G_1] \), where \( B \in \mathbb{F}_2^{(K_m - r') \times r'} \). Therefore, we obtain

\[
\mathcal{H}^E_{V_{m+1},V_{m+1}} (G', \mathcal{V}', \mathcal{V}'') = \left\{ \left[ G', G_1 \right]_T , \left[ G_2 \right]_T \right\}_T | A \in \mathbb{F}_2^{r' \times (K_m - r')} , B \in \mathbb{F}_2^{(K_m - r') \times r'} \right\}.
\]

(64)

Then for given \( A \) and \( B \), \( \mathcal{V} (\left[ G', G_1 \right]_T , \left[ G_2 \right]_T )_T \) is the same for all \( G' \) having the same rank. Therefore, the second property holds.

From the above two properties, \( \sum_{H \in \mathcal{H}^E_{V_{m+1},V_{m+1}} (G', \mathcal{V}', \mathcal{V}'')} 1/|V[H]| \) is the same for all \( \mathcal{V}' \) and \( \mathcal{V}'' \), and \( G' \) having the same rank. We also know that \( |V (\text{rank}(G'), \text{rank}(G'), \mathcal{V}_m, \mathcal{V}_m) \) is the same for all \( G' \) having the same rank.

As a result, \( \Pr(G') \) is the same for all \( G' \) having the same rank. Thus, (60) is given by

\[
\Pr(G') = \sum_{G' \in \mathcal{F}_2^{r' \times r}, \text{rank}(G') = r} 1 = \Pr(H) \sum_{H \in \mathcal{F}_2^{K_m+1 \times K_m}, \text{rank}(H) = r} 1.
\]

(65)

Since \( \sum_{G' \in \mathcal{F}_2^{r' \times r}, \text{rank}(G') = r} 1 = N_{r,r}(r) \) and \( \sum_{H \in \mathcal{F}_2^{K_m+1 \times K_m}, \text{rank}(H) = r} 1 = N_{K_m, K_m}(r) \), we finally obtain

\[
\Pr(G) = 2^{-K_m+1} \frac{N_{K_m+1, K_m}(r)}{N_{r,r}(r)},
\]

(66)

which completes the proof.

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TABLE I
ABBREVIATIONS USED IN SECTION V

| Pr($H_m$) | Pr($H_m$) = $H_m$ | Pr($H_m$) = $H_m$ |
|-----------|---------------------|---------------------|
| Pr($V'$, $V''$|Pr($V_{m,rx}[t], V_{m,rx}[t]$) = $H$ | Pr($V_{m,rx}[t], V_{m,rx}[t]$) = $H$ |
| Pr($H_{m}$, $V'$, $V''$) | Pr($V_{m,tx}[t], V_{m,tx}[t]$) = $H$ | Pr($V_{m,tx}[t], V_{m,tx}[t]$) = $H$ |
| Pr($G$) | Pr($V_{m,tx}[t], V_{m,tx}[t]$) = $G$ | Pr($V_{m,tx}[t], V_{m,tx}[t]$) = $G$ |

Fig. 1. Interference mitigation for the single-hop network (a) and for the two-hop network (b), where the dashed lines and the solid lines denote the corresponding channels are zeros and ones, respectively.

Fig. 2. Layered relay network.

Fig. 3. Example of the cut-set bound
Fig. 4. Deterministic pairing of two $H_1$'s.

Fig. 5. Deterministic pairing of $H_1$ and $H_2$.

Fig. 6. 16 possible instances of $H_m[t]$.

Fig. 7. 2-2-2 relay network.
Fig. 8. Randomized pairing of $H_1$, $H_2$, and $H_3$.

Fig. 9. Construction of $\mathcal{H}_{V_m, V_{m+1}}^F (G, V', V'')$, where $A \in \mathbb{F}_2^{r' \times (K_m - r')}$, and $B \in \mathbb{F}_2^{(K_m+1-r') \times r'}$. 
Fig. 10. Sum capacity when $p_{j,i,m} = p$ for all $i$, $j$, and $m$. 

[Diagram showing sum capacity for different network configurations]