Multi-asset Generalised Variance Swaps
in Barndorff-Nielsen and Shephard model

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Abstract

This paper proposes swaps on two important new measures of generalized variance, namely the maximum eigenvalue and trace of the covariance matrix of the assets involved. We price these generalized variance swaps for Barndorff-Nielsen and Shephard model used in financial markets. We consider multiple assets in the portfolio for theoretical purpose and demonstrate our approach with numerical examples taking three stocks in the portfolio. The results obtained in this paper have important implications for the commodity sector where such swaps would be useful for hedging risk.

**Keywords:** Barndorff-Nielsen and Shephard model, Generalized Variance, Swaps, Trace, Maximum Eigenvalue

**AMS Classification:** 91G10, 91G80
1 Introduction

Covariance and correlation swaps are among recent financial products which are useful for volatility hedging and speculation using two different financial underlying assets. For example, option dependent on exchange rate movements, such as those paying in a currency different from the underlying currency, have an exposure to movements of the correlation between the asset and the exchange rate, this risk may be eliminated by using a covariance swap. The literature devoted to the volatility derivatives is growing. The Non-Gaussian Ornstein-Uhlenbeck stochastic volatility model was used by Benth et al. (2007) to study volatility and variance swaps. Broadie and Jain (2008a) evaluated price and hedging strategy for volatility derivatives in the Heston square root stochastic volatility model and in Broadie and Jain (2008b) they compare result from various model in order to investigate the effect of jumps and discrete sampling on variance and volatility swaps. Pure jump process with independent increments return models were used by Carr et al. (2005) to price derivatives written on realized variance, and subsequent development by Carr et al. (2005). This paper also provides a good survey on volatility derivatives. Fonseca et al. (2009) analyzed the influence of variance and covariance swap in a market by solving a portfolio optimization problem in a market with risky assets and volatility derivatives. Correlation swap price has been investigated by Bossu (2005) and Bossu (2007) for component of an equity index using statistical method. Habtemicael and SenGupta (2016) developed the covariance swap price for a pair of assets using the well known Barndorff-Nielsen and Shephard (BNS) model.

By definition, all the covariance based methods discussed above can only consider a combination of two assets at a time. But in today’s complex financial transactions, there is no reason why volatility of three or more assets will not be considered for contracting together. To wit, over the last few decades, financial industry has introduced various products that are directly linked to volatility for the purposes of indexing and hedging volatility risk. Such financial products include variance swap, volatility swap, option on realized variance, etc. CBOE introduced the Volatility Index (symbol VIX), as a reference for the 30-day volatility of the S&P500. It launched the trading of futures on VIX in 2004 and, later in 2006, the trading of options on VIX. Subsequently CBOE also published “VIX” on other indices such as VXN (volatility index of NASDAQ), VXD (volatility of DJIA), and RVX (volatility of Russell 2000), etc. (See Alos et. al. 2019 for a detailed discussion.) In the presence of such multiple volatility linked indices and contracts, where the underlying assets are closely interlinked, it very much makes sense to consider a measure of (multivariate) volatility of these underliers together as a vector.

Thus, in this paper, we extend these methods to a situation where a suitably defined generalized variance of a portfolio of assets can be contracted on. Taking cue from multivariate analysis, we look at two important measures of generalized variance, namely the maximum eigenvalue and trace of the covariance matrix of the assets involved. Biswas and Mukherjee (2019) develops the theory for pricing such assets for a Markov-modulated volatility situation. The objective of the present paper is to extend this analysis to price generalized variance swaps for financial markets with jumps in the return process. Our candidate model for the returns is the well known BNS model.

We outline the problem and the theoretical results in section 2. The application to the pricing of Swaps is done in Section 3. First we work out the price of the trace swap and in a subsequent subsection we discuss pricing of the eigenvalue swap with a target return constraint. The numerical examples are presented with real data in section 4. Finally section 5 concludes.
2 Problem formulation

Let us consider a financial market with two types of securities, the risk free bond and the stock. Barndorff-Nielsen and Shephard assumed that the price process of the stock $S = (S_t)_{t \geq 0}$ is defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$ and is given by

$$S_t = S_0 e^{\mu t}.$$  

$$dX_t = (\mu + \beta \sigma_t^2)dt + \sigma_t dW_t + \rho dZ_{\lambda t},$$

$$d\sigma_t^2 = -\lambda \sigma_t^2 dt + dZ_{\lambda t}, \sigma_t^2 \geq 0.$$  

where the parameters $\mu, \beta, \rho, \lambda \in \mathbb{R}$ with $\lambda > 0$ and $\rho \leq 0$. $W = (W_t)$ is a Brownian motion and the process $Z = (Z_{\lambda t})$ is a subordinator. Barndorff-Nielsen and Shephard refer to $Z$ as the background driving Lévy process (BDLP). $Z$ satisfies the Assumptions [1 - 3] in the paper Habtemicael and SenGupta (2016) that there exists an equivalent martingale measure under which the equations can be written as:

$$dX_t = b_t dt + \sigma_t dW_t + \rho dZ_{\lambda t},$$

$$d\sigma_t^2 = -\lambda \sigma_t^2 dt + dZ_{\lambda t}, \sigma_t^2 \geq 0.$$  

(1)

and $W_t$ and $Z_{\lambda t}$ are Brownian motion and Lévy process respectively with respect to the equivalent martingale measure. In the expression for $b_t$, the cumulant transform for $Z_1$ under the new measure is denoted as $\kappa(\theta)$. The solution of the (1) is given by

$$\sigma_t^2 = e^{-\lambda t} \sigma_0^2 + \int_0^t e^{-\lambda (t-s)} dZ_{\lambda s}.$$  

Here $Z$ is an increasing function and $\sigma_0^2 > 0$, the process $\sigma^2 = (\sigma_t^2)$ is strictly positive and is bounded by $\sigma_0^2 e^{-\lambda t}$. The instantaneous variance of log returns is given by $\sigma_t^2 + \rho^2 \lambda \text{Var}[Z_1] dt$ and therefore simple calculation shows the continuous realized variance in the interval $[0, T]$ is

$$\sigma_R^2 = \frac{1}{T} \int_0^T \sigma_t^2 dt + \rho^2 \lambda \text{Var}[Z_1].$$

Substituting $\sigma_t^2$ to the realized variance $\sigma_R^2$ and using integration by parts of Ito calculus yield

$$\sigma_R^2 = \frac{1}{T} \left( \lambda^{-1} (1 - e^{-\lambda T}) \sigma_0^2 + \lambda^{-1} \int_0^T (1 - e^{-\lambda (T-t)}) dZ_{\lambda t} \right) + \rho^2 \lambda \text{Var}[Z_1].$$

A portfolio consists of 3 stocks with the corresponding returns given by $dX^1$, $dX^2$ and $dX^3$. The risk-neutral dynamics of the assets are given as:

$$dX^i_t = b^i_t dt + (\sigma^i_t) dW^i_t + \rho dZ^i_{\lambda t},$$  

(2)

where $i = \{1, 2, 3\}$

$$d(\sigma^1_t)^2 = -\lambda (\sigma^1_t)^2 dt + dZ^1_{\lambda t}, \quad (\sigma^1_t)^2 \geq 0$$

$$d(\sigma^2_t)^2 = -\lambda (\sigma^2_t)^2 dt + dZ^2_{\lambda t}, \quad (\sigma^2_t)^2 \geq 0$$

$$d(\sigma^3_t)^2 = -\lambda (\sigma^3_t)^2 dt + dZ^3_{\lambda t}, \quad (\sigma^3_t)^2 \geq 0.$$  

(3)
where \( W_i' \)'s are the Wiener processes and correlated with other Wiener process as \( \text{Cov}(W_i', W_j') = \gamma_{ij} \) and \( b_i' \)'s are deterministic functions of \( \sigma_i^2 \). Here \( \rho_i \)'s are the leverage parameter corresponding to \( S' \)'s respectively.

**Proposition 1.** 

**Habtemicael and SenGupta (2016):** Let \( (Z_1^i, Z^*_i), (Z_2^i, Z_3^i) \) and \( (Z_4^i, Z_5^i) \) be three pairs of independent Lévy subordinators. Here the independence of the Lévy processes is understood in the sense of Cont and Tankov (2004) (Proposition 5.3). If \( (X_i, Y_i) \) is a Lévy process with Lévy measure \( \nu_{(X,Y)} \) and without Gaussian part then its components are independent if and only if the support of \( \nu_{(X,Y)} \) is contained in the set \( \{(x,y): xy = 0\} \), that is, if and only if they never jump together with probability one. In this case \( \nu_{(X,Y)}(A) = \nu_X(A_X) + \nu_Y(A_Y) \), where \( A_X = \{ x: (x,0) \in A \} \) and \( A_Y = \{ y: (y,0) \in A \} \), and \( \nu_X \) and \( \nu_Y \) are Lévy measures of \( X \) and \( Y \). So we define,

\[
\begin{align*}
  dZ_2^i &= r_2 dZ_4^i + \sqrt{1 - (r_2)^2} dZ_5^i, \quad (4) \\
  dZ_3^i &= r_3 dZ_4^i + \sqrt{1 - (r_3)^2} dZ_5^i. \quad (5)
\end{align*}
\]

From the above two equations we can write,

\[
\begin{align*}
  r_3 dZ_2^i - r_2 dZ_3^i &= r_3 \sqrt{1 - (r_2)^2} dZ_4^i - r_2 \sqrt{1 - (r_3)^2} dZ_5^i, \\
  dZ_3^i &= \frac{r_3}{r_2} dZ_2^i - \frac{r_2}{r_3} \sqrt{1 - (r_2)^2} dZ_4^i + \sqrt{1 - (r_3)^2} dZ_5^i. \quad (6)
\end{align*}
\]

which is also a Lévy subordinator provided \( 0 \leq r_2, r_3 \leq 1 \).

Thus, for \( 0 \leq r_2, r_3 \leq 1 \), \( (Z_2^i, Z_3^i), (Z_4^i, Z_5^i) \) and \( (Z_2^i, Z_3^i) \) are positively correlated Lévy subordinators SenGupta (2016). It is clear that \( \text{Var}(Z_1^i) = r_2 \kappa_2^2 \), \( \text{Var}(Z_2^i) = r_3 \kappa_3^2 \), \( \text{Cov}(Z_1^i, Z_2^i) = r_2 r_3 \kappa_2 \kappa_3 \), \( \text{Cov}(Z_1^i, Z_3^i) = r_2 r_3 \kappa_2 \kappa_3 \), where \( \kappa_2^2 \) and \( \kappa_3^2 \) are the variances (second cumulant) of \( Z_2^i \) and \( Z_3^i \) respectively. Therefore the correlation coefficient between \( Z_2^i \) and \( Z_3^i \) at any \( t \) is given by \( r_2 \sqrt{\frac{\kappa_2}{\kappa_3}} \). Similarly, the correlation coefficient between \( Z_2^i \) and \( Z_3^i \) at any \( t \) is given by \( r_3 \sqrt{\frac{\kappa_3}{\kappa_2}} \) and the correlation coefficient between \( Z_2^i \) and \( Z_3^i \) at any \( t \) is given by \( \frac{r_2 r_3 \kappa_1}{\sqrt{\kappa_2 \kappa_3}} \).

Notation: \([\cdot, \cdot] \) represents the quadratic covariation.

Let the portfolio return covariance matrix be given by

\[
\Omega = \begin{bmatrix}
\text{Cov}_R(S^1_1, S^1_2) & \text{Cov}_R(S^1_1, S^2_2) & \text{Cov}_R(S^1_1, S^3_2) \\
\text{Cov}_R(S^2_2, S^1_2) & \text{Cov}_R(S^2_2, S^2_2) & \text{Cov}_R(S^2_2, S^3_2) \\
\text{Cov}_R(S^3_2, S^1_2) & \text{Cov}_R(S^3_2, S^2_2) & \text{Cov}_R(S^3_2, S^3_2)
\end{bmatrix}
\]

From the definition of \( \text{Cov}_R(S^1_1, S^2_2) \) we have

\[
\text{Cov}(S^1_1, S^2_2) = \frac{1}{T} [\ln S^1_T, \ln S^2_T] = \frac{1}{T} [X^1_T, X^2_T] = \text{Cov}(S^2_2, S^1_1).
\]

Similarly we can write,

\[
\text{Cov}(S^2_2, S^3_2) = \frac{1}{T} [\ln S^2_T, \ln S^3_T] = \frac{1}{T} [X^2_T, X^3_T] = \text{Cov}(S^3_2, S^2_2).
\]
\[
\text{Cov}(S^1, S^3) = \frac{1}{T} \text{ln} S^1_T \text{ln} S^3_T = \frac{1}{T} [X^1_T, X^3_T] = \text{Cov}(S^3, S^1).
\]

Considering the Assumption 4 and Remark (3.2) from Habtemicael and SenGupta (2016) we can write the solution of Equation (2) as

\[
X_t^i = \int_0^T b^i \, dt + \int_0^T (\sigma_i) \, dW^i_t + \int_0^T \int_0^T \rho \, J_{21} (ds, dy).
\]

Therefore \([X^1_T, X^2_T]\) is given by Cont and Tankov (2004) (Section 8.2.2)

\[
[X^1_T, X^2_T] = \int_0^T \gamma_2 (\sigma_1) r (\sigma_2) \, dt + \rho_1 \rho_2 \int_0^T \int_0^T \lambda^2 J_{21} (ds, dy).
\]

Similarly,

\[
[X^2_T, X^3_T] = \int_0^T \gamma_3 (\sigma_2) r (\sigma_3) \, dt + \rho_2 \rho_3 \int_0^T \int_0^T \lambda^3 J_{21} (ds, dy).
\]

\[
[X^1_T, X^3_T] = \int_0^T \gamma_3 (\sigma_1) r (\sigma_3) \, dt + \rho_1 \rho_3 \int_0^T \int_0^T \lambda^3 J_{21} (ds, dy).
\]

We will now try to find the value for

\[
E[\text{Cov}(S^1, S^2)] = \frac{1}{T} E[X^1_T, X^2_T].
\]

\[
E[\text{Cov}(S^2, S^3)] = \frac{1}{T} E[X^2_T, X^3_T].
\]

\[
E[\text{Cov}(S^3, S^1)] = \frac{1}{T} E[X^3_T, X^1_T].
\]

Stating Lemma 3.3 from Habtemicael and SenGupta (2016) as a theorem

**Theorem 1.** If \(B_t\) has been defined as

\[
B_t = \alpha_1 + \int_0^\lambda T e^s \, dV_s.
\]

where \(\lambda > 0\), \(\alpha_1, \lambda \in R\) are constants and \(0 \leq t \leq T\) and \(V\) is a Lévy subordinator with no deterministic drift. Then,

\[
\phi_{B_t} (\theta) = \exp \left( i \theta \alpha_1 + \int_0^\lambda T k_{V_1} (i \theta e^s) \, ds \right).
\]

where \(k_{V_1} (\cdot)\) is the cumulant generating function for \(V_1\). The moment of \(B_t\) are given by

\[
E(B_t^k) = (-i)^k \hat{g}_k (0), \quad k = 1, 2, \ldots
\]

where

\[
\hat{g}_1 (\theta) = i \left( \alpha_1 + \int_0^\lambda T e^s \, k_{V_1} (i \theta e^s) \, ds \right),
\]

and

\[
\hat{g}_k (\theta) = \hat{g}_1 (\theta) \hat{g}_k (\theta) + \hat{g}_k (\theta), \quad k = 1, 2, \ldots
\]

In the above formulas prime represents the derivative with respect to the parameter in parenthesis.

**Theorem 2.** The correlation between \((\sigma_2)^2\) & \((\sigma_3)^2\), \((\sigma_2)^2\) & \((\sigma_3)^2\) and \((\sigma_2)^2\) & \((\sigma_3)^2\) can be calculated as;

1. Suppose \(Z^2, Z^3, Z^*\) and \(Z^{**}\) are related by (6) then the correlation coefficient between \((\sigma_2)^2\) and \((\sigma_3)^2\) is independent of time and is given by \(\frac{k_{21} \sigma_2^2 \kappa_3^2}{\sqrt{k_2^2 \kappa_2^2}}\) where \(\kappa_2^1, \kappa_2^2\) and \(\kappa_2^3\) are the variances (second cumulant) of \(Z_t^1\),
Using Theorem 1 we get,

Similarly,

It is clear from Equation (3) that,

Proof: It is clear from Equation (3) that,

Using Theorem 1 we get,

Similarly,

Now,

Simplifying it,


\[ Cov((\sigma_2)_t^2, (\sigma_3)_t^2) = r_{23}e^{-2\lambda t}\frac{k_1^4}{2}(e^{2\lambda t} - 1). \]
Proof: 

Theorem 4. We can similarly prove the second and third part of the Theorem. Hence Theorem 2 is proved. \(\square\)

Theorem 3. (Lemma 3.5 from Habtemicael and SenGupta (2016))

1. Suppose that \((\sigma_1)^2\) and \((\sigma_2)^2\) defined respectively by (3) and \(Z^1, Z^2\) and \(Z^*\) are related by (4). Let there exists \(\beta_{12} > 0\) such that for \(0 \leq t \leq T\), \((\sigma_1)^2, (\sigma_2)^2 \leq \beta_{12}^2\). Then,

\[
E[(\sigma_1)(\sigma_2)] = e^{-\lambda t} \sum_{k=0}^{\infty} \sum_{p=0}^{k} \sum_{u=0}^{p} \frac{(-1)^{p+1}(2K)^!}{4^k(k!)^2(2k-1)} \left(\frac{k}{p}\right) \left(\frac{p}{u}\right) (\beta_{12})^{-4p+2} (r_2)^u (1 - (r_2)^2)^{\frac{p-u}{2}} \mathcal{N}^{12}(p,u). \tag{9}
\]

where

\[
\mathcal{N}^{12}(p,u) = E \left[ \left( (\sigma_1)^0 + \int_0^{\lambda T} e^s dZ_s \right)^{p+u} \right] \mathcal{E} \left[ \left( \frac{(\sigma_2)^0 - r_2 (\sigma_2)^0}{\sqrt{1 - r_2}} + \int_0^{\lambda T} e^s dZ_s \right)^{p-u} \right].
\]

\(\mathcal{N}^{12}(p,u)\) can be calculated using Theorem 1.

2. Similarly \((\sigma_1)^2\) and \((\sigma_3)^2\) are defined respectively by (3) and \(Z^1, Z^3\) and \(Z^{**}\) are related by (5). Let there exists \(\beta_{31} > 0\) such that for \(0 \leq t \leq T\), \((\sigma_1)^2, (\sigma_3)^2 \leq \beta_{31}^2\). Then,

\[
E[(\sigma_1)(\sigma_3)] = e^{-\lambda t} \sum_{k=0}^{\infty} \sum_{p=0}^{k} \sum_{u=0}^{p} \frac{(-1)^{p+1}(2K)^!}{4^k(k!)^2(2k-1)} \left(\frac{k}{p}\right) \left(\frac{p}{u}\right) (\beta_{31})^{-4p+2} (r_3)^u (1 - (r_3)^2)^{\frac{p-u}{2}} \mathcal{N}^{31}(p,u). \tag{10}
\]

where

\[
\mathcal{N}^{31}(p,u) = E \left[ \left( (\sigma_1)^0 + \int_0^{\lambda T} e^s dZ_s \right)^{p+u} \right] \mathcal{E} \left[ \left( \frac{(\sigma_3)^0 - r_3 (\sigma_3)^0}{\sqrt{1 - r_3}} + \int_0^{\lambda T} e^s dZ_s \right)^{p-u} \right].
\]

\(\mathcal{N}^{31}(p,u)\) can be calculated using Theorem 1.

Extending Theorem 3 we state the following,

Theorem 4. Suppose that \((\sigma_1)^2\) and \((\sigma_3)^2\) defined respectively by (3) and \(Z^1, Z^3\) and \(Z^{*}\) and \(Z^{**}\) are related by (6). Let there exists \(\beta_{23} > 0\) such that for \(0 \leq t \leq T\), \((\sigma_2)^2, (\sigma_3)^2 \leq \beta_{23}^2\). Then, we can write,

\[
E[(\sigma_2)(\sigma_3)] = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(2K)^!}{4^k(k!)^2(2k-1)} \sum_{p=0}^{k} \left(\frac{k}{p}\right) \sum_{u=0}^{p} \left(\frac{p}{u}\right) \sum_{v=0}^{u} \left(\frac{u}{v}\right) \sum_{w=0}^{v} \left(\frac{u-w}{v-w}\right) (\beta_{23})^{-4p+2} (r_2)^u (1 - (r_2)^2)^{\frac{p-u}{2}} (1 - (r_3)^2)^{\frac{p-u}{2}} \mathcal{N}^{23}(p,u,v,w). \tag{11}
\]

where,

\[
\mathcal{N}^{23}(p,u,v,w) = E \left( F_t^{p+u} \right) E \left( (G_t)^{p-u+v} \right) E \left( (G_t)^{p-v-w} \right).
\]

Proof: It is clear from Equation (3) that,

\[
(\sigma_2)^2 = e^{-\lambda t} (\sigma_2)^0 + e^{-\lambda t} \int_0^t e^{\lambda t} dZ_{k,t},
\]

\[
(\sigma_2)^2 = e^{-\lambda t} (\sigma_2)^0 + e^{-\lambda t} \int_0^{\lambda t} e^{r_2 dZ_t + 1 - (r_2)^2 dZ_t}.
\]
\[(\sigma_2)_t = e^{-\lambda t} \beta_{23}^2 \left( (\sigma_2)_0^2 + r_2 \int_0^t e^t dZ_t^1 + \sqrt{1-(r_2)^2} \int_0^t e^t dZ_t^2 \right) \].

Likewise,
\[(\sigma_3)_t = e^{-\lambda t} \beta_{23}^2 \left( (\sigma_3)_0^2 + r_3 \int_0^t e^t dZ_t^1 + \sqrt{1-(r_3)^2} \int_0^t e^t dZ_t^2 \right) \].

We denote \(F_t = (\sigma_2)_0^2 + r_2 \int_0^t e^t dZ_t^1, \ G_t = \int_0^t e^t dZ_t^2, \ G_t^{**} = \int_0^t e^t dZ_t^{**}\) and \(c = (\sigma_3)_0^2 - \frac{r_2}{r_3} (\sigma_2)_0^2\).

Therefore, we can write,
\[(\sigma_2)_t (\sigma_3)_t = e^{-\lambda t} \beta_{23}^2 \sqrt{1 + \frac{F_t c + \frac{r_3}{r_2} F_t^2 + \sqrt{1-r_2^2 F_t G_t^{**} + \sqrt{1-r_2^2 G_t c + \frac{r_3}{r_2} \sqrt{1-r_2^2 G_t F_t + \sqrt{1-r_2^2 G_t G_t^{**} - \beta_{23}^4}}}}}{\beta_{23}^2}} < 1.\]

Using the following convergence,
\[
\sqrt{1+x} = \sum_{k=0}^{\infty} \frac{(-1)^{p+1}(2K)!}{4^k (k!)^2} x^k, \quad |x| < 1
\]

We get,
\[\begin{align*}
(\sigma_2)_t (\sigma_3)_t & = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(-1)^{p+1}(2K)!}{4^k (k!)^2 (k+1)(2k-1)} \beta_{23}^{2k-2} \left( F_t c + \frac{r_3}{r_2} F_t^2 + \sqrt{1-r_2^2 F_t G_t^{**} + \sqrt{1-r_2^2 G_t c + \frac{r_3}{r_2} \sqrt{1-r_2^2 G_t F_t + \sqrt{1-r_2^2 G_t G_t^{**} - \beta_{23}^4}}}} \right)^k, \\
(\sigma_2)_t (\sigma_3)_t & = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(-1)^{p+1}(2K)!}{4^k (k!)^2 (k+1)(2k-1)} \beta_{23}^{2k-2} \sum_{p=0}^{k} \sum_{u=0}^{k} \binom{k}{p} \binom{k}{u} \left( F_t c + \frac{r_3}{r_2} F_t^2 + \sqrt{1-r_2^2 F_t G_t^{**} + \sqrt{1-r_2^2 G_t c + \frac{r_3}{r_2} \sqrt{1-r_2^2 G_t F_t + \sqrt{1-r_2^2 G_t G_t^{**} - \beta_{23}^4}}}} \right)^u \left( \sqrt{1-r_2^2 G_t c + \sqrt{1-r_2^2 G_t G_t^{**}}} \right)^{p-u} \left( \beta_{23} \right)^{-4p+2}, \\
& = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(-1)^{p+1}(2K)!}{4^k (k!)^2 (k+1)(2k-1)} \sum_{p=0}^{k} \sum_{u=0}^{k} \binom{k}{p} \binom{k}{u} \left( F_t c + \frac{r_3}{r_2} F_t^2 + \sqrt{1-r_2^2 F_t G_t^{**} + \sqrt{1-r_2^2 G_t c + \frac{r_3}{r_2} \sqrt{1-r_2^2 G_t F_t + \sqrt{1-r_2^2 G_t G_t^{**} - \beta_{23}^4}}}} \right)^u \left( \sqrt{1-r_2^2 G_t c + \sqrt{1-r_2^2 G_t G_t^{**}}} \right)^{p-u} \left( \beta_{23} \right)^{-4p+2},
\end{align*}\]
\[ e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(-1)^{p+1}(2K)!}{4^k(k!)^2(2k-1)} \sum_{p=0}^{k} \binom{k}{p} \sum_{u=0}^{p} \binom{p}{u} F_i^u \left( c + \frac{r_3}{r_2} F_i + \sqrt{1 - r_2^2 G_t} \right)^u \left( 1 - r_2^2 G_t \right)^{p-u} \left( \sqrt{1 - r_2^2 G_t} \right)^{p-u} \left( \frac{r_3}{r_2} \right)^{4p+2}, \]

\[ = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(-1)^{p+1}(2K)!}{4^k(k!)^2(2k-1)} \sum_{p=0}^{k} \binom{k}{p} \sum_{u=0}^{p} \binom{p}{u} F_i^u \sum_{v=0}^{u} \binom{u}{v} \left( \frac{r_3}{r_2} \right)^{v} \left( 1 - r_2^2 G_t \right)^{p-u} \left( \sqrt{1 - r_2^2 G_t} \right)^{p-u} \left( \frac{r_3}{r_2} \right)^{4p+2}, \]

\[ = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(-1)^{p+1}(2K)!}{4^k(k!)^2(2k-1)} \sum_{p=0}^{k} \binom{k}{p} \sum_{u=0}^{p} \binom{p}{u} F_i^u \sum_{v=0}^{u} \binom{u}{v} \left( \frac{r_3}{r_2} \right)^{v} \left( 1 - r_2^2 G_t \right)^{p-u} \left( \sqrt{1 - r_2^2 G_t} \right)^{p-u} \left( \frac{r_3}{r_2} \right)^{4p+2}, \]

\[ \frac{G_t^{**}}{1 - r_3^2} = G_t^{**} + \frac{c}{\sqrt{1 - r_3^2}}. \]

Taking the expectation of Equation \((12)\):

\[ E[(\sigma_2)_t, (\sigma_3)_t] = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(-1)^{p+1}(2K)!}{4^k(k!)^2(2k-1)} \sum_{p=0}^{k} \binom{k}{p} \sum_{u=0}^{p} \binom{p}{u} \sum_{v=0}^{u} \binom{u}{v} \left( \frac{r_3}{r_2} \right)^{v} \left( 1 - r_2^2 G_t \right)^{p-u} \left( \frac{r_3}{r_2} \right)^{4p+2} \frac{p+v+w}{p+u+w} \left( \beta_{23} \right)^{4p+2} E \left( F_i^{p+v} \right) E \left( (G_t^{**})^{p-u+w} \right) E \left( (G_t^{*})^{p-v-w} \right). \]

Assigning,

\[ N_{23}^{23}(p,u,v,w) = \frac{E(\sigma_2)_t}{E(\sigma_3)_t}, \]

\[ N_{23}^{23}(p,u,v,w) \] can be computed by using Theorem 1. Comparing the Equation \((8)\) with different terms of \(N_{23}^{23}(p,u,v,w)\), we get,

For \(F_i\),

\[ \alpha_1 = \frac{(\sigma_2)_0^2}{r_2^3}, \]

\[ V_i = Z_i \]

For \(G_t^{**}\),

\[ \alpha_1 = \frac{(\sigma_3)_0^2 - \frac{r_3}{r_2^3} (\sigma_2)_0^2}{\sqrt{1 - r_3^2}}. \]
which concludes the proof of Theorem 4.

For \( G^* \),

\[ \alpha_1 = 0. \]

\[ V_t = Z^*_t. \]

Thus, we can write \( N_{T^2}^{23}(p,u,v,w) \) as

\[ N_{T^2}^{23}(p,u,v,w) = r_{2}^{p+2}E \left[ \left( \frac{(\sigma_2)^2}{r_2} + \int_{0}^{T} e^{t}dZ_s^{1} \right)^{u+v} \right] E \left[ \left( \frac{(\sigma_2)^2}{r_2} + \int_{0}^{T} e^{t}dZ_s^{1} \right)^{p-u+w} \right] \]

which concludes the proof of Theorem 4. \( \square \)

Now considering \([X^{1}_t,X^{2}_t]\) of Equation (7) and substituting from Equation (9) we get,

\[ E[\text{Cov}(S^1,S^2)] = \frac{1}{T}E[X^{1}_t,X^{2}_t] = \frac{1}{T} \int_{0}^{T} \gamma_{12} E\left( (\sigma_1)(\sigma_2) \right) dt + \frac{1}{T} \rho_1 \rho_2 E \left( \int_{0}^{T} \int_{0}^{\infty} \gamma^2 J_{21}(ds,dy) \right). \]

\[ \frac{1}{T} \int_{0}^{T} \gamma_{12} E\left( (\sigma_1)(\sigma_2) \right) dt = \frac{\gamma_{12}}{T} \int_{0}^{T} e^{-\lambda t} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-1)^{p+1}(2k)!}{4(k!)^2(2k-1)} \binom{k}{p} \binom{p}{u} \beta_{12}^p (r_2)^{u}(1-(r_2)^2) \frac{e^{-\lambda t} N_{12}^{1}(p,u)}{T}, \]

Observing \( E(J_{21}(ds,dy)) = v_{21}(dy)ds \) we obtain,

\[ \frac{\rho_1 \rho_2}{T} E \left( \int_{0}^{T} \int_{0}^{\infty} \gamma^2 J_{21}(ds,dy) \right) = \rho_1 \rho_2 \lambda \int_{0}^{\infty} y^2 v_{21}(dy). \]

Now from the paper of Cont and Tankov [2004] (proposition 3.13) we can see \( \kappa_{2}^{1} = \text{Var}(Z^1) = \int_{0}^{\infty} y^2 v_{21}(dy). \)

\[ \frac{\rho_1 \rho_2}{T} E \left( \int_{0}^{T} \int_{0}^{\infty} y^2 J_{21}(ds,dy) \right) = \rho_1 \rho_2 \lambda \kappa_{2}^{1}. \]

Therefore,

\[ E[\text{Cov}(S^1,S^2)] = \frac{\gamma_{12}}{T} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-1)^{p+1}(2k)!}{4(k!)^2(2k-1)} \binom{k}{p} \binom{p}{u} \beta_{12}^p (r_2)^{u}(1-(r_2)^2) \frac{e^{-\lambda t} N_{12}^{1}(p,u)}{T}, \]

\[ \int_{0}^{T} e^{-\lambda t} N_{12}^{1}(p,u)dt + \rho_1 \rho_2 \lambda \kappa_{2}^{1} = E[\text{Cov}(S^1,S^1)]. \] (14)

Similarly using Equations (7) and (11),

\[ E[\text{Cov}(S^2,S^3)] = \frac{\gamma_{23}}{T} \sum_{k=0}^{\infty} \frac{(-1)^{p+1}(2k)!}{4(k!)^2(2k-1)} \sum_{p=0}^{\infty} \binom{k}{p} \sum_{u=0}^{\infty} \binom{p}{u} \sum_{v=0}^{\infty} \binom{u}{v} \sum_{w=0}^{\infty} \binom{u-v}{w} \sum_{r_2=0}^{\infty} \frac{r_2^{u-w}}{r_2} \left( 1-(r_2)^2 \right) \frac{e^{-\lambda t} N_{23}^{1}(p,u,v,w)}{T}, \]

\[ (\beta_{23})^{-4p+2} \int_{0}^{T} e^{-\lambda t} N_{23}^{1}(p,u,v,w)dt + \rho_2 \rho_3 \lambda \kappa_{2}^{1} = E[\text{Cov}(S^3,S^3)]. \] (15)
And again using Equations (7) and (10),

\[ E[\text{Cov}(S^3, S^1)] = \frac{\gamma_3}{T} \sum_{k=0}^{\infty} \sum_{p=0}^{k} \sum_{u=0}^{p} (-1)^{p+1}(2k)! \left( \frac{k}{p} \right) \left( \frac{-p}{u} \right) \left( \beta_{31} \right)^{-ap+2}(r_3)^u(1-(r_3)^2)^{p-u} \int_0^T e^{-\lambda_t N_t^3}(p,u)dt + \rho_3 \rho_1 \lambda \kappa_1^3 = E[\text{Cov}(S^1, S^3)]. \] (16)

Now we are going to calculate,

\[ [X^1_t, X^1_t] = \int_0^T (\sigma_1)^2 dt + (\rho_1)^2 \int_0^T \int_0^\infty y^2 J_{21}(ds,dy). \]

Similarly,

\[ [X^2_t, X^2_t] = \int_0^T (\sigma_2)^2 dt + (\rho_2)^2 \int_0^T \int_0^\infty y^2 J_{21}(ds,dy). \]

\[ [X^3_t, X^3_t] = \int_0^T (\sigma_3)^2 dt + (\rho_3)^2 \int_0^T \int_0^\infty y^2 J_{21}(ds,dy). \] (17)

We will now try to find the value for

\[ E[\text{Cov}(S^1, S^1)] = E[\text{Var}(S^1)] = \frac{1}{T} E[X^1_t, X^1_t] \]

\[ \frac{1}{T} E[X^1_t, X^1_t] = \frac{1}{T} \int_0^T E[(\sigma_1)^2]dt + \frac{1}{T}(\rho_1)^2 E\left[ \int_0^\infty \int_0^\infty y^2 J_{21}(ds,dy) \right]. \]

Using the solution of (3)

\[ (\sigma_1)^2 = e^{-\lambda t}(\sigma_1)^2_0 + e^{-\lambda t} \int_0^t e^{\lambda s} dZ^1_{ks}. \]

\[ E[(\sigma_1)^2] = e^{-\lambda t} E[(\sigma_1)^2_0 + \int_0^t e^{\lambda s} dZ^1_{ks}]. \]

Comparing this with (8), we have \( \alpha_1 = (\sigma_1)^2_0 \) and \( V_t = Z^1_t \) so,

\[ N^1_t = E\left[ (\sigma_1)^2 + \int_0^t e^{\lambda s} dZ^1_{ks} \right]. \]

\[ \frac{1}{T} \int_0^T E[(\sigma_1)^2]dt = \frac{1}{T} \int_0^T e^{-\lambda t} N^1_t dt. \]

\[ \frac{1}{T}(\rho_1)^2 E\left[ \int_0^\infty \int_0^\infty y^2 J_{21}(ds,dy) \right] = \frac{(\rho_1)^2}{T} \lambda \kappa_1^3. \]

Therefore the value obtained is,

\[ E[\text{Var}(S^1)] = \frac{1}{T} E[X^1_t, X^1_t] = \frac{1}{T} \int_0^T e^{-\lambda t} N^1_t dt + \frac{(\rho_1)^2}{T} \lambda \kappa_1^3. \]

Similarly,

\[ E[\text{Cov}(S^2, S^2)] = E[\text{Var}(S^2)] = \frac{1}{T} E[X^2_t, X^2_t] = \frac{1}{T} \int_0^T e^{-\lambda t} N^2_t dt + \frac{(\rho_2)^2}{T} \lambda \kappa_1^3. \]

where

\[ N^2_t = E\left[ (\sigma_2)^2 + \int_0^t e^{\lambda s} dZ^1_{ks} \right]. \]
Theorem 5. (Lemma 3.9 from Habtemicael and SenGupta (2016))

2. Similarly when $Z^{(2)} \sigma_t^1 + Z^{(2)} \sigma_t^1 dZ^1_t$, we have

\[
N_t^1 = E \left[ \frac{1}{T} \int_0^T e^{\lambda t} N_t^1 \right].
\]

To avoid complexity of the numerical computations we derive an alternative approximate version of Equations (14), (15) and (16).

Result 1. A useful estimate approximation regarding the expected value of realized volatility ($\sigma_R$) is obtained in Brockhaus and Londo (2000) and is given by

\[
E[\sigma_R] = \sqrt{E[\sigma_R^2] - \frac{Var[\sigma_R^2]}{8(Var[\sigma_R^2])^2}}.
\]  

Theorem 5. (Lemma 3.9 from Habtemicael and SenGupta (2016))

1. For $Z_t^1$, $Z_t^2$ and $Z_t^*$ related by (4) we can calculate

\[
E \left[ (\sigma_1^2)^2(\sigma_2^2) \right] = e^{-2\lambda t} (\sigma_1^2(\sigma_2^2) + e^{-2\lambda t} \left( (\sigma_1^2)^2 \sqrt{1 - (r_1^2)^2} \right) + \left( (\sigma_1^2)^2 + (\sigma_2^2)^2 \right) e^{\lambda t} - 1) + e^{-2\lambda t} \left( (\sigma_1^2)^2 (e^{\lambda t} - 1)^2 + \frac{\lambda t}{2} (e^{2\lambda t} - 1) \right) + \sqrt{1 - (r_1^2)^2} \right) \right].
\]  

2. Similarly when $Z_t^1$, $Z_t^2$ and $Z_t^*$ are related as (5) we can calculate,

\[
E \left[ (\sigma_1^2)^2(\sigma_3^2) \right] = e^{-2\lambda t} (\sigma_1^2(\sigma_3^2) + e^{-2\lambda t} \left( (\sigma_1^2)^2 \sqrt{1 - (r_3^2)^2} \right) + \left( (\sigma_1^2)^2 + (\sigma_3^2)^2 \right) e^{\lambda t} - 1) + e^{-2\lambda t} \left( (\sigma_1^2)^2 (e^{\lambda t} - 1)^2 + \frac{\lambda t}{2} (e^{2\lambda t} - 1) \right) + \sqrt{1 - (r_3^2)^2} \right) \right].
\]  

\[
Var \left[ (\sigma_1^2)^2(\sigma_2^2) \right] = \left( (\sigma_1^2)^2 \sqrt{1 - (r_2^2)^2} \right) \left( Var(Y_2) + \left( (\sigma_1^2)^2 + (\sigma_2^2)^2 \right) Var(Y_1) + r_2 Var(Y_2) + (1 - r_2)^2 Var(Y_2) + \right) \left( (\sigma_1^2)^2 + (\sigma_2^2)^2 \right) Cov(Y_2, Y_1) + 2 \left( (\sigma_1^2)^2 + (\sigma_2^2)^2 \right) Cov(Y_1, Y_2) + 2 \left( (\sigma_1^2)^2 + (\sigma_2^2)^2 \right) Cov(Y_1, Y_2, Y_2). \right)
\]  

\[
Var \left[ (\sigma_1^2)^2(\sigma_3^2) \right] = \left( (\sigma_1^2)^2 \sqrt{1 - (r_3^2)^2} \right) \left( Var(Y_3) + \left( (\sigma_1^2)^2 + (\sigma_3^2)^2 \right) Var(Y_1) + r_3 Var(Y_2) + (1 - r_3)^2 Var(Y_1) + \right) \left( (\sigma_1^2)^2 + (\sigma_3^2)^2 \right) Cov(Y_3, Y_1) + 2 \left( (\sigma_1^2)^2 + (\sigma_3^2)^2 \right) Cov(Y_1, Y_2) + 2 \left( (\sigma_1^2)^2 + (\sigma_3^2)^2 \right) Cov(Y_1, Y_2, Y_2). \right)
\]
Extending Theorem 5 we state a theorem pertaining to our set up.

**Theorem 6.** For $Z_2^2, Z_3^3, Z_4^4$ and $Z_5^{**}$ related by (6) we get.

$$E\left[\left(\sigma_2 \right)^2 \left(\sigma_3 \right)^2 \right] = e^{-2\lambda t} \left(\sigma_2 \right)^2 \left(\sigma_3 \right)^2 + e^{-2\lambda t} \left(\left(\sigma_2 \right)^2 \left( r_1 \sigma_1^2 + \sqrt{1-(r_2)^2} \kappa_1^* \right) \right) \left( \left( e^{\lambda t} - 1 \right) + \left( \sigma_2 \right)^2 \left( r_2 \sigma_1^2 + \sqrt{1-(r_2)^2} \kappa_1^* \right) \right)$$

$$\left( e^{\lambda t} - 1 \right) + e^{-2\lambda t} r_2 \left( \left( \kappa_1^2 \right) \left( e^{\lambda t} - 1 \right)^2 + \frac{\kappa_1^4}{2} \left( e^{2\lambda t} - 1 \right) \right) + r_2 \sqrt{1-(r_3)^2} \kappa_1^* \left( e^{\lambda t} - 1 \right)^2 + r_3 \sqrt{1-(r_4)^2} \kappa_1^* \left( e^{\lambda t} - 1 \right)^2.$$

(23)

$$\text{Var} \left[\left(\sigma_2 \right)^2 \left(\sigma_3 \right)^2 \right] = a_1^2 \text{Var}(Y_1) + a_1^2 \text{Var}(Y_3) + a_2^2 \text{Var}(Y_2) + a_3^2 \text{Var}(Y_4) + a_1^2 \text{Var}(Y_1 Y_2) + 2 a_2 a_4 \text{Cov}(Y_1, Y_4) + 2 a_2 a_6 \text{Cov}(Y_1, Y_3) + 2 a_4 a_7 \text{Cov}(Y_2, Y_1 Y_3) + 2 a_5 a_7 \text{Cov}(Y_2, Y_1 Y_4) + 2 a_6 a_8 \text{Cov}(Y_4, Y_1 Y_3) + 2 a_8 a_7 \text{Cov}(Y_3, Y_1 Y_4).$$

(24)

**Proof:** To start with we are going to find the product of the pairs of variances.

Calculating the product of $(\sigma_2)^2$ and $(\sigma_3)^2$ defined in Equation (5)

$$\left(\sigma_2 \right)^2 \left(\sigma_3 \right)^2 = e^{-2\lambda t} \left(\sigma_2 \right)^2 \left(\sigma_3 \right)^2 + e^{-2\lambda t} \left(\left(\sigma_2 \right)^2 \int_0^{\lambda t} e^e dZ_1^1 + \left(\sigma_3 \right)^2 \int_0^{\lambda t} e^e dZ_1^2 \right) + e^{-2\lambda t} \left(\int_0^{\lambda t} e^e dZ_1^3 \right)^2. \tag{25}$$

$$E \left(\int_0^{\lambda t} e^e dZ_1^3 \right) = E \left(\int_0^{\lambda t} e^e \left( r_2 dZ_1^1 + \sqrt{1-(r_2)^2} dZ_1^2 \right) \right) = \left( r_2 \kappa_1^* + \sqrt{1-(r_2)^2} \kappa_1^* \right) \left( e^{\lambda t} - 1 \right).$$

$$E \left(\int_0^{\lambda t} e^e dZ_1^3 \right) = E \left(\int_0^{\lambda t} e^e \left( r_3 dZ_1^1 + \sqrt{1-(r_3)^2} dZ_1^2 \right) \right) = \left( r_3 \kappa_1^* + \sqrt{1-(r_3)^2} \kappa_1^* \right) \left( e^{\lambda t} - 1 \right).$$

$$E \left(\int_0^{\lambda t} e^e dZ_1^3 \right)^2 = E \left(\int_0^{\lambda t} e^e \left( r_2 dZ_1^1 + \sqrt{1-(r_2)^2} dZ_1^2 \right) \right) \int_0^{\lambda t} e^e \left( r_3 dZ_1^1 + \sqrt{1-(r_3)^2} dZ_1^2 \right) + \left( r_2 \kappa_1^* \right)^2 \left( e^{2\lambda t} - 1 \right) + \left( r_3 \kappa_1^* \right)^2 \left( e^{2\lambda t} - 1 \right) + r_2 \kappa_1^* r_3 \kappa_1^* \left( e^{\lambda t} - 1 \right)^2 +$$

$$+ \sqrt{1-(r_2)^2} \sqrt{1-(r_3)^2} \left( r_2 \kappa_1^* \right)^2 \left( e^{2\lambda t} - 1 \right) + \sqrt{1-(r_2)^2} \sqrt{1-(r_3)^2} \left( r_3 \kappa_1^* \right)^2 \left( e^{2\lambda t} - 1 \right) +$$

$$+ \sqrt{1-(r_2)^2} \sqrt{1-(r_3)^2} \left( r_2 \kappa_1^* \right) \left( r_3 \kappa_1^* \right) \left( e^{\lambda t} - 1 \right)^2.$$
\[(\sigma_0^2)^2 = e^{-2\lambda t}(\sigma_3^2)^2 + e^{-2\lambda t}[r_3^{\lambda t} \int e^dZ^1_2 + (\sigma_0^2)\int e^dZ^1_3 + (\sigma_0^2)\int e^dZ^2 + (\sigma_0^2)\int e^dZ^3 + r_3^{\lambda t} \int e^dZ^4_5 + r_3^{\lambda t} \int e^dZ^5_3 + e^{-2\lambda t}(r_3^{\lambda t} \int e^dZ^6_4 + r_3^{\lambda t} \int e^dZ^7_5 + r_3^{\lambda t} \int e^dZ^8_3).
\]

Let us define
\[
Y_1 = \int_0^{\lambda t} e^dZ^1_1.
\]
\[
Y_2 = \int_0^{\lambda t} e^dZ^2.
\]
\[
Y_3 = \int_0^{\lambda t} e^dZ^3.
\]
\[
a_1 = (\sigma_0^2)^2.
\]
\[
a_2 = (\sigma_0^2)^2 + (\sigma_3^2)^2.
\]
\[
a_3 = (\sigma_0^2)^2 \int e^dZ^1_3.
\]
\[
a_4 = (\sigma_0^2)^2 \int e^dZ^2.
\]
\[
a_5 = r_3^{\lambda t}.
\]
\[
a_6 = r_3^{\lambda t} \int e^dZ^3.
\]
\[
a_7 = r_3^{\lambda t} \int e^dZ^4.
\]

So, Variance of Equation (26) looks like
\[
Var[(\sigma_0^2)^2] = Var[a_1 + a_2 Y_1 + a_3 Y_3 + a_4 Y_2 + a_5 Y_5 + a_6 Y_1 Y_3 + a_7 Y_1 Y_2].
\]

Using the independence of $Y_1, Y_2$ and $Y_3$ and comparing with (8), it is easy to show
\[
Var(Y_1) = E(Y_1^2) - (E(Y_1))^2.
\]
\[
Var(Y_2) = E(Y_2^2) - (E(Y_2))^2.
\]
\[
Var(Y_3) = E(Y_3^2) - (E(Y_3))^2.
\]
\[
E(Y_1^2) = \left(\kappa_1^4(e^{\lambda t} - 1) + 3(\kappa_1^4)(e^{\lambda t} - 1)^2(\kappa_1^4)) + \frac{3}{4}(\kappa_1^4)^2(e^{2\lambda t} - 1)^2 + \frac{4}{3}(\kappa_1^4)(e^{\lambda t} - 1)(e^{3\lambda t} - 1) + \frac{\kappa_1^4}{3}(e^{4\lambda t} - 1).
\]
\[
E(Y_3^3) = \left(\kappa_1^3(e^{\lambda t} - 1)^3 + \frac{3}{2}(\kappa_1^3)(e^{\lambda t} - 1)(e^{2\lambda t} - 1) + \frac{\kappa_1^3}{3}(e^{3\lambda t} - 1).
\]
Using (18) and considering
\[E(Y_1^2) = (\kappa_1^1)^2(e^{\lambda t} - 1)^2 + \frac{\kappa_1^1}{2}(e^{2\lambda t} - 1).\]
\[E(Y_2^2) = (\kappa_1^2)^2(e^{\lambda t} - 1)^2 + \frac{\kappa_2^2}{2}(e^{2\lambda t} - 1).\]
\[E(Y_3^2) = (\kappa_1^3)^2(e^{\lambda t} - 1)^2 + \frac{\kappa_2^2}{2}(e^{2\lambda t} - 1).\]
\[E(Y_1) = \kappa_1^1(e^{\lambda t} - 1).\]
\[E(Y_2) = \kappa_1^2(e^{\lambda t} - 1).\]
\[E(Y_3) = \kappa_1^3(e^{\lambda t} - 1).\]
\[Var(Y_1,Y_2) = E(Y_1^2)E(Y_2^2) - (E(Y_1))^2(E(Y_2))^2.\]
\[Var(Y_1,Y_3) = E(Y_1^2)E(Y_3^2) - (E(Y_1))^2(E(Y_3))^2.\]
\[Cov(Y_1,Y_1) = Var(Y_1)E(Y_3) = \frac{\kappa_1^1}{2}\kappa_1^{**}(e^{2\lambda t} - 1)(e^{\lambda t} - 1).\]
\[Cov(Y_1,Y_2) = \frac{\kappa_1^1}{2}\kappa_1^2(e^{2\lambda t} - 1)(e^{\lambda t} - 1).\]
\[Cov(Y_1,Y_3) = \frac{\kappa_1^1}{2}\kappa_1^3(e^{2\lambda t} - 1)(e^{\lambda t} - 1).\]
\[Cov(Y_2,Y_1) = \frac{\kappa_1^2}{2}\kappa_1^1(e^{2\lambda t} - 1)(e^{\lambda t} - 1).\]
\[Cov(Y_2,Y_3) = \frac{\kappa_1^2}{2}\kappa_1^3(e^{2\lambda t} - 1)(e^{\lambda t} - 1).\]
\[Cov(Y_3,Y_1) = \frac{\kappa_1^3}{2}\kappa_1^1(e^{2\lambda t} - 1)(e^{\lambda t} - 1).\]
\[Cov(Y_3,Y_2) = \frac{\kappa_1^3}{2}\kappa_1^2(e^{2\lambda t} - 1)(e^{\lambda t} - 1).\]
\[Cov(Y_2,Y_3) = \frac{\kappa_1^2}{2}\kappa_1^3(e^{2\lambda t} - 1)(e^{\lambda t} - 1).\]
\[Cov(Y_1,Y_2,Y_3) = E(Y_1^2)E(Y_2^2)E(Y_3^2) - (E(Y_1))^2(E(Y_2))^2(E(Y_3))^2 = \frac{\kappa_1^1}{2}\kappa_1^{**}(e^{2\lambda t} - 1)(e^{\lambda t} - 1)^2.\]
Using all the above covariances and variance terms in the (24) we can calculate the value of \[Var([\sigma_2]^2(\sigma_3)^2)]\], which concludes the proof of Theorem 6.

Using (18) and considering \([X_1^2,X_3^2]\) of Equation (7) we substitute from Equation (23) & (24) to get,
\[E[Cov(S^3,S^2)] = \frac{\eta^2}{T} \int_0^T \left( \sqrt{E([\sigma_2]^2(\sigma_3)^2]} - \frac{Var([\sigma_2]^2(\sigma_3)^2]}{8(E([\sigma_2]^2(\sigma_3)^2]}^2 \right) dt + \rho_1\rho_2\lambda \kappa_1^2 = E[Cov(S^2,S^3)].\]
Using (18) and considering \([X_1^3,X_1^3]\) of Equation (7) we substitute from Equation (19) & (20) to get,
\[E[Cov(S^1,S^2)] = \frac{\eta^2}{T} \int_0^T \left( \sqrt{E([\sigma_1]^2(\sigma_2)^2]} - \frac{Var([\sigma_1]^2(\sigma_2)^2]}{8(E([\sigma_1]^2(\sigma_2)^2]}^2 \right) dt + \rho_1\rho_2\lambda \kappa_1^2 = E[Cov(S^2,S^1)].\]
Using (18) and considering \([X_1^3,X_1^3]\) of Equation (7) we substitute from Equation (21) & (22) to get,
The following are various results related to expected covariances and variances for multiple assets:

1. The expected covariance between \((S^1, S^2)\) is given by

\[
E[\text{Cov}(S^1, S^2)] = \frac{\gamma_3}{T} \int_0^T \left( \sqrt{E[(\sigma_1)^2(\sigma_2)^2]} - \frac{\text{Var}[(\sigma_1)^2(\sigma_2)^2]}{8(E[(\sigma_1)^2(\sigma_2)^2])^2} \right) dt + \rho_1 \rho_2 \lambda \kappa_1^2 = E[\text{Cov}(S^2, S^1)].
\]

2. The expected covariance between \((S^2, S^3)\) is given by

\[
E[\text{Cov}(S^2, S^3)] = \frac{\gamma_2}{T} \int_0^T \left( \sqrt{E[(\sigma_1)^2(\sigma_2)^2]} - \frac{\text{Var}[(\sigma_1)^2(\sigma_2)^2]}{8(E[(\sigma_1)^2(\sigma_2)^2])^2} \right) dt + \rho_1 \rho_2 \lambda \kappa_1^2 = E[\text{Cov}(S^2, S^3)].
\]

3. The expected covariance between \((S^3, S^1)\) is given by

\[
E[\text{Cov}(S^3, S^1)] = \frac{\gamma_1}{T} \int_0^T \left( \sqrt{E[(\sigma_1)^2(\sigma_2)^2]} - \frac{\text{Var}[(\sigma_1)^2(\sigma_2)^2]}{8(E[(\sigma_1)^2(\sigma_2)^2])^2} \right) dt + \rho_3 \rho_1 \lambda \kappa_1^2 = E[\text{Cov}(S^3, S^1)].
\]
4. The expected variance of $S^1$ is given by

$$E[\text{Var}(S^1)] = \frac{1}{T} \int_0^T e^{-\lambda t} N_1^t dt + \frac{(\rho_1)^2}{T} \lambda \kappa_2^1.$$  

5. The expected variance of $S^2$ is given by

$$E[\text{Var}(S^2)] = \frac{1}{T} \int_0^T e^{-\lambda t} N_2^t dt + \frac{(\rho_2)^2}{T} \lambda \kappa_2^1.$$  

6. The expected variance of $S^3$ is given by

$$E[\text{Var}(S^3)] = \frac{1}{T} \int_0^T e^{-\lambda t} N_3^t dt + \frac{(\rho_3)^2}{T} \lambda \kappa_2^1.$$  

These results help us to derive the probability distribution of the eigenvalue that we discuss subsequently. We first look at the derivation of the trace swap price.

3 Swap using the Trace and the largest Eigenvalue of the Covariance Matrix

3.1 Swap using Trace

As the first proposal, we consider the investor using the trace of the covariance matrix to develop the swap. The trace is given by

$$tr \Omega = \text{Var}(S^1) + \text{Var}(S^2) + \text{Var}(S^3)$$

where

$$\Omega = \begin{bmatrix}
\text{Cov}_R(S^1, S^1) & \text{Cov}_R(S^1, S^2) & \text{Cov}_R(S^1, S^3) \\
\text{Cov}_R(S^2, S^1) & \text{Cov}_R(S^2, S^2) & \text{Cov}_R(S^2, S^3) \\
\text{Cov}_R(S^3, S^1) & \text{Cov}_R(S^3, S^2) & \text{Cov}_R(S^3, S^3)
\end{bmatrix}$$

Now the price of the swap on trace is the expected present value of the payoff in the risk neutral world for the assets we have considered

$$P_{\text{trace}}(x) = E\{e^{-rT} \{ tr \Omega - K_{\text{strike price}} \} \}.$$

$$P_{\text{trace}}(x) = e^{-rT} E\{ \{ tr \Omega - K_{\text{strike price}} \} \}.$$

$$P_{\text{trace}}(x) = e^{-rT} E[tr \Omega] - e^{-rT} K_{\text{strike price}}.$$

$$P_{\text{trace}}(x) = e^{-rT} E[\text{Var}(S^1) + \text{Var}(S^2) + \text{Var}(S^3)] - e^{-rT} K_{\text{strike price}}.$$

Using the values from results 4, 5 and 6 from Theorem 7 we can write

$$P_{\text{trace}}(x) = e^{-rT} \left[ \frac{1}{T} \int_0^T e^{-\lambda t} \left( \sum_{i=1}^3 N_i^t \right) dt + \frac{\lambda \kappa_2^1}{T} \left( \sum_{i=1}^3 \rho_i^2 \right) - K_{\text{strike price}} \right].$$
3.2 Swap using the largest Eigenvalue

The objective here is to define and derive the price of an eigenvalue swap. But we do not address the problem without an efficiency consideration as combinations of underlying assets for unconstrained variance may not be interesting as an investment destination. So, here we assume that the investor considers the maximum eigenvalue of the covariance matrix, for a given expected mean return. For which we have to find the distribution. We are going to use the concept of quadratic optimization taken from the paper Biswas and Mukherjee (2019). Using the same notation, we execute the following constrained optimisation programme:

\[
\begin{align*}
\text{maximize} & \quad w(t)^T \Omega w(t) \\
\text{subject to} & \quad w(t)^T w(t) = 1, \\
& \quad \Gamma w(t) = 1, \\
& \quad E(R)^T w(t) = k,
\end{align*}
\]

where \(w(t)\) is the weight vector corresponding to each of the stocks and \(R\) is the vector containing the expected return of the stocks. Overall the constraint can be combined as

\[
\begin{align*}
\text{maximize} & \quad w(t)^T \Omega w(t) \\
\text{subject to} & \quad w(t)^T I w(t) = 1, \\
& \quad A^T w(t) = b,
\end{align*}
\]

where

\[
\begin{align*}
w(t) &= \begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{bmatrix} \\
A &= \begin{bmatrix} E(R) \\ I \end{bmatrix} \\
b &= \begin{bmatrix} k \\ 1 \end{bmatrix}.
\end{align*}
\]

In our situation we have three assets so,

\[
\Omega = \begin{bmatrix}
\text{Cov}_R(S^1, S^1) & \text{Cov}_R(S^1, S^2) & \text{Cov}_R(S^1, S^3) \\
\text{Cov}_R(S^2, S^1) & \text{Cov}_R(S^2, S^2) & \text{Cov}_R(S^2, S^3) \\
\text{Cov}_R(S^3, S^1) & \text{Cov}_R(S^3, S^2) & \text{Cov}_R(S^3, S^3)
\end{bmatrix}
\]

and \(A = \begin{bmatrix} 0 \\ \mu_2 \\ \mu_3 \end{bmatrix}\).

We have considered mean of one asset as 0, now doing a QR decomposition of the matrix \(A\) as defined in the paper by Gander et al. (1991), we get

\[
P = \begin{bmatrix} P_1 & P_2 \end{bmatrix},
\]
where \( P_1 = \)
\[
\begin{bmatrix}
0 & \frac{\mu_3}{\sqrt{\mu_1^2 + \mu_3^2}} \\
\frac{\mu_2}{\sqrt{\mu_1^2 + \mu_2^2}} & \frac{\mu_2 + \mu_3}{\mu_1} \\
\frac{\mu_2}{\sqrt{\mu_1^2 + \mu_2^2}} & \frac{\mu_2 + \mu_3}{\mu_1}
\end{bmatrix}
\]
and \( P_2 = \)
\[
\begin{bmatrix}
\frac{\mu_3}{\sqrt{2(\mu_1^2 + \mu^2_3) - \mu_2 \mu_3}} \\
\frac{\mu_2}{\sqrt{2(\mu_1^2 + \mu^2_3) - \mu_2 \mu_3}} \\
-\frac{\mu_1}{\sqrt{2(\mu_1^2 + \mu_2^2) - \mu_2 \mu_3}}
\end{bmatrix}
\].

\[
A = \begin{bmatrix} P_1 & P_2 \end{bmatrix} [R]
\]

where \( R = \)
\[
\begin{bmatrix}
\sqrt{\mu_1^2 + \mu_3^2} & \frac{\mu_2 + \mu_3}{\sqrt{\mu_1^2 + \mu_3^2}} \\
0 & \frac{\mu_3}{\sqrt{2(\mu_2^2 + \mu_2 \mu_3)}}
\end{bmatrix}
\].

Let us consider the following definitions,
\[
\sqrt{\mu_1^2 + \mu_3^2} = U.
\]
\[
\sqrt{2(\mu_1^2 + \mu_3^2) - \mu_2 \mu_3)} = W.
\]
\[
\sqrt{2(\mu_2^2 + \mu_2 \mu_3)} = Z.
\]
\[
\mu_2 - \mu_3 = V.
\]

We finally want to compute
\[
\begin{bmatrix} P_1 & P_2 \end{bmatrix}^T \begin{bmatrix} \text{Cov}_R(S^1, S^1) & \text{Cov}_R(S^1, S^2) & \text{Cov}_R(S^1, S^3) \\ \text{Cov}_R(S^2, S^1) & \text{Cov}_R(S^2, S^2) & \text{Cov}_R(S^2, S^3) \\ \text{Cov}_R(S^3, S^1) & \text{Cov}_R(S^3, S^2) & \text{Cov}_R(S^3, S^3) \end{bmatrix} \begin{bmatrix} P_1 & P_2 \end{bmatrix}.
\]

**Theorem 8.** From the paper (Biswa and Mukherjee [2019]) we use value of the matrix \( F \)
\[
F = \begin{bmatrix}
\frac{1}{\det R} \left( \frac{kZ}{U} \right) \\
\frac{1}{\det R} \left( -k \left( \frac{\mu_2 + \mu_3}{U} \right) + U \right) \\
\sqrt{1 - \frac{1}{\det R^2} \left( \frac{k^2 Z^2}{U^2} + \left( -k \left( \frac{\mu_2 + \mu_3}{U} \right) + U \right)^2 \right)}
\end{bmatrix}
\]
which is used to calculate \( w(t) = PF \)

Now using this theorem 8, we can write the maximum eigenvalue as,
\[
\lambda = w(t)^T \Omega w(t).
\]
\[
\lambda = F^T P^T \Omega PF. \tag{34}
\]
With the previous definition we can simplify $\mathbf{P}$ as

$$
\mathbf{P} = \begin{bmatrix}
\mu_2 & \mu_2 + \mu_3 & \mu_2 - \mu_3 \\
\mu_2 & \mu_2 - \mu_3 & \mu_2 - \mu_3 \\
\mu_2 & \mu_2 - \mu_3 & \mu_2 - \mu_3
\end{bmatrix}
= \begin{bmatrix}
0 & \frac{\mu_2^2 + \mu_3^2}{W} & \frac{\mu_2 - \mu_3}{Z} \\
\frac{\mu_2^2}{W} & \frac{\mu_2^2 - \mu_3^2}{W} & \frac{\mu_2 - \mu_3}{Z} \\
\frac{\mu_2^2}{W} & \frac{\mu_2^2 - \mu_3^2}{W} & \frac{\mu_2 - \mu_3}{Z}
\end{bmatrix}
$$

Then, $\mathbf{P}^T \mathbf{\Omega} \mathbf{P} =$

$$
\begin{bmatrix}
0 & \frac{\mu_2^2}{W} & \frac{\mu_2^2 - \mu_3^2}{W} \\
\frac{\mu_2^2}{W} & \frac{\mu_2^2 - \mu_3^2}{W} & \frac{\mu_2 - \mu_3}{Z} \\
\frac{\mu_2^2}{W} & \frac{\mu_2^2 - \mu_3^2}{W} & \frac{\mu_2 - \mu_3}{Z}
\end{bmatrix}
\begin{bmatrix}
\text{Cov}(S^1, S^1) & \text{Cov}(S^1, S^2) & \text{Cov}(S^1, S^3) \\
\text{Cov}(S^2, S^1) & \text{Cov}(S^2, S^2) & \text{Cov}(S^2, S^3) \\
\text{Cov}(S^3, S^1) & \text{Cov}(S^3, S^2) & \text{Cov}(S^3, S^3)
\end{bmatrix}
\begin{bmatrix}
0 & \frac{\mu_2^2}{W} & \frac{\mu_2^2 - \mu_3^2}{W} \\
\frac{\mu_2^2}{W} & \frac{\mu_2^2 - \mu_3^2}{W} & \frac{\mu_2 - \mu_3}{Z} \\
\frac{\mu_2^2}{W} & \frac{\mu_2^2 - \mu_3^2}{W} & \frac{\mu_2 - \mu_3}{Z}
\end{bmatrix}
= \begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix}
$$

Multiplying out, we get the individual vectors $g_1$ etc. as

$$
g_1 = \begin{bmatrix}
\frac{\mu_2^2}{W} \text{Cov}(S^2, S^2) + \frac{\mu_2}{W} \text{Cov}(S^2, S^3) + \frac{\mu_3}{W} \text{Cov}(S^3, S^3)
\end{bmatrix}
$$

$$
g_2 = \begin{bmatrix}
\frac{\mu_2^2}{W} \text{Cov}(S^2, S^2) + \frac{\mu_2}{W} \text{Cov}(S^2, S^3) + \frac{\mu_3}{W} \text{Cov}(S^3, S^3)
\end{bmatrix}
$$

$$
g_3 = \begin{bmatrix}
\frac{\mu_2^2}{W} \text{Cov}(S^2, S^2) + \frac{\mu_2}{W} \text{Cov}(S^2, S^3) + \frac{\mu_3}{W} \text{Cov}(S^3, S^3)
\end{bmatrix}
$$

So, using (34) we can calculate the eigenvalue as,

$$
\lambda = \mathbf{F}^T \begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix} \mathbf{F}
$$

where let us assign

$$
\begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix} = \begin{bmatrix} d_1 & d_2 & d_3 \end{bmatrix}
$$

$$
d_1 = \left( \frac{\mu_2^2}{U_2} \text{Cov}(S^2, S^2) + \frac{\mu_2 \mu_2}{U_2} \text{Cov}(S^2, S^3) + \frac{\mu_3^2}{U_2} \text{Cov}(S^3, S^3) \right) \left( \frac{1}{\det R} \left( \frac{\text{Trace}}{U} \right) \right) + \frac{U \mu_2}{W} \text{Cov}(S^2, S^2) + \ldots
$$

19
Now the price of the largest eigenvalue swap is the expected present value of the payoff in the risk neutral world for this three asset we have considered

$$P_{\text{eigenvalue}} = E\{e^{-RT}(\lambda - K)\}.$$

Replacing the Equation (35) in to the above equation,

$$P_{\text{eigenvalue}} = e^{-RT}E\left(d_1\left(\frac{1}{\det R}\left(\frac{kZ}{U}\right)\right) + d_2\left(\frac{1}{\det R}\left(-k\left(\frac{\mu_2 + \mu_3}{U}\right) + Y\right)\right) +$$
\[ \mathbf{d}_3 \sqrt{1 - \frac{1}{(\det R)^2} \left( k^2 \frac{Z^2}{U^2} + \left((-k \frac{\mu_2 + \mu_3}{U}) + U \right)^2 \right)} - (e^{-\tau T} \times K_{\text{strike price}}). \]

\[ P_{\text{eigenvalue}} = e^{-\tau T} \left( E(d_1) \left( \frac{1}{\det R} \left( \frac{kZ}{U} \right) \right) + E(d_2) \left( \frac{1}{\det R} \left(-k \left( \frac{\mu_2 + \mu_3}{U} \right) + Y \right) \right) + E(d_3) \sqrt{1 - \frac{1}{(\det R)^2} \left( k^2 \frac{Z^2}{U^2} + \left((-k \frac{\mu_2 + \mu_3}{U}) + U \right)^2 \right)} - K_{\text{strike price}} \right). \] (36)

Using the values (1-6) obtained in Theorem 7, we can calculate the values of \( E(d_1), E(d_2) \) and \( E(d_3) \) which will finally give the \( P_{\text{eigenvalue}} \) in (36).

4 Numerical Example

Spot Price of three agricultural commodities are considered for our numerical illustration. The first one is Potato (AgroMarket [2020], spot Price of Mustard is chosen as the second commodity and spot price of Rice is chosen as third. The data is taken from the Multi-Commodity Exchange of India and we have used the daily closing price (which we have considered the Spot price) of these three commodities, denoted by \( S_1, S_2 \) and \( S_3 \) respectively, in the time range 1st June, 2019 till 31st May, 2020. We computed the return of the spot prices using the formula

\[ \text{return at time } t; \tau_t \mathbf{r} = \frac{S_t - S_{t-1}}{S_{t-1}}, i = 1, 2, 3. \]

We estimate the relevant parameters from the descriptive statistics for the three commodities presented in the table below. From the statistics we can observe that range of the returns of Mustard is least whereas that of Rice is highest. Also the median value of the returns of the three commodities are not close to the mean showing that the data is skewed in nature. The volatility of Rice is least whereas for Potato it is maximum which goes with the intuition that rice has less volatility in the market.

| Commodity     | Mustard | Potato | Rice   |
|---------------|---------|--------|--------|
| No. of Observation | 252     | 252    | 252    |
| Mean          | -0.0038 | 0.0317 | -0.0002|
| Std. Error    | 0.052   | 0.044  | 0.027  |
| 95% CI of Mean| 0.095   | 0.0205 | 0.0115 |
| Min           | -0.0526 | -0.0916| -0.990 |
| Max           | 0.01465 | 0.0618 | 0.0432 |
| Range         | 0.0941  | 0.1598 | 1.033  |
| Median        | 0       | 0.0005 | 0.0018 |
| Std. Deviation| 0.0502  | 0.0267 | 0.0058 |

Plotting the histogram of the return for Mustard, Potato and Rice we can see that for Mustard and Potato it is evenly spread [Refer Figure (1a & 1b)] but for Rice it is highly skewed [Refer Figure (1c)].

\[^1\text{We thank Suranjana Joarder for help with data access.}\]
Plotting the line diagrams of the spot return for the three commodities over the entire time period we can see there are sudden movements indicating that there are clear Jumps in all the commodities [Refer Figure 2].

Below we also show the required parameter estimates, derived from the descriptive statistics, which are going to be used in the pricing calculations.
Figure 2: Line Plots of the Spot Return for the 3 commodities

- The mean, standard deviation and leverage parameters of returns of the commodities are estimated as (standard error of the estimate in bracket),

| Parameter | Value     | Standard Error |
|-----------|-----------|----------------|
| $\mu_1$   | -0.0038   | 0.052          |
| $\mu_2$   | 0.0317    | 0.044          |
| $\mu_3$   | -0.0002   | 0.027          |
| $(\sigma_1)_0$ | 0.0502 | 0.0643         |
| $(\sigma_2)_0$ | 0.0267 | 0.0761         |
| $(\sigma_3)_0$ | 0.0058 | 0.0097         |
| $\rho_1$  | 0.8       | 0.065          |
| $\rho_2$  | 0.5       | 0.034          |
| $\rho_3$  | 0.6       | 0.087          |

- From the paper by Hellström and Lönnbark (2011) we can calculate the correlation of returns among the commodities as,

| Correlation | Value     |
|-------------|-----------|
| $\gamma_{12}$ | -0.0216   |
| $\gamma_{23}$ | -0.0862   |
| $\gamma_{31}$ | -0.0276   |

- The correlation value of the Lévy subordinators and $\lambda$ of the commodities are calculated as (Ait-Sahalia and Jacod 2009),

| Parameter | Value |
|-----------|-------|
| $r_2$     | 0.2319|
| $r_3$     | 0.5721|
| $\lambda$ | 0.4   |
| $r$       | 0.00014|

We can evaluate the expected value of the variance term and the covariance term using Theorem 7 and the above values to obtain the matrix as:
\[
\begin{bmatrix}
E[\text{Var}(S^1, S^1)] & E[\text{Cov}(S^1, S^2)] & E[\text{Cov}(S^1, S^3)] \\
E[\text{Cov}(S^2, S^1)] & E[\text{Var}(S^2, S^2)] & E[\text{Cov}(S^2, S^3)] \\
E[\text{Cov}(S^3, S^1)] & E[\text{Cov}(S^3, S^2)] & E[\text{Var}(S^3, S^3)]
\end{bmatrix} = \begin{bmatrix}
0.00736 & 0.00065 & 0.00082 \\
0.00065 & 0.00498 & 0.00039 \\
0.00082 & 0.00039 & 0.00217
\end{bmatrix}
\]

As mentioned in the introduction, our first candidate measure of generalized variance is the trace of the covariance matrix which is nothing but the sum of the individual variances. Intuitively, this is the variance of the return of a portfolio comprising one unit of each of the stocks, assuming them to be uncorrelated. So we can calculate the price of the swap given by Equation (33) as,

\[
P_{\text{trace}}(x) = e^{-rT} \times 0.01451 - e^{-rT} K_{\text{strike price}}
\]

The swap is written of the trace. Considering the strike price as 0.01 and duration is for 1 year i.e. \(T = 252\) we can finally calculate the trace swap as,

\[
P_{\text{trace}} = 0.00435
\]

The second candidate measure of generalised variance considered here is the maximum eigenvalue which is the magnitude of the biggest component of the orthogonalised system for the return covariance matrix. As the return distributions are correlated (the covariance matrix is not diagonal), this biggest component will be significantly larger than the individual variances. This is considered as we are interested in managing the variance, so swapping for the biggest component is a safe strategy to adopt.

Considering \(T = 252\) we do the following calculations,

\[
A = \begin{bmatrix}
-0.0038 & 1 \\
0.0317 & 1 \\
-0.0002 & 1
\end{bmatrix}
\]

Doing a QR decomposition of \(A\), we get

\[
P_1 = \begin{bmatrix}
-0.119 & 0.7359 \\
0.9929 & 0.0925 \\
-0.0063 & 0.6707
\end{bmatrix}
\]

\[
R = \begin{bmatrix}
0.0319 & 0.8676 \\
0 & 1.4991
\end{bmatrix}
\]

\[
P_2 = \begin{bmatrix}
0.6874 \\
0.0747 \\
0.7417
\end{bmatrix}
\]

Therefore \(P\) is,

\[
P = \begin{bmatrix}
-0.119 & 0.7359 & 0.6874 \\
0.9929 & 0.0925 & 0.0747 \\
-0.0063 & 0.6707 & 0.7417
\end{bmatrix}
\]

\[
R^{-T} = \begin{bmatrix}
31.3479 & 0 \\
-18.1425 & 0.6670
\end{bmatrix}
\]
Given that,

\[ b = \begin{bmatrix} 0.0007 \\ 1 \end{bmatrix} \]

\[ q = \begin{bmatrix} 0.0219 \\ 0.6543 \end{bmatrix} \]

\[ r = \begin{bmatrix} 0.7559 \end{bmatrix} \]

Therefore we can calculate \( w(t) \)

\[ w(t) = \begin{bmatrix} 0.9985 \\ 0.1387 \\ 0.9994 \end{bmatrix} \]

Using the Equation (56) we can calculate the price of the eigenvalue swap is given by,

\[ P_{\text{eigenvalue}} = e^{-rT} 0.0115 - (e^{-rT} \times K_{\text{strike price}}) \]

The swap is written of the eigenvalue. Considering the strike price as 0.01 we can finally calculate the eigenvalue swap as,

\[ P_{\text{eigenvalue}} = 0.00145 \]

5 Conclusion

In this paper we have presented a new approach for pricing swaps for a Barndorff-Nielsen and Shephard model defined on two important measures of generalized variance, namely the maximum eigenvalue and trace of the covariance matrix of the returns on assets involved. The objective is to price generalized variance swaps for financial markets guided by the BNS model for asset returns. We have considered multiple assets in the portfolio for theoretical purpose and demonstrated the theoretical approach with the help of numerical examples taking three commodities in the portfolio. The results derived in this paper are the comparison between the swaps defined by the trace and the eigenvalue. Moreover, the results obtained in this paper have important implications for their use in the commodity sector as volatility in the commodity markets, agricultural in particular, are often related through natural causes. This would be an important area where such swaps would be useful for hedging risk.

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