Spectra of sub-Dirac operators on certain nilmanifolds

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Abstract

We study sub-Dirac operators that are associated with left-invariant bracket-generating sub-Riemannian structures on compact quotients of nilpotent semi-direct products $G = \mathbb{R}^n \rtimes_A \mathbb{R}$. We will prove that these operators admit an $L^2$-basis of eigenfunctions. Explicit examples show that the spectrum of these operators can be non-discrete and that eigenvalues may have infinite multiplicity.

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1 Introduction

Spectra of sub-Laplace operators on sub-Riemannian manifolds are intensely studied. Especially interesting is the case where the distribution defining the sub-Riemannian structure is bracket generating, what we shall assume in the following. In this case the sub-Laplacian is known to be hypo-elliptic [H].

Many explicit calculations of the spectrum have been done in the situation where the underlying manifold is a compact Lie group or a quotient of a Lie group by a discrete cocompact subgroup, see, for example, [BF1, BFI, BF2, P]. In [BF1, BFI, P] the authors study spectral properties of sub-Laplace operators on nilpotent groups of step two and on compact quotients by discrete subgroups. They determine the heat kernels of these operators. This allows an explicit determination of the spectrum of the sub-Laplacian, which is discrete in this situation.

The aim of this paper is to study spectra of sub-Riemannian analogs of the classical Dirac operator. In the definition of the sub-Dirac operator the following difficulty occurs: In contrast to the Riemannian case where we have the Levi-Civita connection as a preferred connection, in general, there is no connection canonically associated with a sub-Riemannian structure. Only in special geometric situations a canonical connection exists. Hence the following definition of the sub-Dirac operator depends on the choice of a connection.

Let $(M, \mathcal{H}, g)$ be a sub-Riemannian manifold, $\dim \mathcal{H} = d$. Suppose that $\nabla$ is a metric connection on $\mathcal{H}$. Moreover, assume that $\mathcal{H}$ is oriented and that the bundle of orthonormal frames of $\mathcal{H}$ admits a reduction to $\text{Spin}(d)$. Such a reduction will be called a
spin structure of $\mathcal{H}$. Then we can associate a spinor bundle $S$ with this spin structure. Moreover, using the connection $\nabla$ we can define a sub-Riemannian Dirac operator, which acts on sections in $S$.

Only few results for sub-Riemannian analogs of the Dirac operator are known. For the case of a sub-Riemannian manifold of contact type such an operator was introduced and studied by Petit [Pe], who called this operator Kohn-Dirac operator. More exactly, this was done for a Spin$^c$-structure.

Studying the sub-Riemannian Dirac operator the following natural questions arise: Which structure does its spectrum have? How does the spectrum depend on the sub-Riemannian geometry of the manifold and on the spin structure of $\mathcal{H}$? How do the sub-Dirac operator and its spectrum depend on the chosen connection?

In general, i.e., for arbitrary metric connections in $\mathcal{H}$, the sub-Dirac is not symmetric. We will characterize the symmetry of this operator by a simple condition on the connection.

Here we focus on nilmanifolds. More precisely, we study sub-Dirac operators on manifolds of the form $M = \Gamma \backslash G$ where $G = \mathbb{R}^n \rtimes_A \mathbb{R}$ is a semi-direct product defined by a one-parameter subgroup $A(t)$ of unipotent matrices in $\text{GL}(n, \mathbb{R})$ and $\Gamma$ is the subgroup $\mathbb{Z}^n \rtimes_A \mathbb{Z}$. These manifolds $M$ can be interpreted as a suspension of the diffeomorphism of the torus $\mathbb{R}^n/\mathbb{Z}^n$ induced by $A(1)$. This is also the starting point of [I] where the spectrum of the Laplacian on left-invariant differential forms on $M$ is considered.

Our sub-Dirac operators will be associated with sub-Riemannian structures $(\mathcal{H}, \dot{g})$ on $\Gamma \backslash G$ coming from a left-invariant and bracket-generating distribution $(\mathcal{H}, g)$ on $G$. We choose a metric connection in $H$ such that $D$ is symmetric.

Our approach is to give an explicit decomposition of the regular representation of $G$. Roughly speaking, it turns out that the sub-Dirac operator is an orthogonal sum of elliptic operators on the real line, each having a discrete spectrum. This shows that $D$ on $\Gamma \backslash G$ has pure point spectrum.

We apply our results to compute the spectrum of $D$ explicitly for two classes of two-step nilmanifolds of the above form. First we consider three-dimensional Heisenberg manifolds. Secondly, we study a class of five-dimensional two-step nilpotent nilmanifolds with a three-dimensional distribution. The latter example shows that the spectrum of the sub-Dirac operator is not necessarily a discrete subset of $\mathbb{R}$ and that its eigenvalues may have infinite multiplicity, contrary to the results for the spectrum of the sub-Laplacian on compact 2-step nilmanifolds.

Finally, we discuss a three-step nilpotent example of dimension four with a two-dimensional distribution. In this case the spectrum can be expressed in terms of the spectra of the family of operators $P_c = \partial^2 + (t^2 + c)^2 \pm 2t, \ c \in \mathbb{R}$.

In all three examples, the multiplicities of the eigenvalues of $D$ can be read off from the coadjoint orbit picture.

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2 Sub-Riemannian Dirac operators

2.1 Definition of sub-Dirac operators

Let $M$ be a smooth manifold and let $\mathcal{H} \subset TM$ be a smooth distribution, where $\dim \mathcal{H}_x = d$ for all $x \in M$. Let $\Gamma(\mathcal{H})$ denote the space of smooth sections of $\mathcal{H}$. We assume that $\mathcal{H}$ is bracket-generating. That means, that for each $x \in M$ there is a $J \in \mathbb{N}$ such that the sequence

$$\Gamma_0 := \Gamma(\mathcal{H}), \quad \Gamma_{j+1} := \Gamma_j + [\Gamma_0, \Gamma_j]$$

satisfies $\{X(x) \mid X \in \Gamma_J\} = T_x M$. If $g$ is a Riemannian metric on $\mathcal{H}$, then the pair $(\mathcal{H}, g)$ is called a sub-Riemannian structure on $M$ and $(M, \mathcal{H}, g)$ is called a sub-Riemannian manifold.

Let $\nabla : \Gamma(\mathcal{H}) \otimes \Gamma(\mathcal{H}) \to \Gamma(\mathcal{H})$ be a metric connection on $\mathcal{H}$. Note that here we consider only derivations by vector fields in $\mathcal{H}$. Suppose that $\mathcal{H}$ is oriented and that it admits a spin structure, i.e., that there is a $\text{Spin}(d)$-reduction $P_{\text{Spin}}(\mathcal{H})$ of the principal $\text{SO}(d)$-bundle $P_{\text{SO}}(\mathcal{H})$ of oriented orthonormal frames of $(\mathcal{H}, g)$. We consider the complex representation of $\text{Spin}(d)$ which is obtained by restriction of (one of) the complex irreducible representation(s) of the Clifford algebra $\text{Cl}(d) := \text{Cl}(\mathbb{R}^d)$. We will call it spinor representation and denote it by $\Delta_d$. The associated bundle $P_{\text{Spin}}(\mathcal{H}) \times_{\text{Spin}(d)} \Delta_d$ is called spinor bundle $S$ of $(\mathcal{H}, g)$. The space of smooth sections in $S$ is denoted by $\Gamma(S)$. The connection $\nabla$ defines a connection $\nabla^S : \Gamma(\mathcal{H}) \times \Gamma(S) \to \Gamma(S)$ in the following way. Let $s_1, \ldots, s_d$ be a local orthonormal frame of $\mathcal{H}$ and consider the local connection forms $\omega_{ij} = g(\nabla s_i, s_j)$. Then we define

$$\nabla^S_X \varphi := X(\varphi) + \frac{1}{2} \sum_{i<j} \omega_{ji}(X) s_i \cdot s_j \cdot \varphi,$$

where $\cdot \cdot$ denotes the Clifford multiplication.

Now we can define a sub-Riemannian Dirac operator, or sub-Dirac operator for short, by

$$D = \sum_i s_i \cdot \nabla^S_{s_i} : \Gamma(S) \longrightarrow \Gamma(S),$$

where again $s_1, \ldots, s_d$ is a local orthonormal frame of $\mathcal{H}$. Note, that the definition of $D$ depends on the choice of the connection $\nabla$ on $\mathcal{H}$ and that, in general, this choice is far from being canonical in contrast to the Riemannian case, where we have the Levi-Civita connection as a preferred connection.

A large class of metric connections in $\mathcal{H}$ can be obtained in the following way. Suppose we are given a further distribution $\mathcal{V} \subset TM$ such that $TM = \mathcal{H} \oplus \mathcal{V}$. Then this decomposition of $TM$ gives us a projection $\text{pr} : TM \to \mathcal{H}$ and we can define a connection $\nabla$ by the Koszul formula

$$2g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y))$$

$$+ g(\text{pr}[X, Y], Z) - g(\text{pr}[X, Z], Y) - g(\text{pr}[Y, Z], X),$$

where $X, Y, Z \in \Gamma(\mathcal{H})$. In this case $\nabla$ is uniquely determined by the vanishing of $\nabla_X Y - \nabla_Y X - \text{pr}[X, Y]$. 

3
2.2 Symmetry of the sub-Dirac operator

A sub-Riemannian manifold is said to be regular if for each \( j = 1, \ldots, J \) the dimension of \( \{ X(x) \mid X \in \Gamma_j \} \) does not depend on the point \( x \in M \). Let \((M, \mathcal{H}, g)\) be an orientable regular sub-Riemannian manifold. Then \((M, \mathcal{H}, g)\) admits an intrinsic volume form \( \omega_0 \) on \( M \), see [M], Section 10.5 and [ABGR].

Let \((M, \mathcal{H}, g)\) be an oriented and regular sub-Riemannian manifold. Consider any volume form \( \omega \) on \( M \), e.g., the intrinsic one. Define the divergence of a vector field \( X \) on \( M \) by \( \mathcal{L}_X \omega = (\text{div} X) \cdot \omega \). Let \( \nabla \) be a metric connection on \( \mathcal{H} \). Suppose that \( \mathcal{H} \) admits a spin structure and define \( D : \Gamma(S) \to \Gamma(S) \) as above. Let \( \langle \cdot , \cdot \rangle \) be a hermitian inner product on \( \Delta \) for which the Clifford multiplication is antisymmetric. This inner product is unique up to scale. It induces a hermitian inner product on \( S \), which together with \( \omega \) gives an \( L^2 \)-inner product \( (\cdot , \cdot) \) on the space \( \Gamma_0(S) \) of sections in \( S \) with compact support.

It is easy to find examples of three-dimensional Heisenberg manifolds with two-dimensional distribution \( \mathcal{H} \) and metric connection for which \( D \) is not symmetric, see Section 4.2.

The following lemma states that the sub-Dirac operator is symmetric if and only if the divergence defined by the sub-Riemannian structure coincides with the divergence given by the connection, compare also [FS] for the Riemannian case.

**Lemma 2.1** Under the above conditions, \( D \) is symmetric if and only if

\[
\text{div} X = \sum_{i=1}^d \langle \nabla_{s_i} X, s_i \rangle \quad (3)
\]

holds for one (and therefore for every) local orthonormal basis \( s_1, \ldots, s_d \) of \( \mathcal{H} \).

If, in addition, \( \mathcal{V} \) is a complement of \( \mathcal{H} \) in \( TM \) and \( \nabla \) is defined as in (2), then (3) is equivalent to the following condition. For one (and therefore for all) sets \( \{ \xi_1, \ldots, \xi_l \} \), \( l = \dim M - k \), of local sections of \( \mathcal{V} \) that satisfy \( \omega(s_1, \ldots, s_d, \xi_1, \ldots, \xi_l) = 1 \) the equation

\[
\eta_1([X, \xi_1]) + \cdots + \eta_l([X, \xi_l]) = 0
\]

holds for all \( X \in \Gamma(\mathcal{H}) \), where \( \eta_1, \ldots, \eta_l \in \Gamma(T^*M) \) are defined to be zero on \( \mathcal{H} \) and dual to \( \xi_1, \ldots, \xi_l \).

In particular, if \( \text{codim} \mathcal{H} = 1 \), then \( D \) is symmetric if and only if \( [\Gamma(\mathcal{H}), \xi_1] \subset \Gamma(\mathcal{H}) \).

**Proof.** Consider sections \( \varphi, \psi \in \Gamma_0(S) \) and define \( f : \mathcal{H} \to \mathbb{C} \) by

\[
f(w) := \langle \varphi, w \cdot \psi \rangle. \quad (4)
\]

Moreover, define \( u \in \Gamma_0(\mathcal{H} \otimes \mathbb{C}) \) by

\[
g^c(u, w) = f(w) \quad (5)
\]

for all \( w \in \Gamma(\mathcal{H}) \), where \( g^c \) denotes the the complex bilinear extension of \( g \). Choose a local orthonormal frame \( s_1, \ldots, s_d \) of \( \mathcal{H} \). Then

\[
\langle D\varphi, \psi \rangle - \langle \varphi, D\psi \rangle = \sum_{i=1}^d \left( f(\nabla_{s_i} s_i) - s_i(f(s_i)) \right) = \sum_{i=1}^d g^c(\nabla_{s_i} u, s_i),
\]
thus
\[
(D\varphi,\psi) - (\varphi, D\psi) = \int_M \left( \sum_{i=1}^d g^C(\nabla_{s_i}u, s_i) \right) \omega = \int_M \left( \sum_{i=1}^d g^C(\nabla_{s_i}u, s_i) - \text{div}(u) \right) \omega.
\]

In particular, (3) is sufficient for the symmetry of $D$. On the other hand, any section $u_1 \in \Gamma_0(H)$ is the real part of a section $u \in \Gamma_0(H \otimes \mathbb{C})$ that satisfies (4) and (5) for some $\varphi, \psi \in \Gamma_0(S)$. Indeed, choose $\psi$ such that $\langle \psi(x), \psi(x) \rangle = 1$ for all $x \in \text{supp } u_1$ and put $\varphi := u_1 \cdot \psi$. Define $u$ by (4) and (5). Then
\[
g(u_1, w) = \text{Re} \langle u_1 \cdot \psi, w \cdot \psi \rangle = \text{Re} \langle \varphi, w \cdot \psi \rangle = \text{Re} f(w)
\]
for all $w \in \Gamma(H)$, hence $u_1 = \text{Re } u$. Consequently, the symmetry of $D$ implies
\[
\int_M \left( \sum_{i=1}^d g(\nabla_{s_i}u, s_i) - \text{div}(u) \right) \omega = 0
\]
for all $u \in \Gamma_0(H)$. Since the integrand is $C^\infty_0(M)$-linear in $u$, Equation (3) follows.

The second part of the lemma now follows from
\[
\text{div}(u) = (L_u \omega)(s_1, \ldots, s_d, \xi_1, \ldots, \xi_l)
\]
\[
= - \sum_{i=1}^d \omega(s_1, \ldots, [u, s_i], \ldots, s_d, \xi_1, \ldots, \xi_l) - \sum_{j=1}^l \omega(s_1, \ldots, s_d, \xi_1, \ldots, [u, \xi_j], \ldots, \xi_l)
\]
\[
= - \sum_{i=1}^d g(\text{pr}[u, s_i], s_i) - \sum_{j=1}^l \eta_j([u, \xi_j]) = \sum_{i=1}^d g(\nabla_{s_i}u, s_i) - \sum_{j=1}^l \eta_j([u, \xi_j]),
\]
where the last equality is a consequence of Equation (2).

\[\square\]

2.3 Sub-Dirac operators on Lie groups and compact quotients

Let $G$ be a simply connected Lie group and $\Gamma \subset G$ a uniform discrete subgroup. Let $H \subset TG$ be a left-invariant distribution and $g$ a left-invariant Riemannian metric on $H$. Obviously, $H$ is spanned by orthonormal left-invariant vector fields $s_1, \ldots, s_d$. In particular, the frame bundle $P_{SO}(H)$ is a trivial bundle and the unique spin structure of $H$ equals $P_{\text{Spin}}(H) = G \times \text{Spin}(d)$.

The pair $(H, g)$ induces a sub-Riemannian structure on $\Gamma \backslash G$, which we will denote by $(\bar{H}, \bar{g})$. The frame bundle $P_{SO}(\bar{H})$ can be identified with
\[
P_{SO}(\bar{H}) = G \times_\Gamma \text{SO}(d),
\]
where $\Gamma$ acts by left multiplication on $G$ and trivially on $\text{SO}(d)$. There is a one-to-one correspondence between homomorphisms $\varepsilon : \Gamma \rightarrow \mathbb{Z}_2 = \{0, 1\}$ and spin structures of $H$ given by
\[
\varepsilon \mapsto P_{\text{Spin},\varepsilon}(\bar{H}) = G \times_\Gamma \text{Spin}(d),
\]
where $\gamma \in \Gamma$ acts by multiplication by $e^{i\pi \xi(\gamma)}$ on $\text{Spin}(d)$. Spinor fields are sections of the associated spinor bundle $P_{\text{Spin}(d)}(\mathcal{H}) \times_{\text{Spin}(d)} \Delta_d \cong G \times_{\Gamma} \Delta_d$ or, equivalently, maps $\psi: G \to \Delta_d$ that satisfy $\psi(\gamma g) = e^{i\pi \xi(\gamma)} \psi(g)$ for all $\gamma \in \Gamma$, $g \in G$.

The intrinsic volume form $\omega_0$ on $\Gamma \backslash G$ introduced in \cite{ABGR} and discussed in Section 2.2 is left-invariant.

Now let $\nabla$ be a left-invariant metric connection on $\mathcal{H}$. Let $s_1, \ldots, s_d$ be an orthonormal basis of $\mathcal{H}$ consisting of left-invariant vector fields. As for the symmetry of the Dirac operator discussed in Lemma 2.1, note that Equation (3) is equivalent to

$$0 = \sum_{i=1}^{d} g(\nabla_{s_i} s_j, s_i)$$

for all $j = 1, \ldots, d$. Indeed, since the intrinsic volume form is left-invariant and $G$ must be unimodular the divergence of any left-invariant vector field vanishes.

### 3 Sub-Riemannian structures on $\Gamma \backslash (\mathbb{R}^n \ltimes_A \mathbb{R})$

#### 3.1 The standard model

Let $A(t) = \exp(tB)$ be a one-parameter subgroup of $\text{GL}(n, \mathbb{R})$. We consider the simply-connected solvable Lie group $\mathbb{R}^n \ltimes_A \mathbb{R}$ with group law

$$(x, s) (y, t) = (x + A(s)y, s + t) .$$

In particular, $(0, t) (x, 0) (0, t)^{-1} = (A(t)x, 0)$. In addition, we assume $A(1) \in \text{SL}(n, \mathbb{Z})$ so that the set $\mathbb{Z}^n \times \mathbb{Z}$ becomes a uniform discrete subgroup.

The pair $(\mathbb{R}^n \ltimes_A \mathbb{R}, \mathbb{Z}^n \ltimes_A \mathbb{Z})$ serves as a standard model in the following sense.

**Lemma 3.1** Let $G$ be an exponential Lie group admitting a connected abelian normal subgroup $N$ of codimension one. Let $\Gamma$ be a uniform discrete subgroup of $G$ such that $\Gamma \cap N$ is uniform in $N$. Then there exists a one-parameter subgroup $A$ of $\text{GL}(n, \mathbb{R})$, $n = \text{dim} \ N$, with $A(1) \in \text{SL}(n, \mathbb{Z})$, and an isomorphism $\Phi$ of $\mathbb{R}^n \ltimes_A \mathbb{R}$ onto $G$ mapping $\mathbb{Z}^n \ltimes_A \mathbb{Z}$ onto $\Gamma$.

**Proof.** We fix generators $v_1, \ldots, v_n$ of the lattice $\Gamma \cap N$ of the vector group $N$ and consider the linear isomorphism $M$ of $\mathbb{R}^n$ onto $N$ given by $M(v_j) = e_j$. On the other hand, the assumption on $\Gamma$ implies that $\Gamma N$ is closed in $G$ and that $\Gamma N/N$ is a discrete subgroup of $G/N$. Hence there exists $b \in \mathfrak{g}$ with $\exp(b) \in \Gamma$ and such that $\exp(b)N$ is a generator of $\Gamma N/N$.

Put $A(t)x = M^{-1}(\exp(tb)M(x)\exp(tb)^{-1})$. Now it follows that $\Phi(x, t) = M(x)\exp(tb)$ is an isomorphism of $\mathbb{R}^n \ltimes_A \mathbb{R}$ onto $G$ with $\Phi(\mathbb{Z}^n \ltimes A) = \Gamma$. In particular, $\mathbb{Z}^n \ltimes A \mathbb{Z}$ is a subgroup of $\mathbb{R}^n \ltimes A \mathbb{R}$. This means that $\mathbb{Z}^n$ is $A(l)$-invariant for all $l \in \mathbb{Z}$ so that $A(l) \in \text{SL}(n, \mathbb{Z})$. \qed
The condition \( A(1) \in \operatorname{SL}(n, \mathbb{Z}) \) implies \( B \in \mathfrak{sl}(n, \mathbb{R}) \) and \( A(t) \in \operatorname{SL}(n, \mathbb{R}) \) for all \( t \in \mathbb{R} \). This reflects the fact that locally compact groups admitting a uniform discrete subgroup are unimodular, compare Theorem 7.1.7 of [W].

The Lie algebra of \( G := \mathbb{R}^n \rtimes_A \mathbb{R} \) is isomorphic to \( \mathfrak{g} = \mathbb{R}^n \rtimes_B \mathbb{R} \), and \( B = \operatorname{ad}(b) | \mathfrak{n} \), where \( b = (0, 1) \) and \( \mathfrak{n} = \mathbb{R}^n \times \{0\} \). Note that \( G \) is exponential if and only if \( B \) has no purely imaginary eigenvalues, compare Theorem 1 of [LL].

It is evident that
\[
\pi(x, t) = \begin{pmatrix}
x_1 \\
A(t) \vdots x_n \\
0 \ldots \ldots 0 \ 1
\end{pmatrix}
\]
defines a representation, which is faithful provided that \( G \) is exponential and not abelian.

**Example 3.2** Fix \( r \in \mathbb{Z}_+ \) and set \( B = \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix} \) so that \( A(t) = \exp(tB) = I + tB \). Since \( A(l) \in \operatorname{SL}(2, \mathbb{Z}) \) for \( l \in \mathbb{Z} \), \( \Gamma = \mathbb{Z}^2 \rtimes_A \mathbb{Z} \) is a subgroup of \( G = \mathbb{R}^2 \rtimes_A \mathbb{R} \). On the other hand,
\[
\pi(x_1, x_2, t) = \begin{pmatrix} 1 & rt & x_1 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix}
\]
gives an isomorphism from \( G \) onto the three-dimensional Heisenberg group \( H(1) \) in its standard realisation as a group of matrices mapping \( \Gamma \) onto
\[
\Gamma_r = \left\{ \begin{pmatrix} 1 & rl & k_1 \\ 0 & 1 & k_2 \\ 0 & 0 & 1 \end{pmatrix} : l, k_1, k_2 \in \mathbb{Z} \right\}.
\]
In particular, the above construction yields all uniform discrete subgroups of \( H(1) \) and hence all three-dimensional Heisenberg manifolds, compare Section 2 of [GW].

Heisenberg manifolds and certain generalisations of them will be discussed in Section 4.2 and 4.3 in greater detail.

For the subgroup \( \Gamma := \mathbb{Z}^n \rtimes_A \mathbb{Z} \), the spin structures of any distribution of \( T(\Gamma \setminus G) \) induced by a left-invariant distribution of \( T(G) \) are determined as follows.

**Lemma 3.3** A map \( \varepsilon : \Gamma \rightarrow \mathbb{Z}_2 \) is a homomorphism if and only if \( \varepsilon(k, l) = \varepsilon'(k) + \varepsilon(l) \) for some homomorphism \( \hat{\varepsilon} : \mathbb{Z} \rightarrow \mathbb{Z}_2 \) and a homomorphism \( \varepsilon' : \mathbb{Z}^n \rightarrow \mathbb{Z}_2 \) satisfying
\[
\sum \varepsilon'(e_{\mu})(A(1) - I)_{\mu \nu} \in 2\mathbb{Z} \quad (7)
\]
for all \( \nu \).
Since $\Gamma$ is uniform and discrete, there exists a unique normal ised right to (7). It is easy to check that the converse holds also true.

\[
\text{Radon measure } \mu \text{ definition of } (F \psi G) \aim \text{ is to decompose the right regular representation of the unit cube of } \mathbb{R}^n.
\]

Let $C$ be the right-regular representation of $\Gamma$. Then

\[
\int_{\Gamma \setminus G} \phi \ d\mu = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \phi(x, s) \ dx \right) ds
\]

for all $\phi \in C(\Gamma \setminus G)$. This can be proved using that the Haar measure of $G$ equals the Lebesgue measure of $\mathbb{R}^{n+1}$ and that $F = [0, 1]^{n+1}$ is a fundamental set for $\Gamma$ on $G$.

### 3.2 Decomposition of the right-regular representation

Let $A(t)$ be one-parameter subgroup of $\text{GL}(n, \mathbb{R})$ such that $G = \mathbb{R}^n \rtimes_A \mathbb{R}$ is exponential and $\Gamma = \mathbb{Z}^n \rtimes_A \mathbb{Z}$ is a subgroup of $G$. Let $\varepsilon : \Gamma \to \mathbb{Z}_2$ be a group homomorphism. Our aim is to decompose the right regular representation of $G$ on $L^2(G, \varepsilon)$.

Let $C(G, \varepsilon)$ denote the space of all continuous $\mathbb{C}$-valued functions $\phi$ on $G$ satisfying $\phi(gy) = e^{i\pi\varepsilon(y)} \phi(g)$ for all $g \in \Gamma$ and $y \in G$, and $L^2(G, \varepsilon)$ the completion of $C(G, \varepsilon)$ with respect to the Hilbert norm

\[
|\phi|^2_{L^2} = \int_{\Gamma \setminus G} |\phi|^2 \ d\mu.
\]

Now right translation $\rho(x) \phi(y) = \phi(yx)$ gives rise to a unitary representation of $G$ on $L^2(G, \varepsilon)$. This is precisely the definition of the induced representation $\rho = \text{ind}_{\Gamma}^{G} e^{i\pi\varepsilon}$ of the unitary character $e^{i\pi\varepsilon}$ of $\Gamma$. By Theorem 7.2.5 of [W], $\rho$ can be written as a countable orthogonal sum of irreducible subrepresentations with finite multiplicity.

We will give such a decomposition explicitly, generalizing the results of [AB] for the three-dimensional Heisenberg group, which motivated this article.

To this end, we consider partial Fourier transformation with respect to the first $n$ variables: If $\phi \in C(G, \varepsilon)$, then $\phi(k + A(l)x, l + t) = e^{i\pi\varepsilon(k,l)} \phi(x, t)$ for all $(k, l) \in \Gamma$. In particular, $\phi(2k + x, t) = e^{i\pi\varepsilon(2k,0)} \phi(x, t) = \phi(x, t)$ which shows that $x \mapsto \phi(x, t)$ is $2\mathbb{Z}^n$-invariant. For such functions it is natural to consider

\[
\hat{\phi}(\xi, t) = \int_{\mathbb{R}^n} \phi(2x, t) e^{-2\pi i \langle \xi, x \rangle} \ dx
\]

for $\xi \in \mathbb{Z}^n$. Clearly $\phi$ is uniquely determined by its Fourier coefficient functions.

It follows from [S] that the restriction of $\phi \in L^2(G, \varepsilon)$ to $2I_n \times [0, 1]$ is $L^2$- and hence $L^1$-integrable with respect to the Lebesgue measure. In particular, the integral in the definition of $(F_\xi \cdot \phi)(t) := \hat{\phi}(\xi, t)$ makes sense for almost all $t$. 

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Proof. Any homomorphisms $\varepsilon : \Gamma \to \mathbb{Z}_2$ defines homomorphisms $\varepsilon : \mathbb{Z} \to \mathbb{Z}_2$, $\varepsilon(l) = \varepsilon(0, l)$, and $\varepsilon' : \mathbb{Z}^n \to \mathbb{Z}_2$, $\varepsilon'(k) = \varepsilon(k, 0)$, where $\varepsilon'$ satisfies

\[
\varepsilon'(A(l)k) = \varepsilon(A(l)k, 0) = \varepsilon(0, l)(k, 0)(0, l)^{-1} = \varepsilon(k, 0) = \varepsilon'(k)
\]

for all $l \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$. The latter condition reduces to $\varepsilon'(A(1)k) = \varepsilon'(k)$ for all $k$, and hence to $\varepsilon'(\varepsilon_l) = \varepsilon'(A(1)\varepsilon_l) = \sum_{\mu} A_{\mu}(1) \varepsilon'(\varepsilon_l)$ in $\mathbb{Z}_2$ for all $\nu$, which is equivalent to [7]. It is easy to check that the converse holds also true. 

Since $\Gamma$ is uniform and discrete, there exists a unique normalised right $G$-invariant Radon measure $\mu$ on $\Gamma \setminus G$ which can be obtained as follows: Let $I_n = [0, 1]^n$ denote the unit cube of $\mathbb{R}^n$. Then

\[
\int_{\Gamma \setminus G} \phi \ d\mu = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \phi(x, s) \ dx \right) ds
\]

for all $\phi \in C(\Gamma \setminus G)$. This can be proved using that the Haar measure of $G$ equals the Lebesgue measure of $\mathbb{R}^{n+1}$ and that $F = [0, 1]^{n+1}$ is a fundamental set for $\Gamma$ on $G$. 

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Proposition 3.4 For $\varphi \in L^2(G, \varepsilon)$ there holds the Plancherel formula

$$|\varphi|_{L^2}^2 = \sum_{\xi \in \mathbb{Z}^n} \int_0^1 |\hat{\varphi}(\xi, t)|^2 dt .$$

Proof. By the Plancherel theorem for $L^2$-functions on the torus, we obtain

$$|\varphi|_{L^2}^2 = \int_0^1 \left( \int_{I_n} |\varphi(x,t)|^2 dx \right) dt = \int_0^1 \sum_{\xi \in \mathbb{Z}^n} |\hat{\varphi}(\xi, t)|^2 dt ,$$

where summation and integration can be interchanged. \qed

As a corollary we get $F_\xi \cdot \varphi \in L^2([0,1])$. Moreover, the inequality $|F_\xi \cdot \varphi|_{L^2} \leq |\varphi|_{L^2}$ shows that $F_\xi : L^2(G, \varepsilon) \to L^2([0,1])$ is a continuous linear operator.

Taking into account

$$\int_{I_n} g(Mx) \, dx = \int_{I_n} g(x) \, dx$$

for integrable $\mathbb{Z}^n$-invariant functions $g$ on $\mathbb{R}^n$ and $M \in \text{SL}(n, \mathbb{Z})$, one can prove that the $\varepsilon$-equivariance of $\varphi$ entails the following conditions on its Fourier transform.

Lemma 3.5 For $\varphi \in L^2(G, \varepsilon)$ and $(k, l) \in \Gamma$ it holds

$$e^{i\pi \varepsilon(k,l)} \hat{\varphi}(\xi, t) = e^{i\pi A(-l)^\top \xi, k} \hat{\varphi}(A(-l)^\top \xi, l + t) ,$$

where $A(l)^\top$ denotes the transpose of the operator $A(l)$ with respect to the standard inner product on $\mathbb{R}^n$.

The one-parameter subgroup $A$ represents the adjoint representation of the subgroup $\mathbb{R} \cong \{0\} \times \mathbb{R}$ of $G$ on the Lie algebra $\mathbb{R}^n$ of the normal subgroup $\mathbb{R}^n \times \{0\}$. Identifying the linear dual of this Lie algebra with $\mathbb{R}^n$ by means of the standard inner product, we see that $t \mapsto A(t)^\top$ is the coadjoint representation. The preceding lemma reveals the importance of this group action in the present context. Any $\xi \in \mathbb{R}^n$ has a $\mathbb{Z}$-orbit $\theta = \{A(l)^\top \xi : l \in \mathbb{Z}\}$ and an $\mathbb{R}$-orbit $\omega = \{A(t)^\top \xi : t \in \mathbb{R}\}$.

Let $\varphi \in L^2(G, \varepsilon)$ and $\xi \in \mathbb{Z}^n$. The equality $\hat{\varphi}(A(l)^\top \xi, t) = e^{i\pi \varepsilon(0, -l)} \hat{\varphi}(\xi, l + t)$ shows that $\hat{\varphi}(\xi, \cdot)$ determines $\hat{\varphi}(\eta, \cdot)$ for all $\eta \in \theta$. In particular, $\text{supp} \hat{\varphi} := \{\xi \in \mathbb{Z}^n : \hat{\varphi}(\xi, \cdot) \neq 0\}$ is a $\mathbb{Z}$-invariant subset.

Lemma 3.6 The set

$$\Sigma_{\varepsilon'} := \{\xi \in \mathbb{Z}^n : \varepsilon'_{\nu} \in 2\mathbb{Z} + \varepsilon'(e_{\nu}) \text{ for all } 1 \leq \nu \leq n \}$$

is $\mathbb{Z}$-invariant and contains $\text{supp} \hat{\varphi}$ for every $\varphi \in L^2(G, \varepsilon)$.

Proof. Let $\xi \in \Sigma_{\varepsilon'}$ and $l \in \mathbb{Z}$. Since $\varepsilon'(A(l)^\top e_{\nu}) = \varepsilon'(e_{\nu})$ and $\langle \xi, k \rangle \in 2\mathbb{Z} + \varepsilon'(k)$ for all $k \in \mathbb{Z}^n$, it follows $\langle A(l)^\top \xi, e_{\nu} \rangle = \langle \xi, A(l)e_{\nu} \rangle \in 2\mathbb{Z} + \varepsilon'(e_{\nu})$ and $A(l)^\top \xi \in \Sigma_{\varepsilon'}$. This proves
In particular, \( \tilde{\varphi} \) is a smooth function in \( U \), that the direct sum of the \( U \) is dense. Since \( C^\infty(G, \varepsilon) \) is dense in \( L^2(G, \varepsilon) \), it suffices to prove that every smooth \( \varepsilon \)-equivariant function can be approximated by a finite sum of functions in the \( U \). By the decay of the Fourier transform of \( \varphi \in C^\infty(G, \varepsilon) \), it follows that
\[
\varphi_\theta(x, t) = \sum_{\xi \in \theta} \hat{\varphi}(\xi, t) e^{i\pi(\xi, x)}
\]
is a smooth function in \( U \) and that \( \varphi = \sum_{\theta \in \mathbb{Z} \setminus \Sigma_{\varepsilon'}} \varphi_\theta \) converges uniformly on \( \mathbb{R}^n \times [0, 1] \). In particular, \( |\varphi - \sum_{\theta \in \mathbb{Z} \setminus \Sigma_{\varepsilon'}} \varphi_\theta|_{L^2} \to 0 \) for \( J \subset \mathbb{Z} \setminus \Sigma_{\varepsilon'} \) finite and increasing. (This also proves that \( U_\theta \cap C^\infty(G, \varepsilon) \) is dense in \( U_\theta \).)

The right regular representation \( \rho(x, s) \varphi (y, t) = \varphi(y + A(t)x, t + s) \) on \( L^2(G, \varepsilon) \) is compatible with partial Fourier transform in the sense that
\[
(\rho(x, s) \varphi) \tilde{} (\xi, t) = e^{i\pi(\xi, x)} \tilde{\varphi}(\xi, t + s).
\]
In particular, \( \tilde{\varphi}(\xi, \cdot) = 0 \) implies \( \tilde{\rho(x, s) \varphi}(\xi, \cdot) = 0 \) for all \( (x, s) \in G \) which proves \( U_\theta \) to be \( \rho(G) \)-invariant. We define \( \rho_\theta = \rho | U_\theta \).

For any \( \mathbb{R} \)-orbit \( \omega \subset \mathbb{R}^n \), we consider the stabilizer \( \tilde{H}_\omega = \{ t \in \mathbb{R} : A(t)^\top \xi = \xi \} \) whose definition does not depend on the choice of the point \( \xi \in \omega \). Since \( t \mapsto A(t)^\top \) is the coadjoint representation of an exponential Lie group, we know that the closed subgroup \( \tilde{H}_\omega \) is connected, see p. 49 of [LL]. Thus there are only two possibilities, either \( \tilde{H}_\omega = \mathbb{R} \) or \( \tilde{H}_\omega = \{0\} \).
Suppose that \( \theta \subset \Sigma_{\omega'} \) is a \( \mathbb{Z} \)-orbit such that \( \hat{H}_{\omega} = \mathbb{R} \) where \( \omega \) is the unique \( \mathbb{R} \)-orbit containing \( \theta \). Then \( \omega = \theta = \{ \xi \} \) is a fixed point. We claim that \((T_{\xi} \cdot \varphi)(t) = \hat{\varphi}(\xi, t)\) gives a unitary isomorphism of \( U_{\theta} \) onto \( L^2(\mathbb{R}, \hat{\varepsilon}) \). First
\[
\hat{\varphi}(\xi, l + t) = \hat{\varphi}(A(-l)^{\top} \xi, l + t) = e^{i\pi\hat{\varepsilon}(l)} \hat{\varphi}(\xi, t)
\]
by Lemma 3.5, which shows \( T_{\xi} \cdot \varphi \in L^2(\mathbb{R}, \hat{\varepsilon}) \) for \( \varphi \in U_{\theta} \). Clearly \( T_{\xi} \) is linear and
\[
|T_{\xi} \cdot \varphi|^2_{L^2} = \int_{-1}^{1} |\hat{\varphi}(\xi, t)|^2 \, dt = |\varphi|^2_{L^2}
\]
by Proposition 3.4. If \( \psi \in C(\mathbb{R}, \hat{\varepsilon}) \), then \( \varphi(x, t) = \psi(t) e^{\pi i \langle \xi, x \rangle} \) is in \( C(G, \varepsilon) \) by Lemma 3.6, and \( T_{\xi} \cdot \varphi = \psi \). Thus \( T_{\xi} \) is onto. From Equation (10) it follows that the representation \( \rho_{\xi}(x, s) = T_{\xi} \rho_{\theta}(x, s) T_{\xi}^* \) on \( L^2(\mathbb{R}, \varepsilon) \) is given by
\[
\rho_{\xi}(x, s) \psi(t) = e^{\pi i \langle \xi, x \rangle} \varepsilon^x(t) \psi(t + s)
\]
opt \( L^2(\mathbb{T}) \). For \( m \in \mathbb{Z} \) we consider the unitary character of \( G \) given by
\[
\chi_{\varepsilon^x, \omega, m}(x, s) = e^{\pi i \langle \xi, x \rangle} e^{2\pi ims} \varepsilon^x(s).
\]
Finally, using the Fourier transformation and the Plancherel theorem for \( L^2 \)-functions on the torus, we conclude that \( \rho_{\theta} \) is unitarily equivalent to an orthogonal sum
\[
\rho_{\theta} \cong \bigoplus_{m \in \mathbb{Z}} \chi_{\varepsilon^x, \omega, m}
\]
of 1-dimensional subrepresentations. Note that up to isomorphism this decomposition does not depend on the choice of the extension \( \varepsilon^x \).

Now we suppose that \( \theta \subset \Sigma_{\omega'} \) is a \( \mathbb{Z} \)-orbit such that \( \hat{H}_{\omega} = \{0\} \). This means that \( \theta \) is an infinite set and that the unique \( \mathbb{R} \)-orbit \( \omega \) containing \( \theta \) is not relatively compact. In this case we have

**Lemma 3.8** For every \( \eta \in \omega \) there exists a unitary isomorphism \( T_{\eta} \) of \( U_{\theta} \) onto \( L^2(\mathbb{R}) \) which intertwines \( \rho_{\theta} \) and
\[
\rho_{\eta}(x, s) \psi(t) = e^{\pi i \langle A(t)^{\top} \eta, x \rangle} \psi(t + s).
\]

**Proof.** Let \( \xi \in \theta \) and \( r \in \mathbb{R} \) such that \( \eta = A(r)^{\top} \xi \). We claim that \((T_{\eta} \varphi)(t) = \hat{\varphi}(\xi, t+r)\) is a unitary isomorphism satisfying our needs: First of all, Proposition 3.4 implies
\[
|T_{\eta} \varphi|^2_{L^2} = \int_{-\infty}^{+\infty} |\hat{\varphi}(\xi, t+r)|^2 \, dt = \int_{-\infty}^{+\infty} |\hat{\varphi}(\xi, t)|^2 \, dt
\]
\[
= \sum_{l \in \mathbb{Z}} \int_{0}^{1} |\hat{\varphi}(\xi, l + t)|^2 \, dt = \sum_{l \in \mathbb{Z}} \int_{0}^{1} |\hat{\varphi}(A(l)^{\top} \xi, t)|^2 \, dt = |\varphi|^2_{L^2}
\]

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which shows that $T_\eta \varphi \in L^2(\mathbb{R})$ is well-defined and that $T_\eta$ is isometric. If $\psi \in C_0(\mathbb{R})$, then the sum
\[
\varphi(x, t) = \sum_{l \in \mathbb{Z}} e^{-i\pi \xi(l)} \psi(l + t - r) e^{i\pi l (A(l) \top \xi, x)}
\]
is locally finite in $t$ and $\varphi \in C(G, \varepsilon)$ by Lemma 3.6. Using $A(l) \top \xi \neq \xi$ for $l \neq 0$, we conclude $T_\eta \varphi = \psi$. This proves $T_\eta$ to be surjective. Finally, we observe that (11) is a consequence of (9). \hfill \Box

Let $X_{\varepsilon}^\infty$ denote the set of all $\mathbb{R}$-orbits which intersect the subset $\Sigma_{\varepsilon'}$ of $\mathbb{Z}^n$ and which are not relatively compact. Let $X_{\varepsilon}^0$ be the set of all orbits of the form $\omega = \{ \xi \}$ for a fixed point $\xi \in \Sigma_{\varepsilon'}$. If $\omega \in X_{\varepsilon}^\infty$, then $\rho_{\omega}$ is the common unitary equivalence class of the representations $\rho_\theta$ for $\theta \in \mathbb{Z} \setminus (\omega \cap \Sigma_{\varepsilon'})$.

Summing up the preceding conclusions, we obtain

**Theorem 3.9** Let $A(t)$ be one-parameter group of $\text{GL}(n, \mathbb{R})$ with $A(1) \in \text{SL}(n, \mathbb{Z})$ and such that $G = \mathbb{R}^n \rtimes_A \mathbb{R}$ is exponential. Let $\varepsilon : \Gamma \to \mathbb{Z}_2$ be a homomorphism. Then the right regular representation $\rho$ of $G$ in $L^2(G, \varepsilon)$ decomposes as follows:

\[
\rho \cong \bigoplus_{\theta \in \mathbb{Z} \setminus \Sigma_{\varepsilon'}} \rho_\theta \cong \left( \bigoplus_{\omega \in X_{\varepsilon}^0} \bigoplus_{m \in \mathbb{Z}} \chi_{\omega, m} \right) \bigoplus \left( \bigoplus_{\omega \in X_{\varepsilon}^\infty} m_{\omega, \omega} \rho_\omega \right)
\]

where the multiplicities $m_{\omega, \omega} = \# \mathbb{Z} \setminus (\omega \cap \Sigma_{\varepsilon'})$ are finite, the

\[
\chi_{\omega, m}(x, s) = e^{i\pi (\xi, x)} e^{i\pi (2m + \xi(\varepsilon')_1)s}
\]

are characters of $G$, and the $\rho_\omega$ are irreducible on $L^2(\mathbb{R})$. For every $\eta \in \omega$,

\[
\rho_\eta(x, s)\psi(t) = e^{i\pi(A(t) \top \eta, x)} \psi(t + s)
\]

is a representative for the unitary equivalence class of $\rho_\omega$. Moreover, the representations
\[
\{ \chi_{\omega, m} : \omega \in X_{\varepsilon}^0, m \in \mathbb{Z} \} \cup \{ \rho_\omega : \omega \in X_{\varepsilon}^\infty \}
\]
are mutually inequivalent.

**Proof.** It remains to verify the last assertion and the irreducibility of $\rho_\omega$. Clearly characters are unitarily equivalent if and only if they are equal, and not unitarily equivalent to a representation on $L^2(\mathbb{R})$. Let $C^*(N)$ be the enveloping $C^*$-algebra of the group algebra $L^1(N)$ of $N = \mathbb{R}^n \rtimes \{ 0 \}$. Recall that $C^*(N)$ is isomorphic to $C^*_\text{c}r(\hat{N})$ via Fourier transformation. The above formula for $\rho_\eta$ shows that the $C^*$-kernel of the integrated form of $\rho_\omega | N$ consists of all $g \in C^*(N)$ whose Fourier transform vanishes.
on $\omega$. Since the $\mathbb{R}$-orbits are locally closed, it follows that $\rho_{\omega_1}$ and $\rho_{\omega_2}$ are inequivalent whenever $\omega_1 \neq \omega_2$. - If $U$ is a closed $\rho_\eta(G)$-invariant subspace of $L^2(\mathbb{R})$, then $U$ is invariant under translations and multiplication by bounded continuous functions. Thus it follows $U = \{0\}$ or $U = L^2(\mathbb{R})$. 

\section*{Lemma 3.10} Suppose that $G = \mathbb{R}^n \rtimes A \mathbb{R}$ is a nilpotent Lie group. Let $\omega \in X^*_\sigma^\infty$ and $\theta \in \mathbb{Z} \setminus (\omega \cap \Sigma_v)$. For $\eta \in \omega$ let $T_\eta$ denote the unitary isomorphism of $U_\theta$ onto $L^2(\mathbb{R})$ defined in the proof of Lemma \ref{lem:3.8}. Then for every Schwartz function $\psi \in \mathcal{S}(\mathbb{R})$ there exists a (unique) $\varphi \in U_\theta \cap C^\infty(G, \varepsilon)$ such that $T_\eta \varphi = \psi$.

\textbf{Proof.} Given $\psi \in \mathcal{S}(\mathbb{R})$ we consider

$$
\varphi(x, t) = \sum_{l \in \mathbb{Z}} e^{-i\pi \varepsilon(l)} \psi(l + t - r) e^{\pi i(A(l)^T \xi, x)}
$$

whose formal derivatives are

$$(\partial^\alpha \partial^\beta_x \varphi)(x, t) = \sum_{l \in \mathbb{Z}} e^{-i\pi \varepsilon(l)} (\pi i)^{\beta} (A(l)^T \xi)^\beta (\partial^\alpha \psi)(l + t - r) e^{\pi i(A(l)^T \xi, x)}.
$$

Since $B$ is nilpotent, the expression $A(l)^T \xi = \exp(lB)^T \xi$ is polynomial in $l$. Hence for each multi-index $\beta$ there exist constants $N \in \mathbb{N}$ and $C_0 > 0$ such that

$$
| (A(l)^T \xi)^\beta | \leq C_0 (1 + l^2)^N.
$$

for all $l \in \mathbb{Z}$. On the other hand, since $\psi \in \mathcal{S}(\mathbb{R})$, for each $\alpha$ there are $C_1, C_2 > 0$ such that

$$
| (\partial^\alpha \psi)(l + t - r) | \leq C_1 | 1 + (l + t - r)^2 |^{-(N+1)} \leq C_2 | 1 + l^2 |^{-(N+1)}
$$

for all $l$, and for $t$ ranging over a compact subset $K$ of $\mathbb{R}$. This implies that the above series converge absolutely and uniformly on $\mathbb{R}^n \times K$ so that $\varphi \in C^\infty(G, \varepsilon)$ is well-defined. Clearly $\varphi \in U_\theta$ and $T_\eta \varphi = \psi$. \hfill \Box

\subsection*{3.3 Sub-Dirac operators with discrete spectrum}

Let $(H, \langle \cdot, \cdot \rangle)$ be a real vector space with inner product and $(\Delta, \langle \cdot, \cdot \rangle)$ a complex vector space with a hermitian inner product. Suppose that $\Delta$ carries a $\text{Cl}(H)$-module structure such that $\langle x \cdot v, w \rangle = -\langle v, x \cdot w \rangle$ for all $x \in H \subset \text{Cl}(H)$ and $v, w \in \Delta$. Let $s_1, \ldots, s_d$ be an orthonormal basis of $H$ and $a \in H$ a non-zero multiple of $s_d$. Furthermore, let $\Omega : \mathbb{R} \to \text{span}\{s_1, \ldots, s_{d-1}\}$ be a non-constant polynomial function. We consider the operator

$$
P = a \partial_t + i \Omega(t)
$$

on the domain $\mathcal{S}(\mathbb{R}, \Delta)$. Here $a, \Omega(t) \in \text{Cl}(H)$ are understood as operators acting by pointwise multiplication. Clearly $P$ is symmetric with respect to the $L^2$-inner product.
and densely defined in the Hilbert space $L^2(\mathbb{R}, \Delta)$. Thus $P$ is closable. The closure $\hat{P}$ of $P$ is a symmetric operator. Let $P^*$ denote the adjoint of $P$. On its domain
\[
\text{dom}(P^*) = \{ \psi \in L^2(\mathbb{R}, \Delta) : \varphi \mapsto \langle P \varphi, \psi \rangle_{L^2} \text{ is continuous w.r.t. to the } L^2\text{-norm} \}
\]
we consider the norm $|\cdot|_P$ given by $|\psi|_P^2 = |\psi|_{L^2}^2 + |P^* \psi|_{L^2}^2$. Our aim is to prove the following result.

**Proposition 3.11** The operator $P$ is essentially self-adjoint and its closure $\hat{P}$ has discrete spectrum.

**Proof.** We can assume $|a| = 1$ what will simplify the estimates below.

To prove the first assertion, we imitate the proof of the essential selfadjointness of the Dirac operator, compare Theorem 5.7 of [LM] and Proposition 1.3.5 of [G]. As a basic fact we know that it suffices to verify ker$(P^* \pm iI) = \{0\}$. Moreover, since $\hat{P}$ is symmetric, it is enough to show that ker$(P^* \pm iI) \subset \text{dom} (\hat{P})$. To begin with, we note that, if $f \in \mathcal{S}(\mathbb{R})$ and $\psi \in \text{dom}(P^*)$, then $f\psi \in \text{dom}(P^*)$ and
\[
P^*(f\psi) = f(P^*\psi) + (\partial_t f)a \cdot \psi.
\]
Let $\psi \in$ ker$(P^* \pm iI)$. If $\hat{P}$ denotes the extension of $P$ to tempered distributions, then we get, as $P$ is symmetric, $(\hat{P} \pm iI)\psi = P^*\psi \pm i\psi = 0$. Since the principal symbol $p(\xi) = \xi a$ of $\hat{P} \pm iI$ is invertible for $\xi \neq 0$, the regularity theorem for elliptic differential operators implies that $\psi$ is a smooth function. Choose $h \in C_0^\infty(\mathbb{R})$ satisfying $0 \leq h \leq 1$ and $h(0) = 1$, and put $h_k(t) = h(t/k)$ for $k \geq 1$. By definition $h_k \to 1$ and $h_k\psi \in C_0^\infty(\mathbb{R}, \Delta) \subset \text{dom}(P)$. Since
\[
\left| (\partial_t h_k)a \cdot \psi \right|_{L^2} \leq |\partial_t h_k|_\infty |a| |\psi|_{L^2} \leq \frac{1}{k} |\partial_t h|_\infty |\psi|_{L^2},
\]
it follows that
\[
|\psi - h_k\psi|_P^2 = |\psi - h_k\psi|_{L^2}^2 + |P^*\psi - P^*(h_k\psi)|_{L^2}^2 \leq |\psi - h_k\psi|_{L^2}^2 + \left( |P^*\psi - h_k(P^*\psi)|_{L^2} + |(\partial_t h_k)a \cdot \psi|_{L^2} \right)^2
\]
converges to 0 for $k \to \infty$ by dominated convergence. Hence $\psi \in \text{dom}(\hat{P})$. This establishes the essential selfadjointness of $P$.

To prove that the spectrum of $\hat{P}$ is discrete, we need the following two lemmata.

**Lemma 3.12** Let $(\Delta, \langle \cdot, \cdot \rangle)$ be a $\text{Cl}(H)$-module as above and $\Omega : \mathbb{R} \to H$ a continuous function satisfying $|\Omega(t)| \to \infty$ for $|t| \to \infty$. Then
\[
X := \{ \varphi \in L^2(\mathbb{R}, \Delta) : \partial_t \varphi \in L^2(\mathbb{R}, \Delta) \text{ and } \Omega \cdot \varphi \in L^2(\mathbb{R}, \Delta) \}
\]
becomes a Hilbert space when endowed with the norm $||\varphi||^2 = |\varphi|_{L^2}^2 + |\partial_t \varphi|_{L^2}^2 + |\Omega \cdot \varphi|_{L^2}^2$, and the inclusion $X \to L^2(\mathbb{R}, \Delta)$ is a compact operator.
Lemma 3.13. The domain of $\bar{P}$ is contained in $X$ and the inclusion $\text{dom}(\bar{P}) \to X$ is continuous.

Proof. Since $\text{dom}(P) = S(\mathbb{R}, \Delta)$ is contained in $X$ and dense in $\text{dom}(\bar{P})$ w.r.t. the norm $|\cdot|_p$, it suffices to prove that there exists $K > 0$ such that $|\varphi| \leq K |\varphi|_P$ for all $\varphi \in S(\mathbb{R}, \Delta)$. Using $a \cdot (\partial_t \varphi) = \partial_t (a \cdot \varphi)$ and $\Omega a = -a \Omega$ in $C(\Omega)$, we compute

$$
\langle a \cdot (\partial_t \varphi), i\Omega \cdot \varphi \rangle_{L^2} = -\langle a \cdot \varphi, i\partial_t (\Omega \cdot \varphi) \rangle_{L^2},
$$

and

$$
\langle i\Omega \cdot \varphi, a \cdot (\partial_t \varphi) \rangle_{L^2} = \langle a \cdot \varphi, i\Omega \cdot (\partial_t \varphi) \rangle_{L^2}.
$$

As $\partial_t (\Omega \cdot \varphi) = (\partial_t \Omega) \cdot \varphi + \Omega \cdot (\partial_t \varphi)$, it follows

$$
|P\varphi|^2_{L^2} = |a \cdot (\partial_t \varphi) + i\Omega \cdot \varphi|^2_{L^2} = |\partial_t \varphi|^2_{L^2} - \langle a \cdot \varphi, i\partial_t (\Omega \cdot \varphi) \rangle_{L^2} + |\Omega \cdot \varphi|^2_{L^2}
$$

for all Schwartz functions. Since $\Omega$ is a polynomial function, there exists $r > 0$ such that $|\partial_t (\Omega)(t)| \leq |\Omega(t)|$ for all $|t| \geq r$. Fix $C > 1$ such that $|\partial_t (\Omega)(t)| \leq C$ for $|t| \leq r$. From

$$
|\langle a \cdot \varphi, i(\partial_t \Omega) \cdot \varphi \rangle_{L^2}| \leq \int_{-r}^{r} |(\partial_t \Omega)(t)||\varphi(t)|^2 dt + \int_{|t| \geq r} |(\partial_t \Omega)(t)||\varphi(t)|^2 dt
$$

$$
\leq C \int_{-r}^{r} |\varphi(t)|^2 dt + \int_{|t| \geq r} |\varphi(t)||\Omega(t) \cdot \varphi(t)| dt
$$

$$
\leq C |\varphi|^2_{L^2} + |\varphi|_{L^2} |\Omega \cdot \varphi|_{L^2}
$$

it then follows

$$
|\varphi|^2_{P} = |\varphi|^2_{L^2} + |P\varphi|^2_{L^2} \geq \frac{1}{2C} \left( C |\varphi|^2_{L^2} + \frac{1}{4} |\varphi|^2_{L^2} + |P\varphi|^2_{L^2} \right)
$$

$$
\geq \frac{1}{2C} \left( \frac{1}{4} |\varphi|^2_{L^2} + |\partial_t \varphi|^2_{L^2} + |\Omega \cdot \varphi|^2_{L^2} - |\varphi|_{L^2}|\Omega \cdot \varphi|_{L^2} \right) \geq \frac{1}{2C} |\partial_t \varphi|^2_{L^2}
$$
for $\varphi \in \mathcal{S}(\mathbb{R}, \Delta)$. Moreover, $i\Omega \cdot \varphi = P\varphi - a \cdot (\partial_t \varphi)$ gives

$$|\Omega \cdot \varphi|_{L^2} \leq |P\varphi|_{L^2} + |\partial_t \varphi|_{L^2} \leq (1+\sqrt{2C})|\varphi|_P.$$  

Altogether, we obtain

$$||\varphi||^2 = |\varphi|^2_{L^2} + |\partial_t \varphi|^2_{L^2} + |\Omega \cdot \varphi|^2_{L^2} \leq \left(1+2C+(1+\sqrt{2C})^2\right)|\varphi|^2_P$$

proving the lemma. \qed

Now we can prove the second assertion of Proposition 3.11. As $\sigma(P) \subset \mathbb{R}$, we know that $P - iI$ is bijective. Put $R := (P - iI)^{-1}$. Since $P - iI$ is continuous w.r.t. the complete norm $| \cdot |_P$ on dom($P$) and the $L^2$-norm, the open-mapping theorem implies that $R : L^2(\mathbb{R}, \Delta) \to \text{dom}(P)$ is continuous. Moreover, since the inclusion dom($P$) $\to X \to L^2(\mathbb{R}, \Delta)$ is compact by Lemma 3.12 and 3.13, it follows that $R$ is a compact normal operator on $L^2(\mathbb{R}, \Delta)$ with $\ker R = \{0\}$. By the spectral theorem there exists an orthonormal basis $\{\varphi_n : n \in \mathbb{N}\}$ of $L^2(\mathbb{R}, \Delta)$ with $R\varphi_n = \mu_n \varphi_n$ for suitable $0 \neq \mu_n \in \mathbb{C}$. This implies $\bar{P}\varphi_n = (\lambda + \frac{1}{\mu_n})\varphi_n$ for all $n$. If $\{\mu_n : n \in \mathbb{N}\}$ happens to be an infinite set, then $\mu_n \to 0$ and hence $|\lambda + \frac{1}{\mu_n}| \to \infty$ for $n \to \infty$. This proves the spectrum of $\bar{P}$ to be discrete. \qed

Now we resume the assumptions of Section 3.4. Let $A(t) = \exp(tB)$ be a one-parameter group of $\text{GL}(n, \mathbb{R})$ with $A(1) \in \text{SL}(n, \mathbb{Z})$. Suppose that $B$ does not possess any purely imaginary eigenvalues. Let $\mathfrak{g}$ denote the Lie algebra of $G$, and $\mathfrak{n}$ the Lie algebra of $N = \mathbb{R}^n \times \{0\}$. As before we identify $\mathfrak{g}$ with the tangent space at the identity element $e = (0,0) \in G$ and denote by $b$ the $(n+1)$th basis vector of $\mathfrak{g} \cong \mathbb{R}^n \times_B \mathbb{R}$.

Let $\mathcal{H}$ be a left-invariant oriented distribution on $G$ such that $b \in \mathcal{H}_e$. Suppose that $\mathcal{H}$ carries a left-invariant Riemannian metric $g$ such that $b$ is orthogonal to $\mathcal{H}_e \cap \mathfrak{n}$ w.r.t. the inner product $\langle \cdot, \cdot \rangle := g_e$ on $\mathcal{H}_e$. We assume that $\mathcal{H}$ is bracket-generating. With $C^0(\mathcal{H}_e) = \mathcal{H}_e$ and $C^k(\mathcal{H}_e) = [\mathcal{H}_e, C^{k-1}(\mathcal{H}_e)]$ for $k \geq 1$, this means $\mathcal{H} = \sum_{k=0}^n C^k(\mathcal{H}_e)$. In particular, it follows

$$[\mathfrak{g}, \mathfrak{g}] = \sum_{k=1}^n C^k(\mathcal{H}_e).$$

(12)

The latter condition is crucial for the proof of Theorem 3.14 but we do not claim that (12) has significance for left-invariant distributions $\mathcal{H}$ on Lie groups $G$ which do not have the form $G = \mathbb{R}^n \times_A \mathbb{R}$.

Let $\nabla$ be a left-invariant metric connection on $\mathcal{H}$ satisfying condition (3) of Lemma 2.11 which guarantees the symmetry of the sub-Dirac operator. Here the divergence in (3) is defined w.r.t. the left-invariant volume form corresponding to the Haar measure of $G$.

If $\mathcal{H}$ is the distribution on $\Gamma \setminus G$ defined by $\mathcal{H}$, then the Riemannian metric on $\mathcal{H}$ induced by $g$ is denoted by $\tilde{g}$, and the connection on $\mathcal{H}$ by $\tilde{\nabla}$. Let $\varepsilon : \Gamma \to \mathbb{Z}_2$ be a homomorphism defining a spin structure $P_{\text{Spin}, \varepsilon}(\mathcal{H}) \cong G \times_\Gamma \text{Spin}(d)$ of $(\mathcal{H}, \tilde{g})$, where $d = \text{dim} \mathcal{H}$.

We fix a positively oriented orthonormal basis $s_1, \ldots, s_d$ of $\mathcal{H}_e$ with $s_1, \ldots, s_{d-1} \in \mathcal{H}_e \cap \mathfrak{n}$ and such that $s_d$ is a positive multiple of $b$. Denoting the corresponding left-invariant
vector fields again by \( s_1, \ldots, s_d \), the sub-Dirac operator, as acting on smooth sections of the spinor bundle \( S(\mathcal{H}, \varepsilon) \), is given by \( D = \sum_i s_i \cdot \nabla_{s_i}^S \).

Let \( \Delta \) be the representation space of the complex spinor representation of \( \text{Spin}(\mathcal{H}_e) \). Identifying \( \Gamma(S(\mathcal{H}, \varepsilon)) \) with \( C^\infty(G, \varepsilon) \otimes \Delta \), we see that \( D \) is given by

\[
D = \sum_i d\rho(s_i) \otimes s_i + \frac{1}{4} \sum_{i,j,k} \Gamma^k_{ij} I \otimes s_i s_j s_k =: P + W
\]

where the \( s_i \)'s in the second factor of the tensor products are understood as operators on \( \Delta \). Furthermore, the constants \( \Gamma^k_{ij} = g(\nabla_{s_i} s_j, s_k) \) are the Christoffel symbols of \( \nabla \) w.r.t. the orthonormal frame \( s_1, \ldots, s_d \) of \( \mathcal{H} \), and \( d\rho \) is the derivative of the right regular representation \( \rho \) of \( G \) on \( L^2(G, \varepsilon) \). By (3), the second sum in (13) reduces to a sum over all pairwise distinct indices \( i, j, k \).

**Theorem 3.14** If, in addition to the above assumptions, \( G = \mathbb{R}^n \times_A \mathbb{R} \) is nilpotent, then the closure of the operator \( D \) on \( \Gamma \setminus \Delta \) has a pure point spectrum.

**Proof.** By Proposition 3.7, we know that \( L^2(G, \varepsilon) \otimes \Delta \) is a direct sum of the orthogonal subspaces \( \{ U_{\theta} \otimes \Delta : \theta \in \mathbb{Z} \setminus \Sigma^c \} \) which are invariant under the action of \( \rho(G) \otimes \text{Cl}(\mathcal{H}_e) \). Let \( U_{\theta} := U_{\theta} \cap C^\infty(G, \varepsilon) \). Then \( U_{\theta} \otimes \Delta \) is \( D \)-invariant. To prove that \( \tilde{D} \) has a pure point spectrum, it suffices to prove that the closure of \( D_{\theta} := D | \tilde{D}_{\theta} \otimes \Delta \) has a pure point spectrum for all \( \theta \). If \( \theta = \{ \xi \} \) is a fixed point, then, according to Equation (10), the subspace \( U_{\theta} \) is an orthogonal sum of one-dimensional \( \rho(G) \)-invariant subspaces of \( U_{\theta} \). Thus \( U_{\theta} \otimes \Delta \) is an orthogonal sum of two-dimensional \( D_{\theta} \)-invariant subspaces of \( \text{dom}(D_{\theta}) \) which shows that \( D_{\theta} \) has a pure point spectrum. Thus we are left with the case where \( \theta \) is an infinite set and \( U_{\theta} \) is isomorphic to \( L^2(\mathbb{R}) \). Fix \( \xi \in \theta \). Lemma 3.8 implies that there exists a unitary isomorphism \( T_\xi : U_{\theta} \to L^2(\mathbb{R}) \) such that \( \rho_\xi := T_\xi \rho \theta T_\xi^* \) is given by

\[
\rho_\xi(x, s) \psi(t) = e^{\pi i(A(t)^\top \xi, x)} \psi(t + s).
\]

By Lemma 3.10 we may define \( D_\xi = (T_\xi \otimes I) D_{\theta} (T_\xi^* \otimes I) | \mathcal{S}(\mathbb{R}, \Delta) \). Since \( d\rho_\xi(s_j) = \pi i \langle A(t)^\top \xi, s_j \rangle \) for \( 1 \leq j \leq d - 1 \) and \( d\rho_\xi(s_d) = | b | \partial_t \), it follows that the operator \( P_\xi := \sum_{j=1}^d d\rho_\xi(s_j) \otimes s_j \) on \( \mathcal{S}(\mathbb{R}, \Delta) \) has the form

\[
P_\xi = a \partial_t + i \Omega_\xi(t)
\]

with \( a = | b |^{-1} s_d \) and \( \Omega_\xi(t) = \pi \sum_{j=1}^{d-1} \langle A(t)^\top \xi, s_j \rangle s_j \in \mathcal{H}_e \subset \text{Cl}(\mathcal{H}_e) \). Note that \( \langle A(t)^\top \xi, s_j \rangle = \langle \xi, \exp(tB)s_j \rangle \) is a polynomial in \( t \) because \( B \) is nilpotent. Since

\[
[g, g] = \sum_{k=1}^n B^k(\mathcal{H}_e \cap n) = \sum_{k=1}^{d-1} \sum_{j=1}^d \mathbb{R} \cdot (B^k s_j)
\]

by (12) and \( \langle \xi, [g, g] \rangle \neq 0 \) for non-fixed points, it follows that at least one of the components of \( \Omega_\xi \) is not constant. Thus Proposition 3.11 implies that \( \hat{P}_\xi \) has discrete spectrum. In other words, the essential spectrum of \( \hat{P}_\xi \) is empty.
The operator \( W_\xi := \frac{1}{4} \sum_{i,j,k} \Gamma_{ij}^k s_i s_j s_k \) is bounded on \( L^2(\mathbb{R}, \Delta) \). In particular, \( W_\xi \) is relatively \( \bar{P}_\xi \)-compact in the sense that \( \text{dom}(P_\xi) \subset \text{dom}(W_\xi) \) and \( W_\xi(\bar{P}_\xi - iI)^{-1} \) is compact. By the Kato-Rellich theorem we know that \( \bar{D}_\xi = \bar{P}_\xi + W_\xi \) is selfadjoint. Moreover, Weyl’s theorem which asserts the stability of the essential spectrum under relatively compact perturbations, and for which we refer to Theorem 14.6 of [HS], implies that the essential spectrum of \( \bar{D}_\xi = \bar{P}_\xi + W_\xi \) is empty. This shows that \( \bar{D}_\xi \) and hence \( \bar{D}_\theta \) have discrete spectrum. The proof of the theorem is complete. \( \square \)

**Remark 3.15** In general, the eigenvalues of the sub-Dirac operator \( D \) do not have finite multiplicity and the spectrum of \( D \) is not a discrete subset of \( \mathbb{R} \). A relevant example is given in Section 4.3.

### 3.4 Two- and three-dimensional distributions

In this subsection we will compute the spectra of the operators \( D_\theta \) arising in the proof of Theorem 3.14 provided that \( G = \mathbb{R}^n \rtimes_A \mathbb{R} \) is 2-step nilpotent and \( \dim H = 2 \) or 3. The explicit formulas that will be given below in the non-fixed point case are a consequence of the following result.

**Proposition 3.16** Let \( \alpha, \beta \in \mathbb{R} \) and \( \omega(t) = a\omega_1 t + \omega_0 \) where \( a > 0 \), \( \omega_0, \omega_1 \in \mathbb{C} \) and \( |\omega_1| = 1 \). Then the spectrum of the operator

\[
D = \alpha I + \beta \begin{pmatrix}
i \partial_t & \bar{\omega} \\
\omega & -i \partial_t \end{pmatrix}
\]

on \( \mathcal{S}(\mathbb{R}, \mathbb{C}^2) \) is discrete. More precisely, \( \sigma(D) = \{ \lambda_0 \} \cup \{ \lambda_k^\pm : k \in \mathbb{N} \setminus \{0\} \} \), where

\[
\lambda_0 := \alpha + \beta \text{Im}(\omega_0 \bar{\omega}_1) \quad \text{and} \quad \lambda_k^\pm := \alpha \pm \beta \left(2ak + \text{Im}(\omega_0 \bar{\omega}_1)^2\right)^{1/2}.
\]

If \( \lambda_0 \) and the \( \lambda_k^\pm \) are pairwise distinct, then all eigenvalues are simple.

**Proof.** We can assume \( \alpha = 0 \) and \( \beta = 1 \). Instead of \( D \), we consider the operator \( S := Q^* D Q \), where \( Q \) is the unitary matrix

\[
Q = \frac{1}{\sqrt{2}} \begin{pmatrix}1 & -i \omega_1 \\
-i \omega_1 & 1 \end{pmatrix},
\]

which diagonalizes \( D^2 \) and does not depend on \( t \). Obviously the spectra of \( D \) and \( S \) coincide. We have

\[
S = \begin{pmatrix} \text{Im}(\omega_1 \bar{\omega}) & \bar{\omega}_1 (\partial_t + \text{Re}(\omega_1 \bar{\omega})) \\
\omega_1 (-\partial_t + \text{Re}(\omega_1 \bar{\omega})) & -\text{Im}(\omega_1 \bar{\omega}) \end{pmatrix} = \begin{pmatrix} -\text{Im}(\omega_0 \bar{\omega}_1) & \bar{\omega}_1 (\partial_t + a t + \text{Re}(\omega_0 \bar{\omega}_1)) \\
\omega_1 (-\partial_t + a t + \text{Re}(\omega_0 \bar{\omega}_1)) & \text{Im}(\omega_0 \bar{\omega}_1) \end{pmatrix}.
\]
To detect $S$-invariant subspaces, we start with the orthonormal basis $\{h_k : k \in \mathbb{N}\}$ of $L^2(\mathbb{R})$ given by the (normalized) Hermite functions

$$h_k(x) = (2^k \pi^{1/2} k!)^{-1/2} H_k(x) e^{-x^2/2}.$$  

Here $H_k(x) = (-1)^k e^{x^2} \partial_x^k e^{-x^2}$ is the $k$th Hermite polynomial. Put $b = 2a \text{Re}(\omega_0 \bar{\omega}_1)$. Using the unitary isomorphism

$$(U \cdot w)(t) := a^{1/4} w(a^{1/2}(t + \frac{b}{2a^2}))$$

of $L^2(\mathbb{R})$, we then define $u_k = U \cdot h_k$. Recall that the creation operator $\Lambda_+ = -\partial_x + x$ and the annihilation operator $\Lambda_- = \partial_x + x$ satisfy $\Lambda_+ (h_k) = \sqrt{2(k+1)} h_{k+1}$ and $\Lambda_- (h_k) = \sqrt{2k} h_{k-1}$. Since

$$U \Lambda_+ U^* = a^{-1/2} \left( \mp \partial_t + a t + \frac{b}{2a} \right),$$

we thus obtain

$$\begin{align*}
(-\partial_t + a t + \frac{b}{2a}) u_k &= \sqrt{2a(k+1)} u_{k+1} \text{ for } k \geq 0, \\
(\partial_t + a t + \frac{b}{2a}) u_k &= \sqrt{2ak} u_{k-1} \text{ for } k \geq 1, \\
(\partial_t + a t + \frac{b}{2a}) u_0 &= 0.
\end{align*}$$

This shows

$$S \cdot \begin{pmatrix} 0 \\ u_0 \end{pmatrix} = \begin{pmatrix} \bar{\omega}_1 (\partial_t + a t + \text{Re}(\omega_0 \bar{\omega}_1)) u_0 \\ \text{Im}(\omega_0 \bar{\omega}_1) u_0 \end{pmatrix} = \text{Im}(\omega_0 \bar{\omega}_1) \begin{pmatrix} 0 \\ u_0 \end{pmatrix},$$

$$S \cdot \begin{pmatrix} u_{k-1} \\ 0 \end{pmatrix} = \begin{pmatrix} -\text{Im}(\omega_0 \bar{\omega}_1) u_{k-1} \\ \omega_1 (\partial_t + a t + \text{Re}(\omega_0 \bar{\omega}_1)) u_{k-1} \end{pmatrix} = \frac{-\text{Im}(\omega_0 \bar{\omega}_1) u_{k-1}}{\sqrt{2ak} \omega_1 u_k},$$

$$S \cdot \begin{pmatrix} 0 \\ u_k \end{pmatrix} = \begin{pmatrix} \bar{\omega}_1 (\partial_t + a t + \text{Re}(\omega_0 \bar{\omega}_1)) u_k \\ \text{Im}(\omega_0 \bar{\omega}_1) u_k \end{pmatrix} = \frac{\sqrt{2ak} \bar{\omega}_1 u_{k-1}}{\text{Im}(\omega_0 \bar{\omega}_1) u_k}.$$ 

In particular, the subspaces

$$V_0 := \mathbb{C} \begin{pmatrix} 0 \\ u_0 \end{pmatrix} \quad \text{and} \quad V_k := \text{span}\left\{ \varphi_k := \begin{pmatrix} u_{k-1} \\ 0 \end{pmatrix}, \psi_k := \begin{pmatrix} 0 \\ u_k \end{pmatrix} \right\}, \quad k \geq 1,$$

are $S$-invariant, and the restriction of $S$ to $V_k$, $k \geq 1$, is given by the matrix

$$\begin{pmatrix} -\text{Im}(\omega_0 \bar{\omega}_1) & \sqrt{2ak} \bar{\omega}_1 \\ \sqrt{2ak} \omega_1 & \text{Im}(\omega_0 \bar{\omega}_1) \end{pmatrix}$$

with respect to the basis $\varphi_k, \psi_k$. Since $L^2(\mathbb{R}, \mathbb{C}^2)$ is the direct sum of the $V_k$, $k \geq 0$, the assertion follows. \qed
Assume that \( G = \mathbb{R}^n \times_A \mathbb{R} \) is 2-step nilpotent. Let \((H, g, \nabla)\) be as in the preceding subsection with \(2 \leq \dim H \leq 3\). First we will determine the spectrum of \( D_\theta := D_\big| U_\theta^\infty \otimes \Delta \) when \( \theta \) does not consist of a single point.

Suppose that \( \dim H = 2 \). Let \( s_1 \in H_e \cap n \) and \( s_2 \in H_e \) be a positive multiple of \( b \) such that \( s_1, s_2 \) is a positively oriented orthonormal basis of \( H_e \). In particular, \( s_2 = |b|^{-1} b \).

By \(^{31}\) we have \( \Gamma_{11}^1 + \Gamma_{21}^2 = 0 \) and \( \Gamma_{12}^1 + \Gamma_{22}^2 = 0 \) which implies that all Christoffel symbols of \( \nabla \) vanish. Up to isomorphism, there exists only one simple \( \mathcal{C}(H_e) \)-module.

Let \( \Delta = \mathbb{C}^2 \) be the one such that \( s_1 \) and \( s_2 \), represented as operators on \( \Delta \), are given by

\[
s_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad s_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\]

Let \( \theta \in \mathbb{Z} \setminus \Sigma_e \) and \( \xi \in \theta \) be a non-fixed point. If \( T_\xi : U_\theta \rightarrow L^2(\mathbb{R}) \) is a unitary isomorphism as in the proof of Theorem \(^{3.14}\) and \( D_\xi = (T_\xi \otimes I)D_\theta(T_\xi^* \otimes I) | S(\mathbb{R}, \Delta) \), then we know by \(^{15}\) that \( D_\xi \) has the form

\[
D_\xi = |b|^{-1} \begin{pmatrix} i\partial_t & \tilde{\omega}_\xi \\ \omega_\xi & -i\partial_t \end{pmatrix}
\]

with \( \omega_\xi(t) = \pi i |b| \langle A(t)^\top \xi, s_1 \rangle \). Since \( B^2 = 0 \), we have \( A(t) = I + t B \) so that

\[
\omega_\xi(t) = \pi i |b| \left( \langle B^\top \xi, s_1 \rangle t + \langle \xi, s_1 \rangle \right)
\]

is a non-constant affine-linear function. Thus Proposition \(^{3.16}\) implies that \( \tilde{D}_\xi \) has discrete spectrum. Moreover, the eigenvalues of \( D_\xi \) can be computed as follows: Put \( a = 0, b = |b|^{-1}, a = \pi |b| |\langle B^\top \xi, s_1 \rangle|, \omega_1 = i \text{sgn} \langle B^\top \xi, s_1 \rangle \) and \( \omega_0 = \pi i \langle \xi, s_1 \rangle \). Note that \( \text{Im} (\omega_0 \tilde{\omega}_1) = 0 \). Hence it follows that

\[
\lambda_0(\xi) = 0 \quad \text{and} \quad \lambda_k^\pm(\xi) = \pm \left( 2 \pi |b|^{-1} |\langle B^\top \xi, s_1 \rangle| k \right)^{1/2}
\]

with \( k \in \mathbb{N} \setminus \{0\} \) are the eigenvalues of \( D_\xi \). This completes the case \( \dim H = 2 \).

Suppose that \( \dim H = 3 \). Choose \( s_1, s_2 \in H_e \cap n \) and \( s_3 = |b|^{-1} b \) such that \( s_1, s_2, s_3 \) becomes a positively oriented orthonormal basis of \( H_e \). Up to isomorphism, there exist two simple \( \mathcal{C}(H_e) \)-modules. Let \( \Delta = \mathbb{C}^2 \) be the one given by the representation

\[
s_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\]

Note that \( s_1 s_2 = s_3 \) and \( s_i s_j + s_j s_i = -2 \delta_{ij} \) for all \( 1 \leq i, j \leq 3 \), as operators on \( \Delta \).

Using this and that \( \nabla \) is metric, we conclude that the second sum in \(^{13}\) simplifies to \( W = -\frac{1}{2} (\Gamma_{12}^3 + \Gamma_{23}^1 + \Gamma_{31}^2) I \otimes I \).

Let \( \theta \in \mathbb{Z} \setminus \Sigma_e \) and \( \xi \in \theta \) be a non-fixed point. If \( T_\xi \) is a unitary isomorphism of \( U_\theta \) onto \( L^2(\mathbb{R}) \) such that \( \rho_\xi = T_\xi \rho_\theta T_\xi^* \) is given by Equation \(^{31}\), then the restriction \( D_\xi \) of \( (T_\xi \otimes I)D_\theta(T_\xi^* \otimes I) \), when realized in \( L^2(\mathbb{R}, \Delta) \), to Schwartz functions has the form

\[
D_\xi = -\frac{1}{2} (\Gamma_{12}^3 + \Gamma_{23}^1 + \Gamma_{31}^2) I + |b|^{-1} \begin{pmatrix} i \partial_t & \tilde{\omega}_\xi \\ \omega_\xi & -i \partial_t \end{pmatrix}
\]

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where $\omega_\xi$, as $G$ is 2-step nilpotent, is a non-constant affine linear function given by

$$\omega_\xi(t) = \pi i |b| \left(\langle A(t)^\top \xi, s_1 \rangle + \langle A(t)^\top \xi, s_2 \rangle \right)$$

$$= -\pi |b| \left(\langle B^\top \xi, s_1 \rangle - i\langle B^\top \xi, s_2 \rangle \right) t + \langle \xi, s_1 \rangle - i\langle \xi, s_2 \rangle.$$ 

Hence Proposition 5.16 implies that the eigenvalues of $D_\xi$ are

$$\lambda_0(\xi) = -\frac{1}{2}(\Gamma_1^3 + \Gamma_2^1 + \Gamma_3^2) - \pi \left(\frac{\langle B^\top \xi, s_1 \rangle \langle \xi, s_2 \rangle - \langle \xi, s_1 \rangle \langle B^\top \xi, s_2 \rangle}{(\langle B^\top \xi, s_1 \rangle^2 + \langle B^\top \xi, s_2 \rangle^2)^{1/2}} \right)$$

and

$$\lambda_\pm(\xi) = -\frac{1}{2}(\Gamma_1^3 + \Gamma_2^1 + \Gamma_3^2) \pm \left(\frac{2\pi k |b|^{-1} (\langle B^\top \xi, s_1 \rangle^2 + \langle B^\top \xi, s_2 \rangle^2)^{1/2}}{\langle B^\top \xi, s_1 \rangle^2 + \langle B^\top \xi, s_2 \rangle^2} \right)^{1/2}.$$  

In (17) - (19) we rediscover the fact that the eigenvalues $\lambda_k(\xi)$ do not depend on the choice of the point $\xi$ on the orbit. More precisely, since $B^2 = 0$ and $A(t)B = B$, it follows, in accordance with Lemma 5.8, that $\lambda_k^\pm(A(t)^\top \xi) = \lambda_k^\pm(\xi)$ for all $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^3 \setminus \{0\}$. This completes the case $\dim H = 3$.

Finally we compute the spectrum of $D_\theta$ when $\theta = \{\xi\}$ is a fixed point. For this purpose, we can drop the assumption that $G$ is nilpotent.

By (10) we know that $U_\theta$ is an orthogonal sum of 1-dimensional subspaces $\{U_{\theta,k} : k \in \mathbb{Z}\}$ of $U_\theta \cap C^\infty(G, \varepsilon)$ on which $(x, s) \in G$ acts by multiplication with

$$\chi_{\xi^\top, \theta, k}(x, s) = e^{\pi i \langle x, \xi \rangle} e^{\pi i (2k + \hat{\varepsilon}(1)) s}.$$

Suppose that $\dim H = 2$. Let $s_1, s_2$ and $\Delta$ be as above. In this case the sub-Dirac operator reads $D = \partial \rho(1) \otimes s_1 + \partial \rho(2) \otimes s_2$. Since $d\chi_{\xi^\top, \theta, k}(s_1) = \pi i \langle \xi, s_1 \rangle$ and $d\chi_{\xi^\top, \theta, k}(s_2) = \pi i |b|^{-1} (2k + \hat{\varepsilon}(1))$, it follows that $D_{\theta, k} := D_{\theta, k} |U_{\theta, k} \otimes \mathbb{C}^2$ is unitarily equivalent to

$$D_{\xi, k} := \alpha I + \beta \begin{pmatrix} 2k + \hat{\varepsilon}(1) & \bar{\omega}_\xi \\ \omega_\xi & -2k - \hat{\varepsilon}(1) \end{pmatrix}$$

on $\mathbb{C}^2$, where $\alpha = 0$, $\beta = -\pi |b|^{-1}$ and $\omega_\xi = -i |b| (\langle \xi, s_1 \rangle)$ are constants. Obviously, $D_{\xi, k}$ admits the eigenvalues

$$\mu_\pm^k(\xi) = \pm \pi \left(|b|^{-2}(2k + \hat{\varepsilon}(1)^2 + \langle \xi, s_1 \rangle^2) \right)^{1/2}, \quad k \in \mathbb{Z}.$$  

Suppose that $\dim H = 3$. Let $s_1, s_2, s_3$ and $\Delta$ be as above. In this case $D = P + W$ where $W = \alpha I \otimes I$ and $\alpha = -\frac{1}{2}(\Gamma_1^3 + \Gamma_2^1 + \Gamma_3^2)$. Hence $D_{\theta, k}$ is unitarily equivalent to $D_{\xi, k}$ as in (20) with $\beta = -\pi |b|^{-1}$ and $\omega_\xi = |b| (\langle \xi, s_1 \rangle - i\langle \xi, s_2 \rangle)$. Thus $D_{\xi, k}$ has the eigenvalues

$$\mu_\pm^k(\xi) = -\frac{1}{2}(\Gamma_1^3 + \Gamma_2^1 + \Gamma_3^2) \pm \pi \left(|b|^{-2}(2k + \hat{\varepsilon}(1)^2 + \langle \xi, s_1 \rangle^2 + \langle \xi, s_2 \rangle^2) \right)^{1/2}.$$  

This shows that $D_\theta$ has discrete spectrum in the fixed point case.
4 Examples of spectra of sub-Dirac operators

4.1 A preliminary remark

To compute the spectrum of the sub-Dirac operator $D$, it remains, by the results in the preceding section for the spectra of the $D_\theta$, to determine a set of representatives for the set of all $\mathbb{Z}$-orbits contained in $\Sigma_\varepsilon'$. More precisely, in view of Theorem 3.9, we carry out the following steps:

1. Describe all homomorphisms $\varepsilon : \Gamma \to \mathbb{Z}_2$.
2. Find a set of representatives $R_\varepsilon'$ for all $\mathbb{R}$-orbits $\omega$ intersecting $\Sigma_\varepsilon'$.
3. Compute the number of $\mathbb{Z}$-orbits contained in $\omega \cap \Sigma_\varepsilon'$.
4. Determine the spectrum of $D_\xi$ for some $\xi \in \omega$.

This requires a detailed knowledge of the orbit picture of the coadjoint representation. In the following examples, the eigenvalues of the sub-Dirac operator including their multiplicities will be determined completely.

4.2 Three-dimensional Heisenberg manifolds

As we will see next, the results of this section comprise Theorem 3.1 of [AB] concerning the spectrum of the Dirac operator on three-dimensional Heisenberg manifolds. Let $G = \mathbb{R}^2 \rtimes_A \mathbb{R}$ be the Heisenberg group as discussed in Example 3.2 and $\Gamma = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$. Then $\mathfrak{g} = \text{span}\{e_1, e_2, b\}$ with $[b, e_2] = re_1$. For positive real numbers $d$ and $T$ we consider the orientation and the Riemannian metric $g$ on $H := TG$ such that $s_1 = \frac{1}{T}e_1$, $s_2 = -de_2$ and $s_3 = \frac{4}{d}b$ becomes a positively oriented orthonormal frame. The constants are chosen in accordance with [AB], where the collapse of Heisenberg manifolds $M(r, d, T)$ for $T \to 0$ is studied. Let $\nabla$ be the Levi-Civita connection of $g$. In particular, $\nabla$ satisfies \[\Gamma_{12}^3 = \Gamma_{23}^1 = \Gamma_{31}^2 = \frac{dT}{2} \] Let $\varepsilon' : \mathbb{Z}^2 \to \mathbb{Z}_2$ be a homomorphism. We abbreviate $\varepsilon'(e_\mu)$ to $e_\mu$. Then (7) is satisfied if and only if $\varepsilon_1 r$ is even. It is easy to see that the disjoint union $R_\varepsilon'$ of

$R^{(1)}_\varepsilon' = \{ \xi \in \Sigma_\varepsilon' : \xi_1 = 0 \}$ and $R^{(2)}_\varepsilon' = \{ \xi \in \Sigma_\varepsilon' : \xi_1 \neq 0 \text{ and } \xi_2 = e_2 \}$

is a set of representatives for the set of all $\mathbb{R}$-orbits intersecting $\Sigma_\varepsilon'$. The set $R^{(1)}_\varepsilon'$ consists of all fixed points in $\Sigma_\varepsilon'$. If $\omega$ is the $\mathbb{R}$-orbit represented by $(\xi_1, e_2) \in R^{(2)}_\varepsilon'$, then $\omega \cap \Sigma_\varepsilon'$ contains $|\xi_1 r|/2$ distinct $\mathbb{Z}$-orbits.

We compute $|b| = \frac{r^2}{d^2}$, $\langle \xi, s_1 \rangle = \frac{1}{T} \xi_1$, $\langle \xi, s_2 \rangle = -d \xi_2$, $\langle B^\top \xi, s_1 \rangle = 0$ and $\langle B^\top \xi, s_2 \rangle = -dr \xi_1$. Inserting this into (22) gives

$$\mu_k^\pm(0, \varepsilon_2) = -\frac{d^2 T}{4} \pm \pi \left( (2k + \varepsilon(1))^2 \frac{d^2}{r^2} + d^2 \varepsilon_2^2 \right)^{1/2}.$$ (23)
Similarly, for $\xi \in \Sigma_{e'}$ with $\xi_1 \neq 0$, Equations (18) and (19) yield
\[
\lambda_0(\xi_1, \varepsilon_2) = -\frac{d^2 T}{4} - \frac{\pi}{T} |\xi_1| \tag{24}
\]
and
\[
\lambda^\pm_k(\xi_1, \varepsilon_2) = -\frac{d^2 T}{4} \pm \left( 2\pi d^2 k|\xi_1| + \frac{\pi^2}{T^2} \xi_1^2 \right)^{1/2}. \tag{25}
\]
We will use the following notation for the description of the spectrum of the Dirac operator $D$ on $\Gamma \setminus G$. We define the spectral multiplicity function
\[
m(D) : \mathbb{R} \to \mathbb{N} \cup \{\infty\}, \quad m(D)(\lambda) = \dim \ker(D - \lambda I).
\]
Moreover, $\delta = \delta(\lambda)$ denotes the function that takes the value 1 in $\lambda$ and that is zero on $\mathbb{R} \setminus \{\lambda\}$.

Suppose we are given a spin structure corresponding to a homomorphism $\varepsilon : \mathbb{Z}^2 \rtimes_A \mathbb{Z} \to \mathbb{Z}_2$ with $\varepsilon_1 = 0$. Summation of (23), (24) and (25) over $\xi_\nu \in 2\mathbb{Z} + \varepsilon_\nu$ gives
\[
m(D) = m_1^+(D) + m_1^-(D) + m_2^+(D) + m_2^0(D) + m_2^-(D),
\]
where
\[
m_1^+(D) = \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \delta \left( -\frac{d^2 T}{4} \pm \frac{\pi d}{r} \left( (2k + \varepsilon(1))^2 + r^2 (2l + \varepsilon_2)^2 \right)^{1/2} \right),
\]
\[
m_2^0(D) = \sum_{l=1}^{\infty} 2rl \delta \left( -\frac{d^2 T}{4} - \frac{2\pi l}{T} \right),
\]
\[
m_2^\pm(D) = \sum_{l=1}^{\infty} 2rl \sum_{k=1}^{\infty} \delta \left( -\frac{d^2 T}{4} \pm \left( 4\pi d^2 kl + \frac{4\pi^2 l^2}{T^2} \right)^{1/2} \right).
\]
Now assume $\varepsilon_1 = 1$, which is only possible if $r$ is even. Then $\mathcal{R}_{e'}^{(1)} = \emptyset$ and we obtain
\[
m(D) = m_2^+(D) + m_2^0(D) + m_2^-(D),
\]
where now
\[
m_2^0(D) = \sum_{l=0}^{\infty} (2l + 1)r \delta \left( -\frac{d^2 T}{4} - \frac{\pi(2l + 1)}{T} \right),
\]
\[
m_2^\pm(D) = \sum_{l=0}^{\infty} (2l + 1)r \sum_{k=1}^{\infty} \delta \left( -\frac{d^2 T}{4} \pm \left( 2\pi d^2 k(2l + 1) + \frac{\pi^2(2l + 1)^2}{T^2} \right)^{1/2} \right).
\]
Now let us turn to the sub-Riemannian case and suppose $\mathcal{H} = \text{span}\{s_2, s_3\}$, where again $s_2$ and $s_3$ are orthonormal. Let $\nabla$ be defined by (2) for a leftinvariant complement $\mathcal{V} := \mathbb{R} \cdot u$, $u \in \mathfrak{g}$ of $\mathcal{H}$. Since we wish to get a symmetric sub-Dirac operator, the only possible choice is $\mathcal{V} := \mathbb{R} \cdot s_1$. Indeed, otherwise $[\Gamma(\mathcal{H}), u] \not\subset \Gamma(\mathcal{H})$, thus $D$ is not symmetric by Lemma 2.1. We proceed as above, now using Equations (21) and (17).
For a spin structure that corresponds to a homomorphism $\varepsilon : \mathbb{Z}^2 \times_A \mathbb{Z} \to \mathbb{Z}_2$ with $\varepsilon_1 = 0$ we obtain

$$m(D) = |N| \cdot \delta(0) + m_+^1(D) + m_-^1(D) + m_+^2(D) + m_-^2(D),$$

where

$$m_+^1(D) = \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \delta \left( \pm \frac{\pi d}{r} \left( (2k + \varepsilon(1))^2 + r^2(2l + \varepsilon_2)^2 \right)^{1/2} \right),$$

$$m_-^2(D) = \sum_{l=1}^{\infty} 2rl \sum_{k=1}^{\infty} \delta \left( \pm \left( 4\pi d^2 kl \right)^{1/2} \right).$$

If $\varepsilon_1 = 1$, then

$$m(D) = |N| \cdot \delta(0) + \sum_{l=0}^{\infty} (2l + 1)r \sum_{k=1}^{\infty} \left( \delta \left( (2\pi d^2 k(2l + 1))^{1/2} \right) + \delta \left( -(2\pi d^2 k(2l + 1))^{1/2} \right) \right).$$

### 4.3 A five-dimensional two-step nilpotent example

We start by considering 2-step nilpotent Lie groups which are isomorphic to a standard model $G = \mathbb{R}^{2p} \times_A \mathbb{R}$ as described in Lemma 4.1 and therefore generalise the example from the preceding subsection. We will describe the orbits of $\mathbb{R}$ and $\mathbb{Z}$ acting on $\mathbb{R}^{2p}$ by $A^1$. Then we will specialise to $\dim G = 5$ for the computation of the spectrum of the sub-Dirac operator on $\Gamma \setminus G$.

**Lemma 4.1** Let $G$ be a simply connected Lie group satisfying $[G, G] = Z(G)$ and admitting a connected abelian normal subgroup $N$ of codimension 1. Let $\Gamma$ be a uniform discrete subgroup of $G$ such that $\Gamma \cap N$ is uniform in $N$. Then there exist $p \geq 1$, a one-parameter subgroup of $\text{GL}(2p, \mathbb{R})$ of the form

$$A(t) = \begin{pmatrix} I & tR \\ 0 & I \end{pmatrix}$$

with $R = \text{diag}(r_1, \ldots, r_p)$ and positive integers $r_\nu$ such that $r_{\nu+1} | r_\nu$ for $\nu = 1, \ldots, p-1$, and an isomorphism $\Phi$ of $G$ onto $\mathbb{R}^{2p} \times_A \mathbb{R}$ such that $\Phi(\Gamma) = \mathbb{Z}^{2p} \times_A \mathbb{Z}$.

**Proof.** Put $p = \dim Z(G) = \frac{1}{2} \dim N$. Since $\Gamma \cap Z(G)$ is uniform in $Z(G)$, we find generators $v_1, \ldots, v_{2p}$ of $\Gamma \cap N$ such that $\Gamma \cap Z(G) = Zv_1 + \ldots + Zv_p$. As in the proof of Lemma 5.1 we consider the linear isomorphism $M : \mathbb{R}^{2p} \to N$ given by $M(e_\nu) = v_\nu$, choose $b \in g$ such that $\exp(b) \in \Gamma$ and $\exp(b)N$ generates $\Gamma N/N$, and define $A_0(t) \in \text{GL}(2p, \mathbb{R})$ such that $\Phi_0(x, t) = M(x) \exp(tb)$ becomes an isomorphism of $G$ onto $\mathbb{R}^{2p} \times_A \mathbb{R}$. Since $\Phi_0(Z(G)) = \mathbb{R}^p \times \{0\} \times \{0\}$, we have

$$A_0(t) = \begin{pmatrix} I & tR_0 \\ 0 & I \end{pmatrix}$$

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with \( R_0 \in \text{GL}(p, \mathbb{Z}) \). Recall that \( R_0 \) can be brought into Smith normal form, i.e., there exist \( Q_1, Q_2 \in \text{GL}(p, \mathbb{Z}) \) such that \( R := Q_1 R_0 Q_2^{-1} = \text{diag}(r_1, \ldots, r_p) \) is diagonal with positive integers \( r_\nu \) such that \( r_{\nu+1} | r_\nu \). Clearly \( \Psi(x', x'', t) := (Q_1 x', Q_2 x'', t) \) gives an isomorphism of \( \mathbb{R}^{2p} \times_A \mathbb{R} \) onto \( \mathbb{R}^{2p} \times_A \mathbb{R} \) with \( A(t)(x', x'') = (x' + tR x'', x'') \). Finally, it follows that the assertion of the lemma holds for \( \Phi := \Psi \Phi_0. \)

Let \( G = \mathbb{R}^{2p} \times_A \mathbb{R} \) be as in Lemma 4.1 with uniform discrete subgroup \( \Gamma = \mathbb{Z}^2 \mathbb{Z} \) \( \mathbb{Z} \). In particular, \( A(t)e_\nu = e_\nu \) and \( A(t)e_{p+\nu} = e_{p+\nu} + r_\nu t e_\nu \) for all \( 1 \leq \nu \leq p \).

Let \( \varepsilon : \mathbb{Z} \to \mathbb{Z}_2 \) and \( \varepsilon' : \mathbb{Z}^2 \mathbb{Z} \to \mathbb{Z}_2 \) be homomorphisms. As before, we put \( \varepsilon_\nu = \varepsilon'(e_\nu) \) for \( \nu = 1, \ldots, 2p \). By Lemma 4.1 it follows that \( \varepsilon(k, l) := \varepsilon'(k) + \varepsilon(l) \) is a homomorphism of \( \Gamma \) if and only if \( r_\nu \varepsilon_\nu \in 2\mathbb{Z} \) for \( 1 \leq \nu \leq p \). Note that the latter condition implies \( \varepsilon_\nu = 0 \) whenever \( r_\nu \) is odd.

Next we will describe the coadjoint orbits. First of all,

\[
\langle A(t)^\top \xi, e_\nu \rangle = \xi_\nu \quad \text{and} \quad \langle A(t)^\top \xi, e_{p+\nu} \rangle = \xi_{p+\nu} + r_\nu \xi_\nu t
\]

for \( 1 \leq \nu \leq p \). To formulate the subsequent result, a little more notation is needed. If \( \xi \in \mathbb{Z}^p \), then \( \bar{\xi} \in \mathbb{Z}^p \) denotes the projection of \( \xi \) onto the first \( p \) variables. For \( \eta \in \mathbb{Z}^p \), the subset \( \{ \xi \in \mathbb{Z}^p : \bar{\xi} = \bar{\eta} \} \) is \( \mathbb{Z} \)-invariant. In particular, \( \{ \xi : \bar{\xi} = 0 \} \) is the set of all points remaining fixed under the coadjoint action. Put \( J_\eta = \{ \nu : \eta_\nu \neq 0 \} \). For \( \eta \neq 0 \), let \( d_\eta > 0 \) be the greatest common divisor of the integers \( |r_1 \eta_1|, \ldots, |r_p \eta_p| \). We choose \( j_\eta = \min J_\eta \) and set \( q_\eta = |r_{j_\eta} \eta_{j_\eta}| / d_\eta \).

Let \( \bar{\Sigma}_\xi \) be the image of \( \Sigma_\xi \) under projection. If \( \eta \in \bar{\Sigma}_\xi \setminus \{0\} \), then \( d_\eta \) is even because \( \eta_\nu \) is even whenever \( r_\nu \) is odd. Furthermore, we define \( \mathcal{R}_{\xi, 0} = \{ \xi \in \Sigma_\xi : \bar{\xi} = 0 \} \) and \( \mathcal{R}_{\xi, \eta} = \{ \xi \in \Sigma_\xi : \bar{\xi} = \bar{\eta} \) and \( 0 \leq \xi_{p+j_\eta} \leq 2q_\eta - 1 \} \) for \( \eta \in \bar{\Sigma}_\xi \) non-zero. Note that \( \mathcal{R}_{\xi, 0} \) is empty if \( \varepsilon_\nu = 1 \) for some \( 1 \leq \nu \leq p \).

**Lemma 4.2** In this situation, the following holds true:

(i) The disjoint union \( \mathcal{R}_{\xi, \eta} := \bigcup_{\eta \in \bar{\Sigma}_\xi} \mathcal{R}_{\xi, \eta} \) is a set of representatives for the set of all \( \mathbb{R} \)-orbits intersecting \( \Sigma_\xi \).

(ii) Let \( \omega \) be an \( \mathbb{R} \)-orbit which intersects \( \Sigma_\xi \). Then \( \eta := \bar{\xi} \) does not depend on the choice of \( \xi \in \omega \cap \Sigma_\xi \). If \( \omega \) is not a fixed point, then \( \omega \cap \Sigma_\xi \) consists of \( d_\eta / 2 \) distinct \( \mathbb{Z} \)-orbits.

**Proof.** Let \( \xi \in \Sigma_\xi \) such that \( \bar{\xi} \neq 0 \). By (26) we know that \( A(t)^\top \xi \in \Sigma_\xi \) if and only if \( r_\nu \xi_\nu t \in 2\mathbb{Z} \) for all \( \nu \in J_\xi \). This proves

\[
\{ t \in \mathbb{R} : A(t)^\top \xi \in \Sigma_\xi \} = \bigcap_{\nu \in J_\xi} \{ t \in \mathbb{R} : 2 \xi_\nu t \in 2\mathbb{Z} / r_\nu \xi_\nu \mathbb{Z} \} = \frac{2}{d_\xi} \mathbb{Z}
\]

To prove (i), let \( \omega \) be an \( \mathbb{R} \)-orbit and \( \xi \in \omega \cap \Sigma_\xi \). Clearly \( \eta := \bar{\xi} \) does not depend on the choice of \( \xi \). We can assume \( \bar{\xi} \neq 0 \). Define \( d_\eta \) and \( j = j_\eta \) as above. Since \( \langle A(t)^\top \xi, e_{p+j} \rangle = e_{p+j} + r_j \xi_j t \), it follows from (27) that there exists \( t \in \frac{2}{d_\eta} \mathbb{Z} \) such that
Now let us restrict ourselves to algebra. Put $s$ according to (2), satisfies $\Gamma_{\xi}$ is given the orientation and Riemannian metric $g$ of indices that the coadjoint representation is given by $\langle \xi, s \rangle \in \omega \cap \mathcal{R}_{\xi}$. We claim that $\omega \cap \mathcal{R}_{\xi}$ consists of a single point: If $\xi, \xi^* \in \omega \cap \mathcal{R}_{\xi}$, then, again by (2), there exists a $t \in \frac{d_\eta}{2} \mathbb{Z}$ such that $\xi^* = A(t)^\top \xi$. In particular $\xi_p + j = \xi_p + j + \eta_j t$. Since $0 \leq \xi_p + j, \xi^*_p + j \leq 2q_\eta - 1$, it follows $t = 0$ and hence $\xi^* = \xi$. This proves $\mathcal{R}_{\xi}$ to be a set of representatives.

Let $\xi \in \omega \cap \Sigma_\xi$ be an arbitrary non-fixed point. Then $f : \mathbb{R} \to \omega$, $f(t) = A(t)^\top \xi$, is bijective and $\mathbb{R}$-equivariant. By (27) it holds $f^{-1}(\omega \cap \Sigma_\xi) = \frac{2}{d_\eta} \mathbb{Z}$. Since $d_\eta/2$ is an integer, it follows

$$\# \mathbb{Z} \setminus \omega \cap \Sigma_\xi = \# \mathbb{Z} \setminus \frac{2}{d_\eta} \mathbb{Z} = \frac{d_\eta}{2}.$$ 

More precisely, the points $\{ A(\frac{2k}{d_\eta})^\top \xi : 0 \leq k < \frac{d_\eta}{2} \}$ are representatives for the set of all $\mathbb{Z}$-orbits in $\omega \cap \Sigma_\xi$.

We point out that the choice of the set $\mathcal{R}_{\xi}$ is in no way canonical. For example, any choice of indices $j_\eta \in J_\eta$ leads to a set of representatives.

Now let us restrict ourselves to $p = 2$. Then canonical basis $e_1, \ldots, e_4, b$ of the Lie algebra $g \equiv \mathbb{R}^{2p} \times_B \mathbb{R}$ of $G$ satisfies the relations $[b, e_3] = r_1 e_1$ and $[b, e_4] = r_2 e_2$.

Put $s_1 = e_3, s_2 = e_4$ and $s_3 = b$. As before, the corresponding left-invariant vector fields are denoted by the same symbol. The left-invariant distribution $\mathcal{H} := \text{span}\{s_1, s_2, s_3\}$ is given the orientation and Riemannian metric $g$ such that $s_1, s_2, s_3$ is a positively oriented, orthonormal frame. In particular, $|b| = 1$. Note that $\mathcal{H}$ is bracket-generating.

\begin{remark}
In general, when $\mathcal{H}$ is a left-invariant 3-dimensional distribution on a Lie group $G$, the affine space of all left-invariant metric connections in $\mathcal{H}$ satisfying (3) has dimension 6. However, in the present example, the left-invariant connections which are defined by a left-invariant projection $\pi$ onto $\mathcal{H}$ and the Koszul formula (2) and which satisfy (3) form a 3-dimensional space.
\end{remark}

Let $\nabla$ be a left-invariant metric connection in $\mathcal{H}$ satisfying (3). For example, we could take the connection given by projection onto $\mathcal{H}$ along $\mathcal{V} := \text{span}\{e_1, e_2\}$, which, according to (2), satisfies $\Gamma^i_{\alpha j} = 0$ for all $i, j, k$ because $[g, g] \subset \mathcal{V}$. Let $\varepsilon : \Gamma \to \mathbb{Z}$ be a homomorphism giving a spin structure of $\mathcal{H}$. By Lemma 2.1 the sub-Dirac operator $D$ defined by $(\mathcal{H}, g, \nabla, \varepsilon)$ is symmetric. We compute its spectrum. To this end, we note that the coadjoint representation is given by

$$B^\top \xi = \begin{pmatrix} 0 & \xi_1 \\ 0 & \xi_2 \\ r_1 \xi_1 & r_2 \xi_2 \end{pmatrix} \quad \text{and} \quad A(t)^\top \xi = \begin{pmatrix} \xi_3 \\ \xi_4 + r_1 \xi_1 t \\ \xi_4 + r_2 \xi_2 t \end{pmatrix}.$$ 

In particular, we get $\langle \xi, s_\nu \rangle = \langle \xi, e_{2+\nu} \rangle = \xi_{2+\nu}$ and $\langle B^\top \xi, s_\nu \rangle = r_\nu \xi_\nu$ for $\nu = 1, 2$. Put $\alpha = -\frac{1}{2} (\Gamma_{12}^3 + \Gamma_{23}^1 + \Gamma_{31}^2)$. By (22), (18) and (19), the eigenvalues of $D_\xi$ are of the form

$$\mu_k^\pm (\xi) = \alpha \pm \pi \left( (2k + \varepsilon(1))^2 + \xi_3^2 + \xi_4^2 \right)^{1/2}.$$ 

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for fixed points, and
\[ \lambda_0(\xi) = \alpha - \pi \frac{r_1 \xi_1 \xi_4 - r_2 \xi_2 \xi_3}{(r_1^2 \xi_1^2 + r_2^2 \xi_2^2)^{1/2}} \]
or
\[ \lambda_2^\pm(\xi) = \alpha \pm \left( 2\pi k(r_1^2 \xi_1^2 + r_2^2 \xi_2^2)^{1/2} + \pi^2 \frac{(r_1 \xi_1 \xi_4 - r_2 \xi_2 \xi_3)^2}{r_1^2 \xi_1^2 + r_2^2 \xi_2^2} \right)^{1/2} \]
ext. We want to decompose the set \( \mathcal{R}_{\nu'} \) of representatives into a disjoint union of sets that we can describe explicitly. To this end, consider \( \eta = \xi \in \Sigma_{\nu'} \) and assume \( \eta \neq 0 \). If \( \eta_1 \neq 0 \) and \( \eta_2 = 0 \), then \( j_\eta = 1 \), \( d_\eta = |r_1 \eta_1| \) and \( q_\eta = 1 \). Similarly, if \( \eta_1 = 0 \) and \( \eta_2 \neq 0 \), then \( j_\eta = 2 \), \( d_\eta = |r_2 \eta_2| \) and \( q_\eta = 1 \). For \( \eta_1 \eta_2 \neq 0 \) we get \( j_\eta = 1 \), and obtain \( d_\eta = \text{gcd}(|r_1 \eta_1|, |r_2 \eta_2|) \) and \( q_\eta = |r_1 \eta_1|/d_\eta \). This leads to a decomposition of \( \mathcal{R}_{\nu'} \) into the following subsets:
\[
\begin{align*}
\mathcal{R}_{\nu'}^{(1)} &= \mathcal{R}_{\nu',0}, \\
\mathcal{R}_{\nu'}^{(2)} &= \{ \xi \in \Sigma_{\nu'} : \xi_1 \neq 0, \xi_2 = 0, \xi_3 = \varepsilon_3 \}, \\
\mathcal{R}_{\nu'}^{(3)} &= \{ \xi \in \Sigma_{\nu'} : \xi_1 = 0, \xi_2 \neq 0, \xi_4 = \varepsilon_4 \}, \\
\mathcal{R}_{\nu'}^{(4)} &= \{ \xi \in \Sigma_{\nu'} : \xi_1 \neq 0, \xi_2 \neq 0, 0 \leq \xi_3 \leq 2q_{(\xi_1, \xi_2)} - 1 \}.
\end{align*}
\]
We have \( \mathcal{R}_{\nu'}^{(1)} = \emptyset \) if \( \varepsilon_1 = 1 \) or \( \varepsilon_2 = 1 \), \( \mathcal{R}_{\nu'}^{(2)} = \emptyset \) if \( \varepsilon_2 = 1 \), and \( \mathcal{R}_{\nu'}^{(3)} = \emptyset \) if \( \varepsilon_1 = 1 \). Recall that for \( \nu = 1, 2 \) the case \( \varepsilon_{\nu'} = 1 \) can occur only if \( r_\nu \) is even.

The spectrum of \( D \) depends on the spin structure given by \( \nu \). It holds \( m(D) = \sum_{i=1}^4 m_i \) where \( m_i = \sum_{\xi \in \mathcal{R}_{\nu}^{(i)}} m(D_\xi) \). Note that \( m_i = 0 \) if \( \mathcal{R}_{\nu}^{(i)} = \emptyset \). Otherwise, \( m_i \) is given as follows, where the sums are meant to be taken over \( \xi_{\nu} \in 2\mathbb{Z} + \varepsilon_{\nu} \) and \( \xi_5 \in 2\mathbb{Z} + \varepsilon(1) \).
\[
\begin{align*}
m_1 &= \sum_{\xi_3, \xi_4, \xi_5} \delta(\alpha + \pi(\xi_3^2 + \xi_4^2 + \xi_5^2)^{1/2}) + \delta(\alpha - \pi(\xi_3^2 + \xi_4^2 + \xi_5^2)^{1/2}) \\
m_2 &= \sum_{\xi_1 \neq 0} \frac{|r_1 \xi_1|}{2} \sum_{\xi_4} \left( \delta(\alpha - \pi \text{sgn}(r_1 \xi_1)) \xi_4 + \sum_{k=1}^\infty \left( \delta(\alpha + (2\pi k|r_1 \xi_1| + \pi^2 \xi_4^2)^{1/2}) + \delta(\alpha - (2\pi k|r_1 \xi_1| + \pi^2 \xi_4^2)^{1/2}) \right) \right) \\
m_3 &= \sum_{\xi_2 \neq 0} \frac{|r_2 \xi_2|}{2} \sum_{\xi_3} \left( \delta(\alpha + \pi \text{sgn}(r_2 \xi_2)) \xi_3 + \sum_{k=1}^\infty \left( \delta(\alpha + (2\pi k|r_2 \xi_2| + \pi^2 \xi_3^2)^{1/2}) + \delta(\alpha - (2\pi k|r_2 \xi_2| + \pi^2 \xi_3^2)^{1/2}) \right) \right) \\
m_4 &= \sum_{\xi_1 \neq 0} \sum_{\xi_2 \neq 0} \text{gcd}(|r_1 \xi_1|, |r_2 \xi_2|) \sum_{k=1}^\infty \delta(\lambda_0(\xi)) + \sum_{k=1}^\infty \left( \delta(\lambda_k^+(\xi)) + \delta(\lambda_k^-(\xi)) \right)
\end{align*}
\]
In particular, if \( \varepsilon_{\nu'} = 0 \) for \( \nu = 1 \) or \( 2 \), then the numbers \( \{\alpha + (2k + \varepsilon_{2+\nu})\pi : k \in \mathbb{Z}\} \) are eigenvalues of \( D \) and each of them has infinite multiplicity.
In this example, the spectrum of $D$ is a non-discrete subset of $\mathbb{R}$, no matter which homomorphism $\varepsilon : \Gamma \to \mathbb{Z}_2$ defining the underlying spin structure is chosen. Indeed, $\alpha^* := \alpha + \pi \text{sgn}(r_2)\varepsilon_3$ is an accumulation point of $\sigma(D)$. To see this, we consider the sequence $\xi_n \in \mathcal{R}_s^{[4]}$ given by $\xi_{n1} = 2 + \varepsilon_1$, $\xi_{n2} = 2n + \varepsilon_2$, $\xi_{n3} = \varepsilon_3$ and $\xi_{n4} = \text{sgn}(r_1r_2)(2 + \varepsilon_4)$. Then $\lambda_0(\xi_n) \neq \alpha^*$ and $\lambda_0(\xi_n) \to \alpha^*$ for $n \to +\infty$.

### 4.4 A three-step nilpotent example

Let $r_1, r_2 \in \mathbb{Z} \setminus \{0\}$ be such that $r_1r_2$ is even. Define a Lie algebra structure on $\mathfrak{g} := \text{span}\{e_1, e_2, e_3, b\}$ such that $\mathfrak{n} := \text{span}\{e_1, e_2, e_3\}$ is an abelian ideal and $[b, X] = B(X)$ for $X \in \mathfrak{n}$, where $B : \mathfrak{n} \to \mathfrak{n}$ is given by

$$B = \begin{pmatrix} 0 & r_1 & 0 \\ 0 & 0 & r_2 \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to the basis $e_1, e_2, e_3$ of $\mathfrak{n}$. Let $G$ be the simply-connected Lie group with Lie algebra $\mathfrak{g}$. Then $G = \mathbb{R}^3 \rtimes_A \mathbb{R}$ with

$$A(t) = \exp tB = \begin{pmatrix} 1 & tr_1 & t^2 r_1r_2/2 \\ 0 & 1 & tr_2 \\ 0 & 0 & 1 \end{pmatrix}.$$  

Since $A(1)$ is in $SL(2, \mathbb{Z})$, the subset $\Gamma := \mathbb{Z}^3 \rtimes_A \mathbb{Z}$ a uniform discrete subgroup of $G$. Let $(\mathcal{H}, g)$ be the oriented sub-Riemannian structure having $s_1 := e_3$, $s_2 := b$ as a positively oriented orthonormal frame. Then $\mathcal{H}$ is bracket generating.

The spin structures of $\mathcal{H}$ correspond to homomorphisms $\varepsilon : \Gamma \to \mathbb{Z}_2$. As above we write $\varepsilon(k, l) = \varepsilon'(k) \cdot \varepsilon(l)$, where $\varepsilon : \mathbb{Z} \to \mathbb{Z}_2$ is an arbitrary homomorphism and $\varepsilon' : \mathbb{Z}^3 \to \mathbb{Z}_2$ is a homomorphism satisfying (7), which, in this example, means that $r_1\varepsilon_1$ and $r_1r_2\varepsilon_1/2 + r_2\varepsilon_2$ are both even. More precisely, this shows: If $r_1$ and $r_2$ are both even, then $\varepsilon_1$ and $\varepsilon_2$ are arbitrary. If $r_1$ is odd and $r_2$ is even, then $\varepsilon_1 = 0$ and $\varepsilon_2$ is arbitrary. Now suppose that $r_2$ is odd. If, in addition, $r_1$ is odd, then $\varepsilon_1 = \varepsilon_2 = 0$. If $r_1$ is even but not divisible by 4, then either $\varepsilon_1 = \varepsilon_2 = 0$ or $\varepsilon_1 = \varepsilon_2 = 1$. Finally, if $r_1$ is divisible by 4, then $\varepsilon_2 = 0$.

Clearly $\mathcal{V} := \text{span}\{e_1, e_2\}$ is a complement of $\mathcal{H}$ in the tangent bundle $TG$. Using the projection onto $\mathcal{H}$ along $\mathcal{V}$, we define a left-invariant connection $\nabla$ in $\mathcal{H}$ by the Koszul formula (3). Since $\text{pr}[s_1, s_2] = 0$, all Christoffel symbols $\Gamma^k_{ij}$ vanish. In particular, the sub-Dirac operator is symmetric and equals

$$D = s_1 \cdot \partial_{s_1} + s_2 \cdot \partial_{s_2},$$

where we use the simple $\text{Cl}(\mathcal{H}_e)$-module structure on $\mathbb{C}^2$ defined by

$$s_1 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad s_2 \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$
On the other hand, we have

$$A^\top (t)\xi = \begin{pmatrix} \xi_1 \\ \xi_2 + tr_1 \xi_1 \\ \xi_3 + tr_2 \xi_2 + t^2 r_1 r_2 \xi_1/2 \end{pmatrix}. \quad (28)$$

In particular, the sets

$$R^{(1)} := \{ \xi \in \mathbb{R}^3 \mid \xi_1 = \xi_2 = 0 \},$$
$$R^{(2)} := \{ \xi \in \mathbb{R}^3 \mid \xi_1 = 0, \xi_2 \neq 0 \},$$
$$R^{(3)} := \{ \xi \in \mathbb{R}^3 \mid \xi_1 \neq 0 \}$$

are invariant under $A^\top (t)$ for all $t \in \mathbb{R}$.

Let us first consider $D_\theta$ for the orbit $\theta = \{ \xi \}$ of an element $\xi \in R^{(1)}$. Then, according to (21), the spectrum of $D_\theta$ consists of the eigenvalues

$$\mu_k^\pm (\xi) = \pm \pi \left( (2k + \dot{\epsilon}(1))^2 + 4 \xi^2_3 \right)^{1/2}, \quad k \in \mathbb{Z}.$$ 

Now consider $\xi \in R^{(2)}$. Then $D_\xi$ has the form

$$\begin{pmatrix} i\partial_t & \bar{\omega} \\ \omega & -i\partial_t \end{pmatrix} \quad (29)$$

with $\omega(t) = a \omega_1 t + \omega_0$, where

$$a = \pi |r_2 \xi_2|, \quad \omega_1 = \text{sgn}(r_2 \xi_2) \cdot i, \quad \omega_0 = \pi i \xi_3.$$ 

According to (17) the spectrum of $D_\xi$ consists of the eigenvalues $\lambda_0 = 0$ and

$$\lambda_k^\pm = \pm (2\pi |r_2 \xi_2| k)^{1/2}, \quad k \in \mathbb{N} \setminus \{0\}.$$ 

Finally, take $\xi \in R^{(3)}$. Then $D_\xi$ is of the form (29) where $\omega(t) = i\pi (\xi_1 r_1 r_2 t^2/2 + \xi_2 r_2 t + \xi_3)$. Hence

$$D_\xi^2 = \begin{pmatrix} -\partial_t^2 - \omega(t)^2 & -i\omega(t) \\ -i\omega(t) & -\partial_t^2 - \omega(t)^2 \end{pmatrix}.$$ 

Obviously, $D_\xi^2$ is time-independent diagonalisable. More exactly, $D_\xi^2$ is conjugate to

$$\begin{pmatrix} -\partial_t^2 - \omega(t)^2 - i\omega'(t) & 0 \\ 0 & -\partial_t^2 - \omega(t)^2 + i\omega'(t) \end{pmatrix}.$$ 

The operators $-\partial_t^2 - \omega(t)^2 \mp i\omega'(t)$ are of the form

$$P_{a,b,c}^\pm := \partial_t^2 + (at^2 + bt + c)^2 \pm (2at + b)$$

for

$$a = \pi \xi_1 r_1 r_2 \neq 0, \quad b = \pi \xi_2 r_2, \quad c = \pi \xi_3.$$ 

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We consider the bijection

\[ L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}), \quad \varphi \mapsto \tilde{\varphi}, \quad \tilde{\varphi}(t) = \frac{1}{x^2} \varphi(xt + y), \]

where \( x = a^{1/3}, \ y = ba^{-2/3}/2. \)

We define \( P^\pm_c := P^\pm_{1,0,c}. \)

**Claim.** The equation \( P^\pm_{a,b,c} \tilde{\varphi} = \tilde{\lambda} \tilde{\varphi} \) is equivalent to \( P^\pm_{c_1} \varphi = \lambda \varphi, \) where

\[ c_1 = -b^2a^{-4/3}/2 + ca^{-1/3}, \quad \tilde{\lambda} = a^{2/3} \lambda. \]

Indeed, assume that \( P^\pm_{c_1} \varphi = \lambda \varphi. \) Then \( \varphi''(t) = ((t^2 + c_1)^2 \pm 2t - \lambda) \varphi(t) \) holds. Hence

\[
(P^\pm_{a,b,c} \tilde{\varphi})(t) = - (\partial_t^2 \tilde{\varphi})(t) + ((at^2 + bt + c)^2 \pm (2at + b)) \tilde{\varphi}(t) \\
= \left(-x^2 \left( ((x^2 + y^2)^2 + c_1^2 \pm 2(xt + y) - \lambda \right) + (at^2 + bt + c)^2 \pm (2at + b) \right) \tilde{\varphi}(t) \\
= x^2 \lambda \tilde{\varphi}(t) = a^{2/3} \lambda \tilde{\varphi}.
\]

The converse can be proven similarly using \( \varphi(t) = x^2 \tilde{\varphi}(t/x^2 - y/x). \)

It is well known that the Schrödinger operator \( P^\pm_c \) having a polynomial potential of degree 4 has the following properties [EGS, T]. The spectrum of \( P^\pm_c \) is discrete. All eigenvalues are real and simple. They can be arranged into an increasing sequence \( \lambda_0 < \lambda_1 < \cdots \rightarrow \infty \) and satisfy

\[
\lambda_k \sim \left( \frac{\sqrt{\pi} \Gamma(7/4) \cdot k}{\Gamma(5/4)} \right)^{4/3}.
\]

Obviously, \( P^+_c \) and \( P^-_c \) have the same eigenvalues. We will denote these eigenvalues by \( \lambda_k(c), \ k \in \mathbb{N}. \)

Since \( \text{dim} \ H \) is even the spectrum of \( D_\xi \) is symmetric. We conclude that \( \text{spec}(D_\xi) \) consists of the eigenvalues

\[
\pm \left( a^{2/3} \lambda_k(-4b^2a^{-4/3} + ca^{-1/3}) \right)^{1/2}, \quad k \in \mathbb{N},
\]

where \( a = \pi \xi_1 r_1 r_2/2, \ b = \pi \xi_2 r_2, \ c = \pi \xi_3. \)

Next we determine a set of representatives of the \( \mathbb{R} \)-orbits in \( \mathbb{R}^3 \) that intersect \( \Sigma_\varepsilon' \) and the number of \( \mathbb{Z} \)-orbits that are contained in them. Obviously,

\[
\mathcal{R}^{(1)} := R^{(1)} \cap \Sigma_\varepsilon'
\]

is the set of fixed points in \( \Sigma_\varepsilon' \) and

\[
\mathcal{R}^{(2)} := \{ \xi \in \Sigma_\varepsilon' \mid \xi_1 = 0, \ \xi_2 \neq 0, \ \xi_3 = \varepsilon_3 \}
\]

is a set of representatives of the \( \mathbb{R} \)-orbits in \( R^{(2)} \) that intersect \( \Sigma_\varepsilon' \). For \( \xi \in \mathcal{R}^{(2)} \) the \( \mathbb{R} \)-orbit through \( \xi \) contains \( |r_2\xi_2|/2 \) \( \mathbb{Z} \)-orbits. Now we turn to orbits contained in \( R^{(3)}. \)

For a given number \( k \in \mathbb{Z} \setminus \{0\} \) let \( p, q \in \mathbb{Z}, \ q > 0 \) be such that

\[
\frac{|r_2|}{r_1k} = \frac{p}{q}, \quad (p, q) = 1
\]

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and put \( q(k) := q \). Moreover, for \( l, q \in \mathbb{N}, q > 0 \) we define

\[
M(l, q) := \{(m_1, m_2) \mid m_1, m_2 \in \mathbb{N} \setminus \{0\}, m_1 + m_2 = l, q|m_1m_2\}.
\]

We will show:

1. The set

\[
\mathcal{R}^{(3)} := \{\xi \in \Sigma_{\epsilon'} \mid 0 \leq \xi_2 < |\lambda_1|, M(\xi_2, q(\xi_1)) = \emptyset\}
\]

is a set of representatives of \( \mathbb{R} \)-orbits in \( \mathcal{R}^{(3)} \) that intersect \( \Sigma_{\epsilon'} \).

2. For \( \xi \in \mathcal{R}^{(3)} \) the number of \( \mathbb{Z} \)-orbits contained in the \( \mathbb{R} \)-orbit of \( \xi \) equals

\[
m(\xi_1, \xi_2) := \#\{k \in \mathbb{N} \mid \xi_2 + 2k < |\lambda_1|, q(\xi_1)|k(k + \xi_2)\}.
\]

Take \( \xi \in R^{(3)} \cap \Sigma_{\epsilon'} \) and denote by \( \theta \) the \( \mathbb{R} \)-orbit of \( \xi \). Using (28) we see that \( A^\top(t)\xi \) is in \( \Sigma_{\epsilon'} \) if and only if \( tr_1\xi_1 \) and \( t^2r_1r_2\xi_1/2 + tr_2\xi_2 \) are in \( 2\mathbb{Z} \). The latter condition is equivalent to

\[
t = \frac{2k}{r_1\xi_1}, \quad q(\xi_1)|k(k + \xi_2) \tag{30}
\]

for some \( k \in \mathbb{Z} \). Obviously, we may choose \( \hat{\xi} = (\xi_1, \xi_2, \xi_3) \in \theta \) such that \( 0 \leq \hat{\xi}_2 < |\lambda_1| \).

Now we want to choose \( \hat{\xi} \) is such a way that \( \hat{\xi}_2 \geq 0 \) is minimal, which ensures the uniqueness of the representative. By (30), \( \hat{\xi}_2 \) is minimal if and only if there does not exist an integer \( k \), \(-[\hat{\xi}_2/2] \leq k \leq -1 \), such that \( q(\xi_1)|k(k + \hat{\xi}_2) \). The latter condition is equivalent to \( q(\xi_1)|(-k)(k + \hat{\xi}_2) \). Hence \( \hat{\xi}_2 \) is minimal if and only if \( \hat{\xi}_2 \) does not decompose as a sum \( \hat{\xi}_2 = m_1 + m_2 \) with \( m_1, m_2 \in \mathbb{N} \setminus \{0\} \) and \( q(\xi_1)|m_1m_2 \). This proves the first assertion. The second one follows from (30).

Now we can give an expression for \( m(D) \). In the following sums are taken over \( \xi_i \in \epsilon_i + 2\mathbb{Z}, \ i = 1, 2, 3 \). Moreover, we will take another index of summation, namely \( \xi_4 \in \hat{\epsilon}(1) + 2\mathbb{Z} \). Furthermore, \( \kappa \in \{1, -1\} \). Then

\[
m(D) = \sum_{\xi_3, \xi_4} \sum_\kappa \delta \left( \kappa\pi (\xi_3^2 + \xi_4^2)^{1/2} \right)
\]

\[
+ \sum_{\xi_2 > 0} |r_2\xi_2| \left( \delta(0) + \sum_{k=1}^{\infty} \sum_\kappa \delta(k2\pi|\xi_2|) \right)
\]

\[
+ \sum_{\xi_1 \neq 0, \xi_2, \xi_3} \sum_{k=0}^{\infty} \sum_\kappa \delta \left( \kappa\pi \xi_3\frac{r_1r_2}{2} \right)^{1/3} \lambda_k \left( \frac{\pi\xi_3 r_1 r_2}{2} \right)^{-1/3} \pi(\xi_3 - \frac{8\xi_2^2 r_2}{\xi_1 r_1})^{1/2} \right).
\]

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