Min-plus eigenvalue of tridiagonal matrices
in terms of the ultradiscrete Toda equation

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Abstract
The discrete Toda molecule equation can be used to compute eigenvalues of
tridiagonal matrices over conventional linear algebra, and is the recursion
formula of the well-known quotient difference algorithm for tridiagonal
eigenvalues. An ultradiscretization of the discrete Toda equation leads to the
ultradiscrete Toda (udToda) equation, which describes motions of balls in
the box and ball system. In this paper, we associate the udToda equation with
eigenvalues of tridiagonal matrices over min-plus algebra, which is a
semiring with two operation types: ⊕ := min and ⊗ := +. We also clarify
an interpretation of the udToda variables in weighted and directed graphs
consisting of vertices and edges.

Keywords: eigenvalue, min-plus algebra, ultradiscrete Toda equation,
weighted directed graph

(Some figures may appear in colour only in the online journal)
1. Introduction

The Toda lattice equation [20] is a mass-spring system with exponential decay. It is expressed as

\[
\frac{d^2x_k(t)}{dt^2} = e^{x_{k-1}(t) - x_k(t)} - e^{x_k(t) - x_{k+1}(t)},
\]

where \(x_k(t)\) denotes displacement of the \(k\)th mass from the equilibrium position at continuous time \(t\). The Toda lattice equation (1) is a completely integrable system with many good properties, such as soliton solutions, Lax pair, conserved quantities, and the Bäcklund transformation. The Toda lattice equation (1) has contributed to the development of various other integrable systems and it has applications to numerical algorithms. The Toda lattice equation (1) with boundary conditions \(x_0 = -\infty\) and \(x_{m+1} = +\infty\), which is called the Toda molecule equation, has an interesting relationship to the well-known QR algorithm for computing eigenvalues of \(m\)-by-\(m\) square matrices. The continuous time \(t\) and subscript \(k\) correspond to the iteration number of the QR algorithm and indexes of entries in \(m\)-by-\(m\) tridiagonal matrices, respectively. Introducing the dependent variable transformations \(a_k = (1/2)e^{(x_{k-1} - x_k)/2}\) and \(b_k = (1/2)d x_k/dt\) and the boundary conditions \(a_0(t) = 0\) and \(b_m(t) = 0\), the Toda molecule equation can be written as

\[
\begin{align*}
\frac{da_k(t)}{dt} &= a_k(t)(b_k(t) - b_{k+1}(t)), & k &= 1, 2, \ldots, m, \\
\frac{db_k(t)}{dt} &= 2((a_{k-1}(t))^2 - (a_k(t))^2), & k &= 1, 2, \ldots, m - 1, \\
a_0(t) &= 0, & b_m(t) &= 0, & t &\geq 0.
\end{align*}
\]

Specifically, one step of the QR algorithm for the exponential of a symmetric matrix coincides with the time evolution from \(t\) to \(t + 1\) in the Toda molecule equation (2) [17]. Furthermore, one step of the QR algorithm is identical to two steps of the LR algorithm [3], which is mathematically equivalent to Rutishauser’s quotient difference (qd) algorithm for computing tridiagonal eigenvalues. Hirota et al [8] showed that the recursion formula of the qd algorithm is a time discretization of the Toda molecule equation (2), i.e. the discrete time Toda (dToda) equation:

\[
\begin{align*}
q_{k}^{(n+1)} &= q_{k}^{(n)} - e_{k-1}^{(n+1)} + e_{k}^{(n)}, & k &= 1, 2, \ldots, m, \\
e_{k}^{(n+1)} &= \frac{q_{k+1}^{(n)}e_{k}^{(n)}}{q_{k}^{(n)}}, & k &= 1, 2, \ldots, m - 1, \\
e_{0}^{(n)} &= 0, & e_{m}^{(n)} &= 0, & n &= 0, 1, \ldots.
\end{align*}
\]

The dToda variables \(q_{k}^{(n)}\) and \(e_{k}^{(n)}\) denote values of \(q_k\) and \(e_k\) respectively at discrete time \(n\). Furthermore, it is known that certain extensions of the dToda equation are also related to the numerical algorithms. For example, the discrete relativistic Toda equation can generate the Laplace transformation [14] and the expansion into continued fractions [12]. The dToda equation (3) can be extended to the discrete hungry Toda (dhToda) equations [16, 22], which are used to compute the eigenvalues of totally nonnegative matrices whose minors are all nonnegative [4, 16].

Tokihiro et al [21] proposed an ultradiscretization technique, sometimes called the tropicalization, for transforming equations into piecewise linear equations with one of two operators \(\min\) and \(\max\). The ultradiscretization is associated with min-plus, which we focus on in this paper, and max-plus algebras. Min-plus algebra has two binary operations \(\oplus := \min\) and \(\otimes := +\) in the set \(\mathbb{R}_{\min} := \mathbb{R} \cup \{\infty\}\). Min-plus algebra has a close relationship to weighted
directed graphs constructed with sets of nodes and directed edges with weights. Weighted directed graphs naturally appear in various fields such as railway systems, automata, and Petri nets used to model discrete event systems [6].

The so-called ultradiscrete Toda (udToda) equation [10, 11] is, of course, an ultradiscretization of the dToda equation (3). The udToda equation describes motions of a finite number of balls in an array of infinite number of boxes [13]. For analysing the box and ball system (BBS) [18], conserved quantities of the udToda equation and the dToda equation (3) are useful. Conserved quantities of the dToda equation (3) are also useful in proving that the dToda equation (3) is applicable to computing tridiagonal eigenvalues. However, to the best of our knowledge, conserved quantities of the udToda equation have not yet been considered in matrix eigenvalue analysis. In this paper, we propose an application of the udToda equation to compute the matrix eigenvalue over min-plus algebra rather than linear algebra. We also show that the udToda equation can solve the minimum circuit problem in weighted and directed graphs. In [19], Tavakoliipour and Shakeri analysed eigenvalues of tridiagonal Toeplitz matrices over max-plus algebra. They give explicit formulas for the eigenvalues under certain conditions. The present paper also treats eigenvalues of tridiagonal matrices but is not restricted to Toeplitz matrices: diagonal and subdiagonal entries are not fixed to the same constants. Moreover, the analytical approach is quite different from [19] in that the udToda equation can be applied to eigenvalue computation over min-plus algebra.

The remainder of this paper is organized as follows. In section 2, we give a brief explanation of the eigenvalues of matrices over min-plus algebra and describe the significance of the eigenvalues in weighted and directed graphs corresponding to these matrices. In section 3, we first introduce an ultradiscretization of the dToda equation (3), and then demonstrate how the udToda equation enables calculation of an eigenvalue of a tridiagonal matrix over min-plus algebra. We also relate one of the udToda variables to the minimum circuit in weighted directed graphs. Finally, in section 4, we give concluding remarks.

2. Min-plus algebra and associated graph

In this section, we briefly review scalar and matrix arithmetics over min-plus algebra, graphs corresponding to matrices over min-plus algebra, and some known results about relationships of min-plus matrices to associated graphs.

We begin by explaining scalar arithmetic and its elementary properties over min-plus algebra. For two \( \mathbb{R}_{\text{min}} \) numbers \( a \) and \( b \), min-plus algebra has the following two binary operations:

\[
\begin{align*}
    a \oplus b &= \min\{a, b\}, \\
    a \otimes b &= a + b.
\end{align*}
\]

We can easily check that \( \oplus \) and \( \otimes \) are both associative and commutative, \( \otimes \) is distributive with respect to \( \oplus \), and \( \varepsilon := +\infty \) and \( e := 0 \) are identities with respect to \( \oplus \) and \( \otimes \), respectively. If the \( \mathbb{R}_{\text{min}} \) number \( b \) satisfies

\[
a \otimes b = e,
\]

then \( b \) is the inverse of \( a \) with respect to \( \otimes \). For convenience, we hereafter employ an auxiliary operator \( \otimes \) as the inverse of \( \otimes \) such that \( a \otimes b \otimes b = a \).

Matrices whose entries are all \( \mathbb{R}_{\text{min}} \) numbers are called min-plus matrices, and the set of all \( m \times n \) min-plus matrices is denoted by \( \mathbb{R}_{\text{min}}^{m \times n} \). Since the min-plus matrices appearing in the next section are all \( m \times m \) square matrices, we hereinafter focus on the case of \( \mathbb{R}_{\text{min}}^{m \times m} \) matrices. For two \( \mathbb{R}_{\text{min}}^{m \times m} \) matrices \( A = (a_{ij}) \) and \( B = (b_{ij}) \), the matrix sum \( A \oplus B = ([A \oplus B]_{ij}) \),
the matrix product \( A \otimes B = ([A \otimes B]_{ij}) \), and the multiplication of \( A \) by the \( \mathbb{R}_{\min} \) number \( \alpha \), namely, \( \alpha \otimes A = ([\alpha \otimes A]_{ij}) \), are respectively given as:

\[
[A \oplus B]_{ij} = a_{ij} \oplus b_{ij} = \min\{a_{ij}, b_{ij}\},
\]

\[
[A \otimes B]_{ij} = \bigoplus_{\ell=1}^{m} (a_{i\ell} \otimes b_{\ell j}) = \min_{\ell=1,2,\ldots,m} \{a_{i\ell} + b_{\ell j}\},
\]

\[
[\alpha \otimes A]_{ij} = \alpha \otimes a_{ij}.
\]

These operations are intuitively similar to those over conventional linear algebra. With respect to eigenvalues and eigenvectors, the following definition gives reasonable min-plus analogues.

**Definition 2.1.** For an \( \mathbb{R}_{\min}^{m \times m} \) matrix \( A \), if there exists an \( \mathbb{R}_{\min} \) number \( \lambda \) and an \( \mathbb{R}_{\min}^{m} \) vector \( x \neq (\varepsilon, \varepsilon, \ldots, \varepsilon)^{T} \) satisfying

\[
A \otimes x = \lambda \otimes x,
\]

then \( \lambda \) and \( x \) are an eigenvalue of \( A \) and its corresponding eigenvector, respectively.

Determinants of \( \mathbb{R}_{\min}^{m \times m} \) matrices, however, are not directly defined as min-plus analogues of linear determinants. This is because min-plus algebra has no operation corresponding to subtraction over linear algebra. The following definition gives the min-plus analogue of determinants over linear algebra.

**Definition 2.2.** The tropical determinant \( \text{tropdet}(A) \) of an \( \mathbb{R}_{\min}^{m \times m} \) matrix \( A \) is defined as

\[
\text{tropdet}(A) := \bigoplus_{\alpha \in S_{m}} a_{1\sigma(1)} \otimes a_{2\sigma(2)} \otimes \cdots \otimes a_{m\sigma(m)},
\]

where \( S_{m} \) is the symmetric group of permutations of \( \{1, 2, \ldots, m\} \). Moreover, for an \( \mathbb{R}_{\min}^{m \times m} \) matrix \( A \), the characteristic polynomial \( g_{A}(x) \) is given by

\[
g_{A}(x) := \text{tropdet}(A \oplus x \otimes I),
\]

where \( I \) denotes the \( m \)-by-\( m \) identity matrix whose \( (i,j) \) entries are 0 if \( i = j \), or \( \varepsilon \) otherwise.

We now consider the so-called min-plus polynomial of degree \( n \) with respect to \( x \),

\[
p(x) = x^{k} \oplus c_{1} \otimes x^{n-1} \oplus \cdots \oplus c_{n-1} \otimes x \oplus c_{n},
\]

where \( x^{k} := x \otimes x \otimes \cdots \otimes x \) \( k \) times \( k \times k \) and \( c_{1}, c_{2}, \ldots, c_{n} \in \mathbb{R}_{\min} \) are the coefficients. Regarding \( p(x) \) as the min-plus function with respect to \( x \), we see that \( p(x) \) is piecewise linear because

\[
p(x) = \min\{ax, c_{1} + (n - 1)x, \ldots, c_{n-1} + x, c_{n}\}.
\]

It is emphasized here that, in contrast to linear algebra, two distinct min-plus polynomials \( p(x) \) and \( \tilde{p}(x) \) are sometimes factorized using common linear factors [2]. If the linear factorizations of \( p(x) \) are the same as those of \( \tilde{p}(x) \), then \( p(x) \) is considered ‘equivalent’ to \( \tilde{p}(x) \). To distinguish this from equality, namely, \( p(x) = \tilde{p}(x) \), we denote equivalent by \( p(x) \equiv \tilde{p}(x) \), where \( p(x) \) is equivalent to \( \tilde{p}(x) \). Figure 1 shows the graphs of the min-plus polynomial functions \( p(x) = x^{2} \oplus 6 \oplus x \oplus 8 \) and \( \tilde{p}(x) = x^{2} \oplus 4 \oplus x \oplus 8 \). Those min-plus polynomials \( p(x) \) and \( \tilde{p}(x) \) are not equal but equivalent since \( p(x) \) and \( \tilde{p}(x) \) define the same function. So we represent this relation as \( p(x) \equiv \tilde{p}(x) \).
The following proposition describes the factorization and the minimum root of the characteristic polynomial \( g_A(t) \).

**Proposition 2.3 (Maclagan–Sturmfels [9])**. The characteristic polynomial \( g_A(t) \) of the \( \mathbb{R}^{m \times m} \) matrix \( A \) is factorized as

\[
g_A(t) \equiv (t \oplus p_1)^{q_1} \otimes (t \oplus p_2)^{q_2} \otimes \cdots \otimes (t \oplus p_k)^{q_k},
\]

where \( p_1 < p_2 < \cdots < p_k \) and \( q_1 + q_2 + \cdots + q_k = m \). Then, the minimum root \( p_1 \) of \( g_A(t) \) coincides with the eigenvalue of \( A \).

Min-plus algebra differs from linear algebra in that all the roots of characteristic polynomials of \( \mathbb{R}^{m \times m} \) matrices do not always coincide with eigenvalues of the \( \mathbb{R}^{m \times m} \) matrices. Over min-plus algebra, only the minimum roots are certainly eigenvalues.

A directed graph (digraph) \( G = (V, E) \) consists of a vertex set \( V \) and a directed edge set \( E \subset V \times V \). An element \( v \in V \) is called a vertex and an element \( e = (v_i, v_j) \in E \) is called an edge of the digraph \( G \). An edge of the form \( (v, v) \) is called a loop. A path \( P \) in the digraph \( G = (V, E) \) is an alternating sequence \( P = (v_0, e_1, v_1, e_2, \ldots, e_s, v_s) \) of pairwise distinct vertices such that \( e_k = (v_{k-1}, v_k) \in E \) for \( k = 1, 2, \ldots, s \). Vertices \( v_0 \) and \( v_s \) are respectively called the initial and the terminal vertices of the path \( P \). If the initial vertex of a path \( P \) coincides with the terminal vertex, then \( P \) is called a circuit. A digraph \( G \) is called strongly connected if, for any vertices \( v_i \) and \( v_j \), there exists at least one path from \( v_i \) to \( v_j \). Furthermore, if \( G \) is a weighted digraph, then a real number \( w(e) \) is assigned to each edge \( e \), and is called the weight. The following definition gives the so-called weighted adjacency matrices associated with weighted digraphs.

**Definition 2.4.** For a weighted digraph \( G \) involving \( m \) vertices, entries of the \( m \)-by-\( m \) weighted adjacency matrix \( A(G) = (a_{ij}) \) are given using \( \mathbb{R}_{\min} \) numbers as

\[
a_{ij} = \begin{cases} w(e), & \text{if } e = (v_i, v_j) \in E, \\ \varepsilon, & \text{otherwise.} \end{cases}
\]
It is clear that the weighted adjacency matrix $A(G)$ is a min-plus matrix. Conversely, for any $\mathbb{R}_{\text{min}}^{m \times m}$ matrix $A$, there exists a weighted digraph whose weighted adjacency matrix is $A$. We hereinafter denote such a weighted digraph by $G(A)$.

Moreover, in a circuit $C$ on a weighted digraph $G(A)$, we refer to the number of edges and the sum of edge weights as the length $\ell(C)$ and the weight sum $w(C)$, respectively. These values are used to define the average weight of the circuit $C$ as follows.

**Definition 2.5.** For a circuit $C$, the average weight $w_{\text{ave}}(C)$ is the ratio of the weight sum $w(C)$ to the length $\ell(C)$, namely:

$$w_{\text{ave}}(C) = \frac{w(C)}{\ell(C)}.$$

Then the following theorem gives interesting relationships between an eigenvalue of the $\mathbb{R}_{\text{min}}^{m \times m}$ matrix $A$ and the average weights of circuits on the weighted digraph $G(A)$.

**Theorem 2.6 (Baccelli et al [1] and Gondran–Minoux [5]).** Let a weighted digraph $G(A)$ be strongly connected. Then, the weighted adjacency matrix $A(G)$ possesses one and only one eigenvalue. Moreover, the minimum value of the average weights of circuits in $G(A)$ coincides with the eigenvalue of matrix $A(G)$.

### 3. The ultradiscrete Toda equation and min-plus eigenvalue

In this section, after referring to the basic properties of the dToda equation (3) related to computing eigenvalues of tridiagonal matrices, we show that an ultradiscrete analogue of the dToda equation (3) has an application to computing eigenvalues of tridiagonal min-plus matrices.

Now, we prepare the $m$-by-$m$ lower and upper bidiagonal matrices $L^{(n)}$ and $R^{(n)}$ involving the dToda variables $q^{(n)}_i > 0$ and $e^{(n)}_i > 0$ as

$$L^{(n)} := \begin{bmatrix} q^{(n)}_1 \\ & q^{(n)}_2 \\ & & \ddots \\ & & & q^{(n)}_m \\ 1 & & & & \\ \end{bmatrix}, \quad R^{(n)} := \begin{bmatrix} 1 \\ e^{(n)}_1 \\ & 1 \\ & & \ddots \\ & & & e^{(n)}_{m-1} \\ & & & & 1 \end{bmatrix}.$$  

By observing the entries in $L^{(n+1)}R^{(n+1)}$ and $R^{(n)}L^{(n)}$, we can easily check that

$$L^{(n+1)}R^{(n+1)} = R^{(n)}L^{(n)}, \quad n = 0, 1, \ldots.$$  

Considering that $R^{(n)}L^{(n)} = R^{(n)}(L^{(n)}R^{(n)})^{-1}$, we see that the dToda equation (3) generates the similarity transformation from $A^{(n)} := L^{(n)}R^{(n)}$ to $A^{(n+1)}$ as

$$A^{(n+1)} = R^{(n)}A^{(n)}R^{(n)}^{-1}.$$  

With respect to the asymptotic behaviour of the dToda equation (3) as $n \to \infty$, Henrici [7] and Rutishauser [15] showed that

$$\lim_{n \to \infty} q^{(n)}_k = \lambda_k, \quad k = 1, 2, \ldots, m,$$

$$\lim_{n \to \infty} e^{(n)}_k = 0, \quad k = 1, 2, \ldots, m - 1.$$
where $\lambda_1, \lambda_2, \ldots, \lambda_m$ denote eigenvalues of $A := A^{(0)}$ satisfying $\lambda_1 > \lambda_2 > \cdots > \lambda_m > 0$. This suggests that the dToda equation (3) with initial settings $q_1^{(0)}, q_2^{(0)}, \ldots, q_m^{(0)}$ and $e_1^{(0)}, e_2^{(0)}, \ldots, e_{m-1}^{(0)}$ given from tridiagonal entries of $A$ can be applied to compute the eigenvalues of $A$. In fact, the dToda equation (3) is just the recursion formula of the qd algorithm.

Applying the variable transformations $q_k^{(n)} = e^{-Q_k^{(n)}/\epsilon}$ and $e_k^{(n)} = e^{-E_k^{(n)}/\epsilon}$ to the dToda equation (3), taking the logarithm, multiplying $\epsilon$ on both sides, and taking the limit $\epsilon \to +0$, we obtain the ultradiscrete Toda (udToda) equation:

$$
\begin{align*}
Q_k^{(n+1)} &= \bigotimes_{j=1}^{k-1} Q_j^{(n)} \bigotimes_{j=k+1}^{m} E_j^{(n)}, \quad k = 1, 2, \ldots, m, \\
E_k^{(n+1)} &= Q_k^{(n)} \bigotimes_{j=k+1}^{m} E_j^{(n)}, \quad k = 1, 2, \ldots, m-1, \\
E_0^{(n)} &= \epsilon, \quad E_m^{(n)} := \epsilon, \quad n = 0, 1, \ldots
\end{align*}
$$

(4)

According to Nagai et al [13], the udToda equation (4) describes dynamics of the BBS introduced in Takahashi and Satsuma [18]. The BBS is considered as a dynamics of solitons in which $Q_k^{(n)}$ and $E_k^{(n)}$ respectively correspond to the number of balls appearing in the $k$th sequence of balls from the left and the number of empty boxes between the $k$th and $(k+1)$th sequential balls at discrete time $n$. As ultradiscrete analogues of $L^{(n)}$ and $R^{(n)}$, we introduce two $\mathbb{R}^{m \times m}_{\text{min}}$ matrices:

$$
\mathcal{L}^{(n)} = \begin{bmatrix}
Q_1^{(n)} & \epsilon & \cdots & \cdots & \epsilon \\
\epsilon & Q_2^{(n)} & \cdots & \cdots & \epsilon \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
\epsilon & \cdots & \epsilon & Q_m^{(n)} & \epsilon \\
\epsilon & \cdots & \epsilon & \epsilon & e
\end{bmatrix}, \quad
\mathcal{R}^{(n)} = \begin{bmatrix}
e & E_1^{(n)} & \epsilon & \cdots & \epsilon \\
\epsilon & e & E_2^{(n)} & \cdots & \epsilon \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
\epsilon & \cdots & \epsilon & e & E_{m-1}^{(n)} \\
\epsilon & \cdots & \epsilon & \epsilon & e
\end{bmatrix}.
$$

Similarly to the discrete case, we define tridiagonal $\mathbb{R}^{m \times m}_{\text{min}}$ matrices $A^{(n)}$ as the product of $\mathcal{L}^{(n)}$ and $\mathcal{R}^{(n)}$, namely:

$$
A^{(n)} := \mathcal{L}^{(n)} \otimes \mathcal{R}^{(n)} = \begin{bmatrix}
Q_1^{(n)} & Q_1^{(n)} \otimes E_1^{(n)} & \epsilon & \cdots & \epsilon \\
e & Q_2^{(n)} \otimes E_2^{(n)} & Q_2^{(n)} \otimes E_2^{(n)} & \cdots & \epsilon \\
\epsilon & \cdots & \cdots & \cdots & \epsilon \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\epsilon & \cdots & \epsilon & \epsilon & Q_m^{(n)} \otimes E_m^{(n)} \\
\epsilon & \cdots & \epsilon & \epsilon & e
\end{bmatrix}.
$$

Figure 2 shows the weighted digraph $G(A^{(n)})$ in the case where $A^{(n)}$ is the weighted adjacency matrix. From figure 2, we observe that $G(A^{(n)})$ is strongly connected. Furthermore, $C_1, C_2, \ldots, C_m$ are loops, $C_{12}, C_{23}, \ldots, C_{m-1,m}$ are circuits, and their weights are expressed by

$$
w(C_1) = Q_1^{(n)} \otimes E_1^{(n)}, \quad w(C_2) = Q_2^{(n)} \otimes E_2^{(n)}, \ldots, \quad w(C_m) = Q_m^{(n)} \otimes E_m^{(n)}, \\
w(C_{12}) = Q_1^{(n)} \otimes E_1^{(n)}, \quad w(C_{23}) = Q_2^{(n)} \otimes E_2^{(n)}, \ldots, \quad w(C_{m-1,m}) = Q_m^{(n)} \otimes E_m^{(n)}.
$$

(5)
Thus, from theorem 2.6, the tridiagonal $\mathbb{R}^{n \times n}$ matrix $A^{(n)}$ has an eigenvalue that coincides with the minimum value of the average weights of the loops $C_1, C_2, \ldots, C_m$, and the circuits $C_{12}, C_{23}, \ldots, C_{m-1,m}$.

To grasp an eigenvalue of matrix $A^{(n)}$ without the corresponding weighted digraph, we consider the characteristic polynomials of min-plus tridiagonal matrices $A^{(n)}$.

**Theorem 3.1.** The characteristic polynomials $g_{A^{(n)}}(t)$ of the tridiagonal $\mathbb{R}^{n \times n}$ matrices $A^{(n)}$ are factorized as

$$g_{A^{(n)}}(t) = (t \oplus Q_1^{(n)}) \otimes (t \oplus Q_2^{(n)} \oplus E_1^{(n)}) \otimes \cdots \otimes (t \oplus Q_m^{(n)} \oplus E_{m-1}^{(n)}).$$ (6)

**Proof.** If $m = 1$, then, because $E_0^{(n)} = \varepsilon$, we can easily derive $g_{A^{(n)}}(t) = \text{tropdet}(A^{(n)} \oplus t \otimes I) = t \oplus Q_1^{(n)}$. If $m = 2$, the characteristic polynomials $g_{A^{(n)}}(t)$ can be factorized as

$$g_{A^{(n)}}(t) = \begin{vmatrix} t \oplus Q_1^{(n)} & Q_1^{(n)} \otimes E_1^{(n)} \\ e & t \oplus Q_2^{(n)} \oplus E_1^{(n)} \end{vmatrix}$$

$$= (t \oplus Q_1^{(n)}) \otimes (t \oplus Q_2^{(n)} \oplus E_1^{(n)}) \otimes (Q_1^{(n)} \otimes E_1^{(n)})$$

$$= (t \oplus Q_1^{(n)}) \otimes (t \oplus Q_2^{(n)} \oplus E_1^{(n)}).$$

It is worth noting that the factors $Q_1^{(n)} \otimes E_1^{(n)}$ eventually vanish in the factorizations of $g_{A^{(n)}}(t)$. According to definition 2.2, cofactor expansions of tropical determinants can be considered as analogues of linear cofactor expansions. Applying the min-plus cofactor expansions along the $m$th column to $\text{tropdet}(A^{(n)} \oplus t \otimes I)$, we obtain

$$g_{A^{(n)}}(t) = (t \oplus Q_m^{(n)} \oplus E_{m-1}^{(n)}) \otimes \text{tropdet}(A_{m-1}^{(n)} \oplus t \otimes I)$$

$$\oplus (Q_{m-1}^{(n)} \otimes E_{m-2}^{(n)}) \otimes \text{tropdet}(A_{m-2}^{(n)} \oplus t \otimes I)$$

where $A_{k}^{(n)}$ are the principal $\mathbb{R}^{k \times k}$ submatrices of $A^{(n)}$, namely:

$$A_k^{(n)} := \begin{bmatrix} Q_k^{(n)} & Q_k^{(n)} \otimes E_k^{(n)} & \varepsilon & \cdots & \varepsilon \\
\varepsilon & Q_2^{(n)} \oplus E_1^{(n)} & Q_2^{(n)} \otimes E_2^{(n)} & \cdots & \varepsilon \\
\varepsilon & \varepsilon & Q_1^{(n)} \otimes E_1^{(n)} & \cdots & \varepsilon \\
\varepsilon & \cdots & \varepsilon & Q_{k-1}^{(n)} \otimes E_{k-1}^{(n)} & \varepsilon \\
\varepsilon & \cdots & \varepsilon & \varepsilon & Q_k^{(n)} \oplus E_{k-1}^{(n)} \end{bmatrix}$$
Under the assumption that \( \text{tropdet}(\mathcal{A}^{(n)}_{m-1} \oplus t \otimes I) = (t \oplus Q_1^{(n)}) \otimes (t \oplus Q_2^{(n)} \oplus E_1^{(n)}) \otimes \cdots \otimes (t \oplus Q_m^{(n)} \oplus E_{m-2}^{(n)}) \) follows that

\[
g_{\mathcal{A}^{(n)}}(t) \equiv (t \oplus Q_1^{(n)} \oplus E_{m-1}^{(n)}) \otimes \text{tropdet}(\mathcal{A}^{(n)}_{m-1} \oplus t \otimes I)
= (t \oplus Q_1^{(n)}) \otimes (t \oplus Q_2^{(n)} \oplus E_1^{(n)}) \otimes \cdots \otimes (t \oplus Q_m^{(n)} \oplus E_{m-1}^{(n)}),
\]

which implies that (6) holds for any \( m \). □

Theorem 3.1 suggests that \( Q_1^{(n)}, Q_2^{(n)} \oplus E_1^{(n)}, \ldots, Q_m^{(n)} \oplus E_{m-1}^{(n)} \) are the roots of the characteristic polynomials \( g_{\mathcal{A}^{(n)}}(t) \). Combining this with (5), we realize that the roots of the characteristic polynomials \( g_{\mathcal{A}^{(n)}}(t) \) coincide with the loop weights \( w(C_1), w(C_2), \ldots, w(C_m) \). Using proposition 2.3, we see that the minimum root of \( g_{\mathcal{A}^{(n)}}(t) \) is an eigenvalue \( \lambda^{(n)} \) of \( \mathcal{A}^{(n)} \). Thus, we can express the eigenvalues \( \lambda^{(n)} \) using the udToda variables \( Q_1^{(n)}, Q_2^{(n)}, \ldots, Q_m^{(n)} \) and \( E_1^{(n)}, E_2^{(n)}, \ldots, E_{m-1}^{(n)} \) as

\[
\lambda^{(n)} = \bigoplus_{k=1}^{m} Q_k^{(n)} \oplus \bigoplus_{k=1}^{m-1} E_k^{(n)}.
\]  

(7)

A study on the BBS also gives the significance of (7) as follows.

**Proposition 3.2 (Tokihiro et al [22])**. For any discrete time \( n \), conserved quantities of the udToda equation (4) are given by

\[
\mathcal{C} = \bigoplus_{k=1}^{m} Q_k^{(n)} \oplus \bigoplus_{k=1}^{m-1} E_k^{(n)}.
\]  

(8)

In the BBS, the conserved quantities \( \mathcal{C} \) are equal to the minimum number of sequential balls corresponding to the amplitude of the shortest soliton. Equations (7) and (8) imply that \( \lambda^{(n)} \) are constants independent of discrete time \( n \). In other words, the tridiagonal \( \mathbb{R}_{\min}^{m \times m} \) matrices \( \mathcal{A}^{(n)} \) have the same eigenvalue \( \lambda = \lambda^{(0)} \) as \( \mathcal{A} = \mathcal{A}^{(0)} \).

**Theorem 3.3**. For any discrete time \( n \), an eigenvalue \( \lambda = \lambda^{(0)} \) of the tridiagonal \( \mathbb{R}_{\min}^{m \times m} \) matrices \( \mathcal{A} = \mathcal{A}^{(0)} \) is expressed as

\[
\lambda = \bigoplus_{k=1}^{m} Q_k^{(n)} \oplus \bigoplus_{k=1}^{m-1} E_k^{(n)}.
\]

Theorem 3.3 also suggests that the udToda equation (4) gives the so-called min-plus similarity transformations of the \( \mathbb{R}_{\min}^{m \times m} \) matrices \( \mathcal{A}^{(n)} \).

Reconsidering the BBS properties is the key to analyzing the asymptotic behaviour as \( n \to \infty \) of \( Q_k^{(n)} \) and \( E_k^{(n)} \). In the BBS, the interchange of solitons’ amplitudes is completed at sufficiently large discrete time \( n \). Simultaneously, all the numbers of empty boxes between two neighboring solitons monotonically increase as \( n \) increases. Recalling here that \( Q_k^{(n)} \) correspond to the amplitudes of the solitons, the property concerning the magnitude of \( Q_k^{(n)} \) and \( E_k^{(n)} \) is known as follows.

**Proposition 3.4 (Tokihiro et al [22])**. In the udToda equation (4), there exists a discrete time \( N \) such that, for any \( n \geq N \),
Proposition 3.4 implies that $Q^{(n)}_1$ with sufficiently large $n$ is equal to the minimum value of all the udToda variables $Q^{(n)}_1, Q^{(n)}_2, \ldots, Q^{(n)}_m$ and $E^{(n)}_1, E^{(n)}_2, \ldots, E^{(n)}_{m-1}$. Combining this with theorem 3.3 leads to the main theorem of this paper.

**Theorem 3.5.** For an eigenvalue $\lambda$ of an $R_{m \times m}^{\text{min}}$ matrix $A$, it holds that, for sufficiently large $n$

$$Q^{(n)}_1 = \lambda.$$ 

Theorems 3.3 and 3.5 conclude that, under the initial settings that $Q^{(0)}_1, Q^{(0)}_2, \ldots, Q^{(0)}_m$ and $E^{(0)}_1, E^{(0)}_2, \ldots, E^{(0)}_{m-1}$ are given from the tridiagonal $R_{m \times m}^{\text{min}}$ matrix $A = A^{(0)}$, the udToda equation (4) generates matrix transformations of $A$ without changing an eigenvalue, and then makes the values of $Q^{(n)}_1$ be the eigenvalue after a sufficient discrete time lapse. Furthermore, in the weighted digraph $G(A^{(0)})$, the value of $Q^{(n)}_1$ with sufficiently large $n$ coincides with the minimum value of the average weights of all circuits.

Here, we present a numerical example to verify that the udToda equation (4) enables us to find an eigenvalue of $A = A^{(0)}$, which coincides with the minimum values of average weights of circuits in a weighted digraph corresponding to $A$, without considering tropical determinants and weighted digraphs. We set the initial values of the udToda equation (4) as $Q^{(0)}_1 = 6, Q^{(0)}_2 = 3, Q^{(0)}_3 = 1, E^{(0)}_1 = 18, \text{ and } E^{(0)}_2 = 15$. From (4), $E^{(0)}_0 = E^{(0)}_3 = \varepsilon$. The $R_{3 \times 3}^{\text{min}}$ tridiagonal matrix $A$ is given by

![Figure 3. The BBS (left) and weighted digraphs (right) corresponding to the $R_{3 \times 3}^{\text{min}}$ matrices $A^{(n)}$ at $n = 0, 6, 12$ under the initial values $Q^{(0)}_1 = 6, Q^{(0)}_2 = 3, Q^{(0)}_3 = 1,$ $E^{(0)}_1 = 18, \text{ and } E^{(0)}_2 = 15$ in the udToda equation (4).]
Figure 3 illustrates the position of the balls in the BBS and weighted digraphs corresponding to $A^{(n)}$ at discrete times $n = 0, 6, 12$. From figure 3, we observe that all the minimum values of the average weights of the circuits at $n = 0, 6, 12$ are 1 which is exactly the same as the eigenvalue of $A^{(n)}$. Table 1 shows the values of $Q_1^{(n)}$, $E_1^{(n)}$, $Q_2^{(n)}$, $E_2^{(n)}$, $Q_3^{(n)}$ at $n = 0, 1, \ldots, 12$, and suggests that the eigenvalue 1 corresponds to one of the udToda variables at any $n$ and appears in the values of $Q_1^{(n)}$ at $n \geqslant 10$.

### 4. Concluding remarks

In this paper, we showed that the ultradiscrete Toda (udToda) equation can be used to compute an eigenvalue of a tridiagonal $R_{m \times m}$ matrix where the eigenvalue is equal to the minimum value of the average weights of all circuits in the corresponding directed and weighted graph. Considering cofactor expansions of tropical determinants, we presented the factorization of the min-plus characteristic polynomial of the tridiagonal $R_{m \times m}$ matrix associated with the udToda equation. By relating the roots of the characteristic polynomial to the minimum value of the average weights of all circuits in the directed and weighted graph corresponding to the tridiagonal $R_{m \times m}$ matrix, we expressed an eigenvalue using the udToda variables at an arbitrary discrete time. Next, using known properties of the BBS, we proved that all tridiagonal $R_{m \times m}$ matrices generated using the udToda equation have at least a common eigenvalue, and that one of the udToda variables coincides with the eigenvalue after a sufficient discrete time lapse.

The discrete hungry Toda (dhToda) equations are extensions of the discrete Toda equation. The dhToda equations differ from the discrete Toda equation in that they are applicable to computing the eigenvalues of band matrices including tridiagonal matrices [4, 16]. Our future
work aims to relate the ultradiscrete hungry Toda equation [4] to eigenvalues of $R_{\text{m min}}^{m \times m}$ band matrices. This will be discussed in a separate paper.

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