Research Article

Approximation in weighted spaces of vector functions

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ABSTRACT. In this paper, we present the duality theory for general weighted space of vector functions. We mention that a characterization of the dual of a weighted space of vector functions in the particular case $V \subset C^+(X)$ is mentioned by J. B. Prolla in [6]. Also, we extend de Branges lemma in this new setting for convex cones of a weighted spaces of vector functions (Theorem 4.2). Using this theorem, we find various approximations results for weighted spaces of vector functions: Theorems 4.2-4.6 as well as Corollary 4.3. We mention also that a brief version of this paper, in the particular case $V \subset C^+(X)$, is presented in [3], Chapter 2, subparagraph 2.5.

Keywords: Nachbin family, weighted space of vector functions, $p-$Radon measure, polar set, extreme point, convex cone, antialgebraic set with respect to a pair $(M, C)$.

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Dedicated to Professor Francesco Altomare on the occasion of his 70th birthday.

1. INTRODUCTION

The weighted spaces of scalar functions was introduced and studied by L. Nachbin in [4] (see also [5]). We recall that if $V$ is a Nachbin family of upper semi-continuous functions on the locally compact spaces $X$, then the weighted space associated to $V$, denoted by $CV_0(X)$, is the set of all continuous functions $f \in V$ such that the function $f \cdot v$ vanishes at infinity. Any weight $v \in V$ generate a seminorm $p_v : CV_0(X) \rightarrow \mathbb{R}_+$ defined by $p_v(f) = \sup \{ v(x) \cdot |f(x)| : x \in X \}$. The locally convex topology defined by this family of seminorms is denoted by $\omega_V$ and it will be called the weighted topology on $CV_0(X)$. For some specific families of weights $V$, some different classes of continuous functions on a locally compact space are obtained, namely the functions with compact support, bounded functions, the functions vanishing at infinity, the rapidly decreasing functions at infinity and so on. A characterization of the dual space of the locally convex spaces $(CV_0(X), \omega_V)$ was obtained by W. H. Summers in [7]. More precisely, he showed that if $V \subset C^+(X)$ then, the dual space $[CV_0(X)]^*$ is isomorphic with the space $V \cdot M_b(X)$, where $M_b(X)$ is the space of all bounded Radon measure on $X$. A similar result for weighted spaces of vector functions, in the particular case $V \subset C^+(X)$, is mentioned by J. B. Prolla in [6]. In Theorem 3.1 of this paper, we obtain a characterization of the dual of a weighted space of vector functions in the general case of the upper semi-continuous weights. The key to getting this result is a new result of Measure Theory, namely Proposition 2.1, in which it is proved that if $U : K(X, E) \rightarrow \mathbb{R}$ is a $p-$Radon measure, then there exists a smallest

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positive Radon measure on $X$, denoted by $|U|$, such that

$$|U(f)| \leq \int p \circ fd|U|, \forall f \in K(X,E).$$

Using two fundamental tools in functional analysis: Hahn-Banach and Krein–Milman theorems, in 1959, Louis de Branges [1] give a nice proof of Stone-Weierstrass theorem on algebras of real continuous functions on a compact Hausdorff space. Some generalizations of de Branges lemma for weighted space of scalar functions was obtained in [2]. In the last part of this paper, we present a generalization of de Branges lemma for a convex cone in a weighted spaces of vector functions (Theorem 4.2). Using this theorem, we obtain various approximations results for weighted spaces of vector functions: Theorems 4.2-4.6 as well as Corollary 4.3.

2. WEIGHTED SPACES OF VECTOR FUNCTIONS

Let $X$ be a locally compact Hausdorff space, let $E$ be a locally convex complete space endowed with a family $P$ of seminorms of $E$. We denote by $C(X,E)$ the set of all continuous functions $f : X \to E$ and by $C_0(X,E)$ respectively $K(X,E)$, the set of continuous functions vanishing at infinity, respectively having compact support. We recall that a function $f : X \to E$ vanishes at infinity if $\lim_{x \to \infty} f(x) = 0$, i.e., for any $p \in P$ and any $\varepsilon > 0$, there exists a compact subset $K_{\varepsilon,p}$ of $X$ such that

$$p[f(x)] < \varepsilon, \forall x \in X \setminus K_{\varepsilon,p}.$$  

Further, we shall denote by $F_0(X,E)$ the set of all functions $f : X \to E$ vanishing at infinity.

**Definition 2.1.** A family $V$ of upper semi-continuous, non-negative functions on $X$ such that for any $v_1, v_2 \in V$ and any $\lambda \in \mathbb{R}$, $\lambda > 0$ there exists $w \in V$ such that

$$v_i(x) \leq \lambda \cdot w(x), \forall x \in X, i = 1, 2$$

will be called a Nachbin family on $X$. Any element of $V$ will be called a weight.

If $V$ is a Nachbin family of weights on $X$, we denote by

$$CV_0(X,E) = \{ f \in C(X,E); v \cdot f \in C_0(X,E), \forall v \in V \}.$$  

We endow this linear space with so called the weighted topology $\omega_{V,P}$, given by the family of seminorms $\|\cdot\|_{v,p}$ or $\|\cdot\|_{p_v}$ defined by

$$\|f\|_{p_v} = \|f\|_{v,p} = \sup \{ v(x) \cdot p[f(x)], \forall x \in X \}, \forall f \in CV_0(X,E).$$

A base of neighborhoods of the origin in $CV_0(X,E)$ is the family $(B_{v,p})_{v \in V, p \in P}$ given by

$$B_{v,p} = \left\{ f \in CV_0(X,E); \|f\|_{v,p} \leq 1 \right\}.$$  

Further, the space $CV_0(X,E)$ endowed with the weighted topology $\omega_{V,P}$ will be called the weighted space of vector functions. As in the scalar case, one can see that $K(X,E)$ is a dense subset of $CV_0(X,E)$ with respect to the weighted topology $\omega_{V,P}$. For any $p \in P$ and any $f \in K(X,E)$, we denote

$$\|f\|_p = \sup_{x \in X} p[f(x)].$$

Obviously, $\|f\|_p < \infty$ since $p : E \to \mathbb{R}_+$ is a continuous function on the locally compact space $E$ and $f(X) = f(K_f) \cup \{0\}$ is a compact subset of $E$, where $K_f$ denotes the support of $f$. If we endow $K(X,E)$ with the family of seminorms $\left( \|\cdot\|_p \right)_{p \in P}$, then $K(X,E)$ becomes a locally convex space and we shall denote by $\tau_P$ the topology given by these seminorms $\left( \|\cdot\|_p \right)_{p \in P}$. 

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Definition 2.2. A linear map $U : K(X, E) \to \mathbb{R}$ is called a $p$–Radon measure, where $p \in \mathbb{P}$, if for any compact subset $K \subset X$ there exists a positive number $\alpha_K$ such that for any $f \in K(X, E)$, $f = 0$ on $X \setminus K$, we have

$$|U(f)| \leq \alpha_K \cdot \|f\|_p.$$ 

If $\alpha_K$ does not depend on the compact $K$, then $U$ is called a $p$– bounded Radon measure. The smallest $\alpha \in \mathbb{R}_+$, such that $|U(f)| \leq \alpha \cdot \|f\|_p$ will be denoted by $\|U\|_p$.

Proposition 2.1. If $U : K(X, E) \to \mathbb{R}$ is a $p$–Radon measure, then there exists a smallest positive Radon measure on $X$, denoted by $|U|$, such that

$$|U(f)| \leq \int p \circ f d|U|, \quad \forall f \in K(X, E).$$

Moreover, for any function $\varphi \in K(X, \mathbb{R})$, the map $\varphi U : K(X, E) \to \mathbb{R}$ given by

$$\varphi U(\psi) = U(\varphi \cdot \psi), \quad \forall \psi \in K(X, E)$$

is a $p$– bounded Radon measure and we have

a) $\|\varphi U\|_p = |\varphi U(1)|$ and generally $\|U\|_p = |U(1)|$ if $U$ is $p$– bounded,

b) $|\varphi U| = |\varphi| \cdot |U|$, $\|\varphi U\|_p = |\varphi U(1)| = (|\varphi| \cdot |U(1)|)(1) = \int |\varphi| d|U|$.

Proof. Passing to a factorization, we may suppose that $p$ is a norm on $X$. We consider a relatively compact open subset $D$ of the locally compact space $X$ and for any $\varphi \in K(X, \mathbb{R})$, $\varphi \geq 0$ and $\text{supp}\varphi \subset D$, we put by definition

$$|U|(\varphi) = \sup \{U(\psi) ; \psi \in K(X, E), p \circ \psi \leq \varphi\} = \sup \{|U(\psi)| ; \psi \in K(X, E), p \circ \psi \leq \varphi\}.$$

Since $\overline{D}$ is compact and $\psi(x) = 0$, if $\varphi(x) = 0$, we deduce that $\psi = 0$ outside $\overline{D}$ and therefore there exists $\alpha \in \mathbb{R}_+$ such that $|U(\psi)| \leq \alpha \cdot \|\psi\|_p \leq \alpha \cdot \|\psi\|$, where $\|\varphi\|$ is the uniform norm of $\varphi$ on $X$. Hence $|U|(\varphi) \leq \alpha \cdot \|\varphi\|$ for all $\varphi \in K(X, \mathbb{R})$, $\varphi \geq 0$ and $\text{supp}\varphi \subset D$. We show now that for any $\varphi_i \in K(X, \mathbb{R})$, $\varphi_i \geq 0$, $\text{supp}\varphi_i \subset D$, $i = 1, 2$, we have

$$|U|(\varphi_1 + \varphi_2) = |U|(\varphi_1) + |U|(\varphi_2).$$

The inequality $|U|(\varphi_1 + \varphi_2) \geq |U|(\varphi_1) + |U|(\varphi_2)$ follows just from the definition. Let $\psi \in K(X, E), p(\psi) \leq \varphi_1 + \varphi_2$. For any $n \in \mathbb{N}^*$, we consider the functions $\psi_i \in K(X, E)$ given by

$$\psi_i = \frac{\varphi_i}{\varphi_1 + \varphi_2 + \frac{1}{n}} \cdot \psi, \quad i = 1, 2.$$ 

Obviously, we have successively

$$p(\psi_i) = \varphi_i \cdot \frac{p(\psi)}{\varphi_1 + \varphi_2 + \frac{1}{n}} \leq \varphi_i, \quad i = 1, 2,$$

$$\psi - (\psi_1 + \psi_2) = \frac{\psi}{\varphi_1 + \varphi_2 + \frac{1}{n}},$$ 

$$p(\psi - (\psi_1 + \psi_2)) \leq \frac{1}{n} \cdot p\left(\frac{\psi}{\varphi_1 + \varphi_2 + \frac{1}{n}}\right),$$ 

$$\text{supp}\left(\frac{\psi}{\varphi_1 + \varphi_2 + \frac{1}{n}}\right) \subset D, \quad p\left(\frac{\psi}{\varphi_1 + \varphi_2 + \frac{1}{n}}\right) \leq 1, \quad \left|U\left(\frac{\psi}{\varphi_1 + \varphi_2 + \frac{1}{n}}\right)\right| \leq \alpha,$$

$$|U(\psi) - U(\psi_1) - U(\psi_2)| \leq \frac{\alpha}{n}, \quad U(\psi) \leq U(\psi_1) + U(\psi_2) + \frac{\alpha}{n},$$

$$U(\psi) \leq |U|(\varphi_1) + |U|(\varphi_2) + \frac{\alpha}{n}, \quad \forall n \in \mathbb{N}^*,$$

$$U(\psi) \leq |U|(\varphi_1) + |U|(\varphi_2), \quad |U|(\varphi_1 + \varphi_2) = \sup \{U(\psi) ; \psi \in K(X, E), p(\psi) \leq \varphi_1 + \varphi_2\},$$

$$|U(\psi)| \leq |U|(\varphi_1) + |U|(\varphi_2).$$
Obviously, we have
\[ |U| (\lambda \cdot \varphi) = \lambda \cdot |U| (\varphi), \quad \forall \lambda \in \mathbb{R}_+ \]
and the map \( |U| : K^+(X, \mathbb{R}) \to \mathbb{R}_+ \) is a positive Radon measure on \( X \). Just from the definition, we have
\[ |U(\psi)| \leq |U| (p(\psi)), \quad \forall \psi \in K(X, E). \]
On the other hand, taking a positive Radon measure \( \mu \) on \( X \) such that \( |U(\psi)| \leq \int p(\psi) d\mu \) then for any \( \varphi \in K(X, \mathbb{R}), \varphi \geq 0 \), we have
\[
\begin{align*}
\int \varphi d\mu & \geq \int p(\psi) d\mu, \quad \forall \psi \in K(X, E), \; p(\psi) \leq \varphi, \\
\int \varphi d\mu & \geq |U(\psi)|, \quad \forall \psi \in K(X, E), \; p(\psi) \leq \varphi, \\
\int \varphi d\mu & \geq |U|(\varphi), \quad |U| \leq \mu \text{ on } K^+(X, \mathbb{R}).
\end{align*}
\]
a) For any \( \varphi \in K(X, \mathbb{R}) \), the map \( \varphi U : K(X, E) \to \mathbb{R} \) defined by \( \varphi U(\psi) = U(\varphi \cdot \psi) \) is linear and we have
\[ |\varphi U(\psi)| \leq \alpha_K \cdot \| \varphi \cdot \psi \|_p \leq \alpha_K \cdot \| \varphi \| \cdot \| \psi \|_p, \]
where \( K = \text{supp} \varphi \) and therefore \( \varphi U \) is a \( p- \) bounded Radon measure on \( K(X, E) \). Further, we have
\[
|\varphi U| (1) = \int 1 d|\varphi U| = \sup \left\{ \int h d|\varphi U| ; \ 0 \leq h \leq 1, \ h \in K(X, \mathbb{R}) \right\} = \sup \left\{ \left( \varphi U \right)(\psi) ; \ \psi \in K(X, \mathbb{R}), \ p(\psi) \leq 1 \right\} = \| \varphi U \|_p.
\]
(In fact, for any \( p- \) bounded Radon measure \( U' : K(X, E) \to \mathbb{R} \) we have, using the definition of \( |U'| \):
\[
\| U' \|_p = |U'| (1) = \int_X d|U'|,
\]
\[ |\varphi U| \leq |\varphi| \cdot |U| \]
follows immediately. Indeed, if \( h \in K(X, \mathbb{R}), \ h \geq 0 \) then,
\[
|\varphi U| (h) = \sup \left\{ U(\varphi \cdot \psi) ; \ p(\psi) \leq h \right\} \leq \sup \left\{ \| U \| (p(\varphi \cdot \psi)) ; \ p(\psi) \leq h \right\} = \sup \left\{ (|\varphi| \cdot |U|)(p(\psi)) ; \ p(\psi) \leq h \right\} = (|\varphi| \cdot |U|) (h).
\]
Hence \( |\varphi U| (h) \leq |\varphi| \cdot |U| (h) \) for any \( h \in K(X, \mathbb{R}), \ h \geq 0 \). For the converse inequality, we restrict ourself to the case \( \varphi \geq 0 \). Let us consider \( \psi \in K(X, E) \) such that \( p(\psi) \leq h \cdot \varphi \) and for any \( n \in \mathbb{N}^* \), we consider the function \( f_n \in K(X, E) \) defined by
\[ f_n = \frac{\psi}{\varphi + \frac{1}{n}}. \]
Obviously, \( p(f_n) \leq h \) and therefore
\[ |\varphi U| (h) \geq U(\varphi \cdot f_n), \ p(\varphi \cdot f_n) \leq h \cdot \varphi, \ p(\psi - \varphi \cdot f_n) \leq \frac{1}{n} \cdot p(h). \]
Since $\psi = 0$ outside $K = \text{supp}\varphi$, we have

$$\psi - \varphi \cdot f_n = 0 \text{ on } X \setminus K, \quad p(\psi - \varphi \cdot f_n) \leq \frac{1}{n} \cdot \|h\|, \quad |U(\psi - \varphi \cdot f_n)| \leq \alpha_K \cdot \frac{1}{n} \cdot \|h\|$$

and therefore

$$|\varphi U(h)| \geq U(\varphi \cdot f_n) \geq U(\psi) - \alpha_K \cdot \|h\| \cdot \frac{1}{n}, \quad |\varphi U(h)| \geq U(\psi).$$

But

$$(\varphi |U|)(h) = |U|(\varphi \cdot h) = \sup\{U(\psi); \psi \in K(X, E), \quad p(\psi) \leq h \cdot \varphi\}.$$ 

From the preceding two lines, we get $|\varphi U(h)| \geq (\varphi |U|)(h)$ and finally $|\varphi U| = |\varphi| \cdot |U|$. □

**Proposition 2.2.** Let $U : K(X, E) \to E$ be a $p-$ Radon measure, $f : X \to \mathbb{R}$ be an integrable function with respect to the positive Radon measure $|U|$ (i.e., $f \in L^1(|U|)$) and let $(\varphi_n)_n$ be a sequence in $K(X, \mathbb{R})$ such that $\lim_{n \to \infty} \varphi_n(x) = f(x), |U| - a.e.$ on $X$ and such that

$$\lim_{n \to \infty} \int |f - \varphi_n| \, d|U| = 0.$$ 

Then, the sequence of $p-$ bounded Radon measures $(\varphi_n U)_n$ is convergent to a $p-$ bounded Radon measure (depending of $f$ only), denoted by $fU$, i.e., $\lim_{n \to \infty} \|fU - \varphi_n U\|_p = 0$. Moreover, we have $|fU| = |f| \cdot |U|$.

**Proof.** Since $\lim_{n \to \infty} \int |f - \varphi_n| \, d|U| = 0$, we deduce that $\lim_{n,m \to \infty} \int |\varphi_n - \varphi_m| \, d|U| = 0$ and therefore, using Proposition 2.1, we have

$$\lim_{n,m \to \infty} \|\varphi_n U - \varphi_m U\|_p = \lim_{n,m \to \infty} \int |\varphi_n - \varphi_m| \, d|U| = 0.$$ 

Hence for any $\psi \in K(X, E)$, the sequence $(\varphi_n U(\psi))_n$ of real numbers is convergent to a number denoted $fU(\psi)$ and for any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}^*$ such that

$$|\varphi_n U(\psi) - \varphi_m U(\psi)| \leq \|\varphi_n U - \varphi_m U\|_p \cdot \|\psi\|_p \leq \varepsilon \cdot \|\psi\|_p, \quad \forall n, m \geq n_\varepsilon,$$

$$|fU(\psi) - \varphi_m U(\psi)| \leq \varepsilon \cdot \|\psi\|_p, \quad \forall m \geq n_\varepsilon,$$

$$|fU(\psi)| \leq |\varphi_m U(\psi)| + \varepsilon \cdot \|\psi\|_p \leq \left(\|\varphi_m U\|_p + \varepsilon\right) \cdot \|\psi\|_p.$$ 

Hence $fU$ is a $p-$ bounded Radon measure on $K(X, E)$, $\lim_{m \to \infty} \|fU - \varphi_m U\|_p = 0$ (Particularly if $f = 0$ $|U|$ a.e., from the relation $\lim_{n \to \infty} \int |f - \varphi_n| \, d|U| = 0$, we deduce $\lim_{n \to \infty} \int |\varphi_n| \, d|U| = 0$ and therefore $\lim_{n \to \infty} \|\varphi_n U\|_p = \lim_{n \to \infty} \int |\varphi_n| \, d|U| = 0$, $\lim_{n \to \infty} (\varphi_n U)(\psi) = 0$, $\forall \psi \in K(X, E)$. This shows that the element $fU$, previously defined, depends only on $f$, does not depend on the choice of the sequence $(\varphi_n)_n$ tending to $f$). Let now $h \in K(X, \mathbb{R})$, $0 \leq h \leq 1$ and let $\psi \in K(X, E)$ be such that $p(\psi) \leq h$. We have

$$|fU(\psi) - \varphi_n U(\psi)| \leq \|fU - \varphi_n U\|_p \cdot \|\psi\|_p \leq \|fU - \varphi_n U\|_p,$$

$$(\varphi_n U)(\psi) - \|fU - \varphi_n U\|_p \leq fU(\psi) \leq \varphi_n U(\psi) + \|fU - \varphi_n U\|_p,$$

$$|\varphi_n U| (h) - \|fU - \varphi_n U\|_p \leq |fU|(h) \leq |\varphi_n| \cdot |U|(h) + \|fU - \varphi_n U\|_p.$$ 

Using Proposition 2.1 b), we deduce that

$$|\varphi_n \cdot |U|(h) - \|fU - \varphi_n U\|_p \leq |fU|(h) \leq |\varphi_n| \cdot |U|(h) + \|fU - \varphi_n U\|_p.$$
\begin{align*}
\int |\varphi_n| \cdot \text{hd} |U| - \|fU - \varphi_n U\|_p & \leq |fU| (h) \leq \int |\varphi_n| \cdot \text{hd} |U| + \|fU - \varphi_n U\|_p,
\end{align*}

Passing to the limit on \(n\), we get
\[
\int |f| \cdot \text{hd} |U| \leq |fU| (h) \leq \int |f| \cdot \text{hd} |U|,
\]
\[
|fU| (h) = \int |f| \cdot \text{hd} |U| = |f| \cdot |U| (h).
\]
The last equality holds for \(0 \leq h \leq 1\) and therefore for all \(h \in K(X, \mathbb{R})\), \(h \geq 0\), i.e.,
\[
|fU| = |f| \cdot |U|.
\]

3. ON THE DUAL OF WEIGHTED SPACES OF VECTOR FUNCTIONS

Let \(E, P, X, V\) and \(V\) as in the preceding section. For any \(p \in P\) and \(v \in V\), let
\[
B_{v,p} = \{ f \in CV_0(X,E); p_v(f) \leq 1 \},
\]
where \(p_v(f) = \sup \{ v(x) \cdot p[f(x)]; \forall x \in X \} = \| f \|_{v,p}, \forall f \in CV_0(X,E)\). The linear vector space \(CV_0(X,E)\) endowed with the family \((p_v)_{p \in P, v \in V}\) of seminorms is a locally convex space whose fundamental system of neighborhoods of the origin is just the family \((B_{v,p})_{v \in V, p \in P}\). We recall that we have denoted by \(\omega_{V,P}\) the weighted topology on \(CV_0(X,E)\) given by the family of seminorms \((p_v)_{p \in P, v \in V}\). It is no lost of generality if we suppose that for any real number \(\alpha, \alpha > 0\), we have \(\alpha \cdot p \in P, \alpha \cdot v \in V\) for any \(p \in P\) and any \(v \in V\). So the dual of the locally convex space \((CV_0(X,E), \omega_{V,P})\) is the set \(\bigcup_{v \in V, p \in P} B^0_{v,p}\), where
\[
B^0_{v,p} = \{ T : CV_0(X,E) \rightarrow \mathbb{R}; T\text{ linear, } T(f) \leq 1, \forall f \in B_{v,p} \}.
\]
If we denote by \([CV_0(X,E)]^*\) this dual, then for any subset \(M\) of \(CV_0(X,E)\) (respectively of \([CV_0(X,E)]^*\)), we denote by \(M^0\) the polar of \(M\) i.e.,
\[
M^0 = \{ T \in [CV_0(X,E)]^*; T(m) \leq 1, \forall m \in M \}
\]
respectively
\[
M^0 = \{ f \in CV_0(X,E); m(f) \leq 1, \forall m \in M \}.
\]
The map on \(CV_0(X,E) \times [CV_0(X,E)]^* \rightarrow \mathbb{R}, (f,T) \rightarrow \langle f,T \rangle = T(f)\) is a natural duality between the linear space \(CV_0(X,E)\) and \([CV_0(X,E)]^*\). The smallest topology on \([CV_0(X,E)]^*\) making continuous the maps
\[
T \rightarrow \langle f,T \rangle : [CV_0(X,E)]^* \rightarrow \mathbb{R}, \forall f \in CV_0(X,E),
\]
is the weak topology on \([CV_0(X,E)]^*\). It is known (Alaoglu’s Theorem) that for any \((p, v) \in P \times V\), the set \(B^0_{p,v}\) is a weakly compact subset of \([CV_0(X,E)]^*\). We know also that the topological space \([CV_0(X,E)]^*\) is a Hausdorff one with respect to this weak topology. Moreover, since \(K(X,E)\) is a dense subset of \(CV_0(X,E)\) with respect to the weighted topology \(\omega_{V,P}\), we deduce that
1) any continuous linear functional \(L : CV_0(X,E) \rightarrow \mathbb{R}\) is completely determined by its restriction to \(K(X,E)\),
2) the smallest topology on \([CV_0(X,E)]^*\) making continuous all linear functionals
\[
T \rightarrow \langle f,T \rangle : [CV_0(X,E)]^* \rightarrow \mathbb{R}, \forall f \in K(X,E),
\]
is also a Hausdorff one and therefore its restriction to \(B^0_{p,v}\) coincides with the restriction to \(B^0_{p,v}\) of the weak topology on \([CV_0(X,E)]^*\).
We conclude that any element of the dual of the locally convex space \((K(X, E), \omega_{V,P} | K(X, E))\) may be uniquely extended to an element of \([CV_0(X, E)]^*\). The following assertion characterizes the elements of \([CV_0(X, E)]^*\) in terms of Radon measures on \(K(X, E)\). With the above notations, we have

**Theorem 3.1.** For any \((p, v) \in P \times V\), we have

a) The restriction of any element \(T \in B^0_{p,v} \) to \(K(X, E)\) is a \(p\)-Radon measure on \(K(X, E)\) such that the function \(1/T\) is integrable with respect to the positive Radon measure \(|T|\) on \(X\). Moreover, the following relation holds:

\[
\int \frac{1}{v} d|T| = \|T\|_{p,v} = \sup \{T(f); \ f \in B_{p,v}\},
\]

b) For any \(p\)-Radon measure \(U\) on \(K(X, E)\) such that the function \(1/U\) is \(|U|\) – integrable, there exists \(T \in B^0_{p,v}\) such that \(U\) is the restriction of \(T\) to \(K(X, E)\).

**Proof.** a) Let \(T \in B^0_{p,v}\) and let \(K\) be a compact subset of \(X\). Since \(v : X \rightarrow [0, \infty)\) is an upper semi-continuous function, its upper bound \(\alpha_K\) on \(K\) is finite. Let \(\varphi \in K(X, E)\) such that \(\varphi = 0\) on \(X \setminus K\). We have

\[
\sup \{v(x) \cdot p(\varphi(x)) : x \in X\} \leq \alpha_K \cdot \sup \{p(\varphi(x)) : x \in X\} = \alpha_K \cdot \|\varphi\|_p,
\]

\[
\frac{\varphi}{\alpha_K \cdot \|\varphi\|_p} \in B_{p,v}, \quad T \left(\frac{\varphi}{\alpha_K \cdot \|\varphi\|_p}\right) \leq 1, \quad |T(\varphi)| \leq \alpha_K \cdot \|\varphi\|_p,
\]

i.e., the restriction of \(T\) to \(K(X, E)\), denoted also by \(T\), is a \(p\)-Radon measure. We have

\[
\|T\|_{p,v} = \sup \{T(f); \ f \in CV_0(X, E), \ p_v(f) \leq 1\} = \sup \{T(f); \ f \in K(X, E), \ p_v(f) \leq 1\} = \sup \left\{T(f); \ f \in K(X, E), \ p(f) \leq \frac{1}{v}\right\} = \int \frac{1}{v} d|T|.
\]

b) Let \(U\) be a \(p\)-Radon measure on \(K(X, E)\) such that the function \(1/U\) is \(|U|\) – integrable. Then, we have

\[
\infty > \int \frac{1}{v} d|U| = \sup \left\{\int \varphi d|U| \ ; \ \varphi \in K(X, \mathbb{R}), \ 0 \leq \varphi \leq \frac{1}{v}\right\} = \sup \left\{U(\psi); \ \psi \in K(X, E), \ p(\psi) \leq \varphi\right\} = \sup \left\{U(\psi); \ \psi \in K(X, E), \ p(\psi) \leq \frac{1}{v}\right\} = \sup \left\{U(\psi); \ \psi \in K(X, E), \ v(x) \cdot p(\varphi(x)) \leq 1\right\} = \|U\|_{p,v}.
\]

□

**Remark 3.1.** From the above considerations, we deduce that:

The elements \(T \in B^0_{p,v}\) are \(p\)-Radon measure on \(K(X, E)\) such that the function \(1/v\) is \(|T|\) – integrable and \(\|T\|_{p,v} = \int \frac{1}{v} d|T| \leq 1\).
Proposition 3.3. Let $T$ be a $p-$ Radon measure, $T \in B_{p,v}^0$. If $f \in CV_0(X,E)$, then

$$|T(f)| \leq \int p(f)d|T|.$$ 

Proof. Let $(\psi_n)_n$ be a sequence in $K(X,E)$ such that $\lim_{n \to \infty} \|f - \psi_n\|_{p,v} = 0$. We know that $|T(\psi_n)| \leq \int p(\psi_n)d|T|$ and $T(f) = \lim_{n \to \infty} T(\psi_n)$. On the other hand

$$p(f - \psi_n) \leq \frac{\|f - \psi_n\|_{p,v}}{v} \text{ on } X,$$

$$\int p(f - \psi_n)d|T| \leq \|f - \psi_n\|_{p,v} \cdot \int \frac{1}{v}d|T| \leq \|f - \psi_n\|_{p,v},$$

$$\int |p(f) - p(\psi_n)|d|T| \leq \int p(f - \psi_n)d|T| \leq \|f - \psi_n\|_{p,v},$$

$$\int p(f)d|T| = \lim_{n \to \infty} \int p(\psi_n)d|T|.$$ 

Hence

$$|T(f)| = \lim_{n \to \infty} |T(\psi_n)| \leq \lim_{n \to \infty} \int p(\psi_n)d|T| = \int p(f)d|T|.$$ 

Corollary 3.1. If $T \in B_{p,v}^0$ and $f \in CV_0(X,E)$ is such that $f = 0$ on supp $|T|$, then $T(f) = 0.$

4. Lemma de Branges and approximation results

In this section, we preserve all notations used in the preceding paragraphs. For any subset $A \subset CV_0(X,E)$, we denote by $A^0$ the polar of $A$, i.e.,

$$A^0 = \{T \in [CV_0(X,E)]^*; \ T(a) \leq 1, \ \forall a \in A\}.$$

If $C$ is a convex cone of the real vector space $CV_0(X,E)$ then, one can see that

$$C^0 = \{T \in [CV_0(X,E)]^*; \ T(c) \leq 0, \ \forall c \in C\}.$$

Theorem 4.2. Let $C$ be a convex cone in $CV_0(X,E)$, $p \in P$, $v \in V$ and let $L \in B_{p,v}^0 \cap C^0$, $L \neq 0$ be an extreme point of the convex and compact subset $B_{p,v}^0 \cap C^0$. If $h \in C(X,[0,1])$ is such for any $c \in C$, we have $h \cdot c|\sigma(|L|) \in C|\sigma(|L|)$ and $(1 - h) \cdot c|\sigma(|L|) \in C|\sigma(|L|)$, then $h$ is constant on $\sigma(|L|)$ the support of the positive Radon measure $|L|$ on $X$.

Proof. Since $L \neq 0$ and $L$ is an extreme point of the subset $B_{p,v}^0 \cap C^0$, we have $\|L\|_{p,v} = \int \frac{1}{v}d|L|$. If $h$ is an arbitrary element in$C(X,[0,1])$, then the map $hL : K(X,E) \to \mathbb{R}$, given by $hL(\psi) = L(h \cdot \psi)$, is a $p-$Radon measure on $K(X,E)$. It is not so difficult to show, using the definition, that $|hL| = |h| \cdot |L|$. Obviously, the function $\frac{1}{v}$ is $|h| \cdot |L|$ - integrable and using Remark 3.1 and the relations

$$\|hL\|_{p,v} = \int \frac{1}{v}d|hL| = \int \frac{h}{v}d|L| \leq \int \frac{1}{v}d|L| \leq 1,$$

we get $hL \in B_{p,v}^0$. Analogously, the map $(1 - h)L : K(X,E) \to \mathbb{R}$ given by $(1 - h)L(\psi) = L((1 - h) \cdot (\psi))$ is a $p-$Radon measure and

$$\|(1 - h)L\|_{p,v} = \int \frac{1 - h}{v}d|L| \leq \int \frac{1}{v}d|L| = 1, \ (1 - h)L \in B_{p,v}^0.$$ 

If we denote $\alpha = \|hL\|_{p,v} = \int \frac{h}{v}d|L|$, $\beta = \|(1 - h)L\|_{p,v} = \int \frac{1 - h}{v}d|L|$, we have $\alpha + \beta = \int \frac{1}{v}d|L| = 1$. We remark also that the function $\frac{1}{v}$ is strictly positive on $X$. If $\alpha = 0$, then
\( h = 0 \) \( |L| \) a.e. on \( \sigma(\overline{|L|}) \). Since the function \( h \) is continuous, it results that \( h = 0 \) on \( \sigma(\overline{|L|}) \), i.e., \( h \) is constant on \( \sigma(\overline{|L|}) \). Analogously, if \( \beta = 0 \), we obtain \( h = 1 \) on \( \sigma(\overline{|L|}) \), i.e., \( h \) is constant on \( \sigma(|L|) \). We suppose further \( \alpha \neq 0 \), \( \beta \neq 0 \) and we denote

\[
L_1 = \frac{1}{\alpha} \cdot hL, \quad L_2 = \frac{1}{\beta} \cdot (1 - h)L.
\]

Obviously, \( \|L_i\|_{p,v} = 1 \), \( i = 1, 2 \) and \( \alpha \cdot L_1 + \beta \cdot L_2 = L \). We show now that \( L_i \in C^0 \), \( i = 1, 2 \), if for any \( c \in C \) there exist \( c_1, c_2 \in C \) such that \( h \cdot c = c_1 \), \( (1 - h) \cdot c = c_2 \) on \( \sigma(|L|) \). Since the functions \( h \cdot c \), \( (1 - h) \cdot c \), \( c_1 \), \( c_2 \) belong to \( CV_0(X,E) \) and \( h \cdot c = c_1 \) on \( \sigma(|L|) \), respectively \( (1 - h) \cdot c = c_2 \) on \( \sigma(|L|) \), using Corollary 3.1, we get

\[
L(h \cdot c) = L(c_1) \leq 0, \quad L((1 - h) \cdot c) = L(c_2) \leq 0,
\]

\[
L_1(c) = \frac{1}{\alpha} \cdot L(h \cdot c) = \frac{1}{\alpha} \cdot L(c_1) \leq 0, \quad L_2(c) = \frac{1}{\beta} \cdot L((1 - h) \cdot c) = \frac{1}{\beta} \cdot L(c_2) \leq 0.
\]

Hence \( L_1, L_2 \) belong to the set \( B^0_{p,v} \cap C^0 \) and since \( L = \alpha \cdot L_1 + \beta \cdot L_2 \), we get \( L_1 = L_2 = L \). Hence \( |L_i| = |L| \), i.e., the measures \( \frac{h}{\alpha} \cdot |L| \) and \( |L| \) coincide and therefore \( \frac{h}{\alpha} = 1 \) almost everywhere on \( \sigma(|L|) \). But \( h \) is continuous and hence \( h = \alpha \) on \( \sigma(|L|) \). \( \square \)

**Definition 4.3.** A subset \( M \subset C(X, [0,1]) \) is called complemented, if for any \( h \in M \), the function \( 1 - h \) belongs to \( M \). If \( C \subset CV_0(X,E) \) is a convex cone and \( M \subset C(X, [0,1]) \) is a complemented family, then a subset \( S \subset X \) is called antialgebraic with respect to the pair \( (M,C) \) (or simpler \( (M,C) \)-antialgebraic), if any \( h \in M \) such that the restriction to \( S \) of the functions \( h \cdot c \) and \( (1 - h) \cdot c \) belong to the restriction of \( C \) to \( S \) (i.e., \( h \cdot c |S \in C |S \), \( (1 - h) \cdot c |S \in C |S \) for any \( c \in C \), is a constant function on \( S \).

We can reformulate de Branges lemma (Theorem 4.2) as follows:

**Corollary 4.2.** For any extreme point \( L \) of \( B^0_{p,v} \cap C^0 \), the support \( \sigma(|L|) \) of the positive Radon measure \( |L| \) on \( X \) is an antialgebraic subset with respect to the pair \( (C(X, [0,1]), C) \). Further, we denote by \( S \) the family of all subsets of \( X \) antialgebraic with respect to the pair \( (M,C) \).

The following assertions are almost obvious.

i) \( \{x\} \subset S, \forall x \in X \),

ii) \( S_1, S_2 \subset S, S_1 \cap S_2 \neq \emptyset \Rightarrow S_1 \cup S_2 \subset S \),

iii) \( S \subset S \Rightarrow \overline{S} \subset S \),

iv) For any upper directed family \( (S_\alpha)_{\alpha \in I} \) from \( S \), we have \( \bigcup_{\alpha \in I} S_\alpha \subset S \).

If for any \( x \in X \), we denote by \( S_x = \overline{\{s \in S; s \in S, x \in s\}} \), then we have

\[
S_x = \overline{S_x} \subset S, \quad S_x \cap S_y = \emptyset \quad \text{if} \quad S_x \neq S_y.
\]

The family \( (S_x)_{x \in X} \) is a partition of \( X \) and for any \( S \subset S \) there exists \( x \in X \) such that \( S \subset S_x \). For the general theory of duality, we have for any convex cone \( C, C \subset CV_0(X,E) \), the closure \( \overline{C} \) in \( CV_0(X,E) \) with respect to the weighted topology \( \omega_{p,v} \) coincides with the bipolar of \( C \) i.e., \( \overline{C} = C^{00} \). In the our special case, we have the following general approximation theorem.

**Theorem 4.3.** If \( C \subset CV_0(X,E) \) is a convex cone, then the closure of \( C \) in \( (CV_0(X,E), \omega_{p,v}) \) is given by

\[
\overline{C} = \left\{ f \in CV_0(X,E); \ f |\sigma(|L|) \in \overline{C} |\sigma(|L|)}, \forall L \in Ext \left( B^0_{p,v} \cap C^0 \right), \forall v \in V, \forall p \in P \right\}.
\]
Proof. We show only that for any function \( g \in CV_0(X, E) \setminus \mathcal{C} \) there exist \( p \in P \), \( v \in V \) and \( L \in Ext(B_{p,v}^0 \cap C^0) \) such that \( g \sigma(|L|) \notin \overline{C|\sigma(L)|} \). Indeed, using Hahn-Banach separation theorem, there exists \( T \in [CV_0(X, E)]^* \) such that \( T \in C^0 \) and \( T(g) > 0 \). Let \( p \in P \) and \( v \in V \) be such that \( |T(f)| \leq \|f\|_{p,v} \), \( \forall f \in CV_0(X, E) \) i.e., \( |T| \left( \frac{1}{L} \right) \leq 1 \). Hence \( T \in B_{p,v}^0 \cap C^0 \). Since \( B_{p,v}^0 \cap C^0 \) is a compact convex subset of \([CV_0(X, E)]^* \) with respect to the weak topology and \( T(g) > 0 \), it follows from Krein-Milman theorem that there exists \( L \in Ext(B_{p,v}^0 \cap C^0) \) such that \( L(g) > 0 \). Since \( L \in C^0 \), we deduce that \( \int \varphi d|L| \geq 0 \) for any \( \varphi \in C|\sigma(|L|)|. \) Hence \( g \sigma(|L|) \notin \overline{C|\sigma(|L|)} \). 

Let now \( M \subset C(X, [0,1]) \) be a complemented family and for any \( x \in X \) let \( S_x \) be the greatest \((M, C)\)–antialgebraic subset of \( X \) containing \( x \).

**Theorem 4.4.** If \( C \subset CV_0(X, E) \) is a convex cone, then the closure of \( C \) in \((CV_0(X, E), \omega_{p,v})\) is given by 
\[
\overline{C} = \left\{ f \in CV_0(X, E); \ f \big| S_x \in \overline{C|S_x|}, \forall x \in X \right\}.
\]

Proof. For any \( p \in P \), \( v \in V \) and any extreme point \( L \) of the compact convex subset \( B_{p,v}^0 \cap C^0 \), the support \( \sigma(|L|) \) is a \((M, C)\)–antialgebraic subset of \( X \). If we choose a point \( x \in \sigma(|L|) \), then \( \sigma(|L|) \subset S_x \), and therefore if \( f \big| S_x \in \overline{C|S_x|} \), we have also \( f \big| \sigma(L) \in \overline{C|\sigma(L)} \). Further, we may use Theorem 4.3. 

**Theorem 4.5.** If \( M \subset C(X, [0,1]) \) is a complemented family and the convex cone \( C \subset CV_0(X, E) \) is stable with respect to the multiplication of \( M \) (i.e., \( c \cdot m \in C \), \( \forall c \in C \), \( m \in M \)), then we have 
\[
\overline{C} = \left\{ f \in CV_0(X, E); \ f \big| [x]_M \in \overline{C| [x]|_M}, \forall x \in X \right\},
\]
where for any \( x \in X \) we denote \([x]_M = \{ y \in X; \ m(y) = m(x), \forall m \in M \} \).

Proof. Using just the definitions and previous notations, we deduce that for any \( x \in X \) we have \([x]_M = S_x \). Further, we use Theorem 4.4. 

The following assertion needs to define so called “section in \( C \)” by the points of \( X \), namely to consider the following convex cone \( C(x) \in E \) given by 
\[
C(x) = \{ c(x); \ c \in C \}
\]
and also its closure \( \overline{C(x)} \) in \( E \). Certainly the starting convex cone \( C \) in \( CV_0(X, E) \) may be a linear subspace and in this case \( C(x) \) is a linear subspace in \( E \).

**Theorem 4.6.** If \( M \subset C(X, [0,1]) \) is a complemented family and the convex cone \( C \subset CV_0(X, E) \) is stable with respect to the multiplication with elements of \( M \) and \( M \) separates the points of \( X \), i.e., for any \( x, y \in X \) there exists \( m \in M \) such that \( m(x) \neq m(y) \), then we have 
\[
\overline{C} = \left\{ f \in CV_0(X, E); \ f(x) \in \overline{C(x)}, \forall x \in X \right\}.
\]
Indeed, in this case, for any \( x \in X \), we have \([x]_M = \{ x \} \) and we close the proof applying Theorem 4.5.

**Corollary 4.3.** If \( M \subset C(X, [0,1]) \) is a complemented family, separating the points of \( X \) and \( W \subset CV_0(X, E) \) is a linear subspace which is stable with respect to the multiplication with elements of \( M \) and for any \( x \in X \) the section \( W(x) \) is a dense subset of the locally convex space \((E, P)\), then 
\[
\overline{W} = CV_0(X, E).
\]
Remark 4.2. For the scalar case $E = \mathbb{R}$, the density of $W(x)$ in $\mathbb{R}$ is automatically fulfilled unless the case where $W(x) = \{0\}$ for the points $x$ of a closed subset $F \subset X$. In this case, we have

$$\overline{W} = \{f \in CV_0(X); f = 0 \text{ on } F\}.$$ 

Even this assertion may be drawn from Theorem 4.6 as a particular case where there exists $F \subset X$ such that the section of $C$ by $x$ is trivial for all $x \in F$ i.e., $C(x) = \{0_E\} \quad \forall x \in F$. Anyway Theorem 4.6 may be used in different manners to obtain density results.

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