Spacetime Variable Superstring Vacua

(Calabi-Yau Cosmic Yarn)

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ABSTRACT
In a general superstring vacuum configuration, the ‘internal’ space (sector) varies in spacetime. When this variation is non-trivial only in two space-like dimensions, the vacuum contains static cosmic strings with finite energy per unit length and which is, up to interactions with matter, an easily computed topological invariant. The total spacetime is smooth although the ‘internal’ space is singular at the center of each cosmic string. In a similar analysis of the Wick-rotated Euclidean model, these cosmic strings acquire expected self-interactions. Also, a possibility emerges to define a global time in order to rotate back to the Lorentzian case.

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1. Introduction, Results, and Summary

The past several years have witnessed a demographic explosion of superstring vacua with N=1 supergravity in 4-dimensional Minkowski spacetime. To enable various anomaly cancellations, the model must include a $c_L = c_R = 9$ ‘internal’ model, for which a geometric description may not be known, but if it is—it must be a compact complex Ricci-flat (Calabi-Yau) 3-fold, possibly singular but in a rather restricted way; see below. Most of these constructions are constant, i.e., the ‘internal’ sector of the theory is the same throughout the 4-dimensional spacetime. Clearly, this is a rather special Ansatz and attempts have been made to construct spacetime variable vacua (see [1] and the references therein); moreover, these may lead to cosmologically more interesting and possibly realistic effective models.

In the most general situation, the internal space varies throughout 4-dimensional spacetime and couples to the geometry of the spacetime.

In this article, we will present some straightforward but general ways of endowing families of Calabi-Yau vacua with spacetime dependence. We explore a class of superstring vacua in which the 10-dimensional spacetime $M$ has the structure of a fibre space over the 4-dimensional spacetime $X$. That is, there exists a projection map

$$\pi : M \longrightarrow X,$$

(1.1)

which associates a fibre $M_x \subset M$ to every point $x \in X$. All fibres $M_x = \pi^{-1}(x)$ are compact complex 3-spaces, they vary over the 4-dimensional spacetime and we require the total spacetime $M$ to be smooth. It turns out that in general $M_x$ are smooth at every point $x \in X$ in the 4-dimensional spacetime, except over a two-dimensional subset $S \subset X$.

In such an Ansatz, we interpret $S$ as the union of the world sheets of one or more cosmic strings in the 4-dimensional spacetime $X$. In the full 10-dimensional spacetime this looks as follows. Consider the collection of fibres $M_x$ over the cosmic string, that is, for which $x \in S \subset X$. Generically, each fibre has one or perhaps several isolated singular points and a little thought reveals that these singular points sweep out a 2-dimensional surface $\tilde{S} \subset M$, of which $S = \pi(\tilde{S})$ is the projection to the 4-dimensional spacetime $X$; see Fig. 1.

If the total 10-dimensional spacetime $M$ is smooth and has a Ricci-flat metric, $g$, it qualifies as a superstring vacuum [1]. At every point $m \in M$ where the projection $\pi$ is non-singular and using the metric $g$, we can define the ‘horizontal’ tangent space to be the orthogonal complement of the tangent space to the ‘vertical’ fibre $M_x$. Furthermore, each (compact, Calabi-Yau) fibre $M_x$ for which $x$ is not a singular value of $\pi$ admits a holomorphic 3-form, and therefore a Ricci-flat metric. However the restriction of $g$ to $M_x$ is not in general Ricci flat. Finally, we can define a 4-dimensional space-time metric by restricting the 10-dimensional metric to the horizontal subspaces and integrating over the fibers.

\[1\] This is easy to show along the arguments of [2,3]; in fact, Ricci-flatness appears to be a general requirement for string theories [4].
It turns out that vacua of this kind can be constructed in abundance using certain techniques of (complex) algebraic geometry. In particular, following the analysis of Ref. [1], we first focus on the case where the ‘internal’ Calabi-Yau space varies only in two space-like dimensions. The ‘internal’ space then becomes singular at a prescribed number of points of this surface, each of which sweeps out a static cosmic string in the third spatial dimension. Certain physically relevant properties of these cosmic strings can be determined straightforwardly.

The presence of these strings curves the spacetime and induces a spacetime metric, which in fact turns out to be related to a flat metric by a conformal factor whose logarithm is the Kähler potential of the Weil-Petersson-Zamolodchikov metric. Several local properties of this spacetime metric are then computed easily. In the general case, when the ‘internal’ space varies over the entire 4-dimensional spacetime, we have not found any comparably simple characterization of the 4-dimensional space-time metric.

We proceed as follows: Section 2 presents the problem from the point of view of a “4-dimensional physicist”, analyzing the effect of varying the ‘internal’ space over spacetime as a 4-dimensional field theory problem. Section 3 offers a more general point of view by considering the geometrical structure of the total 10-dimensional spacetime. Some simple calculations are described in Section 4 and Section 5 presents some concrete models and corresponding calculations. Section 6 remarks on certain unusual global properties of such configurations and a preliminary analysis of the fully variable 10-dimensional spacetime case and our closing remarks are presented in Section 7.

2. Varying Vacua in Spacetime

While spacetime variable superstring vacua are clearly more general solutions of the superstring theory, such a variation may produce physically unacceptable effects: physical observables such as the mass of the electron may vary too much. Nevertheless, we consider varying the ‘internal’ sector over spacetime and leave checking the physical acceptability a posteriori.

2.1. Variable ‘internal’ space field theory

Suitable ‘internal’ models are 2-dimensional (2, 2)-superconformal $c_L = c_R = 9$ field theories, all of which have marginal operators. Through these, any such model may be deformed into a neighboring one, that is, such operators chart a local deformation space. In spacetime variable superstring vacua, these marginal deformations acquire spacetime dependence.

Once the parameters of the ‘internal’ model have become spacetime variable, so-called moduli fields $t^\alpha(x)$, they are naturally interpreted as maps that immerse the spacetime into the moduli space $\mathcal{B}$ of the ‘internal’ model. Their dynamics is then governed by the usual $\sigma$-model (harmonic map) action. To discuss this, focus on the natural interaction of

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2) Unless otherwise specified, by ‘spacetime’ we mean the 4-dimensional one.
with gravity which follows simply by requiring the action to be invariant under general coordinate reparametrizations. So consider

\[ A_{\text{eff}} = \int d^4x \sqrt{-g} \left[ -\frac{1}{2}R - G_{\alpha\bar{\beta}}(t,\bar{t}) g^{\mu\nu} \partial_\mu t^\alpha \partial_\nu t^{\bar{\beta}} \right] , \]  

(2.1)

using the facts that the moduli space of the ‘internal’ models is complex and that the natural (Weil-Petersson-Zamolodchikov) metric \( G \) on the moduli space is Hermitian. \( g^{\mu\nu} \) is the spacetime inverse metric, \( g \overset{\text{def}}{=} \det[g_{\mu\nu}] \) and \( R \) is the spacetime scalar curvature. As usual, the equations of motion for \( t^\alpha \) are

\[ g^{\mu\nu} \left[ \nabla_\mu \nabla_\nu t^\alpha + \Gamma^\alpha_{\beta\gamma}(t,\bar{t}) \partial_\mu t^\beta \partial_\nu t^\gamma \right] = 0 , \]  

(2.2)

where \( \nabla_\mu \) is the spacetime gravitationally covariant derivative and \( \Gamma^\alpha_{\beta\gamma} = \frac{1}{2}G^\alpha_{\beta\delta} \partial_{(\beta}G_{\gamma)\delta} \) is the Cristoffel connection on the moduli space derived from the Weil-Petersson-Zamolodchikov metric. Generally, solutions of Eq. (2.2) are minimal submanifolds of moduli space, i.e., spacetimes embedded in the moduli space with least action. Given such a solution, we then need to solve the Einstein equations

\[ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} R = T_{\mu\nu}[t,\bar{t}] , \]  

(2.3)

for the spacetime metric \( g_{\mu\nu} \). Here, as usual,

\[ T_{\mu\nu}[t,\bar{t}] = -G_{\alpha\bar{\beta}} \left[ \partial_{(\mu} t^\alpha \partial_{\nu)} t^{\bar{\beta}} - \frac{1}{2}g_{\mu\nu}(g^{\rho\sigma} \partial_\rho t^\alpha \partial_\sigma t^{\bar{\beta}}) \right] \]  

(2.4)

is the energy-momentum tensor of the moduli fields \( t^\alpha(x) \), which are being treated as matter from the point of view of gravity even though they describe the vacuum. In fact, because of the gravitationally covariant derivatives \( \nabla_\mu \) in (2.2), which involve a choice of a connection over the full 4-dimensional spacetime, Eqs. (2.2)–(2.4) form a nontrivially coupled system. As usual, we would treat them perturbatively, solving first Eq. (2.2) in the flat background \( (\nabla_\mu = \partial_\mu) \) to determine the vacuum, then solving Eq. (2.3) for the metric and finally verifying that the feedback of Eq. (2.3) into Eq. (2.2) is negligible, or as in our main case—zero.

For a simple but rather abundant class, assume that \( t^\alpha \) is static and varies only in two space-like dimensions, which for each fixed \( x^3 \) span a space-like infinite surface \( \mathcal{Z} \). This simplifies the problem, as seen by combining these two space-like coordinates into a complex coordinate, whereupon the action (2.1) and Eq. (2.2) become

\[ A_{\text{eff}} = -\frac{1}{2} \int d^4x \sqrt{-g} R - \int dt dx^3 \mathcal{E} , \]  

(2.5a)

\[ \mathcal{E} \overset{\text{def}}{=} \iint_{\mathcal{Z}} d^2z \sqrt{-g} G_{\alpha\bar{\beta}} g^{\bar{z}\bar{\zeta}} \left( \partial_\zeta t^\alpha \partial_\bar{\zeta} t^{\bar{\beta}} + \partial_\bar{\zeta} t^\alpha \partial_\zeta t^{\bar{\beta}} \right) , \]  

(2.5b)

and, respectively (neglecting gravity),

\[ \partial_\zeta \partial_\bar{\zeta} t^\alpha + \partial_\bar{\zeta} t^\beta \Gamma^\alpha_{\beta\gamma} \partial_\zeta t^\gamma = 0 . \]  

(2.6)
Indeed, this equation of motion for the moduli fields is most easily solved by taking the $t^\alpha$ to be arbitrary (anti)holomorphic functions of $z$. We thus consider holomorphic maps of a complex plane into the moduli space of the ‘internal’ $c_L = c_R = 9$ model. By adding a point at spatial infinity, we compactify the surface $\mathcal{Z}$ into a 2-sphere, $S^2 = \mathbb{P}^1$, and will hence consider holomorphic mappings $\mathcal{Z}^c = \mathbb{P}^1 \to \bar{\mathcal{B}}$, where $\bar{\mathcal{B}}$ is a suitable (partial) compactification of the moduli space $\mathcal{B}$.

For holomorphic $t^\alpha$, the expression (2.5b) simplifies in an important way: the second term drops out and $E$ is the integral over $\mathcal{Z}^c$ of the (1,1)-form associated to the pull-back metric $t^* (G)_{\bar{z} \bar{z}} = [\partial_z t^\alpha G_{\alpha \bar{\beta}} \partial_{\bar{z}} t^\bar{\beta}]$. Since the Weil-Petersson-Zamolodchikov metric $G_{\alpha \bar{\beta}}$, is actually Kähler,

$$E_{\bar{z} \bar{z}} \doteq \partial_z t^\alpha \partial_{\bar{z}} t^\bar{\beta} G_{\alpha \bar{\beta}} = \partial_z \partial_{\bar{z}} K,$$

and so

$$E = i \int_{\mathcal{Z}^c} \partial \bar{\partial} K \doteq 2\pi \deg t(\mathcal{Z}^c). \tag{2.8}$$

Looking back at Eq. (2.6), we see that the trivial solutions are $t = \text{const.}$, for which $E = 0$. In less trivial cases, we see that $E \neq 0$ is the total energy per unit length (along $x_3$) contained in the fields $t^\alpha$. As long as the singularities of the integrand are not too bad, (2.8) is invariant against holomorphic deformations and $\deg t$ is an integer. In most of the cases we will consider, $t$ factors through a projective space (e.g., the space of quintics in $\mathbb{P}^4$), and $\deg t$ is the degree of $t$ as a polynomial mapping.

Having tentatively compactified $\mathcal{Z}$ into a $\mathbb{P}^1$, $t(\mathcal{Z}^c)$ represents a rational curve (or its multiple cover) in some partial compactification of the moduli space; the constant (trivial) solutions corresponding to mapping the whole $\mathcal{Z}^c$ into a single point of the moduli space. Eventually, one point on this (complex) curve will be identified with the infinity and removed, restoring our open space-like surface $\mathcal{Z}$ and we will later obtain some sufficient conditions for this to produce a consistent superstring solution.

### 2.2. Spacetime structure in the vacuum

The moduli space of $c_L = c_R = 9$ (2,2)-supersymmetric models is spanned by two types of variations, corresponding to deformations of the complex structure and (complexified) variations of the Kähler class. The mirror-map equates the physically natural metric structure, so-called ‘special geometry’, of the two sectors in this moduli space. It suffices therefore to consider only one of them and we choose to discuss deformations of the complex structure, these being simpler to describe exactly. The anticipated rigorous establishment of mirror symmetry will then ensure that the analogous description holds for variations of the (complexified) Kähler class.

For a deformation class of Calabi-Yau 3-folds $\{\mathcal{M}_t, \ t \in \mathcal{B}\}$, the space $\mathcal{B}$ is complex and moreover Kähler. Locally at $t$, it is parametrized by $H^1(\mathcal{M}_t, T_{\mathcal{M}_t})$, which in turn

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3) We adhere to the Calabi-Yau geometrical vernacular although not every $c_L = c_R = 9$ (2,2)-supersymmetric model has a known geometrical interpretation. Our analysis easily extends to such models also.
equals $H^{2,1}(\mathcal{M}_t)$ since $\mathcal{M}$ is Calabi-Yau. That is, $H^1(\mathcal{M}_t, \mathcal{TM}_t)$ is tangent to $\mathcal{B}$ at $t$. There are no non-constant holomorphic maps from $Z^c$ to $\mathcal{B}$. However if we enlarge $\mathcal{B}$ to $\mathcal{B}$, the completion of $\mathcal{B}$ in the Weil-Petersson-Zamolodchikov metric, the situation changes. As we shall see below, there are many cases of interesting holomorphic maps from $Z^c$ (in some cases also from $Z$) to $\mathcal{B}$. Because $\mathcal{B}^{\sharp} \overset{\text{def}}{=} \mathcal{B} - \mathcal{B}$ has codimension one in $\mathcal{B}$, the set of points mapping to $\mathcal{B}^{\sharp}$ must be finite when the domain is $Z^c$ and is so in the cases we consider even when the domain is the noncompact $Z$. $\mathcal{B}^{\sharp}$ has a subset $\mathcal{B}^{\sharp}_{\text{con}}$ whose points correspond to conifolds, varieties which are smooth except for a single node. $\mathcal{B}^{\sharp}_{\text{con}}$ is open in $\mathcal{B}^{\sharp}$ and can be shown in most cases to be dense in $\mathcal{B}^{\sharp}$. When this is the case, it can be arranged that all the exceptional points map to $\mathcal{B}^{\sharp}_{\text{con}}$, and in fact this will then be the generic situation.

For sub-generic choices of the holomorphic $t$, the ‘internal’ Calabi-Yau 3-fold will acquire more and/or worse singularities. For example, if the image $t(Z^c)$ is arranged to intersect $\mathcal{B}^{\sharp}$ at an $n$-fold normal crossing $t^{\sharp}$, the conifold $\mathcal{M}_t^{\sharp}$ will have $n$ nodes.

These special points in $Z$, where the ‘internal’ space singularizes, sweep a filamentary structure along the third space-like coordinate, identified as cosmic strings. Note that these substantially differ from the cosmic strings in Grand-unified theories: there $E$ is divergent, although typically only logarithmically. Here, $E$ is finite and moreover a topological invariant (2.8).

Ref. [1] analyzed in great detail the toy model in which the Calabi-Yau 3-fold is chosen to be $T \times K3$ (or $T \times T \times T$) and only the complex structure of the torus $T$ is allowed to vary. There, $T$ can have only nodes. Also, we have that $G_{\tau \bar{\tau}} = \frac{1}{4}(3m \tau)^{-2} = -(\tau - \bar{\tau})^{-2}$, so

$$E_{\text{torus}} = \frac{i}{2} \int_Z \frac{d\tau \ d\bar{\tau}}{(\tau - \bar{\tau})^2} = \frac{m}{24} \cdot 4\pi ,$$

where $m$ is the multiplicity of the mapping $\tau : Z \to F$ of the compactified space-like surface $Z$ to the fundamental region $F$ of the torus moduli space. Suffice it here to note that for consistency reasons, $m$ is a multiple of 6 [4].

For the general case, where the variable ‘internal’ space $\mathcal{M}$ is an irreducible Calabi-Yau 3-fold, the Weil-Petersson-Zamolodchikov metric equals

$$G_{\alpha \bar{\beta}}(t, \bar{t}) \overset{\text{def}}{=} \partial_{\alpha} \partial_{\bar{\beta}} K , \quad K \overset{\text{def}}{=} - \log \left( i \int_{\mathcal{M}_t} \Omega_t \wedge \overline{\Omega}_t \right) ,$$

\begin{enumerate}
\item Our use of the term “generic” in the sequel will refer to the situation in which there exist maps from $Z^c$ to $\mathcal{B} \cup \mathcal{B}^{\sharp}_{\text{con}}$, although much of the analysis carries over into cases where worse singularities than nodes, or even clusters of nodes, are unavoidable.
\item However, requiring additional symmetries of the deformed potential may be required, e.g., for construction of the mirror model [6] and may very well enforce the occurrence of singularities worse than nodes.
\end{enumerate}
where $\Omega_t$ is the holomorphic 3-form on $\mathcal{M}_t$ at $t \in \mathcal{B}$. Consider the generic case when $\mathcal{M}$ develops one or perhaps a few nodes at some special points $z_i \in \mathcal{Z}$. We then have that locally around each special point $t_i = t(z_i^2)$ where $\mathcal{M}_t$ becomes singular,

$$e^{-K} \sim a(\tau, \bar{\tau}) + b(\tau, \bar{\tau})|\tau_i|^2 \log |\tau_i|, \quad \tau_i \overset{\text{def}}{=} t - t_i, \quad (2.11)$$

and where we know that $a(\tau, \bar{\tau})$ and $b(\tau, \bar{\tau})$ are bounded, non-zero and smooth $C^\infty$ functions around $\tau_i = 0$. For example, in a concrete model of Ref. [4], $a(\tau, \bar{\tau}) = a_0(1 - |\tau|^2)$ and $b(\tau, \bar{\tau}) = b_0$ to leading order in $\tau_i$ and $a_0$ and $b_0$ are readily calculable constants. This implies that $G_{\tau_i \bar{\tau}_i} \sim \log |\tau_i|^2$. So, in the total energy per unit length

$$\mathcal{E} = -i \int_{\mathcal{Z}} \partial \bar{\partial} \log \left( i \int_{\mathcal{M}_t(z)} \Omega_t(z) \wedge \overline{\Omega_t(z)} \right), \quad (2.12)$$

the integrand

$$\mathcal{E}_{zz} = \partial_z \bar{\partial}_z K = -\partial_z \bar{\partial}_z \log \left( i \int_{\mathcal{M}_t(z)} \Omega_t(z) \wedge \overline{\Omega_t(z)} \right) \quad (2.13)$$

diverges logarithmically at the locus of each string. So far, the behavior of $G_{\alpha \beta}(t, \bar{t})$ was in general not known away from $t_i$, the value(s) of moduli corresponding to conifold singularization(s). However, the techniques of Ref. [3] lend themselves for straightforward generalization and a powerful and rather general machinery is currently being developed to this end [13].

So far, $G_{\alpha \beta}$ has been computed only for one one-parameter family of models, in Ref. [3]. Fig. 4.1 there plots the Weil-Petersson-Zamolodchikov metric against the only parameter $\psi$. From this plot, it is clear that the regions $|\psi| > 1$ and $|\psi| < 1$ provide commensurate contributions to $\mathcal{E}$ ($\int_{|\psi| > 1} \partial \bar{\partial} K \approx 4 \int_{|\psi| < 1} \partial \bar{\partial} K$), and that the small region $|1 - \psi| \leq \frac{1}{10}$ contributes a similar (convergent) amount owing to the logarithmic divergence at $\psi = 1$ where the Calabi-Yau 3-fold develops 125 nodes. So, the energy per unit length of the field $t(z)$ is stored partly away from the string and partly around it. This contradicts the naive expectation that the energy density of a string ought to be concentrated around its locus. As we shall see below, this is due in part to the fact that there is a highly charged string “hiding” at infinity, and in part to the fact that we are not yet comparing the curvature to the physical space-time metric.

In a generic situation, there are $m$ points $z_i^2 \in \mathcal{Z}$ where $\mathcal{M}_{t(z_i)}$ is a conifold with a single node and worse singularizations of $\mathcal{M}$ do not occur. Assuming that the energy per unit length is distributed similarly to the case of Ref. [3], we expect that there would be a more or less uniform ‘background’ energy density with logarithmically divergent spikes at the loci of the cosmic strings. Now, if the point at infinity corresponds to a smooth rather than a singular Calabi-Yau 3-fold, then the background energy density will fall off exponentially with the distance. This follows easily from two observations pertaining to large $|z|$, i.e., far away from the strings. The first is that the energy density falls of in this case as $\frac{1}{|z|^2}$; the second is that, as we shall see below, the physical metric is asymptotically proportional to $\frac{dz \, dz}{\log |z|}$. It follows that the physical distance is asymptotically proportional to $\log |z|$, so that the energy density falls of exponentially with the physical distance, as asserted.
From Eq. (2.8), the total energy per unit length is a deformation invariant, \( \frac{1}{2} C_1 \cdot 4\pi \), where \( C_1 \) is the degree of the mapping from our projective line to the projective space parametrizing the complex structures. For example, in the torus case, the \( m = 12 \)-string case corresponds to a \( C_1 = 1 \) (linear) mapping and can easily be described by embedding the torus as a cubic curve in \( \mathbb{P}^2 \)

\[
Q_0(u, v, w) + z Q_1(u, v, w) = 0, \quad (u, v, w) \in \mathbb{P}^2,
\]

where \( z \in \mathbb{Z} = \mathbb{P}^1 \) and \( Q_0 \) and \( Q_1 \) are homogeneous cubics.

2.3. The spacetime metric

Quite naturally, one assumes the spacetime metric to be specified as

\[
ds^2 = -dt^2 + e^{\phi(z, \bar{z})} dz d\bar{z} + dx_3^2.
\]

This Ansatz has the virtue of leaving the expression (2.5b) unchanged, since the relation \( \sqrt{-g} g^{z\bar{z}} = 1 \) obtains. So does therefore also Eq. (2.6), and \( t^\alpha(z) \) can again be chosen to be arbitrary holomorphic functions. With this, the energy-momentum tensor and the Einstein equations (2.3) simplify, the latter of which reduce to a single independent equation

\[
\left[ R_{z\bar{z}} - \frac{1}{2} g_{z\bar{z}} R = \frac{1}{2} \partial_z \partial_{\bar{z}} \phi \right] = \left[ T_{z\bar{z}} = -\frac{1}{2} \partial_z \partial_{\bar{z}} K \right].
\]

This suggests the simple solution \( \phi = -K \), that is, with (2.10),

\[
ds^2 = -dt^2 + \left( i \int_{\mathcal{M}_t} \Omega_t \wedge \overline{\Omega_t} \right) dz d\bar{z} + dx_3^2.
\]

Now, as the logarithm of the Kähler potential for the Weil-Petersson-Zamolodchikov metric, \( (i \int \Omega_t \overline{\Omega_t}) \) is defined only up to the modulus of a holomorphic function in \( t \). In other words, there is a ‘gauge transformation’, generated by the freedom to rescale \( \Omega_t \rightarrow f(t) \Omega_t \) by any non-zero holomorphic function \( f(t) \). As a component of the spacetime metric, however, this ambiguity is rather undesired. In the torus case, as discussed in Ref. [1], the remedy was found by multiplying \( e^{-K} = \Im m \tau \) by \( \eta^2 \eta^2 \), where \( \eta \) is the Dedekind modular function. Since we do not yet have a detailed Teichmüller theory of Calabi-Yau spaces, we must proceed differently. Surprisingly perhaps, the ambiguity will be resolved up to a constant factor by requiring that the integral in (2.17) be finite and non-vanishing except at the point corresponding to spatial infinity, as we shall see in more detail in the next section.

Finally, we note that, unlike the torus case where \( e^{-K} \) blows up at the cosmic strings [1], here \( g_{z\bar{z}} = e^{-K} \) has a finite value and a continuous first derivative at any \( t_i \). It is straightforward that the distance function \( \int_0^r |z| \sqrt{g_{z\bar{z}}} \), with the results in Eqs. (2.17), (2.10) and (2.11), is nearly linear for small \( r \). This property is in sharp contrast with the torus case and is at the root of most novel features of the irreducible [6] Calabi-Yau 3-fold moduli space.

[6] This ‘irreducibility’ means that the holonomy of the ‘internal’ space is \( SU(3) \) rather then a subgroup thereof; this is also the condition for spacetime \( N=1 \) rather then extended supersymmetry.
2.4. Curvature and Yukawa couplings

Since the spacetime metric behaves rather nicely at the locus of a cosmic string, one may doubt the validity of the identification of this filamentary structure with cosmic strings. Recall, however, that the energy density does diverge at the loci $t_i$ of the strings, although only logarithmically.

In fact, there are two important identities. On one hand, practically by definition, the energy distribution in (2.7), $E_{\bar{z}z} \triangleq -\partial_{\bar{z}}\partial_{z} \log \left(i \int \Omega \wedge \bar{\Omega} \right)$, equals the Weil-Petersson-Zamolodchikov metric. Therefore, it is everywhere positive. On the other hand, $E_{\bar{z}z}$ is precisely the (Ricci) curvature for our spacetime metric (2.17). The spacetime scalar curvature is then $g^{a\bar{a}}E_{a\bar{a}} = e^K E_{\bar{z}z}$ and diverges logarithmically at the locus of these cosmic strings.

As easily seen from Eq. (2.17), the spacetime (Ricci) curvature equals the Weil-Petersson-Zamolodchikov area and so is positive over the space-like surface $\mathcal{Z}$. Note also that the extrinsic curvature of this surface in the 4-dimensional spacetime vanishes; transversally to $\mathcal{Z}$, spacetime is flat.

Besides this metric structure, there is another quantity which (very effectively) detects the locus of the cosmic strings—the Yukawa couplings. One of the $b_{2,1}$ 27-fields of the effective model, $\Psi_i$, will correspond to the deformation of the complex structure proportional to $\tau_i = (t - t_i)$. As in Ref. [7], one may compute that the (unnormalized, but otherwise exact) Yukawa coupling behaves like $\kappa_{iii} \sim (C/\tau_i)$, for some constant $C$. It can also be shown that, to leading order, the Yukawa coupling of $\Psi_i$ with any other 27-field vanishes. The kinetic term of $\Psi_i$ is the same as for $t^a$ in (2.5a) and so the leading term is $\log |\tau_i|$. Redefining $\Psi_i \rightarrow \Psi_i (\log |\tau_i|)^{-1/2}$, the kinetic term becomes the canonical one. The Yukawa coupling then becomes $(C/|\tau_i (\log |\tau_i|)^{3/2}|)$ and still diverges: $\Psi_i$ is (infinitely) strongly coupled at $\tau_i = 0$. Alternatively, rescale $\Psi_i \rightarrow \tau_i^{1/3} \Psi_i$, so that the Yukawa coupling becomes constant. Then, however, the kinetic term becomes $|\tau_i|^{2/3} \log |\tau_i|$ and dies out as $\tau_i \rightarrow 0$, turning $\Psi_i$ into a non-propagating field.

Either way, the dynamics of the $\Psi_i$ detects the cosmic string rather drastically\footnote{It has been suggested that the singularizations of the ‘internal’ space such as at those that occur at the locus of these cosmic strings may interface to ‘internal’ spaces of distinct topology \cite{7,8}. More recently, this possibility finds support in some exact results \cite{9}.}—it becomes “frozen”. Again, this is rather different than in the cosmic strings of Grand-unified theories, where some (Higgs) scalars become massless at the core of the string. Finally, the above conclusions rely on the exceptional features of ‘special geometry’ to protect the general relation between Yukawa couplings and the Weil-Petersson-Zamolodchikov metric. However, the spacetime singularity at the cosmic strings indicates that there the vacuum configuration no longer makes sense as a compactification, limiting the validity of the above analysis. We thus seek a more general description.
3. Advantages of a Bigger Vantage Point

Reflect for the moment on the vacua described above. In essence, the 10-dimensional spacetime $M$ has become the product of a 2-dimensional Minkowski spacetime (spanned by $x_3$ and time) and an 8-dimensional compact space $M^c$. This 8-dimensional space has the structure of a fibre space, where the compactified space-like surface, $Z^c = \mathbb{P}^1$, serves as a base and the Calabi-Yau 3-folds $M_z$ are associated to each $z \in Z^c$ in such a way that the complex structure of $M_z$ varies holomorphically with $z$. We now turn to some general methods of constructing suitable Kähler complex 4-dimensional spaces $M^c$, from which then non-compact Calabi-Yau 4-folds can be manufactured with ease.

3.1. Constructing spacetime variable vacua

Start with a non-singular complex projective four-fold $X$ the anti-canonical bundle $K^*_X$ of which is non-trivial and is generated by global holomorphic sections. That is, if $\phi(x)$ is such a section (‘polynomial’) of $K^*_X$, the zero-set in $X$ of $\phi(x) = 0$ is a Calabi-Yau 3-fold. Numerous simple examples of such spaces are provided in the collection of Calabi-Yau 3-folds constructed as complete intersections of hypersurfaces in products of (weighted) complex projective spaces $[3,10]$. Each of these may be thought of (typically in several different ways) as a simple hypersurface in some 4-fold $X$, obtained as the intersection of all but one hypersurface.

The above conditions on $K^*_X$ ensure that it has at least two independent global holomorphic sections; let $\phi_1, \phi_2$ be two such sections. We then form a 2-dimensional linear space $V = \{z_1\phi_1 + z_2\phi_2\}$. Each element from this space defines a Calabi-Yau 3-fold as the zero-set, which remains the same if we rescale the section by an arbitrary complex number. The parameters $z_1, z_2$ may therefore be considered as projective coordinates on $P(V) = \mathbb{P}^1_z$, readily identified with the compactified space-like surface $Z^c$.

In other words, we consider $M^c$, the space of solutions of

$$\Phi(x, z) \overset{\text{def}}{=} z_1 \phi_1(x) + z_2 \phi_2(x) = 0, \quad x \in X, \quad z \in \mathbb{P}^1 = Z^c, \quad (3.1)$$

which is simply a 4-fold in $X \times \mathbb{P}^1$. If $\pi$ denotes the projection on $Z^c = \mathbb{P}^1$, the inverse image $\pi^{-1}(z)$ is the subspace in $M^c$ which is projected to $z$ and is easily seen to be the Calabi-Yau 3-fold $M_z \subset X$. This provides the 4-fold $M^c$ with the structure of a fibre space with Calabi-Yau 3-folds $M_z$ fibred over $Z^c$.

In general, most choices of $\phi_1, \phi_2$ will yield a non-singular $M^c$. However, there will always be a finite set of singular fibres $M^*_i$ at some $z^*_i$, the number of which (counting with appropriate multiplicities) can be predicted from the choice of $X$, using elementary techniques of algebraic geometry.
The resulting 4-fold $M^c \subset X \times \mathbb{P}^1$ itself is not Calabi-Yau. Its first Chern class equals one half of the first Chern class of $\mathcal{Z}^c = \mathbb{P}^1$. That is, the canonical bundle of $M^c$, $K_{M^c}$, is isomorphic\(^8\) to $\pi^*(\mathcal{O}_{\mathbb{P}^1}(-1))$, where $\pi : M^c \rightarrow \mathbb{P}^1$ denotes the projection on the $z$-coordinates. Let
\[
\omega : \pi^*(\mathcal{O}_{\mathbb{P}^1}(-1)) \cong K_{M^c}
\]
denote this isomorphism (soon to be made more explicit). Recall that $K_{M^c} \overset{\text{def}}{=} \det \mathcal{T}_{M^c}^* = \Omega^4_{M^c}$ is the bundle of holomorphic 4-forms over $M^c$.

From this compact 4-fold $M^c$, we will obtain a non-compact 4-fold $M$ by excising a fibre $\mathcal{M}_{z_\infty} \subset X$ and the corresponding point $z_\infty \in \mathcal{Z}^c$, identified as the spatial infinity, and show that $M$ is a non-compact Calabi-Yau 4-fold by constructing explicitly a nowhere zero holomorphic 4-form.

Let $z_1, z_2$ be homogeneous coordinates on $\mathcal{Z}^c = \mathbb{P}^1$, chosen so that the point $z_\infty = (1, 0)$ corresponds to the spatial infinity. Then there is a unique meromorphic section of $\mathcal{O}_{\mathbb{P}^1}(-1)$ with a pole precisely at $z_\infty$, which may be written as $\mu = \frac{c}{z^2}$, where $c$ is an irrelevant constant. Note that $\mu$ is non-zero over $\mathcal{Z}^c$ and blows up only at spacetime infinity. Then, $\omega(\mu) = \Omega$ is a nowhere zero holomorphic 4-form over $M \overset{\text{def}}{=} M^c - \pi^{-1}(z_\infty)$. Thus, as promised, the non-compact space $M$ admits a nowhere-zero (and finite) holomorphic 4-form, which ensures that there is a Ricci-flat Kähler metric on it \([11]\), provided $z_\infty$ is a non-singular value of $\pi$.

As the above argument may appear too abstract, we also construct a more explicit representative for $\Omega$, using a by now well known residue formula \([12]\). On $\mathbb{P}^4 \times \mathbb{P}^1$, there is a holomorphic 5-form, $\omega \overset{\text{def}}{=} \frac{1}{5!} \epsilon_{ijklm} x^i dx^j dx^k dx^m \frac{1}{2!} \epsilon_{ab} z^a dz^b$. The quantity $\omega/\Phi(x, z)$ has a simple pole at $M^c \subset \mathbb{P}^4 \times \mathbb{P}^1$, where $\Phi(x, z) = 0$. This then serves as an integral kernel to define forms on the compact submanifold $M^c$. That is, for suitable $f(x, z)$, one calculates the residue of $f(x, z) \omega/\Phi(x, z)$ at the locus of $\Phi(x, z) = 0$, $M^c$. Now, note that $\omega/\Phi(x, z)$ is homogeneous of degree 0 over $\mathbb{P}^4$, but of degree +1 over $\mathbb{P}^1$. To make it into an invariant form, $f(x, z)$ must be chosen homogeneous of degree $(0, -1)$. Indeed, the meromorphic section $\mu$ exactly suits this purpose. If point by point $(x, z) \in M^c$, $\Gamma$ is a circuit in $\mathbb{P}^4 \times \mathbb{P}^1$ about $(x, z)$, the quantity
\[
\Omega(x, z) \overset{\text{def}}{=} \text{Res}_{x, z} \left[ \frac{c}{z_2} \frac{\omega}{\Phi(x, z)} \right], \quad (x, z) \in M^c,
\]
is easily seen to be a nowhere zero holomorphic 4-form, which blows up precisely over $z_\infty$.

This residue calculation explicitly realizes the isomorphism \((3.2)\): to every (meromorphic) section of $\mathcal{O}_{\mathbb{P}^1}(-1)$ over $\mathcal{Z}^c$, it assigns a meromorphic 4-form over $M^c$, that is section of $K_{M^c}$. The only pole of $\Omega$ is precisely at spatial infinity, $z_\infty \in \mathcal{Z}^c$. Thus, over
\[
M \overset{\text{def}}{=} M^c - \mathcal{M}_\infty; \quad \mathcal{M}_\infty = \pi^{-1}(z_\infty)
\]
\({\text{8}}\) For $\Phi(x, z) = 0$ to define a Calabi-Yau 4-fold, $\Phi(x, z)$ would have to be quadratic in $z$, since $c_1(\mathbb{P}^1) = 2J_{\mathbb{P}^1}$; our $\Phi(x, z)$ is only linear, so that $c_1(M^c) = J_{\mathbb{P}^1}$.
\( \Omega \) is nowhere-zero and finite—as promised.

On the other hand, \( \mu^2 = e^2(z_2)^{-2} \) lives in \( \mathcal{O}_\mathbb{P}^1(-2) = K_{\mathbb{P}^1} = \Omega_{\mathbb{P}^1}^1 \), and may therefore be identified with \( dz \) over \( \mathcal{Z} \overset{\text{def}}{=} \mathcal{Z}^c - z_\infty \). Then for each non-singular fibre \( \mathcal{M}_z = \pi^{-1}(z) \), \( z \neq z_\infty \), there is a holomorphic 3-form \( \Omega_z \), defined by

\[
\mu^2 \wedge \Omega_z = dz \wedge \Omega_z = \varpi(\mu) = \Omega.
\]

This clearly defines the nowhere-zero 3-form \( \Omega_z \) over each fibre \( \mathcal{M}_z \) as the ‘vertical’ (fibre-wise) factor of the nowhere-zero holomorphic 4-form over the non-compact 4-fold \( \mathcal{M} \). So, both the total (non-compact) space \( \mathcal{M} \) and the (compact) fibres \( \mathcal{M}_z \) are Calabi-Yau.

Bertini’s theorem assures us that, provided \( X \) is smooth, so is \( \mathcal{M}_c \), and, therefore, \( \mathcal{M} \) for a generic choice of \( \phi_1 \) and \( \phi_2 \). See [5,10] for more detail on this issue.

3.2. The spacetime metric revisited

Next, by a theorem of Tian and Yau [11], provided \( \mathcal{M}_\infty \) is smooth, there is a unique Ricci-flat Kähler metric in each \((1,1)\)-cohomology class, for which the associated Kähler form \( J \) satisfies

\[
J^4 = dz \wedge \Omega_z \wedge \Omega_{\bar{z}} = -dz \wedge d\bar{z} \wedge \Omega_z \wedge \Omega_{\bar{z}}. \tag{3.6}
\]

This Ricci-flat Kähler metric defines, at every fibre \( \mathcal{M}_z = \pi^{-1}(z) \) the orthogonal complement. With respect to this decomposition, we may write

\[
J = \frac{i}{4} V(x,z) dz \wedge d\bar{z} + \gamma(x,z), \tag{3.7}
\]

where \( \gamma(x,z) \) is a \((1,1)\)-form along \( \mathcal{M}_z \). Taking the fourth wedge power and equating with the expression (3.6), we obtain

\[
\gamma^3 = \frac{i}{V(x,z)} \Omega_z \wedge \Omega_{\bar{z}}. \tag{3.8}
\]

Finally, note that \( \mathcal{Z}^c = \mathbb{P}^1 \) is not a holomorphic subspace of \( \mathcal{M}_c \) in any natural way and hence does not acquire a metric by restriction. Rather, the projection \( \pi : \mathcal{M}_c \to \mathcal{Z}^c \) specifies \( \mathcal{Z}^c = \mathbb{P}^1 \) as a quotient. This leaves open another natural possibility, using the fact that on a complex one-dimensional space, the volume form defines a metric. Thus, we integrate the volume form of the non-compact 4-fold \( \mathcal{M} \) over the fibre \( \mathcal{M}_z \), point by point in \( \mathcal{Z} \), and obtain the induced volume form at \( z \in \mathcal{Z} \):

\[
\int_{\mathcal{M}_z} J^4 = \int_{\mathcal{M}_z} iV(x,z)dz \wedge d\bar{z} \wedge \gamma^3(x,z) = -\left( \int_{\mathcal{M}_z} \Omega_z \wedge \Omega_{\bar{z}} \right) dz \wedge d\bar{z}. \tag{3.9}
\]

Since \( dz \wedge d\bar{z} = -id^2z \), we obtain the associated spacetime metric

\[
ds^2 = -dt^2 + (i \int_{\mathcal{M}_z} \Omega_z \wedge \Omega_{\bar{z}}) |dz|^2 + dx_3^2, \tag{3.10}
\]

matching exactly, and independently, our earlier result (2.17).
3.3. Gauge-fixing and some related properties

From this bigger perspective, we can now address the gauge-fixing which we mentioned in the previous section. From the point of view of 4-dimensional field theory in which the vacuum (as defined by the choice of the ‘internal’ space) varies in spacetime, we could redefine the holomorphic 3-form \( \Omega_z \rightarrow f(z)\Omega_z \), where \( f(z) \) is an arbitrary non-zero holomorphic function over the space-like surface \( Z^c \). This indeed seems as too much freedom for \( (i \int \Omega \bar{\Omega}) \) to be interpreted as a metric component. We have remarked in the previous section that there is a (class, perhaps, of) natural gauge(s) in which \( f(z) \) is fixed so that \( (i \int \Omega \bar{\Omega}) \) is non-zero over all of the non-compact \( Z \).

Most importantly in the foregoing analysis, \( \Omega_z \) comes out by definition as non-zero over all of \( Z \), being the 3-form factor in the nowhere-zero holomorphic 4-form \( \Omega \) on \( M^c \); see (3.5). True, this 4-form itself is defined only up to a rescaling \( \Omega \rightarrow f\Omega \), but this \( f \) must be a constant over all of \( M^c \)!

It is of no physical consequence, for it specifies the relative proportion of distances in the \( Z \)-plane, such as the distance between two cosmic strings, \( \text{vs} \) some characteristic length scale in the \((t, x^3)\)-plane, where there is no structure to compare with. In other words, the vacua with such static cosmic strings are invariant under rescalings in the \((t, x^3)\)-plane and the constant \( |f| \) may be absorbed by these symmetries. In retrospect, we see that \( f \) in fact equals the inverse of the overall scale \( c \) in the meromorphic section \( \mu \) used in (3.3).

It therefore follows that the metric (3.10) has no physically relevant free parameters. Also, the parameter \( |f| \) which determines the characteristic distance between the cosmic strings is thus unrelated to the compactification theory and is free to assume astronomical proportions as hopefully governed by cosmological mechanisms.

Reviewing the above analysis, we see that the formula (3.10) follows from requiring that the complex structure of \( M_z \) vary holomorphically with \( z \in Z \) and upon declaring a particular point \( z_\infty \) of \( Z^c \) to be the spatial infinity. Let us remind the reader here that, in order to use the above stated theorem of Tian and Yau [11], \( z_\infty \) must be chosen so that the corresponding fibre \( M_\infty \) is smooth. It is not unlikely that this condition can be relaxed to a certain extent, but it is not clear how much.

For the spacetime metric (3.10), the Ricci tensor equals the Weil-Petersson-Zamolodchikov metric and therefore diverges logarithmically at the cosmic string loci. Owing to Eq. (3.8), the geometry defined by the \((1, 1)\)-form \( J \) in (3.7) is also singular. The arguments of [3], however, ensure the existence of a perturbative solution to superstring theory only for smooth geometries. We therefore need to replace the metric (3.7) on the 4-fold \( M \) with a smooth one.

Here our situation is qualitatively the same as that discussed in Ref. [1], although we cannot specify explicitly the fibre-wise metric component better than in Eq. (3.8). That is,
Eq. (3.8) simply states that the volume form of \( \gamma(x,z) \) equals the \( \Omega_z \wedge \overline{\Omega_z} \) one, normalized by \(-iV(x,z)\). Unfortunately, we have found no reason to expect any simple form for \(-iV(x,z)\) in general.

Just as in Ref. [1], we temporarily re-compactify the 4-fold into \( M^c \) and recall that this is a compact Kähler submanifold in \( X \times \mathbb{P}^1 \). It therefore inherits a compact Kähler metric from \( X \times \mathbb{P}^1 \) by reduction and let \( J_c \) denote the associated Kähler class. We form

\[
J_\lambda(x,z) \overset{\text{def}}{=} v(z,\bar{z})J(x,z) + \lambda J_c(x,z),
\]

where \( v(z,\bar{z}) \) is a “notch” function which vanishes sufficiently fast at the loci of cosmic strings\(^9\) in \( Z \) and equals 1 outside a suitably small compact region around these loci, while \( \lambda \) is a suitably large number. The correction term \( \lambda J_c \) offsets the asymptotic value of \( J_\lambda \) merely by an irrelevant constant since \( J_c \) is a compact metric. Near the locus of the cosmic strings, however, \( J_\lambda \) is smooth and positive, as required. While the metric associated to \( J_\lambda \) is not Ricci-flat, it has the advantage of being smooth, making the arguments of Refs. [3] available and we conclude that at least perturbatively, these vacua with cosmic strings are valid solutions of superstring theory.

### 3.4. 1-dimensional vs. more-dimensional moduli space

We find it necessary to point out an essential difference between the situation encountered with a typical \( b_{2,1} \)-dimensional moduli space where \( b_{2,1} > 1 \) and the 1-dimensional special case. In many ways, the developing theory of Calabi-Yau modular geometry borrows from the well understood modular geometry the torus. However, the moduli space of the torus is 1-dimensional and care has to be taken when relying on parallels.

In the present context, we are considering (harmonic) maps from a punctured \( \mathbb{P}^1 \) (the puncture corresponding to the spatial infinity) to the Calabi-Yau moduli space. The trivial (constant) map excluded, the case \( b_{2,1} = 1 \) differs in a very important way from the \( b_{2,1} > 1 \) case: When \( b_{2,1} = 1 \), any holomorphic map from \( Z^c \) to the moduli space must be surjective (onto) and will in general be \( m \)-to-1. Therefore, the space-like surface \( Z \) becomes carved up in \( m \) isomorphic preimages of the moduli space, that is, fundamental domains. Of necessity, then, the various quantities defined above (and likewise in Ref. [1]) must be identified over corresponding points in these different copies of the fundamental domain, which places additional non-trivial requirements on the spacetime metric structure.

In the typical case, when \( b_{2,1} > 1 \), a holomorphic map from the (compactified) space-like surface \( Z^c \) cannot possibly be surjective (simply since \( \dim Z^c < b_{2,1} \)). Instead, the map will be \( m \)-to-1, with \( m = 1 \) generically. Therefore, unlike when \( b_{2,1} = 1 \), in the typical case no additional conditions arise and our above discussion is complete in this regard.

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\(^9\) Since the Ricci tensor diverges only logarithmically at the loci of the strings, \( z_i \), near these points \( v(z) \sim |z - z_i|^\epsilon \) suffices for however small positive \( \epsilon \).
4. The Number of Cosmic Strings and the Local Geometry

4.1. Counting the strings

There is one global property of these vacua with cosmic strings which can be obtained easily—their total number. More precisely, there is a notion of ‘charge’ carried by these cosmic strings which may be defined as follows. Assume that in the 4-fold $M$ each singular fibre $M_i^\sharp$ at $z_i^\sharp$ has precisely one node, with $z_i^\sharp$ thus locating the cosmic strings. By varying the defining equation(s) of $M$ some of these strings may be brought to coincide. Suppose that at $z_i^\sharp$, two cosmic strings from the initial setting have been brought together. Then, the ‘internal’ space at $z_i^\sharp$, $M_i^\sharp$ will have two nodes, generically at different points in $M_i^\sharp$. We will say that the cosmic string at $z_i^\sharp$ is doubly charged.

In a more special case, when the locations of the nodes inside $M_i^\sharp$ are also brought to coincide, $M_i^\sharp$ will have developed an $A_2$-singularity (nodes are $A_1$-singularities). The cosmic string occurring where the ‘internal’ space develops a single $A_2$ singularity therefore will also be doubly charged, for by deformation, that cosmic string decomposes into two singly charged ones. It is easy to see that this provides a conserved additive charge.

The total charge of a vacuum configuration with such cosmic strings can be obtained for the above collection of models very easily. Consider again the compactified 4-fold $M^c$. By standard techniques, we straightforwardly compute its Euler characteristic. Now, suppose for a moment that $M^c$ was in fact a fibration of only smooth 3-folds $M$ over $\mathbb{P}^1$, with not a single singular fibre. The Euler characteristic would then simply be $\chi_E(M)\cdot \chi_E(\mathbb{P}^1)$. As there exist singular fibres in $M^c$, the latter result ought to be corrected for the (fibre-wise) singular set:

$$\chi_E(M^c) = \chi_E(M)\cdot \chi_E(\mathbb{P}^1) + \sum_{z_i^\sharp} \chi_E(\text{Sing}(M_i^\sharp)) .$$

Since each singular fibre $M_i^\sharp$ is a conifold with isolated singular points and $\chi_E(\text{point}) = 1$ and $\chi_E(\mathbb{P}^1) = 2$, the correction term simply becomes the number of (fibre-wise) nodes, that is, the total ‘charge’:

$$\#_{\text{total}} = \chi_E(M^c) - 2\chi_E(M) .$$

If we wish to rely only on the stronger Tian-Yau condition [11], which requires the fibre at infinity to be smooth, we cannot push any of the cosmic strings off to spatial infinity. Therefore $\#_{\text{total}}$ will characterize the non-compact vacuum $M$ also. Moreover, the Euler characteristic formula (4.2) is valid even if singularities worse than nodes cannot be avoided owing to insufficient generality of the model considered.

4.2. The Local Geometry: Deficit Angle and Curvature

The space-time metric $e^{-K}|dz|^2$ differs from the flat metric $|dz|^2$ by a conformal factor which is $C^1$ and bounded both above and below in a neighborhood of each string. More explicitly, it has the form

$$a(\tau_i, \bar{\tau}_i) + b(\tau_i, \bar{\tau}_i)|\tau_i|^2 \log |\tau_i| ,$$

14
where \( z_i \) is the location of a string, \( \tau_i \overset{\text{def}}{=} z - z_i \), and \( a(0) > 0 \). From this it is readily calculated that there is no cone-like angular deficit at \( z_i \).

It now follows, by Gauss-Bonnet, that the curvature of the spacetime metric integrates to \( 2\pi \chi_E(\mathbb{P}^1) = 4\pi \). On the average, each singly charged string may be thought of as contributing \( 4\pi/\#_{\text{total}} \). However, in this regard our situation is markedly different from that in Ref. [1]. As remarked earlier, with a (complex) 1-dimensional space of the moduli which vary over the space-like surface \( \mathcal{Z} \), the mapping of \( \mathcal{Z} \) to the moduli space is surjective and \( m \rightarrow 1 \). Thus, \( \mathcal{Z}^c \) is naturally carved up into \( m \) preimages of the moduli space. If there is one cosmic string locus as in the torus case, integration over each one of these preimages defines the charge of the corresponding string. Furthermore, each preimage contributes the same amount to the total curvature \( 4\pi \), whence the charge (energy per unit length) of each string is a well-defined quantity and equals \( 4\pi/\#_{\text{total}} \).

With a \( b_{2,1} \)-dimensional space of moduli (some of) which are varied over the space-like surface \( \mathcal{Z} \), where \( b_{2,1} > 1 \), this is no longer true. The mapping \( t^\alpha \) of \( \mathcal{Z} \) into the moduli space \( \mathcal{B} \) is typically injective and the various cosmic strings are the various different intersection points of \( t(\mathcal{Z}) \subset \mathcal{B} \) and \( \text{Sing}(\mathcal{B}) \). There is no natural way to equipartition \( \mathcal{Z} \) into \( \#_{\text{total}} \) regions, each of which contains one singly charged string and contributes the same amount to the total charge \( 4\pi \). In this respect, the typical situation in our case is much less determined in general.

In a concrete model, the precise locations of the strings can be computed. Then, relying on sufficient knowledge of the periods\(^ {10} \)

\[
\int_{A^a} \Omega_z , \quad \int_{B^a} \Omega_z , \tag{4.4}
\]

and so the metric component \( (i \int \Omega \overline{\Omega}) \), one may naturally carve up \( \mathcal{Z} \) into regions at the partial extrema of \( (i \int \Omega \overline{\Omega}) \) each containing a single string. Without this detailed model-dependent information, for which techniques are being developed [13], we are not aware of any further generally valid results.

### 4.3. A simple reference metric

The general picture of the space-like surface \( \mathcal{Z} \), with the geometry obtained for it as above and with the cosmic strings passing through it is sketched in Fig. 2.

Note, however, that this is not how the space-like surface \( \mathcal{Z} \) is embedded in the spacetime. The extrinsic curvature of \( \mathcal{Z} \) in spacetime vanishes, as should be clear from the form of the metric (3.10). Also, the sketch in Fig. 2 does not show the geometry in any fine detail; roughly, the surface \( \mathcal{Z} \) appears conical around the locus of a cosmic string, although there is no conical deficit angle at the string.

Motivated by this, it is reasonable to introduce an auxiliary geometry, which is indeed flat, except for a conical singularity at the locus of each node. In order to do this, we write

\(^{10} \) Here, \( A \) and \( B \) form a simplectic basis of \( H_3 \): \( A \cap B = 1 \), \( A \cap A = 0 \), \( B \cap B = 0 \).
\( z_1, \ldots, z_n \) for values of \( z \) locating the nodal varieties (accounting for multiplicities). We set

\[
P(z) = \prod_{i=1}^{n} (z - z_i) , \quad n = \#_{\text{total}} ,
\]

and we choose for our auxiliary flat metric

\[
d s_{\text{aux.}}^2 \overset{\text{def}}{=} -d t^2 + \frac{d z \, d \bar{z}}{(P \bar{P})^{\frac{1}{n}}} + d x_j^2 .
\]

By straightforward computation, it is easily seen that this metric is flat and asymptotically cylindrical at infinity, and that the deficit angle at each \( z_i \) is \( \frac{2\pi}{n} \). It is tempting to conjecture that our physical metric \( (3.10) \) is a smoothing of \( (4.6) \), with the curvature concentrated around the locations of the strings. In any case, since the energy density of the cosmic strings falls off exponentially, the metric \( (4.6) \) appears to be a reasonable and useful approximation. To appreciate the temptation, consider the scattering problem for \( (4.6) \):

The geodesic equation for \( (4.6) \) is given

\[
\frac{d z}{d t} = c P(z)^{\frac{1}{n}} ,
\]

where \( c \) is a complex constant in general. For large \( z \), this is asymptotic to

\[
\frac{d z}{d t} = c' z ,
\]

on the understanding that the correspondence between the values of \( c \) and \( c' \) is \( n \)-to-\( n \), rather than 1–1. For example, if \( c \) is real and negative, the geodesic is axial and incoming. Exceptionally, such a geodesic may eventually hit one of the cone points. Generically, it must come back out. When it does so, \( c' \) must be replaced by \( ac \), where \( a \) is an \( n \)th root of 1 with negative real part. Geometrically, this means that the outgoing ray has acquired a circumferential component through scattering among the strings and where the ratio between the axial and circumferential velocities of the outgoing ray is quantized.

5. Concrete Models

The general description of model construction in the preceding sections may have been too vague except for the experts. We therefore include some sample models just to illustrate the general method of endowing families of Calabi-Yau 3-folds with spacetime dependence.

Consider firstly the one-parameter family of quintics \( M_\psi \) studied recently in Ref. [6]:

\[
P_{\psi}(x) \overset{\text{def}}{=} \left( \sum_{i=1}^{5} x_i^5 \right) - 5\psi \left( \prod_{i=1}^{5} x_i \right) .
\]

16
It possesses a $\mathbb{Z}_5^5/\mathbb{Z}_5 = \mathbb{Z}_5^4$ symmetry, which acts on the $x_i$ simply by multiplying them with fifth roots of 1. The space of complex structures of the quotient $\mathcal{W}_\psi \overset{\text{def}}{=} \mathcal{M}_\psi/\mathbb{Z}_5^3$ is 1-dimensional and parametrized by $\psi$. To turn this into a vacuum configuration with cosmic strings, simply allow $\psi$ to vary over spacetime $X$. If, following the preceding analysis, $\psi$ becomes a holomorphic function over a space-like surface $Z \subset X$ and we need to consider mappings from the compactified space-like surface $Z^c$ to the $\psi$-moduli space; the moduli space is seen as the $0 \leq \text{Arg}(\psi) \leq \frac{2\pi}{5}$ wedge in the $\psi$-plane [4].

At $\psi = 1$, $\mathcal{M}_\psi$ develops 125 nodes and so $\mathcal{W}_\psi$ is likewise singular—thus $\psi(z) = 1$ specifies the locus of the cosmic string (of charge 125). At $\psi = \infty$, $\mathcal{M}_\infty$ and so also $\mathcal{W}_\infty$ becomes rather badly singular: $\mathcal{M}_\infty$ is in fact the union of five $\mathbb{P}^3$s, meeting (and singular) at ten $\mathbb{P}^2$’s, which in turn have ten $\mathbb{P}^1$’s in common and which intersect in five points. $\mathcal{W}_\infty$ is then just the $\mathbb{Z}_5^3$ quotient of this and is none less singular. Since $\dim \text{Sing}(\mathcal{W}_\infty) > 0$, the total space $M^c$ of this $\psi$-family of Calabi-Yau 3-folds is singular at $\psi = \infty$. To construct a smooth non-compact $M$, we therefore must excise the singular fibre and then the theorem of Tian and Yau [11] does not guarantee the existence of a Ricci-flat metric $X$ spacetime; this then remains an open question. Nevertheless, the metric on the 4-dimensional spacetime $X$ is still determined by (3.10). Hoping that Ricci-flatness can eventually be proved, and because detailed computations are available from Ref. [6], we have chosen to include this example here, and indeed to refer to it throughout the paper.

Consider next another family of quintics in $\mathbb{P}^4$, one that has been studied in Refs. [7,8]:

$$\Delta_t(x) \overset{\text{def}}{=} (S(x)P(x) - Q(x)R(x)) - tT(x) = 0 ,$$

(5.2)

where $S, Q, P, Q$ are generic polynomials with $\deg S = \deg Q = q$, $\deg P = \deg R = (5 - q)$ and $T(x)$ is a generic quintic in $\mathbb{P}^4$. At $t = 0$, $\Delta_0$ is singular and has precisely $q^2(5 - q)^2$ nodes. This is easily seen since

$$d\Delta_0(x) = dSP + SdP - dQR - QdR$$

(5.3)

vanishes when $S = P = Q = R = 0$. This being four (generically) independent conditions in $\mathbb{P}^4$, they will be satisfied at isolated points. The number of these points follows by Bézout’s theorem and equals the product of degrees. Near $t = 0$, the quintics $\Delta_t = 0$ are smooth, but eventually, as we let $t$ vary linearly over a $Z^c = \mathbb{P}^1$, there will be other singular loci. Interpreting $t$ as a spacetime field varying holomorphically over the space-like surface $Z$, we obtain a vacuum configuration with cosmic strings.

To use the formula (4.2), we need that the Euler characteristics of smooth quintics in $\mathbb{P}^4$ is $-200$. Likewise, the Euler characteristic of 4-folds of bi-degree $5, 1$ in $\mathbb{P}^4 \times \mathbb{P}^1$ (corresponding to a linear fibration over $Z$) is easily found to be $+880$, whence the total cosmic string charge becomes $\#_{\text{total}} = 1280$. The precise location, in $\mathbb{P}^1$, of the remaining 1264 strings (several subsets of which may be coinciding in location) depends on the specific choice of the polynomials $S, P, Q, R$ and $T$. Since $b_{2,1} = 101$ for smooth quintics in $\mathbb{P}^4$, we see that there is a wealth of possible vacuum configurations with (static) cosmic strings. As long as the fibration is linear as above, $\#_{\text{total}}$ remains at 1280.
The well known Tian-Yau family of Calabi-Yau 3-folds is defined as the common zero-set of
\[3 \sum_{i=0}^{3} x_i^3 - 3 \sum_{i,j,k=0}^{3} \alpha_{ijk} x_i x_j x_k = 0\, ,\]
\[3 \sum_{i=0}^{3} y_i^3 - 3 \sum_{i,j,k=0}^{3} \beta_{ijk} y_i y_j y_k = 0\, ,\]
\[3 \sum_{i=0}^{3} x_i y_i = 0\, .\]  

(5.4)

For suitable choices of the constants \(\alpha\) and \(\beta\), these 3-folds, \(\mathcal{M}_{0,\alpha,\beta}^0\), are smooth and admit a free \(\mathbb{Z}_3\) action and the quotients \(\mathcal{M}_{\alpha,\beta} = \mathcal{M}_{0,\alpha,\beta}/\mathbb{Z}_3\) are smooth Calabi-Yau 3-folds with \(\chi_E = -6\). We may then make any of \(\alpha, \beta\) into a \(\mathbb{Z}\)-dependent spacetime field and obtain thereby a vacuum configuration with cosmic strings. Again, making any of \(\alpha, \beta\) into a linear function over \(\mathbb{Z}\) yields a linear fibration. Temporarily compactifying the space-like surface into \(\mathcal{Z}\) = \(\mathbb{P}^1\), we have just constructed a compact 4-fold embedded as the common zero set of a system of three equations in \(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^1\) and of tri-degree \((3, 0, 1), (0, 3, 0)\) and \((1, 1, 0)\). The Euler characteristic of this 4-fold is +135, whence \(#_{total} = 135 - 2(-18) = 171\). As the whole family admits the \(\mathbb{Z}_3\) symmetry, by passing to the \(\mathbb{Z}_3\)-quotient, we obtain a 3-generation vacuum configuration with 57 (possibly coinciding) cosmic strings.  

\[\circ\]

The simplicity of these constructions is we hope clear. Of course, more detailed information on the spacetime geometry, such as an exact expression for the spacetime metric throughout \(\mathcal{Z}\), depends on the knowledge of periods (4.4). Certain preliminary and general results are already known in this respect and may soon become routine calculation [13].

6. Some Unusual Global Properties

We have so far seen several unusual properties of the cosmic strings that arise in the present context. Below we discuss another feature which is caused by certain delicate relations between complex structure moduli and (complexified) Kähler class moduli at certain singular Calabi-Yau 3-folds.

Consider the family of Calabi-Yau 3-folds embedded in \(\mathbb{P}^5\) by means of the two equations
\n\begin{align*}
\mathcal{M}_\gamma : &\quad \sum_{i=0}^{5} x_i^4 = 0\, , \\
&\quad x_2 x_3 - x_4 x_5 + \gamma Q(x) = 0\, ,
\end{align*}

(6.1)

where \(Q(x)\) is a generic quadric in \(\mathbb{P}^5\). At \(\gamma = 0\), \(\mathcal{M}_0\) becomes singular and has four nodes, located in \(\mathbb{P}^5\) at \((1, \xi, 0, 0, 0, 0)\), where \(\xi^4 = -1\). Near \(\gamma = 0\), \(\mathcal{M}_{\gamma \neq 0}\) are smooth and a
simple local calculation \(^6\) finds four small 3-spheres \(S^3 \subset M_{\gamma \neq 0}\), each of which collapses produced if the singular fibre is smoothed by small resolutions instead of deforming \(^5\). This number is also known as the ‘deficit’ \(^1\).\(^4\).

Another unusual property of these fibrations is the fact that the whole family of Calabi-Yau 3-folds over the space-like surface \(Z\) is given globally by means of some system of equations as above. As defined so far, these constructions provide static vacuum configurations. However, if we try to change the family so as to relocate some of this remark is to note that not all nodes in a fibre are independently smoothed by deformations in fibrations of the kind we are examining. In fact, the number of relations among these deformations by smoothing equals the number of vanishing cycles which are introduced if the singular fibre is smoothed by small resolutions instead of deforming \(^5\),\(^7\),\(^8\). This number is also known as the ‘deficit’ \(^1\).\(^4\).

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7. The Full (10-Dimensional) Picture

In full generality, the moduli of the ‘internal’ space should have full spacetime dependence and we need to solve the equations of motion for the moduli \(^2\),\(^2\) and the Einstein equations \(^2\),\(^3\) for the metric. If one performed a Wick-rotation into imaginary time, spacetime becomes Euclidean and if we adopt the Ansatz that it also admits a complex structure, we are able to proceed in close analogy with the static case.

We start, as before, with a (complex) 4-fold \(X\) whose anti-canonical bundle is non-trivial and generated by global sections. We now assume that \(K_X^*\) has at least three linearly independent global holomorphic sections, which constitute a basis of a 3-dimensional space of linear combinations \(U = \{\xi_1 \phi_1 + \xi_2 \phi_2 + \xi_3 \phi_3\}\), which we projectivize into a \(\mathbb{P}^2\)-family of sections of \(K_X^*\). The projective space \(\mathbb{P}^2_\xi = X^c\) will be interpreted as the compactified and Wick-rotated spacetime, endowed moreover with a complex structure. A generic section \(\Phi(x, \xi)\) from \(U\) then defines a smooth Calabi-Yau 3-fold \(M_\xi\) lying, therefore, over a generic point of \(\xi \in X^c\). The total space of this (projective) family is a non-singular complex 5-fold \(M_X^c\), with the projection

\[
\pi : M_X^c \longrightarrow \mathbb{P}^2_\xi = X^c ,
\]

of which the generic fibre is a smooth Calabi-Yau 3-fold. That is, the compact 5-fold \(M_X^c\) has the structure of a fibre space, with \(\mathbb{P}^2_\xi\) as the base an Calabi-Yau 3-folds \(M_\xi\) defined for each \(\xi \in X^c = \mathbb{P}^2_\xi\) by \(\Phi(x, \xi) = 0\).
Again, the rôle of the world sheet of the cosmic string(s) is played by the singular locus in \( M_\mathcal{X} \) of the projection \( \pi : M_\mathcal{X} \to \mathbb{P}^2_\xi = \mathcal{X} \), which in this case is an algebraic curve (Riemann surface). We will consider this first for the compact 5-fold \( M_\mathcal{Y} \) and later interpret its intersection with a non-compact total 10-dimensional spacetime \( \mathcal{M} \).

Let \( P^2_\mathcal{X} \) denote the projectivized space of global sections of \( K_\mathcal{X} \) for which the zero locus is singular. Recall that it is a hypersurface in \( P_\mathcal{X} \), the projective space of all global sections of \( K_\mathcal{X} \). The degree \( d \) of this hypersurface is given by

\[
d = \chi_E(M_\mathcal{X}^c) - 2\chi_E(\mathcal{M}) ,
\]  

(7.2)

where \( Z = \mathbb{P}^1 \) is a linear subspace of \( P_\mathcal{X} \), \( M_\mathcal{Z} \) is a \( \mathbb{P}^1 \)-family of Calabi-Yau 3-folds, defined in Eq. (3.1), while \( \mathcal{M} \) is the generic Calabi-Yau 3-fold defined as the zero locus of a non-singular global section of \( K_\mathcal{X} \). This is simply a reinterpretation of the computation of the number of strings for the static case, Eq. (4.2).

Now, by identifying \( \mathbb{P}^2_\xi \subset P_\mathcal{X} \) with the compactified (real 4-dimensional) spacetime \( X^c \), the latter has been mapped into the projective space of sections of \( K_\mathcal{X} \), which is a moduli space for Calabi-Yau 3-folds \( \mathcal{M}_\xi \subset \mathcal{X} \), up to reparametrizations\(^{11}\). As \( P^2_\mathcal{X} \) is a hypersurface in \( P_\mathcal{X} \), the image of \( X^c \) in \( P_\mathcal{X} \) is bound to intersect \( P^2_\mathcal{X} \) in a complex 1-dimensional space. The inverse image of this is a projective plane curve \( S \subset X^c \) of degree \( d \).

In other words, at the spacetime points \( \xi \in S \subset X^c \) the ‘internal’ space \( \mathcal{M}_\xi \) develops a node and fibres with two nodes correspond to transverse self intersections of \( S \). For a generic choice of \( \mathbb{P}^2_\xi \subset P_\mathcal{X} \) and if the number of global holomorphic sections is sufficiently big, fibres with more than two nodes or with worse singularities will not occur. It follows that the set \( \tilde{S} \subset M_\mathcal{X}^c \) of singular points of the mapping \( \pi \) is a non-singular curve, in fact the desingularization of \( S \).

Remembering that the Euler characteristic of each nodal fibre exceeds the Euler characteristic of a smooth fibre by the number of nodes on the fibre, we obtain the equation (cf. (4.2) and (7.2))

\[
\chi_E(\tilde{S}) = \chi_E(M_\mathcal{X}^c) - 3\chi_E(\mathcal{M}) ,
\]  

(7.3)

since \( \chi_E(\mathbb{P}^2) = 3 \); the number of handles on the cosmic string world sheet is then \( g = 1 - \frac{1}{2}\chi_E(\tilde{S}) \). There is good technology for computing the Euler characteristics of smooth 3-folds \( \mathcal{M} \), the total space of their linear \( \mathbb{P}^1 \)-families \( M_\mathcal{Z} \), and the total space of their linear \( \mathbb{P}^2 \)-families \( M_\mathcal{X} \). Therefore, Eqs. (7.2) and (7.3) yield the degree \( d \) and the genus \( g = 1 - \frac{1}{2}\chi_E(\tilde{S}) \) of the world sheet of the cosmic string.

The number \( n^\sharp \) of self-intersections of \( S \) (nodes of the cosmic string world sheet) is then determined by the equation

\[
n^\sharp = \frac{1}{2} \left[ \chi_E(\tilde{S}) - (3d - d^2) \right] .
\]  

(7.4)

\(^{11}\) Please note that \( X \) denotes the (real) 4-dimensional spacetime which is here Wick-rotated, while the boldface symbol \( \mathcal{X} \) refers to the complex 4-fold in which the ‘internal’ Calabi-Yau 3-fold is embedded.
This follows from the fact that a non-singular plane curve of degree \(d\) has Euler characteristic \(3d - d^2\) and that the Euler characteristic of the desingularization of a plane curve of the same degree with transverse self intersections exceeds that of a non-singular plane curve by twice the number of self-intersections.

7.2. Decompactification and Metric

In order to decompactify \(M_X^c\) and obtain a non-compact 5-fold \(M_X\) with a Ricci-flat metric, we choose a projective line \(\mathbb{P}_\infty^1 \subset \mathbb{P}^2\), recalling that \((\mathbb{P}^2_\xi - \mathbb{P}_\infty^1) = \mathbb{C}^2 \approx \mathbb{R}^4\). For a generic choice, the total space of the family of Calabi-Yau 3-fold fibres over this \(\mathbb{P}_\infty^1\), \(\pi^{-1}(\mathbb{P}_\infty^1)\), is a non-singular hypersurface in \(M_X^c\). This hypersurface is the locus of the pole of an otherwise holomorphic and non-vanishing 5-form

\[
\Omega(x, \xi) \overset{\text{def}}{=} \text{Res}_{(x, \xi)} \left[ \frac{c}{C(\xi_1, \xi_2, \xi_3)} \frac{\omega}{\Phi(x, \xi)} \right],
\]

\[
= \lim_{r \to (x, \xi)} \int_{S^2} \frac{c}{C(\xi_1, \xi_2, \xi_3)} \frac{\omega}{\Phi(x, \xi)}, \quad (x, \xi) \in M_X^c,
\]

where \(C(\xi) = L^2(\xi)\), with \(\Delta = 3 - \deg \Phi(x, \xi)\) and \(L(\xi)\) is linear in \((\xi_1, \xi_2, \xi_3)\). This is of course very much like \(\mathcal{M}_{z_\infty}\) over spatial infinity \(z_\infty \in \mathbb{P}_z^1\), which was the locus of the pole of the otherwise holomorphic and nowhere-zero 4-form \(\Omega (3.3)\). We note also that a generic choice of \(\mathbb{P}_\infty^1\) will meet \(S\), the set of singular points of the singular fibres, transversely in \(d\) points on distinct fibres.

Now we can invoke the theorem of Yau and Tian to deduce the existence of a Ricci-flat metric on \(M_X = M_X^c - \pi^{-1}(\mathbb{P}_\infty^1)\).

We take our four dimensional spacetime to be \(X = (\mathbb{P}^2_\xi - \mathbb{P}_\infty^1) \approx \mathbb{R}^4\). The metric on \(M\) induces both a horizontal subspace wherever \(\pi\) is non-singular and a metric on each fibre. By integrating the horizontal metric over the fibres, we produce a metric at each point of \(X\) which is not a singular value of \(\pi\). Unfortunately, there is no reason to believe either that this metric is Kähler or that the logarithm of its volume form is a Kähler potential for the Weil-Petersson-Zamolodchikov metric.

The fact that \(c_1(\mathbb{P}^2) = 3J_{\mathbb{P}^2}\), allows another construction. With \(\Phi(x, \xi)\) linear in \(\xi\), the quadratic function \(C(\xi)\) in Eq. (7.5) could be replaced by a product of distinct linear factors, say \(L_1 L_2\), whereby the singular locus becomes the union of two \(\mathbb{P}^1\)’s meeting transversely in a single point. This offers an interesting possibility for defining Wick-rotation back to Lorentzian spacetime, by choosing \(t = \log |L_1/L_2|\) as a time coordinate. Ignoring (or perhaps blowing up) the point \(L_1 = L_2 = 0\), we may identify \(L_1 = 0\) as the light-cone 2-sphere at the infinite past and \(L_2 = 0\) as the light-cone 2-sphere at infinite future. For generic choices, each of these two light-cone 2-spheres will have \(d\) points corresponding to singular fibres, and \(S\) will exemplify the by now standard picture of the world sheet of an interacting string as advanced by Mandelstam \[15\] and others, in which \(d\) cylinders emerge from the remote past, they join and diverge finitely many times, and \(d\)
cylinders continue into the remote future. The authors hope to explore this idea in more
detail in a future publication. One difficulty to be overcome is that fact that $dt = 0$ at
finitely many points of $M_X$, all of which are on the string.

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We close with the following remark. At least in the main case under study here, when
the moduli fields vary only over a space-like surface, the spacetime metric (3.10) differs
from the flat metric by a conformal factor which in fact is the Kähler potential of the
Weil-Petersson-Zamolodchikov metric, in a suitable gauge:

$$ g_{z\bar{z}} = e^{-K}. \tag{7.6} $$

On the other hand, the image of the 4-dimensional spacetime in the moduli space acquires
a metric by restriction, and so there is another `natural' metric: the pull-back of the
Weil-Petersson-Zamolodchikov metric:

$$ G_{z\bar{z}} = \partial_z \partial_{\bar{z}} e^{-K}. \tag{7.7} $$

Besides a possible application of the above analysis in cosmology, the relation between
$g_{z\bar{z}}$ and $G_{z\bar{z}}$ enables one to use the spacetime variable vacua as a laboratory for studying
the Weil-Petersson-Zamolodchikov geometry of the moduli space. Of course, results of this
modular geometry have already been used in our study of the spacetime variable vacua and
further results will again acquire their cosmological interpretation through these cosmic
string models. It is our hope that perhaps this interpretation will bring about a cross-
disciplinary communication in the other direction also.

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