Plane strain bending under tension of a functionally graded sheet at large strains as an ideal flow process

Sergei Alexandrov1,2 and Alexander Pirumov3
Division of Computational Mathematics and Engineering, Institute for Computational Science, Ton Duc Thang University, Ho Chi Minh City, Vietnam
Faculty of Civil Engineering, Ton Duc Thang University, Ho Chi Minh City, Vietnam
Department of Elektrotechnics and Mechanics, Moscow Technological University, Moscow, Russia

sergeialexandrov@tdt.edu.vn

Abstract. Ideal plastic deformations (ideal flows) have been defined as solenoidal smooth deformations in which an eigenvector field associated everywhere with the greatest (major) principal rate of deformation is fixed in the material. In the case of plane strain deformation of rigid perfectly plastic material obeying an arbitrary isotropic yield criterion and its associated flow rule, it is always possible to find an equilibrium stress field which is compatible with an ideal deformation. It is shown in the present paper that an ideal deformation is possible for functionally graded sheets in the process of plane strain bending under tension. In contrast to the general process, the tensile force and bending moment cannot be prescribed arbitrary but should be found from the solution.

1. Introduction
Ideal plastic flows are those for which all material elements follow minimum work paths. The ideal flow theory has long been associated with the Tresca yield criterion [1, 2]. Recently, it has been demonstrated that ideal flow solutions exist in anisotropic plasticity [3]. In particular, the theory of anisotropic plasticity proposed in [4] has been adopted. A comprehensive overview of the ideal flow theory and corresponding solutions has been provided in [5]. This theory has been widely used as the basis for inverse methods for the preliminary design of bulk metal forming processes driven by minimum plastic work (see, for example, [6]). The process of bending under tension is widely used in metal forming technologies. Using a number of assumptions concerning the through-thickness distribution of strains analytic plane-strain solutions at large strains have been proposed in [7, 8]. Ignoring the transverse stress a solution has been obtained in [9]. An efficient semi-analytic approach to analysis of plane strain bending under tension has been proposed in [10]. This approach is an extension of the corresponding approach for the process of pure bending [11]. In the present paper, the approach developed in [10] is used in conjunction with the ideal flow theory to design the process of plane strain bending under tension of functionally graded sheets. Other ideal flow solutions for this process have been proposed in [12, 13].

2. Kinematics
For incompressible materials an approach to analysis of plane strain pure bending at large strains has been proposed in [11]. This approach has been extended to bending under tension in [10]. For isotropic material models kinematics of the process of bending under tension is independent of constitutive equations other than the equation of incompressibility. For completeness, the basic equations derived in [10] are presented in this section. The process of bending under tension is described by the following mapping between Eulerian Cartesian coordinates \((x, y)\) and Lagrangian coordinates \((\zeta, \eta)\):

\[
\frac{x}{H} = \sqrt{\frac{\zeta + s}{a}} \cos(2\eta) - \frac{\sqrt{s}}{a}, \quad \frac{y}{H} = \sqrt{\frac{\zeta + s}{a}} \sin(2\eta)
\]  

where \(H\) is the initial thickness of the sheet, \(s\) is an arbitrary function of \(a\), \(a\) is a function of the time, \(t\), and \(a = 0\) at \(t = 0\). The function \(s(a)\) should satisfy the condition

\[
s = \frac{1}{4}
\]  

for \(a = 0\). Using (1) and (2) it is possible to verify by inspection that \(x \rightarrow \zeta H\) and \(y \rightarrow \eta H\) as \(a \rightarrow 0\). The initial shape of the sheet is a rectangular. With no loss of generality it is possible to assume that the sides of this rectangular are given by the equations \(x = -H\) (or \(\zeta = -1\)) , \(x = 0\) (or \(\zeta = 0\)) and \(y = \pm L\) (or \(\eta = \pm L/H\)). It is convenient to introduce a moving plane polar coordinate system \((r, \theta)\) with its origin \(O_1\) at \(x = -H\sqrt{s}/a\) and \(y = 0\). Then, it is possible to find from (1) that

\[
\frac{r}{H} = \sqrt{\frac{\zeta + s}{a}} a, \quad \theta = 2\eta a
\]  

Since \((\zeta, \eta)\) is a Lagrangian coordinate system, it follows from this equation that the shape of the sheet after any amount of deformation is a circular sector. Its concave, \(r = r_1\), and convex, \(r = r_2\), boundaries are determined from (3) at \(\zeta = -1\) and \(\zeta = 0\), respectively. As a result,

\[
\frac{r_1}{H} = \sqrt{\frac{s-a}{a}}, \quad \frac{r_2}{H} = \sqrt{\frac{s}{a}}, \quad \frac{h}{H} = \sqrt{\frac{s-a}{2a}}.
\]  

Here \(h\) is the current thickness of the sheet. The straight boundaries of this sector are given by

\[
\theta = \pm \frac{2aL}{H}.
\]  

The coordinate curves of the Lagrangian coordinate system are principal strain rate trajectories. The principal strain rates can be found from (1) as

\[
\xi_\zeta = -\frac{(\zeta + ds/da)}{2(\zeta a + s)} \frac{da}{dt}, \quad \xi_\eta = \frac{(\zeta + ds/da)}{2(\zeta a + s)} \frac{da}{dt}
\]  

It is evident from this equation that the equation of incompressibility, \(\xi_\zeta + \xi_\eta = 0\), is satisfied. Since \(\xi_\zeta = \xi_\eta = 0\) at the neutral line, the equation for this line follows from (6) as

\[
\zeta = \zeta_n = -\frac{ds}{da}.
\]
It will be shown in subsequent sections that this description of kinematics is compatible with stress equations for the model chosen.

3. Material model

The material is supposed to be rigid plastic. In particular, the elastic portion of the strain tensor is neglected. It is assumed that the principal axes of the strain rate and stress tensors coincide. Then, the coordinate curves of the Lagrangian coordinate system are principal stress trajectories. The plane strain yield criterion at the initial instant is

$$|\tau_x| = |\tau_y| = \frac{\sigma_0 g(x)}{\sqrt{3}}$$

(8)

where $\tau_x$ and $\tau_y$ are the deviatoric stresses in the $(x, y)$ coordinate system, $\sigma_0 g(x)$ is the yield stress in uniaxial tension, $\sigma_0$ is a constant with dimension of stress and $g(x)$ is a given non-dimensional function of its argument. It is assumed that plastic deformation does not change material properties at material particles. Therefore, using (8) the yield criterion after any amount of deformation can be written as

$$|\tau_{\xi}| = |\tau_\eta| = \frac{\sigma_0 g(H\zeta)}{\sqrt{3}}$$

(9)

where $\tau_{\xi}$ and $\tau_\eta$ are the deviatoric stresses in the Lagrangian coordinate system. Moreover, it is seen from (3) that

$$r_{\xi} = r_\eta, \quad \tau_\eta = \tau_\theta$$

(10)

where $\tau_\xi$ and $\tau_\theta$ are the deviatoric stresses in the polar coordinate system. The associated flow rule is automatically satisfied because the mapping (1) satisfies the equation of incompressibility and, by assumption, the principal axes of the strain rate and stress tensors coincide. Thus the only constitutive equation to be satisfied is the yield criterion.

4. General stress solution

The sheet is loaded by bending moment $M$ and force $F$ per unit length. A consequence of the equilibrium equations is that some pressure $P$ should be applied to the concave surface of the sheet in the process of deformation. Assuming that the pressure is uniformly distributed over the surface it is possible to find that $F = \kappa P$. Using equation (4) this relation can be transformed to

$$\frac{P}{\sigma_0} = \frac{fa}{\sqrt{s-a}}, \quad f = \frac{F}{\sigma_0 H}.$$  

(11)

Let $\sigma_{\xi}$ and $\sigma_\eta$ be the normal stresses in the Lagrangian coordinates. It is evident that these stresses are the principal stresses. In the case under consideration $\tau_\eta = -\tau_\eta$. Substituting this equation into the identity $\sigma_{\xi} - \tau_\xi = \sigma_\eta - \tau_\eta$ leads to

$$\sigma_\eta = \sigma_{\xi} - 2\tau_\xi.$$  

(12)

The bending moment, $M$, and its dimensionless representation, $m$, are

$$M = \frac{1}{h} \int \left( \sigma_\eta - \frac{F}{h} \right) r dr, \quad m = \frac{2\sqrt{3} M}{\sigma_0 H^2}.$$  

(13)
Note that \( m = 1 \) in the process of pure bending of a homogeneous rigid perfectly plastic sheet if \( \sigma_0 \) is the yield stress in uniaxial tension [14]. Replacing in (13) integration with respect to \( r \) with integration with respect to \( \zeta \) by means of (3) results in

\[
m = \frac{\sqrt{3}}{a} \int_{-1}^{0} \left( \frac{\sigma_n}{\sigma_0} - \frac{f}{h} \right) d\zeta. \tag{14}\]

Taking into account (10) the only non-trivial equilibrium equation in the polar coordinates can be written as

\[
\frac{\partial \sigma_\zeta}{\partial r} + \frac{\tau_\zeta - \tau_\eta}{r} = 0.
\]

Replacing in this equation the derivative with respect to \( r \) with the derivative with respect to \( \zeta \) by means of (3) leads to

\[
\frac{\partial \sigma_\zeta}{\partial \zeta} + \frac{a(\tau_\zeta - \tau_\eta)}{2(\zeta a + s)} = 0. \tag{15}\]

The inequalities \( \tau_\zeta > 0 \) and \( \tau_\eta < 0 \) are valid in the region \(-1 \leq \zeta < \zeta_n\). Then, substituting (9) into (15) gives

\[
\frac{\partial \sigma_\zeta}{\sigma_0 \partial \zeta} + \frac{ag(H\zeta)}{\sqrt{3}(\zeta a + s)} = 0. \tag{16}\]

The boundary condition to this equation is \( \sigma_\zeta = -P \) for \( \zeta = -1 \) where \( P \) is given in (11). The solution of equation (16) satisfying this boundary condition is

\[
\frac{\sigma_\zeta}{\sigma_0} = -\frac{fa}{\sqrt{s-a}} - \frac{a}{\sqrt{3}} \int_{-1}^{\zeta_n} \frac{g(H\chi)}{(\chi a + s)} d\chi. \tag{17}\]

Here \( \chi \) is a dummy variable of integration. Let \( \sigma_n \) be the value of \( \sigma_\zeta \) at \( \zeta = \zeta_n \). Then, it follows from (17) that

\[
\frac{\sigma_n}{\sigma_0} = -\frac{fa}{\sqrt{s-a}} - \frac{a}{\sqrt{3}} \int_{-1}^{\zeta_n} \frac{g(H\chi)}{(\chi a + s)} d\chi. \tag{18}\]

The inequalities \( \tau_\zeta < 0 \) and \( \tau_\eta > 0 \) are valid in the region \( \zeta_n < \zeta \leq 0 \). Then, substituting (9) into (15) gives

\[
\frac{\partial \sigma_\zeta}{\sigma_0 \partial \zeta} - \frac{ag(H\zeta)}{\sqrt{3}(\zeta a + s)} = 0. \tag{19}\]

The boundary condition to this equation is \( \sigma_\zeta = 0 \) for \( \zeta = 0 \). The solution of equation (19) satisfying this boundary condition is

\[
\frac{\sigma_\zeta}{\sigma_0} = \frac{a}{\sqrt{3}} \int_{0}^{\zeta} \frac{g(H\chi)}{(\chi a + s)} d\chi. \tag{20}\]

It follows from this equation that
\[ \frac{\sigma_n}{\sigma_0} = \frac{a}{\sqrt{3}} \int_{0}^{\zeta} g(H\chi) \, d\chi. \]  

Equations (18) and (21) combine to give
\[ -\frac{\sqrt{3}f}{\sqrt{s-a}} = \int_{-1}^{0} g(H\chi) \, d\chi = \int_{0}^{\zeta} g(H\chi) \, d\chi. \]

In this equation, \( \zeta_n \) can be replaced with \( -ds/da \) by means of (7). If \( f \) is prescribed then (22) is an ordinary differential equation for \( s \) as a function of \( a \). This equation should satisfy the boundary condition (2). However, in the case of ideal deformation \( f \) should be found from the solution.

5. Ideal deformation
It follows from the conditions imposed on ideal deformation (see, for example, [5]) that \( \zeta_n \) is independent of \( a \). At the initial instant, the distribution of the normal stress along the surface \( y = L \) is

\[ \sigma_{n} = \sigma_0 g(H\zeta) \quad \text{in the range} \quad \zeta_n < \zeta \leq 0 \]
\[ \sigma_{n} = -\sigma_0 g(H\zeta) \quad \text{in the range} \quad -1 \leq \zeta < \zeta_n. \]  

The initial value of \( F \) should be prescribed, \( F = F_0 \). The corresponding value of \( f \) is denoted by \( f_0 \). Then, it follows from (11) and (23) that
\[ \int_{-1}^{0} g(H\zeta) \, d\zeta - \int_{0}^{\zeta_n} g(H\zeta) \, d\zeta = f_0. \]  

This equation determines \( \zeta_n \) at the initial instant and, therefore, throughout the process of deformation. Substituting this value of \( \zeta_n \) into (22) gives the variation of \( f \) with \( a \) that ensures ideal deformation. The variation of \( m \) with \( a \) is then found from (14) in which the ratio \( \sigma_{n}/\sigma_0 \) should be eliminated by means of (7), (12), (17) and (20).

6. Conclusions
A new ideal flow solution for the process of bending under tension of functionally graded sheets has been presented. The through – thickness distribution of the yield stress is characterized by the function \( g(x) \) involved in (8) and is quite arbitrary. A review of material property distributions in functionally graded sheets used in the mechanics of functionally graded material structures has been provided in [15]. Any of these distributions can be combined with the new solution. The solution is practically analytic. A numerical technique is only necessary to evaluate ordinary integrals in (14), (17), (20) and (24).

Acknowledgments
The second author was partly supported by Moscow Technological University [16].

References
[1] R. Hill, “Ideal Forming Operations for Perfectly Plastic Solids,” J. Mech. Psys. Solids, vol. 15, pp. 223–227, 1967.
[2] O. Richmond and S. Alexandrov, “The Theory of General and Ideal Plastic Deformations of Tresca Solids,” Acta Mech., vol. 158, pp. 33–42, 2002.
[3] S. Alexandrov, Y. Mustafa, and E. Lyamina, “Steady Planar Ideal Flow of Anisotropic Materials,” Meccanica, vol. 51, pp. 2235–2241, 2016.
[4] I.F. Collins and S.A. Meguid, “On the Influence of Hardening and Anisotropy on the Plane-Strain Compression of Thin Metal Strip,” Trans. ASME J. Appl. Mech., vol. 44, pp. 271–278, 1977.

[5] K. Chung and S. Alexandrov, “Ideal Flow in Plasticity,” Appl. Mech. Rev., vol. 60, pp. 316–335, 2007.

[6] O. Richmond and H.L. Morrison, “Streamlined Wire Drawing Dies of Minimum Length,” J. Mech. Phys. Solids, vol. 15, pp. 195–203, 1967.

[7] M. Yoshida, F. Yoshida, H. Konishi and K. Fukumoto, “Fracture Limits of Sheet Metals under Stretch Bending,” Int. J. Mech. Sci., vol. 47, pp. 1885–1896, 2005.

[8] M.H. Parsa and S.N.A. Ahkami, “Bending of Work Hardening Sheet Metals Subjected to Tension,” Int. J. Mater. Form., suppl. 1, pp. 173–176, 2008.

[9] M.A. Guler, F. Ozer, M. Yenice, and M. Kaya, “Springback Prediction of DP600 Steels for Various Material Models,” Steel Res. Int., vol. 81, pp. 801–804, 2010.

[10] S. Alexandrov, K. Manabe, and T. Furushima, “A General Analytic Solution for Plane Strain Bending under Tension for Strain-Hardening Material at Large Strains,” Arch. Appl. Mech., vol. 81, pp. 1935–1952, 2011.

[11] S. Alexandrov, J.-H. Kim, K. Chung, and T.-J. Kang, “An alternative Approach to Analysis of Plane-Strain Pure Bending at Large Strains,” J. Strain Anal. Eng. Des., vol. 41, pp. 397–410, 2006.

[12] S. Alexandrov, W. Lee, and K. Chung, “Design of Plane Strain Bending Based on Ideal Flow Theory,” Proc. 8th Int. Conf. NUMIFORM2004, American Institute of Physics, pp. 1664-1669, 2004.

[13] E. Lyamina and S. Alexandrov, “An Elastic Plastic Non-Steady Ideal Flow Solution and Its Application to Bending under Tension,” J. Technol. Plast., vol. 34, pp. 39–47, 2009.

[14] R. Hill, The Mathematical Theory of Plasticity. Oxford: Oxford University Press, 1950.

[15] I. Elishakoff, D. Pentaras, and C. Gentilini, Mechanics of Functionally Graded Material Structures. Singapore: World Scientific, 2016.

[16] V.L. Pankov, “Effectiveness of Incentive Mechanism and the Potential Level Meeting the Needs of an Employee,” Herald of MSTU MIREA, vol. 1, № 4 (9), pp. 288, 2015.