Coordinate-invariant Path Integral Methods in Conformal Field Theory

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We present a coordinate-invariant approach, based on a Pauli-Villars measure, to the definition of the path integral in two-dimensional conformal field theory. We discuss some advantages of this approach compared to the operator formalism and alternative path integral approaches. We show that our path integral measure is invariant under conformal transformations and field reparametrizations, in contrast to the measure used in the Fujikawa calculation, and we show the agreement, despite different origins, of the conformal anomaly in the two approaches. The natural energy-momentum in the Pauli-Villars approach is a true coordinate-invariant tensor quantity, and we discuss its nontrivial relationship to the corresponding non-tensor object arising in the operator formalism, thus providing a novel explanation within a path integral context for the anomalous Ward identities of the latter. We provide a direct calculation of the nontrivial contact terms arising in expectation values of certain energy-momentum products, and we use these to perform a simple consistency check confirming the validity of the change of variables formula for the path integral. Finally, we review the relationship between the conformal anomaly and the energy-momentum two-point functions in our formalism.

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1. Introduction

In the conformal field theory literature, path integrals are often used for heuristic derivations. Operator methods or axiomatics are then invoked to give operational meaning to the resulting expressions.

Doing this is highly dangerous. It is easy to overlook the fact that identities derived using path integral methods may be invalidated by replacing insertions by the obvious operator quantities. In fact, the relationship between natural path integral and natural operator quantities can be highly nontrivial. Some lack of appreciation of this issue in the literature clearly calls for a more careful comparative analysis.

There are few direct path integral calculations in the conformal field theory literature. Apart from some derivations of the conformal anomaly, done for example using the Fujikawa approach, careful path integral calculations of many nontrivial results, including Ward identities, seem to be absent. Here we provide a few such
calculations.

A disadvantage of the operator formalism is that the definitions of the operators typically depend on a choice of coordinates, due to the chosen regularizations. The coordinate-invariance of the original Lagrangian is neither respected nor exploited. Changes of coordinates are conflated with implicit anomalous gauge transformations, complicating the transformation laws and obscuring the physical origin of the anomalies.

Some common approaches to path-integrals suffer from comparable problems. Dimensional, zeta-function and heat kernel regularization techniques all introduce implicit dependencies on extra structure on the world sheet via choices of coordinates, cutoff, complex structure, or through the lack of a canonical way of regulating interesting operator insertions. In addition, neither dimensional nor zeta-function techniques are adaptable to a non-perturbative definition of the path integral measure.

The Fujikawa approach provides a way of defining a regulated path integral measure, but does not come with a corresponding prescription for regulating interesting operator insertions. It is therefore incomplete.

In this paper we address all these issues by presenting a coordinate-invariant approach, based on a Pauli-Villars regularization, to the definition of the path integral measure and the calculation of anomalies in a two-dimensional scalar conformal field theory. The measure does not depend on a choice of coordinates or complex structure on the world sheet and is largely insensitive to the details of the cutoff procedure. In contrast to the Fujikawa measure, it is invariant under conformal transformations and field reparametrizations, and it simultaneously regulates also the interesting operator insertions. In contrast to dimensional or zeta-function techniques, the Pauli-Villars measure is suitable for a non-perturbative definition of the path integral. In contrast to the operator approach, the natural insertions are regulated in a coordinate-invariant way, and are true tensors.

By comparing the path integral measures in the two cases, we show the consistency, despite apparently different origins, of the conformal anomaly in the Pauli-Villars and the Fujikawa approaches. In the Fujikawa approach, the anomaly arises as a dependence of the path integral measure on the background metric. On the other hand, we show how the full Pauli-Villars path integral measure is invariant under conformal transformations of the background, and the anomaly is shifted to the expectation value of the trace of the energy-momentum tensor, which becomes nonzero. While the conformal anomaly has been derived before using Pauli-Villars regulators, previous work has either been silent on the proper definition of the measure or introduced new, nonstandard measures for the Pauli-Villars auxiliary fields. Here we only use ordinary bosonic and Grassmann auxiliary fields.

We point out that there is a nontrivial relationship between the energy-momentum tensor in the path-integral formalism and the corresponding quantity in the operator formalism. We show that the natural energy-momentum in the Pauli-Villars approach is a true coordinate-invariant tensor quantity, which we then relate
to the more familiar non-tensor object arising in the operator approach, obtaining a new explanation within the context of the path integral for the anomalous properties of the latter.

We confirm by explicit calculation that the full energy-momentum tensor satisfies classical Ward identities, and we explain how these are related to the anomalous Ward identities found in the operator formalism.

We provide a first direct path-integral calculation of the nontrivial contact terms arising in expectation values of certain energy-momentum products, previously derived only using axiomatic considerations, and we use these to perform a simple consistency check confirming the change of variables formula for the Pauli-Villars path integral. Finally, we review the relationship between the conformal anomaly and the energy-momentum two-point functions in our formalism. Here the contact terms are essential to obtaining the correct results.

2. The Fujikawa approach

In this section we derive the conformal anomaly using an adaptation of the methods of Fujikawa, according to which anomalies arise as a non-invariance of the path integral measure under classical symmetries. We present the full calculation, both because our method is simpler than existing presentations, and because we will reuse aspects of it in our calculation of the transformation of the Pauli-Villars measure in the next section. There we will provide a derivation of the anomaly using a Pauli-Villars definition of the path integral measure, showing that the origin of the anomaly is shifted from the measure to the expectation value of the trace of the energy-momentum tensor.

We are interested in the dependence of the partition function

\[ Z(g) \equiv \int [d\phi]_g e^{-S(g, \phi)} \]

on the background metric \( g_{ij} \). Here \([d\phi]_g\) denotes the \( \infty \)-dimensional differential form

\[ [d\phi]_g \equiv \bigwedge_{n=0}^{\infty} da^0_n = da^0_n \wedge da^1_n \wedge \cdots, \]

where the coefficients \( a^0_n : \phi \to \mathbb{R} \) depend on the metric via the expansion

\[ \phi(z) = \sum_{n=0}^{\infty} a^0_n \phi^g_n(z), \]

and where \( \phi^g_n(z) \) denotes an orthonormal basis of field configurations satisfying

\[ \int d^2 x \sqrt{g} \phi^g_m(x) \phi^g_n(x) = \delta_{mn}. \]  \( \text{(1)} \)
To calculate the variation \( \delta_g[\phi]_g \) of the path integral measure with respect to deformations of the metric, we will need the variation

\[
\delta_g a^g_m = \delta_g \int d^2 x \sqrt{g} \phi^g_m \phi
\]

(2)

\[
= \int d^2 x \sqrt{g} \left( \frac{1}{2} g^{ij} \delta g_{ij} \right) \phi^g_m \phi + \int d^2 x \sqrt{g} (\delta \phi^g_m) \phi.
\]

(3)

The metric dependence of \( \phi^g_m \) is not uniquely fixed by the above orthonormality requirement. However, given a metric \( g \), any two orthonormal bases are related by a unitary transformation that will leave the form \( [\phi]_g \) invariant. Therefore we can, without loss of generality, choose one particular metric dependence for \( \phi^g_m \) compatible with orthonormality. To find such a choice, we vary both sides of (1) with respect to the metric, obtaining

\[
0 = \int d^2 x \sqrt{g} \left( \frac{1}{2} g^{ij} \delta g_{ij} \right) \phi^g_m \phi^g_n + \int d^2 x \sqrt{g} (\delta \phi^g_m) \phi^g_n + \int d^2 x \sqrt{g} \phi^g_m (\delta \phi^g_n).
\]

A suitable choice for \( \delta_g \phi^g_m \) is therefore

\[
\delta_g \phi^g_m = -\frac{1}{4} g^{ij} \delta g_{ij} \phi^g_m.
\]

Inserting this in (3) gives

\[
\delta_g a^g_m = \int d^2 x \sqrt{g} \left( \frac{1}{4} g^{ij} \delta g_{ij} \right) \phi^g_m \phi
\]

\[
= \sum_n a^g_n \cdot \int d^2 x \sqrt{g} \left( \frac{1}{4} g^{ij} \delta g_{ij} \right) \phi^g_m \phi^g_n
\]

\[
= \sum_n a^g_n C_{mn},
\]

where

\[
C_{mn} = \int d^2 x \sqrt{g} \phi^g_m \left( \frac{1}{4} g^{ij} \delta g_{ij} \right) \phi^g_n
\]

We then find, using the normal rules for manipulating differential forms,

\[
\delta_g[\phi]_g = \delta_g \left( \bigwedge_n d a^g_n \right)
\]

\[
= \sum_m \cdots \wedge d a^g_{m-1} \wedge \delta (d a^g_m) \wedge d a^g_{m+1} \wedge \cdots
\]

\[
= \sum_m \cdots \wedge d a^g_{m-1} \wedge \left( \sum_n d a^g_n C_{nm} \right) \wedge d a^g_{m+1} \wedge \cdots
\]

\[
= \left( \sum_m C_{mm} \right) \wedge d a^g_n
\]

\[
= (\text{Tr} C)_g \ [\phi]_g,
\]

(4)
where $C$ is the operator

$$C \equiv \frac{1}{4} g^{ij} \delta g_{ij} = \frac{1}{2} \frac{1}{\sqrt{g}} \delta \sqrt{g} = \delta \omega.$$  

Notice that $\delta \sqrt{g}$ is the local change of volume. In other words, the path integral measure will be scale dependent. This is the origin of the conformal anomaly.

We now need to calculate the trace

$$(\text{Tr} C)_{g} = \int d^{2} x \sqrt{g} \delta \omega(x) A(x),$$

where the infinite sum

$$A(x) \equiv \sum_{m} \phi_{m}^{g}(x) \phi_{m}^{g}(x)$$

does not in general converge. A natural short-distance regularization, which can be taken as part of the definition of the path integral, is obtained by considering instead the limit as $\epsilon \rightarrow 0$ of the sum

$$\sum_{m} \phi_{m}^{g}(x) e^{\epsilon \Delta} \phi_{m}^{g}(x) = \langle x | e^{\epsilon \Delta} | x \rangle$$

where $\Delta$ denotes the Laplacian

$$\Delta \equiv \frac{1}{\sqrt{g}} \partial_{i} \left[ \sqrt{g} g^{ij} \partial_{j} (\cdot) \right].$$

Here the position basis bras and kets are defined via $\langle x | y \rangle \equiv \delta(x - y) / \sqrt{g}$, to make $\langle x | \phi \rangle = \phi(x)$ with respect to the inner product $\langle \phi_{1} | \phi_{2} \rangle = \int \sqrt{g} \phi_{1} \phi_{2}$.

As required for consistency with our previous remark that a unitary transformation leave $[d\phi]_{g}$ invariant, the regularized expression $\langle x | e^{\epsilon \Delta} | x \rangle$ depends only on the metric and not on the particular orthonormal basis $\phi_{m}^{g}$ that we have chosen. Also notice that if we insert an eigenbasis $| m \rangle$ of $\Delta$, the above becomes

$$\langle x | e^{\epsilon \Delta} | x \rangle = \sum_{m} \langle m | x \rangle \langle x | m \rangle e^{\epsilon \lambda_{m}}$$

where the $\lambda_{m}$ are the corresponding (negative) eigenvalues, making it clear that the regularization corresponds to a large-momentum suppression.

We therefore need to calculate the small $t$ behaviour of the function

$$G(x, t | y, 0) \equiv \theta(t) \langle x | e^{t \Delta} | y \rangle,$$

where $\theta(t)$ denotes the step function, inserted to avoid the regime $t < 0$ where $\langle x | e^{t \Delta} | y \rangle$ diverges. It is easy to check that

$$(\partial_{t} - \Delta_{x}) G(x, t | y, 0) = \delta(t) \langle x | y \rangle = \frac{1}{\sqrt{g}} \delta(t) \delta(x - y).$$

In other words, the distribution $G(x, t | y, 0)$ solves the heat or diffusion equation given a point source at $y$ at time $t = 0$, and is known as a heat kernel. Without loss
of generality, we can choose coordinates so that \( y = 0 \). In two dimensions, we can further choose the coordinate system so that
\[
\begin{align*}
g_{ij}(x) &= e^{2\omega} \delta_{ij}, \\
\omega(0) &= 0.
\end{align*}
\]
so that the equation for \( G(x, t|y, 0) \) becomes
\[
(\partial_t - e^{-2\omega(x)} \Delta_0) G(x, t|0, 0) = \delta(t) \delta(x),
\]
where \( \Delta_0 = \delta^{ij} \partial_i \partial_j \). Regarding this as an operator equation in the space \( \mathcal{L}^2(\mathbb{R}^2) \), we write the solution as
\[
G(x, t|0, 0) = \langle x, t \rangle \left( \frac{1}{\partial_t - e^{-2\omega(x)} \Delta_0} \right) |0, 0\rangle.
\]
Since we are interested in the limit as \( t \to 0 \), for which the diffusion becomes increasingly short-ranged, we will develop an expansion for \( G(0, t|0, 0) \) in terms of derivatives of \( \omega \) at the origin. Expanding around \( x^i = 0 \), we have
\[
e^{-2\omega(x)} = 1 - 2(\partial_i \omega) x^i + \left[-(\partial_i \partial_j \omega) + 2(\partial_i \omega)(\partial_j \omega)\right] x^i x^j + \cdots,
\]
where the derivatives are all evaluated at \( x^i = 0 \). We can then write
\[
G = \frac{1}{A - B} = \frac{1}{A} + \frac{1}{A} B \frac{1}{A} + \frac{1}{A} B \frac{1}{A} B + \cdots
\]
where \( A = \partial_t - \Delta_0 \), so that \( 1/A \) is the flat space solution
\[
\langle x, t \rangle \left( \frac{1}{A} \right) |0, 0\rangle = \frac{1}{4\pi t} e^{-x^2/4t} \theta(t)
\]
while
\[
B = \left\{ -2(\partial_i \omega) x^i + \left[-(\partial_i \partial_j \omega) + 2(\partial_i \omega)(\partial_j \omega)\right] x^i x^j + \cdots \right\} \Delta_0.
\]
Inserting this expansion in the first-order contribution \( A^{-1}BA^{-1} \) to \( G(0, \epsilon|0, 0) \), the first term, with odd integrand proportional to \( x^i \), vanishes. The next term in \( A^{-1}BA^{-1} \) is proportional to
\[
\int_0^\epsilon dt \int d^2 x \frac{1}{4\pi (\epsilon - t)} e^{-x^2/4(\epsilon - t)} x^i x^j \Delta_0 \frac{1}{4\pi t} e^{-x^2/4t} x^i x^j \\
= \int_0^\epsilon dt \int d^2 x \frac{1}{4\pi (\epsilon - t)} \frac{1}{4\pi t} e^{-x^2/4(\epsilon - t)} x^i x^j \left[ \frac{1}{t} + \frac{x^2}{4t^2} \right] e^{-x^2/4t} x^i x^j \\
= \frac{1}{2} \delta^{ij} \int_0^\epsilon dt \int d^2 x \frac{1}{4\pi (\epsilon - t)} \frac{1}{4\pi t} e^{-x^2/4(\epsilon - t)} x^2 \left[ -\frac{1}{t} + \frac{x^2}{4t^2} \right] e^{-x^2/4t} x^i x^j \\
= \frac{1}{2} \delta^{ij} \int_0^\epsilon dt \int d^2 x \frac{1}{4\pi (\epsilon - t)} \frac{1}{4\pi t} x^2 \left[ -\frac{1}{t} + \frac{x^2}{4t^2} \right] e^{-x^2/4t(\epsilon - t)}.
\]
Writing the terms containing \( x^2 \) and \( x^4 \) as derivatives with respect to \( \lambda \equiv \epsilon/4t (\epsilon - t) \) of the Gaussian integral
\[
\int d^2 x e^{-\lambda x^2} = \frac{\pi}{\lambda},
\]
this simplifies to

$$\frac{1}{2\pi} \delta^{ij} \int_0^\epsilon dt \left\{ -\frac{(\epsilon - t)}{\epsilon^2} + \frac{2(\epsilon - t)^2}{\epsilon^3} \right\} = \frac{1}{12\pi} \delta^{ij}$$

The next nonzero term in the contribution $A^{-1}BA^{-1}$ has integrand proportional to $x^i x^j x^k x^l$. This may be checked to be of order $\epsilon$. Further terms are of even higher order in $\epsilon$, so that the contribution to $G$ from the term $A^{-1}BA^{-1}$ can be written

$$G_1(0, \epsilon|0, 0) = \frac{1}{12\pi} \left[ -\Delta \omega + 2 \partial_i \omega \partial^i \omega \right] + o(\epsilon).$$

There is one additional contribution of order $\epsilon^0$ to $G(0, \epsilon|0, 0)$. It comes from the second-order term $A^{-1}BA^{-1}BA^{-1}$ in the above expansion. It is

$$4 (\partial_i \omega) (\partial_j \omega) \int_0^\epsilon dt \int_0^t du \int d^2x \int d^2y \times$$

$$\times \frac{1}{4\pi (\epsilon - t)} e^{-x^2/4(\epsilon - t)} x^i \Delta_0^x \frac{1}{4\pi (t - u)} e^{-(x-y)^2/4(t-u)} y^j \Delta_0^u \frac{1}{4\pi u} e^{-y^2/4u}$$

By similar manipulations, this becomes

$$-\frac{1}{6\pi} \partial_i \omega \partial^i \omega,$$

so that

$$G_2(0, \epsilon|0, 0) = -\frac{1}{6\pi} \partial_i \omega \partial^i \omega + o(\epsilon)$$

Notice that this contributes a term that exactly cancels the term of the same form in $G^2(0, \epsilon|0, 0)$. We find the result

$$G(0, \epsilon|0, 0) = \frac{1}{4\pi \epsilon} - \frac{1}{12\pi} \Delta \omega + o(\epsilon)$$

$$= \frac{1}{4\pi \epsilon} + \frac{1}{24\pi} R + o(\epsilon),$$

where we have used

$$R = -2 e^{-2\omega} \Delta \omega$$

and $\omega(0) = 0$. Inserting our result into the formula for the variation of the measure, we obtain

$$\delta_g \int [d\phi]_g e^{-S(\phi,g)} = \left( \frac{1}{24\pi} \int d^2x \sqrt{g} \delta \omega(x) R \right) \int [d\phi]_g e^{-S(\phi,g)}. \quad (5)$$
3. A Pauli-Villars derivation of the anomaly

In the previous section the conformal anomaly was obtained from the dependence of the path integral measure on the metric. In the case of a Weyl transformation $g_{ij} \rightarrow e^{2\omega}g_{ij}$, we could absorb the anomalous dependence of the measure into the energy-momentum tensor by defining a modified energy-momentum tensor $\tilde{T}_{ij}$ via

$$
\delta \int [d\phi]_g e^{-S(g,\phi)} = \frac{1}{4\pi} \int d^2x \sqrt{g} \delta g^{ij} \langle \tilde{T}_{ij} \rangle_g = \frac{1}{4\pi} \int d^2x \sqrt{g} (-2\delta \omega) \langle \tilde{T}_{ij} \rangle_g.
$$

Given the action

$$
S(\phi) = \frac{1}{2} \int d^2x \sqrt{g} g^{ij} \partial_i \phi \partial_j \phi,
$$

we have

$$
T_{ij} = -2\pi \left( \partial_i \phi \partial_j \phi - \frac{1}{2} g_{ij} g^{kl} \partial_k \phi \partial_l \phi \right).
$$

Comparing with (5), this would require that we define

$$
\tilde{T}^i_i \equiv T^i_i - \frac{1}{12} R = -\frac{1}{12} R,
$$

since $T^i_i$ is identically zero. The curvature term was entirely due to the variation $\delta [d\phi]_g$ of the integration measure.

Note, however, that the Fujikawa method is incomplete, since it does not provide a canonical choice for regulating $T_{ij}$. Without specifying such a choice, we cannot assume, as we have above, that $\langle \tilde{T}^i_i \rangle$ will indeed be zero, or even finite.

It is therefore very instructive to derive the conformal anomaly using a Pauli-Villars regularization, which provides a complete, coordinate-invariant way of regulating both the measure and the energy-momentum tensor. This was first done by Vilenkin in Ref. 7, though only up to first order in the expansion of the curvature around flat space, and ignoring possible nontrivial transformations of the path integral measure. For related work, including comparisons of the Pauli-Villars method to the $\zeta$-function and point-splitting methods, see for example Refs. 8–12. These works are either silent on the proper definition of the measure or introduce new, nonstandard measures for the Pauli-Villars auxiliary fields. In the following we carefully define the measure using only ordinary bosonic and Grassmann auxiliary fields.

With a Pauli-Villars definition of the path integral, we shall see that the integration measure will in fact be invariant under variations of $\tilde{T}^i_i$ and the curvature anomaly will instead be due to the fact that the Pauli-Villars fields are massive, so that $T^i_i \neq 0$, and we shall indeed find that

$$
\langle \tilde{T}^i_i \rangle_g \rightarrow -\frac{1}{12} R
$$

(7)
in the limit where the Pauli-Villars masses go to infinity.

In the following discussions, we will have need to carefully regulate the theory in both the infrared and ultraviolet. For our infrared regularization we give a small mass \( m \) to the field \( \phi \). The ultraviolet regularization will consist in adding auxiliary Pauli-Villars fields that are either real scalars \( \chi_m \) with bosonic statistics and action

\[
S(\chi_m) = \frac{1}{2} \int d^2x \sqrt{g} \left( g^{ij} \partial_i \chi_m \partial_j \chi_m + M^2_m \chi_m^2 \right)
\]
or complex scalars \( \chi_m, \bar{\chi}_m \) with Grassmann statistics and action

\[
S(\chi_m, \bar{\chi}_m) = \frac{1}{2} \int d^2x \sqrt{g} \left( g^{ij} \partial_i \bar{\chi}_m \partial_j \chi_m + M^2_m \bar{\chi}_m \chi_m \right).
\]

Denoting \( \bar{\chi}_m = \chi_m \) for the bosonic fields, we can write the Pauli-Villars action in unified form

\[
S_{PV} = \sum_m \frac{1}{2} \int d^2x \sqrt{g} \left( g^{ij} \partial_i \chi_m \partial_j \chi_m + M^2_m \bar{\chi}_m \chi_m \right).
\]

The masses \( M_m \) and statistics of the fields will be chosen to obtain finite expectation values for interesting quantities. In the end, we will be interested in the limits \( m \to 0 \) where the field \( \phi \) becomes massless, and \( M_m \to \infty \) where the Pauli-Villars fields become non-dynamical.

We first calculate the variation

\[
\delta_g \left( [d\phi]_{g}^{PV} \right) = \delta_g \left( [d\phi]_g \wedge [d\chi_1]_g \wedge \cdots \wedge [d\chi_n]_g \right)
\]
of the full path integral measure \([d\phi]_{g}^{PV}\) including matter and auxiliary fields, where for a bosonic field \( \chi \) the form \([d\chi]_g\) is defined just like \([d\phi]_g\) in the previous section, while for a Grassmann field \( \chi, \bar{\chi} \), we define

\[
[d\chi]_g \equiv d\chi_0^g \wedge d\bar{\chi}_0^g \wedge d\chi_1^g \wedge d\bar{\chi}_1^g \wedge \cdots.
\]

Referring back to our calculation (4), each real scalar field \( \chi_i \) will contribute a term

\[
(\text{Tr} \, C)_g \left[ [d\chi_i]_g \right]
\]
to the variation. The contribution of each Grassmann field \( \chi_i \) will be

\[
-2 \left( \text{Tr} \, C \right)_g \left[ [d\chi_i]_g \right].
\]

Indeed, the reader may easily check that with Grassmann integration rules, if \( \delta \chi = \epsilon \chi \) where \( \epsilon \) is a real parameter, then for the change of variables formula \( \int d\chi' f(\chi') = \int d\chi f(\chi) \) to hold (equivalent to \( \delta \int d\chi f(\chi) = 0 \)) one needs \( \delta (d\chi) \equiv -\epsilon d\chi \). Doing this for each factor in \( d\chi_0^g \wedge d\bar{\chi}_0^g \), we find a factor of \(-2\) relative to the result (4) of the previous section.
Defining $c_i = 1$ for $\chi_i$ bosonic and $c_i = -2$ for $\chi_i$ Grassmann, we find

$$
\delta_g \left( [d\phi]_g \wedge [d\chi_1]_g \wedge \cdots \wedge [d\chi_n]_g \right) 
= \left( 1 + \sum_i c_i \right) \left( \frac{1}{4\pi \epsilon} \int d^2 x \sqrt{g} \delta \omega(x) + \frac{1}{24\pi} \int d^2 x \sqrt{g} \delta \omega(x) R \right) \times 
\times \left( [d\phi]_g \wedge [d\chi_1]_g \wedge \cdots \wedge [d\chi_n]_g \right)
$$

We therefore can choose our Pauli-Villars field statistics to cancel the variation of $[d\phi]_g$. In particular, if we choose

$$
1 + \sum_i c_i = 0,
$$

the variation of the full measure is zero

$$
\delta_g \left( [d\phi]_{PV} \right) \equiv \delta_g \left( [d\phi]_g \wedge [d\chi_1]_g \wedge \cdots \wedge [d\chi_n]_g \right) = 0.
$$

Since the measure is now invariant, the anomaly will have to be due entirely to the expectation value of the trace of the energy-momentum tensor, i.e.,

$$
\delta \int [d\phi]_{PV} e^{-S(g,\phi,\bar{\chi}_i,\chi_i)} = \frac{1}{4\pi} \int d^2 x \sqrt{g} \left( -2 \delta \omega \right) \left\langle T_{ii} \right\rangle_g.
$$

where, with our convention

$$
\left\langle T_{ii} \right\rangle_g = 2\pi \left\langle m^2 \phi^2 + \sum_i M_i^2 \bar{\chi}_i \chi_i \right\rangle_g,
$$

and the expectation values are finite due to conditions on the Pauli-Villars masses to be specified below.

The propagator of a complex scalar is double that of a real scalar, while each Grassmann loop takes an additional factor $-1$. We therefore find on the plane, with $g_{ij} = \delta_{ij}$,

$$
\left\langle T_{ii} \right\rangle_\delta = 2\pi \int \frac{d^2 p}{(2\pi)^2} \left\{ \frac{m^2}{p^2 + m^2} + \sum_i c_i \frac{M_i^2}{p^2 + M_i^2} \right\}.
$$

As long as we impose the condition

$$
m^2 + \sum c_i M_i^2 = 0,
$$

on the Pauli-Villars masses, the integral converges. The answer in this case is

$$
2\pi \cdot \frac{1}{4\pi} \lim_{\Lambda \to \infty} \left( m^2 \ln \frac{\Lambda^2}{m^2} + \sum_i c_i M_i^2 \ln \frac{\Lambda^2}{M_i^2} \right) 
= - \frac{1}{2} \left( m^2 \ln \frac{m^2}{\mu^2} + \sum_i c_i M_i^2 \ln \frac{M_i^2}{\mu^2} \right),
$$
where the limit is finite due to the Pauli-Villars condition (12). Here $\mu$ is an arbitrary parameter with dimension of mass, irrelevant due to the condition (9). As $m \to 0$, the first term in (11) falls away and we get

$$-\frac{1}{2} \sum_i c_i M_i^2 \ln \frac{M_i^2}{\mu^2},$$

This diverges in the limit $M_i \to \infty$ but can be exactly compensated by a counterterm of the form

$$\frac{1}{8\pi} \sum_i c_i M_i^2 \ln \frac{M_i^2}{\mu^2} \int d^2x \sqrt{g},$$

in the original action. We could also avoid the need for the counterterm by imposing the additional Pauli-Villars condition

$$m^2 \ln \frac{m^2}{\mu^2} + \sum_i c_i M_i^2 \ln \frac{M_i^2}{\mu^2} = 0. \quad (13)$$

If we do this, note that to satisfy the three Pauli-Villars conditions (9), (12) and (13) with each $c_i = 1$ or $c_i = -2$, while keeping the ability to take the masses $M_i \to \infty$, we will need at least five auxiliary fields. If we use exactly five, three of these should be bosonic and two Grassmann.

We now consider the case of a curved space, where we need to calculate the limit

$$\langle T^{\mu \nu}_i \rangle_g = 2\pi \lim_{\Lambda \to \infty} \left\{ m^2 G_{m,\Lambda}(x, x) + \sum_i c_i M_i^2 G_{M_i,\Lambda}(x, x) \right\} (1) \bigg|_g.$$  

Here $G_{m,\Lambda}(x, x)$ denotes the two-point function of a scalar field computed with large-momentum cutoff $\Lambda$. The Pauli-Villars conditions are precisely what are needed to make the limit $\Lambda \to \infty$ finite.

In a background $g_{ij}$, the propagator $G_M(x, x')$ satisfies

$$-\partial_i \left[ \sqrt{g} g^{ij} \partial_j G_M(x, x') \right] + \sqrt{g} M^2 G_M(x, x') = \delta(x - x').$$

Given $x'$, we can change coordinates so that $x'$ lies at the origin. In two dimensions, we can furthermore choose the coordinate system so that

$$g_{ij}(x) = e^{2\omega} \delta_{ij},$$

$$\omega(0) = 0.$$

so that the equation for the propagator becomes

$$\left( -e^{-2\omega(x)} \Delta + M^2 \right) G_M(x, 0) = e^{-2\omega(x)} \delta(x) = e^{-2\omega(0)} \delta(x) = \delta(x),$$

since $\omega(0) = 0$. Here $\Delta \equiv \delta^{ij} \partial_i \partial_j$. As we increase the Pauli-Villars mass $M$, the propagator becomes increasingly short-ranged, and we will develop an expansion
for $G_M(x)$ in terms of derivatives of $\omega$ at the origin. Expanding around $x^i = 0$, we have

$$e^{-2\omega(x)} = 1 - 2 (\partial_i \omega) x^i + \left[-(\partial_i \partial_j \omega) + 2 (\partial_i \omega) (\partial_j \omega)\right]x^i x^j + \cdots,$$

where the derivatives are all evaluated at $x^i = 0$. We can then write

$$G_M = \frac{1}{A - B} = \frac{1}{A} + \frac{1}{A} B \frac{1}{A} B \frac{1}{A} + \cdots,$$

where $A \equiv -\Delta + M^2$, so that $1/A$ is the flat space propagator, while

$$B \equiv -\left\{2 (\partial_i \omega) x^i + \left[\partial_i \partial_j \omega - 2 (\partial_i \omega) (\partial_j \omega)\right]x^i x^j + \cdots\right\} \Delta.$$ 

In the first-order contribution $A^{-1} BA^{-1}$ to $G_M(0, 0)$, the term with odd integrand proportional to $x^i$ vanishes. The next term in $A^{-1} BA^{-1}$ is proportional to

$$\int d^2x d^3x^3 \int \frac{d^2k}{(2\pi)^2} \frac{e^{ikx}}{k^2 + M^2} \int \frac{d^2p}{(2\pi)^2} \frac{p^2 e^{-ipx}}{p^2 + M^2}.$$

The next nonzero term in $A^{-1} BA^{-1}$ has integrand proportional to $x^i x^j x^k x^l$. This may be checked to be of order $1/M^4$. Further terms are of even higher order in $1/M$, so that the full first-order contribution is

$$G_{M,1}(0, 0) = -\frac{1}{12\pi} \frac{1}{M^2} \left[\Delta \omega - 2 \partial_i \omega \partial^i \omega\right] + o\left(\frac{1}{M^4}\right),$$

which will contribute a term of zeroth order in $1/M^2$ to the expectation value $\langle T^i_{\text{\scriptsize g}} \rangle$.

There is one additional contribution of order $1/M^2$ to $G_M(0, 0)$. It comes from the second-order term $A^{-1} BA^{-1} BA^{-1}$ in the above expansion. It is

$$4 (\partial_i \omega) (\partial_j \omega) \int d^2x \int d^2y x^i y^j \times$$

$$\times \int \frac{d^2k}{(2\pi)^2} \frac{e^{ikx}}{k^2 + M^2} \int \frac{d^2p}{(2\pi)^2} \frac{p^2 e^{ip(x-y)}}{p^2 + M^2} \int \frac{d^2q}{(2\pi)^2} \frac{q^2 e^{-iqy}}{q^2 + M^2}.$$
By similar manipulations, this becomes
\[-8M^2 \delta^{ij} (\partial_i \omega) (\partial_j \omega) \int \frac{d^2 k}{(2\pi)^2} \frac{k^4}{(k^2 + M^2)^5} = -\frac{1}{6\pi} \frac{1}{M^2} \partial_i \omega \partial^i \omega,\]
so that
\[G_{M,2}(0,0) = -\frac{1}{6\pi} \frac{1}{M^2} \partial_i \omega \partial^i \omega + o \left( \frac{1}{M^4} \right).\]

Notice that this contributes a term that exactly cancels the term of the same form in \(G_M(0,0)\). We find
\[G_\Lambda_M(0,0) = \frac{1}{4\pi} \ln \frac{\Lambda^2}{M^2} - \frac{1}{12\pi} \frac{1}{M^2} \Delta \omega + o \left( \frac{1}{M^4} \right)\]
\[= \frac{1}{4\pi} \ln \frac{\Lambda^2}{M^2} + \frac{1}{12\pi} \frac{1}{M^2} R + o \left( \frac{1}{M^4} \right).\]

The matter contribution can be expanded in \(m\) by writing the equation for the propagator as
\[\left( -\Delta + e^{2\omega(x)} m^2 \right) G_m(x,0) = \delta(x),\]
so that
\[G_m = \frac{1}{A - B} = \frac{1}{A} + \frac{1}{A} B \frac{1}{A} + \frac{1}{A} B \frac{1}{A} B \frac{1}{A} + \cdots\]
where \(A \equiv -\Delta + m^2\) and\( B \equiv m^2 \left( 1 - e^{2\omega(x)} \right) \equiv m^2 \gamma(x).\)

Therefore
\[G_m^\Lambda(0,0) = \frac{1}{4\pi} \ln \frac{\Lambda^2}{m^2} \]
\[+ m^2 \int d^2 x \frac{1}{2\pi} K_0(m|x|) \gamma(x) \frac{1}{2\pi} K_0(m|x|)\]
\[+ m^4 \int d^2 x \int d^2 y \frac{1}{2\pi} K_0(m|x|) \gamma(x) \frac{1}{2\pi} K_0(m|y - x|) \gamma(y) \frac{1}{2\pi} K_0(m|y|)\]
\[+ \cdots,\]
where the Bessel function
\[\frac{1}{2\pi} K_0(m|x|) = \int \frac{d^2 k}{(2\pi)^2} \frac{e^{ikx}}{k^2 + m^2}\]
decays exponentially for large \(x\) and
\[K_0(m|x|) \to -\ln(m|x|)\]
as \(m|x| \to 0\), so that the integrals are well-defined for a large class of functions \(\gamma(x)\).

Since
\[\lim_{m \to 0} m \ln(m|x|) \to 0,\]
all terms but the first in the above series vanish as $m \to 0$ as long as $\gamma(x)$ is sufficiently well-behaved that we may exchange the integral and the limit. We find

$$G^A_m(0, 0) \to \frac{1}{4\pi} \ln \frac{\Lambda^2}{m^2}$$

as $m \to 0$. Including the Pauli-Villars contributions, we find

$$\langle T^i \rangle_g = \frac{1}{2} \left( m^2 \ln \frac{\Lambda^2}{m^2} + \sum_i c_i M_i^2 \ln \frac{\Lambda^2}{M_i^2} \right) - \frac{1}{12} R + o \left( \frac{1}{M^2} \right),$$

where we have used the Pauli-Villars condition $\mathcal{O}$ to obtain the negative sign for the curvature term. As before, the terms in parentheses can be compensated by a counterterm or made zero by the condition $\mathcal{P}$ on the Pauli-Villars masses. After doing this, we find, as we set out to prove, that

$$\langle T^i \rangle_g \to -\frac{1}{12} R \quad (14)$$

as the masses $m \to 0$ and $M_i \to \infty$. As a result, $\mathcal{H}$ becomes

$$\delta \int [d\phi]^P \mathcal{V} e^{-S(g, \phi, \bar{\chi}, \chi)} = \left( \frac{1}{24\pi} \int d^2x \sqrt{g} \langle \delta \omega \rangle R \right) \langle 1 \rangle_g. \quad (15)$$

This agrees with the formula $\mathcal{E}$ calculated in the Fujikawa approach, but here the anomaly comes from the Pauli-Villars mass terms, not the measure, which is invariant under $\delta$.

4. A study of $T_{zz}$

It is instructive to observe the effect of the Pauli-Villars regularization on the expectation values of $T_{zz}$ and $T_{\bar{z}\bar{z}}$. As a side benefit, we shall identify the nontrivial relationship between the energy-momentum tensor $T_{ij}$ in the path integral formalism, which is a true coordinate invariant and finite tensor object, and the anomalous nontensor object in the operator formalism. By identifying explicitly the path integral expression corresponding to the latter, we shall obtain an explanation, within the context of the path integral, for the nontensor transformation property.

Writing $z \equiv x^1 + i x^2$, we have

$$\langle T_{zz} \rangle_g = -2\pi \lim_{\Lambda \to \infty} \left\{ \frac{\partial_z \partial_w G^A_m(z, w) + \sum_i \partial_z \partial_w G^A_{M_i}(z, w)}{w \to z} \right\}.$$
of the Pauli-Villars fields. We shall see explicit examples of this the next section. So although the full \( \langle T_{zz} \rangle_g \) is finite and covariant, let us also introduce a name for the combination

\[
\hat{T}_{zz} \equiv T_{zz}^{m=0} + \lim_{M_i \to \infty} \sum_i \langle T_{zz}^{M_i} \rangle_0.
\]  

This may be regarded as defining a particular minimal subtraction renormalization prescription. But note that it is coordinate-dependent. As we shall see, it is this combination that will turn out to correspond to the renormalized energy-momentum operator occurring in the operator formalism. Obviously, it differs from the full energy-momentum tensor \( T_{zz} \) of our system. But it only differs by a finite amount.

The Pauli-Villars contributions may be expanded in \( 1/M^2 \) as in our calculation of the trace anomaly in the previous section. We find first-order contributions of the form

\[
-2\pi \left[ -\partial_i \partial_j \omega + 2 \partial_i (\partial_j \omega) \right] \times
\int d^2x \, x^i x^j \frac{1}{2\pi} \partial_z K_0(|x|) \frac{1}{2\pi} \partial_z \Delta K_0(|x|).
\]

The term with integrand proportional to \( \bar{z} \bar{z} \) vanishes

\[
\int d^2x \, \bar{z} \bar{z} \partial_z K_0(Mr) \partial_z \Delta K_0(Mr)
= M^2 \int d^2x \, \bar{z} \bar{z} \partial_r K_0(Mr) \partial_r \Delta K_0(Mr)
= 0,
\]

since the integrand is odd under \( z \to e^{i\pi/4} z \). Similarly, the term with integrand proportional to \( z \bar{z} \) vanishes, and we are left with the term

\[
\int d^2x \, z^2 \frac{1}{2\pi} \partial_z K_0(Mr) \frac{1}{2\pi} \partial_z \Delta K_0(Mr)
\]

\[
= \int d^2x \, z^2 \int \frac{d^2k}{(2\pi)^2} \frac{k}{kk + M^2} \int \frac{d^2p}{(2\pi)^2} \frac{1}{2\pi} \bar{p} (pp) \frac{e^{-\frac{i}{2} (k^2 + pp)}}{pp + M^2} \partial^2 \delta^2(k - p)
\]

\[
= \int \frac{d^2k}{(2\pi)^2} \frac{kk}{kk + M^2} \partial_k^2 \left( \frac{k}{kk + M^2} \right),
\]

which is of the form \( \int d^2k \, f \left( \partial^2_k g \right) \). In the last step, we could just as well have used the identity

\[
\partial_k^2 \delta^2(k - p) = \partial_k^2 \delta^2(k - p)
\]

to write this in the form \( \int d^2k \, \left( \partial_k^2 f \right) g \). However, the reader may check by explicit calculation that

\[
\int d^2k \, f \left( \partial_k^2 g \right) \neq \int d^2k \, \left( \partial_k^2 f \right) g,
\]
an ambiguity due to naively exchanging the integral over $x$ with those over $k$ and $p$. In general, integrals may be exchanged only after a careful analysis of their uniformity of convergence. Here we see an order dependence proportional to the quantities $f(\partial_4 y)$ or $(\partial_k f)g$ integrated over the surface of the integration region. Since these surface integrands are of order $1/r$, they do contribute in two dimensions (the corresponding surface terms may be checked to be of order $1/r^2$ in our previous calculation of the trace anomaly, and therefore did not threaten the validity of that calculation). Fortunately we can avoid these uniformity issues by using the equation of motion $(-\Delta + M^2) K_0(Mr) = \delta^2(x)$ in position space first as follows:

\[
\int d^2x \, z^2 \frac{1}{2\pi} \partial_z K_0(Mr) \frac{1}{2\pi} \partial_z \Delta K_0(Mr)
\]

\[
= \int d^2x \, z^2 \frac{1}{2\pi} \partial_z K_0(Mr) \partial_z \left( -\delta^2(x) + \frac{1}{2\pi} M^2 K_0(Mr) \right)
\]

\[
= M^2 \int d^2x \, z^2 \frac{1}{2\pi} \partial_z K_0(Mr) \frac{1}{2\pi} \partial_z K_0(Mr)
\]

where we have used the behaviour $\partial_z K_0(Mr) = o(1/r)$ as $r \to 0$ to drop the term containing $\partial_z \delta^2(x)$. This then becomes

\[
M^2 \int d^2x \, z^2 \int \frac{d^2k}{(2\pi)^2} \frac{1}{kk+M^2} \int \frac{d^2p}{(2\pi)^2} \frac{1}{pp+M^2} \frac{1}{kk} e^{\frac{i}{2} (kz + k \omega)} e^{-\frac{i}{2} (p \bar{z} + p \bar{\omega})}
\]

\[
= -M^2 \int \frac{d^2k}{(2\pi)^2} \frac{k}{kk+M^2} \partial^2_k \left( \frac{k}{kk+M^2} \right)
\]

\[
= 2M^2 \int \frac{d^2k}{(2\pi)^2} \left\{ \frac{kk}{(kk+M^2)^2} - \frac{(kk)^2}{(kk+M^2)^4} \right\}
\]

\[
= \frac{1}{12\pi}.
\]

The first-order contribution is therefore

\[
\langle T_{zz}^M \rangle_1 = -\frac{1}{6} \left[ - (\partial^2_z \omega) + 2 (\partial_z \omega)^2 \right] + o \left( \frac{1}{M^2} \right).
\]

The only other contribution of order 0 in $1/M^2$ is the second-order term

\[
-2\pi \cdot 4 (\partial_z \omega)^2 \int d^2x \int d^2y \, zw \frac{1}{2\pi} \partial_z K_0(M|x|) \frac{1}{2\pi} \Delta K_0(M|y - x|) \frac{1}{2\pi} \partial_u \Delta K_0(M|y|)
\]

\[
= -8\pi (\partial_z \omega)^2 \int d^2x \int d^2y \, zw \frac{1}{2\pi} \partial_z K_0(M|x|) \left( -\delta(y - x) + \frac{1}{2\pi} M^2 K_0(M|y - x|) \right) \times
\]

\[
\times \partial_u \left( -\delta(y) + \frac{1}{2\pi} M^2 K_0(M|y|) \right)
\]

\[
= -8\pi (\partial_z \omega)^2 \left\{ M^4 \int d^2x \int d^2y \, zw \frac{1}{2\pi} \partial_z K_0(M|x|) \frac{1}{2\pi} K_0(M|y - x|) \frac{1}{2\pi} \partial_u K_0(M|y|)
\]

\[
- M^2 \int d^2x \, z^2 \frac{1}{2\pi} \partial_z K_0(M|x|) \frac{1}{2\pi} \partial_z K_0(M|x|) \right\},
\]
where again we have manipulated the integrals in position space into a form where the surface terms in momentum space will vanish. This becomes

\[-8\pi (\partial_z^2 \omega)^2 \left\{ -M^4 \int \frac{d^2k}{(2\pi)^2} \partial_k \left( \frac{\bar{k}}{kk + M^2} \right) \frac{1}{kk + M^2} \partial_k \left( \frac{\bar{k}}{kk + M^2} \right) \right. \]
\[+ M^2 \int \frac{d^2k}{(2\pi)^2} \left[ \partial_k \left( \frac{\bar{k}}{kk + M^2} \right) \right]^2 \left\{ \right. \]
\[= \frac{1}{6} (\partial_z^2 \omega)^2, \]

so that the second order contribution is

\[\langle T^{M}_{zz} \rangle_2 = \frac{1}{6} (\partial_z^2 \omega)^2 + o \left( \frac{1}{M^2} \right). \]

As \( M \to \infty \), we get for each Pauli-Villars field

\[\langle T^{M}_{zz} \rangle = \langle T^{M}_{zz} \rangle_0 + \frac{1}{6} c_i (\partial_z^2 \omega)^2. \]

Including the matter contribution and using the Pauli-Villars condition, we find

\[\langle T_{zz} \rangle = \langle \hat{T}_{zz} \rangle - \frac{1}{6} \left[ (\partial_z^2 \omega) - (\partial_z^2 \omega)^2 \right] \]
\[\equiv \langle \hat{T}_{zz} \rangle - \frac{1}{12} t_{zz}. \]

Since \( \Gamma^z_{zz} = 2 \partial_z \omega \), we may write \( t_{zz} \) in the form

\[t_{zz} = \partial_z \Gamma^z_{zz} - \frac{1}{2} (\Gamma^z_{zz})^2 \]

used by Eguchi and Ooguri in the context of the axiomatic approach to conformal field theory.\(^{14,15}\) One may verify that \( t_{zz} \) transforms as

\[\delta_v t_{zz} = \partial_z^3 v^z + v^z \partial_z t_{zz} + 2 (\partial_z v^z) t_{zz} \]

under the action of a holomorphic vector field \( v \). Since \( T_{zz} \) is a component of a true tensor, the term \( \langle \hat{T}_{zz} \rangle \) in \( 18 \) is therefore not a tensor in isolation. From the transformation laws for \( T_{zz} \) and \( t_{zz} \), it follows that \( \langle \hat{T}_{zz} \rangle \) transforms as

\[\delta_v \langle \hat{T}_{zz} \rangle = \frac{1}{12} \partial_z^3 v^z + v^z \partial_z \langle \hat{T}_{zz} \rangle + 2 (\partial_z v^z) \langle \hat{T}_{zz} \rangle , \]

which coincides with the transformation law obtained in the operator product formalism after point-splitting renormalization.\(^{14,15,16,17,18,19,20}\) The above result provides a straightforward explanation, in the context of the path integral formalism, for the familiar anomalous transformation law of \( \hat{T}_{zz} \). In the path integral formalism, the more natural object is in fact the true tensor \( T_{zz} \), which includes the full Pauli-Villars correction and, as we shall see, satisfies simpler Ward identities. The non-covariant split \( 18 \) is rather less natural from this point of view.
Note that it would be incorrect to try to absorb the Pauli-Villars effects into an operator redefinition by replacing

\[ T_{zz} \to \hat{T}_{zz} - \frac{1}{12} t_{zz} \]

in general calculations of expectation values. For example, with such a redefinition one would lose essential contact terms in the Ward identities that we consider in section 8.

The above transformation law for \( t_{zz} \) can be integrated to give, for \( z' = f(z) \),

\[ t_{zz'} \, dz' \otimes dz' = t_{zz} \, dz \otimes dz - \{ f, z \} \, dz \otimes dz, \]

where

\[ \{ f, z \} \equiv -6 \lim_{w \to z} \left( \frac{f'(w) f'(z)}{(f(w) - f(z))^2} - \frac{1}{(w - z)^2} \right), \]

is called the Schwarzian derivative. In this form, the transformation is easily seen to be precisely the difference between the renormalization subtractions needed in the operator product formalism for energy-momentum tensors defined in different coordinate systems, which provides a simple way to see that \( \hat{T}_{zz} \) corresponds to the usual operator formalism definitions.

Defining \( t_{\overline{z}z} = t_{z\overline{z}} = 0 \) and \( t_{\overline{z}\overline{z}} = \partial_{\overline{z}} \Gamma_{\overline{z}z} - \frac{1}{2} (\Gamma_{\overline{z}z})^2 \), we may summarize the results of this section as

\[ \langle T_{ij} \rangle_g = \langle \hat{T}_{ij} \rangle_g - \frac{1}{12} t_{ij} - \frac{1}{24} g_{ij} R. \quad (21) \]

5. Examples

In this section we provide some example computations of energy-momentum tensor expectation values on a couple of simple manifolds.

First we consider the plane, for which

\[ \langle T_{mzz} \rangle_g = -\frac{2\pi}{4} \int \frac{d^2k}{(2\pi)^2} \frac{(k_1 - ik_2)^2}{k^2 + m^2}. \]

The numerator in the integrand is \( k_1^2 - 2ik_1k_2 - k_2^2 \). The cross term is odd in \( k_1 \) and \( k_2 \) separately, while the sum of the first and the last term is odd under \( k_1 \leftrightarrow k_2 \), so that the integral vanishes when integrated over the particular choice \( 0 \leq k \leq \Lambda \) of cutoff region in momentum space, suitable for manifolds for which \( k_1 \) and \( k_2 \) are continuous. With this cutoff region, we get the contribution

\[ \langle T_{m=0} \rangle = 0. \]

Similarly,

\[ \langle T_{Mz} \rangle = 0. \]

for each Pauli-Villars field, and we find

\[ \langle T_{zz} \rangle = 0 = \langle \hat{T}_{zz} \rangle. \]
In section 3 we showed, using a Pauli-Villars definition of the path integral measure, that the expectation value
\[ \langle T_{zz} \rangle = -\frac{1}{48} R = 0 \]
on the plane.

In a later section, we will also need the expectation value
\[ \langle \partial_z \phi \partial_{\bar{z}} \phi \rangle = \frac{1}{16} \int \frac{d^2 k}{(2\pi)^2} \frac{k^2}{k^2 + m^2}. \]

This integrand is not odd, so that our simple argument for \( T_{zz} \) cannot be used. Computing this with a large-momentum cutoff \( \Lambda \), we find
\[ \langle \partial_z \phi \partial_{\bar{z}} \phi \rangle = \frac{1}{16\pi} \Lambda^2 - \frac{1}{16\pi} m^2 \ln \frac{\Lambda^2}{m^2}. \]
As we saw, both these terms can be exactly canceled by introducing Pauli-Villars fields with the appropriate statistics and mass conditions, and we obtain
\[ \langle \partial_z \phi \partial_{\bar{z}} \phi + PV \rangle = 0 \]
on the plane.

Let us employ the Pauli-Villars formalism to perform a path-integral calculation of the expectation value \( \langle T_{zz} \rangle \) on an infinite cylinder of circumference \( L \), a simple nontrivial manifold with a length scale that will give rise to a Casimir energy. We have, for a field of mass \( M \),
\[ \langle T_{zz}^M \rangle = -2\pi \cdot \frac{1}{4} \cdot \frac{1}{L} \cdot \sum_n \int \frac{dk_2}{2\pi} \frac{(2\pi n)^2 - k_2^2}{(2\pi/L)^2 + k_2^2 + M^2} \]
\[ = -\frac{\pi}{2} \cdot \frac{1}{L} \cdot \sum_n \left\{ \sqrt{(2\pi n)^2 + M^2} - \frac{M^2/2}{\sqrt{(2\pi n)^2 + M^2}} \right\}. \]
We have performed the integral over \( k_2 \) by closing the integration contour either above or below the real line. Although the resulting contour integral strictly diverges as we move the contour to infinity, this is one term in the full integrand containing both the matter and all Pauli-Villars, the integral of which will converge due to the Pauli-Villars conditions, validating the contour integral argument.

We first evaluate the matter contribution, for which \( M = 0 \). Inserting a convergence factor, the dependence on which will be canceled by the Pauli-Villars contributions, we find
\[ \langle T_{zz}^{m=0} \rangle = -\frac{\pi}{L} \cdot \sum_{n>0} \frac{2\pi n}{L} e^{-\epsilon 2\pi n/L} \]
\[ = -\frac{\pi}{L} \cdot \frac{2\pi}{L} e^{-\epsilon \pi} \cdot \frac{e^{-\epsilon 2\pi/L} - 1}{(1 - e^{-\epsilon 2\pi/L})^2}. \]
Using the expansion
\[
e^{-\nu} \left( 1 - e^{-\nu} \right)^2 = \frac{1}{\nu^2} - \frac{1}{12} + o(\nu),
\]
we get
\[
\langle T^{m=0}_{zz} \rangle = -\frac{1}{2\epsilon^2} + \frac{1}{24} \left( \frac{2\pi}{L} \right)^2 + o(\epsilon).
\]

It remains to show that, as promised, the Pauli-Villars fields will cancel the dependence on \( \epsilon \) as \( \epsilon \to 0 \). As \( M \to \infty \), we can write each Pauli-Villars contribution as a continuous integral as follows.

\[
\langle T^M_{zz} \rangle = -\frac{1}{2} \cdot M^2 \cdot \frac{2\pi}{ML} \sum_{n \geq 0} \left( \sqrt{\frac{4\pi n}{M^2}} - \frac{1/2}{\sqrt{\frac{4\pi n}{M^2}} + 1} \right) e^{-\epsilon M \cdot 2\pi n/ML}
\]

\[
\to -\frac{1}{2} \cdot M^2 \int_0^\infty dx \left\{ \sqrt{x^2 + 1} - \frac{1/2}{\sqrt{x^2 + 1}} \right\} e^{-\epsilon M x}
\]

\[
= -\frac{1}{2} \cdot M^2 \cdot \frac{\sqrt{\pi}}{2} \left\{ \frac{2}{\epsilon M} \Gamma \left( \frac{3}{2} \right) [H_1(\epsilon M) - N_1(\epsilon M)]
\right.
\]

\[
\left. + \frac{1}{2} \Gamma \left( \frac{1}{2} \right) [H_0(\epsilon M) - N_0(\epsilon M)] \right\},
\]

where \( H_\nu \) denotes Struve functions and \( N_\nu \) Bessel functions of the second kind.\(^{21}\)

Using the small \( z \) expansions
\[
H_0(z) = o(z),
\]
\[
H_1(z) = o(z^2),
\]
\[
\pi N_0(z) = 2 \left( \ln \frac{z}{2} + C \right) + o(z^2)
\]
\[
\pi N_1(z) = -\frac{2}{z} - \frac{z}{2} + z \left( \ln \frac{z}{2} + C \right) + o(z^3),
\]

we find
\[
\langle T^M_{zz} \rangle = -\frac{1}{2\epsilon^2} - \frac{1}{8} M^2 + o(\epsilon).
\]

By the Pauli-Villars conditions \( \sum c_i = 1 \) and \( \sum c_i M_i^2 = 0 \) from \(^{23}\) and \(^{12}\), the first term cancels the \( \epsilon \) dependence of the matter field and the second term falls away. Our final answer for the full expectation value becomes
\[
\langle T_{zz} \rangle = \frac{1}{24} \left( \frac{2\pi}{L} \right)^2.
\]

This is finite without further renormalization.
6. Reparametrization-invariance of the measure

Consider a family of reparametrizations $f_\lambda : M \rightarrow M$, $f_0 = \text{id}$ where $\lambda$ is a real parameter, acting on scalar fields via push-forward

$$\phi(z) \rightarrow \phi^\lambda(z) \equiv \phi(f_{-\lambda}(z)).$$

In this section, we will study the transformation of the path integral measure under such deformations acting only on the fields, not on the metric. Given a metric $g$ on $M$, let $\phi_n$ be a basis of field configurations satisfying

$$\delta_{mn} = \int d^2 x \sqrt{g} \phi_m(x) \phi_n(x)$$

and expand $\phi^\lambda$ as

$$\phi^\lambda(x) = \sum_n a^\lambda_n \phi_n(z).$$

For each $\lambda$, the map $\phi \mapsto (a^\lambda_0, a^\lambda_1, \ldots)$ may be regarded as a coordinate chart on the space of field configurations $\phi$. Denoting the $\infty$-form

$$\bigwedge_n da^\lambda_n \equiv da^\lambda_0 \wedge da^\lambda_1 \wedge \cdots \equiv [d\phi^\lambda],$$

the change of variables theorem tells us that

$$\int [d\phi^\lambda] e^{-S(\phi^\lambda)} = \int [d\phi] e^{-S(\phi)}.$$

Note that this had better be true if the concept of integration is to make sense in a chart-independent way. In preparation for our derivation of the Ward identities in the next section, let us determine how the form $[d\phi^\lambda]$ depends on $\lambda$. We can project out the coefficient

$$a^\lambda_m = \int d^2 x \sqrt{g} \phi_m(x) \phi^\lambda(x),$$

and calculate

$$\frac{d}{d\lambda} a^\lambda_m = \int d^2 x \sqrt{g} \phi_m(x) (-v^i) \partial_i \phi^\lambda(x)$$

$$= -\sum_n a^\lambda_n \int d^2 x \sqrt{g} \phi_m(x) v^i \partial_i \phi_n(x)$$

$$\equiv \sum_n C_{mn} a^\lambda_n$$

where $v^i$ is the vector field whose flow gives $f_\lambda$, in other words

$$v^i(x) \equiv \left. \frac{df_\lambda^i(x)}{d\lambda} \right|_{\lambda=0}$$

The normal rules for manipulating differential forms now give

$$\frac{d}{d\lambda} [d\phi^\lambda] \equiv \frac{d}{d\lambda} \bigwedge_n da^\lambda_n = \left( \sum_n C_{mn} \right) \bigwedge_n da^\lambda_n = (\text{Tr } C) [d\phi^\lambda].$$
Now

\[
C_{mm} = - \int d^2 x \sqrt{g} \phi_m(x) v^i \partial_i \phi_m(x)
= \int d^2 x \sqrt{g} \left( v^i \partial_i \phi_m(x) \right) \phi_m(x)
+ \int d^2 x \phi_m(x) \partial_i \left( \sqrt{g} v^i \right) \phi_m(x)
= \frac{1}{2} \int d^2 x \phi_m(x) \partial_i \left( \sqrt{g} v^i \right) \phi_m(x)
= \frac{1}{2} \int d^2 x \sqrt{g} \phi_m(x) \left( \nabla_i v^i \right) \phi_m(x)
\]

where \( \nabla_i \) denotes the covariant derivative on the worldsheet with respect to the deformed metric \( f^*_\lambda g \). Similar to the calculation of the Weyl anomaly, we get the trace of an operator \( \nabla_i v^i \) representing a change in area, in this case along the flow of \( v \). The calculation of the trace can be done via a heat kernel regularization and is identical to the calculation of the Weyl anomaly, with \( \nabla_i v^i \) replacing \( 2 \delta \omega \). We get

\[
\frac{d}{d \lambda} \left[ d\phi^\lambda \right] = \left( \frac{1}{8\pi} \int d^2 x \sqrt{g} \nabla_i v^i + \frac{1}{48\pi} \int d^2 x \sqrt{g} \left( \nabla_i v^i \right) R_g \right) \left[ d\phi^\lambda \right]
\]

The integrand \( \sqrt{g} \nabla_i v^i = \partial_i \left( \sqrt{g} v^i \right) \) in the first term is a total derivative, and will integrate to zero for suitable boundary conditions on \( v^i \), which we will assume. A similar partial integration in the second term, and remembering that \( \nabla_i = \partial_i \) on scalars, leads to

\[
\frac{d}{d \lambda} \left[ d\phi^\lambda \right] = \left( -\frac{1}{48\pi} \int d^2 x \sqrt{g} v^i \nabla_i R_g \right) \left[ d\phi^\lambda \right].
\]

In particular, on a flat manifold, the measure will be invariant under reparametrizations of \( \phi \). It is also interesting to note that the prefactor is independent of the deformation parameter \( \lambda \).

As in the section 3, a similar analysis applies to the Pauli-Villars auxiliary fields. As a result, the full measure is reparametrization-invariant:

\[
\frac{d}{d \lambda} \left[ d\phi^\lambda \right]_{PV} = \left( 1 + \sum_i c_i \right) \left( -\frac{1}{48\pi} \int d^2 x \sqrt{g} v^i \nabla_i R_g \right) \left[ d\phi^\lambda \right]_{PV}
= 0.
\]

7. Ward identities

In this section we derive the Ward identity for covariant conservation of energy-momentum. At first we do not use Pauli-Villars fields, and we therefore need to keep careful account of the transformation of the measure in the spirit of Fujikawa, obtaining an anomalous conservation law. We then discuss why this procedure is
not quite correct, motivating the introduction of a Pauli-Villars measure. The Pauli-Villars measure transforms trivially, but now the energy-momentum tensor has extra terms making the insertion finite. We find that the full, finite energy-momentum insertion satisfies the classical conservation law, which we then show to be consistent with the anomalous conservation law for the corresponding quantity in the operator formalism.

The starting point for deriving the energy-momentum conservation Ward identities is the change of variables theorem

\[
\int [d\phi^\lambda] e^{-S(\phi^\lambda)} = \int [d\phi] e^{-S(\phi)},
\]

(25)

where

\[
S(\phi^\lambda) = \frac{1}{2} \int d^2x \sqrt{g} g^{ij} \partial_i \phi^\lambda \partial_j \phi^\lambda, \quad \phi^\lambda(z) = \phi(f(z)).
\]

Note that the deformation \( \phi \to \phi^\lambda \) does not act on the metric.

The above formula extrapolates a well-known property of finite-dimensional integration to the infinite-dimensional case. It says that the path integral is independent of the choice of coordinates on the space of fields, an essential property that must be satisfied by any reasonable definition of integration. In chapter 9 we will perform a consistency check on this formula for our particular choice.

Classically, covariant conservation of energy-momentum is a consequence of the observation that, since the action density in \( S \) is a coordinate-invariant expression, it would be invariant if the deformation \( f_\lambda \) acted not only on the fields but also on the metric. However, since the deformation acts only on the fields, the variation of \( S \) is not in general zero but instead proportional to the omitted variation \( \delta g^{ij} = \nabla^i v^j + \nabla^j v^i \) of the metric (see the calculation of \( dS/d\lambda \) below). This allows one to obtain a Noether current via standard methods, even though the corresponding conservation law involves covariant derivatives and cannot in general be integrated to give conserved charges.

We now provide a path integral derivation of the current by varying (25) with respect to a generic deformation \( f_\lambda \). Differentiating with respect to \( \lambda \), the right hand side gives zero. For the left hand side, we have already calculated the variation of
The variation of $S$ is
\[
\frac{dS}{d\lambda} = \frac{1}{2} \cdot \frac{d}{d\lambda} \int d^2 x \sqrt{g} g^{ij} \partial_i \phi^\lambda \partial_j \phi^\lambda
\]
(by pullback)
\[
= \frac{1}{2} \cdot \frac{d}{d\lambda} \int d^2 x \sqrt{f_\lambda^* g} (f_\lambda^* g)^{ij} \partial_i \phi^\lambda \partial_j \phi^\lambda
\]
(definition of $T$)
\[
= \frac{1}{4\pi} \int d^2 x \sqrt{f_\lambda^* g} \frac{d}{d\lambda} (f_\lambda^* g)^{ij} T^{f_\lambda^* g}_{ij}(\phi)
\]
(by pull-back)
\[
= \frac{1}{2\pi} \int d^2 x \sqrt{g} (\nabla^i v^j + \nabla^j v^i) T^g_{ij}(\phi^\lambda)
\]
(symmetry of $T$)
\[
= \frac{1}{2\pi} \int d^2 x \sqrt{g} (\nabla^i v^j T^g_{ij}(\phi^\lambda)) - v^i \nabla^i T^g_{ij}(\phi^\lambda))
\]
\[= \frac{1}{2\pi} \int d^2 x \sqrt{g} v^j \nabla^i T^g_{ij}(\phi^\lambda),\]

where
\[
T^g_{ij}(\phi^\lambda) \equiv -2\pi \left( \partial_i \phi^\lambda \partial_j \phi^\lambda - \frac{1}{2} g_{ij} g^{kl} \partial_k \phi^\lambda \partial_l \phi^\lambda \right).
\]

and where we have used the property $\sqrt{g} \nabla^i J_i = \partial^i (\sqrt{g} J_i)$ of the covariant derivative to get a total derivative which we have dropped in the last line. Differentiation of equation (24) with respect to $\lambda$ then gives
\[
0 = \int d^2 x \sqrt{g} \left( \frac{1}{2\pi} v^j \nabla^i T^g_{ij}(\phi^\lambda) - \frac{1}{48\pi} v^i \nabla_i R \right)_\lambda
\]
\[= \frac{1}{2\pi} \int d^2 x \sqrt{g} v^j \nabla^i \left( T^g_{ij}(\phi^\lambda) - \frac{1}{24} g_{ij} R \right)_\lambda.
\]

Since $v^j$ is arbitrary, we get the conservation law
\[
\nabla^i \left( T^g_{ij}(\phi^\lambda) - \frac{1}{24} g_{ij} R \right)_\lambda = 0,
\]
where the variation of the measure has contributed a curvature term not present in the classical conservation law. Remembering our definition (10) of $\tilde{T}_{ij}$, this is the same as
\[
\nabla^i \left( \tilde{T}_{ij}(\phi^\lambda) \right)_\lambda = 0.
\]

In this derivation, which assumed a single massless scalar field for which $T^i_i$ is identically zero, the curvature term in the conservation law was due to the variation of the path integral measure under a reparametrization.

As it stands, though, this formula is not quite meaningful. In fact, as we have seen, the expectation values $\langle T^g_{ij} \rangle$ for a single matter field are not generally finite,
and we needed to introduce Pauli-Villars fields to obtain finite values for the full energy-momentum, including matter and auxiliary fields, in a coordinate-invariant way. As shown in (24), the combined matter-Pauli-Villars path integral measure is invariant under reparametrizations, and we find by the above argument that

$$\nabla^i \langle T^\lambda_{ij} \rangle_\lambda = \nabla^i \langle T_{ij} (\phi^\lambda) + PV \rangle_\lambda = 0.$$  \hspace{1cm} (27)

In other words, the full $T^\lambda_{ij}$ satisfies the classical covariant conservation law. Note that, by (24), we may now drop the subscript $\lambda$ denoting the measure used to calculate the expectation values. Repeating the definition (21) of $\hat{T}_{ij}$,

$$\langle T^\lambda_{ij} \rangle \equiv \langle \hat{T}^\lambda_{ij} \rangle - \frac{1}{12} t_{ij} - \frac{1}{24} g_{ij} R,$$

we see that the quantity $\hat{T}_{ij}$, which corresponds to the operator formalism energy-momentum, does satisfy an anomalous conservation law depending on the curvature, consistent with the corresponding operator-formalism calculations.

How about conformal symmetry? A conformal vector field satisfies $\delta g^{ij} \equiv -\nabla^i v^j - \nabla^j v^i - 2 \delta \omega g^{ij} = 0$. By symmetry of $T_{ij}$ and the product rule for the covariant derivative, we find from (27) and (7) that

$$\nabla^i \langle v^j T^\lambda_{ij} \rangle = -\delta \omega \langle T^\lambda_i \rangle = \frac{1}{12} \delta \omega R.$$  \hspace{1cm} (28)

If the right hand side were zero, $v^j T^\lambda_{ij}$ would span an infinite-dimensional family of covariantly conserved currents. Instead, these currents have an effective source proportional to the local conformal factor and the curvature. In other words, background curvature breaks conformal invariance. Since curvature gives an intrinsic local scale to a manifold that can be observed by quantum diffusion processes, this is not surprising.

8. The Ward identity for $T_{ij}$, or, where is the anomaly?

In this section we study the Ward identities for the transformation law of the energy-momentum tensor. We confirm by explicit calculation that the full energy-momentum tensor satisfies the classical Ward identities, and we explain how these are related to the anomalous Ward identities found in the operator formalism.

In the process we perform a very careful, direct path-integral calculation of nontrivial contact terms arising in expectation values of certain energy-momentum products, previously derived only using axiomatic considerations. The results of this section will then be used in the following section to perform a consistency check supporting the validity of the change of variables formula for the infinite-dimensional integration used to define the path integral. The contact terms will also be essential to our review of the relationship between the conformal anomaly and the energy-momentum two-point functions in the last section.

The Ward identity for an insertion of $T_{ij}$ follows from the change of variables

$$\int [d\phi^\lambda]_{PV} T^\lambda_{ij}(x) e^{-S(g, \phi^\lambda, \bar{\chi}^\lambda, \chi^\lambda)} = \int [d\phi]_{PV} T_{ij}(x) e^{-S(g, \phi, \bar{\chi}, \chi)}.$$  \hspace{1cm} (29)
Here $T^\lambda_{ij}$ includes the contribution of the Pauli-Villars auxiliary fields. Since $[d\phi^\lambda]_{PV}$ is independent of $\lambda$, as explained in section 7, we find the following Ward identity after differentiating both sides with respect to $\lambda$,

$$\left\langle \frac{d}{d\lambda} T^\lambda_{ij}(x) \right\rangle + \frac{1}{4\pi} \int d^2 y \sqrt{g} h^{kl}(y) \langle T^\lambda_{ij}(x) T^\lambda_{kl}(y) \rangle = 0,$$

where

$$h^{kl} = -\nabla^k v^l - \nabla^l v^k.$$

and

$$\frac{d}{d\lambda} T_{ij} = -\mathcal{L}_v T_{ij} + \pi \mathcal{L}_v (g_{ij} g^{kl}) \left( \partial_k \phi \partial_l \phi + \sum_i \partial_k \bar{\chi} \partial_l \chi \right)$$

$$+ \pi \mathcal{L}_v g_{ij} \left( m^2 \phi^2 + \sum_i M_i^2 \bar{\chi}_i \chi_i \right),$$

where $\mathcal{L}_v$ denotes the Lie derivative. For example,

$$\mathcal{L}_v T_{ij} = v^i \partial_i T_{ij} + (\partial_i v^m) T_{mj} + (\partial_j v^m) T_{im},$$

and where $\mathcal{L}_v (g_{ij} g^{kl}) \equiv \mathcal{L}_v (g \otimes g^{-1})_{ij}^{kl}$. The second and the third term in (31) subtract the contributions to $\mathcal{L}_v T_{ij}$ coming from varying the metric, since in $dT^\lambda_{ij}/d\lambda$ only the fields are varied, not the metric.

In the special case where the vector field $v$ is holomorphic in a neighbourhood of the point $x$ of the insertion (though not necessarily everywhere), generating a conformal deformation of that neighbourhood, the second term in (31) vanishes. In this case, and only in this case, we may write, in a local conformal coordinate system with $g_{ij} = e^{2\omega} \delta_{ij}$,

$$\frac{d}{d\lambda} T_{zz} = -v^i \partial_i T_{zz} - 2(\partial_k v^z) T_{zz}$$

$$\frac{d}{d\lambda} T_{\bar{z}z} = -v^i \partial_i T_{\bar{z}z} + v^i \partial_i (2\omega) T_{zz}. $$

It is important to note that the second term in the Ward identity (30) generates the transformation $\mathcal{L}_v T_{ij}$ for the components $T_{zz}$ and $T_{\bar{z}z}$ only when $v$ is holomorphic in a neighbourhood of the insertion. In other cases, the extra terms in (31) cannot be ignored.

It is also important to note that the transformation $\mathcal{L}_v T_{ij}$ appearing in the Ward identity is the classical one. This expresses the fact that, since the Pauli-Villars regularization is coordinate-invariant, the full energy-momentum tensor $T_{ij}$, including Pauli-Villars contributions, is finite and a true coordinate invariant tensor quantity. As discussed in section 4, it is the quantity $\hat{T}_{zz}$ introduced in (10) that satisfies an anomalous transformation law. We shall see by explicit calculation that the above, non-anomalous Ward identity is indeed correct, but contains contact terms that can be compensated by a redefinition of $T_{ij}$ to obtain the familiar anomalous Ward identity for the modified insertion $\hat{T}_{zz}$. 
For simplicity, we restrict attention to the plane and consider
\[
\left\langle \frac{d}{d\lambda} T_{zz}^\lambda \right\rangle = -\frac{1}{4\pi} \int d^2 w \ h_{ww} \left\langle T_{zz}^\lambda T_{ww}^\lambda \right\rangle \\
- 2 \cdot \frac{1}{4\pi} \int d^2 w \ h_{\bar{w} \bar{w}} \left\langle T_{zz}^\lambda T_{\bar{w} \bar{w}}^\lambda \right\rangle \\
- \frac{1}{4\pi} \int d^2 w \ h_{\bar{w} \bar{w}} \left\langle T_{zz}^\lambda T_{\bar{w} \bar{w}}^\lambda \right\rangle
\]
The first expectation value on the right hand side is easily calculated by a double contraction to be
\[
\left\langle T_{zz}^\lambda T_{ww}^\lambda \right\rangle = \frac{1}{2} \left( \frac{1}{(z - w)^4} \right),
\]
where self-contractions vanish due to the Pauli-Villars conditions as in section 5. The expectation values in the second and third terms above are not discussed in many standard treatments, but in fact contribute contact terms, in the absence of which the above identity would be untrue. The presence of the contact terms are inferred using axiomatic frameworks in Refs. [18, 22] and [23] but we have been unable to find a calculation from first principles as presented below.

The contact terms are nontrivial to calculate. Consider for example \( \left\langle T_{zz} T_{\bar{w} \bar{w}} \right\rangle \). Naively taking appropriate derivatives of the double contraction for a massless field would give the square of the delta function, which does not exist as a well-defined distribution. Also troublesome is the expectation value \( \left\langle T_{zz} T_{w \bar{w}} \right\rangle \). Since \( T_{w \bar{w}} \) is identically zero in a massless theory, one might expect the answer to be zero. With our careful definition of the path integral, we shall see that this is only true up to a contact term.

Let us therefore calculate these expectation values more carefully using our regularized path integral. We realize an infrared regularization by introducing a mass \( m \) for the field \( \phi \), eventually to be taken to zero, while the ultraviolet regularization is taken care of, as before, by the Pauli-Villars auxiliary fields whose masses we eventually take to infinity.

We start by considering the expectation value \( \left\langle T_{zz} T_{\bar{w} \bar{w}} \right\rangle \). Writing the contractions in terms of derivatives of the propagator
\[
\langle \phi(x) \phi(0) \rangle = \int \frac{d^2 p}{(2\pi)^2} \frac{e^{-ipx}}{p^2 + m^2}
\]
and Fourier transforming the result gives the familiar one-loop Feynman integral
\[
\langle T_{zz}(x) T_{zz}(0) \rangle = \frac{2}{16} \int \frac{d^2 p}{(2\pi)^2} e^{-ipx} \int \frac{d^2 k}{(2\pi)^2} \left( F(m) + \sum_i c_i F(M_i) \right)
\]
where
\[
F(m) \equiv \frac{k^2 (p - k)^2}{[k^2 + m^2] [(p - k)^2 + m^2]}.
\]
In the absence of the Pauli-Villars field contributions, the integral over \( k \) would have both a quadratic and a logarithmic divergence. The regularization consists in choosing the coefficients \( c_i \) and masses \( M_i \) so as to make the integral finite. Assuming this has been done, we can then write, using identities such as \( k^2 = k^2 + m^2 - m^2 \),

\[
\int \frac{d^2 k}{(2\pi)^2} \left\{ \frac{k^2 (p-k)^2}{|k^2 + m^2| [(p-k)^2 + m^2]} + PV \right\} = \int \frac{d^2 k}{(2\pi)^2} \left\{ 1 - \frac{m^2}{(p-k)^2 + m^2} - \frac{m^2}{k^2 + m^2} \right. \\
+ \frac{m^4}{|k^2 + m^2| [(p-k)^2 + m^2]} + PV \right\}
\]

\[
= \int \frac{d^2 k}{(2\pi)^2} \left\{ 1 - \frac{2 m^2}{k^2 + m^2} + \frac{m^4}{|k^2 + m^2| [(p-k)^2 + m^2]} + PV \right\},
\]

where the shift of \( k \) in the last line is permitted since the integral converges. Integrating the first two terms between 0 and \( \Lambda \), we get, for large \( \Lambda \),

\[
\frac{1}{4\pi} \frac{\Lambda^2}{2\pi} - \frac{1}{2\pi} m^2 \ln \frac{\Lambda^2}{m^2} + \int \frac{d^2 k}{(2\pi)^2} \frac{m^4}{|k^2 + m^2| [(p-k)^2 + m^2]} + PV
\]

By the Pauli-Villars conditions (9), (12) and (13),

\[
1 + \sum_i c_i = 0, \\
m^2 + \sum_i c_i M_i^2 = 0, \\
m^2 \ln \frac{m^2}{\mu^2} + \sum_i c_i M_i^2 \ln \frac{M_i^2}{\mu^2} = 0,
\]

the first two terms cancel entirely. What remains is the finite integral

\[
\langle T_{zz}(x) T_{\bar{z}z}(0) \rangle = \frac{2 m^4 (2\pi)^2}{16} \int \frac{d^2 p}{(2\pi)^2} e^{-ip \cdot x} \int \frac{d^2 k}{(2\pi)^2} \frac{1}{|k^2 + m^2| [(p-k)^2 + m^2]} + PV
\]

Feynman’s trick gives

\[
\int \frac{d^2 k}{(2\pi)^2} \frac{1}{k^2 + m^2} \frac{1}{(p-k)^2 + m^2} = \int \frac{d^2 k}{(2\pi)^2} \int_0^1 dx \frac{1}{x (k^2 + m^2) + (1-x) [(p-k)^2 + m^2]}
\]

\[
= \int \frac{d^2 k}{(2\pi)^2} \int_0^1 dx \frac{1}{(k^2 + x(1-x) p^2 + m^2)^2},
\]

where we have redefined \( k - (x-1)p \to k \) to complete the square in the last line. Performing the straightforward integration over \( k \), we find

\[
\frac{1}{4\pi} \int_0^1 dx \frac{1}{x(1-x) p^2 + m^2},
\]
Changing variables from $x$ to

$$s = \frac{m^2}{x(1-x)},$$

and including the prefactor, we find the spectral representation

$$\frac{1}{4\pi} \cdot \frac{2}{16} \cdot \frac{4}{2} \int_{4m^2}^{\infty} \frac{ds}{s} \frac{2m^4}{\sqrt{1 - 4m^2/s}} \cdot \frac{1}{p^2 + s}$$

$$= \frac{1}{16} \cdot 4\pi \int_{2m}^{\infty} \frac{d\mu}{\mu} \frac{2m^4}{\sqrt{1 - 4m^2/\mu^2}} \cdot \frac{1}{p^2 + \mu^2}$$

$$= \frac{1}{16} \cdot \frac{\pi}{3} \int_{2m}^{\infty} \frac{d\mu}{\mu^5} \frac{24m^4}{\sqrt{1 - 4m^2/\mu^2}} \cdot \frac{1}{p^2 + \mu^2}$$

$$= \frac{1}{16} \cdot \frac{\pi}{3} \int d\mu c(\mu, m) \cdot \frac{\mu^4}{p^2 + \mu^2}$$

for the Fourier transform of the expectation value. Here the spectral function

$$c(\mu, m) = \frac{24 m^4}{\mu^5 \sqrt{1 - 4m^2/\mu^2}} \theta(\mu - 2m)$$

is dimensionless and has area equal to 1, independent of $m$. Contributions to the expectation value come from two-particle intermediate states propagating between 0 and $x$. The lowest of these has energy $2m$, which explains the lower bound on the integral.

To confirm that the area is one, we calculate

$$24 m^4 \int_{2m}^{\infty} \frac{d\mu}{\mu^5} \frac{1}{\sqrt{1 - 4m^2/\mu^2}} = \frac{3}{2} \int_1^{\infty} \frac{d\eta}{\eta} \frac{1}{\sqrt{\eta^2 - 1}}$$

$$= \frac{3}{2} \cdot \frac{1}{2} \cdot B(2, \frac{1}{2})$$

$$= 1.$$ 

As $m \to 0$, $c(\mu, m)$ develops a spike at $2m$ and goes to zero elsewhere. It follows that

$$c(\mu, m) \to \delta(\mu) \quad \text{as} \quad m \to 0.$$ 

The Fourier transformed expectation value, including the Pauli-Villars contributions, is then

$$\frac{1}{16} \cdot \frac{\pi}{3} \int d\mu \left\{ c(\mu, m) + \sum_i c_i c(\mu, M_i) \right\} \cdot \frac{\mu^4}{p^2 + \mu^2}$$

$$= \frac{1}{16} \cdot \frac{\pi}{3} \int d\mu \left\{ c(\mu, m) + \sum_i c_i c(\mu, M_i) \right\} \cdot \left( \mu^2 - p^2 + \frac{p^4}{p^2 + \mu^2} \right)$$
A change of variables from $\mu$ to $\eta$ as above shows that the contribution
\[ \int d\mu \left\{ \frac{c(\mu, m) + \sum_i c_i c(\mu, M_i)}{m^2} \right\} \mu^2 \]
is proportional to $m^2 + \sum_i c_i M_i^2$, which is zero by the Pauli-Villars conditions. So is the contribution
\[ \int d\mu \left\{ \frac{c(\mu, m) + \sum_i c_i c(\mu, M_i)}{p^2} \right\} p^2, \]
which is proportional to $1 + \sum_i c_i$ by the fact that $c(\mu, \cdot)$ has unit area. We are therefore left with
\[ \frac{1}{16} \frac{\pi}{3} \int d\mu \left\{ \frac{c(\mu, m) + \sum_i c_i c(\mu, M_i)}{p^2 + \mu^2} \right\}. \]
Here, as we take the Pauli-Villars masses to infinity, we find
\[ \int_{2M_i} d\mu c(\mu, M_i) \cdot \frac{p^4}{p^2 + \mu^2} \rightarrow 0 \quad \text{as} \quad M_i \rightarrow \infty \]
because of the lower bound on the integration and the unit area property of $c(\mu, \cdot)$ making the integrand of order $1/M_i^2$. All that remains is the matter contribution which, as we remove the infrared cutoff, is
\[ \frac{1}{16} \frac{\pi}{3} \int d\mu c(\mu, m) \cdot \frac{p^4}{p^2 + \mu^2} \rightarrow \frac{1}{16} \frac{\pi}{3} \cdot p^2 \quad \text{as} \quad m \rightarrow 0, \]
since in this limit $c(\mu, m) \rightarrow \delta(\mu)$. Fourier transforming, we find
\[ \langle T_{zz}(x) T_{zz}(0) \rangle \rightarrow -\frac{\pi}{12} \partial_x \partial_{\bar{z}} \delta(x) \quad (35) \]
as $m \rightarrow 0$.

It is important to point out that, in addition to the above contribution, we would expect additional terms due to self-contractions. However, as discussed in section 5, these all vanish on the plane.

Next we calculate the expectation value
\[ \langle T_{zz} T_{zz} \rangle. \]
Since the mass term breaks conformal invariance, $T_{zz}$ is not zero. In fact
\[ T_{zz} = \frac{\pi}{2} m^2 \phi^2. \]
Although this indeed goes identically to zero as $m \rightarrow 0$, a contact term survives in the limit $m \rightarrow 0$. Indeed,
\[ \langle T_{zz}(x) T_{zz}(0) \rangle = 2m^4 \left( \frac{\pi}{2} \right)^2 \int \frac{d^2 p}{(2\pi)^2} e^{-ip \cdot x} \int \frac{d^2 k}{(2\pi)^2} e^{ik \cdot x} \left\{ \frac{1}{k^2 + m^2} \frac{1}{(p-k)^2 + m^2} + PV \right\}. \]
which is the same expression we obtained above for \( \langle T_{zz}(x) T_{zz}(0) \rangle \). Therefore
\[
\langle T_{zz}(x) T_{zz}(0) \rangle \rightarrow -\frac{\pi}{12} \partial_x \partial_z \delta(x)
\] (36)
as \( m \rightarrow 0 \).

The remaining expectation value \( \langle T_{zz}(x) T_{zz}(0) \rangle \) has Fourier transform (with a slight abuse of notation we denote \( \bar{k} \equiv k_1 - ik_2 \) but keep \( k^2 = k_1^2 + k_2^2 \))
\[
-\frac{1}{4} \cdot 2m^2 \cdot (-2\pi) \cdot \left( \frac{\pi}{2} \right) \int \frac{d^2 k}{(2\pi)^2} \left\{ \frac{\bar{k} \cdot (\bar{p} - \bar{k})}{[k^2 + m^2] \cdot [(p - k)^2 + m^2]} + PV \right\}
\]
\[
= \frac{2m^2 (2\pi)^2}{16} \int \frac{d^2 k}{(2\pi)^2} \int_0^1 dx \left\{ \frac{[\bar{k} + (1 - x)\bar{p}] \cdot [\bar{p} - \bar{k} - (1 - x)\bar{p}]}{[k^2 + x(1 - x)p^2 + \mu^2]^2} + PV \right\}
\]
\[
= \frac{2m^2 (2\pi)^2}{16} \int \frac{d^2 k}{(2\pi)^2} \int_0^1 dx \left\{ \frac{x(1 - x)p^2}{[k^2 + x(1 - x)p^2 + \mu^2]^2} + PV \right\},
\]
where in the last line we have dropped odd integrands, including terms proportional to \( k_1^2 - k_2^2 \) and \( k_1k_2 \). Performing the integration over \( k \) and changing variables from \( x \) to \( \mu \) as before, we find
\[
\frac{1}{16} \cdot \frac{\pi}{3} \int_{2m}^\infty d\mu \cdot \frac{24 m^4}{\sqrt{1 - 4m^2/\mu^2}} \cdot \frac{\mu^2 \bar{p}^2}{\bar{p}^2 + \mu^2} + PV
\]
\[
= \frac{1}{16} \cdot \frac{\pi}{3} \int d\mu \left\{ c(\mu, m) \cdot \frac{\mu^2 \bar{p}^2}{\bar{p}^2 + \mu^2} + PV \right\}
\]
\[
= \frac{1}{16} \cdot \frac{\pi}{3} \int d\mu \left\{ c(\mu, m) \cdot \bar{p}^2 - c(\mu, m) \cdot \frac{\bar{p}^2 \bar{p}^2}{\bar{p}^2 + \mu^2} + PV \right\}.
\]
As before, the first integrand will cancel due to the Pauli-Villars condition \( 1 + \sum_i c_i = 0 \), while the second term will vanish for the Pauli-Villars fields in the limit of infinite mass. Again, all that remains is the matter contribution
\[
-\frac{1}{16} \cdot \frac{\pi}{3} \int d\mu \cdot c(\mu, m) \cdot \frac{\bar{p}^2 \bar{p}^2}{\bar{p}^2 + \mu^2} \rightarrow -\frac{1}{16} \cdot \frac{\pi}{3} \bar{p}^2
\]
as \( m \rightarrow 0 \). Fourier transforming, we get
\[
\langle T_{zz}(x) T_{zz}(0) \rangle \rightarrow \frac{\pi}{12} \partial_z^2 \delta(x)
\] (37)

Summarizing, we have
\[
\langle T_{zz}(x) T_{zz}(0) \rangle = \frac{1}{2} \frac{1}{(z - w)^4},
\] (38)
\[
\langle T_{zz}(x) T_{zz}(0) \rangle = -\frac{\pi}{12} \partial_z \partial_z \delta(x),
\] (39)
\[
\langle T_{zz}(x) T_{zz}(0) \rangle = -\frac{\pi}{12} \partial_z \partial_z \delta(x),
\] (40)
\[
\langle T_{zz}(x) T_{zz}(0) \rangle = \frac{\pi}{12} \partial_z^2 \delta(x).
\] (41)

The same formulae were obtained in the axiomatic approach to conformal field theory in Ref. 18. A separate argument provided in Ref. 23 motivates them as
follows: If we knew that conservation of energy-momentum held even in the limit of coinciding points (which we actually do not know without explicit calculation), we could have expected the form of these correlation functions by inserting a spectral decomposition of the unit operator between the two $T$s. By conservation of $T_{ij}$, the correlator must then have the form

$$\langle T_{\mu \nu}(x) T_{\rho \sigma}(0) \rangle = \frac{\pi}{3} \int d\mu c(\mu) \int \frac{d^2p}{(2\pi)^2} e^{ipx} \frac{(g_{\mu \nu}p^2 - p_\mu p_\nu)(g_{\rho \sigma}p^2 - p_\rho p_\sigma)}{p^2 + \mu^2},$$

which, noting that the Fourier transform of $1/z^4$ is $(\pi^2/24) \bar{p}^4/|p|^2$, indeed coincides with our results upon guessing $c(\mu) \propto \delta(\mu)$, as expected for a massless theory.

Our calculation confirms this explicitly. Further work on contact terms of energy-momentum tensors is reported in Ref. 24.

As a consistency check, even without performing the full calculation above, a simple algebraic argument applied to the original integrands, combined with a shift of variables, which is permitted in the presence of the Pauli-Villars fields, shows that, for example,

$$\partial_\xi \langle T_{zz}(x) T_{\bar{z}\bar{z}}(0) \rangle + \partial_\bar{z} \langle T_{\bar{z}z}(x) T_{zz}(0) \rangle = 0.$$

In other words, the Pauli-Villars regularization does not break conservation of energy-momentum, even in the limit of coinciding points.

We are now finally ready to verify

$$\left\langle \frac{d}{d\lambda} T_{zz}^\lambda \right\rangle = -\frac{1}{12} \partial_\lambda v^z + \partial_\lambda \partial_\bar{z} v^z + \frac{1}{12} \partial_\bar{z} \partial^2 v^z = 0$$

(42)

In other words, the contact terms neatly cancel the anomalous contribution coming from the $1/(z - w)^4$ term in $\langle T_{zz}^\lambda T_{ww}^\lambda \rangle$. We may also calculate the left hand side directly. We have, by (31),

$$\left\langle \frac{d}{d\lambda} T_{zz} \right\rangle = -\mathcal{L} \langle T_{zz} \rangle + \pi \mathcal{L} (g_{zz}g^{kl}) \langle \partial_k \phi \partial_l \phi + PV \rangle + \pi g_{zz} \langle m^2 \phi^2 + PV \rangle = 0,$$
since all the expectation values on the right hand side were shown to vanish on the plane in sections 3 and 5. This confirms the validity of the classical Ward identity for the component $T_{zz}$. In the special case where $v$ is holomorphic in a neighbourhood of the insertion $T_{zz}$, this may be simplified using (32) and we find that we have verified the expression

$$\langle -v^i \partial_i T^\lambda_{zz} - 2 (\partial_z v^z) T^\lambda_{zz} \rangle = -\frac{1}{4\pi} \int d^2 w \ h^{ww} \langle T^\lambda_{zz} T^\lambda_{ww} \rangle - 2 \cdot \frac{1}{4\pi} \int d^2 w \ h^{ww} \langle T^\lambda_{zz} T^\lambda_{w\bar{w}} \rangle - \frac{1}{4\pi} \int d^2 w \ h^{\bar{w}\bar{w}} \langle T^\lambda_{zz} T^\lambda_{\bar{w}\bar{w}} \rangle.$$  

As promised, this Ward identity has no anomaly.

Let us also verify the Ward identity for an insertion of $T_{z\bar{z}}$. A similar calculation to the above gives

$$\left\langle \frac{d}{d\lambda} T^\lambda_{z\bar{z}} \right\rangle = + \frac{1}{12} \partial_z^2 \partial_{\bar{z}} v^z - \frac{1}{12} \partial_{\bar{z}}^2 \partial_z v^z + \frac{1}{12} \partial_z \partial_{\bar{z}}^2 v^z = 0$$

and, similar to the case of $T_{zz}$ above, the left hand side may also be shown to be 0 by the results of sections 3 and 5.

How does one reconcile our non-anomalous Ward identity for $T_{zz}$ with the anomalous identity appearing in the operator formalism literature? We note that if we define the quantity $\hat{T}_{zz}$ to coincide with $T_{zz}$

$$\hat{T}_{zz} = T_{zz}$$

on the plane with trivial metric, and deform $\hat{T}_{zz}$ according to the nontensor transformation law

$$\delta_v \hat{T}_{zz} = \frac{1}{12} \partial_z^2 v^z + v^i \partial_i T_{zz} + 2 (\partial_z v^z) \hat{T}_{zz},$$

as we deform the metric along the flow of a vector field $v$ holomorphic in a neighbourhood of the insertion, the extra term in the transformation law of $\hat{T}_{zz}$ will exactly cancel the contributions coming from the contact terms (second and third lines) in the derivation (42). In terms of $\hat{T}_{zz}$, the Ward identity can therefore be expressed as

$$- \left\langle \delta_v \hat{T}_{zz} \right\rangle = -\frac{1}{4\pi} \int d^2 w \ h^{ww} \langle \hat{T}_{zz} \hat{T}_{ww} \rangle = \frac{1}{\pi} \int d^2 w \langle \hat{w} w \rangle \langle \hat{T}_{zz} \hat{T}_{ww} \rangle,$$
which is the form familiar from the operator formalism. Indeed, the energy-momentum tensor obtained from the common point-splitting renormalization of the operator product coincides with \( \hat{T}_{zz} \), as can be seen from its transformation law.

It is possible to express the metric-dependence of \( \hat{T}_{zz} \) generated by the above transformation law directly as \[ \hat{T}_{zz} = T_{zz} + \frac{1}{12} t_{zz}, \]

where

\[ t_{zz} \equiv \partial_z \Gamma^z_{zz} - \frac{1}{2} (\Gamma^z_{zz})^2. \]

Looking back to section 4, we see that \( \hat{T}_{zz} \) here coincides with the corresponding \( \hat{T}_{zz} \) of equation (18).

Using the property \[ \partial_{\bar{z}} t_{zz} = -\frac{1}{2} g_{zz} \partial_z R, \]

the insertion \( \hat{T}_{zz} \) is easily seen to still satisfy the conservation law

\[ \langle \partial_{\bar{z}} \hat{T}_{zz} \rangle = 0 \]

on a flat manifold.

9. Checking the change of variables theorem

Fundamental to path-integral derivations of conservation laws and Ward identities is the generalization (29)

\[ \int [d\phi^\lambda]_{PV} e^{-S(g,\phi^\lambda,\bar{\chi}^i,\chi^i)} = \int [d\phi]_{PV} e^{-S(g,\phi,\bar{\chi}^i,\chi^i)} \]

of the change of variables theorem from finite to infinite dimensions. It says that the path integral is independent of the choice of coordinates on the space of fields, a statement that must be satisfied if the concept of integration is to make sense in a chart-independent way. In the absence of a general proof, we here perform a small consistency check in support of this statement.

In particular, let us check this formula to second order around \( \lambda = 0 \) on the plane. As explained in section 7, \( [d\phi^\lambda]_{PV} \) is independent of \( \lambda \), so that differentiating the left hand side of (45) twice with respect to \( \lambda \) gives

\[ \left. \frac{d^2}{d\lambda^2} \right|_{\lambda=0} \int [d\phi^\lambda]_{PV} e^{-S(g,\phi^\lambda,\bar{\chi}^i,\chi^i)} = \frac{1}{4\pi} \int d^2 x \ h^{ij}(x) \left. \left( \frac{d}{d\lambda} \right|_{\lambda=0} T_{ij}^\lambda(x) \right) \]

\[ + \left( \frac{1}{4\pi} \right)^2 \int d^2 x \int d^2 y \ h^{ij}(x) \ h^{kl}(y) \langle T_{ij}(x) T_{kl}(y) \rangle, \]

where

\[ h^{ij} \equiv \delta_\lambda(f_\lambda g)^{ij} \big|_{\lambda=0} = -\partial^i v^j - \partial^j v^i. \]
For the change of variables formula to be valid to second order, this expression should be zero. But notice that this expression is just the integral over \( x \) of the Ward identity \( (30) \), laboriously verified in the previous section. We therefore find the required result

\[
\left. \frac{d^2}{d\lambda^2} \right|_{\lambda=0} \int [d\phi^\lambda]_{PV} e^{-S(g,\phi^\lambda,\bar{\chi}^\lambda,\chi^\lambda)} = 0.
\]

10. Relating the conformal anomaly and the Ward identity

By changing from an active to a passive point of view, the result of the previous section can also be interpreted as telling us that the second-order variation of the partition function is zero when we pull the metric, as opposed to the fields, along the flow of a vector field. Such a deformation does not change the curvature of an initially flat surface, and therefore, as expected, the Weyl anomaly did not contribute.

Let us now instead consider the change of

\[
\int \left[ d\phi \right]_{PV} e^{-S(g,\phi,\bar{\chi},\chi)}
\]
to second order under a deformation \( h_{ij} \equiv \delta g_{ij} \) of the trivial metric \( g_{ij} = \delta_{ij} \) on the plane, not necessarily generated by a vector field. Since in general this cannot be compensated by a change of variables, we do not expect the variation to be zero. In general, a second derivative will bring down up to two instances of the energy-momentum tensor from the exponent, so that we will have to calculate terms of the form \( \langle T_{ij} T_{kl} \rangle \), for which we are forced to use the Pauli-Villars regularization of the previous sections to obtain the correct contact terms. These contact terms are essential to obtaining the correct result.

Since the Pauli-Villars measure \( [d\phi]_{PV} \) is invariant under variations of \( g \), we can write

\[
\delta_g^2 \int \left[ d\phi \right]_{PV} e^{-S} = \frac{1}{4\pi} \frac{1}{2} \int d^2 x \sqrt{g} g_{kl} h^{kl} \langle T_{ij} \rangle + \frac{1}{4\pi} \int d^2 x \sqrt{g} h^{ij} \langle \delta g T_{ij} \rangle + \left( \frac{1}{4\pi} \right)^2 \int d^2 x \sqrt{g} \int d^2 y \sqrt{g} h^{ij}(x) h^{kl}(y) \langle T_{ij}(x) T_{kl}(y) \rangle.
\]

On the plane, \( \sqrt{g} = 1 \) and, as shown in section 5, \( \langle T_{zz} \rangle \), \( \langle T_{\bar{z}\bar{z}} \rangle \), and \( \langle T_{z\bar{z}} \rangle \) are all zero with the Pauli-Villars measure, so that the first term vanishes. Remembering that \( T_{ij} \) depends on the metric, we calculate

\[
\delta_g \langle T_{ij} \rangle = 2\pi \cdot \frac{1}{2} \delta g \langle g_{ij} g^{kl} \rangle \langle \partial_k \phi \partial_l \phi + PV \rangle + \pi \delta g_{ij} \langle m^2 \phi^2 + PV \rangle.
\]

The expectation values on the right hand side are all linear combinations of \( \langle T_{zz} \rangle \), \( \langle T_{\bar{z}\bar{z}} \rangle \), \( \langle T_{z\bar{z}} \rangle \) and \( \langle \partial_z \phi \partial_{\bar{z}} \phi + PV \rangle \), all of which vanish on the plane as shown in sections 3 and 5. So

\[
\delta_g \langle T_{ij} \rangle = 0.
\]
Let us now see how this Ward identity is related to the conformal anomaly discussed in sections 2 and 3. For simplicity we first consider a Weyl variation

$$\delta_\omega g_{ij} = 2 \delta \omega g_{ij}$$

of the flat metric. Then $h^{zz} = \bar{h}^{zz} = -4 \delta \omega$, and the above formula becomes

$$\delta^2 \omega \int [d\phi]^P e^{-S} = \left( \frac{1}{4\pi} \right)^2 \int d^2 z \int d^2 w \, 4 h^{zz} h^{\bar{w}w} \langle T_{zz} T_{w\bar{w}} \rangle$$

$$= \left( \frac{1}{4\pi} \right)^2 \int d^2 z \int d^2 w \, 4 h^{zz} h^{\bar{w}w} \left( -\frac{\pi}{12} \right) \partial_z \partial_{\bar{z}} \delta(z - w) \langle 1 \rangle$$

$$= -\frac{1}{12\pi} \int d^2 z \, \delta \omega \Delta \delta \omega \langle 1 \rangle$$

$$= \frac{1}{24\pi} \int d^2 x \, \delta \omega \delta R \langle 1 \rangle,$$

where we used

$$R = -2 e^{-2\omega} \Delta \omega \quad \text{for} \quad g_{ij} = e^{2\omega} \delta_{ij}.$$ 

This formula coincides precisely with the second variation around the flat metric ($R = 0$) of the formula for the conformal anomaly

$$\delta_\omega \int [d\phi]^P e^{-S(g, \phi, \bar{\chi}, \chi_i)} = \left( \frac{1}{24\pi} \int d^2 x \, \sqrt{g} \delta \omega(x) R \right) \int [d\phi]^P e^{-S(g, \phi, \bar{\chi}, \chi_i)}.$$

derived in section 3.

For more generic deformations of the metric, we have

$$\delta^2 \omega \int [d\phi]^P e^{-S(g, \phi, \bar{\chi}, \chi_i)} = \left( \frac{1}{4\pi} \right)^2 \int d^2 x \sqrt{g} \int d^2 y \sqrt{g} h^{ij}(x) h^{kl}(y) \langle T_{ij}(x) T_{kl}(y) \rangle$$

$$= \left( \frac{1}{4\pi} \right)^2 \int d^2 z \int d^2 w \left\{ h^{zz} h^{\bar{w}w} \left( \frac{1}{7} \right) \frac{1}{(z - w)^2} \right.$$ 

$$+ 2 h^{zz} h^{\bar{w}w} \left( \frac{\pi}{12} \right) \partial_z \partial_{\bar{z}} \delta(z - w)$$

$$+ h^{zz} h^{\bar{w}w} \left( -\frac{\pi}{12} \right) \partial_z \partial_{\bar{z}} \delta(z - w)$$

$$+ \ldots \right\} \langle 1 \rangle,$$

where we have inserted the contact term two-point functions derived before. To save space, we only wrote out the first three terms. Now, using

$$\pi \delta(z - w) = \partial_z \partial_{\bar{z}} \ln |z - w|^2,$$
and performing partial integrations, we find

\[-\frac{1}{12} \left(\frac{1}{4\pi}\right)^2 \int d^2 z \int d^2 w \left(-\partial_z^2 h^{zz} - \partial_{\bar{z}}^2 h^{\bar{z}\bar{z}} + 2 \partial_z \partial_{\bar{z}} h^{zz}\right) \times \]
\[\times \ln |z - w|^2 \left(-\partial_w^2 h^{ww} - \partial_{\bar{w}}^2 h^{\bar{w}\bar{w}} + 2 \partial_w \partial_{\bar{w}} h^{ww}\right)\]
\[\times \ln |z - w|^2 \delta_g R(z) \ln |z - w|^2 \delta_g R(w).\]

Notice that all the \(\langle TT\rangle\) contact terms were necessary to obtain the correct curvature factors \(\delta R\).

Our final result for the plane is

\[\delta_g^2 \int [d\phi]^P e^{-S} = -\frac{1}{12} \left(\frac{1}{4\pi}\right)^2 \int d^2 z \int d^2 w \delta_g R(z) \ln |z - w|^2 \delta_g R(w) \langle 1 \rangle.\]

Notice that this result is entirely consistent with that of the previous section, since for a deformation of \(g\) by a vector field we have \(\delta R = 0\), so that the right hand side vanishes.

11. Conclusion

In this paper we presented a coordinate-invariant Pauli-Villars-based approach to the definition of the path integral measure and the calculation of anomalies in two-dimensional scalar conformal field theory.

We showed the agreement, despite seemingly different origins, of the conformal anomaly in the Pauli-Villars and the Fujikawa approaches.

The natural, fully regularized energy-momentum in our coordinate-invariant approach is a true tensor quantity satisfying classical Ward identities. We related this quantity to the more familiar non-tensor object arising in the operator formalism.

We provided a direct path-integral calculation of the nontrivial contact terms arising in expectation values of certain energy-momentum products, previously derived only using axiomatic considerations. We used these in a simple consistency check confirming the change of variables formula for the path integral measure. We also showed that the contact terms are essential to obtaining the correct relationship between the conformal anomaly and the energy-momentum two-point functions in our formalism.

It is our hope that this work may have some inherent interest as an illustration, in a simple model, of the issues involved in defining a coordinate-invariant path integral and energy-momentum tensor in a matter theory on a nontrivial gravitational background.

We also hope that this work may be helpful in illustrating the origin, often physically opaque in the operator formalism, of some simple anomalous formulas in conformal field theory. It is important to understand to what extent one can trust straightforward manipulations of path integrals to obtain potentially anomalous conservation laws and Ward identities. The conclusion of this paper is that, given
a suitably coordinate-invariant regularization such as the one defined here, one can trust these manipulations a great deal. However, as we have seen, the translation of the resulting formulas to other formalisms may be non-trivial and subtle.

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