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Vascular blood flow reconstruction from tomographic projections with the adjoint method and receding optimal control strategy

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Abstract. In this work, we study the measurement of blood velocity with contrast enhanced computed tomography. The inverse problem is formulated as an optimal control problem with the transport equation as constraint. The velocity field is reconstructed with a receding optimal control strategy and the adjoint method. The convergence of the method is fast.

1. Introduction
The blood velocity in arteries is an important clinical parameter [1]. Recently, we have proposed a method to evaluate this velocity with contrast agent enhanced tomography [2]. In this work, the inverse problem of contrast enhanced computerized tomography (CT) reconstruction of vascular blood flow is formulated as an optimization problem with partial differential equation constraints. The reconstruction is based on the acquisition of 2D CT projections with a projection angle perpendicular to main flow propagation direction and increasing linearly with time. The inversion is regularized with the partial differential equation constraint describing the transport of the contrast agent coupling its density \( f(x,t) \) and the flow field \( w(x,t) \). In this former approach, the pde constraint is used as a side constraint in the optimization problem and the variables \( (f, w) \) are treated as independent optimization variables. Yet, the efficient solution of the optimality system is a very challenging task. Good results have been obtained for the density but not for the velocity field. Moreover, the computing time is very long. In this new work, a suboptimal receding optimal control strategy is used to get a simpler optimization problem and to reduce the computing time. This type of approach consists in replacing the initial optimization problem by a sequence of optimization problems on smaller time intervals. It has been used for the optimal control of fluid flows [3, 4]. Moreover, we consider here a new approach for solving the optimization problem. The pde constraint is eliminated with a control to state operator \( f = f(w) \), and the only optimization variable is the fluid velocity \( w \). A gradient type method is used to minimize the reduced functional. The efficient evaluation of the derivative uses the adjoint method [5]. This paper is organized as follows. In Section 1, the dynamic tomography problem considered is summarized and it is formulated as an optimal control problem. The receding optimal control strategy and the adjoint method are presented in the second section. We display numerical results with a simple synthetic fluid flow and we show that this new formulation improves the reconstruction of the flow field.
2. The contrast enhanced tomography inverse problem

2.1. Optimal control formulation

We summarize here the basic optimal control formulation presented in [2] which is the starting point of our new approach. The phantom considered and the CT acquisition geometry are shown in Figure 1. This phantom simulates a straight arterial vessel segment oriented along the z axis. During the propagation into the phantom, two dimensional projections are acquired with a projection direction perpendicular to the z axis and a projection angle \( \theta(t) \) increasing linearly with time \( t \).

\[ \begin{align*}
E(f, w) &= \frac{\partial f(x,t)}{\partial t} + u \frac{\partial f(x,t)}{\partial x} + v \frac{\partial f(x,t)}{\partial y} + w \frac{\partial f(x,t)}{\partial z} - D \Delta f(x,t) = 0 \quad \forall t \in [0,T] \quad \forall x \in \Omega \\
f(x, t) &= 1 \quad \text{on} \ (D) \quad \forall t \in [0,T] \\
f(x, 0) &= 0 \quad \text{on} \ \Omega - (D)
\end{align*} \]

Compared to our former work [2], the addition of a diffusion term ensures the unicity of \( f \) for a smooth field \( w \). For the sake of simplicity, we have suppressed the incompressibility constraint. Let \( X = L_2(\Omega) \), our aim is to find \( f : [0, T] \rightarrow \mathbb{R}^3 \) from the linear equations, \( R_{\theta(t)} f(u_1, u_2) = p^\delta_{\theta(t)} \) where \( p^\delta_{\theta(t)} \) are the noisy measured projections, \( R_{\theta(t)} \) is the linear Radon operator for the projection angles \( \theta(t) \), and \( u_1 \) and \( u_2 \) are the detector coordinates. The cost functional considered is:

\[ J(f, w) = \frac{1}{2} \int_0^T \| R_{\theta(t)} f(u_1, u_2) - p^\delta_{\theta(t)} \|^2 dt + \frac{\alpha}{2} \int_0^T \| \nabla w \|_2^2 dt \]

where \( \| \nabla w \|_2 \) the Frobenius norm of the Jacobian matrix of the flow field \( w \) and \( \alpha \) a regularization parameter. In our former work, we considered the following optimization problem, \( \min_{f, w} J(f, w) \) with \( E(f, w) = 0 \).

2.2. The receding horizon technique and the new optimal control formulation

Solving the time dependent optimal control problem governed by the former PDE requires a significant computing time and a large storage power. Several suboptimal solutions have been proposed [3, 4] to make large problems feasible and to reduce the computing time. We consider here the receding horizon control method. The idea of the receding horizon control is to replace the optimal control problem on the full time horizon by a succession of simpler optimal
control problems on short control horizons. Let \([0, T]\) denote the time interval over which the optimization is performed. We chose time values \(0 = T_0 < \ldots T_K = T\) to partition this time interval. On each time sub-interval, we assume that we have constant velocity fields \(w_k\).

For \(k=0, \ldots, K-1\), we minimize sequentially the functionals

\[
\dot{J}_k(w_k) = J_k(f(w_k), w_k)
\]

with

\[
J_k(f, w) = \frac{1}{2} \int_{T_k}^{T_{k+1}} \|R_{\theta(t)} f(u_1, u_2) - p_{\theta(t)}^\delta(t)\|^2 dt + \frac{1}{2} \int_{T_k}^{T_{k+1}} \|\nabla w\|^2 dt = \int_{T_k}^{T_{k+1}} H(f, w) dt
\]

subject to the partial differential equation (1) over \([T_k, T_{k+1}] \times \Omega\). On each of the intervals \([T_k, T_{k+1}]\) we have to consider a vector \(q\) of unknown parameters \(q = [a, w]\) with \(f(T_k, \cdot) = a\) and \(a \in L_2(\Omega)\), corresponding to the initial condition. The initial condition can be written as

\[
geq_k = f(T_k) - a = 0.
\]

### 2.3. The adjoint method

In our first approach [2], the variables \((f, w)\) are both considered as optimization variables. We present here an alternative approach. We consider here a parabolic equation (1) with a diffusion term which admits a unique solution for sufficiently smooth \(w\) [6, 7]. The former optimal control problem is equivalent to the reduced optimization problem where only the control variable \(w\) appears:

\[
\min \dot{J}(w) = J(f(w), w)
\]

where \(f(w)\) satisfies \(E(f(w), \dot{f}(w), w) = 0\). To derive the optimality conditions and to calculate the gradient of the reduced functional \(\dot{J}_k\), we have to introduce the Lagrangian corresponding to the optimisation problem:

\[
L_k(f, q) = J(f(w_k), w_k) + \int_{T_k}^{T_{k+1}} \left< E(f(t), p_k(t)) > dt + \mu_k, g(f(T_k), q_k) > \right.
\]

We consider two dual variables \(p_k : [T_k, T_{k+1}] \times \Omega \rightarrow \mathbb{R}\) and \(\mu_k : [T_k, T_{k+1}] \times \Omega \rightarrow \mathbb{R}\) for the constraints. In order to apply the adjoint method, we compute the directional derivative of \(L_k\) with respect to \(q_k\) in the direction \(r\):

\[
L_{k, q_k}(f, q_k) r = \int_{T_k}^{T_{k+1}} \left[ \partial f H d_q q_k f r + \partial_q H r \right] dr + \left< p_k, \partial f E d_q q_k f r + \partial_q E d_q q_k f r \right.
\]

\[
+ \partial q_k E r > dt + \left< \mu, \partial f(T_k) g d_q q_k f(T_k) r + \partial_q g r \right>
\]

With an integration by part,

\[
\int_{T_k}^{T_{k+1}} < p_k, \partial f E d_q q_k f r > dt = \left< p_k, \partial f E d_q q_k f r \right>_{T_{k+1}}^{T_k} - \int_{T_k}^{T_{k+1}} \left< \partial p_k, \partial f E r > + < p_k, \partial_q E r > \right] dr
\]

Substituting in (10), we obtain:

\[
L_{k, q_k}(f, q_k) r = \int_{T_k}^{T_{k+1}} \left[ \partial f H d_q q_k f r + \partial_q H r \right] dr + \left< p_k, \partial f E - d_t \partial f E d_q q_k f r \right>
\]

\[
- \left< \hat{p}_k, \partial f E d_q q_k f r \right> + \left< p_k, \partial_q E r > \right] dt
\]

\[
- \left< p_k, \partial f E d_q q_k f r \right|_{T_{k+1}}^{T_k} - \left< p(T_k), \partial f E |_T d_q q_k f(T_k) r \right> + \left< \mu, g f(T_k) |_T d_q q_k f(T_k) r \right> + \left< \mu, g q e r \right>
\]
The principle of the adjoint method is to avoid calculating the total derivative \( d_{q_k} f \) by setting:
\[
\partial f H + (\partial f E - d_t \partial f E)^* p_k - \partial f E^* \dot{p}_k = 0
\] (12)

For the problem considered,
\[
\partial f H = R^*_\theta(t) (R_\theta(t) f(x, t) - \hat{p}^*_\theta(t)) \quad \partial f E = \mathbf{w}_k, \nabla - D \Delta
\] (13)

We obtain the following adjoint partial differential equation:
\[
R^*_\theta(t) (R_\theta(t) f(x, t) - \hat{p}^*_\theta(t)) + (-\mathbf{w}_k, \nabla - D \Delta) p_k - \dot{p}_k = 0 \quad p_k(T_{k+1}) = 0
\] (14)

Similarly the optimality with respect to \( f(T_k) \) gives \( \partial f E|_{T_k}^* p_k(T_k) = g\|f(T_k)\|_{T_k}^* \mu \) or
\[
\mu = g\|f(T_k)\|_{T_k}^* \partial f E|_{T_k}^* p_k(T_k) = -p_k(T_k)
\] (15)

The gradient of the Lagrangian is then given by:
\[
\nabla J^*(q_k) = \nabla L_{q_k}(q_k) = \int_{T_k}^{T_{k+1}} (H_{q_k} + \partial q_k E^* p_k) dt + g_{q_k}^* g\|f(T_k)\|_{T_k}^* \partial f E|_{T_k}^* p_k(T_k)
\] (16)

The first term \( \int_{T_k}^{T_{k+1}} (-\alpha \Delta \mathbf{w}_k + p_k \nabla f) dt \) corresponds to the gradient for the \( \mathbf{w} \) component of the vector \( q_k \) and the second one, \(-p_k(T_k)\) to the initial value component \( f(T_k) \). The evaluation of the gradient \( \nabla J(q_k) \) for a given control function \( \mathbf{w}_k \) can be summarized in the algorithm:
1) Solve the forward flow equation to obtain \( f(x, t) \) from the current approximation \( \mathbf{w}_k \) (Eq.1).
2) Solve the backward adjoint equation to obtain the adjoint variable \( p_k \) (Eq.14).
3) Return the gradient, \( \int_{T_k}^{T_{k+1}} (-\alpha \Delta \mathbf{w}_k + p_k \nabla f) dt, -p_k(T_k) \)

3. Numerical algorithm
3.1. Simulation details
The values of the components \((u, v, w)\) of the flow field \( \mathbf{w} \) and of the density \( f \) are discretized on a grid of size \( N_s = 53 \times 53 \times 53 \) for the spatial domain \( \Omega \) and on a grid of size \( N_t = 100 \) for the time domain \([0, T]\). For each time \( t_k \) in the time domain \([0, T]\), a projection angle \( \theta_k = 2\pi t_k/T \) is used for the Radon projection operator. During the CT gantry rotation, 100 parallel projections are thus acquired. Moreover, 77 projections rays are acquired for each value of the projection angle. The ground truth solution \( f^* \) is obtained from the scalar transport equation with a uniform velocity along the \( z \) axis, slowly decreasing with time, \( \mathbf{w}^* = (u^*, v^*, w^*) = (0, 0, 10 + 5 \exp(-t/10)) \). The step size for the receding control strategy \( T_{k+1} - T_k \) is 6 five times the time step used for the projection \( t_{k+1} - t_k \). The total horizon time is thus divided in 16 subintervals. The initial density of the contrast agent is \( f(x, z = 0, t = 1) = 1 \) for the pixels inside a disk of diameter 17 around the \( z \) axis. A Gaussian white noise with peak-to-peak signal-to-noise ratio (PPSNR) of 48 dB was added to the projections. The regularization parameter value \( \alpha = 2.10^{-1} \) is chosen to have the best decrease of the discrepancy term. For the integration of the partial differential equation, the problem is discretized by finite difference and we use an explicit scheme of order 2 with numerical diffusion. For this numerical scheme, the CFL condition is verified, \( CFL = \|\mathbf{w}\|dt/dx < 1 \). For the simulation of the transport equation, the time step \( dt = 0.005 \) is used. The adjoint equation is solved by the same method as described before by reversing time. The reduced functional \( \min J \) is minimized iteratively with a steepest descent method on each subinterval with an inner loop sing the negative of the gradient of \( J \) as search direction. The step length \( \rho_k \) is chosen as, \( \rho_k = \arg \min J(\mathbf{w}_k + \rho \nabla J(\mathbf{w}_k)) \).
3.2. Results

Sections of the reconstructed $w$ and of the contrast agent density obtained at the end of the optimization process for $z = 5$ and $t = 10s$ are displayed in Figure 2. In order to validate the method, on each subinterval $[T_k, T_{k+1}]$, we have studied the evolution of the $L_2$ error term for the velocity field $w^j_k$ and the reconstructed density $f^j_k$ with the inner iterations $j$ during the steepest descent:

$$D(f^j_k) = \sum_{t_i=T_k}^{T_{k+1}} \|f^j_k(x, t_i) - f^*(x, t_i)\|^2$$  \hspace{1cm} (17)

$$D(w^j_k) = \sum_{t_i=T_k}^{T_{k+1}} \|u^*(x, t_i) - u^j_k(x, t_i)\|^2 + \|v^*(x, t_i) - v^j_k(x, t_i)\|^2 + \|w^*(x, t_i) - w^j_k(x, t_i)\|^2$$  \hspace{1cm} (18)

where the sum is extended to the times $t_i \in [T_k, T_{k+1}]$. Some error curves are presented on Figure 3 for a selected time interval $[T_k, T_{k+1}] = [6, 12]$. A large and fast reduction of the error is obtained for the reconstructed density and the velocity field. We have also calculated the total error terms on the whole horizon time $[0, T]$ as a function of the total number of outer iterations. The decreases for the reconstruction errors for the new method are compared to the ones of our former approach [2] in Figure 4. The simulation takes 800 s for the case investigated for the receding horizon technique and 20000 for the former method. The new approach presented in this work clearly outperforms the former in terms of speed for similar reconstruction results.

4. Conclusion

In this work we have presented a new method for the reconstruction of the flow velocity of a contrast agent from CT measurements with projections perpendicular to the main axis of propagation. The propagation equation of the contrast agent is used to formulate an unconstrained optimization problem where the unknown is the velocity field. The optimal control problem is solved with a receding horizon technique and with the adjoint method. This new method is tested on a simple phantom. It is much faster than our former approach where the full time interval is considered and where the velocity field and the density field are both considered as optimization variables. In the following, we will investigate second order optimization methods on more complex phantoms and velocity fields.

![Figure 2. Sections of the reconstructed velocity and density for z=5, t=10s at the end of the optimization process.](image)
Figure 3. Evolution with the iterations of the RMSE for the velocity and the density on a sub-interval $[T_k, T_{k+1}] = [6, 12]$.

Figure 4. Evolution with the iterations of the RMSE for the velocity and the density. Black curve: receding horizon technique and adjoint method, blue curve: optimal control strategy detailed in [2].

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