Blow-up rates
for the general curve shortening flow

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Abstract

The blow-up rates of derivatives of the curvature function will be presented when
the closed curves contract to a point in finite time under the general curve shortening
flow. In particular, this generalizes a theorem of M.E. Gage and R.S. Hamilton about
mean curvature flow in $\mathbb{R}^2$.

Key words: the general curve shortening flow; Young inequality; Wirtinger
inequality; Sobolev inequality

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1 Introduction

The curve shortening flow has been studied extensively in the last thirty
years (cf. [1], [2]). As for the newest development both on expansion and
contraction of convex closed curves in $\mathbb{R}^2$, see [3], [4], [5], [6], etc.

It is the asymptotic behavior of the general curve shortening flow that we
will mostly be concerned with. Let $S^1$ be an unit circle in the plane, and define

$$
\gamma_0: \ S^1 \rightarrow \mathbb{R}^2,
$$

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as the closed convex curve in the plane. We look for a family of closed curves
\[ \gamma(u, t) : S^1 \times [0, T) \to \mathbb{R}^2, \]
which satisfies
\[
\begin{cases}
\frac{\partial \gamma}{\partial t}(u, t) = G(k)kN, & u \in S^1, \ t \in [0, T), \\
\gamma(u, 0) = \gamma_0(u), & u \in S^1, \ t = 0,
\end{cases}
\]  
(1.1)
where \( G \) is a positive, non-decreasing smooth function on \((0, \infty)\), \( k(\cdot, t) \) is the inward curvature of the plane curve \( \gamma(\cdot, t) \) and \( N(\cdot, t) \) is the unit inward normal vector.

Assume that \( A(t) \) is the area of a bounded domain enclosed by the curve \( \gamma(\cdot, t) \), \( L(t) \) is the length of \( \gamma(\cdot, t) \), \( r_{out}(t) \) and \( r_{in}(t) \) are respectively the radii of the largest circumscribed circle and the smallest circumscribed circle of \( \gamma(\cdot, t) \). Define
\[ k_{\text{max}}(t) = \max \{ k(u, t) \mid u \in S^1 \}, \]
\[ k_{\text{min}}(t) = \min \{ k(u, t) \mid u \in S^1 \}. \]

The following existence theorem of (1.1) belongs to Ben. Andrews (cf. Theorem II.4.1, Proposition II.4.4 in [1]).

**Proposition 1.1** Let \( \gamma_0 \) be a closed strictly convex curve. Then the unique classical solution \( \gamma(\cdot, t) \) of (1.1) exists only at finite time interval \([0, \omega)\), and the solution \( \gamma(\cdot, t) \) converges to a point \( \vartheta \) as \( t \to \omega \) and \( A(t), k_{\text{max}}(t) \) satisfy the following properties:
\[
\forall t \in [0, \omega), \ A(t) > 0, \ k_{\text{max}}(t) < +\infty, \\
\lim_{t \to \omega} A(t) = 0, \quad \lim_{t \to \omega} k_{\text{max}}(t) = +\infty.
\]
As \( t \to \omega \), the normalized curves
\[ \eta(\cdot, t) = \sqrt{\frac{\pi}{A(t)}} \gamma(\cdot, t) \]
converges to the unit circle centered at the point \( \vartheta \).

Furthermore, if we assume that \( G(x) \) is a function on \((0, \infty)\) satisfying
(H1) \( G(x) \in C^3(0, \infty), G'(x) \geq 0 \) and \( G(x) > 0 \) for \( x \in (0, \infty) \),
(H2) \( G(x)x^2 \) is convex in \((0, \infty)\) and there is a positive constant \( C_0 \) such that
\[ G'(x)x \leq C_0 G(x) \quad \text{for sufficiently large } x. \]
Then Rong-Li Huang and Ji-Guang Bao had obtained the following theorem (cf. [7]).

**Proposition 1.2** Suppose that \( G(x) \) satisfies (H1) and (H2). Let \( \gamma(\cdot, t) \) be the solution for Proposition 1.1. Then there holds:

i) \[
\lim_{t \to \omega} \frac{r_{in}(t)}{r_{out}(t)} = 1.
\]

ii) \[
\lim_{t \to \omega} \frac{k_{\text{min}}(t)}{k_{\text{max}}(t)} = 1.
\]

iii) \[
\lim_{t \to \omega} \frac{1}{\omega - t} \int_{k(\theta, t)}^{+\infty} \frac{dx}{G(x)x^3} = 1 \text{ is uniformly convergent on } S^1.
\]

Especially, set \( G(x) = |x|^{p-1} \) with \( p \geq 1 \). Then

\[
k(\theta, t)[(p + 1)(\omega - t)]^{\frac{1}{p+1}} \text{ converges uniformly to } 1 \text{ as } t \to \omega. \quad (1.2)
\]

Ben. Andrews classifies the limiting shapes for the isotropic curve flows (cf [3]). However, the blow-up rates of derivatives of the curvature function for the general curve shortening flow are not concerned. We now state the main theorem of this paper.

**Theorem 1.1** Suppose that \( G(x) = |x|^{p-1} \) with \( p \geq 1 \) and \( k \) is the curvature function of \( \gamma(\cdot, t) \) according to the flow (1.1). \( \gamma_0 \) is a smooth and closed convex curve. If \( \gamma(\cdot, t) \) converges to a point \( \vartheta \) as \( t \to \omega \). Then for each \( l \in \{1, 2, \cdots\} \), there exists some positive constant \( C(l, p) \) depending only on \( l, p \), such that

\[
\| \frac{\partial^l k}{\partial \theta^l} \|_{L^\infty(S^1)} \leq C(l, p)(\omega - t)^{2\alpha \frac{p}{p+1} - \frac{1}{p+1}}, \quad (1.3)
\]

where \( \alpha \) is any constant satisfying \( 0 < \alpha < 1 \).

**Remark 1.1** By (1.3) we generalize Corollary 5.7.2 in [8].

This paper is organized as follows: In the next section we obtain an equivalent proposition of Theorem 1.1 by means of transferring the flow (1.1) into an Cauchy PDE’s problem. Finally in the third section, it proves Proposition 2.1 by making use of Gage-Hamilton’s method.
2 Preliminaries

By Lemma 2.4 in [7], the general curve shortening problem (1.1) with initial convex curve \( \gamma_0(\theta) \) is equivalent to the cauchy problem

\[
\begin{aligned}
\frac{\partial k}{\partial t} &= k^2 \left( \frac{\partial^2}{\partial \theta^2} (G(k)k + G(k)) \right), \quad \theta \in S^1, \quad t \in (0, \omega), \\
k &= k_0(\theta), \quad \theta \in S^1, \quad t = 0.
\end{aligned}
\] (2.1)

where \( 0 < \alpha < 1 \), \( k \in C^{2+\alpha,1+\frac{\alpha}{2}}(S^1 \times (0, \omega)) \), \( k_0(\theta) \) denotes the curvature function of the curve \( \gamma_0(\theta) \).

Assume that \( G(x) = |x|^{p-1} \) with \( p \geq 1 \). Set

\[
\tau = -\frac{1}{p+1} \ln \frac{\omega - t}{\omega}, \quad \tilde{k}(\theta, \tau) = k^p(\theta, t)[(p+1)(\omega - t)]^{\frac{p}{p+1}}.
\]

It follows from (2.1) that \( \tilde{k}(\theta, \tau) \) satisfies

\[
\begin{aligned}
\frac{\partial \tilde{k}}{\partial \tau} &= p\tilde{k}^{1+p} \frac{\partial^2 \tilde{k}}{\partial \theta^2} + p\tilde{k}^{2+p} \frac{\partial \tilde{k}}{\partial \theta} - p\tilde{k}, \quad \theta \in S^1, \quad \tau \in (0, +\infty), \\
\tilde{k} &= \tilde{k}_0(\theta), \quad \theta \in S^1, \quad \tau = 0,
\end{aligned}
\] (2.2)

where \( \tilde{k}_0(\theta) = k_0^p(\theta)((p+1)\omega)^{\frac{p}{p+1}} \) is a positive smooth function. By (1.2), there holds

\[
\lim_{\tau \to +\infty} \tilde{k}(\theta, \tau) = 1 \text{ uniformly on } S^1.
\] (2.3)

Define

\[
\|\tilde{k}(l)\|_q = \left[ \int_{S^1} \left| \frac{\partial^l \tilde{k}}{\partial \theta^l} \right|^q \right]^\frac{1}{q} \quad \text{for } l \in \{1, 2, \cdots \} \text{ and } q \geq 1.
\]

Here, \( \tilde{k}', \tilde{k}'' , \cdots, \tilde{k}(l), \cdots \), denote partial differentiation by \( \theta \). Then Theorem 1.1 is equivalent to the following proposition.

**Proposition 2.1** If \( \tilde{k}(\theta, \tau) \) is a smooth solution of problem (2.2) with \( p \geq 1 \) and satisfies (2.3). Then for each \( l \in \{1, 2, \cdots \} \) there exists some constant \( C(l, p) \) depending only on \( l \), \( p \) and the initial curve \( \gamma_0 \) such that

\[
\|\tilde{k}(l)\|_\infty \leq C(l, p) \exp(-2\alpha p \tau), \quad \forall \tau > 0,
\] (2.4)

where \( \alpha \) is any constant satisfying \( 0 < \alpha < 1 \).

Obviously our task in the following can be changed over to prove Proposition 2.1. The forging listing of three facts and two lemmas can be used repeatedly (cf. [8]).
(Ⅰ). Young inequality. For all positive $\epsilon, p, q$, $ab \leq \frac{\epsilon^p a^p}{p} + \frac{b^q}{\epsilon^q q}$ where $\frac{1}{p} + \frac{1}{q} = 1$.

(Ⅱ). Wirtinger inequality. If $\int_{S^1} f = 0$, then $\int_{S^1} f^2 \leq \int_{S^1} f'^2$.

(Ⅲ). Sobolev inequality. If $\| f \|_2 \leq C$ and $\| f' \|_2 \leq C$, then $\| f \|_{\infty} \leq \left( \frac{1}{\sqrt{2\pi}} + \sqrt{2\pi} \right) C$.

Lemma 2.1 (Gage-Hamilton) Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy

$$\frac{df}{d\tau} \leq C f^{1-\frac{1}{q}} - 2C(q) f \quad \text{where} \quad q \geq 1.$$ 

Then

$$f^{\frac{1}{q}}(\tau) \leq \left( \frac{C}{2q} + D \exp(-2\tau) \right) \leq \tilde{C}(q).$$

Lemma 2.2 (Gage-Hamilton) If $\frac{df}{d\tau} \leq -\alpha f + C \exp(-\beta\tau)$, then

$$f(\tau) \leq D \exp(-\alpha\tau) + \frac{C}{\alpha - \beta} \exp(-\beta\tau) \quad \text{if} \quad \alpha \neq \beta,$$

or

$$f(\tau) \leq D \exp(-\alpha\tau) + C\tau \exp(-\alpha\tau) \quad \text{if} \quad \alpha = \beta.$$ 

3 Decay estimates of the scaling flow

In this section we establish some formal estimates on the analogy of [8] in order to obtain (2.4). In addition, those heuristic deductions bring out the decay estimates of the scaling curvature function. Hereafter we always suppose that $\tilde{k}$ is the smooth positive solution of (2.2) and satisfies (2.3).

Lemma 3.1 There exists a constant $C_1$ depending only on $p$ and the initial curve $\gamma_0$ such that there holds

$$\| \tilde{k}' \|_2 \leq C_1, \quad \| \tilde{k}' \|_4 \leq C_1.$$ 

(3.1)

Proof. For $n \in \{1, 3, 5, 7, \cdots \}$, we write

$$f(\tau) = \int_{S^1} \tilde{k}^{n+1}. $$
In view of the equation (2.1), it gives

$$\frac{f'(\tau)}{(n + 1)p} = \int_{S^1} \tilde{k}'(\tilde{k}\tilde{k}' + \tilde{k}^2 \tilde{k}' - \tilde{k}')$$

$$= -\int_{S^1} \tilde{k}^{m+1} - n \int_{S^1} \tilde{k}^{m-1} \tilde{k}^{n+1}$$

$$- n \int_{S^1} \tilde{k}^{n+1} \tilde{k}^{n+1} \tilde{k}'^{n+1} - \tilde{k}^{n+1} \tilde{k}'^{n+1}$$

$$\leq -f(\tau) - n \int_{S^1} \tilde{k}^{m-1} \tilde{k}^{n+1}$$

$$+ n \int_{S^1} \tilde{k}^{m-1} \tilde{k}^{n+1} + \frac{n}{4} \int_{S^1} \tilde{k}^{m-1} \tilde{k}^{n+1}$$

$$= -f(\tau) + \frac{n}{4} \int_{S^1} \tilde{k}^{m-1} \tilde{k}^{n+1}$$

$$\leq -f(\tau) + \epsilon^{n+1} \frac{n}{4} \int_{S^1} \tilde{k}^{m+1}$$

$$+ \epsilon^{n+1} \frac{n}{4} \int_{S^1} \tilde{k}^{m+1}$$

in terms of calculating the derivative of $f(\tau)$ and integrating by parts.

We take $\epsilon$ such that

$$\epsilon^{n+1} \frac{n}{4} \int_{S^1} \tilde{k}^{m+1} = \frac{1}{2}$$

and use (2.3) then (3.2) implies

$$f'(\tau) \leq -\frac{(n + 1)p}{2} f(\tau) + C(n, p).$$

By Lemma 2.1 we get the upper bound

$$f(\tau) \leq C(n, p),$$

and this yields (3.1) for $n = 1, n = 3$. □

**Lemma 3.2** There exists some constant $C_2$ depending only on $p$ and the initial curve $\gamma_0$ such that

$$\| \tilde{k}'' \|_2 \leq C_2. \quad (3.3)$$

**Proof.** Define

$$g(\tau) = \int_{S^1} \tilde{k}^2.$$
Owing to (2.3) and by analogy with (3.2) we have

\[
\frac{g'(\tau)}{2p} = \int_{S^1} \tilde{k}''(\tilde{k}^2 + \tilde{k}^2 + \tilde{k} - \tilde{k})''
\]

\[
= - \int_{S^1} \tilde{k}'' - \int_{S^1} \tilde{k}''\tilde{k}''\tilde{k}'' - \frac{p+1}{p} \int_{S^1} \tilde{k}''\tilde{k}''\tilde{k}''\tilde{k}'' - \frac{2p+1}{p} \int_{S^1} \tilde{k}''\tilde{k}''\tilde{k}''\tilde{k}''
\]

\[
\leq -g(\tau) - \int_{S^1} \tilde{k}''\tilde{k}''\tilde{k}''\tilde{k}''
\]

\[
+ \frac{1}{2} \epsilon_1^2 \frac{p+1}{p} \int_{S^1} \tilde{k}''\tilde{k}''\tilde{k}''\tilde{k}'' + \frac{1}{2} \epsilon_1^2 \frac{p+1}{p} \int_{S^1} \tilde{k}''\tilde{k}''\tilde{k}''\tilde{k}''
\]

\[
+ \frac{1}{2} \epsilon_2^2 \frac{2p+1}{p} \int_{S^1} \tilde{k}''\tilde{k}''\tilde{k}''\tilde{k}'' + \frac{1}{2} \epsilon_2^2 \frac{2p+1}{p} \int_{S^1} \tilde{k}''\tilde{k}''\tilde{k}''\tilde{k}''.
\]  

(3.4)

Choose \(\epsilon_1, \epsilon_2\) such that

\[
\frac{1}{2} \epsilon_1^2 \frac{p+1}{p} = \frac{1}{2}, \quad \frac{1}{2} \epsilon_2^2 \frac{2p+1}{p} = \frac{1}{2}.
\]

Consequently, putting (2.4), (3.4) and Lemma 3.1 together, it shows

\[
g'(\tau) \leq -2pg(\tau) + C_1(p) \int_{S^1} \tilde{k}''\tilde{k}'' + C_2(p)
\]  

(3.5)

From (3.2) we see that for \(n = 3\) there holds

\[
3 \int_{S^1} \tilde{k}''\tilde{k}''\tilde{k}''\tilde{k}'' = -\frac{1}{4p} \frac{\partial}{\partial \tau} \int_{S^1} \tilde{k}'' - \int_{S^1} \tilde{k}'' - 3 \int_{S^1} \tilde{k}''\tilde{k}''\tilde{k}''\tilde{k}''
\]

\[
\leq -\frac{1}{4p} \frac{\partial}{\partial \tau} \int_{S^1} \tilde{k}'' + \frac{3}{2} \epsilon_2 \int_{S^1} \tilde{k}'' + \frac{3}{2} \epsilon_2 \int_{S^1} \tilde{k}''\tilde{k}''\tilde{k}''\tilde{k}''.
\]  

(3.6)

By Lemma 2.6 in [7] and (2.3), \(\tilde{k}\) has positive lower bound and upper bound. Thus combining (3.5) with (3.6) we deduce that

\[
g'(\tau) \leq -2pg(\tau) - C_3(p) \frac{\partial}{\partial \tau} \int_{S^1} \tilde{k}'' + \epsilon_3^2 C_4(p) \int_{S^1} \tilde{k}''
\]

\[
+ \epsilon_3^2 C_5(p) \int_{S^1} \tilde{k}''\tilde{k}''\tilde{k}''\tilde{k}'' + C_6(p).
\]

Taking \(\epsilon_3 = \sqrt{\frac{p}{C_4(p)}}\) and using (3.1) we obtain

\[
g'(\tau) \leq -pg(\tau) - C_3(p) \frac{\partial}{\partial \tau} \int_{S^1} \tilde{k}'' + C_7(p).
\]  

(3.7)

Then the inequality (3.7) implies

\[
\frac{\partial}{\partial \tau} [e^{pr} g(\tau)] \leq -C_3(p)e^{pr} \frac{\partial}{\partial \tau} \int_{S^1} \tilde{k}'' + C_7(p)e^{pr}.
\]
Integrating from 0 to $A$ on both sides and using (3.1) we find

$$e^{pA}g(A) \leq C_8(p)e^{pA} + C_9(p, \gamma_0).$$

So that this establish (3.3). □

Similarly, using the methods from the proof of Lemma 5.7.8 in [8], we have

**Corollary 3.1** $\| \tilde{k}' \|_\infty$ converges to zero as $\tau \to +\infty$.

It is convenient to introduce the following proposition (cf. Lemma 5.7.9 in [8]) which will provide us with a key means to obtain the estimates (2.4).

**Proposition 3.1 (Gage-Hamilton)** For any $0 < \alpha < 1$ we can choose $A$ so that for $\tau \geq A$

$$4\alpha \int_{S^1} \tilde{k}'^2 \leq \int_{S^1} \tilde{k}''^2.$$

We can next obtain the exponential decay estimates which are similar to Lemma 5.7.10 in [8].

**Lemma 3.3** For any $\alpha$, $0 < \alpha < 1$, there is a constant $C_3$ depending only on $p$ and the initial curve $\gamma_0$ such that

$$\| \tilde{k}' \|_2 \leq C_3 \exp(-2\alpha p \tau). \quad (3.8)$$

**Proof.** Without loss we may assume $p > 1$ and $\frac{1}{2}(1 + \frac{1}{p}) < \alpha < 1$. As same as (3.2) we calculate that

$$\frac{\partial}{\partial \tau} \int_{S^1} \tilde{k}'^2 = \int_{S^1} \tilde{k}'(\tilde{k} \tilde{k}' \tilde{k}' + \tilde{k}^2 \tilde{k}'') - \tilde{k}' = -\int_{S^1} \tilde{k}'^2 - \int_{S^1} \tilde{k}''^2 \tilde{k}^p + \frac{2p + 1}{p} \int_{S^1} \tilde{k}'^2 \tilde{k}^p \quad (3.9)$$

It follows from (2.3) and Proposition 3.1 that there is a positive constant $A$ so large that for $\tau \geq A$ we have

$$1 - \epsilon \leq \tilde{k}^p \leq 1 + \epsilon, \quad 4\alpha \int_{S^1} \tilde{k}'^2 \leq \int_{S^1} \tilde{k}''^2 \quad (3.10)$$

where

$$\epsilon = \frac{2\alpha - 1 - \frac{p}{2} + \frac{1}{p} + 4\alpha}{2 + \frac{1}{p} + 4\alpha}.$$
Combining (3.9) with (3.10) for \( \tau \geq A \) we obtain
\[
\frac{\partial}{\partial \tau} \int_{S_1} \tilde{k}'^2 \leq -\int_{S_1} \tilde{k}'^2 - (1 - \epsilon)4\alpha \int_{S_1} \tilde{k}'^2 + \frac{2p+1}{p}(1 + \epsilon) \int_{S_1} \tilde{k}'^2
\]
\[
= -2\alpha \int_{S_1} \tilde{k}'^2.
\]
By Gronwall’s inequality the assertion follows easily. \( \square \)

Further, we have the conclusions below.

**Lemma 3.4** For any \( \alpha, \ 0 < \alpha < 1 \), there is a constant \( C_4 \) depending only on \( p \) and the initial curve \( \gamma_0 \) such that
\[
\| \tilde{k}'' \|_2 \leq C_4 \exp(-2\alpha p\tau).
\]  
(3.11)

**Proof.** Similar to the calculation of (3.2), we arrive at
\[
\frac{\partial}{\partial \tau} \int_{S_1} \tilde{k}'^2 = \int_{S_1} \tilde{k}''(\tilde{k}'k' \tilde{k}' + \tilde{k}'k' - \tilde{k})''
\]
\[
= -\int_{S_1} \tilde{k}'^2 - \int_{S_1} \tilde{k}''^2 \tilde{k}' \tilde{k}' - \frac{p+1}{p} \int_{S_1} \tilde{k}''^2 \tilde{k}' \tilde{k}' - (2 + \frac{1}{p}) \int_{S_1} \tilde{k}''^2 \tilde{k}' \tilde{k}' \tilde{k}'^\epsilon.
\]  
(3.12)

Using (2.3), (3.12), Corollary 3.1 and letting \( \tau \) so large that we have the estimate
\[
\frac{\partial}{\partial \tau} \int_{S_1} \tilde{k}'^2 \leq -\int_{S_1} \tilde{k}'^2 - (1 - \frac{\epsilon}{2}) \int_{S_1} \tilde{k}'^2 + \frac{\epsilon}{4} \int_{S_1} \tilde{k}'^2 + \frac{\epsilon}{4} \int_{S_1} \tilde{k}'^2 + \frac{C_3}{\epsilon} \exp(-4\alpha p\tau)
\]
\[
\leq -(1 - \epsilon) \int_{S_1} \tilde{k}'^2 - (1 - \epsilon) \int_{S_1} \tilde{k}'^2 + \frac{C_3}{\epsilon} \exp(-4\alpha p\tau)
\]
\[
\leq -(2 - 2\epsilon) \int_{S_1} \tilde{k}'^2 + \frac{C_3}{\epsilon} \exp(-4\alpha p\tau)
\]
provided any positive constants \( \epsilon < 1 \). Here we use Young inequality and Wirtinger inequality. Taking \( \epsilon = 1 - \alpha \) we obtain
\[
\frac{\partial}{\partial \tau} \int_{S_1} \tilde{k}'^2 \leq -2\alpha \int_{S_1} \tilde{k}'^2 + \frac{C_3}{1 - \alpha} \exp(-4\alpha p\tau)
\]

Furthermore, based on Lemma 2.2, the assertion follows. \( \square \)

Applying Sobolev inequality to (3.8) and (3.11) implies the following conclusions.
**Corollary 3.2** For any $\alpha$, $0 < \alpha < 1$, there is a constant $C_5$ depending only on $p$ and the initial curve $\gamma_0$ such that

$$
\| \tilde{k}' \|_\infty \leq C_5 \exp(-2\alpha p \tau).
$$

**Proof of Proposition 2.1.**

We will give the complete proof of this proposition by Mathematical Induction. First we get

$$
\| \tilde{k}^{(L)} \|_\infty \leq C(L, p) \exp(-2\alpha p \tau), \quad \forall \tau > 0,
$$

and

$$
\| \tilde{k}^{(L+1)} \|_2 \leq C(L, p) \exp(-2\alpha p \tau), \quad \forall \tau > 0,
$$

for $L = 1$ by Corollary 3.2 and Lemma 3.4. Next we assume that

$$
\| \tilde{k}^{(l)} \|_\infty \leq C(l, p) \exp(-2\alpha p \tau), \quad \forall \tau > 0, \quad (3.13)
$$

and

$$
\| \tilde{k}^{(l+1)} \|_2 \leq C(l, p) \exp(-2\alpha p \tau), \quad \forall \tau > 0, \quad (3.14)
$$

for $l = 1, 2, \cdots, L$ provided $L > 1$. If we can deduce that

$$
\| \tilde{k}^{(L+1)} \|_\infty \leq C(L + 1, p) \exp(-2\alpha p \tau), \quad \forall \tau > 0, \quad (3.15)
$$

by the assumption of (3.13) and (3.14) then we obtain the desired results.

To prove (3.15) we need only

$$
\| \tilde{k}^{(L+2)} \|_2 \leq C(L + 1, p) \exp(-2\alpha p \tau), \quad \forall \tau > 0, \quad (3.16)
$$

by the Sobolev inequality.

In order to verify (3.16) we show that

$$
\frac{\partial}{\partial \tau} \int_{S^1} [\tilde{k}^{(L+2)}]^2 \frac{2}{2p} = \int_{S^1} \tilde{k}^{(L+2)} \left( \tilde{k} \frac{\partial}{\partial \tau} \frac{1}{\tilde{k}^p} + \tilde{k}^2 \frac{1}{\tilde{k}^p} - \tilde{k} \right)^{(L+2)}
$$

$$
= - \int_{S^1} [\tilde{k}^{(L+2)}]^2 - \int_{S^1} [\tilde{k}^{(L+3)}]^2 \frac{\tilde{k}^{L+1}}{\tilde{k}^p}
$$

$$
- \sum_{i=1}^L \int_{S^1} \tilde{k}^{(L+3)} \tilde{k}^{(L+1-i)} \left( \tilde{k}^{L+1} \right)^{(i)}
$$

$$
= - \int_{S^1} \tilde{k}^{(L+3)} \left( \tilde{k}^{L+1} \right)^{(L+1)} - \int_{S^1} \tilde{k}^{(L+3)} \left( \tilde{k}^{L+1} \right)^{(L+1)}. \quad (3.17)
$$

Let $\epsilon$ be small constant to be determined and take $\tau$ to be large enough. Then by (2.3) we have

$$
\int_{S^1} [\tilde{k}^{(L+3)}]^2 \frac{\tilde{k}^{L+1}}{\tilde{k}^p} \geq (1 - \epsilon) \int_{S^1} [\tilde{k}^{(L+3)}]^2. \quad (3.18)
$$
Hence by (3.13) and the Young inequality,

\[- \sum_{i=1}^{L} \int_{S^1} \tilde{k}^{(L+3)}(\tilde{k}^{(L+1-i)}(\tilde{k}^{1+1/p})^{(i)} \leq \frac{\epsilon}{2} \int_{S^1} [\tilde{k}^{(L+3)}]^2 + \frac{C(L,p)}{\epsilon} \exp(-4\alpha p\tau). \tag{3.19}\]

By (3.14), using the Young inequality again we obtain

\[- \int_{S^1} \tilde{k}^{(L+3)}(\tilde{k}^{1+\frac{1}{p}})^{(L+1)} \leq \frac{\epsilon}{4} \int_{S^1} [\tilde{k}^{(L+3)}]^2 + \frac{C(L,p)}{\epsilon} \exp(-4\alpha p\tau), \tag{3.20}\]

\[- \int_{S^1} \tilde{k}^{(L+3)}(\tilde{k}^{2+\frac{1}{p}})^{(L+1)} \leq \frac{\epsilon}{4} \int_{S^1} [\tilde{k}^{(L+3)}]^2 + \frac{C(L,p)}{\epsilon} \exp(-4\alpha p\tau). \]

Substituting (3.18)-(3.20) to (3.17) and using the Wirtinger inequality we get

\[\frac{\partial}{\partial \tau} \int_{S^1} [\tilde{k}^{(L+2)}]^2 \leq - \int_{S^1} [\tilde{k}^{(L+2)}]^2 - (1 - 2\epsilon) \int_{S^1} [\tilde{k}^{(L+3)}]^2 + \frac{C(L,p)}{\epsilon} \exp(-4\alpha p\tau) \]

\[\leq - \int_{S^1} [\tilde{k}^{(L+2)}]^2 - (1 - 2\epsilon) \int_{S^1} [\tilde{k}^{(L+2)}]^2 + \frac{C(L,p)}{\epsilon} \exp(-4\alpha p\tau) \]

\[= -(2 - 2\epsilon) \int_{S^1} [\tilde{k}^{(L+2)}]^2 + \frac{C(L,p)}{\epsilon} \exp(-4\alpha p\tau). \]

Choose \( \epsilon = 1 - \alpha \) then there holds

\[\frac{\partial}{\partial \tau} \int_{S^1} [\tilde{k}^{(L+2)}]^2 \leq -2\alpha \int_{S^1} [\tilde{k}^{(L+2)}]^2 + \frac{C(L,p)}{1 - \alpha} \exp(-4\alpha p\tau). \tag{3.21}\]

Applying Lemma 2.2 to (3.21) we obtain (3.16). This completes the proof. □

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