Fault-tolerant conversion between adjacent Reed–Muller quantum codes based on gauge fixing

Dong-Xiao Quan\textsuperscript{1,2,\textcopyright}, Li-Li Zhu\textsuperscript{1}, Chang-Xing Pei\textsuperscript{1} and Barry C Sanders\textsuperscript{2,3,4,\textcopyright}

\textsuperscript{1} State Key Laboratory of Integrated Services Networks, Xidian University, Xi’an, Shaanxi 710071, People’s Republic of China
\textsuperscript{2} Institute for Quantum Science and Technology, University of Calgary, Alberta T2N 1N4, Canada
\textsuperscript{3} Program in Quantum Information Science, Canadian Institute for Advanced Research, Toronto, Ontario M5G 1Z8, Canada
\textsuperscript{4} Hefei National Laboratory for Physical Sciences at Microscale, University of Science and Technology of China, Hefei, Anhui 230026, People’s Republic of China
\textsuperscript{5} Shanghai Branch, CAS Center for Excellence and Synergetic Innovation Center in Quantum Information and Quantum Physics, University of Science and Technology of China, Shanghai 201315, People’s Republic of China

E-mail: dxquan@xidian.edu.cn

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Abstract

We design forward and backward fault-tolerant conversion circuits, which convert between the Steane code and the 15-qubit Reed–Muller quantum code so as to provide a universal transversal gate set. In our method, only seven out of a total 14 code stabilizers need to be measured, and we further enhance the circuit by simplifying some stabilizers; thus, we need only to measure eight weight-4 stabilizers for one round of forward conversion and seven weight-4 stabilizers for one round of backward conversion. For conversion, we treat random single-qubit errors and their influence on syndromes of gauge operators, and our novel single-step process enables more efficient fault-tolerant conversion between these two codes. We make our method quite general by showing how to convert between any two adjacent Reed–Muller quantum codes $\text{RM}(1, m)$ and $\text{RM}(1, m + 1)$, for which we need only measure stabilizers whose number scales linearly with $m$ rather than exponentially with $m$ obtained in previous work. We provide the explicit mathematical expression for the necessary stabilizers and the concomitant resources required.
Keywords: quantum error correction, Reed–Muller quantum code, code conversion, gauge fixing

1. Introduction

Quantum information technology, with applications such as secure quantum communication or universal quantum computing, is extremely powerful but challenging due to the fragility of quantum information in the presence of noise, loss and decoherence. Fault-tolerant quantum error correction [1–3] ameliorates this problem of fragility by encoding plain-text quantum information into cipher text, processing this encoded information while also measuring error syndromes and correcting, and finally turning back to plain text. For fault-tolerant error correction, transversal gates are especially valuable as qubits in each code block act bitwise between corresponding qubits in each code block, thereby naturally preventing error propagation [4]. Unfortunately, no code can simply enable a universal set of transversal gates [4–6] without invoking a technique involving ancillary qubits; thus, complicated strategies are employed to produce universal transversal gate sets. One strategy to circumvent the non-universal gate set problem is to employ magic-state distillation [7–12]. Although magic-state distillation has the advantage of a higher fault-tolerant threshold, the overhead in preparation and distillation is a major bottleneck for scalable quantum computing [13–16].

Alternative techniques can provide a universal fault-tolerant gate set. The Steane code [17] provides transversal Clifford gates but not the transversal $T := \text{diag}(1, \exp\{i\pi/4\})$ gate whereas the 15-qubit Reed–Muller quantum code (RMQC) [18] is transversal for $T$, controlled-NOT (CNOT), controlled-$S$ (CS), controlled-$Z$ (CZ), and controlled-controlled-$Z$ (CCZ) gates, but not for the Hadamard (H) gate. One approach to achieving a universal set of gates employs just one code such as the Steane code or the 15-qubit RMQC and does not invoke code conversion. For example, for the Steane code, the fault-tolerant $T$ gate can be realized with the help of an ancillary logical qubit, Pauli $Z$-measurement and transversal $S := \text{diag}(1,1)$ and $X$ operations, which we call the standard method to construct $T$ gate for the Steane code [19]. Alternatively, for the 15-qubit RMQC, which does not include the transversal $H$ gate in the universal gate set, a Hadamard gadget can be introduced: this Hadamard gadget exploits the transversality of the CS gate, which yields a CZ gate if applied twice, and, combined with an ancillary logical qubit and a Pauli $X$-measurement and post-selection, yields a Hadamard gate [20]. A universal gate set for 15-qubit RMQC can also be achieved by using transversal operations and the gauge-fixing method to realize the fault-tolerant $H$ gates [21–23].

Another technique is code concatenation, for which the qubits that make up the code are subsequently encoded into a second code so combining the 7-qubit Steane code and 15-qubit RMQC requires 105 qubits [13, 24].

An alternative method is by code conversion [25–30], which converts between two codes to provide a universal transversal gate set. Direct conversion can be realized via Clifford operations, which is general for converting between different kinds of codes but requires 13 single-qubit and 74 two-qubit gates. Furthermore, this direct conversion requires error correction at every step to ensure fault tolerance [26]. Another direct conversion method is by gauge fixing as both codes correspond to the same subsystem code with different gauge qubits [25, 27], which is more efficient compared to the method of Clifford operations with respect to the number of qubits and gates required. This method is also studied from the view of color codes [27] and furthermore generalized to the dimensional jump between two-dimensional
and three-dimensional color codes or even higher dimensions [30]. This code conversion can also be realized by code teleportation [29, 31].

As both the Steane code and the RMQC are of the same code distance 3, they can only correct one error. For fixed distance, fewer qubits are better as fewer errors can happen so smaller codes are easier to implement and need fewer resources to perform syndrome measurements. Thus, the Steane code is more efficient than the 15-qubit RMQC. Detailed analysis shows that the overhead of using only the Steane code to realize a fault-tolerant T gate is lower compared with direct conversion [25] and with the teleportation scheme [29], since implementation of the T gate is feasible in the Steane code with ancilla, and the necessary stabilizer measurements in the direct conversion method require many gates [29, 31]. In this paper we find that some stabilizers are not necessary for fault-tolerant conversion between adjacent RMQCs based on gauge fixing, especially for higher $m$, so that we can significantly decrease resource requirement.

Here we significantly improve the fault-tolerant code-conversion method of Anderson, Duclos-Cianci and Poulin (ADP14) [25] by exponentially reducing the number of stabilizer measurements, which thus reduces the resource requirement. Our circuits and method have the following advantages. First we split some of the stabilizers into two gauge operators, which reduces the resource requirement for the measurements of the syndromes. Second in the single-step process we consider random single-qubit errors and their influence on the syndromes of the gauge operators, using fixed syndromes to choose the operations so that we can make fault-tolerant conversion. Third we generalize the method to any adjacent RMQCs for which we need only measure $2m + 1$ stabilizers of the RMQC code. In contrast ADP14’s approach discusses measuring all $2^{m+1} - 2$ stabilizers and requiring $2^{m+1} - m - 2$ stabilizers. Our approach significantly reduces this stabilizer overhead.

The rest of this paper is organized as follows. In section 2, we first give a brief review of the order-one RMQC and then give the stabilizer groups for the 7-qubit Steane code, the 15-qubit RMQC and extended Reed–Muller quantum code (ERMQC) and, in section 3, we describe the ADP14 method. In section 4 we describe forward conversion from the 7-qubit Steane code to the 15-qubit RMQC in detail, and we deal with backward conversion in section 5. In section 6, we explain how to convert between adjacent order-one RMQCs and, in section 7, we elaborate on our simulation process and cost analysis. We conclude in section 8.

2. Basic knowledge

In this section, we give the basic concept of quantum error correction (QEC) and the stabilizer groups for the Steane code, the 15-qubit RMQC and ERMQC. From the relations for the stabilizer groups, we design the conversion scheme in the following sections.

QEC [1–3] is an effective way to protect the information from the influence of noise. The quantum block notation $[[n, k, d]]$ refers to encoding $k$ logical qubits into $n$ physical qubits with code distance $d$ so that $t < d/2$ random errors can be detected and corrected by syndrome measurements and error correction processes. The order-one RMQC [18] is written as $\mathbb{RM}(1, m) = [[M - 1, 1, 3]]$, which encodes 1 logical qubit into $M - 1$ qubits with distance 3 for $M := 2^m$. Adjacent codes are $\mathbb{RM}(1, m)$ and $\mathbb{RM}(1, m + 1)$. The Steane code is written as $\mathbb{RM}(1, 3)$ and the 15-qubit RMQC is written as $\mathbb{RM}(1, 4)$; in the framework of RMQC, they are adjacent RMQCs.

RMQCs are derived from the recursively defined classical order-one Reed–Muller code [32]. By deleting the first row and column of the generator matrix for the classical Reed–Muller code, we obtain $\mathbb{RM}(1, 3)$’s generator matrix
\[
\bar{G}(1, 3) := \begin{bmatrix}
\bar{G}(1, 3)_1 \\
\bar{G}(1, 3)_2 \\
\bar{G}(1, 3)_3
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix},
\]  

which can be used recursively to obtain the generator matrix for \(RM(1, m + 1)\) given by

\[
\bar{G}(1, m + 1) = \begin{bmatrix}
\bar{G}(1, m) & 0 \\
\bar{G}(1, m) & 1
\end{bmatrix}_{M-1}, \quad \bar{G}(1, m) = \begin{bmatrix}
\bar{G}(1, m)_1 \\
\vdots \\
\bar{G}(1, m)_m
\end{bmatrix},
\]  

where

\[
b_M := b \otimes M, \quad b \in \{0, 1\}.
\]

As all RMQCs are Calderbank–Shor–Steane codes [33, 34], we can subdivide their stabilizers into two parts corresponding to phase errors (X stabilizers) and flip errors (Z stabilizers). The X parts are obtained from the generator matrix of the code, and the Z parts are obtained from the generator matrix of the dual code, which is also the parity-check matrix of the code. As \(RM(1, 3)\) is self dual [32], the stabilizers of the Steane code can be written as

\[
S(1, 3) = \langle \bar{G}(1, 3)^X, \bar{G}(1, 3)^Z \rangle,
\]

with

\[
\bar{G}(1, 3)_1^{BE(X,Z)} = B_1B_3B_4B_7, \\
\bar{G}(1, 3)_2^{BE(X,Z)} = B_2B_3B_6B_7, \\
\bar{G}(1, 3)_3^{BE(X,Z)} = B_2B_4B_6B_7,
\]

where \(G^{BE(X,Z)}\) is a matrix obtained from \(G\) by substituting 1 with \(B\) and substituting 0 with \(I\). As the Steane code is self dual, it is transversal for logical

\[
\bar{H} := H^\otimes n,
\]

where \(n\) is the number of qubits for the code [27, 35].

When \(m \geq 3\), \(RM(1, m)\) is contained in its dual [32]. For the case \(m = 4\), we define a matrix

\[
\tilde{H}(1, 4) := \begin{bmatrix}
\tilde{H}(1, 4)_1 \\
\tilde{H}(1, 4)_2 \\
\tilde{H}(1, 4)_3 \\
\tilde{H}(1, 4)_4 \\
\tilde{H}(1, 4)_5 \\
\tilde{H}(1, 4)_6
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
\bar{G}_{1,3} \\
\tilde{0}_b
\end{bmatrix},
\]

and we have

\[
\tilde{H}(1, 4)_1^{BE(X,Z)} = B_1B_3B_4B_{11}, \\
\tilde{H}(1, 4)_2^{BE(X,Z)} = B_2B_3B_{10}B_{11}, \\
\tilde{H}(1, 4)_3^{BE(X,Z)} = B_3B_4B_{11}B_{15},
\]

with \(\tilde{H}\) and \(\tilde{H}^{BE(X,Z)}\) related in the same way as \(G\) and \(G^{BE(X,Z)}\) earlier.
Obviously
\[ \begin{align*}
\hat{H}(1,4)\hat{G}(1,4)^T &= 0 \\
\end{align*} \] (9)
and the parity-check matrix for RMQC(1,4) is the vertical concatenation of the two matrices \( \hat{G}(1,4) \) and \( \hat{H}(1,4) \). Thus, the stabilizers of the 15-qubit RMQC are
\[ \begin{align*}
S(1,4) &= \langle \hat{G}(1,4)^X, \hat{G}(1,4)^Z, \hat{H}(1,4)^Z \rangle \\
\end{align*} \] (10)
with
\[ \begin{align*}
\hat{G}(1,4)^{BE}_{1,4} &= B_1B_2B_3B_7B_9B_{11}B_{13}B_{15}, \\
\hat{G}(1,4)^{BE}_{2,4} &= B_2B_3B_6B_7B_{10}B_{11}B_{14}B_{15}, \\
\hat{G}(1,4)^{BE}_{3,4} &= B_3B_6B_7B_{12}B_{13}B_{14}B_{15}, \\
\hat{G}(1,4)^{BE}_{4,4} &= B_4B_6B_{10}B_{12}B_{13}B_{14}B_{15}. \\
\end{align*} \] (11)

As the syndromes of \( \hat{G}(1,4)^Z \) can unambiguously discriminate all single-qubit \( X \) errors, the syndromes of \( \hat{H}(1,4)^Z \) are often omitted. The code is triply-even as its weight is 8 so it is transversal for logical \( T := T^{\text{gen}} [27, 35] \).

From equation (2) we can see the 15-qubit RMQC comprises two blocks of Steane code for the first and last 7-qubit blocks plus another interconnecting qubit labeled 8. This interconnecting qubit entangles with the last 7-qubit block of Steane code. Thus, we can prepare an 8-qubit quantum state \(|\phi\rangle = 1/\sqrt{2} (|00\rangle_3 + |11\rangle_3) \) in the last 8 qubits [25], where \(|\bar{b}\rangle_m, b \in \{0, 1\}, \) is the logical bit \( b \) encoded into the RMQC(1, m) code.

Together with the Steane code \(|\psi\rangle = \alpha|0\rangle_3 + \beta|1\rangle_3, \) we construct a 15-qubit ERMQC
\[ \langle \phi\rangle_{3,4} = 1/\sqrt{2} (\alpha|0\rangle_3 + \beta|1\rangle_3) (|0\rangle_0 + |1\rangle_1). \] (12)

We refer to equation (12) as the no-error case for preparation of the 15-qubit ERMQC, and this ideal input preparation (12) is justified in the realistic case by introducing fault-tolerant preparation using gadgets [36]. The stabilizer group for this code can be expressed as
\[ \begin{align*}
S_{3,4} &= \langle (\hat{G}(1,3) \otimes \hat{0}_b)^X, (\hat{G}(1, 3) \otimes \hat{0}_b)^Z, (\hat{G}(1,4) \otimes \hat{0}_b)^Z, (\hat{G}(1,4)^Z) \rangle. \\
\end{align*} \] (13)

It is also transversal for logical \( H \) as it is self dual. There is no information in the last 8 qubits so it has the same logical operations as the Steane code.

We define a subsystem code with the stabilizers of
\[ \begin{align*}
S_{3,4}^{\text{sub}} &= \langle (\hat{G}(1,4)^X, (\hat{G}(1,3) \otimes \hat{0}_b)^Z, \hat{G}(1,4)^Z) \rangle. \\
\end{align*} \] (14)
Pauli operators that commute with both the stabilizers and logical operators generate the gauge group [27] so the gauge group of this subsystem code is
\[ \begin{align*}
G_{3,4}^{\text{sub}} &= \langle (\hat{G}(1,4)^X, (\hat{G}(1,3) \otimes \hat{0}_b)^Z, \hat{G}(1,4)^Z, \hat{H}(1,4)^Z) \rangle. \\
\end{align*} \] (15)

Both the 15-qubit RMQC and the ERMQC belong to this subsystem code with different gauge operators so they can convert to each other by gauge fixing.
Table 1. Relation between stabilizer measurement results and corresponding operations for forward conversion in the no-error case (12) with $S_i = M (\tilde{H}(1, 4)^i_2)$.

| $S_1$ | $S_2$ | $S_3$ | Operation | $S_1$ | $S_2$ | $S_3$ | Operation |
|-------|-------|-------|-----------|-------|-------|-------|-----------|
| 0     | 0     | 0     | I         | 1     | 0     | 0     | $X_{10}X_{11}X_{14}X_{15}$ |
| 0     | 0     | 1     | $X_{12}X_{13}X_{14}X_{15}$ | 1     | 0     | 1     | $X_{10}X_{11}X_{12}X_{13}$ |
| 0     | 1     | 0     | $X_9X_{11}X_{13}X_{15}$ | 1     | 1     | 0     | $X_9X_{10}X_{13}X_{14}$ |
| 0     | 1     | 1     | $X_9X_{11}X_{12}X_{14}$ | 1     | 1     | 1     | $X_9X_{10}X_{12}X_{15}$ |

3. ADP14

Based on the theory of subsystem codes, ADP14 show how to convert from $\mathbb{RM}(1, m)$ to $\mathbb{RM}(1, m + 1)$. They first fault-tolerantly prepare the ERMQC, fault-tolerantly measure the $2^{m+1} - 2$ stabilizer generators of $\mathbb{RM}(1, m + 1)$, error-correct the first $2^{m+1} - m - 2$ syndrome bits, and restore the last $m$ syndrome bits using their associated pure errors. To convert from $\mathbb{RM}(1, m + 1)$ to $\mathbb{RM}(1, m)$, they simply fault-tolerantly measure the $2^{m+1} - 2$ stabilizer generators of ERMQC, use the first $2^{m+1} - m - 2$ syndrome bits to diagnose errors, and restore the last $m$ syndrome bits using the associated pure errors [25].

Here we improve the fault-tolerant code conversion method of ADP14 by exponentially reducing the number of stabilizer measurements from $2^{m+1} - 2$ to $2m + 1$, which thereby significantly reduces the resource requirement. Furthermore we use the relations between these stabilizers to simplify the stabilizer measurements. For code conversion, we consider single-qubit errors and their influence on the gauge operators so as to achieve fault-tolerant conversion. In the following we first use conversion between the Steane code and the 15-qubit RMQC as an example to describe conversion, simplification of stabilizer measurements, the single-step error-correction and code-conversion process, and then generalize this conversion to any adjacent RMQCs.

4. Conversion from the Steane code to the 15-qubit RMQC

In this section we deal with forward conversion, which converts from the Steane code to the 15-qubit RMQC. From equations (10) and (13), we see that the ERMQC satisfies all the other stabilizers of the 15-qubit RMQC except for $\tilde{H}(1, 4)^i_2$, $i = 1, 2, 3$. Our procedure entails measuring the stabilizers of $\tilde{H}(1, 4)^i_2$, $i = 1, 2, 3$, given in equation (8). The $i$th Z syndrome is

$$S_i = M (\tilde{H}(1, 4)^i_2) \in \{0, 1\}, \ i = 1, 2, 3,$$

(16)

for $M(\tilde{H})$ denoting the syndrome measurement result of stabilizer $\tilde{H}$. Operations are then performed on the collapsed state according to syndromes as shown in table 1 in order to get the 15-qubit RMQC.

The operations shown in table 1 are chosen as follows. For example, if we obtain $S_1 = S_2 = 0$ and $S_3 = 1$, then we need to use X-type operations, which commute with all Z-type stabilizers except $\tilde{H}(1, 4)^3_2$ so we can choose $X_{12}X_{13}X_{14}X_{15}$ as the operations. If we obtain $S_1 = S_2 = 1$ and $S_3 = 0$, then we can choose the operations as $X_9X_{10}X_{13}X_{14}$, which anti-commute with $\tilde{H}(1, 4)^3_2$ and $\tilde{H}(1, 4)^2_2$ but commute with all the others.

We can divide our analysis into two cases: the no-error case, which assumes ideal state preparation (12), and the case of a single-qubit error occurring before the code conversion. In the case of a single-qubit error, this error can influence the syndromes of table 1 and...
accordingly result in wrong operations so we need to distinguish single-qubit errors and fix the syndromes before the corrections in table 1. The syndromes of $S(1, 4)$ are
\[ S(1, 4)_{\text{BE} (XZ)} = M (\bar{G}(1, 4)_{\text{BE} (XZ)}) , \quad i = 1, 2, 3, 4, \] (17)
and the single-qubit errors are shown in table 2.

Using the $X$-type syndromes of table 2, we can identify the single-qubit $Z$ error represented as $Z_i$. The error $Z_i$ has no influence on the syndromes of table 1 so we need only add the $Z_i$ operation in the correction process. Similarly, using the $Z$-type syndromes of table 2, we can identify the single-qubit $X$ error written as $X_j$. However, this $X$ error could also result in wrong syndromes of table 1 so we need to consider the influence of this error when choosing the operations.

Let us consider the point in figure 1 subsequent to the $S_3$ measurement at which point we have measured all three syndromes $S_1$, $S_2$ and $S_3$. Then we can simplify the measurement according to
\[ S(1, 4)_{\bar{G}} = M (\bar{G}(1, 4)) \]
\[ = M (Z_1 Z_2 Z_3 Z_4 Z_5 Z_6 Z_7 Z_8 Z_9 Z_{10} Z_{11} Z_{12} Z_{13} Z_{14} Z_{15}) \]
\[ = M (Z_1 Z_2 Z_3 Z_4 Z_5 Z_6 Z_7 Z_8 Z_9 Z_{10} Z_{11} Z_{12} Z_{13} Z_{14} Z_{15}) \]
\[ = S_3 \oplus S_4, \] (18)
where we have used the commutativity of $Z$ operators in the second step and, for the third step, we have used the syndrome definition (16) and the expression for $H(1, 4)_{\bar{G}}$ given in equation (8). Generalizing equation (18), we obtain the simplifying assignments
\[ M (\bar{G}(1, 4)) \rightarrow S_4 = M (Z_1 Z_2 Z_3 Z_4 Z_5 Z_6 Z_7 Z_8 Z_9 Z_{10} Z_{11} Z_{12} Z_{13} Z_{14} Z_{15}) \]
\[ M (\bar{G}(1, 4)) \rightarrow S_5 = M (Z_1 Z_2 Z_3 Z_4 Z_5 Z_6 Z_7 Z_8 Z_9 Z_{10} Z_{11} Z_{12} Z_{13} Z_{14} Z_{15}) \]
\[ M (\bar{G}(1, 4)) \rightarrow S_6 = M (Z_1 Z_2 Z_3 Z_4 Z_5 Z_6 Z_7 Z_8 Z_9 Z_{10} Z_{11} Z_{12} Z_{13} Z_{14} Z_{15}) \]
\[ M (\bar{G}(1, 4)) \rightarrow S_7 = M (Z_1 Z_2 Z_3 Z_4 Z_5 Z_6 Z_7 Z_8 Z_9 Z_{10} Z_{11} Z_{12} Z_{13} Z_{14} Z_{15}) \]
\[ \text{Unfortunately} \]
\[ M (\bar{G}(1, 4)) \rightarrow S_8 = M (Z_1 Z_2 Z_3 Z_4 Z_5 Z_6 Z_7 Z_8 Z_9 Z_{10} Z_{11} Z_{12} Z_{13} Z_{14} Z_{15}) \]
\[ \text{cannot be simplified in this way. However, it can be divided into two weight-4 stabilizers} \]
in order to be experimentally realistic since both of them commute with all the stabilizers.

The total set of $Z$-type syndromes is \{\(S_1, S_2, \ldots, S_8\)\}. Using the technique shown in equation (18), we obtain three additional relations

\[ S(1, 4)_Z = S_3 \oplus S_5, \quad S(1, 4)_Z = S_5 \oplus S_2 \oplus S_6, \quad S(1, 4)_Z = S_7 \oplus S_8 \]

and use table 2 to deduce the single-qubit $X$ error $X_j$. Then we modify the first three syndromes $S_{1,2,3}$ using the relation

\[ Z_j \in H(1, 4)_Z \Rightarrow S_n = S_n \oplus 1(n = 1, 2, 3). \]

Now we use the modified $S_{1,2,3}$ and table 1 to deduce the operations written as $X_{\text{sub}}$. Thus, the total operations we need to perform for the correction process are $X_{\text{sub}}X_iZ_j$, by these operations we obtain the 15-qubit RMQC.

The forward conversion circuit is shown in figure 1, where we first prepare the entangled state of a single qubit with the Steane-code qubits, these ancillary qubits together with the initial Steane code form the ERMQC. Then we measure four weight-8 $X$-type stabilizers and
eight weight-4 $Z$-type stabilizers. Preparation and syndrome measurements shown in figure 1 are not fault-tolerant as the single error may spread into multiple errors. The fault-tolerant preparation can be realized by using gadgets [36]. In order to ensure fault-tolerant measurement, we use the Shor state [37], or use encoded Bell pairs [38], use encoded blocks [39, 40], or by adding flags [41] which reduces requisite resources significantly, to perform syndrome measurement, repeat the syndrome measurement three times and then make a majority vote to ensure fault-tolerant measurement.

Recall that we separately consider the no-error case (12) and the case of a single-qubit error. Even in the no-error case, subsequent to the $S_1$, $S_2$ and $S_3$ syndrome measurements, the ERMQC collapses to the 15-qubit RMQC with probability $\frac{1}{8}$. Thus, we need to apply the fixing operation shown in table 1 with probability $\frac{7}{8}$. This fixing operation is shown as the correct box in figure 1. After this correction we need another round of syndrome measurements; this round needs 8 ancillary qubits to measure the stabilizers of $\bar{G}(1, 4)_X$ and $\bar{G}(1, 4)_Z$ in order to judge whether an error happened during the correction process. If an error happened in the correction phase, then we need to repeat the correction and syndrome measurement one more time. As we assume the error probability is sufficiently small that we can ignore more than one error occurring during the entire process, we are sure that all eight syndromes are zero in this final step, which we verify by this final syndrome measurement.

If we only want to fault-tolerantly convert from the Steane code to the 15-qubit RMQC, as the code distance is 3 one error is allowed in the final code block. Single $Z$ errors do not have any influence on the gauge operators, summarized in table 1, which we have used to do conversion, so these single $Z$ errors do not propagate to multiple errors. Thus, we do not need to measure the four $X$-type syndromes that are used to diagnose the $Z$ errors; the ancillary qubits 9–12 in figure 1 can be omitted, this time we need only measure eight weight-4 $Z$-type stabilizers. After the correct operation, there may be one single-qubit error in the code, but fault-tolerant conversion is fulfilled.

5. Conversion from the 15-qubit RMQC to the Steane code

In this section we deal with the backward conversion, which first converts from the 15-qubit RMQC to the ERMQC then drops the last 8 qubits to get the Steane code.

As in section 4, we first deal with the ideal case of no error in the 15-qubit RMQC. From equations (10) and (13), we see that the 15-qubit RMQC satisfies all the other stabilizers of the ERMQC except for $\bar{G}(1, 3)_X^i$, $i = 1, 2, 3$. Thus, we measure these three stabilizers $\bar{G}(1, 3)_X^i$, $i = 1, 2, 3$, given in equation (5), and write the $i$th $X$ syndrome as

$$S'_i = M(\bar{G}(1, 3)_X^i) \in \{0, 1\}, \ i = 1, 2, 3.$$  \hspace{1cm} (24)

Following the syndrome measurements, we perform the fixing operations on the collapsed state according to table 3. In table 3 the operations are chosen similar to before in section 4. For example, if we obtain $S'_1 = S'_2 = 0$ and $S'_3 = 1$, then we can choose $Z_1Z_7Z_{11}Z_{15}$ as the fixing operations, which anti-commute with $\bar{G}(1, 3)_X^3$ but commute with all the other stabilizers of $S'_{1-4}$.

We now consider random single-qubit errors and their influence on the syndromes of table 3 using an approach similar to that used in section 4. First we measure the $Z$-type syndromes of table 2 to deduce the single-qubit $X$ error on qubit $i$, written as $X_i$. This single-qubit error has no influence on the syndromes of table 3 so we need only add the $X_i$ operation in the correction process. For the $X$-type syndromes, as we already have the syndromes $S'_i$, $S'_j$ and $S'_k$, similar to section 4, by generalizing equation (18), we obtain the simplifying assignments...
The relations between these $X$-type syndromes and $S(1,4)^X_{1,2,3,4}$ are

\begin{align}
S(1,4)^X_1 &= S'_1 \oplus S'_2, \quad S(1,4)^X_2 = S'_2 \oplus S'_3, \\
S(1,4)^X_3 &= S'_3 \oplus S'_4, \quad S(1,4)^X_4 = S'_4 \oplus S'_5.
\end{align}

(26)

We use the syndrome set $\{S'_1, S'_2, \ldots, S'_5\}$ and the relations in equation (26) as well as table 2 to deduce the single-qubit $Z$ error denoted $Z_j$. Then, we modify the first three syndromes according to

\[ X'_j \in G(1,3)^X \Rightarrow S'_n = S'_{n} \oplus 1(n = 1, 2, 3). \]

(27)

Now we use the fixed syndromes $S'_{1,2,3}$ and table 3 to get the operations written as $Z_{\text{sub}}$, the total operations we need to perform in the correction process is $Z_{\text{sub}}Z_i$, by these operations we get the 15-qubit ERMQC. Then we can discard the last 8 qubits to get the Steane code.

The backward conversion circuit is shown in figure 2, which shows that we need to measure four weight-8 $Z$-type stabilizers and seven weight-4 $X$-type stabilizers. To guarantee fault-tolerance, the method used in section 4 to make fault-tolerant measurements should also be applied here. The probability of obtaining the correct Steane code without applying the fixing operation is also 1/8 for the no-error case, as is the case for forward conversion. Thus, with probability 1/8, the procedure terminates here with no error. Otherwise, in order to obtain the correct Steane code without error, then, after the correction process, we undertake another round of syndrome measurements; this additional round needs six ancillary qubits to measure the stabilizers of $G(1,3)^X$ and $G(1,3)^Z$ to determine whether an error happened during the correction process.

If an error is detected, we repeat the correction and syndrome measurement one more time. Then, because we allow only one error, the procedure is complete. As in section 4, if we only want to fulfill fault-tolerant conversion, the ancillary qubits 8–11 in figure 2, which are used to diagnose the single $X$ errors, can be omitted. Thus we need only measure seven weight-4 $X$-type stabilizers.

### 6. Conversion between adjacent RMQCs

As the recursive relation (2) exists for all adjacent RMQCs, denoted $\overline{RM}(1,m)$ and $\overline{RM}(1,m+1)$, our method is suitable for all conversions between these adjacent codes. Now we explain how to convert between adjacent RMQCs and the resources needed.
Beginning with the case $m = 3$ and equation (2), we obtain the generator matrix $\bar{G}(1, m)$ and $\bar{G}(1, m + 1)$ for $\overline{\text{RM}}(1, m)$ and $\overline{\text{RM}}(1, m + 1)$. Accordingly we can define a matrix

$$
\bar{H}(1, m + 1) := \begin{bmatrix}
\bar{H}(1, m + 1)_1 \\
\vdots \\
\bar{H}(1, m) \Theta_{2^m} \\
\bar{H}(1, m) \Theta_{2^m} \\
\end{bmatrix}, \quad (28)
$$

from $m = 4$ where

$$
\bar{H}(1, m + 1)_1 = \mathbf{1}(1, 3), \bar{H}(1, m + 1)_2 = \mathbf{1}(2, 3) \quad (29)
$$

and

$$
\bar{H}(1, m + 1)_n = \mathbf{1}(3, 3 + 2^{n-1}), \quad 3 \leq n \leq m, \quad (30)
$$

and $\mathbf{1}(x, y)$ means a vector is 1 at the positions of $x, y, x + 2^m, y + 2^m$, and 0 for the other elements. We can verify

$$
\bar{H}(1, m + 1) \bar{G}(1, m + 1)^f = 0 \quad (31)
$$
and the parity-check matrix for $\mathbb{R}M(1, m+1)$, namely, $H(1, m+1)$, is the vertical concatenation of $\tilde{G}(1, m+1)$ and $\tilde{H}(1, m+1)$. Thus, the stabilizer group for $\mathbb{R}M(1, m+1)$ can be written as

$$S(1, m+1) = \{ \tilde{G}(1, m+1)^X, \tilde{G}(1, m+1)^Z, \tilde{H}(1, m+1)^Z \}. \quad (32)$$

The ERMQC for conversion from $m$ to $m+1$ can be written as

$$|\phi\rangle_{m\rightarrow m+1} = 1/\sqrt{2} (\alpha|0\rangle_m + \beta|1\rangle_m) (|0\rangle_m + |1\rangle_m). \quad (33)$$

The stabilizer group for this code can be written as

$$S_{m\rightarrow m+1} = \{ (G(1, m) \otimes \bar{0}_{2^n})^X, G(1, m+1)^X, (G(1, m) \otimes \bar{0}_{2^n})^Z, \tilde{G}(1, m+1)^Z, (\tilde{H}(1, m) \otimes \bar{0}_{2^n})^Z, (\bar{0}_{2^n} \otimes \tilde{H}(1, m))^Z \}. \quad (34)$$

Both the RMQC and ERMQC correspond to a subsystem code with stabilizer group of

$$S_{m\rightarrow m+1}^{\text{sub}} = \{ \tilde{G}(1, m+1)^X, G(1, m+1)^Z, (\tilde{G}(1, m) \otimes \bar{0}_{2^n})^Z, (\tilde{H}(1, m) \otimes \bar{0}_{2^n})^Z, (\bar{0}_{2^n} \otimes \tilde{H}(1, m))^Z \} \quad (35)$$

with different gauge operators so they can convert to each other by gauge fixing.

For forward conversion, which converts from $\mathbb{R}M(1, m)$ to $\mathbb{R}M(1, m+1)$, we compare equation (34) with equation (32) to discover that the first $m$ stabilizers of $\tilde{H}(1, m+1)^Z$ may not be satisfied. Thus, in order to convert forward, we need only measure these $m$ stabilizers and choose the corresponding operations to correct the collapsed state to $\mathbb{R}M(1, m+1)$. These $X$-type operations can be chosen similar to those in section 4, which commute with all the other stabilizers except for the stabilizers that were measured to be $I$.

For backward conversion, which converts from $\mathbb{R}M(1, m+1)$ to $\mathbb{R}M(1, m)$, we discover that the stabilizers $(G(1, m) \otimes \bar{0}_{2^n})^X$ of ERMQC may not be satisfied by comparing equation (32) with equation (34). Thus, we need $m$ ancillary qubits to measure these stabilizers and do corresponding operations to get the ERMQC. Finally we drop the last $2^n$ qubits to obtain $\mathbb{R}M(1, m)$.

Consider the random single-qubit errors; for both directions of conversion we need also measure the stabilizers of $G(1, m+1)^X$ and $G(1, m+1)^Z$ to distinguish the single-qubit errors, and fix the first $m$ syndromes using the relations in equations (23) and (27) with $n = 1, 2, \ldots, m$ respectively. Then, by using the fixed syndromes, we choose the correct fixing operations and perform error correction and fixing operations in one single step.

Following the above description, we need to separately measure $3m+2$ stabilizers for both forward and backward conversion. For forward conversion we need to measure separately $m+1$ weight-$2^n$ $X$-type stabilizers and $m$ weight-$4$ $Z$-type stabilizers, and, for backward conversion, we need to measure separately $m+1$ weight-$2^n$ $X$-type stabilizers and $m$ weight-$2^{m-1}$ $X$-type stabilizers. However, in [25], they measure all the $2^{m+1} - 2$ stabilizers and refer to $2^{m+1} - m - 2$ syndromes to do error correction. For the case $m = 3$, the number of useful syndromes is the same, but for larger $m$, our method reduces the number exponentially with $m$.

Using the simplified method shown in equation (18), the resources can be further reduced. For forward conversion we need to measure $m+1$ weight-$2^n$ $X$-type stabilizers, $m$ weight-$4$ $Z$-type stabilizers, $m$ weight-$2^n - 4$ $Z$-type stabilizers plus one weight-$2^n$ $Z$-type stabilizer;
for the backward conversion we need to measure $m + 1$ weight-$2^m$ $Z$-type stabilizers and $2m + 1$ weight-$2^{m-1}$ $X$-type stabilizers. For the case $m = 3$, if we divide the only weight-$2^m$ $Z$-type stabilizer into two weight-4 stabilizers, the number of syndromes is 12 and the total weights of the syndromes are 64 and 60, respectively, for forward and backward conversion, the same as shown in figures 1 and 2. If we only want to fulfill fault-tolerant conversion, the $m + 1$ $X$-type stabilizer measurements for the forward conversion and the $m + 1$ $Z$-type stabilizer measurements for the backward conversion can be omitted, and the number of stabilizers needed to be measured is reduced to $2m + 1$.

7. Simulation and cost analysis

We simulate both directions of conversion using MATLAB® for the case of $m = 3$ and $m = 4$, where we introduce random single-qubit errors. The $X$-type stabilizer measurements are realized by first making a transversal $\bar{H}$, defined in equation (6), to the code, which converts an $X$ measurement to a $Z$ measurement and vice versa. Thus, we can measure the corresponding $Z$-type stabilizers in the dual code. Then we use the transversal $H$ to convert back to the original code space.

The simulation process is depicted in figure 3. The computer programme is used to verify that, for any single-qubit errors in the input, our single-step error-correction and code-conversion process is performing correctly. The single-step process includes first combining the syndromes according to equations (18), (22) and (26). Then, based on these syndromes, we diagnose the single-qubit errors and use relations (23) and (27) to fix the first $m$ syndromes and choose the corresponding operations. Then we implement the operations and compare the actual output with the ideal output to check whether the circuit and the single-step process is correct.

The result shows that, if we use all $3m + 2$ syndromes, we obtain the correct output. If we only use $2m + 1$ syndromes, we obtain the output with at most only one error. This error does not propagate during conversion as it is fault-tolerant.

In order to see the huge decrease in requisite resources, we compare the average resource requirement for three methods to realize the fault-tolerant logical $\bar{T}$ gate for the Steane code. These three methods are the standard method [19, 20, 42], the ADP14 method [25] and our simplified method. We assess using the average cost employed by Choi [29], which includes our previous counting of CNOT gates but also incorporates all gates, measurements and error rate. Specifically, we apply Choi’s average-cost assessment [29] to the standard method. The average cost for ADP14 to realize $\bar{T}$ is shown in table 4, the average cost for our method to realize $\bar{T}$ is shown in table 5.

In our calculation, we treat all physical gates and single-qubit measurements as having fixed unit cost. We treat the error rates of the qubit, gate and measurement as all being equal to $\epsilon \in (10^{-6} \rightarrow 10^{-3})$ [43]. The calculation results for these three methods are shown in table 6.
From these results, we see that our method reduces the resource overhead significantly compared with ADP14. The main reason for this cost reduction is that we only measure eight weight-4 syndromes for the forward conversion and seven weight-4 syndromes for the backward conversion, whereas ADP14 requires measurement of all eight weight-8 syndromes and six weight-4 syndromes. For larger $m$, the advantage of our method is evident as the syndromes are exponentially reduced from $2^{m+1} - 2$ to $2m + 1$.

Compared with the standard method for fault-tolerantly effecting a single $T$ gate, our method needs approximately twice the resource of it, as we first need to convert to the 15-qubit RMQC and then convert back to the Steane code after performing the transversal $T$ gate. This double overhead is not a problem, though. We do not expect to convert back and forth between codes for each execution of $T$. Once in the 15-qubit RMQC, we can execute many gates as long as the sequence is not interrupted by the $H$ gate, which is not transversal in that code. Quantum-code conversion is valuable precisely for this reason: order the operations in the circuit so as many gates as possible can be performed transversally in one code before converting to the other code for a sequence of gates that can be performed transversally in that code. Moreover, the conversion between the two codes is also extremely important because

| Table 4. Cost (ADP14), where Cost (entangleS) can be found in table 26 in [29] plus the cost for $|0\rangle_{\text{Steane}}$, AvgCost$(S_{1,4})$ can be found in table 12 in [29]. |
|---|---|---|
| Cost or probability | Value or reference | Explanation |
| Cost (ancillary) | Cost(entangleS) | Cost for the input |
| Cost($QEC_{RM}$) | $8 \times \text{AvgCost}(S_{8,8}) + 6 \times \text{AvgCost}(S_{4,4})$ | 14 stabilizer measurements |
| Cost (fix operation) | $0.875 \times 4 \times \text{Cost}(X)$ | Fix operation, table 1 |
| Cost ($T$) | $15 \times \text{Cost}(T)$ | Transversal $T$ on RM code |
| Cost ($QEC_{RM}$) | $8 \times \text{AvgCost}(S_{8,8}) + 6 \times \text{AvgCost}(S_{4,4})$ | 14 stabilizer measurements |
| Cost (fix operation) | $0.75 \times 4 \times \text{Cost}(Z) + 0.125 \times 8 \times \text{Cost}(Z)$ | Fix operation, table 3 |
| Average cost | Sum of all costs | |

| Table 5. Cost (our method), where cost (entangleS) can be found in table 26 in [29] plus the cost for $|0\rangle_{\text{Steane}}$, AvgCost$(S_{1,4})$ can be found in table 12 in [29]. |
|---|---|---|
| Cost or probability | Value or reference | Explanation |
| Cost (ancillary) | Cost(entangleS) | Cost for the input |
| Cost($QEC_{RM}$) | $8 \times \text{AvgCost}(S_{8,8})$ | 8 stabilizer measurements |
| Cost (fix operation) | $0.875 \times 4 \times \text{Cost}(X)$ | Fix operation, table 1 |
| Cost ($T$) | $15 \times \text{Cost}(T)$ | Transversal $T$ on RM code |
| Cost ($QEC_{RM}$) | $7 \times \text{AvgCost}(S_{8,8})$ | 7 stabilizer measurements |
| Cost (fix operation) | $0.75 \times 4 \times \text{Cost}(Z) + 0.125 \times 8 \times \text{Cost}(Z)$ | Fix operation, table 3 |
| Average cost | Sum of all costs | |

| Table 6. Average cost of fault-tolerant logical $\bar{T}$ for the standard method, ADP14 method and our method. |
|---|---|---|---|
| Error rate ($\epsilon$) | Standard method | ADP14 | Our method |
| 0.000001 | 932 | 4009 | 1882 |
| 0.00001 | 932 | 4010 | 1883 |
| 0.0001 | 937 | 4020 | 1887 |
| 0.001 | 981 | 4127 | 1928 |
we could convert the 15-qubit RMQC to the smaller Steane code so as to operate most of the gates more efficiently.

8. Conclusions

We use the special case of converting from the Steane code to the 15-qubit RMQC as an example to explain the simplified code-conversion method, in which the novel single-step process can diagnose the single qubit errors and also consider their influence on the gauge operators so as to fulfill fault-tolerant conversion. The required resource is further decreased by the relations between gauge operators and stabilizers.

We extend this special case to the general case of adjacent RMQCs, where the required number of stabilizer measurements is decreased to $2^m + 1$ from $2^{m+1} - 2$. In particular, we provide explicit mathematical expression for all stabilizers that need to be measured for conversion, and we discuss all requisite ancillary qubits and stabilizers weights.

Conversion between $\text{RM}(1, 3)$ and $\text{RM}(1, 4)$ enables a universal transversal set of gates. Furthermore we can also convert between any adjacent order-one RMQCs to non-adjacent RMQCs conversion, which enables all gates of the type $\gamma^m \sqrt{T}$ to be transversal for $m \geq 4$ [4].

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ORCID IDs

Dong-Xiao Quan  @  https://orcid.org/0000-0001-5420-3880
Barry C Sanders  @  https://orcid.org/0000-0002-8326-8912

References

[1] Shor P W 1995 Phys. Rev. A 52 R2493
[2] Gottesman D 1997 Stabilizer codes and quantum error correction PhD Thesis Caltech, Pasadena
[3] Shor P W 1996 Proc. 37th Annual Symp. on Foundations of Computer Science (Washington, DC: IEEE Computer Society) pp 56–65
[4] Zeng B, Cross A and Chuang I L 2011 IEEE Trans. Inf. Theory 57 6272
[5] Eastin B and Knill E 2009 Phys. Rev. Lett. 102 110502
[6] Chen X, Chung H, Cross A W, Zeng B and Chuang I L 2008 Phys. Rev. A 78 012353
[7] Bravyi S and Kitaev A 2005 Phys. Rev. A 71 022316
[8] Reichardt B W 2005 Quantum Inf. Process. 4 251
[9] Meier A M, Eastin B and Knill E 2013 Quantum Inf. Comput. 13 195
[10] Bravyi S and Haah J 2012 Phys. Rev. A 86 052329
[11] Jones C 2014 Phys. Rev. A 87 042305
[12] Fowler A G, Devitt S J and Jones C 2013 Sci. Rep. 3 1939
[13] Jochym-O’Connor T and Laflamme R 2014 Phys. Rev. Lett. 112 010505
16

[14] Goto H 2014 Sci. Rep. A 7501
[15] Ducloscianci G and Poulin D 2015 Phys. Rev. A 91 042315
[16] Beverland M E, Buerschaper O, Koenig R, Pastawski F, Preskill J and Sijher S 2016 J. Math. Phys. 57 022201
[17] Steane A M 1996 Phys. Rev. Lett. 77 793
[18] Steane A M 1999 IEEE Trans. Inf. Theory 45 1701
[19] Nielsen M A and Chuang I L 2000 Quantum Computation and Quantum Information (Cambridge: Cambridge University Press)
[20] Bremner M J, Jozsa R and Shepherd D J 2011 Proc. R. Soc. A 467 459
[21] Paetznick A and Reichardt B W 2013 Phys. Rev. Lett. 111 090505
[22] Bombin H 2015 New J. Phys. 17 083002
[23] Kubica A and Beverland M E 2015 Phys. Rev. A 91 032330
[24] Jochym-O’Connor T and Bartlett S D 2016 Phys. Rev. A 93 022323
[25] Anderson J T, Duclos-Cianci G and Poulin D 2014 Phys. Rev. Lett. 113 080501
[26] Hwang Y, Choi B S, Ko Y C and Heo J 2015 arXiv:1511.02596
[27] Bravyi S and Cross A 2015 arXiv:1509.03239
[28] Hill C D, Fowler A G, Wang D S and Hollenberg L C L 2013 Quantum Inf. Comput. 13 439
[29] Choi B S 2015 Quantum Inf. Process. 14 2775
[30] Bombin H 2016 New J. Phys. 18 043038
[31] Choi B S 2013 2013 Int. Conf. on ICT Convergence pp 1083–7
[32] MacWilliams F J and Sloane N J A 1977 The Theory of Error-Correcting Codes (North-Holland Mathematical Library vol 16) (Amsterdam: North-Holland)
[33] Steane A M 1996 Proc. R. Soc. A 452 2551
[34] Calderbank A R and Shor P W 1996 Phys. Rev. A 54 1098
[35] Betsuhiyama K and Munemasa A 2012 J. London Math. Soc. 86 1
[36] Aliferis P, Gottesman D and Preskill J 2006 Quantum Inf. Comput. 6 97
[37] Weinstein Y S 2015 Quantum Inf. Process. 14 1841
[38] Knill E 2005 Nature 434 39
[39] Steane A M 1997 Phys. Rev. Lett. 78 2252
[40] Weinstein Y S 2016 Quantum Inf. Process. 15 1263
[41] Chao R and Reichardt B W 2017 arXiv:1705.02329
[42] Fowler A G 2011 Quantum Inf. Comput. 11 867
[43] Steane A M 2003 Phys. Rev. A 68 042322