Abstract. We consider renormalization group (RG) transformations for classical Ising-type lattice spin systems in the infinite-volume limit. Formally, the RG maps a Hamiltonian $H$ into a renormalized Hamiltonian $H'$:

$$
\exp(-H'((\sigma'))) = \sum_{\sigma} T(\sigma,\sigma') \exp(-H(\sigma)),
$$

where $T(\sigma,\sigma')$ denotes a specific RG probability kernel, $\sum_{\sigma} T(\sigma,\sigma') = 1$, for every configuration $\sigma$. With the help of the Dobrushin uniqueness condition and standard results on the polymer expansion, Haller and Kennedy gave a sufficient condition for the existence of the renormalized Hamiltonian in a neighborhood of the critical point. By a more complicated but reasonably straightforward application of the cluster expansion machinery, the present investigation shows that their condition would further imply a band structure on the matrix of partial derivatives of the renormalized interaction with respect to the original interaction. This in turn gives an upper bound for the RG linearization.

1. Introduction

We consider renormalization group (RG) transformations of finite-range and translation-invariant Hamiltonians. Among possible RG transformations, there may be some that are good, in the sense that they have a non-trivial fixed point with desirable properties. This fixed point would represent a critical state and is invariant under the RG transformation. It sits on a critical surface, which consists of all those distinct Hamiltonians whose critical trajectories under the RG map converge to it. For a point not on the critical surface but very close to being critical, the RG map will first drive it towards the fixed point for a large number of iterations, but eventually will drive it away. In the critical region, the thermodynamic systems are characterized by long-range correlations among microscopic fluctuations of local quantities that persist out to macroscopic wavelengths. There is the astonishing empirical fact that certain exponents associated with critical phenomena are universal, and in particular, they are related by the scaling laws to eigenvalues of the linearized RG map near the fixed point [11]. We are therefore interested in studying the behavior of the RG transformation when the Hamiltonian is on or near the critical surface, and most preferably in a neighborhood of the fixed point.

The relevant mathematical work is extensive, but it is fair to say that many if not most questions remain unanswered. One could worry about the various issues raised by van Enter, Fernández, and Sokal [19], questioning whether the transformation is even defined. Fortunately, if one thinks of the expansions defining the RG transformation as analogous to other expansions in statistical mechanics, then even if the original system is at the critical point, under certain conditions, the systems that must be studied to define the RG map need not be critical. Therefore there is hope of using convergent expansions to define and analyze the properties of the RG transformation.
Aizenman [1] proved and explained a number of basic features of the critical behavior of Ising models and $\phi^4$ fields in high and low dimensions by a non-perturbative analysis of the field variables. Rivasseau [17] used perturbative and constructive renormalization to investigate rigorously the phenomenon of asymptotic freedom. Martinelli and Olivieri [14, 15] investigated the stability and instability of pathologies of RG transformations under decimation. Haller and Kennedy [7] showed that a single RG transformation could map an area including a critical point to a set of well-defined renormalized interactions. (See Fisher [6] for a brief introduction to the historical developments in RG theory.)

Triggered by the realization that a single-site stochastic RG map, say Kadanoff transformation with time-dependent parameter $p(t)$, could be viewed as an infinite-temperature Glauber dynamics, further inquiries followed. Van Enter, Fernández, den Hollander, and Redig [18] studied the time evolution of a low non-zero temperature Gibbs state of Ising spins under finite temperature Glauber dynamics and showed that the evolved state is Gibbsian for short time always, but non-Gibbsian for long enough times. Le Ny and Redig [10] proved that for a short interval of time a Gibbs measure with a finite range interaction evolved under a general local stochastic dynamics would always remain Gibbsian. Maes and Netočný [12] considered classes of both discrete time and continuous time interacting particle systems in the weak coupling regime and identified sufficient conditions for which the time-evolved measure is Gibbsian for all (even infinite) times. Külske and Opoku [9] extended the notion of Gibbinessness for mean-field systems to the setup of continuous local state spaces and generalized previous case studies made for spins taking finitely many values. (See [20] for a review on some recent developments in the study of Gibbs and non-Gibbs properties of transformed $n$-vector lattice and mean-field models under various transformations). More references to rigorous results inspired by, or implementing RG ideas, may be found in Brydges [2], Faris [4], Feldman et al. [5], and Mitter [16].

Formally, the RG maps a Hamiltonian $H(\sigma) = -\sum_X J(X) \sigma_X$ into a renormalized Hamiltonian $H'(\sigma') = -\sum_Y J'(Y) \sigma'_Y$:

$$\exp(-H'(\sigma')) = \sum_{\sigma} T(\sigma, \sigma') \exp(-H(\sigma)), \quad (1)$$

where $J$ is the original interaction, $J'$ is the renormalized interaction, and $T(\sigma, \sigma')$ is a probability kernel, $\sum_{\sigma'} T(\sigma, \sigma') = 1$, for every configuration $\sigma$. Our basic assumption is that the original interaction $J$ lies in a Banach space $B$, with norm

$$||J|| = \sum_{X \ni 0} |J(X)|, \quad (2)$$

Since the interaction space $B$ does not put any additional restrictions on the interactions other than absolute summability, it is the largest physically reasonable space of interactions. However, as argued in [19], it may be too large for a useful implementation of RG ideas near a critical point as some rather strange phenomena would occur. To study the behavior of the RG transformation when the system is at criticality, extra assumptions are needed. A key tenet of the renormalization group is that the introduction of the block spins would shift the location of the critical point, thus making the critical situation treatable by analytic methods. Haller and Kennedy [7], using the Dobrushin-Shlosman machinery, adopted this approach. They defined a probability measure $\mu_{\sigma', V, \tau}$ which depends on the finite volume $V$, the boundary condition $\tau$.
and the block spin configuration $\sigma'$ by

$$
\mu_{\sigma', V, \tau}(F) = \frac{\sum_{\sigma} F(\sigma) T(\sigma, \sigma') e^{-H(\sigma)}}{\sum_{\sigma} T(\sigma, \sigma') e^{-H(\sigma)}},
$$

(3)

where $F(\sigma)$ is a function on the original spin configuration $\sigma$. With the help of the Dobrushin uniqueness condition and standard results on the polymer expansion, they gave a condition which is sufficient to imply that the renormalized Hamiltonian $H'$ is defined. Roughly speaking, the condition is that the collection of measures $\mu_{\sigma'}$ is in the high-temperature phase uniformly in the block spin configuration $\sigma'$.

**Hypothesis:** There exist constants $c < \infty$ and $m > 0$ such that for every finite subset $V$ of the lattice, every two sites $i, j \in V$, every boundary condition $\tau$, and every block spin configuration $\sigma'$,

$$
|\mu_{\sigma', V, \tau}(\sigma_i \sigma_j) - \mu_{\sigma', V, \tau}(\sigma_i) \mu_{\sigma', V, \tau}(\sigma_j)| \leq ce^{-m|i-j|}.
$$

(4)

They verified this condition numerically in two special cases [7]: Decimation with spacing $b = 2$ on the square lattice for $\beta < 1.363\beta_c$, and the Kadanoff transformation with parameter $p$ on the triangular lattice in a subset of the $(\beta, p)$ plane that includes values of $\beta$ around $\beta_c$. Although critical fixed points do not arise after infinite iterations of the decimation transformation, and the limiting behavior of the system under the Kadanoff transformation still seems to be an open question, this problem does not show up after a finite number of applications of these maps, so we will not worry too much about it in our present investigation. Furthermore, since this hypothesis is similar to the Dobrushin-Shlosman complete analyticity condition, one would expect that it holds in a more general setting, and provides a reasonable assumption for inquiry into the behavior of the RG transformation near the critical surface.

Using cluster expansion techniques as in [22], we show that the Hypothesis also guarantees the existence of the partial derivatives of the RG transformation (Theorem 4.6). By a more careful analysis, we will then show that the partial derivative decays sub-exponentially as the distance between the set in the original lattice and the set in the image lattice gets large. It follows that the matrix of partial derivatives displays an approximate band property (Theorem 5.3). This in turn gives an upper bound for the RG linearization (Theorem 6.2). These results extend my previous work on the behavior of the RG transformation at infinite temperature [21], which shows that the RG spectrum corresponding to decimation and majority rule is of an unusual kind: dense point spectrum for which the adjoint operators have no point spectrum at all, but only residual spectrum. Thus, although the RG transformation exists, its properties appear not at all to be what one would expect from the physics literature predictions.

For notational convenience, we will denote $T(\sigma, \sigma') e^{-H(\sigma)}$ by $e^{-H}$ in the following. As shown in [7], this modified Hamiltonian $H$ is also finite-range.

**Proposition 1.1.** For every subset $W$ of the original lattice and every subset $Z$ of the image lattice, the partial derivative $\frac{\partial J'(Z)}{\partial J(W)}$ of the RG transformation is given by the expression

$$
\frac{\partial J'(Z)}{\partial J(W)} = \sum_{\sigma'} \sigma'_Z \sum_{\sigma} \exp(-H) \sigma_W \sum_{\sigma} \exp(-H).
$$

(5)

**Proof.** The renormalized coupling constants $J'$ are given by

$$
J'(Z) = \sum_{\sigma'} \sigma'_Z \log \left( \sum_{\sigma} T(\sigma, \sigma') e^{\sum_X J(X) \sigma_X} \right).
$$

(6)

We take the derivative of both sides of (6) with respect to $J(W)$. □
Definition 1.2. For every subset $Z$ of the image lattice, the linearization $L(J_c)$ of the RG transformation at a critical point $J = J_c$ is given by a linear function of the deviation $K$:

$$L(J_c)K(Z) = \sum_W \frac{\partial J'(Z)}{\partial J(W)} \bigg|_{J=J_c} K(W),$$

where $W$ ranges over all finite subsets of the original lattice.

Remark. The above calculations are only rigorous for finite lattices, but may be interpreted in some more sophisticated limiting sense for infinite lattices, following standard interpretation of statistical mechanics, as will be shown in later sections.

Many important physical properties emerge from spectral properties of the linearization of the RG map. For an interaction very close to being critical ($J = J_c + K$ with $K$ small), its behavior under the RG transformation will be governed by the linearization $L(J_c)$:

$$(J_c + K)'(Z) = J'_c(Z) + L(J_c)K(Z) + \text{corrections}.$$  

The main difficulty in this approximation is that the original interaction $J$ is not small near the critical point, thus a direct cluster expansion is not applicable. However, there is a marvellous estimate on long range energies that will provide us with the smallness needed. We first review relevant results [7], where the existence of the renormalized interaction $J'_c$ was justified.

2. Review of relevant results

The key idea [7] is to divide the original lattice $L$ into blocks that are $L$ sites long on each side. Haller and Kennedy referred to them as $L$-blocks (indexed by $\bar{L}$), and chose $L$ large enough so that these $L$-blocks are commensurate with the blocks in the RG transformation, i.e., each RG block is a subset of an $L$-block. A subset $X$ of $L$ defines a subset $\bar{X}$ of $\bar{L}$, corresponding to the set of $L$-blocks that have non-empty intersection with $X$. Conversely, for each site $y$ in $\bar{L}$, there is a corresponding $L$-block $\bar{y}$ that is a subset of $L$. They divided these $L$-blocks into $2^d$ types, where $d$ is the number of dimensions of the lattice system. For illustration purposes, they restricted their attention to two dimensions, so there would be 4 types of $L$-blocks, labelled by $i = 1, 2, 3, 4$. Let $\sum_i$ denote the summation over the spins which are in a type-$i$ $L$-block. Then trivially,

$$\sum_i \exp(-H) = \sum_1 \sum_3 \sum_2 \sum_1 \exp(-H).$$

They started by considering $\sum_1 \exp(-H)$ and defined $F^1$ by $\exp(-F^1) = \sum_1 \exp(-H)$. The sum $\sum_1$ would factor into a product over type 1 blocks of the sum over the spins in that block, $F^1$ is therefore a function of the spins in blocks of types 2, 3, 4 and the boundary spins. However, when they tried to compute $\sum_2 \exp(-F^1)$ in a similar fashion, they ran into difficulty: $F^1$ can contain terms which involve spins in more than one type 2 block, so the sum $\sum_2$ does not factor into a product of independent sums over the type 2 blocks. To proceed, they distinguished long-range terms $F^1_{LR} = \sum_{B:LR} F^1_B$ supported on sets of sites with diameter greater than $L$ that prevent the factorization from short-range terms $F^1_{SR} = \sum_{B:SR} F^1_B$ that do not. Then $\sum_2 \exp(-F^1_{SR})$ would factor into a product over the type 2 blocks, and they defined $F^2$ by $\exp(-F^2) = \sum_2 \exp(-F^1_{SR})$.

They continued the above constructions iteratively, always throwing out the long-range terms that prevent the factorization. Eventually, after performing all the summations, they obtained...
For each allowable long-range $B$ (small enough to fit inside $L$-blocks with side length not exceeding $3L$, thus consisting of at most $p = 3^d$ $L$-blocks), they defined

$$K(B) = \exp(-F_B^1 - F_B^2 - F_B^3) - 1.$$  

They then defined a modified expectation $E$, given by

$$Ef = \exp(F_4) \sum_{4} \sum_{3} \sum_{2} \sum_{1} \exp(-H + F_{LR}^1 + F_{LR}^2 + F_{LR}^3) f.$$  

### 3. Cluster expansion

We introduce some combinatorial concepts. A hypergraph is a set of sites together with a collection $\Gamma$ of nonempty subsets. Such a nonempty set is referred to as a hyper-edge or link. Two links are $L$-connected if the $L$-blocks they occupy are within $aL$-distance apart, where $a$ is a constant that only depends on the number of dimensions $d$ as shown in [7]. A hypergraph $\Gamma$ is $L$-connected if the support of $\Gamma$ is nonempty and cannot be partitioned into nonempty sets with no $L$-connected links. We use $\Gamma_C$ to indicate $L$-connectivity of the hypergraph $\Gamma_C$, and write $\Gamma_C^* = \bigcup_{\Gamma_C}$ for the support of $\Gamma_C$ in $L$.

As is usual for expansion methods, we work in a finite volume, but as explained in [7], all estimates are uniform in the volume and insensitive to boundary conditions, thus the infinite-volume limit exists according to standard interpretation of statistical mechanics. In the following most quantities depend on the finite volume $V$, the choice of boundary condition $\tau$, and the block spin configuration $\sigma'$, but this dependence is made implicit.

Haller and Kennedy [7] argued that the denominator of (5) has the following cluster representation:

$$\sum_{\sigma} \exp(-H) = e^{-F_4} \sum_{\Delta} \prod_{N \in \Delta} w_N,$$

where $\Delta$ is a set of subsets $N$’s of $\bar{L}$ (pairwise at least $a$-distance apart), and

$$w_N = \sum_{\Gamma_C^* = N} E \left( \prod_{B \in \Gamma_C} K(B) \right).$$

They justified this by first noticing that it is possible to bound (13) by

$$|w_N| \leq v_N = \sum_{\Gamma_C^* = N} \prod_{B \in \Gamma_C} ||K(B)||_\infty,$$

and then showing that under their Hypothesis there is a function $\epsilon(L)$ such that

$$||K(B)||_\infty \leq \epsilon(L)$$

for every allowable $B$, with $\epsilon(L) \to 0$ as $L \to \infty$.

We now examine the effect of multiplying $\sigma_W$ to the above cluster representation as in the numerator of (5). There will be two kinds of terms. In some of these, none of the $L$-connected components intersect $W$, so for these terms one gets a product of $\sigma_W$ with a product of independent $w_N$’s. For the other terms, one decomposes $\Delta$ into a set of $L$-connected components that is also $L$-connected to $W$ and remaining ones that are not. We arrive at the representation

$$\sum_{\sigma} \exp(-H) \sigma_W = e^{-F_4} \sum_{R, \Delta'} \tilde{w}_R \prod_{N \in \Delta'} w_N,$$
where \( R = \emptyset \) or \( R \) and \( \bar{W} \) are within \( a \)-distance apart, and \( \tilde{w}_R \) is a sum over hypergraphs \( \Delta_R \) with \( \cup \Delta_R = R \) such that \( W \) and \( \Delta_R \) are \( L \)-connected. \( \Delta_R \) itself may not be \( L \)-connected. Therefore

\[
\frac{\partial J'(Z)}{\partial J(W)} = \sum_{\sigma'_Z} \sigma'_Z \frac{\sum_{R \Delta'} \tilde{w}_R \prod_{N \in \Delta'} w_N}{\sum_{\Delta} \prod_{N \in \Delta} w_N}.
\]  

(17)

We will justify this formal result in the following section.

4. Existence of the partial derivatives

**Theorem 4.1** (Kotecký-Preiss). For subsets \( N_i \)'s of \( \bar{L} \), define

\[
c(N_1, N_2) = \begin{cases} 1 & \text{if } N_1 \text{ and } N_2 \text{ are within } a \text{-distance apart;} \\ 0 & \text{otherwise,} \end{cases}
\]

(18)

and

\[
C(N_1, ..., N_n) = \sum_{G_c} \prod_{(i,j) \in G_c} (-c(N_i, N_j)),
\]

(19)

where \( G_c \) is a connected graph with vertex set \( \{1, ..., n\} \). Take \( M > 1 \). Suppose that for each site \( y \) in \( \bar{L} \),

\[
\sum_{N'} c(N, N')v_{N'}M^{|N'|} \leq |N| \log(M).
\]

(20)

Then the avoidance probability for every \( Y \subset \bar{L} \) has a convergent power series expansion,

\[
\left| \sum_{\Delta'} \prod_{N \in \Delta'} w_N / \sum_{\Delta} \prod_{N \in \Delta} w_N \right| = \exp \left( -\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{N_1, ..., N_n} C(N_1, ..., N_n) c(Y, \cup_i N_i) w_{N_1} \cdots w_{N_n} \right)
\]

\[
\leq \exp \left( \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{N_1, ..., N_n} |C(N_1, ..., N_n)| |c(Y, \cup_i N_i)v_{N_1} \cdots v_{N_n}| \right) \leq M^{|Y|},
\]

(21)

where \( \Delta' \) is a set of subsets of \( \bar{L} \) (pairwise at least \( a \)-distance apart) that is also at least \( a \)-distance away from \( Y \), and \( \Delta \) is a set of subsets of \( L \) (pairwise at least \( a \)-distance apart).

**Proposition 4.2.** Take \( M > 1 \). Suppose Hypothesis holds. Suppose \( L \) is sufficiently large so that \( \epsilon(L) \) is sufficiently small,

\[
\epsilon(L) \leq \frac{\log(M)(p-1)^p}{rc(M)^p(1 + (p-1) \log(M))},
\]

(22)

where

\[
c = \sum_{m=1}^{p} \sup_{x \in L} \# \{B : x \in B, |\overline{B}| = m\} < \infty
\]

(23)

due to finite-range and translation-invariant assumptions on the Hamiltonian, and \( r \) is a constant that only depends on the distance \( a \) and the number of dimensions \( d \):

\[
r = \sup_{y \in L} \# \{z : \text{dist}(y, z) \leq a\}.
\]

(24)
For each site \( y \) in \( \bar{L} \), let \( a_y(N) \) be the collection of subsets \( N \subset \bar{L} \) that satisfy \( \text{dist}(y, N) \leq a \). Then we have
\[
\sum_{N \in a_y(N)} v_N M^{|N|} \leq \log(M). \tag{25}
\]

**Remark.** The inequality (25) is a standard sufficient condition for (20). It will be applied in the following Theorem 4.6.

**Proof.** For a fixed but arbitrarily chosen \( y \) in \( \bar{L} \), we estimate (25).
\[
\sum_{N \in a_y(N)} v_N M^{|N|} = \sum_{N \in a_y(N)} \sum_{\Gamma_C = N} M^{|N|} \prod_{B \in \Gamma_C} ||K(B)||_\infty \tag{26}
\leq \sum_{N \in a_y(N)} \sum_{\Gamma_C = N} M^{|\Gamma_C|} (\epsilon(L))^{||\Gamma_C||}
= \sum_{\Gamma_C : \text{dist}(y, \Gamma_C) \leq a} \sum_{n \in \mathbb{N}} (M^{|\Gamma_C|} \epsilon(L))^n. \tag{27}
\]

We say that a hypergraph \( \Gamma_C \) is \( L \)-rooted at \( y \) if \( \Gamma_C^* \) and \( y \) are within \( a \)-distance apart. Let \( a_n(y) \) be the number of all \( L \)-connected hypergraphs with \( n \) links that are \( L \)-rooted at \( y \),
\[
a_n(y) = \#\{\Gamma_C : |\Gamma_C| = n \text{ and dist}(y, \Gamma_C^*) \leq a\}. \tag{29}
\]
Let \( a_n \) be the supremum over \( y \) of the number of \( L \)-connected hypergraphs with \( n \) links that are \( L \)-rooted at \( y \), i.e., \( a_n = \sup_{y \in \bar{L}} a_n(y) \). Then
\[
\sum_{N \in a_y(N)} v_N M^{|N|} \leq \sum_{n=1}^{\infty} a_n (M^p \epsilon(L))^n. \tag{30}
\]

It seems that once we show that \( a_n \) grows at most exponentially with \( n \), the geometric series above will converge for small enough \( \epsilon(L) \), and our claim might follow. To estimate \( a_n \), we relate to some standard combinatorial facts [13]. The rest of the proof follows from a series of lemmas.

**Lemma 4.3.** Let \( a_n \) be the supremum over \( y \) of the number of \( L \)-connected hypergraphs with \( n \) links that are \( L \)-rooted at \( y \). Then \( a_n \) satisfies the recursive bound
\[
a_n \leq r c \sum_{k=0}^{p} \binom{p}{k} \sum_{a_{n_1},...,a_{n_k} : n_1 + \cdots + n_k + 1 = n} a_{n_1} \cdots a_{n_k} \tag{31}
\]
for \( n \geq 1 \), where \( \binom{p}{k} \) is the binomial coefficient.

**Proof.** We first linearly order the points \( x \) in \( L \) and also linearly order the allowable \( L \)-blocks \( B \) of \( L \). This naturally induces a linear ordering of the points \( y \) in \( \bar{L} \). For a fixed but arbitrarily chosen \( y \) in \( \bar{L} \), we examine (29). Write \( \Gamma_C = \{B_1\} \cup \mathcal{C}_1 \), where \( B_1 \) is the least \( B \) in \( \Gamma_C \) with \( \text{dist}(y, B_1) \leq a \). There must be such an allowable \( B_1 \), since \( \text{dist}(y, \Gamma_C^*) \leq a \). Moreover, there must be some \( z \in B_1 \) such that \( \text{dist}(y, z) \leq a \), of which there are \( r \) possibilities. Also notice that every \( x \in z^o \) will satisfy \( x \in B_1 \). Thus
\[
a_n(y) \leq r \sum_{m=1}^{p} \sup_{x \in L} \sum_{B_1 : x \in B_1, |B_1| = m} \#\{\mathcal{C}_1\}. \tag{32}
\]
As a consequence,
\[ a_n(y) \leq rc\#\{\Gamma^1_C\}. \]  
(33)

The remaining hypergraph \( \Gamma_1^1 \) has \( n - 1 \) subsets and breaks into \( k \) \( L \)-connected components \( \Gamma_1, ..., \Gamma_k \) of sizes \( n_1, ..., n_k \), with \( n_1 + \cdots + n_k = n - 1 \). For each component \( \Gamma_i \), there is a least \( L \)-block \( y_i \) through which it is \( L \)-connected to \( B_1 \), and the map from the components \( \Gamma_i \) to the \( L \)-sites \( \{y_i\} \) is injective. We have
\[ a_n(y) \leq rc \sum_{k=0}^{p} \binom{p}{k} \sum_{a_{n_1}, ..., a_{n_k}; n_1 + \cdots + n_k = n + 1} a_{n_1} \cdots a_{n_k}. \]  
(34)

Our inductive claim follows by taking the supremum over all \( y \) in \( \tilde{L} \). Finally, we look at the base step: \( n = 1 \). In this simple case, as reasoned above, we have
\[
\begin{align*}
a_1 &= \sup_{y \in \tilde{L}} \#\{\Gamma_C : |\Gamma_C| = 1 \text{ and } \text{dist}(y, \Gamma_C^*) \leq a\} \\
&\leq r \sum_{m=1}^{p} \sup_{x \in L} \#\{B : x \in B, |\bar{B}| = m\} \\
&= rc,
\end{align*}
\]  
(35)

and this verifies our claim. \( \square \)

Clearly, \( \sum_{N \in a_n(N)} v_N M^{[N]} \) will be bounded above by \( \sum_{n=1}^{\infty} \bar{a}_n \left(M^p e(L)\right)^n \), if
\[
\bar{a}_n = rc \sum_{k=0}^{p} \binom{p}{k} \sum_{a_{n_1}, ..., a_{n_k}; n_1 + \cdots + n_k = n + 1} \bar{a}_{n_1} \cdots \bar{a}_{n_k}
\]  
(36)

for \( n \geq 1 \), i.e., equality is obtained in the above lemma.

**Lemma 4.4.** Consider the coefficients \( \bar{a}_n \) that bound the number of \( L \)-connected and \( L \)-rooted hypergraphs with \( n \) links. Let \( w = \sum_{n=1}^{\infty} \bar{a}_n z^n \) be the generating function of these coefficients. Then the recursion relation (36) for the coefficients is equivalent to the formal power series generating function identity
\[ w = rc(1 + w)^p. \]  
(37)

**Proof.** Notice that \((1 + w)^p = \sum_{k=0}^{p} \binom{p}{k} w^k\), thus
\[ w = rcz \sum_{k=0}^{p} \binom{p}{k} w^k. \]  
(38)

Writing completely in terms of \( z \), we have
\[
\sum_{n=1}^{\infty} \bar{a}_n z^n = rc \sum_{k=0}^{p} \binom{p}{k} \sum_{a_{n_1}, ..., a_{n_k}; n_1 + \cdots + n_k + 1 = n} \bar{a}_{n_1} \cdots \bar{a}_{n_k} z^n.
\]  
(39)

Our claim follows from term-by-term comparison. \( \square \)

**Lemma 4.5.** If \( w \) is given as a function of \( z \) as a formal power series by the generating function identity (37), then this power series has a nonzero radius of convergence \( |z| \leq \frac{(p-1)^{p-1}}{r cp^p} \).
Proof. Without loss of generality, assume \( z \geq 0 \). Set \( z_1 = rcz \). Solving (37) for \( z_1 \) gives
\[
z_1 = \frac{w}{(1 + w)^p}.
\]
(40)
By elementary calculus, this increases as \( w \) goes from 0 to \( 1/(p - 1) \) to have values \( z_1 \) from 0 to \( (p - 1)^{-1}(p - 1) / p \). It follows that as \( z_1 \) goes from 0 to \( (p - 1)^{-1}(p - 1) / p \), the \( w \) values range from 0 to \( 1/(p - 1) \).
\( Q.E.D. \)

Proof of Proposition 4.2 continued. We notice that in the above lemma, \( w = \sum_{n=1}^{\infty} \bar{a}_n z^n = 1/(p - 1) \) corresponds to \( z_1 = rcz = (p - 1)^{-1}/p \), which implies that for each \( n \),
\[
\bar{a}_n \leq (rcp)^n (p - 1)^{-1 + (p - 1)n}.
\]
(41)
Gathering all the information we have obtained so far,
\[
\sum_{N \in a_y(N)} v_M M_{\{N\}} \leq \sum_{n=1}^{\infty} (rc(Mp)^p \epsilon(L))^n (p - 1)^{-1 + (p - 1)n} \leq \sum_{n=1}^{\infty} \frac{rc(Mp)^p \epsilon(L)}{(p - 1)^p} (p - 1)^{-1} \leq \log(M)
\]
(42)
by (22).
\( Q.E.D. \)

Theorem 4.6. Suppose Hypothesis holds. Then for every subset \( W \) of the original lattice and every subset \( Z \) of the image lattice, the power series expansion of the partial derivative \( \partial_{J(Z)} / \partial_{J(W)} \) of the RG transformation (17) converges absolutely.

Proof. The proof of this theorem is an application of the Kotecký-Preiss result [8]. Recall that \( N \in \Delta' \) implies \( N \) and \( R \cup \bar{W} \) are at least \( \alpha \)-distance apart. By the Kotecký-Preiss theorem (Theorem 4.1), (25) implies
\[
\left| \sum_{N \in \Delta'} \prod_{N \in \Delta} w_N / \prod_{N \in \Delta} w_N \right| \leq M^{\{R \cup \bar{W}\}}.
\]
(44)
To verify our claim, we need to estimate
\[
\sum_R |\tilde{w}_R| M^{\{R \cup \bar{W}\}} \leq \sum_R M^p \prod_{Y \in \Delta_R} v_Y M^{\{Y\}}.
\]
(45)
But this is easy, remove \( W \), the remaining hypergraph breaks up into \( k : 0 \leq k \leq p L \)-connected components. So this last quantity is bounded by
\[
M^p \sum_{k=0}^{p} \binom{p}{k} (\log(M))^k = M^p (1 + \log(M))^p.
\]
(46)
\( Q.E.D. \)

5. Band structure

By a more complicated application of the cluster expansion machinery, we show that when Hypothesis holds, the matrix of partial derivatives displays an approximate band property.
Proposition 5.1. Suppose Hypothesis holds. Suppose $L$ is sufficiently large (cf. (22)). Then for every site $y$ in $\bar{L}$, we have
\[
\sum_{N \in a_y(N) \text{ and } |N| > P} v_{N} M_{\bar{N}^{|N|}} \leq \delta(P),
\]
where
\[
\delta(P) = \frac{\left(\frac{rc(Mp)\epsilon(L)}{(p-1)^{p-1}}\right)^{\frac{P}{p}}}{(p-1) \left(1 - \frac{rc(Mp)\epsilon(L)}{(p-1)^{p-1}}\right)}.
\]
It is clear that $\delta(P) \to 0$ as $P \to \infty$.

Proof. An $L$-connected hypergraph that is $L$-rooted at $y$ and with cardinality greater than $P$ will have at least $\frac{P}{p}$ links. This implies
\[
\sum_{N \in a_y(N) \text{ and } |N| > P} v_{N} M_{\bar{N}^{|N|}} \leq \sum_{n=P/p}^{\infty} (rc(Mp)^p \epsilon(L))^n (p-1)^{-(1+(p-1)n)} = \delta(P).
\]

Proposition 5.2. Suppose Hypothesis holds. Suppose $L$ is sufficiently large (cf. (22)). Then for every subset $Y$ of $\bar{L}$, we have
\[
\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{N_1, \ldots, N_n \text{ and } |\cup_i N_i| > P} |C(N_1, \ldots, N_n)| c(Y, \cup_i N_i) v_{N_1} \cdots v_{N_n} \leq |Y| \delta(P).
\]

Proof. This follows from Proposition 5.1. Remove $Y$, the remaining hypergraph is still $L$-connected by (19). Moreover, there can be at most $|Y|$ choices for where it is pinned down.

Theorem 5.3. Suppose Hypothesis holds. Then there is an approximate band property for the matrix of partial derivatives: For subset $W$ of the original lattice and subset $Z$ of the image lattice that are sufficiently far apart, the partial derivative $\frac{\partial J'(Z)}{\partial J(W)}$ of the RG transformation (17) is arbitrarily small. Let
\[
l(W, Z) = \inf \{|dist(w, z) : w \in \bar{W}, z \in \bar{Z}\}
\]
be the distance between $W$ and $Z$ measured in $\bar{L}$. For fixed but arbitrary $S, Q$ and $K$, if
\[
l(W, Z) > (3 + a)(pS + QK),
\]
then
\[
\frac{\partial J'(Z)}{\partial J(W)} \leq M^p (1 + \log(M))^p \left(\frac{\delta(S)}{\log(M)} + (\delta(Q) + \delta(K))p(1 + S)M^{p(1+S)}\right).
\]

Before starting the proof of Theorem 5.3, let us try to understand this band property better.

Proposition 5.4. Suppose Hypothesis holds. Then for subset $W$ of the original lattice and subset $Z$ of the image lattice, as the distance $l(W, Z)$ between $W$ and $Z$ gets large, the partial derivative $\frac{\partial J'(Z)}{\partial J(W)}$ decays sub-exponentially, a little slower than $\exp(-l(W, Z)^{1/2})$.

Proof. For notational convenience, we denote $l(W, Z)$ simply by $l$. Take
\[
S = \frac{1}{p} \left(\frac{l}{2(3 + a)}\right)^{\alpha},
\]
where
\[
\alpha = \frac{1}{p} \left(\frac{l}{2(3 + a)}\right)^{\alpha},
\]
and so on.
and
\[ Q = K = \left( \frac{l}{2(3 + a)} \right)^\beta, \]
where \( 0 < \alpha < \beta \leq 1/2 \). We examine (53). The first factor, \( M^p (1 + \log(M))^p \), is just a constant. The second factor is more complicated and thus merits more attention. The first term, \( \delta(S)/\log(M) \), decays as \( \exp(-l^\alpha) \), whereas the second term, \( (\delta(Q) + \delta(K)) p(1 + S)M^{p(1+S)} \), decays as \( \exp(-l^\beta + l^\alpha) \sim \exp(-l^\beta) \). Piecing it all together, \( |\frac{\partial J'(W)}{\partial J(W)}| \) decays sub-exponentially, as \( \exp(-l^\alpha) \).

**Proof of Theorem 5.3.** Fix an \( S \) that is large enough. We rewrite (17) as
\[
\frac{\partial J'(Z)}{\partial J(W)} = \sum_{\sigma'_Z} \sum_{|R| > pS, \Delta'} \bar{w}_R \prod_{N \in \Delta'} w_N + \sum_{\sigma'_Z} \sum_{|R| \leq pS, \Delta'} \bar{w}_R \prod_{N \in \Delta'} w_N. \quad (56)
\]
Following, we will verify the smallness of (56) by examining the two terms on the right-hand side separately.

Case 1: \( |R| > pS \). Similarly as in the proof of Theorem 4.6 we estimate (45). Remove \( W \), the remaining hypergraph (with cardinality greater than \( pS \)) breaks up into \( k : 0 \leq k \leq p \) \( L \)-connected components, so at least one of them has cardinality greater than \( S \). By (46) and (13), the contribution of this hypergraph is bounded by
\[
M^p \delta(S) \sum_{k=0}^p \left( \frac{p}{k} \right) (\log(M))^{k-1} = \frac{\delta(S)}{\log(M)} M^p (1 + \log(M))^p. \quad (57)
\]
Case 2: \( |R| \leq pS \). We need to do a more careful analysis for this case. Recall that \( N \in \Delta' \) implies \( N \) and \( R \cup \bar{W} \) are at least \( a \)-distance apart. By the Kotecký-Preiss theorem (Theorem 4.1), (25) implies
\[
\sum_{\Delta'} \prod_{N \in \Delta'} w_N / \sum_{\Delta} \prod_{N \in \Delta} w_N = \exp \left( -\sum_{n=1}^\infty \frac{1}{n!} \sum_{N_1, \ldots, N_n} C(N_1, \ldots, N_n) c(R \cup \bar{W}, \cup^\infty N_i) w_{N_1} \cdots w_{N_n} \right). \quad (58)
\]
For notational convenience, we will denote the right-hand side of (58) by \( F(\infty, \infty) \), where the first parameter of \( F \) indicates the maximum number of subsets \( N_i \)'s allowed in the expansion, and the second parameter of \( F \) indicates the cardinality restriction over these \( N_i \)'s. It is straightforward that for fixed \( Q \) and \( K \),
\[
F(\infty, \infty) = F(\infty, \infty) - F(Q, \infty) + F(Q, \infty) - F(Q, K) + F(Q, K). \quad (59)
\]
We first examine \( F(\infty, \infty) - F(Q, \infty) \). This difference can be regarded as the tail of the convergent series (58), thus should be small when \( Q \) is large. In fact, it is bounded by \( p(1 + S)M^{p(1+S)} \delta(Q) \) by the mean value theorem, applied to (21) and (50). Fix such a \( Q \). We next examine \( F(Q, \infty) - F(Q, K) \). For every subset \( N \) of \( L \), define
\[
u_N = \begin{cases} w_N & \text{if } |N| \leq K; \\ 0 & \text{otherwise.} \end{cases} \quad (60)
\]
The difference in \( F \) can then be interpreted as induced by evaluating (58) using two sets of parameters \( w_N \) and \( u_N \). These two parameter sets both lie in the region of analyticity of (58), thus intuitively, the difference can be as small as desired when \( K \) is large enough. We again
refer to (21) and (51), and conclude that it is bounded by $p(1 + S)M^{p(1+S)}\delta(K)$. Fix such a $K$. For these two situations, the only thing left to show now is that
\[ \sum_{|R| \leq pS} |\tilde{w}_R| \]

is finite, but this naturally follows from (46).

Finally, we examine $F(Q, K)$. As $R \cup \bar{W}$ and $\cup_{i=1}^n N_i$ are within $a$-distance apart, $F(Q, K)$ will only depend on $L$-sites in a finite region (roughly a ball with radius $(3 + a)(pS + QK)$). If $Z$ is outside this region, then
\[ \sum_{|R| \leq pS} \tilde{w}_R F(Q, K) \]

is a constant with respect to $\sigma'_Z$, thus, when summing over all possible image configurations $\sigma'$ as in (60), it vanishes.

\[ \Box \]

6. Upper bound for the RG linearization

**Proposition 6.1.** Fix a subset $Z$ of the image lattice. Let $n(E)$ be the number of subsets $W$ of the original lattice that are at most $E$-distance away from $Z$ (measured in $L$),
\[ n(E) = \# \{W : l(W, Z) \leq E\}. \]

Then $n(E)$ grows polynomially in $E$.

**Proof.** Due to our finite-range and translation-invariant assumptions on the Hamiltonian,
\[ n = \sup_{y \in \bar{L}} \# \{W : y \in \bar{W}\} < \infty. \]

Thus $n(E)$ grows at the same rate as the volume of a $d$-dimensional ball with radius $E$, i.e., polynomial growth $E^d$.

\[ \Box \]

**Theorem 6.2.** Suppose Hypothesis holds. Then the linearization $L(J_c)$ of the RG transformation (7) is well-defined and has an upper bound.

**Proof.** This is mainly due to the fact that sub-exponential decay dominates polynomial growth. Take $K$ with $||K||$ small (and so, a fortiori, $||K||_{\infty}$ is small). By Propositions 5.4 and 6.1
\[ |L(J_c)K(Z)| \leq \sum_{n=0}^{\infty} \sum_{n \leq l(W, Z) < n+1} \left| \frac{\partial J'(Z)}{\partial J(W)} \right|_{J=J_c} |K(W)| \]
\[ \lesssim ||K||_{\infty} \sum_{n=0}^{\infty} \exp(-n^\alpha)(n+1)^d. \]

Our claim then follows from the integral test.

\[ \Box \]

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS, AUSTIN, TX 78712, USA
E-mail address: myin@math.utexas.edu