Role of the compensating current in the weak Josephson coupling regime: An extended study on excitonic Josephson junctions

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Huang’s experiment [Phys. Rev. Lett. \textbf{109}, 156802 (2012).] found, in the quantum Hall bilayer of the Corbino geometry, the interlayer tunneling currents at two edges are coupled to each other and one of two tunneling currents is referred to as the compensating current of the other. Our another work\textsuperscript{46} has explained this exotic coupling phenomenon as a result of excitonic Josephson effect induced by interlayer tunneling current. In this paper, we study the same setup—excitonic Josephson junction—but in the weak Josephson coupling regime, which occurs for large junction length. Interestingly, we find the compensating current drives the other edge to undergo a nonequilibrium phase transition from a superfluid to resistive state, which is signaled by an abrupt jump of the critical tunneling current. We also identify the critical exponent and furthermore offer more experimental prediction.

I. INTRODUCTION

Josephson effect is particularly attractive to condensed matter researchers because it serves as the striking manifestation of condensation and the promising candidate for quantum technology. The unrelenting and strong attention has been recently received in optically-excited exciton or exciton-polariton cold gas and graphene electron-hole bilayer exciton\textsuperscript{27,28}. The quantum Hall bilayer, which is the most practicable one to achieve the exciton condensation\textsuperscript{9–31}, however, remains not studied extensively in the land of Josephson effect to date. Actually, the search for Josephson effect in quantum Hall bilayer ever arouse intense interest since the observation of Josephson-like tunneling\textsuperscript{32,33}, which is is signaled by a dramatically enhanced interlayer conductance occurring near zero bias and an abrupt increase of interlayer voltage once exceeding a critical interlayer tunneling current\textsuperscript{11,14,18,20,22,25}. In the end, however, the Josephson-like tunneling is attributed to a mixture of coherent and incoherent interlayer tunneling\textsuperscript{34–36} instead of the “real” Josephson effect. Once exceeding a critical current, the incoherent tunneling dominates over the coherent one.

The scattering approach by solving the Bogoliubov-de Gennes Hamiltonian\textsuperscript{37,38} is the standard one to explore the Josephson effect but it is difficult to access in the context of quantum Hall bilayer. In our previous work\textsuperscript{39,41}, we therefore turn to a new method within the frame of pseudospin dynamics, which is originated from the idea that layer degree can be treated as pseudospin\textsuperscript{36,42,43}. We firstly employ this new method to study the exciton-condensate/exciton-condensate (EC/EC)\textsuperscript{40} and exciton-condensate/normal-barrier/exciton-condensate (EC/N/EC) junctions\textsuperscript{41} with a constant relative phase between two ECs that can be generated by perpendicular electric field, as suggested in Ref.\textsuperscript{44}. We found that excitonic Josephson effect occurs only when \( d_J \leq \xi \) and new transport mechanism—tunneling-assisted Andreev reflection at a single N/EC interface—emerges when \( d_J > \xi \), where \( d_J \) and \( \xi \) are barrier length and correlation length\textsuperscript{40,41} (the EC/EC junction is in the strong Josephson coupling regime). The excitonic Josephson effect gives rise to novel fractional solitons\textsuperscript{40} while the new mechanism leads to a half portion of fractional solitons\textsuperscript{41}. Notably, these new types of solitons have potential to improve the stability and efficiency of quantum logic circuits\textsuperscript{42}. We next study another setup suggested to have a relative phase by externally applying interlayer tunneling current\textsuperscript{46}.

Inspired by Huang’s experiment\textsuperscript{27}, we consider the setup of interlayer tunneling currents exerted on two edges of quantum Hall bilayer, as shown in Fig. 1(a). The tunneling currents \((J_{IL},J_{IR})\) twist the condensate phases of two edges so as to create the relative phases between three condensates: EC1, EC2, and EC3 (more detail on how the interlayer tunneling current changes the condensate phase is illustrated in the supplementary material of Ref.\textsuperscript{39}). Such structure is regarded as two condensates (EC1 and EC3) sandwiched by a superfluid barrier (EC3), which is equivalent to an excitonic Josephson junction\textsuperscript{48}. Our another work\textsuperscript{49} has explored this setup but focuses on the short junction whose junction length \( L \) is smaller than Josephson length \( \lambda \). Its results demonstrated that the exotic coupling phenomenon of edge tunneling currents observed by Huang et al\textsuperscript{47} is originated from excitonic Josephson effect and Huang’s experiment may be by far the most robust evidence for quantum Hall bilayer exciton condensation.

In this paper, we turn our attention to the opposite case—long junction of \( L \sim 10a \), which corresponds to the typical quantum Hall bilayer\textsuperscript{11,55}. Our calculation of the condensate phase [see Fig. 1(b)] reflects that the Josephson current is essentially negligible in the bulk since the phase goes to zero and becomes flat there (this inference is based on that supercurrent is proportional to the slope of the condensate phase). The two edges are weakly Josephson coupled and the long junction can be
approximated as two independent EC/EC junctions. It is therefore highly desirable that the long junction can display entirely different properties from the short junction in which two edges are strongly Josephson coupled. It turns out that the long junction indeed exhibits an unique property: one edge undergoes a nonequilibrium phase transition with increasing the tunneling current on the other edge, namely, the compensating current. During this phase transition, the critical interlayer tunneling current of the edge sharply falls and the corresponding critical exponent is indentified as $\gamma \sim 0.5$. Since the Josephson coupling is weak, we wonder why the compensating current can influence the other edge so largely? According to our analysis, it is because the compensating current reduces the effective junction length of the constituent EC/EC junction on the opposite side. We furthermore calculate the magnetic field induced by Josephson current (denoted by $B_J$) in the Corbino-geometry excitonic Josephson junction as illustrated in Fig. 1(c). We find the length reduction effect of the compensating current is revealed by the crossover of the $B_J$ versus $J_{tR} - J_{tL}$ curve into the linear one (that is a characteristic of the short junction). The induced magnetic field is estimated at $\sim 100\mu$T that is large enough to be detected by the scanning superconducting interference device (SQUID). In the main body of this paper, we show the results of the short junction in Figs. 3-6 while that of the Corbino-geometry junction in Fig. 7.

II. MODEL AND METHOD

Burkov and MacDonald treated two layers of the quantum Hall bilayer as pseudospin quantum degrees of freedom and accordingly deduced a lattice model Hamiltonian:

$$H = \frac{1}{2} \sum_{i,j} (2H_{ij} - F_{i,j}^{\text{intra}}) S_i^z S_j^z - F_{i,j}^{\text{inter}} (S_i^x S_j^x + S_i^y S_j^y),$$

$$\tilde{S}_i^z = \frac{1}{2} \sum_{\sigma,\sigma'} a_{i,\sigma}^{\dagger} \tau_{\sigma,\sigma'} a_{i,\sigma'}.$$

(1)

Here $a_{i,\sigma}^{\dagger}$ ($a_{i,\sigma}$) is the Schwinger boson creation (annihilation) operators where $i$ and $\sigma$ label the site and layer indexes and $\tau$ is the Pauli matrix vector. The Hartree term $H_{ij}$ describes the direct Coulomb interaction while the Fock term $F_{i,j}^{\text{intra}}$ ($F_{i,j}^{\text{inter}}$) serves the intralayer (interlayer) exchange interaction. This lattice Hamiltonian possesses the eigenstate wave function which can be generally expressed as

$$|\Psi\rangle = \prod_i \left[ \cos \left( \frac{\theta(X_i)}{2} \right) + \sin \left( \frac{\theta(X_i)}{2} \right) e^{i\phi(X_i)} c_{i\uparrow}^{\dagger} \right] |0\rangle$$

(2)

The operator $c_{i\uparrow}^{\dagger}$ ($c_{i\downarrow}^{\dagger}$) creates an electron at the lattice site location $X_i$ in the top (bottom) layer. It is difficult to study the present issue through quantum scattering

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**FIG. 1.** (color online) (a) Schematic layout of an excitonic Josephson junction induced by interlayer tunneling current. The relative phases between three condensate regions: EC1, EC2 and EC3, are generated by externally applying tunneling currents $J_{tL}$ and $J_{tR}$. $\theta_B$ and $L$ denote the magnetic and junction length. (b) The calculated phase profiles for parallel polarity $\oplus$ ($J_{tR} = J_{tL}$) and anti-parallel polarity $\ominus$ ($J_{tR} = -J_{tL}$) with $L = 12\lambda$. The green (black) and pink (grey) lines correspond to the parallel and anti-parallel polarity, respectively. The employed values of $J_{tL}$ are 5, 10, 15, 20, 25 $J_0$ and, with increasing $J_{tL}$, the phase $\phi$ departs from the $x$ axis. The length and current units are the Josephson length $\lambda$ and $J_0 = e\Delta_1/2\hbar$. The orange (grey) and blue (black) arrows denote the flow direction of Josephson current for the parallel and anti-parallel polarity, respectively. The dots indicate Josephson current is essentially negligible in the deep of the bulk. Such a long junction is similar to two weakly coupled exciton-condensate/exciton-condensate (EC/EC) junctions. The cross is the breakpoint of the two EC/EC junctions and occurs at where the Josephson current approaches zero. The left (right) part of the bulk combines with the left (right) edge forming an EC/EC junction. (c) Schematic layout of a Corbino-geometry exciton Josephson junction. The two tunneling currents $J_{tL}$ and $J_{tR}$ are exerted on the blue (upper) and orange (lower) shadow zones. $R_{\text{min}}$ and $R_{\text{max}}$ are the minimum and maximum radius.
approach which is based on the wave function since we cannot simply write down the explicit forms of \( \theta(X_1) \) and \( \phi(X_1) \).

We therefore request a SU(2) to O(3) mapping and the wave function is transformed into a classical pseudospin

\[
m(X_1) = (m_\perp \cos \phi, m_\perp \sin \phi, m_z),
\]

\[m_\perp = \sin \theta, \quad m_z = \cos \theta. \tag{3}\]

Accordingly, the dynamics of the quantum Hall bilayer can be described by the Landau-Lifshitz-Gilbert (LLG) equation

\[
d\vec{m}/dt = \vec{m} \times (2/nh)(\delta E[\vec{m}]/\delta \vec{m}) - \alpha \left( \vec{m} \times d\vec{m}/dt \right),
\]

\[E[\vec{m}] = A_{\text{unit}} \sum_i \left[ \beta m_z^2 + \rho_s m_\perp^2 |\nabla X_i \phi|^2 - n\Delta m_\perp \cos \phi \right] \tag{4},
\]

where \( A_{\text{unit}} \) is the area of the unit cell for the pseudospin lattice and \( n \) is the pseudospin density. The excitonic superfluid loses its coherence after traveling over one correlation length \( \xi \) so the size of the unit cell is equal to \( \xi \), which is estimated at \( \sim 20 \text{nm} \). In unit of the magnetic length \( l_B \), \( \xi \sim 10 l_B \) \( (l_B \text{ has the typical value of } \sim 20 \text{nm}) \). On the other hand, the energy functional \( E[\vec{m}] \) is composed of the capacitive penalty, the exchange correlation, and the interlayer tunneling energy, which are characterized by the parameters: anistropic energy \( \beta \), pseudospin stiffness \( \rho_s \), and interlayer tunneling \( \Delta t \), respectively. These model parameters is up to which kind of samples we are discussing and their values will be given later. The second term for the LLG equation is the Gilbert damping which relaxes the energy toward the minimum.

A. Modeling excitonic Josephson junctions

The key breakthrough of the present work is to introduce the effect of external tunneling currents. When exerting the \(+z\)-direction tunneling current \( J_t \) on a area of \( A \) over a short duration of \( dt \), there are electrons as many as \( J_t Adt/e \) pouring out of the top layer and trickling into the bottom layer simultaneously (see Fig. 2), giving rise to the change of \(-2J_t Adt/e\) in the total pseudospin \( nAm_z \). Under the effect of tunneling current, the \( z\)-component LLG equation thus can be modified as

\[
dm_z/dt = -2p/nh m_\perp^2 \nabla^2 \phi + \Delta /h m_\perp \sin \phi - 2J_t/e + \alpha m_\perp^2 \phi/Adt. \tag{5}\]

In the rectangle-shaped excitonic Josephson junction as shown in Fig. 2(a), two tunneling current \( J_{tL} \) and \( J_{tR} \) are applied to two edges over a length as large as one lattice size. We can therefore model the junction through setting \( J_t \) to

\[
J_t = J_{tL} \Theta(x + L/2)\Theta(L/2 - 10l_B - x) + J_{tR} \Theta(L/2 - x)\Theta(x - L/2 + 10l_B). \tag{6}\]

Notice we from here on use the continuous varying \( x \) instead of the discrete \( X_i \), for convenience in presentation and \( \Theta(x) \) is the Heaviside step function. The origin \( x = 0 \) is defined to be located at the center of the junction. After evolving with time, we ultimately acquire the static solutions for \( \phi \), \( m_\perp \), and \( m_z \) that specify the pseudospin orientation. The Josephson current is furthermore calculated by

\[
J_s = e\rho_s \nabla \phi/h. \tag{7}\]

B. Calculation of induced magnetic field due to excitonic Josephson effect

We next consider a Corbino-geometry excitonic Josephson junction that can generate circular Josephson current [see Fig. 2(c)]. The Corbino can be divided into a set of rings with radius which ranges from \( R_{\text{min}} \) to \( R_{\text{max}} \). A single ring of the specific radius \( r \) can be viewed as a bent Josephson junction with \( L = 2\pi r \). We firstly calculate the phase profile for the junction of \( L = 2\pi R_{\text{min}} \) by the LLG equation and then acquire the phase profile for other values of \( r \) by taking the azimuthal symmetry into account. The Josephson current is similarly calculated by Eq. (7). By using the Biot-Savart Law, we finally
obtain the induced magnetic field:

\[
B(z) = \mu_0 \langle J_a(R_{\text{min}}, \theta) \rangle_\theta z dR_{\text{min}} \frac{1}{2} \left[ \frac{1}{(R_{\text{min}}^2 + z^2)^{3/2}} - \frac{1}{(R_{\text{max}}^2 + z^2)^{3/2}} \right]
\]

where \( d \) is the interlayer separation, \( z \) is the distance above the center of the bilayer, and \( \langle \cdot \cdot \cdot \rangle_\theta \) is the average over the angular axis of polar coordinate.

C. Identification of critical current and determination of parameters

Both two geometries we consider are discussed based on a length scale, namely, Josephson length:

\[
\lambda = \sqrt{2 \mu_0 \rho_s n \Delta_t}
\]

and we identify the critical interlayer tunneling by finding the upper and lower boundaries at which the junction departs from the condensation phase, i.e., \( m_z \) begins to become nonzero. The main focus of the present work is the typical quantum Hall bilayer of \( \lambda \sim 45 \mu \text{m} (\Delta_t = 10^{-8} E_0) \), which corresponds to the samples fabricated by Eisenstein’s group. Here the Coulomb interaction \( E_0 = e^2/\varepsilon \lambda \) serves as the energy scale and \( E_0 \sim 7 \text{meV} \).

The other parameters we use are listed as follows: \( \beta = 0.02 E_0 \) and \( \rho_s = 0.005 E_0 \), which were derived from the mean-field calculation.

III. ANALYSIS OF ROLE OF THE COMPENSATING CURRENT

The electric circuit equipment of Huang’s Corbino can be regarded as a rectangle-shaped excitonic Josephson junction (the detail reason is given in our another work). To serve the goal of expelling the contribution of edge-state current, we here consider the same equipment on the basis of the Corbino geometry but with small \( \lambda \) because of the difference in the fabrication of samples. The corresponding junction length \( L \) is 0.54 mm and equivalent to 12\( \lambda \).

A. Nonequilibrium phase transition

Huang’s experiments demonstrated that the edges of a quantum Hall bilayer exhibit Josephson-like behavior: the interlayer voltage suddenly emerges when the applied tunneling current exceeds some critical values so there exists upper and lower limits, within which the transport is coherent (see Figs. 1 and 2 of Ref.\( ^{47} \)). The upper and lower critical values of the external tunneling current were shown to depend on its compensating current — the tunneling current exerted on the other edge. We therefore discuss this dependence for the long junction of \( L = 12 \lambda \) in Fig. \( ^{3} \) Over a wide range of \( J_{1R} \), both the upper and lower critical currents nearly keep constant [see Fig. \( ^{3} \) a)]. Near \( J_{1R} = \pm 30.692 \), however, the critical currents rapidly fall to zero. The sharp jump of critical currents \( J^c \) indicates the left edge is switched from a superfluid to resistive state. The left edge undergoes a phase transition in the condition of compensating-current-driven nonequilibrium. With slowly adjusting \( J_{1R} \), it is indentified as a first-order phase transition since \( |J^c(J_{1R} = \pm 30.692 J_0)| = 15.999 J_0 \) and \( |J^c(J_{1R} = \pm 30.6925 J_0)| = 0 \) (The giant change in critical currents hints possible incontinuity). We futher-
more define new critical exponents:
\[
\Delta J_{tL} \propto \begin{cases} 
(30.692 - J_{tR})^{\gamma^+} & \text{for } J_{tR} \lesssim 30.692, \\
(J_{tR} + 30.692)^{\gamma^-} & \text{for } J_{tR} \gtrsim -30.692,
\end{cases} \tag{10}
\]
where \(\Delta J_{tL} = J^{c}(J_{tR}) - J^{c}(\pm 30.692 J_0)\). The fits to our numerical results extract the values of exponents [see Fig. 3(b)]: \(\gamma^+ = 0.4939, \gamma^- = 0.4999\) for the upper \(J^c\) curve. For the lower \(J^c\) curve, the values of \(\gamma^+\) and \(\gamma^-\) are exactly exchanged because of electron-hole symmetry.

**B. Junction-length reduction effect**

Why the compensating current can largely reduce the critical currents as \(J_{tR} \approx \pm 30.692\) even if the Josephson coupling is so weak? As have been illustrated in Fig. 4(b), the long junction can be decomposed into two nearly independent EC/EC junctions. We here indentify the breakpoint occurring at \(J_s = 0\) or where \(J_s\) reaches its minimum and determine the effective length of the left junction as shown in Figs. 4(a) and (b). We find, regardless of the polarity, the compensating current decreases the effective length and hence leads to the jump of the critical currents. It is quite intuitive or shown in Fig. 4(c) that the critical current would decrease with decreasing the junction length.

**IV. OTHER INTERESTING PREDICTION**

**A. Discussion on Josephson breakdown effect**

Another excitement observed by Huang et al is that, with increasing the compensating current beyond \(\pm 16nA\), the upper and lower \(J^c\) curves suddenly become symmetric and a finite interedge voltage emerges. Our another work regards Huang’s device as a short excitonic Josephson junction and attributes this phenomenon to the breakdown of Josephson effect—Josephson coupling collapses when the induced Josephson current attains some critical value, in which the external tunneling currents will prefer to converting into edge-state currents. We here comment whether this breakdown effect occurs also in the long junction or not. Differing
from the short junction, the upper and lower \( J_c \) curves are originally symmetric with respect to \( J_{\text{LL}} = 0 \) and the applied compensating current is limited to a range of \( J_{\text{LR}} = -30.692J_{\text{LO}} \sim 30.692J_{\text{LO}} \) beyond which coherent interlayer tunneling disappears [see Fig. 3(a)]. We have performed numerical calculation demonstrating that, over the range of \( J_{\text{LR}} = -30.692J_{\text{LO}} \sim 30.692J_{\text{RO}} \), static solutions can exist and did not see any critical variation. We therefore believe that the breakdown effect does not occur in the long junction.

We furthermore give more detail analysis through Fig. 6 and discuss how to distinguish the weak Josephson coupling from the breakdown effect. Huang’s experiment ever discussed this disappearance of the compensating phenomenon based on the difference of two tunneling currents \( \Delta J_\ell \) and from theoretical side \( \Delta J_i \) plays the similar role as the phase difference in the Josephson junction. On the other hand, it is easier to compare with the experiment directly based on the compensating current \( J_{\text{LR}} \). In Fig. 6 we therefore plot the maximum value of supercurrent in the spatial distribution \( J_{\text{c}}^{\text{max}} \) as a function of not only \( \Delta J_i \) but also \( J_{\text{LR}} \). We find that \( J_{\text{c}}^{\text{max}} \) rises or drops to saturation over the range of \( \Delta J_i = -20J_{\text{LO}} \sim -40J_{\text{LO}} \) or \( \Delta J_i = 20J_{\text{LO}} \sim 40J_{\text{LO}} \) [see Fig. 6(a)], which corresponds to \( J_{\text{LR}} = -20J_{\text{LO}} \sim 20J_{\text{LO}} \) [see Fig. 6(b)]. With increasing the compensating current, if the Josephson-breakdown regime is achieved, it necessarily occurs at \( J_{\text{LR}} = -20 \sim 20J_{\text{LO}} \) where the \( J_c \) curves hold horizontal [see Fig. 6(a)]. Measuring the interedge voltage will help us clarify the junction being in the weak Josephson coupling regime or Josephson-breakdown regime. Alternatively, after increasing the compensating current beyond \( \pm 20J_{\text{LO}} \), \( |J'| \) begins to fall [see Fig. 6(a)], providing an unique signature for the weakly Josephson coupling, namely, Josephson fall.

**B. The crossover behavior with varying junction length**

Since the dependence of the critical currients on the compensating current is so distinct for the short and long junctions, we next want to understand the crossover behavior for increasing junction length through Fig. 6. Because the lower \( J_c \) curve can be produced through doing the electron-hole transformation: \( J_{\text{LL}} \rightarrow -J_{\text{LL}}, J_{\text{LR}} \rightarrow -J_{\text{LR}} \) on the upper \( J_c \) curve, in Fig. 6 we display only the upper \( J_c \) curve for conciseness. Fig. 6 shows that, with increasing the junction length, the curve is gradually skew and no abrupt change occurs. Moreover, the Josephson fall already can be found as \( L = 4\lambda \) while the weakly symmetric Josephson region can be achieved as \( L \sim 5\lambda \). The values of \( 5\lambda \) happens to meet the junction length for the typical quantum Hall bilayer of Hall-bar geometry, although Hall-bar geometry may be difficult to coincide with our calculation due to the influence of edge-state current.

**C. The induced magnetic field due to Josephson current in a Corbino geometry**

Next Fig. 7 shows the results for the Corbino-geometry excitonic Josephson junction, which is depicted in Fig. 11(c). Except for the minimum radius \( R_{\text{min}} \), here the other parameters are determined according to the actual situation of realistic experiments. The minimum radius for the typical Corbino is roughly 0.16mm or equivalently \( R_{\text{min}} \approx 3.56\lambda \) instead of \( R_{\text{min}} = 1.9\lambda \) we choose for increasing the numerical efficiency. But, the investigated Corbino of \( \lambda < 2\pi R_{\text{min}} < 2\pi R_{\text{max}} \) can already capture the physics of the long junction to a qualitative level and
such a Corbino with smaller $R_{\text{min}}$ is easily realized by etching. We find, differing from the short junction\cite{9}, the dependence of the induced magnetic field $B_J$ on the difference of two tunneling currents $\Delta J_i$ can have apparent curvature. The curve however becomes linear when $J_{1R}$ reaches $\pm 30 J_0$. It is because $J_{1R}$ decreases the effective length of the EC/EC junction on the opposite side and drives the investigated Corbino into the short-junction regime of a linear dependence. Moreover, the extremely subtle magnetic field must be measured by the scanning superconducting quantum interference device (SQUID). To our best knowledge, the resolution of the typical scanning SQUID is up to $\sim 10 \text{pT}$ at a sensor-to-sample distance of $\sim 100 \text{nm}$ and the current technology even improves the resolution to $\sim 1 \text{pT}$. We estimate $B_J$ on the scale $\sim 100 \text{pT}$ and it is measurable without doubts.

V. CONCLUSION

In conclusion, we predict a nonequilibrium phase transition occurring in the long junction of weak Josephson coupling and find the effective length reduction effect of the compensating current. The sample size is not highly tunable in experimental measurement and therefore this length reduction effect will be largely helpful in observing the interesting crossover behavior predicted in Ref.\cite{24}. We furthermore discuss the possibility of the breakdown of Josephson effect and suggest measuring the interedge voltage and Josephson fall\cite{24} to distinguish the Josephson breakdown effect from weak Josephson coupling. We also calculate the induced magnetic field in the Corbino-geometry Josephson junction to suggest the detection of Josephson current. The present work is devoted to offering experimental prediction but more details on experimental realization are given in our another work\cite{24} that exhibits correspondence between theory and experiment. It should be noted that there are still very much theoretical effort called for, such as developing Bogolubov-deGennes description, exactly indentifying phase transition (especially for it being first-order or second-order), systematically exploring the Josephson breakdown effect and etc. We believe the present work together with Ref.\cite{24}—excitonic Josephson effect induced by interlayer tunneling current—will bring new attention to the condensed matter physics community.

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