Abstract. Reciprocity is an important characteristic of directed networks and has been widely used in the modeling of World Wide Web, email, social, and other complex networks. In this paper, we take a statistical physics point of view and study the limiting entropy and free energy densities from the microcanonical ensemble, the canonical ensemble, and the grand canonical ensemble whose sufficient statistics are given by edge and reciprocal densities. The sparse case is also studied for the grand canonical ensemble. Extensions to more general reciprocal models including reciprocal triangle and star densities will likewise be discussed.

1. Introduction

Reciprocity measures the tendency of vertex pairs to form mutual connections between each other and is an important object to study in complex networks, such as email networks, see e.g. Newman et al. [25], World Wide Web, see e.g. Albert et al. [1], World Trade Web, see e.g. Gleditsch [14], social networks, see e.g. Wasserman and Faust [34], and cellular networks, see e.g. Jeong et al. [16]. In networks that aggregate temporal information, reciprocity provides a measure of the simplest feedback process occurring on the network, i.e., the tendency of one stimulus, a vertex, to respond to another stimulus, another vertex. Reciprocity is important because most complex networks are directed and it is the main quantity characterizing feasible dyadic patterns, namely possible types of connections between two nodes. One example is the email network. Just because user B’s email address appears in user A’s address book does not necessarily mean that the reverse is also true, although it often is, see e.g. Newman et al. [25]. Another example is the social network. Reciprocity captures a basic way in which different forms of interaction take place on a social network like Twitter. When two users A and B interact as peers, one expects that messages will be exchanged between them in both directions. However, if user A sends messages to user B, who is a celebrity or news source, it is likely that user B will not send messages in return, see e.g. Cheng et al. [8]. Therefore, it is not enough to just understand the edge density of a directed network, the reciprocal density needs to be studied as well. In Garlaschelli and Loffredo [12], it was discovered that detecting nontrivial patterns of reciprocity can reveal mechanisms and organizing principles that help understand the topology of the observed network. They also proposed a measure of reciprocity and studied how strong the reciprocity is for different complex networks and found that reciprocity is strongest in the World Trade Web. People often treat complex networks as undirected ones for simplicity, and reciprocity can help quantify the information.

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loss induced by projecting a directed network into an undirected one. Using the knowledge of reciprocity, significant directed information can be retrieved from an undirected projection, and the error introduced when a directed network is treated as undirected may be estimated, see e.g. Garlaschelli and Loffredo [13].

Directed networks consisting of \( n \) nodes can be modeled by directed graphs on \( n \) vertices, where a graph is represented by a matrix \( X = (X_{ij})_{1 \leq i,j \leq n} \) with each \( X_{ij} \in \{0, 1\} \). Here, \( X_{ij} = 1 \) means there is a directed edge from vertex \( i \) to vertex \( j \); otherwise, \( X_{ij} = 0 \). We assume that \( (X_{ii})_{1 \leq i \leq n} = 0 \) so that there are no self loops. Give the set of such graphs the probability

\[
P_n^{\beta_1, \beta_2}(X) = Z_n(\beta_1, \beta_2)^{-1} \exp \left[ n^2 (\beta_1 e(X) + \beta_2 r(X)) \right],
\]

(1.1)

where

\[
e(X) := n^{-2} \sum_{1 \leq i,j \leq n} X_{ij}, \quad r(X) := n^{-2} \sum_{1 \leq i,j \leq n} X_{ij} X_{ji},
\]

(1.2)

\( \beta_1 \) and \( \beta_2 \) are parameters, and \( Z_n(\beta_1, \beta_2) \) is the appropriate normalization. Note that \( e(X) \) and \( r(X) \), defined in (1.2), respectively represent the directed edge density and the reciprocal density.

In the literature, \( X_{ij} \) and \( X_{ij} X_{ji} \) are sometimes referred to as the single edge and the reciprocal edge. This belongs to class of exponential random graph models called \( p_1 \) models of Holland and Leinhardt [15]. Further extensions include \( p_2 \) models, see e.g. Lazega and Van Duijin [19] and Van Duijin et al. [33]. More general types of exponential models have also been introduced and studied. See Besag [4], Newman [24], Rinaldo et al. [30], Robins et al. [31], Snijders et al. [32], Wasserman and Faust [34], and Fienberg [10, 11] for history and a review of developments. The exponential random graph models have popular counterparts in statistical physics: a hierarchy of models ranging from the grand canonical ensemble, the canonical ensemble, to the microcanonical ensemble, with particle density and energy density in place of \( e(X) \) and \( r(X) \), and temperature and chemical potential in place of \( \beta_1 \) and \( \beta_2 \). In the grand canonical ensemble, the reciprocal model (1.1) in this case, no prior knowledge of the graph is assumed. In the canonical ensemble, partial information of the graph is given. For instance, the edge density of the graph is close to \( 1/2 \) or the reciprocal density is close to \( 1/4 \). In the microcanonical ensemble, complete information of the graph is observed beforehand, say in the reciprocal model, both the edge density and the reciprocal density are specified.

It is well-known that models in this hierarchy have a very simple relationship involving Legendre transforms and, more importantly, the free energy density (of the grand canonical ensemble), the conditional free energy density (of the canonical ensemble), and the entropy density (of the microcanonical ensemble) encode important information of a random graph drawn from the model. See illustration below. In particular, these quantities serve as a measure of how close the system is to equilibrium, namely perfect internal disorder, and their monotonicity shed light on the relative likelihood of each configuration following the philosophy that the higher the entropy the greater the disorder. Since real-world networks are often very large in size, the infinite limit asymptotics of these densities have received exponentially growing attention in recent years. See e.g. Aristoff and Zhu [2, 3], Chatterjee and Dembo [5], Chatterjee and Diaconis [6], Kenyon et al. [17], Kenyon and Yin [18], Lubetzky and Zhao [22, 23], Radin and Sadun [27, 28], Radin et al. [26], Radin and Yin [29], Yin [35], Yin et al. [38], and Zhu [38]. It may be worth pointing out
that most of these papers utilize the theory of graph limits as developed by Lovász and coworkers [20, 21].

The hierarchy

\[
\begin{array}{ccc}
\text{grand canonical ensemble} & \text{free energy density} \\
\downarrow & \downarrow \\
\text{canonical ensemble} & \text{conditional free energy density} \\
\downarrow & \downarrow \\
\text{microcanonical ensemble} & \text{entropy density}
\end{array}
\]

The rest of this paper is organized as follows. In Section 2 we derive the exact expression for the normalization constant (partition function) of the reciprocal model (the grand canonical ensemble) and analyze the asymptotic features of its associated microcanonical ensemble. Our main results are: an exact expression for the limiting entropy density (Theorem 3), a joint central limit theorem describing convergence of the edge density and the reciprocal density (Proposition 5), and some discussions on the monotonicity of the limiting entropy density (Remark 9). In Section 3 we investigate the asymptotic features of two canonical ensembles associated with the reciprocal model, one conditional on the edge density and the other conditional on the reciprocal density. Our main results are: exact expressions for the two limiting conditional free energy densities (Theorem 10) and some discussions on their monotonicity (Remark 11). In Section 4 we take another look at the reciprocal model and examine its asymptotic features in the sparse regime. Our main results are: exact scalings for the limiting normalization constant (Theorem 12), the mean (Proposition 13) and variance (Remark 14) of the limiting probability distribution. Lastly, in Section 5 we extend our analysis to more general reciprocal models whose sufficient statistics, besides single edge and reciprocal edge, also include reciprocal $p$-star and reciprocal triangle. Large deviations techniques are used throughout this paper. We refer the readers to the works of Chatterjee and Diaconis [6] and Chatterjee and Varadhan [7] for more details of this framework.

2. THE MICROCANONICAL ENSEMBLE

Let \((X_{ij})_{1 \leq i \neq j \leq n}\) be iid Bernoulli random variables taking values 1 and 0 each with probability \(\frac{1}{2}\). Denote the associated probability measure and the associated expectation by \(\mathbb{P}\) and \(\mathbb{E}\) respectively. Define

\[
\psi_{n,\delta}(\epsilon, r) = \frac{1}{n^2} \log \mathbb{P}(e(X) \in (\epsilon - \delta, \epsilon + \delta), r(X) \in (r - \delta, r + \delta)).
\]

(2.1)

Shrink the intervals around \(\epsilon\) and \(r\) by letting \(\delta\) go to zero, we are interested in the limit

\[
\psi(\epsilon, r) := \lim_{\delta \to 0} \lim_{n \to \infty} \psi_{n,\delta}(\epsilon, r).
\]

(2.2)

The quantity in (2.2) will be called the **limiting entropy density**. Via the theory of large deviations, it is directly connected to the **limiting free energy density**

\[
\psi(\beta_1, \beta_2) := \lim_{n \to \infty} \frac{1}{n^2} \log Z_n(\beta_1, \beta_2).
\]

(2.3)
Theorem 1.

\[ \psi(\beta_1, \beta_2) = \frac{1}{2} \log \left( 1 + 2e^{\beta_1} + e^{2\beta_1+2\beta_2} \right). \]  \hfill (2.4)

Proof of Theorem 1

From the proof of Theorem 1, we have

\[ Z_{n}(\beta_1, \beta_2) = 2^{n(n-1)}E \left[ e^{\beta_1 \sum_{1 \leq i < j \leq n} X_{ij} + \beta_2 \sum_{1 \leq i < j \leq n} X_{ij} X_{ji}} \right] \]  \hfill (2.5)

\[ = 2^{n(n-1)} \prod_{1 \leq i < j \leq n} E \left[ e^{\beta_1 (X_{ij} + X_{ji}) + 2\beta_2 \sum_{1 \leq i < j \leq n} X_{ij} X_{ji}} \right] \]

\[ = 2^{n(n-1)} \prod_{1 \leq i < j \leq n} E \left[ e^{\beta_1 X_{ij} + X_{ji}} \right] \]

\[ = (1 + 2e^{\beta_1} + e^{2\beta_1+2\beta_2})^\binom{n}{2}. \]

Hence we draw the conclusion. \hfill \Box

Corollary 2.

\[ \psi(\epsilon, r) = -\sup_{\beta_1, \beta_2 \in \mathbb{R}} \left\{ \beta_1 \epsilon + \beta_2 r - \frac{1}{2} \log \left( \frac{1}{4} + \frac{1}{4} e^{\beta_1} + \frac{1}{4} e^{2\beta_1+2\beta_2} \right) \right\}. \]  \hfill (2.6)

Proof of Corollary 2

From the proof of Theorem 1, we have

\[ \lim_{n \to \infty} \frac{1}{n^2} \log E \left[ e^{\beta_1 \sum_{1 \leq i < j \leq n} X_{ij} + \beta_2 \sum_{1 \leq i < j \leq n} X_{ij} X_{ji}} \right] = \frac{1}{2} \log \left( \frac{1}{4} + \frac{1}{4} e^{\beta_1} + \frac{1}{4} e^{2\beta_1+2\beta_2} \right), \]  \hfill (2.7)

which is finite for any \( \beta_1, \beta_2 \in \mathbb{R} \) and is differentiable in both \( \beta_1 \) and \( \beta_2 \). The result then follows from Gärtner-Ellis theorem in large deviations theory, see e.g. Dembo and Zeitouni [3]. \hfill \Box

Remark 3. (i) Note that \( 0 \leq \frac{1}{n} \sum_{1 \leq i < j \leq n} X_{ij} \leq 1 \) and \( 0 \leq \frac{1}{n} \sum_{1 \leq i < j \leq n} X_{ij} X_{ji} \leq 1 \), which implies that \( \psi(\epsilon, r) = -\infty \) if \( \epsilon \notin [0, 1] \) or \( r \notin [0, 1] \).

(ii) Note that \( \sum_{1 \leq i < j \leq n} X_{ij} X_{ji} \leq \sum_{1 \leq i, j \leq n} X_{ij} \), which implies that \( \psi(\epsilon, r) = -\infty \) if \( r > \epsilon \).

(iii) Note that

\[ \sum_{1 \leq i < j \leq n} (X_{ij} X_{ji} + 1) - 2 \sum_{1 \leq i < j \leq n} X_{ij} = \sum_{1 \leq i < j \leq n} (X_{ij} X_{ji} + 1 - X_{ij} - X_{ji}) \]

\[ = \sum_{1 \leq i, j \leq n} (X_{ij} - 1)(X_{ji} - 1) \geq 0, \]

which implies that \( \psi(\epsilon, r) = -\infty \) if \( 1 + r - 2 \epsilon < 0 \).

Theorem 4. For \( \epsilon, r \in [0, 1] \), \( \epsilon \geq r \) and \( 1 + r - 2 \epsilon \geq 0 \),

\[ \psi(\epsilon, r) = -\epsilon \log \left( \frac{\epsilon - r}{1 + r - 2 \epsilon} \right) - \frac{r}{2} \log \left( \frac{r(1 + r - 2 \epsilon)}{\epsilon - r} \right) \]

\[ + \frac{1}{2} \log \left( \frac{1}{4(1 + r - 2 \epsilon)} \right), \]  \hfill (2.8)

and otherwise \( \psi(\epsilon, r) = -\infty \).
Proof of Theorem 4. Under the assumption that $\epsilon, r \in [0, 1]$, it is easy to see that supremum in (2.6) can not be obtained at $\beta_1, \beta_2 = \pm \infty$, and $\psi(\epsilon, r)$ must attain its extremum at finite $\beta_1, \beta_2$. At optimality, 

$$\epsilon = \frac{e^{\beta_1} + e^{2\beta_1 + 2\beta_2}}{1 + 2e^{\beta_1} + e^{2\beta_1 + 2\beta_2}},$$

(2.9)

$$r = \frac{e^{2\beta_1 + 2\beta_2}}{1 + 2e^{\beta_1} + e^{2\beta_1 + 2\beta_2}}.$$  

(2.10)

Dividing (2.10) into (2.9), we get

$$\frac{\epsilon}{r} = 1 + e^{-\beta_1 - 2\beta_2},$$

(2.11)

Substitute this back into (2.9),

$$\epsilon = \frac{e^{-\beta_1 - 2\beta_2} + 1}{e^{-2\beta_1 - 2\beta_2} + 2e^{-\beta_1 - 2\beta_2} + 1} = \frac{\epsilon}{e^{-\beta_1}(\frac{\epsilon}{r} - 1) + \frac{2r}{r} - 1},$$

(2.12)

which implies that

$$e^{\beta_1} = \frac{\epsilon - r}{1 + r - 2\epsilon}, \quad e^{2\beta_2} = \frac{r(1 + r - 2\epsilon)}{(\epsilon - r)^2}.$$  

(2.13)

The conclusion thus follows. □

The entropy $\psi(\epsilon, r)$ in the microcanonical ensemble is essentially the negation of the rate function from large deviations principle. In the microcanonical model, maximal entropy is attained at the averaged edge density $\frac{1}{2}$ and the averaged reciprocal density $\frac{1}{4}$. One can further study the fluctuations of the edge and reciprocal densities, i.e., the central limit theorem.

Proposition 5. In the microcanonical model,

$$n \left( e(X) - \frac{1}{2}, r(X) - \frac{1}{4} \right) \to N(\mu, \Sigma)$$

(2.14)

in distribution as $n \to \infty$, where

$$\mu := \left( \begin{array}{c} -\frac{1}{4} \\ \frac{1}{2} \end{array} \right), \quad \Sigma := \left( \begin{array}{cc} \frac{3}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{array} \right).$$

(2.15)

Remark 6. The drift term $\mu$ in Proposition 5 is due to the definition of $e(X)$ and $r(X)$ in (1.2). If one defines $e(X)$ and $r(X)$ as

$$e(X) = \frac{1}{n(n-1)} \sum_{1 \leq i,j \leq n} X_{ij}, \quad r(X) = \frac{1}{n(n-1)} \sum_{1 \leq i,j \leq n} X_{ij}X_{ji}$$

(2.16)

instead, then Proposition 5 will hold with minor modifications:

$$(n-1) \left( e(X) - \frac{1}{2}, r(X) - \frac{1}{4} \right) \to N(0, \Sigma).$$

(2.17)

Though definitions (1.2) and (2.16) lead to a difference of the drift term in the central limit theorem, they are indistinguishable for the limiting entropy and free energy densities.
Figure 1. On the left hand side, we have the contour plot of the limiting entropy density \( \psi(\epsilon, r) \) obtained from Theorem 4. On the right hand side, we specify the regions of monotonicity as obtained in Remark 9. In region \( \downarrow \), \( \psi \) is decreasing in both \( \epsilon \) and \( r \); in region \( \rightarrow \), \( \psi \) is increasing in \( \epsilon \) and decreasing in \( r \); in region \( \rightarrow \), \( \psi \) is increasing in both \( \epsilon \) and \( r \); in region \( \downarrow \), \( \psi \) is decreasing in \( \epsilon \) and increasing in \( r \). The boundaries are given by \( 1 + 2r = 3\epsilon \) and \( r = \frac{\epsilon}{2} \).

Proof of Proposition 5. For any \( \theta_1, \theta_2 \in \mathbb{R} \),

\[
E \left[ e^{\theta_1 n(e(X) - \frac{1}{2}) + \theta_2 n(r(X) - \frac{1}{4})} \right]
\]

\[
= E \left[ e^{\frac{\theta_1}{n} \sum_{1 \leq i,j \leq n} X_{ij} + \frac{\theta_2}{n} \sum_{1 \leq i,j \leq n} X_{ij} X_{ji}} \right] e^{-\frac{\theta_1}{2} n - \frac{\theta_2}{4} n}
\]

\[
= \left( \frac{1}{4} + \frac{1}{2} e^{\frac{\theta_1}{n}} + \frac{1}{4} e^{\frac{\theta_1}{2n}} + \frac{2\theta_1}{n} \right) e^{-\frac{\theta_1}{2} n - \frac{\theta_2}{4} n}
\]

\[
= \left( 1 + \frac{\theta_1}{n} + \frac{\theta_2}{2n} + \frac{3\theta_1^2}{4n^2} + \frac{\theta_1 \theta_2}{n^2} + \frac{\theta_2^2}{2n^2} + O(n^{-3}) \right) e^{-\frac{\theta_1}{2} n - \frac{\theta_2}{4} n}
\]

\[
\rightarrow \exp \left\{ -\frac{\theta_1}{2} - \frac{\theta_2}{4} + \frac{1}{2} \left( \frac{3\theta_1^2}{4} + \frac{2\theta_1 \theta_2}{2} + \frac{\theta_2^2}{2} \right) \right\}
\]

as \( n \to \infty \). Since convergence of moment generating functions implies convergence in distribution, the proof is complete. \( \square \)

Remark 7. It is straightforward to compute that

\[
\psi \left( \frac{1}{2}, \frac{1}{4} \right) = -\frac{1}{2} \log \left( \frac{3}{4} \right) - \frac{1}{8} \log \left( \frac{\frac{1}{2} + \frac{1}{4}}{\frac{1}{2}} \right) + \frac{1}{2} \log \left( \frac{1}{\sqrt{4}} \right) = 0.
\] (2.19)

This is consistent with the law of large numbers and the maximal entropy principle.

Remark 8. Along the Erdős-Rényi curve \( r = \epsilon^2 \), \( 0 \leq \epsilon \leq 1 \),

\[
\psi(\epsilon, \epsilon^2) = -\epsilon \log \epsilon - (1 - \epsilon) \log(1 - \epsilon) - \log 2.
\] (2.20)

This is the entropy of a Bernoulli random variable and is the negation of the rate function of the large deviations for the edge density.
Remark 9. Let us analyze the monotonicity of the limiting entropy density. On one hand,

$$ \frac{\partial \psi}{\partial \epsilon} = -\log \left( \frac{\epsilon - r}{1 + r - 2\epsilon} \right), $$

(2.21)

which implies that $\frac{\partial \psi}{\partial \epsilon} \geq 0$ if and only if $\epsilon - r \leq 1 + r - 2\epsilon$, which is equivalent to $1 + 2r \geq 3\epsilon$. On the other hand,

$$ \frac{\partial \psi}{\partial r} = -\frac{1}{2} \log \left( \frac{r(1 + r - 2\epsilon)}{(\epsilon - r)^2} \right), $$

(2.22)

which implies that $\frac{\partial \psi}{\partial r} \geq 0$ if and only if $r(1 + r - 2\epsilon) \leq (\epsilon - r)^2$, which is equivalent to $r \leq \epsilon^2$, i.e., $\psi(\epsilon, r)$ is increasing in $r$ below the Erdős-Rényi curve $r = \epsilon^2$ and decreasing in $r$ above the Erdős-Rényi curve $r = \epsilon^2$. (See [28] for a similar phenomenon across the Erdős-Rényi curve in the (undirected) edge-triangle model.)

3. The canonical ensemble

As in Aristoff and Zhu [3], Kenyon and Yin [18], and Zhu [38], we are interested in the asymptotic features of constrained exponential random graph models. Let us study the exponential random graph models conditional on the edge density

$$ \psi_{n, \delta}(\epsilon, \beta_2) = \frac{1}{n^2} \log \mathbb{E} \left[ \exp \left\{ \beta_2 \sum_{1 \leq i, j \leq n} X_{ij}X_{ji} \right\} 1_{|e(X) - \epsilon| < \delta} \right], $$

(3.1)

and on the reciprocal density

$$ \psi_{n, \delta}(\beta_1, r) = \frac{1}{n^2} \log \mathbb{E} \left[ \exp \left\{ \beta_1 \sum_{1 \leq i, j \leq n} X_{ij} \right\} 1_{|r(X) - r| < \delta} \right]. $$

(3.2)

The corresponding conditional probability measure to (3.1) is given by

$$ P_{n, \delta}^{\epsilon, \beta_2}(X) = \frac{1}{2^{n(n-1)}} \exp \left\{ -n^2 \psi_{n, \delta}(\epsilon, \beta_2) + \beta_2 \sum_{1 \leq i, j \leq n} X_{ij}X_{ji} \right\} 1_{|e(X) - \epsilon| < \delta}, $$

(3.3)

and the corresponding conditional probability measure to (3.2) is given by

$$ P_{n, \delta}^{\beta_1, r}(X) = \frac{1}{2^{n(n-1)}} \exp \left\{ -n^2 \psi_{n, \delta}(\beta_1, r) + \beta_1 \sum_{1 \leq i, j \leq n} X_{ij} \right\} 1_{|r(X) - r| < \delta}. $$

(3.4)

We can shrink the interval around $\epsilon$ (or $r$) by letting $\delta$ go to zero:

$$ \psi(\epsilon, \beta_2) := \lim_{\delta \to 0} \lim_{n \to \infty} \psi_{n, \delta}(\epsilon, \beta_2), $$

(3.5)

$$ \psi(\beta_1, r) := \lim_{\delta \to 0} \lim_{n \to \infty} \psi_{n, \delta}(\beta_1, r). $$

The quantities in (3.5) will be called the limiting conditional free energy densities.

Theorem 10. For any $\beta_2 \in \mathbb{R}$, $0 \leq \epsilon \leq 1$,

$$ \psi(\epsilon, \beta_2) = -\epsilon \log \left( \frac{\epsilon - r^*}{1 + r^* - 2\epsilon} \right) + \frac{1}{2} \log \left( \frac{1}{4(1 + r^* - 2\epsilon)} \right), $$

(3.6)
where
\[ r^* = \begin{cases} \frac{(2e^{2\beta_2} - 2\epsilon - 1) - \sqrt{(2e^{2\beta_2} - 2\epsilon + 1)^2 - 4e^{2\beta_2}(2e^{2\beta_2} - 1)}}{2(e^{2\beta_2} - 1)} & \text{if } \beta_2 \neq 0, \\ \epsilon^2 & \text{if } \beta_2 = 0, \end{cases} \] (3.7)

and for any \( \beta_1 \in \mathbb{R}, 0 \leq r \leq 1, \)
\[ \psi(\beta_1, r) = -\frac{r}{2} \log r - \log 2 - \frac{1 + r}{2} \log \left( \frac{1 - r}{2e^{\beta_1} + 1} \right) + r \log \left( \frac{e^{\beta_1}(1 - r)}{2e^{\beta_1} + 1} \right). \] (3.8)

Proof of Theorem 10 By using Varadhan’s lemma, see e.g. Dembo and Zeitouni [9],
\[ \psi(\epsilon, \beta_2) = \sup_{2\epsilon - 1 \leq r \leq \epsilon} \{ \beta_2 r + \psi(\epsilon, r) \}, \] (3.9)
\[ \psi(\beta_1, r) = \sup_{r \leq r \leq r \frac{1}{2}} \{ \beta_1 r + \psi(\epsilon, r) \}. \] (3.10)

By (2.22), the optimal \( r \) in (3.9) satisfies
\[ 0 = \beta_2 + \frac{\partial \psi}{\partial r} = \beta_2 - \frac{1}{2} \log \left( \frac{r(1 + r - 2\epsilon)}{(\epsilon - r)^2} \right), \] (3.11)
which is equivalent to
\[ (e^{2\beta_2} - 1)r^2 - (2\epsilon e^{2\beta_2} - 2\epsilon + 1)r + \epsilon^2 e^{2\beta_2} = 0. \] (3.12)

When \( \beta_2 = 0, (3.12) \) has one solution \( r^* = \epsilon^2 \), and when \( \beta_2 \neq 0 \), since
\[ (2\epsilon e^{2\beta_2} - 2\epsilon + 1)^2 - 4\epsilon e^{2\beta_2}(e^{2\beta_2} - 1) = 4\epsilon^2 + 1 - 4\epsilon + 4\epsilon e^{2\beta_2} - 4\epsilon e^{2\beta_2} \]
\[ = (2\epsilon - 1)^2 + 4\epsilon(1 - \epsilon)e^{2\beta_2} \geq 0, \] (3.13)
(3.12) has two solutions
\[ r^\pm = \frac{(2\epsilon e^{2\beta_2} - 2\epsilon + 1) \pm \sqrt{(2\epsilon e^{2\beta_2} - 2\epsilon + 1)^2 - 4\epsilon e^{2\beta_2}(e^{2\beta_2} - 1)}}{2(e^{2\beta_2} - 1)}. \] (3.14)
Therefore we have mean value theorem, since $r^+ + r^- = \left(\frac{2e^{2\beta_2} - 2\epsilon + 1}{e^{2\beta_2} - 1}\right) > 2\epsilon$.

When $\beta_2 < 0$, we have $e^{2\beta_2} - 1 < 0$ and one solution of $r^+$ and the other is negative. We check that $r^- > 0$ and $r^+ < 0$ and thus the optimal $r^* = r^-$. When $\beta_2 > 0$, both solutions $r^\pm$ of $r^+$ are positive. We check that

$$r^+ + r^- = \frac{(2e^{2\beta_2} - 2\epsilon + 1)}{(e^{2\beta_2} - 1)} > 2\epsilon$$

and $r^+ \geq \frac{r^+ + r^-}{2} > \epsilon$ and thus the optimal $r^* = r^-$. This is indeed the optimizer following the mean value theorem, since $\frac{\partial \psi}{\partial r}\big|_{r=\epsilon^-} = -\infty$ and $\frac{\partial \psi}{\partial r}\big|_{r=2\epsilon^-} = +\infty$. By (2.21), the optimal $\epsilon$ in (3.10) satisfies

$$0 = \beta_1 + \frac{\partial \psi}{\partial \epsilon} = \beta_1 - \log \left(\frac{\epsilon - r}{1 + r - 2\epsilon}\right),$$

which has one solution $\epsilon^* = \frac{e^{\beta_1(1+r)+r}}{2\epsilon^{\beta_1}+1}$. This is indeed the optimizer following the mean value theorem, since $\frac{\partial \psi}{\partial \epsilon}\big|_{\epsilon=r^+} = \infty$ and $\frac{\partial \psi}{\partial \epsilon}\big|_{\epsilon=(\frac{1+r}{2})^-} = -\infty$.

**Remark 11.** Let us analyze the monotonicity of the two limiting conditional free energy densities. We have

$$\frac{\partial \psi(\epsilon, \beta_2)}{\partial \epsilon} = r^* + \left[\beta_2 + \frac{\partial \psi(e, r^*)}{\partial r^*}\right] \frac{\partial r^*}{\partial \beta_2} = r^*;$$

$$\frac{\partial \psi(\beta_1, r)}{\partial \beta_1} = \epsilon^* + \left[\beta_1 + \frac{\partial \psi(e^*, r)}{\partial e^*}\right] \frac{\partial e^*}{\partial \beta_1} = \epsilon^*. $$

Therefore $\psi(\epsilon, \beta_2)$ and $\psi(\beta_1, r)$ are increasing in $\beta_2$ and $\beta_1$, respectively. Moreover, we have

$$\frac{\partial \psi(\epsilon, \beta_2)}{\partial \epsilon} = \left[\beta_2 + \frac{\partial \psi(e, r^*)}{\partial r^*}\right] \frac{\partial r^*}{\partial \epsilon} + \frac{\partial \psi(e, r^*)}{\partial \epsilon}$$

$$= -\log \left(\frac{\epsilon - r^*}{1 + r^* - 2\epsilon}\right).$$

Therefore $\psi(\epsilon, \beta_2)$ is increasing in $\epsilon$ if and only if $1 + 2r^* \geq 3\epsilon$. This is equivalent to $\epsilon \leq \frac{1}{2}$ when $\beta_2 = 0$; while for $\beta_2 \neq 0$, this is equivalent to

$$\frac{(2e^{2\beta_2} - 2\epsilon + 1) - \sqrt{(2e^{2\beta_2} - 2\epsilon + 1)^2 - 4e^2e^{2\beta_2}(e^{2\beta_2} - 1)}}{(e^{2\beta_2} - 1)} \geq 3\epsilon - 1,$$

which can be simplified to

$$-e^{2\beta_2} + e + e^{2\beta_2} \geq \sqrt{(2e^{2\beta_2} - 2\epsilon + 1)^2 - 4e^2e^{2\beta_2}(e^{2\beta_2} - 1)} \quad \text{if } \beta_2 > 0,$$

$$-e^{2\beta_2} + e + e^{2\beta_2} \leq \sqrt{(2e^{2\beta_2} - 2\epsilon + 1)^2 - 4e^2e^{2\beta_2}(e^{2\beta_2} - 1)} \quad \text{if } \beta_2 < 0,$$

and can be further simplified to

$$(1 - \epsilon)e^{4\beta_2} - 2\epsilon e^{2\beta_2} + (3\epsilon - 1) \geq 0 \quad \text{if } \beta_2 > 0,$$

$$(1 - \epsilon)e^{4\beta_2} - 2\epsilon e^{2\beta_2} + (3\epsilon - 1) \leq 0 \quad \text{if } \beta_2 < 0,$$

or alternatively

$$\epsilon \leq \frac{e^{2\beta_2} + 1}{e^{2\beta_2} + 3}. $$
Figure 3. On the left hand side, we have the contour plot of the limiting conditional free energy density \( \psi(\beta_1, r) \) obtained from Theorem 10. On the right hand side, we specify the regions of monotonicity as obtained in Remark 11. \( \psi \) is always increasing in \( \beta_1 \). In region \( _+^− \), \( \psi \) is decreasing in \( r \) and in region \( _+^+ \), \( \psi \) is increasing in \( r \). The boundary is specified in Remark 11.

Similarly,

\[
\frac{\partial \psi(\beta_1, r)}{\partial r} = \left[ \beta_1 + \frac{\partial \psi(e^*, r)}{\partial e^*} \right] \frac{\partial e^*}{\partial r} + \frac{\partial \psi(e^*, r)}{\partial r}
\]

(3.23)

Therefore \( \psi(\beta_1, r) \) is increasing in \( r \) if and only if \( r \leq (e^*)^2 \). This is equivalent to

\[
\sqrt{r} \leq \frac{e^{\beta_1}(1 + r) + r}{2e^{\beta_1} + 1},
\]

(3.24)

or alternatively

\[
\beta_1 \geq \log \left( \frac{\sqrt{r}}{1 - \sqrt{r}} \right).
\]

(3.25)

4. Another look at the grand canonical ensemble

Similar to the analysis in Yin and Zhu [37], we can also study directed graphs where the parameters may depend on the number of vertices. Assume that \( \beta_1^{(n)} = \beta_1 \alpha_n \) and \( \beta_2^{(n)} = \beta_2 \alpha_n \), where \( \beta_1 \) and \( \beta_2 \) are fixed, \( \alpha_n \) is positive and \( \alpha_n \to \infty \) as \( n \to \infty \). With some abuse of notation, we will still denote the associated normalization constant and probability measure by \( Z_n(\beta_1, \beta_2) \) and \( P_n^{\beta_1, \beta_2} \) respectively. From the proof of Theorem 1

\[
Z_n(\beta_1, \beta_2)^{1/n} = 1 + 2e^{\beta_1(n)} + e^{2\beta_1(n) + 2\beta_2(n)},
\]

(4.1)

which yields the following optimal asymptotics for the normalization constant.

**Proposition 12.** (i) When \( \beta_1 < 0 \) and \( \beta_1 + \beta_2 < 0 \), \( \lim_{n \to \infty} (Z_n(\beta_1, \beta_2))^{1/n} = 1 \).
(ii) When \( \beta_1 < 0 \) and \( \beta_1 + \beta_2 = 0 \), \( \lim_{n \to \infty} (Z_n(\beta_1, \beta_2))^{1/n} = \sqrt{2} \).
(iii) When \( \beta_1 \leq 0 \) and \( \beta_1 + \beta_2 > 0 \), \( \lim_{n \to \infty} \frac{(Z_n(\beta_1, \beta_2))^{1/n}}{e^{\alpha_n(\beta_1 + \beta_2)}} = 1 \).
(iv) When $\beta_1 = 0$ and $\beta_2 < 0$, $\lim_{n \to \infty} (Z_n(\beta_1, \beta_2))^{\frac{1}{n}} = \sqrt{3}$.

(v) When $\beta_1 = 0$ and $\beta_2 = 0$, $\lim_{n \to \infty} (Z_n(\beta_1, \beta_2))^{\frac{1}{n}} = 2$.

(vi) When $\beta_1 > 0$ and $\beta_1 + 2\beta_2 < 0$, $\lim_{n \to \infty} (Z_n(\beta_1, \beta_2))^{\frac{1}{n}} = \sqrt{2}$.

(vii) When $\beta_1 > 0$ and $\beta_1 + 2\beta_2 = 0$, $\lim_{n \to \infty} (Z_n(\beta_1, \beta_2))^{\frac{1}{n}} = \sqrt{3}$.

(viii) When $\beta_1 > 0$ and $\beta_1 + 2\beta_2 > 0$, $\lim_{n \to \infty} (Z_n(\beta_1, \beta_2))^{\frac{1}{n}} = 1$.

Since many networks data are sparse in the real world, we are more interested in the situation where a random graph sampled from this modified model is sparse, i.e., the probability that there is an edge between vertex $i$ and vertex $j$ goes to 0 as $n \to \infty$. One natural question to ask is for what set of parameters $(\beta_1, \beta_2)$ will this happen? And a natural follow-up question is what is the speed of the graph towards sparsity when this indeed happens? We now give some concrete answers to these questions.

**Proposition 13.** For any $i \neq 1$,

(i) When $\beta_1 < 2(\beta_1 + \beta_2) < 0$, $\lim_{n \to \infty} \frac{P_{n}^{\beta_1, \beta_2}(X_{1i} = 1)}{e^{\alpha n} \beta_{1}^{\beta_{1}}(X_{1i} = 1)} = \frac{1}{4}$.

(ii) When $2(\beta_1 + \beta_2) < \beta_1 < 0$, $\lim_{n \to \infty} \frac{P_{n}^{\beta_1, \beta_2}(X_{1i} = 1)}{e^{\alpha n} \beta_{1}^{\beta_{1}}(X_{1i} = 1)} = \frac{1}{2}$.

(iii) When $\beta_1 = 2(\beta_1 + \beta_2) < 0$, $\lim_{n \to \infty} \frac{P_{n}^{\beta_1, \beta_2}(X_{1i} = 1)}{e^{\alpha n} \beta_{1}^{\beta_{1}}(X_{1i} = 1)} = 1$.

**Proof of Proposition 13**

\[
\Pr_{n}^{\beta_1, \beta_2}(X_{1i} = 1) = \mathbb{E}\left[ X_{1i} e^{\beta_1 \alpha_n \sum_{1 \leq i, j \leq n} X_{ij} + \beta_2 \alpha_n \sum_{1 \leq i, j \leq n} X_{ij} X_{ji}} \right]
= \mathbb{E}\left[ e^{\beta_1 \alpha_n \sum_{1 \leq i, j \leq n} X_{ij}} e^{\beta_2 \alpha_n \sum_{1 \leq i, j \leq n} X_{ij} X_{ji}} \right]
= \mathbb{E}\left[ e^{\beta_1 \alpha_n \sum_{1 \leq i, j \leq n} X_{ij} + 2\beta_2 \alpha_n \sum_{1 \leq i, j \leq n} X_{ij} X_{ji}} \right]
= \mathbb{E}\left[ e^{\beta_1 \alpha_n (X_{1i} + X_{ii}) + 2\beta_2 \alpha_n X_{1i} X_{1i}} \right]
= e^{\beta_1 \alpha_n + 2\beta_2 \alpha_n X_{1i} X_{1i}}
= 1 + 2e^{\beta_1 \alpha_n + 2\beta_2 \alpha_n X_{1i} X_{1i}}.
\]

Therefore, when $\beta_1 < 0$ and $\beta_1 + \beta_2 < 0$, $\Pr_{n}^{\beta_1, \beta_2}(X_{1i} = 1) \to 0$ as $n \to \infty$. The rest of the proof easily follows. \[\square\]
Remark 14. For any \( i \neq j \) and \( i, j \neq 1 \),
\[
\mathbb{P}_n^{\beta_1, \beta_2}(X_{1i} = 1, X_{1j} = 1)
\begin{align*}
&= \mathbb{E}[X_{1i}X_{1j}e^{\beta_1\alpha_n \sum_{1 \leq i, j \leq n} X_{ij} + \beta_2\alpha_n \sum_{1 \leq i, j \leq n} X_{ij}X_{ji}}] \\
&= \frac{\mathbb{E}[X_{1i}X_{1j}e^{\beta_1\alpha_n \sum_{1 \leq i, j \leq n} X_{ij} + \beta_2\alpha_n \sum_{1 \leq i, j \leq n} X_{ij}X_{ji}}]}{\mathbb{E}[e^{\beta_1\alpha_n \sum_{1 \leq i, j \leq n} X_{ij} + \beta_2\alpha_n \sum_{1 \leq i, j \leq n} X_{ij}X_{ji}}]} \\
&= \frac{\mathbb{E}[e^{\beta_1\alpha_n (X_{1i} + X_{1j}) + 2\beta_2\alpha_n \sum_{1 \leq i, j \leq n} X_{ij}X_{ji}}]}{\mathbb{E}[e^{\beta_1\alpha_n (X_{1i} + X_{1j}) + 2\beta_2\alpha_n X_{ij}X_{ji}}]} \\
&= \mathbb{P}_n^{\beta_1, \beta_2}(X_{1i} = 1)\mathbb{P}_n^{\beta_1, \beta_2}(X_{1j} = 1).
\end{align*}
\]

5. Further Discussions

We have studied directed graphs whose sufficient statistics are given by edge and reciprocal densities. Now let us generalize these ideas and analyze directed graphs whose sufficient statistics also include densities of \textit{reciprocal p-stars} and \textit{reciprocal triangles}. Reciprocal triangles are sometimes called \textit{cyclic triads} in the literature, see e.g. Robins et al. [31]. They are used to model the situation where you have three vertices \( i, j \) and \( k \) and there are bilateral relations between \( i \) and \( j \) and \( k \), and \( i \), \( j \), \( k \), i.e., \( X_{ij} = X_{ji} = X_{jk} = X_{kj} = X_{ki} = X_{ik} = 1 \). Similarly, reciprocal p-stars have generated significant interest as well. We define the densities of reciprocal triangles and reciprocal p-stars respectively as
\[
t(X) := \frac{1}{n^3} \sum_{1 \leq i, j, k \leq n} X_{ij}X_{ji}X_{jk}X_{kj}X_{ki}X_{ik} \tag{5.1}
\]
and
\[
s(X) := \frac{1}{n^{p+1}} \sum_{i=1}^{n} \left( \sum_{j=1}^{n} X_{ij}X_{ji} \right)^p. \tag{5.2}
\]

As for the less complicated model investigated earlier, we are interested in the \textit{limiting free energy density}
\[
\psi(\beta_1, \beta_2, \beta_3, \beta_4) := \lim_{n \to \infty} \frac{1}{n^2} \log 2^n \mathbb{E}[e^{n^2(\beta_1 \epsilon(X) + \beta_2 r(X) + \beta_3 t(X) + \beta_4 s(X))}] \tag{5.3}
\]
for the grand canonical ensemble and the \textit{limiting entropy density}
\[
\psi(\epsilon, r, t, s) := \lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n^2} \log \mathbb{P}(X \in B_\delta(\epsilon), r(X) \in B_\delta(r), t(X) \in B_\delta(t), s(X) \in B_\delta(s)) \tag{5.4}
\]
for the microcanonical ensemble, where \( B_\delta(x) := \{ y : |y - x| < \delta \} \). The key observation here is that we can define \( Z_{ij} = Z_{ji} = X_{ij}X_{ji} \) so that \( Z_{ij} \) are iid random variables with \( \mathbb{P}(Z_{ij} = 1) = \frac{1}{4} \) and \( \mathbb{P}(Z_{ij} = 0) = \frac{3}{4} \). Then densities of reciprocal edges, reciprocal triangles, and reciprocal p-stars may be alternatively
written as
\[ r(X) = \frac{1}{n^2} \sum_{1 \leq i,j \leq n} Z_{ij}, \quad (5.5) \]
\[ t(X) = \frac{1}{n^3} \sum_{1 \leq i,j,k \leq n} Z_{ij} Z_{jk} Z_{ki}, \]
\[ s(X) = \frac{1}{n^{p+1}} \sum_{i=1}^{n} \left( \sum_{j=1}^{n} Z_{ij} \right)^p. \]

Using Chatterjee and Varadhan’s large deviations results for the Erdős-Rényi random graph, see e.g. Chatterjee and Varadhan [7], this gives
\[ \lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n^2} \log \mathbb{P}(r(X) \in B_\delta(r), t(X) \in B_\delta(t), s(X) \in B_\delta(s)) \quad (5.6) \]
\[ = -\inf_{g: [0,1]^2 \to [0,1], g(x,y)=g(y,x), r(g)=r, t(g)=t, s(g)=s} \frac{1}{2} I_\frac{1}{4}(g), \]
where
\[ r(g) := \int_{[0,1]^2} g(x,y) dxdy, \quad (5.7) \]
\[ t(g) := \int_{[0,1]^3} g(x,y) g(y,z) g(z,x) dxdydz, \]
\[ s(g) := \int_0^1 \left( \int_0^1 g(x,y) dy \right)^p dx, \]
and \[ I_\frac{1}{4}(g) := \int_{[0,1]^2} I_\frac{1}{4}(g(x,y)) dxdy, \]
where
\[ I_\frac{1}{4}(x) : = x \log \left( \frac{x}{1/4} \right) + (1-x) \log \left( \frac{1-x}{1-1/4} \right) \]
\[ = x \log 3 + x \log x + (1-x) \log(1-x) - \log(3/4). \]

We examine the limiting entropy density \((5.4)\) first. Another key observation here is that the distribution of \(e(X)\) conditional on \((Z_{ij})_{1 \leq i,j \leq n}\) is the same as conditional on \(r(X)\). Thus, the distribution of \(e(X)\) conditional on \(r(X), t(X), s(X)\) is the same as conditional on \(r(X)\). We compute
\[ \lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n^2} \log \mathbb{P}(r(X) \in B_\delta(r)) \quad (5.9) \]
\[ = -\frac{1}{2} r \log \left( \frac{r}{1/4} \right) - \frac{1}{2} (1-r) \log \left( \frac{1-r}{1-1/4} \right), \]
which, combined with \((2.8)\), implies that
\[ \phi(\epsilon, r) := \lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n^2} \log \mathbb{P}(e(X) \in B_\delta(\epsilon) | r(X) \in B_\delta(r)) \quad (5.10) \]
\[ = -\epsilon \log \left( \frac{\epsilon - r}{1 + r - 2\epsilon} \right) - \frac{r}{2} \log \left( \frac{(1-r)(1+r-2\epsilon)}{3(\epsilon - r)^2} \right) \]
\[ + \frac{1}{2} \log \left( \frac{1-r}{3(1+r-2\epsilon)} \right). \]
Together with (5.6), we hence conclude that
\[
\psi(\epsilon, r, t, s) = \phi(\epsilon, r) - \inf_{g: [0,1]^2 \rightarrow [0,1], g(x,y) = g(y,x)} \frac{1}{2} I_{\frac{1}{2}} (g). \tag{5.11}
\]

Next we examine the limiting free energy density (5.3). From Varadhan’s lemma in large deviations theory, see e.g. Dembo and Zeitouni [9],
\[
\psi(\beta_1, \beta_2, \beta_3, \beta_4)
\]
\[
= \sup_{0 \leq \epsilon, r, t, s \leq 1} \left\{ \beta_1 \epsilon + \beta_2 r + \beta_3 t + \beta_4 s + \psi(\epsilon, r, t, s) \right\} + \log 2
\]
\[
= \sup_{0 \leq \epsilon, r, t, s \leq 1} \left\{ \beta_1 \epsilon + \beta_2 r + \beta_3 t + \beta_4 s + \phi(\epsilon, r) - \frac{1}{2} \left( I_{\frac{1}{2}} (g) - 2 \log 2 \right) \right\}.
\tag{5.12}
\]

Consider the optimization problem \( \eta(\beta_1, r) := \sup_{0 \leq \epsilon \leq 1} \{ \beta_1 \epsilon + \phi(\epsilon, r) \} \). Note that at optimality
\[
\frac{\partial \phi(\epsilon, r)}{\partial \epsilon} = \frac{\partial \psi(\epsilon, r)}{\partial \epsilon} = -\log \left( \frac{\epsilon - r}{1 + r - 2 \epsilon} \right) = -\beta_1,
\tag{5.13}
\]
which implies that \( \epsilon = \frac{1+r+e^{\beta_1} r}{1+2 e^{\beta_1}} \). Therefore, we have
\[
\eta(\beta_1, r) = \beta_1 \frac{1+r+e^{\beta_1} r}{1+2 e^{\beta_1}} + \phi \left( \frac{1+r+e^{\beta_1} r}{1+2 e^{\beta_1}}, r \right),
\tag{5.14}
\]
and hence
\[
\psi(\beta_1, \beta_2, \beta_3, \beta_4)
\]
\[
= \sup_{0 \leq \epsilon, r, t, s \leq 1} \left\{ \eta(\beta_1, r) + \beta_2 r + \beta_3 t + \beta_4 s - \frac{1}{2} \left( I_{\frac{1}{2}} (g) - 2 \log 2 \right) \right\}
\]
\[
= \sup_{g: [0,1]^2 \rightarrow [0,1], g(x,y)=g(y,x)} \left\{ \eta(\beta_1, r(g)) + \beta_2 r(g) + \beta_3 t(g) + \beta_4 s(g) - \frac{1}{2} \left( I_{\frac{1}{2}} (g) - 2 \log 2 \right) \right\}.
\tag{5.15}
\]

This is a complicated variational problem that is hard to solve in general, however we can proceed further in two special situations.

The first special situation is when \( \beta_1 = 0 \),
\[
\psi(0, \beta_2, \beta_3, \beta_4)
\]
\[
= \sup_{g: [0,1]^2 \rightarrow [0,1], g(x,y)=g(y,x)} \left\{ \beta_2 r(g) + \beta_3 t(g) + \beta_4 s(g) - \frac{1}{2} \left( I_{\frac{1}{2}} (g) - 2 \log 2 \right) \right\}
\]
\[
= \sup_{g: [0,1]^2 \rightarrow [0,1], g(x,y)=g(y,x)} \left\{ \left( \beta_2 - \frac{\log 3}{2} \right) r(g) + \beta_3 t(g) + \beta_4 s(g) - \frac{1}{2} I_0 (g) \right\} + \frac{1}{2} \log 3,
\]
where \( I_0 (g) := \int_{[0,1]^2} I_0 (g(x,y)) \, dx \, dy \) and \( I_0 (x) := x \log x + (1-x) \log (1-x) \). This shows that \( \psi(0, \beta_2, \beta_3, \beta_4) \) may be equivalently viewed as the limiting free energy density of an undirected model whose sufficient statistics are given by (undirected) edge, triangle, and p-star densities. The 3 parameters \( \beta_2, \beta_3, \beta_4 \) allow one to adjust
the influence of these different local features on the limiting probability distribution and thus expectedly should impact the global structure of a random graph drawn from the model. It is therefore important to understand if and when the supremum in (5.16) is attained and whether it is unique. Many people have delved into this area. A particularly significant discovery was made by Chatterjee and Diaconis [6], who showed that the supremum in (5.16) is always attained and a random graph drawn from the model must lie close to the maximizing set with probability vanishing in $n$. When $\beta_3, \beta_4 \geq 0$, Yin [35] further showed that the 3-parameter space would consist of a single phase with first order phase transition(s) across one (or more) surfaces, where all the first derivatives of $\psi$ exhibit (jump) discontinuities, and second order phase transition(s) along one (or more) critical curves, where all the second derivatives of $\psi$ diverge.

The second special situation is when $\beta_3 = 0$,

$$\psi(\beta_1, \beta_2, 0, \beta_4) = \sup_{g: [0,1]^2 \rightarrow [0,1]} \left\{ \eta(\beta_1, r(g)) + \beta_2 r(g) + \beta_4 s(g) - \frac{1}{2} \left( I_4(g) - 2 \log 2 \right) \right\}. \quad (5.17)$$

We can derive the Euler-Lagrange equation for this variational problem, and it is given by

$$\frac{\partial \eta}{\partial r}(\beta_1, r(g)) + 2\beta_2 + \beta_4 pd(x)^{p-1} + \beta_4 pd(y)^{p-1} = \log \left( \frac{g(x, y)}{1 - g(x, y)} \right) + \log 3, \quad (5.18)$$

where $d(x) := \int_0^1 g(x, y) dy$. Solving for $g(x, y)$ and then integrating over $y$, we get

$$d(x) = \int_0^1 \frac{dy}{1 + 3e^{-2\frac{\eta}{\beta_1} (\beta_1, r(g)) - 2\beta_2 - \beta_4 pd(x)^{p-1} - \beta_4 pd(y)^{p-1}}}. \quad (5.19)$$

Following similar arguments as in Kenyon et al. [17], we conclude that $d(x)$ can take only finitely many values, and hence the optimal graphon $g$ is multipodal.

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