DAGGER COMPLETIONS AND
BORNOLOGICAL TORSION-FREENESS

RALF MEYER AND DEVARSHI MUKHERJEE

Abstract. We define a dagger algebra as a bornological algebra over a discrete valuation ring with three properties that are typical of Monsky–Washnitzer algebras, namely, completeness, bornological torsion-freeness and a certain spectral radius condition. We study inheritance properties of the three properties that define a dagger algebra. We describe dagger completions of bornological algebras in general and compute some noncommutative examples.

1. Introduction

In [6], Monsky and Washnitzer introduce a cohomology theory for non-singular varieties defined over a field $k$ of nonzero characteristic. Let $V$ be a discrete valuation ring with residue field $k = V/\pi V$, such that the fraction field $K$ of $V$ has characteristic 0. Let $\pi \in V$ be a uniformiser. Monsky and Washnitzer lift the coordinate ring of a smooth affine variety $X$ over $k$ to a smooth commutative algebra $A$ over $V$. The dagger completion $A^\dagger$ of $A$ is a certain subalgebra of the $\pi$-adic completion of $A$. If $A$ is the polynomial algebra over $V$, then $A^\dagger$ is the ring of overconvergent power series. The Monsky–Washnitzer cohomology is defined as the de Rham cohomology of the algebra $K \otimes V A^\dagger$.

The dagger completion is interpreted in [4] in the setting of bornological algebras, based on considerations about the joint spectral radius of bounded subsets. The main achievement in [4] is the construction of a chain complex that computes the rigid cohomology of the original variety $X$ and that is strictly functorial. In addition, this chain complex is related to periodic cyclic homology. Here we continue the study of dagger completions. We define dagger algebras by adding a bornological torsion-freeness condition to the completeness and spectral radius conditions already present in [4]. We also show that the category of dagger algebras is closed under extensions, subalgebras, and certain quotients, by showing that all three properties that define them are hereditary for these constructions.

The results in this article should help to reach the following important goal: define an analytic cyclic cohomology theory for algebras over the finite field $k$ that specialises to Monsky–Washnitzer or rigid cohomology for the coordinate rings of smooth varieties over $k$. A general machine for defining such cyclic cohomology theories is developed in [5]. It is based on a class of nilpotent algebras, which must be closed under extensions. This is why we are particularly interested in properties hereditary for extensions.

If $S$ is a bounded subset of a $K$-algebra $A$, then its spectral radius $\rho(S) \in [0, \infty]$ is defined in [4]. If $A$ is a bornological $V$-algebra, then only the inequalities $\rho(S) \leq s$ for $s > 1$ make sense. This suffices, however, to characterise the linear growth bornology on a bornological $V$-algebra: it is the smallest $V$-algebra bornology with $\rho(S) \leq 1$ for all its bounded subsets $S$. We call a bornological algebra $A$ with this property semi-dagger because this is the main feature of dagger algebras. Any bornological algebra $A$ carries a smallest bornology with linear growth. This defines a semi-dagger algebra $A_{lg}$. If $A$ is a torsion-free, finitely generated, commutative
V-algebra with the fine bornology, then the bornological completion $\overline{A}_{lg}$ of $A_{lg}$ is the Monsky–Washnitzer completion of $A$.

Any algebra over $k$ is also an algebra over $V$. Equipped with the fine bornology, it is complete and semi-dagger. We prefer, however, not to call such algebras “dagger algebras.” The feature of Monsky–Washnitzer algebras that they lack is bornological embedding bounded if $f \in M$. Thus $\overline{A}_{lg}$ is a bornological embedding. This property is very important. On the one hand, we must keep working with modules over $V$ in order to keep the original algebra over $k$ in sight and because the linear growth bornology only makes sense for algebras over $V$. On the other hand, we often need to pass to the $K$-vector space $K \otimes_V M$ – this is how de Rham cohomology is defined. Bornological vector spaces over $K$ have been used recently to do analytic geometry in [1, 3]. The spectral radius of a bounded subset of a bornological $V$-algebra $A$ is defined in [4] by working in $K \otimes_V A$, which only works well if $A$ is bornologically torsion-free. Here we define a truncated spectral radius in $[1, \infty]$ without reference to $K \otimes_V A$, in order to define semi-dagger algebras independently of torsion issues.

We prove that the properties of being complete, semi-dagger, or bornologically torsion-free are hereditary for extensions. Hence an extension of dagger algebras is again a dagger algebra.

To illustrate our theory, we describe the dagger completions of monoid algebras and crossed products. Dagger completions of monoid algebras are straightforward generalisations of Monsky–Washnitzer completions of polynomial algebras.

2. Basic notions

In this section, we recall some basic notions on bornological modules and bounded linear maps between them. See [4] for more details. We also study the inheritance properties of separatedness and completeness for submodules, quotients and extensions.

Let $V$ be a complete discrete valuation ring. A bornological $V$-module $M$ is a $V$-module with a bornology $B_M$ such that every bounded subset is contained in a bounded submodule. We always write $B_M$ for the bornology on $M$.

Let $M' \subseteq M$ be a $V$-submodule. The subspace bornology on $M'$ consists of all subsets of $M'$ that are bounded in $M$. The quotient bornology on $M/M'$ consists of all subsets of the form $q(S)$ with $S \in B_M$, where $q : M \to M/M'$ is the canonical projection. We always equip submodules and quotients with these canonical bornologies.

Let $M$ and $N$ be two bornological $V$-modules. A $V$-module map $f : M \to N$ is bounded if $f(S) \in B_N$ for all $S \in B_M$. The bornological $V$-modules and the bounded $V$-module maps form an additive category. The isomorphisms in this category are called bornological isomorphisms. A bounded $V$-module map $f : M \to N$ is a bornological embedding if the induced map $M \to f(M)$ is a bornological isomorphism, where $f(M) \subseteq N$ carries the subspace bornology. It is a bornological quotient map.
if the induced map $M/\ker f \to N$ is a bornological isomorphism. Equivalently, for each $T \in \mathcal{B}_N$ there is $S \in \mathcal{B}_M$ with $f(S) = T$.

An extension of bornological $V$-modules is a diagram of $V$-modules

$$M' \xrightarrow{f} M \xrightarrow{g} M''$$

that is algebraically exact and such that $f$ is a bornological embedding and $g$ a bornological quotient map. Equivalently, $g$ is a cokernel of $f$ and $f$ a kernel of $g$ in the additive category of bornological $V$-modules. A split extension is an extension with a bounded $V$-linear map $s: M'' \to M$ such that $g \circ s = \text{id}_{M''}$.

Let $M$ be a bornological $V$-module. A sequence $(x_n)_{n \in \mathbb{N}}$ in $M$ converges towards $x \in M$ if there are $S \in \mathcal{B}_M$ and a sequence $(\delta_n)_{n \in \mathbb{N}}$ in $V$ with $\lim \delta_n = 0$ and $x_n - x \in \delta_n \cdot S$ for all $n \in \mathbb{N}$. It is a Cauchy sequence if there are $S \in \mathcal{B}_M$ and a sequence $(\delta_n)_{n \in \mathbb{N}}$ in $V$ with $\lim \delta_n = 0$ and $x_n - x_m \in \delta_j \cdot S$ for all $n, m, j \in \mathbb{N}$ with $n, m \geq j$.

We call a subset $S$ of $M$ closed if $x \in S$ for any sequence in $S$ that converges in $M$ to $x \in M$. These are the closed subsets of a topology on $M$. Bounded maps preserve convergent sequences and Cauchy sequences. Thus they are continuous for these canonical topologies.

2.1. Separated bornological modules. We call $M$ separated if limits of convergent sequences in $M$ are unique. If $M$ is not separated, then the constant sequence $0$ has a non-zero limit. Therefore, $M$ is separated if and only if $\{0\} \subseteq M$ is closed. And $M$ is separated if and only if any $S \in \mathcal{B}_M$ is contained in a $\pi$-adically separated bounded $V$-submodule.

**Lemma 2.1.** Let $M' \xrightarrow{f} M \xrightarrow{g} M''$ be an extension of bornological $V$-modules.

1. If $M$ is separated, so is $M'$.
2. The quotient $M''$ is separated if and only if $f(M')$ is closed in $M$.
3. If $M'$ and $M''$ are separated and $M''$ is torsion-free, then $M$ is separated.

**Proof.** Assertion (1) is trivial.

If $M''$ is separated, then $\{0\} \subseteq M''$ is closed. Hence $g^{-1}(\{0\}) = f(M')$ is closed in $M$. If $M''$ is not separated, then the constant sequence $0$ in $M''$ converges to some non-zero $x'' \in M''$. That is, there are a bounded subset $S'' \subseteq M''$ and a null sequence $(\delta_n)_{n \in \mathbb{N}}$ in $V$ with $x'' - 0 \in \delta_n \cdot S''$ for all $n \in \mathbb{N}$. Since $g$ is a bornological quotient map, there are $x \in M$ and $S \in \mathcal{B}_M$ with $g(x) = x''$ and $g(S) = S''$. We may choose $y'' \in S''$ with $x'' = \delta_n \cdot y''$ and $y_n \in S$ with $y_n = y''$. So $g(x - \delta_n y_n) = 0$. Thus the sequence $(x - \delta_n y_n)$ lies in $f(M')$. It converges to $x$, which does not belong to $f(M')$ because $x'' \neq 0$. So $f(M')$ is not closed. This finishes the proof of (2).

We prove (3). Let $x \in M$ belong to the closure of $\{0\}$ in $M$. That is, there are $S \in \mathcal{B}_M$ and a null sequence $(\delta_n)_{n \in \mathbb{N}}$ in $V$ with $x \in \delta_n \cdot S$ for all $n \in \mathbb{N}$. Then $g(x) \in \delta_n \cdot g(S)$ for all $n \in \mathbb{N}$. This implies $g(x) = 0$ because $M''$ is separated. So there is $y \in M'$ with $f(y) = x$. And $f(y) = x \in \delta_n \cdot S$. Choose $x_n \in S$ with $f(y) = \delta_n \cdot x_n$. We may assume $\delta_n \neq 0$ for all $n \in \mathbb{N}$ because otherwise $x \in S$ is 0. Since $M''$ is torsion-free, $\delta_n \cdot x_n \in f(M')$ implies $g(x_n) = 0$. So we may write $x_n = f(y_n)$ for some $y_n \in M'$. Since $f$ is a bornological embedding, the set $\{y_n : n \in \mathbb{N}\}$ in $M'$ is bounded. Since $M'$ is separated and $y = \delta_n \cdot y_n$, we get $y = 0$. Hence $x = 0$. So $\{0\}$ is closed in $M$. □

The quotient $M/\{0\}$ of a bornological $V$-module $M$ by the closure of $0$ is called the separated quotient of $M$. It is separated by Lemma 2.1, and it is the largest separated quotient of $M$. Even more, the quotient map $M \to M/\{0\}$ is the universal arrow to a separated bornological $V$-module, that is, any bounded $V$-linear map from $M$ to a separated bornological $V$-module factors uniquely through $M/\{0\}$. 

The following example shows that Lemma 2.1 (3) fails without the torsion-freeness assumption.

Example 2.2. Let $M' = V$ and let $M = V[x]/S$, where $S$ is the $V$-submodule of $M$ generated by $1 - \pi^n x^n$ for all $n \in \mathbb{N}$. We embed $M' = V$ as multiples of $1 = x^0$. Then

$$M/M' = \bigoplus_{n=1}^{\infty} V/(\pi^n),$$

We endow $M$, $M'$ and $M/M'$ with the bornologies where all subsets are bounded. We get an extension of bornological $V$-modules $V \to M \to \bigoplus_{n=1}^{\infty} V/(\pi^n)$. Here $V$ and $\bigoplus_{n=1}^{\infty} V/(\pi^n)$ are $\pi$-adically separated, but $M$ is not: the constant sequence 1 in $M$ converges to 0 because $1 = 1 - \pi^n x^n + \pi^n x^n \equiv \pi^n x^n$ in $M$.

2.2. Completeness. We call a bornological $V$-module $M$ complete if it is separated and for any $S \in B_M$ there is $T \in B_M$ so that all $S$-Cauchy sequences are $T$-convergent. Equivalently, any $S \in B_M$ is contained in a $\pi$-adically complete bounded $V$-submodule (see [4]). By definition, any Cauchy sequence in a complete bornological $V$-module has a unique limit.

Theorem 2.3. Let $M' \xrightarrow{f} M \xrightarrow{g} M''$ be an extension of bornological $V$-modules.

1. If $M$ is complete and $f(M')$ is closed in $M$, then $M'$ is complete.
2. If $M'$ is complete, $M$ separated, and $M''$ torsion-free, then $f(M')$ is closed in $M$.
3. Let $M$ be complete. Then $M''$ is complete if and only if $f(M')$ is closed in $M$.
4. If $M'$ and $M''$ are complete and $M$ is separated, then $M$ is complete. If $M'$ and $M''$ are complete and $M'$ is torsion-free, then $M$ is complete.

Proof. Statement (1) is [4, Lemma 2.3], and there is no need to repeat the proof here. It is somewhat similar to the proof of (4). Next we prove (2). Assume that $M'$ is complete, that $M''$ is torsion-free, and that $f(M')$ is not closed in $M$. We are going to prove that $M$ is not separated. There is a sequence $(x_n)_{n \in \mathbb{N}}$ in $M'$ for which $f(x_n)_{n \in \mathbb{N}}$ converges in $M$ towards some $x \notin f(M')$. So there is a bounded set $S \subseteq M$ and a sequence $(\delta_k)_{k \in \mathbb{N}}$ in $V$ with $\delta_k = 0$ and $f(x_n) - x \in \delta_n \cdot S$ for all $n \in \mathbb{N}$. We may assume without loss of generality that the sequence of norms $|\delta_n|$ is decreasing: let $\delta^*_n$ be the $\delta_n$ for $m \geq n$ with maximal norm. Then $f(x_n) - x \in \delta_n \cdot S \subseteq \delta^*_n \cdot S$ and still $\delta_n^* = 0$. We may write $f(x_n) - x = \delta^*_n y_n$ for $y_n \in S$. If $m < n$, then $\delta^*_m g(y_m) = -g(x) = \delta^*_n g(y_n)$ and hence $\delta^*_m g(y_m) = g(y_n) \delta^*_n / \delta^*_m = 0$. Since $M''$ is torsion-free, this implies $g(y_m) = g(y_n) \delta^*_n / \delta^*_m$ for all $n > m$. So there is $z_{m,n} \in M'$ with $y_m + f(z_{m,n}) = y_n \delta^*_m / \delta^*_n$. We even have $z_{m,n} \in f^{-1}(S)$, which is bounded because $f$ is a bornological embedding. We get $f(x_n) - f(x_m) = \delta^*_n y_n - \delta^*_m y_m = f(\delta^*_m z_{m,n})$ and hence $x_n - x_m = \delta^*_n z_{m,n}$ for $n > m$. This witnesses that the sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy in $M'$. Since $M'$ is complete, it converges towards some $y \in M'$. Then $f(x_n)$ converges both towards $f(y) \in f(M')$ and towards $x \notin f(M')$. So $M$ is not separated. This finishes the proof of (2).

Next we prove (3) if $f(M')$ is not closed, then Lemma 2.1 shows that $M''$ is not separated and hence not complete. Conversely, we claim that $M''$ is complete if $M'(M')$ is closed. Lemma 2.1 shows that $M''$ is separated. Let $S'' \in B_{M''}$. There is $S \in B_M$ with $g(S) = S''$ because $g$ is a bornological quotient map. And there is $T \in B_M$ so that any $S$-Cauchy sequence is $T$-convergent. We claim that any $S''$-Cauchy sequence is $g(T)$-convergent. So let $(x'_n)_{n \in \mathbb{N}}$ be an $S''$-Cauchy sequence. Thus there is a null sequence $(\delta_n)_{n \in \mathbb{N}}$ in $V$ with $x'_n - x'_m \in \delta_j \cdot S''$ for all $n, m, j \in \mathbb{N}$ with $n, m \geq j$. As above, we may assume without loss of generality that the sequence
of norms $|\delta_n|$ is decreasing. Choose any $x_0 \in M$ with $g(x_0) = x'_0$. For each $n \in \mathbb{N}$, choose $y_n \in S$ with $x''_{n+1} - x''_n = \delta_n \cdot g(y_n)$. Let

$$x_n := x_0 + \delta_0 \cdot y_0 + \cdots + \delta_{n-1} \cdot y_{n-1}.$$ 

Then $g(x_n) = x''_n$. And $x_{n+1} - x_n = \delta_n \cdot y_n \in \delta_n \cdot S$. Since $|\delta_n|$ is decreasing, this implies $x_m - x_n \in \delta_n \cdot S$ for all $m \geq n$. So the sequence $(x_n)_{n \in \mathbb{N}}$ is S-Cauchy. Hence it is $T$-convergent. Thus $g(x_n) = x''_n$ is $g(T)$-convergent as asserted. This finishes the proof of (3).

Finally, we prove (4). So we assume $M'$ and $M''$ to be complete. If $M''$ is torsion-free, then $M$ is separated by Lemma 2.1. Hence the second statement in (4) is a special case of the first one. Let $S \in \mathcal{B}_M$. We must find $T \in \mathcal{B}_M$ so that every $S$-Cauchy sequence is $T$-convergent. Since $M$ is separated, this says that it is complete. Since $M''$ is complete, there is a $\pi$-adically complete $V$-submodule $T_0 \in \mathcal{B}_{M''}$ that contains $g(S)$. Since $g$ is a bornological quotient map, there is $T_1 \in \mathcal{B}_M$ with $g(T_1) = T_0$. Replacing it by $T_1 + S$, we may arrange, in addition, that $S \subseteq T_1$. Since $f$ is a bornological embedding, $T_2 := f^{-1}(T_1)$ is bounded in $M'$. As $M'$ is complete, there is $T_3 \in \mathcal{B}_{M'}$ so that every $T_2$-Cauchy sequence is $T_3$-convergent. We claim that any $S$-Cauchy sequence is $T_1 + f(T_3)$-convergent. The proof of this claim will finish the proof of the theorem.

Let $(x_n)_{n \in \mathbb{N}}$ be an $S$-Cauchy sequence. So there are $\delta_n \in V$ and $y_n \in S$ with $\lim |\delta_n| = 0$ and $x_{n+1} - x_n = \delta_n \cdot y_n$. As above, we may assume that $|\delta_n|$ is decreasing and that $\delta_0 = 1$. Since $g(y_{n+k}) \in g(S) \subseteq T_0$ and $T_0$ is $\pi$-adically complete, the following series converges in $T_0$:

$$\bar{w}_n := -\sum_{k=0}^{\infty} \frac{\delta_n k}{\delta_n} g(y_{n+k}).$$

Since $\bar{w}_n \in T_0$, there is $w_n \in T_1$ with $g(w_n) = \bar{w}_n$. So

$$\delta_k g(w_k) = \lim_{N \to \infty} \frac{g(\sum_{k=0}^{N} \delta_{n+k} y_{n+k})}{g(\sum_{k=0}^{N} \delta_{n+k} y_{n+k})} = \lim_{N \to \infty} g(x_k - x_{N+k+1}) = g(x_k) - g(x_{N+k}) = \lim_{N \to \infty} g(x_N) - g(x_0).$$

In particular, $g(w_0) = g(x_0) - \lim_{N \to \infty} g(x_N)$. Now let

$$\tilde{x}_k := x_k - \delta_k w_k + w_0 - x_0.$$ 

Then

$$g(\tilde{x}_k) = g(x_k) - g(x_k) + \lim_{N \to \infty} g(x_N) + g(x_0) - \lim_{N \to \infty} g(x_N) - g(x_0) = 0.$$ 

So $\tilde{x}_k \in f(M')$ for all $k \in \mathbb{N}$.

(2.5) $\tilde{x}_{n+1} - \tilde{x}_n = x_{n+1} - x_n - \delta_{n+1} w_{n+1} + \delta_n w_n$

$$= \delta_n y_n + \delta_n w_n - \delta_{n+1} w_{n+1} + \delta_n \cdot \left(y_n + w_n - \frac{\delta_{n+1}}{\delta_n} w_{n+1}\right).$$

Let $z_n := y_n + w_n - \frac{\delta_{n+1}}{\delta_n} w_{n+1}$. A telescoping sum argument shows that

(2.6) $g(z_n) = g(y_n) + \bar{w}_n - \frac{\delta_{n+1}}{\delta_n} \bar{w}_{n+1} = 0$.

So $z_n \in f(M')$. And $z_n \in S + T_1 + T_1 = T_1$. Thus there is $\tilde{z}_n \in f^{-1}(T_1) = T_2$ with $z_n = f(\tilde{z}_n)$. Equation (2.5) means that the sequence $f^{-1}(\tilde{x}_n)$ is $T_2$-Cauchy. Hence it is $T_3$-convergent. So $(\tilde{x}_n)$ is $f(T_3)$-convergent. Then $(x_n)$ is $T_1 + f(T_3)$-convergent. □
The following examples show that the technical extra assumptions in [2] and [4] in Theorem 2.3 are necessary. They only involve extensions of $V$-modules with the bornology where all subsets are bounded. For this bornology, bornological completeness and separatedness are the same as $\pi$-adic completeness and separatedness, respectively, and any extension of $V$-modules is a bornological extension.

Example 2.7. Let $M' := \{0\}$ and $M := K$ with the bornology of all subsets. Then $M'$ is bornologically complete, but not closed in $M$, and $M/M' = M$ is torsion-free. So Theorem 2.3(2) needs the assumption that $M$ be separated.

Example 2.8. Let $M$ be the space of all power series $\sum_{n=0}^\infty c_n x^n$ with $\lim |c_n| = 0$. This is the $\pi$-adic completion of the polynomial algebra $V[x]$. Let $M' = M$ and define $f: M' \to M$, $f(\sum_{n=0}^\infty c_n x^n) := \sum_{n=0}^\infty c_n \pi^n x^n$. This is a bornological embedding simply because all subsets of $M = M'$ are bounded. Let $p_n := \sum_{j=0}^n x^j$. This sequence in $M' = M$ does not converge. Nevertheless, the sequence $f(p_n) = \sum_{j=0}^n \pi^j x^j$ converges in $M$ to $\sum_{j=0}^\infty \pi^j x^j$. Thus $f(M')$ is not closed in $M$, although $M$ and $M'$ are complete and $f$ is a bornological embedding. So Theorem 2.3(2) needs the assumption that $M'$ be bornologically complete, but not closed in $M$.

Example 2.9. We modify Example 2.2 to produce an extension of $V$-modules $N' \to N \to N''$ where $N'$ and $N''$ are $\pi$-adically complete, but $N$ is not $\pi$-adically separated and hence not $\pi$-adically complete. We let $N' := V/(\pi) = k$. We let $N''$ be the $\pi$-adic completion of the $V$-module $M''$ of Example 2.2. That is,

$$N'' := \left\{(c_n)_{n \in \mathbb{N}} \in \prod_{n=0}^\infty V/(\pi^n) : \lim |c_n| = 0 \right\}.$$ 

This is indeed $\pi$-adically complete. So is

$$N_1 := \left\{(c_n)_{n \in \mathbb{N}} \in \prod_{n=0}^\infty V/(\pi^{n+1}) : \lim |c_n| = 0 \right\}.$$ 

The kernel of the quotient map $q: N_1 \to N''$ is isomorphic to $\prod_{n=0}^\infty V/(\pi) = \prod_{n=0}^\infty k$. This is a $k$-vector space, and it contains the $k$-vector space $\sum_{n=0}^\infty k$. Since any $k$-vector space has a basis, we may extend the linear functional $\sum_{n=0}^\infty k \to k$, $(c_n)_{n \in \mathbb{N}} \mapsto \sum_{n=0}^\infty c_n$, to a $k$-linear functional $\sigma: \prod_{n} k \to k$. Let $L := \ker \sigma \subseteq \ker q$ and let $N := N_1/L$. The map $q$ descends to a surjective $\pi$-linear map $N \to N''$. Its kernel is isomorphic to $\prod_{n} k/k \ker \sigma \cong k = N'$. The functional $\sigma: \prod_{n} k \to k$ vanishes on $\delta_0 - \delta_k$ for all $k \in \mathbb{N}$, but not on $\delta_0$. When we identify $\prod_{n} k \cong \ker q$, we map $\delta_k$ to $\pi^k \delta_k \in N_1$. So $\delta_0$ and $\pi^k \delta_k$ get identified in $N$, but $\delta_0$ does not become 0: it is the generator of $N' = V/(\pi)$ inside $N$. Since $[\delta_0] = [\pi^k \delta_k]$ in $N$, the $V$-module $N$ is not $\pi$-adically separated.

The completion $\overline{M}$ of a bornological $V$-module $M$ is a complete bornological $V$-module with a bounded $V$-linear map $M \to \overline{M}$ that is universal in the sense that any bounded $V$-linear map from $M$ to a complete bornological $V$-module $X$ factors uniquely through $\overline{M}$. Such a completion exists and is unique up to isomorphism (see [4]). We shall describe it more concretely later when we need the details of its construction.

2.3. Vector spaces over the fraction field. Recall that $K$ denotes the quotient field of $V$. Any $V$-linear map between two $K$-vector spaces is also $K$-linear. So $K$-vector spaces with $K$-linear maps form a full subcategory in the category of $V$-modules. A $V$-module $M$ comes from a $K$-vector space if and only if the map

$$\pi_M: M \to M, \quad m \mapsto \pi \cdot m,$$

(2.10)
is invertible. We could define bornological $K$-vector spaces without reference to $V$. Instead, we realise them as bornological $V$-modules with an extra property:

**Definition 2.11.** A bornological $V$-module $M$ is a bornological $K$-vector space if the map $\pi_M$ in (2.10) is a bornological isomorphism, that is, an invertible map with bounded inverse.

Given a bornological $V$-module $M$, the tensor product $K \otimes M := K \otimes_V M$ with the tensor product bornology (see [4]) is a bornological $K$-vector space because multiplication by $\pi$ is a bornological isomorphism on $K$.

**Lemma 2.12.** The canonical bounded $V$-linear map $t_M : M \to K \otimes M$, $m \mapsto 1 \otimes m$, is the universal arrow from $M$ to a bornological $V$-vector space, that is, any bounded $V$-linear map $f : M \to N$ to a bornological $K$-vector space $N$ factors uniquely through a bounded $V$-linear map $f^\# : K \otimes M \to N$, and this map is also $K$-linear.

**Proof.** A $V$-linear map $f^\# : K \otimes M \to N$ must be $K$-linear. Hence the only possible candidate is the $K$-linear map defined by $f^\# (x \otimes m) := x \cdot f(m)$ for $m \in M$, $x \in K$. Any bounded submodule of $K \otimes M$ is contained in $\pi^{-k} V \otimes S$ for some bounded submodule $S \subseteq M$ and some $k \in \mathbb{N}$, and $f^\# (\pi^{-k} V \otimes S) = \pi_N^k f(S)$ is bounded in $N$ because $\pi_N$ is a bornological isomorphism. Thus $f^\#$ is bounded. □

3. Spectral radius and semi-dagger algebras

A bornological $V$-algebra is a bornological $V$-module $A$ with a bounded, $V$-linear, associative multiplication. We do not assume $A$ to have a unit element. We fix a bornological algebra $A$ throughout this section.

We recall some definitions from [4]. Let $\varepsilon = |\pi|$. Let $S \in B_A$ and let $r \leq 1$. There is a smallest integer $j$ with $\varepsilon^j \leq r$, namely, $[\log_\varepsilon(r)]$. Define $r \star S := \pi^{[\log_\varepsilon(r)]} S$.

Let $\sum_{n=1}^\infty r^n S^n$ be the $V$-submodule generated by $\bigcup_{n=1}^\infty r^n S^n$. That is, its elements are finite $V$-linear combinations of elements in $\bigcup_{n=1}^\infty r^n S^n$.

**Definition 3.1.** The truncated spectral radius $\varrho_1(S) = \varrho_1(S; B_A)$ of $S \in B_A$ is the infimum of all $r \geq 1$ for which $\sum_{n=1}^\infty r^{-n} S^n$ is bounded. It is $\infty$ if no such $r$ exists.

By definition, $\varrho_1(S) \in [1, \infty]$. If $A$ is an algebra over the fraction field $K$ of $V$, then we may define $\sum_{n=1}^\infty r^{-n} S^n$ also for $0 < r < 1$. Then the full spectral radius $\varrho(S) \in [0, \infty]$ is defined like $\varrho_1(S)$, but without the restriction to $r \geq 1$. If $A$ is bornologically torsion-free, then it is safe to define $\varrho(S; B_A) := \varrho(S; B_K \otimes A)$ for $S \in B_A$. This is useful to study tube algebras, but shall not be needed in this article.

**Definition 3.2.** A bornological $V$-algebra $A$ is semi-dagger if $\varrho_1(S) = 1$ for all $S \in B_A$.

**Definition 3.3 ([4]).** The linear growth bornology on a bornological $V$-algebra $A$ is the smallest semi-dagger bornology on $A$. That is, it is the smallest bornology $B_A'$ with $\varrho_1(S; B_A') = 1$ for all $S \in B_A'$. Let $A_{lg}$ be $A$ with the linear growth bornology.

The algebra $A_{lg}$ satisfies the following universal property: if $B$ is a semi-dagger $V$-algebra, then an algebra homomorphism $A \to B$ is bounded if and only if it is bounded on $A_{lg}$.

**Lemma 3.4 ([4]).** Let $T \subseteq A$. The following are equivalent:
(1) \( T \) is bounded in \( A_{lg} \);
(2) \( T \subseteq \sum_{i=0}^{\infty} \pi^1 S^n + 1 \) for some \( S \in B_A \).
(3) \( T \subseteq \sum_{i=0}^{\infty} \pi^i S^n + d \) for some \( S \in B_A \) and \( c, d \in \mathbb{N} \) with \( d \geq 1 \).

The algebra \( A \) is semi-dagger if and only if \( A = A_{lg} \). So the lemma implies that \( A \) is semi-dagger if and only if \( \sum_{i=0}^{\infty} \pi^1 S^n + 1 \) is bounded for all \( S \in B_A \). We shall need the following strengthening of this statement, which is implicit in the proofs in [1].

**Lemma 3.5.** Let \( S \in B_A, m \in \mathbb{N}_{\geq 1} \). Then \( q_1(S) = 1 \) if and only if \( \bigcup_{j=1}^{\infty} (\pi^m S)^j \) is bounded for all \( j \in \mathbb{N}_{\geq 1} \).

**Proof.** Let \( q_1(S) = 1 \) and \( j \in \mathbb{N}_{\geq 1} \). Then \( \sum_{i=1}^{\infty} \pi^i S^n j \subseteq \sum_{k=1}^{\infty} (\varepsilon^{m/j})^k \ast S^k \) is bounded because \( \varepsilon^{m/j} < 1 \). Conversely, let \( \bigcup_{j=1}^{\infty} (\pi^m S)^j \) be bounded. Then \( q_1(\pi^m S^j) = 1 \). The same argument as in the proof of [1] Lemma 3.1.2 shows that \( \varepsilon^{m/j} q_1(S) = q_1(\pi^m S^j) = 1 \), that is, \( q_1(S) \leq \varepsilon^{-m/j} \). This inequality for all \( j \in \mathbb{N}_{\geq 1} \) implies \( q_1(S) = 1 \) \( \square \).

**Theorem 3.6.** Let \( A \xrightarrow{i} B \xrightarrow{q} C \) be an extension of bornological V-algebras. Then \( B \) is a semi-dagger algebra if and only if both \( A \) and \( C \) are.

**Proof.** First assume \( B \) to be semi-dagger. Let \( S \in B_A \). Then \( \sum_{j=0}^{\infty} \pi^i(S)^j+1 \) is bounded in \( B \). Since \( i \) is a bornological embedding, it follows that \( \sum_{j=0}^{\infty} \pi^i(S)^j+1 \) is bounded in \( A \). That is, \( q_1(S; B_A) = 1 \). So \( A \) is semi-dagger. Now let \( S \in B_C \). Since \( q \) is a bornological quotient map, there is \( T \in B_B \) with \( q(T) = S \). The subset \( \sum_{j=0}^{\infty} \pi^i(T)^{j+1} \) is bounded in \( B \) because \( B \) is semi-dagger. Its image under \( q \) is also bounded, and this is \( \sum_{j=0}^{\infty} \pi^i(T)^{j+1} \). So \( q_1(S; B_C) = 1 \) and \( C \) is semi-dagger.

Now assume that \( A \) and \( C \) are semi-dagger. We show that \( \bigcup_{j=1}^{\infty} (\pi^2 S)^j \) is bounded in \( B \) for all \( S \in B_B, j \in \mathbb{N}_{\geq 1} \). This implies \( q_1(S; B_B) = 1 \) by Lemma 3.5.

Since \( C \) is semi-dagger, \( q_1(q(S); B_C) = 1 \). Thus \( S_2 := \bigcup_{j=1}^{\infty} q(\pi^j S)^j \) is bounded in \( C \) by Lemma 3.5. Since \( q \) is a quotient map, there is \( T \in B_B \) with \( q(T) = S_2 \). We may choose \( T \) with \( \pi S^j \subseteq T \). For each \( x, y \in T \), we have \( q(x \cdot y) \in S_2 \). \( S_2 \subseteq S_2 = q(T) \). Hence there is \( \omega(x, y) \in T \) with \( x \cdot y - \omega(x, y) \in i(A) \). Let \( \Omega := \{ x \cdot y - \omega(x, y) : x, y \in T \} \).

This is contained in \( T^2 - T \). So \( \Omega \subseteq B_B \). And \( T^2 \subseteq T + \Omega \). By construction, \( \Omega \) is also contained in \( i(A) \). Since \( i \) is a bornological embedding, \( i^{-1}(\Omega) \) is bounded in \( A \).

Since \( A \) is semi-dagger, we have \( q_1(i^{-1}(\Omega); B_A) = 1 \). So \( \sum_{n=1}^{\infty} (\pi \cdot \Omega)^n \) is bounded.

Thus the subset
\[
U := \sum_{n=1}^{\infty} (\pi \cdot \Omega)^n + \sum_{n=0}^{\infty} T \cdot (\pi \cdot \Omega)^n = \sum_{n=1}^{\infty} (\pi \cdot \Omega)^n + T + \sum_{n=1}^{\infty} T \cdot (\pi \cdot \Omega)^n
\]
of \( B \) is bounded. Using \( T^2 \subseteq T + \Omega \), we prove that \( \pi T \cdot U \subseteq U \). Hence \( (\pi T)^n \cdot U \subseteq U \) for all \( n \in \mathbb{N}_{\geq 1} \) by induction. Since \( T \subseteq U \), this implies \( \bigcup_{n=1}^{\infty} \pi^j T^{n+1} \subseteq U \). Hence \( \bigcup_{j=1}^{\infty} (\pi T)^j \) is bounded. Since \( \pi^j S^j \subseteq \pi T \), it follows that \( \bigcup_{j=1}^{\infty} (\pi^2 S)^j \) is bounded for all \( j \), as desired. \( \square \)

**Proposition 3.7 ([1] Lemma 3.1.12).** If \( A \) is semi-dagger, then so is its completion \( \overline{A} \).

Let \( \overline{A_{lg}} \) be the completion of \( A_{lg} \). This algebra is both complete and semi-dagger by Proposition 3.7. The canonical bounded homomorphism \( A \rightarrow \overline{A_{lg}} \) is the universal arrow from \( A \) to a complete semi-dagger algebra, that is, any bounded homomorphism \( A \rightarrow B \) for a complete semi-dagger algebra \( B \) factors uniquely through it. This follows immediately from the universal properties of the linear growth bornology and the completion.
4. Bornological torsion-freeness

Let $M$ be a bornological module over $V$. Recall the bounded linear map $\pi_M : M \to M$, $m \mapsto \pi \cdot m$, defined in (2.10).

**Definition 4.1.** A bornological $V$-module $M$ is bornologically torsion-free if $\pi_M$ is a bornological embedding. Equivalently, $\pi \cdot m = 0$ for $m \in M$ only happens for $m = 0$ and any bounded subset of $M$ that is contained in $\pi \cdot M$ is of the form $S = \pi \cdot T$ for some $T \in \mathcal{B}_M$.

Bornological $K$-vector spaces are bornologically torsion-free because bornological isomorphisms are bornological embeddings. We are going to show that $M$ is bornologically torsion-free if and only if the canonical map $\iota_M : M \to K \otimes M$ defined in Lemma [2.12] is a bornological embedding. The proof uses the following easy permanence property:

**Lemma 4.2.** Let $M$ be a bornological $V$-module and let $N \subseteq M$ be a $V$-submodule with the subspace bornology. If $M$ is bornologically torsion-free, then so is $N$.

**Proof.** Let $j : N \to M$ be the inclusion map, which is a bornological embedding by assumption. Since $\pi_M$ is a bornological embedding, so is $\pi_M \circ j = j \circ \pi_N$. Since $j$ is a bornological embedding, this implies that $\pi_N$ is a bornological embedding. That is, $N$ is bornologically torsion-free. $\square$

**Proposition 4.3.** A bornological $V$-module $M$ is bornologically torsion-free if and only if the canonical map $\iota_M : M \to K \otimes M$ is a bornological embedding.

**Proof.** As a bornological $K$-vector space, $K \otimes M$ is bornologically torsion-free. Hence $M$ is bornologically torsion-free by Lemma 4.2 if $\iota_M$ is a bornological embedding. Conversely, assume that $M$ is bornologically torsion-free. The map $\iota_M$ is injective because $M$ is algebraically torsion-free. It remains to show that a subset $S$ of $M$ is bounded if $\iota_M(S) \subseteq K \otimes M$ is bounded. If $\iota_M(S)$ is bounded, then it is contained in $\pi^{-k} \cdot V \otimes T$ for some $k \in \mathbb{N}$ and some $T \in \mathcal{B}_M$. Equivalently, $\pi^k_M(S) = \pi^k \cdot S$ is bounded in $M$. Since $\pi_M$ is a bornological embedding, induction shows that $\pi^k_M : M \to M$, $m \mapsto \pi^k \cdot m$, is a bornological embedding as well. So the boundedness of $\pi^k_M(S)$ implies that $S$ is bounded. $\square$

**Proposition 4.4.** Let $M_{\#} := \iota_M(M) \subseteq K \otimes M$ equipped with the subspace bornology and the surjective bounded linear map $\iota_M : M \to M_{\#}$. This is the universal arrow from $M$ to a bornologically torsion-free module, that is, any bounded linear map $f : M \to N$ into a bornologically torsion-free module $N$ factors uniquely through a bounded linear map $f^\# : M_{\#} \to N$.

**Proof.** Since $K \otimes M$ is bornologically torsion-free as a bornological $K$-vector space, $M_{\#}$ is bornologically torsion-free as well by Lemma 4.2. We prove the universality of the canonical map $\iota_M : M \to M_{\#}$. Let $N$ be a bornologically torsion-free $V$-module and let $f : M \to N$ be a bornological $V$-module map. Then $N \hookrightarrow K \otimes N$ is a bornological embedding by Proposition 4.3 and we may compose to get a bounded $V$-linear map $M \to K \otimes N$. By Lemma 2.12 there is a unique bounded $K$-linear map $f^\# : K \otimes M \to K \otimes N$ with $f^\#(\iota_M(m)) = f(m)$ for all $m \in M$. Since $f^\#(\iota_M(M)) \subseteq N$, $f^\#$ maps the submodule $M_{\#} \subseteq K \otimes M$ to the submodule $N \subseteq K \otimes N$. The restricted map $f^\# : M_{\#} \to N$ is bounded because both submodules carry the subspace bornology. This is the required factorisation of $f$. It is unique because $\iota_M : M \to M_{\#}$ is surjective. $\square$

We have seen that being bornologically torsion-free is hereditary for submodules. The obvious counterexample $k = V / \pi V$ shows that it cannot be hereditary for quotients. Next we show that it is hereditary for extensions:
Theorem 4.5. Let \( M' \xrightarrow{i} M \xrightarrow{q} M'' \) be an extension of bornological \( V \)-modules. If \( M' \) and \( M'' \) are bornologically torsion-free, then so is \( M \).

Proof. The exactness of the sequence \( 0 \to \ker \pi_M \xrightarrow{\pi} \ker \pi_{M'} \xrightarrow{\pi} \ker \pi_{M''} \), shows that \( \pi_M \) is injective. Let \( S \in B_M \) be contained in \( \pi M \). We want a bounded subset \( S' \in B_{M'} \) with \( \pi \cdot S' = S \). We have \( q(S) \subseteq q(\pi \cdot M) \subseteq \pi \cdot M'' \), and \( q(S) \in B_{M''} \) because \( q \) is bounded. Since \( M'' \) is torsion-free, there is \( T'' \in B_{M''} \) with \( \pi \cdot T'' = q(S) \).

Since \( q \) is a bornological quotient map, there is \( T \in B_M \) with \( q(T) = T'' \). Thus \( q(\pi \cdot T) = \pi(S) \). So for any \( x \in S \) there is \( y \in T \) with \( q(\pi \cdot y) = q(x) \). Since \( i = \ker(q) \), there is a unique \( z \in M' \) with \( x - \pi y = \iota(z) \). Let \( T' \) be the set of these \( z \).

Since \( x \in \pi \cdot M \) by assumption and \( M'' \) is bornologically torsion-free, we have \( z \in \pi \cdot M' \). So \( T' \subseteq \pi \cdot M' \). And \( T' \) is bounded because \( T' \subseteq \iota^{-1}(S - \pi \cdot T) \) and \( i \) is a bornological embedding. Since \( M' \) is bornologically torsion-free, there is a bounded subset \( U' \in B_{M'} \) with \( \pi \cdot U' = T' \). Then \( S \subseteq \pi \cdot T + \iota(\pi \cdot U') = \pi \cdot (T + \iota(U')) \).

Next we prove that bornological torsion-freeness is inherited by completions:

Theorem 4.6. If \( M \) is bornologically torsion-free, then so is its bornological completion \( \hat{M} \).

The proof requires some preparation. We must look closely at the construction of completions of bornological \( V \)-modules.

Proposition 4.7 ([4]). A completion of a bornological \( V \)-module \( M \) exists and is constructed as follows. Write \( M = \varinjlim M_i \) as an inductive limit of the directed set of its bounded \( V \)-submodules. Let \( \hat{M}_i \) denote the \( \pi \)-adic completion of \( M_i \). These form an inductive system as well, and \( \hat{M} = \left( \varinjlim \hat{M}_i \right) / \{0\} \) is the separated quotient of their bornological inductive limit.

The completion functor commutes with colimits, that is, the completion of a colimit of a diagram of bornological \( V \)-modules is the separated quotient of the colimit of the diagram of completions.

Since taking quotients may create torsion, the information above is not yet precise enough to show that completions inherit bornological torsion-freeness. This requires some more work. First we write \( M \) in a certain way as an inductive limit, using that it is bornologically torsion-free. For a bounded submodule \( S \) in \( M \), let

\[
\pi^{-n} S := \{ x \in M : \pi^n \cdot x \in S \} \subseteq M
\]

\[
M_S := \bigcup_{n \in \mathbb{N}} \pi^{-n} S \subseteq M.
\]

The gauge semi-norm of \( S \) is defined by \( \|x\|_S := \inf \{ \varepsilon^n : x \in \pi^n S \} \), where \( \varepsilon = |\pi| \) (see [4] Example 2.4). A subset is bounded for this semi-norm if and only if it is contained in \( \pi^{-n} S \) for some \( n \in \mathbb{N} \). Since \( M \) is bornologically torsion-free, \( \pi^{-n} S \in B_M \) for \( n \in \mathbb{N} \). So subsets that are bounded in the gauge semi-norm on \( M_S \) are bounded in \( M \). If \( S \subseteq T \), then \( M_S \subseteq M_T \) and the inclusion is contracting and hence bounded. The bornological inductive limit of this inductive system is naturally isomorphic to \( M \) because any bounded subset of \( M \) is bounded in \( M_S \) for some bounded submodule \( S \subseteq M \), compare the proof of [4] Proposition 2.5).

The bornological completion \( \hat{M}_S \) of \( M_S \) as a bornological \( V \)-module is canonically isomorphic to its Hausdorff completion as a semi-normed \( V \)-module. We call this a Banach \( V \)-module. Both completions are isomorphic to the increasing union of the \( \pi \)-adic completions \( \pi^{-n} \hat{S} \). If \( S \subseteq T \), then \( M_S \subseteq M_T \) and this inclusion is norm-contracting. So we get an induced contractive linear map \( \hat{M}_S \to \hat{M}_T \). This map need not be injective any more (see [4] Example 2.15]). Hence the canonical maps \( i_{T,S} : M_S \to \hat{M} \) need not be injective. The bornological completion commutes
with (separated) inductive limits by Proposition 4.7. So the completion of $M$ is isomorphic to the separated quotient of the colimit of the inductive system formed by the Banach $V$-modules $\hat{M}_S$ and the norm-contracting maps $i_{T,S}$ for $S \subseteq T$.

**Lemma 4.8.** The submodules

$$Z_S := \ker(i_{\infty,S} : \hat{M}_S \to \hat{M}) = \hat{i}_{\infty,S}^{-1}\{0\} \subseteq \hat{M}_S$$

are norm-closed and satisfy $\hat{i}_{T,S}^{-1}(Z_T) = Z_S$ if $S \subseteq T$. They are minimal with these properties in the sense that if $L_S \subseteq \hat{M}_S$ are norm-closed and satisfy $\hat{i}_{T,S}^{-1}(L_T) = L_S$ for $S \subseteq T$, then $Z_S \subseteq L_S$ for all bounded $V$-submodules $S \subseteq M$.

**Proof.** The property $i_{\infty,S}^{-1}(Z_T) = Z_S$ is trivial. The map $i_{\infty,S}$ is bounded and hence preserves convergence of sequences. Since $\hat{M}$ is separated, the subset $\{0\} \subseteq \hat{M}$ is bornologically closed. Therefore, its preimage $Z_S$ in $\hat{M}_S$ is also closed. Let $(L_S)$ be any family of closed submodules with $i_{T,S}^{-1}(L_T) = L_S$. The quotient seminorm on $\hat{M}_S / L_S$ is again a norm because $L_S$ is closed. And $\hat{M}_S / L_S$ inherits completeness from $\hat{M}_S$ by Theorem 2.3. If $S \subseteq T$, then $i_{T,S}$ induces an injective map $\hat{i}_{T,S} : \hat{M}_S / L_S \to \hat{M}_T / L_T$ because $L_S = i_{T,S}^{-1}(L_T)$. Hence the colimit of the inductive system $(\hat{M}_S / L_S, i_{T,S})$ is like a directed union of subspaces, and each $\hat{M}_S / L_S$ maps faithfully into it. Thus this colimit is separated. It is even complete because each $\hat{M}_S / L_S$ is complete. Hence the map from $M$ to this colimit induces a map on the completion $\hat{M}$. This implies $Z_S \subseteq L_S$. \(\square\)

Next we link $\hat{M}$ to the $\pi$-adic completion $\hat{M} := \varprojlim M / \pi^j M$. Equip the quotients $M / \pi^j M$ with the quotient bornology. Since $\pi^j \cdot (\hat{M} / \pi^j)$ is $0$, any Cauchy sequence in $M / \pi^j M$ is eventually constant. So each $M / \pi^j M$ is complete. Hence the quotient map $M \to M / \pi^j M$ induces a bounded $V$-module homomorphism $\hat{M} \to M / \pi^j M$. Putting them all together gives a map $\hat{M} \to \hat{M}$, which is bounded if we give $\hat{M}$ the projective limit bornology.

Let $S \subseteq M$ be a bounded $V$-submodule and let $j \in \mathbb{N}$. We have defined the submodules $M_S$ so that $M_S / \pi^j M = \pi^j M_S$. That is, the map $M_S / \pi^j M_S \to M / \pi^j M$ is injective. Since $M_S$ is dense in its norm-completion $\hat{M}_S$, we have $\hat{M}_S = M_S + \pi^j \hat{S}$ and hence $\hat{M}_S = M_S + \pi^j \hat{M}_S$. Thus the inclusion $M_S \to \hat{M}_S$ induces an isomorphism

$$M_S / \pi^j M_S \cong \hat{M}_S / \pi^j \hat{M}_S.$$ 

Letting $j$ vary, we get an injective map $\hat{M}_S \to \hat{M}$ and an isomorphism between the $\pi$-adic completions of $M_S$ and $\hat{M}_S$.

**Proof of Theorem 4.6.** For each bounded $V$-submodule $S \subseteq M$, define $L_S := \bigcap_{j \in \mathbb{N}} \pi^j \hat{M}_S \subseteq \hat{M}_S$. This is the kernel of the canonical map to the $\pi$-adic completion of $\hat{M}_S$. The completion $\hat{M}_S$ is torsion-free because it carries a norm. Hence $L_S$ is also the largest $K$-vector space contained in $\hat{M}_S$. The subspace $L_S$ is closed because the maps $\hat{M}_S \to \hat{M}_S / \pi^j \hat{M}_S$ for $j \in \mathbb{N}$ are bounded and their target spaces are separable, even complete.

Let $S \subseteq T$. The maps $M_S / \pi^j M_S \to M_T / \pi^j M_T$ are injective for all $j \in \mathbb{N}$, and $\hat{M}_S / \pi^j \hat{M}_S \cong M_S / \pi^j M_S$, $\hat{M}_T / \pi^j \hat{M}_T \cong M_T / \pi^j M_T$. So $i_{T,S}$ induces an injective map $\hat{M}_S / \pi^j \hat{M}_S \to \hat{M}_T / \pi^j \hat{M}_T$. This implies $i_{T,S}^{-1}(\pi^j \hat{M}_S) = \pi^j \hat{M}_S$ for all $j \in \mathbb{N}$ and then $i_{T,S}^{-1}(L_T) = L_S$.

By Lemma 4.8, the kernel $Z_S = \ker(i_{\infty,S})$ is contained in $L_S$ for all $S$. Since $\pi L_S$ is a bornological isomorphism, the subsets $\pi \cdot Z_S \subseteq Z_S$ are also bornologically closed, and they satisfy $i_{T,S}(\pi \cdot Z_T) = \pi \cdot i_{T,S}^{-1}(Z_T) = \pi \cdot Z_S$. Hence $Z_S \subseteq \pi \cdot Z_S$. 


for all $S$ by Lemma 4.8. Thus $Z_S \subseteq L_S$ is a $K$-vector subspace in $M_S$. So the quotient $M_S/Z_S$ is still bornologically torsion-free. And any element of $M_S/Z_S$ that is divisible by $\pi^j$ lifts to an element in $\pi^jM_S$.

Any bounded subset of $\hat{M}$ is contained in $i_{\infty,S}(\hat{S})$ for some bounded $V$-submodule $S \subseteq M$, where we view $\hat{S}$ as a subset of $M_S$. Let $j \in \mathbb{N}$. To prove that $\hat{M}$ is bornologically torsion-free, we must show that $\pi^{-j}i_{\infty,S}(\hat{S})$ is bounded. Let $x \in \hat{M}$ satisfy $\pi^jx \in i_{\infty,S}(\hat{S})$. We claim that $x = i_{\infty,S}(y)$ for some $y \in M_S$ with $\pi^jy \in \hat{S}$. This implies that $\pi^{-j}i_{\infty,S}(\hat{S})$ is bounded in $\hat{M}$. It remains to prove the claim. There are a bounded $V$-submodule $T \subseteq M$ and $z \in M_T$ with $x = i_{\infty,T}(z)$. We may replace $T$ by $T + S$ to arrange that $T \supseteq S$. Let $w \in \hat{S}$ satisfy $\pi^jx = i_{\infty,S}(w)$. This is equivalent to $\pi^{j}z - i_{T,S}(w) \in \ker i_{\infty,T} = Z_T$. Since $Z_T$ is a $K$-vector space, there is $z_0 \in Z_T$ with $\pi^{j}z - i_{T,S}(w) = \pi^{j}z_0$. Since $x = i_{\infty,T}(z - z_0)$, we may replace $z$ by $z - z_0$ to arrange that $\pi^{j}z = i_{T,S}(w)$. Since $i_{T,S}(\pi^{j}M_T) = \pi^{j}M_S$, there is $y \in M_S$ with $\pi^jy = w$. Then $\pi^{j}z = \pi^{j}i_{T,S}(y)$. This implies $z = i_{T,S}(y)$ because $\hat{M}$ is torsion-free. This proves the claim.

Proposition 4.9. Let $M$ be a bornologically torsion-free bornological $V$-module. Then $K \otimes \hat{M} \cong \hat{K} \otimes \hat{M}$ with an isomorphism compatible with the canonical maps from $M$ to both spaces.

Proof. The canonical map $M \to \hat{M}$ is the universal arrow from $M$ to a complete $V$-module. The canonical map $M \to K \otimes \hat{M}$ is the universal arrow from $M$ to a bornological $K$-vector space by Lemma 2.12. Since $K \otimes \hat{M}$ is again complete, the canonical map $M \to K \otimes \hat{M}$ is the universal arrow from $M$ to a complete bornological $K$-vector space. The completion $K \otimes \hat{M}$ is also a bornological $K$-vector space. The canonical map $M \to K \otimes \hat{M}$ is another universal arrow from $M$ to a complete bornological $K$-vector space. So both arrows are isomorphic.

Corollary 4.10. If $M$ is bornologically torsion-free, then the canonical map $\hat{M} \to K \otimes \hat{M}$ is a bornological embedding.

Proof. Use the isomorphism $K \otimes \hat{M} \cong \hat{K} \otimes \hat{M}$ to replace the canonical map $\hat{M} \to K \otimes \hat{M}$ by the canonical map $\hat{M} \to K \otimes \hat{M}$. This is a bornological embedding if and only if $\hat{M}$ is bornologically torsion-free by Proposition 4.3. And this is true by Theorem 4.6.

Finally, we show that being bornologically torsion-free is compatible with linear bornologies:

Proposition 4.11. If $A$ is a bornologically torsion-free *$V$*-algebra, then so is $A_{\lg}$.

Proof. Let $S \subseteq \pi \cdot A$ be bounded in $A_{\lg}$. Then there is $T \in B_A$ with $S \subseteq T_1 := \sum_{i=0}^{\infty} \pi^iT^{i+1}$ by Lemma 3.4. The subset $T_2 := \sum_{i=0}^{\infty} \pi^iT^{i+2}$ also has linear growth. And

$$T_1 = T + \sum_{i=1}^{\infty} \pi^iT^{i+1} = T + \sum_{i=0}^{\infty} \pi^{i+1}T^{i+2} = T + \pi T_2.$$  

Since $T$ is bounded in $A$ and $A$ is bornologically torsion-free, $\pi^{-1} \cdot T := \{x \in A : \pi \cdot x \in T\}$ is also bounded. We have $\pi^{-1}S \subseteq \pi^{-1}T_1 \subseteq \pi^{-1} \cdot T + T_2$. This is bounded in $A_{\lg}$.

5. DAGGER ALGEBRAS

Definition 5.1. A dagger algebra is a complete, bornologically torsion-free, semi-dagger algebra.
Theorem 5.2. Let \( A \xrightarrow{\cdot} B \xrightarrow{\rho} C \) be an extension of bornological \( V \)-algebras. If \( A \) and \( C \) are dagger algebras, so is \( B \).

Proof. All three properties defining dagger algebras are hereditary for extensions by Theorems 2.3, 3.6 and 4.5.

We have already seen that there are universal arrows \( A \to A_{\text{tf}} \subseteq K \otimes A \), \( A \to A_{\text{lg}} \) from a bornological algebra \( A \) to a bornologically torsion-free algebra, to a semi-dagger algebra, and to a complete bornological algebra, respectively. We now combine them to a universal arrow to a dagger algebra:

Theorem 5.3. Let \( A \) be a bornologically torsion-free algebra. Then the canonical map from \( A \) to \( A^! := (A_{\text{tf}})_{\text{lg}} \) is the universal arrow from \( A \) to a dagger algebra. That is, any bounded algebra homomorphism from \( A \) to a dagger algebra factors uniquely through \( A^! \). If \( A \) is already bornologically torsion-free, then \( A^! \cong A_{\text{lg}} \).

Proof. The bornological algebra \( A^! \) is complete by construction. It is semi-dagger by Proposition 4.7. And it is bornologically torsion-free by Proposition 4.11 and Theorem 4.6. So it is a dagger algebra. Let \( B \) be a dagger algebra. A bounded homomorphism \( A \to B \) factors uniquely through a bounded homomorphism \( A_{\text{tf}} \to B \) by Proposition 4.4 because \( B \) is bornologically torsion-free. This factors uniquely through a bounded homomorphism \( (A_{\text{tf}})_{\text{lg}} \to B \) because \( B \) is semi-dagger. And this factors uniquely through a bounded homomorphism \( (A_{\text{tf}})_{\text{lg}} \to B \) because \( B \) is complete. So \( A^! \) has the asserted universal property. If \( A \) is bornologically torsion-free, then \( A \cong A_{\text{tf}} \) and hence \( A^! \cong A_{\text{lg}} \).

Definition 5.4. We call \( A^! \) the dagger completion of the bornological \( V \)-algebra \( A \).

6. Dagger completions of monoid algebras

As a simple illustration, we describe the dagger completions of monoid algebras. The monoid algebra of \( \mathbb{N}^j \) is the algebra of polynomials in \( j \) variables, and its dagger completion is the Munkres–Washburn algebra of overconvergent power series equipped with a canonical bornology (see [4]). The case of general monoids is similar.

The monoid algebra \( V[S] \) of \( S \) over \( V \) is defined by its universal property: if \( B \) is a unital \( V \)-algebra, then there is a natural bijection between algebra homomorphisms \( V[S] \to B \) and monoid homomorphisms \( S \to (B, \cdot) \) into the multiplicative monoid of \( B \). More concretely, \( V[S] \) is the free \( V \)-module with basis \( S \) or, equivalently, the \( V \)-module of formal linear combinations of the form

\[
\sum_{s \in S} x_s \delta_s, \quad x_s \in V, \ s \in S,
\]

with \( x_s = 0 \) for all but finitely many \( s \), and equipped with the multiplication

\[
\sum_{s \in S} x_s \delta_s \ast \sum_{t \in S} y_t \delta_t = \sum_{s,t \in S} x_s y_t \delta_{s \cdot t}.
\]

We give \( V[S] \) the fine bornology. Then it has an analogous universal property in the category of bornological \( V \)-algebras. So the dagger completion \( V[S]^! \) is a dagger algebra with the property that bounded algebra homomorphisms \( V[S]^! \to B \) for a dagger algebra \( B \) are in natural bijection with monoid homomorphisms \( S \to (B, \cdot) \).

Assume first that \( S \) has a finite generating set \( F \). Let \( F^n \subseteq S \) be the set of all words \( s_1 \cdots s_k \) with \( s_1, \ldots, s_k \in F \) and \( k \leq n \). This gives an increasing filtration on \( S \) with \( F^n = \{1\} \) and \( S = \bigcup_{n=0}^{\infty} F^n \). For \( s \in S \), we define \( \ell(s) \in \mathbb{N} \) as the smallest \( n \) with \( s \in F^n \). This is the word length generated by \( F \). Let \( V[F^n] \subseteq V[S] \) be the free \( V \)-submodule of \( V[S] \) spanned by \( F^n \). Any finitely generated \( V \)-submodule
of \(V[S]\) is contained in \(V[F^n]\) for some \(n \in \mathbb{N}\). By Lemma 3.4, a subset of \(V[S]\) has linear growth if and only if it is contained in \(M_0 := \sum_{j=0}^{\infty} \pi^j (V[F^n])^{j+1}\) for some \(n \in \mathbb{N}_{\geq 1}\). That is, \(M_\text{lg} = \lim_{n \rightarrow \infty} M_n\).

Recall the valuation \(\nu: V \rightarrow \mathbb{N} \cup \{\infty\}\) defined by

\(\nu(x) := \sup \{n \in \mathbb{N}: x \in \pi^n V\}\).

By definition, the submodule \(M_n\) consists of all finite sums of terms \(x_s \delta_s\) with \(x_s \in \pi^j V\) and \(\ell(s) \leq n(j+1)\) for some \(j \in \mathbb{N}\) or, equivalently, \(\ell(s)/n \leq j + 1 \leq \nu(x_s) + 1\). That is, \(M_n\) contains a finite sum \(\sum_{s \in S} x_s \delta_s\) with \(x_s \in V\) and \(x_s = 0\) for all but finitely many \(s \in S\) if and only if \(\nu(x_s) + 1 \geq \ell(s)/n\) for all \(s \in S\). The \(\pi\)-adic completion \(\hat{M}_n\) of \(M_n\) is the set of all formal power series \(\sum_{s \in S} x_s \delta_s\) such that \(x_s \in V\), \(\nu(x_s) + 1 \geq \ell(s)/n\) for all \(s \in S\) and \(\lim_{s \rightarrow \infty} \nu(x_s) + 1 - \ell(s)/n = \infty\). This implies \(x_s \rightarrow 0\) in the \(\pi\)-adic norm, so that \(\hat{M}_n \subseteq \hat{V}[S]\). So the extension \(\hat{M}_n \rightarrow \hat{M}_{n+1}\) of the inclusion map \(M_n \rightarrow M_{n+1}\) remains injective. Therefore, \(\hat{M}_n\) is separated, and it is contained in \(V[S]\). Proposition 4.7 implies \(V[S]^! = \lim_{n \rightarrow \infty} \hat{M}_n\).

Elements of \(\hat{V}[S]\) are formal series \(\sum_{s \in S} x_s \delta_s\) with \(x_s \in V\) for all \(s \in S\) and \(\lim [x_s] = 0\). We have seen above that such a formal series belongs to \(\hat{M}_n\) if and only if \(\nu(x_s) + 1 \geq \ell(s)/n\) for all \(s \in S\) and \(\lim_{\ell(s) \rightarrow \infty} \nu(x_s) + 1 - \ell(s)/n = \infty\). If \(0 < 1/n < c\), then \(\nu(x_s) + 1 \geq c \ell(s)\) implies \(\nu(x_s) + 1 \geq \ell(s)/n\) and \(\lim_{\ell(s) \rightarrow \infty} \nu(x_s) + 1 - \ell(s)/n = \infty\). Thus all \(\sum_{s \in S} x_s \delta_s \in \hat{V}[S]\) with \(\nu(x_s) + 1 \geq c \ell(s)\) belong to \(\hat{M}_n\). Conversely, all elements of \(\hat{M}_n\) satisfy this for \(c = 1/n\). Letting \(c\) and \(n\) vary, we see that \(V[S]^!\) is the set of all \(\sum_{s \in S} x_s \delta_s\) in \(\hat{V}[S]\) for which there is \(c > 0\) with

\(\nu(x_s) + 1 \geq c \ell(s)\) for all \(s \in S\),

and that a subset of \(V[S]^!\) is bounded if and only if all its elements satisfy (6.1) for the same \(c > 0\). The growth condition (6.1) does not depend on the word length function \(\ell\) because the word length functions for two different generating sets of \(S\) are related by linear inequalities \(\ell \leq a \ell'\) and \(\ell' \leq a \ell\) for some \(a > 0\).

Now we drop the assumption that \(S\) be finitely generated. Then we may write \(S\) as the increasing union of its finitely generated submonoids. By the universal property, the monoid algebra of \(S\) with the fine bornology is a similar inductive limit in the category of bornological \(V\)-algebras, and its dagger algebra is the inductive limit in the category of dagger algebras. Since \(V[S]^! \subseteq \hat{V}[S] \subseteq \hat{V}[S]\) for any finitely generated \(S' \subseteq S\), we may identify this inductive limit with a subalgebra of \(\hat{V}[S]\) as well, namely, the union of \(V[S']^!\) over all finitely generated submonoids \(S' \subseteq S\). That is, \(V[S]^!\) is the set of elements of \(\hat{V}[S]\) that are supported in some finitely generated submonoid \(S' \subseteq S\) and that satisfy (6.1) for some length function on \(S'\).

We may also twist the monoid algebra. Let \(V^\times = \{x \in V: |x| = 1\}\) and let \(c: S \times S \rightarrow V^\times\) be a normalised 2-cocycle, that is,

\(c(r, s \cdot t) \cdot c(s, t) = c(r \cdot s, t) \cdot c(r, s)\), \(c(s, 1) = c(1, s) = 1\)

for all \(r, s, t \in S\). The \(c\)-twisted monoid algebra of \(S\), \(V[S, c]\), is the \(V\)-module \(V[S]\) with the twisted multiplication

\(\sum_{s \in S} x_s \delta_s \ast \sum_{t \in S} y_t \delta_t = \sum_{s, t \in S} x_s y_t c(s, t) \cdot \delta_{s \cdot t}\).

The condition (6.2) is exactly what is needed to make this associative and unital with unit \(\delta_1\). Since we assume \(c\) to have values in \(V^\times\), the twist does not change the
linear growth bornology. Therefore, the dagger completion $V[S, c]$ consists of all infinite sums $\sum_{s \in S} x_s \delta_s$ that are supported in a finitely generated submonoid of $S$ and satisfy the growth condition (6.1), and a subset is bounded if and only if all its elements satisfy these two conditions uniformly. Only the multiplication changes and is now given by (6.3).

Example 6.4. Let $S = \mathbb{Z}^2 \times \mathbb{Z}$ with the unit element 0. Define $c((s_1, s_2), (t_1, t_2)) := \lambda^{s_2 \cdot t_1}$ for some $\lambda \in V^\times$. This satisfies (6.2). The resulting twisted convolution algebra is an analogue of a noncommutative torus over $V$. Indeed, let $U_1 := \delta_{(1,0)}$ and $U_2 := \delta_{(0,1)}$ as elements of $V[\mathbb{Z}^2, c]$. Then $\delta_{(-1,0)} = U_1^{-1}$ and $\delta_{(0,-1)} = U_2^{-1}$ are inverse to them, and $\delta_{(s_1, s_2)} = U_1^{s_1} \cdot U_2^{s_2}$. So $U_1, U_2$ generate $V[\mathbb{Z}^2, c]$ as a $V$-algebra. They satisfy the commutation relation

$$U_2 \cdot U_1 = \lambda \cdot U_1 \cdot U_2.$$  

And this already dictates the multiplication table in $V[\mathbb{Z}^2, c]$. The dagger completion $V[\mathbb{Z}^2, c]^!$ is isomorphic as a bornological $V$-module to the Monsky–Washnitzer completion of the Laurent polynomial algebra $V[U_1^{\pm 1}, U_2^{\pm 1}]$, equipped with a twisted multiplication satisfying (6.5).

7. Dagger completions of crossed products

Let $A$ be a unital, bornological $V$-algebra, let $S$ be a finitely generated monoid and let $\alpha: S \to \text{End}(A)$ be an action of $S$ on $A$ by bounded algebra homomorphisms. The crossed product $A \rtimes_\alpha S$ is defined as follows. Its underlying bornological $V$-module is $A \rtimes_\alpha S = \bigoplus_{s \in S} A$ with the direct sum bornology. So elements of $A \rtimes_\alpha S$ are formal linear combinations $\sum_{s \in S} a_s \delta_s$ with $a_s \in A$ and $a_s = 0$ for all but finitely many $s \in S$. The multiplication on $A \rtimes_\alpha S$ is defined by

$$\left( \sum_{s \in S} a_s \delta_s \right) \cdot \left( \sum_{t \in S} b_t \delta_t \right) := \sum_{s,t \in S} a_s \alpha_s(b_t) \delta_{st}.$$  

This makes $A \rtimes_\alpha S$ a bornological $V$-algebra. What is its dagger completion?

It follows easily from the universal property that defines $A \subseteq A \rtimes_\alpha S$ that

$$(A \rtimes_\alpha S)^! \cong (A^! \rtimes_{\alpha^!} S)^!;$$

here we use the canonical extension of $\alpha$ to the dagger completion $A^!$, which exists because the latter is functorial for bounded algebra homomorphisms. Therefore, it is no loss of generality to assume that $A$ is already a dagger algebra. It is easy to show that $(A \rtimes S)^!$ is the inductive limit of the dagger completions $(A \rtimes S')^!$, where $S'$ runs through the directed set of finitely generated submonoids of $S$. Hence we may also assume that $S$ is finitely generated to simplify. First we consider the following special case:

Definition 7.1. The action $\alpha: S \to \text{End}(A)$ is called uniformly bounded if any $U \in \mathcal{B}_A$ is contained in an $\alpha$-invariant $T \in \mathcal{B}_A$; $\alpha$-invariance means $\alpha_s(T) = T$ for all $s \in S$.

If $T$ is $\alpha$-invariant, so is the $V$-module generated by $T$. Therefore, $\alpha$ is uniformly bounded if and only if any bounded subset of $A$ is contained in a bounded, $\alpha$-invariant $V$-submodule. If $A$ is complete, then the image of $\hat{T}$ in $A$ is also $\alpha$-invariant because the maps $\alpha_s$ are bornological isomorphisms. Hence we may assume in this case that $T$ in Definition 7.1 is a bounded, $\alpha$-invariant $\pi$-adic complete $V$-submodule.

Proposition 7.2. Let $A$ carry a uniformly bounded action $\alpha$ of $S$. Then the induced actions on $\hat{A}$, $A_\mathbb{Q}$, and $A_{\mathbb{Q}}$ are uniformly bounded as well. Hence so is the induced action on $A^!$. 

Proof. If $\alpha$ is uniformly bounded, then $A$ is the bornological inductive limit of its $\alpha$-invariant bounded $V$-submodules. The action of $\alpha$ restricts to any such submodule $T$ and then extends canonically to its $\pi$-adic completion $\hat{T}$. Then the image of $\hat{T}$ in $\hat{A}$ is $S$-invariant as well. This gives enough $S$-invariant bounded $V$-submodules in $\hat{A}$. So the induced action on $\hat{A}$ is uniformly bounded.

If the action $\alpha$ on $A$ is uniformly bounded, then so is the action $\text{id}_B \circ \alpha$ on $B \otimes A$ for any bornological algebra $B$. In particular, the induced action on $K \otimes A$ is uniformly bounded. Since the canonical map $A \to K \otimes A$ is $S$-equivariant, the image $A_{tf}$ of $A$ in $K \otimes A$ is $S$-invariant. The restriction of the uniformly bounded action of $S$ on $K \otimes A$ to this invariant subalgebra inherits uniform boundedness. So the induced action on $A_{tf}$ is uniformly bounded.

Any subset of linear growth in $A$ is contained in $\sum_{j=0}^{\infty} \pi^j T^{j+1}$ for a bounded $V$-submodule $T$. Since $\alpha$ is uniformly bounded, $T$ is contained in an $\alpha$-invariant bounded $V$-submodule $U$. Then $\sum_{j=0}^{\infty} \pi^j U^{j+1} \supseteq \sum_{j=0}^{\infty} \pi^j T^{j+1}$ is $\alpha$-invariant and has linear growth. So $\alpha$ remains uniformly bounded for the linear growth bornology.

The uniform boundedness of the induced action on the dagger completion $A^\dagger$ follows from the inheritance properties above and Theorem 5.3.

Example 7.3. Let $S$ be a finite monoid. Any bounded action of $S$ by bornological algebra endomorphisms is uniformly bounded because we may take $T = \sum_{s \in S} \alpha_s(U)$ in Definition 7.1.

Example 7.4. We describe a uniformly bounded action of $\mathbb{Z}$ on the polynomial algebra $A := V[x_1, \ldots, x_n]$ with the fine bornology. So a subset of $A$ is bounded if and only if it is contained in $(V + Vx_1 + \cdots + Vx_n)^k$ for some $k \in \mathbb{N}_{\geq 1}$. Let $a \in \text{GL}_n(V) \subseteq \text{End}(V^n)$ and $b \in V^n$. Then

$$\alpha_1 : V[x_1, \ldots, x_n] \to V[x_1, \ldots, x_n], \quad (\alpha_1 f)(x) := f(ax + b),$$

is an algebra automorphism $\alpha_1$ of $A$ with inverse $(\alpha_1^{-1} f)(x) := f(a^{-1}(x - b))$. This generates an action of the group $\mathbb{Z}$ by $\alpha_n := \alpha_1^n$ for $n \in \mathbb{Z}$. If a polynomial $f$ has degree at most $m$, then the same is true for $\alpha_1 f$ and $\alpha_{-1} f$, and hence for $\alpha_n f$ for all $n \in \mathbb{Z}$. That is, the subsets $(V + Vx_1 + \cdots + Vx_n)^k$ in $A$ for $k \in \mathbb{N}_{\geq 1}$ are $\alpha$-invariant. So the action $\alpha$ on $A$ is uniformly bounded. Proposition 7.2 implies that the induced action on $V[x_1, \ldots, x_n]^k$ is uniformly bounded as well.

Proposition 7.5. Let $S$ be a finitely generated monoid with word length function $\ell$. Let $A$ be a dagger algebra and let $\alpha : S \to \text{End}(A)$ be a uniformly bounded action by algebra endomorphisms. Then $(A \rtimes S)^I \subseteq \prod_{s \in S} A$. A formal series $\sum_{s \in S} a_s \delta_s$ with $a_s \in A$ for all $s \in S$ belongs to $(A \rtimes S)^I$ if and only if there are $\varepsilon > 0$ and $T \in B_A$ with $a_s \in \pi^{|\varepsilon|} T$ for all $s \in S$, and a set of formal series is bounded in $(A \rtimes S)^I$ if and only if $\varepsilon > 0$ and $T \in B_A$ for its elements may be chosen uniformly.

Proof. We first describe the linear growth bornology on $A \rtimes V[S]$. Let $B'$ be the set of all subsets $U \subseteq A \rtimes S$ for which there are $T \in B_A$ and $\varepsilon > 0$ such that any element of $U$ is of the form $\sum_{s \in S} a_s \delta_s$ with $a_s \in \pi^{|\varepsilon|} T$ for all $s \in S$. We claim that $B'$ is the linear growth bornology on $A \rtimes S$. The inclusion $V[S] \subseteq A \rtimes V[S]$ induces a bounded algebra homomorphism $V[S]_{\varepsilon} \to (A \rtimes V[S])_{\varepsilon}$. We have already described the linear growth bornology on $V[S]$ in Section 6. This implies easily that all subsets in $B'$ have linear growth: write $\pi^{|\varepsilon|} a_s \delta_s = a'_s \cdot \pi^{|\varepsilon|} \delta_s$. We claim, conversely, that any subset of $A \rtimes S$ of linear growth is contained in $B'$. All bounded subsets of $A \rtimes S$ are contained in $B'$. If $T \in B_A$ is contained in a bounded, $\alpha$-invariant
V-submodule $T_2$. Then $T_3 := \sum_{j=0}^{\infty} \pi^j T_2^{j+1}$ is a bounded, $\alpha$-invariant V-submodule with $\pi \cdot T_2^j \subseteq T_3$ and $T \subseteq T_3$. If $a_s \in \pi^{[\varepsilon(s)]} T_3$ and $a_t \in \pi^{[\varepsilon(t)]} T_3$, then

$$\pi^2 \cdot a_s \cdot a_t \in \pi^{2 + [\varepsilon(s)] + [\varepsilon(t)]} T_3^2 \subseteq \pi^{[\varepsilon(s)] + [\varepsilon(t)]} T_3 \subseteq \pi^{[\varepsilon(t)]} T_3$$

because $1 + [\varepsilon(s)] + [\varepsilon(t)] \geq [\varepsilon(s \cdot t)]$. This implies

$$\pi^2 \cdot \sum_{s \in S} \pi^{[\varepsilon(s)]} T_3 \delta_s + \sum_{t \in S} \pi^{[\varepsilon(t)]} T_3 \delta_t \subseteq \sum_{s,j \in S} \pi^{[\varepsilon(s)]} T_3 \delta_s \subseteq \sum_{s \in S} \pi^{[\varepsilon(s)]} T_3 \delta_s.$$ 

So any subset in $B'$ is contained in $U \in B'$ with $\pi^2 \cdot U^2 \subseteq U$. By induction, this implies $(\pi^2 U)^k \cdot U \subseteq U$ for all $k \in \mathbb{N}$. Hence $\sum_{j=0}^{\infty} \pi^{2k} U^{k+1}$ is in $B'$. Now Lemma 3.4 shows that the bornology $B'$ has linear growth. This proves the claim that $B'$ is the linear growth bornology on $A \times S$.

Since $A$ as a dagger algebra is bornologically torsion-free, so is $A \times S$. So $(A \times S)^I$ is the completion of $(A \times S)_U = (A \times S, B')$. It is routine to identify this completion with the bornological V-module described in the statement.

Propositions 7.2 and 7.5 describe the dagger completion of $A \times S$ for a uniformly bounded action of $S$ on $A$ even if $A$ is not a dagger algebra. Namely, the universal properties of the crossed product and the dagger completion imply

$$(A \times S)^I \cong (A^I \times S)^I.$$ 

Example 7.7. Let $\alpha$ be the uniformly bounded action of $Z$ on $V[x_1, \ldots, x_k]$ from Example 7.3. The induced action $\alpha^I$ on $V[x_1, \ldots, x_k]^I$ is also uniformly bounded by Proposition 7.2. And (7.6) implies

$$(V[x_1, \ldots, x_k] \times_{\alpha^I} Z)^I \cong (V[x_1, \ldots, x_k]^I \times_{\alpha^I} Z)^I.$$ 

The latter is described in Proposition 7.5. Namely, $(V[x_1, \ldots, x_k]^I \times_{\alpha^I} Z)^I$ consists of those formal series $\sum_n a_n \delta_n$ with $a_n \in V[x_1, \ldots, x_k]^I$ for which there are $\varepsilon > 0$ and a bounded V-submodule $T$ in $V[x_1, \ldots, x_k]^I$ such that $a_n \in \pi^{[\varepsilon |n|]} T$ for all $n \in \mathbb{Z}$; notice that $|n|$ is indeed a length function on $Z$. And a subset is bounded if some pair $\varepsilon, T$ works for all its elements.

We combine this with the description of bounded subsets of $V[x_1, \ldots, x_k]^I$ in Section 6. There is some $\delta > 0$ so that a formal power series $\sum_{m \in \mathbb{N}} b_m x^m$ belongs to $T$ if and only if $b_m \in \pi^{[\varepsilon |m|]} V$ for all $m \in \mathbb{N}$. Here we use the length function $|m| = \sum_{j=1}^{\infty} m_j$. We may merge the parameters $\varepsilon, \delta > 0$ above, taking their minimum. So $(V[x_1, \ldots, x_k] \times Z)^I$ consists of the formal series $\sum_{n \in \mathbb{Z}, m \in \mathbb{N}} a_{n,m} x^m \delta_n$ with $a_{n,m} \in \pi^{[\varepsilon (|n| + |m|)]} V$ or, equivalently, $\nu(a_{n,m}) + 1 > \varepsilon (|n| + |m|)$ for all $n \in \mathbb{Z}$, $m \in \mathbb{N}$.

If the action of $S$ on $A$ is not uniformly bounded, then the linear growth bornology on $A \times S$ becomes much more complicated. It seems unclear whether the description below helps much in practice. Let $F \subseteq S$ be a finite generating subset containing 1. Any bounded subset of $A \times S$ is contained in $\left( \sum_{s \in F} T \cdot \delta_s \right)^N$ for some $N \in \mathbb{N}$ and some $T \in B_A$ with 1 $\in T$. Therefore, a subset of $A \times S$ has linear growth if and only if it is contained in the $V$-submodule generated by

$$\bigcup_{n=1}^{\infty} \pi^{[\varepsilon n]} (T \cdot \{\delta_s : s \in F\})^n$$ 

for some $\varepsilon > 0$, $T \in B_A$. Using the definition of the convolution, we may rewrite the latter set as

$$\bigcup_{n=1}^{\infty} \bigcup_{s_1, \ldots, s_n \in F} \pi^{[\varepsilon n]} \cdot T \cdot \alpha_{s_1}(T) \cdot \alpha_{s_1 s_2}(T) \cdot \ldots \cdot \alpha_{s_1 \ldots s_{n-1}}(T) \cdot \delta_{s_1} \ldots \delta_{s_n}.$$
The resulting $V$-module is the sum $\sum_{s \in S} U_s \delta_s$, where $U_s$ is the $V$-submodule of $A$ generated by finite products

$$\{ \pi^{[\epsilon n]} \cdot T \cdot \alpha_{s_1}(T) \cdots \alpha_{s_{n-1}}(T) : n \in \mathbb{N} \cap \{1, \ldots, s \}, s_1, \ldots, s_n \in F, s_1 \cdots s_n = s \}.$$ 

Here taking a factor $1 \in T$ is allowed. Thus we may leave out a factor $\alpha_{s_{n-1}}(T)$. This has the same effect as increasing $n$ by 1 and putting $s_i = s_i^1 \cdot s_i^2$ with $s_i^1, s_i^2 \in F$.

Since $F$ generates $S$ as a monoid, we may allow arbitrary $s_i \in S$ when we change the exponent of $\pi$ appropriately. Namely, we must then replace $n$ in the exponent of $\pi$ by the number of factors in $F$ that are needed to produce the desired elements $s_i$, which is $\ell_{\geq 1}(s_1) + \cdots + \ell_{\geq 1}(s_n)$, where $\ell_{\geq 1}(1) = 1$ and $\ell_{\geq 1}(s) = \ell(s)$ for $s \in S \setminus \{1\}$.

As a result, $U_s$ is the $V$-submodule of $A$ generated by

$$\pi^{[\epsilon \ell_{\geq 1}(s_1) + \cdots + \ell_{\geq 1}(s_n)]} \cdot x_0 \cdot \alpha_{s_1}(x_1) \cdots \alpha_{s_{n-1}}(x_{n-1}),$$

$n \in \mathbb{N} \cap \{1, \ldots, s \}, x_0, \ldots, x_{n-1} \in T, s_1, \ldots, s_n \in S, s_1 \cdots s_n = s$.

Now assume that $S$ is a group, not just a monoid. Then any sequence of elements $g_1, \ldots, g_n \in S$ may be written as $g_i = s_1 \cdots s_i$ by putting $s_i := g_i^{-1} g_i$ with $g_0 := 1$.

$$\text{So } U_g \text{ is the } V\text{-submodule of } A \text{ generated by}$$

$$\pi^{[\epsilon \ell_{\geq 1}(g_1) + \cdots + \ell_{\geq 1}(g_n)]} \cdot \alpha_{g_0}(x_0) \cdot \alpha_{g_1}(x_1) \cdots \alpha_{g_{n-1}}(x_{n-1}),$$

$n \in \mathbb{N} \cap \{1, \ldots, n\}, x_0, \ldots, x_{n-1} \in T, g_0, \ldots, g_n \in S, g_0 = 1, g_n = g$.

These subsets $U_g$ for fixed $T$ and $\epsilon$ depend on $g$ in a complicated way. The bornology on $A \times G$ generated by these subsets is, however, also generated by the sets of the form $\sum_{g \in G} \pi^{\ell(g)} \cdot U \delta_1$, where $U$ is the $V$-submodule of $A$ generated by

$$\pi^{[\epsilon \ell_{\geq 1}(g_1) + \cdots + \ell_{\geq 1}(g_n)]} \cdot \alpha_{g_0}(x_0) \cdot \alpha_{g_1}(x_1) \cdots \alpha_{g_{n-1}}(x_{n-1}),$$

$n \in \mathbb{N} \cap \{1, \ldots, n\}, x_0, \ldots, x_{n-1} \in T, g_0, \ldots, g_n \in S, g_0 = 1, g_n = g$.

for some $T \in B_A, \epsilon > 0$. The reason is that

$$\ell(g_n) - \ell(g_{n-1}) \leq \ell(g_n^{-1} g_n) \leq \ell(g_{n-1}) + \ell(g_n)$$

and $\ell(g_{n-1}) \leq \sum_{j=1}^{n-1} \ell(g_j^{-1} g_j)$. Therefore, replacing the exponents of $\pi$ as above does not change the bornology on $A \times G$ that is generated by the sets above when $\epsilon > 0$ varies.

REFERENCES

[1] Federico Bambozzi, *On a generalization of affinoid varieties*, Ph.D. Thesis, Università degli Studi di Padova, 2014. arXiv: 1401.5702.

[2] Federico Bambozzi and Oren Ben-Bassat, *Dagger geometry as Banach algebraic geometry*, J. Number Theory 162 (2016), 391–462. doi: 10.1016/j.jnt.2015.10.023 MR 3448274

[3] Federico Bambozzi, Oren Ben-Bassat, and Kobi Kremnizer, *Stein domains in Banach algebraic geometry*, J. Funct. Anal. 274 (2018), no. 7, 1865–1927. doi: 10.1016/j.jfa.2018.01.003 MR 3762090

[4] Guillermo Cortiñas, Joachim Cuntz, Ralf Meyer, and Georg Tamme, *Nonarchimedean bornologies, cyclic homology and rigid cohomology* (2017), eprint. arXiv: 1708.00357.

[5] Ralf Meyer, *Local and analytic cyclic homology*, EMS Tracts in Mathematics, vol. 3, European Mathematical Society (EMS), Zürich, 2007. MR 2337277

[6] Paul Monsky and G. Washnitzer, *Formal cohomology. I*, Ann. of Math. (2) 88 (1968), 181–217, doi: 10.2307/1970571 MR 0248141

E-mail address: ralf.meyer@uni-goettingen.de

E-mail address: devarshi.mukherjee@mathematik.uni-goettingen.de

Mathematisches Institut, Georg-August Universität Göttingen, Bunsenstrasse 3-5, 37073 Göttingen, Germany