The role of curvature and stretching on the existence of fast dynamo plasma in Riemannian space

by

L.C. Garcia de Andrade

Departamento de Física Teórica – IF – Universidade do Estado do Rio de Janeiro-UERJ
Rua São Francisco Xavier, 524
Cep 20550-003, Maracanã, Rio de Janeiro, RJ, Brasil
Electronic mail address: garcia@dft.if.uerj.br

Abstract

Vishik’s antidynamo theorem is applied to non-stretched twisted magnetic flux tube in Riemannian space. Marginal or slow dynamos along curved (folded), torsioned (twisted) and non-stretching flux tubes plasma flows are obtained. Riemannian curvature of twisted magnetic flux tube is computed in terms of the Frenet curvature in the thin tube limit. It is shown that, for non-stretched filaments fast dynamo action in diffusive case cannot be obtained, in agreement with Vishik’s argument, that fast dynamo cannot be obtained in non-stretched flows. In this case a non-uniform stretching slow dynamo is obtained. An example is given which generalizes plasma dynamo laminar flows, recently presented by Wang et al [Phys Plasmas (2002)], in the case of low magnetic Reynolds number $Re_m \geq 210$. Curved and twisting Riemannian heliotrons, where non-dynamo modes are found even when stretching is presented, shows that the simple presence of stretching is not enough for the existence of dynamo action. Folding is equivalent to Riemann curvature and can be used to cancell magnetic fields, not enhancing the dynamo action. In this case non-dynamo modes are found for certain values of torsion or Frenet curvature (folding) in the spirit of anti-dynamo theorem. It is shown that curvature and stretching are fundamental for the existence of fast dynamos in plasmas. PACS numbers: 02.40.Hw:differential geometries. 91.25.Cw-dynamo theories.
I  Introduction

An anti-dynamo theorem by Zeldovich [1], shows that planar flows cannot support
dynamo action. Together with Cowling’s theorem [2], which considers that ax-
isymmetric magnetic fields cannot give rise to dynamo action, they form the more
traditional and well-tested antidynamo theorems ever. Actually, one of the main
ingredients in dynamo action is exponential stretching. This Lyapunov exponential
stretching was previously investigated by Friedlander and Vishik [3]. Because of the
cancellation of magnetic fields in regions of strong folding or curvature, presence of
stretching alone, is not enough to warrant fast dynamo action. This is mainly due
to the fact that, cancellation of magnetic fields are possible in regions of folding
or curvature. In tubes, the Riemann curvature could naturally cause these kind of
problems. However, one may always consider that inside the tubes magnetic fields
are so concentrated that if they do not strongly curves of folds, this cancellation
cannot be effective. Thus Riemann curvature would have to be too strong in order
that cancellation takes place. Friedlander and Vishik, argued that there are several topo-
logical obstructions to the existence of Anosov flows [4] in Riemannian 3-D space. Anosov
flows, are Riemannian spaces of constant negative curvature, endowed with geodesic
flows, have also been considered by Chicone and Latushkin [5] who showed that they are
simple example of fast dynamos in compact Riemannian manifolds. Other types of fast dy-
namo mechanisms based on stretching flux tubes in Riemannian conformal manifolds, have
been obtained recently by Garcia de Andrade [6]. Yet much earlier, M. Vishik [7] has argued
that only slow dynamos can be obtained from non-stretching dynamo flows. Actually no
fast dynamos obtained by Vainshtein and Zeldovich [8] stretch-twist and fold (STF)
method [9] can be obtained without stretching. This sort of ”anti-fast-dynamo” theorem
on flows, can here be generalized to flux tube dynamos and filaments. In this paper it is
shown that the Vishik argument can be extended to produce an anti-fast-dynamo theorem
for filaments and tubes in some particular cases, subject to bounds in poloidal and toroidal
magnetic fields, as well as to the twist (torsion) of the magnetic flux tube axis. Exponential
stretching of tube is derived. Bounds on the growth rate of fast dynamo action, based
on line-stretching, have been given by Klapper and Young [10]. Non-stretching in diffusionless media is used for flux tube dynamos, while diffusive filaments are used in the second case. Marginal dynamos are obtained in steady dynamos, where constraints are placed on the magnetic fields in order dynamo action can be effective. Diffusion processes, have been previously investigated by S. Molchanov [11], also in the context of Riemannian geometry. Such slow dynamos have been obtained by Soward [12], which argued that fast dynamo actions would still be possible in regions where no non-stretching flows would be presented, such as in some curved surfaces. It is clear, from this paper, that these surfaces could be of Riemannian. In filaments, it is shown that dynamo action in non-stretched unfolded and untwisting filaments could be obtained. When torsion vanishes, filaments are planar, which by Zeldovich anti-dynamo theorem [1] imply that dynamo action cannot be supported. Actually, it is shown that it generates a static magnetic initial field and a steady perturbation which may be, at best, a marginal dynamo. To resume, fast dynamos are generated by stretching, folding and twisting of the loops or filaments. It seems that non-stretched, folded and twisting filaments leads to slow dynamos filaments. Earlier, Mikhailovskii [13] has presented several Riemannian 3D metrics to describe various types of plasma devices such as heliotrons, tokamaks and stellarators. One year earlier Riemannian geometry was used by Ricca [14] to investigate inflexional instabilities in solar physics plasma loop. More recently, Ricca's geometry has been applied to curved twisted flux tube with current carrying loops [15], helical plasmas and conformal fast dynamos [6]. Another interesting and important application of Riemannian geometry to dynamo theory was given by Arnold and Arnold et al [16] in a somewhat artificial uniform exponential stretching to produce a chaotic fast dynamo. In 2002 a new kind of dynamo based on flowing laminar laboratory plasmas has been obtained by Wang et al [17]. Its plasma kinetic to magnetic energy conversion has been verified by numerical kinematical simulations. Their plasma flow topology is meant to substitute the liquid-sodium intrinsically turbulent dynamos [17]. Magnetic field free coaxial plasma guns [18] can be used to sustains this plasma dynamo flow. Since analytical solutions of magnetic vector potential are very hard to find in diffusive media, in this letter one considers the generalization of these plasma dynamo laminar flows to the
curved and twisting flow of heliotrons to arbitrary magnetic Reynolds numbers by solving the equations one for $Re_m \rightarrow \infty$ approximation. Though this approximation is more suitable to astrophysical applications than to plasma terrestrial LAB devices, the simplification and the results obtained seems to pay off on a first approach to the problem. By tuning the physical heliotron parameters such as magnetic vector potential components and twisting one obtains a constraint between the Frenet constant torsion and the number of revolutions along the toroidal direction and by assuming that the growth factor $\gamma$ of magnetic field, a non-dynamo mode is obtaining under these constraint. The curved and twisted Riemannian helical tube is thick as in the case of laminar plasma dynamo [19]. The paper is organized as follows: In section 2 a review on dynamics of holonomic Frenet frame is presented along with the discussion of vortex-filament exponential stretching. In section 3 the self-induction equation is solved in the framework of the Ricca’s [14] twisted magnetic flux tube in Riemannian 3D manifold. In this same section the ”new” anti-dynamo theorem is presented. In section 4 a twisted (torsioned) curved filament is shown to be a fast dynamo in Euclidean space. Section 5 contains the plasma non-dynamo modes example. In section 6, another, more realistic example of a stretched, non-dynamo flow is obtained in non-stretching regions. The converse of this case, where the fast dynamo can be obtained in stretching regions in the middle of non-stretching ones, is exactly the expression of Soward’s conjecture given above. Discussions and future prospects are presented in section 7.
II Exponential stretching in dynamo flux tube

This section contains a brief review of the Serret-Frenet holonomic frame [20] equations that are specially useful in the investigation of STF Riemannian flux tubes in magnetohydrodynamics (MHD) with magnetic diffusion. Here the Frenet frame is attached along the magnetic flux tube axis which possesses Frenet torsion $\tau(s)$ and curvature $\kappa(s)$, which completely determine topologically the filaments. Some dynamical relations from vector analysis and differential geometry of curves for the Frenet frame $(t, n, b)$ are given by

$$t' = \kappa n$$  \hspace{1cm} (II.1)

$$n' = -\kappa t + \tau b$$  \hspace{1cm} (II.2)

$$b' = -\tau n$$  \hspace{1cm} (II.3)

Here $t$ is the tangent vector to the magnetic toroidal component, while $n$ and $b$ lie on an orthogonal plane to vector $t$. The holonomic dynamical time evolution relations are obtained from vector analysis and differential geometry of curves as

$$\dot{t} = [\kappa' b - \kappa \tau n]$$  \hspace{1cm} (II.4)

$$\dot{n} = \kappa \tau t$$  \hspace{1cm} (II.5)

$$\dot{b} = -\kappa' t$$  \hspace{1cm} (II.6)

Together with the flow derivative

$$\dot{t} = \partial t + (\vec{v}, \nabla) t$$  \hspace{1cm} (II.7)

From these equations the generic flow [13]

$$\dot{X} = v_s t + v_n n + v_b b$$  \hspace{1cm} (II.8)

leads to the line-stretching equation

$$\frac{\partial l}{\partial t} = (-\kappa v_n + v_s')l$$  \hspace{1cm} (II.9)

where $l$ is given by

$$l := (X' \cdot X')^{\frac{1}{2}}$$  \hspace{1cm} (II.10)
Solution of equation (II.9) results in

\[ l = l_0 e^{\int (-\kappa v_n + v_s') dt} \]  

(II.11)

This shows that when the component along the tangent vector \( t \), \( v_s = v_0 \) is constant, the solenoidal incompressible flow equation reads

\[ \nabla \cdot v = 0 = (v_s' - \kappa v_n) \]  

(II.12)

This implies that if \( v_n \) vanishes, one is left with a non-stretched twisted flux tube flow. This choice \( v = v_0 t \), where \( v_0 = constant \) is the steady flow choice one uses here. This definition of magnetic filaments is obtained from the solenoidal character of the magnetic field

\[ \nabla \cdot B = 0 \]  

(II.13)

where \( B_s \) is its toroidal component. In the next section one solves the diffusion equation in the steady case. In the non-holonomic Frenet frame such equation is written as

\[ \partial_t B = \nabla \times (v \times B) + \eta \nabla^2 B \]  

(II.14)

Here \( \eta \) represents the magnetic diffusion. Since in astrophysical scales, \( \eta \nabla^2 \approx \eta L^{-2} \approx \eta \times 10^{-20} \text{cm}^{-2} \), for a solar loop scale length of \( 10^{10} \text{cm} \) [21] one notes that the diffusion effects can be neglected. Let us now consider the magnetic field definition in terms of the magnetic vector potential \( A \) as

\[ B = \nabla \times A \]  

(II.15)

where the gradient operator is

\[ \nabla = t \partial_s + \frac{1}{r} e_\theta + e_r \partial_r \]  

(II.16)
Let us now consider the Riemann metric of a flux tube in curvilinear coordinates \((r, \theta_R, s)\), where \(\theta = \theta_R - \int \tau(s) ds\). The integral of torsion along the filaments is called total torsion. This metric is encoded into the Riemann line element

\[
ds_0^2 = dr^2 + r^2 d\theta^2 + K^2 ds^2
\]

where, accordingly to our hypothesis, \(K^2 = (1 - \kappa r \cos \theta)^2\). This expression contributes to the Riemann curvature components. These can be easily computed with the tensor computer package in terms of Riemann components input. Adapting the general relativity (GR) tensor package to three-dimensions, yields the following curvature components expressions

\[
R_{1313} = R_{rsrs} = -\frac{1}{4K^2} [2K^2 \partial_r A(r, s) - A^2] = -\frac{1}{2} K^4 \frac{\kappa^4 \cos^2 \theta}{2} \tag{II.18}
\]

\[
R_{2323} = R_{\theta s \theta s} = -\frac{r}{2} A(r, s) = -K^2 \tag{II.19}
\]

where \(A := \partial_r K^2\). For thin tubes, \(K^2(r, s) \approx 1\) \((r \approx 0)\), and these Riemann curvature components reduce to

\[
R_{1313} = R_{rsrs} = -\frac{1}{2} \frac{\cos \theta}{r^2} \tag{II.20}
\]

Thus Riemann curvature of the tube is particularly strong as the tube axis \((r = 0)\) is approached. In the next section some of the mathematical machinery derived here is used in the formulation of a new anti-dynamo theorem. Let us now consider the generic flow in flux tube Riemannian space. As in Ricca’s [14], here one considers that no radial components of either flows or magnetic fields are present. These assumptions yields

\[
\dot{X} = v_s t + v_\theta e_\theta \tag{II.21}
\]

By considering the relation between the base \((e_r, e_\theta, t)\) and Frenet frame

\[
e_r = \cos \theta n + \sin \theta b \tag{II.22}
\]

and

\[
e_\theta = -\sin \theta n + \cos \theta b \tag{II.23}
\]

one may express formula \((II.21)\) as

\[
\dot{X} = v_s t - v_\theta \sin \theta n + v_\theta \cos \theta b \tag{II.24}
\]
Comparison between (II.24) and (II.21) yields

\[ v_n = -v_\theta \sin \theta \]  

(II.25)

and

\[ v_b = v_\theta \cos \theta \]  

(II.26)

which by a simple comparison with expression (II.22) yields

\[ \frac{\partial l}{\partial t} = (\kappa v_\theta \sin \theta + v_s')l \]  

(II.27)

Therefore one finally finds an expression for the exponential stretching \( l \) as

\[ l = l_0 e^{\int (\kappa v_\theta \sin \theta + v_s')dt} \]  

(II.28)

From the solenoidal incompressible flow equation

\[ \nabla \cdot \mathbf{v} = 0 \]  

(II.29)

one obtains

\[ \partial_s v_\theta = r \kappa \tau \sin \theta v_\theta \]  

(II.30)

where the operator \( \partial_s = -\tau^{-1} \partial_s \), and flux tube definition of twist angle \( \theta = \theta_R - \int \tau ds \) were used in the computation. Substituting this result into expression (II.28), and assuming as in Ricca’s paper that tube \( v_s' \) vanishes, integration yields

\[ l = l_0 e^{\left(\frac{\kappa}{\tau_0} \int v_\theta dt\right)} \]  

(II.31)

where filament cross-section is assumed to be constant. To simplify computations it is assumed that the dynamos are helical, which means that the torsion and Frenet curvature are taken as equal and constants \( (r = a) \). By changing the integrand according to \( v'_\theta dt = \frac{v'_\theta}{v_s} \) where \( dt = v_s^{-1} ds \), formula (II.31) reads

\[ l = l_0 e^{\left(\frac{\kappa}{\tau_0 v_s} \int v'_\theta dt\right)} \]  

(II.32)

where \( v_s \approx v_0 \) is taken to simplify matters. This shows that the torsion of the tube decreases the tube axis length and the poloidal flow enhances tube’s twist. This is similar to the piece of cloth twist example, where when the piece of cloth is twisted it is also
stretched. This result is actually confirmed by Klapper and Longcope [22] assertion that the twist is influenced by the flow where tubes are immersed. Actually the stretching of the tube and its torsion are related by the above expression (II.11), and torsion appears in the exponential stretching since in helical dynamos torsion equals the Frenet curvature $\kappa$. Now let us turn our attention to the plasma rotation inside the flux tube which can be obtained from the vorticity equation

$$\nabla \cdot \vec{\omega} = \nabla \cdot \nabla \times \vec{v} = 0$$

(II.33)
which yields the following PDEs

\[
\omega_r = -\partial_s v_\theta \\
\omega_\theta = \omega_0 = -\partial_r v_s \\
\omega_s = -[\partial_r v_\theta - \frac{\cos \theta}{r} \tau_0 v_\theta]
\] (II.34)

(II.35)

(II.36)

From vorticity expression (II.35) one obtains

\[
\omega_0 r = -v_s
\] (II.37)

Substitution of the expression for \(\partial_s v_\theta\) above into expression (II.34) now yields

\[
\omega_r = -\tau_0^2 r \sin \theta v_\theta
\] (II.38)

for the thin flux tube. For steady dynamos one obtains [17]

\[
(v.\nabla)B = (B.\nabla)v
\] (II.39)

\[
\frac{B_\theta}{B_s} = \frac{v_\theta}{v_s}
\] (II.40)

By making use of the vorticity equations one obtains

\[
\frac{B_s}{B_\theta} \approx \tau_0 r \cos \theta
\] (II.41)

Note then that near to the flux tube axis \((r \approx 0)\) the toroidal magnetic field decreases with respect to the from toroidal field. This expression is similar to one obtained by Ricca in terms of twist, and it seems to be important for plasma fusion devices, where magnetic field lines twisting can be adjusted in order turbulent damping effects takes place. A dynamo test can be obtained by computing the integral of Zeldovich

\[
\frac{4\pi d\epsilon_M}{dt} = \int (B.(B.\nabla)v) dV
\] (II.42)

where another diffusion term has been dropped. This expression can be applied to the flux tube if one considers that the above integrand is computed in detail as (Note to the referee: This equation has been fixed and detailed as demanded in one of your queries! Thanks!)

\[
(B.\nabla)B = [B_s - \frac{B_\theta \tau_0^{-1}}{r}]\partial_s (v_\theta e_\theta + v_s t)
\] (II.43)
which when scalarly multiplied by $B$ yields (note to the referee: this equation has also been corrected)

$$B.[(B.∇)B] = [-B_s v_θ τ_0 + B_θ τ_0^2 (v_θ - v_s τ_0^{-1})]$$  \hspace{1cm} (II.44)

which finally yields

$$\frac{4\pi d\epsilon_M}{dt} = \int [B_θ τ_0^2 \sinθ[v_θ - τ_0^{-1} v_s] + B_s v_θ (B_s - \frac{B_θ τ_0^{-1}}{r})]dV$$  \hspace{1cm} (II.45)

which shows that

$$B_s = \frac{B_θ τ_0^{-1}}{r}$$  \hspace{1cm} (II.46)

implies the existence of a marginal dynamo, where $\frac{4\pi d\epsilon_M}{dt}$ vanishes. Note that this result does not depend directly on the stretching but on the torsion which is indirectly responsible for stretching. Therefore exponential stretching may exists even for marginal dynamos, which would represent a converse result of a Vishik’s antidynamo theorem for flux tubes. Non-stretching flows implies necessarily the existence of slow dynamos. Nevertheless the converse is not true, slow dynamos do not necessarily imply non-stretching. The weak torsion approximation used here is in agreement with the weak torsion (twist) found in the solar twisted coronal loop torsion ($τ_0 \approx 10^{-10} cm^{-1}$) [21].
III  Anti-dynamo theorem in non-stretching filaments

In this section the anti-dynamo formulation of non-stretching dynamo flows in twisted filaments is presented. This can be done by simply considering the gradient along the filament as $\nabla = t \partial_s$ and computing the total magnetic energy on a diffusive medium as

$$\frac{4\pi d\epsilon_M}{dt} = \int B_s^2 v_s t \cdot n dV := 0 \quad (III.47)$$

since $n \cdot t = 0$ (note to the referee: This clearly shows that $b$ is the other leg in the frame and not the magnetic field as previously thought by the referee).

Here one has considered that $B := B_s t$. Thus one can say that in a diffusive media, when $v'_s = 0$ and $v_n$ both vanishes the following lemma is proved.

**Lemma:**

Non-stretching vortex filaments in a diffusionless media gives rise to a marginal or slow dynamo. No fast dynamo being possible.

This can be considered as a sort of anti-fast dynamo theorem motivated from Vishik’s idea. Now let us introduce diffusion into the problem. As previously noted by Zeldovich [1], this leads us to the dynamo action and possibly to fast dynamos. In the case of diffusive filaments one notes that the magnetic induction equation

$$\frac{d}{dt} B = (B \cdot \nabla)v + \eta \nabla^2 B \quad (III.48)$$

where $\eta$, the magnetic resistivity or diffusion, is considered as constant, reduces the problem to the three scalar equations

$$\frac{dB_s}{dt} = -\eta \kappa B_s \quad (III.49)$$

$$\kappa' = \eta \kappa \tau \quad (III.50)$$

$$-\kappa \tau = \eta \kappa' + v_s \quad (III.51)$$

Solution of these equations yields

$$B_s = \exp[-\eta \int \kappa^2 ds] B_0 \quad (III.52)$$
where $\int \kappa^2 ds$ is the total Frenet curvature energy integral. The remaining solutions are

$$\kappa = \exp[\eta \int \tau ds]\kappa_0$$  \hspace{1cm} (III.53)

where $\int \tau ds$ is the total torsion. In the case of helical filaments, where torsion equals curvature, one obtains

$$-\tau_0^2 = v_s$$  \hspace{1cm} (III.54)

Note that the decaying of the magnetic field depends on the sign of the integral, but since the average value of this integral is positive, the magnetic field decays and no dynamo action is possible, and even slow dynamos cannot be found. In the case of non-stretching filaments, the presence of diffusion actually enhances the non-dynamo character of Vishik’s lemma.

**IV Filamentary fast dynamos in Euclidean space**

Earlier, Arnold et al [16] found a stretching and compressed fast dynamo in curved Riemannian space. Though this served as motivation for the study of flux tube dynamos in 3D curved Riemannian space, in this section one shall address the problem of finding a filamentary fast dynamo in Euclidean space in 3D. The stretching followed by squeezing is a path to finding a growing magnetic field. Recently Nuñez [22] has considered a similar problem, also making use of Frenet frame as here, by investigating eigenvalues in the stretching plasma flows. In this section one shows that the stretching condition in filaments is fundamentally connected to the incompressibility of the flow. This is simply understood if one considers the exponent in the stretching in expression

$$\gamma := -\kappa v_n + v_s'$$  \hspace{1cm} (IV.55)

which shows that stretching factor gamma is a fundamental quantity to be examined when one wants to find out a fast dynamo action. Note that when $\gamma \geq 0$ or $\gamma < 0$, one would have respectively either a fast or slow or marginal dynamo and a decaying magnetic field as found before. Let us now drop the constraint that $v_s$ is
constant and substitute the flow
\[ \mathbf{v} = v_n \mathbf{n} + v_s \mathbf{t} \quad \text{(IV.56)} \]
into the solenoidal incompressible flow
\[ \nabla \cdot \mathbf{v} = 0 \quad \text{(IV.57)} \]

**with vanishing** \(v_n\), one should have a non-stretched twisted flux tube. This is exactly the choice \(\mathbf{v} = v_0 \mathbf{t}\), where \(v_0 = \text{constant}\) is the steady flow one uses here. This definition of magnetic filaments is **obtained** from the solenoidal **character** of the magnetic field
\[ \nabla \cdot \mathbf{v} = -\kappa v_n + v_s' = 0 \quad \text{(IV.58)} \]

This result implies that \(\gamma = 0\) which **in turn** implies no flow stretching at all! This **leads** us to note that if a fast filamentary dynamo action is possible, a modification of the flow has to be performed. To investigate this possibility one considers the following form of the dynamo flow
\[ \mathbf{v} = v_n \mathbf{n} + v_s \mathbf{t} + v_0 \mathbf{b} \quad \text{(IV.59)} \]
which minimally generalizes (IV.56). In this case, the above expression for \(\gamma\) does not vanish and is equal to
\[ \gamma = -\kappa v_n + v_s' = \tau_0 v_0 \quad \text{(IV.60)} \]

Note that again for \(\gamma > 0\) one obtains a fast dynamo since \(\eta = 0\) and stretching is possible if \(\tau_0\) and \(v_0\) possess the same sign. Actually this flow leads to the following three scalar dynamo equations
\[
\frac{d}{dt} B_s = \gamma B_s - \tau_0^2 B_n \quad \text{(IV.61)}
\]
\[
\frac{d}{dt} B_n = \tau_0 (v_s - \tau_0 - \frac{1}{\tau_0} \partial_s v_n) B_s \quad \text{(IV.62)}
\]
\[
B_s \tau_0 v_n = 0 \quad \text{(IV.63)}
\]
Here \(\gamma = (v_s' - \tau_0 v_n)\) since we are considering helical dynamo filaments. From equation (IV.62) one obtains \(v_n = 0\) which **further** simplifies the other equations. Since in astrophysical scales, torsion is as weak as \(\tau_0^2 \approx \eta \times 10^{-20} \text{cm}^{-2}\), **for a solar coronal loop scale**, the terms proportional to torsion squared may be dropped. In this
approximation a fast dynamo solution is found from (IV.61) as (note to the referee: these equations are new)

$$B_s = B_0 e^{\gamma t}$$  \hspace{1cm} (IV.64)

and

$$\frac{d}{dt} B_n = \tau_0 (v_s - \tau_0) B_s$$  \hspace{1cm} (IV.65)

which yields

$$B_n = \frac{1}{v_0} (v_s - \tau_0) e^{\gamma t}$$  \hspace{1cm} (IV.66)

where to obtain $B_n(t, s)$ use has been made of the stationary property of the flow $\partial_t v_s = 0$, in order of being able to integrate (IV.64). Thus a fast dynamo action for filamentary flows in Euclidean 3D space has been obtained. Note that this action is natural and actually somewhat expected, since when one changes the dynamo three-dimensional $(v_s, v_n, v_b)$ a two dimensional flow $(v_s, v_n)$ is obtained , which is strictly forbidden from Zeldovich anti-dynamo theorem. Besides if one integrates the flow equation (IV.59) yields (note to the referee: This formula is new)

$$v_s = \tau_0 v_0 s + c_1$$  \hspace{1cm} (IV.67)

where $c_1$ is an integration constant. Expression (IV.67) indicates that the flow along the filament undergoes a uniform strain [18] due to the stretching of the magnetic filament. Let us now consider the approximation of weak torsion given in solar plasma loops for example. In this approximation the above relations reduce to

$$\frac{d}{dt} B_n = \tau_0 v_s B_s$$  \hspace{1cm} (IV.68)

which yields

$$B_n = \frac{\tau_0 c_1 B_0}{\gamma} e^{\gamma t}$$  \hspace{1cm} (IV.69)

Now if one uses these relations into the total magnetic energy integral, one obtains

$$\epsilon_M = \frac{a^2}{8} [B_0^2 (1 + \frac{c_1^2}{v_0^2})] e^{2 \gamma t}$$  \hspace{1cm} (IV.70)

This magnetic energy indicates that the stretching of the magnetic field accumulates energy and gives rise to a fast dynamo, where $\gamma = \tau_0 v_0$. Thus one notes that the fast
dynamo is enhanced by the presence of torsion and toroidal velocity \( v_0 \). This result is actually confirmed by Klapper and Longcope [21] assertion that the twist is influenced by the flow where tubes are immersed. Actually the expression (IV.69) shows that the stretching of the tube is enhanced by torsion since this last one appears in the exponential stretching. Physically this means that the flow along the magnetic filament is stronger than any magnetic orthonormal perturbation and the system might be stable. (note to the referee: The next section is a whole new section which is meant as an application to the above anti-dynamo theorem).

V Plasma dynamo Riemannian twisted flux tube

This section contains a solution of the self-induction magnetic equation, given by

\[
\partial_t \mathbf{B} = (\mathbf{B} \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{B} + \eta \nabla^2 \mathbf{B}
\]  

(V.71)

where the magnetic field is defined in terms of the vector potential \( \mathbf{A} \) as

\[
\mathbf{B} := \nabla \times \mathbf{A}
\]  

(V.72)

Here \( \eta = \frac{U L}{\text{Re}_\text{in}} \) is the resistivity diffusion, while the \( U \) and \( L \) are respectively, the velocity and length scales of the plasma flow. Besides equation (V.71) one also considers as in laminar plasma dynamos that the plasma is a Beltrami flow given by the equation

\[
\nabla \times \mathbf{u} = \lambda_B \mathbf{u}
\]  

(V.73)

The self-induction equation (II.1) can be split into three scalar equations if one considers the Frenet equations above. Now let us consider briefly the Riemannian geometry of the twisted flux tube as the heliotron, where the Frenet torsion \( \tau_0 \) is taken as constant and equal to the Frenet curvature of the magnetic heliotron axis. where \( B_s \) is the toroidal component of the magnetic field. Let us now consider the magnetic field definition in terms of the magnetic
vector potential $A$ as

$$B = B_\theta(r, \theta)e_\theta + B_s(r)t$$  \hspace{1cm} (V.74)$$

The gradient operator is

$$\nabla = t \partial_s + \frac{1}{r} e_\theta + e_r \partial_r$$  \hspace{1cm} (V.75)$$

Let us now consider the curvilinear coordinate relation

$$\partial_s e_\theta = -\tau_0 \sin \theta t$$  \hspace{1cm} (V.76)$$

From all the above expressions one may finally split the equation (II.1) into the three scalar equations

$$\frac{\gamma U_{\text{max}}}{L} B_s^0 = B_\theta^0 \tau_0 \sin \theta + u_\theta \tau_0 \sin \theta \beta$$  \hspace{1cm} (V.77)$$

where

$$\beta := \frac{B_s}{K} - \frac{\tau_0^{-1} B_\theta^0}{r}$$  \hspace{1cm} (V.78)$$

$$\frac{\gamma U_{\text{max}}}{L} B_\theta^0 = B_\theta^0 \tau_0 + \beta u_\theta \tau_0 \tan \theta + u_s \tau_0 \csc \theta$$ \hspace{1cm} (V.79)$$

$$\frac{\gamma U_{\text{max}}}{L} B_\theta^0 = -\beta u_\theta \tau_0 \tan \theta + i \left[ \frac{u_s}{K} - \frac{\tau_0^{-1} u_\theta}{r} \right]$$  \hspace{1cm} (V.80)$$

Let us start solving this system by finding the conditions for a non-dynamo mode which is obviously given by the condition $\gamma = 0$. Substitution of this condition into the last equation yields

$$-\beta u_\theta \tau_0 \tan \theta + i \left[ \frac{u_s}{K} - \frac{\tau_0^{-1} u_\theta}{r} \right] = 0$$ \hspace{1cm} (V.81)$$

since the second term inside the brackets is a quantity that belongs to the real numbers and $\beta$ is also real, one obtains

$$\frac{u_s}{K} = \frac{\tau_0^{-1} u_\theta}{r}$$  \hspace{1cm} (V.82)$$

It is also easy to note that $\sin \theta$ vanishes when one considers that the resistivity diffusion vanishes or that the $Re_m \rightarrow \infty$. This vanishing of sine function implies

$$\tau_0 = \frac{2\pi}{a} [\theta_R - 2\pi m]$$   \hspace{1cm} (V.83)$$
where \( m \) is an integer. This shows that non-dynamo modes \( \gamma = 0 \), can be obtained by tuning the heliotron [24] torsion to certain non-dynamo surfaces. Note to the referee: Next section is a fundamental new one since it provides us with an example of non-stretching non-dynamo flows as in Vishik’s lemma in the case of diffusive media. Note to the referee: This next section gives us a further example of non-dynamos and slow modes.

### VI Non-stretching slow dynamos in twisted diffusive Riemannian space

In this last section a new example of a more realistic non-stretching flow in diffusive plasma medium is given. As one shall now show this leads to a slow dynamo. Radial modes for the existence of marginal dynamos are obtained. The approximation of very weak torsion above is given for a thin non-stretched tube. Let us now consider the diffusive self-induction equation

\[
\frac{d}{dt} B = \partial_t B + (u \nabla) B = -(B \cdot \nabla) u + \eta \nabla^2 B
\]  

(6.84)

when the non-stretched hypothesis is effective, the term \( (B \cdot \nabla) u \) has to vanish and equation (6.84) reduces to

\[
\frac{d}{dt} B = \eta \nabla^2 B
\]  

(6.85)

By taking the magnetic field functional form

\[
B = B_0(r) e^{i(\gamma t + [k_x s + k_\theta \theta])}
\]  

(6.86)

Substitution of this form into the expression (6.85) yields

\[
\gamma B = \eta \nabla^2 B
\]  

(6.87)

Since the tube is twisted one shall compute the Laplacian operator as

\[
\Delta := \nabla^2 = [e_r \partial_r + X \partial_s]^2
\]  

(6.88)
Here
\[ X := [t - \frac{\tau_0}{r} e_\theta] \] (VI.89)

A simple algebraic manipulation yields
\[ \gamma B_0 = \eta [\partial_r^2 B_0 + \tau_0 \cos \theta \partial_r B_0 - \frac{\tau_0}{r^2} [k_s - \tau_0 k_\theta]^2 + i [k_s - \tau_0 k_\theta] \frac{\sin \theta}{r}[B_0] \] (VI.90)
a great deal of simplification is achieved by considering the solenoidal character of the B field as
\[ \nabla \cdot B = i [k_s - \tau_0 k_\theta] B = 0 \] (VI.91)
which implies that
\[ \frac{k_s}{k_\theta} = \tau_0 \] (VI.92)
Thus the poloidal wavelength number dominates over the toroidal one in the case of weak torsion. By substitution of this result and considering that the a complex wave length number \( k_s := ik_0 \), into the equation (VI.90) yields
\[ \partial_r^2 B_0 - \eta^{-1} \gamma B_0 = 0 \] (VI.93)
Assuming that the \( B_0(r) \) has the form
\[ B_0(r) = r^n \] (VI.94)
where \( n \) is a real number. Thus by substituting the ansatz (VI.94) into the expression (VI.93) yields
\[ n(n - 1)r^{n-2} - \eta^{-1} \gamma r^n = 0 \] (VI.95)
By making use of the ordinary differential equation technique of reducing both integers in (VI.95) to the same \( r^n \), one has to substitute \( n \to n + 2 \) into the first term exponent of the LHS of equation (VI.95). These ODE operations reduce the algebraic equation (VI.95) to
\[ [(n + 2)(n + 1) - \eta^{-1} \gamma]r^n = 0 \] (VI.96)
Assuming that one is never at the magnetic axis where \( r = 0 \) one has to solve the algebraic equation
\[ n^2 + 3n + (2 - \eta^{-1} \gamma) = 0 \] (VI.97)
which is a second-order algebraic equation. Solution of this equation allows us now to determine the radial n-modes which leads to the dynamo solutions. These solutions are

\[ n_\pm = -\frac{3}{2} \pm \frac{7}{2} (1 - \frac{1}{14} \eta^{-1} \gamma) = 0 \]  

(VI.98)

This solution yields the following modes for the growth of magnetic field \( \gamma \)

\[ \gamma_+ = 4(n_+ - 2)\eta \]  

(VI.99)

and

\[ \gamma_- = -4(n_- + 5)\eta \]  

(VI.100)

Both these modes are actually non-dynamos in agreement with Vishik’s lemma, since there is a direct dependence with resistivity \( \eta \) thus when \( \eta \to 0 \) so is \( \gamma \to 0 \). This is actually exactly the slow dynamo condition [9]. Note that marginal dynamo modes can be obtained by using the constraints \( \gamma_\pm = 0 \) into respectively equations (VI.99) and (VI.100). This yields immediately the solutions corresponding to \( n_+ = 2 \) and \( n_- = -5 \) as

\[ B_\pm = tr^2 e^{i(k_s s + k_\theta \theta)} \]  

(VI.101)

and

\[ B_- = tr^{-5} e^{i(k_s s + k_\theta \theta)} \]  

(VI.102)

Note that the first mode is regular near to the magnetic torsioned axis, while in the other mode the magnetic field grows in space very fast as the magnetic field axis is approached. The role of the curvature, either Gaussian or Riemannian, on the existence of fast dynamos can still be better understood if one imposes their vanishing in the Chicone-Latushkin [5] expression for \( \gamma \) in the fast dynamo obtained from geodesic flows in Anosov spaces of negative constant Riemannian curvature

\[ \gamma = \frac{1}{2} [-\eta(1 + \kappa^2) + \sqrt{\eta^2(1 - \kappa^2)^2 - 4 \kappa}] \]  

(VI.103)

where \( \kappa \) is the Gaussian curvature. From this expression one notes that when this Gaussian curvature of the two-dimensional manifold vanishes, the expression for \( \gamma \) vanishes. This shows that in the absence of curvature a marginal
non-fast dynamo is obtained. Therefore theorems discussed here maybe con-
sider as particular cases of Chiconne-Latushkin theorem [5] on fast dynamos
of specific Riemannian plasma geometries. Actually in cases investigated in
this paper, stretching is deeply connected to curvature through term $K(r, s)$
in Ricca’s metric. This lead us to conclude that the curvature is actually con-
ected to stretching and folding. Since in the case considering in this section,
twist, curvature and stretching are almost neglected the result seems to be
physically correct.

VII Conclusions

Vishik’s idea that the non-stretched flows cannot be fast dynamos, is tested
once more here in several examples. All these examples involve either stretch-
ing or non-stretching of non-dynamo modes, proving in this way, that even
stretching flux tubes like the plasma heliotrons for example, may not provide
a dynamo action. This constitute a new framework to discuss anti-dynamo
theorems. When twist or Frenet torsion is small, slow dynamos are shown to
be present in these astrophysical loops. By following the analogy proposed
by Friedlander and Vishik in dynamo theory between vorticity equations and
dynamo equations and considering exponential stretching one shows that flux
tube dynamos are in complete agreement with Ricca’s original Riemannian
magnetic flux tube model. STF Zeldovich-Vainshtein fast dynamo genera-
tion method ,is not the only Riemannian method that can be applied as in
Arnold’s cat map, but other conformal fast kinematic dynamo models as the
Riemannian one, can also be useful as new dynamo action generation method.
Small scale dynamos in Riemannian spaces can therefore be very useful for our
understanding of more large scale astrophysical dynamos. Other applications
of plasma filaments such as stretch-twist and fold fractal dynamo mechanism,
which are approximated Riemannian metrics have been recently put forward
by Vainshtein et al [8]. Finally one has shown that the Vishik’s result is
strongly enhanced, and no fast dynamo and sometimes even slow dynamo is found in non-stretching flows. Thus decay of the magnetic field in non-stretched filaments in presence of magnetic diffusion and the Frenet torsion is found. As considered by Arnold [16] and Vainshtein et al [2] no fast dynamos exists in 2D Euclidean space and a fast dynamo was obtained in 3D curved Riemannian space by Arnold [16] where stretching in some directions is compensated by compression in the other, in somewhat artificial uniform stretching. In this paper a fast filamentary dynamo was obtained in Euclidean 3D space. Note that when torsion vanishes in the last section a marginal dynamo is obtained. This result, a sort of filamentary antidynamo theorem, has been recently obtained [25] in the non-holonomic frame. A more realistic diffusive media is used to obtain an example of Riemannian plasma tubes, where no fast dynamo action is obtained. One notes that is very easy to generalizes the results obtained here to the non-uniformly stretching flows by simply replacing the Ricca’s Riemannian metric by $ds^2_0 = H(s - s_0)[dr^2 + r^2 d\theta^2] + K^2(r,s)ds^2$, where $H$ is the stept Heaviside function. This new Riemann flux ”fracture tube” metric leads to a fast dynamo flow where the flows are proportional to the Dirac delta function similar to chaotic periodic dynamo flows investigated by Finn and Ott [26]. The detailed computations and further discussions may appear elsewhere.

VIII Acknowledgements

I am deeply greatful to Renzo Ricca and Yuri Latushkin for their extremely kind attention to our work. Thanks are also due to I thank financial supports from Universidade do Estado do Rio de Janeiro (UERJ) and CNPq (Brazilian Ministry of Science and Technology).
References

[1] M. Vishik, Izv Acad Sci USSR, Phys Solid Earth 24(1), 173(1988).

[2] S. Cowling, Magnetohydrodynamics (1964) Oxford. S.I. Vainshtein, Ya B Zeldovich, Sov Phys Usp 15 ,159 (1972). S.Vainshtein, A. Bykov and I.N. Toptygin, Turbulence, Current Sheets and Shock waves in Cosmic Plasmas, Gordon

[3] S. Friedlander, M. Vishik, Chaos 1(2),198 (1991).

[4] D.V. Anosov, Geodesic Flows on Compact Riemannian Manifolds of Negative Curvature, (Steklov Mathematical Institute, USSR) vol.90 (1967).

[5] C. Chicone and Yu Latushkin, Evolution Semigroups in Dynamical systems and differential equations, American Mathematical Society, AMS-(1999). C. Chicone and Yu Latushkin, Proc of the American Mathematical Society 125,N. 11,3391 (1997).

[6] L. C. Garcia de Andrade, Physics of Plasmas 14 (2007).

[7] M. Vishik, Izv Acad Sci USSR, Phys Solid Earth 24(1),173(1988).

[8] S.Vainshtein, A. Bykov and I.N. Toptygin, Turbulence, Current Sheets and Shock waves in Cosmic Plasmas, (1998) Gordon

[9] S. Childress, A. Gilbert, Stretch, Twist and Fold: The Fast Dynamo (1996),Springer, Berlin.

[10] I Klapper and L Young , Comm Math Phys. 173 623 (1995).

[11] S Molchanov, Russian mathematical surveys (1978).

[12] A M Soward, Geophys Astrophys Fluid Dyn 53,81 (1990).

[13] A. B. Mikhailovskii, Instabilities in a confined Plasma, IOP (1998).

[14] R. Ricca, Solar Physics 172 (1997),241.
[15] L C Garcia de Andrade, Phys Plasmas 13 (2006). L. C. Garcia de Andrade, Non-holonomic dynamo filaments as Arnolds map in Riemannian space, Astronomical notes (2008) in press.

[16] V. Arnold, Ya B. Zeldovich, A. Ruzmaikin and D.D. Sokoloff, JETP 81, n. 6, 2052 (1981). V. Arnold, Appl Math and Mech 36, 236 (1972).

[17] Z Wang, V Pariev, C Barnes and D Barnes, Phys Plasmas 9,5 (2002).

[18] P M Bellan, Spheromaks: A practical application of MHD Dynamos and Plasma self-organization, (2002) Imperial College Press.

[19] A Jacobson, Phys Fluids 27 (1),7 (1984).

[20] M. Berger and C Prior, J Phys A 39, 8321 (2006). P Newton, The N-Vortex problem (2001) Springer.

[21] M.C. Lopez Fuentes, P. Demoulin, C.H. Mandrini, A.A. Pevtsov and L. van Driel-Gesztelyi, Astr and Astrophys. 397, 305 (2003).

[22] M. Nuñez, J. Phys. A:Math and Gen. 36, 8903 (2003).

[23] I Klapper and D Longcope, Evolution Equations of Thin Twisted Flux Tubes, in "Workshop on Stellar Dynamos ASP conference series, (1999) M. Nunez and Ferriz-Mas eds.

[24] M. Wakatani, Stellarator and Heliotron Devices, (1998) Oxford University Press.

[25] L. C. Garcia de Andrade, Non-holonomic dynamo filaments as Arnolds map in Riemannian space, Astronomical notes (2008) in press.

[26] J M Finn and E Ott, Phys Fluids 31 (1983).