Quantum particle production near the big rip revisited

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Abstract
The effect of quantum particle production near the big rip singularity has been investigated previously, with the conclusion that the energy of the produced particle decreases as the future singularity is approached. Hence, the effect of particle production would not be effective to avoid the big rip singularity. That calculation was performed by introducing an ultra-violet cut-off. In the present work we consider a renormalization of the energy-momentum tensor, obtaining a different expression for the particle production. The new expression seems to indicate that the effect of particle production may be dominant as the singularity is approached.

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1 Introduction
The particle creation in curved space-time is a quantum effect associated with the non-unicity of the vacuum state when the space-time does not have a time-like killing vector [1, 2]. In this situation the curvature changes with time, and the characterization of the vacuum state depends on time. Hence, if we fix an initial vacuum state at, let us say, $t = 0$, this vacuüm state becomes later a multiparticle state. This is just another way of saying the particles are created as the space-time evolves with time.

There are two main situations where the particle creation as a gravitational phenomena has been especially studied: the space-time of a black hole and the expanding universe. If we consider the fact that the black hole must be formed
dynamically by the gravitational collapse process, there is no time invariance of the configuration, the curvature grows with time, and particle must be created as the event horizon is formed. On the other hand, the time invariance does not exist when the universe expands, and we must also expect that the phenomena of particle creation must appear in the usual cosmological scenarios. We must expect that the particle creation rate must be linked with the curvature of the four-dimensional space-time.

Considering the usual Friedmann universe, with homogeneity and isotropy, there are two situations where the particle creation phenomena does not occur. The first one, is when the universe is in a de Sitter state. The de Sitter space-time is maximally symmetric. Hence, it contains a timelike Killing vector, and the vacuum state must be unique. The other situation is a radiative universe. Since the electromagnetic field is conformal invariant, the particle production is zero; in fact, an isotropic conformal field does not lead to any particle creation.

For other types of Friedmann universes, the situation is more complex. In the cases where the strong energy condition $\rho + 3p \geq 0$ ($\rho$ the energy density and $p$ the pressure) is satisfied, the curvature is initially infinite, decreasing later with time. In such a situation, the greatest problem is to define an initial vacuum state: such definition becomes quite arbitrary, and the vacuum state appears naturally only in the asymptotic future. When the strong energy condition is violated, however, it is possible to define uniquely a vacuum state, and the problem of particle creation seems to be well posed.

The problem of particle creation in a Friedmann universe has been analysed recently in reference [3]. The main motivation of that analysis is the fact that the universe today seems to be dominated by a phantom fluid [4], a fluid that violates the dominant energy condition $\rho + p \geq 0$. A universe dominated by a phantom fluid will end inevitably in a future singularity, in a finite proper time, called big-rip: as the universe expands, the energy density of the phantom fluid increases, becoming divergent at a given moment in the future [5]. If this is so, the question it was tried to be answered in that work was: can the particle creation to be so effective that it leads to the avoidance of the big rip? It was used a massless scalar field leaving in a universe dominated by a fluid with an equation of state $p = \alpha \rho$. In particular, it was payed special attention to the case $\alpha < -1/3$, the situation where the strong energy condition is violated.

The analysis of this problem becomes cumbersome due to the fact that the energy-momentum tensor for the scalar field becomes divergent in the ultraviolet limit. Hence, a renormalization procedure must be employed. In the case of curved space-time this is not a simple task. In reference [3] it was chosen to introduce an ultraviolet cut-off connected with the Planckian frequency. The reason for that is the fact that such problem involves gravitation, and above the Planck frequency we enter in the quantum gravity domain, and very probable the dispersion relation connecting the frequency $\omega$ and the wavenumber $k$ must be changed.

The answer for the question of the avoidance of the big-rip due to particle creation found in [3] using the procedure described above was negative: the number of particle created goes to infinity, but the associated energy density
goes to zero, as the big-rip is approached. This seems to agree with the result obtaining in reference [6], where a similar study was performed in the case of the sudden singularity [7], a singularity that occurs also in the future but without violation neither of the strong energy condition nor the dominant energy condition: the sudden singularity is robust with respect to the particle creation phenomena.

However, it must be remarked that, in contrast with what happens in the case of reference [3], in the sudden singularity scenario the subtraction of the infinities in the energy-momentum tensor is a much easier task, and is possible to obtain a renormalized energy-momentum tensor (ignoring all transplanckian problems). Here, we would like to come back to the problem treated in reference [3], and ignore the transplanckian problem also in big-rip scenario, trying to renormalize the energy-momentum tensor for the massless scalar field. We employ the \( n \)-wave method, which is quite convenient for an isotropic universe. However, this task becomes more complicated due to the existence of logarithmic divergence in the ultraviolet limit. We propose a simple way to deal with this problem, without introducing any new arbitrary parameter. We obtain a finite energy-momentum tensor. However, using now this renormalized energy, we verify that the energy associated with the particle production diverges near the big rip, and it can become the dominant energy component, altering the final state of the universe. The unicity of the answer and its modification due to transplanckian considerations remain open questions. However, our result points out that the problem can be much more complex than it seems at first glance.

This paper is organized as follows. In next section we revise the general problem of renormalization of a quantum scalar field living in a FRW spacetime. In section 3 we apply the formalism to the case of a radiative universe, and in section 4 to the case of a de Sitter universe. In section 5 the general inflationary case is discussed, showing that the method has some limitations due to the appearance of an ultraviolet divergence. A full renormalisation is proposed in section 6, obtaining as result that the big rip may be avoid due to quantum effects. In section 7 we present our conclusions.

2 General problem of renormalization

The Friedmann-Lemaître-Robertson-Walker (FLRW) metric

\[
ds^2 = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2)
\]

(1)

describes a spatially flat, expanding universe, with \( a(t) \) being the time-dependent scale factor. In terms of the conformal time defined by \( dt = ad\eta \), the FLRW metric takes the form

\[
ds^2 = a^2(d\eta^2 - dx^2 - dy^2 - dz^2).
\]

(2)
If the universe is filled with a barotropic perfect fluid, with the equation of state \( p = \alpha \rho \), the Friedmann equation
\[
\left( \frac{a'}{a} \right)^2 = \frac{8\pi G}{3} \rho a^2
\]
(primes meaning derivative with respect to the conformal time \( \eta \)) implies
\[
a \propto \eta^\beta, \quad \beta = \frac{2}{1 + 3\alpha}.
\]
For \( \alpha > -1/3 \), the strong energy condition \( \rho + 3p \geq 0 \) is satisfied and the universe expands desacelerating as \( \eta \rightarrow \infty \); for \( \alpha < -1/3 \) the universe expands accelerating as \( \eta \rightarrow 0^- \). In particular, \( \alpha = -1 \) corresponds to a de Sitter universe, while \( \alpha = 1/3 \) implies a radiation-dominated universe. If \( \alpha < -1 \), the fluid is called phantom, and the universe must faces a big-rip in the future.

The fundamental equation for a massless scalar field is living in a FLRW space-time is,
\[
\phi''_k + 2\frac{a'}{a} \phi'_k + k^2 \phi_k = 0.
\]

The general expressions for the energy and pressure of the quantum scalar field are given by the following expressions:
\[
\rho = \frac{1}{a^2} \int_0^\infty \left\{ \phi'_k \phi'^*_k + k^2 \phi_k \phi^*_k \right\} k^2 dk;
\]
\[
p = \frac{1}{a^2} \int_0^\infty \left\{ \phi'_k \phi'^*_k - \frac{k^2}{3} \phi_k \phi^*_k \right\} k^2 dk.
\]

These expressions can be written as
\[
\rho = \int_0^\infty k^2 E_k dk, \quad E_k = \frac{1}{a^2} \left\{ \phi'_k \phi'^*_k + k^2 \phi_k \phi^*_k \right\},
\]
\[
p = \int_0^\infty k^2 P_k dk, \quad P_k = \frac{1}{a^2} \left\{ \phi'_k \phi'^*_k - \frac{k^2}{3} \phi_k \phi^*_k \right\}.
\]

In general, the above expressions for the energy and the pressure have quartic, quadratic and logarithmic divergencies in the ultraviolet limit, while it is regular in the infrared limit. In order to cope with these divergencies, we employ the \( n \)-wave method described in the reference [8]. Essentially, this method consists in subtracting terms obtained by expanding \( E_k \) and \( P_k \) in powers of \( k^{-2} \):
\[
E_k^{\text{ren}} = E_k - E_k^0 - E_k^1 - \frac{1}{2} E_k^2,
\]
\[
P_k^{\text{ren}} = P_k - P_k^0 - P_k^1 - \frac{1}{2} P_k^2,
\]
where
\[
E_k^p = \lim_{n \to \infty} \frac{\partial^p E_k^n}{\partial (n^{-2})^p}, \quad P_k^p = \lim_{n \to \infty} \frac{\partial^p P_k^n}{\partial (n^{-2})^p}.
\]
with the definitions,

\[ E_k^n = \frac{1}{n} E_k(nk), \quad P_k^n = \frac{1}{n} P_k(nk). \]  \hfill (13)

It will come out more convenient to express the derivatives as

\[ \frac{\partial f}{\partial n^{-2}} = \frac{\partial n}{\partial n^{-2}} \frac{\partial f}{\partial n} = -\frac{n^3}{2} \frac{\partial f}{\partial n}, \]  \hfill (14)

and subsequently for the higher derivatives, \( f \) being either \( E_k \) or \( P_k \). Hence, we find:

\[ \frac{\partial f}{\partial n^{-2}} = -\frac{n^3}{2} \frac{\partial f}{\partial n}, \]  \hfill (15)

\[ \frac{\partial^2 f}{\partial n^{-2}} = \frac{3}{4} n^5 \frac{\partial f}{\partial n} + \frac{n^6}{4} \frac{\partial^2 f}{\partial n^2}. \]  \hfill (16)

It will be necessary later to use the following expressions for the Hankel’s functions in the limit of large values for the argument:

\[ H^{(1)}_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \left\{ P(\nu, x) + iQ(\nu, x) \right\} e^{i\chi}; \]  \hfill (17)

\[ H^{(2)}_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \left\{ P(\nu, x) - iQ(\nu, x) \right\} e^{-i\chi}, \]  \hfill (18)

where \( \chi = \left[ x - \left( \nu + \frac{1}{2} \right) \frac{\pi}{2} \right] \) and

\[ P(\nu, x) = \sum_{k=0}^{\infty} (-1)^k \frac{\nu}{(2x)^{2k}} = 1 - \frac{(4\nu^2 - 1)(4\nu^2 - 9)}{2!(8x)^2} + \frac{(4\nu^2 - 1)(4\nu^2 - 9)(4\nu^2 - 25)(4\nu^2 - 49)}{4!(8x)^4} + \ldots, \]  \hfill (19)

\[ Q(\nu, x) = \sum_{k=0}^{\infty} (-1)^k \frac{(\nu + 1)}{(2x)^{2k+1}} = \frac{4\nu^2 - 1}{8x} - \frac{(4\nu^2 - 1)(4\nu^2 - 9)(4\nu^2 - 25)}{3!(8x)^3} + \ldots, \]  \hfill (20)

\[ (\nu, k) = \frac{\Gamma(1/2 + n + k)}{k! \Gamma(1/2 + n - k)}. \]  \hfill (21)

We will consider from now on two particular cases, the radiative case and the de Sitter case, as well as the general inflationary and phantom cases.
3 Radiative case

Let us consider now the particular case of the flat universe dominated by the radiative fluid. Since the radiative fluid is conformal invariant, and the universe is isotropic and homogenous, we must expect that the rate of particle creation is zero, as it is stated in reference [8] (see also reference [9]).

For a radiative universe, the scale factor behaves as $a = a_0 \eta$. Hence, the equation (5) reads now,

$$\phi'' + 2 \frac{\phi'}{\eta} + k^2 \phi = 0.$$  \hspace{1cm} (22)

The general solution is

$$\phi = c_1(k) \frac{e^{i(k\eta - \vec{k} \cdot \vec{x})}}{\eta} + c_2(k) \frac{e^{i(k\eta + \vec{k} \cdot \vec{x})}}{\eta}. $$  \hspace{1cm} (23)

We find the initial Bunch-Davies vacuum if, for example,

$$c_1 = \sqrt{\frac{1}{2k}}, \quad c_2 = 0. $$  \hspace{1cm} (24)

Now the energy density and the pressure read,

$$\rho = \frac{1}{2a^2} \int_0^\infty \left\{ 2k^2 + \frac{1}{\eta^2} \right\} \frac{k}{\eta^4} dk = \frac{1}{2a_0^2} \int_0^\infty \left\{ 2k^2 + \frac{1}{\eta^2} \right\} \frac{k}{\eta^4} dk, $$

$$p = \frac{1}{2a^2} \int_0^\infty \left\{ \frac{2}{3} k^2 + \frac{1}{\eta^2} \right\} \frac{k}{\eta^4} dk = \frac{1}{2a_0^2} \int_0^\infty \left\{ \frac{2}{3} k^2 + \frac{1}{\eta^2} \right\} \frac{k}{\eta^4} dk. $$  \hspace{1cm} (25)

These expressions for the energy and pressure obey the conservation law,

$$\rho' + 3 \frac{a'}{a} (\rho + p) = 0.$$  \hspace{1cm} (27)

From (8,9), we find:

$$E_k = \left\{ 2k^2 + \frac{1}{\eta^2} \right\} \frac{1}{kn^4}, $$

$$P_k = \left\{ \frac{2}{3} k^2 + \frac{1}{\eta^2} \right\} \frac{1}{kn^4}. $$  \hspace{1cm} (28)

There is a quartic and a quadratic divergences both for the energy and for the pressure.

In order to give sense to the energy and pressure expression, let us proceed with the renormalization schema described in section 2. Using the expressions for $E_k^p$ e $P_k^p$, we obtain the following relations:

$$E_k^0 = 2 \frac{k}{\eta^4}, \quad E_k^1 = \frac{1}{k\eta^5}, \quad E_k^2 = 0,$$

$$P_k^0 = 2 \frac{k}{3 \eta^4}, \quad P_k^1 = \frac{1}{k\eta^5}, \quad P_k^2 = 0.$$  \hspace{1cm} (30)

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Hence, we have,

\[ E^\text{ren}_k = E_k - E^0_k - E^1_k - \frac{1}{2} E^2_k = 0, \]  
\[ P^\text{ren}_k = P_k - P^0_k - P^1_k - \frac{1}{2} P^2_k = 0. \]  
(32)  
(33)

Consequently, the initial expression for the energy diverges but, after renormalization, it is equal to zero, in agreement with the fact that no particle is created during the radiative phase.

Note that the equation (22), after the redefinition \( \phi = \frac{\xi}{a} \), takes the form

\[ \xi'' + k^2 \xi = 0. \]  
(34)

From this expression it is clear that we could expect no particle creation from the beginning.

4 Inflation: de Sitter phase

In a de Sitter phase, resulting from the equation of state \( p = -\rho \) (\( \alpha = -1 \)), the scale factor is given by

\[ a = -\frac{1}{\eta}. \]  
(35)

The Klein-Gordon equation reads,

\[ \phi'' - 2\frac{\phi'}{\eta} + k^2 \phi = 0. \]  
(36)

The solution is,

\[ \phi = \eta^{3/2} \left\{ c_1 H^{(1)}_{3/2}(k\eta) + c_2 H^{(2)}_{3/2}(k\eta) \right\}. \]  
(37)

Notice that, from now on, we have made the substitution \( \eta \rightarrow -\eta \), in order to avoid the repetitive use of the absolute value of the original \( \eta \) parameter which is defined in the negative real axis.

The initial condition is imposed at \( \eta \rightarrow \infty \), where

\[ \phi = \eta^{3/2} \sqrt{\frac{2}{\pi k\eta}} \left\{ c_1 e^{i(k\eta-\pi)} + c_2 e^{-i(k\eta-\pi)} \right\}. \]  
(38)

We can obtain an initial Bunch-Davies vacuum state if

\[ c_1 = -\frac{\sqrt{\pi}}{2}, \quad c_2 = 0. \]  
(39)

Hence, we have the final solution

\[ \phi = \eta^{3/2} c_1 H^{(1)}_{3/2}(k\eta). \]  
(40)
Using the recurrence relations for the Hankel’s functions, we obtain the following expression for the energy and the pressure:

\[
\rho = 4\pi c^2 \eta^5 \int_0^{\infty} k^4 \left\{ H^{(1)}_{1/2}(k\eta) H^{(2)}_{1/2}(k\eta) + H^{(1)}_{3/2}(k\eta) H^{(2)}_{3/2}(k\eta) \right\} dk, \quad (41)
\]

\[
p = 4\pi c^2 \eta^5 \int_0^{\infty} k^4 \left\{ H^{(1)}_{1/2}(k\eta) H^{(2)}_{1/2}(k\eta) - \frac{1}{3} H^{(1)}_{3/2}(k\eta) H^{(2)}_{3/2}(k\eta) \right\} dk. \quad (42)
\]

These expressions contain a quartic and a quadratic divergences as in the radiative case. In fact, these expressions can be simplified using the following forms for the Hankel’s functions:

\[
H^{(1)}_{1/2}(x) = -i \sqrt{\frac{2}{\pi x}} e^{ix}, \quad H^{(2)}_{1/2}(x) = i \sqrt{\frac{2}{\pi x}} e^{-ix}; \quad (43)
\]

\[
H^{(1)}_{3/2}(x) = -\sqrt{\frac{2}{\pi x}} \left(1 + \frac{i}{x}\right) e^{ix}, \quad H^{(2)}_{3/2}(x) = -\sqrt{\frac{2}{\pi x}} \left(1 - \frac{i}{x}\right) e^{-ix}. \quad (44)
\]

With these expressions for the Hankel’s functions, the expression for the energy and the pressure reduces to:

\[
\rho = 8c^2 \eta^4 \int_0^{\infty} k^3 \left\{ 2 + \frac{1}{(k\eta)^2} \right\} dk; \quad (45)
\]

\[
p = \frac{8}{3} c^2 \eta^4 \int_0^{\infty} k^3 \left\{ 2 - \frac{1}{(k\eta)^2} \right\} dk. \quad (46)
\]

We see to appear the usual quartic and quadratic divergencies in the ultraviolet limit. There is no divergence in infrared limit.

We proceed with the renormalization of the energy and the pressure associated with the quantum field.

• Renormalization of the energy.

\[
E^n_k = \frac{\eta^4}{2\pi^2} \left\{ 2k + \frac{1}{k\eta^2 n^2} \right\}; \quad (47)
\]

\[
\frac{\partial E^n_k}{\partial n^{-2}} = \frac{\eta^2}{2\pi^2 k}, \quad (48)
\]

\[
\frac{\partial^2 E^n_k}{\partial n^{-2^2}} = 0. \quad (49)
\]

In the limit \( n \to \infty \), we have

\[
E^0_k = \lim_{n \to \infty} E^n_k = \frac{\eta^4 k}{\pi^2}; \quad (50)
\]

\[
E^1_k = \lim_{n \to \infty} \frac{\partial E^n_k}{\partial n^{-2}} = \frac{\eta^2}{2\pi^2 k}; \quad (51)
\]

\[
E^2_k = \lim_{n \to \infty} \frac{\partial^2 E^n_k}{\partial n^{-2^2}} = 0. \quad (52)
\]
Hence, the renormalized energy is:

\[ E_{k}^{\text{ren}} = \frac{2}{\pi \eta^4} \left\{ 2k + \frac{1}{k \eta^2} - 2k - \frac{1}{k \eta^2} \right\} = 0. \]  

\( 53 \)

- Renormalization of the pressure:

\[ P_{k}^{n} = \frac{\eta^4}{6\pi^2} \left\{ 2k - \frac{1}{k \eta^2 n^2} \right\}; \]  

\( 54 \)

\[ \frac{\partial P_{k}^{n}}{\partial n^{-2}} = -\frac{\eta^2}{6\pi^2 k}; \]  

\( 55 \)

\[ \frac{\partial^2 P_{k}^{n}}{\partial n^{-2}} = 0. \]  

\( 56 \)

In the limit \( n \to \infty \), we have

\[ P_{k}^{0} = \lim_{n \to \infty} P_{k}^{n} = \frac{\eta^4 k}{3\pi^2}; \]  

\( 57 \)

\[ P_{k}^{1} = \lim_{n \to \infty} \frac{\partial P_{k}^{n}}{\partial n^{-2}} = -\frac{\eta^2}{6\pi^2 k}; \]  

\( 58 \)

\[ P_{k}^{2} = \lim_{n \to \infty} \frac{\partial^2 P_{k}^{n}}{\partial n^{-2}} = 0. \]  

\( 59 \)

Hence, the renormalized pressure is:

\[ P_{k}^{\text{ren}} = \frac{2}{3\pi \eta^4} \left\{ 2k - \frac{1}{k \eta^2} - 2k + \frac{1}{k \eta^2} \right\} = 0. \]  

\( 60 \)

Again this is a result that we could expect from the beginning since the de Sitter space-time has maximal symmetry. As consequence, there is a time-like Killing vector, and no particle can be created, and the energy and pressure associated to the quantum field must be zero.

## 5 Inflation: the general case

Let us consider now the general inflationary case. The scale factor reads,

\[ a = a_0 \eta^{2/(1+3\alpha)}. \]  

\( 61 \)

There is inflation if \( \alpha < -1/3 \). The Klein-Gordon equation reads,

\[ \phi'' + 2\beta \frac{\phi'}{\eta} + k^2 \phi = 0, \quad \beta = \frac{2}{1+3\alpha}. \]  

\( 62 \)

The general solution is

\[ \phi = \eta^{\nu} \left\{ c_1 H^{(1)}_{\nu}(k\eta) + c_2 H^{(2)}_{\nu}(k\eta) \right\}, \quad \nu = \frac{1}{2} - \beta. \]  

\( 63 \)
As before, we can choose \( c_1 \) in order to have the Bunch-Davies vacuum in the limit \( k \to \infty \), and \( c_2 = 0 \). Hence, the final expression is

\[
\phi = \eta' c_1 H^{(1)}_\nu(k\eta),
\]

where \( c_1 \) does not depend on \( k \).

The corresponding expressions for the energy and pressure are

\[
\rho = A\eta^{2\nu-2\beta} \int_0^\infty k^4 \left\{ H^{(1)}_{\nu-1}(k\eta) H^{(2)}_\nu(k\eta) + H^{(1)}_\nu(k\eta) H^{(2)}_{\nu-1}(k\eta) \right\} dk, \quad (65)
\]

\[
p = A\eta^{2\nu-2\beta} \int_0^\infty k^4 \left\{ \frac{1}{3} H^{(1)}_\nu(k\eta) H^{(2)}_\nu(k\eta) - \frac{2}{\eta} \right\} dk, \quad (66)
\]

where \( A = 4\pi c_1^2 \).

The conservation law

\[
\rho' + 3\frac{\beta}{\eta}(\rho + p) = 0 \quad (67)
\]

is verified.

As in the preceding cases, there is no divergence in the infrared limit. Let us find first the divergences that may occur at the limit \( k \to \infty \). Using the expansion (17,18), we must go until order \( k^{-5} \) in the brackets of the integrands of (65,66): due to the term \( k^4 \) in the integral of the energy and pressure, all terms up to this order in the product of Hankel’s functions are divergent. In particular, the term of order \( k^{-5} \) gives a logarithmic divergence, the term \( k^{-3} \) a quadratic divergence, and the term \( k^{-1} \) a quartic divergence, and all subsequent terms in the expansion are zero in the limit \( k \to \infty \). Note that in both examples treated before, the logarithmic divergence is absent.

Using (17,18), we find

\[
H^{(1)}_{\nu}(x)H^{(2)}_{\nu-1}(x) = \frac{2}{\pi x} [-iP_\nu(x)P_{\nu-1}(x) - P_\nu(x)Q_{\nu-1}(x) + Q_\nu(x)P_{\nu-1}(x)]; \quad (68)
\]

\[
H^{(1)}_{\nu-1}(x)H^{(2)}_\nu(x) = \frac{2}{\pi x} [iP_{\nu-1}(x)P_\nu(x) + P_{\nu-1}(x)Q_\nu(x) - P_\nu(x)Q_{\nu-1}(x) + iQ_\nu(x)Q_{\nu-1}(x)]; \quad (69)
\]

\[
H^{(1)}_{\nu-1}(x)H^{(2)}_{\nu}(x) + H^{(1)}_\nu(x)H^{(2)}_{\nu-1}(x) = \frac{4}{\pi x} [Q_\nu(x)P_{\nu-1}(x) - P_\nu(x)Q_{\nu-1}(x)]; \quad (70)
\]

Using the expansion for the functions \( P \) and \( Q \), the expression for the energy and the pressure become:

\[
E_k = A\eta^{2\nu-2\beta} \frac{2}{\pi} \left\{ \frac{2 + (2\nu - 1)^2}{4 x^2} \right\}
+ \frac{3(2\nu - 3)(2\nu - 1)^2(2\nu + 1)}{64 x^4} + \ldots \}, \quad (71)
\]

10
\[ P_k = A \eta^{2\nu - 2\beta - 2} \frac{2}{\pi} \left\{ \frac{2}{3} + \frac{4\nu^2 - 12\nu + 5}{12x^2} + \frac{(2\nu - 3)(2\nu - 1)(2\nu - 9)(2\nu + 1)}{64x^4} + \cdots \right\}, \] (72)

where \( x = k\eta \). It is clear, at this approximation, that there are a quadratic, a quartic and a logarithmic divergences in the ultraviolet limit. Let us compute the different components to be subtracted from the above expressions, as we have done before.

### 5.1 Computation of the energy

The energy reads,

\[ E_k = \eta^\gamma k^2 \left\{ H_{\nu - 1}^{(1)}(k\eta) H_{\nu - 1}^{(2)}(k\eta) + H_{\nu}^{(1)}(k\eta) H_{\nu}^{(2)}(k\eta) \right\}, \] (73)

where \( \gamma = 2(\nu - \beta) \). As consequence, we find

\[ E_k^0 = \eta \gamma^{-1} k f(x), \quad f(x) = x \left\{ H_{\nu - 1}^{(1)}(x) H_{\nu - 1}^{(2)}(x) + H_{\nu}^{(1)}(x) H_{\nu}^{(2)}(x) \right\}. \] (75)

Hence, we find:

\[ E_k^0 = \lim_{x \to \infty} \eta \gamma^{-1} k x \left\{ H_{\nu - 1}^{(1)}(x) H_{\nu - 1}^{(2)}(x) + H_{\nu}^{(1)}(x) H_{\nu}^{(2)}(x) \right\}; \] (76)

\[ E_k^1 = \lim_{x \to \infty} \left\{ -\frac{2\nu - 1}{2k} \eta \gamma^{-3} x^3 \left[ H_{\nu - 1}^{(1)}(x) H_{\nu - 1}^{(2)}(x) - H_{\nu}^{(1)}(x) H_{\nu}^{(2)}(x) \right] \right\}; \] (77)

\[ E_k^2 = \lim_{x \to \infty} \frac{2\nu - 1}{4k^3} \eta \gamma^{-5} x^5 \left\{ (2\nu + 1) H_{\nu - 1}^{(1)}(x) H_{\nu - 1}^{(2)}(x) + (2\nu - 3) H_{\nu}^{(1)}(x) H_{\nu}^{(2)}(x) \right\} - 2x \left[ H_{\nu}^{(1)}(x) H_{\nu}^{(2)}(x) + H_{\nu - 1}^{(1)}(x) H_{\nu - 1}^{(2)}(x) \right]. \] (78)

Now we are read to compute \( E_k^0, E_k^1, E_k^2 \) and \( E_k \). Using first the relation

\[ H_{\nu - 1}^{(1)}(x) H_{\nu - 1}^{(2)}(x) + H_{\nu}^{(1)}(x) H_{\nu}^{(2)}(x) = \]
\[ \frac{2}{\pi x} \left\{ 2 + \frac{4\nu^4 - 4\nu + 1}{4x^2} + \frac{48\nu^4 - 96\nu^3 + 24\nu^2 + 24\nu - 9}{64x^4} \right\}, \] (79)

\[ H_{\nu}^{(1)}(x) H_{\nu - 1}^{(2)}(x) + H_{\nu - 1}^{(1)}(x) H_{\nu}^{(2)}(x) = \]
\[ \left\{ \frac{2}{\pi x} \left( \frac{2\nu - 1}{x} + 4 \frac{384 \nu^3 - 576 \nu^2 - 96 \nu + 144}{3(8x)^3} \right) - \frac{1}{3(8x)^5} \right\} \left\{ (4\nu^2 - 1)(4\nu^2 - 8\nu + 3)(2\nu - 1)(16\nu^4 - 32\nu^3 - 140\nu^2 + 456\nu + 1736) \right\}, \] (80)

we obtain from (76),

\[ E^0_k = \frac{4k}{\pi} \eta^{\gamma - 1}. \] (81)

On the other hand, using

\[ H^{(1)}_{\nu-1}(x) H^{(2)}_{\nu}(x) - H^{(1)}_{\nu}(x) H^{(2)}_{\nu-1}(x) = \frac{2}{\pi x} \left\{ \frac{-8\nu + 4}{8x^2} + \frac{3}{2} \left( -64\nu^3 + 96\nu^2 + 16\nu - 24 \right) \right\}, \] (82)

and inserting in (77) we find

\[ E^1_k = \frac{1}{2\pi k} \eta^{\gamma - 3}(2\nu - 1)^2. \] (83)

In the same way it is possible to show that

\[ E^2_k = \frac{2\nu - 1}{2\pi k^3} f(\nu) \eta^{\gamma - 5}, \] (84)

where

\[ f(\nu) = \frac{1}{12288} \left( 128\nu^9 - 576\nu^7 - 256\nu^7 + 5968\nu^6 + 2127\nu^5 \right. \]
\[ - \left. 102040\nu^4 + 14432\nu^3 + 554155\nu^2 - 1978\nu - 10794 \right). \] (85)

From (17,18) it can be written,

\[ E_k = \frac{4k}{\pi} \eta^{\gamma - 1} + \frac{(2\nu - 1)^2}{2\pi k^{3\gamma - 3}} \eta^{\gamma - 3} + \frac{48\nu^4 - 96\nu^3 + 24\nu^2 + 24\nu - 9}{32\pi k^3} \eta^{\gamma - 5}. \] (86)

Consequently

\[ E^{ren} = E_k - E^0_k - E^1_k - \frac{1}{2} E^2_k = \frac{h(\nu)}{349152\pi k^3} \eta^{\gamma - 5}, \] (87)

where

\[ h(\nu) = -256\nu^{10} + 1280\nu^9 - 64\nu^8 - 12192\nu^7 - 3657\nu^6 \]
\[ - 225352\nu^5 - 57176\nu^4 - 1241334\nu^3 \]
\[ + 5954975\nu^2 + 37044\nu - 14903. \] (88)

Finally, we find

\[ \rho^{ren} = \frac{h(\nu) \eta^{\gamma - 5}}{49152\pi} \int_0^{\infty} \frac{dk}{k}, \] (89)

revealing that a logarithmic divergence remain.
5.2 Computation for the pressure

The expression for the pressure is

\[ P_k = \eta^2 k^2 \left\{ H^{(1)}_{\nu-1}(k\eta)H^{(2)}_{\nu-1}(k\eta) - \frac{1}{3}H^{(1)}_{\nu}(k\eta)H^{(2)}_{\nu}(k\eta) \right\}. \tag{90} \]

From the definition \ref{eq:13}, it comes out,

\[ P_k^n = \eta^2 k^2 n \left\{ H^{(1)}_{\nu-1}(x)H^{(2)}_{\nu-1}(x) - \frac{1}{3}H^{(1)}_{\nu}(x)H^{(2)}_{\nu}(x) \right\}, \tag{91} \]

where \( x = nk\eta \). Hence,

\begin{align*}
P_k^0 &= \lim_{x \to \infty} \eta^{-1} k x \left\{ H^{(1)}_{\nu-1}(x)H^{(2)}_{\nu-1}(x) - \frac{1}{3}H^{(1)}_{\nu}(x)H^{(2)}_{\nu}(x) \right\}, \tag{92} \\
P_k^1 &= \lim_{x \to \infty} - \frac{\eta^{-3} x^3}{6k} \left\{ (2\nu - 1) \left[ 3H^{(1)}_{\nu-1}(x)H^{(2)}_{\nu-1}(x) + H^{(1)}_{\nu}(x)H^{(2)}_{\nu}(x) \right] \\
&\quad - 4x \left[ H^{(1)}_{\nu}(x)H^{(2)}_{\nu-1}(x) + H^{(1)}_{\nu-1}(x)H^{(2)}_{\nu}(x) \right] \right\}, \tag{93} \\
P_k^2 &= \lim_{x \to \infty} \frac{\eta^{-5} x^5}{12k^5} \left\{ (12\nu^2 - 3)H^{(1)}_{\nu-1}(x)H^{(2)}_{\nu-1}(x) \\
&\quad + (-4\nu^2 + 8\nu - 3)H^{(1)}_{\nu}(x)H^{(2)}_{\nu}(x) \\
&\quad + x(-4\nu - 10) \left[ H^{(1)}_{\nu}(x)H^{(2)}_{\nu-1}(x) + H^{(1)}_{\nu-1}(x)H^{(2)}_{\nu}(x) \right] \\
&\quad - 8x^2 \left[ H^{(1)}_{\nu-1}(x)H^{(2)}_{\nu}(x) - H^{(1)}_{\nu}(x)H^{(2)}_{\nu}(x) \right] \right\}. \tag{94} \end{align*}

Using \ref{eq:17} and \ref{eq:18} we find,

\[ H^{(1)}_{\nu-1}(x)H^{(2)}_{\nu-1}(x) - \frac{1}{3}H^{(1)}_{\nu}(x)H^{(2)}_{\nu}(x) = \frac{2}{\pi x} \left\{ \frac{2}{3} x^3 + \frac{4\nu^2 - 12\nu + 5}{12x^2} + \frac{16\nu^4 - 96\nu^3 + 104\nu^2 + 24\nu - 27}{64x^4} \right\}, \tag{95} \]

leading to

\[ P_k^0 = \frac{4}{3\pi} \eta^{-1} k. \tag{96} \]

In a similar way, we find

\begin{align*}
P_k^1 &= \frac{4\nu^2 - 12\nu + 5}{6\pi k} \eta^{-1}, \tag{97} \\
P_k^2 &= \frac{g(\nu)}{73728\pi k^3} \eta^{-5}, \tag{98} \end{align*}

where

\begin{align*}
g(\nu) &= 256\nu^{10} - 512\nu^9 - 3456\nu^8 + 12672\nu^7 + 239168\nu^6 - 642912\nu^5 \\
&\quad - 278328\nu^4 - 67762\nu^3 - 246453\nu^2 - 150032\nu + 29838. \tag{99} \end{align*}
From (90) we have,
\[ P_k = \frac{4}{3\pi} \eta^{\gamma-1} k + \frac{4\nu^2 - 12\nu + 5}{6\pi k^3} \eta^{\gamma-3} + \frac{16\nu^4 - 96\nu^3 + 104\nu^2 + 24\nu - 27}{32\pi k^3} \eta^{\gamma-5}. \] (100)

Consequently,
\[ P^{ren} = P_k - P_0^k - P_1^k - \frac{1}{2} P_2^k = \frac{l(\nu)}{147456\pi k^3} \eta^{\gamma-5}, \] (101)

where
\[ l(\nu) = -256\nu^{10} + 512\nu^9 + 3456\nu^8 - 12672\nu^7 - 239168\nu^6 + 642912\nu^5 \]
\[ - 204600\nu^4 + 235256\nu^3 + 725685\nu^2 + 260624\nu - 154254. \] (102)

The renormalized pressure takes again the form,
\[ p^{ren} = \frac{l(\nu)}{31474456\pi} \eta^{\gamma-5} \int \frac{dk}{k}, \] (103)

with also a logarithmic divergence.

### 6 Renormalizing the general inflationary case

The presence of the logarithmic divergence in the original expression for the energy and pressure seems to be a general feature of the problem of particle production in cosmology. See, for example, the reference [1]. One important aspect concerning the logarithmic divergence is that the method of renormalization employed seems inefficient in the sense that not only it does not eliminate the logarithmic divergence in the ultraviolet limit, but also it adds a logarithmic divergence in the infrared limit. Hence, we propose an alternative scheme to cope with this undesirable feature.

The proposed procedure is the following. First of all, we write the expressions for the energy and pressure as
\[ \rho = A\eta^{2\nu-2\beta-5} \int_0^{\infty} x^4 \left\{ H_{\nu-1}^{(1)}(x) H_{\nu-1}^{(2)}(x) + H_{\nu}^{(1)}(x) H_{\nu}^{(2)}(x) \right\} dx, \] (104)
\[ p = A\eta^{2\nu-2\beta-5} \int_0^{\infty} x^4 \left\{ H_{\nu-1}^{(1)}(x) H_{\nu+1}^{(2)}(x) - \frac{1}{3} H_{\nu}^{(1)}(x) H_{\nu}^{(2)}(x) \right\} dx, \] (105)

where \( x = k\eta \). This simple redefinition is well justified if \( \eta \neq 0 \), that is, the energy and pressure are computed out of the singularity. But the computation can be carried out as near of the singularity as we want. Hence, we have
\[ E_k = A\eta^{2\nu-2\beta-5} x^2 \left\{ H_{\nu-1}^{(1)}(x) H_{\nu-1}^{(2)}(x) + H_{\nu}^{(1)}(x) H_{\nu}^{(2)}(x) \right\}, \] (106)
\[ P_k = A\eta^{2\nu-2\beta-5} x^2 \left\{ H_{\nu-1}^{(1)}(x) H_{\nu-1}^{(2)}(x) - \frac{1}{3} H_{\nu}^{(1)}(x) H_{\nu}^{(2)}(x) \right\}. \] (107)
The divergencies can then be written as

\[
E_k = A\eta^{2\nu-2\beta-5}\frac{2}{\pi} \left\{ 2 + \frac{(2\nu-1)^2}{4x^2} \right. \\
+ \left. \frac{3(2\nu-3)(2\nu-1)^2(2\nu+1)}{64x^4} + \cdots \right\}, \quad (108)
\]

\[
P_k = A\eta^{2\nu-2\beta-5}\frac{2}{\pi} \left\{ \frac{2}{3} + \frac{4\nu^2-12\nu+5}{12x^2} \\
+ \frac{(2\nu-3)(2\nu-1)(2\nu-9)(2\nu+1)}{64x^4} + \cdots \right\}. \quad (109)
\]

We subtract the (divergent) expressions for energy and pressure as

\[
E_{k\text{ren}} = E_k - E^0_k - E^2_k - E^{log}_k, \quad (110)
\]

\[
P_{k\text{ren}} = P_k - P^0_k - E^2_k - P^{log}_k, \quad (111)
\]

with the definitions

\[
E^{log}_k = A\eta^{2\nu-2\beta-5}\frac{6}{\pi} \frac{(2\nu-3)(2\nu-1)^2(2\nu+1)}{64x^3} (1 - e^{-\sigma x}), \quad (112)
\]

\[
P^{log}_k = A\eta^{2\nu-2\beta-5}\frac{2}{\pi} \frac{(2\nu-3)(2\nu-1)(2\nu-9)(2\nu+1)}{64x^3} (1 - e^{-\sigma x}), \quad (113)
\]

where an extra term has been added in order to assure that the divergence is eliminated in the ultraviolet limit without creating a new divergence in the infrared limit. The parameter \(\sigma\) defines the efficiency of this mechanism, and the final result must be independent of it in order to guarantee the consistency of the procedure proposed.

The energy and the pressure are now given by

\[
\rho^{ren} = \int_0^\infty x^2(E_k - E^0_k - E^2_k - E^{log}_k)dx - \int_1^{1/\sigma} x^2E^{log}_k dx, \quad (114)
\]

\[
p^{ren} = \int_0^\infty x^2(P_k - P^0_k - E^2_k - P^{log}_k)dx - \int_1^{1/\sigma} x^2P^{log}_k dx. \quad (115)
\]

Essentially we must compute an integral of the type

\[
I = \int_1^{1/\sigma} \frac{1 - e^{-\sigma x}}{x} dx, \quad (116)
\]

showing that the logarithmic divergence can be eliminate in the ultraviolet limit, when \(\tau \to \infty\), remaining a final result independent of \(\sigma\). In fact, this expression can be defined as

\[
I = \int_1^{\tau} \frac{1 - e^{-y}}{y} dy, \quad (117)
\]

where \(y = \sigma x\). The result is

\[
I = \ln \tau - \int_1^{\tau} \frac{e^{-y}}{y} dy. \quad (118)
\]
The first term eliminates the ultraviolet logarithmic divergence in the limit \( \tau \to \infty \) while the second one remains finite in the same limit. Moreover, the result is independent of \( \sigma \). This procedure can be easily generalized if the cut-off is introduced as

\[
I = \int_{1/\sigma^{1/n}}^{\tau} (1 - e^{-\sigma x^n}) \frac{dx}{x},
\]

(119)

where \( n \) is any positive number.

As a result, we remain with the expressions for the energy and pressure

\[
\rho_{\text{ren}} = \bar{A}_1 \eta^{2\nu-2\beta-5} I_1 = \bar{A}_1 \eta^{-\frac{12(1+n)}{3+\alpha}} I_1,
\]

(120)

\[
p_{\text{ren}} = \bar{A}_2 \eta^{2\nu-2\beta-5} I_2 = \bar{A}_2 \eta^{-\frac{12(1+n)}{3+\alpha}} I_2,
\]

(121)

where \( \bar{A}_1 \) and \( \bar{A}_2 \) are constants and \( I_1 \) and \( I_2 \) are the (finite) integrals resulting from the renormalization procedure sketched before.

Notice that, since the Ricci scalar can be written as

\[
R = 6 \frac{a''}{a^3} \propto \eta^{-\frac{6+\alpha}{1+3\alpha}},
\]

(122)

the pressure and energy density are proportional to the square of the Ricci curvature. Hence, when the null energy condition is violated (\( \alpha < -1 \)), the energy of the created particles diverges as the singularity is approached, while in the case the strong energy condition is violated but the null energy condition is satisfied, the energy of the created particle is very high in the beginning decreasing as the universe expands.

Hence, using the renormalization procedure proposed here, the energy of created particle may be enough to avoid the big rip, in opposition to what has been stated in [3]. In fact the ratio of the renormalized energy of the created particle to the energy density of the dark energy fluid \( \rho_x \) is

\[
\frac{\rho_{\text{ren}}}{\rho_x} \propto \eta^{-\frac{6+\alpha}{1+3\alpha}},
\]

(123)

going to zero when \(-1/3 > \alpha > -1\) and to infinity when \( \alpha < -1 \). The case of the de Sitter space-time \( \alpha = -1 \), is just the limiting one, the energy of the created particle neither decreasing nor increasing, remaining always equal to zero.

7 Conclusions

In this work we have re-analyzed the problem of particle creation in a universe dominated by dark energy. Special attention has been given to the case where dark energy is represented by a phantom fluid. This problem has already been addressed in reference [3]. There the conclusion was that the energy associated to the created particles, as the universe approaches the big rip, goes to zero in spite of the fact that the number of created particles goes to infinity.
The conclusion presented in reference [3] has been obtained by introducing a cut-off in the energy integral at Planck's scale. This seems to be natural because it can be expected that the usual Klein-Gordon equation must be modified beyond the Planck's scale, and some proposals in this sense suggest the introduction of an exponential decreasing term with the wavenumber $k$, suppressing the contribution due to the transplanckian scales [10].

Here, instead of introducing an upper cut-off, we have chosen to renormalize the energy integral using the $n$ wave method described in reference [8]. First we have shown that the employment of this method leads to the expected conclusions concerning the radiative and de Sitter universes: the energy expression is divergent, but after renormalization it becomes zero.

Applying the method to the general dark energy model, a difficulty appears due to a logarithmic divergence in the ultraviolet limit. We have proposed a method to cope with this difficulty, introducing a new cut-off. The final result is independent of the cut-off. Moreover, the final expressions for energy and pressure are covariantly conserved.

The result for the energy density differs from that found in reference [3], since now the energy density of the created particles diverges as the big rip is approached, and becomes the dominant component. Hence, following this result, quantum effects can be effective to avoid the big rip. In fact, the renormalized energy comes out to be proportional to the square of the Ricci scalar. Another consequence of the computation made in the present work is that in the non phantom dark energy models the renormalized energy goes asymptotically to zero. The de Sitter case is the separatrix of these two different behaviour.

An important point concerning this result is its unicity. A comparison must be made with other renormalization methods, for example those described in [11] and references therein. We hope to address this problem in a future work. However, the absence of any final dependence on the cut-off employed to renormalize the energy, together with the fact that the renormalized energy and pressure obeys a covariant conservation law, indicates that this result is quite consistent.

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