Zeta function method and repulsive Casimir forces for an unusual pair of plates at finite temperature

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We apply the generalized Zeta function method to compute the Casimir energy and pressure between an unusual pair of parallel plates at finite temperature, namely, a perfectly conducting plate ($\epsilon \to \infty$) and an infinitely permeable one ($\mu \to \infty$). The high and low temperature limits of these quantities are discussed. Relationships between high and low temperature limits for the free energy are established by means of a modified version of the temperature inversion symmetry.

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I. INTRODUCTION
Since Casimir’s paper on the attraction between two parallel perfectly conducting plates due to the vacuum fluctuations of the electromagnetic field was published, a considerable amount of effort, which varies from the investigation of new geometries and theories to the application of the Casimir effect to alternative technologies, has been put into the study of this important subject. (For a review see, for example, Mostepanenko and Trunov or Plunien et al). Recently, the experimental observation of this effect was greatly improved by the experiments due to Lamoreaux and Mohideen and Roy. From the theoretical viewpoint Casimir’s approach to this problem essentially consisted in computing the interaction energy between the plates as the regularized difference between the zero point energies with and without boundary conditions dictated by the physical situation at hand, for instance, the perfectly conducting character of the plates. The great novelty of Casimir’s 1948 paper was not the fact that two neutral object were attracted towards each other, which was familiar to those studying dispersive van der Waals forces, but the simplicity of the method of calculating this attraction in the framework of quantum field theory.

Casimir’s definition of the vacuum energy requires a regularization recipe for its implementation. Many regularization techniques are available nowadays and, depending on the specific physical situation at hand, one of them may be more suitable than the others. Particularly, methods of computing effective actions are in general very powerful to give physical meaning to the divergent quantities we must deal with. Here we will be concerned with one of these methods, namely, the so-called generalized zeta function method. There are several examples of the application of this method to the evaluation of the Casimir effect at zero and finite temperature, in its global and also in its local version, see for example. Here we will apply it to the case of a pair of parallel infinite plates one of which is perfectly conducting ($\epsilon \to \infty$), while the other is infinitely permeable ($\mu \to \infty$). The setup will be considered to be in thermal equilibrium with a heat reservoir at finite temperature $T$. This problem at zero temperature was analyzed by Boyer two decades ago in the framework of random electrodynamics, a kind of classical electrodynamics which includes the zero-point electromagnetic radiation. Boyer was able to show that for this unusual pair of plates the Casimir effect is positive which results in a repulsive force per unit area between the plates. Recently Boyer’s result at zero temperature was rederived by zeta function methods. Repulsive electromagnetic Casimir forces can arise in geometrically more complicated setups as for instance a spherical shell. More complicated geometries may lead to extremely involved calculations. Though for a given geometry it is possible to infer on dimensional grounds only, the form of the Casimir energy, its correct algebraic sign, that is: the attractive or repulsive character of the associated Casimir force, and numerical factors are obtained only after complex calculations. More generally, the non-trivial dependence of the algebraic sign of the Casimir force on the type of quantum field being studied, type of spacetime, on the dimensions of the spacetime and on the type of boundary imposed on the quantum field was denoted by some authors as ‘the mystery of the Casimir force’.

Boyer’s unusual pair of plates is the simplest example where we can find repulsive Casimir forces at work. They were recently employed by Hushwater as a counterexample in order to show that the naive interpretation of the standard Casimir attraction between two parallel conducting plates as being due to a difference between the number of vacuum modes in the region between the plates and the region outside the plates does not apply. Boyer’s plates were also used in connection with the Scharnhorst effect where they provided one more example in which the propagation of a light signal in the confined electromagnetic vacuum is modified with respect to propagation in the unconstrained vacuum. While thermal corrections to the standard Casimir effect were calculated by several authors and constitute a large body of literature on this subject, see for rep-
representative examples, there are to our knowledge no calculations of thermal corrections to the repulsive Casimir effect associated with Boyer’s setup. Here, in order to remedy this situation we take into account the thermal effects of the equilibrium state, and study this problem within the framework of finite temperature QFT. Moreover, we show that, though boundary conditions are not symmetric, it is still possible to discuss temperature inversion symmetry for this system. We take advantage of the fact that in the case of the simple geometry we are considering the electromagnetic field can be simulated by a scalar massless field. The insertion of a multiplicative factor equal to 2 will take into account the two possible polarizations of the electromagnetic field. The article is divided as follows: Firstly we derive general expressions for the free energy and pressure. Secondly, we consider the low and high temperatures limits of these quantities. Thirdly, we show that Boyer’s setup is equivalent to the difference between two Casimir’s setups — a fact that allows us to discuss the temperature inversion symmetry associated with this system. Finally, the last section is devoted to concluding remarks. We will employ units such that Boltzmann constant, the speed of light and \( h = h/2\pi \) are set equal to unity.

II. EVALUATION OF THE FREE ENERGY

Since we will be dealing with a system in thermal equilibrium, the imaginary time formalism will be convenient. In order to apply the generalized zeta function method, let us introduce the partition function \( Z \) for a bosonic theory [13]:

\[
Z = N \int_{\text{Periodic}} \left[ D\phi \right] \exp \left( \int_0^\beta d\tau \int d^3x \mathcal{L} \right),
\]

where \( \mathcal{L} \) is the Lagrangian density for the theory under consideration, \( N \) is a constant and ‘periodic’ means that \( \phi(t, x, y, z, 0) = \phi(t, x, y, z, \beta) \),

\[
\phi(x, y, z, 0) = \phi(x, y, z, \beta),
\]

where \( \beta = T^{-1} \), the reciprocal of the temperature, is the periodic length in the Euclidean time axis. The Helmholtz free energy \( F(\beta) \) is related to the partition function \( Z(\beta) \) through the relation \( F(\beta) = -\beta^{-1} \log Z(\beta) \). Other than the periodic conditions given by (1), we must also consider boundary conditions which are determined by the geometry and the nature of the physical system under study. An example is the configuration mentioned above. Choosing Cartesian axes such that the axis \( OZ \) is perpendicular to both plates with the perfectly conducting plate at \( z = 0 \) and the infinitely permeable one at \( z = d \), the boundary conditions on the vacuum oscillations of the electromagnetic field are the following: the tangential components of the electric field as well as the normal component of magnetic field must vanish at \( z = 0 \), while the tangential components of the magnetic field and the normal component of the electric field must vanish at \( z = d \). As mentioned before, for the plate geometry that we are considering, the electromagnetic field can be mimicked by a scalar massless field \( \phi \). The boundary conditions stated above can be translated into:

\[
\frac{\partial\phi(\tau, x, y, z = d)}{\partial z} = 0, \quad (3)
\]

where \( \tau \) is the Euclidean time. The insertion at the end of the calculation of a factor 2 will take into account the two possible transverse polarizations of the electromagnetic field. Thus we write log \( Z(\beta) \) as:

\[
\log Z(\beta) = \left(-\frac{1}{2}\right) \log \det (-\partial_{\beta}[\mathcal{F}_d]), \quad (4)
\]

where \( \partial_{\beta} = \partial^2/\partial\tau^2 + \nabla^2 \), and the symbol \( \mathcal{F}_d \) stands for the set of functions which satisfy conditions (3) and (4). The generalized zeta function method basically consists of the following three steps: (i) first, we compute the eigenvalues of the operator \( -\partial_{\beta} \) whose eigenfunctions obey the appropriate boundary conditions and write \( \zeta(s; -\partial_{\beta}) = \text{Tr}(-\partial_{\beta})^{-s} \); (ii) second, we perform an analytical continuation of \( \zeta(s; -\partial_{\beta}) \) to a meromorphic function on the whole complex \( s \)-plane; (iii) finally, we compute \( \det (-\partial_{\beta}[\mathcal{F}_d]) = \exp \left(-\frac{\partial\zeta(s=0; -\partial_{\beta})}{\partial s} \right) \). Combining equation (4) with the definition of free energy we obtain:

\[
F(\beta) = -\beta^{-1} \frac{\partial\zeta(s=0; -\partial_{\beta})}{\partial s}. \quad (5)
\]

The eigenvalues of \( -\partial_{\beta} \) whose eigenfunctions satisfy (2) and (3) are:

\[
\left\{ k_x^2 + k_y^2 + \left(n + \frac{1}{2}\right)^2 \pi^2 \pm \frac{4\pi^2 m^2}{\beta^2} \right\}, \quad (6)
\]

where \( k_x, k_y \in \mathbb{R} \), \( n \in \{0, 1, 2, 3, ...\} \) and \( m \in \{0, \pm 1, \pm 2, ...\} \). The generalized zeta function then reads:

\[
\zeta(s; -\partial_{\beta}) = L^2 \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \int \frac{dk_x dk_y}{(2\pi)^2} \times \left[k_x^2 + k_y^2 + (2n + 1)^2 \pi^2 - \frac{4\pi^2 m^2}{\beta^2} \right]^{-(s-1)} \quad (7)
\]

where \( L^2 \) is the area of the plates. After rearranging terms in the summations, changing to polar coordinates and integrating the angular part out, we can rewrite this last equation as:

\[
\zeta(s; -\partial_{\beta}) = \frac{L^2}{2\pi} \left[ \sum_{n=1}^{\infty} \int_0^\infty dk \left[k^2 + \frac{n^2\pi^2}{4d^2} + \frac{4\pi^2 m^2}{\beta^2} \right]^{-(s-1)} + 2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} dk \left[k^2 + \frac{n^2\pi^2}{4d^2} + \frac{4\pi^2 m^2}{\beta^2} \right]^{-(s-1)} \right], \quad (8)
\]
where $k^2 = k_1^2 + k_2^2$ and the prime on the summation symbol remind us that the integer $n$ assumes odd values only. Using the following representation for the Euler beta function, c.f. formula 3.251.2 [14]:

$$\int_0^\infty dx x^{\mu-1} (x^2 + a^2)^{\nu-1} = \frac{1}{2} B\left(\frac{\mu}{2}, 1 - \nu - \frac{\mu}{2}\right) \times a^{2\nu-2},$$

(9)

where

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

(10)

which holds for $\Re(\nu + \frac{\mu}{2}) < 1$ and $\Re\mu > 0$, we obtain:

$$\zeta(s, -\partial_\mu) = \frac{L^2}{4\pi} \Gamma(s-1) \sum_{n=1}^{\infty} \left( \frac{\pi}{2d} \right)^{2-2s} \sum_{n=1}^{\prime} n^{2-2s}$$

$$+ 2n^{2-2s} \sum_{m=1}^{\infty} \sum_{n=1}^{\prime} \left[ \frac{n^2}{4d^2} + \frac{4m^2}{\beta^2} \right]^{1-s}$$

(11)

In order to connect the simple sum on the r.h.s. of the above equation to the Riemann zeta function $\zeta_R$ we write:

$$\sum_{n=1}^{\infty} n^{2-2s} = (1 - 2^{-2s} \zeta_R(2s) - 2).$$

(12)

On the other hand, the double sum can be expressed in terms of Epstein functions which for any positive integer $N$ and $\Re z$ large enough are defined by [14, 16]:

$$E_{N}^{M^2}(z; a_1, a_2, ..., a_N) := \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} ... \sum_{n_N=1}^{\infty} \frac{1}{(a_1n_1^2 + a_2n_2^2 + ... + a_Nn_N^2 + M^2)^z},$$

(13)

where $a_1, ..., a_N$ and $M^2 > 0$ and writing:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\prime} \left[ \frac{n^2}{4d^2} + \frac{4m^2}{\beta^2} \right]^{1-s} = \sum_{m,n=1}^{\infty} \left[ \frac{n^2}{4d^2} + \frac{4m^2}{\beta^2} \right]^{1-s}$$

$$- \sum_{m,n=1}^{\infty} \left[ \frac{(2n)^2}{4d^2} + \frac{4m^2}{\beta^2} \right]^{1-s},$$

(14)

we can write:

$$\zeta(s, -\partial_\mu) = \frac{L^2}{4\pi} \Gamma(s-1) \sum_{n=1}^{\infty} \left( \frac{\pi}{2d} \right)^{2-2s} (1 - 2^{-2s} \zeta_R(2s) - 2)$$

$$+ 2E_2 \left( s - 1; \frac{1}{4d^2}, \frac{1}{\beta^2} \right) - 2E_2 \left( s - 1; \frac{1}{4d^2}, \frac{4}{\beta^2} \right).$$

(15)

The Epstein functions can be analytically continued to a meromorphic function in the complex plane, (see for example [16]). For $N = 2$ and $M^2 = 0$ the analytic continuation is given by:

$$E_2(z; a_1, a_2) = -\frac{\alpha(z)}{2} \zeta_R(2z) + \frac{1}{2} \pi \Gamma(z - \frac{1}{4})$$

$$\times E_1(z - \frac{1}{2}; a_1) + \frac{2\pi^2}{\Gamma(z) a_1^{\frac{1}{2} + \frac{1}{2}}}$$

$$\times \sum_{n,m=1}^{\infty} \frac{m^2 - (1/2)}{(a_1n^2 - (1/2))z} K_{\frac{1}{z}} \left( 2\pi m \frac{\sqrt{a_1n^2}}{a_2} \right).$$

(16)

Here $K_{\nu}(z)$ is a Macdonald’s function [14] defined on the complex $z$-plane cut along the negative real axis, $[-\infty, 0]$. Performing the appropriate substitutions for $z$, $a_1$ and $a_2$ and taking advantage of the useful fact that the derivative of the function $G(s)/\Gamma(s)$ at $s = 0$ is simply $G(0)$ for a well-behaved $G(s)$ we obtain:

$$\zeta'(s, -\partial_\mu) = -\frac{7}{8} \times \frac{\pi^2 \beta L^2}{720d^3} + \frac{L^2 \sqrt{\pi}}{\beta^2} \sum_{n,m=1}^{\infty} \left( \frac{md}{n} \right)^{-\frac{1}{2}}$$

$$\times \left[ 2^{-\frac{3}{2}} K_{3/2} \left( \frac{\beta \pi nm}{2d} \right) - K_{3/2} \left( \frac{2\beta \pi nm}{2d} \right) \right].$$

(17)

It follows that the Helmholtz free energy per unit area for this peculiar arrangement is given by:

$$\frac{F}{L^2} = \frac{7}{8} \times \frac{\pi^2}{720d^3} - \frac{\sqrt{\pi}}{\beta^2} \sum_{n,m=1}^{\infty} \left( \frac{md}{n} \right)^{-\frac{1}{2}}$$

$$\times \left[ 2^{-\frac{3}{2}} K_{3/2} \left( \frac{\beta \pi nm}{2d} \right) - K_{3/2} \left( \frac{2\beta \pi nm}{2d} \right) \right],$$

(18)

where we have already accounted for the two possible polarization states. The first term in (18) represents the regularized repulsive Casimir energy at zero temperature found by Boyer [14]. Notice that this term is $-7/8$ times the result obtained for the Casimir effect with Dirichlet boundary conditions at zero temperature. The second term in (18) is the contribution to the free energy due to thermal effects and we can recast it into a more manageable form as is shown next.

The Macdonald’s functions $K_{\nu}(z)$ of half-integral order are given by (c.f. formula 8.468 in [14]):

$$K_{\nu}(z) = \left( \frac{\pi}{2z} \right)^{\frac{1}{4}} e^{-z} \sum_{k=0}^{\infty} \frac{(n+k)!}{k!(n-k)!(2z)^k}.$$  

(19)

Hence, defining the dimensionless variable $\xi$ by $\xi := d/\pi\beta = Td/\pi$ and making use of (19), we can recast (18) into the form:

*We use the terminology employed by N. N. Lebedev in Special Functions and Their Applications, Dover Publications, New York, 1972. The function $K_{\nu}(z)$ is also known as modified Bessel function of the third kind and Bessel function of imaginary argument.
\[
F(\beta) = \frac{7}{8} \times \frac{\pi^2}{720d^3} - \frac{1}{\pi \beta^3} f(\xi),
\]
(20)

where \( f(\xi) \) is a dimensionless function, also referred to as scaled free energy, defined by the double sum:

\[
f(\xi) := \sum_{n,m=1}^{\infty} \left[ \left( \frac{1}{m^2} + \frac{n}{2 \xi_m^2} \right) e^{-nm/2\xi} - \left( \frac{1}{m^3} + \frac{n}{\xi_m^2} \right) e^{-nm/(\xi)} \right].
\]
(21)

The sum over \( n \) can be readily evaluated and after some manipulations we end up with:

\[
f(\xi) = \frac{1}{4\xi} \sum_{n=1}^{\infty} \frac{2\xi}{n} + \coth \left( \frac{\eta}{2\xi} \right) \right] \]
(22)

Equation (22) summarizes all thermodynamical information concerning the bosonic excitations confined between the plates. From (22) we can easily obtain the low temperature regime of the free energy. It suffices to set \( \coth \left( \frac{\eta}{2\xi} \right) \approx 1, \ \sinh \left( \frac{\eta}{2\xi} \right) \approx 2/\exp \left( \frac{\eta}{2\xi} \right) \) and keep the term corresponding to \( n = 1 \):

\[
f(\xi \ll 1) \approx \left( 1 + \frac{1}{2\xi} \right) \exp \left( -\frac{1}{2\xi} \right).
\]
(23)

This yields the low temperature limit

\[
\frac{F(\beta)}{L^2} = \frac{7}{8} \times \frac{\pi^2}{720d^3} - \left( \frac{1}{\pi \beta^3} + \frac{1}{2d\beta^2} \right) e^{-\pi \beta/2d}.
\]
(24)

In the low temperature limit the Helmholtz free energy for the original Casimir’s setup is given by, see e.g. Brown and Maclay [12]:

\[
\frac{F(\beta)}{L^2} = -\frac{\pi^2}{2\beta^3} - \frac{\zeta(3)}{2\beta^3} - \left( \frac{1}{\pi \beta^3} + \frac{1}{d\beta^2} \right) e^{-\pi \beta/d}.
\]
(25)

Notice the absence of the factor proportional to \( 1/\beta^3 \) in the case of Boyer’s setup. The reason for the absence of this factor will become clear later on when we discuss the question of the temperature inversion symmetry associated with the problem at hand. The very high temperature limit is obtained by setting \( \coth \left( \frac{\eta}{2\xi} \right) \approx \frac{\eta}{2\xi} \) and \( \sinh \left( \frac{\eta}{2\xi} \right) \approx \frac{\eta}{2\xi} \) and evaluating the sum. The result is:

\[
f(\xi \gg 1) \approx \frac{1}{45} \pi^4 \xi.
\]
(26)

This will lead to the Stefan-Boltzmann term corresponding to a slice of vacuum of volume \( L^2d \). More accurate results at high temperature demand that we transform the slowly convergent sum over \( m \) in (21) into a more rapidly convergent one. This can be accomplished with the help of Poisson summation formula as we shall see next.

III. THE PRESSURE

Let us go back to the scaled free energy \( f(\xi) \) defined by the double sum (21). Each of the double sums in equation (21) can be written in the form

\[
\sum_{n,m=1}^{\infty} \left( \frac{a}{m^3} + \frac{bn}{m^2} \right) e^{-nm/c} = -a\beta^2 \sum_{n=1}^{\infty} \int_{\frac{\pi}{2}}^{\infty} d\omega \ln \left( 1 - e^{-\beta \omega} \right)
\]
(27)

where \( a, b \) and \( c \) are constant satisfying the condition \( \alpha = bc, \ \kappa = 1/\beta c \). The scaled free energy can be recast into the form

\[
f(\xi) = -\beta^2 \sum_{n=1}^{\infty} \int_{\frac{\pi}{2}}^{\infty} d\omega \omega \ln \left( 1 - e^{-\beta \omega} \right)
\]
(28)

\[
+ \beta^2 \sum_{n=1}^{\infty} \int_{\frac{\pi}{2}}^{\infty} d\omega \ln \left( 1 - e^{-\beta \omega} \right)
\]

\[
\kappa_1 = \pi/2d \quad \kappa_2 = \pi/d.
\]

The thermal contribution reads:

\[
P_{\text{thermal}} = -\frac{1}{\pi^2/\beta^4/\xi^3} \left[ \frac{1}{4} \sum_{n=1}^{\infty} n^2 \ln \left( 1 - e^{-n/2\xi} \right) - \sum_{n=1}^{\infty} n^2 \ln \left( 1 - e^{-n/\xi} \right) \right].
\]
(30)

Now we are ready to make use of one of the several versions of Poisson summation formula [17]. The particular version suitable for our purposes reads:

\[
\sum_{n=1}^{\infty} G(n) = -\frac{G(0)}{2} + \sum_{l=-\infty}^{\infty} \int_{0}^{\infty} dx \ G(x) \ \cos \left( 2\pi lx \right),
\]
(31)

If we use (31) in (30) and add the result to the zero temperature contribution we obtain
we obtain:

\[
P_{\text{net}} = -\frac{\pi^2}{45\beta^4} + \frac{1}{32\pi^4\beta^4} \delta^2 \frac{1}{\xi^2} \sum_{m=1}^{\infty} \frac{\coth(4\pi^2m\xi)}{m^3}
+ \frac{1}{8\pi^4\beta^4} \delta^2 \frac{1}{\xi^2} \sum_{m=1}^{\infty} \frac{\coth(2\pi^2m\xi)}{m^3}.
\]

(32)

This result holds for all temperatures. Notice that the zero temperature pressure is apparently missing in our final result. This happens because upon the application of Poisson summation formula we obtain, besides the Stefan-Boltzmann term and the two sums, a term with a negative sign which exactly cancels out the repulsive zero temperature contribution. A similar cancellation occurs also in the case of the Casimir effect for confined massless fermions at finite temperature [13] and in the high temperature limit of the standard electromagnetic Casimir effect as shown, for instance, in Plunien et al [2]. Nevertheless, it is a straightforward matter to show that if we take the zero temperature limit of (32) we recover the zero temperature term.

The high temperature limit is also easily obtained from (22). Approximating conveniently the hyperbolic cotangents in the sums and evaluating the second partial derivatives and keeping the leading correction terms only we obtain:

\[
P_{\text{net}} \approx -\frac{\pi^2}{\beta^4 \xi^4} + \frac{3\zeta(3)}{16d^2\beta}
+ \frac{1}{2\pi^2 \beta^4} e^{-4\pi d/\beta} \left( 1 + \frac{4\pi d}{\beta} + \frac{8\pi^2 d^2}{\beta^2} \right).
\]

(33)

A simple integration of (32) yields another possible representation for the Helmholz free energy of this setup:

\[
\frac{F(\beta)}{L^2} \approx -\frac{\pi^2 d}{45\beta^4} + \frac{1}{32\pi^4\beta^4} \delta^2 \frac{1}{\xi^2} \sum_{m=1}^{\infty} \frac{\coth(4\pi^2m\xi)}{m^3}
+ \frac{1}{8\pi^4\beta^4} \delta^2 \frac{1}{\xi^2} \sum_{m=1}^{\infty} \frac{\coth(2\pi^2m\xi)}{m^3}.
\]

(34)

where the integration constant is determined by demanding that in the very high temperature limit the only surviving term in (31) must be the Stefan-Boltzmann term. Notice that we can also determine this integration constant analyzing the zero temperature limit of (34). The high temperature limit of (34) is given by

\[
\frac{F(\beta)}{L^2} \approx -\frac{\pi^2 d}{45\beta^4} + \frac{3\zeta(3)}{32\pi d^2\beta}
+ \left( \frac{1}{4\pi^2 d^2\beta} + \frac{1}{4d^3\beta^2} \right) e^{-4\pi d/\beta}.
\]

(35)

Apart from the all-important signs and numerical factors, these results compare with those obtained in this limit for the attractive case. From Brown and Maclay’s results [12], for example, we can infer the following expression for the free energy per unit area in the case of two infinite parallel perfectly conducting plates:

\[
\frac{F(\beta)}{L^2} \approx -\frac{\pi^2 d}{45\beta^4} - \frac{\zeta(3)}{8\pi d^2\beta} \left( \frac{1}{4\pi^2 d^2} + \frac{1}{d^3\beta^2} \right) e^{-4\pi d/\beta}.
\]

(36)

In both cases in the very high temperature limit the dominant term is the Stefan-Boltzmann term. Figures (4) and (5) show the behavior of the scaled free energy function \( f(\xi) := [F(\beta)/L^2d] \times d^4 \), where \( F(\beta)/L^2 \) is given by equation (34) as a function of the scaled temperature \( \xi \). Calculations involved up to ten terms in the summations required by (34). We also show the scaled free energy corresponding to two perfectly conducting parallel plates as given by equation (39). Though exact, equation (34) is specially suited for the high temperature regime, convergence getting slower and slower for small values of \( \xi \). This can be seen if we examine in greater detail the behavior of equation (34) for very small values of \( \xi \). To obtain accurate results in this region it is necessary to include many more terms in the required sums. Figure (6) shows the behavior of the scaled pressure \( p(\xi) \), which is given by equation (22) multiplied by \( d^4 \), as a function of \( \xi \). All curves were plotted using MATHEMATICA version 2.2.1 [13].

IV. TEMPERATURE INVERSION SYMMETRY

Temperature inversion symmetry is a symmetry occurring in the free energy associated with the Casimir effect at finite temperature that depends on the nature of the boundary conditions imposed on the quantum oscillations of the massless field under study. In fact, as it was shown by Ravndal and Tollefsen [20], temperature inversion symmetry obtains for massless bosonic fields and symmetric boundary conditions and for massless fermionic fields and antisymmetric boundary conditions. One of the most remarkable feature of the temperature inversion symmetry is the possibility of relating the Stefan-Boltzmann term to the zero temperature Casimir effect. Temperature inversion symmetry appears already in Brown and Maclay’s work [12] in their evaluation of the standard Casimir effect at finite temperature. Brown and Maclay were able to write the scaled free energy as a sum of three contributions: a zero temperature contribution \( i. e., \) the Casimir energy density at zero temperature, a Stefan-Boltzmann contribution proportional the fourth power of the scaled temperature \( \xi \), and a non-trivial contribution. This non-trivial term is given by

\[
f_{\text{non-trivial}}(\xi) = -\frac{1}{4\pi^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(2\pi\xi)^4}{m^2 + (2\pi n\xi)^2}.
\]

(37)

\(^\dagger\xi \) have the definition we have given herein. It differs from \( \xi \) in Brown and Maclay’s paper by a factor \( \pi \).
This function has the following property:

\[
(2\pi \xi)^4 f(1/4\pi \xi) = f(\xi),
\]

which is the mathematical statement of the temperature inversion symmetry. It turns out that the three contributions to the scaled free energy can be combined and recasted into one piece as the double sum below:

\[
\tilde{f}(\xi) = -\frac{1}{16\pi^2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(2\pi \xi)^4}{[m^2 + (2\pi n \xi)^2]^2},
\]

where in the double sum we must exclude the term corresponding to \(n = m = 0\). If we set \(n = 0\) and sum over \(m\) with \(m \neq 0\), we will obtain the Stefan-Boltzmann term \(-\pi^d \xi^4/45\). On the other hand, if we set \(m = 0\) and sum over \(n\) with \(n \neq 0\), we will obtain the Casimir term corresponding to zero temperature \(-\pi^2/720\). As proved by Ravndal and Tollefsen [21], equation (38) obeys the same symmetry under temperature inversion as the one originally obtained by Brown and Maclay. It was also shown by Gundersen and Ravndal [18] that the scaled free energy associated with massless fermions fields at finite temperature submitted to MIT boundary conditions satisfy the relation given by equation (35) and therefore exhibits temperature inversion symmetry. Tadaki and Takagi [22] have calculated Casimir free energies for a massless scalar field obeying Dirichlet or Neumann boundary conditions on both plates and found this symmetry. However, if the massless scalar field must satisfy mixed boundary conditions, say, Dirichlet boundary conditions on one plate and Neumann boundary conditions on the other, the temperature inversion symmetry is lost. In the case of a massless scalar field at finite temperature and periodic boundary conditions, it is possible to show that the partition function, and consequently the free energy, can be written in a closed form such that the temperature inversion symmetry becomes explicit [21].

In the repulsive case we are dealing with for which the boundary conditions are not symmetric we should not expect to find this symmetry; nonetheless, we will show that it is possible to extract a somewhat more complicated version of the inversion temperature symmetry which will allow us to establish a relationship between the high and the low temperature limits for the repulsive case. We will also show that the underlying reason why it is still possible to discuss the temperature inversion symmetry in our case is the fact that Boyer’s setup is equivalent to two original Casimir’s setups, a feature that we have already noted earlier (see equation 23).

Our starting point is equation (22), which defines the dimensionless function \(f(\xi)\), and the identity:

\[
\sum_{m=-\infty}^{\infty} \frac{(-1)^m}{[m^2 + (b^2)^2]^2} = \frac{\pi^2}{2b^2} \left[ \frac{1}{\pi b} + \coth(\pi b) \right],
\]

where here \(b := n/2\pi \xi\). Making use of the identity above we can rewrite equation (22) in the form

\[
f(\xi) = \frac{1}{8\pi^4 \xi^3} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^m(2\pi \xi)^4}{[n^2 + (2\pi m \xi)^2]^2}
\]

\[
= \frac{1}{16\pi^4 \xi^3} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^m(2\pi \xi)^4}{[n^2 + (2\pi m \xi)^2]^2}
\]

\[
- \frac{1}{16\pi^4 \xi^3} \sum_{m=-\infty}^{\infty} \frac{(-1)^m}{m^4},
\]

where in the double sum the term corresponding to \(m = n = 0\) and in the single sum the term corresponding to \(m = 0\) must be both excluded. If we take the expression above into equation (24) we will be able to write the free energy per unit area as

\[
\frac{F}{L^2} = -\frac{1}{16\pi^2 \xi^3} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^m(2\pi \xi)^4}{[n^2 + (2\pi m \xi)^2]^2}.
\]

Now we write down equation (42) as a sum over even terms plus a sum over odd terms in \(m\). If we do this we can separate the free energy per unit area in two terms

\[
\frac{F}{L^2} = \frac{F_1}{L^2} - \frac{F_2}{L^2},
\]

where \(F_1\) and \(F_2\) are defined by

\[
\frac{F_1}{L^2} := -\frac{1}{128\pi^2 \xi^3} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{(4\pi \xi)^4}{[n^2 + (4\pi m \xi)^2]^2},
\]

and

\[
\frac{F_2}{L^2} := -\frac{1}{16\pi^2 \xi^3} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{(2\pi \xi)^4}{[n^2 + (2\pi m \xi)^2]^2}.
\]

Now it is possible to verify directly that the free energies \(F_1\) and \(F_2\) satisfy the relations

\[
(4\pi \xi)^4 F_1 \left( \frac{1}{16\pi^2 \xi} \right) = F_1 (\xi),
\]

and

\[
(2\pi \xi)^4 F_2 \left( \frac{1}{4\pi^2 \xi} \right) = F_2 (\xi).
\]

Equation (43) can be construed as follows: First notice that \(F_1/L^2\) and \(F_2/L^2\) can be respectively interpreted as the free energies per unit area associated with two infinite parallel conducting plates setups, one corresponding to a separation distance between the plates equal to \(2d\) and the other to a separation distance equal to \(d\). In this way we can say that the symmetries given by equations (16) and (17) are induced by these correspondences, in accordance with Ravndal and Tollefsen’s [21] result, in the sense that both setups require symmetric boundary conditions.
Observe also that by making use of the identity
\[
\sum_{l=-\infty}^{\infty} \frac{1}{|l^2 + l^2|} = \frac{\pi}{2b} \coth (\pi b) + \frac{\pi^2}{2b^2 \sinh^2 (\pi b)},
\]
we can also write \( F_1 \) and \( F_2 \) as single sums
\[
\frac{F_1}{L^2} = -\frac{\pi^2}{16d^3} \sum_{n=-\infty}^{\infty} \left[ \frac{4\xi^3}{n^3} \coth \left( \frac{n}{2\xi} \right) + \frac{2\xi^2}{n^2} \right] ,
\]
and
\[
\frac{F_2}{L^2} = -\frac{\pi^2}{16d^3} \sum_{n=-\infty}^{\infty} \left[ \frac{4\xi^3}{n^3} \coth \left( \frac{n}{2\xi} \right) + \frac{2\xi^2}{n^2} \right] .
\]

As a first application of equations (43), (46) and (47) let us relate the Stefan-Boltzmann term, which is the dominant term in the very high temperature limit, to the zero temperature Casimir energy. In the very high temperature limit we can write for each setup
\[
\frac{F_1(\infty)}{L^2} = -\frac{2}{45\beta^4} \pi^2 d ,
\]
and
\[
\frac{F_2(\infty)}{L^2} = -\frac{1}{45\beta^4} \pi^2 d .
\]
Making use of (46) and (47) we obtain
\[
\frac{F_1(0)}{L^2} = -\frac{\pi^2}{4^3 \times 90 d^3} ,
\]
and
\[
\frac{F_2(0)}{L^2} = -\frac{\pi^2}{2^3 \times 90 d^3} .
\]
Hence, making use of (43) we obtain
\[
\frac{F_0}{L^2} = \frac{7}{8} \times \frac{\pi^2}{720 d^3} ,
\]
which is the Casimir energy at zero temperature associated with our original setup.

As a second application of our version of the temperature inversion symmetry we now establish the relationship between the low and the high temperature limits. In the high temperature limit the Helmholtz free energies \( F_1 \) and \( F_2 \) corresponding to two setups each one of them formed by two infinite parallel conducting plates kept at a distance 2\(d\) and 2\(d\) apart, respectively read
\[
\frac{F_1}{L^2} \approx -\frac{2\pi^6 \xi^4}{45d^3} - \frac{\zeta(3)\xi}{32d^3} \left( -\frac{\xi}{16d^3} + \frac{\pi^2 \xi^2}{2d^3} \right) \text{e}^{-8\pi^2 \xi} ,
\]
where we have made use of results obtained by Brown and Maclay for the standard Casimir effect in the high temperature limit [12]. Notice that if in accordance with (43) we subtract (52) from (51) we will obtain the high temperature limit of the Helmholtz free energy corresponding to Boyer’s setup, equation (53). Making use of (44) and (47) we obtain
\[
\frac{F_1}{L^2} = -\frac{\pi^2}{5750d^3} - \frac{\zeta(3)\pi^2 \xi^3}{2d^3} - \left( \frac{\pi^2 \xi^3}{d^3} + \frac{\pi^2 \xi^2}{2d^3} \right) \text{e}^{-\frac{4\pi^2}{\beta}} ,
\]
and
\[
\frac{F_2}{L^2} = -\frac{\pi^2}{720d^3} - \frac{\zeta(3)\pi^2 \xi^3}{2d^3} - \left( \frac{\pi^2 \xi^3}{d^3} + \frac{\pi^2 \xi^2}{2d^3} \right) \text{e}^{-\frac{4\pi^2}{\beta}} .
\]
Hence, upon making use of (43) we obtain
\[
\frac{F}{L^2} \approx \frac{7}{8} \times \frac{\pi^2}{720d^3} - \left( \frac{\pi^2 \xi^3}{d^3} + \frac{\pi^2 \xi^2}{2d^3} \right) \text{e}^{-\frac{4\pi^2}{\beta}} ,
\]
which is the low temperature approximation to the Helmholtz free energy corresponding to Boyer’s setup, equation (43).

V. CONCLUSIONS

In this paper we have shown how neatly the generalized zeta function regularization method applies to the repulsive electromagnetic Casimir effect at finite temperature for the simple geometry of a pair of infinite parallel plates, each one of them endowed with a certain special physical property, namely, perfect electric conduction and infinite magnetic permeability. Advantage was taken from the fact that for this simple geometry, the electromagnetic field can be simulated by a massless uncharged scalar field. As a follow-up to the application of this method we have obtained expressions for the Helmholtz free energy and the force per unit area acting on any one of the two plates which comprise this peculiar system. We have obtained the low and high temperature limits of those two quantities. We have also shown that though Boyer’s plates demand the imposition of non-symmetric boundary conditions on the scalar massless field that simulates the electromagnetic field, it is still possible to take advantage of the temperature inversion symmetry, a symmetry which in principle does not hold for the case we have studied here, and relate the high and the low temperature limits, particularly the Stefan-Boltzmann free energy and the Casimir energy at zero temperature.

It is also worth noticing that our high temperature limit result is compatible with ideas of dimensional reduction that occurs for \( \beta \to 0 \). Remark that in this
The interesting similarities and differences of the Casimir effect associated with a massless bosonic field, which arise when we compare the consequences of imposing Dirichlet boundary conditions in one case and mixed ones in the other, indicate that a similar investigation in other theories, such as the massive scalar field at zero as well as at finite temperature might be rewarding. This investigation is being carried out and results will be published elsewhere.

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FIG. 1. The scaled free energy per unit area as a function of the scaled temperature for two perfectly conducting parallel plates and for Boyer’s unusual setup. The scaled free energy is given by $f(\xi) := d^4 \times [F(\beta)/L^2 d]$, where $F(\beta)/L^2$ is given by equation (34). The scaled temperature $\xi$ is defined by $\xi := d/\pi \beta$. The Casimir energy at zero temperature for the parallel conducting plates setup and Boyer’s setup are represented respectively by the straight lines parallel to the scaled temperature axis intercepting the vertical axis at $f(0) = -\pi^2/720 \approx -0.014$ and $f(0) = (7/8) \times \pi^2/720 \approx 0.012$. The scaled free energy curve for Boyer’s setup is represented by longer dashes and tends to $(7/8)\pi^2/720 \approx 0.012$ when $\xi \to 0$.

FIG. 2. The scaled free energy per unit area as a function of the scaled temperature for two perfectly conducting parallel plates and Boyer’s setup for higher values of the scaled temperature. The scaled free energy for Boyer’s setup is the curve represented by longer dashes. In the high $\xi$ limit both curves tend to the Stefan-Boltzmann curve, $-\pi^6 \xi^4/45$.

FIG. 3. The scaled pressure for Boyer’s setup. The vertical axis represents the dimensionless function $p(\xi) := d^4 \times P_{\text{net}}(\beta)$, where $P_{\text{net}}$ is given by equation (32). The straight line parallel to the $\xi$-axis intercepts the scaled pressure axis at $p(0) = (7/8)\pi^2/240 \approx 0.036$. 

9
Conducting plates and Boyer's setup
Conducting plates and Boyer's setup
Scaled temperature

Scaled pressure

Boyer's setup