LINEAR OPTIMAL CONTROL OF TIME DELAY SYSTEMS VIA HERMITE WAVELET

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ABSTRACT. To solve the time delay optimal control problem with quadratic performance index, a direct numerical method based on Hermite wavelet has been proposed in the present study. The idea is to convert the time delay optimal control problem into a quadratic programming problem. To do so, various time functions in the system are expanded as their truncated series and the properties of the operational matrices of integration, delay and product of two Hermite wavelet vectors are used as well. These matrices are utilized to reduce the solution of optimal control with time delay system, to the solution of a quadratic programming with linear constraints. Finally, three examples of time varying and time invariant coefficients are given to compare the results with some of the existing methods.

1. Introduction. A time delay (TD) system is a system in which time delays occur between the application of the delayed variables in the system and their resulting effects on it. The TD system can be constructed either through inherent delays in the components or with a deliberate application of time delays into the system for different purposes [15]. Delays occur frequently in chemical processes, electronic, aerospace and mechanical systems, transmission lines and industrial processes. Mathematical model of systems such as population growth, epidemic growth, economic growth and neural networks results in delay differential equations [5]. Control and optimization of delay systems have attracted the interest of many researchers. Also considerable attention has been paid to the development of efficient numerical methods for solving such systems. Direct and indirect methods are two major methods for solving optimal control problems. Pontryagins maximum principle as an indirect method to optimization of control systems with time delays is based on converting the original problem into a two-point boundary value problem which is very difficult to solve [6, 7, 10, 12, 13]. The main objective of all computational aspects of time delay optimal control systems is to avoid the solution of two-point

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boundary value problem. Direct methods (parameterization, discretization) are based on converting the original optimal control problem into nonlinear optimization problem [3, 17, 18, 29]. Analytical solving of delay systems is difficult. Based on orthogonal functions and polynomial series different direct numerical methods have been used to solve optimal control problems, they have received considerable attention in dealing with time delay optimal control problems.

Some signals may have breakpoints or jumps in some time intervals for example in delay systems, the states might have some breakpoints or jumps. Signals with jumps in the time domain often have higher frequencies and the truncated approximation of these signals have low accuracy. Neither the continuous basis functions nor piecewise constant basis functions taken alone would form an efficient basis in the representation of such signals [2]. Wavelets are very accurate to solve these problems. They are flexible and have limited duration, varying frequency and the possibility of time shifting, therefore they are accurate to solve such problems.

A brief review of related literature on the method developed in this paper, is presented here. As mentioned above orthogonal functions have received significant attention in dealing with time delay optimal control problems. Orthogonal functions or polynomial series such as block pulse functions [20], Walsh functions [25], Fourier series [26], Legendre polynomials [14, 22], Legendre wavelet [27], Chebyshev polynomials [11, 19], Chebyshev wavelet [16, 23], Cosine wavelet [2], Laguerre series [28], hybrid of block pulse and orthogonal functions [4, 9, 21, 24, 31] were applied to find the optimal control of TD systems.

According to the above discussions, in this work, a direct method based on Hermite wavelet is used to solve the time delay optimal control problem with quadratic performance index. This method is based on transforming the time delay optimal control problem to a quadratic programming (QP), in which there are many numerical methods for solving a QP. We used quadprog function which is available in optimization toolbox in MATLAB.

The outline of this paper is as follows. It has six sections and the introduction section is the first of them. In Section 2, the basic formulation of Hermite wavelets required for our subsequent development is described. Section 3 is dedicated to introducing operational matrices of Hermite wavelet. In section 4, there is a brief discussion concerning the problem statement. In Section 5, results of our method are reported and the accuracy of the method is demonstrated by considering three numerical examples. The paper ends with the conclusion.

2. Hermite Wavelet. A brief description of Hermite wavelet is presented in this section. Wavelets constitute a family of functions constructed from dilatation and translation of a single function $\psi(t)$ called mother wavelet. When the dilation parameter $a$ and the translation parameter $b$ vary continuously, we have the following family of continuous wavelets as

$$
\psi(a, b) = |a|^{-\frac{1}{2}} \psi\left(\frac{t-b}{a}\right), \quad a, b \in \mathbb{R}, \ a \neq 0,
$$

if the parameters $a$ and $b$ are restricted to discrete values as $a = a_0^{-k}, b = nb_0a_0^{-k}$, $a_0 > 1, b_0 > 0$ we have the following family of discrete wavelets

$$
\psi_{k,n}(t) = |a_0|^{\frac{k}{2}} \psi(a_0^kt - nb_0), \quad k, n \in \mathbb{R},
$$

where they form a wavelet basis for $L^2(\mathbb{R})$. In particular, when $a_0 = 2$ and $b_0 = 1$, then $\psi_{k,n}(t)$ form an orthonormal basis [8]. The Hermite wavelets $\psi_{n,m}(t) = \psi_{n,m}(t)$
ψ(k, n, m, t) involve four arguments n = 1, 2, ⋯, 2^{k-1}, where k is any positive integer, m is the degree of Hermite polynomials and t, the normalized time. They are defined on the interval [0, 1] by [1]:
\[
\psi_{n,m}(t) = \begin{cases} 
2^n \sqrt{\pi} H_m(2^n t - 2n + 1), & \frac{n-1}{2^{k-1}} \leq t < \frac{n}{2^{k-1}}, \\
0, & \text{otherwise},
\end{cases}
\]
(3)
where m = 0, 1, 2, ⋯, M − 1. Here H_m is the Hermite polynomial of degree m with respect to the weight function w(t) = e^{−t^2} in the interval (−∞, ∞) and satisfies the following recursive formula
\[
H_0(t) = 1, \quad H_1(t) = 2t,
H_{m+2}(t) = 2tH_{m+1}(t) − 2(m + 1)H_m(t), \quad m = 0, 1, 2, 3, ⋯.
\]
(4)
Hermite wavelets are an orthogonal set, with respect to the weight function w(t) = e^{−(2^k t−2n+1)^2}. The orthogonality of Hermite wavelets is proven at the end of this section in detail. Function f(t) ∈ L^2(\mathbb{R}) can be expanded as:
\[
f(t) = \sum_{n,m=1}^{\infty} f_{n,m} \psi_{n,m}(t),
\]
(5)
\[f_{n,m} = \langle f, \psi_{n,m} \rangle\] where \(\langle , , \rangle\) denotes the inner product. From the orthogonality of Hermite wavelets, the coefficients \(f_{n,m}\) are given by
\[
f_{n,m} = \frac{\sqrt{\pi}}{2^{m+1} m!} \sqrt{\frac{e}{e-1}} \int_0^1 e^{−(2^k t−2n+1)^2} f(t) \psi_{n,m}(t) dt.
\]
If the infinite series in (5) are truncated, then it can be written as
\[
f(t) \approx \sum_{n=1}^{2^{k-1}-1} \sum_{m=0}^{M-1} f_{n,m} \psi_{n,m}(t) = F^T \Psi(t),
\]
(6)
where \(F\) and \(\Psi(t)\) are \(M 2^{k-1} \times 1\) matrices as below:
\[
F = [f_1, f_2, \cdots, f_{2^{k-1}-1}],
\]
\[
\Psi = [\psi_1, \psi_2, \cdots, \psi_{2^{k-1}-1}].
\]
(7)
Here, the orthogonality property of Hermite wavelet with respect to the weight function w(t) = e^{−(2^k t−2n+1)^2} is proven. Indeed \(\int_0^1 w(t) \psi_{n,m}(t) \psi_{p,q}(t) dt = 0\) if n ≠ p. Below, the orthogonality of Hermite wavelet is proved for n = p. Two cases are considered.

**Case 1:** m ≠ q
We want to show \(\int_0^1 e^{−(2^k t−2n+1)^2} \psi_{n,m}(t) \psi_{n,q}(t) dt = 0\)
\[
\int_0^1 e^{−(2^k t−2n+1)^2} \psi_{n,m}(t) \psi_{n,q}(t) dt = \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} e^{−(2^k t−2n+1)^2} \psi_{n,m} \psi_{n,q} dt
\]
(8)
\[= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} e^{−(2^k t−2n+1)^2} \frac{2^{k+1}}{\pi} H_m(2^k t − 2n + 1)H_q(2^k t − 2n + 1) dt,
\]

we use the change of variable technique as follows

\[ z = 2^k t - 2n + 1 \, dz = 2^k dt, \]
\[ t = \frac{n - 1}{2^k - 1} \rightarrow z = -1, \number{9} \]
\[ t = \frac{n}{2^k - 1} \rightarrow z = 1, \]

thus

\[ \int_0^1 e^{-(2^k t - 2n + 1)^2} \psi_{n,m}^2(t) \psi_{n,q}(t) dt = \frac{2}{\pi} \int_{-1}^1 e^{-z^2} H_m(z) H_q(z) dz. \number{10} \]

The generating function of Hermite polynomial gives

\[ \sum_{q=0}^{\infty} H_q(z) \frac{r^q}{q!} e^{2rz - r^2}, \number{11} \]
\[ \sum_{m=0}^{\infty} H_m(z) \frac{s^m}{m!} e^{2sz - s^2}, \number{12} \]

multiplying these two relations, we get

\[ \sum_{q=0}^{\infty} \sum_{m=0}^{\infty} H_q(z) H_m(z) \frac{r^q s^m}{q! m!} e^{-z^2} = e^{-z^2 + 2rz - r^2 + 2sz - s^2}, \number{13} \]

now, multiply both sides of Eq. (13) by \( e^{-z^2} \)

\[ \sum_{q=0}^{\infty} \sum_{m=0}^{\infty} H_q(z) H_m(z) \frac{r^q s^m}{q! m!} e^{-z^2} = e^{-z^2 + 2rz - r^2 + 2sz - s^2}, \number{14} \]

integrate Eq. (14) with respect to \( z \) on the interval \([-1, 1]\)

\[ \sum_{q=0}^{\infty} \sum_{m=0}^{\infty} \int_{-1}^{1} H_q(z) H_m(z) \frac{r^q s^m}{q! m!} e^{-z^2} dz = \int_{-1}^{1} e^{-z^2 + 2rz - r^2 + 2sz - s^2} dz \]
\[ = \int_{-1}^{1} e^{-z^2 + 2rz - r^2 + 2sz - s^2} dz = e^{2rs} \int_{-1}^{1} e^{-[z-(r+s)]^2} dz, \number{15} \]

indeed \( \int_{-1}^{1} e^{-[z-(r+s)]^2} dz \) has finite value, suppose it is equal to \( \ell \), thus we get

\[ \sum_{q=0}^{\infty} \sum_{m=0}^{\infty} \int_{-1}^{1} H_q(z) H_m(z) \frac{r^q s^m}{q! m!} e^{-z^2} dz = \ell e^{2rs} = \ell \sum_{m=0}^{\infty} \frac{(2rs)^m}{m!}, \number{16} \]

clearly when \( q \neq m \), equating the coefficients of \( r^m s^m \) on both sides of Eq. (16), we obtain

\[ \int_{-1}^{1} e^{-z^2} H_q(z) H_m(z) dz = 0, \number{17} \]

thus

\[ \int_{0}^{1} e^{-(2^k t - 2n + 1)^2} \psi_{n,m}^2(t) \psi_{n,q}(t) dt = \frac{2}{\pi} \int_{-1}^{1} e^{-z^2} H_m(z) H_q(z) dz = 0 \number{18} \]

**Case 2:** \( m = q \)

Equating the coefficients of \( r^m s^m \) on both sides of Eq. (16), gives:

\[ \frac{1}{m! m!} \int_{-1}^{1} e^{-z^2} H_m^2(z) dz = \ell \frac{2^m}{m!} \Rightarrow \int_{-1}^{1} e^{-z^2} H_m^2(z) dz = \ell 2^m m!, \number{19} \]
therefore
\[ \int_{0}^{1} e^{-(2^k t-2n+1)^2} \psi_{n,m}(t)\psi_{n,q}(t)dt = \]
\[ \frac{2}{\pi} \int_{-1}^{1} e^{-z^2} H_m(z)H_q(z)dz = \frac{2}{\pi} \ell 2^m m! = \frac{2}{\pi} 2^m m! \sqrt{\pi \left(1 - \frac{1}{e}\right)} \]

(20)

3. Operational Matrices of Hermite Wavelets. In this section, four new Hermite wavelets operational matrices are introduced.

3.1. Hermite wavelets operational matrix of derivative. The derivative of the vector \( \Psi(t) \) can be expressed by
\[
\frac{d\Psi(t)}{dt} \simeq D\Psi(t),
\]
where \( D \) is the \( M2^{k-1} \times M2^{k-1} \) operational matrix of derivative which is extracted as follows. In order to find the mentioned matrix, the following relation is used

\[ \dot{H}_{m+1}(t) = 2(m+1)H_m(t) \Rightarrow \dot{\psi}_{n,m+1}(t) = 2^{k+1}(m+1)\psi_{n,m}(t), \]

thus
\[ D = \begin{bmatrix}
W & 0 & \ldots & 0 \\
0 & W & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & W
\end{bmatrix}_{2^{k-1} \times 2^{k-1}}, \]

(23)

where \( W \) is as follows
\[ W = 2^{k+1} \begin{bmatrix}
0 & 0 & \ldots & 0 & 0 \\
1 & 0 & \ldots & 0 & 0 \\
0 & 2 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & M-1 & 0
\end{bmatrix}_{M \times M}. \]

(24)

3.2. Hermite wavelets operational matrix of integration. Here we extract the operational matrix of integration \( P \) for Hermite wavelet using the derivative of Hermite wavelet. The operational matrix of integration is defined as follows
\[
\int_{0}^{t} \Psi(\tau)d\tau \simeq P\Psi(t),
\]
where \( P \) is a \( M2^{k-1} \times M2^{k-1} \) matrix given by
\[ P = \begin{bmatrix}
S & F & F & \ldots & F \\
0 & S & F & \ldots & F \\
0 & 0 & S & \ldots & F \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & S
\end{bmatrix}, \]

(26)
where \( S \) and \( F \) are \( M \times M \) matrices as given below

\[
S = \begin{bmatrix}
    a_1 & b_1 & 0 & 0 & \cdots & 0 \\
    a_2 & 0 & b_2 & 0 & \cdots & 0 \\
    a_3 & 0 & 0 & b_3 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    a_{M-1} & 0 & 0 & 0 & \cdots & b_{M-1} \\
    a_M & 0 & 0 & 0 & \cdots & 0
\end{bmatrix}, \quad F = \begin{bmatrix}
    c_1 & 0 & \cdots & 0 \\
    c_2 & 0 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{M} & 0 & \cdots & 0
\end{bmatrix}, \quad (27)
\]

where

\[
a_m = \frac{H_m(-1)}{2^{k+1}m}, \quad b_m = \frac{1}{2^{k+1}m}, \quad c_m = \frac{H_m(1) - H_m(-1)}{2^{k+1}m}. \quad (28)
\]

The procedure of finding \( S \) and \( F \) is as follows

\[
\dot{\psi}_{n,m+1}(t) = 2^{k+1}(m+1)\psi_{n,m}(t)
\]

\[
\Rightarrow \int_0^t \dot{\psi}_{n,m+1}(\tau)d\tau = 2^{k+1}(m+1)\int_0^t \psi_{n,m}(\tau)d\tau
\]

\[
\Rightarrow \int_0^t \psi_{n,m}(\tau)d\tau = \frac{1}{2^{k+1}(m+1)}\int_0^t \dot{\psi}_{n,m+1}(\tau)d\tau. \quad (29)
\]

According to the definition of \( \psi_{n,m} \), we considered three cases for \( t \)

1. \( t < \frac{n-1}{2^{k-1}} \) \( \Rightarrow \int_0^t \psi_{n,m}(\tau)d\tau = 0, \)

2. \( \frac{n-1}{2^{k-1}} \leq t < \frac{n}{2^{k-1}} \) \( \Rightarrow \int_0^t \psi_{n,m}(\tau)d\tau \)

\[
= \frac{1}{2^{k+1}(m+1)} \int_{\frac{n-1}{2^{k-1}}}^{t} \dot{\psi}_{n,m+1}(\tau)d\tau
\]

\[
= \frac{1}{2^{k+1}(m+1)} \psi_{n,m+1}(t) - \frac{H_{m+1}(-1)}{2^{k+1}(m+1)} \psi_{n,0}(t)
\]

\[
= b_{m+1}\psi_{n,m+1}(t) + a_{m+1}\psi_{n,0}(t),
\]

3. \( t \geq \frac{n}{2^{k-1}} \) \( \Rightarrow \int_0^t \psi_{n,m}(\tau)d\tau \)

\[
= \frac{1}{2^{k+1}(m+1)} \int_{\frac{n}{2^{k-1}}}^{t} \dot{\psi}_{n,m+1}(\tau)d\tau
\]

\[
= (H_{m+1}(1) - H_{m+1}(-1))\psi_{n,0}(t) = c_{m+1}\psi_{n',0}(t).
\]

\( n' \) depends on \( t \).

### 3.3. Hermite wavelets operational matrix of production.

One of the operators that play an important role in modelling of equations is operational matrix of production (\( \tilde{F} \)), which is defined as follows:

\[
\Psi(t)\Psi^T(t)F \simeq \tilde{F}\Psi(t), \quad (30)
\]

here \( F \) is a given \( M2^{k-1} \times 1 \) vector and \( \tilde{F} \) is a \( M2^{k-1} \times M2^{k-1} \) matrix. We use the following property of Hermite polynomials for finding \( \tilde{F} \).

\[
H_m(t)H_q(t) = \sum_{r=0}^{\min(m,q)} r!2^r \binom{m}{r} \binom{q}{r} H_{m+q-2r}(t), \quad (31)
\]
In general, \( T \) is a block diagonal matrix and takes the following form

\[
\begin{pmatrix}
B_1 & 0 & \cdots & 0 \\
0 & B_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_{2^{k-1}}
\end{pmatrix}
\]

(35)

where the method of obtaining each \( B_i \), \( i = 1, 2, \cdots, 2^{k-1} \), is similar to the one described in (33)-(34).

3.4. The delay operational matrix of the Hermite wavelets. The delay function \( \Psi(t-\tau) \) is the shift of the function \( \Psi(t) \) along the time axis by \( \tau = l \frac{\pi}{2k-1} \), \( l \in \mathbb{N} \). Below, the expression \( T_d \) is the delay operational matrix of Hermite wavelets:

\[
\Psi(t-\tau) = T_d \Psi(t) \quad t > \tau, \quad 0 \leq t \leq 1,
\]

(36)

\( T_d \) can be derived by the following process

\[
\psi_{n,m}(t-\tau) = 2^{\frac{k}{2}} \sqrt{\frac{\pi}{2}} H_{m}(2^{k}(t-\tau) - 2n + 1), \quad \frac{n-1}{2k-1} \leq t-\tau < \frac{n}{2k-1},
\]

\[
= 2^{\frac{k}{2}} \sqrt{\frac{\pi}{2}} H_{m}(2^{k}(t-\frac{l}{2k-1}) - 2n + 1), \quad \frac{n-1}{2k-1} \leq t - \frac{l}{2k-1} < \frac{n}{2k-1},
\]

\[
= 2^{\frac{k}{2}} \sqrt{\frac{\pi}{2}} H_{m}(2^{k}t - 2(n + l) + 1), \quad \frac{n+l-1}{2k-1} \leq t < \frac{n+l}{2k-1} = \psi_{n+l,m}(t).
\]

Thus \( T_d \) is a constant matrix given by

\[
T_d = \begin{bmatrix}
0_{(\beta,\alpha)} & I_{(\beta,\beta)} \\
0_{(\alpha,\alpha)} & 0_{(\alpha,\beta)}
\end{bmatrix}_{M2^{k-1} \times M2^{k-1}},
\]

(37)

where \( \alpha = lM \), \( \beta = M2^{k-1} - \alpha \).
4. problem statement. Consider the following quadratic optimal control with delay system

\[
\begin{align*}
\min J &= \frac{1}{2} \int_0^1 [X^T(t)QQ(t) + U^T(t)RU(t)]dt, \\
s.t. &= X(t) = A(t)X(t) + B(t)X(t - \tau) + E(t)U(t) + F(t)U(t - \tau), \quad 0 \leq t \leq 1, \\
X(0) &= X_0, \\
X(t) &= g(t), \quad -\tau \leq t < 0, \\
U(t) &= h(t), \quad -\tau \leq t < 0,
\end{align*}
\]

where \( R \) is a symmetric positive definite and \( Q \) is positive semi-definite matrices, \( X(t) \in \mathbb{R}^s \), \( U(t) \in \mathbb{R}^q \), are state and control vectors respectively and \( X_0 \) is a constant specified vector, \( g(t) \) and \( h(t) \) are arbitrary known functions. The problem is to find \( X(t) \) and \( U(t) \) satisfying Eqs. (39)-(42) while minimizing (38). Let

\[
X(t) = [x_1(t), x_2(t), \ldots, x_s(t)]^T,
\]

\[
U(t) = [u_1(t), u_2(t), \ldots, u_q(t)]^T,
\]

each of \( x_i(t), \ i = 1, 2, \ldots, s \) and \( u_j(t), \ j = 1, 2, \ldots, q \) is written in terms of Hermite wavelet as

\[
x_i(t) = \Psi^T(t)X_i, \quad u_j(t) = \Psi^T(t)U_j,
\]

thus we get

\[
X(t) = \hat{\Psi}^T_s(t)X, \quad \hat{\Psi}^T_s(t) = I_s \otimes \Psi(t), \quad X = [X_1^T, X_2^T, \ldots, X_s^T]^T,
\]

\[
U(t) = \hat{\Psi}^T_q(t)U, \quad \hat{\Psi}^T_q(t) = I_q \otimes \Psi(t), \quad U = [U_1^T, U_2^T, \ldots, U_q^T]^T,
\]

where \( I_s \) and \( I_q \) are identity matrices and \( \otimes \) denotes the Kronecker product. Similarly:

\[
g(t) = \hat{\Psi}^T_s(t)G, \quad A(t) = \hat{\Psi}^T_s(t)A, \quad B(t) = \hat{\Psi}^T_s(t)B,
\]

\[
h(t) = \hat{\Psi}^T_q(t)H, \quad E(t) = \hat{\Psi}^T_q(t)E, \quad F(t) = \hat{\Psi}^T_q(t)F.
\]

\( X(t - \tau) \) and \( U(t - \tau) \) are written in terms of Hermite wavelet as follows

\[
X(t - \tau) = \begin{cases} \hat{\Psi}^T_s(t)G, & 0 \leq t < \tau, \\ \hat{\Psi}^T_s(t)\hat{T}_dX, & \tau \leq t \leq 1, \end{cases}
\]

\[
U(t - \tau) = \begin{cases} \hat{\Psi}^T_q(t)H, & 0 \leq t < \tau, \\ \hat{\Psi}^T_q(t)\hat{T}_dU, & \tau \leq t \leq 1, \end{cases}
\]

where \( \hat{T}^T_d = I \otimes T^T_d \). Also we have

\[
A(t)X(t) = X^T\hat{\Psi}(t)\hat{\Psi}^T(t)A = X^T\hat{A}\hat{\Psi}(t),
\]

\[
E(t)U(t) = U^T\hat{\Psi}(t)\hat{\Psi}^T(t)E = U^T\hat{E}\hat{\Psi}(t).
\]

By integrating Eq. (39) from 0 to \( t \) and using Eqs. (40)-(50) we get a set of algebraic equations. Similarly for \( J \) in (38) we get

\[
J(X, U) = \frac{1}{2} [X^T(\Lambda \otimes Q)X + U^T(\Lambda \otimes R)U],
\]
where $\Lambda = \int_0^1 \Psi(t)\Psi^T(t)dt$. Now the delay optimal control problem has been reduced to an optimization problem. The details of the method are described in the numerical examples section.

5. **Numerical examples.** In this section, three examples are given to demonstrate the applicability and accuracy of the method based on Hermite wavelet. We will compare our findings with the numerical results in other papers.

**Example 5.1.** Consider the following time delay optimal control problem [9][30]

$$
\min J = \frac{1}{2} \int_0^2 [x^2(t) + u^2(t)]dt,
$$

(52)

s.t. $\dot{x}(t) = x(t - \frac{1}{2}) + u(t), \quad 0 \leq t \leq 2,$

(53)

$x(t) = 1, \quad -1 \leq t \leq 0,$

(54)

we want to find the control and state variables which minimize the cost functional (52).

By rescaling the time into $[0,1]$, with $t \rightarrow t/2$, we have

$$
\min J = \int_0^1 [x^2(t) + u^2(t)]dt,
$$

(55)

s.t. $\dot{x}(t) = 2(x(t - \frac{1}{2}) + u(t)), \quad 0 \leq t \leq 1,$

(56)

$x(t) = 1, \quad -\frac{1}{2} \leq t \leq 0.$

(57)

We expanded $x(t)$ and $u(t)$ in terms of Hermite wavelets to solve the above problem

$$
x(t) = X^T(t)\Psi(t), \quad u(t) = U^T\Psi(t).
$$

(58)

By integrating $x(t - \frac{1}{2})$ from 0 to 1 we get

$$
\int_0^1 x(t - \frac{1}{2})dt = K^T_1\Psi(t) + K^T_2\Psi(t) + X^TT_pP\Psi(t),
$$

(59)

where $T_p$ and $P$ are delay and integration operational matrices, respectively. $K_1$ and $K_2$ are obtained as follows:

1. $0 \leq t \leq \frac{1}{2}$

$$
\int_0^t x(s - \frac{1}{2})ds = K^T_1\Psi(t),
$$

(60)

$$
K^T_1 = \Big[\frac{a_1}{b}, \frac{1}{b}, 0, \cdots, 0 \Big] \cdots \Big[\frac{a_l}{b}, \frac{1}{b}, 0, \cdots, 0\Big] 0, \cdots, 0 \cdots 0, 0, \cdots, 0, 0\Big]^T,
$$

(61)

$$
a_i = -2(-2i + 1), \quad i = 1, 2, \cdots, l, \quad l = \tau 2^{k-1} \in \mathbb{N}
$$

$$
b = 2^{\frac{k}{2}} \sqrt{\frac{2\pi}{\tau}} 2^{k+1},
$$

2. $\frac{1}{2} \leq t \leq 1$

$$
\int_0^t x(s - \frac{1}{2})ds = K^T_2\Psi(t) + X^TT_pP\Psi(t),
$$

(62)

$$
K^T_2 = \frac{1}{2} \Big[0, \cdots, 0 \cdots 0, \cdots, 0 \Big] \frac{1}{2^{\frac{k}{2}} \sqrt{\frac{2\pi}{\tau}} 2^{k+1}} 0, 0, \cdots, 0 \cdots 0, 0, \cdots, 0\Big]^T.
$$

(63)
By integrating (56) from 0 to 1 we get

\[ X^T \Psi(t) - X_0^T \Psi(t) = 2K_1^T \Psi(t) + 2K_2^T \Psi(t) + 2X^T T_d P \Psi(t) + 2U^T P \Psi(t), \]  

(64)

where \( X_0^T = \frac{1}{2} \sqrt{\frac{2}{\pi}} [1, 0, \ldots, 0] \cdots [1, 0, \cdots, 0] \). From Eq. (64) we get

\[ (I - 2P^T T_d^T) X - 2P^T U = X_0 + 2K_1 + 2K_2. \]  

(65)

The cost functional \( J \) in (55) changes to the following form

\[ J(X, U) = X^T \Lambda X + U^T \Lambda U, \]  

(66)

where

\[ \Lambda = \int_0^1 \Psi^T(t) \Psi(t) dt. \]  

(67)

Now, the following reduced system is obtained

\[
\begin{align*}
\min J(X, U) &= X^T \Lambda X + U^T \Lambda U, \\
\text{s.t.} \quad (I - 2P^T T_d^T) X - 2P^T U &= X_0 + 2K_1 + 2K_2.
\end{align*}
\]

(68)

(69)

In Table 1, we have shown a comparison between the result obtained through this method and the results from other papers.

| Proposed method | Dadkhah [4] | Palanisamy [25] | Wang [30] |
|-----------------|-------------|-----------------|-----------|
| 1.64787419      | 1.64787419  | 1.6497          | 0.85124283 |

**Example 5.2.** Consider the following time delay optimal control problem [25]

\[
\begin{align*}
\min J &= \int_0^2 [x^2(t) + u^2(t)] dt, \\
\text{s.t.} \quad \dot{x}(t) &= 4tx(t) + 2x(t - \frac{1}{2}) + 2u(t), \quad 0 \leq t \leq 1, \\
x(t) &= 1, \quad -\frac{1}{2} \leq t \leq 0.
\end{align*}
\]

(70)

(71)

(72)

The problem is to find the optimal control \( u(t) \) and state \( x(t) \), which minimize \( J \) subject to (71) and (72).

By rescaling the time into \([0, 1]\) there is

\[
\begin{align*}
\min J &= 2 \int_0^1 [x^2(t) + u^2(t)] dt, \\
\text{s.t.} \quad \dot{x}(t) &= 4tx(t) + 2x(t - \frac{1}{2}) + 2u(t), \quad 0 \leq t \leq 1, \\
x(t) &= 1, \quad -\frac{1}{2} \leq t \leq 0.
\end{align*}
\]

(73)

(74)

(75)

First, \( x(t) \) and \( u(t) \) are expanded in terms of Hermite wavelet

\[ x(t) = X^T \Psi(t), \quad u(t) = U^T \Psi(t), \]  

(76)

also

\[ tx(t) = X^T \Psi(t) \Psi^T(t) d = X^T d \Psi(t), \]  

(77)
where
\[
d^T = \left[ \frac{a_1}{b}, \frac{1}{b}, 0, \ldots, 0 \right]^{T},
\]
\[
d_T = \left[ \frac{a_2}{b}, \frac{1}{b}, 0, \ldots, 0 \right] \cdots \left[ \frac{a_{2^k-1}}{b}, \frac{1}{b}, 0, \ldots, 0 \right]^T, \quad (78)
\]
\[
\tilde{d} \text{ is the same as in section 3.3, } a_i, i = 1, 2, \ldots, 2^k-1 \text{ and } b \text{ are similar to those in example (5.1). By substituting (76), (77) and (59) in (74) we get}
\]
\[
J(X, U) = 2X^T \Lambda X + 2U^T \Lambda U, \quad (79)
\]
\[
(I - 4\tilde{d}^T - 2P^T T_d^T)X - 2P^T U = X_0 + 2K_1 + 2K_2. \quad (80)
\]
\Lambda, X_0, K_1, K_2 \text{ are as Example (5.1). In Table 2, a comparison is made between the value of } J \text{ obtained by the present method with the value of } J \text{ reported in some papers.}

### Table 2. Estimated values for } J \text{ for Example 5.2}

| Proposed method | Haddadi [9] | Khellat [14] | Palanisamy [25] |
|-----------------|-------------|--------------|-----------------|
| 4.6192          | 4.7404      | 5.1713       | 6.0079          |

Example 5.3. Consider the delay system described by
\[
\begin{aligned}
\dot{x}_1(t) &= x_1(t) + x_2(t - \frac{1}{4}), \\
\dot{x}_2(t) &= x_2(t) - 5x_1(t - \frac{1}{4}) - x_2(t - \frac{1}{4}) - u(t), \\
x_1(t) &= x_2(t) = 1, \quad -\frac{1}{4} \leq t \leq 0,
\end{aligned} \quad (81)
\]
we want to find the optimal control which minimizes the cost functional [31]
\[
J = \frac{1}{2} \int_0^1 ([x_1(t) + x_2(t)]^2 + u^2(t))dt. \quad (82)
\]
Suppose that
\[
x_1(t) = X_1^T \Psi(t), \quad x_2(t) = X_2^T \Psi(t), \quad u(t) = U^T \Psi(t), \quad (83)
\]
also we have
\[
\int_0^t x_1(s - \frac{1}{4})ds = \begin{cases} 
K_{1}^T \Psi(t) & 0 \leq t \leq \frac{1}{4}, \\
K_{2}^T \Psi(t) + X_1^T T_d \Psi(t) & \frac{1}{4} \leq t \leq 1,
\end{cases} \quad (84)
\]
and
\[
\int_0^t x_2(s - \frac{1}{4})ds = \begin{cases} 
K_1^T \Psi(t) & 0 \leq t \leq \frac{1}{4}, \\
K_2^T \Psi(t) + X_2^T T_d \Psi(t) & \frac{1}{4} \leq t \leq 1,
\end{cases} \quad (85)
\]
where
\[
K_1^T = \left[ \frac{a_1}{b}, \frac{1}{b}, 0, \ldots, 0 \right] \cdots \left[ \frac{a_i}{b}, \frac{1}{b}, 0, \ldots, 0 \right] \cdots \left[ \frac{a_{2^k-1}}{b}, \frac{1}{b}, 0, \ldots, 0 \right], \quad (86)
\]
\[
K_2^T = \frac{1}{4} \frac{1}{2^k \sqrt{\frac{2}{\pi}}} \left[ \frac{0, \ldots, 0}{[0, \ldots, 0]} \cdots \frac{0, \ldots, 0}{[0, \ldots, 0]} \right], \quad (87)
\]
\[ l = \frac{1}{2^{k-1}}, a_i \text{ and } b \text{ are described in example (5.1). If differential equations in (81) are integrated from 0 to } t \text{ and (83)-(87) are used, we get}
\]
\[
(I - P^T)X_1 - P^T T_d^T X_2 = X_0 + K_1 + K_2, \tag{88}
\]
\[ 5P^T T_d^T X_1 + (I - P^T + P^T T_d^T) X_2 + P^T U = X_0 - 6K_1 - 6K_2. \tag{89}
\]

\(X_0\) is similar to example (5.1). Suppose \(\xi = [X_1^T \ X_2^T \ U^T]^T\), the cost functional in (82) changes to the following form

\[ J = \frac{1}{2} \xi^T Q \xi, \quad Q = \begin{bmatrix}
\Lambda & \Lambda & 0 \\
\Lambda & \Lambda & 0 \\
0 & 0 & \Lambda 
\end{bmatrix}, \tag{90}
\]

we could rewrite equations (88) and (89) as follows

\[
\begin{bmatrix}
I - P^T & -P^T T_d^T & 0 \\
5P^T T_d^T & I - P^T + P^T T_d^T & P^T
\end{bmatrix} \xi = \begin{bmatrix}
X_0 + K_1 + K_2 \\
X_0 - 6K_1 - 6K_2
\end{bmatrix}. \tag{91}
\]

Now, we have a quadratic optimization problem. The obtained result given in Table 3 is compared to the result obtained by Wang [31].

**Table 3. Estimated value for \(J\) for Example 5.3**

| Proposed method | Wang [31] |
|-----------------|-----------|
| 2.7930174564    | 2.7930174564 |

6. **Conclusion.** In this study, Hermite wavelets and the excellent properties of operational matrices are used to solve optimal control of delay systems. The idea is to transform the delay differential equation into algebraic equations and also to transform quadratic cost functional to a function of variables \(X\) and \(U\). The obtained QP model of a time delay optimal control problem has a very simple structure and its implementation is convenient. Three test examples were used to observe the efficiency and applicability of using Hermite wavelets and obtained matrices. The properties of operational matrices and Hermite wavelets, can be used for the linear inverse time system too.

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