RIBAUCOUR PARTIAL TUBES AND HYPERSURFACES OF ENNEPER TYPE

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Abstract. In this article we introduce the notion of a Ribaucour partial tube and use it to derive several applications. These are based on a characterization of Ribaucour partial tubes as the immersions of a product of two manifolds into a space form such that the distributions given by the tangent spaces of the factors are orthogonal to each other with respect to the induced metric, are invariant under all shape operators, and one of them is spherical. Our first application is a classification of all hypersurfaces with dimension at least three of a space form that carry a spherical foliation of codimension one, extending previous results by Dajczer, Rovenski and the second author for the totally geodesic case. We proceed to prove a general decomposition theorem for immersions of product manifolds, which extends several related results. Other main applications concern the class of hypersurfaces of $\mathbb{R}^{n+1}$ that are of Enneper type, that is, hypersurfaces that carry a family of lines of curvature, correspondent to a simple principal curvature, whose orthogonal $(n-1)$-dimensional distribution is integrable and whose leaves are contained in hyperspheres or affine hyperplanes of $\mathbb{R}^{n+1}$. We show how Ribaucour partial tubes in the sphere can be used to describe all $n$-dimensional hypersurfaces of Enneper type for which the leaves of the $(n-1)$-dimensional distribution are contained in affine hyperplanes of $\mathbb{R}^{n+1}$, and then show how a general hypersurface of Enneper type can be constructed in terms of a hypersurface in the latter class. We give an explicit description of some special hypersurfaces of Enneper type, among which are natural generalizations of the so called Joachimsthal surfaces.

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1. Introduction

In [5] and [6], the authors studied the problem of determining the hypersurfaces of dimension at least three of a space form that carry a totally geodesic foliation of codimension one. The initial motivation of this work was to investigate the similar problem that one can pose by assuming the foliation to be spherical instead of being totally geodesic. A foliation being spherical means that each leaf is an umbilical submanifold whose mean curvature vector field is parallel with respect to its normal connection.

In the solution of the problem addressed in [5] and [6], one main example of a hypersurface of a space form that carries a totally geodesic foliation of codimension...
one is a partial tube over a smooth regular curve. Partial tubes are submanifolds that are generated by starting with any submanifold whose normal bundle has a parallel and flat subbundle, taking a submanifold in the fiber of that subbundle at a given point, and then parallel transporting it along the former submanifold with respect to its normal connection. They are characterized by the property that the tangent spaces to the submanifold that is parallel transported along the starting submanifold give rise to a totally geodesic distribution that is invariant under all of its shape operators.

In this paper we introduce a class of submanifolds that play the role of partial tubes in our context, and which are the basis for some of our main results. They are immersions of product manifolds with two factors that contain partial tubes as special cases and whose construction is based on the extension of the Ribaucour transformation for submanifolds developed in [3] and [4], so we call them Ribaucour partial tubes (see Section 3 for the precise definition). They extend the notion of an \( \mathcal{N} \)-Ribaucour transform defined in [7].

It is a basic result of this paper that Ribaucour partial tubes are precisely the immersions of product manifolds with two factors such that the distributions given by the tangent spaces of the factors are orthogonal to each other with respect to the induced metric, are invariant under all shape operators, and one of them is spherical (Theorem 8). This immediately implies that any submanifold of a space form that carries a spherical distribution that is invariant under all shape operators and whose orthogonal distribution is integrable is locally a Ribaucour partial tube (Corollary 10). In particular, this yields an explicit description of all surfaces with flat normal bundle of a space form such that the lines of curvature of one the two families have constant geodesic curvature (Corollary 11). Another immediate application is a description of all foliations of a space form by spherical submanifolds whose orthogonal distributions are integrable, in particular of all foliations of a space form whose leaves are spherical hypersurfaces (Corollary 14).

In the hypersurface case, we show that the condition of being invariant by the shape operator is automatically satisfied if the rank of the distribution is greater than half of the dimension of the hypersurface. In particular, it follows that any hypersurface with dimension \( \geq 3 \) of a space form that carries a spherical foliation of codimension one is locally a Ribaucour partial tube over a curve, giving a complete answer to the problem that was one of the initial motivations of this paper and leading to a rather explicit description of such hypersurfaces (Corollary 18).

Hypersurfaces of \( \mathbb{R}^{n+1} \) that are Ribaucour partial tubes over curves can be also characterized by the fact that they carry a family of lines of curvature, correspondent to a simple principal curvature, whose orthogonal \((n - 1)\)-dimensional distribution is integrable and whose leaves are contained in hyperspheres of \( \mathbb{R}^{n+1} \) that intersect the hypersurface orthogonally. This has led us to investigate the more general class of hypersurfaces for which such leaves are contained in hyperspheres that do not necessarily intersect the hypersurface orthogonally. For \( n = 2 \), this reduces to studying the class of surfaces with spherical lines of curvature correspondent to one of the principal curvatures, which was widely investigated by many geometers since the second half of the nineteenth century. The interest in such surfaces has been renewed in connection with the construction of immersed constant mean curvature tori in Euclidean three space by Wente [18] and others. In [18], surfaces with spherical lines of curvature correspondent to one of the principal curvatures
were called *surfaces of Enneper type*. Accordingly, we say that a hypersurface of \( \mathbb{R}^{n+1} \) is of Enneper type if it carries a family of lines of curvature, correspondent to a simple principal curvature, whose orthogonal \((n-1)\)-dimensional distribution is integrable and whose leaves are contained in hyperspheres or affine hyperplanes of \( \mathbb{R}^{n+1} \).

Our approach to studying hypersurfaces of Enneper type was inspired by that presented in Bianchi’s book \[1\] for the case \( n = 2 \). First we show how Ribaucour partial tubes in \( S^n \) can be used to describe all hypersurfaces of Enneper type for which the leaves of the \((n-1)\)-dimensional distribution are contained in affine hyperplanes of \( \mathbb{R}^{n+1} \) (Theorem \[28\]). For \( n = 2 \), these correspond to surfaces with planar lines of curvature associated with one of the principal curvatures. Then we prove that any hypersurface of Enneper type can be constructed in terms of a hypersurface in the latter class (Theorem \[24\]). For that, we first show how to parametrize any hypersurface of Enneper type in \( \mathbb{R}^{n+1} \) in terms of its Gauss map and a triple \((\gamma, \alpha, \beta)\), where \( \gamma : I \to \mathbb{R}^{n+1} \) is a smooth curve and \( \alpha, \beta \in C^\infty(I) \) (Theorem \[26\]). Then we determine all the triples \((\bar{\gamma}, \bar{\alpha}, \bar{\beta})\) that give rise to hypersurfaces of Enneper type in \( \mathbb{R}^{n+1} \) with the same Gauss map as a given one (Proposition \[27\]). It turns out that, among them, there always exists a hypersurface for which the hyperspheres containing the leaves of the \((n-1)\)-dimensional distribution all pass through a common point. An inversion with respect to a hypersphere centered at that point then maps such hyperspheres into affine hyperplanes, and hence, maps the hypersurface into a hypersurface of Enneper type whose leaves are contained in affine hyperplanes of \( \mathbb{R}^{n+1} \).

We also give an explicit description of hypersurfaces of Enneper type in \( \mathbb{R}^{n+1} \) for which the leaves of the \((n-1)\)-dimensional distribution are contained either in concentric hyperspheres, parallel affine hyperplanes or affine hyperplanes that intersect along a common affine \((n-1)\)-dimensional subspace (Theorem \[29\]). Surfaces in \( \mathbb{R}^3 \) with the last of these properties are classically known as Joachimsthal surfaces, and our result yields a new description of these surfaces (Corollary \[30\]). It was recently shown in \[12\] that this last property is also shared by the so called cyclic conformally flat hypersurfaces of \( \mathbb{R}^4 \) with three distinct principal curvatures.

In the last part of the article, the aforementioned characterization of Ribaucour partial tubes is applied to the program of investigating the geometry of an isometric immersion with high codimension by trying to “decompose it” into simpler “components”. This program has similar counterparts in many branches of mathematics, with the decomposition of an integer into prime factors as its most basic example. In differential geometry, from an intrinsic point of view it has led to several de Rham-type theorems, which provide conditions under which a certain Riemannian manifold is (locally or globally) isometric to a product manifold whose metric is of a certain type.

Here we first derive such a de Rham-type theorem that gives conditions for a Riemannian manifold to be locally isometric to a product manifold whose metric is conformal to a polar metric (Theorem \[21\]). Recall that a metric \( g \) on a product manifold \( M = \prod_{i=0}^r M_i \) is said to be polar if there exist a metric \( g_0 \) on \( M_0 \) and, for each \( 1 \leq a \leq r \), a family of metrics on \( M_a \) smoothly parametrized by \( M_0 \), such that

\[
g = \pi_0^*g_0 + \sum_{a=1}^r \pi_a^*(g_a \circ \pi_0),
\]
where $\pi_i : M \to M_i$ is the projection for $0 \leq i \leq r$. Polar metrics include as special cases the warped product of metrics $g_0, \ldots, g_r$ on $M_0, \ldots, M_r$, respectively, with smooth warping functions $\rho_a : M_a \to \mathbb{R}_+$, $1 \leq a \leq r$, that is, metrics given by

$$g = \pi_0^*g_0 + \sum_{a=1}^{r}(\rho_a \circ \pi_0)^2\pi_a^*g_a,$$

in particular the Riemannian product of $g_0, \ldots, g_r$, for which the warping functions $\rho_a$, $1 \leq a \leq r$, are identically one. Warped (respectively, Riemannian) product metrics correspond to polar metrics for which all metrics $g_a(x_0)$ on $M_a$, $1 \leq a \leq r$, $x_0 \in M_0$, are homothetical (respectively, isometric) to a fixed Riemannian metric. Our result extends previous results in [8], [13] and [15] for warped product metrics, metrics that are conformal to Riemannian and warped product metrics, and polar metrics, respectively.

From an extrinsic point of view, several decomposition theorems for immersions of product manifolds have been obtained under the assumption that the tangent spaces to the factors are invariant by all shape operators, starting from Moore’s basic result characterizing extrinsic products of immersions among isometric immersions of Riemannian product manifolds that satisfy that condition. Moore’s theorem has been generalized in [9], [14], [15] and [16] for product manifolds endowed with more general types of metrics, namely, warped product metrics, metrics that are conformal to Riemannian product and warped product metrics, and polar metrics, respectively. Here we use the notion of a Ribaucour partial tube to provide a further generalization for the fairly general class of metrics that are conformal to polar metrics. Namely, we give a complete description of all conformal immersions of a product manifold endowed with a polar metric under the assumption that the tangent spaces of the factors are invariant by all shape operators (Theorem 31).

2. The Ribaucour Transformation

This section is devoted to review some basic facts on the Ribaucour transformation. For further details we refer to [3] and [4].

Let $f : M^n \to \mathbb{R}^m$ be an isometric immersion of a Riemannian manifold $M^n$. We denote by $\mathcal{S}(M)$ the module of symmetric sections of the vector bundle of endomorphisms of $TM$, that is, those elements of $\Gamma(\text{End}(TM))$ such that $\langle \Phi X, Y \rangle = \langle X, \Phi Y \rangle$ for all $X, Y \in \mathcal{X}(M)$.

A map $F : M^n \to \mathbb{R}^m$ is called a Combescure transform of an isometric immersion $f : M^n \to \mathbb{R}^m$ determined by $\Phi \in \mathcal{S}(M)$ when $F_* = f_* \circ \Phi$. This condition forces $\Phi$ to satisfy the Codazzi equation

$$(\nabla_X \Phi)Y = (\nabla_Y \Phi)X$$

and to commute with the second fundamental form of $f$, in the sense that

$$\alpha(X, \Phi Y) = \alpha(\Phi X, Y)$$

for all $X, Y \in \mathcal{X}(M)$. Conversely, if $M^n$ is simply connected then any $\Phi \in \mathcal{S}(M)$ satisfying these two conditions determines a Combescure transform $F : M^n \to \mathbb{R}^m$ of $f : M^n \to \mathbb{R}^m$ such that $F_* = f_* \circ \Phi$.

For any Combescure transform $F : M^n \to \mathbb{R}^m$ of an isometric immersion $f : M^n \to \mathbb{R}^m$ of a simply connected Riemannian manifold, there exist $\varphi \in C^\infty(M)$ and
\[ \beta \in \Gamma(N_f M) \] satisfying

\[ \alpha(\text{grad } \varphi, X) + \nabla_X^\perp \beta = 0 \]

such that

\[ \mathcal{F} = f \star \text{grad } \varphi + \beta \quad \text{and} \quad \Phi = \text{Hess } \varphi - A_\beta. \]

Conversely, any solution \((\varphi, \beta)\) of (1) determines a Combescure transform \(\mathcal{F}\) of \(f\) defined by (2). We denote by \(D(f)\) the space of solutions \((\varphi, \beta)\) of (1).

Given an isometric immersion \(f: M^n \to \mathbb{R}^m\), an immersion \(\tilde{f}: M^n \to \mathbb{R}^m\) is said to be a Ribaucour transform of \(f\) when \(||f - \tilde{f}|| \neq 0\) everywhere and there exists a triple \((\mathcal{P}, D, \delta)\), with \(\mathcal{P}: f^* T\mathbb{R}^m \to \tilde{f}^* T\mathbb{R}^m\) a vector bundle isometry, \(D \in \mathcal{S}(M)\) and \(\delta \in \Gamma(f^* T\mathbb{R}^m)\) nowhere vanishing, such that

\begin{itemize}
  \item[(i)] \(\mathcal{P}Z - Z = \langle Z, \delta \rangle (f - \tilde{f})\)
  \item[(ii)] \(\tilde{f} = \mathcal{P}f \star D\).
\end{itemize}

Geometrically, \(f\) and \(\tilde{f}\) envelop a common congruence of \(n\)-dimensional spheres, with \(\mathcal{P}(x), x \in M^n\), being the reflection with respect to the hyperplane orthogonal to \(\tilde{f}(x) - f(x), x \in M^n\). The requirement that the tensor \(D\) be symmetric implies that the shape operators \(A^f_\xi\) and \(A^\tilde{f}_\xi\) with respect to corresponding normal directions commute for every \(\xi \in \Gamma(N_f M)\). For surfaces in \(\mathbb{R}^3\), this is equivalent to requiring \(f\) and \(\tilde{f}\) to share the same lines of curvature.

A nice feature of the Ribaucour transformation is that all Ribaucour transforms of a given isometric immersion \(f: M^n \to \mathbb{R}^m\) can be explicitly parametrized as follows in terms of \(f\) and the pairs \((\varphi, \beta) \in D(f)\).

**Theorem 1 (H).** Let \(f: M^n \to \mathbb{R}^m\) be an isometric immersion of a simply connected Riemannian manifold and let \(\tilde{f}: M^n \to \mathbb{R}^m\) be a Ribaucour transform of \(f\) with data \((\mathcal{P}, D, \delta)\). Then, there exists \((\varphi, \beta) \in D(f)\) such that

\[ \tilde{f} = f - 2\nu \varphi \mathcal{F}, \]  

where \(\mathcal{F} = f \star \text{grad } \varphi + \beta\) is the Combescure transform determined by \((\varphi, \beta)\) and \(\nu = \langle \mathcal{F}, \mathcal{F} \rangle^{-1}\). Moreover,

\[ \mathcal{P} = I - 2\nu \mathcal{F}^* \mathcal{F}, \quad D = I - 2\nu \varphi \Phi \quad \text{and} \quad \delta = -\varphi^{-1} \mathcal{F}, \]

where \(\Phi = \text{Hess } \varphi - A_\beta\) and \(\mathcal{F}^* Z = (\mathcal{F}, Z)\).

Conversely, given \((\varphi, \beta) \in D(f)\) and an open subset \(U \subset M^n\) where \(\varphi\) and \(\mathcal{F} = f \star \text{grad } \varphi + \beta\) are nowhere vanishing and \(D\) is invertible, then \(\tilde{f}: U \to \mathbb{R}^m\) given by (3) is a Ribaucour transform of \(f|_U\).

The Ribaucour transform determined by \((\varphi, \beta)\) is denoted by \(R_{(\varphi, \beta)} f\).

**Example 2.** (i) Given a point \(P_0 \in \mathbb{R}^m\) and \(r > 0\), set \(2\varphi_1 = ||f - P_0||^2 - r^2\) and \(\beta_1 = (f - P_0)_{N_f M}\). Then \(\mathcal{F} = f - P_0\), \(\Phi = I\), and

\[ \tilde{f} = R_{(\varphi_1, \beta_1)} (f) = P_0 + r^2 ||f - P_0||^{-2}(f - P_0) \]

is the composition of \(f\) with an inversion with respect to the sphere of radius \(r\) centered at \(P_0\).

(ii) Given a parallel normal vector field \(\xi\), define \((\varphi_2, \beta_2)\) by \(2\varphi_2 = ||\xi||^2\) and \(\beta_2 = -\xi\). Then \(\mathcal{F} = -\xi\) and

\[ \tilde{f} = R_{(\varphi_2, \beta_2)} (f) = f + \xi \]
is the parallel translation of $f$ by $\xi$.

The Ribaucour transformation can be easily extended for submanifolds of any space form $Q^n$. Namely, an immersion $\tilde{f} : M^n \to Q^n$ is a Ribaucour transform of an isometric immersion $f : M^n \to Q^n$ with data $(\mathcal{P}, D, \delta)$ if $\tilde{F} := i \circ f : M^n \to \mathbb{R}_c^{m+1}$, where $i : Q^n \to \mathbb{R}_c^{m+1}$ is the umbilical inclusion, is a Ribaucour transform of $F = i \circ f$ with data $(\mathcal{P}, D, \tilde{\delta})$, where $\tilde{\delta} = \delta - cF$ and $\tilde{\mathcal{P}} : F^*\mathbb{R}_c^{m+1} \to F^*\mathbb{R}_c^{m+1}$ is the extension of $\mathcal{P}$ defined by setting $\tilde{\mathcal{P}}(F) = \tilde{F}$. In this setting, Theorem 1 reads as follows.

**Theorem 3 ([1]).** Let $f : M^n \to Q^n$ be an isometric immersion of a simply connected Riemannian manifold and let $\tilde{f} : M^n \to Q^n$ be a Ribaucour transform of $f$ with data $(\mathcal{P}, D, \delta)$. Then there exists $(\varphi, \beta) \in \mathcal{D}(f)$ such that

\begin{equation}
\tilde{F} = F - 2\nu \varphi \mathcal{G},
\end{equation}

where $\mathcal{G} = F \times \text{grad} \varphi + \beta + c\varphi F$ and $\nu = (\mathcal{G}, \mathcal{G})^{-1}$. Moreover,

\begin{equation}
\tilde{\mathcal{P}} = I - 2\nu \mathcal{G}^* \mathcal{G}^*, \quad D = I - 2\nu \varphi (\text{Hess} \varphi + c\varphi I - A_\beta) \quad \text{and} \quad \tilde{\delta} = -\varphi^{-1}\mathcal{G}.
\end{equation}

Conversely, given $(\varphi, \beta) \in \mathcal{D}(f)$ and an open subset $U \subset M^n$ where $\varphi \nu \neq 0$ and the tensor $D$ given by (5) is invertible, let $\tilde{F} : U \to \mathbb{R}_c^{m+1}$ be defined by (4). Then $\tilde{F} = i \circ \tilde{f}$, where $\tilde{f}$ is a Ribaucour transform of $f$.

### 3. Ribaucour Partial Tubes

In this section we introduce the concept of a Ribaucour partial tube, on which some of the main results of this article are based.

Let $f_1 : M_1 \to \mathbb{R}^k$ be an isometric immersion along which there is an orthonormal set $\{\xi_1, \ldots, \xi_k\}$ of normal vector fields that are parallel in the normal connection. The subbundle $\mathcal{L} = \text{span} \{\xi_1, \ldots, \xi_k\}$ of $N_{f_1}M_1$ is thus parallel and flat. Hence the map $\Psi : M_1 \times \mathbb{R}^k \to \mathcal{L}$, defined by

\begin{equation}
\Psi_{x_1}(y) = \Psi(x_1, y) = \sum_{i=1}^k y_i \xi_i(x_1)
\end{equation}

for all $x_1 \in M_1$ and $y = (y_1, \ldots, y_k) \in \mathbb{R}^k$, is a parallel vector bundle isometry.

For a fixed $y \in \mathbb{R}^k$, we denote by $\Psi(y)$ the parallel section of $\mathcal{L}$ given by $\Psi(y)(x_1) = \Psi(x_1, y)$ for all $x_1 \in M_1$. Given an isometric immersion $f_0 : M_0 \to \mathbb{R}^k$, first recall that the partial tube over $f_1$ with $f_0$ as fiber is the map $g : M_0 \times M_1 \to \mathbb{R}^n$ given by

\begin{equation}
g(x_0, x_1) = f_1(x_1) + \Psi_{x_1}(f_0(x_0)).
\end{equation}

Geometrically, $g(M_0 \times M_1)$ is generated by taking the image of $f_0(M_0)$ under a fixed $\Psi_{x_1}$, $x_1 \in M_1$, and parallel translating it along $f_1$ with respect to its normal connection (see, e.g., Chapter 10 of [2] for details).

Now take $(\varphi, \beta) \in \mathcal{D}(f_1)$ and define $f : M_0 \times M_1 \to \mathbb{R}^m$ by

\begin{equation}
f(x_0, x_1) = (\mathcal{R}_{(\varphi, \beta, \Psi(f_0(x_0)))}f_1)(x_1) = (f_1 - 2\nu_{x_0} \varphi \mathcal{F}_{x_0})(x_1),
\end{equation}

where $\mathcal{F}_{x_0} = f_1 \times \text{grad} \varphi + \beta + \Psi(f_0(x_0))$ and $\nu_{x_0} = ||\mathcal{F}_{x_0}||^{-2}$. For each $x_0 \in M_0$, we denote by

\begin{equation}
\mathcal{P}_{x_0} = I - 2\nu_{x_0} \mathcal{F}_{x_0}^* \mathcal{F}_{x_0} \quad \text{and} \quad D_{x_0} = I - 2\nu_{x_0} \varphi \Psi_{x_0},
\end{equation}

where $x_0 \in M_0$. Then $\mathcal{F}_{x_0} \mathcal{G}_{x_0} = \nu_{x_0}$ and $\varphi_{x_0} = \mathcal{G}_{x_0}$.
with \( \Phi_{x_0} = \text{Hess } \varphi - A_{f_1}^1 + \Psi(f_0(x_0)) \), the vector bundle isometry and the symmetric endomorphism associated with the Ribaucour transform \( R(\varphi, \beta + \Psi(f_0(x_0)))f_1 \) of \( f_1 \).

If, in particular, \((\varphi, \beta) \in D(f_1)\) is given by \(2\varphi = -1\) and \(\beta = 0\), then

\[
R(\varphi, \beta + \Psi(f_0(x_0)))f_1 = f_1 - 2\nu_{x_0}\varphi f_0 = f_1 + \Psi \left( \frac{f_0(x_0)}{\|f_0(x_0)\|^2} \right),
\]

thus \( R(\varphi, \beta + \Psi(f_0(x_0)))f_1 \) reduces to the partial tube over \( f_1 \) whose fiber is the composition of \( f_0 \) with an inversion with respect a hypersphere of unit radius centered at the origin.

Given a product manifold \( M = \prod_{i=0}^r M_i \) and \( x = (x_0, \ldots, x_r) \in M \), we denote by \( \tau_i^x : M_i \to M \) the inclusion given by \( \tau_i^x(y_i) = (x_0, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_r) \).

**Proposition 4.** The differential of the map \( f \) in (6) at \( x = (x_0, x_1) \) is given by

\[
f_*\tau^x_0 X_0 = -2\nu_{x_0}\varphi P_{x_0}(\Psi_{x_1}(f_0, X_0))
\]

and

\[
f_*\tau^x_1 X_1 = P_{x_0}f_1*D_{x_0}X_1,
\]

for all \( X_i \in T_x M_i, 0 \leq i \leq 1 \).

**Proof.** Differentiating (6) we obtain

\[
f_*\tau^x_0 X_0 = 4\nu_{x_0}\varphi (\Psi_{x_1}(f_0, X_0), F_{x_0}) F_{x_0} - 2\nu_{x_0}\varphi \Psi_{x_1}(f_0, X_0)
\]

\[
= -2\nu_{x_0}\varphi P_{x_0}(\Psi_{x_1}(f_0, X_0)).
\]

Equation (8) is part of the assertions in Theorem 1. \(\square\)

If the map \( f \) given by (6) is an immersion at any point, then it is called the *Ribaucour partial tube* over \( f_1 \) with fiber \( f_0 \) associated with \( \Psi \) and \((\varphi, \beta) \in D(f_1)\), or the Ribaucour partial tube determined by \((f_0, f_1, \Psi, \varphi, \beta)\). It follows from Proposition 4 that \( f \) is an immersion at \((x_0, x_1) \in M\) if and only if neither \( \varphi \) nor \( F_{x_0} \) vanish at \( x_1 \) and the endomorphism \( D_{x_0} T_{x_2} M_1 \) is invertible. We always assume that \( f_0 : M_0 \to \mathbb{R}^k \) is a substantial immersion, for if this is not the case, say, \( f_0(M_0) \subset v + \mathbb{R}^l \) for some \( v \in \mathbb{R}^k \) and \( l < k \), we may replace \((\varphi, \beta)\) by \((\varphi, \beta + \Psi_{x_1}(v))\) and restrict \( \Psi \) to \( M_1 \times \mathbb{R}^l \).

**Proposition 5.** The following assertions on the Ribaucour partial tube \( f \) determined by \((f_0, f_1, \Psi, \varphi, \beta)\) hold:

(i) The induced metric is given by

\[
\langle \tau^x_0 X_0 + \tau^x_1 X_1, \tau^x_0 Y_0 + \tau^x_1 Y_1 \rangle f = 4\nu_{x_0}\varphi^2 \langle X_0, Y_0 \rangle f_0 + \langle D_{x_0}^2 X_1, Y_1 \rangle f_1.
\]

(ii) The normal space of \( f \) at \((x_0, x_1)\) is

\[
N_f M(x_0, x_1) = \mathcal{P}_{x_0}(\mathcal{L}^+(x_1) \oplus \Psi_{x_1}(N_{f_0} M_0(x_0))).
\]

(iii) Given \( \delta \in \Gamma(\mathcal{L}^+) \) and \( \zeta \in \Gamma(N_{f_0} M_0) \), the shape operators of \( f \) with respect to \( \delta, \zeta \in N_f M \), defined by

\[
\delta(x_0, x_1) = \mathcal{P}_{x_0}\delta(x_1) \quad \text{and} \quad \zeta(x_0, x_1) = \mathcal{P}_{x_0}(\Psi_{x_1}(\zeta(x_0))'),
\]

are

\[
A_{\delta}^f \tau^x_0 = -\varphi^{-1} \langle \delta, \beta \rangle \tau^x_0,
\]

\[
A_{\zeta}^f \tau^x_1 = \tau^x_1 D_{x_0}^{-1}(A_{\delta}^f + 2\nu_{x_0}(\beta, \delta) \Phi_{x_0}),
\]

for all \( \delta \in \Gamma(\mathcal{L}^+) \) and \( \zeta \in \Gamma(N_{f_0} M_0) \).
and

\[ A^f_{\xi} \tau^\xi_{\ast} = \frac{1}{2\nu_{x_0}} A^\xi_{\ast} + (\Psi_{x_1}(\xi), \mathcal{F}_{x_0}) \tau^\xi_{\ast} \]

(11)

and

\[ A^f_{\xi} \tau^\xi_{\ast} = \tau^\xi_{\ast} D_{x_0}^{-1}(A^f_{\Psi_{x_1}(\xi)} + 2\nu_{x_0}(\Psi_{x_1}(\xi), \mathcal{F}_{x_0}) \Phi_{x_0}). \]

(12)

(iv) The normal connection of \( f \) is given by

\[ f\nabla^\perp_{\tau^0_{\ast}} X_0 \hat{\delta} = 0, \]

(13)

\[ f\nabla^\perp_{\tau^0_{\ast}} X_1 \hat{\delta} = \mathcal{P}_{x_0}(f_0 \nabla^\perp_{x_1} \zeta), \]

(14)

\[ f\nabla^\perp_{\tau^0_{\ast}} X_0 \hat{\zeta} = \mathcal{P}_{x_0}(\Psi_{x_1}(f_0 \nabla^\perp_{X_0} \zeta)), \]

(15)

and

\[ f\nabla^\perp_{\tau^0_{\ast}} X_1 \hat{\zeta} = 0. \]

(16)

Proof. Items (i) and (ii) are immediate consequences of (7) and (8). For (iii) and (iv), on one hand we have

\[ \nabla_{\tau^0_{\ast}} X_0 \hat{\delta} = -f_0 A^f_{\lambda} \tau^\xi_{\ast} \Phi_{x_0} X_0 + f\nabla^\perp_{\tau^0_{\ast}} X_0 \hat{\delta} \quad \text{and} \quad \nabla_{\tau^0_{\ast}} X_1 \hat{\delta} = -f_0 A^f_{\lambda} \tau^\xi_{\ast} X_1 + f\nabla^\perp_{\tau^0_{\ast}} X_1 \hat{\delta}. \]

On the other hand, using (7) and (8) we obtain

\[ \nabla_{\tau^0_{\ast}} X_0 \hat{\delta} = \nabla_{\tau^0_{\ast}} X_0 (\delta - 2\nu_{x_0}(\mathcal{F}_{x_0}, \delta) \mathcal{F}_{x_0}) \]

\[ = -2\nu_{x_0}(\beta, \delta) \mathcal{P}_{x_0}(f_0 \Psi_{x_1}(X_0)) \]

\[ = \varphi^{-1}(\beta, \delta) f_0 \tau^0_{\ast} X_0 \]

and

\[ \nabla_{\tau^0_{\ast}} X_1 \hat{\delta} = \nabla_{\tau^0_{\ast}} X_1 (\delta - 2\nu_{x_0}(\mathcal{F}_{x_0}, \delta) \mathcal{F}_{x_0}) \]

\[ = -\mathcal{P}_{x_0}(f_0 \mathcal{A}_{\xi}^f X_1 + 2\nu_{x_0}(\beta, \delta) \mathcal{F}_{x_0} X_1) + \mathcal{P}_{x_0}(f_0 \nabla^\perp_{X_1} \delta) \]

\[ = -f_0 A^f_{\lambda} \tau^\xi_{\ast} X_1 + f\nabla^\perp_{\tau^0_{\ast}} X_1 \hat{\delta}, \]

which yield (9), (10), (13) and (14). Repeating the argument for \( \hat{\zeta} \in \Gamma(N_f \mathcal{M}) \) gives

\[ \nabla_{\tau^0_{\ast}} X_0 \hat{\zeta} = -f_0 A^f_{\lambda} \tau^\xi_{\ast} X_0 + f\nabla^\perp_{\tau^0_{\ast}} X_0 \hat{\zeta} \quad \text{and} \quad \nabla_{\tau^0_{\ast}} X_1 \hat{\zeta} = -f_0 A^f_{\lambda} \tau^\xi_{\ast} X_1 + f\nabla^\perp_{\tau^0_{\ast}} X_1 \hat{\zeta}. \]

Using (7), (8) and the fact that \( \Psi \) is a parallel vector bundle isometry, we obtain

\[ \nabla_{\tau^0_{\ast}} X_0 \hat{\zeta} = \nabla_{\tau^0_{\ast}} X_0 (\Psi(\zeta) - 2\nu_{x_0}(\Psi(\zeta), \mathcal{F}_{x_0}) \mathcal{F}_{x_0}) \]

\[ = -\mathcal{P}_{x_0}(f_0 \mathcal{A}_{\xi}^f X_0 + 2\nu_{x_0}(\Psi_{x_1}(\zeta), \mathcal{F}_{x_0}) X_0) + \mathcal{P}_{x_0}(\Psi_{x_1}(f_0 \nabla^\perp_{X_0} \zeta)) \]

\[ = (2\nu_{x_0}(\varphi)^{-1} f_0 \tau^0_{\ast} A_{\xi}^f X_0 + \varphi^{-1}(\Psi_{x_1}(\zeta), \mathcal{F}_{x_0}) f_0 \tau^0_{\ast} X_0 \]

\[ + \mathcal{P}_{x_0}(\Psi_{x_1}(f_0 \nabla^\perp_{X_0} \zeta)) \]

and

\[ \nabla_{\tau^0_{\ast}} X_1 \hat{\zeta} = \nabla_{\tau^0_{\ast}} X_1 (\Psi(\zeta) - 2\nu_{x_0}(\Psi(\zeta), \mathcal{F}_{x_0}) \mathcal{F}_{x_0}) \]

\[ = -\mathcal{P}_{x_0}(f_0 \mathcal{A}_{\xi}^f X_1 + 2\nu_{x_0}(\Psi(\zeta), \mathcal{F}_{x_0}) \Phi_{x_0} X_1) \]

\[ = -f_0 A^f_{\lambda} \tau^\xi_{\ast} X_1 + f\nabla^\perp_{\tau^0_{\ast}} X_1 \hat{\zeta}. \]

Thus (11), (12), (15) and (16) follow. \(\square\)
Before we state some consequences of the preceding proposition, we first recall some terminology.

A net $\mathcal{E} = (E_i)_{i=0,\ldots,r}$ on a differentiable manifold $M$ is a decomposition of its tangent bundle $TM = \oplus_{i=0}^{r} E_i$ as a Whitney sum of integrable distributions. If $M$ is a Riemannian manifold, the net $\mathcal{E} = (E_i)_{i=0,\ldots,r}$ is said to be an orthogonal net if the distributions $E_i$ are mutually orthogonal. Given an isometric immersion $f: M \to \mathbb{R}^m$ of a Riemannian manifold $M$ equipped with a net $\mathcal{E} = (E_i)_{i=0,\ldots,r}$, then the second fundamental form $\alpha_f$ of $f$ is said to be adapted to the net $\mathcal{E}$ if $\alpha_f(X_i, X_j) = 0$ whenever $X_i \in \Gamma(E_i)$ and $X_j \in \Gamma(E_j)$ with $1 \leq i \neq j \leq r$.

In a product manifold $M = \prod_{i=0}^{r} M_i$, we have a natural net $\mathcal{E} = (E_i)_{i=0,\ldots,r}$, called its product net, given by the tangent bundles of its factors, that is, $E_i(x) = \tau_i^* T_x M_i$ for all $x = (x_0, \ldots, x_r) \in M$. For each $0 \leq i \leq r$ we denote

$$M_{\perp_i} = \prod_{j=0}^{r} M_j \quad \text{for} \quad j \neq i$$

and define the projection $\pi_{\perp_i}: M \to M_{\perp_i}$ by

$$\pi_{\perp_i}(x_0, \ldots, x_r) = (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r).$$

Given $x_i \in M_i$, the map $\mu_{x_i}: M_{\perp_i} \to M$ stands for the inclusion

$$\mu_{x_i}(y_0, \ldots, y_{i-1}, y_{i+1}, \ldots, y_r) = (y_0, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_r).$$

A tangent subbundle $E \subset TM$ is umbilical if there exists $Z \in \Gamma(E^\perp)$ such that

$$\langle \nabla_T S \rangle_{E^\perp} = \langle T, S \rangle Z$$

for all $T, S \in \Gamma(E)$, where the subscript $E^\perp$ denotes the orthogonal projection onto $E^\perp$. Umbilical distributions are integrable and the leaves are totally umbilical submanifolds with mean curvature vector field $Z$. If, in addition, $Z$ satisfies

$$\langle \nabla_T Z \rangle_{E^\perp} = 0 \quad \text{for any} \quad T \in \Gamma(E),$$

then $E$ is said to be spherical and its leaves are extrinsic spheres.

**Corollary 6.** Let $f: M_0 \times M_1 \to \mathbb{R}^m$ be the Ribaucour partial tube determined by $(f_0, f_1, \Psi, \varphi, \beta)$ and let $\mathcal{E} = (E_0, E_1)$ be the product net of $M$. Then the following assertions hold:

(i) The net $\mathcal{E}$ is orthogonal with respect to the metric induced by $f$.

(ii) The second fundamental form of $f$ is adapted to $\mathcal{E}$.

(iii) $E_0$ is a spherical distribution.

**Proof.** The assertions in items (i) and (ii) are immediate consequences of parts (i) and (iii) of Proposition 5 respectively. To prove item (iii), we must show that there exists $Z \in \Gamma(E_1)$ such that

$$\langle \nabla_S T, X \rangle = \langle S, T \rangle \langle X, Z \rangle$$

and

$$\langle \nabla_S Z, X \rangle = 0$$
for all $S, T \in \Gamma(E_0)$ and $X \in \Gamma(E_1)$. We have
\[
\overline{\nabla}_{\tau^*_0 \times_0} f \tau^*_0 \times_0 Y_0
= -\nabla_{\tau^*_0 \times_0} 2 \nu_{x_0} \varphi \mathcal{P}_{x_0}(\nabla f_0, Y_0)
= \nabla_{\tau^*_0 \times_0} (-2 \nu_{x_0} \varphi \nabla f_0, Y_0 + 4 \nu_{x_0} \varphi \nabla f_0, Y_0, \mathcal{F}_{x_0}, \mathcal{F}_{x_0})
= 4 \nu_{x_0}^2 \varphi \nabla f_0, (f_0, Y_0, \mathcal{F}_{x_0}) \mathcal{P}_{x_0}(\nabla f_0, Y_0, \mathcal{F}_{x_0})
+ 4 \nu_{x_0} \varphi \nabla f_0, (f_0, Y_0, \mathcal{F}_{x_0}) \mathcal{P}_{x_0}(\nabla f_0, Y_0, \mathcal{F}_{x_0})
- 2 \nu_{x_0} \varphi \mathcal{P}_{x_0}(\nabla f_0, Y_0 + f_0, Y_0)
- \varphi^{-1} \langle \tau^*_0 \times_0, \tau^*_0 \times_0 \rangle_f \mathcal{P}_{x_0}(f_1, \text{grad} \varphi + \Psi f_0, Y_0),
\]
where grad stands for the gradient with respect to the metric induced by $f_1$. Using (7) and (8) we obtain
\[
\langle \nabla_{\tau^*_0 \times_0} f \tau^*_0 \times_0 Y_0, \tau^*_0 \times_1 Y_1 \rangle
= -\varphi^{-1} \langle \tau^*_0 \times_0, \tau^*_0 \times_0 \rangle_f \langle \tau^*_0, D_{x_0} \text{grad} \varphi, \tau^*_0 \times_1 Y_1 \rangle_f.
\]
Hence (19) holds with $Z(x_0, x_1) = -\tau^*_1 \times\text{grad} \varphi (x_1)$. Now, (20) follows from
\[
\overline{\nabla}_{\tau^*_0 \times_0} f_1, Z = -\overline{\nabla}_{\tau^*_0 \times_0} f_1, \text{grad} \varphi
= -4 \nu_{x_0}^2 \langle \nabla f_0, Y_0, \mathcal{F}_{x_0} \rangle \langle f_1, \text{grad} \varphi, \mathcal{F}_{x_0}, \mathcal{F}_{x_0} \rangle + 2 \nu_{x_0} \langle f_1, \text{grad} \varphi, \mathcal{F}_{x_0} \rangle \mathcal{P}_{x_0}(\nabla f_0, Y_0, \mathcal{F}_{x_0})
= -\varphi^{-1} \langle f_1, \text{grad} \varphi, \mathcal{F}_{x_0}, \tau^*_0 \times_0 X_0 \rangle
= -\langle f_1, Z, f_1, Z \rangle_f \tau^*_0 \times_0 X_0.
\]

4. A characterization of Ribaucour partial tubes

The aim of this section is to prove that conditions (i) to (iii) in Corollary (6) characterize Ribaucour partial tubes among immersions $f: M_0 \times M_1 \to \mathbb{R}^m$ of product manifolds. We make use of the following lemma (see Proposition 9 of [15]).

**Lemma 7.** Let $f: M^n \to \mathbb{R}^m$ and $g: L^k \to M^n$ be isometric immersions. Then the following assertions are equivalent:

(i) $g$ is an extrinsic sphere whose mean curvature vector has length $1/r$, $r > 0$, and $\alpha f = 0$ for all $X \in \mathfrak{X}(L)$ and $Z \in \Gamma(N_g L)$.

(ii) There exists $\zeta \in \Gamma(N_g L)$ of length $1/r$, $r > 0$, such that the subbundle of $f_* N_g L$ orthogonal to $f_* \zeta$ is constant in $\mathbb{R}^m$ and the map $f \circ g + r^2 f_* \zeta$ is constant on $L^k$.

(iii) $f(g(L))$ is contained in a sphere of radius $r$ and dimension $(m - n + k)$ in $\mathbb{R}^m$ whose normal space along $f(g(L))$ is $f_* N_g L$.

The characterization of Ribaucour partial tubes is as follows.

**Theorem 8.** Let $f: M = M_0 \times M_1 \to \mathbb{R}^m$ be an immersion satisfying conditions (i) to (iii) in Corollary (4). Then, for any fixed $x_0 \in M_0$, the map $f_1: M_1 \to \mathbb{R}^m$ given by $f_1 = f \circ \mu_{x_0}$ is an immersion whose normal bundle $N_{f_1} M_1$ carries a parallel flat vector subbundle $\mathcal{L}$, and there exist a parallel vector bundle isometry $\Psi: M_1 \times \mathbb{R}^k \to \mathcal{L}$, an immersion $f_0: M_0 \to \mathbb{R}^k$ and $(\varphi, \beta) \in \mathcal{D}(f_1)$ such that $f$ is the Ribaucour partial tube determined by $(f_0, f_1, \Psi, \varphi, \beta)$. 
Proof. The normal space of $f_1$ at $x_1 \in M_1$ splits orthogonally as

$$N_{f_1}M_1(x_1) = f_1^*E_0(x_0, x_1) \oplus N_fM(x_0, x_1).$$

Let $Z \in \Gamma(E_1)$ be the mean curvature vector field of $E_0$ and let $\mu: M_1 \to \mathbb{R}^m$ be defined by $\mu(x_1) = f_1^*Z(x_0, x_1) = f_1xZ_1(x_1)$, where $\mu_0, \mu_1 = Z \circ \mu_{\bar{x}_0}$. By Lemma 7, for each $x_1 \in M_1$ the image by $f$ of the leaf $\sigma(x_1) = M_0 \times \{x_1\} = \mu_0(M_0)$ of $E_0$ is contained in an $(m - m_1)$-dimensional sphere $S(x_1)$ through $f_1(x_1)$ having $f_1E_1$ as its normal bundle along $\sigma(x_1)$, and whose center is the constant value

$$f_1(x_1) + \frac{\mu(x_1)}{||\mu(x_1)||^2}$$

along $\sigma(x_1)$ of the map $f + f_1^*Z$. Thus we can parametrize $S(x_1)$ by

$$\mu(x_1) + t \in (\mu(x_1) + N_{f_1}M_1(x_1)) \mapsto f_1(x_1) + \frac{2}{||\mu(x_1) + t||^2} (\mu(x_1) + t),$$

which can be thought of as the restriction to the affine subspace $\mu(x_1) \oplus N_{f_1}M_1(x_1)$ of the composition of an inversion with respect to a sphere of radius $\sqrt{2}$ centered at the origin and a translation by $f_1(x_1)$. Notice that the image of this parametrization misses the point $f_1(x_1) = f(x_0, x_1) \in S(x_1)$ itself, which is achieved by letting $||t||$ go to infinity.

For each $x_1 \in M_1$, since $f(\sigma(x_1)) \subset S(x_1)$ there exists an immersion $h^{\mu_1}: M_0 \to N_{f_1}M_1(x_1)$ such that

$$f(x_0, x_1) = f_1(x_1) + \frac{2}{||\mu(x_1) + h^{\mu_1}(x_0)||^2} (\mu(x_1) + h^{\mu_1}(x_0)).$$

Denoting

$$\rho(x_0, x_1) = \mu(x_1) + h^{\mu_1}(x_0),$$

we may write

$$f = f_1 \circ \pi_1 + \frac{2}{||\rho||^2} \rho,$$

where $\pi_1: M \to M_1$ is the projection.

Given $\theta \in N_{f_1}M_1(x_1) = f_1^*E_0(x_0, x_1) \oplus N_fM(x_0, x_1)$, for each $x_0 \in M_0$ let

$$\tilde{\theta} = \theta - \frac{2 \langle \theta, \rho(x_0, x_1) \rangle}{||\rho(x_0, x_1)||^2} \rho(x_0, x_1)$$

be the reflection of $\theta$ with respect to the hyperplane orthogonal to the vector $\rho(x_0, x_1)$. We claim that $\tilde{\theta} \in T_{f(x_0, x_1)}S(x_1) = (f_1^*E_1(x_0, x_1))^\perp$, that is, $\langle \tilde{\theta}, \gamma \rangle = 0$ for all $\gamma \in f_1^*E_1(x_0, x_1)$. To prove this, consider the decomposition

$$f_1E_1(x_0, x_1) = (f_1E_1(x_0, x_1) \cap \{f_1^*Z(x_0, x_1)\}^\perp) \oplus \text{span} \{f_1^*Z(x_0, x_1)\}.$$

Since the subbundle $f_1^*E_1 \cap \{f_1^*Z\}$ is parallel along the leaves of $E_0$ by Lemma 7

$$f_1E_1(x_0, x_1) \cap \{f_1^*Z(x_0, x_1)\}^\perp = f_1E_1(x_0, x_1) \cap \{f_1^*Z(x_0, x_1)\}^\perp = f_1T_xM_1 \cap \{\mu(x_1)\}^\perp.$$

Thus $\langle \tilde{\theta}, \gamma \rangle = 0$ for all $\gamma \in f_1E_1(x_0, x_1) \cap \{f_1^*Z(x_0, x_1)\}^\perp$. On the other hand, Lemma 7 also says that

$$f(x_0, x_1) + \frac{f_1^*Z(x_0, x_1)}{||Z(x_0, x_1)||^2} = f_1(x_1) + \frac{\mu(x_1)}{||\mu(x_1)||^2}.$$
for all \( x_0 \in M_0 \). Moreover, \( \|Z(x_0, x_1)\| \) also does not depend on \( x_0 \), and hence coincides with \( \|\mu(x_1)\| \). Hence, substituting (21) in the preceding equation gives

\[
f_*Z(x_0, x_1) = \frac{2\|\mu(x_1)\|^2}{\|\rho(x_0, x_1)\|^2}\rho(x_0, x_1).
\]

It follows that \( \langle \bar{\theta}, f_*Z(x_0, x_1) \rangle = 0 \), and the claim follows.

Now, differentiating (21) at \( x = (x_0, x_1) \) gives

\[
f_*\tau^x_{0,0}X_0 = \frac{2}{\|\rho\|^2}(h^x_{x_0}X_0 - \frac{2}{\|\rho\|^2}\rho)\rho
\]

and

\[
f_*\tau^x_{1,0}X_1 = f_1X_1 + \frac{2}{\|\rho\|^2}(\rho_x\tau^x_{1,0}X_1 - \frac{2}{\|\rho\|^2}\rho)
\]

Given \( \theta \in N_{f_1}M_1(x_1), x_0 \in M_0 \) and \( X_1 \in T_{x_1}M_1 \), endowing \( M_1 \) with the metric induced by \( f_1 \) we obtain

\[
\langle \rho_x\tau^x_{1,0}X_1, \theta \rangle = \langle Z_1, X_1 \rangle h^x_{x_0}(x_0), \theta,
\]

bearing in mind that \( \langle \bar{\theta}, f_*\tau^x_{1,0}X_1 \rangle = 0 \). Thus

(22)

\[
\alpha^{f_1}(X_1, Z_1) + \nabla^h_{X_1} h(x_0) = \langle Z_1, X_1 \rangle h^x_{x_0}(x_0)
\]

for all \( x_0 \in M_0 \) and \( X_1 \in T_{x_1}M_1 \).

For a fixed \( z_0 \in M_0 \), define \( \xi^{x_0} = h(x_0) - h(z_0) \). Then

\[
\nabla^h_{X_1} \xi^{x_0} = (X_1, Z_1) \xi^{x_0}
\]

for all \( X_1 \in T_{x_1}M_1 \). In particular, this implies that

\[
\langle X_1, Z_1 \rangle = X_1(\log \|\xi^{x_0}\|)
\]

for all \( X_1 \in T_{x_1}M_1 \), that is, \( Z_1 = \text{grad} \tau \), where \( \tau = \log \|\xi^{x_0}\| \). Hence \( e^{-\tau}\xi^{x_0} \) is a parallel section of \( N_{f_1}M_1 \) for any \( x_0 \in M_0 \). It follows that the subspaces

\[
L(x_1) = \text{span} \{ \xi^{x_0}(x_1) : x_0 \in M_0 \} \subset N_{f_1}M_1(x_1)
\]

define a parallel and flat subbundle of \( N_{f_1}M_1 \).

Let \( \Psi : M_1 \times \mathbb{R}^k \to \mathcal{L} \) be a parallel vector bundle isometry. Since \( e^{-\tau}\xi^{x_0} \) is a parallel section of \( \mathcal{L} \), there exists \( f_0 : M_0 \to \mathbb{R}^k \) such that \( \Psi(f_0(x_0)) = e^{-\tau}\xi^{x_0} \).

Let \( \varphi \in C^\infty(M_1) \) be given by \( \varphi = -e^{-\tau} \) and let \( \beta = e^{-\tau}h(z_0) \in \Gamma(N_{f_1}M_1) \).

Then (22) becomes

\[
\alpha^{f_1}(X_1, \text{grad} \varphi) + \nabla^h_{X_1} \beta = 0,
\]

that is, \( (\varphi, \beta) \in \mathcal{D}(f_1) \). Finally,

\[
\mathcal{R}_{(\varphi, \beta+\Psi(f_0(x_0)))}f_1 = f_1 - \frac{2(-e^{-\tau})}{e^{-2\tau}\|\mu + h(x_0)\|^2}(e^{-\tau}(\mu + h(x_0)))
\]
\[
= f_1 + \frac{2}{\|\mu + h(x_0)\|^2}(\mu + h(x_0)),
\]

thus \( f \) is the Ribaucour partial tube determined by \( (f_0, f_1, \Psi, \varphi, \beta) \). \( \square \)
Remarks 9. 1) Theorem 8 can be regarded as a conformal counterpart of Theorem 3.5 in [16], which characterizes partial tubes as the immersions \( g: M_0 \times M_1 \to \mathbb{R}^m \) of product manifolds whose induced metrics have properties (i) and (ii) of Corollary 4 and, in addition, the distribution \( E_0 \) in the product net \( \mathcal{E} = (E_0, E_1) \) of \( M_0 \times M_1 \) is totally geodesic. A preliminary step in the proof of that result is the version of Lemma 7 which states that, if \( g: M_0 \to \mathbb{R}^m \) and \( h: L^k \to M_0 \) are isometric immersions, then \( g \) is totally geodesic and \( \alpha_g(h_*, X, Z) = 0 \) for all \( X, Z \in \Gamma(N_0 L) \) if and only if \( g(h(L)) \) is contained in an affine subspace of \( \mathbb{R}^m \) whose normal space along \( g(h(L)) \) is \( g_* N_0 L \). In particular, if \( g: M_0 \times M_1 \to \mathbb{R}^m \) is a partial tube, then the image \( g(M_0 \times \{ x_1 \}) \) by \( g \) of each leaf of \( E_0 \) is contained in an affine subspace of \( \mathbb{R}^m \) whose normal space along \( g(M_0 \times \{ x_1 \}) \) is \( g_* E_1 \). Similarly, it follows from Corollary 4 and Lemma 7 that if \( f: M_0 \times M_1 \to \mathbb{R}^m \) is a Ribaucour partial tube, then the image \( f(M_0 \times \{ x_1 \}) \) by \( f \) of each leaf of \( E_0 \) is contained in a sphere of \( \mathbb{R}^m \) whose normal space along \( f(M_0 \times \{ x_1 \}) \) is \( f_* E_1 \). As a consequence, the composition of a partial tube \( g: M_0 \times M_1 \to \mathbb{R}^m \) with an inversion \( I \) with respect to a hypersphere of \( \mathbb{R}^m \) is a Ribaucour partial tube, and a Ribaucour partial tube \( f: M_0 \times M_1 \to \mathbb{R}^m \) is given in this way if and only if the spheres containing the images \( f(M_0 \times \{ x_1 \}) \) by \( f \) of the leaves of \( E_0 \) all pass through a common point.

2) The definition of a Ribaucour partial tube, as well as Proposition 5 Corollary 6 and Theorem 8 can be easily extended for the case in which the ambient space is any space form \( \mathbb{Q}^m_c \), by making use of the extension of the Ribaucour transformation to this setting discussed at the end of Section 2. The details are left to the reader.

Corollary 10. Let \( M \) be a Riemannian manifold carrying a spherical distribution \( D \) whose orthogonal distribution \( D^\perp \) is integrable. Then any isometric immersion \( f: M \to \mathbb{R}^m \) whose second fundamental form is adapted to the net \( \mathcal{E} = (D, D^\perp) \) is locally a Ribaucour partial tube over the restriction of \( f \) to a leaf of \( D^\perp \).

In particular, Corollary 6 and Theorem 8 yield the following explicit parametrization of any umbilic-free surface with flat normal bundle of a surface form such that the lines of curvature of one of the two families have constant geodesic curvature.

Corollary 11. Let \( \gamma: I \to \mathbb{R}^{n+1} \) be a unit-speed curve. Let \( \xi_1, \ldots, \xi_n \) be a parallel orthonormal frame of \( N_\gamma I \) and let \( \varphi, \beta \in C^\infty(I) \), \( 1 \leq i \leq n \), satisfy \( \beta_i + \varphi' k_i = 0 \), where \( \gamma'' = \sum_{i=1}^n \xi_i k_i \). Let \( \alpha: J \to \mathbb{R}^n \) be a unit-speed curve. Then the map \( f: I \times J \to \mathbb{R}^{n+1} \) given by

\[
(23) \quad f(s, t) = (\gamma(s) - 2\varphi(s) \frac{\varphi'(s) \gamma'(s) + \sum_{i=1}^n (\beta_i(s) + \alpha_i(t)) \xi_i(s)}{(\varphi'(s))^2 + \sum_{i=1}^n (\beta_i(s) + \alpha_i(t))^2})
\]

parametrizes, at regular points, a surface with flat normal bundle whose coordinate curves are lines of curvature and such that the \( t \)-coordinate curves have constant geodesic curvature.

Conversely, any umbilic-free surface with flat normal bundle whose lines of curvature of one family have constant geodesic curvature can be locally parametrized in this way.

Proof. That \( \xi_i \) is parallel along \( \gamma \) in the normal connection means that there exists \( k_i \in C^\infty(I) \) such that \( \xi'_i = -k_i \gamma' \). This implies that \( \gamma'' = \sum_{i=1}^n k_i \xi_i \). Therefore, for \( \varphi \in C^\infty(I) \) and \( \beta = \sum_{i=1}^n \beta_i \xi_i \), Eq. (14) reduces to the set of ODEs \( \beta'_i + \varphi' k_i = 0 \), \( 1 \leq i \leq n \). Since the map \( f: I \times J \to \mathbb{R}^{n+1} \) given by (23) is the Ribaucour partial
tube over \( \gamma \) with \( \alpha : J \to \mathbb{R}^n \) as fiber, the assertion in the direct statement is a consequence of Corollary 6 while the converse follows from Theorem 8.

Given an isometric immersion \( f : M^n \to \tilde{M}^m \), a vector \( \eta \in N_f M(x) \) at \( x \in M^n \) is called a principal normal vector field if the subspace

\[
E_\eta(x) = \{ T \in T_x M : \alpha(T, X) = \langle T, X \rangle \eta \text{ for all } X \in T_x M \}
\]

is nontrivial. A normal vector field \( \eta \in \Gamma(N_f M) \) is called a principal normal vector field of \( f \) with multiplicity \( q \geq 2 \) if \( E_\eta(x) \) has dimension \( q \) at any point \( x \in M^n \). A principal normal vector field \( \eta \in \Gamma(N_f M) \) is said to be a Dupin principal normal vector field if \( \eta \) is parallel in the normal connection along \( E_\eta \).

As a particular case of Corollary 10, we recover one of the main results in [7].

**Corollary 12.** Let \( f : M^n \to \mathbb{R}^m \) be an isometric immersion that carries a Dupin principal normal vector field \( \eta \) with multiplicity \( q \geq 2 \). If the subbundle \( E_\eta^q \) is integrable, then the restriction \( N = E_\eta|_{M_1} \) of \( E_\eta \) to any leaf \( M_1^{\eta \leq q} \) of \( E_\eta^q \) is a flat parallel subbundle of \( N_f M_1 \), where \( f_1 = f|_{M_1} \), and there exist \( (\varphi, \beta) \in D(f_1) \) and a vector bundle isometry \( \Psi : M_1 \times \mathbb{R}^q \to N \) such that \( f \) is locally a Ribaucour partial tube determined by \( (f_0, f_1, \Psi, \varphi, \beta) \), where \( f_0 : \mathbb{R}^q \to \mathbb{R}^q \) is the identity map.

Corollary 12 yields as a special case the following description of all channel hypersurfaces, that is, hypersurfaces \( f : M^n \to \mathbb{R}^{n+1} \), \( n \geq 2 \), carrying a principal curvature \( \lambda \) with multiplicity \( n - 1 \), which is constant along the corresponding lines of curvature if \( n = 2 \) (\( \lambda \) is automatically constant along the leaves of its eigendistribution if \( n \geq 3 \)). If \( n \geq 4 \), these are precisely the conformally flat hypersurfaces of \( \mathbb{R}^{n+1} \) (see, e.g., Theorem 9.6 in [2] for an alternative description based on the conformal Gauss parametrization).

**Corollary 13.** Let \( f : M^n \to \mathbb{R}^{n+1} \), \( n \geq 2 \), be a hypersurface carrying a principal curvature \( \lambda \) with multiplicity \( n - 1 \), which is constant along the correspondent lines of curvature if \( n = 2 \). Then the restriction \( N = E_\lambda|_{M_1} \) of the eigendistribution \( E_\lambda \) to any integral curve \( M_1 \) of \( E_\lambda^q \) is a flat parallel subbundle of \( N_f M_1 \), where \( f_1 = f|_{M_1} \), and there exist \( (\varphi, \beta) \in D(f_1) \) and a vector bundle isometry \( \Psi : M_1 \times \mathbb{R}^{n-1} \to N \) such that \( f \) is locally a Ribaucour partial tube determined by \( (f_0, f_1, \Psi, \varphi, \beta) \), where \( f_0 : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \) is the identity map.

Ribaucour partial tubes also provide a parametrization of all foliations of a space form by spherical submanifolds whose orthogonal distributions are integrable, in particular of all foliations of a space form whose leaves are spherical hypersurfaces.

**Corollary 14.** Let \( f : M^m \to Q^n \) be a local diffeomorphism of a product manifold \( M^m = M_0^m \times M_1^m \) such that the product net \( E = (E_0, E_1) \) of \( M^m \) is orthogonal and \( E_0 \) is spherical with respect to the metric induced by \( f \). Then, for any fixed \( \tilde{x}_0 \in M_0 \), the map \( f_1 : M_1 \to \tilde{M}^m \) given by \( f_1 = f \circ \mu_{\tilde{x}_0} \) is an immersion with flat normal bundle and there exist a parallel vector bundle isometry \( \Psi : M_1^m \times \mathbb{R}^m \to N_{f_1} M_1 \), a local isometry \( f_0 : M_0 \to \mathbb{R}^m \) and \( (\varphi, \beta) \in D(f_1) \) such that \( f \) is the Ribaucour partial tube determined by \( (f_0, f_1, \Psi, \varphi, \beta) \).

5. Hypersurfaces of space forms carrying a spherical foliation

In this section we derive some consequences of Theorem 8 for hypersurfaces of space forms, in particular we provide a complete solution of the problem that was the initial motivation of this work.
A smooth distribution \(D\) on a Riemannian manifold \(M^n\) is said to be curvature invariant if \(R(X,Y)Z \in \Gamma(D)\) for all \(X,Y,Z \in \Gamma(D)\), where \(R\) denotes the curvature tensor of \(M^n\). A basic observation for us is the following fact.

**Lemma 15.** Any spherical distribution on a Riemannian manifold is curvature invariant.

**Proof.** Let \(D\) be a spherical distribution on a Riemannian manifold \(M^n\) with mean curvature vector field \(Z\). Given \(T,S,U \in \Gamma(D)\) and \(X \in \Gamma(D^\perp)\) with \(\langle X,Z \rangle = 0\), since \(\langle \nabla_S U,X \rangle = 0\) and \(\nabla_T X = 0\) by (17) and (18), respectively, it follows that \(\langle R(T,S)U,X \rangle = 0\). On the other hand, again from (17) and (18) we obtain

\[
\langle \nabla_T \nabla_S U,Z \rangle = T(\langle S,U \rangle)\|Z\|^2 - \langle \nabla_S U, \nabla_T Z \rangle
= (T(\langle S,U \rangle) + \langle \nabla_S U, T \rangle)\|Z\|^2.
\]

Similarly,

\[
\langle \nabla_S \nabla_T U,Z \rangle = S(\langle T,U \rangle) + \langle \nabla_T U, S \rangle\|Z\|^2
\]

and

\[
\langle \nabla_{[T,S]} U, Z \rangle = \langle [T,S], U \rangle\|Z\|^2.
\]

Subtracting the last two equations from the first one gives \(\langle R(T,S)U, Z \rangle = 0\). □

For an oriented hypersurface \(f: M^n \rightarrow Q^n_{+}\), we denote by \(A\) its shape operator with respect to the Gauss map \(N\) and by \(\Delta = \ker A\) its relative nullity distribution. The next algebraic lemma was proved in [6].

**Lemma 16.** Let \(f: M^n \rightarrow Q^n_{+}\) be an oriented hypersurface carrying a curvature invariant distribution \(D\) of rank \(k\). Then one of the following holds pointwise:

(i) \(A(D) \subset D^\perp\),
(ii) \(D^\perp \subset A(D)\),
(iii) \(\text{rank } D \cap \Delta = k - 1\).

The two preceding lemmas have the following consequence for hypersurfaces of dimension \(n \geq 3\) that carry a spherical foliation of high rank.

**Corollary 17.** If \(f: M^n \rightarrow Q^n_{+}\), \(n \geq 3\), is an oriented hypersurface and \(D\) is a spherical distribution on \(M^n\) of rank \(k > n/2\) that is not totally geodesic on any open subset, then \(D\) is invariant by the shape operator of \(f\).

**Proof.** Since the subset of \(M^n\) where \(D\) is invariant by the shape operator of \(f\) is closed, there is no loss of generality in assuming that its mean curvature vector field never vanishes. Then \(D \cap \Delta\) must be trivial, for \(\Delta\) is totally geodesic. On the other hand, the distribution \(D\) being curvature invariant by Lemma 15 it follows from Lemma 15 that one of conditions (i) to (iii) must hold at any point of \(M^n\). Since \(k > n/2\), then \(\text{rank } D(x) \cap \Delta(x) > 0\) if \(A(D(x)) \subset D^\perp(x)\) at some \(x \in M^n\), and because \(k - 1 > 0\), then \(\text{rank } D(x) \cap \Delta(x) > 0\) also if (iii) holds at \(x\). Thus neither condition (i) nor condition (iii) can hold at any point of \(M^n\). □

It follows from Corollaries 10 and 17 that any hypersurface \(f: M^n \rightarrow Q^n_{+}\), \(n \geq 3\), that carries a spherical distribution \(D\) of rank \(k > n/2\) whose mean curvature vector field never vanishes and such that \(D^\perp\) is integrable is locally a Ribaucour partial tube over a leaf of \(D^\perp\). In particular, this leads to the following explicit description of such hypersurfaces when \(k = n - 1\) (in which case integrability of \(D^\perp\) is automatic).
Corollary 18. Let $\gamma : I \to \mathbb{R}^{n+1}$ be a unit-speed curve. Let $\xi_1, \ldots, \xi_n$ be a parallel orthonormal frame of $N, I$ and let $\varphi, \beta_i \in C^\infty(I), 1 \leq i \leq n$, satisfy $\beta_i' + \varphi' k_i = 0$, where $\gamma'' = \sum_{i=1}^{n-1} k_i \xi_i$. Let $g : M_1^{n-1} \to \mathbb{R}^n$ be a hypersurface. Then the map $f : M^n = I \times M_1^{n-1} \to \mathbb{R}^{n+1}$ given by

$$f(s, t) = \gamma(s) - 2\varphi(s)\gamma'(s) + \sum_{i=1}^{n} (\beta_i(s) + g_1(t)) \xi_i(s)$$

parametrizes, at regular points, a hypersurface for which the tangent spaces to $\gamma$ where $\gamma'$ of codimension one can be locally parametrized in this way.

Conversely, any hypersurface of $\mathbb{R}^{n+1}, n \geq 3$, carrying a spherical distribution of codimension one can be locally parametrized in this way.

6. A de Rham-type theorem for product manifolds

In the last section of this article we will derive from Theorem 8 a decomposition theorem for conformal immersions of product manifolds endowed with polar metrics (see the end of the introduction for the definition of a polar metric). As a preliminary step of independent interest, in this section we prove a de Rham-type theorem of an intrinsic nature which characterizes the Riemannian manifolds that are conformal to product manifolds endowed with polar metrics.

It was shown in Proposition 4 in [13] that a Riemannian metric on a product manifold $M = \prod_{i=0}^{r} M_i$ is polar if and only if the product net $E = (E_i)_{i=0, \ldots, r}$ of $M$ is an orthogonal net such that $E_a^\perp$ is totally geodesic for all $1 \leq a \leq r$. Our first goal is to obtain a similar characterization of metrics on a product manifold that are conformal to a polar metric. Recall that two Riemannian metrics $g_1$ and $g_2$ on a manifold $M$ are conformal if there exists a positive $\lambda \in C^\infty(M)$ such that $g_2 = \lambda^2 g_1$. The function $\lambda$ is called the conformal factor of $g_2$ with respect to $g_1$.

Theorem 19. A Riemannian metric on a product $M = \prod_{i=0}^{r} M_i$ of connected manifolds is conformal to a polar metric if and only if the product net $E = (E_i)_{i=0, \ldots, r}$ of $M$ is an orthogonal net such that $E_a^\perp$ is totally umbilical for $1 \leq a \leq r$.

Proof. Let $g_1$ be a polar metric on $M = \prod_{i=0}^{r} M_i$ and let $g_2$ be conformal to $g_1$ with conformal factor $\lambda \in C^\infty$. It is well known that the Levi-Civita connections $\nabla^1$ and $\nabla^2$ of $g_1$ and $g_2$, respectively, are related by

$$\nabla^2_X Y = \nabla^1_X Y + \frac{1}{\lambda} (Y(\lambda) X + X(\lambda) Y - g_2(X, Y) \text{grad}_2 \lambda),$$

where $\text{grad}_2$ denotes the gradient with respect to $g_2$. Since $E_a^\perp$ is a totally geodesic distribution with respect to $g_1$ for $1 \leq a \leq r$, it follows from (24) that $E_a^\perp$ is totally umbilical with respect to $g_2$ with mean curvature vector field $-(\text{grad}_2 \log \lambda)_{E_a}$.

To prove the converse statement, for each $1 \leq a \leq r$ decompose $M = M_a \times M_{\perp a}$. Fix $p = (p_0, \ldots, p_r) \in M$ and endow $M_{\perp a}$ with the metric $g_{\perp a} = \mu^*_{p_a} g$. Given any $x = (x_0, \ldots, x_r)$, denote $p^a = (x_0, \ldots, x_{a-1}, p_a, x_{r+1}, \ldots, x_r), \ x^a = (p_0, \ldots, p_{a-1}, x_a, p_{a+1}, \ldots, p_r)$ and $x^{0, a} = (x_0, p_1, \ldots, p_{a-1}, x_a, p_{a+1}, \ldots, p_r)$.

Since $E_a^\perp$ is totally umbilical, it follows from Proposition 1 in [10] that there exists a positive $\lambda_a \in C^\infty(M)$ such that

$$\mu^*_{x_a} g = \lambda^2_{x_a} g_{\perp a} = \lambda^2_{x_a} \mu^*_{p_a} g, \quad \text{for } 1 \leq a \leq r,$$
where \( \lambda_{x_a} = \lambda_a \circ \mu_{x_a} \). Thus, for all \( 0 \leq b \neq a \leq r \) and \( X_b, Y_b \in T_{x_a} M_b \) we have

\[
g(x_0, \ldots, x_r)(\tau^x_b, X_b, \tau^x_b, Y_b) = \lambda^2_{x_a}(\pi_{\pm_a}(x))g(p^a)(\tau^x_b, X_b, \tau^x_b, Y_b).
\]

(25)

For a fixed \( 1 \leq a \leq r \), applying (25) for each \( 1 \leq b \neq a \leq r \) we obtain

\[
g(x_0, \ldots, x_r)(\tau^x_a, X_a, \tau^x_a, Y_a)
= \lambda^2_{x_a}(x_0, \ldots, x_{r-1}) \cdots \lambda^2_{x_{a+1}}(x_0, \ldots, x_a, p_{a+2}, \ldots, p_r)
\]

(26)

\[
\lambda^2_{x_{a-1}}(x_0, \ldots, x_{a-2}, x_a, p_{a+1}, \ldots, p_r) \cdots
\]

\[
\cdots \lambda^2_{x_1}(x_0, p_2, \ldots, p_{a-1}, x_a, p_{a+1}, \ldots, p_r)
\]

\[
g(x^0)(\tau^x_0, X_0, \tau^x_0, Y_0).
\]

(27)

On the other hand, for \( b = 0 \), using recursively (25) following the order given by an arbitrary permutation \( s: \{1, \ldots, r\} \rightarrow \{1, \ldots, r\} \), we obtain

\[
g(x_0, \ldots, x_r)(\tau^x_s, X_s, \tau^x_s, Y_s)
= \lambda^2_{x_{s(1)}}(x_0, \ldots, x_{s(1)-1}, x_{s(1)+1}, \ldots, x_r) \cdots
\]

(28)

\[
\cdots \lambda^2_{x_{s(r)}}(x_0, p_1, \ldots, p_{s(r)-1}, p_{s(r)+1}, \ldots, p_r)g(x^0)(\tau^x_0, X_0, \tau^x_0, Y_0).
\]

Since the permutation \( s \) in (27) is arbitrary, then for any two such permutations \( s, t \) we have

\[
\lambda^2_{x_{s(1)}}(x_0, \ldots, x_{s(1)-1}, x_{s(1)+1}, \ldots, x_r) \cdots \lambda^2_{x_{s(r)}}(x_0, p_1, \ldots, p_{s(r)-1}, p_{s(r)+1}, \ldots, p_r)
\]

\[
= \lambda^2_{x_{t(1)}}(x_0, \ldots, x_{t(1)-1}, x_{t(1)+1}, \ldots, x_r) \cdots \lambda^2_{x_{t(r)}}(x_0, p_1, \ldots, p_{t(r)-1}, p_{t(r)+1}, \ldots, p_r)
\]

Consider the metric \( g_0 = \pi_0^* g \) on \( M_0 \) and, for each \( 1 \leq a \leq r \) and \( x_0 \in M_0 \), let \( g_a(x_0) \) be the metric on \( M_a \) given by

\[
g_a(x_0)(x_a) = \lambda_{x_a}^{-2}(x_0, p_1, \ldots, p_a, \ldots, p_r)(\tau^0_a, g)(x_a).
\]

The proof will be completed as soon as we show that

\[
g = \varphi^2(\pi_0^* g_0 + \sum_{b=1}^{r} \pi_0^*(g_b \circ \pi_0)).
\]

where

\[
\varphi^2(x_0, \ldots, x_r) = \lambda^2_{x_1}(x_0, \ldots, x_{r-1}) \cdots \lambda^2_{x_1}(x_0, p_2, \ldots, p_r).
\]

Using (27) for the permutation \( s(\kappa) = r - 1 - \kappa \) we obtain

\[
g(x)(\tau^x_0, X_0, \tau^x_0, Y_0)
\]

\[
= \lambda^2_{x_1}(x_0, \ldots, x_{r-1}) \cdots \lambda^2_{x_1}(x_0, p_2, \ldots, p_r)g(x^0)(\tau^x_0, X_0, \tau^x_0, Y_0),
\]

while from the definition of the \( g_0 \) we have

\[
\varphi^2(\pi_0^* g_0)(x)(\tau^x_0, X_0, \tau^x_0, Y_0) = \varphi^2(x)g_0(x)(X_0, Y_0)
\]

\[
= \varphi^2(x)g(x^0)(\tau^x_0, X_0, \tau^x_0, Y_0)
\]

\[
= \varphi^2(x)g(x^0)(\tau^x_0, X_0, \tau^x_0, Y_0).
\]
Thus \([29]\) holds for pairs of vectors that belong to \(E_0\). Since the product net is orthogonal, we only have to show that \([29]\) holds for pairs of vectors in \(E_0\) for any \(1 \leq a \leq r\). From the definition of the metric \(g_a(x_0)\) for \(x_0 \in M_0\) we have

\[
\pi_a^*(g_a \circ \pi_a)(x)(\tau_a^x X_a, \tau_a^x Y_a) = g_a(x_0)(X_a, Y_a)
\]

\[
= \lambda_a^{-2}(x_0, p_1, \ldots, p_r)(\tau_a^{x_0} g)(x_0)(X_a, Y_a)
\]

\[
= \lambda_a^{-2}(x_0, p_1, \ldots, p_r)g(x^{0,a})(\tau_a^{x_0} X_a, \tau_a^{x_0} Y_a)
\]

\[
= \lambda_a^{-2}(x_0, p_1, \ldots, p_r)g(x^{0,a})(\tau_a^{x_0} X_a, \tau_a^{x_0} Y_a),
\]

whereas from the definition of \(\varphi\) and \([25]\) we obtain

\[
\varphi^2(x) = \lambda^2_{x_1}(x_0) \cdots \lambda^2_{x_{r-1}}(x_0) \cdot \lambda^2_{x_{a+1}}(x_0, x_a, p_{a+2}, \ldots, p_r)\]

\[
\cdots \lambda^2_{x_1}(x_0, p_2, \ldots, p_{a-1}, x_a, p_{a+1}, \ldots, p_r) \lambda^2_{x_a}(x_0, p_1, \ldots, p_r).
\]

The last two equalities give

\[
\varphi^2(x)(\pi^*_a(g_a \circ \pi_a)(x)(\tau_a^x X_a, \tau_a^x Y_a))
\]

\[
= \lambda^2_{x_r}(x_0, \ldots, x_{r-1}) \cdots \lambda^2_{x_{a+1}}(x_0, x_a, p_{a+2}, \ldots, p_r)
\]

\[
\lambda^2_{x_{a-1}}(x_0, \ldots, x_{a-2}, x_a, p_{a+1}, \ldots, p_r) \cdots
\]

\[
\cdots \lambda^2_{x_1}(x_0, p_2, \ldots, p_{a-1}, x_a, p_{a+1}, \ldots, p_r) \lambda^2_{x_a}(x_0, p_1, \ldots, p_r)g(x^{0,a})(\tau_a^{x_0} X_a, \tau_a^{x_0} Y_a),
\]

thus proving \([20]\) and completing the proof. \(\Box\)

The following additional fact will be needed in the proof of Theorem \([31]\) below.

**Proposition 20.** If \(g_1\) is a polar metric on a product manifold \(M = \prod_{i=0}^r M_i\) with product net \(E = (E_i)_{i=0}^r\) and \(g_2\) is conformal to \(g_1\) with conformal factor \(\lambda \in C^\infty(M)\), then \(E_0\) is a spherical distribution if and only if \(\text{Hess} \lambda\) is adapted to the net \((E_0, E_0^+)\).

**Proof.** Since \(E_0\) is totally geodesic with respect to \(g_1\), by \([24]\) it is umbilical with mean curvature vector field \(\eta = -(\text{grad} \log \lambda)E_0^+\) with respect to \(g_2\). Now, for \(X_0 \in E_0\) and \(Y_j \in E_j \subset E_0^+, j \neq 0\), we have

\[
- \langle \nabla X_0, \eta, Y_j \rangle = \langle \nabla X_0, (\text{grad} \log \lambda), Y_j \rangle - \langle \nabla X_0, (\text{grad} \log \lambda) E_0^+, Y_j \rangle
\]

\[
= \langle \nabla X_0, (\lambda^{-1} \text{grad} \lambda), Y_j \rangle - \langle X_0, (\text{grad} \log \lambda) E_0^+, Y_j \rangle
\]

\[
= -\lambda^{-2} X_0(\lambda^{-1} \text{grad} \lambda) Y_j(\lambda) + \lambda^{-1} \text{Hess} \lambda(X_0, Y_j) + \lambda^{-2} X_0(\lambda) Y_j(\lambda)
\]

\[
= \lambda^{-1} \text{Hess} \lambda(X_0, Y_j),
\]

and the statement follows. \(\Box\)

Given a net \(F = (F_i)_{i=0}^r\) on a manifold \(M\), a diffeomorphism \(\Psi : \prod_{i=0}^r M_i \rightarrow M\) from a product manifold with product net \(E = (E_i)_{i=0}^r\) is called a **product representation** of \(F\) if \(\Psi, E_i(p) = F_i(\Psi(p))\) for \(0 \leq i \leq r\). Combining Theorem 1 in \([11]\) and Theorem \([19]\) yields the following de Rham-type result.

**Theorem 21.** Let \(M\) be a Riemannian manifold and let \(E = (E_i)_{i=0}^r\) be an orthogonal net such that \(E_a^+\) is totally umbilical for each \(1 \leq a \leq r\). Then for every point \(p \in M\) there exists a local product representation \(\Psi : \prod_{i=0}^r M_i \rightarrow U\) of \(E\), with \(p \in U \subset M\), which is conformal with respect to a polar metric on \(\prod_{i=0}^r M_i\).
The decomposition in Theorem 21 is of a local nature. Indeed, an example provided before Theorem 1 in [10] shows that a global representation cannot always be achieved.

Remark 22. Observe that any orthogonal net $\mathcal{E} = (E_0, E_1)$ with only two factors such that $E_0$ has rank one satisfies the conditions in Theorem 21 for any one-dimensional distribution is umbilical. Moreover, if $Z$ is a vector field spanning $E_0$ on a simply connected open subset $U \subset M^n$, then the integrability of $E_1 = E_0^\perp$ is equivalent to $Z$ being the gradient of a smooth function $\varphi \in C^\infty(U)$, the leaves of $E_1$ being the level sets of $\varphi$. Therefore, on any Riemannian manifold $M$ one can find as many such orthogonal nets $\mathcal{E}$, and hence as many local product representations of them that are conformal with respect to a polar metric, as smooth functions on open subsets of $M$.

7. Hypersurfaces of Enneper Type

According to Theorem 3 hypersurfaces $f : M^n = M_0^{n-1} \times I \to \mathbb{R}^{n+1}$ that are Ribaucour partial tubes over curves $\gamma : I \to \mathbb{R}^{n+1}$ are characterized by the fact that their product nets $\mathcal{E} = (E_0, E_1)$ satisfy conditions (i) to (iii) in Corollary 6, with $E_1$ of rank one. By Corollary 13 a special class of such hypersurfaces consists of channel hypersurfaces.

Notice that, by Lemma 7 conditions (ii) and (iii) in Corollary 6 can be replaced by the requirement that the image by $f$ of each leaf $\sigma$ of $E_0$ be contained in a hypersphere of $\mathbb{R}^{n+1}$ that intersects $f(M)$ orthogonally along $f(\sigma)$.

It is a natural problem to investigate the more general class of hypersurfaces $f : M^n = M_0^{n-1} \times I \to \mathbb{R}^{n+1}$ for which condition (iii) is replaced by the following:

(iii$'$) The image by $f$ of each leaf of $E_0$ is contained in a hypersphere of $\mathbb{R}^{n+1}$ (which does not necessarily intersect $f(M)$ orthogonally along $f(\sigma)$).

The next lemma shows that, under condition (i), conditions (ii) and (iii$'$) together are equivalent to requiring the image by $f$ of each leaf $\sigma$ of $E_0$ to be contained in a hypersphere of $\mathbb{R}^{n+1}$ that intersects $f(M)$ at a constant angle along $f(\sigma)$.

Lemma 23. Let $f : M^n \to \mathbb{R}^{n+1}$ be a hypersurface and let $g : L^{n-1} \to M^n$ be a hypersurface of $M^n$ such that $f(g(L))$ is contained in a hypersphere $S$ of $\mathbb{R}^{n+1}$. Then $S$ intersects $f(M)$ at a constant angle along $f(g(L))$ if and only if the shape operator $A$ of $f$ leaves $g_*TL$ invariant.

Proof. Let $P_0 \in \mathbb{R}^{n+1}$ and $R > 0$ be the center and the radius of $S$, respectively, and let $\theta$ be the angle between its unit normal vector field $(f \circ g - P_0)/R$ and a unit normal vector field $N$ of $f$ along $f(g(L))$. Then

$$f \circ g - P_0 = R \cos \theta(N \circ g) + R \sin \theta f_*\delta,$$

where $\delta$ is a unit normal vector field to $g$. Hence, for all $T \in \mathfrak{X}(L)$ we have

$$f_*T = RT(\theta)(-\sin \theta N \circ g + \cos \theta f_*\delta) - R \cos \theta f_*A g_*T + R \sin \theta (f_*\nabla T \delta + \langle Ag_*T, \delta \rangle N).$$

Thus $T(\theta) = 0$ for all $T \in \mathfrak{X}(L)$ if and only if $\langle Ag_*T, \delta \rangle = 0$ for all $T \in \mathfrak{X}(L)$. \hfill $\square$

For $n = 2$, conditions (i) and (ii) in Corollary 6 mean that the leaves of $E_0$ and $E_1$ are lines of curvature of $f$. Condition (iii$'$) says that those correspondent to $E_0$ are contained in spheres. Surfaces in $\mathbb{R}^3$ with these properties were called surfaces of Enneper type in [18]. Accordingly, we call a hypersurface $f : M^n = M_0^{n-1} \times I \to \mathbb{R}^{n+1}$, with product net $\mathcal{E} = (E_0, E_1)$, that satisfies conditions (i), (ii)
Theorem 24. \( \gamma : (30) \in \gamma \beta \).

In the next subsection we first consider the case in which condition (iii) is replaced by the following:

(iii') The image by \( f \) of each leaf of \( E_0 \) is contained in an affine hyperplane of \( \mathbb{R}^{n+1} \).

For \( n = 2 \), surfaces that satisfy conditions (i), (ii) and (iii') are surfaces with planar lines of curvature correspondent to one of their principal curvatures.

7.1. Hypersurfaces of Enneper type with extrinsically planar leaves. The next result shows how all hypersurfaces \( f : M^n = M_0^{n-1} \times I \rightarrow \mathbb{R}^{n+1} \) that satisfy conditions (i), (ii) and (iii') can be constructed in terms of Ribaucour partial tubes \( N : M^n \rightarrow S^n \) over curves in \( S^n \).

Notice that, for a Ribaucour partial tube \( N : M^n \rightarrow S^n \) over a curve in \( S^n \), the product net \( E = (E_0, E_1) \) of \( M^n \) is a twisted product net with respect to the metric \( ds^2 \) induced on \( M^n = M_0^{n-1} \times I \) by \( N \), that is, both \( E_0 \) and \( E_1 \) are umbilical distributions (with \( E_0 \) being, in fact, spherical), and

\[
ds^2 = v_0^2 d\sigma_0^2 + \nu^2 ds^2,
\]

where \( d\sigma_0^2 \) is a metric of constant curvature 1 on \( M_0 \) and \( ds^2 \) is the standard metric on \( I \). The mean curvature vector field of \( E_0 \) is

\[
H_0 = -(\text{grad } \log v_0)_{E_1} = -\frac{1}{\nu^2} \frac{\partial (\log v_0)}{\partial s} \nu ,
\]

where \( \nu \) is a unit vector field (with respect to the metric \( ds^2 \)) along \( I \). Writing

\[
\varphi = -\frac{1}{\nu} \frac{\partial (\log v_0)}{\partial s},
\]

that \( E_0 \) is spherical is equivalent to \( \varphi \) depending only on \( s \).

**Theorem 24.** Let \( N : M^n = M_0^{n-1} \times I \rightarrow S^n \) be a Ribaucour partial tube over a unit-speed curve \( \beta : I \rightarrow S^n \). Given \( V \in C^\infty (I) \) and \( U \in C^\infty (M_0) \), define \( \gamma \in C^\infty (M) \) by

\[
(30) \quad \gamma (x, s) = v_0 (x, s) \left( U (x) + \int_0^s \frac{V (\tau) \nu (x, \tau)}{v_0 (x, \tau)} d\tau \right).
\]

Then the map \( f : M^n \rightarrow \mathbb{R}^{n+1} \) given by

\[
(31) \quad f = \gamma (i \circ N) + i_* N_* \text{grad } \gamma,
\]

where \( i : S^n \rightarrow \mathbb{R}^{n+1} \) is the inclusion and \( \text{grad } \gamma \) is computed with respect to the metric \( d\sigma^2 \) on \( M^n \) induced by \( N \), defines, on the subset of its regular points, a hypersurface satisfying conditions (i), (ii) and (iii').

Conversely, any hypersurface \( f : M_0^{n-1} \times I \rightarrow \mathbb{R}^{n+1} \) satisfying conditions (i), (ii) and (iii') whose shape operator has rank \( n \) everywhere is given locally in this way.

**Proof.** Let \( f : M^n \rightarrow \mathbb{R}^{n+1} \) be given by (31) for some \( \gamma \in C^\infty (M) \). Differentiating (31) gives

\[
(32) \quad f_* = i_* N_* P,
\]

where \( P = \text{Hess } \gamma + \gamma I \), the Hessian being computed with respect to \( d\sigma^2 \).
Moreover, on the open subset where $P$ is invertible, that is, on the open subset of regular points of $f$, it follows from (32) that the map $N$ is the Gauss map of $f$ and that the shape operator of $f$ with respect to $N$ is

\[
A = -P^{-1}.
\]

We claim that $Hess \, \gamma$ is adapted to $E = (E_0, E_1)$ if and only if $\gamma$ is given by (30) for some $V \in C^\infty(I)$ and $U \in C^\infty(M_0)$. Using that $E_0$ is umbilical with mean curvature vector field $H_0 = \nu^{-1} \partial_s$, we obtain

\[
\nabla_X \partial_s = \nabla_X \nu (\nu^{-1} \partial_s)
\]

\[
= X(\nu)\nu^{-1} \partial_s - \nu \|X\|^{-2} \langle \nabla_X X, \nu^{-1} \partial_s \rangle X
\]

\[
= X(\log \nu) \partial_s - \nu \varphi X
\]

for all $X \in \Gamma(E_0)$. Hence

\[
Hess \, \gamma(X, \partial_s) = X \left( \frac{\partial \gamma}{\partial s} \right) - \left( \nabla_X \partial_s \right) (\gamma)
\]

\[
= X \left( \frac{\partial \gamma}{\partial s} \right) - X(\log \nu) \frac{\partial \gamma}{\partial s} + \nu \varphi X(\gamma).
\]

Therefore $Hess \, \gamma(X, \partial_s) = 0$ if and only if

\[
X \left( \frac{\partial \gamma}{\partial s} \right) + \nu \varphi X(\gamma) = X(\log \nu) \frac{\partial \gamma}{\partial s}
\]

Since $\varphi$ depends only on $s$, this can also be written as

\[
X \left( \frac{\partial \gamma}{\partial s} + \varphi \nu \gamma \right) = X(\log \nu) \left( \frac{\partial \gamma}{\partial s} + \varphi \nu \gamma \right).
\]

Thus $\frac{\partial \gamma}{\partial s} + \varphi \nu \gamma = \nu V$ for some $V \in C^\infty(I)$, which can be written as

\[
\frac{\partial (\gamma v_0^{-1})}{\partial s} = \nu V v_0^{-1},
\]

taking into account that $\varphi \nu = -\frac{\partial (\log v_0)}{\partial s}$. This proves our claim.

Notice that $Hess \, \gamma$ being adapted to $E = (E_0, E_1)$ implies both $P$ and $P^{-1}$ to be also adapted to $E$. This, together with (32), (33) and the fact that the product net $E = (E_0, E_1)$ of $M^n$ is orthogonal with respect to the metric induced by $N$, implies that $E$ is also orthogonal with respect to the metric induced by $f$ and that the second fundamental form of $f$ is adapted to $E$. Finally, the image by $N$ of each leaf $\sigma$ of $E_0$ is a small hypersphere of $\mathbb{S}^n$. Hence, if $\mathcal{H}$ is the hyperplane of $\mathbb{R}^{n+1}$ that is parallel to the affine hyperplane that contains $N(\sigma)$, then $N_* T \in \mathcal{H}$ for all $x \in \sigma$ and $T \in T_x \sigma$. It follows from (32) and the fact that $P$ leaves $E_0$ invariant that $f_* T \in \mathcal{H}$ for all $x \in \sigma$ and $T \in T_x \sigma$. Therefore $f(\sigma)$ is also contained in an affine hyperplane parallel to $\mathcal{H}$.

Conversely, let $f: M^n = M_0^{n-1} \times I \to \mathbb{R}^{n+1}$ be a hypersurface satisfying conditions (i), (ii) and (iii') above and let $N: M^n \to \mathbb{S}^n$ be its Gauss map. If $\mathcal{H}$ is the hyperplane that is parallel to the affine hyperplane containing the image by $f$ of a leaf $\sigma$ of $E_0$, by condition (ii) we have

\[
N_* T = \nabla_T N = -f_* A T \in f_* T_x \sigma \in \mathcal{H}
\]
for all $T \in T_x\sigma$. Hence $N(\sigma)$ is also contained in an affine hyperplane parallel to $\mathcal{H}$, and consequently it is an open subset of the small hypersphere of $S^n$ given by its intersection with $S^n$. Therefore, $N: M_0 \times I \to S^n$ is a local diffeomorphism (by the assumption that the shape operator of $f$ has rank $n$ everywhere) with the property that the image by $N$ of any leaf of $E_0$ is a small hypersphere of $S^n$. Thus $E_0$ is a spherical distribution with respect to the metric induced by $N$. Moreover, by condition $(ii)$ the images by $N$ of the integral curves of $E_1$ are orthogonal trajectories of the foliation of $S^n$ given by the images of the leaves of $E_0$. In other words, the net $\mathcal{E}$ is also an orthogonal net with respect to the metric induced by $N$.

It follows from Corollary 14 that $N$ is a Ribaucour partial tube over a unit-speed curve $\beta: I \to S^n$.

Now, the Gauss parametrization allows to recover $f$ in terms of $N$ and the support function $\gamma$ by means of (31). Since the second fundamental form of $f$ is a unit vector field (with respect to the metric induced by $N$), the net $\mathcal{E}$ is also orthogonal with respect to the metric induced by $N$. Moreover, by condition $(ii)$ the images by $N$ of the integral curves of $E_1$ are orthogonal trajectories of the foliation of $S^n$ given by the images of the leaves of $E_0$. In other words, the net $\mathcal{E}$ is also an orthogonal net with respect to the metric induced by $N$. It follows from Corollary 14 that $N$ is a Ribaucour partial tube over a unit-speed curve $\beta: I \to S^n$.

For $n = 2$, Theorem 24 reads as follows.

**Corollary 25.** Let $N: J \times I \to S^2$ be a Ribaucour partial tube over a unit-speed curve $\beta: I \to S^2$ and let $ds^2 = v_1^2 du_1^2 + v_2^2 du_2^2$ be the metric induced by $N$. Given $U \in C^\infty(J)$ and $V \in C^\infty(I)$, let $\gamma \in C^\infty(J \times I)$ be given by

$$
\gamma(u_1, u_2) = v_1(u_1, u_2) \left(U(u_1) + \int_0^{u_2} \frac{V(\tau)v_2(u_1, \tau)}{v_1(u_1, \tau)} d\tau\right).
$$

Then the map $f: J \times I \to \mathbb{R}^3$ given by

$$
f(u_1, u_2) = \gamma(u_1, u_2)N(u_1, u_2) + \frac{1}{v_1^2} \frac{\partial \gamma}{\partial u_1} \frac{\partial N}{\partial u_1} + \frac{1}{v_2^2} \frac{\partial \gamma}{\partial u_2} \frac{\partial N}{\partial u_2}
$$
defines, on the open subset of its regular points, a surface parametrized by lines of curvature whose $u_1$-lines of curvature are planar.

Conversely, any surface in $\mathbb{R}^3$ free of flat points whose lines of curvature correspond to one of its principal curvatures are planar can be locally parametrized in this way.

7.2. The general case. We now address the general problem of describing all hypersurfaces $f: M^n = M_0^{n-1} \times I \to \mathbb{R}^{n+1}$ of Enneper type.

If $N: M^n \to S^n$ is the Gauss map of a hypersurface of Enneper type, it follows from conditions $(i)$ and $(ii)$ that the product net $\mathcal{E} = (E_0, E_1)$ of $M^n$ is orthogonal also with respect to the metric $d\sigma^2$ induced by $N$. Hence, by Theorem 21 (see also Remark 22), we can write

$$
d\sigma^2 = g_s + \rho^2 ds^2
$$
for some $\rho \in C^\infty(M)$ and for some smooth family of metrics $g_s$ on $M_0^{n-1}$ indexed on $I$. In particular, $\rho^{-1} \partial_s$ spans $E_1$ and has unit length with respect to $d\sigma^2$.

Let us assume that, for each $s_0 \in I$, the image by $f$ of the leaf $s = s_0$ of $E_0$ is contained in a hypersphere $S^n(\gamma(s_0), R(s_0))$ of $\mathbb{R}^{n+1}$ with center $\gamma(s_0) \in \mathbb{R}^{n+1}$ and radius $R(s_0)$. The position vector $\zeta(x, s) = f(x, s) - \gamma(s)$ of $S^n(\gamma(s), R(s))$ at $f(x, s)$ with respect to $\gamma(s)$ can be written as $\zeta = R \cos \theta N + R \sin \theta f_\delta$, where $\delta$ is a unit vector field (with respect to the metric induced by $f$) spanning $E_1$ and $\theta \in C^\infty(M)$ is the angle between $\zeta$ and $N$, which depends only on $s$ by Lemma 23.
After changing $\delta$ by $-\delta$, if necessary, we can assume that $f_\delta = \rho^{-1}N, \partial_\delta$. Thus we can write
\begin{equation}
(35) \quad f = \gamma + \alpha N + \beta \rho^{-1}N, \partial_\delta,
\end{equation}
where $\alpha = \alpha(s) = R\cos \theta$ and $\beta = \beta(s) = R\sin \theta$. Now we impose $N$ to be normal to $f$. Since $\alpha$ and $\beta$ depend only on $s$, the condition $0 = \langle f, T, N \rangle$ is identically satisfied. On the other hand, $0 = \langle f, \partial_\delta, N \rangle$ if and only if
\begin{equation}
(36) \quad \langle \gamma', N \rangle + \alpha' = \beta \rho,
\end{equation}
where the prime means derivative with respect to $s$. We have thus proved the converse statement of the following result.

**Theorem 26.** Let $N: M^n \to S^n$ be a local diffeomorphism of a product manifold $M^n = M_0^{n-1} \times I$ such that the product net $\mathcal{E} = (E_0, E_1)$ of $M^n$ is orthogonal with respect to the metric $d\sigma^2_{\gamma}$ induced by $N$ (equivalently, $N$ is a local diffeomorphism whose induced metric $d\sigma^2_{\gamma}$ is given as in (34) for some $\rho \in C^\infty(M)$ and for some smooth family of metrics $g_s$ on $M_0^{n-1}$ indexed on $I$). If there exist a smooth curve $\gamma: I \to \mathbb{R}^{n+1}$ and $\alpha, \beta \in C^\infty(I)$ such that (36) holds, then the map $f: M^n \to \mathbb{R}^{n+1}$ given by (35) parametrizes a hypersurface of Enneper type with respect to $\mathcal{E}$ having $N$ as a Gauss map.

Conversely, if $f: M^n = M_0^{n-1} \times I \to \mathbb{R}^{n+1}$ is a hypersurface of Enneper type with respect to the product net $\mathcal{E} = (E_0, E_1)$ of $M^n$ having $N: M^n \to S^n$ as a Gauss map, then there exist a smooth curve $\gamma: I \to \mathbb{R}^{n+1}$ and $\alpha, \beta \in C^\infty(I)$ satisfying (36) such that $f$ is given by (35).

**Proof.** If $f: M^n \to \mathbb{R}^{n+1}$ is given by (35) in terms of $N$ and $(\gamma, \alpha, \beta)$, then (36) is precisely the condition for $N$ to be a Gauss map of $f$. Moreover, if that condition is satisfied, then
\begin{equation}
(37) \quad \|f - \gamma\|^2 = \alpha^2 + \beta^2 \quad \text{and} \quad \langle \frac{f - \gamma}{\|f - \gamma\|}, N \rangle = \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}.
\end{equation}

The preceding equations show that the image by $f$ of each leaf of $E_0$ is contained in a hypersphere of $\mathbb{R}^{n+1}$ that intersects $f(M)$ at a constant angle. By Lemma (23), the distribution $E_0$ is invariant under the shape operator of $f$, that is, the second fundamental form of $f$ is adapted to the net $\mathcal{E} = (E_0, E_1)$. Since the net $\mathcal{E}$ is orthogonal with respect to the metric induced by $N$, this implies that it is also orthogonal with respect to the metric induced by $f$. Thus $f$ is a hypersurface of Enneper type with respect to $\mathcal{E}$ having $N$ as a Gauss map.

Given a hypersurface of Enneper type $f: M^n = M_0^{n-1} \times I \to \mathbb{R}^{n+1}$ with respect to the product net $\mathcal{E} = (E_0, E_1)$ of $M^n$, we determine next all hypersurfaces of Enneper type with respect to $\mathcal{E}$ sharing the same Gauss map with $f$.

**Proposition 27.** Let $f: M^n = M_0^{n-1} \times I \to \mathbb{R}^{n+1}$ be a hypersurface of Enneper type with respect to the product net $\mathcal{E} = (E_0, E_1)$ of $M^n$. Assume that its Gauss map $N: M^n \to S^n$ is a local diffeomorphism whose induced metric $d\sigma^2$ is given by (34) and that $f$ is parametrized by (35) in terms of $N$, a smooth regular curve $\gamma: I \to \mathbb{R}^{n+1}$ and $\alpha, \beta \in C^\infty(I)$ satisfying (36). Suppose also that there does not exist any leaf $s = s_0$ of $E_0$ whose image by $f$ is (an open piece of) a round $(n-1)$-dimensional sphere. Then any other hypersurface $\tilde{f}: M^n \to \mathbb{R}^{n+1}$ of Enneper type
with respect to $E$ having $N$ as a Gauss map is parametrized by (35) in terms of a smooth curve $\tilde{\gamma}: [0, 1] \to \mathbb{R}^{n+1}$ and $\gamma, \lambda, \beta$ by

$$\tilde{\beta} = \lambda \beta, \quad \tilde{\alpha}' = \lambda \alpha' \quad \text{and} \quad \tilde{\gamma}' = \lambda \gamma'$$

for some $\lambda \in C^\infty([0, 1])$.

**Proof.** Let $\tilde{f}: M^n \to \mathbb{R}^{n+1}$ be another hypersurface of Enneper type with respect to $E$ sharing the same Gauss map $N$ with $f$. By Theorem 26 it can be parametrized by (38) in terms of a smooth curve $\tilde{\gamma}: [0, 1] \to \mathbb{R}^{n+1}$ and $\tilde{\alpha}, \tilde{\beta} \in C^\infty([0, 1])$ satisfying

$$\langle \tilde{\gamma}', N \rangle + \tilde{\alpha}' = \tilde{\beta} \rho.$$

Notice that if $\beta$ vanishes at some $s_0 \in I$, since $\gamma'$ is nowhere vanishing by assumption then (36) implies that the image by $N$ of the leaf $s = s_0$ of $E_0$ is contained in an affine hyperplane of $\mathbb{R}^{n+1}$ (hence a $(n - 1)$-dimensional round hypersphere of $\mathbb{S}^n$). Hence also the image by $f$ of the leaf $s = s_0$ of $E_0$ is contained in an affine hyperplane of $\mathbb{R}^{n+1}$, and therefore it is an open piece of an $(n - 1)$-dimensional round hypersphere given by its intersection with the hypersphere $\mathbb{S}^n(\gamma(s_0), R(s_0))$ containing such image, contradicting our assumption. Thus $\beta$ is nowhere vanishing and we can write $\tilde{\beta} = \lambda \beta$ for some $\lambda \in C^\infty([0, 1])$. Comparing (38) with (36) yields

$$\langle \tilde{\gamma}' - \lambda \gamma', N \rangle + \tilde{\alpha}' - \lambda \alpha' = 0.$$

If $\tilde{\gamma}' - \lambda \gamma'$ was nonzero for some $s_0 \in I$, then arguing as before we would conclude that the image by $f$ of the leaf $s = s_0$ of $E_0$ would be an open piece of an $(n - 1)$-dimensional round hypersphere, a contradiction. Thus $\tilde{\gamma}' - \lambda \gamma'$, and hence also $\tilde{\alpha}' - \lambda \alpha'$ by (39), must vanish everywhere. \qed

We are now able to show how any hypersurface of Enneper type in $\mathbb{R}^{n+1}$ satisfying the assumptions of Proposition 27 can be constructed by means of a hypersurface of Enneper type with extrinsically planar leaves.

**Theorem 28.** Let $f: M^n = M_0^{n-1} \times I \to \mathbb{R}^{n+1}$ be a hypersurface given as in Theorem 27, let $\tilde{f} = I \circ f: M^n = M_0^{n-1} \times I \to \mathbb{R}^{n+1}$ be its composition with an inversion with respect to a hypersphere of unit radius centered at the origin and let $N: \tilde{M}^n \to \mathbb{S}^n$ be the Gauss map of $\tilde{f}$. Let $\tilde{f}$ be parametrized by (35) in terms of $\tilde{N}$ and a triple $(\tilde{\gamma}, \tilde{\alpha}, \tilde{\beta})$ satisfying (36), where $\tilde{\alpha}, \tilde{\beta} \in C^\infty([0, 1])$ and $\tilde{\gamma}: [0, 1] \to \mathbb{R}^{n+1}$ is a smooth curve. Define a new triple $(\gamma, \alpha, \beta)$ by

$$\gamma = \lambda \tilde{\gamma}, \quad \alpha' = \lambda \tilde{\alpha}' \quad \text{and} \quad \gamma' = \lambda \tilde{\gamma}'$$

for some $\lambda \in C^\infty([0, 1])$. Then the hypersurface $\tilde{f}: \tilde{M}^n = M_0^{n-1} \times I \to \mathbb{R}^{n+1}$ parametrized by (35) in terms of $\tilde{N}$ and $(\tilde{\gamma}, \tilde{\alpha}, \tilde{\beta})$ is of Enneper type.

Conversely, any hypersurface of Enneper type in $\mathbb{R}^{n+1}$ satisfying the assumptions of Proposition 27 can be constructed as above.

**Proof.** Since $f$ satisfies conditions (i), (ii) and (iii)” above, it is clear that $\tilde{f} = I \circ f$ satisfies conditions (i), (ii) and (iii)’, and hence is a hypersurface of Enneper type. If the triple $(\gamma, \alpha, \beta)$ satisfies (36), then the same holds for the new triple $(\gamma, \alpha, \beta)$ defined by (40). Then the hypersurface $\tilde{f}: \tilde{M}^n = M_0^{n-1} \times I \to \mathbb{R}^{n+1}$ parametrized by (35) in terms of $\tilde{N}$ and $(\tilde{\gamma}, \tilde{\alpha}, \tilde{\beta})$ is of Enneper type by Theorem 26.

To prove the converse, let $f: M^n = M_0^{n-1} \times I \to \mathbb{R}^{n+1}$ be a hypersurface of Enneper type with respect to the product net $E = (E_0, E_1)$ of $M^n$, parametrized by (35) in terms of its Gauss map $\tilde{N}: \tilde{M}^n \to \mathbb{S}^n$, a smooth regular curve $\tilde{\gamma}: [0, 1] \to \mathbb{S}^n$ sharing the same Gauss map $\tilde{N}$ as $M^n$. A similar construction to the one above shows that $f$ is also of Enneper type.
Let \( \tilde{\Phi} : M^n \to \mathbb{R}^{n+1} \) be the hypersurface given by \( (55) \) in terms of \( \tilde{N} \) and \( (\tilde{\gamma}, \tilde{\alpha}, \tilde{\beta}) \). By Theorem 28, \( \tilde{f} \) also satisfies conditions (i), (ii) and (iii) and has \( \tilde{N} \) as a Gauss map. Moreover, Eqs. \( (37) \) and \( (41) \) imply that the hyperspheres containing the images by \( \tilde{f} \) of the leaves of \( E_0 \) all pass through the origin. Therefore, the composition \( f = I \circ \tilde{f} \) has with an inversion with respect to a hypersphere of unit radius centered at the origin satisfies conditions (i), (ii) and (iii), and hence is given as in Theorem 24.

\[ \int \lambda \tilde{\gamma}' \parallel^2 = \lambda \tilde{\beta}^2 + (\int \lambda \tilde{\alpha}')^2. \]

Let \( \tilde{f} : M^n \to \mathbb{R}^{n+1} \) be some special hypersurfaces of Enneper type. In this subsection we discuss some special hypersurfaces \( f : M^n = M_0^{n-1} \times I \to \mathbb{R}^{n+1} \) of Enneper type with respect to the product net \( \mathcal{E} = (E_0, E_1) \) of \( M^n \), in particular those with the property that the hyperspheres containing the images by \( f \) of the leaves of \( E_0 \) are concentric, which are ruled out in the converse statement of Theorem 28.

First recall that there exists a conformal diffeomorphism \( \Phi : S^{n-1} \times \mathbb{R} \to \mathbb{R}^n \setminus \{0\} \) given by \( (x, t) \mapsto e^t x \). Similarly, there is also a conformal diffeomorphism between \( \mathbb{H}^n \times S^1 \) and \( \mathbb{R}^{n+1} \setminus \mathbb{R}^{n-1} \) given as follows. Let \( e_0, e_1, \ldots, e_{n-1}, e_n \) be a pseudo-orthonormal basis of the Lorentzian space \( \mathbb{R}_1^{n+1} \) satisfying \( \langle e_0, e_0 \rangle = 0 = \langle e_n, e_n \rangle \), \( \langle e_i, e_{n-i} \rangle = -1/2 \) and \( \langle e_i, e_j \rangle = \delta_{ij} \) for \( 1 \leq i \leq n-1 \) and \( 0 \leq j \leq n \). Then the map \( \Phi : \mathbb{H}^n \times S^1 \subset \mathbb{R}_1^{n+1} \times \mathbb{R}^2 \to \mathbb{R}^{n+1} \setminus \mathbb{R}^{n-1} \) given by

\[ \Phi(x_0 e_0 + \ldots + x_n e_n, (y_1, y_2)) = \frac{1}{x_0} (x_1, \ldots, x_{n-1}, y_1, y_2) \]

is a conformal diffeomorphism. Composing \( \Phi \) with the isometric covering map

\[ \pi : \mathbb{H}^n \times \mathbb{R} \to \mathbb{H}^n \times S^1 : (x, t) \mapsto (x, (\cos t, \sin t)) \]

gives rise to a conformal covering map \( \Phi : \mathbb{H}^n \times \mathbb{R} \to \mathbb{R}^{n+1} \setminus \mathbb{R}^{n-1} \) given by

\[ \Phi(x_0 e_0 + \ldots + x_n e_n, t) = \frac{1}{x_0} (x_1, \ldots, x_{n-1}, \cos t, \sin t). \]

Now let \( g : M^{n-1} \to \mathbb{Q}^n \) be a hypersurface, where \( \mathbb{Q}^n \) stands for \( \mathbb{S}^n \) if \( \epsilon = 1 \), \( \mathbb{R}^n \) if \( \epsilon = 0 \) and \( \mathbb{H}^n \) if \( \epsilon = -1 \), and let \( g_s : M^{n-1} \to \mathbb{Q}^n \subset \mathbb{R}^{n+\left|\epsilon\right|} \), where \( \mu = 0 \) if \( \epsilon = 0 \) or \( \epsilon = 1 \), and \( \mu = 1 \) if \( \epsilon = -1 \), be the family of its parallel hypersurfaces, that is,

\[ g_s(x) = C_\epsilon(s) g(x) + S_\epsilon(s) N(x), \]

where \( N \) is a unit normal vector field to \( g \),

\[ C_\epsilon(s) = \begin{cases} \cos s, & \text{if } \epsilon = 1 \\ 1, & \text{if } \epsilon = 0 \\ \cosh s, & \text{if } \epsilon = -1 \end{cases} \quad \text{and} \quad S_\epsilon(s) = \begin{cases} \sin s, & \text{if } \epsilon = 1 \\ s, & \text{if } \epsilon = 0 \\ \sinh s, & \text{if } \epsilon = -1. \end{cases} \]
Define

\[ F': M^n = M^{n-1} \times I \to Q^\epsilon_n \times \mathbb{R} \subset \mathbb{R}^{n+1+|x|} \]

by

\[ F(x, s) = g_s(x) + a(s) \frac{\partial}{\partial t} \]

for some smooth function \( a: I \to \mathbb{R} \) with positive derivative on an open interval \( I \subset \mathbb{R} \). Here, we regard \( F \) as taking values in the underlying flat space.

In the next statement, we denote by \( \Phi \) either the conformal diffeomorphism \( \Phi: Q^\epsilon_n \times \mathbb{R} \to \mathbb{R}^{n+1} \setminus \{0\} \) if \( \epsilon = 1 \), the conformal covering map \( \Phi: Q^\epsilon_n \times \mathbb{R} \to \mathbb{R}^{n+1} \setminus \mathbb{R}^n \) if \( \epsilon = -1 \) or the standard isometry \( \Phi: Q^\epsilon_n \times \mathbb{R} \to \mathbb{R}^{n+1} \) if \( \epsilon = 0 \).

**Theorem 29.** Let \( f: M^n = M^{n-1} \times I \to \mathbb{R}^{n+1} \) be a hypersurface of Enneper type with respect to the product net \( \mathcal{E} = (E_0, E_1) \) of \( M^n \). Assume that the images by \( f \) of the leaves of \( E_0 \) are contained in either

- (a) concentric hyperspheres.
- (b) parallel affine hyperplanes.
- (c) affine hyperplanes intersecting along an affine \((n-1)\)-dimensional subspace.

Then, assuming \( M^n \) simply connected in case (c), there exists \( F: M^n \to Q^\epsilon_n \times \mathbb{R} \) given by (44) in terms of a hypersurface \( g: M^{n-1} \to Q^\epsilon_n \), with \( \epsilon = 1 \) in case (a), \( \epsilon = 0 \) in case (b) and \( \epsilon = -1 \) in case (c), such that \( f = \Phi \circ F \).

**Proof.** Let \( \Psi: \mathbb{R}^{n+1} \setminus \mathbb{R}^{k-1} \to \mathbb{H}^k \times \mathbb{S}^{n-k+1} \subset \mathbb{H}^1 \times \mathbb{R}^{n-k+2} = \mathbb{R}^{n+3} \) be the map given by

\[ \Psi(y_1, \ldots, y_{n+1}) = \sqrt{\frac{1}{y_2^2 + \cdots + y_{n+1}^2}} \left( e_0 + \sum_{i=1}^{k-1} y_i e_i + \sum_{i=1}^{n+1} y_i^2 e_k, (y_k, \ldots, y_{n+1}) \right), \]

where \( \mathbb{R}^{k-1} = \{(y_1, \ldots, y_{n+1}) \in \mathbb{R}^{n+1} : y_k = \cdots = y_{n+1} = 0\} \) and \( e_0, \ldots, e_k \) is a pseudo-orthonormal basis of \( \mathbb{H}^{k+1} \) with \( \langle e_0, e_0 \rangle = 0 = \langle e_k, e_k \rangle \), \( \langle e_i, e_k \rangle = -1/2 \) and \( \langle e_i, e_j \rangle = \delta_{ij} \) for \( 1 \leq i \leq k-1 \) and \( 0 \leq j \leq k \). It is a conformal diffeomorphism whose conformal factor \( \varphi \in C^\infty(\mathbb{R}^{n+1} \setminus \mathbb{R}^{k-1}) \) is \( \varphi(y_1, \ldots, y_{n+1}) = (\sum_{i=1}^{n+1} y_i^2)^{1/2}. \) For \( k = n, \) \( \Psi \) is the inverse of the conformal diffeomorphism \( \Phi: \mathbb{H}^n \times \mathbb{S}^1 \to \mathbb{R}^{n+1} \setminus \mathbb{R}^n \) given by (42). Notice that \( \Psi \) takes each half-space of a \( k \)-dimensional subspace \( \mathbb{R}^k \) containing \( \mathbb{R}^{k-1} \) onto a slice \( \mathbb{H}^k \times \{t\} \) of \( \mathbb{H}^k \times \mathbb{S}^{n-k+1} \), while \((n-k+1)\)-dimensional spheres centered at \( \mathbb{R}^{k-1} \) lying in subspaces \( \mathbb{R}^{n-k+2} \) orthogonal to \( \mathbb{R}^{k-1} \) are mapped onto slices \( \{t\} \times \mathbb{S}^{n-k+1} \) of \( \mathbb{H}^k \times \mathbb{S}^{n-k+1} \).

Assume first that condition (a) is satisfied. Let \( \Psi \) be the diffeomorphism defined above for \( k = 1 \), and let \( \tilde{\Psi} \) be its composition with the isometry

\[(y^{-1}e_0 + ye_1, x) \in \mathbb{H}^1 \times \mathbb{S}^n (y > 0) \mapsto (x, \log y) \in \mathbb{S}^n \times \mathbb{R}.\]

We can assume that the hypersurfaces containing the images by \( f \) of the leaves of \( E_0 \) are centered at the origin. Then the images by \( F = \tilde{\Psi} \circ f: M^n \to \mathbb{S}^n \times \mathbb{R} \) of the leaves of \( E_0 \) are contained in the slices \( \mathbb{S}^n \times \{t\}, t \in \mathbb{R} \). This means that the height function \( (x,s) \mapsto \langle F(x,s), \partial_k \rangle \) depends only on \( s \), where \( \partial_k \) is a unit vector field tangent to the factor \( \mathbb{R} \). Differentiating with respect to \( X \in E_0 \) gives

\[ 0 = \langle F_X, \partial_k \rangle = \langle F_X, T \rangle, \]

where \( \partial_k = F_0 + (\partial_t, N) N \). Since \( \Psi \) is a conformal diffeomorphism, the metrics induced by \( f \) and \( F \) are conformal, hence \( T \) spans \( E_1 \).

Moreover, since conformal diffeomorphisms preserve principal directions and the
integral curves of $E_1$ are lines of curvature of $f$, it follows that $T$ is a principal direction of $F$.

Now assume that condition (b) holds. Denoting by $\Psi: \mathbb{R}^{n+1} \to \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n+1}$ the standard isometry, the images under $F = \Psi \circ f$ of the leaves of $E_0$ are contained in slices $\mathbb{R}^n \times \{t\}$, $t \in \mathbb{R}$. Arguing as in the preceding paragraph, this means that the tangent component of the unit vector field tangent to the factor $\mathbb{R}$ is a principal direction of $F$.

Finally, suppose that condition (c) holds and let $\Psi$ be the diffeomorphism defined in the preceding paragraph for $k = n$. Then the images by $\hat{F} = \Psi \circ f : M^n \to \mathbb{H}^n \times \mathbb{S}^1$ of the leaves of $E_0$ are contained in slices $\mathbb{H}^n \times \{x\}$, $x \in \mathbb{S}^1$. Let $F : M^n \to \mathbb{H}^n \times \mathbb{R}$ be such that $\hat{F} = \pi \circ F$, where $\pi : \mathbb{H}^n \times \mathbb{R} \to \mathbb{H}^n \times \mathbb{S}^1$ is the covering map $(x, t) \mapsto (x, (\cos t, \sin t))$. Then the images by $F$ of the leaves of $E_0$ are contained in slices $\mathbb{H}^n \times \{t\}$, $t \in \mathbb{R}$. Arguing as in case (a) we conclude that $T$ is a principal direction of $F$, where $\partial_t = F_* T + (\partial_t, N) N$, with $\partial_t$ a unit vector field tangent to $\mathbb{R}$.

In either of the preceding cases, it follows from Theorem 1 in [17] that the map $F$ is given by (4.1) in terms of a hypersurface $g : M^{n-1} \to \mathbb{Q}^n$, with $\epsilon = 1$ in case (a), $\epsilon = 0$ in case (b) and $\epsilon = -1$ in case (c). In either case we have $f = \Phi \circ F$, thus the statement follows.

**Surfaces of Enneper type in $\mathbb{R}^3$ satisfying condition (c) in the preceding theorem are known in the literature as Joachimsthal surfaces. These are the surfaces whose lines of curvature correspond to one of the principal curvatures are contained in planes that intersect along a common line, whereas the lines of curvature correspondent to the other principal curvature lie on spheres centered on that line. The following consequence of Theorem 29 shows how any such surface arises.**

**Corollary 30.** Let $\gamma : J \subset \mathbb{R} \to \mathbb{H}^2$ be a unit-speed curve and let $\gamma_s : J \subset \mathbb{R} \to \mathbb{H}^2 \subset \mathbb{R}^4_1$ be the family of its parallel curves, that is,

\[ \gamma_s(t) = \cosh(s) \gamma(t) + \sinh(s) \gamma(t) \wedge \gamma'(t). \]

Define $F : J \times I \to \mathbb{H}^2 \times \mathbb{R} \subset \mathbb{R}^4_1$ by

\[ F(t, s) = \gamma_s(t) + a(s) \frac{\partial}{\partial t}, \]

where $I \subset \mathbb{R}$ is an open interval and $a \in C^\infty(I)$ has positive derivative. Then, on the subset $M^2 \subset J \times I$ of its regular points, the map $f = \Phi \circ F : M^2 \to \mathbb{R}^3 \setminus \mathbb{R}$, where $\Phi : \mathbb{H}^2 \times \mathbb{R} \to \mathbb{R}^3 \setminus \mathbb{R}$ is the conformal covering map given by (4.3), defines a Joachimsthal surface.

Conversely, any Joachimsthal surface in $\mathbb{R}^3$ can be parametrized in this way.

**8. A decomposition theorem**

The aim of this last section is to prove the following decomposition theorem for immersions of product manifolds.

**Theorem 31.** Let $f : M = \prod_{i=0}^r M_i \to \mathbb{R}^m$ be a conformal immersion with conformal factor $\lambda \in C^\infty(M)$ of a product manifold $M$ endowed with a polar metric. Assume that the second fundamental form of $f$ is adapted to the product net $E = (E_i)_{i=0, \ldots, r}$ of $M$. If $\dim M_0 = 1$, suppose further that $\text{Hess}\lambda$ is adapted to the net $(E_0, E_0^1)$. Then $f$ is a Ribaucour partial tube over an immersion $\hat{f} : \hat{M} = \prod_{a=1}^r M_a \to \mathbb{R}^m$ given in one of the following ways:
Proof. First, suppose that \( r = 1 \). In this case, by Proposition [19] the assumption that the metric induced by \( f \) is conformal to a polar metric says that \( E_0 \) is an umbilical distribution. It follows from Lemma 12 of [14] if \( \dim M_0 = 2 \), or the assumption that \( \Hess \lambda \) is adapted to the net \((E_0, E_1)\) combined with Proposition [20] if \( \dim M_0 = 1 \), that \( E_0 \) is indeed spherical. Since the second fundamental form of \( f \) is adapted with respect to the product net of \( M \), the statement in this case is a consequence of Theorem [8].

If \( r \geq 2 \) is arbitrary, apply the case \( r = 1 \) just proved to \( f \) regarded as an immersion of \( M_0 \times M \) into \( \mathbb{R}^m \), where \( \bar{M} = \prod_{a=1}^r M_i \). It follows that \( f \) is a Ribaucour partial tube over \( \tilde{f} : \bar{M} \to \mathbb{R}^m \) given by \( \tilde{f} = f \circ \mu_{\tilde{x}_0} \) for some \( \tilde{x}_0 \in M_0 \), where \( \mu_{\tilde{x}_0} : \bar{M} \to M \) is the inclusion given by \( \mu_{\tilde{x}_0}(\tilde{x}) = (\tilde{x}_0, \tilde{x}) \). The metric \( \tilde{g} \) induced on \( \bar{M} \) by \( \mu_{\tilde{x}_0} \) from the metric \( g \) of \( M \) is
\[
\tilde{g} = \mu_{\bar{x}_0}^* g = (\lambda \circ \mu_{\tilde{x}_0}) \mu_{\bar{x}_0}^* (\pi_0^* g_0 + \sum_{a=1}^r \pi_a^* (g_a \circ \pi_0)) = (\lambda \circ \mu_{\tilde{x}_0}) \sum_{a=1}^r \tilde{\pi}_a^* g_a(\tilde{x}_0),
\]
where \( \tilde{\pi}_a : \bar{M} \to M_a \) is the projection. Hence \( \tilde{g} \) is conformal to a Riemannian product metric.

We now claim that the second fundamental form of \( \tilde{f} = f \circ \mu_{\tilde{x}_0} \) is adapted to the product net on \( M \). We have
\[
\alpha^\tilde{f}(\tau_a^x \times X_a, \tau_b^x \times X_b) = \alpha^f(\tau_{\tilde{x}_0}^{\tilde{x}_0}, X_0, \tau_{\tilde{x}_0}^{\tilde{x}_0}, X_0) + f_* \alpha^\mu_{\tilde{x}_0}(\tau_a^x \times X_a, \tau_b^x \times X_b),
\]
since the second fundamental form of \( f \) is adapted to the product net. Because \( E_0^1 \) is totally umbilic,
\[
\langle \nabla_{\tau_a^x \times X_a} \tau_b^x \times X_b, \tau_0^x \times X_0 \rangle = - \langle \tau_b^x \times X_b, \nabla_{\tau_a^x \times X_a} \tau_0^x \times X_0 \rangle = 0.
\]
Thus \( \alpha_{\bar{x}_0}^\mu(\tau_a^x \times X_a, \tau_b^x \times X_b) = 0 \), and our claim follows.

To complete the proof, it remains to show that \( \tilde{f} : \prod_{a=1}^r M_i \to \mathbb{R}^m \) is given as in the statement. But this follows from Theorem 5 in [14], where conformal immersions of a Riemannian product whose second fundamental forms are adapted to the product net of the manifold have been classified.

\[\square\]

**Corollary 32.** Let \( f : M^m = \prod_{i=0}^r M_i \to \mathbb{R}^m \) be a conformal local diffeomorphism with conformal factor \( \lambda \in C^\infty(M) \) of a product manifold \( M \) endowed with a polar metric. If \( n_0 = \dim M_0 = 1 \), suppose further that \( \Hess \lambda \) is adapted to the net \((E_0, E_0^1)\). Then \( f \) is a Ribaucour partial tube over an immersion \( \tilde{f} : \bar{M} = \prod_{a=1}^r M_a \to \mathbb{R}^m \) given as in Theorem [21] with each of the immersions \( f_a, 1 \leq a \leq r \), having flat normal bundle and \( f_0 : M_0^{n_0} \to \mathbb{R}^{n_0} \) being a local isometry.
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