Malliavin differentiability of fractional Heston-type model and applications to option pricing

Marc Mukendi Mpanda

Department of Decision Sciences
University of South Africa, P. O. Box 392, Pretoria, 0003. South Africa
mpandmm@unisa.ac.za

Abstract
This paper defines fractional Heston-type ($fHt$) model as an arbitrage-free financial market model with the infinitesimal return volatility described by the square of a single stochastic equation with respect to fractional Brownian motion with Hurst parameter $H \in (0, 1)$. We extend the idea of Alos and Ewald (2008) [Alos, E., & Ewald, C. O. (2008). Malliavin differentiability of the Heston volatility and applications to option pricing. Advances in Applied Probability, 40(1), 144-162.] to prove that $fHt$ model is Malliavin differentiable and deduce an expression of expected payoff function having discontinuity of any kind. Some simulations of stock price process and option prices are performed.

Keywords: Fractional Heston-type model, Fractional Brownian motion, Fractional Cox-Ingersoll-Ross process and Malliavin differentiability.

1 Introduction

Allowing volatility to be stochastic in a financial market model was one of the great achievement in the history of quantitative finance. This yields stochastic volatility modelling that was previously discussed by Heston (1993) and several other researchers to overcome shortfalls in the standard Black-Schole model (See e.g. Alòs et al. (2019) for a summary). In the sense of Heston (1993), the stock price process is described by a geometric Brownian motion

$$dS_t = \eta S_t dt + \sqrt{Y_t} S_t dB_t,$$

where $\eta$ and $\sqrt{Y_t}$ represent the drift and stochastic variance of the infinitesimal return $X_t := \log S_t$. The stochastic process $(Y_t)_{t \geq 0}$ takes the form of standard Cox-Ingersoll-Ross process that satisfies the following stochastic differential equation:

$$dY_t = \theta(\mu - Y_t) dt + \nu \sqrt{Y_t} d\tilde{B}(t).$$
The parameter $\theta$ represents the speed of reversion of the stochastic process $(Y_t)_{t\geq 0}$ towards its long-run mean $\mu$ and the parameter $\nu$ represents the volatility of $(Y_t)_{t\geq 0}$. The Brownian motions $(B_t)_{t\geq 0}$ and $(\tilde{B}_t)_{t\geq 0}$ are assumed to be correlated. This model is well known in the literature as the “Heston model”.

The standard Heston model comes with three main drawbacks: (1) the spot volatility is driven by a standard Brownian motion which does not display memory. New findings show roughness in volatility time-series (see e.g. Comte and Renault (1998), Chronopoulou and Viens (2010) for long-range dependency or Gatheral et al. (2018), Livieri et al. (2018) and subsequent results for short range dependency known as “rough volatility”). (2) Perfect calibration may not be possible as the stochastic volatility parameters are constants. It was proven that dependent parameters reduce the calibration error sensibly (See e.g. Benhamou et al. (2010)) and (3) the analytical solution of option price is very complex especially for exotic payoff functions.

This paper addresses these issues by defining the stock price process as a geometric Brownian motion $(S_t)_{t\geq 0}$ that satisfies the following stochastic differential equation:

$$dS_t = \eta S_t dt + \sigma(Y_t)S_t dB_t,$$

where $\sigma(Y_t)$ represents the volatility of the infinitesimal log-return $dX_t := dS_t/S_t$ with $(Y_t)_{t\geq 0}$ a fractional Cox-Ingersoll-Ross (fCIR) process that captures both long and short range dependency. We opt for the definition of Mishura and Yurchenko-Tytarenko (2018) and describe the stochastic process $(Y_t)_{t\geq 0}$ as

$$Y_t(\omega) = Z_t^2(\omega)1_{[0,\tau(\omega)]}, \quad \forall t \geq 0, \quad \omega \in \Omega,$$

where the stochastic process $(Z_t)_{t\geq 0}$ is referred to a general form of fCIR process that satisfies the following differential equation:

$$dZ_t = \frac{1}{2}\left( f(t, Z_t)Z_t^{-1} dt + \nu dW_t^H \right), \quad \nu > 0,$$

and $\tau$ is the first time the process $(Z_t)_{t\geq 0}$ hits zero defined by

$$\tau(\omega) = \inf \{ t > 0 : Z_t(\omega) = 0 \}.$$ 

In (1.3), the function $f(t, z)$ represents the drift of the volatility process $(Y_t)_{t\geq 0}$ and the stochastic process $(W_t^H)_{t\geq 0, H \in (0,1)}$ is well-known as fractional Brownian motion (fBm) of Hurst parameter $H$ defined as a centered Gaussian process with covariance function

$$E[W_t^H W_s^H] = \frac{1}{2}\left( t^{2H} + s^{2H} - |t - s|^{2H} \right), \quad \forall s, t \geq 0.$$

Recall that fBm can be represented in terms of stochastic integral in at least three different ways: time representation, Harmonisable representation and Volterra
representation (See Nourdin (2012) for more details). In what follows, we shall consider the Volterra representation of \( fBm \) given by

\[
W_t^H = \int_0^t \kappa_H(s,t)dV_t,
\]

(1.6)

where \((V_t)_{t \in [0,T]}\) is a standard Brownian motion and where \(\kappa_H(s,t)\) is a square integrable kernel defined by

\[
\kappa_H(t,s) = \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})} \, _2F_1 \left( H - \frac{1}{2}; \frac{1}{2}; H + \frac{1}{2}; 1 - \frac{t}{s} \right) 1_{[0,t]}(s), \quad \forall s \in [0,t],
\]

(1.7)

with \(\Gamma(\cdot)\) and \(_2F_1(a,b,c;d)\) the gamma and Gaussian hypergeometric functions respectively. The standard Brownian motions \((B_t)_{t \in [0,T]}\) and \((V_t)_{t \in [0,T]}\) are assumed to be correlated, that is, there exists \(\rho \in [-1,1]\) such that \(E[B_tV_t] = \rho t\). This means that there exists a Brownian motion \((\tilde{V}_t)_{t \in [0,T]}\) independent to \((V_t)_{t \in [0,T]}\), that is \(E[V_t, \tilde{V}_t] = 0\), such that

\[
B_t = \rho V_t + \sqrt{1-\rho^2} \tilde{V}_t.
\]

(1.8)

Now taking into consideration the risk-free asset process \((A_t)_{t \geq 0}\), the fractional Heston-type \((fHt)\) model is given by the following system

\[
\begin{cases}
  dA_t = rA_t dt, \\
  dX_t = \eta dt + \sigma(Y_t)dB_t, \\
  Y_t = Z_t^2 1_{[0,\tau(\omega)]} \\
  dZ_t = \frac{1}{2} f(t, Z_t) Z_t^{-1} dt + \frac{1}{2} \nu dW_t^H \\
  W_t^H = \int_0^t \kappa_H(s,t)dV_t \\
  B_t = \rho V_t + \sqrt{1-\rho^2} \tilde{V}_t,
\end{cases}
\]

(1.9)

The existence of stochastic process \((Z_t)_{t \geq 0}\) in (1.3) was previously discussed by Nualart and Ouknine (2002). They proposed that for \(H < 1/2\), the drift function \(g(t,z) := f(t,z)z^{-1}\) must satisfy the linear growth condition and for \(H > 1/2\), \(g(t,z)\) must verify the Hölder continuity condition.

Particular cases of \((fHt)\) model (1.9) has been previously investigate by Alòs and Yang (2017), Bezborodov et al. (2019) and Mishura and Yurchenko-Tytarenko (2020) for \(H > 1/2\).

One can use the same idea of Bezborodov et al. (2019, Theorem 4) to show that the \((fHt)\) model is free of arbitrage. In this paper, we also show that both stock
price and fractional volatility processes are Malliavin differentiable through their approximating sequences, and deduce the expected payoff function.

The remainder of this paper is structured as follows: Section 2 constructs an approximating sequences of stock prices and \( f_{CIR} \) processes. The Malliavin differentiability within the \( f_{Ht} \) model is discussed in Section 3. Finally, Section 4 derives the expected payoff function and perform some simulations of option prices.

2 Approximating sequences in \( f_{Ht} \) model.

The main purpose of introducing approximating sequences of both fractional volatility and stock price processes relies on their positiveness. The following theorems discuss the positiveness of \((Z_t)_{t \geq 0}\) and before this, we consider the following assumption.

Assumption 2.1.

(i) The function \( g : [0, \infty) \times (0, \infty) \rightarrow (-\infty, \infty) \) defined by \( g(t, z) := f(t, z)/z \) is continuous and admits a continuous partial derivative with respect to \( x \) on \((0, \infty)\).

(ii) for any \( T > 0 \), there exists \( z_T > 0 \) such that

\[
    f(t, z) > 0 \quad \text{for all} \quad 0 < t \leq T \quad \text{and} \quad 0 \leq z \leq z_T.
\]

Under this assumption, the following theorems were proved by Mishura and Yurchenko-Tytarenko (2018) or Mpanda et al. (2020).

**Theorem 2.1.** Let \((Z_t)_{t \geq 0}\) be a stochastic process that verifies (1.3) with \( H > 1/2 \) and \( f : [0, \infty) \times [0, \infty) \) is a continuous function that satisfies Assumption 2.1. Then

\[
    \mathbb{P}(\tau = \infty) = 1,
\]

where \( \tau(\omega) = \inf\{t > 0 : Z_t(\omega) = 0\} \).

**Theorem 2.2.** Consider for each \( k > 0 \), the stochastic process \((Z_t^{(k)})_{t \geq 0}\) defined by

\[
    Z_t^{(k)} = \begin{cases} 
    Z_0 + \int_0^t f_k(t, Z_s^{(k)}) \, ds + \frac{\nu}{2} W_t^H & \text{if } t < \tau^{(k)}(\omega) \\
    0 & \text{otherwise},
    \end{cases}
\]

where \( \tau^{(k)}(\omega) = \inf\{t \geq 0 : Z_t^{(k)}(\omega) = 0\} \). Then for any \( T > 0 \) and \( H < 1/2 \),

\[
    \mathbb{P}(\omega \in \Omega : \tau^{(k)}(\omega) > T) \to 1 \quad \text{as} \quad k \to \infty.
\]
2.1 Approximating sequences of \(( Z_t^\epsilon )_{t \geq 0} \)

Inspired by Alos and Ewald (2008), we construct an approximating sequence \(( Z_t^\epsilon )_{t \geq 0, \epsilon > 0} \) of the \( fCIR \) process that satisfies the following differential equation:

\[
dZ_t^\epsilon = \frac{1}{2} f(t, Z_t^\epsilon) \Lambda_\epsilon( Z_t^\epsilon ) dt + \frac{\sigma}{2} dW_t^H, \quad Z_0^\epsilon = Z_0 > 0, \quad (2.1)
\]

where the function \( \Lambda_\epsilon(z) \) in (2.1) is defined by

\[
\Lambda_\epsilon(z) = (z1_{\{z>0\}} + \epsilon)^{-1}. \quad (2.2)
\]

It is easy to verify that \( \Lambda_\epsilon(z) > 0 \) for all \( \epsilon > 0 \). As a straight consequence, the drift of \(( Z_t^\epsilon )_{t \geq 0, \epsilon > 0} \) is also positive. In addition, \( \lim_{z \to 0} \Lambda_\epsilon(z) = \epsilon^{-1} \), \( \lim_{z \to \infty} \Lambda_\epsilon(z) = 0 \) and

\[
\Lambda'_\epsilon(z) = \begin{cases} 
0, & \text{if } z < 0 \\
-\frac{1}{(z+\epsilon)^2}, & \text{if } z \geq 0 
\end{cases} \quad (2.3)
\]

The next step is to show that for every \( t \geq 0 \), the sequence \( Z_t^\epsilon \) converges to \( Z_t \) in \( L^p \) as \( \epsilon \to 0 \).

**Proposition 2.3.** The sequence of estimated random variables \( Z_t^\epsilon \) converges to \( Z_t \) in \( L^p(\Omega) \) for all \( p \geq 1 \).

**Proof.**

**Case 1.** \( H = 1/2 \). This case was discussed previously by Alos and Ewald (2008, Proposition 2.1) and can be easily extended to the case where \( \Lambda_\epsilon(z) \) is defined by (2.2).

**Case 2.** For \( H > 1/2 \), the dominated convergence theorem shall be applied. Firstly, we need to show the pointwise convergence of the approximated stochastic process \(( Z_t^\epsilon )_{t \geq 0} \) towards \(( Z_t )_{t \geq 0} \), that is \( \lim_{\epsilon \to 0} Z_t^\epsilon = Z_t \). For this, let \( \tau_\epsilon(\omega) = \inf\{ t \geq 0 : Z_t(\omega) \leq \epsilon \} \) be the first time the process \(( Z_t^\epsilon )_{t \geq 0} \) hits \( \epsilon \). Since the sample paths of the stochastic process \(( Z_t )_{t \geq 0} \) are positive everywhere almost surely as in Theorem 2.1, then \( P(\omega \in \Omega : \tau_0 = \infty) = 1 \) as and consequently, \( \lim_{\epsilon \to 0} \tau_\epsilon = \infty \) almost surely.

Next, denote \( Z_t^{\tau_\tau} \) the stochastic process \(( Z_t )_{t \geq 0} \) up to stopping time \( \tau_\epsilon \). Then, for all \( t \in [0, \tau_\epsilon] \) and using the definition of \( \Lambda_\epsilon(z) \) given by (2.3), \( Z_t^{\tau_\epsilon} = Z_t^\epsilon \) almost surely when \( \epsilon \to 0 \) since the drift function \( f(t, z) \) is monotonic.

Again, the positiveness of \(( Z_t )_{t \geq 0} \) means that \( \lim_{\epsilon \to 0} Z_t^{\tau_\epsilon} = Z_t \) a.s. We may conclude that \( \lim_{\epsilon \to 0} Z_t^{\tau_\epsilon} = \lim_{\epsilon \to 0} Z_t^\epsilon = Z_t \) almost surely and for all \( t \geq 0 \).
On the other hand, the result from Hu et al. (2008, Theorem 3.1) shows that for a fixed \( T > 0 \) and for all \( p \geq 1 \),

\[
E \left[ \sup_{t \in [0,T]} |Z_t|^p \right] = C < \infty,
\]

where \( C = C(p, H, \gamma, \beta, T, Z_0) \) is a non-random constant taking the form

\[
C = C_1(1 + Z_0) \exp \left[ C_2 \left( 1 + \|W^H\|^{2(\gamma-1)} \right) \right],
\]

where \( \beta \in (\frac{1}{2}, H) \), \( \gamma > \frac{2\beta}{2\beta - 1} \), \( C_1 = C_1(\gamma, \beta, T) \) and \( C_2 = C_2(\gamma, \beta, T) \) are nonrandom constants depending on parameters \( \gamma, \beta, T \), and

\[
\|W^H\| = \sup_{s \geq 0, t \leq T} \left\{ \frac{|W_s^H - W_t^H|}{|s - t|^{\beta}} \right\}.
\]

This result also implies that

\[
E \left[ \sup_{t \in [0,T]} |Z_t|^p \right] = C(p, H, \gamma, \beta, T, Z_0) < \infty.
\]

It follows that \( \sup_{t \in [0,T]} \{ |Z_t(\omega)| \} \in L^p(\Omega) \) which yields the desired \( L^p \) convergence.

**Case 3.** For \( H < 1/2 \), we consider a sequence of an increasing drift functions \( f_k(t, z), k \in \mathbb{N} \) and define the stochastic process \((Z_t^{(e,k)})_{t \geq 0}\) as follows:

\[
Z_t^{(e,k)} = \begin{cases} 
Z_0 + \frac{1}{2} \int_0^t f_k \left( s, Z_s^{(e,k)} \right) \Lambda \left( Z_s^{(e,k)} \right) \, ds + \frac{\nu}{2} W^H_t & \text{if } t < \tau^{(k)}(\omega) \\
0 & \text{otherwise},
\end{cases}
\]

where \( \Lambda(z) \) is defined by (2.2) and \( \tau^{(k)}(\omega) = \inf \{ t \geq 0 : Z_t^{(e,k)}(\omega) = 0 \} \) is the first time that the stochastic process \((Z_t^{(e,k)})_{t \geq 0}\) hits zero. If we now define \( \tau^{(e,k)}(\omega) = \inf \{ t \geq 0 : Z_t^{(e,k)}(\omega) \leq \epsilon \} \) be the first time the process \((Z_t^{(e,k)})_{t \geq 0}\) hits \( \epsilon \), then from Theorem 2.2, for any fixed \( T > 0 \), \( P(\omega \in \Omega : \tau^{(e,k)} > T) \to 1 \) as \( k \to \infty \). This implies that \( \lim_{(e,k) \to (0,\infty)} \tau^{(e,k)} = \tilde{T} > T \) a.s. This is because the process \((Z_t^{(e,k)})_{t \geq 0}\) remains positive up to time \( \tilde{T} \) which is not necessary equal to infinity unlike the previous case.

After using similar arguments of **Case 2**, one may conclude that \( \lim_{\epsilon \to 0} Z_t^{(e)} = \lim_{\epsilon \to 0} Z_t = Z_t \) for all \( t \in [0, \tilde{T}] \). Next, we need to show that \( E \left[ \sup_{t \in [0,T]} |Z_t|^p \right] < \infty \). To achieve this, we borrow some ideas from Mishura and Yurchenko-Tytarenko (2019).

Firstly, let \( \tilde{Z}_0 \) be a small positive value less than the initial value \( Z_0 \) such that \( 0 < \tilde{Z}_0 < Z_0 \) and let \( \tau_1 = \tau_1(\epsilon, \omega) \) be the last time the stochastic process
(\text{2.7}) hits (or before hits) \hat{Z}_0, that is,

\[ \tau_1(\epsilon, \omega) = \sup \{ t \geq 0 : Z_t^{\epsilon}(\omega) \geq \hat{Z}_0, \forall t \in [0, T] \}. \] 

(2.4)

Technically, there exists a constant \( M \geq 2 \) such that \( \hat{Z}_0 = \frac{Z_0}{M} \). Now we can consider two cases: \( t \in [0, \tau_1] \) and \( t \in (\tau_1, T] \).

**Case 3.1:** \( t \in [0, \tau_1] \). By triangle inequality, we have

\[ |Z^t_t|^p = \left| Z_0 + \frac{1}{2} \int_0^t f(s, Z_s^{\epsilon}) \Lambda_\epsilon(Z_s^{\epsilon}) ds + \frac{\nu}{2} W_t^H \right|^p \]

\[ \leq \left( Z_0 + \frac{1}{2} \int_0^t f(s, Z_s^{\epsilon}) \Lambda_\epsilon(Z_s^{\epsilon}) ds + \frac{\nu}{2} |W_t^H| \right)^p \]

\[ \leq \left( Z_0 + \frac{1}{2} \int_0^t |f(s, Z_s^{\epsilon}) \Lambda_\epsilon(Z_s^{\epsilon})| ds + \frac{\nu}{2} |W_t^H| \right)^p. \] 

(2.5)

By applying the Callebaut's inequality theorem, it will be easy to show that for all \( p \geq 1 \),

\[ \left( Z_0 + \frac{1}{2} \int_0^t |f(s, Z_s^{\epsilon}) \Lambda_\epsilon(Z_s^{\epsilon})| ds + \frac{\nu}{2} |W_t^H| \right)^p \]

\[ \leq 3p \left( Z_0^p + \frac{1}{2} \int_0^t |f(s, Z_s^{\epsilon}) \Lambda_\epsilon(Z_s^{\epsilon})| ds \right)^p \left( \frac{\nu}{2} |W_t^H| \right)^p. \] 

(2.6)

From (2.4), we may deduce that \( Z^t_t \geq \tilde{Z}_0 > 0 \), with \( t \) on \( [0, \tau_1] \). This yields \( \Lambda_\epsilon(Z^t_t) < MZ_0^{-1}, M \geq 2 \) and

\[ \int_0^\epsilon |f(s, Z_s^{\epsilon}) \Lambda_\epsilon(Z_s^{\epsilon})| ds \leq \frac{M}{Z_0} \int_0^\epsilon |f(s, Z_s^{\epsilon})| ds. \] 

(2.7)

Since the drift function satisfies the linear growth condition, this means there exists a positive constant \( k \) such that \( f(t, z) \leq k(1 + |z|) \). It follows that

\[ \int_0^\epsilon |f(s, Z_s^{\epsilon})| ds \leq \int_0^\epsilon |k(1 + |Z_s^{\epsilon}|)| ds \leq k \left( T + \int_0^\epsilon |Z_s^{\epsilon}| ds \right). \] 

(2.8)

Inequalities (2.6), (2.7) and (2.8) yield the following:

\[ |Z^t_t|^p \leq 3p \left( Z_0^p + \left( \frac{kM}{2Z_0} \right)^p \left( T + \int_0^\epsilon |Z_s^{\epsilon}| ds \right)^p \right) \left( \frac{\nu}{2} \right)^p |W_t^H|^p. \]
On the other hand, recall that $|W_{t}^{H}| < \sup_{s \in [0, T]} |W_{s}^{H}| < \infty$ (See e.g. Nourdin (2012)) and since

$$
\left( T + \int_{0}^{t} |Z_{s}'| ds \right)^{p} \leq 2^{p} \left( T^{p} + \int_{0}^{t} |Z_{s}'|^{p} ds \right),
$$

then it follows that

$$
|Z_{t}'|^{p} \leq (3Z_{0})^{p} + \left( \frac{3kMT}{Z_{0}} \right)^{p} + (3\nu)^{p} \sup_{s \in [0, T]} |W_{s}^{H}|^{p} + \left( \frac{3kM}{Z_{0}} \right) \int_{0}^{t} |Z_{s}'| ds \right)^{p}.
$$

From the Grönwall-Bellman inequality theorem, we obtain

$$
|Z_{t}'|^{p} \leq \left( (3Z_{0})^{p} + \left( \frac{3kMT}{Z_{0}} \right)^{p} + (4\nu)^{p} \sup_{s \in [0, T]} |W_{s}^{H}|^{p} \right) \exp \left( \left( \frac{3kM}{Z_{0}} \right)^{p} t \right)
$$

which can be shortly written as $|Z_{t}'|^{p} \leq C$, where $C = C(r, k, T, Z_{0}, \nu, H)$ is a non-random constant in parameters $r, k, T, Z_{0}, \nu$ and $H$ taking the following form

$$
C \leq C_{1} + C_{2} \sup_{s \in [0, T]} |W_{s}^{H}|^{p},
$$

with $C_{1} = C_{1}(p, k, T, Z_{0})$ and $C_{2} = C_{2}(p, k, T, Z_{0}, \nu)$ are non-random constants defined respectively by

$$
C_{1} = (3Z_{0})^{p} \left( 1 + \left( \frac{kMT}{Z_{0}^{2}} \right)^{p} \right) \exp \left( \left( \frac{3kM}{Z_{0}} \right)^{p} T \right),
$$

and

$$
C_{2} = (4\nu)^{p} \exp \left( \left( \frac{3kM}{Z_{0}} \right)^{p} T \right).
$$

**Case 3.2:** $t \in (\tau_{1}, T]$, with $T > \tau_{1} > 0$. Define

$$
\tau_{2} = \tau_{2}(\epsilon, \omega) = \sup\{ s \in (\tau_{1}, t) : |Z_{s}'(\omega)| < \tilde{Z}_{0} \}.
$$
Then we have:

\[
|Z_t^c|^p \leq |Z_t^c - Z_{\tau_2}^c|^p + |Z_{\tau_2}^c|^p
\]

\[
\leq Z_0^p + |Z_t^c - Z_{\tau_2}^c|^p
\]

\[
\leq Z_0^p + \left(\frac{1}{2}\right)^p \left| \int_{\tau_2}^t f(s, Z_s^c)\Lambda_c(Z_t^c)\,ds + \nu(W_t^H - W_{\tau_2}^H) \right|^p
\]

\[
\leq Z_0^p + \left( \int_{\tau_2}^t |f(s, Z_s^c)\Lambda_c(Z_t^c)|\,ds \right)^p + (2\nu)^p \left| W_t^H \right|^p + |W_{\tau_2}^H|^p.
\]

As previously, the integral in the last inequality of (2.11) can be expressed as follows

\[
\int_{\tau_2}^t |f(s, Z_s^c)\Lambda_c(Z_t^c)|\,ds \leq \frac{k}{Z_0} \left( T + \int_{\tau_2}^t |Z_t^c|\,ds \right), \quad \forall t \in [0, T].
\]

On the other hand, we may observe that

\[
\left| W_t^H \right|^p + |W_{\tau_2}^H|^p \leq 2 \sup_{s \in [0, T]} \left| W_s^H \right|^p.
\]

It follows that,

\[
|Z_t^c|^p \leq Z_0^p + \left( \frac{2kT}{Z_0} \right)^p + \left( \frac{2k}{Z_0} \int_0^t |Z_s^c|^r \,ds \right)^p + 2(2\nu)^p \sup_{s \in [0, T]} \left| W_s^H \right|^p
\]

\[
\leq (3Z_0)^p + \left( \frac{3kMT}{Z_0} \right)^p + (4\nu)^p \sup_{s \in [0, T]} \left| W_s^H \right|^p + \left( \frac{3kM}{Z_0} \int_0^t |Z_s|\,ds \right)^p.
\]

From this expression, we may also conclude that \( |Z_t^c|^p \leq C \), where \( C = C(C_1, C_2) \) where \( C_1 \) and \( C_2 \) are non-random constants defined by (2.9) and (2.10) respectively. This shows that \( \mathbb{E}[|Z_t^c|^p] < \infty \) and consequently, \( \mathbb{E}[\sup_{t \in [0, T]} |Z_t|^p] < \infty \). This concludes the proof of the proposition. \( \square \)

**Corollary 2.4.** Fix \( p \geq 1 \). If \( \sigma(y) \) satisfies the linear growth condition, then

\[
\lim_{\epsilon \to 0} \mathbb{E} \left[ \sup_{t \geq 0} |\sigma(Y_t^\epsilon) - \sigma(Y_t)|^p \right] = 0 \quad a.s.
\]

**Proof.** This follows immediately from the previous proposition.

**Remark.** One may use similar arguments of Mishura and Yurchenko-Tytarenko (2019) to show that the stochastic process \( (Z_t^c)_{t \geq 0, c > 0} \) is strictly positive almost surely for all \( H \in (0, 1) \). Consequently, it is also well suitable for rough volatility processes, that is, fractional volatility process with \( H < 1/2 \).
2.2 Approximating sequences of stock price process

With \((Z_t^i)_{t \geq 0, \epsilon > 0}\), let us construct the approximating sequence \((S_t^i)_{t \geq 0, \epsilon > 0}\) of the stock price process \((S_t)_{t \geq 0}\) defined by the following geometric Brownian motion:

\[
dS_t^i = \eta S_t^i dt + \sigma(Y_t^i) S_t^i dB_t,
\]

where

\[
Y_t^i = (Z_t^i)^2,
\]

with \((Z_t^i)_{t \geq 0, \epsilon > 0}\) the approximating sequence that satisfies (2.1). The solution to (2.12) is unique and can be found by using the standard Itô formula and it is given by. Next step is to show that \(S_t^i\) converges to \(S_t\) in \(L^p\), \(p \geq 1\).

**Proposition 2.5.** Set \(X_t := \log S_t\) and \(X_t^i := \log S_t^i\). Then the sequence \(X_t^i\) converges to \(X_t\) in \(L^p(\Omega)\) for all \(p \geq 1\).

**Proof.** Firstly, we have from Itô formula that

\[
X_t^i = X_0 + \eta t - \frac{1}{2} \int_0^t \sigma^2(Y_s^i) ds + \int_0^t \sigma(Y_s^i) dB_s,
\]

where \(X_0 := \log S_0\). Then for some non-random constant \(C > 0\), one may have:

\[
\mathbb{E} \left[ \sup_{t \geq 0} |X_t^i - X_t|^p \right] \leq \frac{C}{2^p} \mathbb{E} \left[ \sup_{t \geq 0} \left| \int_0^t (\sigma^2(Y_s^i) - \sigma^2(Y_s)) ds \right|^p \right]
+ C \mathbb{E} \left[ \sup_{t \geq 0} \left| \int_0^t (\sigma(Y_s^i) - \sigma(Y_s)) dB_s \right|^p \right]
\]

Set

\[
T_1 := \mathbb{E} \left[ \sup_{t \geq 0} \left| \int_0^t (\sigma^2(Y_s^i) - \sigma^2(Y_s)) ds \right|^p \right]
\]

and

\[
T_2 := \mathbb{E} \left[ \sup_{t \geq 0} \left| \int_0^t (\sigma(Y_s^i) - \sigma(Y_s)) dB_s \right|^p \right].
\]

Then it follows firstly that \(T_1 \to 0\) from Corollary 2.4. To analyse convergence of \(T_2\), the Burkholder-Davis-Gundy inequality can be used and one may deduce that

\[
T_2 \leq c(p) \mathbb{E} \left[ \sup_{t \geq 0} \left| \int_0^t (\sigma(Y_s^i) - \sigma(Y_s)) ds \right|^{\frac{p}{2}} \right],
\]

which also converges to zero from Corollary 2.4. It follows that

\[
\lim_{\epsilon \to 0} \sup_{t \geq 0} |X_t^i - X_t|^p = 0, \, \forall p > 0
\]

that implies the desired \(L^p\) convergence of \(X_t^i\) to \(X_t\) and \(S_t^i\) to \(S_t\).

**Remarks.**
Malliavin differentiability of $fHt$ model and applications to option pricing

(1) The approximated stochastic volatility and stock price processes will be compulsory for $H \leq 1/2$ and optional for $H > 1/2$. However, for the sake of consistency, we shall use the approximated sequences (2.1) with $\epsilon = 0$ for $H > 1/2$ and with $\epsilon > 0$ for $H \leq 1/2$.

(2) For the simulations of stock price process, one may use the Euler-Maruyama approximation scheme. This can be done by considering the time interval $[0, T]$ that is subdivided into $N$ sub-intervals of equal length such that $0 = t_0, t_1, \ldots, t_N = T$ with $t_i = iT/N$ and the lag $\Delta t = T/N$. The estimated stock price at time $t_i$ denoted by $\hat{S}_{t_i}$ and the volatility $(\hat{Y}_{t_i})_{i=1,\ldots,N}$ are respectively given by

$$
\begin{align*}
\hat{S}_{t_{i+1}} &= \hat{S}_{t_i} \left( 1 + \eta \Delta t + \sigma(\hat{Y}_{t_i}) \left( \rho \Delta V_{t_i} + \sqrt{1 - \rho^2} \Delta \tilde{V}_{t_i} \right) \right) \\
\hat{Y}_{t_i} &= \hat{Z}_{t_i}^2 1_{[0,\tau(\omega)]} \\
\tilde{Z}_{t_{i+1}} &= \tilde{Z}_{t_i} + \frac{1}{2} \int_{t_i}^{t_{i+1}} f(s, \hat{Z}_s) \Lambda(\hat{Z}_s) ds + \frac{1}{2} \nu \Delta W_{t_{i+1}}^H.
\end{align*}
$$

(2.14)

where $\Delta V_{t_i} = V_{t_{i+1}} - V_{t_i}$, $\Delta \tilde{V}_{t_i} = \tilde{V}_{t_{i+1}} - \tilde{V}_{t_i}$, and $\Delta W_{t_{i+1}}^H = W_{t_{i+1}}^H - W_{t_i}^H$ are respectively the increment of Brownian motions $V_{t\in[0,T]}$, $\tilde{V}_{t\in[0,T]}$ and fBm $W_{t\in[0,T]}$.

As an illustrative example, the following figures represent 10 sample paths of the stock price process on the interval $[0, T]$ with $N = 1000$, $\rho = 0.6$, $X_0 = 100$, $\eta = r = 0.05$, $\nu = 0.1$, $\sigma(\hat{Y}_{t_i}) = 0.8\hat{Y}_{t_i} + 0.1$. The drift of the fractional volatility process is defined by

$$
f(t, y) = \frac{\sigma^2}{2} \left( 1 - e^{-2\kappa t} \right) + \kappa (c - y^2), \quad t \geq 0, y \geq 0,
$$

(2.15)

with $\kappa = 1$, $c = 2$. For $H > 1/2$, we use $\epsilon = 0$ (See e.g. Figures 2.3 and 2.4) and for $H \leq 1/2$, we set $\epsilon = 0.01$ as shown in Figures 2.1 and 2.2.

![Figure 2.1: $H = 0.15$, $\epsilon = 0.01$](image1.png)

![Figure 2.2: $H = 0.5$, $\epsilon = 0.01$](image2.png)
3 Malliavin differentiability

In what follows, we show that the stochastic processes \((Z_t)_{t \geq 0}\) and \((S_t)_{t \geq 0}\) are Malliavin differentiable with respect to the Brownian motions \((V_t)_{t \geq 0}\), \((\tilde{V}_t)_{t \geq 0}\) and \(fBm\) \((W^H_t)_{t \geq 0}\). We refer the reader to Nualart (2006) for a background in Malliavin calculus.

3.1 Differentiability of the stochastic process \((Z_t)_{t \geq 0}\)

**Proposition 3.1.** Let \((Z^\epsilon_t)_{t \geq 0, \epsilon > 0}\) be a stochastic process that verifies the stochastic differential equation \((2.1)\) driven by a \(fBm\) \((W^H_t)_{t \in [0,T]}\) that takes the Volterra representation form given by

\[
W^H_t = \int_0^t \kappa_H(s,t)dB_s,
\]

where \((B_t)_{t \geq 0}\) is a standard Brownian motion and \(\kappa_H(s,t)\) is a square integrable kernel given by \((1.7)\). Assume that the drift function \(f(t,z)\) is differentiable and define

\[
F_z(t,z) = \frac{\partial f(t,z)}{\partial z} \Lambda(z) + f(t,z)\Lambda'(z),
\]

where \(\Lambda'(z)\) is defined by \((2.3)\). Moreover, let \(\mathcal{D}_u^B\) and \(\mathcal{D}_u^W\) be the Malliavin derivatives at the time \(u \in [0,T]\) with respect to \((B_t)_{t \geq 0}\) and \((W^H_t)_{t \geq 0}\) respectively. Then it follows that \(Z^\epsilon_t \in \mathbb{D}^{1,p}\),

\[
\mathcal{D}_u^B Z^\epsilon_t = \frac{\nu}{2} \left( \kappa_H(t,u) + \int_u^t \kappa_H(s,u)F_z(s,Z^\epsilon_s) \exp \left( \int_s^t F_z(u,Z^\epsilon_u)du \right)ds \right)1_{[0,t]}(u)
\]

and

\[
\mathcal{D}_u^W Z^\epsilon_t = \frac{\nu}{2} \left( \exp \left( \int_s^t F_z(u,Z^\epsilon_u)du \right) \right)1_{[0,t]}(u).
\]
Malliavin differentiability of $fHt$ model and applications to option pricing

**Proof.** The Malliavin derivative $\mathcal{D}_u Z_t^\epsilon$ can be found as follows:

$$
\mathcal{D}_u Z_t^\epsilon = \frac{1}{2} \int_0^t \mathcal{D}_u (f(s, Z_s^\epsilon) \Lambda\epsilon(Z_s^\epsilon)) \, ds + \frac{\nu}{2} \mathcal{D}_u W_t^H
$$

$$
= \frac{1}{2} \int_0^t F_s(s, Z_s^\epsilon) \mathcal{D}_u Z_s^\epsilon \, ds + \frac{\nu}{2} \kappa_H(t, u) 1_{[0,t]}(u)
$$

The function $F_s(s, z)$ exists indeed since $\Lambda\epsilon'(z)$ is well-defined for all Hurst parameter $H \in (0, 1)$. Next, by letting $D_t = \mathcal{D}_u Z_t^\epsilon$, we obtain a Volterra integral equation given by

$$
D_t = \frac{1}{2} \int_0^t F_s(s, Z_s^\epsilon) D_s ds + \frac{\nu}{2} \kappa_H(t, u) 1_{[0,t]}(u),
$$

to which a solution is given by

$$
D_t = \frac{\nu}{2} \left( \kappa_H(t, u) + \int_u^t \kappa_H(s, u) F_s(s, Z_s^\epsilon) \exp \left( \int_s^t F_u du \right) ds \right) 1_{[0,t]}(u).
$$

Since $D_t \in L^p(\Omega)$, then it follows that the stochastic process $Z_t^\epsilon \in \mathcal{D}_1^{L^p}$ from Nualart (2006). The proof of (3.2) can be deduced in a similar way or by following the idea of Hu et al. (2008, Theorem 3.3). □
Malliavin differentiability of \( fHt \) model and applications to option pricing

Remarks

(1) This proposition holds for all \( H \in (0, 1) \). However, for \( H > 1/2 \) one may use directly \((Z_t)_{t \geq 0}\) given by (1.3) without going through its approximating sequence \((Z^\epsilon_t)_{t \geq 0, \epsilon > 0}\) since the sample paths of \((Z_t)_{t \geq 0}\) are strictly positive everywhere almost surely as in Theorem 2.1.

(2) As a straight consequence of Proposition 2.3, we have

\[
\lim_{\epsilon \to 0} F_\epsilon(t, z) = F(t, z)
\]

where

\[
F(t, z) = \left( \frac{\partial f(t, z)}{\partial z} - f(t, z) \right) z^{-2}.
\]

It follows from Proposition 3.1 that \( Z_t \in D^{1,p} \), and

\[
D^V_u Z_t = \frac{\nu}{2} \left( \kappa_H(t, u) + \int_u^t \kappa_H(s, u) F(s, Z_s) \exp \left( \int_s^t F(u, Z_u) du \right) ds \right) 1_{[0,t]}(u),
\]

and

\[
D^W_u Z_t = \frac{\nu}{2} \left( \exp \left( \int_s^t F(u, Z_u) du \right) \right) 1_{[0,t]}(u).
\]

3.2 Differentiability of the stock price process \((S_t)_{t \geq 0}\)

The following proposition shows that the stock price process and its estimation are Malliavin differentiable.

Proposition 3.2. Assume that the volatility \( \sigma(y) \) is Lipschitz and differentiable. Then \( S^\epsilon_t, X^\epsilon_t \in D^{1,p} \) and for all \( u \leq t \), we have

\[
D^B_u S^\epsilon_t = S^\epsilon_t D^B_u X^\epsilon_t, \quad D^V_u S^\epsilon_t = S^\epsilon_t D^V_u X^\epsilon_t \quad \text{and} \quad D^\tilde{V}_u S^\epsilon_t = S^\epsilon_t \sqrt{1 - \rho^2} \sigma(Y^\epsilon_t) 1_{[0,t]}(u),
\]

where

\[
D^B_u X^\epsilon_t = \left( \int_u^t \sigma'(Y^\epsilon_s) D^B_u Y^\epsilon_s dB_s - \int_u^t \sigma(Y^\epsilon_s) \sigma'(Y^\epsilon_s) D^B_u Y^\epsilon_s ds \right) 1_{[0,t]}(u)
\]

and

\[
D^V_u X^\epsilon_t = \left( \rho \int_u^t \sigma'(Y^\epsilon_s) D^V_u Y^\epsilon_s dB_s + \sqrt{1 - \rho^2} \int_u^t \sigma'(Y^\epsilon_s) D^V_u \tilde{V}^\epsilon_s ds \right) 1_{[0,t]}(u)
\]

\[\quad - \int_u^t \sigma(Y^\epsilon_s) \sigma'(Y^\epsilon_s) D^V_u \sigma(Y^\epsilon_s) ds \right) 1_{[0,t]}(u)
\]

14
In addition,

\[ \sup_{u, t \geq 0} \left| D_u X^e_t - D_u X^e_t \right| \to 0, \]

where \( D_u \) represents a Malliavin derivative with respect to \( B_t, V_t \) or \( \tilde{V} \).

**Proof.** The equations (3.6) follows immediately from chain rule formula for Malliavin derivatives. Expressions of derivatives \( D^B u X^e_t \) and \( D^V u X^e_t \) are straight consequences of Nualart (2006, Theorem 1.2.4).

**Corollary 3.3.** The laws of both stock price process \( (S_t)_{t \geq 0} \) and its log-return \( (X_t)_{t \geq 0} \) are absolutely continuous.

**Proof.** One may verify that \( \left| D^B u X_t \right|_{L^2(\Omega)} > 0 \) and \( \left| D^B u S_t \right|_{L^2(\Omega)} > 0 \) almost surely, then the absolutely continuity with respect to the Lebesgue measure on \( \mathbb{R} \) follows immediately from Nualart (2006, Theorem 2.1.3).

**Remark.** The Malliavin differentiability property of both stochastic volatility and stock price processes will be crucial for the derivation of the expected payoff function that will be discussed in the next chapter.

## 4 Application to option pricing

The aim of this section is to derive the expected payoff function \( \mathbb{E}[h(S_T)] \) by using some results from Malliavin calculus and deduce its option price. We follow Altmayer and Neuenkirch (2015) closely.

### 4.1 The Expected Payoff function

Let \( h : \mathbb{R} \to \mathbb{R} \) be the payoff function that satisfies the following assumption.

**Assumption 4.1.** The payoff function \( h : \mathbb{R} \to \mathbb{R} \) and its antiderivative denoted by \( L(x) \) (such that \( L(x) = h(x) \)) are bounded and verify the Lipschitz condition.

**Proposition 4.1.** \( L(S_T) \in D^{1,2} \).

**Proof.** Firstly, it is straightforward to check that \( \mathbb{E}[L^2(S_T)] < \infty \) since \( L(x) \) also verifies the linear growth condition and the law of stock price process \( (S_t)_{t \in [0,T]} \) are bounded almost surely. On the other hand, since \( L \) verifies Assumption 4.1 and the sample paths of the stock price process \( (S_t)_{t \in [0,T]} \) is absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R} \) (See Corollary 3.3), then from the chain rule formula for Malliavin derivatives, we may deduce

\[ D^V L(S_T) = L'(S_T)D^V S_T = h(S_T)D^V S_T. \]

It follows that
Malliavin differentiability of $fHt$ model and applications to option pricing

\[
E \left[ \int_0^T (D^V_s L(S_T))^2 ds \right] = E \left[ \int_0^T \left( h(S_T) D^V_s S_T \right)^2 ds \right] \\
= E \left[ h^2(S_T) \int_0^T (D^V_s S_T)^2 ds \right] \\
\leq \left( E \left[ h^4(S_T) \right] \int_0^T E \left[ (D^V_s S_T)^4 \right] ds \right)^{\frac{1}{2}} < \infty.
\]

The first inequality is due to Holder inequality and the finiteness of the last expression makes sense since $S_t \in \mathbb{D}^{1,2}$ as discussed previously. It follows that $||L||_{1,2} < \infty$ which concludes the proof. \[\square\]

As now $L(S_T)$ is Malliavin differentiable, then the following lemma that discusses the expected payoff follows.

**Lemma 4.2.** Let $h(x)$, $x \in \mathbb{R}$ be a payoff function that satisfies Assumption 4.1 and denote $h(e^x) := g(x)$ with its antiderivative $G(x)$ that also satisfies the Lipschitz condition. Set

\[
I_T := \frac{1}{T \sqrt{1 - \rho^2}} \int_0^T \frac{1}{\sigma(Y_u)} d\tilde{V}.
\]

Then

\[
E [g(X_T)] = E \left[ G(X_T) I_T \right],
\]

and

\[
E [h(S_T)] = E \left[ \frac{L(S_T)}{S_T} \left( 1 + I_T \right) \right].
\]

where $X_T := \log S_T$ and

\[
L(S_T) = \int_0^{S_T} h(x) dx.
\]

**Proof.** We follow the idea of Altmayer and Neuenkirch (2015). To establish the equality (4.2), we rewrite $E[g(X_T)]$ as

\[
E[g(X_T)] = E \left[ \frac{1}{T} \int_0^T g(X_T) du \right] = E \left[ \frac{1}{T} \int_0^T g(X_T) D^V \frac{1}{D^V X_T} du \right].
\]

From Proposition 4.1, we may deduce that $G(X_T) \in \mathbb{D}^{1,2}$ and

\[
D^V G(X_T) = g(X_T) D^V X_T.
\]

We now obtain

\[
E[g(X_T)] = E \left[ \frac{1}{T} \int_0^T D^V G(X_T) \frac{1}{D^V X_T} du \right].
\]
In addition, from Proposition 3.2,

\[ D_{u}^{V}X_T = \sqrt{1 - \rho^2} \sigma(Y_u) 1_{[0,T]}(u) \]

and since the integral \( \int_{0}^{T} \frac{1}{\sigma(Y_u)} du \) is well defined from Assumption ??, then we have:

\[ E[g(X_T)] = E \left[ \frac{G(X_T)}{T \sqrt{1 - \rho^2}} \int_{0}^{T} \frac{1}{\sigma(Y_u)} d\tilde{V}_u \right], \]

and defining \( I_T \) by (4.1), we obtain (4.2). To establish (4.3), we rewrite the function \( G(x) \) (which is the antiderivative of \( g(x) \)) as follows

\[ G(x) = \int_{0}^{x} g(u) du + C, \]

where \( C \) is a constant taking the form \( C = \int_{0}^{1} h(u) du \) and by using the standard integration by part formula, one may obtain

\[ G(x) = \frac{L(e^x)}{e^x} + \int_{0}^{x} \frac{L(e^u)}{e^u} du. \]

With this setting, we have

\[ E[h(S_T)] = E[g(X_T)] \]

\[ = E \left[ G(X_T) I_T \right] \]

\[ = E \left[ \left( \frac{L(S_T)}{S_T} \right) I_T + \left( \int_{0}^{X_T} \frac{L(e^u)}{e^u} du \right) I_T \right] \]

\[ = E \left[ \frac{L(S_T)}{S_T} I_T \right] + E \left[ \left( \int_{0}^{X_T} \frac{L(e^u)}{e^u} du \right) I_T \right] \]

\[ = E \left[ \frac{L(S_T)}{S_T} I_T \right] + E \left[ \frac{L(S_T)}{S_T} \right]. \]

We may use again the Euler-Maruyama approximation scheme to compute the expected payoff numerically. We may use the following approximations:
Malliavin differentiability of $fHt$ model and applications to option pricing

\[
\begin{align*}
\hat{S}_{t+1} &= \hat{S}_t \left(1 + \eta \Delta t + \sigma(\hat{Y}_t) \left(\rho \Delta V_t + \sqrt{1 - \rho^2} \Delta \hat{V}_t\right)\right) \\
\hat{Y}_t &= \hat{Z}_t^2 1_{[0,\tau(\omega)]} \\
\hat{Z}_{t+1} &= \hat{Z}_t + \frac{1}{2} \int_{t_i}^{t_{i+1}} f(s, \hat{Z}_s) \Lambda(\hat{Z}_s) ds + \frac{1}{2} \nu \Delta W_{t+1}^H \\
\hat{I}_T &= \frac{1}{T \sqrt{1 - \rho^2}} \sum_{i=0}^{N} \frac{1}{\sigma(\hat{Y}_i)} \Delta \hat{V}_i 
\end{align*}
\]  

(4.5)

with $0 = t_0, t_1, \cdots, t_N = T$ with $t_i = iT/N$ and the lag $\Delta t = T/N$.

4.2 Some simulations

Pricing options with volatility taking the form of Ornstein-Uhlenbeck and standard $fCIR$ process

Firstly, we consider the stochastic process $(Z_t)_{t \geq 0}$ defined as a Ornstein-Uhlenbeck process, that is with $f(t, z) = -\theta z^2$, where $\theta$ is a positive parameter, $\nu = 2$ and $H > 1/2$. Under these settings, one may recover the model discussed by Bezborodov et al. (2019) with $Y_t = Z_t^2$ instead. In this case, the volatility process will not be necessary positive almost surely since it violates the Assumption 2.1 and consequently the Theorems 2.1 and 2.2 do not apply. To compensate this, the volatility function $\sigma(y)$ is chosen to be strictly positive.

In addition, we define the payoff function $h(x)$ as a combination of European and binary options with the same strike price $K$ and time to maturity $T$, that is $h(S_T) = (S_T - K)^+ + 1_{S_T > K}$. It is easy to check that the strike price $K$ is a removable discontinuity of the payoff function $h$. In addition, the expression of $L(S_T)$ can be deduced from (4.4) as

\[
L(S_T) = \begin{cases} \frac{1}{2} \left((S_T - K)(S_T - K + 2)\right) & \text{if } S_T \geq S \\ 0 & \text{otherwise.} \end{cases}
\]

(4.6)

We use the same parameters ($\eta = r = 0.2, \theta = 0.6, T = 1, H = 0.6$) with different forms of volatility process $\sigma(Y_t)$ of the infinitesimal return process $dS_t/S_t$ as in Bezborodov et al. (2019). Since the $fCIR$ process of the form 1.2 and 1.3 cannot be used, we consider the direct form of the stochastic volatility $(Y_t)_{t \geq 0}$ driven by a $fBm$ represented by the Volterra stochastic integral (1.5) which can be discretised as follows:

\[
W_{i,j}^H = \sum_{i=0}^{j-1} \left( \int_{t_{i-1}}^{t_i} \kappa_H(t_j, s) ds \right) \delta V_i,
\]

(4.7)
Malliavin differentiability of $fH_t$ model and applications to option pricing

Table 1: Option prices using Direct Estimations

| $H$ | Mean/CV | Mean | CV  | Mean/CV | Mean | CV  | Mean/CV | Mean | CV  |
|-----|---------|------|-----|---------|------|-----|---------|------|-----|
| 0.1 | 0.74320051 | 0.01059408 | 0.17875572 | 0.03865630 | 0.37607682 | 0.00261751 | 0.37607682 | 0.00261751 | 0.37607682 |
| 0.3 | 0.81294186 | 0.03145479 | 0.93531247 | 0.01536297 | 1.06074049 | 0.00498927 | 0.93531247 | 0.01536297 | 1.06074049 |
| 0.5 | 0.91514365 | 0.02579565 | 0.74343656 | 0.02579565 | 0.55539556 | 0.02579565 | 0.55539556 | 0.02579565 | 0.55539556 |
| 0.7 | 0.99999004 | 0.00000000 | 0.99999004 | 0.00000000 | 0.99999004 | 0.00000000 | 0.99999004 | 0.00000000 | 0.99999004 |
| 0.9 | 0.81294186 | 0.03145479 | 0.93531247 | 0.01536297 | 1.06074049 | 0.00498927 | 0.93531247 | 0.01536297 | 1.06074049 |

Table 2: Option prices using $(4.3)$

| $H$ | Mean/CV | Mean | CV  | Mean/CV | Mean | CV  | Mean/CV | Mean | CV  |
|-----|---------|------|-----|---------|------|-----|---------|------|-----|
| 0.1 | 0.79340973 | 0.07560649 | 0.81121348 | 0.04028921 | 0.78827183 | 0.11421244 | 0.76642501 | 0.08935762 | 0.77047340 |
| 0.3 | 0.99999004 | 0.00000000 | 0.99999004 | 0.00000000 | 0.99999004 | 0.00000000 | 0.99999004 | 0.00000000 | 0.99999004 |
| 0.5 | 0.99999004 | 0.00000000 | 0.99999004 | 0.00000000 | 0.99999004 | 0.00000000 | 0.99999004 | 0.00000000 | 0.99999004 |
| 0.7 | 0.99999004 | 0.00000000 | 0.99999004 | 0.00000000 | 0.99999004 | 0.00000000 | 0.99999004 | 0.00000000 | 0.99999004 |
| 0.9 | 0.99999004 | 0.00000000 | 0.99999004 | 0.00000000 | 0.99999004 | 0.00000000 | 0.99999004 | 0.00000000 | 0.99999004 |

for all $j = 1, \cdots, N; \ i = 0, \cdots, j$ and where $\delta V_i = V_i - V_{i-1}$ is the increment of standard Brownian motion with $W_H^0 = 0$. Here $\kappa_H(t_j, s)$ is a discretised square integrable kernel (1.7) given by

$$
\kappa_H(t_j, s) = \frac{(t_j - s)^{H - \frac{1}{2}}}{\Gamma(H + \frac{1}{2})} 2F_1 \left( H - \frac{1}{2} + \frac{1}{2}; 1 + \frac{1}{2}; s, \frac{t_j - s}{s} \right) 1_{[0, t_j]}(s), \ \forall s \in [0, t_j].
$$

(4.8)

In this case, we observe that the values of option prices are not remarkably different for $\rho = 0$ and $H \geq 1/2$. The option prices are increasing or decreasing when $\rho$ is positive or negative respectively.

Now, consider the fractional volatility process described by a standard fCIR process, that is, with $f(t, z) = \mu - \theta z^2$ and correlation $\rho$ between infinitesimal returns and volatility, the option prices are simulated with $\rho = 0.5$ and $\mu = 0.1$.

We do 100 trials for 500 simulations and 500 time-steps on the time interval $[0, 1]$. We get the mean of option prices (that is, expected payoff function discounted by the net present value) with their corresponding coefficient of variations. Table 1 corresponds to the formula $(4.3)$ and Table 2 to direct estimation of expected payoff function.
Malliavin differentiability of $fHt$ model and applications to option pricing

Table 3: Option prices using Direct Estimations

| $H$ | $0.1$ | $0.3$ | $0.5$ | $0.7$ | $0.9$ |
|-----|-------|-------|-------|-------|-------|
| Mean/CV | Mean/CV | Mean/CV | Mean/CV | Mean/CV | Mean/CV |
| $\sigma_t = \sqrt{\theta \nu t}$ | $0.73728349$ | $0.46727731$ | $0.57840496$ | $0.75642781$ | $0.85662286$ | $0.76065572$ | $0.71243413$ | $0.76333388$ | $0.63585714$ |
| $\lambda_t = 1 + \nu t$ | $0.52140189$ | $0.32359582$ | $0.34333784$ | $0.16242491$ | $0.53521232$ | $0.341487$ | $0.30670632$ | $0.54490039$ | $0.62533212$ | $0.64070707$ |
| $\lambda_t = 1 + \nu t$ | $0.75804142$ | $0.68434942$ | $0.72435428$ | $0.16246471$ | $0.75413457$ | $0.68031396$ | $0.70756831$ | $0.60494052$ |

Table 4: Option prices using (4.3)

| $H$ | $0.1$ | $0.3$ | $0.5$ | $0.7$ | $0.9$ |
|-----|-------|-------|-------|-------|-------|
| Mean/CV | Mean/CV | Mean/CV | Mean/CV | Mean/CV | Mean/CV |
| $\sigma_t = \sqrt{\theta \nu t}$ | $0.70017421$ | $0.15348432$ | $0.57299817$ | $0.13664831$ | $0.76196204$ | $0.15710785$ | $0.74581301$ | $0.16525256$ | $0.75713214$ | $0.06593613$ |
| $\lambda_t = 1 + \nu t$ | $0.56056941$ | $0.03240347$ | $0.12290812$ | $0.11325305$ | $0.31734288$ | $0.11407129$ | $0.30678781$ | $0.06546034$ | $0.60134103$ | $0.16524489$ |
| $\lambda_t = 1 + \nu t$ | $0.80173037$ | $0.27221512$ | $0.75278272$ | $0.01034481$ | $0.75416026$ | $0.20199940$ | $0.74362583$ | $0.20331144$ | $0.83591968$ | $0.18426555$ |

Pricing options with volatility taking the form of $fCIR$ process with time varying parameters

In this section, we perform some simulations of option prices under the fractional Heston model with time varying parameters. For this, the drift function is given by $f(t, z) = (\mu_t - \theta_t z^2)$, where $\theta_t = \theta > 0$ and $\mu_t = c + \nu_t \theta_t (1 - e^{-2t})$. It follows that

$$f(t, z) = \frac{\nu_t^2}{2\theta} (1 - e^{-2t}) + (c - \theta t^2).$$

We shall use $Z_0 = 1$, $\nu = 0.4$, $c = 0.02$, $\theta = 1$. To keep positiveness of the stochastic process $(Z_t)_{t \geq 0}$ for all $H \in (0, 1)$, we shall rather use its approximated stochastic process $(\tilde{Z}_t)_{t \geq 0}$ defined by (2.1), that is

$$dZ_t = \frac{1}{2} f(t, Z_t) \Lambda_t (Z_t) dt + \sigma_t dW^H_t, \quad Z_0 = Z_0 > 0,$$

where the function $\Lambda_t(z)$ is defined by

$$\Lambda_t(z) = (z 1_{\{z > 0\}} + \epsilon)^{-1}$$

with $\epsilon = 0.01$ for $H \leq 1/2$ and $\epsilon = 0$ for $H > 1/2$. As previously, the $fBm$ is simulated by using the formula (4.7) and (4.8). We perform again 100 trials for 500 simulations and 500 time-steps on the time interval $[0, 1]$. We get the mean of option prices with their corresponding coefficient of variations for different volatility functions $\sigma(y)$ under the European-Binary option as given in Table 3 for direct estimations and in Table 4 by using (4.3).

5 Conclusion

In this paper, we have constructed an arbitrage-free and incomplete financial market model that consists of a risk-free asset with prices $A_t$ that verifies $dA_t = r A_t dt$ and the risky asset with price given as a geometric Brownian motion $dS_t = \eta S_t dt + \sigma(Y_t) S_t dB_t$. The volatility of infinitesimal return $dS_t/S_t$ given by $\sigma(Y_t)$ is
Malliavin differentiability of $fHt$ model and applications to option pricing

a function of the generalised $fCIR$ process $(Y_t)_{t\geq 0}$ defined by $Y_t^2 = Z_t^2 1_{[0,\tau)}$ with $dZ_t = \frac{1}{2} f(t, Z_t) Z_t^{-1} dt + \frac{1}{2} \sigma dW^H_t$, $Z_0 > 0$, where $f(t, x)$ is a continuous function on $\mathbb{R}_+^2$ that satisfies two mild conditions. We have investigated different properties of each components of this financial market model.

To support the results, we perform simulations firstly when the volatility takes the form of Ornstein-Uhlenbeck process and recover the results in Bezborodov et al. (2019) and secondly, we consider the fractional Cox-Ingersoll-Ross process with time-varying parameters. We have observed that option prices are more stable. Calibration of parameters will be subject to further investigations.

References

Alos, E. and Ewald, C.-O. (2008). Malliavin differentiability of the heston volatility and applications to option pricing, *Advances in Applied Probability* 40(1): 144–162.

Alòs, E., Mancino, M. E. and Wang, T.-H. (2019). Volatility and volatility-linked derivatives: estimation, modeling, and pricing, *Decisions in Economics and Finance* 42(2): 321–349.

Alòs, E. and Yang, Y. (2017). A fractional Heston model with $H > 1/2$, *Stochastics* 89(1): 384–399.

Altmayer, M. and Neuenkirch, A. (2015). Multilevel Monte Carlo quadrature of discontinuous payoffs in the generalized Heston model using Malliavin integration by parts, *SIAM Journal on Financial Mathematics* 6(1): 22–52.

Benhamou, E., Gobet, E. and Miri, M. (2010). Time dependent Heston model, *SIAM Journal on Financial Mathematics* 1(1): 289–325.

Bezborodov, V., Di Persio, L. and Mishura, Y. (2019). Option pricing with fractional stochastic volatility and discontinuous payoff function of polynomial growth, *Methodology and Computing in Applied Probability* 21(1): 331–366.

Chronopoulou, A. and Viens, F. G. (2010). Hurst index estimation for self-similar processes with long-memory, *Recent Development in Stochastic Dynamics and Stochastic Analysis*, World Scientific, pp. 91–117.

Comte, F. and Renault, E. (1998). Long memory in continuous-time stochastic volatility models, *Mathematical finance* 8(4): 291–323.

Gatheral, J., Jaisson, T. and Rosenbaum, M. (2018). Volatility is rough, *Quantitative Finance* 18(6): 933–949.

Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options, *The review of financial studies* 6(2): 327–343.
Hu, Y., Nualart, D. and Song, X. (2008). A singular stochastic differential equation driven by fractional Brownian motion, *Statistics & Probability Letters* **78**(14): 2075–2085.

Livieri, G., Mouti, S., Pallavicini, A. and Rosenbaum, M. (2018). Rough volatility: evidence from option prices, *IISE Transactions* **50**(9): 767–776.

Mishura, Y. and Yurchenko-Tytarenko, A. (2018). Fractional Cox–Ingersoll–Ross process with non-zero “mean”, *Modern Stochastics: Theory and Applications* **5**: 99–111.

Mishura, Y. and Yurchenko-Tytarenko, A. (2019). Fractional Cox–Ingersoll–Ross process with small Hurst indices, *Modern Stochastics: Theory and Applications* **6**(1): 13–39.

Mishura, Y. and Yurchenko-Tytarenko, A. (2020). Approximating Expected Value of an Option with Non-Lipschitz Payoff in Fractional Heston-Type Model, *International Journal of Theoretical and Applied Finance* .

Mpanda, M. M., Mukeru, S. and Mulaudzi, M. (2020). Generalisation of Fractional-Cox-Ingersoll-Ross Process, *arXiv preprint arXiv:2008.07798* .

Nourdin, I. (2012). *Selected aspects of fractional Brownian motion*, Vol. 4, Springer.

Nualart, D. (2006). *The Malliavin calculus and related topics*, Vol. 1995, Springer.

Nualart, D. and Ouknine, Y. (2002). Regularization of differential equations by fractional noise, *Stochastic Processes and their Applications* **102**(1): 103–116.