AN EQUIVARIANT PPV THEOREM AND
PASCHKE-HIGSON DUALITY

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Abstract. We state the Paschke-Higson duality theorem for a transformation groupoid. Our proof relies on an equivariant localized and norm-controlled version of the Pimsner-Popa-Voiculescu theorem. The main consequence is the existence of a Higson-Roe exact sequence, involving the Baum-Connes assembly map for such groupoid.

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Introduction

In their groundbreaking work on coarse geometry and the Baum-Connes conjecture, Higson and Roe established the existence of a long exact sequence in $K$-theory, now called the Higson-Roe sequence. Informally, this sequence “measures” the potential failure of the Baum-Connes conjecture, i.e. the failure of the Baum-Connes assembly map to be an isomorphism [HR1:05, HR2:05, HR3:05]. The Higson-Roe sequence uses the Paschke-Higson duality isomorphism in a fundamental way, which allows to express the $K$-homology of a compact space $Y$ as the $K$-theory of some dual $C^*$-algebra $Q(Y, H)$, which can be taken to be the commutant of $C(Y)$ in the Calkin algebra of any ample representation of $C(Y)$, see [P:81, Hig:95]. More generally, Paschke showed in [P:81] that for any separable unital $C^*$-algebra $A$, the $K$-theory of the commutant of $A$ in the Calkin algebra (for any ample representation) is isomorphic to the group of invertibles in Ext($A$), see also [HPR:97, HR:00] and [V:83] for further developments. In this context, Paschke-Higson duality as applied to the Higson-Roe sequence, relies on a remarkable and deep theorem of Voiculescu [V:76]. Indeed, the commutative case of Voiculescu’s theorem provides a crucial step in identifying the boundary map in the Higson-Roe exact sequence with the Baum-Connes assembly map.

Recall that Voiculescu’s theorem was originally carried out in order to solve some then open questions in operator theory [H:70], it also implied a noncommutative version of the Brown-Douglas-Fillmore theorem about the existence of the trivial element of Ext($Y$) [BDF:73], a far-reaching generalization of the classical Weyl-von Neumann classification theorem [W:09, vN:35, B:71]. It is worth pointing out that Kasparov studied, in the early eighties, representations of unital, nuclear $C^*$-algebras on Hilbert $C^*$-modules,
proved a generalized Voiculescu theorem which played a crucial part in establishing his powerful $KK$-theory, see \cite{Ka:80, Ka:81}.

In the present paper, we shall be interested in the statement and applications of an equivariant family version of the Voiculescu theorem. We shall only be interested in the Paschke-Higson duality corollary and the resulting Higson-Roe exact sequence for transformation groupoids. Other applications will be investigated elsewhere. The equivariant Paschke-Higson duality includes a proper and cocompact action of a countable discrete group $\Gamma$, and allows to express the classical Baum-Connes assembly map \cite{BC:00} using boundary maps in $K$-theory of $C^*$-algebras, thereby providing a $K$-theory obstruction group. This latter group turns out to be a natural receptacle for some defect secondary eta invariants of Dirac operators. Paschke-Higson duality therefore provides a bridge between equivariant $K$-homology and the $K$-theory of appropriate coarse algebras associated with proper, co-compact group actions on non-compact spaces, see for instance \cite{Roe:16}. In \cite{HR:10}, this approach allowed to give a nice proof of the Keswani rigidity of reduced eta invariants, for the spin Dirac operator in the presence of positive scalar curvature metrics and for the signature operator in the presence of a homotopy equivalence. Other results were obtained following the same line of ideas, in relation with the Novikov and Gromov-Lawson-Rosenberg conjectures, see for instance \cite{HR1:05, HR2:05, HR3:05, BM:15, BR:15, BEKW:18, PS:13, XY:14, Zen:19, Zei:16} as well as some slightly different approaches in \cite{HPS:15, STY:02, Yu1:97, Yu2:00}.

So our first goal is to prove the equivariant family version of the Paschke-Higson duality theorem, which contains as a special case the Higson-Roe equivariant version recalled above. The countable discrete group $\Gamma$ must now be replaced by a transformation groupoid $X \rtimes \Gamma$ and, guided by the family version and the equivariant version as well as the Baum-Connes assembly map for étale groupoids, we had to provide for any proper cocompact $\Gamma$-space $Z$, some dual $C^*$-algebra $Q(X, \Gamma, Z, H)$ associated with an “ample” representation $H$ whose $K$-theory is intended to be isomorphic to the $\Gamma$-equivariant bivariant group $KK^*_\Gamma(Z, X)$. Notice indeed that this latter group is the building block of the LHS of the Baum-Connes assembly map for the transformation groupoid $X \rtimes \Gamma$. See Theorem \cite{LS} which states the precise Paschke-Higson duality for the transformation groupoid $X \rtimes \Gamma$. When $X$ is reduced to the point, the Paschke-Higson Theorem \cite{LS} reduces as expected to the classical Higson-Roe result \cite{HR:10, BR:15}. When the group is trivial and all involved spaces are compact, one recovers the Pimsner-Popa-Voiculescu (PPV) setting and the Paschke-Higson result is already stated in a different form by Connes and Skandalis in \cite{CS:84} using $KK$-theory. The dual algebra $Q(X, \Gamma, Z, H)$ is more precisely constructed using a generalized version of the Roe algebras generated by finite propagation operators, and we had to precisely keep track of the finite propagation properties of the intertwining unitaries appearing in the PPV work and also to take care of the extra equivariance properties. With the extra action of the countable discrete group, we show that one can indeed ensure the localization of the supports of these intertwining unitaries, a notion that generalizes the finite propagation property to the non-cocompact case, see Definition \cite{A1}. The invariance of the unitaries is fulfilled by using standard averagings, which in turn only converge in the strong topology and hence tend to violate the desired intertwining up to compacts property. In the cocompact and metric case, we eventually obtain the needed finite propagation property and the proof of the expected equivariant family Paschke-Higson theorem.

As explained above, our main application of the Paschke-Higson theorem for $X \rtimes \Gamma$ is to deduce a Higson-Roe exact sequence, encompassing the corresponding Baum-Connes assembly map for this groupoid, see the companion paper \cite{BR2:21} and also \cite{BR:15}. This exact sequence yields, as in the classical case, to rigidity applications of some secondary invariants of Dirac operators on suspended foliated spaces \cite{BP:09}, especially laminations such as the principal solenoidal tori \cite{CC:00} used in \cite{BM:20}. Exactly as for the classical Paschke-Higson duality, we were naturally led to the statement of an equivariant family version of the Voiculescu theorem, an independent result. In the non-equivariant case, the family Voiculescu theorem is an extension of a classical theorem due to Pimsner-Popa-Voiculescu \cite{PPV:79}.

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1. Statement of the main theorems

All the spaces considered in the present paper are assumed second-countable. We devote this preliminary section to the detailed statement of the main results. Let $X$ be a compact metrizable space of finite dimension, and let $\Gamma$ be a discrete infinite countable group acting by homeomorphisms on $X$. Given a locally compact Hausdorff space $Z$ with a proper (not necessarily cocompact) $\Gamma$-action, recall the usual $C^\ast$-algebra $C_0(Z)$ of continuous functions vanishing at infinity. We shall also need the following $\Gamma$-equivariant $C^\ast$-ideal $C_{0[1]}(Z)$ of $C_b(Z)$. If $C_{\omega[1]}(Z)$ denotes the $\Gamma$-algebra of those continuous bounded functions $f$ on $Z$ such that only a finite number of elements $g \in \Gamma$ may satisfy $g \text{supp}(f) \cap \text{supp}(f) \neq \emptyset$, the algebra $C_{0[1]}(Z)$ will be the closure of $C_{\omega[1]}(Z)$ in $C_b(Z)$. Notice that $C_{0[1]}(Z)$ reduces to $\Gamma$-algebra $C_0(Z)$ when the action is cocompact (say when $Z/\Gamma$ is compact), and is thus separable in this case. In general, it will obviously not be separable. Consider now a separable $\Gamma$-algebra $A$ which is a $\Gamma$-proper $C^\ast$-algebra over a locally compact Hausdorff space $Z$. Recall that this means that $Z$ is a proper $\Gamma$-space in the usual sense and that there exists a $\Gamma$-equivariant

morphism $C_0(Z) \rightarrow ZM(A)$ from $C_0(Z)$ to the center $ZM(A)$ of the multiplier algebra $M(A)$ of $A$, such that $C_0(Z)A$ is dense in $A$. We denote for simplicity by $fa \in A$ the resulting action of $f \in C_0(Z)$ on $a \in A$. The first example of such algebra $A$ is $C_0(Z)$ itself but given such $A$ for any extra separable $\Gamma$-algebra $B$, the $\Gamma$-algebra $A \otimes B$ is then again $\Gamma$-proper over $Z$. Since we are mainly interested in examples like $C_0(Z,B)$ where $B$ is a given separable unital $\Gamma$-algebra, we shall always assume that $C_0(Z)$ maps to the center $ZA$ of the $C^\ast$-algebra $A$ itself. Notice that this can be ensured by replacing $A$ by $A + C_0(Z)$ where $C_0(Z)$ is meant as its multiplier algebra $M(A)$.

When the $\Gamma$-space $Z$ is cocompact, it will sometimes be convenient to express the properties of our operators in terms of propagation with respect to a given proper $\Gamma$-invariant distance $d$ on $Z$, so as to compare with the litterature. All our results are though valid without reference to such distance by using the new notion of localized operators expanded in Appendix A. The diagonal action of $\Gamma$ on $X \times Z$ then endows $X \times Z$ with a proper action. Let $G$ denote the transformation groupoid $X \rtimes \Gamma$. If a Hilbert space $H$ is endowed with a unitary action of $\Gamma$, then a given $C(X)$-representation $\tilde{\pi}: C(X,A) \rightarrow \mathcal{L}_{C(X)}(C(X) \otimes H)$ is a $G$-equivariant representation if the corresponding field $(\pi_x)_{x \in X}$ of representations of $A$ is $\Gamma$-equivariant. The same comment applies to a $G$-operator from $\mathcal{L}_{C(X)}(C(X) \otimes H)$ which then corresponds to a $\Gamma$-equivariant $*$-strongly continuous field of operators in $H$.

1.1. An extended PPV theorem. We fix the proper $\Gamma$-algebra $A$ over $Z$ as above. Recall again that we have assumed that $C_0(Z)$ maps inside the center $ZA$ of $A$. The $C^\ast$-algebra $C(X,A)$ of continuous functions from $X$ to $A$ is naturally equipped with a $C(X)$-algebra structure and the action of $\Gamma$ endows it

with the structure of a $\Gamma$-algebra, see [LeGall:99, BR1:20]. Suppose that $E$ is a countably-generated Hilbert $C(X)$-module. We shall denote abusively by $\mathcal{L}_{C(X)}(E)$ the $C^\ast$-algebra of adjointable operators in $E$ and by $K_{C(X)}(E)$ its ideal of $\ast$-compact operators [Kai:80].

A given representation $\tilde{\pi}: C(X,A) \rightarrow \mathcal{L}_{C(X)}(E)$ is called a $C(X)$-representation if the action of $C(X)$ on $C(X,A)$ is compatible with the right $C(X)$-module structure on $E$. Such a $C(X)$-representation then corresponds to a $\ast$-homomorphism $\pi: A \rightarrow \mathcal{L}_{C(X)}(E)$, which in turn corresponds to a field of representations $\pi_x : A \rightarrow \mathcal{L}(E_x)$, where $E_x := E \otimes_{ev_x} C$ is the Hilbert space fibre over $x$ associated with the Hilbert module $E$. Recall that the field $(E_x)_{x \in X}$ is then a continuous field of Hilbert spaces in the sense of [Dix:77]. Only the $C(X)$-algebra $C(X,A)$ will be needed in the present paper, meaning a constant field, and we shall always use in the sequel this notation of adding a hat for the $C(X)$-representation of $C(X,A)$ associated with a given $\ast$-homomorphism from $A$ to $\mathcal{L}_{C(X)}(E)$. We have chosen to state our results in this language of $C(X)$-representations for the sake of possible generalizations, see [BR1:20, BR2:21].
Once such representation $\pi$ is fixed and $Z$ is metric-proper as above, an operator $T \in \mathcal{L}_{C(X)}(E)$ will be said to have finite propagation $\leq R$ (with respect to $\pi$) if

$$\pi(a_1)T\pi(a_2) = 0 \text{ for any } a_1, a_2 \in A \text{ such that } d(\text{Supp}(a_1), \text{Supp}(a_2)) > R.$$ 

Recall that the support $\text{Supp}(a)$ of an element $a \in A$ is the complement of the largest open subspace $U$ of $Z$ such that $fa = 0$ for any $f \in C_0(U)$. When $Z$ is not necessarily a proper-metric space, the support $\text{Supp}(T)$ of the operator $T \in \mathcal{L}_{C(X)}(E)$ itself with respect to the representation $\pi$ can still be defined as the complement in $Z \times Z$ of the union of all open sets of the form $U \times V$, where $U$ and $V$ are open in $Z$, such that $\pi(a_1)T\pi(a_2) = 0$ for any $a_1 \in C_0(U)A$ and $a_2 \in C_0(V)A$.

**Definition 1.1.** A cutoff function on the $\Gamma$-proper space $Z$ will be any continuous function $\chi : Z \to [0,1]$ such that

1. $\chi$ belongs to $C\Gamma(Z)$;
2. $\sum_{g \in \Gamma} g\chi = 1$.

Recall that the first item means that $\{g \in \Gamma, g\text{supp}(\chi) \cap \text{supp}(\chi) \neq \emptyset\}$ is finite. The second item means that for any $z \in Z$, $\sum_{g \in \Gamma} \chi(gz) = 1$, the latter sum being finite by the first item. Moreover, if $W_\chi = \{\chi \neq 0\}$ then $Z = \bigcup_{g \in \Gamma} gW_\chi$, and for any compact subspace $K \subset Z$, the set $\{g \in \Gamma, gK \cap \text{supp}(\chi) \neq \emptyset\}$ is also finite. We denote from now on for $k \geq 1$ and for any continuous cutoff function $\chi \in C(Z)$ for the proper $\Gamma$-action on $Z$ with $W_\chi := \{\chi \neq 0\}$, by $\Gamma^{(k)}_\chi$ the subset of $\Gamma^2$ given by

$$\Gamma^{(k)}_\chi := \{(g_0, g_k) \in \Gamma^2 | \exists (g_i)_{1 \leq i \leq k-1} \text{ such that } g_iW_\chi \cap g_{i+1}W_\chi \neq \emptyset \text{ for } 0 \leq i < k\}.$$ 

For $k = 0$, we simply take for $\Gamma^{(0)}_\chi$ the diagonal in $\Gamma^2$ which is isomorphic to $\Gamma$. We point out that the first and second projections $\Gamma^{(k)}_\chi \to \Gamma$ are proper, this is indeed equivalent to the same statement for $k = 1$ where the statement is clear from the above definitions.

**Definition 1.2.** Let $Z$ be a given proper $\Gamma$-space. We shall say that the (non-negative) cutoff function $\chi$ is uniform if $\{g \in \Gamma, g\text{supp}(\chi) \cap \text{supp}(\chi) \neq \emptyset\}$ generates the group $\Gamma$.

In particular, the existence of such uniform cutoff function implies that $\Gamma$ is finitely generated. Conversely, if $\Gamma$ is finitely generated, then every proper $\Gamma$-space admits uniform cutoff functions. More precisely:

**Lemma 1.3.** Let $\Gamma$ be a countable group, then the following are equivalent:

1. $\Gamma$ is finitely generated;
2. There exists a proper $\Gamma$-space with a uniform cutoff function;
3. Every proper $\Gamma$-space has a uniform cutoff function.

**Proof.** It is clear from Definition 1.2 that (ii) implies (i). Since $\Gamma$ has proper actions (for instance $Z = \Gamma$ itself), it is also obvious that (iii) implies (ii). Assume now that $\Gamma$ is finitely generated and that $Z$ is a proper $\Gamma$-space. Choose some non-negative cutoff function $\chi_0$ on $Z$. Let $F$ be a finite symmetric generating subset of $\Gamma$. Recall that this means that the finite subset $F$ generates the group $\Gamma$ and satisfies that $e \in S$ and for any $g \in F$, $g^{-1} \in F$. We then set

$$\chi := \frac{1}{|F|} \sum_{g \in F} g\chi_0.$$ 

It is then clear that $\chi$ is a new non-negative cutoff function and that $F \subset \{g \in \Gamma, g\text{supp}(\chi) \cap \text{supp}(\chi) \neq \emptyset\}$.

**Lemma 1.4.** Assume that $\Gamma$ is finitely generated and that $\chi$ is a uniform cutoff function on the proper $\Gamma$-space $Z$. Then

$$\Gamma^2 = \bigcup_{k \geq 0} \Gamma^{(k)}_\chi.$$
Proof. Set \( A = \{ g \in \Gamma, gV_\chi \cap V_\chi \neq \emptyset \} \). Notice that 
\[
(g, g') \in \Gamma^{(k)}_\chi \iff \exists (g_i)_{1 \leq i \leq k-1} \text{ such that } g_{i+1}^{-1}g_i \in A \text{ for } 0 \leq i \leq k-1, \text{ with } g_0 = g \text{ and } g_k = g'.
\]
This shows that \( g'^{-1}g \in A^k \), and this latter condition is clearly equivalent to the condition \( (g, g') \in \Gamma^{(k)}_\chi \).

Indeed, given \( (a_1, \ldots, a_k) \in A^k \) such that \( g'^{-1}g = a_k \cdots a_1 \), by setting \( g_0 = g \) and \( g_{i+1} = g_i a_i^{-1} \) one deduces immediately that \( (g, g') \in \Gamma^{(k)}_\chi \).

Now, since we assume that \( A \) generates \( \Gamma \), we can write any \( (g, g') \in \Gamma^2 \) as a product \( (k_1, k'_1) \cdots (k_r, k'_r) \) for some elements \( (k_j)_{1 \leq j \leq r} \) and \( (k'_j)_{1 \leq j \leq r} \) in \( A \). Using that \( A \) is symmetric, we conclude that \( g'^{-1}g \in A^{2r} \). \( \square \)

In the present paper we shall need to work with uniform cutoff functions and will therefore assume from now on that \( \Gamma \) is finitely generated. Then, all cutoff functions will be chosen uniform. Then a given operator \( T \) will have localized support if there exists an integer \( k \geq 0 \) such that the support of \( T \) is contained in the closure of 
\[
\bigcup_{(g, g') \in \Gamma^{(k)}_\chi} gW_\chi \times g'W_\chi.
\]
As explained in Appendix A this is equivalent to the existence of \( k' \geq 0 \) such that the support of \( T \) is contained in \( \bigcup_{(g, g') \in \Gamma^{(k')}_\chi} gW_\chi \times g'W_\chi \). The propagation index of \( T \) is then the least such \( k' \).

When the action of \( \Gamma \) is cocompact and \( Z \) is endowed with a \( \Gamma \)-invariant metric which endows it with the topology of a proper-metric space, it is easy to see that localized operators coincide with finite propagation operators, see again Appendix A.

Let us recall now the notion of fibrewise ample representation, see [PPV:79, BR2:21].

**Definition 1.5.** A \( C(X) \)-representation \( \hat{\pi} : C(X, A) \to \mathcal{L}_{C(X)}(E) \) will be called a fibrewise ample representation if for any \( x \in X \), the representation \( \pi_x : A \to \mathcal{L}(E_x) \) is ample, i.e. for any \( x \in X \), \( \pi_x \) is non-degenerate and one has for \( a \in A \):
\[
\pi_x(a) \in \mathcal{K}(E_x) \implies a = 0.
\]
Here and as usual \( \mathcal{K}(E_x) \) denotes the elementary \( C^* \)-algebra of compact operators on the Hilbert space \( E_x \).

Given a Hilbert space unitary representation \( U : \Gamma \to U(H) \), we denote as usual by \( H^\infty \) the Hilbert space \( H \otimes \ell^2\mathbb{N} \) endowed with the unitary representation \( U \otimes \text{id}_{\ell^2\mathbb{N}} \). Unless otherwise specified, the Hilbert space \( \ell^2\Gamma \) will be endowed with the right regular representation of \( \Gamma \), so \( \ell^2\Gamma^\infty \) is endowed with the corresponding representation. Our extended PPV theorem can be stated as follows:

**Theorem 1.6.** Assume that the action of \( \Gamma \) on \( Z \) is proper with a uniform cutoff function \( \chi \). Let \( H_1 \) and \( H_2 \) be two infinite-dimensional separable complex Hilbert spaces, endowed with unitary representations of \( \Gamma \). Let \( \hat{\pi}_1 \) and \( \hat{\pi}_2 \) be as above two fibrewise ample \( \Gamma \)-equivariant \( C(X) \)-representations of \( C(X, A) \) in the Hilbert \( \Gamma \)-modules \( C(X) \otimes H_1 \) and \( C(X) \otimes H_2 \) respectively. Then, identifying each \( \hat{\pi}_i \) with the trivially extended representation 
\[
\hat{\pi}_i = \begin{pmatrix} \pi_i & 0 \\ 0 & 0 \end{pmatrix}
\]
that is further tensored by the identity of \( \ell^2\Gamma^\infty \) there exists a sequence \( \{ W_n \}_{n \in \mathbb{N}} \) of \( \Gamma \)-invariant unitary operators
\[
W_n \in \mathcal{L}_{C(X)} \left( ([H_1 \oplus H_2] \otimes \ell^2\Gamma^\infty) \otimes C(X), ([H_2 \oplus H_1] \otimes \ell^2\Gamma^\infty) \otimes C(X) \right),
\]
such that 
\[
W_n^* \hat{\pi}_2(\varphi)W_n - \hat{\pi}_1(\varphi) \text{ is compact, and } \lim_{n \to \infty} ||W_n^* \hat{\pi}_2(\varphi)W_n - \hat{\pi}_1(\varphi)|| = 0.
\]
Moreover, we can ensure that the operators \( W_n \) are localized with uniform propagation index, actually \( \leq 7 \). In particular, if \( Z \) is proper-metric such that \( Z/\Gamma \) is compact then we can always ensure that the unitaries \( W_n \) have (uniform) finite propagation.
In the next section, Theorem 1.6 is first partially proved, more precisely we prove the weaker version stated as Theorem 2.1 which only constructs one unitary W with the allowed properties. It is only later on in Subsection 2.2 that the construction of the sequence (W_n)_n is carried out with the norm-control. In the sequel, an isometry (resp. unitary) S satisfying the (up to compact operators) intertwining property (2.2) will be referred to as a PPV-isometry (resp. PPV-unitary).

1.2. Equivariant Paschke-Higson duality. As an important application, we deduce the Paschke-Higson duality isomorphism for Γ-families. More precisely, we assume now and for simplicity that the action of Γ on Z is cocompact and that Z is a proper-metric space with a chosen Γ-invariant distance. In [BR2:21], we defined a generalization of the classical equivariant Roe algebras of pseudolocal and locally compact operators associated with a fiberwise ample representation of C(X, A) on the Hilbert C(X)-module (ℓ^2Γ_∞ ⊗ H) ⊗ C(X) induced by a given ample representation of A in a fixed H. The Roe algebra of pseudolocal operators is denoted by D^Γ_Γ(X, A; (ℓ^2Γ_∞ ⊗ H)), and the Roe algebra of locally compact operators is denoted C^Γ_Γ(X, A; (ℓ^2Γ_∞ ⊗ H)). More precisely, D^Γ_Γ(X, A; (ℓ^2Γ_∞ ⊗ H)) is by definition the norm closure in L_C^Γ_Γ(X) (C(X) ⊗ (ℓ^2Γ_∞ ⊗ H)) of the space of Γ-invariant operators with finite propagation and whose commutators with the elements of C(X, A) are compact operators. The C^∗-algebra C^Γ_Γ(X, A; (ℓ^2Γ_∞ ⊗ H)) is on the other hand its subspace which is composed of those operators which satisfy the additional condition that their composition with the elements of C(X, A) are already compact operators.

An obvious observation is that C^Γ_Γ(X, A; (ℓ^2Γ_∞ ⊗ H)) is a 2-sided closed ideal in the unital C^∗-algebra D^Γ_Γ(X, A; (ℓ^2Γ_∞ ⊗ H)). Our Paschke-Higson duality theorem identifies the K-theory of the quotient Roe algebra Q^Γ_Γ(X, A; (Z, ℓ^2Γ_∞ ⊗ L^2Z)) with the Γ-equivariant KK-theory of the pair of Γ-algebras (A, C(X)). More precisely:

Theorem 1.7. Suppose again that the isometric action of Γ on Z is proper and cocompact. Then we have a group isomorphism

\[ \mathcal{P}_*: K^*_n(Q^Γ_Γ(X, A; (Z, ℓ^2Γ_∞ ⊗ H))) \overset{\sim}{\longrightarrow} K^{Γ}_{n+1}(A, C(X)), \quad * = 0, 1. \]

Notice that Z does not appear in the RHS, only its existence is supposed so that the LHS does not depend on the choice of such Z. The fact that the ample representation does not appear in the RHS is standard due to our PPV theorem. In the case of trivial Γ, this theorem is well known, see for instance [Hig:95, V:83]. An interesting case corresponds to the case A = C_0(Z). Then we get using the notations from [BR2:21] the following theorem which was fully used there to deduce the Higson-Roe sequence for the groupoid G = X ⋊ Γ:

Theorem 1.8. Suppose again that the isometric action of Γ on Z is proper and cocompact. Then we have a group isomorphism

\[ \mathcal{P}_*: K^*_n(Q^Γ_Γ(X; (Z, ℓ^2Γ_∞ ⊗ H))) \overset{\sim}{\longrightarrow} K^{Γ}_{n+1}(Z, X), \quad * = 0, 1. \]

The proof of our PPV theorem as well as the deduction of the Paschke-Higson isomorphism, are carried out in the next sections.

2. Proof of the extended PPV theorem

We devote this section to the proof of our G-equivariant, norm-controlled and support-localized, version of the PPV theorem, say Theorem 1.6. In order to simplify the reading of this technical proof, we have first given the proof of a weaker version which does not adress the norm-control question.
2.1. The support-localized PPV theorem. We first forget the norm-control and prove the following weaker version of Theorem 1.6.

**Theorem 2.1** (Extended PPV theorem). Assume that the action of \( \Gamma \) on \( Z \) is proper with a uniform cutoff function \( \chi \). Let \( H_1 \) and \( H_2 \) be two infinite-dimensional separable complex Hilbert spaces, endowed with unitary representations of \( \Gamma \). Let \( \tilde{\pi}_1 \) and \( \tilde{\pi}_2 \) be as above two fiberwise ample \( \Gamma \)-equivariant \( C(X) \)-representations of \( C(X,A) \) in the Hilbert \( \Gamma \)-modules \( C(X) \otimes H_1 \) and \( C(X) \otimes H_2 \) respectively. Then, identifying each \( \tilde{\pi}_1 \) with the trivially extended representation \( \begin{pmatrix} \tilde{\pi}_1 & 0 \\ 0 & 0 \end{pmatrix} \) that is further tensored by the identity of \( \ell^2 \Gamma^\infty \), there exists a \( \Gamma \)-invariant unitary operator

\[
W \in \mathcal{L}_{C(X)} \left( \left( (H_1 + H_2) \otimes \ell^2 \Gamma^\infty \right) \otimes C(X), \left( (H_2 + H_1) \otimes \ell^2 \Gamma^\infty \right) \otimes C(X) \right),
\]

which essentially intertwines the extended representations, i.e. such that

\[
W^\ast \tilde{\pi}_2(\varphi)W - \tilde{\pi}_1(\varphi) \in \mathcal{K}_{C(X)} \left( \left( (H_1 + H_2) \otimes \ell^2 \Gamma^\infty \right) \otimes C(X) \right), \quad \text{for all } \varphi \in C(X,A).
\]

Moreover, we can ensure that the operator \( W \) is localized. In particular, if the proper \( \Gamma \)-space \( Z \) is cocompact then we can ensure that the unitary \( W \) has finite propagation.

Under the assumption that \( A = C_0(Z) \) for a proper and cocompact \( \Gamma \)-space \( Z \), a striking application of Theorem 2.1 is to the equivariant family Paschke-Higson duality Theorem 1.7, as stated in Section 3 and which allows to incorporate the Baum-Connes map for the groupoid \( G = X \rtimes \Gamma \) in a long six-term exact sequence, see [BR2:21]. Notice that if \( \Gamma \) is a finite group then any separable \( \Gamma \)-algebra is a proper \( \Gamma \)-algebra over the trivial space \( Z = \{ \ast \} \), and the theorem is valid for any such \( \Gamma \)-algebra. This is well known, see [PPV:79] for trivial \( \Gamma \) and unital \( A \), and [Ka:S1] for the general case of compact group actions. Forgetting first the \( \Gamma \)-invariance of the intertwining unitary, we shall first prove the following independent result:

**Theorem 2.2.** Under the assumption and notations of Theorem 2.1 but for any proper \( \Gamma \)-action on \( Z \), there exists a unitary

\[
U \in \mathcal{L}_{C(X)} \left( \left[ \ell^2 \Gamma^\infty \otimes (H_1 + H_2) \right] \otimes C(X), \left[ \ell^2 \Gamma^\infty \otimes (H_2 + H_1) \right] \otimes C(X) \right)
\]

such that

\[
U^\ast \tilde{\pi}_2(\varphi)U - \tilde{\pi}_1(\varphi) \in \mathcal{K}_{C(X)} \left( \left( \ell^2 \Gamma^\infty \otimes (H_1 + H_2) \right) \otimes C(X) \right), \quad \text{for all } \varphi \in C(X,A).
\]

Moreover, we can ensure that the operator \( U \) is localized. In particular, if the proper \( \Gamma \)-space \( Z \) is cocompact, then we can ensure that \( U \) has finite propagation.

We thus fix two fiberwise ample \( G \)-equivariant representations \( \tilde{\pi}_1 \) and \( \tilde{\pi}_2 \) of \( C(X,A) \) in the Hilbert \( G \)-modules \( H_1 \otimes C(X) \) and \( H_2 \otimes C(X) \) respectively. In the sequel, we shall denote for \( x \in X \), by \( q_x \) the composite map

\[
q_x : \mathcal{L}_{C(X)}(H_1 \otimes C(X)) \xrightarrow{d_x^\ast} \mathcal{L}(H_1) \xrightarrow{pr} Q(H_1),
\]

where \( d_x \) is evaluation at \( x \) while the map \( \mathcal{L}(H_1) \xrightarrow{pr} Q(H_1) \) is the quotient projection onto the Calkin algebra \( Q(H_1) = \mathcal{L}(H_1)/\mathcal{K}(H_1) \). We begin with the construction of a finite-propagation PPV-isometry.

**Lemma 2.3.** Under the assumptions of Theorem 2.2 there exists an isometry

\[
\hat{S} \in \mathcal{L}_{C(X)}((H_1 + H_2) \otimes C(X), (H_2 + H_1) \otimes \ell^2 \Gamma \otimes C(X))
\]

such that

\[
\hat{S}^\ast((\tilde{\pi}_2(f) \otimes \text{id}_{\ell^2 \Gamma}) \oplus 0)\hat{S} - (\tilde{\pi}_1(f) \otimes 0) \in \mathcal{K}_{C(X)}((H_1 + H_2) \otimes C(X)), \quad \forall f \in C_0(X \times Z)
\]

Moreover, we can ensure that the operator \( \hat{S} \) is localized. In particular, when \( Z/\Gamma \) is compact, we can ensure that \( \hat{S} \) has finite propagation.
Proof: When $\Gamma$ is a finite group, this result is well known, see for instance [Ka:81], and we give the proof under the assumption that $\Gamma$ is infinite. Fix a cutoff function $\chi \in C(Z)$ for the proper $\Gamma$-action on $Z$. The quotient projection $Z \to Z/\Gamma$ then restricts into a proper map $\text{Supp}(\chi) \to Z/\Gamma$. Denote by $V_\chi$ the interior of $\text{Supp}(\chi)$ and let $C_0(\chi)$ be the $C^*$-subalgebra of $C_0(Z)$ generated by $C_0(V_\chi)$ and $\chi$, and in the same way let $C_0(\chi)$ be the $C^*$-subalgebra of $C_0(Z)$ generated by $C_0(V_\chi)$ and $\chi$. Elements of $C_0(\chi)$ (resp. of $C_0(\chi)$) are then (uniform limits of) functions of type $f + \varphi \circ \chi$ where $f \in C_0(Z)$ (resp. $f \in C_0(V_\chi)$) and $\varphi$ is any continuous function on $\mathbb{R}$, with $\varphi(0) = 0$. It is an obvious observation that $C_0(\chi) \subset C_0(\Gamma)$ and that $C_0(\chi)$ is an ideal in $C_0(Z, \chi)$. Notice that the $C^*$-algebra $C_0(Z, \chi)$ as well as its ideal $C_0(\chi)$ are separable. Recall that the $*$-homomorphism $\pi_i : A \to \mathcal{L}(H_i \otimes C(X))$ is non-degenerate and extends to a unital $*$-homomorphism, still denoted $\pi_i$, from $M(A)$ to $\mathcal{L}(H_i \otimes C(X))$. We shall identify $C_0(Z, \chi)$ with its range in the center of the multiplier algebra of $A$ when no confusion can occur.

We now set $A_\chi := C_0(\chi)A^+$, then $A_\chi$ is a two-sided self-adjoint ideal in $C_0(Z, \chi)A^+$. The proof of Lemma 2.3 will be split into 3 steps.

**Step 1 (Apply PPV):** Consider the Hilbert submodules $E_i^\chi := \overline{\pi_i(A_\chi)(H_i \otimes C(X))}$ for $i = 1, 2$.

**Claim:** The $*$-homomorphism $\pi_i$, extended to $C_0(Z, \chi)A^+$, preserves $E_i^\chi$ since the image of $C_0(Z, \chi)$ in $M(A)$ is valued in the center $ZM(A)$, and its restriction to $A_\chi$ (acting on $E_i^\chi$) is denoted $\pi_\chi^i$. This is again associated with a fibrewise ample representation $\hat{\pi}_\chi^i$.

To check this, note that the field of Hilbert spaces associated to the Hilbert module $E_i^\chi$ is given by $(R_{i,x} := [\pi_{i,x}(A_\chi)H_i]_{x \in X})$, where $(\pi_{i,x})_{x \in X}$ is the field of representations associated with the $*$-homomorphism $\pi_i : M(A)A \to \mathcal{L}(H_i \otimes C(X))$. Then, the field of Hilbert space representations associated with $\pi_\chi^i$ is given at $x \in X$ by the restriction of the representations $\pi_{i,x}$ and denoted $\pi_{\chi, i,x}^i : A_\chi \to \mathcal{L}(R_{i,x})$. For each $x \in X$, the restricted representations $\pi_{\chi, i,x}^i$ are clearly ample.

The $(C(X), A^+)$-bimodule $E_i^\chi$ is countably generated and we may assume that it is an orthocomplemented sub-bimodule of $H_i \otimes C(X)$, respectively for $i = 1, 2$. The $*$-homomorphism $\pi_\chi^i$ will be extended by zero on the orthocomplement, this corresponds to extending each $\pi_{\chi, i,x}^i$ by zero on the orthogonal Hilbert subspace of $R_{i,x}$. The corresponding extended representation $\hat{\pi}_\chi^i \oplus 0$ of $(C(X), A^+_X)$ then satisfies the following properties [PPV:79]:

1. it is lower semi-continuous, i.e. for any convergent sequence $x_n \to x_0$ in $X$, we have $\bigcap_n I_{x_n} \subseteq I_{x_0}$, where $I_x := \ker(q_x \circ (\hat{\pi}_\chi^i \oplus 0)) \subseteq A_\chi$;  
2. it is exact, i.e. $\cap_{x \in X} I_x = \{0\}$, and  
3. it is trivial, i.e. $\ker(q_x \circ (\hat{\pi}_\chi^i \oplus 0)) = \ker(d_x \circ (\hat{\pi}_\chi^i \oplus 0))$.

First note that (3) is automatically satisfied since $\hat{\pi}_\chi^i$ is fibrewise ample. To check (1), just notice that $$d_x \circ (\pi_\chi^i(a) \oplus 0) = (\pi_{\chi, i,x}^i(a) \oplus 0)(a),$$ and that each $\pi_{\chi, i,x}^i$ is ample here. A similar but easier argument can be used to prove (2).

The $(C(X))$-representation $\hat{\pi}_\chi^i \oplus 0$ is given by the $*$-homomorphism $\pi_\chi^i : A_\chi \to \mathcal{L}(C(X))(H_i \otimes C(X))$. In order to apply the main PPV theorem about trivial $X$-extensions, we check now the same properties for the unique extension of $\pi_\chi^i \oplus 0$ to the $C^*$-algebra unitalization $A_\chi \oplus C$ of $A_\chi$. In terms of $(C(X))$-representations, we thus obtain the extended representation to $(C(X), A_\chi) \oplus C(X)$ given by $$(\hat{\pi}_\chi^i \oplus 0)^* (f + \lambda) := (\hat{\pi}_\chi^i \oplus 0)(f) + \rho(\lambda)$$ for $f \in C(X, A_\chi)$ and $\lambda \in C(X)$.

Here $\lambda$ acts on the Hilbert $C(X)$-module $H_i \otimes C(X)$ by the adjointable operator $\rho(\lambda)$ corresponding to the right module multiplication.

Since $\Gamma$ is infinite all the Hilbert spaces $R_{i,x}$ are infinite dimensional separable Hilbert spaces, we may use the Kasparov stabilisation theorem to replace $(E_i^\chi)^\perp$ by the standard infinite dimensional countably generated Hilbert $C(X)$-module $H_i \otimes C(X)$, so as to be able to apply the PPV theorem, see [Ka:81].

Hence the verification of the three properties for $(\hat{\pi}_\chi^i \oplus 0)^*$ is obvious. If for instance $(f, \lambda) \in \ker(q_x \circ (\hat{\pi}_\chi^i \oplus 0)^*)$, then $\lambda = 0$ since $(E_i^\chi)^\perp$ is infinite-dimensional. Thus, we again get $\pi_{\chi, i,x}^i(f(x)) \in K(R_{i,x}) \iff f(x) = 0$, so $(f, \lambda) \in \ker(d_x \circ (\hat{\pi}_\chi^i \oplus 0)^*)$. Therefore, we get the triviality property. Lower-semicontinuity and exactness are proved similarly and are left as an exercise.
Therefore, the representations \((\hat{\pi}_1 \oplus 0)^\dagger\) are essentially unitarily equivalent by the PPV theorem [PPV79] (Theorem 2.10), i.e. there exists a unitary \(S_\chi \in \mathcal{L}_{C(X)}(H_1 \otimes C(X), H_2 \otimes C(X))\) such that we have in particular for any \(f \in C(X,A_\chi)\):

\[
S_\chi \begin{bmatrix}
\hat{\pi}_2^\chi(f) \\
0 \\
0
\end{bmatrix} 
S_\chi - \begin{bmatrix}
\hat{\pi}_1^\chi(f) \\
0 \\
0
\end{bmatrix} \in \mathcal{K}_{C(X)}(H_1 \otimes C(X)).
\]

Notice that we then have the same relation for \(\hat{\pi}_2^\chi\), i.e. there exists a unitary \(S_\chi \in \mathcal{L}_{C(X)}(H_1 \otimes C(X), H_2 \otimes C(X))\) such that we have in particular for any \(f \in C(X,A_\chi)\):

\[
S_\chi \begin{bmatrix}
\hat{\pi}_2^\chi(f) \\
0 \\
0
\end{bmatrix} 
S_\chi - \begin{bmatrix}
\hat{\pi}_1^\chi(f) \\
0 \\
0
\end{bmatrix} \in \mathcal{K}_{C(X)}(H_1 \otimes C(X)).
\]

Step 2 (First modification): Consider the operator \(s_\chi \in \mathcal{L}(\mathcal{E}_{\chi}^1, \mathcal{E}_{\chi}^2)\) which is the \((1,1)\)-entry in the matrix decomposition of \(S_\chi : \mathcal{E}_{\chi}^1 \oplus (\mathcal{E}_{\chi}^1)^\perp \rightarrow \mathcal{E}_{\chi}^2 \oplus (\mathcal{E}_{\chi}^2)^\perp\). It satisfies the following properties for any \(f \in C(X,A_\chi)\):

1. \(s_\chi^* \hat{\pi}_2^\chi(f) s_\chi - \hat{\pi}_1^\chi(f) \sim 0\),
2. \(\hat{\pi}_2^\chi(f)(s_\chi^* s_\chi - \text{id}) \sim 0\) and \((s_\chi^* s_\chi - \text{id}) \hat{\pi}_1^\chi(f) \sim 0\),
3. \([s_\chi^* s_\chi, \hat{\pi}_2^\chi(f)] \sim 0\) and \([s_\chi^* s_\chi, \hat{\pi}_1^\chi(f)] \sim 0\),
4. \((1 - s_\chi^* s_\chi)\) and \((1 - s_\chi^* s_\chi)\) are positive operators.

Recall that the notation \(A \sim B\) means that the difference \(A - B\) is compact. Therefore we can form the unitary \(\hat{s}_\chi : \mathcal{E}_{\chi}^1 \oplus \mathcal{E}_{\chi}^2 \rightarrow \mathcal{E}_{\chi}^2 \oplus \mathcal{E}_{\chi}^1\) given by the matrix

\[
\hat{s}_\chi := \begin{bmatrix}
s_\chi & (1 - s_\chi^* s_\chi)^{1/2} \\
-(1 - s_\chi^* s_\chi)^{1/2} & s_\chi
\end{bmatrix}
\]

Properties (1), (2) and (3) above, imply that \(\hat{s}_\chi\) intertwines the representations \(\hat{\pi}_1^\chi \oplus \hat{\pi}_1^\chi\) and \(\hat{\pi}_2^\chi \oplus \hat{\pi}_2^\chi\) up to compacts on \(\mathcal{E}_{\chi}^1 \oplus \mathcal{E}_{\chi}^2\) and \(\mathcal{E}_{\chi}^2 \oplus \mathcal{E}_{\chi}^1\), respectively. Extending the unitary \(\hat{s}_\chi\) by zero, we get a partial isometry in \(\mathcal{L}_{C(X)}((H_1 \oplus H_2) \otimes C(X), (H_2 \oplus H_1) \otimes C(X))\), that we still denote by \(\hat{s}_\chi\), given by:

\[
\hat{s}_\chi := \begin{pmatrix}
s_\chi & 0 & \sqrt{1 - s_\chi^* s_\chi} \\
0 & 0 & 0 \\
-\sqrt{1 - s_\chi^* s_\chi} & s_\chi & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

where we have written

\[
(H_1 \oplus H_2) \otimes C(X) = [\mathcal{E}_{\chi}^1 \oplus (\mathcal{E}_{\chi}^1)^\perp] \oplus [\mathcal{E}_{\chi}^2 \oplus (\mathcal{E}_{\chi}^2)^\perp],
\]

and similarly for \((H_2 \oplus H_1) \otimes C(X)\).

Step 3 (Second modification): For \((x,g) \in G\), we denote by \(V_{(x,g)}^i\) the unitary implementing the \(G\)-action on \(H_i \otimes C(X)\). Then we define an operator \(\hat{S} \in \mathcal{L}_{C(X)}((H_1 \oplus H_2) \otimes C(X), \mathcal{E}_{\chi}^2 \otimes C(X))\) by setting the following pointwise formula:

\[
\hat{S}_g := \left(V_{(x,g)} \oplus V_{(x,g)}^i\right) \hat{s}_\chi \left(\hat{\pi}_1(x^{1/2}) \oplus \hat{\pi}_2(x^{1/2})\right) \left(V_{(x,g)}^i \oplus V_{(x,g)}\right)^{-1}.
\]

The operator \(\hat{S}\) then satisfies the allowed properties in the statement Lemma 2.3 as we prove it in Lemma 2.4 below. Therefore, the proof of Lemma 2.3 is now complete. \(\square\)

Properties of finite propagation operators as well as the notion of propagation index, are expanded in Appendix A, see in particular Definition A.1.

Lemma 2.4. The operator \(\hat{S}\) satisfies the following properties:

1. \(\hat{S}\) is an isometry.
2. \(\hat{S}\) intertwines \(\hat{\pi}_1 \oplus 0\) and \((\text{id}_{\mathcal{E}_{\chi}^i} \oplus \hat{\pi}_2) \oplus 0\) up to compacts.
3. The support of \(\hat{S}\) is contained in the closure of \(\mathcal{A}_0 = \bigcup_{g \in \Gamma} gW_\chi \times gW_\chi\) and hence has propagation index \(\leq 2\). In particular, if \(Z/\Gamma\) is compact then \(\hat{S}\) has finite propagation (bounded above by \(\text{diam}_Z(V_\chi)\)).
Proof. (1) Since \( \hat{s}_\chi \) is an isometry in restriction to the range of \( \hat{\pi}_1(\chi^{1/2}) \oplus \hat{\pi}_2(\chi^{1/2}) \), a straightforward verification using the relation \( \sum_{g \in \Gamma} g^* \chi = 1_Z \) shows that \( \hat{S} \) is an isometry.

(2) It suffices to check this condition for elements \( f \) in \( C(X, A_c) \) where \( A_c := C_c(Z)A \subset A \). Then, using the previously listed properties of \( \hat{s}_\chi \), we have:

\[
[S^* (\id_{L^2} \otimes \hat{\pi}_2(f) \oplus 0) \hat{S}]_ξ = \sum_{g \in \Gamma} (V^1_{(x,g)} \oplus V^2_{(x,g)})(\pi_1(\chi^{1/2}) \oplus \pi_2(\chi^{1/2}))\hat{s}_\chi^*(\hat{\pi}_2(f) \oplus 0)
\]

\[
\sim \sum_{g \in \Gamma} (V^1_{(x,g)} \oplus V^2_{(x,g)})(\pi_1(\chi^{1/2}) \oplus \pi_2(\chi^{1/2}))\hat{\pi}_2(g^* f) \oplus 0)\hat{s}_\chi(\hat{\pi}_2(\chi^{1/2}) \oplus 0)\hat{s}_\chi(\hat{\pi}_2(\chi^{1/2}) \oplus 0)
\]

The last equivalence is a consequence of the fact that \( \hat{s}_\chi \) commutes up to compacts with \( \pi_1(\chi^{1/2}) \oplus \pi_2(\chi^{1/2}) \).

Recall that \( \chi^{1/2} \) belongs to \( C_0(\chi) \) and hence to \( A_\chi \). Therefore we deduce

\[
[S^* (\id_{L^2} \otimes \hat{\pi}_2(f) \oplus 0) \hat{S}]_ξ 
\sim \sum_{g \in \Gamma} (V^1_{(x,g)} \oplus V^2_{(x,g)})(\pi_1(\chi^{1/2}) \oplus \pi_2(\chi^{1/2}))\hat{\pi}_2(g^* f) \oplus 0)(\pi_1(\chi^{1/2}) \oplus \pi_2(\chi^{1/2}))((\hat{\pi}_2(\chi^{1/2}) \oplus 0)\hat{s}_\chi((\hat{\pi}_2(\chi^{1/2}) \oplus 0)\hat{s}_\chi((\hat{\pi}_2(\chi^{1/2}) \oplus 0)\hat{s}_\chi
\]

The above computation is legal because the number of elements \( g \in \Gamma \) such that

\[
\text{Supp}(g^* f) \cap (X \times \text{Supp}(\chi)) \neq \emptyset
\]

is finite, due to the properness of the \( \Gamma \)-action and the fact that \( \chi \) is a cut-off function. Indeed, we know from the very definition of \( \chi \) that for any compact subspace \( K \) of \( Z \), the subset \( \{ g \in \Gamma | \text{Supp}(\chi) \cap gK \neq \emptyset \} \) is finite, see for instance [11,99].

(3) Assume now that \( W_1 \) and \( W_2 \) are two open subspaces of \( Z \) such that \( W_1 \times W_2 \) does not intersect any subspace of \( Z^2 \) of the form \( g\text{W}_1 \times g\text{W}_\chi \), where \( g \) runs over \( \Gamma \), then for \( a_1 \in C_0(W_1)A \), we can compute

\[
[(\id_{L^2} \oplus \pi_2(a_2) \oplus 0)\hat{S}(\pi_1(a_1) \oplus 0)]_g 
= (\pi_2(a_2) \oplus 0)(V^1_{(x,g)} \oplus V^2_{(x,g)})\hat{s}_\chi(\pi_1(\chi^{1/2}) \oplus \pi_2(\chi^{1/2}))(V^1_{(x,g)} \oplus V^2_{(x,g)})^{-1}(\pi_1(a_1) \oplus 0)
\]

\[
= (\pi_2(a_2) \oplus 0)(\pi_1(\chi^{1/2}) \oplus \pi_2(\chi^{1/2}))(V^1_{(x,g)} \oplus V^2_{(x,g)})^{-1}(\pi_1(a_1) \oplus 0)
\]

Therefore, we see that if for a given \( g \in \Gamma \), we have \( \chi^{1/2}g^{-1}a_1 \) is non-zero then \( \text{Supp}(a_1) \cap g\text{Supp}(\chi) \neq \emptyset \).

But then by hypothesis we know that since \( W_1 \) and \( W_2 \) are open we also have

\[
W_1 \times W_2 \cap g\text{Supp}(\chi) \times g\text{Supp}(\chi) = \emptyset,
\]
and hence necessarily \( \text{Supp}(a_2) \cap g \text{Supp}(\chi) = \emptyset \), say that \( \text{Supp}(g^{-1} a_2) \cap \text{Supp}(\chi) = \emptyset \). This in turn implies that
\[
(\pi_2(g^{-1} a_2) \oplus 0) \delta_\chi = 0
\]
since the range of \( \delta_\chi \) is contained in \( E^2_\chi \oplus E^1_\chi \).

Therefore we conclude that the operator \((\text{id}_{\ell^2 \Gamma} \otimes \pi_2(a_2) \oplus 0)\hat{S}(\pi_1(a_1) \oplus 0)\) is trivial.

If we assume that \( Z/\Gamma \) is compact and that \( Z \) is a metric-proper space with the above properties, then setting \( \kappa := \text{diam}_Z(\text{Supp}(\chi)) \) which is now a finite positive number, we can deduce by the same calculation that whenever \( a_1, a_2 \in C_c(Z)A \) are such that \( d(\text{Supp}(a_1), \text{Supp}(a_2)) > \kappa \), one has by the \( \Gamma \)-invariance of the distance \( d \) the same relation
\[
(\text{id}_{\ell^2 \Gamma} \otimes \pi_2(a_2) \oplus 0)\hat{S}(\pi_1(a_1) \oplus 0) = 0.
\]

\[\square\]

**Corollary 2.5.** There exists an isometry \( S \in \mathcal{L}_{C(X)}(\ell^2 \Gamma^\infty \otimes (H_1 \oplus H_2) \otimes C(X), \ell^2 \Gamma^\infty \otimes (H_2 \oplus H_1) \otimes C(X)) \) such that
\[
S^* (\hat{\pi}^\infty_2 (f) \oplus 0) S - (\hat{\pi}^\infty_1 (f) \oplus 0) \in K_{C(X)} \left( [\ell^2 \Gamma^\infty \otimes (H_1 \oplus H_2)] \otimes C(X) \right).
\]

Moreover, we can ensure that the operator \( S \) is localized with support contained in the closure of \( \mathcal{A}_0 = \bigcup_{g \in \Gamma} gW_\chi \times gW_\chi \) and hence with propagation index \( \leq 2 \). In particular, if \( Z/\Gamma \) is compact then \( S \) has finite propagation (bounded above by \( \text{diam}(V_\chi) \)).

**Proof.** From Lemma 2.23 we deduce an isometry \( \hat{S} \in \mathcal{L}_{C(X)}(\ell^2 \Gamma^\infty \otimes (H_1 \oplus H_2) \otimes C(X), \ell^2 \Gamma \otimes \ell^2 \Gamma^\infty \otimes (H_2 \oplus H_1) \otimes C(X)) \), such that \( \hat{S} \) intertwines the representations \( \hat{\pi}^\infty_1 \oplus 0 \) and \( (\text{id}_{\ell^2 \Gamma} \otimes \hat{\pi}^\infty_2) \oplus 0 \) up to compacts. Consider a unitary \( u_\infty : \ell^2 \Gamma \otimes \ell^2 \Gamma^\infty \to \ell^2 \Gamma^\infty \). The isometry \( S_1 := (u_\infty \otimes \text{id}_{(H_2 \oplus H_1) \otimes C(X)}) \circ \hat{S} \) then intertwines \( \hat{\pi}^\infty_1 \oplus 0 \) and \( \hat{\pi}^\infty_2 \oplus 0 \), up to compacts and still has the same support as \( \hat{S} \). In particular, it has uniform finite propagation when \( Z/\Gamma \) is compact. \( \square \)

We are now ready to prove Theorem 2.2.

**Proof.** (of Theorem 2.2) Replacing, in the statement of Corollary 2.5, \( H_i \) by \( \ell^2 \mathbb{N} \otimes H_i \) and \( \hat{\pi}_i \) by \( \hat{\pi}_i^\infty = \text{id}_{\ell^2 \mathbb{N}} \otimes \hat{\pi}_i \) for \( i = 1, 2 \), we obtain an isometry with the prescribed support condition (finite-propagation when \( Z/\Gamma \) is compact and \( Z \) is metric-proper)
\[
S_0 \in \mathcal{L}_{C(X)}( (\ell^2 \Gamma^\infty)^\infty \otimes (H_1 \oplus H_2) \otimes C(X), (\ell^{2 \Gamma^\infty})^\infty \otimes (H_2 \oplus H_1) \otimes C(X) )
\]
such that for any \( f \in C(X,A) \):
\[
S_0^* (\hat{\pi}^\infty_2 (f) \oplus 0) S_0 - (\hat{\pi}^\infty_1 (f) \oplus 0) \in K_{C(X)} \left( (\ell^{2 \Gamma^\infty})^\infty \otimes (H_1 \oplus H_2) \otimes C(X) \right).
\]

The support of \( S_0 \) is more precisely contained in \( \bigcup_{g \in \Gamma} \text{Supp}(g\chi) \times \text{Supp}(g\chi) \) and hence \( S_0 \) has finite propagation in the metric and cocompact case. Indeed, in this case and since the distance is \( \Gamma \)-invariant, the propagation is \( \leq \) the diameter of \( \text{Supp}(\chi) \). Let \( r_\infty : \ell^2 \mathbb{N} \otimes \ell^2 \mathbb{N} \to \ell^2 \mathbb{N} \) be a unitary. Composing \( S_0 \) with \( r_\infty \otimes \text{id}_{\ell^2 \Gamma \otimes (H_1 \oplus H_2) \otimes C(X)} \), we get an isometry
\[
S_1 := (r_\infty \otimes \text{id}_{\ell^2 \Gamma \otimes (H_1 \oplus H_2) \otimes C(X)}) \circ S_0 \in \mathcal{L}_{C(X)} \left( (\ell^2 \Gamma^\infty)^\infty \otimes (H_1 \oplus H_2) \otimes C(X), (\ell^{2 \Gamma^\infty})^\infty \otimes (H_2 \oplus H_1) \otimes C(X) \right)
\]
which satisfies
\[
S_1^* (\hat{\pi}^\infty_2 (f) \oplus 0) S_1 - (\hat{\pi}^\infty_1 (f) \oplus 0) \in K_{C(X)} \left( (\ell^{2 \Gamma^\infty})^\infty \otimes (H_1 \oplus H_2) \otimes C(X) \right)
\]
and has the same support.

Consider the operator \( R_1 : \ell^2 \mathbb{N} \otimes \ell^2 \Gamma^\infty \otimes (H_1 \oplus H_2) \to \ell^2 \mathbb{N} \otimes \ell^2 \Gamma^\infty \otimes (H_1 \oplus H_2) \) defined by the following formula:
\[
R_1(h_1 \oplus h_2 \oplus \cdots) = 0 \oplus h_1 \oplus h_2 \oplus \cdots
\]

Then \( R_1 \) induces a \( C(X) \)-linear isometry on \( \ell^2 \mathbb{N} \otimes \ell^2 \Gamma^\infty \otimes (H_1 \oplus H_2) \otimes C(X) \). Consider also the operator \( R_2 : \ell^2 \mathbb{N} \otimes \ell^2 \Gamma^\infty \otimes (H_1 \oplus H_2) \to \ell^2 \Gamma^\infty \otimes (H_1 \oplus H_2) \) defined by the formula:
\[
R_2(h_1 \oplus h_2 \oplus \cdots) = h_1
\]
Then $R_2$ induces a $C(X)$-adjointable co-isometry

$$R_2 \in L_{C(X)} \left( (\ell^2\Gamma)^\infty \otimes (H_1 \oplus H_2) \otimes C(X), \ell^2\Gamma^\infty \otimes (H_1 \oplus H_2) \otimes C(X) \right).$$

Notice that we have the convenient relations

$$R_2R_1 = 0 \text{ and } R_1R_1^* + R_2^*R_2 = \text{id}_{\ell^2\mathbb{N} \otimes \ell^2\Gamma^\infty \otimes (H_1 \oplus H_2) \otimes C(X)}.$$

We are now in position to define the unitary

$$S \in L_{C(X)} \left( (\ell^2\Gamma)^\infty \otimes (H_2 \oplus H_1) \otimes C(X), \ell^2\Gamma^\infty \otimes ((H_2 \oplus H_1) \oplus (H_1 \oplus H_2)) \otimes C(X) \right)$$

by using the following formula:

$$S := \left( \begin{array}{cc}
I - S_1S_1^* + S_1R_1^*S_1^* & R_2S_1^* \\
R_2^*S_1 & 0
\end{array} \right).$$

It is a straightforward computation to show that $S$ is a unitary and that it intertwines $\hat{\pi}_2^\infty \oplus 0$ and $\hat{\pi}_1^\infty \oplus 0 \oplus \hat{\pi}_1^\infty \oplus 0$ up to compacts. The operator $R_1$ commutes with the representation $\text{id}_{\ell^2\mathbb{N} \otimes \hat{\pi}_1^\infty} \oplus 0$, and $R_2$ intertwines (exactly) the representations $\hat{\pi}_2^\infty \oplus 0$ and $(\text{id}_{\ell^2\mathbb{N}} \otimes \hat{\pi}_2^\infty) \oplus 0$, and therefore have support contained in the diagonal of $Z^2$. Whence, the operator $S$ is localized by composition with the propagation index which is $\leq 7$. We refer the reader again to Appendix A for the properties of localized operators. Again in the cocompact case with the $\Gamma$-invariant distance on $Z$, we see that the operator $S$ has finite propagation which is $\leq 7$. As a consequence in the cocompact case with the $\Gamma$-invariant distance, we conclude again by an easy verification that the unitary $U$ has finite propagation as desired.

Let us now take into account the action of our discrete countable group $\Gamma$ by homeomorphisms on $X$. Recall that $A$ is a proper $\Gamma$-algebra over $Z$ and that $C_0(Z)$ maps inside the center of $A$ itself. We denote as before by $G$ the action groupoid $X \rtimes \Gamma$ or its space of arrows, since no confusion can occur. A specific unitary representation of $\Gamma$ is the (right) regular representation $\rho$ in the Hilbert space $\ell^2\Gamma$, which can be tensored by the identity of $\ell^2\mathbb{N}$ to get the unitary representation $\rho^\infty$ of $\Gamma$ in $\ell^2\Gamma^\infty = \ell^2\Gamma \otimes \ell^2\mathbb{N}$. Recall that given the proper $\Gamma$-space, we always choose a uniform cutoff function $\chi \in C(Z)$, and that the first (and also the second) projection $\Gamma^{(1)} \to \Gamma$ is proper. We are now in position to state Theorem 2.1 that we restate using the groupoid language, so as to fit with possible generalizations, as follows:

**Theorem 2.6.** Assume that the action groupoid $G = X \rtimes \Gamma$ acts properly on the $G$-space $Y = X \times Z$, meaning here that $\Gamma$ acts properly on $Z$ with a fixed uniform cutoff function $\chi \in C(Z)$. Let $\hat{\pi}_1$ and $\hat{\pi}_2$ be two fiberwise ample $G$-equivariant representations of $C(X,A)$ in the Hilbert $G$-modules $H_1 \otimes C(X)$ and $H_2 \otimes C(X)$ respectively. Then, identifying each $\hat{\pi}_i$ with the trivially extended representation $\left( \begin{array}{cc}
\hat{\pi}_i & 0 \\
0 & 0
\end{array} \right)$ that is further tensored by the identity of $\ell^2\Gamma^\infty$, there exists a $G$-invariant unitary operator

$$W \in L_{C(X)} \left( ([\ell^2\Gamma^\infty \otimes (H_1 \oplus H_2)] \otimes C(X), [\ell^2\Gamma^\infty \otimes (H_2 \oplus H_1)] \otimes C(X)) \right),$$

such that for any $\varphi \in C(X,A)$

$$W^*\hat{\pi}_2(\varphi)W - \hat{\pi}_1(\varphi) \in K_{C(X)} \left( ([\ell^2\Gamma^\infty \otimes (H_1 \oplus H_2)] \otimes C(X)) \right).$$

Moreover, we can ensure that the operator $W$ is localized with the propagation index $\leq 7$. In particular, if $Z/\Gamma$ is compact with the previous metric assumption on $Z$, then we can ensure that $W$ has finite propagation.
Proof. Since the extended representations (of the unitilization $C(X, A^+)$) are fiberwise ample (say homogeneous in the terminology used in \cite{PPV79}), by “forgetting” the right regular $\Gamma$-action on $\ell^2\Gamma^\infty$, from Theorem \ref{thm:main}, we deduce again the existence of a unitary that we rather denote in this proof by $S$ ($U$ will denote below another family of isometries) with support within $\mathcal{A}_\Gamma$ (so with finite propagation when $Z/\Gamma$ is compact and $Z$ is metric-proper):

$$S \in \mathcal{L}_{C(X)} ([\ell^2\Gamma^\infty \otimes (H_1 \oplus H_2)] \otimes C(X), [\ell^2\Gamma^\infty \otimes (H_2 \oplus H_1)] \otimes C(X))$$

such that for any $f \in C(X, A^+)$,

$$S^* \tilde{\pi}_2(f) S - \tilde{\pi}_1(f) \in K_{C(X)} ([\ell^2\Gamma^\infty \otimes (H_1 \oplus H_2)] \otimes C(X)).$$

In particular this property holds for the restrictions of $\tilde{\pi}_1$ and $\tilde{\pi}_2$ to $C(X, A)$.

The unitary $U$ obtained in this way is of course a priori not $\Gamma$-invariant. To remedy this, we shall use a classical trick which allows to “average”. Using Fell’s trick, one can construct a family of operators $(U_g)_{g \in \Gamma}$ acting on $\ell^2(\Gamma)^\infty$, such that (see for instance \cite{Gwy16} or \cite{BR22}):

- for $g, g' \in \Gamma$, $U_g U_{g'} = \delta_{g,g'} \text{id}_{\ell^2(\Gamma)^\infty}$, in particular each $U_g$ is an isometry;
- $\sum_{g \in \Gamma} U_g U_g^* = \text{id}_{\ell^2(\Gamma)^\infty}$; and
- ($\Gamma$-equivariance) $U_g^* U_g = \rho_g^\infty$ $\rho_{g^{-1}}^\infty$ for any $(g, g') \in \Gamma^2$.

Here of course $\rho$ is the right regular representation of $\Gamma$. Recall the cutoff function $\chi \in C(Z)$ defined using the properness of the $\Gamma$-action on $Z$ and which is compactly supported when $Z/\Gamma$ is assumed compact. We proceed now to define the allowed field $W_x : (H_1 \oplus H_2) \otimes \ell^2(\Gamma)^\infty \to (H_2 \oplus H_1) \otimes \ell^2(\Gamma)^\infty$ or equivalently the corresponding operator $W$ obtained by the averaging trick.

Consider the dense submodule $E'_1$ of $[\ell^2\Gamma^\infty \otimes (H_1 \oplus H_2)] \otimes C(X)$ which is given by

$$E'_1 := (\tilde{\pi}_1 \oplus \tilde{\pi}_2)(C(X, A_c)) ([\ell^2\Gamma^\infty \otimes (H_1 \oplus H_2)] \otimes C(X)).$$

In this notation, $A_c = C_c(Z)A$ as before, and $\tilde{\pi}_i$ is the original representation of $C(X, A)$ on $C(X) \otimes H_i$ that we have tensored with the identity in $\ell^2\Gamma^\infty$. We similarly define $E'_2$.

Notice that, $\tilde{\pi}_i$ also denotes the extended representation of $C(X, A)$ in $[\ell^2\Gamma^\infty \otimes (H_1 \oplus H_2)] \otimes C(X)$ (resp. $[\ell^2\Gamma^\infty \otimes (H_2 \oplus H_1)] \otimes C(X)$) obtained as $\tilde{\pi}_i \otimes 0$ for $i = 1, 2$, respectively. On the other hand, an operator

$$T \in \mathcal{L}_{C(X)} ([\ell^2\Gamma^\infty \otimes (H_1 \oplus H_2)] \otimes C(X), [\ell^2\Gamma^\infty \otimes (H_2 \oplus H_1)] \otimes C(X))$$

is $G$-invariant if for any $g \in \Gamma$, we have for $(x, g) \in X \times \Gamma$:

$$T_x = (gT)_x := (V_{(x, g)} V_{(x, g)}^1) T_{xg} (V_{(x, g)}^1 V_{(x, g)}^2)^{-1}.$$

Recall that $V^i$ denotes the extensions of the $X \times \Gamma$-actions on $H_i \otimes C(X)$ by tensoring with the right regular representation of $\Gamma$, $\rho^\infty$ on $\ell^2\Gamma^\infty$. So, choosing a cutoff function $\chi$ as before with the extra property that the first projection $\Gamma\chi^{(1)} \to \Gamma$ is proper, we now replace $S$ by the (well defined) average operator

$$W = \sum_{g \in \Gamma} (\text{id}_{H_2 \oplus H_1} \otimes U_g) \circ g (S\pi_1(\sqrt{\chi})).$$

Here $(U_g)_{g \in \Gamma}$ is the family of isometric operators on the Hilbert space $\ell^2(\Gamma)^\infty$ defined above. For $e \in E'_1$, we thus have defined

$$W_x(e_x) := \sum_{g \in \Gamma} (\text{id}_{H_2 \oplus H_1} \otimes U_g) \left( (V^2 \oplus V^1)_{(x, g)} \right) S_{xg} (V^2 \oplus V^1)_{(x, g^{-1})} \pi_{1,x}(g' \sqrt{\chi})(e_x)$$

The sum defining $W_x$ is then finite since for any $\varphi \in C_c(Z)$ that is viewed in $A$, we have

$$(\pi_{1,x} \otimes 0)(g' \sqrt{\chi})(\pi_{1,x} \otimes \pi_{2,x})(\varphi) = (\pi_{1,x}(\varphi \sqrt{\chi^2}) \otimes 0),$$

and the number of $g \in \Gamma$ such that $\varphi \sqrt{\chi^2} \neq 0$ is finite by the properness of the $\Gamma$-action on $Z$. Hence, $W(e)$ is well defined on the elements $e \in E'_1$. Moreover, an easy inspection, using the properties of the family $(U_g)_{g \in \Gamma}$, shows that the relation $W_x W_x = \text{id}$ holds on $E'_1$. 


This shows that $W_x$ automatically extends to an isometry between the corresponding Hilbert spaces that we still denote $W_x$. Moreover, when $e = [\hat{\pi}_1(f) \oplus \hat{\pi}_2(f)] e_1$ with $f \in C(X, A_c)$, there is a finite subset $I_e$ of $\Gamma$, which does not depend on the variable $x \in X$, such that,

$$W(e)_x = \sum_{g \in I_e} T_{g,x}(e_x) \quad \forall x \in X.$$ 

Here each of the maps $x \mapsto T_{g,x}(e_x)$ and $x \mapsto T_{g,x}^*(e_x)$ is of course norm-continuous. We thus end up with the adjointable isometry, still denoted $\hat{W}$ since the sum is finite, the operator $A \hat{W}$ is finite independently of the test vector $e$, and therefore makes sense in the uniform operator topology.

Therefore, we may compute using the $G$-equivariance of the representations $\hat{\pi}_i$:

$$\hat{\pi}_2(f)W = \sum_{g \in \Gamma} (\text{id}_{H_2 \oplus H_1} \otimes U_g) \circ g \left( \hat{\pi}_2(g^{-1}f)S\pi_1(\sqrt{\chi}) \right),$$

But an easy inspection of the sum of the representations allows to find in place of the isometry $W$, a unitary which will also satisfy the same two properties. Note (see the notation in the proof of Theorem 2.2) that if the initial isometry $W$ is $G$-invariant, then the operators $S_1$, as well as $R_1$ and $R_2$ appearing in the proof of Theorem 2.2 are all $G$-invariant by construction.

Regarding the first item, notice that we have for any $f \in C(X, A_c)$:

$$\hat{\pi}_2(f)W = \sum_{g \in \Gamma} (\text{id}_{H_2 \oplus H_1} \otimes U_g) \circ g \left( \hat{\pi}_2(g^{-1}f)S\pi_1(\sqrt{\chi}) \right),$$

The sign $\sim$ again refers to equality modulo the compact operators of the corresponding Hilbert modules and since the sum is finite, the operator $\hat{\pi}_2(f)W - W\hat{\pi}_1(f)$ is clearly compact. Now, since $W^* \pi_i$ is an adjointable operator, composing with $W^*$ on the left yields to the conclusion.

Finally, $W$ was indeed born to be $G$-invariant. Since the submodule $\mathcal{E}'_1$ is a $G$-submodule, we may prove $G$-invariance strongly on the vectors of $\mathcal{E}'_1$. Let us denote the $G$-actions on $H_1 \oplus H_2$ by $V^1 := V^1 \oplus V^2$ and similarly by $\hat{V}^2 := V^2 \oplus V^1$ the $G$-action on $H_2 \oplus H_1$. We then compute for any $(x, h) \in G$:

$$W_2\hat{V}^1_{(x,h)} = \sum_{g \in \Gamma} (\text{id} \otimes U_g)(\hat{V}^2_{(x,g)}S_{xg\pi_1} \sqrt{\chi} \hat{V}^1_{(xg,g^{-1})})_{(x,h)}$$

$$= \sum_{g \in \Gamma} (\text{id} \otimes U_g)(\hat{V}^2_{(x,g)}S_{xg\pi_1} \sqrt{\chi} \hat{V}^1_{(xg,g^{-1}h)})_{(x,h)}$$

$$= \sum_{l \in \Gamma} (\text{id} \otimes U_{hl})(\hat{V}^2_{(x,hl)}S_{xhl\pi_1} \sqrt{\chi} \hat{V}^1_{(xhl,l^{-1})})_{(x,h,l)}.$$
with the same support as in Lemma 2.3 such that \( \varepsilon > 0 \). Under the assumptions of Theorem 2.7 and given Lemma 2.8.

We thus start by stating the following Lemma which generalizes Lemma 2.3.

By obtaining the \( \varepsilon \)-invariant unitary operator obtained in that theorem, satisfies in addition the following control condition:

\[ W = W_\varepsilon \in \mathcal{L}_{C(X)} \left( [(H_1 \oplus H_2) \otimes \ell^2 \Gamma \setminus \infty] \otimes C(X), [(H_2 \oplus H_1) \otimes \ell^2 \Gamma \setminus \infty] \otimes C(X) \right) , \]

obtained in that theorem, satisfies in addition the following control condition:

\[ \forall \varphi \in \Sigma, \exists C_\varphi \text{ independent of } \varepsilon \text{ such that } \| W_\varphi^* \tilde{\pi}_2(\varphi) W_\varepsilon - \tilde{\pi}_1(\varphi) \| \leq C_\varphi \varepsilon . \]

Said differently, \( W_\varepsilon \) satisfies the support condition plus the relation

\[ W_\varepsilon^* \tilde{\pi}_2(\varphi) W_\varepsilon - \tilde{\pi}_1(\varphi) \sim 0, \text{ for all } \varphi \in \Sigma. \]

It is worth pointing out that all the previous theorems apply to the case of \( A = C_0(Z, B) \) where \( B \) is any separable unital \( \Gamma \)-algebra. An already interesting application is when \( A = C_0(Z) \) as we shall see in the next section.

### 2.2. The General Case

By using an easy generalization of the PPV work, expanded in Appendix [B] we now state the norm-controlled version of our main Theorem 2.1 say Theorem 1.6 which gives the precise generalization of results in [V:76], compare also with [Ka:80]. So the goal of this section is to explain how to adapt the proof of the previous section so as to construct the sequence of unitaries of Theorem 1.6. For a family \( T = (T_\varepsilon)_{0 < \varepsilon \leq \varepsilon_0} \) of operators \( T_\varepsilon \in C(X, K(H)) \), we shall use the notation \( T \sim 0 \) to denote the fact that the family \( T \) (is composed of compact operators \( T_\varepsilon \) which) have uniform norm at most a constant multiple of \( \varepsilon \) as \( \varepsilon \to 0 \); the constant may depend on the family \( T \). More precisely, such a family \( T := (T_\varepsilon)_{0 < \varepsilon \leq \varepsilon_0} \) in \( C(X, K(H)) \) satisfies \( T \sim 0 \) if there is some constant \( C(T) > 0 \) such that \( \| T_\varepsilon \| \leq C(T) \varepsilon \) for any \( \varepsilon \). Recall that \( A \) is a separable proper \( \Gamma \)-algebra over \( Z \).

Let \( \Sigma \) be a countable dense subset of the separable \( \Gamma^* \)-algebra \( C(X, A) \), which contains 0, is closed under the involution \( a \mapsto a^* \), and globally \( \Gamma \)-invariant. Such \( \Sigma \) always exists since we can for instance take the union of the \( \Gamma \)-orbits of a countable dense self-adjoint subset of \( C(X, A) \).

**Theorem 2.7.** [Controlled version of Theorem 2.1]

Under the assumptions of Theorem 2.7 if we fix \( \varepsilon > 0 \) then we can ensure that the \( \Gamma \)-invariant unitary operator

\[ W = W_\varepsilon \in \mathcal{L}_{C(X)} \left( [(H_1 \oplus H_2) \otimes \ell^2 \Gamma \setminus \infty] \otimes C(X), [(H_2 \oplus H_1) \otimes \ell^2 \Gamma \setminus \infty] \otimes C(X) \right) , \]

obtained in that theorem, satisfies in addition the following control condition:

\[ \forall \varphi \in \Sigma, \exists C_\varphi \text{ independent of } \varepsilon \text{ such that } \| W_\varphi^* \tilde{\pi}_2(\varphi) W_\varepsilon - \tilde{\pi}_1(\varphi) \| \leq C_\varphi \varepsilon . \]

Said differently, \( W_\varepsilon \) satisfies the support condition plus the relation

\[ W_\varepsilon^* \tilde{\pi}_2(\varphi) W_\varepsilon - \tilde{\pi}_1(\varphi) \sim 0, \text{ for all } \varphi \in \Sigma. \]

We only need to explain how to complete the proof given for Theorem 2.1 so that the control is ensured. We thus start by stating the following Lemma which generalizes Lemma 2.3.

**Lemma 2.8.** [Controlled version of Lemma 2.3]

Under the assumptions of Theorem 2.7 and given \( \varepsilon > 0 \), there exists an isometry

\[ \hat{S}_\varepsilon \in \mathcal{L}_{C(X)} \left( [(H_1 \oplus H_2) \otimes C(X), (H_2 \oplus H_1) \otimes \ell^2 \Gamma \otimes C(X) \right) \]

with the same support as in Lemma 2.3 such that

\[ \hat{S}_\varepsilon^* (\tilde{\pi}_2(\varphi) \otimes \id_{\ell^2 \Gamma}) \oplus 0) \hat{S}_\varepsilon - (\tilde{\pi}_1(\varphi) \oplus 0) \sim 0, \quad \forall \varphi \in \Sigma. \]
Proof. We explain the needed complements to the steps in the proof of Lemma 2.3 which exploit the norm control on the residual compact operators as in Theorem B.4 of the appendix. We also forget the support condition for $\tilde{S}$ which is again satisfied as one can check easily. We again fix a cut-off function $\chi \in C(Z)$. Let $V_\chi$ be the non-empty interior of the support of $\chi$.

**Step 1:** We need to apply Corollary B.7. Let $\Lambda$ be a countable dense subset of $C(X)$, containing 0 and 1, which is closed under adjoints. Let $\Sigma^0$ be a countable dense subset of $C(X, C_0(Z, \chi)A^+)$ which contains $\Sigma$, is closed under adjoints and such that (modifying $\Sigma$ if necessary) $\Sigma^0 \cap C(X, A_\chi)$ is dense in $C(X, A_\chi)$. This latter condition can be easily achieved. One can for instance choose an extra countable dense self-adjoint subset $\Sigma^1$ of $C(X, A_\chi)$ and replace $\Sigma^0$ by $\Sigma^0 \cup \Sigma^1$. Alternatively one can join to $\Sigma$ from the beginning the self-adjoint $\Gamma$-subset of all $\Gamma$-orbits of some dense countable self-adjoint subset of $C(X, C_0(V_\chi)A)$, and then easily build up $\Sigma^0$ with the above properties. Consider then the subset $\Sigma^0_\chi := \{ r_n \}_{n \in \mathbb{N}}$ of $C(X, A_\chi) \oplus C(X)$ composed of elements $r_n$ which either belong to $(\Sigma^0 \cap C(X, A_\chi), \Lambda)$ or are of the form $(f \chi^{1/2}, 0)$, where $f \in \Sigma$. Since the sequence

$$r'_n = r_n/(1 + n, |r_n|)$$

is convergent to 0 in $C(X, A_\chi) \oplus C(X)$, the collection $\Sigma_\chi = \{(\chi^{1/2}, 0), (0, 1)\} \cup \{ r'_n \}_{n \in \mathbb{N}}$ is a compact self-adjoint total subset of $C(X, A_\chi) \oplus C(X)$.

Then the image $(\pi_1 \oplus 0)^+ (\Sigma^0_\chi)$ is a self-adjoint compact subset of $C_\chi := (\pi_1 \oplus 0)^+ (C(X, A_\chi) \oplus C(X))$, which is total in $C_\chi$ and contains the identity. Let $B_\chi$ be the algebra generated by $C_\chi$ and $(C(X, K(H_\chi)))$, which defines an $X$-extension algebra for the unital algebra $C(X, A_\chi) \oplus C(X)$. Since $(\pi_1 \oplus 0)^+$ is fibrewise ample, using the same arguments as in the proof of Lemma 2.3 we conclude that $B_\chi$ is a trivial $X$-extension. Using the separability of $(C(X, K(H_\chi)))$, we fix a compact self-adjoint total subset $F_\chi$ of $B_\chi$ which contains $\Sigma_\chi$.

Consider also the trivial $X$-extension obtained analogously by $(\pi_2 \oplus 0)^+$. Then we get, using the notations in the proof of Lemma 2.3 and Corollary B.7 (for the compact subset $F_\chi$), a unitary $S_{\chi, \epsilon} \in L_{C(X)}((H_1 \oplus H_2) \otimes C(X), (H_2 \oplus H_1) \otimes C(X))$ depending on $\epsilon$, such that we have in particular for any $\hat{f} \in \Sigma_\chi$ of the form $\hat{f} = (f, 0)$:

$$S_{\chi, \epsilon}^* \begin{bmatrix} \tilde{\pi}_2^\chi(f) & 0 \\ 0 & 0 \end{bmatrix} S_{\chi, \epsilon} - \begin{bmatrix} \tilde{\pi}_1^\chi(f) & 0 \\ 0 & 0 \end{bmatrix} \tilde{\pi}_1^\chi(f) \tilde{\pi}_1^\chi(f) \sim 0$$

where the Hilbert $C(X)$-modules $H_i \otimes C(X)$ are decomposed as $E_i^\chi \oplus (H_i \otimes C(X))$ for $i = 1, 2$.

**Step 2:** Let $s_{\chi, \epsilon} : E_1^\chi \to E_1^\chi$ be the $(1,1)$-entry in the matrix decomposition of $S_{\chi, \epsilon}$. Then we have for $\hat{f} = (f, 0) \in \Sigma_\chi$:

1. $s_{\chi, \epsilon}^* \tilde{\pi}_2^\chi(f) s_{\chi, \epsilon} - \tilde{\pi}_1^\chi(f) \sim 0$,
2. $\tilde{\pi}_2^\chi(f) (s_{\chi, \epsilon} s_{\chi, \epsilon}^* - \text{id}) \sim 0$ and $(s_{\chi, \epsilon}^* s_{\chi, \epsilon} - \text{id}) \tilde{\pi}_1^\chi(f) \sim 0$,
3. $[s_{\chi, \epsilon} s_{\chi, \epsilon}^* \tilde{\pi}_2^\chi(f)] \sim 0$,
4. $(1 - s_{\chi, \epsilon} s_{\chi, \epsilon}^*)$ and $(1 - s_{\chi, \epsilon}^* s_{\chi, \epsilon})$ are positive operators.

Indeed, for the first item, it suffices to observe that $s_{\chi, \epsilon}^* \tilde{\pi}_2^\chi(f) s_{\chi, \epsilon} - \tilde{\pi}_1^\chi(f)$ is the $(1,1)$-entry in the matrix given by:

$$S_{\chi, \epsilon}^* \begin{bmatrix} \tilde{\pi}_2^\chi(f) & 0 \\ 0 & 0 \end{bmatrix} S_{\chi, \epsilon} - \begin{bmatrix} \tilde{\pi}_1^\chi(f) & 0 \\ 0 & 0 \end{bmatrix}$$

Since the norms of the elements constituting a $2 \times 2$ matrix is bounded above by the norm of the matrix itself, we are done. The proof of the other properties is similar. As in Lemma 2.3 we define the unitary $\dot{s}_{\chi, \epsilon} : E_1^\chi \oplus E_2^\chi \to E_1^\chi \oplus E_1^\chi$ as follows:

$$\dot{s}_{\chi, \epsilon} := \begin{bmatrix} s_{\chi, \epsilon} & 0 \\ 0 & s_{\chi, \epsilon} \end{bmatrix} (1 - s_{\chi, \epsilon}^* s_{\chi, \epsilon})^{1/2}$$
We have the following formula for any \((f,0) \in \Sigma_\chi\):

\[
\hat{s}^*_\chi,\epsilon(\hat{\pi}_2^\chi(f) \oplus 0) \hat{s}_\chi,\epsilon - (\hat{\pi}_1^\chi(f) \oplus 0) = \left[ s^*_\chi,\epsilon \hat{\pi}_2^\chi(f) s_\chi,\epsilon - \hat{\pi}_1^\chi(f) \right] \left( 1 - s_\chi,\epsilon s^*_\chi,\epsilon \right)^{1/2} \left[ s^*_\chi,\epsilon \hat{\pi}_2^\chi(f) \right] \left( 1 - s_\chi,\epsilon s^*_\chi,\epsilon \right)^{1/2}
\]

Note that \(||s_\chi,\epsilon|| \leq 1\) and \(||1 - s_\chi,\epsilon s^*_\chi,\epsilon||^{1/2} \leq 1\). We also have

\[
||1 - s_\chi,\epsilon s^*_\chi,\epsilon||^{1/2} = ||\hat{\pi}_1^\chi(f^*) (1 - s_\chi,\epsilon s^*_\chi,\epsilon) \hat{\pi}_2^\chi(f)|| \leq ||f|| ||1 - s_\chi,\epsilon s^*_\chi,\epsilon||^{1/2} \hat{\pi}_2^\chi(f)||
\]

Thus we get from properties (2) and (3) above that \((1 - s_\chi,\epsilon s^*_\chi,\epsilon)^{1/2} \hat{\pi}_2^\chi(f) \sim 0\). Therefore all the matrix entries \(A_{ij}\) in the above matrix satisfy \(A_{ij} \sim 0\). Thus we get:

\[
\hat{s}^*_\chi,\epsilon(\hat{\pi}_2^\chi(f) \oplus 0) \hat{s}_\chi,\epsilon - (\hat{\pi}_1^\chi(f) \oplus 0) \sim 0
\]

Extending the unitary \(\hat{s}_\chi,\epsilon\) by zero, we get a partial isometry in \(L_{C(\chi)}((H_1 \oplus H_2) \otimes C(\chi), (H_2 \oplus H_1) \otimes C(\chi))\), that we still denote by \(\hat{s}_\chi,\epsilon\).

**Step 3** : For \((x,g) \in G\), we denote by \(V_{(x,g)}\) the unitary implementing the \(G\)-action on \(H_1 \otimes C(\chi)\). Then we define an operator \(\hat{S}_\epsilon \in L_{C(\chi)}((H_1 \oplus H_2) \otimes C(\chi), \ell^2 \Gamma \otimes (H_2 \oplus H_1) \otimes C(\chi))\) by setting the following pointwise formula:

\[
\left[ \hat{S}_{\epsilon,\chi} \right]_g := \left( V_{(x,g)}^2 \oplus V_{(x,g)}^1 \right) \hat{s}_{\chi,\epsilon} \left( \hat{\pi}_1(\chi^{1/2}) \oplus \hat{\pi}_2(\chi^{1/2}) \right) \left( V_{(x,g)}^1 \oplus V_{(x,g)}^2 \right)^{-1}.
\]

The operator \(\hat{S}_\epsilon\) then satisfies the allowed properties in the statement of Lemma 2.3 as we prove below. Let us show that we also have:

\[
\hat{S}^*_\epsilon(\hat{\pi}_2(f) \otimes \text{id}_{\ell^2 \Gamma}) \sim 0 \hat{S}_\epsilon - (\hat{\pi}_1(f) \oplus 0) \sim 0, \quad \forall f \in \Sigma.
\]

Replacing \(f\) by a compactly supported element which is as uniformly close as we please to \(f\), we may assume that \(f\) is itself compactly supported. Denote then by \(\Gamma(\chi, f)\) the set of \(g \in \Gamma\) such that \(\text{Supp}(g^* f) \cap \text{Supp}(\chi) \neq \emptyset\). Due to the properness of the \(\Gamma\)-action, this is a finite set. Consider the compact operators for \(g \in \Gamma, f \in \Sigma\):

\[
K_{\chi,\epsilon} := \hat{s}_\chi,\epsilon \left( \hat{\pi}_1(\chi^{1/2}) \oplus \hat{\pi}_2(\chi^{1/2}) \right) - \left( \hat{\pi}_2(\chi^{1/2}) \oplus \hat{\pi}_1(\chi^{1/2}) \right) \hat{s}_\chi,\epsilon, \quad \text{and}
\]

\[
K_{\epsilon,\chi}(x,f) := s^*_\chi,\epsilon(\hat{\pi}_2^\chi(g^* f) \chi^{1/2} \oplus 0) \hat{s}_\chi,\epsilon - (\hat{\pi}_1^\chi((g^* f) \chi^{1/2} \oplus 0))
\]

Note that we have \(||K_{\epsilon,\chi}(\chi,f)|| \leq C_{\epsilon,\chi}\) for some constant \(C_{\epsilon,\chi} > 0\) independent of \(\epsilon\) and similarly for \(K_{\chi,\epsilon}\) with constant \(C_{\chi,\epsilon}\).

Then from the computation in item (2) in the proof of Lemma 2.3 in the form of Lemma 2.3 we get:

\[
\left[ \hat{S}^*_\epsilon(\hat{\pi}_2(f) \otimes \text{id}_{\ell^2 \Gamma}) \right]_* = \sum_{g \in \Gamma(\chi,f)} \left( V_{(x,g)}^1 \oplus V_{(x,g)}^2 \right) \left( \hat{\pi}_1(\chi^{1/2}) \oplus \hat{\pi}_2(\chi^{1/2}) \right) \left( s^*_\chi,\epsilon(\hat{\pi}_2(g^* f) \oplus 0)K_{\chi} + K_{\epsilon,\chi}(x,f) \right) \left( V_{(x,g)}^1 \oplus V_{(x,g)}^2 \right)^{-1}
\]

Recall that when \(\text{supp}(\varphi) \cap \text{supp}(\chi) = \emptyset\) we have \(\hat{s}_\chi,\epsilon(\hat{\pi}_1^\chi(\varphi) \oplus 0) = 0\) since the range of \(\hat{s}_\chi,\epsilon\) is contained in \(E_{\chi}^2 \oplus E_{\chi}^1\).

Thus one gets for any \(f \in \Sigma\),

\[
||\hat{S}^*_\epsilon(\hat{\pi}_2(f) \otimes \text{id}_{\ell^2 \Gamma}) \hat{S}_\epsilon - (\hat{\pi}_1(f) \otimes 0)|| \leq \left( C_{\chi} + \max_{g \in \Gamma(\chi,f)} C_g \right) ||\Gamma(\chi,f)|| \epsilon.
\]

This proves the claim. \(\square\)

**Corollary 2.9** (Norm-controlled version of Theorem 2.2). There exists a unitary \(\hat{S}_\epsilon \in L_{C(\chi)}(\ell^2 \Gamma^\infty \otimes (H_1 \oplus H_2) \otimes C(\chi), \ell^2 \Gamma^\infty \otimes (H_2 \oplus H_1) \otimes C(\chi))\) as in Theorem 2.2 such that \(\hat{S}^*_\epsilon(\hat{\pi}_2(f) \otimes \text{id}_{\ell^2 \Gamma}) \hat{S}_\epsilon - (\hat{\pi}_1(f) \otimes 0) \sim 0\) for any \(f \in \Sigma\).
Proof. By directly verifying the constructions in Corollary 2.3 and the proof of Theorem 2.2, we see that if the initial isometry is chosen to satisfy the conditions in Lemma 2.8, then all the intertwining isometries and unitaries that appear in the proofs of Corollary 2.5 and Theorem 2.2 must also satisfy the analogous condition on the norms of the residual compact operators. □

We are now ready to prove Theorem 2.7.

Proof of Theorem 2.7. Let \( S_\epsilon \in \mathcal{L}(\mathcal{C}(X))\) be a unitary, obtained from Corollary 2.3 such that
\[
(2.2) \quad S_\epsilon^* \hat{\pi}_2(f) S_\epsilon - \hat{\pi}_1(f) \lesssim 0 \quad \forall f \in \Sigma.
\]
Observe that if \( f' \in \mathcal{C}(X, A_\epsilon) \), with \( A_\epsilon := C_c(Z) A \), satisfies \( \|f - f'\|_\infty \leq \epsilon \), then the analogous relation to (2.2) also holds for \( f' \), and vice versa, if the relation holds for \( f' \) it also holds for \( f \). Also note that since \( \Sigma \) is globally \( \Gamma \)-invariant, the construction of the \( \Gamma \)-invariant unitary \( W_\epsilon \) which intertwines the representations \( \hat{\pi}_1 \) and \( \hat{\pi}_2 \) then follows from Theorem 2.1 using the norm-controlled operator \( S_\epsilon \). The only thing to check is that for all \( f \in \Sigma \),
\[
\hat{\pi}_2(f) W_\epsilon - W_\epsilon \hat{\pi}_1(f) \lesssim 0.
\]
Let \( f \in \Sigma \), we first show that the required relation holds for any \( f' \in \mathcal{C}(X, A_\epsilon) \) such that \( \|f - f'\|_\infty \leq \epsilon \).
First note that by the localization of the support of \( S_\epsilon \), for \( f' \in \mathcal{C}(X, A_\epsilon) \), the sum defining \( \hat{\pi}_2(f') W_\epsilon \) is again finite. Moreover, the number of terms in the finite sum is independent of the operator \( S_\epsilon \) itself, and therefore independent of \( \epsilon \).

We have from the computations in the proof of Theorem 2.1 for \( f' \) as above,
\[
\hat{\pi}_2(f') W - W \hat{\pi}_1(f') = \sum_{g \in \Gamma} (\text{id}_{H_2 \oplus H_1} \otimes U_g) \circ g \left[ (\hat{\pi}_2(g^{-1}f') S - S \hat{\pi}_1((g^{-1})^*f')) \hat{\pi}_1(\sqrt{\chi}) \right]
\]
This sum is again over a finite subset \( \Gamma(\chi, f') \) of \( \Gamma \), due to the assumption of uniform proper action and given the support condition for \( S \). As before, if we let \( K_\epsilon(\chi, f') \) denote the compact operator \( [\hat{\pi}_2((g^{-1})^*f') S_\epsilon - S_\epsilon \hat{\pi}_1((g^{-1})^*f')] \), we have \( K_\epsilon(\chi, f') \lesssim 0 \) for each \( g \in \Gamma \), say with the constant of inequality \( C_\epsilon > 0 \), and hence we have
\[
\|\hat{\pi}_2(f') W - W \hat{\pi}_1(f')\| \leq \left( \max_{g \in \Gamma(\chi, f')} C_\epsilon \right) \| \Gamma(\chi, f') \| \epsilon.
\]
Now as \( \|f - f'\| \leq \epsilon \), we also have:
\[
\hat{\pi}_2(f) W - W \hat{\pi}_1(f) \lesssim 0.
\]
In the cocompact and metric-proper case, notice that \( \rchi(f, f') \) is contained in the set of \( g \in \Gamma \) such that \( \text{Supp}(g^{-1} f') \cap \overline{\text{B}_\epsilon(\text{Supp}(\chi)} \neq \emptyset \), where \( \epsilon \) is the diameter of the cutoff function \( \chi \). This ends the proof. □

We have now completed the proof of Theorem 1.6.

3. Application to Equivariant Paschke Duality

As an application of our equivariant version of the PPV theorem, stated in Theorem 2.4, we now prove the Paschke-Higson duality theorem in this context. We assume in this section that the proper \( \Gamma \)-space \( Z \) is cocompact and endowed as before with the \( \Gamma \)-equivariant metric \( d \) so that closed balls are compact subspaces of \( Z \), said differently the metric space \((Z, d)\) is proper. Recall that \( A \) is a proper \( \Gamma \)-algebra over \( Z \) and that we have assumed that \( C_0(Z) \) maps inside \( A \) itself. Recall from [BR2-21], that associated with the proper metric space \( Z \) and a proper action of the groupoid \( G = X \rtimes \Gamma \) on the \( C^* \)-algebra \( C(X, A) \), we can define the \( G \)-equivariant Roe algebras, which will be denoted as \( D^*_G(X, A; \ell^2 \Gamma \infty \otimes H) \) and \( C^*_G(X, A; \ell^2 \Gamma \infty \otimes H) \) associated with a given ample \( \Gamma \)-equivariant representation of \( A \) in \( H \). The first one is the closure of the space of pseudo-local \( \Gamma \)-invariant operators, while the second one is the ideal in the first one composed of those operators that are moreover locally compact. The quotient algebra is denoted as \( Q^*_G(X; (Z, \ell^2 \Gamma \infty \otimes L^2 Z)) \).

Let us recall the precise definitions which are the immediate generalizations of the ones given in [BR2-20] and [BR2-21] when \( A = C_0(Z) \). Let \((H, U)\) be a unitary Hilbert space representation of \( \Gamma \) together with an ample \( \Gamma \)-equivariant representation \( \pi \) of \( A \). Recall that any adjointable operator \( T \) of \( \mathcal{L}(\mathcal{C}(X))(C(X) \otimes H) \)
is endowed with the transported $G$-action and is self-adjoint. Using Kasparov’s stabilization theorem, we obtain a cycle of the form $[\sigma_1, H \otimes C(X), F_1]$, which is endowed with the transported $G$-action via the Kasparov isomorphism $E \oplus (H \otimes C(X)) \cong H \otimes C(X)$. Note that the summand $H \otimes C(X)$ appearing on the left side of the isomorphism is endowed with its canonical $G$-action induced by the action of $G$ on $C(X, A)$. It is easy to check that the latter cycle lies in the same $KK^G_1$-class as $[\sigma, E, F]$. 

Step 1: Let $[(\sigma, E, F)] \in KK^G_1(A, C(X))$. We may assume as usual that $\sigma$ is non-degenerate and that $F$ is self-adjoint. Using Kasparov’s stabilization theorem, we obtain a cycle of the form $[\sigma_1, H \otimes C(X), F_1]$, which is endowed with the transported $G$-action via the Kasparov isomorphism $E \oplus (H \otimes C(X)) \cong H \otimes C(X)$. Note that the summand $H \otimes C(X)$ appearing on the left side of the isomorphism is endowed with its canonical $G$-action induced by the action of $G$ on $C(X, A)$. It is easy to check that the latter cycle lies in the same $KK^G_1$-class as $[\sigma, E, F]$. 

We thus have the short exact sequence of $C^*$-algebras 

$$0 \rightarrow C^*_\Gamma(X, A; H) \rightarrow D^\Gamma(X, A; H) \rightarrow Q^\Gamma(X, A; H) \rightarrow 0,$$

where we have denoted by $Q^\Gamma(X, A; H)$ the quotient $C^*$-algebra of $D^\Gamma(X, A; H)$ by its two-sided closed involutive ideal $C^*_\Gamma(X, A; H)$. The notation here is ambiguous as we don’t mention the space $Z$ while the notion of propagation with respect to the representation of $A$ depends a priori on the choice of $(Z, d)$. The reason for this simplified notation is that the $K$-groups will not depend on this choice as we shall see below, although the identifications are not natural.

The Paschke-Higson duality theorem identifies the $K$-theory of the quotient algebra $Q^\Gamma(X, A; \ell^2 \Gamma^\infty \otimes H)$ with the $G$-equivariant $KK$-theory of the pair $(C(X, A), C(X))$. For details about the definition of $G$-equivariant $KK$-theory the reader is referred to the fundamental paper of Le Gall [LeGal:99]. Since $X$ is compact here, notice though that the latter group is naturally isomorphic to the $\Gamma$-equivariant $KK$-theory of the pair $(A, C(X))$, see [BR2:21], section 4, for more details.

When $A = C_0(Z)$, we can for instance make use of the representation $C_0(X, Z) \otimes H$, which is induced by multiplication on $\ell^2 \Gamma^\infty \otimes L^2(Z) \otimes C(X)$, where $L^2(Z) = L^2(Z, \mu_Z)$ is defined for a choice of a Borel $\Gamma$-invariant measure $\mu_Z$ on $Z$, which we shall always assume to be fully supported. This representation is fibrewise ample in the sense of Definition 1.5.

We are now in position to prove Theorem 1.7. We need to construct a group isomorphism 

$$P_* : K_*(Q^\Gamma(X, A; \ell^2 \Gamma^\infty \otimes H)) \cong KK^\Gamma_{*+1}(A, C(X)), \quad * = 0, 1.$$ 

We only treat the case $* = 0$. The proof is again a repetition of the proof given in [BR2:21], Theorem 4.1, and adapted to the more general proper $\Gamma$-algebra $A$; we sketch it here only for completeness. We construct a group homomorphism $P'_0 : KK^\Gamma_0(A, C(X)) \rightarrow K_0(Q^\Gamma(X, A; \ell^2 \Gamma^\infty \otimes H))$, using the equivariant PPV Theorem 1.1. The homomorphism $P'$ will then be an inverse to the natural Paschke-Higson map 

$$P_0 : K_0(Q^\Gamma(X, A; \ell^2 \Gamma^\infty \otimes H)) \rightarrow KK^\Gamma_1(A, C(X))$$

We are now in position to prove Theorem 1.7. We need to construct a group isomorphism 

$$P_* : K_*(Q^\Gamma(X, A; \ell^2 \Gamma^\infty \otimes H)) \cong KK^\Gamma_{*+1}(A, C(X)), \quad * = 0, 1.$$ 

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$$P_0 : K_0(Q^\Gamma(X, A; \ell^2 \Gamma^\infty \otimes H)) \rightarrow KK^\Gamma_1(A, C(X))$$
Step 2: Embed $H \otimes C(X)$ equivariantly in $\ell^2 \Gamma \otimes H \otimes C(X)$ via an equivariant isometry $S : H \otimes C(X) \to \ell^2(\Gamma) \otimes H \otimes C(X)$, defined by the following formula which uses the cut-off function $\chi \in C_c(Z)$ as in the previous section:

$$S(e) = \sum_{g \in \Gamma} \delta_{g^{-1}} \otimes \sigma_1(g\sqrt{\chi})(e) \quad \text{for } e \in \mathcal{E},$$

where $\mathcal{E} = \pi(A_e)(H \otimes C(X))$, the $G$-action on $H \otimes C(X)$ is given by the action $V$ from Step 1, while the $G$-action on $\ell^2(\Gamma) \otimes H \otimes C(X)$ is given by the right regular representation of $\Gamma$ on $\ell^2 \Gamma$ tensored by the same action $V$.

Now, $(S\sigma_1(\bullet)S^*, SS^*(\ell^2 \Gamma \otimes H \otimes C(X)), SF_1S^*)$ is equivalent to $(\sigma_1, H \otimes C(X), F_1)$, and after adding a suitable degenerate cycle, we get the cycle $(\sigma_2 := \text{id}_{\ell^2 \Gamma} \otimes \sigma_1, \ell^2 \Gamma \otimes H \otimes C(X), F_2 := (F_1 \oplus \text{id}))$ which is still in the same $KK_1$-class as $(\sigma_1, H \otimes C(X), F_1)$. For details of this construction we refer the reader to [BR21], see Step 2 of the proof of Theorem 4.1 there.

Step 3: Add further degenerate cycles to $[\sigma_2, \ell^2 \Gamma \otimes H \otimes C(X), F_2]$ we may pass to a new $\Gamma$-equivariant Kasparov cycle $(\sigma_3^\infty := \text{id}_{\ell^2 \Gamma} \otimes \sigma_2, \ell^2 \Gamma^\infty \otimes H \otimes C(X), F_3^\infty := \text{diag}(F_2, \text{id}, \text{id}, \ldots))$ which represents the same $KK_1$-class. We further add the degenerate cycle $(0, \ell^2 \Gamma^\infty \otimes H \otimes C(X), 0)$ to $[\sigma_3^\infty, \ell^2 \Gamma^\infty \otimes H \otimes C(X), F_2^\infty]$ with the $\Gamma$-action now taken as the one coming canonically from the $\Gamma$-action on $H \otimes C(X)$ tensored with the right regular representation on the factor $\ell^2 \Gamma$ and extended trivially on $\ell^2 \mathbb{N}$. We obtain in this way a new $\Gamma$-equivariant Kasparov cycle

$$(\sigma_3 := \sigma_3^\infty \oplus 0, \ell^2 \Gamma^\infty \otimes (H \oplus H) \otimes C(X), F_3 := F_2^\infty \oplus 0)$$

still remaining in the same $KK_1$-class.

Step 4: We can now apply Theorem 2.1 to get a $\Gamma$-invariant $C(X)$-adjointable unitary $W$ such that

$$W \sigma_3(f)W^* - (\tilde{\pi}^\infty(f) \oplus 0) \in K_{C(X)}(\ell^2 \Gamma^\infty \otimes (H \oplus H) \otimes C(X)), \quad \text{for all } f \in C(X, A),$$

where $\pi^\infty : C(X, A) \to \mathcal{L}_{C(X)}(\ell^2 \Gamma^\infty \otimes H \otimes C(X))$ is induced by the ample representation $\pi$ of $A$ in the Hilbert module $H \otimes C(X)$ and extended by the identity on $\ell^2 \Gamma^\infty$. By Kasparov’s homological equivalence Lemma (see [BR21, Appendix B]), the cycles

$$(\sigma_3, \ell^2 \Gamma^\infty \otimes (H \oplus H) \otimes C(X), F_3) \quad \text{and} \quad (\pi^\infty \oplus 0, \ell^2 \Gamma^\infty \otimes (H \oplus H) \otimes C(X), F_4),$$

live in the same $KK_1$-class, where $F_4 := WF_3W^*$.

Step 5: Let $F_5$ be the $(1, 1)$-entry in the $2 \times 2$-matrix decomposition of $F_4$, corresponding to the direct sum $\ell^2 \Gamma^\infty \otimes (H \oplus H) \otimes C(X)$. Then the cycle

$$[\tilde{\pi}^\infty \oplus 0, \ell^2 \Gamma^\infty \otimes (H \oplus H) \otimes C(X), F_4]$$

is in the same $KK_1$-class as the cycle $[\tilde{\pi}^\infty, \ell^2 \Gamma^\infty \otimes H \otimes C(X), F_5]$.

Step 6: Replace the operator $F_5$ by a $\Gamma$-invariant finite propagation operator $F_6$ as usual by averaging $\sqrt{\chi}F_5\sqrt{\chi}$. We define the inverse map $\mathcal{P}' : KK_1(C(X, A), C(X)) \to K_0(Q^*_F(X, A; \ell^2 \Gamma^\infty \otimes H))$ by setting

$$\mathcal{P}'(\mathcal{P}(\sigma_3, E, F)) := \left[ \frac{1}{2}(\text{Id}_{\ell^2 \Gamma^\infty \otimes H \otimes C(X)} + F_6) \right]$$

where $q : D^*_F(X, A; \ell^2 \Gamma^\infty \otimes H) \to Q^*_F(X, A; \ell^2 \Gamma^\infty \otimes H)$ is the quotient projection.

The map $\mathcal{P}'$ is well-defined and a bijective group homomorphism, following the same arguments as in the compact case in [BR21], Theorem 4.1. Hence the proof of our Paschke-Higson theorem is now complete.

APPENDIX A. LOCALIZED OPERATORS ON UNIFORMLY PROPER $\Gamma$-SPACES

We prove in this appendix some standard results about supports of our localized operators that are used in some proofs. Let us fix a non-degenerate $*$-representation $\pi : C_0(Z) \to \mathcal{L}(H)$ of the $C^*$-algebra $C_0(Z)$ in the separable Hilbert space $H$, that we extend to $C_b(Z)$ as usual. Recall that $\Gamma$ acts uniformly properly on $Z$ and that $\chi$ is a chosen uniform continuous cutoff function.

We shall use the following notations for an operator $T \in \mathcal{L}(H)$:

$$\text{Supp}(T)_z := \{ z' \in Z | (z', z) \in \text{Supp}(T) \}, \quad \text{Supp}(T)^z := \{ z \in Z | (z', z) \in \text{Supp}(T) \}$$
Notice that if \( W_\chi = \{ \chi \neq 0 \} \) then \( Z = \bigcup_{g \in \Gamma} gW_\chi \). We denote as in Section \( \text{Section} \) for any \( k \geq 1 \):
\[
\Gamma_\chi^{(k)} := \{(g,g') \in \Gamma^2 | \exists (g_i)_{0 \leq i \leq k-1} \text{ such that } g_iW_\chi \cap g_{i+1}W_\chi \neq \emptyset \text{ and } g_0 = g, g_k = g' \}.
\]
For \( k = 0 \), we set \( \Gamma_\chi^{(0)} = \Gamma \) viewed as the diagonal of \( \Gamma^2 \). Notice that \( \Gamma_\chi^{(k)} \subseteq \Gamma_\chi^{(k+1)} \) for any \( k \), and that \( \bigcup_{k \geq 0} \Gamma_\chi^{(k)} = \Gamma^2 \). Recall that the uniform properness of the action means that the first (or the second) projection \( \Gamma^2 \to \Gamma \) becomes proper when restricted to \( \Gamma_\chi^{(1)} \). It is an obvious observation that if the proper \( \Gamma \)-space \( Z \) is cocompact, then the action of \( \Gamma \) on \( Z \) is automatically uniformly proper since the support of \( \chi \) can then be taken compact, so that \( \{ g \in \Gamma | g\text{Supp}(\chi) \cap \text{Supp}(\chi) \neq \emptyset \} \) is finite.

Set \( A_k := \bigcup_{(g,g') \in \Gamma_\chi^{(k)}} gW_\chi \times g'W_\chi \), then it is easy to check using the properties of \( W_\chi \) that for any \( k \geq 0 \) the closure of \( A_k \) is contained \( A_{k+2} \).

**Definition A.1** (Localized operators). An operator \( T \in \mathcal{L}(H) \) is said to have localized support if there exists \( k \geq 0 \) so that \( \text{Supp}(T) \) is contained in (the closure of) some \( A_k \) with \( k \geq 0 \).

The least \( k \) such that the support of \( T \) is contained in \( A_k \) will be called the propagation index of \( T \) (with respect to \( \chi \)). For brevity, we shall call an operator with finite propagation index a localized operator.

For a localized operator \( T \) with propagation index \( k \) and if we denote by \( \Gamma_z \) the finite subset of \( \Gamma \) composed of those \( g \) for which \( z \in gW_\chi \), then for any \( z \in Z \) we have:
\[
\text{Supp}(T)_z \subseteq \bigcup_{g \in \Gamma_z} \bigcup_{g'(g,g') \in \Gamma_\chi^{(k)}} g'W_\chi.
\]

**Proposition A.2.** Assume that \( Z \) is a proper cocompact \( \Gamma \)-space with a \( \Gamma \)-invariant distance \( d \) such that \( Z \) is a metric-proper space. Then localized operators coincide with finite propagation operators.

**Proof.** We can find a cutoff function \( \chi \) which is compactly supported in \( Z \) and hence whose support has finite diameter. An operator \( T \) is localized with propagation index \( \leq k \) if and only if its support is contained in \( A_k \). Hence denoting by \( d_\chi \) the diameter of \( W_\chi \) in \( Z \) which is equal to the diameter of any translate \( gW_\chi \) for \( g \in \Gamma \), we see that for any \( (z,z') \in \text{Supp}(T) \), we have by
\[
d(z,z') \leq kd_\chi.
\]
Hence \( T \) has finite propagation \( \leq kd_\chi \). If conversely \( T \) has finite propagation \( \kappa \). For any \( (z,z') \in \text{Supp}(T) \), we have \( d(z,z') \leq \kappa \) and we also know that there exists \( g_1 \in \Gamma \) such that \( z \in g_1W_\chi \). Since \( Z \) is metric-proper, there exists a finite subset \( \Gamma_z \) of \( \Gamma \) such that the closed ball neighborhood \( B_\chi := \{ z \in \bigcup_{g \in \Gamma_z} gW_\chi \} \) is contained in \( \bigcup_{g \in \Gamma_z} gW_\chi \). Moreover, let us denote by \( k \) the least integer such that for any \( g \in \Gamma_z \), we have \( (e,g) \in \Gamma_\chi^{(k)} \), with \( e \) being the neutral element of \( \Gamma \). To sum up we know that \( z \in g_1W_\chi \) while \( d(z,z') \leq \kappa \) so that \( z' \in \bigcup_{g \in \Gamma_z} g_1W_\chi \), and henceforth
\[
(z,z') \subseteq \bigcup_{g \in \Gamma_z} g_1W_\chi \times g_1W_\chi \subseteq \bigcup_{(g,g') \in \Gamma_\chi^{(k)}} gW_\chi \times g'W_\chi = A_k,
\]
and \( k \) is of course independent of the chosen \( (z,z') \in \text{Supp}(T) \). \( \square \)

**Proposition A.3.** The space of localized operators is unital \( \ast \)-subalgebra of \( \mathcal{L}(H) \). Moreover,

1. the propagation index of the adjoint is equal to the propagation index of the given localized operator.
2. the propagation index of the sum of two localized operators is \( \leq \) to the maximum of the propagation indices.
3. the propagation index of the composition of two localized operators is \( \leq 3 + \) the sum of the propagation indices.

**Proof.** The first item is clear since the one has the relation \( \text{Supp}(T^*) = \sigma(\text{Supp}(T)) \), where \( \sigma : Z \times Z \to Z \times Z \) is the involution \((z,z') \mapsto (z',z)\). The support of the identity operator is the diagonal in \( Z^2 \) which is contained
in $\Gamma^{(0)}_\chi$. Take two localized operators $T$ and $S$ with propagation indices $k$ and $k'$ respectively. The support of the sum $T + S$ is obviously contained in $\text{Supp}(T) \cup \text{Supp}(S)$. Therefore

$$\text{Supp}(T + S) \subset A_k \cup A_{k'} = A_{\max(k,k')}.$$ 

On the other hand, for any $(z,z'')$ such that $\text{Supp}(T)_z \cap \text{Supp}(S)^{z''} \neq \emptyset$, and denoting the propagation index of $T$ by $k$ and the propagation index of $S$ by $k'$, there exists $(g_0, \cdots , g_k) \in \Gamma^k$ and $(g'_0, \cdots , g'_{k'}) \in \Gamma^{k'}$ such that

$$z \in g_0 W_\chi, z'' \in g'_0 W_\chi, g_0 W_\chi \cap g_i W_\chi \neq \emptyset \text{ for } 0 \leq i \leq k - 1,$$

$$g_k W_\chi \cap g'_0 W_\chi \neq \emptyset \text{ and } g'_j W_\chi \cap g'_{j+1} W_\chi \neq \emptyset \text{ for } 0 \leq j \leq k' - 1.$$ 

Hence $(z,z'') \in A_{k+k' + 1}$. Hence, using that the support of $TS$ is contained the closure of $\{(z,z'') \in Z^2 \mid \text{Supp}(T)_z \cap \text{Supp}(S)^{z''} \neq \emptyset\}$ and the inclusion

$$A_{k+k' + 1} \subset A_{k+k' + 3}$$

we deduce that the support of $TS$ is contained in $A_{k+k' + 3}$. $\square$

**Remark A.4.** The analogously defined Roe $C^*$-algebras of locally compact and pseudolocal operators with localized support, can hence be defined in our more general setting of non-cocompact uniformly proper actions.

**Appendix B. The norm-controlled PPV theorem**

In this section we give a norm-controlled version of the PPV theorem [PPV:79][Theorem 2.10]. This is a folklore-type result which nevertheless is not found in the literature to the best of our knowledge.

Let $X$ be a finite dimensional compact metrizable space, $A$ a unital separable $C^*$-algebra and $H$ an infinite-dimensional separable Hilbert space. Denote $UCP(A,M_n)$ the space of unital, completely positive maps from $A$ to $M_n(\mathbb{C})$, equipped with the point-norm topology. We shall denote by $\mathcal{L}(H)_{\ast\ast}$ the algebra of bounded linear operators on $H$ equipped with the strong-$\ast$ topology.

**Proposition B.1** (Proposition 2.8 in [PPV:79]). Consider an exact $X$-extension

$$0 \to C(X,K(H)) \to B \xrightarrow{\varepsilon} A \to 0$$

with ideal symbol $\{I_x\}_{x \in X}$ and $\Psi : X \to UCP(A,M_n)$ be a continuous map such that $\Psi(x)|_{I_x} = 0$ for all $x \in X$. Then, given $\epsilon > 0$, $V \subset H$ and $1 \in W \subset B$ finite-dimensional subspaces, there exists a norm-continuous map $U : X \to \mathcal{L}(\mathbb{C}^n,H)$ such that

$$U^*(x)U(x) = \text{id}_{\mathbb{C}^n}, \quad U(x)(\mathbb{C}^n) \perp V, \quad \forall x \in X$$

and

$$||\Psi(x)(\sigma(b)) - U^*(x)b(x)U(x)|| \leq \epsilon|b| \quad \forall x \in X, b \in W.$$ 

The linear span of $\{U(x)\mathbb{C}^n\}_{x \in X}$ in $H$ is finite-dimensional.

Using Proposition B.1 one gets the following:

**Corollary B.2.** Consider an $X$-extension

$$0 \to C(X,K(H)) \to B \xrightarrow{\varepsilon} A \to 0$$

with exact ideal symbol $\{I_x\}_{x \in X}$ and let $\Psi : X \to UCP(A,M_n)$ be a continuous map such that $\Psi(x)|_{I_x} = 0$ for all $x \in X$. Let $V$ be a finite-dimensional subspace of $H$. Then there exists a sequence of norm-continuous maps $U_k : X \to \mathcal{L}(\mathbb{C}^n,H)$ such that

1. $U_k(x)U_k(x) = \text{id}_{\mathbb{C}^n}, \quad U_k(x)(\mathbb{C}^n) \perp V \quad \forall x \in X, \forall k \in \mathbb{N}$
2. $\lim_{k \to \infty} \sup_{x \in X} ||\Psi(x)(\sigma(b)) - U_k^*(x)b(x)U_k(x)|| = 0 \quad \forall b \in B$, and
3. $\lim_{k \to \infty} \sup_{x \in X} ||U_k^*(x)\eta(x)|| = 0 \quad \forall \eta \in C(X,H)$.
4. The linear span of $\{U_k(x)\mathbb{C}^n\}_{x \in X}$ in $H$ is finite-dimensional for each $k \in \mathbb{N}$. 

Proof. Fix a convergent sequence $F_0 = \{b_i\}_{i \in \mathbb{N}}$ in $B$ containing $1 \in B$, such that $F_0 = F_0^\circ$, $||b_i|| \leq 1$, for all $i$, and the linear span of $F_0$ is dense in $B$. Recall that $B$ is separable here since it is an extension algebra. Set $F := \{b_{\infty}, b^*_\infty\} \cup F_0$ where $b_{\infty}$ is lim $b_i$. Since $F$ is compact, for each $k \in \mathbb{N}$, there exists $N_k \in \mathbb{N}$ and a finite set $F_k := \{b_{i_m}\}_{m=1}^{N_k}$ which includes $1 \in B$, such that for any $b_i \in F$ there exists an index $m \in \{1, 2, \ldots, N_k\}$ such that $||b_{i_m} - b_i|| < 1/3k$.

Let $\{e_n\}$ be an orthonormal basis for $H$. Denote by $P_j$ the linear span of $\{e_1, e_2, \ldots, e_j\}$. For each $k \in \mathbb{N}$, we iteratively apply Proposition B.1 by taking $V_k = \text{span}\{V, F_k\}$ and $W_k = F_k$ and $\epsilon_k = 1/3k$. We thus obtain norm-continuous maps $U_k : X \to \mathcal{L}(C^n, H)$ such that

$$U^*_k(x)U_k(x) = \text{id}_{C^n} \quad U_k(x)(C^n) \perp V_k \quad \forall x \in X, \forall k \in \mathbb{N}$$

which shows that (1) is satisfied. We also have

$$\sup_{x \in X} ||\Psi(x)\sigma(b) - U^*_k(x)b(x)U_k(x)|| \leq 1/3k \quad \forall b \in W_k$$

Now, for any $b \in F$, there exists an element $b' \in W_k$ such that $||b - b'|| < 1/3k$, then we get for any $x \in X$,

$$||\Psi(x)(\sigma(b)) - U^*_k(x)b(x)U_k(x)|| \leq ||\Psi(x)(\sigma(b)) - \Psi(x)(\sigma(b'))|| + ||\Psi(x)(\sigma(b')) - U^*_k(x)b(x)U_k(x)||$$

$$+ ||U^*_k(x)b(x)U_k(x)||$$

$$\leq 1/3k + 1/3k + 1/3k = 1/k$$

where we have used the fact that $||\Psi|| = 1$ (since $A$ is unital) and $U_k(x)U^*_k(x)$ is an orthogonal projection for all $x \in X$, so $||U_k(x)||=1$. Thus (2) is established for all $b \in F$. Since $F$ spans $B$, another density argument then gives the result for all $b \in B$.

To check (3), let $\epsilon > 0$ and note that if $\eta \in P_j$ for some $j$, then $<\eta, U_k(x)U^*_k(x)\eta> = 0$ for all $k > j$, since range of $U_k(x)$ is perpendicular to $P_k$. Now let $\eta = \sum_{i=1}^{\infty} \alpha_i e_i$, choose $N_0$ such that $||\eta - \sum_{i=1}^{N_0} \alpha_i e_i|| < \epsilon$. Then for any $k$, we have

$$||U^*_k(x)\eta|| \leq ||U^*_k(x)(\eta - \sum_{i=1}^{N_0} \alpha_i e_i)|| + ||U^*_k(x)(\sum_{i=1}^{N_0} \alpha_i e_i)||$$

since $\sum_{i=1}^{N_0} \alpha_i e_i \in P_{N_0}$, the second term above is zero for $k > N_0$ for all $x \in X$. Therefore one gets

$$\sup_{x \in X} ||U^*_k(x)\eta|| \leq \epsilon \quad \forall k > N_0, \forall x \in X.$$

which establishes (3) in the case when $\eta \in C(X, H)$ is constant in the $X$-variable. To deal with the general case, let for each $x \in X$, $W_x$ be an open neighbourhood of $x$ such that for any $x' \in W_x$, we have:

$$||\eta(x) - \eta(x')|| \leq \epsilon/2$$

Since $X$ is compact we get a finite collection $\{W_{x_i}\}_{i=1}^{m}$ of such open neighbourhoods with centers $\{x_i\}_{i=1}^{m}$. Choose $N_0$ large enough such that

$$\sup_{x \in X} ||U^*_k(x)\eta(x)|| \leq \epsilon/2$$

for all $k \geq N_0$ and for all $i = 1, 2, \cdots, m$.

Then we have for any $k \geq N_0$ and $x \in W_{x_i}$ for some $i$,

$$||U^*_k(x)\eta(x)|| \leq ||U^*_k(x)(\eta(x) - \eta(x_i))|| + ||U^*_k(x)\eta(x_i)||$$

This proves (3). The last item (4) follows from the last line of Proposition B.1. \qed

**Remark B.3.** In the proof above one can also take any countable approximate unit $\{a_k\}_{k \in \mathbb{N}}$ for $C(X, K(H))$ consisting of increasing sequence of finite-rank operators which are constant in $X$, and take $V_k = \text{span}\{V, P_k\}$

where $P_k$ is the projection onto the range of $a_k$.

Using the above result, we can now give a strengthening of Proposition 2.9 in [PPV79]. Denote by $d_x : C(X, \mathcal{L}(H))_{\alpha_x} \to \mathcal{L}(H)$ the evaluation map. We keep the notations used above.
Theorem B.4. Given a trivial $X$-extension by $A$ with exact lsc ideal symbol $\{I_x\}_{x \in X}$:

$$0 \to C(X,K(H)) \to B_1 \xrightarrow{\pi} A \to 0$$

which is implemented by a unital $*$-homomorphism $\mu_1 : A \to C(X,\mathcal{L}(H)_{sa})$ and another arbitrary $X$-extension with same ideal symbol $\{I_x\}_{x \in X}$, whose extension algebra is $B \subseteq C(X,\mathcal{L}(H)_{sa})$ for some infinite-dimensional separable Hilbert space $H$:

$$0 \to C(X,K(H)) \to B \xrightarrow{\pi} A \to 0$$

Let $F$ be a compact subset of $B$ such that $F = F^*$, $1 \in F$ and the linear span of $F$ is dense in $B$. Given $\epsilon > 0$, there exists an isometry $S \in C(X,\mathcal{L}(H)_{sa})$ such that

(i) $bS - S\mu_1(\sigma(b)) \in C(X,K(H))$ for all $b \in B$.

(ii) $\forall b \in F, \exists C$ independent of $\epsilon$ such that $\|S\mu_1(\sigma(b)) - bS\| \leq C\epsilon$.

Proof. Recall that $B_1$ be the unital $C^*$-algebra generated by the image of $\mu_1$ and $C(X,K(H))$. Let $\{a_k\}_{k=0}^\infty$ be a quasi-compact approximate unit for $(X,K(H))$ consisting of an increasing sequence of constant (in the $X$-variable) finite-rank operators $0 = a_0 \leq a_1 \leq a_2 \cdots$, $\|a_k\| \leq 1$, and

$$\lim_{k} \|a_k l - l\| = 0, \forall l \in C(X,K(H)) \quad \text{and} \quad \lim_{k} \|(a_k, h)\| = 0, \forall h \in B_1.$$ 

where $[x, y]$ denotes the commutator $xy - yx$. Let $F$ be a compact, self-adjoint subset of the unit ball of $B$ whose span is $B$. Passing to a subsequence if necessary, we may assume that

$$\|\mu_1(\sigma(b)), (a_k - a_{k-1})^{1/2}\| \leq \epsilon/2^k \quad \forall b \in F, k \geq 1.$$

Let $Q_k$ be the constant orthogonal projection onto the range of $a_k$ for each $k \geq 1$. Using Corollary B.2 we iteratively define a sequence of compact operators $U_k \in C(X,K(H)), k \in \mathbb{N}$ whose initial projections are the range of $a_k$ and final projections are of uniformly finite rank, and an increasing sequence of finite-rank projections $R_k, k \in \mathbb{N}$ on $H$, converging strongly to the identity, such that we have for all $k \geq 1$:

1. $U_k^*(x)U_k(x) = Q_k$, for all $x \in X$.
2. $\text{Range}(U_k(x)) \subseteq (R_{k+1} - R_k)(H)$, for all $x \in X$.
3. $\text{Range}(U_k(x)) \perp \text{Range}(U_{k'}(x'))$ for all $x, x' \in X$ for all $k' < k$.
4. $\|Q_k \mu_1(\sigma(b))(x)Q_k - U_k^*(x)b(x)U_k(x)\| \leq \epsilon/2^k$, for all $x \in X, b \in F$.
5. $\|U_k^*(x)b(x)U_j(x)\| \leq \epsilon/2^{k+j}$, for all $b \in F, x \in X, j \neq i$.

Some remarks are in order. The first property is clear from the construction in Corollary B.2, the existence of the finite-rank operators $R_k$ in the property (2) also follows from the fact that the $U_k$ themselves are of uniformly finite-rank. The third property can be obtained in the construction of $U_k$ by adding the ranges of all the $U_{k'}$ for $k' < k$ in the choice of the finite-dimensional space $V$ in Corollary B.2. The fourth property is simply obtained by taking the completely positive map $\Psi$ in Corollary B.2 to be $Q_k \mu_1(\bullet)Q_k$. The last property (5) can be obtained from item (3) in Corollary B.2 since the initial space of each $U_j$ for $j < i$ is of uniformly finite-dimension.

Define the operator $S \in C(X,\mathcal{L}(H)_{sa})$ pointwise in the following way:

$$S(x) := \sum_{k=1}^{\infty} U_k(x)(a_k - a_{k-1})^{1/2}$$

Indeed, it suffices to use properties (1), (2), and (3) above to show that $S(x)$ is uniformly convergent in $X$ with respect to the strong-* topology on $\mathcal{L}(H)$. It can also be verified directly that $S^*(x)S(x) = \text{id}_H$, thus $S(x)$ is an isometry, using the fact that $\text{Range}(a_k - a_{k-1})^{1/2} \subseteq \text{Range}(Q_k) = \text{Range}(U_k^*(x)U_k(x))$ for all $x \in X$.

Let $f_k = (a_k - a_{k-1})^{1/2}$. Using the fact that $\mu_1(\sigma(b)) = \sum_{k=1}^{\infty} \mu_1(\sigma(b))f_k^2$, where the series converges in the strict topology, we get:

$$\mu_1(\sigma(b))(x) - \sum_{k=1}^{\infty} f_k \mu_1(\sigma(b))(x)f_k = \sum_{k=1}^{\infty} [\mu_1(\sigma(b))(x), f_k]f_k$$

(B.1)
Thus, by the assumptions on \( f_k = (a_k - a_{k-1})^{1/2} \), we get \( \|\mu_1(\sigma(b))(x) - \sum_{k=1}^{\infty} f_k \mu_1(\sigma(b))(x)f_k\| \leq \epsilon \), for all \( b \in F \).

Therefore, we finally get for all \( b \in F \),

\[
(B.2) \quad \|S^*(x)b(x)S(x) - \mu_1(\sigma(b))(x)\| \leq \|\mu_1(\sigma(b))(x) - \sum_{k=1}^{\infty} f_k \mu_1(\sigma(b))(x)f_k\|
\]

\[
\quad + \sum_{k=1}^{\infty} \|f_k (\mu_1(\sigma(b))(x) - U_k^*(x)b(x)U_k(x)) f_k\| + \sum_{i \neq j} \|U_i^*(x)b(x)U_j(x)\| \leq 3\epsilon
\]

by properties (4) and (5) above, and noticing that \( Q_kf_k = f_kQ_k = f_k \).

On the other hand, we also get \( \mu_1(\sigma(b)) - S^*bS \in C(X, K(H)) \) for all \( b \in F \). Indeed, writing the expression pointwise as follows:

\[
\mu_1(\sigma(b))(x) - S^*(x)b(x)S(x) = \mu_1(\sigma(b))(x) - \sum_{k=1}^{\infty} f_k \mu_1(\sigma(b))(x)f_k \quad (I_1)
\]

\[
\quad + \sum_{k=1}^{\infty} f_k (\mu_1(\sigma(b))(x) - U_k^*(x)b(x)U_k(x)) f_k \quad (I_2)
\]

\[
\quad + \sum_{i \neq j} f_i U_i^*(x)b(x)U_j(x)f_j \quad (I_3)
\]

Using the fact that \( f_k \) and \( U_k(x) \) are uniformly finite-rank for each \( k \), we deduce the compactness of each term in the equation above as follows. The first term \( I_1 \) is compact due to equation \([B.1]\), whose right hand side is compact since it is norm-convergent and each of its partial sum is compact. Similarly, \( I_2 \) and \( I_3 \) are compact because the partial sums for each series are all compact and each series is norm-convergent. Since the norm continuity in the \( x \)-variable of the left-hand side of the above equation is clear from the construction, we conclude that \( \mu_1(\sigma(b)) - S^*bS \in C(X, K(H)) \) for all \( b \in F \).

Using a density argument as before, one can finish the proof of (i) by establishing the desired properties for all \( b \in B \).

Notice also that we have the following relation for any \( b \in B \),

\[
(B.3) \quad (S\mu_1(\sigma(b)) - bS)^* (S\mu_1(\sigma(b)) - bS) = (S^*b^*bS - \mu_1(\sigma(b^*)) + \mu_1(\sigma(b^*)) (\mu_1(\sigma(b)) - S^*bS)\mu_1(\sigma(b))
\]

from which the claim follows.

\[\square\]

**Remark B.5.** set for \( b \in B \), \( K_1(b) := bS - \mu_1(\sigma(b)) \) then the operator \( S \in C(X, \mathcal{L}(H)) \) constructed in the proof above also satisfies

\[
K_1(b)^*S = SK_1(b^*)^* = [b, SS^*].
\]

Hence \([b, SS^*] \sim 0\) for all \( b \in F \).

We may rewrite for \( b \in F \), the relations in Theorem \([B.4]\) using as well the previous remark, as

\[
S^*bS - \mu_1(\sigma(b)) \sim 0, bS - \mu_1(\sigma(b)) \sim 0, [b, SS^*] \sim 0.
\]

**Corollary B.6.** Let \( H_x \) be a separable infinite-dimensional Hilbert space. Given a trivial \( X \)-extension by \( A \) with exact lsc ideal symbol \( \{I_x\}_{x \in X} \):

\[
0 \to C(X, K(H_1)) \to B_1 \xrightarrow{\sigma_1} A \to 0
\]
which is implemented by a unital $*$-homomorphism $\mu_1 : A \to C(X, \mathcal{L}(H_1))$ and another arbitrary $X$-extension with same ideal symbol $\{I_x\}_{x \in X}$, whose extension algebra is $B \subseteq C(X, \mathcal{L}(H))$ for some infinite-dimensional separable Hilbert space $H$:

$$0 \to C(X, \mathcal{K}(H)) \hookrightarrow B \xrightarrow{\mu_1} A \to 0,$$

there exists a sequence of operators $\{S_n\}_{n \in \mathbb{N}}, S_n \in C(X, \mathcal{L}(H_1))$ for all $n \in \mathbb{N}$, such that we have for all $b \in B$:

1. $S_n \mu_1(\sigma(b)) - b S_n \in C(X, \mathcal{K}(H_1))$ for any $n \in \mathbb{N}$,
2. $\lim_{n \to \infty} \|S_n \mu_1(\sigma(b)) - b S_n\| = 0$, and
3. $S_n^* S_n = \text{id}_{H_1} \otimes C(X)$ for all $n \in \mathbb{N}$.

Proof. This is an immediate application of Theorem 3.4 by reducing to the case $H_1 = H$ via a unitary isomorphism $u : H_1 \to H$. \hfill $\square$

**Corollary B.7.** Given a trivial $X$-extension by $A$ with exact lsc ideal symbol $\{I_x\}_{x \in X}$:

$$0 \to C(X, \mathcal{K}(H)) \hookrightarrow B_1 \xrightarrow{\mu} A \to 0$$

which is implemented by a unital $*$-homomorphism $\mu_1 : A \to C(X, \mathcal{L}(H))$, and another trivial $X$-extension with the same ideal symbol $\{I_x\}_{x \in X}$, implemented similarly by $\mu$ and whose extension algebra is $B \subseteq C(X, \mathcal{L}(H))$:

$$0 \to C(X, \mathcal{K}(H)) \hookrightarrow B \xrightarrow{\mu_1} A \to 0,$$

there exists for any $\epsilon > 0$ and any compact subset $F$ in $A$ such that $F = F^*$, $1 \in F$ a unitary operator $S_\epsilon \in C(X, \mathcal{L}(H))$ such that we have

1. $\mu_1(a) - (S_\epsilon)^* \mu(a) S_\epsilon \in C(X, \mathcal{K}(H))$ for any $a \in A$;
2. The family $S = (S_\epsilon)$ satisfies that for any $a \in F$, $\mu_1(a) - (S_\epsilon)^* \mu(a) S_\epsilon \subseteq 0$.

Proof. This is done by the usual PPV trick to pass from isometries to unitaries, as in Theorem 2.10 in [PV:79]. Let us give some details for the convenience of the reader. Starting with any trivial $X$-extension, one obtains a new trivial $X$-extension with same ideal symbol, by amplifying as follows. Consider for instance the trivial $X$-extension given by $\sigma_1$, and set $H' := H \otimes C_\infty$ and the $X$-extension associated with the unital $*$-homomorphism

$$\mu_1' : A \to C(X, \mathcal{L}(H'))$$

given by $\mu_1'(a) := \mu_1(a) \otimes \text{id}_{C_\infty}$.

So, the $X$-extension is $0 \to C(X, \mathcal{K}(H')) \hookrightarrow B_1' \xrightarrow{\sigma_1'} A \to 0$ with $B_1' = \mu_1'(A) + C(X, \mathcal{K}(H'))$. Applying the previous corollary, we deduce the existence of a sequence $(S_{0n})_{n \in \mathbb{N}}$ of isometries in $C(X, \mathcal{L}(H', H'))$ such that

$$S_{0n} \mu_1'(\sigma(b)) - b S_{0n} \in C(X, \mathcal{K}(H', H))$$

and $\lim_{n \to \infty} \|S_{0n} \mu_1'(\sigma(b)) - b S_{0n}\| = 0$.

As in [PV:79] page 71, we then construct out of each isometry $S_{0n}$ a unitary $U_{0n} \in C(X, \mathcal{L}(H, H \oplus H))$ which intertwines the original $X$-extension given by $B$ and its direct sum with the one given by $B_1$. More precisely, we consider the shift isometry $V$ on $H'$ that we view as constant in the $X$-variable, and similarly the projection $P : H' \to H$ onto the first component seen as a constant co-isometry. It is an obvious observation that $PV = 0$, hence also $V^* P^* = 0$, and we can define following again [PV:79] the unitaries $(U_{0n} \in C(X, \mathcal{L}(H, H \oplus H)))_{n \in \mathbb{N}}$ by setting

$$U_{0n} := \left( \text{id} - S_{0n}(S_{0n})^* + S_{0n}V^*(S_{0n})^* \right) P(S_{0n})^*.$$

A straightforward computation using the properties of the family $(S_{0n})_n$, shows that

$$\begin{pmatrix} b & 0 \\ 0 & \mu_1(\sigma(b)) \end{pmatrix} U_{0n} - U_{0n}b \in C(X, \mathcal{K}(H, H \oplus H)), \quad \forall b \in B.$$

and that $\left\| \begin{pmatrix} b & 0 \\ 0 & \mu_1(\sigma(b)) \end{pmatrix} U_{0n} - U_{0n}b \right\| \to 0$. 

Applying the same construction using now that the original $X$-extension given by $B$ is also $X$-trivial, one deduces a sequence $(U_n)_{n \in \mathbb{N}}$ of unitaries in $C(X, \mathcal{L}(H, H \oplus H))$ such that (applying as well the unitary interchanging the two copies of $H$):

$$
\begin{pmatrix}
\mu(\sigma_1(b_1)) & 0 \\
0 & b_1
\end{pmatrix} U_n^1 - U_n^1 b_1 \in C(X, \mathcal{K}(H, H \oplus H)), \quad \forall b_1 \in B_1.
$$

and that $\left\| \begin{pmatrix}
\mu(\sigma_1(b_1)) & 0 \\
0 & b_1
\end{pmatrix} U_n^1 - U_n^1 b_1 \right\| \to 0$. Applying the above properties to $b = \mu(a)$ and $b_1 = \mu_1(a)$ for $a \in A$, we get

$$U_n^0 \mu(a)(U_n^0)^* - U_n^1 \mu_1(a)(U_n^1)^* \in C(X, \mathcal{K}(H)) \quad \text{and} \quad ||U_n^0 \mu(a)(U_n^0)^* - U_n^1 \mu_1(a)(U_n^1)^*|| \to 0.$$

Hence the conclusion using the unitaries $S_n := (U_n^0)^* U_n^1$. It is then easy to check that the argument gives the same estimates over the compact subspace $F$. 

\[\square\]

**References**

[BC:00] P. Baum, A. Connes Geometric $K$-theory for Lie groups and foliations, Enseign. Math. (2) 46 (2000), no. 1-2, 3-42.

[BM:15] M.-T. Benameur and V. Mathai, Spectral sections, twisted rho invariants and positive scalar curvature. J. Noncommut. Geom. 9 (2015), no. 3, 821-850.

[BR:15] M.-T. Benameur and I. Roy, The Higson-Roe exact sequence and $\ell^2$ eta invariants. J. Funct. Anal. 268 (2015), no. 4, 974-1031.

[BDF:73] L. G. Brown, R. G. Douglas and P. A. Fillmore, Unitary equivalence modulo the compact operators and extensions of $C^*$-algebras, Proc. Conf. on Operator Theory, Springer Lecture Notes 345 (1973), 58-128.

[BEK:08] U. Bunke, A. Engel, D. Kasprowski and C. Winges, Equivariant coarse homotopy theory and coarse algebraic $K$-homology. arXiv preprint: [arXiv:1710.04935]

[CC:00] A. Candel and L. Conlon Foliations (Volume 1), Graduate studies in mathematics, American Mathematical Soc., 2000

[CS:84] A. Connes , G. Skandalis, The longitudinal index theorem for foliations. Publ. Res. Inst. Math. Sci. 20 (1984), no. 6, 1139–1183.

[Dix:77] J. Dixmier, Les $C^*$-algèbres et leur representations, 1977.

[GWY:16] E. Guentner, R. Willett and G. Yu, Dynamic asymptotic dimension and controlled operator $K$-theory, arXiv:1609.02093, 2016

[H:70] P. R. Halmos, Ten problems in Hilbert space. Bull. Amer. Math. Soc. 76 (1970), 887-933.

[Hig95] N. Higson, $C^*$-algebra extension theory and duality, Journal of Functional Analysis, 129, 349-363, 1995

[HPS:15] Hanke, B.; Pape, D.; Schick, T. Codimension two index obstructions to positive scalar curvature, Annales de l’Institut Fourier, Volume 65 (2015) no. 6, pp. 2681-2710.

[HR:00] N. Higson and J. Roe, Analytic $K$-homology, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.

[HPR:97] N. Higson, E.K. Pedersen, J. Roe, $C^*$-algebras and controlled topology, K-theory 11 (1997) 209–239.

[HR1:05] N. Higson and J. Roe, Mapping surgery to analysis. I. Analytic signatures. K-Theory 33 (2005), no. 4, 277-299.

[HR2:05] N. Higson and J. Roe, Mapping surgery to analysis. II. Geometric signatures. K-Theory 33 (2005), no. 4, 301-324.

[HRS:05] N. Higson and J. Roe, Mapping surgery to analysis. III. Exact sequences. K-Theory 33 (2005), no. 4, 325-346.

[HR:10] N. Higson and J. Roe, $K$-homology, assembly and rigidity theorems for relative eta invariants. Pure Appl. Math. Q. 6 (2010), no. 2, Special Issue: In honor of Michael Atiyah and Isadore Singer, 555-601.

[Ka:80] G. G. Kasparov, Hilbert $C^*$-modules: theorems of Stinespring and Voiculescu. J. Operator Theory 4 (1980), no. 1, 133-150.

[Ka:81] G. G. Kasparov, Operator $K$-Functor and extension of $C^*$-algebras, Mathematics of the USSR-Izvestiya 16. 513 (reprinted in English IOPScience, 2007)

[Ka:88] G. G. Kasparov, Equivariant $KK$-theory and the Novikov conjecture, Invent. Math. 91, 147-201, 1988
[La:95] E. Lance, Hilbert $C^*$-modules: a toolkit for operator algebraists, Lon. Math. Soc. Lect. Notes Series 210

[LeGall:99] P.-Y. Le Gall, Théorie de Kasparov équivariante et groupoides I, K-theory, 16 (1999), 361-390.

[Osin:16] D. Osin, Acylindrically hyperbolic groups, Trans. Amer. Math. Soc. 368 (2016), no. 2, 851-888.

[P:81] W. L. Paschke, $K$-theory for commutants in the Calkin algebra. Pacific J. Math. 95 (1981), no. 2, 427-434.

[PPV:79] M. Pimsner, S. Popa and D. Voiculescu, Homogeneous $C^*$-extensions of $C(X) \otimes K(H)$, Part I, J. Operator Theory 1 (1979), 55-108.

[PS:13] P. Piazza and T. Schick, Rho-classes, index theory and Stolz’ positive scalar curvature sequence Journal of Topology, 7(4), 965-1004, 2013.

[Roe:16] J. Roe, Positive Curvature, Partial Vanishing Theorems and Coarse Indices, Proceedings of the Edinburgh Mathematical Society, 59(1), 223-233, 2016.

[RS:81] J. Rosenberg and C. Schochet, Comparing functors classifying extensions of $C^*$-algebras, J. Operator Theory, Vol. 5 (1981), pp. 267-282.

[STY:02] G. Skandalis, J.-L. Tu, and G. Yu, The coarse Baum–Connes conjecture and groupoids, Topology 41.4 (2002): 807-834.

[V:83] A. Valette, A remark on the Kasparov groups $\text{Ext}(A,B)$, Pacific J. Math. 109 (1983), no. 1, 247-255.

[Tu:99] J.-L. Tu, La conjecture de Novikov pour les feuilletages hyperboliques, K theory 16: 129-184, 1999

[V:76] D. Voiculescu, A noncommutative Weyl-von Neumann theorem, Rev. Roum. Math. Pures Appl., 21 (1976), 97-113.

[vN:35] J. von Neumann, Charakterisierung des Spektrums eines Integraloperators, Actualités Sci. Indust., no. 229, Hermann, Paris, 1935.

[W:09] H. Weyl, Über beschränkte quadratischen Formen deren Differenz vollstetig ist, Rend. Circ. Mat. Palermo 27 (1909), 373-392.

[XY:14] Z. Xie, G. Yu Positive scalar curvature, higher rho invariants and localization algebras, Advances in Mathematics Volume 262, 2014, Pages 823-866

[Yul:97] G. Yu, The Novikov conjecture for groups with finite asymptotic dimension, Annals of Mathematics 147 (1998): 325-355.

[Yu2:00] G. Yu, The coarse Baum–Connes conjecture for spaces which admit a uniform embedding into Hilbert space, Inventiones Mathematicae 139.1 (2000): 201-240.

[Zeidler:16] R. Zeidler, Positive scalar curvature and product formulas for secondary index invariants, J. Topol., 9(3):687–724, 2016.

[Zen:19] V. F. Zenobi, Adiabatic groupoid and secondary invariants in $K$-theory, Adv. Math. 347 (2019), 940-1001.

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