SPECIAL VALUES OF ANTICYCLOTONIC L-FUNCTIONS MODULO $\lambda$

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Abstract. The purpose of this article is to generalize some results of Vatsal on the special values of Rankin-Selberg L-functions in an anticyclotomic $\mathbb{Z}_p$-extension. Let $g$ be a cuspidal Hilbert modular form of parallel weight $(2, ..., 2)$ and level $N$ over a totally real field $F$, and let $K/F$ be a totally imaginary quadratic extension of relative discriminant $D$. We study the $l$-adic valuation of the special values $L(g, \chi, \frac{1}{2})$ as $\chi$ varies over the ring class characters of $K$ of $\mathcal{P}$-power conductor, for some fixed prime ideal $\mathcal{P}$. We prove our results under the only assumption that the prime to $\mathcal{P}$ part of $N$ is relatively prime to $D$.

0. Introduction

Let $E$ be an elliptic curve over $\mathbb{Q}$ of conductor $N$, and let $K/\mathbb{Q}$ be an imaginary quadratic field extension of discriminant $D$ such that $N$ and $D$ are relatively prime. Denote by $K_{\infty}$ the anticyclotomic $\mathbb{Z}_p$-extension of $K$ where $p$ is a given prime number with $p \nmid ND$. In 2002, Vatsal succeeded in settling a conjecture of Mazur pertaining to the size of the Mordell-Weil group $E(K_{\infty})$. In fact, Mazur’s conjecture predicts that the group $E(K_{\infty})$ is finitely generated, and Vatsal proved in [9] that this is true, at least when $E$ is ordinary at $p$, or when the class number of $K$ is prime to $p$.

In more concrete terms, Vatsal considered the modular form $g$ associated to $E$ and the family of Rankin-Selberg $L$-functions $L(g, \chi, s)$ as $\chi$ varies over ring class characters of $K$ of $p$-power conductor. Under certain conditions on $g$ and $\chi$, the result of Vatsal asserts that the special values $L(g, \chi, 1)$ are non-vanishing for all but finitely many $\chi$, provided that $p$ is an ordinary prime for $g$ or $p$ does not divide the class number of $K$. One consequence of this result is the non-triviality of certain Euler systems as formulated by Bertolini-Darmon in [1] which in its turn implies that the desired statement about the Mordell-Weil group is true.

In 2004, Cornut and Vatsal generalized in [3] the above mentioned work of Vatsal to totally real fields. Numerous technical complications arise due to the fact that a more general number field $F$ is considered. However, the basic arguments are ultimately the same, as the authors invoke deep theorems of Ratner [8] on uniform distribution of unipotent orbits on $p$-adic Lie groups to deduce the desired result.

In 2003, Vatsal extended the results and methods of [9] to study the variation of the $\lambda$-adic absolute value of $L^\text{al}(g, \chi, 1)$ as a function of $\chi$, where $\lambda$ is a fixed prime of $\overline{\mathbb{Q}}$ with residue characteristic $l$. The object of this paper is to generalize this work to totally real fields.
fields while removing most of the restrictions on $N$, $p$, $D$ and $l$ (Theorem 6.10). We use the improved formalism developed in [3] to achieve this purpose.

We now give a brief account of the results in this work. Let $\pi$ be an irreducible automorphic representation of $GL_2$ over a totally real field $F$ corresponding to a cuspidal Hilbert modular newform $g$ of level $\mathcal{N}$, trivial character and parallel weight $(2, \ldots, 2)$. The Hecke eigenvalues of $g$ are denoted by $a_q \ (T_v g = a_v g)$. Let $\mathcal{P}$ be a prime ideal of $F$ such that $\mathcal{P}$ lies over an odd rational prime $p$. Let $l$ be a rational prime, and denote by $E_l$ the $l$-adically complete discrete valuation ring containing the Hecke eigenvalues of $g$, $\lambda$ its maximal ideal and $E_\lambda$ the residue field $E_l/\lambda$. To simplify the exposition of the introduction, we assume that $l \neq p$ although we should mention that the case $l = p$ does not give rise to significant complications. Let $\chi$ be a ring class character of $K$ of conductor $\mathcal{P}^n$ such that $\chi = 1$ when restricted to $\mathcal{A}_{F}^* \subset \mathcal{A}_K^*$. We assume that the prime to $\mathcal{P}$ part of $N$ is relatively prime to the discriminant $D$ of $K/F$. We also impose sufficient conditions to make the sign in the functional equation of $L(\pi, \chi, s)$ equal $+1$ for all but finitely many characters $\chi$ of the type considered above.

Let $G(n)$ be the Galois group of the ring class field of conductor $\mathcal{P}^n$ over $K$, and let $G_0$ be the torsion subgroup of $G(\infty) = \bigcup_{n \to \infty} G(n)$. It is shown in [3] that $G_0$ is finite and can be identified with its image in $G(n)$ if $n \gg 0$, in which case we put $H(n) = G(n)/G_0$. By class field theory, we can view $\chi$ as a character of $G(n)$. Hence, we can write $\chi = \chi_0 \chi_1$ for some character $\chi_0$ of $G_0$ and a primitive character $\chi_1$ of $H(n)$.

It can be shown that $G(n)$ acts simply transitively on the set of CM points of conductor $\mathcal{P}^n$ on the Shimura curve associated to some carefully chosen totally definite quaternion algebra $B$. Given a CM point $x$ of conductor $\mathcal{P}^n$ and a ring class character $\chi$ of the same conductor, we define the Gross-Zagier sum

$$ a(x, \chi) = \frac{1}{|Pic(O_F)|} \sum_{\sigma \in G(n)} \chi(\sigma) \psi(\sigma x), $$

where $\psi$ is the $E_l$-valued function on the set of CM points, associated to $g$ via the Jacquet-Langlands correspondence. In the light of the existing Gross-Zagier formula (see Section 4), our job is reduced to studying the $l$-adic valuation of this sum. More precisely, our goal is to prove an analogue of Proposition 4.1 and Corollary 4.2 in [10] for a Hilbert modular form $g$ over a totally real field $F$, while removing the assumptions on $l$ and the order of $\chi_0$. Before we describe the results we obtained in this direction, we state Vatsal’s result in which $\pi$ corresponds to a weight 2 newform $g$ for $\Gamma_0(N)$ such that $N$, $p$, and $D$ are pairwise relatively prime.

**Theorem 0.1.** Let $\chi_0$ be a given character of $G_0$ such that its order is prime to $p$. Then, under some restrictions on $l$ and the Hecke field of $g$, we have

$$ \ord_{\lambda}(a(x, \chi)) < \mu $$

for all $n \gg 0$ and $\chi = \chi_0 \chi_1$, where $\mu$ is the smallest integer such that $a_q \not\equiv 1 + q \mod \lambda^\mu$ for some $q \nmid pND$.  

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Note that the restriction on the Hecke field of $g$, which will be made clear in Section 5, was overlooked in [10]. More importantly, we remark that (1) is mistakenly given as an equality in [10] due to an error made in the proof of Proposition 5.3 part (2). We mention here that in order to make the necessary corrections, we modify the definition of the constant $\mu$ from the one given in [10]. We say a bit more about these issues in Section 5.

We may assume without loss of generality that $E_l$ contains the values of $\chi_0$ and the $p^{th}$ roots of unity. We consider the trace of $a(x, \chi)$ taken from $E_\lambda(\chi_1)$ to $E_\lambda$. This trace expression is different than the average expression

$$b(x, \chi_0) = \sum_{\chi_1 \in H(n)} a(x, \chi_0 \chi_1)$$

considered in the work of Vatsal and Cornut-Vatsal. In particular, given any $\chi = \chi_0 \chi_1$, the non-vanishing of $\text{Tr}(a(x, \chi))$ implies that of $a(x, \chi)$. After a series of reductions in Section 5, we arrive at the following result.

**Proposition 0.2.** The trace expression simplifies to

$$\text{Tr}(a(x, \chi)) = [E_\lambda(\chi_1) : E_\lambda] \sum_{\sigma \in G_0/G_1} \chi_0(\sigma) \psi_{m, \mathcal{D}}(\sigma.x_{m, \mathcal{D}}),$$

where $G_1$ is the genus subgroup of $G(n)$, $\psi_{m, \mathcal{D}}$ is a function of higher level induced by $\psi$, and $x_{m, \mathcal{D}}$ is a CM point of higher level and conductor $\mathcal{P}^n$.

Hence, the problem is reduced to studying the $\lambda$-adic valuation of

$$\sum_{\sigma \in G_0/G_1} \chi_0(\sigma) \psi_{m, \mathcal{D}}(\sigma.x_{m, \mathcal{D}}).$$

Finally, we arrive at the following theorem (Theorem 6.10 in Section 4).

**Theorem 0.3.** Let $\chi_0$ be any character of $G_0$. For any CM point $x$ of conductor $\mathcal{P}^n$ with $n \gg 0$, there exists some $y \in G(n).x$ such that

$$\text{ord}_\lambda \left(\sum_{\sigma \in G_0/G_1} \chi_0(\sigma) \psi_{m, \mathcal{D}}(\sigma.y)\right) < \mu,$$

where $\mu$ is precisely given in Section 6 Definition 6.4.

1. Preliminaries and Notations

Let us first fix some notation. We write $\mathbb{A}$ (resp. $\mathbb{A}_f$) for the ring of adeles (resp. finite adeles) of $\mathbb{Q}$. Let $F$ be a totally real number field with $[F : \mathbb{Q}] = d$, and let $K$ be a totally imaginary quadratic extension of $F$ the discriminant of which we denote by $\mathcal{D}$. The ring of adeles of $F$ is $\mathbb{A}_F = \mathbb{A} \otimes_{\mathbb{Q}} F$, and the ring of finite adeles of $F$ is $\mathbb{F} = \mathbb{A}_f \otimes_{\mathbb{Q}} F$. Similarly,
we write $\mathbb{A}_K$ (resp. $\hat{K}$) for the ring of adeles (resp. finite adeles) of $K$. We denote by $\hat{M} = M \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ the profinite completion of a finitely generated $\mathbb{Z}$-module $M$.

We consider an adelic Hilbert modular newform $g$ of level $N$, trivial central character and parallel weight $(2, ..., 2)$. We denote by $a_v$ the Hecke eigenvalues of $g$ ($T_v g = a_v g$). Let $\pi$ be the automorphic irreducible representation of $GL_2$ over $F$ corresponding to $g$, and denote by $\pi_v$ the local component of $\pi$ at $v$ for any finite place $v$ of $F$.

We fix a prime ideal $\mathcal{P}$ of $F$ which lies above an odd rational prime $p$ with ramification index $e$ and inertia degree $f$, so that $|O_F/\mathcal{P}| = p^f q$. Let $\varpi_{\mathcal{P}}$ be a uniformizer of $F_{\mathcal{P}}$. We consider finite-order Hecke characters $\chi$ of $K$ of conductor $\mathcal{P}^n$ with $n \geq 0$.

The data of the previous paragraph is to remain fixed and the following hypotheses are assumed throughout this article:

1. The representations $\pi$ and $\pi \otimes \eta$ are distinct, where $\eta$ is the quadratic character associated to the extension $K/F$. We say that the pair $(\pi, K)$ is non-exceptional.

2. The prime to $\mathcal{P}$ part $\mathcal{N}'$ of $\mathcal{N}$ is relatively prime to the discriminant $\mathcal{D}$ of $K/F$.

3. We let $\chi$ vary through the collection of ring class characters of $\mathcal{P}$-power conductor which satisfy: $\chi$ is trivial when restricted to $\mathbb{A}_F^* \subset \mathbb{A}_K^*$.

4. Let $S$ be the set of all the Archimedean places of $F$, together with those finite places of $F$ which do not divide $\mathcal{P}$, are inert in $K$, and divide $\mathcal{N}$ to an odd power. We require $S$ to have an even cardinality.

Recall that $L(\pi, \chi, s)$ is the Rankin-Selberg $L$-function associated to $\pi$ and $\pi(\chi)$, where $\pi(\chi)$ is the automorphic representation of $GL_2$ attached to $\chi$. It follows from the last condition that the sign in the functional equation of $L(\pi, \chi, s)$ is $+1$ for all but finitely many characters $\chi$ that satisfy condition (3) (see Lemma 1.1 in [3]).

Let $l$ be any rational prime. Fix an embedding $\overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_l$, and denote by $E$ the subalgebra of $\overline{\mathbb{Q}}_l$ generated by the images of the Hecke eigenvalues of $g$. Write $E_l$ for the integral closure of $E$ in its field of fractions and $\lambda$ for the maximal ideal in $E_l$.

2. CM Points and Galois Action

Let $B$ be the totally definite quaternion algebra over $F$ such that $\text{Ram}(B) = S$. Let $G = \text{Res}_{F/Q}(B^*)$ be the algebraic group over $Q$ associated to $B^*$. Thus, the center of $G$ is $Z = \text{Res}_{F/Q}(F^*)$. Since every place in $F$ that ramifies in $B$ is inert in $K$, there exists an $F$-embedding $K \hookrightarrow B$. After fixing such an embedding, the group $T = \text{Res}_{F/Q}(K^*)$ can be viewed as a maximal sub-torus of $G$ defined over $Q$.

In what follows, we sketch the construction of an $O_F$-order $R$ of reduced discriminant $\mathcal{N}$ in $B$ following [3] and [15]. Let $\mathcal{N}'$ be the prime to $\mathcal{P}$ part of $\mathcal{N}$, and write $\mathcal{N} = \mathcal{P}^\delta \mathcal{N}'$. Let $R_0$ be an Eichler order of level $\mathcal{P}^\delta$ in $B$. We choose $R_0$ such that the $O_F$-order $O = O_K \cap R_0$ has a $\mathcal{P}$-power conductor. Denote by $\mathcal{N}_B$ the discriminant of $B/F$, and let $\mathcal{M}_K$ be an ideal in $O_K$ which has relative norm $\mathcal{N}'/\mathcal{N}_B$. We may find such an ideal $\mathcal{M}_K$ as follows. For each
prime \( \mathfrak{p} \) dividing \( N' \), let \( \mathfrak{p}_K \) be a prime of \( O_K \) dividing \( \mathfrak{p} \). If we put

\[
\mathcal{M} = \prod_{\mathfrak{p} \mid N} \mathfrak{p}_K^{\operatorname{ord}_p(N)/2} \cdot \prod_{\mathfrak{p} \mid N'} \mathfrak{p}_K^{\operatorname{ord}_p(N)},
\]

then

\[
\mathcal{M}_K = \prod_{\mathfrak{p}} \mathfrak{p}_K^{\operatorname{ord}_p(M)}.
\]

Finally, we obtain \( R \) by the following formula:

\[
R = O + (O \cap \mathcal{M}_K).R_0.
\]

In particular, \( R_P = R_{0,P} \) is an Eichler order of level \( P^\delta \) in \( B_P \simeq M_2(F_P) \). Without loss of generality, we make the identification

\[
(2) \quad R_P = h_PM^0(P^\delta)h_P^{-1}
\]

where

\[
M^0(P^\delta) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in M_2(O_F) : b \equiv 0 \pmod{P^\delta} \right\}.
\]

Hence, we also identify \( R_P^* \) with the subgroup \( h_PK^0(P^\delta)h_P^{-1} \), where

\[
K^0(P^\delta) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in GL_2(O_F) : b \equiv 0 \pmod{P^\delta} \right\}.
\]

Define an open compact subgroup \( H \) of \( G(\mathbb{A}_f) \) by \( H = \hat{R}^* \). The subgroup \( H \) is sometimes referred to as the level structure. This gives rise to the finite sets

\[
M_H = G(\mathbb{Q}) \backslash G(\mathbb{A}_f)/H,
\]

and

\[
N_H = Z(\mathbb{Q})^+ \backslash Z(\mathbb{A}_f)/\operatorname{nr}(H).
\]

It also gives rise to the set of CM points

\[
\operatorname{CM}_H = T(\mathbb{Q}) \backslash G(\mathbb{A}_f)/H.
\]

Notice that any function on \( M_H \) induces a function on \( \operatorname{CM}_H \) via the obvious reduction map \( \operatorname{red} : \operatorname{CM}_H \rightarrow M_H \).

The action of \( T(\mathbb{A}_f) \) on \( \operatorname{CM}_H \) by left multiplication in \( G(\mathbb{A}_f) \) factors through the reciprocity map

\[
\operatorname{rec}_K : T(\mathbb{A}_f) \rightarrow \operatorname{Gal}_{K}^\text{ab}
\]

and thus induces an action of \( \operatorname{Gal}_{K}^\text{ab} \) on \( \operatorname{CM}_H \). Hence, for \( x = [g] \in \operatorname{CM}_H \) and \( \sigma \in \operatorname{Gal}_{K}^\text{ab} \), we have \( \sigma.x = [\beta g] \) where \( \beta \in T(\mathbb{A}_f) \) is such that \( \operatorname{rec}_K(\beta) = \sigma \).

Moreover, the reduced norm map on \( G(\mathbb{A}_f) \) induces the map \( c : M_H \rightarrow N_H \). Hence, the action of \( \operatorname{Gal}_{K}^\text{ab} \) on \( N_H \) induces an action of \( \operatorname{Gal}_{K}^\text{ab} \) on \( N_H \). For \( x = [z] \in N_H \) and \( \sigma \in \operatorname{Gal}_{K}^\text{ab} \), we have \( \sigma.x = [\operatorname{nr}(\beta)g] \) where \( \beta \in T(\mathbb{A}_f) \) is such that \( \operatorname{rec}_K(\beta) = \sigma \).

We now introduce the notion of a CM point with a \( P \)-power conductor.
Definition 2.1. We say that \( x = [g] \in CM_H \) is a CM point of conductor \( \mathcal{P}^n \) and write \( x \in CM_H(\mathcal{P}^n) \) if \( T(\mathfrak{A}_f) \cap gHg^{-1} = \mathcal{O}_{\mathcal{P}^n}^* \), where \( \mathcal{O}_{\mathcal{P}^n} \subset O_K \) is the \( \mathcal{P}_F \)-order of conductor \( \mathcal{P}^n \).

Choose \( \alpha_\mathcal{P} \in O_K, \mathcal{P} \) such that \( \{1, \alpha_\mathcal{P}\} \) is an \( \mathcal{O}_F, \mathcal{P} \)-basis of \( O_K, \mathcal{P} \). Since \( \mathcal{O}_{\mathcal{P}^n, \mathcal{P}} = \mathcal{O}_F, \mathcal{P} + \mathcal{P}^n O_K, \mathcal{P} \), the set \( \{1, \varpi_\mathcal{P}^n \alpha_\mathcal{P}\} \) is an \( \mathcal{O}_F, \mathcal{P} \)-basis of \( O_{\mathcal{P}^n, \mathcal{P}} \). We fix the embedding \( \mathcal{K}_\mathcal{P} \hookrightarrow M_2(F_\mathcal{P}) \) defined by

\[
\begin{pmatrix}
    a + b \alpha_\mathcal{P} & b \\
    -b \alpha_\mathcal{P} & a
\end{pmatrix},
\]

where \( \text{Tr} \) and \( \text{N} \) denote the trace and norm maps.

Lemma 2.2. Consider \( k_\mathcal{P} \in B_\mathcal{P} \simeq M_2(F_\mathcal{P}) \) specified as:

\[
k_\mathcal{P} = \begin{pmatrix}
    \varpi_\mathcal{P}^{n-\delta} & 0 \\
    0 & 1
\end{pmatrix}.
\]

Then, for \( n \) large enough, the order \( k_\mathcal{P} h_\mathcal{P}^{-1} R_\mathcal{P} h_\mathcal{P} k_\mathcal{P}^{-1} \) in \( B_\mathcal{P} \) optimally contains the \( \mathcal{O}_F, \mathcal{P} \)-order in \( \mathcal{K}_\mathcal{P} \) of conductor \( \mathcal{P}^n \).

Proof. Let \( \tau = a + b \alpha_\mathcal{P} \) be any element in \( \mathcal{K}_\mathcal{P} \). We have

\[
k_\mathcal{P}^{-1} \tau k_\mathcal{P} = \begin{pmatrix}
    a + b \text{Tr} \alpha_\mathcal{P} & b \varpi_\mathcal{P}^{\delta-n} \\
    -b \varpi_\mathcal{P}^{\delta-n} \alpha_\mathcal{P} & a
\end{pmatrix}.
\]

Then, for all \( \mathcal{P}^n \gg 0 \), we have \( k_\mathcal{P}^{-1} \tau k_\mathcal{P} \in M^0(\varpi_\mathcal{P}) \) if and only if \( b \in \mathcal{P} \). Hence, the order \( k_\mathcal{P} h_\mathcal{P}^{-1} R_\mathcal{P} h_\mathcal{P} k_\mathcal{P}^{-1} \) in \( B_\mathcal{P} \) optimally contains the \( \mathcal{O}_F, \mathcal{P} \)-order in \( \mathcal{K}_\mathcal{P} \) of conductor \( \mathcal{P}^n \). \( \square \)

We choose \( z = [g_0] \in CM_H(\mathcal{P}^n) \) such that the \( \mathcal{P}^n \)-component \( g_0, \mathcal{P} \) of \( g_0 \) is equal to \( k_\mathcal{P} h_\mathcal{P}^{-1} \).

This choice of a CM point proves useful in Section 4.

3. Uniform Distribution of CM Points

The CM points are uniformly distributed on the components of the Shimura curve associated to \( B \). This was the most crucial idea behind Vatsal’s proof of Mazur’s conjecture for weight two modular forms over \( \mathbb{Q} \). In this section, we recall a crucial result on the uniform distribution of CM points due to Cornut-Vatsal, which we use to prove our main theorem. To describe this result, we need to introduce some more notation.

Let \( K[\mathcal{P}^n] \) be the ring class field over \( K \) of conductor \( \mathcal{P}^n \). In other words, \( K[\mathcal{P}^n] \) is the abelian extension of \( K \) associated by class field theory to the subgroup \( K^* \mathcal{O}_{\mathcal{P}^n}^* \) of \( \mathbb{A}_K^* \), where \( K_\infty = K \otimes_{\mathbb{Q}} \mathbb{R} \). Let \( G(n) \) denote the Galois group of this extension. We have

\[
G(n) = \text{Gal}(K[\mathcal{P}^n]/K) \simeq \mathbb{A}_K^*/(K^* \mathcal{O}_{\mathcal{P}^n}^*)
\]

via the reciprocity map of \( K \).

Set \( K[\mathcal{P}^\infty] = \bigcup_{n \geq 0} K[\mathcal{P}^n] \), so that \( G(\infty) = \text{Gal}(K[\mathcal{P}^\infty]/K) \). The torsion subgroup of \( G(\infty) \) is denoted by \( G_0 \). It is finite and \( G(\infty)/G_0 \) is a free \( \mathbb{Z}_p \)-module of rank \( [F_\mathcal{P} : \mathbb{Q}_p] \). The reciprocity map of \( K \) maps \( \mathbb{A}_F^* \subset \mathbb{A}_K^* \) onto the subgroup \( G_2 \simeq \text{Pic}(O_F) \) of \( G_0 \).
Let $G(\infty)'$ be the subgroup of $G(\infty)$ generated by the Frobeniuses of the primes of $K$ which are not above $\mathcal{P}$. Write $G_1 = G_0 \cap G(\infty)'$. Let $\mathcal{D}'$ be the square-free product of the primes $Q \neq \mathcal{P}$ of $F$ which ramify in $K$. Then $G_1/G_2$ is an $\mathbb{F}_2$-vector space with basis

$$\{\sigma_Q \mod G_2 : Q|D\},$$

where $\sigma_Q = \text{Frob}_Q$ and $Q$ is the prime of $K$ above $Q$.

Loosely speaking, the uniform distribution result in [9] states the following. Let $p_1$ and $p_2$ be arbitrary double cosets in $M_{H\mathcal{P}}$, and let $\sigma$ be an arbitrary nontrivial element of $G_0$ with $\sigma \notin G_1$. Then there exists a CM point $x \in \text{CM}_{H}(\mathcal{P}^n)$ such that $\text{red}(x) = p_1$ and $\text{red}(\sigma.x) = p_2$ whenever $n$ is sufficiently large.

In what follows, we describe the result of Cornut and Vatsal which extends and refines Vatsal’s theorem alluded to in the previous paragraph.

Let $R$ be a set of representatives for $G_0/G_1$ containing 1. We have the following maps:

$$\text{RED} : \text{CM}_{H}(\mathcal{P}^\infty) \to M_{H}^R, \quad x \mapsto (\text{red}(\tau.x))_{\tau \in R}$$

$$C : M_{H}^R \to N_{H}^R, \quad (a_{\tau})_{\tau \in R} \mapsto (c(a_{\tau}))_{\tau \in R}$$

and the composite map

$$C \circ \text{RED} : \text{CM}_{H}(\mathcal{P}^\infty) \to N_{H}^R,$$

which is $G(\infty)$-equivariant.

The following is the key theorem of Cornut-Vatsal as stated in [3]. However, the reader is referred to [2] for a proof of this result.

**Theorem 3.1** ([3]). For all but finitely many $x \in \text{CM}_{H}(\mathcal{P}^\infty)$,

$$\text{RED}(G(\infty).x) = C^{-1}(G(\infty).C \circ \text{RED}(x))$$

4. A Special Value Formula

**Definition 4.1.** An automorphic form of weight 2 on $G$ is a smooth (=locally constant) function $\theta : G(\mathbb{Q}) \backslash G(\mathbb{A}_f) \to \mathbb{C}$.

Adapting the notation from [3], we denote the space of all automorphic forms of weight 2 on $G$ by $S_2$. The group $G(\mathbb{A}_f)$ acts on $S_2$ by right translation:

$$(g.\phi)(x) = \phi(xg), \quad g \in G(\mathbb{A}_f) \text{ and } \phi \in S_2.$$

Let $\pi'$ be the unique cuspidal automorphic representation on $B$ that is associated to $\pi$ by the Jacquet-Langlands correspondence ($\pi = JL(\pi')$), and let $S_2(\pi')$ be the realization of $\pi'$ in $S_2$. We associate to $g$ a unique function $\theta$ in the one-dimesional subspace $S_2(\pi')^H$:

$$\theta : M_{H} \to \mathbb{C}.$$

This yields the function $\psi = \theta \circ \text{red}$ on $\text{CM}_{H}$. The space of functions on $M_{H}$ is endowed by an action of Hecke operators $T_v$. This action agrees with the classical Hecke action on the
space of adelic Hilbert modular forms in the sense that \( \theta \) has the same eigenvalues as \( g \) for all \( T_v, v \nmid \mathcal{N} \). Hence, we can write

\[
\theta : M_H \to E_g,
\]

where \( E_g \) is the Hecke field of \( g \).

Consider the linear form \( l_\chi : S_2(\pi') \to \mathbb{C} \) defined by the period integral:

\[
l_\chi(\phi) = \int_{Z(A)T(Q)/T(A)} \chi(t)\phi(t) \, dt,
\]

where \( dt \) is the choice of Haar measure used in [4]. In 1985, Waldspurger proved a fundamental theorem (Théorème 2 in [13]) which implies that for a ring class character \( \chi \) of \( P \)-power conductor satisfying \( \chi = 1 \) on \( \mathbb{A}_F^* \), we have

\[
L(\pi, \chi, \frac{1}{2}) \neq 0 \Leftrightarrow \exists \phi \in S_2(\pi') : l_\chi(\phi) \neq 0.
\]

However, the result of Waldspurger doesn’t give an explicit formula for the special value \( L(\pi, \chi, \frac{1}{2}) \). Several authors have subsequently taken up the task of specifying a test vector \( \phi \) on which to evaluate the linear functional \( l_\chi \) and finding a formula for \( L(\pi, \chi, \frac{1}{2}) \) in terms of \( l_\chi(\phi) \). Such results have numerous applications to arithmetic and Iwasa wa theory. In the rest of this section, we briefly present some of this work in the framework of this article.

In [14], Zhang has obtained a precise formula for \( L(\pi, \chi, \frac{1}{2}) \) under the assumption that \( \mathcal{N}, \mathcal{P}^n \) and \( \mathcal{D} \) are pairwise co-prime. A stronger result in this direction has been made in [7] using the relative trace formula of Jacquet and Chen [6]. Under the only assumption that \( \mathcal{N} \) and \( \mathcal{P}^n \) are co-prime, Martin and Whitehouse established an explicit formula for \( L(\pi, \chi, \frac{1}{2}) \) in terms of \( l_\chi(\phi) \), where \( \phi \) is a nice test vector extracted from the work of Gross and Prasad [5]. The authors noted that the restriction on the conductors of \( \pi \) and \( \chi \) was not essential to their approach but was made only for the sake of the Gross-Prasad test vector. In [4], the authors succeeded in removing this restriction by constructing a suitable test vector when \( n \geq 1 \) and \( \mathcal{P} \) divides \( \mathcal{N} \) under the following assumption:

\[
(3) \quad \text{if } \mathcal{P} \text{ is inert in } K, \text{ then } n \geq \delta.
\]

This allowed the authors to obtain a more general version of the main theorem in [7]. We mention here that although the main formula in [4] applies to a more general setting than the one imposed in our present work, we only state it under the hypotheses fixed in Section 1. Before doing so, we need to introduce some notation.

Let \( S(\pi) \) (resp. \( S(\chi) \)) be the set of finite places of \( F \) where \( \pi \) (resp. \( \chi \)) is ramified and \( S_2(\pi) \) the set of places where \( c(\pi_v) \geq 2 \) (\( c(\pi_v) \) is the exponent of the conductor of \( \pi_v \)). Denote the absolute value of the discriminants of \( F \) and \( K \) by \( \Delta_F \) and \( \Delta_K \).

**Theorem 4.2** ([4]). There exists a test vector \( \theta_\chi \in S_2(\pi') \) such that

\[
\frac{|l_\chi(\theta_\chi)|^2}{(\theta_\chi, \theta_\chi)} = \frac{1}{2\pi^d q^n} \frac{\Delta_F}{\Delta_K} L_{S(\pi) \cup S(\chi)}(1, 1) L_{S(\pi) \cap S(\chi)}(1_F, 1) L_{S_2(\pi)}(\pi, 1, 1) \left( \frac{L_{S(\pi)}(\pi, \chi, \frac{1}{2})}{L_{S_2(\pi)}(\pi, Ad, 1)} \right).
\]

Here \((.,.)\) is the standard inner product on \( \pi' \) with respect to the measure on \( G(\mathbb{A}) \) which is the product of the local Tamagawa measures multiplied by \( L^{S(\pi)}(1_F, 2) \).
Our next step is to establish a relation between the test vector $\theta_\chi$ and the form $\theta$ which obviously does not depend on $\chi$. In doing so, we produce a CM point $x \in \text{CM}_H(\mathcal{P}^n)$. This step allows us to conveniently express $l_\chi(\theta_\chi)$ in terms of the finite sum:

$$a(x, \chi) = \frac{1}{|G_2|} \sum_{\sigma \in G(n)} \chi(\sigma) \psi(\sigma x),$$

which is an essential ingredient in relating questions about the special values $L(\pi, \chi, \frac{1}{2})$ to the distribution of CM points. The basic idea behind this step is extracted from the work of Vatsal in [11].

The global test vector $\theta_\chi$ is constructed as an element of the restricted tensor product $\otimes \pi'_v$ by specifying that $\theta_\chi = \otimes \theta_{\chi, v}$. The local vectors $\theta_{\chi, v}$ are chosen such that $l_v(\theta_{\chi, v}) \neq 0$ for any nonzero $l_v \in \text{Hom}_{K_v^*}(\pi'_v, \mathbb{C}(\chi_v))$.

Let $v$ be a finite place of $F$. If $\pi_v$ is an unramified principal series representation, we let $R_{\chi, v} \subset B_v$ denote a maximal order which optimally contains the order of $K_v$ with the same conductor as $\chi_v$. It follows from [3] that $R_{\chi, v}^*$ fixes a unique line in $\pi'_v$. We let $\theta_{\chi, v}$ be any nonzero vector which is right invariant under $R_{\chi, v}^*$. If $\pi_v$ is ramified and $\chi_v$ is unramified, we let $R_{\chi, v} \subset B_v$ denote an Eichler order of reduced discriminant $N_v$ which contains $O_{K_v}$. Again, it is shown in [5] that $R_{\chi, v}^*$ fixes a unique line in $\pi'_v$, and we let $\theta_{\chi, v}$ be any nonzero vector on this line. However, the work of Gross and Prasad does not provide a local test vector when $\pi_v$ and $\chi_v$ are both ramified. In this case the desired test vector is obtained from the main local results of [4] (Theorems 1.6, 1.7). Notice that under the hypothesis of this paper, if $\pi_v$ and $\chi_v$ are both ramified, then the place $v$ necessarily corresponds to the prime ideal $\mathcal{P}$. In particular, we have $\pi_v \simeq \pi'_v$. Assuming condition (3), it is shown in [4] that there exists an element $g_v \in GL_2(F_v)$ such that, for any non-zero $l_v \in \text{Hom}_{K_v^*}(\pi_v, \mathbb{C}(\chi_v))$, the subgroup $g_v K^0(\mathcal{P}^d) g_v^{-1}$ fixes a 1-dimensional subspace of $\pi_v$ consisting of test vectors for $l_v$. We let $\theta_{\chi, v} \in \pi'_v$ be the image under $\pi_v \simeq \pi'_v$ of the unique (up to scalar multiples) vector fixed by this subgroup. We note that the determination of $g_v$ depends on whether $K_v/F_v$ is split or inert, but in both cases the subgroup $g_v K^0(\mathcal{P}^d) g_v^{-1}$ is an Eichler order of reduced discriminant $\mathcal{P}^d$ which optimally contains the order of $K_v$ of conductor $\mathcal{P}^n$. We mention here that our choice of $k_\mathcal{P}$ in Lemma 2.2 is motivated by Theorem 1.7 in [4].

Recall that $\theta$ is right invariant under $H = \hat{R}^*$ where for each finite place $v$ of $F$, the order $R_v \subset B_v$ is an Eichler order of reduced discriminant $\mathcal{N}_v$ which optimally contains the order $O_v$ of $K_v$ ($O \subset K$ is the $\mathcal{O}_F$-order of $\mathcal{P}$-power conductor specified at the beginning of Section 2). There exists $b_v \in B_v^*$ such that $b_v R_v b_v^{-1} = R_{\chi, v}$, since $R_v$ and $R_{\chi, v}$ are Eichler orders of the same reduced discriminant in $B_v$. In other words, the $O_{F, v}$-order $K_v \cap b_v R_v b_v^{-1}$ has conductor equal to that of $\chi_v$ for every finite place $v$ of $F$ so that $x = [b] \in \text{CM}_H(\mathcal{P}^n)$. It follows that $\theta_\chi = b \theta$ (up to scalar multiples). Since the quantity $\frac{|L_v(\theta_\chi, \frac{1}{2})|^2}{\theta_\chi(\frac{1}{2})}$ that appears in the special value formula (Theorem 4.2) is invariant under scaling, we can assume without loss of generality that $\theta_\chi(\frac{1}{2}) = \theta(z)$ for all $z \in G(A_F)$.

**Lemma 4.3.** There exists $x \in \text{CM}_H(\mathcal{P}^n)$ such that

$$l_\chi(\theta_\chi) = \mu_\chi a(x, \chi),$$

where $\mu_\chi$ is a constant depending only on $\chi$. In other words, $\theta_\chi$ is related to the local test vector $\theta$ through the action of $x$. This completes the proof of Theorem 1.1.

**Remark.** The proof of Lemma 4.3 is based on the work of Vatsal and the local results of [4]. It involves a detailed analysis of the local and global test vectors, taking into account the ramification and splitting behavior of the places of $F$. The key idea is to construct a test vector $\theta_\chi$ that is right invariant under $R_{\chi, v}$ for each finite place $v$ of $F$, and to relate this test vector to the global test vector $\theta$ through the action of a CM point $x \in \text{CM}_H(\mathcal{P}^n)$. This allows us to express $l_\chi(\theta_\chi)$ in terms of the finite sum $a(x, \chi)$, which is a natural generalization of the special values $L(\pi, \chi, \frac{1}{2})$. The proof of Lemma 4.3 is quite technical and requires a deep understanding of the local and global representations involved.
where $\mu_\chi$ is the volume of the image of $U_{P^n} = \mathcal{O}_{\mathfrak{p}^n}^*, K_\infty^*$ in $Z(\mathbb{A})T(\mathbb{Q})/T(\mathbb{A})$.

**Proof.** Since $\theta_\chi$ is right invariant under the subgroup $\hat{R}_\chi^*$ where for each finite place $v$ of $F$, the order $R_{\chi,v} \subset B_v$ is an Eichler order of reduced discriminant $\mathcal{N}_v$ which optimally contains the order of $K_v$ with the same conductor as $\chi_v$, it follows that $l_\chi(\theta_\chi)$ can be written as

$$l_\chi(\theta_\chi) = \mu_\chi \sum_{\chi(t) \theta_\chi(t)} \chi(t)\theta_\chi(t)$$

$$= \mu_\chi \sum_{\chi(t) \theta_\chi(t)} \chi(t)(tb)$$

$$= \frac{\mu_\chi}{|G_2|} \sum_{\sigma \in G(n)} \chi(\sigma)\psi(\sigma,x)$$

The last equality follows from that fact that $Z(\mathbb{A})T(\mathbb{Q})/T(\mathbb{A})$ can be identified with the quotient $G(n)/G_2$. □

Using the fact that $\mu_\chi = \text{vol}(U_{P^n}, dt) = \text{vol}(U_K, dt)L_P(\eta, 1)q^{-n}$, we arrive at the following formula:

$$|a(x, \chi)|^2 = \frac{q^n(\theta, \theta)}{2\pi^n\text{vol}(U_K, dt)^2} \sqrt{\frac{\Delta_{E}}{\Delta_{K}}} L_{S(\pi)\cap S(\chi)}(\eta, 1)L_{S(\pi)\cap S(\chi)}(1_F, 1)\frac{L^{S(\pi)}(\pi, \chi, \frac{1}{2})}{L^{S(\pi)}(\pi, Ad, 1)}.$$

**Remark 4.4.** Since $a(\gamma.x, \chi) = \chi^{-1}(\gamma)a(x, \chi)$ for any $\gamma \in G(n)$, and $G(n)$ acts simply and transitively on $\text{CM}_{H}(P^n)$, it suffices to study the $\lambda$-adic valuation of $a(y, \chi)$ for any $y \in \text{CM}_{H}(P^n)$.

5. Two Level Raising Steps

Adapting the notation from [3], we identify ring class characters of $P$-power conductor with finite-order characters of $G(\infty)$. Hence, given a character $\chi_0$ of $G_0$ such that $\chi_0 = 1$ on $G_2$, denote by $P(n, \chi_0)$ the set of primitive characters of $G(n)$ (do not factor through $G(n-1)$) and induce $\chi_0$ on $G_0$. In [9] and [3], the authors proved that for each character $\chi_0$ of $G_0$ and all but finitely many $n$, there exists a character $\chi \in P(n, \chi_0)$ such that $L(\pi, \chi, \frac{1}{2}) \neq 0$. Moreover, Vatsal showed in [9] that if $\chi_0$ has order prime to $p$ and the Hecke field of $g$ is linearly disjoint from the field generated over $\mathbb{Q}$ by the $p^{th}$-roots of unity, then $L(\pi, \chi, \frac{1}{2}) \neq 0$ for all $\chi \in P(n, \chi_0)$ with $n$ sufficiently large. We remark that this statement differs slightly from the statement given in [9] since the condition on the Hecke field of $g$ was overlooked there. In [10], Vatsal extended the results and methods of [9] to study the algebraic part of the special value $L(\pi, \chi, \frac{1}{2})$ modulo a given prime ideal $\lambda$ in $\mathbb{Q}$ of characteristic $l$. Vatsal proved that for all $n \gg 0$, there exists $\chi \in P(n, \chi_0)$ such that

$$\text{ord}_\lambda \left( \frac{L(\pi, \chi, \frac{1}{2})}{\Omega_{g}^{\text{can}}C_{\text{csp}}} \right) < \text{ord}_\lambda(C_{\text{Eis}}^2),$$

where $C_{\text{Eis}}$ is a constant that measures the congruence between $g$ and the space of Eisenstein Series, and $C_{\text{csp}}$ is a constant that measures the congruence between $g$ and some cusp forms.
of lower levels. Moreover, if \( \chi_0 \) has order prime to \( p \), the Hecke field of \( g \) is linearly disjoint from the field generated over \( \mathbb{Q} \) by the \( p \)-th roots of unity, and \( l \) satisfies certain conditions (see next paragraph), then (4) is true for all \( \chi \in P(n, \chi_0) \) with \( n \) sufficiently large. We remark that (4) is mistakenly given as an equality in [10], the source of the mistake being an error made in the proof of Proposition 5.3 part (2). We provide a correct version of this result in Proposition 6.8 below. We mention here that establishing the correct statement involves modifying the choice of the constant \( C_{\text{Eis}} \) as in Definition 6.4 below.

Given \( x = [g] \in CM_H(P^n) \) and a ring class character \( \chi \) of conductor \( P^n \), we define the Gross-Zagier sum

\[
a(x, \chi) = \frac{1}{|G_2|} \sum_{\sigma \in G(n)} \chi(\sigma) \psi(\sigma \cdot x).
\]

In order to prove that a family of values is non-vanishing, it is a standard technique to compute their average. Hence, in order to show that \( a(x, \chi) \neq 0 \) is non-vanishing for some \( \chi \in P(n, \chi_0) \), it suffices to show that

\[
b(x, \chi_0) = \sum_{\chi \in P(n, \chi_0)} a(x, \chi) \neq 0.
\]

Vatsal observed in [9] that if the order of \( \chi_0 \) is prime to \( p \), then all the characters in \( P(n, \chi_0) \) are in fact conjugates under the action of \( \text{Aut}(\mathbb{C}) \). It follows that the sums \( a(x, \chi) \) are also conjugates for all \( \chi \in P(n, \chi_0) \). Hence, the non-vanishing of \( a(x, \chi) \) for some \( \chi \in P(n, \chi_0) \) forces the non-vanishing of \( a(x, \chi) \) for all \( \chi \in P(n, \chi_0) \). In addition, Vatsal noticed in [10] that if \( l \) splits completely in the field \( \mathbb{Q}(\chi_0) \) generated by the values of \( \chi_0 \), and if it is inert in the field \( \mathbb{Q}(\mu_{p^\infty}) \) generated by all \( p \)-power roots of unity, then all the characters in \( P(n, \chi_0) \) are conjugates under the action of a decomposition group \( D_\lambda \). Thus, the sums \( a(x, \chi) \) have the same \( \lambda \)-adic valuation for all \( \chi \in P(n, \chi_0) \).

Our goal is to prove an analogue of Theorem 1.2 in [10] (see also Proposition 4.1 and Corollary 4.2) for a Hilbert modular form \( g \) over a totally real field \( F \), while removing the above mentioned assumptions on \( l \) and the order of \( \chi_0 \).

Recall that \( E_l \) is an \( l \)-adically complete discrete valuation ring containing the Fourier coefficients of \( g \), and \( \lambda \) is the maximal ideal in \( E_l \). Enlarge \( E_l \) if necessary to contain the values of \( \chi_0 \) and the \( p \)-th roots of unity. Denote by \( E_\lambda \) the residue field \( E_l/\lambda \) and let \( E_\lambda(\chi_1) \) be the field obtained by adjoining to \( E_\lambda \) the values of \( \chi_1 \). We may view \( \theta \) as taking values in \( E_\lambda \):

\[
(5) \quad \theta : M_H \to E_\lambda.
\]

We assume that \( \theta([g]) \) is a \( \lambda \)-adic unit for some \( [g] \in M_H \).

Consider the trace of \( a(x, \chi) \) taken from \( E_\lambda(\chi_1) \) to \( E_\lambda \):

\[
\text{Tr}(a(x, \chi)) = \sum_{\sigma \in \text{Gal}(E_\lambda(\chi_1)/E_\lambda)} \sigma(a(x, \chi)).
\]
This trace expression is different than the average expression

\[ b(x, \chi) = \sum_{\chi \in \mathcal{P}(n,\chi)} a(x, \chi) \]

considered in the work of Vatsal and Cornut-Vatsal. Nevertheless, the same approach is followed to study both expressions. Evidently, given any \( \chi \in \mathcal{P}(n,\chi_0) \), the non-vanishing of \( \text{Tr}(a(x, \chi)) \) would then imply the non-vanishing of \( a(x, \chi) \).

We know by Lemma 2.8 in [3] that we can identify \( G_0 \) with its image \( G_0(n) \) in \( G(n) \) whenever \( n \) is sufficiently large. We then denote the quotient group \( G(n)/G_0(n) \) by \( H(n) \). Suppose that \( \chi_0 \) is a fixed character of \( G_0 \) and let \( \chi \in \mathcal{P}(n,\chi_0) \). One can express \( \chi \) as \( \chi = \chi_0 \chi_1 \), where \( \chi_0 \) is some character of \( G(n) \) inducing \( \chi_0 \) on \( G_0(n) \cong G_0 \), and \( \chi_1 \) is some primitive character of \( H(n) \).

Notice that

\[
\text{Tr}(a(x, \chi)) = \frac{1}{|G_2|} \sum_{\sigma \in G_0(n)} \sum_{\tau \in H(n)} \chi_0'(\sigma)\chi_1(\tau)\psi(\sigma\tau.x)
\]

\[
= \frac{1}{|G_2|} \sum_{\sigma \in G_0} \sum_{\tau \in H(n)} \psi(\sigma\tau.x)\text{Tr}_{E_\chi(\chi_1)/E_\lambda} \chi_0(\sigma)\chi_1(\tau)
\]

\[
= [E_\lambda(\chi_1) : E_\lambda] \sum_{\sigma \in G_0} \sum_{\tau \in H(n)} \psi(\sigma\tau.x)\chi_1(\tau).\]

Let \( p^o \) be the highest power of \( p \) dividing the order of \( \chi_0 \), and set \( r = \max\{1,o\} \). Notice that an element \( \tau \) in \( H(n) \) satisfies \( \chi_1(\tau) \in E_\lambda \) if and only if the order of \( \tau \) divides \( p^r \).

For \( m \geq 1 \) and \( n \gg 0 \), we put \( Z(n,m) := \ker(H(n) \hookrightarrow H(n-m)) \). The following lemma is crucial in the sequel.

**Lemma 5.1.** For \( n \gg 0 \), \( Z(n,m) \cong O_{F,p}/\mathcal{P}^m \) as a group.

**Proof.** If \( n \) is sufficiently large, the natural quotient map \( G(n) \to H(n) \) induces an isomorphism between \( \ker(H(n) \to H(n-m)) \) and \( \ker(G(n) \to G(n-m)) \). We also have an isomorphism between \( \ker(G(n) \to G(n-m)) \) and

\[
K^*\hat{\mathcal{O}}_{p^m-n}/K^*\hat{\mathcal{O}}_{p^m} \cong O_{p^m-n,p}/O_{p^m-n,m}O_{p^m,p},
\]

induced by the reciprocity map. Notice that \( O_{p^m-n,m} = O_{p^m} \) is contained in \( O_{p^m,p} \) for sufficiently large \( n \), so that

\[
\ker(G(n) \to G(n-m)) \cong O_{p^m-n,p}/O_{p^m,p}.
\]

On the other hand,

\[
O_{p^m-n,p}/O_{p^m,p} = \{1 + a\alpha_p \omega_p^{n-m} \mod O_{p^m,p} : a \in O_{F,p}/\mathcal{P}^m\},
\]

thus yielding the desired isomorphism. \( \square \)
In what follows, we set $m = er$. Hence, we get
\[
\text{Tr}(a(x, \chi)) = \frac{[E_\lambda(\chi_1) : E_\lambda]}{[G_2]} \sum_{\sigma \in G_0} \chi_0(\sigma) \sum_{\tau \in Z(n, m)} \chi_1(\tau) \psi(\sigma \tau.x)
\]
\[
= \frac{[E_\lambda(\chi_1) : E_\lambda]}{[G_2]} \sum_{\sigma \in G_0} \chi_0(\sigma) \sum_{a \in O_{F,P}/P^m} \chi_1(\tau_a) \psi(\sigma \tau_a.x)
\]

In view of the preceding lemma, each element $\tau_a \in Z(n, m)$ corresponds to the class of $\lambda_{a,P} = 1 + a\alpha_\tau \varpi_P^{-m}$ in $O_{P^n-m,P}/O_{P^n,P}$ via the reciprocity map. In the following computations, we identify $\lambda_{a,P}$ with its image in $GL_2(O_{F,P})$:
\[
\lambda_{a,P} \mapsto \begin{pmatrix} 1 + a\varpi_P^{-m} \text{Tr} \alpha_\tau & a\varpi_P^{-m} \\ -a\varpi_P^{-m}N\alpha_\tau & 1 \end{pmatrix}.
\]
We denote by $\lambda_a$ the image of $\lambda_{a,P}$ in $T(A_f)$. We now analyze the term $\psi(\tau_a.x)$ that appears in the above expression for $\text{Tr}(a(x, \chi))$.

Let $x = [g]$ be any CM point in $CM_H(P^n)$. There exists $\sigma \in G(n)$ such that $x = \sigma.z$ with $z = [g_0]$ being the CM point specified at the end of Section 2. By definition, $\sigma.z = [\mu g_0]$ where $\mu \in T(A_f)$ is such that $\text{rec}_K(\mu) = \sigma$. It follows that if $x = [g] \in CM(P^n)$, then
\[
g = \eta \mu g_0 r \text{ for some } \eta \in T(Q) \text{ and } r \in H.
\]
Hence,
\[
\psi(\tau_a.x) = \theta(\lambda_a g) = \theta(\mu \lambda_a g_0).
\]
However, notice that
\[
\lambda_{a,P} k_P = k_P \begin{pmatrix} 1 + a\varpi_P^{-m} \text{Tr} \alpha_\tau & a\varpi_P^{-m} \\ -a\varpi_P^{-2n-m-\delta}N\alpha_\tau & 1 \end{pmatrix}
\]
\[
= k_P \begin{pmatrix} 1 & a\varpi_P^{-\delta-m} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 + a\varpi_P^{-m} \text{Tr} \alpha_\tau + a^2 \varpi_P^{-2n-2m}N\alpha_\tau & 0 \\ -a\varpi_P^{-2n-m-\delta}N\alpha_\tau & 1 \end{pmatrix}.
\]
For every $a \in O_{F,P}$ and $m \geq 1$, we put
\[
\alpha_{a,m} = (1, 1, ..., h_P \begin{pmatrix} 1 & a\varpi_P^{-m} \\ 0 & 1 \end{pmatrix} h_P^{-1}, ..., 1, 1)
\]
so that $\theta(\mu \lambda_a g_0) = \theta(\mu g_0 \alpha_{a,m})$. Consequently,
\[
\psi(\tau_a.x) = \theta(gr^{-1} \alpha_{a,m}).
\]
We now introduce a level structure $H_m \subset H$ which agrees with $H$ outside $P$ and corresponds to an $O_F$-order $R_m \subset B$. Let us first recall the definitions of a couple of specific open and compact subgroups of $GL_2(F_P)$. For an ideal $\mathcal{I}$ of $O_{F,P}$, we have
\[
K_1(\mathcal{I}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(O_{F,P}) : c \equiv 0, a \equiv d \mod \mathcal{I} \right\},
\]

and
\[ K^1(\mathcal{I}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(O_{F,P}) : b \equiv 0, a \equiv d \mod\mathcal{I} \right\}. \]

We put
\[ H_{m,P} = R^*_m = \begin{cases} R^*_P \cap h_P K_1(\mathcal{P}^{2m-\delta})h_P^{-1}, & \text{if } \delta < m \\
R^*_P \cap h_P K_1(\mathcal{P}^m)h_P^{-1}, & \text{if } m \leq \delta < 2m \\
R^*_P \cap h_P K_1(\mathcal{P}^m)h_P^{-1}, & \text{otherwise.} \end{cases} \]

Define a function \( \theta_m \) on \( G(\mathbb{Q}) \backslash G(\mathbb{A}_f) \) by:
\[ \theta_m = \sum_{a \in O_{F,P}/\mathcal{P}^m} \chi_1(\tau_a)(\alpha_{a,m} \theta). \]

Lemma 5.2. The function \( \theta_m \) has level \( H_m = \widehat{R^*_m} \).

Proof. We need to prove that \( \gamma \cdot \theta_m = \theta_m \) for all \( \gamma \in H_m \). Let \( \gamma \) be any element in \( H_m \), and write
\[ \gamma = h_P \begin{pmatrix} w & x \\ y & z \end{pmatrix} h_P^{-1}. \]

In particular, we have \( w \equiv z \mod\mathcal{P}^m \). Notice that
\[
\begin{pmatrix} w & x \\ y & z \end{pmatrix} \begin{pmatrix} 1 & a\omega^{\delta-m}_P \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & xz^{-1} \\ yw^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & wz^{-1}a\omega^{\delta-m}_P \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w & 0 \\ 0 & z \end{pmatrix} \\
= \begin{pmatrix} 1 & wz^{-1}a\omega^{\delta-m}_P \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - yz^{-1}a\omega^{\delta-m}_P & xz^{-1} + wz^{-2}ya^2\omega^{2\delta-2m}_P \\ yw^{-1} & 1 + yz^{-1}a\omega^{\delta-m}_P \end{pmatrix} \\
\times \begin{pmatrix} w & 0 \\ 0 & z \end{pmatrix}.
\]

Since \( \gamma \in H_{m,P} \), we know that
\[
h_P \begin{pmatrix} 1 - yz^{-1}a\omega^{\delta-m}_P & xz^{-1} + wz^{-2}ya^2\omega^{2\delta-2m}_P \\ yw^{-1} & 1 + yz^{-1}a\omega^{\delta-m}_P \end{pmatrix} \begin{pmatrix} w & 0 \\ 0 & z \end{pmatrix} h_P^{-1} \in R^*_P.
\]

Hence, we get
\[
\gamma \cdot \theta_m = \sum_{a \in O_{F,P}/\mathcal{P}^m} \chi_1(\tau_a)(\alpha_{wz^{-1}a,m} \theta) \\
= \sum_{a \in O_{F,P}/\mathcal{P}^m} \chi_1(\tau_{azw^{-1}})(\alpha_{a,m} \theta) \\
= \sum_{a \in O_{F,P}/\mathcal{P}^m} \chi_1(\tau_a)(\alpha_{a,m} \theta) = \theta_m.
\]

\[ \square \]
Proposition 5.3. Let $\psi_m$ be the function induced by $\theta_m$ on $CM_{H_m}$. If $n$ is sufficiently large, then for every $x \in CM_H(P^n)$, there exists $x_m \in CM_H(P^n)$ such that

$$\psi_m(x_m) = \sum_{\tau \in \mathbb{Z}(n,m)} \chi_1(\tau)\psi(\tau.x).$$

Proof. Consider some $x = [g] \in CM_H(P^n)$ with $g \in G(\mathbb{A}_f)$ and $n \geq m$. Let $x_m$ be the class of $gr^{-1}$ in $CM_{H_m}$, where $r$ is as specified in (6). It is easy to check that the CM point $x_m = [gr^{-1}] \in CM_{H_m}$ has conductor $P_n$.

To prove that formula (8) holds for all $x \in CM_H(P^n)$ with $n \gg 0$, we simply have to observe that

$$\psi_m(x_m) = \theta_m(gr^{-1})$$

$$= \sum_{a \in \mathcal{O}_F/P^m} \chi_1(\tau_a)\theta(gr^{-1}\alpha_{a,m})$$

$$= \sum_{a \in \mathcal{O}_F/P^m} \chi_1(\tau_a)\psi(\tau_a.x)$$

$$= \sum_{\tau \in \mathbb{Z}(n,m)} \chi_1(\tau)\psi(\tau.x),$$

where the third equation follows trivially from (7).

Since $\chi_0 = 1$ on $G_2$, we have:

$$\text{Tr}(a(x,\chi)) = [E_\lambda(\chi_1) : E_\lambda] \sum_{\sigma \in G_0/G_2} \chi_0(\sigma)\psi_m(\sigma.x_m).$$

We can reduce the above sum into something even simpler by means of another level raising step.

Proposition 5.4. There exists an $\mathcal{O}_F$-order $R_{m,D}$, a non-zero function $\theta_{m,D}$ of level $H_{m,D} = \widehat{R_{m,D}}^*$ on $G(\mathbb{A}_f)$, and for each $n \geq 0$, a Galois equivariant map $x_m \mapsto x_{m,D}$ from $CM_{H_m}(P^n)$ to $CM_{H_{m,D}}(P^n)$ such that

$$\psi_{m,D}(x_{m,D}) = \sum_{\tau \in G_1/G_2} \chi_0(\tau)\psi_m(\tau.x_m),$$

where $\psi_{m,D} = \theta_{m,D} \circ \text{red}$.

Proof. The reader is referred to the proof of Lemma 5.9 in [3] \qed

Hence, the trace expression simplifies to

$$\text{Tr}(a(x,\chi)) = [E_\lambda(\chi_1) : E_\lambda] \sum_{\sigma \in G_0/G_1} \chi_0(\sigma)\psi_{m,D}(\sigma.x_{m,D}).$$
We now study the $\lambda$-adic valuation of the sum

$$\sum_{\sigma \in \mathcal{R}} \chi_0(\sigma)\psi_{m, D}(\sigma, x_{m, D}),$$

where $\mathcal{R}$ is a set of representatives for $G_0/G_1$ containing 1.

**Definition 6.1.** Let $k$ be any ring. A $k$-valued function $\phi$ on $M_H$ is said to be Eisenstein if it factors through $N_H$ via the map $c$, where as $\phi$ is said to be exceptional if there exists $z \in N_H$ such that $\phi$ is constant on $c^{-1}(\sigma, z)$ for all $\sigma \in \text{Gal}^{ab}_K$.

Choose an ideal $C$ in $O_F$ such that $nrd(H)$ contains all elements of $\hat{O}_F^*$ congruent to 1 modulo $C$. Such an integral ideal exists because $nrd(H)$ is open in $\hat{F}^*$.

**Lemma 6.2.** Let $\phi$ be an $E_l$-valued function on $M_H$ such that $\phi(x)$ is a unit for some $x \in M_H$. If $\phi$ is Eisenstein modulo $\lambda^r$ for some positive integer $r$, then $a_v \equiv q_v + 1 \mod \lambda^r$ for all $v \in S$.

Proof. Let $v$ be a finite place in $F$ corresponding to a principal prime ideal $Q = aO_F$ with $a \equiv 1 \mod C$ and $a$ is totally positive. Choose $x = [g] \in M_H$ such that $\phi(x)$ is a $\lambda$-adic unit. By definition, we know that

$$T_v \phi(x) = \sum_{i \in I_v} \phi([g n_{v, i}]).$$

Here $H_v \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} H_v = \prod_{i \in I_v} n_{v, i} H_v$. Notice that $c([g n_{v, i}]) = c([g n_v])$, where

$$n_v = (1, \ldots, 1, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, 1, \ldots, 1).$$

Since $\phi$ is Eisenstein modulo $\lambda^r$, we get

$$T_v \phi(x) \equiv (1 + q_v) \phi([g n_v]) \mod \lambda^r.$$ 

Choose $d \in G(Q)$ such that $nrd(d) = a$. Notice that $nrd(n_v^{-1} d) = (a, \ldots, a, 1, a, \ldots, a) \equiv 1 \mod C$. We thus obtain an element $h \in H$ such that $nrd(h) = nrd(n_v^{-1} d)$. Hence,

$$\phi([g n_v]) \equiv \phi([g n_v d^{-1}]) \equiv \phi([g h^{-1}]) \equiv \phi(x) \mod \lambda^r.$$ 

On the other hand, we know that $T_v \phi(x) = a_v \phi(x)$. Putting all of this together gives $a_v \phi(x) \equiv (1 + q_v) \phi(x) \mod \lambda^r$, which implies that $a_v \equiv 1 + q_v \mod \lambda^r$ since $\phi(x)$ is a $\lambda$-adic unit. \qed
Lemma 6.3. If \( \phi \) is exceptional but non-Eisenstein modulo \( \lambda^r \) for some positive integer \( r \), then \( a_v \) is a \( \lambda \)-adic non-unit for all finite places \( v \) of \( F \) that are inert in \( K \) and do not divide \( \mathcal{N} \).

Proof. The argument given here is drawn from [3]. Recall that we have an action of the group \( \text{Gal}_{K}^{\text{ab}} \) on \( N_H \), and one can show that there are at most two \( \text{Gal}_{K}^{\text{ab}} \)-orbit in \( N_H \). If there were only one \( \text{Gal}_{K}^{\text{ab}} \)-orbit in \( N_H \), then any exceptional function on \( M_H \) would also be Eisenstein. Since \( \phi \) is exceptional and non-Eisenstein modulo \( \lambda^r \), we know there must be exactly two \( \text{Gal}_{K}^{\text{ab}} \)-orbits in \( N_H \), which we denote by \( X \) and \( Y \) with \( \phi \) being constant modulo \( \lambda^r \) on \( c^{-1}(z) \) for all \( z \in X \). Since \( \phi \) is non-Eisenstein modulo \( \lambda^r \), there exist \( y \in Y \) and some \( x_1, x_2 \in c^{-1}(y) \) such that \( \phi(x_1) \not\equiv \phi(x_2) \) mod \( \lambda^r \).

Let \( v \) be a finite place of \( F \) that is inert in \( K \) and does not divide \( \mathcal{N} \). For any \( x = [g] \in M_H \), we know that

\[
T_v \phi(x) = a_v \phi(x) = \sum_{i \in I_v} \phi([g_\eta_{v,i}]).
\]

We also know that if \( x \in c^{-1}(y) \) then \([g_\eta_{v,i}] \in c^{-1}(\text{Frob}_v,y) \). Since \( v \) is inert in \( K \), we get \( \text{Frob}_v \cdot y \in X \), so that \( \phi \) is constant modulo \( \lambda^r \) on \( c^{-1}(\text{Frob}_v,y) \) with \( \phi(v, y) \) being the common value. Hence,

\[
a_v \phi(x_1) \equiv (1 + q_v) \phi(v, y) \equiv a_v \phi(x_2) \mod \lambda^r.
\]

It follows that \( a_v \) is a \( \lambda \)-adic non-unit, since otherwise \( \phi(x_1) \) and \( \phi(x_2) \) would be congruent modulo \( \lambda^r \).

We shall assume henceforth that \( g \) satisfies the condition: \( a_v \) is a \( \lambda \)-adic unit for some \( v \) inert in \( K \), \( v \not\mid \mathcal{N} \).

Definition 6.4. Let \( \mu \) be the smallest integer such that \( a_v \not\equiv 1 + q_v \mod \lambda^\mu \) for some \( v \in S, v \not\mid \mathcal{D} \).

It follows immediately from the definition of \( \mu \) that the function \( \theta \) as specified in [5] is non-exceptional modulo \( \lambda^\mu \).

Recall that we denote by \( f \) the inertia degree of \( \mathcal{P} \) over \( p \). If \( \lambda \) lies above \( p \) \((l = p) \), we let \( s \) be the corresponding ramification index. In this case, we denote by \( k \) the ring \( E_l/\lambda^{smf+\mu}E_l \). If \( \lambda \) does not lie above \( p \) \((l \neq p) \), we denote by \( k \) the ring \( E_l/\lambda^\mu E_l \). We shall view \( \theta, \theta_m \) and \( \theta_{m,\mathcal{D}} \) as \( k \)-valued functions.

Proposition 6.5. The function \( \theta_m : M_{H_m} \to k \) is a non-zero eigenfunction for all Hecke operators \( T_v (v \not\mid \mathcal{P}\mathcal{N}'\mathcal{D}') \) with \( T_v \theta_m = a_v \theta_m \).

Proof. It is clear that \( \theta_m \) is an eigenfunction for all Hecke operators \( T_v (v \not\mid \mathcal{P}\mathcal{N}'\mathcal{D}') \) with \( T_v \theta_m = a_v \theta_m \).

For every \( u \in (O_F,\mathcal{P}/\mathcal{P}^m)^* \), we put

\[
\beta_u = (1, 1, ..., h_\mathcal{P} \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} h_\mathcal{P}^{-1}, ..., 1, 1),
\]

\( \mathcal{P}^{th\text{ place}} \)
where the element $h_P$ is as specified in (2). If $\theta_m = 0$ as a $k$-valued function, then

$$0 = \sum_{u \in (O_F/P^m)^*} \beta_u \theta_m$$

$$= \sum_{a \in O_F/P^m} \sum_{u \in (O_F/P^m)^*} \chi_1(\tau_{ua}) \alpha_{a,m} \theta$$

$$= \sum_{a \in O_F/P^m} \alpha_{a,m} \theta \sum_{u \in (O_F/P^m)^*} \chi_1(\tau_{ua})$$

Let $q = p^f$ denote the cardinality of the residue class field $O_F/P$. By means of Lemma 6.7 below, we get

$$0 = q^{m-1}(q - 1)\theta - q^{m-1} \sum_{a \in (O_F/P)^*} \alpha_{a,1} \theta$$

$$= q^m \theta - q^{m-1} \sum_{a \in O_F/P} \alpha_{a,1} \theta$$

$$= q^{m-1} \left( q\theta - \sum_{a \in O_F/P} \alpha_{a,1} \theta \right)$$

$$= q^{m-1} \theta^+.$$ 

This yields a contradiction since $q^{m-1} \theta^+$ is non-zero by Lemma 4.12 in [3]; the proof of this lemma uses the fact that $\theta$ is non-eisenstein modulo $\lambda^\mu$. The reader is referred to [3] for a description of the function $\theta^+$ and its properties (see, for example, Section 1.6, Theorem 5.10 and the Appendix).

**Corollary 6.6.** $\theta_m$ is non-exceptional as a $k$-valued function.

**Lemma 6.7.** For $a \in O_F/P^m$, we have

$$\sum_{u \in (O_F/P^m)^*} \chi_1(\tau_{ua}) = \left\{ \begin{array}{ll} q^{m-1}(q - 1) & a \in P^m \\ -q^{m-1} & a \in P^{m-1}/P^m \text{ and } a \notin P^m \\ 0 & \text{otherwise} \end{array} \right.$$ 

**Proof.** The statement of the lemma follows trivially for $a \equiv 0 \mod P^m$. For the remaining cases, we write

$$\sum_{u \in (O_F/P^m)^*} \chi_1(\tau_{ua}) = \sum_{u \in O_F/P^m} \chi_1(\tau_{ua}) - \sum_{u \in P/P^m} \chi_1(\tau_{ua})$$

Notice that if $a \in P^{m-1}/P^m$, we have

$$\sum_{u \in O_F/P^m} \chi_1(\tau_{ua}) = 0 \text{ and } \sum_{u \in P/P^m} \chi_1(\tau_{ua}) = q^{m-1}. $$
Proposition 6.8. The function \( \theta_{m,D} : M_{H_{m,D}} \to k \) is a non-zero eigenfunction for all Hecke operators \( T_v \) away from \( \mathcal{PN}D' \) with \( T_v \theta_{m,D} = a_v \theta_{m,D} \).

Proof. By definition (see [3] p. 57),

\[
\theta_{m,D} = \sum_{d \mid D'} \chi_0(\sigma_d)(\alpha_d \theta_m),
\]

where \( \alpha_d = \prod_{Q \mid d} \alpha_Q \), and \( \alpha_Q \) is an element in \( R_Q \sim M_2(O_{F,Q}) \) whose reduced norm is a uniformizer in \( O_{F,Q} \). Notice that \( \theta_{m,D} \) is left-invariant under \( H_{m,D} = \hat{R}_{m,D} \) where \( R_{m,D} \) is the unique \( O_F \)-order which agrees with \( R_m \) outside \( D' \) and equals \( R_Q \cap \alpha_Q R_Q \alpha_Q^{-1} \) at \( Q \mid D' \).

If we fix a prime divisor \( Q \) of \( D' \), it is easy to see that \( \theta_{m,D} \) can be rewritten as

\[
\theta_{m,D} = \sum_{d \mid D'} \chi_0(\sigma_d)(\alpha_d \theta_m) + \chi_0(\sigma_Q)\alpha_Q \sum_{d \mid D'} \chi_0(\sigma_d)(\alpha_d \theta_m).
\]

Let \( \vartheta_1 \) and \( \vartheta_2 \) be \( k \)-valued functions on \( M_{H_m} \) satisfying \( T_v \vartheta_i = a_v \vartheta_i \) for all \( v \nmid \mathcal{PN}D' \). We claim that any nontrivial linear combination \( a \vartheta_1 + b \alpha_Q \vartheta_2 \) is non-zero in \( k \). If \( a \vartheta_1 + b \alpha_Q \vartheta_2 = 0 \) for some scalars \( a \) and \( b \), then \( a \vartheta_1 = -b \alpha_Q \vartheta_2 \) is fixed under the group spanned by \( R_Q^* \) and \( \alpha_Q R_Q \alpha_Q^{-1} \) which contains the kernel of the reduced norm map \( B_P \to F_P \). It follows from the strong approximation theorem ([12] p. 81) that \( \vartheta_1 \) factors through the norm map as a \( k \)-valued function, which is a contradiction to the fact that \( a_v \neq q_v + 1 \mod \lambda^u \) for some \( v \in S \) (Lemma [6.2]). Hence, \( a \vartheta_1 + b \alpha_Q \vartheta_2 \) is non-zero. Not only this, but \( a \vartheta_1 + b \alpha_Q \vartheta_2 \) is also an eigenfunction for all \( T_v (v \nmid \mathcal{PN}D') \) with the same eigenvalues as \( \vartheta_1 \) and \( \vartheta_2 \).

In light of the above observation, we proceed by induction on the number of prime ideal divisors of \( D' \) to prove that \( \theta_{m,D} \) is non-zero and satisfies \( T_v \theta_{m,D} = a_v \theta_{m,D} \) for all \( v \nmid \mathcal{PN}D' \). This reduces the problem to the case of \( \vartheta_m \) which satisfies the required hypothesis by Proposition [5.3].

Corollary 6.9. \( \theta_{m,D} \) is non-exceptional as a \( k \)-valued function.

Now we state and prove the main result in this paper. This result gives an upper bound for the \( l \)-adic valuation of the sum

\[
\sum_{\tau \in \mathcal{R}} \chi_0(\tau) \psi_{m,D}(\tau,x_{m,D}),
\]

which we recall is related to the Gross-Zagier sum \( a(x,\chi) \) by the formula

\[
\text{Tr}(a(x,\chi)) = [E_\lambda(\chi_1) : E_\lambda] \sum_{\tau \in \mathcal{R}} \chi_0(\tau) \psi_{m,D}(\tau,x_{m,D}).
\]
**Theorem 6.10.** Let \( \chi_0 \) be any character of \( G_0 \). For any \( x \in CM_{H_{m,D}}(\mathcal{P}^n) \) with \( n \gg 0 \), there exists some \( y \in G(\infty).x \) such that

\[
\sum_{\tau \in \mathcal{R}} \chi_0(\tau) \psi_{m,D}(\tau.y) \not\equiv 0 \quad (\text{in } k).
\]

**Proof.** We follow the proof of Corollary 5.7 in [3]. Since \( \theta_{m,D} \) is non-exceptional as a \( k \)-valued function, there exists \( \sigma \in G(\infty) \) such that \( \theta_{m,D} \) is non-constant as a \( k \)-valued function on \( c^{-1}(c \circ \text{red}(\sigma.x)) \). Choose \( p_1, p_2 \in c^{-1}(c \circ \text{red}(\sigma.x)) \) such that \( \theta_{m,D}(p_1) \not\equiv \theta_{m,D}(p_2) \) (in \( k \)). If \( n \) is sufficiently large, Theorem 5.1 guarantees the existence of \( y_1, y_2 \in G(\infty).x \) such that

\[
\text{red}(y_1) = p_1, \quad \text{red}(y_2) = p_2
\]

and

\[
\text{red}(\tau.y_1) = \text{red}(\tau.x) = \text{red}(\tau.y_2) \quad \text{for all } \tau \neq 1 \text{ in } \mathcal{R}.
\]

We thus obtain

\[
\sum_{\tau \in \mathcal{R}} \chi_0(\tau) \psi_{m,D}(\tau.y_1) - \sum_{\tau \in \mathcal{R}} \chi_0(\tau) \psi_{m,D}(\tau.y_2) = \theta_{m,D}(p_1) - \theta_{m,D}(p_2)
\]

\[
\not\equiv 0 \quad (\text{in } k).
\]

Therefore, at least one of the sums \( \sum_{\tau \in \mathcal{R}} \chi_0(\tau) \psi_{m,D}(\tau.y_1) \) or \( \sum_{\tau \in \mathcal{R}} \chi_0(\tau) \psi_{m,D}(\tau.y_2) \) is non-zero in \( k \).

Finally, we remark that one can easily obtain a lower bound on the \( l \)-adic valuation of the Gross-Zagier sum \( a(x, \chi) \). In fact, let \( \nu \) be the largest integer such that \( \theta \) is Eisenstein modulo \( \lambda^{\nu} \). Then

\[
\sum_{\sigma \in G(n)} \chi(\sigma) \theta \circ \text{red}(\sigma.x) = \sum_{\sigma \in G(n)} \chi(\sigma) \theta(\text{red}(\sigma.x))
\]

\[
\equiv \sum_{\sigma \in G(n)} \chi(\sigma) \theta(c \circ \text{red}(\sigma.x))
\]

\[
\equiv \sum_{\sigma \in G(n)} \chi(\sigma) \theta(\sigma.c \circ \text{red}(x))
\]

\[
\equiv \sum_{\sigma \in G(n)} \chi(\sigma) \theta(\text{rd}(\beta)c \circ \text{red}(x))
\]

\[
\equiv 0 \mod \lambda^{\nu},
\]

where the last line follows from the orthogonality property of group characters. Hence,

\[
\text{ord}_\lambda \left( \sum_{\sigma \in G(n)} \chi(\sigma) \psi(\sigma.x) \right) \geq \nu.
\]
It is obvious that this simple observation combined with Theorem 6.10 would give an exact value for the \( l \)-adic valuation of \( a(x, \chi) \) if we have \( \nu + 1 = \mu \). However, it is not clear to us whether the statement \( \nu + 1 = \mu \) is true or not. This is a very interesting question, but we choose not to discuss it in this work. We remark only that the answer seems to be connected to multiplicity-one-type results for the component group of a Shimura curve at Eisenstein primes.

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