ON A FRACTIONAL VERSION OF HAEMERS’ BOUND

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Abstract. In this note, we present a fractional version of Haemers’ bound on the Shannon capacity of a graph, which is originally due to Blasiak. This bound is a common strengthening of both Haemers’ bound and the fractional chromatic number of a graph. We show that this fractional version outperforms any bound on the Shannon capacity that could be attained through Haemers’ bound. We show also that this bound is multiplicative, unlike Haemers’ bound.

1. Introduction

For graphs \(G_1, \ldots, G_n\), the strong product of \(G_1, \ldots, G_n\), denoted \(G_1 \boxtimes \cdots \boxtimes G_n\), is the graph on vertex set \(V(G_1) \times \cdots \times V(G_n)\) where \((v_1, \ldots, v_n) \sim (u_1, \ldots, u_n)\) if and only if for every \(i \in [n]\), either \(v_i = u_i\) or \(v_i u_i \in E(G_i)\). For brevity, we write \(G^{\boxtimes n} = G \boxtimes \cdots \boxtimes G\).

The Shannon capacity of a graph \(G\), introduced by Shannon in [8], is

\[
\Theta(G) := \sup_n \alpha(G^{\boxtimes n})^{1/n} = \lim_{n \to \infty} \alpha(G^{\boxtimes n})^{1/n},
\]

where \(\alpha(G)\) denotes the independence number of \(G\). Despite the fact that Shannon defined this parameter in 1956, very little is known about it in general. For example, \(\Theta(C_7)\) is still unknown.

There are two general upper bounds on \(\Theta(G)\). Firstly, the theta function, \(\vartheta(G)\), is a bound on \(\Theta(G)\) which is the solution to a semi-definite program dealing with arrangements of vectors associated with \(G\). Introduced by Lovász in [6], the theta function was used to verify that \(\Theta(C_5) = \vartheta(C_5) = \sqrt{5}\).

Secondly, Haemers’ bound, \(\mathcal{H}(G; \mathbb{F})\), is a bound on \(\Theta(G)\) which considers the rank of particular matrices over the field \(\mathbb{F}\) associated with the graph \(G\). Introduced by Haemers in [4, 3], \(\mathcal{H}(G; \mathbb{F})\) was used to provide negative answers to three questions put forward by Lovász in [6].

In this paper, we present a strengthening of Haemers’ bound by defining a parameter \(\mathcal{H}_f(G; \mathbb{F})\), to which we refer to as the fractional Haemers bound. After we wrote the paper, we learned from Ron Holzman, that this parameter previously appeared in a thesis of Anna Blasiak [2, Section 2.2]. We defer the definition of this parameter to Section 2. We show the following results:

**Theorem 1** (First proved by Blasiak [2]). For any graph \(G\) and a field \(\mathbb{F}\),

\[
\Theta(G) \leq \mathcal{H}_f(G; \mathbb{F}) \leq \mathcal{H}(G; \mathbb{F}).
\]
The following results are new.

**Theorem 2.** For any field $F$ of nonzero characteristic, there exists an explicit graph $G = G(F)$ with

$$
\mathcal{H}_f(G; F) < \min\{\mathcal{H}(G; F'), \vartheta(G)\},
$$

for every field $F'$.

Therefore, $\mathcal{H}_f$ is a strict improvement over both $\mathcal{H}$ and $\vartheta$ for some graphs.

**Remark.** Recently, Hu, Tamo and Shayevitz in \[5\] constructed a different generalization of $\mathcal{H}(G; F)$ using linear programming. Their bound, $\minrk_F(G)$, satisfies $\mathcal{H}_f(G; F) \leq \minrk_F(G)$ for every graph $G$ and every field $F$. They show that there exists a graph $G$ for which $\minrk_F(G) < \min\{\mathcal{H}(G; F'), \vartheta(G)\}$ for every field $F'$, so Theorem 2 holds also when $F = \mathbb{R}$.

Recall that Lovász showed $\vartheta(G \boxtimes H) = \vartheta(G) \cdot \vartheta(H)$ for any graphs $G, H$; the fractional Haemers bound shares this property.

**Theorem 3.** For graphs $G, H$ and a field $F$,

$$
\mathcal{H}_f(G \boxtimes H; F) = \mathcal{H}_f(G; F) \cdot \mathcal{H}_f(H; F).
$$

This is in contrast to $\mathcal{H}(G; F)$. As we will show in Proposition 4 for any field $F$, $\mathcal{H}(C_5; F) \geq 3$, yet $\mathcal{H}(C_5^{\otimes 2}; F) \leq 8$.

Sadly, $\mathcal{H}_f$ does not improve upon the known bounds for the Shannon capacity of odd cycles.

**Proposition 4.** For any positive integer $k$ and any field $F$, $\mathcal{H}_f(C_{2k+1}; F) = k + \frac{1}{2}$.

The organization of this paper is as follows: in Section 2 we will define $\mathcal{H}_f(G; F)$; in fact, we will provide four equivalent definitions, each of which will be useful. In Section 3, we will prove Theorems 1 and 2 and Proposition 4. In Section 4, we will prove Theorem 2 and also show that $\mathcal{H}_f(G; F)$ and $\mathcal{H}_f(G; F')$ can differ when $F$ and $F'$ are different fields. We will then briefly look at two attempts to “fractionalize” the Lovász theta function in Section 5. We conclude with a list of open problems in Section 6.

2. **The fractional Haemers bound**

For a graph $G$ and a field $F$, a matrix $M = (m_{uv}) \in F^{V \times V}$ is said to fit $G$ if $m_{vv} = 1$ for all $v \in V$ and $m_{uv} = m_{vu} = 0$ whenever $uv \notin E$. Define $\mathcal{M}_F(G)$ to be the set of all matrices over $F$ that fit $G$. The Haemers bound \[4\] \[3\] of $G$ is then defined as

$$
\mathcal{H}(G; F) := \min\{\text{rank}(M) : M \in \mathcal{M}_F(G)\}.
$$

Haemers introduced $\mathcal{H}(G; F)$ as an upper bound on $\Theta(G)$ in order to provide negative answers to three questions put forward by Lovász in \[6\]. When the field $F$ is understood or arbitrary, we will condense the notation and write $\mathcal{H}(G) = \mathcal{H}(G; F)$. 
A drawback of Haemers’ bound is that it is always an integer. To combat this, we introduce a fractional version.

Let \( F \) be a field, \( d \) be a positive integer and \( G \) be a graph, and consider matrices \( M \) over \( F \) whose rows and columns are indexed by \( V \times [d] \). We can consider \( M \) as a block matrix where for \( u, v \in V \), the \( uv \) block, \( M_{uv} \), consists of the entries with indices \( (u, i), (v, j) \) for all \( i, j \in [d] \). We say that \( M \) is a \( d \)-representation of \( G \) over \( F \) if

1. \( M_{uv} = I_d \) for all \( v \in V \), where \( I_d \) is the \( d \times d \) identity matrix, and
2. \( M_{uv} = M_{vu} = O_d \) for all \( uv \notin E \), where \( O_d \) is the \( d \times d \) zero matrix.

Define \( M^d_f(G) \) to be the set of all \( d \)-representations of \( G \) over \( F \). We then define the fractional Haemers bound to be

\[
\mathcal{H}_f(G; F) := \inf \left\{ \frac{\text{rank}(M)}{d} : M \in M^d_f(G), d \in \mathbb{Z}^+ \right\}.
\]

Notice that \( M_F(G) = M^1_F(G) \), so \( \mathcal{H}_f(G; F) \leq \mathcal{H}(G; F) \). Again, when the field is understood or arbitrary, we will condense the notation and simply write \( \mathcal{H}_f(G) \). More specifically, if we were to write, e.g., \( \mathcal{H}_f(G) \leq \mathcal{H}_f(H) \), it is assumed that the field is the same in both instances.

2.1. Alternative formulations. We now set out three equivalent ways to define \( \mathcal{H}_f(G) \), each of which will be useful going forward.

For positive integers \( d \leq n \) (unrelated to the graph \( G \)), consider assigning to each \( v \in V \) a pair of matrices \((A_v, B_v) \in (\mathbb{F}^{n \times d})^2 \). We say that such an assignment is an \((n, d)\)-representation of \( G \) over \( \mathbb{F} \) if

1. \( A_v^T B_v = I_d \) for every \( v \in V \), and
2. \( A_v^T B_v = A_v^T B_u = O_d \) whenever \( uv \notin E \).

**Proposition 5.** For a graph \( G \) and a field \( F \),

\[
\mathcal{H}_f(G; F) = \inf_{n,d} \left\{ \frac{n}{d} : G \text{ has an } (n, d)\text{-representation over } F \right\}.
\]

**Proof.** A matrix \( M \in \mathbb{F}^{(V \times [d]) \times (V \times [d])} \) has rank(M) \( \leq n \) if and only if it is of the form \( M = A^T B \) where \( A, B \in \mathbb{F}^{n \times (V \times [d])} \). Let \( A_v \) be the submatrix of \( A \) consisting of all entries with indices \( (i, (v, j)) \) for \( i \in [n] \) and \( j \in [d] \), and let \( B_v \) be defined similarly for \( B \). With this, for any \( u, v \in V \), \( A_u^T B_v = M_{uv} \). Therefore, \( M \in M^d_F(G) \) if and only if \( A_v^T B_v = I_d \) for every \( v \in V \) and \( A_u^T B_u = A_v^T B_u = O_d \) whenever \( uv \notin E \). \( \square \)

A second way to understand \( \mathcal{H}_f(G) \) is by considering the lexicographic product, \( G \ltimes H \), which is formed by “blowing up” each vertex of \( G \) into a copy of \( H \). More formally, \( V(G \ltimes H) = V(G) \times V(H) \) and \((u, x) \sim (v, y) \) in \( G \ltimes H \) whenever either \( uv \in E(G) \) or \( u = v \) and \( xy \in E(H) \). In this context, it easy to verify that \( M^d_F(G) = M_F(G \ltimes K_d) \), so:
Proposition 6. For a graph $G$, 
\[
\mathcal{H}_f(G) = \inf_d \frac{\mathcal{H}(G \times K_d)}{d}.
\]

The last equivalent formulation of $\mathcal{H}_f(G)$ is, in some sense, the most general. Consider matrices $M$ over $\mathbb{F}$ whose rows and columns are indexed by $\{(v, i) : v \in V, i \in [d_v]\}$ where $d_v$ is some positive integer assigned to $v$. As with $d$-representations, we consider $M$ as a block matrix where for $u, v \in V$, the $uv$ block, $M_{uv}$, consists of those entries with indices $(u, i), (v, j)$ for all $i \in [d_u], j \in [d_v]$. We say that $M$ is a rank-$r$-representation of $G$ over $\mathbb{F}$ if

1. $\text{rank}(M_{uv}) \geq r$ for all $v \in V$, and
2. $M_{uv} = M_{vu}^T = O_{d_u \times d_v}$ whenever $uv \notin E$, where $O_{d_u \times d_v}$ is the $d_u \times d_v$ zero matrix.

Proposition 7. For a graph $G$ and a field $\mathbb{F}$, 
\[
\mathcal{H}_f(G; \mathbb{F}) = \inf_{M, r} \left\{ \frac{\text{rank}(M)}{r} : M \text{ is a rank-}$r$-representation of $G$ over $\mathbb{F} \right\}.
\]

Proof. The lower bound is immediate as an $r$-representation of $G$ is also a rank-$r$-representation. For the other direction, let $M$ be a rank-$r$-representation of $G$ over $\mathbb{F}$ for some $r$. As $\text{rank}(M_{uv}) \geq r$ for all $v \in V$, we can find an $r \times r$ submatrix of $M_{uv}$ of full rank, call this submatrix $M'_{uv}$. Let $M'$ be the submatrix of $M$ induced by the blocks $\{M'_{uv} : v \in V\}$; we index the rows and columns of $M'$ by $V \times [r]$. For any fixed $v \in V$, $M'_{uv}$ has full rank, so we may perform row operations on $M'$ using only the rows indexed by $\{v\} \times [r]$ to transform $M'_{uv}$ into $I_r$. Let $M''$ be the matrix formed by doing this for every $v \in V$. Therefore, $M''_{uv} = I_r$ for every $v \in V$. Further, as all row operations occurred only between rows corresponding to the same vertex, if $M_{uv} = O_{d_u \times d_v}$, then we also have $M''_{uv} = O_r$. Thus, as $M$ was a rank-$r$-representation of $G$, $M''$ is an $r$-representation of $G$. We conclude that
\[
\mathcal{H}_f(G) \leq \frac{\text{rank}(M'')}{r} = \frac{\text{rank}(M')}{r} \leq \frac{\text{rank}(M)}{r},
\]
so the same is true of the infimum over all $M$ and $r$. \hfill $\square$

Remark. While this paper was in submission, Lex Schrijver introduced us to the following equivalent definition of $\mathcal{H}_f(G; \mathbb{F})$. For fixed positive integers $d \leq n$, a collection of subspaces $\{S_v \leq \mathbb{F}^n : v \in V\}$ is called an $(n, d)$-subspace-representation of $G$ over $\mathbb{F}$ if

1. $\dim S_v = d$ for all $v \in V$, and
2. $S_v \cap \left( \sum_{u \neq v} S_u \right) = \{0\}$, where the summation is over $u$ satisfying $uv \notin E$.

With this, for a graph $G$ and a field $\mathbb{F}$, we have
\[
\mathcal{H}_f(G; \mathbb{F}) = \inf \left\{ \frac{n}{d} : G \text{ has an } (n, d)\text{-subspace-representation over } \mathbb{F} \right\}.
\]
This formulation can be used to give a coordinate-free proof of Theorem 3.
3. Proofs of Theorems 1 and 3 and Proposition 4

We now set out to prove that \( \Theta(G) \leq H_f(G) \) and explore some basic properties.

Recalling the lexicographic product of graphs, the \textit{fractional chromatic number} of a graph \( G \) is defined as
\[
\chi_f(G) := \inf_d \frac{\chi(G \times K_d)}{d}.
\]
In his original paper, Shannon \cite{Shannon} established \( \Theta(G) \leq \chi_f(G) \).

In the same spirit, Lovász showed that \( \alpha(G) \leq \vartheta(G) \leq \chi_f(G) \). Similarly, Haemers established \( \alpha(G) \leq \mathcal{H}(G) \leq \chi(G) \); note that, in general, \( \mathcal{H}(G) \not\leq \chi_f(G) \), e.g. \( \chi_f(C_5) = 5/2 \) whereas \( \mathcal{H}(C_5) \geq 3 \).

**Theorem 8.** For any graph \( G \),
\[
\alpha(G) \leq \mathcal{H}_f(G) \leq \chi_f(G).
\]

**Proof.** Notice that for any graphs \( G, H \), \( \alpha(G \times H) = \alpha(G) \cdot \alpha(H) \) and \( \overline{G} \times \overline{H} = \overline{G \times H} \). Thus, as \( \alpha(G) \leq \mathcal{H}(G) \leq \chi(G) \), we find
\[
\alpha(G) = \inf_d \frac{d \cdot \alpha(G)}{d} \leq \inf_d \frac{\mathcal{H}(G \times K_d)}{d} = \mathcal{H}_f(G),
\]
and
\[
\mathcal{H}_f(G) = \inf_d \frac{\mathcal{H}(G \times K_d)}{d} \leq \inf_d \frac{\chi(G \times K_d)}{d} = \chi_f(G).
\]

We now provide a proof of Theorem 3. Before we do so, recall that \( \vartheta(G \boxtimes H) = \vartheta(G) \cdot \vartheta(H) \); however, the same is not true of \( \mathcal{H}(G) \).

**Proposition 9.** For any field \( \mathbb{F} \), \( \mathcal{H}(C_5; \mathbb{F}) \geq 3 \), yet \( \mathcal{H}(C_5 \boxtimes C_5; \mathbb{F}) \leq 8 \).

**Proof.** As shown by Lovász, \( \Theta(C_5) = \sqrt{5} \). Thus, as \( \mathcal{H} \) is always an integer, \( \mathcal{H}(C_5; \mathbb{F}) \geq \lceil \sqrt{5} \rceil = 3 \).

On the other hand, it is not difficult to verify that \( \chi(C_5 \boxtimes C_5) \leq 8 \); indeed, Figure 1 provides such a coloring. Therefore, \( \mathcal{H}(C_5 \boxtimes C_5; \mathbb{F}) \leq 8 \) for any field \( \mathbb{F} \).

![Figure 1](image.png)  
Figure 1. A coloring of \( C_5 \boxtimes C_5 \) using 8 colors in which each color class is a clique.
Proof of Theorem 4. Upper bound. Let $M \in \mathcal{M}_{d_1}^d(G)$ and $N \in \mathcal{M}_{d_2}^d(H)$, and set $M^* = M \otimes N$ where $\otimes$ is the tensor/Kronecker product. We claim that $M^* \in \mathcal{M}_{d_1 d_2}^d(G \boxtimes H)$. Indeed, the rows and columns of $M^*$ are indexed by $(V(G) \times [d_1]) \times (V(H) \times [d_2])$, so we may in fact suppose they are indexed by $V(G \boxtimes H) \times [d_1 d_2]$. Further, for any $(u, x), (v, y) \in V(G \boxtimes H)$, the $(u, x)(v, y)$ block of $M^*$ satisfies $M_{(u, x)(v, y)}^* = M_{uv} \otimes N_{xy}$. As such, $M^*_{(u, x)(v, y)} = I_{d_1} \otimes I_{d_2} = I_{d_1 d_2}$. Further, if $(u, x) \not\sim (v, y)$ in $G \boxtimes H$, then either $uv \not\in E(G)$ or $xy \not\in E(H)$, so either $M_{uv} = M_{vv} = 0$ or $N_{xy} = N_{yy} = 0$. In either case, $M^*_{(u, x)(v, y)} = M^*_{(v, y)(u, x)} = 0_{d_1 d_2}$, so we have verified $M^* \in \mathcal{M}_{d_1 d_2}^d(G \boxtimes H)$.

Finally, $\text{rank}(M^*) = \text{rank}(M) \cdot \text{rank}(N)$, so

$$\mathcal{H}_f(G \boxtimes H) \leq \frac{\text{rank}(M^*)}{d_1 d_2} = \frac{\text{rank}(M)}{d_1} \cdot \frac{\text{rank}(N)}{d_2}.$$ Taking infimums establishes $\mathcal{H}_f(G \boxtimes H) \leq \mathcal{H}_f(G) \cdot \mathcal{H}_f(H)$.

Lower bound. Let $M \in \mathcal{M}_d^d(G \boxtimes H)$ for some $d$. For $u, v \in V(G)$, let $[M]_{uv}$ denote the submatrix of $M$ consisting of all blocks of the form $M_{(u, x)(v, y)}$ for $x, y \in V(H)$. Certainly we can consider $M$ as a matrix with blocks $[M]_{uv}$, which is a $d_u \times d_v$ matrix for positive integers $d_u, d_v$. Additionally, for every $v \in V(G)$, the submatrix $[M]_{vv}$ can be considered a $d$-representation of the graph $H$, where the $xy$ block is $([M]_{vv})_{xy} = M_{(v, x)(v, y)}$. Set $r = \min_{v \in V(G)} \text{rank}([M]_{vv})$, so $\mathcal{H}_f(H) \leq r/d$.

On the other hand, if $uv \not\in E(G)$, then $[M]_{uv} = 0_{d_u \times d_v}$ and $[M]_{vu} = 0_{d_v \times d_u}$ as $(u, x) \not\sim (v, y)$ in $G \boxtimes H$ for every $x, y \in V(H)$. Thus, as $\text{rank}([M]_{uv}) \geq r$ for all $v \in V(G)$, $M$ is a rank-$r$ representation of $G$, so $\mathcal{H}_f(G) \leq \text{rank}(M)/r$.

Putting the bounds on $\mathcal{H}_f(G)$ and $\mathcal{H}_f(H)$ together, we have

$$\mathcal{H}_f(G) \cdot \mathcal{H}_f(H) \leq \frac{\text{rank}(M)}{r} \cdot \frac{r}{d} = \frac{\text{rank}(M)}{d},$$

so taking infimums yields $\mathcal{H}_f(G) \cdot \mathcal{H}_f(H) \leq \mathcal{H}_f(G \boxtimes H)$. \qed

Putting together Theorems 3 and 8 we arrive at a proof of Theorem 1.

Proof of Theorem 5. We have already noted that $\mathcal{H}_f(G) \leq \mathcal{H}(G)$. On the other hand, by Theorems 3 and 8

$$\Theta(G) = \sup_n \alpha(G^{\otimes n})^{1/n} \leq \sup_n \mathcal{H}_f(G^{\otimes n})^{1/n} = \sup_n \mathcal{H}_f(G) = \mathcal{H}_f(G).$$

Theorem 3 has another nice corollary. Certainly $\Theta(G) \leq \mathcal{H}(G)$, but one could additionally attain bounds on $\Theta(G)$ by using Haemers’ bound on large powers of $G$, i.e. $\Theta(G) \leq \mathcal{H}(G^{\otimes n})^{1/n}$. This could lead to improved bounds as in general $\mathcal{H}(G^{\otimes 2}) < \mathcal{H}(G)^2$, e.g. $G = C_5$. It turns out that $\mathcal{H}_f(G)$ outperforms any bound attained in this fashion.

Corollary 10. For any positive integer $n$ and graph $G$, $\mathcal{H}_f(G) \leq \mathcal{H}(G^{\otimes n})^{1/n}$.

Proof. By Theorem 3 we calculate $\mathcal{H}_f(G) = \mathcal{H}_f(G^{\otimes n})^{1/n} \leq \mathcal{H}(G^{\otimes n})^{1/n}$. \qed
To end this section, we show that the fractional Haemers bound cannot improve upon the known bounds for the Shannon capacity of odd cycles. We require the following observation about \((n, d)\)-representations of a graph.

**Proposition 11.** Let \(G\) be a graph and \(\{(A_v, B_v) : v \in V\} \subseteq (\mathbb{F}^{n \times d})^2\) be an \((n, d)\)-representation of \(G\). For \(v \in V\), let \(X_v\) denote the column space of \(A_v\). If \(S, T \subseteq V\) are disjoint sets of vertices where \(S\) is an independent set and there are no edges between \(S\) and \(T\), then the subspaces \(\sum_{v \in S} X_v\) and \(\sum_{v \in T} X_v\) are linearly independent.

**Proof.** Let \(\{a_{v, i} : i \in [d]\}\) be the columns of \(A_v\) and \(\{b_{v, i} : i \in [d]\}\) be the columns of \(B_v\). As \(\{(A_v, B_v) : v \in V\}\) is an \((n, d)\)-representation of \(G\), we know that

1. \(\langle a_{v, i}, b_{v, i} \rangle = 1\) for every \(v \in V, i \in [d]\),
2. \(\langle a_{v, i}, b_{v, j} \rangle = 0\) for every \(v \in V\) and \(i \neq j\), and
3. \(\langle a_{u, i}, b_{v, j} \rangle = 0\) for every \(i, j \in [d]\) whenever \(uv \notin E\).

Let \(B\) be a basis for \(\sum_{v \in T} X_v\) and consider a linear combination

\[
\sum_{v \in S} \sum_{i=1}^{d} c_{v, i}a_{v, i} + \sum_{x \in B} d_{x}x = 0,
\]

where \(c_{v, i}, d_{x} \in \mathbb{F}\) for every \(v \in S, i \in [d], x \in B\). As \(S\) is an independent set and there are no edges between \(S\) and \(T\), we find that for any \(u \in S, j \in [d]\),

\[
0 = \left\langle \sum_{v \in S} \sum_{i=1}^{d} c_{v, i}a_{v, i} + \sum_{x \in B} d_{x}x, b_{u, j} \right\rangle = c_{u, j}.
\]

Therefore \(c_{v, i} = 0\) for every \(v \in S, i \in [d]\), so we must have \(\sum_{x \in B} d_{x}x = 0\). As \(B\) is a basis, this implies \(d_{x} = 0\) for every \(x \in B\).

Thus, as \(\sum_{v \in S} X_v = \text{span}_{\mathbb{F}} \{a_{v, i} : v \in S, i \in [d]\}\) and \(\sum_{v \in T} X_v = \text{span}_{\mathbb{F}} B\), we have shown that \(\sum_{v \in S} X_v\) and \(\sum_{v \in T} X_v\) are linearly independent subspaces. \(\square\)

**Proof of Proposition 11.** We will show that \(\mathcal{H}_f(C_{2k+1}) = \chi_f(C_{2k+1}) = k + \frac{1}{2}\). It is well-known that \(\chi_f(C_{2k+1}) = k + \frac{1}{2}\), so we will focus only on the lower bound.

Identify the vertices of \(C_{2k+1}\) with \(\mathbb{Z}_{2k+1}\) in the natural way and let \(\{(A_i, B_i) \in (\mathbb{F}^{n \times d})^2 : i \in \mathbb{Z}_{2k+1}\}\) be an \((n, d)\)-representation of \(C_{2k+1}\) for any \(n, d\). Let \(X_i\) denote the column space of \(A_i\). As \(A_i^T B_i = I_d\), we observe that \(\dim(X_i) = d\).

We observe that \(I = \{3, 5, 7, \ldots, 2k - 1\}\) is an independent set in \(C_{2k+1}\) and further that the edge \(0 \sim 1\) is not adjacent to any vertex in \(I\). By iterating Proposition 11, we find that

\[
\dim \left( (X_0 + X_1) + \sum_{i \in I} X_i \right) = \dim(X_0 + X_1) + \dim \left( \sum_{i \in I} X_i \right) = \dim(X_0 + X_1) + \sum_{i \in I} \dim(X_i),
\]
so

\[ n \geq \dim\left( (X_0 + X_1) + \sum_{i \in I} X_i \right) \]
\[ = \dim(X_0 + X_1) + \sum_{i \in I} \dim(X_i) \]
\[ = \dim(X_0) + \dim(X_1) - \dim(X_0 \cap X_1) + \sum_{i \in I} \dim(X_i) \]
\[ = (k + 1)d - \dim(X_0 \cap X_1). \]

From this, we have \( \dim(X_0 \cap X_1) \geq (k + 1)d - n \), and so by symmetry, for any \( i \in \mathbb{Z}_{2k+1} \),
\( \dim(X_i \cap X_{i+1}) \geq (k + 1)d - n \).

Because \( 1 \neq 2k \) in \( C_{2k+1} \), by Proposition 11 it follows that \( X_1 \cap X_{2k} = \{0\} \). Since we also have \( (X_0 \cap X_1) + (X_0 \cap X_{2k}) \leq X_0 \), we conclude that
\[ d = \dim(X_0) \geq \dim((X_0 \cap X_1) + (X_0 \cap X_{2k})) \]
\[ = \dim(X_0 \cap X_1) + \dim(X_0 \cap X_{2k}) \geq 2((k + 1)d - n), \]
which implies that \( \frac{n}{d} \geq k + \frac{1}{2} \). Taking the infimum yields \( \mathcal{H}_f(C_{2k+1}) \geq k + \frac{1}{2} \).

\[ \square \]

4. PROOF OF THEOREM 2 AND FURTHER SEPARATION

In this section, we first give a proof of Theorem 2, namely, for every field \( F \) of nonzero characteristic, we need to find a graph \( G = G(F) \) for which \( \mathcal{H}_f(G; F) < \min\{\mathcal{H}(G; F'), \vartheta(G)\} \) for every field \( F' \). After this, we provide further separation of \( \mathcal{H}_f \) over fields of different characteristics.

For the proof of Theorem 2, we need the following result. The first part of the following lemma was provided by Haemers in [4]; we provide a full proof for completeness.

**Lemma 12.** For a prime \( p \) and an integer \( n \), let \( J_p^n \) be the graph with vertex set \( \left[ \begin{array}{c} n \\ p+1 \end{array} \right] \) where \( X \sim Y \) in \( J_p^n \) whenever \( |X \cap Y| \neq 0 \) (mod \( p \)).

1. If \( (p + 2) \mid n \) and \( F \) is a field of characteristic \( p \), then
\[ \mathcal{H}(J_p^n; F) = \alpha(J_p^n) = n. \]

2. For fixed \( p \) and all large \( n \),
\[ \vartheta(J_p^n) = \left( \frac{p}{(p + 1)^2} + o(1) \right) n^2. \]

Before continuing with the proof, it is important to point out a typo in [4] in which it is stated that \( \vartheta(J_n^2) = \frac{n(n-2)(2n-1)}{3(n-1)} \). The correct formula is \( \vartheta(J_n^2) = \frac{n(n-2)(2n-11)}{3(3n-14)} \), though for brevity’s sake we prove only \( \vartheta(J_n^2) \sim \frac{2}{3} n^2 \).

**Proof.**
1. Let \( M \in \mathbb{F}^{[n] \times [p+1]} \) be the incidence matrix of all \((p + 1)\) -subsets of \([n]\), i.e. the matrix with entries \( M_{i,X} = 1[i \in X] \). Certainly the matrix \( M^T M \) fits \( J_p^n \) over \( \mathbb{F} \) as \( \mathbb{F} \) has characteristic...
p and \( p + 1 \equiv 1 \pmod{p} \). Thus, \( \mathcal{H}(J_n^p; \mathbb{F}) \leq \text{rank}(M^TM) = \text{rank}(M) \leq n \). On the other hand, as \((p + 2) \mid n\), partition \([n]\) into sets \(I_1, \ldots, I_k\) where \(|I_i| = p + 2\) for all \(i\). If \(X, Y \subseteq I_i\) are sets of size \(p + 1\), then either \(X = Y\) or \(|X \cap Y| = p\). Thus, the collection \(\binom{I_1}{p+1} \cup \cdots \cup \binom{I_k}{p+1}\) is an independent set in \(J_n^p\) and has size \((\frac{p+2}{p+1})^n p^{(p+1)^2} = n\).

(2) We require the following fact which can be deduced quickly from \([7]\) (see specifically items \((12),\ (13)\) and \((27))\): provided \(n \geq 2(p+1)\),

\[
\vartheta(J_n^p) = \max \quad 1 + a_1 + a_{p+1}
\text{s.t.} \quad \frac{(p+1-u)(p-u-1)-u}{(p+1)(p+1)} a_1 + (-1)^u \left( \frac{-p+u-1}{p+1} \right) a_{p+1} \geq -1, \quad \text{for } u \in \{0, \ldots, p+1\}.
\]

For large \(n\), the preceding inequality can be written as

\[
\begin{align*}
\left( \frac{p+1-u}{p+1} + o(1) \right) a_1 + \left( \frac{(p+1)!}{p+1} + o(1) \right) (-1)^u \left( \frac{p+u-1}{p+1} \right) a_{p+1} & \geq -1 \quad \text{for } u \in \{0, \ldots, p\} \\
-(1 + o(1)) \left( \frac{1}{n} \right) a_1 + \left( (p+1)! + o(1) \right) (-1)^{p+1} \left( \frac{p+1}{p+1} \right) a_{p+1} & \geq -1 \quad \text{for } u = p+1,
\end{align*}
\]

where \(o(1) \rightarrow 0\) as \(n \rightarrow \infty\).

Set \(a_1 = n + o(n)\) where the precise value of the \(o(n)\) term is chosen so that \((4.2)\) is satisfied. Then set \(a_{p+1} = \frac{p}{(p+1)^2} n^2 + o(n^2)\) where the value of \(o(n^2)\) is chosen so the inequality \((4.1)\) with \(u = 1\) is satisfied. The remaining inequalities are then satisfied as well. Indeed, for even \(u\), both terms on the left side of \((4.1)\) are positive, whereas for odd \(u \geq 3\) the second term is \(o(1)\) for \(n \rightarrow \infty\). Hence, \(\vartheta(J_n^p) \geq \frac{p}{(p+1)^2} + o(1)n^2\).

On the other hand, when \(p = 2\), \((4.2)\) immediately implies that \(a_1 \leq n + o(n)\). For \(p > 2\), we also find that \(a_1 \leq n + o(n)\) by putting together \((4.2)\) and the \(u = 1\) case of \((4.1)\).

In either case, the \(u = 1\) case of \((4.1)\) implies that

\[
a_{p+1} \leq \left( 1 + o(1) \right) \left( \frac{1}{n} \left( \frac{p+1}{p+1} + o(1) \right) \right) a_1 \leq \left( \frac{p}{(p+1)^2} + o(1) \right) n^2.
\]

Hence, \(\vartheta(J_n^p) \leq 1 + a_1 + a_{p+1} \leq \left( \frac{p}{(p+1)^2} + o(1) \right) n^2\).

\(\square\)

Proof of Theorem \([2]\). Let \(\mathbb{F}\) be a field of characteristic \(p\) and set \(G = J_n^p \boxtimes C_{53}^5\) where \((p+2) \mid n\) and \(8 \nmid n\). By Proposition \([4]\) and Lemma \([12]\) we calculate

\[
\mathcal{H}_f(G; \mathbb{F}) = \mathcal{H}_f(J_n^p; \mathbb{F}) \cdot \mathcal{H}_f(C_5^5; \mathbb{F})^3 = \frac{125}{8} n.
\]

For any field \(\mathbb{F}'\), as \(n = o(J_n^p) \leq \mathcal{H}_f(J_n^p; \mathbb{F}')\) and \(\mathcal{H}_f(C_5^5; \mathbb{F}') = \frac{5}{2}\), we have

\[
\mathcal{H}(G; \mathbb{F}') \geq \mathcal{H}_f(G; \mathbb{F}') = \mathcal{H}_f(J_n^p; \mathbb{F}') \cdot \mathcal{H}_f(C_5^5; \mathbb{F}')^3 \geq \frac{125}{8} n.
\]

However, as \(8 \nmid n\) and \(\mathcal{H}\) is always an integer, we have \(\mathcal{H}(G; \mathbb{F}') \geq \lceil \frac{125}{8} n \rceil > \mathcal{H}_f(G; \mathbb{F})\).

Further, by Lemma \([12]\) and the fact that \(\vartheta(C_5^5) = \sqrt{5}\), we have

\[
\vartheta(G) = \vartheta(J_n^p) \cdot \vartheta(C_5^5)^3 = 5^{3/2} \left( \frac{p}{(p+1)^2} + o(1) \right) n^2.
\]

Thus, for sufficiently large \(n\) with \((p+2) \mid n\) and \(8 \nmid n\),

\[
\mathcal{H}_f(G; \mathbb{F}) < \min\{\mathcal{H}(G; \mathbb{F}'), \vartheta(G)\},
\]
for every field $F'$. \hfill \square

We next show that the choice of field matters when evaluating $\mathcal{H}_f$. In particular, for any field $F$ of nonzero characteristic, we will show that there is an explicit graph $G = G(F)$ for which $\mathcal{H}_f(G; F) < \mathcal{H}_f(G; F')$ for any field $F'$ with $\text{char}(F') \neq \text{char}(F)$.

First, we define a “universal graph” for $\mathcal{H}_f(G; F)$. For a field $F$ and positive integers $d \leq n$, define the graph $G_F(n, d)$ as follows: $V(G_F(n, d)) = \{ (A, B) \in (F^{n \times d})^2 : A^TB = I_d \}$ where $(A, B) \neq (C, D)$ in $G_F(n, d)$ if and only if $A^TD = C^TB = O_d$. We require the following two facts.

**Observation 13.** For graphs $G$ and $H$, if there is a graph homomorphism from $G$ to $H$, then $\mathcal{H}_f(G) \leq \mathcal{H}_f(H)$.

**Observation 14.** A graph $G$ has an $(n, d)$-representation over $F$ if and only if there is a graph homomorphism from $G$ to $G_F(n, d)$. In particular, if $\mathcal{H}(G; F) = n$, then there is a graph homomorphism from $G$ to $G_F(n, 1)$.

Further, we can essentially pin down the Shannon capacity of $G_F(n, d)$.

**Proposition 15.** For positive integers $d \leq n$ and a field $F$,

$$\left\lceil \frac{n}{d} \right\rceil = \alpha(G_F(n, d)) \leq \Theta(G_F(n, d)) \leq \mathcal{H}_f(G_F(n, d); F) \leq \frac{n}{d}.$$

**Proof.** By definition, the vertices of $G_F(n, d)$ are their own $(n, d)$-representation over $F$; therefore $\mathcal{H}_f(G_F(n, d); F) \leq \frac{n}{d}$.

On the other hand, certainly $\left\lceil \frac{n}{d} \right\rceil \leq \alpha(G_F(n, d))$ by considering the independent set made of the pairs $\{(A_i, A_i) : 1 \leq i \leq \lfloor n/d \rfloor \}$ where $A_i = [e_{(i-1)d+1} \cdots e_{id}]$.

The full claim follows from the fact that $\alpha(G) \leq \Theta(G) \leq \mathcal{H}_f(G; F)$. \hfill \square

We also require the following formulation of $\mathcal{H}(G; F)$ given by Alon in [1].

**Observation 16.** For a graph $G$ and a field $F$, consider assigning to each $v \in V$ both a polynomial and vector $(P_v, x_v) \in F[x_1, \ldots, x_n] \times F^n$. Such a collection of pairs is said to represent $G$ if $P_v(x_v) \neq 0$ for every $v \in V$ and $P_u(x_v) = P_v(x_u) = 0$ whenever $uv \notin E$. If $\{ (P_v, x_v) : v \in V \}$ represents $G$, then $\mathcal{H}(G; F) \leq \dim \text{span}_F\{P_v : v \in V \}$.

A proof of the above observation follows from considering the matrix $M = (m_{uv}) \in F^{V \times V}$ with entries $m_{uv} = (P_u(x_u))^{-1}P_v(x_v)$.

We will also require a graph used by Alon in [1]. For primes $p, q$ and a positive integer $n$, define the graph $B_n^{p, q}$ as follows: $V(B_n^{p, q}) = \{ [n] \}$ and $X \sim Y$ in $B_n^{p, q}$ if and only if $|X \cap Y| \equiv -1 \pmod{p}$. We will make use of $B_n^{p, q}$ both when $p = q$ and when $p$ and $q$ are distinct.

The majority of the following was proved in [1], but we provide a full proof for completeness.

**Proposition 17.** Let $p$ and $q$ be (not necessarily distinct) primes.
(1) If \( \mathbb{F} \) is a field of characteristic \( p \), then
\[
\mathcal{H}(B_n^{p,q}; \mathbb{F}) \leq \sum_{i=0}^{p-1} \binom{n}{i}.
\]

(2) If \( q \neq p \) and \( \mathbb{F}' \) is a field of characteristic \( q \), then
\[
\mathcal{H}(B_n^{p,q}; \mathbb{F}') \leq \sum_{i=0}^{q-1} \binom{n}{i}.
\]

(3) If \( \mathbb{F}'' \) is any field with either \( \text{char}(\mathbb{F}'') = 0 \) or \( \text{char}(\mathbb{F}'') > p \), then
\[
\mathcal{H}(\overline{B}_n^{p,q}; \mathbb{F}'') \leq \sum_{i=0}^{p-1} \binom{n}{i}.
\]

Proof. (1) For \( X \in \binom{[n]}{pq-1} \), define the polynomial \( P_X \in \mathbb{F}[x] \) in \( n \) variables by
\[
P_X(x) = \prod_{i=0}^{p-2} ((1_X, x) - i),
\]
where \( 1_X \) is the indicator vector of the set \( X \). Notice that \( |X| = pq - 1 \equiv -1 \pmod{p} \), so \( P_X(1_X) \neq 0 \). Additionally, if \( Y \in \binom{[n]}{pq-1} \) has \( |X \cap Y| \neq -1 \pmod{p} \), then \( P_X(1_Y) = 0 \) over \( \mathbb{F} \). Reducing \( P_X \) to the multilinear polynomial \( \widehat{P}_X \) by repeatedly applying the identity \( x^2 = x \), we notice that \( \widehat{P}_X \) also has the properties stated above. Thus, \( \{(\widehat{P}_X, 1_X) : X \in \binom{[n]}{pq-1}\} \) represents \( B_n^{p,q} \). Each \( \widehat{P}_X \) is a multilinear polynomial in \( n \) variables of degree at most \( p-1 \), so
\[
\mathcal{H}(B_n^{p,q}; \mathbb{F}) \leq \dim \text{span}_{\mathbb{F}} \left\{ \widehat{P}_X : X \in \binom{[n]}{pq-1} \right\} \leq \sum_{i=0}^{p-1} \binom{n}{i}.
\]

(2) For \( X \in \binom{[n]}{pq-1} \), define the polynomial \( Q_X \in \mathbb{F}'[x] \) in \( n \) variables by
\[
Q_X(x) = \prod_{i=0}^{q-2} ((1_X, x) - i).
\]
As \( |X| = pq - 1 \equiv -1 \pmod{q} \), we have \( Q_X(1_X) \neq 0 \). Additionally, if \( Y \in \binom{[n]}{pq-1} \setminus \{X\} \) has \( |X \cap Y| \equiv -1 \pmod{p} \), then as \( |X \cap Y| < pq - 1 \) and \( q \neq p \), we must have \( |X \cap Y| \equiv -1 \pmod{q} \). Therefore, \( Q_X(1_Y) = 0 \) over \( \mathbb{F}' \). Again, reducing \( Q_X \) to the multilinear polynomial \( \widehat{Q}_X \), we find that \( \{(\widehat{Q}_X, 1_X) : X \in \binom{[n]}{pq-1}\} \) represents \( \overline{B}_n^{p,q} \). As each \( \widehat{Q}_X \) is a multilinear polynomial in \( n \) variables of degree at most \( q-1 \),
\[
\mathcal{H}(\overline{B}_n^{p,q}; \mathbb{F}') \leq \dim \text{span}_{\mathbb{F}'} \left\{ \widehat{Q}_X : X \in \binom{[n]}{pq-1} \right\} \leq \sum_{i=0}^{q-1} \binom{n}{i}.
\]

(3) For \( X \in \binom{[n]}{p^2-1} \), define the polynomial \( R_X \in \mathbb{F}''[x] \) in \( n \) variables by
\[
R_X(x) = \prod_{i=1}^{p-1} ((1_X, x) - (ip - 1)).
\]
We notice that $R_X(1_X) = \prod_{i=1}^{p-1} ((p^2 - 1) - (ip - 1)) = p^{p-1}(p-1)!$. Thus, as $\text{char}(\mathbb{F}''') = 0$ or $\text{char}(\mathbb{F}''') > p$, we have $R_X(1_X) \neq 0$ over $\mathbb{F}''$. Furthermore, whenever $Y \in \binom{[n]}{p^2-1} \setminus \{X\}$ and $|X \cap Y| \equiv -1 \pmod{p}$, we have $R_X(1_Y) = 0$. Finally, reducing $R_X$ to the multilinear polynomial $\tilde{R}_X$, we know that $\{ (\tilde{R}_X, 1_X) : X \in \binom{[n]}{p^2-1} \}$ represents $\overline{B_n^{p,p}}$, so

$$\mathcal{H}(\overline{B_n^{p,p}}; \mathbb{F}'') \leq \dim \text{span}_{\mathbb{F}''} \left\{ \tilde{R}_X : X \in \left( \binom{[n]}{p^2-1} \right) \right\} \leq \sum_{i=0}^{p-1} \binom{n}{i}. \quad \square$$

**Lemma 18.** For distinct primes $p, q$ and a number $\epsilon > 0$, there is an integer $n_{p,q} = n_{p,q}(\epsilon)$ so that if $\mathbb{F}$ is a field of characteristic $p$ and $\mathbb{F}'$ is a field of characteristic $q$, then whenever $n \geq n_{p,q},$

$$\mathcal{H}(B_n^{p,q}; \mathbb{F}) < \epsilon \cdot \mathcal{H}_f(B_n^{p,q}; \mathbb{F}').$$

Further, there is an integer $n_{p,p} = n_{p,p}(\epsilon)$ so that if $\mathbb{F}''$ is any field with either $\text{char}(\mathbb{F}'') = 0$ or $\text{char}(\mathbb{F}'') > p$, then whenever $n \geq n_{p,p},$

$$\mathcal{H}(B_n^{p,p}; \mathbb{F}) < \epsilon \cdot \mathcal{H}_f(B_n^{p,p}; \mathbb{F}'').$$

**Proof.** Set $B = B_n^{p,q}$ and suppose that $\mathcal{H}(B; \mathbb{F}) \geq \epsilon \cdot \mathcal{H}_f(B; \mathbb{F}')$. For any graph $G$, $\alpha(G \boxtimes \overline{G}) \geq |V(G)|$, so by Proposition 17

$$\left( \binom{n}{pq - 1} \right) \leq \alpha(B \boxtimes \overline{B}) \leq \mathcal{H}_f(B \boxtimes \overline{B}; \mathbb{F}') = \mathcal{H}_f(B; \mathbb{F}') \cdot \mathcal{H}_f(\overline{B}; \mathbb{F}') \leq 1 + \epsilon \sum_{i=0}^{p-1} \binom{n}{i} \sum_{i=0}^{q-1} \binom{n}{i}. \quad \square$$

The left-hand side is a polynomial of degree $pq - 1$ whereas the right-hand side is a polynomial of degree $p + q - 2 < pq - 1$; a contradiction for all sufficiently large $n$ compared to $p, q$.

Similarly, set $B = B_n^{p,p}$ and suppose that $\mathcal{H}(B; \mathbb{F}) \geq \epsilon \cdot \mathcal{H}_f(B; \mathbb{F}'')$, then by Proposition 17

$$\left( \binom{n}{p^2 - 1} \right) \leq \alpha(B \boxtimes \overline{B}) \leq \mathcal{H}_f(B \boxtimes \overline{B}; \mathbb{F}'') = \mathcal{H}_f(B; \mathbb{F}'') \cdot \mathcal{H}_f(\overline{B}; \mathbb{F}'') \leq 1 + \epsilon \left( \sum_{i=0}^{p-1} \binom{n}{i} \right)^2;$$

another contradiction for all sufficiently large $n$ compared to $p$. \quad \square

**Theorem 19.** For any field $\mathbb{F}$ of nonzero characteristic and $\epsilon > 0$, there exists an explicit graph $G = G(\mathbb{F}, \epsilon)$ so that if $\mathbb{F}'$ is any field with $\text{char}(\mathbb{F}') \neq \text{char}(\mathbb{F})$, then

$$\mathcal{H}_f(G; \mathbb{F}) < \epsilon \cdot \mathcal{H}_f(G; \mathbb{F}').$$

**Proof.** Suppose that $\text{char}(\mathbb{F}) = p$ for some prime $p$ and set $n = \max\{n_{p,q} : q \leq p, \text{ q prime}\}$ where $n_{p,q} = n_{p,q}(\epsilon)$ is as in Lemma 18. For this $n$, set $N_{p,q} = \mathcal{H}(B_n^{p,q}; \mathbb{F})$ and set $G_{p,q} = \mathcal{G}_F(N_{p,q}, 1)$. We know that

$$N_{p,q} = \alpha(G_{p,q}) = \mathcal{H}(G_{p,q}; \mathbb{F}) = \mathcal{H}_f(G_{p,q}; \mathbb{F}).$$

Define

$$G = \prod_{q \leq p, \text{ q prime}} G_{p,q},$$

where the product is the strong product. Notice that $n_{p,q}$ depends only on $p, q, \epsilon$, so $G$ depends only on $\epsilon$ and on the field $\mathbb{F}$. 

Now, if $F'$ is any field with $\text{char}(F') \neq p$, then by Lemma 18 and the choice of $n$, there is some prime $q^* \leq p$, for which $N_{p,q^*} = H(B_n^{p,q^*}; F') < \epsilon \cdot H_f(B_n^{p,q^*}; F') \leq \epsilon \cdot H_f(G(p,q); F')$, where the last inequality follows from the fact that $H(B_n^{p,q^*}; F) = N_{p,q^*}$, so there is a graph homomorphism from $\overline{B_n^{p,q^*}}$ to $\overline{G(p,q)}$ (see Observations 13 and 14).

Additionally, for all other $q \leq p$, we have $H_f(G(p,q); F') \geq \alpha(G(p,q)) = N_{p,q}$. Therefore,

$$H_f(G; F') = \prod_{q \leq p, \text{prime}} H_f(G(p,q); F') = \prod_{q \leq p, \text{prime}} N_{p,q} \leq \epsilon \cdot \prod_{q \leq p, q \neq q^*} \prod_{q \text{ prime}} N_{p,q} \leq \epsilon \cdot \prod_{q \leq p} H_f(G(p,q); F') = \epsilon \cdot H_f(G; F').$$

\[\Box\]

5. Fractionalizing Lovász’s theta function

One could attempt to fractionalize Lovász’s theta function in ways similar to how we fractionalized Haemers’ bound. In this section, we provide two attempts and show that neither yields any improvements.

Recall that for a graph $G$, a collection of unit vectors $\{x_v \in \mathbb{R}^n : v \in V\}$ is said to be an orthonormal representation of $G$ if $\langle x_u, x_v \rangle = 0$ whenever $uv / E$. A handle is simply a unit vector $h$. The theta function of $G$ is defined to be

$$\vartheta(G) = \min \max \frac{1}{\langle x_v, h \rangle^2}$$

where the minimum is taken over all $\{x_v : v \in V\}$, which are orthonormal representations of $G$, and all handles $h$.

Recall also that $\vartheta(G) = \max \sum_{v \in V} \langle x_v, h \rangle^2$ where the maximum is taken over all $\{x_v : v \in V\}$, which are orthonormal representations of $\overline{G}$, and all handles $h$. This “dual form” of the theta function will be essential below.

A first attempt at fractionalizing the theta function is to define

$$\vartheta_f(G) := \inf_d \frac{\vartheta(G \times \overline{K_d})}{d}.$$ 

We recover $\vartheta(G)$ when $d = 1$, so certainly $\vartheta_f(G) \leq \vartheta(G)$. Unfortunately, it turns out that $\vartheta_f$ is equal to $\vartheta$.

**Theorem 20.** For any graph $G$, $\vartheta_f(G) = \vartheta(G)$.

**Proof.** As noted above, $\vartheta_f(G) \leq \vartheta(G)$. On the other hand, let $\{x_v : v \in V\}$ be an orthonormal representation of $\overline{G}$ and $h$ be a handle for which $\vartheta(G) = \sum_{v \in V} \langle x_v, h \rangle^2$. Let $d$ be any positive integer and for $i \in [d]$ and $v \in V$ define $\overline{x}_{(v,i)} = \overline{x}_v$. Certainly $\{\overline{x}_{(v,i)} : v \in V, i \in [d]\}$ is an
Lemma 22. The proof hinges on the following lemma.

We have already noted that\

Therefore,\

As a brief note, the above provides a quick proof that \( \vartheta(G \times \overline{K_d}) = d \cdot \vartheta(G) \) for every positive integer \( d \).

A second attempt may proceed by replacing vectors by matrices. In particular, for some positive integer \( n \), we say that a collection of matrices \( \{ M_v \in \mathbb{R}^{n \times d_v} : v \in V \} \), where \( d_v \) is some positive integer assigned to \( v \), is a matrix representation of \( G \) if \( M_v^T M_v = I_d \) for every \( v \in V \) and \( M_v^T M_v = O_d \) whenever \( uv \notin E \). A \( k \)-handle is a matrix \( H \in \mathbb{R}^{n \times k} \) with \( \|h_i\| = 1 \) for all \( i \in [k] \) where \( h_i \) is the \( i \)th column of \( H \).

For a positive integer \( k \), we define

Unlike in the definition of \( \vartheta(G) \), it is not clear, a priori, that the infimums can be replaced by minimums.

We recover \( \vartheta(G) \) when \( k = d_v = 1 \) for all \( v \in V \), so \( \vartheta_f(G) \leq \vartheta(G) \). Sadly, yet again, it turns out that \( \vartheta_f \) is equal to \( \vartheta \).

Theorem 21. For any graph \( G \), \( \vartheta_f(G) = \vartheta(G) \).

Proof. We have already noted that \( \vartheta_f(G) \leq \vartheta(G) \), so we need only establish the opposite inequality. The proof hinges on the following lemma.

Lemma 22. Let \( \{ M_v \in \mathbb{R}^{n \times d_v} : v \in V \} \) be a matrix representation of \( G \) and \( \{ \overline{M_v} \in \mathbb{R}^{\overline{n} \times \overline{d}_v} : v \in V \} \) be a matrix representation of \( \overline{G} \). For any \( k \)-handle \( H \in \mathbb{R}^{n \times k} \) and any \( \overline{k} \)-handle \( \overline{H} \in \mathbb{R}^{\overline{n} \times \overline{k}} \),

Proof of Lemma. Let \( \{ h_i : i \in [k] \} \) be the columns of \( H \), \( \{ \overline{h}_i : i \in [\overline{k}] \} \) be the columns of \( \overline{H} \), \( \{ m_{v,i} : i \in [d_v] \} \) be the columns of \( M_v \) and \( \{ \overline{m}_{v,i} : i \in [\overline{d}_v] \} \) be the columns of \( \overline{M}_v \). We begin by noticing that for any \( u, v \in V \) and \( s \in [d_u], s' \in [d_v], t \in [\overline{d}_u], t' \in [\overline{d}_v] \), we have

Firstly, if \( u = v \), the claim follows from the fact that \( \{ m_{v,i} : i \in [d_v] \} \) and \( \{ \overline{m}_{v,i} : i \in [\overline{d}_v] \} \) are orthonormal. If \( u \neq v \), then either \( uv \notin E(G) \) or \( uv \notin E(\overline{G}) \), so either \( \langle m_{u,s}, m_{v,s'} \rangle = 0 \) or \( \langle m_{u,t}, m_{v,t'} \rangle = 0 \).
Therefore, the vectors \( \{m_{v,s} \otimes \overline{m}_{v,t} : v \in V, s \in [d_v], t \in [\overline{d}_v] \} \) are orthonormal. From this, we calculate

\[
k \overline{k} = \left( \sum_{i=1}^{k} \langle h_i, h_i \rangle^2 \right) \left( \sum_{j=1}^{k} \langle \overline{h}_j, \overline{h}_j \rangle^2 \right)
\]

\[
= \sum_{i=1}^{k} \sum_{j=1}^{k} \langle h_i, h_i \rangle^2 \langle \overline{h}_j, \overline{h}_j \rangle^2
\]

\[
= \sum_{i=1}^{k} \sum_{j=1}^{k} (h_i \otimes \overline{h}_j, h_i \otimes \overline{h}_j)^2
\]

\[
\geq \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{s=1}^{d_v} \sum_{t=1}^{d_v} (h_i \otimes \overline{h}_j, m_{v,s} \otimes \overline{m}_{v,t})^2
\]

\[
= \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{s=1}^{d_v} \sum_{t=1}^{d_v} \sum_{v=1}^{V} \sum_{s=1}^{d_v} \sum_{t=1}^{d_v} \text{tr}((h_i \otimes \overline{h}_j)(h_i \otimes \overline{h}_j)^T(m_{v,s} \otimes \overline{m}_{v,t})(m_{v,s} \otimes \overline{m}_{v,t})^T)
\]

\[
= \sum_{v=1}^{V} \text{tr}\left(\left(\sum_{i=1}^{k} \sum_{j=1}^{k} (h_i \otimes \overline{h}_j)(h_i \otimes \overline{h}_j)^T\right)\left(\sum_{s=1}^{d_v} \sum_{t=1}^{d_v} (m_{v,s} \otimes \overline{m}_{v,t})(m_{v,s} \otimes \overline{m}_{v,t})^T\right)\right)
\]

Using the above lemma, the theorem follows quickly. Let \( \{\overline{x}_v : v \in V\} \) be an orthonormal representation of \( \overline{G} \) and \( \overline{h} \) be a handle with \( \vartheta(G) = \sum_{v \in V} \langle \overline{x}_v, \overline{h} \rangle^2 \). Also, for any \( \epsilon > 0 \), let \( \{M_v : v \in V\} \) be a matrix representation of \( G \) and \( H \) be a \( k \)-handle with \( \vartheta_f(G) + \epsilon \geq \max_{v \in V} \frac{k}{\text{tr}(M_v)^2}. \) By the lemma,

\[
k \cdot 1 \geq \sum_{v \in V} \text{tr}(M_v^T H H^T M_v) \text{tr}(\overline{x}_v^T \overline{h} \overline{h}^T \overline{x}_v) \geq \frac{k}{\vartheta_f(G) + \epsilon} \sum_{v \in V} \langle \overline{x}_v, \overline{h} \rangle^2 = k \cdot \frac{\vartheta(G)}{\vartheta_f(G) + \epsilon}.
\]

Thus, \( \vartheta_f(G) + \epsilon \geq \vartheta(G) \) for every \( \epsilon > 0 \), so \( \vartheta_f(G) \geq \vartheta(G) \).

Although it turns out that the matrix formulation of \( \vartheta_f(G) \) does not provide any improvements on \( \vartheta(G) \), it could still be useful in providing bounds on \( \vartheta(G) \) for large graphs or establishing general theorems. For example, a theorem of Lovász in \( [6] \) states that if \( G \) has an orthonormal
representation in $\mathbb{R}^N$, then $\vartheta(G) \leq N$. While this is not difficult to prove directly from the definition of $\vartheta(G)$, it does require some creativity; however, it follows essentially by definition for $\vartheta_f(G)$. In fact, we can quickly show something stronger. For a graph $G$, a collection of subspaces \( \{S_v \subseteq \mathbb{R}^N : v \in V \} \) is said to be a $d$-dimensional representation over $\mathbb{R}^N$ if $\dim(S_v) = d$ for every $v \in V$ and $S_u \perp S_v$ whenever $uv \notin E$. Of course, a $1$-dimensional representation is equivalent to an orthonormal representation.

**Proposition 23.** If $G$ has a $d$-dimensional representation over $\mathbb{R}^N$, then $\vartheta(G) \leq N/d$.

**Proof.** Let $\{S_v \subseteq \mathbb{R}^N : v \in V \}$ be a $d$-dimensional representation of $G$ over $\mathbb{R}^N$. For each $v \in V$, let $M_v \in \mathbb{R}^{N \times d}$ be a matrix whose columns form an orthonormal basis for $S_v$; thus, $M_v^T M_v = I_d$. Further, as $S_u \perp S_v$ whenever $uv \notin E$, we have $M_u^T M_v = O_d$ whenever $uv \notin E$, so $\{M_v : v \in V\}$ is a matrix representation of $G$. Let $H = I_N$, which is an $N$-handle, so

$$
\vartheta(G) = \vartheta_f(G) \leq \max_{v \in V} \frac{N}{\operatorname{tr}(M_v^T H H^T M_v)} = \max_{v \in V} \frac{N}{\operatorname{tr}(I_d)} = \frac{N}{d}.
$$

\hfill \Box

6. Conclusion

We conclude with a list of open questions related to our study of $\mathcal{H}_f(G)$.

- For a graph $G$ and a field $\mathbb{F}$, is $\mathcal{H}_f(G; \mathbb{F})$ attained? That is to ask: is the infimum really a minimum? Beyond this, is $\mathcal{H}_f(G; \mathbb{F})$ computable?
- Theorem 2 shows that for any field $\mathbb{F}$ of nonzero characteristic, there is a graph $G = G(\mathbb{F})$ with $\mathcal{H}_f(G; \mathbb{F}) < \min\{\mathcal{H}(G; \mathbb{F}'), \vartheta(G)\}$ for every field $\mathbb{F}'$. While this $G$ satisfied $\mathcal{H}_f(G; \mathbb{F}) < \epsilon \cdot \vartheta(G)$, we only verified that $\mathcal{H}_f(G; \mathbb{F}) \leq \mathcal{H}(G; \mathbb{F}) - \frac{1}{9}$. It would nice to construct a graph $G$ with $\mathcal{H}_f(G; \mathbb{F}) < \epsilon \cdot \min\{\mathcal{H}(G; \mathbb{F}'), \vartheta(G)\}$ for every field $\mathbb{F}'$. We believe that such a graph does indeed exist.
- Theorem 13 shows that for any field $\mathbb{F}$ of nonzero characteristic, there is a graph $G = G(\mathbb{F})$ with $\mathcal{H}_f(G; \mathbb{F}) < \mathcal{H}_f(G; \mathbb{F}')$ for every field $\mathbb{F}'$ with $\operatorname{char}(\mathbb{F}') \neq \operatorname{char}(\mathbb{F})$. We were unable to prove a similar separation for fields of equal characteristic. Namely, given a finite field $\mathbb{F}$, is there a graph $G = G(\mathbb{F})$ with $\mathcal{H}_f(G; \mathbb{F}) < \mathcal{H}_f(G; \mathbb{F}')$ for every field $\mathbb{F}'$ that is not an extension of $\mathbb{F}$? We suspect that the graph $G_{\mathbb{F}}(n, d)$ provides such an example for appropriately chosen $n, d$.
- Are there graphs for which $\vartheta(G) < \mathcal{H}(G)$, yet $\mathcal{H}_f(G) < \vartheta(G)$? While this paper was in submission, this question was answered affirmatively by Hu, Tamo and Shayevitz in [5] using their parameter $\minrk^*_f(G)$. They construct a graph $G$ with $\vartheta(G) = 9 + 7\sqrt{5} < 28 = \mathcal{H}(G; \mathbb{F})$ for every field $\mathbb{F}$, yet $\mathcal{H}_f(G; \mathbb{F}_{11}) \leq \minrk^*_f_{11}(G) \leq 24.5 < 9 + 7\sqrt{5}$.
- For a graph $G$, is $\mathcal{H}_f(G) = \lim_{n \to \infty} \mathcal{H}(G^{\otimes n})^{1/n}$? Corollary 13 shows that $\mathcal{H}_f(G) \leq \mathcal{H}(G^{\otimes n})^{1/n}$ for every positive integer $n$, so it is only necessary to verify the reverse inequality.

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