Theoretical expectations for 
the top-quark mass

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Abstract

I review the theoretical expectations for the top-quark mass in a variety of models: the Standard Model, unified models (GUTs), low-energy supersymmetric models (SUSY), unified supersymmetric models (SUSY GUTs), supergravity models, and superstring models. In all instances I consider the constraints on the top-quark mass which arise by demanding that these theories be weakly interacting. This assumption is quantified by the use of partial-wave unitarity or triviality. The resulting upper bounds on the top-quark mass are most stringent in SUSY GUTs models ($m_{\text{pole}}^t \lesssim 200 \sin \beta \text{ GeV}$). I also discuss a class of $SU(5) \times U(1)$ superstring models where $m_{\text{pole}}^t \sim (170 - 195) \text{ GeV}$ is predicted. I conclude that experimental determinations of the top-quark mass can be naturally understood in SUSY GUTs and superstring models. (Lecture presented at the International School of Subnuclear Physics, 32nd Course: From Superstring to Present-Day Physics, Erice–Sicily: July 3–11, 1994.)

1 Introduction

On April 26, 1994 the CDF Collaboration announced “evidence” for the existence of the top quark in $p\bar{p}$ collisions at $\sqrt{s} = 1.8 \text{ TeV}$ [1]. In a lengthy paper CDF argued that they had observed an excess of dilepton and lepton+jets events which were most naturally explained as coming from $t\bar{t}$ production. A kinematical fit to the candidate event masses yields $m_t = 174 \pm 17 \text{ GeV}$, when systematic and statistical errors are combined in quadrature. In reality, the issue of the existence of the top quark is not settled since the statistical significance of the measurement is not compelling, there are annoying discrepancies among various observables (e.g., the theoretically predicted QCD cross section for the determined top-quark mass and the actually measured cross section), and the other detector at the Tevatron (D0) does not observe any
meaningful excess of top-like events. Fortunately the Tevatron is currently running (Run IB) and should accumulate five times as much data by the end of the run. This increased data sample should shed a lot of light on the present top-quark controversy.

Indirect evidence for the existence of the top quark has been mounting through fits to the LEP electroweak observables \[2\] which depend on the value of \( m_t \). These fits can be compared with the theoretical calculations \[3\] and a value of \( m_t \) is deduced within the framework of the Standard Model. This value has a Higgs-boson mass uncertainty which is now comparable to the experimental uncertainty. For light Higgs-boson masses (i.e., those consistent with a supersymmetric Standard Model) the latest fit gives \( m_t = 162 \pm 9 \text{ GeV} \) \[4\] (this value includes the CDF measurement). Heavier Higgs-boson masses (e.g., \( m_H = 300 \text{ GeV} \) as is many times assumed) increase the central value by as much as 20 GeV. In any event, either directly (CDF) or indirectly (LEP) there seems to be evidence for the existence of the top quark. Moreover, its presumed mass makes it the heaviest elementary particle ever discovered. In fact, we have \( m_t = \mathcal{O}(M_{W,Z}) \), which hints that the top quark may have something to do with the breaking of the electroweak symmetry (e.g., as in the radiative electroweak breaking mechanism in a supergravity theory).

In what follows I will review the theoretical expectations for the top-quark mass in a variety of theoretical frameworks: the Standard Model, unified models (GUTs), low-energy supersymmetric models (SUSY), supersymmetric unified models (SUSY GUTs), supergravity, and superstrings. The motivation for considering such a sequence of theoretical frameworks is well known \[5\] and will not be elaborated any further. However, to refresh your memory in Fig. 1 I show a “road map” starting from the Standard Model and ending at very high energies with superstring models.

The theoretical expectations which I consider are mostly in the form of upper bounds on the top-quark mass. These follow from the practical and usually tacit assumption that the various theories be weakly interacting in their regime of applicability. Violation of these bounds entails a strongly interacting theory which cannot be analyzed in the usual way. That is, we have a new “phase” of the theory with new physical consequences – a different theory. In the case of superstring models, one is able to calculate in principle the top-quark mass and the expectations we discuss are not just upper bounds, but actual predictions.

## 2 The Standard Model

For our present purposes I will view the Standard Model as an effective theory valid for energies \( \lesssim 1 \text{ TeV} \). The top-quark mass is related to the top-quark Yukawa coupling in the Yukawa Lagrangian via

\[
m_t = \lambda_t \frac{v_0}{\sqrt{2}} = 174 \lambda_t \text{ GeV},
\]

where \( v_0 = 1/\sqrt{\sqrt{2}G_F} = 246 \text{ GeV} \) is the Higgs vacuum expectation value. This relation shows that large values of \( m_t \) originate from large values of \( \lambda_t \). However, the
Figure 1: A “road map” showing the successes (left arrows) and problems (right arrows) of a sequence of theoretical frameworks from the Standard Model up to superstring models.
interactions among top quarks, or top quarks with Higgs bosons and the longitudinal components of the electroweak gauge bosons (i.e., the would-be Goldstone bosons) are all proportional to $\lambda_t$. Therefore, a large value of $m_t \leftrightarrow \lambda_t$ leads to strong interactions in the Yukawa sector of the Standard Model. This new “phase” of the Standard Model has been investigated in the context of $tt$ condensates [6], and may be fine in its own right, but it definitely leads to new physical consequences which are not part of the usual weakly interacting Standard Model.

We are thus led to find an upper bound on $m_t \leftrightarrow \lambda_t$ such that if it is violated, we would say that the Standard Model has become a strongly interacting theory in the Yukawa sector. We will use the concept of partial-wave unitarity to address this question. This is an arcane subject that is of general applicability, but which is probably unfamiliar to present-day students.

Consider two-body to two-body (i.e., $2 \rightarrow 2$) scattering processes. These are called “elastic” processes, as opposed to the $2 \rightarrow n$ ($n > 2$) inelastic processes. The elastic $2 \rightarrow 2$ amplitudes can be decomposed in a partial-wave expansion, which for scalar particles in the high-energy limit is given by

$$a_{2\rightarrow 2}^j(s) = \frac{1}{32\pi} \int_{-1}^{1} d\cos \theta \ T(s, \cos \theta) P_j(\cos \theta),$$

where $T(s, \cos \theta)$ are the usual Feynman amplitudes, $\theta$ is the scattering angle, and $P_j$ are the Legendre polynomials. These partial-wave amplitudes are related to the partial-wave projections of the $S$ matrix by

$$a_{2\rightarrow 2}^j = \frac{1}{2i} (S_{2\rightarrow 2}^j - 1).$$

For inelastic processes we have instead $a_{2\rightarrow n}^j = S_{2\rightarrow n}^j / 2i$, $n > 2$ (i.e., no “$-1$” since one necessarily has particle production). The unitarity of the $S$ matrix (or of any of its partial waves $S_j$), $SS^\dagger = 1$ implies that (summing over a row of $S$)

$$1 = \sum_b S_{ab} S_{ba}^\dagger = \sum_b |S_{ab}|^2$$

$$= |S_{2\rightarrow 2}^j|^2 + \sum_{n>2} |S_{2\rightarrow n}^j|^2$$

$$= |2ia_{2\rightarrow 2}^j + 1|^2 + \sum_{n>2} |2ia_{2\rightarrow n}^j|^2 .$$

Or, as it usually written

$$|a_{2\rightarrow 2}^j - \frac{i}{2}|^2 + \sum_{n>2} |a_{2\rightarrow n}^j|^2 = \frac{1}{4} .$$

1. Until the end of the lecture we will not worry about whether the top-quark masses we are considering may already be in conflict with experimental observations. We would like to focus on the theoretical constraints that may exist on $m_t$, irrespective of any phenomenological biases.

2. An analogous strongly interacting Standard Model in the Higgs sector is obtained for heavy Higgs boson masses. Corresponding upper bounds on $M_H$ can be obtained, as described in Ref. [7].
That is, the elastic amplitude $a_{j}^{2\rightarrow 2}$ must lie on a circle (the “unitarity circle”) of radius $\eta/2$ centered at $(0, \frac{1}{2})$ in the complex plane, with $\eta^2 = 1 - 4 \sum_{n>2} |a_{j}^{2\rightarrow n}|^2$. This gives the Argand diagram for the scattering amplitude, as shown below for the case where inelastic processes are neglected (i.e., $\eta = 1$).

The Unitarity Circle

The partial-wave amplitudes $a_{j}^{2\rightarrow 2}$ can be calculated in perturbation theory for any desired process. (Below we will be interested in $t\bar{t} \rightarrow t\bar{t}$ scattering.) However, it is the all-orders amplitude that must lie on the unitarity circle. The idea of “unitarity bounds” is that if a given partial-wave amplitude (usually $j=0$), calculated to some low-order in perturbation theory (usually tree-level), does not lie on the unitarity circle, then it must be that higher-loop corrections are important in order to restore unitarity – i.e., there is a breakdown of perturbation theory.

Without thinking too much about the unitarity circle, it is common practice to demand $|a_{j}^{2\rightarrow 2}| \leq 1$ (i.e., to be inside the dashed semi-circle) or $\text{Re} |a_{j}^{2\rightarrow 2}| \leq \frac{1}{2}$ (to be between the dotted vertical lines), which are clearly necessary conditions but may not be sufficient. Taking advantage of the unitarity circle requires a one-loop calculation of the partial-wave amplitudes, since at tree-level all amplitudes are real. The above discussion indicates that if one is limited to a tree-level calculation, then a more realistic unitarity constraint would be $\text{Re} |a_{j}^{2\rightarrow 2}| \ll \frac{1}{2}$, and perhaps $\text{Re} |a_{j}^{2\rightarrow 2}| \leq \frac{1}{4}$ is a sensible educated guess of the actual one-loop constraint.

In the Standard Model we are interested in constraining the top-quark Yukawa coupling. Therefore we need to consider processes which depend strongly on $\lambda_t$, such as $t\bar{t} \rightarrow t\bar{t}$ (which proceeds through $s$- and $t$-channel Higgs-boson and $Z$ exchanges). In the high-energy limit (i.e., $s \gg m^2_t, m^2_{H}$) this amplitude is given by $T = -3\lambda^2_t$.

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3One-loop unitarity analyses in the Higgs sector support this expectation.

4When considering the high-energy limit, only coefficients of dimension-four operators in the

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The $j = 0$ partial wave follows from Eq. (2): $a_0 = -3\lambda_t^2 / 16\pi$. The “unitarity bound” arising from $|\text{Re} a_0| \leq \frac{1}{2}$ is then

$$\lambda_t \leq \sqrt{\frac{8\pi}{3}} \approx 2.89.$$  \hspace{1cm} (5)

Requiring instead $|\text{Re} a_0| \leq \frac{1}{4}$ as our “one-loop” guess, gives $\lambda_t \leq \sqrt{\frac{\pi}{3}} \approx 2.05$. From Eq. (\[\text{I}\]) we then obtain an upper bound on $m_t$

$$m_t = \lambda_t \frac{v_0}{\sqrt{2}} < \lambda_t^{\max} \frac{v_0}{\sqrt{2}} \approx 500 \text{ GeV} \quad \text{(Standard Model)},$$  \hspace{1cm} (6)

or $m_t \lesssim 355$ GeV using the “one-loop” bound.

Within the Standard Model there is another theoretical constraint on $m_t$ that one can find by requiring stability of the minimum of the Higgs potential (“vacuum stability”). The one-loop effective potential is given by \[\text{II}\]

$$V = V_0 + \frac{B}{64\pi^2} \phi^4 \ln \frac{\phi^2}{Q^2},$$  \hspace{1cm} (7)

with $B = (6m_W^4 + 3m_Z^4 - 12m_t^4)/v^4$. If $m_t$ is too large, $B$ will turn negative and the potential becomes unbounded from below. For large values of $\phi$, the expansion parameter is enhanced by large logarithms, and one must use renormalization group methods to obtain a reliable result. For a given upper bound on the mass scale where the Standard Model breaks down, there is a constraint in the $m_t - m_H$ plane \[\text{II}\]

$$\text{For } \phi \lesssim 1 \text{ TeV} \quad \Rightarrow \quad m_t \lesssim 2m_H + 60 \text{ GeV}.$$  \hspace{1cm} (8)

This constraint is not very useful, although it can become strictier is the scale of new physics is pushed up.

### 3 Grand Unified Theories (GUTs)

For our purposes it will suffice to assume that the Standard Model is valid up to scales $M_U = \mathcal{O}(10^{15} \text{ GeV})$ where the new GUT physics comes in. The constraint derived in Eq. (\[\text{I}\]) for a weakly interacting theory remains valid, except that now it must be satisfied at all mass scales $Q$, i.e.,

$$\lambda_t(Q) \lesssim 2.89, \quad M_Z \lesssim Q \lesssim M_U.$$  \hspace{1cm} (9)

Our weakly interacting assumption required some justification within the Standard Model, whereas in GUTs this assumption is usually (and tacitly) made in order to Lagrangian (such as Yukawa couplings) will survive. Mass parameters (e.g., soft-supersymmetry-breaking parameters) cannot be bounded in this fashion.
be able to connect the low- and high-mass scales via the (one-loop) renormalization group equations (RGEs). In fact, in order to enforce the constraint in Eq. (9) we need to use the RGEs for $\lambda_t$ and the gauge couplings (valid for $M_Z < Q < M_U$)

$$\frac{d\lambda_t}{dt} = \frac{\lambda_t}{8\pi^2} \left[ a\lambda_t^2 - \sum_i c_i g_i^2 \right]$$  \hspace{1cm} (10)

$$\frac{dg_i}{dt} = \frac{b_i}{16\pi^2} g_i^3$$  \hspace{1cm} (11)

where the various coefficients depend on the theory at hand, i.e., the Standard Model or its supersymmetric version. These coefficients are given below.

|       | $a$  | $c_1$  | $c_2$  | $c_3$  | $b_1$  | $b_2$  | $b_3$  |
|-------|------|--------|--------|--------|--------|--------|--------|
| SM    | 9/4  | 17/40  | 9/8    | 4      | 41/10  | -19/6  | -7     |
| SUSY  | 3    | 13/30  | 3/2    | 8/3    | 33/5   | 1      | -3     |

If all other Yukawa couplings can be neglected (as is the case in the Standard Model), then the above RGEs can be solved analytically. We obtain \[12\]

$$\lambda_t^2(t) = \frac{\lambda_t^2(0) F(t)}{1 - \frac{a}{4\pi^2} \lambda_t^2(0) T(t)} ,$$  \hspace{1cm} (12)

where $t = \ln(Q/M_Z)$ and $T(t) = \int_0^t F(t') dt'$ with

$$F(t) = \prod_{i=1}^3 \left[ 1 - \frac{b_i}{8\pi^2} g_i^2(0) t \right]^{2c_i/b_i} .$$  \hspace{1cm} (13)

To evaluate these expressions we use:

$$g_1^2(0) = \left( \frac{5}{3} \right) \frac{4\pi\alpha_e}{\cos^2\theta_W} = 0.212 ,$$  \hspace{1cm} (14)

$$g_2^2(0) = \frac{4\pi\alpha_e}{\sin^2\theta_W} = 0.426 ,$$  \hspace{1cm} (15)

$$g_3^2(0) = 4\pi\alpha_3 = 1.508 ,$$  \hspace{1cm} (16)

with \[\footnote{To be consistent with one-loop gauge coupling unification, we should use the predicted value of $\sin^2\theta_W$, i.e., $\sin^2\theta_W = 0.2 + (7/15)(\alpha_e/\alpha_3)$.} \]

$$\alpha_e^{-1} = 127.9 , \hspace{1cm} \sin^2\theta_W = 0.2304 , \hspace{1cm} \alpha_3 = 0.120 .$$  \hspace{1cm} (17)

The calculated values of $\lambda_t(Q)$ versus the scale $Q$ are shown in the following plot. These numbers only depend on the assumed value of $\lambda_t(0) = \lambda_t(Q = M_Z)$. The horizontal dashed line represents the constraint in Eq. (9).
Note that as $\lambda_t(0)$ increases, the unitarity limit (dashed line) is crossed for lower and lower values of $Q$: for $\lambda_t(0) = 1.4 \ (1.5)$, $Q_{\text{max}}^{\text{max}} \sim 10^{12} \ (10^9) \ \text{GeV}$. This is the maximum scale for which such theory remains weakly interacting. If we want $Q_{\text{max}}^{\text{max}} > M_U = 10^{15} \ \text{GeV}$, then there is an upper bound on $\lambda_t(0)$:

$$\lambda_t^{\text{max}}(0) \approx 1.357.$$  \hspace{1cm} (18)

From Eq. (11) we then get the corresponding upper bound on $m_t$

$$m_t < \lambda_t^{\text{max}}(0) \frac{v_0}{\sqrt{2}} \approx 236 \ \text{GeV} \quad \text{(GUTs – unitarity)}. \hspace{1cm} (19)$$

This result is to be compared with that for the Standard Model without any “RGE improvement” (in Eq. (2)): the value of $\lambda_t^{\text{max}}$ is reduced from 2.89 down to 1.36, i.e., by more than a factor of two! Using the “one-loop” value of 2.05 instead of 2.89, reduces the 1.357 result to 1.342. Thus, a 30% reduction at the low scale only amounts to a 1% reduction at the high scale.

Note that given a sufficiently high mass scale, the Yukawa coupling will always blow up (a “Landau pole”), i.e., the denominator of Eq. (12) becomes zero for $T(t_c) = 4\pi^2/\alpha \lambda^2(0)$ which determines the critical scale $t_c$ for any value of $\lambda(0)$.

\[ \text{In practice, } Q_c = M_Z e^{t_c} \text{ exceeds the Planck scale for } \lambda(0) < 1.3. \]

\[ ^6\text{This behavior} \]
of the theory is called triviality \[13\], since the only allowed value of the \(\lambda(0)\) parameter appears to be zero, \textit{i.e.}, the theory is trivial. This result would be worrisome if the theory had no \textit{cutoff}, that is, no scale beyond which the theory changes and the previous reasoning does not apply. However, all possible theories of this kind have a natural cutoff at the Planck scale: no matter what happens at the GUT scale or beyond, the theory must change at the Planck scale to accommodate the gravitational interactions. Moreover, the change which is needed is not just into another gauge theory, since the same problem would reappear above the cutoff scale. In the case of string theory, the theory becomes finite and the parameters do not run anymore. Therefore, a theory of extended objects (like string theory) appears to be also motivated as a final solution to the triviality problem of the Yukawa sector.

In this connection, instead of demanding that the GUT theory be weakly interacting all the way up the unification scale, one can simply restrict the theory such that the Yukawa coupling does not blow up before the unification scale \[14\]. So the question now is: for what \(\lambda_t^{\text{max}}(0)\) is \(\lambda_t(Q) < \infty\) for \(Q = M_U\)? We find

\[
\lambda_t^{\text{max}}(0) \approx 1.373 ,
\]

and therefore

\[
m_t < \lambda_t^{\text{max}}(0) \frac{v_0}{\sqrt{2}} \approx 239 \text{ GeV} \quad \text{(GUTs – triviality)}.
\]

Note that there is little difference between the \(m_t\) upper bounds with the unitarity \[19\] or the triviality \[21\] requirements. The reason is that large values of Yukawa coupling at the high scale are driven at low energies to an “infrared fixed point” which depends on the scale where \(\lambda_t(Q) \to \infty\), \textit{i.e.}, for \(Q = 10^{15}\) GeV, the fixed point is \(\lambda_t(0) = 1.373\).

Finally, we can re-consider the constraint coming from the stability of the Higgs vacuum, but this time allowing the Higgs field to take values as large as the unification scale. The result is \[11\]

\[
m_t \lesssim \frac{1}{2} m_H + 100 \text{ GeV}, \quad (22)
\]

which is much stronger than the one obtained in Eq. \[8\] because of the “RGE strengthening” of the limit when allowing higher mass scales.

## 4 Supersymmetric unified theories (SUSY GUTs)

The introduction of supersymmetry into GUTs is basically required to make sense of the gauge hierarchy problem. The analysis performed above for the case of (non-supersymmetric) GUTs has a direct parallel in the case of SUSY GUTs, except for a crucial difference: in a supersymmetric theory one must have at least two Higgs doublets in order to have Yukawa couplings (and therefore masses) for both up-type and down-type quarks. In the minimal case (which we will focus on) the Higgs sector
consists of only two Higgs doublets \((H_1, H_2)\), and therefore there are two vacuum expectation values \(v_1, v_2\). The sums of the squares of these is constrained (i.e., \(v_1^2 + v_2^2 = v_0^2\)) but their ratio (\(\tan \beta = v_2/v_1\)) is not. The fermion Yukawa couplings then depend on this new parameter:

\[
m_t = \frac{\lambda_t v_0}{\sqrt{2}} \sin \beta, \quad (23)
\]
\[
m_b = \frac{\lambda_b v_0}{\sqrt{2}} \cos \beta, \quad (24)
\]
\[
m_\tau = \frac{\lambda_\tau v_0}{\sqrt{2}} \cos \beta. \quad (25)
\]

When \(\tan \beta\) is large (\(\gtrsim 40\)) the Yukawa couplings of the bottom quark and tau lepton are enhanced, and one should include these also in the RGE scaling.

The partial-wave unitarity analysis performed for the Standard Model is also applicable to a low-energy supersymmetric theory, as follows. In the supersymmetric theory one has the same

\[
t\bar{t} \rightarrow \gamma, Z \rightarrow t\bar{t} \quad (26)
\]

amplitude (since supersymmetric particles cannot be exchanged in tree-level diagrams when \(R\)-parity is conserved). In addition, one has amplitudes involving the superpartners to the top quarks (the top-squarks \(\tilde{t}\))

\[
\tilde{t}\tilde{t} \rightarrow \gamma, Z \rightarrow \tilde{t}\tilde{t}, \quad (27)
\]

and amplitudes involving superpartners of the \(\gamma, Z\) (the neutralinos \(\chi_i^0\))

\[
\tilde{t}\tilde{t} \rightarrow \chi_i^0 \rightarrow \tilde{t}\tilde{t}. \quad (28)
\]

There are also “crossed” amplitudes, like \(t\gamma \rightarrow t \rightarrow t\gamma\) and \(t\chi_i^0 \rightarrow \tilde{t} \rightarrow t\chi_i^0\), etc. All of these amplitudes are proportional to \(\lambda_t^2\) and should be taken into account in obtaining the unitarity limit. A great simplification occurs when one considers the high-energy limit, in which all mass parameters decouple and only the \(\lambda_t\) dependence survives \([12]\). In this case supersymmetry is effectively restored (since all soft-supersymmetry-breaking parameters become negligible) and the various amplitudes get correlated. The unitarity constraint \(|\text{Re } a_0| \leq \frac{1}{2}\) applies to all eigenvalues of the matrix of coupled channels. This coupling usually results in a strengthening of the unitarity limit by a factor of order 1. If one nonetheless neglects the channel coupling and only considers the channels in Eqs. (26,27), the result is the same as in the Standard Model case (i.e., \(\lambda_t < \sqrt{8\pi/3}\)) since the effectively restored supersymmetry entails same amplitudes for these two channels.

Despite the bound on \(\lambda_t\) being the same as in the Standard Model case, the corresponding upper bound on \(m_t\) can be stronger because of the \(\tan \beta\) dependence

\[
m_t < \lambda_t^{\text{max}} \cdot 174 \cdot \sin \beta \approx 500 \sin \beta \text{ GeV} \quad (\text{SUSY}). \quad (29)
\]

For \(\tan \beta = 1.0, 1.3, 1.5, 2, 3\) we get \(m_t \lesssim 355, 400, 420, 450, 475\text{ GeV}\).
On the other hand, in a weakly interacting SUSY GUT, the bound $\lambda_t \lesssim 2.89$ must hold for all scales up to the unification scale $M_U \sim 10^{16}$ GeV in this case. We can repeat the exercise for the GUT case and run the supersymmetric RGEs for the gauge and Yukawa couplings. Neglecting the $\lambda_b$ and $\lambda_\tau$ contributions (for not too large values of $\tan \beta$) we find for the running top-quark Yukawa coupling the following.

Note that for increasingly larger values of $\lambda_t(0)$, the critical scale $Q_c$ where the unitarity limit is crossed, decreases. Requiring that this scale be above the unification scale $M_U \sim 10^{16}$ GeV gives the upper limit on $\lambda_t(0)$

$$\lambda_t^{\text{max}}(0) \approx 1.132,$$  

which entails the following upper bound on $m_t$

$$m_t < \lambda_t^{\text{max}}(0) \frac{v_0}{\sqrt{2}} \sin \beta \approx 197 \sin \beta \text{ GeV} \quad \text{(SUSY GUTs – unitarity).}$$  

In this case the “RGE improvement” decreases the maximum value of $\lambda_t$ by 60%, making the upper bound on $m_t$ that much stronger.
As in the GUTs case, we can also determine the critical value of $\lambda_t(0)$ above which $\lambda_t(Q)$ would blow up before the unification scale. We find

$$\lambda_t^{\text{max}}(0) \approx 1.138,$$  

(32)

which entails the following upper bound on $m_t$

$$m_t < \lambda_t^{\text{max}}(0) \frac{v_0}{\sqrt{2}} \sin \beta \approx 198 \sin \beta \text{ GeV} \quad \text{(SUSY GUTs – triviality).}$$  

(33)

The above results are sensitive to various “thresholds” encountered between $M_Z$ and $M_{\tilde{t}}$, as well as the value of $\alpha_3$, and the contribution from $\lambda_b$. A more refined calculation gives the same qualitative results, but slightly stronger bounds [15]

$$m_t \lesssim 190 \sin \beta \text{ GeV.}$$  

(34)

This upper bound on $m_t$ was of “academic” interest when originally derived [13], but has since become very relevant given the experimental trend towards increasingly heavier top-quark masses since then.

**Subtlety.** The above-quoted top-quark masses are the “running” masses at the scale $Q = m_t$, i.e., $m_t(m_t)$. On the other hand, the experimentally observable top-quark mass is the “pole” mass, which is related to the running mass through [17]

$$m_t^{\text{pole}} = m_t \left[ 1 + \frac{4}{3} \frac{\alpha_s(m_t)}{\pi} + K_t \left( \frac{\alpha_s(m_t)}{\pi} \right)^2 \right]$$  

(35)

where

$$K_t = 16.11 - 1.04 \sum_{m_i < m_t} \left( 1 - \frac{m_i}{m_t} \right) \approx 11.$$  

(36)

We thus obtain

$$m_t^{\text{pole}} \approx 1.067 m_t.$$  

(37)

Distinguishing between these top-quark masses is unnecessary at lowest order, but becomes essential in a next-to-leading order calculation. Equation (34) then gives

$$m_t^{\text{pole}} \lesssim \begin{cases} 143 \text{ GeV} & \text{for } \tan \beta = 1, \\ 169 \text{ GeV} & \text{for } \tan \beta = 1.5, \\ 181 \text{ GeV} & \text{for } \tan \beta = 2, \\ 198 \text{ GeV} & \text{for } \tan \beta = 5, \\ 202 \text{ GeV} & \text{for } \tan \beta = 10. \end{cases}$$  

(38, 39, 40, 41, 42)

Note that $\tan \beta = 1$ is almost ruled out experimentally: at the 95% C.L. the CDF result requires $m_t \gtrsim 141 \text{ GeV}$ and the global fit requires $m_t \gtrsim 144 \text{ GeV.}$ Should
the top-quark mass actually be $m_t = 162 (174) \text{GeV}$, we would deduce that $\tan \beta > 1.33 (1.67)$.

The fact that the unitarity and triviality SUSY GUTs (or GUTs) upper bounds on $\lambda_t$ and $m_t$ are so close is related to an *infrared fixed point* in the RGE for the Yukawa coupling. We can see this phenomenon clearly if we plot the low-energy value of the Yukawa coupling $\lambda_t(M_Z)$ which is obtained for a given high-energy value $\lambda_t(M_U)$:

![SUSY GUTs](image)

The picture is clear: for sufficiently large values of the high-energy Yukawa coupling, its low-energy counterpart will be driven to the same “fixed point value”. This phenomenon explains why it does not matter what we choose for $\lambda_t(M_U)$ if it exceeds $\sim 1.5$. Thus the little difference between the unitarity and triviality constraints. Also, since the tree-level unitarity constraint is not very precise, as discussed above, one could consider a “one-loop” guess for it by taking half of its value. This brings 2.89 down to 2.05, which gives a very similar value of $\lambda_t(M_Z)$, as the figure shows. If one decides that the fixed point value of $\lambda_t$ is the preferred one, then one obtains the “fixed point value” of the top-quark mass, *i.e.*, $m_t^{\text{pole}} \approx (203 \text{GeV}) \sin \beta$, and the
upper bounds in Eqs. (38)–(42) are actually attained.

Further constraints on $m_t$ can be obtained by assuming special relations among the Yukawa couplings at the unification scale.

- $SU(5)$-like: $\lambda_b = \lambda_r$ at $M_U$ gives a definite curve in the $(m_t, \tan \beta)$ plane [18].
- $SO(10)$-like: $\lambda_t = \lambda_b = \lambda_r$ at $M_U$ gives a point in the $(m_t, \tan \beta)$ plane [19].

Both relations are quite sensitive to the choice of $m_b$ and $\alpha_3$, and also depend on the threshold structure at the GUT and electroweak scales. Below we show typical predictions from these relations (data from Refs. [20, 21]). Note that the $SO(10)$-like relations require large values of $\tan \beta$, whereas the $SU(5)$-like relation prefers small ($\sim 1$) or large values; intermediate values are allowed only for large values of $m_t$.

5 Supergravity

Local supersymmetry or supergravity allows the spontaneous breaking of supersymmetry via the super-Higgs effect, and thus the calculation of the soft-supersymmetry-breaking parameters in terms of a few input functions (the superpotential, the Kähler
potential, and the gauge kinetic function). Usually one assumes that the resulting soft-supersymmetry-breaking parameters are universal at the unification scale. The running of the various scalar and gaugino masses from the unification scale down to the electroweak scale is governed by a set of coupled RGEs. Of particular relevance are the squared Higgs-doublet masses, one of which can turn negative at sufficiently low scales, signaling the breaking of the electroweak symmetry by radiative effects – the radiative electroweak breaking mechanism \[22, 23\].

Consider one of these RGEs schematically

\[
\frac{d\tilde{m}^2}{dt} = \frac{1}{(4\pi)^2} \left\{ - \sum_i c_i g_i^2 M_i^2 + a\lambda_t^2 \left( \sum_i \tilde{m}_i^2 \right) \right\}
\]

(43)

where \(g_i\) are the gauge couplings, \(M_i\) are the gaugino masses, and \(c_i\) and \(a\) are various positive coefficients. If \(\lambda_t\) is “not small” compared to the gauge couplings, then \(\tilde{m}^2\) could turn negative at low energies. For concreteness let’s say \(\lambda_t \gtrsim g \sim 0.6\), which implies

\[
m_t = 174\lambda_t \sin \beta \gtrsim 100 \sin \beta \gtrsim 75 \text{ GeV.}
\]

(44)

This qualitative result is what people have in mind when stating that radiative breaking works only for “heavy top quarks”. Heavy here means heavier than the prevailing prejudice a decade ago, \(i.e., m_t \gg m_b\). All imaginable top-quark masses today should satisfy this constraint automatically.

Nonetheless, radiative electroweak breaking does impose constraints in the \((m_t, \tan \beta)\) plane \[13\], as the following figure shows. The allowed region is bounded completely:

- **Top boundary:** the vacuum of the Higgs potential is untable above this line. This in effect is the analogue of the vacuum stability constraint discussed in connection with the Standard Model.

- **Upper corner:** this corresponds to a fixed point in \(\lambda_b\), which translates into an upper bound on \(\tan \beta\) since \(m_b \propto \lambda_b \cos \beta\).

- **Right boundary:** the fixed point in \(\lambda_t\) produces an upper bound on \(m_t\) as a function of \(\tan \beta\), as discussed above.

- **Bottom boundary:** \(\tan \beta > 1\) is required by radiative electroweak symmetry breaking.

- **Left boundary:** the radiative breaking mechanism does not work to the left of this line, \(i.e., \mu^2 < 0\) would be required.

As the figure shows, there is always a minimum value of \(m_t\) which is required. Note that for \(\mu > 0\), it may be possible to reach small values of \(m_t\) for values of \(\tan \beta\) sufficiently close to \(1\).\[13\] However, such region of parameter space is highly disfavored.

\[1\] I would like to thank Chris Kolda for pointing this out to me.
since the tree-level contribution to the lightest Higgs-boson mass nearly vanishes and
the one-loop contribution is small because of the small values of $m_t$. In the figure one
can also appreciate the effect of using the one-loop effective potential in determining
the value of $\mu$: the largest deviation from the tree-level result occurs at the left
boundary.

6 Superstrings

Superstrings provide the only known consistent theory of quantum gravity. They also
provide means for calculating all parameters in a string model in terms of dimension-
less constants or dynamically determined parameters. In particular, the Yukawa cou-
plings can be calculated in a given string "vacuum" in terms of the string gauge cou-
pling and possibly the expectation values of some "moduli" fields which parametrize
deformations of the chosen vacuum. This calculational property should be regarded
as unique and fundamental as that of finite quantum gravitational interactions.

In the free-fermionic formulation of the heterotic string in four dimensions \[24\], a typical result of such calculations of Yukawa couplings is \[25, 26\]

\[
\lambda_t(M_U) = \sqrt{2} g \cos \theta_t, \tag{45}
\]

where \(\cos \theta_t\) is an “effective” coefficient which may arise from some mixing of states leading to the physical states, or is simply another coefficient which appears in the actual calculation. Typical values of this effective coefficient are \(\cos \theta_t = 1, \frac{1}{\sqrt{2}}, \frac{1}{2}\). Therefore, the typical string predictions for \(\lambda_t\) are not small, but since \(\lambda_t = \sqrt{2} g \cos \theta_t \lesssim 1\), these are always below the unitarity limit of 2.89. Moreover, this prediction is also generically not small, therefore large values of \(m_t\) are typical predictions of string models. In the following figure we show the calculated values of \(\lambda_t(M_U)\) for given (running) top-quark masses and fixed values of \(\tan \beta\) \[27\]. Two typical string predictions are denoted by horizontal dashed lines. Note that one expects \(m_t \sim 160 - 185\) GeV (or \(m_t^{\text{pole}} \sim 170 - 195\) GeV).

\[\text{SU}(5) \times \text{U}(1)\] supergravity

\[
\begin{array}{c}
\lambda_t(M_U) \\
\end{array}
\]

\[
\begin{array}{c}
\tan \beta = 2 \\
\lambda_t = \sqrt{2} g \\
\lambda_t = \sqrt{2} g / 2
\end{array}
\]

\[
\begin{array}{c}
130 \\
140 \\
150 \\
160 \\
170 \\
180 \\
190
\end{array}
\]

\[
\begin{array}{c}
0.0 \\
0.5 \\
1.0 \\
1.5 \\
2.0 \\
2.5
\end{array}
\]

\[
\begin{array}{c}
10 \\
6
\end{array}
\]

\[m_t\ (\text{GeV})\]

17
7 Conclusions

In an attempt to make some sense of the mounting evidence for the top-quark mass, we have recalled the various theoretical constraints that should be satisfied in a sensible theory. These constraints depend on the assumptions made about the theory, i.e., the energy range where it is expected to hold, its matter content, the larger theory which may embed it, etc.

Most of the constraints which we derived were based on the assumption that the theory remained weakly interacting in its presumed regime of validity. This is an assumption that does not need to hold, i.e., the theory could have a strongly interacting phase with new physical predictions. However, if one decides to ignore this possibility (as it is tacitly done all the time), then one must be consistent in requiring that no sector of the theory violates this tacit assumption.

The requirement of weakly interacting theories results in upper bounds on $m_t$:

- Standard Model: $m_t \lesssim 500 \text{GeV}$;
- SUSY: $m_t \lesssim 500 \sin \beta \text{GeV}$;
- GUTs: $m_t \lesssim 240 \text{GeV}$;
- SUSY GUTs: $m_t \lesssim 200 \sin \beta \text{GeV}$.

On the other hand, radiative electroweak symmetry breaking requires a not-too-small value of $m_t$, typically $m_t \gtrsim 75 \text{GeV}$. We also discussed predictions for the top-quark mass in superstring models based on the gauge group $SU(5) \times U(1)$. These predictions satisfy all the theoretical constraints discussed previously and typically require $m_t^{\text{pole}} \sim 170 - 195 \text{GeV}$.

All of these expectations are based solely on theoretical concepts. On the other hand, phenomenological expectations appear to favor $m_t \sim 160 \pm 10 \text{GeV}$. We therefore conclude that SUSY GUTs expectations and superstring predictions are in good agreement with present experimental expectations. Moreover, in these theoretical frameworks one can naturally understand the apparently large value of the top-quark mass.

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