Gravity Dual for Very Special Conformal Field Theories in type IIB Supergravity

Yu Nakayama

Department of Physics, Rikkyo University, Toshima, Tokyo 171-8501, Japan

Abstract

We study holographic dual descriptions of very special conformal field theories with the T(2) symmetry. After constructing solutions in effective five dimensional Einstein gravity coupled with massive two-form fields, we uplift them to the ten dimensional type IIB supergravity, via a consistent truncation ansatz, to derive new analytical solutions in string theory. From the Kaluza-Klein ansatz in terms of the internal Sasaki-Einstein space, we obtain their field theory interpretation with concrete realizations in a large class of holographic $\mathcal{N} = 1$ supersymmetric conformal field theories. Null compactification of these theories yields holographic dual descriptions of non-relativistic critical systems with translational invariance but without rotational invariance such as the ones induced from a constant electromagnetic field.
1 Introduction

In high energy physics, the Poincaré symmetry is often regarded as a fundamental assumption, but it is theoretically possible that it is merely an accidental low energy symmetry. One way to break the Lorentz symmetry while preserving the translational symmetry is to choose a particular direction in space-time. Certainly in our universe, there is a special “time direction” which emerges from the evolution of the universe, resulting in the breaking of the Lorentz symmetry down to a spatial rotation. Mathematically, however, it is more interesting to choose a particular null direction to break the Lorentz symmetry, leading to more non-trivial residual symmetries than the mere spatial rotation. This was suggested by Cohen and Glashow sometime ago \[1\][2] and dubbed “very special relativity”.

To address the question of the emergent Lorentz symmetry, we recall that our modern understanding of effective field theories is based on the idea of renormalization group, and it is instructive to see how quantum field theories with the very special relativity behaves under the renormalization group flow. In order to study the renormalization group flow, it is crucial to understand the structure of renormalization group fixed points with the very special relativity. In \[3\], we have classified all the conformal extension of the very special relativity as possible candidates of the renormalization group fixed points that are consistent with the very special relativity. Such theories were named “very special conformal field theories”.

A remarkable feature of the construction of very special conformal field theories is that if we start with the four-dimensional Poincaré invariant conformal field theories, the deformation that preserves the very special conformal symmetry has the Poincaré conformal dimension of five, which means (i) it is an irrelevant deformation from the point of view of the Poincaré conformal scaling and (ii) it may appear to be “non-renormalizable” from the naive power-counting. Therefore, while it is easy to write down the effective action that preserves the very special conformal symmetry at the tree level, the study of their quantum nature is difficult by design.

In this paper, we offer an alternative view on the strongly interacting very special conformal field theories possibly realized at the ultraviolet fixed point by using the holography. The holographic dual or gravity dual descriptions of strongly coupled conformal field theories are successful not only with Poincaré invariance but also without Poincaré invariance. In our previous paper \[3\], we have found that one of such examples, i.e. holographic
dual description of the Schrödinger invariant conformal field theories [4][5][6][7][8][9], has many common features with the holographic dual description of a particular kind of very special conformal field theories with the E(2) symmetry. See also earlier works [10][11] on the holographic descriptions of these theories with the same symmetry. The main focus of this paper is to study the least symmetric case of the T(2) invariant very special conformal field theories and their gravity duals which have not been addressed before. As we will see, null compactification of these theories yields holographic dual descriptions of non-relativistic critical systems with translational invariance but without rotational invariance and may have potential applications in condensed matter physics.

The organization of the paper is follows. In section 2, we review the basic facts about the very special conformal field theories. In section 3, we construct the holographic duals in the effective five dimensional Einstein gravity coupled with two-form matter fields. In section 4, we uplift the solutions to ten dimensional type IIB supergravity and obtain new analytical solutions. In section 5, we give the field theory interpretation of our holographic dual solutions. In section 6, we conclude with further discussions.

2 Very Special Conformal Field Theories

Very special conformal symmetry is a subgroup of the Poincaré conformal symmetry of \(SO(4,2)\) that preserves a particular null direction and include at least one special conformal transformation. For our purpose, let us choose this null direction to be \(x^-\), where \(x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^i)\) with the metric convention \(ds^2 = -2dx^+dx^- + (dx^i)^2\). Here \(i = 2,3\).

Based on the classification without the conformal symmetry [1][2], in [3] we argued that there are four interesting classes of very special conformal symmetry. The maximal case is \(\text{SIM}(2)\) very special conformal algebra spanned by \(\{P_+, P_-, P_i, J_{+i}, J_{+-}, J_{ij}, \tilde{D}, K_+\}\), whose commutation relation can be inferred from the original Poincaré conformal group with the conventional notation of \(P_\mu\) being translation, \(J_{\mu\nu}\) being Lorentz transformation, \(K_\mu\) being special conformal transformation, and \(D\) being dilatation (while \(\tilde{D} = D + J_{+-}\)). If we abandon \(J_{ij}\), we obtain \(\text{HOM}(2)\) very special conformal algebra, and if we abandon \(J_{+-}\), we obtain \(\text{E}(2)\) very special conformal algebra. Finally, if we abandon \(J_{+-}\) and \(J_{ij}\) we obtain \(\text{T}(2)\) very special conformal algebra, which will be the main focus of this paper.
Note that the names only refer to the commutation relations of the Lorentz part of the algebra. For completeness the commutation relations of T(2) very special algebra are summarized in Table 1.

|     | $P_+$ | $P_-$ | $P_i$ | $J_{+i}$ | $K_+$ | $\tilde{D}$ |
|-----|-------|-------|-------|----------|-------|-----------|
| $P_+$ | 0     | 0     | 0     | 0        | 0     | 0         |
| $P_-$ | 0     | 0     | 0     | $P_i$    | $-\tilde{D}$ | 2$P_-$ |
| $P_i$ | 0     | 0     | 0     | $P_+$    | $J_{+i}$ | $P_i$     |
| $J_{+i}$ | 0     | $-P_i$ | $-P_+$ | 0        | 0     | $-J_{+i}$ |
| $K_+$ | 0     | $\tilde{D}$ | $-J_{+i}$ | 0        | 0     | $-2K_+$   |
| $\tilde{D}$ | 0     | $-2P_-$ | $-P_i$ | $J_{+i}$ | 2$K_+$ | 0         |

Table 1: The commutation relation of the T(2) very special conformal algebra.

Very special conformal field theories possess a local energy-momentum tensor that satisfies the following conditions

\[
\begin{align*}
\partial_+ T_+^+ + \partial_- T_-^- + \partial_i T_i^i &= 0 \\
\partial_+ T_-^- + \partial_- T_-^- + \partial_i T_i^i &= 0 \\
\partial_+ T_j^+ + \partial_- T_j^- + \partial_i T_j^i &= 0 \\
T_i^- + T_i^i &= 0 \\
2T_-^- + T_i^i &= 0 .
\end{align*}
\]  

(1)

so that one can construct the local current associated with the dilatation $\tilde{D}$ and the special conformal transformation $K_+$.

3 Holographic realizations

The main purpose of this paper is to construct holographic descriptions of the T(2) very special conformal field theories while the corresponding holographic models for E(2) very special conformal field theories were constructed in [3]. In the field theory side, one way

\footnote{The cases with SIM(2) and HOM(2) very special conformal field theories are much more challenging because we have to abandon either unitarity or locality. See some ideas to construct holographic duals in [12].}
to construct very special conformal field theories with T(2) symmetry is to start with the
Poincaré invariant conformal field theories and deform them by using the dimension five
primary operator in the anti-symmetric tensor representation of the Poincaré group.

\[ S = S_0 + \int d^4x O^{-x}, \quad (2) \]

where \( S_0 \) is the action with Poincaré conformal invariance, and \( O^{-x} = -O^{x-} \) transforms
as an anti-symmetry tensor and it has \( \Delta = 5 \) under the Poincaré dilatation \( D \).

To mimic the construction in the gravity side, we begin with the five dimensional
Einstein-Hilbert action with negative cosmological constant, which supports the AdS
space, coupled with a massive two-form tensor field, which is dual to the dimension five
anti-symmetric tensor operator. The explicit action is

\[ S_B = \int d^5x \sqrt{g} \left( \frac{1}{2} R - \Lambda - \frac{1}{6} H_{MNL} H^{MNL} - \frac{m^2}{2} B_{MN} B^{MN} \right) \quad (3) \]

where \( B_{MN} = -B_{NM} \) and \( H_{MNL} = \partial_M B_{NL} + \partial_N B_{LM} + \partial_L B_{MN} \). We will set \( \Lambda = -6 \) and
\( m^2 = 9 \), where we recall that with this cosmological constant the mass and the Poincaré
conformal dimension \( \Delta \) of the two-form field are related as \( m^2 = (\Delta - 2)^2 \).

We find the solution of the equations motion that have the symmetry corresponding
to the \( T(2) \) very special conformal symmetry. The metric is given by

\[ ds^2 = -cdx^{-2} + \frac{dz^2}{z^4} + \frac{2dx^+ dx^- + (dx^2)^2 + (dx^3)^2}{z^2}, \quad (4) \]

where it is invariant under the dilatation \( \tilde{D} \)

\[ x^- \rightarrow \lambda^2 x^-, \quad x^i \rightarrow \lambda x^i, \quad z \rightarrow \lambda z, \quad x^+ \rightarrow x^+ \quad (5) \]

as well as the very special conformal transformation \( K_+ \)

\[ x^- \rightarrow \frac{x^-}{1 + ax^-}, \quad x^i \rightarrow \frac{x^i}{1 + ax^-}, \quad z \rightarrow \frac{z}{1 + ax^-}, \quad x^+ \rightarrow x^+ - \frac{a x_i^2 + z^2}{21 + ax^-}. \quad (6) \]

Note that this metric has the larger symmetry than the T(2) symmetry because the
rotation in \( ij \) directions is not broken: it is invariant under the E(2) symmetry. One can
argue that whenever one wants to preserve the T(2) symmetry in the five dimensional
metric, it must possess the enhanced E(2) symmetry.
In order to break the isometry further down to the $T(2)$ very special conformal symmetry, we need a non-trivial matter configuration. In our case, the massive two-form tensor has the condensation

$$B = \frac{1}{2} B_{MN} dx^M dx^N = b \frac{dx^- dx^2}{z^3},$$

which is invariant only under the isometry of the $T(2)$ very special conformal symmetry: $J_{ij}$ rotation is obviously broken by choosing a particular $x^2$ direction in the two-form. We can readily compute the matter energy-momentum tensor and it has only component in $--$. Then the $--$ component of the Einstein equation demands $c = 6b^2$. Note that $c$ can be changed by the rescaling of $dx^-$ and $dx^+$ unless it vanishes, so its magnitude is not physically significant. We also note that although $B$ breaks the $ij$ rotation, the energy-momentum tensor constructed out of $B$ does not break the $ij$ rotation so that the Einstein equation is consistent with the above metric ansatz.

In $d = 5$ dimensions, we may use an alternative way to introduce a massive two-form field from the topological coupling. Let us consider a complex two-form field $q_2$ with the action

$$S_q = \int \left( \frac{1}{2} R \ast 1 - \Lambda \ast 1 - \frac{i}{3} \bar{q}_2 \wedge dq_2 + \frac{i}{3} q_2 \wedge d\bar{q}_2 - 2q_2 \wedge \ast \bar{q}_2 \right),$$

where $*$ is the five dimensional Hodge dual operator. The equation of motion of $q_2$ gives the self-dual equation

$$dq_2 = 3i \ast q_2$$

which describes a propagating (real) massive two-form with $m^2 = 9$. To see this, we may substitute $q_2 = k_2 + il_2$ with real two-form fields $k_2$ and $l_2$. The equations of motion tell that $l_2 = \frac{1}{3} \ast dk_2$ and $k_2$ (or accordingly $l_2$ as well) satisfies the massive two-form equation with $m^2 = 9$. The Einstein equation is

$$R_{MN} = -4g_{MN} + 2(q_M^L \bar{q}_{NL} + q_N^L \bar{q}_{ML} - \frac{1}{12} g_{MN} q_{LK} \bar{q}^L \bar{q}^K)$$

We find the following solutions with the symmetry corresponding to the $T(2)$ very special conformal symmetry: the metric is given by

$$ds^2 = -c \frac{dx^-}{z^4} + \frac{dz^2 - 2dx^+dx^- + (dx^2)^2 + (dx^3)}{z^2}$$
and the massive two-form has the condensation
\[ q_2 = q \left( \frac{dx^- dx^2}{z^3} + i \frac{dx^- dx^3}{z^3} \right). \] (12)

The \(-\) component of the Einstein equation demands \( c = \frac{4}{3} q^2 \).

## 4 Uplifting to type IIB supergravity

In the previous section, we constructed the gravity dual for the \( T(2) \) very special conformal field theories in the bottom up approach. Here we would like to uplift them to the ten dimensional type IIB supergravity, allowing us to interpret them from the string theory viewpoint as we will discuss in the next section.

Our starting point is the \( AdS_5 \times SE_5 \) flux compactification of type IIB supergravity
\[
ds_{10}^2 = ds^2(AdS_5) + ds^2(SE_5)
F(5) = 4\text{vol}(SE_5) + 4\text{vol}(AdS_5)
H(3) = F(3) = 0
\] (13)

with a constant axion-dilaton field. Here the Sasaki-Einstein space \( SE_5 \) has a preferred \( U(1) \) isometry called Reeb Killing vector and the metric has the local decomposition of
\[
ds^2(SE_5) = ds^2(KE_4) + \eta^2,
\] (14)

where \( \eta = d\psi + A \) is the (contact) one-form dual to the Reeb Killing vector. On the Sasaki-Einstein space, there is an \( SU(2) \) structure \((\eta, J, \Omega)\) which satisfies
\[
J \wedge \Omega = 0
\Omega \wedge \bar{\Omega} = 2J \wedge J = 4\text{vol}(KE_4)
*_{4}J = J
*_{4}\Omega = \Omega
d\eta = 2J
d\Omega = 3i\eta \wedge \Omega.
\] (15)

Here \( *_{4} \) is the Hodge dual on the (locally) Kahler Einstein base \( KE_4 \). The Kahler form \( J \) and \((2,0)\) form \( \Omega \) are induced from those of the Kahler-Einstein base \( KE_4 \). The holographic interpretation of these gravity solutions have been studied in the literature and
they are given by $\mathcal{N} = 1$ superconformal field theories realized as large $N$ quiver gauge theories \cite{15,16}.

With these ingredients, our Kaluza-Klein ansatz is to deform the five-form flux as

$$
\begin{align*}
 ds_{10}^2 &= ds^2(E_5) + ds^2(K E_4) + \eta^2 \\
 F_5 &= 4\text{vol}(SE_5) + 4\text{vol}(E_5) + (\ast q_2 \wedge \Omega + q_2 \wedge \Omega \wedge \eta + \text{c.c.}) \\
 F_3 &= 0 \\
 H_3 &= 0 \\
 C_0 &= 0 \\
 \Phi &= \phi ,
\end{align*}
$$

(16)

where $\ast$ is the Hodge dual with respect to the five dimensional metric $ds^2(E_5)$. For the complete consistent truncation, we have to keep the other scalar degrees of freedom such as the breathing mode, but for our particular solution, we are not going to excite any scalar degrees of freedom as we will argue so that this ansatz will become consistent.

As discussed in \cite{13,14}, the Kaluza-Klein reduction gives us the five dimensional Einstein gravity coupled with massive (complex) two-form field $q_2$ that has the topological mass term with $m^2 = 9$, which we have already discussed in the previous section. The non-trivial claim is that the Kaluza-Klein reduction is a consistent truncation and the five dimensional solution

$$
\begin{align*}
 ds^2 &= -c \frac{dx^{-2}}{z^4} + \frac{dz^2 - 2dx^+dx^- + (dx^2)^2 + (dx^3)}{z^2} \\
 q_2 &= \bar{q}(\frac{dx^-dx^2}{z^3} + i\frac{dx^-dx^3}{z^3}) .
\end{align*}
$$

(17)

substituted into (16) solves the ten-dimensional type IIB supergravity equations of motions when $c = \frac{4}{3}q^2$. Most of the relevant computation has been done in the literature \cite{13,14}, but since it is a relatively simple solution, one can directly check that the ten dimensional equations of motion hold.

Explicitly, the five form equations of motion gives

$$
dq_2 = 3i \ast q_2
$$

(18)

and it is readily solved by our ansatz since it is the same equation as the self-dual equation of the complex two-form field. The self-dual condition of the five form was already encoded
in our Kaluza-Klein ansatz. Then the $MN$ (i.e. five dimensional) component of the ten
dimensional Einstein equation gives

$$R_{MN} = -4g_{MN} + 2(q^L_M q^L_N + q^L_N q^L_M - \frac{1}{12} g_{MN} q^L_K q^L_K)$$  \hspace{1cm} (19)$$

and this is exactly the same equations that we have already solved in the effective five
dimensional gravity. The crucial point here is that the other equations of motion are not
affected because in those equations we have to contract $q_2$ to make a scalar, but since $q_2$
has only a null component, the scalar contraction always gives zero. Another point is that
the Einstein equation (19) holds not only in average but pointwise in the internal space
due to the special geometric feature of our Kaluza-Klein ansatz.

## 5 Field theory interpretation

A comparison of the Kaluza-Klein ansatz with the linearized harmonic analysis on the flux
compactification of the Sasaki-Einstein space with the AdS/CFT dictionary allows us to
interpret the holographic solutions we constructed from the dual field theory perspective.
As we have already mentioned two-form fields with the mass $m^2 = 9$ in the gravity side
correspond to dimension five anti-symmetric tensor operators in the dual field theories.
More explicitly, in the Sasaki-Einstein compactification, it is given by

$$O = \sum_I c_I \text{Tr} F_I^{x-}(\lambda^I \lambda^I + cc) ,$$  \hspace{1cm} (20)$$

where $\lambda^I$ are gauginos and $I$ runs over the gauge groups of the quiver gauge theory. See
for example [17] for the case of $SE_5 = T_{1,1}$ and [18][19] for the case of $SE_5 = S_5$. In the general
flux compactification of the Sasaki-Einstein space, there is one particular choice of $c_I$ over
different gauge groups which have a protected scaling dimension so that one can use it
as our dimension five anti-symmetric tensor deformations. In the case of $\mathcal{N} = 4$ super
Yang-Mills theory, it is in the $10_c$ rep of SU(4) R-symmetry from the choice of the four
different fermions, and all of them can be used.

The operator $O$ is charged under the R-symmetry as we can also see from the Kaluza-
Klein reduction (since $\Omega$ appearing in the ansatz transforms under the Lie derivative with
respect to the Reeb Killing vector), and it breaks the $R$-symmetry once added. Note
also that the operator breaks (at least part of) the supersymmetry because the protected
operator Tr$F_{\alpha\beta}\lambda\bar{\lambda}$ and its conjugate Tr$F_{\alpha\beta}\bar{\lambda}\lambda$ preserves the different supersymmetry. Even this is the case, the deformation is an exactly marginal deformation with respect to the very special conformal symmetry as demonstrated in the holographic description since the deformation parameter $c$ is arbitrary.

6 Discussions

In this paper, we constructed the gravity dual for T(2) invariant very special conformal field theories. In particular, we have presented new exact solutions in type IIB supergravity with the isometry corresponding to the T(2) very special conformal symmetry. Such top-down constructions give precise field theory interpretation of our dual descriptions in terms of explicit $\mathcal{N} = 1$ supersymmetric conformal field theories.

Our field theory interpretation reveals that we can construct very special conformal field theories from the Poincaré conformal field theory by introducing dimension five operator such as the gauge field strength multiplied by gaugino bilinear. The similar operator can be found in the standard model of particle physics. For example, in QED one can readily construct the dimension five anti-symmetric tensor operator such as $F^{x-}\bar{\Psi}_e\Psi_e$, where $\Psi_e$ is the electron field, and in this way one may imagine the Lorentz violating interaction with the T(2) very special conformal symmetry may exist in the (ultraviolet completion of the) nature.

Beside the particle physics application, we would like to point out that the gravity dual description of T(2) symmetric very special conformal field theories have potential applications in condensed matter physics. Suppose we compactify the null direction $x^+$, and regard the Kaluza-Klein momentum as a particle number. Then we obtain non-relativistic conformal field theories with $z = 2$ scaling. Here $x^-$ direction is identified with the non-relativistic time, and $J_{+i}$ is regarded as the Galilean boost symmetry. Furthermore $K_+$ is regarded as the non-relativistic special conformal transformation. Now the difference between our (compactified) T(2) invariant non-relativistic critical systems and the ones studied in the literature (e.g. the Schödinger conformal field theories) is that we do not have the spatial rotation as a symmetry. Therefore, our theory is suitable to describe the system without rotational symmetry (but with the translational symmetry). Such critical systems are ubiquitous: one may realize them by introducing background
electric field or magnetic field along a particular spatial direction. It would be interesting to investigate their properties such as thermal properties by constructing corresponding blackhole solutions in our setup.

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