Conformal symmetry wormholes and the null energy condition

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Abstract

In this paper we seek a relationship between the assumption of conformal symmetry and the exotic matter needed to hold a wormhole open. By starting with a Morris-Thorne wormhole having a constant energy density, it is shown that the conformal factor provides the extra degree of freedom sufficient to account for the exotic matter. The same holds for Morris-Thorne wormholes in a noncommutative-geometry setting. Applied to thin shells, there would exist a radius that results in a wormhole with positive surface density and negative surface pressure and which violates the null energy condition on the thin shell.

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1 Introduction

Wormholes are handles or tunnels in spacetime that connect different regions of our Universe or completely different universes. That wormholes could be macroscopic structures allowing interstellar travel was first proposed by Morris and Thorne [1]. With the Schwarzschild line element in mind, such a wormhole could be described by the static and spherically symmetric line element:

\[ ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \]

(1)

using units in which \( c = G = 1 \). Here \( \Phi = \Phi(r) \) is referred to as the redshift function, which must be everywhere finite to avoid an event horizon. The function \( b = b(r) \) is called the shape function since it helps to determine the spatial shape of the wormhole [2]. The spherical surface \( r = r_0 \) is the throat of the wormhole. At the throat, \( b = b(r) \) must satisfy the following conditions: \( b(r_0) = r_0, b(r) < r \) for \( r > r_0 \), and \( b'(r_0) \leq 1 \), usually called

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the flare-out condition. This condition can only be satisfied by violating the null energy condition, defined as follows: for the stress-energy tensor $T_{\alpha\beta}$, we must have

$$T_{\alpha\beta}\mu^\alpha\mu^\beta \geq 0 \quad (2)$$

for all null vectors. By Ref. [1], the violation is equivalent to the condition

$$\frac{b'(r_0) - b(r_0)/r_0}{2[b(r_0)]^2} < 0. \quad (3)$$

For a Morris-Thorne wormhole, matter that violates the null energy condition is called “exotic.”

In this paper, we are going to seek a relationship between exotic matter and conformal symmetry, by which is meant the existence of a conformal Killing vector $\xi$ defined by the action of $L_\xi$ on the metric tensor

$$L_\xi g_{\mu\nu} = \psi(r) g_{\mu\nu}; \quad (4)$$

here $L_\xi$ is the Lie derivative operator and $\psi(r)$ is the conformal factor.

It is shown that $\psi(r)$ provides the extra degree of freedom to account for the exotic matter for certain types of wormholes. Applied to thin-shell wormholes, an appropriate choice of the radius avoids the usual negative surface density typical of thin shells.

## 2 Conformal Killing vectors

We assume in this paper that our static spherically symmetric spacetime admits a one-parameter group of conformal motions, i.e., motions along which the metric tensor of a spacetime remains invariant up to a scale factor. Equivalently, there exist conformal Killing vectors such that

$$L_\xi g_{\mu\nu} = g_{\eta\nu} \xi^\eta_{;\mu} + g_{\mu\eta} \xi^\eta_{;\nu} = \psi(r) g_{\mu\nu}; \quad (5)$$

where the left-hand side is the Lie derivative of the metric tensor and $\psi(r)$ is the conformal factor [2, 3]. The vector $\xi$ generates the conformal symmetry and the metric tensor $g_{\mu\nu}$ is conformally mapped into itself along $\xi$. This type of symmetry has proved to be effective in describing relativistic stellar-type objects [4, 5]. Moreover, conformal symmetry has led to new solutions, as well as to new geometric and kinematical insights [6, 7, 8, 9]. Two earlier studies assumed non-static conformal symmetry [3, 10].

To study the effect of conformal symmetry, it is convenient to use an alternate form of the metric [11]:

$$ds^2 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (6)$$

Using this line element, the Einstein field equations become

$$e^{-\lambda} \left( \frac{\lambda'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2} = 8\pi \rho, \quad (7)$$
\[ e^{-\lambda} \left( \frac{1}{r^2} + \frac{\nu'}{r} \right) - \frac{1}{r^2} = 8\pi p_r, \]  

and

\[ \frac{1}{2} e^{-\lambda} \left[ \frac{1}{2} \left( \nu' \right)^2 + \nu'' - \frac{1}{2} \lambda' \nu' + \frac{1}{r} (\nu' - \lambda') \right] = 8\pi p_t. \]

Here \( \rho \) is the energy density, while \( p_r \) and \( p_t \) are the radial and transverse pressures, respectively. Eq. (9) could actually be obtained from the conservation of the stress-energy tensor, i.e., \( T_{\mu\nu} \cdot \; ; \nu = 0 \). So we need to use only Eqs. (7) and (8).

The subsequent analysis can be simplified somewhat by following Herrera and Ponce de León [4] and restricting the vector field by requiring that \( \xi^\alpha U_\alpha = 0 \), where \( U_\alpha \) is the four-velocity of the perfect fluid distribution. As a result, fluid flow lines are mapped conformally onto fluid flow lines. The assumption of spherical symmetry then implies that \( \xi^0 = \xi^2 = \xi^3 = 0 \) [4]. Eq. (5) now yields the following results:

\[ \xi^1 \nu' = \psi, \]  

\[ \xi^1 = \frac{\psi r}{2}, \]  

and

\[ \xi^1 \lambda' + 2 \xi^1,1 = \psi. \]

From Eqs. (10) and (11), we then obtain

\[ e^\nu = c_1 r^2, \]  

which, combined with Eq. (12), produces another important result:

\[ e^\lambda = \left( \frac{c_2}{\psi} \right)^2; \]

\( c_1 \) and \( c_2 \) are integration constants.

The field equations (7) and (8) can be rewritten as follows:

\[ \frac{1}{r^2} \left( 1 - \frac{\psi^2}{c_2^2} \right) - \frac{(\psi^2)'}{c_2^2 r} = 8\pi \rho \]  

and

\[ \frac{1}{r^2} \left( 3\psi^2 - 1 \right) = 8\pi p_r. \]

It now becomes apparent that \( c_2 \) is merely a scale factor in Eqs. (14)-(16); so we may assume that \( c_2 = 1 \). The constant \( c_1 \), however, has to be obtained from the junction conditions, the need for which can be seen from Eq. (13): since our wormhole spacetime is not asymptotically flat, the wormhole material must be cut off at some \( r = a \) and joined to an exterior Schwarzschild solution,

\[ ds^2 = - \left( 1 - \frac{2M}{r} \right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \]
so that \( e^{\nu(a)} = c_1 a^2 = 1 - 2M/a \), whence
\[
c_1 = \frac{1 - 2M/a}{a^2},
\] where \( M \) is the mass of the wormhole as seen by a distant observer. It also follows that \( b(a) = 2M \).

3 Wormhole structure

To simplify the analysis in the next section, we will assume that the energy density \( \rho \) is constant and that the wormhole material is confined to the spherical shell \( r_0 \leq r \leq a \), where \( a \) is the cut-off in Eq. (18). (This form of \( \rho(r) \) was also assumed by Sushkov \[12\] in his discussion of wormholes supported by phantom energy.) So from Eq. (15) (with \( c_2 = 1 \)), we obtain the following differential equation:
\[
\frac{1}{r^2} (1 - \psi^2) - \frac{(\psi^2)'}{r} = 8\pi \rho_0, \quad r_0 \leq r \leq a.
\] After multiplying by \( r \) and rearranging, we obtain
\[
(\psi^2)' + \frac{1}{r} \psi^2 = \frac{1}{r} - 8\pi \rho_0 r.
\] This equation is linear in \( \psi^2 \) and can readily be solved to obtain
\[
\psi^2 = 1 - \frac{8}{3} \pi \rho_0 r^2 \frac{c}{r},
\] where \( c \) is a constant of integration. From Eqs. (11) and (6), we get \( e^{-\lambda} = 1 - b(r)/r \), whence
\[
b(r) = r(1 - \psi^2).
\] The requirement \( b(r_0) = r_0 \) now implies that \( \psi^2(r_0) = 0 \). So by Eq. (21),
\[
c = \frac{8}{3} \pi \rho_0 r_0^3 - r_0,
\] and from Eq. (22),
\[
b(r) = r \left( \frac{8}{3} \pi \rho_0 r^2 - \frac{8}{3} \pi \rho_0 r_0^3 \frac{1}{r} + \frac{r_0}{r} \right), \quad r_0 \leq r \leq a.
\] Next, to meet the flare-out condition, we require that
\[
b'(r) = \frac{8}{3} \pi \rho_0 (3r^2) \bigg|_{r=r_0} = \frac{8}{3} \pi \rho_0 (3r_0^2) < 1,
\] which implies that
\[
\rho_0 < \frac{1}{8\pi r_0^2}.
\]
4 The null energy condition

Returning to the null energy condition (2), $T_{\alpha\beta}u^\alpha u^\beta \geq 0$, observe that for the radial outgoing null vector $(1, 1, 0, 0)$, we obtain $\rho + p_r < 0$ whenever the condition is violated. As noted earlier, for a Morris-Thorne wormhole, matter that violates the null energy condition is usually called “exotic.” Moreover, we saw in the previous section that $\psi'(r_0) < 1$ whenever $\rho_0 < 1/(8\pi r_0^2)$. The extra assumption of conformal symmetry now yields by Eq. (16)

$$p_r = \frac{1}{8\pi} \frac{1}{r^2} [3\psi^2(r) - 1]$$

and by Eq. (26), as expected,

$$\rho + p_r |_{r=r_0} = \rho_0 + \frac{1}{8\pi} \frac{1}{r_0^2} [3\psi^2(r_0) - 1] < 0$$

since $\psi^2(r_0) = 0$. This result suggests that the assumption of conformal symmetry helps explain the violation of the null energy condition by accounting for the exotic matter. More precisely, while the physical requirements impose some severe constraints on the geometry, they do not determine the conformal factor $\psi(r)$. Such a determination depends on other important geometric notions such as shape characteristics and shape deformations that arise in various fields such as computer graphics. (For further discussion, see Refs. 13, 14, 15, 16, 17.) These geometric factors provide an extra degree of freedom via $\psi(r)$ that is not available for the usual Morris-Thorne wormholes.

The case for eliminating (in the above sense) exotic matter in certain cases can be strengthened in the context of noncommutative geometry, which replaces point-like structures by smeared objects. The smearing effect can be accomplished by assuming that the energy density of a static and spherically symmetric and particle-like gravitational source has the form 18, 19, 20, 21

$$\rho(r) = \frac{M\sqrt{\theta}}{\pi^2(r^2 + \theta)^2}.$$  

Here the mass $M$ is diffused throughout the region of linear dimension $\sqrt{\theta}$ due to the uncertainty. Observe that $\rho + p_r$ now becomes

$$\rho(r) + p_r(r) = \frac{M\sqrt{\theta}}{\pi^2(r^2 + \theta)^2} + \frac{1}{8\pi} \frac{1}{r^2} [3\psi^2(r) - 1]$$

and

$$\rho(r_0) + p_r(r_0) = \frac{M\sqrt{\theta}}{\pi^2(r_0^2 + \theta)^2} - \frac{1}{8\pi} \frac{1}{r_0^2} < 0$$

since $\sqrt{\theta} \ll 1$. So the violation of the null energy condition can be attributed to a combination of physical and geometric factors.
5 Thin-shell wormholes

Using the now standard cut-and-paste technique, a thin-shell wormhole is constructed by taking two copies of a Schwarzschild spacetime and removing from each the four-dimensional region

$$\Omega = \{ r \leq a \mid a > 2M \}$$

where $a$ is a constant \[22\]. By identifying the boundaries, i.e., letting

$$\partial \Omega = \{ r = a \mid a > 2M \}$$

we obtain a manifold that is geodesically complete, while possessing two asymptotically flat regions connected by a wormhole. The throat is the surface $\partial \Omega$.

Since the shell is infinitely thin, the radial pressure is zero. So if the surface density is denoted by $\sigma$, then $\sigma + p_r < 0$ implies that $\sigma$ is negative. The goal in this section is to show that under the assumption of conformal symmetry, $\sigma$ can be positive. The null energy condition will then be violated on the thin shell itself, even though it is met for the radial outgoing null vector $(1, 1, 0, 0)$.

To that end, let us consider the surface stresses. So we need to recall the Lanczos equations \[23\]

$$\sigma = -\frac{1}{4\pi} \kappa^\theta_\theta$$

and

$$\mathcal{P} = \frac{1}{8\pi} (\kappa^\tau_\tau + \kappa^\theta_\theta),$$

where $\kappa_{ij} = K^+_{ij} - K^-_{ij}$ and $K_{ij}$ is the extrinsic curvature. According to Ref. \[23\],

$$\kappa^\theta_\theta = \frac{1}{a} \sqrt{1 - \frac{2M}{a}} - \frac{1}{a} \sqrt{1 - \frac{b(a)}{a}}.$$

So by Eq. (34),

$$\sigma = -\frac{1}{4\pi a} \left( \sqrt{1 - \frac{2M}{a}} - \sqrt{1 - \frac{b(a)}{a}} \right).$$

In view of the assumption $b(a) = 2M$, one could reasonably expect that $\sigma = 0$. However, part of the junction formalism is to assume that the junction surface $r = a$ is an infinitely thin surface having a nonzero density that may be positive or negative. So we have instead, $b(a) \approx 2M$. Again following Ref \[23\],

$$K^\tau_\tau^+ = \frac{M/a^2}{\sqrt{1 - 2M/a}}$$

and

$$K^\tau_\tau^- = \Phi'(a) \sqrt{1 - \frac{b(a)}{a}}.$$

The surface pressure is therefore given by

$$\mathcal{P} = \frac{1}{8\pi} \left[ \frac{M/a^2}{\sqrt{1 - 2M/a}} - \Phi'(a) \sqrt{1 - \frac{b(a)}{a}} + \frac{1}{a} \sqrt{1 - \frac{2M}{a}} - \frac{1}{a} \sqrt{1 - \frac{b(a)}{a}} \right].$$
To apply these ideas, let us describe a thin shell by starting with a typical shape function and letting \( r_0 \) be arbitrarily close to \( a \), again taken to be the cut-off. This cut-off results in the redshift function \( \nu = 2\Phi = \ln c_1 r^2 \) by Eq. (13); here \( c_1 = (1 - 2M/a)/a^2 \) from Eq. (18). As a result, \( \Phi'(a) = 1/a \).

Since \( r_0 \) is arbitrarily close to \( a \), the junction surface itself becomes the thin shell with \( r = a \). So if \( b(r) \) is a typical shape function, then, ignoring the cut-off for now, we have \( b(r_0) = r_0 < b(r) < r \) for \( r > r_0 \), and \( \lim_{r \to \infty} b(r)/r = 0 \). So \( b(a)/a \) assumes all values on the interval \((0, 1]\) at least once. It follows that \( a \) can be chosen to yield any desired value for \( b(a)/a \) in the interval \((0, 1]\).

Returning now to Eq. (37), since \( \Phi'(a) = 1/a \),

\[
P = \frac{1}{8\pi} \left( \frac{M}{a^2} - \frac{1}{a} \sqrt{1 - \frac{b(a)}{a}} \sqrt{1 - \frac{2M}{a}} \right) + \frac{1}{2} \frac{1}{4\pi a} \left( \sqrt{1 - \frac{2M}{a}} - \sqrt{1 - \frac{b(a)}{a}} \right). \tag{38}
\]

Observe that the last term is equal to \( \frac{1}{2} \sigma \) in absolute value. Moreover, if \( \sigma \) is positive, then the last term is negative.

To see how \( P \) can be negative (while \( \sigma > 0 \)), let us assume for a moment that \( b(a) = 2M \), making \( \sigma = 0 \). Then

\[
\frac{M}{a^2} - \frac{1}{a} \sqrt{1 - \frac{b(a)}{a}} \sqrt{1 - \frac{2M}{a}} = \frac{b(a)}{a^2} - \frac{1}{a} \left( 1 - \frac{b(a)}{a} \right) = \frac{1}{a} \left( -1 + \frac{3}{2} \frac{b(a)}{a} \right) = 0 \tag{39}
\]

whenever \( b(a)/a = 2/3 \), but if \( b(a) \approx 2M \), then we only have \( b(a)/a \approx 2/3 \). As already noted, since \( b(a)/a \) assumes all values on the interval \((0, 1]\) at least once, such a choice can be made. Referring to Eq. (38), it now follows that

\[
\frac{1}{a} \left( -1 + \frac{3}{2} \frac{b(a)}{a} \right) < 0
\]

whenever \( b(a)/a \leq 2/3 \). More precisely, suppose \( b(a)/a = 2/3 - k_a \) for some \( k_a > 0 \). Then with Eq. (38) in mind, consider a value for \( k_a \) for which

\[
\frac{1}{8\pi} \frac{1}{\sqrt{1 - 2M/a}} \left[ -1 + \frac{3}{2} \left( \frac{2}{3} - k_a \right) \right] < \frac{1}{8\pi} \frac{2}{a} \left( \sqrt{1 - \frac{2M}{a}} - \sqrt{1 - \frac{b(a)}{a}} \right); \tag{40}
\]

observe that the right side is equal to \( \sigma \) in absolute value. Solving for \( k_a \) shows that such a \( k_a \) exists:

\[
k_a > -\frac{4}{3} \left( \sqrt{1 - \frac{2M}{a}} - \sqrt{1 - \frac{b(a)}{a}} \right) \sqrt{1 - \frac{2M}{a}} > 0. \tag{41}
\]

As a result, the first term on the right-hand side of Eq. (38) is negative and bounded away from zero, while the second term can be arbitrarily close to zero. The same is true for

\[
P + \sigma = \frac{1}{8\pi} \left( \frac{M}{a^2} - \frac{1}{a} \sqrt{1 - \frac{b(a)}{a}} \sqrt{1 - \frac{2M}{a}} \right) - \frac{1}{2} \frac{1}{4\pi a} \left( \sqrt{1 - \frac{2M}{a}} - \sqrt{1 - \frac{b(a)}{a}} \right). \tag{42}
\]
Hence

\[ \sigma + P < 0, \]

which was to be shown.

6 Conclusion

The purpose of this paper is to seek a relationship between the assumption of conformal symmetry and the exotic matter needed to hold a wormhole open. It was concluded that the conformal factor \( \psi(r) \) provides an extra degree of freedom sufficient to account for the exotic matter for certain types of wormholes, those having a constant energy density on the spherical shell \( r_0 \leq r \leq a \) and wormholes in a noncommutative-geometry setting. The extra degree of freedom does not exist for the usual Morris-Thorne wormholes.

Applied to thin shells, the assumption of conformal symmetry implies that the surface density can be positive and the surface pressure negative for some radius \( r = a \) and that the null energy condition is violated on the thin shell.

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