**RESEARCH ARTICLE**

**Twin peaks**

Krzysztof Burdzy | Soumik Pal

Department of Mathematics, University of Washington, Seattle, Washington

**Correspondence**
Krzysztof Burdzy, Department of Mathematics, University of Washington, Box 354350, Seattle, WA 98195. Email: burdzy@uw.edu

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**Abstract**
We study random labelings of graphs conditioned on a small number (typically one or two) peaks, that is, local maxima. We show that the boundaries of level sets of a random labeling of a square with a single peak have dimension 2, in a suitable asymptotic sense. The gradient line of a random labeling of a long ladder graph conditioned on a single peak consists mostly of straight line segments. We show that for some tree-graphs, if a random labeling is conditioned on exactly two peaks then the peaks can be very close to each other. We also study random labelings of regular trees conditioned on having exactly two peaks. Our results suggest that the top peak is likely to be at the root and the second peak is equally likely, more or less, to be any vertex not adjacent to the root.

**KEYWORDS**
graph labeling, peak, random growth model, random labeling

1 | INTRODUCTION

This paper is the first part of a project motivated by our desire to understand random labelings of graphs conditioned on having a small number of peaks (local maxima). The second part of the project [7] will be briefly discussed below. The roots of this project lie in the paper [4], where some new results on peaks of random permutations were proved. These were later applied in [3].

1.1 | Overview of the main results

We will keep our overview at the heuristic level because the rigorous presentation of the main results requires a number of technical definitions. We will call our informal statements “Claims.”
A labeling $L$ of a finite graph with $n$ vertices is a one-to-one function mapping the set of vertices onto $\{1, 2, \ldots, n\}$. We say that the labeling has a peak at $v$ if $L(v)$ is larger than the label at any adjacent vertex. A random labeling is a labeling chosen uniformly from all labelings of a given graph.

Our paper is inspired by the following open problem.

**Problem 1.1.** Consider a random labeling of a large discrete torus with size $n \times n$. If we condition the random labeling on the event that there are exactly two peaks, what is the likely distance between the locations of the peaks? Is it of order $n$?

In the case of the one-dimensional torus (i.e., the discrete circle with $n$ vertices), the locations of the two peaks are almost antipodal, with high probability, for large $n$ (see [3, 4]).

Although we were not able to make any progress on Problem 1.1, we will present two theorems on labelings of discrete rectangles. These were inspired by simulations, to be discussed in Section 1.2.

**Claim 1.2.** Consider a random labeling of a discrete $n \times n$ square conditioned on having exactly one peak. Let $C_k$ be the set of vertices to which the top $k$ labels are assigned and let $\partial C_k$ be the set of vertices outside $C_k$ which are adjacent to $C_k$. When $n \to \infty$, the dimension of $\partial C_k$ converges to 2, for most $k$.

**Claim 1.3.** Consider a random labeling of a discrete $m \times n$ rectangle conditioned on having exactly one peak at one of its corners. Let $\lambda$ be any path from the location of the peak to the side of the rectangle that is at distance $n$, such that the labeling is monotone along $\lambda$. When $m$ is fixed and $n \to \infty$, the path $\lambda$ is almost straight.

We have already mentioned that if a random labeling of a one dimensional connected graph (line segment or circle) with $n$ vertices is conditioned on having exactly two peaks then the locations of the two peaks will be at a distance of order $n$, with high probability, for large $n$. Our next claim implies that this is not the case for arbitrary connected graphs.

**Claim 1.4.** There is a sequence of trees with sizes $n$ such that if a random labeling of one of these trees is conditioned on having exactly two peaks then the locations of the two peaks will be at a distance less than 5, with high probability, for large $n$.

Our remaining results are concerned with regular trees of constant depth. We summarize them as follows.

**Claim 1.5.** Consider a regular tree, that is, a tree whose all vertices have the same degree except for leaves, and all leaves have the same distance to the root. If the degree is greater than 10 and a random labeling of the tree is conditioned on having exactly two peaks then the highest peak is most likely to be at the root and the location of the second peak is almost uniformly distributed among other vertices.

A subdivision graph is obtained from a given graph by inserting a large number of vertices between any two adjacent vertices of the original graph (roughly speaking). Random labelings of large subdivision graphs, conditioned on a small number of peaks, are investigated in a follow up paper [7]. The methods applied in that paper are considerably different from and more analytic than those in the present article.
1.2 Motivating models and simulations

We will define a random labeling of \( N \) vertices of \( \mathbb{Z}^d \) and a random labeling of \( d \)-dimensional torus with \( N \) vertices. The two algorithms generate similarly looking random labelings in dimension \( d = 1 \), but computer simulations and some rigorous results (to be presented later in this paper) show that the two algorithms generate strikingly different random labelings in \( \mathbb{Z}^2 \).

(i) Fix an integer \( N > 0 \). We will define inductively a random function \( \tilde{L} : \mathbb{Z}^d \rightarrow \{0, 1, 2, \ldots, N\} \) such that the origin is mapped onto \( N \) and \( \tilde{L}^{-1}(j) \) is a singleton for \( 1 \leq j \leq N \). Let \( \tilde{C}_k \) be the inverse image of \( \{N, N-1, \ldots, N-k\} \) by \( \tilde{L} \) for \( 0 \leq k \leq N-1 \). Given \( \tilde{C}_k \) for some \( 0 \leq k \leq N-2 \), the vertex \( \tilde{L}^{-1}(N-k-1) \) is distributed uniformly among all vertices in \( \mathbb{Z}^d \setminus \tilde{C}_k \) which have a neighbor in \( \tilde{C}_k \). We let \( \tilde{L}(x) = 0 \) for all \( x \notin \tilde{C}_{N-1} \). It is easy to see that the only peak (strict local maximum) of \( \tilde{L} \) is at \( \tilde{L}^{-1}(N) \).

(ii) Suppose that \( N_1 > 0, N = N_1^d \) and let \( \mathcal{G} \) be the \( d \)-dimensional torus with side length \( N_1 \) and vertex set \( \mathcal{V} \). Let \( L : \mathcal{V} \rightarrow \{1, 2, \ldots, N\} \) be a random (uniformly chosen) bijection conditioned on \( N \) being the only peak. Let \( C_k \) be the inverse image of \( \{N, N-1, \ldots, N-k\} \) by \( L \) for \( 0 \leq k \leq N-1 \).

Note that both \( \tilde{C}_k \) and \( C_k \) are connected for every \( 0 \leq k \leq N-1 \). We also have \( \tilde{C}_k \subset \tilde{C}_{k+1} \) and \( C_k \subset C_{k+1} \) for all \( k \).

Methods applied in [4] and other standard techniques show that in dimension \( d = 1 \), for large \( N \) and \( k \) comparable with \( N \), both \( \tilde{C}_k \) and \( C_k \) are intervals centered, more or less, at the vertex labeled \( N \), with high probability.

In dimension \( d = 2 \), the cluster growth process \( \{\tilde{C}_k, k \geq 1\} \) is the Eden model [11]. See Figure 1. Recent results on a related model (see [9] and Remark 3.1) strongly suggest that the boundary of \( \tilde{C}_k \) has the size of the order \( k^{1/2} \).

The contrast between Figures 1 and 2 could not be more dramatic. Figure 2 is a simulation of the process \( \{C_k, k \geq 1\} \) in two dimensions. We will show that the boundaries of the sets \( C_k \) have dimension 2, in an asymptotic sense.

The simulation of a random labeling presented in Figure 2 is a sample from the stationary distribution of the following Markov chain. The state space of the Markov chain is the set of all labelings of the torus with a single peak. The initial state was generated by a simulation of the Eden model (see Figure 1). For a single step, two vertices of the torus were chosen randomly (uniformly) and their labels were exchanged if and only if doing this did not create an extra peak. We do not have a rigorous proof that our algorithm generates a uniform random labeling conditioned on a single peak.

The following elementary example shows that models (i) and (ii) are not equivalent for general graphs. In other words, the sets \( C_k \) associated with the (uniformly chosen) random labeling with a single peak do not grow by attaching vertices uniformly on the boundary.

**Example 1.6.** Consider a graph with vertices \( \{a, b, c, d\} \) and edges \( \{ab, bc, cd\} \) (a linear graph, with \( a, b, c, \) and \( d \) arranged along a line in this order). Let \( L \) be the uniformly random labeling conditioned on having a single peak at \( b \). Let \( R = L^{-1} \) be the inverse function of \( L \). The following three realizations of \( L \) are possible and equally likely.

\[
L(a) = 3, L(b) = 4, L(c) = 2, L(d) = 1; \\
L(a) = 2, L(b) = 4, L(c) = 3, L(d) = 1; \\
L(a) = 1, L(b) = 4, L(c) = 3, L(d) = 2.
\]

We see that \( R(3) \) can be equal to \( a \) or \( c \) but

\[
1/3 = \mathbb{P}(R(3) = a \mid R(4) = b) \neq \mathbb{P}(R(3) = c \mid R(4) = b) = 2/3.
\]
In other words, if $R(3)$ is chosen uniformly from all vertices adjacent to $R(4)$ and this is followed by any choice of $R(2)$ and $R(1)$ consistent with the condition that $R(4) = b$ is the only peak of $L$ then the resulting labeling is not uniformly random (conditional on $\{R(4) = b\}$).

1.3 | Related articles

We are grateful to Nicolas Curien and an anonymous referee for pointing out to us the following related articles. Local maxima of labelings are analyzed in [1]. The labels (ages) of vertices in a random tree generated according to the Barabási and Albert “preferential attachment model” (see [14, Chap. 8]), conditioned on a given tree structure, have the distribution of a random labeling conditioned on having a single peak at the root. The paper [5] is devoted to the enumeration of parameters on increasing trees, that is, labeled rooted trees in which labels along any branch from the root go in increasing order. All of these research directions are significantly different from ours.
Section 2 contains some preliminaries (notation and basic formulas). In Sections 3 and 4 we will prove two results on random labelings of tori inspired by simulations. Section 5 will present results on random labelings of trees. One can prove more precise results for random labelings of regular trees—these will be given in Section 6. Section 7 offers some open problems.

2 | PRELIMINARIES

For an integer \( N > 0 \), let \([N] = \{1, 2, \ldots, N\}\). By an abuse of notation, we will use \(|\cdot|\) to denote the absolute value of a real number and cardinality of a finite set.

We will usually denote graphs by \( G \), their vertex sets by \( \mathcal{V} \) and edge sets by \( \mathcal{E} \). We will indicate adjacency of vertices \( x \) and \( y \) by \( x \leftrightarrow y \).

Suppose that \( 1 < |\mathcal{V}| = N < \infty \). We will call a function \( L : \mathcal{V} \to [N] \) a labeling if it is a bijection.
We will say that a vertex \( x \in V \) is a peak (of \( L \)) if and only if it is a local maximum of \( L \), that is, \( L(x) > L(y) \) for all \( y \leftrightarrow x \). The event that the number of peaks of a random labeling of graph \( G \) is equal to \( k \) will be denoted \( \mathcal{N}(G,k) \).

We will use \( \mathbb{P} \) to denote the distribution of a random (uniform, unconditioned) labeling. The symbol \( \mathbb{P}_1 \) (\( \mathbb{P}_2 \)) will stand for the distribution \( \mathbb{P} \) conditioned on existence of exactly one peak (exactly two peaks). If an argument involves multiple graphs, we may indicate the graph in the notation by writing, for example, \( \mathbb{P}_G \) or \( \mathbb{P}_1,G \).

Recall the Stirling formula for \( n \geq 1 \),

\[
\log n! = n \log n - n + O(\log n). \tag{2.1}
\]

This obviously implies that,

\[
\log n! \leq n \log n + O(\log n). \tag{2.2}
\]

A more accurate version of the Stirling formula says that, for \( n \geq 1 \),

\[
(n + 1/2) \log n - n + \log \sqrt{2\pi} \leq \log n! \leq (n + 1/2) \log n - n + 1. \tag{2.3}
\]

For every \( k = 0, 1, \ldots, n \), we have \( \binom{n}{k} \leq \sum_{j=0}^{n} \binom{n}{j} = 2^n \), so

\[
\log \binom{n}{k} \leq n \log 2. \tag{2.4}
\]

3 | ROUGHNESS OF LEVEL BOUNDARIES

Recall notation from Section 1.2. We will show that boundaries of clusters \( C_k \) (“level sets”) for random labelings of two-dimensional tori, conditioned on having a single peak, have dimension 2, in a suitable asymptotic sense.

Remark 3.1. Recall the Eden model, a sequence of randomly growing clusters \( \tilde{C}_k \), from Section 1.2. The model was introduced in [11]. The site version of the Eden model is identical to the site version of the “first passage percolation” model introduced in [12]. A shape theorem was proved in [8] for the edge version of the model; it says that when the size of the cluster goes to infinity, the rescaled cluster converges to a convex set \( B \). For a connection between the Eden model and the KPZ equation, see, for example, [13].

Theorem 3.1 of [2] (see also a stronger version in [10, Prop. 3.1]) implies that with probability 1, for all large \( n \), the boundary of the set \( \tilde{C}_n \) lies in the “annulus” of width \( c_1 n^{1/4} \log n \) containing \( n^{1/2} \partial B \) or, more precisely, a neighborhood of width \( c_1 n^{1/4} \log n \) of the boundary of \( n^{1/2} B \). This implies that the cardinality of the boundary of \( \tilde{C}_n \) is bounded by the volume of the annulus, that is, \( c_2 n^{3/4} \log n \).

A result recently proved in [9] for the edge version of the first passage percolation model suggests that for “most” \( n \), the boundary of \( \tilde{C}_n \) contains at most \( c_3 n^{1/2} \) vertices. We note parenthetically that the research presented in [9] was partly inspired by the discussions of cluster boundaries with the authors of the present paper.

Even the weaker upper bound, \( c_2 n^{3/4} \log n \), on the size of cluster boundaries in the Eden model and Theorem 3.2 show that the clusters \( C_k \) in the random labeling model have much different character from those in the Eden model. The comparison of Figures 1 and 2 seems to support this claim.
Let $G_{n,n}$ denote the Cartesian product of path (linear) graphs, each with $n$ vertices. In other words, $G_{n,n}$ is a discrete square with side length $n$. The set of vertices of $G_{n,n}$ is denoted $V_{n,n}$.

Recall clusters $C_k$ defined in Section 1.2. Let $\partial C_k$ denote the set of vertices outside $C_k$ that are adjacent to a vertex in $C_k$.

**Theorem 3.2.** Consider a random labeling of $G_{n,n}$ conditioned on having a single peak. Let $f(n) = (\log n)^{-1/2} \log \log n$. Let $F$ be the event that the number of $k \in [n^2]$ such that $|\partial C_k| \leq n^2 - f(n)$ is greater than or equal to $f(n)n^2$. For large $n$,

$$\mathbb{P}_1(F) \leq \exp(-(1/2)n^2(\log \log n)^2).$$

**Proof.** Step 1. In this step, we will prove that for any $1 \leq m_1, m_2 \leq n$, the number of labelings of $G_{n,n}$ which have only one peak at $(m_1, m_2)$ is greater than

$$\exp(2n^2 \log n + O(n^2 \log \log n)).$$

Let $\ell = \ell(n) = \lfloor \log n \rfloor$. Let $H_n$ be the graph whose vertex set is a subset of $V_{n,n}$ and contains all points with coordinates $(k_1, k_2)$ satisfying at least one of the following four conditions: (i) $k_1 = m_1$, (ii) $k_2 = m_2$, (iii) $k_1 = j\ell$ for some integer $j$, and (iv) $k_2 = j\ell$ for some integer $j$. Two vertices of $H_n$ are connected by an edge if and only if they are adjacent in $G_{n,n}$. The graph $H_n$ has the shape of a lattice with the edge length $\ell$ and a pair of extra lines passing through $(m_1, m_2)$.

Let $N$ be the number of vertices in $H_n$. We have $N \leq 3n^2 / \log n$ for large $n$. Let $V_n^H$ denote the set of vertices of $H_n$.

We will define a family of deterministic labelings $L$ of $G_{n,n}$. We will first label $V_n^H$. It is easy to see that one can label $V_n^H$ in such a way that $L(m_1, m_2) = n^2$, $(m_1, m_2)$ is the only peak of $L$, and $L(V_n^H) = \{n^2, n^2 - 1, \ldots, n^2 - N + 1\}$. Fix any such labeling of $V_n^H$ and consider labelings $L$ of $V_{n,n}$ which agree with this labeling on $V_n^H$.

Consider labelings with no restriction on peaks outside $V_n^H$. There are $n^2 - N$ numbers in $[n^2]$ not used for labeling $V_n^H$ so there are $(n^2 - N)!$ labelings of $G_{n,n}$ which extend a given labeling of $V_n^H$. Let this family be called $\mathcal{A}$.

Consider a maximal horizontal line segment $\Gamma \subset \bigcup V_{n,n} \setminus V_n^H$. The length of $\Gamma$ is at most $\ell - 1$. At least one of the permutations of $L(\Gamma)$ is monotone with the largest number adjacent to $V_n^H$. So there are at least $(n^2 - N)!/((\ell - 1)!)^2$! labelings in $\mathcal{A}$ with no peaks on $\Gamma$. Let $\Gamma_1$ be a maximal horizontal line segment in $\bigcup V_{n,n} \setminus V_n^H$, disjoint from $\Gamma$. We can apply the same argument using this time permutations of $L(\Gamma_1)$ to see that the number of labelings in $\mathcal{A}$ with no peaks on $\Gamma \cup \Gamma_1$ is greater than or equal to $(n^2 - N)!((\ell - 1)!)^2$! The set $V_{n,n} \setminus V_n^H$ can be partitioned into at most $n(n+1)/\ell \leq (n^2 / \log n)(1 + 1/n)(1 + 2/ \log n)$ disjoint maximal horizontal line segments of length at most $\ell$, for large $n$. So the number of labelings in $\mathcal{A}$ with no peaks is greater than or equal to $(n^2 - N)!((\ell - 1)!)^2(n^2 / \log n)(1 + 1/n)(1 + 2/ \log n)$. For large $n$, the logarithm of this number is greater than or equal to, using (2.3),

\[
(n^2 - N + 1/2) \log(n^2 - N) - (n^2 - N) + c_1 \\
= (n^2 / \log n)(1 + 1/n)(1 + 2/ \log n)(\ell - 1 + 1/2) \log(\ell - 1) \\
\geq (n^2 - 3n^2 / \log n + 1/2) \log(n^2 - 3n^2 / \log n) - n^2 \\
- (n^2 / \log n)(1 + 1/n)(1 + 2/ \log n) n \log n \\
\geq 2n^2(1 - 3/ \log n) n \log n + n^2(1 - 3/ \log n) \log(1 - 3/ \log n) - n^2
\]
-  n^2(1 + 1/n)(1 + 2/ \log n) \log \log n \\
\geq 2n^2 \log n + O(n^2 \log \log n).

This proves (3.1).

Step 2. Suppose that \(B_1 \subset B_2 \subset \ldots \subset B_n\) is a sequence of sets in \(V_{n,n}\), starting with a singleton \(B_1\) and such that \(|B_k| = |B_{k-1}| + 1\) for all \(k\). Let \(\partial B_k\) denote the set of vertices outside \(B_k\) that are adjacent to a vertex in \(B_k\). We also require that \(B_{k+1} \setminus B_k \subset \partial B_k\) for all \(k\). Recall that \(f(n) = (\log n)^{-1/2} \log \log n\).

Let \(M\) be the number of sequences \((B_k)_{k \in [n^2]}\) with the property that the number \(R\) of \(k \in [n^2]\) such that \(|\partial B_k| \leq n^{2-f(n)}\) is greater than or equal to \(f(n)n^2\).

The number of possible values of \(R\) is bounded by \(n^2\). Given \(R \geq f(n)n^2\), the number of possible sequences of \(k\)'s such that \(|\partial B_k| \leq n^{2-f(n)}\) is bounded by \(\binom{n^2}{R}\). Given a specific set \(\Lambda\) of \(k\)'s in \([n^2]\) with cardinality greater than or equal to \(f(n)n^2\), the number of sequences \((B_k)_{k \in [n^2]}\) with the property that \(|\partial B_k| \leq n^{2-f(n)}\) for all \(k \in \Lambda\) is less than or equal to

\[
(n^2-f(n))^{f(n)n^2} (n^2)^{(1-f(n))n^2+1},
\]

so

\[
M \leq n^2 \left( \frac{n^2}{R} \right)^{(n^2-f(n))^{f(n)n^2} (n^2)^{(1-f(n))n^2+1}}.
\]

By (2.4), for some \(c\) and all \(n\) and \(R\), \(\binom{n^2}{R} \leq cn^2\). Hence,

\[
\log M \leq 2 \log n + cn^2 + f(n)n^2 (2 - f(n)) \log n + 2((1 - f(n))n^2 + 1) \log n \\
= cn^2 - n^2 f^2(n) \log n + 2n^2 \log n + 4 \log n \\
= cn^2 - n^2 (\log n)^{-1} (\log \log n)^2 \log n + 2n^2 \log n + 4 \log n \\
= cn^2 - n^2 (\log \log n)^2 + 2n^2 \log n + 4 \log n.
\]

The probability of \(F\) is less than or equal to \(M\) divided by the number in (3.1), so for some \(c\) this probability is bounded above by

\[
\frac{\exp(cn^2 - n^2 (\log \log n)^2 + 2n^2 \log n + 4 \log n)}{\exp(2n^2 \log n - cn^2 \log \log n)} \\
= \exp(-n^2 (\log \log n)^2 + cn^2 + 4 \log n + cn^2 \log \log n) \\
\leq \exp(-(1/2)n^2 (\log \log n)^2),
\]

for all large \(n\).

4 | LADDER GRAPHS

The results of a simulation presented in Figure 2 suggest that gradient lines for the random labeling contain long straight stretches. We cannot prove this feature of random labelings of tori but we will prove a similar result for “ladder” graphs. Theorem 4.1 will show that the longest of all gradient lines of a random labeling with a single peak of a ladder graph mostly consists of straight stretches, with the exception of a logarithmically small percentage of steps.
A generalized ladder graph \( G_{m,n} \) is the Cartesian product of path (linear) graphs with \( m \) and \( n \) vertices. In other words, \( G_{m,n} \) is a discrete rectangle with side lengths \( m \) and \( n \). We will consider \( m \) a fixed parameter and we will state an asymptotic theorem when \( n \) goes to infinity.

We will identify the set of vertices \( \mathcal{V}_{m,n} \) of the graph \( G_{m,n} \) with points of \( \mathbb{Z}^2 \), so that we can refer to them using Cartesian coordinates \((j, k)\), \( 1 \leq j \leq n, 1 \leq k \leq m \). Let \( B \) be the family of all (deterministic) labelings of \( G_{m,n} \) with only one peak at \((1, 1)\). Let \( \Lambda \) be the set of all “continuous” non-self-intersecting paths taking values in \( \mathcal{V}_{m,n} \), that is, \( \lambda \in \Lambda \) if and only if there is some \( k \) such that \( \lambda : [k] \to \mathcal{V}_{m,n} \), \( \lambda(j) \leftrightarrow \lambda(j + 1) \) for \( j \in [k - 1] \), and \( \lambda(i) \neq \lambda(j) \) for \( i \neq j \). We will write \( |\lambda| = k \).

We will argue that for any \( L \in B \) there exists \( \lambda_L = (\lambda_L^1, \lambda_L^2) \in \Lambda \) such that \( \lambda_L(1) = (1, 1) \), \( \lambda_L^1(|\lambda_L^1|) = n \), and \( j \to L(\lambda_L(j)) \) is a decreasing function. Fix any ordering of \( \mathcal{V}_{m,n} \). We will construct a tree. Let \( v_1 = (1, 1) \) be the root of the tree. Suppose that vertices \( \{v_1, v_2, \ldots, v_j\} \) of the tree have been chosen. Consider the following cases. (a) The set \( \mathcal{V}_{m,n} \setminus \{v_1, v_2, \ldots, v_j\} \) contains a vertex \( w \) such that \( w \) is adjacent to a vertex in \( \{v_1, v_2, \ldots, v_j\} \), say, \( v_i \), and \( L(w) < L(v_i) \). Then we take the largest \( w \) with this property and add this \( w \) to \( \{v_1, v_2, \ldots, v_j\} \) as a new element \( v_{j+1} \). We also choose the largest \( v_i \in \{v_1, v_2, \ldots, v_j\} \) with the property that \( v_{j+1} \leftrightarrow v_i \) and add an edge to the tree between \( v_{j+1} \) and \( v_i \). (b) Suppose that the set \( \mathcal{V}_{m,n} \setminus \{v_1, v_2, \ldots, v_j\} \) does not contain a vertex \( w \) such that \( w \) is adjacent to a vertex in \( \{v_1, v_2, \ldots, v_j\} \) and has the property that \( L(w) < L(v_i) \) for some \( v_i \). If \( v_1, v_2, \ldots, v_j \) contains a vertex of the form \((n, k)\) then the unique path within the tree from \( v_1 = (1, 1) \) to \((n, k)\) satisfies all conditions that \( \lambda_L \) is supposed to satisfy. In the opposite case, \( L \) restricted to the set \( \mathcal{V}_{m,n} \setminus \{v_1, v_2, \ldots, v_j\} \), must attain the maximum, say, at \( x \). If \( x \) is adjacent to any vertex \( y \in \{v_1, v_2, \ldots, v_j\} \) then \( L(x) > L(y) \) for every such \( y \) because of the assumption made at the beginning of case (b). This implies that \( x \) is a peak of \( L \) in the whole graph \( G_{m,n} \). Since this contradicts the assumption that \( L \) has only one peak at \((1, 1)\), the proof of existence of \( \lambda_L \) is complete.

If there is more than one path \( \lambda_L \) satisfying the properties stated above then we let \( \lambda_L \) be the path that, in addition, is minimal in some fixed arbitrary ordering of \( \Lambda \).

**Theorem 4.1.** Let \( L \) be a random labeling of \( \mathcal{V}_{m,n} \) chosen uniformly from \( B \) and let \( P_B \) denote the corresponding probability. For every \( m \geq 3 \) there exists \( n_1 \) such that for \( n \geq n_1 \),

\[
P_B \left( |\lambda_L| \geq n \left( 1 + \frac{\log \log n}{\log n} \right) \right) \leq \exp(-(1/2)n \log \log n).
\]

**Proof.** First we will estimate the total number of deterministic labelings of \( \mathcal{V}_{m,n} \) with a single peak at \((1, 1)\).

Let \( A \) be the family of all deterministic labelings \( L \) of \( \mathcal{V}_{m,n} \) such that

\[
L(1, 1) = nm, L(2, 1) = nm - 1, \ldots, L(n, 1) = nm - n + 1.
\]

Note that labelings in the family \( A \) may have multiple peaks. There are \((n(m - 1))!\) labelings in \( A \).

Let \( \Gamma_k = \{(k, 2), (k, 3), \ldots, (k, m)\} \) for \( k \in [n] \). There exists a unique permutation of \( L(k, 2), L(k, 3), \ldots, L(k, m) \) which is decreasing so the number of labelings in \( A \) which do not have peaks on \( \Gamma_k \) is at least \((n(m - 1))!/((m - 1))!^n \). For the same reason, the number of labelings in \( A \) which do not have peaks on \( \Gamma_1 \cup \Gamma_2 \) is at least \((n(m - 1))!/((m - 1))!^n \). Extending the argument to all \( \Gamma_k \)'s, we conclude that the number of labelings in \( A \) which do not have a peak on any \( \Gamma_k \) is at least \((n(m - 1))!/((m - 1))!^n \). This implies that the number of labelings with a single peak at \((1, 1)\) is bounded below by \((n(m - 1))!/((m - 1))!^n \).
Let \( f(n) = \log \log n / \log n \). Consider any integer \( j \) such that \( n(1 + f(n)) \leq j \leq nm \). We will count the number of labeled paths \( \lambda \in \Lambda \) such that \( |\lambda| = j \) and \( \lambda = \lambda_L \) for some \( L \in B \). We must have \( L(1, 1) = nm \) and \( \lambda(1) = (1, 1) \). There are at most \( \binom{nm}{j} \) choices for the values of \( L(\lambda(k)) \), \( 2 \leq k \leq j \). Once these values are chosen, we arrange them along a “continuous” path by choosing inductively a vertex \( \lambda(k+1) \), given \( \lambda(1), \ldots, \lambda(k) \). Going from the top value monotonically to the lowest value, there are at most 3 choices at every step so the number of \( \lambda \) such that \( |\lambda| = j \) and \( \lambda = \lambda_L \) for some \( L \in B \) is bounded by \( \binom{nm}{j-1}3^{j-1} \).

If \( |\lambda| = j \) then there are \( nm - j \) vertices outside the range of \( \lambda \). Hence, the number of labelings \( L \in B \) such that \( \lambda_L \) is equal to a fixed \( \lambda \) with \( |\lambda| = j \) and the values of \( L(\lambda(k)) \) are fixed for \( 1 \leq k \leq j \), is bounded by \( (nm - j)! \). Let \( M(m, n, j) \) be the number of labelings \( L \in B \) such that \( |\lambda_L| = j \). We have

\[
M(m, n, j) \leq \binom{nm}{j-1}3^{j-1} (nm - j)! \leq \binom{nm}{j-1}3^{j-1} (nm - j + 1)! = \frac{(nm)!}{(j-1)!} 3^{j-1}.
\]

For \( n \geq 4 \), the expression on the right-hand side, considered to be a function of \( j \) on the interval \( n(1 + f(n)) \leq j \leq nm \), is maximized by \( j = \lceil n(1 + f(n)) \rceil \). Let \( n_1 = \lceil n(1 + f(n)) \rceil - 1 \). We obtain the following bound,

\[
M(m, n, j) \leq (nm)! 3^{n_1} / n_1!.
\]

Next we sum over \( j \) such that \( n(1 + f(n)) \leq j \leq nm \). The number of labelings \( L \in B \) such that \( |\lambda_L| \geq n(1 + f(n)) \) is bounded above by

\[
n(m - f(n))(nm)! 3^{n_1} / n_1! \leq nm(nm)! 3^{n_1} / n_1!.
\]  \( \text{(4.2)} \)

The probability that \( |\lambda_L| \geq n(1 + f(n)) \) for a labeling chosen uniformly from \( B \) is bounded above by the ratio of the right-hand side of (4.2) and (4.1). We use the Stirling formula (2.1) and (2.2) to see that for all sufficiently large \( n \),

\[
\Pr_B(\|\lambda_L\| \geq n(1 + f(n))) \leq \frac{nm(nm)! 3^{n_1} / n_1!}{(nm - 1)! (m - 1)!} = \frac{nm(nm)! 3^{n_1} (m - 1)! n_1!}{(nm - 1)! n_1!}
\]

\[
\leq \exp \left( \log n + \log m + nm \log(nm) + O(\log(nm)) + n(1 + f(n)) \log 3 + n(m - 1) \log(m - 1) + O(n \log(m - 1)) - n(m - 1) \log(nm - 1) + n(m - 1) + O(\log(nm - 1)) - n(1 + f(n)) \log(n(1 + f(n)) - 1) + n(1 + f(n)) + O(\log(n(1 + f(n)))) \right)
\]

\[
\leq \exp \left( nm \log(nm) - n(m - 1) \log(n - n(1 + f(n)) \log(n(1 + f(n)) - 1) + O(n) \right)
\]

\[
= \exp \left( nm \log n + nm \log m - n(m - 1) \log n - n \log n - nf(n) \log n - n \log(1 + f(n) - 1/n) - nf(n) \log(1 + f(n) - 1/n) + O(n) \right)
\]

\[
= \exp \left( - nf(n) \log n - n \log(1 + f(n) - 1/n) - nf(n) \log(1 + f(n) - 1/n) + O(n) \right)
\]

\[
= \exp \left( - nf(n) \log n + O(n) \right)
\]
\begin{align*}
= \exp \left( -n (\log \log n / \log n) \log n + O(n) \right) \\
\leq \exp(- (1/2) n \log \log n).
\end{align*}

\section{Random Labelings of Trees}

It often happens that trees are more tractable than general graphs. This is indeed the case when we consider random labelings conditioned on a small number of peaks. We will start this section with some preliminary results.

\textbf{Definition 5.1.}  
(i) Suppose that a graph $\mathcal{G}$ is a tree with $N$ vertices. We will say that $x$ is a centroid of $\mathcal{G}$ if each subtree of $\mathcal{G}$ which does not contain $x$ has at most $N/2$ vertices.

(ii) Suppose that $\mathcal{G}$ is a tree and $x$ is one of its vertices. Define a partial order “$\leq$” on the set of vertices $\mathcal{V}$ by declaring that for $y, z \in \mathcal{V}$, we have $y \leq z$ if and only if $y$ lies on the unique path joining $x$ and $z$. If $y \leq z$ then we will say that $y$ is an ancestor of $z$ and $z$ is a descendant of $y$. We will write $D_x^y$ to denote the family of all descendants of $y$, including $y$. We will also use the notation $n_x(y) = |D_x^y|$. If needed, we will indicate the dependence on the graph by adding a superscript, for example, $n_x^G(y)$.

\textbf{Remark 5.2.}  
The following results can be found in [6, Thm. 2.3]. (i) A finite tree graph has at least one and at most two centroids. (ii) If it has two centroids then they are adjacent.

Part (i) cannot be strengthened to say that all trees have only one centroid. A linear graph with $N$ vertices has one or two centroids depending on the parity of $N$.

\textbf{Proposition 5.3.}  
Suppose that $L$ is a random labeling of a tree $\mathcal{G}$ and let $K$ denote the location of the highest label. The function $x \mapsto \mathbb{P}(\mathcal{N}(\mathcal{G}, 1) \cap \{K = x\})$ attains the maximum at all centroids of $\mathcal{G}$ and only at the centroids.

\textbf{Proof.}  
Consider any $x \in \mathcal{V}$. Recall the notation $D_x^y$ and $n_x(y)$ from Definition 5.1 (ii). Let

\[ F_v = \left\{ L(v) \geq \max_{z \in D_v^y} L(z) \right\}. \]

Note that

\[ \mathcal{N}(\mathcal{G}, 1) \cap \{K = x\} = \bigcap_{v \in \mathcal{V}} F_v. \]

Let $\mathcal{V}_0 = \{x\}$, $\partial \mathcal{V}_k = \{v \in \mathcal{V} \setminus \mathcal{V}_k : \exists z \in \mathcal{V}_k \text{ such that } v \leftrightarrow z\}$ and $\mathcal{V}_k = \mathcal{V}_{k-1} \cup \partial \mathcal{V}_{k-1}$ for $k \geq 1$. Let $N = |\mathcal{V}|$ and note that $\mathbb{P}(F_x) = 1/N = 1/n_x(x)$. It is easy to see that

\[ \mathbb{P} \left( \bigcap_{v \in \mathcal{V}} F_v \bigg| F_x \right) = \frac{1}{\prod_{v \in \partial \mathcal{V}_k} n_x(y)}. \]

In general, for $k \geq 0$ such that $\partial \mathcal{V}_k \neq \emptyset$,

\[ \mathbb{P} \left( \bigcap_{v \in \partial \mathcal{V}_k} F_v \bigg| \bigcap_{v \in \mathcal{V}_k} F_v \right) = \frac{1}{\prod_{v \in \partial \mathcal{V}_k} n_x(y)}. \]
We have

\[ P(\mathcal{N}(G, 1) \cap \{K = x\}) = P(F_x) \prod_{k \geq 0, \partial V_k \neq \emptyset} P\left( \bigcap_{v \in \partial V_k} F_v \bigg\vert \bigcap_{v \in V_k} F_v \right) \]

\[ = \frac{1}{n_x(x)} \prod_{k \geq 0, \partial V_k \neq \emptyset} \frac{1}{\prod_{v \in \partial V_k} n_v(v)} = \frac{1}{\prod_{z \in V} n_z(z)}. \]  

(5.1)

We will now vary \( x \). Consider two adjacent vertices \( x \) and \( y \). Note that \( n_v(v) = n_x(v) \) for \( v \neq x, y \). We have

\[ n_y(x) = n_x(x) - n_x(y), \quad n_x(y) = N = n_x(x). \]

Therefore,

\[ \frac{P(\mathcal{N}(G, 1) \cap \{K = x\})}{P(\mathcal{N}(G, 1) \cap \{K = y\})} = \frac{n_y(x)n_x(y)}{n_x(x)n_y(y)} = \frac{n_x(x) - n_x(y)}{n_x(y)} = \frac{n_x(x) - n_x(y)}{n_x(y)}. \]  

(5.2)

It follows that \( P(\mathcal{N}(G, 1) \cap \{K = x\}) \geq P(\mathcal{N}(G, 1) \cap \{K = y\}) \) iff \( n_x(x) - n_x(y) \geq n_x(y) \) iff \( N = n_x(x) \geq 2n_x(y) \) iff \( n_x(y) \leq N/2 \). This can be easily rephrased as the statement of the proposition.

Consider a tree \( G \) with the vertex set \( V \). Let \( V \) be the family of all (unordered) pairs of subsets \( \{V', V''\} \) of the vertex set \( V \) such that \( V' \) and \( V'' \) are vertex sets of nonempty subtrees of \( G \), \( V' \cap V'' = \emptyset \) and \( V' \cup V'' = V \). In other words, \( V \) is the family of all partitions of \( G \) into two subtrees that can be generated by removing an edge.

**Lemma 5.4.** Consider a (deterministic) labeling \( L \) of a tree \( G \) that has exactly two peaks at \( y_1 \) and \( y_2 \).

(i) There exist exactly two pairs \( \{V', V''\} \in V \) such that \( y_1 \) is the only peak of \( L \) restricted to \( V' \) and \( y_2 \) is the only peak of \( L \) restricted to \( V'' \).

(ii) If the distance between \( y_1 \) and \( y_2 \) is greater than \( 2 \) then there exist at least three pairs \( \{V', V''\} \in V \) such that \( y_1 \in V' \) and \( y_2 \in V'' \). Suppose that \( \{V', V''\} \in V \) is one of such pairs and it does not satisfy the condition stated in (i). Then either (a) \( L \) restricted to \( V' \) has exactly two peaks, one of them located at \( y_1 \) and the other one adjacent to \( V'' \), and \( L \) restricted to \( V'' \) has exactly one peak at \( y_2 \), or (b) \( L \) restricted to \( V'' \) has exactly two peaks, one of them located at \( y_2 \) and the other one adjacent to \( V' \), and \( L \) restricted to \( V' \) has exactly one peak at \( y_1 \).

**Proof.** (i) Since \( y_1 \) and \( y_2 \) are peaks, the distance between them must be equal to or greater than \( 2 \). For \( x, z \in V \), let \( [x, z] \) denote the geodesic between \( x \) and \( z \), including both vertices. If there are exactly two peaks \( y_1 \) and \( y_2 \) then there exists a unique \( y_3 \in [y_1, y_2] \) such that \( y_1 \neq y_3 \neq y_2 \) and the labeling is monotone on both \([y_1, y_3]\) and \([y_2, y_3]\). Let \( y_4 \) be the neighbor of \( y_3 \) in \([y_1, y_3]\) and let \( y_5 \) be the neighbor of \( y_3 \) in \([y_2, y_3]\). Note that \( y_4 \) could be \( y_1 \) or \( y_5 \) could be \( y_2 \). Let \( \{V', V''\} \) be subtrees obtained by removing the edge between \( y_4 \) and \( y_3 \) from \( G \). Let \( \{\tilde{V}', \tilde{V}''\} \) be subtrees obtained by removing the edge between \( y_5 \) and \( y_3 \) from \( G \). It is easy to see that \( \{V', V''\} \) and \( \{\tilde{V}', \tilde{V}''\} \) are the only elements of \( V \) such that \( y_1 \) is the only peak in one of the subtrees in the pair and \( y_2 \) is the only peak in the other subtree in the same pair.

(ii) If the distance between \( y_1 \) and \( y_2 \) is greater than \( 2 \) then there are at least three edges along the geodesic \([y_1, y_2]\). Removing an edge not adjacent to \( y_3 \) will generate a pair \( \{V', V''\} \in V \) with the properties specified in part (ii).
The next result is related to Problem 1.1. It follows easily from the arguments used in the proof of [3, Thm. 4.9] that if a random labeling of a large linear graph is conditioned to have exactly two peaks then the peaks are likely to be at a distance about one half of the length of the graph. Going in the opposite direction, we will show that the ratio of the distance between the twin peaks and the tree diameter can be arbitrarily close to zero.

In the following proposition, if the random labeling has exactly two peaks, we will denote the location of the highest peak $K_1$ and the location of the other one $K_2$.

**Proposition 5.5.** Fix arbitrarily small $p > 0$. There exists $m_0$ such that for every $m \geq m_0$ there exists a tree with diameter greater than $m$ such that the function $f(x, y) = P_2(K_1 = x, K_2 = y)$ attains the unique maximum for some $x$ and $y$ at the distance 3. Moreover, $P_2(\text{dist}(K_1, K_2) \geq 8) < p$.

**Proof.** Suppose that $m > 100$ and let $n = (m!)^2$. Let the vertex set $\mathcal{V}$ of the graph $G$ consist of points $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, z_1, z_2, \ldots, z_m$. The only pairs of vertices connected by edges are $(z_k, z_{k+1})$ for $k \in [m - 1], (z_1, x_k)$ for $k \in [n]$ and $(z_4, y_k)$ for $k \in [n]$. We will find a lower estimate for

$$P_G(\mathcal{N}(G, 2) \cap ((K_1 = z_1, K_2 = z_4) \cup (K_2 = z_1, K_1 = z_4))).$$

Let $G'$ be the subgraph of $G$ consisting of vertices $z_1, z_2, x_1, x_2, \ldots, x_n$. Let $G''$ be the subgraph of $G$ consisting of all the remaining vertices. Let $K_1'$ be the location of the highest peak of a random labeling of $G'$ and let $K_1''$ be the location of the highest peak of a random labeling of $G''$. We have

$$P_G(\mathcal{N}(G, 2) \cap ((K_1 = z_1, K_2 = z_4) \cup (K_2 = z_1, K_1 = z_4))) = P_G(\mathcal{N}(G', 1) \cap \{(K_1' = z_1) \cap \mathcal{N}(G'', 1) \cap (K_1'' = z_4)) = P_G(\mathcal{N}(G', 1) \cap (K_1' = z_1))P_G(\mathcal{N}(G'', 1) \cap (K_1'' = z_4)).$$

We have equality on the second line above because if $z_1$ and $z_4$ are peaks of a labeling of $G$ then the same labeling restricted to $G'$ cannot have a peak at $z_2$ and the labeling restricted to $G''$ cannot have a peak at $z_3$.

By (5.1) applied to $G'$,

$$P_G(\mathcal{N}(G', 1) \cap (K_1' = z_1)) = \frac{1}{n + 2}.$$  

The same formula applied to $G''$ yields

$$P_G(\mathcal{N}(G'', 1) \cap (K_1'' = z_4)) = \frac{1}{(n + m - 2)(m - 4)(m - 5) \ldots 2 \cdot 1 = \frac{1}{(n + m - 2)(m - 4)!}}.$$ 

It follows that

$$P_G(\mathcal{N}(G, 2) \cap ((K_1 = z_1, K_2 = z_4) \cup (K_2 = z_1, K_1 = z_4))) \geq \frac{1}{(n + 2)(n + m - 2)(m - 4)!}. \quad (5.3)$$

Next we will find an upper bound for $P_G(\mathcal{N}(G, 2) \cap (K_1 = v_1, K_2 = v_2))$, assuming that $(v_1, v_2) \neq (z_1, z_4)$ and $(v_1, v_2) \neq (z_4, z_1).$ Let $\mathcal{G}$ be the family of all pairs $(G', G'')$ of disjoint subtrees of $G$, with
vertex sets $\mathcal{V}'$ and $\mathcal{V}''$, such that $\mathcal{V}' \cup \mathcal{V}'' = \mathcal{V}, v_1 \in \mathcal{V}'$ and $v_2 \in \mathcal{V}'$. Let $K'_1$ be the location of the highest peak of a random labeling of $G'$; similar notation will be used for $G''$. We will use Lemma 5.4.

Consider a pair $(G', G'') \in \mathcal{G}$. Note that if $z_1$ is in $G'$ then at least $n - 1$ of vertices $x_1, x_2, \ldots, x_n$ are also in $G'$. If $z_4$ is in $G'$ then at least $n - 1$ of vertices $y_1, y_2, \ldots, y_n$ are also in $G'$. Analogous claims hold for $z_1$ and $G''$, and for $z_4$ and $G''$.

Suppose that $z_1$ and $z_4$ are in two different graphs $G'$ and $G''$. Suppose without loss of generality that $z_1$ is in $G'$ and $z_4$ is in $G''$. Recall that $(v_1, v_2) \neq (z_1, z_4)$. Consider the case when $v_1 \neq z_1$. Note that $n^{G'}_1(v_1) \geq n$ and $n^{G''}_1(z_1) \geq n - 1$ so, by (5.1),

$$
P_G(\mathcal{N}(G', 1) \cap \{K'_1 = v_1\}) \leq \frac{1}{n(n - 1)}.$$

Since $n^{G''}_2(v_2) \geq n$ so, by (5.1),

$$
P_{G''}(\mathcal{N}(G'', 1) \cap \{K''_1 = v_2\}) = \frac{1}{n}.$$

A completely analogous argument shows that, if $v_1 = z_1$ and $v_2 \neq z_4$ then

$$
P_G(\mathcal{N}(G', 1) \cap \{K'_1 = v_1\}) \leq \frac{1}{n},$$

$$
P_{G''}(\mathcal{N}(G'', 1) \cap \{K''_1 = v_2\}) \leq \frac{1}{n(n - 1)}.$$

We conclude that if $z_1 \in \mathcal{V}'$, $z_4 \in \mathcal{V}''$ and $(v_1, v_2) \neq (z_1, z_4)$ then,

$$
P_G(\mathcal{N}(G', 1) \cap \mathcal{N}(G'', 1) \cap \{K'_1 = v_1, K''_1 = v_2\}) \leq \frac{1}{n^2(n - 1)}.$$  \hfill (5.4)

Next suppose that $z_1$ and $z_4$ are in the same of two graphs $G'$ and $G''$. Without loss of generality, suppose that they are in $G'$. It is easy to see that $n^{G'}_1(v_1) \geq n$ and $n^{G'}_2(x) \geq n$ for at least two other $x$ in the set $\{z_1, z_2, z_3, z_4\}$ so, by (5.1),

$$
P_G(\mathcal{N}(G', 1) \cap \mathcal{N}(G'', 1) \cap \{K'_1 = v_1, K''_1 = v_2\}) \leq P_G(\mathcal{N}(G', 1) \cap \{K'_1 = v_1\}) \leq \frac{1}{n^3}.$$

This and (5.5) imply that for any fixed $(G', G'') \in \mathcal{G}$,

$$
P_G(\mathcal{N}(G', 1) \cap \mathcal{N}(G'', 1) \cap \{K'_1 = v_1, K''_1 = v_2\}) \leq \frac{1}{n^2(n - 1)}.$$  \hfill (5.5)

It is easy to see that for any $v_1$ and $v_2$ which are not adjacent, the family $\mathcal{G}$ has at most $m$ elements. Lemma 5.4 (i) implies that

$$
P_G(\mathcal{N}(G, 2) \cap \{K_1 = v_1, K_2 = v_2\}) \leq \sum_{(G', G'') \in \mathcal{G}} P_G(\mathcal{N}(G', 1) \cap \mathcal{N}(G'', 1) \cap \{K'_1 = v_1, K''_1 = v_2\}) \leq m \frac{1}{n^2(n - 1)} = \frac{m}{n(n - 1)(m!)^2}.$$
Comparing this bound to (5.3), we conclude that \( P_2(K_1 = x, K_2 = y) \) is maximized at \((x, y) = (z_1, z_4)\) or \((x, y) = (z_4, z_1)\), for all large \( m \).

We will now strengthen our estimates assuming that \( \text{dist}(v_1, v_2) \geq 8 \).

First suppose that \( z_1 \) and \( z_4 \) are in the same of two graphs \( G' \) and \( G'' \). Without loss of generality, suppose that they are in \( G' \). Then \( n_{v_1}(x) \geq n - 1 \) for all \( x \) in the set \( \{z_1, z_2, z_3, z_4\} \) (this is true whether \( v_1 \) belongs to this set or not) so, by (5.1),

\[
P_{G}(\mathcal{N}(G', 1) \cap \mathcal{N}(G'', 1) \cap \{K_1' = v_1, K_1'' = v_2\}) \leq P_{G}(\mathcal{N}(G', 1) \cap \{K_1' = v_1\}) \leq \frac{1}{(n - 1)^4}. \tag{5.6}
\]

Since \( z_1 \) is in \( G' \), the condition \( \text{dist}(v_1, v_2) \geq 8 \) implies that \( v_2 = z_k \) for some \( k \geq 7 \). It follows that there are at most \( 6nm^2 \) families \( (v_1, G', v_2, G'') \) such that \( \text{dist}(v_1, v_2) \geq 8 \) and \( z_1 \) and \( z_4 \) are in the same of the two graphs \( G' \) and \( G'' \).

Suppose that \( z_1 \) and \( z_4 \) are in two different graphs \( G' \) and \( G'' \). Consider the case when \( z_1 \) is in \( G' \). Recall that \( \text{dist}(v_1, v_2) \geq 8 \) implies that \( v_2 = z_k \) for some \( k \geq 7 \). We have \( n_{v_1}(v_2) \geq n \) and \( n_{v_2}(z_k) \geq n \) for \( k = 4, 5, 6 \), so, by (5.1),

\[
P_{G}(\mathcal{N}(G', 1) \cap \mathcal{N}(G'', 1) \cap \{K_1' = v_1, K_1'' = v_2\}) \leq P_{G'}(\mathcal{N}(G'', 1) \cap \{K_1'' = v_2\}) \leq \frac{1}{n^4}. \tag{5.7}
\]

There are at most \( 18nm \) families \( (v_1, G', v_2, G'') \) such that \( \text{dist}(v_1, v_2) \geq 8 \) and \( z_1 \) and \( z_4 \) are in two different graphs \( G' \) and \( G'' \).

We now appeal to Lemma 5.4 (i) and use (5.6) and (5.7) to conclude that, for large \( m \),

\[
\sum_{\text{dist}(v_1, v_2) \geq 8} P_{G}(\mathcal{N}(G, 2) \cap \{K_1 = v_1, K_2 = v_2\}) \\
\leq 6nm^2 \frac{1}{(n - 1)^4} + 18nm \frac{1}{n^4} \leq \frac{12m^2}{(n - 1)^3}.
\]

Therefore, in view of (5.3),

\[
P_2(\text{dist}(K_1, K_2) \geq 8) \leq \frac{\sum_{\text{dist}(v_1, v_2) \geq 8} P_{G}(\mathcal{N}(G, 2) \cap \{K_1 = v_1, K_2 = v_2\})}{P_{G}(\mathcal{N}(G, 2) \cap ((\{K_1 = z_1, K_2 = z_4\} \cup \{K_2 = z_1, K_1 = z_4\}))} \\
\leq \frac{12m^2}{(n - 1)^3(n + 1)(n + m - 2)(m - 4)!}.
\]

Recall that \( n = (m!)^2 \). The above bound goes to 0 when \( n \to \infty \) so for any \( p > 0 \), we have \( P_2(\text{dist}(K_1, K_2) \geq 8) < p \), if \( m \) is sufficiently large.

\section{TWIN PEAKS ON REGULAR TREES}

This section investigates twin peaks on \((d + 1)\)-regular trees of constant depth.

\textbf{Definition 6.1.} A finite rooted tree will be called a \((d + 1)\)-regular tree of depth \( k \) if the root, denoted \( v_s \), has \( d + 1 \) children, every child has further \( d \) children and so on, continuing till \( k \) generations. That is, the distance of each leaf to the root is exactly \( k \), and the degree of every non-leaf vertex is \( d + 1 \).
Our main result in this section is not as complete as we wish it had been. Theorem 6.3 (i) requires the assumption that \( d \geq 10 \); we believe that the assumption could be weakened.

We will briefly outline heuristic reasons why finding the location of the two peaks on a \((d + 1)\)-regular tree seems to be particularly challenging. If there are only two peaks on a tree graph, the graph can be divided into two subtrees such that the labeling restricted to each of the subtrees has only one peak (see Lemma 5.4). If a \((d + 1)\)-regular tree, for \( d \geq 2 \), of constant depth is divided into two subtrees then it is easy to see that the centroid of the smaller subtree is adjacent to a vertex in the other subtree. Hence, in view of Proposition 5.3, one would expect the top peak in the smaller of the two subtrees to be close to the other subtree. This seems to contradict the tendency for two peaks to be far away on “well-structured” graphs (see the remarks preceding Proposition 5.5).

First we will prove a general estimate similar to the “total probability formula.”

**Lemma 6.2.** Suppose that \( A_1, A_2, \ldots, A_n \) satisfy the following condition,
\[
A_i \cap A_j \cap A_m = \emptyset \quad \text{if } i \neq j \neq m 
\]
These events need not be pairwise disjoint but no triplet has a nonempty intersection. Suppose that for some \( c_1 > 0 \), some events \( B \) and \( C \), and all \( i \),
\[
P(B \mid A_i) \leq c_1 P(C \mid A_i).
\]
Assume that \( B \subset \bigcup_{i=1}^{n} A_i \). Then
\[
P(B) \leq 2c_1 P(C). \tag{6.1}
\]

**Proof.** The proof is contained in the following calculation,
\[
P(B) = P\left( B \cap \bigcup_{i=1}^{n} A_i \right) = P\left( \bigcup_{i=1}^{n} (B \cap A_i) \right) \leq \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(B \mid A_i) P(A_i)
\]
\[
\leq \sum_{i=1}^{n} c_1 P(C \mid A_i) P(A_i) = \sum_{i=1}^{n} c_1 P(C \cap A_i) = c_1 \sum_{i=1}^{n} \mathbf{1}_{C \cap A_i}
\]
\[
= c_1 \mathbf{E}\left( \sum_{i=1}^{n} \mathbf{1}_{C \cap A_i} \right) \leq c_1 \mathbf{E}(2 \cdot \mathbf{1}_C) = 2c_1 P(C).
\]

**Theorem 6.3.** Consider a random labeling of a \((d + 1)\)-regular tree \( G \) of depth \( k \). Let the locations of the highest and second highest peaks of the labeling be denoted \( K_1 \) and \( K_2 \), respectively.

(a) If \( d \geq 10 \) and \( k \geq 2 \) then the function \((y_1, y_2) \to P_2(K_1 = y_1, K_2 = y_2)\) takes the maximum value only if \( y_1 = v_\ast \).

(b) If \( d \geq 3 \), \( k \geq 2 \) and \( m \geq \sqrt{8k} \) then
\[
P_2\left( \min_{i=1,2} \text{dist}(K_i, v_\ast) \geq m \right) \leq 8 \exp(-(m/2) \log(d - 1)).
\]

(c) For every \( d \geq 3 \) there exists \( c_1 > 0 \) such that for all \( k \geq 2 \) and \( y_1, y_2 \in V \) with \( \text{dist}(y_i, v_\ast) \geq 2 \) for \( i = 1, 2 \), we have
\[
P_2(K_1 = v_\ast, K_2 = y_1) \geq c_1 P_2(K_1 = v_\ast, K_2 = y_2).
\]
(d) For every \( d \geq 3 \) there exist \( c \) and \( c' \) such that for \( k \geq 2 \) and \( n \geq 2\sqrt{8k} \),

\[
\mathbb{P}_2 (|\text{dist}(K_1, K_2) - k| \geq n) \\
\leq 8 \exp(-(n/4) \log(d - 1)) + c \exp(-(n/2) \log d) \\
\leq c' \exp(-(\sqrt{k}/64) \log d).
\]

**Proof.** (a) The proof of part (a) will be subdivided into several steps. We will now provide a summary of the steps. Step 1 contains a proof that if a random labeling of a subtree of a \((d + 1)\)-regular tree has only one peak then the peak is likely to be close to the root (even if the root is not in the subtree). A quantitative version of this claim, that is, an estimate of the relevant probabilities, is supplied.

Step 2 contains a formal presentation of the idea that the event that there are two peaks in the whole tree is contained in the union, over a family of partitions of the tree into two subtrees, of events that there is exactly one peak in each of the two subtrees.

Step 3 shows that if two peaks have prescribed locations in two subtrees then the higher peak is likely to be in the larger subtree.

Step 4 shows that if there are two peaks than the largest peak is more likely to be at the root than at some other vertex (roughly speaking).

Step 5 deals with the case, set aside in Steps 3 and 4, when two prescribed locations of peaks are at the distance 2. Claims and estimates of Steps 3 and 4 are extended to this case.

Step 6 is devoted to the analysis of the event when one of the peaks is at the root and the other is at a distance greater than or equal to 2 from the root.

Step 7 analyzes the case when the locations of the peaks are adjacent to the root.

**Step 1.** If we remove an edge (but retain all vertices) of \( G \), we obtain two subtrees. Consider a subtree \( G' \) constructed in this way and let \( v_1 \) be the vertex in \( G' \) closest to the root \( v_s \) in \( G \). If \( v_s \) is in \( G' \) then \( v_1 = v_s \). Consider a random labeling of \( G' \) conditioned on having exactly one peak. Let \( K \) denote the position of the peak.

We create a branching structure in \( G' \) by declaring that \( v_1 \) is the ancestor of all vertices in \( G' \). Suppose that a vertex \( x_1 \in G' \) is a parent of \( x_2 \in G' \). Recall the notation \( n^{G'}_j(y) \) from Definition 5.1. It is easy to see that \( n^{G'}_1(x_1) \geq (d - 1)n^{G'}_1(x_2) \). It follows from (5.2) that

\[
\mathbb{P}_1(K = x_1) \geq (d - 1) \mathbb{P}_1(K = x_2). \tag{6.2}
\]

By induction, for any \( x \in G' \) such that \( x \neq v_1 \),

\[
\mathbb{P}_1(K = v_1) \geq (d - 1) \mathbb{P}_1(K = x). \tag{6.3}
\]

We will also need sharper versions of the above estimates. Suppose that \( v_1 = v_s \), that is, \( v_s \in G' \).

Suppose that a vertex \( x_1 \in G' \) is a parent of \( x_2 \in G' \) and the distance from \( x_1 \) to \( v_s \) is \( j \). Then \( n^{G'}_{s_2}(x_1) \geq (d - 1)^{j+1} n^{G'}_{s_1}(x_2) \). It follows from (5.2) that

\[
\mathbb{P}_1(K = x_1) \geq (d - 1)^{j+1} \mathbb{P}_1(K = x_2). \tag{6.4}
\]

By induction, for any \( x \in G' \) such that \( \text{dist}(x, v_s) = m \geq 1 \),

\[
\mathbb{P}_1(K = v_s) \geq \mathbb{P}_1(K = x) \prod_{j=0}^{m-1} (d - 1)^{j+1} = (d - 1)^{m(m+1)/2} \mathbb{P}_1(K = x). \tag{6.5}
\]
Step 2. Consider the following conditions on vertices $y_1, y_2 \in V$ such that none of these vertices is the root.

(A1) The distance between $y_1$ and $y_2$ is larger than 2.

(A2) The distance between $y_1$ and $y_2$ is equal to 2 and either $y_1$ lies between $v_*$ and $y_2$, or $y_2$ lies between $v_*$ and $y_1$.

(A3) The distance between $y_1$ and $y_2$ is equal to 2 and none of the vertices $y_1, y_2$ and $v_*$ lies between the other two.

We will focus on (A1) and (A2) for a couple of steps of this proof. We will return to (A3) later.

Let $V$ be the family of all (unordered) pairs of subsets $\{V', V''\}$ of the vertex set $V$ such that $V'$ and $V''$ are the vertex sets of nonempty subtrees, $V' \cap V'' = \emptyset$ and $V' \cup V'' = V$. In other words, $V$ is the family of all partitions of $G$ into two subtrees that can be generated by removing an edge. Graphs corresponding to $V'$ and $V''$ will be denoted $G'$ and $G''$.

In case (A1), let $V(y_1, y_2) \subset V$ be the family of all $\{V', V''\}$ such that $y_1 \in V', y_2 \in V'', y_1$ is not adjacent to a vertex in $V'$, and $y_2$ is not adjacent to a vertex in $V'$.

In case (A2), we define the family $V(y_1, y_2)$ as a set containing only one pair $\{V', V''\}$ constructed by removing one edge from $G$, as follows. Let $v_3$ be the vertex between $y_1$ and $y_2$. If $y_1$ lies between $v_*$ and $v_3$ then we remove the edge joining $y_1$ and $v_3$. If $y_2$ lies between $v_*$ and $v_3$ then we remove the edge joining $y_2$ and $v_3$. We label $V'$ and $V''$ so that $y_1 \in V'$ and $y_2 \in V''$.

For $\{V', V''\} \in V$ and a random labeling, let $A(V', V'') = \mathcal{N}(G', 1) \cap \mathcal{N}(G'', 1)$, that is, $A(V', V'')$ is the event that the labeling restricted to $V'$ has exactly one peak and the same holds for $V''$.

It follows from Lemma 5.4 that in cases (A1) and (A2),

$$\mathcal{N}(G, 2) \cap \{K_1 = y_1, K_2 = y_2\} \subseteq \bigcup_{\{V', V''\} \in V(y_1, y_2)} A(V', V''). \quad (6.6)$$

Step 3. Let

$$B_1 = \mathcal{N}(G, 2) \cap \{K_1 = y_1, K_2 = y_2\}, \quad B_2 = \mathcal{N}(G, 2) \cap \{K_1 = y_2, K_2 = y_1\}.$$

Assume that (A1) or (A2) holds. Suppose that $\{V', V''\} \in V(y_1, y_2)$. Recall that $y_1 \in V'$. We will show that if $v_* \in V'$ then

$$\mathbb{P}(B_1 \cap A(V', V'')) \geq \mathbb{P}(B_2 \cap A(V', V'')). \quad (6.7)$$

Let $F$ be the event that the highest label is given to a vertex in $V'$. Let $K'_1$ be the location of the highest peak in $G'$ and let $K''_1$ be the location of the highest peak in $G''$.

Since $v_* \in V'$, we have $|V'| > |V''|$, and thus, $p_1 := \mathbb{P}(F) > 1/2$. Note that

$$\mathbb{P}(\mathcal{N}(G', 1) \cap \mathcal{N}(G'', 1) \cap \{K'_1 = y_1, K''_1 = y_2\} | F)$$

$$= \mathbb{P}(\mathcal{N}(G', 1) \cap \mathcal{N}(G'', 1) \cap \{K'_1 = y_1, K''_1 = y_2\} | F^c),$$

and call the common value of the two conditional probabilities $p_2$.

Since $y_2$ is not adjacent to a vertex in $V'$, the event $\{B_1 \cap A(V', V'')\}$ is the same as the intersection of the events (i) the highest label is given to a vertex in $V'$, (ii) there is a single peak at $y_1$ in $V'$ and (iii) there is a single peak at $y_2$ in $V''$. Hence $\mathbb{P}(B_1 \cap A(V', V'')) = p_1p_2$. 


If the event \( \{ B_2 \cap A(\mathcal{V}', \mathcal{V}'') \} \) holds then the intersection of the following events also holds: (i) the highest label is given to a vertex in \( \mathcal{V}'' \), (ii) there is a single peak at \( y_1 \) in \( \mathcal{V}' \) and (iii) there is a single peak at \( y_2 \) in \( \mathcal{V}'' \) (but \( B_2 \cap A(\mathcal{V}', \mathcal{V}'') \) is not equal to the intersection of (i)–(iii) because \( y_1 \) is adjacent to \( \mathcal{V}'' \) in case (A2)). This implies that \( \mathbb{P}(B_2 \cap A(\mathcal{V}', \mathcal{V}'')) \leq (1 - p_1)p_2. \) Since \( p_1 > 1/2, \)

\[
\mathbb{P}(B_1 \cap A(\mathcal{V}', \mathcal{V}'')) = p_1p_2 > (1 - p_1)p_2 \geq \mathbb{P}(B_2 \cap A(\mathcal{V}', \mathcal{V}'')),
\]

so (6.7) is proved.

**Step 4.** Consider \( y_1, y_2 \in \mathcal{V} \) such that either (A1) or (A2) is satisfied and none of these vertices is the root. Let

\[
\begin{align*}
B_1 &= \mathcal{N}(\mathcal{G}, 2) \cap \{ K_1 = y_1, K_2 = y_2 \}, & C_1 &= \mathcal{N}(\mathcal{G}, 2) \cap \{ K_1 = \nu_*, K_2 = y_2 \}, \\
B_2 &= \mathcal{N}(\mathcal{G}, 2) \cap \{ K_1 = y_2, K_2 = y_1 \}, & C_2 &= \mathcal{N}(\mathcal{G}, 2) \cap \{ K_1 = \nu_*, K_2 = y_1 \}, \\
B &= B_1 \cup B_2, & C &= C_1 \cup C_2.
\end{align*}
\]

Consider the event \( A(\mathcal{V}', \mathcal{V}'') \) for some \( \{ \mathcal{V}', \mathcal{V}''' \} \in \mathcal{V}(y_1, y_2) \) and suppose that \( \nu_* \in \mathcal{V}' \). If we condition on \( A(\mathcal{V}', \mathcal{V}'') \), the distribution of the order statistics of labels in \( \mathcal{V}' \) is independent of the values of the labels in \( \mathcal{V}''' \). Hence, (6.3) shows that conditional on \( A(\mathcal{V}', \mathcal{V}'') \), the probability that the only peak in \( \mathcal{V}' \) is at \( \nu_* \) is at least \( d - 1 \) times larger than the probability that it is at \( y_1 \). Since \( y_2 \) is not adjacent to a vertex in \( \mathcal{V}' \), if the only peak in \( \mathcal{V}''' \) is at \( y_2 \), the only peak in \( \mathcal{V}' \) is at \( \nu_* \) and the highest label is in \( \mathcal{V}' \) then \( C_1 \) holds. This and (6.7) imply that,

\[
\mathbb{P}(B | A(\mathcal{V}', \mathcal{V}'')) \leq 2\mathbb{P}(B_1 | A(\mathcal{V}', \mathcal{V}'')) \leq \frac{2}{d - 1} \mathbb{P}(C_1 | A(\mathcal{V}', \mathcal{V}'')). \tag{6.10}
\]

By symmetry, if \( \nu_* \in \mathcal{V}''' \) then

\[
\mathbb{P}(B | A(\mathcal{V}', \mathcal{V}'')) \leq 2\mathbb{P}(B_2 | A(\mathcal{V}', \mathcal{V}'')) \leq \frac{2}{d - 1} \mathbb{P}(C_2 | A(\mathcal{V}', \mathcal{V}'')). \tag{6.11}
\]

It follows from (6.10)–(6.11) that for all \( \{ \mathcal{V}', \mathcal{V}''' \} \in \mathcal{V}(y_1, y_2) \),

\[
\mathbb{P}(B | A(\mathcal{V}', \mathcal{V}'')) \leq \frac{2}{d - 1} \mathbb{P}(C | A(\mathcal{V}', \mathcal{V}'')). \tag{6.12}
\]

We will apply Lemma 6.2 with the family \( \{ A(\mathcal{V}', \mathcal{V}'') \}_{(\mathcal{V}', \mathcal{V}'') \in \mathcal{V}(y_1, y_2)} \) (for fixed \( y_1 \) and \( y_2 \)) playing the role of the family \( \{ A_k \} \). In view of (6.6), we see that (6.1) and (6.12) imply that

\[
\mathbb{P}(B) \leq \frac{4}{d - 1} \mathbb{P}(C).
\]

Recall events \( C_1 \) and \( C_2 \) from the definition (6.9) of \( C \). We have assumed that \( d \geq 10 \). The last estimate implies that for at least one \( i \) we must have

\[
\mathbb{P}(C_i) \geq \frac{d - 1}{8} \mathbb{P}(B) \geq \frac{10 - 1}{8} \mathbb{P}(B) \geq \mathbb{P}(\mathcal{N}(\mathcal{G}, 2) \cap \{ K_1 = y_1, K_2 = y_2 \}). \tag{6.13}
\]

In preparation for proofs of other parts of the theorem, we present a stronger version of the last estimate under stronger assumptions. Suppose that for some \( m \geq 2 \) we have \( \min_{i=1,2} \text{dist}(y_i, \nu_*) \geq m \).
Assume that \( \{ \mathcal{V}', \mathcal{V}'' \} \subseteq \mathcal{V}(y_1, y_2) \) and \( v_* \in \mathcal{V}' \). If we condition on \( A(\mathcal{V}', \mathcal{V}'') \), the distribution of the order statistics of labels in \( \mathcal{V}' \) (i.e., the joint distribution of the location of the highest label in \( \mathcal{V}' \), the location of the second highest label in \( \mathcal{V}' \), etc.) is independent of the values of the labels in \( \mathcal{V}' \). Hence, (6.5) shows that conditional on \( A(\mathcal{V}', \mathcal{V}'') \), the probability that the only peak in \( \mathcal{V}' \) is at \( v_* \) is at least \((d - 1)^{m(m+1)/2} \) times larger than the probability that it is at \( y_1 \). If the only peak in \( \mathcal{V}'' \) is at \( y_2 \), the only peak in \( \mathcal{V}' \) is at \( v_* \) and the highest label is in \( \mathcal{V}' \) then \( K_1 = v_* \). We obtain using (6.7),

\[
P(B \mid A(\mathcal{V}', \mathcal{V}'')) \leq 2P(B_1 \mid A(\mathcal{V}', \mathcal{V}'')) \leq 2(d - 1)^{-m(m+1)/2} P(C_1 \mid A(\mathcal{V}', \mathcal{V}'')).
\]  

(6.14)

By symmetry, if \( v_* \in \mathcal{V}'' \) then

\[
P(B \mid A(\mathcal{V}', \mathcal{V}'')) \leq 2P(B_2 \mid A(\mathcal{V}', \mathcal{V}'')) \leq 2(d - 1)^{-m(m+1)/2} P(C_2 \mid A(\mathcal{V}', \mathcal{V}'')).
\]

(6.15)

It follows from (6.14) and (6.15) that for all \( \{\mathcal{V}', \mathcal{V}''\} \subseteq \mathcal{V}(y_1, y_2) \),

\[
P(B \mid A(\mathcal{V}', \mathcal{V}'')) \leq 2(d - 1)^{-m(m+1)/2} P(C \mid A(\mathcal{V}', \mathcal{V}'')).
\]

(6.16)

In view of (6.1), (6.6), and (6.16),

\[
P(B) \leq 4(d - 1)^{-m(m+1)/2} P(C) \leq 4(d - 1)^{-m(m+1)/2} P(\mathcal{N}(G, 2)).
\]

(6.17)

**Step 5.** We will now discuss case (A3) (see the beginning of Step 2 for the definition).

Let \( y_3 \) be the vertex between \( y_1 \) and \( y_2 \). We define \( \{\mathcal{V}', \mathcal{V}''\} \subseteq \mathcal{V} \) by removing the edge joining \( y_2 \) and \( y_3 \). We label \( \mathcal{V}' \) and \( \mathcal{V}'' \) so that \( y_1 \in \mathcal{V}' \) and \( y_2 \in \mathcal{V}'' \). Note that \( v_* \in \mathcal{V}' \).

If we remove a vertex \( v \) from \( \mathcal{V}' \) then we obtain \( d + 1 \) trees \( \mathcal{W}_1^v, \mathcal{W}_2^v, \ldots, \mathcal{W}_{d+1}^v \). If \( v \) is a leaf of \( G \) then \( d \) of these trees are empty but this does not present a problem with the following argument. If \( v = v_* \) then \( \mathcal{W}^v_1, \ldots, \mathcal{W}^v_{d+1} \) are labeled so that \( y_1 \in \mathcal{W}^v_1 \). If \( v = y_1 \) then \( \mathcal{W}^v_1, \ldots, \mathcal{W}^v_{d+1} \) are labeled so that \( v_* \in \mathcal{W}^v_{d+1} \).

A random labeling of \( G \) can be constructed in the following stages. At each stage our choices should be understood to be random, that is, uniform in the usual sense.

(S1) Choose a vertex in \( \mathcal{V} \) for the highest label.

(S2) If the highest label is not assigned to \( y_1 \) or \( v_* \) then assign all other labels randomly.

For the remaining stages, assume that the highest label is assigned to \( w \), where \( w = y_1 \) or \( w = v_* \).

(S3) Assign labels, chosen from the set of all labels except the highest one, to all elements of \( \mathcal{V}' \).

(S4) Choose an “ordering” of labels for each \( \mathcal{W}_k^w, k = 1, \ldots, d + 1 \), in the sense that once the values of the labels for \( \mathcal{W}_k^w \) are chosen, we already have the place for the highest label, second highest label, etc.

(S5) Partition all remaining labels (i.e., those that are not assigned to \( \mathcal{V}'' \), and are not the highest label assigned to \( w \)) into (unordered) subsets \( R_k, k = 1, \ldots, d + 1 \), of sizes matching \( |\mathcal{W}_k^w| \), \( k = 1, \ldots, d + 1 \), without assigning them to specific vertices.

(S6) For each \( k = 1, \ldots, d + 1 \), assign labels in the subset \( R_k \), constructed in (S5) and corresponding to \( \mathcal{W}_k^w \), to specific vertices according to “ordering” chosen in (S4).

After completing (S4), we already know which, if any, of the events \( \mathcal{N}(G', 1) \cap \{ K'_1 = y_1 \} \), \( \mathcal{N}(G', 1) \cap \{ K'_1 = v_* \} \) and \( A(\mathcal{V}', \mathcal{V}'') \) hold but we do not know whether \( B_1 \) or \( C_1 \) hold.
The following argument presented in Step 4 applies in the current setting as well. Suppose that for some \( m \geq 2 \) we have \( \min_{i=1,2} \text{dist}(y_i, v_*) \geq m \). If we condition on \( A(V, V') \), the distribution of the order statistics of labels in \( V' \) (i.e., the joint distribution of the location of the highest label in \( V' \), the location of the second highest label in \( V' \), etc.) is independent of the values of the labels in \( V' \). Hence, (6.5) shows that conditional on \( A(V', V'') \), the probability that the only peak in \( V' \) is at \( v_* \) is at least \((d - 1)^{m(m+1)/2} \) times larger than the probability that it is at \( y_1 \).

In Step 4, the above argument led rather easily to (6.17) because \( y_2 \) was assumed not to be adjacent to \( G' \). In the present case, \( y_2 \) is adjacent to \( y_3 \) and, therefore, it is adjacent to \( G' \).

Recall notation from (6.8). We will argue that for every \( k \),

\[
\mathbb{P}(B_1 | \mathcal{N}(G', 1) \cap \{K'_1 = y_1\} \cap \mathcal{N}(G'', 1) \cap \{K''_1 = y_2\} \cap \{L(y_2) = k\}) \\
= \mathbb{P}(L(y_2) < L(y_1) | \mathcal{N}(G', 1) \cap \{K'_1 = y_1\} \cap \mathcal{N}(G'', 1) \cap \{K''_1 = y_2\} \cap \{L(y_2) = k\}) \\
< \mathbb{P}(L(y_2) < L(y_1) | \mathcal{N}(G', 1) \cap \{K'_1 = v_*\} \cap \mathcal{N}(G'', 1) \cap \{K''_1 = y_2\} \cap \{L(y_2) = k\}) \\
= \mathbb{P}(C_1 | \mathcal{N}(G', 1) \cap \{K'_1 = v_*\} \cap \mathcal{N}(G'', 1) \cap \{K''_1 = y_2\} \cap \{L(y_2) = k\}) \tag{6.18}
\]

Both equalities follow from the definitions of the events appearing in the formula. The inequality follows from the following: (i) the events on the right-hand sides of the last formula are determined by stages (S1)–(S4), and (ii) \( |W^0_{1,i}| < \omega^0_{1,i} \). At stage (S5), if \( w = y_1 \), the highest label assigned to \( W^0_{1,i} \) will go to \( y_3 \) (because we are conditioning on \( \mathcal{N}(G', 1) \cap \{K'_1 = y_1\} \)), while the highest label assigned to \( W^0_{1,i} \) might not go to \( y_3 \), in the case \( w = v_* \).

Recall that we have shown that conditional on \( A(V', V'') \), the probability that the only peak in \( V' \) is at \( v_* \) is at least \((d - 1)^{m(m+1)/2} \) times larger than the probability that it is at \( y_1 \). This and (6.18) imply that (6.14)–(6.17) hold in case (A3).

**Step 6.** We will show that for any \( y \) whose distance from the root is greater than or equal to 2,

\[
\mathbb{P}(\mathcal{N}(G, 2) \cap \{K_1 = v_*, K_2 = y\}) > \mathbb{P}(\mathcal{N}(G, 2) \cap \{K_1 = y, K_2 = v_*\}) \tag{6.19}
\]

Let \( V_i, 1 \leq i \leq d + 1 \), be the subtrees obtained by removing the root from \( G \). Without loss of generality, we assume that \( y \in V_1 \).

Let \( N = |V| \). A random labeling of \( G \) can be generated in the following way. First, we generate independent random labelings \( L_j \) of \( V_j \), for \( 1 \leq j \leq d + 1 \). Then we assign a random label from the set \([N]\) to \( v_* \) and divide randomly the labels remaining in the set \([N]\) into \( d + 1 \) (unordered) families \( \Lambda_j \) of equal sizes, independently from \( L_j \)’s. Next, labels in \( L_j \) are placed at vertices in \( V_j \) in such a way that the order structure is the same as for \( L_j \), for every \( j \). Let \( L \) be the name of the resulting labeling of \( G \).

Let \( F_1 \) denote the event that

(i) For \( j \neq 1 \), the labeling \( L_j \) has only one peak and it is located at a vertex adjacent to \( v_* \), and
(ii) \( L_1 \) has either only one peak at \( y \) or it has exactly two peaks, one at \( y \) and one at the neighbor of \( v_* \).

It follows from Lemma 5.4 that

\[
\mathcal{N}(G, 2) \cap \{K_1 = v_*, K_2 = y\} \subset F_1 \quad \text{and} \quad \mathcal{N}(G, 2) \cap \{K_1 = y, K_2 = v_*\} \subset F_1. \tag{6.20}
\]

If \( L(v_*) = N \) then (i)–(ii) are not only necessary but also sufficient conditions for the event that there are exactly two peaks at \( v_* \) and \( y \). More formally,

\[
\{L(v_*) = N\} \cap F_1 = \{L(v_*) = N\} \cap \mathcal{N}(G, 2) \cap \{K_1 = v_*, K_2 = y\}. \tag{6.21}
\]
If \( F_1 \) holds, \( L(v_a) = N - j \) and \( L \) has exactly two peaks, with one of them at \( v_a \) and the other in \( Y_1 \), then all labels \( N, N - 1, \ldots, N - j + 1 \) must be in \( Y_1 \). Otherwise, because of (i), at least one of the numbers \( N, N - 1, \ldots, N - j + 1 \) would be the label of a vertex adjacent to \( v_a \) and, therefore, \( v_a \) would not be a peak. Given \( \{L(v) = N - j\} \cap F_1 \), the probability that all labels \( N, N - 1, \ldots, N - j + 1 \) are in \( Y_1 \) is less than \( 1/(d + 1)^j \) for \( j > 1 \) and equal to \( 1/(d + 1) \) for \( j = 1 \). We use (6.20) and (6.21) in the following calculation,

\[
\mathbb{P}(\mathcal{N}(G, 2) \cap \{K_1 = y, K_2 = v_a\}) = \mathbb{P}(\mathcal{N}(G, 2) \cap \{K_1 = y, K_2 = v_a\} \cap F_1)
\]

\[
= \sum_{j=1}^{N-1} \mathbb{P}(\mathcal{N}(G, 2) \cap \{K_1 = y, K_2 = v_a, L(v_a) = N - j\} \cap F_1)
\]

\[
= \sum_{j=1}^{N-1} \mathbb{P}\left(\mathcal{N}(G, 2) \cap F_1 \cap \{K_1 = y, K_2 = v_a, L(v_a) = N - j, \{N, N - 1, \ldots, N - j + 1\} \subset L(Y_1)\}\right)
\]

\[
\leq \sum_{j=1}^{N-1} \mathbb{P}(\{L(v) = N - j, \{N, N - 1, \ldots, N - j + 1\} \subset L(Y_1)\} \cap F_1)
\]

\[
\leq \sum_{j=1}^{N-1} \frac{1}{(d + 1)^j} \mathbb{P}(\{L(v) = N - j\} \cap F_1)
\]

\[
= \sum_{j=1}^{N-1} \frac{1}{(d + 1)^j} \mathbb{P}(\{L(v_a) = N\} \cap F_1) < \frac{1}{d} \mathbb{P}(\{L(v_a) = N\} \cap F_1)
\]

\[
= \frac{1}{d} \mathbb{P}(\{L(v_a) = N\} \cap \mathcal{N}(G, 2) \cap \{K_1 = v_a, K_2 = y\})
\]

\[
= \frac{1}{d} \mathbb{P}(\mathcal{N}(G, 2) \cap \{K_1 = v_a, K_2 = y\}).
\]

Since \( 1/d < 1 \) for \( d \geq 2 \), we conclude that (6.19) holds.

**Step 7.** It follows from (6.13) and (6.19) that \( \mathbb{P}(\mathcal{N}(G, 2) \cap \{K_1 = y_1, K_2 = y_2\}) \) is maximized either when \( y_1 = v_a \) or if \( y_1 \) and \( y_2 \) are neighbors of \( v_a \).

Assume that \( y_1 \) and \( y_2 \) are neighbors of \( v_a \) and \( y_3 \) is a neighbor of \( y_2 \), but \( y_3 \neq v_a \). We will show that

\[
\mathbb{P}(\mathcal{N}(G, 2) \cap \{K_1 = v_a, K_2 = y_3\}) > \mathbb{P}(\mathcal{N}(G, 2) \cap \{K_1 = y_1, K_2 = y_2\}). \quad (6.22)
\]

The proof will use the same ideas as the proof of Proposition 5.3. Recall the notation from Definition 5.1.

The probability \( \mathbb{P}(\mathcal{N}(G, 2) \cap \{K_1 = v_a, K_2 = y_3\}) \) is the product of the following five factors, (i)–(v).

(i) The probability that the label of \( v_a \) is the highest among all labels. This probability is equal to \( 1/|\mathcal{Y}| \), where \( \mathcal{Y} \) is the vertex set of \( G \).

(ii) If we remove \( v_a \) from \( G \), we obtain \( d + 1 \) subtrees. Let \( G' \) be the subtree which contains \( y_3 \). Its vertex set will be denoted \( \mathcal{Y}' \). The second factor is the probability that \( y_3 \) has the largest label in \( G' \). This probability is equal to \( 1/|\mathcal{Y}'| \).

(iii) The third factor is the probability that the label of \( y_2 \) is larger than the labels of all of its descendants in \( G' \), if we consider \( y_3 \) to be the root of \( G' \). This probability is equal to \( 1/n_{y_2}^G \).

(iv) The fourth factor is the probability that the label of \( y_1 \) is larger than the labels of all of its descendants in \( G \), with the usual root \( v_a \). This probability is equal to \( 1/n_{y_1}^G \).
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(v) Just like in the proof of Proposition 5.3, we have to multiply by probabilities corresponding to all other vertices in \( G \), that is, the last factor is \( \prod_{x \neq y_1, y_2, y_3} 1/n_{v_s}^G(x) \).

The probability \( \mathbb{P}(\mathcal{N}(G, 2) \cap \{ K_1 = y_1, K_2 = y_2 \}) \) is the product of the following five factors, (1)–(5).

1. The probability that the label of \( y_1 \) is the highest among all labels. This probability is equal to \( 1/|\mathcal{V}| \), where \( \mathcal{V} \) is the vertex set of \( G \).
2. If we remove \( y_1 \) from \( G \), we obtain \( d + 1 \) subtrees. Let \( G'' \) be the subtree which contains \( y_2 \). Its vertex set will be denoted \( \mathcal{V}'' \). The second factor is the probability that \( y_2 \) has the largest label in \( G'' \). This probability is equal to \( 1/|\mathcal{V}''| \).
3. The third factor is the probability that the label of \( v_s \) is larger than the labels of all of its descendants in \( G'' \), if we consider \( y_2 \) to be the root of \( G'' \). This probability is equal to \( 1/n_{v_s}^{G''}(v_s) \).
4. The fourth factor is the probability that the label of \( y_3 \) is larger than the labels of all of its descendants in \( G \), with the usual root \( v_s \). This probability is equal to \( 1/n_{v_s}^G(y_3) \).
5. We have to multiply by probabilities corresponding to all other vertices in \( G \), that is, the last factor is \( \prod_{x \neq y_1, y_2, y_3} 1/n_{v_s}^G(x) \).

Note that the factors described in (i) and (1) are identical. The same applies to (v) and (5). Hence, it will suffice to show that

\[
\frac{1}{|\mathcal{V}'|} \frac{1}{n_{y_2}^G(y_2)} \frac{1}{n_{v_s}^G(y_1)} > \frac{1}{|\mathcal{V}''|} \frac{1}{n_{y_2}^{G''}(v_s)} \frac{1}{n_{v_s}^G(y_3)}.
\]

The above inequality is equivalent to each of the following inequalities.

\[
\begin{align*}
|\mathcal{V}'| \cdot n_{y_2}^G(y_2) \cdot n_{v_s}^G(y_1) &< |\mathcal{V}''| \cdot n_{y_2}^{G''}(v_s) \cdot n_{v_s}^G(y_3), \\
\frac{d^k - 1}{d - 1} \left( \frac{d^{k-1} - 1}{d - 1} \cdot (d - 1) + 1 \right) &< \left( \frac{d^k - 1}{d - 1} \cdot d + 1 \right) \left( \frac{d^{k-1} - 1}{d - 1} \cdot (d - 1) + 1 \right) \frac{d^{k-1} - 1}{d - 1}, \\
\frac{(d^k - 1)d^{k-1}(d^k - 1)}{(d - 1)^2} &< \frac{((d^k - 1)d + d - 1)d^k(d^{k-1} - 1)}{(d - 1)^2}, \\
(d^k - 1)(d^k - 1) &< (d^{k+1} - 1)d(d^{k-1} - 1), \\
-(d - 1)(d^k(d^k - d - 1) + 1) &< 0.
\end{align*}
\]

(6.24)

It is easy to see that (6.24) is true for all \( d \geq 3 \) and \( k \geq 2 \). It follows that (6.23) is also true. This completes the proof of (6.22) and, therefore, the proof of part (a) of the theorem.

(b) We have

\[
\mathbb{P}\left(\mathcal{N}(G, 2) \cap \left\{ \min_{i=1,2} \text{dist}(K_i, v_s) \geq m \right\} \right) = \frac{1}{2} \sum_{y_1, y_2} \mathbb{P}(\mathcal{N}(G, 2) \cap \{ K_1 = y_1, K_2 = y_2 \} \cup \{ K_1 = y_2, K_2 = y_1 \}).
\]

(6.25)

The factor 1/2 on the second line appears because we include both \((y_1, y_2)\) and \((y_2, y_1)\) in the sum.
The number of vertices in $G$ is $(d + 1)(d^k - 1)/(d - 1) + 1$ so, assuming that $d \geq 3$, the number of pairs $(y_1, y_2)$ is bounded by
\[
\left( \frac{(d + 1)(d^k - 1)}{d - 1} + 1 \right) \left( \frac{(d + 1)(d^k - 1)}{d - 1} \right) \leq 4d^{2k}.
\] (6.26)

It follows from (6.17) that for $m \geq 2$,
\[
P(\mathcal{N}(G, 2) \cap (\{K_1 = y_1, K_2 = y_2\} \cup \{K_1 = y_2, K_2 = y_1\})) \\
\leq 4(d - 1)^{-m(m+1)/2} P(\mathcal{N}(G, 2)).
\]

This, (6.25) and (6.26) imply that
\[
P \left( \mathcal{N}(G, 2) \cap \left\{ \min_{i=1,2} \text{dist}(K_i, v_*) \geq m \right\} \right) \leq 2d^{2k}4(d - 1)^{-m(m+1)/2} P(\mathcal{N}(G, 2)).
\] (6.27)

If $d \geq 3$ and $k \leq \frac{1}{8} m^2$ then $k \leq \frac{1}{4} m^2 \log(d - 1)/\log d$ and
\[
2d^{2k}4(d - 1)^{-m(m+1)/2} = 8 \exp(\log(d^k(d - 1)^{m(m+1)/2})) \\
= 8 \exp(2k \log d - \frac{1}{2} m(m + 1) \log(d - 1)) \\
\leq 8 \exp(-(m/2) \log(d - 1)).
\]

We use this and (6.27) to see that if $d \geq 3$, $k \geq 2$ and $m \geq \sqrt{8k}$ then
\[
P \left( \mathcal{N}(G, 2) \cap \left\{ \min_{i=1,2} \text{dist}(K_i, v_*) \geq m \right\} \right) \leq 8 \exp(-(m/2) \log(d - 1)) P(\mathcal{N}(G, 2)).
\]

This proves part (b) of the theorem.

(c) The argument will be based on the comparison of $P(\mathcal{N}(G, 2) \cap \{K_1 = v_*, K_2 = y\})$ and $P(\mathcal{N}(G, 1) \cap \{K_1 = v_*\})$.

Suppose that $y \in \mathcal{V}$ and $m := \text{dist}(y, v_*) \geq 2$. Let $\Gamma$ be the geodesic between $y$ and $v_*$ and let $z_j \in \Gamma$, $0 \leq j \leq m$, be such that $\text{dist}(z_j, y) = j$. Note that it is possible that $z_2 = v_*$. Suppose that $\mathcal{V}'$ and $\mathcal{V}''$ are obtained by removing the edge between $z_1$ and $z_2$ from $G$; we assume without loss of generality that $v_* \in \mathcal{V}'$. The corresponding graphs will be denoted $G'$ and $G''$. If a labeling $L$ of $G$ has the following properties: (i) $L$ restricted to $\mathcal{V}'$ has only one peak at $v_*$, (ii) $L$ restricted to $\mathcal{V}''$ has only one peak at $y$, and (iii) the largest label is in $\mathcal{V}'$, then $L$ has exactly two peaks at $v_*$ and $y$. Since $v_* \in \mathcal{V}'$, we have $|\mathcal{V}'| > |\mathcal{V}''|$ and so the probability that the largest label is assigned to a vertex in $\mathcal{V}'$ is greater than $1/2$. The distribution of the order statistics of $L$ restricted to $\mathcal{V}'$ is independent of this event, and the same holds for $\mathcal{V}''$. These observations and (5.1) imply that
\[
\frac{P(\mathcal{N}(G, 2) \cap \{K_1 = v_*, K_2 = y\})}{P(\mathcal{N}(G, 1) \cap \{K_1 = v_*\})} \geq \frac{1}{2} \prod_{x \in \mathcal{V}'} \left( \frac{1}{n_x^G(x)} \right) \prod_{x \in \mathcal{V}''} \left( \frac{1}{n_x^G(x)} \right) \frac{1}{\prod_{x \in \mathcal{V}} \left( \frac{1}{n_x^G(x)} \right)}.
\]

Note that the factors corresponding to $x \notin \Gamma$ are identical in the numerator and denominator, so
\[
\frac{P(\mathcal{N}(G, 2) \cap \{K_1 = v_*, K_2 = y\})}{P(\mathcal{N}(G, 1) \cap \{K_1 = v_*\})} \geq \frac{1}{2} \prod_{j=2}^{m} \frac{n_{v_*}^G(z_j)}{n_{v_*}^G(z_j)} \prod_{j=0}^{1} \frac{n_{y}^{G''}(z_j)}{n_{y}^{G''}(z_j)}.
\] (6.28)
Since \( m \leq k \),
\[
\frac{n_v^G(z_0)}{n_v^{G''}(z_0)} = \frac{n_v^G(y)(n_v^{G''}(y))^{-1}}{d(d^{k-m+1} - 1)/(d - 1)} \left( \frac{d(d^{k-m+2} - 1)}{d - 1} \right)^{-1}
= d^{-1}(1 - d^{-(k-m+1)})(1 - d^{-(k-m+2)})^{-1} \geq d^{-1}(1 - d^{-(k-m+1)}) \geq \frac{d - 1}{d^2}.
\]  

(6.29)

The set of vertices corresponding to \( n_v^G(z_1) \) contains the set corresponding to \( n_v^{G''}(z_1) \) so we have the following bound for the second factor in (6.28) corresponding to \( \Psi'' \),
\[
\frac{n_v^G(z_1)}{n_v^{G''}(z_1)} \geq 1.
\]  

(6.30)

Since \( \Psi' \subset \Psi \), \( n_v^G(z_j) \geq n_v^{G'}(z_j) \), for all \( j \). This implies that
\[
\prod_{j=2}^{m} \frac{n_v^G(z_j)}{n_v^{G''}(z_j)} \geq 1.
\]  

(6.31)

We combine (6.28)–(6.31) to obtain for \( d \geq 3 \),
\[
\frac{\mathbb{P}(\mathcal{N}(G, 2) \cap \{ K_1 = v_s, K_2 = y \})}{\mathbb{P}(\mathcal{N}(G, 1) \cap \{ K_1 = v_s \})} \geq \frac{1}{2} \frac{d - 1}{d^2} \geq \frac{1}{4d}.
\]  

(6.32)

We will now derive an upper bound. For \( 1 \leq j \leq m - 2 \), let \( \Psi_j^i \) and \( \Psi_j^i \) be obtained by removing the edge between \( z_j \) and \( z_{j+1} \) from \( G \); we assume that \( v_s \in \Psi_j^i \). The corresponding graphs will be called \( G_j^i \) and \( G_j^i \). Recall the following notation, \( A(\Psi_j^i, \Psi_j^i) = \mathcal{N}(G_j^i, 1) \cap \mathcal{N}(G_j^i, 1) \). Let \( K_j^i \) be the location of the highest peak in \( G_j^i \). It follows from the proof of Lemma 5.4 that
\[
\mathcal{N}(G, 2) \cap \{ K_1 = v_s, K_2 = y \} = \bigcup_{j=1}^{m-2} (A(\Psi_j^i, \Psi_j^i) \cap \{ K_j^{i,j} = v_s, K_j^{i,2} = y \}).
\]

These observations and (5.1) imply that
\[
\frac{\mathbb{P}(\mathcal{N}(G, 2) \cap \{ K_1 = v_s, K_2 = y \})}{\mathbb{P}(\mathcal{N}(G, 1) \cap \{ K_1 = v_s \})} \leq \sum_{i=1}^{m-2} \left( \prod_{x \in \Psi_j^i} (1/n_v^G(x)) \prod_{x \in \Psi_j^i} (1/n_v^{G''}(x)) \right) \prod_{x \in \Psi} (1/n_v^G(x)).
\]

The factors corresponding to \( x \notin \Gamma \) are identical in the numerator and denominator, so
\[
\frac{\mathbb{P}(\mathcal{N}(G, 2) \cap \{ K_1 = v_s, K_2 = y \})}{\mathbb{P}(\mathcal{N}(G, 1) \cap \{ K_1 = v_s \})} \leq \sum_{i=1}^{m-2} \prod_{j=i+1}^{m} \frac{n_v^G(z_j)}{n_v^{G''}(z_j)} \prod_{j=0}^{i-1} n_v^{G''}(z_j).
\]  

(6.33)

For \( m \leq k, d \geq 3 \) and \( i \geq 1 \),
\[
\frac{n_v^G(z_0)}{n_v^{G''}(z_0)} = \frac{d(d^{k-m+1} - 1)}{d - 1} \left( \frac{d(d^{k-m+1+i} - 1)}{d - 1} \right)^{-1}
\]
\[
\]
\[ \begin{align*}
&= d^{-i}(1 - d^{-(k-m+1)})(1 - d^{-(k-m+1+i)})^{-1} \\
&\leq d^{-i}(1 - d^{-(k-m+1)})(1 + 2d^{-(k-m+1+i)}) \leq 3d^{-i}.
\end{align*} \]

We estimate the other factors corresponding to \( V_j \) as follows, for \( 1 \leq j \leq i \),

\[ \frac{n^G_j(z_j)}{n^G_i(z_j)} = \frac{d(d^{k-m+1+j} - 1)}{d - 1} \left( \frac{d(d^{k-m+1+i} - 1)}{d - 1} - \frac{d(d^{k-m+1+j} - 1)}{d - 1} \right)^{-1} \]

\[ = d^{-j-i}(1 - d^{-(k-m+1+j)})(1 - d^{-j-i+1})^{-1} \leq d^{-j-i}(1 + 2d^{-j-i+1}) \leq 3d^{-i}.
\]

Hence

\[ \prod_{j=0}^{i} \frac{n^G_j(z_j)}{n^G_i(z_j)} \leq \prod_{j=0}^{i} 3d^j = 3^{i+1}d^{-i(i+1)/2}. \quad (6.34) \]

Next we deal with factors corresponding to \( V_j \), for \( i + 1 \leq j \leq m \),

\[ \frac{n^G_j(z_j)}{n^G_i(z_j)} = \frac{d(d^{k-m+1+j} - 1)}{d - 1} \left( \frac{d(d^{k-m+1+j} - 1)}{d - 1} - \frac{d(d^{k-m+1+i} - 1)}{d - 1} \right)^{-1} \]

\[ = (1 - d^{-(k-m+1+j)})(1 - d^{-j})^{-1} \leq 1 + 2d^{-j}.
\]

This implies that

\[ \prod_{j=i+1}^{m} \frac{n^G_j(z_j)}{n^G_i(z_j)} \leq \prod_{j=i+1}^{m} (1 + 2d^{j-i}) = \exp \left( \sum_{j=i+1}^{m} \log(1 + 2d^{j-i}) \right) \leq \exp \left( \sum_{j=i+1}^{m} 2d^{j-i} \right) \]

\[ \leq \exp \left( \sum_{j=i+1}^{\infty} 2d^{j-i} \right) = \exp \left( \frac{2}{d-1} \right). \quad (6.35) \]

We combine (6.33)–(6.35) to obtain for \( d \geq 3 \),

\[ \frac{P(\mathcal{N}(G, 2) \cap \{K_1 = v_x, K_2 = y\})}{P(\mathcal{N}(G, 1) \cap \{K_1 = v_x\})} \leq \sum_{i=1}^{m-2} \exp \left( \frac{2}{d-1} \right) 3^{i+1}d^{-i(i+1)/2} \]

\[ \leq \sum_{i=1}^{\infty} \exp \left( \frac{2}{d-1} \right) 3^{i+1}d^{-i(i+1)/2} < \infty. \quad (6.36) \]

Since the lower and upper bounds in (6.32) and (6.36) do not depend on \( m = \text{dist}(y, v_x) \), part (c) of the theorem follows.

(d) The number of \( y \in V \) with \( \text{dist}(y, v_x) \geq 2 \) is equal to \( (d^k - d)(d + 1)/(d - 1) \). Since

\[ d^k/2 \leq (d^k - d)(d + 1)/(d - 1) \leq 2d^k, \]

for \( d \geq 3 \) and \( k \geq 2 \), part (c) of the lemma implies that for every \( d \geq 3 \) there exists \( c_2 \) such that for all \( k \geq 2 \) and \( y \in V \) with \( \text{dist}(y, v_x) \geq 2 \), we have

\[ d^{-k}/c_2 \leq P_2(K_1 = v_x, K_2 = y) \leq c_2d^{-k}. \quad (6.37) \]
Consider $y_1, y_2 \in V$ and let $m_i = \text{dist}(y_i, v_*)$ for $i = 1, 2$. Let $m = \min(m_1, m_2)$. We use (6.17) and (6.37) to see that

$$P(N(G, 2) \cap \{K_1 = y_1, K_2 = y_2\} \cup \{K_1 = y_2, K_2 = y_1\})$$

$$\leq 4(d - 1)^{-m(m+1)/2} P(N(G, 2) \cap \{K_1 = v_*, K_2 = y_2\} \cup \{K_1 = v_*, K_2 = y_1\})$$

$$= 4(d - 1)^{-m(m+1)/2} P_2(\{K_1 = v_*, K_2 = y_2\} \cup \{K_1 = v_*, K_2 = y_1\}) P(N(G, 2))$$

$$\leq 8(d - 1)^{-m(m+1)/2} c_2 d^{-k} P(N(G, 2)).$$

This implies that for some $c_3$ depending on $d$ but not on $k$,

$$P_2(K_1 = y_1, \text{dist}(K_2, v_*) \leq m_1) = \sum_{j=0}^{m_1} \sum_{\{y_2: \text{dist}(y_2, v_*) = j, \text{dist}(y_1, v_*) \geq 2\}} P_2(K_1 = y_1, K_2 = y_2)$$

$$\leq \sum_{j=0}^{m_1} (d + 1)d^{j-1} 8(d - 1)^{-j(j+1)/2} c_2 d^{-k} \leq c_3 d^{-k}.$$  \hspace{1cm} (6.38)

A similar argument yields

$$P_2(K_2 = y_2, \text{dist}(K_1, v_*) \leq m_2) \leq c_3 d^{-k}.$$  \hspace{1cm} (6.39)

We use (6.38) and (6.39) to obtain

$$P_2 \left( \max_{i=1,2} \text{dist}(K_i, v_*) \leq k - n/2 \right)$$

$$= \sum_{j=0}^{k-n/2} \sum_{\{y_1: \text{dist}(y_1, v_*) = j\}} P_2(K_1 = y_1, \text{dist}(K_2, v_*) \leq m_1)$$

$$+ \sum_{j=0}^{k-n/2} \sum_{\{y_2: \text{dist}(y_2, v_*) = j\}} P_2(K_2 = y_2, \text{dist}(K_1, v_*) \leq m_2)$$

$$\leq \sum_{j=0}^{k-n/2} \sum_{\{y: \text{dist}(y, v_*) = j\}} 2c_3 d^{-k} \leq \left( 1 + \sum_{j=1}^{k-n/2} (d + 1)d^{j-1} \right) 2c_3 d^{-k}$$

$$= \left( 1 + (d + 1)d^{k-n/2} - 1 \right) \frac{2c_3 d^{-k}}{d - 1} \leq c_4 d^{-n/2}.$$  \hspace{1cm} (6.40)

Suppose that $| \text{dist}(K_1, K_2) - k | \geq n$. It is possible that $\min_{i=1,2} \text{dist}(K_i, v_*) \geq n/2$. Suppose that the last condition is not satisfied and consider the case when $\text{dist}(K_1, v_*) < n/2$. If $\text{dist}(K_2, v_*) > k - n/2$ then, by the triangle inequality, $\text{dist}(K_1, K_2) \geq k - n$, contradicting the assumption that $| \text{dist}(K_1, K_2) - k | \geq n$. It follows that $\text{dist}(K_2, v_*) \leq k - n/2$. Similarly, if $\text{dist}(K_2, v_*) < n/2$ then $\text{dist}(K_1, v_*) \leq k - n/2$. This argument proves that

$$\{ | \text{dist}(K_1, K_2) - k | \geq n \}$$

$$\subset \left\{ \min_{i=1,2} \text{dist}(K_i, v_*) \geq n/2 \right\} \cup \left\{ \max_{i=1,2} \text{dist}(K_i, v_*) \leq k - n/2 \right\}.$$
It follows from the last formula, (6.40) and part (b) of the theorem that for every $d \geq 3$ there exist $c_4$ and $c_5$ such that for $k \geq 2$ and $n \geq 2\sqrt{8k},$

$$\mathbb{P}_2\left(|\text{dist}(K_1, K_2) - k| \geq n\right) \leq 8 \exp(-\frac{n}{4} \log(d - 1)) + c_4 d^{-n/2} \leq 8 \exp(-\frac{n}{4} \log(d - 1)) + c_4 \exp(-\frac{n}{2} \log d) \leq c_5 \exp\left(-\frac{\sqrt{k}}{64} \log d\right).$$

7 | OPEN PROBLEMS

We collect here a few unanswered questions scattered throughout the paper.

(i) See Problem 1.1 concerned with the distance of two peaks on a torus.
(ii) Theorems 3.2 and 4.1 give only upper bounds for probabilities of relevant events. Can one find tight lower and upper bounds for these probabilities?
(iii) Can one weaken assumptions on the degree $d$ in Theorem 6.3 (assumptions are different in different parts of the theorem)? Can one find tight lower and upper bounds for probabilities in parts (b) and (d)?
(iv) For which families of graphs can one find (explicit) formulas for the number of (nonrandom) labelings with a given number of peaks or with prescribed peak locations, analogous to formulas developed for permutations in [4]?

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