ASYMPTOTIC SYMMETRIES IN 3-DIM GENERAL RELATIVITY: THE B(2,1) CASE

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Abstract. The ordinary Bondi–Metzner–Sachs (BMS) group \( B \) is the common asymptotic symmetry group of all asymptotically flat Lorentzian space–times. As such, \( B \) is the best candidate for the universal symmetry group of General Relativity (G.R.). Strongly continuous unitary irreducible representations (IRs) of \( B(2,1) \), the analogue of \( B \) in three space–time dimensions, are analysed in the Hilbert topology. It is proved that all IRs of \( B(2,1) \) are induced from IRs of compact ‘little groups’, which are the closed subgroups of \( SO(2) \). It is proved that all IRs of \( B(2,1) \) are obtained by Wigner–Mackey’s inducing construction notwithstanding the fact that \( B(2,1) \) is not locally compact in the employed Hilbert topology.

1. Introduction
The Bondi–Metzner–Sachs (BMS) group \( B \) is the common asymptotic group of all curved real Lorentzian space–times which are asymptotically flat in future null directions [1, 2], and is the best candidate for the universal symmetry group of G.R..

In 1939 Wigner laid the foundations of special relativistic quantum mechanics [3] and relativistic quantum field theory by constructing the Hilbert space strongly continuous unitary irreducible representations (IRs) of the (universal cover) of the Poincare group \( P \).

The universal property of \( B \) for G.R. makes it reasonable to attempt to lay a similarly firm foundation for quantum gravity by following through the analogue of Wigner’s programme with \( B \) replacing \( P \). Some years ago McCarthy constructed explicitly [4, 5, 6, 7, 8, 9, 10, 11] the IRs of \( B \) for exactly this purpose. This work was based on G.W.Mackey’s pioneering work on group representations [3, 12, 13, 14, 15]; in particular McCarthy’s work extended G.W.Mackey’s work to the relevant infinite–dimensional case.

It is difficult to overemphasize the importance of Piard’s results [16, 17] who soon afterwards proved that all the IRs of \( B \), when this is equipped with the Hilbert topology, are derivable by the inducing construction. This proves the exhaustivity of McCarthy’s list of representations and renders his results even more important.

Here, we follow this programme for 3–dim G.R. and construct in the Hilbert topology the IRs of \( B(2,1) \), the analogue of \( B \) in three space–time dimensions. It is proved that all IRs of \( B(2,1) \) are induced from IRs of compact ‘little groups’. It follows that some IRS of \( B(2,1) \) are controlled by IRs of the finite symmetry groups of regular polygons in ordinary euclidean 2–space. It is proved that all IRS of \( B(2,1) \) are induced by the IRS of its little groups notwithstanding the
fact that $B(2, 1)$ is not locally compact in the employed Hilbert topology. The paper closes with the explicit construction of the IRs of $B(2, 1)$.

2. The group $B(2, 1)$

We turn now to the study of to $B(2, 1)$, the analogue of $B$ in three space–time dimensions.

2.1. The group $B^{2,1}(N^+)$

Recall that the $2 + 1$ Minkowski space is the vector space $R^3$ of row vectors with 3 real components, with the inner product defined as follows. Let $x, y \in R^3$ have components $x^\mu$ and $y^\mu$ respectively, where $\mu = 0, 1, 2$. Define the inner product $x.y$ between $x$ and $y$ by

$$x.y = x^0 y^0 - x^1 y^1 - x^2 y^2.$$ (1)

Then the $2 + 1$ Minkowski space, sometimes written $R^{2,1}$, is just $R^3$ with this inner product. The “2,1” refers to the one plus and two minus signs in the inner product. Let $SO(2, 1)$ be the (connected component of the identity element of the) group of linear transformations preserving the inner product. Matrices $\Lambda \in SO(2, 1)$ are taken as acting by matrix multiplication from the right, $x \mapsto x\Lambda$, on row vectors $x \in R^{2,1}$.

The future null cone $N^+ \subset R^{2,1}$ is just the set of nonzero vectors with zero length and $x^0 > 0$:

$$N^+ = \{ x \in R^{2,1} | x.x = 0, x^0 > 0 \}.$$ (2)

Let $R^*_+$ denote the multiplicative group of all positive real numbers. Obviously, if $x \in N^+$, then $tx \in N^+$ for any $t \in R^*_+$. Let $F_1(N^+)$ denote the vector space (under pointwise addition) of all functions $f : N^+ \to R$ satisfying the homogeneity condition

$$f(tx) = tf(x)$$ (3)

for all $x \in N^+$ and $t \in R^*_+$. Define a representation $T$ of $SO(2, 1)$ on $F_1(N^+)$ by setting, for each $x \in N^+$ and $\Lambda \in SO(2, 1)$,

$$(T(\Lambda)f)(x) = f(x\Lambda).$$ (4)

Now let $B^{2,1}(N^+)$ be the semi–direct product

$$B^{2,1}(N^+) = F_1(N^+) \rtimes_T SO(2, 1).$$ (5)

That is to say, $B^{2,1}(N^+)$ is, as a set, just the product $F_1(N^+) \times SO(2, 1)$, and the group multiplication law for pairs is

$$(f_1, \Lambda_1)(f_2, \Lambda_2) = (f_1 + T(\Lambda_1)f_2, \Lambda_1\Lambda_2).$$ (6)

2.2. The double cover $B^{2,1}(N^+)_c$

Let $SL(2, R)$ be the group of all real $2 \times 2$ matrices with determinant one. $SL(2, R)$ is sometimes denoted by $G$ below. Let $M_s(2, R)$ be the set of all $2 \times 2$ symmetric real matrices. We define a right action of $G$ on $M_s(2, R)$ by $M_s(2, R) \times G \to M_s(2, R)$ with

$$(m, g) \mapsto g^\top mg,$$ (7)

where the superscript $\top$ means transpose. Clearly any element $\mu \in M_s(2, R)$ can be parameterized as follows:

$$\mu = \begin{bmatrix} x^0 - x^1 & x^2 \\ x^2 & x^0 + x^1 \end{bmatrix}.$$
where $x^o, x^1, x^2 \in R$. We now consider the map $b : R^3 \rightarrow M_s(2, R)$ defined by

$$b(x) = \begin{bmatrix} x^o - x^1 & x^2 \\ x^2 & x^o + x^1 \end{bmatrix},$$

where the $x^\mu$ are the components of $x \in R^3$. This map is a linear bijection, so the right action of $G$ on $M_s(2, R)$ induces a linear right action of $G$ on $R^3$. Since

$$\det(b(x)) = x \cdot x$$

and the $G$ action preserves determinants (indeed $\det g = 1$) in $M_s(2, R)$, $G$ acts as transformations from $SO(2, 1)$. In fact, this construction gives an homomorphism

$$\gamma : G \rightarrow SO(2, 1)$$

which is onto, and has kernel $Z_2 = \{ \text{Id}, -\text{Id} \}$ in $G$, $\text{Id}$ denoting the identity element of $G$. Thus $\gamma$ identifies $G$ as the double cover of $SO(2, 1)$

$$G = SO(2, 1)_{\gamma}.$$

Therefore, the double cover of the group $B^{2,1}(N^+)$, given in (5), has the form

$$B^{2,1}(N^+)_{\gamma} = F_1(N^+) \ltimes T \text{SL}(2, R).$$

Strictly speaking, “$T$” should read “$T\gamma$”, but the notation is simpler as above.

2.3. The group $B(2, 1)$

So far, the supertranslation space $F_1(N^+)$ has been defined as a space of truly arbitrary homogeneous functions of degree one. This has been merely for clarity; for physical applications, it is necessary to give this space additional structure. For reasons discussed in detail in McCarthy [18], we now give a new realization of $B^{2,1}(N^+)_{\gamma}$ where the supertranslation space is restricted to be the separable Hilbert space $L^2(P_1(R), \lambda, R)$ of real—valued functions defined on $P_1(R) \simeq S^1$; functions square integrable with respect to the standard normalized (Lesbegue) measure $\lambda$ on $P_1(R) \simeq S^1$; $P_1(R) \equiv S^1/Z_2$ is the one—dimensional real projective space (the circle quotient the antipodal map).

In particular in [19] the following Theorem is proved:

**Theorem 1** The group $B^{2,1}(N^+)_{\gamma}$ can be realised as

$$B(2, 1) = L^2(P_1(R), \lambda, R) \otimes_T G$$

with semi—direct product specified by

$$(T(g)\alpha)(x) = \kappa_g(x)\alpha(xg),$$

where $G = \text{SL}(2, R)$, $g \in G$, $\alpha \in L^2(P_1(R), \lambda, R)$. Moreover, if

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then the components $x_1, x_2$ of $x \in R^2 - \{0\}$ transform linearly under $g$, so that the ratio $x = x_1/x_2, x_2 \neq 0$, transforms fraction linearly under $g$. Writing $xg$ for the transformed ratio,

$$xg = \frac{(xg)_1}{(xg)_2} = \frac{x_1a + x_2c}{x_1b + x_2d} = \frac{xa + c}{xb + d}.$$
The ratio \( x = x_1/x_2 \), \( x_2 \neq 0 \), is a local inhomogeneous coordinate of \( P_1(R) \). We denote our final realization of our group by \( B(2,1) \) to distinguish it from the previous realizations \( B^{2,1}(N^+) \) and \( B^{2,1}(N^+)_c \). In analogy to \( B \), it is natural to choose a measure \( \lambda \) on \( P_1(R) \) which is invariant under the maximal compact subgroup \( SO(2) \) of \( G \); we choose \( \lambda \) to be the standard normalized Lebesgue measure \( d\lambda = dx/2\pi \).

The factor \( \kappa_g(x) \) on the right hand side of (14) is defined by

\[
\kappa_g(x) = \frac{(xb+\theta)^2 + (xa+\phi)^2}{1+x^2}.
\]

It is well known [20] that the topological dual of a Hilbert space can be identified with the Hilbert space itself, so that we have \( L^2(P_1(R),\lambda,R) \approx L^2(P_1(R),\lambda,R) \). In fact, given a continuous linear functional \( \phi \in L^2(P_1(R),\lambda,R) \), we can write, for \( \alpha \in L^2(P_1(R),\lambda,R) \)

\[
(\phi,\alpha) = <\phi,\alpha>,
\]

where the function \( \phi \in L^2(P_1(R),\lambda,R) \) on the right is uniquely determined by (and denoted by the same symbol as) the linear functional \( \phi \in L^2(P_1(R),\lambda,R) \) on the left. The representation theory of \( B(2,1) \) is governed by the dual action \( T' \) of \( G \) on the topological dual \( L^2(P_1(R),\lambda,R) \) of \( L^2(P_1(R),\lambda,R) \). The dual action \( T' \) is defined by:

\[
<T'(g)\phi,\alpha> = <\phi,T(g^{-1})\alpha>.
\]

A short calculation gives

\[
(T'(g)\phi)(x) = \kappa_g^{-2}(x)\phi(xg).
\]

Now, this action \( T' \) of \( G \) on \( L^2(P_1(R),\lambda,R) \), given explicitly above is, like the action \( T \) of \( G \) on \( L^2(P_1(R),\lambda,R) \), continuous. The ‘little group’ \( L_{\phi} \) of any \( \phi \in L^2(P_1(R),\lambda,R) \) is the stabilizer

\[
L_{\phi} = \{g \in G \mid T'(g)\phi = \phi\}.
\]

By continuity, \( L_{\phi} \subset G \) is a closed subgroup.

3. Representation theory

Let \( A \) and \( G \) be topological groups, and let \( T \) be a given homomorphism from \( G \) into the group of automorphisms \( \text{Aut}(A) \) of \( A \). Suppose \( A \) is abelian and \( H = A \rtimes_T G \) is the semi-direct product of \( A \) and \( G \), specified by the continuous action \( T : G \to \text{Aut}(A) \). In the product topology of \( A \times G \), \( H \) then becomes a topological group. It is assumed that it becomes a separable locally compact topological group.

In order to give the operators of the induced representations explicitly it is necessary ([3], [12], [13], [14], [15] and references therein) to give the following information:

(i) An irreducible unitary representation \( U \) of \( L_{\phi_o} \) on a Hilbert space \( D \) for each \( L_{\phi_o} \).

(ii) A \( G \)-quasi-invariant measure \( \mu \) on each orbit \( G\phi \cong G/L_{\phi_i} \), where \( L_{\phi_i} \) denotes the little group of the base point \( \phi_o \in A' \) of the orbit \( G\phi \); \( A' \) is the topological dual of \( A \).

Let \( D_\mu \) be the space of functions \( \psi : G \to D \) which satisfy the conditions

\[
(a) \quad \psi(gl) = U(l^{-1})\psi(g) \quad (g \in G, \ l \in L_{\phi})
\]

\[
(b) \quad \int_{G_{\phi_o}} |\psi(q)|^2 \, dq < \infty,
\]
Two remarks are in order regarding the representations of $B$. Note, that the constraint (a) implies that $< \psi(gl), \psi(gl) > = < \psi(g), \psi(g) >$, and therefore the inner product $< \psi(g), \psi(g) >$, $g \in \mathcal{G}$, is constant along every element $g$ of the coset space $\mathcal{G} / L_{\phi_0} \approx \mathcal{G}_{\phi_0}$. This allows to assign a meaning to $< \psi(q), \psi(q) >$, where $q = gL_{\phi_0}$, by defining $< \psi(q), \psi(q) > := < \psi(g), \psi(g) >$. Thus the integrand in (b) becomes meaningful due to the condition (a). A pre–Hilbert space structure can now be given to $D_{\mu}$ by defining the scalar product

$$< \psi_1, \psi_2 > = \int_{G_{\phi_0}} < \psi_1(q), \psi_2(q) > d\mu(q),$$

where $\psi_1, \psi_2 \in D_{\mu}$. It is convenient to complete the space $D_{\mu}$ with respect to the norm defined by the scalar product (22). In the resulting Hilbert space, functions are identified whenever they differ, at most, on a set of $\mu$–measure zero. Thus our Hilbert space is

$$D_{\mu} = L^2(G_{\phi_0}, \mu, D).$$

Define now an action of $\mathcal{H} = A \rtimes_T G$ on $D_{\mu}$ by

$$(g_o \psi)(q) = \sqrt{\frac{d\mu_{g_o}}{d\mu}}(q) \psi(g_o^{-1}q),$$

$$\alpha \psi(q) = e^{i<q_{\phi_0}, a>} \psi(q)$$

where, $g_o \in G$, $q \in G_{\phi_0}$, and $\alpha \in A$. Eqs. (24) and (25) define the IRs of $B(2,1)$ induced for each $\phi_0 \in A'$ and each irreducible representation $U$ of $L_{\phi_0}$. The ‘Jacobian’ $\frac{d\mu_{g_o}}{d\mu}$ of the group transformation is known as the Radon–Nikodym derivative of $\mu_{g_o}$ with respect to $\mu$ and ensures that the resulting IRs of $B(2,1)$ are unitary.

The central results of induced representation theory ([3], [12], [13], [14], [15] and references therein) are the following:

(i) Given the topological restrictions on $\mathcal{H} = A \rtimes_T G$ (separability and local compactness), any representation of $\mathcal{H}$, constructed by the method above, is irreducible if the representation $U$ of $L_{\phi_0}$ on $D$ is irreducible. Thus an irreducible representation of $\mathcal{H}$ is obtained for each $\phi_0 \in A'$ and each irreducible representation $U$ of $L_{\phi_0}$.

(ii) If $\mathcal{H} = A \rtimes_T G$ is a regular semi–direct product (i.e., $A'$ contains a Borel subset which meets each orbit in $A'$ under $\mathcal{H}$ in just one point) then all of its irreducible representations can be obtained in this way.

4. Obstructions and resolutions

Two remarks are in order regarding the representations of $B(2,1)$ obtained by the above construction:

(i) As it is explained in [19] the subgroup $L^2(P_1(R), \lambda, R)$ of $B(2,1) = L^2(P_1(R), \lambda, R) \rtimes_T G$ is topologised as a (pre) Hilbert space by using a natural measure on $P_1(R)$ and by introducing a scalar product into $L^2(P_1(R), \lambda, R)$. If $R^4$ is endowed with the natural metric topology then the group $G = SL(2, R)$, considered as a subset of $R^4$, inherits the induced topology on $G$. In the product topology of $L^2(P_1(R), \lambda, R) \times G$, $B(2,1)$ is a non–locally compact group (the proof follows, without substantial change, Cantoni’s proof [21], see also [4]). In fact the subgroup $L^2(P_1(R), \lambda, R)$, and therefore the group $B(2,1)$ can be employed with many different topologies. The Hilbert type topology employed here appears to describe quantum mechanical systems in asymptotically flat space–times [10]). Since in the Hilbert type topology $B(2,1) = L^2(P_1(R), \lambda, R) \rtimes_T G$ is not locally compact
the theorems dealing with the reducibility of the representations obtained by the above construction no longer apply (see e.g. [13]). However, it can be proved that the induced representations obtained above are irreducible. The proof follows very closely the one given in [7] for the case of the original BMS group $B$.

(ii) Here it is assumed that $B(2,1)$ is equipped with the Hilbert topology. It is of utmost significance that it can be proved [19] that in this topology $B(2,1)$ is a regular semi–direct–product. The proof follows the corresponding proof [16, 17] for the group $B$. Regularity amounts to the fact [12] that $L^2(P_1(R), \lambda, R)$ can have no equivalent classes of quasi–invariant measures $\mu$ such that the action of $G$ is strictly ergodic with respect to $\mu$. When such measures $\mu$ do exist it can be proved [12] that an irreducible representation of the group, with the semi–direct–product structure at hand, may be associated with each that is not equivalent to any of the IRs constructed by the Wigner–Mackey’s inducing method. In a different topology it is not known if $B(2,1)$ is a regular or irregular semi–direct–product. Irregularity of $B(2,1)$ in a topology different from the Hilbert topology would imply that there are IRs of $B(2,1)$ that are not equivalent to any of the IRs obtained above by the inducing construction. Strictly ergodic actions are notoriously hard to deal with even in the locally compact case. Indeed, for locally compact non–regular semi–direct products, there is no known example for which all inequivalent irreducibles arising from strictly ergodic actions have been found. For the other 41 groups defined in [18] regularity has only been proved for $B$ [16, 17] when $B$ is equipped with the Hilbert topology. Similar remarks apply to all of them regarding IRs arising from strictly ergodic actions in a given topology.

5. Results

In [19] it is proved that when $B(2,1)$ is employed with the Hilbert topology all little groups of $B(2,1)$ are compact. In particular the following Theorem is proved

**Theorem 2** The little groups for $B(2,1)$ are precisely the closed subgroups of $K = SO(2)$ which contain the element $-I$, $I$ being the identity element of $G$. These are (A) $K$ itself, and (B) the cyclic groups of even order.

Moreover in [19] it is shown that the Wigner–Mackey’s inducing construction is exhaustive despite the fact that $B(2,1)$ is not locally compact in the employed Hilbert topology. This result is rather important because other group theoretical approaches to quantum gravity which invoke Wigner–Mackey’s inducing construction (see e.g. [22]) are typically plagued by the non–exhaustiveness of the inducing construction which results precisely from the fact that the group in question is not locally compact in the prescribed topology. Exhaustiveness is not just a mathematical nicety: If the inducing construction is not exhaustive one cannot simply know if the most interesting information or part of it is coded in the irreducibles which cannot be found by the Wigner–Mackey’s inducing procedure. These results, i.e. compactness of the little groups and exhaustiveness of the inducing construction, not only are significant for the group theoretical approach to quantum gravity advocated here, but also they have repercussions [19] for other approaches to quantum gravity.

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