ON A DESIGN FROM PRIMITIVE REPRESENTATIONS OF THE
FINITE SIMPLE GROUPS

Ali Reza Rahimipour

Abstract. In this paper we present a design construction from primitive permutation representations of a finite simple group \( G \). The group \( G \) acts primitively on the points and transitively on the blocks of the design. The construction has this property that with some conditions we can obtain \( t \)-design for \( t \geq 2 \). We examine our design for fourteen sporadic simple groups. As a result we found a 2-(176,5,4) design with full automorphism group \( M_{22} \).

Keywords. primitive permutation; group; finite simple group; automorphism group.

1. Introduction

Designs are interesting combinatorial objects. They have important applications in coding theory and information theory. Constructing combinatorial designs by using finite permutation groups is a well-studied subject. Well-known methods to construct 1-design from primitive permutation representations of finite simple groups were introduced by Moori and Key [10, 11]. Also in [5] a generalization of the construction in [10] was described. Here we present a design construction from primitive representations of a finite simple group. The groups we consider are primitive on the points and transitive on the blocks of constructed designs. In some conditions we can construct \( t \)-designs for \( t \geq 2 \). We employ this method on some simple groups and calculate full automorphism groups of constructed designs.

Sporadic simple groups are interesting family of the finite simple groups. Designs that are invariant under sporadic groups or have full automorphism group equal to sporadic groups are very interesting. Some of these designs were presented in [2, 8, 10, 12, 13]. Here we consider fourteen sporadic groups for our purpose. For these sporadic groups we obtained some designs for which the full automorphism groups are the same as the sporadic group or double cover of that. These constructed
designs from sporadic groups are usually 1-design or 2-design and in some cases groups act primitively on the points and the blocks of the designs.

In Section 2, first we present some preliminary definitions and lemmas which will be used in the proof of our main results. In Theorem 2.1 we give our design construction from primitive permutation representations of a finite simple group. Then some properties of these design are considered. One advantage of our design in comparison with the designs presented in [5, 10, 11] is that we can determine some conditions to construct $t$-design for $t \geq 2$. In Proposition 2.3 these conditions are determined. Applying this result on some finite simple groups we found some 2-designs from 1-transitive actions of these simple groups. In Section 3 we describe constructed designs from fourteen sporadic simple groups. Especially we make use of large sporadic simple groups $Co_3$ and $Fi_{23}$. These groups are full automorphism group of the constructed designs which act primitively on the points and the blocks.

In Section 4 we present a 2-(176,5,4) design from Mathieu group $M_{22}$. With the best of knowledge, this design is new and group $M_{22}$ acts primitively on the points and transitively on the blocks of this design. The full automorphism group of this design is isomorphic to $M_{22}$.

2. Design Construction

In this paper all groups are assumed to be finite. Our notations are standard and for design are from [3] and for group theory and character theory are from [6, 9]. For the name and structure of finite simple groups we use the Atlas notation [4]. All computations were done with GAP [14] and Magma [1]. All programs are accessible from the author upon request.

Let $t, \lambda, v$ and $k$ be integers such that $1 \leq t \leq k \leq v$ and $\lambda > 0$. Let $X$ be an $v$-set. A $t$-$(v,k,\lambda)$ design is a pair $D = (X,B)$ such that $B$ is a collection of $k$-subsets of $X$ called blocks such that every $t$-subset of $X$ appears in exactly $\lambda$ blocks. A design is called simple if it has no repeated blocks. All designs in this paper are simple. The design $D$ is called symmetric if the number of points and the blocks are equal.

An automorphism of $D$ is a permutation $f$ on $X$ such that $f(b) \in B$ for each $b \in B$. A group whose elements are automorphism of $D$ is called an automorphism group of $D$. We use $\text{Aut}(D)$ to denote the full automorphism group of $D$.

Let $G$ be a finite permutation group acting on a set $X$. The orbit of $x \in X$ is defined as $O(x) = \{x^g | g \in G\}$ and the stabilizer subgroup of $x$ is $G_x = \{g \in G | x^g = x\}$. It is well-known that $|G| = |O(x)||G_x|$. For $g \in G$, the conjugacy class of $g$ is $\text{cl}(g) = \{a^{-1}ga | a \in G\}$. It is well-known that $|G| = |\text{cl}(g)||C_G(g)|$ such that $C_G(g) = \{a \in G | ag = ga\}$ is centralizer subgroup of $g$ in $G$.

Let $G$ be a finite group and $H$ be a subgroup of $G$. Assume that $\Omega$ is the set of all conjugates of $H$ in $G$. Let $\chi_H = \chi(G|H)$ be the permutation character corresponding to the action of $G$ on $\Omega$. For $g \in G$ if $\text{cl}(g) \cap H = \emptyset$ then $\chi_H(g) = 0$.

In what follows we present some lemmas that are used in constructing design.
Lemma 2.1. [6, Corollary 1.5A] Let $G$ be a group acting transitively on a set $\Omega$ with at least two points. Then action of $G$ on $\Omega$ is primitive if and only if for each $x \in \Omega$, $G_x$ is a maximal subgroup of $G$.

Lemma 2.2. [7, Corollary 3.1.3] Let $G$ be a finite group and $H$ a subgroup of $G$ containing a fixed element $x$. Then the number $h$ of conjugates of $H$ in $G$ containing $x$ is given by

$$h = [N_G(H) : H]^{-1} \sum_{i=1}^{m} \frac{|C_G(x)|}{|C_H(x_i)|}$$

where $x_1, \ldots, x_m$ are representatives of $H$-conjugacy classes that fuse to the $G$-class $cl(x)$.

Lemma 2.3. [9, Corollary 5.14] Let $G$ be a finite group and $H$ be a subgroup of $G$. Let $nX$ be the set of all conjugates of $H$ in $G$. Then for all $g \in G$, $\chi_H(g)$ is equal to the number of points in $\Omega$ fixed by $g$.

Let $G$ be a finite simple group and $M$ be a maximal subgroup of $G$. Let $\Omega$ be the set of all conjugates of $M$ in $G$. In the action of $G$ on $\Omega$, the stabilizer subgroup of each point of $\Omega$ is conjugate to $M$. For $g \in G$ let $cl(g) \cap M \neq \emptyset$. Then by Lemma 2.2 each $g \in G$ is in $r = \sum_{i=1}^{m} \frac{|C_G(g_i)|}{|C_M(g_i)|}$ conjugates of $M$, where $g_1, \ldots, g_m$ are representatives of the $M$-conjugacy classes that fuse to the $G$-class $cl(g)$. Then by Lemma 2.3, $\chi_M(g) = r$. Let $nX$ be a conjugacy class of the elements of order $n$ such that $nX \cap M \neq \emptyset$. Consider $g \in nX$. Then $g$ is contained in $r$ conjugates of $M$. We define $B_{g,M} = \{M_1, \ldots, M_r\}$ such that $M_i$ for $i \in \{1, 2, \ldots, r\}$, is a conjugate of $M$ for which $g \in M_i$. Also set $S_{g,M} = \bigcap_{i=1}^{r} M_i$. Clearly for $g, g' \in nX$ if the subgroups $S_{g,M}$ and $S_{g',M}$ are conjugate, then $|S_{g,M} \cap nX| = |S_{g',M} \cap nX|$.

Lemma 2.4. Let $G$ be a finite simple group and $M$ be a maximal subgroup of $G$. Let $nX$ be a conjugacy class of the elements of order $n$ such that $nX \cap M \neq \emptyset$. Then for each $g, g' \in nX$ the following hold:

1. $B_{g,M} = B_{g',M}$ if and only if $S_{g,M} \cap nX = S_{g',M} \cap nX$;
2. if $S_{g,M} \cap S_{g',M} \cap nX \neq \emptyset$ then $S_{g,M} \cap nX = S_{g',M} \cap nX$;
3. $\frac{|nX|}{|S_{g,M} \cap nX|}$ is a positive integer,
4. $\frac{|M \cap nX|}{|S_{g,M} \cap nX|}$ is a positive integer.

Proof. (1) Let $B_{g,M} = B_{g',M}$. Then obviously $S_{g,M} \cap nX = S_{g',M} \cap nX$. Now suppose $S_{g,M} \cap nX = S_{g',M} \cap nX$. Then $g, g' \in S_{g,M}$ and $g, g' \in S_{g',M}$. If $B_{g,M} \neq B_{g',M}$ without lose of generally, there exists $M'$, a conjugate of $M$ such that $M' \in B_{g,M}$ but $M' \notin B_{g',M}$. Now since $g' \in S_{g,M}$ then $g' \in M'$ and $M' \in B_{g',M}$ which is a contradiction. (2) Suppose $x \in S_{g,M} \cap S_{g',M} \cap nX$. So $x \in S_{g,M}$. 


and \( x \in S_{g',M} \). Then we have \( B_{x,M} = B_{g,M} = B_{g',M} \) and by (1) the result is achieved. (3) By (2) each element of \( nX \) is contained in a unique \( S_{g,M} \cap nX \). Then \( nX = \bigcup_{i=1}^{k}(S_{g,M} \cap nX) \), which is a disjoint union of the elements of class \( nX \) for some \( S_{g_i,M} \) for \( i \in \{1, 2, ..., k\} \). Therefore \( k = \frac{|nX|}{|S_{g,M} \cap nX|} \) is a positive integer. (4) According to the proof of (3) we have \( nX = \bigcup_{i=1}^{k}(S_{g_i,M} \cap nX) \). For each \( i \in \{1, 2, ..., k\} \), \( S_{g_i,M} \cap M \cap nX = S_{g_i,M} \cap nX \) or \( S_{g_i,M} \cap M \cap nX = \emptyset \). Therefore \( nX \cap M = \bigcup_{j=1}^{h}(S_{g_j,M} \cap nX) \) such that for each \( j \in \{1, 2, ..., h\} \) we have \( S_{g_j,M} \cap M \cap nX = S_{g_j,M} \cap nX \) hence \( h = \frac{|M \cap nX|}{|S_{g,M} \cap nX|} \) is a positive integer. 

Now we are ready to present a design construction from primitive permutation representations of a finite simple group.

**Theorem 2.1.** Let \( G \) be a finite simple group. Let \( M \) be a maximal subgroup of \( G \) and \( \Omega \) be the set of all conjugates of \( M \) in \( G \). Let \( nX \) be a conjugacy class of the elements of order \( n \) such that \( M \cap nX \neq \emptyset \) and \( g \in nX \). Set \( B = \{B_{x,M}|x \in nX\} \). Then \( D = (\Omega, B) \) is a,

\[
1 - \left( |G : M|, \chi_M(g), \frac{|M \cap nX|}{|S_{g,M} \cap nX|} \right)
\]

design which has \( \frac{|nX|}{|S_{g,M} \cap nX|} \) blocks. Also \( G \) is an automorphism group of \( D \) which acts primitively on the points and transitively on the blocks of \( D \).

**Proof.** In the action of \( G \) on \( \Omega \), the stabilizer of each point of \( \Omega \) is conjugate to \( M \). Since \( B_{g,M} \) is the collection of all conjugates of \( M \) that contain \( g \), by Lemma 2.3 the length of a block is \( \chi_M(g) \). Clearly \( |S_{g,M} \cap nX| \geq 1 \). If \( |S_{g,M} \cap nX| = 1 \) then by Lemma 2.4(1) for every \( g', g'' \in nX \) we have \( B_{g',M} \neq B_{g'',M} \) and so \( |B| = |nX| \). If \( |S_{g,M} \cap nX| = m > 1 \) then by Lemma 2.4(1) for \( m \) elements \( g_1, ..., g_m \) of \( S_{g,M} \cap nX \) we have \( B_{g_1,M} = B_{g_2,M} = ... = B_{g_m,M} \). So according to the proof of Lemma 2.4(3), \( nX = \bigcup_{i=1}^{k}(S_{g_i,M} \cap nX) \) for \( i \in \{1, 2, ..., k\} \), which is a disjoint union of the elements of \( S_{g_i,M} \cap nX \). Then for each \( S_{g_i,M} \cap nX \) there exists a unique block. Hence the number of different blocks is equal to \( \frac{|nX|}{|S_{g,M} \cap nX|} \) which by Lemma 2.4(3) is a positive integer. By the proof of Lemma 2.4(4) \( M \cap nX = \bigcup_{j=1}^{h}(S_{g_j,M} \cap nX) \) for \( j \in \{1, 2, ..., h\} \), which is a disjoint union of the elements of \( S_{g_j,M} \cap nX \). Then for each \( j \in \{1, 2, ..., h\} \), subgroup \( M \) is in a unique block. Then each point appears in exactly \( \frac{|M \cap nX|}{|S_{g_j,M} \cap nX|} \) blocks which by Lemma 2.4(4) is a positive integer. The group \( G \) acts on the points and the blocks with conjugation. Since \( M \) is maximal then by Lemma 2.1, \( G \) acts primitively on the points. Each block is corresponding to an element of \( nX \) and since \( G \) is transitive on \( nX \) then \( G \) is transitive on the blocks.

We denote the constructed design in Theorem 2.1 by \( D(G, M, nX) \). The design \( D(G, M, nX) \) is not necessary symmetric and the action of \( G \) on the blocks is not necessary primitive. In the following propositions we consider some conditions to achieve these properties.
Proposition 2.1. Let $G$ be a finite simple group. Let $M$ be a maximal subgroup of $G$ and $\Omega$ be the set of all conjugates of $M$ in $G$. Let $nX$ be a conjugacy class of the elements of order $n$ such that $M \cap nX \neq \emptyset$ and $g \in nX$. If $C_G(g)$ is maximal subgroup then the action of $G$ on the blocks of $D(G, M, nX)$ is primitive.

**Proof.** Since $C_G(g)$ is maximal, by Lemma 2.1 the action of $G$ on right cosets of $C_G(g)$ is primitive. Each block is corresponding to an element of $nX$ and respectively to a right coset of $C_G(g)$. Then the action of $G$ on the blocks of $D(G, M, nX)$ is primitive. 

Let $g \in nX$ and consider $D(G, M, nX)$. According to the definition, for every $x \in C_G(g)$ we have $B_{g,M} = B_{g^x,M} = (B_{g,M})^x$. Therefore $C_G(g)$ is a subgroup of stabilizer of the block $B_{g,M}$.

Proposition 2.2. Let $G$ be a finite simple group. Let $M$ be a maximal subgroup of $G$ and $\Omega$ be the set of all conjugates of $M$ in $G$. Let $nX$ be a conjugacy class of the elements of order $n$ such that $M \cap nX \neq \emptyset$ and $g \in nX$. Then the following hold,

1. if $C_G(g)$ is conjugate to $M$ and $|S_{g,M} \cap nX| = 1$ then $D(G, M, nX)$ is symmetric and $G$ acts on the blocks primitively,

2. if $|S_{g,M} \cap nX|, |C_G(g)| = |M|$ then $D(G, M, nX)$ is symmetric.

**Proof.** (1) The number of points and blocks in $D(G, M, nX)$ are $\frac{|G|}{|M|}$ and $\frac{|nX|}{|S_{g,M} \cap nX|, |C_G(g)|}$, respectively. Since $C_G(g)$ is conjugate to $M$ then $|C_G(g)| = |M|$ and the number of points and blocks are equal. Also by Proposition 2.1 the action of $G$ on blocks is primitive. (2) In this case clearly the number of points and blocks are equal and $D(G, M, nX)$ is symmetric. 

In Table 2.1, 2.2 and 2.3 the columns from left are: number of row, the considered finite simple group $G$, maximal subgroup $M$ of $G$, a conjugacy class of $G$, properties of the constructed design from $G$, number of the blocks of the design, full automorphism group of the design and symmetric property of the design.

Example 2.1. In Table 2.1 $D(G, M, nX)$ for some finite simple groups was constructed.

By [4] in $L_2(11)$ centralizer of an element of class 2A is maximal subgroup $D_{12}$. The design $D(L_2(11), D_{12}, 2A)$ in row 3 of the table is a 1-(55,7,7) design with 55 blocks then is symmetric and $L_2(11)$ acts on the blocks primitively. This design is an example such that satisfies Lemma 2.1(1). In $G_2(3)$ for maximal subgroup $M = (3^{1+2} \times 3^2) : 2S_4$, consider $D(G_2(3), M, 3A)$. Let $g \in 3A$, in this case $|S_{g,M} \cap 3A| = 2$ and $|C_{G_2(3)}(g)| = \frac{|M|}{2}$ then $D(G_2(3), M, 3A)$ in row 2 of the table is a 1-(364,13,13) symmetric design. This is an example that satisfies Lemma 2.1(2).

Always $t$-designs with $t \geq 2$ are interesting. In two following propositions we consider some conditions to construct $t$-designs for $t \geq 2$. 

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On a Design from the Finite Simple Groups
Table 2.1: $D(G, M, nX)$ for some finite simple groups

| No. | $G$  | $M$                | $nX$          | $D(G, M, nX)$ | No. Blocks | $\text{Aut}(D)$ | Symm |
|-----|------|--------------------|---------------|---------------|------------|-----------------|------|
| 1   | $A_5$| $L_2(8)$ : 3       | 2B            | 1-(120,8,9)   | 36         | $A_5$           | NO   |
| 2   | $G_2(3)$| $(3^{3+7} \times 3^3)$ : $2S_4$ | 3A | 1-(364,13,13) | 364 | $G_2(3)$ | YES |
| 3   | $L_2(11)$| $D_{12}$          | 2A            | 1-(55,7,7)    | 55         | $L_2(11)$ : 2   | YES |
| 4   | $S_6(2)$| $2^2$ : $(2S_3 \times S_3)$ | 2B            | 1-(315,43,43) | 315 | $S_6(2)$ | YES |
| 5   | $L_3(2)$| $2^{1+3} \times 3^{1+3}$ : $2A_4$ | 2A | 1-(166,37,37) | 166 | $L_3(2)$ | YES |
| 6   | $L_4(2)$| $4^3 : S_4$       | 3B            | 1-(63,3,16)   | 336 | $S_6(2)$ | NO |
| 7   | $O_5^+(2)$| $(3 \times U_4(2)) : 2$ | 2A            | 1-(1120,8,360)| 50400 | $O_5^+(2)$ | NO |
| 8   | $O_6^+(2)$| $(3 \times U_4(2)) : 2$ | 2A            | 1-(1120,40,40)| 1120 | $O_6^+(3)$ : 4 | YES |

Proposition 2.3. Let $G$ be a finite simple group, $M$ be a maximal subgroup of $G$ and $\Omega$ be the set of all conjugates of $M$ in $G$. Let $nX$ be a conjugacy class of the elements of order $n$ such that $M \cap nX \neq \emptyset$ and $g \in nX$. Let $m \in \{1, 2, ..., \chi_M(g)\}$. If intersection of every $m$ different conjugates of $M$ has $f \geq 1$ elements of the class $nX$ then $D(G, M, nX)$ is an $m - ([G : M], \chi_M(g), \frac{f}{|S_{g,M} \cap nX|})$ design. Also $G$ is an automorphism group of $D(G, M, nX)$ such that acts primitively on the points and transitively on the blocks of $D(G, M, nX)$.

Proof. Consider an $m$-set of different conjugates of $M$, set $S$ intersection of these subgroups. By the proof of Lemma 2.4(4) we have this partition $S \cap nX = \bigcup_{j=1}^h \{S_{g_j,M} \cap nX\}$. Therefore for each $j \in \{1, 2, ..., h\}$ these $m$ conjugates of $M$ are in a unique block. Then every $m$ conjugates of $M$ appears in exactly $\frac{f}{|S_{g,M} \cap nX|}$ blocks and result is concluded. $\Box$

Example 2.2. In Table 2.2 we consider some finite simple groups in their 1-transitively action such that satisfy conditions of Proposition 2.3.

Table 2.2: Constructed $D(G, M, nX)$ from Proposition 2.3

| No. | $G$  | $M$                | $nX$          | $D(G, M, nX)$ | No. Blocks | $\text{Aut}(D)$ | Symm |
|-----|------|--------------------|---------------|---------------|------------|-----------------|------|
| 1   | $S_4(3)$| $3^{1+7} : 2A_4$  | 3A            | 2-(40,14,4)   | 40         | $L_4(3)$ : 2    | YES |
| 2   | $S_4(4)$| $2^2 : (3 \times A_4)$ | 2A | 2-(85,21,5)   | 85         | $L_4(4)$ : 2    | YES |
| 3   | $S_6(5)$| $5^{1+7} : 4A_4$  | 5A            | 2-(156,31,6)  | 156 | $L_4(5)$ : 4    | YES |
| 4   | $L_2(8)$| $L_{16}$           | 2A            | 2-(28,4,1)    | 63         | $L_2(8)$ : 3    | NO   |
| 5   | $L_2(16)$| $D_{34}$           | 2A            | 2-(120,8,1)   | 255        | $L_2(16)$ : 4   | NO   |

Corollary 2.1. Let $G$ be a finite simple group. Let $M$ be a maximal subgroup of $G$ and $\Omega$ be the set of all conjugates of $M$ in $G$. Let $nX$ be a conjugacy class of the elements of order $n$ such that $M \cap nX \neq \emptyset$ and $g \in nX$. For $t > 1$ let the action of $G$ on $\Omega$ be $t$-transitive. For $m \in \{1, 2, ..., t\}$ consider an $m$-set of different conjugates of $M$ and set $S$ intersection of these subgroups. If $S \neq \{1\}$ and $|S \cap nX| = k \geq 1$ then $D(G, M, nX)$ is an $m - ([G : M], \chi_M(g), \frac{k}{|S_{g,M} \cap nX|})$ design.
and $G$ is an automorphism group of $D(G, M, nX)$ that acts $t$-transitively on the points and transitively on the blocks.

**Proof.** Since $G$ is $t$-transitive on $\Omega$ then intersection of any $m$-set of different conjugates of $M$ is conjugate to $S$ and result is concluded by Proposition 2.3.

**Example 2.3.** In Table 2.3 we consider some finite simple groups in their $2$-transitively action such that satisfy conditions of Corollary 2.1.

| No. | $G$  | $M$      | $nX$   | $D(G, M, nX)$        | Aut($D$) | Symm |
|-----|------|----------|--------|----------------------|----------|------|
| 1   | $L_2(11)$ | $A_5$ | 2A | 2-(11,1,1) | $L_2(11)$ | NO   |
| 2   | $L_3(3)$ | $3^2 \cdot 2S_4$ | 2A | 2-(135,15) | $L_3(3)$ | NO   |
| 3   | $L_3(3)$ | $3^2 \cdot 2S_4$ | 2A | 2-(134,14) | $L_3(3)$ | YES  |
| 4   | $L_3(5)$ | $5^2 \cdot GL_2(5)$ | 2A | 2-(31,7,35) | $L_3(5)$ | NO   |
| 5   | $L_3(5)$ | $5^2 \cdot GL_2(5)$ | 2A | 2-(31,6,1) | $L_3(5)$ | YES  |
| 6   | $S_6(2)$ | $S_4(2)$ | 2A | 2-(28,10,20) | $S_6(2)$ | NO   |

3. **Constructed Designs from Sporadic Groups**

In this section we construct some designs from fourteen sporadic simple groups. For each considered sporadic group, we present one or two designs that their full automorphism groups are as the same sporadic group. These results are presented in Table 3.1. For information on the sporadic simple groups and their maximal subgroups we use Atlas [4].

In Table 3.1 the columns from left are: number of row, group $G$, maximal subgroup $M$ of $G$, a conjugacy class of $G$, properties of the constructed design from $G$, number of the blocks of the design, full automorphism group of the design and symmetric property of the design.

For instance we study properties of the designs in row 15 and 18 of Table 3.1.

The Conway group $Co_3$ has order 495766656000 = $2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$. The group $Co_3$ has forty two conjugacy classes of elements and fourteen conjugacy classes of maximal subgroups. The centralizer subgroup of an elements of class $2A$ in $Co_3$ is maximal subgroup isomorphic to $2.S_6(2)$. The group $McL : 2$ is maximal subgroup of index 276 and $Co_3$ acts 2-transitive on conjugates of $McL : 2$. We consider design $D = D(Co_3, McL : 2, 2A)$. Intersection of every two different maximal subgroups conjugate to $McL : 2$ have 2835 elements of class $2A$ therefore by Corollary 2.1 $D$ is a 2-(276,36,2835) design. The design $D$ has 170775 blocks and since centralizer of an element of the class $2A$ is maximal hence by Proposition 2.1 group $Co_3$ acts primitively on the blocks of this design. Also $Co_3$ acts 2-transitively on the points of $D$. The full automorphism group of $D$ is $Co_3$.  

Table 3.1: Designs constructed from some sporadic groups

| No. | G             | M       | nX          | D(M_{11}, M, nX) | No. Blocks | Aut(D) | Symm |
|-----|---------------|---------|-------------|------------------|------------|--------|------|
| 1   | M_{11}        | L_{2}(11) | 2A         | 3-(12,4,3)      | 165        | M_{11} | NO   |
| 2   | M_{11}        | 2, S_{4} | 2A         | 1-(165,13,13)   | 165        | M_{11} | YES  |
| 3   | M_{12}        | M_{9} : S_{4} | 4A | 1-(220,4,4) | 220        | M_{12} | YES  |
| 4   | M_{22}        | A_{7}    | 4A         | 1-(416,4,310)   | 13860      | M_{22} | NO   |
| 5   | M_{23}        | M_{22}   | 4A         | 1-(443,5,180)   | 28336      | M_{23} | NO   |
| 6   | M_{23}        | L_{3}(4) : 2 | 2A | 1-(253,29,435) | 3795       | M_{23} | NO   |
| 7   | J_{1}         | 2 \times A_{5} | 2A | 1-(1463,31,31) | 1463       | J_{1}  | YES  |
| 8   | J_{1}         | D_{5} \times D_{10} | 2A | 1-(2926,46,23) | 1463       | J_{1}  | NO   |
| 9   | J_{2}         | L_{2}(19) | 2A         | 1-(4988,96,171) | 26163      | J_{2}  | NO   |
| 10  | HS            | U_{3}(5) : 2 | 2B | 2-(176,12,66) | 15400      | HS     | NO   |
| 11  | McL           | M_{22}   | 2A         | 1-(2025,105,1155) | 22275    | McL   | NO   |
| 12  | McL           | 2^{3} : A_{7} | 2A | 1-(22275,435,435) | 22275    | McL   | YES  |
| 13  | He            | 2^{3} : S_{5} | 2A | 1-(29155,651,558) | 24990    | He    | NO   |
| 14  | S_{6}         | U_{6}(2) | 2A         | 1-(32760,252,176) | 22880    | S_{6}  | NO   |
| 15  | S_{6}         | M_{11}   | 2A         | 1-(31671,3511,3511) | 31671    | S_{6}  | YES  |
| 16  | S_{6}         | U_{6}(2) | 2A         | 1-(31671,3511,3511) | 31671    | S_{6}  | YES  |

The Fischer sporadic simple group $F_{i23}$ has order $4089470473293004800 = 2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$. The group $F_{i23}$ has ninety eight conjugacy classes of elements and fourteen conjugacy classes of maximal subgroups. The group $2.F_{i22}$ is maximal subgroup of index 31671 and also is centralizer subgroup of an element of the class $2.A$. By Proposition 2.2(1) $D(F_{i23}, 2.F_{i22}, 2A)$ is a symmetric $1-(31671, 3511, 3511)$ design. The group $F_{i23}$ acts primitively on the points and the blocks of $D(F_{i23}, 2.F_{i22}, 2A)$. The full automorphism group of $D(F_{i23}, 2.F_{i22}, 2A)$ is isomorphic to $F_{i23}$.

4. A 2-design invariant under $M_{22}$

The Mathieu sporadic group $M_{22}$ has order 443520 = $2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$. The group $M_{22}$ has twelve conjugacy classes of elements and eight conjugacy classes of maximal subgroups [4].

Consider permutation representation of $M_{22}$ on 176 conjugates of maximal subgroup $A_7$. The group $M_{22}$ has one conjugacy class of elements of order 3. The centralizer of an element of the class $3.A$ is isomorphic to $3 \times A_4$. Each elements of class $3.A$ is contained in 5 conjugates of maximal subgroup $A_7$.

**Proposition 4.1.** The $D(M_{22}, A_7, 3A)$ is a $2-(176,5,4)$ design with full automorphism group isomorphic to $M_{22}$.

**Proof.** Let $M_{1}$ and $M_{2}$ be two different maximal subgroups isomorphic to $A_7$. Then $M_{1} \cap M_{2}$ is isomorphic to $3^2:4$ or $S_4$. Both subgroups $3^2:4$ and $S_4$ have 8 elements of order 3. Also for each $g \in 3.A, S_{g,A_7}$ is subgroup of order 3. Hence by Proposition...
2.3 $D(M_{22}, A_7, 3A)$ is a 2-(176,5,4) design. The automorphism group is calculated by GAP [14]. □

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Ali Reza Rahimipour
Faculty of Science
Department of Mathematics
University of Qom
P. O. Box 37161-46611
Qom, Iran
a.rahimipour@qom.ac.ir