MATHER MEASURES FOR SPACE-TIME PERIODICAL NONCONVEX HAMILTONIANS

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Abstract. In [G] Diogo Gomes developed techniques and tools with the purpose of extending the Aubry-Mather theory in a stochastic setting, namely he proved the existence of stochastic Mather measures and their properties. These results were also extended in the time-dependent setting in the doctoral thesis the author [GV]. However, to construct analogs to the Aubry-Mather measures for nonconvex Hamiltonians it is necessary to use the adjoint method introduced by Evans [E1] and H. V. Tran [T], the construction of the measures is in [CGT]. The main goal of this paper is to construct Mather measures for space-time periodical nonconvex Hamiltonians using the techniques in [E1], [T] and [CGT].

1. INTRODUCTION.

1.1. The Convex Case. Let $\mathbb{T}^{d+1}$ be the $d+1$-torus and consider a smooth periodic Tonelli Hamiltonian $H : \mathbb{T}^{d+1} \times \mathbb{R}^d \to \mathbb{R}$.

Let $L : \mathbb{T}^{d+1} \times \mathbb{R}^d \to \mathbb{R}$ be the Lagrangian associated to the Hamiltonian:

$$L(x,v,t) = \max_p pv - H(x,p,t),$$

for every $(x,v,t) \in \mathbb{T}^{d+1} \times \mathbb{R}^d$.

Now we consider the corresponding flow of the time dependent Hamiltonian:

$$\begin{align*}
\dot{x} &= D_p H(x,p,t), \\
\dot{p} &= -D_x H(x,p,t).
\end{align*}$$

Now the dynamics transforms to

$$\begin{align*}
\dot{X} &= D\tilde{H}(P), \\
\dot{P} &= 0.
\end{align*}$$

under the change of variables

$$(p,x) \to (P,X)$$

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where \( p = P + D_x u(x, P, t) \), \( X = x + D_P u(x, P, t) \) if we suppose that both \( u(x, P, t) \) and \( \bar{H}(P) \) are smooth functions and \( u(x, P, t) \) satisfies the time dependent Hamilton-Jacobi equation

\[
(4) \quad u_t + H(x, D_x u, t) = \bar{H}(P).
\]

**Definition 1.** A continuous function \( u : \mathbb{T}^{d+1} \to \mathbb{R} \) is called a *forward viscosity solution* of (4) if it satisfies the two properties.

1. If \( v \) is a \( C^1 \) function and \( u - v \) has a local maximum at \( (x, t) \), then
   \[
   v_t + H(x, D_x v(x, t), t) \geq \bar{H}(P),
   \]
2. If \( v \) is a \( C^1 \) function and \( u - v \) has a local minimum at \( (x, t) \), then
   \[
   v_t + H(x, D_x v(x, t), t) \leq \bar{H}(P).
   \]

*Backward viscosity solutions* are defined by reversing both inequalities.

It is known ([CIS], [EG]) that there is only one value \( \bar{H}(P) \), such that (4) has a time periodic viscosity solution.

As a consequence of the semilinearity and convexity there is a consequence map \( \Phi : \mathbb{T}^{d+1} \times \mathbb{R}^{d} \to \mathbb{T}^{d+1} \times \mathbb{R}^{d} \), given by \( \Phi(x, v, t) = (x, D_v L(x, v, t), t) \), which is well defined and one-to-one.

Recall the Poisson bracket,

\[
\{F, G\} := D_p F \cdot D_x G - D_x F \cdot D_p G.
\]

In Hamiltonian coordinates, the property of invariance for a probability measure \( \nu \) can be written as

\[
\int_{\mathbb{T}^{d+1} \times \mathbb{R}^{d}} \phi + \{H, \phi\} d\mu = 0
\]

for every \( \phi \in C^1_c(\mathbb{T}^{d+1} \times \mathbb{R}^{d}) \), where \( \mu = \Phi_# \nu \) is the push-forward of the measure \( \nu \) with respect to the map \( \Phi \), i.e. the measure \( \mu \) such that

\[
\int_{\mathbb{T}^{d+1} \times \mathbb{R}^{d}} \phi(x, p, t) d\mu(x, p, t) = \int_{\mathbb{T}^{d+1} \times \mathbb{R}^{d}} \phi(x, D_v L(x, v, t), t) d\nu(x, v, t)
\]

for every \( \phi \in C^1_c(\mathbb{T}^{d+1} \times \mathbb{R}^{d}) \).

Denoting by \( \mathcal{P}(\mathbb{T}^{d+1} \times \mathbb{R}^{d}) \) the class of probability measures on \( \mathbb{T}^{d+1} \times \mathbb{R}^{d} \), and taking \( \Omega = \mathbb{T}^{d+1} \times \mathbb{R}^{d} \), where \( (x, v, t) = z \) represents a generic point \( z \in \Omega \) with \( (x, t) \in \mathbb{T}^{d+1} \) and \( v \in \mathbb{R}^{d} \).

Now let \( \mathcal{D} \) be the class of probability measures in \( \Omega \) that are invariant under the Euler-Lagrange flow, so we have

\[
\mathcal{D} = \left\{ \nu \in \mathcal{P}(\Omega) : \int_{\Omega} \phi_t + \{H, \phi\} d\Phi_# \nu(x, p, t) = 0 \text{ for every } \phi \in C^1_c(\Omega) \right\},
\]

and the set of holonomic measures

\[
\mathcal{F} = \left\{ \nu \in \mathcal{P}(\Omega) : \int_{\Omega} \psi_t + v \cdot D\psi(x, t) d\nu(x, v, t) = 0, \text{ for every } \psi \in C^1(\mathbb{T}^{d+1}) \right\}.
\]
We recall the Mather problem
\[
\min_{v \in F} \int_{\Omega} L(x,v,t) \, dv(x,v,t),
\]
a more general version of (5) consists in studying for each \( P \in \mathbb{R}^d \) fixed
\[
\min_{\nu \in F} \int_{\Omega} (L(x,v,t) - P \cdot v) \, dv.
\]
Any minimizer of (6) is a Mather measure, now the following proposition will be helpful to prove an important result.

**Proposition 2.** Let \( H : \mathbb{T}^{d+1} \times \mathbb{R}^d \to \mathbb{R} \) be a smooth function that satisfies the classical hypotheses. Let \( P \in \mathbb{R}^d, \nu \in \mathcal{P}(\Omega) \) be a minimizer of (6) and set \( \mu = \Phi_\# \nu \).

(i) \( \mu \) is invariant under the Hamiltonian dynamics, i.e.,
\[
\int_{\Omega} \phi_t + \{H, \phi\} \, d\mu(x,p,t) = 0, \quad \text{for every } \phi \in C^1_c(\Omega)
\]
(ii) \( \mu \) is supported on the graph
\[
\Sigma := \{(x,p,t) \in \mathbb{T}^{d+1} \times \mathbb{R}^d : p = P + D_x u(x,P,t)\}
\]
where \( u \) is any viscosity solution of (4).

The proof of the proposition is a consequence of results in [B] and [CI S].

As in [CGT] the following theorem gives a characterization of Mather measures in the time dependent convex case.

**Theorem 3.** Assume \( H : \mathbb{T}^{d+1} \times \mathbb{R}^d \to \mathbb{R} \) is a smooth function that satisfies the classical hypotheses of convexity, superlinearity, and periodicity and let \( P \in \mathbb{R} \). Then \( \nu \in \mathcal{P}(\Omega) \) is a solution of
\[
-\min_{\nu \in F} \int_{\Omega} (L(x,v,t) - P \cdot v) \, dv(x,v,t),
\]
if and only if
(a) \( \int_{\Omega} \phi_t + H(x,p,t) \, d\mu = \bar{H}(P) = H(x,p,t) \) \( \mu \) a.e.,
(b) \( \int_{\Omega} \phi_t + (p + P) \cdot D_p H(x,p,t) \, d\mu(x,p,t) = 0, \)
(c) \( \int_{\Omega} \phi_t + D_p H(x,p,t) \cdot D\phi(x,p,t) = 0, \quad \text{for every } \phi \in C^1(\mathbb{T}^{d+1}). \)

where \( \mu = \Phi_\# \nu \) and \( \bar{H}(P) \) is the unique value such that (4) has a time periodic viscosity solution.

**Proof.** To simplify, we will assume \( P = 0 \). Let us prove that \( \mu = \Phi_\# \nu \) satisfies (a)-(c). From (ii) of the last proposition, and (i), we have that
\[
\int_{\Omega} \phi_t + H(x,p,t) \, d\mu = \bar{H}(0),
\]
so (a) holds.

Now, we know that
\[ H(x, p, t) = p \cdot D_p H(x, p, t) - L(x, D_p H(x, p, t), t), \]
and from (a) it follows that
\[ \int_{\Omega} \phi_t + p \cdot D_p H(x, p, t) \, d\mu = 0. \]

Finally (c) follows from that \( \nu \in F \).

Reciprocally let \( \mu \in P(\Omega) \) such that (a), (b) and (c) holds, and we will show that \( \nu = \Phi_{\#} \mu \) is a minimizer of (6).

Now observe that \( \nu \in F \), then
\[ \int_{\Omega} \psi_t + v \cdot D\psi(x, t) \, d\nu = \int_{\Omega} \psi_t + D_p H(x, p, t) \cdot D\psi(x, t) = 0, \]
for every \( \psi \in C^1(T^{d+1}) \).

The fact that \( \nu \) is a minimizer is obtained by using (a) and (b)
\[ \bar{H}(0) = \int_{\Omega} \phi_t + H(x, p, t) \, d\mu = \int_{\Omega} \phi_t + p \cdot D_p H - Ld\mu = \int_{\Omega} -Ld\mu. \]

The previous characterization will help us to define Mather measures in the non-convex case.

1.2. The Nonconvex Case. Throughout the paper, we will assume that

i. \( H \) is smooth,

ii. \( H(\cdot, p, t) \) is \( \mathbb{Z}^{d+1} \)-periodic for \((p, t) \in \mathbb{R}^{d+1} \),

iii. There exists a continuous function \( \chi : [0, +\infty) \rightarrow \mathbb{R} \) such that

\[
\int_0^\infty \chi(u)^{-1} \, du = \infty \text{ and } |H(x, p, t)| \leq \chi(|p|). 
\]

Example 1. Consider
\[ H(x, p, t) = |p_1| - |p_2| + V(x, t) \]
If we take \( \chi(u) = 2u + C \),
\[ \int_0^\infty \chi(u)^{-1} \, du = \infty \]
and \( |H(x, p, t)| = ||p_1| - |p_2| + V(x, t)| \leq \chi(|p|) \).

We extend the definition of Mather measure in the nonconvex and time dependent setting:

Definition 4. We say that a measure \( \mu \in \mathcal{P}(\Omega) \) is a Mather measure if there exists \( P \in \mathbb{R}^d \) such that properties (a)-(c) in Theorem 3 are satisfied.
Our main result is:

**Theorem 5.** Assume that the Hamiltonian is a smooth function that satisfies the conditions (i.)-(iii.) and let \( \{ \mu^\varepsilon \}_{\varepsilon > 0} \) be the family of measures defined in (16). Then there exist a Mather measure \( \mu \) and a nonnegative and symmetric \( d \times d \) matrix \((m_{kj})_{k,j=1,...,d}\) of Borel measures called the dissipation measure, such that:

1. \( \mu^\varepsilon \rightharpoonup \mu \) in the sense of measures up to subsequences,
2. \( \int_\Omega \varphi t + \{ H, \varphi \} d\mu + \int_\Omega \varphi p_k p_l d m_{kj} = 0 \) for all \( \varphi \in C_c(\Omega) \),
3. \( \text{supp} \mu \) and \( \text{supp} m \) are compact.

2. Uniform Derivate Bounds

Let us consider the equation:

\[
\phi^\varepsilon_t + \varepsilon \Delta \phi^\varepsilon + H(x, P + D\phi^\varepsilon(x,t), t) = \bar{H}(P).
\]

**Lemma 6.** The periodic solutions of (8) have first derivatives, uniformly bounded in \( \varepsilon \).

**Sketch of the proof.** For every \( \varepsilon > 0 \) let us consider the following problem

\[
\phi^\varepsilon_t + \Delta \phi^\varepsilon + H(x, D\phi^\varepsilon, t) - \varepsilon \phi^\varepsilon = 0
\]

The above equation has a unique smooth solution \( \phi^\varepsilon \) in \( \mathbb{R}^{d+1} \) which is \( \mathbb{Z}^{d+1} \) periodic. First, we proved that \( D\phi^\varepsilon \) is uniformly bounded, by following [BS] we proved that there exists \( K > 0 \) depending only on \( H \) such that

\[
\sup \| D\phi^\varepsilon(\cdot, t) \|_\infty \leq K
\]

Finally if we take \( g = d_t^2 \phi^\varepsilon + |D\phi^\varepsilon|^2 \) and using the Bernstein’s method we prove that \( d_t \phi^\varepsilon \) is uniformly bounded.

\[\Box\]

**Theorem 7.** For every \( \varepsilon > 0 \) and every \( P \in \mathbb{R}^d \), there exists a unique number \( H^\varepsilon(P) \in \mathbb{R} \) such that the equation (8) admits a unique (up to constants) \( \mathbb{Z}^{d+1} \) periodic viscosity solution. Moreover, for every \( P \in \mathbb{R}^d \) \( \lim_{\varepsilon \to 0^+} \bar{H}^\varepsilon(P) \to \bar{H}(P) \) and \( \phi^\varepsilon \to \phi_0 \) uniformly (up to subsequences), where \( \phi_0 : \mathbb{T}^{d+1} \to \mathbb{R} \) is that (4) is satisfied in the viscosity sense.

**Proof.** The theorem follows by Lemma 6 the stability theorem for viscosity solutions and the Arzela-Ascoli Theorem. \[\Box\]

3. Stochastic Measures

**Definition 8.** Let \( \varepsilon > 0 \) and \( P \in \mathbb{R}^d \). The linearized operator associated to (8) is defined as \( L_{\varepsilon,P} : C^2(\mathbb{T}^{d+1}) \to C(\mathbb{T}^{d+1}) \):

\[
L_{\varepsilon,P} \psi = \psi_t + \varepsilon \Delta \psi + D_p H(x, P + D\phi^\varepsilon(x,t), t) D\psi,
\]
for every $\psi \in C^2(\mathbb{T}^{d+1})$.

As in [CGT], we denote by $\beta$ either a direction in $\mathbb{R}^d$ (i.e., $\beta \in \mathbb{R}^d$ with $|\beta| = 1$) or a parameter (for example $\beta = P_i$ for some $i \in \{1, \ldots, d\}$). When $\beta = P_i$ for some $i \in \{1, \ldots, d\}$ the symbols $H_\beta$ and $H_{\beta\beta}$ have to be understood as $H_{p_i}$ and $H_{p_ip_i}$ respectively. If we derive (8) with respect to $\beta$ and recalling (11) we get

$$L_{\epsilon,P}\phi_\beta = \phi_{\beta} + \epsilon\Delta\phi_\beta + D_pH(x,P + D\phi(x,t),t)D\phi_\beta + H_\beta = \tilde{H}_\epsilon,$$

so

$$L_{\epsilon,P}\phi_\beta = \tilde{H}_\epsilon - H_\beta. \tag{12}$$

As before, let $\Omega = \mathbb{T}^{d+1} \times \mathbb{R}^d$, where $(x,v,t)$ represents a generic point with $(x,t) \in \mathbb{T}^{d+1}$ and $v \in \mathbb{R}^d$. We need to introduce a probability space $(\Omega,\mathcal{B},\mathbb{P})$ endowed with a Brownian motion $W(t) : \Omega \to \mathbb{T}^d$ on the flat $d$-torus. Let $\epsilon > 0$, to simplify we set $P = 0$ and we introduce the time dependent vector field $\tilde{F}$, $U_\epsilon(x,t) = D_pH(x,D\phi(x,t),t)$ and consider the solution $X_\epsilon(s)$ of the stochastic differential equation

$$\begin{align*}
\begin{cases}
    dX_\epsilon(s) &= U_\epsilon(X_\epsilon(s),s)ds + \sqrt{2\epsilon}dW(s), \\
    X_\epsilon(t) &= x.
\end{cases}
\end{align*} \tag{13}$$

And the momentum variable is defined as

$$p_\epsilon(t) = D\phi(X_\epsilon(t),t).$$

Now suppose $z : [0, +\infty) \to \mathbb{R}^d$ is a solution to the stochastic differential equation

$$dz_i = a_i \, ds + b_{ij}dW(s)$$

with $a_i$ and $b_{ij}$ bounded and progressively measurable processes. Let $\varphi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ be a smooth function where $\varphi(z,t)$ satisfies the Itô formula:

$$d\varphi = \varphi_z \, dz_i + (\varphi_t + \frac{1}{2}b_{ij}b_{jk} \varphi_{zj}z_i)dt. \tag{14}$$

From hereafter, we will use Einstein’s convention for repeated indices in a sum. Here, we have $a_i = D_pH(x,D\phi(x,t),t)$ and $b_{ij} = \sqrt{2\epsilon}\delta_{ij}$.

Therefore, from (13), (14) and (12),

$$dp_i = \phi_{xixj} [D_pH(X_\epsilon, D\phi(x,t), t)dt + \sqrt{2\epsilon}dW^j] + \phi_{xi,t}(X_\epsilon(x,t) + \epsilon\Delta\phi_{xi} dt,
\begin{align*}
    &= L_{\epsilon,P}\phi_{xi} dt + \sqrt{2\epsilon} \phi_{xixj}dW^j \\
    &= H_{xi} dt + \sqrt{2\epsilon} \phi_{xixj}dW^j.
\end{align*}$$

Thus $(X_\epsilon, p_\epsilon)$ satisfies the following stochastic version of the Hamiltonian dynamics

$$\begin{align*}
\begin{cases}
    dX_\epsilon(s) &= U_\epsilon(X_\epsilon(s),s)ds + \sqrt{2\epsilon}dW(s), \\
    dp_\epsilon(s) &= -DH(X_\epsilon, p_\epsilon, s)ds + \sqrt{2\epsilon} D^2\phi \, dW(s).
\end{cases} \tag{15}
\end{align*}$$
Now we are going to study the solution \( \phi^\varepsilon \) of (15) along the trajectory \( X_\varepsilon(s) \). Due to the Itô formula, and the equations (8) and (13),
\[
d\phi^\varepsilon(X_\varepsilon(s)) = D\phi^\varepsilon dX_\varepsilon + (\phi^\varepsilon_t + \varepsilon \Delta \phi^\varepsilon) ds,
\]
\[
= D\phi^\varepsilon[D_pH(x, D\phi^\varepsilon(x, s), s) ds + \sqrt{2\varepsilon} dW(s)] + (\phi^\varepsilon_t + \varepsilon \Delta \phi^\varepsilon) ds,
\]
\[
= (\phi^\varepsilon_t + \varepsilon \Delta \phi^\varepsilon + D_pH(x, D\phi^\varepsilon(x, s), s) \cdot D\phi^\varepsilon) ds + \sqrt{2\varepsilon} D\phi^\varepsilon dW(s),
\]
\[
= L_{\varepsilon,p}\phi^\varepsilon ds + \sqrt{2\varepsilon} dW(s) = (H_\varepsilon(P) - H + D_pH \cdot D\phi^\varepsilon) ds + \sqrt{2\varepsilon} dW(s).
\]

And using the Dynkin formula, we obtain
\[
E[\phi^\varepsilon(X_\varepsilon(T)) - \phi^\varepsilon(X_\varepsilon(0))] = E\left[ \int_0^T (D_pH(X_\varepsilon, D\phi^\varepsilon(x, t), t) D\phi^\varepsilon + \varepsilon \Delta \phi^\varepsilon) dt \right]
\]
\[
= E\left[ \int_0^T (D_pH(X_\varepsilon, D\phi^\varepsilon(x, t), t) D\phi^\varepsilon + \bar{H}_\varepsilon(P) - H + \phi^\varepsilon_t) dt \right]
\]

Now we will associate to each trajectory \((X_\varepsilon, \rho_\varepsilon, t)\) of (15) a probability measure \(\mu_\varepsilon \in \mathcal{P}(T^{d+1} \times \mathbb{R}^d)\) defined by
\[
\int_{T^{d+1} \times \mathbb{R}^d} \phi(x, p, t) d\mu_\varepsilon(x, p, t) := \lim_{T \to \infty} \frac{1}{T} E\left[ \int_0^T \phi(X_\varepsilon(t), \rho^\varepsilon(t), t) dt \right]
\]
for every \(\phi \in C_c(T^{d+1} \times \mathbb{R}^d)\). Here, the definition makes sense provided the limit is taken over an appropriate subsequence. Then using Dynkin’s formula, we have that for every \(\varphi \in C_c(T^{d+1} \times \mathbb{R}^d)\),
\[
E\left[ \varphi(X^\varepsilon(T), \rho^\varepsilon(T), T) - \varphi(X^\varepsilon(0), \rho^\varepsilon(0), 0) \right]
\]
\[
= E\left[ \int_0^T \varphi_t + (D_x \varphi \cdot D_pH - D_p\varphi \cdot D_x H) dt \right]
\]
\[
+ E\left[ \int_0^T (\varepsilon \varphi_{xx,i} + 2\varepsilon \varphi_{x,i} \varphi_{x,p_j} + \varepsilon \varphi_{x,i} \varphi_{x,p_j} + \varepsilon \varphi_{x,i} \varphi_{x,j} \varphi_{p_kp_j}) dt \right].
\]

Dividing the equation (17) by \(T\) and taking the limit when \(T \to \infty\) along a suitable subsequence we obtain:
\[
\int_{T^{d+1} \times \mathbb{R}^d} \varphi_t + \{H, \varphi\} d\mu_\varepsilon + \int_{T^{d+1} \times \mathbb{R}^d} (\varepsilon \varphi_{xx,i} + 2\varepsilon \varphi_{x,i} \varphi_{x,p_j} + \varepsilon \varphi_{x,i} \varphi_{x,j} \varphi_{p_kp_j}) d\mu_\varepsilon
\]

Let us define the projected measure \(\theta_{\mu_\varepsilon} \in \mathcal{P}(T^{d+1})\) as follows
\[
\int_{T^{d+1}} \varphi(x, t) d\theta_{\mu_\varepsilon}(x, t) := \int_{T^{d+1} \times \mathbb{R}^d} \varphi(x, t) d\mu_\varepsilon(x, p, t)
\]
for all \(\varphi \in C(T^{d+1})\). And using test functions that do not depend on \(p\) in the last definition:
\[
- \int_{T^{d+1}} D_pH \cdot D_x \varphi d\theta_{\mu_\varepsilon} = \int_{T^{d+1}} (\varphi_t + \varepsilon \Delta \varphi) d\theta_{\mu_\varepsilon}
\]
for all \( \varphi \in C^2(\mathbb{T}^{d+1}) \).

Given \( \phi^\varepsilon \), let us consider the partial differential equation

\[
\theta^\varepsilon_t - \varepsilon \Delta \theta^\varepsilon + \text{div}(\theta^\varepsilon \cdot D_p H(x, D\phi^\varepsilon, t)) = 0,
\]

From lemma 32 and lemma 33 in [GV], we have that 0 is the principal value of Fokker-Planck operator

\[
N(\theta^\varepsilon) = -\theta^\varepsilon_t + \Delta \theta^\varepsilon - \text{div}(\theta^\varepsilon \cdot D_p H(x, D\phi^\varepsilon, t))
\]

and so \( \mu^\varepsilon \) can be defined as a unique measure such that

\[
\int_{\mathbb{T}^{d+1} \times \mathbb{R}^d} \psi(x, p, t) d\mu^\varepsilon(x, p, t) = \int_{\mathbb{T}^{d+1}} \psi(x, D\phi^\varepsilon(x, t), t) d\theta^\varepsilon(x, t),
\]

for every \( \psi \in C_c(\mathbb{T}^{d+1} \times \mathbb{R}^d) \).

3.1. Uniform Estimates.

Lemma 9. We have the following estimates:

\[
(20) \quad \varepsilon \int_{\mathbb{T}^{d+1}} |D\phi^\varepsilon_\beta|^2 d\theta^\varepsilon = 2 \int_{\mathbb{T}^{d+1}} \phi^\varepsilon_\beta(H - H_\beta) d\theta^\varepsilon = -2 \int_{\mathbb{T}^{d+1}} \phi^\varepsilon_\beta(H - H_\beta) d\theta^\varepsilon,
\]

\[
(21) \quad \int_{\mathbb{T}^{d+1}} (\bar{H}^\varepsilon_{\beta \beta} - H_{\beta \beta} - 2D_p H_{\beta} \cdot D\phi^\varepsilon_\beta - D_{pp}^2 H D\phi^\varepsilon_\beta \cdot D\phi^\varepsilon_\beta) d\theta^\varepsilon = 0,
\]

\[
(22) \quad \varepsilon \int_{\mathbb{T}^{d+1}} |D\phi^\varepsilon_{\beta \beta}|^2 d\theta^\varepsilon = -2 \int_{\mathbb{T}^{d+1}} \phi^\varepsilon_{\beta \beta}(\bar{H}^\varepsilon_{\beta \beta} - H_{\beta \beta} - 2D_p H_{\beta} \cdot D\phi^\varepsilon_\beta - D_{pp}^2 H : D\phi^\varepsilon_{\beta} \otimes D\phi^\varepsilon_{\beta}) d\theta^\varepsilon.
\]

Proof. Recalling (11), we obtain

\[
L_{\varepsilon, p} |\phi^\varepsilon_\beta|^2 = \frac{\partial}{\partial t} < \phi^\varepsilon_\beta, \phi^\varepsilon_\beta > + \varepsilon \Delta < \phi^\varepsilon_\beta, \phi^\varepsilon_\beta > + D_p H(x, P + D < \phi^\varepsilon_\beta, \phi^\varepsilon_\beta >, t) + H_{\beta} = \bar{H}^\varepsilon_{\beta \beta}
\]

\[
= 2\phi^\varepsilon_\beta \cdot \phi^\varepsilon_\beta + 2\varepsilon D\phi^\varepsilon_\beta \cdot \phi^\varepsilon_\beta + 2\varepsilon \Delta \phi^\varepsilon_\beta \cdot \phi^\varepsilon_\beta + D_p H(x, P + D\phi^\varepsilon(x, t), t) \cdot 2D\phi^\varepsilon_\beta \cdot \phi^\varepsilon_\beta.
\]

Thus

\[
L_{\varepsilon, p} |\phi^\varepsilon_\beta|^2 = 2L_{\varepsilon, p} \phi^\varepsilon_\beta \cdot \phi^\varepsilon_\beta + \varepsilon |D\phi^\varepsilon_\beta|^2 = 2\phi^\varepsilon_\beta(\bar{H}^\varepsilon_{\beta \beta} - H_{\beta \beta}) + \varepsilon |D\phi^\varepsilon_\beta|^2.
\]

Integrating with respect to \( \theta^\varepsilon \) and using (19), we get (20).

To obtain (21) we differentiate (12) with respect to \( \beta \), we have:

\[
L_{\varepsilon, p} \phi^\varepsilon_{\beta \beta} = \bar{H}^\varepsilon_{\beta \beta} - H_{\beta \beta} - 2D_p H_{\beta} \cdot D\phi^\varepsilon_{\beta} - D_{pp}^2 H : D\phi^\varepsilon_{\beta} \otimes D\phi^\varepsilon_{\beta},
\]

Integrating again with respect to \( \theta^\varepsilon \) and using (19) we get (21).

On the other hand

\[
L_{\varepsilon, p} |\phi^\varepsilon_{\beta \beta}|^2 = 2L_{\varepsilon, p} \phi^\varepsilon_{\beta \beta} \cdot \phi^\varepsilon_{\beta \beta} + \varepsilon |D\phi^\varepsilon_{\beta \beta}|^2,
\]

using (23) we obtain

\[
\frac{1}{2} L_{\varepsilon, p} |\phi^\varepsilon_{\beta \beta}|^2 = \phi^\varepsilon_{\beta \beta}(\bar{H}^\varepsilon_{\beta \beta} - H_{\beta \beta} - 2D_p H_{\beta} \cdot D\phi^\varepsilon_{\beta} - D_{pp}^2 H : D\phi^\varepsilon_{\beta} \otimes D\phi^\varepsilon_{\beta}) + \frac{\varepsilon}{2} |D\phi^\varepsilon_{\beta \beta}|^2,
\]
once again, integrating with respect to $\theta_{\mu^\varepsilon}$ and by (19) we get (22).

Following the techniques of [CGT], [E1] and [T], we will obtain several estimates that will be useful in the future.

**Proposition 10.** We have the following

$$\varepsilon \int_{T_\varepsilon} |D_{xx}^2 \phi^\varepsilon|^2 d\theta_{\mu^\varepsilon} \leq C$$

$$\varepsilon \int_{T_\varepsilon} |D_{P_{xx}}^2 \phi^\varepsilon|^2 d\theta_{\mu^\varepsilon} \leq \int_{T_\varepsilon} |D_P \phi^\varepsilon|^2 d\theta_{\mu^\varepsilon} + \int_{T_\varepsilon} |D_P H - D_P \bar{H}^\varepsilon|^2 d\theta_{\mu^\varepsilon}.$$

$$\varepsilon \int_{T_\varepsilon} |D_{xx} \phi^\varepsilon|^2 \leq C(1 + \int_{T_\varepsilon} |D_{xx}^2 \phi^\varepsilon|^3 d\theta_{\mu^\varepsilon}).$$

**Proof.** Taking $\beta = x_1, x_2, \ldots, x_{d+1}$ respectively in (20), we have

$$\varepsilon \int_{T_\varepsilon} |D_{xx} \phi^\varepsilon|^2 d\theta_{\mu^\varepsilon} = 2 \int_{T_\varepsilon} \phi_{x_1}^\varepsilon H_{x_1} d\theta_{\mu^\varepsilon},$$

and adding these $d + 1$ identities we obtain

$$\varepsilon \int_{T_\varepsilon} |D_{xx}^2 \phi^\varepsilon|^2 d\theta_{\mu^\varepsilon} = 2 \int_{T_\varepsilon} D_x \phi^\varepsilon \cdot D_x H d\theta_{\mu^\varepsilon},$$

now, due to Lemma, $|D_x \phi^\varepsilon|, |D_x H|$ are uniformly bounded, thus we get (21).

Now, the relation (25) follows by taking $\beta = P_1, P_2, \ldots, P_{d+1}$ in (20), adding the $d + 1$ identities

$$\varepsilon \int_{T_\varepsilon} |D_{P_{xx}}^2 \phi^\varepsilon|^2 d\theta_{\mu^\varepsilon} = 2 \int_{T_\varepsilon} D_P \phi^\varepsilon [D_P H - D_P H^\varepsilon] d\theta_{\mu^\varepsilon}.$$

And using the Young’s inequality.

To obtain (26), taking $\beta = x_i$ in (22)

$$\varepsilon \int_{T_\varepsilon} |D_{xx} \phi^\varepsilon|^2 d\theta_{\mu^\varepsilon} = 2 \int_{T_\varepsilon} \phi_{x_i}^\varepsilon (H_{x_i} + 2D_P H_{x_i} \cdot D_{xx} \phi^\varepsilon - D_{pp}^2 H : D_{xx} \phi^\varepsilon \otimes D_{xx} \phi^\varepsilon) d\theta_{\mu^\varepsilon}$$

Due to Lemma, we have that $|H_{x_i}^\varepsilon|, |D_{xx} H_{x_i}^\varepsilon|, |D_{pp}^2 H| \leq C$ on the support of $d\theta_{\mu^\varepsilon}$, so

$$\varepsilon \int_{T_\varepsilon} |D_{xx} \phi^\varepsilon|^2 \leq C \left( \int_{T_\varepsilon} |D_{xx}^2 \phi^\varepsilon|^2 d\theta_{\mu^\varepsilon} + \int_{T_\varepsilon} |D_{xx} \phi^\varepsilon|^2 d\theta_{\mu^\varepsilon} + \int_{T_\varepsilon} |D_{xx}^2 \phi^\varepsilon|^3 d\theta_{\mu^\varepsilon} \right)$$

$$\leq C \left( 1 + \int_{T_\varepsilon} |D_{xx}^2 \phi^\varepsilon|^3 d\theta_{\mu^\varepsilon} \right).$$

$\square$
Now we are able to prove the existence of Mather measures.

**Proof of Theorem 5.** The proof straightforward noticing that \( \{ \phi^\varepsilon \} \) have a uniform Lipschitz estimate, therefore there exists a compact set \( K \subset \Omega \) such that \( \text{supp} \mu^\varepsilon \subset K \forall \varepsilon > 0 \). Moreover, up to subsequences, we have \( \mu^\varepsilon \rightharpoonup \mu \), that is
\[
\lim_{\varepsilon \to 0} \int_\Omega \phi \, d\mu^\varepsilon \to \int_\Omega \phi \, d\mu
\]
for every function \( \phi \in C_c(\Omega) \), for some probability measure \( \mu \in \mathcal{P}(\Omega) \), and also it follows that \( \text{supp} \mu \subset K \).

To obtain (2), let us remember (18) particularly the second term
\[
\int_\Omega (\varepsilon \varphi_{x_i}x_i + 2\varepsilon \phi^\varepsilon_{x_i}x_j \varphi_{x_i}p_j + \varepsilon \phi^\varepsilon_{x_i}x_k \phi^\varepsilon_{x_i}x_j \varphi_{x_i}p_kp_j) \, d\mu^\varepsilon
\]
But
\[
\int_\Omega (\varepsilon \varphi_{x_i}x_i + 2\varepsilon \phi^\varepsilon_{x_i}x_j \varphi_{x_i}p_j) \, d\mu^\varepsilon \leq C\varepsilon + C\varepsilon \int_\Omega |\phi^\varepsilon_{x_i}x_j| \, d\mu^\varepsilon
\]
\[
\leq C\varepsilon + C\varepsilon^{1/2}.
\]
by using the estimates in Proposition 10, so the
\[
\lim_{\varepsilon \to 0} \int_\Omega (\varepsilon \varphi_{x_i}x_i + 2\varepsilon \phi^\varepsilon_{x_i}x_j \varphi_{x_i}p_j) \, d\mu^\varepsilon = 0
\]
Note that \( \varepsilon \int_\Omega (\phi^\varepsilon_{x_i}x_k \phi^\varepsilon_{x_i}x_j \varphi_{x_i}p_kp_j) \, d\mu^\varepsilon \) does not vanish in the limit, through a subsequence for every \( k, j = 1, \ldots, n \) we have
\[
\varepsilon \int_\Omega (\phi^\varepsilon_{x_i}x_k \phi^\varepsilon_{x_i}x_j \varphi_{x_i}p_kp_j) \, d\mu^\varepsilon \to \int_\Omega \varphi_{x_i}p_kp_j \, dm_{kj}, \ \forall \varphi \in C_c(\Omega)
\]
for some nonnegative, symmetric \( d \times d \) matrix \( (m_{kj}) \) \( k, j = 1, \ldots, n \) of Borel measures, so condition 2 follows. To obtain (3), recall that \( \sup |\varphi^\varepsilon(t)| < \infty \) and the periodicity in time.

Now we will prove that \( \mu \) satisfies the conditions (a)-(c) in Definition 4 with \( P = 0 \). Following [CGT], [EG], and [T] consider
\[
\int_\Omega (\dot{\phi}^\varepsilon + H(x, p, t) - H^\varepsilon)^2 \, d\mu^\varepsilon = \varepsilon^2 \int_\Omega |\Delta \phi^\varepsilon(x, t)|^2 \, d\mu^\varepsilon \to 0
\]
when \( \varepsilon \to 0 \) due to (8) and (24), thus (a) occurs.

Recalling the equation (18) and choosing as a test function \( \varphi = \psi(\phi^\varepsilon(x, t)) \)
\[
\int_\Omega \psi'(\phi^\varepsilon)[\phi^\varepsilon_t + D_x \phi^\varepsilon \cdot D_p H] \, d\mu^\varepsilon + \varepsilon \int_\Omega \psi'(\phi^\varepsilon)[\phi^\varepsilon_{x_i}x_i + (\phi^\varepsilon_{x_i})^2] \, d\mu^\varepsilon = 0,
\]
if $\varepsilon$ goes to zero, we obtain $\int_{\Omega} \psi'(\phi)[\phi_t + p \cdot D_p H]d\mu = 0$, choosing $\psi(\phi) = \phi$, we obtain b). Now part c) follows by choosing in (2) test functions $\varphi$ that do not depend on the variable $p$. □

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