LOCAL AND GLOBAL ANALYSIS OF EIGENFUNCTIONS ON RIEMANNIAN MANIFOLDS

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Abstract. This is a survey on eigenfunctions of the Laplacian on Riemannian manifolds (mainly compact and without boundary). We discuss both local results obtained by analyzing eigenfunctions on small balls, and global results obtained by wave equation methods. Among the main topics are nodal sets, quantum limits, and $L^p$ norms of global eigenfunctions. The emphasis is on the connection between the behavior of eigenfunctions and the dynamics of the geodesic flow, reflecting the relation between quantum mechanics and the underlying classical mechanics. We also discuss the analytic continuation of eigenfunctions of real analytic Riemannian manifolds $(M, g)$ to the complexification of $M$ and its applications to nodal geometry. Besides eigenfunctions, we also consider quasi-modes and random linear combinations of eigenfunctions with close eigenvalues. Many examples are discussed.

Key Words: Laplacian, eigenvalues and eigenfunctions, quasi-mode, wave equation, frequency function, doubling estimate, nodal set, quantum limit, $L^p$ norm, geodesic flow, quantum complete integrable, ergodic, Anosov, Riemannian random wave.

AMS subject classification: 34L20, 35P20, 35J05, 35L05, 53D25, 58J40, 58J50, 60G60.

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Date: March 23, 2009.
Research partially supported by NSF grant DMS-0603850.
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The aim of this article is to survey both classical and recent results on the eigenfunctions \( \Delta_g \varphi_{\lambda_j} = \lambda_j^2 \varphi_{\lambda_j} \) of the (positive) Laplacian \( \Delta_g \) on a (mainly compact) Riemannian manifold \((M, g)\). We concentrate on the boundaryless case \( \partial M = \emptyset \) for simplicity; when \( \partial M \neq \emptyset \) we impose standard boundary conditions. When \((M, g)\) is compact, the spectrum is discrete and we arrange the eigenvalues in non-decreasing order \( \lambda_0 < \lambda_1 \leq \lambda_2 \uparrow \infty \). We denote by \( \{\varphi_{\lambda_j}\} \) an orthonormal...
basis of eigenfunctions with respect to the inner product $\langle \varphi_{\lambda_j}, \varphi_{\lambda_k} \rangle = \int_M \varphi_{\lambda_j}(x) \varphi_{\lambda_k}(x) dV$. The ‘topography’ of an eigenfunction ideally encompasses the the shapes of pits and peaks in the graph of $\varphi_{\lambda}$, the geometry and connectivity of ‘excursion sets’ $\{x : \varphi_{\lambda_j}(x) > h(\lambda_j)\}$, the $L^p$ norms and distribution function of $\varphi_{\lambda_j}$, the distribution of its nodal sets and other level sets, the number of nodal components of different sizes, the number and distribution of its critical points, and the concentration, oscillation and vanishing order properties of $\varphi_{\lambda_j}$, often encapsulated by the so-called quantum limits (or microlocal defect measures), i.e. limits of quantum expectation values $\langle A \varphi_{\lambda_j}, \varphi_{\lambda_k} \rangle$ as the eigenvalues tend to infinity.

Eigenfunctions of Laplacians arise in physics as modes of periodic vibration of drums and membranes. They also represent stationary states of a free quantum particle on a Riemannian manifold. More generally, eigenfunctions of Schrödinger operators represent stationary energy states of atoms and molecules in quantum mechanics $\text{Sch}$. The topography of modes of vibration and stationary states began with Ernst Chladni $\text{C, C2}$, who raised the prospect of ‘visualizing sound’ by bowing plates and observing the patterns of nodal lines (zero sets) of these modes. Over the last thirty years, chemists and physicists have used computers rather than bowed plates to visualize the energy states of atoms. Some computer graphics of eigenfunctions may be found in such articles as $\text{SHM, H}$. In commemoration of the 200th anniversary of Chladni’s diagrams, a recent volume $\text{SS}$ has appeared which connects his work with that of contemporary physicists, chemists and mathematicians.

In mathematics, studies of eigenfunctions tend to fall into two categories:

- (i) analyses of ground states, i.e. $\varphi_{\lambda_0}$ and $\varphi_{\lambda_1}$;
- (ii) analyses of high frequency (or semi-classical) limits of eigenfunctions, i.e. the limit as $\lambda_j \to \infty$.

Behavior of ground states is very relevant to behavior of highly excited states, since an eigenfunction $\varphi_{\lambda}$ is always the ground state Dirichlet eigenfunction in any of its nodal domains. But our main emphasis in this survey is on the high frequency behavior of eigenfunctions rather than on the ground states.

Studies of high frequency behavior eigenfunctions also fall into two categories:

- Local results, which often hold for any solution of (1) on a (small) ball $B_r(x)$, often $r = O(\lambda^{-1})$, irrespective of whether the eigenfunction extends to a global eigenfunction on $M$. Doubling estimates and vanishing order estimates in terms of the frequency function, exponential decay bounds and nodal volume estimates often fall into this category. These methods often apply to large classes of functions: harmonic functions, polynomials, eigenfunctions, and more general solutions of elliptic equations.
- Global results, which do use this global extension to $M$. The typical global assumption is that $\varphi_\lambda$ is an eigenfunction of the wave group $U_t = e^{it\sqrt{\Delta}}$. Global properties generally reflect the relation of the wave group and geodesic flow, particularly the long time behavior of waves and geodesics on the manifold.

We aim to cover both sides of the subject. There already exists a very well written survey of the local aspects (the book of Q. Han and F.H. Lin $\text{H}$), so we give more details on the global aspects. But one of our purposes in this survey is to state the main local and global results so that the reader can compare approaches. As we have recently written a
survey on inverse spectral problems [Z1], we concentrate on eigenfunctions and do not discuss
eigenvalue asymptotics very much.

The global behavior of eigenfunctions can only be fully understood by making a phase
space analysis, where the phase space is the co-tangent bundle $T^* M$ or an energy surface
$S^*_p M$. For example, one often wishes to construct highly localized eigenfunctions or approxi-
mate eigenfunctions (quasi-modes) of $\Delta_g$ or to prove that they do not exist. Zonal spherical
harmonics on the standard sphere (or any surface of revolution) are examples of eigenfunc-
tions which are most highly localized at a point $p$ (the poles). But it is more illuminating to
observe that such modes are actually concentrating on certain Lagrangian submanifolds in
phase space (namely, the one obtained by flowing out $S^*_p M$ under the geodesic flow). With
this picture in mind, one would not expect to find modes which are highly concentrated at
a point unless there exists a $g^t$ invariant Lagrangian manifold which has a large singularity
over that point, and unless the eigenfunction concentrates in the phase space (microlocal)
sense on this Lagrangian manifold. One similarly expect modes which are extremal for lower
$L^p$ norms to concentrate along elliptic closed geodesics. In the case of integrable systems
(see [12]) one can prove that such expectations are correct. One further expects such special
modes in the integrable case to be extremals for concentration.

To obtain global phase space results relating the behavior of eigenfunctions to the behavior
of geodesics, it is necessary to use microlocal analysis, i.e. the calculus of pseudo-differential
and Fourier integral operators. Microlocal analysis is in part the mathematically precise
formulation of the semi-classical limit in quantum mechanics: Pseudo-differential operators
are ‘quantizations’ $Op(a)$ of functions on the phase space $T^* M$ while Fourier integral oper-
ators are quantizations of Hamiltonian flows (and more general canonical transformations).
To motivate the phase space analysis, consider that one often studies the concentration and
oscillation properties of eigenfunctions through the linear functionals $\rho_{\lambda_j}(A) = \langle A \varphi_{\lambda_j}, \varphi_{\lambda_j} \rangle$
on the space of zeroth order pseudo-differential operators $A$. The possible limits of the family
$\{\rho_{\lambda_j}\}$, known as quantum limits or microlocal defect measures, are probability measures on
$S^* M$ which are invariant under the geodesic flow. It is difficult but important to determine
or at least constrain the possible limits. This aim would not even be visible without a phase
space perspective.

We briefly review some of the basic methods and results of microlocal analysis in §6. Due
to space limitations, we cannot go over the definitions and methods in much detail. For
background in microlocal analysis, we refer to [EZ, DS, GS] and for systematic treatises
to [Ho-IV] (see especially Volumes III and IV) and to [SV] (in particular for boundary
problems). Other articles with expositions of the wave kernel of a Riemannian manifold are
[Be, Su, Ta2, Z4]. Since microlocal methods and results are fundamental to the global
theory, we use them in somewhat detailed arguments in the second half of this survey.
Although the ideas and methods may be unfamiliar to some geometric analysts interested in
eigenfunctions, they are quite geometric and concrete, and in a somewhat less precise form
are common-place among physicists and chemists. While writing this survey, the author
found it interesting to compare the range and power of the local and global methods, and
hopes that experts in each side of the subject will find the other side of the story stimulating.

We would like to thank H. Christianson, Q. Han, H. Hezari, D. Mangoubi, S. Nonnen-
macher, M. Sodin, J. Toth and the referee for corrections on earlier versions of this survey.
1. Basic Definitions and Notations

The Laplacian $\Delta_g$ of a Riemannian Riemannian $(M, g)$ is the self-adjoint operator associated to the Dirichlet form $Q(f) = \int_M |df|^2_g dV_g$ where $dV_g$ is the volume form of $(M, g)$ and $|\cdot|_g$ is the metric on one-forms. It is given in local coordinates by the expression,

\begin{equation}
\Delta = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( g^{ij} \sqrt{g} \frac{\partial}{\partial x_j} \right).
\end{equation}

It is a natural geometric operator in the sense that $\Delta_g$ commutes with all isometries $h$ of $(M, g)$ where $h$ acts on functions by translation $f^h(x) = f(h^{-1}x)$.

The eigenvalue problem (1) is a stationary version of the homogeneous wave equation

\begin{equation}
\Box_g u(t, x) = 0, \quad \text{where} \quad \Box_g = \frac{\partial^2}{\partial t^2} + \Delta_g.
\end{equation}

The eigenvalue problem (1) is equivalent to finding the periodic solutions $e^{i\lambda t} \varphi_\lambda(x)$ in time.

The reader should carefully note two conventions we use throughout this article:

1. We define the Laplacian to be positive in (2). This is opposite to the convention of many authors, but saves us from writing many minus signs in wave and heat operators;
2. We denote eigenvalues by $\lambda^2$ in (1), so that $\lambda$ is the eigenvalue of $\sqrt{\Delta}$. This again saves many square root signs in wave operators. Physicists often denote $\lambda$ by $k$.

1.1. Planck’s constant and eigenvalue asymptotics. The eigenvalue problem for fixed $\lambda$ is a model elliptic equation. A standard fact (elliptic regularity) is that $\varphi_\lambda \in C^\infty(M)$ for any $C^\infty$ metric $g$ and that $\varphi_\lambda \in C^\omega(M)$ (real analytic) when $(M, g)$ is real analytic.

However, it is more illuminating to regard $\lambda$ as the inverse of a semi-classical parameter

$$\lambda = \hbar^{-1},$$

where $\lambda$ is regarded as an ‘operator’ of order 1. Often one writes

$$\lambda^{-2} \Delta - 1 = \hbar^2 \Delta - 1,$$

in particular when constructing approximate eigenfunctions and normal forms.

1.2. Spectral kernels. The individual eigenfunctions are very difficult to study directly. One generally approaches them through various kernel functions, i.e. Schwartz kernels of functions of $\Delta_g$. A basic one is the spectral projections kernel,

\begin{equation}
E(\lambda, x, y) = \sum_{j: \lambda_j \leq \lambda} \varphi_j(x) \varphi_j(y).
\end{equation}

Semi-classical asymptotics is the study of the $\lambda \to \infty$ limit of the spectral data $\{\varphi_j, \lambda_j\}$ or of $E(\lambda, x, y)$. The (Schwartz) kernel of the wave group $U_t = e^{it\sqrt{\Delta}}$ can be represented in terms of the spectral data by

$$U_t(x, y) = U(t, x, y) := \sum_j e^{it\lambda_j} \varphi_j(x) \varphi_j(y),$$

or equivalently as the Fourier transform $\int_{\mathbb{R}} e^{it\lambda} dE(\lambda, x, y)$ of the spectral projections. (Note that we sometimes write $U_t(x, y)$ and sometimes $U(t, x, y)$ for the wave kernel, whichever
Hence spectral asymptotics is often studied through the large time behavior of the wave group. It is more or less equivalent to study the resolvent kernel
\[ G(\lambda, x, y) = \sum_j \frac{\varphi_j(x)\varphi_j(y)}{\lambda^2 - \lambda_j^2} \]
for \( \lambda \in \mathbb{C} \) lying on horizontal or logarithmic curves in \( \mathbb{C} \).

To obtain relations between geometry and eigenfunctions, it is necessary to give approximate formulae for these kernel functions in terms of geometric invariants. Such formulae originate in works of Hadamard and Riesz and are often called Hadamard-Riesz parametrices for manifolds without boundary. Other useful parametrices have been constructed by Lax and Hörmander. The constructions are reviewed in §5, based on the exposition in [D.G, Be]. Another excellent reference is [HoI-IV].

We only discuss wave kernel parametrices for manifolds without boundary. Parametrices for the wave kernel in the boundary case are very difficult due to ‘grazing rays’. For further discussion we refer to [HoI-IV, SV].

1.3. Geodesic flow. The geodesic flow is the Hamiltonian flow of \( |\xi|_g \) on the cotangent bundle \( T^*M \) of \( M \), equipped with its canonical symplectic form \( \sum_i dx_i \wedge d\xi_i \). By definition, \( g^t(x, \xi) = (x_t, \xi_t) \), where \( (x_t, \xi_t) \) is the terminal tangent vector at time \( t \) of the unit speed geodesic starting at \( x \) in the direction \( \xi \). Here and below, we often identify \( T^*M \) with the tangent bundle \( TM \) using the metric to simplify the geometric description. The geodesic flow preserves the energy surfaces \( \{|\xi|_g = E\} \), i.e., the co-sphere bundles \( S_E^*M \). Due to the homogeneity of \( H \), the flow on any energy surface \( \{|\xi|_g = E\} \) is equivalent to that on the co-sphere bundle \( S^*M \).

This definition of the geodesic flow makes \( g^t \) homogeneous of degree one, and is slightly different from the geometer’s definition as the Hamiltonian flow of \( |\xi|^2_g \). The homogeneous geodesic flow is not well-defined on the zero section of \( T^*M \), and so it is punctured out. The punctured cotangent bundle is denoted \( T^*M \setminus 0 \).

1.4. Closed geodesics. By a closed geodesic \( \gamma \) one means a periodic orbit of \( g^t \) on \( S^*_gM \). The period is the length \( L_\gamma \) of \( \gamma \) viewed as a curve in \( M \) (i.e., projected to \( M \)), thus \( g^{L_\gamma}(x, \xi) = (x, \xi) \) for all \( (x, \xi) \in \gamma \). The set of all periodic points \( (x, \xi) \), of all possible periods, is often called ‘the set of closed geodesics’. A basic dichotomy in spectral theory is as follows:

1. Aperiodic; The Liouville measure of the closed orbits of \( g^t \) is zero; or
2. Positive measure of periodic orbits: the Liouville measure of the closed orbits is positive.
3. Periodic : If the entire geodesic flow is periodic, \( g^T = id \) for some \( T > 0 \), the metric is said to be Zoll. The common Morse index of the \( T \)-periodic geodesics will be denoted by \( \beta \).

In the real analytic case, \( (M, g) \) is automatically of type either (1) or (3), since a positive measure of closed geodesics implies that all geodesics are closed. In the \( C^\infty \) case, it is simple to construct examples with a positive but not full measure of closed geodesics (e.g., a pimpled sphere).
1.5. Jacobi fields and linear Poincaré map along a closed geodesic. We recall that Jacobi’s equation for a normal vector field $Y$ along $\gamma$ is $Y'' + R(T,Y)T = 0$ where $T = \dot{\gamma}$ and where $R$ is the curvature tensor. We denote the space of normal Jacobi fields along a closed geodesic $\gamma$ by $\mathcal{J}_\gamma$. It is a symplectic vector space of dimension $2(\dim M - 1)$ with respect to the Wronskian

$$\omega(X,Y) = g(X, \frac{D}{ds}Y) - g(\frac{D}{ds}X, Y).$$

The linear Poincare map $P_\gamma$ is the (real) linear symplectic monodromy map on $(\mathcal{J}_\gamma, \omega)$ defined by $P_\gamma Y(t) = Y(t + L_\gamma)$.

To diagonalize it, we complexify $P_\gamma$ to obtain a linear complex symplectic map on the space $\mathcal{J}_\gamma \otimes \mathbb{C}$ of complex normal Jacobi fields. Since $P_\gamma^c \in Sp(\mathcal{J}_\gamma \otimes \mathbb{C}, \omega)$ (the symplectic group), its spectrum $\sigma(P_\gamma^c)$ is stable under inverse and complex conjugation: thus, if $\rho \in \sigma(P_\gamma^c)$, then also $\rho^{-1}, \bar{\rho}, \bar{\rho}^{-1} \in \sigma(P_\gamma^c)$. The closed geodesic is non-degenerate if

$$\rho_1^{m_1} \ldots \rho_n^{m_n} = 1 \Rightarrow m_i = 0 \quad (\forall i, m_i \in \mathbb{N}).$$

In particular, the eigenvalues are simple and $\pm 1 \notin \sigma(P_\gamma^c)$. It is called elliptic if all eigenvalues of $P_\gamma$ are of modulus one. We then denote them by $\{e^{\pm i\alpha_j}, j = 1, ..., n\}$. Thus, $\{\alpha_1, ..., \alpha_n\}$, together with $\pi$, are independent over $\mathbb{Q}$. The closed geodesic is called hyperbolic if all of the eigenvalues are real. They then come in inverse pairs and we denote them by $\{e^{\pm i\mu_j}, j = 1, ..., n\}$. When $\dim M > 2$, there are mixed hyperbolic and elliptic geodesics, and more general ones where the eigenvalues are complex and not of modulus one. For simplicity of exposition, we only consider elliptic and hyperbolic geodesics.

The associated normalized eigenvectors will be denoted $\{Y_j, \bar{Y}_j, j = 1, ..., n\}$,

$$P_\gamma Y_j = e^{i\alpha_j}Y_j \quad P_\gamma \bar{Y}_j = e^{-i\alpha_j}\bar{Y}_j \quad \omega(Y_j, \bar{Y}_k) = \delta_{jk}$$

and relative to a fixed parallel normal frame $e(s) := (e_1(s), ..., e_n(s))$ along $\gamma$ they will be written in the form $Y_j(s) = \sum_{k=1}^{n} y_{jk}(s)e_k(s)$.

1.6. Geodesic flow as a unitary operator. Since the Hamiltonian flow of $|\xi|_g$ preserves the canonical symplectic form $\omega = \sum_j dx_j \wedge d\xi_j$ of $T^*M$, it preserves the volume form $\omega^m$ where $m = \dim M$. It also preserves the one form $d|\xi|$ and hence it preserves the Liouville form $d\mu_L := \frac{d\omega^m}{d|\xi|}$ on $S^*M$. By definition, the quotient is the unique $2m - 1$ form whose wedge product with $d|\xi|_g$ equals $\omega^m$.

We define the unitary operator $V_t$ on $L^2(S^*M, d\mu_L)$

$$(5) \quad V_t(a) := a \circ g^t.$$ 

It is sometimes called the Koopman operator associated to the geodesic flow.

1.7. Spectrum and geodesic flow. One of the principal emphases in this survey is on the relations between the global dynamics of the geodesic flow and the eigenfunctions of the wave group. These relations have a long tradition in quantum mechanics, and are referred to as semi-classical analysis or study of the classical limit. There exists a large speculative physics literature on relations between eigenfunctions and eigenvalues and the underlying classical Hamiltonian system. Very little is understood, even conjecturally, outside of the following model types of geodesic flow:
• Ergodic or weak mixing geodesic flows, reviewed in §15. More quantitative results hold if the geodesic flow is Anosov (e.g. rates of quantum ergodicity §15; entropy of quantum limits §16).

• Integrable systems, reviewed in §12-14.

• KAM systems, i.e. small perturbations of integrable systems. There are few if any results on eigenfunctions, but there exist results on approximate eigenfunctions or quasi-modes [L, CV2, Pop].

• Special systems with a ‘divided phase space’ possessing an open set of periodic orbits [MOZ].

• Special cases where group theory is available (quotients of reductive, or solvable, or nilpotent groups by discrete subgroups).

1.8. Ergodic, weak mixing and Anosov geodesic flows. Ergodicity and weak mixing are spectral conditions on the geodesic flow. The geodesic flow is called ergodic if the only invariant functions, \( V_t f = f \) with \( f \in L^2(S^*M, d\mu) \) are the constant functions. Equivalently, 1 is an eigenvalue of multiplicity one. It is called weak mixing if the spectrum of \( V_t \) is continuous on the orthogonal complement of the constant functions. Mixing systems add a ‘smoothness’ assumption on the spectral measures and is also a spectral condition.

1.8.1. Examples.

• The most familiar examples are \((M, g)\) of strictly negative sectional curvatures. Their geodesic flows are Anosov. Model examples are compact quotients of hyperbolic space.

• In [BD], many embedded surfaces in \( \mathbb{R}^3 \) with ergodic geodesic flow are described. They may have any genus. There exist real analytic metrics on \( S^2 \) with ergodic geodesic flow.

• Although we do not discuss manifolds with boundary in detail, there are many examples of domains in \( \mathbb{R}^m \) or other Riemannian manifolds with ergodic geodesic flow. The most famous are the Bunimovich stadium (a rectangle with semi-circular ends) and a Sinai billiard table (a rectangle or torus with a disc removed). References to the literature are given in [HZ, ZZw, GL], where ergodicity on the quantum level is studied.

We recall that a geodesic flow \( g^t \) is Anosov if, for each \( \rho \in S^*_g M \), the tangent space \( T_\rho S^*_g M \) splits into \( g^t \) invariant sub-bundles

\[
T_\rho S^*_g M = E^u(\rho) \oplus E^s(\rho) \oplus \mathbb{R} X_H(\rho)
\]

where \( E^u \) is the unstable subspace and \( E^s \) the stable subspace and where \( X_H \) is the Hamiltonian vector field of the function \( H(x, \xi) = |\xi|_g \) on \( T^*M \). The unstable Jacobian \( J^u(\rho) \) at the point \( \rho \) is defined as the Jacobian of the map \( g^{-1} \), restricted to the unstable subspace at the point \( g^t \rho : J^u(\rho) = \det \left( dg^{-1}_{E^u(\rho)} \right) \) (the unstable spaces at \( \rho \) and \( g^t \rho \) are equipped with the induced Riemannian metric).

1.9. Completely integrable geodesic flow. By a (classical) completely integrable system on \( T^*M \) with \( \dim M = n \), we mean a set of \( n \) independent, \( C^\infty \) functions \( p_1, \ldots, p_n \), on \( T^*M \) satisfying:
• \{p_i, p_j\} = 0 \text{ for all } 1 \leq i, j \leq n;
• dp_1 \land dp_2 \land \cdots \land dp_n \neq 0 \text{ on an open dense subset of } T^*M.

The associated moment map is defined by
\begin{equation}
\mathcal{P} = (p_1, \ldots, p_n) : T^*M \to B \subset \mathbb{R}^n.
\end{equation}

We refer to the set \( B \) as the ‘image of the moment map,’ and denote the set of regular values of \( \mathcal{P} \) by \( B_{\text{reg}} \). When \( \mathcal{P} \) is proper and \( b \) is a regular value of \( \mathcal{P} \),
\begin{equation}
\mathcal{P}^{-1}(b) = \Lambda^{(1)}(b) \cup \cdots \cup \Lambda^{(m_{\text{cl}})}(b), \quad (b \in B_{\text{reg}})
\end{equation}
where each \( \Lambda^{(i)}(b) \simeq T^n \) is an \( n \)-dimensional Lagrangian torus. The number \( m_{\text{cl}}(b) = \# \mathcal{P}^{-1}(b) \) of orbits on the level set \( \mathcal{P}^{-1}(b) \) is constant on connected components of \( B_{\text{reg}} \) and the moment map (6) is a fibration over each component with fiber (7).

The Hamiltonians \( p_j \) generate an action of \( \mathbb{R}^n \) defined by
\begin{equation}
\Phi_t = \exp t_1 \Xi_{p_1} \circ \exp t_2 \Xi_{p_2} \cdots \circ \exp t_n \Xi_{p_n}.
\end{equation}

We denote the \( \Phi_t \)-orbits by \( \mathbb{R}^n \cdot (x, \xi) \), and the isotropy group of \( (x, \xi) \) by \( I(x, \xi) \). When \( \mathbb{R}^n \cdot (x, \xi) \) is a compact Lagrangian orbit, then \( I(x, \xi) \) is a lattice of full rank in \( \mathbb{R}^n \), and is known as the ‘period lattice’, since it consists of the ‘times’ \( T \in \mathbb{R}^n \) such that \( \Phi_T |_{\Lambda^{(i)}(b)} = Id \).

By the Liouville-Arnold theorem, the orbits of \( \Phi_t \) are diffeomorphic to \( \mathbb{R}^k \times T^m \) for some \( (k, m), k + m \leq n \). In sufficiently small neighbourhoods \( \Omega^{(i)}(b) \) of each component torus, \( \Lambda^{(i)}(b) \), the Liouville-Arnold theorem also gives the existence of local action-angle variables \( (I_1^{(i)}, \ldots, I_n^{(i)}, \theta_1^{(i)}, \ldots, \theta_n^{(i)}) \) in terms of which the joint flow of \( \Xi_{p_1}, \ldots, \Xi_{p_n} \) is linearized. For convenience, we henceforth normalize the action variables \( I_1^{(i)}, \ldots, I_n^{(i)} \) so that \( I_j^{(i)} = 0; \ j = 1, \ldots, n \) on the torus \( \Lambda^{(i)}(b) \).

Some examples of integrable systems are as follows.

1.9.1. \( T^n = \mathbb{R}^n / \mathbb{Z}^n \). In the case of a flat torus, the \( \xi_j \) variables are action variables and the moment map is the projection \( \mathcal{P} : T^*T^n \to \mathbb{R}^n \). The joint eigenvalues of the action operators \( \hat{I}_j = \frac{\partial}{\partial x_j} \) is the standard integer lattice \( \mathbb{Z}^n \) and rays are multiples of lattice points.

1.9.2. \( S^2 \). The classical action variables are \( I_1 = p_y(x, \xi) = \langle \xi, \frac{\partial}{\partial y} \rangle \), known as the Clairaut integral, and \( I_2 = |\xi|_g \). The moment map \( \mathcal{I} = (I_1, I_2) \) maps \( T^*S^2 \) to the triangular cone \( \{(x, y) : y > 0, |x| \leq y\} \subset \mathbb{R}^2 \).

1.9.3. Simple surfaces of revolution. These are metrics on \( S^2 \) for which the geodesic flow and eigenfunctions are almost as simple as for the standard sphere. The key feature is that they are ‘toric integrable’.

To explain this notion, we assume that the \( S^1 \) action is given by rotations around the \( x_3 \)-axis. We are not primarily interested in this \( S^1 \) action but rather the \( S^1 \times \mathbb{R} \) action on \( T^*M \setminus 0 \) generated by rotations and by the geodesic flow. The basic assumption defining simple surfaces of revolution is that the \( \mathbb{R} \times S^1 \) action simplifies to a \( T^2 \) (2-torus) action on \( T^*S^2 \setminus 0 \). That is, can define two global action variables \( I_1, I_2 \) with \( 2\pi \)-periodic Hamiltonian flows such that \( |\xi|_g = H(I_1, I_2) \). If we write the metric as \( dr^2 + a(r)^2d\theta^2 \) in geodesic polar coordinates centered at a fixed point, which we visualize as the north pole, then a sufficient condition that the geodesic flow be toric is that the distance function to the axis of revolution possesses precisely one local maximum. For instance, a convex surface of revolution is simple.
A simple surface of revolution possesses precisely one \( S^1 \) invariant closed geodesic, which we refer to as the equator.

The first action function is the Clairaut integral generating the \( S^1 \) action, defined by \( p_\theta(v) := \langle v, \frac{\partial}{\partial \theta} \rangle \). To define the second action variable, we need to consider the moment map

\[
\mathcal{P} = (|\xi|_g, p_\theta) : T^* S^2 \to B := \{(b_1, b_2) : |b_2| \leq a(r_\theta)b_1 \} \subset \mathbb{R} \times \mathbb{R}^+
\]

of the Hamiltonian \( \mathbb{R} \times S^1 \)-action defined by the geodesic flow and by rotation. The singular set of \( \mathcal{P} \) is the closed conic set \( Z := \{(r_\theta, \theta, 0, p_\theta) : \theta \in [0, 2\pi), p_\theta \in \mathbb{R} \} \), i.e. \( Z \) is the cone through the equatorial geodesic (in either orientation). The map \( \mathcal{P}|_{T^* S^2 - Z} \) is a trivial \( S^1 \times S^1 \) bundle over the open convex cone \( B_0 \) (the interior of \( B \)), so that \( \mathcal{P}^{-1}(b) \) is an invariant torus for the geodesic flow and for rotation. The second action function is the function \( I_2(\mathcal{P}) \) given by

\[
I_2(b_1, b_2) = \frac{1}{\pi} \int_{r_-(b)}^{r_+(b)} \sqrt{b_1^2 - \frac{b_2^2}{a(r)^2}} dr + |b_2|
\]

where \( r_{\pm}(b) \) (with \( b = (b_1, b_2) \) are the extremal values of \( r \) on the annulus \( \pi \circ \mathcal{P}^{-1}(b) \) (with \( \pi : S^1 \times S^2 \to S^2 \) the standard projection).

These action variables are best thought of in the following classical way (Liouville, Jacobi, Arnold): For each \( b \in B_0 \), let \( H_1(F_b, \mathbb{Z}) \) denote the homology of the fiber \( \mathcal{P}^{-1}(b) \). This lattice bundle is trivial since \( B \) is contractible, so there exists a smoothly varying homology basis \( \{\gamma_1(b), \gamma_2(b)\} \in H_1(\mathcal{P}^{-1}(b), \mathbb{Z}) \) where \( \gamma_1 \) are the orbits of the \( S^1 \) action, and \( \gamma_2 \) is a fixed closed meridian \( \gamma_M \) when \( b \) is on the center line \( \mathbb{R}^+ \cdot (1, 0) \). Then the actions are the integrals of the ‘action form’ over this moving homology basis:

\[
I_1(b) = \int_{\gamma_1(b)} \xi dx = p_\theta, \quad I_2(b) = \int_{\gamma_2(b)} \xi dx = \frac{1}{\pi} \int_{r_-(b)}^{r_+(b)} \sqrt{b_1^2 - \frac{b_2^2}{a(r)^2}} dr + |b_2|.
\]

On the torus of meridians in \( S^1 \times S^2 \), the value of \( I_2 \) equals \( \frac{y}{\pi} \) and it equals one on the equatorial geodesic. So extended, \( I_1, I_2 \) are smooth homogeneous functions of degree 1 on \( T^* S^2 \), and generate \( 2\pi \)-periodic Hamilton flows.

The pair \( \mathcal{I} := (I_1, I_2) \) generate a global Hamiltonian torus \( (S^1 \times S^1) \)-action commuting with the geodesic flow. The singular set of \( \mathcal{I} \) equals \( \mathcal{Z} := \{I_2 = \pm p_\theta\} \), corresponding to the equatorial geodesics. The map

\[
\mathcal{I} : T^* S^2 - \mathcal{Z} \to \Gamma_0 := \{(x, y) \in \mathbb{R} \times \mathbb{R}^+ : |x| < y \}
\]

is a trivial torus fibration.

1.10. Quantum mechanics: Wave group and pseudo-differential operators. The quantization of the Hamiltonian \( |\xi|_g \) is the square root \( \sqrt{\Delta} \) of the positive Laplacian, of \( (M, g) \). It generates the wave

\[
U_t = e^{it\sqrt{\Delta}},
\]

which is a group of Fourier integral operators which propagates singularities along geodesics. In particular, for fixed \( t, x \), the singular support of the wave kernel \( e^{it\sqrt{\Delta}}(t, x, \cdot) \) lies on the distance sphere centered at \( x \) of radius \( t \). This is the wave front of a spherical wave launched at \( x \). Moreover, the wave front propagates outward along geodesics normal to the distance spheres. Thus, the ‘singular directions’ lie along the geodesic rays emanating from \( x \).
Above, \( \sqrt{\Delta} \), resp. \( e^{it\sqrt{\Delta}} \) are defined by the spectral theorem: i.e. they have the same eigenfunctions \( \varphi_\lambda \) as \( \Delta \) on \( (M, g) \) and with eigenvalues \( e^{it\lambda_j} \) resp. \( \lambda_j \).

1.10.1. **Pseudo-differential operators and expected values.** The relations between the wave group and geodesic flow are fundamental to the global analysis and will be discussed at greater length in §6. They originated in Dirac’s Principles of Quantum Mechanics [Dir] as the relations between a classical Hamiltonian flow and its quantization as a unitary group on a Hilbert space. The theory of pseudo-differential and Fourier integral operators is the rigorous mathematical framework for the somewhat heuristic ideas of Dirac’s book. We refer the reader to textbook treatments [DSj, EZ, GSj] and to the comprehensive treatise [HoI-IV] for background.

We denote the class of pseudo-differential operators of order \( m \) on \( M \) by \( \Psi^m \). We fix a quantization \( a \rightarrow \text{Op}^\hbar(a) \) of pseudo-differential operators to functions \( a(x, \xi) \in C^\infty(T^*M \setminus 0) \) which are symbols of order \( m \) in the sense of admitting poly-homogeneous expansions, \( a \sim \sum_{j=0}^\infty a_{m-j} \) where \( a_{m-j} \) is homogeneous of order \( m-j \) on \( T^*M \setminus 0 \). It is the behavior at infinity and not at \( \xi = 0 \) of a symbol which is important and one usually cuts off the homogeneous functions in a small ball around 0. The principal symbol \( \sigma_A \) of \( A = \text{Op}^\hbar(a) \) is the leading homogeneous term \( \sigma_A = a_m \).

On \( \mathbb{R}^n \), \( \text{Op}^\hbar(a) \) is given by the formula

\[
\text{Op}^\hbar(a)f(x) = \int_{\mathbb{R}^n} a(x, \xi)e^{i(x-y, \xi)}f(y)dyd\xi.
\]

Equivalently, \( \text{Op}^\hbar(a) \) is defined by its actions on exponentials,

\[
(9) \quad \text{Op}^\hbar(a)e^{i(x, \xi)} = a(x, \xi)e^{i(x, \xi)}.
\]

Since \( a \) is assumed to be a symbol, the right side is a perturbation expansion with leading term \( a_0(x, \xi)e^{i(x, \xi)} \). When \( a \) is independent of \( \xi \) one has a multiplication operator and when it is independent of \( x \) one has a convolution operator.

We also follow the notation of Dimassi-Sjöstrand [DSj] for operator classes: Given an open \( U \subset \mathbb{R}^n \), we say that \( a(x, \xi; \hbar) \in C^\infty(U \times \mathbb{R}^n) \) is in the symbol class \( S^{m,k}(U \times \mathbb{R}^n) \), provided

\[
|\partial_x^\alpha \partial_\xi^\beta a(x, \xi; \hbar)| \leq C_{\alpha\beta} \hbar^{-m}(1 + |\xi|)^{k-|\beta|}.
\]

We say that \( a \in S_{cl}^{m,k}(U \times \mathbb{R}^n) \) provided there exists an asymptotic expansion:

\[
a(x, \xi; \hbar) \sim \hbar^{-m} \sum_{j=0}^\infty a_j(x, \xi)\hbar^j,
\]

valid for \( |\xi| \geq \frac{1}{\hbar} > 0 \) with \( a_j(x, \xi) \in S^{0,k-j}(U \times \mathbb{R}^n) \) on this set. The associated \( \hbar \) Kohn-Nirenberg quantization is given by

\[
\text{Op}_\hbar(a)(x, y) = (2\pi\hbar)^{-n} \int_{\mathbb{R}^n} e^{i(x-y, \xi)/\hbar} a(x, \xi; \hbar) d\xi.
\]

As is well-known, the definition can be globalized to \( M \) using a partition of unity. We denote this class by \( \text{Op}_\hbar(S^{m,k})(T^*M) \). The symbol of the composition is given by the usual formula:
Given $a \in S^{m_1,k_1}$ and $b \in S^{m_2,k_2}$, the composition $Op_\hbar(a) \circ Op_\hbar(b) = Op_\hbar(c) + \mathcal{O}(\hbar^\infty)$ in $L^2(M)$ where locally,
\[
c(x, \xi; \hbar) \sim \hbar^{-(m_1+m_2)} \sum_{|\alpha|=0}^\infty \frac{(-i\hbar)^{|\alpha|}}{\alpha!} (\partial_\xi^\alpha a) \cdot (\partial_x^\beta b).
\]
For further details, we refer to [DSj].

On a manifold one patches together such local expressions using a partition of unity. There is no unique way to do this, and in fact there is no unique definition of $Op(a)$ on $\mathbb{R}^m$. The one above is the ‘Kohn-Nirenberg’ definition; other natural choices are the Weyl definition and the Friedrichs positive quantization.

On special manifolds, such as symmetric spaces, one may define $Op(a)$ using the Fourier transform of the symmetric space in place of the Euclidean Fourier transform in local coordinates. We will discuss the definition in the case of hyperbolic space.

Pseudo-differential operators greatly enlarge the class of operators which can be used to test eigenfunctions for their properties. One often uses matrix elements $\langle A\varphi_\lambda, \varphi_\lambda \rangle$ for this purpose. The diagonal matrix element is viewed as the ‘expected value of the observable $A$ in the state $\varphi$’ in quantum mechanics. Such quadratic forms in $\varphi_\lambda$ often arise in the study of eigenfunctions, usually with $Op(a)$ being just multiplication by a function, or a gradient operator. It is useful to analyze quadratic expressions of a more general kind.

1.11. **Modes and quasi-modes.** Quasi-modes are approximate eigenfunctions. It is important to discuss quasi-modes along with modes (true eigenfunctions) for the following reasons:

- Most theorems and proofs concerning modes also apply to quasi-modes. There are few techniques that distinguish modes from quasi-modes.
- It is often possible to construct quasi-modes with special properties. In many applications, they are just as useful as modes, and often have clearer geometric properties.

There are several ways to define ‘approximate eigenfunction’. The classical definition of Keller [K], Babich [B] (see also [B.B], Lazutkin [L], Arnold [Ar], Ralston [Ra, Ra2], Guillemin-Weinstein [GW], Colin de Verdière [CV2] (see also Popov [Pop]) and others is that a quasi-mode $\{\psi_k\}$ of order zero is a sequence of $L^2$-normalized functions satisfying
\[
||| \Delta - \mu_k || \psi_k ||_{L^2} = O(1),
\]
for a sequence of quasi-eigenvalues $\mu_k$. By the spectral theorem it follows that there must exist true eigenvalues in the interval $[\mu_k - \delta, \mu_k + \delta]$ for some $\delta > 0$. Moreover, if $E_{k,\delta}$ denotes the spectral projection for the Laplacian corresponding to this interval, then
\[
||| E_{k,\delta} \psi_k - \psi_k ||_{L^2} = O(k^{-1}).
\]

One can refine the definition by demanding that the remainder in (10) is of order $O(\mu_k^{-s})$ and define a quasi-mode of order $s$ by
\[
||| \Delta - \mu_k || \psi_k ||_{L^2} = O(\mu_k^{-s}).
\]

We refer to [CV2, Z9] for other modifications.
In the ‘classical’ work on quasi-modes of Babich, Lazutkin, Arnol’d, Ralston and others, quasi-modes are often constructed as oscillatory integrals (or Lagrangian quasi-modes)

\[ \psi_\lambda(x) = \int_{\mathbb{R}^k} e^{i\lambda S(x,\xi)} a(x, \xi) d\xi \]

with special phases and amplitudes designed so that \(||(-\Delta - \lambda^2)\psi_\lambda||| is small. An important example is the construction of a quasi-mode associated to a stable elliptic orbit, reviewed in \[10\]. Another important quasi-mode of this type is the quasi-mode associated to the central rectangle of a Bunimovich stadium, or more precisely to the Lagrangian cylinder with boundary consisting of unit tangent vectors to the ‘bouncing ball orbits’ in the vertical direction in the rectangle. A recent study of such quasi-modes and of the possible behavior of actual modes is given in the articles [BZ, BZ2] of Burq-Zworski. However, the definition (10) does not force quasi-modes to be Lagrangian and includes ‘random’ combinations of eigenfunctions with eigenvalues in a small interval around \(\lambda^2\), which need not have such an oscillatory integral structure.

An important example of such a quasi-mode is a sequence of “shrinking spectral projections”, i.e. the \(L^2\)-normalized projection kernels

\[ \Phi_j^*(x) = \frac{\chi_{[\lambda_j, \lambda_j+\epsilon_j]}(x, z)}{\sqrt{\chi_{[\lambda_j, \lambda_j+\epsilon_j]}(z, z)}} \]

with second point frozen at a point \(z \in M\) and with width \(\epsilon_j \to 0\). Here, \(\chi_{[\lambda_j, \lambda_j+\epsilon_j]}(x, z)\) is the orthogonal projection onto the sum of the eigenspaces \(V_\lambda\) with \(\lambda \in [\lambda_j, \lambda_j+\epsilon_j]\). The zonal eigenfunctions of a surface of revolution are examples of such shrinking spectral projections for a sufficiently small \(\epsilon_j\), and when \(z\) is a partial focus such \(\Phi_j^*(x)\) are generalizations of zonal eigenfunctions. On a general Zoll manifold, shrinking spectral projections of widths \(\epsilon_j = O(\lambda_j^{-1})\) are the direct analogues of zonal spherical harmonics, and are quasi-modes of order 1.

1.12. **Heuristics and intuitions.** There are several key intuitions to keep in mind from the outset:

- **Local intuition:** Eigenfunctions of eigenvalue \(\lambda^2\) resemble polynomials of degree \(\sim C\lambda\) in terms of their local complexity and growth, e.g. vanishing order at zeros, volumes of nodal hypersurfaces, growth rates on small balls.

- **Global intuition:** Eigenfunctions are stationary states of the quantization \(U(t) = \exp it\sqrt{\Delta}\) of the geodesic flow. Their high-frequency limits \(\lambda \to \infty\) should reflect the dynamics of the classical geodesic flow. When the geodesic flow is integrable, eigenfunctions should localize on the invariant tori (or more correctly, on level sets of the moment map). When the geodesic flow is ergodic, eigenfunctions should be diffuse (i.e. not localize).

- **Modes versus quasi-modes and random waves:** Most results about eigenfunctions apply to quasi-modes, i.e. linear combination of eigenfunctions with very close by eigenvalues. More precisely, when \(|\lambda_j - \lambda_k| \leq \frac{C}{\log \lambda}\). In integrable cases, one can spectrally separate out true eigenfunctions from such ‘random waves’ but in general one cannot. In ergodic cases, eigenfunctions in many respects resemble random waves.
The key difficulty in relating classical to quantum mechanics (e.g. in quantum chaos) is that it involves a comparison between long-time dynamical properties of \( g^t \) and \( U_t \) through the symbol map and similar classical limits. The classical dynamics defines the ‘principal symbol’ behavior of \( U_t \) and the ‘error’ \( U_t AU^*_t - Op(\sigma_A \circ g^t) \) typically grows exponentially in time. This illustrates the ubiquitous ‘exponential barrier’ in the subject. The classical approximation is not clearly valid after the so-called ‘Heisenberg time’ (see [16]). The articles [AI] [AN] [ANK] have excellent discussions of these problems.

1.13. Notational Index.
- \( g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) \) is the inverse matrix to \([g_{ij}]\).
- \( r = r(x, y) \) denotes the distance function of \((M, g)\).
- \( B(x_0, r) \subset M \) denotes the geodesic ball of radius \( r \) centered at \( x_0 \).
- \( dV_g \) denotes the volume density of \((M, g)\).
- \( \Theta(x, y) \) denotes the volume density in normal coordinates at \( x \), i.e. \( dV_g(y) = \Theta(x, y)dy \).
- \( T^*M \) is the cotangent bundle of \( M \), and \( T^*M \setminus \{0\} \) is the puncture cotangent bundle where the zero section is deleted.
- \( |\xi|_g = \sqrt{\sum_{ij=1}^n g^{ij}(x)\xi_i\xi_j} \) is the length of a \((co-)\)vector.
- The unit \((co-)\)ball bundle is denoted \( B^*M = \{(x, \xi) : |\xi|_g \leq 1\} \). Its boundary \( S^*M = \{|\xi|_g = 1\} \) is the unit cosphere bundle.
- \( \mu \) is the Liouville measure on \( S^*M \), i.e. the surface measure \( d\mu = \frac{dx d\xi}{\mu(S^*M)} \) induced by the Hamiltonian \( H = |\xi|_g^2 \) and by the symplectic volume measure \( dx d\xi \) on \( T^*M \).
- \( \omega \) is the linear functional on \( C(S^*M) \) defined by \( \omega(\sigma) = \frac{1}{\mu(S^*M)} \int_{S^*M} \sigma d\mu \). The same notation is used for the functional (state) on the algebra \( \Psi^0(M) \) of zeroth order pseudo-differential operators defined by \( \omega(A) = \omega(\sigma_A) \).
- \( g^t : T^*M \setminus \{0\} \rightarrow T^*M \setminus \{0\} \) denotes the geodesic flow, i.e. the Hamilton flow of \( |\xi|_g^2 \).
- \( \gamma \) denotes a closed geodesic, i.e. closed orbit of \( g^t \) in \( S^*M \). Thus, \( \gamma(t) = g^t(x_0, \xi_0) \in S^*M \) where \( g^L(x_0, \xi_0) = (x_0, \xi_0) \). \( L = L_\gamma \) is the period of the closed geodesic. By abuse of notation, we sometimes also use \( \gamma \) to denote its projection to \( M \), where \( L_\gamma \) is the length of \( \gamma \).
- Geodesic loops versus closed geodesics: Viewed as curves on \( M \) both satisfy \( \alpha(0) = \alpha(L) \), but the latter also satisfy \( \alpha'(0) = \alpha'(L) \).
- \( \text{inj}(M, g) \) denote the injectivity radius.
- \( \Lambda(M) \) denote the \( H^1 \) loopspace of \( M \);
- \( \mathcal{G}(M) \) denote the subset of closed geodesics in \( \Lambda(M) \);
- \( \mathcal{G}_[\gamma] \) denote the set of closed geodesics in \( \mathcal{G}(M) \) whose free homotopy class is \( [\gamma] \);

2. Explicitly solvable eigenfunctions.

There are only a few Riemannian manifolds \((M, g)\) where one has explicit formulae for eigenfunctions. In this section, we briefly review these examples. What they have in common is that in each case \( \Delta_\gamma \) is completely integrable, i.e. commutes with a maximal family of (pseudo-differential operators). The joint eigenfunctions of such quantum integrable Laplacians have very special properties reflecting the complete integrability of the geodesic flow of
\((M, g)\). Our intuition is that such integrable eigenfunctions should be models of *extremal* behavior: for instance, extremal growth and concentration. Quantum integrable eigenfunctions will be discussed in depth in (12). Here, we only wish to go through the simplest examples.

### 2.1. \(\mathbb{R}^n\)

The eigenspaces of the Laplacian on \(\mathbb{R}^n\) are defined by

\[
\mathcal{E}_\lambda = \{ \varphi_\lambda \in \mathcal{S}'(\mathbb{R}^n) : \Delta \varphi_\lambda = \lambda^2 \varphi_\lambda \},
\]

where \(\mathcal{S}'(\mathbb{R}^n)\) is the space of tempered distributions. Since eigenfunctions are \(C^\infty\) the temperedness only constrains the growth to be polynomial, i.e. rules out exponentially growing eigenfunctions such as \(e^{(\lambda x, \xi)}\).

Since the Euclidean motion group \(E_n\) commutes with the flat Laplacian \(\Delta = \Delta_{\mathbb{R}^n}\) it preserves the eigenspaces. Hence the infinitesimal translations \(\frac{\partial}{\partial x_j}\) and the infinitesimal rotations commute with \(\Delta\). The joint complex valued eigenfunctions of the translations are the Euclidean plane waves \(e^{i(\xi, x)}\) with \(\xi \in \mathbb{R}^n\). The eigenspaces \(\mathcal{E}_\lambda\) are infinite dimensional but are spanned in the following sense by the \(e^{i(\xi, x)}\) with \(|\xi| = \lambda\) (i.e. of frequency \(\lambda\)). There is a Poisson type integral formula for eigenfunctions: for any \(\varphi_\lambda \in \mathcal{E}_\lambda\) there exists a distribution \(d\mu \in \mathcal{D}'(S^{n-1})\) such that

\[
\varphi_\lambda(x) = \int_{S^{n-1}} e^{i\lambda(\xi, x)} d\mu(\xi).
\]

We refer to [Hel](#) for further background.

Let \(\varphi_\lambda \in \mathcal{E}_\lambda\). Since \(SO(n)\) acts on \(\mathcal{E}_\lambda\) we may decompose it into isotypic subspaces,

\[
\mathcal{E}_\lambda = \bigoplus_{N=0}^{\infty} \mathcal{E}_\lambda(N),
\]

where \(\mathcal{E}_\lambda(N)\) is the subspace of eigenfunctions transforming by the \(N\)th irreducible representation of \(SO(n)\), realized by the space \(\mathcal{H}_N\) of spherical harmonics of degree \(N\) on \(S^{n-1}\) (see [2, 3]). As this implies, the elements of \(\mathcal{E}_\lambda(N)\) may be expressed as sums of of separation of variable eigenfunctions \(J_{N,n}(\lambda r)\varphi_N(\omega)\) where \(\varphi_N \in \mathcal{H}_N(S^{n-1})\). The radial factor is a temperate solution of the \(n\) dimensional spherical Bessel equation

\[
\left( r^2 \frac{d^2}{dr^2} + (n-1)r \frac{d}{dr} + (r^2 - N(N + n - 1)) \right) J_{N,n}(\lambda r) = 0,
\]

and is given explicitly by

\[
J_{N,n}(\lambda r) = (\lambda r)^{-\frac{n-2}{2}} J_{|N|+\frac{n-2}{2}}(\lambda r) = C_{N+\frac{n-2}{2}}(\lambda r)^N \int_0^\pi \cos(\lambda r \cos \theta) \sin^{2N+n-2} \theta d\theta,
\]

where \(C_{N+\frac{n-2}{2}} = \frac{1}{2^\frac{n-2}{2} \Gamma(\frac{n-2}{2}+\frac{1}{2})\Gamma(\frac{N+\frac{n-2}{2}}{2})}\) and where \(J_\nu\) is the Bessel function of order \(\nu\),

\[
J_{N+\frac{n-2}{2}}(\lambda r) = C_{N+\frac{n-2}{2}}(\lambda r)^{N+\frac{n-2}{2}} \int_0^\pi \cos(\lambda r \cos \theta) \sin^{2N+n-2} \theta d\theta.
\]

The space \(\mathcal{E}_\lambda\) carries an inner product which is invariant under \(E_{n+1}\). An orthonormal basis is given by

\[
u_\alpha(x) = \int_{S^{n-1}} e^{i\lambda(\xi, x)} \chi_\alpha(\xi) dV_0(\xi),
\]
where $\{\chi_\alpha\}$ is an orthonormal basis of spherical harmonics of $L^2(S^{n+1})$. The notation is simplest when $n = 2$: Then an orthonormal basis of $E_\lambda$ is given by $\{e_\lambda^\ell := J_\ell(\lambda r)e^{i\ell \theta}\}_{\ell = -\infty}^\infty$. Equivalently, the norm of $u$ in $E_\lambda$ is the norm of the distribution $d\mu$ in $L^2(S^{n-1})$. We denote by $E_\lambda^{(2)}$ the subspace of elements of finite norm.

In view of the infinite dimensionality of $E_\lambda$, it is possible to construct eigenfunctions with very special properties. First, we consider the orthogonal projection onto $E_\lambda^{(2)}$, which is given by the Bessel kernel

$$ E_\lambda f(x) = \int_{\mathbb{R}^n} J_{n-2}(\lambda|x - y|) f(y) dy. $$

(14)

In the notation of [Hol-IV] (vol III, Chapter XVII), the spectral projection for $\Delta_{\mathbb{R}^n}$ for the spectral interval $[0, \lambda^2]$ is given by

$$ e_0(x, \lambda^2) = (2\pi)^{-n} \int_{|\xi| < \lambda} e^{i(x, \xi)} d\xi. $$

(15)

Hence, the spectral projection onto $E_\lambda$ is given by

$$ \frac{d}{d\lambda} e_0(x, \lambda^2) = (2\pi)^{-n} \int_{|\xi| = \lambda} e^{i(x, \xi)} dS, $$

(16)

where $S$ is the standard surface measure.

If we fix $y$ we obtain a normalized Euclidean coherent state $\varphi^y_\lambda = J_{n-2}(\lambda|x - y|)$. By the standard asymptotics of Bessel’s function, $|\varphi^y_\lambda| \sim (r(\cdot, y)\lambda)^{-1/2}$ as $r(\cdot, y)\lambda \to \infty$. Thus, $\varphi^y_\lambda$ peaks at $x = y$ and decays at a rate $r(x, y)^{-1/2}$ away from the peak point. Also, in the high frequency limit it decays like $\lambda^{-1/2}$ for fixed $x \neq y$.

2.2. Flat tori. A flat torus is a compact quotients $\mathbb{R}^n/L$ where $L$ is lattice such as $\mathbb{Z}^n$. The Laplacian $\Delta$ of the flat metric again commutes with the $n$ vector fields $\frac{\partial}{\partial x_j}$. In the compact case, an orthonormal basis of joint eigenfunctions $\varphi_\lambda$ is provided by the exponentials $e^{i(\lambda, x)}$ where $\lambda \in L^*$, the dual lattice. The corresponding Laplace eigenvalue is $|\lambda|^2$. Usually (for instance when studying nodal sets), we prefer the real orthonormal basis $\sin\langle \lambda, x \rangle, \cos\langle \lambda, x \rangle$. A key feature of these eigenfunctions is that they are linear combinations of a finite number (two) functions of the form $a(x)e^{i\xi(x)/h}$ with $h = |\lambda|^{-1}$. Such functions are known as WKB modes or Lagrangian states. The heuristic scaling (16) is exactly true, with

$$ e^{i(\lambda, x_0 + \frac{x}{|x|})} = e^{i(\lambda, x_0)} e^{i(\frac{\lambda}{|x|}, u)}, $$

i.e. with $dT_\lambda^{x_0} = e^{i(\lambda, x_0)} \delta_\lambda$. The ‘phase function’ $S(x) = \langle x, \frac{\lambda}{|\lambda|} \rangle$ is the generating function of a Lagrangian submanifold $\Lambda = \{(x, \xi = \frac{\lambda}{|\lambda|})\}$, which is a Lagrangian torus in $T^*_g(\mathbb{R}^n/L)$. The eigenfunctions have norm one, $|e^{i(\lambda, x)}| = 1$. We also note that

$$ \langle Op(a)e_\lambda, e_\lambda \rangle = \int_{\mathbb{R}^n/L} a(x, \lambda) dx = \int_{T^*_\lambda} a d\mu_L. $$
2.3. Standard Sphere. Eigenfunctions of the Laplacian $\Delta_{S^n}$ on the standard sphere $S^n$ are restrictions of harmonic homogeneous polynomials on $\mathbb{R}^{n+1}$.

Let $\Delta_{\mathbb{R}^{n+1}} = -(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{n+1}^2})$ denote the Euclidean Laplacian. In polar coordinates $(r, \omega)$ on $\mathbb{R}^{n+1}$, we have $\Delta_{\mathbb{R}^{n+1}} = -\left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\Delta_{S^n}$. A polynomial $P(x) = P(x_1, \ldots, x_{n+1})$ on $\mathbb{R}^{n+1}$ is called:

- homogeneous of degree $k$ if $P(rx) = r^kP(x)$. We denote the space of such polynomials by $\mathcal{P}_k$. A basis is given by the monomials $x^\alpha = x_1^{\alpha_1} \cdots x_{n+1}^{\alpha_{n+1}}$, $|\alpha| = \alpha_1 + \cdots + \alpha_{n+1} = k$.

- Harmonic if $\Delta_{\mathbb{R}^{n+1}} P(x) = 0$. We denote the space of harmonic homogeneous polynomials of degree $k$ by $\mathcal{H}_k$.

Suppose that $P(x)$ is a homogeneous harmonic polynomial of degree $k$ on $\mathbb{R}^{n+1}$. Then,

$$0 = \Delta_{\mathbb{R}^{n+1}} P = -\left\{\frac{\partial^2}{\partial r^2} + \frac{n}{r}\frac{\partial}{\partial r}\right\} r^k P(\omega) + \frac{1}{r^2}\Delta_{S^n} P(\omega)$$

$$\implies \Delta_{S^n} P(\omega) = (k(k-1) + nk) P(\omega).$$

Thus, if we restrict $P(x)$ to the unit sphere $S^n$ we obtain an eigenfunction of eigenvalue $k(n + k) - 1$. Let $\mathcal{H}_k \subset L^2(S^n)$ denote the space of spherical harmonics of degree $k$. Then:

- $L^2(S^n) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$. The sum is orthogonal.
- $Sp(\Delta_{S^n}) = \{\lambda_k^2 = k(n + k - 1)\}$.
- $\dim \mathcal{H}_k$ is given by

$$d_k = \binom{n+k-1}{k} - \binom{n+k-3}{k-2}$$

The Laplacian $\Delta_{S^n}$ is quantum integrable. For simplicity, we restrict to $S^2$. Then the group $SO(2) \subset SO(3)$ of rotations around the $x_3$-axis commutes with the Laplacian. We denote its infinitesimal generator by $L_3 = \frac{\partial}{\partial \theta}$. The standard basis of spherical harmonics is given by the joint eigenfunctions $(|m| \leq k)$

$$\left\{\begin{array}{l}
\Delta_{S^2} Y^k_m = k(k+1) Y^k_m, \\
\frac{\partial}{\partial \theta} Y^k_m = m Y^k_m.
\end{array}\right.$$

Two basic spherical harmonics are:

- The highest weight spherical harmonic $Y^k_k$. As a homogeneous polynomial it is given up to a normalizing constant by $(x_1 + ix_2)^k$ in $\mathbb{R}^3$ with coordinates $(x_1, x_2, x_3)$. It is a ‘Gaussian beam’ along the equator $\{x_3 = 0\}$, and is also a quasi-mode associated to this stable elliptic orbit. These general notions will be discussed in §110.

- The zonal spherical harmonic $Y^k_0$. It may be expressed in terms of the orthogonal projection $\Pi_k : L^2(S^2) \to \mathcal{H}_k$.

We now explain the last statement: For any $n$, the kernel $\Pi_k(x, y)$ of $\Pi_k$ is defined by

$$\Pi_k f(x) = \int_{S^n} \Pi_k(x, y) f(y) dS(y),$$

where $\Pi_k(x, y)$ is the orthogonal projection $\Pi_k : L^2(S^2) \to \mathcal{H}_k$. 

where $dS$ is the standard surface measure. If $\{Y^k_m\}$ is an orthonormal basis of $H_k$ then

$$\Pi_k(x, y) = \sum_{m=1}^{d_k} Y^k_m(x)Y^k_m(y).$$

Thus for each $y$, $\Pi_k(x, y) \in H_k$. We can $L^2$ normalize this function by dividing by the square root of

$$||\Pi_k(\cdot, y)||^2_{L^2} = \int_{S^n} \Pi_k(x, y)\Pi_k(y, x)dS(x) = \Pi_k(y, y).$$

We note that $\Pi_k(y, y) = C_k$ since it is rotationally invariant and $O(n+1)$ acts transitively on $S^n$. Its integral is $\dim H_k$, hence, $\Pi_k(y, y) = \frac{1}{Vol(S^n)} \dim H_k$. Hence the normalized projection kernel with ‘peak’ at $y_0$ is

$$Y^k_0(x) = \frac{\Pi_k(x, y_0)\sqrt{Vol(S^n)}}{\sqrt{\dim H_k}}.$$ 

Here, we put $y_0$ equal to the north pole $(0, 0 \cdots, 1)$. The resulting function is called a zonal spherical harmonic since it is invariant under the group $O(n+1)$ of rotations fixing $y_0$.

One can rotate $Y^k_0(x)\rightarrow Y^k_0(g \cdot x)$ with $g \in O(n+1)$ to place the ‘pole’ or ‘peak point’ at any point in $S^2$.

2.4. Surface of revolution. By a surface of revolution is meant a surface, necessarily $M = S^2$ or $M = \mathbb{R}^2/\mathbb{Z}^2$, whose metric $g$ is invariant under an $S^1$ action by isometries. In this case, one can separate variables to analyze eigenfunctions, i.e. the joint eigenfunctions of $\Delta_g$ and $\frac{\partial}{\partial \theta}$ have the form $e^{in\theta} \varphi_{n,j}(r)$. But this is not necessarily the best way to analyze eigenfunctions. In §12.2.2, we will discuss how to express $\Delta$ in terms of action operators and how to obtain results on eigenfunctions that would be awkward if one used separation of variables. In the case of simple surfaces of revolution, the existence of a ‘global Birkhoff normal form’ for $\Delta_g$ gives much more control than separating variables.

2.5. $H^n$. Much of the discussion of §2.1 has an analogue on hyperbolic space. For the sake of simplicity we assume $n = 2$, so that the group of isometries is $SL(2,\mathbb{R})$. Then the analogue of Euclidean plane waves are the horocyclic (or hyperbolic) plane waves $e^{(i\lambda+1)\langle z,b \rangle}$ where $\langle z, b \rangle$ is the function on $H^2 \times B$ (with $B = S^1$ the ideal boundary of $H^2$) equal to the signed distance from 0 to the horocycle passing through $z$ and $b$. Equivalently,

$$e^{\langle z,b \rangle} = \frac{1 - |z|^2}{|z - b|^2} = P_D(z, b),$$

where $P_D(z, b)$ is the Poisson kernel of the unit disc. (We caution again that $e^{\langle z,b \rangle}$ is written $e^{2(z,b)}$ in [H, Z2]). Helgason defines a non-Euclidean Fourier transform by means of these plane waves, and there exist analogues of the objects and results of §2.1. Further one can define a covariant calculus of pseudo-differential operators by using the non-Euclidean Fourier transform by replacing (9) by

$$(17)\quad Op(a)e^{(i\lambda+1)\langle z,b \rangle} = a(z, b, \lambda)e^{(i\lambda+1)\langle z,b \rangle}.$$
The most interesting aspect of $\mathbb{H}^2$ lies in its quotients by discrete groups $\Gamma \subset SL(2, \mathbb{R})$. Eigenfunctions on the quotient $\mathbb{H}^2/\Gamma$ are the same as automorphic eigenfunctions on $\mathbb{H}^2$ satisfying $\varphi(\gamma z) = \varphi(z)$. Helgason has introduced a generalized Poisson formula for eigenfunctions of exponential growth in the sense that there exists $C > 0$ such that $|\varphi(z)| \leq Ce^{Cd_2(0,z)}$ for all $z$. We assume $\Delta \varphi = \lambda^2 \varphi$ and following a traditional notation put $\lambda^2 = \frac{1}{4} + r^2$. Then in [H], Theorems 4.3 and 4.29, it is proved that there exists a distribution $T_{ir,\varphi_{ir}} \in \mathcal{D}'(B)$ such that

$$\varphi_{ir}(z) = \int_B e^{\left(\frac{1}{2} + ir\right)(z,b)} T_{ir,\varphi_{ir}}(db),$$

for all $z \in D$. The distribution is unique if $\frac{1}{2} + ir \neq 0, -1, -2, \ldots$.

The distribution $T_{ir,\varphi_{ir}}$ is called the boundary value of $\varphi_{ir}$ and may be obtained from $\varphi_{ir}$ in several explicit ways. One is to expand the eigenfunction into the “Fourier series”

$$\varphi_{ir}(z) = \sum_{n \in \mathbb{Z}} a_n \Phi_{r,n}(z),$$

in the disc model in terms of the generalized spherical functions $\Phi_{r,n}$ defined by ([H], Theorem 4.16)

$$e^{\left(\frac{1}{2} + ir\right)(z,b)} = \sum_{n \in \mathbb{Z}} \Phi_{r,n}(z)b^n, \quad b \in B.$$

Then (cf. [H], p. 113)

$$T_{ir,\varphi_{ir}}(db) = \sum_{n \in \mathbb{Z}} a_n b^n |db|.$$

A second way is that, at least when $\text{Re}(ir) > 0$, the boundary value is given by the limit ([H], Theorem 4.27)

$$\lim_{d(0,z) \to \infty} e^{\left(\frac{1}{2} + ir\right)d(0,z)} \varphi_{ir}(z) = c(ir)T_{ir,\varphi_{ir}},$$

where $c$ is the Harish-Chandra $c$-function and $d(0,z)$ is the hyperbolic distance.

In particular, eigenfunctions on the quotient $\mathbb{H}^2/\Gamma$ are generalized Poisson transforms of boundary values. We fix an orthonormal basis $\{\varphi_{ir_j}\}$ of eigenfunctions on $\mathbb{H}^2/\Gamma$ and denote their boundary values by $T_{ir_j}$. As observed in [Z2], when $\varphi_{ir_j}$ is a $\Gamma$-invariant eigenfunction, the boundary values $T_{ir_j}(db)$ have the following invariance property:

$$\varphi_{ir_j}(\gamma z) = \varphi_{ir_j}(z) \quad \Rightarrow \quad e^{\left(\frac{1}{2} + ir_j\right)(\gamma z, \gamma b)} T_{ir_j}(d\gamma b) = e^{\left(\frac{1}{2} + ir_j\right)(z,b)} T_{ir_j}(db)$$

$$\quad \Rightarrow \quad T_{ir_j}(d\gamma b) = e^{-\left(\frac{1}{2} + ir_j\right)(\gamma 0, \gamma b)} T_{ir_j}(db)$$

(22)

This follows from the uniqueness of the Helgason representation and by the identity $\langle \gamma z, \gamma b \rangle = \langle z, b \rangle + \langle \gamma 0, \gamma b \rangle$. Conversely, any distribution with such a $\Gamma$-invariance property defines an eigenfunction on $\mathbb{H}^2/\Gamma$. It follows that the study of eigenfunctions in this setting is equivalent to the study of such boundary values. It is proved in [GO] that the boundary values are derivatives of $C^2$ functions in the co-compact case. Similar results are proved in [MS] with graphs of the boundary values. The boundary values will be discussed again in [15.8] in relation to quantum chaos and in [17] in relation to analytic continuation.
2.6. The Euclidean unit disc $D$. Although our emphasis is on manifolds without boundary, we review the standard orthonormal basis of eigenfunctions for the disc.

The standard orthonormal basis of real valued Neumann eigenfunctions is given in polar coordinates by $\varphi_{m,n}(r,\theta) = C_{m,n} \sin m\theta J_m(j_{m,n} r)$, (resp. $C_{m,n} \cos m\theta J_m(j_{m,n} r)$) where $j_{m,n}$ is the $n$th critical point of the Bessel function $J_m$ and where $C_{m,n}$ is the normalizing constant. The $\Delta$-eigenvalue is $\lambda_{m,n}^2 = (j_{m,n})^2$. The parameter $m$ is referred to as the angular momentum. Dirichlet eigenfunctions have a similar form with $j_{m,n}^D$ replaced by the $n$th zero $j_{m,n}^D$ of $J_m$. Nodal loops correspond to zeros of the radial factor while open nodal lines correspond to zeros of the angular factor.

If we fix $m$ and let $\lambda_{m,n} \to \infty$ we obtain a sequence of eigenfunctions of bounded angular momentum but high energy. At the opposite extreme are the whispering gallery modes which concentrate along the boundary. These are eigenfunctions of maximal angular momentum (with given energy), and $\lambda_m \sim m$. As discussed in [B.B], they are asymptotically given by the real and imaginary parts of $e^{i\lambda_n s} A_i(p^{-1/3} \lambda_{m,n}^{2/3})$. Here, $A_i(y) := A_i(-t_p + y)$ where $A_i$ is the Airy function and $\{-t_p\}$ are its negative zeros. Also, $s$ is arc-length along $\partial D$, $\rho$ is a normalizing constant and $y = 1 - r$.

2.7. An ellipse. Ellipses have several special sequences of eigenfunctions. One is a sequence of eigenfunctions concentrating in a Gaussian fashion along the minor axis. Such eigenfunctions are known as Gaussian beams, or as bouncing ball modes, and quite some effort has gone into their generalizations to more general domains and to manifolds without boundary. Hence, we briefly review the existence of exact eigenfunction of this kind. We will discuss the construction of approximate eigenfunctions of such Gaussian beams in [10].

One can separate variables in special coordinates on the ellipse and obtain exact, if rather esoteric, formulae for eigenfunctions of the Dirichlet or Neumann problem. We express an ellipse in the form $x^2 + \frac{y^2}{1-a^2} = 1$, $0 < a < 1$, with foci at $(x, y) = (\pm a, 0)$. We define elliptical coordinates $(\varphi, \rho)$ by $(x, y) = (a \cos \varphi \cosh \rho, a \sin \varphi \sinh \rho)$. Here, $0 \leq \rho \leq \rho_{\text{max}} = \cosh^{-1} a^{-1}$, $0 \leq \varphi \leq 2\pi$. The lines $\rho = \text{const}$ are confocal ellipses and the lines $\varphi = \text{const}$ are confocal hyperbolae. The foci occur at $\varphi = 0, \pi$ while the origin occurs at $\rho = 0, \varphi = \pm \frac{\pi}{2}$.

The eigenvalue problem separates into a pair of Mathieu equations,

$$\begin{align*}
\partial^2_{\varphi} G_{m,n} - c^2 \cos^2 \varphi G_{m,n} &= -\lambda_{m,n}^2 G_{m,n} \\
\partial^2_{\rho} F_{m,n} - c^2 \cosh^2 \rho F_{m,n} &= \lambda_{m,n}^2 F_{m,n}
\end{align*}$$

where $c$ is a certain parameter. The eigenfunctions have the form $\Psi_{m,n}(\varphi, \rho) = C_{m,n} F_{m,n}(\rho) \cdot G_{m,n}(\varphi)$ where, $F_{m,n}(\rho) = C e_m(\rho, \frac{k_n c}{2})$ and $G_{m,n}(\varphi) = c e_m(\varphi, \frac{k_n c}{2})$ (and their sin analogues). Here, $c e_m, C e_m$ are special Mathieu functions (cf. [C] (3.10)-(3.2)). The Neumann or Dirichlet boundary conditions determine the eigenvalue parameters $k_n c$.

There exists a special sequence of eigenfunctions which are like Gaussian beams along the minor axis. For this special sequence, $G = G_{m,n}$ are asymptotic to ground state Hermite functions. More precisely,

$$G_{m,n}(\varphi, \lambda_{m,n}) = c_{m,n}(\lambda_{m,n}) e^{-\lambda_{m,n} \cos^2 \varphi} (1 + O(\lambda_{m,n}^{-1})),$$

while

$$F_{m,n}(\rho, \lambda_{m,n}) = e^{i\lambda_{m,n} \int_0^\rho \sqrt{\cosh^2 x + 1} dx} a_+(\rho; \lambda_{m,n}) + e^{-i\lambda_{m,n} \int_0^\rho \sqrt{\cosh^2 x + 1} dx} a_-(\rho; \lambda_{m,n})$$
where \( a_\pm(\rho; \lambda_{m,n}) \sim \sum_{j=0}^\infty a_{\pm,j}(\rho)\lambda_{m,n}^{-j} \) are determined by the Dirichlet or Neumann boundary conditions. Moreover, from the \( L^2 \)-normalization condition \( \int \left| \Psi_{m,n}(\rho, \frac{\pi}{2}) \right|^2 d\rho = 1 \) it follows that \( c_{m,n}(\lambda_{m,n}) \sim \lambda_{m,n}^{1/4} \).

From (24) and (25), the Gaussian beams are roughly asymptotic to superpositions of \( e^{\pm is} e^{-\lambda_{m,n}y^2} \) (cf. [B.B]), where \( s \) denotes arc-length along the bouncing ball orbit and \( y \) denotes the Fermi normal coordinate. It follows that outside a tube of any given radius \( \epsilon > 0 \), the Gaussian beam decays on the order \( O(e^{-\lambda_{m,n}\epsilon^2}) \).

Before leaving this example, we should point out two further interesting sequences of eigenfunctions. As in the case of the disc, there exists a ‘whispering gallery’ sequence which concentrates on the boundary of the domain. As with Gaussian beams, there exists a generalization of this sequence to any convex smooth domain in the form of ‘quasi-modes’. A second interesting sequence becomes highly enhanced at the two foci.

3. Local behavior of eigenfunctions

By the local behavior of eigenfunctions, we often mean methods and results which pertain to all solutions of \( \Delta u = \lambda^2 u \) on a ball \( B(x_0, R) \), not just to solutions which extend to global eigenfunctions on \((M, g)\). More generally, we consider results which are obtained from a small ball analysis and which use covering arguments to draw global conclusions from local arguments. But we emphasize that some aspects of eigenfunctions discussed in this section are truly global, for instance the estimate of the frequency function in terms of the eigenvalue. Thus, we do not aim to segregate local from global results in this section, but do try to indicate when a result assumes that the eigenfunction is global.

Local properties include:

- Frequency function estimates; we also include Carleman estimates, although they often require integration of global eigenfunctions over all of \( M \);
- Vanishing order estimates at points;
- Doubling estimates;
- Bernstein estimates;
- Lower bounds on masses in small balls.
- Local structure of nodal sets.

For background and references on general elliptic estimates, see [GiTr, HL].

3.1. Eigenfunctions and harmonic functions on a cone. One can easily convert eigenfunctions \( \varphi_\lambda \) to harmonic functions in a space of one higher dimension. Two closely related options are:

- Form the cone \( \mathbb{R}^+ \times M \) and consider the metric \( \dot{g} = dr^2 + r^2 g \). Let \( \dot{\varphi}_\lambda = r^\alpha u \) where
  \[
  \alpha = \frac{1}{2} \left( \sqrt{4\lambda + (n-1)^2} - (n-1) \right).
  \]

  Let \( \dot{\Delta} \) be the Laplacian on the cone. Then,
  \[
  \dot{\Delta} \dot{\varphi}_\lambda = 0.
  \]

- Form \( \mathbb{R}^+ \times M \) and consider \( e^{\lambda t} \varphi_\lambda \). Then \( (\partial_t^2 + \Delta)(e^{-\lambda t} \varphi_\lambda) = 0 \).
This approach was first taken in [GaL, Lin] and used further in [Ku, CM], among other places. It is a useful approach when the frequency function is employed, since the latter has its best properties for harmonic functions. In [CM] it allowed for the use of Harnack inequalities on balls of $\mathbb{R}_+ \times M$ where $\varphi_\lambda(x)$ is positive.

### 3.2. Frequency function.

The frequency function $N(a, r)$ of a function $u$ is a local measure of its 'degree' as a polynomial like function in $B_r(a)$. More precisely, it controls the local growth rate of $u$. In the case of harmonic functions, it is given by

$$N(a, r) = \frac{rD(a, r)}{H(a, r)},$$

where

$$H(a, r) = \int_{\partial B_r(a)} u^2 d\sigma, \quad D(a, r) = \int_{B_r(a)} |\nabla u|^2 dx.$$

A well-written detailed treatment of the frequency function and its applications can be found in [H, Ku], following the original treatments in [GaL, GaL2, Lin].

To motivate the frequency function, let us calculate it in the special case of a global harmonic function $u$ on $\mathbb{R}^n$. Then $u$ may be decomposed into a sum of homogeneous components of non-negative integral order:

$$u = \sum_{N=0}^\infty u_N, \quad u_N(r\omega) = a_N r^N \varphi_N(\omega).$$

Then $\varphi_N$ is a spherical harmonic of degree $N$ on $S^{n-1}$, and we choose $a_N$ so that its $L^2$ norm is equal to one. From the fact that the $\varphi_N$ are orthogonal on $S^{n-1}$, one easily calculates that

$$N(r) = \sum_{N=0}^\infty N |a_N|^2 r^{2N} = \frac{d}{2d \log r} \log \sum_{N=0}^\infty |a_N|^2 r^{2N},$$

i.e.

$$\frac{d}{dr} \left( \log \frac{H(r)}{r^{n-1}} \right) = 2 \frac{D(r)}{H(r)}.$$

Here, one also uses that $\int_{B_r} |\nabla u|^2 = \int_{\partial B_r} u \partial_\nu u$, where $\partial_\nu$ is the unit normal. From the formula it is immediate that if $u$ is homogeneous of degree $N$, i.e. has only one component $a_N r^N \varphi_N$, then $N(r)$ is the constant function $N(r) \equiv N$ equal to its degree.

We note that $N(r)$ is analogous to the average energy of a statistical ensemble with partition function $Z(r) = \sum_{N=0}^\infty |a_N|^2 r^N$. We recall a partition function for the canonical ensemble at temperature $\beta$ is given by $Z(\beta) = \int e^{-\beta E} d\omega(E)$ where $\omega(E)$ is the density of states. The average energy is then $\langle E \rangle = -\frac{d}{d\beta} \log Z(\beta)$. Monotonicity of the average energy follows from the fact its derivative $\langle (E - \langle E \rangle)^2 \rangle = -\frac{d^2}{d\beta^2} \log Z(\beta)$ is the variance of the energy.

Frequency functions may also be defined for eigenfunctions. At least two variations have been studied: (i) where the eigenfunctions are converted into harmonic functions on the cone $\mathbb{R}^+ \times M$ as in §3.1; (ii) where a frequency function adapted to eigenfunctions on $M$ is defined.

We first consider method (i) in the case of an eigenfunction $\varphi_\lambda$ on $S^n$. The associated harmonic function on the cone is precisely the homogeneous harmonic polynomial on $\mathbb{R}^{n+1}$
which restricted to $S^n$ gives $\varphi_\lambda$. By the previous calculation, $N(0, r) \equiv N$ where $\lambda = N(N + n - 1)$. We note that on the cone $\mathbb{R}^{n+1}$, the ball of radius $r$ has the form $[0, r] \times S^n$, i.e. the frequency function is global on $S^n$. On a general manifold, the analogous global calculation is cleanest if we define $N(r)$ with $B_r(0)$ everywhere replaced by $[0, r] \times M$. If we ‘harmonize’ an eigenfunction $\varphi_\lambda \rightarrow r^\alpha \varphi_\lambda$ as in §3.1, we obtain $N(r) \equiv \alpha$.

The second method is to define a frequency function on balls of $M$ itself. The generalization to eigenfunctions (1) is as follows (see [GaL, GaL2, Ku]). Fix a point $a \in M$ and choose geodesic normal coordinates centered at $a$ so that $a = 0$. Put $\mu(x) = g^{ij} x_i x_j |x|^2$, and put

$$D(a, r) := \int_{B_r} \left( g^{ij} \frac{\partial \varphi_\lambda}{\partial x_i} \frac{\partial \varphi_\lambda}{\partial x_j} + \lambda^2 \varphi_\lambda^2 \right) dV, \quad \text{resp.} \quad H(a, r) := \int_{\partial B_r} \mu \varphi_\lambda^2.$$  

By the divergence theorem, one has

$$D(a, r) = \int_{\partial B_r} \varphi_\lambda \frac{\partial \varphi_\lambda}{\partial \nu}.$$  

Define the frequency function of $\varphi_\lambda$ by

$$N(a, r) := \frac{rD(r)}{H(r)}.$$  

As in the case of harmonic functions, the main properties of the frequency function of an eigenfunction are a certain monotonicity in $r$ in small balls of radius $O(\frac{1}{\lambda})$ (see Theorem 3.1) and the fact that $N(a, r)$ is commensurate with $N(b, r)$ when $a$ and $b$ are close.

Simple examples show that, despite its name, the frequency function measures local growth but not frequency of oscillations of eigenfunctions, and therefore is not necessarily comparable to $\lambda$. For instance, the frequency function of $\sin nx$ in a ball or radius $\leq \frac{1}{|n|}$ is bounded. An example considered in [DF] are the global eigenfunctions $e^{sx} \sin ty$ on $\mathbb{R}^2$ of $\Delta$-eigenvalue $s^2 - t^2$ and frequency function of size $s$. One could let $s, t \rightarrow \infty$ with $s^2 - t^2$ bounded and obtain a high frequency function but a low eigenvalue. However, as discussed in [L] (p. 291), if one forms the harmonic function from a global eigenfunction as in §3.1, then one has $N(0, 2) \leq C\lambda$ where $C$ depends only on the metric. This is a global estimate since a ball centered at 0 in the cone will cover all of $M$. An application is given in Theorem 3.2.

Let us work out the frequency function for a global eigenfunction on $\mathbb{R}^n$, parallel to the discussion above for harmonic functions. We use the notation of §2.1. Since the tempered solution is unique up to constant multiples, we have $\mathcal{E}_\lambda(N) \simeq \mathcal{H}_N(S^{n-1})$, as with harmonic functions. In the notation §2.1 of may therefore write

$$\varphi_\lambda(r\omega) = \sum_{N=0}^{\infty} a_N J_{N,n}(\lambda r) \varphi_N(\omega),$$

where as before $||\varphi_N||_{L^2(S^{n-1})} = 1$. Then,

$$rD(0, r) = \lambda r^n \sum_{N=0}^{\infty} |a_N|^2 J_{N,n}'(\lambda r), \quad H(0, r) = r^{n-1} \sum_{N=0}^{\infty} |a_N|^2 (J_{N,n})^2(\lambda r),$$
The monotonicity of $N$ and $r$

There exists a partition function $Z(r) = \sum_{N=0}^{\infty} |a_N|^2 (J_{N,n})^2(\lambda r)$ which no longer has the canonical form (i.e., it involves Bessel functions rather than powers). Unlike $r^n$, Bessel functions oscillate on the scale $\frac{1}{r}$ and hence the frequency function is generally not monotone decreasing in $r$; but it is monotone for $r$ in a small interval $[0, r_0(\lambda)]$ with $r_0(\lambda) = \frac{A}{b}$ for some $a > 0$. The following weaker monotonicity result is however good enough for applications to vanishing order and doubling estimates.

**Theorem 3.1.** Theorem 2.3 of [Gal] (see also [Gal2, Lin, H] and [Ku] (Th. 2.3, 2.4): There exists $C > 0$ such that $e^{Cr}(N(r) + \lambda^2 + 1)$ is a non-decreasing function of $r$ in some interval $[0, r_0(\lambda)]$.

Another basic fact is that the frequency of $\varphi_{\lambda}$ in $B_{a}(a)$ is comparable to its frequency in $B_{R}(b)$ if $a, b$ are close and $r, R$ are close. More precisely, there exists $N_0(R) << 1$ such that if $N(0, 1) \leq N_0(R)$, then $\varphi_{\lambda}$ does not vanish in $B_{R}$, while if $N(0, 1) \geq N_0(R)$, then

$$N(p, \frac{1}{2}(1 - R)) \leq C N(0, 1), \quad \forall p \in B_{R}.$$  

3.3. **Doubling estimate, vanishing order estimate and lower bound estimate.** Doubling estimates and vanishing order estimates give quantitative versions of unique continuation theorems. Given a partial differential operator $L$ and a solution $u$ of $Lu = 0$, the unique continuation problems asks whether $u$ is uniquely determined in a ball $B$ by its values in a smaller set $E \subset B$? That is, if $u \equiv 0$ in $E$, must $u \equiv 0$ in $B$? In the limiting case where $K = \{x_0\}$ is a point, if $u$ vanishes to high enough order $k$ at $x_0$ must $u \equiv 0$? A related question is the ‘proximity to zero’ of $u$ in the sense of Nevanlinna theory, i.e., how small can $\sup_{x \in E} |u(x)|$ be in a ball $B$ (see [3,4]? These questions are answered by frequency function estimates and by Carleman estimates.

An early doubling inequalities was proved by Bernstein for polynomials of one variable:

$$\max_{-R \leq x \leq R} |p_N(x)| \leq R^N \max_{-1 \leq x \leq 1} |p_N(x)|$$

for any polynomial of degree $N$. A generalization known as Remez’s inequality allows one to compare the growth on an interval to growth on any measurable set in the interior. Let $\sigma \in \mathbb{R}_+$ and denote by $P_N(\sigma)$ to be those polynomials $p_N$ of degree $N$ such that $||p_N|| \leq 1$ for some subset $E \subset [-1, 1 + \sigma]$. Then

$$\sup_{p_N \in P_N(\sigma)} \|p\|_{[-1,1+\sigma]} \leq \|T_N\|_{[-1,1+\sigma]}.$$  

Here, $\|f\|_E = \sup_{x \in E} |f(x)|$ and $T_N$ is Tchebychev’s polynomial.

In the case of harmonic functions on $\mathbb{R}^n$ one may set

$$\bar{H}(a, r) = \frac{H(a, r)}{r^{n-1}}.$$  

The monotonicity of $N(a, r)$ immediately implies the doubling formula

$$\bar{H}(a, 2R) = \bar{H}(a, R) \exp \left( \int_R^{2R} \frac{2N(a, r)}{r} dr \right) \leq 4^{N(a,1-a)} \bar{H}(a, R).$$
Integrating in $R$ and using (32) gives

$$
\frac{1}{\text{Vol}(B_{2r}(a))} \int_{B_{2r}(a)} u^2 dx \leq C(n) 4^{2N(0,1)} \frac{1}{\text{Vol}(B_{2r}(a))} \int_{B_{2r}(a)} u^2.
$$

As mentioned in §3.2, the frequency function of a global eigenfunction may be estimated in terms of the eigenvalue.

**Theorem 3.2.** [DF; Lin] and [H] (Lemma 6.1.1) Let $\varphi_\lambda$ be a global eigenfunction of a $C^\infty (M, g)$ there exists $C = C(M, g)$ and $r_0$ such that for $0 < r < r_0$,

$$
\frac{1}{\text{Vol}(B_{2r}(a))} \int_{B_{2r}(a)} |\varphi_\lambda|^2 dV_g \leq e^{C\lambda} \frac{1}{\text{Vol}(B_r(a))} \int_{B_r(a)} |\varphi_\lambda|^2 dV_g.
$$

Further,

$$
\max_{B(p, r)} |\varphi_\lambda(x)| \leq \left( \frac{r}{r'} \right)^{C\lambda} \max_{x \in B(p, r')} |\varphi_\lambda(x)|, \ (0 < r' < r).
$$

The doubling estimates imply the vanishing order estimates. Let $a \in M$ and suppose that $u(a) = 0$. By the vanishing order $\nu(u, a)$ of $u$ at $a$ is meant the largest positive integer such that $D^n u(a) = 0$ for all $|a| \leq \nu$. The vanishing order of an eigenfunction at each zero is of course finite since eigenfunctions cannot vanish to infinite order without being identically zero. The following estimate is a quantitative version of this fact.

**Theorem 3.3.** (see [DF]; Lin Proposition 1.2 and Corollary 1.4; and [H] Theorem 2.1.8.) Suppose that $M$ is compact and of dimension $n$. Then there exist constants $C(n), C_2(n)$ depending only on the dimension such that the vanishing order $\nu(u, a)$ of $u$ at $a \in M$ satisfies $\nu(u, a) \leq C(n) N(0, 1) + C_2(n)$ for all $a \in B_{1/4}(0)$. In the case of a global eigenfunction, $\nu(\varphi_\lambda, a) \leq C(M, g)\lambda$.

In the case of harmonic functions, one may write $u = P_\nu + \psi_\nu$ where $P_\nu$ is a homogeneous harmonic polynomial of degree $\nu$ and where $\psi_\nu$ vanishes to order $\nu + 1$ at $a$. We note that highest weight spherical harmonics $C_n(x_1 + ix_2)^N$ on $S^2$ are examples which vanish at the maximal order of vanishing at the poles $x_1 = x_2 = 0, x_3 = \pm 1$.

### 3.4. Semi-classical Lacunas.

For the purposes of this article, we define a ‘semi-classical lacuna’ to be an open subset $U \subset M$ for which there exist a sequence $\{\varphi_{\lambda_{jk}}\}$ of $L^2$-normalized eigenfunctions $(M, g)$ and constants $C, a > 0$ so that

$$
\int_U |\varphi_{\lambda_{jk}}|^2 dV_g \leq C e^{-a\lambda_{jk}}.
$$

The sup-norm could be used in place of the $L^2$ norm. Another descriptive term is ‘exponential trough’.

Lacunae often arise as ‘classically forbidden regions’ of quantum mechanical systems, and in fact we do not know of any other examples. Consider for instance a semi-classical Schrödinger operators $h^2 \Delta + V$ for which the classical energy level $\xi^2 + V(x) = E$ projects over a compact subset $K_E = \{x : V(x) \leq E\} \subset M$. Then eigenfunctions of $h^2 \Delta + V$ with eigenvalues $E_j(h) \in [E - O(h), E + O(h)]$ decay exponentially outside $K_E$ at a rate given by $O(e^{-\frac{1}{h}d(x, K_E)})$ where $d(x, K_E)$ is the distance from $x$ to $K_E$ in the Agmon metric, i.e. the metric $(E - V(x))dx^2$. 
Classically forbidden regions also occur on compact Riemannian manifolds with integrable geodesic flow. For instance, on the round sphere, we consider sequences \( \{Y_m^N\} \) of joint eigenfunctions for which \( m/N \to E \). As will be discussed below, this sequence concentrates on the Lagrangian torus \( \Lambda_E \subset S^*S^2 \) for which \( \langle \frac{x}{|x|}, \frac{\partial}{\partial x} \rangle = E \). This torus projects to an \( S^1 \) invariant annulus \( K_E \) on \( S^2 \). This annulus is the ‘classically allowed region’ and its complement is the classically forbidden region. It is not hard to show that \( |Y_m^N(x)| \leq e^{-N d_A(x,K_E)} \) in this example, where \( d_A \) is a suitable Agmon distance. We will discuss this and related examples in more detail in [12].

To our knowledge, there are no converse results characterizing lacunae in terms of classically allowed or forbidden regions. For instance, can lacunae occur if the geodesic flow of \( (M,g) \) is ergodic? Theorem 15.1 shows that lacunae cannot occur in the full density sequence of ‘ergodic eigenfunctions’, but might occur in a possible sparse subsequence.

A more refined notion is that of microlocal lacunae, i.e. open subsets \( U \subset S_g^*M \) such that \( \langle \chi(x,D)\varphi_{\lambda_j}, \varphi_{\lambda_k} \rangle \leq C \lambda_j^{-R} \) for all \( R > 0 \) and all \( \chi(x,D) \in \Psi^0(M) \) which are microsupported in \( U \). In the real analytic case, one can insist on exponential decay, but in the general \( C^\infty \) case one could always add a smoothing operator to \( \chi(x,D) \) to ruin exponential decay.

One could also define lacunae depending on \( \lambda \). Namely, one could term a sublevel set of the form \( \{x : |\varphi_{\lambda}(x)| \leq C e^{-A\lambda}\} \) a lacunae if its volume \( \text{Vol}\{x : |\varphi_{\lambda}(x)| \leq C e^{-A\lambda}\} \) is bounded below by some constant \( \epsilon > 0 \) for some \( C,A \). Eigenfunctions with a semi-classical lacuna clearly have this property, but there could exist other examples. It would be interesting to know if ergodic eigenfunctions can have lacunae of this form, or even if their sublevel sets can be larger than a \( \frac{1}{\lambda} \) tube around the nodal set.

The doubling estimates and Carleman estimates give quantitative lower bounds on the exponential decay rate of eigenfunctions in balls as the eigenvalue tends to infinity and show that the rate is never faster than in the definition above of semi-classical lacunae:

**Corollary 3.4.** [DF] Suppose that \( M \) is compact and that \( \varphi_{\lambda} \) is a global eigenfunction, \( \Delta \varphi_{\lambda} = \lambda^2 \varphi_{\lambda} \). Then

\[
\max_{x \in B(p,h)} |\varphi_{\lambda}(x)| \geq C' e^{-C\lambda}.
\]

As an illustration, Gaussian beams such as highest weight spherical harmonics decay at a rate \( e^{-C\lambda d(x,\gamma)} \) away from a stable elliptic orbit \( \gamma \). Hence if the closure of an open set is disjoint from \( \gamma \), one has a uniform exponential decay rate which saturate the lower bounds. To our knowledge, it is unknown whether semi-classical lacunae can occur in more general situations than ‘classically forbidden regions’. It even seems to be unknown whether semi-classical lacunae can occur on \( (M,g) \) with classically chaotic (i.e. highly ergodic) geodesic flows.

### 3.5. Three ball inequalities and propagation of smallness

These inequalities generalize Hadamard’s three circle theorem and Nevanlinna’s two constants theorem. We only briefly mention the results and refer to [M,Ku] for discussion of the work of Nadirashvili, Korevaar-Meyers and others and for further background.

Let \( \Omega \) be a domain in \( \mathbb{R}^n (n \geq 2) \), let \( \Omega_0 \subset \Omega \) be a non-empty subdomain, and let \( E \subset \Omega \) be a non-empty compact set. Define \( \|u\|_A = \sup_{x \in A} |u(x)| \). Then there exists a constant \( \alpha = \alpha(E,\Omega_0,\Omega) \in (0,1] \) such that \( \|u\|_E \leq \|u\|_{\Omega_0}^{\alpha} \|u\|_{\Omega}^{1-\alpha} \) for all complex-valued harmonic
functions $u$ on $\Omega$. If $\|u\|_\Omega \leq C$ and if $\|u\|_{\Omega_0} << 1$, the smallness of $u$ on the subdomain $\Omega_0$ propagates to any compact subset $E \subset \Omega$.

In particular, one has a three spheres inequality (see (29) for notation)

**Theorem 3.5.** (see [Ku]) Let $0 < r_1 < r_2 < r_3$. Then,

$$H(r_2) \leq C_1 \left( \frac{r_1}{r_2} \right)^{C_2}\ H(r_1)^{\alpha_0/\alpha_0 + \beta_0} \ H(r_3)^{\beta_0/\alpha_0 + \beta_0};$$

where

$$\alpha_0 = \log \frac{r_3}{r_2}, \quad \beta_0 = C \log \frac{r_2}{r_1}.$$

### 3.6. Bernstein inequalities.

There are a variety of types of inequalities known as Bernstein inequalities. The original inequalities were proved for polynomials of one variable; a survey is given in [RY]. Further results in this direction (for analytic functions) are in [Bru1, Bru2]. We will discuss these further in §17.

Among the classic Bernstein inequalities are the gradient estimates on polynomials of degree $N$:

1. $|p'_N(x)| \leq N(1 - x^2)^{-1/2}||p_N||_{[-1,1]}$ for $x \in [-1,1]$ (the Bernstein-Markov inequality);
   it implies that $\int_{-1}^1 |p'_N(x)|dx \leq \pi N||p_N||_{[-1,1]}$.
2. $||p'_N(x)||_{[-1,1]} \leq N^2||p_N||_{[-1,1]}$ for $x \in [-1,1]$ (Markov’s inequality);
3. $|p'_N(x)| \leq N(1 - x^2)^{-1/2} \left(||p_N||^2_{[-1,1]} - p_N(x)^2\right)$ for $x \in [-1,1]$.

The following result of Donnelly-Fefferman generalizes Bernstein’s gradient inequality to eigenfunctions:

**Theorem 3.6.** [DF3] Local eigenfunctions of a Riemannian manifold satisfy:

1. $L^2$ Bernstein estimate:

   $$\left( \int_{B(p,r)} |\nabla \varphi_\lambda|^2 dV \right)^{1/2} \leq \frac{C\lambda}{r} \left( \int_{B(p,r)} |\varphi_\lambda|^2 dV \right)^{1/2}.$$

2. $L^\infty$ Bernstein estimate: There exists $K > 0$ so that

   $$\max_{x \in B(p,r)} |\nabla \varphi_\lambda(x)| \leq \frac{C\lambda K}{r} \max_{x \in B(p,r)} |\varphi_\lambda(x)|.$$

3. Dong’s improved bound:

   $$\max_{B_r(p)} |\nabla \varphi_\lambda| \leq \frac{C_1 \sqrt{\lambda}}{r} \max_{B_r(p)} |\varphi_\lambda|$$

   for $r \leq C_2 \lambda^{-1/4}$.

### 3.7. Carleman inequalities.

Carleman inequalities are weighted integral inequalities which are an alternative to frequency function estimates in giving quantitative unique continuation results. We only indicate some results in this section and refer to the original articles [An, DF, Ta] and more expository articles [Ta2, IL, EZ] for further discussion. As discussed in [IL], the idea of Carleman estimates is to use weights $\psi$ which are largest on the set from which one wants uniqueness (or other features) to propagate. On the other hand, there is
a constraint on $\psi$ in order that the Carleman estimate be true: it needs to be convex in a suitable sense.

We first follow [Ts2, I]. Let

$$A(x, \partial) = \sum_{j,k} a_{jk}(x) \partial_j \partial_k + \sum_j a_j \partial_j + a(x).$$

One searches for weights $\varphi = e^{\lambda \psi}$ so that the weighted $L^2$ Carleman estimate holds:

$$(39) \quad \int_{\Omega} e^{2\tau \varphi} \left( \tau^3 |u|^2 + \tau |u|^2 \right) dx \leq C_1 \left( \int_{\Omega} e^{2\tau \varphi} |Au|^2 + \int_{\partial\Omega} e^{2\tau \varphi} \left( \tau^3 |u|^2 + \tau |\nabla u|^2 \right) dx \right),$$

for all $\tau > C_1$ and all $u \in H^2(\Omega)$.

**Theorem 3.7.** [I] If $\psi$ is pseudo-convex with respect to $A$ on $\overline{\Omega}$, then there exist constants $C_1(\lambda), C_2$ so that for $\lambda > C_2$ and $\tau > C_1(\lambda)$, the Carleman estimate (39) holds.

Here, a function $\psi$ is called pseudo-convex with respect to $A$ on $\Omega$ if

$$A(x, \xi) = 0; \sum_j \frac{\partial A}{\partial \xi_j}(x, \xi) \partial_j \psi(x) = 0, \quad \forall x \in \Omega, \xi \in \mathbb{R}^n \backslash \{0\},$$

imply that

$$\sum \partial_j \partial_k \psi \frac{\partial A}{\partial \xi_j} \frac{\partial A}{\partial \xi_k} + \sum \left( \frac{\partial_k A}{\partial \xi_j} \frac{\partial A}{\partial \xi_k} - \partial_j A \frac{\partial^2 A}{\partial \xi_j \partial \xi_k} \right) \partial_j \psi > 0$$

and if

$$A(x, \nabla \psi(x)) \neq 0, \quad \forall x \in \overline{\Omega}.$$

(1) If $A$ is elliptic then any $\psi \in C^2(\overline{\Omega})$ with $\nabla \psi \neq 0$ in $\overline{\Omega}$ is pseudo-convex with respect to $A$ on $\overline{\Omega}$.

We are particularly interested in the Helmholtz operator $A = \Delta - \lambda^2$, and wish to estimate the dependence of the constants on $\lambda$. In this case one can put $\varphi(x) = r(x, b)^2$.

**Theorem 3.8.** [I]. Theorem 3.1: Let $\varphi(x) = |x - b|^2$. Then

$$\int_{\Omega} e^{2\tau \varphi} \left( \tau^3 |u|^2 + \tau |u|^2 \right) dx \leq C \left( \int_{\Omega} e^{2\tau \varphi} (\Delta - k^2) |u|^2 dx + \int_{\partial\Omega} e^{2\tau \varphi} \left( \tau^3 |u|^2 + \tau |\nabla u|^2 dS \right) \right)$$

for $\tau > C$ and all real $k$ and $u \in H^2(\Omega)$.

There is a microlocal interpretation of the convexity condition. Let $P(x, D)$ be a pseudodifferential operator, and let $h \in [0, h_0)$ be a small parameter, and put

$$(40) \quad P_\varphi = e^{\frac{i}{h} \nabla \varphi}.$$

Then,

$$P_\varphi = P(x, D + \frac{i}{h} \nabla \varphi).$$

We denote the principal symbol of $P_\varphi$ by $p_\varphi$. The Carleman weight condition (in the Hörmander formulation) is

$$(41) \quad p_\varphi(x, \xi) = 0 \quad \Rightarrow \quad \{\text{Re} p_\varphi, \text{Im} p_\varphi\}(x, \xi) > 0,$$
and in the boundary case,
\[
\partial_\nu \varphi(x) \neq 0.
\]
\[
||P_\varphi f||^2 = ||Op(\text{Re} p_\varphi)f||^2 + ||Op(\text{Im} p_\varphi)f||^2 + (f, [Op(\text{Re} p_\varphi), Op(\text{Im} p_\varphi)]f).
\]
One has,
\[
[Op(\text{Re} p_\varphi), Op(\text{Im} p_\varphi)] >> 0 \quad \text{modulo terms depending on } Op(\text{Re} p_\varphi), Op(\text{Im} p_\varphi),
\]
and this is the condition \{\text{Re} p_\varphi, \text{Im} p_\varphi\}(x, \xi) > 0.

In [DF] [DF2] (see also [JL]), Donnelly-Fefferman use singular weight functions to prove vanishing order estimates and doubling estimates (Theorem 3.4). Their Carleman inequalities involve a function \(\bar{r}\) which is comparable to the distance \(r\). Let \(u \in C_c(B(p,h))\) and let \(r(x) = d(x,p)\). Put \(\bar{r}(x) = \int_{0}^{r(x)} e^{-\nu s^2} ds = r(x) + O(r(x)^3)\) for some constant \(\nu >> 0\).

**Theorem 3.9. [A1] [DF3] (Lemma A)** There exist constants \(C, B, \delta\) such that, if \(\alpha > C(1 + \lambda)\) and if \(u\) vanishes in a ball \(B(p, \delta) \subset B(p, h)\), then

\[
\int_{B(p,h)} \bar{r}^{-2\alpha}|(\Delta - \lambda^2)u|^2 dV \geq 2\alpha^4 \int_{B(p,\delta + \frac{\bar{r}}{\alpha})} \bar{r}^{-2\alpha - 4} u^2 dV.
\]

3.8. **Geometric comparison inequalities.** We now state a theorem due to P. Kröger and Bacry-Qian which compares eigenfunctions of \((M, g)\) to eigenfunctions of model operators \(L_{R,m}\) on an interval \((a, b)\). We follow the exposition in [BQ]. The model operators are essentially the radial parts of the Laplacian in geodesic polar coordinates on spaces of constant curvature. To motivate them, we note that the radial part equals

- \(\frac{d^2}{dr^2} + \frac{m-1}{r} \frac{d}{dr}\) on \([0, \infty]\) on \(\mathbb{R}^m\);
- \(\frac{d^2}{dr^2} + (m-1) \cot r \frac{d}{dr}\) on \([0, \pi]\) on the standard sphere \(S^m\). A better comparison interval is to shift the pole to the center of the interval to obtain \(\frac{d^2}{dr^2} - (m-1) \tan r \frac{d}{dr}\) on \([-\frac{\pi}{2}, \frac{\pi}{2}]\);
- \(\frac{d^2}{dr^2} + (m-1) \coth r \frac{d}{dr}\) on \([0, \infty]\) on hyperbolic space \(\mathcal{H}^m\).

We put
\[
I_1 \cup I_2 \cup I_3 = (0, \infty) \cup (-\infty, \infty) \cup (-\infty, 0).
\]
The model operators are defined as follows:

1. If \(R > 0, m > 1\),

\[
L_{r,m}(v)(x) = v''(x) - \sqrt{R(m-1)} \tan \left( \sqrt{\frac{R}{m-1}} x \right) v'(x)
\]
on \([-\frac{\pi}{2} \sqrt{\frac{R}{m-1}}, \frac{\pi}{2} \sqrt{\frac{R}{m-1}}]\).

2. If \(R < 0, m > 1\) then

\[
L_{r,m}(v)(x) = v''(x) + \sqrt{-R(m-1)} \coth \left( \sqrt{\frac{-R}{m-1}} x \right) v'(x), \quad \text{on } I_1 \cup I_3
\]

\[
L_{r,m}(v)(x) = v''(x) + \sqrt{-R(m-1)} \tanh \left( \sqrt{\frac{-R}{m-1}} x \right) v'(x), \quad \text{on } I_2,
\]
Corollary 3.13. Hence for any smooth function vanishing on $\partial \Omega$ is given in [PS].

Theorem 3.10. Let $(M, g)$ be a compact Riemannian manifold of dimension $m$ with Ricci curvature bounded below by $R$. Let $\varphi_\lambda$ be an eigenfunction of eigenvalue $\lambda^2$, let $(a, b) \subset \mathbb{R}$ be a finite interval, and let $v_\lambda$ be a solution of

$$L_{R,m}v = -\lambda^2 v, \text{ on } (a, b)$$

satisfying:

- $v'(a) = v'(b) = 0.$
- $v' \neq 0$ on $(a, b)$.
- $[\min \varphi_\lambda, \max \varphi_\lambda] \subset [\min v_\lambda, \max v_\lambda].$

Then

$$|\nabla (v_\lambda^{-1} \circ \varphi_\lambda)| \leq 1.$$ 

In other words, consider an interval $(a, b)$ so that $\lambda^2$ is the lowest eigenvalue of the Neumann problem for $L_{R,m}$ on $(a, b)$ and let $v_\lambda$ be the corresponding ground state eigenfunction. Then if the range of $\varphi_\lambda$ is contained in the range of $v_\lambda$ then at any points $x \in M$ and $y \in (a, b)$ such that $\varphi_\lambda(x) = v_\lambda(y)$, $|\nabla \varphi_\lambda(x)| \leq |\nabla v_\lambda(y)|$. 

3.9. Symmetry of positive and negative sets. Let $M$ be a smooth compact manifold and let $\varphi_\lambda$ be a real nonconstant eigenfunction of the Laplacian on $M$. Let $\varphi_\lambda^+$, resp. $\varphi_\lambda^-$ denote the positive and the negative part of $\varphi_\lambda$. Since $\int_M \varphi_\lambda dV_g = 0$, it is obvious that $\int_M \varphi_\lambda^+ dV_g = \int_M \varphi_\lambda^- dV_g$. This represents a symmetry in $L^1$ between the positive and negative parts. On the other hand, eigenfunctions are not necessarily symmetric in this sense for higher $L^p$ norms. The article [JN2] of Jakobson-Nadriashvili discusses the extent of such symmetries and proves:

Theorem 3.11. [JN2] Then for $p \geq 1$ there exists a positive constant $C$, depending only on $p$ and $M$ such that $1/C \leq \|\varphi_\lambda^+\|_{L^p}/\|\varphi_\lambda^-\|_{L^p} \leq C$.

3.10. Alexandroff-Bakelman-Pucci-Cabré inequality. The following is Cabré’s improvement of the Alexandroff-Bakelman-Pucci estimate. In the following, $\|u\|$ denotes the $L^2$ norm in $\Omega$.

Theorem 3.12. (see [Cab], Theorem 1.4) There exists a constant $C = C(M, g)$ independent of $\Omega$ so that, for any subdomain $\Omega \subset M$,

$$\|u\|_{L^\infty(\Omega)} \leq \limsup_{x \to \partial \Omega} |u| + C|\Omega|^{\frac{1}{2}}\|\Delta u\|.$$ 

Hence for any smooth function vanishing on $\partial \Omega$, $\|u\|_{L^\infty(\Omega)} \leq C|\Omega|^{\frac{1}{2}}\|\Delta u\|.$

Corollary 3.13. If $\varphi_\lambda$ satisfies $\Delta u_\lambda = \lambda^2 u$ and $\varphi_\lambda|_{\partial\Omega} = 0$, then $\|\varphi_\lambda\|_{L^\infty(\Omega)} \leq C\lambda^2|\Omega|^\frac{1}{2}\|\varphi_\lambda\|.$

An interesting application of this inequality to extrema of eigenfunctions on nodal domains is given in [PS].
3.11. Bers scaling near zeros. By Theorem 3.3 we know that at each \( x_0 \in \mathcal{N}_{\varphi_\lambda} \), the order of vanishing \( k \) of \( \varphi_\lambda \) is finite.

Proposition 3.14. [Bers, HW2] Assume that \( \varphi_\lambda \) vanishes to order \( k \) at \( x_0 \). Let \( \varphi_\lambda(x) = \varphi^{x_0}_k(x) + \varphi^{x_0}_{k+1} + \cdots \) denote the \( C^\infty \) Taylor expansion of \( \varphi_\lambda \) into homogeneous terms in normal coordinates \( x \) centered at \( x_0 \). Then \( \varphi^{x_0}_k(x) \) is a Euclidean harmonic homogeneous polynomial of degree \( k \).

To prove this, one substitutes the homogeneous expansion into the equation \( \Delta \varphi_\lambda = \lambda^2 \varphi_\lambda \) and rescales \( x \to \lambda x \), i.e. one applies the dilation operator
\[
D^{x_0}_\lambda \varphi_\lambda(u) = \varphi(x_0 + \frac{u}{\lambda}).
\]
The rescaled eigenfunction is an eigenfunction of the locally rescaled Laplacian
\[
\Delta^{x_0}_\lambda := \lambda^{-2} D^{x_0}_\lambda \Delta_g (D^{x_0}_\lambda)^{-1} = \sum_{j=1}^n \frac{\partial^2}{\partial u_j^2} + \cdots
\]
in Riemannian normal coordinates \( u \) at \( x_0 \) but now with eigenvalue 1,
\[
D^{x_0}_\lambda \Delta_q (D^{x_0}_\lambda)^{-1} \varphi(x_0 + \frac{u}{\lambda}) = \lambda^2 \varphi(x_0 + \frac{u}{\lambda})
\]
(45)
implies
\[
\Delta^{x_0}_\lambda \varphi(x_0 + \frac{u}{\lambda}) = \varphi(x_0 + \frac{u}{\lambda}).
\]
Since \( \varphi(x_0 + \frac{u}{\lambda}) \) is, modulo lower order terms, an eigenfunction of a standard flat Laplacian on \( \mathbb{R}^n \), it behaves near a zero as a sum of homogeneous Euclidean harmonic polynomials.

The Bers scaling is used by S.Y. Cheng (see also earlier results of Hartman-Wintner [HW, Ch1, Ch2]) to prove that at a singular point of \( \varphi_\lambda \) does not vanish. Since \( \varphi(x_0 + \frac{u}{\lambda}) \) is, modulo lower order terms, an eigenfunction of a standard flat Laplacian on \( \mathbb{R}^n \), it is reasonable to think that asymptotically \( \varphi_\lambda \) behaves like a sum of plane waves, i.e. that there exists a distribution \( dT_\lambda \) on the unit sphere in momentum space such that
\[
\varphi_\lambda(x_0 + \frac{u}{\lambda}) \sim \int |\xi| = 1 e^{i\langle \xi, u \rangle} dT^{x_0}_\lambda(\xi).
\]
(46)
This is not a rigorous definition of \( dT_\lambda \) since \( \sim \) has not been precisely defined.

However, examples suggest that there do exist rigorous plane wave approximations. For instance, if we rescale the Helgason representation [IS] for an eigenfunction on a hyperbolic quotient, we obtain
\[
\varphi_\lambda(z_0 + \frac{u}{\lambda}) = \int_B e^{i\lambda + 1}(z_0 + \frac{u}{\lambda}, b) dT_\lambda.
\]
We have \( \langle z_0 + \frac{u}{\lambda}, b \rangle = \langle z_0, b \rangle + \frac{u}{\lambda} \cdot b + O(\lambda^{-1}) \) where \( u \cdot b \) denotes the Euclidean inner product of \( u \) with the element of \( S^1 \) represented by \( b \in B = S^1 \). Hence, \( e^{i\lambda + 1}(z_0, b) dT_\lambda \) is a plausible candidate for the distribution. In this heuristic discussion, we neglect the remainder estimate which requires an estimate of the \( \lambda \) dependence of \( dT_\lambda \).
4. Nodal sets on $C^\infty$ Riemannian manifolds

In this section we review results on the nodal, critical and singular sets of eigenfunctions in the smooth case. In §17 we review the much stronger results in the real analytic case.

The nodal set of an eigenfunction $\varphi_\lambda$ is the zero set
\begin{equation}
Z_{\varphi_\lambda} = \{ x \in M : \varphi_\lambda(x) = 0 \}.
\end{equation}
The global structure of the nodal set is determined by integrals $\int_{Z_{\varphi_\lambda}} fd\mathcal{H}^{n-1}$ of continuous functions (or characteristic functions of nice sets) over $Z_{\varphi_\lambda}$. This seems very difficult, so we study first the local structure of the set, e.g. its Hausdorff dimension and local Hausdorff measure. In §17 we present some global results on the nodal set when the geodesic flow is ergodic.

The following theorem, due to Brüning after an observation of R. Courant (see [Br]), is used to obtain lower bounds on volumes of nodal sets:

**Theorem 4.1.** If $(M^n, g)$ is a $C^\infty$ compact Riemannian manifold without boundary, then there exists $C(M, g) > 0$ so that, in each ball of radius $\geq \frac{C}{\lambda}$ there exists a point where $\varphi_\lambda$ vanishes.

**Proof.** Fix $x_0, r$ and consider $B(x_0, r)$. If $\varphi_\lambda$ has no zeros in $B(x_0, r)$, then $B(x_0, r) \subseteq D_{j,\lambda}$ must be contained in the interior of a nodal domain $D_{j,\lambda}$ of $\varphi_\lambda$. Now $\lambda^2 = \lambda^2_j(D_{j,\lambda})$ where $\lambda^2_j(D_{j,\lambda})$ is the smallest Dirichlet eigenvalue for the nodal domain. By domain monotonicity of the lowest Dirichlet eigenvalue (i.e. $\lambda_1(\Omega)$ decreases as $\Omega$ increases), $\lambda^2 \leq \lambda^2_j(D_{j,\lambda}) \leq \lambda^2_j(B(x_0, r))$. To complete the proof we show that $\lambda^2_j(B(x_0, r)) \leq \frac{C}{\lambda}$ where $C$ depends only on the metric. This is proved by comparing $\lambda^2_j(B(x_0, r))$ for the metric $g$ with the lowest Dirichlet Eigenvalue $\lambda^2_j(B(x_0, cr); g_0)$ for the Euclidean ball $B(x_0, cr; g_0)$ centered at $x_0$ of radius $cr$ with Euclidean metric $g_0$ equal to $g$ with coefficients frozen at $x_0$; $c$ is chosen so that $B(x_0, cr; g_0) \subseteq B(x_0, r, g)$. Again by domain monotonicity, $\lambda^2_j(B(x_0, r, g)) \leq \lambda^2_j(B(x_0, cr; g))$ for $c < 1$. By comparing Rayleigh quotients $\frac{\int_{B(x_0, r, g)} |\nabla \varphi_\lambda|^2 dV_g}{\int_{B(x_0, r, g)} |\varphi_\lambda|^2 dV_g}$ one easily sees that $\lambda^2_j(B(x_0, cr; g)) \leq C \lambda^2_j(B(x_0, cr; g_0))$ for some $C$ depending only on the metric. But by explicit calculation with Bessel functions, $\lambda^2_j(B(x_0, cr; g_0)) \leq \frac{C}{\lambda}$. Thus, $\lambda^2 \leq \frac{C}{\lambda}$.

For background we refer to [Ch]. A nice variation on the proof is given in [CM], where eigenfunctions are converted to harmonic functions as in §11 and Harnack’s inequality on positive harmonic functions is used to prove Theorem 4.1. Another use of positivity is given in [H] to prove $\lambda_1(B(x_0, r)) \geq \lambda$. Let $u_r$ denote the ground state Dirichlet eigenfunction for $B(x_0, r)$. Then $u_r > 0$ on the interior of $B(x_0, r)$. If $B(x_0, r) \subseteq D_{j,\lambda}$ then also $\varphi_\lambda > 0$ in $B(x_0, r)$. Hence the ratio $\frac{u_r}{\varphi_\lambda}$ is smooth and non-negative, vanishes only on $\partial B(x_0, r)$, and must have its maximum at a point $y$ in the interior of $B(x_0, r)$. At this point (recalling that our $\Delta$ is minus the sum of squares),
\[ \nabla \left( \frac{u_r}{\varphi_\lambda} \right)(y) = 0, \quad -\Delta \left( \frac{u_r}{\varphi_\lambda} \right)(y) \leq 0, \]
so at $y$,
\[ 0 \geq -\Delta \left( \frac{u_r}{\varphi_\lambda} \right) = -\varphi_\lambda \Delta u_r - u_r \Delta \varphi_\lambda = \frac{(\lambda^2_j(B(x_0, r)) - \lambda^2) \varphi_\lambda u_r}{\varphi_\lambda^2}. \]
Since $\varphi_{\lambda}^{\omega_k} > 0$, this is possible only if $\lambda_1(B(x_0, r)) \geq \lambda$.

We recall that the nodal set of an eigenfunction $\varphi_{\lambda}$ is its zero set. When zero is a regular value of $\varphi_{\lambda}$ the nodal set is a smooth hypersurface. This is a generic property of eigenfunctions [U]. It is pointed out in [Bac] that eigenfunctions can always be locally represented in the form

$$\varphi_{\lambda}(x) = v(x) \left( x_1^k + \sum_{j=0}^{k-1} x_1^j u_j(x') \right),$$

in suitable coordinates $(x_1, x')$ near $p$, where $\varphi_{\lambda}$ vanishes to order $k$ at $p$, where $u_j(x')$ vanishes to order $k - j$ at $x' = 0$, and where $v(x) \neq 0$ in a ball around $p$. It follows that the nodal set is always countably $n - 1$ rectifiable when $\dim M = n$.

For a general $C^\infty(M, g)$ of dimension $n$, S.T. Yau [Y1, Y2] has conjectured that

$$c_{M, g} \lambda \leq H^{n-1}(Z_{\varphi_{\lambda}}) \leq C_{M, g} \lambda.$$  

Here, $H^k$ is the $k$ dimensional Hausdorff measure. The conjecture was proved in [DF] in the real analytic case, which will be discussed in detail in §17; see Theorem 17.1. In the $C^\infty$ case, the lower bound was proved in dimension 2 by J. Brüning [Br] and S.T. Yau. Because of this result, and lower bounds in some other calculable cases, Yau conjectured the same lower bound in all dimensions. It remains an open problem in dimensions $\geq 3$.

Regarding upper bounds, in dimension two one has

**Theorem 4.2.** [DF4, Dong] Suppose $(M, g)$ is $C^\infty$ and that $\dim M = 2$. Then,

$$H^1(Z_{\varphi_{\lambda}}) \leq C_{M, g} \lambda^{3/2}.$$  

Dong’s proof was based on the following integral formula:

**Theorem 4.3.** [Dong] Let $q = |\nabla u|^2 + \frac{\lambda^2}{n} u^2$. Then

$$H^{n-1}(Z_u) = \frac{1}{2} \int_M \frac{(\Delta + \lambda^2)|u|}{q} dV.$$  

In higher dimensions, the best estimate to date is the following:

**Theorem 4.4.** [HS] For any $C^\infty (M, g)$ of dimension $n$ one has

$$H^{n-1}(Z_{\varphi_{\lambda}}) \leq C_{M, g} e^{CM, \lambda \log \lambda}.$$  

4.1. Courant and Pleijel bounds on nodal domains. Another well-known result is Courant’s nodal domain theorem:

**Theorem 4.5.** The number $n_k$ of nodal domains of the $k$ satisfies $n_k \leq k$.

Here, a nodal domain is a component of $M \setminus Z_{\varphi_{\lambda_k}}$. This estimate is not sharp. Pleijel [P] proved:

**Theorem 4.6.** For any plane domain with Dirichlet boundary conditions, $\limsup_{k \to \infty} \frac{n_k}{k} \leq \frac{1}{j_1} \simeq 0.691..., \text{ where } j_1 \text{ is the first zero of the } J_0 \text{ Bessel function.}$

He conjectured that the same result should be true for a free membrane, i.e. for Neumann boundary conditions. This was recently proved in the real analytic case by I. Polterovich [Pd], using a result [TZ3] counting the number of nodal lines which touch the boundary. Another recent result is the proof that there exists an asymptotic mean number of nodal domains of random spherical harmonics [NS]; see [18.6].
4.2. Critical and singular sets of eigenfunctions on $C^\infty$ Riemannian manifolds. Critical points are points where $\nabla \varphi_\lambda(x) = 0$. We denote the critical set by $\Sigma = \nabla \varphi_\lambda^{-1}(0)$. Singular points are critical nodal points, i.e. critical points lying on the nodal hypersurface, $\Sigma_0 = \Sigma \cap N$. The reader is warned that the term 'critical point' is sometimes used for singular points. In general, the singular set is simpler to study than the critical set.

There exist simple examples such as surfaces of revolution where the critical point sets of eigenfunctions are of codimension one. For instance, the rotationally invariant eigenfunctions on a surface of revolution have $S^1$ invariant critical point sets, and not all consist of fixed points. Thus, zonal spherical harmonics have critical point sets consisting of unions of latitude circles.

For generic metrics, all of the eigenfunctions are Morse functions and consequently their critical point sets are discrete [U]. However, there are no known bounds on the number of critical points. In the analogy of eigenfunctions of eigenvalue $\lambda^2$ to polynomials of degree $\lambda$, a very naive application of Bézout’s theorem suggests that the number of critical points should be bounded above by $C\lambda^n$, since the critical point equation is a system of $n$ equations of degree $\lambda - 1$ in $n$ unknowns. Even for real analytic metrics, no rigorous results in this direction are known. Any such bounds would have to reflect the non-degeneracy of the critical points; a simple count could be unstable if a small perturbation of the metric had a sequence of eigenfunctions with codimension one critical point sets. On the other hand, no lower bound on the number of critical points exists: it is proved in [JN] that there exists a Riemannian surface possessing a sequence of eigenfunctions with a fixed finite number of critical points, answering a question of S.T. Yau [Y3], which asks if the number of critical point for the eigenfunction grows when the eigenvalue grows (Yau believes that the answer is positive for most metrics, and that it is interesting to understand it and to give some estimate of the growth.) In [Y3], Yau also proved the existence of at least one nontrivial critical point for surfaces that can not be guaranteed by standard Morse theory. Up to now, this is the only result of proving existence of nontrivial critical points.

Singular sets have been studied for harmonic functions, eigenfunctions, and more general solutions of elliptic equations. The singular set is somewhat simpler than the critical point set. In the real analytic case, the singular set is known to be at most of codimension 2 and $\mathcal{H}^{n-2}(\Sigma) < \infty$ (see [Fed], 3.4.8). This was extended to $C^\infty$ metrics in [HHL]. One has:

**Theorem 4.7.** [HNOO, HHL] Let $\dim M = n$. There exists a constant $C_{N_R}$ depending only on the frequency function $N_R$ of $\varphi$ in $B_R(0)$ such that the singular set $\Sigma_0$ satisfies:

$$\mathcal{H}^{n-2}(\Sigma_0 \cap B_{R/2}) \leq C_{N_R}.$$  

A key point in the proof is that the complex critical set of a homogeneous harmonic polynomial in dimension $n$ is of dimension $\leq n - 2$.

Lin has conjectured:

**Conjecture 4.8.** If $\dim M = n$, the singular set $\Sigma$ satisfies:

$$\mathcal{H}^{n-2}(\Sigma \cap B_{R/2}) \leq C N_R^2.$$  

**Theorem 4.9.** [Dong] Suppose that $\dim M = 2$ and let $m(\lambda_j)$ denote the number of singular points of $\varphi_{\lambda_j}$ counted with multiplicity (i.e. $m(\lambda_j)$ equals the sum over all singular points of
the order of vanishing minus one). Then

\[ m(\lambda_j) \leq \frac{1}{4\pi} \left( \lambda^2 \text{Vol}(M, g) - 2 \int_M \min(K, 0) dV_g \right), \]

where \( K \) is the Gaussian curvature.

5. The wave kernel of a compact Riemannian manifold

Global properties of eigenfunctions often arise from the fact that global eigenfunction are eigenfunction of the wave group \( U_t = e^{it \sqrt{\Delta}} \). We begin by reviewing some basic theory of the wave equation on a compact Riemannian manifold.

The wave group of a Riemannian manifold is the unitary group \( U_t = e^{it \sqrt{\Delta}} \) is defined by the spectral theorem,

\[ U(t, x, y) = \sum_j e^{it\lambda_j} \varphi_j(x) \varphi_j(y). \]

Closely related but simpler wave kernels are the even part of the wave kernel, \( \cos t \sqrt{\Delta} \) which solves the initial value problem

\[ \begin{cases} \left( \frac{\partial^2}{\partial x^2} - \Delta \right) u = 0 \\ u|_{t=0} = f \\ \frac{\partial}{\partial t} u|_{t=0} = 0 \end{cases} \]

Similar, the odd part of the wave kernel, \( \sin t \sqrt{\Delta} \sqrt{\Delta} \) is the operator solving

\[ \begin{cases} \left( \frac{\partial^2}{\partial x^2} - \Delta \right) u = 0 \\ u|_{t=0} = 0 \\ \frac{\partial}{\partial t} u|_{t=0} = g \end{cases} \]

To employ wave kernels in spectral geometry it is indispensi ble to have approximations defined in terms of geometric data. They were first constructed by Hadamard and Riesz and are usually referred to as Hadamard(-Riesz) parametrices. There are alternative parametrices due to Lax and Hörmander.

We begin with the small-time Hadamard Riesz parametrices for \( \cos t \sqrt{\Delta} \), \( \sin t \sqrt{\Delta} \sqrt{\Delta} \). These kernels only involve \( \Delta \) and their kernels can be constructed in the form

\[ \int_0^\infty e^{it(\nu^2-\nu^2)} \sum_{j=0}^\infty W_j(x, y) \theta_{\text{reg}}^{n-1} d\theta \mod C^\infty \]

where \( W_j \) are the Hadamard-Riesz coefficients determined inductively by the transport equations

\[ \Theta_{\nu^2} W_0 + \frac{\partial W_0}{\partial r} = 0 \]

\[ 4ir(x, y) \left\{ (\frac{k+1}{r(x, y)} + \frac{\Theta_{\nu^2}}{2r}) W_{k+1} + \frac{\partial W_{k+1}}{\partial r} \right\} = \Delta_y W_k. \]

Here, \( r = r(x, y) \) is the geodesic distance and \( \theta_{\text{reg}} \) is a regularization of \( \theta^s \) at \( \theta = 0 \); \( t^{-n} \) is the distribution defined by \( t^{-n} = \text{Re}((t + i0)^{-n}) \) (see [Be]). We recall that \((t + i0)^{-n} = e^{-i\pi \frac{n}{4} \frac{1}{\Gamma(n)}} \int_0^\infty e^{itx} x^{n-1} dx \) and also that \( \int_0^\infty e^{itx} x^{n-1} dx \) has precisely the same singularity at \( t = 0 \) as the sum \( \sum_{k=0}^\infty e^{it(k+\frac{1}{4})} (k + \frac{\beta}{4})^{n-1} \).
The solutions are given by:

\[ W_0(x, y) = \Theta^{-\frac{1}{2}}(x, y) \]

\[ W_{j+1}(x, y) = \Theta^{-\frac{1}{2}}(x, y) \int_0^1 s^k \Theta(x, x_s) \Delta_2 W_j(x, x_s) ds \]

where \( x_s \) is the geodesic from \( x \) to \( y \) parametrized proportionately to arc-length and where \( \Delta_2 \) operates in the second variable.

Performing the integrals, one finds that

\[ \cos t \sqrt{\Delta}(x, y) \sim C_0 |t| \sum_{j=0}^{\infty} (-1)^j w_j(x, y) \frac{(r^2 - t^2)^{j - \frac{d-3}{2}}}{4j! \Gamma(j - \frac{d-3}{2} - 1)} \mod \mathcal{C}^\infty \]

where \( C_0 \) is a universal constant and where \( W_j = \tilde{C}_0 e^{-ij \frac{\pi}{4}} 4^{-j} w_j(x, y) \). Similarly

\[ \sin t \sqrt{\Delta}(x, y) \sim C_0 \text{sgn}(t) \sum_{j=0}^{\infty} (-1)^j w_j(x, y) \frac{(r^2 - t^2)^{j - \frac{d-3}{2}}}{4j! \Gamma(j - \frac{d-3}{2} - 1)} \mod \mathcal{C}^\infty \]

Here, \( \sim \) means that the difference of the two sides is a \( \mathcal{C}^\infty \) function on \( M \times M \), or more precisely, that if one truncates the sum after a number \( n_R \) of terms, the difference lies in \( \mathcal{C}^{R}(M \times M) \). The formulae are only valid for times \( t < \text{inj}(M, g) \) and for this reason are called small-time parametrices. When the metric is real analytic, the series for the amplitude converges for \( t \) sufficiently small and \((x, y)\) sufficiently near the diagonal [Be].

To obtain truly global results on eigenfunctions, one actually needs large time parametrices. These are very complicated and in many respects are not understood. The simplest way to obtain one is to use the group property of \( U(t) = U(t/N)^N \) to determine the wave kernel for all times from the wave kernel at a small time. It shows that for fixed \((x, t)\) the kernel \( U(t)(x, y) \) is singular along the distance sphere \( S_t(x) \) of radius \( t \) centered at \( x \), with singularities propagating along geodesics.

By a similar but more complicated calculation (using the action of \( \sqrt{\Delta} \) on the oscillatory integral for \( \frac{\sin t \sqrt{\Delta}}{\sqrt{\Delta}} \)), one has

\[ U(t, x, y) = \int_0^\infty e^{i\theta(r^2(x, y) - t^2)} \sum_{k=0}^{\infty} W_k(x, y) \theta^{\frac{d-3}{2} - k} d\theta \quad (t < \text{inj}(M, g)) \]

where \( U_0(x, y) = \Theta^{-\frac{1}{2}}(x, y) \) is the volume 1/2-density, where the higher coefficients are determined by transport equations, and where again \( \theta^r \) is regularized at 0.

An alternative parametrix has the form

\[ U(t, x, y) = \int_{T_\xi M} e^{i|\xi|_{g_x} y} e^{-i(\xi, \exp^{-1}(x))} A(t, x, y, \xi) d\xi \]

where \( |\xi|_{g_x} \) is the metric norm function at \( x \), and where \( A(t, x, y, \xi) \) is a polyhomogeneous amplitude of order 0. The expression \( \exp^{-1}(y) \) is again only defined in a sufficiently small neighborhood of the diagonal. For background, we refer to [Hol-IV, D.G].

The existence of these parametrices is sufficient to prove:
Theorem 5.1. \( U(t, x, y) \in \mathcal{D}'(\mathbb{R} \times M \times M) \) is a Fourier integral operator of order \(-\frac{1}{4}\) associated to the canonical relation,
\[
\Gamma = \{(t, \tau, x, y) : \tau + |\xi|_g = 0, g^t(x, \xi) = (y, \eta)\} \subset T^*(\mathbb{R} \times M \times M).
\]

This means that \( U(t, x, y) \) can be locally written as a finite sum of oscillatory integrals
\[
\int e^{it \xi \cdot \eta} a(t, x, y, \xi) d\xi
\]
whose phases \( \varphi \) locally parametrize \( \Gamma \) in the following sense: Define the critical set along the fibers by
\[
C_\varphi = \{(t, x, y, \xi) \in \mathbb{R} \times M \times M : d_\xi \varphi = 0\}
\]
and define the immersion \( i_\varphi : C_\varphi \rightarrow \Gamma \subset T^*(\mathbb{R} \times M \times M) \) by
\[
i_\varphi(t, x, y, \xi) = (t, d_t \varphi, x, d_x \varphi, y, -d_y \varphi).
\]
For instance, when
\[
\varphi(t, x, y, \xi) = -t|\xi|_g + \langle \xi, \exp_y^{-1}(x) \rangle
\]
one has
\[
C_\varphi = \{(t, x, y, \xi) : \exp_y(t \xi / |\xi|_g) = x\},
\]
so that \( x, y \) are linked by a geodesic segment of length \( t \) and
\[
\Gamma_\varphi = \{(t, -|\xi|_g, x, \xi, g^t(x, \xi))\}.
\]

5.0.1. Manifolds without conjugate points. The Riemannian manifolds with the simplest wave groups are those without conjugate points. A clear exposition is given in [Be]. We recall that a Riemannian manifold \((M, g)\) is without conjugate points if the exponential maps \( \exp_x : T_xM \rightarrow M \) have no singular points. In this case, the universal Riemannian cover \( \tilde{M} \), the total space of covering map \( \pi : (\tilde{M}, \tilde{g}) \rightarrow (M, g) \), is diffeomorphic to \( \mathbb{R}^n \) \((n = \dim M)\) and the exponential maps \( \exp_x : T_x\tilde{M} \rightarrow \tilde{M} \) are diffeomorphisms for all \( x \). The distance function \( \tilde{r} \) is globally well defined and real analytic away from the diagonal \( \Delta_{\tilde{g}} \) of \( \tilde{M} \times \tilde{M} \).

On a manifold without conjugate points, the Hadamard and Lax-Hörmander parametrices for the wave kernel \( \tilde{U}(t, x, y) \) on \( \tilde{M} \times \tilde{M} \) are well-defined for all \( t \). The wave kernel on the quotient \( M \) can be expressed in the form
\[
U(t, x, y) = \sum_{\gamma \in \Gamma} \tilde{U}(t, x, \gamma y)
\]
where \( \Gamma \) is the deck transformation group of \( \tilde{M} \rightarrow M \) and where we implicitly identify \( x, y \) with one of their lifts to \( \tilde{M} \). The series converges when one takes for \( \tilde{U} \) the \( \cos t \sqrt{\Delta} \) or \( \sin t \sqrt{\Delta} \): by finite propagation speed, there are only a finite number of terms for each \( t \) namely terms where \( \tilde{r}(x, \gamma y) \leq t \). The unitary kernel \( e^{it \sqrt{\Delta}}(t, x, y) \) does not have finite propagation speed, but outside the light cone the kernel is smooth and the same finite number of terms of the sum determine the singularity completely.
6. METHODS FOR GLOBAL ANALYSIS

The global analysis is based on properties of eigenfunctions which derive from the fact that they extend globally to \( M \) and hence solve the wave equation eigenvalue problem \( e^{it\sqrt{\Delta}}\varphi_\lambda = e^{i\lambda t}\varphi_\lambda \). The wave operator is very useful because it propagates singularities along geodesics. Studying singularities of various expressions involving eigenfunctions and eigenvalues gives further information that the local methods do not provide.

To take full advantage of the wave group it is important to use pseudo-differential operators, which transform well under conjugation by the wave group. We briefly review some of the key properties of pseudo-differential operators in this section. For a detailed treatment, we refer to [HoI-IV, EZ, DSj, GSj, T2].

We denote by \( \Psi^m(M) \) the space of pseudo-differential operators of order \( m \) on \( M \). The principal symbol of \( A \) is denoted \( \sigma_A \).

6.1. **Egorov’s theorem.** Egorov’s theorem for the wave group concerns the conjugations
\[
\alpha_t(A) := U_tAU_t^*, \quad A \in \Psi^m(M).
\]
Such a conjugation defines the quantum evolution of observables in the Heisenberg picture. Egorov’s theorem is the following:

**Theorem 6.1.** \( \alpha_t \) defines an order-preserving automorphism of \( \Psi^*(M) \), i.e. \( \alpha_t(A) \in \Psi^m(M) \) if \( A \in \Psi^m(M) \), and that
\[
\sigma_{U_tAU_t^*}(x,\xi) = \sigma_A(\Psi^t(x,\xi)) := V_t(\sigma_A), \quad (x,\xi) \in T^*M\setminus 0,
\]
where \( V_t \) is the unitary operator (5).

6.2. **Sharp Garding inequality.** The Garding inequality addresses the question: To what extent is the quantization \( Op(a) \) of a positive symbol a positive pseudo-differential operator? Can one give lower bounds for expressions \( \langle Op(a)\varphi_j,\varphi_j \rangle \) in terms of \( a \)?

The answer depends on the precise definition of \( Op(a) \). There is a somewhat complicated quantization \( Op^F(a) \) due to Friedrichs in the case of \( \mathbb{R}^m \) such that \( Op^F(a) \geq 0 \) if \( a \geq 0 \). But for a general definition of \( Op(a) \) one can only expect the following sharp Garding inequality:

**Theorem 6.2.** For any \( f \in C^\infty(M) \), we have
\[
\langle Op(a)f,f \rangle \geq (\inf a)||f||^2 - C||\Delta^{-1/2}f||^2,
\]
where \( ||f|| \) is the \( L^2 \)-norm of \( f \).

This immediately implies the

**Corollary 6.3.** If \( a \geq 0 \), then
\[
\langle Op(a)\varphi_\lambda,\varphi_\lambda \rangle \geq -C\lambda^{-1}.
\]

6.3. **Operator norm and symbol norm.** Another natural question comparing the properties of the classical observable \( a \) to its quantization \( Op(a) \) concerns traces and norms. For instance,

**Theorem 6.4.** For any \( A \in \Psi^0 \), \( ||\sigma_A||_{L^\infty} = \inf_K ||A+K|| \) where the infimum is taken over the set of compact operators \( K \).
6.4. Quantum Limits (Microlocal defect measures). One of the principal problems in the global analysis of eigenfunctions is to determine the possible weak limits of the Wigner measures, or microlocal defect measures, which arise from matrix elements.

To define them, we regard diagonal matrix elements as linear functionals on $\Psi^0$
\begin{equation}
\rho_k(A) = \langle A\varphi_k, \varphi_k \rangle.
\end{equation}
We observe that $\rho_k(I) = 1$, that $\rho_k(A) \geq 0$ if $A = \geq 0$ and that
\begin{equation}
\rho_k(U_t A U_t^*) = \rho_k(A).
\end{equation}
Indeed, if $A \geq 0$ then $A = B^*B$ for some $B \in \Psi^0$ and we can move $B^*$ to the right side. Similarly (63) is proved by moving $U_t$ to the right side and using the fact that the eigenvalues of $U_t$ are of modulus one. In quantum statistical mechanics, these properties are summarized by saying that $\rho_j$ is an invariant state on the algebra $\Psi^0$, or more precisely, on its closure in the operator norm. An invariant state is the analogue in quantum statistical mechanics of an invariant probability measure.

We denote by $\mathcal{M}_I$ the convex set of invariant probability measures for the geodesic flow. Further, we say that a measure is time-reversal invariant if it is invariant under the anti-symplectic involution $(x, \xi) \mapsto (x, -\xi)$ on $T^*M$. We denote the time-reversal invariant elements of $\mathcal{M}_I$ by $\mathcal{M}_I^+$. \[\text{Proposition 6.5. Any weak limit of the sequence } \{\rho_k\} \text{ on } \Psi^0 \text{ is a time-reversal invariant, } g^t \text{ invariant probability measure on } S^*M, \text{ i.e. is an element of } \mathcal{M}_I^+.\]

\textbf{Proof.} For any compact operator $K$, $\langle K\varphi_j, \varphi_j \rangle \to 0$. Hence, any limit of $\langle A\varphi_k,\varphi_k \rangle$ is equally a limit of $\langle (A + K)\varphi_k, \varphi_k \rangle$. By the norm estimate, the limit is bounded by $\inf_K ||A + K||$ (the infimum taken over compact operators). Hence any weak limit is bounded by a constant times $||\sigma_A||_{L^\infty}$ and is therefore continuous on $C(S^*M)$. It is a positive functional since each $\rho_j$ is and hence any limit is a probability measure. By Egorov’s theorem and the invariance of the $\rho_k$, any limit of $\rho_k(A)$ is a limit of $\rho_k(Op(\sigma_A \circ \Phi^t))$ and hence the limit measure is invariant. It is also time-reversal since the eigenfunctions are real-valued, i.e. complex conjugation invariant. \[\Box\]

\textbf{Problem 6.6. Determine the set } \mathcal{Q} \text{ of ‘quantum limits’, i.e. weak* limit points of the sequence } \{\Phi_k\} \text{ of distributions on the classical phase space } S^*M, \text{ defined by}
\begin{equation}
\int_X ad\Phi_k := \langle Op(a)\varphi_{\lambda_k}, \varphi_{\lambda_k} \rangle
\end{equation}
where $a \in C^\infty(S^*M)$.\[\text{The set } \mathcal{Q} \text{ is independent of the definition of } Op. \text{ The simplest examples are the exponentials on a flat torus } \mathbb{R}^m/\mathbb{Z}^m. \text{ By definition of pseudodifferential operator, } Ae^{2\pi i(k,x)} = a(x, k)e^{2\pi i(k,x)} \text{ where } a(x, k) \text{ is the complete symbol. Thus,}
\begin{equation}
\langle Ae^{2\pi i(k,x)}, e^{2\pi i(k,x)} \rangle = \int_{\mathbb{R}^n/\mathbb{Z}^n} a(x, k)dx \sim \int_{\mathbb{R}^n/\mathbb{Z}^n} \sigma_A(x, \frac{k}{|k|})dx.
\end{equation}
\text{A subsequence } e^{2\pi i(k, x)} \text{ of eigenfunctions has a weak limit if and only if } \frac{k_j}{|k_j|} \text{ tends to a limit vector } \xi_0 \text{ in the unit sphere in } \mathbb{R}^n. \text{ In this case, the associated weak* limit is}\]
\[ \int_{\mathbb{R}^n/\mathbb{Z}^n} \sigma_A(x, \xi_0) dx, \]

i.e. the delta-function on the invariant torus \( T_{\xi_0} \subset S^*M \) for \( g^t \), defined by the constant momentum condition \( \xi = \xi_0 \). The eigenfunctions are said to localize on this invariant torus. Given \( \xi_0 \), we can always define a sequence \( k_j \) so that \( \frac{k_j}{|k_j|} \to \xi_0 \), and thus, every invariant torus measure arises as a quantum limit.

In general, there are many possible limit measures. The most important are:

1. Normalized Liouville measure. In fact, the functional \( \omega \) of (72) is also a state on \( \Psi^0 \) for the reason explained above. A subsequence \( \{ \varphi_{jk} \} \) of eigenfunctions is considered diffuse if \( \rho_{jk} \to \omega \).
2. A periodic orbit measure \( \mu_\gamma \) defined by \( \mu_\gamma(A) = \frac{1}{L_\gamma} \int_A \sigma_A ds \) where \( L_\gamma \) is the length of \( \gamma \). A sequence of eigenfunctions for which \( \rho_{kj} \to \mu_\gamma \) obviously concentrates (or strongly ‘scars’) on the closed geodesic.
3. A finite sum of periodic orbit measures.
4. A delta-function along an invariant Lagrangian manifold \( \Lambda \subset S^*M \). The associated eigenfunctions are viewed as 'localizing' along \( \Lambda \).
5. A more general measure which is singular with respect to \( d\mu \).

All of these possibilities arise as \( (M,g) \) varies among Riemannian manifolds. Indeed, the standard sphere provides an extreme example:

**Theorem 6.7.** [JZ] For the standard round sphere \( S^n \), \( Q = \mathcal{M}_I^n \).

An important case is when \( \rho_{kj} \to \omega \), i.e. the limit measure is Liouville measure. The corresponding eigenfunctions become uniformly distributed on the energy surface \( S_g^*M \) and are sometimes called 'diffuse'. By testing against multiplication operators, one gets

\[
\frac{1}{Vol(M)} \int_E |\varphi_{kj}(x)|^2 dVol \to \frac{Vol(E)}{Vol(M)}
\]

for any measurable set \( E \) whose boundary has measure zero. In the interpretation of \( |\varphi_{kj}(x)|^2 dVol \) as the probability density of finding a particle of energy \( \lambda_k^2 \) at \( x \), this says that the sequence of probabilities tends to uniform measure. However, \( \rho_{kj} \to \omega \) is much stronger since it says that the eigenfunctions become uniformly distributed on \( S^*M \) and not just on the configuration space \( M \). For instance, on the flat torus \( \mathbb{R}^n/\mathbb{Z}^n \), the standard exponentials \( e^{2\pi i(k,x)} \) satisfy \( |e^{2\pi i(k,x)}|^2 = 1 \), and are thus uniformly distributed in configuration space. On the other hand, as seen above, in phase space they localize on invariant Lagrange tori in \( S^*M \).

The flat torus is a model of a completely integrable system, on both the classical and quantum levels. On the other hand, if the geodesic flow is ergodic one would expect the eigenfunctions to be diffuse in phase space. The statement that the all eigenfunctions are diffuse, i.e. \( Q = \{ \omega \} \), is known as quantum unique ergodicity. It will be discussed in §15.

Off-diagonal matrix elements

\[ \rho_{jk}(A) = \langle A \varphi_{\lambda_j}, \varphi_{\lambda_k} \rangle \]

are also important as transition amplitudes between states. They no longer define states since \( \rho_{jk}(I) = 0 \), are no longer positive, and are no longer invariant. Indeed, \( \rho_{jk}(U_tA U^*_t) = e^{it(\lambda_j - \lambda_k)} \rho_{jk}(A) \), so they are eigenvectors of the automorphism \( \alpha_t \) of (60). A sequence of such matrix elements cannot have a weak limit unless the spectral gap \( \lambda_j - \lambda_k \) tends to a limit \( \tau \in \mathbb{R} \). In this case, by the same discussion as above, any weak limit of the functionals
\( \rho_{jk} \) will be a time-reversal invariant eigenmeasure of the geodesic flow which transforms by \( e^{i\tau t} \) under the action of \( g^t \). Examples of such eigenmeasures are orbital Fourier coefficients \( \frac{1}{L^\gamma} \int_0^{L^\gamma} e^{-i\tau t} \sigma_A(g^t(x, \xi)) dt \) along a periodic orbit. Here \( \tau \in \frac{2\pi}{L^\gamma} \mathbb{Z} \). We denote by \( Q_\tau \) such eigenmeasures of the geodesic flow. Problem 1 has the following extension to off-diagonal elements:

**Problem 6.8.** Determine the set \( Q_\tau \) of ‘quantum limits’, i.e. weak* limit points of the sequence \( \{\rho_{kj}\} \) on the classical phase space \( S^*M \).

As will be discussed in §15.3, the asymptotics of off-diagonal elements depends on the weak mixing properties of the geodesic flow and not just its ergodicity.

7. **Singularities pre-trace formulae**

One of the principal methods for relating eigenfunctions and geodesic flow are the local Weyl laws. One form of the local Weyl law is to find the asymptotics and remainder for the pointwise sums,

\[
N(\lambda; x) = \sum_{j: \lambda_j \leq \lambda} |\varphi_{\lambda_j}(x)|^2.
\]

Another is to find, for \( A \in \Psi^0 \), the asymptotics and remainder for

\[
N_A(\lambda) = \sum_{j: \lambda_j \leq \lambda} \langle A\varphi_{\lambda_j}, \varphi_{\lambda_j} \rangle.
\]

In both cases, the asymptotics are determined by a Fourier Tauberian method, by studying the singularities of the dual trace,

\[
S(t, x) = \sum_j e^{it\lambda_j} |\varphi_{\lambda_j}(x)|^2,
\]

resp.

\[
S_A(t) = \sum_j e^{it\lambda_j} \langle A\varphi_{\lambda_j}, \varphi_{\lambda_j} \rangle.
\]

7.1. **Duistermaat-Guillemin short time pre-trace formula.** By a pre-trace formula one means a ‘geometric’ singularities formula for the wave kernels

\[
U(t, x, x) = \sum_j e^{it\lambda_j} \varphi_{\lambda_j}(x)^2
\]
on the diagonal. The actual trace formula is the integral over \( M \) of the pre-trace formula. The trace formula clearly gives only spectral information, while the pre-trace formula gives information on eigenfunctions.

**Proposition 7.1.** [D.G] Let \( (M, g) \) be a \( C^\infty \) compact Riemannian manifold of dimension \( n \). Then there exists a sequence \( \omega_1, \omega_2, \ldots \) of real valued smooth densities on \( M \) such that, for every \( \rho \in \mathcal{S}([0, \infty)) \) with \( \text{supp } \hat{\rho} \) contained in a sufficiently small neighborhood of 0 and \( \hat{\rho} \equiv 1 \) in a small neighborhood of 0,

\[
\sum_j \rho(\lambda - \lambda_j)|\varphi_{\lambda_j}(x)|^2 \sim \sum_{k=0}^{\infty} \omega_k \lambda^{n-k-1}
\]
as $\lambda \to \infty$ (and rapidly decaying as $\lambda \to -\infty$) with
\[\omega_0(x) = \text{Vol}(S^*_x M), \quad \omega_1 = 0 = \omega_\eta; \omega_k = 0 \text{ for odd } k.\]

**Proof.** One uses a short-time parametrix,
\[U(t, x, y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\varphi(t, x, y, \eta)} a(t, x, y, \eta) d\eta\]
where $\alpha$ is a classical symbol of order $m$ and where $\varphi(t, x, y, \eta) = \psi(x, y, \eta) - t|\eta|$, with $\psi(x, y, \eta) = 0$ if $\langle x - y, \eta \rangle = 0$. Hence,
\[\rho \ast dN(\lambda, x) = \int e^{i\lambda t} \hat{\rho}(t) U(t, x, x) dt = (2\pi)^{-n} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{i\lambda t} \hat{\rho}(t) e^{-it|\eta|} a(t, x, x) d\eta dt.\]
One now changes variables $\eta \to \lambda \eta$, puts the $d\xi$ integral into polar coordinates $\xi = r\omega, |\omega| = 1$ and carries out the $dt dr$ integral by the method of stationary phase. 

The same kind of argument applies to $N_A(\lambda)$:

**Proposition 7.2.** For $A \in \Psi^m(M)$, let $N_A(\lambda) = \sum_{j} A \varphi_j \varphi_j$. Then for any $\rho \in \mathcal{S}(\mathbb{R})$ with $\hat{\rho} \in C_0^\infty(\mathbb{R})$, $\text{supp} \hat{\rho} \cap\text{Lsp}(M, g) = \{0\}$ and with $\hat{\rho} \equiv 1$ in some interval around 0, we have:
\[\rho \ast dN_A(\lambda) \sim \sum_{k=0}^\infty \alpha_k \lambda^{n+m-k-1} \quad (\lambda \to +\infty)\]

where:
\[n = \text{dim } M, \quad \alpha_0 = \int \sigma_A d\mu, \quad \alpha_k = \int \omega_k d\mu\]
where $\omega_k$ is determined from the $k$-jet of the complete symbol $a$ of $A$. 

**Proof.** The only new step is to apply $A$ to the parametrix for $U_t$. Applying a $\psi DO$ to $ae^{i\varphi}$ produces an expression $\alpha e^{i\varphi}$ with the same phase and only a change in the amplitude. Hence,
\[AU(t, x, y) = (2\pi)^{-n} \int_{\mathbb{R}^n} \alpha(t, x, y, \eta) e^{i\varphi(t, x, y, \eta)} d\eta\]
where $\alpha$ is a classical symbol of order $m$. Now proceed as before.

### 7.2. Long time pre-trace formulae.
We now discuss pre-trace formulae for $U(t, x, x)$ for long times $t$. The main difficulty is that there is no better parametrix for $U(t, x, x)$ than to use the group formula $U(t) = U(t/N)^N$ to reduce to times $< \text{inj}(M, g)$. But this will require $N \times m$ integral signs and amplitudes that are difficult to control. Moreover, the singularities become very difficult to control.

One can get a good idea of the difficulties by studying the simplest case, that of manifolds without conjugate points. This case is a rather straightforward generalization of the Selberg pre-trace formula for compact hyperbolic manifolds. The key feature is that there exists a global in time parametrix on the universal cover, so it is not necessary to assume that the
support of $\hat{\rho}$ is small. As in the classical trace formula, one organizes the elements $\gamma \in \Gamma$ (the deck transformation group) into conjugacy classes $\hat{\gamma}$. One then has

$$U(t, x, x) = \sum_{\hat{\gamma}} U_{\hat{\gamma}}(t, x, x),$$

where

$$U_{\hat{\gamma}}(t, x, y) = \sum_{\alpha \in \Gamma \backslash \Gamma_{\gamma}} \tilde{U}(t, x, \alpha^{-1} \gamma \alpha x) = \sum_{\alpha \in \Gamma \backslash \Gamma_{\gamma}} \tilde{U}(t, \alpha x, \gamma \alpha x),$$

since $\alpha$ is an isometry. Here, $\Gamma_{\gamma}$ is the stabilizer in $\Gamma$ of $\gamma$.

In this case, it is best to use the Hadamard parametrix. Since we are interested in the long time singularities, we can use the phase function $r - t$ instead of $r^2 - t^2$. Either one parameterizes the graph of the geodesic flow away from $r = 0$. One then has,

$$\rho * dN(\lambda, x) = \sum_{\gamma \in \Gamma} \int_{\mathbb{R}} e^{i\lambda t} \hat{\rho}(t) \tilde{U}(t, x, \gamma x) dt$$

$$= (2\pi)^{-n} \sum_{\hat{\gamma}} \sum_{\alpha \in \Gamma \backslash \Gamma_{\gamma}} \int_{\mathbb{R}_+} \int e^{i\lambda t} \hat{\rho}(t)e^{i\theta (r(\alpha x, \gamma \alpha x) - t)} a(\alpha x, \gamma \alpha x, \theta) d\theta dt$$

$$= (2\pi)^{-n} \sum_{\hat{\gamma}} \sum_{\alpha \in \Gamma \backslash \Gamma_{\gamma}} \int_{\mathbb{R}_+} \hat{\rho}(t)e^{i\lambda (t + \theta (r(\alpha x, \gamma \alpha x) - t))} a(\alpha x, \gamma \alpha x, \theta) d\theta dt .$$

To determine the asymptotics as $\lambda \to \infty$ one again applies stationary phase. The phase function is critical when $\theta = 1, t = r(\alpha x, \gamma \alpha x)$. This corresponds to the times $t$ when there exists a geodesic loop at $x \in M$ (i.e. at $\alpha x \in \tilde{M}$). The geodesic loops are in one-one correspondence with conjugacy classes in $\Gamma$ and hence form a countable set. However, the growth rate of this set as $t \to \infty$ is often exponentially large. Thus, if $\text{supp} \hat{\rho} \subset [-T, T]$, then there are often $e^{CT}$ terms in the sum over $\gamma$. This makes it difficult to control the remainder terms.

### 7.3. Safarov trace formula.

Safarov added some precision to the Duistermaat-Guillemin singularities pre-trace formula. For fixed $x$, Given $x \in M$, we let $\mathcal{L}_x$ denote the set of loop directions at $x$:

$$\mathcal{L}_x = \{ \xi \in S_x^* M : \exists T : \exp_x T \xi = x \} .$$

We let $T_x : S_x^* M \to \mathbb{R}_+ \cup \{ \infty \}$ denote the return time function to $x$,

$$T_x(\xi) = \begin{cases} \inf \{ t > 0 : \exp_x t \xi = x \}, & \text{if } \xi \in \mathcal{L}_x; \\ +\infty, & \text{if no such } t \text{ exists.} \end{cases}$$

We then define the first return map by $\Phi_x = \varphi_x^{T_x} : \mathcal{L}_x \to S_x^* M$. We also define $T^{(k)}(\xi)$ to be the time of $k$th return for directions which loop back at least $k$ times.

We then consider the positive partially unitary operator (the Perron-Frobenius operator)

$$U_x : L^2(\mathcal{L}_x, |d\omega|) \to L^2(S_x^*, |d\omega|), \quad U_x f(\xi) = \begin{cases} f((\Phi_x)(\xi)) \sqrt{J_x(\xi)}, & \xi \in \mathcal{L}_x; \\ 0, & \xi \notin \mathcal{L}_x. \end{cases}$$
Here, $J_x$ is the Jacobian of the map $\Phi_x$, i.e. $\Phi_x^*|d\xi| = J_x(\xi)|d\xi|$. We have:

$\text{ker } U_x = \{ f \in L^2(S^*_x) : \text{supp } f \cap \Phi_x(\mathcal{L}_x) = \emptyset \}; \text{ Im } U_x = \{ f \in L^2(S^*_x) : \text{supp } f \subset \mathcal{L}_x \}.$

We further define

$$U_x^\pm(\lambda) = e^{i\lambda T^\pm}U_x^\pm.$$

Let $\rho_T$ be the dilated test function satisfying $\hat{\rho}_T(\tau) = \hat{\rho}(\frac{\tau}{T})$. The pre-trace formula then has the form,

$$(67) \quad \rho_T * dN(\lambda, x) = a_0(x)\lambda^{n-1} + \lambda^{n-1}\int_{L_x} \sum_{k=1}^\infty \hat{\rho}(\frac{T_x^{(k)}(\xi)}{T})U_x(\lambda)^k|d\xi| + o_T(\lambda^{n-1}).$$

A key point in the proof is that the phase of the oscillatory integral for the left side only has a stationary phase point at $\xi \in \mathcal{L}_x$. That reduces the integral to one over $\mathcal{L}_x$ modulo an error $o_T(\lambda^{n-1})$. For the details, we refer to Proposition 4.1.16 of [SV].

8. Weyl law and local Weyl law

The classical Weyl law asymptotically counts the number of eigenvalues less than $\lambda$,

$$(68) \quad N(\lambda) = \# \{ j : \lambda_j \leq \lambda \} = \frac{|B_n|}{(2\pi)^n}Vol(M, g)\lambda^n + O(\lambda^{n-1}).$$

Here, $|B_n|$ is the Euclidean volume of the unit ball and $Vol(M, g)$ is the volume of $M$ with respect to the metric $g$. Equivalently,

$$(69) \quad Tr E_\lambda = \frac{Vol(|\xi|_g \leq \lambda)}{(2\pi)^n} + O(\lambda^{n-1}),$$

where $Vol$ is the symplectic volume measure relative to the natural symplectic form $\sum_{j=1}^n dx_j \wedge d\xi_j$ on $T^*M$. Thus, the dimension of the space where $H = \sqrt{\Delta}$ is $\leq \lambda$ is asymptotically the volume where its symbol $|\xi|_g \leq \lambda$.

The Weyl law with remainder is proved by using the integrated version of Proposition 7.1 together with the Fourier Tauberian estimate $N(\lambda) - \rho * N(\lambda) = O(\lambda^{n-1})$, valid when $\rho * dN(\lambda) = O(\lambda^{n-1})$. The latter is visibly true from Proposition 7.1. See the Appendix (19.1) Theorem 19.1

Proposition 7.1 assumes that $\hat{\rho}$ is supported in a small interval around $t = 0$. Thus it only takes into account the generic singularity at $t = 0$. An improved, two-term Weyl law has been proved which takes into account the singularities of $Tr \cos t\sqrt{\Delta}$ for larger values of $t$.

The singular $t \neq 0$ are the lengths of the closed geodesics $\gamma$ of $g^t$. The size of the remainder reflects the measure of closed geodesics.

(1) In the aperiodic case, Ivrii’s two term Weyl law states

$$N(\lambda) = \# \{ j : \lambda_j \leq \lambda \} = c_m Vol(M, g) \lambda^m + o(\lambda^{m-1})$$

where $m = \dim M$ and where $c_m$ is a universal constant.

(2) In the periodic case, the spectrum of $\sqrt{\Delta}$ is a union of eigenvalue clusters $C_N$ of the form

$$C_N = \{(\frac{2\pi}{T})(N + \frac{\beta}{4}) + \mu_N, i = 1 \ldots d_N \}$$
with $\mu_{N_1} = 0(N^{-1})$. The number $d_N$ of eigenvalues in $C_N$ is a polynomial of degree $m − 1$.

We refer to [D.G, HoI-IV, SV, Z1] for background and further discussion.

The integrated Weyl law is relevant to spectral theory, while the pointwise local Weyl law is relevant to eigenfunctions. By the same Fourier Tauberian theorem, one has (D.G),

\begin{equation}
\sum_{\lambda_j \leq \lambda} |\varphi_j(x)|^2 = \frac{1}{(2\pi)^n} |B^n| \lambda^n + R(\lambda, x),
\end{equation}

where $R(\lambda, x) = O(\lambda^{n-1})$ uniformly in $x$. In this case one obtains a $o(\lambda^{n-1})$ remainder if the set of geodesic loops at $x$ has measure zero. Such refinements will be discussed in §14.2.

The other local Weyl law concerns the traces $\text{Tr}_A E(\lambda)$ where $A \in \Psi^m(M)$. It asserts that

\begin{equation}
\sum_{\lambda_j \leq \lambda} \langle A \varphi_j, \varphi_j \rangle = \frac{1}{(2\pi)^n} \left( \int_{B^*M} \sigma_A dx d\xi \right) \lambda^n + O(\lambda^{n-1}).
\end{equation}

When the periodic geodesics form a set of measure zero in $S^*M$, one could average over the shorter interval $[\lambda, \lambda + 1]$. Combining the Weyl and local Weyl law, we find the surface average of $\sigma_A$ is a limit of traces:

\begin{equation}
\omega(A) := \frac{1}{\mu(S^*M)} \int_{S^*M} \sigma_A d\mu = \lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \langle A \varphi_j, \varphi_j \rangle
\end{equation}

Here, $\mu$ is the Liouville measure on $S^*M$.

9. LOCAL AND GLOBAL $L^p$ ESTIMATES OF EIGENFUNCTIONS

One of the applications of pointwise and local Weyl laws is to obtain bounds on $L^p$ norms of $L^2$-normalized eigenfunctions, when $p$ is sufficiently large. Estimates of the $L^\infty$ norms can be obtained from the local Weyl law (12.4). We use the notation $f(\lambda) = O(g(\lambda))$ if there exists a constant $C$ such that $f(\lambda) \leq C g(\lambda)$, and $f(\lambda) = \Omega(g(\lambda))$ if there exists a constant $C$ such that $f(\lambda_j) \geq C g(\lambda_j)$ for some sequence $\lambda_j \to \infty$ (in other words, the negation of $f = o(g(\lambda))$).

**Proposition 9.1.** Let $(M, g)$ be a compact $m$-dimensional $C^\infty$ Riemannian manifold. Then $||\varphi_{\lambda_j}||_{L^\infty} = O(\lambda_j^{m-1})$.

**Proof.** Since the jump in the the left hand side of (70) at $\lambda$ is $\sum_{j: \lambda_j = \lambda} |\varphi_{\lambda_j}(x)|^2$ and the jump in the right hand side is the jump of $R(\lambda, x)$, we have

\begin{equation}
\sum_{j: \lambda_j = \lambda} |\varphi_{\lambda_j}(x)|^2 = O(\lambda_j^{m-1}) \implies ||\varphi_{\lambda_j}||_{L^\infty} = O(\lambda_j^{m-1}).
\end{equation}
We note that this estimate is stronger by the factor $\lambda^{-\frac{1}{2}}$ than the estimate obtained by Sobolev estimates of eigenfunctions: If $\dim M = m$, then
\[
\|\varphi_\lambda\|_{L^p} \leq \lambda^{m\left(\frac{1}{2} - \frac{1}{p}\right)}\|\varphi_\lambda\|_{L^q}.
\]
Thus, when $p = \infty$, the Sobolev estimate gives $\|\varphi_\lambda\|_{L^\infty} \leq \lambda^{\frac{m}{2}}\|\varphi_\lambda\|_{L^2}$.

For general $L^p$-norms, the following bounds hold on any compact Riemannian manifold

**Theorem 9.2.** [Sog:],

(74) \[
\frac{\|\varphi_\lambda\|_p}{\|\varphi_\lambda\|_2} = O(\lambda^{\delta(p)}), \quad 2 \leq p \leq \infty.
\]

where

(75) \[
\delta(p) = \begin{cases} 
  n\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2}, & \frac{(n+1)}{n-1} \leq p \leq \infty \\
  \frac{n-1}{2} \left(\frac{1}{2} - \frac{1}{p}\right), & 2 \leq p \leq \frac{2(n+1)}{n-1}.
\end{cases}
\]

These estimates are sharp for some $(M, g)$. For instance, on the unit sphere $S^n \subset \mathbb{R}^{n+1}$, the zonal spherical harmonics achieve the maximal $L^\infty$ norm. Moreover, the highest weight spherical harmonics on $S^2$ saturate the bounds for small $L^p$ norms: using Gaussian integrals, one easily finds that $\|\varphi_\lambda\|_{L^2(S^2)} \sim k^{-1/4}$. For instance,

\[
\int_{\mathbb{R}^3} (x^2 + y^2)^k e^{-(x^2 + y^2 + z^2)} \, dx \, dy \, dz = \|\varphi_\lambda\|_{L^2(S^2)}^2 \int_0^\infty r^2 e^{-r^2} \, dr,
\]

\[
\Rightarrow \quad \|\varphi_\lambda\|_{L^2(S^2)}^2 = \frac{\Gamma(k+1)}{\Gamma(\frac{k+2}{2})} \sim k^{-1/2}.
\]

Thus, the $L^2$ normalized highest weight vector has the form $k^{1/4}(x + iy)^k$. It achieves its $L^\infty$ norm at $(1, 0, 0)$ where it has size $k^{1/4}$. More importantly, it is an extremal for $L^p$ for $2 \leq p \leq 6$. For instance,

\[
\int_{\mathbb{R}^3} (x^2 + y^2)^{3k} e^{-(x^2 + y^2 + z^2)} \, dx \, dy \, dz = \|\varphi_\lambda\|_{L^6(S^2)}^6 \int_0^\infty r^{6k} e^{-r^2} \, dr,
\]

\[
\Rightarrow \quad \|\varphi_\lambda\|_{L^6(S^2)}^6 = \frac{\Gamma(6k+1)}{\Gamma(6k+\frac{3}{2})} \sim k^{-1/2}.
\]

Hence, the $L^6$ norm of $k^{1/4}(x + iy)^k$ equals

\[
k^{1/4}k^{-1/12} = k^{1/6}.
\]

Since $\lambda_k \sim k$ and $\delta(6) = \frac{1}{6}$ in dimension 2, we see that it is an extremal.

It is natural to ask if extremals (in order of magnitude) for the $L^p$ norms necessarily resemble these examples. This motivates the problems:

1. What is the structure of eigenfunctions which are maximal for the $L^p$ norms?
2. Determine the $(M, g)$ for which $L^\infty(\lambda, g) = \Omega(\lambda^\frac{1}{4})$.
3. At the other extreme, determine the structure of eigenfunctions which minimize the $L^p$ norms, and determine the $(M, g)$ with a uniformly bounded orthonormal basis of eigenfunctions. Irrational flat tori are examples.
9.1. Sketch of proof of the Sogge $L^p$ estimate. We now sketch the proof of the Sogge estimates, following the approach in [KTZ].

The estimates are proved by interpolation from three estimates: $p = \infty$, $\frac{2(n+1)}{n-1}$ (for $n \neq 1$). The $L^2$ estimate is of course trivial. The $L^\infty$ estimate already followed from the un-integrated local Weyl law with remainder estimated by the Fourier Tauberian method. We will nevertheless sketch an alternative proof which has more in common with the $L^\frac{2(n+1)}{n-1}$ estimate.

9.1.1. The $L^\infty$ estimate.

Proof. The first observation is that the estimates are local in phase space $T^*M$. That is, the estimate holds for $\varphi_\lambda$ if it holds for $\chi(x, \lambda^{-1}D)\varphi_\lambda$ for any microlocal cutoff $\chi(x, \lambda^{-1}D) = Op_{\lambda^{-1}}(\chi)$, where $\chi$ is a smooth function on $T^*M\setminus 0$ which is supported in a small conic subset (i.e. the cone through a small open subset of $S^*_0M$.)

The semi-classical symbol of $\lambda^{-2}\Delta - 1$ equals $p(x, \xi) = |\xi|^2 - 1$. We may choose coordinates in the support of $\chi$ so that $\partial_{\xi_1}((|\xi|^2 - 1) \neq 0$. Locally write $p = p(x, \xi', \xi_1)$, where $\xi' = (\xi_2, \ldots, \xi_m)$. By the implicit function theorem, there exists a local function $a(x, \xi')$ such that $\xi_1 = a(x, \xi')$ when $p = 0$, and there exists a symbol $g(x, \xi) > 0$, on the support of $\chi$ so that

\begin{equation}
\tag{76}
p(x, \xi) = g(x, \xi)(\xi_1 - a(x, \xi')).
\end{equation}

Quantizing the right factor and using the support properties of the symbol, we get

\begin{equation}
\tag{77}
(q(x, \lambda^{-1}D)(\lambda^{-1}D_{x_1}) - a(x, \lambda^{-1}D'))(\chi(x, \lambda^{-1}D)\varphi_\lambda) = (\lambda^{-2}\Delta - 1)\chi(x, \lambda^{-1}D)\varphi_\lambda
\end{equation}

\begin{equation*}
= [\lambda^{-2}\Delta, \chi(x, \lambda^{-1}D)]\varphi_\lambda = O(\lambda^{-1}).
\end{equation*}

Since $q(x, \lambda^{-1}D)$ is elliptic we may invert it to obtain,

\begin{equation}
\tag{78}
f_\lambda(x_1, x') := (\lambda^{-1}D_{x_1} - a(x, \lambda^{-1}D'))\chi(x, \lambda^{-1}D)\varphi_\lambda = O(\lambda^{-1}).
\end{equation}

Let $E(t)$ be the unitary evolution operator which solves

\begin{equation}
\tag{79}
(\lambda^{-1}D_t - a(t, x', \lambda^{-1}D'))\chi(t, x', \lambda^{-1}D)u = f
\end{equation}

with $E(0) = Id$. The $L^\infty$ estimate then follows from the Sobolev bound in dimension $m - 1$ as long as

\begin{equation}
\tag{80}
||\chi(x, \lambda^{-1}D)\varphi_\lambda(x_1, \cdot)||_{L^2(\mathbb{R}^{m-1})} = O(1).
\end{equation}

This follows by solving the ODE \((79)\) in the $x_1$ variable. If the right side were equal to zero, it would be solved by a unitary evolution operator $E(t)\chi(x, D)\varphi_\lambda\psi_\lambda(0, x')$. One then solves the inhomogeneous equation by the Duhamel formula,

\begin{equation*}
(\chi(x, \lambda^{-1}D)\varphi_\lambda)(x_1, x') = E(x_1)(\chi(x, \lambda^{-1}D)\varphi_\lambda)(0, x') + i\lambda \int_0^{x_1} E(t - s)O(\lambda^{-1})ds,
\end{equation*}

proving \((80)\) and completing the proof of the Theorem. \qed
9.1.2. The $L^{\frac{2(n+1)}{n-1}}$ estimate. The estimate to be proved is that

\begin{equation}
\|\varphi_\lambda\|_{L^{\frac{2(n+1)}{n-1}}} \leq C \lambda^{\frac{n-1}{2(n+1)}}.
\end{equation}

We start again with the symbol factorization (76) and the estimate (77), which implies

\begin{equation}
\int_{\mathbb{R}} \| f(x_1, \cdot) \|_{L^2(\mathbb{R}^{n-1})} dx_1 \leq C \| f \|_{L^2(\mathbb{R}^n)} = O(\lambda^{-1}).
\end{equation}

We now use the evolution operator $E(t)$ of (79). The equation is time dependent, so the solution operator solves an operator ODE of the form

\begin{equation}
\lambda^{-1} D_t E(t, s) + A(t) E(t, s) = 0, \quad E(t, t) = Id.
\end{equation}

Then there exists a parametrix for $E(t)$ of the type (140). More generally, suppose that $F(t, s)$ solves the equation (19.1) with initial condition $F(s, s) = G(s)(x, \lambda^{-1} D)$, a smoothing operator. Then for small times $t \in (0, t_0)$ there exists an oscillatory integral parametrix

\[ F(t, s) u(x) = \lambda^{-k} \int_{\mathbb{R}^k} e^{i\lambda(t, s, x, \eta) - (y, \eta)} b(t, x, \eta; \lambda) dyd\eta + R(t, s) u(x), \]

where $R(t, x)$ is a smoothing operator.

Let $U(t, s) = \psi(t) F(t, s) \chi(x, \lambda^{-1} D)$ where $\chi \in C_\infty^\infty(\mathbb{R})$. Then the main point is to prove,

\begin{equation}
\sup_{s \in I} \left( \int_{\mathbb{R}} \| U(t, s) f \|_{L^q(\mathbb{R}^n)}^q dt \right)^{\frac{1}{q}} \leq C \lambda^{-\frac{1}{p}} \| f \|_{L^2(\mathbb{R}^n)}.
\end{equation}

For the application to the $L^{\frac{2(n+1)}{n-1}}$ estimate one takes $p = q = \frac{2(n+1)}{n-2}$, and we assume this from now on. The idea of the proof is to use the oscillatory integral parametrix for $F$ and $U(t, s)$ and the stationary phase method to prove,

\begin{equation}
\| U(t, r) U(s, r)^* f \|_{L^\infty} \leq C \lambda^{(n-1)/2}(\lambda^{-1} + |t - s|)^{-(n-1)/2}.
\end{equation}

Then (84) follows by an abstract Strichartz estimate (see [KTZ] for references).

One can choose $\psi$ and $\chi$ so that

\[ \chi(x, \lambda^{-1} D) \varphi_\lambda(x_1, x') = i\lambda \int_0^{x_1} U(x_1, s) f_\lambda(s, x') ds + O(\lambda^{-\infty}). \]

Then,

\[ \| \chi(x, \lambda^{-1} D) \varphi_\lambda(x_1, x') \|_{L^{\frac{2(n+1)}{n-1}}} \leq \lambda \lambda^{\frac{n-1}{2(n+1)}} \int_{\mathbb{R}} \| f_\lambda(x, \cdot) \|_{L^2(\mathbb{R}^{n-1})} ds + O(\lambda^{-\infty}) \]

\begin{equation}
\leq C \lambda^{\frac{n-1}{2(n+1)}}.
\end{equation}

9.2. Generic non-sharpness of Sogge estimates. As the proof indicates, the Sogge bounds are ‘(micro-) local results’, and do not take the global dynamics of the geodesic flow into account. The dynamics of the geodesic flow has a strong impact on the growth of $L^p$ norms of eigenfunctions. In [SoZ], it is shown that the Sogge $L^p$ estimates for large $p$ are
very rarely sharp. To state the result, we need some notation. Let \( V_\lambda := \{ \varphi : \Delta \varphi = \lambda \varphi \} \) denote the \( \lambda \)-eigenspace for \( \lambda \in \text{Spec}(\Delta) \) and define

\[
L^\infty(\lambda, g) = \sup_{\varphi \in V_\lambda} ||\varphi||_{L^\infty}, \quad \ell^\infty(\lambda, g) = \inf_{\text{ONB}(\varphi_j) \in V_\lambda} \left( \sup_{j=1, \ldots, \dim V_\lambda} ||\varphi_j||_{L^\infty} \right).
\]

Thus, \( L^\infty(\lambda, g) = O(\lambda^{n-1}) \) for any \((M, g)\).

We define the measure \( |L_x| \) of loops at \( x \) (see (66) for the definition) as the surface measure on \( S^*_g M \) induced by the metric \( g_x \) on \( T^*_x M \). For instance, the poles \( x_N, x_S \) of a surface of revolution \((S^2, g)\) satisfy \( |L_x| = 2\pi \).

**Theorem 9.3.** If \( L^\infty(\lambda, g) = \Omega(\lambda^{n-1}) \), then there exists a point \( x_0 \) such that \( |L_{x_0}| > 0 \). In particular, if \( g \) is analytic, then all geodesics leaving \( x_0 \) must return to \( x_0 \) at the same time.

The proof is based on a study of the remainder term in the local Weyl law. The main step is:

**Lemma 9.4.** Let \( R(\lambda, x) \) denote the remainder for the local Weyl law at \( x \). Then

\[
R(\lambda, x) = o_x(\lambda^{n-1}) \text{ if } |L_x| = 0.
\]

Additionally, if \( |L_x| = 0 \) then, given \( \varepsilon > 0 \), there exists a neighborhood \( \mathcal{N} \) of \( x \) and \( \Lambda < \infty \), both depending on \( \varepsilon \) so that

\[
|R(\lambda, y)| \leq \varepsilon \lambda^{n-1}, \ y \in \mathcal{N}, \ \lambda \geq \Lambda.
\]

Thus, there are topological conditions on \( M \) which are necessary for \( M \) to possess a real analytic metric such that some sequence of eigenfunctions has the maximally growing \( L^\infty \) norms. In dimension two, only the sphere possesses such a metric. Moreover, the maximal growth rate of \( L^\infty \) exhibited by zonal spherical harmonics can never occur for a metric on \( S^2 \) with ergodic geodesic flow.

Theorem 9.3 is certainly not the end of the story. In all known cases where ‘maximal eigenfunction growth’ occurs, all geodesics leaving the ‘pole’ \( x_0 \) return to it at the same time \( T \), not just a set of positive measure. Further, the ‘first return map’ \( G^T : S^*_x M \to S^*_x M \) is the identity map. One would hope to prove that the second property holds at least on a set of positive measure.

When the geodesic flow is ‘chaotic’ (i.e. highly mixing), it is expected that the eigenfunctions resemble Gaussian random functions. The random wave model (see §18) then predicts that eigenfunctions of Riemannian manifolds with chaotic geodesic flow should have the bounds \( ||\varphi_\lambda||_{L^p} = O(1) \) for \( p < \infty \) and that \( ||\varphi_\lambda||_{L^\infty} < C\sqrt{\log \lambda} \). But no rigorous PDE methods to date have done better than \( O(\lambda^{n-1}/\log \lambda) \). There also exist counterexamples to the logarithmic estimate on special arithmetic hyperbolic quotients (see [RS, IS, Don2]). As in the case of zonal spherical harmonics on surfaces of revolution (but using Hecke operators in place of the rotations), there exist special eigenfunctions which take large values at ‘fixed points’ of the Hecke action or theta-correspondence. In general, the exponential growth of the geodesic flow is a huge obstacle to improving the estimate beyond the logarithm. Further discussion of \( L^\infty \)-norms, as well as zeros, will be given at the end of §15 for ergodic systems.
10. Gaussian beams and quasi-modes associated to stable closed geodesics

We noted above that highest weight spherical harmonics maximized $L^p$ norms for $2 \leq p \leq \frac{2(n+1)}{n-1}$, and we noted that they had the shape of an oscillating bump along the equator with Gaussian decay in the transverse direction. Such eigenfunctions are only known in a few cases, all of them quantum completely integrable: convex surfaces of revolution, ellipsoids and ellipses. What they have in common is the existence of a stable elliptic closed geodesic along which to construct a Gaussian beam.

The construction of Gaussian beams is possible on any compact Riemannian manifold with a stable elliptic closed geodesic. However, in general it only produces an approximate eigenfunction or quasi-mode. We review them in this section since they are a good introduction to quantum Birkhoff normal forms and local models. In the following section we will consider quantum integrable systems where the construction produces true eigenfunctions.

References for Gaussian beams and more general quasi-modes are [B.B, Ra, Ra2, TZ2, W] among many other places. We follow [Z8] and [B.B] Chapter 9.

Quasi-modes along stable elliptic orbits are approximate eigenfunctions (sometimes exact) which have the form of Gaussian beams along geodesics. A Gaussian beam is a simple oscillatory function $e^{iks}$ along the geodesic times a transverse Gaussian or (more generally) Hermite function. Hence, the quasi-modes are very localized along closed geodesics and in a sense are the best localized approximate eigenfunctions.

Let $(M, g)$ be a Riemannian manifold of dimension $m = n + 1$ with a stable elliptic closed geodesic $\gamma$ of length $L$. We assume $\gamma$ is an embedded (non self-intersecting) curve with an orientable normal bundle, and use Fermi normal coordinates $(s, y)$ along $\gamma$. That is, we fix an origin $\gamma(0)$ and let $s$ denote arc-length on $\gamma$. The exponential map $\exp : N_\gamma \rightarrow T_\epsilon(\gamma)$ from the $\epsilon$-ball in the normal bundle along $\gamma$ to a tubular neighborhood of radius $\epsilon$ is a diffeomorphism and any choice of linear coordinates on $N_\gamma$ endows $T_\epsilon(\gamma)$ with Fermi normal coordinates.

We denote the eigenvalues of the linear Poincaré map by $\{e^{i\alpha_j}\}$. For $q \in \mathbb{N}^n$ we put

$$r_{kq} = \frac{1}{L}(2\pi k + \sum_{j=1}^{n}(q_j + \frac{1}{2})\alpha_j)$$

In this section, Planck’s constant takes the form,

$$h = r_{kq}^{-1}.$$ 

10.1. Local model. The model space is the normal bundle $N_\gamma$ along $\gamma$, which may be identified with $\mathbb{R}^n \times S^1$. On the quantum level, the model Hilbert space is isomorphic to $\mathcal{H} = H^2(S^1_L) \otimes L^2(\mathbb{R}^n)$, where $H^2(S^1_L)$ is the Hardy space of the circle of length $L = L_\gamma$. An orthonormal basis of $L^2(\mathbb{R}^n)$ of joint eigenfunctions of Harmonic oscillators

$$\hat{I}_j = \hat{I}_j(y, D_y) := \frac{1}{2}(D^2_{y_j} + y_j^2)$$

is provided by the Hermite functions $D_q, q \in \mathbb{N}^n$. Here, $D_0$ is the Gaussian $D_0(y) = e^{-\frac{1}{2}|y|^2}$, while $D_q := C_q A_1^{q_1} \ldots A_n^{q_n} D_0(q \in \mathbb{N}^n)$, with $C_q = (2\pi)^{-n/2}(q!)(q!)^{-1/2}, q! = q_1! \ldots q_n!$ and where

$$A_j := y_j + iD_{y_j}, \quad A_j^* = y_j - iD_{y_j}$$
are the creation/annihilation operators. An orthonormal basis of $H^2(S^1_L) \otimes L^2(\mathbb{R}^n)$ of joint eigenfunctions of $\frac{d}{ds}$ and the $\hat{I}_j$ is then furnished by

$$\varphi^o_{kq}(s, y) := e_k(s) \otimes D_q(y), \quad e_k(s) := e^{2\pi i k s}.$$ 

This model needs to be adapted to $\gamma$ in the sense that $\mathbb{R}^n \times S^1$ needs to be converted to the normal bundle $N_\gamma$. Without explaining this in detail, it may be illustrated by the fact that the adapted model transverse ground state Hermite function is

$$U_o(s, u) = (det Y(s))^{-1/2} e^{i\frac{1}{2} \Gamma(s) u, u},$$

where $Y = (y_{jk})$ is the matrix whose columns are a basis of vertical Jacobi fields along $\gamma$ and where $\Gamma(s) := \frac{dY}{ds} Y^{-1}$. Higher transverse Hermite functions are obtained by applying standard creation/annihilation operators

$$\Lambda_j = \sum_{k=1}^n (iy_{jk} D_{uk} - \frac{dy_{jk}}{ds} u_k), \quad \Lambda^*_j = \sum_{k=1}^n (-i\overline{y_{jk}} D_{uk} - \frac{\overline{dy}_{jk}}{ds} u_k),$$

(adapted to $\gamma$):

$$U_q = \Lambda_1^{q_1} ... \Lambda_n^{q_n} U_o.$$ 

Roughly speaking, the metaplectic representation $\mu$ maps the model eigenfunctions on $\mathbb{R}^n \times S^1$ to the normal bundle $N_\gamma$. Introducing the symplectic matrix

$$a_s := \begin{pmatrix} Im \dot{Y}(s)^* & Im Y(s)^* \\ Re \dot{Y}(s)^* & Re Y(s)^* \end{pmatrix},$$

with $s \in S^1_L$, we introduce the unitary metaplectic operator (depending on the parameter $s$)

$$\mu(a) := \int_{S^1_L} \mu(a_s) ds \text{ on } \int_{S^1_L} \mathcal{L}^2(\mathbb{R}^n) ds.$$ 

Then $\mu(a)$ conjugates standard creation/annihilation operators to the adapted ones, and thus explains why the full quasi-mode is obtained by applying adapted creation/annihilation operators. We refer to [ZS] [B.B] for further details.

10.2. **WKB ansatz for a Gaussian beam.** The most direct way to construct the Gaussian beam is to try to construct an approximate solution $u = e^{i\delta} U(s, y_1, \ldots, y_n)$ of $(\Delta - \lambda^2) u = 0$. Assuming that $U$ is localized in a $\lambda^{-1/2}$ neighborhood of 0, one obtains a parabolic transport equation for $U$,

$$LU := 2i\lambda \frac{\partial u}{\partial s} + \sum_{j=1}^n \frac{\partial^2 u}{\partial y_j^2} - \lambda \sum_{j,h=1}^n K_{jh}(s) y_j y_h U = 0,$$

where $K_{jk}$ are components of $R(T, Y) T$ where $R$ is the curvature tensor and where $T = \dot{\gamma}$.

The detailed ansatz $\{ \Phi_{kq}(s, \sqrt{T} q) \}$ of [B.B] is defined by

$$\Phi_{kq}(s, \sqrt{T} q) = e^{ir_{kq}^s} \sum_{j=0}^\infty r_{kq}^{-j} U_q^j(s, \sqrt{T} q, r_{kq}^{-1})$$

with $U_q^0 = U_q$ and where $U_q^j$ are constructed inductively so that

$$\Delta \Phi_{kq}(s, \sqrt{T} q, r_{kq}^{-1}) \sim \lambda_{kq}^2 \Phi_{kq}(s, \sqrt{T} q),$$

where

$$\lambda_{kq} \equiv r_{kq} + \frac{p_1(q)}{r_{kq}^2} + \frac{p_2(q)}{r_{kq}^2} + \ldots.$$
The numerators $p_j(q)$ are polynomials of degrees $j + 1$. Thus one simultaneously constructs approximate eigenfunctions and eigenvalues around a stable elliptic geodesic. For each $q$ and large $k$, the right side solves the eigenvalue problem to high approximation with a sequence of functions concentrated near $\gamma$.

Instead of scaling the WKB quasi-mode, one can scale the Laplacian. More precisely (see [Z8]), one rescales the Laplacian around $\gamma$ in $T_\epsilon(\gamma)$ and conjugates by $e^{irqk}\bar{s}$ using the operators (with $h = r_{kq}^{-1}$),

$$T_h(f(s, u)) := h^{-n/2}f(s, h^{-\frac{1}{2}}u), \quad M_h(f(s, u)) := e^{i\frac{h}{2}s}f(s, y).$$

We define the scaling of an operator $A$ as $A_h := T_h^*M_h^*AT_hM_h$. Then a simple computation gives

$$-\Delta_h = -(hL)^{-2}g_{[h]}^{oo}+2i(hL)^{-1}g_{[h]}^{oo}\partial_s+i(hL)\Gamma_{[h]}^o+h^{-1}\sum_{ij=1}^ng_{[h]}^{ij}\partial_{u_i}\partial_{u_j}+h^{-\frac{1}{2}}\sum_{i=1}^n\Gamma_{[h]}^i\partial_{u_i}+(\sigma)_{[h]},$$

where the subscript $[h]$ indicates to dilate the coefficients of the operator in the form, $f_h(s, u) := f(s, h\frac{1}{2}u)$. One then has,

$$\Delta_h \sim \sum_{m=0}^\infty h^{(-2+m/2)}L_{2-m/2}$$

where $L_2 = L^{-2}$, $L_{3/2} = 0$ and where

$$L_1 = 2L^{-1}[i\frac{\partial}{\partial s} + \frac{1}{2}\{\sum_{j=1}^n\partial_{u_j}^2 - \sum_{ij=1}^nK_{ij}(s)u_iu_j\}].$$

One then plugs into the ansatz (91) to get

$$\left(\sum_{m=0}^\infty h^{(-2+m/2)}L_{2-m/2}\right)\sum_{j=0}^\infty h^{j/2}U_q^{j/2} \sim (hL)^{-2}\sum_{j=0}^\infty h^{j/2}U_q^{j/2}$$

and solves recursively for $U_q^{j/2}$. A key point is that the metaplectic operator $\mu$ intertwines $D_s$ to the operator

$$\mathcal{L} := \mu(a)^*D_s\mu(a) = D_s - \frac{1}{2}\sum_{j=1}^nD_{u_j}^2 + \sum_{ij=1}^nK_{ij}(s)u_iu_j$$

of (90).

10.3. Quantum Birkhoff normal form: intertwining to the model. A second approach, which perhaps originates in [W] and which has been developed in many articles including [Sj, Z8, TZ, TZ2], is the method of intertwining operators. In this approach, the construction of quasi-modes follows from the dual construction of an interwining operator $W_\gamma$ to a normal form of $\Delta$ around $\gamma$. The intertwining operator maps the model eigenfunctions to the detailed quasi-mode,

$$W_\gamma\varphi_{kq}(s, y) = \Phi_{kq}(s, \sqrt{r_{kq}}y).$$
where $\Phi_{kq}$ solve (92). Rather than constructing the quasi-mode, one instead constructs $W_\gamma$ so that it conjugates $\sqrt{\Delta}$ to a function of the model operators on the model space. We are assuming that $\gamma$ is elliptic, so the model operators are harmonic oscillators.

This function is the quantum Birkhoff normal form. The normal form is implicit in (92), i.e. the normal form is the same as the quasi-eigenvalues (93) interpreted on the operator level. The precise statement is:

**Theorem 10.1.** There exists a microlocally unitary Fourier integral operator $W_\gamma$ defined near the cone $R^+\gamma$ generated by $\gamma$ in $T^*(S^1_L \times \mathbb{R}^n)$ such that

$$W_\gamma^{-1}\sqrt{\Delta}W_\gamma \equiv P_1(\mathcal{L}, I_{\gamma 1}, ..., I_{\gamma n}) + P_o(\mathcal{L}, I_{\gamma 1}, ..., I_{\gamma n}) + \ldots,$$

where

$$P_1(\mathcal{L}, I_{\gamma 1}, ..., I_{\gamma n}) \equiv \mathcal{L} + \frac{p^{[2]}_{1}(I_{\gamma 1}, ..., I_{\gamma n})}{L\mathcal{L}} + \frac{p^{[3]}_{2}(I_{\gamma 1}, ..., I_{\gamma n})}{(L\mathcal{L})^2} + \ldots$$

$$P_{-m}(\mathcal{L}, I_{\gamma 1}, ..., I_{\gamma n}) \equiv \sum_{k=m}^{\infty} \frac{p^{[k-m]}_{k}(I_{\gamma 1}, ..., I_{\gamma n})}{(L\mathcal{L})^j}$$

with $p^{[k-m]}_{k}$, for $m=-1,0,1,...$, homogenous of degree $k-m$ in the variables $(I_{\gamma 1}, ..., I_{\gamma n})$ and of weight -1. The $k$th remainder term lies in the space $\bigoplus_{j=0}^{k+2} O_{2(k+2-j)} \Psi^{1-j}$

Here, $O_n \Psi^r$ is the space of pseudodifferential operators of order $r$ whose complete symbols vanish to order $n$ at $(y, \eta) = (0,0)$. Thus, the remainder terms are ‘small’ in that they combine in some mixture a low pseudodifferential order or a high vanishing order along $\gamma$.

If one plugs in the eigenvalues $\in \mathbb{Z}^n$ for the harmonic oscillators $(I_{\gamma i}$, the operator expansion in Theorem (10.1) becomes the eigenvalue expansion in (92).

**11. Birkhoff normal forms around closed geodesics**

The classical Birkhoff normal form of a Hamiltonian around a closed geodesic (or an invariant torus) is a local approximation around $\gamma$ of the Hamiltonian by completely integrable Hamiltonians modulo errors which vanish to higher order around the orbit. One constructs local action variables $I_j$ (Hamiltonians with $2\pi$-periodic flows) and approximates the Hamiltonian by a function $H(I)$ up to an error $I^k$.

On the quantum level one has two notions of order: ‘order in $\hbar$’ and vanishing order at $\gamma$. As in Theorem 10.1, one constructs local ‘action operators’ and expresses $\hat{H} = \sqrt{\Delta}$ as a polyhomogeneous function of the action operators modulo small remainders.

In the general non-degenerate case, the normal form involves a greater variety of quadratic normal forms or ‘action operators’ than in Theorem 10.1. In addition to the elliptic action operator $\hat{I}_j^e$ there can also occur the real hyperbolic action operators $\hat{I}_j^h$ and complex hyperbolic (or loxodromic) action operators $\hat{I}_j^{ch,Re}, \hat{I}_j^{ch,Im}$.

In the elliptic case $P_\gamma$ was a direct sum of rotations, and the quantum normal form of $\mathcal{L}$ had the form

$$\mathcal{R}^e = D_s + \frac{1}{L} H_{\alpha}, \quad H_{\alpha} = \sum_{j=1}^{n} \alpha_j \hat{I}_j^e$$
where the spectrum \( \sigma(P_\gamma) = \{e^{\pm i\alpha}\} \). In the general non-degenerate case the normal form will similarly depend on the spectral decomposition of \( P_\gamma \). Recall that, since \( P_\gamma \) is symplectic, its eigenvalues \( \rho_j \) come in three types: (i) pairs \( \rho, \bar{\rho} \) of conjugate eigenvalues of modulus 1; (ii) pairs \( \rho, \rho^{-1} \) of inverse real eigenvalues; and (iii) 4-tuples \( \rho, \bar{\rho}, \rho^{-1}, \bar{\rho}^{-1} \) of complex eigenvalues. We will often write them in the forms: (i) \( e^{\pm i\alpha} \), (ii) \( e^{\pm \lambda} \), (iii) \( e^{\pm \mu, \pm \nu} \) respectively (with \( \alpha, \lambda, \mu, \nu \in \mathbb{R} \)), although a pair of inverse real eigenvalues \( \{-\bar{\epsilon}^{\pm \lambda}\} \) could be negative. Here, and throughout, we make the assumption that \( P_\gamma \) is non-degenerate in the sense that

\[
\Pi_{i=1}^{2n} \rho_i^{m_i} \neq 1, \quad (\forall \rho_i \in \sigma(P_\gamma), \quad (m_1, \ldots, m_{2n}) \in \mathbb{N}^{2n}).
\]

Each type of eigenvalue then determines a different type of quadratic action, both on the classical and quantum levels (cf. [Ho, Theorem 3.1], [Ar]):

| Eigenvalue type | Classical Normal form | Quantum normal form |
|-----------------|-----------------------|---------------------|
| (i) Elliptic type \( \{e^{\pm i\alpha}\} \) | \( I^c = \frac{1}{2} \alpha(\eta^2 + y^2) \) | \( I^c := \frac{1}{2} \alpha(D_y^2 + y^2) \) |
| (ii) Real hyperbolic type \( \{e^{\pm \lambda}\} \) | \( I^h = 2\lambda \eta \eta \) | \( I^h := \lambda(yD_y + D_y y) \) |
| (iii) Complex hyperbolic (or loxodromic type) \( \{e^{\pm \mu, \pm \nu}\} \) | \( I^{ch, Re} = 2\mu(y_1 \eta_1 + y_2 \eta_2) \) | \( I^{ch, Re} = \mu(y_1 D_{y_1} + D_{y_1} y_1 + y_2 D_{y_2} + D_{y_2} y_2), \) |
| \( I^{ch, Im} = \nu(y_1 \eta_2 - y_2 \eta_1) \) | \( I^{ch, Im} = \nu(y_1 D_{y_2} - y_2 D_{y_1}) \) |

In the case where the Poincare map \( P_\gamma \) has \( p \) pairs of complex conjugate eigenvalues of modulus 1, \( q \) pairs of inverse real eigenvalues and \( c \) quadruplets of complex hyperbolic eigenvalues, the linearized \( \sqrt{\Delta} \) will have the form:

\[
\mathcal{R} = D_s + \frac{1}{\bar{r}} \left[ \sum_{j=1}^{p} \alpha_j \hat{I}_j^e + \sum_{j=1}^{q} \lambda_j \hat{I}_j^h + \sum_{j=1}^{c} \mu_j \hat{I}_j^{ch, Re} + \nu_j \hat{I}_j^{ch, Im} \right].
\]

The full quantum Birkhoff normal form is then given by the analogue of Theorem B of [Z.1]:

**Theorem B** Assuming \( \gamma \) non-degenerate, there exists a microlocally elliptic Fourier integral operator \( W \) from the conic neighborhood of \( \mathbb{R}^+ \gamma \) in \( T^*(N_\gamma) \) to the corresponding cone in \( T^*_+S^1 \) in \( T^*(S^1 \times \mathbb{R}^n) \) such that

\[
W \sqrt{\Delta} W^{-1} \equiv D_s + \frac{1}{\bar{r}} \left[ \sum_{j=1}^{p} \alpha_j \hat{I}_j^e + \sum_{j=1}^{q} \lambda_j \hat{I}_j^h + \sum_{j=1}^{c} \mu_j \hat{I}_j^{ch, Re} + \nu_j \hat{I}_j^{ch, Im} \right] + \ldots
\]

\[
\frac{p_1(\hat{I}_1^e, \ldots, \hat{I}_p^e, \hat{I}_1^h, \ldots, \hat{I}_q^h, \hat{I}_1^{ch, Re}, \hat{I}_1^{ch, Im}, \ldots, \hat{I}_c^{ch, Re}, \hat{I}_c^{ch, Im})}{D_s} + \ldots
\]

\[
\frac{p_{k+1}(\hat{I}_1^e, \ldots, \hat{I}_c^{ch, Im})}{D_k^a} + \ldots
\]

where the numerators \( p_j(\hat{I}_1^e, \ldots, \hat{I}_p^e, \hat{I}_1^h, \ldots, \hat{I}_c^{ch, Im}) \) are polynomials of degree \( j+1 \) in the variables \( (\hat{I}_1^e, \ldots, \hat{I}_c^{ch, Im}) \) and where the kth remainder term lies in the space \( \oplus_{j=o}^{k+2} O_2(k+2-j) \Psi^{1-j} \).
Here, $O_n\Psi'$ is the space of pseudodifferential operators of order $r$ whose complete symbols vanish to order $n$ at $(y,\eta) = (0,0)$. Thus, the remainder terms are ‘small’ in that they combine in some mixture a low pseudodifferential order or a high vanishing order along $\gamma$.

### 11.1. Local quantum Birkhoff normal forms.

Readers unfamiliar with microlocal Birkhoff normal forms might keep in mind a comparison to the local models used in §3.8 to study gradient estimates and ranges of eigenfunctions. The local models were spaces of constant curvature, a classical kind of comparison in geometric analysis. The quantum Birkhoff normal forms are of a different nature: they are more global, since they are local around closed orbits rather than points. Moreover, they are dynamical and geodesic-based rather than curvature-based. The local models around regular orbits will be used in §11.2 to obtain localization results for quantum limits and in §11.3 normal forms around singular closed orbits will be used to deduce $L^p$ estimates on quantum integrable eigenfunctions. The local models are enumerated and described in §11.2. Note that they model operators are often Schrödinger operators with quadratic potentials rather than Laplacians for special metrics.

Throughout this section, the parameter $\hbar = \lambda^{-1}$ plays the role of Planck’s constant.

### 11.2. Model eigenfunctions around closed geodesics.

Model quantum completely integrable systems are direct sums of the quadratic Hamiltonians:

- $\hat{I}_{\hbar}^h := \hbar(D_{y}y + yD_{y})$ on $L^2(\mathbb{R}^n)$ (hyperbolic Hamiltonian),
- $\hat{I}_{\ell}^c := \hbar^2D_y^2 + y^2$ on $L^2(\mathbb{R})$ (elliptic harmonic oscillator Hamiltonian),
- $\hat{I}_{ch} := \hbar \{ (y_1D_{g_1} + y_2D_{g_2}) + \sqrt{-1}(y_1D_{2} - y_2D_{g_1}) \}$ on $L^2(\mathbb{R}^2)$ (complex hyperbolic Hamiltonian),
- $\hat{I} := \hbar D_{\theta}$, on $S^1$ (regular Hamiltonian).

The corresponding model eigenfunctions are:

- $u_h(y; \lambda, \hbar) = |\log \hbar|^{-1/2} \left[ c_+(\hbar)Y(y)^{\hbar} - 1^{2+i\lambda(\hbar)/\hbar} + c_-(\hbar)Y(-y)^{\hbar} - 1^{2+i\lambda(\hbar)/\hbar} \right]; |c_+(\hbar)|^2 + |c_-(\hbar)|^2 = 1; \lambda(\hbar) \in \mathbb{R}$.
- $u_{e}(y; n, \hbar) = \hbar^{-1/4} \exp(-y^2/\hbar) \Phi_n(\hbar^{-1/2}y); n \in \mathbb{N}$.
- $u_{ch}(r, \theta; t_1, t_2, \hbar) = |\log \hbar|^{-1/2} r(\hbar^{-1}i(h)/\hbar) e^{i\theta(t_2)}; t_1(h), t_2(h) \in \mathbb{R}$.
- $u_{reg}(\theta; m, \hbar) = e^{im\hbar \theta}; m \in \mathbb{Z}$.

Here, $Y(x)$ denotes the Heaviside function, $\Phi_n(y)$ the $n$-th Hermite polynomial and $(r, \theta)$ polar variables in the $(y_1, y_2)$ complex hyperbolic plane.

The important part of a model eigenfunctions is its microlocalization to a neighborhood of $x = \xi = 0$, so we put:

$$\psi(x; \hbar) := Op_{\hbar}(\chi(x)\chi(y)\chi(\xi)) \cdot u(y; \hbar),$$

where $\epsilon > 0$ and $\chi \in C^\infty_0([\epsilon, \epsilon])$. In the hyperbolic, complex hyperbolic, elliptic and regular cases, we write $\psi_h(x; \hbar), \psi_{ch}(y; \hbar), \psi_e(y; \hbar)$ and $\psi_{reg}(y; \hbar)$ respectively. A straightforward computation [112] shows that when $t_1(h), t_2(h), nh, m\hbar = O(\hbar)$ the model quasimodes are $L^2$-normalized; that is

$$\|Op_{\hbar}(\chi(x)\chi(y)\chi(\xi)) \cdot u(y; \hbar)\|_{L^2} \sim 1$$

as $\hbar \to 0$. Note that, although the model eigenfunctions above are not in general smooth functions, the microlocalizations are $C^\infty$ and supported near the origin.
12. Quantum integrable Laplacians

We now go into more detail on (globally) integrable Laplacians. As discussed in §2, quantum integrable Laplacians are the only examples where one has explicit formulae for eigenfunctions. They are also examples where one can explicitly determine the \( L^p \) norms of the eigenfunctions. Moreover, there are geometric properties of the geodesic flow which account for the behavior of the \( L^p \) norms.

Quantum integrable Laplacians are fundamental for two reasons besides the relatively explicit computability of their eigenfunctions. The first reason, which is rigorous and now well-understood, is that quantum integrable Laplacians provide local models for all Laplacians around closed geodesics. The sense in which they are local models is made precise by quantum Birkhoff normal forms. The second reason why they are fundamental is heuristic: to date, all extremals for \( L^p \) norms occur in integrable systems and are explained by the geometry of the geodesic flow. It is possible that eigenfunction sequences which achieve maximal \( L^p \) growth rates must resemble sequences of eigenfunctions of integrable systems. At least, one hopes that they must do so locally, e.g. in the vicinity of a closed geodesic.

In this section, we define quantum complete integrability and give some examples. In succeeding sections, we survey results on the norms and concentration properties of the joint eigenfunctions.

12.1. Quantum integrability and ladders of eigenfunctions. Definition: We say that the operators \( P_j \in \Psi^1(M); j = 1, \ldots, n \), generate a semiclassical quantum completely integrable system if

\[
[ P_i, P_j ] = 0; \quad \forall 1 \leq i, j \leq n,
\]

and the respective semiclassical principal symbols \( p_1, \ldots, p_n \) generate a classical integrable system with \( dp_1 \wedge dp_2 \wedge \cdots \wedge dp_n \neq 0 \) on a dense open set \( \Omega \subset T^*M \setminus \{0\} \).

Semiclassical limits are taken along ladders in the joint spectrum. For fixed \( b = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n \), we define a ladder of joint eigenvalues of the original homogeneous problem \( P_1 = \sqrt{\Delta}, P_2, \ldots, P_n \) by:

\[
\Sigma_b := \{ (\lambda_{1k}, \ldots, \lambda_{nk}) \in \text{Spec}(P_1, \ldots, P_n); \forall j = 1, \ldots, n, \lim_{k \to \infty} \frac{\lambda_{jk}}{\lambda_k} = b_j \},
\]

where \( |\lambda_k| := \sqrt{\lambda_{1k}^2 + \cdots + \lambda_{nk}^2} \).

We define the joint eigenspace corresponding to \( \Sigma_b \) as follows: For \( b \in B := \mathcal{P}(T^*M \setminus \{0\}) \), where \( \mathcal{P} \) is the classical moment map, define

\[
V_b := \{ \varphi_\mu; \| \varphi_\mu \|_{L^2} = 1 \text{ with } \mu \in \Sigma_b \}.
\]

12.2. Geometric examples.

12.2.1. The round sphere. On the quantum level we define \( \hat{I}_1 = \frac{\partial}{\partial \theta} \) and \( \hat{I}_2 = \sqrt{\Delta + \frac{1}{4}} \), so that \( \hat{I}_2 Y_m^k = k + \frac{1}{2} \). The joint spectrum of \( \hat{I}_1, \hat{I}_2 \) equals \( \{ (m, k + \frac{1}{2}); |m| \leq k \} \).

Let us compare classical and quantum values. The image of the classical moment map \( \langle p_\theta(x, \xi), |\xi| \rangle \) is the triangular region of points \( (x, y) \) satisfying \( |x| \leq y, y \geq 0 \). It is easy to see that the inverse image of any point in the interior is invariant under the \( x_3 \)-axis rotations and under the geodesic flow; this is a Lagrangian torus. The boundary of the image corresponds
to singular points of the moment map, namely the equatorial geodesic, traversed in either of its two orientations.

The joint eigenfunction $Y^k_m$, with joint eigenvalue $(m, k + \frac{1}{2})$, corresponds to the Lagrangian torus with $p_{\theta}(x, \xi) = m$ and $|\xi| = k$. If we rescale back to $S^*S^2$ we obtain the Lagrangian torus $p_{\theta} = m/k$. The Lagrangian corresponding to zonal spherical harmonics is the meridian torus $p_{\theta} = 0$, i.e. to longitudinal great circles which depart from the north pole, converge at the south pole and then return to the north pole. The highest weight spherical harmonics $Y^k_k$ correspond to the boundary points of the triangular image of the moment map, hence to the equatorial great circle. Indeed, $Y^k_k$ is the restriction of the harmonic polynomial $(x_1 + ix_2)^k$ (up to normalization). This polynomial is independent of $x_3$ and is a holomorphic function of $x_1, x_2$ so it is certainly a harmonic homogeneous polynomial. Also, by its form it clearly is largest on the unit circle in the $(x_1, x_2)$ plane and tends to zero as $(x_1, x_2) \to (0, 0)$. Hence on the sphere it defines a spherical harmonic which is large on the equator and tends to zero at the poles.

12.2.2. Simple surfaces of revolution [CV3]. The classical action variables can be quantized to produce quantum action variables $\hat{I}_j$, which are first order pseudodifferential operators with the property that $e^{2\pi i \hat{I}_j} = C_j \text{Id}$ for some constant $C_j$ of modulus one. From the fact that $e^{2\pi i \hat{I}_2} = C_j \text{Id}$ for a quantum torus action, it follows that the joint spectrum of the quantum moment map $Sp(\mathcal{I}) \subset \mathbb{Z}^2 \cap \Gamma + \{\mu\}$ is the set of integral lattice points, translated by $\mu$, in the closed convex conic subset $\Gamma \subset \mathbb{R}^2$. The vector $\mu = (\mu_1, \mu_2)$ can be identified with the Maslov indices of the homology basis of the invariant torii. In the case of the round sphere it is the projection onto the eigenspace $H_N$ of $\Delta$ of spherical harmonics of degree $N$, but in general it does not project onto eigenspaces of the Laplacian and is a rather novel feature of the pseudo-differential approach.

One may express $\sqrt{\Delta_g}$ as a function $\hat{H}(\hat{I}_1, \hat{I}_2)$ where $\hat{H}(\xi_1, \xi_2)$ is a polyhomogeneous function on $\Gamma$. Its principal symbol is the function $|\xi|_g = H(\mathcal{I})$ expressing the metric norm function in terms of the action variables. These expressions are known as the Birkhoff normal forms of the classical, resp. quantum, Hamiltonian. In generic situations, the action variables are only locally defined and there exists only a local Birkhoff normal form. The interesting feature of simple surfaces of revolution is that the Birkhoff normal forms are global.

One can then characterize the the packet of eigenvalues of $\sqrt{\Delta}$ which corresponds to the eigenfunctions with eigenvalue $N$ for $\hat{I}_2$: the eigenvalues form the set $\{\hat{H}(m, N) : |m| \leq N\}$.

12.2.3. Hyperbolic cylinders with boundary [CVP]. A hyperbolic cylinder with boundary is a metric tube $X$ around the unique closed geodesic $\gamma$ of a hyperbolic cylinder $H/\mathbb{Z}$ where $\mathbb{Z}$ acts by the cyclic group generated by a hyperbolic element. Its axis then projects to the closed geodesic $\gamma$. 

As with any surface of revolution, the Laplacian commutes with the generator \( \frac{\partial}{\partial \theta} \) of the \( S^1 \) action. Here, one puts either Dirichlet or Neumann boundary conditions on the boundary of the tube \( X \). The novel feature is that the closed geodesic is hyperbolic and lies on a singular level of the moment map and the normal form of \( \sqrt{\Delta} \) around \( \gamma \) is given by \( \hat{I}^h := \hbar(D_y y + y D_y) \).

The (not very close) analogue of the Gaussian beam or highest weight spherical harmonic is the sequence of eigenfunctions where the weight, i.e. \( \partial_i \partial_\theta \)-eigenvalue, is asymptotically the same as the \( \sqrt{\Delta} \) eigenvalue. Equivalently, if one separates variables to obtain Sturm-Liouville operators \( L_m \) for each weight, then the sequence is obtained by taking the smallest \( L_m \)-eigenvalue for each weight.

12.3. Localization of integrable eigenfunctions. In this section, we consider the construction of highly localized or concentrated eigenfunctions (and quasi-modes).

12.3.1. Toric integrable systems. Let \( A \in \Psi^0(M) \) denote any zeroth order pseudodifferential operator and \( d\mu_\lambda \) denote Lebesgue measure on the Lagrangian torus \( T_\lambda \). In the toric case we have the following localization theorem:

**Proposition 12.1.** \([Z1]\) For any ladder \( \{k\lambda + \nu : k = 0, 1, 2, \ldots \} \) of joint eigenvalues, we have:

\[
(A \varphi_{k\lambda}, \varphi_{k\lambda}) = \int_{T_\lambda} \sigma_A d\mu_\lambda + O(k^{-1}).
\]

We thus have:

**Corollary 12.2.** For any invariant torus \( T_\lambda \subset S^*M \), there exists a ladder \( \{\varphi_{k\lambda}, k = 0, 1, 2, \ldots \} \) of eigenfunctions localizing on \( T_\lambda \).

We illustrate the localization result on \( S^2 \). The proof is the same in all toric quantum integrable cases, but is harder for general integrable systems for reasons explained in the next section.

The image of \( T^*S^2 - 0 \) under the moment map \( \mu(x, \xi) = (p_\theta(x, \xi), |\xi|) \) is a vertical triangular wedge. It is a cone, reflecting that \( \mu(x, r\xi) = r\mu(x, \xi) \) is homogeneous. We can break the homogeneity by taking a base for the cone with \( |\xi| = 1 \), i.e. by considering points \( (x, 1) \). This corresponds to looking at \( p_\theta : S^*S^2 \to \mathbb{R} \).

Thus, we consider pairs \( (m_j, k_j) \) in the joint spectrum of \( D_\theta, A = \sqrt{\Delta + 1/2} - 1/2 \) whose projection to the base of the cone has a limit \( (c, 1) \).

**Theorem 12.3.** Suppose that \( m_j/k_j \to c \). Then

\[
\langle Op(a)Y^k_m, Y^k_m \rangle \to \int_{\mu^{-1}(c, 1)} a_0 dx,
\]

where \( a_0 \) is the principal symbol of \( Op(a) \).

Thus, the eigenfunctions in this ray localize on the invariant torus \( p^{-1}_\theta(c) \).

We define \( U(t_1, t_2) = e^{i(t_1 D_\theta + t_2 A)} \) and note that it is a unitary representation of the 2-torus \( T^2 \) on \( L^2(S^2) \). Further

\[
\langle Op(a)Y^k_m, Y^k_m \rangle = \langle U(t_1, t_2)^* Op(a) U(t_1, t_2) Y^k_m, Y^k_m \rangle.
\]
Indeed, the eigenvalues cancel out. Average this formula over $T^2$. We note that
\[
\langle A \rangle := \int_{T^2} U(t_1, t_2)^* Op(a) U(t_1, t_2) dt_1 dt_2
\]
commutes with both $D_\theta$ and $A$. Indeed, the commutator with $A$ gives $\frac{d}{dt_2}$ under the integral sign, and the integral of this derivative equals zero.

But $D_\theta, A$ have a simple joint spectrum: the dimension of the joint eigenspace equals one. Hence, any operator which commutes with them is a function of them. Thus,
\[
\langle A \rangle = F(D_\theta, A).
\]
The function $F$ must be homogeneous of degree zero. Also, the right side is a $\Psi D_0$. Its symbol is
\[
\langle a_0 \rangle : \int_{T^2} a_0(\Phi^{t_1, t_2}(x, \xi)) dt_1 dt_2.
\]
It follows first that
\[
\langle Op(a) Y_m^k, Y_m^k \rangle = \langle \langle Op(a) \rangle Y_m^k, Y_m^k \rangle
\]
\[
= F(m, k).
\]
Secondly, as $(m_j, k_j) \to \infty$ with $m_j/k_j \to c$, we have $F(m_j, k_j) \to F(c, 1)$. But also, the limit is the integral of $a_0$ against an invariant measure. The principal symbol of $F$ is $\langle a_0 \rangle$, which is a function on the image of the moment map. Its value at $(c, 1)$ is by definition $\int_{\mu^{-1}(c, 1)} a_0 dx$, concluding the proof.

12.3.2. $\mathbb{R}^n$-integrable systems. It is a surprisingly large step from toric integrable systems such as the torus, sphere or simple surface of revolution to a general $\mathbb{R}^n$ integrable system, even such as a surface of revolution with a hyperbolic orbit to a ‘peanut’ shaped surface in $\mathbb{R}^3$ or an undulating surface of revolution with several ‘waists’. From the dynamical point of view, there now exists a hyperbolic closed orbit (the waist of the peanut), and a cylinder of geodesics which asymptotically spiral toward the hyperbolic closed geodesic. Together they form a singular level set with several components. Eigenfunctions on this singular level Proposition (12.1) localize on level sets of the moment map rather than on individual tori or components. We prove this and further analyze the degree of concentration by putting the Laplacian into a quantum normal form around each component. We follow [TZ2] and use its semi-classical notation where $h = \lambda^{-1}$.

Let $b$ be a regular value of the moment map $\mathcal{P}$, let
\[
\mathcal{P}^{-1}(b) = \Lambda^{(1)}(b) \cup \cdots \cup \Lambda^{(m_{cl})}(b),
\]
where the $\Lambda^{(l)}(b); l = 1, \ldots, m$ are $n$-dimensional Lagrangian tori, and $d\mu_{\Lambda^{(j)}(b)}$ denote the normalized Lebesgue measure on the torus $\Lambda^{(j)}(b)$. Define the semiclassical ladders
\[
L_{b_{\delta}}(h) := \{ b_{j_1}(h), b_{j_2}(h), \ldots, b_{j_{n_{\delta}}}(h) \} \in \text{Spec}(P_1, \ldots, P_n); |b_{j_1}(h) - b| \leq C h^{1-\delta} \}.
\]
Taking a sequence $h \to 0$, the joint eigenvalues in $L_{b_{\delta}}(h)$ form a sequence tending to $b$ which is the analogue of a homogeneous ladder.

Define
\[
c_l(h; b_{j}(h)) := \langle Op_h(\chi_l) \varphi_{b_{j}(n_l)}, \varphi_{b_{j}(n_l)} \rangle; \quad l = 1, \ldots, m_{cl}(b).
\]
Here, $\chi_l$ is a cutoff function which is equal to 1 in a neighbourhood $\Omega^{(l)}(b)$ of the torus $\Lambda^{(l)}(b)$ and vanishes on $\bigcup_{l \neq b} \Omega^{(k)}(b)$.

**Proposition 12.4.** Let $b \in B_{\text{reg}}$, and let $\{\varphi_{b_j(b)}\}$ be a sequence of $L^2$-normalized joint eigenfunctions of $P_1, \ldots, P_n$ with joint eigenvalues in the ladder $L_{b, \delta}(\hbar)$. Then, for any $a \in S^{0,-\infty}$, we have that as $\hbar \to 0$:

$$\langle Op_{\hbar}(a) \varphi_{b_j(b)}, \varphi_{b_j(b)} \rangle = \left( \sum_{l=1}^{m} c_l(h; b_j(h)) \right) \int_{\Lambda^{(l)}(b)} a \, d\mu_{\Lambda^{(l)}(b)} + O(\hbar^{1-\delta}).$$

Here, $d\mu_{\Lambda^{(l)}(b)}$ denotes Lebesgue measure on $\Lambda^{(l)}(b)$.

This result says that a sequence of eigenfunctions of a quantum integrable system localizes on the level set of the moment map corresponding to the limit point in the image of the classical moment map, but it is not very precise about the weight attached to each component.

### 12.4. Conjugation to normal form around torus orbits

We sketch the proof of Proposition 12.4 as an example of the use of normal forms.

Let $\Omega^{(l)}$ be a small neighbourhood of $\Lambda^{(l)}$ on which there exist action-angle variables $(\theta^{(l)}, \epsilon^{(l)})$. Then there exists a ‘microlocal’ conjugation to quantum Birkhoff normal form. That is, for $\ell = 1, \ldots, k$ and $j = 1, \ldots, n$, there exist microlocally unitary $\hbar$-Fourier integral operators, $U_{b, \hbar}^{(l)} : C^\infty(M) \to C^\infty(\mathbb{T}^n; \mathcal{L}^{(l)})$, together with $C^\infty$ symbols, $f_j^{(l)}(x; \hbar) \sim \sum_{k=0}^{\infty} f_{jk}^{(l)}(x) \hbar^k$, with $f_{j0}(0) = 0$ such that:

$$U_{b, \hbar}^{(l)} \left( f_j^{(l)}(P_1 - b^{(1)}, \ldots, P_n - b^{(n)}; \hbar) \right) U_{b, \hbar}^{(l)*} = \frac{\hbar}{i} \frac{\partial}{\partial \theta_j}.$$

The simple operators on the right side are the normal forms.

Conjugation to normal form shows that the space of (microlocal) solutions of equations

$$P_k \varphi_{b_j(h)} = \Omega^{(l)}(b) b_j^{(k)}(\hbar) \varphi_{b_j(h)}$$

is one-dimensional. Indeed, such solutions are the same as solutions of

$$f_k^{(l)}(P_1 - b^{(1)}, \ldots, P_n - b^{(n)}; \hbar) \varphi_j = \Omega^{(l)}(b) f_k^{(l)}(b_j^{(1)} - b^{(1)}, \ldots, b_j^{(n)} - b^{(n)}; \hbar) \varphi_j.$$

After conjugation to Birkhoff normal form (99), the equation becomes

$$\frac{\hbar}{i} \frac{\partial}{\partial \theta_j} u_j = m_j u_j$$

and the solutions are just multiples of $\exp[i(n + \pi \gamma/4)\theta]$, where $\gamma$ is the Maslov index and $n \in \mathbb{Z}$. Thus, the joint eigenfunctions $\varphi_{b_j(h)}$ are given microlocally by

$$\varphi_{b_j(h)} = \Omega^{(l)}(b) \sqrt{c_l(h; b_j(h))} U_{b, h}^{(l)}(e^{(n_j + \pi \gamma/4)\theta}).$$

The right sides of (101) are the usual quasimodes or semiclassical Lagrangian distributions [CV2].

We further use the normal form to prove the localization statement. Let $\chi_l(x, \xi) \in C_0^\infty(T^*M); \ell = 1, \ldots, m_c(b)$ be a cutoff function which is identically equal to one on the
neighborhood \( \Omega^{(l)}(b) \) and vanishes on \( \Omega^{(k)}(b) \) for \( k \neq l \). For \( \hbar \) sufficiently small, we then have

\[
\langle Op(\varphi)_{\hbar} \varphi_{b_j(h)}, \varphi_{b_j(h)} \rangle = \sum_{l=1}^{m_c(b)} \langle Op(\varphi)_{\hbar} \varphi_{b_j(h)}, \varphi_{b_j(h)} \rangle + O(\hbar^\infty).
\]

In each term, we now conjugate to normal form. Recalling the definition (98) of the weights \( c_j(h, b_j(h)) \), we have

\[
(102) \langle Op(\varphi)_{\hbar} \varphi_{b_j(h)}, \varphi_{b_j(h)} \rangle = c_l(h, b_j(h)) \langle Op(\varphi)_{\hbar} \varphi_{b_j(h)}, \varphi_{b_j(h)} \rangle
\]

In the last step we used the Egorov theorem to recognize \( U^{(l)}_{\hbar} \) as a pseudo-differential operator on the standard torus and then used the calculation in (64) to complete the proof. For the details of the remainder estimate we refer to [TZ2].

### 13. Concentration and non-concentration for general \((M, g)\)

In this section we go over results about concentration on, and \( L^p \) norms along, submanifolds of general Riemannian manifolds. Then we again compare the results to the quantum integrable case.

#### 13.1. \( L^p \) norms for restrictions to submanifolds.

A measure of concentration near a submanifold is given by the \( L^p \) norms of the restrictions of eigenfunctions to the submanifold. Early results were obtained by Reznikov [R] for curves on hyperbolic surfaces. The most general results are due to Burq-Gerard-Tzvetkov [BGT] and pertain to submanifolds of general Riemannian manifolds. On curves, the estimates in [BGT] are as follows:

**Theorem 13.1.** [BGT] Let \( \| \varphi_{\lambda_j} \|_{L^2} = 1 \). Then

- (i) If \( \gamma \) is a unit-length geodesic, then

\[
\int_\gamma |\varphi_{\lambda_j}(s)|^p ds = O(\lambda_j^{\tilde{\delta}(p)}),
\]

where

\[
(103) \tilde{\delta}(p) = \begin{cases} \frac{1}{2} - \frac{1}{p}, & 4 \leq p \leq \infty \\ \frac{1}{4}, & 2 \leq p \leq 4. \end{cases}
\]

- (ii) If \( \gamma \) is a curve with strictly-positive geodesic curvature,

\[
\int_\gamma |\varphi_{\lambda_j}(s)|^p ds = O(\lambda_j^{\frac{p}{2} - \frac{1}{2}}).
\]
13.2. Non-concentration in tubes around hyperbolic closed geodesics. We now consider results which give upper bounds on the concentration of eigenfunctions in tubes around hyperbolic closed geodesics and other invariant sets. The article [CVP] gave such a bound and a general principle was then proved in [BZ] (see also [Chr]). It is stated in terms of lower bounds of eigenfunction mass outside of a tube around a hyperbolic closed geodesic.

**Theorem 13.2.** [BZ, Chr] Let \((M,g)\) be a compact Riemannian manifold, and let \(\gamma\) be a hyperbolic closed geodesic. Let \(U\) be any tubular neighborhood of \(\gamma\) in \(M\). Then for any eigenfunction \(\varphi_\lambda\), there exists a constant \(C\) depending only on \(U\) such that

\[
\int_{M\setminus U} |\varphi_\lambda|^2 dV_g \geq \frac{C}{\log \lambda} \|\varphi_\lambda\|_{L^2}^2.
\]

More generally, let \(A \in \Psi^0(M)\) be a pseudo-differential orbit whose symbol equals one in a neighborhood of \(\gamma\) in \(S^*_\gamma M\) and equals zero outside another neighborhood. Then for any eigenfunction \(\varphi_\lambda\)

\[
||(I - A)\varphi_\lambda||_{L^2} \geq \frac{C}{\sqrt{\log \lambda}} \|\varphi_\lambda\|_{L^2}.
\]

These theorems are proved in [BZ] using a rather abstract observability estimate:

**Theorem 13.3.** Let \(P(h)\) be a family of self-adjoint operators on a Hilbert space \(\mathcal{H}\), with a fixed domain \(\mathcal{D}\). Let \(\mathcal{H}_1\) be a second Hilbert space and suppose there exists a family of bounded operators \(A(h): \mathcal{D} \to \mathcal{H}_1\) satisfying

\[
\|u\|_{\mathcal{H}} \leq \frac{G(h)}{h} (\|u + g(h)\|_{\mathcal{H}} + A(h)u)_{\mathcal{H}_1},
\]

for all \(\tau \in (-b, -a)\) and with \(1 \leq G(h) \leq Ch^{-N_0}\) for some \(N_0\). Let \(\chi \in C_c^\infty(-b, -a)\). Then there exist constants \(c_0, C_0\) and \(h_0 > 0\) so that for any function \(T(h)\) satisfying \(\frac{G(h)}{T(h)} < c_0\),

\[
\|\chi(P(h))u\|_{\mathcal{H}}^2 \leq C_0 \frac{g(h)^2}{T(h)} \int_0^{T(h)} \|A(h)e^{-it P(h)/h}\|_{\mathcal{H}_1}||\chi(P(h))u||_{\mathcal{H}_1}^2 dt.
\]

This formulation has applications to non-concentration of eigenfunctions in the Buminovich stadium as well.

13.3. Non-concentration around closed geodesics on compact hyperbolic surfaces. We sketch the proof in [CVP] in the case of hyperbolic closed geodesics on a compact hyperbolic surface \(X = H\setminus \Gamma\).

Suppose \(\gamma\) is a simple closed geodesic whose normal bundle is trivial. Then it possesses a tubular (collar) neighborhood \(X_0\) which is isometric to a collar around a closed geodesic \(\gamma\) in a hyperbolic surface of revolution \(H\setminus \Gamma \simeq S^1_\theta \times \mathbb{R}_y\). We noted above that such a collar is quantum integrable and that the Birkhoff normal form of \(\sqrt{\Delta}\) around \(\gamma\) is the hyperbolic operator \(qD_y + D_y q\). Using the explicit formulae for the eigenfunctions in the model domain, one can calculate the amount of mass of the model eigenfunction near \(\gamma\), and finds that the mass increases logarithmically in \(\lambda\) (cf. [CVP], Proposition 16). The mass for the singular energy level \(E\) is explicitly given by

\[
\int_0^{\frac{\pi}{\theta}} \frac{dy}{\pi} \left| \int_0^\infty e^{-iy(\frac{E}{\hbar} - \ln y)} \chi(\frac{y}{\hbar}) \frac{dy}{\sqrt{y}} \right|^2 \sim \left| \Gamma(\frac{1}{2} + \frac{iE}{\hbar}) \right|^2 e^{\frac{E}{\hbar} \ln \hbar}.
\]
The mass of the sequence of eigenmodes of the hyperbolic surface of revolution with boundary grows at the same logarithmic rate, since it is microlocally unitarily equivalent to the model eigenfunctions.

Now consider the eigenfunctions of the compact hyperbolic surface without boundary. Unlike the hyperbolic cylinder with boundary, its geodesic flow is ergodic and far from completely integrable. Let \( \varphi_{\lambda_j} \) denote a sequence of eigenfunctions, and let

\[
\varphi_{\lambda_j}(y, \theta) = \sum_k a_{k;\lambda_j}(y) e^{ik\theta}
\]

be the Fourier series expansion of \( \varphi_{\lambda_j} \) for each \( y \). Let \( ||a_{k;\lambda_j}||_{[-1,1]} \) denote the \( L^2 \) norm of the \( k \) component in \([-1,1]\] and let \( ||\varphi_j||_{X_0} \) denote the \( L^2 \) norm in the collar. Then we have

\[
||\varphi_{\lambda_j}||_{X_0}^2 = \sum_k ||a_{k;\lambda_j}||_{[-1,1]}^2.
\]

Suppose now that the sequence \( \{\varphi_{\lambda_j}\} \) concentrates in \( X_0 \) in the sense that

\[
||\varphi_{\lambda_j}||_{X_0}^2 \geq C_0 > 0, \quad ||\varphi_j||_K^2 \to 0, \quad \forall K \subset X \setminus \gamma.
\]

Then,

\[
(105) \quad \sum_k ||a_{k;\lambda_j}||_{[-1,1]}^2 \geq C_0 > 0
\]

uniformly in \( j \). Then the only possible quantum limit measures of the sequence must have the form \( C \mu + \nu \) where \( \nu = 0 \) in the tube around \( \gamma \) and \( C > 0 \). For a time-reversal symmetric symbol \( a \) with essential support in the collar, we must have

\[
\langle Op(a)\varphi_{\lambda_j}, \varphi_{\lambda_j} \rangle \to C \int_{\gamma} a_0 ds.
\]

Now let \( P = \Delta^{-1} \frac{\partial^2}{\partial \theta^2} + 1 \), a pseudo-differential operator of order 0 whose principal symbol equals \( 1 - \frac{k^2}{\lambda^2} \). The symbol vanishes along \( \gamma \). Then

\[
(106) \quad \int_{X_0} \varphi_{\lambda_j} P \varphi_{\lambda_j} dV = \sum_k (1 - \frac{k^2}{\lambda^2}) ||a_{k;\lambda_j}||_{[-1,1]}^2 \to 0.
\]

Let \( Z_j = \{k : \left| \frac{k^2}{\lambda_j} - 1 \right| \leq \frac{1}{2}\} \). Combining (105) and (106), we have

\[
(107) \quad \sum_{k \in Z_j} ||a_{k;\lambda_j}||^2 \geq C > 0.
\]

Thus, the Fourier coefficients of a sequence of eigenfunctions concentrating near \( \gamma \) concentrate around the joint eigenvalues of the modes on the hyperbolic surface of revolution which concentrate on the central geodesic. The mass around \( \gamma \) of the associated modes can only grow logarithmically must charge the complement of any collar around \( \gamma \) with mass at least of order \( \frac{1}{\log \lambda} \). It follows from (107) that any sequence of eigenfunctions on the compact surface without boundary must also charge the complement of the collar by at least \( \frac{1}{\log \lambda} \).
14. $L^p$ Norms and Concentration in the Quantum Integrable Case

$L^p$ norms of eigenfunctions in the quantum integrable case have a much more geometric theory than in general. Explicit examples show that they are often extremals for $L^p$ norm and concentration inequalities.

One extremal problem raised in [Y1, Y2] is to determine the Riemannian manifolds which possess orthonormal bases of eigenfunctions with uniformly bounded $L^\infty$ norms. An obvious example is a flat torus. The question arises whether any others exist. The following result shows that flat tori are the unique minimizers in the class of quantum integrable Laplacians.

**Theorem 14.1.** [TZ] Suppose that $\Delta$ is a quantum completely integrable Laplacian on a compact Riemannian manifold $(M, g)$. Then

(a) If $L^\infty(\lambda, g) = O(1)$ then $(M, g)$ is flat.
(b) If $\ell^\infty(\lambda, g) = O(1)$, then $(M, g)$ is flat.

It is possible that flat tori are the only compact Riemannian manifolds with a uniformly bounded orthonormal basis of eigenfunctions. But at this time, it is not even known if the standard sphere possesses such an orthonormal basis. The idea of the proof of Theorem 14.1 is that the joint eigenfunctions concentrate on level sets of the moment map, and therefore develop singularities at points where the projection of the level set to the base manifold is singular. The only case where no singularities occur is when $(M, g)$ is a torus without conjugate points, and in this case Burago-Ivanov proved (the Hopf conjecture) that $(M, g)$ must be flat.

There also exists a quantitative improvement of Theorem 14.1 which gives blow-up rates for $L^p$ norms for quantum integrable eigenfunctions concentrating on singular level sets, i.e. level sets which are not regular in the sense of (7). These eigenfunctions are the extremals for $L^p$ blow-up and mass concentration. In the following there is an additional technical assumption (Eliasson non-degeneracy) which we omit for simplicity.

**Theorem 14.2.** [TZ2] Suppose that $(M, g)$ is a compact Riemannian manifold whose Laplacian $\Delta$ is quantum completely integrable as in (12). Then, unless $(M, g)$ is a flat torus, this action must have a singular orbit of dimension $< n$. If the minimal dimension of the singular orbits is $\ell$, then for every $\epsilon > 0$, there exists a sequence of eigenfunctions satisfying:

\[
\begin{align*}
\|\varphi_k\|_{L^\infty} &\geq C(\epsilon)\lambda_k^{\frac{n-\ell}{4}-\epsilon} \\
\|\varphi_k\|_{L^p} &\geq C(\epsilon)\lambda_k^{\frac{(n-\ell)(p-2)}{4p}-\epsilon}, \quad 2 < p < \infty.
\end{align*}
\]

Here,

- A point $(x, \xi)$ is called a singular point of the moment map $\mathcal{P}$ if $dp_1 \wedge \cdots \wedge dp_n(x, \xi) = 0$.
- A level set $\mathcal{P}^{-1}(c)$ of the moment map is called a singular level if it contains a singular point $(x, \xi) \in \mathcal{P}^{-1}(c)$.
- An orbit $\mathbb{R}^n \cdot (x, \xi)$ of $\Phi_t$ is singular if it is non-Lagrangian, i.e. has dimension $< n$.

The idea in the proof is to consider singular orbits and to conjugate to a quantum Birkhoff normal form around the orbit. Hence, one calculates the mass in the normal form space.
Conjugation to normal form does not preserve $L^p$ norms, and what is really calculated are not $L^p$ norms but rather $L^2$ norms in shrinking tubes. Interestingly, this method produces optimal results. The proof does not determine the minimal dimension $\ell$. By taking products of lower dimensional manifolds, it is easy to construct examples with any value of $\ell = 1, \ldots, n - 1$.

14.1. Mass concentration on small length scales. We sketch the proof of Theorem 14.2 as an illustration of mass estimates in shrinking tubes. We follow the semi-classical notation $\hbar = \lambda^{-1}$ of [172].

Let $\Lambda := \mathbb{R}^n \cdot v$ be a compact, $k < m$-dimensional singular orbit of the Hamiltonian $\mathbb{R}^n$-action generated by $(p_1, \ldots, p_n)$. In this section, we study mass concentration of modes in shrinking tubes of radius $\sim 1/4$. Mass concentration on small length scales.

Conjugation to normal form does not preserve $L^p$ norms. Also, $\hbar = \hbar$. Theorem 14.3. Let $\varphi_\mu \in V_c(\hbar)$ satisfy the bounds in Lemma 92. Then for any $0 < \delta < 1/2$, $(\mathcal{O}_h(\chi^1_\delta(x; \hbar))\varphi_\mu, \varphi_\mu) \gg |\log \hbar|^{-m}$.

We briefly sketch the proof. Let $\chi^1_\delta(x, \xi; \hbar) \in C^\infty_0(T^*M; [0, 1])$ be a second cutoff supported in a radius $\hbar^c$ tube, $\Omega(\hbar)$, around $\Lambda$ with $\Omega(\hbar) \subset \text{supp}\chi^1_\delta$ and such that $\chi^1_1 = 1$ on $\text{supp}\chi^2_\delta$. Then, clearly

$$\chi^1_\delta(x, \xi) \geq \chi^2_\delta(x, \xi),$$

for any $(x, \xi) \in T^*M$. By Garding’s inequality, (109) implies

$$\left( \mathcal{O}_h(\chi^1_\delta) \varphi_\mu, \varphi_\mu \right) \gg \left( \mathcal{O}_h(\chi^2_\delta) \varphi_\mu, \varphi_\mu \right).$$

We now conjugate the right side to the model by the $\hbar$- Fourier integral operator $F$ of Lemma 99. Since $F$ is a microlocally elliptic $\hbar$-Fourier integral operator associated to a canonical transformation $\kappa$, it follows by Egorov’s theorem

$$\left( \mathcal{O}_h(\chi^2_\delta) \varphi_\mu, \varphi_\mu \right) = |\tilde{c}(\hbar)|^2 \left( \mathcal{O}_h(\chi^2_\delta \circ \kappa) u_\mu, u_\mu \right) - C_3 \hbar^{1-2\delta}$$

where $c(\hbar) u_\mu(y, \theta; \hbar)$ is the microlocal normal form for the eigenfunction $\varphi_\mu$. Since $\varphi_\mu \in V_c(\hbar)$ satisfies the bounds in Lemma 93, it follows that $|\tilde{c}(\hbar)|^2 \gg |\log \hbar|^{-m}$ and from (111) we are left with estimating the matrix elements $(\mathcal{O}_h(\chi^2_\delta \circ \kappa) u_\mu, u_\mu)$ from below. As in the localization proof, the matrix elements are now in terms of elementary model eigenfunctions and the calculation has become easy. The normal form eigenfunctions separate into a product
of factors and one only has to calculate one (or two) dimensional integrals. As an example, in the hyperbolic case the integral has the form

\[ M_h = \frac{1}{\log h} \left( \int_0^{\infty} \chi(h\xi/h^3) \left| \int_0^{\infty} e^{-ix\cdot x-1/2+i\lambda/h} \chi(x/h^3\xi)dx \right|^2 \frac{d\xi}{\xi} \right) \]

(112)

\[ \geq \frac{1}{c_0} (\log h)^{-1} \int_0^{h^{-1}} \frac{d\xi}{\xi} \left| \int_0^{h^3\xi} e^{-ix\cdot x-1/2+i\lambda/h} dx \right|^2 + O(|\log h|^{-1}) \]

\[ \gg |\Gamma(1/2+i\lambda/h)|^2 (1 - 2\delta) + O(|\log h|^{-1}) \geq C(\epsilon) > 0 \]

uniformly for \( h \in (0, h_0(\epsilon)] \).

**14.1.1. Completion of the proof of Theorem 14.2.** The small scale mass estimates immediately imply lower bounds on \( L^\infty \) norms and \( L^p \) norms due to the shrinking volumes of the tubes. For instance,

\[ \int_M |\varphi_\mu(x)|^2 \chi_1(x; h) dvol(x) \leq \sup_{x \in \mathcal{T}_{\alpha}(\pi(\Lambda))} |\varphi_\mu(x)|^2 \int_M \chi_1^4(x; h) dvol(x) \]

(113)

\[ \leq \|\varphi_\mu\|_{L^\infty}^2 \cdot \int_M \chi_1^4(x; h) dvol(x) \]

and it follows from Lemma 14.3 that

\[ \|\varphi_\mu\|_{L^\infty}^2 \cdot \left( \int_M \chi_1^4(x; h) dvol(x) \right) \geq C(\epsilon) |\log h|^{-m}, \]

(114)

uniformly for \( h \in (0, h_0(\epsilon)] \). Since

\[ \int_M \chi_1^4(x; h) dvol(x) = O(h^{\delta(n-\ell)}), \]

(115)

implies

\[ \|\varphi_\mu\|_{L^\infty}^2 \geq C(\epsilon) h^{-\frac{1}{2}(n-\ell)+\epsilon} |\log h|^{-m}. \]

Recalling that \( h^{-1} \in \{\lambda_j; \lambda_j \in Spec - \sqrt{\Delta} \} \), this gives:

\[ \|\varphi_{\lambda_j}\|_{L^\infty} \geq C(\epsilon) \lambda_j^{\frac{n-\ell}{2}-\epsilon}. \]

**14.1.2. Concentration of quantum integrable eigenfunctions on submanifolds.** Similar methods were used in [102] to obtain sharp bounds on \( L^2 \) norms for restrictions to submanifolds in the quantum integrable case, making more precise the results of [BGT] in this special case. For simplicity, let us consider curves on surfaces. First is the generic upper bound:

**Theorem 14.4.** ([To2] Let \( \varphi_{\lambda_j}; j = 1, 2, 3, \ldots \) be the \( L^2 \)-normalized joint Laplace eigenfunctions of the commuting operators \( P_1 = -\Delta \) and \( P_2 \) on a Riemannian surface \( (M^2, g) \). Then for a generic curve \( \gamma \) such that \( i^*P_2|_{S^*_\gamma M} \) is Morse, we have

\[ \int_\gamma |\varphi_{\lambda_j}|^2 ds = O(|\gamma| (\log \lambda_j)). \]

When the curve is a geodesic, the bounds depend on the type of level set the geodesic lies on:

**Theorem 14.5.** ([To2] Let \( P_j(h); j = 1, 2 \) be a non-degenerate quantum integrable system on a surface, \( (M, g) \). Then,
• (i) When $\gamma$ is the projection of a geodesic segment contained in $P^{-1}(B_{reg})$,
\[
\int_{\gamma} |\varphi_{\lambda_j}(s)|^2 ds = O_{|\gamma|}(1),
\]

• (ii) When $\gamma$ is the projection of a singular joint orbit in $P^{-1}(B_{sing})$,
\[
\int_{\gamma} |\varphi_{\lambda_j}(s)|^2 ds = O_{|\gamma|}(\lambda_j^{1/2}).
\]

Moreover, there exists a constant $c_\gamma > 0$ depending only on the curve $\gamma$, and a subsequence of joint eigenfunctions, $\varphi_{\lambda_j} = j_1, j_2, \ldots$ such that
\[
\int_{\gamma} |\varphi_{\lambda_j}(s)|^2 ds \geq c_\gamma \lambda_j^{1/2} \quad \text{when } \gamma \text{ is stable},
\]
\[
\int_{\gamma} |\varphi_{\lambda_j}(s)|^2 ds \geq c_\gamma \lambda_j^{1/2} |\log \lambda_j|^{-1} \quad \text{when } \gamma \text{ is unstable}.
\]

Thus the exact bound depends on the nature of the geodesic. In the general quantum integrable case, most geodesics lie on regular Lagrangian tori in $P^{-1}(B_{reg})$ and these geodesics do not support large $L^2$-bounds. But as in Theorems 14.1 and 14.2, there always exists a subsequence of joint eigenfunctions of $P_1$ and $P_2$ with mass concentrated along (singular) orbits contained in $P^{-1}(B_{sing})$, and the associated eigenfunctions saturate the upper bounds. For instance in the case of a simple surface of revolution, the equator is the projection of a singular orbit of the $\mathbb{R}^2$ action generated by geodesic flow and rotation. The corresponding joint eigenfunctions (the analogs of highest weight spherical harmonics) satisfy $\int_{\gamma} |\varphi_{\lambda_j}|^2 ds \sim \lambda_j^{1/2}$ along the equator, $\gamma$. The equatorial geodesic is singular and the $L^2$ norms along it had singular blowup. In the case of the meridian great circles, the closed geodesic lies in the base space projection of a maximal Lagrangian torus. The zonal harmonics have $\hbar$-microsupport on this torus and have $L^2$-restriction bound $\sim \log \lambda$ along any meridian great circle.

15. Delocalization in quantum ergodic systems, I

In this section, we discuss general results on eigenfunctions when the geodesic flow of $(M, g)$ is assumed to be ergodic (see §1 and §6.4 for definition and notation). The study of eigenfunctions and eigenvalues of Laplacians on manifolds with ergodic (or more highly mixing) geodesic flows is generally known as ‘quantum chaos’. The basic question is, what impact do dynamical properties of the geodesic flow $g^t$ have on eigenvalues and eigenfunctions of its quantization $U_t$? This question has been studied over the last three decades by a large collection of mathematicians and physicists, using both theoretical and computational methods. In this section, we largely follow our recent survey [Z3]. Another exposition with emphasis on arithmetic hyperbolic quotients is [Sar2]. For recent computational results, we refer to [Bar].

One of the basic and most studied problem is Problem 6.6 for quantizations of classically ergodic systems. The main result is that there exists a subsequence $\{\varphi_{j_k}\}$ of eigenfunctions whose indices $j_k$ have counting density one for which $\rho_{j_k}(A) := \langle A\varphi_{j_k}, \varphi_{j_k} \rangle \to \omega(A)$ (where as above $\omega(A) = \frac{1}{\mu(S^*M)} \int_{S^*M} \sigma d\mu$ is the normalized Liouville average of $\sigma(A)$). Such a sequence
of eigenfunctions is called a sequence of ‘ergodic eigenfunctions’. The key quantities to study are the quantum variances

\[ V_A(\lambda) := \frac{1}{N(\lambda)} \sum_{j: \lambda_j \leq \lambda} |\langle A\varphi_j, \varphi_j \rangle - \omega(A)|^2. \]

**Theorem 15.1.** [Sh.1, Sh.2, Z2, CV, Su, ZZw, GL, Z3] Let \((M, g)\) be a compact Riemannian manifold (possibly with boundary), and let \(\{\lambda_j, \varphi_j\}\) be the spectral data of its Laplacian \(\Delta\). Then the geodesic flow \(G^t\) is ergodic on \((S^*M, d\mu)\) if and only if, for every \(A \in \Psi^o(M)\), we have:

1. \(\lim_{\lambda \to \infty} V_A(\lambda) = 0.\)
2. \(\forall \varepsilon \exists \delta \lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{j \neq k, \lambda_j, \lambda_k \leq \lambda} |(A\varphi_j, \varphi_k)|^2 < \varepsilon.\)

Since all the terms in (1) are positive, no cancellation is possible, hence (1) is equivalent to the existence of a subset \(S \subset \mathbb{N}\) of density one such that \(Q_S := \{d\Phi_k : k \in S\}\) has only \(\omega\) as a weak* limit point.

As explained in [Z9], this ergodicity of eigenfunctions may be viewed as a convexity theorem: By the Banach-Alaoglu theorem, the set of invariant probability measures for the geodesic flow is a compact convex set \(\mathcal{M}_I\). An invariant measure is ergodic if it is an extreme point of the compact convex set. The same is true on the quantum level: The set of \(\mathcal{E}_R\) of invariant states for \(\alpha_t\) is a convex set. A classical invariant measure is an invariant state, and if it is ergodic classically it is also ergodic quantum mechanically, i.e. it is an extreme point of \(\mathcal{E}_R\). Hence Liouville measure \(\omega\) is an extreme point of this convex set. But the local Weyl law says that \(\omega\) is the limit of the convex combination \(\frac{1}{N(E)} \sum_{\lambda_j \leq E} \rho_j\). An extreme point cannot be written as a convex combination of other states unless all the states in the combination are equal to it. In our case, \(\omega\) is only a limit of an infinite sequence of convex combinations, and the result is that almost all terms in the sequence tend to \(\omega\), and that is equivalent to (1).

**Sketch of Proof of (1)**

Let

\[ \langle A \rangle_T := \frac{1}{2T} \int_{-T}^T U_t^*AU_t dt. \]

Then,

\[ \sum_{\lambda_j \leq \lambda} |\langle A\varphi_{\lambda_j}, \varphi_{\lambda_j} \rangle - \omega(A)|^2 = \sum_{\lambda_j \leq \lambda} |\langle \langle A \rangle_T - \omega(A) \varphi_{\lambda_j}, \varphi_{\lambda_j} \rangle|^2. \]

Apply the Schwartz inequality for states,

\[ \sum_{j: \lambda_j \leq \lambda} |(B\varphi_{\lambda_j}, \varphi_{\lambda_j})|^2 \leq \text{Tr} \Pi_{[0,\lambda]} B^*B, \]
where $\Pi_{[0,\lambda]}$ is the spectral projection for $\sqrt{\Delta}$ corresponding to the interval $[0,\lambda]$, to the operator $B = \Pi_{[0,\lambda]}(A) - \omega(A)\Pi_{[0,\lambda]}$. We then have

$$\sum_{\lambda_j \leq \lambda} |\langle A \rangle_T - \omega(A)\varphi_{\lambda_j}, \varphi_{\lambda_j} |^2 \leq \text{Tr} (\Pi_{[0,\lambda]}(A) - \omega(A)\Pi_{[0,\lambda]})^* [\Pi_{[0,\lambda]}(A) - \omega(A)\Pi_{[0,\lambda]}]$$

$$\leq \text{Tr} (\Pi_{[0,\lambda]}(A) - \omega(A))^* (\langle A \rangle_T - \omega(A))\Pi_{[0,\lambda]}$$

$$= \omega((\langle A \rangle_T - \omega(A))^*(\langle A \rangle_T - \omega(A))).$$

Here, we used the Jensen inequality,

$$\frac{1}{N(\lambda)} \text{Tr} \varphi(\Pi_{[0,\lambda]}(A) - \omega(A))\Pi_{[0,\lambda]} \leq \frac{1}{N(\lambda)} \text{Tr} \Pi_{[0,\lambda]} \varphi((\langle A \rangle_T - \omega(A)))\Pi_{[0,\lambda]},$$

valid for any convex function \(\varphi\) in the case \(\varphi(x) = x^2\). By the local Weyl law, we get

$$\lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle A \varphi_{\lambda_j}, \varphi_{\lambda_j} \rangle - \omega(A) |^2 \leq \int_{S^*M} |\langle \sigma_A \rangle_T - \omega(A) |^2 d\mu.$$

As \(T \to \infty\) the right side approaches 0 by the dominated convergence theorem and by Birkhoff’s ergodic theorem. Since the left hand side is independent of \(T\), this implies the stated theorem.

15.1. Quantum ergodicity in terms of operator time and space averages. To explain the term ‘quantum ergodicity’, we reformulate the result in terms of space and time averages. We assume for simplicity the generic condition that all eigenvalues are of multiplicity one (cf. [U]). The space and time averages are defined as follows:

**Definition** Let \(A \in \Psi^0\) be an observable and define its time average to be:

$$\langle A \rangle := w - \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T U_t^* AU_t dt$$

and its space average to be scalar operator

$$\omega(A) \cdot I$$

The limit is taken in the weak operator topology, i.e. in the sense of matrix elements. To verify it we observe that

$$\langle \frac{1}{2T} \int_{-T}^T U_t^* AU_t dt, \varphi_i, \varphi_j \rangle = \frac{\sin T(\lambda_i - \lambda_j)}{T(\lambda_i - \lambda_j)} (A\varphi_i, \varphi_j),$$

hence the matrix element tends to zero as \(T \to \infty\) unless \(\lambda_i = \lambda_j\). The limit only occurs in the weak sense since the rate is clearly not uniform, as the spacings \(\lambda_i - \lambda_j\) could be arbitrarily small.

Quantum ergodicity can thus be reformulated as the condition,

$$\langle A \rangle = \omega(A)I + K,$$

where \(\omega(\lambda) = Tr E(\lambda)A\). Thus, the time average equals the space average moduli a term \(K\) whose Hilbert-Schmidt norm in the range of \(\Pi_\lambda\) is \(o(N(\lambda))\). Note that \(\langle A \rangle\) commutes with \(\sqrt{\Delta}\), hence is diagonal in the basis \(\{\varphi_j\}\) of joint eigenfunctions of \(\langle A \rangle\) and of \(U_t\). Also, \(K\) is
the diagonal matrix with entries $\langle A\varphi_k, \varphi_k \rangle - \omega(A)$. The condition is therefore equivalent to $V_A(\lambda) \to 0$

15.2. **Quantum unique ergodicity and converse quantum ergodicity.** A Laplacian is said to be QUE (quantum uniquely ergodic) if Liouville measure is the only weak * limit point of the semi-classical Wigner measures. The terminology was introduced in [RS].

The condition may be reformulated in terms of $K$: Namely, QUE is equivalent to the compactness of $K$. Indeed, this would imply that $\langle K\varphi_k, \varphi_k \rangle \to 0$, hence $\langle A\varphi_k, \varphi_k \rangle \to \omega(A)$ along the entire sequence. A key difficulty of settling the question whether $K$ is compact is that the time averaged operator $\langle A \rangle$ no longer belongs to the class of pseudo-differential operators, due to the very weak nature of the weak operator limit.

It is widely conjectured there exist non QUE $(M, g)$ with ergodic geodesic flow. We refer to [Bar] [BZ2] [Z9] for some recent discussions and references to the literature. The simplest example is that of the Bunimovich stadium, which possesses quasi-modes concentrating on the invariant Lagrangian cylinder (with boundary) formed by bouncing ball orbits in the middle rectangle. An analogue among Riemannian manifolds without boundary is that of a non-positively curved surface with a flat cylindrical part and ergodic geodesic flow [Don1], which also carries product quasi-modes of the same kind. The existence of such quasi-mode suggests that there are nearby modes which are not quantum ergodic (see [Z9] for the precise suggestion). Faure-Nonnenmacher-de Bièvre [FNB] have shown that QUE does not hold for the hyperbolic system defined by a quantum cat map on the torus, and since the methods available for studying eigenfunctions of quantum maps and of Laplacians are very similar, this negative result shows that there cannot exist a universal structural proof of QUE. A QUE result has been proved by E. Lindenstrauss, namely the QUE property for the orthonormal basis of Laplace-Hecke eigenfunctions eigenfunctions on arithmetic hyperbolic surfaces (see [LIND]).

To the author, an interesting and almost completely open problem is the converse of quantum ergodicity: does quantum ergodicity imply classical ergodicity? More precisely, are there natural sufficient conditions for this? For instance, if $\langle A \rangle = \omega(A) + K$ where $K$ is compact, is the geodesic flow ergodic? Very little is known on this converse problem at present. One may imagine a system which is non-ergodic but in which ‘tunnelling’ causes eigenfunctions to become uniformly distributed in phase space. A simple model is the one-dimensional Schrödinger operator with the even double well potential $V(x) = (1 - x^2)^2$. Level sets $\xi^2 + V(x) = E$ with $E$ near zero have two components and hence the Hamiltonian flow is not ergodic on this level; yet the eigenfunctions are either even or odd and hence they are quantum ergodic. No such example is known in the case of Laplacians $\Delta$ of compact Riemannian manifolds. However, the example shows that no general ‘abstract’ proof of $QE \implies CE$ is possible.

In [Z5] [MOZ] it is shown that if there exists an open set in $S^*M$ filled by periodic orbits, then the Laplacian cannot be quantum ergodic. But it has not even been proved that KAM systems, which have Cantor-like positive measure invariant sets, are not quantum ergodic although there exists a positive density of quasi-modes concentrating on invariant tori [Pop]. The problem is to prove that eigenfunctions are linear combinations of not too many quasi-modes.
15.3. **Quantum weak mixing.** Quantum weak mixing concerns the off-diagonal matrix elements.

**Theorem 15.2. (see [Z3] for references)** The geodesic flow $\Phi^t$ of $(M, g)$ is weak mixing if and only if the conditions (1)-(2) of Theorem 15.1 hold and additionally, for any $A \in \Psi^o(M)$,

$$\forall \epsilon \exists \delta \lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda, \lambda_j - \lambda_k - \tau < \delta} |\langle A \varphi_j, \varphi_k \rangle|^2 < \epsilon \quad (\forall \tau \in \mathbb{R})$$

The restriction $j \neq k$ is of course redundant unless $\tau = 0$, in which case the statement coincides with quantum ergodicity. This result follows from the general asymptotic formula, valid for any compact Riemannian manifold $(M, g)$, that

$$\lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle A \varphi_i, \varphi_j \rangle|^2 \frac{\sin T(\lambda_i - \lambda_j - \tau)}{T} \sim \| \frac{1}{T} \int_{-T}^T e^{it\tau} V_t(\sigma_A) \|^2 - |\sin T\tau| \omega(A)^2.$$  

In the case of weak-mixing geodesic flows, the right hand side $\to 0$ as $T \to \infty$.

15.4. **Spectral measures and matrix elements.** Theorem 15.2 is based on expressing the spectral measures of the geodesic flow in terms of matrix elements. The main limit formula is:

$$\int_{\tau = -\epsilon}^{\tau + \epsilon} d\mu_{\sigma_A} := \lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{i,j: \lambda_i, \lambda_j \leq \lambda, |\lambda_i - \lambda_j - \tau| < \epsilon} |\langle A \varphi_i, \varphi_j \rangle|^2 ,$$

where $d\mu_{\sigma_A}$ is the spectral measure for the geodesic flow corresponding to the principal symbol of $A$, $\sigma_A \in C^\infty(S^*M, d\mu)$. Recall that the spectral measure of $V_t$ corresponding to $f \in L^2$ is the measure $d\mu_f$ defined by

$$\langle V_t f, f \rangle_{L^2(S^*M)} = \int_{\mathbb{R}} e^{it\tau} d\mu_f(\tau) .$$

The limit formula (122) is equivalent to the dual formula (under the Fourier transform)

$$\lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{i,j: \lambda_i, \lambda_j \leq \lambda} e^{it(\lambda_i - \lambda_j)} |\langle A \varphi_i, \varphi_j \rangle|^2 = \langle V_t \sigma_A, \sigma_A \rangle_{L^2(S^*M)} .$$

The proof of (123) is to consider, for $A \in \Psi^o$, the operator $A^*_t A \in \Psi^o$ with $A_t = U_t^* A U_t$. By the local Weyl law,

$$\lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \text{Tr} E(\lambda) A^*_t A = \langle V_t \sigma_A, \sigma_A \rangle_{L^2(S^*M)} .$$

The right side of (122) defines a measure $d\mu_A$ on $\mathbb{R}$ and (123) says

$$\int_{\mathbb{R}} e^{it\tau} d\mu_A(\tau) = \langle V_t \sigma_A, \sigma_A \rangle_{L^2(S^*M)} = \int_{\mathbb{R}} e^{it\tau} d\mu_{\sigma_A}(\tau) .$$

Since weak mixing systems are ergodic, it is not necessary to average in both indices along an ergodic subsequence:
(124) \[ \lim_{\lambda_j \to \infty} \langle A^*_t A \varphi_j, \varphi_j \rangle = \sum_j e^{it(\lambda_i - \lambda_j)} |\langle A \varphi_i, \varphi_j \rangle|^2 = \langle V_t \sigma_A, \sigma_A \rangle_{L^2(S^*M)}. \]

Dually, one has

(125) \[ \lim_{\lambda_j \to \infty} \sum_{i : |\lambda_i - \lambda_j - \tau| < \epsilon} |\langle A \varphi_i, \varphi_j \rangle|^2 = \int_{\tau - \epsilon}^{\tau + \epsilon} d\mu_{\sigma_A}. \]

For QUE systems, these limit formulae are valid for the full sequence of eigenfunctions.

15.5. **Rate of quantum ergodicity and mixing.** A quantitative refinement of quantum ergodicity is to ask at what rate the sums in Theorem 15.1(1) tend to zero, i.e. to establish a rate of quantum ergodicity. In the off-diagonal case one may view \( |\langle A \varphi_i, \varphi_j \rangle|^2 \) as analogous to \( |\langle A \varphi_j, \varphi_j \rangle - \omega(A)|^2 \). However, the sums in (122) are double sums while those of (116) are single. One may also average over the shorter intervals \([\lambda, \lambda + 1]\).

The only rigorous result valid on general Riemannian manifolds with hyperbolic geodesic flow is the logarithmic decay:

**Theorem 15.3.** \[ (Z5) \text{(see also Schu2 for } p = 2) \text{ For any } (M, g) \text{ with hyperbolic geodesic flow,} \]

\[ \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle A \varphi_j, \varphi_j \rangle - \omega(A)|^{2p} = O \left( \frac{1}{(\log \lambda)^p} \right). \]

The proof uses the central limit theorem for geodesic flows of M. Ratner. The logarithm reflects the exponential blow up in time of remainder estimates for traces involving the wave group associated to hyperbolic flows. It would be surprising if the logarithmic decay is sharp for Laplacians. It was shown by R. Schubert [Schu] that the estimate is sharp in the case of two-dimensional hyperbolic quantum cat maps. Hence the estimate cannot be improved by semi-classical arguments that hold in both settings.

A stunning asymptotic formula for \( V_A(\lambda) \) was proved by Luo-Sarnak for holomorphic forms of arithmetic hyperbolic quotients. Before stating the result, we review some conjectures in the physics literature.

15.6. **Quantum chaos conjectures.** First, consider off-diagonal matrix elements. One conjecture is that it is not necessary to sum in \( j \) in (125): each individual term has the asymptotics consistent with (125). This is implicitly conjectured by Feingold-Peres in [FP] (11) in the form

\[ |\langle A \varphi_i, \varphi_j \rangle|^2 \simeq \frac{C_A(\frac{E_i - E_j}{\hbar})}{2\pi \rho(E)}, \]

where \( C_A(\tau) = \int_{-\infty}^{\infty} e^{-\tau t} \langle V_t \sigma_A, \sigma_A \rangle dt \). In our notation, \( \lambda_j = \hbar^{-1} E_j \) and \( \rho(E) dE \sim dN(\lambda) \).

There are \( \sim \lambda^{n-1} \) eigenvalues \( \lambda_i \) in the interval \([\lambda_j - \tau - \epsilon, \lambda_j - \tau + \epsilon]\), so (126) says that individual terms have the asymptotics of (125).
On the basis of the analogy between \(|\langle A\varphi_i, \varphi_j \rangle|^2\) and \(|\langle A\varphi_i, \varphi_j \rangle - \omega(A)\)|, it is conjectured in [FP] that

\[
V_A(\lambda) \sim \frac{C_{A-\omega(A)} I(0)}{\lambda^{n-1} \text{vol}(\Omega)}.
\]

The idea is that \(\varphi_{\pm} = \frac{1}{\sqrt{2}}(\varphi_i \pm \varphi_j)\) have the same matrix element asymptotics as eigenfunctions when \(\lambda_i - \lambda_j\) is sufficiently small. But then \(2\langle A\varphi_{\pm}, \varphi_{\pm} \rangle = \langle A\varphi_i, \varphi_i \rangle - \langle A\varphi_j, \varphi_j \rangle\) when \(A^* = A\). Since we are taking a difference, we may replace each matrix element by \(\langle A\varphi_i, \varphi_i \rangle\) by \(\langle A\varphi_i, \varphi_i \rangle - \omega(A)\) (and also for \(\varphi_j\)). The conjecture then assumes that \(\langle A\varphi_i, \varphi_i \rangle - \omega(A)\) has the same order of magnitude as \(\langle A\varphi_i, \varphi_i \rangle - \langle A\varphi_j, \varphi_j \rangle\). The order of magnitude is predicted by some natural random wave models, as discussed below in [Ls].

15.7. Rigorous results. At this time, the strongest variance result is an asymptotic formula for the diagonal variance proved by Luo-Sarnak for special Hecke eigenfunctions on the quotient \(H^2/SL(2, \mathbb{Z})\) of the upper half plane by the modular group [LsS2]. What they prove is an asymptotic variance formula for holomorphic Hecke eigenforms. One expects that their proof, suitably modified, extends to smooth Maass-Hecke eigenfunctions. Therefore, as in [Z3], we describe the statement for smooth eigenfunctions as a Theorem/Conjecture – i.e. it is a Theorem for holomorphic forms, but still a conjecture for non-holomorphic forms. Note that \(H^2/SL(2, \mathbb{Z})\) is a non-compact finite area surface whose Laplacian \(\Delta\) has both a discrete and a continuous spectrum. The discrete Hecke eigenfunctions are joint eigenfunctions of \(\Delta\) and the Hecke operators \(T_p\).

**Theorem/Conjecture 1.** [LsS2] Let \(\{\varphi_k\}\) denote the orthonormal basis of Hecke eigenfunctions for \(H^2/SL(2, \mathbb{Z})\). Then there exists a quadratic form \(B(f)\) on \(C_0^\infty(H^2/SL(2, \mathbb{Z}))\) such that

\[
\frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \int_X |f|_{\varphi_j}^2 d\text{vol} - \frac{1}{\text{vol}(X)} \int_X f d\text{Vol}|^2 = \frac{B(f, f)}{\lambda} + o(\frac{1}{\lambda}).
\]

When the multiplier \(f = \varphi_\lambda\) is itself an eigenfunction, Luo-Sarnak have shown that

\[
B(\varphi_\lambda, \varphi_\lambda) = C_{\varphi_\lambda}(0) L(\frac{1}{2}, \varphi_\lambda)
\]

where \(L(\frac{1}{2}, \varphi_\lambda)\) is a certain \(L\)-function. Thus, the conjectured classical variance is multiplied by an arithmetic factor depending on the multiplier which has no dynamical significance. At this time, it is unknown whether variance asymptotics exist in the non-arithmetic case. From numerical experiments, it is believed that arithmetic settings behave somewhat differently from non-arithmetic ones (see [Sar2]), and this could be another example of the non-generic behavior of arithmetic quantum chaos.

15.8. Quantum limits on a hyperbolic surface and Patterson-Sullivan distributions. In this section, we mention a curious link between quantum limits and classical dynamics on a hyperbolic quotient that was observed in [AZ]. It is related to the invariant triple products studied in [Sar, BR, MS].

We write \(G = PSU(1, 1) := SU(1, 1)/\pm I \equiv PSL(2, \mathbb{R}), K = PSO(2)\) and identify the quotient \(G/K\) with the hyperbolic disc \(D\). We let \(\Gamma \subset G\) denote a co-compact discrete group and let \(X_\Gamma = \Gamma \backslash D\) denote the associated hyperbolic surface. In this context it is standard
to denote eigenvalues by \( \lambda_j^2 = s_j(1 - s_j) = \frac{1}{4} + r_j^2 \) \((s_j = \frac{1}{2} + ir_j)\) and eigenfunctions by \( \{ \varphi_{ir_j} \}_{j=0,1,2,...} \).

We recall that on a hyperbolic quotient there exists a quantization \( a \to Op(a) \) of symbols which is adapted to the non-Euclidean Fourier transform of Helgason (see [17] and [AZ] for references and details). We then define the Wigner distributions by

\[
\langle a, W_{ir_j} \rangle = \int_{S^*X_{\Gamma}} a(g)W_{ir_j}(dg) := \langle Op(a)\varphi_{ir_j}, \varphi_{ir_j} \rangle_{L^2(X_{\Gamma})}, \quad a \in C^\infty(S^*X_{\Gamma})
\]

On the other hand, one can define a second sequence of phase space distributions, the Patterson-Sullivan distributions \( \{ PS_{ir_j} \} \) associated to the eigenfunctions \( \{ \varphi_{ir_j} \} \), by the expression

\[
PS_{ir_j}(dg) = PS_{ir_j}(db', db, dt) := \frac{T_{ir_j}(db)T_{ir_j}(db')}{|b - b'|^{1 + 2ir_j}} \otimes |dt|.
\]

In this definition, \( T_{ir_j} \) is the boundary values of \( \varphi_{ir_j} \) in the sense of Helgason (see [21]). The parameters \( (b', b) \) \((b \neq b')\) vary in \( B \times B \), where \( B = \partial D \) is the boundary of the hyperbolic disc, and \( t \) varies in \( \mathbb{R} \); \( (b', b) \) parametrize the space of oriented geodesics, \( t \) is the time parameter along geodesics, and the three parameters \( (b', b, t) \) are used to parametrize the unit tangent bundle \( SD \). The Patterson-Sullivan distributions \( PS_{ir_j} \) are by construction invariant under the geodesic flow \( (g^t) \) on \( SD \), i.e.

\[
(g^t)^*PS_{ir_j} = PS_{ir_j},
\]

and by using [22] they can be shown to be \( \Gamma \)-invariant. Hence to each eigenfunction one obtains a geodesic-flow invariant distributions on \( S^{X_{\Gamma}} \). We also introduce normalized Patterson-Sullivan distributions

\[
\widehat{PS}_{ir_j} := \frac{1}{\langle 1, PS_{ir_j} \rangle_{S^{X_{\Gamma}}}} PS_{ir_j},
\]

which satisfy the same normalization condition \( \langle 1, \widehat{PS}_{ir_j} \rangle = 1 \) as \( W_{ir_j} \) on the quotient \( S^{X_{\Gamma}} \). In [AZ] the following is proved:

**Theorem 15.4.** For any \( a \in C^\infty(\Gamma \backslash G) \),

\[
\int_{S^{X_{\Gamma}}} a(g)W_{ir_j}(dg) = \int_{S^{X_{\Gamma}}} a(g)\widehat{PS}_{ir_j}(dg) + O(r_j^{-1}).
\]

It follows that the Wigner distributions are equivalent to the Patterson-Sullivan distributions in the study of quantum ergodicity. Yet, the Patterson-Sullivan distributions have a purely classical dynamical definition: Define the classical dynamical zeta functions,

\[
(i) \quad \mathcal{L}_2(a, s) = \sum_\gamma \frac{e^{s(a - 1)\gamma}}{sinh(\gamma / 2)^2} \left( \int_{\gamma_0} a \right),
\]

\[
(ii) \quad \mathcal{L}(s; a) := \sum_\gamma \frac{e^{-s\gamma}}{1 - e^{-\gamma}} \left( \int_{\gamma_0} a \right), \quad (Re s > 1)
\]

where the sum runs over all closed orbits, and \( \gamma_0 \) is the primitive closed orbit traced out by \( \gamma \). The sum converges absolutely for \( Re s > 1 \).
Theorem 15.5. Let $a$ be a real analytic function on the unit tangent bundle. Then $\mathcal{L}(s;a)$ and $\mathcal{L}_2(s;a)$ admit meromorphic extensions to $\mathbb{C}$. The poles in the critical strip $0 < \Re e s < 1$, appear at $s = 1/2 + ir$, where as above $1/4 + r^2$ is an eigenvalue of $\Delta$. For each zeta function, the residue is

$$\sum_{j: r_j^2 = r^2} \langle a, \hat{P} S_{ir_j} \rangle_{SX},$$

where $\{ \hat{P} S_{ir_j} \}$ are the normalized Patterson-Sullivan distributions associated to an orthonormal eigenbasis $\{ \varphi_{ir_j} \}$.

Thus, the quantum limit problem is the same (for compact hyperbolic surfaces) as the problem of finding the limiting behavior of the residues of classical dynamical zeta functions as the pole moves up the critical line.

16. Delocalization of eigenfunctions: II: Entropy of quantum limits on manifolds with Anosov geodesic flow

We now describe the recent results of Anantharaman [A], Anantharaman-Nonnenmacher [AN] (see also [ANK]) on entropy of quantum limits for $(M,g)$ with Anosov geodesic flow. These articles give lower bounds on entropies of the quantum limit measures that arise from sequences of eigenfunctions. We closely follow the presentation in [AN, ANK], and refer in particular to the partly expository article [ANK] and to the recent Bourbaki seminar report [CV4] for an exposition of the results.

As discussed above, it is known that quantum limits are invariant probability measures for $g^t$, but many such measures exist for any geodesic flow. To pin down the possible quantum limits, one needs to add constraints on the possible limit measures. The entropy bounds of Anantharaman et al provide almost the only additional constraints known at this time. The lower bound on entropy rules out such possible limit measures as periodic orbit measures $\mu_\gamma$ (which have entropy zero) or finite sums of such measures. However, the entropy results leave open the possibility that a sequence of eigenfunctions could tend to a limit of the form $a \mu_\gamma + (1 - a)d\mu$ if $a$ is small enough (here, $\mu$ is Liouville measure).

We recall (see §1) that a geodesic flow $g^t$ is Anosov on $S^*_gM$ if the tangent bundle $T S^*_gM$ splits into $g^t$ invariant sub-bundles $E^u(\rho) \oplus E^s(\rho) \oplus \mathbb{R} X_H(\rho)$ where $E^u$ is the unstable subspace and $E^s$ the stable subspace. The unstable Jacobian $J^u(\rho)$ at $\rho$ is defined by $J^u(\rho) = \det \left( dg^{-1}_{E^u(g^t\rho)} \right)$.

The goal is to give a lower bound for entropy of quantum limits. Entropy is complicated to define, and we only provide a brief sketch here. Classically, entropies are defined for an invariant probability measure $\mu$ for the geodesic flow and measures the average complexity of $\mu$-typical orbits. In the Kolmogorov-Sinai entropy, one starts with a partition $\mathcal{P} = (E_1, \ldots, E_k)$ of $S^*_gM$ and defines the Shannon entropy of the partition by $h_\mathcal{P}(\mu) = \sum_{j=1}^k \mu(E_j) \log \mu(E_j)$. Under iterates of the time one map $g$ of the geodesic flow, one refines the partition to $\mathcal{P}^m = \{ E_{a_0} \cap g^{-1}E_{a_1} \cap \cdots \cap g^{-n+1}E_{a_{n-1}} \}$.

One defines $h_m(\mathcal{P}, \mu)$ to be the Shannon entropy of this partition and then defines $h_{KS}(\mu, \mathcal{P}) = \lim_{n \to \infty} \frac{1}{n} h_m(\mu, \mathcal{P})$. Then $h_{KS}(\mu) = \sup_\mathcal{P} h_{KS}(\mu, \mathcal{P})$.

The main result of [AN, ANK] is the following
Theorem 16.1. Let \( \mu \) be a semiclassical measure associated to the eigenfunctions of the Laplacian on \( M \). Then its metric entropy satisfies

\[
\int_{S^*M} \log J^\mu(p) \, d\mu(p) \geq \frac{(d-1)}{2} \lambda_{\text{max}},
\]

where \( d = \dim M \) and \( \lambda_{\text{max}} = \lim_{|\xi| \to \infty} \frac{1}{2} \log \sup_{\rho \in \mathcal{E}} |d\rho^f| \) is the maximal expansion rate of the geodesic flow on \( \mathcal{E} \).

In particular, if \( M \) has constant sectional curvature \(-1\), this means that

\[
\int_{S^*M} \log J^\mu(p) \, d\mu(p) \geq \frac{d-1}{2}.
\]

The proof is based on a quantum analogue of the metric entropy, and in particular on a development of the ‘entropic uncertainty principle’ of Maassen-Uffink. There are several notions of quantum or non-commutative entropy, but for applications to eigenfunctions it is important to find one with good semi-classical properties.

Let \( (\mathcal{H}, \langle \cdot, \cdot \rangle) \) be a complex Hilbert space, and let \( \|\psi\| = \sqrt{\langle \psi, \psi \rangle} \) denote the associated norm. The quantum notion of partition is a family \( \pi = (\pi_k)_{k=1, \ldots, N} \) of operators on \( \mathcal{H} \) such that \( \sum_{k=1}^{N} \pi_k \pi_k^* = \text{Id} \). If \( \|\psi\| = 1 \), the entropy of \( \psi \) with respect to the partition \( \pi \) is define by

\[
h_{\pi}(\psi) = -\sum_{k=1}^{N} \|\pi_k^* \psi\|^2 \log \|\pi_k^* \psi\|^2.
\]

We note that the quantum analogue of an invariant probability measure \( \mu \) is an invariant state \( \rho \), and the direct analogue of the entropy of the partition would be \( \sum \rho(\pi_k \pi_k^*) \log \rho(\pi_k \pi_k^*) \). If the state is \( \rho(A) = \langle A\psi, \psi \rangle \) then \( \rho(\pi_k \pi_k^*) = \|\pi_k^* \psi\|^2 \).

The dynamics is generated by a unitary operator \( U \) on \( \mathcal{H} \). We now state a simple version of the entropy uncertainty inequality of Maassen-Uffink. A more elaborate version in \[A,\ AN,\ ANK\] gives a lower bound for a certain ‘pressure’.

Theorem 16.2. For any \( \epsilon \geq 0 \), for any normalized \( \psi \in \mathcal{H} \),

\[
h_{\pi}(U\psi) + h_{\pi}(\psi) \geq -2 \log c(U),
\]

where

\[
c(U) = \sup_{j,k} |\langle e_k, U e_j \rangle|
\]

is the supremum of all matrix elements in the orthonormal basis \( \{e_j\} \). In particular, \( h_{\pi}(\psi) \geq -\log c(U) \) if \( \psi \) is an eigenfunction of \( U \).

In the application to eigenfunctions, one fixes a partition \( \{M_k\} \) of \( M \) and a corresponding partition \( T^*M_k \) of \( T^*M \). One then defines a smooth quantum partition of unity \( P_k \) by smoothing out the characteristic functions of \( M_k \). The partition is refined by

\[
P_{\alpha} = P_{\alpha_{n-1}}(n-1)P_{\alpha_{n-2}}(n-2) \cdots P_{\alpha_0},
\]

where \( \alpha = (\alpha_0, \ldots, \alpha_{n-1}) \) and where \( P(k) = U^{*k} P U^k \). One then specifies:

(1) \( U = e^{i T_E \sqrt{\Delta}} \) is the wave operator at the ‘Ehrenfest time’ \( T_E = \frac{\log \lambda}{\lambda_{\text{max}}} \). Or from a semi-classical (where \( h = \frac{1}{\lambda} \), where the Hamiltonian is \( H = h^2 \Delta \) and where the time evolution is \( e^{i \frac{t}{\lambda} \Delta} \), \( U = e^{i n_E(h) \Delta} \) with \( n_E(h) = \frac{\log \lambda}{\lambda_{\text{max}}} \).
(2) $\psi_\hbar$ is an eigenfunction of $U$;
(3) $h_\pi(\psi_\hbar) = \sum_{|\alpha|=n_\hbar} ||P^*_\alpha \psi_\hbar||^2 \log ||P^*_\alpha \psi_\hbar||^2$.

With these specifications,

$$c(U) = \max_{|\alpha|=|\alpha'|=n_\hbar} ||P^*\alpha U^{n_\hbar} P^*_\alpha Op(\chi^{(nE)})||$$

where $\chi^{(nE)}$ is a very sharp energy cutoff supported in a tubular neighborhood $E^\epsilon := H^{-1}(1-\epsilon, 1+\epsilon)$ of $E = S^*M$ of width $2h^{1-\delta} e^{\epsilon d}$ for a given $\delta > 0$.

We now give a very sketchy outline of how the entropy uncertainty inequality (Theorem 16.1) is used to prove the lower bound on the entropy of the limit measure (Theorem 16.2). The argument is technical and difficult and the outline only gives the flavor of the estimates; the interested reader should consult [AN] for a complete exposition.

The next step is to link $c(U)$ to the classical dynamics. The authors introduce a discrete ‘coarse-grained’ unstable Jacobian

$$J^u_1(\alpha_0, \alpha_1) := \sup\{J^u(\rho) : \rho \in T^*\Omega_{\alpha_0} \cap E^\epsilon : g^t\rho \in T^*\Omega_{\alpha_1}\},$$

for $\alpha_0, \alpha_1 = 1, \ldots, K$. Here, $\Omega_j$ are small open neighborhoods of the partition sets $M_j$. For a sequence $\alpha = (\alpha_0, \ldots, \alpha_{n-1})$ of symbols of length $n$, one defines

$$J^u_n(\alpha) := J^u_1(\alpha_0, \alpha_1) \cdots J^u_1(\alpha_{n-2}, \alpha_{n-1}).$$

**Theorem 16.3.** Given a partition $P^{(0)}$ and $\delta, \delta' > 0$ small enough, there exists $h_{P^{(0)}, \delta, \delta'}$ such that, for any $\hbar \leq h_{P^{(0)}, \delta, \delta'}$, for any positive integer $n \leq n_\hbar(\hbar)$, and any pair of sequences $\alpha, \alpha'$ of length $n$,

$$||P^*_\alpha U^n P^*_\alpha Op(\chi^{(n)})|| \leq C \hbar^{-(d-1+c\delta)} \sqrt{J^u_n(\alpha)J^u_n(\alpha')}.$$  

Here, $d = \dim M$ and the constants $c, C$ only depend on $(M, g)$.

To prove this, one shows that any state of the form $Op(\chi^{(n)})\Psi$ can be decomposed as a superposition of essentially $h^{-\frac{d-1}{2}}$ normalized Lagrangian states, supported on Lagrangian manifolds transverse to the stable leaves of the flow. The action of the operator $P^*_\alpha$ on such Lagrangian states is intuitively as follows: each application of $U$ stretches the Lagrangian in the unstable direction (the rate of elongation being described by the unstable Jacobian) whereas each multiplication by $P^*_\alpha$ projects onto a small piece of the Lagrangian. This iteration of stretching and cutting accounts for the exponential decay.

Combined with the entropy uncertainty inequality, one obtains

**Proposition 16.4.** Let $d = \dim M$ and let $(\psi_\hbar)_{\hbar \to 0}$ be a sequence of eigenfunctions. Then there exist $\delta, \delta'$ so that, at time $n = n_\hbar(\hbar)$,

$$h_n(\psi_\hbar) \geq 2(d-1+c\delta) \log \hbar + O(1) \geq -2\frac{(d-1+c\delta)\lambda_{\max}}{(1-\delta')} n + O(1).$$

Now suppose that the Wigner measures $W^{\psi_\hbar}_{\hbar}$ of a subsequence $(\psi_\hbar)_{\hbar \to 0}$ of eigenfunctions converges to the semiclassical measure $\mu$ on $\mathcal{E}$. Consider the limit $\hbar \to 0$ of $h_\pi(\psi_\hbar)$ (so that $n_\hbar(\hbar) \to \infty$). For any sequence $\alpha$ of length $n_\hbar$, each $||P^*_\alpha \psi_\hbar||^2$ converges to $\mu(\{\alpha\})$, where $\{\alpha\}$ is the function $P^2_\alpha (P^2_{\alpha_1} \circ g^1) \cdots (P^2_{\alpha_n} \circ g^{nE})$ on $T^*M$. Then for any $n_0 \leq n_\hbar$, $h_{n_0}(\psi_\hbar)$
semiclassically converges to the classical entropy
\[ h_{n_0}(\mu, P_{\text{sm}}) = h_{n_0}(\mu, (P^2_k)) = - \sum_{|\alpha| = n_0} \mu(\{ 1^{\text{sm}}_{M_{ao}} \})^2 \log J_n^u(\alpha), \]
where
\[ 1^{\text{sm}}_{M_{ao}} = (1^{\text{sm}}_{M_{ao}^{-1}} \circ g^{n_0-1}) \cdots (1^{\text{sm}}_{M_{ao+1}} \circ g) 1^{\text{sm}}_{M_{ao}}. \]
Here, \( 1^{\text{sm}}_{M_{ao}} \) is a smoothing of the characteristic function of the indicated set.

Using Proposition 16.4, one obtains the lower bound
\[ \lim_{n_0 \to \infty} \frac{h_{n_0}(\mu, P_{\text{sm}})}{n_0} \geq - \frac{n_0 - 1}{n_0} - \sum_{\alpha_0, \alpha_1} \mu(\{ 1^{\text{sm}}_{M_{ao}} \})^2 \log J_n^u(\alpha_0, \alpha_1) - \frac{(d - 1 + c\delta) \lambda_{\text{max}}}{(1 - \delta')} - \frac{2R}{n_0}. \]
Here, \( \delta \) and \( \delta' \) could be taken arbitrarily small, and at this stage they can be set equal to zero.

The Kolmogorov–Sinai entropy of \( \mu \) is by definition the limit of the left side of (138) when \( n_0 \to \infty \). Then let \( n_0 \to \infty \), and let the diameter \( \varepsilon/2 \) of the partition tend to 0. Then the first term in the right hand side of (138) converges to the integral \( -\int_{\mathcal{E}} \log J^u(\rho) d\mu(\rho) \) as \( \varepsilon \to 0 \), proving (132).

\[ \square \]

17. **Real analytic manifolds and their complexifications**

In this section, we consider eigenfunctions on a real analytic compact Riemannian manifold \((M, g)\). The advantage of real analyticity is that one can complexify the manifold and analytically continue the eigenfunctions to the complexification of \( M \). This allows one to use methods of holomorphic and pluri-subharmonic function theory to obtain sharper results on volumes and distribution of nodal hypersurfaces than are possible for \( C^\infty(M, g) \). The gain in simplicity is two fold, reflecting the relative simplicity of real polynomials over smooth functions, and of complex zeros of polynomials over real zeros. This point of view has been taken in [DF], [Lin], [Z6] among other articles.

A real analytic manifold \( M \) always possesses a unique complexification \( M_\mathbb{C} \) generalizing the complexification of \( \mathbb{R}^m \) as \( \mathbb{C}^m \). The complexification is an open complex manifold in which \( M \) embeds \( \iota: M \to M_\mathbb{C} \) as a totally real submanifold (Bruhat-Whitney). As examples, we have:

- \( M = \mathbb{R}^m/\mathbb{Z}^m \) is \( M_\mathbb{C} = \mathbb{C}^m/\mathbb{Z}^m \).
- The unit sphere \( S^n \) defined by \( x_1^2 + \cdots + x_{n+1}^2 = 1 \) in \( \mathbb{R}^{n+1} \) is complexified as the complex quadric \( S^n_\mathbb{C} = \{(z_1, \ldots, z_n) \in \mathbb{C}^{n+1} : z_1^2 + \cdots + z_{n+1}^2 = 1 \} \).
- The hyperboloid model of hyperbolic space is the hypersurface in \( \mathbb{R}^{n+1} \) defined by
  \[ \mathbb{H}^n = \{x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1, \quad x_n > 0 \}. \]
Then,
  \[ H^n_\mathbb{C} = \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} : z_1^2 + \cdots + z_n^2 - z_{n+1}^2 = -1 \}. \]
- Any real algebraic subvariety of \( \mathbb{R}^m \) has a similar complexification.
- Any Lie group \( G \) (or symmetric space) admits a complexification \( G_\mathbb{C} \).
The Riemannian metric determines a special kind of distance function on $M_C$. The metric $g$ determines a plurisubharmonic function $\sqrt{p} = \sqrt{p}_g$ on $M_C$ as the unique solution of the Monge-Ampère equation

$$(\partial\bar{\partial}\sqrt{p})^m = \delta_{M_g, dV_g}, \quad i^*(i\partial\bar{\partial}\rho) = g.$$  

Here, $\delta_{M_g, dV_g}$ is the delta-function on the real $M$ with respect to the volume form $dV_g$, i.e. $f \to \int_M f dV_g$. In fact, $\sqrt{p}(\zeta) = i\sqrt{r^2(\zeta, \zeta)}$ where $r^2(x, y)$ is the squared distance function in a neighborhood of the diagonal in $M \times M$.

One defines the Grauert tubes $M_\tau = \{\zeta \in M_C : \sqrt{p}(\zeta) \leq \tau\}$. There exists a maximal $\tau_0$ for which $\sqrt{p}$ is well defined, known as the Grauert tube radius. For $\tau \leq \tau_0$, $M_\tau$ is a strictly pseudo-convex domain in $M_C$.

The complexified exponential map $(x, \xi) \to \exp_{|\xi|}i\xi$ defines a diffeomorphism from $B^*_\tau M$ to $M_\tau$ and pulls back $\sqrt{p}$ to $|\xi|_g$. The one-complex dimensional null foliation of $\partial\bar{\partial}\sqrt{p}$, known as the ‘Monge-Ampère’ or Riemann foliation, are the complex curves $t + i\tau \to \tau\gamma(t)$, where $\gamma$ is a geodesic, where $\tau > 0$ and where $\tau\gamma(t)$ denotes multiplication of the tangent vector to $\gamma$ by $\tau$. We refer to [LS1] for further discussion.

17.1. **Analytic Continuation of eigenfunctions.** Let $A(\tau)$ denote the operator of analytic continuation of a function on $M$ to the Grauert tube $M_\tau$. It is simple to see that $A(\tau) = U_C(i\tau)e^{r\sqrt{A}}$ where $U(i\tau, x, y) = e^{-r\sqrt{A}}(x, y)$ is the Poisson operator of $(M, g)$, i.e. the wave operator at positive imaginary time, and $U_C(i\tau, \zeta, y)$ is its analytic continuation in $x$ to $M_\tau$. In terms of the eigenfunction expansion, one has

$$U(i\tau, \zeta, y) = \sum_{j=0}^{\infty} e^{-\tau\lambda_j}\varphi_j^C(\zeta)\varphi_j(y), \quad (\zeta, y) \in M_C \times M.$$  

To understand the analytic continuability of the wave kernel, we first consider Euclidean $\mathbb{R}^n$ and its wave kernel $U(t, x, y) = \int_{\mathbb{R}^n} e^{it|\xi|} e^{i\langle\xi, x-y\rangle} d\xi$ which analytically continues to $t + i\tau$, $\zeta = x + ip \in \mathbb{C}_+ \times \mathbb{C}^n$ as the integral

$$U_C(t + i\tau, x + ip, y) = \int_{\mathbb{R}^n} e^{i(t+\tau)|\xi|} e^{i\langle\xi, x+ip-y\rangle} d\xi.$$  

The integral clearly converges absolutely for $|p| < \tau$.

Exact formulæ of this kind exist for $S^m$ and $\mathbb{H}^m$. For a general real analytic Riemannian manifold, there exists an oscillatry integral expression for the wave kernel of the form,

$$U(t, x, y) = \int_{T_y^* M} e^{i|\xi|_g} e^{i\langle\xi, \exp^{-1}_g(\cdot)\rangle} A(t, x, y, \xi) d\xi$$  

where $A(t, x, y, \xi)$ is a polyhomogeneous amplitude of order 0. For background, we refer to [Be], [DG], [T]. The holomorphic extension of (140) to the Grauert tube $|\zeta| < \tau$ in $x$ at time $t = i\tau$ then has the form

$$U_C(i\tau, \zeta, y) = \int_{T_y^*} e^{-\tau|\xi|_g} e^{i\langle\xi, \exp^{-1}_g(\cdot)\rangle} A(t, \zeta, y, \xi) d\xi \quad (\zeta = x + ip).$$  

Since

$$U_C(i\tau)\varphi_\lambda = e^{-\tau\lambda}\varphi_\lambda^C,$$
the analytic continuability of the Poisson operator to $M_\tau$ implies that every eigenfunction analytically continues to the same Grauert tube. It follows that the analytic continuation operator to $M_\tau$ is given by

$$A_C(\tau) = U_C(i\tau) \circ e^{\tau \Delta}.$$  

Thus, a function $f \in C^\infty(M)$ has a holomorphic extension to the closed tube $\sqrt{\rho}(\zeta) \leq \tau$ if and only if $f \in \text{Dom}(e^{\tau \Delta})$, where $e^{\tau \Delta}$ is the backwards `heat operator’ generated by $\sqrt{\Delta}$ (rather than $\Delta$). That is, $f = \sum_{n=0}^{\infty} a_n \varphi_{\lambda n}$ admits an analytic continuation to the open Grauert tube $M_\tau$ if and only if $f$ is in the domain of $e^{\tau \Delta}$, i.e. if $\sum_n |a_n|^2 e^{2\tau \lambda n} < \infty$. Indeed, the analytic continuation is $U_C(i\tau)e^{\tau \Delta}f$. The subtlety is in the nature of the restriction to the boundary of the maximal Grauert tube.

This result generalizes one of the classical Paley-Wiener theorems to real analytic Riemannian manifolds [Bou, GS2]. In the simplest case of $M = S^1$, $f \sim \sum_{n \in \mathbb{Z}} a_n e^{in\theta} \in C^\infty(S^1)$ is the restriction of a holomorphic function $F \sim \sum_{n \in \mathbb{Z}} a_n z^n$ on the annulus $S^1_\tau = \{|\log|z|| < \tau\}$ and with $F \in L^2(\partial S^1_\tau)$ if and only if $\sum_n |\hat{f}(n)|^2 e^{2n|\tau|} < \infty$. The case of $\mathbb{R}^m$ is more complicated since it is non-compact. We are mainly concerned with compact manifolds and so the complications are not very relevant here. But we recall that one of the classical Paley-Wiener theorems states that a real analytic function $f$ on $\mathbb{R}^n$ is the restriction of a holomorphic function on the closed tube $|\text{Im}\, \zeta| \leq \tau$ which satisfies $\int_{\mathbb{R}^m} |F(x + i\xi)|^2 dx \leq C$ for $\xi \leq \tau$ if and only if $\hat{f}e^{\tau|\text{Im}\, \zeta|} \in L^2(\mathbb{R}^n)$.

Let us consider examples of holomorphic continuations of eigenfunctions:

- On the flat torus $\mathbb{R}^m/\mathbb{Z}^m$, the real eigenfunctions are $\cos(k,x), \sin(k,x)$ with $k \in 2\pi\mathbb{Z}^m$. The complexified torus is $\mathbb{C}^m/\mathbb{Z}^m$ and the complexified eigenfunctions are $\cos(k,\zeta), \sin(k,\zeta)$ with $\zeta = x + i\xi$.

- On the unit sphere $S^m$, eigenfunctions are restrictions of homogeneous harmonic functions on $\mathbb{R}^{m+1}$. The latter extend holomorphically to holomorphic harmonic polynomials on $\mathbb{C}^{m+1}$ and restrict to holomorphic function on $S^m_\tau$.

- On $\mathbb{H}^m$, one may use the hyperbolic plane waves $e^{i(\lambda+1)(z,b)}$, where $(z,b)$ is the (signed) hyperbolic distance of the horocycle passing through $z$ and $b$ to 0. They may be holomorphically extended to the maximal tube of radius $\pi/4$.

- On compact hyperbolic quotients $\mathbb{H}^m/\Gamma$, eigenfunctions can be then represented by Helgason’s generalized Poisson integral formula [H],

$$\varphi_\lambda(z) = \int_B e^{i(\lambda+1)(z,b)} dT_\lambda(b).$$

Here, $z \in D$ (the unit disc), $B = \partial D$, and $dT_\lambda \in \mathcal{D}'(B)$ is the boundary value of $\varphi_\lambda$, taken in a weak sense along circles centered at the origin 0. To analytically continue $\varphi_\lambda$ it suffices to analytically continue $\langle z,b \rangle$. Writing the latter as $\langle \zeta,b \rangle$, we have:

$$\varphi_\lambda^C(\zeta) = \int_B e^{i(\lambda+1)\langle \zeta,b \rangle} dT_\lambda(b).$$

The results of Sarnak [Sar] on the exponential decay of integrals $\int_M \varphi_\lambda \varphi_\rho^2 dV_g$ and subsequent results of Miller-Schmidt [MS] and Bernstein-Reznikov [BR] are closely related to the Paley-Wiener theory and this analytic continuation formula for eigenfunctions on hyperbolic quotients.
17.2. Maximal plurisubharmonic functions and growth of $\varphi^C_\lambda$. There are natural analogues in the setting of Grauert tubes for the basic notions of pluripotential theory on domains in $\mathbb{C}^m$. One may view the Grauert tube function $\sqrt{\rho}$ as the analogue of the pluri-complex Green’s function or Siciak maximal PSH (pluri-subharmonic) function.

In the case of domains $\Omega \subset \mathbb{C}^m$, we recall that the maximal PSH function (or pluri-complex Green’s function) relative to a subset $E \subset \Omega$ is defined by

$$V_E(\zeta) = \sup\{u(\zeta) : u \in PSH(\Omega), u|_E \leq 0, u|_{\partial \Omega} \leq 1\}.$$ 

This maximal function controls the Bernstein constant $B(f, E, \Omega) = \max_{\Omega} |f| / \max_E |f|$ for any holomorphic $f$.

An alternative construction of the maximal PSH function due to Siciak is defined by taking the supremum only with respect to polynomials $p$. We denote by $\mathcal{P}_N^\lambda$ the space of all complex analytic polynomials of degree $\lambda$ and put $\mathcal{P}_N^\lambda_{\mathcal{M}} = \{p \in \mathcal{P}_N^\lambda : ||p||_{\mathcal{M}} \leq 1, ||p||_{\Omega} \leq e\}$. Then define

$$\log \Phi^\lambda_E(\zeta) = \sup \left\{ \frac{1}{N} \log |p_N(\zeta)| : p \in \mathcal{P}_N^\lambda \right\}, \quad \log \Phi_E = \lim_{N \to \infty} \sup \log \Phi^\lambda_E.$$ 

Here, $||f||_{\mathcal{K}} = \sup_{z \in \mathcal{K}} |f(z)|$. Siciak proved that $\log \Phi_E = V_E$ (see [K], Theorem 5.1.7). Intuitively, there are enough polynomials that one can obtain the sup by restricting to polynomials.

On a real analytic Riemannian manifold, the natural analogue of $\mathcal{P}_N^\lambda$ is the space

$$\mathcal{H}_\lambda = \{p = \sum_{j : \lambda_j \leq \lambda} a_j \varphi_{\lambda_j}, a_1, \ldots, a_{N(\lambda)} \in \mathbb{R}\}$$

spanned by eigenfunctions with frequencies $\leq \lambda$. Rather than using the sup norm, it is convenient to work with $L^2$ based norms than sup norms, and so we define

$$\mathcal{H}_M^\lambda = \{p = \sum_{j : \lambda_j \leq \lambda} a_j \varphi_{\lambda_j}, ||p||_{L^2(M)} = \sum_{j=1}^{N(\lambda)} |a_j|^2 = 1\}.$$ 

We define the $\lambda$-Siciak extremal function by

$$\Phi^\lambda_M(z) = \sup \{|\psi(z)|^{1/\lambda} : \psi \in \mathcal{H}_\lambda, ||\psi||_M \leq 1\},$$

and the extremal function by

$$\Phi_M(z) = \sup_\lambda \Phi^\lambda_M(z).$$

We may also define a natural analogue of the pluri-complex Green’s function by putting $E = M$ and $\Omega = M_{\tau}$ and defining

$$V_g(\zeta; \tau) = \sup\{u(\zeta) : u \in PSH(M_{\tau}), u|_M \leq 0, u|_{\partial M_{\tau}} \leq \tau\}.$$ 

Although it does not seem to have been proved at this time, it is easy to guess that $V_g = \sqrt{\rho}$ since the latter solves the homogeneous Monge-Ampère equation $(\partial \bar{\partial} \sqrt{\rho})^m = 0$ on $M_{\tau} \setminus M$. Moreover, it is not hard to prove that

$$\Phi_M = V_g,$$

generalizing the so-called Siciak-Zaharjuta theorem in the special case where the boundary conditions are placed on all of $M$. To see this, we consider the analytic continuation of
the spectral projections kernels $\Pi_{[0,\lambda]}(x, y) = \sum_{j: \lambda_j \in [0, \lambda]} \varphi_j(x) \varphi_j(y)$. Its complexification evaluated on the anti-diagonal equals,

$$
\Pi_{[0,\lambda]}(\zeta, \bar{\zeta}) = \sum_{j: \lambda_j \in [0, \lambda]} |\varphi^C_j(\zeta)|^2.
$$

By using a Bernstein-Walsh inequality

$$
\frac{1}{N(\lambda)} \leq \frac{\Pi_{[0,\lambda]}(\zeta, \bar{\zeta})}{\Phi^\lambda_M(\zeta)} \leq CN(\lambda) e^{eN(\lambda)},
$$

it is not hard to show that

$$
\Phi^M_M(z) = \lim_{\lambda \to \infty} \frac{1}{\lambda} \log \Pi_{[0,\lambda]}(\zeta, \bar{\zeta}).
$$

To evaluate the logarithm, one can show that the kernel is essentially $e^{N\sqrt{\rho}}$ times the temperate projection defined by the Poisson operator,

$$
P_{[0,\lambda]}(\zeta, \bar{\zeta}) = \sum_{j: \lambda_j \in [0, \lambda]} e^{-2\sqrt{\rho} \lambda_j} |\varphi^C_j(\zeta)|^2.
$$

The equality (145) follows from the fact that $\lim_{\lambda \to \infty} \frac{1}{\lambda} \log P_{[0,\lambda]}(\zeta, \bar{\zeta}) = 0$.

### 17.3. Analytic continuation and nodal hypersurfaces.

We now survey some methods and results on nodal hypersurfaces of eigenfunctions on real analytic compact Riemannian manifolds $(M, g)$ of dimension $m$. The principal result on volumes is due to Donnelly-Fefferman [DF]:

**Theorem 17.1.** [DF] (see also [Lin]) Let $(M, g)$ be a compact real analytic Riemannian manifold, with or without boundary. Then there exist $c_1, C_2$ depending only on $(M, g)$ such that

$$
c_1 \lambda \leq \mathcal{H}^{m-1}(Z_{\varphi_\lambda}) \leq C_2 \lambda, \quad (\Delta \varphi_\lambda = \lambda^2 \varphi_\lambda; c_1, C_2 > 0).
$$

A very readable exposition of the proof is contained in [H]. The upper and lower bounds require rather different arguments. The upper bound is simpler and may be sketched as follows: By a local Crofton’s formula, the real volume of the nodal hypersurface equals the mean number of intersections it has with a random line in a coordinate chart, i.e. by the number of zeros of $\varphi_\lambda$ along each line. This number is obviously bounded above by the number of complex zeros of $\varphi^C_\lambda$ on the complexification of the line. Hence, the upper bound is reduced to bounding the number of complex zeros of a family of complex analytic functions of one variable in a disc. For each disc, the number of zeros is in principle estimated by Jensen’s formula, which bounds the number of zeros in a disc of a holomorphic function by the logarithm of the modulus of the holomorphic function. One can then use the frequency function estimates, Carleman estimates or Bernstein-Walsh type inequalities to obtain the doubling estimates in Theorem 3.2 and they bound the growth of the log modulus of $\varphi^C_\lambda$ by $C\lambda$. Jensen’s formula does not directly apply, since it concerns the growth of a single holomorphic function in an expanding family of domains, while we are interested in a family of holomorphic functions $\varphi^C_\lambda$ in a single domain $M$. Donnelly-Fefferman find a suitable replacement using Blaschke product factorizations. Lin gives an alternative bound on zeros in terms of growth using the frequency function for functions of one complex variable. Thus,
the upper bound reflects the growth rate of $\varphi_\lambda$ combined with the upper bound on the number of complex zeros by the growth rate.

The lower bound is of a somewhat different nature. A general analytic function of exponential growth (e.g. $e^z$) need not have any zeros, i.e. zero might be an exceptional value. Jensen’s formula equates the growth of the log modulus to the sum of the number of zeros and the ‘proximity to zero’, i.e. in our setting to values where $|\varphi_\lambda(\zeta)| \leq e^{-\lambda \epsilon}$. It is necessary to rule out the possibility that zero is an exceptional value of complexified eigenfunctions. In effect, this is possible because real eigenfunction must have a zero in any ball of radius $\frac{C}{\lambda}$ by Theorem 4.1. But then one needs a lower bound on the hypersurface volume of the nodal set in such a small ball. More precisely, in [DF] a lower bound for the hypersurface volume is proved for a certain $\lambda$-independent proportion of a covering by small balls of radius $\frac{C}{\lambda}$. The global result then follows by summing the volume in the these small balls. Besides [DF], we refer to [HL] for a detailed discussion.

17.4. **Nodal hypersurfaces in the case of ergodic geodesic flow.** We now consider global results when hypotheses are made on the dynamics of the geodesic flow. Use of the global wave operator brings into play the relation between the geodesic flow and the complexified eigenfunctions, and this allows one to prove global results on nodal hypersurfaces that reflect the dynamics of the geodesic flow. In some cases, one can determine not just the volume, but the limit distribution of complex nodal hypersurfaces.

The complex nodal hypersurface of an eigenfunction is defined by

$$Z_{\varphi_\lambda}^c = \{ \zeta \in B^*_{\epsilon_0}M : \varphi_\lambda^c(\zeta) = 0 \}.$$  

There exists a natural current of integration over the nodal hypersurface in any ball bundle $B^*_{\epsilon}M$ with $\epsilon < \epsilon_0$, given by

$$\langle [Z_{\varphi_\lambda}^c], \varphi \rangle = \frac{i}{2\pi} \int_{B^*_{\epsilon}M} \partial\bar{\partial} \log |\varphi_\lambda^c|^2 \wedge \varphi = \int_{Z_{\varphi_\lambda}^c} \varphi, \quad \varphi \in D^{(m-1,m-1)}(B^*_{\epsilon}M).$$

In the second equality we used the Poincaré-Lelong formula. The notation $D^{(m-1,m-1)}(B^*_{\epsilon}M)$ stands for smooth test $(m-1, m-1)$-forms with support in $B^*_{\epsilon}M$.

The nodal hypersurface $Z_{\varphi_\lambda}^c$ also carries a natural volume form $|Z_{\varphi_\lambda}^c|$ as a complex hypersurface in a Kähler manifold. By Wirtinger’s formula, it equals the restriction of $\frac{\omega_{m-1}}{(m-1)!}$ to $Z_{\varphi_\lambda}^c$. Hence, one can regard $Z_{\varphi_\lambda}^c$ as defining the measure

$$\langle |Z_{\varphi_\lambda}^c|, \varphi \rangle = \int_{Z_{\varphi_\lambda}^c} \varphi \frac{\omega_{m-1}}{(m-1)!}, \quad \varphi \in C(B^*_{\epsilon}M).$$

We prefer to state results in terms of the current $[Z_{\varphi_\lambda}^c]$ since it carries more information.

We will say that a sequence $\{\varphi_j\}$ of $L^2$-normalized eigenfunctions is **quantum ergodic** if

$$\langle A\varphi_j, \varphi_j \rangle \to \frac{1}{\mu(S^*M)} \int_{S^*M} \sigma_A d\mu, \quad \forall A \in \Psi^0(M).$$

Here, $\Psi^s(M)$ denotes the space of pseudodifferential operators of order $s$, and $d\mu$ denotes Liouville measure on the unit cosphere bundle $S^*M$ of $(M, g)$. More generally, we denote by
that the similar (but more complicated) description of the zeros exists in all quantum integrable cases.

One takes in the limit. The result reflects the quantum integrability of the flat torus, and a lattice point. The exact limit distribution depends on which ray or ladder of lattice points in the sense that, for any continuous test form \( \psi \) on a real hypersurface in

We also denote by \( \alpha \) the canonical action 1-form of \( T^*M \).

**Theorem 17.2.** Let \((M, g)\) be real analytic, and let \( \{ \varphi_{jk} \} \) denote a quantum ergodic sequence of eigenfunctions of its Laplacian \( \Delta \). Let \((B^*_0 M, J)\) be the maximal Grauert tube around \( M \) with complex structure \( J_g \) adapted to \( g \). Let \( \epsilon < \epsilon_0 \). Then:

\[
\frac{1}{\lambda_{jk}} [Z_{\varphi_{jk}^c}] \to \frac{i}{\pi} \partial \bar{\partial} \sqrt{\rho} \text{ weakly in } \mathcal{D}^{(1,1)}(B^*_\epsilon M),
\]

in the sense that, for any continuous test form \( \psi \in \mathcal{D}^{(m-1,m-1)}(B^*_\epsilon M) \), we have

\[
\frac{1}{\lambda_{jk}} \int_{Z_{\varphi_{jk}^c}} \psi \to \frac{i}{\pi} \int_{B^*_\epsilon M} \psi \wedge \partial \bar{\partial} \sqrt{\rho}.
\]

Equivalently, for any \( \varphi \in C(B^*_\epsilon M) \),

\[
\frac{1}{\lambda_{jk}} \int_{Z_{\varphi_{jk}^c}} \frac{\omega_g^{m-1}}{(m-1)!} \varphi \to \frac{i}{\pi} \int_{B^*_\epsilon M} \varphi \partial \bar{\partial} \sqrt{\rho} \wedge \frac{\omega_g^{m-1}}{(m-1)!}.
\]

**Corollary 17.3.** Let \((M, g)\) be a real analytic with ergodic geodesic flow. Let \( \{ \varphi_{jk} \} \) denote a full density ergodic sequence. Then for all \( \epsilon < \epsilon_0 \),

\[
\frac{1}{\lambda_{jk}} [Z_{\varphi_{jk}^c}] \to \frac{i}{\pi} \partial \bar{\partial} \sqrt{\rho}, \text{ weakly in } \mathcal{D}^{(1,1)}(B^*_\epsilon M).
\]

The proof consists of three ingredients:

1. By the Poincaré-Lelong formula, \( [Z_{\varphi_{jk}^c}] = i \partial \bar{\partial} \log |\varphi_{jk}^c| \). This reduces the theorem to determining the limit of \( \frac{1}{\lambda} \log |\varphi_{\lambda}^c| \).
2. \( \frac{1}{\lambda} \log |\varphi_{\lambda}^c| \) is a sequence of PSH functions which are uniformly bounded above by \( \sqrt{\rho} \). By a standard compactness theorem, the sequence is pre-compact in \( L^1 \): every sequence from the family has an \( L^1 \) convergent subsequence.
3. \( |\varphi_{\lambda}^c|^2 \), when properly \( L^2 \) normalized on each \( \partial M_\epsilon \) is a quantum ergodic sequence on \( \partial M_\epsilon \). This property implies that the \( L^2 \) norm of \( |\varphi_{\lambda}^c|^2 \) on \( \partial \Omega \) is asymptotically \( \sqrt{\rho} \).
4. Ergodicity and the calculation of the \( L^2 \) norm imply that the only possible \( L^1 \) limit of \( \frac{1}{\lambda} \log |\varphi_{\lambda}^c| \). This concludes the proof.

We note that the first two steps are valid on any real analytic \((M, g)\). The difference is that the \( L^2 \) norms of \( \varphi_{\lambda}^c \) may depend on the subsequence and can often not equal \( \sqrt{\rho} \). That is, \( \frac{1}{\lambda} |\varphi_{\lambda}^c| \) behaves like the maximal PSH function in the ergodic case, but not in general. For instance, on a flat torus, the complex zero sets of ladders of eigenfunctions concentrate on a real hypersurface in \( M_\mathbb{C} \). This may be seen from the complexified real eigenfunctions \( \sin(k, x + i\xi) \), which vanish if and only if \( \langle k, x \rangle \in 2\pi\mathbb{Z} \) and \( \langle k, \xi \rangle = 0 \). Here, \( k \in \mathbb{N}^m \) is a lattice point. The exact limit distribution depends on which ray or ladder of lattice points one takes in the limit. The result reflects the quantum integrability of the flat torus, and a similar (but more complicated) description of the zeros exists in all quantum integrable cases.
The fact that $\frac{1}{\lambda} \log |\varphi^C_\lambda|$ is pre-compact on a Grauert tube of any real analytic Riemannian manifold confirms the upper bound on complex nodal hypersurface volumes.

17.5. **Analytic domains with boundary.** Many of the basic estimates on nodal hypersurfaces, such as Theorem 149, apply to real analytic manifolds with boundary as well as boundaryless manifolds. We are concentrating on manifolds without boundary, but mention one result on nodal lines where the boundary effects are central. Namely, in the case of real analytic plane domains, it is visible from computer graphics (see [FGS] for references) that only a small proportion of the nodal components touch the boundary. Most are small nodal loops in the interior.

We thus consider Neumann (resp. Dirichlet) eigenfunctions $\varphi_{\lambda_j}$ on (piecewise) real analytic plane domains $\Omega \subset \mathbb{R}^2$, i.e. solutions of (I) with boundary conditions

$$\partial_\nu \varphi_{\lambda_j} = 0 \ (\text{resp. } \varphi_{\lambda_j} = 0) \text{ on } \partial \Omega,$$

**Theorem 17.4.** [TZ3] Let $\Omega$ be a piecewise analytic domain and let $n_{\partial \Omega}(\lambda_j)$ be the number of components of the nodal set of the $j$th Neumann or Dirichlet eigenfunction which intersect $\partial \Omega$. Then there exists $C_\Omega$ such that $n_{\partial \Omega}(\lambda_j) \leq C_\Omega \lambda_j$.

In the Dirichlet case, we remove the boundary before counting components. For a generic piecewise analytic plane domains, zero is a regular value of all eigenfunctions $\varphi_{\lambda_j}$, i.e. $\nabla \varphi_{\lambda_j} \neq 0$ on $Z_{\varphi_{\lambda_j}}$, and the nodal set is a disjoint union of connected components which are homeomorphic either to circles contained in the interior $\Omega^o$ of $\Omega$ or to intervals intersecting the boundary in two points. We call the former ‘closed nodal loops’ and the latter ‘open nodal lines’. Thus, the theorem states that the number of open nodal lines is $O(\lambda_j)$. As mentioned above, the Courant nodal domain theorem implies that the number of nodal components is of order $O(\lambda_j^2)$. When the upper bound is achieved, the number of open nodal lines in dimension 2 is of one lower order in $\lambda_j$ than the number of closed nodal loops. This effect is known from numerical experiments of eigenfunctions and random waves [FGS].

The proof of Theorem 17.4 is to complexify the Cauchy data (i.e. $\varphi_{\lambda_j}|_{\partial \Omega}$ in the Neumann case or $\partial_\nu \varphi_{\lambda_j}|_{\partial \Omega}$ in the Dirichlet case) as holomorphic functions on the complexification of the boundary, and to count the number of its complex zeros. This number is bounded by the growth rate of the log modulus, which can be directly estimated in terms of the eigenvalue. Thus, the proof is of a similar form to the upper bound half of [DF]. On the other hand, no comparable lower bound exists: in the case of the disc (see §2.6), the boundary values are of the form $\sin m\theta$ or $\cos m\theta$, where $m$ is the angular momentum rather than the frequency $\lambda$. So the number of boundary zeros remains bounded for some sequences of eigenfunctions as the eigenvalue tends to infinity.

As previously mentioned, I. Polterovich [Po] used Theorem 17.4 to resolve an old conjecture of Pleijel, to the effect that Pleijel’s bound on the number of nodal domains (see Theorem 4.6) is valid for Neumann as well as Dirichlet boundary conditions. Polterovich observes that Pleijel’s argument applies to any nodal domain that does not touch the boundary. By Theorem 17.4, the nodal domains which do touch the boundary are of lower order than the bound, concluding the proof.
18. Riemannian random waves

We have mentioned that the random wave model provides a kind of guideline for what to conjecture about eigenfunctions of quantum chaotic system. In this final section, we briefly discuss random wave models and what they predict.

To define Riemannian random waves, we first consider the standard sphere. We choose an orthonormal basis \( \{ \varphi_{Nj} \}_{j=1}^{d_N} \) for \( \mathcal{H}_N \). We endow the real vector space \( \mathcal{H}_N \) with the Gaussian probability measure \( \gamma_N \) defined by

\[
\gamma_N(s) = \left( \frac{d_N}{\pi} \right)^{d_N/2} e^{-d_N|c|^2} dc, \quad \psi = \sum_{j=1}^{d_N} c_j \varphi_{Nj}, \quad d_N = \dim \mathcal{H}_N.
\]

Here, \( dc \) is \( d_N \)-dimensional real Lebesgue measure. The normalization is chosen so that \( \mathbb{E}_{\gamma_N} \langle \psi, \psi \rangle = 1 \), where \( \mathbb{E}_{\gamma_N} \) is the expected value with respect to \( \gamma_N \). Equivalently, the \( d_N \) real variables \( c_j \) \( (j = 1, \ldots, d_N) \) are independent identically distributed (i.i.d.) random variables with mean 0 and variance \( \frac{1}{2d_N} \); i.e.,

\[
\mathbb{E}_{\gamma_N} c_j = 0, \quad \mathbb{E}_{\gamma_N} c_j c_k = \frac{1}{d_N} \delta_{jk}.
\]

We note that the Gaussian ensemble is equivalent to picking \( \psi_N \in \mathcal{H}_N \) at random from the unit sphere in \( \mathcal{H}_N \) with respect to the \( L^2 \) inner product. The latter description is more intuitive but it is technically more convenient to work with Gaussian measures. In the cutoff ensemble, we put the product Gaussian measure \( \Pi_{n=1}^{N} \gamma_n \) on \( \bigoplus_{n=1}^{N} \mathcal{H}_n \).

On a general compact Riemannian manifold \((M, g)\) of dimension \( m \), the analogue of the space \( \mathcal{H}_N \) of spherical harmonics of degree \( N \) is played by the space \( \mathcal{H}_{I_N} \) of linear combinations of eigenfunctions with frequencies in an interval \( I_N := [N, N+1] \). The precise decomposition of \( \mathbb{R} \) into intervals is not canonical on a generic Riemannian manifold and the results do not depend on the choice. We choose \( I_N = [N, N+1] \) only for notational simplicity. In the special case of Zoll manifolds (all of whose geodesics are closed), there is a canonical choice where the intervals are centered in the middle of the eigenvalue clusters, i.e. the points \( \frac{2N + k}{L} \), where \( L \) (resp. \( \beta \)) is the common length (resp. Morse index) of the geodesics. Henceforth we abbreviate \( \mathcal{H}_N = \mathcal{H}_{I_N} \) on general Riemannian manifolds. We continue to denote by \( \{ \varphi_{Nj} \}_{j=1}^{d_N} \) an orthonormal basis of \( \mathcal{H}_N \) where \( d_N = \dim \mathcal{H}_N \). We equip it with the Gaussian measure (156) and again denote the expected value with respect to \( (\mathcal{H}_N, \gamma_N) \) by \( \mathbb{E}_{\gamma_N} \).

18.1. Levy concentration of measure. Levy concentration of measure occurs when Lipschitz continuous functions \( f \) on a metric probability space \((X, d, \mu)\) of large dimension \( d \) are highly concentrated around their median values \( M_f \). In the fundamental case where \( X \) is the unit \( N \)-sphere \( S^N \) with the usual distance function, and \( \mu \) is the \( \text{SO}(N+1) \)-invariant probability measure, the concentration of measure inequality says that

\[
\text{Prob} \{ x \in S^N : |f(x) - M_f| \geq r \} \leq \exp \left( -\frac{(N-1)^2}{2 \| f \|_{Lip}^2} r^2 \right),
\]

where

\[
\| f \|_{Lip} = \sup_{d(x,y)>0} \frac{|f(x) - f(y)|}{d(x,y)}.
\]
is the Lipschitz norm. (See, e.g. [L].)

A key point in the proof is the following well-known and simple fact that the mass the sphere concentrates in a small tube around any ‘equatorial’ sphere.

**Lemma 18.1.** Let \( A \in S^{2d-1} \subset \mathbb{C}^d \), and give \( S^{2d-1} \) Haar probability measure. Then

\[
\text{Prob}\{P \in S^{2d-1} : |\langle P, A \rangle| > \lambda\} \leq e^{-(d-1)\lambda^2}.
\]

Another basic fact about concentration of Gaussian measures is the following Levy concentration of Gaussian measure result due to Sudakov-Tsirelson and Borell:

**Lemma 18.2.** [ST] Let \( F \subset \mathcal{H}_N \) and let \( d_N = \dim \mathcal{H}_N \). Let \( F + \rho \) be the \( \rho \)-tube around \( F \). Then

\[
\mathbb{P}(F + \rho) \leq \frac{3}{4} \implies \mathbb{P}(F) \leq 2e^{-C\rho d_N}.
\]

### 18.2. Concentration of measure and \( L^p \) norms.

The purpose of this section is to illustrate the use of concentration of measure inequalities in one of its simplest applications to random waves: to determine the asymptotic behavior of \( L^p \) norms of random spherical harmonics of degree \( N \). The main functionals we consider are the norms on \( \mathcal{SH}_N \):

\[
L^p(\psi) = \|\psi\|_p \quad (2 \leq p \leq \infty).
\]

We only consider the case \( p = \infty \) in detail; for \( p < \infty \) see [SZ].

**Theorem 18.3.** For each of the above complex ensembles, there exist constants \( C > 0 \) such that:

\[
\nu_N \left\{ s_N \in \mathcal{SH}_N : \sup_{S^m} |\psi_N| > C\sqrt{\log N} \right\} < O\left(\frac{1}{N^2}\right).
\]

In fact, for any \( k > 0 \), we can bound the probabilities by \( O(N^{-k}) \) by choosing \( C \) to be sufficiently large.

As a corollary we obtain almost sure bounds on the growth of \( L^\infty \) norms for independent random sequences of \( \mathcal{L}^2 \)-normalized spherical harmonics. To state the result, we introduce the probability sequence space \( \mathcal{S} = \prod_{N=1}^\infty \mathcal{SH}_N \) with the measure \( \nu = \prod_{N=1}^\infty \nu_N \). The estimate of Theorem 18.3 immediately implies that

\[
\limsup_{N \to \infty} \sup_X \frac{\sup_{S^m} |\psi_N|}{\sqrt{\log N}} \leq C \quad \text{almost surely}.
\]

Hence we have:

**Corollary 18.4.** Sequences of sections \( \psi_N \in \mathcal{SH}_N \) satisfy:

\[
\|\psi_N\|_\infty = O(\sqrt{\log N}) \quad \text{almost surely}.
\]

Results of the latter type were first proved by Salem-Zygmund in the case of random trigonometric polynomials on the circle, and by Kahane for random trigonometric polynomials on tori. Vanderkam [Van] generalized the results to the case of random spherical harmonics by a geometric method that seems special to the sphere. Neuheisel [Neu] adapted a method of Nonnenmacher-Voros on holomorphic sections to simplify the proof of the sup norm estimates of [Van]. Here we follow an approach of [SZ] to derive them again from Levy concentration of measure. It also gives concentration results on \( L^p \) norms:
**Theorem 18.5.** Let $2 \leq p < \infty$. Then the median values of the $L^p$ norm on the unit spheres $S\mathcal{H}_N$ are bounded by a constant $\alpha = \alpha(p, m)$, and

$$
\nu_N \{ \psi_N \in S\mathcal{H}_N : L^p(\psi_N) > r + \alpha \} \leq \exp(-C r^2 N^{2m/p}) ,
$$

for some constant $C > 0$. Hence, sequences of sections $\psi_N \in S\mathcal{H}_N$ satisfy $\| \psi_N \|_p = O(1)$ almost surely.

18.3. $L^\infty$ norms: Proof of Theorem 18.3

**Proof.** We refer to §2.3 for notation, but drop the dimensional subscript. Throughout this section we assume that $\| \psi_N \|_{L^2} = 1$.

We define $\Phi_N(x) = (\varphi_{N,1}, \ldots, \varphi_{N,d_N}) : S^m \rightarrow \mathbb{R}^{d_N}$. It is well known (and easy to prove) that this eigenmap is an isometric minimal embedding of $(S^m, g_0)$ as a subsphere of a sphere of $(\mathbb{R}^{d_N}, ds^2)$ where $ds^2$ is the standard Euclidean metric. In fact,

$$
\Phi^*_N ds^2 = \sum_{j=1}^{d_N} d\varphi_j^N(x) \otimes d\varphi_j^N(x) = \frac{\lambda_j^2 d_N}{m \operatorname{Vol}(S^m)} g_0 \sim C_m N^{m+1} g_0.
$$

Indeed, by $SO(m + 1)$ invariance, the metric (158) is a constant multiple of the standard metric $g_0$. The constant can be calculated using the fact that $\Pi_N(x, x)$ is a constant function. By integrating over $S^m$ one sees that $\Pi_N(x, x) = \frac{d_N}{\operatorname{Vol}(S^m)} \sim \frac{1}{\operatorname{Vol}(S^m)} N^{m-1}$, and by taking $\Delta$ of this formula one can determine the constants in (158) (see e.g. [Ch] for background). Then

$$
\Pi_N(x, y) = \sum_{j=1}^{d_N} \varphi_j^N(x) \varphi_j^N(y) = \langle \Phi_N(x), \Phi_N(y) \rangle .
$$

Let $\psi_N = \sum_{j=1}^{d_N} c_j \varphi_j^N \ (\sum |c_j|^2 = 1)$ denote a random element of $S\mathcal{H}_N$, and write $c = (c_1, \ldots, c_{d_N})$. Recall that

$$
\psi_N(x) = \int_X \Pi_N(x, y) s_N(y) dy = \sum_{j=1}^{d_N} c_j \varphi_j^N(x) = c \cdot \Phi_N(x) .
$$

Thus

$$
|\psi_N(x)| = \| \Phi_N(x) \| \cos \theta_x , \quad \text{where } \cos \theta_x = \frac{|c \cdot \Phi_N(x)|}{\| \Phi_N(x) \|} .
$$

The angle $\theta_x$ can be interpreted as the distance in $S^{d_N-1}$ between $[\bar{c}]$ and $\tilde{\Phi}_N(x)$. Also, $\| \Phi_N(x) \| = \sqrt{\Pi_N(x, x)} \sim C_m N^{(m-1)/2}$.

Now fix a point $x \in S^m$. By Lemma 18.1,

$$
\nu_N \{ \psi_N : \cos \theta_x \geq CN^{-(m-1)/2} \sqrt{\log N} \} \leq \exp \left( - \frac{C^2 \log N}{N^{m-1}} \right) = N^{-C^2 N^{-(m-1)/(d_N-1)}} .
$$

We can cover $S^m$ by a collection of $k_N$ balls $B(z^i)$ of radius

$$
R_N := \frac{1}{N^m}
$$
centered at points \( z^1, \ldots, z^{k_N} \), where
\[
  k_N \leq O(R^{-m}) \leq O(N^{\frac{m^2}{2}}).
\]

By (162), we have
\[
  \nu_N \left\{ \psi_N : \max_j \cos \theta_{x^j} \geq C N^{-(m-1)/2} \sqrt{\log N} \right\} \leq k_N N^{-C^2 N^{-(m-1)}(d_N-1)},
\]

where \( x^j \) denotes a point in \( X \) lying above \( z^j \).

Equation (164) together with (161) implies that the desired sup-norm estimate holds at the centers of the small balls with high probability. To extend (164) to points within the balls, we consider an arbitrary point \( w^j \in B(z^j, N^{-m}) \). To estimate the distance \( \delta_N^j \), between \( \Phi_N(z^j) \) and \( \Phi_N(w^j) \) in \( S^{d_N-1} \) we let \( \gamma \) denote the geodesic in \( S^m \) from \( z^j \) to \( w^j \) and use (158) to get
\[
  \delta_N^j \leq \int_{\Phi_N \circ \gamma} ds = \int_{\gamma} \sqrt{\Phi_N^* ds} = C_m N^{\frac{m+1}{2}} \int_{\gamma} ds \leq C_m N^{-\frac{m+1}{2}} N^{-m} = C_m N^{-(m-1)/2}.
\]

By the triangle inequality in \( S^{d_N-1} \), we have \( |\theta_{x^j} - \theta_{y^j}| \leq \delta_N^j \). Therefore by (165),
\[
  \cos \theta_{x^j} \geq \cos \theta_{y^j} - \delta_N^j \geq \cos \theta_{y^j} - C_m N^{-(m-1)/2}.
\]

By (166),
\[
  \cos \theta_{y^j} \geq \frac{(C+1)\sqrt{\log N}}{N^{(m-1)/2}} \Rightarrow \cos \theta_{x^j} \geq \frac{(C+1)\sqrt{\log N} - C_m}{N^{(m-1)/2}} \geq \frac{C \sqrt{\log N}}{N^{(m-1)/2}}
\]

and thus
\[
  \{ \psi_N \in SH_N : \sup \cos \theta \geq (C+1)N^{-(m-1)/2} \sqrt{\log N} \} \subset \{ s_N \in SH_N : \max_j \cos \theta_{x^j} \geq C N^{-(m-1)/2} \sqrt{\log N} \}.
\]

Hence by (164),
\[
  \nu_N \left\{ \psi_N \in SH_N : \sup \cos \theta \geq (C+1)N^{-(m-1)/2} \sqrt{\log N} \right\} \leq k_N N^{-C^2 N^{-(m-1)}(d_N-1)}.
\]

There exists \( A_M > 0 \) so that \( d_N \sim A_m N^{m-1} \), so one has
\[
  \nu_N \left\{ \psi_N \in SH_N : \sup_M |\psi_N| \geq (C+2) \sqrt{\log N} \right\} \leq k_N N^{-C^2 N^{-(m-1)}(d_N-1)} \leq O \left( N^{m^2 - C^2 A_m} \right).
\]

Choosing \( C \) so that \( C^2 A_m - m^2 > 0 \) is sufficiently large concludes the proof.
18.4. **Sup norms on small balls.** We use the same method to prove a claim in \[\text{[NS]}\] on the sup norm in small balls on \(S^2\). This corresponds to \(m = 1\)

**Corollary 18.6** (Estimate of the maximum). Given \(\rho > 0\), there exists \(A\) such that, for any \(x_0 \in \mathbb{S}\), \(\mathbb{P}\{\max_{B(x_0, \rho/N)} |f| \geq A\} \leq \frac{1}{\rho}\).

Again by Lemma [18.1] for any \(x \in B(x_0, \rho/N)\),

\[
\nu_N \left\{ \psi_N : \cos \theta_x \geq N^{-1}A \right\} \leq \exp \left( -2N - 2 \frac{A^2}{N} \right).
\]

We can cover \(B(x_0, \rho/N)\) by a collection of \(k_0(\epsilon)\) of balls \(B(x^i, \frac{\epsilon}{N})\). By (168), we have

\[
(168) \quad \nu_N \left\{ \psi_N : \max_i \cos \theta_{x^i} \geq CN^{-1/2}A \right\} \leq k_0(\epsilon)e^{-A^2}.
\]

Again, we need that (168) implies that the desired sup-norm estimate holds on all of \(B(x_0, \frac{\epsilon}{N})\), so we consider an arbitrary point \(y^i \in B(z^j)\) and estimate the distance \(\delta_N\), between \(\Phi_N(x^i)\) and \(\Phi_N(y^i)\) in \(S^{dN-1}\). As before, we have

\[
\cos \theta_{y^i} \geq \frac{A}{N} \Rightarrow \cos \theta_{x^i} \geq \frac{A}{N}
\]

and thus

\[
\left\{ \psi_N \in \mathcal{SH}_N : \sup_{y \in (B, \frac{\epsilon}{N})} \cos \theta_y \geq A/N \right\} 
\subset \left\{ \psi_N \in \mathcal{SH}_N : \max_i \cos \theta_{x^i} \geq A/N \right\}
\]

or

\[
\nu_N \left\{ \psi_N \in \mathcal{SH}_N : \sup_{y \in (B, \frac{\epsilon}{N})} \theta_y \geq A/N \right\} \leq k_0(\epsilon)e^{-A^2}.
\]

It follows that

\[
\nu_N \left\{ \psi_N \in \mathcal{SH}_N : \sup_{B(x_0, 1/N)} |\psi_N| \geq A \right\} \leq k_0(\epsilon)e^{-A^2}.
\]

Choosing \(A\) sufficiently large will make the right side \( \leq \frac{1}{\rho} \).

18.5. **Relation to Levy concentration.** The estimate in Theorem [18.3] is very closely related to Levy’s estimate. The proof shows that

(i) \(\mathcal{L}_\infty^N\) is Lipschitz continuous with norm \(\frac{N^{(m-1)/2}}{\log N} \leq \|\mathcal{L}_\infty^N\|_{Lip} \leq N^{(m-1)/2}\).

(ii) The median of \(\mathcal{L}_\infty^N\) satisfies: \(\mathcal{M}_{\mathcal{L}_\infty^N} \leq C_m \sqrt{\log N}\) for sufficiently large \(N\).

Indeed, Lipschitz continuity follows from equivalence of norms on finite dimensional vector spaces. To estimate the Lipschitz norm, we recall the well-known fact that the \(\mathcal{L}^2\)-normalized ‘coherent states’ \(\Phi_N^w(z) = \frac{\Pi_N(z, w)}{\Pi_N(w, w)}\) are the global maxima of \(\mathcal{L}_\infty^N\) on \(SH^0(M, L^N)\), as follows from the Schwartz inequality applied to the reproducing identity \(s(z) = \int_M \Pi_N(z, w)s(w)dV(w)\).

Moreover, \(\|\Phi_N^w(z)\|_{\mathcal{L}_\infty^N} = \sqrt{\Pi_N(w, w)} \sim N^{(m-1)/2}\). It follows that

\[
\|s_1 + s_2\|_{\infty} - \|s_1\|_{\infty} \leq 3N^{(m-1)/2}.
\]
Now let $s_1$ have $L^\infty$ norm $\leq C\sqrt{\log N}$ and let $s_1 = \Phi_N^w$ for some $w$. Then we see that
\[
\left\| s_1 + s_2 \right\|_\infty - \left\| s_1 \right\|_\infty \geq \frac{N^{(m-1)/2}}{\sqrt{\log N}}.
\]

It obviously follows from (i)–(ii) combined with the Levy estimate (157) that (for any $C > 0$)
\begin{equation}
\nu_N \{ \psi \in \mathcal{SH}_N : L^\infty_N(\psi) \geq C\sqrt{\log N} \} \leq \exp\left( -C(d_N - 1) \log N / 2N^{m-1} \right).
\end{equation}

Since $d_N \sim N^{m-1}$, this is essentially the same estimate as in Theorem 18.3.

18.6. Nazarov-Sodin Theorem on the mean number of nodal domains of random spherical harmonics. In this section we review a recent theorem of Nazarov-Sodin [NS] on the mean number of nodal domains of random spherical harmonics on $S^2$. We closely follow their exposition.

We recall that a nodal domain of an eigenfunction $f$ is a connected component of $M \setminus Z_f$ where $Z_f$ is the zero set of $f$, i.e. the nodal hypersurface. The number of nodal domains of $f$ is denoted $N(f)$. The main result of [NS] is:

**Theorem 18.7.** There exists a constant $a > 0$ such that, for every $e > 0$, we have
\[
P \left\{ \left| \frac{N(f)}{n^2} - a \right| > e \right\} \leq C(e) e^{-c(e)n}
\]
where $c(e)$ and $C(e)$ are some positive constants depending on $e$ only.

**Remark:** As noted above, the total length of the nodal line $Z(f)$ of any spherical harmonic $f \in \mathcal{H}_n$ does not exceed $Cn$ for some $C > 0$ [DF]. Since the typical spherical harmonic has $\sim an^2$ nodal domains, the perimeter of most of its nodal domains must be of order $\frac{1}{n}$ and the diameters of the nodal domains must be comparable to $\frac{1}{n}$.

The proof of Theorem 18.7 consists of the following steps:

1. Proof that the lower bound $\mathbb{E} N(f) \geq \text{const} n^2$.
2. Proof of exponential concentration of the random variable $N(f)/n^2$ around its median. The proof uses the uniform lower continuity of the functional $f \mapsto N(f)$ with respect to the $L^2$-norm outside of an exceptional set $E \subset \mathcal{H}$ of exponentially small measure and Levy’s concentration of measure principle.
3. Proof of the existence of the limit $\lim_{n \to \infty} \mathbb{E} N(f)/n^2$. In this part, we use existence of the scaling limit for the covariance function $\mathbb{E} \{ f(x) f(y) \}$.

18.6.1. Lower bound for $\mathbb{E} N(f)$. The first step is to show that $\mathbb{E} N(f) \gtrsim n^2$. By Corollary 18.6, for any $\rho > 0$, there exists $A > 0$ such that, for all $x \in S$,
\[
P \left\{ \max_{D(x,\rho/n)} |f| \geq A \right\} \leq \frac{1}{3}.
\]

Furthermore, the zonal spherical harmonic $b_x^N$ with pole at $x$, i.e. $b_x = \frac{\Pi_N(x,\cdot)}{\sqrt{\Pi_N(x,x)}}$, has the following property: there exists $\rho$ and $c_1$ such that, for each sufficiently large $N$
\[
\|b_x\| = 1, \quad b_x(x) \geq c_1 \sqrt{N}, \quad \text{and} \quad b_x|_{\partial D(x,\rho/N)} \leq -c_1 \sqrt{N}.
\]
Proof of the lower bound for $N(f)$: Fix $x \in \mathbb{S}$. We have $f = \xi_0 b_x + f_x$ where $\xi_0$ is a Gaussian random variable with $E \xi_0^2 = \frac{1}{2N+1}$, and $f_x$ is a Gaussian spherical harmonic built over the orthogonal complement to $b_x$ in $\mathcal{H}_N$ and normalized by $E \|f_x\|^2 = \frac{2N}{2N+1}$. We choose a Gaussian random variable $\tilde{\xi}_0$ independent of $\xi_0$ and of $f_x$ with $E \tilde{\xi}_0^2 = E \xi_0^2 = \frac{1}{2N+1}$, and set $f_\pm = \pm \tilde{\xi}_0 b_x + f_x$. These are Gaussian spherical harmonics having the same distribution as $f$. Note that

$$f = \xi_0 b_x + \frac{1}{2} (f_+ + f_-),$$

and that by Corollary 18.6

$$P\left( \max_{D(\rho,x)} |f_+| \leq A \right) \cap \left\{ \max_{D(\rho,x)} |f_-| \leq A \right\} \geq 1 - \left( \frac{1}{3} + \frac{1}{3} \right) = \frac{1}{3}.$$

Now, consider the event $\Omega_x$ that $f(x) \geq A$ and $f|_{\partial D(\rho,x/N)} \leq -A$. Clearly, $\Omega_x$ must contain a nodal component of $f$. The event $\Omega_x$ happens provided that

$$\xi_0 \sqrt{N} \geq 2c_1^{-1} A \quad \text{and} \quad \max_{D(\rho,x)} |f_\pm| \leq A.$$ 

Since the variance of the Gaussian random variable $\xi_0 \sqrt{N}$ is of constant size, there exists $\kappa > 0$ such that

$$P(\Omega_x) \geq P(\xi_0 \sqrt{N} \geq 2c_1^{-1} A) \cdot P\left( \max_{D(\rho,x)} |f_+| \leq A \right) \cap \left\{ \max_{D(\rho,x)} |f_-| \leq A \right\} \geq \kappa > 0.$$

Now pack $S^2$ with $\simeq N^2$ disjoint disks $D_j$ of radius $2\rho/N$ centered at points $x_j$. Each of them contains a component of $Z(f)$ with probability at least $\kappa$, and since the discs are disjoint, $N(f) \geq \sum_j N(f;D_j)$ where $N(f,D_j)$ is the number of nodal components in the disc $D_j$. Hence,

$$EN(f) \geq \sum_j EN(f,D_j) \geq \sum_j P(\Omega_{x_j}) \gtrsim C_{\kappa,\rho} N^2.$$

18.6.2. Exponential concentration near the median. This exponential concentration follows from Levy (-Sudakov-Tsirelson) concentration of measure plus the delicate fact that $N(f)$ is stable under slight perturbations of $f$ for all but an exponentially rare set of $f$, called ‘unstable’ spherical harmonics.

**Lemma 18.8.** For every $e > 0$, there exists $\rho > 0$ and an exceptional set $E \subset \mathcal{H}_N$ of probability $P(E) \leq C(e)e^{-c(e)N}$ such that for all $f \in \mathcal{H}_N \setminus E$ and for all $g \in \mathcal{H}_N$ satisfying $\|g\| \leq \rho$, we have $N(f + g) \geq N(f) - eN^2$.

We refer to [NS] for the proof. Lemma 18.8 in conjunction with the Sudakov-Tsirelson concentration of measure theorem (Lemma 18.2) implies an exponential concentration of

$$\frac{N(f)}{N^2} \quad \text{near its median} \quad a_N.$$ 

Recall that $A_{\rho}$ is the $\rho$-tube around $A$.

**Corollary 18.9.** $P\{ f \in \mathcal{H}_N : |N(f) - a_N| > eN^2 \} \leq 2e^{-c\rho^2N}$. 
Proof. Let $F = \{ f \in \mathcal{H}_N : N(f) > (a_N + e)N^2 \}$. Then for $f \in (F \setminus E)_{+\rho}$, we have $N(f) > a_NN^2$, and therefore, $\mathbb{P}((F \setminus E)_{+\rho}) \leq \frac{1}{2}$. By the Sudakov-Tsirelson inequality (Lemma 18.2), $\mathbb{P}(F \setminus E) \leq 2e^{-c\rho^2N}$, hence

$$\mathbb{P}(F) \leq 2e^{-c\rho^2N} + C(e)e^{-c(e)N} \leq C(e)e^{-c(e)N}.$$  

On the other hand, let $G = \{ f \in \mathcal{H}_N : N(f) < (a_N - e)N^2 \}$. Then

$$G_{+\rho} \subset \{ f \in \mathcal{H}_N : N(f) < a_N^2 \} \cup E$$

and so

$$\mathbb{P}(G_{+\rho}) \leq \frac{1}{2} + C(e)e^{-c(e)N} < \frac{3}{4}$$

for large $N$. It follows again that $\mathbb{P}(G) \leq 2e^{-c\rho^2N}$ for large $N$. \hfill \Box

Further, it is proved in [NS] that

**Proposition 18.10.** The sequence of medians $\{a_N\}$ converges.

Since the random variable $N(f)/N^2$ exponentially concentrates near its median $a_N$ and is uniformly bounded, it suffices to show that the sequence of means $\mathbb{E}N(f)/n^2$ converges; the sequence of medians $\{a_N\}$ converges to the same limit. The authors of [NS] show that $\{\mathbb{E}N(f)/n^2\}$ is a Cauchy sequence.

19. Appendix on Tauberian Theorems

We record here the statements of the Tauberian theorems that we use in the article. Our main reference is [SV], Appendix B and we follow their notation.

We denote by $F_+$ the class of real-valued, monotone nondecreasing functions $N(\lambda)$ of polynomial growth supported on $\mathbb{R}_+$. The following Tauberian theorem uses only the singularity at $t = 0$ of $\hat{dN}$ to obtain a one term asymptotic of $N(\lambda)$ as $\lambda \to \infty$:

**Theorem 19.1.** Let $N \in F_+$ and let $\psi \in \mathcal{S}(\mathbb{R})$ satisfy the conditions: $\psi$ is even, $\psi(\lambda) > 0$ for all $\lambda \in \mathbb{R}$, $\hat{\psi} \in C_0^\infty$, and $\hat{\psi}(0) = 1$. Then,

$$\psi \ast dN(\lambda) \leq A\lambda^\nu \implies |N(\lambda) - N \ast \psi(\lambda)| \leq C A\lambda^\nu,$$

where $C$ is independent of $A, \lambda$.

To obtain a two-term asymptotic formula, one needs to take into account the other singularities of $\hat{dN}$. We let $\psi$ be as above, and also introduce a second test function $\gamma \in \mathcal{S}$ with $\hat{\gamma} \in C_0^\infty$ and with the supp $\hat{\gamma} \subset (0, \infty)$.

**Theorem 19.2.** Let $N_1, N_2 \in F_+$ and assume:

1. $N_j \ast \psi(\lambda) = O(\lambda^\nu), (j = 1, 2)$;
2. $N_2 \ast \psi(\lambda) = N_1 \ast \psi(\lambda) + o(\lambda^\nu)$;
3. $\gamma \ast dN_2(\lambda) = \gamma \ast dN_1(\lambda) + o(\lambda^\nu)$.

Then,

$$N_1(\lambda - o(1)) - o(\lambda^\nu) \leq N_2(\lambda) \leq N_1(\lambda + o(1)) + o(\lambda^\nu).$$
This Tauberian theorem is useful when the non-zero singularities of \( \hat{dN}_2 \) are as strong as the singularity at \( t = 0 \) and \( N_2 \) does not have two term polynomial asymptotics.

Alternatively, we may use the Tauberian lemma of \[ \text{Hol-IV} \]:

**Lemma 19.3.** Suppose that \( \mu \) is a non-decreasing temperate function satisfying \( \mu(0) = 0 \) and that \( \nu \) is a function of locally bounded variation such that \( \nu(0) = 0 \). Suppose also that \( m \geq 1 \) and that \( \varphi \in S(\mathbb{R}) \) is a fixed positive function satisfying
\[
\int \varphi(\lambda) d\lambda = 1
\]
and \( \hat{\varphi}(t) = 0 \), \( t \not\in [-1, 1] \). If \( \varphi_\sigma(\lambda) = \sigma^{-1}\varphi(\lambda/\sigma) \), \( 0 < \sigma \leq \sigma_0 \), assume that for \( \lambda \in \mathbb{R} \)
\[
|d\nu(\lambda)| \leq (A_0(1 + |\lambda|)^m + A_1(1 + |\lambda|)^{m-1}) d\lambda,
\]
and that
\[
|((d\mu - d\nu) * \varphi_\sigma)(\lambda)| \leq B(1 + |\lambda|)^{-2}.
\]
Then
\[
|\mu(\lambda) - \nu(\lambda)| \leq C_m\left(A_0 \sigma(1 + |\lambda|)^m + A_1 \sigma(1 + |\lambda|)^{m-1} + B\right),
\]
where \( C_m \) is a uniform constant depending only on \( \sigma_0 \) and our \( m \geq 1 \).

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