A proof of the convexity of a set of lamination parameters

Jean-Luc Akian

ONERA, Université Paris Saclay F-92322 Châtillon, France

Correspondence
Jean-Luc Akian, ONERA, Université Paris Saclay F-92322 Châtillon, France.
Email: jean-luc.akian@onera.fr

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In this paper we show that the proof of the convexity of the set of lamination parameters given by J.L. Grenestedt and P. Gudmundson, which is extensively cited in the literature, is not correct. We give a proof of the convexity of this set when the class of layup functions is the set of step functions. Moreover we give a proof of the non-convexity of this set when the class of layup functions is the set of step functions where the layers have the same thickness and the number of layers is not fixed.

KEYWORDS
composites, convexity, lamination parameters, optimization

MSC CLASSIFICATION
52B55; 74A40; 74Pxx

1 INTRODUCTION

Lamination parameters are extensively used for the layup optimization of laminated composite structures instead of the ply thicknesses and the layup angles. These parameters are integrals through the thickness of functions of the layup angles of the different plies of the composite laminate and their number is small (12). A key result for layup optimization is the convexity of the set of lamination parameters (more precisely the set of families of 12 lamination parameters, but for simplicity we shall use instead the expression “set of lamination parameters”). A fundamental remark is that when one speaks of the convexity of the set of lamination parameters, one must specify the class of layup functions (that is, the functions which give the layup angle as a function of the through-the-thickness coordinate). For a composite laminate these functions should be step functions (that is to say piecewise constant functions).

The paper gives a proof of the convexity of the set of lamination parameters and is extensively cited in the literature for this proof (see, e.g., Bendsøe & Sigmund, p.209 Vannucci, p.265 Bloomfield et al., p.1123 Diaconu et al., p.559 Bordogna et al., Section 3.1.1 Albazzan et al., p.5 Macquart et al., p.5 Duan et al., p.2112 Serhat & Basdogan, p.3 Serhat & Basdogan, p.3 Viquerat, p.2 to name just a few, see also the list of articles quoting on Google Scholar). But the proof of this paper is not correct as will be shown in the present paper (see Sections 2 and 4 for a detailed proof of the result of Grenestedt & Gudmundson).

In the present paper we give a proof of the convexity of the set of lamination parameters when the class of layup functions is the set of step functions (Section 3, Proposition 3.1). This means in particular that in the class of composite laminates we consider the number of layers and their thicknesses are not fixed. Moreover the proof is constructive: given two families of 12 lamination parameters corresponding to two layup functions, the proof gives formulas for a layup function corresponding to any convex combination of the two families of 12 lamination parameters.

In the literature most of the papers about lamination parameters deal with laminates composed of layers of the same thickness (it is the “uniform thickness approach”), although some of them concern laminates composed of layers of not necessarily the same thickness (it is the “variable thickness approach”), see, for example. As in the paper, the present paper considers laminates with layers of not necessarily the same thickness. The contribution of the present paper is
to give a rigorous mathematical proof of the convexity of the set of lamination parameters within this framework. It is an improvement compared with the proof of Grenestedt and Gudmundson\(^1\) which is false, although largely cited in the literature.

In a very recent paper,\(^{15}\) it is shown that the set of lamination parameters is non-convex when the aforementioned class of layup functions is the set of step functions where the number of layers is fixed (\(= N > 1\)) and the layers have the same thickness (see Scardaoni & Montemurro,\(^{15}\) Proposition 3.4). The results of Scardaoni and Montemurro\(^{15}\) and of the present paper are not incompatible since the classes of layup functions, therefore also the set of lamination parameters, are different in the two studies (more precisely the set of lamination parameters in Scardaoni and Montemurro\(^{15}\) is included in the set of lamination parameters in the present paper).

In Section 5 we give a proof of Proposition 3.4 of Scardaoni and Montemurro\(^{15}\) which is true for any number of layers \(N \geq 1\) (Proposition 5.1). Moreover, we show that when the class of layup functions is the set of step functions where the layers have the same thickness but the number of layers is not fixed, the associated set of lamination parameters is non-convex (Proposition 5.2). This proves that when one considers the set of lamination parameters associated to a class of step functions, in order to get a convex set, the condition “the layers have the same thickness” must be discarded.

This all shows once again that it is very important to precise the class of layup functions when one speaks of the convexity of the set of lamination parameters and the present paper clarifies the matter about this subject.

Finally, in Appendix A1, thanks to Lemma A.1, we give a little simpler proof of Proposition 3.2 of Scardaoni and Montemurro,\(^{15}\) which shows the non-convexity of the set of polar parameters when the number of layers \(N\) is \(> 1\).

## 2 | PROBLEM STATEMENT AND INACCURACY OF THE PROOF IN [1]

In the sequel the set of natural numbers will be denoted by \(\mathbb{N}\) (containing 0) and the set of positive natural numbers \(\mathbb{N} \setminus \{0\}\) by \(\mathbb{N}^*\). Recall that the word “iff” means “if and only if.”

The lamination parameters are given by the formulas:

\[
\xi_{1,2,3,4}^A[\theta] = \frac{1}{2} \int_{-1}^{1} \cos 2\theta(z), \cos 4\theta(z), \sin 2\theta(z), \sin 4\theta(z) dz, \quad (2.1)
\]

\[
\xi_{1,2,3,4}^R[\theta] = \int_{-1}^{1} \cos 2\theta(z), \cos 4\theta(z), \sin 2\theta(z), \sin 4\theta(z) dz, \quad (2.2)
\]

\[
\xi_{1,2,3,4}^D[\theta] = \frac{2}{2} \int_{-1}^{1} \cos 2\theta(z), \cos 4\theta(z), \sin 2\theta(z), \sin 4\theta(z) z^2 dz, \quad (2.3)
\]

(Grenestedt & Gudmundson\(^1\) Equation (13)), where \(z\) is the normalized through-the-thickness coordinate and \(\theta\): \(z \in [-1, 1] \mapsto \theta(z) \in \mathbb{R}\) (where \(\mathbb{R}\) is the set of real numbers) is the (measurable) layup function. If \(\theta\) is a measurable function from \([-1, 1]\) into \(\mathbb{R}\), let us call \(\xi[\theta]\) the family of the corresponding 12 lamination parameters. If \(C\) is a class of real measurable functions defined on the interval \([-1, 1]\) set

\[
LP_C = \{\xi[\theta] \text{ such that } \theta \in C\}, \quad (2.4)
\]

that is to say \(LP_C\) is the set of the families of 12 lamination parameters associated to all the \(\theta\) in the class \(C\). For a given class of functions \(C\) the convexity of \(LP_C\) reads as follows: if \(\theta_1\) and \(\theta_2 \in C\), and if \(a \in [0, 1]\) (or \(a \in (0, 1)\)), then \((1 - a)\xi[\theta_1] + a\xi[\theta_2] \in LP_C\), that is to say \((1 - a)\xi[\theta_1] + a\xi[\theta_2]\) can be written under the form

\[
(1 - a)\xi[\theta_1] + a\xi[\theta_2] = \xi[\theta] \quad (2.5)
\]

with \(\theta \in C\).

In Grenestedt and Gudmundson,\(^1\) p.317 given \(a \in [0, 1]\) and two functions \(\theta_1\) and \(\theta_2\) from \([-1, 1]\) into \(\mathbb{R}\), whose regularity is not specified, a sequence of functions \(\theta^n\) is constructed such that

\[
\xi[\theta^n] \rightarrow (1 - a)\xi[\theta_1] + a\xi[\theta_2] \quad (2.6)
\]
when \( n \to +\infty \). The proof of (2.6) in Grenestedt and Gudmundson\(^1\) is right if \( \theta_1 \) and \( \theta_2 \) are in the class of piecewise continuous functions on \([-1, 1]\) that have a continuous extension on each of the subintervals of their definition. This result of Grenestedt and Gudmundson\(^1\) for this class of functions is detailed in Section 4.

But by no means (2.6) proves the convexity of the set of lamination parameters because it does not prove the existence of a function \( \theta \) such that (2.5) is satisfied.

On the other hand in Scardaoni and Montemurro\(^{15}\) p.6 it is written: “In (Grenestedt and Gudmundson (1993)) it was claimed that the feasible domain in LPs space is convex. The thesis and the proof of this claim are erroneous. The authors consider one of the twelve components of \( p \) at time.”

But this is not at all the real reason why the proof of\(^2\) is erroneous. The problem in Grenestedt and Gudmundson\(^1\) is not that “authors consider one of the twelve components of \( p \) at time.” The proof of (33) in Grenestedt and Gudmundson\(^1\) is true for all the functions of the form \( z \in [-1, 1] \mapsto f(\theta(z))z^j, \ j = 0, 1, 2 \) (and even \( j \in \mathbb{N} \)) where \( f \) is a continuous function on \( \mathbb{R} \), as shown in Section 4. As already mentioned, in Grenestedt and Gudmundson\(^1\) a sequence of functions \((\theta^n)\) is constructed such that (2.6) is satisfied and this does not show the convexity of the set of lamination parameters.

Recall that a function \( \theta: z \in [-1, 1] \mapsto \theta(z) \in \mathbb{R} \) is a step function if there exists \( N \in \mathbb{N}^+ \) and a sequence of real numbers \((a_1), \ldots, a_N, a_0 = -1, a_N = 1, a_i < a_{i+1}, i = 0, \ldots, N-1\) such that \( \theta \) is constant on each of the intervals \((a_i, a_{i+1}), i = 0, \ldots, N \equiv 0 \) (the value of \( \theta \) at the points \( a_i \)). This is possible thanks to Lemma 3.1.

In Scardaoni and Montemurro\(^{15}\) if \( N \in \mathbb{N}^+ \), one considers the class of step functions on \([-1, 1]\) that have a continuous extension on each of the subintervals of their definition. This result is non-convex (Proposition 5.2). In Scardaoni and Montemurro\(^{15}\) Proposition 3.4 it is shown that for \( N \) fixed \( \geq 1 \), \( \text{LP}_{C_{\theta}} \) is non-convex. In Section 5 we give a proof of Proposition 3.4 of Scardaoni and Montemurro\(^{15}\) which is true for all \( N \geq 1 \) (Proposition 5.1).

Obviously, for \( N \) fixed \( C_N \) is a subset of \( C_S \); therefore \( \text{LP}_{C_N} \) is a subset of \( \text{LP}_{C_{S}} \). But since \( \text{LP}_{C_N} \) (for \( N \) fixed) and \( \text{LP}_{C_{S}} \) are different there is absolutely no contradiction between the result of Scardaoni and Montemurro\(^{15}\) and the result of the present paper (even if \( \text{LP}_{C_N} \) is a subset of \( \text{LP}_{C_{S}} \)).

The result of Scardaoni and Montemurro\(^{15}\) is understandable because in order to show the convexity in this case, if \( \theta_1 \) and \( \theta_2 \in C_N \), and if \( \alpha \in [0, 1] \), one must show that \((1 - \alpha)\varepsilon[\theta_1] + \alpha\varepsilon[\theta_2] \in \text{LP}_{C_N} \), that is to say one must construct a step function \( \theta \in C_N \) such that Equation (2.5) is satisfied. This means in particular that this \( \theta \) must correspond to \( N \) layers with the same thickness (even if the values of \( \theta \) over several neighboring layers may be identical and the corresponding layers can be gathered to form a layer with a different thickness): this constraint is stronger than in the situation of the present paper where the step function \( \theta \) to be constructed may correspond to any number of layers and to any thicknesses: that is the fundamental difference.

Now let us consider the class of step functions on \([-1, 1]\) such that the layers have the same thickness but the number of layers is not fixed. Mathematically this class denoted \( C_s \) is defined by

\[
C_s = \bigcup_{N \in \mathbb{N}^+} C_N
\]

(2.7)

In Section 5, we show that \( \text{LP}_{C_s} \) is non-convex (Proposition 5.2).

In summary, the sets of lamination parameters associated to the set of step functions on \([-1, 1]\) (\( = \text{LP}_{C_{S}} \)), to the set of step functions on \([-1, 1]\) such that the layers have the same thickness but the number of layers is not fixed (\( = \text{LP}_{C_s} \)) and to the set of step functions on \([-1, 1]\) such that the layers have the same thickness and the number of layers is fixed to a value \( N \in \mathbb{N}^+ \) (\( = \text{LP}_{C_N} \)) are such that

\[
\text{LP}_{C_N} \subset \text{LP}_{C_s} \subset \text{LP}_{C_{S}}
\]

(2.8)
3 | PROOF OF THE CONVEXITY OF $LP_{C_{SF}}$

We shall prove the convexity of the set of lamination parameters when the functions $\theta$ are in the class of step functions. Let $\theta_1$ and $\theta_2$ be two step functions on $[-1, 1]$. There exists $N \in \mathbb{N}^*$ and a sequence of real numbers $(a_i)_{i=0, \ldots, N}$, $a_0 = -1$, $a_N = 1$, $a_i < a_{i+1}$, $i = 0, \ldots, N-1$ such that the two step functions $\theta_1$ and $\theta_2$ are constant on each of the intervals $(a_i, a_{i+1})$, taking the values $\theta_1^i$ and $\theta_2^i$, $i = 0, \ldots, N-1$: the sequence $(a_i)_{i=0, \ldots, N}$ is simply the union of the corresponding sequences associated to $\theta_1$ and $\theta_2$.

The lamination parameters are all under the form

$$E_i = \{\theta(z) \in \mathcal{F} \mid \theta(x) \in \mathcal{F}_i \text{ for } x \in \mathcal{E}_i \},$$

where $\mathcal{F}$ is the class of step functions, $\mathcal{F}_i$ is the class of step functions such that $\theta(x) \in \mathcal{F}_i$ for $x \in \mathcal{E}_i$, and $\mathcal{E}_i$ is a (disjoint) union of open intervals.

In (3.5), we have used the following notation: if $E$ is a subset of $\mathbb{R}$, $1_E$ is the indicator function of the subset $E$, that is the function such that $1_E(x) = 1$ if $x \in E$ and $1_E(x) = 0$ if $x \notin E$. Equations (3.5) and (3.6) imply that for all continuous function $f$ on $\mathbb{R}$, we have

$$\int_{a_i}^{a_{i+1}} f(\theta(z))z'dz = f(\theta_1^i) \int_{E_1} z'dz + f(\theta_2^i) \int_{E_2} z'dz,$$

$$i = 0, \ldots, N-1, \quad j = 0, 1, 2.$$  

From (3.4) and (3.7), for all $i = 0, \ldots, N-1$, it is sufficient to find $E_1^i$ ($E_2^i$ will be determined by (3.6)) such that for all continuous function $f$ on $\mathbb{R}$,

$$[(1 - \alpha)f(\theta_1^i) + \alpha f(\theta_2^i)] \int_{a_i}^{a_{i+1}} z'dz = f(\theta_1^i) \int_{E_1} z'dz + f(\theta_2^i) \int_{E_2} z'dz,$$

$$j = 0, 1, 2.$$
It is enough to satisfy
\[
(1 - \alpha) \int_{a_i}^{b_i} z^j dz = \int_{E_i} z^j dz, \quad i = 0, \ldots, N - 1, \ j = 0, 1, 2
\]  
(3.9)
and
\[
\alpha \int_{a_i}^{b_i} z^j dz = \int_{E_i} z^j dz, \quad i = 0, \ldots, N - 1, \ j = 0, 1, 2.
\]  
(3.10)

We stress that if we find $\theta$ of the form (3.5), (3.6) such that (3.9), (3.10) is satisfied, then $\theta$ will be independent of $f$ and $j = 0, 1, 2$, so that $\theta$ will be the same for all the lamination parameters.

But if $E_i^1$ and $E_i^2$ satisfy (3.6) it results that for all $i = 0, \ldots, N - 1, \ j = 0, 1, 2$,
\[
\int_{E_i^1} z^j dz + \int_{E_i^2} z^j dz = \int_{a_i}^{b_i} z^j dz = (1 - \alpha) \int_{a_i}^{b_i} z^j dz + \alpha \int_{a_i}^{b_i} z^j dz.
\]  
(3.11)
thus it suffices to find $E_1^1$ satisfying (3.9) and (3.10) will be necessarily satisfied. The answer to this problem is given by the following lemma:

**Lemma 3.1.** Let $A, B \in \mathbb{R}, A < B$ and let $\alpha \in (0, 1)$. Then there exists a union of intervals $\mathcal{C}(A, B)$ (denoted $E$) such that
\[
\alpha \int_{A}^{B} z^j dz = \int_{E} z^j dz, \quad j = 0, 1, 2.
\]  
(3.12)

**Proof.** Since $E$ must meet the three conditions (3.12), we seek $E$ under the form $E = (a, b) \cup (c, d)$ with $a, b, c, d \in \mathbb{R}, A < a < b < c < d < B$ and from the beginning we enforce the condition $b - a = d - c$ (denoted $e$), so that $E$ will depend on three parameters to be determined. The following conditions must be satisfied:
\[
\alpha \int_{A}^{B} z^j dz = \int_{a}^{b} z^j dz + \int_{c}^{d} z^j dz, \quad j = 0, 1, 2,
\]  
(3.13)
that is
\[
\alpha(B - A) = b - a + d - c = 2e,
\]  
(3.14)
\[
\alpha(B^2 - A^2) = b^2 - a^2 + d^2 - c^2,
\]  
(3.15)
\[
\alpha(B^3 - A^3) = b^3 - a^3 + d^3 - c^3.
\]  
(3.16)
Equation (3.14) makes it possible to determine $e$: $e = \alpha(B - A)/2$. Setting $X = (a + b)/2$ and $Y = (c + d)/2$, we get
\[
a = X - e/2, \ b = X + e/2, \ c = Y - e/2, \ d = Y + e/2,
\]  
(3.17)
\[
\alpha(B^2 - A^2) = \alpha(B - A)(B + A) = 2e(A + B),
\]  
(3.18)
\[
b^2 - a^2 + d^2 - c^2 = (b - a)(a + b) + (d - c)(c + d) = 2e(X + Y).
\]  
(3.19)
Equations 3.15, 3.18, (3.19) give
\[
X + Y = A + B.
\]  
(3.20)
On the other hand
\[
b^3 - a^3 = (X + e/2)^3 - (X - e/2)^3 = 2[3X^2(e/2) + (e/2)^3] = e(3X^2 + e^3/4),
\]  
(3.21)
\[
d^3 - c^3 = (Y + e/2)^3 - (Y - e/2)^3 = 2[3Y^2(e/2) + (e/2)^3] = e(3Y^2 + e^3/4),
\]  
(3.22)
\[
A = (A + B)/2 - (B - A)/2 = (X + Y)/2 - e/\alpha,
\]  
(3.23)
\[
B = (A + B)/2 + (B - A)/2 = (X + Y)/2 + e/\alpha,
\]  
(3.24)
\[ B^3 - A^3 = [(X + Y)/2 + e/\alpha]^3 - [(X + Y)/2 - e/\alpha]^3 \]
\[ = 2[(X + Y)/2]^3(e/\alpha + e/\alpha)^3] = 2e/\alpha[3/4(X + Y)^2 + (e/\alpha)^2]. \quad (3.25) \]

Therefore Equation (3.16) can be written
\[ 2e[3/4(X + Y)^2 + (e/\alpha)^2] = e(3X^2 + e^2/4) + e(3Y^2 + e^2/4) \quad (3.26) \]
that is
\[ 2[3/4(X + Y)^2 + (e/\alpha)^2] = 3(X^2 + Y^2) + e^2/2 \quad (3.27) \]
or
\[ 3[X^2 + Y^2 - (X + Y)^2/2] = e^2(2/\alpha^2 - 1/2) \quad (3.28) \]
and
\[ (X - Y)^2 = (e^2/3)(4/\alpha^2 - 1). \quad (3.29) \]

Since \(4/\alpha^2 - 1 > 0\), let us set
\[ \beta = \left( e/\sqrt{3} \right) \sqrt{4/\alpha^2 - 1} > 0. \quad (3.30) \]

Since \(X\) and \(Y\) must meet the condition \(Y > X\), from (3.20) and (3.29) we get
\[ Y - X = \beta, \quad (3.31) \]
\[ X = [A + B - \beta]/2, \quad (3.32) \]
\[ Y = [A + B + \beta]/2. \quad (3.33) \]

One must now verify that the solution \(a, b, c, d\) we have obtained (see formulas (3.17)) is suitable, that is \(A < a < b < c < d < B\). This amounts to show that \(Y - X > e\) and \(Y - X + e < B - A\). Since \(0 < \alpha < 1\) we obtain \(Y - X = \beta > e\). On the other hand the relation \(Y - X + e < B - A\) is equivalent to
\[ \left( e/\sqrt{3} \right) \sqrt{4/\alpha^2 - 1} + e < 2e/\alpha \quad (3.34) \]
that is
\[ \sqrt{4/\alpha^2 - 1} < \sqrt{3(2/\alpha - 1)} \quad (3.35) \]
or
\[ 4/\alpha^2 - 1 < 3(2/\alpha - 1)^2. \quad (3.36) \]

Equation (3.36) is equivalent to the condition
\[ \alpha^2 - 3\alpha + 2 > 0 \quad (3.37) \]
which is satisfied since the condition \(0 < \alpha < 1\) implies \(\alpha^2 - 3\alpha + 2 = (1 - \alpha)(2 - \alpha) > 0\).

We have proved

**Proposition 3.1.** The set \(LP_{C_{sp}}\) is convex.
4 | DETAILED PROOF OF THE RESULT OF [1]

In Grenestedt and Gudmundson, (2.6) is shown for functions $\theta_1$ and $\theta_1$ whose regularity is not specified. The proof of Grenestedt and Gudmundson is true for piecewise continuous functions on $[-1, 1]$ that have a continuous extension on each of the subintervals of their definition and we shall detail this proof here. A function $\theta: z \in [-1, 1] \mapsto \theta(z) \in \mathbb{R}$ is a piecewise continuous function if there exists $N \in \mathbb{N}^+$ and a sequence of real numbers $(a_i)_{i=0}^{N}$, $a_0 = -1$, $a_N = 1$, $a_i < a_{i+1}$, $i = 0, \ldots, N - 1$ such that $\theta$ is continuous on each of the intervals $(a_i, a_{i+1})$, $i = 0, \ldots, N - 1$ (the value of $\theta$ at the points $a_i$, $i = 0, \ldots, N$ does not matter). We shall consider the class of piecewise continuous functions $\theta$ on $[-1, 1]$ such that on each of the open intervals $(a_i, a_{i+1})$, $i = 1, \ldots, N - 1$, $\theta$ has a continuous extension on the closed interval $[a_i, a_{i+1}]$ and we shall denote by $C_{PC}$ this class of functions. In this context when we shall speak of the value of $\theta \in C_{PC}$ on $[a_i, a_{i+1}]$, $i = 0, \ldots, N - 1$, it will mean the value of the extension of $\theta$ on $[a_i, a_{i+1}]$, $i = 0, \ldots, N - 1$.

Assume that $\theta_1, \theta_2 \in C_{PC}$ and $\alpha \in (0, 1)$. Let us recall the construction of the sequence $(\theta^n)$ in Grenestedt and Gudmundson. If $n \in \mathbb{N}^+$, consider a sequence of real numbers $(z^n_0, z^n_1, \ldots, z^n_{N-1}, \Delta z^n_0 = -1, z^n_1 = 1, z^n_i < z^n_{i+1}, i = 0, \ldots, n - 1$. The sequence $(z^n_i)_{i=0}^{N}$ is chosen such that $\theta_1$ and $\theta_2$ are continuous on each of the intervals $[z^n_i, z^n_{i+1}]$, $i = 0, \ldots, n - 1$. This can be done as follows (this is not explicit in the paper). First choose a sequence of real numbers $(a_i)_{i=0}^{N}$, $a_0 = -1$, $a_N = 1$, $a_i < a_{i+1}$, $i = 0, \ldots, N - 1$ such that $\theta_1$ and $\theta_2$ are continuous on each of the intervals $[a_i, a_{i+1}]$. This can be achieved by choosing the union of the corresponding sequences in the characterization of $\theta_1$ and $\theta_2$. Then the sequence $(z^n_i)_{i=0}^{N}$ is chosen such that each of the points $a_i, i = 1, \ldots, N - 1$ is one of the points $z^n_j$, $j = 1, \ldots, n, i - 1$. This assumes that $n \geq N$. With this choice of the points $z^n_j$, $j = 1, \ldots, n - 1$, $\theta_1$ and $\theta_2$ are continuous on each of the intervals $[z^n_i, z^n_{i+1}]$, $i = 0, \ldots, n - 1$.

Let $\alpha \in [0, 1]$. For all $n \in \mathbb{N}^+$, with the notation $\Delta z^n_i = z^n_{i+1} - z^n_i$, $i = 0, \ldots, n - 1$, on each interval $(z^n_i, z^n_{i+1})$, $\theta^n$ is defined by $\theta^n(z) = \theta(z)$ if $z \in (z^n_i, z^n_{i+1} + \alpha \Delta z^n_i)$ and $\theta^n(z) = \theta(z)$ if $z \in (z^n_i + \alpha \Delta z^n_i, z^n_{i+1})$, $i = 0, \ldots, n - 1$. The function $\theta^n$ is not defined for other points of the interval $[-1, 1]$ (in finite number), that is the points $z^n_i$ ($i = 0, \ldots, n$) and $z^n_i + \alpha \Delta z^n_i$ ($i = 0, \ldots, n - 1$). The sequence $(\theta^n)_{i=0}^{n}$ is chosen such that $\max \Delta z^n_i \to 0$ when $n \to +\infty$.

As mentioned in Section 3 the lamination parameters are all under the form (3.1) where $f$ is a continuous function on $\mathbb{R}$. If $f$ is a continuous function on $\mathbb{R}$ for all $n \in \mathbb{N}$, $n \geq N$, we can write

$$
\int_{-1}^{1} f(\theta^n(z))z^n'dz = \sum_{i=0}^{n-1} \int_{z^n_i}^{z^n_{i+1}} f(\theta^n(z))z^n'dz = \sum_{i=0}^{n-1} \int_{z^n_i}^{z^n_{i+1} + \alpha \Delta z^n_i} f(\theta_1(z))z^n'dz + \sum_{i=0}^{n-1} \int_{z^n_i + \alpha \Delta z^n_i}^{z^n_{i+1}} f(\theta_2(z))z^n'dz, j = 0, 1, 2.
$$

(4.1)

At that point, in Grenestedt and Gudmundson, the Mean Value Theorem is applied to each of the terms in the last line of (4.1). This theorem asserts that if $f$ is a continuous function on an interval $[a, b]$ ($a, b \in \mathbb{R}$, $a < b$), then there exists $c \in (a, b)$ such that

$$
\int_{a}^{b} f(z)dz = (b - a) f(c).
$$

(4.2)

The functions $\theta_1$ and $\theta_2$ are continuous on the intervals $[z^n_i, z^n_{i+1}]$, $i = 0, \ldots, n - 1$, then the functions $z \in [-1, 1] \mapsto g^n_k(z) = f(\theta_k(z))z^n$, $k = 1, 2$, $j = 0, 1, 2$ are also continuous on the intervals $[z^n_i, z^n_{i+1}]$, $i = 0, \ldots, n - 1$, and using the Mean Value Theorem we obtain (as in Grenestedt & Gudmundson)

$$
\int_{z^n_i}^{z^n_{i+1} + \alpha \Delta z^n_i} f(\theta_1(z))z^n'dz = \int_{z^n_i}^{z^n_{i+1} + \alpha \Delta z^n_i} g^n_1(z)dz = \alpha \Delta z^n_i g^n_1(z^n_i + \zeta^j_1 \alpha \Delta z^n_i).
$$

(4.3)

where $\zeta^j_1 \in (0, 1)$, $j = 0, 1, 2$.

For each $n \in \mathbb{N}$, $n \geq N$ and $l = 0, \ldots, N$, there exists a unique $k_l \in \mathbb{N}$, $0 \leq k_l \leq n$ such that $z^n_{k_l} = a_l$. Now consider the first term in the last line of (4.1). This term can be written under the form

$$
\sum_{i=0}^{n-1} \int_{z^n_i}^{z^n_{i+1} + \alpha \Delta z^n_i} f(\theta_1(z))z^n'dz = \sum_{i=0}^{N-1} A^n_i
$$

(4.4)
where, using (4.3),
\[
A^n_i = \sum_{i=k_i}^{k_i+1-1} \int_{\xi_i}^{\xi_{i+1}} f(\theta(z))z'\,dz = \alpha \sum_{i=k_i}^{k_i-1} \Delta \xi^n g'_i(\xi_i + \varepsilon_i \alpha \Delta \xi^n).
\]  
(4.5)

On the other hand \(\sum_{i=k_i}^{k_i-1} \Delta \xi^n g'_i(\xi_i + \varepsilon_i \alpha \Delta \xi^n)\) is a Riemann sum on \([a_i, a_{i+1}]\) of the function \(g'_i\) which is continuous on \([a_i, a_{i+1}]\) thus (see Knapp,16 Theorem 1.35)
\[
\sum_{i=k_i}^{k_i-1} \Delta \xi^n g'_i(\xi_i + \varepsilon_i \alpha \Delta \xi^n) \rightarrow \int_{a_i}^{a_{i+1}} g'_i(z)\,dz = \int_{a_i}^{a_{i+1}} f(\theta(z))z'\,dz, \quad j = 0, 1, 2
\]  
(4.6)
when \(n \rightarrow +\infty\) (since \(\max \Delta \xi^n \rightarrow 0\) when \(n \rightarrow +\infty\)). From (4.5) and (4.6), one obtains: for all \(l = 0, \ldots, N-1,\)
\[
A^n_i \rightarrow \alpha \int_{a_i}^{a_{i+1}} f(\theta(z))z'\,dz
\]  
(4.7)
when \(n \rightarrow +\infty\), so that by (4.4) and (4.7)
\[
\sum_{i=0}^{n-1} \int_{\xi_i}^{\xi_{i+1}} f(\theta^n(z))z'\,dz \rightarrow \alpha \int_{-1}^{1} f(\theta(z))z'\,dz
\]  
(4.8)
when \(n \rightarrow +\infty\). A similar result can be proved for the second term of the last line of (4.1). This shows that when \(\theta_1\) and \(\theta_2\) \(\in C_{PCE}\),
\[
\int_{-1}^{1} f(\theta^n(z))z'\,dz \rightarrow \alpha \int_{-1}^{1} f(\theta(z))z'\,dz + (1 - \alpha) \int_{-1}^{1} f(\theta_2(z))z'\,dz, \quad j = 0, 1, 2
\]  
(4.9)
when \(n \rightarrow +\infty\). This shows (2.6) when \(\theta_1\) and \(\theta_2\) \(\in C_{PCE}\).

5 PROOF OF THE NON-CONVEXITY OF \(LP_{C_N}\) AND \(LP_{C_e}\)

Let \(N \in \mathbb{N}^*\) and for all \(j = 1, \ldots, N\), set \(I_j = (-1 + 2(j - 1)/N, -1 + 2j/N)\). The elements of \(C_N\) are under the form
\[
\theta = \sum_{j=1}^{N} \theta_j 1_{I_j}
\]  
(5.1)
where \((\theta_1, \ldots, \theta_N) \in \mathbb{R}^N\).

**Proposition 5.1.** For all \(N \in \mathbb{N}^*\), the set \(LP_{C_N}\) is non-convex.

**Proof.** Let \(N \in \mathbb{N}^*\). With the notation (5.1), for all \(\theta \in C_N,\)
\[
\xi_1^A[\theta] = \frac{\sum_{j=1}^{N} \cos 2\theta_j}{N}, \quad \xi_2^A[\theta] = \frac{\sum_{j=1}^{N} \cos 4\theta_j}{N}, \quad \xi_3^A[\theta] = \frac{\sum_{j=1}^{N} \sin 2\theta_j}{N}, \quad \xi_4^A[\theta] = \frac{\sum_{j=1}^{N} \sin 4\theta_j}{N}.
\]  
(5.2)
Let us choose the following elements of \(C_N:\)
\[
\theta^a = 0, \quad \theta^b = (\pi/2)1_{I_1}.
\]  
(5.3)
We have
\[
\xi_1^A[\theta^a] = 1, \quad \xi_2^A[\theta^a] = 1, \quad \xi_3^A[\theta^b] = 1 - 2/N, \quad \xi_4^A[\theta^b] = 1.
\]  
(5.4)
Let \( \alpha \in [0, 1] \). According to (5.4) we have
\[
(1 - \alpha)\xi_1^A[\theta^a] + \alpha \xi_1^A[\theta^b] = 1 - 2\alpha/N, \quad (1 - \alpha)\xi_2^A[\theta^a] + \alpha \xi_2^A[\theta^b] = 1.
\]
(5.5)

If \( LP_{CN} \) is assumed to be convex, there exists \( \theta^c \in C_N \) such that \( (1 - \alpha)\xi[\theta^a] + \alpha \xi[\theta^b] = \xi[\theta^c] \). In that case
\[
\xi_1^A[\theta^c] = 1 - 2\alpha/N, \quad \xi_2^A[\theta^c] = 1.
\]
(5.6)

From Equation (5.2) and the second Equation (5.6), it follows that for all \( j = 1, \ldots, N \), \( \cos 4\theta_j^c = 1 \), that is \( \theta_j^c \in (\pi/2)\mathbb{Z} \) (where \( \mathbb{Z} \) is the set of integers). Let us denote by \( M \in \mathbb{N} \), \( 0 \leq M \leq N \) the number of indices \( j, j = 1, \ldots, N \) such that \( \theta_j^c \in \pi\mathbb{Z} \) (hence \( \cos 2\theta_j^c = 1 \)), thus the other indices \( j, j = 1, \ldots, N \) (in number \( N - M \)) are such that \( \theta_j^c \in \pi\mathbb{Z} + (\pi/2)\mathbb{Z} \) (hence \( \cos 2\theta_j^c = -1 \)). From (5.2), we obtain
\[
\xi_1^A[\theta^c] = 2M/N - 1.
\]
(5.7)

The condition \( \xi_1^A[\theta^c] = 1 - 2\alpha/N = 2M/N - 1 \) is equivalent to \( M = N - \alpha \). This gives a contradiction by taking \( \alpha \in (0, 1) \).

By a proof similar to that of Proposition 5.1, we obtain the following:

**Proposition 5.2.** The set \( LP_C \) is non-convex.

**Proof.** The proof is the same as that of Proposition 5.1 except some minor modifications. It is exactly the same up to Equation (5.5). Then if \( LP_C \) is assumed to be convex, there exists \( \theta^c \in C_e \) such that \( (1 - \alpha)\xi[\theta^a] + \alpha \xi[\theta^b] = \xi[\theta^c] \). But \( \theta^c \in C_m \) means that there exists \( N_1 \in \mathbb{N}^+ \) such that \( \theta^c \in C_{N_1} \). Then \( \theta^c \) satisfies Equation (5.6). The paragraph after Equation (5.6) up to Equation (5.7) is the same as that of Proposition 5.1 except that \( N \) and \( M \) must be replaced by \( N_1 \) and \( M_1 \). We obtain \( \xi_1^A[\theta^c] = 1 - 2\alpha/N = 2M_1/N_1 - 1 \), that is \( \alpha = N(1 - M_1/N_1) \). This gives a contradiction by taking \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) (where \( \mathbb{Q} \) is the set of rational numbers).

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This work does not have any conflicts of interest.

**ORCID**

Jean-Luc Akian
https://orcid.org/0000-0001-9748-6577

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APPENDIX A: PROOF OF THE NON-CONVEXITY OF THE SET OF POLAR PARAMETERS

In Scardaoni and Montemurro,\textsuperscript{15} Proposition 3.2 it is shown that for $N \in \mathbb{N}, N \geq 2$ the set of polar parameters is non-convex. We propose here a proof of this result which is a little simpler, thanks to Lemma A.1. Let us state Proposition 3.2 of Scardaoni and Montemurro\textsuperscript{15}:

**Proposition A.1.** For all $N \in \mathbb{N}, N \geq 2$, the set of polar parameters is non-convex.

**Proof.** With the notations (5.1), (5.2), the terms $\rho_0$ and $\rho_1$ defined in Equations (14) and (15) of Scardaoni and Montemurro\textsuperscript{15} (we shall denote them by $\rho_0(\theta)$ and $\rho_1(\theta)$) read as follows: for all $\theta \in C_N$,

$$
\rho_0(\theta) = \sqrt{(\xi_2^A[\theta])^2 + (\xi_4^A[\theta])^2}
$$

(A.8)

and

$$
\rho_1(\theta) = \sqrt{(\xi_1^A[\theta])^2 + (\xi_3^A[\theta])^2}.
$$

(A.9)

Define the function $g$ from $\mathbb{R}^N$ into $\mathbb{R}$ by: for all $(\theta_1, \ldots, \theta_N) \in \mathbb{R}^N$,

$$
g(\theta_1, \ldots, \theta_N) = \frac{1}{N} \sqrt{\left( \sum_{j=1}^{N} \cos \theta_j \right)^2 + \left( \sum_{j=1}^{N} \sin \theta_j \right)^2} = \frac{1}{N} \left\| \sum_{j=1}^{N} \epsilon^{\theta_j} \right\|.
$$

(A.10)

With this definition of $g$ and from (5.2), (A.8), (A.9), we have for all $\theta \in C_N$,

$$
\rho_0(\theta) = g(4(\theta_1, \theta_2, \ldots, \theta_N)), \rho_1(\theta) = g(2(\theta_1, \theta_2, \ldots, \theta_N)).
$$

(A.11)

Choose the following elements of $C_N$:

$$
\theta^a = 0, \theta^b = (\pi/2)1_{[a]}.
$$

(A.12)
We have
\[\xi_2^A[\theta^a] = 1, \xi_3^A[\theta^a] = 0, \xi_4^A[\theta^a] = 1, \xi_5^A[\theta^a] = 0,\] (A.13)
\[\xi_2^A[\theta^b] = 1, \xi_3^A[\theta^b] = 0, \xi_4^A[\theta^b] = 1 - 2/N, \xi_5^A[\theta^b] = 0,\] (A.14)
and since \(N \geq 2\), we have \(1 - 2/N \geq 0\) and
\[\rho_0(\theta^a) = 1, \rho_0(\theta^b) = 1, \rho_1(\theta^a) = 1, \rho_1(\theta^b) = 1 - 2/N.\] (A.15)

Let \(\alpha \in [0, 1]\). If the set of polar parameters is assumed to be convex, there exists \(\theta^c \in \mathcal{C}_N\) such that
\[(1 - \alpha)\rho_i(\theta^a) + \alpha \rho_i(\theta^b) = \rho_i(\theta^c), \ i = 0, 1\] (A.16)
(this is Equation (24) of Scardaoni and Montemurro\textsuperscript{15}). Due to (A.16) and (A.15) we have \(\rho_0(\theta^c) = 1\) and \(\rho_1(\theta^c) = 1 - 2\alpha/N\). Before proceeding further with the proof of the proposition, let us state the following lemma.

**Lemma A.1.** Let \(N \in \mathbb{N}, N \geq 2\). With the definition of \(g\) given by (A.10), for all \((\theta_1, \ldots, \theta_N) \in \mathbb{R}^N\),
\[0 \leq g(\theta_1, \ldots, \theta_N) \leq 1\] (A.17)
and \(g(\theta_1, \ldots, \theta_N) = 1\) iff for all \(j = 2, \ldots, N, \theta_j - \theta_1 \in 2\pi\mathbb{Z}\).

**Proof.** By the triangle inequality, for all \((\theta_1, \ldots, \theta_N) \in \mathbb{R}^N\),
\[g(\theta_1, \ldots, \theta_N) = \frac{1}{N} \left\| \sum_{j=1}^N e^{i\theta_j} \right\| \leq \frac{1}{N} \sum_{j=1}^N \left\| e^{i\theta_j} \right\| = 1.\] (A.18)
On the other hand the triangle inequality in (A.18) is an equality iff for all \(j = 2, \ldots, N\), there exists \(\lambda_j \in \mathbb{R}, \lambda_j > 0\) such that \(e^{i\theta_j} = \lambda_j e^{i\theta_1}\), that is for all \(j = 2, \ldots, N, e^{i\theta_j} = e^{i\theta_1}\). This shows the lemma. \(\square\)

Applying (A.11) and Lemma A.1, we have \(0 \leq \rho_0(\theta^c) \leq 1, \ 0 \leq \rho_1(\theta^c) \leq 1\), and since \(\rho_0(\theta^c) = 1\) it follows that for all \(j = 2, \ldots, N, \theta_j^c - \theta_1^c \in (\pi/2)/\mathbb{Z}\). Let us denote by \(M \in \mathbb{N}, 1 \leq M \leq N\) the number of indices \(j, j = 1, \ldots, N\) such that \(\theta_j^c - \theta_1^c \in \pi\mathbb{Z}\) (hence \(\cos 2\theta_j^c = \cos 2\theta_1^c\) and \(\sin 2\theta_j^c = \sin 2\theta_1^c\)), thus the other indices \(j, j = 1, \ldots, N\) (in number \(N - M\)) are such that \(\theta_j^c - \theta_1^c \in \pi\mathbb{Z} + (\pi/2)/\mathbb{Z}\) (hence \(\cos 2\theta_j^c = -\cos 2\theta_1^c\), \(\sin 2\theta_j^c = -\sin 2\theta_1^c\)). We obtain therefrom
\[\rho_1(\theta^c) = \frac{1}{N} \sqrt{(N - 2M) \cos 2\theta_1^c + ((N - 2M) \sin 2\theta_1^c)^2} = |1 - 2M/N|.\] (A.19)

Equation (A.19) corresponds to Equation (26) of Scardaoni and Montemurro\textsuperscript{15} but with a far simpler proof. The rest of the proof is as in Scardaoni and Montemurro,\textsuperscript{15} p.5. The condition \(\rho_1(\theta^c) = 1 - 2\alpha/N = |1 - 2M/N|\) is equivalent to \(M = \alpha\) or \(M = N - \alpha\). This gives a contradiction by taking \(\alpha \in (0, 1)\). \(\square\)