SOME UNIFIED INTEGRALS ASSOCIATED WITH
BESSEL-STRUVE KERNEL FUNCTION

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Abstract. In this paper, we discuss the generalized integral formula involving
Bessel-Struve kernel function \( S_\alpha (\lambda z) \), which expressed in terms of generalized
Wright functions. Many interesting special cases also obtained in this study.

1. Introduction

In 1888 Pincherle studied the integrals involving product of Gamma functions
along vertical lines (see [1, 2, 3]). Latterly, Barnes [4], Mellin [5] and Cahen [6]
extended the study and applied some of these integrals in the study of Riemann
zeta function and other Drichlet’s series. The integral formulas involving special
functions have been developed by many researchers ([7],[8]). In [9] presented unified
integral representation of Fox H-functions and in [10] hypergeometric \( 2F_1 \) functions.
Recently J. Choi and P. Agarwal [11] obtained two unified integral representations
of Bessel functions \( J_v (z) \). Also, many interesting integral formula involving \( J_v (z) \)
is given in [7] and [12].

The Bessel-Struve kernel \( S_\alpha (\lambda z), \lambda \in \mathbb{C}, [14] \) which is unique solution of the
initial value problem \( l_\alpha u(z) = \lambda^2 u(z) \) with the initial conditions \( u(0) = 1 \) and
\( u'(0) = \lambda \Gamma (\alpha + 1) / \sqrt{\pi} \Gamma (\alpha + 3/2) \) is given by

\[
S_\alpha (\lambda z) = j_\alpha (i\lambda z) - ih_\alpha (i\lambda z), \forall z \in \mathbb{C}
\]

where \( j_\alpha \) and \( h_\alpha \) are the normalized Bessel and Struve functions.

Moreover, the Bessel-Struve kernel is a holomorphic function on \( \mathbb{C} \times \mathbb{C} \) and it
can be expanded in a power series in the form

\[
S_\alpha (\lambda z) = \sum_{n=0}^{\infty} \frac{(\lambda z)^n \Gamma (\alpha + 1) \Gamma ((n+1)/2)}{\sqrt{\pi n!} \Gamma (n/2 + \alpha + 1)}.
\]

The generalized Wright hypergeometric function \( p\psi_q(z) \) is given by the series

\[
p\psi_q(z) = p\psi_q \left[ \frac{(a_i, \alpha_i)_{1,p}}{(b_j, \beta_j)_{1,q}} \right] z^k = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma (a_i + \alpha_i k)}{\prod_{j=1}^{q} \Gamma (b_j + \beta_j k)} k!
\]

where \( a_i, b_j \in \mathbb{C} \), and real \( \alpha_i, \beta_j \in \mathbb{R} \) \( (i = 1, 2, \ldots, p; j = 1, 2, \ldots, q) \). Asymptotic
behavior of this function for large values of argument of \( z \in \mathbb{C} \) were studied in [21]
and under the condition
\[ q \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > -1 \]
was found in the work of [22, 23]. Properties of this generalized Wright function were investigated in [25], (see also [26, 27]). In particular, it was proved [25] that $p\psi_q(z)$, $z \in \mathbb{C}$ is an entire function under the condition (1.3).

The generalized hypergeometric function represented as follows [28]:
\begin{equation}
_{p}F_{q} \left[ \begin{array}{c}
\alpha_1, ..., \alpha_p, \\
\beta_1, ..., \beta_q
\end{array} ; \ z \right] = \sum_{n=0}^{\infty} \frac{\Pi_{j=1}^{p} (\alpha_j)_n z^n}{\Pi_{j=1}^{q} (\beta_j)_n n!},
\end{equation}
provided $p \leq q; p = q + 1$ and $|z| < 1$ where $(\lambda)_n$ is well known Pochhammer symbol defined for (for $\lambda \in \mathbb{C}$) (see [28])
\begin{equation}
(\lambda)_n := \begin{cases}
1 & (n = 0) \\
\lambda (\lambda + 1) \cdots (\lambda + n - 1) & (n \in \mathbb{N} := \{1, 2, 3, \ldots\})
\end{cases}
\end{equation}
\begin{equation}
(\lambda)_n = \frac{\Gamma (\lambda + n)}{\Gamma (\lambda)} (\lambda \in \mathbb{C}\setminus \mathbb{Z}_0).
\end{equation}
where $\mathbb{Z}_0^-$ is the set of nonpositive integers.

If we put $\alpha_1 = \ldots = \alpha_p = \beta_1 = \ldots = \beta_q$ in (1.2), then (1.4) is a special case of the generalized Wright function:
\begin{equation}
p\psi_q(z) = p\psi_q \left[ \begin{array}{c}
\alpha_1, 1, ..., \alpha_p, 1, \\
\beta_1, 1, ..., \beta_q, 1
\end{array} ; \ z \right] = \frac{\Pi_{j=1}^{p} \Gamma (\alpha_j)}{\Pi_{j=1}^{q} \Gamma (\beta_j)} \ _{p}F_{q} \left[ \begin{array}{c}
\alpha_1, ..., \alpha_p, \\
\beta_1, ..., \beta_q
\end{array} ; \ z \right].
\end{equation}

For the present investigation, we need the following result of Oberbettinger [30]
\begin{equation}
\int_{0}^{\infty} x^{\mu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} dx = 2\lambda a^{-\lambda} \left( \frac{a}{2} \right)^{\mu} \frac{\Gamma (2\mu) \Gamma (\lambda - \mu)}{\Gamma (1 + \lambda + \mu)}
\end{equation}
provided $0 < \text{Re} (\mu) < \text{Re} (\lambda)$

Motivated by the work of [8], here we present the integral formulas of Bessel-Struve Kernel function of first kind $S_\alpha (\lambda z), \lambda \in \mathbb{C}$, which expressed in terms of generalized Wright or generalized hypergeometric functions.

2. MAIN RESULTS

Two generalized integral formulas established here, which expressed in terms of generalized (Wright) hypergeometric functions (1.7) by inserting the Bessel-Struve kernel function of the first kind (1.1) with the suitable argument in the integrand of (1.5).

**Theorem 1.** For $\lambda, \mu, \nu, \gamma \in \mathbb{C}$, and $x > 0$, $\text{Re} (\lambda) > \text{Re} (\mu) > 0$, then the following integral formula holds true:
\[
\int_0^\infty x^{\mu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} S_\alpha \left( \frac{\gamma y}{x + a + \sqrt{x^2 + 2ax}} \right) \, dx
= \frac{2^{1-\mu} a^{\alpha-\lambda} \Gamma (\alpha + 1) \Gamma (2\mu)}{\sqrt{\pi}}
\times_3 \Psi_2 \left[ \left( \frac{1}{2}, \frac{1}{2} \right), (\lambda + 1, 1), (\lambda - \mu, 1) ; \frac{\gamma y}{a} \right]
\]

(2.1)

**Proof.** Consider the series representation of \( S_\alpha \left( \frac{\gamma y}{x + a + \sqrt{x^2 + 2ax}} \right) \) and applying \((1.8)\). By interchanging the order of integration and summation, which verified by uniform convergence of the involved series under the given conditions, we get

\[
\int_0^\infty x^{\mu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} S_\alpha \left( \frac{\gamma y}{x + a + \sqrt{x^2 + 2ax}} \right) \, dx
= \int_0^\infty x^{\mu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} \sum_{n=0}^\infty \left( \frac{\gamma y}{x + a + \sqrt{x^2 + 2ax}} \right)^n \frac{\Gamma (\alpha + 1) \Gamma (\frac{n+1}{2})}{\sqrt{\pi} \Gamma (\frac{n}{2} + \alpha + 1) n!} \, dx
= \sum_{n=0}^\infty (\gamma y)^n \frac{\Gamma (\alpha + 1) \Gamma (\frac{n+1}{2})}{\sqrt{\pi} \Gamma (\frac{n}{2} + \alpha + 1) n!} \int_0^\infty x^{\mu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda+n} \, dx
\]

in view of the conditions give in Theorem 1 and applying the integral formula \((1.8)\), we obtain the following integral representation:

\[
\int_0^\infty x^{\mu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} S_\alpha \left( \frac{\gamma y}{x + a + \sqrt{x^2 + 2ax}} \right) \, dx
= 2^{1-\mu} a^{\alpha-\lambda} \Gamma (\alpha + 1) \Gamma (2\mu) \pi^{-1/2} \sum_{n=0}^\infty \frac{\Gamma (\lambda + n + 1)}{\Gamma (\lambda + n)} \frac{\Gamma (\frac{n+1}{2})}{\Gamma (\frac{n}{2} + \alpha + 1) n!} \gamma^n y^n
\]

which, upon using \((1.7)\) , yields \((2.1)\). This completes proof of theorem 1 \(\square\)

**Theorem 2.** For \( \lambda, \mu, \nu, \gamma \in \mathbb{C} \) with \( 0 < \Re (\mu) < \Re (\lambda + \nu) \) and \( x > 0 \). The following integral formula hold true:

\[
\int_0^\infty x^{\mu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} S_\alpha \left( \frac{\gamma xy}{x + a + \sqrt{x^2 + 2ax}} \right) \, dx
= 2^{1+\mu} a^{\alpha-\lambda} \Gamma (\alpha + 1) \Gamma (\lambda - \mu) \sqrt{\pi} \Gamma (1 + \lambda + \mu) \times_3 \Psi_2 \left[ \left( \frac{1}{2}, \frac{1}{2} \right), (2\mu, 2), (\lambda + 1, 1) ; \gamma y \right]
\]
Proof. Interchanging the order of integration and summation and the series representation of Bessel Struve kernel function, we get

\[
\int_0^\infty x^{\mu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} S_\alpha \left( \frac{\gamma xy}{x + a + \sqrt{x^2 + 2ax}} \right) \, dx
\]

\[
= \int_0^\infty x^{\mu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} \times \sum_{n=0}^{\infty} \left( \frac{\gamma xy}{x + a + \sqrt{x^2 + 2ax}} \right)^n \frac{\Gamma (\alpha + 1) \Gamma \left( \frac{n+1}{2} \right)}{\sqrt{\pi} \Gamma \left( \frac{n}{2} + \alpha + 1 \right)} \, dx
\]

\[
= \sum_{n=0}^{\infty} \frac{\gamma^n y^n \Gamma (\alpha + 1) \Gamma \left( \frac{n+1}{2} \right)}{\sqrt{\pi} \Gamma \left( \frac{n}{2} + \alpha + 1 \right)} \int_0^\infty x^{\mu+n-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-(\lambda+n)} \, dx
\]

in view of the condition give in theorem 1, we can apply the integral formula (1.8) and obtain the following integral representation:

\[
\int_0^\infty x^{\mu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} S_\alpha \left( \frac{\gamma xy}{x + a + \sqrt{x^2 + 2ax}} \right) \, dx
\]

\[
= 2^{1+\mu-\lambda \mu} a^{-\lambda} \Gamma (2\mu) \sum_{n=0}^{\infty} \Gamma \left( \frac{n+1}{2} \right) \frac{\Gamma (\lambda + n + 1) \Gamma (2\mu + 2n) \gamma^n y^n}{\Gamma (\lambda + n) \Gamma \left( \frac{n}{2} + \alpha + 1 \right) n! a^n}
\]

which gives the desired result. □

2.1. Representation of Bessel Struve kernel function in terms of exponential function. In this subsection we represent the Bessel Struve function in terms of exponential function. Also, we derive the Marichev Saigo Maeda operator representation of special cases. The representation Bessel Struve Kernel function in terms of exponential function as:

\[
S_{-\frac{1}{2}} (x) = e^x,
\]

(2.2)

\[
S_{\frac{1}{2}} (x) = \frac{-1 + e^x}{x}.
\]

(2.3)

Now, we give the the following corollaries:

**Corollary 1.** For \( \lambda, \mu \in \mathbb{C} \) with \( 0 < \Re (\mu) < \Re (\lambda) \) and \( x > 0 \) The following integral formula holds true

\[
\int_0^\infty x^{\mu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} e^{\frac{y}{x+a+\sqrt{x^2+2ax}}} \, dx
\]

\[
= 2^{1-\mu} a^{-\lambda} \Gamma (2\mu) \Psi_2 \left[ \begin{array}{c} (\lambda + 1, 1), (\lambda - \mu, 1) \\ (\lambda, 1), (1 + \lambda - \mu) \end{array} ; \frac{y}{a} \right]
\]

(2.4)

**Proof.** As same as in theorem 1 and theorem 2, using the formula (1.8) and (2.2), one can easily reach the result □
Corollary 2. Let the conditions given in Corollary 1 satisfied. Then the following integral formula holds true
\[
\int_0^\infty x^{\mu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} e^{-\frac{y}{x + a + \sqrt{x^2 + 2ax}}} \, dx
= \frac{2^{1-\mu} a^{-\lambda} \Gamma(2\mu) \Gamma(\lambda - \mu)}{\Gamma(\lambda) \Gamma(1 + \lambda - \mu)} \, _2F_2 \left[ \begin{array}{c} \lambda + 1, \lambda - \mu; \\ \lambda, 1 + \lambda - \mu; \frac{y}{a} \end{array} \right]
\]

Proof. In the view of equations (1.4), (1.5) and (2.4), we obtain the required result.

Corollary 3. For \( \lambda, \mu, \in \mathbb{C} \) with \( 0 < \Re(\mu) < \Re(\lambda) \) and \( x > 0 \). The following integral formula holds true
\[
\int_0^\infty x^{\mu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} e^{-\frac{y}{x + a + \sqrt{x^2 + 2ax}}} \, dx
= \frac{2^{-\mu} a^{-\lambda} \Gamma(2\mu)}{\Gamma(\lambda - \mu)} \, _3\Psi_3 \left[ \begin{array}{c} \left( \frac{1}{2}, \frac{1}{2}, 1 \right), (\lambda + 1, 1), (\lambda - \mu, 1); \\ \frac{y}{a} \end{array} \right]
\]

2.2. Relation between Bessel Struve kernel function and Bessel and Struve function of first kind. In this subsection we show the relation between \( S_\alpha(x) \) and Bessel function \( I_v(x) \) and Struve function \( L_v(x) \) by choosing particular values of \( \alpha \)

(2.5) \[ S_0(x) = I_0(x) + L_0(x), \]

(2.6) \[ S_1(x) = \frac{2I_1(x) + L_1(x)}{x}. \]

In the light of above relations ,we have the following theorems:

Theorem 3. For \( \lambda, \mu, \in \mathbb{C} \) with \( 0 < \Re(\mu) < \Re(\lambda) \) and \( x > 0 \). Then the following integral formula holds true:
\[
\int_0^\infty x^{\mu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} \times \left[ I_0 \left( \frac{y}{x + a + \sqrt{x^2 + 2ax}} \right) + L_0 \left( \frac{y}{x + a + \sqrt{x^2 + 2ax}} \right) \right] \, dx
= \frac{2^{1-\mu} a^{-\lambda} \pi^{-1/2} \Gamma(2\mu)}{\Gamma(\lambda - \mu)} \, _3\Psi_3 \left[ \begin{array}{c} \left( \frac{3}{2}, \frac{3}{2}, 1 \right), (\lambda + 1, 1), (\lambda - \mu, 1); \\ \frac{y}{a} \end{array} \right]
\]

Proof. Consider the relation given in (2.5) and applying (1.8). By interchanging the order of integration and summation,which verified by uniform convergence of
the involved series under the given conditions, we get
\[
\int_0^\infty x^{\mu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} dx
\times \left[ I_0 \left( \frac{y}{x + a + \sqrt{x^2 + 2ax}} \right) + L_0 \left( \frac{y}{x + a + \sqrt{x^2 + 2ax}} \right) \right]
= \int_0^\infty x^{\mu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} \sum_{n=0}^\infty \left( \frac{y}{x + a + \sqrt{x^2 + 2ax}} \right)^n \frac{\Gamma \left( \frac{n+1}{2} \right)}{\sqrt{\pi} n! \Gamma \left( \frac{n}{2} + 1 \right)} dx
= \sum_{n=0}^\infty \frac{\Gamma \left( \frac{n+1}{2} \right)}{\sqrt{\pi} n! \Gamma \left( \frac{n}{2} + 1 \right)} \int_0^\infty x^{\mu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} \left( \frac{y}{x + a + \sqrt{x^2 + 2ax}} \right)^n dx
= 2^{1-\mu} a^{\mu-\lambda\pi^{-1/2}} \Gamma (2\mu) \sum_{n=0}^\infty \frac{\Gamma \left( \frac{n+1}{2} \right) \Gamma (\lambda + n - \mu) \Gamma (\lambda + n + 1 + \mu)}{n! \Gamma \left( \frac{n}{2} + 1 \right) a^n \Gamma (\lambda + n)}
\]
which gives the desired result.

\[\]  

**Theorem 4.** The following integral formula holds true with \( \lambda, \mu, \in \mathbb{C} \) with \( 0 < \Re (\mu) < \Re (\lambda) \) and \( x > 0 \).
\[
\int_0^\infty x^{\mu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} dx
\times \left[ 2I_1 \left( \frac{y}{x + a + \sqrt{x^2 + 2ax}} \right) + L_1 \left( \frac{y}{x + a + \sqrt{x^2 + 2ax}} \right) \right]
= 2^{1-\mu} a^{\mu-\lambda\pi^{-1/2}} \Gamma (2\mu) \sum_{n=0}^\infty \frac{\Gamma \left( \frac{n+1}{2} \right) \Gamma (\lambda + n - \mu) \Gamma (\lambda + n + 1 + \mu)}{n! \Gamma \left( \frac{n}{2} + 1 \right) a^n \Gamma (\lambda + n)}
\]

**Proof.** Consider the series representation of \( S_1 \left( \frac{y}{x + a + \sqrt{x^2 + 2ax}} \right) \) and applying (1.8) and interchanging the order of integration and summation, we get
\[
\int_0^\infty x^{\mu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} dx
\times \left[ 2I_1 \left( \frac{y}{x + a + \sqrt{x^2 + 2ax}} \right) + L_1 \left( \frac{y}{x + a + \sqrt{x^2 + 2ax}} \right) \right]
= 2^{1-\mu} a^{\mu-\lambda\pi^{-1/2}} \Gamma (2\mu) \sum_{n=0}^\infty \frac{\Gamma \left( \frac{n+1}{2} \right) \Gamma (\lambda + n - \mu) \Gamma (\lambda + n + 1 + \mu)}{n! \Gamma \left( \frac{n}{2} + 1 \right) a^n \Gamma (\lambda + n)}
\]
which gives the required result.

\[\]

**Conclusion**

The generalized integral formula involving Bessel-Struve kernel function \( S_\alpha (\lambda z) \), which expressed in terms of generalized Wright functions are given in this paper. Also the relation between exponential function, Bessel function and Struve function with Bessel-Struve kernel function is also discussed with particular cases.
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