Exact correlation functions of the BCS model in the canonical ensemble

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We evaluate correlation functions of the BCS model for finite number of particles. The integrability of the Hamiltonian relates it with the Gaudin algebra \( G[sl(2)] \). Therefore, a theorem that Sklyanin proved for the Gaudin model, can be applied. Several diagonal and off-diagonal correlators are calculated. The finite size scaling behavior of the pairing correlation function is studied.

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Introduction. The BCS model expresses the competition between the kinetic energy and the tendency of Cooper pairs to condense \[. \] This idea has been successfully employed to a large variety of different physical context such as superconductivity \[, \] nuclear physics \[, \] QCD \[ and astrophysics \[. \] The BCS Hamiltonian is

\[ H = \sum_{j=1}^{\Omega} \varepsilon_j n_{j\sigma} - g \sum_{i<j} \Omega \epsilon_i^+ c_i^+ c_j c_{j+}^i. \]

\( g \) is the pairing coupling constant; the quantum numbers \( j \in \{1...\Omega\} \) label the single particle energy levels \( \varepsilon_j \) which are doubly degenerate since \( \sigma \in \{\uparrow, \downarrow\} \) labels time reversed electron states; \( c_{j,\sigma} \) and \( n_{j,\sigma} := c_{j,\sigma}^+ c_{j,\sigma} \) are annihilation and number operators.

In the grand canonical ensemble, the mean field approximation of the BCS model has been successfully applied \[. \] However, the grand canonical analysis is not appropriate in many important physical situations (noticeably in nuclei). Recent experiments on small metallic grains \[ evidenced this limitation also in condensed matter \[. \] Indeed, the typically very low capacitance of metals of nanoscale size fixes the number of particles in the grain. This has constituted a conceptual challenge \[, \] since physical quantities must be studied in the \textit{canonical ensemble}. In particular, the system cannot undergo the superconducting phase transition, but a cross-over regime dominated by superconducting fluctuations \[. \] The quantity playing the role of the “order parameter” is the pairing correlation function \( u_{ij} := \langle c_i^+ c_j + c_j^+ c_i \rangle \). Canonical pairing fluctuations were studied in the recent literature employing numerical and analytical techniques \[. \] Since the system is characterized by strong quantum fluctuations, its physical behavior is very sensitive to the approximations employed and therefore exact results play an important role. The BCS model is Bethe Ansatz (BA) solvable \[ (see also Ref. \[) and more recently found to be integrable \[. \] The integrals of motion obtained in Ref. \[ can be also generated within the Quantum Inverse Scattering (QIS) scenario through the quasi classical limit of the \( R \)-matrix of disordered six vertex models \[. \] By these integrals, the model becomes connected with the Gaudin Hamiltonians. The BA solution played a central role for theoretical predictions in particular of the low temperature behavior of thermodynamic quantities of grains \[. \] Indeed, for systems with a fixed number of particles (metallic grains but also nuclei) the BA solution is particularly useful and feasible even beyond the thermodynamic limit (which is rare in condensed matter) due to its simplicity. Using the exact eigenstates of the Hamiltonian, the \textit{diagonal} pairing correlation function was obtained by Richardson although it was not evaluated explicitly \[. \] The aim of the present work is to fill these gaps; we evaluate exactly \textit{diagonal} and \textit{off-diagonal} correlation functions of the BCS model for finite number of particles.

In the realm of exactly solvable models Correlation Functions (CFs) are studied resorting various approaches. For large systems (in the thermodynamic limit) they have been extracted from analytic properties of form factors \[. \] For spin models, CFs have been studied by Korepin using the operator algebra from the QIS method \[ to calculate the scalar products of Bethe states. In a recent paper \[ Sklyanin suggested how the combinatoric complications involved in this calculations can be overcome resorting the Generating Function (GF) technique. He applied it to the \( sl(2) \) Gaudin model \[. \] The key role in his approach is played by an analog of the Gauss decomposition or Baker–Campbell–Hausdorff (BCH) formula for elements of the \( SL(2) \) loop group associated to the Gaudin algebra \( G[sl(2)] \). In the present work, the latter approach is pursued. We exploit the common algebraic root of the Gaudin and BCS models to extend the Sklyanin theorem to the BCS model. The GF of exact CFs for the BCS model is obtained. The CFs are suitable residues of the GF (see Eqs. \[). \] The \textit{M}-point charge and pairing correlations

\[ \pi_0(E,M,F) := \langle E \rangle M \prod_{k=1}^{M} (n_{jk,\uparrow} + n_{jk,\downarrow} - 1)/2 \langle F \rangle \]  

\[ u_{ij}(E,F) := \langle E | c_i^+ c_j + c_j^+ c_i \rangle \langle F \rangle \]  

are computed exactly in the canonical ensemble (see Eqs. \[). \] The vectors \( \langle E \rangle \) and \( \langle F \rangle \) are exact \( N \)-pair eigenstates of \( \) (see Eqs. \[. \] \).
The present paper is laid out as follows. First we fix the algebraic aspects we employ in the paper. This motivates the extension of the Sklyanin theorem to the BCS model. Then we evaluate the charge and the pairing CFs. Finally we will draw our conclusions.

The BCS and the Gaudin algebra. We first sketch the connection of the BCS model with the \( sl(2) \)-Gaudin model. For this goal we introduce the fundamental realization of \( su(2) \simeq sl(2) \) in terms of electron pairs is

\[
S_j^- := c_{j+1} c_{j+1}^\dagger, \quad S_j^+ := (S_j^-)^\dagger = c_{j,1}^\dagger c_{j,1}, \quad S_j^z := (c_{j,1}^\dagger c_{j,1} + c_{j,1} c_{j,1}^\dagger - 1)/2. \tag{1}
\]

The \( sl(2) \) “lowest” weight module is generated by the vacuum vector \( |0\rangle_j, S_j^- |0\rangle_j = 0 \), \( S_j^z |0\rangle_j = s_j |0\rangle_j \) where \( s_j \) is the “lowest” weight (\( s_j = -1/2 \) for spin 1/2, which is the case of interest here [26]). The quadratic Casimir operator is:

\[
G := (S_j^+)^2 + \frac{1}{2} (S_j^+ S_j^- + S_j^- S_j^+), \quad S_j^z |0\rangle_j = s_j |0\rangle_j. \tag{2}
\]

The bilinear combinations \( S_j^+ S_j^- \) and \( S_j^- S_j^+ \) can be expressed in terms of Casimir and Cartan operators

\[
S_j^+ S_j^- = S_j^2 - (S_j^z)^2 \pm S_j^z. \tag{3}
\]

The integrals of motion \( \tau_j \) of the BCS model are \( \tau_j = S_j^z \exp (\epsilon_j - e_j) \), that is: \( \left[ H, \tau_j \right] = \{ \tau_j, \tau_l \} = 0 \) for \( j, l = 1, \ldots, \Omega \) [8]. By these integrals of motion, the model becomes connected with the Gaudin Hamiltonians

\[
\sum_{j \neq l} S_j \cdot S_l / (\epsilon_j - \epsilon_l), \quad S_j := (S_j^+, S_j^0, S_j^z), \quad S_j^\pm := 1/\sqrt{2} (S_j^\pm \pm i S_j^z)) \quad \text{are spin vectors.} \tag{4}
\]

The Hamiltonian \( \mathcal{H} \) can be expressed in terms of \( \tau_j : H = g \sum_{j=1}^\Omega \tau_j + g^3 \sum_{j=1}^\Omega \tau_j - \text{const} \). However, the relation is deeper and the integrability of the BCS model is founded in its connection with the infinite dimensional Gaudin algebra \( \mathcal{G}[sl(2)] \) that is constructed from \( sl(2) \) as

\[
S^\pm(u) := \sum_{j=1}^\Omega \frac{S_j^\pm}{u - 2 \epsilon_j}, \quad S^z(u) := \sum_{j=1}^\Omega \frac{S_j^z}{u - 2 \epsilon_j}. \tag{5}
\]

The lowest weight module of \( \mathcal{G}[sl(2)] \) is generated by the vacuum \( |0\rangle = \otimes_{j=1}^\Omega |0\rangle_j \); \( S^- (u) |0\rangle = 0 \), \( S^z (u) |0\rangle = s(u) |0\rangle \), where \( s(u) := \sum_{j=1}^\Omega s_j / (u - 2 \epsilon_j) \) is the lowest weight of \( \mathcal{G}[sl(2)] \). The mutual commutativity of \( \tau_j \) descends from the relation between \( \tau(u) := \sum_{j=1}^\Omega \tau_j / (u - 2 \epsilon_j) \) and invariants (trace and quantum determinant [1]) of \( \mathcal{G}[sl(2)] \). It can be written as \( \tau(u) := t(u) + s(u) \) [27] where \( s(u) := \sum_{j=1}^\Omega s_j / (u - 2 \epsilon_j)^2 \); \( t(u) := -2S(u) + S^z(u)/g \) and \( S(u) := S^z(u)S(u)^z + \frac{1}{4} (S^+ u)S(u)^- + S(u)^- (u)S^z(u) \).

The property \( \{ t(u), t(v) \} = 0 \) is the ultimate reason for the integrability of the BCS model. Accordingly, the exact eigenstates of both the BCS model [13,14] and the set of operators \( \tau_j \) [16] are constructed from \( \mathcal{G}[sl(2)] \) generators

\[
|\mathcal{E} \rangle_N = \prod_{\alpha=1}^N S^+(\epsilon_\alpha) |0\rangle, \tag{6}
\]

\[
H(\mathcal{E})_N = E(\mathcal{E})_N; \quad \text{the energy } E = \sum_{\alpha=1}^N \epsilon_\alpha \text{ is given in terms of the set } \mathcal{E} \text{ of the spectral parameters } \epsilon_\alpha \text{ satisfying the algebraic Richardson-Sherman (RS) equations }
\]

\[
s(\epsilon_\alpha) = \frac{1}{2g} + \sum_{\beta \neq \alpha}^N \frac{1}{\epsilon_\beta - \epsilon_\alpha}, \quad \alpha = 1, \ldots, N. \tag{7}
\]

We note that RS equations (7) are intimately related to the algebraic structure of \( \mathcal{G}[sl(2)] \) since they act as constraints on the lowest weight \( s(e_a) \). Thus, the difference between the BCS and Gaudin model results in a different constraint imposed on the lowest weight vector of \( \mathcal{G}[sl(2)] \) which leads to different sets \( \mathcal{E}, \mathcal{E}' \) (\( \mathcal{E}' \) is spanned by the solutions of (7) when \( g \rightarrow \infty \)). We will use this fact to extend the Sklyanin theorem to the BCS model.

Generating functions. CFs of elements in \( \mathcal{G}[sl(2)] \) can be expressed in terms of the following GF

\[
C(\mathcal{E}, \mathcal{H}, F) := \left( F \right) \prod_{h \in \mathcal{H}} S^2(h) |\mathcal{E}\rangle \tag{8}
\]

where \( F := \langle 0 | \prod_{f \in \mathcal{F}} S^-(f) \rangle \) and the sets \( \mathcal{E}, \mathcal{F} \subset \mathcal{C} \setminus \mathcal{E}_0 \) are (in general distinct) sets of solutions of the RS equations (7): \( \mathcal{E}_0 := \{ 2\epsilon_j, j = 1 \ldots \Omega \}; \mathcal{H} \subset \mathcal{C} \setminus \{ \mathcal{E} \cup \mathcal{F} \cup \mathcal{E}_0 \} \). The order of the correlation is the cardinality of \( \mathcal{H} : |\mathcal{H}| \) and \( |\mathcal{F}| \) are fixed by the number of pairs \( N \). For instance, the one and two point CFs correspond to \( |\mathcal{H}| = 1 \) and \( |\mathcal{H}| = 2 \) respectively.

Now we present the Sklyanin theorem for GF of \( sl(2) \) Gaudin model and apply it to the BCS model. Therefore we need the notation of the set of \( \text{coordinated partitions} \mathcal{P} = \{ P_l : l = 1 \ldots |\mathcal{P}| \} \) of the sets \( \mathcal{E}, \mathcal{F}, \mathcal{H} \) (see Ref. [24]): the partition \( P \in \mathcal{P} \) is a set of triplets \( \{ T_1, \ldots, T_p \} \); the triplet \( T \in P = (\mathcal{E}_T, \mathcal{F}_T, \mathcal{H}_T) \), where \( \emptyset \neq \mathcal{E}_T \subset \mathcal{E}, \emptyset \neq \mathcal{F}_T \subset \mathcal{F} \) and \( \mathcal{H}_T \subset \mathcal{H} \) such that \( |\mathcal{E}_T| = |\mathcal{F}_T| > 0 \); \( |\mathcal{H}_T| \geq 0 \).

The GF has been evaluated for the \( sl(2) \) Gaudin model exploiting the BCH formula for the \( SL(2) \) loop group generated by \( S^-_{\phi(x)} := \sum_{j \in \mathcal{F}} \phi_j S^-(j), \quad S^z_{\eta(x)} := \sum_{h \in \mathcal{H}} \eta_h S^z(h), \quad S^+_{\psi(x)} := \sum_{e \in \mathcal{E}} \psi_e S^+(e) \) where \( \{ S^z(u), S^\pm(u) \} = \mathcal{G}[sl(2)] \) and \( \phi(x), \eta_h, \psi_e \) are meromorphic functions for \( x \in \mathcal{C} \) with residues \( \phi_j, \eta_h, \psi_e \) respectively [25]. This formula allows to reorder the products between loop group elements in [13]: \( \langle \exp S^-_{\phi(x)} \exp S^z_{\eta(x)} \exp S^+_{\psi(x)} \rangle = \langle \exp S^+_{\psi(x)} \exp S^z_{\eta(x)} \exp S^-_{\phi(x)} \rangle \). Sklyanin proved the following theorem [20].

**Theorem.** \( C(\mathcal{E}, \mathcal{H}, F) \) is given by the formula

\[
C(\mathcal{E}, \mathcal{H}, F) = (-1)^N \times \frac{1}{\prod_{h \in \mathcal{H}_p}} \sum_{p} \left( \prod_{T \in P} n_T (|\mathcal{E}_T|)^{\mathcal{H}_T} S(W_T \cup \mathcal{H}_T) \right) \prod_{h \in \mathcal{H}_p} s(h) \tag{9}
\]
where \( S(\mathcal{L}) = 1/2\pi i \int_{\Gamma} s(z) \prod_{y \in \mathcal{L}} (z - y)^{-1} dz \langle \mathcal{B} \rangle \), \( n_T := -2|E_T|!(|E_T| - 1)! \), \( \mathcal{W}_T := \mathcal{E}_T \cup \mathcal{F}_T \), and \( \mathcal{H}_p := \mathcal{H} \setminus \bigcup_{T \in \mathcal{P}} \mathcal{H}_T \). \( C(\mathcal{E}, \mathcal{H}, \mathcal{F}) \) is a polynomial in \( S \) with integer coefficients.

Expression (2) depends only on the sets \( \mathcal{W} := \mathcal{E} \cup \mathcal{F} \) and \( \mathcal{H} \) for the Gaudin model \( \mathcal{W} \) is a set of solutions of (4) for \( g \to \infty \); for the BCS model \( \mathcal{W} \) is a set of solutions of the RS (3) for generic \( g \). The scalar products of Bethe states (and their norms) are a corollary of the Sklyanin theorem (1) for \( \mathcal{H} = \emptyset \). \( \langle \mathcal{E}|\mathcal{F} \rangle = C(\mathcal{E}, \emptyset, \mathcal{F}) \).

We point out that the GF (8) has simple poles in the set \( \mathcal{E}_0 \). Its concet with the determinant formulas (14) has been elucidated in Refs. \([25,29]\).

We point out that the GF (3) has simple poles in the set \( \mathcal{E}_0 \). This will play a key role in the following.

**Correlation functions.** The charge and the pairing CFs are matrix elements of \( su(2) \) Lie algebra (instead of elements of \( G[sl(2)] \)) using vector states of \( G[sl(2)] \). The projection from the \( sl(2) \) loop algebra on its Lie algebra is performed by taking the residue of \( C(\mathcal{E}, \mathcal{H}, \mathcal{F}) \) in the poles \( h_l = 2\varepsilon_{jl} \) for \( h_l \in \mathcal{H} \), \( l \in \{1 \ldots M \} \). The charge CFs (3) are

\[
\pi_0(\mathcal{E}, \mathcal{M}, \mathcal{F}) = \lim_{\mathcal{H} \to \mathcal{E}_0} (\mathcal{H} - \mathcal{E}_0) \cdot C(\mathcal{E}, \mathcal{H}, \mathcal{F})
\]

(10)

where \( \mathcal{H} \to \mathcal{E}_0 \) and \( \mathcal{H} - \mathcal{E}_0 \) mean \( h_l \to 2\varepsilon_{jl} \) \( \forall l \) and \( \prod_l (h_l - 2\varepsilon_{jl}) \) respectively. Using (2) yields

\[
\pi_0(\mathcal{E}, \mathcal{M}, \mathcal{F}) = (-1)^N \prod_{l=1}^{M} s_{jl} \times
\]

\[
\sum_{p \in \mathcal{P}_k} \left( \prod_{T \in \mathcal{T}_0} n_T S(W_T) \right) \left( \prod_{T \in \mathcal{T}_0} \frac{n_T|E_T|}{P |h_T - y|} \right)
\]

(11)

where \( \mathcal{P}_k \equiv \{ P \in \mathcal{P} : \max_{T \in \mathcal{P}} |H_T| = k \} \), \( T_k \equiv \{ T \in P : |H_T| = k \} \). The quantity \( S(W_T) \) is

\[
S(W_T) = \sum_{e \in W_T} \frac{s(e)}{\prod_{x \in W_T} (e - x)} - \sum_{d \in W_T} \frac{s(d)}{\prod_{x \in W_T} (d - x)} - \frac{1}{\prod_{x \in W_T} (d - y)} + \frac{s^{[2]}(d)}{\prod_{x \in W_T} (d - x)}
\]

(12)

where \( e \) and \( d \) are elements appearing singly and doubly in \( W_T \) respectively. The pairing CF (3) can be extracted from \( C(\mathcal{E}, \emptyset, \mathcal{F}) \) where the vectors in (3) are \( |\mathcal{E}| := \langle \mathcal{E}|S^{-}(z_1) \rangle \) and \( |\mathcal{F}| := S^{+}(z_2) \). Then \( u_{lm}(\mathcal{E}, \mathcal{F}) \) is

\[
u_{lm}(\mathcal{E}, \mathcal{F}) = \lim_{z_1 \to z_2 \varepsilon_l} (z_2 - 2\varepsilon_l) (z_2 - 2\varepsilon_m) C(\mathcal{E}, \emptyset, \mathcal{F})
\]

(13)

\( C(\mathcal{E}, \emptyset, \mathcal{F}) \) can be calculated using the Sklyanin theorem. For \( l \neq m \) formula (13) gives

\[
\sum_{p \in \mathcal{P}_1} \left( \prod_{T \in \mathcal{T}_0} n_T S(W_T) \right) \left( \prod_{T \in \mathcal{T}_0} \frac{n_T|E_T|}{P |2\varepsilon_T - y|} \right)
\]

where \( Z := \{ z_1, z_2 \} \), \( \tilde{\mathcal{P}}_k \equiv \{ P \in \mathcal{P} : \max_{T \in \mathcal{P}} |Z_T| = k \} \), \( \tilde{T}_k \equiv \{ T \in P : |Z_T| = k \} \). The pairing CF for \( l = m \) can be achieved by a variation of the procedure depicted above. Namely by \( \lim_{z_1 \to z_2 - \varepsilon_l} (z_2 - 2\varepsilon_l) C(\mathcal{E}, \emptyset, \mathcal{F}) \).

But in the present case \( \theta_j = -1/2 \forall j \) it is more convenient employing the formula (4) which simplifies in \( S^T_\eta S^T_\eta = 1/2 \pm S^T_\eta \) Then the CF \( \Psi := \sum_{j} u_jv_j := \sum_{j} \sqrt{S^T_j S^T_\eta} f \) can be calculated evaluating \( S^T_j \) by the formula (11) for \( M = 1 \).

**Conclusions.** We have evaluated exactly charge and pairing CFs of the BCS model in the canonical ensemble. They are the main results of this paper (Eqs. (11), (14), (16). 3
Figs. (1), (2)). For this goal the algebraic connection of the BCS model with the Gaudin model is crucial and it is emphasized and exploited for the first time. General charge and pairing correlations are obtained as certain residues of the GF (a “correlator” within $\mathcal{G}[sl(2)]$) via a theorem for CFs of $su(2)$ operators in the lowest weight module of $\mathcal{G}[sl(2)]$. The limitation of the numerics is the vastly increasing number of partitions, which depends on the number of pairs $|\mathcal{E}| = N$ and the order of the CF $|\mathcal{H}|$. We want to emphasize that it does not depend on the dimension of the Hilbert space $\Omega$.

We roughly estimated the complexity involved in the evaluation of our formulas and compared it with the complexity of the corresponding diagonal quantities calculated in Ref. [22]. We found that for diagonal form factors it is favorable to use the expression in Ref. [22], whereas the complexity of the formula presented here becomes much lower for the 2-point functions. Our results might have immediate physical relevance for metallic grains, in view of the universality of their physical properties for very small systems already, as shown in Ref. [11]. In particular, our results apply to arbitrary filling.

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[28] $s(z)$ is assumed to be analytic inside the contour $\Gamma$ encircling $C \subset \mathbb{C}$; therefore, $\mathcal{E}_0$ must be outside $\Gamma$, $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{H}$ are to be inside $\Gamma$; $\Gamma$ placed in the open set $\{\mathcal{E}, \mathcal{C}, \mathcal{H}\}$ pairwise disjoint. Afterwards, limits of $\mathcal{E}, \mathcal{F}, \mathcal{H}$ towards $\mathcal{E}_0$ can be taken for the analytic GF.
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