MULTISCALE NONLOCAL FLOW IN A FRACTURED POROUS MEDIUM

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ABSTRACT. We study the flow generated by an incompressible viscoelastic fluid in a fractured porous medium. The model consists of a fluid flow governed by Stokes-Volterra equations evolving in a periodic double-porosity medium. Using the multiscale convergence method associated to some recent tools about the convergence of convolution sequences, we show that the equivalent macroscopic model is of the same type as the microscopic one, but in a fixed domain.

1. INTRODUCTION

The Stokes equations have been for a long time widely used to describe the flow at moderate velocity of incompressible viscous fluids. However, models of viscoelastic fluids have been proposed in the twentieth century. Some of these models take into account the history of the flow and are not subject to the Newtonian effects.

Among these models is the one proposed by Oldroyd in 1956, and which is commonly known as Oldroyd model. For details about the physical background and its mathematical modelling, we refer e.g., to Oldroyd, Joseph, Oskolkov, Agranovich et al., and Sobolevskii.

In this work we focus on flows of an incompressible viscoelastic fluid of Oldroyd type in a multiscale porous medium. The equations of motion arising from that model give rise to a system of integro-differential equations of Volterra-Stokes type (1.1) and (1.3) below; see e.g., Agranovich et al. and Sobolevskii. In this work we focus on flows of an incompressible viscoelastic fluid of Oldroyd type in a multiscale porous medium. The equations of motion arising from that model give rise to a system of integro-differential equations of Volterra-Stokes type (1.1) and (1.3) below; see e.g., Agranovich et al. and Sobolevskii. In this work we focus on flows of an incompressible viscoelastic fluid of Oldroyd type in a multiscale porous medium. The equations of motion arising from that model give rise to a system of integro-differential equations of Volterra-Stokes type (1.1) and (1.3) below; see e.g., Agranovich et al. and Sobolevskii.

2000 Mathematics Subject Classification. 35B27, 76Bxx, 76D05.
Key words and phrases. Multiscale convergence, Oldroyd equations, double-porosity medium, convolution.
earlier in [17, 18, 29, 33, 36]. In such media, the blocks behave as sources of fluid alimenting the fissures (fractures) which are characterized by substantially higher flow rates and lower relative volume. For that reason, the average flow in the block is always delayed with respect to the flow in the fractures. This phenomenon is analytically characterized by the appearance of memory in the resulting model equation, which is the justification of the main physical expected feature of such media. To be more precise, the domain is described as follows.

Let \( N \geq 3 \) be an integer. Let \( Y = [0,1]^N = \overline{Y}_1 \cup Y_2 \) where \( Y_1 \) and \( Y_2 \) are two disjoint open sets representing the local structure of the porous matrix for \( Y_1 \), and the local structure of cracks for \( Y_2 \). We assume that \( Y_2 \) is connected and that the boundary of \( Y_1 \) is Lipschitz continuous. We set \( G_1 = \cup_{k \in \mathbb{Z}^N}(k+Y_1) \) and \( G_2 = \mathbb{R}^N \setminus \overline{Y}_1 \). Next, let \( Z_1 \) and \( Z_2 \) be two disjoint open subsets of \( Y_1 \) such that \( Y_1 = Z_1 \cup Z_2 \), \( Z_2 \) being connected. \( Z_1 \) is the local structure of the skeleton while \( Z_2 \) is the one of the pores. Set \( H_1 = \cup_{k \in \mathbb{Z}^N}(k+Z_1) \) and \( H_2 = G_1 \setminus \overline{H}_1 \). \( H_2 \) is open and connected, representing the effective pore space. The crack space \( G_2 \) is also connected. Finally we assume that \( Y_2 \) and \( Z_2 \) have positive Lebesgue measure.

With this in mind, let \( \Omega \) be an open bounded connected subset of \( \mathbb{R}^N \) with Lipschitz boundary \( \partial \Omega \), and let \( \varepsilon > 0 \) be a small parameter. We define the fluid domain to be the union of both pores and cracks domains as follows. First set \( \Omega_p^\varepsilon = \Omega \cap \varepsilon^2 H_2 \) (the pores domain), \( \Omega_c^\varepsilon = \Omega \cap \varepsilon G_2 \) (the cracks domain), and define the fluid domain \( \Omega_p^\varepsilon = \Omega_p^\varepsilon \cup \Omega_c^\varepsilon \), and the skeleton \( \Omega_1^\varepsilon = \Omega \setminus \overline{\Omega}_2 \). It holds that \( \Omega = \Omega_1^\varepsilon \cup \Gamma_1^\varepsilon \cup \Omega_2^\varepsilon \) (disjoint union) where \( \Gamma_1^\varepsilon = \partial \Omega_1^\varepsilon \cap \partial \Omega_2^\varepsilon \) is the interface of \( \Omega_1^\varepsilon \) and \( \Omega_2^\varepsilon \). Let \( \nu_j \) \((j = 1, 2)\) denote the unit outward normal on \( \partial \Omega_j^\varepsilon \).

As an illustration of the analytic construction made above, see figure 1 below.

Fig.1: Fractured porous medium

There are several ways to model fractured porous media. The classical and most studied model named double porosity model was introduced by Barenblatt et al. [4], and has been further developed in [39, 44, 8, 13, 11, 22, 38, 31, 34, 46]. In contrast, very few works are devoted to multiscale media with the geometry similar to the one considered in this paper. We may cite [17, 18, 29, 33, 36].

(2) The flow equations. Motivated by the above-mentioned phenomenological feature of our medium (the memory appearance), we found necessary to study in such media, the flow of fluid having memory. That is why we consider a generalized class of Oldroyd incompressible viscoelastic fluids including as a special case, the classical Newtonian flow of Stokes’ type. They are modeled by a system of integro-differential equations in which all the coefficients and memory kernels depend on both fast and slow space– and time– variables. From the physical point of view, it means that the memory effects arising by meeting an obstacle decay in the surrounding of the next obstacle. For that simple reason, we can not expect using the Laplace transform to perform the homogenization process since the memory kernels depend on the fast time variable. Therefore, to achieve our goal, instead of using the Laplace transform, we use a direct method involving some results about the reiterated convergence of sequences defined by convolution (see Theorems 3.6 which is new in the context). Let us emphasize that although nonlocal in time terms can appear in some homogenization problems (see e.g. [29, 45]), our approach can handle more complicated homogenization problems with nonlocal terms in both time and space variables; see e.g. [44]. As far as we know, this is the first time that such a problem is considered in the literature. Therefore, taking into account both the
structure of the media and the model equations, we can see that our main result is new.

To be more precise, we consider a non-stationary flow of an incompressible viscoelastic non-Newtonian fluid governed by the Stokes system. The viscoelastic constitutive law associated to the momentum balance, and the continuity equations of the normal stress and velocity at the interface are given by (for a.e. \(0 < t < T, T\) being given)

\[
\begin{align*}
\rho_1^s \frac{\partial \mathbf{u}_1}{\partial t} - \text{div} \sigma_1^s &= \rho_1^s f_1 \quad \text{in } \Omega_1^s \quad (1.1) \\
\text{div} \mathbf{u}_1 &= 0 \quad \text{in } \Omega_1^s \quad (1.2) \\
\rho_2^s \frac{\partial \mathbf{u}_2}{\partial t} - \text{div} \sigma_2^s &= \rho_2^s f_2 \quad \text{in } \Omega_2^s \quad (1.3) \\
\text{div} \mathbf{u}_2 &= 0 \quad \text{in } \Omega_2^s \quad (1.4) \\
\mathbf{u}_1 &= \mathbf{u}_2 \quad \text{on } \Gamma_1^s \quad (1.5) \\
\sigma_1^s \cdot \nu_1 &= \sigma_2^s \cdot \nu_2 \quad \text{on } \Gamma_1^s \quad (1.6) \\
\mathbf{u}_1(x,0) &= \mathbf{u}_0^1(x) \quad \text{in } \Omega_1^s \quad \text{and } \mathbf{u}_2(x,0) = \mathbf{u}_0^2(x) \quad \text{in } \Omega_2^s. \quad (1.8)
\end{align*}
\]

In the above equations, denoting by \(I\) is the identity tensor, \(\sigma_1^s = -p_1 \mathbf{I} + A_1^s \nabla \mathbf{u}_1 + \int_0^t A_1^s(x,t-\tau) \nabla \mathbf{u}_1(x,\tau) d\tau\) is the stress tensor of the fluid in \(\Omega_1^s\) with density \(\rho_1^s\), velocity \(\mathbf{u}_1\) and pressure \(p_1\); \(\rho_2^s f_1\) and \(\rho_2^s f_2\) are the external body forces per volume. The functions \(\rho_j^s, A_j^s, \text{ and } B_j^s (j = 0, 1)\) are periodic in the following sense:

\((A1)\quad A_i, B_i \in C(\overline{Q}; L^\infty(\mathbb{R}_p^N \times \mathbb{R}^+_0)^{N \times N})\) are \(N \times N\) symmetric matrices satisfying the following assumption:

\[A_0 \xi \cdot \xi \geq \alpha |\xi|^2, \quad B_0 \xi \cdot \xi \geq \alpha |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^N \text{ and a.e. in } \overline{Q} \times \mathbb{R}_p^{N+1}\]

where \(\alpha > 0\) is a given constant not depending on \(x, t, y, \tau\) and \(\xi\), \(\mathbb{R}_p^m\) (integer \(m \geq 1\)) being denoting the numerical space \(\mathbb{R}^m\) with variables \(\zeta \in \mathbb{R}^m\) and \(\mathbb{R}_p^+ = \mathbb{R}_p \cap [0, \infty)\);

\((A2)\quad p_j \in C(\overline{T}; L^\infty(\mathbb{R}_p^N))\) and there exists \(\Lambda > 0\) such that \(\Lambda^{-1} \leq p_j \leq \Lambda\) a.e. in \(\overline{T} \times \mathbb{R}_p^N\).

\((A3)\quad \text{Periodicity.}\) The characteristic functions \(\chi_{Y_2}\) and \(\chi_{Z_2}\) of the sets \(Y_2\) and \(Z_2\) are \([0,1]^N\)-periodic, and the functions \(A_i(x, t, \cdot, \cdot), B_i(x, t, \cdot, \cdot)\) and \(p_j(x, \cdot)\) are periodic in the following sense:

\[
\begin{align*}
A_i(x, t, \cdot, \cdot), B_i(x, t, \cdot, \cdot) \in L^\infty_{\text{per}}(Y \times \mathcal{T})^{N \times N} \quad \text{for all } (x, t) \in \overline{Q} \quad (1.9) \\
p_j(x, \cdot) \in C^\text{per}(Y) \quad \text{for all } x \in \overline{Q} \quad (1.10)
\end{align*}
\]

where \(\mathcal{T} = [0,1]\) and the spaces \(L^p_{\text{per}} (1 \leq p \leq \infty)\) and \(C^\text{per}\) are defined below.
Remark 1.1. If we denote by $\chi^i_\varepsilon$ ($i=1,2$) the characteristic function of the set $\Omega^i_\varepsilon$, then it is important to express $\chi^i_\varepsilon$ in terms of the characteristic functions of the sets $Y_2$ and $Z_2$. Denoting by $\chi^c_\varepsilon$ and $\chi^p_\varepsilon$ the characteristic functions of the cracks and pores spaces in $\mathbb{R}^N$ respectively, we have

$$\chi^c_\varepsilon(x) = \chi_{G_2} \left( \frac{x}{\varepsilon} \right) = \chi_{Y_2} \left( \frac{x}{\varepsilon} \right) \quad \text{(obtained by $Y$-periodicity)}$$

$$\chi^p_\varepsilon(x) = \left( 1 - \chi_{G_2} \left( \frac{x}{\varepsilon} \right) \right) \chi_{H_2} \left( \frac{x}{\varepsilon^2} \right)$$

$$= \left( 1 - \chi_{Y_2} \left( \frac{x}{\varepsilon} \right) \right) \chi_{Z_2} \left( \frac{x}{\varepsilon^2} \right) \quad \text{(obtained by $Z$-periodicity)},$$

hence

$$\chi^c_\varepsilon(x) = \chi^c_\varepsilon(x) + \chi^p_\varepsilon(x) = \chi_{Y_2} \left( \frac{x}{\varepsilon} \right) + \left( 1 - \chi_{Y_2} \left( \frac{x}{\varepsilon} \right) \right) \chi_{Z_2} \left( \frac{x}{\varepsilon^2} \right)$$

and

$$\chi^i_\varepsilon(x) = 1 - \chi^c_\varepsilon(x), \quad \text{for } x \in \Omega.$$__

Finally we assume that the functions $f_1, f_2 \in L^2(Q)^N$ and $u^0, v^0 \in L^2(\Omega)^N$ with div $u^0 = \text{div} v^0 = 0$ in the sense of the distributions in $\Omega$.

Our main objective in this work is to find the limiting behavior when $\varepsilon \to 0$, of the sequence of solutions to the system \((\text{1.1})-(\text{1.8})\). In this respect, we prove the following result.

- Assuming that (A1)-(A3) hold true, let (for any $\varepsilon > 0$) let $u_\varepsilon$ (resp. $v_\varepsilon$) be the velocity field of the fluid in $\Omega^1_\varepsilon$ (resp. in $\Omega^2_\varepsilon$). Let $\pi_\varepsilon = \chi^1_\varepsilon \rho_\varepsilon + \chi^2_\varepsilon q_\varepsilon$ (where $\chi^j_\varepsilon$ is the characteristic function of the set $\Omega^j_\varepsilon$) be the global pressure and set

$$f(x,t) = \int \int_{Y \times Z} (\chi^1(y,z)\rho_1(x,y)\rho_1(x,t) + \chi^2(y,z)\rho_2(x,y)\rho_2(x,t)) dy dz, \quad \text{a.e. } (x,t) \in Q.$$__

There exist $u \in L^\infty(0,T;L^2(\Omega)^N)$ – the velocity of the fluid in the skeleton, $v \in L^\infty(0,T;L^2(\Omega)^N)$ – the velocity of the fluid in the pores and cracks system, and $p \in L^2(0,T;L^2(\Omega)/\mathbb{R})$ such that, as $\varepsilon \to 0$, $\chi^1_\varepsilon u_\varepsilon \to u$ in $L^2(Q)^N$-weak, $\chi^2_\varepsilon v_\varepsilon \to v$ in $L^2(Q)^N$-weak and $\pi_\varepsilon \to p$ in $L^2(Q)$-weak. Moreover $u = (1-m_c)(1-m_p)u_0$ and $v = v_c + v_p$ where $v_c = m_c u_0$ is the velocity of the fluid in the crack space and $v_p = (1-m_c)m_p u_0$ is the velocity of the fluid in the pore space, and $m_p$ (resp. $m_c$) is the porosity of the pore (resp. crack) space and $(u_0, p)$ is the unique solution to problem

$$\begin{cases}
\rho \frac{\partial u_0}{\partial t} - \text{div} \left( A_0 \nabla u_0 + \int_0^t A_1(x,t-\tau)\nabla u_0(x,\tau)d\tau \right) + \nabla p = f \quad \text{in } Q \\
\text{div } u_0 = 0 \quad \text{in } Q \\
u_0 = 0 \quad \text{on } \partial \Omega \times (0,T) \\
u_0(x,0) = (1-m_c)(1-m_p)u^0(x) + (m_c + m_p(1-m_c))v^0(x), \quad x \in \Omega.
\end{cases}$$__

The rest of the paper is organized as follows. In Section 2, we prove an existence result and a compactness result. In Section 3, we give necessary material about multiscale convergence together with its connection to convolution. Finally, Section 4 deals with the derivation of the limiting model in which we prove the main result of the paper.

We end this section with some notations. All functions are assumed real valued and all function spaces are considered over $\mathbb{R}$. Let $Y = [0,1)^N$ and let $F(\mathbb{R}^N)$ be a given function space. In the case when $F$ is either $L^p$ or $W^{1,p}$ ($1 \leq p \leq \infty$),
we denote by $F_{\text{per}}(Y)$ the space of functions in $F_{\text{loc}}(\mathbb{R}^N)$ that are $Y$-periodic. For $F = \mathcal{C}$, we denote by $C_{\text{per}}(Y)$ the space of continuous functions over $\mathbb{R}^N$ which are $Y$-periodic. We denote by $F_{\#}(Y)$ the subspace of $F_{\text{per}}(Y)$ consisting of functions $u$ having mean value zero: $\int_Y u(y)dy = 0$. To wit, $W_{\#}^{1,p}(Y)$ stands for the space of those functions $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^N)$ which are $Y$-periodic and satisfy $\int_Y u(y)dy = 0$. As special case, $C_{\text{per}}^\infty(Y) = C_{\text{per}}(Y) \cap C^\infty(\mathbb{R}^N)$. Accordingly, we set $Z = [0, 1)^N$, $\mathcal{T} = [0, 1)$, and we define the corresponding spaces. Let $A$ be a $m \times m$ matrix whose entries are functions of unknowns $w$, and let $u$ be either a $m \times 1$ vector function or a $m \times m$ matrix. We will denote by $A(w)$ the value of $A$ at $w$, while either $A u$ or $A[u]$ will stand for the product of the $m \times m$ matrix $A$ by $u$. If $\xi = (\xi_{ij})_{1 \leq i,j \leq N}$ and $\eta = (\eta_{ij})_{1 \leq i,j \leq N} \in \mathbb{R}^{N^2}$, we define the product $\xi \cdot \eta$ by $\xi \cdot \eta = \sum_{i,j=1}^N \xi_{ij} \eta_{ij}$.

2. Existence result and uniform estimates

Our first aim is to give an existence result. The main classical spaces involved in the mathematical study of incompressible fluid flows are spaces connected to kinetic energy, the boundary conditions and the conservation of mass. These spaces are here defined as follows:

$$V_\varepsilon = \{(v_1, v_2) \in V_\varepsilon^1 \times V_\varepsilon^2 : \gamma^1 v_1 = \gamma^2 v_2 \text{ on } \Gamma_{12}^\varepsilon\}$$

and

$$H_\varepsilon = \{(v_1, v_2) \in L^2(\Omega_1^\varepsilon)^N \times L^2(\Omega_2^\varepsilon)^N : \text{div } v_j = 0 \text{ in } \Omega_j^\varepsilon \text{ and } v_j \cdot \nu_j = 0 \text{ on } \partial \Omega_j^\varepsilon\}$$

where

$$V_\varepsilon^j = \{v \in H^1(\Omega_j^\varepsilon)^N : \text{div } v_j = 0 \text{ in } \Omega_j^\varepsilon \text{ and } \gamma^j v_j = 0 \text{ on } \partial \Omega_j^\varepsilon \cap \partial \Omega_j\} \quad (j = 1, 2),$$

$\gamma^j$ being denoting the zero order trace on the boundary $\partial \Omega_j^\varepsilon$ of $\Omega_j^\varepsilon$. The space $H_\varepsilon$ is a Hilbert space with Hilbertian norm

$$\|(v_1, v_2)\|_{H_\varepsilon} = \left(\sum_{j=1}^2 \|v_j\|_{L^2(\Omega_j^\varepsilon)}^2\right)^{\frac{1}{2}}.$$

For $(v_1, v_2) \in V_\varepsilon$ we set

$$\|(v_1, v_2)\|_{V_\varepsilon} = \left(\sum_{j=1}^2 \|\nabla v_j\|_{L^2(\Omega_j^\varepsilon)^N}^2\right)^{\frac{1}{2}}.$$

The following holds true.

**Lemma 2.1.** Equipped with $\|(\cdot)\|_{V_\varepsilon}$, $V_\varepsilon$ is a Hilbert space.

**Proof.** It is sufficient to verify that $\|(\cdot)\|_{V_\varepsilon}$ is a norm on $V_\varepsilon$. To that end, let $(v_1, v_2) \in V_\varepsilon$, and set $u = \chi_1^1 v_1 + \chi_2^2 v_2$. Since $\gamma^1 v_1 = \gamma^2 v_2$ on $\Gamma_{12}^\varepsilon$ and $\gamma^j v_j = 0$ on $\partial \Omega_j^\varepsilon \cap \partial \Omega$ $(j = 1, 2)$, it holds that $\nabla u = \chi_1^1 \nabla v_1 + \chi_2^2 \nabla v_2$ and $u = 0$ on $\partial \Omega$. Thus $u \in H^1_0(\Omega)^N$. Using the general Poincaré inequality for punctured domains [28], Theorem 4, there exists a positive constant $C$ depending only on $\Omega$ such that

$$\left\|\frac{1}{|\Omega|} \int_\Omega u dx\right\|_{L^2(\Omega)} \leq C \left[\|\nabla u\|_{L^2(\Omega_1^\varepsilon)} + \|\nabla u\|_{L^2(\Omega_2^\varepsilon)}\right]$$

$$= C \left[\|\nabla v_1\|_{L^2(\Omega_1^\varepsilon)} + \|\nabla v_2\|_{L^2(\Omega_2^\varepsilon)}\right].$$
Now assume \( \|(v_1, v_2)\|_{V_\varepsilon} = 0 \); then \( \|\nabla v_j\|_{L^2(\Omega)} = 0 \) \((j = 1, 2)\), hence
\[
\frac{u - 1}{|\Omega|} \int_\Omega u dx = 0 \text{ a.e. in } \Omega.
\]
We infer that \( u = 0 \) a.e. in \( \Omega \) since \( u \in H^1_0(\Omega)^N \). This entails \( \chi_\varepsilon^j u = 0 \), i.e. \( v_j = 0 \) a.e. in \( \Omega^\varepsilon \). This is sufficient to conclude that \( \|\cdot\|_{V_\varepsilon} \) is a norm on \( V_\varepsilon \) since the other properties are easily verified.

Now, we assume in the sequel that \( H_\varepsilon \) is rather equipped with the inner product
\[
((u, v)) = \int_\Omega (\chi_1^\varepsilon \rho_1^\varepsilon u_1 \cdot v_1 + \chi_2^\varepsilon \rho_2^\varepsilon u_2 \cdot v_2) dx
\]
which makes it a Hilbert space. This stems from the inequality \( \Lambda^{-1} \leq \rho^\varepsilon \leq \Lambda \) a.e. in \( \Omega \). Keeping this in mind, the following continuous embeddings \( V_\varepsilon \hookrightarrow H_\varepsilon \hookrightarrow V'_\varepsilon \) hold true.

Now, set \( U_\varepsilon = (u_\varepsilon, v_\varepsilon) \), \( F = (f_1, f_2) \) and \( U^0 = (u^0, v^0) \). If we choose \( v = (v_1, v_2) \in V_\varepsilon \) and multiply Eqns (1.1) and (1.3) by \( v_1 \) and \( v_2 \) respectively, and next sum up the resulting equations, we get
\[
\frac{d}{dt} ((U_\varepsilon(t), v)) + \int_\Omega (\chi_1^\varepsilon A_0^\varepsilon(t) \nabla u_\varepsilon(t) \cdot \nabla v_1 + \chi_2^\varepsilon B_0^\varepsilon(t) \nabla v_\varepsilon(t) \cdot \nabla v_2) dx
\]
\[
+ \int_0^t \left( \int_\Omega (\chi_1^\varepsilon A_1^\varepsilon(t - \tau) \nabla u_\varepsilon(\tau) \cdot \nabla v_1 + \chi_2^\varepsilon B_1^\varepsilon(t - \tau) \nabla v_\varepsilon(\tau) \cdot \nabla v_2) dx \right) d\tau
\]
\[
= ((F, v)).
\]
The linear mappings
\[
V_\varepsilon \ni v = (v_1, v_2) \rightarrow \int_\Omega (\chi_1^\varepsilon A_0^\varepsilon(t) \nabla u_\varepsilon(t) \cdot \nabla v_1 + \chi_2^\varepsilon B_0^\varepsilon(t) \nabla v_\varepsilon(t) \cdot \nabla v_2) dx
\]
and
\[
V_\varepsilon \ni v \rightarrow - \int_\Omega (\chi_1^\varepsilon A_1^\varepsilon(t - \tau) \nabla u_\varepsilon(\tau) \cdot \nabla v_1 + \chi_2^\varepsilon B_1^\varepsilon(t - \tau) \nabla v_\varepsilon(\tau) \cdot \nabla v_2) dx
\]
belong to \( V'_\varepsilon \) and hence define two bounded linear operators \( A_\varepsilon(t) \) and \( B_\varepsilon(t, \tau) \) (for a.e. \( 0 \leq t \leq T \) and \( 0 \leq \tau \leq t \leq T \)) from \( V'_\varepsilon \) into \( V'_\varepsilon \) as follows:
\[
\langle A_\varepsilon(t) U_\varepsilon(t), v \rangle = \int_\Omega (\chi_1^\varepsilon A_0^\varepsilon(t) \nabla u_\varepsilon(t) \cdot \nabla v_1 + \chi_2^\varepsilon B_0^\varepsilon(t) \nabla v_\varepsilon(t) \cdot \nabla v_2) dx,
\]
\[
\langle B_\varepsilon(t, \tau) U_\varepsilon(\tau), v \rangle = - \int_\Omega (\chi_1^\varepsilon A_1^\varepsilon(t - \tau) \nabla u_\varepsilon(\tau) \cdot \nabla v_1 + \chi_2^\varepsilon B_1^\varepsilon(t - \tau) \nabla v_\varepsilon(\tau) \cdot \nabla v_2) dx
\]
for \( v \in V_\varepsilon \).

Therefore (2.1) tantamount to
\[
\frac{dU_\varepsilon}{dt}(t) + A_\varepsilon(t) U_\varepsilon(t) = \int_0^t B_\varepsilon(t, \tau) U_\varepsilon(\tau) d\tau + F \text{ in } V'_\varepsilon \tag{2.2}
\]
\[
U_\varepsilon(0) = U^0 \text{ in } H_\varepsilon. \tag{2.3}
\]
The following result holds.

**Proposition 2.1.** It holds that
\[
\langle A_\varepsilon(t) v, v \rangle \geq \alpha \|v\|^2_{V_\varepsilon} \text{ for all } v \in V_\varepsilon. \tag{2.4}
\]
Proof. For \( v = (v_1, v_2) \in V_{\varepsilon} \) we have

\[
(A_{\varepsilon}(t)v, v) = \int_{\Omega} \left( \chi_1 A_0(t) \nabla v_1 \cdot \nabla v_1 + \chi_2 B_0(t) \nabla v_2 \cdot \nabla v_2 \right) dx \geq \alpha \int_{\Omega} \left( \chi_1 |\nabla v_1|^2 + \chi_2 |\nabla v_2|^2 \right) dx = \alpha \|v\|_{V_{\varepsilon}}^2.
\]

\[
\square
\]

We can now state and prove the existence result.

**Theorem 2.1.** For any \( \varepsilon > 0 \), there exist two pairs \((u_{\varepsilon}, v_{\varepsilon}) \in L^2(0, T; V_{\varepsilon}) \cap L^\infty(0, T; H_{\varepsilon})\) and \((p_{\varepsilon}, q_{\varepsilon}) \in L^2(0, T; L^2(\Omega_{\varepsilon}^1) \times L^2(\Omega_{\varepsilon}^2))\) that solve (2.1) - (2.8). Moreover, the vector-function \((u_{\varepsilon}, v_{\varepsilon})\) is unique and belongs to \( C([0, T]; H_{\varepsilon}) \), and \((p_{\varepsilon}, q_{\varepsilon})\) is unique up to a constant in the following sense:

\[
\int_{\Omega_{\varepsilon}^1} p_{\varepsilon} dx = 0 \quad \text{and} \quad \int_{\Omega_{\varepsilon}^2} q_{\varepsilon} dx = 0. \tag{2.5}
\]

**Proof.** We infer from Proposition 2.1 that the hypotheses of Theorem 3.2 in [24] are fulfilled. Therefore, appealing to the above cited result we obtain the existence and uniqueness of \((u_{\varepsilon}, v_{\varepsilon}) \in L^2(0, T; V_{\varepsilon}) \cap C([0, T]; H_{\varepsilon})\) satisfying (2.2) - (2.8). The existence of \( p_{\varepsilon} \) and \( q_{\varepsilon} \) satisfying (2.5) follows by the use of Propositions 1.1 and 1.2 of [24].

The next result provides us with uniform estimates.

**Lemma 2.2.** Under assumptions (A1)-(A2) it holds that

\[
\sup_{0 \leq t \leq T} \left( \|u_{\varepsilon}(t)\|_{L^2(\Omega_{\varepsilon}^1)}^2 + \|v_{\varepsilon}(t)\|_{L^2(\Omega_{\varepsilon}^2)}^2 \right) \leq C, \tag{2.6}
\]

\[
\int_0^T \left( \|\nabla u_{\varepsilon}(t)\|_{L^2(\Omega_{\varepsilon}^1)}^2 + \|\nabla v_{\varepsilon}(t)\|_{L^2(\Omega_{\varepsilon}^2)}^2 \right) dt \leq C, \tag{2.7}
\]

\[
\|p_{\varepsilon}\|_{L^2(\Omega_{\varepsilon}^1 \times (0, T))} \leq C \quad \text{and} \quad \|q_{\varepsilon}\|_{L^2(\Omega_{\varepsilon}^2 \times (0, T))} \leq C \tag{2.8}
\]

\[
\left\| \rho_1 \frac{\partial u_{\varepsilon}}{\partial t} \right\|_{L^2(0, T; (V_{\varepsilon}^1)^*)} \leq C \quad \text{and} \quad \left\| \rho_2 \frac{\partial v_{\varepsilon}}{\partial t} \right\|_{L^2(0, T; (V_{\varepsilon}^2)^*)} \leq C \tag{2.9}
\]

where \( C \) is a positive constant not depending on \( \varepsilon \).

**Proof.** We multiply (1.1) and (1.3) respectively by \( u_{\varepsilon} \) and \( v_{\varepsilon} \). Then denoting by * the convolution with respect to the time variable \( t \), it holds that

\[
(\rho_1^2 \frac{\partial u_{\varepsilon}}{\partial t}, u_{\varepsilon}) + (\rho_2^2 \frac{\partial v_{\varepsilon}}{\partial t}, v_{\varepsilon}) + (A_0^2 \nabla u_{\varepsilon} + A_1^2 * \nabla u_{\varepsilon}, \nabla u_{\varepsilon})
\]

\[
+ (B_0^2 \nabla v_{\varepsilon} + B_1^2 * \nabla v_{\varepsilon}, \nabla v_{\varepsilon}) = (\rho_1^2 f_1, u_{\varepsilon}) + (\rho_2^2 f_2, v_{\varepsilon}),
\]

or equivalently,

\[
\frac{1}{2} \frac{d}{dt} \left\| (\rho_1^2)^{\frac{1}{2}} u_{\varepsilon}(t) \right\|_{L^2(\Omega_{\varepsilon}^1)}^2 + \frac{1}{2} \frac{d}{dt} \left\| (\rho_2^2)^{\frac{1}{2}} v_{\varepsilon}(t) \right\|_{L^2(\Omega_{\varepsilon}^2)}^2
\]

\[
+ (A_0^2 \nabla u_{\varepsilon} + A_1^2 * \nabla u_{\varepsilon}, \nabla u_{\varepsilon}) + (B_0^2 \nabla v_{\varepsilon} + B_1^2 * \nabla v_{\varepsilon}, \nabla v_{\varepsilon})
\]

\[
= (\rho_1^2 f_1, u_{\varepsilon}) + (\rho_2^2 f_2, v_{\varepsilon}).
\]
Integrating with respect to $t$,

$$
\left\| (\rho_1^t)^2 u_e(t) \right\|^2_{L^2(\Omega^t_1)} + \left\| (\rho_2^t)^2 v_e(t) \right\|^2_{L^2(\Omega^t_2)} + 2 \int_0^t (A_0^* \nabla u_e + A_1^* \nabla u_e, \nabla u_e) \, dt + 2 \int_0^t (B_0^* \nabla v_e + B_1^* \nabla v_e, \nabla v_e) \, dt
$$

$$
= 2 \int_0^t [(\rho_1^t f_1, u_e) + (\rho_2^t f_2, v_e)] \, dt + \left\| (\rho_1^t)^2 u^0 \right\|^2_{L^2(\Omega^1_1)} + \left\| (\rho_2^t)^2 v^0 \right\|^2_{L^2(\Omega^1_2)}.
$$

But

$$
2 \int_0^t [(\rho_1^t f_1, u_e) + (\rho_2^t f_2, v_e)] \, dt
$$

$$
\leq \int_0^t \left( \left\| (\rho_1^t)^2 f_1(\tau) \right\|^2_{L^2(\Omega^1_1)} + \left\| (\rho_2^t)^2 f_2(\tau) \right\|^2_{L^2(\Omega^1_2)} \right) \, d\tau
$$

Making use of (A1) we get

$$
\left\| (\rho_1^t)^2 u_e(t) \right\|^2_{L^2(\Omega^t_1)} + \left\| (\rho_2^t)^2 v_e(t) \right\|^2_{L^2(\Omega^t_2)}
$$

$$
+ 2\alpha \int_0^t \left( \left\| \nabla u_e(\tau) \right\|^2_{L^2(\Omega^t_1)} + \left\| \nabla v_e(\tau) \right\|^2_{L^2(\Omega^t_2)} \right) \, d\tau
$$

$$
\leq -2 \int_0^t [(A_0^* \nabla u_e, \nabla u_e) + (B_0^* \nabla v_e, \nabla v_e)] \, dt
$$

$$
+ \int_0^t \left( \left\| (\rho_1^t)^2 u_e(\tau) \right\|^2_{L^2(\Omega^t_1)} + \left\| (\rho_2^t)^2 v_e(\tau) \right\|^2_{L^2(\Omega^t_2)} \right) \, d\tau
$$

$$
+ \int_0^t \left( \left\| (\rho_1^t)^2 f_1(t) \right\|^2_{L^2(\Omega^t_1)} + \left\| (\rho_2^t)^2 f_2(t) \right\|^2_{L^2(\Omega^t_2)} \right) \, dt.
$$

Now, using Young’s inequality,

$$
2 \int_0^t (A_0^* \nabla u_e, \nabla u_e) \, dt
$$

$$
= 2 \int_0^t \left( \int_0^\tau \left( \int_0^{\Omega^t_1} A_0^*(\tau-s) \nabla u_e(s) \cdot \nabla u_e(\tau) \, dx \right) \, ds \right) \, d\tau
$$

$$
\leq 2 \int_0^t \left( C \int_0^\tau \left\| A_0^*(\tau-s) \nabla u_e(s) \right\|^2_{L^2(\Omega^t_1)} \, d\tau + \frac{\alpha}{2\tau} \left\| \nabla u_e(\tau) \right\|^2_{L^2(\Omega^t_1)} \, d\tau \right)
$$

$$
\leq \alpha \int_0^t \left\| \nabla u_e(\tau) \right\|^2_{L^2(\Omega^t_1)} \, d\tau + C \int_0^t \left( \int_0^\tau \left\| \nabla u_e(s) \right\|^2_{L^2(\Omega^t_1)} \, ds \right) \, d\tau,
$$
hence
\begin{align*}
2 \int_0^t \left[ (A^*_1 \ast \nabla u_\varepsilon, \nabla u_\varepsilon) + (B^*_1 \ast \nabla v_\varepsilon, \nabla v_\varepsilon) \right] \, d\tau \\
\leq \alpha \int_0^t \left( \| \nabla u_\varepsilon(\tau) \|^2_{L^2(\Omega^1_\varepsilon)} + \| \nabla v_\varepsilon(\tau) \|^2_{L^2(\Omega^2_\varepsilon)} \right) \, d\tau \\
+ C \int_0^t \left( \int_0^\tau \left( \| \nabla u_\varepsilon(s) \|^2_{L^2(\Omega^1_\varepsilon)} + \| \nabla v_\varepsilon(s) \|^2_{L^2(\Omega^2_\varepsilon)} \right) \, ds \right) \, d\tau.
\end{align*}

Therefore
\begin{align*}
\left\| (\rho^2_1)^{\frac{1}{2}} u_\varepsilon(t) \right\|^2_{L^2(\Omega^1_\varepsilon)} + \left\| (\rho^2_2)^{\frac{1}{2}} v_\varepsilon(t) \right\|^2_{L^2(\Omega^2_\varepsilon)} \\
+ \alpha \int_0^t \left( \| \nabla u_\varepsilon(\tau) \|^2_{L^2(\Omega^1_\varepsilon)} + \| \nabla v_\varepsilon(\tau) \|^2_{L^2(\Omega^2_\varepsilon)} \right) \, d\tau \\
\leq C + \int_0^t \left( \| (\rho^2_1)^{\frac{1}{2}} u_\varepsilon(\tau) \|^2_{L^2(\Omega^1_\varepsilon)} + \| (\rho^2_2)^{\frac{1}{2}} v_\varepsilon(\tau) \|^2_{L^2(\Omega^2_\varepsilon)} \right) \, d\tau \\
+ C \int_0^t \left( \int_0^\tau \left( \| \nabla u_\varepsilon(s) \|^2_{L^2(\Omega^1_\varepsilon)} + \| \nabla v_\varepsilon(s) \|^2_{L^2(\Omega^2_\varepsilon)} \right) \, ds \right) \, d\tau.
\end{align*}

It follows from Gronwall’s inequality that
\begin{align*}
\left\| (\rho^2_1)^{\frac{1}{2}} u_\varepsilon(t) \right\|^2_{L^2(\Omega^1_\varepsilon)} + \left\| (\rho^2_2)^{\frac{1}{2}} v_\varepsilon(t) \right\|^2_{L^2(\Omega^2_\varepsilon)} \\
+ \alpha \int_0^t \left( \| \nabla u_\varepsilon(\tau) \|^2_{L^2(\Omega^1_\varepsilon)} + \| \nabla v_\varepsilon(\tau) \|^2_{L^2(\Omega^2_\varepsilon)} \right) \, d\tau \leq C
\end{align*}
for all \(0 \leq t \leq T\) and all \(\varepsilon > 0\), where \(C\) is independent of \(\varepsilon\) and \(t\). We therefore deduce (2.6) and (2.7). (2.9) follows immediately, and (2.8) is obtained by repeating the same arguments used in [7, Section 2.2].

The next result deals with the compactness of the global velocity defined below by
\begin{equation}
\mu^\varepsilon = \chi_1^1 u_\varepsilon + \chi_2^1 v_\varepsilon. 
\end{equation}

We also set \(\rho^\varepsilon = \chi_1^1 \rho_1^\varepsilon + \chi_2^1 \rho_2^\varepsilon\), recalling that \(\chi_1^j, j = 1, 2\) denotes the characteristic function of the open set \(\Omega_1^j\).

**Proposition 2.2.** Assume that the sequence \((\rho^\varepsilon)_{\varepsilon > 0}\) weakly \(\ast\)-converges in \(L^\infty(\Omega)\) to some real function \(\rho\) as \(\varepsilon \to 0\), with \(\rho(x) \neq 0\) for a.e. \(x \in \Omega\). Then the sequence \((\mu^\varepsilon)_{\varepsilon > 0}\) is relatively compact in the space \(L^2(Q)^N\).

**Proof.** The proof is copied on that of [8, Proposition 2.1].

3. Multiscale convergence and related convolution results

The letter \(E\) denotes throughout any ordinary sequence \((\varepsilon_n)_{n \in \mathbb{N}}\) with \(0 < \varepsilon_n \leq 1\) such that \(\varepsilon_n \to 0\) as \(n \to \infty\). \(\varepsilon\) will denote a generic element of \(E\), and \(\varepsilon_n \to 0\) as \(n \to \infty\) will henceforth be merely denoted by \(\varepsilon \to 0\). We assume throughout this section that \(\Omega\) is an open subset of \(\mathbb{R}^N\). We also set \(Y = Z = [0, 1]^N\).
Definition 3.1. Let $1 \leq p < \infty$. (1) A sequence $(u_\varepsilon)_{\varepsilon > 0} \subset L^p(\Omega)$ is said to \textit{weakly multiscale converge in} $L^p(\Omega)$ to some $u_0 \in L^p(\Omega \times Y \times Z)$ if as $\varepsilon \to 0$,
\[
\int_\Omega u_\varepsilon(x) \psi \left( x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2} \right) dx \to \int_{\Omega \times Y \times Z} u_0(x, y, z) \psi(x, y, z) dx dy dz
\]
for every $\psi \in L^{p'}(\Omega; C_{\text{per}}(Y \times Z))$ ($1/p' = 1 - 1/p$). We express this by writing $u_\varepsilon \to u_0$ reit. in $L^p(\Omega)$-weak.

(2) The sequence $(u_\varepsilon)_{\varepsilon > 0} \subset L^p(\Omega)$ is said to \textit{strongly multiscale converge in} $L^p(\Omega)$ to some $u_0 \in L^p(\Omega \times Y \times Z)$ if it is weakly multiscale converge towards $u_0$ and further satisfies the following condition:
\[
\|u_\varepsilon\|_{L^p(\Omega)} \to \|u_0\|_{L^p(\Omega \times Y \times Z)} \text{ as } \varepsilon \to 0.
\]
We denote this by writing $u_\varepsilon \to u_0$ reit. in $L^p(\Omega)$-strong.

The above definition has been introduced in [2] for the case $p = 2$ and later generalized to any $1 < p < \infty$ in [13] (see also [15, 16]).

The following two results are worth recalling; see e.g. [21] Theorems 3.1 and 3.5] (see also [2] for the case $p = 2$).

Theorem 3.1. Any bounded sequence $(u_\varepsilon)_{\varepsilon \in E} \subset L^p(\Omega)$ ($1 < p < \infty$) possesses a subsequence which is weakly multiscale convergent in $L^p(\Omega)$.

Theorem 3.2. Let $(u_\varepsilon)_{\varepsilon \in E}$ be a bounded in $W^{1,p}(\Omega)$ ($1 < p < \infty$). Then there exist a subsequence $E'$ of $E$ and a triple of functions $u_0 \in W^{1,p}(\Omega)$, $u_1 \in L^p(\Omega; W_{#}^{1,p}(Y))$ and $u_2 \in L^p(\Omega \times Y; W_{#}^{1,p}(Z))$ such that, as $\varepsilon \to 0$,
\[
u_\varepsilon \to u_0 \text{ in } W^{1,p}(\Omega)-\text{weak,}
\]
\[
\frac{\partial u_\varepsilon}{\partial x_i} \to \frac{\partial u_0}{\partial x_i} + \frac{\partial u_1}{\partial y_i} + \frac{\partial u_2}{\partial z_i} \text{ reit. in } L^p(\Omega)-\text{weak, } 1 \leq i \leq N.
\]

The proof of the next result is copied on that of [32, Theorem 6].

Theorem 3.3. Let $1 < p, q < \infty$ and $r \geq 1$ be such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} \leq 1$. Suppose that $u_\varepsilon \to u_0$ reit. in $L^p(\Omega)$-weak and $v_\varepsilon \to v_0$ reit. in $L^q(\Omega)$-strong, where $u_0 \in L^p(\Omega \times Y \times Z)$ and $v_0 \in L^q(\Omega \times Y \times Z)$. Then $u_\varepsilon v_\varepsilon \to u_0 v_0$ reit. in $L^r(\Omega)$-weak.

As an immediate consequence of the preceding result, the following holds true.

Corollary 3.1. Let $(u_\varepsilon)_{\varepsilon \in E} \subset L^p(\Omega)$ and $(v_\varepsilon)_{\varepsilon \in E} \subset L^{p'}(\Omega) \cap L^\infty(\Omega)$ ($1 < p < \infty$ and $p' = p/(p - 1)$) be two sequences such that: (i) $u_\varepsilon \to u_0$ reit. in $L^p(\Omega)$-weak; (ii) $v_\varepsilon \to v_0$ reit. in $L^{p'}(\Omega)$-strong; (iii) $(v_\varepsilon)_{\varepsilon \in E}$ is bounded in $L^\infty(\Omega)$. Then $u_\varepsilon v_\varepsilon \to u_0 v_0$ reit. in $L^r(\Omega)$-weak.

The following result establishes a relationship between the limit of a multiscale convergent sequence and the limit of its translates.

Theorem 3.4. Let $(u_\varepsilon)_{\varepsilon \in E} \subset L^p(\Omega)$ be a sequence such that $u_\varepsilon \to u_0$ reit. in $L^p(\Omega)$-weak, where $u_0 \in L^p(\Omega \times Y \times Z)$. Let $\xi \in \mathbb{R}^N$ and define $v_\varepsilon(x) = u_\varepsilon(x + \xi)$ for $x \in \Omega - \xi$. Then there exist a subsequence $E'$ of $E$ and a couple $(r, s) \in Y \times Z$ such that $v_\varepsilon \to v_0$ reit. in $L^r(\Omega)$-weak when $E' \ni \varepsilon \to 0$, where $v_0 \in L^r((\Omega - \xi) \times Y \times Z)$ is defined by $v_0(x, y, z) = u_0(x + \xi, y + r, z + s)$ for $(x, y, z) \in (\Omega - \xi) \times Y \times Z$. 


Proof. Let $R \left( \xi / \varepsilon^i \right) = \xi / \varepsilon^i - \left[ \xi / \varepsilon^i \right]$ \((i = 1, 2)\) where $\left[ \xi / \varepsilon^i \right]$ denotes the integer part of $\xi / \varepsilon^i$. Then $\left( R \left( \xi / \varepsilon^i \right) \right)_{\varepsilon \in E}$ is a bounded sequence in \([0,1)^N\), hence there exist a subsequence $E'$ of $E$ and $(r,s) \in Y \times Z$ \((Y = Z = [0,1)^N)\) such that the following convergence results hold in the usual topology of $\mathbb{R}$:

\[
R \left( \frac{\xi}{\varepsilon^i} \right) \rightarrow r \quad \text{and} \quad R \left( \frac{\xi}{\varepsilon^2} \right) \rightarrow s \quad \text{as} \quad E' \ni \varepsilon \rightarrow 0. \tag{3.1}
\]

This being so, let $\varphi \in \mathcal{C}_\infty^\infty (\Omega - \xi)$ and $\psi \in \mathcal{C}_{\text{per}} (Y \times Z)$; then

\[
\int_{\Omega - \xi} u_\varepsilon (x + \xi) \varphi (x) \psi \left( \frac{x - \xi}{\varepsilon^2} \right) dx = \int_{\Omega} u_\varepsilon (x) \varphi (x - \xi) \psi \left( \frac{x - \xi}{\varepsilon^2} \right) dx = \int_{\Omega} u_\varepsilon (x) \varphi (x - \xi) \left[ \psi \left( \frac{x - \xi}{\varepsilon} - \frac{x - \xi}{\varepsilon^2} \right) - \psi \left( \frac{x - \xi}{\varepsilon} - r - \frac{x - \xi}{\varepsilon^2} - s \right) \right] dx
\]

\[+ \int_{\Omega} u_\varepsilon (x) \varphi (x - \xi) \psi \left( \frac{x - \xi}{\varepsilon} - r - \frac{x - \xi}{\varepsilon^2} - s \right) dx\]

\[= (I) + (II).\]

Set $\phi(y,z) = \psi(y - r, z - s), \ (y,z) \in Y \times Z$. Then $\phi \in \mathcal{C}_{\text{per}} (Y \times Z)$, and thus, as $E' \ni \varepsilon \rightarrow 0$,

\[(II) \rightarrow \iint_{\Omega \times Y \times Z} u_0 (x,y,z) \varphi (x - \xi) \psi (y - r, z - s) dxdydz = \iint_{\Omega - \xi \times Y \times Z} u_0 (x + \xi, y + r, z + s) \varphi (x) \psi (y, z) dxdydz.\]

On the other hand, \(|(I)| \leq C \left\| \psi \left( \cdot - \frac{\xi}{\varepsilon^2} \right) - \psi \left( \cdot - r - \frac{\xi}{\varepsilon^2} - s \right) \right\|_\infty \leq C \left\| \psi \left( \cdot - \frac{\xi}{\varepsilon^2} \right) - \psi \left( \cdot - r - \frac{\xi}{\varepsilon^2} - s \right) \right\|_\infty\]

since $\psi$ is $Y \times Z$-periodic, where $\| \cdot \|_\infty$ stands for the supremum norm. The uniform continuity of $\psi$ and the convergence results \((3.1)\) yield $(I) \rightarrow 0$ as $E' \ni \varepsilon \rightarrow 0$. The result follows from the boundedness of $(v_\varepsilon)_{\varepsilon \in E}$ in $L^p (\Omega - \xi)$.

The next result deals with the convergence of convolution of sequences. Before we can proceed further, let $p,q,m \geq 1$ be real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{m}$. Let $(u_\varepsilon)_{\varepsilon \in E} \subset L^p (\Omega)$ and $(v_\varepsilon)_{\varepsilon \in E} \subset L^q (\mathbb{R}^N)$ be two sequences. Viewing $u_\varepsilon$ as defined on the whole $\mathbb{R}^N$ (by taking its zero-extension off $\Omega$) we define $u_\varepsilon * v_\varepsilon$ on $\mathbb{R}^N$ by

\[
(u_\varepsilon * v_\varepsilon)(x) = \int_{\mathbb{R}^N} u_\varepsilon (\xi) v_\varepsilon (x - \xi) d\xi \quad (x \in \mathbb{R}^N).
\]

Then $u_\varepsilon * v_\varepsilon \in L^m (\mathbb{R}^N)$ and further

\[
\|u_\varepsilon * v_\varepsilon\|_{L^m (\mathbb{R}^N)} \leq \|u_\varepsilon\|_{L^p (\Omega)} \|v_\varepsilon\|_{L^q (\mathbb{R}^N)}.
\]

As in \cite{38} where the double convolution is defined, we define the triple convolution denoted by ** as follows. For $u \in L^p (\mathbb{R}^N; L^p_{\text{per}} (Y \times Z)) \equiv L^p (\mathbb{R}^N \times Y \times Z)$ and
v ∈ L^q(\mathbb{R}^N \times Y \times Z), u * v stands for the function

\[ (u * v)(x, y, z) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} u(\xi, r, s) v(x - \xi, y - r, z - s) \, d\xi \, dr \, ds, \quad (x, y, z) \in \mathbb{R}^N \times Y \times Z. \]

Then u * v is well-defined and belongs to \( L^m(\mathbb{R}^N \times Y \times Z) \), and further satisfies

\[ \|u * v\|_{L^m(\mathbb{R}^N \times Y \times Z)} \leq \|u\|_{L^p(\mathbb{R}^N \times Y \times Z)} \|v\|_{L^q(\mathbb{R}^N \times Y \times Z)}. \]

Now, if \( u \in L^p(\Omega \times Y \times Z) \) and \( v \in L^q(\Omega \times Y \times Z) \), we define \( u * v \) just by viewing \( u \) and \( v \) as defined in the whole \( \mathbb{R}^N \times Y \times Z \); just take the zero-extension off \( \Omega \).

Bearing this in mind, the next result is in order.

**Theorem 3.5.** Let \((u_\varepsilon)_{\varepsilon \in E}\) and \((v_\varepsilon)_{\varepsilon \in E}\) be as above. Suppose that, as \( E \ni \varepsilon \to 0 \), \( u_\varepsilon \to u_0 \) reit. in \( L^p(\Omega) \)-weak and \( v_\varepsilon \to v_0 \) reit. in \( L^q(\mathbb{R}^N) \)-strong, where \( u_0 \in L^p(\Omega \times Y \times Z) \) and \( v_0 \in L^q(\Omega \times Y \times Z) \). Then, as \( E \ni \varepsilon \to 0 \),

\[ u_\varepsilon * v_\varepsilon \to u_0 * v_0 \text{ reit. in } L^p(\Omega)\text{-weak.} \]

**Proof.** The proof is very similar to the one of its homologue Theorem 2.6 in [44] (see also [43]). Since Theorem 2.6 in [44] involves almost periodicity and moreover is checked in the two-scale sense, it is suitable to repeat the proof here in the periodicity and multiscale frameworks for completeness. First and foremost, it is easy to see that the sequence \((u_\varepsilon * v_\varepsilon)_{\varepsilon \in E}\) is bounded in \( L^m(\Omega) \). Now, let \( \eta > 0 \) and let \( \psi_0 \in \mathcal{K}(\mathbb{R}^N; C_{\text{per}}(Y \times Z)) \) (the space of continuous functions from \( \mathbb{R}^N \) into \( C_{\text{per}}(Y \times Z) \) with compact support in \( \mathbb{R}^N \)) be such that \( \|v_0 - \psi_0\|_{L^q(\mathbb{R}^N \times Y \times Z)} \leq \frac{\eta}{2} \).

Since \( v_\varepsilon \to v_0 \) reit. in \( L^q(\mathbb{R}^N)\)-strong, we have that \( v_\varepsilon - \psi_0 \to v_0 - \psi_0 \) reit. in \( L^q(\mathbb{R}^N)\)-strong, hence \( \|v_\varepsilon - \psi_0\|_{L^q(\mathbb{R}^N)} \to \|v_0 - \psi_0\|_{L^q(\mathbb{R}^N \times Y \times Z)} \) as \( E \ni \varepsilon \to 0 \). So, there is \( \alpha > 0 \) such that

\[ \|v_\varepsilon - \psi_0\|_{L^q(\mathbb{R}^N)} \leq \eta \text{ for } E \ni \varepsilon \leq \alpha. \] (3.2)

If we still denote by \( u_\varepsilon \) the extension by zero of \( u_\varepsilon \) outside \( \Omega \), then we have, for any \( f \in \mathcal{K}(\Omega; C_{\text{per}}(Y \times Z)) \),

\[
\int_{\Omega} (u_\varepsilon * v_\varepsilon)(x) f\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) \, dx = \int_{\Omega} \left( \int_{\mathbb{R}^N} u_\varepsilon(t) v_\varepsilon(x - t) \, dt \right) f\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) \, dx \\
= \int_{\mathbb{R}^N} u_\varepsilon(t) \left[ \int_{\mathbb{R}^N} v_\varepsilon(x - t) f\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) \, dx \right] \, dt \\
= \int_{\mathbb{R}^N} u_\varepsilon(t) \left[ \int_{\mathbb{R}^N} v_\varepsilon(x) f\left(\frac{x + t}{\varepsilon}, \frac{x + t}{\varepsilon^2} + \frac{t}{\varepsilon^2}\right) \, dx \right] \, dt \\
= \int_{\mathbb{R}^N} u_\varepsilon(t) \left( \int_{\mathbb{R}^N} (v_\varepsilon(x) - \psi_0(x)) f_\varepsilon(x + t) \, dx \right) \, dt \\
+ \int_{\mathbb{R}^N} u_\varepsilon(t) \left( \int_{\mathbb{R}^N} \psi_0(x) f_\varepsilon(x + t) \, dx \right) \, dt \\
= (I) + (II).
\]

Firstly, \((I) = \int_{\Omega} [u_\varepsilon * (v_\varepsilon - \psi_0)] f_\varepsilon(x) \, dx \) and

\[ |(I)| \leq \|u_\varepsilon\|_{L^p(\Omega)} \|v_\varepsilon - \psi_0\|_{L^q(\mathbb{R}^N)} \|f_\varepsilon\|_{L^m(\Omega)} \]

\[ \leq C \|v_\varepsilon - \psi_0\|_{L^q(\mathbb{R}^N)}, \]

\( C > 0 \) being independent of \( \varepsilon \). It follows from (3.2) that

\[ |(I)| \leq \varepsilon \eta \text{ for } 0 < \varepsilon \leq \alpha. \] (3.3)
Secondly, owing to Theorem 3.3 we have, as $E \ni \varepsilon \to 0$,

$$\int_{\mathbb{R}^N} \psi_0^\varepsilon(x)f^\varepsilon(x+t)dx \to \int_{\mathbb{R}^N \times Y \times Z} \psi_0(x,y,z)f(x+t,y+r,z+s)dx dy dz$$

where $r = \lim R\left(\frac{t}{\varepsilon}\right)$ and $s = \lim R\left(\frac{1}{\varepsilon}\right)$ (for a suitable subsequence $E'$ of $E$). So let $\Phi : \mathbb{R}^N \times Y \times Z \to \mathbb{R}$ be defined by

$$\Phi(t,r,s) = \int_{\mathbb{R}^N \times Y \times Z} \psi_0(x,y,z)f(x+t,y+r,z+s)dx dy dz, \quad (t,r,s) \in \mathbb{R}^N \times Y \times Z.$$ 

Then it is easy to see that $\Phi \in \mathcal{K}(\mathbb{R}^N; C_{per}(Y \times Z))$, so that the trace $\Phi^\varepsilon(t) = \Phi\left(t,\frac{t}{\varepsilon},\frac{1}{\varepsilon}\right)$ ($t \in \mathbb{R}^N$) is well-defined by

$$\Phi^\varepsilon(t) = \int_{\mathbb{R}^N \times Y \times Z} \psi_0(x,y,z)f\left(x+t,y+\frac{t}{\varepsilon},z+\frac{1}{\varepsilon}\right) dx dy dz.$$ 

Next, we have

$$(II) = \int_{\mathbb{R}^N} u_\varepsilon(t)\left(\int_{\mathbb{R}^N} \psi_0^\varepsilon(x)f^\varepsilon(x+t)dx - \Phi^\varepsilon(t)\right) dt + \int_{\mathbb{R}^N} u_\varepsilon(t)\Phi^\varepsilon(t) dt = (II_1) + (II_2).$$

Dealing with $(II_1)$, let

$$V_\varepsilon(t) = \int_{\mathbb{R}^N} \psi_0^\varepsilon(x)f^\varepsilon(x+t)dx - \Phi^\varepsilon(t)$$

for a.e. $t \in \mathbb{R}^N$.

Then the following holds true:

- (P1) For a.e. $t$, $V_\varepsilon(t) \to 0$ as $E' \ni \varepsilon \to 0$ (possibly up to a subsequence)

- (P2) $\int_{\mathbb{R}^N} u_\varepsilon(t)V_\varepsilon(t) dt \to 0$ as $E' \ni \varepsilon \to 0$.

Indeed, for (P1), appealing to Theorem 3.3,

$$\int_{\mathbb{R}^N} \psi_0^\varepsilon(x)f^\varepsilon(x+t)dx \to \int_{\mathbb{R}^N \times Y \times Z} \psi_0(x,y,z)f(x+t,y+r,z+s)dx dy dz$$

as $E' \ni \varepsilon \to 0$

where $r$ and $s$ are such that $R(\frac{t}{\varepsilon}) \to r$ and $R(\frac{1}{\varepsilon}) \to s$ for some subsequence of $E'$ (not relabeled). Moreover, since $\Phi^\varepsilon(t) = \Phi(t,R(\frac{t}{\varepsilon}),R(\frac{1}{\varepsilon}))$, we have by the continuity of $\Phi(t,\cdot,\cdot)$ that, for the same subsequence,

$$\Phi^\varepsilon(t) \to \int_{\mathbb{R}^N \times Y \times Z} \psi_0(x,y,z)f(x+t,y+r,z+s)dx dy dz.$$ 

Thus (P1) is justified. As for (P2), first and foremost we have

$$|V_\varepsilon(t)| \leq C$$

for a.e. $t \in \mathbb{R}^N$,

$C > 0$ being independent of $t$ and $\varepsilon$. Since $f$ and $\psi_0$ belong to $\mathcal{K}(\mathbb{R}^N; C_{per}(Y \times Z))$ we have that $f^\varepsilon$ and $\psi_0^\varepsilon$ lie in $\mathcal{K}(\mathbb{R}^N)$ and their supports are contained in a fixed compact set of $\mathbb{R}^N$. Therefore $\psi_0^\varepsilon \ast f^\varepsilon \in \mathcal{K}(\mathbb{R}^N)$. As a result, $V_\varepsilon \in \mathcal{K}(\mathbb{R}^N)$ and further its support is contained in a fixed compact set $L \subset \mathbb{R}^N$ independent of $\varepsilon$.

With this in mind, let $\gamma > 0$. We infer from Egorov’s theorem that there exists $D \subset \mathbb{R}^N$ such that $\text{meas}(\mathbb{R}^N \backslash D) < \gamma$ and $V_\varepsilon$ converges uniformly to 0 on $D$. We
have the following series of inequalities

\[
\left| \int_{\mathbb{R}^N} u_\varepsilon(t) V_\varepsilon(t) dt \right| \leq \| u_\varepsilon \|_{L^p(D)} \| V_\varepsilon \|_{L^p'(D)} + \| u_\varepsilon \|_{L^p(\mathbb{R}^N \setminus D)} \| V_\varepsilon \|_{L^p'(\mathbb{R}^N \setminus D)}
\]

\[
\leq C \| V_\varepsilon \|_{L^p'(D \setminus \Omega)} + C \text{meas}(\mathbb{R}^N \setminus D)
\]

\[
\leq C_1 \text{meas}(L) \sup_{t \in D} \| V_\varepsilon(t) \| + C_1 \gamma
\]

where \( C_1 \) > 0 is independent of both \( \varepsilon \) and \( D \). It emerges from the uniform continuity of \( V_\varepsilon \) in \( D \) that there exists \( \alpha_1 > 0 \) with \( \alpha_1 \leq \alpha \) such that

\[
\left| \int_{\mathbb{R}^N} u_\varepsilon(t) V_\varepsilon(t) dt \right| \leq C_2 \gamma \text{ provided } E' \ni \varepsilon \leq \alpha_1,
\]

where \( C_2 \) > 0 is independent of \( \varepsilon \). This shows (P2). We concludes that \((I I_1) \to 0\) as \( E' \ni \varepsilon \to 0 \).

As for \((I I_2)\),

\[
\int_\Omega u_\varepsilon(t) \Phi_\varepsilon(t) dt \to \iint_{\Omega \times Y \times Z} u_0(t, r, s) \Phi(t, r, s) dt dr ds,
\]

and

\[
\iint_{\Omega \times Y \times Z} u_0(t, r, s) \Phi(t, r, s) dt dr ds
\]

\[
= \iint_{\Omega \times Y \times Z} u_0(t, r, s) \left[ \iint_{\mathbb{R}^N \times Y} \psi_0(x, y, z) f(x + t, y + r, z + s) dx dy dz \right] dt dr ds
\]

\[
= \iint_{\Omega \times Y \times Z} \left[ \iint_{\mathbb{R}^N \times Y} u_0(t, r, s) \psi_0(x - t, y - r, z - s) dt dr ds \right] f(x, y, z) dx dy dz
\]

\[
= \iint_{\Omega \times Y \times Z} (u_0 * \psi_0)(x, y, z) f(x, y, z) dx dy dz.
\]

Thus, there is \( 0 < \alpha_2 \leq \alpha_1 \) such that

\[
\left| \int_\Omega (u_\varepsilon * \psi_0^\varepsilon) f^\varepsilon dx - \iint_{\Omega \times Y \times Z} (u_0 * \psi_0) f dx dy dz \right| \leq \frac{\eta}{2} \text{ for } E' \ni \varepsilon \leq \alpha_2. \quad (3.4)
\]

Now, let \( 0 < \varepsilon \leq \alpha_2 \) be fixed. The decomposition

\[
\int_\Omega (u_\varepsilon * v_\varepsilon) f^\varepsilon dx - \iint_{\Omega \times Y \times Z} (u_0 * v_0) f dx dy dz
\]

\[
= \int_\Omega [u_\varepsilon * (v_\varepsilon - \psi_0^\varepsilon)] f^\varepsilon dx + \iint_{\Omega \times Y \times Z} [u_0 * (\psi_0 - v_0)] f dx dy dz
\]

\[
+ \int_\Omega (u_\varepsilon * \psi_0^\varepsilon) f^\varepsilon dx - \iint_{\Omega \times Y \times Z} (u_0 * \psi_0^\varepsilon) f dx dy dz,
\]

associated to \((3.2)-(3.4)\) lead us to

\[
\left| \int_\Omega (u_\varepsilon * v_\varepsilon) f^\varepsilon dx - \iint_{\Omega \times Y \times Z} (u_0 * v_0) f dx dy dz \right| \leq C\eta \text{ for } E' \ni \varepsilon \leq \alpha_2.
\]

Here \( C \) is a positive constant independent of \( \varepsilon \). This concludes the proof. \( \square \)

In the present work, we will deal with the following time-dependent version of multiscale convergence. It has been for the first time considered by Holmbom [9] (see also [10 41 42]). It reads as: A sequence \((u_\varepsilon)_{\varepsilon > 0} \subset L^p(Q) \quad (Q = \Omega \times (0, T), \)
Let $\varepsilon > 0$. The micro-translation $r$ of the characteristic function of the open set $\Omega$ for all $f \in L^p(Q; C_{\mathrm{per}}(Y \times Z \times T))$, where $T = [0,1)$. Let $\varepsilon > 0$. Then $u_0 \in L^p(Y \times Z \times T)$. Define $\varepsilon(x,y)$ for $f \in L^p(Q; C_{\mathrm{per}}(Y \times Z \times T))$. Then $u_0 \to u_0$ reit. in $L^p(Q; C_{\mathrm{per}}(Y \times Z \times T))$. Then $v_0 \to v_0$ reit. in $L^p(\Omega \times (-a,T-a))$-weak, where $v_0(x,y,z,\tau) = u_0(x, t+a, y, z, \tau + \tau)$. (x,t,y,z,\tau) \in \Omega \times (-a,T-a) \times Y \times Z \times T$, the micro-translation $r \in T$ being a cluster point of the sequence $(\mathcal{R}_{\varepsilon}(\omega))_{\varepsilon > 0}$.

4. Derivation of the limiting problem

Our aim in this section is to pass to the limit in (1.1)-(1.8) as $\varepsilon \to 0$. Before we can do this, we need a few preliminary results.

4.1. Preliminaries. Let the functions $A_i$, $B_i$ and $\rho_j$ be as in Section 1. Assuming (A3) (see Section 1), we get that

$$A_i, B_i \in C^2(Q; L^p(\Omega \times T)^{N^2}); \quad \rho_j \in C^2(\Omega; C_{\mathrm{per}}(Y)) \quad (4.1)$$

$$\langle \chi_{Y^2}, \chi_{Z^2} \rangle \in L^2(\Omega \times Z) \quad \langle \chi_{Y^2} \rangle > 0 \quad \langle \chi_{Z^2} \rangle > 0 \quad (4.2)$$

where $\langle \chi_{Y^2} \rangle = \int_{Y^2} dy$ and $\langle \chi_{Z^2} \rangle = \int_{Z^2} dz$. Recalling that $\chi^j_\varepsilon$ (j = 1, 2) stands for the characteristic function of the open set $\Omega^j_\varepsilon$, it comes from (4.2) that $\chi^j_\varepsilon \to \chi^j$ reit. in $L^2(\Omega)$-weak, where $\chi_2(y,z) = \chi_{Y^2}(y) + (1 - \chi_{Y^2}(y)) \chi_{Z^2}(z)$ and $\chi_1(y,z) = 1 - \chi_2(y,z) = (1 - \chi_{Y^2}(y))(1 - \chi_{Z^2}(z))$. This stems from the fact that $\chi^j_\varepsilon(x) = \chi_{Y^2}(\varepsilon) + (1 - \chi_{Y^2}(\varepsilon)) \chi_{Z^2}(\varepsilon)$ and $\chi_1(x) = 1 - \chi_2(x)$ for $x \in \Omega$.

The following result holds true; see e.g. [20] for its proof.

Lemma 4.1. Let $(u_\varepsilon)_{\varepsilon > 0} \subset L^p(Q)$ be such that $u_\varepsilon \to u_0$ reit. in $L^p(Q)$-weak. Then $\chi_\varepsilon u_\varepsilon \to \chi u_0$ reit. in $L^p(Q)$-weak.

Let $h \in L^1(\Omega)$ and let $E$ be an ordinary sequence. For $j = 1, 2$, the sequence $(h\rho_j^2)_{\varepsilon \in E}$ defined by

$$(h\rho_j^2)(x) = h(x)\rho_j\left(\frac{x}{\varepsilon}\right) \quad x \in \Omega$$

lies in $L^1(\Omega)$ and multiscale converges weakly in $L^1(\Omega)$ towards the function $h \circ \rho_j \in L^1(\Omega; C_{\mathrm{per}}(Y))$ defined by

$$(h \circ \rho_j)(x,y) = h(x)\rho_j(x,y) \quad \text{for a.e.} \quad (x,y) \in \Omega \times Y;$$

see e.g., [19] Example 4.1]. Thus,

$$\int_\Omega h(x)(\chi_1^2 \rho_1 + \chi_2^2 \rho_2) \, dx$$

$$\Rightarrow \int_{\Omega \times Y \times Z} \[ \chi_1(y,z) \rho_1(x,y) + \chi_2(y,z) \rho_2(x,y) \] h(x) \, dxdydz$$
so that, as $E \ni \varepsilon \to 0$,

$$
\rho(\varepsilon) \to \int_{Y \times Z} \left[ \chi_1(y, z) \rho_1(x, y) + \chi_2(y, z) \rho_2(x, y) \right] \, dy \, dz = \rho(x) \text{ in } L^\infty(\Omega)\text{-weak}.
$$

(recall that $\rho(\varepsilon) = \chi_1(\frac{\varepsilon}{2}, \frac{\varepsilon}{2}) \rho_1(x, \frac{\varepsilon}{2}) + \chi_2(\frac{\varepsilon}{2}, \frac{\varepsilon}{2}) \rho_2(x, \frac{\varepsilon}{2})$).

But

$$
\rho(x) \geq \Lambda^{-1} \int_{Y \times Z} \left[ \chi_1(y, z) + \chi_2(y, z) \right] \, dy \, dz = \Lambda^{-1} > 0.
$$

Thus Proposition 2.2 applies and there exist a subsequence $E'$ of $E$ and a function $u_0 \in L^2(Q)^N$ such that, as $E' \ni \varepsilon \to 0$,

$$
u^\varepsilon \to u_0 \text{ in } L^2(Q)^N\text{-strong. (4.3)}
$$

We can extract by a diagonal process a subsequence of $(\nu^\varepsilon)_{\varepsilon \in E'}$ not relabeled, that weakly converges to $u_0$ in $L^2(0, T; H^1_0(\Omega)^N)$, so that $u_0 \in L^2(0, T; H^1_0(\Omega)^N)$.

Here below and in the sequel we shall use the following notation. For $w = (w_0, w_1, w_2)$ with $w_0 \in L^2(0, T; H^1_0(\Omega)^N)$, $w_1 \in L^2(Q \times T; W_{E_0}^{1,2}(Y)^N)$, and $w_2 \in L^2(Q \times Y \times T; W_{E_0}^{1,2}(Z)^N)$, we set

$$
\mathbb{D}w = \nabla w_0 + \nabla_y w_1 + \nabla_z w_2 = (\mathbb{D}_j w)_{1 \leq j \leq N}
$$

where $\mathbb{D}_j w = (\mathbb{D}_j w^k)_{1 \leq k \leq N}$ and $\mathbb{D}_j w^k = \frac{\partial w^k_0}{\partial x_j} + \frac{\partial w^k_1}{\partial y_j}$ for $w_l = (w^k_l)_{1 \leq k \leq N}$, $l = 0, 1, 2$.

Bearing this in mind, the following preliminary result holds.

**Proposition 4.1.** There exist a subsequence $E'$ of $E$ and vector functions $(u_1, v_1) \in L^2(Q \times T; W_{E_0}^{1,2}(Y)^N)^2$, $(u_2, v_2) \in L^2(Q \times Y \times T; W_{E_0}^{1,2}(Z)^N)^2$ such that, as $E' \ni \varepsilon \to 0$,

$$
\chi^\varepsilon_1 \nabla u_\varepsilon \rightharpoonup \chi_1 \mathbb{D} u \text{ reit. in } L^2(Q)^N\text{-weak} \quad (4.4)
$$

and

$$
\chi^\varepsilon_2 \nabla v_\varepsilon \rightharpoonup \chi_2 \mathbb{D} v \text{ reit. in } L^2(Q)^N\text{-weak} \quad (4.5)
$$

where $u_0$ is that function defined by (4.3), $u = (u_0, u_1, u_2)$ and $v = (u_0, v_1, v_2)$.

**Proof.** We deduce from Lemma 4.1 and the convergence result (4.3) that $\chi^\varepsilon_1 u_\varepsilon \rightharpoonup \chi_1 u_0$ reit. in $L^2(Q)^N$-weak as $E' \ni \varepsilon \to 0$, $E'$ being the same as in (4.3). Let us start by showing (4.4). The same arguments will suffice to verify (4.5). To do this, let us set $u_\varepsilon = (u^k_\varepsilon)_{1 \leq k \leq N}$ and $u_\varepsilon = (u^k_\varepsilon)_{1 \leq k \leq N}$. Fix $1 \leq i \leq N$ arbitrarily. If we denote by $\mathbb{D} u^i_\varepsilon$ the zero-extension of $\nabla u^i_\varepsilon$ on the whole $\Omega$, then

$$
\mathbb{D} u^i_\varepsilon = \chi^\varepsilon_1 \nabla u^i_\varepsilon.
$$

Thus the sequence $(\mathbb{D} u^i_\varepsilon)_{\varepsilon \in E'}$ is bounded in $L^2(Q)^N$, and hence there exist a subsequence of $E'$ (that we still denote by $E'$) and a function $w^i \in L^2(Q \times Y \times T)^N$ such that $\mathbb{D} u^i_\varepsilon \rightharpoonup w^i$ reit. in $L^2(Q)^N$-weak as $E' \ni \varepsilon \to 0$. In view of Lemma 4.1, we have $\chi^\varepsilon_1 \nabla u^i_\varepsilon \rightharpoonup \chi_1 w^i$ reit. in $L^2(Q)^N$-weak as $E' \ni \varepsilon \to 0$. It follows that $w^i = \chi_1 w^i$. Now, let $\Phi \in (C_0^\infty(Q) \otimes C_0^\infty(Y \times Z \times T))^N$ be such that $\text{div} \Phi \Phi = \text{div}_y \Phi = 0$ and $\Phi(x, t, y, z, \tau) = 0$ for $y \in Y_2$ or $z \in Z_2$ and $\Phi \cdot \nu_1 = 0$.

Then

$$
\int_Q \chi_1 \nabla u^i_\varepsilon \cdot \Phi^i \, dx \, dt = - \int_Q \chi_1 u^i_\varepsilon (\text{div} \Phi)^i \, dx \, dt
$$

and letting $E' \ni \varepsilon \to 0$ yields

$$
\iint_{Q \times Y \times Z \times T} (w^i - \chi_1 \nabla u_0) \cdot \Phi \, dx \, dy \, dz \, d\tau = 0
$$
for all $\Phi \in (C_c^\infty (Q) \otimes C_c^\infty_p (Y \times Z \times T))^N$ satisfying $\text{div}_y \Phi = \text{div}_z \Phi = 0$ and $\Phi(x,t,y,z,\tau) = 0$ for $y \in Y_2$ or $z \in Z_2$. Lemma 4.14 of [2] entails the existence of $u_1^0 \in L^2 (Q \times T; W^{1,2}_\# (Y))$ and $u_2^1 \in L^2 (Q \times Y \times T; W^{1,2}_\# (Z))$ such that $w^i = (\nabla u_1^0 + \nabla_y u_1^1 + \nabla_z u_2^1) \chi_1$. Setting $u_1 = (u_1^0)_{1 \leq i \leq N}$ and $u_2 = (u_2^1)_{1 \leq i \leq N}$, (4.14) holds true.

4.2. Homogenization result. Assume that the functions $u_0, u_1, u_2, v_1$ and $v_2$ are as in Proposition 4.1. Let $\psi_0 = (\psi_0^k)_{1 \leq k \leq N} \in C_c^\infty (Q)^N$, $\psi_1 = (\psi_1^k)_{1 \leq k \leq N}$, $\phi_1 = (\phi_{1,k})_{1 \leq k \leq N} \in (C_c^\infty (Q) \otimes C_c^\infty_p (Y \times T))^N$ and $\psi_2 = (\psi_2^k)_{1 \leq k \leq N}$, $\phi_2 = (\phi_{2,k})_{1 \leq k \leq N} \in (C_c^\infty (Q) \otimes C_c^\infty_p (Y \times Z \times T))^N$ be such that $\psi_1 (x,t,y,\tau) = 0$ for $y \in Y_2, \phi_1 (x,t,y,\tau) = 0$ for $y \in Y \setminus Y_2, \psi_2 (x,t,y,z,\tau) = 0$ for $y \in Y_2$ or $z \in Z_2$ and $\phi_2 (x,t,y,z,\tau) = 0$ for $y \in Y \setminus Y_2$ and $z \in Z \setminus Z_2$. Set $\Psi = (\psi_0, \psi_1, \psi_2)$ and $\Phi = (\psi_0, \phi_1, \phi_2)$, and define $\Psi_\varepsilon = \psi_0 + \varepsilon \psi_1^\varepsilon + \varepsilon^2 \psi_2^\varepsilon$ and $\Phi_\varepsilon = \psi_0 + \varepsilon \phi_1^\varepsilon + \varepsilon^2 \phi_2^\varepsilon$ by

$$
\Psi_\varepsilon (x,t) = \psi_0 (x,t) + \varepsilon \psi_1 (x,t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon}) + \varepsilon^2 \psi_2 (x,t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon}),
$$

$$
\Phi_\varepsilon (x,t) = \psi_0 (x,t) + \varepsilon \phi_1 (x,t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon}) + \varepsilon^2 \phi_2 (x,t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon}),
$$

for $(x,t) \in Q$.

Then $\Psi_\varepsilon$ and $\Phi_\varepsilon \in C_c^\infty (Q)^N$ and further $\gamma_\varepsilon^\Psi \Psi_\varepsilon = \gamma_\varepsilon^\Phi \Phi_\varepsilon$ on $\Gamma_{1,2}$ (we have in fact $\varepsilon \psi_1^\varepsilon + \varepsilon^2 \psi_2^\varepsilon \in C_c^\infty (\Omega_1 \times (0,T))^N$ and $\varepsilon \phi_1^\varepsilon + \varepsilon^2 \phi_2^\varepsilon \in C_c^\infty (\Omega_2 \times (0,T))^N$) and $\gamma_\varepsilon^\Psi \Psi_\varepsilon = 0$ on $\partial \Omega_1 \cap \partial \Omega, \gamma_\varepsilon^\Phi \Phi_\varepsilon = 0$ on $\partial \Omega_2 \cap \partial \Omega$. Thus the vector-function $(\Psi_\varepsilon, \Phi_\varepsilon)$ can be taken as a test function in the variational form of (4.1)–(4.3):

$$
- \int_Q \left( \chi_1^\varepsilon \rho_1^\varepsilon u_x \cdot \frac{\partial \Psi_\varepsilon}{\partial t} + \chi_2^\varepsilon \rho_2^\varepsilon v_x \cdot \frac{\partial \Phi_\varepsilon}{\partial t} \right) dxdt + \int_Q \left( \chi_1^\varepsilon (A_0^\varepsilon \nabla u_x + A_1^\varepsilon \ast \nabla u_x) \cdot \nabla \Psi_\varepsilon dxdt 
+ \int_Q \chi_2^\varepsilon (B_0^\varepsilon \nabla v_x + B_1^\varepsilon \ast \nabla v_x) \cdot \nabla \Phi_\varepsilon dxdt 
- \int_Q (\chi_1^\varepsilon \rho_1 \div \Psi_\varepsilon + \chi_2^\varepsilon \rho_2 \div \Phi_\varepsilon) dxdt 
= \int_Q (\chi_1^\varepsilon \rho_1^\varepsilon f_1 \cdot \Psi_\varepsilon + \chi_2^\varepsilon \rho_2^\varepsilon f_2 \cdot \Phi_\varepsilon) dxdt.
$$

(4.6)

Our aim is to pass to the limit in (4.6) as $E' \ni \varepsilon \to 0$ ($E'$ being as in Proposition 4.1). Considering the first integral term in the left-hand side of (4.6), we have

$$
\int_Q \left( \chi_1^\varepsilon \rho_1^\varepsilon u_x \cdot \frac{\partial \Psi_\varepsilon}{\partial t} + \chi_2^\varepsilon \rho_2^\varepsilon v_x \cdot \frac{\partial \Phi_\varepsilon}{\partial t} \right) dxdt

\to \int_Q \left( \int_{Y \times Z} (\chi_1 \rho_1 + \chi_2 \rho_2) dydz \right) u_0 \cdot \frac{\partial \psi_0}{\partial t} dxdt.
$$

This stems from (4.3) associated to Lemma 4.1. Next, it is easy to see that

$$
\nabla \Psi_\varepsilon \to \mathbb{D} \Psi \text{ reit. in } L^2 (Q)^{N^2}\text{-strong},
$$

$$
\nabla \Phi_\varepsilon \to \mathbb{D} \Phi \text{ reit. in } L^2 (Q)^{N^2}\text{-strong}.
$$
Appealing to (4.4)-(4.5), it follows from a suitable statement of Corollary 3.1 that

\[ \chi_1^e \nabla u_\varepsilon \cdot \nabla \Psi_\varepsilon \to \chi_1^e \mathbb{D} u \cdot \mathbb{D} \Psi \text{ reit. in } L^2(Q)^{N^2}-\text{weak} \]

and

\[ \chi_2^e \nabla v_\varepsilon \cdot \nabla \Phi_\varepsilon \to \chi_2^e \mathbb{D} v \cdot \mathbb{D} \Phi \text{ reit. in } L^2(Q)^{N^2}-\text{weak}. \]

Thus using \( A_0 \) and \( B_0 \) as test functions entails

\[
\int_Q (\chi_1^e A_0^* \nabla u_\varepsilon \cdot \nabla \Psi_\varepsilon + \chi_2^e B_0^* \nabla v_\varepsilon \cdot \nabla \Phi_\varepsilon) dx \, dt \\
\to \int_{Q \times Y} \int_{Z \times T} (\chi_1 A_0^* u_\varepsilon \cdot \mathbb{D} \Psi + \chi_2 B_0^* v_\varepsilon \cdot \mathbb{D} \Phi) \, dx \, dt \, dy \, dz \, d\tau.
\]

Now, for the terms involving convolution, it is a fact that \( A_1^* \to A_1 \) reit. in \( L^1(Q)^{N^2} \)-strong. Thus, owing to Theorem 3.5 we get

\[ A_1^* \ast \chi_1^e \nabla u_\varepsilon \to A_1 \ast \chi_1^e \mathbb{D} u \text{ reit. in } L^2(Q)^{N^2}-\text{weak}. \]

We also have

\[ B_1^* \ast \chi_2^e \nabla v_\varepsilon \to B_1 \ast \chi_2^e \mathbb{D} v \text{ reit. in } L^2(Q)^{N^2}-\text{weak}. \]

Therefore the same reasoning conducted before leads to

\[
\int_Q \int_{Q \times Y} \int_{Z \times T} [\chi_1 (A_1 \ast \mathbb{D} u_\varepsilon \cdot \mathbb{D} \Psi + \chi_2 (B_1 \ast \mathbb{D} v_\varepsilon) \cdot \mathbb{D} \Phi)] dx \, dt \, dy \, dz \, d\tau.
\]

Now, dealing with the terms with pressure, we have

\[
\int_Q (\chi_1^e \pi_\varepsilon \text{ div } \Psi_\varepsilon + \chi_2^e \varrho_\varepsilon \text{ div } \Phi_\varepsilon) dx \, dt \\
= \int_Q \pi_\varepsilon \text{ div } \psi_\varepsilon dx \, dt + \int_Q (\chi_1^e p_\varepsilon (\text{div}_y \psi_1)^\varepsilon + \chi_2^e \varrho_\varepsilon (\text{div}_y \phi_1)^\varepsilon) dx \, dt \\
+ \int_Q (\chi_1^e p_\varepsilon (\text{div}_z \psi_2)^\varepsilon + \chi_2^e \varrho_\varepsilon (\text{div}_z \phi_2)^\varepsilon) dx \, dt \\
+ \varepsilon \int_Q (\chi_1^e p_\varepsilon (\text{div}_y \psi_1)^\varepsilon + \chi_2^e \varrho_\varepsilon (\text{div}_y \phi_1)^\varepsilon) dx \, dt \\
+ \varepsilon^2 \int_Q (\chi_1^e p_\varepsilon (\text{div}_z \psi_2)^\varepsilon + \chi_2^e \varrho_\varepsilon (\text{div}_z \phi_2)^\varepsilon) dx \, dt
\]

where \( \text{div}_y \) (resp. \( \text{div}_z \)) stands for the divergence operator in \( \mathbb{R}^N \) (resp. \( \mathbb{R}^N \)), and \( (\text{div}_y \psi_1)^\varepsilon \) and \( (\text{div}_y \phi_1)^\varepsilon \) are defined for \( (x,t) \in Q \) by

\[
(\text{div}_y \psi_1)^\varepsilon(x,t) = (\text{div}_y \psi_1) \left( x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon} \right), \quad (\text{div}_y \phi_1)^\varepsilon(x,t) = (\text{div}_y \phi_1) \left( x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon} \right),
\]

and where

\[ \pi_\varepsilon = \chi_1^e p_\varepsilon + \chi_2^e \varrho_\varepsilon \] (4.7)

is the global pressure. By virtue of (2.8), the sequence \((\pi_\varepsilon)_{\varepsilon \in E^r}\) is bounded in \( L^2(Q) \), so that there exist \( p \in L^2(Q), p_1, q_1 \in L^2(Q \times Y \times Z \times T) \) such that, up to
a subsequence of $E'$, $\pi_\varepsilon \to p$ in $L^2(Q)$-weak, $\chi_1^\varepsilon p_\varepsilon \to p_1$ reit. in $L^2(Q)$-weak and $\chi_2^\varepsilon q_\varepsilon \to q_1$ reit. in $L^2(Q)$-weak. Thus

$$\int_Q (\chi_1^\varepsilon p_\varepsilon \div \Psi_\varepsilon + \chi_2^\varepsilon q_\varepsilon \div \Phi_\varepsilon)dxdt \to \int_Q \left( \int_{Y \times Z} (\chi_1^\varepsilon p_1 \div_y \psi_1 + \chi_2^\varepsilon q_1 \div_y \phi_1) \right) dxdt dydz \cdot \psi_0 dxdt.$$

Finally

$$\int_Q (\chi_1^\varepsilon \rho_1^\varepsilon f_1 \cdot \Psi_\varepsilon^\ast + \chi_2^\varepsilon \rho_2^\varepsilon f_2 \cdot \Phi_\varepsilon^\ast)dxdt \to \int_Q \left( \int_{Y \times Z} (\chi_1^\varepsilon \rho_1 f_1 + \chi_2^\varepsilon \rho_2 f_2) \right) dydz \cdot \psi_0 dxdt.$$

In order to formulate the result that we have just proved, we note that $\div \mathbf{u}^\varepsilon = 0$ in $\Omega$ since $\div \mathbf{u}_\varepsilon = 0$ in $\Omega_1^\varepsilon$ and $\div \mathbf{v}_\varepsilon = 0$ in $\Omega_2^\varepsilon$. This implies that $\div \mathbf{u}_0 = 0$ in $\Omega$, $\div_y \mathbf{u}_1 = 0$ in $Y \setminus Y_2$, $\div_y \mathbf{u}_2 = 0$ in $Y$ and $\div_y \mathbf{u}_1 = 0$ in $Y_2$ and $\div_z \mathbf{v}_2 = 0$ if $y \in Y_2$ or $z \in Z_2$. So we set

$$\mathbb{F}^1 = \{(u_0, u_1, v_1, u_2, v_2) \in F^1 : \div u_0 = 0 \text{ in } \Omega, \div_y u_1 = 0 \text{ in } Y \setminus Y_2, \\ \div_y v_1 = 0 \text{ in } Y_2, \div_z u_2 = 0 \text{ in } Y \times Z \setminus (Y_2 \times Z_2) \text{ and } \div_z v_2 = 0 \text{ in } \{(y, z) \in Y \times Z : y \in Y_2 \text{ or } z \in Z_2\} \};$$

$$\rho(x) = \int_{Y \times Z} [\chi_1(y, z) \rho_1(x, y) + \chi_2(y, z) \rho_2(x, y)] dydz, \quad x \in \overline{\Omega} \quad (4.8)$$

and

$$\mathbf{f}(x, t) = \int_{Y \times Z} (\chi_1(y, z) \rho_1(x, y) f_1(x, t) + \chi_2(y, z) \rho_2(x, y) f_2(x, t)) dydz, \quad \text{a.e. } (x, t) \in Q$$

where

$$F^1 = L^2(0, T; H^1_0(\Omega)^N) \times [L^2(Q \times T; W^{1, 2}_\#(Y))^2 \times [L^2(Q \times Y \times T; W^{1, 2}_\#(Z))^2$$

and

$$\mathcal{F} = C^\infty_0(Q)^N \times [(C^\infty_0(Q) \otimes C^\infty_{\mathrm{per}}(Y \times T))^N]^2 \times [(C^\infty_0(Q) \otimes C^\infty_{\mathrm{per}}(Y \times Z \times T))^N]^2$$

its smooth counterpart. We have that $\rho \in \mathcal{C}^{\infty}(\overline{\Omega})$ and $\mathbf{f} \in L^2(Q)^N$. We have just proved the following result.
Theorem 4.1. The vector function \((u_0, u_1, v_1, u_2, v_2) \in \mathbb{F}^1\) solves the variational problem

\[
\begin{align*}
-\int_Q \rho u_0 \cdot \frac{\partial \psi_0}{\partial t} dx dt &+ \iint_{Q \times Y \times Z \times T} \chi_1 [A_0 \nabla u + A_1 \ast \nabla u] \cdot \nabla \psi_0 dx dt dy dz d\tau \\
+ \iint_{Q \times Y \times Z \times T} \chi_2 [B_0 \nabla v + B_1 \ast \nabla v] \cdot \nabla \psi_0 dx dt dy dz d\tau \\
- \int_Q p \div \psi_0 dx dt &- \iint_{Q \times Y \times Z \times T} (\chi_1 p_1 \div y \psi_1 + \chi_2 q_1 \div y \phi_1) dx dt dy dz d\tau \\
- \iint_{Q \times Y \times Z \times T} (\chi_1 p_1 \div y \psi_2 + \chi_2 q_1 \div y \phi_2) dx dt dy dz d\tau &= \int_Q f \cdot \psi_0 dx dt
\end{align*}
\]

for all \((\psi_0, \psi_1, \phi_1, \psi_2, \phi_2) \in \mathcal{F}^\infty\) with \(\psi_1 = 0\) for \(y \in Y_2\), \(\phi_1 = 0\) for \(y \in Y_2\) or \(z \in Z_2\) and \(\phi_2 = 0\) for \(y \notin Y_2\) and \(z \notin Z_2\).

Our next purpose is to find the equation satisfied by the function \(u_0\). To do so, we need to construct the effective homogenized viscosity tensor. Before we can do that, let us however recall that the functions \(\chi_1\) and \(\chi_2\) are expressed as follows:

\[
\begin{align*}
\chi_2(y, z) &= \chi_{Y_2}(y) + (1 - \chi_{Y_2}(y))\chi_{Z_2}(z) \\
\chi_1(y, z) &= 1 - \chi_2(y, z) = (1 - \chi_{Y_2}(y))(1 - \chi_{Z_2}(z)) \text{ for } (y, z) \in Y \times Z.
\end{align*}
\]

With this in mind, if in (4.9) we choose consecutively the functions \((\psi_0, \psi_1, \phi_1, \psi_2, \phi_2) \in \mathcal{F}^\infty\) such that: 1) \(\psi_1 = \phi_1 = \psi_2 = \phi_2 = 0\), 2) \(\psi_0 = \phi_1 = \psi_2 = \phi_2 = 0\), 3) \(\psi_0 = \psi_1 = \phi_1 = \phi_2 = 0\) and 4) \(\psi_0 = \psi_1 = \phi_1 = \psi_2 = 0\), then we get the system consisting of problems (4.11)-(4.15) below:

\[
\begin{align*}
-\int_Q \rho u_0 \cdot \frac{\partial \psi_0}{\partial t} dx dt &+ \iint_{Q \times Y \times Z \times T} \chi_1 (A_0 \nabla u + A_1 \ast \nabla u) \cdot \nabla \psi_0 dx dt dy dz d\tau \\
+ \iint_{Q \times Y \times Z \times T} \chi_2 (B_0 \nabla v + B_1 \ast \nabla v) \cdot \nabla \psi_0 dx dt dy dz d\tau - \int_Q p \div \psi_0 dx dt &= \int_Q f \cdot \psi_0 dx dt \quad \text{for all } \psi_0 \in C_0^\infty(Q)^N,
\end{align*}
\]

\[
\begin{align*}
\iint_{Q \times Y \times Z \times T} \chi_1 (A_0 \nabla u + A_1 \ast \nabla u) \cdot \nabla y \psi_1 dx dt dy dz d\tau \\
- \iint_{Q \times Y \times Z \times T} \chi_1 p_1 \div y \psi_1 dx dt dy dz d\tau &= 0
\end{align*}
\]

for all \(\psi_1 \in (C_0^\infty(Q) \otimes C^\infty(Y \times T))^N\) with \(\psi_1 = 0\) for \(y \in Y_2\).
\[ \begin{aligned}
&\iint_{Q \times Y \times T} \chi_1 (A_0 \nabla u + A_1 * \nabla u) \cdot \nabla_z \psi_2 dxdt \, dy \, dz \, d\tau \\
&\quad - \iint_{Q \times Y \times T} \chi_1 p_1 \div_z \psi_2 dxdt \, dy \, dz \, d\tau = 0
\end{aligned} \] (4.13)

for all \( \psi_2 \in (C_0^\infty (Q) \otimes C_{\text{per}}^\infty (Y \times Z \times T))^N \) with \( \psi_2 = 0 \)

for \( y \in Y_2 \) or \( z \in Z_2 \),

\[ \begin{aligned}
&\iint_{Q \times Y \times T} \chi_2 (B_0 \nabla v + B_1 * \nabla v) \cdot \nabla_y \phi_1 dx \, dt \, dy \, dz \, d\tau \\
&\quad - \iint_{Q \times Y \times T} \chi_2 q_1 \div_y \phi_1 dx \, dt \, dy \, dz \, d\tau = 0
\end{aligned} \] (4.14)

for all \( \phi_1 \in (C_0^\infty (Q) \otimes C_{\text{per}}^\infty (Y \times T))^N \) with \( \phi_1 = 0 \) for \( y \in Y \setminus Y_2 \),

and

\[ \begin{aligned}
&\iint_{Q \times Y \times T} \chi_2 (B_0 \nabla \bar{w} + B_1 * \nabla \bar{w}) \cdot \nabla_z \phi_2 dx \, dt \, dy \, dz \, d\tau \\
&\quad - \iint_{Q \times Y \times T} \chi_2 q_1 \div_z \phi_2 dx \, dt \, dy \, dz \, d\tau = 0
\end{aligned} \] (4.15)

for all \( \phi_2 \in (C_0^\infty (Q) \otimes C_{\text{per}}^\infty (Y \times Z \times T))^N \) with \( \phi_2 = 0 \)

for \( y \notin Y_2 \) and \( z \notin Z_2 \).

Conversely, since (4.11)-(4.15) are made of linear equations, summing them up we get (4.9). Thus, (4.9) is equivalent to (4.11)-(4.15).

This being so, let us observe that the problems (4.12) (resp. (4.13)) and (4.14) (resp. (4.15)) are very similar, so that the analysis that will be made for the couple (4.12)-(4.13), will be exactly the same for the couple (4.14)-(4.15), and will therefore be omitted for the latter couple. With this in mind, let us first and foremost deal with (4.13). If in (4.13) we choose \( \psi_2(x,t,y,z,\tau) = \varphi(x,t)\theta(y)w(z)\chi(\tau) \) with \( \varphi \in C_0^\infty (Q) \), \( \theta \in C_{\text{per}}^\infty (Y) \), \( w \in C_{\text{per}}^\infty (Z)^N \), \( \chi \in C_{\text{per}}^\infty (T) \) and \( \theta = 0 \) in \( Y_2 \) or \( w = 0 \) in \( Z_2 \), then (4.13) becomes (owing to (4.10))

\[ \int_{Y \times Z} (1 - \chi_{Y_2}(y))(1 - \chi_{Z_2}(z))((A_0 \nabla u + A_1 * \nabla u) \cdot \nabla_z w - p_1 \div_z w) \theta \, dy \, dz = 0 \]

or equivalently,

\[ \int_{Z \setminus Z_2} (A_0 \nabla u + A_1 * \nabla u) \cdot \nabla_z w \, dz - \int_{Z \setminus Z_2} p_1 \div_z w \, dz = 0 \quad \text{a.e. in } Q \times (Y \setminus Y_2) \times T. \] (4.16)

This being so, let \( \mathcal{V}_{2,\text{div}} = \{ \psi \in C_{\text{per}}^\infty (Z)^N : \div_z \psi = 0 \text{ in } Z \setminus Z_2 \text{ and } \psi = 0 \text{ in } Z_2 \} \), and define the space \( B^1_{\text{per}}(Z \setminus Z_2) \) to be the strong closure in \( W^{1,2}_{\text{per}}(Z)^N \) of \( \mathcal{V}_{2,\text{div}} \).

Next, let \( \xi \in \mathbb{R}^N \) and consider the variational cell problem for \( u_2 \):

\[ \begin{aligned}
&\text{Find } u^2(\xi) \in B^1_{\text{per}}(Z \setminus Z_2) \text{ such that } \\
&\int_{Z \setminus Z_2} (A_0 [\xi + \nabla_z u^2(\xi)] + A_1 * (\xi + \nabla_z u^2(\xi))) \cdot \nabla_z w \, dz = 0
\end{aligned} \] (4.17)

for all \( w \in \mathcal{V}_{2,\text{div}} \).
Then (4.17) is the equivalent version of (4.16) but with test functions $w$ taken in $V_{2,\text{div}}$. In view of the properties of the matrices $A_0$ and $A_1$, if we proceed exactly as in the proof of Theorem 2.1, then we infer from [24, Theorem 3.2] the existence of $u^2(\xi)$ solution (4.17) which is unique up to an additive constant. On the other hand, if in (4.17) we choose $\xi = \nabla u_0(x,t) + \nabla_y u_1(x,t,y,\tau)$ for a fixed $(x,t,y,\tau) \in Q \times (Y \setminus Y_2) \times T$ and compare the resulting equation with (4.16) (for test functions taken in $V_{2,\text{div}}$), then we get from the uniqueness argument that $u_2 = u^2(\nabla u_0 + \nabla_y u_1)$, where the right-hand side of the preceding equality stands for the function $(x,t,y,\tau) \mapsto u^2(\nabla u_0(x,t) + \nabla_y u_1(x,t,y,\tau))$ from $Q \times (Y \setminus Y_2) \times T$ into $B_{1,2}^0(Z \setminus Z_2)$.

Let us now consider the variational problem for (4.12). If we define the matrices $C_0$ and $C_1$ by setting (for $\xi \in \mathbb{R}^{N^2}$)

$$C_0 \xi = \int_Z (1 - \chi_{Z_2}(z)) A_0(\xi + \nabla_z u^2(\xi))dz$$

and

$$C_1 \xi = \int_Z (1 - \chi_{Z_2}(z))(A_1 * * (\xi + \nabla_z u^2(\xi)))dz = A_1 * * \int_Z (1 - \chi_{Z_2}(z))(\xi + \nabla_z u^2(\xi))dz$$

for a.e. $(x,t,y,\tau) \in Q \times \mathbb{R}^{N+1}$, then we see that $u_1(x,t,\cdot,\tau)$ is the solution to the equation

$$\int_Y (1 - \chi_{Y_2}(y))[C_0[\nabla u_0 + \nabla_y u_1] + C_1 * * (\nabla u_0 + \nabla_y u_1)] \cdot \nabla_y w dy = 0$$

for all $w \in V_{1,\text{div}} = \{ \psi \in C^\infty_0(Y)^N : \text{div}_z \psi = 0 \text{ in } Y \setminus Y_2 \text{ and } \psi = 0 \text{ in } Y_2 \}$. So, by fixing once again $\xi \in \mathbb{R}^{N^2}$, a similar study conducted for (4.12) reveals that the cell problem

$$\begin{align*}
\text{Find } & u^1(\xi) \in B_{1,2}^{1,2}(Y \setminus Y_2) \text{ such that } \\
& \int_{Y \setminus Y_2} \left(C_0[\xi + \nabla_y u^1(\xi)] + C_1 * * (\xi + \nabla_y u^1(\xi))\right) \cdot \nabla_y w dz = 0
\end{align*}$$

for all $w \in V_{1,\text{div}}$ possesses a unique solution in $B_{1,2}^{1,2}(Y \setminus Y_2)$ (the strong closure in $W_{\text{per}}^{1,2}(Y)^N$ of $V_{1,\text{div}}$) up to a constant. One also obtains that $u_1 = u^1(\nabla u_0)$ in $Q \times T$. We also set, for $\xi \in \mathbb{R}^{N^2}$ and $(x,t) \in Q$,

$$D_0 \xi = \int_{Y \times T} (1 - \chi_{Y_2}(y)) C_0[\xi + \nabla_y u^1(\xi)]dyd\tau, \quad (4.18)$$

$$D_1 \xi = \int_{Y \times T} (1 - \chi_{Y_2}(y)) C_1[\xi + \nabla_y u^1(\xi)]dyd\tau. \quad (4.19)$$

Then we easily see that, for $(x,t) \in Q$,

$$D_0 \xi = \iint_{Y \times Z \times T} \chi_1 A_0[\xi + \nabla_y u^1(\xi)] + \nabla_z u^2(\xi + \nabla_y u^1(\xi))dydzd\tau$$
and
\[ D_1 \xi = \iint_{Y \times Z \times T} \chi_1 \{ A_1 * (\xi + \nabla_y v^1(\xi)) + \nabla_z u^2(\xi + \nabla_y v^1(\xi)) \} \, dydzd\tau. \]

Similar arguments used for (4.14) and (4.15) lead to the existence of unique \( v^1(\xi) \) and \( v^2(\xi) \) (for \( \xi \in \mathbb{R}^{2N} \)), solutions to the cell problems for (4.14) and (4.15) respectively, so that \( \mathbf{v}_2 = \nabla \mathbf{v}_0 + \nabla_y \mathbf{v}_1 \) and \( \mathbf{v}_1 = \nabla \mathbf{v}_0 \). We also define the corresponding homogenized matrices
\[ E_0 \xi = \iint_{Y \times Z \times T} \chi_2 B_0[\xi + \nabla_y v^1(\xi)] + \nabla_z u^2[\xi + \nabla_y v^1(\xi)] \, dydzd\tau, \quad (x, t) \in Q \]
and
\[ E_1 \xi = \iint_{Y \times Z \times T} \chi_2 B_1[\xi + \nabla_y v^1(\xi)] + \nabla_z u^2[\xi + \nabla_y v^1(\xi)] \, dydzd\tau, \quad (x, t) \in Q. \]

It is worth noticing that the matrices \( D_0 \) and \( D_1 \) (and the same for \( E_0 \) and \( E_1 \)) are defined by \( D_0 = (d^0_{ij})_{1 \leq i, j \leq N} \) and \( D_1 = (d^1_{ij})_{1 \leq i, j \leq N} \) where the \( d^0_{ij} \) and \( d^1_{ij} \) are obtained by choosing in (4.13) and (4.19) \( \xi = (\delta_{ij})_{1 \leq i, j \leq N} \) (the identity matrix), \( \delta_{ij} \) being the Kronecker delta. Finally, set \( \mathbf{A}_0 = D_0 + E_0 \) and \( \mathbf{A}_1 = D_1 + E_1 \), that is, for any \( \xi \in \mathbb{R}^{2N} \), \( \mathbf{A}_0 \xi = D_0 \xi + E_0 \xi \) and \( \mathbf{A}_1 \xi = D_1 \xi + E_1 \xi \). Set also
\[ m_c = \int_{y_2} dy \quad \text{and} \quad m_p = \int_{z_2} dz. \]
The positive constants \( m_c \) and \( m_p \) are the porosity of the crack and pore spaces respectively. The function \( \rho \) defined by (4.15) is the effective homogenized density while the matrices \( \mathbf{A}_0 \) and \( \mathbf{A}_1 \) are the effective homogenized elasticity tensors which depend continuously on \( (x, t) \in Q \) as seen in the next result whose easy and classical proof is left to the reader.

**Proposition 4.2.** It holds that
(i) \( \mathbf{A}_i \ (i = 0, 1) \) are symmetric and further \( \mathbf{A}_i \in \mathcal{C}(Q)^{N^2} \);
(ii) \( \mathbf{A}_0 \cdot \lambda \geq \alpha |\lambda|^2 \) for all \((x, t) \in Q\) and all \( \lambda \in \mathbb{R}^N \), where \( \alpha \) is the same as in assumption (A1);
(iii) \( \rho \in \mathcal{C}(\overline{Q}) \) and further \( \Lambda^{-1} \leq \rho(x) \leq \Lambda \) for all \( x \in \overline{Q} \).

Now we consider the anisotropic nonlocal Stokes system
\[
\begin{aligned}
\frac{\partial \mathbf{u}_0}{\partial t} - \text{div} \left( \mathbf{A}_0 \nabla \mathbf{u}_0 + \int_0^t \mathbf{A}_1(x, t - \tau) \nabla \mathbf{u}_0(x, \tau) \, d\tau \right) + \nabla p &= \mathbf{f} \quad \text{in} \ Q \\
\text{div} \mathbf{u}_0 &= 0 \quad \text{in} \ Q \\
\mathbf{u}_0 &= 0 \quad \text{on} \ \partial \Omega \times (0, T) \\
\mathbf{u}_0(x, 0) &= \left(1 - m_c\right)(1 - m_p)\mathbf{u}^0(x) + \left(m_c + m_p(1 - m_c)\right)\mathbf{v}^0(x), \ x \in \Omega.
\end{aligned}
\]
In (4.20) the function \( \mathbf{u}_0 \) is the strong limit of the global velocity field \( \mathbf{u}^e = \chi_1^e \mathbf{u}_e + \chi_2^e \mathbf{v}_e \) while \( p \) is the weak limit of the global pressure \( \pi^e = \chi_1^e \pi_e + \chi_2^e \pi_v \). Moreover, since \( \int_\Omega \mathbf{t} \cdot \mathbf{d}x = \int_\Omega q \, dx = 0 \), we have \( \int_\Omega \pi_e \, dx = 0 \), so that \( \int_\Omega \pi_v \, dx = 0 \).

Now, in view of (i)-(iii) in Proposition 4.2 and owing to the fact that \( \int_\Omega p \, dx = 0 \), we can argue as in the proof of Theorem 2.1 to show that Problem (4.20) possesses a
unique solution \((u_0, p)\) such that \(u_0 \in L^2(0, T; H^1_0(\Omega))^N\) and \(p \in L^2(0, T; L^2(\Omega)/\mathbb{R})\) where \(L^2(\Omega)/\mathbb{R}\) stands for the space of \(v \in L^2(\Omega)\) satisfying \(\int_{\Omega} v dx = 0\). We can therefore state the main homogenization result.

**Theorem 4.2.** Assume (A1)-(A3) hold. For any \(\varepsilon > 0\), let \(u_\varepsilon\) (resp. \(v_\varepsilon\)) the velocity field of the fluid in \(U^1\) (resp. \(\Omega_2\)) be given by the system (1.1)-(1.8). Let \(\pi_\varepsilon\) be the global pressure given by (1.7). There exist \(u \in L^{\infty}(0, T; L^2(\Omega)^N)\) – the velocity of the fluid in the skeleton, \(v \in L^{\infty}(0, T; L^2(\Omega)^N)\) – the velocity of the fluid in the pores and cracks system, and \(p \in L^2(0, T; L^2(\Omega)/\mathbb{R})\) such that, as \(\varepsilon \to 0\), \(\chi_1^\varepsilon u_\varepsilon \to u\) in \(L^2(Q)^N\)-weak, \(\chi_2^\varepsilon v_\varepsilon \to v\) in \(L^2(Q)^N\)-weak and \(\pi_\varepsilon \to p\) in \(L^2(Q)\)-weak. Moreover \(u = (1-m_c)(1-m_p)u_0\) and \(v = v_c + v_p\) where \(v_c = m_c u_0\) is the velocity of the fluid in the crack space and \(v_p = (1-m_c)m_p u_0\) is the velocity of the fluid in the pore space, and \(m_p\) (resp. \(m_c\)) is the porosity of the pore (resp. crack) space and \((u_0, p)\) is the unique solution to Problem (4.20).

**Proof.** First, if we substitute in (4.11) \(u_1 = u^1(\nabla u_0), u_2 = u^2(\nabla u_0 + \nabla_y u_1), v_1 = v^1(\nabla u_0)\) and \(v_2 = v^2(\nabla u_0 + \nabla_y v_1)\), we get, after mere computations, the variational formulation of (4.20). Moreover, owing to the uniqueness of the solution to (4.20), we infer that the whole sequence \((u^\varepsilon, \pi_\varepsilon)\) (where \(u^\varepsilon\) is the global velocity field defined by (2.10)) converges as \(\varepsilon \to 0\), in the following way: \(u^\varepsilon \to u_0\) in \(L^2(Q)^N\)-strong and \(\pi_\varepsilon \to p\) in \(L^2(Q)\)-weak. Second, because of both Lemma 4.1 and the convergence result (4.3), we have \(\chi_1^\varepsilon u_\varepsilon \to \chi_1^\varepsilon u_0 \) reit. in \(L^2(Q)^N\)-weak when \(\varepsilon \to 0\), hence \(\chi_1^\varepsilon u_\varepsilon \to \left(\int_{Y \times Z} \chi_1 dydz\right) u_0\) in \(L^2(Q)^N\)-weak, and

\[
\int_{Y \times Z} \chi_1 dydz = (1-m_c)(1-m_p).
\]

Also, as \(\varepsilon \to 0\), \(\chi_2^\varepsilon v_\varepsilon \to \chi_2^\varepsilon u_0\) reit. in \(L^2(Q)^N\)-weak, hence \(\chi_2^\varepsilon v_\varepsilon \to \left(\int_{Y \times Z} \chi_2 dydz\right) u_0\) in \(L^2(Q)^N\)-weak, and

\[
\int_{Y \times Z} \chi_2 dydz = m_c + (1-m_c)m_p.
\]

We may therefore set \(u = (1-m_c)(1-m_p)u_0\), \(v = v_c + v_p\) with \(v_c = m_c u_0\) and \(v_p = (1-m_c)m_p u_0\). The fact that \(u\) and \(v\) belong to \(L^\infty(0, T; L^2(\Omega)^N)\) follows from both the boundedness of the sequences \((\chi_1^\varepsilon u_\varepsilon)_{\varepsilon > 0}\) and \((\chi_2^\varepsilon v_\varepsilon)_{\varepsilon > 0}\) in \(L^\infty(0, T; L^2(\Omega)^N)\) (see Lemma 2.2; see especially (2.6) therein) and uniqueness of the weak limit. This concludes the proof of the theorem. \(\square\)

**Remark 4.1.** We see from the statement of Theorem 4.2 that the limiting velocity in skeleton as well as in pores and cracks are both proportional.

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