K-THEORY OF SMOOTH COMPLETE TORIC VARIETIES AND RELATED SPACES

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Abstract. The $K$-rings of non-singular complex projective varieties as well as quasitoric manifolds were described in terms of generators and relations in an earlier work of the author with V. Uma. In this paper we obtain a similar description for complete non-singular toric varieties. Indeed, our approach enables us to obtain such a description for the more general class of torus manifolds with locally standard torus action and orbit space a homology polytope.

1. Introduction

This paper consists of two parts, the first of which gives a description of $K$-ring of a non-singular complete toric variety in terms of generators and relations. In the second part, which subsumes the first, we obtain the same result for the class of torus manifolds with locally standard action whose orbit space is a homology polytope. Although the proofs in both cases involve the same steps, they are easier to establish or well known in the context of toric varieties and helps to focus ideas. Besides, the language used in the two proofs are different. For these reasons, we have separated out the algebraic geometric case from the topological one, taking care to avoid unnecessary repetitions.

Let $N \cong \mathbb{Z}^n$ and let $N_R := N \otimes_{\mathbb{Z}} \mathbb{R}$. We shall denote by $M$ the dual lattice $\text{Hom}_\mathbb{Z}(N, \mathbb{Z})$ and by $\mathbb{T}$ the $n$-dimensional complex algebraic torus whose coordinate ring is the group algebra $\mathbb{C}[M]$. The lattice $M$ is identified with the group of characters of $\mathbb{T}$ and $N$ with the group of 1-parameter subgroups of $\mathbb{T}$. Let $\Delta$ be a fan in $N$ and let $X(\Delta)$ the corresponding complex $\mathbb{T}$-toric variety. When $\Delta$ is complete the variety $X(\Delta)$ is complete. When $\Delta$ is regular, (i.e the cones of $\Delta$ are generated by part of a $\mathbb{Z}$-basis for $N$) the variety $X(\Delta)$ is smooth. We refer the reader to [11] for an introduction to toric varieties.

Let $\Delta$ be a complete regular fan in $N$ and let $X := X(\Delta)$ be the corresponding non-singular complete variety. Recall that one has an inclusion-reversing correspondence between cones of $\Delta$ and $\mathbb{T}$-orbit closures of $X$. In fact the orbit closure $V(\sigma)$ of a cone $\sigma \in \Delta$ equals the union of $\mathbb{T}$-orbits of a certain (closed) points $p_\gamma \in X$ as $\gamma$ varies over those cones of $\Delta$ which contain $\sigma$. In particular, if $\sigma, \sigma' \in \Delta$ are not faces of a cone $\gamma \in \Delta$,
then \( V(\sigma) \cap V(\sigma') = \emptyset \). The dimension of \( V(\sigma) \) equals the co-dimension of \( \sigma \) in \( N_\mathbb{R} \). The orbit closures are themselves toric varieties (under the action of suitable quotients of \( \mathbb{T} \)) and are smooth as we assumed \( X \) to be smooth.

We shall denote by \( \Delta(k) \) the set of all \( k \)-dimensional cones of \( \Delta \).

We shall now recall the description of the Chow ring \( A^*(X) \) of \( X \). The ring \( A^*(X) \) is generated by the classes of divisors \( [V(\rho)] \) as \( \rho \) varies over the edges of \( \Delta \). Also one has the following relations among these classes:

(i) if \( \rho_1, \cdots , \rho_k \in \Delta(1) \) do not span a cone in \( \Delta \), then

\[
[V(\rho_1)] \cdots [V(\rho_k)] = 0
\]  

(ii) For any \( u \in M \) and \( v_\rho \in N \) is the primitive vector on \( \rho \in \Delta \) one has the relation

\[
\sum_{\rho \in \Delta(1)} \langle u, v_\rho \rangle [V(\rho)] = 0.
\]

Danilov’s theorem \[2, \text{Theorem 10.8}\] asserts that these are no further generating relations and that \( A^*(X) \) has no torsion. It turns out that the cycle class map \( A^*(X) \rightarrow H^*(X) \) is an isomorphism that doubles the gradation. So Danilov’s result also yields a description of the singular cohomology (with \( \mathbb{Z} \)-coefficients). In the case of non-singular projective toric varieties these results are due to Jurkiewicz \[7\].

In this paper we give a description of the ‘topological’ \( K \)-ring of \( X(\Delta) \), denoted \( K(X(\Delta)) \) in terms of generators and relations. Denoting the Grothendieck ring of algebraic vector bundle by \( \mathcal{K}(X(\Delta)) \), it turns out that the forgetful homomorphism \( \mathcal{K}(X(\Delta)) \rightarrow K(X(\Delta)) \) is an isomorphism and hence we obtain a similar description of \( \mathcal{K}(X(\Delta)) \) as well. We shall also establish a similar description of the \( K \)-ring of a torus manifolds with locally standard action whose orbit space is a homology polytope. These results were established for non-singular projective toric varieties in \[11\] and for quasitoric manifolds in \[12\]. Recently V. Uma \[13\] has extended the results of \[12\] to the case of torus manifolds under an additional shellability hypothesis on the orbit space which is assumed to be locally standard with quotient a homology polytope. The method of proof adopted here, which involves only elementary considerations, is quite different in spirit and applies equally well to previously established cases.

### 2. \( K \)-theory of smooth complete toric varieties

Let \( \rho \) be any edge in \( \Delta \). As \( X \) is smooth, the Weil divisor \( V(\rho) \) determines a \( \mathbb{T} \)-equivariant line bundle \( \mathcal{O}(V(\rho)) \) which will be denoted \( L_\rho \). The bundle \( L_\rho \) admits a \( (\mathbb{T} \)-equivariant) algebraic cross-section \( s_\rho: X \rightarrow L_\rho \) which vanishes to order 1 along \( V(\rho) \),
that is, the zero scheme of \( s \) equals the variety \( V(\rho) \). (This is a general fact concerning the line bundle \( \mathcal{O}(D) \) associated to an effective Weil divisor \( D \) on a smooth variety. Any line bundle can be regarded as a subsheaf of \( \kappa_X \), the sheaf of ‘total quotient rings’ of \( X \). When \( D \) is effective, \( \mathcal{O}(-D) \hookrightarrow \kappa_X \) is the ideal sheaf of the subscheme \( D \). See [6, Ch. II, Proposition 6.18].) The inclusion \( \mathcal{O}(-D) \hookrightarrow \mathcal{O}_X \) yields, upon taking duals, a morphism \( s_D : \mathcal{O}_X \to \mathcal{O}(D) \) which defines the required global section with scheme \( D \). Applying these considerations to \( L_\rho \) we obtain the required section \( s_\rho \). Furthermore, \( c_1(\rho) \in A^1(X) \) equals the divisor class \([V(\rho)]\). Topologically, the first Chern class \( c_1(\rho) \in H^2(X) \) can also be described as the cohomology class – also denoted \([V(\rho)]\) – dual to the submanifold \( V(\rho) \subset X \), which is the image of \([V(\rho)] \in A^1(X) \) under the isomorphism \( A^*(X) \cong H^*(X) \).

If \( \rho_1, \ldots, \rho_k \) are edges of \( \Delta \) which do not span a cone of \( \Delta \), then the section \( s : X \to L_{\rho_1} \oplus \cdots \oplus L_{\rho_k} \) is nowhere vanishing since \( V(\rho_1) \cap \cdots \cap V(\rho_k) = \emptyset \). Therefore \( s \) defines a monomorphism \( \tilde{s} : \mathcal{O} \to L_{\rho_1} \oplus \cdots \oplus L_{\rho_k} \) of vector bundles. Denote by \( E \) the quotient of \( L_{\rho_1} \oplus \cdots \oplus L_{\rho_k} \) by the image of \( \tilde{s} \). Applying the \( \gamma \)-operation in \( K(X) \), we see that \( \gamma^k([L_{\rho_1} \oplus \cdots \oplus L_{\rho_k}] - k) = \lambda^k([E]) = 0 \) as rank of \( E \) equals \( k - 1 \). On the other hand \( \gamma^k([L_{\rho_1} \oplus \cdots \oplus L_{\rho_k}] - k) = \prod_{1 \leq i \leq k} \gamma^1([L_{\rho_i}] - 1) \). That is,

\[
\prod_{1 \leq i \leq k} ([L_{\rho_i}] - 1) = 0. \tag{3}
\]

Let \( u \in M \). Set \( L_u := \prod_{\rho \in \Delta(1)} L_{\rho}^{(u, v_{\rho})} \). The first Chern class of the line bundle \( L_u \) can be readily calculated to be \( \sum_{\rho \in \Delta(1)} \langle u, v_{\rho} \rangle c_1(\rho) \mathcal{O} = \sum \langle u, v_{\rho} \rangle [V(\rho)] = 0 \) in view of equation \((2)\) above. Since the isomorphism class of a line bundle is determined by its first Chern class, we have the following equation in \( K(X) \):

\[
\prod_{\rho \in \Delta(1)} [L_{\rho}]^{(u, v_{\rho})} = 1. \tag{4}
\]

**Definition 2.1.** Let \( \Delta \) be a complete regular fan in \( N \). Let \( R(\Delta) \) denote the ring \( \mathbb{Z}[x_{\rho} \mid \rho \in \Delta(1)] / \mathfrak{I} \) where \( \mathfrak{I} \) is ideal generated by the elements:

(i) \( x_{\rho_1} \cdots x_{\rho_k} = 0 \) whenever \( \rho_1, \ldots, \rho_k \in \Delta(1) \) do not span a cone of \( \Delta \).

(ii) \( z_u := \prod_{\rho \in \Delta(1) \mid \langle u, v_{\rho} \rangle > 0} (1 - x_{\rho})^{(u, v_{\rho})} - \prod_{\rho \in \Delta(1) \mid \langle u, v_{\rho} \rangle < 0} (1 - x_{\rho})^{-(u, v_{\rho})} \).

We are now ready to state the main theorem.

**Theorem 2.2.** The \( K \)-ring of the complete non-singular toric variety \( X(\Delta) \) is isomorphic to \( R(\Delta) \) under an isomorphism \( \psi \) which maps \( x_{\rho} \) to \((1 - [L_{\rho}])\).

In view of equations \((3)\) and \((4)\) it is clear that there is a homomorphism of rings \( \psi : R(\Delta) \to K(X(\Delta)) \). In §3 we shall show that \( \psi \) is onto. In §4 we complete the proof by showing that both the abelian groups \( R(\Delta) \) and \( K(X(\Delta)) \) are free of the same rank. In
§5 we shall extend the results to the class of torus manifolds with orbit space a homology polytope.

3. LINE BUNDLES AND $K$-THEORY

In this section $X$ denotes a path connected finite CW complex. Assume that $H^*(X)$ is generated by $H^2(X)$. Then the $K$-ring of $X$ is generated (as a ring) by the classes of line bundles on $X$. This was proved in [11] under the hypothesis that $X$ has cells only in even dimensions. However essentially the same proof works under our weaker hypothesis. Indeed, suppose that $\dim(X) < 2n$ and that $H^2(X)$ is generated as an abelian group by $k$ elements. Then one has a continuous map $f: X \to (\mathbb{CP}^n)^k$ which induces a surjection in cohomology in dimension 2 and hence, by our hypothesis on $X$, in all dimensions. Since the cohomology of $X$ vanishes in odd dimensions, it follows that the Atiyah-Hirzebruch sequence collapses. Since $f^*$ induces surjection in cohomology, the naturality of the spectral sequence implies that it $f^*$ induces surjection in $K$-theory. Since $K(\mathbb{CP}^n)$ is generated by line bundles, it follows by the Künneth theorem for $K$-theory that $K((\mathbb{CP}^n)^k)$ is generated by line bundles. Hence $K(X)$ is also generated by line bundles. As a consequence we obtain

**Proposition 3.1.** Let $X(\Delta)$ be a complete non-singular toric variety. With notations as in §1, the ring homomorphism $\psi: R(\Delta) \to K(X(\Delta))$ is a surjection. \(\Box\)

**Remark 3.2.** Suppose that $H^*(X) = H^{ev}(X)$ is a free abelian group. A straightforward argument involving the Atiyah-Hirzebruch spectral sequence shows that, as far as the additive structure is concerned, $K(X)$ is a free abelian group of rank equal to the Euler characteristic $\chi(X)$ of $X$.

It well-known that $\chi(X(\Delta))$ equals $\#\Delta(n)$, the number of $n$-dimensional cones in $\Delta$ [4]. Since $\psi$ is a surjection we conclude that as an abelian group, the rank of $R$ is at least $\#\Delta(n)$.

4. PROOF OF THEOREM [2.2]

Let $\Delta$ be a complete regular fan in $N$ and let $R(\Delta)$ be the ring defined in §1. It is clear that the set of all monomials $x(\sigma) := x_{\rho_1} \cdots x_{\rho_k}, \sigma \in \Delta$, where $\sigma \in \Delta(k)$ is spanned by edges $\rho_1, \cdots, \rho_k$ forms a generating set for $R(\Delta)$. We shall show that $R(\Delta)$ is a free abelian group and describe a monomial basis for it.

**Filtered rings**

Let $S$ be the polynomial ring $\mathbb{Z}[x_1, \cdots, x_d]$ and let $\mathcal{I} \subset S$ be an ideal generated by elements $f_1, \cdots, f_m$ where each $f_j$ has constant term zero. The ring $S$ is graded where we set $\deg(x_i) = 1$. We denote the abelian group of all homogeneous polynomials of degree
groups \[ \eta \]

at most abelian of rank \( A \)

Chow ring \( h \)

\# \( \Delta \).

\[ S \]

\[ f \]

\[ i.e., \]

\[ n \]

\# \( \Delta (f) \) elements. Since \[ R \]

\[ f \]

is isomorphic to \( R / S \)

\[ \Delta \]

\[ X \]

\[ m \].

\[ \rho \]

\[ h_u \]

\[ u \]

\[ M \].

\[ [2] \]

Theorem 10.8] we see that the ring \( \tilde{R} (\Delta) = S / I \)

isomorphic to the Chow ring \( A^* (X (\Delta)) \)

under an isomorphism which maps \( x_\rho \) to \([V(\rho)]\). We shall identify \( \tilde{R} (\Delta) \) with \( A^* (X (\Delta)) \).

Thus we obtain a surjective homomorphism of graded abelian groups \( \eta : \tilde{R} (\Delta) \twoheadrightarrow \text{gr} (R (\Delta)) \).

\textbf{Proof of Theorem 2.2} We shall abbreviate \( X (\Delta) \) to \( X \) etc. Recall that \( A^* (X) \cong H^* (X) \)

is a free abelian group, its rank being equal to \# \( \Delta (n) \), the number of \( n \)-dimensional cones of \( \Delta \). Since \( \eta \) is a surjection, \( \text{gr} (R (\Delta)) \)

is an abelian group generated by at most \# \( \Delta (n) \) elements. Since \( R \) and \( \text{gr} (R) \) have equal rank, it follows that \( R \) must be free abelian of rank \textit{at most} \# \( \Delta (n) \). In view of Remark 3.2 we conclude that rank of \( R \) equals \# \( \Delta \).

Since \( \psi : R \twoheadrightarrow K (X) \)

is a surjection between free abelian groups of same rank, it follows that \( \psi \) is an isomorphism. This completes the proof.

Let \( X \) be a smooth complete variety over \( C \). Consider the ‘forgetful’ homomorphism \( f : \mathcal{K} (X) \twoheadrightarrow K (X) \)

where \( \mathcal{K} (X) \) denotes the Grothendieck \( K \)-ring of algebraic vector bundles over \( X \). Since \( X \) is smooth, \( \mathcal{K} (X) \)

is isomorphic to the Grothendieck group \( \mathcal{K}' (X) \)

of coherent sheaves over \( X \). One has a ‘topological filtration’ on \( \mathcal{K}' (X) \cong \mathcal{K} (X) \)

and one has a well-defined surjective homomorphism \( \phi : A_* (X) \twoheadrightarrow \text{gr} (\mathcal{K} (X)) \)

of graded abelian groups. When \( X = X (\Delta) \), it follows that \( \text{gr} (\mathcal{K} (X)) \)

is a quotient of the free abelian group \( A_* (X) = A^{n-*} (X) \).

It follows that \( \mathcal{K} (X) \)

is generated by at most \( \chi (X) \) elements. Since
\( K(X) \) has rank equal to \( \chi(X) \), it follows that \( f: K(X) \rightarrow K(X) \) is an isomorphism of rings. We record this as

**Theorem 4.1.** Let \( X = X(\Delta) \) be a smooth complete toric variety. Then, the forgetful morphism of rings \( f: K(X) \rightarrow K(X) \) is an isomorphism. In particular, \( K(X(\Delta)) \) is isomorphic to \( R(\Delta) \).

**Remark 4.2.** (i) It is immediate from the main theorem that \( x_\rho \in R \) are nilpotent since \([L_\rho]\) are invertible. Indeed, the surjectivity of \( \eta \) already implies that the filtration \( R_j = R_{n+1} \) for all \( j > n \). Since \( \cap_{j \geq 0} R_j = 0 \), it follows that \( R_j = 0 \) for all \( j > n \).

(ii) It follows from our proof that \( \eta: A^*(X) \rightarrow \text{gr}(R(\Delta)) \) is an isomorphism. Recall that if \( \sigma \in \Delta(k) \) is spanned by edges \( \rho_1, \ldots, \rho_k \), then \( [V(\sigma)] = [V(\rho_1)] \cdots [V(\rho_k)] \in A^{n-k}(X) \). If \( \Gamma \subset \Delta \) is a collection of cones such that \( \{[V(\sigma)]\}_{\sigma \in \Gamma} \) is \( \mathbb{Z} \)-basis for \( A^*(X) \), then \( \{x(\sigma)\}_{\sigma \in \Gamma} \) is a monomial basis for \( R(\Delta) \).

5. **K-theory of torus manifolds**

The algebraic geometric notion a non-singular projective toric variety has been brought home to realm of topology by Davis and Januszkiewicz [3] who introduced the class of quasi-toric manifolds. (Davis and Januszkiewicz called them ‘toric manifolds’; the present terminology of ‘quasi-toric manifolds’ is due to Buchstaber and Panov [1].) See also Masuda [8]. Recently, Masuda and Panov [9] introduced a new class of manifolds called *torus manifolds* in their efforts to develop the analogue of a non-singular complete toric variety. The class of torus manifolds, which includes all complete non-singular complex toric varieties, is much more general than that of quasi-toric manifold as there are non-singular complete toric varieties which are not quasi-toric manifolds.

In this section we obtain a description of the \( K \)-ring of a torus manifold \( X \) assuming that the orbit space \( X/T = Q \) is a homology polytope. Most of the ingredients needed for the proof of Theorem 5.3 can be found in [9].

We shall recall here the results of [9] concerning the cohomology of toric manifolds with locally standard action and orbit space a homology polytope that are needed for our purposes. We recall below the definition and some basic facts relevant for our purposes, referring the reader to [9] for a detailed exposition of torus manifolds. As mentioned in the Introduction, our main result in this section, namely, Theorem 5.3 subsumes Theorem 2.2. In fact, as we shall see, the steps involved in the proof in both cases are the same; we need only to establish those steps whose proofs in the algebraic geometric situation are either unavailable or are not obvious in the topological context of torus manifolds.

We begin by recalling the basic definition and properties of torus manifolds. Let \( T := (S^1)^n \) denote the compact torus and let \( X \) be a smooth compact oriented connected
manifold of dimension $2n$ on which $T$ acts effectively with a finite non-empty set of $T$-fixed points. Such a manifold $X$ is called a torus manifold. The orbit space $Q =: X/T$ is a ‘manifold with corners’. The $T$-action on $X$ is called \textit{locally standard} if $X$ is covered by $T$-invariant open sets $U$ such that $U$ is equivariantly diffeomorphic to an invariant open subset contained in a $T$-representation $U \cong \mathbb{C}^n$ whose characters form a $\mathbb{Z}$-basis for $\text{Hom}(T, S^1) \cong \mathbb{Z}^n$. For example, $X = X(\Delta)$ a complete non-singular toric variety with action of $\mathbb{T}$ restricted to $T \subset \mathbb{T}$ is a torus manifold with locally standard $T$-action.

Also, any quasi-toric manifold is a torus manifold. A characteristic submanifold of $X$ is a codimension 2 submanifold which is pointwise fixed by a one-dimensional subgroup of $T$. There are only finitely many characteristic submanifolds; we denote them by $V_1, \cdots, V_d$.

In the case when $X$ is a smooth projective toric variety or a quasi-toric manifold the orbit space $Q$ is a simple polytope. However, for a general torus manifold, this is not so. Assume that $X$ is torus manifold with locally standard $T$-action. Define the boundary of $Q$, denoted $\partial Q$, to be the set of all points which do \textit{not} have a neighbourhood homeomorphic to an open set of $\mathbb{R}^n$. Then $\partial Q$ is the image under the quotient map $\pi: X \rightarrow Q$ of the union of all characteristic submanifolds of $X$. Denote the image of $V_i$ by $Q_i \subset \partial Q, 1 \leq i \leq d$; these are the \textit{facets} of $Q$. A non-empty intersection of facets are called \textit{prefaces} of $Q$; a \textit{face} of $Q$ is a connected component of a preface. The space $Q$ is called \textit{homology polytope} if every preface $F$, including $Q$ itself, is acyclic, i.e., $\tilde{H}_*(F; \mathbb{Z}) = 0$, in particular, every preface of $Q$ is path connected. One of the main results of [9] is that the integral cohomology of $X$ is generated by degree 2 elements if and only if the action is locally standard and $Q$ is a homology polytope. \textit{In this paper we shall only consider torus manifolds with locally standard $T$-action whose orbit space is a homology polytope.}

Note that any characteristic submanifold of such a torus manifold inherits these properties; see [9, Lemma 2.3]. In particular, characteristic submanifolds are orientable and have $T$-fixed points. We fix an omni-orientation of $X$, i.e., orientations of $X$ and all its characteristic submanifolds. Thus the normal bundle $\nu_i$ to the imbedding $V_i \hookrightarrow X$ is oriented by the requirement that $\nu_i \oplus TV_i$ be oriented-isomorphic to $TX|V$, the tangent bundle of $X$ restricted to $V$.

\textbf{Line bundles over $X$}

Let $V \subset X$ a codimension 2 oriented closed connected submanifold of an \textit{arbitrary} oriented closed connected manifold of dimension $m$. Let $[V] \in H^2(X; \mathbb{Z})$ denote the cohomology class dual to $V$ and let $L$ denote a complex line bundle over $X$ with $c_1(L) = [V]$. Assume that $H^1(V; \mathbb{Z}) = 0$.

\textbf{Lemma 5.1.} \textit{Suppose that} $c_1(L) = [V] \neq 0$. \textit{Then the complex line bundle} $L$ \textit{admits a section} $s: X \rightarrow L$ \textit{such that the zero locus of} $s$ \textit{is precisely} $V$. 

Proof. Assume that $X$ has been endowed with a Riemannian metric and let $N \subset X$ denote a tubular neighbourhood of $V$ which is identified with the disk bundle associated to the normal bundle $\nu$ to the imbedding $V \hookrightarrow X$. The normal bundle is canonically oriented by the requirement that $TV \oplus \nu$ is orientated isomorphic to $TX|V$. Since $\nu$ is an oriented real 2-plane bundle, we regard it as a complex line bundle. We shall denote by $\pi: N \to V$ the projection of the disk bundle.

The complex line bundle $\pi^*(\nu)$ over $N$ evidently admits a canonical cross-section $\eta: N \to \pi^*(\nu)$ which vanishes precisely along $V$. Take the trivial complex line bundle $\varepsilon$ over $(X \setminus \text{int}(N))$ and consider the vector bundle map $\tilde{\eta}: \varepsilon|\partial N \to \pi^*(\nu)|\partial N$ defined by $\tilde{\eta}(x, 1) = \eta(x)$ for all $x \in \partial N$. Gluing $\varepsilon|\partial N$ along $\pi^*(\nu)|\partial N$ using the identification $\tilde{\eta}$ yields a vector bundle $\xi$ over $X$ which clearly admits a cross-section $s$ (which restricts to $\eta$ on $N$ and $x \mapsto (x, 1)$ on $X \setminus \text{int}(N)$) that vanishes precisely along $V$. Now it suffices to establish that $\xi$ is isomorphic to $L$.

Both $\xi$ and $L$ restrict to trivial bundles over $X \setminus \text{int}(N)$. The bundle $L$ restricts to the normal bundle $\nu$ over $V$ (cf. [10, Theorem 11.3]). Since $\pi: N \to V$ and $V \hookrightarrow N$ are homotopy inverses, it follows that $\xi$ and $L$ restrict to isomorphic bundles over $N$ as well. Let $\sigma: \varepsilon|\partial N \to L|\partial N \cong \pi^*(\nu)|\partial N$ be the gluing data for $L$. The vector bundle maps $\sigma$ and $\tilde{\eta}$ are related by a map $\partial N \to GL_1(\mathbb{C})$. The homotopy classes of such maps is isomorphic to $H^1(\partial N; \mathbb{Z})$. Using the Serre spectral sequence of the circle bundle $\partial N \to V$ hypotheses that $H^1(V) = 0$ and $c_1(L) \neq 0$ we see that $E_2^{0,1} = 0, E_2^{1,0} = 0$. Therefore $H^1(\partial N; \mathbb{Z}) = 0$. Hence $\sigma$ and $\tilde{\eta}$ are homotopic via bundle isomorphisms. It follows that $L$ is isomorphic to $\xi$. 

Now applying the above lemma to characteristic submanifolds $V_1, \cdots, V_d$ of a torus manifold $X$ with locally standard $T$-action and orbit space a homology polytope, we see that there exist complex line bundles $L_1, \cdots, L_d$ such that $c_1(L_i) = [V_i]$ and each $L_i$ admits a section $s_i: X \to L_i$ which vanishes precisely along $V_i$, $1 \leq i \leq d$. We proceed as we did in obtaining Equation (3) in §2, to obtain the following equation in $K(X)$

$$\prod_{1 \leq j \leq r} (1 - [L_{i_k}]) = 0$$  \hspace{1cm} (5)

whenever $\cap_{1 \leq k \leq r} V_{i_k} = \emptyset$.

The characteristic map

Recall that the characteristic submanifolds $V_i$ are, by definition, fixed by one dimensional subgroups $S_i$ of $T$. Our assumption on $X$ (local standardness and orbit space being a homology polytope) implies that every characteristic submanifold has a $T$-fixed point. There is unique 1-parameter subgroup $\upsilon_i \in \text{Hom}(S^1, T) \cong \mathbb{Z}^n$ by the following requirements, the first of which determines $\upsilon_i$ upto sign: (i) $\upsilon_i$ is a primitive element in $\text{Hom}(S^1, T)$ with image $S_i$, and, (ii) the sign ($+\upsilon_i$ or $-\upsilon_i$) is determined by orienting $S_i$ so that at any point
Let \( p \in V \), the oriented normal plane \( \nu_p \) is oriented isomorphic to the tangent space to \( S_i \) at the identity element.

The map \( \Lambda: \{Q_1, \ldots, Q_d\} \rightarrow \text{Hom}(S^1, T) \) which maps \( Q_i \) to \( v_i \) is called the characteristic map. Under our hypothesis of local standardness and \( Q \) being a homology polytope, the manifold \( X \) is determined up to equivariant diffeomorphism by the pair \( (Q, \Lambda) \). (See \[9\] Lemma 4.5.)

Local standardness of the \( T \)-action implies that if \( V_{i_1} \cap \cdots \cap V_{i_r} \neq \emptyset \), then \( \{v_{i_1}, \ldots, v_{i_r}\} \) is part of a \( \mathbb{Z} \) basis for \( \text{Hom}(S^1; T) \cong \mathbb{Z}^n \).

**Cohomology of \( X \)**

We now recall from \[9\] Corollary 7.8 the description of the integral cohomology ring of \( X \).

Let \( Q \) denote the set of all faces of \( Q \). Then the cohomology of \( X \) has the following description:

\[
H^*(X; \mathbb{Z}) \cong \mathbb{Z}[x_F; F \in Q]/I
\]

where \( I \) is the ideal generated by the following two types of elements:

(i) \( x_Ax_B = x_{A \vee B}x_{A \cap B} \) where \( A \vee B \) denotes the smallest face of \( Q \) which contains both \( A \) and \( B \),

(ii) \( \sum_{1 \leq i \leq d} \langle u, v_i \rangle x_{Q_i}, \ u \in \text{Hom}(T, S^1) \) where \( v_i = \Lambda(Q_i) \in \text{Hom}(S^1; T) \).

The element \( x_{Q_i} \) corresponds under the isomorphism to \( [V_i] \in H^2(X; \mathbb{Z}) \). (It is understood that \( x_\emptyset = 1 \), and \( x_\emptyset = 0 \).)

From our hypothesis, \( H^*(X; \mathbb{Z}) \) is generated by degree two elements. Set \( x_i := x_{Q_i} \), for \( 1 \leq i \leq d \). Any face \( F \) of \( Q \) is the intersection of those facets of \( Q \) which contain \( F \). If \( F \) is of codimension \( r \), then it is contained in exactly \( r \) distinct facets, say, \( Q_{i_1}, \ldots, Q_{i_r} \) and so the intersection \( F = Q_{i_1} \cap \cdots \cap Q_{i_r} \) is transversal. Therefore \( x_F = x_{i_1}, \ldots, x_{i_r} \). Thus, we see that

\[
H^*(X; \mathbb{Z}) = \mathbb{Z}[x_1, \ldots, x_d]/I
\]

where the ideal \( I \) of relations is generated by the elements:

(iii) \( x_{i_1} \cdots x_{i_r} \) whenever \( V_{i_1} \cap \cdots \cap V_{i_r} = \emptyset \)

(iv) \( \sum_{1 \leq i \leq d} \langle u, v_i \rangle x_i \).

**Proposition 5.2.** (Masuda-Panov \[9\])

Let \( X \) be a \( T \)-torus manifold with locally standard action whose orbit space is a homology polytope. With the above notations, \( H^*(X; \mathbb{Z}) \cong \mathbb{Z}[x_1, \ldots, x_d]/I \) where \( x_i \) represents the cohomology class dual to the characteristic manifolds \( V_i, 1 \leq i \leq d \). In particular, \( H^*(X; \mathbb{Z}) \) is a free abelian group.

**Proof.** The only part of the theorem that needs to be established is that \( H^*(X; \mathbb{Z}) \) is a free abelian group as other assertions follow from \[9\] Corollary 7.8 as noted above. By \[9\] Theorem 7.7, the cohomology of \( X \) with coefficients in \( \mathbb{Z} \) or \( \mathbb{Z}/p\mathbb{Z} \) for any prime \( p \)
vanishes in odd dimensions, it follows that $H^*(X; \mathbb{Z})$ is torsion free. As $X$ is a compact manifold, $H^*(X; \mathbb{Z})$ is finitely generated. It follows that $H^*(X; \mathbb{Z})$ is free abelian. □

Recall that $L_i$ is the complex line bundle over $X$ with $c_1(L_i) = [V_i]$. As an immediate consequence of the above description of $H^*(X; \mathbb{Z})$, we obtain the following equality in $K(X)$, analogous to equation (4):

$$\prod_{1 \leq i \leq d} [L_i]^{(u, v_i)} = 1.$$ (6)

We are now ready to state the main result of this section.

**Theorem 5.3.** Let $X$ be any $T$-torus manifold with locally standard action whose orbit space is $Q$ a homology polytope. Then $K(X)$ is the ring isomorphic to the ring $R(Q, \Lambda) := \mathbb{Z}[y_1, \ldots, y_d]/\mathfrak{J}$ where $\mathfrak{J}$ is the ideal generated by the elements

(i) $y_{i_1} \cdots y_{i_r}$ whenever $Q_{i_1} \cap \cdots \cap Q_{i_r} = \emptyset$

(ii) $\prod_{i \leq d: (u, v_i) > 0} (1 - y_i)^{(u, v_i)} - \prod_{j \leq d: (u, v_j) < 0} (1 - y_j)^{-(u, v_j)}$ for each $u \in \text{Hom}(T; \mathbb{S}^1)$, where $v_i := \Lambda(Q_i), 1 \leq i \leq d$. The isomorphism is established by sending $y_i$ to $1 - [L_i]$.

**Proof.** The proof follows exactly as in the case of non-singular complex toric varieties. In view of Proposition 5.2, $K(X)$ is generated by $[L_i], 1 \leq i \leq d$; see §3. Furthermore, $K(X)$ is a free abelian group of rank $\chi(X)$. Set $R := R(Q, \Lambda)$. From equations (5) and (6), there is a ring homomorphism $\psi: R \rightarrow K(X)$ which maps $y_i$ to $1 - [L_i]$ which is surjective. Arguing as in §4, we see that there is a decreasing filtration on $R$ and a surjective homomorphism of abelian groups $H^*(X; \mathbb{Z}) \rightarrow \text{gr}(R)$, showing that as an abelian group, $R$ generated by at most $\chi(X)$ many elements. Since $\psi$ is surjective, it follows that $R$ also a free abelian group of rank $\chi(X)$ and that $\psi$ is an isomorphism of rings. □

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