Quintessential Axion: Inflation and Dark energy

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Abstract. The natural inflation fails to explain low tensor-to-scalar ratio with current observational data and the cosine inflaton potential is not valid at large-N limit SU(N) Yang-Mills theory accord with Witten’s conjecture. We address a simple quintessence axionic inflationary model in which the axionic inflaton couples to the gauge fields which are belongs to a large-N limit pure Yang-Mills theory. This quintessence pure natural inflation satisfies the current Planck data as well as the smallness of the dark energy density(vacuum energy density) at a non-zero minimum field value of quintessence axion. Further, we show that the quintessence pure natural inflation obeys the strong scalar weak gravity conjecture which follows from the de Sitter distance conjecture for any quasi de Sitter background. Unlike ΛCDM, the energy density of the dark energy ρφ and the equation of the state ωφ is dynamical in our thawing quintessence axion model.

Keywords: Classical theory of gravity, Cosmological theory of BSM.
1 Introduction

It is believed that the universe started with initial conditions seeded by cosmic inflation, which grew up the cosmic scale early universe within very short time\cite{1, 2}. The inflationary approach remarkably alleviates the infamous horizon and flatness problem of hot Big-Bang universe at early time. At the end of inflation, the energy density of the inflaton transform to other degrees of freedom which are standard model particles, dark matter and dark energy. Here the baryonic matter and pressureless dark matter are responsible for the galaxy formation at low redshift era which has constrained by nucleosynthesis(BBN) and initial conditions of the cosmic inflation at early time. Though the dark matter drives the growth of the structure formation faster, dark energy slows down in it due to their property of negative pressure at late time nearly low redshift; larger the acceleration, higher the suppression of growth of structure \cite{3–6}. Thus the radiation dominated and matter dominated phases of universe sandwiched by early and late time cosmic acceleration, both acceleration drives the universe by negative pressure through violating the strong energy condition(SEC). Such kind of late time cosmic acceleration confirmed the existence of dark energy by type Ia supernovae distance measurement as a cosmic candle\cite{7, 8}.

Importantly the measurement of the baryon acoustic oscillation(BAO) is the standard ruler which exhibit the distribution(not random) of the energy density fluctuation of the matter at a very large scale\cite{9, 10}. Even though the matter distributed in a non-linear way as clumps at small scale($< 10$ Mpc) which is not surrounded by empty space instead such void space occupied by dark energy homogeneously and isotropically. Two different kinds of candidate explain the nature of dark energy with negative pressure which are cosmological constant and quintessence\cite{11–15}; the vacuum energy density is constant with fixed equation of state(EOS) $\omega = -1$ throughout universe evolution and other candidates dynamically vary the energy density with $\omega$ close to $-1$ (quintessence) which varies accord with cosmic scale factor respectively. Apart from equation of state $\omega$ of the dark energy, the sound speed $C_s$ is also important to distinguish whether the dark energy belongs to cosmological constant or quintessence. The sound speed affects the quintessence energy density distribution as well as power spectrum of cosmic micro wave background(CMB), if it differ from unity\cite{16}. But most of the quintessence model takes $C_s = 1$ which means sound speed equal to the speed of light,
thereby the effect of this property is confined to very large scale, this can be manifested in the large-angle multipoles of the CMB anisotropies. Such probes of large angular multipoles (low l) in the CMB anisotropies and the equation of state measurement of supernovae type-Ia, large scale structure (LSS) and CMB which is $\omega \neq -1$ with error, could be favored to dynamical dark energy density model.

Similar to CMB photon, the cosmic neutrino decouples from cosmic plasma far earlier than the photons and their last scattering surface much closer than CMB when the mass of the neutrino greater than $10^{-4}$ eV\cite{17}. After early cosmic acceleration, if the neutrino has mass then it doesn’t change the evolution of the dark energy though they have interaction with dark energy field, but in the late time, growing of neutrino mass would stops the dynamical evolution of dark energy\cite{18–21}. In contrast, the interaction strength of the neutrino and quintessence field is large compared to gravitational strength, the tiny neutrino energy density itself would dominate the time evolution of the quintessence field. Thereby, the neutrino mass feasibly explains why dark energy dominates at present and how it can behave like a trigger for the crossover to a new cosmological epoch.

In this perspective, one can unify the early and late time cosmic acceleration by constructing the quintessence field model. There are few models proposed in this direction which satisfies slow-roll inflation data as well as provides the quintessence nature of dark energy. In this regard, most importantly Starobinsky model\cite{22, 23}, Natural inflation\cite{24, 25} and Axion monodromy model\cite{26, 27} satisfy the recent inflationary data (Planck2018) within the $3\sigma$.

Unfortunately, cosmological field space relies on the super-Planckian scale according to Lyth bound by the observed upper limit of the tensor-scalar ratio. Therefore one needs to classify whether the inflationary models are UV completed or not, in addition to that we have to examine whether (meta)stable vacua exist or not because of slow-roll condition and swampland de-Sitter conjecture conflicts each other.

In this paper, we have explore the aspects of natural inflation with large-N limit pure Yang-Mills theory. In section 2, we present the analytical expression for quintessence axionic potential which is based on Witten’s conjecture, then we perform the numerical analysis of our model by satisfying Planck 2018 results. Further we have discussed about sub-planckian as well as trans-planckian field physics through Lyth bound. Section 3 includes the weak gravity conjecture. Here we get the constraint for tensor-scalar ratio from swampland conjecture though it fails to explain slow-roll condition. Then we have analyzed the strong scalar weak gravity conjecture in terms of slow-roll parameter up to fourth order which provides the criteria whether the inflation model valid up to trans-planckian field space or not. In section 4, we have explained about the solution of dynamical dark energy parameters, depending on nature of parameter we classified our model as thawing frozen model. Here we have explained the nature of thawing dynamical dark energy by using our inflationary model results which are obtained from section 2. Finally we have concluded our results in section 5.

2 Large N dynamics potential and inflation

According to t’Hooft mechanism, the $SU(N)$ Yang-Mills (YM) theory with the infinite number of color ($N \to \infty$), the t’Hooft coupling $\lambda = g^2 N$ is fixed when the gauge coupling $g$ vanishes \cite{28}. Thus the pure Yang-Mills interaction can be written in terms of expansions of $1/N$. Accommodating the strong CP Lagrangian in the large-N limit, the instanton potential of the vacuum energy protects the quantum corrections thereby non-perturbative. Such potential takes the Cosine form which is not valid under Witten’s conjecture \cite{29} in the large-N limit.
The Witten’s conjecture states that the potential of the SU(N) vacuum energy has to be multi valued due to the existence of several metastable vacua and the quadratic term (mass term) should be independent of number of color N. The potential has to be continuous, smooth, periodic and CP invariant assuming when no spontaneous breaking in the Vacuum. The SU(N) Lagrangian in the large-N limit is

\[ \mathcal{L} = -\frac{1}{4} \left( \frac{N}{\lambda} \right) F_{\mu \nu} F^{\mu \nu} + \frac{\theta}{32\pi^2} F_{\mu \nu} \tilde{F}^{\mu \nu} \]  

(2.1)

Where \( \tilde{F}^{\mu \nu} = (1/2) \epsilon^{\rho \sigma \mu \nu} F_{\rho \sigma} \). Here \( \theta \) is the angular variable. When \( N \to \infty \), the vacuum energy is minimum at \( \theta \to 0 \) and the Euclidean path integrals gets real and positive. Importantly, when \( \theta \neq 0 \), the vacuum energy is minimized by maximizing the Euclidean space path maximal thereby the vacuum energy \( E(\theta) \) computed by expanding the exponential \( \exp(-S_E) \). Where \( S_E \) is the Euclidean action in the \( \mathbb{R}^4 \).

\[ \exp[-E(\theta) \ \text{Vol}(\mathbb{R}^4)] = \int [DA_\mu] \exp(-S_E) \]  

(2.2)

Thus the conjectured vacuum energy can be written as

\[ E(\theta) = C \min_k (\theta + 2\pi k)^2 + O(1/N) \]  

\[ \Rightarrow N^2 \min_k \left( \frac{\theta + 2\pi k}{N} \right) \]  

(2.3)

The true vacuum of the multi branched vacuum energy is determined by certain branch value of \( k \). According to Peccei-Quinn mechanism [30], the angular variable \( \theta \) behave like a shift parameter to the axion field known as \( a(x) \)-pseudo Nambu-Goldstone boson with the periodicity of \( 2\pi f_a \). Here \( f_a \) is the axion decay constant. Consequently, the instanton generates the SU(N)YM potential for axion-like inflaton field \( a(x) \) by replacing \( \theta = a(x)/f_a \) in the normalized equation.(2.1) with \( N/\lambda \) for single branch can be written as follows [29]

\[ V(a(x)) = N^2 \Lambda^4 \xi(\xi) \]  

Where \( \xi = \frac{\lambda a(x)}{8\pi^2 N f_a} \)  

(2.4)

The inflaton field in equation.(2.4) is not periodic under \( a(x) \to a(x) + 2\pi f_a \), thereby the large-N limit field parameter \( \xi \) allows to construct a axionic inflaton model. Aside from lattice gauge theory, the string theory is the good candidate for constructing such axionic inflation model. Recently, the multivalued branch axionic model was proposed in ref.[31–33] known as pure natural inflation inspired by axion monodromy [26, 27]. On the other hand, one can think Quintessence models [13, 34] which explains inflation as well as dark energy with wide range of field excursions. In this paper, we infer such kind of pure natural inflation model which is inspired by Quintessence in the context of large-N dynamics of SU(N)YM. Accommodating the Witten’s conjecture equation.(2.3) and avoiding domain wall problem by setting domain wall number \( (N_{DW}) \) equal to one [35], in general the axionic potential can be written as follows

\[ V(a(x)) = M^4 \left[ 1 - \exp \left( -N_{DW} \left( \frac{a(x)}{F_a} \right)^2 \right) \right] \]  

(2.5)

Here \( M = \sqrt{N}\Lambda \) and \( F_a = N f_a \). Here the ’t’Hooft coupling \( \lambda \) taken to be \( 8\pi^2 \) and then the mass of the inflaton at large-N limit obtained as \( m_a = \sqrt{2}\Lambda^2/f_a \). Here \( \Lambda \) is the scale where
the global symmetry has broken explicitly at lower energy scale compared to the spontaneous symmetry breaking scale \( f_a \). The potential is extremely flat and the required curvature for slow roll inflation constrained by Lyth bound \([36–38]\) which is
\[
\frac{\Delta \phi}{M_{Pl}} \gtrsim N_e \sqrt{\frac{r}{8}}. \tag{2.6}
\]
Here \( N_e \) is the required amount of e-folds for slow roll inflation and \( r \) is the tensor-scalar ratio. The non-zero \( r \) emphasis that the possible existence of B-modes in CMB. The current Planck 2018 data \([7, 8]\) has observed the upper bound of tensor-scalar ratio that is \( r_{0.002} < 0.064 \) at 95\%C.L. This upper bound breaks the effective field theory and Lyth bound takes super-

\textbf{Figure 1.} The Quintessence pure natural inflation (QPNI) is consistent with Planck 2018 results with a wide range of tensor-scalar ratio for 60 e-folds. The natural inflation may be disfavour near the future for low \( r \) value.

Planckian at 60 e-folds which is \( \Delta \phi \gtrsim 5.4 M_{Pl} \). In order to satisfy \( r \approx 10^{-2} \) the axion decay constant \( f_a \) require 10\( M_{Pl} \) in natural inflation, thereby the single field natural inflation gets super-Planckian instead of sub-plankian. Observationally favored Starobinsky model satisfies spectral tilt \((n_s)\) nearly \( r \sim 0.003 \) for 55e-folds. One can argue that if the upper bound on \( r \) lie beyond \( 10^{-2} \) or non-observation of B-modes in CMB or pointedly \( r \) lie in the Big Bang Observer (BBO) near future \([39–42]\) which is \( 10^{-3} \) – \( 10^{-4} \) then one cannot have super-Planckian in the slow roll inflation. Once we get into sub-Planckian, we realize that the flatness and stability of the small field excursion inflation potential spoiled by radiative

| Tensor-scalar ratio \((r)\) | \(f_a\) (Unit of \(M_{Pl}\)) | Spectral tilt \((n_s)\) |
|-----------------------------|-----------------|----------------|
| \(6.4 \times 10^{-2}\)      | 14.3/\(N\)       | 0.9653         |
| \(10^{-2}\)                | 5.6/\(N\)        | 0.9607         |
| \(10^{-3}\)                | 2.3/\(N\)        | 0.9612         |
| \(10^{-4}\)                | 1.0/\(N\)        | 0.9628         |
| \(10^{-5}\)                | 0.4/\(N\)        | 0.9640         |

\textbf{Table 1.} Our Quintessence pure natural inflation (QPNI) at large-N limit satisfies wide range of tensor-scalar ratio with 60 e-folds. The sub-Planckian scale \( f_a \) achieved by choosing \( N \geq 15 \), which is number of color presented in the \( SU(N)YM \) at present upper bound on tensor-scalar ratio \( r < 0.064 \).
corrections in the low energy EFT. But the single field axionic inflation has global shift symmetry which preserves the flatness and stability of the potential at small field excursions \([43, 44]\). The broken spontaneous and explicit symmetry scales of quintessence axion are well controlled by global symmetry and they rely on sub-planckian. The figure.(1) shows that our QPNI model satisfies the Planck2018 result within \(3\sigma\) similar to the \(\alpha\)-attractor and the Starobinsky\((\alpha = 1)\) model.

3 Weak gravity conjecture

3.1 Swampland conjecture

In general, the low energy effective field theoretic inflationary models when satisfy \(|\Delta \phi|/M_{Pl} \lesssim O(1)\) are amenable for UV-completed theory such condition not yet supported by observation. The recent construction of conjectures \([45–47]\) on string landscape and string swampland namely de Sitter swampland conjectures, classifies all possible low energy effective QFT into UV-completed and non-UV completed in the context of quantum gravity. The necessary condition for the existence of UV completion conjectured as follows \([48]\)

1. Distance conjecture: The range of scalar field traversed in the field space restricted as

\[
\frac{|\Delta \phi|}{M_{Pl}} \lesssim c_0
\] (3.1)

2. Refined de Sitter conjecture: Any scalar field with potential \(V(\phi)\) in the low energy effective theory of consistent quantum gravity must satisfy either

\[
\frac{|\nabla V|}{V} \geq \frac{c_1}{M_{Pl}}
\] (3.2)

or

\[
\min(\nabla_i \nabla_j V) \leq -\frac{c_2}{M_{Pl}^2} V
\] (3.3)

Where \(c_0, c_1\) and \(c_2\) are positive universal and \(O(1)\) parameters. The parameter \(c_1\) depends on the details of string flux compactification and it should be greater than \(\sqrt{2}\). Here the de Sitter conjecture doesn't allow (meta-)stable vacua with positive energy density and which is conflict with slow roll inflationary scenario \([49, 50]\). Thus the parametric constraints does not require to be \(O(1)\) rather it can be used to constrain the inflation paradigm. The distance conjecture do not have significant tension with the present observations but refined swampland conjecture(RSC) has non-trivial implications on potential dominated \((\dot{\phi}^2 \ll V(\phi))\) slow roll inflation. According to RSC, the slow roll parameters defined customarily as

\[
\epsilon_V = \frac{1}{2} \left( \frac{V'}{V} \right)^2 \geq \frac{c_1^2}{2}; \quad \eta_V = \left( \frac{V''}{V} \right) \leq -c_2
\] (3.4)

In contrast to RSC, the slow roll inflation where \(\epsilon_V \ll 1\) and \(\eta_V \ll 1\) put restriction on the universal parameters \(c_1\) and \(c_2\) which cannot have \(O(1)\) parameter value due to current observations and their bounds are \(c_1 < 0.1\) and \(c_2 < 0.01\). Thereby the stability of de Sitter vacuum is questionable not only from observation, also from studies of tensor perturbations.
[51], scalar entropic fluctuations [52–54] and IR instability [55, 56]. Other hands, RSC equation (3.3) leads to the bound on tensor-scalar ratio [57, 58]

$$r_{\text{bound}} \leq \frac{8}{3} (1 - 2c_2 - n_s).$$  \hspace{1cm} (3.5)

The upper bound on $r$ shown in figure (1) for the values of $c_2 = 0$ and 0.01. Similarly when we consider potential (equation (2.5)), RSC equation (3.2) restricts the axion decay constant when $V'' = 0$ at $a(x)/Nf_a = 1/\sqrt{2}$ which is

$$f_a < \frac{2.18}{Nc_1}M_{\text{Pl}}.$$  \hspace{1cm} (3.6)

It is obvious that $Nc_1$ could be achieved greater than 2.18 by demanding higher values of $N$ which keeps natural inflationary behaviour of the axion decay constant.

3.2 Scalar weak gravity conjecture

Interestingly, current Planck 2018 result shows that the scalar field is traverse over the super-Planckian distance (equation (2.6)) which oblige an infinite tower of states with field dependent mass which is $m(\phi)$ accord with scalar weak gravity conjecture (SWGC). The mass of such WGC scalar ($S_{\text{WGC}}$) decreases exponentially as a function of scalar field variation as follows [59–61]

$$m(\phi + \Delta\phi) \leq m(\phi)e^{-\alpha \frac{\Delta\phi}{M_{\text{Pl}}}}$$  \hspace{1cm} (3.7)

Here $\alpha$ is a positive constant determined by direction of $\Delta\phi$ in the field space. The tower of states are nearly massless, such light scalar doesn’t need to have interaction with standard model directly, instead they couple through gravitational interaction. Therefore gravity has to be weakest force at the horizon scale. According to SWGC, the force which is mediated by light scalar $\phi$ must be stronger than graviton mediated force which can demonstrated by trilinear coupling ($\mu$) of WGC scalar $S_{\text{WGC}}$ as follows

$$\left(\frac{m}{\mu}\right)^2 \leq M_{\text{Pl}}^2.$$  \hspace{1cm} (3.8)

The trilinear coupling defined as $\mu \equiv \partial_\phi m$ and using $m^2 = V''$ one can translate equation (3.8) into

$$\frac{1}{2}(V''')^2 \geq \frac{(V'')^2}{M_{\text{Pl}}^2}$$  \hspace{1cm} (3.9)

But this conjecture applicable only for interaction of WGC scalar $S_{\text{WGC}}$ and light scalar $\phi$, further inequality of the equation (3.8) emphasize that the existence of the fifth force which is stronger than gravitational force. Such fifth force is excluded by observational constraint on violation of the weak equivalence principle. Further, the SWGC conjecture (equation (3.9)) is inconsistent with the properties of periodic potential such as for axion. Thereby we require a modified version of SWGC which must be general for any scalar arguably massless or massive which is given as follows [62]

$$2(V'')^2 - V'''V'' \geq \frac{(V'')^2}{M_{\text{Pl}}^2}$$  \hspace{1cm} (3.10)

The conjecture equation (3.10) is valid for any canonically normalized real scalar potential as well as wide range of field value. Here the WGC scalar $S_{\text{WGC}}$ is not necessary to satisfy WGC.
because they play a role in the towers of states when they achieve equality in equation (3.10). Presumably, absence of light scalar particle as a mediator, the massive scalar decay through trilinear interaction can be intuitively thought of the attractive force or appearance of IR in the scalar interaction which is related to $V'''$. The additional term $V''''$ related to the quartic coupling of the scalar interaction which strengthened the interaction strength of the scalar against the strength of the gravity which is stronger than the strength of the scalar interaction given in the equation (3.9) thus it called a strong scalar weak gravity conjecture (SSWGC). Such inclusion of quartic (repulsive or UV) term $V''''$ in the SSWGC encapsulates the UV/IR mixing effects [62–64]. The UV/IR mixing possibly would give a better understanding about naturalness problem [65, 66] in scalar theory and the schematic behavior of UV, IR and UV/IR mixing at weak as well as strong gravity shown in figure (2) pictorially. Therefore,

![Figure 2](image)

**Figure 2.** The schematic diagram represents the interaction strength of the scalar field presence of weak/strong gravity. The trilinear coupling $\mu$ is related to IR decay and quartic interaction related to UV. Here SSWGC generates tower of light scalar states when $\chi_S = 0$ and encapsulates UV/IR mixing effects when $\chi_S \neq 0$. Where $\chi_S = 2(V''')^2 - V''V'''$.

SSWGC is applicable for any low energy effective theory with UV completion as well as entire landscape of the string vacua [62]. One can translate the SSWGC conjecture equation (3.10) comply to slow roll parameters as follows

$$\chi \equiv \frac{\xi_V^4}{\epsilon_V \eta_V^2} - \frac{\omega_V^3}{2 \epsilon_V \eta_V}$$

Where $\chi$ is the order parameter and must satisfy $\chi \geq 1$ units of $M_{Pl}$ and normalized by $\eta_V^2$. Here $\epsilon_V$ and $\eta_V$ are the slow roll parameters of order $n = 2$ of Taylor expansion of inflaton potential $V(\phi)$ which are shown in equation (3.4) and we know that the spectral tilt is $n_s = 1 - 6\epsilon_V + 2\eta_V$. The order $n = 3$ and $n = 4$ slow roll parameters are defined in terms of running of $n_s$ and running of running of $n_s$ parameter $\alpha_s$ and $\beta_s$ respectively as follows

$$\xi_V^2 = \left(\frac{V''V'''}{V^2}\right) = \frac{1}{2} \left[\frac{3r}{16} + n_s - 1 - \alpha_s\right]$$

$$\omega_V^3 = \left(\frac{V''^2V'''}{V^3}\right) = \frac{8\beta_s - 6r\alpha_s + 3r^2(r - 5)}{16} - (\alpha_s + \frac{3r}{16} + 2\xi_V^2)[1 - \frac{n_s}{2} + \frac{r(12r - 41)}{64}]$$

Here we have taken $\epsilon_V = r/16$ and $\eta_V = \frac{1}{2}(n_s - 1 + \frac{3r}{8})$. Using *Planck* TT,TE,EE + lowE + BAO + BK14 results of Taylor expanded 3rd and 4th order potential [7], one can obtain the observational constraints on SSWGC parameter ($\chi$) which must be $\chi \geq 14.27$ and $\chi \geq 49.03$. 

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when it satisfy equation (3.9) and equation (3.10) respectively. Here we considered the central value of slow roll parameters. The constraint $\chi \geq 5.09$ represents when slow roll parameter $\epsilon_4 = 0$ with set of non-zero $\epsilon_V$, $\eta_V$ and $\xi_V^2$ from Planck TT,TE,EE + lowE + lensing + BK14 result 2018 [7]. The theoretical constraints ($\chi \geq 1$) and observational constraints ($\chi \geq 49.03$) on natural inflation which provides $f_a < M_{Pl}$ and it is true for Axion WGC too.

In contrast, to satisfy $n_s - r$ constraints the natural inflation require $f_a > M_{Pl}$. Similarly Starobinsky inflation is inconsistent when $\chi \geq 1$ at trans-Planckian field space and it violated SSWGC with observational constraints when $\chi \geq 49.03$. Indeed, the Starobinsky-like inflation obey the SSWGC when $\alpha \leq 0.012$ shown in figure (3) and (4) which is closely allowed with limitation $-2 < \log_{10} \alpha < 4$ by Planck 2018 result. Thereby, observationally best-fitted model Starobinsky-like($\alpha = 1$, $\alpha > 0.012$) inflation would seem facing severe tension by SSWGC not only for super-Planckian even sub-Planckian too. Therefore such models required large
corrections to their potential to intact SSWGC at large as well as small field excursions.

Our model, QPNI with a large-N limit doesn’t require any corrections in the potential to satisfy SSWGC as well as observational data and naturally it follows always \( f_a < M_{Pl} \) which suggests that the amplitude of \( B \)-modes would perhaps smaller and smaller as shown in the table.(1). According to equation.(3.11) QPNI gets a upper limit on decay constant \( f_a \) which is \( f_a \leq 0.32/N \) shown in the figure.(4) when \( V'' > 0 \). The equation.(3.10) diverges for our model at \( \phi = N f_a / \sqrt{2} \) or in general at \( V'' = 0 \). Essentially, SSWGC is applicable to all kinds of scalar fields, but background not necessarily to be the de Sitter/quasi-de Sitter space. Meanwhile if SSWGC not valid then the string theory motivation towards quantum gravity would get severe tension.

4 Dynamical dark energy

After inflation, Universe enters into the (p)reheating era to energize the inflated fields which is required for particle creation. Our interest is to study the early inflation as well as late time inflation (cosmic acceleration) through quintessence axionic field. Thereby, the usual reheating mechanism doesn’t work, instead the instant reheating [67–70], gravitational preheating [71–73], Ricci reheating [74–78] etc., would be an alternative mechanism for reheating.

However, the quintessence scalar field model could be a natural candidate to unify the inflation and dark energy but it faces the challenge due to quantum corrections which spoil the flatness of the potential and fifth force problem by ultra-light scalar. Further, quintessence field has to survive after inflation to till date to satisfy the property of dynamical dark energy [14, 79, 80]. Axionic models solve such problems by shift symmetry even though when symmetry broken in a controlled manner to get a axion naturally light. Axionic derivative couplings suppresses the fifth-force constraints by nature [34]. Such kind of axionic potential drives early time as well as late time inflation by pseudo-Nambu-Goldstone boson(pNGB) which is slowly rolling on potential curvature. Here the energy density of the quintessence field not necessarily constant indeed, it varies dynamically as the universe evolves with respect to time [79, 81]. In this paper, we concentrate on the aspects of dynamical dark energy of our model. Let us consider the action for quintessence field in the presence of barotropic perfect fluid is given by

\[
S \supset \int d^4x \sqrt{-g} \left[ \frac{1}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]
\]

The dark energy of equation of state reads

\[
\omega = \frac{P_\phi}{\rho_\phi} = \frac{\dot{\phi}^2/2 - V(\phi)}{\dot{\phi}^2/2 + V(\phi)}
\]

The Euler-Lagrange equation of motion of the quintessence is field defined as

\[
\ddot{\phi} + 3H \dot{\phi} + V'(\phi) = 0
\]

Here \( H = \dot{a}/a \) and \( V'(\phi) = dV/d\phi \). Where \( a \) is scale factor and dot denotes derivatives with respect to time. In addition to that

\[
H^2 = \left( \frac{\ddot{a}}{a} \right)^2 = \frac{\rho}{3} \quad \text{and} \quad \frac{\ddot{a}}{a} = -\frac{(\rho + 3P)}{6}
\]
To deal with cosmological dynamics we can introduce following dimensionless variables such as

\[ x = \frac{\phi_{,N}}{\sqrt{6}}; \quad y = \sqrt{\frac{V(\phi)}{3H^2}}; \quad \lambda = -\frac{V'}{V} \]  

(4.5)

Where \( N = \ln a \) and \( \phi_{,N} = d\phi/dN \). Assume that scalar field and matter are only present then \( \Omega_\phi + \Omega_m = 1 \). The scalar field energy density \( \Omega_\phi = \rho_\phi/3H^2 \) can be written as

\[ \Omega_\phi = x^2 + y^2 \]  

(4.6)

and the equation of state

\[ \gamma = 1 + \omega = \frac{2x^2}{x^2 + y^2} \]  

(4.7)

here we work on where \( \omega \) is near to \(-1\), Which means \( \gamma << 1 \). Therefore, in order to find the fixed points of \( x \) and \( y \) for the system which satisfies equation.(4.3) and (4.4) we set \( dx/dN = 0 \) and \( dy/dN = 0 \). But \( x \) and \( y \) are related to the observable quantities \( \Omega_\phi \) and \( \gamma \), therefore we assume that \( dx/dN > 0 \) then we get the evolution equation for \( \Omega_\phi \), \( \gamma \) and \( \lambda \), which are [82–85]

\[
\frac{d\gamma}{dN} = -3\gamma(2 - \gamma) + \lambda(2 - \gamma)\sqrt{3\gamma\Omega_\phi} \\
\frac{d\Omega_\phi}{dN} = 3(1 - \gamma)\Omega_\phi(1 - \Omega_\phi) \\
\frac{d\lambda}{dN} = -\sqrt{3\lambda^2(\Gamma - 1)}\sqrt{\gamma\Omega_\phi} \\
\text{Where} \quad \Gamma = \frac{VV''}{V'^2} 
\]  

(4.8) (4.9) (4.10)

As mentioned earlier, the limit \( \gamma << 1 \) produces the solution of \( \Omega_\phi \) which is given by

\[ \Omega_\phi = [1 + (\Omega_{\phi_0}^{-1} - 1)a^{-3}]^{-1} \]  

(4.12)

Where \( \Omega_{\phi_0} \) is the present value of \( \Omega_\phi \). Importantly, the dynamical dark energy models are classified into thawing and freezing [86] and which are characteristically different behaviour. Depends on value of the \( \Gamma \), we can classify as when \( \Gamma > 1 \) known as Tracker freezing model [80], when \( \Gamma = 1 \) Scaling freezing model [82] and when \( \Gamma < 1 \) called as Thawing frozen model [84, 85]. Our model follows thawing frozen approach when \( V'' > 0 \) and the initial field value should be \( \phi_i < Nf_a/\sqrt{2} \). We have shown that our model diverges when \( V'' = 0 \) which is generally true for all potentials.

4.1 Thawing quintessential axion

In this quintessence model, the field is nearly frozen far away from the minimum of the potential by Hubble friction at early time epoch of the cosmic acceleration. Later, the field starts to roll slowly towards potential minimum due to diminishing of the Hubble friction and such damping declined to \( H_0 \) today and we get energy density parameter \( \Omega_{\phi_0} \) at \( \phi_0 \)(present field value). In such early time epoch, we assume that if the initial energy density of field is \( \rho_{\phi_i} = V(\phi_i) \) then the equation of state can be \(-1 \leq \omega_\phi \leq 0\), generally any model, the kinetic term is dominated by potential then \( \omega \) never evolves very far from \(-1\). Therefore one has to find the solution for equation.(4.3) to infer the initial field value. Let substitute change of
variable $\phi(t) = a^{-3/2}u(t)$ and evaluate the potential up to second order in Taylor expansion around initial value $\phi_i$ with the limit $\gamma << 1$ then we get

$$\ddot{u} - k^2 u \simeq -a^{-3/2} V'(\phi_i)$$  \hspace{1cm} (4.13)$$

Here $k = \sqrt{3V(\phi_i)/4 - V''(\phi_i)}$ and the scale factor approximated from $\Lambda$CDM model which is

$$a(t) = \left( \frac{1 - \Omega_{\phi 0}}{\Omega_{\phi 0}} \right)^{1/3} \sinh^{2/3}(t/t_\Lambda) \quad \text{at} \quad t_\Lambda = 2/\sqrt{3V(\phi_i)}$$  \hspace{1cm} (4.14)$$

Then the solution for $\phi$ with initial conditions $\phi(0) = \phi_i$, $\dot{\phi}(0) = 0$ and $V''(\phi_i) \neq 0$ is

$$\phi(t) = \phi_i + \frac{V'(\phi_i)}{V''(\phi_i)} \left( \frac{\sinh(kt)}{kt_\Lambda \sinh(t/t_\Lambda)} - 1 \right)$$  \hspace{1cm} (4.15)$$

Since we assume the initial energy density of field is $\rho_{\phi_i} = V(\phi_i)$ then the approximated equation of state $1 + \omega \simeq \dot{\phi}^2/V(\phi_i)$ which can be written as follows (known as Scherrer-Sen equation of state [84, 87, 88])

$$\omega(a) = -1 + (1 + \omega_0)a^{-3} \mathcal{F}(\Omega_{\phi})$$  \hspace{1cm} (4.16)$$

From equation (4.12) and (4.14) we get

$$a(t) = \left( \frac{\Omega_{\phi 0}(1 - \Omega_{\phi})}{\Omega_{\phi}(1 - \Omega_{\phi 0})} \right)^{-1/3}$$  \hspace{1cm} (4.17)$$

$$\mathcal{F}(\Omega_{\phi}) = \left[ \frac{K \cos(Kt) - 1}{\sqrt{\Omega_{\phi}}} \sinh(Kt) \right]^2$$  \hspace{1cm} (4.18)$$

$$\frac{t_0}{t_\Lambda} = \tanh^{-1}(\sqrt{\Omega_{\phi 0}})$$  \hspace{1cm} (4.19)$$

$$\frac{t}{t_\Lambda} = \sinh^{-1} \left( \sqrt{\Omega_{\phi}} \right)$$  \hspace{1cm} (4.20)$$

Importantly the potential dependent parameter defined when $K^2 > 0$ and $V''(\phi_i) > 0$ is

$$K = \sqrt{\frac{4V''(\phi_i)}{3V(\phi_i)} - 1}$$  \hspace{1cm} (4.21)$$

From ref. [8] $\omega_0 \neq -1$, thus for example the figure (5) shows that the fixed value of $\omega_0 = -0.97$ and present value of dark energy density $\Omega_{\phi 0} = 0.694$ we have arrived three initial field values due to oscillatory field solution which respects $V'' > 0$ as well as $\phi_i < Nf_a/\sqrt{2}$, it is required for cosmological evolution towards stable potential local minimum. In order to follow lower value of tensor-scalar ratio ($r$), we have chosen $Nf_a$ from table (1) which are $0.4 \times 10 M_{Pl}$ and $1.0 \times 10 M_{Pl}$. The Scherrer-Sen equation of state (equation (4.16)) for our model follows sine and cosine functions instead of hyperbolic functions by definition of $K$ given in equation (4.21). Here the value of $K$ obtained from initial field value at end of the inflation which sets the initial conditions for dynamical scalar energy density ($\rho_{\phi}$) as well as equation of state of the dark energy component ($\omega_{\phi}$).
The initial field value($\phi_i$) computed when the solution of the equation(4.16) has solved based on present $\omega_0$ and $\Omega_{\phi_0}$ as a input. At high redshift nearly $z = 10^{28}$ (where $a^{-1} = 1 + z$) with the initial field value $\phi_i$, the energy density parameter $\Omega_{\phi}$ taken to be $10^{-4}$ when all the components were in equipartition state. Since the thawing models naturally attain the value of the equation of state close to $-1$ at initial field value, then they start to increase when the field rolls towards the potential minimum which is near lower redshift as shown at the figure.(5).

Our result doesn’t affect the BBN by thawing quintessence because of they have negligible energy density parameter $\Omega_{\phi}$ and $\omega_ = -1$ at high red shift. Current value of dark energy density $\rho = \rho_c \Omega_{\phi_0}$ is equal to $(2.25 \times 10^{-5} eV)^4$ where critical density $\rho_c = 3.67 \times 10^{-47} GeV^4$ and $\Omega_{\phi_0} = 0.694$, in order to achieve these values, potential($V(\phi)$) has to meet same value for particular field value($\phi_0$) which need not to equal to zero. In principle, we can get explicit symmetry breaking scale($\Lambda$) nearly reduced Planck scale ($M_{Pl} = 2.4 \times 10^{18}GeV$), at the same time we follow $M < F_a$ thus the height of the potential($V(\phi_i) = \rho_{\phi_i}$) at initial field $\phi_i = 0.195 M_{Pl}$, $M = 10^{-2} F_a$ and $Nf_a = F_a = 0.4 M_{Pl}$ we get $\rho_{\phi_i} = 5.42 \times 10^{61} GeV^4$.

These values are taken from figure.(5) particularly $K = 5.6151$. Eventually it is warping of height of the potential by cosmic acceleration further sake of fine tuning [89, 91] we infer the $\phi_0$ value is approximately $10^{-56} F_a$ with breaking scale $\Lambda$ remains constant and it can be $\Lambda \leq 2 \times 10^{-3} M_{Pl}$ when $N \geq 4$. The mass of the quintessence particle $m_a$ at $\phi_0$ is equal to $\sqrt{2} \rho_{\phi_0}/F_a \simeq 7.5 \times 10^{-33} eV$.

5 Conclusion

In this work we have proposed a quintessence pure natural inflation model in accord with Witten’s large-N limit conjecture. Our model satisfies current Planck 2018 results as well as it will be valid for even very low tensor-scalar ratio which leads sub-planckian cosmic universe. Further we have discussed SSWGC with higher order slow-roll parameters which sets the criteria whether the model valid in both sub-panckian as well as super-planckian field space or not in any spacetime(de sitter or quasi de sitter). We have noticed that Starobinsky
model require large quantum corrections to satisfy sub-planckian and also for super-planckian. Similarly natural inflation lost their natural property being spontaneous breaking scale lesser than planck scale against Planck result but they satisfies the Axion WGC. In the late time evolution, in our model the quintessence field evolves with respect to cosmic time from initial field value($\phi_i$) to present non-zero minimum field value($\phi_0$) which is $10^{-56} F_a$ required to achieve present vacuum energy density $\rho_{\phi_0} = (2.25 \times 10^{-3} \text{eV})^4$. Such required minimum field obtained when the spontaneous breaking scale $F_a = Nf_a$ is equal to $0.4 M_{Pl}$ which maintain the property of natural scale inflation as well as constant explicit breaking scale $M = 10^{-2} F_a$ convince to arrive present vacuum energy density at mentioned $\phi_0$.

Acknowledgments

Author would like to acknowledge Prof. Partha Konar (PRL Ahmedabad) for his valuable discussions and comments. This work is supported by Physical Research Laboratory (PRL), Department of Space, Government of India and also acknowledge the computational support from Vikram-100 HPC at PRL.

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