A Folkman Linear Family

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Abstract
For graphs $F$ and $G$, let $F \rightarrow (G, G)$ signify that any red/blue edge coloring of $F$ contains a monochromatic $G$. Define Folkman number $f(G; p)$ to be the smallest order of a graph $F$ such that $F \rightarrow (G, G)$ and $\omega(F) \leq p$. It is shown that $f(G; p) \leq cn$ for graphs $G$ of order $n$ with $\Delta(G) \leq \Delta$, where $\Delta \geq 3$, $c = c(\Delta)$ and $p = p(\Delta)$ are positive constants.

Keywords: Folkman number; Folkman linear; Multi-partite regularity lemma

1 Introduction

For graphs $F$ and $G$, let $F \rightarrow (G, G)$ signify that any red/blue edge coloring of $F$ contains a monochromatic $G$. The Ramsey number $R(G)$ is the smallest $N$ such that $K_N \rightarrow (G, G)$. For most graphs $G$, it is difficult to determine the behavior of $R(G)$, and even more difficult if the edge-colored graphs are restricted within that of smaller cliques instead of the complete graphs.

Define a family $\mathcal{F}(G; p)$ of graphs as
$$\mathcal{F}(G; p) = \{ F : F \rightarrow (G, G) \text{ and } \omega(F) \leq p \},$$
where $\omega(G)$ is the clique number of $G$, and define
$$f(G; p) = \min \{|V(F)| : F \in \mathcal{F}(G; p)\},$$
which is called the Folkman number. We admit that $f(G; p) = \infty$ if $\mathcal{F}(G; p) = \emptyset$, and thus $f(G; p) = \infty$ if $p < \omega(G)$.

The investigation was motivated by a question of Erdős and Hajnal [9] who asked what was the minimum $p$ such that $\mathcal{F}(K_3; p) \neq \emptyset$. An important result of Folkman [10] states that $\mathcal{F}(K_n; p) \neq \emptyset$ for $p \geq n$, which was generalized by Nešetřil and Rödl [20] as $\mathcal{F}(G; p) \neq \emptyset$ for $p \geq \omega(G)$. The following property is clear.

Lemma 1 The function $f(G; p)$ is decreasing on $p$, and if $p \geq R(G)$, then $f(G; p) = R(G)$.

Graham [14] proved that $f(K_3; 5) = 8$ by showing $K_8 \not\rightarrow (K_3, K_3)$. Irving [15] proved that $f(K_3; 4) \leq 18$, and it was further improved by Khadzhiivanov and Nenov [16] to $f(K_3; 4) \leq 16$. Finally,
Piwakowski, Radziszowski, and Urbanski [13] and Lin [18] proved \( f(K_3; 4) = 15 \). However, both upper bounds of Folkman and of Nešetřil and Rödl for \( f(K_3; 3) \) are extremely large. Frankl and Rödl [11] first gave a reasonable bound \( f(K_3; 3) \leq 7 \times 10^{11} \). Erdős set a prize of $100 for the challenge \( f(K_3; 3) \leq 10^{10} \). This reward was claimed by Spencer [10, 11], who proved that \( f(K_3; 3) < 3 \times 10^9 \). Erdős then offered another $100 prize (see [2], page 64) for the new challenge \( f(K_3; 3) < 10^6 \). Chung and Graham [3] conjectured further \( f(K_3; 3) < 10000 \), which was confirmed by Lu [19] with \( f(K_3; 3) < 9697 \), and by Dudek and Rödl [7] with more computer aid.

Let us call a family \( G \) of graphs \( G_n \) of order \( n \) to be Ramsey linear if there exists a constant \( c = c(G) > 0 \) such that \( R(G_n) \leq cn \) for any \( G_n \in G \). Similarly, we call \( G \) to be Folkman \( p \)-linear if \( f(G_n; p) \leq cn \) for any \( G_n \in G \), where \( p \) is a constant. Let \( \Delta(G_n) \) be the maximum degree of \( G_n \) of order \( n \) and set a family of graphs as

\[
G_{\Delta} = \{ G_n \mid \Delta(G_n) \leq \Delta \}.
\]

A result of Chvátal, Rödl, Szemerédi and Trotter [5] is as follows.

**Theorem 1** The family \( G_{\Delta} \) is Ramsey linear.

The proof of Theorem 1 is a remarkable application of Szemerédi regularity lemma, in which they used the general form of the lemma. In order to generalize Theorem 1 to Folkman number, we shall have a multi-partite regularity lemma as follows.

**Theorem 2** For any \( \epsilon > 0 \) and integers \( m \geq 1 \) and \( p \geq 2 \), there exists an \( M = M(\epsilon, m, p) \) such that each \( p \)-partite graph \( G(V^{(1)}, \ldots, V^{(p)}) \) with \( |V^{(s)}| \geq M, 1 \leq s \leq p \), has a partition \( \{ V_{i}^{(s)}, \ldots, V_{k}^{(s)} \} \) for each \( V^{(s)} \), where \( k \) is same for each part \( V^{(s)} \) and \( m \leq k \leq M \), such that

1. \( |V_{i}^{(s)}| - |V_{j}^{(s)}| \leq 1 \) for each \( s \);
2. All but at most \( ek^2(p) \) pairs \( (V_{i}^{(s)}, V_{j}^{(t)}) \), \( 1 \leq s < t \leq p, 1 \leq i, j \leq k \), are \( \epsilon \)-regular.

Using the above Theorem 2 we can deduce the following result on the Folkman \( p \)-linearity of \( G_{\Delta} \) for some fixed \( p \).

**Theorem 3** Let \( \Delta \geq 3 \) be an integer and \( p = R(K_{\Delta}) \). Then the family \( G_{\Delta} \) is Folkman \( p \)-linear.

Note that for sub-family \( G_{\Delta, \chi} \) consisting of \( G \in G_{\Delta} \) with \( \chi(G) \leq \chi \), we can take \( p = R(K_{\chi}) \) such that \( G_{\Delta, \chi} \) is Folkman \( p \)-linear. A natural problem is asking what is a smaller \( p \) such that \( G_{\Delta} \) is Folkman \( p \)-linear.

For an integer \( r \geq 2 \), we call an edge coloring of a graph by \( r \) colors as an \( r \)-edge coloring of the graph. For graphs \( F \) and \( G \), let \( F \to (G)_r \) signify that any \( r \)-edge coloring of \( F \) contains a monochromatic \( G \). Thus \( R_r(G) \) is the smallest \( N \) such that \( K_N \to (G)_r \), and \( f_r(G; p) \) is the smallest \( N \) such that there exists a graph \( F \) of order \( N \) with \( \omega(F) = p \) satisfying \( F \to (G)_r \). Theorem 3 can be generalized as follows.

**Theorem 4** Let \( \Delta \geq 3 \) and \( r \geq 2 \) be integers and \( p = R_r(K_{\Delta}) \). Then, there is some constant \( c = c(\Delta, r) > 0 \) such that \( f_r(G_n; p) \leq cn \) for any \( G_n \in G_{\Delta} \).

## 2 Multi-partite regularity lemma

Let \( A \) be a set of positive integers and \( A_n = A \cap \{1, \ldots, n\} \). In the 1930s, Erdős and Turán conjectured that if \( \lim_{n \to \infty} \frac{|A_n|}{n} > 0 \), then \( A \) contains arbitrarily long arithmetic progressions. The conjecture in case of length 3 was proved by Roth [22, 23]. The full conjecture was proved by Szemerédi [20] with a deep and complicated combinatorial argument. In the proof he used a result, which is now called the bipartite regularity lemma, and then he proved the general regularity lemma in [24]. The lemma has become a totally new tool in extremal graph theory. Sometimes the regularity lemma is called uniformity lemma,
see e.g., Bollobás [2] and Gowers [13]. For many applications, we refer the readers to the survey of Komlós and Simonovits [17]. In this note, we shall discuss multi-partite regularity lemma in slightly different forms.

Let $G(U, V)$ be a bipartite graph on two color classes $U$ and $V$. For $X \subseteq U$ and $Y \subseteq V$, denote by $e(X, Y)$ the number of edges between $X$ and $Y$ of $G$. The ratio

$$d(X, Y) = \frac{e(X, Y)}{|X||Y|}$$

is called the edge density of $(X, Y)$, which is the probability that any pair $(x, y)$ selected randomly from $X \times Y$ is an edge. Clearly $0 \leq d(X, Y) \leq 1$.

The first form of regularity lemma given by Szemerédi in [26] is as follows, in which corresponding to each subset $U_i$ in the partition of $U$, we have to choose its own partition $V_{ij}$ of $V$.

**Lemma 2 (Bipartite Regularity Lemma-Old Form)** For any positive $\epsilon_1, \epsilon_2, \delta, \rho_1, \rho_2$, there exist $k_1, k_2, M_1, M_2$ such that every bipartite graph $G(U, V)$ with $|U| > M_1$ and $|V| > M_2$, there exist disjoint $U_i \subseteq U$, $i < k_1$, and for each $i < k_1$, disjoint $V_{ij} \subseteq V$, $j < k_2$, such that:

1. $|U_i - \cup_{j<k_2}U_i| < \rho_1|U_i|$, and $|V_j - \cup_{i<k_1}V_{ij}| < \rho_2|V_j|$ for any $i < k_1$;
2. For all $i < k_1$, $j < k_2$, $X \subseteq U_i$ and $Y \subseteq V_{ij}$ with $|X| > \epsilon_1|U_i|$ and $|Y| > \epsilon_2|V_{ij}|$, we have
   $$d(X, Y) \geq d(U_i, V_{ij}) - \delta;$$

3. For all $i < k_1$, $j < k_2$ and $x \in U_i$, $|N(x) \cap V_{ij}| \leq (d(U_i, V_{ij}) + \delta)|V_{ij}|$.

For $\epsilon > 0$, a disjoint pair $(X, Y)$ is called $\epsilon$-regular if any $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| > \epsilon|X|$ and $|Y'| > \epsilon|Y|$ satisfy

$$|d(X, Y) - d(X', Y')| \leq \epsilon.$$

We shall call $U_0 = U - \cup_{i<k_1}U_i$, and $V_0 = V - \cup_{j<k_2}V_{ij}$ in Theorem 2 to be the exceptional sets. The following is the general regularity lemma of Szemerédi [27], in which the partition $C_0, C_1, \ldots, C_k$ is equitable in sense of that all sets $C_i$ other than the exceptional set $C_0$ have the same size.

**Lemma 3 (General Regularity Lemma)** For any $\epsilon > 0$ and any $m \geq 1$, there exists $M = M(\epsilon, m) > m$ such that every graph $G$ of order at least $m$ has a partition $C_0, C_1, \ldots, C_k$ with $m \leq k \leq M$ such that:

1. $|C_0| = |C_2| = \cdots = |C_k|$ and $|C_0| \leq cm$;
2. All but at most $ck^2$ pairs $(C_i, C_j)$ with $1 \leq i < j \leq k$ are $\epsilon$-regular.

There are many generalizations of Szemerédi regularity lemma, in particular, Frankl and Rödl [12] generalized it to hypergraphs and later Chung [4] formulated regularity lemma on $t$-uniform hypergraphs when discussing the problems of quasi-random hypergraphs.

The regularity lemma has numerous applications in various areas, mainly in extremal graph theory such as [5] by Chvátal, Rödl, Szemerédi and Trotter. In an application, Eaton and Rödl [8] obtained a form of the regularity lemma for $p$-partite $p$-uniform hypergraph. To state their result for multi-partite graph, let us have some definitions.

Let $G(V^{(1)}, \ldots, V^{(p)})$ be a $p$-partite graph on vertex set $\cup_{i=1}^p V^{(i)}$. Consider partitions of the set $V^{(1)} \times \cdots \times V^{(p)}$, where each partition class is of the form $W_1 \times \cdots \times W_p$, $W_i \subseteq V^{(i)}$, $1 \leq i \leq p$, which is called cylinders. Let us say that a cylinder $W_1 \times \cdots \times W_p$ is $\epsilon$-regular if the subgraph of $G$ induced on the set $\cup_{i=1}^p W_i$ is such that all pairs $(W_i, W_j)$, $1 \leq i < j \leq p$, are $\epsilon$-regular.

Eaton and Rödl stated their result with exceptional $p$-tuples instead of exceptional sets, for which Alon, Duke, Leffmann, Rödl and Yusterk studied the computational difficulty of finding such a regular partition in [4].
Lemma 4 Let \( G(V^{(1)}, \ldots, V^{(p)}) \) be a \( p \)-partite graph with \( |V^{(i)}| = n, i = 1, \ldots, p \). Then for every \( \epsilon > 0 \) there exists a partition of \( V^{(1)} \times \cdots \times V^{(p)} \) into \( k \) cylinders with \( k \leq 4^h \), where \( h = \frac{\epsilon}{2} \), such that all but at most \( \epsilon n^p \) of the \( p \)-tuples \( (v_1, \ldots, v_p) \) of \( V^{(1)} \times \cdots \times V^{(p)} \) are in \( \epsilon \)-regular cylinders of the partition.

Note that in Lemma 6 the transverse section \( \{W_i\} \) of the partition is a partition of \( V^{(i)} \), which may be not equitable, and for \( i \neq j \), the numbers of subsets in the partitions \( \{W_i\} \) and \( \{W_j\} \) may be different. We shall have a multi-partite regularity lemma as follows.

Lemma 5 For any \( \epsilon > 0 \) and integers \( m \geq 1 \) and \( p \geq 2 \), there exists \( M = M(\epsilon, m, p) \) such that each \( p \)-partite graph \( G(V^{(1)}, \ldots, V^{(p)}) \) with \( |V^{(s)}| \geq M \), \( 1 \leq s \leq p \), has a partition \( \{V^{(s)}_0, V^{(s)}_1, \ldots, V^{(s)}_k\} \) for each \( V^{(s)} \), where \( k \) is same for each part \( V^{(s)} \) and \( m \leq k \leq M \), such that

1. \( |V^{(s)}_i| = \cdots = |V^{(s)}_k| \) and \( |V^{(s)}_0| \leq \epsilon |V^{(s)}| \) for each \( s \);
2. All but at most \( \epsilon k^2 p \) pairs \( (V^{(s)}_i, V^{(t)}_j), 1 \leq s < t \leq p, 1 \leq i, j \leq k \), are \( \epsilon \)-regular.

The following multicolor multi-partite regularity lemma is an analogy of Theorem 2, which is needed for proof of Theorem 4.

Lemma 6 For any \( \epsilon > 0 \) and integers \( m \geq 1 \), \( p \geq 2 \) and \( r \geq 1 \), there exists an \( M = M(\epsilon, m, p, r) \) such that if the edges of a \( p \)-partite graph \( G(V^{(1)}, \ldots, V^{(p)}) \) with \( |V^{(s)}| \geq M \), \( 1 \leq s \leq p \), are \( r \)-colored, then all monochromatic graphs have the same partition \( \{V^{(s)}_1, \ldots, V^{(s)}_k\} \) for each \( V^{(s)} \), where \( k \) is same for each part \( V^{(s)} \) and \( m \leq k \leq M \), such that

1. \( ||V^{(s)}_i| - V^{(s)}_j|| \leq 1 \) for each \( s \);
2. All but at most \( \epsilon k^2 r p \) pairs \( (V^{(s)}_i, V^{(t)}_j), 1 \leq s < t \leq p, 1 \leq i, j \leq k \), are \( \epsilon \)-regular in each monochromatic graph.

3 Proofs for multi-partite regularity lemma

In this section, we prove Lemma 5, Theorem 2, and Lemma 6. To reduce the complicity of notations in the proofs, we shall prove them in case \( p = 2 \), which are bipartite regularity lemmas.

Lemma 7 Let \( G(U, V) \) be a bipartite graph and let \( X \subseteq U \) and \( Y \subseteq V \). If \( X' \subseteq X \) and \( Y' \subseteq Y \) satisfy \( |X'| > (1 - \delta)|X| \) and \( |Y'| > (1 - \delta)|Y| \), then

\[
|d(X', Y') - d(X, Y)| < 2\delta \quad \text{and} \quad |d^2(X', Y') - d^2(X, Y)| < 4\delta.
\]

A crucial point for the regularity lemma in partition is that the number \( k \) of classes in partition is bounded for any graph. For proofs, we need the well-known defect form of Cauchy-Schwarz inequality.

Lemma 8 Let \( d_i \) be reals and \( s > t \geq 1 \) be integers. If

\[
\frac{1}{s} \sum_{i=1}^{s} d_i = \frac{1}{t} \sum_{i=1}^{t} d_i + \delta,
\]

then

\[
\frac{1}{s} \sum_{i=1}^{s} d_i^2 \geq \left( \frac{1}{s} \sum_{i=1}^{s} d_i \right)^2 + \frac{t\delta^2}{s-t}.
\]
Let $G(U, V)$ be a bipartite graph, a partition

$$\mathcal{P} = \left\{ U_i, V_j \mid 0 \leq i, j \leq k \right\},$$

where $U = \bigcup_{i=1}^k U_i$ and $V = \bigcup_{i=1}^k V_i$, is called to be an equitable partition of $U \cup V$ with exceptional classes $U_0$ and $V_0$ if $|U_i| = |U_j|$ and $|V_i| = |V_j|$ for $1 \leq i, j \leq k$. For convenience, we say an equitable partition $\mathcal{P}$ is $\epsilon$-regular if all but at most $\epsilon k^2$ pairs of $(U_i, V_j)$ are $\epsilon$-regular. Define

$$q(\mathcal{P}) = \frac{1}{k^2} \sum_{1 \leq i, j \leq k} d^2(U_i, V_j).$$

It is easy to see that $0 \leq q(\mathcal{P}) \leq 1$ since $0 \leq d(U_i, V_j) \leq 1$.

In the following, we will show that if $\mathcal{P}$ is not $\epsilon$-regular, then there is a partition $\mathcal{P}'$ with the new exceptional classes a bit larger than the old one, but $q(\mathcal{P}') \geq q(\mathcal{P}) + \frac{\epsilon}{2}$. Do this again if $\mathcal{P}'$ is not $\epsilon$-regular yet. The number of iterations is thus at most $4/\epsilon^2$ in order to obtain an $\epsilon$-regular partition. Without loss of generality, we assume that $0 < \epsilon \leq 1/2$ since if $\epsilon > 1/2$, one can take $M(\epsilon, m)$ to be $M(1/2, m)$.

**Lemma 9** Let $G(U, V)$ be a bipartite graph with $|U| = n_1 \geq M$ and $|V| = n_2 \geq M$, which has an equitable partition

$$\mathcal{P} = \left\{ U_i, V_j \mid 0 \leq i, j \leq k \right\}$$

with exceptional classes $U_0$ and $V_0$. Suppose $2^k \geq 16/\epsilon^5$, $|U_i| = c_1 \geq 2^{3k}$ and $|V_j| = c_2 \geq 2^{3k}$. We have if $\mathcal{P}$ is not $\epsilon$-regular; then there is an equitable partition

$$\mathcal{P}' = \left\{ U_i', V_j' \mid 0 \leq i, j \leq \ell \right\}$$

with exceptional class $U_0' \supseteq U_0$ and $V_0' \supseteq V_0$, and $\ell = k(4^k - 2^k)$ satisfying

1. $|U_0'| \leq |U_0| + n_1/2^{k-1}$ and $|V_0'| \leq |V_0| + n_2/2^{k-1}$;
2. $q(\mathcal{P}') \geq q(\mathcal{P}) + \epsilon^5/4$.

**Proof.** Separate all pairs $(i, j)$, $1 \leq i, j \leq k$, of indices into $S$ and $T$, corresponding with that the pair $(U_i, V_j)$ is $\epsilon$-regular or not, respectively. For $(i, j) \in S$, set $U_{ij} = V_{ji} = \emptyset$, and for $(i, j) \in T$, set $U_{ij} \subseteq U_i$ and $V_{ji} \subseteq V_j$ with $|U_{ij}| > \epsilon c_1$, $|V_{ji}| > \epsilon c_2$, and

$$d(U_{ij}, V_{ji}) - d(U_i, V_j) > \epsilon.$$

For fixed $i$, $1 \leq i \leq k$, consider an equivalence relation $\equiv$ on $U_i$ as $x \equiv y$ if and only if both $x$ and $y$ belong to the same $U_{ij}$'s. The equivalence classes are atoms of algebra induced by $U_{ij}$, and each $U_i$ has at most $2^k$ atoms. Similarly, each $V_j$ has at most $2^k$ atoms.

For $p = 1, 2$, set $d_p = [c_p/4^k]$. Let us cut each atom in $U_i$ into pairwise disjoint $d_1$-subsets. Denote by $z$ for the maximal number of these $d_1$-subsets that one can take, clearly $z \geq 4^k - 2^k$ as $zd_1 + 2^k(d_1 - 1) \geq c_1$. Set

$$H = 4^k - 2^k,$$

and take exactly $H$ such $d_1$-subsets and add the remainder to the “rubbish bin” to get a new exceptional set $U_0'$. Label all these $d_1$-subsets in $U_i$ as $D_{i1}, \ldots, D_{iH}$. Set $U_0' = U_0 \cup \left[ \bigcup_{h=1}^{H-1} \left( U_i \setminus \cup_{h=1}^{H-1} D_{ih} \right) \right]$, and so $|U_0'| = |U_0| + k(c_1 - Hd_1)$. Since

$$Hd_1 \geq (4^k - 2^k)(\frac{c_1}{4^k} - 1) > c_1 - \frac{c_1}{2^{k-1}}$$

by noting $c_1 \geq 2^{3k}$, we have $|U_0'| \geq |U_0| + n_1/2^{k-1}$. Rename $D_{ih}$ as $U_s'$ for $1 \leq s \leq \ell$, where $\ell = kH$. 

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Similarly, we can cut each atom in $V_j$ into pairwise disjoint $d_2$-subsets and take $H$ such subsets $E_{j1}, \ldots, E_{jH}$ in $V_j$. Set $V_0' = V_0 \cup \left( \cup_{i=1}^k \left( V_j \setminus \cup_{h=1}^H E_{jh} \right) \right)$, and similarly $|V_0'| \leq |V_0| + n_2/2^{k-1}$. Rename $E_{jh}$ as $V_t'$ for $1 \leq t \leq \ell$.

Denote the new equitable partition by

$$P' = \left\{ U_i', V_j' \mid 0 \leq i, j \leq \ell \right\}$$

of $U \cup V$ with exceptional classes $U_0' \supseteq U_0$ and $V_0' \supseteq V_0$. All that remains is to show $q(P') \geq q(P) + \epsilon^5/4$.

For $1 \leq i, j \leq k$, set

$$U_i = \cup_{h=1}^H D_{ih}, \quad U_{ij} = \cup \{ D_{ih} : D_{ih} \subseteq U_{ij} \}, \quad \text{and} \quad V_j = \cup_{h=1}^H E_{jh}, \quad V_{ji} = \cup \{ E_{jh} : D_{jh} \subseteq V_{ji} \}.$$

Set partition $P = \{ U_0', U_1', \ldots, U_k'; V_0', V_1', \ldots, V_k' \}$ with exceptional class $U_0'$ and $V_0'$.

**Claim 1.** $q(P') \geq q(P) - \epsilon^5/2$.

**Proof of Claim 1.** Note that $\frac{|U_i \setminus U_{ij}|}{|U_{ij}|} < \frac{1}{2} < \frac{\epsilon^4}{8}$ and $\frac{|V_j \setminus V_{ji}|}{|V_{ji}|} < \frac{\epsilon^4}{8}$ for any pair $(U_i, V_j)$, we have

$$|d(U_i, V_j) - d(U_i', V_j')| \leq \frac{\epsilon^5}{4} \quad (1)$$

by Lemma. Hence $d^2(U_i, V_j) \geq d^2(U_i', V_j') - \epsilon^5/2$, which implies that $q(P') \geq q(P) - \epsilon^5/2$ as claimed.

**Claim 2.** If $(i, j) \in T$, then $|d(U_{ij}, V_{ji}) - d(U_{ij}, V_{ji})| > \frac{\epsilon^3}{12} \epsilon$.

**Proof of Claim 2.** Clearly, $\frac{|U_{ij} \setminus U_{ij}'|}{|U_{ij}'|} \leq \frac{\epsilon^3}{8}$ and $\frac{|V_{ji} \setminus V_{ji}'|}{|V_{ji}'|} \leq \frac{\epsilon^3}{8}$, which and Lemma give

$$|d(U_{ij}, V_{ji}) - d(U_{ij}, V_{ji})| \leq \frac{\epsilon^4}{4} \quad (2)$$

Therefore, if $(i, j) \in T$, the bounds (1) and (2) with the fact that $0 < \epsilon \leq 1/2$ will yield the desired inequality.

Let us return to the partition $P'$ in which each class is either a $d_1$-subset $D_{iu}$ or a $d_2$-subset $E_{jv}$ except $U_0'$ and $V_0'$. For any pair $(U_i, V_j)$,

$$d(U_i, V_j) = \frac{1}{H^2} \sum_{1 \leq u, v \leq H} d(D_{iu}, E_{jv})$$

since $|U_i| = Hd_1$ and $|V_j| = Hd_2$. Set

$$A(i, j) = \frac{1}{H^2} \sum_{1 \leq u, v \leq H} d^2(D_{iu}, E_{jv}).$$

Then from Cauchy-Schwarz inequality, for any pair $(i, j)$, we have

$$A(i, j) \geq d^2(U_i, V_j). \quad (3)$$

If $(i, j) \in T$, we have some gain. Let $R = R(i, j)$ be the set of indices $(u, v)$ such that $D_{iu} \in U_{ij}$ and $E_{jv} \in V_{ji}$. Then

$$d(U_{ij}, V_{ji}) = \frac{1}{|R|} \sum_{(u, v) \in R} d(D_{iu}, E_{jv}).$$
Note that \(\frac{|R|}{H^2} = \frac{k}{2} + \frac{4}{k} \geq (1 - 2^{-7})\epsilon^2\), So Lemma 5 and Claim 2 imply

\[
A(i, j) \geq d^2(U_i, V_j) + \frac{|R|}{H^2} (d(U_i, V_j) - d(U_i, V_j))^2 \geq d^2(U_i, V_j) + \frac{3}{4} \epsilon^4. \tag{4}
\]

Noticing that \(\ell = kH\), we have

\[
q(P') = \frac{1}{k^2} \sum_{1 \leq s \neq t \leq \ell} d^2(U_s', V_t') = \frac{1}{k^2} \sum_{1 \leq s \neq t \leq \ell} \sum_{1 \leq i, j \leq \ell} d^2(D_{ii}, E_{jj}) = \frac{1}{k^2} \sum_{1 \leq i, j \leq k} A(i, j).
\]

Now, combine inequalities (3) and (4), and recall Claim 1 and that \(P\) is not \(\epsilon\)-regular, we have

\[
q(P') \geq \frac{1}{k^2} \left[ \sum_{(i,j) \in S} d^2(U_i, V_j) + \sum_{(i,j) \in \mathcal{T}} \left( d^2(U_i, V_j) + \frac{3}{4} \epsilon^4 \right) \right] \geq q(P) + \frac{\epsilon^5}{4}.
\]

This completes the proof of Lemma 6.

\[\square\]

**Proof of Lemma 5**. Let \(k_0 \) be an integer such that \(k_0 \geq m\) and \(2^{-k_0} \leq \epsilon^5/16\), and define \(k_{i+1} = k_i (4^{k_i} - 2^{k_i})\). Set \(M_i = k_i 2^{k_i}\), and \(M = M_i\). Lemma 5 implies that at most \(t = 4|\epsilon^{-5}\) iterations will yield a required partition, which completes the proof of Lemma 6.

\[\square\]

**Proof of Theorem 2**. For given \(\epsilon > 0\) and \(m \geq 1\), Theorem 2 implies that there is an \(M > m\) and an equitable and \(\frac{\epsilon}{4}\)-regular partition \(P = \{U_i, V_j\}, 0 \leq i, j \leq k\) with \(m \leq k \leq M\). Since \(|U_0| < \frac{\epsilon}{4}n_1\), we have \([1 - \epsilon^2/4]n_1/k \leq |U_0| \leq n_1/k\). Partition \(U_0\) into \(k\) classes \(U_{01}, U_{02}, \ldots, U_{0k}\) such that \(|U_{0i}| = [\frac{|U_0|}{k}]\) or \([\frac{|U_0|}{k}]\). Set \(U_i' = U_i \cup U_0\), clearly \(|U_i'| = n_1/k\) or \(|U_i'| = n_1/k\). Similarly, let us partition \(V_0\) into \(k\) classes \(V_{01}, V_{02}, \ldots, V_{0k}\) such that \(|V_0| = [\frac{|V_0|}{k}]\) or \(|V_0| = [\frac{|V_0|}{k}]\). Set \(V_i' = V_i \cup V_0\), we have the sizes of any \(V_i'\) and \(V_j'\) differ at most by one. Then the Partition \(P' = \{U_i', V_j'\}, 0 \leq i, j \leq k\) is as desired by noting that if a pair \((U_i, V_j)\) is \(\frac{\epsilon}{4}\)-regular, then \((U_i', V_j')\) is \(\epsilon\)-regular.

\[\square\]

**Proof of Lemma 6**. A similar proof as Theorem 2 but modify the definition of index by summing the indices for each color,

\[
q(P) = \frac{1}{k^2} \sum_{1 \leq k \leq r} \sum_{1 \leq s < t \leq p} \sum_{1 \leq i, j \leq k} d^2(V_i^{(s)}, V_j^{(t)}).
\]

Then we have analogy of Lemma 5 for multi-color case. Furthermore, we have Lemma 6.

\[\square\]

### 4 A Folkman linear family

In this section, we shall apply multi-partite regularity lemma to the Folkman numbers involving the family \(G_{\Delta}\) of graphs with maximum degree bounded. In order to prove Theorem 3 and Theorem 4, we shall establish the following Lemma, in which \(K_p(k)\) is the complete \(p\)-partite graph with \(k\) vertices in each part.

**Lemma 10**. For integers \(k \geq 1\) and \(p \geq 2\), let \(t_p(k)\) be the maximum number of edges in a subgraph of \(K_p(k)\) that contains no \(K_p\). Then

\[
t_p(k) = \left[ \frac{p}{2} \right] k^2.
\]
**Proof.** By deleting all edges between a pair of parts of $K_p(k)$, we have the lower bound for $t_p(k)$ as required. On the other hand, we shall prove by induction of $k$ that if a subgraph $G = G(V^{(1)}, \ldots, V^{(p)})$ of $K_p(k)$ contains no $K_p$, then $\epsilon(G) \leq \left(\frac{p}{2} - \frac{1}{2}\right) k^2$. Suppose $k \geq 2$ and $p \geq 3$ as it is trivial for $k = 1$ or $p = 2$. Furthermore, suppose that $G$ has the maximum possible number of edges subject to this condition. Then $G$ must contain $K_p - e$ as a subgraph, otherwise we could add an edge and the resulting graph would still not contain $K_p$. Pick a vertex set $X$ consisting of a vertex from each $V^{(i)}$ for $i = 1, 2, \ldots, p$ such that $\epsilon(X)$ is maximum among all such vertex subsets, and so $\epsilon(X) = \left(\frac{p}{2} - \frac{1}{2}\right) k^2$. We may suppose that $X$ induces a complete graph of order $p$ with an edge $v_1v_2$ missing, where $v_1 \in V_1$ and $v_2 \in V_2$. Let $Y = V(G) \setminus X$, clearly each part of $Y$ has $k - 1$ vertices. Now, by noticing the fact that no vertex in $V^{(i)} \cap V(Y)$ is adjacent to all the vertices of $X \setminus \{v_i\}$ for $i = 1, 2$ since $G$ contains no $K_p$, we can safely deduce the desired upper bound of $t_p(k)$ by a simple calculation, which completes the induction hypothesis hence the proof. 

**Lemma 11** Let $(A, B)$ be an $\epsilon$-regular pair of density $d \in (0, 1]$, and $Y \subseteq B$ with $|Y| \geq \epsilon |B|$. Then there exists a subset $A' \subseteq A$ with $|A'| \geq (1 - \epsilon)|A|$, each vertex in $A'$ is adjacent to at least $(d - \epsilon)|Y|$ vertices in $Y$.

**Proof.** Let $X$ be the set of vertices with fewer than $(d - \epsilon)|Y|$ neighbors in $Y$. Then $\epsilon(X, Y) < (d - \epsilon)|X||Y|$, so $d(X, Y) < d - \epsilon$. Since $(A, B)$ is $\epsilon$-regular, this implies that $|X| < \epsilon |A|$. 

**Proof of Theorem 3.** We will consider a red/blue edge coloring of $K_p(cn)$. Denote by $H_R$ and $H_B$ the subgraphs spanned by red edges and blue edges, respectively. Note that a partition obtained by applying Theorem 2 for $\Delta$ is such a partition for $H_B$.

Let $p = R(K_\Delta)$ as defined. Clearly, we can only consider graphs $G = G_n$ in $G_\Delta$ with $n \geq \Delta + 2$. Choose $\epsilon = \min\left\{\frac{1}{p}, \frac{1}{\Delta}\right\}$, where $m$ is a positive integer such that

$$(1 - \Delta \epsilon)(1/2 - \epsilon)^2 m \geq 1 \quad \text{hence} \quad (1 - \Delta \epsilon)(1/2 - \epsilon)^2 \geq \epsilon.$$ 

Let $M = M(\epsilon, m, p) > 2m$ be the integer determined by $\epsilon$ and $p$ in Theorem 2 for $H_R$. Finally, let $c = mM$ which is a constant determined completely by $\Delta$. We shall show that either $H_R$ contains $G$ or $H_B$ contains $G$, hence $f(G; p) \leq c pn$.

Let the vertex set of the $K_p(cn)$ be $V = V^{(1)} \cup \cdots \cup V^{(p)}$ with $|V_\ell| = cn$ for $1 \leq \ell \leq p$. There is a partition of $V$, in which each $V^{(\ell)}$ is partitioned into $\{V_1^{(\ell)}, \ldots, V_{k}^{(\ell)}\}$ with $|V_1^{(\ell)}| = |V_k^{(\ell)}| \leq 1$ and $m \leq k \leq M$, and all but at most $ek^2(\frac{p}{2})$ pairs $(V_i^{(s)}, V_j^{(t)})$, $1 \leq i, j \leq k$, $1 \leq i \neq t \leq p$, are $\epsilon$-regular.

Let $F$ be the subgraph of $K_p(k)$, whose vertices are $\{V_1^{(\ell)} \mid 1 \leq \ell \leq p, 1 \leq i \leq k\}$ in which a pair $(V_i^{(s)}, V_j^{(t)})$ for $s \neq t$ is adjacent if and only if the pair is $\epsilon$-regular in $H_R$. Then the number of edges of $F$ is at least

$$(1 - \epsilon)k^2\left(\frac{p}{2} - \frac{1}{2}\right) k^2 = t_p(k).$$

By Lemma 11, $F$ contains a complete graph $K_p$. Without loss of generality, assume that $V_1^{(1)}, \ldots, V_p^{(p)}$ are pairwise $\epsilon$-regular. Color an edge between a pair $(V_i^{(s)}, V_j^{(t)})$ green if $d(V_i^{(s)}, V_j^{(t)}) \geq 1/2$, or white if $d(V_i^{(s)}, V_j^{(t)}) < 1/2$. As $p = R(K_\Delta)$, we have $\Delta$ sets in $\{V_1^{(1)}, V_1^{(2)}, \ldots, V_1^{(p)}\}$ such that they form a monochromatic $K_\Delta$. We may assume that the color is green since otherwise we consider the graph $H_B$.

Relabeling the sets in the partition if necessary, we assume that $V_1^{(1)}, V_1^{(2)}, \ldots, V_1^{(\Delta)}$ are pairwise $\epsilon$-regular in $H_R$, and $d(V_1^{(s)}, V_1^{(t)}) \geq 1/2$. Write

$$C_1 = V_1^{(1)}, C_2 = V_1^{(2)}, \ldots, C_\Delta = V_1^{(\Delta)}.$$
We thus finished the general step hence the proof of Theorem 3. 

Note that if $Y_i \subseteq C_i$ with $|Y_i| \geq (1 - \Delta \epsilon)(1/2 - \epsilon)\Delta |C_i|$, then $|Y_i| \geq \epsilon |C_i|$, which is the preparation for using Lemma [1] and 

$$|Y_i| \geq (1 - \Delta \epsilon)(1/2 - \epsilon)\Delta \frac{cn}{M} \geq n,$$

which will give us enough room to maneuver for constructing a color class of $G$.

Note that if a graph is neither a complete graph nor an odd cycle, its chromatic number is at most $\Delta(G)$. For considered graph $G = G_n$, as $n \geq \Delta + 2$ and $\Delta \geq 3$, we have $\chi(G) \leq \Delta$.

Assume that $V(G) = \{u_1, u_2, \ldots, u_n\}$. We shall show that the red graph $H_R$ contains $G$ as a subgraph. We will choose $v_1, v_2, \ldots, v_n$ from the sets $C_1, C_2$. Since $\chi(G) \leq \Delta$, so $V(G)$ can be partitioned into $\Delta$ color classes, which defines a map $\phi: \{1, \ldots, n\} \rightarrow \{1, \ldots, \Delta\}$, where $\phi(i)$ is the color of vertex $u_i$. Our aim is to define an embedding $u_i \rightarrow v_i \in C_{\phi(i)}$, such that $v_i, v_j$ is an edge of $H_R$ whenever $u_i, u_j$ is an edge of $G$.

Our plan is to choose the vertices $v_1, \ldots, v_n$ inductively. Throughout the induction, we shall have a target set $Y_i \subseteq C_{\phi(i)}$ assigned to each $i$. Initially, $Y_i$ is the entire $C_{\phi(i)}$. As the embedding proceeds, $Y_i$ will get smaller and smaller. Some vertices will be deleted in procedure. But any $C_{\phi(i)}$ will really have some vertices deleted at most $\Delta$ times. To make this approach work, we have to ensure $Y_i$ do not get too small.

Let us begin the initial step. Set

$$Y_1^0 = C_{\phi(1)}, \ Y_2^0 = C_{\phi(2)}, \ldots, \ Y_n^0 = C_{\phi(n)}.$$

Note that $Y_i^0$ and $Y_j^0$ are not necessarily distinct sets.

We then begin the first step by considering $u_1$, for which $v_1$ will be selected from $Y_1^0$, and its neighbors, $u_\alpha, \ldots, u_\beta$, say. Suppose that the degree of $u_1$ is $d$. By using Lemma [1] repeatedly, we know that there exists a subset $Y_1^1 \subseteq Y_1^0$ with $|Y_1^1| \geq (1 - \Delta \epsilon)d |C_1| \geq n$, such that each vertex in $Y_1^1$ has at least $(1/2 - \epsilon)|Y_1^0|$ neighbors in $Y_1^0$, where $j = \alpha, \ldots, \beta$. Choose an arbitrary vertex $v_1$ from $Y_1^1$. For $j = \alpha, \ldots, \beta$, define $Y_1^j$ be the neighborhood of $v_1$ in $Y_1^0$. For $j \geq 2, j \neq \alpha, \ldots, \beta$, define $Y_1^j = Y_1^0$, that is, no vertices are deleted from such $Y_1^0$. In this step, $v_1$ has been chosen and it completely adjacent to $Y_1^1$ in $H$ whenever $u_1$ and $u_j$ are adjacent in $G$.

In a general step, we consider $u_1$ and its neighbors. We will choose $v_1$ for $u_i$ from $Y_i^{j-1}$. Suppose that $u_i$ has $d_1$ neighbors in $\{u_1, \ldots, u_{i-1}\}$, and $d_2$ neighbors, $u_\alpha, \ldots, u_\beta$, say, in $\{u_{i+1}, \ldots, u_n\}$. Then $d_1 + d_2 \leq \Delta$, and $|Y_i^{j-1}| \geq (1/2 - \epsilon)^{d_1} |Y_i^0|$. That is to say, the current set $Y_i^{j-1}$ are obtained from $Y_i^0$ by deleting some vertices $d$ times before this step. By using Lemma [1] repeatedly again, we have a subset $Y_i^j \subseteq Y_i^{j-1}$ with $|Y_i^j| \geq (1 - d\epsilon) |Y_i^{j-1}|$ such that each vertex in $Y_i^j$ has at least $(1/2 - \epsilon)|Y_i^{j-1}|$ neighbors in $Y_i^{j-1}$, where $j = \alpha, \ldots, \beta$. Since 

$$|Y_i^j| \geq (1 - d\epsilon) |Y_i^{j-1}| \geq (1 - d\epsilon)(1/2 - \epsilon)^{d_2} |Y_i^0| \geq (1 - \Delta \epsilon)(1/2 - \epsilon) \Delta |C_i| \geq n,$$

we can choose a vertex $v_i$ from $Y_i^j$, which is distinct from $v_1, \ldots, v_{i-1}$ that have been chosen before this step, and some may be from $Y_i^j$. For $j = \alpha, \ldots, \beta$, define $Y_i^j$ to be the neighborhood of $v_i$ in $Y_i^{j-1}$. For $j \geq i + 1, j \neq \alpha, \ldots, \beta$, define $Y_i^j = Y_i^{j-1}$, that is, no vertices are deleted from such $Y_i^{j-1}$. Note that $v_i$ is adjacent to any $v_j$, where $j < i$ and $u_j$ is adjacent to $u_i$, and $v_i$ is completely connected with each set $Y_j^i$, in which a neighbor of $v_i$ will be selected after this step.

It is easy to check that the condition for using Lemma [1] can be satisfied since $(1 - \Delta \epsilon)(1/2 - \epsilon)^\Delta \geq \epsilon$. We thus finished the general step hence the proof of Theorem [4].

**Proof of Theorem [4]** For $p = R_r(K_\Delta)$, take $\epsilon = \min\{\frac{1}{r}, \frac{1}{m}\}$, where $m$ is an integer such that 

$$(1 - \Delta \epsilon)(1/r - \epsilon)^\Delta m \geq 1.$$
In the proof, we use Lemma 6. We shall have $p$ sets, say $V_1^{(1)}, \ldots, V_1^{(p)}$, such that every pair $(V_1^{(s)}, V_1^{(t)})$, $1 \leq s < t \leq p$, is $\epsilon$-regular in each monochromatic graph. Connecting this pair with color $\ell$ if its edge density is at least $1/r$ in the monochromatic graph in color $\ell$, $1 \leq \ell \leq r$. Then we have a $r$-edge coloring of $K_p$, which implies a monochromatic $K_\Delta$ in some color, say the color $a$. Hence we obtain $\Delta$ sets, say $V_1^{(1)}, \ldots, V_1^{(\Delta)}$, such that each pair $(V_1^{(s)}, V_1^{(t)})$, $1 \leq s < t \leq \Delta$, is $\epsilon$-regular in monochromatic graph of color $a$, and the edge density of the pair is at least $1/r$ in this color. The remaining proof is similar to that for Theorem 3. 

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