QUANTUM ERROR CORRECTION WITH REFLEXIVE STABILIZER CODES AND CAYLEY GRAPHS

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Abstract. Long distance communication of digital data, whether through a physical medium or a broadcast signal, is often subjected to noise. To deliver data reliably through noisy communication channels, one must use codes that can detect and correct the particular noise of the channel. For transmission of classical data, error correcting schemes can be as simple as the sending of replicates. For quantum data, and in tandem the development of machines that can process quantum data, quantum error correcting codes must be developed. In addition to a larger set of possible errors, quantum error correcting schemes must contend with other peculiarities of quantum mechanics, such as the no-cloning theorem which can prevent the sending of replicate messages. Stabilizer codes are one family of quantum error correcting codes which can protect and correct errors expressed in terms of the Pauli group, exploiting its group structure and utilizing classical codes and the corresponding duals. We develop and examine a family of quantum stabilizer codes which arise from reflexive stabilizers. Moreover, we provide a mapping from our reflexive stabilizer codes to the well-known CSS codes developed by Calderbank, Shor, and Steane. For the case of a 4-state system we show that these codes can obtain the minimal embedding for code which can correct any flip or phase error. We also provide heuristic algorithms for creating reflexive stabilizer codes starting from the noise of a quantum channel. Furthermore, we show that the problem can be posed in terms of finding maximal Cayley subgraphs with restrictions imposed by the set of potential errors.

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1. Introduction

Error correcting codes are essential tools in communication theory as they provide the means for the reliable delivery of data over noisy communication channels. In classical computing theory we have found that the ability to correct single-bit errors is not only fundamental, but sufficient for most purposes [14]. This classical computing mindset has influenced the current approach to quantum error correction, with the majority of work focusing on the correction of single-qubit flip, phase and phase-flip errors [14,16,20,23].

The wealth of research in engineering a quantum computer has resulted in a variety of different systems by companies such as IBM, D-Wave and Google [8,10,13]. With a different system of qubits comes the possibility of errors intrinsic to a particular system. As such, the importance of finding codes that are resilient to a variety of errors is increasingly important. For instance, considering that entanglement is one of the strongest tools of quantum computation, it is reasonable to predict that entangled or correlated errors may be one of the more common errors. Moreover, correlated errors are already difficult to handle in mathematically, for example in toric quantum codes.

This paper explores a novel approach to quantum error correcting codes which we call reflexive stabilizer codes. We develop an adaptable framework that allows one to begin with a pre-described error set, perhaps intrinsic to a particular computer, and build quantum codes that avoid these errors. Reflexive stabilizer codes are stabilizer codes on qudits; i.e. for states represented by a finite field $\mathbb{F}_d^n$, where $d$ is a power of a prime $p$, $\omega$ is the $p^{\text{th}}$ root of unity, and $D_{a,b}$ are single qudit errors defined in Section 2.

Definition 1. Let $C$ be a linear subspace of $\mathbb{F}_d^n$. The reflexive stabilizer of $C$ is the subgroup of the error group $E$ generated by $S_C := \langle \omega^\kappa 1, D_{\pi,\pi} : \pi \in C, \kappa \in \{0,1,\ldots,p-1\}\rangle$.

This simple stabilizer, has one major advantage to most known strategies of quantum error correction, in that it lends itself to a graph theoretic representation in which the errors become an set of edges that the code must avoid. To improve the error correction rate, we then create extended reflexive stabilizer codes, using two linear subspaces $C_2 \subset C_1 \subset \mathbb{F}_d^n$, denoted $R^X_{C_1,C_2}$ and $R^Z_{C_1,C_2}$.

The adaptability of this approach is best illustrated in the main result of this work, a heuristic algorithm, Section 5, built from around the existence of a linear undirected Caley graph [15]. As this algorithm is rooted heavily in graph theory, and on top of that a common theme of edge avoidance, it can be benefit from collaboration from this field.

The basic idea is that we start with an error set $\mathcal{E}$ which contains the quantum errors we would like to be able to correct from our noisy channel, and build a graph, $G_\mathcal{E}$. We then are left with the task to find a reflexive undirected Caley graph which avoids all the edges of $G_\mathcal{E}$. The main Theorem 5.0.4 (Theorem 1 below), is the main ingredient which allows us to make error correcting codes from a graph.

Theorem 1. Let $d = p^m$ for some prime $p$, and let $C_2 \subset C_1 \subset \mathbb{F}_d^n$ be two linear subspaces, then $R^X_{C_1,C_2}$ (or $R^Z_{C_1,C_2}$) can correct any error set $\mathcal{E}$, such that

$$D(\mathcal{E}) \subseteq \{\omega^\kappa D_{a,b} \mid (a,b) \notin E(G(C_1)), a \notin C_2^\perp, \kappa \in \{0,\ldots,p-1\}\}$$
and if
\[ \omega^\kappa D_{a,a} \in \mathcal{E}^2 \]
for some \( \kappa \in \{0, \ldots, p-1\} \) with \( a \neq 0 \) then \( a \notin C_2^\perp \).

In Section 4, we show that these codes have the same capabilities in the standard single qubit/qudit error approach as CSS codes. We do this by using standard methods from the field to produce weights for the classical code underlying the parameter space of our stabilizer code, which will correct \( t \) single qubit/qudit errors. For instance, we show in Theorem 4.2.2 the following result giving equal error correction rates as those for CSS codes.

**Theorem 2.** Let \( C_2 \subset C_1 \subset \mathbb{F}_d^d \) be linear subspaces, then the codes \( R_X^{C_1, C_2} \) and \( R_Z^{C_1, C_2} \) can correct up to \( t \) single-qudit flip, phase and phase-flip errors where
\[
t = \min \left\{ \left\lfloor \frac{\text{wt}(C_1) - 1}{2} \right\rfloor, \left\lfloor \frac{\text{wt}(C_2^\perp \setminus C_1) - 1}{2} \right\rfloor \right\}.
\]

This paper is meant to serve as an introduction to reflexive stabilizer codes. It also provides a new graph theoretic approach to quantum error correction, opening many avenues for further research. As such, we finish with many questions given in Section 6.

2. **Preliminaries**

In what follows, all codings will be done into strings of qubits or, more generally, the superposition of strings of qudits. All codewords will be strings of elements from a quantum \( d \)-ary alphabet, \( \mathbb{C}^d \), that is the computational basis of dimension \( d \). Unlike classical computing, we also consider codewords that are the superposition of those strings from the computational basis. By convention, we choose \( d = p^m \) for some prime \( p \) and integer \( m \), and set the computational basis as \( \mathcal{A} = \{ |x\rangle : x \in \mathbb{F}_d \} \) which is an orthonormal basis of the Hilbert space \( H_{\mathcal{A}} = \mathbb{C}^d \). Note that the field with \( d \) elements \( A = \mathbb{F}_d \) is simply the classical alphabet used to represent dits. For sake of calculations we will assume a fixed basis for \( \mathbb{F}_d \) over \( \mathbb{F}_p \), namely \( \{ f_i : i \in \{1, \ldots, m\} \} \).

Next we recall some of the basics of quantum error correction. For a more thorough introduction, see e.g. [1, 2, 4–6, 12, 16, 21, 23].

2.1. **The error group.** The type of single qudit errors considered in quantum computation most often fall into 3 types, the Pauli spin matrices. For the case of qubits, the authors of [2, 22] show how these generate all possible single qubit errors. We discuss this case in more detail in Example 2.1.6. We generalize the Pauli spin matrices to errors on qudits in the following definition.

**Definition 2.1.1.** We define the following unitary operators \( X(a) \) and \( Z(b) \) for \( a, b \in \mathbb{F}_d \) on any \( |x\rangle \in \mathcal{A} = \mathbb{C}^d \) as
\[
X(a)|x\rangle := |x + a\rangle \quad \text{and} \quad Z(b)|x\rangle := \omega^{bx}|x\rangle,
\]
known as the flip and phase errors respectively, acting on each of the qudits of a code, where \( \omega = \exp(2\pi i/p) \) is the primitive \( p \)-th root of unity and
\[
a * b = \sum_{i=1}^m \alpha_i \beta_i.
\]
for \( a = \sum_{i=1}^m \alpha_i f_i \) and \( b = \sum_{i=1}^m \beta_i f_i \).
Remark 2.1.2. It is notable that for error groups there is a wonderful connection between our inner product of $a \ast b$ and $\text{tr}(ab)$ for the standard trace operator $\mathbb{F}_d \mapsto \mathbb{F}_p$ see [1]. We make the choice to not use this trace operator only for ease of calculations. Also, we chose to use a notation similar to that in [4].

To be able to have the most agile codes we will also consider the following definition.

Definition 2.1.3. For $a \in \mathbb{F}_d$ we define the phase-flip error as,

$$ Y(a) := X(a)Z(a). $$

Thus for $|x\rangle \in \mathcal{A}$ we have

$$ Y(a)|x\rangle = \omega^{ax}|x + a\rangle. $$

Remark 2.1.4. As seen in [1] this definition is equivalent to first defining $X(\cdot)$ and $Z(\cdot)$ for parameters in $\mathbb{F}_p$, which act on $\mathbb{C}^p$. Then for $a = \sum_{i=1}^m \alpha_i f_i \in \mathbb{F}_d$ we define $X(a)$ and $Z(a)$ on $\mathbb{C}^d \cong (\mathbb{C}^p)^\otimes m$, as

$$ X(a) = \bigotimes_{i=1}^m X(\alpha_i), \quad \text{and} \quad Z(a) = \bigotimes_{i=1}^m Z(\alpha_i). $$

Remark 2.1.5. Notice that for $a, b, c, d \in \mathbb{F}_d$ we have

$$ X(a)X(b) = X(a + b), \quad Z(a)Z(b) = Z(a + b), \quad Z(b)X(a) = \omega^{b \bar{a}} X(a)Z(b), \quad X(a)Z(b)X(c)Z(d) = \omega^{bc - ad} X(c)Z(d)X(a)Z(b) $$

Example 2.1.6. For those more familiar with qubits, one quickly notes that when $p = d = 2$ then $X$ and $Z$ are exactly the Pauli matrices:

$$ X(0) = Z(0) = \mathbb{I}_2 \quad X := X(1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Z := Z(1) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad Y := iXZ = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} $$

Notice, that the set $\mathcal{P}_\mathbb{R} := \{ \mathbb{I}_2, X, Y, Z \}$ is linearly independent over both $\mathbb{C}$ or $\mathbb{R}$, and hence span the space of all $2 \times 2$ complex matrices. The set $\mathcal{P}_\mathbb{R}$ also has the property that the real span (i.e. linearly combinations with all real coefficients) forms all hermitian operators, and that each element of $\mathcal{P}_\mathbb{R}$ are themselves hermitian and unitary. Furthermore, the set $\mathcal{P}_\mathbb{C} := \{ i^\kappa \mathbb{I}_2, i^\kappa X, i^\kappa Y, i^\kappa Z \mid \kappa \in \{0, 1, 2, 3\} \}$ form a group with the standard multiplication of matrices.

Remark 2.1.7. In what follows we assume that a computational basis has been chosen and thus flip and phase errors are chosen. The approach taken in this paper is the set up to eventual use a change of basis, as seen in Section [4] to create optimal codes from a purely graph theoretical approach.

Definition 2.1.8. For each $n \in \mathbb{N}$, $\bar{a} := (a_1, \ldots, a_n) , \bar{b} := (b_1, \ldots, b_n) \in \mathbb{F}_d^n$, we define

$$ D_{\bar{a},\bar{b}} := X(a_1)Z(b_1) \otimes \ldots \otimes X(a_n)Z(b_n). $$

For $n \in \mathbb{N}$ the collection

$$ \mathcal{E}_{n,d} = \mathcal{E}_n := \left\{ \omega^\kappa D_{\bar{a},\bar{b}} \mid \bar{a}, \bar{b} \in \mathbb{F}_d^n, \kappa \in \{0, \ldots, p - 1\} \right\} $$
from Remark 2.1.5 this collection is a group of order $p^{2mn+1}$. We will refer to this group as the **Error group**. Furthermore, we will refer to any non-trivial subset $\mathcal{E}$ of $\mathcal{E}_n$ as an **error set**.

It is easily seen that the center, $Z$, of the group $\mathcal{E}_n$ is generated by $\omega^1$ and therefore has order $p$.

**Remark 2.1.9.** Here we enumerate some of the properties of the errors in $\mathcal{E}_n$. For $\overline{a}, \overline{b}, \overline{c}, \overline{d} \in \mathbb{F}_d^n$, we have

\[(2.1) \quad D^{-1}_{\overline{a}, \overline{b}} = D_{-\overline{a}, -\overline{b}}\]

and

\[(2.2) \quad D_{\overline{a}, \overline{b}} D_{\overline{c}, \overline{d}} = \omega^{-\langle \overline{b}, \overline{c} \rangle} D_{\overline{a} + \overline{c}, \overline{b} + \overline{d}}.\]

**Remark 2.1.10.** For $\overline{a}, \overline{b}, \overline{c}, \overline{d} \in \mathbb{F}_d^n$ and $\kappa, \kappa' \in \{0, \ldots, p-1\}$ then we have the following relations

\[(2.3) \quad (\omega^\kappa D_{\overline{a}, \overline{b}}) (\omega^{\kappa'} D_{\overline{c}, \overline{d}}) = \omega^{\langle \overline{a}, \overline{b} \rangle \kappa' \langle \overline{c}, \overline{d} \rangle} (\omega^{\kappa} D_{\overline{a}, \overline{b}})\]

where

\[(2.4) \quad \langle \overline{a}, \overline{b} \rangle \ast \langle \overline{c}, \overline{d} \rangle := \langle \overline{b}, \overline{c} \rangle - \langle \overline{a}, \overline{d} \rangle\]

and

\[(2.5) \quad \langle \overline{a}, \overline{b} \rangle := \sum_{i=1}^{t} a_i b_i.\]

Therefore, two operators $\omega^\kappa D_{\overline{a}, \overline{b}}$ and $\omega^{\kappa'} D_{\overline{c}, \overline{d}}$ commute if and only if $\langle \overline{a}, \overline{b} \rangle \ast \langle \overline{c}, \overline{d} \rangle = 0$.

This operation is a symplectic inner product on the space $\mathbb{F}_d^{2n}$.

The objective of quantum stabilizer codes is to be able to protect from any error of a commutative subgroup of errors, $S \subset \mathcal{E}_n$, and correct any error from a larger set of errors $\mathcal{E} \subset \mathcal{E}_n$.

**Definition 2.1.11.** Let $S \subset \mathcal{E}_n$ be a commutative subgroup such that the center $Z$ is contained in $S$, then a **Quantum stabilizer code** is any joint eigenspace of the operators in $S$.

The reader less familiar with stabilizer codes is referred to [1][6][16][23].

3. **Reflexive Codes**

In this section we will introduce a special class of quantum stabilizer codes, which we refer to as reflexive codes. These reflexive codes will provide the basic building blocks of our new codes.

**Definition 3.0.1.** Let $C$ be a linear subspace of $\mathbb{F}_d^n$. The **reflexive stabilizer** of $C$ is the subgroup of the error group $\mathcal{E}_n$ generated by

$$S_C := \langle \omega^\kappa \mathbb{1}, D_{\overline{a}, \overline{b}} : \overline{a} \in C, \ \kappa \in \{0, 1, \ldots, p-1\} \rangle.$$
Remark 3.0.2. One quickly sees that $S_C$ is commutative as for two elements $\vec{a}, \vec{b} \in C$ we have that

$$(\vec{a}, \vec{a}) \star (\vec{b}, \vec{b}) = (\vec{a}, \vec{b}) - (\vec{a}, \vec{b}) = 0,$$

and hence follows from Remark 2.1.10. Therefore a reflexive stabilizer is indeed a quantum stabilizer.

Definition 3.0.3. Let $C \subset F^n_d$ be a linear subspace and let $S_C$ be the reflexive stabilizer defined by $C$. We define the Reflexive Stabilizer Code (RSC) $R_C$ to be the quantum code generated by

$$R_C := \{ |\Psi_c\rangle := \bigotimes_{i=1}^n |\psi_{c_i}\rangle : c := (c_1, \ldots, c_n) \in C \}$$

where $|\psi_c\rangle$ is a particular choice of eigenvector for each $c \in C$. In what follows we give particular choice of $|\psi_c\rangle$. In Theorem 3.0.5, we will prove that these codes are indeed stabilizer codes.

Fix a prime $p$ and a positive integer $m$, and let $d = p^m$. Given a reflexive stabilizer $S$, we will choose a linear character; i.e. a joint eigenspace, for an RSC. We begin by fixing a computational basis $\{|x\rangle : x \in F_d\}$. For a linear subspace of $C \subset F^n_d$, we denote by $C^\perp$ the orthogonal subspace with respect to the inner product in Equation 2.5; i.e.

$$C^\perp := \{ \vec{v} \in F^n_d : \langle \vec{v}, \vec{a} \rangle = 0, \forall \vec{a} \in C \}$$

Additionally, we let $f_i$ denote the individual basis elements for our $m$-dimensional basis, $\{f_i : i \in \{1, 2, \ldots, m\}\}$, of $F_d$ over $F_p$. Note that for a given $a = \sum_{i=1}^m \alpha_i f_i \in F_d$ that, from Equation (2.3), we have,

$$X(a)Z(a) = \bigotimes_{i=1}^m X(\alpha_i)Z(\alpha_i) = \bigotimes_{i=1}^m \omega^{\alpha_i(\alpha_i - 1)/2} (X(1)Z(1))^{\alpha_i}$$

(3.1)

$$= \omega^{\tau_a} \bigotimes_{i=1}^m (X(1)Z(1))^{\alpha_i}$$

where

$$\tau_a = \frac{1}{2} \sum_{i=1}^m \alpha_i(\alpha_i - 1).$$

(3.2)

Due to Equation (3.1), in order to understand the eigenspaces of $X(a)Z(a)$, it is enough to understand the eigenspaces of $X(1)Z(1)$. Thus, the last piece that we need is to examine the eigenspaces of $X(1)Z(1)$ in $O^n_p$.

First, for $p = 2$, the eigenvalues for $X(1)Z(1)$ are $\pm i$, where each eigenspace is 1-dimensional and spanned, respectively, by

$$|\psi_0\rangle := \frac{1}{\sqrt{2}} (i|0\rangle + |1\rangle) \quad \text{and} \quad |\psi_1\rangle := \frac{1}{\sqrt{2}} (-i|0\rangle + |1\rangle).$$

(3.3)

For $p > 2$, the eigenvalues are $\omega^\kappa$ for $\kappa \in \{0, 1, \ldots, p-1\}$, and again the respective eigenspaces are each one-dimensional. In what follows, we will choose a unit vector to
span each, and denote denote it by $|\psi_0\rangle$. In general, one option is to choose

$$|\psi_0\rangle := \frac{1}{\sqrt{d}} \sum_{a \in \mathbb{F}_p} \alpha_a |a\rangle$$

where $\alpha_0 = 1$, $\alpha_{p-1} = \omega^{\kappa+1}$, and for $1 \leq i \leq p-2$, $\alpha_i = \omega^{T_i^\kappa}$ for

$$T_i^\kappa = \frac{i(i-1-2\kappa)}{2}.$$

The next example illustrates this construction for $p = 3$.

**Example 3.0.4.** Again, we begin by considering the case $p = d = 3$ and work out a general formula for any linear subspace $C \subset \mathbb{F}_3^n$. The eigenspace decomposition for this case is as follows:

$$|\psi_0\rangle := \frac{1}{\sqrt{3}} \left( |0\rangle + \omega^1 |1\rangle + |2\rangle \right)$$

$$|\psi_1\rangle := \frac{1}{\sqrt{3}} \left( |0\rangle + |1\rangle + \omega^2 |2\rangle \right)$$

$$|\psi_2\rangle := \frac{1}{\sqrt{3}} \left( \omega |0\rangle + |1\rangle + |2\rangle \right)$$

Now for any $c = (c_1 \ldots c_n) \in C$ we have the following quantum code word in $R_C$

$$(3.4)$$

$$|\Psi_c\rangle := \bigotimes_{i=1}^n |\psi_{c_i}\rangle = 3^{-\frac{n}{2}} \sum_{\bar{\pi} \in \mathbb{F}_3^n} \omega^{\delta(\bar{\pi},c+\bar{\pi})} |\bar{\pi}\rangle$$

where $\bar{1}$ is the vector with all ones, and $\delta(\bar{a},\bar{b})$ for $\bar{a},\bar{b} \in \mathbb{F}_d^n$ gives the number of matching entries.

For each $a = \sum_{i=1}^m \alpha_i f_i \in \mathbb{F}_d$, we denote by $|\psi_a\rangle$ the eigenvector, given explicitly by

$$|\psi_a\rangle := \bigotimes_{i=1}^m |\psi_{\alpha_i}\rangle$$

which for the eigenvalue $\omega^{\kappa_a}$ where $\kappa_a = \sum_{i=1}^m \alpha_i$. We are now ready to prove that the codes defined in Definition 3.0.3 are indeed stabilizer codes.

**Theorem 3.0.5.** Given a linear subspace $C \subset \mathbb{F}_d^n$ the code $R_C$ is a reflexive stabilizer code for the reflexive stabilizer $S_{C^\perp}$. Thus $R_C$ is a quantum stabilizer code which embeds $k$ logical qudits into $n$ physical qudits, where $k = \dim_{\mathbb{F}_d}(C)$.

**Proof.** To prove this result, we need only show that all of the vectors of $R_C$ lie in the same eigenspace for any operator $E \in S_{C^\perp}$ (by Definition 2.1.11). The result is trivial for $E = \omega^{\kappa} 1$ as every vector lies in the $\omega^{\kappa}$ eigenspace.

For the remainder of the proof we need only show, for each $c \in C^\perp$, that all of the vectors $|\Psi_c\rangle \in R_C$ lie in the same eigenspace of $D_{c,c}$. This follows from Equation (3.4) since for $c = (c_1, \ldots, c_n) \in C^\perp$ and $\bar{d} \in C$ we have that

$$(3.5)$$

$$D_{c,c} |\Psi_c\rangle = \omega^{(c,\bar{d})/\omega^{\tau_c}} |\Psi_c\rangle = \omega^{\tau_c} |\Psi_c\rangle$$

where $\tau_c = \sum_{i=1}^n \tau_{c_i}$ and each $\tau_{c_i}$ defined in Equation (3.2). Equation (3.5) depends only on $c \in C^\perp$, and we are done. □
Example 3.0.6. Let $C \subset \mathbb{F}_2^3$ be the 2-dimensional code consisting of the following vectors $000, 100, 001, 101$.

By taking tensors of the eigenvectors in Equation 3.3, we embed the $k = 2$ logical qubits into the $n = 3$ physical qubits of $R_C$ as follows:

$$|00\rangle \mapsto |\Psi_{000}\rangle = \frac{\sqrt{2}}{4} \left( -|000\rangle - |001\rangle - |010\rangle + |011\rangle - |100\rangle + |101\rangle + |110\rangle + |111\rangle \right)$$

$$|01\rangle \mapsto |\Psi_{001}\rangle = \frac{\sqrt{2}}{4} \left( |000\rangle - |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle - |110\rangle + |111\rangle \right)$$

$$|10\rangle \mapsto |\Psi_{100}\rangle = \frac{\sqrt{2}}{4} \left( |000\rangle + |001\rangle + |010\rangle - |011\rangle - |100\rangle + |101\rangle + |110\rangle + |111\rangle \right)$$

$$|11\rangle \mapsto |\Psi_{101}\rangle = \frac{\sqrt{2}}{4} \left( -|000\rangle + |001\rangle - |010\rangle - |011\rangle + |100\rangle + |101\rangle - |110\rangle + |111\rangle \right)$$

One can check the error correction capabilities of $R_C$ using the results presented in Section 4. Specifically, using Theorem 4.1.4, $R_C$ can correct any error from the error set $\mathcal{E}_R = \{1, D_{010,010}\}$.

3.1. Extension of reflexive codes. In Section 4, we see that reflexive codes, have limitations in error correction, to overcome these limitations we extend these reflexive codes. This extension is similar to CSS codes [4, 6]. We establish two extensions that protect against phase and flip errors, respectively.

Definition 3.1.1. Let $C_2 \subset C_1 \subset \mathbb{F}_d^n$, be linear subspaces and let $R_{C_1} = \{|\Psi_c\rangle\}$ be the Reflexive QECC associated to $C_1$. We then define two extensions of reflexive codes. The first will be referred to as a phase-error extension by $C_2$, and is the quantum code generated by

$$R_{C_1,C_2}^Z := \left\{ |\Phi_{c'}\rangle := \frac{1}{\sqrt{|C_2|}} \left( \sum_{c \in C_2} D_{c',c} |\Psi_c\rangle \right) \bigg| c' \in C_1 \right\}.$$

The second extension will be referred to as a flip-error extension by $C_2$, and is the quantum code generated by

$$R_{C_1,C_2}^X := \left\{ |\Phi_{c'}\rangle := \frac{1}{\sqrt{|C_2|}} \left( \sum_{c \in C_2} D_{c',c} |\Psi_c\rangle \right) \bigg| c' \in C_1 \right\}.$$

One sees that in either case, two of the above code words $|\Phi_c\rangle$ and $|\Phi_{c'}\rangle$ match if and only if $c - c' \in C_2$, and hence the dimension of $R_{C_1,C_2}^X$ (or $R_{C_1,C_2}^Z$) is $\dim_{\mathbb{F}_d}(C_1) - \dim_{\mathbb{F}_d}(C_2)$.

Theorem 3.1.2. Let $C_2 \subset C_1 \subset \mathbb{F}_d^n$ be a linear subspaces, then both $R_{C_1,C_2}^X$ and $R_{C_1,C_2}^Z$ are stabilizer codes, and the stabilizer is generated by the elements of $S_{C_1}^{\perp}$ and $\left\{ D_{\vec{c},c} \bigg| c \in C_2 \right\}$ for $R_{C_1,C_2}^Z$, and $S_{C_1}^{\perp}$ and $\left\{ D_{c,\vec{c}} \bigg| c \in C_2 \right\}$ for $R_{C_1,C_2}^X$.

Proof. We will only show the result for $R_{C_1,C_2}^Z$ as the other follows similarly. First, note that for any $\vec{c} \in C_1^{\perp}$ and $c \in C_2$, the operators $D_{\vec{c},c} \in S_{C_1}^{\perp}$ and $D_{c,\vec{c}}$ commute because

$$(\vec{c}, \vec{c}) \ast (\vec{0}, c) = (\vec{c}, c) = 0.$$
since \(a \in C_1^\perp\). Now we show that for any \(c \in C_2\) and \(c' \in C_1\) that 
\(D_{\tilde{\sigma},c}|\Phi_{c'}\rangle = |\Phi_{c'}\rangle\). This follows from the observation that for \(m = 1\) for any \(\alpha, \beta \in \mathbb{F}_p\) that 
\(Z(\alpha)|\psi_{\beta}\rangle = |\psi_{\beta-\alpha}\rangle\) so 
\(D_{\tilde{\sigma},c}|\Psi_{c'}\rangle = |\Psi_{c'}\rangle\).

The result for general \(m\) is follows and thus the result follows. \(\square\)

The next results is an immediate consequence.

**Corollary 3.1.3.** Let \(C_2 \subset C_1 \subset \mathbb{F}_d^n\) be linear subspaces, and \(S_X\) be the stabilizer for \(R_{C_1,C_2}^X\) and \(S_Z\) be the stabilizer for \(R_{C_1,C_2}^Z\) then we have
\[
S_X^\perp := \{ \omega^kD_{a,b} \mid a - b \in C_1 \text{ and } a \in C_1^\perp \}
\]
\[
S_Z^\perp := \{ \omega^kD_{a,b} \mid a - b \in C_1 \text{ and } b \in C_2^\perp \}
\]
where \(S^\perp\) is the set of operators which commute with operators in \(S\).

**Remark 3.1.4.** One will notice that CSS codes are in one-to-one correspondence with extended reflexive codes via another choice of error basis. That is, an error basis which was instead generated by \(Y\) and \(Z\) from Example 2.1.6 and obtained if we were to define:
\[
\tilde{D}_{\pi,\bar{\pi}} = \prod_{i=1}^n Y^{a_i}Z^{b_i}.
\]
That is to say sticking with the standard basis on \(\mathbb{F}_2^{2n}\), the parameter space of the Error Basis, we can build a linear isomorphism
\[
\mathbb{F}_2^{2n} \mapsto \mathbb{F}_2^{2n} \\
(a, a) \mapsto (a, 0) \\
(b, 0) \mapsto (0, b).
\]
It is now a simple exercise to show that the above isomorphism takes the stabilizer for a reflexive stabilizer code \(R_{C_1,C_2}^X\) (with \(d = 2\)) to a stabilizer of a CSS code. In the following section we show that reflexive stabilizer codes have the same correction capabilities for single-qubit errors.

The next two examples help elucidate the application Definition 3.1.1. Moreover, these examples utilize well-known classical and quantum codes, providing context for reflexive codes.

**Example 3.1.5.** Our next example uses the classical example of the \([7,4,3]\)-Hamming code, that is the linear subspace of \(\mathbb{F}_2^7\) consisting of the following vectors:

\[
\begin{align*}
0000000, & \quad 0001011, \quad 0010110, \quad 0011011, \\
0100111, & \quad 0101100, \quad 0110001, \quad 0111010, \\
1000010, & \quad 1001110, \quad 1010011, \quad 1011000, \\
1100010, & \quad 1101001, \quad 1110100, \quad 1111111.
\end{align*}
\]

Shor used this code in [6] to build a quantum stabilizer code which encoded one qubit into seven qubits.
For reflexive codes, we will embed three qubits into the seven qubits. We set $C_1$ as the classical Hamming code given in Equation (3.6) and we set $C_2 = \text{span}_{F_2}(1111111)$. Then $C_2^\perp$ consists of all elements of $F_2$ with even Hamming weight. It follows from Theorem 4.2.1 that our code will correct any single-qubit flip or phase error. From Definition 3.1.1, we find the code words for the flip-error extended code $R_{X \overline{C_1},C_2}$

$$|\Psi_{000000}\rangle + |\Psi_{111111}\rangle + |\Psi_{111000}\rangle,$$

$$|\Psi_{001011}\rangle + |\Psi_{110110}\rangle + |\Psi_{110001}\rangle,$$

$$|\Psi_{010011}\rangle + |\Psi_{101001}\rangle + |\Psi_{100101}\rangle.$$

One could write out these code words using the computational basis, as in Example 3.0.6, but doing so would be cumbersome as each is a linear combination of 58 basis elements.

Example 3.1.6. Now, we provide an extended RCS which obtains the same rate as the $[5,1,3]$-Perfect Code introduced in [17]. Unlike the Perfect Code, the code presented here is on a 3-state system. Let $C_1 \subset F_3^3$, be the 2-dimensional code consisting of the following vectors:

$$00000, 11111, 22222, 00211, 00122, 11200, 11022, 22100, 22011$$

Similar to Example 3.1.5 above, we choose $C_2 = \text{span}_{F_3}(11111)$. Hence the extended flip-error extended code $R_{X \overline{C_1},C_2}$ has code words:

$$|\Psi_{00000}\rangle + |\Psi_{11111}\rangle + |\Psi_{22222}\rangle,$$

$$|\Psi_{00211}\rangle + |\Psi_{11022}\rangle + |\Psi_{22100}\rangle,$$

$$|\Psi_{00122}\rangle + |\Psi_{11200}\rangle + |\Psi_{22011}\rangle.$$

By Theorem 4.2.1 one easily checks that this code can correct any single-qudit flip or phase error. Thus this code embeds one logical qudit into five physical qubits and corrects any single-qudit flip or phase error. Notice that cannot correct against single-qudit flip-phase errors, by Theorem 4.2.2, and therefore does not match the correction capability of the Perfect Code [17].

4. Error correction

Before we examine the error correction capabilities of reflexive codes we cover a basic property of stabilizer codes. Given a commutative subgroup $S \subset \mathcal{E}_n$ then clearly a quantum stabilizer code associated to $S$ can protect from error in $S$. For a quantum code to be able to protect from any error in an error set $\mathcal{E} \subset \mathcal{E}_n$, it is necessary and sufficient [2][16] for any two code words $|c_1\rangle, |c_2\rangle$ in our quantum code with $\langle c_1|c_2 \rangle = 0$, and any two errors $E_1, E_2 \in \mathcal{E}$, we have

$$\langle c_1 | E_1^{-1} E_2 | c_2 \rangle = 0,$$

$$\langle c_1 | E_1^{-1} E_2 | c_1 \rangle = \langle c_2 | E_1^{-1} E_2 | c_2 \rangle.$$

It is of note that we will always assume $\mathbb{I} \in \mathcal{E}$ as one should always protect from no errors occuring.
Definition 4.0.1. Given any two errors from an error set $E_1, E_2 \in \mathcal{E}$, we will refer to $E_1^{-1}E_2$ as a conjugate error of $\mathcal{E}$. Furthermore, we will denote the set of conjugate errors of $\mathcal{E}$ by

\[
\mathcal{E}^2 := \left\{ E_1^{-1}E_2 \mid E_1, E_2 \in \mathcal{E} \right\}.
\]

Furthermore, we define the avoidance set as

\[
\mathcal{A}_\mathcal{E} := \left\{ a \in \mathbb{F}_d^n \mid \omega^\kappa D_{a,a} \in \mathcal{E}^2 \text{ for some } \kappa \right\}.
\]

To find codes later we will use the power of our parametric treatment of these error operators. Specifically, for any subgroup $S \subset \mathcal{E}_n$ such that $Z \subset S$, we associate a linear subspace $L_S$ of $\mathbb{F}_d^{2n}$ given by

\[
L_S = \left\{ (\overline{a}, \overline{b}) \in \mathbb{F}_d^{2n} \mid D_{a,a} \in S \right\}.
\]

It remains only to define a joint eigenspace of $S$, i.e. we need only find a basis $\{\overline{v}_i : i \in \{1, \ldots, \dim(L_S)\}\}$ for $L_S$, and then choose eigenvalues for $D_{\overline{v}_i}$, denote these as $\lambda_{\overline{v}_i}$. Thus it follows from Equation (2.3) that the errors of $\mathcal{E}_n$ permute the eigenspaces of $\lambda_{\overline{v}_i}$.

The following theorem follows identically to that in [4, Theorem 1] and [1, Theorem 3]. We include it for later reference.

Theorem 4.0.2. Let $S$ be a commutative subgroup of $\mathcal{E}_n$ which contains the center, i.e. $Z \subset S$, and let $S^\perp$ be the subgroup such that $L_{S^\perp} = L_S^\perp$ with respect to the inner product from Equation (2.4). Further, let $\mathcal{E} \subset \mathcal{E}_n$ be an error set. Then any stabilizer code for $S$ is an error-correcting code which will correct any error from $\mathcal{E}$ if and only if every conjugate error $E \in \mathcal{E}^2$ satisfies either $E \in S$ or $E \not\in S^\perp$.

Next we specialize to the case of reflexive codes.

4.1. Reflexive codes. Now we begin exploring the basic error correction capabilities for these new codes.

Definition 4.1.1. Given a linear subspace $C \subset \mathbb{F}_d^n$ we define the non-detection group, denoted $S_C^\perp$ as the subgroup of $\mathcal{E}_n$ generated by

\[
S_C^\perp := \left\{ \omega^\kappa \mathbb{1}, D_{\overline{b},\overline{b}} : b, \overline{b} \in \mathbb{F}_d^n, \overline{b} \in C^\perp, \kappa \in \{0, 1, \ldots, p-1\} \right\}.
\]

where $C^\perp$ is the orthogonal space with respect to the inner product in Equation (2.5). It is easy to check that this satisfies $L_{S_C^\perp} = L_C^\perp$ from the previous theorem, where $L_S$ is given in Equation (4.3).

An immediate consequence of Theorem 4.0.2 and Theorem 3.0.5 is the following.

Corollary 4.1.2. Let $C \subset \mathbb{F}_d^n$ be a linear subspace, and $\mathcal{E} \subset \mathcal{E}_n$ be an error set. Then $R_{C^\perp}$ is an error-correcting code which will correct any single error from $\mathcal{E}$ if and only if, for any conjugate error $E \in \mathcal{E}^2$, either $E \in S_C$ or $E \not\in S_C^\perp$.

We now dive into the error correction capabilities and limitations of a Reflexive QECC $R_C$. We begin with general qudit errors and provide a complete description of the possible error sets which $R_C$ can correct.
For an error set $\mathcal{E} \subseteq \mathcal{E}_n$ we define the difference set of $\mathcal{E}$ as the set

$$D(\mathcal{E}) := \left\{ \overline{b} - \overline{a} \mid \exists \kappa, \ \omega^\kappa D_{\overline{a}, \overline{b}} \in \mathcal{E}^2 \right\}$$

**Theorem 4.1.4.** Let $C \subseteq \mathbb{F}_d^n$ be a non-trivial linear subset. Then $R_C$ can correct any error of an error set $\mathcal{E} \subseteq \mathcal{E}_n$ if and only if

$$D(\mathcal{E}) \cap C = \{ \overline{u} \} \text{ and } A_{\mathcal{E}} \subseteq C^\perp.$$ 

**Proof.** By Theorem 4.0.2 (or Corollary 4.1.2), $R_C$ can correct any error from $\mathcal{E}$ if and only if, for any conjugate error $E \in \mathcal{E}^2$, either $E \in S_C$ or $E \notin S_{C^\perp}$. We see that $\omega^\kappa D_{\overline{b}, \overline{c}} \in S_{C^\perp}$ if and only if $\overline{b} - \overline{c} \in C$, since for any $\overline{b} \in C^\perp$ the following must hold

$$(\overline{a}, \overline{a}) \star (\overline{b}, \overline{a}) = (\overline{a}, \overline{b}) - (\overline{a}, \overline{b}) = 0.$$ 

Hence, $\omega^\kappa D_{\overline{b}, \overline{c}} \in S_{C^\perp} \setminus S_C$ if and only if $\overline{b} - \overline{c} \in C$ and $\overline{b} \notin C$, and thus our theorem follows.$\square$

Now we consider some special cases of the above theorem.

**Definition 4.1.5.** In lieu of the second condition of Theorem 4.1.4, we aim to construct the largest error set so that $D(\mathcal{E})$ has nothing of the form $\omega^\kappa D_{a,a}$. Let $P \subseteq \{1, \ldots, n\}$, we first denote

$$\mathcal{P} = \text{Span}_{\mathbb{F}_d} \left( e_i \mid i \in P \right)$$

where $e_i$ is the $i^\text{th}$ standard basis element of $\mathbb{F}_d^n$. Now define the $t$-qudit error set for $P$ as:

$$\mathcal{A}_t(P) := \left\{ D_{\overline{a}, \overline{b}} \mid w(\overline{a}) + w(\overline{b}) \leq t, \ \overline{a} \in \mathcal{P}, \ \overline{b} \notin \mathcal{P} \right\}$$

where $w(\overline{a})$ is the number of non-zero entries in $\overline{a}$.

**Remark 4.1.6.** One quickly notes that $\mathcal{A}_t(\emptyset)$ consists of all $t$ single-qudit flip errors, and $\mathcal{A}_t(\{1, \ldots, n\})$ consists of all $t$ single-qudit phase errors. In general, $\mathcal{P}$ specifies the expected locations of phase errors, allowing for flip errors on the remaining qudits.

**Lemma 4.1.7.** Let $C \subseteq \mathbb{F}_d^n$ be a non-trivial linear subspace, and $P \subseteq \{1, \ldots, n\}$. Then the avoidance set of $\mathcal{A}_t(P)$ is trivial; i.e.

$$A_{\mathcal{A}_t(P)} = \{ \overline{0} \}.$$

**Proof.** The results follow from the application of Equations (2.1) and (2.2). If $E_1 = D_{a,b}$, $E_2 = D_{c,d} \in \mathcal{A}_t(P)$, then $E_1^{-1} E_2 = \omega^{(b,c)} D_{c-a,d-b}$, where $c-a \in \mathcal{P}$ and $d-b \notin \mathcal{P}$ such that $w(\overline{c} - \overline{a}) + w(\overline{d} - \overline{b}) \leq 2t$. Every conjugate error $E \in \mathcal{A}_t^2(P)$ is of this form. The result follows.$\square$

**Corollary 4.1.8.** Let $C \subseteq \mathbb{F}_d^n$ be a non-trivial linear subspace, and let $\mathcal{E} \subseteq \mathcal{E}_n$ be an error set such that

$$A_{\mathcal{E}} = \{ \overline{0} \}.$$

Then $R_C$ can correct any error of $\mathcal{E}$ if and only if

$$D(\mathcal{E}) \cap C = \{ \overline{0} \}.$$ 

In particular, this result holds for the error set $\mathcal{A}_t(P)$ for every $P \subseteq \{1, \ldots, n\}$. 
Corollary 4.1.12. For all non-trivial linear subspaces $C \subset \mathbb{F}_d^n$, $R_C$ cannot correct all single qudit errors simultaneously.

Proof. For the sake of contradiction, assume one can find a $C \subset \mathbb{F}_d^n$ such that $R_C$ can correct all single qudit errors simultaneously. Denoting by $\mathcal{E}_f$ the set containing all single qudit errors, there exists, for all $e_i$ in the standard basis of $\mathbb{F}_d^n$, $E \in \mathcal{E}_f^2$ such that $E = \omega^n D_{e_i,e_i}$ for some $\kappa \in \{1, \ldots, p-1\}$. Hence, by Theorem 4.1.4 we have that $\mathbb{F}_d^n = C^\perp$ and hence $C = \{\mathbf{0}\}$ a contradiction. \hfill \Box

At first, this result begs the question, “Why study reflexive codes at all?”, but conventional wisdom rarely holds when studying quantum mechanics. In what follows, we will show that the framework of reflexive stabilizer codes provides a straightforward algorithm for correcting a set of typical correlated errors, allowing for better transmission rates in this scenario. Furthermore, quantum error correct codes that correct single qudit errors already exist. \[1,4,17\]
4.2. Extensions of reflexive codes. We built $\mathcal{N}_t(P)$ to keep the phase and flip errors on different qudits. $C_1$ corrects these types of errors. The extension/subcode by $C_2$ helps tackle the same qudit (phase-flip) errors. By lowering $k$ we should be able to fix more errors. The min means a possibly smaller number of errors are guaranteed to be correctable, but we are able to correct from a larger class of errors.

**Theorem 4.2.1.** Let $C_2 \subset C_1 \subset \mathbb{F}_d^n$ be linear subspaces, then the codes $R_{C_1, C_2}^X$ and $R_{C_1, C_2}^Z$ can both correct up to $t$ single-qudit flip or phase errors where

$$t = \min \left\{ \left\lfloor \frac{\text{wt}(C_1) - 1}{2} \right\rfloor, \text{wt} \left( C_2^\perp \setminus C_1 \right) - 1 \right\}.$$ 

**Proof.** We show the case for $R_{C_1, C_2}^Z$. The other follows similarly. From Remark 4.1.10, our code can correct up to $\left\lfloor \frac{\text{wt}(C_1) - 1}{2} \right\rfloor$-qudit errors of any form except $\omega^k D_{\bar{u}, \bar{\pi}}$. On the other hand, any error of the form $\omega^k D_{\bar{u}, \bar{\pi}}$ as in $S_{C_1}^\perp$ Corollary 3.1.3 exactly when $\bar{u} \in C_2^\perp$. To see this, recall from Theorem 3.1.2, that $R_{C_1, C_2}^Z$ is generated by $S_{C_1}^\perp$ and $\left\{ D_{\bar{u}, c} \mid c \in C_2 \right\}$. Then, we see that for any $c \in C_2$, $D_{\bar{u}, c} \in S$ and hence

$$0 = (\bar{u}, \bar{\pi}) \ast (\bar{0}, c) = (\bar{u}, c)$$

Hence $\bar{\pi} \in C_2^\perp$. Furthermore, some requirement from the definition gives us that $w(\pi) \leq w(C_2^\perp \setminus C_1) - 1$, we see that $R_{C_1, C_2}^Z$ can correct up to $w(C_2^\perp \setminus C_1) - 1$ errors of this form. \qed

The next result follows identically.

**Theorem 4.2.2.** Let $C_2 \subset C_1 \subset \mathbb{F}_d^n$ be linear subspaces, then the codes $R_{C_1, C_2}^X$ and $R_{C_1, C_2}^Z$ can correct up to $t$ single-qudit flip, phase and phase-flip errors where

$$t = \min \left\{ \left\lfloor \frac{\text{wt}(C_1) - 1}{2} \right\rfloor, \left\lfloor \frac{\text{wt}(C_2^\perp \setminus C_1) - 1}{2} \right\rfloor \right\}.$$ 

For some simple examples of this theorem in action the reader is referred back to Example 3.1.5 and Example 3.1.6. The following remark and example gives the best possible encoding rate for an extended reflexive code.

**Remark 4.2.3.** The basic principal in quantum error correction is the concept that each error transforms distinct code words into distinct orthogonal subspaces. This becomes quite restrictive on the minimal length in which one can embed into. This is the topic of a later paper. For now, we present this minimal length for a specific example, namely the case of a 4-state system (see Example 4.2.4). To protect from any flip or phase error on a 4-state system, the space requires an orthogonal subspace for each of the 6, 3 flips and 3 phases, errors plus one for the unperturbed state. This makes a total of $6n + 1$. We must quadruple this to have enough space to accommodate for each of the 4 embedded states; i.e. $k = 1$ logical qudits. Thus, the number of subspaces is $4(6n + 1)$. To have enough room in the Hilbert space we have the condition:

$$4(6n + 1) \leq 4^n$$

for which the smallest number satisfying this equation is $n = 4$. The next example gives a reflexive stabilizer code which meets this lower bound.
By Theorem 4.2.1, one easily checks that this code can correct any single-qudit flip or phase vectors:

\[ x \]

where \( x \equiv x^2 + x + 1 \equiv 0 \). Let \( C_1 \subseteq \mathbb{F}_4^2 \) be the 2-dimensional code consisting of the following vectors:

\[
\begin{align*}
0000, & \quad 1x10, \quad xx^2x0, \quad x^21x^20, \quad x^2x^201, \quad 110x, \quad xx0x^2, \quad x111 \\
x^2xx, & \quad 1x^2x^2x^2, \quad 10x1, \quad x0x^2x, \quad x^201x^2, \quad 0x^21x, \quad 01xx^2, \quad 0xx^21.
\end{align*}
\]

We then set \( C_2 = \{0000, 1x11, x^2xx, 1x^2x^2x^2\} \) and hence our code words for the flip-error extended RSC \( R_{C_1,C_2}^X \) are given by

\[
\begin{align*}
|\Psi_{0000}\rangle & + |\Psi_{x111}\rangle + |\Psi_{x^2xx}\rangle + |\Psi_{1x^2x^2x^2}\rangle \\
|\Psi_{1x10}\rangle & + |\Psi_{x^2x^201}\rangle + |\Psi_{x0x^2x}\rangle + |\Psi_{01xx^2}\rangle \\
|\Psi_{10x1}\rangle & + |\Psi_{x^21x^20}\rangle + |\Psi_{xx0x^2}\rangle + |\Psi_{0x^21x}\rangle \\
|\Psi_{xx^2x^20}\rangle & + |\Psi_{0xx^21}\rangle + |\Psi_{110x}\rangle + |\Psi_{x^201x^2}\rangle
\end{align*}
\]

By Theorem 4.2.1 one easily checks that this code can correct any single-qudit flip or phase error which, by Remark 4.2.3 is the minimal such code for a 4-state system.

5. Cayley Graphs

First, we define Cayley graphs. A Cayley graph \( \Gamma = \Gamma(G, S) \) is a graph that is defined by a group \( G \) be a group and a connecting set \( S \subseteq G \). The vertices of \( \Gamma \) are the elements of the group \( G \). The directed edges of \( \Gamma \) are determined by the group action of \( S \) on \( G \) so that the edges are of the form \((g, sg)\) for \( s \in S \) and \( g \in G \). We will consider only undirected Cayley graphs. For more on undirected Cayley graphs see [15].

The Cayley graphs that we will be most concerned with in this presentation are given in the following definition.

**Definition 5.0.1.** For \( d \), a power of a prime, we define a linear undirected Cayley (LUC) graph with connecting set \( C \subset \mathbb{F}_d^\ast \), where \( C \) is a linear subspace over \( \mathbb{F}_d \), denoted \( G(C) \) with vertex set \( V(G(C)) = \mathbb{F}_d^n \) and edge set

\[
E(G(C)) = \left\{(a, b) \in (\mathbb{F}_d^n)^2 \mid a - b \in C \right\}.
\]

It is easy to see that \( G(C) \) is the Cayley graph \( \Gamma(\mathbb{F}_d^n, C) \).

**Example 5.0.2.** For an easy first example of a LUC graph we choose \( d = 3, n = 2 \), and

\[
C = \{00, 11, 22\}
\]

Note that in this example the component of 00 is \( C \) and that the graph is broken into \( 3 = 3^{2-1} \) connected components, we will make this observation more formal in the next theorem.

**Theorem 5.0.3.** Let \( d = p^n \) be a power of a prime \( p \) and \( C \subset \mathbb{F}_d^n \) be a linear subspace. The LUC \( G(C) \) has exactly \( d^n - \dim(C) \) number of connected components, each of which is complete. Further, the connected component which contains \( \overline{0} \) is exactly \( C \).
Proof. Due to linearity of $F$, it is enough to show the connected component, $H$, containing $0$ is complete and contains exactly the elements from $C$. From Definition 5.0.1 it is clear that $0$ is connected to exactly those $c \in C$. By linearity of $C$, if $a, b \in C$, then $b - a \in C$ and thus $(a, b) \in E(G(C))$. Thus $H$ is complete. $\square$

The following theorem shows the connection between LUC graphs and reflexive quantum error correcting codes.

**Theorem 5.0.4.** Let $d = p^m$ for some prime $p$, and let $C_2 \subset C_1 \subset \mathbb{F}_d^n$ be two linear subspaces, then $R^X_{C_1, C_2}$ (or $R^Z_{C_1, C_2}$) can correct any error from the set $\mathcal{E}$, if

$$D(\mathcal{E}) \subseteq \left\{ \omega^\kappa D_{a,b} \left| (a,b) \notin E\left(G(C_1)\right), a \notin C_2^\perp, \kappa \in \{0, \ldots, p - 1\} \right. \right\}$$

and

$$\mathcal{A}_\mathcal{E} \cap C_2^\perp = \{0\}.$$  

Proof. Let $\mathcal{E}$ be an error set such that $R^X_{C_1, C_2}$ can correct $\mathcal{E}$. Next, let $E := \omega^\kappa D_{a,b} \in D(\mathcal{E})$, thus $E \notin S^\perp$ for the $S$, the stabilizer of $R^X_{C_1, C_2}$ by Theorem 4.0.2, and thus by Lemma 3.1.3 we have $a - b \notin C_1$ and $a \notin C_2^\perp$ as desired. Again by Lemma 3.1.3 if $\omega^k D_{a,a} \in \mathcal{E}^2$ and $a \notin C_2^\perp$ then $\omega^k D_{a,b} \notin S^\perp$ and hence by Theorem 4.0.2 our code can correct as desired. $\square$

5.1. **Heuristic algorithm.** In this section we will lay out the concise steps for the heuristic algorithm to build an extended reflexive stabilizer code which will correct a given error set. Specifically, for this section, we will fix an error set $\mathcal{E} \subseteq E_n$ at the beginning and look for an extended RSC which can correct those errors.

**Step 1: Given an error set $\mathcal{E}$, build $G_\mathcal{E}$**

The avoidance graph $G_\mathcal{E}$ has vertex set $V\left(G_\mathcal{E}\right) = \mathbb{F}_d^n$ and edge set

$$E\left(G_\mathcal{E}\right) = \left\{ (a,b) \in (\mathbb{F}_d^n)^2 \left| \omega^\kappa D_{a,b} \in \mathcal{E}^2 \text{ for some } \kappa \right. \right\}.$$
The graph $G_\mathcal{E}$ encodes the errors in $\mathcal{E}$ as edges to be avoided and, in particular, the condition in Theorem 4.1.4 as a group action on the basis elements of $\mathbb{F}_d^n$.

**Step 2: Find LUC graph which avoids $G_\mathcal{E}$**
Next, find a maximum Sub-LUC graph of the compliment of the graph from Step 1, i.e. $G^c(\mathcal{E})$, and denote the connecting set as $C$. This connecting set builds the RSC $R_C$.

**Step 3: Find Sub-Code of the LUC graph connecting set to build extended code**
Finally, for a subcode $C_1 \subset C$ such that the set $\mathcal{A}_\mathcal{E} \cap C_1^\perp \subseteq \{0\}$ where $\mathcal{A}_\mathcal{E}$ is the avoidance set given in Equation (4.2).

**Conclusion**
Following Theorem 5.0.4 the extended reflexive stabilizer code $R_{X,C,C_1}$ will correct any error of $\mathcal{E}$.

The next example shows the step of this algorithm worked out for a small example.

**Example 5.1.1.** We consider qubits using the notation of Example 2.1.6 and let $\mathcal{E} \subset \mathcal{E}_2$ be the following error set:

$$\mathcal{E} = \{D_{e_2,e_3}, D_{e_1,e_1}, D_{e_3,e_2}, 1\}$$

Then the possible parameters in $\mathcal{E}^2$ are,

$$\{(\overline{0},\overline{0}), (010,001), (100,100), (001,010), (110,101), (011,011), (101,110)\}$$

Then we have that $G_\mathcal{E}$ is the following graph (without the loops drawn)

Perhaps, more interestingly, and which will become important in Section 5.1 the compliment, $G^c_\mathcal{E}$, is as follows
The red and the blue edges make the two connected components of a LUC subgraph of $G_{\mathcal{E}}^c$, whose connecting set is

$$C = \{000, 100, 001, 101\}$$

and hence for $C_2 := \{000, 101\} \subset C$ it follows from Theorem 5.0.4 that $R_{C_2}^X$ will correct any error of $\mathcal{E}$, since $C_2^\perp \cap \{100, 011\} = \emptyset$.

**Remark 5.1.2.** In the following section we provide questions which arise from following this algorithm backwards.

### 6. Questions

We foresee many directions this work can be expanded. In this section we will lay out a few of these possible directions. Our first question is perhaps the most important to expanding these codes to solve any error not just those defined by single qudit errors.

**Question 1.** Can one insert CNOT gates into the graph avoidance representation to extend these codes to have universal fault tolerance?

Our next questions are in pairs to encourage the collaboration of quantum information theorist to work with graph theorist.

**Question 2(a).** What properties of the graph, $G_{\mathcal{E}}$, are required for the existence of a Sub-LUC graph?

**Question 2(b).** What properties the error set $\mathcal{E}$ create a graph, $G_{\mathcal{E}}$, have the existence of a Sub-LUC graph?

**Question 3(a).** Which graphs, $G_{\mathcal{E}}$ and $G_{\mathcal{E}'}$, have the same Sub-LUC graph?

**Question 3(b).** Which error sets $\mathcal{E}$ and $\mathcal{E}'$ create graphs, $G_{\mathcal{E}}$ and $G_{\mathcal{E}'}$, that have the same Sub-LUC graph?
Quantum random walks are a field of study already at the intersection of quantum information and algebraic graph theory. Indeed, quantum random walks have been shown to be universal for quantum computation. An important property to exploit for quantum algorithms is perfect or group state transfer [3,7,11]. The next question expands the possible routes taken to study (extended) reflexive stabilizer codes using quantum random walks.

**Question 4.** What LUC graphs have state transfer with quantum random walks, continuous or discrete? [9]

The next question is an option to incorporate graph-theory techniques into the study of quantum error-correcting codes.

**Question 5.** Using limiting properties of graphs or graphons, can one find a GV-Bound for extended reflexive stabilizer codes? [18, 19]

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