Gauge-invariant Cosmological Perturbations in Generalized Einstein Theories

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Abstract

Using the covariant approach and conformal transformations, we present a gauge-invariant formalism for cosmological perturbations in generalized Einstein theories (GETs), including the Brans-Dicke theory, theories with a non-minimally coupled scalar field and certain curvature-squared theories. We find an enhancement in the growth rate of density perturbations in the Brans-Dicke theory, and discuss attractive features of GETs in the structure formation process.

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1 Introduction

The formation of structure in the Universe is one of the most interesting mysteries in nature. The latest observations have revealed that for galaxy formation, density perturbations must grow much faster than the Universe expands. However, the standard theory of gravitational instability in Einstein’s gravity theory does not provide such a high growth rate. Although some modifications of the standard theory with appropriate dark matter, a cosmological constant, or cosmic strings have been proposed, we do not have any completely satisfactory solution yet. Hence it is worthwhile to search for an alternative, in which the gravity sector is modified.

Generalizations of Einstein’s theory have found various interesting applications to astrophysics. The Brans-Dicke (BD) theory or its modified versions may provide a graceful exit from the inflationary era of the Universe (Extended or Hyper-extended inflation, and soft inflation). Non-minimally coupled terms may produce a virtual structure in the Universe. There may also exist an important difference in the growth rate of cosmological density perturbations, which might resolve the problem of structure formation. In a preliminary work, Berkin and one of the present authors investigated the growth of density perturbations in a theory with a non-minimally coupled scalar field and showed a sufficient enhancement in the growth of scalar field perturbations. Although these authors excluded ordinary matter fluids and used only a scalar field, their calculation suggests that such a coupling may play an important role in structure formation in a realistic model with a baryonic matter fluid.

In this paper, we discuss cosmological density perturbations in generalized Einstein theories (GETs), which include the BD theory, theories with a non-minimally coupled scalar field and certain curvature-squared theories. The action of GETs we consider here is

\[
S = \int d^4 x \sqrt{-g} \left[ F(\phi, R) - \frac{\epsilon(\phi)}{2} (\nabla \phi)^2 + L_m \right],
\]

where \( F(\phi, R) \) is an arbitrary function of a scalar field \( \phi \) and a scalar curvature \( R \), \( \epsilon(\phi) \) is an arbitrary function of \( \phi \), and \( L_m \) is the Lagrangian of ordinary matter, for which we assume a perfect fluid.

When we study cosmological perturbations, an appropriate choice of gauge becomes important. In order not to pick up unphysical modes by a bad gauge choice, Bardeen proposed a gauge-invariant (GI) formalism for density perturbations. Although Bardeen’s GI formalism is one of the most elegant and attractive approaches to cosmological density perturbations, its application to GETs requires tedious calculations, which have yet to be performed.

On the other hand, Ellis and his collaborators have recently developed an alternative to Bardeen’s formalism. Their covariant approach is rather simple. Using their formalism and a conformal transformation, we will present GI perturbation

\footnote{Kodama and Sasaki nicely reviewed and extended Bardeen’s formalism in \([10]\). They also tried to formulate GI equations for a theory with a non-minimally coupled scalar field, but owing to the complexity of the problem, they made no attempt to solve it completely.}
equations in GETs. Although we can write down the basic equations schematically for an arbitrary GET (see appendix A), we will present them explicitly only for the following two interesting cases:

**Case (A)**: $F$ is a linear function of $R$, i.e., $F(\phi, R) = f(\phi)R - V(\phi)$ (in §3).

**Case (B)**: $F$ depends only on $R$, i.e., $F(\phi, R) = L(R)$ (in appendix A).

Case (A) includes the BD theory, induced gravity\textsuperscript{[17]}, and other theories with a non-minimally coupled scalar field, while Case (B) includes $R^2$-gravity\textsuperscript{[3]}.

In order to show the possibility of an enhancement in the growth rate of density perturbations in GETs, we analyze perturbations in the BD theory.

The plan of this paper is as follows: A brief review of the covariant approach to density perturbations is given in §2. In §3 we derive the perturbation equations for Case (A), introducing new conformal variables. In §4 we apply these equations to BD cosmology and analyze the evolution of density perturbations of dust fluid both analytically and numerically. Finally in §5 we discuss general features of density perturbations in GETs. Appendix A presents the formulation for the most general type of GETs described by equation (1.1) and gives explicit expressions for Case (B). In appendix B we show our equations reduce to the already-known ones in a scalar field dominated universe. In appendix C perturbations in a BD cosmology are solved analytically.

## 2 Covariant Approach to Cosmological Density Perturbations

The covariant approach to cosmological density perturbations, introduced originally by Hawking\textsuperscript{[18]} and developed by Ellis and Bruni\textsuperscript{[11]}, provides us with a GI formalism, which is an alternative to Bardeen’s approach\textsuperscript{[9]-[10]}. Here we summarize it briefly in order to define our notation and introduce the dynamical variables used in this paper.

In the formalism, first we have to select a frame in which GI quantities are defined. The frame is described by 4-velocity of an observer, $u^a$. The time derivative along the motion of the observer (denoted by a dot) of any rank of tensor $T^{ab\cdots cd\cdots}$ and its spatial covariant derivative in the 3-space $\Sigma$ orthogonal to $u^a$ are defined by

$$\dot{T}^{ab\cdots cd\cdots} \equiv u^e \nabla_e T^{ab\cdots cd\cdots},$$

$$^{(3)}\nabla_a T^{bc\cdots de\cdots} \equiv h_a^m h_b^i h_j^c \cdots h_d^k h_e^l \cdots \nabla_{m} T^{iji\cdots kl\cdots},$$

respectively, where $h_{ab} = g_{ab} + u_a u_b$ is the projection tensor into the 3-space $\Sigma$ and $\nabla_a$ denotes the covariant derivative with respect to $g_{ab}$.

The derivative of $u^a$ is decomposed as

$$\nabla_b u_a = \omega_{ab} + \sigma_{ab} + \frac{1}{3} \theta h_{ab} - a_a u_b,$$

where $a_a \equiv \dot{u}_a$, $\theta \equiv \nabla_a u^a$, $\omega_{ab} \equiv ^{(3)} \nabla_{[a} u_{b]}$, and $\sigma_{ab} \equiv ^{(3)} \nabla_{(a} u_{b)}$ are the acceleration, the expansion, the vorticity tensor and the shear tensor, respectively. We define typical
length scale $a$ and time scale $H^{-1}$ from the expansion $\theta$ as

$$
\frac{1}{3} \theta \equiv \frac{\dot{a}}{a} \equiv H. \tag{2.4}
$$

These $a$ and $H$ turn out to be the scale factor and the Hubble parameter, respectively, when the Universe is homogeneous and isotropic. In inhomogeneous spacetimes, however, those are local quantities defined by each observer.

Next we define the GI perturbation variables. The basic requirement for GI quantities is that they are invariant under a general coordinate transformation, i.e., for any choice of correspondence between a homogeneous and isotropic background spacetime and the physical, inhomogeneous universe. The simplest GI quantities are a scalar field which is homogeneous in the background, and any vector or tensor field which vanishes in the background. For such quantities, all perturbations defined by comparison with background values are free from any gauge dependence. Such quantities as $a_a$, $\omega_{ab}$, and $\sigma_{ab}$ defined in a real lumpy universe are GI perturbations, because their background values vanish.

As for the energy density $\mu$, the spatial variation

$$
X_a \equiv (3\nabla a)\mu \tag{2.5}
$$

is GI because $\mu$ is homogeneous in the background. The GI quantity

$$
D_a \equiv \frac{a}{\mu} X_a, \tag{2.6}
$$

which is the ratio of its spatial gradient to the density at a fixed comoving scale, is convenient in discussing cosmological perturbations. If we are interested simply in density perturbations, however, we have to extract the information of its scalar part from the vector quantity $D_a$, which contains other information as well. In this formalism, a local and unique splitting\cite{12} is attained by the operation of the (comoving) spatial derivative $a(3\nabla b)$ and its decomposition. For $D_a$, we have

$$
a^{(3)}\nabla_b D_a \equiv \Delta_{ab} = \frac{1}{3}\Delta h_{ab} + \Sigma_{ab} + W_{ab}, \tag{2.7}
$$

$$
\Delta \equiv h_{ab}\Delta_{ab} \quad W_{ab} \equiv \Delta_{[ab]}, \quad \Sigma_{ab} \equiv \Delta_{(ab)} - \frac{1}{3}\Delta h_{ab},
$$

where the skew-symmetric tensor $W_{ab}$ contains information about vorticity, the symmetric and trace-free tensor $\Sigma_{ab}$ describes the evolution of anisotropy in the universe, and the trace $\Delta$ is related to the aggregation of matter fluid. This $\Delta$, which corresponds to the density perturbations $\delta\mu/\mu$, is one of the most important GI variables$^\Delta$.

$^\Delta$ $\Delta$ is related to Bardeen’s GI variable $\epsilon_m(= \delta\mu/\mu$ in velocity-orthogonal slicing) in \cite{9} as follows$^\Delta$:

$$
\Delta = -\sum_n n^2 \epsilon_m^{(n)} Q^{(n)}, \quad \epsilon_m = \sum_n \epsilon_m^{(n)} Q^{(n)},
$$

to first order of perturbations, where $n$ is a wave number of the harmonics $Q^{(n)}$.  

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Obtaining the evolution equations for GI variables is rather simple. First, take spatial derivatives (by operating \( a^{(3)}\nabla_a \)) of the fundamental equations, which consist of the energy conservation, the momentum conservation, the Raychaudhuri, and the Gauss-Codacci equations (see e.g. [19] or [20]). Then, take the (comoving) divergence (by operating \( a^{(3)}\nabla^a \) to the gradient equations) and construct scalar equations for GI variables such as \( \Delta \) defined above. The resultant equations describe the evolution of GI scalar perturbations.

3 Formulation of Perturbation Equations in GETs

3.1 Conformal Transformation

We now derive GI perturbation equations in GETs, using the covariant formalism described in §2. In this section, we consider Case (A), i.e., the case where \( F(\phi, R) \) in equation (1.1) is a linear function of \( R \). As for Case (B), i.e., the case where \( F(\phi, R) \) depends only on \( R \), we will present explicit perturbation equations in appendix A, which turn out to be the same as those given here. Furthermore, the derivation of perturbation equations for the most general action (1.1) is straightforward (see also appendix A).

The theories we consider in this section are described by the action

\[
S = \int d^4x \sqrt{-g} \left[ f(\phi)R - \frac{\epsilon(\phi)}{2}(\nabla \phi)^2 - V(\phi) + L_m \right],
\]

where \( f(\phi), \epsilon(\phi) \) and \( V(\phi) \) are arbitrary functions of \( \phi \). \( L_m \) is the Lagrangian of ordinary matter, for which we assume a perfect fluid, i.e.,

\[
T_{ab}^M = \mu u_a^M u_b^M + p h_{ab}^M,
\]

where \( \mu, p \) and \( u_a^M \) represent the energy density, the pressure and the 4-velocity of the matter fluid, respectively, and \( h_{ab}^M \) is the projection tensor into the 3-space orthogonal to \( u_a^M \). Taking variations of equation (3.1) with respect to the metric \( g_{ab} \) and \( \phi \), we find the basic equations

\[
G_{ab} = \frac{1}{f(\phi)} \left\{ \frac{\epsilon(\phi)}{2} \left[ \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab}(\nabla \phi)^2 \right] + [\nabla_a \nabla_b f(\phi) - \Box f(\phi) g_{ab}] - \frac{1}{2} V(\phi) g_{ab} + \frac{1}{2} T_{ab}^M \right\},
\]

\[
\epsilon(\phi) \Box \phi + \frac{1}{2} \frac{d\epsilon(\phi)}{d\phi} (\nabla \phi)^2 - \frac{dV(\phi)}{d\phi} + \frac{df(\phi)/d\phi}{f(\phi)} \left\{ 3 \Box f(\phi) + \frac{\epsilon(\phi)}{2} (\nabla \phi)^2 + 2V(\phi) + \frac{1}{2}(\mu - 3p) \right\} = 0,
\]

where \( G_{ab} = R_{ab} - \frac{1}{2} g_{ab}R \) is the Einstein tensor.

The application of the covariant approach in §2 to the present model is not straightforward because of the presence of higher-derivative terms such as \( \nabla_a \nabla_b f(\phi) \) and \( \Box f(\phi) g_{ab} \).
Although there may be a way to resolve this problem leaving the present variables as they are, we prefer to circumvent it by changing variables using the conformal transformation

\[ \hat{g}_{ab} = e^{2\omega(x)}g_{ab}. \] (3.5)

If we choose the conformal factor as

\[ e^{2\omega} = 2\kappa^2|f(\phi)|, \] (3.6)

the field equations become

\[
\hat{G}_{ab} = \kappa^2 \left\{ \hat{\nabla}_a \hat{\nabla}_b \phi - \frac{1}{2} \hat{g}_{ab} (\hat{\nabla} \phi)^2 - U(\phi) \hat{g}_{ab} \right\} + \kappa^2 N(\phi) \left\{ \mu \hat{u}^M_a \hat{u}_b^M + p \hat{h}^M_{ab} \right\} \\
= \kappa^2 (\hat{T}^a_{ab} + \hat{T}^M_{ab}),
\]

\[ \Box \phi - U'(\phi) + \kappa^2 M(\phi) \{ \mu - 3p \} = 0, \] (3.8)

with

\[
\kappa \varphi = \int d\phi \left[ \frac{\epsilon(\phi) f(\phi) + 3(d\epsilon(\phi)/d\phi)^2}{2f^2(\phi)} \right]^{\frac{1}{2}},
\]

\[
U(\phi) = \frac{(\text{sign})}{(2\kappa^2 f(\phi))^2} V(\phi), \quad \text{(sign)} = \frac{f(\phi)}{|f(\phi)|},
\]

\[
M(\phi) = \frac{d\epsilon(\phi)/d\phi}{(2\kappa^2 f(\phi))^2 [2\kappa^2 \{ \epsilon(\phi) f(\phi) + 3(d\epsilon(\phi)/d\phi)^2 \}]^{\frac{1}{2}},
\]

\[
N(\phi) = \frac{(\text{sign})}{(2\kappa^2 f(\phi))^2},
\]

\[
\hat{u}_a^M = e^{\omega} u_a^M, \quad \hat{u}_M^a = e^{-\omega} u_M^a,
\]

\[
\hat{h}_{ab}^M = \hat{h}_{ab}^M, \quad \hat{h}_a^M = e^{2\omega} h_a^M, \quad \hat{h}_M^{ab} = e^{-2\omega} h_M^{ab},
\] (3.13)

where \( \kappa^2 = 8\pi G, \) \( (\text{a prime}) = d/d\varphi, \) and new variables with \( \hat{\cdot} \) (a caret) denote those with respect to \( \hat{g}_{ab} \). The resultant basic equations in terms of new variables turn out to be familiar ones, that is, those for the Einstein gravity \( (\hat{g}_{ab}) \) plus a minimal scalar field \( \phi \) with a potential \( U(\phi) \) and a perfect fluid, which interact with each other through coupling functions \( M(\phi) \) and \( N(\phi) \). The non-minimal coupling to a scalar curvature has been absorbed into such interactions. Note that we have not changed the original energy density \( \mu \) and pressure \( p \) for a perfect fluid.

We can recover physical variables from new ones through equation (3.5), which gives the relations between the length scales and Hubble parameters

\[
\hat{a} = e^{\omega} a, \quad \hat{H} = e^{-\omega} (H + \dot{\omega}),
\] (3.15)

(3.16)
and equations (3.14). In particular, the density perturbation \( \Delta^M \) observed along fluid flow \( u_a^M \) is obtained from \( \hat{\Delta}^M \) as follows: First, we have the relation

\[
D^M_a \equiv \frac{a}{\mu} (3) \nabla^M_a \mu = \frac{a}{\mu} h^M_a \nabla_b \mu = e^{-\omega} \hat{a} (3) \nabla^M_a \mu = e^{-\omega} \hat{\Delta}^M_a,
\]

(3.17)

to first order in the perturbations. Taking the comoving divergence of this relation by \( a (3) \nabla_a \) yields

\[
\Delta^M = \hat{\Delta}^M,
\]

(3.18)

Thus, there is an advantage to using the conformal variables. The original covariant approach is applied as it is to the simplified conformal equations and it gives the same density perturbations as those in terms of the physical variables. In what follows, we will formulate equations in terms of the new conformal variables. We omit \( \hat{\imath} \) (a caret) for brevity. When we discuss the results, of course, we transform back to the physical variables (see \$4\).

### 3.2 Formulation in terms of Conformal Variables

The total energy-momentum tensor, \( T_{ab}^\ast \equiv T_{ab}^\phi + T_{ab}^M \) in equation (3.8), takes on a fluid form for any observer with 4-velocity \( u_a^O \),

\[
T_{ab} = \mu_s u_a^O u_b^O + p_s h_{ab}^O + q_{ab}^O u_a^O + \pi_{ab}^O,
\]

(3.19)

with

\[
\begin{align*}
\mu_s &\equiv \frac{1}{2} \dot{\varphi}^2 + U(\varphi) + N(\varphi) \mu, \\
p_s &\equiv \frac{1}{2} \dot{\varphi}^2 - U(\varphi) + N(\varphi) p, \\
q_{ab}^O &\equiv -\dot{\varphi}^O (3) \nabla^a \varphi + N(\varphi) (\mu + p) V^M_a, \\
\pi_{ab}^O &\equiv 0,
\end{align*}
\]

(3.20) (3.21) (3.22) (3.23)

where we have neglected the higher than first order of perturbations composed of all the spatial derivatives and the relative velocity of the matter fluid with respect to \( u_a^O \),

\[
V_a^M \equiv u_a^M - u_a^O.
\]

(3.24)

In what follows, we will also neglect higher-order perturbations in all equations, as in the usual first order perturbation theory. In equation (3.19), \( \mu_s, p_s, q_{ab}^O \) and \( \pi_{ab}^O \) represent the total energy density, pressure, energy flux and anisotropic pressure measured by the observer \( u_a^O \). Note that the energy density \( \mu_s \) and the pressure \( p_s \) do not depend on the observer, but only the energy flux does.

There is no physical constraint on \( u_a^O \) other than that \( V_a^M \) be small. However, there exists a preferred choice, i.e., the 4-velocity of the center of mass. In this reference frame, the total energy flux vanishes. Ellis et al. call it the energy frame and we use \( u_a^E \) to
denote its 4-velocity. The total energy-momentum tensor, written in terms of the center
of mass observer \((u_E^a)\), takes the perfect fluid form:

\[
T_{ab} = \mu^* u_E^a u_E^b + p^* h_{ab},
\]

(3.25)

where \(\mu^*\) and \(p^*\) are defined by equations (3.20) and (3.21).

Once a reference frame is fixed, we can derive explicitly the evolution equations from
energy conservation, momentum conservation, the Raychaudhuri equation, the equation
of motion for \(\varphi\), and the Gauss-Codacci equation in the center of mass frame, which are

\[
\dot{\mu}^* + 3H(\mu^* + p^*) = 0,
\]

(3.26)

\[
(\mu^* + p^*) a_a + (3) \nabla_a^E p^* = 0,
\]

(3.27)

\[
3\ddot{H} + 3H^2 - \nabla_a a^a + \frac{1}{2} \kappa^2(\mu^* + 3p^*) + 2(\sigma^2 - \omega^2) = 0,
\]

(3.28)

\[
(3) R = 2(\frac{1}{3} \theta^2 + \mu^* + \sigma^2 - \omega^2),
\]

(3.29)

\[
\nabla^\alpha \varphi - U'(\varphi) + \kappa^2 M(\varphi)(\mu - 3p) = 0,
\]

(3.30)

where \(\sigma^2 \equiv \frac{1}{2} \sigma_{ab} \sigma_{ab}\) and \(\omega^2 \equiv \frac{1}{2} \omega_{ab} \omega_{ab}\).

The equations for the background isotropic and homogeneous fields are derived by
setting \(\sigma\), \(\omega\) and the spatial derivatives equal to zero in equations (3.26)–(3.29) as

\[
\dot{\mu} + 3H(\mu + p) + \frac{N'(\varphi)}{N(\varphi)} \mu \dot{\varphi} + \kappa^2 \frac{M(\varphi)}{N(\varphi)} (\mu - 3p) \dot{\varphi} = 0,
\]

(3.31)

\[
3\dot{H} + 3H^2 + \kappa^2(\varphi^2 - U(\varphi)) + \frac{1}{2} \kappa^2 N(\varphi)(\mu + 3p) = 0,
\]

(3.32)

\[
3H^2 + \frac{3k}{a^2} = \kappa^2 \left( \frac{1}{2} \dot{\varphi}^2 + U(\varphi) + N(\varphi)\mu \right),
\]

(3.33)

\[
\dddot{\varphi} + 3H \ddot{\varphi} + U'(\varphi) - \kappa^2 M(\varphi)(\mu - 3p) = 0,
\]

(3.34)

where \(k(= 0 \text{ or } \pm 1)\) is the so-called spatial curvature constant in the Friedmann-Lemaître-
Robertson-Walker (FLRW) universe.

The scalar GI perturbation variables for the matter and scalar field in the center of
mass frame are defined as

\[
\Delta^E = \frac{a^2}{\mu^*} (3) \nabla_E^2 \mu^*, \quad Y^E = a^2 (3) \nabla_E^2 p^*,
\]

(3.35)

\[
\Delta^E = \frac{a^2}{\mu} (3) \nabla_E^2 \mu, \quad Y^E = a^2 (3) \nabla_E^2 p,
\]

(3.36)

\[
Z^E = a^2 (3) \nabla_E^2 \theta, \quad \Phi^E = a^2 (3) \nabla_E^2 \varphi.
\]

(3.37)

(3.38)
Following the procedure in §2, we find the first order perturbation equations as follows:

\[
\begin{align*}
\dot{\Delta}_E^a & = H \frac{p_a}{\mu_a} \Delta^a_E + (1 + \frac{p_a}{\mu_a}) \mathcal{Z}_E^a = 0, \\
\mathcal{Z}_E^a & = 2H \mathcal{Z}_E^a + \frac{1}{2} \kappa^2 \mu_a \Delta^a_E + \frac{1}{\mu_a + p_a} \left( \frac{k}{a^2} + (3) \nabla^2_E \right) \mathcal{Y}_E^a = 0. \\
\dot{\Phi}_E^a & = 3H \dot{\Phi}_E^a + \left\{ U''(\varphi) - \kappa^2 (M'(\varphi)) (\mu - 3p) \right\} \Phi_E^a \\
& \quad - \kappa^2 M(\varphi) \mu \Delta^a_E + 3 \kappa^2 M(\varphi) \mathcal{Y}_E^a + \dot{\varphi} \mathcal{Z}_E^a - (3) \nabla^2_E \Phi_E^a - a^2 \varphi(3) \nabla^2_E \mathcal{Y}_E^a a_a = 0.
\end{align*}
\]

(3.39), (3.40), (3.41)

Here we have used the following useful relations for \( \omega_{ab} \), \( \varphi \), an arbitrary vector \( \Upsilon_a^E \) orthogonal to \( u^b_a \) (\( \Upsilon_a^E u_b^a = 0 \)) and \( \Upsilon^E \equiv a(3) \nabla a \Upsilon_a^E \):

\[
\begin{align*}
a(3) \nabla_a \nabla_b \omega_{ab} & = 0, \\
\square \varphi & = \frac{1}{a^2} \Phi_E - 3H \dot{\varphi} - \ddot{\varphi}, \\
a(3) \nabla_a \Upsilon_a^E a^b & = \Upsilon^E, \\
a(3) \nabla_a \Upsilon_a^E \Upsilon^b & = \Upsilon^b, \\
a(3) \nabla_a \left[ (3) \nabla^2 - \frac{2k}{a^2} \right] \Upsilon_a^E & = (3) \nabla^2 \Upsilon^E.
\end{align*}
\]

(3.42), (3.43), (3.44)

The explicit equations in terms of \( \Delta^a_E \), \( \gamma^a_E \), \( \Phi_E^a \) and \( \mathcal{Z}_E^a \) are derived from equations (3.39)~(3.41), by using relations (3.20), (3.21) and

\[
\Delta^a_E = \frac{1}{2} \dot{\varphi}^2 + U(\varphi) + N(\varphi) \mu \left[ N(\varphi) \mu \Delta^a_E - a^2 \varphi(3) \nabla^2 \mathcal{Y}_E^a \right]
\]

(3.45)

Finally, we obtain the basic equations for the harmonic components of perturbed quantities as follows (We drop the suffix \( (n) \) for brevity):

\[
\begin{align*}
\dot{\Delta}^a & = 3w \left( H - \kappa^2 \frac{M}{N} \varphi \right) \Delta^a + 3 \kappa^2 \frac{M}{N} \varphi \mathcal{Y} - (1 + w) \mathcal{Z}_E^a \\
& + \left[ \frac{3H \dot{\varphi}}{N \mu} - \kappa^2 \frac{M}{N} (1 - 3w) - \frac{N'}{N} \right] \dot{\Phi}_E^a
\end{align*}
\]

(3.46), (3.47)
where we have used notations $w \equiv p/\mu$, $c_s^2 \equiv \dot{p}/\mu$. This happens in several interesting models (see eqs. (1.15)~(1.17) for the BD theory and also appendix A).

Although the variables in the center of mass frame are easy to handle, when we discuss the density perturbations of matter fluid, we should evaluate the density fluctuations of the matter fluid in the matter frame defined by $u^M_a$. Hence, we must construct the density perturbations in the matter frame from variables in the center of mass frame $\mathbf{13}$. First, the projection tensor $h^M_{ab}$ defined by $u^M_a$ and the energy density of matter $\mu^M$ in
the matter frame are written as
\[ \begin{align*}
  h^M_{ab} &= g_{ab} + u^M_a u^M_b = h_{ab} + u^E_a V^M_b + V^M_a u^E_b, \\
  \mu^M &= \mu,
\end{align*} \]
to first order. Then, using these relations and the condition
\[ V^M_a = \dot{\phi} aN(\mu + p)\Phi E, \]
which is derived from the condition that the total energy flux vanishes in the center of mass frame, the density perturbation in the matter frame \( \Delta^M \) is expressed as
\[ \Delta^M \equiv a^3 \nabla^a_M(D^M_a) = a h^a_b \nabla^b_a(\frac{\mu}{\mu M} h^M_c \nabla^c \mu E) = \Delta^E + \frac{\dot{\mu}}{\mu N(\mu + p)} \Phi E. \]
In some specific theories, we recover already-known basic equations. For example, setting \( f(\phi) = 1/2\kappa^2 \), and \( \epsilon(\phi) = V(\phi) = 0 \), we find immediately the perturbation equations for the dust universe in the Einstein theory. Hwang\[21\] and Bruni et al.\[22\] derived perturbation equations in some scalar field dominated universes. We can also recover these equations by taking the limit \( \mu, p \to 0 \) and rewriting our variables in terms of their variables (see appendix B).

4 Density Perturbations in the Brans-Dicke Theory

In order to see how the coupling term to the curvature scalar in GETs changes the growth rate of density perturbations, we consider the BD theory\[7\] as an example. In the BD theory, the reciprocal of the scalar field, \( 1/\phi \), corresponds to an “effective” gravitational constant \( G(\phi) \). If a flat dust universe \( (k = 0, p = 0) \) satisfies a specific initial condition
\[ \frac{\mu_0 t_0^2}{\phi_0} = \frac{(3 + 2\omega)}{4\pi(4 + 3\omega)}, \]
where \( \omega \) is a constant (the BD parameter) and the suffix 0 represents the present time, the BD scalar field \( \phi \), the energy density of matter \( \mu \) and the scale \( a \) evolve in time as
\[ \begin{align*}
  \phi &= \phi_0 \left( \frac{t}{t_0} \right)^{2(4+3\omega)/(4+3\omega)}, \\
  a &= \left( \frac{t}{t_0} \right)^{(2+2\omega)/(4+3\omega)}, \\
  \mu &= \mu_0 a^{-3} = \mu_0 \left( \frac{t}{t_0} \right)^{-3(2+2\omega)/(4+3\omega)}.
\end{align*} \]
In the limit of large \( \omega \), this solution approaches the Einstein de-Sitter universe \( (\phi = \text{constant}, a \propto t^{2/3}) \). When the condition \( 4.1 \) is not satisfied initially, the behaviors of
φ and a in the early stage may differ considerably from the above solution for the same value of ω [23].

Nariai [24] analyzed density perturbations in the above specific solution (4.2) of the BD cosmology using a specific gauge condition and solved its evolution equations analytically as

\[
\frac{\delta \mu}{\mu} = \frac{c_1}{(1 + \nu)(1 + 2\nu)} \left\{ \frac{1}{2} (2\nu t_0) \left( \frac{t}{t_0} \right)^{\frac{1}{2}} + (3 - 2\nu)(1 + \nu) \right\} + c_2 \left( \frac{t}{t_0} \right)^{-1} \]
\[
+ \left( \frac{t}{t_0} \right)^{-\frac{1}{2}} \left\{ c_3 J_\nu(nT) + c_4 J_{-\nu}(nT) \right\},
\]

where \( c_1, c_2, c_3 \) and \( c_4 \) are arbitrary integration constants, \( \nu \equiv (4 + 3\omega)/2(2 + \omega) \), \( T \equiv 2\nu t_0(t/t_0)^{-\frac{1}{2\nu}} \), \( J_\nu \) is the Bessel function of the \( \nu \)-th order and \( n \) represents the wave number of harmonics.

In our formalism, the BD theory is given by setting

\[
f(\phi) = \frac{\phi}{16\pi}, \quad (4.6)
\]
\[
e(\phi) = \frac{\omega}{8\pi\phi}, \quad (4.7)
\]
\[
V(\phi) = 0, \quad (4.8)
\]

in the action (3.1). New variables, potential and coupling functions in the conformal frame introduced in §3 are given as

\[
\frac{d\hat{t}}{dt} = \sqrt{\frac{\phi}{\phi_0}}, \quad (4.9)
\]
\[
\hat{a} = \sqrt{\frac{\phi}{\phi_0}} a, \quad (4.10)
\]
\[
\varphi = \sqrt{\frac{2(3 + 2\omega)}{2\kappa}} \ln \left| \frac{\phi}{\phi_0} \right|, \quad (4.11)
\]
\[
N(\varphi) = \exp \left[ -\frac{4\kappa}{\sqrt{2(3 + 2\omega)}} \varphi \right], \quad (4.12)
\]
\[
M(\varphi) = \frac{1}{\kappa\sqrt{2(3 + 2\omega)}} N(\varphi), \quad (4.13)
\]
\[
U(\varphi) = 0. \quad (4.14)
\]

The exponential form of \( N \) and \( M \) and vanishing \( U \) rather simplify the perturbation equations (3.48)~(3.50). Moreover, if we assume the ordinary matter is dust (\( p = 0 \)), they are reduced as

\[
\dot{\Delta}^E = -2\dot{E}^E + \left[ \frac{3H\dot{\varphi}}{N\mu} + \frac{3\kappa}{\sqrt{2(3 + 2\omega)}} \right] \dot{\Phi}^E
\]
\[
+ \frac{n^2 \dot{\varphi}}{a^2 N\mu} \Phi^E, \quad (4.15)
\]
\[
\dot{Z}^E = - \frac{1}{2} \kappa^2 N \mu \Delta^E - 2H Z^E \\
- \frac{\kappa^2}{2} \left[ \kappa^2 + \frac{1}{N \mu} \left( \kappa^2 \dot{\varphi}^2 - \frac{2n^2}{a^2} + \frac{6k}{a^2} \right) \right] \dot{\varphi}^E \\
+ \frac{2\kappa^3 N \mu}{\sqrt{2(3 + 2\omega)}} \Phi^E, \tag{4.16}
\]

\[
\ddot{\Phi}^E = - \frac{1}{2} \dot{\varphi}^2 + \frac{N \mu}{\dot{\varphi}^2 + N \mu} \left[ 3H (N \mu - \dot{\varphi}^2) + \frac{\kappa \dot{\varphi}}{\sqrt{2(3 + 2\omega)}} (3N \mu + \dot{\varphi}^2) \right] \Phi^E \\
- \frac{N \mu}{\dot{\varphi}^2 + N \mu} \left[ \frac{2\kappa^2 N \mu}{3 + 2\omega} + \frac{n^2}{a^2} \right] \Phi^E \\
+ \frac{\kappa N^2 \mu^2}{\dot{\varphi}^2 + N \mu} \frac{1}{\sqrt{2(3 + 2\omega)}} \Delta^E - \frac{N \mu \dot{\varphi}}{\dot{\varphi}^2 + N \mu} Z^E. \tag{4.17}
\]

In the case of a flat dust universe \((k = 0, p = 0)\) with the specific initial condition \(\langle 4.1 \rangle\), we can solve these equations analytically (see appendix C). As for the density perturbation of dust, we obtain

\[
\Delta^E = c_1 \left\{ \left( \frac{\dot{t}}{t_0} \right)^{\frac{2(2 + \nu)}{4 + 3\omega}} + 4(8 + 5\omega) \frac{(5 + 3\omega)^{\frac{1 + \nu}{4 + 3\omega}}}{(4 + 3\omega)^2} \frac{1}{t_0^2 n^2} \right\} \\
+ c_2 \left( \frac{5 + 3\omega}{4 + 3\omega} \right) \frac{\dot{t}}{t_0}^{\frac{4 + 3\omega}{4 + 3\omega}} \\
+ c_3 \left( \frac{\dot{t}}{t_0} \right)^{-\frac{2(2 + \nu)}{4 + 3\omega}} J_\nu(nT) + c_4 \left( \frac{\dot{t}}{t_0} \right)^{-\frac{4 + 3\omega}{4(4 + 3\omega)}} J_{-\nu}(nT), \tag{4.18}
\]

where \(c_1, c_2, c_3\) and \(c_4\) are arbitrary integration constants, and

\[
\nu \equiv \frac{4 + 3\omega}{2(2 + \omega)}, \quad T \equiv \frac{4 + 3\omega}{2(2 + \omega)} \left( \frac{5 + 3\omega}{4 + 3\omega} \right)^{\frac{2 + \omega}{4 + 3\omega}} t_0, \tag{4.19}
\]

which are the same as Nariai’s notation. We can easily show that this solution is essentially the same as the gauge-dependent one \(\langle 4.5 \rangle\) by use of the relation \(\langle C.1 \rangle\).

The density perturbation \(\Delta^E\) has four independent modes. The first two terms have their counterparts in the Einstein de-Sitter model in the limit of \(\omega \to \infty\), while the last two originate from the BD scalar field. The growing mode (the first term in equation \(\langle 4.18 \rangle\)) evolves asymptotically

\[
\left( \frac{\dot{t}}{t_0} \right)^{\frac{2(2 + \omega)}{5 + 3\omega}} \propto \left( \frac{\dot{t}}{t_0} \right)^{\frac{2(2 + \omega)}{4 + 3\omega}} \propto a^{\frac{2 + \omega}{1 + \omega}}, \tag{4.20}
\]

which shows that the growth rate of \(\Delta^E\) in this specific BD model is somewhat higher (by \(a^{\frac{2 + \omega}{1 + \omega}}\) times) than that in the Einstein de-Sitter model.

Although we have obtained an analytic solution for the above specific model, in order to see whether such an enhancement in the growth rate of density perturbations
is generic and to find what the key to such an enhancement is, we reanalyze numerically the density perturbations of BD cosmological models with arbitrary initial conditions using our formalism.

We also evaluate the growth of density perturbations quantitatively under constraints from observations. Thus we can confirm that if all the observations are reliable, then the enhancement in the BD theory itself is inadequate to resolve the structure formation problem. The observational constraints on the BD theory may be summarized as follows:

1) The BD parameter \( \omega \): \( \omega > 500 \),

2) The present rate of time variation of \( G \): \( \left| \frac{\dot{G}}{G}_{\text{present}} \right| \leq 1 \times 10^{-11} \text{year}^{-1} \),

3) From nucleosynthesis:

\[
\left| \frac{\Delta G}{G} \right|_{\text{nucleosynthesis}} \leq 40\% ,
\]

where \( \Delta G \equiv G - G|_{\text{present}} \).

Among these constraints, 1) and 2) are obtained directly from present-day observations of gravity, while 3) is a result obtained indirectly, i.e., through the comparison between the theoretically predicted and observationally inferred primordial abundances of light elements. If we adopt 3), which critically depends on the theory of nucleosynthesis, the time variation of gravitational constant in the past is also strongly restricted, resulting in no difference in the evolution of density perturbations from that in the Einstein theory, as we will see shortly.

For the sake of easy comparison with the Einstein de-Sitter model, we first assume the ordinary matter is dust \( (p = 0) \) and the present density parameter \( \Omega_{\text{dust,0}} = \kappa^2 \mu_0 / 3 H_0^2 \) equals unity, which means the universe is closed \( (k > 0) \) in the BD cosmology. Besides, we set the Hubble constant \( H_0 = 100 \text{km/sec/Mpc} \). Then, regarding the observational constraints, we consider the following two cases:

Case (i) : All the constraints 1), 2) and 3) are satisfied.

Case (ii) : Only 1) and 2) are satisfied.

In all the calculations below, we set the BD parameter \( \omega = 500 \), therefore the constraint 1) is always satisfied. As for constraint 3), we calculated the background spacetime back to the nucleosynthesis era where \( a \sim 10^{-9} \) and found \( |\dot{G}/G|_{\text{present}} \leq 2.039 \times 10^{-13} \text{year}^{-1} \) for the universe which satisfies the constraint \( |\Delta G/G|_{\text{nucleosynthesis}} \leq 40\% \). Hence we adopt \( |\dot{G}/G|_{\text{present}} = 2.039 \times 10^{-13} \text{year}^{-1} \) for the model in Case (i). As for the model in Case (ii), we set \( |\dot{G}/G|_{\text{present}} = 2.042 \times 10^{-13} \text{year}^{-1} \), resulting in \( |\Delta G/G|_{\text{nucleosynthesis}} \simeq 200\% \). Both values for \( |\dot{G}/G|_{\text{present}} \) clearly satisfy constraint 2).

For the above two cases, we calculate the evolution of the density perturbations of dust \( \Delta^E \) as the scale of the universe grows \( 10^3 \) times to today. We assume the wave number of the perturbations \( n = 30 \), which means today’s ratio of physical wavelength
of perturbations to the Hubble horizon scale $\lambda_0/\lambda_{H,0} = 1/30$, corresponding to large scale structure on scales of $\lambda_0 \simeq 100\text{Mpc}$. We normalize the initial condition for $\Delta^E (= \delta \mu/\mu)$ to unity at the starting time. Therefore, it is natural to set the initial conditions for $Z^E (= 3\delta \dot{H})$ and $\Phi^E (= \delta \varphi)$ so as to approximate roughly their background values $\dot{H}$ and $\varphi$, respectively. We set $Z^E \simeq -\dot{H} (= -32077\dot{H}_0)$ and $\Phi^E \simeq \varphi (= 0.1903\sqrt{3}/\kappa)$ for Case (i), and $Z^E \simeq -\dot{H} (= -51692\dot{H}_0)$ and $\Phi^E \simeq \varphi (= 1.074\sqrt{3}/\kappa)$ for Case (ii) at the starting time.

The evolution of $\Delta^E$ is shown in Fig. 1. The solid line represents Case (i), while the dashed line represents Case (ii). Although $\Delta^E$ with all the constraints grows in almost the same way as in the Einstein de-Sitter case, that without constraint 3) is greatly enhanced by a factor $\sim 6 \times 10^2$. To see why, consider the time evolution of gravitational “constant” $G$ and its change rate $\dot{G}$, as shown in Fig. 2 and Fig. 3, respectively. Corresponding to the enhancement in the growth rate of $\Delta^E$, we see a great deviation of $\dot{G}$ from $|\dot{G}|_{\text{present}}$ in Case (ii). On the other hand, the variations of gravitational constant itself are within about several percent in both cases. This means that it is not the variation of gravitational constant itself but its change rate that is the key to the enhancement of the growth rate of perturbations.

We emphasize that the enhancement of $\Delta^E$ shown above is a generic feature for a wide range of parameters, such as $\Omega_{\text{dust},0}$ or $|\dot{G}/G|_{\text{present}}$. In order to see that the qualitative behavior is the same for the case of $\Omega_{dust,0} \neq 1$, consider Fig. 4, in which $\Omega_{dust,0} = 0.1$. We set $|\dot{G}/G|_{\text{present}} = 2.747 \times 10^{-14}\text{year}^{-1}$ for Case (i), $|\dot{G}/G|_{\text{present}} = 2.889 \times 10^{-14}\text{year}^{-1}$ for Case (ii), and set at the starting time $Z^E = -19138\dot{H}_0$, $\Phi^E = 1.186\sqrt{3}/\kappa$ and $Z^E = -10023\dot{H}_0$, $\Phi^E = 0.1228\sqrt{3}/\kappa$ for each case. The enhancement is slightly reduced because of the lower density of matter. As for the values of $|\dot{G}/G|_{\text{present}}$, as we mentioned above, constraint 3) strictly confines them. In order to satisfy constraint 3) in the BD theory, $|\dot{G}/G|_{\text{present}}$ must be almost exactly $|\dot{G}/G|_{\text{present}} = 2.039 \times 10^{-14}\text{year}^{-1}$ for $\Omega_{dust,0} = 1.0$ and $|\dot{G}/G|_{\text{present}} = 2.747 \times 10^{-14}\text{year}^{-1}$ for $\Omega_{dust,0} = 0.1$. The other values correspond to large deviations $|\Delta G/G|_{\text{nucleosynthesis}}$, and $\Delta^E$ is greatly enhanced in most of such cases.

5 Concluding Remarks

In this article we have explicitly presented gauge-invariant cosmological perturbation equations in generalized Einstein theories which are characterized as equations (3.1) and (A.21) with a perfect fluid. Although such equations have been of interest to resolve the galaxy formation problem, the complex structure of the system has made the formulation difficult. Therefore, we have first simplified the fundamental equations through conformal transformation. This process reduces the equations to the same form as in the Einstein theory which allows us to apply many well-developed techniques to their solution. This considerable simplification is important not only for this problem, but for several others as well. Then we have applied the covariant approach to derive the perturbation equa-
tions. This approach makes the formulation straightforward, and the intuitive meanings of the naturally introduced gauge-invariant variables are easy to obtain. Now that its correspondence to the conventional Bardeen’s approach is well understood, this approach is becoming established and will be applied in various situations to formulate relativistic perturbation theories.

As a simple example, we have applied the derived equations to perturbations in Brans-Dicke cosmology. We have presented an analytic solution for a specific background model, and shown an enhancement in the growth rate of perturbations. We have also analyzed perturbations in more generic models numerically. The enhancement of density perturbations occurs throughout the range of the initial conditions or parameters, so we have evaluated them assuming the observational constraints. Although the three constraints we consider here are too strict to permit a large enhancement in density perturbations, if we take into account only today’s direct observations, we have found a growth rate high enough to account for galaxy formation.

Our calculations imply that it is not the variation of effective gravitational constant itself but its change rate that is the key to the acceleration of the perturbation evolution. That means that if only our universe experienced an epoch in which the change rate of gravitational constant was high, even if the variation itself was small, then density perturbations could have grown fast enough to form the structure observed today, and that other complex processes would be wholly unnecessary. Hence the structure formation mechanism in generalized Einstein theories is very attractive. It may also be difficult to rule out completely such a possibility from observations. If GETs correctly describe nature, they may predict enhanced growth rates for density perturbations. We are now engaged in further studies of theories with non-minimally coupled scalar fields and perfect fluid matter, to which we plan to apply the results presented here.

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Appendices

A Formulation for the Most General Type of GETs

In this appendix we consider the most general action for GETs, i.e.,

\[ S = \int d^4x \sqrt{-g} \left[ F(\phi, R) - \frac{\epsilon(\phi)}{2} (\nabla \phi)^2 + L_m \right], \]  

(A.1)
where $F(\phi, R)$ is an arbitrary function of a scalar field $\phi$ and of a scalar curvature $R$. Here, we do not include Case (A) (where $F$ is a linear function of $R$) because it has been discussed in the text. Assuming a perfect fluid as (3.2), the basic equations are

$$
G_{ab} = \left( \frac{\partial F}{\partial R} \right)^{-1} \left\{ \frac{\epsilon(\phi)}{2} \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} (\nabla \phi)^2 \right\} + \frac{1}{2} g_{ab} \left( F - \frac{\partial F}{\partial R} R \right) + \left[ \nabla_a \nabla_b \left( \frac{\partial F}{\partial R} \right) - \Box \left( \frac{\partial F}{\partial R} \right) g_{ab} \right] + \frac{1}{2} f_{ab}^M \right\},
$$

$$
(\text{A.}2)
$$

$$
\epsilon(\phi) \Box \phi + \frac{1}{2} \frac{d\epsilon(\phi)}{d\phi} (\nabla \phi)^2 + \frac{\partial F}{\partial \phi} = 0.
$$

(A.3)

Although these involve higher-derivative terms of the metric, the conformal transformation [3.5] with

$$
\omega = \frac{1}{2} \ln \left( 2 \kappa^2 \left| \frac{\partial F}{\partial R} \right| \right),
$$

(A.4)

eliminates those higher-derivatives, and the introduction of a new “scalar” field

$$
\kappa \varphi \equiv \sqrt{6} \omega = \frac{\sqrt{6}}{2} \ln \left( 2 \kappa^2 \left| \frac{\partial F}{\partial R} \right| \right),
$$

(A.5)

enables us to write down the equations in the Einstein form:

$$
\tilde{G}_{ab} = \kappa^2 \left\{ \tilde{\nabla}_a \varphi \tilde{\nabla}_b \varphi - \frac{1}{2} g_{ab} (\tilde{\nabla} \varphi)^2 \right\} + \kappa^2 E(\phi, \varphi) \left\{ \tilde{\nabla}_a \phi \tilde{\nabla}_b \phi - \frac{1}{2} \hat{g}_{ab} (\tilde{\nabla} \phi)^2 \right\} - \kappa^2 U(\phi, \varphi) \hat{g}_{ab} + \kappa^2 N(\varphi) \left\{ \mu \tilde{u}_a^M \tilde{u}_b^M + \rho \tilde{h}_a^M \right\},
$$

(A.6)

$$
\epsilon(\phi) \Box \phi + \frac{1}{2} \frac{d\epsilon(\phi)}{d\phi} (\tilde{\nabla} \phi)^2 - \kappa \epsilon(\phi) \frac{\sqrt{6}}{3} \tilde{\nabla}_a \varphi \tilde{\nabla}^a \phi + \exp \left[ -\frac{\sqrt{6}}{3} \kappa \varphi \right] \frac{\partial F}{\partial \phi} = 0,
$$

(A.7)

where

$$
E(\phi, \varphi) = (\text{sign}) \epsilon(\phi) \exp \left[ -\frac{\sqrt{6}}{3} \kappa \varphi \right], \quad (\text{sign}) = \frac{\partial F}{\partial R} \big/ \left| \frac{\partial F}{\partial R} \right|,
$$

(A.8)

$$
N(\varphi) = (\text{sign}) \exp \left[ -\frac{2}{3} \sqrt{6} \kappa \varphi \right],
$$

(A.9)

$$
U(\phi, \varphi) = N(\varphi) \left( \frac{\partial F}{\partial R} R - F \right),
$$

(A.10)

$$
\tilde{u}_a^M = e^{\omega} u_a^M, \quad \tilde{u}_M \equiv e^{-\omega} u_M^a,
$$

(A.11)

$$
\tilde{h}_a^M = h_a^M, \quad \tilde{h}_M^a = e^{2\omega} h_M^a, \quad \tilde{h}_{ab} = e^{-2\omega} h_{ab}.
$$

(A.12)

The (sign) equals 1 for the present universe, and for a generic spacetime there appears a singularity at $\partial F/\partial R = 0$ beyond which the (sign) turns to be $-1$. Then we should usually assume (sign) = 1 although we leave it in this article. The equation for $\varphi$ is obtained by taking the trace of equation (A.2) and using relation (A.5):

$$
\Box \varphi + \frac{\sqrt{6}}{6} \kappa E(\phi, \varphi) (\tilde{\nabla} \phi)^2 + \frac{\sqrt{6}}{3} \kappa N(\varphi) \left\{ \frac{\partial F}{\partial R} R - F - \frac{1}{2} (\kappa \mu + 3p) \right\} = 0.
$$

(A.13)
In these equations, $R$ and $F$ are functions of $\phi$ and $\varphi$ i.e., $R = R(\phi, \varphi)$ and $F = F(\phi, \varphi) \equiv F(\phi, R(\phi, \varphi))$, which are defined implicitly by the relation

$$\frac{\partial F}{\partial R}(\phi, R) = \text{a given function of } \phi \text{ and } R$$

$$= (\text{sign}) \frac{1}{2\kappa^2} \exp \left[ \frac{2}{\sqrt{6} \kappa \varphi} \right]. \quad (A.14)$$

The gravitational field equation (A.6) corresponds to the Einstein one with two scalar fields and a perfect fluid interacting with each other through $E(\phi, \varphi)$, $U(\phi, \varphi)$ and $N(\varphi)$. The energy-momentum tensor in equation (A.6) is expressed in a fluid form like equation (3.19) as

$$T_{ab} = \mu_* u_a^O u_b^O + p_* h_{ab}^O + q_{sa} u_b^O + \pi_{sab}, \quad (A.15)$$

$$\mu_* \equiv \frac{1}{2} \dot{\varphi}^2 + \frac{1}{2} E(\phi, \varphi)\dot{\varphi}^2 + U(\phi, \varphi) + N(\varphi)\mu, \quad (A.16)$$

$$p_* \equiv \frac{1}{2} \dot{\varphi}^2 + \frac{1}{2} E(\phi, \varphi)\dot{\varphi}^2 - U(\phi, \varphi) + N(\varphi)p, \quad (A.17)$$

$$q_{sa} \equiv -\dot{\varphi}^{(3)} \nabla_a \varphi - E(\phi, \varphi)\dot{\varphi}^{(3)} \nabla_a \phi + N(\varphi)(\mu + p)V_a^M, \quad (A.18)$$

$$\pi_{sab} \equiv 0, \quad (A.19)$$

$$V_a^M \equiv u_a^M - u_a^O, \quad (A.20)$$

to first order. Hence, the derivation of the GI perturbation equations to follow is straightforward, and proceeds as described in §3.

Particularly in the case of $F = L(R)$ and $\epsilon(\phi) = 0$ (Case (B)), namely, when the action is given by

$$S = \int d^4x \sqrt{-g} [L(R) + L_m], \quad (A.21)$$

the basic equations become

$$\hat{G}_{ab} = \kappa^2 \left\{ \nabla_a \varphi \nabla_b \varphi - \frac{1}{2} \hat{g}_{ab}(\nabla \varphi)^2 - U(\varphi)\hat{g}_{ab} \right\}$$

$$+ \kappa^2 N(\varphi) \left\{ \mu_* \hat{u}_a^M \hat{u}_b^M + p_* \hat{h}_{ab}^M \right\} \quad (A.22)$$

$$\Box \varphi = \frac{dU(\varphi)}{d\varphi} + \kappa^2 M(\varphi)\{\mu - 3p\} = 0, \quad (A.23)$$

with

$$N(\varphi) = (\text{sign}) \exp \left[ \frac{2}{3} \sqrt{6} \kappa \varphi \right], \quad (\text{sign}) = \frac{\partial L}{\partial R} / \left| \frac{\partial L}{\partial R} \right|, \quad (A.24)$$

$$U(\varphi) = N(\varphi) \left[ \frac{\partial L}{\partial R} R - L \right], \quad (A.25)$$

$$M(\varphi) = \frac{\sqrt{6}}{6\kappa} N(\varphi). \quad (A.26)$$

Here, $R = R(\varphi)$ and $L = L(\varphi) \equiv L(R(\varphi))$ through the relation

$$\frac{\partial L}{\partial R}(R) = \text{a given function of } R$$

$$= (\text{sign}) \frac{1}{2\kappa^2} \exp \left[ \frac{2}{\sqrt{6} \kappa \varphi} \right]. \quad (A.27)$$
Equations (A.22) and (A.23) have exactly the same form as equations (3.7) and (3.8) in [3]. Hence the perturbation equations are given by equations (3.48), (3.50) and (3.49).

Among the models in Case (B), we are mostly interested in $R^2$ gravity from the astrophysical point of view. By setting $L(R) = \frac{1}{2\kappa^2}(R + CR^2)$ with (sign) = 1, equations (A.24) ~ (A.27) become

\[
N(\phi) = \exp \left[ -\frac{2}{3}\sqrt{6}\kappa\varphi \right],
\]

\[
M(\phi) = \frac{\sqrt{6}}{6\kappa} N(\varphi),
\]

\[
U(\varphi) = \frac{N(\varphi)}{8\kappa^2 C} \left( 1 - \exp \left[ \frac{\sqrt{6}}{3}\kappa\varphi \right] \right)^2,
\]

\[
\kappa\varphi = \frac{\sqrt{6}}{2} \ln \left[ |1 + 2CR| \right].
\]

In [28], perturbations in the $R^2$ gravity were discussed. However, they were concerned only with metric perturbations without matter fluid ($L_m = 0$), and their approach was not GI, although their gauge (conformal-Newtonian gauge) gave physical modes.

**B Perturbations in a Scalar Field Dominated Universe**

In this appendix, taking the limit $\mu, p \to 0$, we show our perturbation equations reduce to the ones in a scalar field dominated universe derived by Bruni et al. [22]. Since they assumed the Einstein theory, we must set

\[
f(\phi) = \frac{1}{2\kappa^2}, \quad \epsilon(\phi) = 1,
\]

in the action (3.1), which immediately gives

\[
N(\varphi) = 1,
\]

\[
M(\varphi) = 0,
\]

\[
U(\varphi) = V(\phi).
\]

with $\varphi = \phi$. Besides, as they used center of mass frame $u^E_a$, we can use equations (3.26) ~ (3.30) as the basic equations.

In a scalar field dominated universe, matter fluid perturbations are absent, so we need consider only two independent GI perturbations. Bruni et al. adopted the total energy density perturbation $\Delta^E E = (a^2/\mu_*)^{(3)}\nabla^2 E \mu_*$ and the curvature perturbation $C^E = a^{2(3)}\nabla^2 [^{(3)} R]$ for them. They are expressed in terms of our GI perturbation variables as

\[
\Delta^E E = \frac{\dot{\phi}^2}{\mu_*} \left( \frac{\dot{\phi} \dot{\Phi}^E}{\dot{\mu}} - \frac{V' \Phi^E}{\mu} \right),
\]

\[
C^E = -4a^2 H Z^E + 2\kappa^2 a^2 \mu_* \Delta^E E.
\]
We can recover their basic equations in the limit of \( \mu, p \to 0 \) in our equations. The background equations (3.31)\( \sim \) (3.34) become

\[
3H + 3H^2 + \kappa^2 (\dot{\phi}^2 - V(\phi)) = 0, \quad (B.8)
\]
\[
3H^2 + \frac{3k}{a^2} = \kappa^2 \mu^*, \quad (B.9)
\]
\[
\ddot{\phi} + 3H \dot{\phi} + V'(\phi) = 0, \quad (B.10)
\]

where \( \mu^* = \frac{1}{2} \dot{\phi}^2 + V(\phi) \) and equation (B.10) is equivalent to

\[
\dot{\mu}^* = -3H \dot{\phi}^2. \quad (B.11)
\]

The condition that total energy flux vanishes in the center of mass frame implies in this limit

\[
u^E_a \to \frac{1}{\dot{\phi}} \nabla_a \phi, \quad (B.12)
\]

which gives \( \Phi^E \to 0 \), so that \( \Delta^E_\star \) has a finite value even in the limit of \( \mu \to 0 \).

In this limit, perturbation equation (3.48) is trivial. Equations (3.49) and (3.50) give respectively

\[
\dot{\Delta}^E_\star = -3H \dot{\phi}^2 \mu^* - \frac{\kappa^2 \dot{\phi}^2}{2H} \Delta^E_\star + \frac{\dot{\phi}^2}{4a^2 H \mu^*} C^E. \quad (B.15)
\]

By using the relation (B.6) and its derivatives, we can eliminate \( \ddot{\phi}/\mu, \dot{\phi}/\mu \) and \( \Phi/\mu \) from equation (B.14) as

\[
\dot{\Delta}^E_\star = \left[ -3H + \frac{3H \dot{\phi}^2}{\mu^*} - \frac{\kappa^2 \dot{\phi}^2}{2H} \right] \Delta^E_\star + \frac{\dot{\phi}^2}{4a^2 H \mu^*} C^E. \quad (B.15)
\]

Rewriting \( 3H \) for \( \theta \) and using the notation \( \gamma \equiv \dot{\phi}^2/\mu^* \), this becomes

\[
\dot{\Delta}^E_\star = \frac{3\gamma}{4a^2 \theta} C^E + \left[ (\gamma - 1)\theta - \frac{3\kappa^2 \mu^* \gamma}{2\theta} \right] \Delta^E_\star, \quad (B.16)
\]

which corresponds to equation (55) in [22]. Moreover we can eliminate \( Z^E \) from equation (B.13) by using equation (B.7) as

\[
\dot{C}^E = 2\kappa^2 a^2 \mu^* \dot{\Delta}^E_\star
= -HC^E - \frac{\kappa^2}{3H} (\dot{\phi}^2 - V) C^E
+ \left[ 8\kappa^2 a^2 H \mu^* + \frac{2\kappa^4 a^2 \mu^*}{3H} (\dot{\phi}^2 - V) - 6\kappa^2 a^2 H \dot{\phi}^2 + 2(-2n^2 + 6k) \frac{H \mu^*}{\phi^2} \right] \Delta^E_\star. \quad (B.17)
\]
Replacing $\Delta^E_*$ by equation \((B.16)\) and by using equation \((3.47)\), this becomes

$$
\dot{C}^E = \frac{3k}{a^2\theta} C^E + \frac{4a^2\theta}{3\gamma} \nabla^2 \Delta^E + 4k \left( \frac{\theta}{\gamma} - \frac{3\kappa^2\mu_*}{2\theta} \right) \Delta^E,
$$

\((B.18)\)

which corresponds to (56) in \([22]\). The equations \((B.16)\) and \((B.18)\) are the basic equations for GI perturbation variables $\Delta^E_*$ and $C^E$ given in \([22]\).

### C Analytic Solution in a BD Cosmology

The solution \((4.2)\sim(4.4)\) of the flat dust BD universe with the condition \((4.1)\) is described in terms of conformal variables as

$$
\hat{t} = \frac{4 + 3\omega}{5 + 3\omega} \left( \frac{t}{t_0} \right)^{\frac{5 + 3\omega}{4 + 3\omega}},
$$

\((C.1)\)

$$
\varphi = \sqrt{2(3 + 2\omega)} \ln \left| \frac{5 + 3\omega}{4 + 3\omega} \frac{\hat{t}}{t_0} \right|,
$$

\((C.2)\)

$$
\hat{a} = \left( \frac{5 + 3\omega}{4 + 3\omega} \right)^{\frac{3 + 2\omega}{4 + 3\omega}} \left( \frac{\hat{t}}{t_0} \right)^{\frac{3 + 2\omega}{4 + 3\omega}},
$$

\((C.3)\)

$$
\mu = \mu_0 \hat{a}^{-3} \left( \frac{\phi}{\phi_0} \right)^{-\frac{1}{3}} = \frac{2(3 + 2\omega)}{\kappa^2 t_0^2 (4 + 3\omega)} \left( \frac{5 + 3\omega}{4 + 3\omega} \right) \left( \frac{\hat{t}}{t_0} \right)^{-\frac{6(1 + \omega)}{5 + 3\omega}}.
$$

\((C.4)\)

Inserting this explicit background solution into equations \((4.15)\sim(4.17)\), we present the perturbation equations as

$$
\dot{\Delta}^E = -Z^E + \frac{3(7 + 5\omega)}{\sqrt{2(3 + 2\omega)(4 + 3\omega)}} \dot{\phi}^E + \frac{\kappa n^2 t_0}{\sqrt{2(3 + 2\omega)}} \left( \frac{5 + 3\omega}{4 + 3\omega} \right)^{-\frac{6(1 + \omega)}{5 + 3\omega}} \phi^E,
$$

\((C.5)\)

$$
\dot{Z}^E = -\frac{(3 + 2\omega)(4 + 3\omega)}{(5 + 3\omega)^2} \frac{1}{\hat{t}^2} \Delta^E
$$

$$
- \frac{2(3 + 2\omega)}{(5 + 3\omega)} \frac{1}{\hat{t}} Z^E
$$

$$
- \left[ \frac{\sqrt{2(3 + 2\omega)} \kappa}{2(4 + 3\omega)} \hat{t} - \frac{\kappa n^2 t_0}{\sqrt{2(3 + 2\omega)}} \left( \frac{5 + 3\omega}{4 + 3\omega} \right) \left( \frac{\hat{t}}{t_0} \right)^{-\frac{6(1 + \omega)}{5 + 3\omega}} \right] \dot{\phi}^E
$$

$$
+ 2 \sqrt{2(3 + 2\omega)} \frac{4 + 3\omega}{(5 + 3\omega)^2} \frac{\kappa}{\hat{t}^2} \dot{\phi}^E,
$$

\((C.6)\)

$$
\ddot{\phi}^E = -\frac{2(4 + 3\omega)}{5 + 3\omega} \frac{1}{\hat{t}} \dot{\phi}^E
$$

$$
- \left[ \frac{4(4 + 3\omega)^2}{(5 + 3\omega)^3} \frac{1}{\hat{t}^2} + \frac{n}{5 + 3\omega} \frac{4 + 3\omega}{(4 + 3\omega)} \left( \frac{5 + 3\omega}{4 + 3\omega} \right) \left( \frac{\hat{t}}{t_0} \right)^{-\frac{2(3 + 2\omega)}{5 + 3\omega}} \right] \phi^E
$$

\((C.7)\)
Then we can solve these equations analytically as

\[
\Delta^E = c_1 \left\{ \left( \frac{\dot{t}}{t_0} \right) \frac{2(2+\omega)}{8+5\omega} + \frac{4(8+5\omega)}{(5+3\omega)^2} \left( \frac{5+3\omega}{4+3\omega} \right) \frac{1}{4\kappa t_0^2} \right\}
\]

\[
+ c_2 \left( \frac{5+3\omega}{4+3\omega} \right) \frac{\dot{t}}{4+3\omega t_0}
\]

\[
+ c_3 \left( \frac{\dot{t}}{t_0} \right) J_{\nu}(nT) + c_4 \left( \frac{\dot{t}}{t_0} \right) J_{-\nu}(nT),
\]

\( C.7 \)

\[
\mathcal{Z}^E = -c_1 \left( 3+2\omega \right) \frac{(4+3\omega)}{(5+3\omega)^2} \frac{1}{t_0^2} \left( \frac{\dot{t}}{t_0} \right) \frac{1}{\sqrt{\kappa n^2}}
\]

\[
+ c_2 t_0 \left( \frac{5+3\omega}{4+3\omega} \right) \frac{1}{4+3\omega t_0}
\]

\[
+ c_3 \left\{ -\frac{n}{4} \left( \frac{5+3\omega}{4+3\omega} \right)^{\frac{2+\omega}{8+5\omega}} \left( \frac{\dot{t}}{t_0} \right) \frac{10+7\omega}{2(5+3\omega)} J_{\nu+1}(nT)
\]

\[
+ \frac{n^2 t_0}{4} \left( \frac{5+3\omega}{4+3\omega} \right)^{-\frac{1+\omega}{8+5\omega}} \left( \frac{\dot{t}}{t_0} \right) \frac{6+5\omega}{2(5+3\omega)} J_{\nu}(nT) \right\}
\]

\[
+ c_4 \left\{ -\frac{n}{4} \left( \frac{5+3\omega}{4+3\omega} \right)^{\frac{2+\omega}{8+5\omega}} \left( \frac{\dot{t}}{t_0} \right) \frac{10+7\omega}{2(5+3\omega)} J_{-\nu-1}(nT)
\]

\[
+ \frac{n^2 t_0}{4} \left( \frac{5+3\omega}{4+3\omega} \right)^{-\frac{1+\omega}{8+5\omega}} \left( \frac{\dot{t}}{t_0} \right) \frac{6+5\omega}{2(5+3\omega)} J_{-\nu}(nT) \right\},
\]

\( C.8 \)

\[
\Phi^E = c_1 \sqrt{2(3+2\omega)} \frac{(8+5\omega)}{(5+3\omega)^2} \left( \frac{5+3\omega}{4+3\omega} \right)^{\frac{1+\omega}{4\kappa t_0^2}} \frac{1}{\kappa n^2 t_0^2}
\]

\[
+ \frac{\sqrt{2(3+2\omega)}}{4\kappa} \left\{ c_3 \left( \frac{\dot{t}}{t_0} \right) \frac{1}{(2+\omega)} J_{\nu}(nT) + c_4 \left( \frac{\dot{t}}{t_0} \right) \frac{1}{(2+\omega)} J_{-\nu}(nT) \right\},
\]

\( C.9 \)

where \( c_1, c_2, c_3 \) and \( c_4 \) are arbitrary constants, and

\[
\nu \equiv \frac{4+3\omega}{2(2+\omega)}, \quad T \equiv \frac{4+3\omega}{2} \left( \frac{5+3\omega}{4+3\omega} \right) \frac{(2+\omega)}{4\kappa t_0^2}
\]

\( C.11 \)
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Figure Captions

Fig. 1:
Density perturbations in the BD theory for the case of $\Omega_{dust,0} = 1.0$. The solid line represents Case (i); all the observational constraints 1), 2) and 3) are satisfied, while the dashed line corresponds to Case (ii); only 1) and 2) are satisfied (see text). The density perturbation in the Einstein theory with the same initial condition coincides with (i).

Fig. 2:
Evolution of effective gravitational constant ($G=1/\phi$). Each line corresponds to the line in Fig. 1. bearing the same number.

Fig. 3:
Evolution of change rate of effective gravitational constant. Each line corresponds to the line in Fig. 1. bearing the same number, and $H_0 = 100\text{km/sec/Mpc}$.

Fig. 4:
Density perturbations in the BD theory for the case of $\Omega_{dust,0} = 0.1$. The solid line represents Case (i); all the observational constraints 1), 2) and 3) are satisfied, while the dashed line corresponds to Case (ii); only 1) and 2) are satisfied (see text). The density perturbation in the Einstein theory with the same initial condition coincides with (i).
This figure "fig1-1.png" is available in "png" format from:

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