A LIOUVILLE TYPE THEOREM TO AN EXTENSION PROBLEM RELATING TO THE HEISENBERG GROUP

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Abstract. We establish a Liouville type theorem for nonnegative cylindrical solutions to the extension problem corresponding to a fractional CR covariant equation on the Heisenberg group by using the generalized CR inversion and the moving plane method.

1. Introduction. We mention that there have been many works for the symmetry property about the solutions of the Laplace equations with the Neumann boundary in a half space, see [1, 17, 32] and the references therein, and then Lou and Zhu [31] established classifications of nonnegative solutions to the Neumann problem on the upper half space $\mathbb{R}^{n}_{+}$ of $\mathbb{R}^{n}$

$$\begin{cases}
\Delta u = 0, & u \geq 0 \text{ in } \mathbb{R}^{n}_{+}, \\
\frac{\partial u}{\partial t} = u^{p}, & \text{on } \partial \mathbb{R}^{n}_{+},
\end{cases}$$

where $\mathbb{R}^{n}_{+} = \{(x', t)|x' = (x_{1}, x_{2}, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}, t > 0\}$ ($n \geq 2$) and $p > 1$, by providing some technical lemmas and using the method of moving planes.

Chen and Zhu in [15] considered the extension problem

$$\begin{cases}
\text{div}(y^{1-\alpha} \nabla w) = 0, & (x, y) \in \mathbb{R}^{n} \times (0, \infty), \\
- \lim_{y \to 0} y^{1-\alpha} \frac{\partial w}{\partial y} = w^{p}(x, 0), & x \in \mathbb{R}^{n}
\end{cases}$$

for the fractional Laplace equation

$$(-\Delta)^{\alpha/2} u(x) = -u^{p},$$

where $0 < \alpha < 2$ and $p > 1$, and obtained classifications of nonnegative solutions to the extension problem by extending the method in [31] to the fractional Laplacian case. Here the fractional Laplacian $(-\Delta)^{\alpha/2}$ is a nonlocal pseudodifferential

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In this paper, we establish a Liouville type theorem for nonnegative cylindrical solutions to the extension problem (1.1). The function $U \in C^2(\hat{H}^n_+) \cap C^1(\hat{H}^n_+) \subset C(\hat{H}^n_+)$ is called the cylindrical solution to (1.1), if

$$U(x,y,t,\lambda) = U(\|x, y\|, t, \lambda), \text{ for } (x,y,t,\lambda) \in \hat{H}^n_+, r_0 = (\|x\|^2 + |y|^2)^{\frac{1}{2}}.$$

Our main result is the following

**Theorem 1.1.** For $0 < \alpha < 2$, $p > 1$, let $U \in C^2(\hat{H}^n_+) \cap C^1(\hat{H}^n_+)$ be a nonnegative cylindrical solution to (1.1), then

$$U = \frac{a}{(\sqrt{2})^\alpha} \lambda^\alpha + b,$$

where $b > 0$, $a = b^p$.

For convenience, we use the notation

$$L_\alpha := \frac{\partial^2}{\partial \lambda^2} + \frac{1 - \alpha}{\lambda} \frac{\partial}{\partial \lambda} + 4\lambda^2 \frac{\partial^2}{\partial t^2} + \Delta_{H^n},$$

by replacing $\lambda$ in (1.1) by $\sqrt{2}\lambda$, and denote

$$\frac{\partial}{\partial u^{\alpha}} := -c_\alpha \lim_{\lambda \to 0} \lambda^{1-\alpha} \frac{\partial}{\partial \lambda}.$$

So Theorem 1.1 can be described as
Theorem 1.2. For $0 < \alpha < 2$, $p > 1$, let $U \in C^2(\mathbb{H}^n_+) \cap C^1(\overline{\mathbb{H}^n_+})$ be a nonnegative cylindrical solution to
\[
\begin{cases}
L_\alpha U = 0, & z \in \mathbb{H}^n_+,
\frac{\partial U}{\partial \nu_\alpha}(x,y,t,\lambda) = -U^p(x,y,t,0), & z \in \partial \mathbb{H}^n_+,
\end{cases}
\]
then
\[U = \frac{a}{\alpha} \lambda^\alpha + b,\]
where $b \geq 0$, $a = b^p$.

The idea of proofs for theorems comes from [15] and [31]. Noting that the structures of $\mathbb{H}^n$ and $\mathbb{R}^n$ are different, we use the H-reflection on $\mathbb{H}^n$ introduced by Birindelli and Prajapat [4] (see (4.4) below). Since the subLaplacian $\Delta_{\mathbb{H}^n}$ is invariance about the H-reflection, we can move planes on $\mathbb{H}^n$ to reach our aim. The more content for the method of moving planes the readers may refer to [12, 13, 33].

The paper is organized as follows. Some well known results for $\mathbb{H}^n$ and $\Delta_{\mathbb{H}^n}$ are collected in Section 2. The generalized CR inversion of cylindrical solutions is given in Section 3. In section 4, we prove Theorem 1.2 by considering two cases: (1) the supercritical case and (2) the subcritical and critical case, and use the method of moving planes. Theorem 1.1 is obtained from Theorem 1.2.

2. Preliminaries. The Heisenberg group $\mathbb{H}^n$ is the Euclidean space $\mathbb{R}^{2n+1}$ ($n \geq 1$) (or $\mathbb{C}^n \times \mathbb{R}$) endowed with the group law $\circ$ defined by
\[
\bar{\xi} \circ \xi = (x + \bar{x}, y + \bar{y}, t + \bar{t} + 2 \sum_{i=1}^{n} (x_i \bar{y}_i - y_i \bar{x}_i)),
\]
where $\xi = (x_1, \cdots, x_n, y_1, \cdots, y_n, t) := (x, y, t) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$, $\bar{\xi} = (\bar{x}, \bar{y}, \bar{t})$. Denote by $\delta_\kappa$ the dilations on $\mathbb{H}^n$, i.e.
\[
\delta_\kappa(\bar{\xi} \circ \xi) = \delta_\kappa(\bar{\xi}) \circ \delta_\kappa(\xi).
\]
which satisfies $\delta_\kappa(\bar{\xi} \circ \xi) = \delta_\kappa(\bar{\xi}) \circ \delta_\kappa(\xi)$.

The left invariant vector fields on $\mathbb{H}^n$ are of the form
\[
X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial \bar{t}}, \quad i = 1, \cdots, n,
Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial \bar{t}}, \quad i = 1, \cdots, n,
T = \frac{\partial}{\partial \bar{t}},
\]
with
\[
[X_i, Y_j] = -4T \delta_{ij}, [X_i, X_j] = [Y_i, Y_j] = 0, i, j = 1, \cdots, n.
\]
The subLaplacian $\Delta_{\mathbb{H}^n}$ on $\mathbb{H}^n$ is
\[
\Delta_{\mathbb{H}^n} := \sum_{i=1}^{n} (X_i^2 + Y_i^2) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial \bar{t}} - 4x_i \frac{\partial^2}{\partial y_i \partial \bar{t}} + 4(x_i^2 + y_i^2) \frac{\partial^2}{\partial \bar{t}^2}.
\]
Noting that $\{X_1, \cdots, X_n, Y_1, \cdots, Y_n\}$ satisfies Hörmander’s rank condition (see [27]), this implies that $\Delta_{\mathbb{H}^n}$ is hypoelliptic (see [27]) and the maximum principle for the solutions of the equation including $\Delta_{\mathbb{H}^n}$ holds (see [7]).
Denote by $|\xi|_{H^n}$ the distance from $\xi$ to the zero (see [20]):

$$|\xi|_{H^n} = \left(\sum_{i=1}^{n} (x_i^2 + y_i^2)^2 + \lambda^2\right)^{\frac{1}{4}}.$$ (2.4)

The distance between two points in $\mathbb{H}^n$ is defined by

$$d_{\mathbb{H}^n}(\xi, \eta) = |\eta^{-1} \circ \xi|_{H^n},$$

where $\eta^{-1}$ denotes the inverse of $\eta \in \mathbb{H}^n$ with respect to $\circ$, that is $\eta^{-1} = -\eta$.

The open ball of radius $R > 0$ centered at $\xi$ is the set

$$B_{\mathbb{H}^n}(\xi, R) = \{ \eta \in \mathbb{H}^n | d_{\mathbb{H}^n}(\eta, \xi) < R \}.$$

By the dilations $\delta_r$ of the group, $\xi \rightarrow |\xi|_{H^n}$ is homogeneous of degree one and

$$|B_{\mathbb{H}^n}(\xi, R)| = |B_{\mathbb{H}^n}(0, R)| = |B_{\mathbb{H}^n}(0, 1)| R^Q,$$

where $|\cdot|$ denotes the Lebesgue measure, $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$.

Existence and uniqueness of solutions to the extension problem of the operator $P_{\alpha/2}$ were given in [21]. More information about the fractional CR covariant operator $P_{\alpha/2}$ ($0 < \alpha < 2$) can be referred to [21]. We will consider cylindrical solutions to the extension problem (1.1) for the equation $P_{\alpha/2} = -u^\prime$ on the extended space $\mathbb{H}^n := \mathbb{H}^n \times \mathbb{R} \subset \mathbb{C}^{n+1}$. The group law $\delta$ on $\mathbb{H}^n$ is defined as

$$\delta z = (x + \bar{x}, y + \bar{y}, t + \bar{t} + 2 \sum_{i=1}^{n} (x_i \bar{y}_i - y_i \bar{x}_i), \lambda + \bar{\lambda}),$$ (2.5)

where $z = (x_1, \cdots, x_n, y_1, \cdots, y_n, \lambda) := (x, y, t, \lambda) \in \mathbb{H}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, $\bar{z} = (\bar{x}, \bar{y}, \bar{t}, \bar{\lambda})$, $\lambda \in \mathbb{R}$. Denote by $|z|_{\mathbb{H}^n}$ the distance from $z \in \mathbb{H}^n$ to the origin:

$$|z|_{\mathbb{H}^n} = (|x|^2 + |y|^2 + \lambda^2)^{\frac{1}{4}}.$$ (2.6)

The distance between $z$ and $\bar{z}$ on $\mathbb{H}^n$ is of the form

$$d_{\mathbb{H}^n}(z, \bar{z}) = |\bar{z}^{-1} \delta z|_{H^n},$$

where $\bar{z}^{-1}$ is the inverse of $z$ with respect to $\delta$. When $\lambda = \bar{\lambda} = 0$, we have $d_{\mathbb{H}^n}(z, \bar{z}) = d_{\mathbb{H}^n}(\xi, \bar{\xi})$, for $z = (x, y, t, 0)$, $\bar{z} = (\bar{x}, \bar{y}, \bar{t}, 0)$, $\xi = (x, y, t)$, $\bar{\xi} = (\bar{x}, \bar{y}, \bar{t})$. The open ball of radius $R$ centered at $\bar{z}$ is the set

$$B_{\mathbb{H}^n}(z, R) = \{ \bar{z} \in \mathbb{C}^{n+1} | d_{\mathbb{H}^n}(z, \bar{z}) < R \}.$$

For any $z_0 \in \mathbb{H}^n \times \{0\}$, we denote

$$B_{\mathbb{H}^n}(z_0, R) = \{ z \in \mathbb{H}^n \times \mathbb{R}^+ | d_{\mathbb{H}^n}(z, z_0) < R \}.$$

Let us denote $\mathbb{H}^n_+ := \mathbb{H}^n \times \mathbb{R}^+$. The operator $L_\alpha$ in (1.2) becomes

$$L_\alpha = \frac{\partial^2}{\partial \lambda^2} + 1 - \frac{\alpha}{\lambda} \frac{\partial}{\partial \lambda} + \sum_{i=1}^{n} \left( \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial t} - 4x_i \frac{\partial^2}{\partial y_i \partial t} \right) + 4(\lambda^2 + \sum_{i=1}^{n} (x_i^2 + y_i^2)) \frac{\partial^2}{\partial t^2}. $$ (2.7)

For any $z = (x, y, t, \lambda) \in \mathbb{H}^n$, denote

$$r_0 = (|x|^2 + |y|^2)^{\frac{1}{2}}; r = (|x|^2 + |y|^2 + \lambda^2)^{\frac{1}{2}}; \rho = |z|_{\mathbb{H}^n} = (r^4 + t^2)^{\frac{1}{4}}.$$
Lemma 2.1. For the cylindrical functions $\phi = \phi(r_0, t, \lambda)$, we have
\[
\mathcal{L}_\alpha \phi = \frac{\partial^2 \phi}{\partial r_0^2} + \frac{2n - 1}{r_0} \frac{\partial \phi}{\partial r_0} + \frac{\partial^2 \phi}{\partial \lambda^2} + \frac{1 - \alpha}{\lambda} \frac{\partial \phi}{\partial \lambda} + 4r_0^2 \frac{\partial^2 \phi}{\partial t^2} + 4\lambda^2 \frac{\partial^2 \phi}{\partial t^2}. \tag{2.8}
\]

Proof. Clearly,
\[
\frac{\partial r_0}{\partial x_i} = \frac{1}{2} \left( \sum_{i=1}^{n} (x_i^2 + y_i^2) \right)^{-\frac{1}{2}} \cdot 2x_i = \frac{x_i}{r_0},
\]
\[
\frac{\partial^2 r_0}{\partial x_i^2} = -\frac{1}{2} \left( \sum_{i=1}^{n} (x_i^2 + y_i^2) \right)^{-\frac{3}{2}} \cdot 2x_i x_i + \frac{1}{r_0} = -\frac{x_i^2}{r_0^3} + \frac{1}{r_0},
\]
\[
\frac{\partial r_0}{\partial y_i} = \frac{y_i}{r_0},
\]
\[
\frac{\partial^2 r_0}{\partial y_i^2} = -\frac{y_i^2}{r_0^3} + \frac{1}{r_0}.
\]
It follows
\[
\frac{\partial \phi}{\partial x_i} = \frac{\partial \phi}{\partial r_0} \frac{\partial r_0}{\partial x_i} = \frac{x_i}{r_0} \frac{\partial \phi}{\partial r_0},
\]
\[
\frac{\partial \phi}{\partial y_i} = \frac{y_i}{r_0} \frac{\partial \phi}{\partial r_0},
\]
\[
\frac{\partial^2 \phi}{\partial x_i^2} = \frac{\partial^2 \phi}{\partial r_0^2} \left( \frac{\partial r_0}{\partial x_i} \right)^2 + \frac{\partial \phi}{\partial r_0} \frac{\partial^2 r_0}{\partial x_i^2} = \frac{x_i^2}{r_0^2} \frac{\partial^2 \phi}{\partial r_0^2} + \left( -\frac{x_i^3}{r_0^3} + \frac{1}{r_0} \right) \frac{\partial \phi}{\partial r_0},
\]
\[
\frac{\partial^2 \phi}{\partial y_i^2} = \frac{y_i^2}{r_0^2} \frac{\partial^2 \phi}{\partial r_0^2} + \left( -\frac{y_i^3}{r_0^3} + \frac{1}{r_0} \right) \frac{\partial \phi}{\partial r_0},
\]
\[
\frac{\partial^2 \phi}{\partial x_i \partial t} = \frac{\partial^2 \phi}{\partial r_0 \partial t} \left( \frac{\partial r_0}{\partial x_i} \right) + \frac{\partial \phi}{\partial x_i} \frac{\partial^2 \phi}{\partial r_0^2} = \frac{x_i}{r_0} \frac{\partial^2 \phi}{\partial r_0 \partial t} + \frac{\partial \phi}{\partial x_i} \frac{\partial^2 \phi}{\partial r_0^2},
\]
and
\[
\frac{\partial^2 \phi}{\partial y_i \partial t} = \frac{y_i}{r_0} \frac{\partial^2 \phi}{\partial r_0 \partial t}.
\]
By (2.7), we arrive at
\[
\mathcal{L}_\alpha \phi = \frac{\partial^2 \phi}{\partial \lambda^2} + \frac{1 - \alpha}{\lambda} \frac{\partial \phi}{\partial \lambda} + \frac{\sum}{\left( \sum_{i=1}^{n} \frac{x_i^2}{r_0^2} \right)} \left[ \frac{x_i}{r_0^2} \frac{\partial^2 \phi}{\partial r_0^2} + \left( -\frac{x_i^3}{r_0^3} + \frac{1}{r_0} \right) \frac{\partial \phi}{\partial r_0} \right]
\]
\[
+ \left[ \frac{y_i^2}{r_0^2} \frac{\partial^2 \phi}{\partial r_0^2} + \left( -\frac{y_i^3}{r_0^3} + \frac{1}{r_0} \right) \frac{\partial \phi}{\partial r_0} \right]
\]
\[
+ 4\lambda \frac{x_i}{r_0^2} \frac{\partial^2 \phi}{\partial r_0 \partial \lambda} - 4x_i \frac{y_i}{r_0} \frac{\partial^2 \phi}{\partial r_0 \partial t} + 4(x_i^2 + y_i^2) \frac{\partial^2 \phi}{\partial t^2} + 4\lambda \frac{\partial^2 \phi}{\partial \lambda \partial t},
\]
\[
= \frac{\partial^2 \phi}{\partial \lambda^2} + \frac{1 - \alpha}{\lambda} \frac{\partial \phi}{\partial \lambda} + \frac{\partial^2 \phi}{\partial r_0^2} + \frac{2n - 1}{r_0} \frac{\partial \phi}{\partial r_0} + 4r_0^2 \frac{\partial^2 \phi}{\partial t^2} + 4\lambda^2 \frac{\partial^2 \phi}{\partial t^2}.
\]
\]

Lemma 2.2 ([7, 10]). Let $V$ be a bounded domain in $\mathbb{R}^n$, $Z$ a smooth vector field on $V$, and $a(z)$ a smooth nonnegative function. Assume that $u \in C^2(V) \cap C^1(\overline{V})$ is a solution to
\[
\begin{cases}
-\mathcal{L}_a u + Zu + a(z)u \geq 0, & z \in V, \\
u \geq 0, & z \in \partial V.
\end{cases} \tag{2.9}
\]
Then $u \geq 0$ in $V$. 

Lemma 2.3 ([33]). For a domain $V$ in $\mathbb{H}^n$, let $P_0 \in \partial V$ satisfy the interior Heisenberg ball condition (see [10], Def. 4.2). Assume that $U \in C^2(V) \cap C^1(\overline{V})$ is a solution to
\[ -\mathcal{L}_\alpha U \geq c_1(z)U, \quad (2.10) \]
for $c_1(z) \in L^\infty(V)$. If $U(z) > U(P_0) = 0, \ z \in V$, then
\[ \frac{\partial U}{\partial \nu}(P_0) < 0, \quad (2.11) \]
where $\nu$ is the outer unit normal to $\partial V$ at $P_0$. If $c_1(z) = 0$, then the above conclusion is also valid when we drop the assumption $U(P_0) = 0$.

Lemma 2.4. Let $V$ be a bounded domain in $\mathbb{H}^n$. Assume that $u \in C^2(V) \cap C^1(\overline{V})$ is a solution to
\[ -\mathcal{L}_\alpha U \geq 0. \quad (2.12) \]
Then the nonpositive minimum of $U$ in $\overline{V}$ can be obtained only on $\partial V$ unless $U$ is a constant.

Proof. Let $M$ be the nonpositive minimum of $U$ in $\overline{V}$, and set $E = \{ z \in V \mid U(z) = M \}$. Then $E$ is relatively closed in $V$. We will show $E = V$.

By contradiction, if $E$ is a proper subset of $V$, then one may find an open ball $B \subset V \setminus E$ with a point on the boundary belong $\partial E$. Suppose $z_0 \in \partial B \cap E$. Obviously, we have $-\mathcal{L}_\alpha U \geq 0$ in $B$ and $U(z) > U(z_0)$ for any $z \in B$ and $U(z_0) = M$.

Lemma 2.3 implies $\frac{\partial U}{\partial \nu}(z_0) < 0$. While $z_0$ is the interior minimum point of $V$, hence $DU(z_0) = 0$. This leads to a contradiction. \qed

3. Generalized CR inversion of cylindrical solutions. We recall the generalized CR inversion on $\mathbb{H}^n_+$, see [10]. For any $(x, y, t, \lambda) \in \mathbb{H}^n_+$, let
\[ \tilde{x}_i = \frac{x_i t + y_i r^2}{\rho^4}, \quad \tilde{y}_i = \frac{y_i t - x_i r^2}{\rho^4}, \quad \tilde{t} = -\frac{t}{\rho^4}, \quad \tilde{\lambda} = \frac{\lambda}{\rho^2}, \quad i = 1, \cdots, n \quad (3.1) \]
then the generalized CR inversion of function $U$ is given by
\[ v(x, y, t, \lambda) = \frac{1}{\rho^{Q-\alpha}} U(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{\lambda}). \quad (3.2) \]

The main result of this section is the following

Lemma 3.1. Let $U \in C^2(\mathbb{H}^n_+) \cap C(\mathbb{H}^n_+)$ be a cylindrical solution to (1.3), then the generalized CR inversion $v$ of $U$ satisfies
\[ \begin{cases} \mathcal{L}_\alpha v = 0, & z \in \mathbb{H}^n_+ \setminus \{0\}, \\ -\frac{\partial v}{\partial \nu^\alpha} = \rho^p(Q-\alpha)-(Q+\alpha)\rho^p, & z \in \mathbb{H}^n \times \{ \lambda = 0 \} \setminus \{0\}. \end{cases} \quad (3.3) \]

Proof. Note that $v$ is also a cylindrical function. We first compute
\[ \mathcal{L}_\alpha v = \frac{\partial^2 v}{\partial r_0^2} + \frac{2n-1}{r_0} \frac{\partial v}{\partial r_0} + \frac{\partial^2 v}{\partial \lambda^2} + \frac{1-\alpha}{\lambda} \frac{\partial v}{\partial \lambda} + 4r_0 \frac{\partial^2 v}{\partial t^2} + 4\lambda^2 \frac{\partial^2 v}{\partial t^2}. \quad (3.4) \]
Due to (3.1), it is easy to see
\[ \tilde{r}_0^2 = |\tilde{x}|^2 + |\tilde{y}|^2 = \frac{r_0^2}{\rho^4}, \quad (3.5) \]
\[ \tilde{\rho} = (\tilde{r}^4 + \tilde{r}_0^2)^{\frac{1}{2}} = \frac{1}{\rho}. \]
and then
\[ \frac{\partial \tilde{r}_0}{\partial r_0} = \frac{\partial}{\partial r_0}(r_0 \rho^{-2}) = \frac{t^2 + \lambda^4 - r_0^4}{\rho^6}, \]
\[ \frac{\partial \tilde{r}_0}{\partial t} = \frac{-r_0 t}{\rho^6}, \]
\[ \frac{\partial \tilde{r}_0}{\partial \lambda} = \frac{-2r_0 \lambda (r_0^2 + \lambda^2)}{\rho^6}. \]

And by (3.1),
\[ \frac{\partial \tilde{t}}{\partial r_0} \frac{\partial \lambda}{\partial r_0} (\lambda \rho^{-2}) = \frac{-2r_0 \lambda (r_0^2 + \lambda^2)}{\rho^6}, \]
\[ \frac{\partial \tilde{t}}{\partial t} = \frac{-\lambda t}{\rho^6}, \]
\[ \frac{\partial \tilde{t}}{\partial \lambda} = \frac{t^2 + r_0^4 - \lambda^4}{\rho^6}, \]
\[ \frac{\partial \tilde{t}}{\partial \xi} = \frac{\partial}{\partial \xi}(r_0 \rho^{-4}) = \frac{4r_0 t(r_0^2 + \lambda^2)}{\rho^8}, \]
\[ \frac{\partial \tilde{t}}{\partial \lambda} = \frac{t^2 - (r_0^2 + \lambda^2)^2}{\rho^6}, \]
\[ \frac{\partial \tilde{t}}{\partial \lambda} = \frac{4(r_0^2 + \lambda^2) t \lambda}{\rho^8}. \]

Therefore, we have
\[
\frac{\partial v}{\partial r_0} = \frac{\partial}{\partial r_0}(\rho^a Q U)
\]
\[
= (\alpha - Q) \rho^a Q^{-1} \frac{1}{4} \left( (r_0^2 + \lambda^2)^2 + t^2 \right)^{-\frac{3}{2}} \cdot 2(r_0^2 + \lambda^2) \cdot 2r_0 U \]
\[
+ \frac{1}{\rho^a} \frac{\partial U}{\partial \tilde{r}_0} \frac{\partial \tilde{r}_0}{\partial r_0} + \frac{\partial U}{\partial \tilde{t}} \frac{\partial \tilde{t}}{\partial r_0} + \frac{\partial U}{\partial \tilde{\lambda}} \frac{\partial \tilde{\lambda}}{\partial r_0}
\]
\[
= (\alpha - Q) r_0^2 (r_0^2 + \lambda^2) r_0 U \rho^{-\alpha+Q+4} + \frac{1}{\rho^{a-4}} \left( \frac{t^2 + \lambda^4 - r_0^4}{\rho^6} \right) \frac{\partial U}{\partial \tilde{r}_0} + \frac{-2r_0 \lambda (r_0^2 + \lambda^2)}{\rho^6} \frac{\partial U}{\partial \tilde{\lambda}} + \frac{4r_0 t(r_0^2 + \lambda^2)}{\rho^8} \frac{\partial U}{\partial \tilde{t}}
\]
\[
+ \frac{4r_0 t(r_0^2 + \lambda^2) \partial U}{\rho^8}, \tag{3.6}
\]

\[
\frac{\partial^2 v}{\partial r_0^2} = \frac{\partial}{\partial r_0} \left( \frac{(\alpha - Q)(r_0^2 + \lambda^2) r_0 U}{\rho^{-\alpha+Q+4}} \right)
\]
\[
+ \frac{\partial}{\partial r_0} \left( \frac{1}{\rho^{a-4}} \left( \frac{t^2 + \lambda^4 - r_0^4}{\rho^6} \frac{\partial U}{\partial \tilde{r}_0} + \frac{-2r_0 \lambda (r_0^2 + \lambda^2)}{\rho^6} \frac{\partial U}{\partial \tilde{\lambda}} + \frac{4r_0 t(r_0^2 + \lambda^2)}{\rho^8} \frac{\partial U}{\partial \tilde{t}} \right) \right)
\]
\[
+ \frac{1}{\rho^{a-4}} \frac{\partial}{\partial r_0} \left( \frac{t^2 + \lambda^4 - r_0^4}{\rho^6} \frac{\partial U}{\partial \tilde{r}_0} + \frac{-2r_0 \lambda (r_0^2 + \lambda^2)}{\rho^6} \frac{\partial U}{\partial \tilde{\lambda}} + \frac{4r_0 t(r_0^2 + \lambda^2)}{\rho^8} \frac{\partial U}{\partial \tilde{t}} \right)
\]
\[
= (\alpha - Q)(\alpha - Q - 4) \frac{r_0^2 (r_0^2 + \lambda^2)^2}{\rho^{-\alpha+Q+8}} + \frac{2(\alpha - Q) r_0^2}{\rho^{-\alpha+Q+4}} + \frac{2(\alpha - Q) r_0^2}{\rho^{-\alpha+Q+4}} \frac{t^2 + \lambda^4 - r_0^4}{\rho^{a+Q+10}}
\]
\[
+ \frac{2(\alpha - Q) (r_0^2 + \lambda^2) r_0 (t^2 + \lambda^4 - r_0^4)}{\rho^{-\alpha+Q+10}}
\]
\[
\frac{\partial v}{\partial \alpha} = \frac{\partial}{\partial \alpha} (\rho^{-Q} U) \\
= (\alpha - Q) \rho^{\alpha - Q - 1} \frac{1}{4} ((r_0^2 + \lambda^2)^2 + t^2)^{-\frac{3}{2}} \cdot 2(r_0^2 + \lambda^2) \cdot 2 \lambda U \\
+ \frac{1}{\rho^{Q-a}} \left( \frac{\partial U}{\partial \lambda} \frac{\partial r_0}{\partial \alpha} + \frac{\partial U}{\partial \lambda} \frac{\partial \lambda}{\partial \alpha} + \frac{\partial U}{\partial \lambda} \frac{\partial \lambda}{\partial \alpha} \right) \\
= (\alpha - Q) (r_0^2 + \lambda^2) \lambda \rho^{\alpha - Q - 4} U \\
+ \frac{1}{\rho^{Q-a}} \left( -2r_0 \lambda (r_0^2 + \lambda^2) \frac{\partial U}{\partial r_0} + \frac{t^2 + r_0^4 - \lambda^4}{\rho^6} \frac{\partial U}{\partial \lambda} + \frac{4t \lambda (r_0^2 + \lambda^2)}{\rho^8} \frac{\partial U}{\partial \lambda} \right), \\
(3.7)
\]

\[
\frac{\partial^2 v}{\partial \lambda^2} = \frac{\partial}{\partial \lambda} ((\alpha - Q) (r_0^2 + \lambda^2) \lambda \rho^{\alpha - Q - 4} U) \\
+ \frac{\partial}{\partial \lambda} \left( \frac{1}{\rho^{Q-a}} \left( -2r_0 \lambda (r_0^2 + \lambda^2) \frac{\partial U}{\partial r_0} + \frac{t^2 + r_0^4 - \lambda^4}{\rho^6} \frac{\partial U}{\partial \lambda} + \frac{4t \lambda (r_0^2 + \lambda^2)}{\rho^8} \frac{\partial U}{\partial \lambda} \right) \right) \\
+ \frac{1}{\rho^{Q-a}} \left( -2r_0 \lambda (r_0^2 + \lambda^2) \frac{\partial U}{\partial r_0} + \frac{t^2 + r_0^4 - \lambda^4}{\rho^6} \frac{\partial U}{\partial \lambda} + \frac{4t \lambda (r_0^2 + \lambda^2)}{\rho^8} \frac{\partial U}{\partial \lambda} \right) \\
= ((\alpha - Q) (\alpha - Q - 4) (r_0^2 + \lambda^2)^2 \lambda^2 + 2(\alpha - Q) \lambda^2 + (\alpha - Q) (r_0^2 + \lambda^2)) \frac{\partial U}{\partial \lambda} \\
+ (-4(\alpha - Q) \frac{r_0 (r_0^2 + \lambda^2)^2 \lambda^2}{\rho^{-a+Q+10}} \\
+ \frac{2(\alpha - Q) (r_0^2 + \lambda^2) \lambda (t^2 + r_0^4 - \lambda^4)}{\rho^{a+Q+10}} \\
+ \frac{-10r_0^4 \lambda^3 - 2r_0^2 \lambda^5 + 2\lambda^7 - 10\lambda^3 t^2 - 6r_0^2 \lambda^2 t^2 - 6r_0^6 \lambda)}{\rho^{a+Q+10}} \frac{\partial U}{\partial \lambda}, \\
(3.8)
\]
\[
\begin{align*}
\frac{\partial v}{\partial t} &= \frac{\partial}{\partial t}(\rho^{\alpha - Q} U) \\
&= (\alpha - Q)\rho^{\alpha - Q - 1} \cdot \frac{1}{4} ((r_0^2 + \lambda^2)^2 + t^2)^{-\frac{3}{2}} \cdot 2U \\
&\quad + \frac{1}{\rho Q - \alpha} \left( \frac{\partial U}{\partial \rho} \frac{\partial r_0}{\partial t} + \frac{\partial U}{\partial \lambda} \frac{\partial \lambda}{\partial t} + \frac{\partial U}{\partial \tilde{t}} \frac{\partial \tilde{t}}{\partial t} \right) \\
&= \frac{(\alpha - Q)U}{2\rho^{\alpha - Q + 4}} + \frac{1}{\rho Q - \alpha} \left( -r_0 t \frac{\partial U}{\partial \rho} \frac{\partial r_0}{\partial t} - \lambda t \frac{\partial U}{\partial \lambda} + t^2 - (r_0^2 + \lambda^2)^2 \frac{\partial U}{\partial \tilde{t}} \right), \\
\frac{\partial^2 v}{\partial t^2} &= \frac{\partial}{\partial t} \left( \frac{(\alpha - Q)t}{2\rho^{\alpha - Q + 4}} U + \frac{(\alpha - Q)t}{2\rho^{\alpha - Q + 4}} U + \frac{1}{\rho Q - \alpha} \left( -r_0 t \frac{\partial U}{\partial \rho} \frac{\partial r_0}{\partial t} - \lambda t \frac{\partial U}{\partial \lambda} + t^2 - (r_0^2 + \lambda^2)^2 \frac{\partial U}{\partial \tilde{t}} \right) \right) \\
&\quad + \frac{t^2 - (r_0^2 + \lambda^2)^2}{\rho^8} \frac{\partial U}{\partial \tilde{t}} \\
&\quad + \frac{1}{\rho Q - \alpha} \left( -r_0 t \frac{\partial U}{\partial \rho} \frac{\partial r_0}{\partial t} - \lambda t \frac{\partial U}{\partial \lambda} + t^2 - (r_0^2 + \lambda^2)^2 \frac{\partial U}{\partial \tilde{t}} \right) \\
&= \frac{(\alpha - Q)(\alpha - Q - 4) t^2}{4\rho^{\alpha - Q + 8}} U + \frac{(\alpha - Q)(\alpha - Q - 4) t^2}{2\rho^{\alpha - Q + 4}} U \\
&\quad + \frac{(-r_0)(r_0^2 + \lambda^2)^2}{\rho^{\alpha - Q + 10}} \frac{\partial U}{\partial r_0} \\
&\quad + \frac{(-\lambda)(r_0^2 + \lambda^2)^2}{\rho^{\alpha - Q + 10}} \frac{\partial U}{\partial \lambda} \\
&\quad + \frac{2((r_0^2 + \lambda^2)^2 + t^2) t - 4t^3 + 4(r_0^2 + \lambda^2)^2 t}{\rho^{\alpha - Q + 12}} \frac{\partial U}{\partial \tilde{t}} \\
&\quad + \frac{r_0^2 t^2}{\rho^{\alpha - Q + 12}} \frac{\partial^2 U}{\partial r_0^2} + \frac{2r_0 \lambda t^2}{\rho^{\alpha - Q + 12}} \frac{\partial^2 U}{\partial r_0 \partial \lambda} + \frac{-2r_0 t^3 + 2r_0(r_0^2 + \lambda^2)^2}{\rho^{\alpha - Q + 14}} \frac{\partial^2 U}{\partial r_0^2} \\
&\quad + \frac{\lambda^2 t^2}{\rho^{\alpha - Q + 12}} \frac{\partial^2 U}{\partial \lambda^2} - \frac{2\lambda t^3 + 2(r_0^2 + \lambda^2)^2 \lambda t}{\rho^{\alpha - Q + 14}} \frac{\partial^2 U}{\partial \lambda \partial \tilde{t}} + \frac{(t^2 - (r_0^2 + \lambda^2)^2)^2 \frac{\partial^2 U}{\partial \tilde{t}^2}}{\rho^{\alpha - Q + 16}},
\end{align*}
\]
Using (3.4) and then combining (3.6), (3.7), (3.8), (3.9) and (3.10), we get
\[ L_\alpha v = a_1 \frac{\partial^2 U}{\partial \tau_0^2} + a_2 \frac{\partial^2 U}{\partial \lambda^2} + a_3 \frac{\partial^2 U}{\partial \tau_0 \partial \lambda} + a_4 \frac{\partial^2 U}{\partial \tau_0^2} + a_5 \frac{\partial^2 U}{\partial \tau_0 \partial \lambda} + a_6 \frac{\partial^2 U}{\partial \lambda^2} + b_1 \frac{\partial U}{\partial \tau_0} + b_2 \frac{\partial U}{\partial \lambda} + b_3 \frac{\partial U}{\partial t} + cU, \]
(3.11)
where \( a_1 = \frac{1}{p^{-\alpha+Q+4}} \), \( a_2 = \frac{1}{p^{-\alpha+Q+4}} \), \( a_3 = \frac{4(\beta^2 + \lambda^2)}{p^{2+Q+4}} \), \( a_4 = a_5 = a_6 = 0 \), \( b_1 = \frac{2n-1}{p^{-\alpha+Q+4}} \), \( b_2 = \frac{1}{p^{-\alpha+Q+4}} \), \( b_3 = \frac{1}{p^{-\alpha+Q+4}} \), and \( c = 0 \).

Hence,
\[ L_\alpha v = \frac{1}{p^{-\alpha+Q+4}} \frac{\partial^2 U}{\partial \tau_0^2} + \frac{1}{p^{-\alpha+Q+4}} \frac{\partial^2 U}{\partial \lambda^2} + \frac{4(\beta^2 + \lambda^2)}{p^{2+Q+4}} \frac{\partial^2 U}{\partial \tau_0^2} + \frac{2n-1}{p^{-\alpha+Q+4}} \frac{1}{\partial \tau_0} \frac{\partial U}{\partial \lambda} \]
\[ = \frac{1}{p^{-\alpha+Q+4}} L_\alpha U(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{\lambda}). \]

Noting \( L_\alpha U(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{\lambda}) = 0 \), it follows
\[ L_\alpha v = 0. \]

Next, in view of the generalized CR inversion, the boundary condition in (1.3) becomes
\[ \frac{\partial v}{\partial \nu^\alpha} = -c_\alpha \lim_{\lambda \to 0} \lambda^{1-\alpha} \frac{\partial v}{\partial \lambda}(x, y, t, \lambda) = -c_\alpha \lim_{\lambda \to 0} (\lambda \rho^2)^{1-\alpha} \frac{1}{\rho^{Q-\alpha}} \frac{\partial U}{\partial \lambda}(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{\lambda}) \]
\[ = -\frac{1}{\rho^{Q-\alpha}} U(\tilde{x}, \tilde{y}, \tilde{t}, 0) \]
\[ = -\rho^{(Q-\alpha)-(Q+\alpha)} \nu^p(x, y, t, 0). \]

\[ \square \]

4. Proof of the main results. In this section, we prove Theorem 1.2 at first. To do so, let us consider the property of solutions to (1.3) in the supercritical case. The subcritical and critical case can be lifted to a suitable higher dimension space which allows us to use the results for the supercritical case. This will enable us to prove Theorem 1.1.

Proof of Theorem 1.2. We will treat two case: (1) the supercritical case \( p > \frac{Q+\alpha}{Q-\alpha} \) and (2) the subcritical and critical case \( 1 < p \leq \frac{Q+\alpha}{Q-\alpha} \), respectively, and give the expressions of the cylindrical solutions to (1.3).

Case 1: \( p > \frac{Q+\alpha}{Q-\alpha} \).

Since the cylindrical solution \( U(z) \) to (1.3) does not own the decay at infinity, we apply Lemma 3.1 to see
\[ \begin{cases} L_\alpha v = 0, & z \in \mathbb{H}^n_+ \setminus \{0\}, \\ \frac{\partial v}{\partial \nu^\alpha} = -|z|^{\beta} \nu^p, & z \in \mathbb{H}^n \times \{\lambda = 0\} \setminus \{0\}, \end{cases} \]
(4.1)
where \( \beta = p(Q-\alpha)-(Q+\alpha) \). The first purpose is to get some symmetry properties of \( v \) by using the method of moving planes; then we show that \( v \) is independent of \( t \) and \( U(z) \) is also independent of \( t \). The expression of \( U(z) \) can be obtained from the result in [15].
When one uses the generalized CR inversion with the centre being the origin, it will appear the singularity at the origin. To handle the singularity, we give a lemma:

**Lemma 4.1.** Assume that $v(z)$ satisfies (4.1). Then for

$$0 < \varepsilon < \min \left\{ 1, \frac{2\alpha}{\alpha + 1} \min_{\partial \mathbb{B}^+_1 \cap \partial \mathbb{B}_1} v \right\},$$

we have $v(z) \geq \frac{\varepsilon}{2}$, $z \in \mathbb{B}^+_1 \setminus \{0\}$.

**Proof.** For $0 < \rho < 1$, introduce an auxiliary function

$$\varphi_1(z) = \frac{\varepsilon}{2} - \frac{(\alpha + 1)\varepsilon}{2\alpha} + \varepsilon \frac{\lambda^\alpha}{2}, \quad z \in \mathbb{B}^+_1 \setminus \mathbb{B}^+_\rho,$$

where $0 < \lambda \leq \rho < 1$, and denote

$$F_1(z) := v(z) - \varphi_1(z).$$

A direct calculation shows that $F_1$ satisfies

$$\begin{cases}
    \mathcal{L}_\alpha F_1 = 0, & z \in \mathbb{B}^+_1 \setminus \mathbb{B}^+_\rho, \\
    \frac{\partial F_1}{\partial \nu^\alpha} = -\varepsilon |z|^{\beta} v^p + \frac{\varepsilon}{2}, & z \in \partial (\mathbb{B}^+_1 \setminus \mathbb{B}^+_\rho) \cap \partial \mathbb{H}^n_+.
\end{cases}$$

We will check by using the contradiction that

$$F_1(z) \geq 0, \quad z \in \mathbb{B}^+_1 \setminus \mathbb{B}^+_\rho.$$

In fact, it follows

$$F_1(z) = v - \left( \frac{\varepsilon}{2} - \frac{(\alpha + 1)\varepsilon}{2\alpha} + \varepsilon \frac{\lambda^\alpha}{2} \right) > 0, \quad \text{on } \partial \mathbb{B}^+_\rho \cap \partial \mathbb{B}_1,$$

and by noting $0 < \varepsilon < \min \left\{ 1, \frac{2\alpha}{\alpha + 1} \min_{\partial \mathbb{B}^+_1 \cap \partial \mathbb{B}_1} v \right\}$ that

$$F_1(z) = v - \left( \frac{\varepsilon}{2} - \frac{(\alpha + 1)\varepsilon}{2\alpha} \right) > 0, \quad \text{on } \partial \mathbb{B}^+_\rho \cap \partial \mathbb{B}_1.$$

Suppose that (4.3) fails, we know by Lemmas 2.2, 2.3 and 2.4 that there exists some $\bar{z} = (\bar{x}, \bar{y}, \bar{t}, 0) \in \partial \mathbb{H}^n_+$ with $\rho < |\bar{z}|^{\beta} < 1$ such that

$$F_1(\bar{z}) = \min_{\mathbb{B}^+_1 \setminus \mathbb{B}^+_\rho} F_1(z) < 0.$$

The boundary condition in (4.2) implies

$$\frac{\partial F_1}{\partial \nu^\alpha} = -\lim_{\lambda \to 0} \lambda^{1-\alpha} \frac{\partial F_1}{\partial \lambda} = -|\bar{z}|^{\beta} v^p(\bar{x}, \bar{y}, \bar{t}, 0) + \frac{\varepsilon}{2} \leq 0,$$

and then for $\beta > 0,$

$$v^p(\bar{x}, \bar{y}, \bar{t}, 0) \geq \frac{\varepsilon}{2}.$$

Therefore,

$$F_1(\bar{z}) = v(\bar{z}) - \left( \frac{\varepsilon}{2} - \frac{(\alpha + 1)\varepsilon}{2\alpha} \right) v(\bar{z}) - \frac{\varepsilon}{2} \geq 0,$$

which contradicts $F_1(\bar{z}) < 0.$ Hence (4.3) is derived.
It yields that for \( z \in \mathcal{B}_+^1 \setminus \{0\} \), if \( 0 < \rho < |z|_{\mathbb{H}^n} \), then \( F_1(z) \geq 0 \) from (4.3). Letting \( \rho \to 0 \), we have \( v(z) \geq \frac{\zeta}{2} \).

For \( \mu < 0 \), denote \( T_\mu = \{ z \in \mathcal{H}_+^n | t = \mu \} \) and \( \Sigma_\mu = \{ z \in \mathcal{H}_+^n | t < \mu \} \). The H-reflection of \( z \in \Sigma_\mu \) is (see [4])

\[
\hat{z} = (y, x, 2\mu - t, \lambda).
\]

(4.4)

Denote \( \hat{\Sigma}_\mu = \Sigma_\mu \setminus \{0^\mu\} \), where \( 0^\mu \) is the H-reflection of origin about \( T_\mu \). Let \( v_\mu(z) = v_\mu([((x, y), t, \lambda) := v([((x, y), 2\mu - t, \lambda) = v(y, x, 2\mu - t, \lambda) = v(z^\mu), \]

and

\[
w_\mu(z) := v_\mu(z) - v(z) = v(z^\mu) - v(z),
\]

then \( w_\mu(z) \) satisfies

\[
\begin{aligned}
&\mathcal{L}_\alpha w_\mu = 0, \quad z \in \mathcal{H}_+^n \setminus \{0\}, \\
&\frac{\partial w_\mu}{\partial \nu^\alpha} \geq -p |z^\mu|_{\mathbb{H}^n}^{\beta} \zeta^{p-1} w_\mu(x, y, t, 0), \quad z \in \partial \mathcal{H}_+^n \setminus \{0\},
\end{aligned}
\]

(4.5)

where \( \zeta \) is a positive function between \( v_\mu(z) \) and \( v(z) \). Now we are ready to apply the method of moving planes for solutions to (4.5) to show that \( v \) is symmetric with respect to \( t = 0 \).

**Proposition 4.2.** If \( \mu \) is sufficiently negative, then \( w_\mu(z) \geq 0 \) for all \( z \in \hat{\Sigma}_\mu \).

**Proof.** Suppose that it fails, then \( w_\mu \) is negative at some point in \( \hat{\Sigma}_\mu \). Since \( v(z) \to 0 \) as \( |z|_{\mathbb{H}^n} \to \infty \), and for any fixed \( \mu \), \( |z^\mu|_{\mathbb{H}^n} \to \infty \) as \( |z|_{\mathbb{H}^n} \to \infty \), we know that \( w_\mu(z) \to 0 \) as \( |z|_{\mathbb{H}^n} \to \infty \). By Lemma 4.1, \( w_\mu(z) > 0 \) for \( z \) near the origin if \( \mu \) is sufficiently negative. Thus there exists some point \( \hat{z} \) such that \( w_\mu(\hat{z}) = \min_{z \in \hat{\Sigma}_\mu} w_\mu(z) < 0 \).

From Lemmas 2.2, 2.3 and 2.4, we have \( \hat{z} \in \partial \mathcal{H}_+^n \cap \hat{\Sigma}_\mu \); furthermore, \( \frac{\partial w_\mu}{\partial \nu^\alpha}(\hat{z}) < 0 \). But this contradicts the boundary condition in (4.5).

**Proposition 4.2** makes sure that the plane \( T_\mu \) can be moved to the right from the negative infinity. Define

\[
\mu_0 = \sup \{ \mu < 0 | w_\sigma(z) \geq 0 \ \forall z \in \hat{\Sigma}_\sigma, \sigma < \mu \}.
\]

**Proposition 4.3.** \( \mu_0 = 0 \).

**Proof.** We prove the conclusion by the contradiction. Suppose \( \mu_0 < 0 \), we claim

\[
w_{\mu_0} \equiv 0.
\]

(4.6)

which will contradict to

\[
\frac{\partial w_{\mu_0}}{\partial \nu^\alpha} = -|z^{\mu_0}|_{\mathbb{H}_0^+}^{\beta} v_{\mu_0}^p(z) + |z|_{\mathbb{H}_0^+}^{\beta} v^p(z).
\]

To check (4.6), we assume \( w_{\mu_0} \neq 0 \), then it follows by Lemmas 2.2 and 2.3 that \( w_{\mu_0} > 0 \) in \( \hat{\Sigma}_{\mu_0} \setminus T_{\mu_0} \). Now we first give the following result.

**Lemma 4.4.** For \( \rho_0 < \min \{ \frac{1}{2}, |\mu_0|, 1 \} \), there exists a positive constant \( C \), such that \( w_{\mu_0}(z) > C \) in \( \mathcal{B}_{\rho_0}^+ (0^\mu_0) \setminus \{0^\mu_0\} \).
Proof. Since $w_{\mu_0}(z) > 0$ in $\Sigma_{\mu_0} \cap B^{+}_{\rho_0}(0^{\mu_0})$, it implies $\min_{\partial B^{+}_{\rho_0}(0^{\mu_0})} w_{\mu_0}(z) \geq \epsilon$ for some $0 < \epsilon < 1$. Due to the continuity of $v$ in $\mathbb{H}^n \setminus \{0\}$, there exists a positive constant $C_1$, such that

$$v(z) \leq C_1, \quad z \in \bar{B}^{+}_{\rho_0}(0^{\mu_0}) \setminus \{0^{\mu_0}\}. \quad (4.7)$$

Denote

$$\varphi_2(z) = \frac{\epsilon \gamma}{2} - \frac{\rho^{Q-\alpha} \epsilon}{|z|_\infty Q-\alpha} + \frac{\epsilon (1 - \gamma) \lambda^\alpha}{2}, \quad z \in B^{+}_{\rho_0}(0^{\mu_0}) \setminus B^{+}_{\rho}(0^{\mu_0}),$$

where $\gamma < 1$ is sufficiently small, and set

$$F_2(z) := w_{\mu_0}(z) - \varphi_2(z).$$

A direct calculation shows that $F_2(z)$ satisfies

$$\begin{cases} L_\alpha F_2 = 0, & z \in B^{+}_{\rho_0}(0^{\mu_0}) \setminus B^{+}_{\rho}(0^{\mu_0}), \\
\frac{\partial F_2}{\partial \nu^\alpha} = -|z^{\mu_0}|_{\infty}^\beta \nu^{\mu_0} + |z|_\infty^\beta \nu^p + \frac{\epsilon (1 - \gamma)}{2}, & z \in \partial (B^{+}_{\rho_0}(0^{\mu_0}) \setminus B^{+}_{\rho}(0^{\mu_0})) \cap \partial \mathbb{H}^{n\beta}. \end{cases} \quad (4.8)$$

We will claim

$$F_2(z) \geq 0, \quad z \in B^{+}_{\rho_0}(0^{\mu_0}) \setminus B^{+}_{\rho}(0^{\mu_0}). \quad (4.9)$$

To prove (4.9), notice that

$$F_2(z) \geq \epsilon - \frac{\epsilon \gamma}{2} - \frac{\rho^{Q-\alpha} \epsilon}{\rho_0 Q-\alpha} + \frac{\epsilon (1 - \gamma)}{2} > 0, \quad \text{on } \partial B^{+}_{\rho_0}(0^{\mu_0}) \setminus \partial B^{+}_{\rho}(0^{\mu_0}),$$

$$F_2(z) > w_{\mu_0}(z) \geq 0, \quad \text{on } \partial B^{+}_{\rho_0}(0^{\mu_0}) \setminus \partial B^{+}_{\rho}(0^{\mu_0}).$$

If there exists some negative minimum point $\bar{z}$ for $F_2(z)$, then we have by Lemmas 2.2, 2.3 and 2.4 that

$$\bar{z} \in \partial (B^{+}_{\rho_0}(0^{\mu_0}) \setminus B^{+}_{\rho}(0^{\mu_0})) \cap \partial \mathbb{H}^{n\beta} \quad \text{and } \frac{\partial F_2}{\partial \nu^\alpha}(\bar{z}) \leq 0.$$

For $F_2(\bar{z}) < 0$, it gets $v_{\mu_0}(\bar{z}) - v(\bar{z}) - \varphi_2(\bar{z}) < 0$, and then

$$v_{\mu_0}(\bar{z}) \leq C_2, \quad (4.10)$$

for some constant $C_2$ depends only on $C_1$. Again by $F_2(\bar{z}) < 0$ and definitions of $\varphi_2$ and $F_2$,

$$w_{\mu_0}(\bar{z}) < \varphi_2(\bar{z}) = \frac{\epsilon \gamma}{2} - \frac{\rho^{Q-\alpha} \epsilon}{\rho_0 Q-\alpha} < \frac{\epsilon \gamma}{2}. \quad (4.11)$$

Together with (4.7) and (4.10), we know

$$|z^{\mu_0}|_{\infty}^\beta \nu^{\mu_0}(\bar{x}, \bar{y}, \bar{t}, 0) - |z|_\infty^\beta \nu^p(\bar{x}, \bar{y}, \bar{t}, 0) \leq |z^{\mu_0}|_{\infty}^\beta (v_{\mu_0} p(\bar{z}) - v^p(\bar{z})) \leq C_3 w_{\mu_0}(\bar{z}),$$

where $C_3$ depends only on $C_1$, $C_2$ and $\mu_0$. By $\frac{\partial F_2}{\partial \nu^\alpha}(\bar{z}) \leq 0$ and the boundary condition in (4.8), it yields

$$w_{\mu_0}(\bar{z}) \geq \frac{\epsilon (1 - \gamma)}{2C_3}. \quad (4.12)$$

Combining (4.11) and (4.12), it follows

$$\frac{\epsilon \gamma}{2} > \frac{\epsilon (1 - \gamma)}{2C_3},$$

i.e. $\gamma > \frac{1}{1 + C_3}$. When we choose $\gamma$ small enough such that $\gamma < \frac{1}{1 + C_3}$, we will reach a contradiction, which proves (4.9).
Let $\rho \to 0$ in (4.9), then $w_{\mu_0}(z) > C = \frac{\gamma^2}{2}$ for $\gamma < \frac{1}{1+\epsilon_\delta}$. \hfill \Box

We continue the proof of Proposition 4.3. By the definition of $\mu_0$, there exists sequences of $\mu_k$ and $z_k \in \bar{\Sigma}_k \setminus \bar{B}_{\rho_0}^+(0^\mu_k)$, such that $\mu_k \to \mu_0$ and $w_{\mu_k}(z_k) = \inf_{\Sigma_{\mu_k}} w_{\mu_k}(z) < 0$. By Lemma 4.4 and the continuity of $w_{\mu_k}$, we have for $k$ large enough,

$$w_{\mu_k}(z) > \frac{C}{2}, \quad z \in \bar{B}_{\rho_0}^+(0^\mu_k) \setminus \{0^\mu_k\}.$$

Note that $w_{\mu_k}(z) \to 0$ as $|z|_{\mathbb{H}_n} \to \infty$. By Lemma 2.2, $z_k \in (\bar{\Sigma}_k \setminus \bar{B}_{\rho_0}^+(0^\mu_k)) \cap \partial \mathbb{H}_n^+$, such that $w_{\mu_k}(z_k) = \min_{\Sigma_{\mu_k}} w_{\mu_k}(z) < 0$. Also, Lemmas 2.2, 2.3 and 2.4 show $z_k \in \partial \mathbb{H}_n^+ \cap \bar{\Sigma}_k$, hence $\partial w_{\mu_k}(z_k) < 0$ from Lemma 2.3, which gets a contradiction with the boundary condition in (4.5). Therefore, $\mu_0 = 0$ and the proof of Proposition 4.3 is ended. \hfill \Box

Similarly to the proofs of Propositions 4.2 and 4.3, we can also move the plane from the right to the left. Then it has claimed that $v((x, y), t, \lambda)$ is symmetric with respect to $t = 0$. The above process is based on that the origin of $\mathbb{H}_n^+$ is the center of the generalized CR inversion. By letting any point on $\mathbb{H}_n^+$ be the center of the generalized CR inversion and repeating the previous proof, we imply that $v((x, y), t, \lambda)$ is symmetric with respect to any point on the $t$ axis and then is independent of $t$, hence $U(z)$ is also independent of $t$. Now we can go back to the case in [15] and have by using the result in [15] that

$$U = \frac{a}{\alpha} \lambda^\alpha + b,$$

where $b > 0$, $a = b^p$.

Case 2. $1 < p \leq \frac{Q+\alpha}{Q-\alpha}$.

To handle this case, we will lift the dimension of the space $\mathbb{H}_n^+$ and reduce Case 2 (including critical and subcritical cases) to Case 1 (the supercritical case). Concretely, we choose a positive integer $m$ so large that $p > \frac{Q+2m+\alpha}{Q-2m-\alpha}$ and set

$$w(x_1, \cdots, x_n, x_{n+1}, \cdots, x_{n+m}, y_1, \cdots, y_n, y_{n+1}, \cdots, y_{n+m}, t, \lambda) = U(x_1, \cdots, x_n, y_1, \cdots, y_n, t, \lambda).$$

Then $w$ satisfies

$$\begin{align*}
\mathcal{L}_\alpha w &= 0, \quad z \in \mathbb{H}_n^{n+m} \setminus \{0\}, \\
\frac{\partial w}{\partial \nu^\alpha} &= -w^p, \quad z \in \mathbb{H}_n^{n+m} \times \{\lambda = 0\} \setminus \{0\}.
\end{align*}$$

(4.13)

By the choice of $m$, it sees that $p$ is a supercritical exponent. Applying the result to Case 1, we know that $w$ is independent of $(x_1, \cdots, x_{n+m}, y_1, \cdots, y_{n+m}, t)$ and finish the proof of Theorem 1.2. \hfill \Box

Proof of Theorem 1.1. We replaced $\lambda$ in (1.1) by $\sqrt{2}\lambda$, and got the expression (1.2). Now we replace $\sqrt{2}\lambda$ by $\lambda$, and see from Theorem 1.2 that $U = \frac{a}{(\sqrt{2})^{\alpha} \lambda^\alpha} + b$, where $b \geq 0$, $a = b^p$. \hfill \Box
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