Error and unsharpness in approximate joint measurements of position and momentum

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Abstract. In recent years, novel quantifications of measurement error in quantum mechanics have for the first time enabled precise formulations of Heisenberg’s famous but often challenged measurement uncertainty relation. These relations take the form of a trade-off for the necessary errors in joint approximate measurements of position and momentum and other incompatible pairs of observables. Here we review some of these error measures, examine their properties and suitability, and compare their relative strengths as criteria for “good” approximations.

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1. Introduction

In recent years, Heisenberg’s uncertainty principle has received renewed attention and scrutiny. The principle is often loosely associated with three sets of ideas – preparation uncertainty, joint measurement error trade-offs, and error-disturbance trade-offs. While the first of these is uncontroversial, the latter two are subjects of an ongoing controversy.

For many decades, the only formally and operationally well-defined form of uncertainty relation known in the physics literature was the familiar preparation uncertainty relation for standard deviations of, say, position and momentum,

\[ \Delta(Q, \rho) \Delta(P, \rho) \geq \frac{1}{2} \hbar. \]  

This relation is a statement about the widths of the probability distributions \( \rho_Q, \rho_P \) of the position \( Q \) and momentum \( P \) in a state \( \rho \), and it can be tested by measuring position and momentum in separate runs of experiments on particles prepared in the same state \( \rho \).

Notwithstanding this clear-cut interpretation, the relation (1) is often paraphrased as constituting a limitation of the accuracies of any attempted joint measurements of position and momentum. This unjustified conflation has equally often been criticised, but then it happened not seldom that the critics (or their readers) jumped to the conclusion that the uncertainty principle has nothing to do with the possibility or impossibility of joint measurements of position and momentum.

The joint measurement uncertainty question was brought into focus with these conflicting views but until recently a rigorous investigation of the problem has remained outstanding. The first seemingly plausible attempt offered at quantifying measurement errors in quantum mechanics is based on the concept of noise operator that was introduced into quantum optical amplifier theory in the 1960s and soon after applied in the measurement context. For a brief history of the development of the notion of noise-(operator) based error, we refer the reader to [1]. On the basis of this state-dependent error measure it appeared that joint measurement error relations are much weaker than suggested by the Heisenberg form (1); this has led to claims of a violation or circumvention of Heisenberg-type measurement uncertainty relations, both theoretically (e.g., [2, 3]) and experimentally (e.g., [4, 5]). As shown in [1], however, the noise-based error measures do not purely quantify errors but also contain contributions of preparation uncertainty; moreover, they are of limited operational significance as state-specific error measures.

In the meantime, different measures of measurement error were introduced that quantify the performance of measuring devices and as such are state-independent. For these measures, joint measurement trade-off relations have been formulated and proven. This development, which is reviewed in [6], was enabled by making full use of the operational possibilities of quantum mechanics, notably by the generalised representation of observables as positive operator valued measures (POVMs).

A key concept for this solution to the joint-measurement problem is that of an approximate measurement of a given observable (represented by a POVM) \( \mathcal{E} \).
which is any measurement whose associated POVM $F$ is close to $E$ in a suitable operationally relevant sense. This has made it possible to overcome the obstacle of the noncommutativity of $Q$ and $P$, which precludes any sharp joint measurement of these observables: there are (generally noncommuting) unsharp observables $M_1$, $M_2$ that are jointly measurable and still constitute reasonable approximations of $Q$ and $P$, respectively.

Two observables $M_1, M_2$ on $\mathcal{B}(\mathbb{R})$ are said to be jointly measurable if there is a third, joint observable $M$ on $\mathcal{B}(\mathbb{R}^2)$ such that $M_1, M_2$ are the Cartesian marginals of $M$, $M_1(X) = M(X \times \mathbb{R})$, $M_2(Y) = M(\mathbb{R} \times Y)$.

Three proposed measures of error for approximate measurements of position and momentum and their associated uncertainty relations were briefly reviewed in [6]: these were referred to as standard error, (Monge) metric error, and error bar width. The first of these is what we called noise-based error (measure) above (due to the limitations of this concept it seems inappropriate to refer to it as “standard”). In the meantime, measurement uncertainty relations have been proven for a wider class of metric error measures [7, 8]: these are based on the so-called Wasserstein distance of order $\alpha$ between probability measures on a metric space; here $\alpha$ is a parameter whose values range from 1 to $\infty$. The Monge metric corresponds to the value $\alpha = 1$, while $\alpha = 2$ is found to provide a natural operational quantum generalisation of the notion of root-mean-square (rms) error.

It is the purpose of the present paper to analyze further the concept of error bar width introduced in [9] and to study its connections with the metric error measures. Some aspects of the noise operator based error will also be considered to the extent that they are useful as estimates of the other measures. In addition to measurement errors, we also review other quantities describing the intrinsic unsharpness of the approximators of $Q, P$.

The proofs of measurement uncertainty relations given in [10] make it evident that the measurement error relations follow mathematically from related preparation uncertainty relations. Versions of these latter relations will be the starting points for the presentations of error and unsharpness measures to be given below.

We begin with a brief introduction of the requisite mathematical tools.

2. Preliminaries

Our considerations will be based on the usual description of a physical system in a separable complex Hilbert space $\mathcal{H}$, with states being represented as positive operators $\rho$ of trace 1 (also called density operators). The convex set of all states will denoted $S$. Pure states correspond to unit vectors $\varphi \in \mathcal{H}$ or rather the associated rank-1 projections $|\varphi\rangle \langle \varphi| \equiv P_\varphi$. Observables are represented as positive operator measures (POVMs) $E$ on a measurable space $(\Omega, \Sigma)$ that are normalised, i.e., $E(\Omega) = I$. In this paper $(\Omega, \Sigma)$ will be one of the Borel spaces $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ or $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$. An observable $E$ is called sharp if it is projection valued; otherwise $E$ is an unsharp observable. We write $\rho^E$
for the probability measure induced by a state \( \rho \) and an observable \( E \) via the formula 
\( \rho^E(X) := \text{tr}[\rho E(X)], \quad X \in \Sigma \). We use the notation \( E[x^k], \ k \in \mathbb{N} \), for the \( k \)-th moment operators \( \int x^k E(dx) \) of an observable \( E \) on \( \mathcal{B} (\mathbb{R}) \). These operators are defined on their natural domains \( \prod D(\mathcal{E}[x^k]) \) of all \( \varphi \in \mathcal{H} \) for which the function \( x \mapsto x^k \) is integrable with respect to the complex measures \( \langle \psi | E(dx) | \varphi \rangle \) for all \( \psi \in \mathcal{H} \); this contains the square-integrability domain \( \{ \varphi \in \mathcal{H} : \int x^{2k} \langle \varphi | E(dx) | \varphi \rangle < \infty \} \). The moments of a probability measure \( \rho \) on \( \mathbb{R} \) will be denoted \( \rho[x^k], \ k \in \mathbb{N} \).

All uncertainty relations to be studied here will be formulated for the case of a quantum particle in one spatial dimension, with Hilbert space \( \mathcal{H} = L^2(\mathbb{R}) \) and canonical position and momentum operators \( Q, P \), defined via \((Q\psi)(x) = x\psi(x), \( (P\psi)(x) = -i\hbar(d\psi/dx)(x) \) on the usual maximal domains ensuring selfadjointness. Generalizations to more degrees of freedom are straightforward. The spectral measures of \( Q \) and \( P \) will be denoted \( \mathcal{Q} \) and \( \mathcal{P} \), respectively.

An important class of POVMs representing approximations of position and momentum are given by smeared position and momentum observables \( Q^\mu, P^\nu \), defined as convolutions of \( Q, P \) with probability measures \( \mu, \nu \) on \( \mathcal{B}(\mathbb{R}) \):

\[
Q^\mu(X) = Q * \mu(X) = \int_{\mathbb{R}} \mu(X-q)Q(dq),
\]

\[
P^\nu(Y) = P * \nu(Y) = \int_{\mathbb{R}} \nu(Y-p)P(dp).
\]

The integrals are defined in the weak operator topology.

We will make use of the important class of covariant phase space observables which is defined as follows. Let \( W(q, p) = \exp(iqp/\hbar) \exp(-iqP/\hbar) \exp(ipQ/\hbar) \) be the Weyl operators comprising an irreducible unitary projective representation of the translations on phase space \( \mathbb{R}^2 \). An observable \( G \) on \( \mathbb{R}^2 \) is called a covariant phase space observable if it satisfies the covariance condition

\[
W(q, p)G(Z)W(q, p)^* = G(Z - (q, p)), \quad Z \in \mathcal{B}(\mathbb{R}^2).
\]

This is satisfied by the following family of observables \( G = G^\tau \) on \( \mathbb{R}^2 \), which are thus covariant phase space observables:

\[
\mathcal{B}(\mathbb{R}^2) \ni Z \mapsto G^\tau(Z) = \frac{1}{2\pi\hbar} \int_{\mathbb{R}^2} W(q, p)\tau W(q, p)^* dq dp; \quad (2)
\]

the integral is defined in the weak operator topology and the operator density is generated by an arbitrary fixed positive operator \( \tau \) of trace 1 (for details of the proof of these properties, see, e.g., [12]). Moreover, every covariant phase space observable is of the form (2) for some positive operator \( \tau \) of trace 1. This fundamental fact is implied by results of [13] and [14] and has been made explicit in [15] using the theory of induced representations and in [16] using the theory of integration with respect to operator measures.

The marginal observables of \( G^\tau \) are smeared position and momentum observables \( Q^{\mu_\tau}, P^{\nu_\tau} \), where \( \mu_\tau := \tau^Q_\Pi \) and \( \nu_\tau := \tau^P_\Pi \) are the probability distributions of \( Q \) and \( P \) in...
the state described by $\tau_\Pi$, that is,

$$G^\tau_1 = Q^* \mu_\tau, \quad G^\tau_2 = P^* \nu_\tau.$$ 

Here $\tau_\Pi = \Pi \tau \Pi^*$ is the operator obtained from $\tau$ under the action of the parity transformation $\Pi$ ($\Pi \varphi(x) = \varphi(-x)$).

There is a simple but fundamental characterization of all pairs of smeared position and momentum observables admitting a joint measurement.

**Theorem 2.1.** A pair of smeared position and momentum observables $Q^\mu, P^\nu$ are jointly measurable exactly when there exists a covariant phase space observable $G^\tau$ of which they are marginals. In that case, $\mu = \mu_\tau$, $\nu = \nu_\tau$.

This result has been obtained in a long series of investigations by various authors, culminating and summarised in [17].

3. Uncertainty: $\alpha$-deviation and overall width

We will make use of the following measures of the widths of a probability distribution $p : B(\mathbb{R}) \to [0, 1]$ on $\mathbb{R}$. The standard deviation $\Delta(p)$ is given by

$$\Delta(p) := \left( \int \left( x - \int xp(dx) \right)^2 p(dx) \right)^{1/2} = \left( p[x^2] - p[x]^2 \right)^{1/2}.$$

The standard deviation of an observable $E$ on $B(\mathbb{R})$ in a state $\rho$ is $\Delta(E, \rho) := \Delta(E^\rho)$.

For vector states $\varphi$ we write $\Delta(E, \varphi) := \Delta(p^E)$.

The standard deviation is a special case of the so-called (Wasserstein) $\alpha$-deviation:

$$\Delta_\alpha(p) := \inf_y \left( \int |x - y|^{\alpha} p(dx) \right)^{1/\alpha}, \quad 1 \leq \alpha < \infty.$$

The uncertainty relation for the standard deviations of position and momentum has recently been generalised to $\alpha$-deviation [7].

**Theorem 3.1** (Preparation Uncertainty). Let $Q$ and $P$ be canonically conjugate position and momentum observables, and $\rho$ a density operator. Then, for any $1 \leq \alpha, \beta < \infty$,

$$\Delta_\alpha(\rho^Q) \Delta_\beta(\rho^P) \geq c_{\alpha\beta} \hbar,$$

The constant $c_{\alpha\beta}$ is connected to the ground state energy $g_{\alpha\beta}$ of the Hamiltonian $H_{\alpha\beta} = |Q|^\alpha + |P|^\beta$ by the equation

$$c_{\alpha\beta} = \alpha^{\frac{1}{\alpha}} \beta^{\frac{1}{\beta}} \left( \frac{g_{\alpha\beta}}{\alpha + \beta} \right)^{\frac{1}{\alpha} + \frac{1}{\beta}}.$$

The lower bound is attained exactly when $\rho$ arises from the ground state of the operator $H_{\alpha\beta}$ by phase space translation and dilatation. For $\alpha = \beta = 2$, $H_{22}$ is twice the harmonic oscillator Hamiltonian with ground state energy $g_{22} = 1$, and $c_{22} = 1/2$. 
The overall width of \( p \) (at confidence level \( 1 - \varepsilon \)) is defined for \( \varepsilon \in [0, 1) \) as

\[
W_\varepsilon(p) := \inf\{w > 0 \mid \exists x \in \mathbb{R} : p([x - \frac{w}{2}, x + \frac{w}{2}]) \geq 1 - \varepsilon\}.
\]

This quantity is finite for any \( \varepsilon > 0 \). For the overall width of the distribution of an observable \( E \) on \( \mathcal{B}(\mathbb{R}) \) in a state \( \rho \) we will write \( W_\varepsilon(E, \rho) := W_\varepsilon(\rho^E) \). This describes the extent to which the quantity described by \( E \) can be approximately localised. As shown in [18], the overall width is generally a more stringent measure of the spread of a distribution than the standard deviation.

In analogy to the uncertainty relation (1) for standard deviations, the overall widths of the position and momentum distributions in a state \( \rho \) also satisfy a trade-off relation:

\[
W_{\varepsilon_1}(Q, \rho) \cdot W_{\varepsilon_2}(P, \rho) \geq 2\pi\hbar K(\varepsilon_1, \varepsilon_2).
\]

(3)

An uncertainty relations of this form was first presented in a somewhat implicit way in the context of signal analysis by Landau and Pollak in 1961 [19]. Its explicit form was given by Uffink in 1990 [20]:

\[
K(\varepsilon_1, \varepsilon_2) = \left( \sqrt{(1 - \varepsilon_1)(1 - \varepsilon_2)} - \sqrt{\varepsilon_1\varepsilon_2} \right)^2.
\]

(4)

A somewhat simpler (but weaker) bound was given in [6] using elementary arguments:

\[
\tilde{K}(\varepsilon_1, \varepsilon_2) = (1 - (\varepsilon_1 + \varepsilon_2))^2 \leq K(\varepsilon_1, \varepsilon_2).
\]

Note that the two expressions coincide on the ‘diagonal’: \( K(\varepsilon, \varepsilon) = \tilde{K}(\varepsilon, \varepsilon) = (1 - 2\varepsilon)^2 \).

4. Intrinsic unsharpness: resolution width

For two noncommuting observables to be jointly measurable, it is necessary that they are unsharp. One expects intuitively that the required degree of their unsharpness depends on the extent of their noncommutativity.

We will see that two unsharp observables which approximate position and momentum, respectively, cannot have arbitrarily small degrees of unsharpness if they are to be jointly measurable. We will use the following measures as indicators of the unsharpness of an observable \( E \) on \( \mathbb{R} \).

For an observable \( E \) with support \( \text{supp}(E) \) given by \( \mathbb{R} \) or a closed interval, the resolution width (at confidence level \( 1 - \varepsilon \)) is defined as [21]:

\[
\gamma_\varepsilon(E) := \inf\{w > 0 \mid \forall x \in \mathbb{R} \exists \rho \in S : \rho^E([x - \frac{w}{2}, x + \frac{w}{2}]) \geq 1 - \varepsilon\}.
\]

For a sharp observable \( E \) on \( \mathcal{B}(\mathbb{R}) \) the resolution width is \( \gamma_\varepsilon(E) = 0 \) for all \( \varepsilon \in (0, 1) \). It is worth noting that vanishing resolution width does not require the observable to be sharp: in fact, any observable whose nonzero effects have norm 1 has zero resolution width; an example is given by the so-called canonical phase observable [22].

For the resolution width of \( Q^\mu, P^\nu \) we obtain (see also [21]):

\[
\gamma_{\varepsilon_1}(Q^\mu) = W_{\varepsilon_1}(\mu), \quad \gamma_{\varepsilon_2}(P^\nu) = W_{\varepsilon_2}(\nu).
\]
If a pair of observables $Q^\mu$, $P^\nu$ is jointly measurable, their resolution widths are determined by the probability measures $\mu = \mu_\tau, \nu = \nu_\tau$ which obey the uncertainty relations (1) and (3); we thus obtain:

$$\gamma_{\varepsilon_1}(Q^\mu_\tau) \gamma_{\varepsilon_2}(P^\nu_\tau) = W_{\varepsilon_1}(Q, \tau) W_{\varepsilon_2}(P, \tau) \geq 2\pi \hbar K(\varepsilon_1, \varepsilon_2).$$

The last inequality holds for any $\varepsilon_1, \varepsilon_2 > 0$ with $\varepsilon_1 + \varepsilon_2 < 1$.

5. Error measures I: Distance between observables

We review three distinct measures of error which quantify the difference between an observable $E$ on $B(\mathbb{R})$ to be approximated and the approximator $F$, which is also a POVM on $B(\mathbb{R})$. Any error measure should be operationally significant in the sense that it quantifies the difference between the distributions $\rho^F$ and $\rho^E$. We begin with a family of metric error measures.

5.1. Wasserstein $\alpha$-distance: definition.

Next we briefly review a family of distances on the set of observables on $\mathbb{R}$ that was used in [7] to formulate measurement uncertainty relations for canonically conjugate pairs of observables such as position and momentum. We adapt the presentation given there for general metric spaces to the case of $\mathbb{R}$.

For any two probability measures $\mu, \nu$ on $\mathbb{R}$ a coupling is defined to be a probability measure $\gamma$ on $\mathbb{R} \times \mathbb{R}$ with $\mu$ and $\nu$ as the Cartesian marginals. The set of couplings between $\mu$ and $\nu$ will be denoted $\Gamma(\mu, \nu)$. Then, for any $\alpha, 1 \leq \alpha < \infty$ the $\alpha$-distance (also Wasserstein $\alpha$-distance [23]) of $\mu$ and $\nu$ is defined as

$$D_\alpha(\mu, \nu) = \inf_{\gamma \in \Gamma(\mu, \nu)} \mathcal{D}_\alpha(\mu, \nu) = \inf_{\gamma \in \Gamma(\mu, \nu)} \left( \int |x - y|^\alpha d\gamma(x, y) \right)^{\frac{1}{\alpha}}$$

For $\alpha = \infty$, one defines $\mathcal{D}_\infty(\mu, \nu) = \gamma - \text{ess sup}\{ |x - y| : (x, y) \in \mathbb{R} \times \mathbb{R} \}$ and thus

$$D_\infty(\mu, \nu) = \inf_{\gamma \in \Gamma(\mu, \nu)} \mathcal{D}_\infty(\mu, \nu).$$

It turns out that $\mathcal{D}_\infty(\mu, \nu)$ actually depends only on the support of $\gamma$, that is, $\mathcal{D}_\infty(\mu, \nu) = \sup\{ |x - y| : (x, y) \in \text{supp}(\gamma) \}$.

The existence of an optimal coupling is known, for $1 \leq \alpha < \infty$, see [23, Theorem 4.1], the case $\alpha = \infty$ is shown in [24, Theorem 2.6], but it does not imply that $D_\alpha(\mu, \nu)$ is finite.

When $\nu = \delta_y$ is a point measure, there is only one coupling between $\mu$ and $\nu$, namely the product measure $\gamma = \mu \times \delta_y$. In that case (5) and (6) describe the deviation of the measure $\mu$ from a point $y$. In particular, the Wasserstein distances between point measures are seen to be extensions of the given metric for points, interpreted as point measures. The metric can become infinite, but the triangle inequality still holds [23, after Example 6.3]. The proof relies on Minkowski’s inequality and the use of a “Gluing
Lemma” [23], which builds a coupling from \( \mu \) to \( \zeta \) out of couplings from \( \mu \) to \( \nu \) and from \( \nu \) to \( \zeta \). It also covers the case \( \alpha = \infty \), which is not otherwise treated in [23].

We can now define the (Wasserstein) \( \alpha \)-distance between observables \( E, F \) on \( \mathbb{R} \):

\[
\Delta_\alpha(E, F) := \sup_{\rho \in \mathcal{S}} \mathcal{D}_\alpha(\rho^E, \rho^F).
\]

These distances are operationally significant and global error measures, taking into account the largest possible deviations between corresponding probability measures of the observables being compared.

5.2. Working with Wasserstein distances: Kantorovich duality.

A powerful tool for working with the distance functions is a dual expression of the infimum over couplings as a supremum over certain other functions obtained by the Kantorovich duality. In this context we exclude the case \( \alpha = \infty \).

First we note that the “gap inequality”

\[
\int \Phi(y) \, d\nu(y) - \int \Psi(x) \, d\mu(x) \leq \int |x - y|^\alpha \, d\gamma(x, y)
\]

holds for any pair of functions \((\Psi, \Phi)\) and any coupling \( \gamma \) whenever the constraint

\[
\Phi(y) - \Psi(x) \leq |x - y|^\alpha
\]

is satisfied. The Kantorovich Duality Theorem asserts that the gap is actually closed:

\[
\mathcal{D}_\alpha(\mu, \nu)^\alpha = \sup_{\Phi, \Psi} \left\{ \int \Phi(y) \, d\nu(y) - \int \Psi(x) \, d\mu(x) \right\}
\]

where functions \( \Phi \) and \( \Psi \) satisfy (8).

When maximizing the left hand side of (7), one can naturally choose \( \Phi \) as large as possible under the constraint (8), i.e., \( \Phi(y) = \inf_x \{ \Psi(x) + |x - y|^\alpha \} \), and similarly for \( \Psi \). Hence one can choose just one variable \( \Phi \) or \( \Psi \) and determine the other by this formula. In the case \( \alpha = 1 \) the triangle inequality for the metric on \( \mathbb{R} \) entails that one can take \( \Phi = \Psi \). In this case (8) just asserts that this function is Lipschitz continuous with respect to the metric on \( \mathbb{R} \), with constant 1. The left hand side of (7) is thus a difference of expectation values of the given measures \( \mu, \nu \).

It is of interest to note that the duality gap still closes if the set of functions \( \Phi, \Psi \) is further restricted. The natural condition is, first of all, that \( \Psi \in L^1(\mu) \). The statement of Kantorovich Duality in [23, Thm. 5.10] includes that the supremum (9) is attained also when one restricts the set of functions to bounded continuous functions. In [7] it is shown that this set can be further restricted to positive continuous functions of compact support without changing the value of the supremum.

5.3. Properties of the 1-distance.

We now specialise to the case of the Wasserstein 1-distance \( (\alpha = 1) \), also known as the Monge metric. This was the choice of metric for the first formulation of a rigorous measurement uncertainty relation for position and momentum in [10].
Denoting by $\Lambda$ the set of Lipschitz functions, that is, the bounded measurable functions $h : \mathbb{R} \to \mathbb{R}$ for which $|h(x) - h(y)| \leq |x - y|$, the Wasserstein-1 distance $D_1$ then becomes
\[
D_1(\mu, \nu) = \sup_{h \in \Lambda} \left| \int h \, d\mu - \int h \, d\nu \right|.
\]
This gives rise to a metric on the set of observables on $\mathcal{B}(\mathbb{R})$ as follows. We first recall that for any bounded measurable function $h : \mathbb{R} \to \mathbb{R}$, the integral $\int_{\mathbb{R}} h \, d\mu$ defines (in the weak sense) a bounded selfadjoint operator, which we denote by $\mathcal{E}[h]$. Thus, for any vector state $\varphi$ the number $\langle \varphi | \mathcal{E}[h] | \varphi \rangle = \int_{\mathbb{R}} h \, d\langle \varphi | \mathcal{E}(x) | \varphi \rangle$ is well-defined.

The Wasserstein 1-distance between observables $\mathcal{E}$ and $\mathcal{F}$ can then be expressed as
\[
\Delta_1(\mathcal{E}, \mathcal{F}) := \sup_{\rho \in \mathcal{S}} \sup_{h \in \Lambda} \left| \text{tr} [\rho \mathcal{E}[h] - \mathcal{F}[h]] \right| = \sup_{h \in \Lambda} \left| \mathcal{E}[h] - \mathcal{F}[h] \right|.
\]

An observable $\mathcal{C}$ will be called a metric approximation to $\mathcal{A}$ if $\Delta_1(\mathcal{A}, \mathcal{C}) < \infty$.

**Example 5.1.** Any trivial observables $\mathcal{E} = \mu I$ on $\mathbb{R}$ has infinite distance from sharp position $\mathcal{Q}$: $d(\mu I, \mathcal{Q}) = \infty$.

Take the family of functions $h_n(x) = n - |x - c_n - n|$ if $|x - c_n - n| \leq n$, and $h_n(x) = 0$ otherwise; here $(c_n)_{n \in \mathbb{N}}$ is an increasing sequence of positive numbers still to be determined. Note that $h_n \in \Lambda$. We have $\| \mathcal{Q}[h_n] \| = h_n(c_n + n) = n$, so this approaches infinity as $n \to \infty$.

For a trivial observable $\mathcal{E}$ we get $\mathcal{E}[h_n] = \int h_n \, d\mu I = : \mu(h_n) I$. We show that for a suitable choice of the sequence $c_n$, one obtains $\| \mathcal{E}[h_n] \| = \mu(h_n) \to 0$ as $n \to \infty$.

Let $c_n$ be such that the set $K_n = (-\infty, c_n]$ has measure $\mu(K_n) > 1 - 1/n^2$, so that $\mu(\mathbb{R} \setminus K_n) = \mu((c_n, \infty)) < 1/n^2$. Then
\[
\| \mathcal{E}[h_n] \| = \mu(h_n) = \int_{c_n}^{c_n+2n} h_n(x) \, d\mu \leq n \mu((c_n, \infty)) < 1/n.
\]

By the triangle inequality for norms we get
\[
\| \mathcal{Q}(h_n) - \mathcal{E}(h_n) \| \geq \| \mathcal{Q}(h_n) \| - \| \mathcal{E}(h_n) \| > n - 1/n.
\]
It follows that the distance $\Delta_1(\mathcal{E}, \mathcal{Q}) = \infty$. \hfill \Box

Our next example exhibits functions of position $\mathcal{Q}$ that may or may not be good metric approximations to $\mathcal{Q}$.

**Example 5.2.** Let $g$ be a bounded measurable function on $\mathbb{R}$. Then the distance of $\mathcal{Q}$ and $\mathcal{Q} \circ g^{-1}$ is infinite, $\Delta_1(\mathcal{Q}, \mathcal{Q} \circ g^{-1}) = \infty$. For a function $f(x) = x + g(x)$ where $g$ is bounded, the distance is finite: $\Delta_1(\mathcal{Q}, \mathcal{Q} \circ f^{-1}) = \sup(|g|)$.

Proof. We will show that $\| \mathcal{Q}[h_n] \| - \| \mathcal{Q} \circ g^{-1}[h_n] \| \to \infty$ as $n \to \infty$ for a suitable sequence of functions $h_n \in \Lambda$. To this end we use the inequality
\[
\| \mathcal{Q}[h_n] \| - \| \mathcal{Q} \circ g^{-1}[h_n] \| \geq \| \mathcal{Q}[h_n] \| - \| \mathcal{Q} \circ g^{-1}[h_n] \|,
\]
and choose $h_n$ such that $\| \mathcal{Q}[h_n] \| \to \infty$ as $n \to \infty$, while $\| \mathcal{Q} \circ g^{-1}[h_n] \|$ will remain bounded.
Let $|g(x)| \leq g_0$. Choose $h_n(x) = n - |x - n|$ if $|x - n| \leq n$ and $h_n(x) = 0$ otherwise. Then we have $h_n \in \Lambda$. Further, $Q[h_n] = \int h_n(x)Q(dx)$, so $||Q[h_n]|| = n \to \infty$ as $n \to \infty$. Next, we see that for $n > g_0$

$$Q \circ g^{-1}[h_n] = \int h_n(t)Q \circ g^{-1}(dt) = \int h_n(g(x))Q(dx)$$

is a bounded operator since then $h_n(g(x)) \leq g_0$, and so $||Q \circ g^{-1}[h_n]|| \leq g_0$.

To verify the second claim, we note that $Q[h] - Q \circ f^{-1}[h] = h(Q) - h(f(Q))$, and so for $h \in \Lambda$ and unit vector $\varphi$,

$$\langle \varphi[h(Q) - h(f(Q))]\varphi \rangle = \int |\varphi(x)|^2 [h(x) - h(f(x))]^2 dx \leq \int |\varphi(x)|^2 (x - f(x))^2 dx = \int |\varphi(x)|^2 g(x)^2 dx \leq ||g(Q)||^2 = (\sup |g|)^2,$$

from which the claim follows.

\[\square\]

\textbf{Example 5.3.} For smeared position and momentum observables $Q^\mu$, $P^\nu$, the distances from $Q$ and $P$ are

$$\Delta_1(Q^\mu, Q) = \int |q| \mu(dq), \quad \Delta_1(P^\nu, P) = \int |p| \nu(dp).$$

(See [7].) Thus, $Q^\mu$ and $P^\nu$ are metric approximations of $Q$ and $P$ exactly when these integrals are finite.

5.4. Measurement uncertainty relations for metric errors

The measurement uncertainty relation for the metric error associated with the 1-deviation (or Monge metric) proven in [10] has recently been generalised to all Wasserstein distances [7].

\textbf{Theorem 5.4.} Let $M$ be a phase space observable and $1 \leq \alpha, \beta \leq \infty$. Then

$$\Delta_\alpha(M_1, Q) \Delta_\beta(M_2, P) \geq c_{\alpha \beta} \hbar$$

provided that the quantities on the left hand side are finite. The constants $c_{\alpha \beta}$ are the same as in Theorem [7].

The proof is analogous to that of Werner's original theorem for 1-deviations: it proceeds by reduction to the covariant case, and the latter is immediately obtained by application of the preparation uncertainty relation of Theorem [3,4].

The case where one of the distances is zero is in fact covered by the Theorem: the other distance must then be infinite. In fact if (in the case of the 1-deviation) one has $\Delta_1(M_1, Q) = 0$, then $M_1 = Q$ and $\Delta_1(M_2, P)$ cannot be finite; otherwise the associated covariant phase space observable would have to have $Q$ as its first marginal, which is impossible. Hence Theorem [5,4] implies that whenever $\Delta_1(M_1, Q) = 0$ then $\Delta_1(M_2, P) = \infty$. It is an instructive exercise to verify this explicitly.
Example 5.5. Let M be an observable on phase space \( \mathbb{R}^2 \) whose first marginal \( M_1 \) is sharp position Q. Then the second marginal \( M_2 \) has infinite 1-distance from sharp momentum P.

Proof. We note first that all positive operators (effects) \( M_2(X) \) in the range of \( M_2 \) commute with Q (see, e.g., [12]) and are thus functions of Q. Thus one can write \( M_2(X) = \int Q(dq)m(q,X) \), where the functions \( m(\cdot,X) \) are defined almost everywhere for all (Borel) subsets \( X \) of \( \mathbb{R} \), and \( X \mapsto m(q,X) \) is then a probability measure. We consider states \( \rho \) with the same fixed position distribution, \( \rho^Q = p \), and compute

\[
\text{tr}[\rho M_2(h)] = \int h(x)m_p(dx), \quad m_p(X) := \int p(dq)m(q,X).
\]

We will let \( h \) run through a family \( h_n \in \Lambda \) and \( p \) through a family \( p_{\rho_n} \in S_q \) such that \( p_{\rho_n}^Q = p \) and \( \text{tr}[\rho_n M_2(h_n)] \to 0 \), while \( \text{tr}[\rho_n P(h_n)] \to \infty \). This shows that \( \Delta_1(M_2,P) = \infty \).

Choose \( h_n \) as in Example 5.1, where we have now \( \mu = m_p \). This gives \( \text{tr}[\rho_n M_2(h_n)] \to 0 \) for any \( \rho_n \) (yet to be specified) with \( p_{\rho_n}^Q = p \).

Let \( \rho_n = W(0,c_n+n-(c_1+1))\rho_1 W(0,c_n+n-(c_1+1))^* \), with \( \rho_1 \) a state whose momentum distribution is centered symmetrically at \( c_1+1 \), the peak location of \( h_1 \). Then the momentum distribution of \( \rho_n \) is centered at the peak location \( c_n+n \) of \( h_n \). Also note that \( p_{\rho_n}^Q = p_{\rho_1}^Q =: p \). Specifically we take \( \rho_n \) such that the densities \( p_{\rho_n}(p) = \chi_{J_n}(p) \), \( J_n = [c_n+n-1/2,c_n+n+1/2] \). Then we have \( \text{tr}[\rho_n P(h_n)] = n - 1/4 \to \infty \) as \( n \to \infty \). \( \square \)

6. Error measures II: error bar width

6.1. Gross error bar.

We now present a definition of measurement error in terms of likely error intervals that follows most closely the usual practice of calibrating measuring instruments. In the process of calibration of a measurement scheme, one seeks to obtain estimates of the likely error and perhaps also the degree of disturbance that the scheme contains. In order to estimate the error, one tests the device by applying it to a sufficiently large family of input states in which the observable one wishes to measure with this setup has fairly sharp values. The error is then characterised as an overall measure of the bias and the width of the output distribution across a range of input values. Error bars give the minimal average interval lengths that one has to allow to contain all output values with a given confidence level.

For simplicity, we give the following definitions only for approximations of a sharp observable \( E \), so that the assumption of localised input states \( \rho \) can be described as \( \rho^E(J_{x,\delta}) = 1 \), for intervals \( J_{x,\delta} := [x-\delta/2,x+\delta/2] \), \( x \in \mathbb{R}, \delta > 0 \).

Let \( E_1,E \) be observables on \( \mathbb{R} \) and \( E \) be sharp. For each \( \varepsilon \in (0,1), \delta > 0 \), we define the error of \( E_1 \) relative to \( E \)
\[ W_{\varepsilon,\delta}(E_1, E) := \inf \{ w > 0 \mid \forall x \in \mathbb{R} \forall \rho \in S : \\
\rho^E(J_{x,\delta}) = 1 \Rightarrow \rho^{E_1}(J_{x,w}) \geq 1 - \varepsilon \}. \]

The error describes the range within which the input values can be inferred from the output distributions, with confidence level \(1 - \varepsilon\), given initial localizations within \(\delta\). \(E_1\) is called an \(\varepsilon\)-approximation to \(E\) if \(W_{\varepsilon,\delta}(E_1, E) < \infty\) for all \(\delta > 0\). Note that the error is an increasing function of \(\delta\), so that one can define the (gross) error bar width of \(E_1\) relative to \(E\):

\[ W_{\varepsilon}(E_1, E) := \inf_{\delta} W_{\varepsilon,\delta}(E_1, E) = \lim_{\delta \to 0} W_{\varepsilon,\delta}(E_1, E). \]

In the case \(W_{\varepsilon,\delta}(E_1, E) = \infty\) for all \(\delta > 0\), we write \(W_{\varepsilon}(E_1, E) = \infty\).

\(E_1\) will be called an approximation (in the sense of finite error bar width) to \(E\) if \(W_{\varepsilon}(E_1, E) < \infty\) for all \(\varepsilon \in (0, \frac{1}{2})\). The restriction to \(\varepsilon < \frac{1}{2}\) reflects the idea that a “good” approximation should have confidence levels greater than \(\frac{1}{2}\).

We note that if \(E_1\) is an approximation to \(E\), the map \(\varepsilon \mapsto W_{\varepsilon,\delta}(E_1, E)\) is a decreasing function of \(\varepsilon \in (0, \frac{1}{2})\) for every \(\delta > 0\).

The following result shows that our definition is not empty.

**Proposition 6.1.** The smeared position and momentum observables \(Q^\mu, P^\nu\) are approximations (in the sense of finite error bar widths) to \(Q\) and \(P\), respectively, for any probability measures \(\mu, \nu\).

**Proof.** It is sufficient to consider the case of the position observable. Let \(\varepsilon \in (0, 1), \delta > 0\) be given. We have to show that there is a finite number \(w > 0\) such that for all \(q \in \mathbb{R}\) one has \(\rho^{Q^\mu}(J_{q,w}) \geq 1 - \varepsilon\) whenever \(\rho^{Q}(J_{q,\delta}) = 1\).

Let \(q_0, w_0\) be such that \(\mu(J_{q_0,w_0}) \geq 1 - \varepsilon\). Then, if \(w \geq 2|q_0| + w_0 + \delta\), it follows that \(J_{q,\delta} \subseteq x + J_{q,w}\) for all \(x \in J_{q_0,w_0}\), that is, \(\rho^{Q}(x + J_{q,w}) = 1\) for all such \(x\). Then:

\[
\rho^{Q^\mu}(J_{q,w}) = \int \rho^{Q}(x + J_{q,w}) \geq \int_{J_{q_0,w_0}} \rho^{Q}(x + J_{q,w}) = \mu(J_{q_0,w_0}) \geq 1 - \varepsilon. \]

\(\square\)

### 6.2. Properties of the error bar width

It is not hard to construct approximations of \(Q\) that do not share the translation covariance of \(Q\).

**Example 6.2.** Let \(f\) be a continuous function on \(\mathbb{R}\) which is one-to-one and such that \(f(q) - q\) is not constant but \(|f(q) - q| < \alpha\) for all \(q \in \mathbb{R}\) and some fixed \(\alpha > 0\). An example is \(f(q) = q + \frac{1}{2}\cos(q)\). Let \(Q^\mu\) be a smeared position observable. Then \(Q^\mu \circ f^{-1}\) is a non-covariant approximation to \(Q\) in the sense of finite error bars.

**Proof.** Let \(\varepsilon \in (0, 1)\) and \(\delta > 0\) be given. We have to show that there is a finite positive \(w\) such that for all \(q \in \mathbb{R}\) and all \(\rho\) with \(\rho^{Q}(J_{q,\delta}) = 1\), then \(\rho^{Q^\mu}(f^{-1}(J_{q,w})) \geq 1 - \varepsilon\).
We know that $Q^\mu$ is an approximation to $Q$. Hence there is $w' > 0$ such that for all $q \in \mathbb{R}$ and all $\rho$ with $\rho^Q(J_{q,\delta}) = 1$, we have $\rho^{Q^\mu}(J_{q,w'}) \geq 1 - \varepsilon$.

Now take $w = w' + 2\alpha$. This entails that $f^{-1}(J_{q,w}) \supseteq J_{q,w'}$ for all $q \in \mathbb{R}$. Then, for $q \in \mathbb{R}$ and $\rho$ such that $\rho^Q(J_{q,\delta}) = 1$ we obtain

$$\rho^{Q^\mu \circ f^{-1}}(J_{q,w}) = \rho^{Q^\mu}(f^{-1}(J_{q,w})) \geq \rho^{Q^\mu}(J_{q,w'}) \geq 1 - \varepsilon.$$ 

Noting that $f^{-1}(J_{q,w} + q') \neq f^{-1}(J_{q,w}) + q'$ (since $f(q) - q$ is not constant) one concludes readily that $Q^\mu \circ f^{-1}$ is not covariant. \hfill\square

**Example 6.3.** For any bounded Borel function $f$ on $\mathbb{R}$, the observable $Q \circ f^{-1}$ has infinite error bars with respect to $Q$.

**Proof.** Let $J$ be a bounded interval which contains the range of $f$. Then for any finite $w > 0$, one can find $q$ such that $J_{q,w} \cap J = \emptyset$. Then $f^{-1}(J_{q,w}) = \emptyset$ and so $\rho^{Q \circ f^{-1}}(J_{q,w}) = \rho^Q(f^{-1}(J_{q,w})) = 0$ for all $\rho$.

It follows that $W_{\varepsilon,\delta}(Q \circ f^{-1}, Q) = \infty$ for all $\varepsilon \in (0,1)$ and all $\delta > 0$. \hfill\square

It is possible to characterise the case of an accurate measurement of the sharp observable $E$.

**Proposition 6.4.** Let $E_1$ be an approximation of the sharp observable $E$. Then the following are equivalent:

(a) $W_{\varepsilon,\delta}(E_1, E) \leq \delta$ for all $\varepsilon \in (0, \frac{1}{2}), \delta > 0$;

(b) $E_1 = E$.

If either of these condition is fulfilled then $W_{\varepsilon}(E_1, E) = 0$ for all $\varepsilon \in (0, \frac{1}{2})$.

**Proof.** Assume (b) holds. Let $\varepsilon \in (0, \frac{1}{2}), \delta > 0$. Choose $w = \delta$; then for any $q \in \mathbb{R}$ and any state $\rho$ with $\rho^E(J_{x,\delta}) = 1$, we also have $\rho^{E_1}(J_{x,\delta}) \geq 1 - \varepsilon$. This shows that $W_{\varepsilon,\delta}(E_1, E) \leq \delta$.

Conversely, assume that (a) holds. Consider any $\varepsilon \in (0, \frac{1}{2}), \delta > 0$. For $w = W_{\varepsilon,\delta}(E_1, E) \leq \delta$, we have, for all $x \in \mathbb{R}$ and all $\rho$ with $\rho^E(J_{x,\delta}) = 1$, that $\rho^{E_1}(J_{x,\delta}) \geq \rho^{E_1}(J_{x,w}) \geq 1 - \varepsilon$. This entails for any vector state $\varphi$ for which $Q(J_{x,\delta})\varphi = \varphi$ that $\langle \varphi | E_1(J_{x,\delta}) | \varphi \rangle \geq 1 - \varepsilon$. As this holds for any $\varepsilon \in (0, \frac{1}{2})$, it follows that $\langle \varphi | E_1(J_{x,\delta}) | \varphi \rangle = 1$. This entails that $E(J_{x,\delta}) \leq E_1(J_{x,\delta})$. Since $x \in \mathbb{R}$ and $\delta > 0$ are arbitrary, this operator inequality holds for any closed interval $J = [a, b]$.

We show that then also $E((a, b)) \leq E_1((a, b))$ for any open interval. Let $J_n$ be an increasing sequence of closed sets which converges to a given open interval $(a, b)$. Put $D_n := E_1(J_n) - E(J_n) \geq O$. For any POVM $N$ on $B(\mathbb{R})$ we have $N(J_n) \to N((a, b))$ (ultraweakly). (This is a consequence of the regularity of Borel measures on $B(\mathbb{R})$, see, e.g., [23]) So we obtain $D_n \to E_1((a, b)) - E((a, b))$ (ultraweakly), and since $D_n \geq O$, this limit operator is also nonnegative. In this way we conclude that $E(K) \leq E_1(K)$ for all open intervals $K$. Similarly we can show that $E((a, b)) \leq E_1((a, b))$. Due to the normalization of both POVMs $E, E_1$, it follows that they must coincide on all intervals and finally, since the intervals generate $B(\mathbb{R})$, that they are identical. \hfill\square

We remark that it is not known whether the condition $W_{\varepsilon}(E_1, E) = 0$ for all $\varepsilon \in (0, \frac{1}{2})$ is sufficient to conclude that $E_1 = E$.  


Proposition 6.5. Let $E_1, E$ be observables with support $\mathbb{R}$, and $E$ be a sharp observable. The error bar width of $E_1$ relative to $E$ is never smaller than the intrinsic resolution width of $E_1$:

$$W_\varepsilon(E_1, E) \geq \gamma_\varepsilon(E_1).$$

The proof is given in [9, Prop. 1].

Corollary 6.6. Let $E$ be a sharp observable on $\mathcal{B}(\mathbb{R})$ with support $\mathbb{R}$. Any $\varepsilon$-approximation $E_1$ (supported on $\mathbb{R}$) of $E$ has finite resolution width, $\gamma_\varepsilon(E_1) < \infty$.

6.3. Bias-free error and bias

We show next how the gross error can be decomposed into a (positive) bias term and a random error. Let $\varepsilon \in (0, 1)$ and $\delta > 0$ be given. Let $E_1, E$ be observables on $\mathbb{R}$ and $E$ be sharp. Note that the condition $\rho^E(J_{x,\delta}) = 1$ (for some $x \in \mathbb{R}$) can be expressed as $W_0(\rho^E) \leq \delta$. We define the bias-free, or random error $W^0_{\varepsilon,\delta}(E_1, E)$ as follows:

$$W^0_{\varepsilon,\delta}(E_1, E) := \sup \{ W_\varepsilon(\rho^{E_1}) \mid W_0(\rho^E) \leq \delta \}.$$  

This is a measure of the overall minimal error, determined by the overall widths of all output distributions, given input distributions supported in intervals $J_{x,\delta}$. If this quantity is finite for some $\delta_0$, it is an increasing function for all $\delta \leq \delta_0$. In that case we can define the bias-free error bar width,

$$W^0_\varepsilon(E_1, E) := \lim_{\delta \to 0} W^0_{\varepsilon,\delta}(E_1, E).$$

The following is obvious:

$$W_{\varepsilon,\delta}(E_1, E) \geq W^0_{\varepsilon,\delta}(E_1, E).$$

If these quantities are finite, one then has in the limit $\delta \to 0$:

$$W_\varepsilon(E_1, E) \geq W^0_\varepsilon(E_1, E).$$  \hspace{1cm} (10)

The difference between $W_{\varepsilon,\delta}(E_1, E)$ and $W^0_{\varepsilon,\delta}(E_1, E)$ disappears when the output distributions are concentrated at the locations of the input distributions, that is, around the intervals $J_{x,\delta}$. This is to say that the difference is a measure of the overall magnitude of the bias $\beta_{\varepsilon,\delta}(E_1, E)$ inherent in $E_1$ relative to $E$:

$$\beta_{\varepsilon,\delta}(E_1, E) := W_{\varepsilon,\delta}(E_1, E) - W^0_{\varepsilon,\delta}(E_1, E) \geq 0.$$  

Rephrasing this as

$$W_{\varepsilon,\delta}(E_1, E) = W^0_{\varepsilon,\delta}(E_1, E) + \beta_{\varepsilon,\delta}(E_1, E),$$

we see that the gross error is decomposed into the bias-free error and the magnitude of the bias. Note that one can take the limit of $\delta \to 0$:

$$\beta_\varepsilon(E_1, E) := W_\varepsilon(E_1, E) - W^0_\varepsilon(E_1, E).$$

As an immediate consequence of these definitions, we can say that $E_1$ is an $\varepsilon$-approximation to $E$ if and only if the bias and random errors are finite for all $\delta > 0$. 
Proposition 6.7. Let $Q^\mu$, $P^\nu$ be smeared position and momentum observables. Then
\begin{equation*}
\mathcal{W}^0_{\varepsilon_1}(Q^\mu, Q) = W_{\varepsilon_1}(\mu), \quad \mathcal{W}^0_{\varepsilon_2}(P^\nu, P) = W_{\varepsilon_2}(\nu).
\end{equation*}

Proof. It suffices to consider the case of position. We show first that $W_{\varepsilon_1}(\rho^{Q^\mu}) \geq W_{\varepsilon_1}(\mu)$. This is equivalent to the following: $w \geq W_{\varepsilon_1}(\rho^{Q^\mu})$ implies $w \geq W_{\varepsilon_1}(\mu)$.

Thus, let $w$ be such that $\rho^{Q^\mu}(J_{q:w}) \geq 1 - \varepsilon$ for some $q \in \mathbb{R}$. Assume $w < W_{\varepsilon_1}(\mu)$; this means that for all $q' \in \mathbb{R}$ one has $\mu(J_{q':w}) < 1 - \varepsilon$. But then
\begin{equation*}
\rho^{Q^\mu}(J_{q':w}) = \int \rho^{Q}(dx)\mu(x - J_{q':w}) < 1 - \varepsilon,
\end{equation*}
which contradicts the premise.

Next we show that whenever $W_0^0(\rho^Q) \leq \delta$, then $W_{\varepsilon_1}(\rho^{Q^\mu}) \leq W_{\varepsilon_1}(\mu) + \delta$. We are given that $\rho^Q(J_{q_0;\delta}) = 1$ for some $q_0 \in \mathbb{R}$. Assume $w \geq W_{\varepsilon_1}(\mu)$, that is, $\mu(J_{q_1:w}) \geq 1 - \varepsilon$ for some $q_1$. We have to show that $w + \delta \geq W_{\varepsilon_1}(\rho^{Q^\mu})$, that is, $\rho^{Q^\mu}(J_{q_2:w+\delta}) \geq 1 - \varepsilon$ for some $q_2 \in \mathbb{R}$.

Let $q_2 = q_0 - q_1$. Then it follows that $q + J_{q_2:w+\delta} \supseteq J_{q_0:\delta}$ for all $q \in J_{q_1:w}$. Then
\begin{align*}
\rho^{Q^\mu}(J_{q_2:w+\delta}) &= \int \mu(dq)\rho^Q(q + J_{q_2:w+\delta}) \\
&\geq \int_{J_{q_2:w}} \mu(dq) = \mu(J_{q_2:w}) \geq 1 - \varepsilon.
\end{align*}
This shows that $w \geq W_{\varepsilon_1}(\mu)$ implies $w + \delta \geq W_{\varepsilon_1}(\rho^{Q^\mu})$ whenever $W_0^0(\rho^Q) \leq \delta$. Thus, under this assumption we let $w$ approach $W_{\varepsilon_1}(\mu)$ to obtain $W_{\varepsilon_1}(\mu) + \delta \geq W_{\varepsilon_1}(\rho^{Q^\mu})$.

To summarise, we have shown: $W_{\varepsilon}(\mu) \leq W_{\varepsilon}(\rho^{Q^\mu}) \leq W_{\varepsilon}(\mu) + \delta$, where the latter inequality holds if $W_0^0(\rho^Q) \leq \delta$. This entails that also $W_{\varepsilon}(\mu) \leq \mathcal{W}^0_{\varepsilon_1}(Q^\mu, Q) \leq W_{\varepsilon}(\mu) + \delta$.

Now we can take the limit $\delta \to 0$ to obtain the result. \hfill \Box

6.4. Measurement uncertainty relations for error bar widths

The following error relations for covariant approximations are special cases of the general result quoted below. Their proofs are straightforward consequences of the considerations of this paper, hence we present them here as separate statements.

Proposition 6.8. Let $G^r$ be a covariant phase space observable. Then the bias-free error bar widths of the marginals relative to $Q$ and $P$ obey the trade-off relation:
\begin{equation}
\mathcal{W}_{\varepsilon_1}(Q^{\mu^r}, Q) \mathcal{W}_{\varepsilon_2}(P^{\nu^r}, P) \geq \mathcal{W}^0_{\varepsilon_1}(Q^{\mu^r}, Q) \mathcal{W}^0_{\varepsilon_2}(P^{\nu^r}, P) \geq 2\pi \hbar K(\varepsilon_1, \varepsilon_2), \tag{11}
\end{equation}
where $K(\varepsilon_1, \varepsilon_2)$ is given by Eq. \[[4]\].

Proof. The first inequality follows from \[[10]\], and the second is a direct consequence of Proposition 6.7 and Eq. \[[4]\].

The corresponding inequality for general phase space observables was proven in \[[9]\].

Theorem 6.9. Let $M$ be an approximate joint observable for $Q$, $P$, in the sense that its marginals have finite error bar widths as approximations of position and momentum,
respectively. Then, for \( \varepsilon_1, \varepsilon_2 \in (0, \frac{1}{2}) \), the error bar widths of \( M_1 \) and \( M_2 \) satisfy the uncertainty relation

\[
\mathcal{W}_{\varepsilon_1}(M_1, P) \cdot \mathcal{W}_{\varepsilon_2}(M_2, P) \geq 2\pi \hbar K(\varepsilon_1, \varepsilon_2),
\]

where \( K(\varepsilon_1, \varepsilon_2) \) is given by Eq. \( \left( \text{4} \right) \).

This result entails the following statement: an approximate joint observable for \( Q, P \) cannot have one of these sharp observables as its marginal. It is instructive to show this explicitly by considering the case \( M_1 = Q \).

**Proposition 6.10.** Let \( M \) be an observable on phase space whose first marginal coincides with sharp position, \( M_1 = Q \) (so that \( \mathcal{W}_{\varepsilon_1}(M_1, Q) = 0 \)). Then the second marginal \( M_2 \) cannot satisfy the condition of an \( \varepsilon_2 \)-approximation to \( P \) for any \( \varepsilon_2 \in (0, \frac{1}{2}) \), that is, \( \mathcal{W}_{\varepsilon_2}(M_2, P) = \infty \). Hence \( M \) cannot be an \( (\varepsilon_1, \varepsilon_2) \)-approximate joint observable to \( Q, P \) for any \( \varepsilon_1, \varepsilon_2 \in (0, \frac{1}{2}) \).

**Proof.** Let \( \varepsilon_2 \in (0, \frac{1}{2}) \) be given and let \( \delta > 0 \) and \( w' > 0 \) be arbitrary. We have to show that there is an interval \( J_{\rho, \delta} \) and a state \( \rho \) localised in \( J_{\rho, \delta} \) so that \( \text{tr} [\rho M_2(J_{\rho, \delta})] < 1 - \varepsilon_2 \).

As noted in Example \( \left( \text{5.5} \right) \), all positive operators (effects) \( M_2(X) \) in the range of \( M_2 \) commute with \( Q \) and are thus functions of \( Q \). Thus we can write: \( M_2(X) = \int m(q, X) Q(dx) \). Consider the sequence of intervals \( J_{n; w}, n = 0, 1, 2, \ldots \). Since \( I = M_2(\mathbb{R}) \), then \( M_2((-\infty, n - w'/2)) \to 1 \) as \( n \to \infty \) (ultraweakly), and it follows that for every state \( \rho \), \( \text{tr} [\rho M_2(J_{n; w})] \leq \text{tr} [\rho M_2([n - w'/2, \infty))] \to 0 \), hence:

\[
\text{tr} [\rho M_2(J_{n; w})] = \int \rho^{Q}(dq)m(q, J_{n; w'}) \to 0 \quad \text{as} \, n \to \infty.
\]

Let \( \rho_0 \) be such that \( \rho_{0}^{Q}(J_{0, \delta}) = 1 \), that is, the distribution \( \rho_{0}^{Q} \) vanishes outside that interval. Then \( \rho_n := W(0, n)\rho_0 \) is localised in \( J_{n; \delta} \), while the position distribution is unchanged, \( \rho_{n}^{Q} = \rho_{0}^{Q} \).

For the given \( \varepsilon_2 \in (0, \frac{1}{2}) \), there is an \( n \in \mathbb{N} \) such that for the fixed state \( \rho_0 \), \( \text{tr} [\rho_0 M_2(J_{n; w'})] < 1 - \varepsilon_2 \). Then, since \( \rho_{0}^{Q} = \rho_{n}^{Q} \), we also have \( \text{tr} [\rho_n M_2(J_{n; w'})] < 1 - \varepsilon_2 \), whereas \( \rho_n \) is localised in \( J_{n; \delta} \).

This result reproduces, in particular, the well-known fact that there is no observable on phase space whose marginals are sharp position and sharp momentum.

**Example 6.11.** Example \( \left( \text{6.2} \right) \) can be used to construct an observable \( M \) on phase space which is not covariant but is still an approximate joint observable for \( Q, P \). Let \( G^{\tau} \) be a covariant phase space observable and define \( M := G^{\tau} \circ \gamma^{-1} \), where \( \gamma(q, p) := (\gamma_{1}(q), \gamma_{2}(p)) \). We assume that \( \gamma_{1}, \gamma_{2} \) are strictly increasing continuous functions such that \( \gamma_{1}(q) - q \) and \( \gamma_{2}(p) - p \) are bounded functions. Then it follows that the marginals \( M_{1}^{\gamma} = G_{1}^{\tau} \circ \gamma_{1}^{-1} \) and \( M_{2}^{\gamma} = G_{2}^{\tau} \circ \gamma_{2}^{-1} \) have finite error bars with respect to \( Q, P \). If \( \gamma \) is a nonlinear function then \( M \) will not be covariant.
Error and unsharpness in approximate joint measurements

It is straightforward to obtain a universal uncertainty relation for the bias-free errors for any approximate joint observable $M$ of $Q, P$. The core of the proof is to show that finite bias-free errors for the marginals entails the existence of a covariant observable $G^\tau$ (obtained by the operation of finite mean used in [10]) such that its marginals are not greater than those of $M$:

$$\mathcal{W}_{\varepsilon_1,\delta}(M_1, Q) \geq \mathcal{W}_{\varepsilon_1,\delta}(G_1^\tau, Q), \quad \mathcal{W}_{\varepsilon_2,\delta}(M_2, P) \geq \mathcal{W}_{\varepsilon_2,\delta}(G_2^\tau, P).$$

The proof of this is similar to that of Lemma 4 of [9] and will be omitted. Using inequality (11), the bias-free errors is then seen to obey the trade-off relation for $\varepsilon_1, \varepsilon_2 < \frac{1}{2}$:

$$\mathcal{W}_{\varepsilon_1}(M_1, Q) \mathcal{W}_{\varepsilon_2}(M_2, P) \geq 2\pi \hbar K(\varepsilon_1, \varepsilon_2).$$

6.5. Trade-off relations for resolution widths

In the work [9] we claimed the validity of an uncertainty relation for resolution widths; a proof was not given explicitly as it was considered to follow closely the steps of the proof of Theorem 6.9. On revisiting this relation, we found that there is no obvious way of adapting that proof. In fact, it may well be that it is only for sufficiently close joint approximations of $Q$ and $P$ that there have to be constraints on the resolution width similar to the error uncertainty relation. Hence we rephrase the claim as a Problem.

**Problem.** Let observable $M$ on $\mathcal{B}(\mathbb{R}^2)$ be an approximate joint observable for $Q, P$ in the sense of finite error bar widths. State conditions on the quality of the approximation (other than the covariance of the joint observable) which entail that the resolution widths must obey the trade-off relation (for $\varepsilon_1 + \varepsilon_2 < 1$):

$$\gamma_{\varepsilon_1}(M_1) \gamma_{\varepsilon_2}(M_2) \geq 2\pi \hbar K(\varepsilon_1, \varepsilon_2)$$

7. Error measures III: Noise-based error

Classical statistical error analysis is prominently based on the use of moments of probability distributions for the quantification of measurement errors. Thus, a widespread approach found in the literature of defining a measure of error is in terms of a formal “root mean square” deviation of an indicator variable $Z$ of the measuring apparatus from the variable $A$ to be measured approximately. Classically, $A$ and $Z$ are given as random variables, and quantum mechanically as selfadjoint operators: this state-dependent noise-based error is given as the root mean square deviation,

$$\epsilon_{\text{NO}}(A, \mathcal{M}, \rho) := \langle (Z_{\text{out}} - A_{\text{in}})^2 \rangle^{1/2}_{\rho \otimes \sigma}.$$ 

Here $Z_{\text{out}}$ denotes the output (pointer) observable at the end of the interaction phase between object system and probe in the measurement $\mathcal{M}$, and $A_{\text{in}}$ is the input object observable to be approximately measured; the object plus probe system is initially in the state $\rho \otimes \sigma$. The choice of name reflects the fact that the operator $Z_{\text{out}} - A_{\text{in}}$ is commonly called noise operator.
A detailed critique of this attempted quantum generalisation of the rms error is given in [1]; the main deficiency is that this quantity fails to be a faithful representation of the absence or magnitude of an approximation error. Therefore this measure has to be used with care; it is operationally significant only in some special circumstances; then it may be used to provide estimates of measurement errors [1]. Here we are concerned with the state-independent upper bound of the quantity $\epsilon_{NO}$, the (global) noise-based error of a measurement $\mathcal{M}$ relative to $A$ as [26]

$$\epsilon_{NO}(A, \mathcal{M}) := \sup_\rho \epsilon_{NO}(A, \mathcal{M}, \rho)$$

where the supremum is taken over all states $\rho$ for which the right hand side is well-defined. We will say that the observable $C$ defined by $\mathcal{M}$ is a finite-noise approximation to $A$ if $C$ has finite global noise-based error relative to $A$.

The noise-based error $\epsilon_{NO}(A, \mathcal{M}, \rho)$ can be expressed in terms of the observable $C$ actually measured by $\mathcal{M}$ [27]:

$$\epsilon_{NO}(A, \mathcal{M}, \rho)^2 = \text{tr} \rho (C[x^2] - C[x]^2) + \text{tr} \rho (C[x] - A)^2.$$  \hspace{1cm} (12)

In order to apply this error measure in the case of joint approximate measurements, we note that if a measurement scheme $\mathcal{M}$ defines an observable $M$ on $\mathcal{B}(\mathbb{R}^2)$, then its marginal observables $M_1, M_2$ can be taken as approximators for, say, position $Q$ and momentum $P$, respectively. In this case the noise-based errors are defined via (12) with $C = M_1$ for $\epsilon_{NO}(Q, \mathcal{M}, \rho) \equiv \epsilon_{NO}(Q, M_1, \rho)$ and with $C = M_2$ for $\epsilon_{NO}(P, \mathcal{M}, \rho) \equiv \epsilon_{NO}(P, M_2, \rho)$. We denote the global errors for the marginals of a general phase space observable by $\epsilon_{NO}(Q, M_1)$ and $\epsilon_{NO}(P, M_2)$, respectively. Then the following general result has been shown [26].

**Theorem 7.1.** Let $\mathcal{M}$ be a measurement realizing an observable $M$ on $\mathcal{B}(\mathbb{R}^2)$. Then the global noise-based errors obey the following trade-off relation.

$$\epsilon_{NO}(Q, M_1) \epsilon_{NO}(P, M_2) \geq \frac{\hbar}{2}.$$  \hspace{1cm} (13)

The lower bound is realised for a covariant phase space observable $G^\tau$ with $\tau$ being the minimum uncertainty state operator with zero means of position and momentum.

8. Connections

We show that the concept of approximation based on finite error bars generalises the notions of finite noise approximation and metric approximations.

**Proposition 8.1.** Any observable $E_1$ on $\mathbb{R}$ that satisfies the condition $\Delta_\alpha(E_1, E) < \infty$ (for some $\alpha \in [1, \infty)$) for a sharp observable $E$ on $\mathbb{R}$ is an approximation to $E$ in the sense of finite error bars. In that case the following inequality holds:

$$\mathcal{W}_\varepsilon(E_1, E) \leq \frac{2}{\varepsilon^\alpha} \Delta_\alpha(E_1, E).$$  \hspace{1cm} (13)
Proof. The proof is a straightforward adaptation of the proof for the case \( \alpha = 1 \) given in [9] Prop. 5.

Using the definition of \( \mathcal{D}_\alpha(\rho^E, \rho^F) \) and equation (9), we are given that

\[
| \text{tr} \rho E_1[\Phi] - \text{tr} \rho E(\Psi) | \leq \Delta_\alpha(E_1, E)^\alpha =: c^\alpha,
\]

which holds for all \( \rho \in S \) and all functions \( \Psi, \Phi \) satisfying the constraint

\[
|\Phi(y) - \Psi(x)| \leq |x - y|^\alpha, \quad x, y \in \mathbb{R}.
\]

Let \( \varepsilon \in (0, 1) \) and \( \delta > 0 \) be given. Put \( w = \delta + 2n \), with \( n \in \mathbb{N} \), \( n^\alpha \geq c^\alpha/\varepsilon \). Consider an interval \( J_{q,\delta} \) and a state \( \rho \) with \( \rho^E(J_{q,\delta}) = 1 \). Define the functions \( \Psi_n = \Phi_n \equiv h_n \) via

\[
h_n(x) := \begin{cases} 
  n^\alpha & \text{if } |x - q| \leq \delta/2; \\
  \left[n + \delta/2 - |x - q|\right]^\alpha & \text{if } \delta/2 < |x - q| \leq \delta/2 + n; \\
  0 & \text{if } \delta/2 + n < |x - q|.
\end{cases}
\]

It is not hard to verify that \( \Psi_n, \Phi_n \) satisfy (13). Condition (14) for \( \Psi_n = \Phi_n = h_n \) entails for \( g_n = h_n/n^\alpha \) that

\[
| \text{tr} \rho E_1[g_n] - \text{tr} \rho E[g_n] | \leq c^\alpha/n^\alpha.
\]

We then have \( \chi_{J_{q,\delta}} \leq g_n \leq \chi_{J_{q,w}} \).

Now \( \rho^E(J_{q,\delta}) = 1 \) implies \( \text{tr} \rho E(g_n) = 1 \), and so, using the assumption \( n^\alpha \geq c^\alpha/\varepsilon \), we obtain

\[
\text{tr} \rho E_1(J_{q,w}) \geq \text{tr} \rho E_1(g_n) \geq \text{tr} \rho E(g_n) - c^\alpha/n^\alpha \geq 1 - \varepsilon.
\]

To prove the inequality (13), we note that on putting \( w = \delta + 2c/\varepsilon^{1/\alpha} \), one still obtains

\[
\text{tr} \rho E_1(J_{q,w}) \geq 1 - \varepsilon.
\]

This yields \( \mathcal{W}_{\varepsilon,\delta}(E_1, E) \leq \delta + 2\Delta_\alpha(E_1, E)/\varepsilon^{1/\alpha} \), and on letting \( \delta \) approach 0, then (13) follows. \( \square \)

**Proposition 8.2.** Any observable \( E_1 \) on \( \mathbb{R} \) that satisfies the condition of finite global noise-based error relative to a sharp observable with self-adjoint operator \( A \) and associated spectral measure \( E \) (such that \( A = E[x] \), \( \epsilon_{\text{NO}}(A, E_1) < \infty \), is an approximation to \( E \) in the sense of finite error bars. In that case, the following inequality holds:

\[
\mathcal{W}_\varepsilon(E_1, E) \leq \epsilon_{\text{NO}}(A, E_1) \left( 1 + \sqrt{\frac{2}{\varepsilon}} \right).
\]

**Proof.** We use the facts that \( A^2 = E[x]^2 = E[x^2] \) and \( \Delta(E, \rho) = \Delta(A, \rho) \).

We begin by rewriting the definition of \( \epsilon_{\text{NO}} \) for general states \( \rho \), denoted \( \epsilon_{\text{NO}}(E_1, E, \rho) \), and expressing the condition of bounded errors: for all \( \rho \) and \( c := \epsilon_{\text{NO}}(E_1, E, \rho) < \infty \),

\[
\epsilon_{\text{NO}}(A, E_1, \rho)^2 = \text{tr} \left[ \rho (E_1[x] - A)^2 \right] + \text{tr} \left[ \rho (E_1[x^2] - E_1[x]^2) \right] \\
= \text{tr} \left[ \rho (E_1[x] - A)^2 \right] + \Delta(E_1, \rho)^2 - \Delta(E_1[x], \rho)^2 \leq c^2.
\]

(This follows readily from the corresponding condition stipulated for all vector states.)

The first term can be estimated as follows: using the inequality

\[
|\text{cov}_\rho(E_1[x], A)| = \frac{1}{2} |\text{tr} \rho E_1[x] A + \text{tr} \rho A E_1[x] - 2 \text{tr} \rho E_1[x] \text{tr} \rho A| \\
\leq \Delta(E_1[x], \rho) \Delta(A, \rho),
\]

and
we see that
\[
\begin{align*}
\text{tr}[\rho(E_1[x] - A)^2] &= \Delta(E_1[x] - A, \rho)^2 + (\text{tr}[\rho(E_1[x] - A)])^2 \\
&= \Delta(E_1[x], \rho)^2 + \Delta(A, \rho)^2 - 2\text{cov}_\rho(E_1[x], A) + (\text{tr}[\rho(E_1[x] - A)])^2 \\
&\geq (\Delta(E_1[x], \rho) - \Delta(A, \rho))^2 + (\text{tr}[\rho(E_1[x] - A)])^2.
\end{align*}
\]
The boundedness of \(\epsilon_{\text{NO}}(E_1, A, \rho)\) then gives:
\[
(\Delta(E[x], \rho) - \Delta(A, \rho))^2 + (\text{tr}[\rho E_1[x]] - \text{tr}[\rho A])^2 + (\Delta(E_1, \rho)^2 - \Delta(E_1[x], \rho)^2) \leq \epsilon_{\text{NO}}(E_1, A, \rho) \leq c^2.
\]
Each of the three bracketed terms is nonnegative and hence bounded above by \(c^2\). This implies:
\[
\begin{align*}
\Delta(E_1[x], \rho)^2 - c^2 &\leq \Delta(E_1, \rho)^2 \leq \Delta(E_1[x], \rho)^2 + c^2, \\
\Delta(A, \rho) - c &\leq \Delta(E_1[x], \rho) \leq \Delta(A, \rho) + c, \\
\text{tr}[\rho A] - c &\leq \text{tr}[\rho E_1[x]] \leq \text{tr}[\rho A] + c; \\
\end{align*}
\]
the first two inequalities taken together yield:
\[
\Delta(E_1, \rho)^2 \leq (\Delta(A, \rho) + c)^2 + c^2. \tag{17}
\]
Now observe that the variance on the l.h.s. is the variance of the distribution \(p := \rho^{E_1}\). We use the following variant of Chebyshev’s inequality, valid for any \(w > 0\):
\[
\Delta(p)^2 = \int (x - p[x])^2 p(dx) \geq \begin{cases} 
\int_{\mathbb{R} \setminus J_{q,w}} (x - p[x])^2 p(dx) \geq \left(\frac{w}{2} - |p[x] - q|\right)^2 (1 - p(J_{q,w})) & \text{if } p[x] \in J_{q,w} \\
\int_{J_{q,w}} (x - p[x])^2 p(dx) \geq \left(\frac{w}{2} - |p[x] - q|\right)^2 p(J_{q,w}) & \text{if } p[x] \notin J_{q,w}.
\end{cases}
\]
We will only be using cases of large \(w\) where \(p[x] \in J_{q,w}\) so that we obtain:
\[
\left(\frac{w}{2} - |p[x] - q|\right)^2 (1 - p(J_{q,w})) \leq \Delta(p)^2. \tag{18}
\]
Combining (17) and (18) yields:
\[
\left(\frac{w}{2} - |\rho^{E_1}[x] - q|\right)^2 (1 - \rho^{E_1}(J_{q,w})) \leq (\Delta(A, \rho) + c)^2 + c^2. \tag{19}
\]
We will only use this in the case of states \(\rho\) for which \(\rho^{E_1}(J_{q,\delta}) = 1\). In this case we have \(\Delta(A, \rho) \leq \delta\) and \(|\text{tr}[\rho A] - q| \leq \delta\), and using (16) we also obtain:
\[
|\rho^{E_1}[x] - q| \leq |\rho^{E_1}[x] - \text{tr}[\rho A]| + |\text{tr}[\rho A] - q| \leq c + \delta.
\]
We will also use only large (finite) \(w\) so that we can assume
\[
\frac{w}{2} - |\rho^{E_1}[x] - q| \geq \frac{w}{2} - (\delta + c) > 0.
\]
Note that this entails, in particular, that $\rho^{E_1}[x] \in J_{q,w}$, so that the use of (19) is justified. Under these conditions (19) entails
\[
(\frac{w}{2} - (\delta + c))^2 (1 - \rho^{E_1}(J_{q,w})) \leq (\delta + c)^2 + c^2.
\]
Now, for any $\varepsilon$ one can choose $w$ large enough such that
\[
(\frac{w}{2} - (\delta + c))^2 = \frac{(\delta + c)^2 + c^2}{\varepsilon}
\]
Then (27) implies that $1 - \rho^{E}(J_{q,w}) \leq \varepsilon$. Moreover, since $\mathcal{W}_{\varepsilon,\delta}(E_1, E) \leq w$, we also have
\[
\mathcal{W}_{\varepsilon,\delta}(E_1, E) \leq \frac{2}{\sqrt{\varepsilon}} \sqrt{(\delta + c)^2 + c^2 + 2(\delta + c)},
\]
which in the limit $\delta \to 0$ yields
\[
\mathcal{W}_{\varepsilon}(E_1, E) \leq \left(1 + \frac{\sqrt{2}}{\sqrt{\varepsilon}}\right) 2\epsilon_{NO}(A, E_1).
\]

An interesting open question is whether finite global noise-based error also implies finite Wasserstein distances.

9. Conclusion

We have reviewed several measures of error and intrinsic unsharpness for measurements of position and momentum (or other observables supported on the real line) and given a detailed investigation of their properties and the relations between them. We then have studied criteria for approximate (joint) measurements of position and momentum, based on three different kinds of error measures: Wasserstein $\alpha$-distances, error bar width (with or without bias), and global noise-based error. We have established two inequalities relating Wasserstein $\alpha$-distance and global noise error to error bar width, respectively, and have concluded that the criterion of finite error bars is satisfied whenever the Wasserstein $\alpha$-distance or the global noise error is finite. Thus the criterion for approximate joint measurability of position and momentum in terms of finite error bars is the most general among the three. It is satisfied by all covariant phase space observables whereas for some of these observables the $\alpha$-distances or global noise errors may be infinite.

For each of the three types of error measures we have reviewed a universal joint-measurement uncertainty relation. Put in geometric terms, these relations state that the marginals $M_1, M_2$ of an observable $M$ on phase space cannot both be arbitrarily close to $Q, P$, respectively.

We also considered the resolution width of an observable on $\mathbb{R}$, introduced in [21], and posed the question under which assumptions on the quality of approximations for approximate joint measurements of position and momentum the resolution widths of the marginals obey a Heisenberg-type uncertainty relation.
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References

[1] P. Busch, P.J. Lahti, and R.F. Werner, Quantum root-mean-square error and measurement uncertainty relations. arXiv:1312.4393, 2013.
[2] M. Ozawa, Phys. Lett. A, 318:21–29, 2003.
[3] M.J.W. Hall, Phys. Rev. A, 69:052113/1–12, 2004.
[4] J. Erhart, S. Sponar, G. Sulyok, G. Badurek, M. Ozawa, and Y. Hasegawa, Nature Phys., 8:185–189, 2012.
[5] L.A. Rozema, A. Darabi, D.H. Mahler, A. Hayat, Y. Soudagar, and A.M. Steinberg, Phys. Rev. Lett., 109:100404, 2012.
[6] P. Busch, T. Heinonen, and P. Lahti, Phys. Rep., 452:155–176, 2007.
[7] P. Busch, P. Lahti, and R.F. Werner, J. Math. Phys., 55:042111, 2014.
[8] P. Busch, P. Lahti, and R.F. Werner, Phys. Rev. Lett., 111:160405, Oct 2013.
[9] P. Busch and D.B. Pearson, J. Math. Phys., 48:082103, 2007.
[10] R.F. Werner, Quant. Inform. Comput., 4:546–562, 2004.
[11] P. Lahti, M. Mączynski, and K. Ylinen, Rep. Math. Phys., 41(3):319–331, 1998.
[12] W. Stulpe, Classical Representations of Quantum Mechanics Related to Statistically Complete Observables. Wissenschaft und Technik Verlag, Berlin, 1997. Also available: quant-ph/0610122.
[13] A.S. Holevo, Probabilistic and Statistical Aspects of Quantum Theory. North-Holland Publishing Co., Amsterdam, 1982.
[14] R Werner, J. Math. Phys. 25:1404, 1984.
[15] G. Cassinelli, E. De Vito, and A. Toigo, J. Math. Phys., 44:4768–4775, 2003.
[16] J. Kiukas, P. Lahti, and K. Ylinen, J. Math. Anal. Appl., 319:783–801, 2006.
[17] C. Carmeli, T. Heinonen, and A. Toigo, J. Phys. A, 38:5253–5266, 2005.
[18] J.B.M. Uffink and J. Hilgevoord, Found. Phys., 15(9):925–944, 1985.
[19] H.J. Landau and H.O. Pollak, Bell System Tech. J., 40:65–84, 1961.
[20] J.B.M. Uffink, Measures of Uncertainty and the Uncertainty Principle. PhD thesis, University of Utrecht, 1990.
[21] C. Carmeli, T. Heinonen, and A. Toigo, J. Phys. A: Math. Theor., 40(6):1303–1323, 2007.
[22] T. Heinonen, P. Lahti, J.-P. Pellenpää, S. Pulmannova, and K. Ylinen, J. Math. Phys., 44(5):1998–2008, 2003.
[23] C. Villani, Optimal Transport: Old and New. Springer, 2009.
[24] H. Jylhä, The \( L^\infty \) optimal transport: infinite cyclical monotonicity and the existence of optimal transport maps. Calculus of Variations and Partial Differential Equations, pages 1–24, February 2014.
[25] S.K. Berberian, Notes on Spectral Theory. D. Van Nostrand Company, Princeton, New Jersey, 1966.
[26] D. M. Appleby, Int. J. Theor. Phys., 37(10):2557–2572, 1998.
[27] M. Ozawa, Ann. Phys., 311:350–416, 2004.