Analog Photonic Computing Engine as Approximate Partial Differential Equation Solver

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The class of partial differential equations are ubiquitous to a plurality of fields including science, engineering, economics and medicine, and require time-iterative algorithms when solved with digital processors. In contrast, analog electronic compute engines have demonstrated to outperform digital systems for special-purpose processing due to their non-iterative operation nature. However, their electronic circuitry sets fundamental challenges in terms of run-time and programmability-speed. Integrated photonic circuits, however, enables both analog compute-hardware while exploiting time parallelism known from optics, while leveraging wafer-scale dense integration. Here, we introduce a photonic partial differential equation solver based on a Manhattan mesh-grid network featuring symmetrical power splitters and arbitrary Dirichlet boundary conditions. Our design numerically and experimentally solves a second-order elliptic partial differential equation with over 97% accuracy against solutions computed through commercially available solvers, achieving a steady state solution in 16 ps and providing a pathway towards real-time, chip-scale, reconfigurable application-specific photonic integrated circuits (ASICs).

**Keywords:** Silicon Photonics, Nanophotonics, Integrated Photonics, Optical Computing, Optical Power Splitter, Partial Differential Equation, Analog Computing

1. Introduction

Partial differential equations (PDEs) represent a wide class of numerical problems spanning multiple scientific and technological fields such as material science, aerodynamics, thermodynamics, physics, chemistry, and even economics with applications in simulating mathematical models, optimizing designs, and formulating multi-variable phenomena. Current (can Neumann) processors solve PDEs through numerical methods involving iterative vector-matrix high-precision operations, which can be both power and
time-costly depending on the complexity and resolution of the problem. Parallelizing hardware (i.e. multi-core processing) also does not offer a significantly different path to accelerate PDEs due to the parallelism overhead and disadvantageous computation complexity scaling\(^1\).

In contrast, analog co-processors, a class of identical dedicated hardware arranged to solve a specific problem category in parallel, represent another compelling solution for ameliorating the computing pressure showing orders of magnitude higher energy and time efficiency\(^2\). Interestingly, concepts of analog co-processors have been perceived and demonstrated since the 1950’s, using continuous signal programmed by changing the interactions between its computing elements with minimum stored programs or algorithms\(^3\). Regarding PDEs, these analog co-processors use continuous signals to emulate the problem by adjusting the interactions between the computing elements, instead of simulating problems with digitalized approximations as is today’s standard. Without clocking limits the parallel process, analog co-processors can break the step-by-step computing manner and provide one-time (non-iterative) computing independent of the problem complexity.

In the context of optical signal processing, photonic integrated circuits (PIC) have established themselves as advanced solution for optical communication\(^4\), quantum information processing\(^5\), neuromorphic computing showing remarkably-reduce energy consumption and accelerate intelligence prediction tasks\(^6\)–\(^9\). Recently, inverse-designed metamaterial platform interfaced by photonics circuit showed the possibility of solving integral equations using monochromatic electromagnetic radiation\(^10\). In the field of PDE solvers, all-optical reconfigurable module based on micro-ring resonators can solve ordinary first and second-order temporal differential-equation\(^11,12\).

Here we experimentally demonstrate a Silicon Photonic Approximate Computing Engine (SPACE) and use it to solve two-dimensional second-order elliptic PDEs. The engine is based on an innovative Silicon Photonic Invertible Cloverleaf Crossing (SPICC) mesh, whose nodes split the incoming light evenly into three remaining directions emulating an optical version of Kirchhoff’s law in analogy to a uniform electronic resistive circuit that solves a PDE via finite difference method (FDM). Boundary conditions are set by controlling the spacial light input intensity and position, while each node’s readout is
time-parallelizable by taking a camera image of all output ports (i.e. grating couplers). We observe an PDE solution accuracy of 97% when compared to a simulated heat transfer problem with the same mesh resolution. This approach features reconfigurability of the input positions and boundary conditions with low-loss network interconnectivity of such distributed networks via PICs and ensures foundry-near cost scaling. This approach could easily adapt another active component, such as electro-optic modulators, photodetectors, and tunable photonic cavities like photonic crystals, to solve more complicated problems with heterogeneous grid and Neumann boundary condition.

2. Results

A representation of our approach for solving PDEs using propagating wave in PIC is shown in Fig. 1. A partial differential equation can be numerically resolved by discretization methods, which approximate the PDE as difference equations, in which finite differences method (FDM) approximate the derivatives. The domain where the PDE is defined is partitioned in both space and in time, and approximations of the

![Figure 1: Solving partial differential equations (PDEs) with coherent laser light in photonic integrated circuits. a) Analytic solution of a partial differential equation for the defined boundary conditions. b) The discretized solution of the same PDE using numerical methods (finite difference). The overplayed mesh denotes the discretization. c) Electrical resistor mesh network which is characteristic for the define PDE and implements a finite difference method. d) A photonic network which imitates the behavior of a lumped circuit obtaining approximate (~97% accuracy) discretized solutions to the PDE. The discretization step for each solver is considered the same (n=4) and the boundary conditions are applied as external bias voltage or optical power for the electrical and the photonic engines, respectively. Arrows indicate the approach adopted and the motivation of this research.](image)
solution are computed at space or time points by replacing derivatives with differential quotients. Similar to a truncated Taylor series, the numerical solution is affected by discretization error, committed by simplifying a differential operator with a difference operator. Therefore, key parameters when solving PDEs are accuracy, stability, and convergence of the FDM, which are all function of the discretization step, in other words, the finite numbers of point in which the entire domain is partitioned. Here, as a proof of principle, we select the following toy-problem; a 2D heat transfer problem represented by a Laplace’s equation at steady state with no extra signal input and which can be mathematically described by Eqn. 1. After applying the FDM to a mesh network, the central node $O_{i,j}$ can be represented by its four adjacent nodes (Eqn. 2), where $h_i$ is the mesh step that describes the discretization level of the problem in the analytical domain (Fig. 2a). Once the discretized mesh node is set with node-to-node correlation function $c_i$ approximate to a constant value when the equidistant mesh step $h$ is small enough, this Laplace’s equation can be relaxed and solved iteratively. However, this usually requires a large amount of compute power, memory, and scales exponentially as the problem size increasing the required accuracy.

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

(1)

$$\nabla^2 f \approx \frac{\varepsilon_1(f_1-f_0)}{h_1^2} + \frac{\varepsilon_2(f_2-f_0)}{h_2^2} + \frac{\varepsilon_3(f_3-f_0)}{h_3^2} + \frac{\varepsilon_4(f_4-f_0)}{h_4^2} \approx \frac{1}{h^2}(f_1 + f_2 + f_3 + f_4 - 4f_0)$$

(2)

As suggested more than 6 decades ago\textsuperscript{13}, electrical resistor networks can mimic finite difference mesh grids. Similar to the analytical model, electrical current $I_i$ and resistance $R_i$ can be mapped to such a node model ruled by Kirchhoff’s law and Ohm’s law written as $I_i=(V_i-V_o)/R_i$. Here, the current injected into the selected node always equals to the current leaving the node (Kirchhoff’s law), while the current splitting ratios to each direction will be automatically ‘adjusted’ by the intrinsic electrical potentials of each path (Fig. 2b). It is worth mentioning that, this model holds only if the circuit can be modeled with lumped components, in other words, if the equivalent signal wavelength (e.g. few tens of centimeters in GHz level) is orders of magnitude larger than the entire circuit. Using integrated photonics provides three immediate advantages over resistive networks. First, the absence of charge/discharge of the wires enables distributed communication and concurrent low power dissipation. Secondly, once the network is set, the picosecond delay is dominated by the time-of-flight of the photon within the PIC. Third, the amplitude and
phase of the traveling wave in the waveguides can be easily tuned using attojoule-efficient modulators through wisely engineered light-matter interactions\textsuperscript{14-17}. Although solving PDEs with FDM and PICs has limitations too; the lumped model condition, in fact, is not valid for PICs operating in the telecommunication wavelength, and due to the distributed nature of the photonic network, mapping a finite-difference circuit applied to a partial differential onto photonic networks is not trivial. Even if in principle, photonics-based platforms based on FDM mesh grids are unable to guarantee an accurate result, we demonstrate that SPACE can provide a sufficiently accurate answer, as an approximate solver, depending on the node to node relations (i.e. optical loss) and the splitting ratios at each node, which determines the light coupling amongst nodes.

Therefore, with this in mind, here we show an initial demonstration of a photonic-based PDE solver to emulate a second-order elliptic PDE. The discussion includes a method for obtaining a resistive-like network behavior and guidelines on building a PDE-solver engine based on photonic circuitry which can effortlessly and non-iteratively solve Laplace PDEs with a high degree of accuracy. The proposed engine concept and our experimental prototype can be expanded upon for other PDE types, by wisely modifying this approach. For instance, one may explore the solution of time-dependent PDE, or parabolic and hyperbolic PDE, using high-speed photodetector, as well as nonlinear PDE by introducing optical nonlinear photonic elements in the network, such as reverse saturable absorbers\textsuperscript{18}, phase change materials\textsuperscript{19} or quantum assemblies\textsuperscript{20}.

The main difficulty in photonic FDM is the lack of lumped features, hence special designs are required to implement a PIC-based FDM mesh network. Considering a two-dimensional mesh, SPACE requires an equal light intensity distribution into all three (remaining) cardinal directions. Each mesh-grid point representing a node consists of directional couplers integrated with ring resonators to perform this light distribution (Fig. 2c). Moreover, due to the distributed nature of the network, we consider the optical power rather than potential energy in analogy to previously electrical FDM analog solvers. The optical loss along each light path in the mesh acts as an equivalent resistances $R_i$ in the electrical model. To achieve equal light slitting, this node design needs to meet the following three criteria. a) The splitter needs to be symmetrical to both x- and y-axis in order to physically build the scalable optical mesh, and needs to provide a 1-to-3 equal
splitting ratio; b) the splitter needs to have tolerance to the fabrication variance since cascading the node will amplify the device variance; c) the segment for light coupling should have the potential to be further integrated with tuning mechanisms (e.g. electro-optic means) in order to ensure reconfigurability of the optical FDM model. This also enables compensating the non-lumped element effect.

The SPICC design considers all the above requirements and includes further chip footprint minimization for a) dense integration, b) smaller FDM discretization step, and c) ideally a higher solution accuracy. To design SPICC we use heuristic process a 1-3 equal power splitter in photonics, and iteratively optimizing the splitting ratio using 3D full-wave numerical simulations (Fig. 3). Nevertheless, other optimized approaches based on inverse design algorithm\textsuperscript{21–23} could lead to a more accurate splitting without trading off in terms of footprint.

**Figure 2. Schematic of Finite Difference Method components** adapted in a) analytical domain, b) electrical domain such as a resistor mesh network, c) photonic integrated circuit which implements four waterdrop-like ring resonators to split light into three ways evenly and one center waveguide crossing to avoid waveguide crosstalks and scattering.
The resulting design comprises of four water-drop shaped rings placed close to two perpendicularly crossed waveguides to couple part of the light coming from one direction into both, the other two perpendicular directions and still let the remaining light pass through to the opposite port (Fig. 3a, the theoretical proof of this design is discussed in the supplementary online material). Instead of using circular rings, the segments close to the straight waveguides are flattened to form a three-waveguide directional coupler. We used directional couplers to couple into the 4 drop-like feedback loops, and refrain from using neither perfectly circular rings and nor high-quality factor cavities to widen the spectral (and thermal) operating window such as to not having to use tuning (e.g. thermal, electro-optic) to control its resonance. In addition, a 4-way waveguide crossing is the center of each SPICC node to reduce the scattering and crosstalk at the intersection (see SOM). After optimizing the bending radii of the water-drop shaped rings, flattened coupler length and the gap between the ring and the straight waveguide, the splitting ratio can be tuned to 22%, 23% and 22% with 12% reflection (Fig. 3b) based on full-wave simulation (Lumerical 3D FDTD). Here the reflection is mainly caused by the return couplings from the three-waveguide couplers at the perpendicular ports (i.e. Output port 1 and 3). Instead of completely coupling to the perpendicular port, the light coming from the first two rings will be partially leaked to the rings on the other side and route the signal back to the input port. In addition, we demonstrate that cutting the shaded area of the rings in Fig. 3b for avoiding back reflection by only using the bending section for light coupling is not a viable alternative since it would increase the port-to-port loss by at least 25% because of the three
tightly coupled fundamental TM supermodes, which would hinder the propagation in larger networks. In terms of the fabrication process, 2% hydrogen silsesquioxane (HSQ) electron-beam resist is used due to its fine resolution and edge contrast, with isotropic dry etching process (SF6 and C4F8) to get the uniform height profile and vertical sidewall profile. More details related to the fabrication are given in the Method section.

In order to test the fabricated device, a 1550 nm continuous wave (CW) laser is used as light source which is coupled to the waveguide by means of a periodical grating coupler (Details regarding the grating coupler are discussed in the Supplementary Online Material). To read out the output value, an InGaAs infrared camera (Xenics 14-bit 2048-level) is used to capture the microscope image of the light outcoupled in each direction. The background noise, such as the arbitrary reflections from the sample surface and ambient light, was minimized using noise-canceling method and post image processing, thence the light intensity from the grating coupler regions in each direction was acquired (More details are provided in the Method section). As the result, the light intensity from the back-reflection port and all three output ports have a ratio of 11.3% : 22.3% : 23% : 22.1% (or 883.2 : 1744.5 : 1801.6 : 1727.2 from the image pixel readout) which is in excellent agreement with the FDTD simulation result. Nevertheless, we envision that high-speed, low noise germanium25–27 or graphene photodetectors28–31 can be integrated into the device and used for improving both detectability and data collection speed and accuracy.

Next, after providing a practical exhibition and guidelines for obtaining an FDM-like node in photonics, we cascade multiple nodes building a 5×5 optical FDM mesh grid to

![Figure 4](image-url) **Figure. 4. A 5×5 SPACE demonstration in solving heat transfer problem.** a) Schematic 3D demonstration of a heat transfer problem with light injected from the central left. Boundary condition is set by using extended waveguides and grating couplers connect to the peripheral nodes. Light can be dissipated without reflections thus can be regarded as a perfect constant temperature boundary condition. b) The block chart shows a top view of the initial setup of the heat transfer problem that can be solved by our 5×5 SPACE design. c) The microscope image captured by infrared camera at 1550 nm wavelength overlaying with a sketch of the optical power splitters. Note, grating couplers, y-branches and bending waveguides are omitted for better visualization.
solve a discretized heat transfer problem. The assembled system maps a symmetric type of heat transfer problem with a heat source injected from the center-left of the mesh grid and surrounded by constant temperature boundary conditions that absorb the heat entering these sections (Fig. 4b). The input signal, which in this case represents the Dirichlet’s boundary conditions may, in general, be any arbitrary laser beam distribution shone on any grating coupler in the circuit. To characterize the performance of the system and obtaining discretized measurements for each node, first, we introduce for each direction of the nodes a set of 50/50 Y-branch splitter followed by a grating coupler in order to estimate the optical power at each node. The power drop at each node represents the heat distribution at each point of the discretized domain, and it is measured, as previously observed, in parallel through a properly calibrated infrared camera using the same microscope magnification to insure the same projected image size on the sensor (Fig. 4c).

In order to get readable data from the furthest node from the input, 39 mW of laser input (as the maximum power output from our laser source) has been applied to the 5×5 SPACE mesh grid. Considering the polarization of the grating coupler and its coupling efficiency, the actual laser power coupled into the mesh is less than 5 mW, which is still far below the nonlinearity energy density limitation of the Silicon Photonic waveguide. Furthermore, it is important to mention that considering the photonic node size, the SPACE engine can be packed with a minimum density of 25 µm/component, although we separate the nodes with 200 µm spacing for less output crosstalk when measuring. This solution induces additional losses (i.e. 1 dB of loss in this case by using waveguide bending) between two adjacent nodes, mirroring a linear node-to-node correlation function $e_i$.

In other words, the light power is split equally in all its propagation directions (i.e. north, south, and east), subsequently half of its power is scattered by the grating couplers, hence collected by the camera, and the rest injected into the neighboring node after experiencing the designated loss in the waveguide bending and three-waveguide coupling.

We verify the accuracy of the approximate solution of the 5×5 SPACE prototype by comparing the obtained experimental measurement of the discretized solutions to four models simulated with COMSOL Multi-physics and Lumerical Interconnect with the same mesh resolution and density. These models include 1) a COMSOL normalized heat transfer problem which can be represented by an FDM network of 1 Ohm resistor mesh-grid; 2) a
photonic integrated circuit simulation of a network which comprises node characterized by a splitting ratio set to 0%: 33%; 33%; 33% with no reflection and no loss, which can be considered as the perfect optical splitter design and the optical accuracy upper bond; 3) a PIC simulation of the engine with a node splitting ratio set to the 3D FDTD simulation result, which can be regarded as the theoretical or expected accuracy that our proposed optical power splitter design could get; 4) a PIC simulation with the experimentally measured node splitting ratio from fabricated single node on the same wafer, which accounts for fabrication variance of the photonic components. In all PIC simulations, the input light source has been set to 1 mW with optical power meter sensitivity set to -100 dBm and simulation time long enough to converge all the signal propagation delays in the network. Despite the different input types (i.e. electrical voltage and optical power) and values (i.e. 1 V in the electrical model, 1 mW optical power in all-optical simulations, and 39 mW optical power in the measurement), normalized data with the same range is able to represent identical heat transfer distribution in the discretized domain when the same equivalent node-to-node correlation functions \( \varepsilon \) are met.

Here, the model simulated by COMSOL is served as the baseline for the others. Note, although it is proverbially that the accuracy of the COMSOL based simulation is proportional to the mesh resolution (e.g. a 5×5 mesh COMSOL simulation has 99.95% accuracy comparing to a 300×300 mesh averaged down to a 5×5 with same initial setup), a 5×5 discretized model is selected to make a fair comparison with other optical 5×5 models simulated and measured (Fig. 5a). Moreover, the error at each of the 9-computing node has been plotted in Fig. 5b with the minimum node accuracy noted by negative error bars in Fig. 5a. Moving towards a more experimentally meaningful simulation, the accuracy of the 5×5 FDM model can still be maintained at very high level (i.e. above 97.5%) for approximate calculations.
Here in both equal splitting node and FDTD node error profiles, symmetric error distribution is observed due to the balanced splitting nature of the nodes and follows with a column-by-column pattern (Fig. 5b-i and 5b-ii). In other words, consider each column as a whole block with sufficient number of nodes, then the incoming light from the previous block is basically a truncated Taylor series of the input factored by the splitting ratio (assume a uniform equal splitting ratio) and the input to the next block is also the same Taylor series but starting from its second order. Thus, a higher splitting ratio will lead to passing more light to the next block with greater errors. And this explains the accuracy of the second column drop slightly when the splitting ratio reduces from 33% (equal node model) to 23% (FDTD node model). Notice that the errors in the first and third column are significantly smaller than the center column due to different reasons. In the first column, the only node accepting input is located at the center. Because of the symmetrical topology of the circuit, this node supposes to have the equal splitting potential to all its three connections. However, in the last column, the light signal arriving at this block is already minimum (starting from the third order of the Taylor series), thus the difference in the splitting ratio could not yield major impact to the result.

**Figure 5. Electrical and optical 5×5 FDM model accuracy after normalization.** a) The averaged error and accuracy comparison among COMSOL simulated electrical model (COMSOL (5×5)), perfect splitting optical model (equal node), FDTD design simulation model (FDTD node), measured single component model (single node), and the 5×5 SPACE measurement model (SPACE). The negative error bars represent the accuracy level from the least accuracy node from the 5×5 FDM model. The electrical model is regarded as the baseline and scaled to 100% accuracy. From the left to right shows the step-by-step approach to the most physical model while the accuracy drops from 98.5% to 97.5%. b) Normalized error heatmap between the baseline model and other models in the scale of (-0.05, 0.05).
In the emulation, which considers the single node and SPACE error profiles, a random error distribution is observed across the light propagation direction (Fig. 5b-iii and 5b-iv). This can be explained as the deviations caused by fabrication variances, such as the grating coupler efficiency, y-branch splitting ratio, and directional coupler gap. Even with a small difference in the splitting ratio, as previously demonstrated for electrical networks\textsuperscript{13}, the error distribution which affects the solution of the SPACE engine is less significant compared to the error which affects the solution of the numerical simulation considering the experimentally derived or numerically computed transfer function of the single node. This is again due to fabrication variances (e.g. resistance tolerance in the resistive network) in the engine which cancels out after the light iterates in the network for several times.

This fabrication robustness could be regarded as an additional benefit of using Silicon Photonic circuit for solving PDEs. Although iterations are still needed in SPACE to allow the light to smooth out in the circuit. Based on PIC simulations performed in Lumerical Interconnect, a full iteration cycle (dominated by the time of flight of the photon) takes only 30 ps considering 100 µm node-to-node (n2n) spacing for a 5×5 SPACE in this sparse design. The iteration time drops to 16 ps with 25 µm n2n spacing which is the minimum packing density of our current design (Fig. 6) achievable when on-chip integrated photodetectors are used as a detection mechanism. It is also worth to mention that, as an approximate computing engine, when the target accuracy relaxed to 90% of its maximum accuracy, the iteration time drops to 1.8 ps which is equivalent to 556 GHz. In terms of the

![Figure 6. Runtime analysis on different network scales from 5×5 to 10×10 with different node-to-node distance varying from 25 µm to 100 µm. Both full accuracy and 90% accuracy runtime show exponential increase in the runtime mainly caused by the node-to-node distance. With closest packing (25 µm), full accuracy and 90% accuracy are able to provide 63 and 556 GHz operating speed respectively. The full accuracy and 90% accuracy are respected to the maximum accuracies that each network scale could get.](image-url)
scalability, the runtime time starts to saturate as the network size scales from 5×5 to 10×10, therefore the light propagation time in the waveguide will contribute even less to the total runtime, thus proving that SPACE lends itself particularly well to further up-scaling.

Our SPICC design and SPACE circuit provide a powerful tool for homogenously to distribute optical power in a defined network similarly to a lumped circuit subjected to Kirchhoff’s Law. When the correlation function between each node of the network is wisely selected or actively tuned via Electro-optic modulation, the network can solve through finite difference approach a general second-order Laplace’s PDE in an analog manner. For instance, if the splitting ratio of nodes adjacent to the boundary conditions could be tuned to 10%: 10%: 80% (i.e. more light routes to the boundaries), a 5×5 modulated SPACE is able to improve the accuracy up to 99.2%.

Similar configurations may potentially be explored to solve time-dependent or nonlinear PDE with arbitrary boundary conditions by introducing time discretization or nonlinear elements, respectively.

Beyond that, Poisson’s equation could also be emulated with the same design if additional light sources are added to the nodes, mimicking the different node potential. Nonetheless, other PDEs like diffusion equations and wave equations would require optical capacitive and inductive elements needed to express the time-dependent variances enabling the one-shot solution.

Different from emulating an optical resistor, which can be easily realized by optical lossy materials or electro-optical modulators, optical capacitors and inductors require specific designs to mimic the behavior of their electrical counterpart. For example, a Fabry-Perot interferometer with chirped Bragg gratings have been demonstrated as an optical capacitive component which can act like a broadband low pass or high pass filter. On the other hand, an optical inductive component can be implemented as a self-electro-optical device with both integrated modulator and detectors that use the photocurrent to back feed the modulator and change the light intensity injected into the detection region as a negative feedback loop.

PDEs applied to non-homogeneous domains can be mapped on SPACE by changing the splitting ratio according to characteristic distribution (e.g. different thermal conductivity mapped as different attenuation for each individual node). Other cases of PDE applied to
non-symmetrical or inhomogeneous domains are reported as PIC numerical simulation in
the Supplementary Online Material.

3. Conclusions

In conclusion, we propose the designs of a photonic network able to replicate the
functionality of a lumped circuit model and incidentally, by using a finite difference
approach, solve partial differential equation effortlessly and noniteratively. Our numerical
and experimental analysis indicates that the steady-state response of a characteristic
SPACE engine may be achievable in 16 ps, obtaining inherently discretized solutions for
each point of the mesh of the domain with a bandwidth up to 63 GHz completely. This
approach could easily adapt another active component, such as electro-optic modulators,
photodetectors, and tunable photonic cavities like photonic crystals, to solve more
complicated problems with heterogeneous grid and Neumann boundary condition. And the
platform may also provide potentiality of reconfigurability, allowing to map multiple PDEs
and domain types, by using energy-efficient electro-optic modulators. Our findings
provided a novel pathway to ultrafast, integrable, and reconfigurable photonic analog
computing engine\textsuperscript{5,13,37-40} based on integrated photonics which can be used as an
approximate solver of partial differential equations.

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Methods

Fabrication Process

All of the single optical power splitters and 5×5 SPACE is fabricated on the same 220 nm Silicon on Insulator
(SOI) chip to minimize the variance during the fabrication process. Raith Voyager 50kV E-beam lithography
system is used with fix beam moving stage (FBMS) feature to allow zero waveguide stitching errors across
multiple write fields. Hydrogen silsesquioxane (HSQ) with 2% concentration is used to provide around 42
nm of mask thickness (4000 rpm for 60 seconds) with high resolution in writing. After the spin coating, the
chip is put on a hotplate for 240 seconds pre-bake at 80-degree centigrade. After the patterning, the chip is
dipped into MF-319 for 70 seconds to develop the unexposed HSQ area including 5 seconds of gentle stirring
to shake off the air bubbles of the chemical reaction. Then 30 seconds of D.I. water rinse will be immediately
applied to stop the development and clean up the residue. To etch down the silicon layer and reveal the
features, a 28 seconds of SF\textsubscript{6} and C\textsubscript{4}F\textsubscript{8} (both at 10 sccm) at 500 W ICP power and 20 W bias etching with
Plasma-Therm Apex SLR Inductively Coupled Plasma Etcher is able to fully etch all the silicon down and provide over 9:1 selectivity for our smallest features.

**Measurement and Data Processing**

To measure the output light intensity, an optical probe station setup is used with a tunable laser at 1550 nm wavelength connecting to a lens fiber to maximize the light coupled onto the chip. Xenics IR camera integrated with the microscope captures the scattering light at each output grating. In addition, a black light shield is applied to cover the entire camera, probe station and microscope to prevent the ambient light. And the thermal noise of the camera is eliminated by capturing the image with no laser input. The last type of noise taken into account in the measurement is the surface reflection including the lens flare, and this is by substituting the averaged background readout that adjacent to the grating coupler. After the noise cancelation, the images are imported into Matlab to integrate the intensity values (0～4095 for our 12-bit depth sensor) of all the pixels of the output region. It is also worth to mention that nodes at different positions have over 3 orders of magnitude difference which is far beyond the dynamic range of the camera. Therefore, lower input laser power with shorter camera integration time is used for nodes closer to the input node and post-processed into the same scale.

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