The Core of a 2-Dimensional Set-Valued Mapping.
Existence Criteria and Efficient Algorithms
for Lipschitz Selections of Low Dimensional Set-Valued Mappings

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Abstract
Let \( \mathcal{M} = (M, \rho) \) be a metric space and let \( X \) be a Banach space. Let \( F \) be a set-valued mapping from \( M \) into the family \( \mathcal{K}_m(X) \) of all compact convex subsets of \( X \) of dimension at most \( m \). The Finiteness Principle [14] provides efficient conditions for the existence of a Lipschitz selection of \( F \), i.e., a Lipschitz mapping \( f : M \to X \) such that \( f(x) \in F(x) \) for every \( x \in M \).

For \( m = 2 \) and \( X = \mathbb{R}^2 \), and for \( m = 1 \) and arbitrary \( X \), we give alternative proofs of the Finiteness Principle for Lipschitz selections. These proofs rely on a simple iteration formula for the “core” of a set-valued mapping \( F \), i.e., for a mapping \( G : M \to \mathcal{K}_m(X) \) which is Lipschitz with respect to the Hausdorff distance, and such that \( G(x) \subset F(x) \) for all \( x \in M \).

We also present several constructive criteria for the existence of Lipschitz selections of set-valued mappings from \( M \) into the family \( \mathcal{HP}(\mathbb{R}^2) \) of all closed half-planes in \( \mathbb{R}^2 \).

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1. Introduction.

Let \( \mathcal{M} = (\mathcal{M}, \rho) \) be a pseudometric space, i.e., \( \rho : \mathcal{M} \times \mathcal{M} \to [0, +\infty] \), \( \rho(x, x) = 0 \), \( \rho(x, y) = \rho(y, x) \), \( \rho(x, y) \leq \rho(x, z) + \rho(z, y) \) for all \( x, y, z \in \mathcal{M} \). Note that \( \rho(x, y) = 0 \) may hold with \( x \neq y \), and \( \rho(x, y) \) may be \( +\infty \).

Let \( (X, \| \cdot \|) \) be a Banach space. Given a non-negative integer \( m \) we let \( \mathcal{K}_m(X) \) denote the family of all nonempty compact convex subsets \( K \subset X \) of dimension at most \( m \). (We say that a convex subset of \( X \) has dimension at most \( m \) if it is contained in an affine subspace of \( X \) of dimension at most \( m \).) We let \( \mathcal{K}(X) = \cup \{ \mathcal{K}_m(X) : m = 0, 1, \ldots \} \) denote the family of all nonempty compact convex finite-dimensional subsets of \( X \).

By \( \text{Lip}(\mathcal{M}, X) \) we denote the space of all Lipschitz mappings from \( \mathcal{M} \) to \( X \) equipped with the Lipschitz seminorm

\[
\| f \|_{\text{Lip}(\mathcal{M}, X)} = \inf \{ \lambda > 0 : \| f(x) - f(y) \| \leq \lambda \rho(x, y) \text{ for all } x, y \in \mathcal{M} \}.
\]

In this paper we study the following problem.

**Problem 1.1** Suppose, we are given a set-valued mapping \( F \) which to each point \( x \in \mathcal{M} \) assigns a set \( F(x) \in \mathcal{K}_m(X) \). A selection of \( F \) is a map \( f : \mathcal{M} \to X \) such that \( f(x) \in F(x) \) for all \( x \in \mathcal{M} \).

We want to know whether there exists a selection \( f \) of \( F \) in the space \( \text{Lip}(\mathcal{M}, X) \). Such an \( f \) is called a Lipschitz selection of the set-valued mapping \( F : \mathcal{M} \to \mathcal{K}_m(X) \).

If a Lipschitz selection \( f \) exists, then we ask how small we can take its Lipschitz seminorm.

See Fig. 1.

The following “Finiteness Principle” provides efficient conditions for the existence of a Lipschitz selection of an arbitrary set-valued mapping from a pseudometric space into the family \( \mathcal{K}_m(X) \).

**Theorem 1.2** (Fefferman, Shvartsman [T4]) Fix \( m \geq 1 \). Let \( (\mathcal{M}, \rho) \) be a pseudometric space, and let \( F : \mathcal{M} \to \mathcal{K}_m(X) \) for a Banach space \( X \). Let

\[
N(m, X) = \min\{2^{m+1}, 2^{\dim X}\}. \tag{1.1}
\]

Suppose that for every \( \mathcal{M}' \subset \mathcal{M} \) consisting of at most \( N = N(m, X) \) points, the restriction \( F|_{\mathcal{M}'} \) of \( F \) to \( \mathcal{M}' \) has a Lipschitz selection \( f_{\mathcal{M}'} \) with Lipschitz seminorm \( \| f_{\mathcal{M}'} \|_{\text{Lip}(\mathcal{M}', X)} \leq 1 \).

Then \( F \) has a Lipschitz selection \( f \) with Lipschitz seminorm \( \| f \|_{\text{Lip}(\mathcal{M}, X)} \leq \gamma \) where \( \gamma = \gamma(m) \) is a positive constant depending only \( m \).
Fig. 1: $f : M \to \mathbb{R}^2$ is a Lipschitz selection of the set-valued mapping $F : M \to \mathcal{K}(\mathbb{R}^2)$.

There is an extensive literature devoted to the Finiteness Principle for Lipschitz selection and related topics. We refer the reader to the papers [1, 3, 5, 11–14, 20–22, 24–28] and references therein for numerous results in this direction.

We note that the “finiteness number” $N(m, X)$ in Theorem 1.2 is optimal; see [24, 26].

If the set $M$ is finite, then we can omit the assumption that the convex sets $F(x)$ ($x \in M$) are compact [14, Theorem 6.2].

For the case of the trivial distance function $\rho \equiv 0$, Theorem 1.2 agrees with the classical Helly’s Theorem [8], except that the optimal finiteness constant for $\rho \equiv 0$ is

$$n(m, X) = \min\{m + 2, \dim X + 1\} \quad \text{in place of} \quad N(m, X) = \min\{2^{m+1}, 2^{\dim X}\}.$$  

Thus, Theorem 1.2 may be regarded as a generalization of Helly’s Theorem.

One of the main ingredients of the proof of Theorem 1.2 is the construction of a special set-valued mapping $G : M \to \mathcal{K}_m(X)$ which we call a “core” of the set-valued mapping $F$. This notion has been introduced in [14]. Below we give its quantitative version.

**Definition 1.3** Let $\gamma$ be a positive constant, and let $F : M \to \mathcal{K}_m(X)$ be a set-valued mapping. A set-valued mapping $G : M \to \mathcal{K}_m(X)$ is said to be a $\gamma$-core of $F$ if

(i). $G(x) \subset F(x)$ for all $x \in M$;

(ii). $G$ is $\gamma$-Lipschitz with respect to the Hausdorff distance, i.e.,

$$d_H(G(x), G(y)) \leq \gamma \rho(x, y) \quad \text{for all} \quad x, y \in M.$$  

We refer to $G$ as a core of $F$ if $G$ is a $\gamma$-core of $F$ for some $\gamma > 0$.

See Fig. 2, 3, 4. Recall that the Hausdorff distance $d_H(A, B)$ between two nonempty bounded sets $A, B \subset X$ is defined as the least $r \geq 0$ such that for each $x \in A$ there exists $y \in B$ such that $\|x - y\| \leq r$, and for each $x \in B$ there exists $y \in A$ such that $\|x - y\| \leq r$. Thus,

$$d_H(A, B) = \inf\{r > 0 : A + B_X(0, r) \supset B \quad \text{and} \quad B + B_X(0, r) \supset A\}. \quad (1.2)$$

Here, given $x \in X$ and $r > 0$ by $B_X(x, r)$, we denote a ball in $X$ with center $x$ and radius $r$. By $B_X = B_X(0, 1)$ we denote the unit ball in $X$, and we write $rB_X$ to denote the ball $B_X(0, r)$.
Fig. 2: A set-valued mapping $F$ into a family of avocados and its Lipschitz selection with Lipschitz seminorm at most $\lambda$.

Fig. 3: The core $G(x)$ is a convex closed subset of $F(x)$.

Fig. 4: The $\gamma$-core $G$ is $\gamma$-Lipschitz with respect to the Hausdorff distance.
Note that in this terminology any Lipschitz selection $f$ of $F$ with Lipschitz seminorm at most $\gamma$ is a 0-dimensional $\gamma$-core of $F$. The next claim shows that a set-valued mapping $F : \mathcal{M} \to \mathcal{K}_m(X)$ has a Lipschitz selection provided $F$ possesses a core (of any dimension).

**Claim 1.4** Let $\gamma$ be a positive constant, and let $G : \mathcal{M} \to \mathcal{K}_m(X)$ be a $\gamma$-core of a set-valued mapping $F : \mathcal{M} \to \mathcal{K}_m(X)$. Then $F$ has a Lipschitz selection $f : \mathcal{M} \to X$ with $\|f\|_{\text{Lip}(\mathcal{M},X)} \leq C\gamma$ where $C = C(m)$ is a constant depending only on $m$.

This claim is immediate from Definition 1.3 and the existence of the so-called “Steiner-type point” map $\text{St} : \mathcal{K}_m(X) \to X$ (Theorem 2.6). See also [14] and Section 2 for more detail.

In [14] given $\lambda > 0$ and a set-valued mapping $F : \mathcal{M} \to \mathcal{K}_m(X)$ satisfying the hypothesis of Theorem 1.2 we construct a $\gamma$-core $G$ of $F$ with $\gamma = \lambda C\lambda$ where $C = C(m)$ is a positive constant depending only on $m$. We produce the core $G$ using a rather delicate and complicated procedure whose main ingredients are families of Basic Convex Sets associated with $F$, metric spaces with bounded Nagata dimension, ideas and methods of work [11] related to the case $\mathcal{M} = \mathbb{R}^n$, Lipschitz selections on finite metric trees. See [14] for more detail.

In the present paper we suggest and discuss a certain new geometrical method for producing a core of a set-valued mapping. Its main ingredient is the so-called a balanced refinement of a set-value mapping which we define below.

**Definition 1.5** Let $\lambda \geq 0$, and let $F : \mathcal{M} \to \mathcal{K}_m(X)$ be a set-valued mapping. Given $x \in \mathcal{M}$ we set

$$\mathcal{BR}[F : \lambda; \rho](x) = \bigcap_{z \in \mathcal{M}} \left[ F(z) + \lambda \rho(x, z) B_X \right]. \quad (1.3)$$

We refer to the set-valued mapping $\mathcal{BR}[F : \lambda; \rho] : \mathcal{M} \to \mathcal{K}_m(X) \cup \{\emptyset\}$ as a $\lambda$-balanced refinement of the mapping $F$.

We note that any Lipschitz selection $f$ of a set-valued mapping $F : \mathcal{M} \to \mathcal{K}_m(X)$ with $\|f\|_{\text{Lip}(\mathcal{M},X)} \leq \lambda$ is simultaneously a Lipschitz selection of the $\lambda$-balanced refinement of $F$, i.e.,

$$f(x) \in \mathcal{BR}[F : \lambda; \rho](x) \quad \text{for all} \quad x \in \mathcal{M}.$$  

(This property is immediate from the definition of a Lipschitz selection and formula (1.3).) Definition (1.3) also tells us that

$$\mathcal{BR}[F : \lambda; \rho](x) \subset F(x) \quad \text{for every} \quad x \in \mathcal{M}.$$  

The set $\mathcal{BR}[F : \lambda; \rho](x)$ may have much more smaller geometrical sizes (such as the diameter, width etc.) than $F(x)$. Thus, in general $\mathcal{BR}[F : \lambda; \rho]$ provides a much more concentrated area for potential Lipschitz selections of $F$.

Of course, the same property holds for consecutive iterations of balanced refinements of $F$. This observation motivates the following definition.

**Definition 1.6** Let $\ell \in \mathbb{N} \cup \{+\infty\}$ be a positive integer (possibly equal to $+\infty$), and let $\vec{\lambda} = \{\lambda_k\}_{k=1}^\ell$ be a sequence with non-negative terms $\lambda_k$. We set $F^{(0)} = F$, and, for every $x \in \mathcal{M}$ and $0 \leq k \leq \ell - 1$, we put

$$F^{(k+1)}(x) = \mathcal{BR}[F^{(k)} : \lambda_{k+1}; \rho](x) = \bigcap_{z \in \mathcal{M}} \left[ F^{(k)}(z) + \lambda_{k+1} \rho(x, z) B_X \right]. \quad (1.4)$$

We refer to the set-valued mapping $F^{(k)} : \mathcal{M} \to \mathcal{K}_m(X) \cup \{\emptyset\}, 1 \leq k \leq \ell$, as the $k$-th order $(\vec{\lambda}, \rho)$-balanced refinement of $F$. 

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Clearly,
\[ F^{[k+1]}(x) \subset F^{[k]}(x) \quad \text{on } M \quad \text{for every } 0 \leq k \leq \ell - 1. \quad (1.5) \]

(Put \( z = x \) in the right hand side of (1.4).)

**Remark 1.7** Of course, the sets \( F^{[k]}(x) \), \( k = 1, \ldots, \ell \), also depend on the vector \( \vec{\lambda} = \{ \lambda_1, \ldots, \lambda_\ell \} \), pseudometric space \( \mathcal{M} = (M, \rho) \) and the Banach space \( X \). However, we use \( F^{[k]} \)'s only in this section and Sections 3-7 where these objects, i.e., \( \vec{\lambda}, \mathcal{M} \) and \( X \), are clear from the context. Therefore, in these cases, we omit \( \vec{\lambda}, \mathcal{M} \) and \( X \) in the notation of \( F^{[k]} \)'s.

Sometimes, for the special case of a constant sequence \( \vec{\lambda} = \{ \lambda, \lambda, \ldots, \lambda \} \) where \( \lambda \) is a positive constant, we will use the notation \( F^{[k]}_\lambda \) for the set \( F^{[k]} \). Thus, thanks to (1.4),
\[ F^{[k+1]}_\lambda(x) = \bigcap_{z \in M} \left[ F^{[k]}_\lambda(z) + \lambda \rho(x, z) B_X \right], \quad \subset \quad (1.6) \]

We formulate the following

**Conjecture 1.8** Let \( (\mathcal{M}, \rho) \) be a pseudometric space, and let \( X \) be a Banach space. Let \( m \in \mathbb{N} \).

There exist constants \( \ell = \ell(m) \in \mathbb{N}, \gamma = \gamma(m) \geq 1 \), and a sequence
\[ \vec{\lambda} = \{ \lambda_1(m), \ldots, \lambda_\ell(m) \} \quad \text{with} \quad \lambda_k \geq 1, \quad k = 1, \ldots, \ell, \]

such that the following holds:

Let \( F : \mathcal{M} \to \mathcal{K}_m(X) \) be a set-valued mapping such that for every subset \( \mathcal{M}' \subset \mathcal{M} \) with \( \# \mathcal{M}' \leq N(m, X) \), the restriction \( F|_{\mathcal{M}'} \) of \( F \) to \( \mathcal{M}' \) has a Lipschitz selection \( f_{\mathcal{M}'} : \mathcal{M}' \to X \) with Lipschitz seminorm \( \| f_{\mathcal{M}'} \|_{\text{Lip}(\mathcal{M}', X)} \leq 1 \). Then the set-valued mapping
\[ F^{[\ell]} : \mathcal{M} \to \mathcal{K}_m(X) \quad \text{is a} \quad \gamma \text{-core of} \quad F. \]

Our main results, Theorem 1.9 and Theorem 1.10 below, state that Conjecture 1.8 holds whenever (i) \( m = 2 \) and \( \dim X = 2 \), or (ii) \( m = 1 \) and \( X \) is an arbitrary Banach space. Note, that in both cases the finiteness number
\[ N(m, X) = \min\{2^{m+1}, 2^{\dim X}\} = 4. \]

See (1.1).

**Theorem 1.9** Let \( \mathcal{M} = (\mathcal{M}, \rho) \) be a pseudometric space. Let \( X \) be a two dimensional Banach space and let \( m = 2 \). In this case Conjecture 1.8 holds for \( \ell = 2 \) and for every \( \lambda_1, \lambda_2 \) and \( \gamma \) such that
\[ \lambda_1 \geq e(\mathcal{M}, X), \quad \lambda_2 \geq 3\lambda_1, \quad \gamma \geq \lambda_2 (3\lambda_2 + \lambda_1)^2 / (\lambda_2 - \lambda_1)^2. \quad (1.7) \]

Here \( e(\mathcal{M}, X) \) denotes the Lipschitz extension constant of \( X \) with respect to \( \mathcal{M} \). (See Definition 3.2.)

Thus, the following statement is true: Let \( F : \mathcal{M} \to \mathcal{K}(X) \) be a set-valued mapping from a pseudometric space \( (\mathcal{M}, \rho) \) into the family \( \mathcal{K}(X) \) of all non-empty convex compact subsets of \( X \). Let
\[ F^{[1]}(x) = \bigcap_{z \in M} \left[ F(z) + \lambda_1 \rho(x, z) B_X \right] \quad \text{and} \quad F^{[2]}(x) = \bigcap_{z \in M} \left[ F^{[1]}(z) + \lambda_2 \rho(x, z) B_X \right], \quad x \in M. \quad (1.8) \]

Suppose that for every subset \( \mathcal{M}' \subset \mathcal{M} \) with \( \# \mathcal{M}' \leq 4 \), the restriction \( F|_{\mathcal{M}'} \) of \( F \) to \( \mathcal{M}' \) has a Lipschitz selection with Lipschitz seminorm at most 1.

See (1.1).
Then for every \( \lambda_1, \lambda_2 \) and \( \gamma \) satisfying (1.7) the set
\[
F^{[2]}(x) \neq \emptyset \quad \text{for every} \quad x \in M. \tag{1.9}
\]
Furthermore,
\[
d_H(F^{[2]}(x), F^{[2]}(y)) \leq \gamma \rho(x, y) \quad \text{for every} \quad x, y \in M. \tag{1.10}
\]
In particular, in these settings the mapping \( F^{[2]} \) satisfies (1.9) and (1.10) provided
\[
\lambda_1 = 4/3, \quad \lambda_2 = 4, \quad \text{and} \quad \gamma = 100. \tag{1.11}
\]
We prove Theorem 1.9 in Section 3. In that section we also show that if \( X \) is a Euclidean two dimensional space, the constant \( \gamma \) from Theorem 1.9 can be slightly decreased. More specifically, we prove that in this case Theorem 1.9 holds provided (1.7) is replaced with
\[
\lambda_1 \geq e(3\pi, X), \quad \lambda_2 \geq 3\lambda_1, \quad \gamma \geq \lambda_2 \left( 1 + 2\lambda_2 / \left( \lambda_2^2 - \lambda_1^2 \right)^{3/2} \right). \tag{1.12}
\]
In particular, one can set \( \lambda_1 = 4/\pi, \lambda_2 = 12/\pi, \text{and} \gamma = 38 \). Furthermore, we prove that if \( M \) is a subset of a Euclidean space \( E, \rho \) is the metric in \( E \), and \( X \) is a two dimensional Euclidean space, properties (1.9) and (1.10) hold for
\[
\lambda_1 = 1, \quad \lambda_2 = 3, \quad \text{and} \quad \gamma = 25. \tag{1.13}
\]
See Theorem 3.1.

In Section 6 we prove Theorem 6.13 which improves the result of Theorem 1.9 for the space \( X = \ell^2_\infty \), i.e., for \( \mathbb{R}^2 \) equipped with the uniform norm
\[
||x|| = \max(|x_1|, |x_2|), \quad x = (x_1, x_2). \]
More specifically, we show that in this case properties (1.9) and (1.10) hold provided
\[
\lambda_1 \geq 1, \quad \lambda_2 \geq 3\lambda_1, \quad \text{and} \quad \gamma \geq \lambda_2 (3\lambda_2 + \lambda_1) / (\lambda_2 - \lambda_1). \tag{1.14}
\]
In particular, these properties hold for \( \lambda_1 = 1, \lambda_2 = 3 \) and \( \gamma = 15 \).

In Section 4 we prove Theorem 1.10 the second main result of the paper related to set-valued mappings from a pseudometric space into the family \( \mathcal{K}_{1}(X) \) of all bounded closed line segments of a Banach space \( X \).

**Theorem 1.10** Let \((M, \rho)\) be a pseudometric space. Let \( m = 1 \) and let \( X \) be a Banach space. In this case Conjecture 1.8 holds for \( \ell = 2 \) and for every \( \lambda_1, \lambda_2 \) and \( \gamma \) such that
\[
\lambda_1 \geq 1, \quad \lambda_2 \geq 3\lambda_1, \quad \gamma \geq \lambda_2 (3\lambda_2 + \lambda_1) / (\lambda_2 - \lambda_1). \tag{1.15}
\]
Thus, the following statement is true: Let \( F : M \to \mathcal{K}_{1}(X) \) be a set-valued mapping such that for every subset \( M' \subset M \) with \( \#M' \leq 4 \), the restriction \( F|_{M'} \) of \( F \) to \( M' \) has a Lipschitz selection with Lipschitz seminorm at most 1.

Let \( F^{[2]} \) be the mapping defined by (1.8). Then properties (1.9) and (1.10) hold provided \( \lambda_1, \lambda_2 \) and \( \gamma \) satisfy (1.12). In particular, one can set \( \lambda_1 = 1, \lambda_2 = 3 \) and \( \gamma = 15 \).

If \( X \) is a Euclidean space, the same statement is true provided (1.12) is replaced with the condition
\[
\lambda_1 \geq 1, \quad \lambda_2 \geq 3\lambda_1, \quad \gamma \geq \lambda_2 + 2\lambda_2^2 / \left( \lambda_2^2 - \lambda_1^2 \right)^{1/2}. \tag{1.16}
\]
In particular, in this case, (1.9) and (1.10) hold provided \( \lambda_1 = 1, \lambda_2 = 3 \) and \( \gamma = 10 \).
Note that Theorem 1.9 tells us that for every set-valued mapping $F$ satisfying the hypothesis of this theorem, the mapping $F^{[2]}$ determined by (1.8) with $\lambda_1 = 4/3$ and $\lambda_2 = 4$ provides a $\gamma$-core of $F$ with $\gamma = 100$. (See Definition 1.3.) In turn, Theorem 1.10 states that the mapping $F^{[2]}$ corresponding to the parameters $\lambda_1 = 1$ and $\lambda_2 = 3$ is a 15-core of any $F$ satisfying the conditions of this theorem.

We note that the proofs of Theorem 1.9 and Theorem 1.10 rely on Helly’s Intersection Theorem and a series of auxiliary results about neighborhoods of intersections of convex sets. See Section 2.

In Section 5 we prove that Conjecture 1.8 holds with

$$\ell = 1, \quad \lambda_1 = 1, \quad \text{and} \quad \gamma = 1,$$

provided $X$ is a one-dimensional Banach space. See Proposition 5.1. In this section we also give several useful formulae for the optimal Lipschitz selection of a set-valued mapping from a pseudometric space into the family of all closed intervals of $X$.

Let us reformulate Conjecture 1.8 in a way which does not require the use of the notion of a core of a set-valued mapping. We recall that the mapping $F^{[\ell]} : M \to \mathcal{K}_m(X)$ from Conjecture 1.8 is a $\gamma$-core of $F$ if

$$d_H(F^{[\ell]}(x), F^{[\ell]}(y)) \leq \gamma \rho(x, y) \quad \text{for all} \quad x, y \in M.$$ 

See part (ii) of Definition 1.3 Hence, given $x \in M$,

$$F^{[\ell]}(x) \subset F^{[\ell]}(y) + \gamma \rho(x, y)B_X \quad \text{for every} \quad y \in M. \quad (1.14)$$

We also recall that

$$F^{[\ell+1]}(x) = BR[F^{[\ell]}; \gamma \rho](x) = \bigcap_{y \in M} \left[ F^{[\ell]}(y) + \gamma \rho(x, y)B_X \right].$$

See (1.4). This and (1.14) imply the inclusion $F^{[\ell+1]}(x) \supset F^{[\ell]}(x), x \in M$. On the other hand, (1.5) tells us that $F^{[\ell+1]}(x) \subset F^{[\ell]}(x)$ proving that

$$F^{[\ell+1]} = F^{[\ell]} \quad \text{on} \quad M.$$

These observations enable us to reformulate Conjecture 1.8 as follows.

**Conjecture 1.11** Given $m \in \mathbb{N}$ there exist a positive integer $\ell = \ell(m) \in \mathbb{N}$ and a sequence of positive numbers $\vec{\lambda} = \{\lambda_1(m), \ldots, \lambda_{\ell}(m)\}$ such that for every set-valued mapping $F : M \to \mathcal{K}_m(X)$ satisfying the hypothesis of the Finiteness Principle (Theorem 1.2), the family of set-valued mappings $\{F^{[k]} : k = 1, \ldots, \ell + 1\}$ constructed by formula (1.4) has the following property:

$$F^{[\ell]}(x) \neq \emptyset \quad \text{and} \quad F^{[\ell+1]}(x) = F^{[\ell]}(x) \quad \text{for all} \quad x \in M. \quad (1.15)$$

We refer to (1.15) as a Stabilization Property of balanced refinements.

Thus, Theorem 1.9 and Theorem 1.10 tell us that the Stabilization Property of balanced refinements holds whenever $\dim X = 2$ or $m = 1$ (and $X$ is an arbitrary). More specifically, Theorem 1.9 shows that if $m = 2$ and $\dim X = 2$, Conjecture 1.11 holds with

$$\ell = 2 \quad \text{and} \quad \vec{\lambda} = \{4/3, 4, 10^2\}.$$

In other words, in this case, $F^{[2]}(x) \neq \emptyset$ for each $x \in M$ and $F^{[3]} = F^{[2]}$ on $M$. In turn, Theorem 1.10 states that the same property holds provided $X$ is an arbitrary Banach space, $m = 1$, and $\vec{\lambda} = \{1, 3, 15\}$. 

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In Sections 7 and 8 we present several explicit criteria for the existence of Lipschitz selections of set-valued mappings from a pseudometric space \( \mathcal{M} = (\mathcal{M}, \rho) \) into the family \( \text{Conv}(\mathbb{R}^2) \) of all convex closed subsets of \( \mathbb{R}^2 \). These criteria develop the ideas and methods of a constructive criterion for Lipschitz selections in \( \mathbb{R}^2 \) given in the paper \([26]\). Let us recall this result.

Let

\[ X = \ell_\infty^2 \quad \text{and let} \quad Q_0 = [-1, 1] \times [-1, 1] \]

be the unit ball of \( X \). Given a set-valued mapping \( F : \mathcal{M} \to \mathcal{K}(\mathbb{R}^2) \), a positive constant \( \lambda \) and elements \( x, x' \in \mathcal{M} \), we introduce a set

\[ R_F[x, x' : \lambda] = \mathcal{H}[F(x) \cap \{F(x') + \lambda \rho(x, x')Q_0\}] \]

See Fig. 5.

Here, given a set \( S \subset \mathbb{R}^2 \), by \( \mathcal{H}[S] \) we denote the smallest rectangular with sides parallel to the coordinate axes containing \( S \).

![Fig. 5: The rectangular \( R_F[x, x' : \lambda] \) for \( x, x' \in \mathcal{M} \) and \( \lambda > 0 \).](image)

We also set

\[ |F|_{\text{Lip}(\mathcal{M}, \ell_\infty^2)} = \inf\{\|f\|_{\text{Lip}(\mathcal{M}, \ell_\infty^2)} : f \text{ is a Lipschitz selection of } F\}. \quad (1.16) \]

**Theorem 1.12** A set-valued mapping \( F : \mathcal{M} \to \mathcal{K}(\mathbb{R}^2) \) has a Lipschitz selection if and only if there exists a constant \( \lambda > 0 \) such that the following conditions are satisfied:

(i) \( \text{dist}(F(x), F(y)) \leq \lambda \rho(x, y) \) for all \( x, y \in \mathcal{M} \);

(ii) for every \( x, x', y, y' \in \mathcal{M} \), we have

\[ \text{dist}( R_F[x, x' : \lambda], R_F[y, y' : \lambda] ) \leq \lambda \rho(x, y). \quad (1.17) \]

Furthermore, the following inequality

\[ \inf \lambda \leq |F|_{\text{Lip}(\mathcal{M}, \ell_\infty^2)} \leq 8 \inf \lambda \quad (1.18) \]

holds. See Fig. 6.
that the following two conditions hold:

\[ \text{Theorem 1.13 (8.15).} \] Finally, by \( \Delta \) explicit formulae for the coordinates of the point boundary is a straight line \( \ell \) the standard inner product in \( \mathbb{R} \). (Here, given \( a \in \mathbb{R} \), by \( \langle a, n(x) \rangle = a_1n_1(x) + a_2n_2(x) \) we denote the standard inner product in \( \mathbb{R}^2 \).) Thus, for each \( x \in M \), the set \( F(x) \) is a half-plane in \( \mathbb{R}^2 \) whose boundary is a straight line \( \ell_F(x) = \{ a \in \mathbb{R}^2 : \langle a, n(x) \rangle + \alpha(x) = 0 \} \). The unit vector \( n(x) \) is directed outside of the half-plane \( F(x) \) and orthogonal to the line \( \ell_F(x) \).

Given \( x, y \in M \) such that \( n(x) \nparallel n(y) \) we set \( w(x, y : F) = \ell_F(x) \cap \ell_F(y) \). (In Section 8 we give explicit formulae for the coordinates of the point \( w(x, y : F) = (w_1(x, y : F), w_2(x, y : F)) \). See (8.13).) Finally, by \( \Delta_n(x, y) \) we denote the determinant

\[
\Delta_n(x, y) = \det\left( \begin{array}{cc} n_1(x) & n_1(y) \\ n_2(x) & n_2(y) \end{array} \right) = n_1(x)n_2(y) - n_2(x)n_1(y).
\]

**Theorem 1.13** Let \( F : M \to \mathcal{HP}(\mathbb{R}^2) \) be a set-valued mapping defined by (1.19). Assume that either \( M \) is finite or there exist elements \( x_1, ..., x_m \in M \) such that the interior of convex hull of points \( n(x_1), ..., n(x_m) \) contains 0.

The set-valued mapping \( F \) has a Lipschitz selection if and only if there exists a constant \( \lambda > 0 \) such that the following two conditions hold:

1. \( \alpha(x) + \alpha(y) \leq \lambda \rho(x, y) \) for every \( x, y \in M \) such that \( n(y) = -n(x) \);
2. For every \( x, x', y, y' \in M \) such that \( n(x) \nparallel n(x') \), \( n(y) \nparallel n(y') \), we have

\[
w_1(x, x' : F) - w_1(y, y' : F) \leq \lambda \left\{ \frac{\rho(x, x')}{|\Delta_n(x, x')|} \min\{|n_2(x)|, |n_2(x')|\} + \frac{\rho(y, y')}{|\Delta_n(y, y')|} \min\{|n_2(y)|, |n_2(y')|\} + \rho(x, y) \right\}
\]  

\[ (1.20) \]
provided
\[ n_2(x) n_2(x') \leq 0, \quad n_1(x) + n_1(x') \leq 0 \quad \text{and} \quad n_2(y) n_2(y') \leq 0, \quad n_1(y) + n_1(y') \geq 0, \]
and
\[
w_2(x, x' : F) - w_2(y, y' : F)
\leq \lambda \left\{ \frac{\rho(x, x')}{|\Delta_n(x, x')|} \min\{|n_1(x)|, |n_1(x')|\} + \frac{\rho(y, y')}{|\Delta_n(y, y')|} \min\{|n_1(y)|, |n_1(y')|\} + \rho(x, y) \right\} \tag{1.21}
\]
provided
\[ n_1(x) n_1(x') \leq 0, \quad n_2(x) + n_2(x') \leq 0, \quad \text{and} \quad n_1(y) n_1(y') \leq 0, \quad n_2(y) + n_2(y') \geq 0. \]

Furthermore,
\[
\frac{1}{\sqrt{2}} \inf \lambda \leq |F|_{\mathcal{H}\mathcal{P}^2} \leq 8 \inf \lambda. \tag{1.22}
\]

Necessary and sufficient conditions for the existence of a Lipschitz selection given in Theorem 1.13 involve Cartesian coordinates of certain geometric objects determined by the set-valued mapping \( F \). Theorem 1.14 below presents another explicit criterion for Lipschitz selections of \( F \). This criterion formulates in terms of geometrical objects which depend only on \( F \) and independent of the coordinate system in \( \mathbb{R}^2 \). We refer to this criterion as a “coordinate-free” Lipschitz selection criterion.

Let us prepare the ingredients that are needed to formulate Theorem 1.14. Let \( F : \mathcal{M} \to \mathcal{H}\mathcal{P}(\mathbb{R}^2) \) be a set-valued mapping defined by formula (1.19). Given \( x, y \in \mathcal{M} \), we let \( \varphi_F(x, y) \in [0, \pi/2] \) denote the angle between the boundaries of \( F(x) \) and \( F(y) \), i.e., between the straight lines \( \ell_F(x) \) and \( \ell_F(y) \). Given a set \( \mathcal{M}' \subset \mathcal{M} \), by \( \text{diam}_\rho(\mathcal{M}') \) we denote the diameter of \( \mathcal{M} \) in \( (\mathcal{M}, \rho) \). Finally, we set \( 0/0 = 0, \quad a/0 = +\infty \) for every \( a > 0 \), and \( \text{dist}(0, A) = 0 \) provided \( A \subset \mathbb{R}^2 \).

**Theorem 1.14** Let \( \mathcal{M} = (\mathcal{M}, \rho) \) be a pseudometric space, and let \( F : \mathcal{M} \to \mathcal{H}\mathcal{P}(\mathbb{R}^2) \) be a set-valued mapping defined by (1.19). Assume that either \( \mathcal{M} \) is finite or there exist elements \( x_1, \ldots, x_m \in \mathcal{M} \) such that the interior of convex hull of points \( n(x_1), \ldots, n(x_m) \) contains 0.

The mapping \( F \) has a Lipschitz selection \( f : \mathcal{M} \to \mathbb{R}^2 \) if and only if there exists a constant \( \lambda > 0 \) such that for every four elements \( x, x', y, y' \in \mathcal{M} \) the following inequality
\[
\text{dist}(F(x) \cap F(x'), F(y) \cap F(y')) \leq \lambda \left\{ \frac{\rho(x, x')}{\sin \varphi_F(x, x')} + \frac{\rho(y, y')}{\sin \varphi_F(y, y')} + \text{diam}_\rho(x, x', y, y') \right\} \tag{1.23}
\]
holds. Furthermore,
\[
\frac{1}{\sqrt{2}} \inf \lambda \leq |F|_{\mathcal{H}\mathcal{P}^2} \leq \gamma \inf \lambda.
\]
Here \( \gamma > 0 \) is an absolute constant, \( \gamma \leq 5 \cdot 10^5 \).

In the forthcoming paper [29] we study the following problem formulated by C. Fefferman [10]:

**Problem 1.15** Let \( (\mathcal{M}, \rho) \) be an \( N \)-point metric space. For each \( x \in \mathcal{M} \), let \( F(x) \subset \mathbb{R}^D \) be a convex polytope.

How can one compute a map \( f : \mathcal{M} \to \mathbb{R}^D \) such that \( f(x) \in F(x) \) for all \( x \in \mathcal{M} \), with Lipschitz norm as small as possible up to a factor \( C(D) \)?

This is a big ill-conditioned linear programming problem. Can we do better than just applying general-purpose linear programming? How does the work of an optimal algorithm scale with the number of points \( N \)?
Let \( \mathcal{M} = (M, \rho) \) be an \( N \)-point pseudometric space (i.e., \( N = \#M \)). In forthcoming paper \([29]\) we present several efficient algorithms for Lipschitz selections of set-valued mappings from \( M \) into the family \( \mathcal{HP}(\mathbb{R}^2) \) of all closed half-planes in \( \mathbb{R}^2 \). These algorithms rely on the methods of proofs of constructive criteria for Lipschitz selections given in Sections 7-9.

In particular, in \([29]\) we exhibit an algorithm which, given a set-valued mapping \( F : M \rightarrow \mathcal{HP}(\mathbb{R}^2) \) computes the order of magnitude of the quantity \( |F|_{\#R^2} \) (i.e., the Lipschitz seminorm of an optimal Lipschitz selection of \( F \), see \((1,16)\)), and a nearly optimal Lipschitz selection \( f \) of \( F \) using work at most \( CN^3 \) and storage at most \( CN \). Here \( C \) is an absolute constant.

Also, let us note a result from \([29]\) related to a set-valued mapping \( F \) from \( M \) into the family \( \mathcal{K}_1(\mathbb{R}^M) \) of all bounded closed line segments of \( \mathbb{R}^M \). (Here \( M \) is a positive integer). In this case, we exhibit an algorithm which computes the order of magnitude of \( |F|_{\#R^M} \) and a nearly optimal Lipschitz selection \( f \) of \( F \) using work at most \( C(M + N^3) \) and storage at most \( C(M + N) \).

The main ingredients of the proofs of these results are linear-time algorithms for linear programming in \( \mathbb{R}^3 \) due to Megiddo \([19]\), and Lipschitz selection criteria of Theorem 7.14 and Theorem 7.17 respectively.

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2. Neighborhoods of intersections of convex sets in a Banach space.

Let us fix some notation. Let \((X, \| \cdot \|) \) be a Banach space. We write

\[
\text{diam } A = \sup \{\|a - b\| : a, b \in A\} \quad \text{and} \quad \text{dist}(A', A'') = \inf \{\|a' - a''\| : a' \in A', a'' \in A''\}
\]

to denote the diameter of a set \( A \subset X \) and the distance between sets \( A', A'' \subset X \) respectively. For \( x \in X \) we also set \( \text{dist}(x, A) = \text{dist}(\{x\}, A) \), and put \( \text{dist}(\emptyset, A) = 0 \) provided \( A \) is an arbitrary (possibly empty) subset of \( X \). If \( A \subset X \) is finite, by \#A we denote the number of elements of \( A \).

Given non-empty sets \( A, B \subset X \) we let \( A + B = \{a + b : a \in A, b \in B\} \) denote the Minkowski sum of these sets. Given a non-negative real number \( \lambda \) by \( \lambda A \) we denote the set \( \lambda A = \{\lambda a : a \in A\} \).

Given \( a, b \in X, a \neq b \), by \([a, b]\) we denote a closed interval (a line segment) with ends in \( a \) and \( b \):

\[
[a, b] = \{x \in X : x = (1 - t)a + tb, 0 \leq t \leq 1\}.
\]

We also write \([a, a]\) = \( \{a\} \) and consider \([a, a]\) as a closed “interval” in \( X \). By \( \mathcal{C}(X) \) we denote the family of all convex closed non-empty subsets of \( X \).

Given a set \( A \subset \mathbb{R} \) we put \( \min A = \{\min x : x \in A\} \) and \( \max A = \{\max x : x \in A\} \) provided \( A \) is a closed subset of \( \mathbb{R} \) bounded from above or below respectively. We let

\[
\mathcal{I}(\mathbb{R}) = \{[a, b] : a, b \in \mathbb{R}, a \leq b\} \cup \{[a, +\infty) : a \in \mathbb{R}\} \cup \{(-\infty, b) : b \in \mathbb{R}\} \cup \{\mathbb{R}\} \quad (2.1)
\]

denote the family of all closed intervals in \( \mathbb{R} \) (bounded or unbounded). We write \( [x]_+ \) for the positive part of the real \( x \), i.e., \( [x]_+ = \max\{x, 0\} \). We set \( \frac{a}{0} = 0 \) and \( \frac{a}{0} = +\infty \) for \( a > 0 \).

Sometimes, given a set \( M \), we will be looking simultaneously at two distinct pseudometrics on \( M \), say \( \rho \) and \( \delta \). In this case we will speak of a \( \rho \)-Lipschitz selection and \( \rho \)-Lipschitz seminorm, or
a $\delta$-Lipschitz selection and $\delta$-Lipschitz seminorm to make clear which pseudometric we are using. Furthermore, given a mapping $f: \mathcal{M} \to X$ we will write $\|f\|_{\text{Lip}((M,\rho),X)}$ or $\|f\|_{\text{Lip}((M,\delta),X)}$ to denote the Lipschitz seminorm of $f$ with respect to the pseudometric $\rho$ or $\delta$ respectively.

We let $\ell^\infty_n$ denote the space $\mathbb{R}^n$ equipped with the uniform norm $\|x\|_\infty = \max\{|x_i| : i = 1, \ldots, n\}$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. By $\ell^n_2$ we denote $\mathbb{R}^n$ equipped with the Euclidean norm $\|x\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$.

By $Ox_1 = \{x = (t, 0) : t \in \mathbb{R}\}$ and $Ox_2 = \{x = (0, t) : t \in \mathbb{R}\}$ we denote coordinate axes in $\mathbb{R}^2$. Let

$$B_0 = \{a \in \mathbb{R}^2 : \|a\|_{\ell_2^1} \leq 1\} \quad \text{and} \quad S_1 = \{a = (a_1, a_2) \in \mathbb{R}^2 : \|a\|_{\ell_2^2} = (a_1^2 + a_2^2)^{1/2} = 1\}$$

be the closed unit disk and the unit circle in $\mathbb{R}^2$ respectively. Given non-zero vectors $u, v \in \mathbb{R}^2$ we write $u \parallel v$ if $u$ and $v$ are collinear, and we write $u \nparallel v$ whenever these vectors are non-collinear. By $\theta(u, v) \in [0, 2\pi)$ we denote

$$\text{the angle of rotation from } u/\|u\|_{\ell_2^1} \text{ to } v/\|v\|_{\ell_2^1} \text{ in the counterclockwise direction.} \quad (2.2)$$

(Thus, $\theta(v, u) = 2\pi - \theta(u, v)$.) We refer to $\theta(u, v)$ as the angle between the vectors $u$ and $v$.

Let us $\ell_1$, $\ell_2$ be two non-parallel straight lines in $\mathbb{R}^2$ (we write $\ell_1 \nparallel \ell_2$), and let $A = \ell_1 \cap \ell_2$. These two lines naturally form two angles $\varphi_1, \varphi_2 \in [0, \pi)$, $\varphi_1 + \varphi_2 = \pi$, with vertex at the point $A$. Let

$$\varphi(\ell_1, \ell_2) = \min(\varphi_1, \varphi_2); \quad \text{clearly, } \varphi(\ell_1, \ell_2) \in [0, \pi/2]. \quad (2.3)$$

Everywhere in this paper we refer to $\varphi(\ell_1, \ell_2)$ as “the angle between straight lines $\ell_1$ and $\ell_2$”. In other words, the angle between two non-parallel lines in $\mathbb{R}^2$ means the smallest angle between these lines. If $\ell_1 \parallel \ell_2$ (i.e., $\ell_1$ and $\ell_2$ are parallel), we set $\varphi(\ell_1, \ell_2) = 0$.

We let

$$\mathcal{R}(\mathbb{R}^2) = \{\Pi = I_1 \times I_2 : I_1, I_2 \in \mathcal{I}(\mathbb{R})\}$$

denote the family of all closed rectangles in $\mathbb{R}^2$ with sides parallel to the coordinate axes. Finally, by $\mathcal{HP}(\mathbb{R}^2)$ we denote the family of all closed half-planes in $\mathbb{R}^2$, and by $\text{Conv}(\mathbb{R}^2)$ the family of all closed convex subsets of $\mathbb{R}^2$.

Given a Banach space $X$, Przesławski and Yost [22] have introduced an important geometrical characteristic of $X$, the so-called modulus of squareness of $X$. Let us recall its definition.

We observe that for any $x, y \in X$ with $\|y\| < 1 < \|x\|$ there exists a unique $z = z(x, y)$ with $\|z\| = 1$ which belongs to the line segment $[x, y]$. We set

$$\omega(x, y) = \frac{\|x - z(x, y)\|}{\|x\| - 1} \quad (2.4)$$

and define a function $\xi : [0, 1) \to [1, \infty)$ by

$$\xi_\beta(\beta) = \sup \{\omega(x, y) : x, y \in X, \|y\| \leq \beta < 1 < \|x\|\}. \quad (2.5)$$

We also put

$$\varphi(\beta) = (1 + \beta)/(1 - \beta) \quad \text{and} \quad \psi(\beta) = (1 - \beta^2)^{-1}, \quad \beta \in [0, 1). \quad (2.6)$$

It is shown in [22], that for any Banach space $X$

$$\xi_\beta(\beta) \leq \varphi(\beta) \quad \text{for every } \beta \in [0, 1), \quad (2.7)$$
Let $\beta \in [0, 1)$, provided $X$ is a Euclidean space.

Theorem 2.1 below recalls an important result from this paper. Recall that $\mathcal{C}(X)$ denotes the family of all nonempty convex closed subsets of $X$; let us equip this family with the Hausdorff distance.

**Theorem 2.1** ([22] Theorem 4) Let $(S, \delta)$ be a metric space, let $X$ be a Banach space, and let $f : S \to X$ and $g : S \to [0, \infty)$ be Lipschitz mappings. Let $F : S \to \mathcal{C}(X)$ be a Lipschitz (with respect to the Hausdorff distance) set-valued mapping.

Suppose that there exists a constant $\gamma > 1$ such that $g(x) \geq \gamma \operatorname{dist}(f(x), F(x))$ for every $x \in S$. Then the intersection mapping $G : S \to \mathcal{C}(X)$ defined by

$$G(x) = f(x) \cap B_X(f(x), g(x))$$

is Lipschitz on $S$ (with respect to $d_H$) with Lipschitz seminorm

$$\|G\|_{\text{Lip}(S, \mathcal{C}(X))} \leq \|F\|_{\text{Lip}(S, \mathcal{C}(X))} + (\|F\|_{\text{Lip}(S, \mathcal{C}(X))} + \|f\|_{\text{Lip}(S, X)} + \|g\|_{\text{Lip}(S, \mathbb{R})}) \xi(1/\gamma).$$

This theorem enables to prove the following

**Proposition 2.2** Let $X$ be a Banach space, $a \in X$, $r \geq 0$, and let $C \subset X$ be a convex set.

Suppose that $C \cap B_X(a, r) \neq \emptyset$. Then for every $s > 0$ and every $L > 1$ the following inequality

$$d_H(C \cap B_X(a, r), (C + sB_X) \cap B_X(a, Lr + s)) \leq \left(1 + 2 \xi_X \left(\frac{1}{L}\right)\right) s$$

holds.

**Proof.** Let $S = \{x, y\} \subset \mathbb{R}$ where $x = 0$ and $y = s$, and let $\delta(x, y) = s$.

We define a mapping $f : S \to X$ and a function $g : S \to \mathbb{R}$ by letting $f(x) = f(y) = a$ and $g(x) = Lr$, $g(y) = Lr + s$. Clearly, $\|f\|_{\text{Lip}(S, X)} = 0$, and $\|g\|_{\text{Lip}(S, \mathbb{R})} = 1$.

We put $\gamma = L$. We note that $C \cap B_X(a, r) \neq \emptyset$ so that

$$\operatorname{dist}(f(x), F(x)) = \operatorname{dist}(a, C) \leq r.$$

Hence, $g(x) = Lr = \gamma r \geq \gamma \operatorname{dist}(f(x), F(x))$.

Then we define a mapping $F : S \to \mathcal{C}(X)$ by setting $F(x) = C$ and $F(y) = C + sB_X$. Clearly,

$$d_H(F(x), F(y)) \leq s = \delta(x, y) \quad \text{so that} \quad \|F\|_{\text{Lip}(S, \mathcal{C}(X))} \leq 1.$$

Thus, the conditions of Theorem 2.1 are satisfied for the metric space $(S, \delta)$ and the mappings $f$, $g$ and $F$. This theorem tells us that the mapping $G : S \to \mathcal{C}(X)$ defined by

$$G(u) = F(u) \cap B_X(f(u), g(u)), \quad u \in S,$$

is Lipschitz on $S$ with respect to the Hausdorff distance. Furthermore,

$$\|G\|_{\text{Lip}(S, \mathcal{C}(X))} \leq \|F\|_{\text{Lip}(S, \mathcal{C}(X))} + (\|F\|_{\text{Lip}(S, \mathcal{C}(X))} + \|f\|_{\text{Lip}(S, X)} + \|g\|_{\text{Lip}(S, \mathbb{R})}) \xi(1/\gamma) \leq 1 + 2 \xi_X \left(\frac{1}{L}\right).$$

Hence,

$$d_H(C \cap B_X(a, Lr), (C + sB_X) \cap B_X(a, Lr + s)) = d_H(G(x), G(y)) \leq \|G\|_{\text{Lip}(S, \mathcal{C}(X))} \delta(x, y) \leq \left(1 + 2 \xi_X \left(\frac{1}{L}\right)\right) s$$

proving the proposition. \hfill \blacksquare

**Proposition 2.2** implies the following important
**Proposition 2.3** Let $X$ be a Banach space, and let $C \subset X$ be a convex set. Let $a \in X$ and let $r \geq 0$. Suppose that

$$C \cap B_X(a, r) \neq \emptyset. \quad (2.9)$$

Then for every $s > 0$ and $L > 1$

$$C \cap B_X(a, Lr) + \theta(L) s B_X \supset (C + sB_X) \cap (B_X(a, Lr) + sB_X) \quad (2.10)$$

where

$$\theta(L) = (3L + 1)/(L - 1). \quad (2.11)$$

If $X$ is a Euclidean space then (2.10) holds with

$$\theta(L) = 1 + \frac{2L}{\sqrt{L^2 - 1}}. \quad (2.12)$$

**Proof.** Let

$$G = C \cap B_X(a, Lr) \quad \text{and} \quad \widetilde{G} = (C + sB_X) \cap (B_X(a, Lr) + sB_X).$$

Definition (1.2) tells us that $\widetilde{G} \subset G + d_H(G, \widetilde{G}) B_X$. In turn, Proposition 2.2 states that

$$d_H(G, \widetilde{G}) \leq (1 + 2 \xi_X \left( \frac{L}{2} \right)) s.$$

Hence,

$$\widetilde{G} \subset G + \Theta(L) s B_X \quad \text{where} \quad \Theta(L) = 1 + 2 \xi_X \left( \frac{1}{L} \right).$$

Now, let $X$ be an arbitrary Banach space. In this case, thanks to (2.6) and (2.7), we have

$$\Theta(L) \leq 1 + 2 \xi_X \left( \frac{1}{L} \right) \leq 1 + 2 \left( 1 + \frac{1/L}{1 - 1/L} \right) = \frac{3L + 1}{L - 1}.$$

This inequality and (2.11) imply inclusion (2.10) in the case under consideration.

Finally, let $X$ be a Euclidean space. In this case, from (2.6), (2.8) and (2.12), we have

$$\Theta(L) = 1 + 2 \xi_X \left( \frac{1}{L} \right) = 1 + 2 \left( 1 - (1/L)^2 \right)^{-\frac{1}{2}} = 1 + \frac{2L}{\sqrt{L^2 - 1}} = \theta(L).$$

The proof of the proposition is complete. $\blacksquare$

For the case of a Banach space, Proposition 2.3 was proved in [20, p. 279]. For similar results we refer the reader to [11, [3, p. 369] and [5, p. 26].

For the sake of completeness, and for the reader’s convenience, below we give

A direct proof of Proposition 2.3 We follow the proof of Lemma 5.3 from [20, p. 279]. If $r = 0$ then (2.10) holds trivially, so we assume that $r > 0$. Without loss of generality, we may also assume that $a = 0$. Thus we should prove that

$$C \cap (LrB_X) + \theta s B_X \supset (C + sB_X) \cap (LrB_X + sB_X)$$

provided $r > 0$, $s > 0$, $L > 1$. Let

$$z \in (C + sB_X) \cap (LrB_X + sB_X) = (C + sB_X) \cap [(Lr + s)B_X]. \quad (2.13)$$
Prove that
\[ z \in C \cap (LrB_X) + \theta s B_X. \]  \hspace{1cm} (2.14)

Thanks to (2.13), \( z \in (C + sB_X) \) so that there exists an element \( v \in C \) such that
\[ \|v - z\| \leq s. \]  \hspace{1cm} (2.15)

If \( \|v\| \leq Lr \), then \( v \in C \cap (LrB_X) \) proving (2.14).

Suppose that
\[ \|v\| > Lr. \]  \hspace{1cm} (2.16)

Property (2.13) tells us that \( \|z\| \leq Lr + s \) so that
\[ \|v\| \leq \|z\| + s \leq Lr + 2s. \]  \hspace{1cm} (2.17)

In turn, assumption (2.9) tells us that there exists an element \( v' \in C \) such that
\[ \|v'\| \leq r < Lr. \]  \hspace{1cm} (2.18)

Choose \( \lambda \in (0, 1) \) such that the element
\[ \tilde{v} = \lambda v' + (1 - \lambda)v \]
has the norm \( \|\tilde{v}\| = Lr. \) We know that \( C \) is convex so that \([v', v] \subset C\) proving that
\[ \tilde{v} \in C \cap (LrB_X). \]  \hspace{1cm} (2.19)

Thanks to (2.18), (2.17) and the triangle inequality,
\[ Lr = \|\tilde{v}\| = \|\lambda v' + (1 - \lambda)v\| \leq \lambda r + (1 - \lambda)(Lr + 2s) \]
proving that
\[ \lambda \leq \frac{2s}{(L - 1)r + 2s}. \]

Consequently, thanks to this inequality, (2.18) and (2.17)
\[ \|v - \tilde{v}\| = \lambda\|v - v'\| \leq \lambda(\|v\| + \|v'\|) \leq \frac{2s}{(L - 1)r + 2s} \cdot (Lr + 2s + r) \leq 2s(L + 1)/(L - 1). \]

From this inequality and (2.15) we have
\[ \|z - \tilde{v}\| \leq \|z - v\| + \|v - \tilde{v}\| \leq s + 2s(L + 1)/(L - 1) = \theta(L) s \]
which together with (2.19) implies (2.14).

Let now \( X \) be a Euclidean space. We modify the above proof after (2.19) as follows.
We put \( \beta = 1/Lr \), and
\[ x = \frac{1}{Lr} v, \quad y = \frac{1}{Lr} v', \quad w = \frac{1}{Lr} \tilde{v}. \]  \hspace{1cm} (2.20)
Then, thanks to (2.16) and (2.18),
\[ \|y\| \leq \beta < 1 < \|x\|. \] (2.21)

We note that for any \( u, \tilde{u} \in X \) such that \( \|\tilde{u}\| < 1 < \|u\| \), there exists a unique \( w = w(u, \tilde{u}) \in [u, \tilde{u}] \) with \( \|w\| = 1 \). Hence, thanks to (2.4),
\[ \omega(u, \tilde{u}) = \frac{\|u - w(u, \tilde{u})\|}{\|u\| - 1}. \] (2.22)

We also recall the definition of the function \( \xi_X \), see (2.5):
\[ \xi_X(\beta) = \sup \{ \omega(u, \tilde{u}) : u, \tilde{u} \in X, \|\tilde{u}\| \leq \beta < 1 < \|u\| \}. \] (2.23)

Prove that
\[ \xi_X(\beta) = (1 - \beta^2)^{-\frac{1}{2}}. \] (2.24)

In fact, fix \( u \) with \( \|u\| > 1 \). One can easily see that \( \sup \{ \omega(u, \tilde{u}) : \|\tilde{u}\| \leq \beta \} \) is attained for some \( \tilde{u} \) with \( \|\tilde{u}\| = \beta \), and the line segment \([\tilde{u}, u]\) is contained in a line tangent to the sphere with center at the origin and radius \( \beta \). Thus, \( u - \tilde{u} \) is perpendicular to \( \tilde{u} \). Hence,
\[ \sup\{\omega(u, \tilde{u}) : \|\tilde{u}\| \leq \beta \} = f(\|u\|) \]
where
\[ f(t) = \frac{\sqrt{t^2 - \beta^2} - \sqrt{1 - \beta^2}}{t} \quad t > 1. \]

The function \( f \) is decreasing on \((1, +\infty)\) so that
\[ \sup_{t>1} f(t) = \lim_{t\to 1^+} f(t) = (1 - \beta^2)^{-\frac{1}{2}} \]
proving (2.24).

We apply formula (2.24) to the points \( x, y, w \) defined by (2.20), and to \( \beta = 1/L \). We have
\[ \frac{\|v - \tilde{v}\|}{\|v\| - Lr} = \frac{\|x - w\|}{\|v\| - 1} = \omega(x, y). \]

See (2.22). Thanks to (2.21),
\[ \frac{\|v - \tilde{v}\|}{\|v\| - Lr} \leq \sup \{ \omega(u, \tilde{u}) : u, \tilde{u} \in X, \|\tilde{u}\| \leq \beta < 1 < \|u\| \} = \xi_X(\beta) = \xi_X(1/L) \]
so that, thanks to (2.24),
\[ \frac{\|v - \tilde{v}\|}{\|v\| - Lr} \leq \xi_X(1/L) = \frac{L}{\sqrt{L^2 - 1}}. \]

In turn, thanks to (2.17), \( \|v\| - Lr \leq 2s \), so that
\[ \|v - \tilde{v}\| \leq \frac{L}{\sqrt{L^2 - 1}} \left( \|v\| - Lr \right) \leq \frac{2sL}{\sqrt{L^2 - 1}}. \]
This inequality and (2.15) imply the following:

\[ \|z - \tilde{v}\| \leq \|z - v\| + \|v - \tilde{v}\| \leq s + \frac{2sL}{\sqrt{L^2 - 1}} = \left(1 + 2L/ \sqrt{L^2 - 1}\right) s. \]

This and (2.19) imply (2.14) with \( \theta = \theta(L) \) defined by (2.12) proving the proposition for a Euclidean space \( X \).

The proof of the proposition is complete. ■

Proposition 2.5 below is one of the main ingredients in the proofs of Theorems 1.9, 1.10, and 3.1. The proof of this proposition relies on Proposition 2.3 and Helly’s Intersection Theorem for two dimensional Banach spaces. We recall this theorem below.

**Theorem 2.4** Let \( \mathcal{K} \) be a collection of convex closed subsets of a two dimensional Banach space \( X \). Suppose that \( \mathcal{K} \) is finite or at least one member of the family \( \mathcal{K} \) is bounded.

If every subfamily of \( \mathcal{K} \) consisting of at most three elements has a common point then there exists a point common to all of the family \( \mathcal{K} \).

**Proposition 2.5** Let \( X \) be a two dimensional Banach space. Let \( C, C_1, C_2 \subset X \) be convex subsets, and let \( r > 0 \). Suppose that

\[ C_1 \cap C_2 \cap (C + rB_X) \neq \emptyset. \tag{2.25} \]

Then for every \( L > 1 \) and every \( \varepsilon > 0 \) the following inclusion

\[ (C_1 \cap C_2 + LrB_X) \cap C + \theta(L)\varepsilon B_X \supset [C_1 \cap C_2 + (Lr + \varepsilon)B_X] \cap [(C_1 + rB_X) \cap C + \varepsilon B_X] \cap [(C_2 + rB_X) \cap C + \varepsilon B_X] \]

holds. Here \( \theta \) is the function from Proposition 2.3 (Thus, \( \theta(L) = (3L + 1)/(L - 1) \) for an arbitrary \( X \), and \( \theta(L) = 1 + 2L/ \sqrt{L^2 - 1} \) for a Euclidean \( X \).)

**Proof.** Suppose that

\[ a \in [C_1 \cap C_2 + (Lr + \varepsilon)B_X] \cap [(C_1 + rB_X) \cap C + \varepsilon B_X] \cap [(C_2 + rB_X) \cap C + \varepsilon B_X] \tag{2.26} \]

and prove that

\[ a \in (C_1 \cap C_2 + LrB_X) \cap C + \theta(L)\varepsilon B_X. \tag{2.27} \]

First, let us show that

\[ C_1 \cap C_2 \cap (C + rB_X) \cap B_X(a, Lr + \varepsilon) \neq \emptyset. \tag{2.28} \]

Helly’s Theorem 2.4 tells us that this statement holds provided any three sets in the left hand size of (2.28) have a common point.

Note that \( C_1, C_2 \) and \( C + rB_X \) have a common point. See (2.25). We also know that

\[ a \in C_1 \cap C_2 + (Lr + \varepsilon)B_X, \]

see (2.26), so that

\[ C_1 \cap C_2 \cap B_X(a, Lr + \varepsilon) \neq \emptyset. \]
Let us prove that
\[ C_1 \cap (C + rB_X) \cap B_X(a, Lr + \varepsilon) \neq \emptyset. \] (2.29)

Property (2.26) tells us that
\[ a \in (C_1 + rB_X) \cap C + \varepsilon B_X. \]

Therefore, there exist elements \( b_1 \in C_1 \) and \( b \in C \) such that
\[ \|b_1 - b\| \leq r \quad \text{and} \quad \|a - b\| \leq \varepsilon. \]

In particular, \( b_1 \in C_1 \cap (C + rB_X) \). Furthermore,
\[ \|a - b_1\| \leq \|a - b\| + \|b - b_1\| \leq \varepsilon + r \leq \varepsilon + Lr. \]

so that \( b_1 \in B_X(a, Lr + \varepsilon) \).

Hence, \( b_1 \in C_1 \cap (C + rB_X) \cap B_X(a, Lr + \varepsilon) \) proving (2.29). In a similar way we show that
\[ C_2 \cap (C + rB_X) \cap B_X(a, Lr + \varepsilon) \neq \emptyset. \]

Thus (2.28) holds proving the existence of a point \( x \in X \) such that
\[ x \in C_1 \cap C_2 \cap (C + rB_X) \cap B_X(a, Lr + \varepsilon). \] (2.30)

In particular, \( x \in C + rB_X \) so that \( B_X(x, r) \cap C \neq \emptyset \) proving that condition (2.9) of Proposition 2.3 holds. This proposition tells us that
\[ C \cap B_X(x, Lr) + \vartheta(L) \varepsilon B_X \supseteq (C + \varepsilon B_X) \cap (B_X(x, Lr) + \varepsilon B_X) = (C + \varepsilon B_X) \cap B_X(x, Lr + \varepsilon). \]

From (2.30) and (2.26) we learn that \( a \in B_X(x, Lr + \varepsilon) \) and \( a \in C + \varepsilon B_X \). Hence,
\[ (C + \varepsilon B_X) \cap B_X(x, Lr + \varepsilon) \ni a \]

proving that
\[ C \cap B_X(x, Lr) + \vartheta(L) \varepsilon B_X \ni a. \]

Finally, property (2.30) tells us that \( x \in C_1 \cap C_2 \) proving the required inclusion (2.27). The proof of the proposition is complete. ■

We finish the section with the proof of Claim [1.4] see [14, Section 5]. For completeness, we give this simple proof here.

Proof of Claim [1.4] The proof relies on the following selection theorem which is a special case of [27, Theorem 1.2].

Theorem 2.6 Let \( X \) be a Banach space, and let \( m \geq 1 \). Then there exists a map \( St : \mathcal{K}_m(X) \to X \) such that
(a) \( St(K) \in K \) for all \( K \in \mathcal{K}_m(X) \)
and
(b) \( \|St(K) - St(K')\| \leq C(m) \cdot d_H(K, K') \) for all \( K, K' \in \mathcal{K}_m(X) \).

Here \( C(m) \) depends only on \( m \).
We refer to \( \text{St}(K) \) as the “Steiner-type point” of \( K \), and we call the mapping \( \text{St} : \mathcal{K}_m(X) \to X \) the “Steiner-type selector”. In the special case \( X = \mathbb{R}^m \), we can take \( \text{St}(K) \) to be the Steiner point of \( K \), see, e.g., [5].

To construct the Lipschitz selection \( f \) and establish the claim, we just set

\[
f(x) = \text{St}(G(x)) \quad \text{for} \quad x \in M.
\]

Since \( G(x) \in \mathcal{K}_m(X) \) for each \( x \in M \), the function \( f \) is well defined on \( M \). By part (i) of Definition 1.3 and by property \((a)\) of the Steiner-type point,

\[
f(x) = \text{St}(G(x)) \in G(x) \subset F(x) \quad \text{for every} \quad x \in M.
\]

On the other hand, part (ii) of Definition 1.3 and property \((\beta)\) of the Steiner-type point imply that

\[
\|f(x) - f(y)\| = \|\text{St}(G(x)) - \text{St}(G(y))\| \leq C(m) \cdot d_H(G(x), G(y)) \leq C(m) \cdot \gamma \rho(x, y)
\]

for all \( x, y \in M \) proving that \( f \) is a Lipschitz selection of the set-valued mapping \( F \) with Lipschitz seminorm at most \( C(m) \cdot \gamma \). \( \blacksquare \)

3. Main Theorem for two dimensional Banach spaces.

In this section we prove Theorem 1.9 and the following

**Theorem 3.1** Let \( \mathcal{M} = (M, \rho) \) be a pseudometric space. Let \( X \) be a two dimensional Euclidean space, and let \( m = 2 \). In this case Conjecture 1.8 holds for \( \ell = 2 \) and for every \( \lambda_1, \lambda_2 \) and \( \gamma \) such that

\[
\lambda_1 \geq e(\mathcal{M}, X), \quad \lambda_2 \geq 3 \lambda_1, \quad \gamma \geq \lambda_2 \left\{ 1 + 2 \lambda_2 / \left( \lambda_2^2 - \lambda_1^2 \right) \right\}^2.
\]  \( (3.1) \)

In other words, the following statement is true: Let \( X \) be a two dimensional Euclidean space, and let \( F \) be a set-valued mapping such that for every subset \( M' \subset M \) with \( \#M' \leq 4 \), the restriction \( F|_{M'} \) of \( F \) to \( M' \) has a Lipschitz selection with Lipschitz seminorm at most \( C(m) \cdot \gamma \).

Then the mapping \( F^{[2]} \) defined by (1.8) have properties (1.9) and (1.10) provided \( \lambda_1, \lambda_2 \) and \( \gamma \) satisfy (3.1).

In particular, one can set

\[
\lambda_1 = 4/\pi, \quad \lambda_2 = 12/\pi \quad \text{and} \quad \gamma = 38.
\]  \( (3.2) \)

Furthermore, if \( M \) is a subset of a Euclidean space \( E \), \( \rho \) is the metric in \( E \), and \( X \) is a two dimensional Euclidean space, properties (1.9) and (1.10) hold for \( \lambda_1 = 1, \lambda_2 = 3, \) and \( \gamma = 25 \).

We begin with the proof of Theorem 1.9. First, we recall the notion of the Lipschitz extension constant \( e(\mathcal{M}, X) \) which we use in the formulation of this theorem.

**Definition 3.2** Let \( \mathcal{M} = (M, \rho) \) be a pseudometric space, and let \( X \) be a Banach space. We define Lipschitz extension constant \( e(\mathcal{M}, X) \) of \( X \) with respect to \( \mathcal{M} \) as the infimum of the constants \( \lambda > 0 \) such that for every subset \( M' \subset M \), and every Lipschitz mapping \( \tilde{f} : M' \to X \), there exists a Lipschitz extension \( \tilde{f} : M \to X \) of \( f \) to all of \( M \) such that \( \|\tilde{f}\|_{\text{Lip}(M', X)} \leq \lambda \|f\|_{\text{Lip}(M', X)} \).
Remark 3.3 It is known that
\[ e(\mathcal{M}, \ell^n_\infty) = 1 \quad \text{for every pseudometric space } \mathcal{M} = (\mathcal{M}, \rho). \quad (3.3) \]
(It is immediate from the case \( n = 1 \) which coincides with the McShane-Whitney extension theorem.)

It follows from [23] and [6] that
\[ e(\mathcal{M}, X) \leq \frac{4}{3}\]
provided \( X \) is an arbitrary two dimensional Banach space. See also [4].

It is also known that \( e(\mathcal{M}, X) \leq \frac{n \Gamma(\frac{n}{2})}{(\sqrt{\pi} \Gamma(\frac{n+1}{2}))} \) provided \( X \) is an \( n \)-dimensional Euclidean space. See [23] and [16]. In particular,
\[ e(\mathcal{M}, X) = \frac{4}{\pi} \quad (3.4) \]
whenever \( X \) is a two dimensional Euclidean space.

We also note that, by Kirszbraun’s extension theorem [18], \( e(\mathcal{M}, X) = 1 \) provided \( X \) is a Euclidean space, \( \mathcal{M} \) is a subset of a Euclidean space \( E \), and \( \rho \) is the metric in \( E \). ▷

Proof of Theorem 1.9 Let \( \mathcal{M} = (\mathcal{M}, \rho) \) be a pseudometric space, and let \( X \) be a two dimensional Banach space. Let \( F : \mathcal{M} \to \mathcal{K}(X) \) be a set-valued mapping satisfying the hypothesis of Theorem 1.9. This enables us to make the following

Assumption 3.4 For every subset \( \mathcal{M}' \subset \mathcal{M} \) with \#\( \mathcal{M}' \leq 4 \), the restriction \( F|_{\mathcal{M}'} \) of \( F \) to \( \mathcal{M}' \) has a \( \rho \)-Lipschitz selection \( f_{\mathcal{M}'} : \mathcal{M}' \to X \) with \( \rho \)-Lipschitz seminorm \( \| f_{\mathcal{M}'} \|_{\text{Lip}(\mathcal{M}', \rho), X} \leq 1 \).

Fix constants
\[ L \geq 3 \quad (3.5) \]
and
\[ \alpha \geq e(\mathcal{M}, X) \quad \text{where } \mathcal{M} = (\mathcal{M}, \rho). \quad (3.6) \]

We introduce a new pseudometric on \( \mathcal{M} \) defined by
\[ d(x, y) = \alpha \rho(x, y), \quad x, y \in \mathcal{M}. \quad (3.7) \]

This definition, Definition 3.2 and inequality 3.6 imply the following

Claim 3.5 Let \( \mathcal{M}' \subset \mathcal{M} \), and let \( f : \mathcal{M}' \to X \) be a \( \rho \)-Lipschitz mapping on \( \mathcal{M}' \). There exists a \( d \)-Lipschitz extension \( \tilde{f} : \mathcal{M} \to X \) of \( f \) to all of \( \mathcal{M} \) with \( d \)-Lipschitz seminorm
\[ \| \tilde{f} \|_{\text{Lip}(\mathcal{M}, d), X} \leq \| f \|_{\text{Lip}(\mathcal{M}', \rho), X}. \]

We introduce set-valued mappings
\[ F^{[1]}(x) = \bigcap_{z \in \mathcal{M}} \left[ F(z) + d(x, z) B_X \right], \quad x \in \mathcal{M}, \quad (3.8) \]
and
\[ F^{[2]}(x) = \bigcap_{z \in \mathcal{M}} \left[ F^{[1]}(z) + L d(x, z) B_X \right], \quad x \in \mathcal{M}. \quad (3.9) \]
Thus, $F^{[1]}$ and $F^{[2]}$ are the first and the second order $([1, L], d)$-balanced refinements of $F$ respectively. See Definition 1.6.

Formulae (3.8) and (3.9) imply an explicit formula for the mapping $F^{[2]}$:

$$F^{[2]}(x) = \bigcap_{z \in M} \left\{ \left( \bigcap_{y \in M} [F(y) + d(z, y) B_X] \right) + L d(x, z) B_X \right\}, \quad x \in M.$$  

We will prove that if $L$ and $\alpha$ are the constants satisfying (3.5) and (3.6) respectively, then the following two facts holds:

$$F^{[2]}(x) \neq \emptyset \quad \text{for every} \quad x \in M, \quad (3.10)$$

and

$$d_H(F^{[2]}(x), F^{[2]}(y)) \leq \gamma_0(L) d(x, y) \quad \text{for all} \quad x, y \in M. \quad (3.11)$$

Here

$$\gamma_0(L) = L \cdot \theta(L)^2 \quad (3.12)$$

where $\theta = \theta(L)$ is the function from Proposition 2.3. See (2.11) and (2.12).

We prove property (3.10) and inequality (3.11) in Proposition 3.9 and Proposition 3.10 respectively.

We begin with the proof of property (3.10). This proof relies on a series of auxiliary lemmas.

Lemma 3.6 Let $\mathcal{K}$ be a collection of convex closed subsets of a two dimensional Banach space $X$. Suppose that

$$\bigcap_{K \in \mathcal{K}} K \neq \emptyset. \quad (3.13)$$

Then for every $r \geq 0$ the following equality

$$\left( \bigcap_{K \in \mathcal{K}} K \right) + rB_X = \bigcap_{K, K' \in \mathcal{K}} \left\{ (K \cap K') + rB_X \right\} \quad (3.14)$$

holds.

Proof. Clearly, the right hand side of (3.14) contains its left hand side. We will prove the converse statement. Let

$$x \in \bigcap_{K, K' \in \mathcal{K}} \left\{ (K \cap K') + rB_X \right\}. \quad (3.15)$$

We prove that $x \in \bigcap\{K : K \in \mathcal{K}\} + rB_X$. Note that this statement is true if and only if

$$B_X(x, r) \bigcap \left( \bigcap_{K \in \mathcal{K}} K \right) \neq \emptyset. \quad (3.16)$$

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Let \( S = K \cup \{ B_X(x, r) \} \). Helly’s Theorem \(^2\) tells us that \((3.16)\) is true provided
\[
\bigcap_{k \in S'} K \neq \emptyset
\]  \hspace{1cm} (3.17)
for every subfamily \( S' \subset S \) consisting of at most three elements.

If \( B_X(x, r) \notin S' \) then \((3.17)\) follows from \((3.13)\). Let \( B_X(x, r) \in S' \). Then \( S' \setminus \{ B_X(x, r) \} \) is a subfamily of \( K \) containing at most two elements. In other words, there exist \( K, K' \in K \) such that \( S' \setminus \{ B_X(x, r) \} = \{ K, K' \} \).

Property \((3.15)\) tells us that \( x \in (K \cap K') + rB_X \). Hence,
\[
\bigcap_{k \in S'} K = B_X(x, r) \cap K \cap K' = \emptyset
\]
proving \((3.17)\) and the lemma. \( \blacksquare \)

**Lemma 3.7** For each \( x \in M \) the set \( F[1] \in K(X) \). Furthermore, for every \( x, z \in M \) the following equality
\[
F[1](z) + Ld(x, z)B_X = \bigcap_{y', y'' \in M} \left[ \{ F(y') + d(z, y')B_X \} \cap \{ F(y'') + d(z, y'')B_X \} \right] + Ld(x, z)B_X \]  \hspace{1cm} (3.18)
holds.

**Proof.** Let \( x \in M \). Formula \((3.8)\) and Helly’s Theorem \(^2\) tell us that \( F[1](x) \neq \emptyset \) provided
\[
[F(z_1) + d(x, z_1)B_X] \cap [F(z_2) + d(x, z_2)B_X] \cap [F(z_3) + d(x, z_3)B_X] \neq \emptyset
\]  \hspace{1cm} (3.19)
for every \( z_1, z_2, z_3 \in M \).

This property easily follows from Assumption \(^3\). Indeed, let \( M' = \{ x, z_1, z_2, z_3 \} \). Then \( \# M' \leq 4 \).

Assumption \(^3\) tells us that there exists a \( \rho \)-Lipschitz selection \( f_{M'} : M' \rightarrow X \) of \( \rho \)-Lipschitz seminorm \( \| f_{M'} \|_{\text{Lip}(M', \rho), X} \leq 1 \). In particular, \( f_{M'}(z_i) \in F(z_i) \) and
\[
\| f_{M'}(x) - f_{M'}(z_i) \| \leq \rho(x, z_i) \leq \alpha \rho(x, z_i) = d(x, z_i) \quad \text{for every} \quad i = 1, 2, 3.
\]
See \((3.7)\). These properties of \( f_{M'} \) and \( F \) tell us that the point \( f_{M'}(x) \) belongs to the left hand side of \((3.19)\). Thus \((3.19)\) holds for arbitrary \( z_i \in M, i = 1, 2, 3 \), proving that \( F[1](x) \neq \emptyset \).

Finally, this property, Lemma \(^3\) and formula \((3.8)\) imply \((3.18)\) proving the lemma. \( \blacksquare \)

**Lemma 3.8** For every \( x \in M \) the following representation
\[
F[2](x) = \bigcap_{u, u', u'' \in M} \left[ \{ F(u') + d(u', u)B_X \} \cap [F(u'') + d(u'', u)B_X] + Ld(u, x)B_X \right]
\]
holds.

**Proof.** The lemma is immediate from \((3.9)\) and \((3.18)\). \( \blacksquare \)

Given \( x, u, u', u'' \in M \) we set
\[
T_x(u, u', u'') = [F(u') + d(u', u)B_X] \cap \{ F(u'') + d(u'', u)B_X \} + Ld(u, x)B_X.
\]  \hspace{1cm} (3.20)
In these settings Lemma \(^3\) reformulates as follows:
\[
F[2](x) = \bigcap_{u, u', u'' \in M} T_x(u, u', u'').
\]  \hspace{1cm} (3.21)
**Proposition 3.9** For every \( x \in \mathcal{M} \) the set \( F^{[2]}(x) \neq \emptyset \).

**Proof.** Formula (3.21) and Helly’s Theorem [2.4] tell us that \( F^{[2]}(x) \neq \emptyset \) provided for every choice of points \( u_i, u'_i, u''_i \in \mathcal{M}, \ i = 1, 2, 3 \), we have

\[
T_x(u_1, u'_1, u''_1) \cap T_x(u_2, u'_2, u''_2) \cap T_x(u_3, u'_3, u''_3) \neq \emptyset. \tag{3.22}
\]

We set

\[
r_i = d(x, u_i), \quad i = 1, 2, 3. \tag{3.23}
\]

Without loss of generality, we may assume that

\[
r_1 \leq r_2 \leq r_3. \tag{3.24}
\]

For each \( i \in \{1, 2, 3\} \) we also set

\[
G(u'_i) = F(u'_i) + d(u'_i, u_i)B_X \quad \text{and} \quad G(u''_i) = F(u''_i) + d(u''_i, u_i)B_X. \tag{3.25}
\]

We will prove that there exist points \( y_i \in X, \ i = 1, 2, 3 \), such that

\[
y_i \in G(u'_i) \cap G(u''_i) \quad \text{for every} \quad i = 1, 2, 3, \tag{3.26}
\]

and

\[
\|y_1 - y_2\| \leq r_1 + r_2 \quad \text{and} \quad \|y_1 - y_3\| \leq r_1 + 2r_2 + r_3. \tag{3.27}
\]

Let us see that the existence of the points \( y_i \) with these properties implies (3.22). In fact, let us set

\[
z = y_1 + \tau(y_3 - y_1) = y_3 + (1 - \tau)(y_1 - y_3)
\]

with \( \tau = r_1/(r_1 + 2r_2 + r_3) \). Then, thanks to (3.27),

\[
\|y_1 - z\| = \tau\|y_3 - y_1\| \leq \frac{r_1}{r_1 + 2r_2 + r_3} \cdot (r_1 + 2r_2 + r_3) = r_1,
\]

and

\[
\|y_3 - z\| = (1 - \tau)\|y_3 - y_1\| \leq \frac{2r_2 + r_3}{r_1 + 2r_2 + r_3} \cdot (r_1 + 2r_2 + r_3) = 2r_2 + r_3.
\]

Hence,

\[
\|y_2 - z\| \leq \|y_2 - y_1\| + \|y_1 - z\| \leq r_1 + r_2 + r_1 = 2r_1 + r_2.
\]

From these inequalities, (3.24) and (3.25) we have

\[
\|z - y_i\| \leq 3r_i = 3d(x, u_i), \quad i = 1, 2, 3. \tag{3.28}
\]

Let us prove that \( z \in T_x(u_1, u'_1, u''_1) \) for each \( i \in \{1, 2, 3\} \). In fact, we know that \( L \geq 3 \), see (3.5). Furthermore, we know that \( y_i \in G(u'_i) \cap G(u''_i) \), see (3.26), so that, thanks to (3.28), (3.25) and (3.20),

\[
z \in G(u'_i) \cap G(u''_i) + 3d(x, u_i)B_X \subset G(u'_i) \cap G(u''_i) + Ld(x, u_i)B_X = T_x(u_i, u'_i, u''_i)
\]

proving (3.22).
Thus, our aim is to prove the existence of points \( y_i \) satisfying (3.26) and (3.27). We will do this in three steps.

**STEP 1.** We introduce sets \( W_i \subset X, i = 1, \ldots, 4 \), defined by

\[
W_1 = G(u'_1), \quad W_2 = G(u''_1), \quad W_3 = G(u'_2) \cap G(u''_2) + (r_1 + r_2)B_X, \quad (3.29)
\]

and

\[
W_4 = G(u'_3) \cap G(u''_3) + (r_1 + 2r_2 + r_3)B_X. \quad (3.30)
\]

Prove that if

\[
W_1 \cap W_2 \cap W_3 \cap W_4 \neq \emptyset \quad (3.31)
\]

then there exist the required points \( y_i \) satisfying (3.26) and (3.27).

In fact, (3.29), (3.30) and the condition (3.31) imply the existence of points \( y_i \in G(u'_i) \cap G(u''_i), \)

\( i = 1, 2, 3 \), such that \( \|y_i - y_2\| \leq r_1 + r_2 \) and \( \|y_i - y_3\| \leq r_1 + 2r_2 + r_3 \). Recall that \( r_1 \leq r_2 \leq r_3 \) so that

\[
\|y_1 - y_2\| \leq 3r_1 + 3r_2 \quad \text{and} \quad \|y_1 - y_3\| \leq r_1 + 2r_2 + r_3 \leq 3r_1 + 3r_3.
\]

Furthermore,

\[
\|y_2 - y_3\| \leq \|y_1 - y_2\| + \|y_1 - y_3\| \leq (r_1 + r_2) + (r_1 + 2r_2 + r_3) \leq 3r_2 + 3r_3.
\]

These inequalities prove that the points \( y_i \)'s satisfy (3.26) and (3.27).

Thus, it suffices to prove property (3.31). Helly’s Theorem 2.4 tells us that (3.31) holds provided the intersection of any three elements of the family of sets \( \{W_1, W_2, W_3, W_4\} \) is non-empty.

**STEP 2.** Prove that

\[
W_1 \cap W_3 \cap W_4 \neq \emptyset. \quad (3.32)
\]

This statement is equivalent to the following one:

\[
G(u'_1) \cap [G(u'_2) \cap G(u''_2) + (r_1 + r_2)B_X] \cap [G(u'_3) \cap G(u''_3) + (r_1 + 2r_2 + r_3)B_X] \neq \emptyset. \quad (3.33)
\]

Let

\[
V_1 = G(u'_1) + (r_1 + r_2)B_X, \quad V_2 = G(u'_2), \quad V_3 = G(u''_2), \quad (3.34)
\]

and let

\[
V_4 = G(u'_3) \cap G(u''_3) + (r_2 + r_3)B_X. \quad (3.35)
\]

Let us see that if

\[
V_1 \cap V_2 \cap V_3 \cap V_4 \neq \emptyset \quad (3.36)
\]

then (3.32) and (3.33) hold.

Indeed, definitions (3.34) and (3.35), and property (3.36) imply the existence of points \( z_1 \in G(u'_1), \)

\( z_2 \in G(u'_2) \cap G(u''_2) \) and \( z_3 \in G(u'_3) \cap G(u''_3) \) such that \( \|z_1 - z_2\| \leq r_1 + r_2 \) and \( \|z_2 - z_3\| \leq r_2 + r_3 \). Hence,

\[
\|z_1 - z_3\| \leq \|z_1 - z_2\| + \|z_2 - z_3\| \leq (r_1 + r_2) + (r_2 + r_3) = r_1 + 2r_2 + r_3.
\]
Thus, thanks to (3.29) and (3.30), the point $z_1 \in W_1 \cap W_3 \cap W_4$ proving (3.33).

Let us prove (3.36). We will again make use of Helly’s Theorem 2.4 which tells us that (3.36) holds provided every three elements of the family $\{V_1, V_2, V_3, V_4\}$ have a common point.

First, let us prove that

$$V_1 \cap V_2 \cap V_4 = [G(u_1') + (r_1 + r_2)B_X] \cap [G(u_2') \cap G(u_3') + (r_2 + r_3)B_X] \neq \emptyset. \quad (3.37)$$

Let $M_1 = \{u_1', u_2', u_3', u_4'\}$. Clearly, $\#M_1 \leq 4$, so that, thanks to Assumption 3.4, there exists a $\rho$-Lipschitz mapping $f_{M_1} : M_1 \to X$ with $\|f_{M_1}\|_{\text{Lip}(M_1, \rho)} \leq 1$ such that

$$f_{M_1}(u_1') \in F(u_1'), \quad f_{M_1}(u_2') \in F(u_2'), \quad f_{M_1}(u_3') \in F(u_3'), \quad \text{and} \quad f_{M_1}(u_4') \in F(u_4').$$

Claim 3.5 tells us that there exists a d-Lipschitz mapping $\tilde{f}_1 : M \to X$ with d-Lipschitz seminorm $\|\tilde{f}_1\|_{\text{Lip}(M, d)} \leq \|f_{M_1}\|_{\text{Lip}(M_1, \rho)} \leq 1$ such that $\tilde{f}_1|_{M_1} = f_{M_1}$.

Prove that

$$\tilde{f}_1(u_2') \in V_1 \cap V_2 \cap V_4.$$

We know that

$$\tilde{f}_1(u_2') = f_{M_1}(u_2') \in F(u_2')$$

and $\|\tilde{f}_1(u_2') - \tilde{f}_1(u_2)\| \leq d(u_2', u_2).

Hence,

$$\tilde{f}_1(u_2) \in F(u_2') + d(u_2', u_2)B_X = G(u_2') = V_2.$$

In the same way we prove that $\tilde{f}_1(u_1) \in G(u_1')$. We also know that

$$\|\tilde{f}_1(u_1) - \tilde{f}_1(u_2)\| \leq d(u_1, u_2)$$

so that $\tilde{f}_1(u_2) \in G(u_1') + d(u_1, u_2)B_X$. By the triangle inequality,

$$d(u_1, u_2) \leq d(u_1, x) + d(x, u_2) = r_1 + r_2$$

proving that $\tilde{f}_1(u_2) \in G(u_1') + (r_1 + r_2)B_X = V_1$.

We also know that

$$\tilde{f}_1(u_3') = f_{M_1}(u_3') \in F(u_3'), \quad \tilde{f}_1(u_4') = f_{M_1}(u_4') \in F(u_4')$$

and

$$\|\tilde{f}_1(u_3) - \tilde{f}_1(u_3')\| \leq d(u_3, u_3'), \quad \|\tilde{f}_1(u_4) - \tilde{f}_1(u_4')\| \leq d(u_4, u_4').$$

Hence,

$$\tilde{f}_1(u_3) \in [F(u_3') + d(u_3', u_3)B_X] \cap [F(u_4') + d(u_4', u_3)B_X] = G(u_3') \cap G(u_3').$$

Furthermore, $\|\tilde{f}_1(u_3) - \tilde{f}_1(u_3')\| \leq d(u_2, u_3)$. These properties of $\tilde{f}_1(u_3)$ and the triangle inequality $d(u_2, u_3) \leq d(u_2, x) + d(x, u_3) = r_2 + r_3$ imply the following:

$$\tilde{f}_1(u_2) \in G(u_3') \cap G(u_3') \cap G(u_4') + (r_2 + r_3)B_X = V_4.$$

Thus, $\tilde{f}_1(u_2) \in V_1 \cap V_2 \cap V_4$ proving (3.37).

In the same fashion we show that $V_1 \cap V_3 \cap V_4 \neq \emptyset$. 

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Prove that

\[ V_2 \cap V_3 \cap V_4 = G(u'_2) \cap G(u'_2) \cap [G(u'_3) \cap G(u'_3) + (r_2 + r_3)B_X] \neq \emptyset. \]  \tag{3.38}  

Following the same scheme as in the proof of (3.37), we introduce a set \( M_2 = \{u'_2, u'_2, u'_3, u'_3\} \). Assumption 3.4 provides the existence of a \( \rho \)-Lipschitz selection \( f_{M_2} : M_2 \to X \) of the restriction \( F|_{M_2} \) with \( \|f_{M_2}\|_{\text{Lip}(M_2, X)} \leq 1 \). In turn, Claim 3.5 tells us that there exists a d-Lipschitz mapping \( \tilde{f}_2 : M \to X \) with d-Lipschitz seminorm \( \|\tilde{f}_2\|_{\text{Lip}(M, X)} \leq \|f_{M_2}\|_{\text{Lip}(M_2, X)} \leq 1 \) such that \( \tilde{f}_2|_{M_2} = f_{M_2} \).

Considerations, similar to those in the proof of (3.37), enable us to show that

\[ \tilde{f}_2(u_2) \in V_2 \cap V_3 \cap V_4 \]

proving (3.38).

Finally, following the same ideas, we prove that

\[ V_1 \cap V_2 \cap V_3 = [G(u'_1) + (r_1 + r_2)B_X] \cap G(u'_2) \cap G(u'_3) \neq \emptyset. \]  \tag{3.39}  

We introduce a set \( M_3 = \{u'_2, u'_2, u'_3, u'_3\} \). Assumption 3.4 guarantees the existence of a \( \rho \)-Lipschitz selection \( f_{M_3} : M_3 \to X \) of the restriction \( F|_{M_3} \) with \( \|f_{M_3}\|_{\text{Lip}(M_3, X)} \leq 1 \). Claim 3.5 provides the existence of a d-Lipschitz mapping \( \tilde{f}_3 : M \to X \) with d-Lipschitz seminorm \( \|\tilde{f}_3\|_{\text{Lip}(M, X)} \leq \|f_{M_3}\|_{\text{Lip}(M_3, X)} \leq 1 \) such that \( \tilde{f}_3|_{M_3} = f_{M_3} \).

Then we show that \( \tilde{f}_3(u_2) \in V_1 \cap V_2 \cap V_3 \) proving (3.39).

We leave the details of the proofs of properties (3.38) and (3.39) to the interested reader.

Thus, we have proved (3.36). As we have seen above, (3.36) implies (3.32). See STEP 2.

STEP 3. We return to the proof of the property (3.31).

We prove that

\[ W_2 \cap W_3 \cap W_4 = G(u'_2) \cap [G(u'_2) \cap G(u'_3) + (r_1 + r_2)B_X] \cap [G(u'_3) \cap G(u'_3) + (r_1 + 2r_2 + r_3)B_X] \neq \emptyset \]

in the same fashion as property (3.32).

Let us show that

\[ W_1 \cap W_2 \cap W_3 = G(u'_1) \cap G(u'_1) \cap [G(u'_3) \cap G(u'_3) + (r_1 + 2r_2 + r_3)B_X] \neq \emptyset. \]  \tag{3.40}  

We put \( M_4 = \{u'_1, u'_1, u'_2, u'_2\} \). This set contains at most 4 points. In this case Assumption 3.4 guarantees us the existence of a \( \rho \)-Lipschitz mapping \( f_{M_4} : M_4 \to X \) with \( \|f_{M_4}\|_{\text{Lip}(M_4, X)} \leq 1 \) such that \( f_{M_4}(u) \in F(u) \) for every \( u \in M_4 \).

Claim 3.5 enables us to extend \( f_{M_4} \) to a d-Lipschitz mapping \( \tilde{f}_4 : M \to X \) with d-Lipschitz seminorm \( \|\tilde{f}_4\|_{\text{Lip}(M, X)} \leq \|f_{M_4}\|_{\text{Lip}(M_4, X)} \leq 1 \). Then we show that \( \tilde{f}_4(u_1) \in W_1 \cap W_2 \cap W_3 \) proving (3.40).

In a similar way we prove that

\[ W_1 \cap W_2 \cap W_3 = G(u'_1) \cap G(u'_1) \cap [G(u'_3) \cap G(u'_3) + (r_1 + 2r_2 + r_3)B_X] \neq \emptyset. \]  \tag{3.41}  

More specifically, we set \( M_5 = \{u'_1, u'_1, u'_2, u'_2\} \). In this case, Assumption 3.4 provides the existence of a \( \rho \)-Lipschitz mapping \( f_{M_5} : M_5 \to X \) with \( \|f_{M_5}\|_{\text{Lip}(M_5, X)} \leq 1 \) such that \( f_{M_5}(u) \in F(u) \) for every \( u \in M_5 \).
Let \( \rho \) be a \( \rho \)-Lipschitz selection \( f_{M_0} \) of the restriction \( F|_{M_0} \) with \( \|f_{M_0}\|_{\text{Lip}(M_0, \rho, X)} \leq 1 \). Finally, we show that \( f_{M_0}(u_1) \in W_1 \cap W_2 \cap W_3 \) proving (3.41).

We leave the details of the proofs of (3.40) and (3.41) to the interested reader.

The proof of the proposition is complete. ■

We turn to the proof of inequality (3.11).

**Proposition 3.10** For every \( x, y \in M \) the following inequality

\[
d_H(F(x), F(y)) \leq \gamma_0(L) d(x, y)
\]  

(3.42)

holds. (We recall that \( \gamma_0(L) = L \theta(L)^2 \) is defined by (3.12).)

**Proof.** Let \( x, y \in M \). Formula (3.21) tells us that

\[
T_x(u) = \bigcap_{u' \in M} T_x(u, u', u'') \quad \text{and} \quad F(x) = \bigcap_{u' \in M} T_x(u, u', u'').
\]  

(3.43)

Let

\[
\tau = \gamma_0(L) d(x, y).
\]

Representation (3.43), Lemma 3.6 and Proposition 3.9 imply the following:

\[
F(x) + \tau B_X = \bigcap \{ T_x(u, u', u'') \cap T_x(v, v', v'') + \tau B_X \}.
\]  

(3.44)

Here the first intersection in the right hand side of this equality is taken over all

\( u, u', u'', v, v', v'' \in M \).

Fix \( u, u', u'', v, v', v'' \in M \) and prove that

\[
A = T_x(u, u', u'') \cap T_x(v, v', v'') + \tau B_X \supset F(x).
\]  

(3.45)

We introduce the following sets:

\[
C_1 = F(u') + d(u', u)B_X, \quad C_2 = F(u'') + d(u'', u)B_X, \quad C = T_x(v, v', v'').
\]  

(3.46)

Let

\[
\varepsilon = L \theta(L) d(x, y) \quad \text{and} \quad r = d(x, u).
\]  

(3.47)

Then \( \tau = \gamma_0(L) d(x, y) = \theta(L) \varepsilon \), and

\[
A = T_x(u, u', u'') \cap T_x(v, v', v'') + \tau B_X = (C_1 \cap C_2 + LrB_X) \cap C + \theta(L) \varepsilon B_X.
\]

We want to apply Proposition 2.5 to the set \( A \). To do this we have to verify condition (2.25) of this proposition, i.e., to show that

\[
C_1 \cap C_2 \cap (C + rB_X) \neq \emptyset.
\]  

(3.48)

Let \( M' = \{ u', u'', v, v'' \} \). This set contains at most four elements so that, thanks to Assumption 3.4, there exists a \( \rho \)-Lipschitz selection \( f_{M'} \) of the restriction \( F|_{M'} \) with \( \|f_{M'}\|_{\text{Lip}(M', \rho, X)} \leq 1 \). Claim 3.5
enables us to extend \( f_{M'} \) to a \( d \)-Lipschitz mapping \( \tilde{f}_1 : M \to X \) defined on all of \( M \) with \( d \)-Lipschitz seminorm

\[
\|\tilde{f}_1\|_{\text{Lip}(M,d,X)} \leq \|f_{M'}\|_{\text{Lip}(M',\rho,X)} \leq 1.
\]

In particular, \( \tilde{f}_1(u') = f_{M'}(u') \in F(u') \), \( \tilde{f}_1(u'') = f_{M'}(u'') \in F(u'') \),

\[
\|\tilde{f}_1(u') - \tilde{f}_1(u'')\| \leq d(u', u), \quad \|\tilde{f}_1(u'') - \tilde{f}_1(u')\| \leq d(u'', u)
\]

and

\[
\|\tilde{f}_1(x) - \tilde{f}_1(u)\| \leq d(x, u) = r. \quad (3.49)
\]

Hence, \( \tilde{f}_1(u) \in C_1 \cap C_2 \).

In a similar way we show that \( \tilde{f}_1(x) \in T_x(v, v', v'') = C \). From this and (3.49) we have \( \tilde{f}_1(u) \in C + rB_X \). Hence,

\[
C_1 \cap C_2 \cap (C + rB_X) \ni \tilde{f}_1(u)
\]

proving (3.48).

Thus, property (2.25) of Proposition 2.5 holds. This proposition tells us that

\[
A = (C_1 \cap C_2 + LrB_X) \cap C + \theta(L) \varepsilon B_X
\]

\[
\sup [C_1 \cap C_2 + (Lr + \varepsilon)B_X] \cap [(C_1 + rB_X) \cap C + \varepsilon B_X] \cap [(C_2 + rB_X) \cap C + \varepsilon B_X]
\]

\[
= S_1 \cap S_2 \cap S_3.
\]

Prove that

\[
S_i \supset F^{[2]}(y) \quad \text{for every} \quad i = 1, 2, 3. \quad (3.50)
\]

We begin with the set \( S_1 = C_1 \cap C_2 + (Lr + \varepsilon)B_X \). Thus,

\[
S_1 = \{F(u') + d(u', u)B_X\} \cap \{F(u'') + d(u'', u)B_X\} + (Ld(u, x) + L\theta(L)d(x, y))B_X.
\]

See (3.46). By the triangle inequality,

\[
d(u, x) + \theta(L)d(x, y) \geq d(u, x) + d(x, y) \geq d(u, y)
\]

so that

\[
S_1 \supset \{F(u') + d(u', u)B_X\} \cap \{F(u'') + d(u'', u)B_X\} + Ld(u, y)B_X = T_y(u, u', u'').
\]

But \( T_y(u, u', u'') \supset F^{[2]}(y) \), see (3.43), which implies the required inclusion \( S_1 \supset F^{[2]}(y) \).

We turn to the proof of the inclusion \( S_2 \supset F^{[2]}(y) \). Note that \( S_2 \) is defined by

\[
S_2 = (C_1 + rB_X) \cap C + \varepsilon B_X. \quad (3.51)
\]

By the triangle inequality,

\[
C_1 + rB_X = F(u') + d(u', u)B_X + d(u, x)B_X \supset F(u') + d(u', x)B_X = F^{[2]}(y).
\]

Let

\[
\overline{C} = F(u') + d(u', x)B_X, \quad \overline{C}_1 = F(v') + d(v', v)B_X, \quad \overline{C}_2 = F(v'') + d(v'', v)B_X,
\]

(3.53)
and let
\[ \tilde{r} = d(v, x). \] (3.54)

In these settings
\[ C = T_x(v, v', v'') = \overline{C}_1 \cap \overline{C}_2 + L\tilde{r}B_X. \]

Let
\[ \overline{A} = (\overline{C}_1 \cap \overline{C}_2 + L\tilde{r}B_X) \cap \overline{C} + \varepsilon B_X. \] (3.55)

Then, thanks to (3.51) and (3.52),
\[ S_2 \supset \{ F(u') + d(u', x)B_X \} \cap C + \varepsilon B_X = (\overline{C}_1 \cap \overline{C}_2 + L\tilde{r}B_X) \cap \overline{C} + \varepsilon B_X = \overline{A}. \] (3.56)

Prove that
\[ \overline{A} \supset F^{[2]}(y). \] (3.57)

As in the previous case, we will do this by applying Proposition 2.5 to the set \( \overline{A} \). But first we have to show that the hypothesis of this proposition holds for \( \overline{A} \), i.e.,
\[ \overline{C}_1 \cap \overline{C}_2 \cap (\overline{C} + \tilde{r}B_X) \neq \emptyset. \] (3.58)

Let \( \hat{M} = \{ u', v', v'' \} \). Assumption \[3.4\] tells us that the restriction \( F|_{\hat{M}} \) of \( F \) to \( \hat{M} \) has a \( \rho \)-Lipschitz selection \( f_{\hat{M}} : \hat{M} \to X \) with \( \| f_{\hat{M}} \|_{\text{Lip}((\hat{M}, \rho), X)} \leq 1 \). In turn, Claim \[3.5\] tells us that there exists a \( d \)-Lipschitz mapping \( \tilde{f}_2 : \hat{M} \to X \) with \( d \)-Lipschitz seminorm \( \| \tilde{f}_2 \|_{\text{Lip}(\hat{M}, d), X} \leq \| f_{\hat{M}} \|_{\text{Lip}((\hat{M}, \rho), X)} \leq 1 \) such that \( \tilde{f}_2|_{\hat{M}} = f_{\hat{M}} \).

In particular,
\[ \tilde{f}_2(u') = f_{\hat{M}}(u') \in F(u'), \quad \tilde{f}_2(v') = f_{\hat{M}}(v') \in F(v'), \quad \tilde{f}_2(v'') = f_{\hat{M}}(v'') \in F(v''). \]

In addition, \( \| \tilde{f}_2(x) - \tilde{f}_2(u') \| \leq d(x, u') \),
\[ \| \tilde{f}_2(v') - \tilde{f}_2(v) \| \leq d(v', v), \quad \| \tilde{f}_2(v'') - \tilde{f}_2(v) \| \leq d(v'', v) \quad \text{and} \quad \| \tilde{f}_2(x) - \tilde{f}_2(v) \| \leq d(x, v). \]

Combining these properties of \( \tilde{f}_2 \) with definitions (3.33) and (3.54), we conclude that
\[ \overline{C}_1 \cap \overline{C}_2 \cap (\overline{C} + \tilde{r}B_X) \ni \tilde{f}_2(v) \]
proving (3.58).

We recall that \( \varepsilon = L \theta(L) d(y, x) \), see (3.47), so that
\[ \overline{A} = (\overline{C}_1 \cap \overline{C}_2 + \tilde{r}B_X) \cap \overline{C} + L \theta(L) d(x, y)B_X, \quad \text{see (3.55)}. \]

We apply Proposition 2.5 to \( \overline{A} \) and obtain the following:
\[ \overline{A} \supset \{ \overline{C}_1 \cap \overline{C}_2 + (L\tilde{r} + Ld(x, y))B_X \} \]
\[ \cap \{ (\overline{C}_1 + \tilde{r}B_X) \cap \overline{C} + Ld(x, y)B_X \} \cap \{ (\overline{C}_2 + \tilde{r}B_X) \cap \overline{C} + Ld(x, y)B_X \} \]
\[ = \overline{S}_1 \cap \overline{S}_2 \cap \overline{S}_3. \]
Prove that
\[ \bar{S}_i \supset F^{[2]}(y) \] for every \( i = 1, 2, 3 \). \hfill (3.59)

First, let us show that
\[ \bar{S}_1 = \bar{C}_1 \cap \bar{C}_2 + (L\bar{r} + L d(x,y))B_X \supset F^{[2]}(y). \] \hfill (3.60)

By (3.54) and the triangle inequality,
\[ \bar{r} + d(x,y) = d(v,x) + d(x,y) \geq d(v,y) \]
so that
\[ \bar{S}_1 \supset \bar{C}_1 \cap \bar{C}_2 + L d(v,y)B_X \]
\[ = \{ F(v') + d(v', v)B_X \} \cap \{ F(v''') + d(v''', v)B_X \} + L d(v,y)B_X \]
\[ = T_y(v', v'''). \]
See (3.53) and (3.20). This inclusion and (3.43) imply (3.60).

Prove that
\[ \bar{S}_2 = (\bar{C}_1 + \bar{r}B_X) \cap \bar{C}_1 + L d(x,y)B_X \supset F^{[2]}(y). \] \hfill (3.61)

Thanks to (3.53), (3.54) and the triangle inequality,
\[ \bar{C}_1 + \bar{r}B_X = F(v') + d(v', v)B_X + d(v,x)B_X \supset F(v') + d(v', x)B_X \]
so that
\[ \bar{S}_2 \supset \{ F(v') + d(v', x)B_X \} \cap \{ F(v''') + d(v''', x)B_X \} + L d(x,y)B_X = T_y(x, u', v'). \]
See (3.20). From this inclusion and (3.43) it follows that \( \bar{S}_2 \supset T_y(x, u', v') \supset F^{[2]}(y) \) proving (3.61).

In the same way we prove that
\[ \bar{S}_3 = (\bar{C}_2 + \bar{r}B_X) \cap \bar{C}_2 + L d(x,y)B_X \supset T_y(x, u', v'') \supset F^{[2]}(y). \]
This inclusion together with (3.60) and (3.61) imply (3.59). Hence,
\[ \bar{A} \supset \bar{S}_1 \cap \bar{S}_2 \cap \bar{S}_3 \supset F^{[2]}(y) \]
proving (3.57).

We know that \( S_2 \supset \bar{A} \), see (3.56), so that \( S_2 \supset F^{[2]}(y) \). In the same fashion we show that
\[ S_3 = (C_2 + rB_X) \cap C + L \varepsilon B_X \supset F^{[2]}(y) \]
proving (3.50). Hence,
\[ A \supset \bar{S}_1 \cap \bar{S}_2 \cap \bar{S}_3 \supset F^{[2]}(y) \]
proving (3.45).

Combining (3.45) with (3.44) we prove that
\[ F^{[2]}(x) + \gamma_0(L) d(x,y)B_X = F^{[2]}(x) + \tau B_X \supset F^{[2]}(y). \]
By interchanging the roles of \( x \) and \( y \) we obtain also

\[
F^{[2]}(y) + \gamma_0(L) d(x,y) B_X \supset F^{[2]}(x) .
\]

These two inclusions imply inequality \((3.42)\) proving the proposition. \(\blacksquare\)

We are in a position to finish the proof of Theorem \(1.9\)

Let \( \lambda_1, \lambda_2 \) and \( \gamma \) be parameters satisfying \((1.7)\). Thus, \( \lambda_1 \geq e(\mathfrak{M}, X), \lambda_2 \geq 3\lambda_1 \) and

\[
\gamma \geq \lambda_2 (3\lambda_2 + \lambda_1)^2 / (\lambda_2 - \lambda_1)^2 . \quad (3.62)
\]

We set \( \alpha = \lambda_1, L = \lambda_2 / \lambda_1 \). Then \( L \) and \( \alpha \) satisfies \((3.5)\) and \((3.6)\) respectively, i.e., \( L \geq 3 \) and \( \alpha \geq e(\mathfrak{M}, X) \). We also recall that

\[
d = \alpha \rho = \lambda_1 \rho, \quad \text{see} \quad (3.7) . \quad (3.63)
\]

In these settings, the set values mappings \( F^{[1]} \) and \( F^{[2]} \) defined by formulae \((3.8)\) and \((3.9)\) has the following representations:

\[
F^{[1]}(x) = \bigcap_{z \in \mathcal{M}} \left[ F(z) + \lambda_1 \rho(x,z) B_X \right] , \quad x \in \mathcal{M} ,
\]

and

\[
F^{[2]}(x) = \bigcap_{z \in \mathcal{M}} \left[ F^{[1]}(z) + \lambda_2 \rho(x,z) B_X \right] , \quad x \in \mathcal{M} .
\]

In other words, \( F^{[1]} \) and \( F^{[2]} \) are the first and the second order \((\lambda_1, \lambda_2, \rho)\)-balanced refinements of \( F \) respectively. See Definition \(1.6\).

Proposition \(3.9\) tells us that, under these conditions \( F^{[2]}(x) \neq \emptyset \) for every \( x \in \mathcal{M} \). In turn, Proposition \(3.10\) states that for every \( x, y \in \mathcal{M} \) the following inequality

\[
d_H( F^{[2]}(x), F^{[2]}(y) ) \leq \gamma_0(L) d(x,y)
\]

holds. Recall that \( \gamma_0(L) = L \cdot \theta(L)^2 \) where \( \theta = \theta(L) = (3L + 1)/(L - 1) \), see \((2.11)\). Hence,

\[
\theta(L) = \frac{3L + 1}{L - 1} = \frac{3(\lambda_2/\lambda_1) + 1}{\lambda_2/\lambda_1 - 1} = \frac{3\lambda_2 + \lambda_1}{\lambda_2 - \lambda_1} .
\]

From this, \((3.64)\) and \((3.63)\), we have

\[
d_H( F^{[2]}(x), F^{[2]}(y) ) \leq L \cdot \theta(L)^2 \ d(x,y) = \frac{\lambda_2}{\lambda_1} \cdot \frac{(3\lambda_2 + \lambda_1)^2}{(\lambda_2 - \lambda_1)^2} (\lambda_1 \rho(x,y)) = \lambda_2 \frac{(3\lambda_2 + \lambda_1)^2}{(\lambda_2 - \lambda_1)^2} \rho(x,y).
\]

This inequality together with \((3.62)\) implies the required inequality \( d_H( F^{[2]}(x), F^{[2]}(y) ) \leq \gamma \rho(x,y) \)

proving Theorem \(1.9\) for \( \lambda_1, \lambda_2 \) and \( \gamma \) satisfying \((1.7)\).

Prove that \( F^{[2]} \) satisfies property \((1.9)\) and inequality \((1.10)\) for \( \lambda_1, \lambda_2 \) and \( \gamma \) satisfying \((1.11)\). Indeed, we know that \( e(\mathfrak{M}, X) \leq 4/3 \) provided \( \mathfrak{M} = (\mathcal{M}, \rho) \) is an arbitrary pseudometric space, and \( X \) is a two dimensional Banach space. We set \( \lambda_1 = 4/3, \lambda_2 = 3\lambda_1 = 4, \) and

\[
\gamma = \lambda_2 (3\lambda_2 + \lambda_1)^2 / (\lambda_2 - \lambda_1)^2 = 100 .
\]

In these settings, inequalities \((1.7)\) for \( \lambda_1, \lambda_2 \) and \( \gamma \) hold proving \((1.9)\) and \((1.10)\) in the case under considerations.
Finally, let $X = \ell^2_\infty$. Then $e(\mathcal{M}, X) = 1$ for an arbitrary pseudometric space $\mathcal{M} = (\mathcal{M}, \rho)$, see (3.3). This and conditions (1.7) enable us to set $\lambda_1 = 1$, $\lambda_2 = 3\lambda_1 = 3$, and
\[
\gamma = \lambda_2 (3\lambda_2 + \lambda_1)^2 / (\lambda_2 - \lambda_1)^2 = 3 \cdot (3 \cdot 3 + 1)^2 / (3 - 1) = 75.
\]
Thus, in this case inequalities (1.7) are satisfied proving that (1.9) and (1.10) hold.

The proof of Theorem 1.9 is complete. ■

**Proof of Theorem 3.7.** We prove Theorem 3.1 by repeating the proof of Theorem 1.9 and replacing in this proof the function $\theta = \theta(L)$ defined by (2.11) with the function $\theta(L) = 1 + 2L/ \sqrt{L^2 - 1}$ defined by (2.12).

Following this scheme of the proof, we set $\alpha = \lambda_1$ and $L = \lambda_2 / \lambda_1$. Again, Proposition 3.9 tells us that $F^{(2)}(x) \neq \emptyset$ for each $x \in \mathcal{M}$. Then we show that inequality (3.64) holds for all $x, y \in \mathcal{M}$ with $\gamma_0(L) = L \cdot \theta(L)^2$ and $\theta(L) = 1 + 2L/ \sqrt{L^2 - 1}$ provided $\lambda_1 \geq e(\mathcal{M}, X)$ and $\lambda_2 \geq 3\lambda_1$.

In these settings,
\[
\gamma_0(L) = L \theta(L)^2 = (\lambda_2 / \lambda_1) \left( 1 + 2\lambda_2 / \left( \lambda_2^2 - \lambda_1^2 \right)^{1/2} \right)^2.
\]
From this equality and (3.64), we have
\[
d_{\mathcal{H}}(F^{(2)}(x), F^{(2)}(y)) \leq \gamma_0(L) (\lambda_1 \rho(x, y)) = \lambda_2 \left( 1 + 2\lambda_2 / \left( \lambda_2^2 - \lambda_1^2 \right)^{1/2} \right)^2 \rho(x, y) \leq \gamma \rho(x, y)
\]
provided $\lambda_1, \lambda_2$ and $\gamma$ satisfy (3.1). This proves that inequalities (3.1) imply (1.9) and (1.10).

We know that $e(\mathcal{M}, X) = 4/\pi$, see (3.4). This and (3.1) enable us to set $\lambda_1 = 4/\pi$, $\lambda_2 = 12/\pi$. In this case, all three inequalities in (3.1) will be satisfied provided
\[
\gamma \geq \lambda_2 \left( 1 + 2\lambda_2 / \left( \lambda_2^2 - \lambda_1^2 \right)^{1/2} \right)^2 = 3 \left( 4/\pi \right) \left( 1 + 6/ \sqrt{8} \right)^2 \approx 37.16.
\]
This shows that (1.9) and (1.10) hold with $\lambda_1, \lambda_2$, and $\gamma$ from (3.2).

Finally, let us assume that $X$ is a Euclidean space, $\mathcal{M}$ is a subset of a Euclidean space $E$, and $\rho$ is the metric in $E$. We know that in this case $e(\mathcal{M}, X) = 1$ (Kirschbraun’s extension theorem [18]). This enables us to set $\lambda_1 = 1$ and $\lambda_2 = 3$. In view of (3.1), for this choice of $\lambda_1$ and $\lambda_2$ one can set
\[
\gamma \geq \lambda_2 \left( 1 + 2\lambda_2 / \left( \lambda_2^2 - \lambda_1^2 \right)^{1/2} \right)^2 = 3 \left( 1 + 6/ \sqrt{8} \right)^2 \approx 24.99.
\]
This proves that (1.9) and (1.10) hold with $\lambda_1 = 1$, $\lambda_2 = 3$, and $\gamma = 25$ provided $X$ is a Euclidean space and $\mathcal{M}$ is a subset of a Euclidean space.

The proof of Theorem 3.1 is complete. ■

4. **Balanced refinements of line segments in a Banach space.**

In this section we prove Theorem 1.10. Let $(\mathcal{M}, \rho)$ be a pseudometric space, and let $(X, \| \cdot \|)$ be a Banach space with $\dim X > 1$. We recall that $\mathcal{K}_1(X)$ is the family of all non-empty compact convex subsets of $X$ of dimension at most 1. In other words, the family $\mathcal{K}_1(X) = \{ [a, b] \subset X : a, b \in X \}$ consists of all points and all closed bounded intervals in $X$.

In this section we need the following version of Helly’s Theorem.
**Theorem 4.1** Let $\mathcal{K}$ be a collection of closed convex subsets of $X$ containing a set $K_0 \in \mathcal{K}_1(X)$. If the intersection of $K_0$ with any two sets from $\mathcal{K}$ is nonempty, then there exists a point common to all of the collection $\mathcal{K}$.

**Proof.** We introduce a family $\mathcal{K'} = \{K \cap K_0 : K \in \mathcal{K}\}$, and apply to $\mathcal{K'}$ one dimensional Helly’s Theorem. (See next section, Lemma 5.2, part (a).) ■

We will also need the following variant of Proposition 2.5 for the family $\mathcal{K}_1(X)$.

**Proposition 4.2** Let $X$ be a Banach space, and let $r \geq 0$. Let $C, C_1, C_2 \subset X$ be convex closed subsets, and let $C_1 \in \mathcal{K}_1(X)$. Suppose that

$$C_1 \cap C_2 \cap (C + rB_X) \neq \emptyset.$$  \hspace{1cm} (4.1)

Then for every $L > 1$ and every $\varepsilon > 0$ the following inclusion

$$(C_1 \cap C_2 + LrB_X) \cap C + \theta(L) \varepsilon B_X \supset [C_1 \cap C_2 + (Lr + \varepsilon)B_X] \cap [(C_1 + rB_X) \cap C + \varepsilon B_X]$$

holds.

Here $\theta(L) = (3L + 1)/(L - 1)$; if $X$ is a Euclidean space, one can set $\theta(L) = 1 + 2L/ \sqrt{L^2 - 1}$.

**Proof.** Let

$$a \in [C_1 \cap C_2 + (Lr + \varepsilon)B_X] \cap [(C_1 + rB_X) \cap C + \varepsilon B_X].$$  \hspace{1cm} (4.2)

Prove that

$$a \in (C_1 \cap C_2 + LrB_X) \cap C + \theta(L) \varepsilon B_X.$$  \hspace{1cm} (4.3)

First, let us show that

$$C_1 \cap C_2 \cap (C + rB_X) \cap B_X(a, Lr + \varepsilon) \neq \emptyset.$$  \hspace{1cm} (4.4)

Recall that $C_1 \in \mathcal{K}_1(X)$. Helly’s Theorem 4.1 tells us that it is suffices to show that any two sets in the left hand size of (4.4) have a common point with $C_1$.

First we note that $C_1, C_2$ and $C + rB_X$ have a common point. See (4.1). We also know that

$$a \in C_1 \cap C_2 + (Lr + \varepsilon)B_X,$$

see (4.2), so that $C_1 \cap C_2 \cap B_X(a, Lr + \varepsilon) \neq \emptyset$.

Let us prove that

$$C_1 \cap (C + rB_X) \cap B_X(a, 2r + \varepsilon) \neq \emptyset.$$  \hspace{1cm} (4.5)

Property (4.2) tells us that $a \in (C_1 + rB_X) \cap C + \varepsilon B_X$. Therefore, there exist points $b_1 \in C_1$ and $b \in C$ such that $\|b_1 - b\| \leq r$ and $\|a - b\| \leq \varepsilon$. In particular, $b_1 \in C_1 \cap (C + rB_X)$. Furthermore,

$$\|a - b_1\| \leq \|a - b\| + \|b - b_1\| \leq \varepsilon + r \leq \varepsilon + 2r,$$

so that $b_1 \in B_X(a, 2r + \varepsilon)$. Hence,

$$b_1 \in C_1 \cap (C + rB_X) \cap B_X(a, 2r + \varepsilon)$$

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Thus, (4.4) holds proving the existence of a point $x \in X$ such that
\[ x \in C_1 \cap C_2 \cap (C + rB_X) \cap B_X(a, 2r + \varepsilon). \] (4.6)

In particular, $x \in C + rB_X$ so that $B_X(x, r) \cap C \neq \emptyset$ proving that condition (2.9) of Proposition 2.3 is satisfied. We apply this proposition to $x$, $r$ and the set $C$ and get:
\[ C \cap B_X(x, Lr) + \theta(L) \varepsilon B_X \supset (C + \varepsilon B_X) \cap (B_X(x, Lr) + \varepsilon B_X) = (C + \varepsilon B_X) \cap B_X(x, Lr + \varepsilon). \]

From (4.6) we learn that $a \in B_X(x, Lr + \varepsilon)$. In turn, (4.2) tells us that
\[ a \in (C_1 + rB_X) \cap C + \varepsilon B_X \subset C + \varepsilon B_X. \]

Hence, $(C + \varepsilon B_X) \cap B_X(x, Lr + \varepsilon) \ni a$ proving that $C \cap B_X(x, Lr) + \theta(L) \varepsilon B_X \ni a$.

Finally, property (4.6) tells us that $x \in C_1 \cap C_2$ proving the required inclusion (4.3). ■

We note that the finiteness number $\dim(1, X) = \min\{2^2, 2\dim X\} = 4$. (Recall that $\dim X > 1$.) Let $F : M \to \mathcal{K}(X)$ be a set-valued mapping. We suppose that $F$ satisfies the hypothesis of Theorem 1.10 i.e., that the following assumption is true.

**Assumption 4.3** For every subset $M' \subset M$ with $\#M' \leq 4$ the restriction $F|_{M'}$ of $F$ to $M'$ has a Lipschitz selection $f_{M'} : M' \to X$ with $\|f\|_{\text{Lip}(M', X)} \leq 1$.

Let $\lambda = \{\lambda_1, \lambda_2\}$. We introduce balanced $(\lambda, \rho)$-refinements of $F$ of the first and the second order, i.e., set-valued mappings
\[ F^{[1]}(x) = \bigcap_{y \in M} \left[ F(y) + \lambda_1 \rho(x, y) B_X \right], \quad x \in M, \]
and
\[ F^{[2]}(x) = \bigcap_{z \in M} \left[ F^{[1]}(z) + \lambda_2 \rho(x, z) B_X \right], \quad x \in M. \]
See Definition 1.6.

Our aim is to prove that if
\[ \lambda_1 \geq 1, \quad \lambda_2 \geq 3\lambda_1, \quad \gamma \geq \lambda_2 (3\lambda_2 + \lambda_1)/(\lambda_2 - \lambda_1), \] (4.7)
then the set-valued mapping $F^{[2]}$ is a $\gamma$-core of $F$ (with respect to $\rho$), i.e.,
\[ F^{[2]}(x) \neq \emptyset \text{ for every } x \in M, \quad \text{and} \quad d_H(F^{[2]}(x), F^{[2]}(y)) \leq \gamma \rho(x, y) \text{ for all } x, y \in M. \]

We set $L = \lambda_2 / \lambda_1$. We also introduce a new pseudometric on $M$ defined by
\[ d(x, y) = \lambda_1 \rho(x, y), \quad x, y \in M. \]
Thus, thanks to (4.7),
\[ L \geq 3 \quad \text{and} \quad \rho \leq d \text{ on } M. \] (4.8)

In these settings,
\[ F^{[1]}(x) = \bigcap_{z \in M} \left[ F(z) + d(x, z) B_X \right] \quad \text{and} \quad F^{[2]}(x) = \bigcap_{z \in M} \left[ F^{[1]}(z) + L d(x, z) B_X \right], \quad x \in M. \] (4.9)

We need the following analog of Lemma 3.6.
Lemma 4.4 Let $\mathcal{K}$ be a collection of convex closed subsets of $X$ containing a set $K_0 \in \mathcal{K}_1(X)$. Suppose that $\cap\{K : K \in \mathcal{K}\} \neq \emptyset$. Then for every $r \geq 0$ the following equality holds:

$$\left( \bigcap_{K \in \mathcal{K}} K \right) + rB_X = \bigcap_{K \in \mathcal{K}} \left\{ K \cap K_0 + rB_X \right\}$$

Proof. Let $\tilde{\mathcal{K}} = \{K \cap K_0 : K \in \mathcal{K}\}$. Clearly, $\tilde{\mathcal{K}} \subset \mathcal{K}_1(X)$. It is also clear that the statement of the lemma is equivalent to the equality

$$\left( \bigcap_{\tilde{K} \in \tilde{\mathcal{K}}} \tilde{K} \right) + rB_X = \bigcap_{\tilde{K} \in \tilde{\mathcal{K}}} \{ \tilde{K} + rB_X \}$$

provided $\cap\{\tilde{K} : \tilde{K} \in \tilde{\mathcal{K}}\} \neq \emptyset$. We prove this equality by a slight modification of the proof of Lemma 3.6. In particular, in this proof we use Helly’s Theorem 4.1 rather than Theorem 2.4. We leave the details to the interested reader. ■

The next lemma is an analog of Lemma 3.7.

Lemma 4.5 For every $x \in M$ the set $F^{[1]}(x) \in \mathcal{K}_1(X)$. Moreover, for every $x, z \in M$ we have

$$F^{[1]}(z) + L d(x, z)B_X = \bigcap_{v \in M} \left\{ [F(v) + d(z, v)B_X] \cap F(z) + L d(x, z)B_X \right\}.$$  \hspace{1cm} (4.10)

Proof. Let $\mathcal{K} = \{F(z) + d(z, x)B_X : z \in M\}$. We know that $\mathcal{K}$ is a family of bounded closed convex subsets of $X$ containing the set $F(x) \in \mathcal{K}_1(X)$. Helly’s Theorem 4.1 tells us that the set $F^{[1]}(x) = \cap\{K : K \in \mathcal{K}\} \neq \emptyset$ provided the set

$$E = F(x) \cap [F(z') + d(z', x)B_X] \cap [F(z'') + d(z'', x)B_X] \neq \emptyset$$  \hspace{1cm} (4.11)

for every $z', z'' \in M$.

Let $M' = \{x, z', z''\}$. Assumption 4.3 tells us that there exists a function $f_{M'} : M' \to X$ satisfying the following conditions: $f_{M'}(x) \in F(x)$, $f_{M'}(z') \in F(z')$, $f_{M'}(z'') \in F(z'')$,

$$\|f_{M'}(z') - f_{M'}(x)\| \leq \rho(z', x) \leq d(z', x), \quad \text{and} \quad \|f_{M'}(z'') - f_{M'}(x)\| \leq \rho(z'', x) \leq d(z'', x).$$

See 4.8. Hence, $f_{M'}(x) \in E$, proving (4.11). Thus, $F^{[1]}(x) \neq \emptyset$.

We also know that $F^{[1]}(x) \in \mathcal{K}_1(X)$. These properties of $F^{[1]}(x)$, Lemma 4.4 and 4.9 imply the required equality (4.10), proving the lemma. ■

Note that, thanks to 4.9, the following explicit representation of the mapping $F^{[2]}$

$$F^{[2]}(x) = \bigcap_{z \in M} \left\{ \left( \bigcap_{y \in M} \left[ F(y) + d(z, y)B_X \right]\right) + L d(x, z)B_X \right\}, \quad x \in M,$$

holds. This representation and Lemma 4.5 imply the following analog of Lemma 3.8.

Lemma 4.6 For every $x \in M$ the following equality

$$F^{[2]}(x) = \bigcap_{u, u' \in M} \left\{ [F(u') + d(u', u)B_X] \cap F(u) + L d(u, x)B_X \right\}$$

holds.
Given $x, u, u' \in \mathcal{M}$ we put
\[
\tilde{T}_x(u, u') = [F(u') + d(u', u)B_X] \cap F(u) + Ld(u, x)B_X.
\tag{4.12}
\]

Now, Lemma 4.6 reformulates as follows:
\[
F^{[2]}(x) = \bigcap_{u, u' \in \mathcal{M}} \tilde{T}_x(u, u').
\tag{4.13}
\]

**Proposition 4.7** For every $x \in \mathcal{M}$ the set $F^{[2]}(x) \neq \emptyset$.

**Proof.** Clearly, $F(x) = \tilde{T}_x(x, x)$, see (3.20). We also know that $F(x) \in \mathcal{K}_1(X)$. Formula (4.13) and Helly’s Theorem 4.1 tell us that $F^{[2]}(x) \neq \emptyset$ provided for every choice of points $u_i, u'_i \in \mathcal{M}$, $i = 1, 2$, we have
\[
F(x) \cap \tilde{T}_x(u_1, u'_1) \cap \tilde{T}_x(u_2, u'_2) \neq \emptyset.
\tag{4.14}
\]

We recall that
\[
\tilde{T}_x(u_i, u'_i) = [F(u'_i) + d(u'_i, u_i)B_X] \cap F(u_i) + Ld(u_i, x)B_X, \quad i = 1, 2.
\tag{4.15}
\]

See (4.12). Without loss of generality, we may assume that
\[
\rho(u_1, x) \geq \rho(u_2, x).
\tag{4.16}
\]

We introduce the following sets:
\[
G_1 = F(u_2), \quad G_2 = F(u'_2) + \rho(u_2, u'_2)B_X, \quad G_3 = F(x) + \rho(u_2, x)B_X,
\tag{4.17}
\]
and
\[
G_4 = [F(u'_1) + \rho(u'_1, u_1)B_X] \cap F(u_1) + \rho(u_1, u_2)B_X.
\tag{4.18}
\]

Prove that
\[
\bigcap_{i=1}^{4} G_i \neq \emptyset.
\tag{4.19}
\]

We know that $G_1 = F(u_2) \in \mathcal{K}_1(X)$. In this case, Helly’s Theorem 4.1 tells us that (4.19) holds provided $G_1 \cap G_i \cap G_j \neq \emptyset$ for every $2 \leq i, j \leq 4, i \neq j$.

First prove that
\[
G_1 \cap G_2 \cap G_3 = F(u_2) \cap [F(u'_2) + \rho(u_2, u'_2)B_X] \cap [F(x) + \rho(u_2, x)B_X] \neq \emptyset.
\tag{4.20}
\]

Let $\mathcal{M}_1 = \{u'_2, u_2, x\}$. Because $\#\mathcal{M}_1 \leq 4$, Assumption 4.3 guarantees the existence of a mapping $f_1 : \mathcal{M}_1 \to X$ with the following properties: $f_1(x) \in F(x), f_1(u_2) \in F(u_2), f_1(u'_2) \in F(u'_2)$,
\[
\|f_1(u_2) - f_1(x)\| \leq \rho(u_2, x) \quad \text{and} \quad \|f_1(u'_2) - f_1(u'_2)\| \leq \rho(u_2, u'_2).
\]

These properties of $f_1$ and definition (4.17) tell us that $f_1(u_2) \in G_1 \cap G_2 \cap G_3$ proving (4.20).
Prove that

\[ G_1 \cap G_2 \cap G_4 = F(u_2) \cap [F(u_2) + \rho(u_2, u_2')B_X] \cap \{F(u_1') + \rho(u_1', u_1)B_X] \cap F(u_1) + \rho(u_1, u_2)B_X \} \neq \emptyset. \]

Let \( M_2 = \{u_1', u_1, u_2', u_2\} \). Clearly, \( \#M_2 \leq 4 \). Assumption \( 4.3 \) tells us that there exists a mapping \( f_2 : M_2 \to X \) with the following properties: \( f_2(u_i) \in F(u_i), f_2(u_i') \in F(u_i'), \ i = 1, 2, \)

\[ \|f_2(u_i) - f_2(u_i')\| \leq \rho(u_1, u_1'), \|f_2(u_i) - f_2(u_2)\| \leq \rho(u_1, u_2), \text{ and } \|f_2(u_2) - f_2(u_2')\| \leq \rho(u_2, u_2'). \]

From these properties of \( f_2 \) and definitions \( (4.17) \) and \( (4.18) \), we have \( f_2(u_2) \in G_1 \cap G_2 \cap G_4 \) proving that \( G_1 \cap G_2 \cap G_4 \neq \emptyset. \)

Finally, prove that

\[ G_1 \cap G_3 \cap G_4 = F(u_2) \cap [F(x) + \rho(u_2, x)B_X] \cap \{F(u_1') + \rho(u_1', u_1)B_X] \cap F(u_1) + \rho(u_1, u_2)B_X \} \neq \emptyset. \]

We introduce a set \( M_3 = \{u_1', u_1, x, u_2, u_2\} \). Because \( \#M_2 \leq 4 \), by Assumption \( 4.3 \) there exists a mapping \( f_3 : M_3 \to X \) with the following properties: \( f_3(v) \in F(v) \) for each \( v \in M_3, \)

\[ \|f_3(u_1) - f_3(u_1')\| \leq \rho(u_1, u_1'), \|f_3(u_1) - f_3(u_2)\| \leq \rho(u_1, u_2), \text{ and } \|f_3(u_2) - f_3(x)\| \leq \rho(u_2, x). \]

These properties of \( f_2, (4.17) \) and \( (4.18) \) tell us that \( f_3(u_2) \in G_1 \cap G_3 \cap G_4 \) proving the required property \( G_1 \cap G_3 \cap G_4 \neq \emptyset. \)

Thus, property \( (4.19) \) is proven. Let \( \widetilde{M} = \{u_1', u_1, x, u_2, u_2\} \). Property \( (4.19) \) and definitions \( (4.17), (4.18) \) imply the existence of a mapping \( g : \widetilde{M} \to X \) with the following properties: \( g(v) \in F(v) \) for every \( v \in \widetilde{M}, \)

\[ \|g(u_1) - g(u_1')\| \leq \rho(u_1, u_1'), \|g(u_1) - g(u_2)\| \leq \rho(u_1, u_2), \|g(u_2) - g(u_2')\| \leq \rho(u_2, u_2'), \quad (4.21) \]

and

\[ \|g(u_2) - g(x)\| \leq \rho(u_2, x). \quad (4.22) \]

Prove \( (4.14) \) by showing that

\[ g(x) \in F(x) \cap \tilde{T}_{x}(u_1, u_1') \cap \tilde{T}_{x}(u_2, u_2'). \quad (4.23) \]

Indeed, from properties of \( g \) we know that \( g(x) \in F(x) \).

We also know that \( g(u_2) \in F(u_2), g(u_2') \in F(u_2') \). Thanks to \( (4.21), (4.22) \) and \( (4.8), \)

\[ \|g(u_2) - g(u_2')\| \leq \rho(u_2, u_2') \leq d(u_2, u_2') \text{ and } \|g(u_2) - g(x)\| \leq \rho(u_2, x) \leq L d(u_2, x). \]

From these properties of \( g \) and definition \( (4.15) \), we have

\[ g(x) \in [F(u_2') + d(u_2', u_2)B_X] \cap F(u_2) + L d(u_2, x) B_X = \tilde{T}_{x}(u_2, u_2'). \]

It remains to show that \( g(x) \in \tilde{T}_{x}(u_1, u_1') \). Indeed, as we know,

\[ g(x) \in F(x), \quad g(u_1) \in F(u_1), \text{ and } g(u_1') \in F(u_1'). \quad (4.24) \]

Furthermore, thanks to \( (4.21) \) and \( (4.8), \)

\[ \|g(u_1) - g(u_1')\| \leq \rho(u_1, u_1') \leq d(u_1, u_1'). \quad (4.25) \]
Let us estimate \( \|g(u_1) - g(x)\| \). Thanks to (4.21), (4.22) and the triangle inequality,

\[
\|g(u_1) - g(x)\| \leq \|g(u_1) - g(u_2)\| + \|g(u_2) - g(x)\| \leq \rho(u_1, u_2) + \rho(u_2, x)
\]

\[ \leq (\rho(u_1, x) + \rho(x, u_2) + \rho(u_2, x) = \rho(u_1, x) + 2\rho(x, u_2) \]

so that, thanks to (4.16) and (4.8),

\[
\|g(u_1) - g(x)\| \leq 3\rho(u_1, x) \leq L d(u_1, x).
\]

From this inequality, (4.25), property (4.24) and definition (4.15) we obtain the required property \( g(x) \in \widetilde{T}_x(u_1, u_1') \) proving (4.23).

The proof of the proposition is complete. \( \square \)

As in Section 3, we again set

\[ \gamma_0 = \gamma_0(L) = L \theta(L)^2 \]

where \( \theta = \theta(L) \) is the function from Proposition 4.2. (Thus \( \theta(L) = (3L+1)/(L-1) \); if \( X \) is a Euclidean space, one can set \( \theta(L) = 1 + 2L/\sqrt{L^2 - 1} \).) Cf. (3.12).

**Proposition 4.8** For every \( x, y \in M \) the following inequality

\[
d_H(F^{[2]}(x), F^{[2]}(y)) \leq \gamma_0(L) d(x, y) \tag{4.26}
\]

holds.

**Proof.** Let \( x, y \in M \). Formula (4.13) tells us that

\[
F^{[2]}(x) = \bigcap_{u, u' \in M} \widetilde{T}_x(u, u') \quad \text{and} \quad F^{[2]}(y) = \bigcap_{u, u' \in M} \widetilde{T}_y(u, u'). \tag{4.27}
\]

Recall that

\[
\widetilde{T}_x(u, u') = [F(u') + d(u', u)B_X] \cap F(u) + L d(u, x) B_X. \tag{4.28}
\]

We also know that the set \( F^{[2]}(x) \neq \emptyset \), see Proposition 4.7 and the set \( \widetilde{T}_x(x, x) = F(x) \in K_1(X) \). These properties, the above formula for \( F^{[2]}(x) \) and Lemma 4.4 tell us that

\[
F^{[2]}(x) + \gamma_0(L) d(x, y) B_X = \bigcap_{u, u' \in M} \left( \widetilde{T}_x(u, u') \cap F(x) + \gamma_0(L) d(x, y) B_X \right). \tag{4.29}
\]

We fix \( u, u' \in M \) and introduce a set

\[
\widetilde{A} = \widetilde{T}_x(u, u') \cap F(x) + \gamma_0(L) d(x, y) B_X.
\]

We also introduce sets

\[
C_1 = F(u), \quad C_2 = F(u') + d(u', u)B_X, \quad \text{and} \quad C = F(x). \tag{4.30}
\]

Let

\[
\varepsilon = L d(x, y) \quad \text{and} \quad r = d(x, u). \tag{4.31}
\]

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In these settings, \( \gamma_0(L) d(x, y) = \theta(L) \varepsilon \) and
\[
\tilde{A} = \overline{T}_y(u, u') \cap F(x) + \gamma_0(L) d(x, y) B_X = (C_1 \cap C_2 + LrB_X) \cap C + \theta(L) \varepsilon B_X.
\]

Let us apply Proposition 4.2 to the set \( \tilde{A} \). First, we have to verify condition (4.1) of this proposition, i.e., to show that
\[
\rho \tilde{M} = \{x, u, u'\}. \tag{4.32}
\]

Let \( \tilde{M} = \{x, u, u'\} \). This set consists of at most three points so that, thanks to Assumption 4.3, there exists a \( \rho \)-Lipschitz selection \( f_{\tilde{M}} \) of the restriction \( F|_{\tilde{M}} \) with \( \|f_{\tilde{M}}\|_{\text{Lip}(\tilde{M}, X)} \leq 1 \). Thus, \( f_{\tilde{M}}(u') \in F(u') \), \( f_{\tilde{M}}(u) \in F(u) \), \( f_{\tilde{M}}(x) \in F(x) \),
\[
\|f_{\tilde{M}}(u') - f_{\tilde{M}}(u)\| \leq \rho(u', u) \quad \text{and} \quad \|f_{\tilde{M}}(x) - f_{\tilde{M}}(u)\| \leq \rho(x, u).
\]

Let us see that
\[
f_{\tilde{M}}(u) \in C_1 \cap C_2 \cap (C + rB_X) \tag{4.33}
\]

Indeed, we know that \( f_{\tilde{M}}(u) \in F(u) = C_1 \), see (4.30). Furthermore, \( f_{\tilde{M}}(u') \in F(u') \) and, thanks to (4.8), \( \rho \leq d \) on \( M \). Hence,
\[
\|f_{\tilde{M}}(u') - f_{\tilde{M}}(u)\| \leq \rho(u', u) \leq d(u', u)
\]
proving that \( f_{\tilde{M}}(u) \in C_2 \), see (4.30).

Finally, thanks to (4.30) and (4.31), \( f_{\tilde{M}}(x) \in F(x) = C \) and
\[
\|f_{\tilde{M}}(x) - f_{\tilde{M}}(u)\| \leq \rho(x, u) \leq d(x, u) = r,
\]
proving that \( f_{\tilde{M}}(u) \in C + rB_X \).

Thus, (4.33) is true, and property (4.32) holds. We also recall that the set \( C_1 = F(u) \in \mathcal{K}_1(X) \).

Now, Proposition 4.2 tells us that
\[
\tilde{A} = (C_1 \cap C_2 + LrB_X) \cap C + \theta(L) \varepsilon B_X
\]
\[
\supseteq [C_1 \cap C_2 + (Lr + \varepsilon)B_X] \cap [(C_1 + rB_X) \cap C + \varepsilon B_X]
\]
\[
= \tilde{S}_1 \cap \tilde{S}_2.
\]

Prove that \( \tilde{S}_i \supset F^{[2]}(y) \) for every \( i = 1, 2 \). We begin with the set \( \tilde{S}_1 = C_1 \cap C_2 + (Lr + \varepsilon)B_X \). Thanks to (4.30) and (4.31),
\[
\tilde{S}_1 = [F(u') + d(u', u)B_X] \cap F(u) + (L d(u, x) + L d(x, y))B_X.
\]

By the triangle inequality, \( \rho(u, x) + \rho(x, y) \geq \rho(u, y) \) so that
\[
\tilde{S}_1 \supset [F(u') + d(u', u)B_X] \cap F(u) + L d(u, y)B_X = \tilde{T}_y(u, u'), \quad \text{see (4.28)}.
\]

But, thanks to (4.27), \( \tilde{T}_y(u, u') \supset F^{[2]}(y) \) which implies the required inclusion \( \tilde{S}_1 \supset F^{[2]}(y) \).

We turn to the set \( \tilde{S}_2 = (C_1 + rB_X) \cap C + \varepsilon B_X \). Definitions (4.12), (4.30) and (4.31) tell us that
\[
\tilde{S}_2 = [F(u) + d(u, x)B_X] \cap F(x) + L d(x, y)B_X = \tilde{T}_y(u, x).
\]

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Thanks to (4.27), \( \overline{T}_x(u, x) \supset F^{[2]}(y) \) proving that \( \overline{S}_2 \supset F^{[2]}(y) \).

Thus,
\[
\overline{A} = \overline{T}_x(u, u') \cap F(x) + \gamma_0(L) \, d(x, y) \, B_x \supset \overline{S}_1 \cap \overline{S}_2 \supset F^{[2]}(y) \quad \text{for every } u, u' \in \mathcal{M}.
\]

From this and representation (4.29), we have
\[
F^{[2]}(x) + \gamma_0(L) \, d(x, y) \, B_x \supset F^{[2]}(y).
\]

By interchanging the roles of \( x \) and \( y \) we obtain also
\[
F^{[2]}(y) + \gamma_0(L) \, d(x, y) \, B_x \supset F^{[2]}(x).
\]

These two inclusions imply the required inequality (4.26) proving the proposition. \( \blacksquare \)

We finish the proof of Theorem 1.10 as follows. We fix \( \lambda_1, \lambda_2 \) and \( \gamma \) satisfying inequalities (1.12). Proposition 4.7 tells us that for such choice of these parameters the set \( F^{[2]}(x) \neq \emptyset \) for every \( x \in \mathcal{M} \).

In turn, Proposition 4.8 tells us that in these settings \( d_H(F^{[2]}(x), F^{[2]}(y)) \leq \gamma_0(L) \, d(x, y) \) for all \( x, y \in \mathcal{M} \). We recall that here \( L = \lambda_2/\lambda_1, \, d = \lambda_1 \rho, \, \gamma_0(L) = L \theta(L) \) and \( \theta(L) = (3L + 1)/(L - 1) \). Hence,
\[
d_H(F^{[2]}(x), F^{[2]}(y)) \leq \gamma_0(L) \, d(x, y) = L \left( \frac{3L + 1}{L - 1} \right) \, d(x, y)
\]
\[
= \left( \frac{\lambda_2}{\lambda_1} \right) \left( \frac{3(\lambda_2/\lambda_1) + 1}{(\lambda_2/\lambda_1) - 1} \right) \cdot (\lambda_1 \rho(x, y)) = \left\{ \lambda_2 (3\lambda_2 + \lambda_1)/(\lambda_2 - \lambda_1) \right\} \rho(x, y).
\]

We recall that \( \gamma \geq \lambda_2 (3\lambda_2 + \lambda_1)/(\lambda_2 - \lambda_1) \), see (1.12), so that \( d_H(F^{[2]}(x), F^{[2]}(y)) \leq \gamma \rho(x, y) \) for all \( x, y \in \mathcal{M} \).

We have proved that property (1.9) and inequality (1.10) hold for the mapping \( F^{[2]} \) provided \( \lambda_1, \lambda_2 \) and \( \gamma \) satisfy inequalities (1.12). In particular, we can set \( \lambda_1 = 1, \lambda_2 = 3 \). For these parameters (1.9) and (1.10) hold provided
\[
\gamma = \lambda_2 (3\lambda_2 + \lambda_1)/(\lambda_2 - \lambda_1) = 3(3 \cdot 3 + 1)/(3 - 1) = 15.
\]

Let now \( X \) be a Euclidean space, and let \( \lambda_1, \lambda_2 \) and \( \gamma \) be parameters satisfying inequalities (1.13). In this case, replacing in the above calculations \( \theta(L) = (3L + 1)/(L - 1) \) with \( \theta(L) = 1 + 2L/ \sqrt{L^2 - 1} \) we obtain the following:
\[
d_H(F^{[2]}(x), F^{[2]}(y)) \leq L \left( 1 + 2L/ \sqrt{L^2 - 1} \right) \, d(x, y)
\]
\[
= \left( \frac{\lambda_2}{\lambda_1} \right) \left( 1 + 2(\lambda_2/\lambda_1)/ \sqrt{(\lambda_2/\lambda_1)^2 - 1} \right) \cdot (\lambda_1 \rho(x, y))
\]
\[
= \left\{ \lambda_2 + 2 \lambda_2^2 \left( \lambda_2^2 - \lambda_1^2 \right)^{1/2} \right\} \rho(x, y) \leq \gamma \rho(x, y).
\]

See (1.13). This proves that (1.9) and (1.10) hold provided \( \lambda_1, \lambda_2 \) and \( \gamma \) satisfy inequalities (1.13). In particular, we can set \( \lambda_1 = 1, \lambda_2 = 3 \) and \( \gamma = 10 \). In fact, in this case
\[
\lambda_2 + 2 \lambda_2^2 \left( \lambda_2^2 - \lambda_1^2 \right)^{1/2} = 3 + 2 \cdot 3^2/ \sqrt{3^2 - 1} = 3 + 18/ \sqrt{8} \approx 9.36 \leq 10 = \gamma.
\]

The proof of Theorem 1.10 is complete. \( \blacksquare \)
5. The case $X = \mathbb{R}$ and related results.

5.1 Main conjecture in the one dimensional case.

In this section we prove Conjecture 1.8 for a one dimensional Banach space $X$. Clearly, we may assume that $X = \mathbb{R}$. Thus, in this case the unit “ball” of $X$ is the interval $I_0 = [-1, 1]$. Given $a \in \mathbb{R}$ and $r \geq 0$, we set $rI_0 = [-r, r]$ and $I(a, r) = [a - r, a + r]$.

Proposition 5.1 Let $(M, \rho)$ be a pseudometric space. Let $X = \mathbb{R}$ and let $m = 1$. In this case Conjecture 1.8 holds with $\ell = 1$, $\lambda_1 = 1$ and $\gamma = 1$.

Thus, the following statement is true: Let $F$ be a set-valued mapping from $M$ into the family $\mathcal{K}(\mathbb{R})$ of all closed bounded intervals in $\mathbb{R}$, and let $\eta$ be a positive constant. Suppose that for every $x, y \in M$ there exist $g(x) \in F(x)$ and $g(y) \in F(y)$ such that $|g(x) - g(y)| \leq \eta \rho(x, y)$.

Let

$$F^{[1]}(x) = \bigcap_{z \in M} \left[ F(z) + \eta \rho(x, z) I_0 \right], \quad x \in M,$$

(5.1)

be the $\eta$-balanced refinement of the mapping $F$.

Then for every $x \in M$ the set $F^{[1]}(x)$ is a non-empty closed subinterval of $F(x)$. Furthermore, for every $x, y \in M$ the following inequality

$$d_H(F^{[1]}(x), F^{[1]}(y)) \leq \eta \rho(x, y)$$

holds.

As in the previous sections, one of the main tools in the proof of Conjecture 1.8 will be Helly’s Theorem. Let us recall its statement in the one-dimensional case. We will also give a formula for a neighborhood of the intersection of intervals in $\mathbb{R}$.

Lemma 5.2 Let $\mathcal{K} \subset I(\mathbb{R})$ be a collection of closed intervals in $\mathbb{R}$. (See (2.1).)

(a) Suppose that either $\mathcal{K}$ is finite or at least one member of $\mathcal{K}$ is bounded. If the intersection of every two intervals from $\mathcal{K}$ is nonempty, then there exists a point in $\mathbb{R}$ common to all of the family $\mathcal{K}$.

(b) Suppose that $\cap \{ K : K \in \mathcal{K} \} \neq \emptyset$. Then for every $r \geq 0$ the following equality

$$\left( \bigcap_{K \in \mathcal{K}} K \right) + rI_0 = \bigcap_{K \in \mathcal{K}} \{ K + rI_0 \}$$

holds.

Proof. In Lemma 3.6 we have proved an analog of property (b) for $\mathbb{R}^2$. The proof of (b) is an obvious modification of that proof where we replace Helly’s Theorem 2.4 in $\mathbb{R}^2$ with Helly’s Theorem in $\mathbb{R}$ formulated in part (a) of the present lemma. We leave the details to the interested reader. $\blacksquare$

Remark 5.3 We can slightly weaken the hypothesis of one dimensional Helly’s theorem given in part (a) of Lemma 5.2 as follows: we may assume that (i) either $\mathcal{K}$ is finite or (ii) there exists a finite subfamily $\tilde{\mathcal{K}} \subset \mathcal{K}$ such that the intersection $\cap\{ I : I \in \tilde{\mathcal{K}} \}$ is nonempty and bounded.

Furthermore, (ii) can be replaced with the following requirement: (ii’) there exist intervals $I, I' \in \mathcal{K}$ such that the intersection $I \cap I'$ is nonempty and bounded. $\blacksquare$
Let $F : M \to \mathcal{K}(R)$ be a set-valued mapping which to every $x \in M$ assigns a closed bounded interval $F(x) = [a(x), b(x)]$. (Thus, $a(x) = \min F(x)$, $b(x) = \max F(x)$, so that $a(x) \leq b(x)$, $x \in M$.) Let
\[ r(x) = \frac{b(x) - a(x)}{2}, \quad c(x) = \frac{a(x) + b(x)}{2}, \quad x \in M. \]
Thus, $F(x) = I(c(x), r(x)) = c(x) + r(x)I_0$. Clearly,
\[ \text{dist}(F(x), F(y)) = |c(x) - c(y)| - r(x) - r(y)| = \max\{[a(x) - b(y)]_+, [a(y) - b(x)]_+\}. \tag{5.2} \]
This formula leads us to the following
\begin{claim}
Given $x, y \in M$, and $\lambda \geq 0$, there exist points $g(x) \in F(x)$, $g(y) \in F(y)$ such that $|g(x) - g(y)| \leq \lambda \rho(x, y)$ if and only if the following inequality
\[ |c(x) - c(y)| \leq r(x) + r(y) + \lambda \rho(x, y) \]
holds. This inequality is equivalent to the inequality
\[ \max\{a(x) - b(y), a(y) - b(x)\} \leq \lambda \rho(x, y). \]
\end{claim}
\begin{proof}
The claim is immediate from formula (5.2) and the following obvious fact: such points $g(x)$, $g(y)$ exist iff dist$(F(x), F(y)) \leq \lambda \rho(x, y)$.

Given a set-valued mapping $F(x) = [a(x), b(x)]$, $x \in M$, we set
\[ \lambda_F = \sup_{x,y \in M} \frac{[a(x) - b(y)]_+}{\rho(x, y)} = \sup_{x,y \in M} \frac{[\min F(x) - \max F(y)]_+}{\rho(x, y)}. \tag{5.3} \]
Note that from inequality (5.2), we have
\[ \lambda_F = \sup_{x,y \in M} \frac{\text{dist}(F(x), F(y))}{\rho(x, y)}. \tag{5.4} \]
Clearly,
\[ \lambda_F = \sup_{x,y \in M} \frac{[\min F(x) + \min(-F(y))]_+}{\rho(x, y)}. \]
(Recall that we set $0_0 = 0$ and $A_0 = +\infty$ for $A > 0$.)

Given $\lambda \geq 0$, we also introduce the following functions on $M$:
\[ f^+[A; F](x) = \inf_{y \in M} \{b(y) + \lambda \rho(x, y)\} = \inf_{y \in M} \{\max F(y) + \lambda \rho(x, y)\}, \tag{5.5} \]
\[ f^-[A; F](x) = \sup_{y \in M} \{a(y) - \lambda \rho(x, y)\} = \sup_{y \in M} \{\min F(y) - \lambda \rho(x, y)\}, \tag{5.6} \]
and
\[ f[A; F](x) = \frac{f^+[A; F](x) + f^-[A; F](x)}{2}. \tag{5.7} \]
Lemma 5.5 Let $\lambda \geq 0$, and let $F : \mathcal{M} \to \mathcal{K}(\mathbb{R})$ be a set-valued mapping.

(i) (The Finiteness Principle for Lipschitz selections in $\mathbb{R}$.) Suppose that for every $x, y \in \mathcal{M}$ the restriction $F_{\{x,y\}}$ of $F$ to $\{x,y\}$ has a Lipschitz selection $f_{\{x,y\}}$ with $\|f_{\{x,y\}}\|_{\text{Lip}(\mathbb{R})} \leq \lambda$. Then $F$ has a Lipschitz selection $f : \mathcal{M} \to \mathbb{R}$ with Lipschitz seminorm $\|f\|_{\text{Lip}(\mathcal{M},\mathbb{R})} \leq \lambda$.

Furthermore, one can set

$$f = f^+[\lambda; F], \quad f = f^-[\lambda; F] \quad \text{or} \quad f = f[\lambda; F].$$

(ii) There exists a Lipschitz selection of $F$ if and only if

$$\lambda_F = \sup_{x,y \in \mathcal{M}} \frac{\left[ \min F(x) - \max F(y) \right]_+}{\rho(x,y)} < \infty,$$  \hspace{1cm} (5.8)

Moreover, if this inequality holds then

$$\lambda_F = \min\{\|f\|_{\text{Lip}(\mathcal{M},\mathbb{R})} : f \text{ is a Lipschitz selection of } F\}.$$  

The above minimum is attained at each of the following functions: $f^+[\lambda; F]$, $f^-[\lambda; F]$ or $f[\lambda; F]$. In other words,

$$\lambda_F = \|f^+[\lambda; F]\|_{\text{Lip}(\mathcal{M},\mathbb{R})} = \|f^-[\lambda; F]\|_{\text{Lip}(\mathcal{M},\mathbb{R})} = \|f[\lambda; F]\|_{\text{Lip}(\mathcal{M},\mathbb{R})}.$$  

Proof. (i) Let $F(x) = [a(x), b(x)]$, $x \in \mathcal{M}$. Prove that the function

$$f(x) = f^+[\lambda; F](x) = \inf_{y \in \mathcal{M}} \{b(y) + \lambda \rho(x,y)\}$$

is a Lipschitz selection of $F$ with $\|f\|_{\text{Lip}(\mathcal{M},\mathbb{R})} \leq \lambda$.

Clearly, $f(x) \leq b(x)$ on $\mathcal{M}$. (Take $y = x$ in the definition of $f$.) The hypothesis of part (i) of the lemma tells us that for every $x, y \in \mathcal{M}$ there exist points $g(x) \in [a(x), b(x)]$, $g(y) \in [a(y), b(y)]$ such that $|g(x) - g(y)| \leq \lambda \rho(x,y)$. Therefore, thanks to Claim 5.4

$$a(x) \leq b(y) + \lambda \rho(x,y).$$

Hence, $a(x) \leq f(x)$. Thus, $f(x) \in [a(x), b(x)] = F(x)$ proving that $f$ is a selection of $F$ on $\mathcal{M}$.

Furthermore, thanks to the triangle inequality,

$$|f(x) - f(y)| = \left| \inf_{u \in \mathcal{M}} \{b(u) + \lambda \rho(x,u)\} - \inf_{u \in \mathcal{M}} \{b(u) + \lambda \rho(y,u)\} \right| \leq \sup_{u \in \mathcal{M}} |\lambda \rho(x,u) - \lambda \rho(y,u)| \leq \lambda \rho(x,y)$$

proving the required inequality $\|f\|_{\text{Lip}(\mathcal{M},\mathbb{R})} \leq \lambda$.

In the same way we show that the function $f = f^-[\lambda; F]$ is a Lipschitz selection of $F$ with Lipschitz seminorm at most $\lambda$. Clearly, the function $f[\lambda; F] = (f^+[\lambda; F] + f^-[\lambda; F])/2$ has the same property.

(ii) Let $f : \mathcal{M} \to \mathbb{R}$ be a Lipschitz selection of $F$ with $\|f\|_{\text{Lip}(\mathcal{M},\mathbb{R})} \leq \lambda$. We know that for every $x, y \in \mathcal{M}$ we have $f(x) \in F(x)$, $f(y) \in F(y)$ and $|f(x) - f(y)| \leq \lambda \rho(x, y)$. In this case Claim 5.4 tells us that

$$[a(x) - b(y)]_+ = \left[ \min F(x) - \max F(y) \right]_+ \leq \lambda \rho(x,y).$$

Hence, $\lambda_F \leq \lambda < \infty$, see (5.8).

Conversely, suppose that $\lambda_F < \infty$. Then, thanks to (5.3), for every $x, y \in \mathcal{M}$ we have

$$a(x) - b(y) \leq \lambda_F \rho(x, y) \quad \text{and} \quad a(y) - b(x) \leq \lambda_F \rho(x, y).$$

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This inequality and Claim 5.4 tell us that there exist points \( g(x) \in F(x), \; g(y) \in F(y) \) such that \(|g(x) - g(y)| \leq \lambda_F \rho(x, y)\). In other words, \( g \) is a Lipschitz selection of the restriction \( F \) to the set \( \{x, y\} \) with Lipschitz seminorm \( \|g\|_{\text{Lip}(\{x, y\}, \mathbb{R})} \leq \lambda_F \). Therefore, thanks to part (i) of the present lemma, there exists a Lipschitz selection \( f : M \to \mathbb{R} \) of \( F \) with Lipschitz seminorm \( \|f\|_{\text{Lip}(M, \mathbb{R})} \leq \lambda_F \).

It remains to note that in the proof of part (i) of the present lemma we have shown that each of the functions \( f^+[\lambda_F, F], \; f^-[\lambda_F, F] \) or \( f|\lambda_F, F \) provides a Lipschitz selection of \( F \) with Lipschitz seminorm at most \( \lambda_F \).

The proof of the lemma is complete. ■

Proof of Proposition 5.1. We have to prove that the set \( F^{[1]}(x) \) is non-empty for each \( x \in M \), and for every \( x, y \in M \)

\[
d_H(F^{[1]}(x), F^{[1]}(y)) \leq \eta \rho(x, y) \tag{5.9}
\]

provided the restriction \( F|_{M'} \) of \( F \) to every two point subset \( M' \subset M \) has a Lipschitz selection \( f_{M'} : M' \to \mathbb{R} \) with \( \|f_{M'}\|_{\text{Lip}(M', \mathbb{R})} \leq \eta \). Thus, for every \( z, z' \in M \) there exist points

\[
g(z) \in F(z), \; g(z') \in F(z') \quad \text{such that} \quad |g(z) - g(z')| \leq \eta \rho(z, z'). \tag{5.10}
\]

We recall that the set-valued mapping \( F^{[1]} \) is defined by formula (5.1). Prove that \( F^{[1]}(x) \neq \emptyset \) for every \( x \in M \). Indeed, thanks to (5.1) and Helly’s Theorem for intervals (part (i) of Lemma 5.2), \( F^{[1]}(x) \neq \emptyset \) provided

\[
(F(z) + \eta \rho(x, z) I_0) \cap (F(z') + \eta \rho(x, z') I_0) \neq \emptyset \tag{5.11}
\]

for every \( z, z' \in M \).

We know that there exist points \( g(z) \) and \( g(z') \) satisfying (5.10). Let

\[
a = \min\{g(z) + \eta \rho(z, x), g(z') + \eta \rho(z', x)\}.
\]

Thanks to the inequality \(|g(z) - g(z')| \leq \eta \rho(z, z')\), we have

\[
g(z) = \min\{g(z), g(z') + \eta \rho(z', z)\}
\]

so that, by the triangle inequality,

\[
|a - g(z)| \leq \max\{\eta \rho(z, x), |\eta \rho(z', x) - \eta \rho(z', z)|\} = \eta \rho(z, x).
\]

We also know that \( g(z) \in F(z) \), see (5.10), so that \( a \in F(z) + \eta \rho(x, z) I_0 \).

In the same way we show that \( a \in F(z') + \eta \rho(x, z') I_0 \) proving the required property (5.11).

Prove that

\[
F^{[1]}(x) + \eta \rho(x, y) I_0 \supset F^{[1]}(y) \tag{5.12}
\]

for every \( x, y \in M \).

We know that \( F^{[1]}(x) \neq \emptyset \) which enables us to apply part (b) of Lemma 5.2 to the left hand side of (5.12). This lemma and definition (5.1) tell us that

\[
F^{[1]}(x) + \eta \rho(x, y) I_0 = \bigcap_{z \in M} \left[ F(z) + \eta \rho(x, z) I_0 \right] + \eta \rho(x, y) I_0 = \bigcap_{z \in M} \left[ F(z) + \left( \eta \rho(x, z) + \eta \rho(x, y) \right) I_0 \right]
\]
so that, thanks to the triangle inequality,

\[ F^{[1]}(x) + \eta \rho(x, y) I_0 \supset \bigcap_{z \in M} [F(z) + \eta \rho(y, z) I_0] = F^{[1]}(y) \]

proving (5.12). By interchanging the roles of \( x \) and \( y \) we obtain also

\[ F^{[1]}(y) + \eta \rho(x, y) I_0 \supset F^{[1]}(x). \]

These two inclusions prove the required inequality (5.9). The proof of Proposition 5.1 is complete. ■

5.2 Several useful formulae for the Hausdorff distance.

Let \( X \) be a Banach space and let \( A, B \subset X \). We recall formula (1.2) for the Hausdorff distance between \( A \) and \( B \):

\[ d_H(A, B) = \inf \{ r > 0 : A + B_X(0, r) \supset B \text{ and } B + B_X(0, r) \supset A \}. \]

We also useful introduce a function

\[ \tilde{d}(A, B) = \inf \{ r > 0 : A + B_X(0, r) \supset B \} = \sup \{ \text{dist}(a, B) : a \in A \}. \]

Then,

\[ d_H(A, B) = \max \{ \tilde{d}(A, B), \tilde{d}(B, A) \}. \]

Let us note the following useful formula for the Hausdorff distance, see [9, p. 144]:

\[ d_H(A, B) = \sup \{ | \text{dist}(x, A) - \text{dist}(x, B) | : x \in X \}. \]

Next, we recall a well known expression for \( d_H \) in terms of support functions. Let \( X^* \) be the dual space of \( X \), and let \( B_{X^*} \) be unit ball of \( X^* \). We recall that the support function \( h_A : X^* \to \mathbb{R} \) is defined by

\[ h_A(f) = \sup \{ f(x) : x \in A \}. \]

One can easily see that for every \( \alpha, \beta \geq 0 \) and every \( A, B \subset X \) we have

\[ h_{\alpha A + \beta B} = \alpha h_A + \beta h_B. \] (5.13)

Furthermore, if \( A \) is a convex closed bounded sets which is symmetric with respect to 0, then for every \( f \in X^* \) we have

\[ h_A(f) = \sup \{ f(x) : x \in A \} = \sup \{ -f(x) : x \in A \} = h_A(-f). \] (5.14)

In these settings, for every convex closed bounded subsets \( A, B \subset X \) the following equality

\[ d_H(A, B) = \sup \{ |h_A(f) - h_B(f)| : f \in B_{X^*} \} \] (5.15)

holds. See, e.g., [7] or [15].

Let us also note the following result proven in [30]: If \( A, B \subset X \) are non-empty, bounded and convex then \( d_H(A, B) \leq d_H(\partial A, \partial B) \). If \( A, B \subset X \) are bounded, convex and have non-empty interior, then

\[ d_H(A, B) = d_H(\partial A, \partial B). \]

Here \( \partial A \) denotes the boundary of the set \( A \).
Lemma 5.6  (i) Let \( I_k = [a_k, b_k], k = 1, 2 \) be two line segments in \( \mathbb{R} \). Then
\[
d_H(I_1, I_2) = \max(|a_1 - a_2|, |b_1 - b_2|).
\]
(ii) Let \( A_1, A_2 \subset X \) be convex closed bounded sets. Suppose that \( A_1, A_2 \) are centrally symmetric with respect to points \( c_1 \) and \( c_2 \) respectively. Then
\[
||c_1 - c_2|| \leq d_H(A_1, A_2).
\]

Proof. (i) Let \( \tilde{I} = [-1, 1] \). Suppose that \( I_1 \neq I_2 \); otherwise the statement (i) is trivial. In this case
\[
\varepsilon = \max(|a_1 - a_2|, |b_1 - b_2|) > 0.
\]
Then \( a_1 - \varepsilon \leq a_2 \leq b_2 \leq b_1 + \varepsilon \) so that
\[
I_1 + \varepsilon \tilde{I} = [a_1 - \varepsilon, b_1 + \varepsilon] \supseteq I_2 = [a_2, b_2].
\]
In the same way we show that and \( I_2 + \varepsilon \tilde{I} \supseteq I_1 \), proving that
\[
d_H(I_1, I_2) \leq \varepsilon = \max(|a_1 - a_2|, |b_1 - b_2|).
\]
Prove the converse inequality. Let \( \varepsilon \) be a positive number such that
\[
I_1 + \varepsilon \tilde{I} = [a_1 - \varepsilon, b_1 + \varepsilon] \supseteq I_2 = [a_2, b_2] \quad \text{and} \quad I_2 + \varepsilon \tilde{I} = [a_2 - \varepsilon, b_2 + \varepsilon] \supseteq I_1 = [a_1, b_1]. \tag{5.16}
\]
Then
\[
a_1 - \varepsilon \leq a_2, \quad b_2 \leq b_1 + \varepsilon, \quad \text{and} \quad a_2 - \varepsilon \leq a_1, \quad b_1 \leq b_2 + \varepsilon.
\]
Hence,
\[
\max(|a_1 - a_2|, |b_1 - b_2|) \leq \varepsilon.
\]
We take the infimum over all \( \varepsilon > 0 \) satisfying (5.16), and obtain the required inequality
\[
\max(|a_1 - a_2|, |b_1 - b_2|) \leq d_H(I_1, I_2).
\]

(ii) Let \( A \subset X \) be a convex closed bounded set. We assume that \( A \) is centrally symmetric with respect to a point \( \bar{a} \in X \). Thus, \( A = \bar{A} + \bar{a} \) where \( \bar{A} \) is a convex closed bounded set with center of symmetry at 0. Therefore, thanks to (5.14), for every \( f \in B_X \), we have \( h_{\bar{A}}(f) = h_{\bar{A}}(-f) \).

Hence, thanks to this property and (5.13),
\[
h_{\bar{A}}(f) = h_{\bar{A}}(f) = f(\bar{a}) + h_{\bar{A}}(f) \quad \text{and} \quad h_{\bar{A}}(-f) = -f(\bar{a}) + h_{\bar{A}}(-f) = -f(\bar{a}) + h_{\bar{A}}(f)
\]
proving that
\[
f(\bar{a}) = \frac{1}{2}(h_{\bar{A}}(f) - h_{\bar{A}}(-f)).
\]
Applying this formula to the sets \( A_1, A_2 \) and their centers \( c_1, c_2 \), we have
\[
|f(c_1) - f(c_2)| = \frac{1}{2}|h_{A_1}(f) - h_{A_1}(-f)| \leq \frac{1}{2}|h_{A_1}(f) - h_{A_1}(-f)| + \frac{1}{2}|h_{A_2}(f) - h_{A_2}(-f)|.
\]
This inequality and (5.15) imply the following:
\[
|f(c_1 - c_2)| = |f(c_1) - f(c_2)| \leq d_H(A_1, A_2).
\]
Hence,
\[ \|c_1 - c_2\| = \sup_{f \in B_x} |f(c_1 - c_2)| \leq d_H(A_1, A_2) \]
proving the lemma. ■

### 5.3 Three criteria for Lipschitz selections.

Lemma 5.6 and Theorem 1.10 imply the following Lipschitz selection theorem.

**Theorem 5.7** Let \((\mathcal{M}, \rho)\) be a pseudometric space, and let \(X\) be a Banach space. Let \(\lambda > 0\) and let \(F : \mathcal{M} \to \mathcal{K}_1(X)\) be a set-valued mapping from \(\mathcal{M}\) into the family \(\mathcal{K}_1(X)\) of all bounded closed line segments in \(X\).

Suppose that for every subset \(\mathcal{M}' \subset \mathcal{M}\) with \(#\mathcal{M}' \leq 4\), the restriction \(F|_{\mathcal{M}'}\) of \(F\) to \(\mathcal{M}'\) has a Lipschitz selection with Lipschitz seminorm at most \(\lambda\).

Then \(F\) has a Lipschitz selection \(f\) with Lipschitz seminorm \(\|f\|_{\text{Lip}(\mathcal{M}, X)} \leq 15\lambda\). If \(X\) is a Euclidean space, there exists a Lipschitz selection \(f\) of \(F\) with \(\|f\|_{\text{Lip}(\mathcal{M}, X)} \leq 10\lambda\).

**Proof.** Let \(\tilde{l} = (1, 3)\) and let \(F^{[1]}\) and \(F^{[2]}\) be the first and the second order \((\tilde{l}, \lambda \rho)\)-balanced refinements of \(F\). See Definition 1.5. Thus,
\[ F^{[1]}(x) = \bigcap_{z \in \mathcal{M}} \left[ F(z) + \lambda \rho(x, z)B_x \right] \quad \text{and} \quad F^{[2]}(x) = \bigcap_{z \in \mathcal{M}} \left[ F^{[1]}(z) + 3\lambda \rho(x, z)B_x \right], \quad x \in \mathcal{M}. \]

Theorem 1.10 tells us that the set-valued mapping \(F^{[2]}\) is a \(\gamma\)-core of \(F\) with \(\gamma = 15\) provided \(X\) is an arbitrary Banach space, and with \(\gamma = 10\) whenever \(X\) is a Euclidean space. In other words, \(F^{[2]}(x) \neq \emptyset\) for every \(x \in \mathcal{M}\), and
\[ d_H(F^{[2]}(x), F^{[2]}(y)) \leq \gamma \lambda \rho(x,y) \quad \text{for all} \quad x, y \in \mathcal{M}. \quad (5.17) \]

Clearly, \(F^{[2]}(x) \in \mathcal{K}_1(X)\), i.e., \(F^{[2]}(x)\) is a closed bounded line segment in \(X\) for each \(x \in \mathcal{M}\). In other words, \(F^{[2]}(x) = [a_1(x), a_2(x)]\), \(x \in \mathcal{M}\), where \(a_i : \mathcal{M} \to X\), \(i = 1, 2\), are certain mappings on \(\mathcal{M}\).

We define a mapping \(f : \mathcal{M} \to X\) by letting
\[ f(x) = \frac{1}{2}(a_1(x) + a_2(x)), \quad x \in \mathcal{M}. \]

Thus, \(f(x)\) is the center of the line segment \(F^{[2]}(x) = [a_1(x), a_2(x)]\) so that \(f(x) \in F^{[2]}(x) \subset F(x)\) proving that \(f\) is a selection of \(F\) on \(\mathcal{M}\). Furthermore, Lemma 5.6 and inequality (5.17) tell us that
\[ \|f(x) - f(y)\| \leq d_H(F^{[2]}(x), F^{[2]}(y)) \leq \gamma \lambda \rho(x,y) \quad \text{for all} \quad x, y \in \mathcal{M}. \]

Thus, \(\|f\|_{\text{Lip}(\mathcal{M}, X)} \leq \gamma \lambda\), and the proof of the theorem is complete. ■

We finish the section with a useful criterion for Lipschitz selections in \(\mathbb{R}\). We recall that, given \(\lambda > 0\) we set
\[ F^{[1]}_{\tilde{l}}(x) = \bigcap_{z \in \mathcal{M}} \left[ F(z) + \lambda \rho(x, z)I_0 \right], \quad x \in \mathcal{M}. \]

See definition (1.6).
Proposition 5.8 Let \( \mathcal{M} = (\mathcal{M}, \rho) \) be a pseudometric space, and let \( F : \mathcal{M} \to \mathcal{I}(\mathbb{R}) \) be a set-valued mapping. Suppose that either \( \mathcal{M} \) is finite or there exist \( x, y \in \mathcal{M} \) and \( \alpha \geq 0 \) such that the set \( F(x) \cap [F(y) + \alpha I_0] \) is non-empty and bounded. Then the following criterion holds: given \( \lambda > 0 \) the mapping \( F \) has a Lipschitz selection \( f : \mathcal{M} \to \mathbb{R} \) with \( \|f\|_{\text{Lip}(\mathcal{M}, \mathbb{R})} \leq \lambda \) if and only if the set \( F^{[1]}_{\lambda}(x) \neq \emptyset \) for every \( x \in \mathcal{M} \).

Furthermore,
\[
|F|_{\mathcal{M}, \mathbb{R}} = \inf \{ \lambda : F^{[1]}_{\lambda}(x) \neq \emptyset \text{ for all } x \in \mathcal{M} \}.
\]

See (1.16).

Proof. The proposition easily follows from Lemma 5.5. Indeed, suppose that \( F \) has a Lipschitz selection \( f : \mathcal{M} \to \mathbb{R} \) with \( \|f\|_{\text{Lip}(\mathcal{M}, \mathbb{R})} \leq \lambda \). Then, given \( x \in \mathcal{M} \), we have \( |f(x) - f(y)| \leq \lambda \rho(x, z) \) for every \( z \in \mathcal{M} \). But \( f(x) \in F(x) \) and \( f(z) \in F(z) \) (because \( f \) is a selection of \( F \)) so that \( f(x) \in F(z) + \lambda \rho(x, z) I_0 \) proving that \( f(x) \in F^{[1]}_{\lambda}(x) \).

Now, suppose that \( F^{[1]}_{\lambda}(x) \neq \emptyset \) for every \( x \in \mathcal{M} \). Then, for every \( x, z \in \mathcal{M} \), we have
\[
F(x) \cap [F(z) + \lambda \rho(x, z) I_0] \neq \emptyset.
\]

Hence, \( \text{dist}(F(x), F(z)) \leq \lambda \rho(x, z) \) so that there exist points \( g(x) \in F(x), g(z) \in F(z) \) such that \( |g(x) - g(y)| \leq \lambda \rho(x, z) \). Part (i) of Lemma 5.5 tells us that in these settings the mapping \( F \) has a Lipschitz selection \( f : \mathcal{M} \to \mathbb{R} \) with \( \|f\|_{\text{Lip}(\mathcal{M}, \mathbb{R})} \leq \lambda \).

The proof of the proposition is complete. \( \blacksquare \)

The following proposition is immediate from Proposition 5.8.

Proposition 5.9 Let \( G : \mathcal{M} \to \mathcal{R}(\mathbb{R}^2) \) be a set-valued mapping from a pseudometric space \( \mathcal{M} = (\mathcal{M}, \rho) \) into the family \( \mathcal{R}(\mathbb{R}^2) \) of all closed rectangles in \( \mathbb{R}^2 \) with sides parallel to the coordinate axes. Suppose that either \( \mathcal{M} \) is finite or there exist \( x, y \in \mathcal{M} \) and \( \alpha \geq 0 \) such that the set \( G(x) \cap [G(y) + \alpha Q_0] \) is non-empty and bounded.

Then the following criterion holds: given \( \lambda > 0 \) the mapping \( G \) has a Lipschitz selection \( g : \mathcal{M} \to \mathbb{R}^2 \) with \( \|g\|_{\text{Lip}(\mathcal{M}, \mathbb{R}^2)} \leq \lambda \) if and only if the set
\[
G^{[1]}_{\lambda}(x) = \bigcap_{z \in \mathcal{M}} [G(z) + \lambda \rho(x, z) Q_0]
\]
is not empty for every \( x \in \mathcal{M} \). (Recall that \( Q_0 = [-1, 1]^2 \).)

Furthermore,
\[
|G|_{\mathcal{M}, \mathbb{R}^2} = \inf \{ \lambda : G^{[1]}_{\lambda}(x) \neq \emptyset \text{ for all } x \in \mathcal{M} \}.
\]

Recall that we measure the distances in \( \mathbb{R}^2 \) in the uniform norm \( \|a\|_{\ell_\infty^2} = \max \{|a_1|, |a_2|\}, a = (a_1, a_2) \).

6. Main Theorem in \( \ell_\infty^2 \).

6.1 Rectangular hulls of plane convex sets.

We recall that by \( \mathcal{I}(\mathbb{R}) \) we denote the family of all closed intervals in \( \mathbb{R} \) (bounded or unbounded). We also recall that \( \mathcal{R}(\mathbb{R}^2) \) is the family of all closed rectangles in \( \mathbb{R}^2 \) with sides parallel to the coordinate axes, i.e.,
\[
\mathcal{R}(\mathbb{R}^2) = \{ \Pi = I_1 \times I_2 : I_1, I_2 \in \mathcal{I}(\mathbb{R}) \}.
\]
We refer to every \( \Pi \in \mathbb{R}(\mathbb{R}^2) \) as a “box” or “rectangle”.

Clearly, each bounded rectangle \( \Pi \in \mathbb{R}(\mathbb{R}^2) \) is a centrally symmetric set. We let \( \text{cntr}(\Pi) \) denote the center of \( \Pi \).

Everywhere in this section we let \( S \) denote a non-empty convex closed subset of \( \mathbb{R}^2 \).

**Definition 6.1** We let \( \mathcal{H}[S] \) denote the smallest (with respect to inclusion) rectangle containing \( S \). Thus,

\[
\mathcal{H}[S] = \cap \{ \Pi : \Pi \in \mathbb{R}(\mathbb{R}^2), \Pi \supset S \}.
\]

We refer to \( \mathcal{H}[S] \) as a “rectangular hull” of the set \( S \).

We let \( \text{Pr}_i \) denote the operator of orthogonal projection onto the axis \( Ox_i, i = 1, 2 \), i.e.,

\[
\text{Pr}_i[x] = x_i \quad \text{for} \quad x = (x_1, x_2) \in \mathbb{R}^2.
\]

Then the rectangular hull of \( S \) has the following representation:

\[
\mathcal{H}[S] = \text{Pr}_1[S] \times \text{Pr}_2[S].
\]

Property (6.2) implies the following: for every rectangle \( H \in \mathbb{R}(\mathbb{R}^2) \) with center 0, we have

\[
\mathcal{H}[S + H] = \mathcal{H}[S] + H.
\]

In particular, for every \( r \geq 0 \) the following equality

\[
\mathcal{H}[S + rQ_0] = \mathcal{H}[S] + rQ_0
\]

holds. This and definition (1.2) imply the following property of rectangular hulls: Let \( S_1, S_2 \) be convex closed subsets of \( \mathbb{R}^2 \). Then

\[
d_H(\mathcal{H}[S_1], \mathcal{H}[S_2]) \leq d_H(S_1, S_2).
\]

Thus, \( \mathcal{H}[S] \) is the only rectangle for which

\[
\text{Pr}_1[\mathcal{H}[S]] = \text{Pr}_1[S] \quad \text{and} \quad \text{Pr}_2[\mathcal{H}[S]] = \text{Pr}_2[S].
\]

We note one more obvious property characterizing the rectangular hull: \( H(S) \) is the only rectangular such that

\[
\mathcal{H}[S] \supset S \quad \text{and each side of} \quad \mathcal{H}[S] \quad \text{has a common point with} \quad S.
\]

Finally, we have the following obvious formula for \( \mathcal{H}[S] \):

\[
\mathcal{H}[S] = (S + Ox_1) \cap (S + Ox_2).
\]

The following important property of rectangular hulls in \( \mathbb{R}^2 \) has been noted in [21, §6]; see also [13, Section 7.2].

**Lemma 6.2** Let \( S \subset \mathbb{R}^2 \) be a convex compact set. Then \( \text{cntr}(\mathcal{H}[S]) \in S \).
Proof. Suppose, given a convex compact set $S \subset \mathbb{R}^2$, its center $\text{cntr}(\mathcal{H}[S]) \notin S$.

Without loss of generality, we may assume that $\text{cntr}(S) = 0$. Thus, $0 \notin S$. In this case the
separation theorem tells us that there exists a vector $a \in \mathbb{R}^2$ such that the inner product $\langle a, x \rangle > 0$ for
every $x \in S$.

Clearly, there exists a side of $\mathcal{H}[S]$, say $[AB]$, such that $\langle a, z \rangle \leq 0$ for every $z \in [AB]$. Then,
$[AB] \cap S = \emptyset$. This contradicts property (6.8) of the rectangular hull proving the lemma. 

We need the following Helly-type theorem in $\mathbb{R}^2$.

**Proposition 6.3** Let $\mathcal{C}$ be a family of nonempty convex closed subsets of $\mathbb{R}^2$. Suppose that either $\mathcal{C}$ is
finite or at least one member of $\mathcal{C}$ is bounded. If

$$\text{Pr}_1[C_1 \cap C_1'] \cap \text{Pr}_1[C_2 \cap C_2'] \neq \emptyset \quad \text{for every} \quad C_1, C_1', C_2, C_2' \in \mathcal{C}, \quad (6.10)$$

then there exists a point common to all of the family $\mathcal{C}$. Furthermore,

$$\text{Pr}_1 \left[ \bigcap_{C \in \mathcal{C}} C \right] = \bigcap_{C, C' \in \mathcal{C}} \text{Pr}_1[C \cap C']. \quad (6.11)$$

**Proof.** Condition (6.10) tells us that for every $C, C' \in \mathcal{C}$ the set $C \cap C'$ is a non-empty. Clearly,
$C \cap C'$ is a convex closed subset of $\mathbb{R}^2$, so that its projection onto $Ox_1$, the set $\text{Pr}_1[C \cap C'] \subset \mathbb{R}$, is a
closed interval in $\mathbb{R}$.

From the lemma’s hypothesis it follows that either the family $\mathcal{K} = \{\text{Pr}_1[C \cap C'] : C, C' \in \mathcal{C}\}$ is finite
or at least one member of $\mathcal{K}$ is bounded. Thus, $\mathcal{K}$ satisfies the hypothesis of the one dimensional Helly’s Theorem
formulated in Lemma 5.2 part (a). Thanks to this lemma,

$$V = \bigcap_{C, C' \in \mathcal{C}} \text{Pr}_1[C \cap C'] \neq \emptyset. \quad (6.12)$$

Fix a point $v \in V$, and set $L = \{w \in \mathbb{R}^2 : \text{Pr}_1[w] = v\}$. Clearly, $L$ is a straight line through $v$
orthogonal to the axis $Ox_1$.

Given $C \in \mathcal{C}$ we set $K(C) = C \cap L$. We know that $\text{Pr}_1[C] \ni v$ so that $K(C) \neq \emptyset$. Furthermore,
because $v \in V$, for every $C, C' \in \mathcal{C}$ we have $v \in \text{Pr}_1[C \cap C']$ so that there exist $\tilde{w} \in C \cap C'$ such that
$\text{Pr}_1[\tilde{w}] = v$. Hence, $\tilde{w} \in L \cap C \cap C' = K(C) \cap K(C')$.

Let $\mathcal{K} = \{K(C) : C \in \mathcal{C}\}$. Clearly, all members of $\mathcal{K}$ are closed intervals in $L$. We have shown that
any two members of $\mathcal{K}$ have a common point, so that $\mathcal{K}$ also satisfies the hypothesis of part (a) of
Lemma 5.2. This proves the existence of a point in $L$, say $u$, common to all of the family $\mathcal{K}$.

Thus, $u \in C \cap L$ for each $C \in \mathcal{C}$ proving that $u \in \cap\{C : C \in \mathcal{C}\}$. At the same time, $u \in L$ so that
$\text{Pr}_1[u] = v$. This shows that (i) $\cap\{C : C \in \mathcal{C}\} \neq \emptyset$, and (ii) the left hand side of (6.11) contains its
right hand side. Obviously, the left hand side of (6.11) is contained in its right hand side, proving that
equality (6.11) holds.

The proof of the proposition is complete. 

**Remark 6.4** Remark 5.3 enables us to slightly modify the hypothesis of Proposition 6.3. Namely, we can replace the requirement “at least one member of $\mathcal{C}$ is bounded” with “there exists a finite subfamily $\mathcal{C} \subset \mathcal{C}$ such that the intersection $\cap\{C : C \in \mathcal{C}\}$ is nonempty and bounded”.

Indeed, suppose that there exists a subfamily $\mathcal{C} \subset \mathcal{C}$ having such a property. Let us see that in this
case (6.12) holds; then the remaining part of the proof holds as well.
We know that Proposition 6.3 is true provided \( \mathcal{C} \) is finite. Applying this proposition to \( \mathcal{C} \) we conclude that formula (6.11) holds for \( \mathcal{C} \), i.e.,

\[
\Pr_1 \left[ \bigcap_{C \in \mathcal{C}} C \right] = \bigcap_{C, C' \in \mathcal{C}} \Pr_1 [C \cap C'] .
\]

Because the set \( \cap \{ C : C \in \mathcal{C} \} \) is nonempty and bounded, the set \( \cap \{ \Pr_1 [C \cap C'] : C, C' \in \mathcal{C} \} \) is nonempty and bounded as well. Therefore, the family \( \mathcal{K} = \{ \Pr_1 [C \cap C'] : C, C' \in \mathcal{C} \} \) satisfies the hypothesis of the one dimensional Helly’s Theorem formulated in Lemma 5.2, part (a), with modification given in Remark 5.3.

This implies the required statement (6.12) proving the proposition. \( \triangleright \)

Proposition 6.3 and properties (6.7), (6.2) of rectangle hulls imply the following

**Corollary 6.5** Let \( \mathcal{C} \) be a family of convex closed subsets of \( \mathbb{R}^2 \). Suppose that either \( \mathcal{C} \) is finite or there exists a finite subfamily \( \mathcal{C} \subset \mathcal{C} \) such that the intersection \( \cap \{ C : C \in \mathcal{C} \} \) is nonempty and bounded. If

\[
\Pr_1 [C_1 \cap C'_1] \cap \Pr_1 [C_2 \cap C'_2] \neq \emptyset \quad \text{for every} \quad C_1, C'_1, C_2, C'_2 \in \mathcal{C},
\]

then \( \cap \{ C : C \in \mathcal{C} \} \neq \emptyset \). Furthermore, in this case

\[
\mathcal{H}[\cap \{ C : C \in \mathcal{C} \}] = \cap \{ \mathcal{H}[C \cap C'] : C, C' \in \mathcal{C} \} .
\]

Let us formulate two useful properties of rectangles from the family \( \mathcal{R}(\mathbb{R}^2) \).

**Lemma 6.6** For every \( r_1, r_2 \geq 0 \) and every two rectangles \( \Pi_1, \Pi_2 \in \mathcal{R}(\mathbb{R}^2) \) we have

\[
\text{dist}(\Pi_1 + r_1 Q_0, \Pi_2 + r_2 Q_0) = \lfloor \text{dist}(\Pi_1, \Pi_2) - r_1 - r_2 \rfloor_+ .
\]

**Lemma 6.7** Let \( \mathcal{R}_1, \mathcal{R}_2 \subset \mathcal{R}(\mathbb{R}^2) \) be two families of rectangles in \( \mathbb{R}^2 \). Suppose that each family has a non-empty intersection. Then

\[
\text{dist} \left( \bigcap_{\Pi \in \mathcal{R}_1} \bigcap_{\Pi' \in \mathcal{R}_2} \Pi \right) = \sup_{\Pi_1 \in \mathcal{R}_1, \Pi_2 \in \mathcal{R}_2} \text{dist}(\Pi_1, \Pi_2) .
\]

We prove both lemmas by projecting onto coordinate axes, i.e., by reduction to the one dimensional case. In this case the first lemma is elementary, while the second lemma easily follows from the one dimensional Helly’s Theorem.

The next lemma is immediate from part (b) of Lemma 5.2

**Lemma 6.8** Let \( \mathcal{K} \subset \mathcal{R}(\mathbb{R}^2) \) be a family of rectangles with non-empty intersection. Let \( H \in \mathcal{R}(\mathbb{R}^2) \) be a rectangle with center 0. Then

\[
\left( \bigcap_{\Pi \in \mathcal{K}} \Pi \right) + H = \bigcap_{\Pi \in \mathcal{K}} \{ \Pi + H \} .
\]

The following three lemmas are certain modifications of Lemma 3.6 for the space \( \ell^2_\infty \).
Lemma 6.9 Let \( \mathcal{K} \) be a collection of convex closed subsets of \( \mathbb{R}^2 \) with non-empty intersection, and let \( \Pi \in \mathcal{H}^n(\mathbb{R}^2) \) be a rectangle with center 0. Then

\[
\left( \bigcap_{K \in \mathcal{K}} K \right) + \Pi = \bigcap_{K,K' \in \mathcal{K}} \left( (K \cap K') + \Pi \right).
\]

Proof. If the rectangle \( \Pi \) is bounded then the lemma is immediate from Lemma 6.6. If \( \Pi \) is unbounded then the lemma is immediate from Proposition 6.3 and (6.4). We leave the details to the interested reader. 

Lemma 6.10 Let \( K_1, K_2 \subset \mathbb{R}^2 \) be convex closed sets with non-empty intersection. Then for every rectangle \( \Pi \in \mathcal{H}^n(\mathbb{R}^2) \) with \( \text{cntr}(\Pi) = 0 \) we have

\[
K_1 \cap K_2 + \Pi = (K_1 + \Pi) \cap (K_2 + \Pi) \cap \mathcal{H}[K_1 \cap K_2 + \Pi].
\] (6.15)

Proof. Clearly, the right hand side of (6.15) contains its left hand side. Let us prove the converse statement. Fix a point

\[
x \in (K_1 + \Pi) \cap (K_2 + \Pi) \cap \mathcal{H}[K_1 \cap K_2 + \Pi]
\]

(6.16)

and prove that \( x \in K_1 \cap K_2 + \Pi \).

Clearly, this property holds if and only if \( (x + \Pi) \cap K_1 \cap K_2 \neq \emptyset \). Let us represent the rectangle \( x + \Pi \) in the form \( x + \Pi = \Pi_1(x) \cap \Pi_2(x) \) where

\[
\Pi_1(x) = x + Ox_1 + \Pi \quad \text{and} \quad \Pi_2(x) = x + Ox_2 + \Pi.
\] (6.17)

(Recall that \( Ox_1 = \{ x = (t,0) : t \in \mathbb{R} \} \) and \( Ox_2 = \{ x = (0,t) : t \in \mathbb{R} \} \) are the coordinate axes.) Thus, \( x \in K_1 \cap K_2 + \Pi \) provided \( K_1 \cap K_2 \cap \Pi_1(x) \cap \Pi_2(x) \neq \emptyset \).

Helly’s Theorem 2.4 tells us that this statement is true provided any three members of the family of sets \( \mathcal{K} = \{ K_1, K_2, \Pi_1(x), \Pi_2(x) \} \) have a common point. Let us see that this property holds for \( x \) satisfying (6.16).

Clearly, for every \( i = 1, 2 \),

\[
K_i \cap \Pi_1(x) \cap \Pi_2(x) = K_i \cap (x + \Pi) \neq \emptyset
\]

because \( x \in K_i + \Pi \). Prove that

\[
K_1 \cap K_2 \cap \Pi_1(x) \neq \emptyset.
\] (6.18)

Indeed, thanks to (6.9),

\[
\mathcal{H}[K_1 \cap K_2] = \{ K_1 \cap K_2 + Ox_1 \} \cap \{ K_1 \cap K_2 + Ox_2 \} \subset K_1 \cap K_2 + Ox_1
\]

so that

\[
\mathcal{H}[K_1 \cap K_2] + \Pi \subset K_1 \cap K_2 + Ox_1 + \Pi
\]

But, thanks to (6.5), \( \mathcal{H}[K_1 \cap K_2] + \Pi = \mathcal{H}[K_1 \cap K_2 + \Pi] \), and, thanks to (6.16), \( x \in \mathcal{H}[K_1 \cap K_2 + \Pi] \). Hence, \( x \in K_1 \cap K_2 + Ox_1 + \Pi \). Clearly, this property is equivalent to (6.18), see (6.17). In the same fashion we prove that \( K_1 \cap K_2 \cap \Pi_2(x) \neq \emptyset \) completing the proof of the lemma. 

This lemma and Lemma 6.9 imply the following result.

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**Lemma 6.11** Let \( \mathcal{K} \subset \mathcal{K}(\mathbb{R}^2) \) be a family of convex closed subsets of \( \mathbb{R}^2 \) with non-empty intersection. Then for every rectangle \( \Pi \in \mathcal{R}(\mathbb{R}^2) \) with center 0, the following equality

\[
\left( \bigcap_{K \in \mathcal{K}} K \right) + \Pi = \left( \bigcap_{K \in \mathcal{K}} (K + \Pi) \right) \bigcap \left\{ \bigcap_{K,K' \in \mathcal{K}} \mathcal{H}[K \cap K' + \Pi] \right\}
\]

holds.

The last result of this section, Proposition 6.12 below, presents the Finiteness Principle for Lipschitz selections for rectangles in \( \mathbb{R}^2 \). Part (i) of this result is immediate from the Finiteness Principle for intervals in \( \mathbb{R} \) given in part (i) of Lemma 5.5, and part (ii) is immediate from Proposition 5.1.

Recall that in the one-dimensional case the finiteness constant \( N(1, \mathbb{R}) = \min(2^2, 2) = 2 \), see (1.1), and the constant \( \gamma \) from Theorem 1.2 equals 1.

**Proposition 6.12** Let \( (M, \rho) \) be a pseudometric space, and let \( \lambda > 0 \). Let \( T : M \to \mathcal{R}(\mathbb{R}^2) \) be a set-valued mapping. Suppose that either \( M \) is finite or \( T(x) \) is bounded for some \( x \in M \). Let us also assume that for every \( x, y \in M \) the restriction \( T_{|x,y} \) of \( T \) to \( \{x, y\} \) has a Lipschitz selection \( g_{x,y}^{\prime} \) with Lipschitz seminorm \( \|g_{x,y}^{\prime}\|_{\text{Lip}(\mathcal{R}(x,y), \mathbb{R})} \leq \lambda \). In these settings, the following statements hold:

(i) The mapping \( T \) has a Lipschitz selection \( g \) with Lipschitz seminorm \( \|g\|_{\text{Lip}(M, \mathbb{R})} \leq \lambda \);

(ii) Let

\[
T^{(1)}(x) = \bigcap_{z \in M} \left[ T(z) + \lambda \rho(x, z) Q_0 \right], \quad x \in M,
\]

be the \( \lambda \)-balanced refinement of the mapping \( T \). Then \( T^{(1)}(x) \neq \emptyset \) for each \( x \in M \), and

\[
\text{d}_H(T^{(1)}(x), T^{(1)}(y)) \leq \lambda \rho(x, y) \quad \text{for every} \quad x, y \in M.
\]

### 6.2. Balanced refinements of set-valued mappings in \( \ell_{\infty}^2 \).

Theorem 1.9 tells us that for the space \( X = \ell_{\infty}^2 \) given \( \lambda_1, \lambda_2, \gamma \geq 0 \) and a set-valued mapping \( F : M \to \mathcal{K}(X) \), the mapping \( F^{[2]} \) defined by (1.8) has properties (1.9) and (1.10) provided \( \lambda_1 \geq 1, \lambda_2 \geq 3\lambda_1 \) and \( \gamma \geq (3\lambda_2 + \lambda_1)^2 / (\lambda_2 - \lambda_1)^2 \).

In this section we show that this result can be improved as follows.

**Theorem 6.13** Let \( \mathfrak{M} = (M, \rho) \) be a pseudometric space. Let \( F : M \to \mathcal{K}(\mathbb{R}^2) \) be a set-valued mapping such that for every \( M' \subset M \) with \( \#M' \leq 4 \), the restriction \( F|_{M'} \) of \( F \) to \( M' \) has a Lipschitz selection \( f : M \to \ell_{\infty}^2 \) with Lipschitz seminorm \( \|f\|_{\text{Lip}(M, \ell_{\infty}^2)} \leq 1 \).

Then for every

\[
\lambda_1 \geq 1, \quad \lambda_2 \geq 3\lambda_1, \quad \text{and} \quad \gamma \geq \lambda_2 (3\lambda_2 + \lambda_1) / (\lambda_2 - \lambda_1)
\]

properties (1.9) and (1.10) hold.

In particular, (1.9) and (1.10) hold provided \( \lambda_1 = 1, \lambda_2 = 3 \), and \( \gamma = 15 \).

**Proof.** We mainly follow the scheme of the proof of Theorem 1.9 given in Section 3. We recall that Lipschitz extension constant \( e(\mathfrak{M}, \ell_{\infty}^2) \) equals 1, see (3.3).

Let \( F : M \to \mathcal{K}(\mathbb{R}^2) \) be a set-valued mapping satisfying the hypothesis of Theorem 6.13. As in Section 3, this enables us to make the following

**Assumption 6.14** For every \( M' \subset M, \#M' \leq 4 \), the restriction \( F|_{M'} \) of \( F \) to \( M' \) has a \( \rho \)-Lipschitz selection \( f_{M'} : M' \to \ell_{\infty}^2 \) with \( \rho \)-Lipschitz seminorm \( \|f_{M'}\|_{\text{Lip}(M', \rho), \ell_{\infty}^2} \leq 1 \).
We fix a constant \( L \geq 3 \) and a constant \( \alpha \geq 1 \), and introduce a pseudometric \( d(x, y) = \alpha \rho(x, y) \), \( x, y \in \mathcal{M} \). Then we introduce set-valued mappings \( F^{[1]} \) and \( F^{[2]} \) defined by

\[
F^{[1]}(x) = \bigcap_{z \in \mathcal{M}} \left[ F(z) + d(x, z)Q_0 \right], \quad x \in \mathcal{M},
\]  

(6.20)

and

\[
F^{[2]}(x) = \bigcap_{z \in \mathcal{M}} \left[ F^{[1]}(z) + L d(x, z)Q_0 \right], \quad x \in \mathcal{M}.
\]  

(6.21)

Recall that \( F^{[1]} \) and \( F^{[2]} \) are the first and the second order \( ([1, L], d) \)-balanced refinements of \( F \) respectively. See Definition 1.6.

Lemma 3.7 and Proposition 3.9 tell us that \( F^{[1]}(x) \neq \emptyset \) and \( F^{[2]}(x) \neq \emptyset \) for every \( x \in \mathcal{M} \). Thus, our aim is to show that for every \( \alpha \geq 1 \), \( L \geq 3 \), and every \( x, y \in \mathcal{M} \) the following inequality

\[
d_H(F^{[2]}(x), F^{[2]}(y)) \leq \tilde{\gamma}(L) \, d(x, y)
\]  

(6.22)

holds with

\[
\tilde{\gamma}(L) = L \theta(L) \quad \text{where} \quad \theta(L) = (3L + 1)/(L - 1)
\]  

(6.23)

is the constant from Proposition 2.5 (for the space \( X = \ell_2^\infty \)).

We prove this inequality with the help of a certain modification of representations (3.21) and (3.44). To its formulation we recall that given \( x, u, u', u'' \in \mathcal{M} \) we set

\[
T_x(u, u', u'') = \{ F(u') + d(u', u)Q_0 \} \cap \{ F(u'') + d(u'', u)Q_0 \} + L d(u, x)Q_0.
\]  

(6.24)

See definition (3.20). Recall also that, thanks to (3.21),

\[
F^{[2]}(x) = \bigcap_{u,u',u'' \in \mathcal{M}} T_x(u, u', u''), \quad x \in \mathcal{M}.
\]  

(6.25)

The next lemma provides another representation of the set \( F^{[2]}(x) \).

**Lemma 6.15** For every \( x \in \mathcal{M} \) the following equalities

\[
F^{[2]}(x) = F^{[1]}(x) \bigcap_{u,u',u'' \in \mathcal{M}} \mathcal{H}[T_x(u, u', u'')] \bigcap_{u \in \mathcal{M}} \mathcal{H}[F^{[1]}(u) + L d(u, x)Q_0],
\]  

(6.26)

\[
F^{[2]}(x) = F^{[1]}(x) \bigcap_{u \in \mathcal{M}} \mathcal{H}[F^{[1]}(u) + L d(u, x)Q_0],
\]  

(6.27)

hold.

**Proof.** Let

\[
\mathcal{K}_x = \{ F(y) + d(x, y)Q_0 : y \in \mathcal{M} \}.
\]  

(6.28)
Then
\[
F^{[1]}(x) = \bigcap_{K \in K_x} K \quad \text{and} \quad F^{[2]}(x) = \bigcap_{y \in M} \left\{ \left( \bigcap_{K \in K_y} K \right) + L d(x, y)Q_0 \right\}.
\]

See (6.20) and (6.21).

Because \( F^{[1]}(y) \neq \emptyset \), the family of sets \( \mathcal{K}_y = \{ F(z) + d(y, z)Q_0 : z \in M \} \) has non-empty intersection for every \( y \in M \). Therefore, thanks to Lemma 6.11,
\[
\left( \bigcap_{K \in K_y} K \right) + L d(x, y)Q_0 = \left\{ \left( \bigcap_{K \in K_y} K \right) + L d(x, y)Q_0 \right\} \cap \left( \bigcap_{K \in K_y} K \right) + L d(x, y)Q_0 \right\}.
\]

Hence,
\[
F^{[2]}(x) = \left\{ \bigcap_{y \in M} \left( \bigcap_{K \in K_y} K \right) + L d(x, y)Q_0 \right\} \bigcap \left( \bigcap_{y \in M} \left( \bigcap_{K \in K_y} K \right) + L d(x, y)Q_0 \right) = A_1 \cap A_2.
\]

Clearly, thanks to the triangle inequality,
\[
A_1 = \bigcap_{y \in M} \left( \bigcap_{K \in K_y} K \right) + L d(x, y)Q_0 = \bigcap_{y \in M} \left( F(z) + d(z, y)Q_0 + L d(x, y)Q_0 \right)
\]
\[
\supset \bigcap_{y \in M} \left( F(z) + d(z, x)Q_0 \right) = F^{[1]}(x).
\]

On the other hand, \( A_1 \subset \cap \{ K : K \in \mathcal{K}_x \} = F^{[1]}(x) \) so that \( A_1 = F^{[1]}(x) \). This equality, definition (6.28) and definition (6.24) imply (6.26). Equality (6.27) is immediate from (6.26), Corollary 6.5 and Lemma 6.8.

The proof of the lemma is complete. \( \blacksquare \)

**Lemma 6.16** For every \( x \in M \) and every rectangle \( \Pi \in \mathcal{R}(\mathbb{R}^2) \) with center 0 we have
\[
F^{[2]}(x) + \Pi = \bigcap_{v,u,u',u'' \in \mathcal{M}} \{ (\mathcal{H}[T_s(u, u', u'')] \cap (F(v) + d(x, v)Q_0)) + \Pi \}.
\]

**Proof.** Let \( \mathcal{K}^{(1)} = \{ F(v) + d(x, v)Q_0 : v \in \mathcal{M} \} \), and let
\[
\mathcal{K}^{(2)} = \{ \mathcal{H}[T_s(u, u', u'')] : u, u', u'' \in \mathcal{M} \}.
\]

We have to prove that
\[
F^{[2]}(x) + \Pi = \cap \{ (K_1 \cap K_2) + \Pi : K_1 \in \mathcal{K}^{(1)}, K_2 \in \mathcal{K}^{(2)} \}.
\]

Formula (6.26) and Lemma 6.9 tell us that
\[
F^{[2]}(x) + \Pi = \cap \{ (K \cap K') + \Pi : K, K' \in \mathcal{K}^{(1)} \cup \mathcal{K}^{(2)} \}.
\]

Let
\[
A = \cap \{ (K_1 \cap K_2) + \Pi : K_1 \in \mathcal{K}^{(1)}, K_2 \in \mathcal{K}^{(2)} \}.
\]

Formula (6.30) tells us that \( A \subset F^{[2]}(x) + \Pi \). Prove the converse inclusion.
It suffices to show that for every \( K, K' \in \mathcal{K}^{(1)} \cup \mathcal{K}^{(2)} \), we have

\[
A \subset (K \cap K') + \Pi. \tag{6.32}
\]

Clearly, thanks to definition (6.31), it is true provided \( K \in \mathcal{K}^{(1)}, K' \in \mathcal{K}^{(2)} \) or \( K \in \mathcal{K}^{(2)}, K' \in \mathcal{K}^{(1)} \).

Prove (6.32) for sets \( K = F(z) + d(z, x)Q_0 \) and \( K' = F(z') + d(z', x)Q_0 \) which belong to the family \( \mathcal{K}^{(1)} \). In this case, thanks to (6.24) and (6.29), the element

\[
\tilde{H} = \mathcal{H} [(F(z) + d(z, x)Q_0) \cap (F(z') + d(z', x)Q_0)] = \mathcal{H} [T_\alpha(x, z, z')] \in \mathcal{K}^{(2)}.
\]

Lemma 6.10 and (6.5) tell us that

\[
(K \cap K') + \Pi = (K + \Pi) \cap (K' + \Pi) \cap (\tilde{H} + \Pi).
\]

Therefore, thanks to (6.31), \( (K \cap K') + \Pi \supset A \).

Let us prove (6.32) for \( K, K' \in \mathcal{K}^{(2)} \). In this case, the sets \( K \) and \( K' \) are rectangles with sides parallel to the coordinate axes and with non-empty intersection, so that, thanks to Lemma 6.8

\[
(K \cap K') + \Pi = (K + \Pi) \cap (K' + \Pi).
\]

It remains to note that, thanks to (6.31), \( K + \Pi \supset A \) and \( K' + \Pi \supset A \) for every \( K, K' \in \mathcal{K}^{(2)} \). This proves (6.32) in the case under consideration completing the proof of the lemma.

\[\blacksquare\]

**Lemma 6.17** For every \( x \in \mathcal{M} \) and every rectangle \( \Pi \in \mathcal{R}(\mathbb{R}^2) \) with center 0 the following representations

\[
F^{[2]}(x) + \Pi = \bigcap_{v, u, u', u'' \in \mathcal{M}} \{ (T_\alpha(u, u'), u'') \cap (F(v) + d(x, v)Q_0) \} + \Pi
\]

hold.

**Proof.** Thanks to (6.24), \( F(v) + d(x, v)Q_0 = T_\alpha(x, v, v) \) which proves the second equality of the lemma. Representation (6.25) and Lemma 6.9 tell us that

\[
F^{[2]}(x) + \Pi = \bigcap_{v, u, u', u'' \in \mathcal{M}} \{ (T_\alpha(u, u', u'') \cap T_\alpha(v, v', v'') \} + \Pi
\]

where the intersection is taken over all \( u, u', u'', v, v', v'' \in \mathcal{M} \). Hence,

\[
F^{[2]}(x) + \Pi \subset \bigcap_{v, u, u', u'' \in \mathcal{M}} \{ (T_\alpha(u, u', u'') \cap T_\alpha(v, v', v'') \} + \Pi
\]

On the other hand, Lemma 6.16 tells us that

\[
F^{[2]}(x) + \Pi = \bigcap_{v, u, u', u'' \in \mathcal{M}} \{ (\mathcal{H}[T_\alpha(u, u', u'')] \cap (F(v) + d(x, v)Q_0)) \} + \Pi
\]

proving the lemma.  

\[\blacksquare\]
Proposition 3.10. proving the lemma.

Hence, for every \( x \in M \), the set \( \mathcal{H}[F^{[2]}(x)] = \bigcap_{v,u,u',u'' \in M} \mathcal{H}[T_x(u,u',u'') \cap (F(v) + d(x,v)Q_0)] \) (6.33)

where the intersection is taken over all \( u, u', u'', v, v' \in M \).

Lemma 6.17 enables us to prove a stronger version of representation (6.33).

Lemma 6.19 For every \( x \in M \), the rectangular hull of \( F^{[2]}(x) \) has the following representation:

\[
\mathcal{H}[F^{[2]}(x)] = \bigcap_{v,u,u',u'' \in M} \mathcal{H}[T_x(u,u',u'') \cap (F(v) + d(x,v)Q_0)].
\]

Recall that \( T_x(u,u',u'') = \{F(u') + d(u',u)Q_0\} \cap \{F(u'') + d(u'',u)Q_0\} + Ld(u,x)Q_0 \); see (6.24).

Proof. Given \( x, v, u, u', u'' \in M \), we have

\[
V_x[v,u,u',u''] = T_x(u,u',u'') \cap (F(v) + d(x,v)Q_0).
\]

Lemma 6.17 tells us that

\[
F^{[2]}(x) + Ox_i = \bigcap_{v,u,u',u'' \in M} (V_x[v,u,u',u''] + Ox_i), \quad i = 1, 2.
\]

In turn, thanks to (6.9), \( \mathcal{H}[F^{[2]}(x)] = (F^{[2]}(x) + Ox_1) \cap (F^{[2]}(x) + Ox_2) \), so that

\[
\mathcal{H}[F^{[2]}(x)] = \left\{ \bigcap_{v,u,u',u'' \in M} (V_x[v,u,u',u''] + Ox_1) \right\} \bigcap \left\{ \bigcap_{v,u,u',u'' \in M} (V_x[v,u,u',u''] + Ox_2) \right\}.
\]

Hence,

\[
\mathcal{H}[F^{[2]}(x)] = \bigcap_{v,u,u',u'' \in M} (V_x[v,u,u',u''] + Ox_1) \cap (V_x[v,u,u',u''] + Ox_2).
\]

From this and (6.9), we obtain the required representation

\[
\mathcal{H}[F^{[2]}(x)] = \bigcap_{v,u,u',u'' \in M} \mathcal{H}[V_x[v,u,u',u'']],
\]

proving the lemma. ✷

We are in a position to prove inequality (6.22). Our proof will follow the scheme of the proof of Proposition 3.10.

Let \( x, y \in M \). We know that

\[
F^{[2]}(y) = \bigcap_{u,u',u'' \in M} T_y(u,u',u''), \quad \text{(see (6.25)).}
\]

(6.34)

Let \( \tau = \hat{y}(L) d(x,y) \). Lemma 6.17 tells us that

\[
F^{[2]}(x) + \tau Q_0 = \bigcap_{v,u,u',u'' \in M} \{ (T_x(u,u',u'') \cap (F(v) + d(x,v)Q_0)) + \tau Q_0 \}.
\]
Let us fix elements \( u, u', u'', v \in M \) and set
\[
A = (T_s(u, u', u'') \cap (F(v) + d(x, v)Q_0)) + \tau Q_0.
\]
Prove that \( A \supset F^{[2]}(y) \). Let
\[
C_1 = F(u') + d(u', u)Q_0, \quad C_2 = F(u'') + d(u'', u)Q_0X, \quad C = F(v) + d(x, v)Q_0,
\]
and let
\[
\varepsilon = L d(x, y) \quad \text{and} \quad r = d(u, x).
\]
See (6.23). Then
\[
\tau = \gamma(L) d(x, y) = L \theta(L) d(x, y) = \theta(L) \varepsilon.
\]
and
\[
A = \{(F(u') + d(u', u)Q_0) \cap (F(u'') + d(u'', u)Q_0) \cap (F(v) + d(x, v)Q_0) + L d(u, x)Q_0
\]
\[
= (C_1 \cap C_2 + LrQ_0) \cap C + \theta(L) \varepsilon Q_0.
\]
Let us verify condition (2.25) of Proposition 2.5 i.e., the condition
\[
C_1 \cap C_2 \cap (C + rQ_0) \neq \emptyset.
\]
Let \( M' = \{u, u', v\} \). Clearly, \#\( M' \) \leq 4, so that, thanks to Assumption 6.14, there exists a \( \rho \)-Lipschitz selection \( f_{M'} : M' \to \ell^2_\infty \) of the restriction \( F|_{M'} \) with \( \|f_{M'}\|_{\text{Lip}(M', \rho), \ell^2_\infty} \leq 1 \).

Because \( \varepsilon(\mathfrak{M}, \ell^2_\infty) = 1 \) (see (5.3)) and \( d = \alpha \rho \geq \rho \), the mapping \( f_{M'} : M' \to \ell^2_\infty \) can be extended to a \( d \)-Lipschitz mapping \( \tilde{f} : M \to \ell^2_\infty \) defined on all of \( M \) with \( d \)-Lipschitz seminorm
\[
\|\tilde{f}\|_{\text{Lip}(M, \delta), \ell^2_\infty} \leq \|f_{M'}\|_{\text{Lip}(M', \rho), \ell^2_\infty} \leq 1.
\]
In particular, \( \tilde{f}(u') = f_{M'}(u') \in F(u') \), \( \tilde{f}(u'') = f_{M'}(u'') \in F(u''), \) \( \tilde{f}(v) = f_{M'}(v) \in F(v) \),
\[
\|\tilde{f}(u') - \tilde{f}(u)\| \leq d(u', u), \quad \|\tilde{f}(u'') - \tilde{f}(u)\| \leq d(u'', u)
\]
and
\[
\|\tilde{f}(x) - \tilde{f}(u)\| \leq d(u, x) = r, \quad \|\tilde{f}(x) - \tilde{f}(v)\| \leq d(v, x).
\]
Hence, \( \tilde{f}(u) \in C_1 \cap C_2 \) and \( \tilde{f}(x) \in C \), so that \( C_1 \cap C_2 \cap (C + rQ_0) \ni \tilde{f}(u) \) proving (6.36).

This enables us to apply Proposition 2.5 to the sets \( C_1, C_2 \) and \( C \). This proposition tells us that
\[
A = (C_1 \cap C_2 + LrQ_0) \cap C + \theta(L) \varepsilon Q_0
\]
\[
\sup \left[ (C_1 \cap C_2 + (Lr + \varepsilon)Q_0) \cap [(C_1 + rQ_0) \cap C + \varepsilon Q_0] \cap [(C_2 + rQ_0) \cap C + \varepsilon Q_0] \right]
\]
\[
= S_1 \cap S_2 \cap S_3.
\]
Prove that \( S_i \supset F^{[2]}(y) \) for every \( i = 1, 2, 3 \). We begin with the set
\[
S_1 = C_1 \cap C_2 + (Lr + \varepsilon)Q_0
\]
\[
= \{F(u') + d(u', u)Q_0\} \cap \{F(u'') + d(u'', u)Q_0\} + (L d(u, x) + L d(x, y))Q_0.
\]
See (6.35). The triangle inequality tells us that \( d(u, x) + d(x, y) \geq d(u, y) \) so that
\[
S_1 \supset \{F(u') + d(u', u)B_x\} \cap \{F(u'') + d(u'', u)Q_0\} + L d(u, y)B_y = T_y(u, u', u'').
\]
Hence, proving (6.22) with $\tilde{\lambda}$ after the proof of Proposition 3.10. Let for every $x$  

$$S_2 = (C_1 + rQ_0) \cap C + \varepsilon Q_0$$  

$$= \{(F(u') + d(u', u)Q_0) + d(x, u)Q_0\} \cap \{F(v) + d(x, v)Q_0\} + L d(x, y)Q_0.$$  

Therefore, thanks to the triangle inequality and (6.34),  

$$S_2 \supset \{(F(u') + d(u', x)Q_0) \cap \{F(v) + d(x, v)Q_0\} + L d(x, y)Q_0 = T_j(x, u', v) \supset F^{[2]}(y).$$  

In the same way we show that $S_3 \supset F^{[2]}(y)$. Hence, $A \supset S_1 \cap S_2 \cap S_3 \supset F^{[2]}(y)$. Thus, we have proved that $F^{[3]}(x) + \tau Q_0 \supset F^{[3]}(y)$. By interchanging the roles of $x$ and $y$ we obtain also $F^{[2]}(y) + \tau Q_0 \supset F^{[2]}(x)$. These two inclusions imply inequality  

$$d_H(F^{[2]}(x), F^{[2]}(y)) \leq \gamma(L) d(x, y)$$  

proving (6.22) with $\tilde{\gamma}(L) = L(3L + 1)/(L - 1)$.  

We finish the proof of Theorem 6.13 in the same fashion as we have finished the proof of Theorem 6.19 (after the proof of Proposition 3.10). Let $\lambda_1$, $\lambda_2$ and $\gamma$ be parameters satisfying (6.19), i.e., $\lambda_1 \geq 1$, $\lambda_2 \geq 3\lambda_1$ and $\gamma \geq \lambda_2 (3\lambda_2 + \lambda_1)/(\lambda_2 - \lambda_1)$. We set $\alpha = \lambda_1$, $L = \lambda_2/\lambda_1$ which provides the required inequalities $L \geq 3$ and $\alpha \geq 1$. We also recall that $d = \alpha \rho = \lambda_1 \rho$.  

In these settings, the mappings $F^{[1]}$ and $F^{[2]}$ are the first and the second order $(\lambda_1, \lambda_2, \rho)$-balanced refinements of $F$ respectively. See Definition 1.6.  

Thanks to Proposition 3.9, under the above conditions on $\alpha = \lambda_1$ and $L = \lambda_2/\lambda_1$, the set $F^{[2]}(x) \neq \emptyset$ for every $x \in M$. Thus, property (1.9) holds.  

In turn, inequality (6.22) tells us that  

$$d_H(F^{[2]}(x), F^{[2]}(y)) \leq \tilde{\gamma}(L) d(x, y) \quad \text{with} \quad \tilde{\gamma}(L) = L(3L + 1)/(L - 1).$$  

Hence,  

$$d_H(F^{[2]}(x), F^{[2]}(y)) \leq L(3L + 1)/(L - 1) d(x, y) = \frac{\lambda_2}{\lambda_1} \cdot \frac{(3\lambda_2 + \lambda_1)}{(\lambda_2 - \lambda_1)} (\lambda_1 \rho(x, y))$$  

$$= \lambda_2 \frac{(3\lambda_2 + \lambda_1)}{(\lambda_2 - \lambda_1)} \rho(x, y) \leq \gamma \rho(x, y)$$  

proving inequality (1.10).  

In particular, this inequality holds provided, $\lambda_1 = 1$, $\lambda_2 = 3$, and  

$$\gamma = \lambda_2 (3\lambda_2 + \lambda_1)/(\lambda_2 - \lambda_1) = 3(3 \cdot 3 + 1)/(3 - 1) = 15.$$  

The proof of Theorem 6.13 is complete.  

6.3 A constructive algorithm for a nearly optimal Lipschitz selection in $\ell^2_{\infty}$.  

The proof of Theorem 6.13 provides a certain constructive algorithm for a Lipschitz selection of a set-valued mapping $F$ satisfying the hypothesis of this theorem. Let us briefly describe main steps of this algorithm and give an explicit formula for a nearly optimal Lipschitz selection of $F$.  

Let $F : M \rightarrow K(\mathbb{R}^2)$ and let $\lambda$ be a positive constant. We make the following
Assumption 6.20 For every subset $M' \subset M$ with $\#M' \leq 4$, the restriction $F|_{M'}$ of $F$ to $M'$ has a Lipschitz selection $f_{M'} : M' \to \ell_\infty^2$ with Lipschitz seminorm $\|f_{M'}\|_{\Lip(M', \ell_\infty^2)} \leq \lambda$.

The following algorithm, given $F$ and $\lambda$ satisfying Assumption 6.20, constructs a Lipschitz mapping $f : M \to \ell_\infty^2$ with $\|f\|_{\Lip(M, \ell_\infty^2)} \leq 15\lambda$ such that $f(x) \in F(x)$ for each $x \in M$.

We construct $f$ in four steps.

**Step 1.** We construct the $\lambda$-balanced refinement of $F$, i.e., the mapping

$$F^{[1]}(x) = \bigcap_{z \in M} \left[ F(z) + \lambda \rho(x, z) Q_0 \right], \quad x \in M,$$

**Step 2.** We construct the second order $((\lambda, 3\lambda), \rho)$-balanced refinement of $F^{[1]}$:

$$F^{[2]}(x) = \bigcap_{z \in M} \left[ F^{[1]}(z) + 3\lambda \rho(x, z) Q_0 \right], \quad x \in M.$$ 

From the proof of Theorem 6.13, we know that (i) $F^{[1]}(x) \neq \emptyset$ and $F^{[2]}(x) \neq \emptyset$ for every $x \in M$, and (ii) for every $x, y \in M$

$$d_H(F^{[2]}(x), F^{[2]}(y)) \leq 15\lambda \rho(x, y). \quad (6.37)$$

**Step 3.** We construct the rectangular hull of $F^{[2]}$, i.e., the mapping

$$H(x) = H[F^{[2]}(x)], \quad x \in M.$$

**Step 4.** We define the required mapping $f$ as the center of the rectangle $H(x)$:

$$f(x) = \text{cntr } H(x) = \text{cntr } H[F^{[2]}(x)], \quad x \in M. \quad (6.38)$$

Lemma 6.2 tells us that $f(x) \in F^{[2]}(x)$ for each $x \in M$. Because $F^{[2]}(x) \subset F^{[1]}(x) \subset F(x)$, $f(x) \in F(x)$ on $M$ proving that $f$ is a selection of $F$. In turn, thanks to (6.6) and (6.37),

$$d_H(H(x), H(y)) = d_H(H[F^{[2]}(x)], H[F^{[2]}(y)]) \leq d_H(F^{[2]}(x), F^{[2]}(y)) \leq 15\lambda \rho(x, y).$$

Finally, thanks to this inequality and part (ii) of Lemma 5.6,

$$\|f(x) - f(y)\| = \|\text{cntr } H(x) - \text{cntr } H(y)\| \leq d_H(H(x), H(y)) \leq 15\lambda \rho(x, y)$$

proving that $f$ is a Lipschitz selection of $F$ with $\|f\|_{\Lip(M, \ell_\infty^2)} \leq 15\lambda$.

These observations and representation (6.33) enable us to give an explicit formula for the selection $f$.

In our settings formula (6.33) looks as follows: set $F^{[2]}(x)$:

$$H[F^{[2]}(x)] = \bigcap H[T_x(u, u', u'') \cap T_x(v, v', v'')] \quad (6.39)$$

Here the intersection is taken over all $u, u', u'', v, v', v'' \in M$, and

$$T_x(u, u', u'') = \{F(u') + \lambda \rho(u', u)Q_0\} \cap \{F(u'') + \lambda \rho(u'', u)Q_0\} + 3\lambda \rho(u, x)Q_0.$$ 

Recall that, thanks to (6.38), $f(x) = (f_1(x), f_2(x)) = \text{cntr } H[F^{[2]}(x)]$. Let us express the coordinates $f_1(x), f_2(x)$ in the explicit form.
Fix a 6-tuple $T = (u, u', u'', v, v', v'')$ with $u, u', u'', v, v', v'' \in \mathcal{M}$. Then, thanks to (6.2) and (6.3),
$$
\mathcal{H}[T_x(u, u', u'') \cap T_x(v, v', v'')] = [a_1(x, T), b_1(x, T)] \times [a_2(x, T), b_2(x, T)]
$$
where for every $j = 1, 2$
$$
b_j(x, T) = \sup\{y_j : y = (y_1, y_2) \in T_x(u, u', u'') \cap T_x(v, v', v'')\}
$$
and
$$
a_j(x, T) = \inf\{y_j : y = (y_1, y_2) \in T_x(u, u', u'') \cap T_x(v, v', v'')\}.
$$
From this and (6.39) it follows that
$$
\mathcal{H}[F^{[2]}(x)] = [\alpha_1(x), \beta_1(x)] \times [\alpha_2(x), \beta_2(x)]
$$
where given $j = 1, 2$,
$$
\beta_j(x) = \inf_T b_j(x, T) = \inf_T \sup\{y_j : y = (y_1, y_2) \in T_x(u, u', u'') \cap T_x(v, v', v'')\}
$$
and
$$
\alpha_j(x) = \sup_T a_j(x, T) = \sup_T \inf\{y_j : y = (y_1, y_2) \in T_x(u, u', u'') \cap T_x(v, v', v'')\}. \tag{6.40}
$$
Thus, $A(x) = (\alpha_1(x), \alpha_2(x))$ is the smallest point, and $B(x) = (\beta_1(x), \beta_2(x))$ is the largest point of the rectangle $\mathcal{H}[F^{[2]}(x)]$. Clearly, its center, the point $\text{cntr} \mathcal{H}[F^{[2]}(x)]$, has the coordinates
$$
\text{cntr} \mathcal{H}[F^{[2]}(x)] = \left(\frac{\alpha_1(x) + \beta_1(x)}{2}, \frac{\alpha_2(x) + \beta_2(x)}{2}\right).
$$
Therefore, according to (6.38),
$$
f_1(x) = \frac{\alpha_1(x) + \beta_1(x)}{2} \quad \text{and} \quad f_2(x) = \frac{\alpha_2(x) + \beta_2(x)}{2}. \tag{6.42}
$$
This formula and formulae (6.40), (6.41) and (6.42) provide explicit formulae for a Lipschitz selection of $F$ (with Lipschitz constant at most 15$\lambda$) whenever $F$ satisfies Assumption (6.20).

We can compare these formulae with corresponding explicit formulae for Lipschitz selection in one dimensional case. See (5.35), (5.6) and (5.7). This comparison shows how grows the complexity of the Lipschitz selection problem in transition from the one dimensional to the two dimensional case.

7. Constructive criteria for Lipschitz selections in $\mathbb{R}^2$.

7.1 Rectangular hulls and a Lipschitz selection criterion in $\ell_\infty^2$.

In this section we discuss constructive criteria for the existence of Lipschitz selections in $\ell_\infty^2$.

Let $\mathcal{M} = (\mathcal{M}, \rho)$ be a pseudometric space, and let $F : \mathcal{M} \to \text{Conv}(\mathbb{R}^2)$ be a set-valued mapping from $\mathcal{M}$ into the family $\text{Conv}(\mathbb{R}^2)$ of all closed convex subsets of $\mathbb{R}^2$. Let $X = (\mathbb{R}^2, \|\cdot\|_X)$ be a two dimensional Banach space.

Let $X = \ell_\infty^2$. Each ball $B_Y(a, r)$ in $\ell_\infty^2$ is a square $Q(a, r)$ (with sides parallel to the coordinate axes) with center $a$ and length of side $2r$. The square $Q_0 = [-1, 1] \times [-1, 1]$ is the unit ball of $X = \ell_\infty^2$. 

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Proposition 7.1 Let \( F : \mathcal{M} \rightarrow \mathcal{K}(\mathbb{R}^2) \) be a set-valued mapping and let \( \lambda > 0 \). If \( F \) has a Lipschitz selection \( f : \mathcal{M} \rightarrow \ell^2 \) with \( \| f \|_{\text{Lip}(\mathcal{M},\ell^2)} \leq \lambda \) then
\[ (i) \text{ dist}(F(x), F(y)) \leq \lambda \rho(x, y) \text{ for every } x, y \in \mathcal{M}; \]
\[ (ii) \text{ dist}(R_F[x, x' : \lambda], R_F[y, y' : \lambda]) \leq \lambda \rho(x, y) \text{ for every } x, x', y, y' \in \mathcal{M}; \]
\[ (iii) \text{ dist}(W_F[x, x', x'' : \lambda], W_F[y, y', y'' : \lambda]) \leq \lambda \rho(x, y) \text{ for every } x, x', x'', y, y', y'' \in \mathcal{M}. \]

Proof. Because \( f \) is a Lipschitz selection of \( F \), for every \( x, x', x'' \in \mathcal{M} \) we have \( f(x) \in F(x) \), \( f(x') \in F(x') \), \( f(x'') \in F(x'') \),
\[ \| f(x) - f(x') \| \leq \lambda \rho(x, x') \quad \text{and} \quad \| f(x) - f(x'') \| \leq \lambda \rho(x, x''). \]

Hence,
\[ \text{dist}(F(x), F(x')) \leq \| f(x) - f(x') \| \leq \lambda \rho(x, x') \]
proving part (i) of the proposition.

The first inequality in (7.3) also implies the inclusion \( f(x) \in F(x') + \lambda \rho(x, x')Q_0 \). Hence,
\[ f(x) \in F(x) \cap \{ F(x') + \lambda \rho(x, x')Q_0 \} \subset \mathcal{H}[F(x) \cap \{ F(x') + \lambda \rho(x, x')Q_0 \}] = R_F[x, x' : \lambda]. \]

See (7.1). Therefore, for every \( x, x', y, y' \in \mathcal{M} \), we have
\[ \text{dist}(R_F[x, x' : \lambda], R_F[y, y' : \lambda]) \leq \| f(x) - f(y) \| \leq \lambda \rho(x, y) \]
proving (ii).

Prove (iii). We know that \( f(x') \in F(x') \) and \( f(x'') \in F(x'') \). From this and (7.3),
\[ f(x) \in \{ F(x') + \lambda \rho(x, x')Q_0 \} \cap \{ F(x'') + \lambda \rho(x, x'')Q_0 \} \]
\[ \subset \mathcal{H}[\{ F(x') + \lambda \rho(x, x')Q_0 \} \cap \{ F(x'') + \lambda \rho(x, x'')Q_0 \}] = W_F[x, x', x'' : \lambda]. \]

Hence, for every \( x, x', x'', y, y', y'' \in \mathcal{M} \), we have
\[ \text{dist}(W_F[x, x', x'' : \lambda], W_F[y, y', y'' : \lambda]) \leq \| f(x) - f(y) \| \leq \lambda \rho(x, y) \]
proving (iii) and the proposition. \( \blacksquare \)

Recall that we measure distances in \( \mathbb{R}^2 \) in the uniform norm (i.e., in \( \ell^\infty \)-norm).

The following two theorem provides sufficient conditions for the existence of Lipschitz selections.

Given a set-valued mapping \( F : \mathcal{M} \rightarrow \mathcal{K}(\mathbb{R}^2) \), a positive constant \( \lambda \) and elements \( x, x' \in \mathcal{M} \), we introduce a set
\[ R_F[x, x' : \lambda] = \mathcal{H}[F(x) \cap \{ F(x') + \lambda \rho(x, x')Q_0 \}]. \]

We recall that \( \mathcal{H}[] \) denotes the rectangular hull of a set in \( \mathbb{R}^2 \), i.e., the smallest rectangular with sides parallel to the coordinate axes containing the set. See Definition 6.1.

Given \( x, x', x'' \in \mathcal{M} \) we also set
\[ W_F[x, x', x'' : \lambda] = \mathcal{H}[\{ F(x') + \lambda \rho(x', x)Q_0 \} \cap \{ F(x'') + \lambda \rho(x'', x)Q_0 \}]. \]
Theorem 7.2 Let $F : \mathcal{M} \to \mathcal{K}(\mathbb{R}^2)$ be a set-valued mapping and let $\lambda > 0$.
Suppose that the following two conditions are satisfied:

(i) for every $x, y \in \mathcal{M}$
\[ \text{dist}(F(x), F(y)) \leq \lambda \rho(x, y); \]

(ii) for every $x, x', y, y' \in \mathcal{M}$, we have
\[ \text{dist}(\mathcal{R}_F[x, x' : \lambda], \mathcal{R}_F[y, y' : \lambda]) \leq \lambda \rho(x, y). \]

Then $F$ has a Lipschitz selection $f : \mathcal{M} \to \ell_2^2$ with $\|f\|_{\text{Lip}(\mathcal{M}, \ell_2^2)} \leq 8\lambda$.

Theorem 7.3 Let $F : \mathcal{M} \to \mathcal{K}(\mathbb{R}^2)$ be a set-valued mapping, and let $\tilde{\lambda} \geq \lambda > 0$. Suppose that the following two conditions are satisfied:

(i) $\text{dist}(F(x), F(y)) \leq \lambda \rho(x, y)$ for every $x, y \in \mathcal{M}$;

(ii) for every $x, x', x'', y, y', y'' \in \mathcal{M}$ the following inequality
\[ \text{dist}\left(\mathcal{W}_F[x, x', x'' : \tilde{\lambda}], \mathcal{W}_F[y, y', y'' : \tilde{\lambda}]\right) \leq \lambda \rho(x, y) \]
holds.

Then $F$ has a Lipschitz selection $f : \mathcal{M} \to \ell_2^2$ with $\|f\|_{\text{Lip}(\mathcal{M}, \ell_2^2)} \leq 2(\lambda + \tilde{\lambda})$.

We prove Theorem 7.2 and Theorem 7.3 following the ideas of the work [26]. For the completeness, in the next section we give detailed proofs of these results.

7.2 Rectangular hulls and nearly optimal Lipschitz selections: sufficiency.

Proof of Theorem 7.3 Suppose that for every $x, x', x'', y, y', y'' \in \mathcal{M}$ conditions (i) and (ii) of Theorem 7.3 hold. Let us construct a Lipschitz selection $f : \mathcal{M} \to \ell_2^2$ of $F$ with Lipschitz seminorm $\|f\|_{\text{Lip}(\mathcal{M}, \ell_2^2)} \leq 2(\lambda + \tilde{\lambda})$. We do this in three steps.

STEP 1. At this step we construct a $\tilde{\lambda}$-balanced refinement of the mapping $F$, i.e., the set
\[ F^{[1]}(x) = \bigcap_{y \in \mathcal{M}} \left[ F(y) + \tilde{\lambda} \rho(x, y) Q_0 \right], \quad x \in \mathcal{M}, \]
and prove that $F^{[1]}(x) \neq \emptyset$ for every $x \in \mathcal{M}$.

First, let us see that
\[ \{F(y) + \tilde{\lambda} \rho(x, y) Q_0\} \cap \{F(z) + \tilde{\lambda} \rho(x, z) Q_0\} \neq \emptyset \quad \text{for every} \quad y, z \in \mathcal{M}. \]

Indeed, part (i) of the theorem tells us that $\text{dist}(F(y), F(z)) \leq \lambda \rho(y, z)$ so that there exist points $a \in F(y)$ and $b \in F(z)$ such that $\|a - b\| \leq \lambda \rho(y, z)$. We set $w = a(= b)$ provided $\rho(y, z) = 0$, and
\[ w = \alpha a + (1 - \alpha) b \quad \text{where} \quad \alpha = \frac{\rho(x, z)}{\rho(y, x) + \rho(x, z)} \quad \text{if} \quad \rho(y, z) > 0. \]

Then, thanks to the triangle inequality, $\rho(y, z) \leq \rho(y, x) + \rho(x, z)$ which implies inequalities
\[ \|w - a\| \leq \lambda \rho(x, y) \quad \text{and} \quad \|w - b\| \leq \lambda \rho(x, z). \]
Recall that $\tilde{\lambda} \geq \lambda$, so that

$$w \in F(y) + \tilde{\lambda} \rho(x,y) Q_0 \quad \text{and} \quad w \in F(y) + \tilde{\lambda} \rho(x,z) Q_0$$

proving (7.7).

Note that, thanks to (7.7), the set

$$\mathcal{W}_F[x, x', x'' : \tilde{\lambda}] \neq \emptyset \quad \text{for all} \quad x, x', x'' \in M.$$  

See definition (7.2).

**Lemma 7.4** For each $x \in M$ the set $F^{[1]}(x)$ defined by (7.6) is a non-empty closed convex subset of $\mathbb{R}^2$. Furthermore,

$$\mathcal{H}[F^{[1]}(x)] = \cap \{ \mathcal{W}_F[x, y, y' : \tilde{\lambda}] : y, y' \in M \}, \quad x \in M. \tag{7.8}$$

**Proof.** Fix $x \in M$ and set

$$\mathfrak{C} = \{ F(y) + \tilde{\lambda} \rho(x,y) Q_0 : y \in M \}. \tag{7.9}$$

Then

$$F^{[1]}(x) = \cap \{ C : C \in \mathfrak{C} \}. \tag{7.10}$$

See (7.6).

Let us show that the family $\mathfrak{C} \subset K(\mathbb{R}^2)$ satisfies the hypothesis of Corollary 6.5. We know that any two elements of $\mathfrak{C}$ have a common point, see (7.7). Also, inequality (7.5) tells us that

$$\text{dist}(\mathcal{W}_F[x, y_1, y'_1 : \tilde{\lambda}], \mathcal{W}_F[x, y_2, y'_2 : \tilde{\lambda}]) \leq \tilde{\lambda} \rho(x, x) = 0$$

proving that

$$\mathcal{W}_F[x, y_1, y'_1] \cap \mathcal{W}_F[x, y_2, y'_2] \neq \emptyset \quad \text{for every} \quad y_1, y'_1, y_2, y'_2 \in M.$$

Hence, it follows that the family $\mathfrak{C}$ satisfies condition (6.13).

Thus, $\mathfrak{C}$ satisfies the hypothesis of Corollary 6.5. This corollary tells us that for every $x \in M$ the set

$$F^{[1]}(x) = \cap \{ C : C \in \mathfrak{C} \} \neq \emptyset.$$

Furthermore, formula (6.14) coincides with formula (7.8) proving the lemma. □

**Remark 7.5** Corollary 6.5 enables us to prove that the statement of Lemma 7.4 holds in the following cases:

(★1) The set $M$ is finite and $F : M \to \text{Conv}(\mathbb{R}^2)$ is a set-valued mapping satisfying conditions (i) and (ii) of Theorem 7.3.

(★2) $F$ is a set-valued mapping from $M$ into $\text{Conv}(\mathbb{R}^2)$ satisfying conditions (i), (ii) of Theorem 7.3 and the following additional condition:

(iii) for every $x \in M$ there exists a finite set $M_x \subset M$ such that the intersection

$$\bigcap_{y \in M_x} \{ F(y) + \tilde{\lambda} \rho(x,y) Q_0 \} \quad \text{is nonempty and bounded}. \tag{7.10}$$

In these settings the family $\mathfrak{C}$ defined by (7.9) satisfies the hypothesis of Corollary 6.5 proving that the set

$$F^{[1]}(x) = \cap \{ C : C \in \mathfrak{C} \} \neq \emptyset$$

and the formula (6.14) holds. □
**STEP 2.** We construct a set-valued mapping \( T : \mathcal{M} \rightarrow \mathbb{R}(\mathbb{R}^2) \) defined by

\[
T(x) = \mathcal{H}[F^{[1]}(x)] = \mathcal{H}\left[\bigcap \left\{ F(y) + \tilde{\lambda} \rho(x, y) Q_0 : y \in \mathcal{M} \right\}\right], \quad x \in \mathcal{M}.
\]

(7.11)

See Fig. 7.

![Figure 7](image)

**Formula (7.8)** provides the following representation of the mapping \( T \):

\[
T(x) = \cap \left\{ W_F[x, x', x'' : \tilde{\lambda}] : x', x'' \in \mathcal{M} \right\}, \quad x \in \mathcal{M}.
\]

(7.12)

**Lemma 7.6** *The set-valued mapping \( T : \mathcal{M} \rightarrow \mathbb{R}(\mathbb{R}^2) \) has a \( \rho \)-Lipschitz selection \( g : \mathcal{M} \rightarrow \ell_\infty^2 \) with \( \rho \)-Lipschitz seminorm \( \|g\|_{\text{Lip}(\mathcal{M}, \ell_\infty^2)} \leq \lambda. \) (See Fig. 8.)*

![Figure 8](image)

**Fig. 8:** \( g(x) \in T(x) \) for each \( x \in \mathcal{M} \), and \( \|g(x) - g(y)\| \leq \lambda \rho(x, y) \) for all \( x, y \in \mathcal{M} \).
Proof. Proposition \[6.12\] tells us that the required Lipschitz selection \( g \) exists provided for every \( x, y \in M \) the restriction \( T_{\{x,y\}} \) of \( T \) to \( \{x,y\} \) has a Lipschitz selection \( g_{x,y} \) with Lipschitz seminorm \( \|g_{x,y}\|_{\text{Lip}(\{x,y\},R^2)} \leq \lambda \). Clearly, this requirement is equivalent to the inequality
\[
dist(T(x), T(y)) \leq \lambda \rho(x, y), \quad x, y \in M.
\] (7.13)

Representation \[7.12\] and Lemma \[6.7\] tell us that
\[
dist(T(x), T(y)) = \sup \{ \text{dist}(W_F[x, x', x'': \lambda'], W_F[y, y', y'': \lambda]) : x', x'', y', y'' \in M \}. \]
In turn, inequality \[7.5\] tells us that
\[
dist(W_F[x, x', x'': \lambda'], W_F[y, y', y'': \lambda]) \leq \lambda \rho(x, y) \quad \text{for every} \quad x', x'', y', y'' \in M
\]
proving the required inequality (7.13). The proof of the lemma is complete. 

**STEP 3.** At this step we construct a Lipschitz selection of \( F \) with Lipschitz constant at most \( 2(\lambda + \lambda') \).

Let \( S \subset R^2 \) be a convex closed set. By \( \text{Pr}(\cdot; S) \) we denote the operator of metric projection onto \( S \) in \( \ell^2_\infty \)-norm. To each \( a \in R^2 \) this operator assigns the set of all points in \( S \) nearest to \( a \) on \( S \) in the uniform norm. Thus,
\[
\text{Pr}(a; S) = S \cap Q(a, \text{dist}(a, S)).
\]
(Recall that \( Q(a, r) = \{ y \in R^2 : \|y - a\| \leq r \} \) is a “ball” in \( \ell^2_\infty \), i.e., a square with center \( a \) and length of side \( 2r \).

We need the following three auxiliary lemmas about properties of metric projections.

**Lemma 7.7** Let \( S \subset R^2 \) be a convex closed set. Then for every \( a \in \mathcal{H}(S) \) the metric projection \( \text{Pr}(a; S) \) is a singleton. Furthermore, \( \text{Pr}(a; S) \) coincides with a vertex of the square \( Q(a, \text{dist}(a, S)) \).

**Proof.** A proof of the lemma is given in [26, p. 301]; for the reader’s convenience, we present this proof here.

If \( a \in S \), nothing to prove. Suppose \( a \not\in S \) so that \( r = \text{dist}(a,S) > 0 \). Because \( S \) is closed, \( \text{Pr}(a; S) \neq \emptyset \). Furthermore, \( \text{Pr}(a; S) = S \cap Q = S \cap \partial Q \) where \( Q = Q(a, r) \).

Because the set \( \text{Pr}(a; S) \) is convex, it is contained in a side of the square \( Q \). In other words, there exist to distinct vertices \( A, B \) of \( Q \) such that \( \text{Pr}(a; S) \subset [A, B] \subset \partial Q \). Prove that
\[
\text{either} \quad \text{Pr}(a; S) = \{A\} \quad \text{or} \quad \text{Pr}(a; S) = \{B\}.
\] (7.14)

Indeed, otherwise there exists a point \( p \in (A, B) \cap \text{Pr}(a; S) \). Let \( \ell \) be a straight line passing through \( A \) and \( B \). Clearly, \( \ell \) is parallel to a coordinate axis. Let \( H_1, H_2 \) be the half-planes determined by \( \ell \). Clearly, \( Q \) is contained in one of these half-planes, say in \( H_1 \).

Prove that in this case \( S \subset H_2 \), i.e., the straight line \( \ell \) separates (not strictly) the square \( Q \) and the set \( S \). Indeed, suppose that there exists a point \( b \in S \cap H_1^{\text{int}} \) where \( H_1^{\text{int}} \) denotes the interior of \( H_1 \). Then also \((p, b) \subset H_1^{\text{int}} \) because \( p \in \partial H_1 = \ell \). But \( p \in (A, B) \) so that \((p, b) \cap Q^{\text{int}} \neq \emptyset \). On the other hand, because \( S \) is convex and \( p \in \partial S \), the interval \((p, b) \subset S \) proving that \( S \subset Q^{\text{int}} \neq \emptyset \). But \( S \subset Q \subset \partial Q \), a contradiction.

Thus, \( S \subset H_2 \) and \( Q \subset H_1 \) so that \( a \not\in H_2 \). But \( H_2 \in \mathfrak{R}(R^2) \), i.e., \( H_2 \) is an (unbounded) rectangle with sides parallel to the coordinate axes. Therefore \( \mathcal{H}(S) \subset H_2 \), see Definition \[6.1\] From this and the lemma’s hypothesis, we have \( a \in \mathcal{H}(S) \subset H_2 \), a contradiction.

This contradiction proves (7.14) completing the proof of the lemma. 

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Lemma 7.8 Let \( A, B \subset \mathbb{R}^2 \) be convex closed sets, \( A \subset B \), and let \( a \in \mathcal{H}[A] \). Then \( \text{Pr}(a; B) \) belongs to the line segment \([\text{Pr}(a; A), a]\). Moreover,

\[
\| \text{Pr}(a; A) - \text{Pr}(a; B) \| = \text{dist}(a, A) - \text{dist}(a, B).
\]

**Proof.** If \( a \in B \), the statement of the lemma is immediate from Lemma 7.7. Suppose that \( a \notin B \).

In this case Lemma 7.7 tells us that \( \text{Pr}(a; A) \) is one of the vertices of the square \( Q(a, r) \) with \( r = \text{dist}(a, A) > 0 \). Because \( A \subset B \), the point \( a \in \mathcal{H}[B] \) so that \( \text{Pr}(a; B) \) is also a vertex of the square \( Q(a, r) \) where \( r = \text{dist}(a, B) > 0 \).

Using a suitable shift and dilation, we can assume that \( a = (0, 0) \), \( r = \text{dist}(a, A) = 1 \) and \( \text{Pr}(a; A) = (1, 1) \). Clearly, in this case \( 0 < \alpha < 1 \). Furthermore, in these settings the statement of the lemma is equivalent to the property

\[
\text{Pr}(a; B) = (\alpha, \alpha). \tag{7.15}
\]

Suppose that this property does not hold, i.e., \( \text{Pr}(a; B) \notin [(\alpha, -\alpha), (-\alpha, \alpha), (-\alpha, -\alpha)] \).

In order to get a contradiction, we construct a straight line \( \ell_A \) which passes through \((1, 1)\) and separates (not strictly) the square \( Q(a, r) = [-1, 1]^2 \) and \( A \). This line determines two half-planes, \( S_A^+ \) and \( S_A^- \). One of them, say \( S_A^+ \), contains \( A \), so that \( S_A^+ \supseteq Q(a, r) \). Because \( S_A^+ \) contains \((1, 1)\) and does not contain intrinsic points of the square \([-1, 1]^2 \), the half-plane \( S_A^- \) can be represented as

\[
S_A^+ = \{ x = (x_1, x_2) \in \mathbb{R}^2 : (x_1 - 1)h_1 + (x_2 - 1)h_2 \geq 0 \} \tag{7.16}
\]

with certain \( h_1, h_2 \geq 0, h_1 + h_2 > 0 \). Because \( a = (0, 0) \in \mathcal{H}[A] \), there exist points \((c_1, 0)\) and \((0, c_2)\) in \( A \). Therefore, thanks to (7.16), \( c_1 > 0 \) and \( c_2 > 0 \).

We let \( \ell_B \) denote a straight line which separates the square \( Q(a, \text{dist}(a, B)) = [\alpha, -\alpha]^2 \) and the set \( B \). Let \( S_B^+ \) be that of two half-planes determined by \( \ell_B \) which contains \( B \). Because \( A \subset B \) and \((c_1, 0), (0, c_2) \in A \), we have \((c_1, 0) \subset S_B^+ \).

Prove that \( S_B^+ \nsubseteq (c_1, 0) \) provided \( \text{Pr}(a; B) = (\alpha, -\alpha) \). We know that \( S_B^+ \) contains \((-\alpha, \alpha)\) and does not contain intrinsic points of the square \([\alpha, -\alpha]^2 \). Therefore, it can be represented in the form

\[
S_B^+ = \{ (x, y) \in \mathbb{R}^2 : -(x + \alpha)s_1 + (y - \alpha)s_2 \geq 0 \}
\]

with certain \( s_1, s_2 \geq 0, s_1 + s_2 > 0 \). Then \((c_1, 0) \notin S_B^+ \), because \((c_1 + \alpha)s_1 + (-\alpha)s_2 \leq 0 \).

In the same way we prove that if \( \text{Pr}(a; B) = (\alpha, -\alpha) \) or \((-\alpha, \alpha) \), then either \((c_1, 0) \) or \((0, c_2) \) does not belong to \( S_B^+ \). This contradiction proves (7.15) completing the proof of the lemma. \( \blacksquare \)

Lemma 7.9 Let \( A \subset \mathbb{R}^2 \) be convex and closed, and let \( a, b \in \mathcal{H}[A] \). Then

\[
\| \text{Pr}(a; A) - \text{Pr}(b; A) \| \leq 2\| a - b \|. \tag{7.17}
\]

**Proof.** Let \( b \in A \). Then

\[
\| \text{Pr}(a; A) - \text{Pr}(b; A) \| = \| \text{Pr}(a; A) - b \| \leq \| \text{Pr}(a; A) - a \| + \| a - b \| \leq 2\| a - b \|.
\]

In the same way we prove (7.17) if \( a \in A \).

Prove (7.17) provided \( a, b \notin A \), \( a, b \in \mathcal{H}[A] \). Let \( u_a = \text{Pr}(a; A) - a \) and \( u_b = \text{Pr}(b; A) - b \). Assume that

\[
\text{there exists } \lambda > 0 \text{ such that } u_b = \lambda u_a. \tag{7.18}
\]
Furthermore, because $H_a$ is a straight line and $H_a^-$ is a closed half-plane which we denote by $H_a$, $H_a^+$, and $H_a^-$, proving (7.17) in the case under consideration.

We know that $r_a = \text{dist}(b, A) = \|a - p_a\|$ and $r_b = \text{dist}(b, A) = \|b - p_b\|$.

It remains to prove (7.17) provided condition (7.18) does not hold. Let us see that in this case we have

$$H_a^+ \subseteq A. \text{ Furthermore, } Q(a, r_a) \cap H_a^+ = \{p_a\}. \text{ In the same way we construct half-planes } H_b^+ \text{ and } H_b^- \text{ such that}$$

$$H_b^+ \subseteq Q(b, r_b), \quad H_b^- \supseteq A \quad \text{and} \quad Q(b, r_b) \cap H_b^- = \{p_b\}.$$ 

Let $\tilde{A} = H_a^- \cap H_b^-$. Then $\tilde{A} \supseteq A$ so that $\mathcal{H}[A] \subseteq \mathcal{H}[\tilde{A}]$ proving that $a, b \in \mathcal{H}[A] \subseteq \mathcal{H}[\tilde{A}]$.

Furthermore, because $\{p_a\} = Q(a, r_a) \cap H_a^-$ and $p_a \in \tilde{A}$, we have

$$\{p_a\} = Q(a, r_a) \cap (H_a^- \cap H_b^-) = Q(a, r_a) \cap \tilde{A}.$$ 

Hence, $r_a = \text{dist}(a, A) = \text{dist}(a, \tilde{A})$ and $p_a = \text{Pr}(a; \tilde{A})$. In the same way we prove that $p_b = \text{Pr}(b; \tilde{A})$.

Thus, it suffices to prove (7.19) provided $A$ is intersection of two half-planes in $\mathbb{R}^2$. In this case, without loss of generality, we may assume that the unique extreme point of $A$ is $0$, so that $A$ is intersection of two half-spaces.

Thus, we may assume that the set $A$ has the following representation: $A = S^- \cap T^-$ where $S^-$ and $T^-$ are half-spaces. In other words, there exist $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{R}^2$ such that

$$S^- = \{x \in \mathbb{R}^2 : \langle x, u \rangle \leq 0\} \quad \text{and} \quad T^- = \{x \in \mathbb{R}^2 : \langle x, v \rangle \leq 0\}.$$ 

We may also assume that

$$Q(a, r_a) \subseteq S^+ = \{x \in \mathbb{R}^2 : \langle x, u \rangle \geq 0\} \quad \text{and} \quad Q(b, r_b) \subseteq T^+ = \{x \in \mathbb{R}^2 : \langle x, v \rangle \geq 0\}.$$ 

One can easily see that if the vectors $a - p_a$ and $b - p_b$ are directed in opposite direction (i.e., $a - p_a$ and $b - p_b$ are antiparallel vectors), then the straight line $\ell_S = \{x \in \mathbb{R}^2 : \langle x, u \rangle = 0\}$ strictly separates the squares $Q(a, r_a)$ and $Q(b, r_b)$. Furthermore, if $(a - p_a) \parallel (b - p_b)$ (i.e., $a - p_a$ and $b - p_b$ are collinear vectors), then one of the coordinate axes strictly separates $Q(a, r_a)$ and $Q(b, r_b)$ proving property (7.19).

We leave the details to the interested reader.
Property (7.19) enables us to finish the proof of the lemma as follows. Thanks to (7.19),
\[ \text{dist}(a, A) + \text{dist}(b, A) \leq \|x - y\| \]
so that
\[
\|\text{Pr}(a; A) - \text{Pr}(b; A)\| 
\leq \|\text{Pr}(a; A) - a\| + \|a - b\| + \|b - \text{Pr}(b; A)\|
= \text{dist}(a, A) + \|a - b\| + \text{dist}(b, A) \leq 2\|a - b\|.
\]
The proof of the lemma is complete. ■

We are in a position to define the required Lipschitz selection \( f \) of \( F \). We set
\[
G(x) = F^{[1]}(x) = \bigcap_{y \in M} \left[ F(y) + \tilde{\lambda} \rho(x, y) Q_{d} \right], \quad x \in M. \tag{7.20}
\]
Lemma 7.6 tells us that there exists a Lipschitz selection \( g : M \rightarrow \ell_{\infty}^{2} \) of the set-valued mapping \( T : M \rightarrow \mathbb{R}(\mathbb{R}^{2}) \) (see (7.11)) with \( \|g\|_{\text{Lip}(M,\ell_{\infty}^{2})} \leq \lambda \). Thus,
\[
g(x) \in T(x) = \mathcal{H}[F^{[1]}(x)] = \mathcal{H}[G(x)] \quad \text{for every} \quad x \in M, \tag{7.21}
\]
and
\[
\|g(x) - g(y)\| \leq \lambda \rho(x, y) \quad \text{for all} \quad x, y \in M. \tag{7.22}
\]
We define a mapping \( f : M \rightarrow \ell_{\infty}^{2} \) by letting
\[
f(x) = \text{Pr}(g(x); G(x)), \quad x \in M, \tag{7.23}
\]
See Fig. 9.

![Fig. 9: The selection \( f(x) \) is the metric projection of \( g(x) \) onto \( F^{[1]}(x) \).](image)

Property (7.21) and Lemma 7.7 tell us that the mapping \( f \) is well defined. Furthermore, because \( F^{[1]}(x) \subseteq F(x) \), the point
\[
f(x) = \text{Pr}(g(x); G(x)) \in G(x) = F^{[1]}(x) \subseteq F(x) \quad \text{for every} \quad x \in M,
\]
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proving that $f$ is a selection of $F$ on $\mathcal{M}$. Prove that $f$ is a Lipschitz selection.

We will follow [26, pp. 302–304]. If $g(x) \in G(x)$ and $g(y) \in G(y)$, then $f(x) = g(x), f(y) = g(y)$, so that, thanks to (7.22), $||f(x) - f(y)|| \leq \lambda \rho(x, y)$.

Suppose that $g(x) \not\in G(x)$. Then for every $\varepsilon, 0 < \varepsilon < 1$, the point $a_\varepsilon = (1 - \varepsilon)f(x) + \varepsilon g(x)$ does not belong to $G(x)$. Hence, thanks to (7.20), there exists an element $x' \in \mathcal{M}$ such that

$$a_\varepsilon \not\in A(x) = F(x') + \tilde{\lambda} \rho(x, x')Q_0.$$

Similarly, if $g(y) \in G(y)$ (i.e., $f(y) = g(y)$), we put $A(y) = F(y)$, otherwise there exists a set $A(y) = F(y') + \tilde{\lambda} \rho(y, y')Q_0$ which does not contain the point $b_\varepsilon = (1 - \varepsilon)f(y) + \varepsilon g(y)$.

Now define a set $U$ by letting

$$U = (A(x) + \tilde{\lambda} \rho(x, y)Q_0) \cap (A(y) + \tilde{\lambda} \rho(x, y)Q_0).$$

Prove that $g(x) \in \mathcal{H}[U]$ and

$$||f(x) - \text{Pr}(g(x); U)|| \leq \tilde{\lambda} \rho(x, y) + \varepsilon \text{dist}(g(x), G(x)).$$

In fact,

$$G(x) \subset \{F(x') + \tilde{\lambda} \rho(x, x')Q_0\} \cap \{F(y') + \tilde{\lambda} \rho(x, y')Q_0\}$$

$$\subseteq A(x) \cap \{F(y') + (\rho(x, y) + \rho(y, y'))Q_0\}$$

$$= A(x) \cap \{A(y) + \tilde{\lambda} \rho(x, y)Q_0\}$$

so that $G(x) \subset U$. Hence $g(x) \in \mathcal{H}[G] \subset \mathcal{H}[U]$.

Furthermore, thanks to Lemma [7.8]

$$||f(x) - \text{Pr}(g(x); U)|| = ||\text{Pr}(g(x); G(x)) - \text{Pr}(g(x); U)||$$

$$= \text{dist}(g(x), G(x)) - \text{dist}(g(x), U)$$

$$\leq \text{dist}(g(x), G(x)) - \text{dist}(g(x), A(x) + \tilde{\lambda} \rho(x, y)Q_0).$$

Clearly,

$$\text{dist}(g(x), A(x)) \leq \text{dist}(g(x), A(x) + \tilde{\lambda} \rho(x, y)Q_0) + \tilde{\lambda} \rho(x, y),$$

so that

$$||f(x) - \text{Pr}(g(x); U)|| \leq \text{dist}(g(x), G(x)) - \text{dist}(g(x), A(x)) + \tilde{\lambda} \rho(x, y).$$

(7.25)

On the other hand, $f(x) \in G(x) \subset A(x)$. Since $a_\varepsilon \not\in A(x)$, this implies the inclusion

$$[f(x), g(x)] \cap A(x) \subset [f(x), a_\varepsilon].$$

From this and Lemma [7.8] we have

$$\text{Pr}(g(x); A(x)) \in [f(x), g(x)] \cap A(x) \subset [f(x), a_\varepsilon]$$

so that

$$||f(x) - \text{Pr}(g(x); A(x)|| \leq ||f(x) - a_\varepsilon|| = \varepsilon||f(x) - g(x)|| = \varepsilon \text{dist}(g(x), G(x)).$$

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Again, using Lemma \[7.8\] we get
\[
dist(g(x), G(x)) - dist(g(x), A(x)) = \|Pr(g(x); G(x)) - Pr(g(x); A(x))\|
\[
= \|f(x) - Pr(g(x); A(x))\| \leq \varepsilon dist(g(x), G(x)).
\]
Combining this inequality with \[7.25\] we prove \[7.24\]. In a similar way we show that
\[
\|f(y) - Pr(g(y); U)\| \leq \lambda \rho(x, y) + \varepsilon dist(g(y), G(y)). \tag{7.26}
\]
Lemma \[7.9\] tells us that
\[
\|Pr(g(x); U) - Pr(g(y); U)\| \leq 2\|g(x) - g(y)\| \leq 2\rho(x, y).
\]
From this inequality, \[7.24\] and \[7.26\], we have
\[
\|f(x) - f(y)\| \leq \|f(x) - Pr(g(x); U)\|
\]
\[
+ \|Pr(g(x); U) - Pr(g(y); U)\| + \|f(y) - Pr(g(y); U)\|
\]
\[
\leq \lambda \rho(x, y) + 2\lambda \rho(x, y) + \lambda \rho(x, y) + \varepsilon (dist(g(x), G(x)) + dist(g(y), G(y))).
\]
Recall that \(\varepsilon\) is an arbitrary number from \((0, 1)\) so that
\[
\|f(x) - f(y)\| \leq 2(\lambda + \lambda) \rho(x, y), \quad x, y \in \mathcal{M},
\]
proving the required inequality
\[
\|f\|_{\text{Lip}(\mathcal{M}, \rho_{\lambda})} \leq 2(\lambda + \lambda).
\]

The proof of Theorem \[7.3\] is complete. \(\blacksquare\)

**Proof of Theorem 7.2.** The proof relies on the following statement.

**Lemma 7.10** Condition (ii) of Theorem \[7.2\] implies condition (ii) of Theorem \[7.3\] with \(\lambda = 3\lambda\).

**Proof.** Let \(y, y', y'' \in \mathcal{M}\). Clearly, \(W_F[y, y', y'': \lambda] = W_F[y, y'', y': \lambda]\), see \[7.2\], so that, without loss of generality, we may assume that \(\rho(y, y') \leq \rho(y, y'')\). By the triangle inequality,
\[
\rho(y, y'') \leq \rho(y, y') + \rho(y, y'') \leq 2\rho(y, y'') \quad \text{so that} \quad \rho(y, y') + \rho(y', y'') \leq 3\rho(y, y'').
\]
Hence,
\[
W_F[y, y', y'': 3\lambda] = \mathcal{H}[(F(y') + 3\lambda \rho(y, y') Q_0] \cap [F(y'') + 3\lambda \rho(y, y'') Q_0]
\]
\[
\supset \mathcal{H}[(F(y') + \lambda \rho(y, y') Q_0] \cap [(F(y'') + \lambda \rho(y', y'') Q_0] + \lambda \rho(y, y') Q_0].
\]
Clearly, for every \(A, B \subset \mathbb{R}^2, A \cap B \neq \emptyset\), and every \(r \geq 0,\)
\[
A \cap B + rQ_0 \subset (A + rQ_0) \cap (B + rQ_0).
\]
From this and \(6.5\), we have
\[
W_F[y, y', y'': 3\lambda] \supset \mathcal{H}[F(y') \cap [F(y'') + \lambda \rho(y', y'') Q_0] + \lambda \rho(y, y') Q_0]
\]
\[
= \mathcal{H}[F(y') \cap [F(y'') + \lambda \rho(y', y'') Q_0] + \lambda \rho(y, y') Q_0].
\]
This and definition (7.1) imply the following inclusion:

$$W_F[y, y', y'': 3\lambda] \supseteq \mathcal{R}_F[y', y'': \lambda] + \lambda \rho(y, y')Q_0.$$

(7.27)

Now, let us consider points $x, x', x'', y, y', y'' \in \mathcal{M}$. We may assume that $\rho(x, x') \leq \rho(x, x'')$ and $\rho(y, y') \leq \rho(y, y'')$. We know that in this case (7.27) holds. In the same way we can prove that

$$W_F[x, x', x'': 3\lambda] \supseteq \mathcal{R}_F[x', x'': \lambda] + \lambda \rho(x, x')Q_0.$$ 

Hence,

$$R = \text{dist} (W_F[x, x', x'': 3\lambda], W_F[y, y', y'': 3\lambda]) $$

$$\leq \text{dist}(\mathcal{R}_F[x', x'': \lambda] + \lambda \rho(x, x')Q_0, \mathcal{R}_F[y', y'': \lambda] + \lambda \rho(y, y')Q_0)$$

so that, thanks to Lemma 6.6

$$R \leq [\text{dist}(\mathcal{R}_F[x', x'': \lambda], \mathcal{R}_F[y', y'': \lambda]) - \lambda \rho(x, x') - \lambda \rho(y, y')]_+.$$ 

Inequality (7.4) tells us that

$$\text{dist}(\mathcal{R}_F[x', x'': \lambda], \mathcal{R}_F[y', y'': \lambda]) \leq \lambda \rho(x', y').$$

(7.28)

Note also that the function $t \to [t - a]_+$ is non-decreasing on $R$. From this, inequality (7.28) and the triangle inequality, we have

$$R \leq [\lambda \rho(x', y') - \lambda \rho(x, x') - \lambda \rho(y, y')]_+ \leq \lambda \rho(x, y),$$

proving (7.5).  

Thus, the conditions (i) and (ii) of Theorem 7.3 are satisfied provided $\bar{\lambda} = 3\lambda$. This theorem tells us that there exists a Lipschitz selection $f : \mathcal{M} \to L^2_{\infty}$ of $F$ with

$$\|f\|_{\text{Lip}(\mathcal{M},L^2_{\infty})} \leq 2(\lambda + \bar{\lambda}) = 2(\lambda + 3\lambda) = 8\lambda.$$ 

The proof of Theorem 7.2 is complete.  

Combining Theorem 7.2 with part (i) and (ii) of Proposition 7.1, we obtain the statement of Theorem 1.12

Remark 7.11 We note that condition (i) of Theorem 1.12 can be replaced with the following equivalent condition:

(i') $\mathcal{R}_F[x, y : \lambda] \neq \emptyset$ for every $x, y \in \mathcal{M}$.

Indeed, inequality dist$(F(x), F(y)) \leq \lambda \rho(x, y)$ is equivalent to $F(x) \cap \{F(y) + \lambda \rho(x, y)Q_0\} \neq \emptyset$. Clearly, the latter is equivalent to the required property $\mathcal{R}_F[x, y : \lambda] \neq \emptyset$, see definition (7.1).

This replacement (of (i) with (i')) shows that the existence of a Lipschitz selection of $F$ is completely determined by the geometric properties of the family of rectangles $\{\mathcal{R}_F[x, y : \lambda] : x, y \in \mathcal{M}\}$.

Theorem 7.3 enables us to give an efficient bound for the constant $\gamma$ from the Finiteness Theorem 1.2 provided $X$ is an arbitrary two dimensional Banach space.
\textbf{Theorem 7.12} Let $\mathfrak{M} = (\mathcal{M}, \rho)$ be a pseudometric space. Let $X$ be a two dimensional Banach space, and let $\lambda > 0$. Given a set-valued mapping $F : \mathcal{M} \to \mathcal{K}(X)$, suppose that for every subset $\mathcal{M}' \subset \mathcal{M}$ consisting of at most four points, the restriction $F|_{\mathcal{M}'}$ of $F$ to $\mathcal{M}'$ has a Lipschitz selection $f_{\mathcal{M}'}$ with Lipschitz seminorm $\|f_{\mathcal{M}'}\|_{\text{Lip}(\mathcal{M}, X)} \leq \lambda$.

Then $F$ has a Lipschitz selection $f$ with Lipschitz seminorm $\|f\|_{\text{Lip}(\mathcal{M}, X)} \leq 6\lambda$. Furthermore, 

$$\|f\|_{\text{Lip}(\mathcal{M}, X)} \leq 4\lambda$$

provided $X = \ell_2^2$. 

\textit{Proof.} First, we show that the theorem holds for $X = \ell_2^\infty$ by proving that conditions (i) and (ii) of Theorem 7.3 are satisfied with $\lambda = \lambda$. Indeed, property (i) is immediate from the fact that for every $x, y \in \mathcal{M}$ the restriction $F|_{\mathcal{M}'}$ of $F$ to the set $\mathcal{M}' = \{x, y\}$ has a Lipschitz selection $f_{\mathcal{M}'}$ with $\|f_{\mathcal{M}'}\|_{\text{Lip}(\mathcal{M}, \mathcal{K}(\mathcal{M}))} \leq \lambda$.

Let us prove property (ii) of Theorem 7.3. Let $x, x', x'', y, y', y'' \in \mathcal{M}$, and let $\mathcal{M}' = \{x', x'', y', y''\}$. We know that the restriction $F|_{\mathcal{M}'}$ of $F$ to $\mathcal{M}'$ has a Lipschitz selection $f_{\mathcal{M}'}$ with $\|f_{\mathcal{M}'}\|_{\text{Lip}(\mathcal{M}, \mathcal{K}(\mathcal{M}))} \leq \lambda$. Therefore, $f_{\mathcal{M}'}(x') \in F(x'), f_{\mathcal{M}'}(x'') \in F(x''), f_{\mathcal{M}'}(y') \in F(y'), f_{\mathcal{M}'}(y'') \in F(y'')$.

Moreover, we know that the mapping $f_{\mathcal{M}'} : \mathcal{M}' \to \ell_2^\infty$ can be extended to a Lipschitz mapping $\tilde{f} : \mathcal{M} \to \ell_2^\infty$ with $\|\tilde{f}\|_{\text{Lip}(\mathcal{M}, \ell_2^\infty)} = \|f_{\mathcal{M}'}\|_{\text{Lip}(\mathcal{M}, \ell_2^\infty)} \leq \lambda$. Thus, the following inequalities hold:

$$\|f_{\mathcal{M}'}(x') - \tilde{f}(x)\| = \|f(x') - \tilde{f}(x)\| \leq \lambda \rho(x, x'), \quad \|f_{\mathcal{M}'}(x'') - \tilde{f}(x)\| = \|f(x'') - \tilde{f}(x)\| \leq \lambda \rho(x, x''),$$

and

$$\|f_{\mathcal{M}'}(y') - \tilde{f}(y)\| = \|f(y') - \tilde{f}(y)\| \leq \lambda \rho(y, y'), \quad \|f_{\mathcal{M}'}(y'') - \tilde{f}(y)\| = \|f(y'') - \tilde{f}(y)\| \leq \lambda \rho(y, y').$$

Hence,

$$\tilde{f}(x) \in \{F(x') + \lambda \rho(x', x) Q_0\} \cap \{F(x'') + \lambda \rho(x'', x) Q_0\}$$

and

$$\tilde{f}(y) \in \{F(y') + \lambda \rho(y', y) Q_0\} \cap \{F(y'') + \lambda \rho(y'', y) Q_0\}$$

so that $\tilde{f}(x) \in \mathcal{W}_F[x, x', x'' : \lambda]$ and $\tilde{f}(y) \in \mathcal{W}_F[y, y', y'' : \lambda]$, see (7.2).

Therefore,

$$\text{dist}\left(\mathcal{W}_F[x, x', x'' : \lambda], \mathcal{W}_F[y, y', y'' : \lambda]\right) \leq \lambda \|\tilde{f}(x) - \tilde{f}(y)\| \leq \lambda \rho(x, y)$$

proving the required inequality (7.5) with $\lambda = \lambda$.

Thus, conditions of part (i) and (ii) of Theorem 7.3 are satisfied. This theorem tells us that the set-valued mapping $F$ has a Lipschitz selection $f : \mathcal{M} \to \ell_2^\infty$ with $\|f\|_{\text{Lip}(\mathcal{M}, \ell_2^\infty)} \leq 2(\lambda + \lambda) = 4\lambda$.

We turn to the proof of the theorem for the general case of an arbitrary two dimensional Banach space $X$. According to a result of Asplund [2], the unit ball $B_X$ of a two-dimensional Banach space $X$ contains a parallelogram $P$ centered at $(0, 0)$ which expanded by $\frac{3}{2}$ will cover $B_X$. Since the Banach space $X_P$ with the unit ball $P$ is linearly isometric to $\ell_2^2$, the set-valued mapping $F$ has a Lipschitz selection $f : \mathcal{M} \to X_P$ with $\|f\|_{\text{Lip}(\mathcal{M}, X_P)} \leq 4\lambda$. Therefore, for an arbitrary $X$ the Lipschitz seminorm $\|f\|_{\text{Lip}(\mathcal{M}, X)} \leq \frac{3}{2} \cdot 4\lambda = 6\lambda$.

The proof of Theorem 7.12 is complete. \hfill \blacksquare

\textbf{Remark 7.13} Part (\textbullet 1) of Remark 7.5 tells us that Theorem 7.3 is true provided $\mathcal{M}$ is finite and $F$ is a set-valued mapping from $\mathcal{M}$ into $\text{Conv}(\mathbb{R}^2)$ satisfying conditions (i), (ii) of this theorem.

Because the proofs of Theorem 1.12 and Theorem 7.12 rely on Theorem 7.3, the statements of Theorems 1.12, 7.12 and 7.14 also hold whenever $\mathcal{M}$ is finite, $F : \mathcal{M} \to \text{Conv}(\mathbb{R}^2)$ and conditions (i) and (ii) of these theorems are satisfied. \hfill <
In the forthcoming paper [29] we present a series of efficient algorithms for Lipschitz selections of set-valued mappings defined on finite pseudometric spaces. These algorithms rely on the results and methods of proofs of Theorem [1.12] and Theorem [1.13] and extension criteria for Lipschitz selections given below. See Theorems [7.14], [7.15], [7.17].

We begin with

**Theorem 7.14** A set-valued mapping $F : M \to \mathcal{K}(\mathbb{R}^2)$ has a Lipschitz selection if and only if there exists a constant $\lambda > 0$ such that for every $x \in M$ the following property holds:

$$\bigcap_{y,y' \in M} \{ R_F[y, y' : \lambda] + \lambda \rho(x, y) Q_0 \} \neq \emptyset. \quad (7.30)$$

Furthermore, in this case inequality (1.18) is satisfied.

**Proof.** (Necessity). Let $f : M \to \mathbb{R}^2$ be a Lipschitz selection of $F$, and let $\lambda = \| f \|_{\text{Lip}(M, \mathbb{R}^2)}$. (Without loss of generality we may assume that $\lambda > 0$.) Thus, for every $x, y, y' \in M$ we have $f(x) \in F(x), f(y) \in F(y), f(y') \in F(y')$. Furthermore,

$$\| f(x) - f(y) \| \leq \lambda \rho(x, y) \quad \text{and} \quad \| f(y) - f(y') \| \leq \lambda \rho(y, y').$$

Hence, thanks to (7.1),

$$f(y) \in F(y) \cap [F(y') + \lambda \rho(y, y') Q_0]$$

We also know that $f(x) \in f(y) + \lambda \rho(x, y) Q_0$ so that

$$f(x) \in \{ F(y) \cap [F(y') + \lambda \rho(y, y') Q_0] \} + \lambda \rho(x, y) Q_0$$

proving that

$$f(x) \in \bigcap_{y,y' \in M} \{ (F(y) \cap [F(y') + \lambda \rho(y', y) Q_0]) + \lambda \rho(x, y) Q_0 \}. \quad (7.31)$$

It remains to note that for all $y, y' \in M$ we have

$$F(y) \cap [F(y') + \lambda \rho(y, y') Q_0] \subset \mathcal{H}[F(y) \cap [F(y') + \lambda \rho(y, y') Q_0]] = R_F[y, y' : \lambda]. \quad (7.32)$$

This equality and (7.31) imply (7.30) and inequality $\inf \lambda \leq \| F \|_{\text{Lip}}$, completing the proof of the necessity.

**(Sufficiency.)** Suppose that property (7.30) of the theorem holds for some $\lambda > 0$. Given $x \in M$ we let $\mathcal{A}(x)$ denote the left hand side of (7.30). This property tells us that $\mathcal{A}(x) \neq \emptyset$.

Clearly, $\mathcal{A}(x) \subset R_F[x, y : \lambda]$ so that $R_F[x, y : \lambda] \neq \emptyset$ proving property (i) of Remark [7.11]. This remark tells us that in this case property (i) of Theorem [1.12] holds as well.

Now fix elements $x, x', y, y' \in M$. From (7.30) we have

$$R_F[x, x' : \lambda] \cap R_F[y, y' : \lambda] + \lambda \rho(x, y) Q_0 \neq \emptyset$$

so that $\text{dist}(R_F[x, x' : \lambda], R_F[y, y' : \lambda]) \leq \lambda \rho(x, y)$. This proves inequality (1.17) and property (ii) of Theorem [1.12].

This theorem tells us that, under these conditions there exists a Lipschitz selection $f$ of $F$ with $\| f \|_{\text{Lip}(M, \mathbb{R}^2)} \leq 8 \lambda$. Hence, $\| F \|_{\text{Lip}} \leq 8 \inf \lambda$, and the proof of the theorem is complete.

In [29] we exhibit an efficient algorithm for Lipschitz selections in $\mathbb{R}^2$ which relies on the following version of Theorem [7.14].

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Theorem 7.15 Let \( \mathcal{M} = (M, \rho) \) be a pseudometric space, and let \( X \) be a two dimensional Banach space. A set-valued mapping \( F : M \to \mathcal{K}(X) \) has a Lipschitz selection if and only if there exists a constant \( \lambda > 0 \) such that for every \( x \in M \) the following property holds:

\[
\bigcap_{y,y'\in M} \{ (F(y) \cap [F(y') + \lambda \rho(y', y)B_X]) + \lambda \rho(x, y)B_X \} \neq \emptyset. \tag{7.33}
\]

(Recall that \( B_X \) denotes the unit ball of \( X \)). Furthermore, in this case

\[
\inf \lambda \leq |F|_{\mathcal{B}_M} \leq \gamma \inf \lambda
\]

with \( \gamma = 12 \) provided \( X \) is arbitrary Banach space, and \( \gamma = 8 \) if \( X = \ell^2 \).

Proof. (Necessity.) In the proof of the property (7.31) we replace the square \( Q_0 \) with the unit ball \( B_X \) of \( X \). This replacement leads us to the following analog of (7.31):

\[
f(x) \in \bigcap_{y,y'\in M} \{ (F(y) \cap [F(y') + \lambda \rho(y', y)B_X]) + \lambda \rho(x, y)B_X \}.
\]

This property imply the necessity part of Theorem 7.15 and the inequality \( \inf \lambda \leq |F|_{\mathcal{B}_M} \).

(Sufficiency.) For \( X = \ell^2 \) the sufficiency part of the theorem and inequality \( |F|_{\mathcal{B}_M, \ell^2} \leq 8 \inf \lambda \) are immediate from (7.32) and the sufficiency of Theorem 7.14.

Now let \( X \) be an arbitrary two dimensional Banach space. We again apply to \( X \) a result of Asplund [2] which states that there exists a parallelogram \( P \) centered at \((0, 0)\) such that \( B_X \subset P \subset (3/2)B_X \). \( \tag{7.35} \)

Let \( X_P \) be a Banach space with the unit ball \( P \). Then, thanks to (7.35), for every \( x \in X \) we have

\[
(2/3) \|x\|_X \leq \|x\|_{X_P} \leq \|x\|_X. \tag{7.36}
\]

It is also clear that \( X_P \) is linearly isometric to \( \ell^2 \) so that the statement of Theorem 7.15 proven for \( X = \ell^2 \) is true for \( X_P \) as well.

Note that, thanks to (7.33) and (7.35), we have

\[
\bigcap_{y,y'\in M} \{ (F(y) \cap [F(y') + \lambda \rho(y', y)P]) + \lambda \rho(x, y)P \} \neq \emptyset.
\]

Therefore, by the result of Theorem 7.15 for \( X_P \) mentioned above, the set-valued mapping \( F \) has a Lipschitz selection \( f : M \to X_P \) with \( \|f\|_{\mathcal{Lip}(M,X_P)} \leq 8\lambda \). Clearly, thanks to (7.36), \( \|f\|_{\mathcal{Lip}(M,X)} \leq \frac{3}{2}\|f\|_{\mathcal{Lip}(M,X_P)} \). Hence,

\[
\|f\|_{\mathcal{Lip}(M,X)} \leq (3/2) \|f\|_{\mathcal{Lip}(M,X_P)} \leq (3/2) \cdot 8\lambda = 12\lambda
\]

proving the theorem.

Let us give an alternative proof of the sufficiency part of Theorem 7.15 (with constant \( \gamma = 42 \) instead \( \gamma = 12 \) in (7.34)) which relies on the result of the Finiteness Theorem 7.12 rather than on Theorem 1.12.

We will need the following combinatorial lemma.
Lemma 7.16 Let \((M, \rho)\) be a pseudometric space, and let \(\#M = 4\). There exists a one-to-one mapping \(\varphi : M \to \mathbb{R}\) such that
\[
\rho(z, z') \leq |\varphi(z) - \varphi(z')| \leq 7 \rho(z, z') \quad \text{for every } \ z, z' \in M. \tag{7.37}
\]

Proof. We let \(z_1, z_2 \in M\) denote elements from \(M\) such that
\[
\rho(z_1, z_2) = \min\{\rho(z, z') : \ z, z' \in M\}. \tag{7.38}
\]

Let \(z_2, z_3 \in M \setminus \{z_1, z_2\}\) be elements of \(M\) such that
\[
\rho(z_1, z_3) \leq \rho(z_1, z_4). \tag{7.39}
\]

We set \(\varphi(z_1) = 0, \varphi(z_2) = \rho(z_1, z_2), \varphi(z_3) = \rho(z_1, z_2) + \rho(z_2, z_3)\) and \(\varphi(z_4) = \rho(z_1, z_2) + \rho(z_2, z_3) + \rho(z_3, z_4)\).

Prove that inequality (7.37) holds, i.e.,
\[
\rho(z_i, z_j) \leq |\varphi(z_i) - \varphi(z_j)| \leq 7 \rho(z_i, z_j) \tag{7.40}
\]
for every \(1 \leq i < j \leq 4\).

We begin with \(i = 1\). Clearly, for \(i = 1\) and \(j = 2\) inequality (7.40) is obvious. Let now \(i = 1\) and \(j = 3\). Then, thanks to (7.38),
\[
\rho(z_1, z_3) \leq \rho(z_1, z_2) + \rho(z_2, z_3) = |\varphi(z_1) - \varphi(z_3)| \leq \rho(z_1, z_2) + \rho(z_2, z_1) + \rho(z_1, z_3) \leq 3 \rho(z_1, z_3)
\]
proving (7.40) in the case under consideration.

Now, consider the case \(i = 1\) and \(j = 4\). We have
\[
\rho(z_1, z_4) \leq \rho(z_1, z_2) + \rho(z_2, z_3) + \rho(z_3, z_4) = |\varphi(z_1) - \varphi(z_4)| \leq \rho(z_1, z_2) + \rho(z_2, z_1) + \rho(z_1, z_3) + \rho(z_3, z_1) + \rho(z_1, z_4)
\]
so that, thanks to (7.38) and (7.39), \(|\varphi(z_1) - \varphi(z_4)| \leq 5 \rho(z_1, z_4)\) proving (7.40) for \(i = 1, j = 4\).

Note that inequality (7.40) is obvious for the case \(i = 2, j = 3\) and for the case \(i = 3, j = 4\). Therefore, it remains to prove (7.40) for \(i = 2\) and \(j = 4\). We have
\[
\rho(z_2, z_4) \leq \rho(z_2, z_3) + \rho(z_3, z_4) = |\varphi(z_2) - \varphi(z_4)| \leq \rho(z_2, z_1) + \rho(z_1, z_3) + \rho(z_3, z_1) + \rho(z_1, z_2) + \rho(z_2, z_4)
\]
so that, thanks to (7.38) and (7.39),
\[
|\varphi(z_2) - \varphi(z_4)| \leq 2 \rho(z_1, z_2) + 2 \rho(z_1, z_3) + \rho(z_2, z_4) \leq 2 \rho(z_1, z_2) + 2 \rho(z_1, z_4) + \rho(z_2, z_4) \leq 2 \rho(z_1, z_2) + 2 \rho(z_1, z_2) + 2 \rho(z_2, z_4) + \rho(z_2, z_4) \leq 7 \rho(z_2, z_4).
\]

The proof of the lemma is complete. \(\blacksquare\)

We are in a position to prove the sufficiency part of Theorem 7.15. Suppose that for every \(x \in M\) property (7.33) holds. Fix a four point subset \(M' \subset M\) and prove that the restriction \(F|_{M'}\) of \(F\) to \(M'\) has a Lipschitz selection \(f_{M'} : M' \to \mathbb{R}^2\) with \(\|f_{M'}\|_{\text{Lip}(M' : X)} \leq 7\lambda\).

Lemma 7.16 tells us that there exists a one-to-one mapping \(\varphi : M' \to \mathbb{R}\) such that
\[
\rho(z, z') \leq |\varphi(z) - \varphi(z')| \leq 7 \rho(z, z') \quad \text{for every } \ z, z' \in M'. \tag{7.41}
\]
Let us order the elements of \( M' = \{ z_1, z_2, z_3, z_4 \} \) in such a way that
\[
\varphi(z_1) \leq \varphi(z_2) \leq \varphi(z_3) \leq \varphi(z_4).
\] (7.42)

Then we apply property (7.33) to \( x = z_2 \) and the elements \( z_1, z_3, z_4 \) proving that
\[
\{ F(z_2) \cap [F(z_1) + \lambda \rho(z_1, z_2) B_X] \} \cap \{ F(z_3) \cap [F(z_4) + \lambda \rho(z_3, z_4) B_X] \} \neq \emptyset.
\]

Therefore, there exist points \( a_i \in F(z_i), \ i = 1, 2, 3, 4, \) such that
\[
\|a_1 - a_2\| \leq \lambda \rho(z_1, z_2), \quad \|a_2 - a_3\| \leq \lambda \rho(z_2, z_3), \quad \|a_3 - a_4\| \leq \lambda \rho(z_3, z_4).
\]

We set \( f_{M'}(z_i) = a_i, \ i = 1, 2, 3, 4. \) Then, \( f_{M'}(z_i) = a_i \in F(z_i) \) for all \( 1 \leq i \leq 4, \) i.e., \( f_{M'} \) is a selection of \( F \) on \( M'. \) Furthermore, thanks to the above inequalities, (7.42) and (7.41),
\[
\|f_{M'}(z_1) - f_{M'}(z_2)\| \leq \lambda \rho(z_1, z_2) \leq \lambda (\varphi(z_2) - \varphi(z_1)), \quad \|f_{M'}(z_2) - f_{M'}(z_3)\| \leq \lambda \rho(z_2, z_3) \leq \lambda (\varphi(z_3) - \varphi(z_2))
\]
and
\[
\|f_{M'}(z_3) - f_{M'}(z_4)\| \leq \lambda \rho(z_3, z_4) \leq \lambda (\varphi(z_4) - \varphi(z_3)).
\]
Hence, for every \( 1 \leq i < j \leq 4, \) the following inequality holds:
\[
\|f_{M'}(z_i) - f_{M'}(z_j)\| \leq \sum_{k=i}^{j-1} \|f_{M'}(z_k) - f_{M'}(z_{k+1})\| \leq \lambda \sum_{k=i}^{j-1} (\varphi(z_{k+1}) - \varphi(z_k)) = \lambda (\varphi(z_j) - \varphi(z_i)) = \lambda \|\varphi(z_j) - \varphi(z_i)\|.
\]
From this and (7.41), we have
\[
\|f_{M'}(z_i) - f_{M'}(z_j)\| \leq 7 \lambda \rho(z_i, z_j), \quad i, j \in \{1, 2, 3, 4\},
\]
proving that \( \|f_{M'}\|_{\text{Lip}(M', X)} \leq 7 \lambda. \)

Thus, the set-valued mapping \( F \) satisfies the hypothesis of the Finiteness Theorem 7.12. This theorem tells us that \( F \) has a Lipschitz selection \( f \) with \( \|f\|_{\text{Lip}(M, X)} \leq 6(7 \lambda) = 42 \lambda \) proving the sufficiency part of Theorem 7.15 and the inequality \( |F|_{M, X} \leq 42 \inf \lambda. \)

The proof of Theorem 7.15 is complete. \( \blacksquare \)

The next theorem is an analog of Theorem 7.15 for set-valued mapping from a pseudometric space to the family \( \mathcal{K}_1(X) \) of all bounded closed line segments in \( X. \)

**Theorem 7.17** Let \( M = (M, \rho) \) be a pseudometric space, and let \( X \) be a Banach space. A set-valued mapping \( F : M \to \mathcal{K}_1(X) \) has a Lipschitz selection if and only if there exists a constant \( \lambda > 0 \) such that for every \( x \in M \) the following property holds:
\[
\bigcap_{y, y' \in M} \{ F(y) \cap [F(y') + \lambda \rho(y', y) B_X] + \lambda \rho(x, y) B_X \} \neq \emptyset.
\]
(Recall that \( B_X \) denotes the unit ball of \( X. \)) Furthermore, in this case
\[
\inf \lambda \leq |F|_{M, X} \leq \gamma \inf \lambda
\]
with \( \gamma = 105 \) provided \( X \) is arbitrary Banach space, and \( \gamma = 70 \) if \( X \) is Euclidean space.
Proof. The proof of the theorem literally follows the proof of Theorem $\text{7.15}$. The only difference is that in the proof of the present theorem we use the Finiteness Theorem $\text{5.7}$ rather than Theorem $\text{7.12}$. This gives the constant \( \gamma = 7 \cdot 15 = 105 \) for an arbitrary Banach space \( X \), and the constant \( \gamma = 7 \cdot 10 = 70 \) for a Euclidean space \( X \). \( \blacksquare \)

7.3 An algorithm for a nearly optimal Lipschitz selection: main formulae.

Let \( \lambda > 0 \), and let \( F : \mathcal{M} \rightarrow \mathcal{K}(R^2) \) be a set-valued mapping. We note that the proof of Theorem \text{1.12} given in Sections 7.1 and 7.2 is constructive. This proof contains two constructive algorithms which we call Algorithm (A) and Algorithm (B).

Algorithm (A) tells us that at least one of the following options is true:

(\( \star \)1) No a Lipschitz selection of \( F \) with Lipschitz seminorm \(< \lambda \);

(\( \star \)2) There exists a Lipschitz selection \( f \) of \( F \) with

\[
\|f\|_{\text{Lip}(\mathcal{M}, R_\lambda^a)} \leq 8 \lambda. \tag{7.43}
\]

In case (\( \star \)2) Algorithm (B) enables us to construct a Lipschitz selection \( f \) of \( F \) satisfying inequality (7.43).

In the present section we describe main steps and theoretical background of Algorithms (A) and (B) and exhibit several useful auxiliary formulae.

Algorithm (A). It includes two main steps.

**Step 1.** Given \( x, y \in \mathcal{M} \) we introduce a set

\[
A(x, y : \lambda) = F(x) \cap \{ F(y) + \lambda \rho(x, y) Q_0 \}.
\]

At this step we verify the condition \( A(x, y : \lambda) \neq \emptyset \) which is equivalent to the condition (i) of Theorem \text{1.12}. If there exist \( x, y \in \mathcal{M} \) such that \( A(x, y : \lambda) = \emptyset \) then, clearly,

\[
\mathcal{R}_F[x, y : \lambda] = \mathcal{H}[A(x, y : \lambda)] = \emptyset
\]
as well. Therefore, in this case condition (i) of Theorem \text{1.12} does not hold. (See also Remark \text{7.11}).

Theorem \text{1.12} and the first inequality in (1.18) tell us that the option (\( \star \)1) holds. Then we stop the algorithm and declare that \( F \) has no a Lipschitz selection with Lipschitz seminorm \(< \lambda \).

If \( A(x, y : \lambda) \neq \emptyset \) for every \( x, y \in \mathcal{M} \), we turn to the next step of the algorithm.

**Step 2.** We know that \( A(x, x' : \lambda) \neq \emptyset \) for all \( x, x' \in \mathcal{M} \) so that \( \mathcal{R}_F[x, x' : \lambda] \neq \emptyset \) as well. Let

\[
I(x, x' : \lambda) = [a_j(x, x' : \lambda), b_j(x, x' : \lambda)] = \text{Pr}_j[A(x, y : \lambda)], \quad j = 1, 2, \tag{7.44}
\]

be the projection of the set \( A(x, x' : \lambda) \) onto the axis \( Ox_j \). Clearly,

\[
a_1(x, x' : \lambda) = \inf\{u : (u, v) \in A(x, x' : \lambda)\}, \quad b_1(x, x' : \lambda) = \sup\{u : (u, v) \in A(x, x' : \lambda)\}
\]

and

\[
a_2(x, x' : \lambda) = \inf\{v : (u, v) \in A(x, x' : \lambda)\}, \quad b_2(x, x' : \lambda) = \sup\{v : (u, v) \in A(x, x' : \lambda)\}.
\]

Then, thanks to (6.2),

\[
\mathcal{R}_F[x, x' : \lambda] = [a_1(x, x' : \lambda), b_1(x, x' : \lambda)] \times [a_2(x, x' : \lambda), b_2(x, x' : \lambda)]. \tag{7.45}
\]
At this step we introduce a 3-step 3. Constructs a selection and stop. If (7.47) holds for all \( x \in \mathcal{M} \), we declare the existence of a Lipschitz selection \( f \) with \( \|f\|_{\text{Lip}(\mathcal{M},\ell_\infty)} \leq 8\lambda \), and stop. If (7.47) holds for all \( x, x', y, y' \in \mathcal{M} \), we declare the existence of a Lipschitz selection \( f \) of \( F \) with \( \|f\|_{\text{Lip}(\mathcal{M},\ell_\infty)} \leq 8\lambda \) and stop.

Algorithm (A) is complete.

**Algorithm (B).** Let \( \lambda > 0 \), and let \( F : \mathcal{M} \to \mathcal{K}(\mathbb{R}^2) \) be a set-valued mapping satisfying conditions (i) and (ii) of Theorem 1.12. Following the proof of Theorem 7.3 (with \( \lambda = 3\lambda \)), Algorithm (B) constructs a selection \( f : \mathcal{M} \to \ell^2_\infty \) of \( F \) with \( \|f\|_{\text{Lip}(\mathcal{M},\ell_\infty)} \leq 8\lambda \). Let us describe the main steps of this construction.

We note that, thanks to Lemma 7.10, condition (ii) of Theorem 7.2 implies condition (ii) of Theorem 7.3 with \( \lambda = 3\lambda \). Thus, for every \( x, x', x'', y, y', y'' \in \mathcal{M} \) we have

\[
\text{dist}(\mathcal{W}_F[x, x', x''] : 3\lambda, \mathcal{W}_F[y, y', y''] : 3\lambda]) \leq \lambda \rho(x, y).
\]  

Recall that

\[
\mathcal{W}_F[x, x', x''] : \lambda] = \mathcal{H}[\mathcal{A}(x, x', x'') : \lambda]
\]

where

\[
\mathcal{A}(x, x', x'') : \lambda] = \{F(x') + \lambda \rho(x', x) Q_0\} \cap \{F(x'') + \lambda \rho(x'', x) Q_0\},
\]

and \( \mathcal{H}[\cdot] \) is the rectangular hull of a set. See (7.2).

Inequality (7.48) and the constructive proof of Theorem 7.3 given in Section 7.2, enable us to construct the required Lipschitz selection \( f \) of \( F \) in two steps.

**Step (i).** At this step we introduce a 3\( \lambda \)-balanced refinement \( F^{[1]} \) of \( F \) defined by formula (7.6):

\[
F^{[1]}(x) = \bigcap_{y \in \mathcal{M}} \{F(y) + 3\lambda \rho(y, x) Q_0\}, \quad x \in \mathcal{M}.
\]

At **Step 1** of the proof of Theorem 7.3 we show that \( F^{[1]}(x) \neq \emptyset \) for every \( x \in \mathcal{M} \), and

\[
\mathcal{H}[F^{[1]}(x)] = \bigcap_{y, y' \in \mathcal{M}} \mathcal{W}_F[x, y, y' : 3\lambda] = \bigcap_{y, y' \in \mathcal{M}} \mathcal{H}[\mathcal{A}(x, x', x'' : 3\lambda)].
\]
See Lemma 7.4 and (7.49).

Lemma 7.6 tells us that the set-valued mapping \( T(x) = \mathcal{H}[F^{[1]}](x), \ x \in \mathcal{M}, \) has a Lipschitz selection \( g = (g_1, g_2) : \mathcal{M} \to \ell^2_\infty \) with \( \|g\|_{Lip(\mathcal{M}, \ell^2_\infty)} \leq \lambda. \)

The aim of the present step of Algorithm (B), i.e., Step (\( \bullet \)1), is to give explicit formulae for the coordinates \( g_1 \) and \( g_2 \) of the mapping \( g. \)

Let

\[ I_j(x, x', x'' : \lambda) = \{ a_j(x, x', x'' : \lambda), b_j(x, x', x'' : \lambda) \} = \text{Pr}_j[A(x, x', x'' : 3\lambda)] , \quad j = 1, 2, \]

be the projection of the set \( A(x, x', x'' : \lambda) \) onto the axes \( Ox_j. \) Thus,

\[
\begin{align*}
a_1(x, x', x'' : \lambda) &= \inf\{ u : (u, v) \in A(x, x', x'' : 3\lambda) \}, \\
b_1(x, x', x'' : \lambda) &= \sup\{ u : (u, v) \in A(x, x', x'' : 3\lambda) \}
\end{align*}
\]

and

\[
\begin{align*}
a_2(x, x', x'' : \lambda) &= \inf\{ v : (u, v) \in A(x, x', x'' : 3\lambda) \}, \\
b_2(x, x', x'' : \lambda) &= \sup\{ v : (u, v) \in A(x, x', x'' : 3\lambda) \}.
\end{align*}
\]

Then, thanks to (6.2),

\[
\mathcal{H}[A(x, x', x'' : 3\lambda)] = \text{Pr}_1[A(x, x', x'' : 3\lambda)] \times \text{Pr}_2[A(x, x', x'' : 3\lambda)],
\]

so that

\[
\mathcal{H}[A(x, x', x'' : 3\lambda)] = [a_1(x, x', x'' : \lambda), b_1(x, x', x'' : \lambda)] \times [a_2(x, x', x'' : \lambda), b_2(x, x', x'' : \lambda)].
\]

From this and (7.50), we have \( \mathcal{H}[F^{[1]}](x) = I_1(x) \times I_2(x) \) where

\[
I_1(x) = \bigcap_{x', x'' \in \mathcal{M}} \{ a_j(x, x', x'' : \lambda), b_j(x, x', x'' : \lambda) \}, \quad x \in \mathcal{M}.
\]

Furthermore, \( I_j(x) = [A_j(x), B_j(x)] \) where

\[
A_j(x) = \sup_{x', x'' \in \mathcal{M}} a_j(x, x', x'' : \lambda) \quad \text{and} \quad B_j(x) = \inf_{x', x'' \in \mathcal{M}} b_j(x, x', x'' : \lambda). \tag{7.51}
\]

We know that the mapping \( g = (g_1, g_2) : \mathcal{M} \to \ell^2_\infty \) is a selection of the set-valued mapping \( T(x) = \mathcal{H}[F^{[1]}](x) = I_1(x) \times I_2(x) \) with \( \|g\|_{Lip(\mathcal{M}, \ell^2_\infty)} \leq \lambda. \) Therefore, the mapping \( g_j : \mathcal{M} \to \mathbb{R}, \) \( j = 1, 2, \) is a selection of the set-valued mapping \( I_j : \mathcal{M} \to \mathbb{R} \) with \( \|g_j\|_{Lip(\mathcal{M}, \mathbb{R})} \leq \lambda. \)

Formula (5.5) tells us that we can set

\[
g_j^+(x) = \inf_{y \in \mathcal{M}} \{ B_j(y) + \lambda \rho(x, y) \}
\]

so that, thanks to (7.51),

\[
g_j^+(x) = \inf_{y \in \mathcal{M}} \{ \inf_{y', y'' \in \mathcal{M}} \{ b_j(y, y', y'' : \lambda) \} + \lambda \rho(x, y) \} = \inf_{y, y', y'' \in \mathcal{M}} \{ b_j(y, y', y'' : \lambda) + \lambda \rho(x, y) \}.
\]

Of course, thanks to (5.6), we can also set

\[
g_j^-(x) = \sup_{y, y', y'' \in \mathcal{M}} \{ a_j(y, y', y'' : \lambda) - \lambda \rho(x, y) \}.
\]

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Following formula (7.23), at this step of the algorithm we construct the required Lipschitz selection $f$ of $F$ as the metric projection (in the norm $\| \cdot \|_{\mathcal{F}}$) of $g$ onto $F^{[1]}$:

$$f(x) = \Pr \left( g(x); F^{[1]}(x) \right) = \Pr \left( g(x); \bigcap_{y \in M} \left[ F(y) + 3\lambda \rho(x,y) Q_0 \right] \right), \quad x \in M.$$ 

Let us describe this procedure of the metric projection as a certain constructive geometrical algorithm for finding of $f(x)$, $x \in M$. At this step we know that $g(x) \in \mathcal{H}[F^{[1]}(x)]$. Lemma 7.7 tells us that in this case the point $f(x) = \Pr \left( g(x); F^{[1]}(x) \right)$ coincides with a vertex of the square $Q(g(x); \text{dist}(g(x), F^{[1]}(x)))$.

Let 

$$h^{(1)} = (1, 1), \quad h^{(2)} = (1, -1),$$

and let 

$$\ell^{(j)} = \{th^{(j)} : t \in \mathbb{R}\}, \quad j = 1, 2.$$ 

Consider the straight line $g(x) + \ell^{(j)}$ with the directional vector $h^{(j)}$ passing through $g(x)$. Then, one of the following options hold: either

(Case 1) \hspace{1cm} $f(x) \in g(x) + \ell^{(1)}$, \hspace{1cm} (7.52)

or

(Case 2) \hspace{1cm} $f(x) \in g(x) + \ell^{(2)}$. \hspace{1cm} (7.53)

Furthermore, in (Case j), $f(x)$ is the point nearest to $g(x)$ on the line segment 

$$T^{(j)}(x) = (g(x) + \ell^{(j)}) \cap F^{[1]}(x).$$

In particular,

$$\| f(x) - g(x) \| = \text{dist}(g(x), F^{[1]}(x)) = \text{dist}(g(x), T^{(j)}(x))$$ \hspace{1cm} (7.54)

provided (Case j) holds.

Given $y \in M$ and $j = 1, 2$, let 

$$Y^{(j)}(x) = (g(x) + \ell^{(j)}) \cap \{ F(y) + 3\lambda \rho(x,y) Q_0 \} = [u^{(j)}(x,y), v^{(j)}(x,y)].$$

Then, 

$$T^{(j)}(x) = \bigcap_{y \in M} [u^{(j)}(x,y), v^{(j)}(x,y)].$$

Therefore, thanks to (7.54) and one dimensional version of Lemma 6.7

$$\text{dist}(g(x), T^{(j)}(x)) = \sup_{y \in M} \text{dist}(g(x), Y^{(j)}(x)) = \sup_{y \in M} \text{dist}(g(x), [u^{(j)}(x,y), v^{(j)}(x,y)]).$$ \hspace{1cm} (7.55)
These observations enable us to fix the parameter $j \in \{1, 2\}$ for which (Case j) holds. See (7.52), (7.53). To do this we simply compare the values of the quantities
\[
D_j = \sup_{y \in M} \text{dist}(g(x), [u^{(j)}(x, y), v^{(j)}(x, y)]), \quad j = 1, 2
\]
Then the smallest $D_j$ corresponds to (Case j). In other words, we set $j = 1$ provided $D_1 \leq D_2$, and we set $j = 2$ whenever $D_2 \leq D_1$.

Suppose that there exists $\tilde{y} \in M$ for which the supremum in (7.55) is attained. (For instance, $\tilde{y}$ always exists provided $M$ is finite.) Then,
\[
R(x) = \text{dist}(g(x), F^{[1]}(x)) = \sup_{y \in M} \text{dist}(g(x), [u^{(j)}(x, \tilde{y}), v^{(j)}(x, \tilde{y})]).
\]
In this case we have
\[
Q(g(x), R(x)) \cap [u^{(j)}(x, \tilde{y}), v^{(j)}(x, \tilde{y})] = f(x).
\]
In other words, $f(x)$ is the nearest point to $g(x)$ on the line segment $[u^{(j)}(x, \tilde{y}), v^{(j)}(x, \tilde{y})]$. Because $g(x) \notin (u^{(j)}(x, \tilde{y}), v^{(j)}(x, \tilde{y}))$, the point $f(x)$ is the nearest $g(x)$ point in two point set $[u^{(j)}(x, \tilde{y}), v^{(j)}(x, \tilde{y})]$.
This remark completes the second and the last step of Algorithm (B).

8. Half-plane-set valued mappings and their Lipschitz selections.

Let $\mathcal{M} = (M, \rho)$ be a pseudometric space, and let $F : M \to \text{Conv}(\mathbb{R}^2)$ be a set-valued mapping from $M$ into the family $\text{Conv}(\mathbb{R}^2)$ of all closed convex subsets of $\mathbb{R}^2$. We recall that
\[
|F|_{\mathbb{R}^2} = \inf \{ ||f||_{\text{Lip}(M, X)} : f \text{ is a Lipschitz selection of } F \}.
\]
See (1.16).

In Section 5.1 we present several formulae for $|F|_{\mathbb{R}^2}$ provided $F$ is a set-valued mapping from $M$ into the family $\mathcal{K}(\mathbb{R})$ of all closed bounded intervals in $\mathbb{R}$. In particular, in this case
\[
|F|_{\mathbb{R}^2} = \sup_{x, y \in M} \frac{\text{dist}(F(x), F(y))}{\rho(x, y)} = \sup_{x, y \in M} \frac{[\min F(x) - \max F(y)]_+}{\rho(x, y)}.
\]
See (5.3), (5.4) and part (ii) of Lemma 5.5.

In the present section we give several explicit formulae for the quantity $|F|_{\mathbb{R}^2}$ whenever $X = \mathbb{R}^2$.

Let us begin with the case of a set-valued mapping $F$ from $M$ into $\mathcal{HP}(\mathbb{R}^2)$. (We recall that $\mathcal{HP}(\mathbb{R}^2)$ denote the family of all closed half-planes in $\mathbb{R}^2$.) Let $n : M \to S_1$ and let $\alpha : M \to \mathbb{R}$ (recall that $S_1$ is the unit circle in $\mathbb{R}^2$). The mappings $n$ and $\alpha$ determine a set valued mapping $F : M \to \mathcal{HP}(\mathbb{R}^2)$ by
\[
F(x) = \{ a \in \mathbb{R}^2 : \langle a, n(x) \rangle + \alpha(x) \leq 0 \}, \quad x \in M.
\]
(8.1)

Recall that given $a = (a_1, a_2)$, $n(x) = (n_1(x), n_2(x)) \in \mathbb{R}^2$ we set $\langle a, n(x) \rangle = a_1 n_1(x) + a_2 n_2(x)$.

Thus, for each $x \in M$, the set $F(x)$ is a half-plane in $\mathbb{R}^2$ whose boundary is a straight line
\[
\ell_F(x) = \{ a \in \mathbb{R}^2 : \langle a, n(x) \rangle + \alpha(x) = 0 \}.
\]
(8.2)
The vector \( n(x) \) is directed outside of the half-plane \( F(x) \) and orthogonal to the line \( \ell_F(x) \). By \( G(x) \) we denote the half-plane

\[
G(x) = \{ a \in \mathbb{R}^2 : \langle a, n(x) \rangle \leq 0 \}.
\]

Let us recall a well-known fact from the plane analytic geometry. Given vectors \( u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{R}^2 \), we let \( (u, v) \) denote a \( 2 \times 2 \) matrix with columns \( u \) and \( v \) respectively, i.e.,

\[
(u, v) = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}.
\]

Recall that \( \theta(u, v) \) denotes the angle between \( u \) and \( v \). See (2.2). Then

\[
\det(u, v) = \|u\| \|v\| \sin \theta(u, v).
\]

Given \( x, y \in M \), let us describe the closed interval \( \Pr_1[G(x) \cap G(y)] \), i.e., the projection of \( G(x) \cap G(y) \) onto the axis \( OX_1 \). (See definition (6.1).) Suppose that the vectors \( n(x) = (n_1(x), n_2(x)), n(y) = (n_1(y), n_2(y)) \) are non-collinear (i.e., \( n(x) \not\parallel n(y) \)). Let

\[
\Delta_n(x, y) = \det(n(x), n(y)) = \det \begin{pmatrix} n_1(x) & n_1(y) \\ n_2(x) & n_2(y) \end{pmatrix} = n_1(x) n_2(y) - n_2(x) n_1(y).
\]

Note that, thanks to (8.3),

\[
\Delta_n(x, y) = \sin \theta(n(x), n(y)).
\]

**Lemma 8.1** (a) Let \( x, y \in M \) and let \( n(x) = (n_1(x), n_2(x)) \not\parallel n(y) = (n_1(y), n_2(y)) \).

(i) \( \Pr_1[G(x) \cap G(y)] = (-\infty, 0] \) if and only if

\[
n_2(x)/\Delta_n(x, y) \leq 0 \quad \text{and} \quad n_2(y)/\Delta_n(x, y) \geq 0.
\]

In turn, \( \Pr_2[G(x) \cap G(y)] = (-\infty, 0] \) if and only if

\[
n_1(x)/\Delta_n(x, y) \geq 0 \quad \text{and} \quad n_1(y)/\Delta_n(x, y) \leq 0.
\]

(ii) \( \Pr_1[G(x) \cap G(y)] = [0, +\infty) \) if and only if

\[
n_2(x)/\Delta_n(x, y) \geq 0 \quad \text{and} \quad n_2(y)/\Delta_n(x, y) \leq 0.
\]

Finally, \( \Pr_2[G(x) \cap G(y)] = [0, +\infty) \) if and only if

\[
n_1(x)/\Delta_n(x, y) \leq 0 \quad \text{and} \quad n_1(y)/\Delta_n(x, y) \geq 0.
\]

(iii) \( \Pr_1[G(x) \cap G(y)] = \mathbb{R} \) if and only if \( n_2(x) n_2(y) > 0 \). In turn, \( \Pr_2[G(x) \cap G(y)] = \mathbb{R} \) if and only if \( n_1(x) n_1(y) > 0 \).

(b) Let \( n(x) \parallel n(y) \), i.e., \( n(x) = \pm n(y) \).

(i) If \( n_2(x) \neq 0 \) then \( \Pr_1[G(x) \cap G(y)] = \mathbb{R} \). In turn, \( \Pr_2[G(x) \cap G(y)] = \mathbb{R} \) provided \( n_1(x) \neq 0 \).

(ii) Let \( n_2(x) = 0 \). If \( n(x) = n(y) \), then \( \Pr_1[G(x) \cap G(y)] = (-\infty, 0] \) provided \( n(x) = n(y) = (1, 0) \), and \( \Pr_1[G(x) \cap G(y)] = [0, +\infty) \) if \( n(x) = n(y) = (-1, 0) \). If \( n(x) = -n(y) \), then \( \Pr_1[G(x) \cap G(y)] = \{0\} \).

(iii) Let \( n_1(x) = 0 \). If \( n(x) = n(y) \), then \( \Pr_2[G(x) \cap G(y)] = (-\infty, 0] \) provided \( n(x) = n(y) = (0, 1) \), and \( \Pr_2[G(x) \cap G(y)] = [0, +\infty) \) if \( n(x) = n(y) = (0, -1) \). If \( n(x) = -n(y) \), then \( \Pr_2[G(x) \cap G(y)] = \{0\} \).
Proof. Statement \((b)\) of the lemma is obvious. Prove statement \((a)\).

Let \(n(x) \not\parallel n(y)\). Then \(a = (a_1, a_2) \in G(x) \cap G(y)\) if and only if

\[
\lambda_1 = a_1n_1(x) + a_2n_2(x) \leq 0 \quad \text{and} \quad \lambda_2 = a_1n_1(y) + a_2n_2(y) \leq 0.
\]

In these settings,

\[
a_1 = \frac{\det \begin{pmatrix} \lambda_1 & \lambda_2 \\ n_2(x) & n_2(y) \end{pmatrix}}{\Delta_n(x, y)} = \lambda_1 \frac{n_2(y)}{\Delta_n(x, y)} - \lambda_2 \frac{n_2(x)}{\Delta_n(x, y)}. \tag{8.10}
\]

Therefore, \(a_1 \leq 0\) for every \(a = (a_1, a_2) \in G(x) \cap G(y)\) if and only if \(n_2(y)/\Delta_n(x, y) \geq 0\) and \(n_2(x)/\Delta_n(x, y) \leq 0\) proving part \((i)\) of the statement \((a)\).

In the same way we prove that \(a_1 \geq 0\) for every \(a = (a_1, a_2) \in G(x) \cap G(y)\) if and only if \(n_2(y)/\Delta_n(x, y) \leq 0\) and \(n_2(x)/\Delta_n(x, y) \geq 0\) proving part \((ii)\) of the statement \((a)\).

Prove part \((iii)\) of the statement \((a)\). Suppose that \(n_2(x)n_2(y) > 0\). In this case, the coordinate \(a_1\) of a point \(a = (a_1, a_2) \in G(x) \cap G(y)\) may take an arbitrary positive and an arbitrary negative value. Indeed, assume that either \(n_2(x)/\Delta_n(x, x') > 0\) and \(n_2(y)/\Delta_n(x, x') > 0\), or \(n_2(x)/\Delta_n(x, x') < 0\) and \(n_2(y)/\Delta_n(x, x') < 0\). In this case, in formula \((8.10)\) we set either \(\lambda_1 < 0\) (arbitrary) and \(\lambda_2 = 0\), or \(\lambda_1 = 0\) and \(\lambda_2 < 0\) (arbitrary) proving the above statement. Thus, \(\Pr_1[G(x) \cap G(y)] = \mathbb{R}\) provided \(n_2(x)n_2(y) > 0\).

Now, assume that \(\Pr_1[G(x) \cap G(y)] = \mathbb{R}\). Thanks to this assumption, there exist points \(a = (1, a_2), b = (-1, b_2) \in G(x) \cap G(y)\). Prove that \(n_2(x)n_2(y) > 0\).

We set \(\Delta = \Delta_n(x, y)\). (Recall that \(\Delta \neq 0\).) Formula \((8.10)\) tells us that there exists

\[
\lambda_1, \lambda_2, \lambda'_1, \lambda'_2 \leq 0 \tag{8.11}
\]

such that

\[
\lambda_1 \frac{n_2(y)}{\Delta} - \lambda_2 \frac{n_2(x)}{\Delta} = 1 \quad \text{and} \quad \lambda'_1 \frac{n_2(y)}{\Delta} - \lambda'_2 \frac{n_2(x)}{\Delta} = -1. \tag{8.12}
\]

Hence \((\lambda_1, \lambda_2) \neq 0, (\lambda_1, \lambda_2) \neq 0\). Moreover, thanks to \((8.11)\), the determinant

\[
\Delta_\lambda = \det \begin{pmatrix} \lambda_1 & -\lambda_2 \\ \lambda'_1 & -\lambda'_2 \end{pmatrix} \neq 0.
\]

Hence,

\[
n_2(y) = \frac{\Delta}{\Delta_\lambda} \det \begin{pmatrix} 1 & -\lambda_2 \\ -1 & -\lambda'_2 \end{pmatrix} = -\frac{\Delta}{\Delta_\lambda} (\lambda_2 + \lambda'_2) \quad \text{and} \quad n_2(x) = \frac{\Delta}{\Delta_\lambda} \det \begin{pmatrix} \lambda_1 & 1 \\ \lambda'_1 & -1 \end{pmatrix} = -\frac{\Delta}{\Delta_\lambda} (\lambda_1 + \lambda'_1).
\]

Thus, \(n_2(x)n_2(y) = (\lambda_1 + \lambda'_1)(\lambda_1 + \lambda'_1) \Delta^2/\Delta_\lambda^2\). From this formula and \((8.11)\), we have \(n_2(x)n_2(y) \geq 0\).

Prove that \(n_2(x)n_2(y) \neq 0\). Otherwise, either \(\lambda'_1 = -\lambda_1\) or \(\lambda'_2 = -\lambda_2\). From this and \((8.11)\) it follows that if \(n_2(x)n_2(y) = 0\) then either \(\lambda_1 = \lambda'_1 = 0\) or \(\lambda_2 = \lambda'_2 = 0\). This contradicts \((8.12)\) proving the first statement of part \((iii)\), \((a)\). In the same way we prove the second statement of part \((iii)\), \((a)\), completing the proof of the lemma. 

Let us note the following useful property of the mapping \(n : M \rightarrow S_1\).
Lemma 8.2 For every \( x, y \in \mathcal{M} \) and every \( i = 1, 2 \) the following inequality holds:

\[
| |n_i(x)| - |n_i(y)|| \leq |\Delta_n(x, y)|.
\]

Proof. Let \( n(x) = (a_1, a_2), n(y) = (b_1, b_2) \). We know that

\[
a_1^2 + a_2^2 = 1 \quad \text{and} \quad b_1^2 + b_2^2 = 1.
\] (8.13)

Prove that \(|a_1| - |b_1|\) \( \leq |a_1 b_2 - a_2 b_1| \). Clearly,

\[
|a_1| |b_1| + \text{sign}(a_1 b_1) a_2 b_2 \leq |a_1| |b_1| + |a_2| |b_2| \leq (a_1^2 + a_2^2)^{\frac{1}{2}} (b_1^2 + b_2^2)^{\frac{1}{2}} = 1.
\]

Hence, \(|a_1| |b_1| - 1 \leq -\text{sign}(a_1 b_1) a_2 b_2\) so that

\[
2 a_1^2 b_2^2 - 2a_1 |b_1| \leq -2 \text{sign}(a_1 b_1) |a_1| |b_1| a_2 b_2 = -2a_1 b_1 a_2 b_2.
\]

From this inequality and (8.13), we have

\[
a_1^2 + b_1^2 - 2a_1 |b_1| |b_1| \leq a_1^2 b_2^2 + a_2^2 b_1^2 - 2a_1 b_1 a_2 b_2
\]

Hence, \(|a_1| |b_1| \leq (a_1^2 b_2^2 - a_2 b_1)^2\) proving that \(|a_1| - |b_1| \leq |a_1 b_2 - a_2 b_1|\). In the same fashion we prove that \(|a_2| - |b_2| \leq |a_1 b_2 - a_2 b_1|\) completing the proof of the lemma. \( \square \)

Remark 8.3 Conditions (8.6) – (8.9) of Lemma 8.1 have the following equivalent reformulations:

Let \( n(x) = (n_1(x), n_2(x)) \) \( \| n(y) = (n_1(y), n_2(y)) \). Then

(a) (8.6) holds iff \( n_2(x) n_2(y) \leq 0 \) and \( n_1(x) + n_1(y) \geq 0 \);

(b) (8.7) holds iff \( n_1(x) n_1(y) \leq 0 \) and \( n_2(x) + n_2(y) \geq 0 \);

(c) (8.8) is true iff \( n_2(x) n_2(y) \leq 0 \) and \( n_1(x) + n_1(y) \leq 0 \);

(d) (8.9) is true iff \( n_1(x) n_1(y) \leq 0 \) and \( n_2(x) + n_2(y) \leq 0 \).

We leave the proofs of these equivalences to the reader as an easy exercise in analytic plane geometry.

\[ \triangleq \]

We return to the set-valued mapping defined by formula (8.1). Recall that the straight line \( \ell_F(x) \) denotes the boundary of \( F(x) \), see (8.2).

Given \( x, y \in \mathcal{M} \) such that \( n(x) \nparallel n(y) \) we set

\[
w(x, y : F) = \ell_F(x) \cap \ell_F(y).
\] (8.14)

Let us give explicit formulae for the coordinates of the point

\[
w(x, y : F) = (w_1(x, y : F), w_2(x, y : F)).
\]

These coordinates are the solution of the linear system of equations

\[
\begin{align*}
n_1(x) w_1 + n_2(x) w_2 &= -\alpha(x) \\
n_1(y) w_1 + n_2(y) w_2 &= -\alpha(y)
\end{align*}
\]

so that

\[
w_1(x, y : F) = -\frac{\det \begin{pmatrix} \alpha(x) & \alpha(y) \\ n_2(x) & n_2(y) \end{pmatrix}}{\Delta_n(x, y)} \quad \text{and} \quad w_2(x, y : F) = -\frac{\det \begin{pmatrix} n_1(x) & n_1(y) \\ \alpha(x) & \alpha(y) \end{pmatrix}}{\Delta_n(x, y)}.
\] (8.15)
Recall that the determinant $\Delta_n(x, y)$ is defined by (8.4). (Note that $\Delta_n(x, y) \neq 0$ because $n(x) \not\parallel n(y)$.)

Recall that $B_0 = \{a \in \mathbb{R}^2 : ||a||_2 \leq 1\}$ denotes the closed unit disk in $\mathbb{R}^2$. Given $\delta \geq 0$ and $y \in M$ we introduce a half-plane $F_\delta(y) = F(y) + \delta B_0$. Clearly,

$$F_\delta(y) = F(y) + \delta B_0 = \{a \in \mathbb{R}^2 : \langle a, n(y) \rangle + \alpha(y) \leq \delta\}. \tag{8.16}$$

See (8.1). Let

$$\ell_F(y : \delta) = \{a \in \mathbb{R}^2 : \langle a, n(y) \rangle + \alpha(y) = \delta\}$$

be the boundary of $F_\delta(y)$. Given $\delta_1, \delta_2 > 0$, let

$$w(x, y : F, \delta_1, \delta_2) = \ell_F(x : \delta_1) \cap \ell_F(y : \delta_2).$$

Then, thanks to (8.15),

$$w_1(x, y : F, \delta_1, \delta_2) = \frac{\det \left( \begin{array}{cc} 1 - \alpha(x) & \delta_2 - \alpha(y) \\ n_2(x) & n_2(y) \end{array} \right)}{\Delta_n(x, y)}, \quad w_2(x, y : F, \delta_1, \delta_2) = \frac{\det \left( \begin{array}{cc} n_1(x) & n_1(y) \\ \delta_1 - \alpha(x) & \delta_2 - \alpha(y) \end{array} \right)}{\Delta_n(x, y)}.$$

Hence,

$$w(x, y : F, \delta_1, \delta_2) = w(x, y : F) + \xi(x, y : \delta_1, \delta_2) \tag{8.17}$$

where

$$\xi(x, y : \delta_1, \delta_2) = \frac{\delta_1}{\Delta_n(x, y)} (n_2(y), -n_1(y)) + \frac{\delta_2}{\Delta_n(x, y)} (-n_2(x), n_1(x)). \tag{8.18}$$

Prove that

$$\{F(x) + \delta_1 B_0\} \cap \{F(y) + \delta_2 B_0\} = F(x) \cap F(y) + \xi(x, y : \delta_1, \delta_2). \tag{8.19}$$

Indeed, let $\xi = \xi(x, y : \delta_1, \delta_2)$. Clearly, thanks to (8.18) and (8.4), $\langle \xi, n(x) \rangle = \delta_1$ and $\langle \xi, n(y) \rangle = \delta_2$. Now, let $u \in F(x) \cap F(y) + \xi$. Then $u = \tilde{u} + \xi$ for some $\tilde{u} \in F(x) \cap F(y)$ so that

$$\langle u, n(x) \rangle = \langle \tilde{u}, n(x) \rangle + \delta_1 \leq -\alpha(x) + \delta_1$$

(because $\tilde{u} \in F(x)$). Therefore, thanks to (8.16), $u \in F(x) + \delta_1 B_0$. Furthermore,

$$\langle u, n(y) \rangle = \langle \tilde{u}, n(y) \rangle + \delta_2 \leq -\alpha(y) + \delta_2$$

(because $\tilde{u} \in F(y)$), so that, thanks to (8.16), $u \in F(y) + \delta_2 B_0$.

Thus, $u \in \{F(x) + \delta_1 B_0\} \cap \{F(y) + \delta_2 B_0\}$ proving that the right hand side of (8.19) is a subset of its left hand side.

Now let $u \in \{F(x) + \delta_1 B_0\} \cap \{F(y) + \delta_2 B_0\}$, and let $\tilde{u} = u - \xi$. Then, $\langle \tilde{u}, n(x) \rangle = \langle u, n(x) \rangle - \delta_1 \leq -\alpha(x)$ (because $u \in F(x) + \delta_1 B_0$), so that $\tilde{u} \in F(x)$. Furthermore,

$$\langle \tilde{u}, n(y) \rangle = \langle u, n(y) \rangle - \delta_2 \leq (-\alpha(y) + \delta_2) - \delta_2 = -\alpha(y)$$

so that $\tilde{u} \in F(y)$. Hence, $\tilde{u} \in F(x) \cap F(y)$. Therefore, $u = \tilde{u} + \xi \in F(x) \cap F(y) + \xi$ proving that the left hand side of (8.19) is a subset of its right hand side. This completes the proof of (8.19).
Lemma 8.4 Let \( \delta_1, \delta_2 \geq 0 \), \( x, y \in \mathcal{M} \) and let \( n(x) \parallel n(y) \).

(i) If \( n_2(x)/\Delta_n(x,y) \leq 0 \) and \( n_2(y)/\Delta_n(x,y) \geq 0 \) then

\[
\Pr_1[[F(x) + \delta_1 B_0] \cap [F(y) + \delta_2 B_0]] = \left( -\infty, w_1(x,y : F) + \frac{\delta_1 |n_2(y)|}{|\Delta_n(x,y)|} + \frac{\delta_2 |n_2(x)|}{|\Delta_n(x,y)|} \right).
\]

(ii) If \( n_2(x)/\Delta_n(x,y) \geq 0 \) and \( n_2(y)/\Delta_n(x,y) \leq 0 \) then

\[
\Pr_1[[F(x) + \delta_1 B_0] \cap [F(y) + \delta_2 B_0]] = \left( w_1(x,y : F) - \frac{\delta_1 |n_1(y)|}{|\Delta_n(x,y)|} - \frac{\delta_2 |n_1(x)|}{|\Delta_n(x,y)|}, +\infty \right).
\]

(iii) If \( n_2(x) n_2(y) > 0 \) then \( \Pr_1[[F(x) + \delta_1 B_0] \cap [F(y) + \delta_2 B_0]] = \mathbb{R} \).

Lemma 8.5 In the settings of Lemma 8.4 the following statements hold:

(i) If \( n_1(x)/\Delta_n(x,y) \geq 0 \) and \( n_1(y)/\Delta_n(x,y) \leq 0 \) then

\[
\Pr_2[[F(x) + \delta_1 B_0] \cap [F(y) + \delta_2 B_0]] = \left( -\infty, w_2(x,y : F) + \frac{\delta_1 |n_1(y)|}{|\Delta_n(x,y)|} + \frac{\delta_2 |n_1(x)|}{|\Delta_n(x,y)|} \right).
\]

(ii) If \( n_1(x)/\Delta_n(x,y) \leq 0 \) and \( n_1(y)/\Delta_n(x,y) \geq 0 \) then

\[
\Pr_2[[F(x) + \delta_1 B_0] \cap [F(y) + \delta_2 B_0]] = \left( w_2(x,y : F) - \frac{\delta_1 |n_1(y)|}{|\Delta_n(x,y)|} - \frac{\delta_2 |n_1(x)|}{|\Delta_n(x,y)|}, +\infty \right).
\]

(iii) If \( n_1(x) n_1(y) > 0 \) then \( \Pr_2[[F(x) + \delta_1 B_0] \cap [F(y) + \delta_2 B_0]] = \mathbb{R} \).

Proofs of Lemmas 8.4 and 8.5 are immediate from formulae (8.17), (8.18) and Lemma 8.1.

Lemma 8.6 Let \( x, y \in \mathcal{M} \), and let \( n(y) = -n(x) \). Then

\[
\text{dist}(F(x), F(y) : \ell_2^2) = [\alpha(x) + \alpha(y)]^1_+.
\]

Here, given sets \( A, B \subset \mathbb{R}^2 \) by \( \text{dist}(A, B : \ell_2^2) \) we denote the distance between \( A \) and \( B \) in \( \ell_2^2 \)-norm.

Proof. Let \( \tilde{a} = -\alpha(x) n(x) \) and let \( \tilde{b} = \alpha(y) n(x) \). Then \( \tilde{a} \in \partial F(x), \tilde{a} \perp \partial F(x) \), and \( \tilde{b} \in \partial F(y), \tilde{b} \perp \partial F(y) \). Clearly,

\[
\text{dist}(F(x), F(y) : \ell_2^2) = 0 \quad \text{if and only if} \quad \tilde{b} \in F(x),
\]

i.e., if \( \langle \alpha(y) n(x), n(x) \rangle \leq -\alpha(x) \). Thus, (8.21) holds if and only if \( -\alpha(x) - \alpha(y) \geq 0 \) proving (8.20) whenever \( F(x) \cap F(y) \neq \emptyset \).

Now suppose that \( F(x) \cap F(y) = \emptyset \); clearly, in this case \( -\alpha(x) - \alpha(y) < 0 \). Then

\[
\text{dist}(F(x), F(y) : \ell_2^2) = ||\tilde{a} - \tilde{b}|| = | -\alpha(x) - \alpha(y)| = \alpha(x) + \alpha(y)
\]

proving (8.20) and the lemma. \( \blacksquare \)

Proposition 8.7 Let \( \lambda \) be a positive constant. Suppose that the set-valued mapping \( F : \mathcal{M} \rightarrow \mathcal{H}_\mathcal{F}(\mathbb{R}^2) \) defined by (8.1) has a Lipschitz selection \( f : \mathcal{M} \rightarrow \ell_2^2 \) with \( ||f||_{\text{Lips}(\mathcal{M},\ell_2^2)} \leq \lambda \). Then

(i) \( \alpha(x) + \alpha(y) \leq \sqrt{2} \lambda \rho(x,y) \) for every \( x, y \in \mathcal{M} \) such that \( n(y) = -n(x) \);
(ii) For every \(x, x', y, y' \in \mathcal{M}\) such that \(n(x) \not| n(x')\), \(n(y) \not| n(y')\), we have

\[
\begin{align*}
&w_1(x, x' : F) - w_1(y, y' : F) \\
&\leq \sqrt{2} \lambda \left\{ \frac{\rho(x, x')}{|\Delta_n(x, x')|} \min\{|n_2(x)|, |n_2(x')|\} + \frac{\rho(y, y')}{|\Delta_n(y, y')|} \min\{|n_2(y)|, |n_2(y')|\} + \rho(x, y) \right\}
\end{align*}
\]

provided

\[
\frac{n_2(x)}{\Delta_n(x, x')} \geq 0, \quad \frac{n_2(x')}{\Delta_n(x, x')} \leq 0, \quad \text{and} \quad \frac{n_2(y)}{\Delta_n(y, y')} \leq 0, \quad \frac{n_2(y')}{\Delta_n(y, y')} \geq 0.
\] (8.22)

Furthermore,

\[
\begin{align*}
&w_2(x, x' : F) - w_2(y, y' : F) \\
&\leq \sqrt{2} \lambda \left\{ \frac{\rho(x, x')}{|\Delta_n(x, x')|} \min\{|n_1(x)|, |n_1(x')|\} + \frac{\rho(y, y')}{|\Delta_n(y, y')|} \min\{|n_1(y)|, |n_1(y')|\} + \rho(x, y) \right\}
\end{align*}
\]

provided

\[
\frac{n_1(x)}{\Delta_n(x, x')} \leq 0, \quad \frac{n_1(x')}{\Delta_n(x, x')} \geq 0, \quad \text{and} \quad \frac{n_1(y)}{\Delta_n(y, y')} \geq 0, \quad \frac{n_1(y')}{\Delta_n(y, y')} \leq 0.
\] (8.23)

**Proof.** Let \(x, y \in \mathcal{M}\), and let \(n(y) = -n(x)\). Lemma 8.6 tells us that in this case

\[
\text{dist}(F(x), F(y) : \ell_2^2) = [\alpha(x) + \alpha(y)]_+.
\]

From this property and part (i) of Proposition 7.1 we have

\[
[\alpha(x) + \alpha(y)]_+ = \text{dist}(F(x), F(y) : \ell_2^2) \leq \sqrt{2} \text{dist}(F(x), F(y)) \leq \sqrt{2} \lambda \rho(x, y)
\]

proving property (i) of the proposition.

Let us prove property (ii). Property (ii) of Proposition 7.1 tells us that given \(x, x', y, y' \in \mathcal{M}\) we have

\[
\text{dist}(R_F[x, x' : \lambda], R_F[y, y' : \lambda]) \leq \lambda \rho(x, y).
\] (8.24)

Recall that \(R_F\) denotes the rectangular hull of a set, see (7.1). We also recall that we measure the distances in \(\mathbf{R}^2\) in \(\ell_\infty^2\)-norm.

Let \(\bar{I}(x, x') = \text{Pr}_1[R_F[x, x' : \lambda]]\) and \(\bar{I}(y, y') = \text{Pr}_1[R_F[y, y' : \lambda]]\). Thanks to (8.24),

\[
\text{dist}(\bar{I}(x, x'), \bar{I}(y, y')) \leq \lambda \rho(x, y).
\] (8.25)

In turn, thanks to (7.1) and property (6.7),

\[
\bar{I}(x, x') = \text{Pr}_1[F(x) \cap \{F(x') + \lambda \rho(x, x')Q_0\}] \quad \text{and} \quad \bar{I}(y, y') = \text{Pr}_1[F(y) \cap \{F(y') + \lambda \rho(y, y')Q_0\}].
\]

Let \(\delta_x = \sqrt{2} \lambda \rho(x, x')\), \(\delta_y = \sqrt{2} \lambda \rho(y, y')\), and let

\[
I(x, x') = \text{Pr}_1[F(x) \cap \{F(x') + \delta_x B_0\}] \quad \text{and} \quad I(y, y') = \text{Pr}_1[F(y) \cap \{F(y') + \delta_y B_0\}].
\] (8.26)

Because \(\sqrt{2} B_0 \supseteq Q_0 = [-1, 1]^2\), we have \(I(x, x') \supseteq \bar{I}(x, x')\) and \(I(y, y') \supseteq \bar{I}(y, y')\). Therefore, thanks to (8.25),

\[
\text{dist}(I(x, x'), I(y, y')) \leq \lambda \rho(x, y).
\] (8.27)
We know that \( n(x) \not\parallel n(x') \) and \( n(y) \not\parallel n(y') \). In this case, Lemma \( \frac{8.4}{8.4} \) definition \( \frac{8.26}{8.26} \) and inequalities \( \frac{8.22}{8.22} \) tell us that

\[
I(x, x') = \left[ w_1(x, x': F) - \frac{\delta_x}{|\Delta_n(x, x')|} |n_2(x)|, +\infty \right]
\]

and

\[
I(y, y') = \left[ -\infty, w_1(y, y': F) + \frac{\delta_y}{|\Delta_n(y, y')|} |n_2(y)| \right].
\]

Inequality \( \frac{8.27}{8.27} \) tells us that there exist points \( v \in I(x, x') \) and \( u \in I(y, y') \) such that \( |u - v| \leq \lambda \rho(x, y) \). Therefore, thanks to \( \frac{8.29}{8.29} \) and \( \frac{8.28}{8.28} \),

\[
w_1(x, x': F) - \frac{\delta_x}{|\Delta_n(x, x')|} |n_2(x)| \leq v \leq u + \lambda \rho(x, y) \leq w_1(y, y': F) + \frac{\delta_y}{|\Delta_n(y, y')|} |n_2(y)| + \lambda \rho(x, y)
\]

proving that

\[
w_1(x, x': F) - w_1(y, y': F) \leq \frac{\delta_x}{|\Delta_n(x, x')|} |n_2(x)| + \frac{\delta_y}{|\Delta_n(y, y')|} |n_2(y)| + \lambda \rho(x, y).
\]

By interchanging the roles of \( x \) and \( x' \), and \( y \) and \( y' \) in \( \frac{8.28}{8.28} \) and \( \frac{8.29}{8.29} \) respectively, we obtain the following:

\[
w_1(x, x': F) - w_1(y, y': F) \leq \frac{\delta_x}{|\Delta_n(x, x')|} \min(|n_2(x)|, |n_2(x')|) + \frac{\delta_y}{|\Delta_n(y, y')|} \min(|n_2(y)|, |n_2(y')|) + \lambda \rho(x, y).
\]

This proves the first inequality of part (ii). In the same way we prove the second inequality of (ii) completing the proof of the proposition.

The following claim provides slight generalizations of Theorems \( \frac{7.2}{7.2} \) and \( \frac{7.12}{7.12} \) which we will need in this and the next sections.

**Claim 8.8** Let \( n: \mathcal{M} \rightarrow S_1 \) and let \( \alpha: \mathcal{M} \rightarrow \mathbb{R} \) be mappings defined on \( \mathcal{M} \), and let \( F: \mathcal{M} \rightarrow \mathcal{H}(\mathbb{R}^2) \) be a set valued mapping defined by formula \( \frac{8.1}{8.1} \). The statements of Theorem \( \frac{7.2}{7.2} \) and Theorem \( \frac{7.12}{7.12} \) are true provided \( F \) satisfies the hypotheses of these theorems and the following condition holds:

Either \( \mathcal{M} \) is finite or there exist elements \( x_1, \ldots, x_m \in \mathcal{M} \) such that the interior of convex hull of points \( n(x_1), \ldots, n(x_m) \) contains \( 0 \).

**Proof.** Theorems \( \frac{7.2}{7.2} \) and \( \frac{7.12}{7.12} \) rely on Theorem \( \frac{7.3}{7.3} \) so that it suffices to show that the above generalization holds for Theorem \( \frac{7.3}{7.3} \).

Part (\( \star \)) of Remark \( \frac{7.5}{7.5} \) tells us that the statement of Theorem \( \frac{7.3}{7.3} \) is true whenever \( \mathcal{M} \) is finite and \( F \) is an arbitrary mapping from \( \mathcal{M} \) into \( \text{Conv}(\mathbb{R}^2) \) satisfying conditions (i), (ii) of this theorem.

Prove that Theorem \( \frac{7.3}{7.3} \) is true for \( F \) satisfying conditions (i), (ii) of this theorem and the second condition of the claim: there exist elements \( x_1, \ldots, x_m \in \mathcal{M} \) such that

the interior of the convex hull of the points \( n(x_1), \ldots, n(x_m) \) contains \( 0 \). \( \frac{8.30}{8.30} \)

Let us show that \( F \) satisfies condition (iii) of part (\( \star \)\)\( \star \)) of Remark \( \frac{7.5}{7.5} \).
Let \( x \in M \). Prove that condition (7.10) holds provided \( M_x = \{x_1, \ldots, x_m\} \), i.e., the set
\[
A = \bigcap_{y \in M_x} \{F(y) + \tilde{\lambda} \rho(x, y)Q_0\}
\]
is nonempty and bounded.

Note that the set \( M_x \) is finite. As we have shown in the proof of Lemma 7.4, a finite collection of sets \( \mathcal{C}' = \{F(y) + \tilde{\lambda} \rho(x, y) Q_0 : y \in M_x\} \) satisfies the hypothesis of Corollary 6.5. This corollary tells us that the set \( A = \cap \{C : C \in \mathcal{C}'\} \) is nonempty.

Prove that the set \( A \) is bounded. Suppose that \( A \) is unbounded. Because \( A \) is convex, it contains a ray. See, e.g., [17, p. 23]. Thus, there exist a point \( a_0 \in A \) and a vector \( h \in \mathbb{R}^2 \) with \( \|h\|_2 = 1 \) such that \( a_0 + th \in A \) for every \( t \geq 0 \). Therefore,
\[
a_0 + th \in F(x_i) + \tilde{\lambda} \rho(x_i, x_i)Q_0 \quad \text{for every} \quad t \geq 0 \quad \text{and every} \quad i = 1, \ldots, m. \tag{8.31}
\]
Recall that \( F(x_i) = \{a \in \mathbb{R}^2 : \langle a, n(x_i) \rangle + \alpha(x_i) \leq 0\} \).

Property (8.31) tells us that for every \( t \geq 0 \) there exist a point \( u_i \in F(x_i) \) and a point \( v_i \in \tilde{\lambda} \rho(x_i, x_i)Q_0 \) such that \( a_0 + th = u_i + v_i \). Hence,
\[
\langle a_0, n(x_i) \rangle + t \langle h, n(x_i) \rangle = \langle u_i, n(x_i) \rangle + \langle v_i, n(x_i) \rangle \leq -\alpha(x_i) + \sqrt{2} \tilde{\lambda} \rho(x_i, x_i).
\]
Because \( t \geq 0 \) is arbitrary, we conclude that \( \langle h, n(x_i) \rangle \leq 0 \) for every \( i = 1, \ldots, m \). Thus, the straight line \( \ell = \{a \in \mathbb{R}^2 : \langle a, h \rangle = 0\} \) (non-strictly) separates 0 and the convex hull of the family of points \( \{n(x_1), \ldots, n(x_m)\} \) proving that 0 is not an intrinsic point of this convex hull. This contradicts assumption (8.30) proving the boundedness of the set \( A \).

Thus, the set-valued mapping \( F \) satisfies conditions (i) and (ii) of Theorem 7.3 and condition (iii) of part (★2) of Remark 7.5. This remark tells us that in this case the statement of Theorem 7.3 holds for \( F \), proving the claim.

\[\square\]

**Theorem 8.9** Let \( \mathfrak{M} = (M, \rho) \) be a pseudometric space, and let \( F : M \to \mathcal{HP}(\mathbb{R}^2) \) be a set-valued mapping defined by [8.1]. Assume that either \( M \) is finite or there exist elements \( x_1, \ldots, x_m \in M \) such that
\[
\text{the interior of convex hull of points} \quad n(x_1), \ldots, n(x_m) \quad \text{contains} \quad 0. \tag{8.32}
\]

Let \( \lambda \) be a positive constant such that the following two conditions hold:

(★1) \( \alpha(x) + \alpha(y) \leq \lambda \rho(x, y) \) for every \( x, y \in M \) such that \( n(y) = -n(x) \);

(★2) For every \( x, x', y, y' \in M \) such that \( n(x) \not\parallel n(x'), n(y) \not\parallel n(y') \), we have
\[
w_1(x, x' : F) - w_1(y, y' : F) 
\leq \lambda \left\{ \frac{\rho(x, x')}{|n_2(x)|} \min\{|n_2(x)|, |n_2(x')|\} + \frac{\rho(y, y')}{|n_2(y)|} \min\{|n_2(y)|, |n_2(y')|\} + \rho(x, y) \right\}
\]
provided condition \( (8.22) \) holds, and
\[
w_2(x, x' : F) - w_2(y, y' : F) 
\leq \lambda \left\{ \frac{\rho(x, x')}{|n_1(x)|} \min\{|n_1(x)|, |n_1(x')|\} + \frac{\rho(y, y')}{|n_1(y)|} \min\{|n_1(y)|, |n_1(y')|\} + \rho(x, y) \right\}
\]
provided condition \( (8.23) \) holds.

Then \( F \) has a Lipschitz selection \( f : M \to \ell_\infty^2 \) with Lipschitz seminorm \( \|f\|_{\text{Lip}(M, \ell_\infty^2)} \leq 8\lambda \).
Proof. Let us show that if the mapping $F : \mathcal{M} \to \mathcal{H}(\mathbb{R}^2)$ defined by (8.1) satisfies conditions (\textcircled{1}), (\textcircled{2}) of the theorem, then conditions (i) and (ii) of Theorem 7.2 hold. More specifically, let us prove that

$$\text{dist}(F(x), F(y)) \leq \lambda \rho(x, y) \quad \text{for every} \quad x, y \in \mathcal{M},$$

(8.33)

and

$$\text{dist}(\mathcal{R}_F[x, x' : \lambda], \mathcal{R}_F[y, y' : \lambda]) \leq \lambda \rho(x, y) \quad \text{for all} \quad x, x', y, y' \in \mathcal{M}.$$  

(8.34)

(Recall that the rectangle $R_i$ is defined by (7.1).)

We begin with (8.33). Clearly, if $n(x) \not\parallel n(y)$ or $n(x) = n(y)$ then $F(x) \cap F(y) \neq \emptyset$, so that (8.33) holds in these cases. Let now $n(x) = -n(y)$. In this case, thanks to Lemma 8.6 and condition (\textcircled{1}) of the theorem,

$$\text{dist}(F(x), F(y)) \leq \text{dist}(F(x), F(y) : \ell_2^F) = [\alpha(x) + \alpha(y)]_+ \leq \lambda \rho(x, y)$$

proving (8.33).

We turn to the proof of inequality (8.34). Thanks to (8.33), given $u, u' \in \mathcal{M}$,

$$F(u) \cap \{F(u') + \lambda \rho(u, u')Q_0\} \neq \emptyset.$$

Therefore, thanks to (7.1),

$$\mathcal{R}_F[u, u' : \lambda] = \mathcal{H}[F(u) \cap \{F(u') + \lambda \rho(u, u')Q_0\}] \neq \emptyset \quad \text{for every} \quad u, u' \in \mathcal{M}.$$

Let

$$\overline{I}_k(u, u') = \text{Pr}_k(F(u) \cap \{F(u') + \lambda \rho(u, u')Q_0\}), \quad k = 1, 2.$$

Then, thanks to (6.7),

$$\text{Pr}_k(\mathcal{R}_F[x, x' : \lambda]) = \text{Pr}_k(\mathcal{H}[F(x) \cap \{F(x') + \lambda \rho(x, x')Q_0\}])$$

$$= \text{Pr}_k(F(x) \cap \{F(x') + \lambda \rho(x, x')Q_0\})$$

$$= \overline{I}_k(x, x'),$$

and $\text{Pr}_k(\mathcal{R}_F[y, y' : \lambda]) = \overline{I}_k(y, y')$. Clearly, for any two rectangles $\Pi_1, \Pi_2 \in \mathfrak{R}(\mathbb{R}^2)$ we have

$$\text{dist}(\Pi_1, \Pi_2) = \max\{\text{dist}(\text{Pr}_1[\Pi_1], \text{Pr}_1[\Pi_2]), \text{dist}(\text{Pr}_2[\Pi_1], \text{Pr}_2[\Pi_2])\}.$$  

(Recall that we measure distances in $\ell_\infty$-norm.) Hence,

$$\text{dist}(\mathcal{R}_F[x, x' : \lambda], \mathcal{R}_F[y, y' : \lambda]) = \max\{\text{dist}(\overline{I}_1(x, x'), \overline{I}_1(y, y')), \text{dist}(\overline{I}_2(x, x'), \overline{I}_2(y, y'))\}.$$  

Let

$$I_k(x, x') = \text{Pr}_k(F(x) \cap \{F(x') + \lambda \rho(x, x')B_0\}),$$

(8.35)

and let

$$I_k(y, y') = \text{Pr}_k(F(y) \cap \{F(y') + \lambda \rho(y, y')B_0\}).$$

(8.36)
Clearly, $I_k(x, x') \subset \overline{I}_k(x, x')$ and $I_k(y, y') \subset \overline{I}_k(y, y')$, $k = 1, 2$, (because $B_0 \subset Q_0 = [-1, 1]^2$), so that

$$\text{dist} (\mathcal{R}_F[x, x' : \lambda], \mathcal{R}_F[y, y' : \lambda]) \leq \max(\text{dist}(I_1(x, x'), I_1(y, y')), \text{dist}(I_2(x, x'), I_2(y, y'))). \quad (8.37)$$

Prove that

$$\text{dist}(I_1(x, x'), I_1(y, y')) \leq \lambda \rho(x, y). \quad (8.38)$$

Let us consider the following cases.

**Case 1.** Suppose that

$$n(x) \nparallel n(x') \quad \text{and} \quad n(y) \nparallel n(y').$$

Prove that in this case (8.38) holds. Consider three possible options:

1. Either $n_2(x) n_2(x') > 0$ or $n_2(y) n_2(y') > 0$. Then, thanks to part (iii) of Lemma 8.4, either $I_1(x, x') = \mathbf{R}$ or $I_1(y, y') = \mathbf{R}$. Clearly, in this case (8.38) holds.

2. Either (a) $n_2(x)/\Delta_n(x, x') \geq 0$, $n_2(x)/\Delta_n(x, x') \geq 0$, $n_2(y)/\Delta_n(y, y') \leq 0$, $n_2(y)/\Delta_n(y, y') \geq 0$, or (b) $n_2(x)/\Delta_n(x, x') \geq 0$, $n_2(x)/\Delta_n(x, x') \leq 0$, $n_2(y)/\Delta_n(y, y') \geq 0$ and $n_2(y)/\Delta_n(y, y') \leq 0$.

   Lemma [8.4] tells us that, in case (a) we have $I_1(x, x') = (-\infty, s]$, $I_1(y, y') = (-\infty, t]$ for some $s, t \in \mathbf{R}$, and, in case (b), $I_1(x, x') = [\tilde{s}, +\infty)$, $I_1(y, y') = [\tilde{t}, +\infty)$ for certain $\tilde{s}, \tilde{t} \in \mathbf{R}$. Clearly, in all these cases, (8.38) trivially holds.

3. Condition (8.22) holds.

Representations (8.28) and (8.29) tell us that in this case

$$I_1(x, x') = \left[ w_1(x, x' : F) - \frac{\lambda \rho(x, x')}{|\Delta_n(x, x')|} |n_2(x)|, +\infty \right)$$

and

$$I_1(y, y') = \left( -\infty, w_1(y, y' : F) + \frac{\lambda \rho(y, y')}{|\Delta_n(y, y')|} |n_2(y)| \right].$$

Cf. (8.26) with (8.35) and (8.36). We also recall that $w_1(x, x' : F)$ and $w_1(y, y' : F)$ are the points defined by (8.15).

Let $r_x = \lambda \rho(x, x') |n_2(x)|/|\Delta_n(x, x')|$ and $r_y = \lambda \rho(y, y') |n_2(y)|/|\Delta_n(y, y')|$. In these settings,

$$I_1(x, x') = [w_1(x, x' : F) - r_x, +\infty) \quad \text{and} \quad I_1(y, y') = (-\infty, w_1(y, y' : F) + r_y].$$

Part (★2) of the theorem’s hypothesis tells us that

$$w_1(x, x' : F) - w_1(y, y' : F) \leq r_x + r_y + \lambda \rho(x, y). \quad (8.39)$$

Clearly,

$$\text{dist}(I_1(x, x'), I_1(y, y')) = [(w_1(x, x' : F) - r_x) - (w_1(y, y' : F) + r_y)]_+.$$ 

Therefore, thanks to (8.39),

$$\text{dist}(I_1(x, x'), I_1(y, y')) \leq \lambda \rho(x, y)$$

proving (8.38) in the case under consideration.

This completes the proof of inequality (8.38) in Case 1.

For proving inequality (8.38) in the remaining cases we need the following.
Claim 8.10 Let $u, u', v, v' \in M$, and let $n_2(u') = n_2(v') = 0$. Let
\[
T_u = \Pr_1[F(u') + \lambda \rho(u, u')B_0] \quad \text{and} \quad T_v = \Pr_1[F(v') + \lambda \rho(v, v')B_0].
\]
Then
\[
\text{dist}(T_u, T_v) \leq \lambda \rho(u, v).
\]

Proof. We know that $n(u') = \pm(1, 0)$ and $n(v') = \pm(1, 0)$. Clearly, if $n(u') = n(v')$ then $\text{dist}(T_u, T_v) = 0$. Therefore, $n(v') = -n(u')$ so that, without loss of generality, we may assume that $n(u') = (1, 0)$ and $n(v') = (-1, 0)$. Hence,
\[
F(u') + \lambda \rho(u, u')B_0 = \{a = (a_1, a_2) \in \mathbb{R}^2 : \langle a, n(u') \rangle \leq -\alpha(u') + \lambda \rho(u, u')\}
\]
so that
\[
F(u') + \lambda \rho(u, u')B_0 = \{a = (a_1, a_2) \in \mathbb{R}^2 : a_1 \leq -\alpha(u') + \lambda \rho(u, u')\}.
\]
Analogously,
\[
F(v') + \lambda \rho(v, v')B_0 = \{a = (a_1, a_2) \in \mathbb{R}^2 : a_1 \geq \alpha(v') - \lambda \rho(v, v')\}.
\]
Hence,
\[
T_u = (-\infty, -\alpha(u') + \lambda \rho(u, u')] \quad \text{and} \quad T_v = [\alpha(v') - \lambda \rho(v, v'), +\infty),
\]
so that
\[
\text{dist}(T_u, T_v) = \left[\left(\alpha(v') - \lambda \rho(v, v')\right) - (-\alpha(u') + \lambda \rho(u, u'))\right]. \tag{8.40}
\]
Property (★1) of the theorem’s hypothesis tells us that $\alpha(u') + \alpha(v') \leq \lambda \rho(u', v')$. From this inequality, (8.40) and the triangle inequality, we have
\[
\text{dist}(T_u, T_v) \leq [\lambda \rho(u', v') - \lambda \rho(v, v') - \lambda \rho(u, u')], \leq \lambda \rho(u, v).
\]
The proof of the claim is complete. ■

We return to the proof of inequality (8.38).

Case 2. $n(x) \parallel n(x')$ and $n(y) \not\parallel n(y')$.

Clearly, in this case $n(x') = \pm n(x)$ (recall that $||n(x)||_2 = ||n(x')||_2 = 1$.) Furthermore, if $n_2(x) \neq 0$ then, thanks to part (b), (i) of Lemma [8.1] $I_1(x, x') = \mathbb{R}$ which obviously implies (8.38).

Thus, we may assume that $n(x) \parallel n(x') \parallel (1, 0)$, i.e., $n(x) = (\pm 1, 0)$ and $n(x') = (\pm 1, 0)$.

Let $J_F(x) = \Pr_1[F(x)] = (-\infty, -\alpha(x)]$ and let
\[
J = \Pr_1[F(x)] \quad \text{and} \quad J' = \Pr_1[F(x') + \lambda \rho(x, x')B_0]. \tag{8.41}
\]
Because $n(x) \parallel n(x') \parallel (1, 0),
\[
I_1(x, x') = \Pr_1[F(x) \cap \{F(x') + \lambda \rho(x, x')B_0\}] = J \cap J'
\]
so that, thanks to Lemma [6.7]
\[
\text{dist}(I_1(x, x'), I_1(y, y')) = \text{dist}(J \cap J', I_1(y, y')) = \max\{\text{dist}(J, I_1(y, y')), \text{dist}(J', I_1(y, y'))\}. \tag{8.42}
\]
Prove (8.38) whenever $n(y) \parallel (1, 0)$, i.e., $n(y) = (\pm 1, 0)$. Because $n(y') \not\parallel n(y)$, we have
\[
I(y, y') = \Pr_1[F(y) \cap \{F(y') + \lambda \rho(y', y')B_0\}] = \Pr_1[F(y)].
\]
Therefore, thanks to \((8.42)\),
\[
\text{dist}(I_1(x, x'), I_1(y, y')) = \max\{\text{dist}(J, \Pr_1[F(y)]), \text{dist}(J', \Pr_1[F(y)])\}.
\] (8.43)

Claim \(8.10\) and definition \((8.41)\) tell us that
\[
\text{dist}(J, \Pr_1[F(y)]) = \text{dist}(\Pr_1[F(x)], \Pr_1[F(y)]) \leq \lambda \rho(x, y)
\]
and
\[
\text{dist}(J', \Pr_1[F(y)]) = \text{dist}(\Pr_1[F(x')] + \lambda \rho(x, x')B_0), \Pr_1[F(y)]) \leq \lambda \rho(x, y).
\]

These inequalities and \((8.43)\) imply the required inequality \((8.38)\) in the case under consideration.

Thus, we may assume that \(n(x) \parallel n(y)\) and \(n(x') \parallel n(y)\). Prove that in this case
\[
\text{dist}(J, I_1(y, y')) \leq \lambda \rho(x, y).
\] (8.44)

We know that the pairs \(x, y\) and \(y, y'\) satisfy the condition of Case 1, i.e., \(n(x) \parallel n(y)\) and \(n(y) \parallel n(y')\). We have proved that in Case 1 inequality \((8.38)\) holds so that
\[
\text{dist}(I_1(x, y), I_1(y, y')) \leq \lambda \rho(x, y).
\]

Recall that
\[
I_1(x, y) = \Pr_1[F(x) \cap \{F(y) + \lambda \rho(x, y)B_0\}].
\]

But \(n(x) \parallel (1, 0)\) and \(n(y) \nparallel n(x)\) so that
\[
I_1(x, y) = \Pr_1[F(x) \cap \{F(y) + \lambda \rho(x, y)B_0\}] = \Pr_1[F(x)] = J.
\]

See \((8.41)\). Hence,
\[
\text{dist}(J, I_1(y, y')) = \text{dist}(I_1(x, y), I_1(y, y')) \leq \lambda \rho(x, y).
\]

proving \((8.44)\).

Prove that \(\text{dist}(J', I_1(y, y')) \leq \lambda \rho(x, y)\). Consider the pairs of elements \(y, x\) and \(y, y'\). We know that the condition of Case 1 holds for these pairs, i.e., \(n(y) \parallel n(x')\) and \(n(y) \nparallel n(y')\). We also know that inequality \((8.38)\) holds in Case 1 so that
\[
\text{dist}(I_1(y, x'), I_1(y, y')) \leq \lambda \rho(y, y) = 0,
\]
i.e., \(I_1(y, x') \cap I_1(y, y') \neq \emptyset\). But \(n(x') \parallel (1, 0)\) while \(n(y) \nparallel n(y')\) so that
\[
I_1(y, x') = \Pr_1[F(y) \cap \{F(x') + \lambda \rho(y, x')B_0\}] = \Pr_1[F(x') + \lambda \rho(y, x')B_0] = \overline{J}.
\]

Thus, \(\overline{J} \cap I_1(y, y') \neq \emptyset\). Thanks to the triangle inequality,
\[
\overline{J} = \Pr_1[F(x') + \lambda \rho(y, x')B_0] \subseteq \Pr_1[F(x') + \lambda \rho(y, x) + \rho(x, x')B_0]
\]
\[
= \Pr_1[F(x') + \lambda \rho(y, x')B_0] + \lambda \rho(y, x)I_0 = J' + \lambda \rho(y, x)I_0.
\]
(Recall that \(I_0 = [-1, 1]\).)

Thus, \(\overline{J} \cap I_1(y, y') \neq \emptyset\) and \(\overline{J} \subseteq J' + \lambda \rho(y, x)I_0\) so that
\[
(J' + \lambda \rho(y, x)I_0) \cap I_1(y, y') \neq \emptyset
\]

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proving the required inequality \( \text{dist}(J', I_1(y, y')) \leq \lambda \rho(x, y) \). From this inequality, (8.44) and (8.43), we have inequality (8.38). This completes the proof of this inequality in Case 2.

**Case 3.** \( n(x) \parallel n(x') \) and \( n(y) \parallel n(y') \).

Thanks to part (b) of Lemma 8.1, inequality (8.38) is obvious if the second coordinate of one of the vectors \( n(x), n(x'), n(y), n(y') \) is not equal to 0. Therefore, we may assume that each of the vectors \( n(x), n(x'), n(y), n(y') \) is either \((1, 0)\) or \((-1, 0)\).

Let

\[
I_1(x, x') = \text{Pr}_1[F(x)], \quad I^{(2)} = \text{Pr}_1[F(x') + \lambda \rho(x, x')B_0]
\]

and

\[
I^{(3)} = \text{Pr}_1[F(y)], \quad I^{(4)} = \text{Pr}_1[F(y') + \lambda \rho(y, y')B_0].
\]

Then

\[
I_1(x, x') = I^{(1)} \cap I^{(2)} \quad \text{and} \quad I_1(y, y') = I^{(3)} \cap I^{(4)}.
\]

Therefore, thanks to Lemma 6.7,

\[
\text{dist}(I_1(x, x'), I_1(y, y')) = \max\{\text{dist}(I^{(i)}, I^{(j)}): i = 1, 2, \quad j = 3, 4\}.
\]

Thanks to Claim 8.10,

\[
\text{dist}(I^{(i)}, I^{(j)}) \leq \lambda \rho(x, y) \quad \text{for every} \quad i = 1, 2 \quad \text{and} \quad j = 3, 4,
\]

proving inequality (8.38) in Case 3.

Thus, we have proved that inequality (8.38) holds. In the same way we show that

\[
\text{dist}(I_2(x, x'), I_2(y, y')) \leq \lambda \rho(x, y)
\]

proving inequality (8.37). This inequality, inequalities (8.38) and (8.37) imply inequality (8.34).

Thus, inequalities (8.33) and (8.34) hold, so that the set-valued mapping \( F \) satisfies conditions (i) and (ii) of Theorem 7.2. Furthermore, we know that either \( M \) is finite or the condition (8.32) holds. In this case, Claim 8.8 tells us that the statement of Theorem 7.2 holds for \( F \), i.e., \( F \) has a Lipschitz selection with Lipschitz seminorm at most 8.6.

The proof of Theorem 8.9 is complete. \( \blacksquare \)

**Remark 8.11** Remark 8.3 enables us to reformulate conditions (8.22) and (8.23) in the following equivalent form: Let \( n(x) \parallel n(x'), n(y) \parallel n(y') \). Then

(i) Condition (8.22) holds if and only if

\[
n_2(x)n_2(x') \leq 0, \quad n_1(x) + n_1(x') \leq 0 \quad \text{and} \quad n_2(y)n_2(y') \leq 0, \quad n_1(y) + n_1(y') \geq 0;
\]

(ii) Condition (8.23) is true if and only if

\[
n_1(x)n_1(x') \leq 0, \quad n_2(x) + n_2(x') \leq 0, \quad \text{and} \quad n_1(y)n_1(y') \leq 0, \quad n_2(y) + n_2(y') \geq 0. \quad \blacksquare
\]

Combining the results of Proposition 8.7, Theorem 8.9 and Remark 8.11, we obtain the statement of Theorem 1.13.
9. Coordinate-free criteria for Lipschitz selections

9.1 A coordinate-free criterion for Lipschitz selections of half-plane-set valued mappings.

In this section we prove Theorem [14]. Let us recall the ingredients that are needed to formulate this result. Let \( M = (M, \rho) \) be a pseudometric space, and let \( n : M \to S_1 \) and \( \alpha : M \to \mathbb{R} \). This two mappings determine a set-valued mapping \( F \) which to every \( x \in M \) assigns a half-plane

\[
F(x) = \{ a \in \mathbb{R}^2 : \langle a, n(x) \rangle + \alpha(x) \leq 0 \}.
\]

We recall that by \( \ell_F(x) \) we denote the boundary of \( F(x) \), i.e., the straight line

\[
\ell_F(x) = \{ a \in \mathbb{R}^2 : \langle a, n(x) \rangle + \alpha(x) = 0 \}.
\]

See (8.2). Recall that, the vector \( n(x) \) is directed outside of the half-plane \( F \) and orthogonal to the line \( \ell_F(x) \).

Given \( x, y \in M \) we set

\[
\varphi_F(x, y) = \{ \text{the angle between } \partial F(x) \text{ and } \partial F(y) \},
\]

i.e., between the straight lines \( \ell_F(x) \) and \( \ell_F(y) \). (Recall that \( \varphi_F(x, y) \in [0, \pi/2] \).) Because \( n(x) \perp \ell_F(x), n(y) \perp \ell_F(y) \), we have

\[
\varphi_F(x, y) = \{ \text{the shortest angle between } n(x) \text{ and } n(y) \}.
\]

In particular, \( \varphi_F(x, y) \in [0, \pi) \) and \( \varphi_F(x, y) = \varphi_F(y, x) \). Furthermore, from (9.2) and (8.5), we have

\[
\sin \varphi_F(x, y) = |\Delta_n(x, y)|.
\]

Recall that

\[
\Delta_n(x, y) = \det(n(x), n(y)) = \det\left( \begin{array}{cc} n_1(x) & n_1(y) \\ n_2(x) & n_2(y) \end{array} \right).
\]

Given a set \( M' \subset M \), by \( \text{diam}_\rho(M') \) we denote the diameter of \( M \) in the pseudometric space \((M, \rho)\):

\[
\text{diam}_\rho(M') = \sup\{ \rho(x, y) : x, y \in M' \}.
\]

We also recall our notational convention adopted in Section 2: we set

\[
\begin{array}{c}
\frac{0}{0} = 0, \quad \frac{a}{0} = +\infty \text{ for every } a > 0, \quad \text{and} \quad \text{dist}(\emptyset, A) = 0 \text{ provided } A \subset \mathbb{R}^2.
\end{array}
\]

(Necessity.) Let \( F : M \to \mathcal{H}P(\mathbb{R}^2) \) be a set-valued mapping satisfying the hypothesis of the theorem, and let \( f : M \to \ell_\infty^2 \) be its Lipschitz selection with \( \| f \|_{\text{Lip}(M, \ell_\infty^2)} \leq \lambda \). According to our notational convention (9.4), we may assume that \( F(x) \cap F(x') \neq \emptyset, F(y) \cap F(y') \neq \emptyset, \) and \( \rho(x, x') = 0 \) if \( n(x) \parallel n(x') \), and \( \rho(y, y') = 0 \) if \( n(y) \parallel n(y') \).

Because \( f \) is a selection of \( F \) on \( M \) with Lipschitz constant at most \( \lambda \), given \( x, x', y, y' \in M \) we have \( f(x) \in F(x), f(x') \in F(x'), f(y) \in F(y), f(y') \in F(y') \), and

\[
\| f(x) - f(x') \| \leq \lambda \rho(x, x'), \quad \| f(x) - f(y) \| \leq \lambda \rho(x, y) \quad \text{and} \quad \| f(y) - f(y') \| \leq \lambda \rho(y, y').
\]
Hence,

\[ f(x) \in F(x) \cap \{F(x') + \lambda \rho(x, x')Q_0\} \quad \text{and} \quad f(y) \in F(y) \cap \{F(y') + \lambda \rho(y, y')Q_0\}. \]

Let \( \delta(x, x') = \sqrt{2} \lambda \rho(x, x') \) and let \( \delta(y, y') = \sqrt{2} \lambda \rho(y, y') \). Because \( Q_0 \subset \sqrt{2}B_0 \), we have

\[ f(x) \in F(x) \cap \{F(x') + \lambda \rho(x, x')Q_0\} \subset F(x) \cap \{F(x') + \delta(x, x')B_0\}, \]

and, correspondingly, \( f(y) \in F(y) \cap \{F(y') + \delta(y, y')B_0\} \). From this and (8.19), we have

\[ f(x) \in F(x) \cap F(x') + \frac{\delta(x, x')}{\Delta_n(x, x')}(-n_2(x), n_1(x)), \quad f(y) \in F(y) \cap F(y') + \frac{\delta(y, y')}{\Delta_n(y, y')}(-n_2(y), n_1(y)). \]

Note that according to (9.4) these properties are valid for any choice of elements \( x, x', y, y' \in \mathcal{M} \) including the cases \( n(x) \parallel n(x') \) or \( n(y) \parallel n(y') \). Thus, there exist points \( g(x) \in F(x) \cap F(x') \) and \( g(y) \in F(y) \cap F(y') \) such that

\[ f(x) = g(x) + \frac{\delta(x, x')}{\Delta_n(x, x')}(-n_2(x), n_1(x)) \quad \text{and} \quad f(y) = g(y) + \frac{\delta(y, y')}{\Delta_n(y, y')}(-n_2(y), n_1(y)). \]

Hence

\[
\text{dist}(F(x) \cap F(x'), F(y) \cap F(y')) \leq \|g(x) - g(y)\| \leq \|g(x) - f(x)\| + \|f(x) - f(y)\| + \|f(y) - g(y)\| \\
\leq \frac{\delta(x, x')}{|\Delta_n(x, x')|} + \lambda \rho(x, y) + \frac{\delta(y, y')}{|\Delta_n(y, y')|}.
\]

Thanks to (8.5) and (9.2), \( |\Delta_n(x, x')| = |\sin \varphi_F(x, x')| \) and \( |\Delta_n(y, y')| = |\sin \varphi_F(y, y')| \) so that

\[
\text{dist}(F(x) \cap F(x'), F(y) \cap F(y')) \leq \sqrt{2} \lambda \left\{ \frac{\rho(x, x')}{|\sin \varphi_F(x, x')|} + \frac{\rho(y, y')}{|\sin \varphi_F(y, y')|} + \rho(x, y) \right\} \\
\leq \sqrt{2} \lambda \left\{ \frac{\rho(x, x')}{|\sin \varphi_F(x, x')|} + \frac{\rho(y, y')}{|\sin \varphi_F(y, y')|} + \text{diam}_F[x, x', y, y'] \right\}.
\]

The proof of the necessity part of the theorem is complete. \( \blacksquare \)

\textbf{(Sufficiency.)} Suppose that inequality (1.23) of the theorem holds for any \( x, x', y, y' \in \mathcal{M} \). Prove that for every set \( \mathcal{M}' \subset \mathcal{M} \) with \( \# \mathcal{M}' \leq 4 \), the restriction \( F|_{\mathcal{M}'} \) of \( F \) to \( \mathcal{M}' \) has a Lipschitz selection \( f : \mathcal{M}' \to \ell^2_{\infty} \) with \( \|f\|_{\text{Lip}(\mathcal{M}', \ell^2_{\infty})} \leq C \lambda \) where \( C > 0 \) is a certain absolute constant.

Let \( \mathcal{M}' = \{x_1, x_2, x_3, x_4\} \). Recall that each set \( F(x_i) \), \( i = 1, ..., 4 \), is a half-plane. Its boundary \( \ell_F(x_i) = \partial F(x_i) \) is a straight line in \( \mathbf{R}^2 \). By \( \tilde{\ell}_F(x_i) \) we denote a straight line parallel to \( \ell_F(x_i) \) and passing through 0. Let \( n^+(x_i) \in S^1 \) be a directional vector of \( \ell_F(x_i) \). Clearly, \( n^+(x_i) \perp n(x_i) \), \( i = 1, ..., 4 \). Thus

\[ \tilde{\ell}_F(x_i) = \{a \in \mathbf{R}^2 : a = t n^+(x_i), t \in \mathbf{R}\}. \quad (9.5) \]

The lines \( \tilde{\ell}_F(x_i) \), \( i = 1, ..., 4 \), divide the unit circle \( S^1 \) into at most 8 arcs \( \mathcal{A}_1, ..., \mathcal{A}_k \) where \( k \in \{1, ..., 8\} \). Let \( \mathcal{A} \in \{\mathcal{A}_1, ..., \mathcal{A}_k\} \) be the arc with the maximal length. We know that

\[ \sum_{i=1}^{k} \text{length}(\mathcal{A}_i) = 2\pi \quad \text{and} \quad k \leq 8 \]

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so that

\[ \text{length}(\tilde{A}) \geq 2\pi/8 = \pi/4. \]

Let us divide the arc \( \tilde{A} \) by points \( u_1, u_2 \in \tilde{A} \) into three equal arcs. Clearly, the length of every such an arc is at least

\[ \frac{1}{3} \text{length}(\tilde{A}) \geq \pi/12. \quad (9.6) \]

By \( p_j, j = 1, 2 \), we denote the straight line passing though \( u_j \) and 0. Then \( u_j \in S_1 \) is the directional vector of the line \( p_j \).

Recall that \( \theta(u_1, u_2) \) denotes the angle between \( u_1 \) and \( u_2 \), see [2.2]. Then, thanks to (9.6),

\[ \sin \frac{\pi}{12} \leq |\sin \theta(u_1, u_2)|. \quad (9.7) \]

We also know that the angle between each straight line \( \ell_F(\xi_i), i = 1, \ldots, 8 \), and each straight line \( p_j, j = 1, 2 \), is at least \( \pi/12 \). (Recall also that \( n^+(\xi_i) \) is the directional vector of \( \ell_F(\xi_i) \).) Therefore,

\[ |\sin \theta(u_j, n^+(\xi_i))| \geq \sin \frac{\pi}{12} \quad \text{for every} \quad \xi_i \in M' \quad \text{and} \quad j = 1, 2. \quad (9.8) \]

Let \( u_1 = (a_1, b_1), u_2 = (a_2, b_2) \), and let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be a linear operator such that

\[ Te_1 = u_1, \quad Te_2 = u_2, \quad \text{where} \quad e_1 = (1, 0), \quad e_2 = (0, 1). \]

Let \( A_T \) be the matrix of \( T \) in the basis \( e_1, e_2 \). Thus,

\[ A_T = (u_1, u_2) = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}. \quad (9.9) \]

Note that, \( u_1, u_2 \in S_1 \) so that

\[ \|u_1\|_{\ell_2^2} = \|u_2\|_{\ell_2^2} = (a_1^2 + b_1^2)^{\frac{1}{2}} = (a_2^2 + b_2^2)^{\frac{1}{2}} = 1. \]

Hence,

\[ \|T\|_{\ell_2^2 \rightarrow \ell_2^2} \leq (a_1^2 + b_1^2 + a_2^2 + b_2^2)^{\frac{1}{2}} = \sqrt{2}. \quad (9.10) \]

We also note that, thanks to (9.7) and (8.3),

\[ 1 \geq |\det(A_T)| = |\det(u_1, u_2)| = |\sin \theta(u_1, u_2)| \geq \sin \frac{\pi}{12}. \quad (9.11) \]

In turn, from (8.3) and (9.8), we have

\[ |\det(u_j, n^+(\xi_i))| = \|u_j\|_{\ell_2^2} \|n^+(\xi_i)\|_{\ell_2^2} |\sin \theta(u_j, n^+(\xi_i))| = |\sin \theta(u_j, n^+(\xi_i))| \geq \sin \frac{\pi}{12}. \quad (9.12) \]

Inequality (9.11) tells us that the inverse operator \( T^{-1} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is well defined. Its matrix \( A_T^{-1} \) is

\[ A_T^{-1} = \frac{1}{\det(A_T)} \begin{pmatrix} b_2 & -a_2 \\ -b_1 & a_1 \end{pmatrix}. \]

In particular, thanks to (9.11),

\[ 1 \leq |\det(A_T^{-1})| = 1/|\det(A_T)| \leq 1/|\sin \frac{\pi}{12}|. \quad (9.13) \]
Furthermore,
\[ \|T^{-1}\|_{\ell^2_1} \leq \frac{1}{|\det(A_T)|} (a_1^2 + b_1^2 + a_2^2 + b_2^2)^\frac{1}{2} = \frac{\sqrt{2}}{|\det(A_T)|} \leq \frac{\sqrt{2}}{\sin \frac{\pi}{12}}. \] (9.14)

See (9.11).

Let
\[ G(x) = T^{-1}(F(x)), \quad x \in \mathcal{M}'. \] (9.15)

Clearly, \( G(x) \) is a half-plane. Therefore, there exist mappings \( g = (g_1, g_2) : \mathcal{M}' \to \mathbb{S}_1 \) and \( \alpha_G : \mathcal{M}' \to \mathbb{R} \) such that
\[ G(x) = \{ a \in \mathbb{R}^2 : \langle a, g(x) \rangle + \alpha_G(x) \leq 0 \}, \quad x \in \mathcal{M}'. \] (9.16)

**Lemma 9.1** (i) For every \( x, y \in \mathcal{M}' \), we have
\[ \eta_1 |\Delta_n(x, y)| \leq |\Delta_g(x, y)| \leq \eta_2 |\Delta_n(x, y)| \] (9.17)
with
\[ \eta_1 = \frac{1}{2} \sin^2 \frac{\pi}{12}, \quad \text{and} \quad \eta_2 = \frac{2}{\sin \frac{\pi}{12}}; \] (9.18)

(ii) For every \( x \in \mathcal{M}' \) the following inequalities
\[ |g_1(x)|, |g_2(x)| \geq \frac{1}{\sqrt{2}} \sin^2 \frac{\pi}{12} \]
hold.

**Proof.** Prove part (i) of the lemma. We recall that for every \( x \in \mathcal{M}' \) the set \( F(x) \) is a half-plane defined by
\[ F(x) = \{ a \in \mathbb{R}^2 : \langle a, n(x) \rangle + \alpha(x) \leq 0 \}. \]
Its boundary \( \ell_F(x) = \partial F(x) \) is the straight line parallel to the line
\[ \tilde{\ell}_F(x) = \{ a \in \mathbb{R}^2 : a = t n^+(x), t \in \mathbb{R} \} \]
with the directional vector \( n^+(x) \in \mathbb{S}_1 \). See (9.5). Clearly, \( n^+(x) \perp n(x) \) for every \( x \in \mathcal{M}' \), so that \( |\sin \theta(n(x), n(y))| = \sin \theta(n^+(x), n^+(y)) \). (Recall that \( \theta(\cdot, \cdot) \) denotes the angle between vectors; see (2.2).) From this, (8.3) and (8.5), we have
\[ |\Delta_n(x, y)| = |\det(n(x), n(y))| = |\sin \theta(n(x), n(y))| \]
\[ = |\sin \theta(n^+(x), n^+(y))| = |\det(n^+(x), n^+(y))|. \] (9.19)

The boundary of the set \( G(x) = T^{-1}(F(x)) \) defined by (9.15) is a straight line parallel to the straight lines
\[ \tilde{\ell}_G(x) = \{ a \in \mathbb{R}^2 : a = t \tilde{h}(x), t \in \mathbb{R} \} \] (9.20)
where
\[ \tilde{h}(x) = T^{-1}(n^+(x)). \] (9.21)

Therefore, the quantity \( \varphi_G(x, y) \), i.e., the angle between the straight lines \( \tilde{l}_G(x) \) and \( \tilde{l}_G(y) \), see (2.3), (9.1), (9.2), has the following properties:

(a) \( \sin \varphi_G(x, y) = | \sin \theta(\tilde{h}(x), \tilde{h}(y)) | \) (see definition (9.1));
(b) \( \sin \varphi_G(x, y) = | \Delta_s(x, y) | = | \det(g(x), g(y)) | \) (thanks to (9.3)).

On the other hand, thanks to (8.3),
\[ | \sin \theta(\tilde{h}(x), \tilde{h}(y)) | = \frac{| \det(B) |}{\|	ilde{h}(x)\|_{\ell_2} \|	ilde{h}(y)\|_{\ell_2}}. \]

where \( \tilde{B} = (\tilde{h}(x), \tilde{h}(y)) \). Hence,
\[ | \Delta_s(x, y) | = \frac{| \det(B) |}{\|	ilde{h}(x)\|_{\ell_2} \|	ilde{h}(y)\|_{\ell_2}}. \]

Let \( B = (n^+(x), n^+(y)) \). Then, thanks to (9.21), \( \tilde{B} = A_T^{-1} B \) (see (9.9)). Therefore, \( \det(\tilde{B}) = \det(A_T^{-1}) \det(B) \).

From this and (9.19), we have
\[ | \det(\tilde{B}) | = | \det(A_T^{-1}) | \cdot | \det(n^+(x), n^+(y)) | = | \det(A_T^{-1}) | \cdot | \Delta_s(x, y) |. \]

Therefore, thanks to (9.13),
\[ | \Delta_s(x, y) | \leq | \det(\tilde{B}) | = | \det(\tilde{h}(x), \tilde{h}(y)) | \leq \frac{1}{\sin \frac{\pi}{12}} | \Delta_s(x, y) |. \] (9.22)

Furthermore, thanks to (9.14),
\[ | \tilde{h}(x) \|_{\ell_2} = | T^{-1}(n^+(x)) |_{\ell_2} \leq | T^{-1} |_{\ell_2 \rightarrow \ell_2} | n^+(x) |_{\ell_2} \leq \sqrt{2} / \sin \frac{\pi}{12}. \]

On the other hand, \( | T |_{\ell_2 \rightarrow \ell_2} \), see (9.10), so that
\[ 1 = | n^+(x) |_{\ell_2} = | T(\tilde{h}(x)) |_{\ell_2} \leq \sqrt{2} | \tilde{h}(x) |_{\ell_2}. \]

Thus,
\[ 1 / \sqrt{2} \leq | \tilde{h}(x) |_{\ell_2} \leq \sqrt{2} / \sin \frac{\pi}{12}. \] (9.23)

From this inequality and (9.22), we have
\[ \left( \frac{1}{2} \sin^2 \frac{\pi}{12} \right) | \Delta_s(x, y) | \leq | \Delta_\varphi(x, y) | \leq \left( \frac{2}{\sin \frac{\pi}{12}} \right) | \Delta_s(x, y) | \]

proving part (i) of the lemma.

Prove part (ii). Inequality (9.8) tells us that
\[ | \sin \theta(u_1, n^+(x)) | \geq \sin \frac{\pi}{12}. \]
Let \( H = (u_1, n^+(x)) \). Then, thanks to (8.3) and (9.12),
\[
|\det(H)| = |\sin \theta(u_1, n^+(x))| \geq \sin \frac{\pi}{12}. \tag{9.24}
\]
We know that \( T^{-1}u_1 = e_1 = (1,0) \) and \( T^{-1}n^+(x) = \vec{h}(x) \), so that
\[
\det\begin{pmatrix}
1 & \vec{h}_1(x) \\
0 & \vec{h}_2(x)
\end{pmatrix} = \det(T^{-1}u_1, T^{-1}n^+(x)) = \det(A_T^{-1}H). \tag{9.9}\]
Hence,
\[
|\vec{h}_2(x)| = |\det(A_T^{-1}H)| = |\det(A_T^{-1})| |\det(H)| = \frac{|\det(H)|}{|\det(A_T)|}. \tag{9.25}
\]
We know that \( |\det(A_T)| \leq 1 \), see (9.11), and \( |\det(H)| \geq \sin \frac{\pi}{12} \), see (9.24), so that
\[
|\vec{h}_2(x)| = \frac{|\det(H)|}{|\det(A_T)|} \geq \sin \frac{\pi}{12}. \tag{9.26}
\]
Recall that \( \vec{h}(x) = T^{-1}(n^+(x)) \) and \( n^+(x) \in S_1 \). Let
\[
g^+(x) = (g_1^+(x), g_2^+(x)) = \vec{h}(x)/||\vec{h}(x)||_{\ell_2^2}.
\]
Then, thanks to (9.23) and (9.25)
\[
|g_2^+(x)| = |\vec{h}_2(x)|/||\vec{h}(x)||_{\ell_2^2} \geq \frac{\sin \frac{\pi}{12}}{\sqrt{2}/ \sin \frac{\pi}{12}} = \frac{1}{\sqrt{2}} \sin^2 \frac{\pi}{12}. \tag{9.27}
\]
In the same fashion we proof that
\[
|g_1^+(x)| \geq \frac{1}{\sqrt{2}} \sin^2 \frac{\pi}{12}. \tag{9.28}
\]
We know that \( \vec{h}(x) \) is the directional vector of the straight line \( \vec{l}_G(x) \) parallel to \( \partial G(x) \). See (9.20).
Thus, the vector \( g^+(x) = \vec{h}(x)/||\vec{h}(x)||_{\ell_2^2} \) is the directional vector of \( \vec{l}_G(x) \) as well. The vector \( g(x) \) from representation (9.16) is orthogonal to \( g^+(x) \). Furthermore, \( g(x), g^+(x) \in S_1 \). Therefore,
\[
|g_1(x)| = |g_2^+(x)| \quad \text{and} \quad |g_2(x)| = |g_1^+(x)|.
\]
From this, (9.26) and (9.27), we have the required estimate
\[
|g_1(x)|, |g_2(x)| \geq \frac{1}{\sqrt{2}} \sin^2 \frac{\pi}{12}.
\]
The proof of the lemma is complete. \( \blacksquare \)

Let us prove that the set-valued mapping \( G : \mathcal{M}' \rightarrow \mathcal{HP}(\mathbb{R}^2) \) satisfies conditions (★1) and (★2) of Theorem 8.9. More specifically, let us show that (★1), (★2) hold provided \( F = G, n = g, \alpha = \alpha_G \), see (9.16). \( \Delta_n(x,y) = \Delta_g(x,y) = \det(g(x), g(y)) \) and \( w_i(x,y : \cdot) = \cdot \), \( i = 1,2 \), are the quantities defined by (8.15) (with \( n = g, \alpha = \alpha_G \)).

We begin with the proof of (★1). Let \( x,y \in \mathcal{M}' \) and let \( g(y) = -g(x) \). Setting \( x = x' \) and \( y = y' \) in inequality (1.23), we get
\[
\text{dist}(F(x), F(y)) \leq \lambda \text{ diam}_{\ell_2}(x,y) = \lambda \rho(x,y). \tag{9.29}
\]
This and definition (9.15) imply the following inequality:

\[ \text{dist}(G(x), G(y) : \ell_2^2) = \text{dist}(T^{-1}(F(x)), T^{-1}(F(y)) : \ell_2^2) \]
\[ \leq \|T^{-1}\|_{\ell_2^2 \to \ell_2^2} \text{dist}(F(x), F(y) : \ell_2^2) \leq \sqrt{2} \|T^{-1}\|_{\ell_2^2 \to \ell_2^2} \text{dist}(F(x), F(y)). \]

Therefore, thanks to (9.14) and (9.28),

\[ \text{dist}(G(x), G(y) : \ell_2^2) \leq \gamma_1 \lambda \rho(x, y) \quad \text{with} \quad \gamma_1 = 2/\sin(\pi/12). \]

From this property and Lemma 8.6, we have

\[ [\alpha_G(x) + \alpha_G(y)]_+ = \text{dist}(G(x), G(y) : \ell_2^2) \leq \gamma_1 \lambda \rho(x, y) \]

proving property (\(\star\)1) of Theorem 8.9.

We turn to the proof of (\(\star\)2). Let \(x, x', y, y' \in M\), and let \(g(x) \parallel g(x'), g(y) \parallel g(y')\). Our aim is to show the existence of an absolute constant \(C > 0\) such that

\[
\begin{align*}
\text{dist}(G(x), G(y) : \ell_2^2) & \leq \gamma_1 \lambda \rho(x, y) \\
\end{align*}
\]

provided

\[ \frac{g_2(x)}{\Delta_\varphi(x, x')} \geq 0, \quad \frac{g_2(x')}{\Delta_\varphi(x, x')} \leq 0, \quad \text{and} \quad \frac{g_2(y)}{\Delta_\varphi(y, y')} \leq 0, \quad \frac{g_2(y')}{\Delta_\varphi(y, y')} \geq 0, \quad (9.30) \]

and

\[
\begin{align*}
\text{dist}(G(x) \cap G(x'), G(y) \cap G(y')) & \leq \gamma_2 \lambda \left\{ \frac{\rho(x, x')}{|\Delta_\varphi(x, x')|} + \frac{\rho(y, y')}{|\Delta_\varphi(y, y')|} + \rho(x, y) \right\}.
\end{align*}
\]

(9.33)

**Lemma 9.2** There exists an absolute constant \(\gamma_2 > 0\) such that

\[
\begin{align*}
\text{dist}(G(x) \cap G(x'), G(y) \cap G(y')) & \leq \gamma_2 \lambda \left\{ \frac{\rho(x, x')}{|\Delta_\varphi(x, x')|} + \frac{\rho(y, y')}{|\Delta_\varphi(y, y')|} + \rho(x, y) \right\}.
\end{align*}
\]

**Proof.** Thanks to (1.23), (9.3) and (9.17),

\[
\begin{align*}
\text{dist}(F(x) \cap F(x'), F(y) \cap F(y')) & \leq \lambda \left\{ \frac{\rho(x, x')}{|\sin \varphi_F(x, x')|} + \frac{\rho(y, y')}{|\sin \varphi_F(y, y')|} + \text{diam}_\rho \{x, x', y, y'\} \right\} \\
& = \lambda \left\{ \frac{\rho(x, x')}{|\Delta_M(x, x')|} + \frac{\rho(y, y')}{|\Delta_M(y, y')|} + \text{diam}_\rho \{x, x', y, y'\} \right\} \\
& \leq (1/\eta_1) \lambda \left\{ \frac{\rho(x, x')}{|\Delta_\varphi(x, x')|} + \frac{\rho(y, y')}{|\Delta_\varphi(y, y')|} + \text{diam}_\rho \{x, x', y, y'\} \right\}.
\end{align*}
\]

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See (9.18). On the other hand, from (9.14) and (9.15), we have
\[
\text{dist}(G(x) \cap G(x'), G(y) \cap G(y')) = \text{dist}(T^{-1}(F(x)) \cap T^{-1}(F(x')), T^{-1}(F(y)) \cap T^{-1}(F(y'))) \\
\leq \|T^{-1}\|_{\ell_2^g \rightarrow \ell_2^g} \text{dist}(F(x) \cap F(x'), F(y) \cap F(y')) \\
\leq \left( \sqrt{2} / \sin \frac{\pi}{12} \right) \text{dist}(F(x) \cap F(x'), F(y) \cap F(y')).
\]

Finally, thanks to the triangle inequality,
\[
diam_p\{x', y', y'\} \leq \rho(x, x') + \rho(y, y') + \rho(x, y) \leq \frac{\rho(x, x')}{|\Delta_g(x, x')|} + \frac{\rho(y, y')}{|\Delta_g(y, y')|} + \rho(x, y).
\]

(Recall that $|\Delta_g(u, v)| = |\sin \varphi_g(u, v)| \leq 1$ for every $u, v \in \mathcal{M}$.)

These inequalities show that (9.33) holds with
\[
\gamma_2 = 2 - \frac{\sqrt{2}}{\sin \frac{\pi}{12} \eta_1} = 4 \sqrt{2} / \sin^3 \frac{\pi}{12}
\]
completing the proof of the lemma. 

We are in a position to finish the proof of the sufficiency. We begin with the proof of inequality (9.29). Suppose that $g(x) \nmid g(x'), g(y) \nmid g(y')$.

Lemma 9.2 implies the following inequality:
\[
dist(\text{Pr}_1[G(x) \cap G(x')], \text{Pr}_1[G(y) \cap G(y')]) \leq \gamma_2 \lambda \left\{ \frac{\rho(x, x')}{|\Delta_g(x, x')|} + \frac{\rho(y, y')}{|\Delta_g(y, y')|} + \rho(x, y) \right\}.
\]

Let us apply Lemma 8.4 to $F = G$, $n = g$, $\delta_1 = \delta_2 = 0$. Thanks to this lemma,
\[
\text{Pr}_1[G(x) \cap G(x')] = [w_1(x, x' : G), +\infty)
\]
provided $g_2(x)/\Delta_g(x, x') \geq 0$, $g_2(x')/\Delta_g(x, x') \leq 0$. In turn,
\[
\text{Pr}_1[G(y) \cap G(y')] = (-\infty, w_1(y, y' : G)]
\]
g_2(y)/\Delta_n(y, y') \leq 0, g_2(y')/\Delta_g(y, y') \geq 0.

From this, (9.35) and inequality (9.34), we have
\[
w_1(x, x' : G) - w_1(y, y' : G) \leq \gamma_2 \lambda \left\{ \frac{\rho(x, x')}{|\Delta_g(x, x')|} + \frac{\rho(y, y')}{|\Delta_g(y, y')|} + \rho(x, y) \right\}.
\]

Part (ii) of Lemma 9.1 tells us that
\[
\min\{g_2(x), |g_2(x')|\} \geq \gamma_3 \quad \text{and} \quad \min\{g_2(y), |g_2(y')|\} \geq \gamma_3
\]
with $\gamma_3 = \frac{1}{\sqrt{2}} \sin^2 \frac{\pi}{12}$. Hence,
\[
w_1(x, x' : G) - w_1(y, y' : G) \leq \left( \frac{\gamma_2}{\gamma_3} \right) \lambda \left\{ \frac{\rho(x, x')}{|\Delta_g(x, x')|} \min\{g_2(x), |g_2(x')|\} + \frac{\rho(y, y')}{|\Delta_g(y, y')|} \min\{g_2(y), |g_2(y')|\} + \rho(x, y) \right\}.
\]
This proves the required inequality (9.29) under the condition (9.30) with
\[ C = \gamma_2/\gamma_3 = 8/\sin^5 \frac{\pi}{12}. \] (9.36)

In a similar way we prove inequality (9.31) provided (9.32) holds. This completes the proof of part (★2) of Theorem 8.9.

Thus, conditions (★1) and (★2) of Theorem 8.9 are satisfied (for \( F = G \), \( n = g \), \( \alpha = \alpha_G \)).

This theorem tells us that in this case the set-valued mapping \( G : M' \to H^\mathcal{P}(\mathbb{R}^2) \) has a Lipschitz selection \( \tilde{f} : M' \to \ell^2_\infty \) with Lipschitz seminorm \( \|\tilde{f}\|_{\text{Lip}(M', \ell^2_\infty)} \leq 8C\lambda \). (Here \( C \) is the constant defined by (9.36).) Thus \( \tilde{f}(x) \in G(x) \) for each \( x \in M' \), and
\[ \|\tilde{f}(x) - \tilde{f}(y)\| \leq 8C\lambda \rho(x, y), \quad x, y \in M'. \] (9.37)

We define a mapping \( f_M : M \to \ell^2_\infty \) by letting
\[ f_M(x) = T(\tilde{f}(x)), \quad x \in M'. \]

Because \( \tilde{f}(x) \in G(x) = T^{-1}(F(x)), x \in M' \), see (9.15), \( f_M(x) \in F(x) \) on \( M' \) proving that \( f_M \) is a selection of \( F|_{M'} \). Furthermore,
\[ \|f_M(x) - f_M(y)\| = \|T(\tilde{f}(x)) - T(\tilde{f}(y))\| \leq \|T(\tilde{f}(x)) - T(\tilde{f}(y))\|_{\ell^2_2} \]
\[ \leq \|T\|_{\ell^2_2 \to \ell^2_2} \|\tilde{f}(x) - \tilde{f}(y)\|_{\ell^2_2} \leq \sqrt{2} \|T\|_{\ell^2_2 \to \ell^2_2} \|\tilde{f}(x) - \tilde{f}(y)\|. \]

We know that \( \|T\|_{\ell^2_2 \to \ell^2_2} \leq \sqrt{2} \), see (9.10). From this and (9.37), we have
\[ \|f_M(x) - f_M(y)\| \leq 16C\lambda \rho(x, y), \quad \text{for all} \quad x, y \in M', \]
proving that \( \|f_M\|_{\text{Lip}(M', \ell^2_\infty)} \leq 16C\lambda \).

We have proved that for every subset \( M' \subset M \) consisting of at most four points, the restriction \( F|_{M'} \) of \( F \) to \( M' \) has a Lipschitz selection \( f_{M'} \) with Lipschitz seminorm \( \|f_{M'}\|_{\text{Lip}(M', x)} \leq 16C\lambda \). Claim 8.8 tells us that Theorem 7.12 is true for \( F \) satisfying the hypotheses of Theorem 1.14.

Thanks to this theorem and (7.29), the set-valued mapping \( F \) has a Lipschitz selection \( f : M \to \ell^2_\infty \) with Lipschitz seminorm \( \|f\|_{\text{Lip}(M, \ell^2_\infty)} \leq \gamma \). Here
\[ \gamma = 4(16C) = 64C = 2^9/\sin^5 \frac{\pi}{12} \leq 5 \cdot 10^5, \quad \text{see} \ (9.36). \]

Theorem 1.14 is completely proved. ■

9.2 Nearly optimal Lipschitz selections in the general case.

In this section we give two explicit formulae for Lipschitz seminorms of nearly optimal Lipschitz selections in \( \mathbb{R}^3 \). We formulate these results as certain constructive criteria for the existence of Lipschitz selections. These criteria easy follow from separation theorem for convex sets and Theorem 1.13 and Theorem 1.14 respectively.

We begin with the first criterion which generalizes Theorem 1.13 to the case of arbitrary set-valued mappings from a pseudometric space into the family \( \mathcal{K}(\mathbb{R}^2) \) of all convex compact subsets of \( \mathbb{R}^2 \). Let us prepare the ingredients that are needed to formulate this result. Given \( n \in S_1 \) and \( \alpha \in \mathbb{R} \), we set
\[ H(n, \alpha) = \{ a \in \mathbb{R}^2 : \langle n, a \rangle + \alpha \leq 0 \}. \]
Thus \(H(n, \alpha)\) belongs to the family \(\mathcal{HP}(\mathbb{R}^2)\) of all closed half-planes in \(\mathbb{R}^2\).

Let \(\mathfrak{M} = (\mathcal{M}, \rho)\) be a pseudometric space, and let \(F : \mathcal{M} \to \mathcal{K}(\mathbb{R}^2)\) be a set-valued mapping. Given \(x \in \mathcal{M}\), we fix a family of half-planes \(\mathcal{H}(x) \subset \mathcal{HP}(\mathbb{R}^2)\) such that

\[
F(x) = \cap \{H : H \in \mathcal{H}(x)\}. \tag{9.38}
\]

Of course, the family \(\mathcal{H}(x)\) can be defined in many ways: for instance, thanks to separation theorem, one can set \(\mathcal{H}(x) = \{H \in \mathcal{HP}(\mathbb{R}^2) : H \supset F(x)\}\). A smaller family \(\mathcal{H}(x)\) satisfying \(\text{(9.38)}\) one can define by

\[
\mathcal{H}(x) = \{H = H(n, h_{F(x)}) : n \in S_1\}.
\]

Recall that given a bounded set \(A \subset \mathbb{R}^2\), we let

\[
h_A = h_A(n) = \sup\{\langle n, a \rangle : a \in A\}
\]

denote the support function of \(A\). Thus, in this case, the above family \(\mathcal{H}(x)\) is the family of all support half-planes of the set \(F(x)\).

**Theorem 9.3** Let \(\mathfrak{M} = (\mathcal{M}, \rho)\) be a pseudometric space, and let \(F : \mathcal{M} \to \mathcal{K}(\mathbb{R}^2)\) be a set-valued mapping. This mapping has a Lipschitz selection if and only if there exists a constant \(\lambda > 0\) such that the following two conditions are satisfied:

(i) \(\text{dist}(F(x), F(y)) \leq \lambda \rho(x, y)\) for every \(x, y \in \mathcal{M}\);

(ii) Condition (\(\star 2\)) of Theorem 1.13 holds for every \(x, x', y, y' \in \mathcal{M}\) and any choice of vectors \(n(x), n(x'), n(y), n(y') \in S_1\), and numbers \(\alpha(x), \alpha(x'), \alpha(y), \alpha(y')\) such that

\[
H(n(x), \alpha(x)) \in \mathcal{H}(x), \quad H(n(x'), \alpha(x')) \in \mathcal{H}(x'), \quad H(n(y), \alpha(y)) \in \mathcal{H}(y), \quad H(n(y'), \alpha(y')) \in \mathcal{H}(y'),
\]

and \(n(x) \nparallel n(x'), n(y) \nparallel n(y')\).

Furthermore, in these settings, inequality (1.22) holds.

**Proof. (Necessity.)** Suppose \(F\) has a Lipschitz selection \(f : \mathcal{M} \to \mathbb{R}^2\) with Lipschitz seminorm

\[
\|f\|_{\text{Lip}(\mathcal{M}, \rho)} \leq \lambda. \tag{9.39}
\]

Then, given \(x, y \in \mathcal{M}\), we have

\[
f(x) \in F(x), \quad f(y) \in F(y), \quad \text{and} \quad \|f(x) - f(y)\| \leq \lambda \rho(x, y),
\]

proving property (i).

Prove property (ii). Let \(\widetilde{\mathcal{M}} = \{x, x', y, y'\}\), and let \(\widetilde{F} : \widetilde{\mathcal{M}} \to \mathcal{HP}(\mathbb{R}^2)\) be a set-valued mapping defined by

\[
\widetilde{F}(z) = H(n(z), \alpha(z)), \quad z \in \widetilde{\mathcal{M}}.
\]

Clearly, the restriction \(\tilde{f} = f_{\mathcal{M}}\) is a selection of \(\widetilde{F}\) because \(\tilde{f}(z) = f(z) \in F(z) \subseteq H(n(z), \alpha(z)) = \widetilde{F}(z)\) for each \(z \in \widetilde{\mathcal{M}}\). Furthermore, thanks to (9.39), \(\|\tilde{f}\|_{\text{Lip}(\widetilde{\mathcal{M}}, \rho')} \leq \lambda\). This enables us to apply part (ii) of Proposition 8.7 to \(\widetilde{F}\). This statement proves condition (\(\star 2\)) of Theorem 1.13 (with \(\sqrt{2} \lambda\) instead of \(\lambda\)) completing the proof of the necessity.

**Sufficiency.** We introduce a new pseudometric space \(\widetilde{\mathfrak{M}} = (\widetilde{\mathcal{M}}, \tilde{\rho})\) where

\[
\widetilde{\mathcal{M}} = \{(x, H) : x \in \mathcal{M}, H \in \mathcal{H}(x)\} \tag{9.40}
\]
and \( \tilde{\rho} : \tilde{M} \times \tilde{M} \to \mathbb{R}_+ \) is a pseudometric on \( \tilde{M} \) defined as follows: given \( (x, H), (x', H') \in \tilde{M} \), we set
\[
\tilde{\rho}(x, H), (x', H')) = \rho(x, x').
\] (9.41)

We introduce a half-plane-set valued mapping \( \tilde{F} : \tilde{M} \to \mathcal{H}(\mathbb{R}^2) \) by letting
\[
\tilde{F}(x) = H, \quad x \in M, \; H \in \mathcal{H}(x).
\] (9.42)

Prove that \( \tilde{F} \) satisfies conditions (★1) and (★2) of Theorem 1.13. We begin with condition (★1). Let \( x \in M, H = H(n(x), \alpha(x)), y \in M, H' = H(n(y), \alpha(y)) \), where \( H \in \mathcal{H}(x) \) and \( H' \in \mathcal{H}(y) \), and let \( n(y) = -n(x) \). Thus, \( \tilde{F}(x, H) = H \) and \( \tilde{F}(y, H') = H' \). Thanks to Lemma 8.6, condition (★1) of Theorem 1.13 for \( \tilde{F} \) and elements \( (x, H), (y, H') \) is equivalent to the inequality
\[
dist(H, H') \leq \lambda \tilde{\rho}(x, H), (y, H')\). (9.43)

We recall that \( \tilde{\rho}(x, H), (y, H') = \rho(x, y) \), see (9.41). We also recall that \( H \in \mathcal{H}(x) \) and \( H' \in \mathcal{H}(y) \) so that \( H \supset F(x) \) and \( H' \supset F(y) \). From this and part (i) of the present theorem, we have
\[
dist(H, H') \leq dist(F(x), F(y) \leq \lambda \rho(x, y) = \lambda \tilde{\rho}(x, H), (y, H'))
\]
proving (9.43) and condition (★1).

Prove that \( \tilde{F} \) satisfies condition (★2) of Theorem 1.13. Let \( x, x', y, y' \in M \) and let the elements \( X = (x, H(n(x), \alpha(x)), X' = (x', H(n(x'), \alpha(x'))), Y = (y, H(n(y), \alpha(y)), Y' = (y', H(n(y'), \alpha(y')) \in \tilde{M}, \)
i.e.,
\[
H(n(x), \alpha(x)) \in \mathcal{H}(x), \quad H(n(x'), \alpha(x')) \in \mathcal{H}(x'), \quad H(n(y), \alpha(y)) \in \mathcal{H}(y), \quad H(n(y'), \alpha(y')) \in \mathcal{H}(y').
\]
Assume also that \( n(x) \not\parallel n(x'), n(y) \not\parallel n(y') \). Following (8.14) we introduce a point \( w(X, Y : \tilde{F}) = (w_1(X, Y : \tilde{F}), w_2(X, Y : \tilde{F})) = \partial \tilde{F}(X) \cap \partial \tilde{F}(Y). \) (9.44)

We have to prove that
\[
(w_1(X, X' : \tilde{F}) - w_1(Y, Y' : \tilde{F}) \leq \lambda \left\{ \frac{\tilde{\rho}(X, X')}{|\Delta_n(x, x')|} \min(|n_2(x)|, |n_2(x')|) + \frac{\rho(Y, Y')}{|\Delta_n(y, y')|} \min(|n_2(y)|, |n_2(y')|) + \tilde{\rho}(X, Y) \right\}
\] (9.45)
provided condition (8.45) holds, and
\[
(w_2(X, X' : \tilde{F}) - w_2(Y, Y' : \tilde{F}) \leq \lambda \left\{ \frac{\tilde{\rho}(X, X')}{|\Delta_n(x, x')|} \min(|n_1(x)|, |n_1(x')|) + \frac{\rho(Y, Y')}{|\Delta_n(y, y')|} \min(|n_1(y)|, |n_1(y')|) + \tilde{\rho}(X, Y) \right\}
\] (9.46)
provided condition (8.46) holds. But these inequalities are immediate from assumption (ii) of the present theorem. Indeed, thanks to (9.42),
\[
\tilde{F}(X) = H(n(x), \alpha(x)), \quad \tilde{F}(X') = H(n(x'), \alpha(x')) \quad \tilde{F}(Y) = H(n(y), \alpha(y)), \quad \tilde{F}(Y') = H(n(y'), \alpha(y')).
\]
so that, thanks to (9.44),
\[ w(X, X' : \tilde{F}) = \partial H(n(x), \alpha(x)) \cap \partial H(n(x'), \alpha(x')) \quad \text{and} \quad w(Y, Y' : \tilde{F}) = \partial H(n(y), \alpha(y)) \cap \partial H(n(y'), \alpha(y')). \]

In other words, the points \( w(X, X' : \tilde{F}), w(Y, Y' : \tilde{F}) \) coincide with corresponding points
\[ w(x, x' : F) = (w_1(x, x' : F), w_2(x, x' : F)) \quad \text{and} \quad w(y, y' : F) = (w_1(y, y' : F), w_1(y, y' : F)) \]
from condition (★2) of Theorem 1.13.

Furthermore, thanks to (9.44), \( \tilde{\rho}(X, X') = \rho(x, x'), \tilde{\rho}(Y, Y') = \rho(y, y') \), and \( \tilde{\rho}(X, Y) = \rho(x, y) \) proving that the right hand sides of inequalities (1.20) and (1.21) coincide with the right hand sides of inequalities (9.45) and (9.46) respectively. Thus, inequality (9.45) coincides with inequality (1.20), and inequality (9.46) coincides with inequality (1.21). But, according to our assumption, inequalities (1.20) and (1.21) hold proving that (9.45) and (9.46) hold as well. This shows that the set-valued mapping \( \tilde{F} \) satisfies condition (★2) of Theorem 1.13.

Thus, conditions (★1) and (★2) of Theorem 1.13 hold for \( \tilde{F} \). This theorem tells us that in this case the mapping \( \tilde{F} \) has a Lipschitz selection
\[ \tilde{f} : \tilde{M} \to \ell^2_\infty \quad \text{with} \quad \|\tilde{f}\|_{\text{Lip}(\tilde{M}, \ell^2_\infty)} \leq 8\lambda. \] (9.47)

In particular, for every \( x \in M \) and any \( H, H' \in \mathcal{H}(x) \), we have
\[ \|\tilde{f}((x, H)) - \tilde{f}((x, H'))\| \leq 8\lambda \tilde{\rho}(x, H), (x, H')) = 8\lambda \rho(x, x) = 0, \]
(see [9.41]), proving that \( \tilde{f}((x, H)) = \tilde{f}((x, H')) \) for all \( H, H' \in \mathcal{H}(x) \). We let \( f(x) \) denote this common value of points \( \tilde{f}((x, H)) \), \( H \in \mathcal{H}(x) \). In other words, we define a mapping \( f : M \to \ell^2_\infty \) with the following property:
\[ f(x) = \tilde{f}((x, H)) \quad \text{for all} \quad H \in \mathcal{H}(x). \]

Prove that \( f \) is a selection of \( F \). Indeed, the mapping \( \tilde{f} \) is a selection of \( \tilde{F} \) so that for every \( H \in \mathcal{H}(x) \), we have \( f(x) = \tilde{f}((x, H)) \in \tilde{F}((x, H)) = H \). See (9.42). Hence,
\[ f(x) \in \cap\{H : H \in \mathcal{H}(x)\} = F(x) \quad \text{(see (9.38))}. \]

Furthermore, given \( x, y \in M \), \( H \in \mathcal{H}(x) \) and \( H' \in \mathcal{H}(y) \), we have
\[ \|f(x) - f(y)\| = \|\tilde{f}((x, H)) - \tilde{f}((y, H'))\| \leq 8\lambda \tilde{\rho}(x, (x, H'), (y, H')) = 8\lambda \rho(x, y) \]
proving that \( \|f\|_{\text{Lip}(M, \ell^2_\infty)} \leq 8\lambda. \)

The proof of Theorem 9.3 is complete.  

We turn to the second criterion for Lipschitz selections in \( \mathbb{R}^2 \) which generalizes Theorem 1.14 to the case of an arbitrary set-valued mapping from \( M \) into \( \mathcal{K}(\mathbb{R}^2) \).

**Theorem 9.4** Let \( \mathfrak{M} = (M, \rho) \) be a pseudometric space, and let \( F : M \to \mathcal{K}(\mathbb{R}^2) \) be a set-valued mapping.

The mapping \( F \) has a Lipschitz selection \( f : M \to \ell^2_\infty \) if and only if there exists a constant \( \lambda > 0 \) such that for every four elements \( x, x', y, y' \in M \) and every four half-planes
\[ G(x) \in \mathcal{H}(x), \quad G(x') \in \mathcal{H}(x'), \quad G(y) \in \mathcal{H}(y), \quad G(y') \in \mathcal{H}(y'), \]
(9.48)
the following inequality

\[ \text{dist}(G(x) \cap G(x'), G(y) \cap G(y')) \leq \lambda \left\{ \frac{\rho(x, x')}{\sin \varphi_G(x, x')} + \frac{\rho(y, y')}{\sin \varphi_G(y, y')} + \text{diam}_\rho\{x, x', y, y'\} \right\} \]  \tag{9.49} 

holds. Furthermore,

\[ \frac{1}{\sqrt{2}} \inf \lambda \leq |F|_{\text{lip} \ell_2^\infty} \leq \gamma \inf \lambda \]

where \( \gamma > 0 \) is an absolute constant. Here \( \varphi_G(x, x') \) (respectively \( \varphi_G(y, y') \)) denotes the angle between the boundaries of \( G(x) \) and \( G(x') \) (respectively \( G(y) \) and \( G(y') \)). See (2.3) and (9.1).

**Proof.** (Necessity.) Let \( x, x', y, y' \in \mathcal{M} \), and let \( G(x), G(x'), G(y), G(y') \in \mathcal{H}\mathcal{P}(\mathbb{R}^2) \) be half-planes satisfying condition (9.48). This condition tells us that

\[ G(x) \ni F(x), \quad G(x') \ni F(x'), \quad G(y) \ni F(y), \quad G(y') \ni F(y'). \]  \tag{9.50} 

Suppose that \( F \) has a Lipschitz selection \( f : \mathcal{M} \to \ell_2^\infty \) with \( \|f\|_{\text{lip} \ell_2^\infty} \leq \lambda \). Let \( \widetilde{\mathcal{M}} = \{x, x', y, y'\} \), and let \( \tilde{f} = f_{\widetilde{\mathcal{M}}} \). Let \( \tilde{G} : \mathcal{M} \to \mathcal{H}\mathcal{P}(\mathbb{R}^2) \) be a half-plane-set valued mapping defined by\n
\[ \tilde{G}(u) = G(u), \quad u \in \widetilde{\mathcal{M}}. \]

Thanks to (9.50), \( F(u) \subset \tilde{G}(u) \) for every \( u \in \widetilde{\mathcal{M}} \), so that \( \tilde{f} \) is a Lipschitz selection of \( \tilde{G} \) with \( \|\tilde{f}\|_{\text{lip} \ell_2^\infty} \leq \lambda \). Therefore, thanks to the necessity part of Theorem 1.14 (applied to \( \widetilde{\mathcal{M}}, \tilde{G} \) and \( \tilde{f} \)), inequality (9.49) holds (with the constant \( \sqrt{2} \lambda \) instead of \( \lambda \)), completing the proof of the necessity part of the present theorem.

(Sufficiency.) We follow the proof of the sufficiency part of Theorem 9.3. More specifically, following formulae (9.40), (9.41) and (9.42), we introduce a pseudometric space \( \mathcal{M} \) equipped with a pseudometric \( \tilde{\rho} \) and a half-plane-set valued mapping \( \tilde{F} \) defined on \( \mathcal{M} \). Then we prove that \( \tilde{F} \) satisfies the hypothesis of Theorem 1.14. Let

\[ X = (x, G(x)), \quad X' = (x', G(x')), \quad Y = (y, G(y)), \quad Y' = (y', G(y')) \in \mathcal{M}, \]

which means that \( x, x', y, y \in \mathcal{M} \) and

\[ G(x) \in \mathcal{H}(x), \quad G(x') \in \mathcal{H}(x'), \quad G(y) \in \mathcal{H}(y), \quad G(y') \in \mathcal{H}(y'). \]

Our aim is to prove that

\[ \text{dist}(\tilde{F}(X) \cap \tilde{F}(X'), \tilde{F}(Y) \cap \tilde{F}(Y')) \leq \lambda \left\{ \frac{\tilde{\rho}(X, X')}{\sin \varphi_{\tilde{F}}(X, X')} + \frac{\tilde{\rho}(Y, Y')}{\sin \varphi_{\tilde{F}}(Y, Y')} + \text{diam}_\rho\{X, X', Y, Y'\} \right\}. \]  \tag{9.51} 

The reader can easily see that this inequality is immediate from the assumption of the sufficiency, i.e., from (9.48) and (9.49). Indeed, definition (9.41) shows that the right hand side of (9.51) coincides with the right hand side of inequality (9.49). In turn, definition (9.42) tells us that the left hand sides of (9.51) and (9.49) are equal.

Thus, the hypothesis of Theorem 1.14 holds for \( \tilde{F} \). Thanks to this theorem, there exists a Lipschitz selection \( \tilde{f} : \mathcal{M} \to \ell_2^\infty \) of \( \tilde{F} \) with \( \|\tilde{f}\|_{\text{lip} \ell_2^\infty} \leq \gamma \lambda \) where \( \gamma > 0 \) is an absolute constant. Then, we literally repeat the proof of Theorem 9.3 after (9.47) (with obvious replacement of constant 8 in this proof with the constant \( \gamma \)). This proves the existence of the required Lipschitz selection \( f \) of \( F \) with Lipschitz seminorm in \( \ell_2^\infty \) at most \( \gamma \lambda \).

The proof of Theorem 9.4 is complete. \( \blacksquare \)
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