Estimating Entropy of Data Streams Using Compressed Counting

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February 6, 2009. April 14, 2009

Abstract

The Shannon entropy is a widely used summary statistic, for example, network traffic measurement, anomaly detection, neural computations, spike trains, etc. This study focuses on estimating Shannon entropy of data streams. It is known that Shannon entropy can be approximated by Rényi entropy or Tsallis entropy, which are both functions of the $\alpha$th frequency moments and approach Shannon entropy as $\alpha \to 1$. Compressed Counting (CC) [24] is a new method for approximating the $\alpha$th frequency moments of data streams. Our contributions include:

- We prove that Rényi entropy is (much) better than Tsallis entropy for approximating Shannon entropy.
- We propose the optimal quantile estimator for CC, which considerably improves the estimators in [24].
- Our experiments demonstrate that CC is indeed highly effective in approximating the moments and entropies. We also demonstrate the crucial importance of utilizing the variance-bias trade-off.

1 Introduction

The problem of “scaling up for high dimensional data and high speed data streams” is among the “ten challenging problems in data mining research” [32]. This paper is devoted to estimating entropy of data streams using a recent algorithm called Compressed Counting (CC) [24]. This work has four components: (1) the theoretical analysis of entropies, (2) a much improved estimator for CC, (3) the bias and variance in estimating entropy, and (4) an empirical study using Web crawl data.

1.1 Relaxed Strict-Turnstile Data Stream Model

While traditional data mining algorithms often assume static data, in reality, data are often constantly updated. Mining data streams [18] [4] [1] [27] in (e.g.,) 100 TB scale databases has become an important area of research, e.g., [10] [11], as network data can easily reach that scale [32]. Search engines are a typical source of data streams [4].

\footnote{Even earlier versions of this paper were submitted in 2008.
We consider the Turnstile stream model\cite{27}. The input stream $a_t = (i_t, I_t)$, $i_t \in [1, D]$ arriving sequentially describes the underlying signal $A$, meaning

$$A_t[i_t] = A_{t-1}[i_t] + I_t,$$

where the increment $I_t$ can be either positive (insertion) or negative (deletion). For example, in an online store, $A_{t-1}[i]$ may record the total number of items that user $i$ has ordered up to time $t - 1$ and $I_t$ denotes the number of items that this user orders ($I_t > 0$) or cancels ($I_t < 0$) at $t$. If each user is identified by the IP address, then potentially $D = 2^{64}$. It is often reasonable to assume $A_t[i] \geq 0$, although $I_t$ may be either negative or positive. Restricting $A_t[i] \geq 0$ results in the strict-Turnstile model, which suffices for describing almost all natural phenomena. For example, in an online store, it is not possible to cancel orders that do not exist.

Compressed Counting (CC) assumes a relaxed strict-Turnstile model by only enforcing $A_t[i] \geq 0$ at the $t$ one cares about. At other times $s \neq t$, $A_s[i]$ can be arbitrary.

1.2 Moments and Entropies of Data Streams

The $\alpha$th frequency moment is a fundamental statistic:

$$F(\alpha) = \sum_{i=1}^{D} A_t[i]^\alpha.$$  

(2)

When $\alpha = 1$, $F(1)$ is the sum of the stream. It is obvious that one can compute $F(1)$ exactly and trivially using a simple counter, because $F(1) = \sum_{i=1}^{D} A_t[i] = \sum_{s=0}^{t} I_s$. $A_t$ is basically a histogram and we can view $p_i = A_t[i] / \sum_{i=1}^{D} A_t[i] = A_t[i] / F(1)$ as probabilities. A useful (e.g., in Web and networks\cite{12, 21, 33, 26} and neural computations\cite{28}) summary statistic is Shannon entropy

$$H = - \sum_{i=1}^{D} \frac{A_t[i]}{F(1)} \log \frac{A_t[i]}{F(1)} = - \sum_{i=1}^{D} p_i \log p_i.$$ 

(4)

Various generalizations of the Shannon entropy exist. The Rényi entropy\cite{29}, denoted by $H_\alpha$, is defined as

$$H_\alpha = \frac{1}{1-\alpha} \log \left( \sum_{i=1}^{D} A_t[i]^\alpha \right)^{1/\alpha} = \frac{1}{1-\alpha} \log \sum_{i=1}^{D} p_i^\alpha.$$ 

(5)

The Tsallis entropy\cite{17, 30}, denoted by $T_\alpha$, is defined as,

$$T_\alpha = \frac{1}{\alpha - 1} \left( 1 - \frac{F(\alpha)}{F(1)} \right) = \frac{1 - \sum_{i=1}^{D} p_i^\alpha}{\alpha - 1}.$$ 

(6)

As $\alpha \to 1$, both Rényi entropy and Tsallis entropy converge to Shannon entropy:

$$\lim_{\alpha \to 1} H_\alpha = \lim_{\alpha \to 1} T_\alpha = H.$$

Thus, both Rényi entropy and Tsallis entropy can be computed from the $\alpha$th frequency moment; and one can approximate Shannon entropy from either $H_\alpha$ or $T_\alpha$ by using $\alpha \approx 1$. Several studies\cite{33, 16, 15} used this idea to approximate Shannon entropy.
1.3 Sample Applications of Shannon Entropy

1.3.1 Real-Time Network Anomaly Detection

Network traffic is a typical example of high-rate data streams. An effective and reliable measurement of network traffic in real-time is crucial for anomaly detection and network diagnosis; and one such measurement metric is Shannon entropy.\(^{[12, 20, 31, 6, 21, 33]}\) The *Turnstile* data stream model is naturally suitable for describing network traffic, especially when the goal is to characterize the statistical distribution of the traffic. In its empirical form, a statistical distribution is described by histograms, \(A_t[i], i = 1 \text{ to } D\). It is possible that \(D = 2^{64}\) (IPV6) if one is interested in measuring the traffic streams of unique source or destination.

The Distributed Denial of Service (DDoS) attack is a representative example of network anomalies. A DDoS attack attempts to make computers unavailable to intended users, either by forcing users to reset the computers or by exhausting the resources of service-hosting sites. For example, hackers may maliciously saturate the victim machines by sending many external communication requests. DDoS attacks typically target sites such as banks, credit card payment gateways, or military sites.

A DDoS attack changes the statistical distribution of network traffic. Therefore, a common practice to detect an attack is to monitor the network traffic using certain summary statics. Since Shannon entropy is a well-suited for characterizing a distribution, a popular detection method is to measure the time-history of entropy and alarm anomalies when the entropy becomes abnormal.\(^{[12, 21]}\)

Entropy measurements do not have to be “perfect” for detecting attacks. It is however crucial that the algorithm should be computationally efficient at low memory cost, because the traffic data generated by large high-speed networks are enormous and transient (e.g., 1 Gbits/second). Algorithms should be real-time and one-pass, as the traffic data will not be stored.\(^{[4]}\) Many algorithms have been proposed for “sampling” the traffic data and estimating entropy over data streams.\(^{[21, 33, 5, 14, 3, 7, 16, 15]}\)

1.3.2 Anomaly Detection by Measuring OD Flows

In high-speed networks, anomaly events including network failures and DDoS attacks may not always be detected by simply monitoring the traditional traffic matrix because the change of the total traffic volume is sometimes small. One strategy is to measure the entropies of all origin-destination (OD) flows. An OD flow is the traffic entering an ingress point (origin) and exiting at an egress point (destination).

\(^{[33]}\) showed that measuring entropies of OD flows involves measuring the intersection of two data streams, whose moments can be decomposed into the moments of individual data streams (to which CC is applicable) and the moments of the absolute difference between two data streams.

1.3.3 Entropy of Query Logs in Web Search

The recent work was devoted to estimating the Shannon entropy of MSN search logs, to help answer some basic problems in Web search, such as, *how big is the web?*

The search logs can be viewed as data streams, and analyzed several “snapshots” of a sample of MSN search logs. The sample used contained 10 million
<Query, URL,IP> triples; each triple corresponded to a click from a particular IP address on a particular URL for a particular query. 26 drew their important conclusions on this (hopefully) representative sample. Alternatively, one could apply data stream algorithms such as CC on the whole history of MSN (or other search engines).

1.3.4 Entropy in Neural Computations

A workshop in NIPS’03 was denoted to entropy estimation, owing to the wide-spread use of Shannon entropy in Neural Computations 28. (http://www.menem.com/~ilya/pages/NIPS03) For example, one application of entropy is to study the underlying structure of spike trains.

1.4 Related Work

Because the elements, $A_t[i]$, are time-varying, a naïve counting mechanism requires a system of $D$ counters to compute $F_\alpha(\alpha)$ exactly (unless $\alpha = 1$). This is not always realistic. Estimating $F_\alpha(\alpha)$ in data streams is heavily studied 2, 11, 13, 19, 23. We have mentioned that computing $F_1(\alpha)$ in strict-Turnstile model is trivial using a simple counter. One might naturally speculate that when $\alpha \approx 1$, computing (approximating) $F_\alpha(\alpha)$ should be also easy. However, before Compressed Counting (CC), none of the prior algorithms could capture this intuition.

CC improves symmetric stable random projections 19, 23 uniformly for all $0 < \alpha \leq 2$ as shown in Figure 3 in Section 4. However, one can still considerably improve CC around $\alpha = 1$, by developing better estimators, as in this study. In addition, no empirical studies on CC were reported.

[33] applied symmetric stable random projections to approximate the moments and Shannon entropy. The nice theoretical work [16, 15] provided the criterion to choose the $\alpha$ so that Shannon entropy can be approximated with a guaranteed accuracy, using the $\alpha$th frequency moment.

1.5 Summary of Our Contributions

• We prove that using Rényi entropy to estimate Shannon entropy has (much) smaller bias than using Tsallis entropy. When data follow a common Zipf distribution, the difference could be a magnitude.

• We provide a much improved estimator. CC boils down to a statistical estimation problem. The new estimator based on optimal quantiles exhibits considerably smaller variance when $\alpha \approx 1$, compared to 24.

• We demonstrate the bias-variance trade-off in estimating Shannon entropy, important for choosing the sample size and how small $|\alpha - 1|$ should be.

• We supply an empirical study.

1.6 Organization

Section 2 illustrates what entropies are like in real data. Section 3 includes some theoretical studies of entropy. The methodologies of CC and two estimators are reviewed in
Section 4 The new estimator based on the optimal quantiles is presented in Section 5. We analyze in Section 6 the biases and variances in estimating entropies. Experiments on real Web crawl data are presented in Section 7.

2 The Data Set

Since the estimation accuracy is what we are interested in, we can simply use static data instead of real data streams. This is because at the time $t$, $F(\alpha) = \sum_{i=1}^{D} A_i[i]^\alpha$ is the same at the end of the stream, regardless whether it is collected at once (i.e., static) or incrementally (i.e., dynamic).

Ten English words are selected from a chunk of Web crawl data with $D = 2^{16} = 65536$ pages. The words are selected fairly randomly, except that we make sure they cover a whole range of sparsity, from function words (e.g., A, THE), to common words (e.g., FRIDAY) to rare words (e.g., TWIST). The data are summarized in Table 1.

Table 1: The data set consists of 10 English words selected from a chunk of $D = 65536$ Web pages, forming 10 vectors of length $D$ whose values are the word occurrences. The table lists their numbers of non-zeros (sparsity), $H$, $H_\alpha$, and $T_\alpha$ (for $\alpha = 0.95$ and 1.05).

| Word  | Nonzero | $H$    | $H_{0.95}$ | $H_{1.05}$ | $T_{0.95}$ | $T_{1.05}$ |
|-------|---------|--------|------------|------------|------------|------------|
| TWIST | 274     | 5.4873 | 5.4962     | 5.4781     | 6.3256     | 4.7919     |
| RICE  | 490     | 5.4474 | 5.4997     | 5.3937     | 6.3302     | 4.7276     |
| FRIDAY| 2237    | 7.0487 | 7.1039     | 6.9901     | 8.5302     | 5.8993     |
| FUN   | 3076    | 7.6519 | 7.6821     | 7.6196     | 9.3660     | 6.3361     |
| BUSINESS | 8284   | 8.3995 | 8.4412     | 8.3566     | 10.502     | 6.8305     |
| NAME  | 9423    | 8.5162 | 8.5677     | 8.4618     | 10.696     | 6.8996     |
| HAVE  | 17522   | 8.9782 | 9.0228     | 9.0335     | 11.402     | 7.2050     |
| THIS  | 27695   | 9.3893 | 9.4370     | 9.3416     | 12.059     | 7.4634     |
| A     | 39063   | 9.5463 | 9.5981     | 9.4950     | 12.318     | 7.5592     |
| THE   | 42754   | 9.4231 | 9.4828     | 9.3641     | 12.133     | 7.4775     |

Figure 1: Two words are selected for comparing three entropies. The Shannon entropy is a constant (horizontal line).

Figure 1 selects two words to compare their Shannon entropies $H$, Rényi entropies $H_\alpha$, and Tsallis entropies $T_\alpha$. Clearly, although both approach Shannon entropy, Rényi entropy is much more accurate than Tsallis entropy.

3 Theoretical Analysis of Entropy

This section presents two Lemmas, proved in the Appendix. Lemma 1 says Rényi entropy has smaller bias than Tsallis entropy for estimating Shannon entropy.
Lemma 1

\[ |H_\alpha - H| \leq |T_\alpha - H|. \]  

(7)

Lemma 1 does not say precisely how much better. Note that when \( \alpha \to 1 \), the magnitudes of \( |H_\alpha - H| \) and \( |T_\alpha - H| \) are largely determined by the first derivatives (slopes) of \( H_\alpha \) and \( T_\alpha \), respectively, evaluated at \( \alpha \to 1 \). Lemma 2 directly compares their first and second derivatives, as \( \alpha \to 1 \).

Lemma 2 As \( \alpha \to 1 \),

\[ T'_\alpha \to \frac{1}{2} \sum_{i=1}^{D} p_i \log^2 p_i, \quad T''_\alpha \to \frac{1}{3} \sum_{i=1}^{D} p_i \log^3 p_i, \]

\[ H'_\alpha \to \frac{1}{2} \left( \sum_{i=1}^{D} p_i \log p_i \right)^2 - \frac{1}{2} \sum_{i=1}^{D} p_i \log^2 p_i, \]

\[ H''_\alpha \to \frac{1}{3} \sum_{i=1}^{D} p_i \log^2 p_i \sum_{i=1}^{D} p_i \log p_i - \frac{1}{3} \sum_{i=1}^{D} p_i \log^3 p_i. \]

Lemma 2 shows that in the limit, \( |H'_1| \leq |T'_1| \), verifying that \( H_\alpha \) should have smaller bias than \( T_\alpha \). Also, \( |H''_1| \leq |T''_1| \). Two special cases are interesting.

3.1 Uniform Data Distribution

In this case, \( p_i = \frac{1}{D} \) for all \( i \). It is easy to show that \( H_\alpha = H \) regardless of \( \alpha \). Thus, when the data distribution is close to be uniform, Rényi entropy will provide nearly perfect estimates of Shannon entropy.

3.2 Zipf Data Distribution

In Web and NLP applications, the Zipf distribution is common: \( p_i \propto \frac{1}{i^\gamma} \).

Figure 2: Ratios of the two first derivatives, for \( D = 10^3, 10^4, 10^5, 10^6, 10^7 \). The curves largely overlap and hence we do not label the curves.

Figure 2 plots the ratio \( \lim_{\alpha \to 1} \frac{T'_\alpha}{H'_\alpha} \). At \( \gamma \approx 1 \) (which is common), the ratio is about 10, meaning that the bias of Rényi entropy could be a magnitude smaller than that of Tsallis entropy, in common data sets.
4 Review Compressed Counting (CC)

Compressed Counting (CC) assumes the relaxed strict-Turnstile data stream model. Its underlying technique is based on maximally-skewed stable random projections.

4.1 Maximally-Skewed Stable Distributions

A random variable \( Z \) follows a maximally-skewed \( \alpha \)-stable distribution if the Fourier transform of its density is

\[
\mathcal{F}_Z(t) = \mathbb{E} \exp \left( \sqrt{-1} Z t \right) = \exp \left( -F|t|^{\alpha} \left( 1 - \sqrt{-1} \beta \text{sign}(t) \tan \left( \frac{\pi \alpha}{2} \right) \right) \right),
\]

where \( 0 < \alpha \leq 2 \), \( F > 0 \), and \( \beta = 1 \). We denote \( Z \sim S(\alpha, \beta = 1, F) \).

The skewness parameter \( \beta \) for general stable distributions ranges in \([-1, 1]\); but CC uses \( \beta = 1 \), i.e., maximally-skewed. Previously, the method of symmetric stable random projections\([19, 23]\) used \( \beta = 0 \).

Consider two independent variables, \( Z_1, Z_2 \sim S(\alpha, \beta = 1, 1) \). For any non-negative constants \( C_1 \) and \( C_2 \), the “\( \alpha \)-stability” follows from properties of Fourier transforms:

\[
Z = C_1 Z_1 + C_2 Z_2 \sim S(\alpha, \beta = 1, C_1^\alpha + C_2^\alpha).
\]

Note that if \( \beta = 0 \), then the above stability holds for any constants \( C_1 \) and \( C_2 \).

We should mention that one can easily generate samples from a stable distribution\([8]\).

4.2 Random Projections

Conceptually, one can generate a matrix \( R \in \mathbb{R}^{D \times k} \) and multiply it with the data stream \( A_t \), i.e., \( X = R^T A_t \in \mathbb{R}^k \). The resultant vector \( X \) is only of length \( k \). The entries of \( R \), \( r_{ij} \), are i.i.d. samples of a stable distribution \( S(\alpha, \beta = 1, 1) \).

By property of Fourier transforms, the entries of \( X \), \( x_j \) \( j = 1 \) to \( k \), are i.i.d. samples of a stable distribution

\[
x_j = \left[ R^T A_t \right]_j = \sum_{i=1}^D r_{ij} A_t[i] \sim S(\alpha, \beta = 1, F(\alpha) = \sum_{i=1}^D A_t[i]^\alpha),
\]

whose scale parameter \( F(\alpha) \) is exactly the \( \alpha \)th moment. Thus, CC boils down to a statistical estimation problem.

For real implementations, one should conduct \( R^T A_t \) incrementally. This is possible because the Turnstile model\([11]\) is a linear updating model. That is, for every incoming \( a_t = (i_t, I_t) \), we update \( x_j \leftarrow x_j + r_{ij} I_t \) for \( j = 1 \) to \( k \). Entries of \( R \) are generated on-demand as necessary.

4.3 The Efficiency in Processing Time

\[13\] commented that, when \( k \) is large, generating entries of \( R \) on-demand and multiplications \( r_{ij} I_t \), \( j = 1 \) to \( k \), can be prohibitive. An easy “fix” is to use \( k \) as small as possible, which is possible with CC when \( \alpha \approx 1 \).
At the same $k$, all procedures of CC and symmetric stable random projections are the same except the entries in $R$ follow different distributions. However, since CC is much more accurate especially when $\alpha \approx 1$, it requires a much smaller $k$ at the same level of accuracy.

4.4 Two Statistical Estimators for CC

CC boils down to estimating $F_{\alpha}$ from $k$ i.i.d. samples $x_j \sim S(\alpha, \beta = 1, F_{\alpha})$. [24] provided two estimators.

4.4.1 The Unbiased Geometric Mean Estimator

$$\hat{F}_{\alpha, gm} = \frac{\prod_{j=1}^{k} |x_j|^{\alpha/k}}{D_{gm}}$$

$$D_{gm} = \left( \cos^k \left( \frac{\kappa(\alpha)\pi}{2k} \right) / \cos \left( \frac{\kappa(\alpha)\pi}{2} \right) \right) \times \left\{ \frac{2}{\pi} \sin \left( \frac{\pi \kappa(\alpha)}{2k} \right) \Gamma \left( \frac{1-1}{k} \right) \Gamma \left( \frac{\alpha}{k} \right) \right\}^k$$

$k = \kappa(\alpha)$, if $\alpha < 1$, $\kappa(\alpha) = 2 - \alpha$ if $\alpha > 1$.

The asymptotic (i.e., as $k \to \infty$) variance is

$$\text{Var} \left( \hat{F}_{\alpha, gm} \right) = \begin{cases} \frac{F_{\alpha}^2}{k} \frac{2 \alpha^2}{(1 + \alpha)} \left( 1 - \frac{\alpha}{k} \right)^2 + O \left( \frac{1}{k^2} \right) , & \alpha < 1 \\ \frac{F_{\alpha}^2}{k} \frac{2 \alpha^2}{(1 + \alpha)} (\alpha - 1)(5 - \alpha) + O \left( \frac{1}{k^2} \right) , & \alpha > 1 \end{cases}$$

As $\alpha \to 1$, the asymptotic variance approaches zero.

4.4.2 The Harmonic Mean Estimator

$$\hat{F}_{\alpha, hm} = \frac{k \cos \left( \frac{\pi \alpha}{1 + \alpha} \right)}{\sum_{j=1}^{k} \left| x_j \right|^{-\alpha}} \left( 1 - \frac{1}{k} \left( \frac{2\Gamma^2(1 + \alpha)}{\Gamma(1 + 2\alpha)} - 1 \right) \right)$$

which is asymptotically unbiased and has variance

$$\text{Var} \left( \hat{F}_{\alpha, hm} \right) = \frac{F_{\alpha}^2}{k} \left( \frac{2\Gamma^2(1 + \alpha)}{\Gamma(1 + 2\alpha)} - 1 \right) + O \left( \frac{1}{k^2} \right).$$

$\hat{F}_{\alpha, hm}$ is defined only for $\alpha < 1$ and is considerably more accurate than the geometric mean estimator $\hat{F}_{\alpha, gm}$.

5 The Optimal Quantile Estimator

The two estimators for CC in [24] dramatically reduce the estimation variances compared to symmetric stable random projections. They are, however, not quite adequate for estimating Shannon entropy using small ($k$) samples.

We discover that an estimator based on the sample quantiles considerably improves [24] when $\alpha \approx 1$. Given $k$ i.i.d samples $x_j \sim S(\alpha, 1, F_{\alpha})$, we define the $q$-quantile is to be the $(q \times k)$th smallest of $|x_j|$. For example, when $k = 100$, then $q = 0.01$th quantile is the smallest among $|x_j|$'s.
To understand why the quantile works, consider the normal \( x_j \sim N(0, \sigma^2) \), which is a special case of stable distribution with \( \alpha = 2 \). We can view \( x_j = \sigma z_j \), where \( z_j \sim N(0, 1) \). Therefore, we can use the ratio of the \( q \)-th quantile of \( x_j \) over the \( q \)-th quantile of \( N(0, 1) \) to estimate \( \sigma \). Note that \( F(\alpha) \) corresponds to \( \sigma^2 \), not \( \sigma \).

### 5.1 The General Quantile Estimator

Assume \( x_j \sim S(\alpha, 1, F(\alpha)) \), \( j = 1 \) to \( k \). One can sort \(|x_j|\) and use the \((q \times k)\)th smallest \(|x_j|\) as the estimate, i.e.,

\[
\hat{F}(\alpha,q) = \left( \frac{q\text{-Quantile}[|x_j|, j = 1, 2, ..., k]}{W_q} \right)^{\alpha},
\]

\( W_q = q\text{-Quantile}[|S(\alpha, \beta = 1, 1)|] \).

Denote \( Z = |X| \), where \( X \sim S(\alpha, 1, F(\alpha)) \). Denote the probability density function of \( Z \) by \( f_Z(z; \alpha, F(\alpha)) \), the probability cumulative function by \( F_Z(z; \alpha, F(\alpha)) \), and the inverse by \( F_Z^{-1}(q; \alpha, F(\alpha)) \). The asymptotic variance of \( \hat{F}(\alpha,q) \) is presented in Lemma 3, which follows directly from known statistics results, e.g., [9, Theorem 9.2].

#### Lemma 3

\[
\text{Var}(\hat{F}(\alpha,q)) = \frac{F_Z(\alpha)}{k} \frac{(q - q^2)\alpha^2}{f_Z^2(F_Z^{-1}(q; \alpha, 1); \alpha, 1)} \left( \frac{1}{k} \right) + O\left( \frac{1}{k^2} \right).
\]

We can then choose \( q = q^* \) to minimize the asymptotic variance.

### 5.2 The Optimal Quantile Estimator

We denote the optimal quantile estimator by \( \hat{F}(\alpha),oq = \hat{F}(\alpha),q^* \). The optimal quantiles, denoted by \( q^* = q^*(\alpha) \), has to be determined by numerically and tabulated (as in Table 2), because the density functions do not have an explicit closed-form. We used the `fBasics` package in R. We, however, found the implementation of those functions had numerical problems when \( 1 < \alpha < 1.011 \) and \( 0.989 < \alpha < 1 \). Table 2 provides the numerical values for \( q^* \), \( W_{q^*} \) (13), and the variance of \( \hat{F}(\alpha),oq \) (without the \( \frac{1}{k} \) term).

### Table 2: In order to use the optimal quantile estimator, we tabulate the constants \( q^* \) and \( W_{q^*} \).

| \( \alpha \) | \( q^* \) | \( \text{Var} \) | \( W_{q^*} \) |
|---|---|---|---|
| 0.90 | 0.101 | 0.04116676 | 5.400842 |
| 0.95 | 0.098 | 0.01059831 | 1.174723 |
| 0.96 | 0.097 | 0.0099821384 | 14.92508 |
| 0.97 | 0.096 | 0.00899153 | 20.22440 |
| 0.98 | 0.0944 | 0.00727439 | 30.82616 |
| 0.989 | 0.0941 | 0.005243589 | 56.86694 |
| 1.011 | 0.8904 | 0.003554749 | 58.83961 |
| 1.02 | 0.8799 | 0.001901498 | 32.76912 |
| 1.03 | 0.8699 | 0.004241489 | 22.12097 |
| 1.04 | 0.8601 | 0.008093239 | 16.80970 |
| 1.05 | 0.855 | 0.01298757 | 13.61799 |
| 1.20 | 0.799 | 0.2516604 | 4.011459 |


5.3 Comparisons of Asymptotic Variances

Figure 3 (left panel) compares the variances of the three estimators for CC. To better illustrate the improvements, Figure 3 (right panel) plots the ratios of the variances. When \( \alpha = 0.989 \), the optimal quantile reduces the variances by a factor of 70 (compared to the geometric mean estimator), or 20 (compared to the harmonic mean estimator).

\[
\text{Figure 3: Let } \hat{F} \text{ be an estimator of } F \text{ with asymptotic variance } \text{Var} \left( \hat{F} \right) = V \frac{k^2}{k} + O \left( \frac{1}{k} \right). \text{ The left panel plots the } V \text{ values for the geometric mean estimator, the harmonic mean estimator (for } \alpha < 1 \text{), and the optimal quantile estimator, along with the } V \text{ values for the geometric mean estimator for symmetric stable random projections in } [23] \text{ ("symmetric GM"). The right panel plots the ratios of the variances to better illustrate the significant improvement of the optimal quantile estimator, near } \alpha = 1. \]

6 Estimating Shannon Entropy, the Bias and Variance

This section analyzes the biases and variances in estimating Shannon entropy. Also, we provide the criterion for choosing the sample size \( k \).

We use \( \hat{H}_\alpha, \hat{H}_\alpha, \text{ and } \hat{T}_\alpha \) to denote generic estimators.

\[
\hat{H}_\alpha = \frac{1}{1 - \alpha} \log \frac{\hat{F}_\alpha(1)}{F_\alpha}, \quad \hat{T}_\alpha = \frac{1}{\alpha - 1} \left( 1 - \frac{\hat{F}_\alpha(1)}{F_\alpha} \right), \quad (15)
\]

Since \( \hat{F}_\alpha(1) \) is (asymptotically) unbiased, \( \hat{H}_\alpha \) and \( \hat{T}_\alpha \) are also asymptotically unbiased. The asymptotic variances of \( \hat{H}_\alpha \) and \( \hat{T}_\alpha \) can be computed by Taylor expansions:

\[
\text{Var} \left( \hat{H}_\alpha \right) = \frac{1}{(1 - \alpha)^2} \text{Var} \left( \log \left( \hat{F}_\alpha(1) \right) \right)
= \frac{1}{(1 - \alpha)^2} \text{Var} \left( \frac{\partial \log F_\alpha}{\partial F_\alpha} \right) \left( \frac{\partial F_\alpha}{\partial F_\alpha} \right)^2 + O \left( \frac{1}{k^2} \right)
= \frac{1}{(1 - \alpha)^2} \frac{1}{F_\alpha^2} \text{Var} \left( \hat{F}_\alpha \right) + O \left( \frac{1}{k^2} \right), \quad (16)
\]

\[
\text{Var} \left( \hat{T}_\alpha \right) = \frac{1}{(\alpha - 1)^2} \frac{1}{F_\alpha^2} \text{Var} \left( \hat{F}_\alpha \right) + O \left( \frac{1}{k^2} \right). \quad (17)
\]

We use \( \hat{H}_{\alpha,R} \) and \( \hat{H}_{\alpha,T} \) to denote the estimators for Shannon entropy using the estimated \( \hat{H}_\alpha \) and \( \hat{T}_\alpha \), respectively. The variances remain unchanged, i.e.,

\[
\text{Var} \left( \hat{H}_{\alpha,R} \right) = \text{Var} \left( \hat{H}_\alpha \right), \quad \text{Var} \left( \hat{H}_{\alpha,T} \right) = \text{Var} \left( \hat{T}_\alpha \right). \quad (18)
\]
However, $\hat{H}_{\alpha,R}$ and $\hat{H}_{\alpha,T}$ are no longer (asymptotically) unbiased, because

$$\text{Bias} \left( \hat{H}_{\alpha,R} \right) = E \left( \hat{H}_{\alpha,R} - H \right) = H_{\alpha} - H + O \left( \frac{1}{k} \right),$$

(19)

$$\text{Bias} \left( \hat{H}_{\alpha,T} \right) = E \left( \hat{T}_{\alpha,R} - H \right) = T_{\alpha} - H + O \left( \frac{1}{k} \right).$$

(20)

The $O \left( \frac{1}{k} \right)$ biases arise from the estimation biases and diminish quickly as $k$ increases. However, the “intrinsic biases,” $H_{\alpha} - H$ and $T_{\alpha} - H$, can not be reduced by increasing $k$; they can only be reduced by letting $\alpha$ close to 1.

The total error is usually measured by the mean square error: $\text{MSE} = \text{Bias}^2 + \text{Var}$. Clearly, there is a variance-bias trade-off in estimating $H$ using $H_{\alpha}$ or $T_{\alpha}$. The optimal $\alpha$ is data-dependent and hence some prior knowledge of the data is needed in order to determine it. The prior knowledge may be accumulated during the data stream process.

7 Experiments

Experiments on real data (i.e., Table 1) can further demonstrates the effectiveness of Compressed Counting (CC) and the new optimal quantile estimator. We could use static data to verify CC because we only care about the estimation accuracy, which is same regardless whether the data are collected at one time (static) or dynamically.

We present the results for estimating frequency moments and Shannon entropy, in terms of the normalized MSEs. We observe that the results are quite similar across different words; and hence only one word is selected for the presentation.

7.1 Estimating Frequency Moments

Figure 4 provides the normalized MSEs (by $F^2_{\alpha}$) for estimating the $\alpha$th frequency moments, $F_{\alpha}$, for word RICE:

- The errors of the three estimators for CC decrease (to zero, potentially) as $\alpha \to 1$. The improvement of CC over symmetric stable random projections is enormous.
- The optimal quantile estimator $\hat{F}_{\alpha,\text{eq}}$ is in general more accurate than the geometric mean and harmonic mean estimators near $\alpha = 1$. However, for small $k$ and $\alpha > 1$, $\hat{F}_{\alpha,\text{eq}}$ exhibits bad behaviors, which disappear when $k \geq 50$.
- The theoretical asymptotic variances in (9), (11), and Table 2 are accurate.

7.2 Estimating Shannon Entropy

Figure 5 provides the MSEs from estimating the Shannon entropy using the Rényi entropy, for word RICE:

- Using symmetric stable random projections with $\alpha$ close to 1 is not a good strategy and not practically feasible because the required sample size is enormous.
- There is clearly a variance-bias trade-off, especially for the geometric mean and harmonic mean estimators. That is, for each $k$, there is an “optimal” $\alpha$ which achieves the smallest MSE.
Using the optimal quantile estimator does not show a strong variance-bias trade-off, because its has very small variance near $\alpha = 1$ and its MSEs are mainly dominated by the (intrinsic) biases, $H_\alpha - H$.

Figure 6 presents the MSEs for estimating Shannon entropy using Tsallis entropy. The effect of the variance-bias trade-off for geometric mean and harmonic mean estimators, is even more significant, because the (intrinsic) bias $T_\alpha - H$ is much larger.

### 8 Conclusion

Web search data and Network data are naturally data streams. The entropy is a useful summary statistic and has numerous applications, e.g., network anomaly detection. Efficiently and accurately computing the entropy in large and frequently updating data streams, in one-pass, is an active topic of research. A recent trend is to use the $\alpha$th frequency moments with $\alpha \approx 1$ to approximate Shannon entropy. We conclude:

- We should use Rényi entropy to approximate Shannon entropy. Using Tsallis entropy will result in about a magnitude larger bias in a common data distribution.

- The optimal quantile estimator for CC reduces the variances by a factor of 20 or 70 when $\alpha = 0.989$, compared to the estimators in [24].

- When symmetric stable random projections must be used, we should exploit the variance-bias trade-off, by not using $\alpha$ very close 1.
A Proof of Lemma 1

\[ T_\alpha = \frac{1}{\alpha} \sum_{i=1}^{D} p_i^\alpha \] and \( H_\alpha = -\log \sum_{i=1}^{D} p_i^\alpha \). Note that \( \sum_{i=1}^{D} p_i = 1, \sum_{i=1}^{D} p_i^\alpha \geq 1 \) if \( \alpha < 1 \) and \( \sum_{i=1}^{D} p_i^\alpha \leq 1 \) if \( \alpha > 1 \).

For \( t > 0, -\log(t) \leq 1 - t \) always holds, with equality when \( t = 1 \). Therefore, \( H_\alpha \leq T_\alpha \) when \( \alpha < 1 \) and \( H_\alpha \geq T_\alpha \) when \( \alpha > 1 \). Also, we know \( \lim_{\alpha \to 1} T_\alpha = \lim_{\alpha \to 1} H_\alpha = H \). Therefore, to show \( |H_\alpha - H| \leq |T_\alpha - H| \), it suffices to show that both \( T_\alpha \) and \( H_\alpha \) are decreasing functions of \( \alpha \in (0, 2) \).

Taking the first derivatives of \( T_\alpha \) and \( H_\alpha \) yields

\[ T'_\alpha = \frac{\sum_{i=1}^{D} p_i^\alpha - (\alpha - 1) \sum_{i=1}^{D} p_i^{\alpha-1} \log p_i}{(\alpha - 1)^2} = \frac{A_\alpha}{(\alpha - 1)^2} \]

\[ H'_\alpha = \frac{\sum_{i=1}^{D} p_i^\alpha \log \sum_{i=1}^{D} p_i^\alpha - (\alpha - 1) \sum_{i=1}^{D} p_i^{\alpha-1} \log p_i}{(\alpha - 1)^2 \sum_{i=1}^{D} p_i^\alpha} = \frac{B_\alpha}{(\alpha - 1)^2} \]

To show \( T'_\alpha \leq 0 \), it suffices to show that \( A_\alpha \leq 0 \). Taking derivative of \( A_\alpha \) yields,

\[ A'_\alpha = - (\alpha - 1) \sum_{i=1}^{D} p_i^{\alpha-1} \log^2 p_i, \]

i.e., \( A'_\alpha \geq 0 \) if \( \alpha < 1 \) and \( A'_\alpha \leq 0 \) if \( \alpha > 1 \). Because \( A_1 = 0 \), we know \( A_\alpha \leq 0 \). This proves \( T'_\alpha \leq 0 \). To show \( H'_\alpha \leq 0 \), it suffices to show that \( B_\alpha \leq 0 \), where

\[ B_\alpha = \log \sum_{i=1}^{D} p_i^\alpha + \sum_{i=1}^{D} q_i \log p_i^{\alpha-1}, \text{ where } q_i = \frac{p_i^\alpha}{\sum_{i=1}^{D} p_i^\alpha}. \]

Note that \( \sum_{i=1}^{D} q_i = 1 \) and hence we can view \( q_i \) as probabilities. Since \( \log() \) is a
concave function, we can use Jensen’s inequality: $E \log(X) \leq \log E(X)$, to obtain

$$B_\alpha \leq \log \sum_{i=1}^{D} p_i^\alpha + \log \sum_{i=1}^{D} q_i 1 - \alpha = \log \sum_{i=1}^{D} p_i^\alpha + \log \sum_{i=1}^{D} p_i^{1-\alpha} = 0.$$ 

### B Proof of Lemma 2

As $\alpha \to 1$, using L’Hôpital’s rule

$$\lim_{\alpha \to 1} T_\alpha' = \lim_{\alpha \to 1} \left[ \frac{\sum_{i=1}^{D} p_i^\alpha - 1 - (\alpha - 1) \sum_{i=1}^{D} p_i^{1-\alpha} \log p_i}{((\alpha - 1)^2)} \right]'$$

$$= \lim_{\alpha \to 1} -\frac{(\alpha - 1) \sum_{i=1}^{D} p_i^{1-\alpha} \log p_i}{2(\alpha - 1)} = -\frac{1}{2} \sum_{i=1}^{D} p_i \log^2 p_i.$$ 

Note that, as $\alpha \to 1$, $\sum_{i=1}^{D} p_i^{1-\alpha}$ converges to $1 - \alpha$ but $\sum_{i=1}^{D} p_i^{1-\alpha} \log^n p_i \to \sum_{i=1}^{D} p_i log^n p_i \neq 0$.

Taking the second derivative of $T_\alpha = \frac{1}{\alpha - 1} \sum_{i=1}^{D} p_i^{1-\alpha}$ yields

$$T_\alpha'' = \frac{2 - 2 \sum_{i=1}^{D} p_i^{1-\alpha} + 2(\alpha - 1) \sum_{i=1}^{D} p_i^{1-\alpha} \log p_i - (\alpha - 1)^2 \sum_{i=1}^{D} p_i^{1-\alpha} \log^2 p_i}{(\alpha - 1)^3}.$$ 

Using L’Hôpital’s rule yields,

$$\lim_{\alpha \to 1} T_\alpha'' = \lim_{\alpha \to 1} -\frac{(\alpha - 1)^2 \sum_{i=1}^{D} p_i^{1-\alpha} \log^3 p_i}{3(\alpha - 1)^2} = -\frac{1}{3} \sum_{i=1}^{D} p_i \log^3 p_i,$$

where we skip the algebra which cancel some terms.

![Figure 6: Shannon entropy, $H$, estimated from Tsallis entropy, $T_\alpha$, for RICE.](image)
Again, applying L’Hospital’s rule yields the expressions for \( \lim_{\alpha \to 1} H'_\alpha \) and \( \lim_{\alpha \to 1} H''_\alpha \):

\[
\lim_{\alpha \to 1} H'_\alpha = \lim_{\alpha \to 1} \left[ \frac{\sum_{i=1}^{D} p_i^\alpha \log \sum_{i=1}^{D} p_i^\alpha - (\alpha - 1) \sum_{i=1}^{D} p_i^\alpha \log p_i}{[(\alpha - 1)^2 \sum_{i=1}^{D} p_i^\alpha]^2} \right] = \lim_{\alpha \to 1} \left[ \frac{\sum_{i=1}^{D} p_i^\alpha \log p_i \log \sum_{i=1}^{D} p_i^\alpha - (\alpha - 1) \sum_{i=1}^{D} p_i^\alpha \log^2 p_i}{2(\alpha - 1) \sum_{i=1}^{D} p_i^\alpha + (\alpha - 1)^2 \sum_{i=1}^{D} p_i^\alpha \log p_i} \right] = \lim_{\alpha \to 1} \left( \frac{\sum_{i=1}^{D} p_i^\alpha \log p_i}{\sum_{i=1}^{D} p_i^\alpha} \right)^2 - \sum_{i=1}^{D} \left[ \frac{p_i^\alpha \log p_i}{p_i^\alpha} \right] + \text{negligible terms} = \frac{1}{2} \left( \sum_{i=1}^{D} p_i \log p_i \right)^2 - \frac{1}{2} \sum_{i=1}^{D} p_i \log^2 p_i
\]

We skip the details for proving the limit of \( H''_\alpha \), where

\[
H''_\alpha = -\frac{C_\alpha}{(\alpha - 1)^3 \left( \sum_{i=1}^{D} p_i^\alpha \right)^2} \tag{21}
\]

\[
C_\alpha = -2(\alpha - 1)^2 \sum_{i=1}^{D} p_i^\alpha \log^2 p_i \sum_{i=1}^{D} p_i^\alpha - 2 \left( \sum_{i=1}^{D} p_i^\alpha \right)^2 \log \sum_{i=1}^{D} p_i^\alpha + (\alpha - 1)^2 \left( \sum_{i=1}^{D} p_i^\alpha \log p_i \right)^2 + 2(\alpha - 1) \sum_{i=1}^{D} p_i^\alpha \log p_i \sum_{i=1}^{D} p_i^\alpha
\]

References

[1] C. C. Aggarwal, J. Han, J. Wang, and P. S. Yu. On demand classification of data streams. In KDD, pages 503–508, Seattle, WA, 2004.

[2] N. Alon, Y. Matias, and M. Szegedy. The space complexity of approximating the frequency moments. In STOC, pages 20–29, Philadelphia, PA, 1996.

[3] K. D. B. Amit Chakrabarti and S. Muthukrishnan. Estimating entropy and entropy norm on data streams. Internet Mathematics, 3(1):63–78, 2006.

[4] B. Babcock, S. Babu, M. Datar, R. Motwani, and J. Widom. Models and issues in data stream systems. In PODS, pages 1–16, Madison, WI, 2002.

[5] L. Bhuvanagiri and S. Ganguly. Estimating entropy over data streams. In ESA, pages 148–159, 2006.

[6] D. Brauckhoff, B. Tellenbach, A. Wagner, M. May, and A. Lakhina. Impact of packet sampling on anomaly detection metrics. In IMC, pages 159–164, 2006.

[7] A. Chakrabarti, G. Cormode, and A. McGregor. A near-optimal algorithm for computing the entropy of a stream. In SODA, pages 328–335, 2007.

[8] J. M. Chambers, C. L. Mallows, and B. W. Stuck. A method for simulating stable random variables. Journal of the American Statistical Association, 71(354):340–344, 1976.

[9] H. A. David. Order Statistics. John Wiley & Sons, Inc., New York, NY, 1981.

[10] C. Domeniconi and D. Gunopulos. Incremental support vector machine construction. In ICDM, pages 589–592, San Jose, CA, 2001.

[11] J. Feigenbaum, S. Kannan, M. Strauss, and M. Viswanathan. An approximate \( l_1 \)-difference algorithm for massive data streams. In FOCS, pages 501–511, New York, 1999.
[12] L. Feinstein, D. Schnackenberg, R. Balupari, and D. Kindred. Statistical approaches to DDoS attack detection and response. In *DARPA Information Survivability Conference and Exposition*, pages 303–314, 2003.

[13] S. Ganguly and G. Cormode. On estimating frequency moments of data streams. In *APPROX-RANDOM*, pages 479–493, Princeton, NJ, 2007.

[14] S. Guha, A. McGregor, and S. Venkatasubramanian. Streaming and sublinear approximation of entropy and information distances. In *SODA*, pages 733 – 742, Miami, FL, 2006.

[15] N. J. A. Harvey, J. Nelson, and K. Onak. Sketching and streaming entropy via approximation theory. In *FOCS*, 2008.

[16] N. J. A. Harvey, J. Nelson, and K. Onak. Streaming algorithms for estimating entropy. In *ITW*, 2008.

[17] M. E. Havrda and F. Charvát. Quantification methods of classification processes: Concept of structural $\alpha$-entropy. *Kybernetika*, 3:30–35, 1967.

[18] M. R. Henzinger, P. Raghavan, and S. Rajagopalan. *Computing on Data Streams*. 1999.

[19] P. Indyk. Stable distributions, pseudorandom generators, embeddings, and data stream computation. *Journal of ACM*, 53(3):307–323, 2006.

[20] A. Lakhina, M. Crovella, and C. Diot. Mining anomalies using traffic feature distributions. In *SIGCOMM*, pages 217–228, Philadelphia, PA, 2005.

[21] A. Lall, V. Sekar, M. Ogihara, J. Xu, and H. Zhang. Data streaming algorithms for estimating entropy of network traffic. In *SIGMETRICS*, pages 145–156, 2006.

[22] P. Li. Computationally efficient estimators for dimension reductions using stable random projections. In *ICDM*, Pisa, Italy, 2008.

[23] P. Li. Estimators and tail bounds for dimension reduction in $l_\alpha$ ($0 < \alpha \leq 2$) using stable random projections. In *SODA*, pages 10 – 19, San Francisco, CA, 2008.

[24] P. Li. Compressed counting. In *SODA*, New York, NY, 2009.

[25] C. D. Manning and H. Schutze. *Foundations of Statistical Natural Language Processing*. The MIT Press, Cambridge, MA, 1999.

[26] Q. Mei and K. Church. Entropy of search logs: How hard is search? with personalization? with backoff? In *WSDM*, pages 45 – 54, Palo Alto, CA, 2008.

[27] S. Muthukrishnan. Data streams: Algorithms and applications. *Foundations and Trends in Theoretical Computer Science*, 1:117–236, 2 2005.

[28] L. Paninski. Estimation of entropy and mutual information. *Neural Comput.*, 15(6):1191–1253, 2003.

[29] A. Rényi. On measures of information and entropy. In *The 4th Berkeley Symposium on Mathematics, Statistics and Probability* 1960, pages 547–561, 1961.

[30] C. Tsallis. Possible generalization of boltzmann-gibbs statistics. *Journal of Statistical Physics*, 52:479–487, 1988.

[31] K. Xu, Z.-L. Zhang, and S. Bhattacharyya. Profiling internet backbone traffic: behavior models and applications. In *SIGCOMM ’05: the conference on Applications, technologies, architectures, and protocols for computer communications*, pages 169–180, 2005.

[32] Q. Yang and X. Wu. 10 challeng problems in data mining research. *International Journal of Information Technology and Decision Making*, 5(4):597–604, 2006.

[33] H. Zhao, A. Lall, M. Ogihara, O. Spatscheck, J. Wang, and J. Xu. A data streaming algorithm for estimating entropies of od flows. In *IMC*, San Diego, CA, 2007.

[34] V. M. Zolotarev. *One-dimensional Stable Distributions*. 1986.