Algorithms to solve the Sutherland model

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Abstract

We give a self-contained presentation and comparison of two different algorithms to explicitly solve quantum many body models of indistinguishable particles moving on a circle and interacting with two-body potentials of $1/\sin^2$-type. The first algorithm is due to Sutherland and well-known; the second one is a limiting case of a novel algorithm to solve the elliptic generalization of the Sutherland model. These two algorithms are different in several details. We show that they are equivalent, i.e., they yield the same solution and are equally simple.

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1 Introduction

We recently presented a novel algorithm to solve the quantum version of the elliptic Calogero-Moser-Sutherland system [L1, L2]. In the trigonometric limit, such an algorithm was discovered already about thirty years ago by Sutherland [Su1, Su2]. Somewhat surprisingly, the former algorithm in that limit reduces to one which is different from Sutherland’s, even though it yields the same solution and is equally simple. The purpose of this paper is to give a detailed and self-contained comparison of these two algorithms, including a proof of their equivalence.

The Sutherland model is defined by the differential operator

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2\lambda(\lambda - 1) \sum_{1 \leq j < k \leq N} V(x_j - x_k)$$  \hspace{1cm} (1)
with $-\pi \leq x_j \leq \pi$, $N = 2, 3, \ldots$, $\lambda > 0$, and

$$V(r) = \frac{1}{4\sin^2(r/2)}.$$  \hspace{1cm} (2)

This differential operator defines a self-adjoint operator on the Hilbert space of square integrable functions on $[-\pi, \pi]^N$, providing a quantum mechanical model for $N$ indistinguishable particles moving on a circle of length $2\pi$ and interacting with a two body potential proportional to $V(r)$ where $\lambda$ determines the coupling strength. (To be precise: This model corresponds to a particularly nice self-adjoint extension of this differential operator which, for $\lambda > 1$, corresponds to the Friedrich’s extension \cite{RS}.) To solve this model amounts to constructing a complete set of eigenfunctions and corresponding eigenvalues of this Hamiltonian.

The starting point for Sutherland’s algorithm is the following

**Fact 1** \cite{Su1}: *The wave function*

$$\Psi_0(x) = \prod_{1 \leq j < k \leq N} \psi(x_k - x_j)^\lambda$$  \hspace{1cm} (3)

with

$$\psi(r) = \sin(r/2)$$  \hspace{1cm} (4)

is the ground state of the Sutherland Hamiltonian, $H\Psi_0 = E_0\Psi_0$.

Exploiting this fact, Sutherland constructed all other eigenfunctions $f$ using the following ansatz,

$$f(x) = \Psi_0(x)\Phi(x)$$  \hspace{1cm} (5)

where $\Phi$ are symmetric polynomials (i.e. non-negative powers) in the variables $z_j = \exp(ix_j)$ \cite{Su2}. The symmetric polynomials thus obtained are the so-called Jack polynomials which have been studied extensively in the mathematics literature, see e.g. \cite{McD, St}.

Our algorithm is based on the following

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$^1$To ease notation, we set the length of space to $2\pi$ from the start. Of course, an arbitrary length $L > 0$ can be easily introduced by rescaling $x_j \rightarrow (2\pi/L)x_j$, $H \rightarrow (2\pi/L)^2H$, etc.

$^2$To fix the phase of $\Psi_0$ unambiguously one can interpret $\sin(r/2)^\lambda$ as $\lim_{\varepsilon \downarrow 0} \sin(r/2 + i\varepsilon)^\lambda$, for example. Anyway, the phase ambiguities associated with the exponentiated sines are irrelevant here. In Appendix B.1 we will have to be more careful about similar phase ambiguities in the functions $F(x; y)$ defined below.
Fact 2 \([\text{L}1]\): The function

\[ F(x; y) = \frac{\prod_{1 \leq j < k \leq N} \psi(x_k - x_j)^\lambda \prod_{1 \leq j < k \leq N} \psi(y_j - y_k)^\lambda}{\prod_{j,k=1}^N \psi(x_j - y_k)^\lambda}, \tag{6} \]

\( \psi(r) \) as in Eq. (4), obeys the following identity,

\[ \sum_{j=1}^N \left( \frac{\partial^2}{\partial x_j^2} - \frac{\partial^2}{\partial y_j^2} \right) F(x; y) = 2\lambda(\lambda - 1) \sum_{1 \leq j < k \leq N} \left( V(x_k - x_j) - V(y_j - y_k) \right) F(x; y) \tag{7} \]

with \( V(r) \) as in Eq. (3).

Note that we can write this latter identity as

\[ H(x)F(x; y) = H(y)F(x; y) \tag{8} \]

where \( H \) is the differential operator in Eq. (1) but acting on different arguments \( x \) and \( y \), as indicated. The idea of our algorithm is to take the Fourier transform of Eq. (8) with respect to the variables \( y \), and this yields an identity allowing to construct eigenfunctions and the corresponding eigenvalues (Proposition 1).

It is interesting to note that Fact 2 holds true in the elliptic case as well (in this case, \( \psi(r) \) is a Jacobi Theta function \( \vartheta_1(r/2) \) with nome \( q = \exp(-\beta/2) \) and \( V(r) \) is Weierstrass’ elliptic function \( \wp(r) \) with periods \( 2\pi \) and \( i\beta \) \([\text{L}1]\), in contrast to Fact 1 \([\text{Su}3]\). For the convenience of the reader, an elementary proof of Fact 2 (in the trigonometric case) is given in Appendix A. (This proof uses Fact 1; a self-contained proof valid also in the elliptic case will be given in \([\text{L}2]\).)

The plan of the rest of this paper is as follows. In Section 2 we review the Sutherland algorithm \([\text{Su}2]\), mainly to introduce our notation. Section 3 contains a detailed description of our algorithm. In the final Section 4 we give the arguments which prove that both algorithms are equivalent, despite of various differences. Lengthy proofs are deferred to two Appendices.

2 Sutherland’s algorithm

We use \( H\Psi_0 = E_0\Psi_0 \) with \([\text{Su}1]\)

\[ E_0 = \frac{1}{12} \lambda^2 N(N^2 - 1) \tag{9} \]
and make the ansatz \( f = \Phi \Psi_0 \). With that the eigenvalue equation \( H f = E f \) becomes \( H' \Phi = E' \Phi \) with \( E' = E - E_0 \) and

\[
H' = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} - i\lambda \sum_{1 \leq j < k \leq N} \left( \frac{e^{ix_j} + e^{ix_k}}{e^{ix_j} - e^{ix_k}} \right) \left( \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k} \right).
\]

One now determines the action of \( H' \) on symmetric polynomials

\[
S_n(x) = \sum_{P} \prod_{j=1}^{N} e^{i n_j x P_j}
\]

where

\[
n_1 \geq n_2 \geq \ldots \geq n_N \geq 0
\]

and the sum is over all permutations \( P \) of \( \{1, 2, \ldots, N\} \). Using the identity \( e^{i x} + e^{i y} \left( \frac{e^{ikx} - e^{iky}}{e^{ix} - e^{iy}} \right) = e^{ikx} + e^{iky} + 2 \sum_{\nu=1}^{k-1} e^{i(k-\nu)x+\nu y} \)

for \( k > 0 \), one obtains

\[
H'S_n = E'_n S_n + \lambda \sum_{1 \leq j < k \leq N} (n_j - n_k) \sum_{\nu=1}^{n_j-n_k-1} S_{n-\nu} E_{jk}
\]

where we introduced the notation

\[
(E_{jk})_{\ell} = \delta_{j\ell} - \delta_{k\ell}, \quad j, k, \ell = 1, 2, \ldots, N
\]

and defined

\[
E'_n = \sum_{j=1}^{N} n_j^2 + \lambda \sum_{1 \leq j < k \leq N} (n_j - n_k) = \sum_{j=1}^{N} \left( n_j^2 + \lambda [N + 1 - 2j]n_j \right).
\]

We now introduce the notation

\[
\mu = \sum_{1 \leq j < k \leq N} \mu_{jk} E_{jk}
\]

\[\text{Note that } n_j \text{ in Ref. Su2 corresponds to } n_{N+1-j} \text{ here.} \]
for non-negative integers \( \mu_{jk} \), and observe that there is a natural order on the set of all these \( \mu \) (which we can identify with \( N_0^{N(N-1)/2} \)),

\[
\mu < \mu' \quad \text{if} \quad \mu_{jk} < \mu'_{jk} \quad \text{for all} \quad j < k.
\]

(17)

It is obvious that \( H'S_n \) is a finite linear combination of symmetrized plane waves \( S_{n-\mu} \) with \( \mu \geq 0 \). We thus can make the following ansatz for the eigenfunctions of \( H' \),

\[
\Phi_n = \sum_{\mu \geq 0} c_{\mu} S_{n-\mu}
\]

(18)

with

\[
c_{\mu} = 0 \quad \text{if} \quad (n - \mu)_j < (n - \mu)_k \quad \text{for at least one} \quad j < k.
\]

(19)

The latter condition shows that there are only a finite number of non-zero \( c_{\mu} \), i.e., the \( \Phi_n \) are polynomials. Then \( H'\Phi_n = E'\Phi_n \) implies \( E' = E'_n \) and the following recursion relations for the coefficients \( c_{\mu} \),

\[
[E'_n - E'_{n-\mu}]c_{\mu} = \lambda \sum_{1 \leq j < k \leq N} \sum_{\nu=1}^{n_j-n_k} [(n - \mu)_j - (n - \mu)_k + 2\nu]c_{\mu-\nu E_{jk}}
\]

(20)

(we used the fact that the functions \( S_n \) are linearly independent). We can set \( c_0 = 1 \) (this fixes the normalization of the eigenfunctions) and then determine the other \( c_{\mu} \) recursively, which is possible provided that there is no resonance, i.e., if \( E'_n - E'_{n+\mu} \) is non-zero. This is the case: we shall prove at the end of this Section that

\[
E'_n - E'_{n-\mu} = \sum_{j<k} \mu_{jk}[(n - \mu)_j - (n - \mu)_k + (n_j - n_k) + 2\lambda(k - j)]
\]

(21)

which is manifestly positive and shows that resonances indeed do not occur. This completes the construction of eigenfunctions and eigenvalues of the Sutherland model: Note that the symmetrized plane waves \( S_n \) provide a complete orthonormal basis of the corresponding non-interacting Hamiltonian (obtained by setting \( \lambda = 0 \)), and we have constructed eigenfunctions \( f_n = \Phi_n \Psi_0 \) and corresponding eigenvalues \( E_n = E'_n + E_0 \) with \( E_0 \) in Eq. (9), which are one-to-one to this free solution which is known to provide a complete basis.

**Remark 1** We can write (cf. Eq. (13)) \( E'_n = \sum_j [n_j + \frac{1}{2}\lambda(N + 1 - 2j)]^2 - E'_0 \) with

\[
E'_0 = \sum_{j=1}^{N} \frac{1}{4} \lambda^2 (N + 1 - 2j)^2.
\]

(22)
It is easy to show that \( E'_0 = E_0 \) (cf. Eq. (9)), which is somewhat remarkable and implies the following simple form of the eigenvalues,

\[
E_n = \sum_{j=1}^{N} \left( n_j + \lambda \left[ \frac{1}{2} (N + 1) - j \right] \right)^2.
\] (23)

The novel algorithm in the next Section will yield this simple form of the eigenvalues directly.

For the convenience of the reader, we conclude this Section with a

**Proof of Eq. (21):**

Eq. (15) implies

\[
E'_n - E'_{n-\mu} = \sum_j \mu_j [-\mu_j + 2n_j + \lambda(N + 1 - 2j)]
\]

with

\[
\mu_j = (\mu)_j = \sum_{k > j} \mu_{jk} - \sum_{k < j} \mu_{kj}.
\] (24)

Using

\[
\sum_j \mu_j a_j = \sum_{j < k} \mu_{jk}(a_j - a_k)
\] (25)

we get

\[
E'_n - E'_{n-\mu} = \sum_{j < k} \mu_{jk}[(\mu_k - \mu_j) + 2(n_j - n_k) + 2\lambda(k - j)]
\]

which proves Eq. (21). \( \square \)

## 3 The novel algorithm

This algorithm is based on the following Proposition which, roughly speaking, is obtained by taking the Fourier transform of the remarkable identity in Eq. (8) with respect to \( y \).

**Proposition 1** Let \( H \) be as in Eqs. (1)–(3). Then

\[
H \hat{F}(\mathbf{x}; \mathbf{n}) = E_n \hat{F}(\mathbf{x}; \mathbf{n}) - \gamma \sum_{1 \leq j < k \leq N} \sum_{\nu=1}^{\infty} \nu \hat{F}(\mathbf{x}; \mathbf{n} + \nu \mathbf{E}_{jk})
\] (26)
where
\[ \hat{F}(x; \mathbf{n}) = \mathcal{P}_n(x) \Psi_0(x), \quad \mathbf{n} \in \mathbb{Z}^N \]  \hfill (27)
with \( \Psi_0 \) as in Eqs. (3)–(4) and
\[ \mathcal{P}_n(x) = \lim_{\varepsilon \downarrow 0} \int_{-\pi}^{\pi} \frac{dy_1}{2\pi i} e^{in_1y_1} \cdots \int_{-\pi}^{\pi} \frac{dy_N}{2\pi i} e^{in_Ny_N} \prod_{1 \leq j < k \leq N} \left( 1 - e^{i(y_j-y_k)-(k-j)\varepsilon} \right)^\lambda, \]  \hfill (28)
E_{jk} as in Eq. (14), \( E_n \) in Eq. (23), and
\[ \gamma = 2\lambda(\lambda - 1). \]  \hfill (29)
(Proof in Appendix B.1.)

Remark 2 To see that these functions \( \mathcal{P}_n \) are well-defined, we note that they can be written as
\[ \mathcal{P}_n(x) = \oint_{C_1} \frac{d\xi_1}{2\pi i \xi_1} \xi_1^{n_1} \cdots \oint_{C_N} \frac{d\xi_N}{2\pi i \xi_N} \xi_N^{n_N} \prod_{j<k} \left( 1 - \xi_j/\xi_k \right)^\lambda, \]  \hfill (30)
with integration paths \( C_j : \xi_j = e^{\varepsilon j} e^{iy_j}, -\pi \leq y_j \leq \pi, \) where \( \varepsilon > 0 \) is arbitrary.

We now show that this proposition provides a solution algorithm: Eq. (26) implies that the action of \( \hat{H} \) on the functions \( \hat{F}(x; \mathbf{n}) \) is triangular, i.e., \( \hat{H}\hat{F}(x; \mathbf{n}) \) is a linear combination of functions \( F(x; \mathbf{n} + \mathbf{\mu}) \) with \( \mathbf{\mu} \geq \mathbf{0} \). We thus can make the following ansatz for eigenfunctions,
\[ f_n(x) = \sum_{\mathbf{\mu} \geq \mathbf{0}} a_{\mathbf{\mu}} F(x; \mathbf{n} + \mathbf{\mu}), \]  \hfill (31)
and then \( Hf_n = Ef_n \) implies
\[ \sum_{\mathbf{\mu} \geq \mathbf{0}} F(x; \mathbf{n} + \mathbf{\mu}) \left[ E_{n+\mathbf{\mu}} - E \right] a_{\mathbf{\mu}} - \gamma \sum_{1 \leq j < k \leq N} \sum_{\nu=1}^{\mu_{jk}} \nu a_{\mathbf{\mu}-\nu} E_{jk} = 0. \]
We thus see that we get a solution of \( Hf_n = Ef_n \) if we set \( E = E_n \) and determine the coefficients \( a_{\mathbf{\mu}} \) by the following recursion relations,
\[ \left[ E_{n+\mathbf{\mu}} - E_n \right] a_{\mathbf{\mu}} = \gamma \sum_{1 \leq j < k \leq N} \sum_{\nu=1}^{\mu_{jk}} \nu a_{\mathbf{\mu}-\nu} E_{jk} \]  \hfill (31)
which has triangular structure: we can set $a_\mathbf{0} = 1$ (this fixes the normalization), and then the other $a_\mathbf{\mu}$ can be determined recursively in terms of the $a_\mathbf{\mu'}$ where $\mathbf{\mu'} < \mathbf{\mu}$, at least if there is no resonance, i.e., if $E_{n+\mathbf{\mu}} - E_n$ does not vanish. This is true due to the following

**Lemma 1**

$$E_{n+\mathbf{\mu}} - E_n = \sum_{j=1}^{N} \mu_j^2 + \sum_{1 \leq j < k \leq N} 2\mu_{jk}[(n_j - n_k) + \lambda(k - j)]$$

(32)

with $\mu_j$ in Eq. (24), which is manifestly positive provided that Eq. (12) holds true.

(Proof in Appendix B.2.)

Moreover, the following Lemma shows that the $f_n$ are in fact symmetric polynomials, i.e., a finite linear combination of the functions $S_n$ in Eq. (11).

**Lemma 2** The functions $P_n$ in Eq. (28) all are symmetric polynomials in the variables $z_j = \exp(ix_j)$ which are non-zero only if

$$n_j + n_{j+1} + \ldots + n_N \geq 0 \quad \forall j = 1, 2, \ldots N.$$ (33)

They can be written as

$$P_n(x) = \sum_m p_{n,m} S_m(x)$$ (34)

with $S_m(x)$ as in Eq. (11), and the coefficients are

$$p_{n,m} = \sum'' \prod_{1 \leq j' < k' \leq N}^{N} \prod_{j,k=1}^{N} \left( \frac{\lambda}{\mu_{j'k'}} \right) (-\lambda)^{\mu_{j'k'} + \nu_{j'k'}} (-1)^{\mu_{j'k'} + \nu_{jk}}$$ (35)

where the sum $\sum''$ is over all non-negative integers $\mu_{jk}, \nu_{jk}$ restricted by the following $2N$ equations,

$$n_j = \sum_{\ell=1}^{N} \nu_{\ell j} + \sum_{\ell=1}^{j-1} \mu_{\ell j} - \sum_{\ell=j+1}^{N} \mu_{j\ell}, \quad m_j = \sum_{\ell=1}^{N} \nu_{j\ell}$$ (36)

and $m_1 \geq m_2 \geq \ldots \geq m_N \geq 0$, implying in particular that there are only terms such that

$$\sum_{j=1}^{N} m_j = \sum_{j=1}^{N} n_j.$$ (37)
Indeed, this Lemma implies the sum in Eq. (33) has only a finite number of non-zero terms (since there is only a finite number of $\mu$ such that $n' = n + \mu$ obeys all the conditions in Eq. (33)), and thus the $f_n$ are a finite number of terms each of which is a finite linear combination of functions $S_n$ in Eq. (11).

4 Conclusions

We can summarize our discussion in the previous Sections as follows.

Proposition 2 For each $n \in \mathbb{Z}^N$ such that $n_1 \geq n_2 \geq \ldots \geq n_N \geq 0$, the standard algorithm reviewed in Sections 3 and the novel one presented in Section 4 both yield an eigenfunction $f_n$ of the Sutherland Hamiltonian $H$ in Eq. (1). In both cases, this eigenfunction is of the form

$$f_n(x) = \Phi_n(x)\Psi_0(x)$$

with $\Psi_0$ in Eqs. 3–4, and $\Phi_n$ a symmetric polynomial in the variables $z_j = \exp(\text{i}x_j)$, and the corresponding eigenvalues $E_n$ are given in Eq. (23).

It thus follows from Theorem 3.1 in Ref. 5 that, for non-degenerate eigenvalues $E_n$, the eigenfunctions $f_n$ obtained with the two algorithms are equal (up to normalization), and the functions $\Phi_n$ are proportional to the so-called Jack polynomials (see Section 2 in Ref. 6 for details). We feel that this is quite remarkable since, even though the two algorithms look somewhat similar and both yield the same solution, there are several differences in details:

- The building blocks of the eigenfunctions in the novel algorithm are the functions $P_n$ defined in Eq. (28) and not the plane waves $S_n$ in Eq. (11).
- With the standard algorithm, one obviously obtains eigenfunctions with polynomials $\Phi_n$ which have the form

$$\Phi_n = \sum_{m \leq n} v_{n,m} S_m$$

$\Phi_n$ and $P_n$.

$4$The latter Reference actually seems to suggest that this is true even for non-degenerate eigenvalues.
where the partial order here is defined as

\[ m \leq n :\Leftrightarrow \sum_{j=1}^{k} m_j \leq \sum_{j=1}^{k} n_j \quad \forall k = 1, 2, \ldots, N \]  

(this is called dominance ordering in [St]; the latter fact follows from Eq. (18) and \( n \geq n - \mu \) for all \( \mu \geq 0 \)). This is not at all obvious for the eigenfunctions obtained with the novel algorithm (but of course should be true as well, at least for non-degenerate eigenfunctions).

- In both algorithms it is important to rule out the occurrence of resonances, but the reason for that is different (cf. Eq. (21) with Lemma 1 above, and observe the different sign of \( \mu \)).
- In the novel algorithm the restriction in Eq. (12) can be dropped, and in fact the solutions thus obtained are relevant in the elliptic case [L1]. There seems no way to drop this restriction in the standard algorithm.
- From Sutherland’s algorithm it seems somewhat surprising that the eigenvalues all can be written in the simple form \( E_n = \sum_j P_j^2 \), but from the novel algorithm this is obvious.
- As discussed in the Introduction, the novel algorithm can be generalized to the elliptic case [L1, L2].

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Appendix A: Proof of Fact 2

We note that,

\[ F(x; y) = \Psi_0(x)\Psi_0(y) \frac{1}{\prod_{j,k=1}^{N} \psi(x_j - y_k)^{\lambda}}, \]

with \( \Psi_0 \) in Eq. (3) and \( \psi \) in Eq. (1). Using \( H\Psi_0 = E_0\Psi_0 \) [Su1] and the Leibniz rule of differentiation we obtain

\[ H(x)F(x; y) = \left( E_0 + 2\lambda^2 \sum_{j,\ell} \sum_{k \neq j} \phi(x_j - x_k)\phi(x_j - y_\ell) + \right. \]

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\[ \sum_{j,k} \lambda \phi'(x_j - y_k) - \sum_{j,k,\ell} \lambda^2 \phi(x_j - y_k) \phi(x_j - y_\ell) \] \( F(x,y) \)

where
\[ \phi(r) = \psi'(r)/\psi(r) = \frac{1}{2} \cot(r/2) \] (A2)

(the prime indicates differentiation). Thus
\[ H(x) - H(y) \] \( F(x,y) = \lambda^2 (\cdot) F(x,y) \)

with
\[ (\cdot) \equiv \left[ \sum_{j,\ell} \sum_{k \neq j} 2 \phi(x_j - x_k) \phi(x_j - y_\ell) - \sum_{j,k} \sum_{\ell \neq k} \phi(x_j - y_k) \phi(x_j - y_\ell) \right] - [x \leftrightarrow y] \]

where \([x \leftrightarrow y]\) means \(\text{the same terms but with } x \text{ and } y \text{ interchanged}\) (we used that all terms which are even under \([x \leftrightarrow y]\) cancel). Relabeling indices and using \(\phi(r) = -\phi(r)\) we rewrite
\[ (\cdot) = \sum_{j,\ell} \sum_{k \neq j} \left[ \phi(x_j - x_k) \phi(x_j - y_\ell) + \phi(x_k - x_j) \phi(x_k - y_\ell) - \phi(x_\ell - y_j) \phi(x_\ell - y_k) \right] - [x \leftrightarrow y] = \sum_{j,\ell} \sum_{k \neq j} \left[ \phi(x_j - x_k) \phi(x_\ell - y_k) + \phi(x_\ell - x_j) \phi(y_\ell - x_k) \right] - [x \leftrightarrow y]. \]

We now can use the trigonometric identity
\[ \cot(x) \cot(y) + \cot(x) \cot(z) + \cot(y) \cot(z) = 1 \quad \text{if } x + y + z = 0, \] (A3)

which shows that
\[ \phi(x_j - x_k) \phi(x_j - y_\ell) + \phi(x_j - x_k) \phi(y_\ell - x_k) + \phi(y_\ell - x_j) \phi(y_\ell - x_k) = -\frac{1}{2} \]

and thus proves \((\cdot) = 0. \)

\[ \Box \]

**Appendix B: Other proofs**

**B.1 Proof of Proposition [1]**

We first observe two simple but useful facts. Firstly, the relation in Eq. (8) remains true if we replace \( F(x,y) \) by
\[ F'(x,y) = c e^{iP \sum_{j=1}^N (x_j - y_j)} F(x,y) \] (B1)
for arbitrary constants \( P \in \mathbb{R} \) and \( c \in \mathbb{C} \). [To see this, introduce center-of-mass coordinates \( X = \sum_{j=1}^{N} x_j/N \) and \( x_j' = (x_j - x_1) \) for \( j = 2, \ldots, N \), and similarly for \( y \). Then \( H(x) = -\partial^2/\partial X^2 + H_c(x') \), and similarly for \( H(y) \). Invariance of Eq. (8) under \( F \to e^{-iP(X-Y)N}F \) thus follows from \( (\partial/\partial X + \partial/\partial Y)F(x,y) = 0 \), and the latter is implied by the obvious invariance of \( F(x,y) \) under \( x_j \to x_j + a, y_j \to y_j + a, a \in \mathbb{R} \).]

The invariance of Eq. (8) under \( F \to cF \) is trivial, of course]. Secondly, the variables \( y_j \) in Eq. (8) need not be real but can be complex.

As mentioned, we intend to perform a Fourier transformation of the identity in Eq. (8), i.e. apply to it \((2\pi)^{-N} \int d^N y e^{iP y} \) with suitable momenta \( P \). We need to do this with care: firstly, the differential operator \( H(y) \) has singularities at points \( y_j = y_k \), and secondly, the function \( F(x,y) \) is not periodic in the variables \( y_j \) but changes by phase factors under \( y_j \to y_j + 2\pi \). We therefore need to specify suitable integration contours for the \( y_j \)’s avoiding the singular points, and we need to choose the \( P_j \) so as to compensate the non-periodicity. To do that, we replace the real coordinates \( y_j \) by \( z_j = y_j - i j \varepsilon \) with \( \varepsilon > 0 \) a regularization parameter: as we will see, we can then integrate along the straight lines from \( y_j = -\pi \) to \( \pi \) and after that perform the limit \( \varepsilon \downarrow 0 \). Since for all \( j < k \), \( z_j - z_k = y_j - y_k + i \varepsilon_{kj} \) with \( \varepsilon_{kj} = (k - j)\varepsilon > 0 \), we can use

\[
\sin[(y + i\varepsilon)/2] = \frac{1}{2} e^{iy}/2 e^{-i\varepsilon/2 + \varepsilon/2}(1 - e^{i\varepsilon})
\]

for \( \varepsilon > 0 \). Taking the log of this identity and differentiating we obtain \( (1/2 \cot[(y + i\varepsilon)/2] = -i/1 - \exp(iy - \varepsilon)) \). Expanding the r.h.s. in a geometric series and differentiating once more yields

\[
\frac{1}{4 \sin^2[(y + i\varepsilon)/2]} = -\sum_{\nu=1}^{\infty} \nu e^{i\nu y - \nu\varepsilon}.
\]

This accounts for all singularities and branch cuts in a consistent way. To determine the suitable \( P \) use Eq. (B3) and compute

\[
F(x; z) = (\cdots) \Psi_0(x) \hat{P}^\varepsilon(x; y)
\]

with \( \Psi_0(x) \) in Eqs. (3)–(4),

\[
\hat{P}^\varepsilon(x; y) = \frac{\prod_{1 \leq j < k \leq N} \left(1 - e^{i(y_j - y_k) - (k - j)\varepsilon}\right)^\lambda}{\prod_{j,k=1}^{N} \left(1 - e^{i(x_j - y_k) - k\varepsilon}\right)^\lambda}
\]

(B5)
a function periodic in all the $y_j$, and

\[
(\ldots) = \left(\frac{1}{2} e^{i\pi/2}\right)^{N(N-1)/2-N^2} \prod_{1 \leq j < k \leq N} e^{-i\lambda (y_j - y_k)/2 + \lambda(k-j)\varepsilon/2} \prod_{j,k=1}^N e^{-i\lambda(x_j - y_k)/2 + \lambda k\varepsilon/2}
\]

\[
= \text{const.} e^{i\lambda N \sum_{j=1}^N (x_j - y_j)/2} e^{-i\lambda \sum_{j=1}^N (N+1-2j)y_j/2}
\]

(we used $\sum_{j<k} (y_j - y_k) = \sum_j (N + 1 - 2j)y_j$). We thus see that we can choose $P$ and $c$ in Eq. (B1) such that

\[
F'(x; z) = e^{-i\lambda \sum_{j=1}^N (N+1)/2 - j} n_j \hat{P}^\varepsilon(x; y) \Psi_0(x).
\]  

(B6)

We need to choose the Fourier variables $P = (P_1, \ldots, P_N)$ such that $e^{iP \cdot y} F'(x; z)$ is periodic in all $y_j$, and this implies

\[
P_j = n_j + \lambda\left[\frac{1}{2}(N + 1) - j\right], \quad n_j \in \mathbb{Z}.
\]  

(B7)

We now can apply $(2\pi)^{-N} \int d^N y e^{iP \cdot y}$ to the identity $H(x) F'(x; z) = H(z) F'(x; z)$. We recall

\[
H(z) = -\sum_j \frac{\partial^2}{\partial y_j^2} + \gamma \sum_{j<k} \frac{1}{4 \sin^2[(y_j - y_k + i(k-j)\varepsilon)/2]}
\]

and use Eq. (B4). After taking the limit $\varepsilon \downarrow 0$ we obtain Eq. (26): the l.h.s. is obvious (note that $\hat{F}$ is the Fourier transform of $F'$). The r.h.s. has two terms. The first one is equal to $\sum_j P_j^2 \hat{F}$ and comes from the derivative terms which we evaluated by partial integration. The second term is obtained from the $1/\sin^2$-terms in $H(z) F'$ which we computed using Eq. (B4). \hfill \square

### B.2 Proof of Lemma 1

We write $(n + \mu)_j = n_j + \mu_j$ with $\mu_j$ in Eq. (16). Thus Eqs. (23) and (24) imply,

\[
E_{n+\mu} - E_n = \sum_j \left(\mu_j^2 + 2\mu_j(n_j + \lambda\left[\frac{1}{2}(N + 1) - j\right])\right),
\]

and with Eq. (25) we obtain Eq. (32). \hfill \square
B.3 Proof of Lemma 2

It is straightforward to evaluate $P_n(x)$ in Eq. (28) by expanding all terms in Taylor series (using the binomial series) and then performing the $y_j$ integrations which corresponds to a projection onto the $y_j$-independent terms. The results is

$$P_n(x) = \sum' \prod_{1 \leq j' < k' \leq N} \left( \frac{\lambda}{\mu_{j'k'}} \right) \left( -\lambda \right)^{\nu_{j'k'}} \left( -1 \right)^{\mu_{j'k'} + \nu_{j'k'}} e^{i\nu_{j'k'} x_j}$$

(B8)

where the sum $\sum'$ is over all non-negative integers $\mu_{kk'}$ and $\nu_{j\ell}$ such that

$$n_j - \sum_{\ell=1}^{N} \nu_{j\ell} - \sum_{\ell=1}^{j-1} \mu_{j\ell} + \sum_{\ell=j+1}^{N} \mu_{j\ell} = 0.$$  (B9)

Recalling the definition of $S_n$ in Eq. (11) we obtain Eqs. (34)–(36).

We now argue that this latter system of equations can have solutions only if the conditions in Eq. (33) all hold, which implies that otherwise $P_n$ is zero. To see this we add up the last $N+1-k$ relation in Eq. (B3) ($k = N, N-1, \ldots, 1$), and by a relabeling of indices we obtain

$$\sum_{j=k}^{N} n_j = \sum_{j=k}^{N} \sum_{\ell=1}^{N} \nu_{j\ell} + \sum_{\ell=1}^{k-1} \sum_{j=k}^{N} \mu_{j\ell}$$

where the r.h.s. is always manifestly positive. This proves Eq. (33). Setting $k = 1$ and comparing with Eq. (B6) we obtain Eq. (B7). Moreover, for fixed $n_j$, there are at most a finite number of different solutions of Eq. (B9), implying that $P_n$ is a polynomial. To see that we write Eq. (B9) as follows,

$$n_j + \sum_{\ell=j+1}^{N} \mu_{j\ell} = \sum_{\ell=1}^{N} \nu_{j\ell} + \sum_{\ell=1}^{j-1} \mu_{j\ell}$$

(B10)

and determine possible solutions for decreasing values of $j$ starting at $j = N$. It is easy to prove by induction that there is only a finite number of solutions

$$\{\nu_{j\ell}\}_{j,\ell=1}^{N}, \{\mu_{j\ell}\}_{1 \leq j < \ell \leq N} \in \mathbb{N}_0^{N^2 + N(N-1)/2}$$

of this system of equations: For $j = N$ we get

$$n_N = \sum_{\ell=1}^{N} \nu_{\ell N} + \sum_{\ell=1}^{N-1} \mu_{\ell N}$$
and there is obviously only a finite number of different solutions \( \{\nu_{\ell N}^{N}\}_{\ell=1} \), \( \{\mu_{\ell N}^{N-1}\}_{\ell=1} \) of that equation. If we consider Eq. (B10) for some \( j = j_0 < N \), the possible solutions for \( \{\mu_{j_0,\ell}\}_{\ell > j_0} \) were already determined by the equations for \( j > j_0 \) and, by the induction hypothesis, there is only a finite number of them. One thus only has to consider a finite number of equations, and each of them obviously has only a finite number of solutions \( \{\nu_{\ell j}^{N}\}_{\ell=1} \), \( \{\mu_{\ell j}^{j-1}\}_{\ell=1} \). □

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