First numerical evidence of global Arnold diffusion in quasi–integrable systems

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Abstract

We provide numerical evidence of global diffusion occurring in slightly perturbed integrable Hamiltonian systems and symplectic maps. We show that even if a system is sufficiently close to be integrable, global diffusion occurs on a set with peculiar topology, the so–called Arnold web, and is qualitatively different from Chirikov diffusion, occurring in more perturbed systems.

1 Introduction

The characterization of mechanisms for diffusion of orbits in quasi–integrable Hamiltonian systems and symplectic maps is a relevant topic for many fields of physics, such as celestial mechanics, dynamical astronomy, statistical physics, plasma physics and particle accelerators. Any dynamical state of an integrable system can be characterized with a set of action–angle conjugate variables \((I_1, \ldots, I_n, \phi_1, \ldots, \phi_n)\), where the actions \(I_j\) are constants of motion, while the angles \(\phi_j\) simply change linearly with time. The properties of the system which are relevant for the stability are determined by the actions.

Small perturbations of integrable systems can produce a slow drift of the initial value of the actions, and after certain time these drifts can cumulate in such a way to drive the system into very different physical state. By global drift we mean that orbits explore macroscopic regions of the action–space. If the drift behaves as a diffusion process (precisely, we mean that the numerically computed mean squared distance of \(I_1, \ldots, I_n\) with respect to their initial values grows, on average, linearly with time), we will refer to it as global diffusion.

In 1979 Chirikov [1] described a possible model for global drift valid when the perturbation is greater than some critical value. Chirikov’s model has so far been successfully used to describe diffusion in systems from different fields of physics (see, for example, [2]). One of the reasons of the broad detection of the Chirikov’s diffusion is that its typical times fall within the simulation abilities of modern computers as far back as the seventies.

For smaller perturbations the systems fall within the range of celebrated perturbation theories such as KAM [3,4,5] and Nekhoroshev theorems [6], which leave the possibility for global drift only on a subset of the possible dynamical states with peculiar topology, the so–called Arnold web, and force diffusion times to be at least exponentially long with an inverse power of the norm of the perturbation. The theoretical possibility of global drift in quasi–integrable systems has been first shown in 1964 by Arnold.
quasi–integrable symplectic map: increases. Any of these straight lines is called resonance, and the integer web and the section proportional to on the frequencies implies that these points are outside a neighborhood of lines that any invariant torus cuts the section S consists of all these straight lines with a neighborhood which decreases as γ/|k| increases. Any of these straight lines is called resonance, and the integer |k| is called resonance order. Therefore, the intersection between the Arnold web and the section S consists of all these straight lines with a neighborhood which decreases as |k| increases. Any of these straight lines is called resonance, and the integer |k| is called resonance order. The Arnold web is open, dense, and has a small relative measure (proportional to γ).

The case of quasi–integrable symplectic maps is analogous. In this paper we consider the following quasi–integrable symplectic map:

\[ \phi'_1 = \phi_1 + I_1, \quad \phi'_2 = \phi_2 + I_2 \]

[7] for a specific system, and is commonly called Arnold diffusion. Recently, Arnold’s result has been much generalized [8]. Being interested to applications to specific systems, and in particular to systems of interest for physics, we use a numerical approach which, avoiding theoretical difficulties, measures directly the quantitative features of eventual long term diffusion. In 2003 we numerically detected a very slow local diffusion confined to the Arnold web [9] in a model perturbed system. In this paper we provide numerical evidence both on quasi–integrable Hamiltonian systems and symplectic maps of a relevant phenomenon of global diffusion of orbits occurring on the Arnold web. More precisely, we show that a set of well chosen initial conditions practically explores the whole web and the process behaves as a global diffusion.

In section 2 we will describe the Arnold web of quasi–integrable Hamiltonian systems and symplectic maps and in section 3 we will describe the numerical methods used to detect the Arnold web and the global diffusion on it.

2 Arnold web of quasi–integrable systems

For definiteness, we refer to the Hamiltonian system with Hamilton function:

\[ H = \frac{I_1^2}{2} + \frac{I_2^2}{2} + I_3 + \epsilon f, \quad f = \frac{1}{\cos(\phi_1) + \cos(\phi_2) + \cos(\phi_3) + 3 + \epsilon} \]

where ε is a small parameter and \( c > 0 \), so that the equations of motion for \( I_1, I_2, I_3 \in \mathbb{R} \) and \( \phi_1, \phi_2, \phi_3 \in S^1 \) are: \( \dot{I}_i = -\frac{\partial H}{\partial \phi_i} \) and \( \dot{\phi}_i = \frac{\partial H}{\partial I_i} \) for any \( i = 1, 2, 3 \). In the integrable case (when \( \epsilon = 0 \)) the actions are constants of motion while the angles \( \phi_1(t) = \phi_1(0) + I_1 t, \phi_2(t) = \phi_2(0) + I_2 t, \phi_3(t) = \phi_3(0) + t \) rotate with frequencies \( \omega_1 = 1, \omega_2 = 2, \omega_3 = 1 \). Therefore, each couple of actions \( I_1, I_2 \) characterizes an invariant torus \( \mathbb{T}^3 \). For any small \( \epsilon \) different from zero, \( H_\epsilon \) is not expected to be integrable. However, if \( \epsilon \) is sufficiently small, the KAM theorem applies (\( H_\epsilon \) is real analytic and \( H_0 \) is isoenergetically non–degenerate): for any invariant torus of the original system with Diophantine non–resonant frequencies\(^1\) there exists an invariant torus in the perturbed system which is a small deformation of the unperturbed one. The complement of the set made of these invariant tori is called Arnold web, and in such a set, in principle, the motions can exhibit chaotic diffusion. Because Nekhoroshev theorem also applies to the system, any eventual chaotic diffusion will occur on very long times that grow at least exponentially with an inverse power of \( \epsilon \). Precisely, because \( H_0 \) is quasi–convex, the following estimates apply: \( |\dot{I}(t) - I(0)| \leq a e^{1/6}, \) for any \( |t| \leq b \exp(e_0/\epsilon)^{1/6} \) with \( a, b \) suitable positive constants (see [10] and also [16]).

To describe the topology of the Arnold web, it is convenient to refer to the subset of the phase space determined by the section \( S = \{(I_1, I_2) \in \mathbb{R}^2, \phi_i = 0, i = 1, 2, 3\} \). From the KAM theorem, it follows that any invariant torus cuts the section \( S \) in only one point \((I_1, I_2)\). Moreover, the Diophantine condition on the frequencies implies that these points are outside a neighborhood of lines \( k_1 I_1 + k_2 I_2 + k_3 = 0 \) proportional to \( \gamma/|k|^\tau \), for any \( k = (k_1, k_2, k_3) \in \mathbb{Z}^3 \backslash 0 \). Therefore, the intersection between the Arnold web and the section \( S \) consists of all these straight lines with a neighborhood which decreases as \( |k| \) increases. Any of these straight lines is called resonance, and the integer \( |k| \) is called resonance order. The Arnold web is open, dense, and has a small relative measure (proportional to \( \gamma \)).

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\(^1\)\( \omega_1, \omega_2, \omega_3 \) are Diophantine if there exist positive constants \( \gamma, \tau \) such that \( |\sum_i k_i \omega_i| > \gamma/|k|^\tau \), with \( |k| = \sum_i |k_i| \), for all \( k = (k_1, k_2, k_3) \in \mathbb{Z}^3 \backslash 0 \). The Diophantine condition considered in KAM theorem requires \( \tau > 2 \) (for 3 degrees of freedom systems) and \( \gamma \) which suitably rescales with \( \epsilon \).
A precise numerical detection of the Arnold web is possible with the so-called Fast Lyapunov Indicator (FLI) method [14]. In [15] we showed that the Arnold web of system (1) indeed corresponds to the above theoretical description. In this paper, we are able to choose initial conditions which are good candidates to diffuse thanks to the accurate detection of the web provided in [15]. In fact, within resonances, both chaotic and regular motions are observed. Of course, regular motions do not diffuse, and therefore diffusive orbits must be chosen in the small subset of the Arnold web made of chaotic motions. But even within chaotic motions, to have the chance to observe the phenomena one has to restrict the choice of initial conditions to the single order resonances, i.e. to the portion of resonant lines which are far from the main crossings between resonances. The reason is that stability times are expected to be much longer at a distance of order $\sqrt{r}$ from these crossings (see, for example, [16]). With this selection of initial conditions, we have been able in [9] to show that indeed after times that increase faster than power law, initial conditions diffuse along the chaotic border of resonant lines. Though the computational time used in [9] was already quite long, due to the slowness of Arnold diffusion, it was not sufficient to evidence a global diffusion, i.e. the wandering of motions from one resonance to the others. In this paper we provide numerical experiments that thanks to a much longer computational time, and proper choice of model parameters, display for the first time such global diffusion.

The crucial parameters to set in the models are $\epsilon$ and the value of constant $c$ appearing in the denominator of the perturbation. In fact, any Fourier harmonic $f_k = \epsilon \int f(\phi) \exp(-i \sum_i k_i \phi) d\phi$ of the perturbing function $f$ is proportional to $\epsilon$ and decreases (asymptotically) exponentially with $c \sum_i |k_i|$. The values of the two parameters must be balanced so that $\epsilon$ is smaller than the critical value for Chirikov diffusion (determined with numerical methods, see [15], [17]) and the value of $c$ is not too large so that harmonics with large order $\sum_i |k_i|$ produce measurable effects on the finite time scale of our numerical computations. We found suitable values for our experiments $\epsilon = 0.6$, $c = 2$ for the map (2), and $\epsilon = 0.01$, $c = 1$ for the Hamiltonian system (1). The Arnold webs detected with the FLI method corresponding to these values are shown in figures 1,2. In the figures, for each point of the action plane $(I_1, I_2)$ we plot a color corresponding to the FLI value according to the color scale reported below the figure. The intermediate value of the FLI (orange in the color scale) corresponds to KAM tori. Darker regions correspond to initial conditions $(I_1, I_2)$ on the section $S$ which produce resonant regular motions, while yellow regions correspond to initial conditions which produce chaotic motions. Each resonance appears as a darker orange region with a yellow border, or as a single yellow line (depending on the resonance and on the value of the angles chosen to define the section $S$; see [15] for more details). Therefore, the yellow region on the FLI map corresponds to the chaotic subset of the Arnold web, where an eventual Arnold diffusion should be confined.

Following the explained criterion of choosing initial conditions within resonant chaotic motions far from the crossings of resonances we have chosen ten initial conditions near $(I_1, I_2) = (0.316, 0.146)$ for the Hamiltonian case (see figure 2a), and twenty initial conditions near $(I_1, I_2) = (1.71, 0.81)$ for the symplectic map (see figure 1a). Then, we computed numerically the map up to $10^{11}$ iterations, and we

$$I'_1 = I_1 + \epsilon \frac{\partial f}{\partial \phi_1}(\phi_1 + I_1, \phi_2 + I_2), \quad I'_2 = I_2 + \epsilon \frac{\partial f}{\partial \phi_2}(\phi_1 + I_1, \phi_2 + I_2)$$

where $f = 1/(\cos(\phi_1) + \cos(\phi_2) + 2 + c)$, with $c > 0$. At small $\epsilon$, the KAM and Nekhoroshev theorems ([11],[12],[13]) apply to this kind of maps. The resonances of this system are defined by the straight lines $k_1 I_1 + k_2 I_2 + 2\pi k_3 = 0$, with $k_1, k_2, k_3 \in \mathbb{Z}\setminus\{0\}$, and the topology of the Arnold web on the section $S = \{(I_1, I_2) \in \mathbb{R}^2, \phi_i = 0, i = 1, 2\}$ is that described for Hamiltonian system (1).

## 3 Global Arnold diffusion: numerical results

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integrated numerically with a symplectic integrator the Hamiltonian system up to a $t \sim 10^{11}$ time units. The results are reported in figures 1 and 2: on the FLI map of the action plane $(I_1, I_2)$ we plotted as black dots all points of the orbits which have returned after some time on the section $S$. Of course, because computed orbits are discrete we represented points on the double section $|\phi_1| \leq 0.005$, $|\phi_2| \leq 0.005$ for both cases, and moreover $\phi_3 = 0$ for the Hamiltonian system; reducing the tolerance 0.005 reduces only the number of points on the section, but does not change their diffusion properties. Figures 1a,2a show only the location of initial conditions (inside the circles), and figures 1b,2b show the result after the intermediate times $2.10^9$ ($10^9$ for figure 2b). Figures 1c,2c,1d,2d show the result after much longer times. To properly display such long term evolutions we needed to use a zoomed out map of the action plane.

In both cases, the orbits filled a macroscopic region of the action plane whose structure is clearly that of the Arnold web. The orbits have moved along the single resonances, and avoided the center of the main resonance crossings, in agreement with the theoretical results which predict longer stability times for motions in these regions. The larger resonances (which correspond to the smallest orders $|k|$) are practically all visited, while this is not the case for the thinnest ones (which correspond to the highest orders $|k|$). This is in agreement with the theoretical results of [18], which predict that the speed of diffusion on each resonance becomes smaller for resonances of high order. Therefore, the possibility of visiting all possible resonances is necessarily limited by finite computational time.

On average, the drift behaves as a diffusion process. In fact, in the case of the map, the average evolution of the squared distance of $(I_1, I_2)$ from the initial datum, reported in fig. 3, increases almost linearly with time, so that we can measure a diffusion coefficient (the average slope of the plot) $D \sim 1.7 \times 10^{-10}$ (we do not report the computation of the diffusion coefficient in the Hamiltonian case, which would require a longer computation time to explore a broader region of action space).

We remark that this diffusion coefficient characterizes the global diffusion process, while diffusion coefficients measured in [9] characterized local diffusion along a specific resonance.

### 4 Conclusions

In this paper we detect a macroscopic diffusion of orbits on the Arnold web of quasi–integrable Hamiltonian systems and symplectic maps. The described diffusion phenomenon is very different from Chirikov diffusion, where the overlapping of resonances allows diffusion in macroscopic regions of phase space in relatively short time scales and without apparent peculiar topological properties of the stochastic region. The study of the long–term evolution in quasi–integrable systems, especially the set up of statistical methods, must therefore take into account that, at small perturbations, in a subset of phase space of peculiar structure there is an important phenomenon of diffusion of orbits. Indeed, it concerns various problems going from the old question of stability of the Solar System to the modern burden of the confinement of particles in accelerators.
Figure 1: The four panels correspond to the FLI map of the action plane \((I_1, I_2)\) for the map (2), with initial condition on the section \(S\) (see [15]), with different magnifications. The yellow region corresponds to the chaotic part of the Arnold web. Moreover, on panel (a) we mark with a circle the location of the twenty initial conditions; on panel (b,c,d) we mark with a black dot all points of the twenty orbits which have returned after some time on the section \(S\). We consider \(2 \times 10^9\) iterations for panel (b); \(2 \times 10^{10}\) iterations for panel (c) and \(10^{11}\) iterations for panel (d).
Figure 2: The four panels correspond to the FLI map of the action plane \((I_1, I_2)\) for the Hamiltonian system (1), with initial condition on the section \(S\) (see [15]), with different magnifications. The yellow region corresponds to the chaotic part of the Arnold web. Moreover, on panel (a) we mark with a circle the location of the ten initial conditions; on panel (b,c,d) we mark with a black dot all points of the twenty orbits which have returned after some time on the section \(S\). We consider \(10^9\) iterations for panel (b); \(1.2 \times 10^{10}\) iterations for panel (c) and \(1.1 \times 10^{11}\) iterations for panel (d).
Figure 3: Average evolution of the squared distance of \((I_1, I_2)\) from the initial datum for the map, measured for the points on the section \(S\). The total computation time \(t = 10^{11}\) iterations has been divided in \(10^3\) intervals. For each initial condition, and for each interval \([(n - 1)10^8, n10^8]\), we have computed the average of the squared distance of \((I_1, I_2)\) from the initial datum, taking into account only points that in the interval \([(n - 1)10^8, n10^8]\) are on the section \(S\). Then, we averaged over all particles.

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