A proof of the birationality of certain BHK-mirrors.

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Abstract

We generalize and give an elementary proof of Kelly’s refinement [9] of Shoemaker’s result [11] on the birationality of certain BHK-mirrors. Our approach uses a construction that is equivalent to the Krawitz generalization [10] of the duality in Berglund-Hübsch [2].

1 Introduction

We consider certain orbifold quotients of Calabi-Yau hypersurfaces in weighted projective spaces. Such a Calabi-Yau has a mirror partner according to the Berglund-Hübsch mirror construction [2]. This construction was introduced as a generalization of Greene-Plesser construction of the mirrors of fermat-type hypersurfaces [7]. Forming the mirror partner to an orbifold quotient of such a hypersurface is best known as Berglund-Hübsch-Krawitz mirror symmetry since a characterization of the group used to quotient the mirror hypersurface was given by Krawitz [10].

In [11], Shoemaker proved that BHK-mirrors of distinct Calabi-Yau orbifolds are birational provided the original Calabi-Yau’s lie in the same weighted projective space and the group used to quotient them is the same. More recently, Kelly was able to prove this result assuming only that the group was the same [9]; in particular the assumption that the hypersurfaces lie in the same projective space was dropped.
The present paper proves the result of [9] (and a generalization: theorem 3.5). The theorem is found to be an easy consequence of some observations about Fourier transforms and transposed matrices. Our main tool is the duality introduced in [4]; this duality is a simple construction, and includes the constructions of [1, 3, 6, 8] as special cases.

BHK-mirror symmetry was shown in [5] to be a special case of the duality of [4] as well. However, since [5] was never published, we include an appendix proving this.

1.1 Acknowledgements

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2 Preliminaries

Matrices multiply from the left. We make the notations

\[ W = \sum_{j=0}^{m} \prod_{i=0}^{n} X_i^{p_{ij}} : \mathbb{C}^{n+1} \to \mathbb{C}, \]

\[ P = (p_{ij})_{ij} : \mathbb{Z}^{m+1} \to \mathbb{Z}^{n+1}, \]

and

\[ (+): \mathbb{C}^{m+1} \longrightarrow \mathbb{C} \]

\[ (Z_0, \ldots, Z_m) \mapsto Z_0 + \ldots + Z_m \]

We assume that the monomials in \( W \) are distinct.

Example 2.1 \( W = X_0^{p_{00}} X_1 + X_1^{p_{11}} X_2 \)

Then

\[ P = \begin{pmatrix} p_{00} & 0 \\ 1 & p_{11} \\ 0 & 1 \end{pmatrix}. \]

\[ F(\mathbb{Z}) = (\mathbb{C}^\times) \] and if we denote \( Z_i: F(\mathbb{Z}^2) \to \mathbb{C} \) the map which sends \( \chi \mapsto \chi(e_i) \), then \( F(P): (\mathbb{C}^\times)^3_\mathbb{Z} \to (\mathbb{C}^\times)^2_\mathbb{Z} \) is given by

\[ (Z_0, Z_1) = (X_0^{p_{00}} X_1, X_1^{p_{11}} X_2). \]

Definition 2.2 We consider abelian groups which are isomorphic to finite direct sums whose summands are algebraic tori and/or finitely generated. For such an abelian group \( H \) we have Fourier transform and transposition functors:

\[ F(H) = \text{Hom}(H, \mathbb{C}^\times), \]
\[ H^* = \text{Hom}(H, \mathbb{Z}). \]

Homomorphisms are assumed to be algebraic (i.e. \( \text{Hom}(\mathbb{C}^*, \mathbb{C}^*) = \mathbb{Z} \)).

**Proposition 2.3**

- For finite \( H \), \( F(H) \cong H \) (non-canonically),
- \( F \) is contravariant, and \( F \circ F = 1 \),
- \( F \) is exact when restricted to finitely generated groups or to affine groups, and it interchanges these two subcategories,
- \( F(\mathbb{Z}^{(\ell+1)}) = (\mathbb{C}^*)^{\ell+1} \) and so \( F((\mathbb{C}^*)^{\ell+1}) = \mathbb{Z}^{\ell+1} \), and
- \( W \) restricted to \( (\mathbb{C}^*)^{n+1} \) is \((+) \circ F(P)\).

**Proof** The first 4 bullets are usual statements about pantyagin duality with \( S^1 \) replaced with \( \mathbb{C}^* \), and are easily checked. The last bullet is proved by elaborating on Example 2.1.

\[ 1 \rightarrow G \rightarrow (\mathbb{C}^*)^{n+1} \rightarrow (\mathbb{C}^*)^{n+1} / G \rightarrow 1 \]

Figure 1: \( G \leq (\mathbb{C}^*)^{n+1} \) is a closed subgroup such that \( W \) descends to the quotient.
Proposition 2.4 If $W$ factors through a quotient $(\mathbb{C}^*)^{n+1}/G$ of $(\mathbb{C}^*)^{n+1}$ by a closed subgroup $G$, then there is a unique homomorphism given by the dashed arrow in figure 1 which makes the diagram commute.

Proof Denote by $R_G$ the Reynolds operator for the action of $G$ on the functions on $(\mathbb{C}^*)^{n+1}$. $R_G$ is a projection, and characters are eigenfunctions for the action of $G$. Characters are also a basis for the space of functions, so $R_G$ evaluated on a function is given by expanding the function in characters and erasing those which are not invariant under $G$. Furthermore, a function descends to $(\mathbb{C}^*)^{n+1}/G$ if and only if it is sent to itself under $R_G$. This means that the characters in the expansion of $W$ must be $G$-invariant, and thus the map exists, and since we have assumed the monomials are distinct, it is unique.

Corollary 2.5 Quotients of $\mathbb{C}^{n+1}$ though which $W$ factors correspond to factorizations

$$P = A \circ B^\tau$$

where $B^\tau: \mathbb{Z}^{m+1} \to M$, $A: M \to \mathbb{Z}^{n+1}$, and $M = F((\mathbb{C}^*)^{n+1}/G)$. Furthermore, the function to which $W$ descends is

$$W_G = (+) \circ F(B^\tau):(C^*)^{n+1}/G \to \mathbb{C}$$

Proof Apply $F$ to the groups in figure 1. $A = F$ (quotient map) and $B^\tau = F$ (dotted arrow).

Definition 2.6 We call $([\mathbb{C}^{n+1}/G], W_G)$ a quotient Landau-Ginzburg model. We assume that such an object includes the data of the presentation: $(\mathbb{C}^{n+1}, W, G)$.

Definition 2.7 Given a quotient landau-ginzburg model $([\mathbb{C}^{n+1}/G], W_G)$, we define the dual quotient landau-ginzburg model $([\mathbb{C}^{m+1}/G^T], W_{GT}^T)$ by the data

- $P^T = P^\tau$, and
- $G^T = \ker F(B) \leq F((\mathbb{Z}^{m+1})^\tau)$.

This definition is illustrated in figure 2.

Remark 2.8 This is the special case of the duality in [4] where the kähler parameter is set to 0, or equivalently all the coefficients of the superpotential are equal to 1. The point of view in loc. cit. is that the dual is obtained by interchanging the roles of the homomorphisms $A$ and $B$. 

Figure 2: The data of a quotient landau-ginzburg model and its dual side-by-side.

3 The theorems of Shoemaker and Kelly

Proposition 3.1 If quotient landau-ginzburg models $([\mathbb{C}^{n+1}/G], W_G)$ and $([\mathbb{C}^{n+1}/G], W'_G)$ are quotients by the same group $G$, then $W_{GT}^T = W'_{GT}^T$ on the torus $F(M^\tau)$.

Proof Since $G$ is the same in both cases, $A = A'$ in the factorization of $P$ and $P'$. The definition of the dual and corollary 2.5 imply $W_{GT}^T = W'_{GT}^T = (\tau \circ F(A'))$. □

Definition 3.2 Given a quotient landau-ginzburg model $([\mathbb{C}^{n+1}/G], W_G)$, a weight vector is a homomorphism $q: M \to \mathbb{Z}$ such that $qB^\tau(e_j - e_k) = 0$ for all $j, k$.

Definition 3.3 Given a weight vector $q$, $W_G = 0$ defines a hypersurface $Z^q_{W,G}$ in the quotient of the quotient $[(\mathbb{C}^*)^{n+1}/G/\mathbb{C}^*]$; where the action of $\mathbb{C}^*$ is through the homomorphism $F(q)(\mathbb{C}^*) \to (\mathbb{C}^*)^{n+1}/G$. $Z^q_{W,G}$ is called the associated non-linear sigma-model.

Remark 3.4 The associated non-linear sigma model defined here is likely wrong for the purposes of mirror symmetry unless $W$ is a Calabi-Yau polynomial (defined below). A more likely candidate, when it exists, is given in Clarke [4].

Theorem 3.5 If quotient landau-ginzburg models $([\mathbb{C}^{n+1}/G], W_G)$ and $([\mathbb{C}^{n+1}/G], W'_G)$ are quotients by the same group $G$, then the groups of weight vectors of $([\mathbb{C}^{n+1}/G^T], W_{GT})$ and $([\mathbb{C}^{n+1}/G^T'], W'_{GT'})$ coincide, and $Z^q_{W^T,G^T}$ is birational to $Z^q_{W'^T,G'^T}$ for each $q$.

Proof As before, $A = A'$. So the set of weight vectors for the duals are those $q$ which satisfy $qA^\tau(e_j - e_k) = 0$ for all $j, k$, and we know that $W_{GT} = W'_{GT}$ on $F(M^\tau)$ by proposition 3.1. □

Definition 3.6 $W$ is called Calabi-Yau if $P$ is square and invertible over $\mathbb{Q}$,

$$(1, \ldots, 1)^\text{ag}(P) > 0 \text{ or } (1, \ldots, 1)^\text{ag}(P) < 0$$

and

$$(1, \ldots, 1)^\text{ag}(P)(1, \ldots, 1)^\tau = \det(P).$$
Where $\text{ag}(P)$ is the adjugate matrix of $P$.

We recover the the main theorems of [11] and [9] here.

**Corollary 3.7** Consider a group $G \leq (\mathbb{C}^*)^{n+1}$ and Calabi-Yau polynomials $W$ and $W'$ which are invariant under the action of $G$. Then weight vectors of the duals cyclic, and $Z^{q_0}_{W^T, G^T}$ and $Z^{q_0}_{W'^T, G'^T}$ are birational for a generator $q_0$.

**Proof** Again $A = A'$, so group of weight vectors is cut out by the $n$-equations $qA^T(e_0 - e_j) = 0$ for $j = 1, \ldots, n$. These equations are linearly independent, because $A$ is invertible, so the group is cyclic. Now we apply the above results. □
APPENDIX

A BHK as Duality for toric LG models

Berglund-Hübsch introduced a mirror construction for certain hypersurfaces in weighted projective spaces given by Calabi-Yau polynomials \([2]\). This was generalized by Krawitz \([10]\) by deeming
\[
\left(\left(\mathbb{C}^n\right)^{n+1}/G\right), W_G \right) \quad \text{and} \quad \left(\left(\mathbb{C}^m\right)^{m+1}/G^T\right), W^T_{G^T}\right)
\]
dual for a Calabi-Yau polynomial \(W\) and a group \(G\) for which \(W\) is invariant. The Krawitz dual \(G^T\) is defined below. The construction of \([2]\) occurs when \(G = \{1\}\). We show that when the BHK-dual is defined (i.e. \(W\) is Calabi-Yau), it is the same as the one above, and give a dictionary between the two.

\(F, -\tau\) and \(\exp\):

**Definition A.1** We write \(\ker(\exp)\) for the functor on algebraic tori which takes an algebraic torus to the kernel of its exponential map.

**Lemma A.2** Considered as functors on algebraic tori, there is a natural isomorphism \(\kappa: F(-)^\tau \Rightarrow \ker(\exp)(-).\)

**Proof** Consider an algebraic torus \(T\). Denote by \(K\) the kernel of its exponential map. In particular, we can write \(T = K \otimes \mathbb{Z} (\mathbb{C}/2\pi i \mathbb{Z})\). Given \(\phi: K \rightarrow \mathbb{Z}\) considered as an element of \(K^\tau\), there is a homomorphism \(\chi: T \rightarrow \mathbb{C}^\times\) by the rule \((k \otimes \mathbb{Z}) \mapsto \phi(k) \cdot \mathbb{Z} \in \mathbb{C}/2\pi i \mathbb{Z} \cong \mathbb{C}^\times\). This defines a isomorphism \(K^\tau \rightarrow F(T)\). Transposing this gives us \(\kappa: F(-)^\tau \Rightarrow \ker(\exp)(-).\)

Krawitz Dual Groups:

**Definition A.3** From \([10]\). Given a polynomial \(W\) as above, and a finite subgroup \(G \leq (\mathbb{C}^\times)^{n+1}\) such that \(W\) is invariant under the action of \(G\). It was assumed in \([10]\) that the exponent matrix \(P\) is square and invertible. We write \(e_j\) as the standard column vector, and define

- the subgroup \(\Lambda_{\text{max}} = \{a \in \mathbb{C}^{n+1} \mid W \text{ is } \exp(a) \text{ invariant}\}\),
- the subgroup \(\Lambda_G = \{a \in \mathbb{C}^{n+1} \mid \exp(a) \in G\}\),
- the inclusion \(\gamma: \Lambda_G \rightarrow \Lambda_{\text{max}}\),
- the isomorphism \(\alpha^{-1}: \Lambda_{\text{max}} \rightarrow (\mathbb{Z}^{m+1})^\tau\) such that \(\alpha^{-1}(2\pi ie_j) = P^\tau e_j\),
- the inclusion \(\lambda = \mathbb{Z}^{m+1} \rightarrow \mathbb{C}^{m+1}\) where \(e_j\) is sent to \(2\pi ie_j\), and
the inclusion \( \theta: \Lambda_G^\tau \rightarrow \mathbb{C}^{m+1} \) such that \( \theta \circ (\alpha^{-1} \circ \gamma)^\tau = \lambda \).

Define the **Krawitz dual group** to be

\[
G^\dagger = \{ \exp(\theta(r)) \mid r \in \Lambda^\tau_G \} \leq (\mathbb{C}^*)^{m+1}.
\]

**Theorem A.4** We have the following identifications

- \( F((\mathbb{C}^*)^{n+1})^\tau = \mathbb{Z}^{n+1} \) and \( F((\mathbb{C}^*)^{m+1}) = \mathbb{Z}^{m+1} \),
- \( \Lambda_G = M^\tau = F((\mathbb{C}^*)^{n+1}/G)^\tau \),
- \( \Lambda_{\text{max}} = F((\mathbb{C}^*)^{m+1})^\tau \),
- \( A^\tau \) is identified with \( F((\mathbb{C}^*)^{n+1})^\tau \rightarrow \Lambda_G \),
- \( \Lambda_G^\tau = M^\tau = F((\mathbb{C}^*)^{m+1}/G^\dagger)^\tau \),
- \( B^\tau \) is identified with \( F((\mathbb{C}^*)^{m+1}) \rightarrow \Lambda_G^\tau \), and
- \( G^\dagger = G^T \).

**Proof** The first bullet is by definition. In light of lemma [A.2] we have second bullet. The third bullet is via \( \alpha^{-1} \). For the fourth, we know that \( F((\mathbb{C}^{n+1})^\tau \rightarrow \Lambda_G \), must be identified with \( A^\tau: F((\mathbb{C}^{n+1})^\tau \rightarrow M^\tau \) since they both have cokernel \( G \) by the first isomorphism theorem. The fifth bullet is by definition. Now for the sixth, from the definition of \( \alpha^{-1} \), we know that the composition \( F((\mathbb{C}^*)^{n+1})^\tau \rightarrow \Lambda_{\text{max}} \xrightarrow{\alpha^{-1}} F((\mathbb{C}^*)^{m+1})^\tau \) is \( P^\tau \). This composition can also be factored \( F((\mathbb{C}^*)^{n+1})^\tau \xrightarrow{A^\tau} \Lambda_G = M^\tau \rightarrow F((\mathbb{C}^*)^{m+1})^\tau \). Since this equals \( P^\tau \), we know that the map \( \Lambda_G \rightarrow F((\mathbb{C}^*)^{m+1})^\tau \) must equal \( B \). Finally, \( \text{cok } B^\tau = G^\dagger = G^T \) where the first equality follows from the sixth bullet.

\[ \square \]
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