Toward parton equilibration with improved parton interaction matrix elements

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Abstract. The Quark-Gluon Plasma can be produced in high energy heavy ion collisions and how it equilibrates is important for the extraction of the properties of strongly interacting matter. A radiative transport model can be used to reveal interesting characteristics of Quark-Gluon Plasma thermalization. For example, screened parton interactions always lead to partial pressure isotropization. Systems with different initial pressure anisotropies evolve toward the same asymptotic evolution. In particular, radiative processes are crucial for the chemical equilibration of the system. Matrix elements under the soft and collinear approximation for these processes, as first derived by Gunion and Bertsch, are widely used. A different approach is to start with the exact matrix elements for the two to three and its inverse processes. General features of this approach will be reviewed and the results will be compared with the Gunion-Bertsch results. We will comment on the possible implications of the exact matrix element approach on Quark-Gluon Plasma thermalization.

1. Introduction

Many interesting discoveries have been made in the quest for the understanding of nuclear matter under extreme conditions [1–9]. Relativistic heavy ion collisions are particularly useful in creating the phase of matter called the Quark-Gluon Plasma (QGP) [10]. During these collisions, radiative processes are important for the thermalization of the Quark-Gluon Plasma. Xu and Greiner first introduced the stochastic method into relativistic transport model simulations [11]. This enabled the microscopic study of the Quark-Gluon Plasma with particle number changing processes. We also developed a similar algorithm for the dynamical study of the Quark-Gluon Plasma. In the following, we will illustrate some interesting features of the thermalization of a gluon system. For the simulation, the perturbative Quantum-Chromo-Dynamics (pQCD) cross section regulated by a Debye screening mass will be used for the two to two process. It is proportional to the strong interaction coupling constant (fine structure constant), $\alpha_s$, squared. The screening mass squared is proportional to $\alpha_s$, inversely proportional to the cell volume, and proportional to the sum of inverses of particle momenta. This helps avoid many conceptual and technical problems associated with large cross sections in dense media. The two to three cross section is taken to be 50% of the two to two cross section. This is in line with a more sophisticated calculation by Xu and Greiner [11]. The outgoing particles will be taken to be isotropic. The three to two reaction integral is determined by detailed balance and in this case is directly proportional to the two to three cross section.
With this setup, various aspects of thermalization of gluons in a box can be studied. For example, for a system having 2000 gluons initially with a temperature of 1 GeV inside a box of dimensions of $5 \times 5 \times 5$ fm$^3$, the three to two rate per unit volume approaches that for the two to three process from below, and the particle energy distribution relaxes to that in thermodynamical equilibrium (kinetic and chemical equilibrium). More details can be found in [12] and [13]. In the following, we will apply the above radiative transport model to the study of the pressure anisotropy and energy density evolutions in relativistic heavy ion collisions. We will then look at the two to three and its inverse processes with exact matrix elements for a more realistic description of relativistic parton dynamics. Finally, a summary will be given together with speculations on the implications of the inelastic processes with improved matrix elements.

2. Pressure anisotropy and energy density evolutions

The initial stage of a relativistic heavy ion collision is dominated by gluons. We will focus on the thermalization of a gluon system in the central cell in central collisions. In this case, kinetic equilibration can be characterized by the pressure anisotropy (i.e., the longitudinal to transverse pressure ratio, $P_L/P_T$) [14–20]. If the system is in equilibrium, the pressure anisotropy equals 1; if the pressure anisotropy is different from 1, the system is not in equilibrium. Fig. 1 shows the pressure anisotropy evolutions for initial conditions similar to those in central Au+Au collisions at the Relativistic Heavy Ion Collider (RHIC). We can start by looking at the isotropic initial condition with only elastic collisions. Even though the initial condition is isotropic, the longitudinal expansion makes it decrease with (proper) time. Expansion dominates the initial period until collisions take over and the pressure anisotropy begins to increase toward isotropy. If the two to three and three to two processes are also included, more thermalization can be achieved. If the initial condition is transverse instead of isotropic, the pressure anisotropy increases as a result of particle collisions. It approaches that starting from an isotropic initial condition. If the initial condition is Color-Glass-Condensate like instead of thermal, more thermalization can be obtained when only elastic collisions are included. If inelastic processes are also allowed, the evolutions are almost identical to those from thermal initial conditions due to fast exponentiation of the momentum spectra. More discussions can be found in [18].

![Figure 1. Pressure anisotropy evolutions in the central cell in central heavy ion collisions. The lines are for evolutions from thermal initial conditions while the points are for idealized Color-Glass-Condensate initial conditions. The dashed lines and the pluses have elastic collisions only while the solid lines and circles include also lowest order inelastic collisions.](image)

The bulk properties of the central cell can be described by the energy density, the longitudinal and transverse pressures. For a gluon system, they are related by the zero trace of the energy momentum tensor. Therefore, there are only two independent variables. The energy density thus
provides additional information relative to the pressure anisotropy. In particular, the early stage energy density evolution reflects the initial anisotropy and the late time evolution is determined by parton interactions. Further discussions on its implications can be found in Ref. [12].

3. Improved matrix elements for inelastic gluonic processes

To get a better description of thermalization, more realistic radiative matrix elements need to be implemented. The exact two to three matrix element was first studied in the late 70’s [21, 22]. The matrix element modulus squared (averaged over initial internal degrees of freedom and summed over final) can be expressed in a very symmetric form as:

\[
|M_{gg \rightarrow ggg}|^2 = \frac{g^6 N_c^3}{2(N_c^2 - 1)} \frac{\sum (ij)^4 \sum (ijklm)}{\prod (ij)}. \tag{1}
\]

In the above equation, the strong interaction coupling constant \(g\) is related to \(\alpha_s\) by \(\alpha_s = g^2 / (4\pi)\). The number of colors is \(N_c = 3\). The sums and the product are over all distinct permutations of the set of particle labels \(\{1, 2, 3, 4, 5\}\). \((ij) = p_i \cdot p_j\) is the product of the four-momenta of particles \(i\) and \(j\), and the string \((ijklm) = (ij)(jk)(kl)(lm)(mi)\). This symmetric form puts all particles on an equal footing. The denominator comes from particle propagators, and we will regulate these propagators by the Debye screening mass squared, \(\mu^2\).

It is instructive to look at some representative numbers. We will set \(\alpha_s = 0.47\). For about 300 MeV temperature, we have approximately \(\mu^2 = 10 \text{ fm}^{-2}\), and \(s = 4 \text{ GeV}^2\) for the center-of-mass energy squared. Then the calculated two to two cross section is \(\sigma_{22} = 0.312 \text{ fm}^2\), and the two to three cross section is \(\sigma_{23} = 0.0523 \text{ fm}^2\). This gives a ratio of \(\sigma_{23}/\sigma_{22} = 0.168\), much smaller than 50%. One can also look at the small coupling limit by taking \(\alpha_s = 0.3\). This leads to a change in \(\mu^2\) to 6.38 \text{ fm}^{-2}. The calculated \(\sigma_{22} = 0.199 \text{ fm}^2\), and \(\sigma_{23} = 0.0504 \text{ fm}^2\). Their ratio is also much smaller than 50%.

\[\text{Figure 2. Normalized Dalitz plot for the outgoing particles in the } gg \rightarrow ggg \text{ process.}\]

In addition to the total cross section which determines the collision rate, it is important to see how different the outgoing particle distribution is from the isotropic distribution. This can be achieved by studying the normalized Dalitz plot. It gives the distribution of outgoing particles as a function of \(m_{12}^2/s\) and \(m_{23}^2/s\) where \(m_{ij}\) is the invariant mass of the subsystem composed of particles \(i\) and \(j\). If the outgoing particles are isotropically distributed, the Dalitz plot is flat at \(\rho = 2\). Fig. 2 shows the distribution when \(\mu^2 = 10 \text{ fm}^{-2}\) and \(s = 4 \text{ GeV}^2\). The three peaks
come from the soft and collinear singularities. With the Debye mass regularization, the outgoing particle distribution is not far from isotropic.

\[
|M_{gg \rightarrow ggg}^{GB}|^2 = \frac{9g^4s^2}{2(q_{\perp}^2 + \mu^2)^2} \frac{12g^2q_{\perp}^2}{k_{\perp}^4((k_{\perp} - q_{\perp})^2 + \mu^2)}. \tag{2}
\]

Here the \( q_{\perp}^2 \) and \((k_{\perp} - q_{\perp})^2 \) poles are already regulated by \( \mu^2 \). We will further use \( \mu^2 \) as a regulator for the \( k_{\perp}^2 \) singularity. Fig. 3 shows some comparisons for \( s = 4 \text{ GeV}^2 \).

The horizontal axis is \( \phi \), the azimuthal angle between \( k_{\perp} \) and \( q_{\perp} \). The vertical axis is \( f(q_{\perp}, k_{\perp}, y, \phi) = \sum y_{1a}y_{1b}|M|^2/\partial F/\partial y_{1}^{\prime}\big|_{F=0} \). In the above expression, \( y \) is the rapidity of the radiated gluon. The rapidity of the outgoing gluon that acquires the transverse momentum transfer is \( y_{1}^{\prime} \), and \( y_{1a} \) and \( y_{1b} \) are the roots of \( F = 0 \) where \( F \) is the four-momentum squared of the particle other than the radiated and the transverse momentum transferred ones. Only results for \( y = 0 \) are shown here. The singularities are not regulated for the top panels. In other

\textbf{Figure 3.} Sums of weighted matrix elements modulus squared as functions of the azimuthal angle \( \phi \).
words, $\bar{\mu}^2 = 0$ for these two cases. In the following, unless stated otherwise, barred symbols are for variables reduced by $\sqrt{s}$. We see that when $k_\perp$ is much smaller than $q_\perp$ and they are both much smaller than the kinematics limit (0.5), the Gunion-Bertsch result is only slightly larger than the exact result. However, if $q_\perp$ is much smaller than $k_\perp$, the Gunion-Bertsch result can be higher than the exact by 25%. Comparison of the left panels shows that the regulator can significantly reduce the magnitudes and the Gunion-Bertsch result can be higher than the exact by as much as 50%. We also looked at other kinematic regions and the two do not always agree.

Besides the two to three process for particle production, its inverse process is also important in the thermalization of a gluon system. In the following, we will calculate the reaction integral and look at the outgoing particle distribution. Fig. 4 gives an example of the initial and final distributions of particles as functions of $\cos(\theta)$ and $\phi$ in the center-of-mass frame. Notice that there is a soft particle in the initial state and the final distribution is for the first outgoing particle with the other balancing the momentum of the first one. The interactions are specified by $\alpha_s = 0.47$ and $\bar{\mu}^2 = 10$ fm$^{-2}$. The calculated reaction integral is $I_{32} = 6.84$ fm$^2$, close to the estimate (6.19 fm$^2$) if the matrix element is isotropic. The outgoing particle distribution has a two-peak structure strongly affected by the two “hard” particles and the soft particle appears to be absorbed. Fig. 5 provides another example. In this case, the three incoming particles have about the same energy. The reaction integral is $I_{32} = 4.85$ fm$^2$, smaller than that estimated from the isotropic matrix element formula. The outgoing particle distribution is a ring determined by the incoming particles. This is also very different from the uniform distribution when the matrix element is isotropic.

![Figure 4](image.png)

**Figure 4.** The incoming (left) and outgoing (right) particle distributions for a three to two process. The numbers in the left panel are for particle energies. The outgoing particle distribution is normalized to 1.

### 4. Summary and speculations

Radiative transport plays an important role in momentum space exponentiation and particle production in relativistic heavy ion collisions. The above calculations of typical two to three and two to two cross sections indicate that elastic collisions may be more important in thermalization than expected from the Gunion-Bertsch formula based calculations. Xu and Greiner obtained the
specific shear viscosity (shear viscosity to entropy density ratio) and showed that it approaches the conjectured quantum limit at large $\alpha_s$ [24]. Their calculations were based on the Gunion-Bertsch formula. As the exact formula based radiative cross section can be much smaller, the specific shear viscosity may be much larger than the quantum limit. If so, this will be in qualitative agreement with calculations by Chen et al. based on the exact matrix elements [25, 26]. The above outgoing particle distributions show that the two to three process is not far from isotropic while the three to two is not quite close. Hence specific viscosity calculations need to be explicitly carried out to see the difference.

The comparison of the exact and Gunion-Bertsch matrix elements shows big differences in key kinematic regions. The Gunion-Bertsch formula was regulated with the screening mass. This is equivalent to replacing the original theta function used by Xu and Greiner by a Lorentzian. In other words, the screened propagator and the theta function are two different ways of saying the same thing, i.e., soft interactions are limited by the Landau-Pomeranchuk-Migdal (LPM) effect. It is also interesting to compare the formation time and the mean free path based on the above calculations. It turns out that they are on the same order for processes that are important for thermalization. This is very different from processes involving jets where the coherent length can be much longer than the mean free path [27]. Therefore, thermalization can be much more sensitive to the space-time evolution of the hot and dense nuclear medium compared to jets. The above picture may not be limited to gluon and light quark processes only. Heavy quark equilibration may also benefit strongly from elastic processes and some other quasi-elastic processes such as the meson dissociation process [28]. Hopefully calculations in the near future will be able to clarify some of these questions.

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Figure 5. Like Fig. 4 but for three different incoming particles.
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