The Gardner Category and Non-local Conservation Laws for $N = 1$ Super KdV

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Abstract

The non-local conserved quantities of $N = 1$ Super KdV are obtained using a complete algebraic framework where the Gardner category is introduced. A fermionic substitution semigroup and the resulting Gardner category are defined and several propositions concerning their algebraic structure are proven. This algebraic framework allows to define general transformations between different nonlinear SUSY differential equations. We then introduce a SUSY ring extension to deal with the non-local conserved quantities of SKdV. The algebraic version of the non-local conserved quantities is solved in terms of the exponential function applied to the $D^{-1}$ of the local conserved quantities of SKdV. Finally the same formulas are shown to work for rapidly decreasing superfields.

1 Introduction

The supersymmetric algebra is the unique extension of the super-Poincaré algebra which is consistent with the $S$-matrix of quantum field theory. The most remarkable SUSY theory explains how superstrings and other extended SUSY objects can be consistently tied together in what also has been called $M$-theory.

Free (string) superstring theory ia a two-dimensional supersymmetric theory whose local symmetry group is generated by the (Virasoro) Super-Virasoro algebra.
These algebras may be realized as algebras of the (potential) superpotential of (KdV) SKdV [1, 2] equations when the second Hamiltonian structure (with the corresponding Poisson structure) is considered [3].

It is then reasonable to think that the hierarchy of (KdV) SKdV is related to the loop expansion of (string) superstring theory in terms of the genus of Riemann surfaces [4].

The SKdV hierarchy also arises from supersymmetric quantum mechanics. In fact, it was proven in [5, 6] that the entire SKdV hierarchy appears in the asymptotic expansion of the Green’s function \( g(x, \theta, y, \theta') \) of the super heat operator, as \( t \to 0^+ \) and \( g(x, \theta, y, \theta') \) is restricted to the diagonal \( x = y, \theta = \theta' \). The same result holds for the pure “bosonic” (non-SUSY) KdV hierarchy arising from the Green’s function of the heat operator with potential, that is, the “euclidean” Schrödinger operator [7].

The KdV equation has an infinite number of discrete conserved quantities (CQ). The SUSY extension of these conserved quantities are also CQ for the SKdV equation; but a remarkable difference between the two equations is that SKdV has a second sequence of CQ, these being non-local and intrinsically supersymmetric in nature. They have been interpreted [8] as the Poisson square root of the local CQ’s, in the sense that

\[ \{ J, J' \} = H \]

where \( J \) and \( J' \) are non-local CQ’s and \( H \) is a local CQ of SKdV.

The conservation laws of KdV and SKdV may be obtained from the Lax representations of these equations; for a review see [8]. The non-local CQ of SKdV were first obtained by analyzing the infinite set of symmetries of SKdV, eg. [9]. Later on they were obtained from the Lax operator in [10].

Another way to obtain these conservation laws is through the supersymmetric extension [1, 11] of Gardner transformation [12]. It may be interpreted as a one-parameter integrable deformation of SKdV. The deformation is

\[ \phi = \chi + \varepsilon D^2 \chi - \varepsilon^2 \chi D \chi, \]

where \( \varepsilon \) is the deformation parameter.

If the superfield \( \chi \) satisfies the S-Gardner equation [11] then \( \phi \) satisfies SKdV. Then, using the fact that \( H = \int dx d\theta \chi \) is a conserved quantity of the S-Gardner equation, it was shown [11] that all the local conserved quantities of SKdV arise in the formal expansion of \( H \) in powers of \( \varepsilon \).

It was left as an open problem, OP1 in the review of P. Mathieu [8], to find the non-local conserved quantities of SKdV from some integrable \( \varepsilon \)-deformation.

In the present paper OP1 is solved, by first rephrasing it in a completely algebraic framework working first in the free SUSY derivation ring constructed in [5], a fermionic substitution semigroup is introduced. The resulting Gardner category is an algebraic construction modelled on the possibility of more general Gardner transforms between different nonlinear SUSY differential equations. In the particular case of SKdV the local conserved quantities are constructed from this formalism.
We then introduce SUSY ring extensions in order to deal with the possibility of non-local conserved quantities. The algebraic version of the non-local CQ problem is solved, using the exponential function applied to the $D^{-1}$ of the local conserved quantities which the ring extensions provide.

Finally the same formulas are shown to work for rapidly decreasing superfields, and the non-local CQ's so obtained are shown to agree with some found in the literature.

2 The Fermionic Substitution Semigroup

Let $A$ be the free SUSY derivation ring on a single fermionic generator $a_1$. This ring is generated by its fermionic elements $a_1, a_3, a_5, \ldots$ and bosonic elements $a_2, a_4, a_6, \ldots$ and its superderivation $D : A \to A$ is determinated by $Da_n = a_{n+1}$ for $n \geq 1$.

The ring extension $A[\varepsilon] \supset A$ consists of all formal power series $h = \sum_{n=0}^{\infty} \varepsilon^n h_n$ with coefficients $h_n(a_1, a_2, \ldots) \in A$. Its involution $h \to \bar{h}$ and superderivation $h \to Dh$ are defined componentwise from the same operations in $A$. The supercommutativity equation $gh = \pm hg$ holds when $\bar{g} = \pm g$ and $\bar{h} = \pm h$, a minus sign when $\bar{g} = -g$ and $\bar{h} = -h$, and a plus sign in the other three cases.

When $f, g \in A[\varepsilon]$ with $\bar{f} = -f$, the substitution of $f$ in $g$ produces another element $g \circ f \in A[\varepsilon]$. It is defined by the formulas

$$g = \sum_{0}^{\infty} \varepsilon^n g_n (a_1, a_2, \ldots)$$

$$g \circ f = \sum_{0}^{\infty} \varepsilon^n g_n (f, Df, \ldots)$$

The following properties are derived.

**Proposition 1** When $f, g, h \in A[\varepsilon]$ with $\bar{f} = -f$ one has

$$(g + h) \circ f = (g \circ f) + (h \circ f)$$

$$(gh) \circ f = (g \circ f) (h \circ f),$$

which is to say that the operation $g \to g \circ f$ is a ring homomorphism $A[\varepsilon] \to A[\varepsilon]$, for any fixed $\bar{f} = -f$.

**Proof of Proposition 1** It suffices to take $g, h \in A$. Since $A[\varepsilon]$ is supercommutative and $f$ is fermionic, there is no ambiguity in passing from $g(a_1, a_2, \ldots)h(a_1, a_2, \ldots)$ to $g(f, Df, \ldots)h(f, Df, \ldots)$. 

\[3\]
**Proposition 2** When \( f, g \in \mathcal{A}[\varepsilon] \) with \( \bar{f} = -f \), one has

\[
D(g \circ f) = (Dg) \circ f.
\]

**Proof of Proposition 2** When \( g \in \mathcal{A} \) and is just some \( a_n \), both sides of the equation give \( D^n f \).

Suppose now that the proposition is true for some \( g, h \in \mathcal{A} \). Then when the Proposition 1 is applied to the equation

\[
D(gh) = (Dg)h + \bar{g}(Dh)
\]

we obtain

\[
(D(gh)) \circ f = ((Dg) \circ f)(h \circ f) + (\bar{g} \circ f)((Dh) \circ f).
\]

On the other hand

\[
D(gh \circ f) = D((g \circ f)(h \circ f)) = ((Dg \circ f))(h \circ f) + (\bar{g} \circ f)D(h \circ f).
\]

Since \( \bar{g} \circ f = \bar{f} \circ f \), the desired equality for \( gh \) follows from its truth for \( g \) and \( h \). It follows that Proposition 2 holds for any element \( g \) of \( \mathcal{A} \), and hence also for \( g \in \mathcal{A}[\varepsilon] \).

Now let \( \mathcal{A}_1[\varepsilon] \subset \mathcal{A}[\varepsilon] \) be the subset of all fermionic elements. The substitution product gives \( g \circ f \in \mathcal{A}_1[\varepsilon] \) if \( g, h \in \mathcal{A}_1[\varepsilon] \).

**Proposition 3** The substitution product is associative:

\[
(h \circ g) \circ f = h \circ (g \circ f).
\]

**Proof of Proposition 3** It suffices to treat the case

\[
h(a_1, a_2, \ldots) \in \mathcal{A}_1[\varepsilon] \supset \mathcal{A}.
\]

By definition

\[
h \circ (g \circ f) = h((g \circ f), D(g \circ f), \ldots).
\]

But from Proposition 2

\[
h \circ (g \circ f) = h((g \circ f), (Dg) \circ f, \ldots)
\]

But \( h \) is just a sum of products of \( a_1, a_2, \ldots \) and the \( f \)-substitution is a ring homomorphism.

Therefore

\[
h \circ (g \circ f) = h(g, Dg, \ldots) \circ f = (h \circ g) \circ f,
\]

and the proof is complete.
Thus $A_1[\varepsilon]$ is made into a semigroup by the substitution construction. Evidently the element $a_1 \in A_1 \subset A_1[\varepsilon]$ acts as the identity element of this semigroup.

When an element of $A_1[\varepsilon]$ has the value $a_1$ when $\varepsilon = 0$ it is invertible:

**Proposition 4** Given $f = a_1 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots \in A_1[\varepsilon]$ there exists $g = a_1 + \varepsilon g_1 + \varepsilon^2 g_2 + \cdots \in A_1[\varepsilon]$ with $g \circ f = a_1$.

**Proof of Proposition 4** For any $h(a_1, a_2, \ldots) \in A$ and $f$ as above, $h \circ f = h + \sum_{k=1}^{\infty} \varepsilon^k h_k$ for some $h_k \in A$. Therefore

$$(a_1 - \varepsilon f_1) \circ f = \sum_{k=2}^{\infty} \varepsilon^k \tilde{h}_k$$

for some $\tilde{h}_k \in A$.

If $g_1, \ldots, g_n \in A$ have been found such that

$$\left(a_1 + \sum_{k=1}^{n} \varepsilon^k g_k\right) \circ f = \varepsilon^{n+1} r + \cdots$$

then the choice $g_{n+1} = -r \in A$ gives the same equation, but for $n + 1$. Thus all the coefficients of $g = a_1 + \sum_{k=1}^{\infty} \varepsilon^k g_k$ are determined recursively, and the proof is complete.

An easy corollary shows that left and right inverses are the same.

**Proposition 5** Given $f = a_1 + \sum_{k=1}^{\infty} \varepsilon^k f_k$ and $g = a_1 + \sum_{k=1}^{\infty} \varepsilon^k g_k$ in $A_1[\varepsilon]$. If $f \circ g = a_1$ then $g \circ f = a_1$.

**Proof of Proposition 5** There exists $h$ with $h \circ f = a_1$. Then $(h \circ f) \circ g = a_1 \circ g = g$ while $h \circ (f \circ g) = h \circ a_1 = h$. Thus $h = g$ and $h \circ f = g \circ f = a_1$, completing the proof.

As an exercise one can compute the inverse of $f = a_1 + \varepsilon a_1 a_2$, obtaining $g = a_1 - \varepsilon a_1 a_2 + 2\varepsilon^2 a_1 a_2^2 - 5\varepsilon^3 a_1 a_2^3 + \cdots$, the coefficient of $\varepsilon^n$ being $(-1)^n \binom{2n+2}{n+1} a_1 a_2^n$.
2.1 Frechet Derivative Operator

Associated with the ring $\mathcal{A}$ and its superderivation $D : \mathcal{A} \to \mathcal{A}$ there is a ring $\mathcal{O}_p\mathcal{A}$ whose elements are the finite order differential operators $L = \sum_{k=0}^{N} l_k D^k$ with $l_k \in \mathcal{A}$. Each $L$ acts linearly in $\mathcal{A}$, and the product of two operators is computed from repeated applications of the SUSY product rule $D(gh) = (Dg)h + \bar{g}(Dh)$.

When $\bar{f} = -f \in \mathcal{A}_1[\varepsilon]$, the substitution of $f$ in $L$ is defined by

$$L \circ f = \sum_{k=0}^{N} (l_k \circ f) D^k.$$ 

Thus $L \circ f$, a formal power series with operator coefficients, is in the ring $(\mathcal{O}_p\mathcal{A})[\varepsilon]$ whose elements are the sums $\sum_{m,n} \varepsilon^m l_{m,n} D^n$ with $l_{m,n} \in \mathcal{A}$ and $l_{m,n} = 0$ for $n >> 0$, at any given $m$.

Given $L \in (\mathcal{O}_p\mathcal{A})[\varepsilon]$ and $h \in \mathcal{A}[\varepsilon]$, the element $Lh \in \mathcal{A}[\varepsilon]$ is well-defined because $(\sum_m \varepsilon^m L_m)(\sum \varepsilon^n h_n)$ is again a power series with coefficients in $\mathcal{A}$. The effect of $f$-substitution is as to be expected.

**Proposition 6** If $L \in (\mathcal{O}_p\mathcal{A})[\varepsilon]$, $h \in \mathcal{A}[\varepsilon]$, and $f \in \mathcal{A}_1[\varepsilon]$, then

$$(Lh) \circ f = (L \circ f)(h \circ f).$$

**Proof of Proposition 6** When $l, h \in \mathcal{A}$ one has

$$(lD^n h) \circ f = (l \circ f) D^n (h \circ f)$$

by Propositions 1 and 2. The general case reduces to linear combinations of this special case.

The foregoing constructions come into play when we ask for the first variation of the substitution operation. Given any $f = \sum_{m=0}^{\infty} \varepsilon^m f_m (a_1, a_2, \ldots)$ in $\mathcal{A}_1[\varepsilon]$, its Frechet derivative operator is

$$f' = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \varepsilon^m \frac{\partial}{\partial a_n} f_m (a_1, a_2, \ldots) D^{n-1} \in (\mathcal{O}_p\mathcal{A})[\varepsilon].$$

Then for any $\psi \in \mathcal{A}_1[\varepsilon]$ the substitution by $a_1 + t\psi \in \mathcal{A}_1[\varepsilon]$ gives

$$f \circ (a_1 + t\psi) = f + tf' \psi + \cdots,$$

the full right side of the equation being a power series in $t$ with $\mathcal{A}[\varepsilon]$ coefficients.
A more general formula appears when $a_1$ is replaced by $\varphi \in A_1[\varepsilon]$ and
\[
f \circ (\varphi + t\psi) = f \circ \varphi + t(f' \circ \varphi)\psi + \cdots,
\]
valid when $f, \varphi, \psi \in A_1[\varepsilon]$.

The chain rule is now immediate.

**Proposition 7** When $f, g \in A_1[\varepsilon]$ one has
\[
(f \circ g)' = (f' \circ g) g'.
\]

**Proof of Proposition 7** From the definition, $(f \circ g)' \in (O_p A)[\varepsilon]$ is given by
\[
f \circ g \circ (a_1 + t\psi) = f \circ g + t(f \circ g)'\psi + \cdots
\]
where $\psi \in A_1[\varepsilon]$ is arbitrary. But
\[
g \circ (a_1 + t\psi) = g + tg'\psi + \cdots,
\]
giving
\[
f \circ g \circ (a_1 + t\psi) = f \circ (g + tg'\psi + \cdots)
\]
\[
= f \circ g + t(f' \circ g) g'\psi + \cdots
\]
Since $\psi \in A_1[\varepsilon]$ is arbitrary, the proof is complete.

### 3 The Gardner Category

An element $f = \sum_{m=0}^{\infty} \varepsilon^m f_m(a_1, a_2, \ldots)$ of $A_1[\varepsilon]$ may be taken to represent a nonlinear differential equation
\[
\frac{\partial}{\partial t} \alpha(x, t) = \sum_{m=0}^{\infty} \varepsilon^m f_m(\alpha(x, t), D\alpha(x, t), \ldots)
\]
if $\alpha(x, t)$ is a fermionic superfield and the superderivation $D_1 = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x}$ is also known as the covariant derivative.

A second element $g \in A_1[\varepsilon]$ represents a second differential equation, for an unknown superfield $\beta(x, t)$.

Then given a third element $r \in A_1[\varepsilon]$, one might want the transformation
\[
\beta(x, t) = \sum_{m=0}^{\infty} \varepsilon^m r_m(\alpha(x, t), D\alpha(x, t), \ldots)
\]
to transform solutions of the first equation into solutions of the second. After some computation one sees that this happens if

\[ g \circ r = r'f. \]

Accordingly, \( f \) and \( g \) can be called “objects” in the Gardner category, and \( r \) a “morphism” from \( f \) to \( g \), written \( g \leftarrow r \), if the above equality holds in \( A_1[\varepsilon] \).

Obviously the choice \( r = a_1 \in A \), \( r' = I \in \mathcal{O}_pA \) gives the identity automorphism of each object.

But the composition of morphisms must be checked.

**Proposition 8** Given \( f, g, h, r, s \in A_1[\varepsilon] \).

If \( h \leftarrow s \) \( g \) and \( g \leftarrow r \) \( f \) then \( h \leftarrow s \) \( r \) \( f \).

**Proof of Proposition 8** From \( h \circ s = s'g \) it follows that

\[ h \circ s \circ r = s'g \circ r \]
\[ = (s' \circ r) (g \circ r) \]

after applying Proposition 6 to \( s' \in \mathcal{O}_pA[\varepsilon] \) and \( g \in A_1[\varepsilon] \). But \( g \circ r = r'f \), giving

\[ h \circ (s \circ r) = (s' \circ r) r'f \]
\[ = (s \circ r)'f \]

by the chain rule, Proposition 7. This completes the proof.

The possibility of isomorphism classes in the Gardner category leads one to examine the invertible elements.

**Proposition 9** Given \( f, g, r, s \in A_1[\varepsilon] \) with \( r \circ s = s \circ r = a_1 \).

If \( g \leftarrow r \) \( f \)

then \( f \leftarrow s \) \( g \).

**Proof of Proposition 9** From \( g \circ r \circ s = r'f \circ s \) one obtains

\[ g = (r' \circ s) (f \circ s) \]
\[ s'g = s' (r' \circ s) (f \circ s). \]

The desired conclusion \( f \circ s' = s'g \) would follow from \( s'(r' \circ s) = I \). But the \( s \)-substitution is also a homomorphism of the ring of operators, and it converts the known \( (s' \circ r)r' = I \) into \( s'(r' \circ s) = I \). This completes the proof.
3.1 The Gardner Transform

It is known that

\[ g \xrightarrow{r} f \]

where \( g = a_7 + 3a_1a_4 + 3a_2a_3 \in \mathcal{A} \) represents the SUSY KdV equation, \( r = a_1 + \varepsilon a_3 - \varepsilon^2 a_1a_2 \in \mathcal{A}[\varepsilon] \) represents the Gardner transform, and \( f \) is a certain modification of the KdV equation.

In general when \( h \in \mathcal{A} \) one has \( h \circ r = h + \varepsilon D^2 h + \cdots \) because \( D^2 : \mathcal{A} \to \mathcal{A} \) is an ordinary derivation. Since

\[ r' = I + \varepsilon D^2 - \varepsilon^2 (a_2 I + a_1 D), \]

the difference \( h \circ r - r'h \) will always have the form \( \sum_{n=2}^{\infty} \varepsilon^n h_n \).

To compute this difference for \( h = a_7 \) we must subtract \( -(a_2 + a_1 D)a_7 \) from \( -D^6(a_1a_2) \), obtaining

\[ a_7 \circ r - r'a_7 = -3\varepsilon^2 (a_3a_6 + a_4a_5). \]

For the second term \( h = a_1a_2 \) we note first that \( (a_2 I + a_1 D)a_1a_2 = 2a_1a_2^2 \), giving

\[ (a_1a_2) \circ r = r Dr = \cdots + \varepsilon^2 (a_3a_4 - 2a_1a_2^2) + \varepsilon^3 (a_1a_2a_4 + a_2^2a_3) + \varepsilon^4 (a_1a_2^3). \]

This gives

\[ (a_1a_2) \circ r - r'(a_1a_2) = \varepsilon^2 a_3a_4 - \varepsilon^3 \chi + \varepsilon^4 \varrho \]

with \( \chi = a_1a_2a_4 + a_2^2a_3, \varrho = a_1a_2^3 \). These elements of \( \mathcal{A} \) satisfy \((a_2 I + a_1 D)\chi = D^2 \varrho \) and \((a_4 I + a_3 D)(a_1a_2) = \chi \). Thus

\[ \varepsilon^2 r' \chi = \varepsilon^2 \chi + \varepsilon^3 D^2 \chi - \varepsilon^4 D^2 \varrho, \]

and

\[ D^2((a_1a_2) \circ r - r'(a_1a_2)) = \varepsilon^2 (a_3a_6 + a_4a_5 + \chi - r'\chi). \]

In order to pass to \( D^2(a_1a_2) = a_1a_4 + a_2a_3 \), we note the operator commutator equation

\[ D^2 r' = r'D^2 - \varepsilon^2 (a_4 I + a_3 D), \]

which gives

\[ D^3 r'(a_1a_2) = r'(a_1a_4 + a_2a_3) - \varepsilon^2 \chi. \]

Together with Proposition 2 this gives

\[ (a_1a_4 + a_2a_3) \circ r - r'(a_1a_4 + a_2a_3) = \varepsilon^2 (a_3a_6 + a_4a_5 - r'\chi). \]

In combination with the formula for \( a_7 \circ r - r'a_7 \) this gives

\[ g \circ r - r'g = -3\varepsilon^2 r' \chi. \]
Taking the modified KdV equation to be represented by
\[ f = g - 3\varepsilon^2 \chi, \]
the preceding equation \( g \circ r = r'f \) shows that \( \xrightarrow{r} f \) as claimed.

By Propositions 4 and 5 there exists \( s = a_1 - \varepsilon a_3 + \cdots \in A[\varepsilon] \)
satisfying \( r \circ s = s \circ r = a_1 \). Then, by Proposition 9,
\[ f \xrightarrow{s} g, \]
that is, \( f \circ s = s'g. \)

Since both \( g \) and \( \chi \) are in \( DA \subset A \), the same is true of all the coefficients of \( \varepsilon^n \) in \( f \circ s \). If \( s_n(a_1, a_2, \ldots) \) is the corresponding coefficient of \( s \), then \( s'_n g \in DA \). This will show that all the \( s_n \) give local conserved quantities for the SUSY KdV equation.

4 Ring Extensions and Non-Local Conservation Laws

In the general situation \( D : B \to B \) of an oriented supercommutative ring and a superderivation, an element \( u \in B \) may or may not have the form \( u = Dv \) for some \( v \in B \). But for a fermionic \( u = -\bar{u} \) one can always pass to the extension \( \tilde{D} : \tilde{B} \to \tilde{B} \) where \( \tilde{B} \) is the ring of polynomials with \( B \) coefficients in a commuting indeterminate \( \lambda \), and the new superderivation is \( \tilde{D} = D + u \frac{\partial}{\partial \lambda} \). (If the extension was unnecessary then \( \tilde{D}v = u \) will have more than one solution in \( \tilde{B} \).)

The natural first example is given by \( u = a_1 \), the generator of \( A(a_1, a_2, \ldots) \), the free SUSY derivation ring on a single fermionic generator. The extension just described is \( \tilde{A}(a_0, a_1, a_2, \ldots) \), the free SUSY derivation ring on a bosonic generator \( a_0 \), with \( Da_n = a_{n+1} \) for \( n \geq 0 \).

The ring of formal power series \( A(a_0, a_1, a_2, \ldots)[\varepsilon] \) has the same universal property seen earlier in the fermionic case. That is, given any \( \tilde{D} : \tilde{B} \to \tilde{B} \) and some formal power series \( b = \sum_{n=0}^{\infty} \varepsilon^n b_n \) with all \( b_n = \bar{b}_n \) \( \in B \), the substitution operation \( g \mapsto g \circ b \) takes \( g = \sum_{n=0}^{\infty} \varepsilon^n g_n(a_0, a_1, a_2, \ldots) \) to \( g \circ b = \sum_{n=0}^{\infty} \varepsilon^n g_m(b, Db, \ldots) \).

Then \( \Phi(g) = g \circ b \) is a well-defined map of the power series ring extensions
\[ \Phi : A(a_0, a_1, a_2, \ldots)[\varepsilon] \to \tilde{B}[\varepsilon]. \]
In fact \( \Phi \) is a ring homomorphism which commutes with the respective involutions and satisfies \( \Phi D = \tilde{D} \Phi \). The proof is the same as for the propositions 1 and 2 given earlier.

Ring extensions of the fermionic ring \( A(a_1, a_2, \ldots) \) are now constructed so as to incorporate \( D^{-1} \) of all the local conserved quantities of the SUSY KdV equation. From the formulas
\[ r = a_1 + \varepsilon a_3 - \varepsilon^2 a_1 a_2 \]
for the Gardner transform and

\[ s = s_0 + \varepsilon s_1 + \varepsilon^2 s_2 + \cdots \]

for its inverse, which satisfy \( r \circ s = s \circ r = a_1 \), one can compute for example

\[
\begin{align*}
  s_0 &= a_1 \\
  s_1 &= -a_3 \\
  s_2 &= a_5 + a_1 a_2 \\
  s_3 &= -a_7 - 2a_1 a_4 - 2a_2 a_3 \\
  s_4 &= a_9 + (3a_1 a_6 + 3a_2 a_5 + 5a_3 a_4) + 2a_1 a_2 a_2.
\end{align*}
\]

It was shown before that

\[ f \circ s = s' g \]

for

\[
\begin{align*}
  g &= a_7 + 3a_1 a_4 + 3a_2 a_3 \\
  f &= g - 3\varepsilon^2 (a_1 a_2 a_4 + a_2 a_2 a_3).
\end{align*}
\]

However \( f = Dh \) for \( h = (a_6 + 3a_2 a_2 - 3a_1 a_3) + \varepsilon^2 (3a_1 a_1 a_3 - 2a_2 a_2 a_2) \).

Therefore \( D(h \circ s) = s' g \).

As pointed out before, this is a proof that \( s_0, s_1, \ldots \) are conserved quantities for the SUSY KdV equation.

For each \( s_n \) the ring extension is made which incorporates \( \lambda_n = D^{-1} s_n \). Done sucessively for \( s_0, s_1, \ldots \) this gives \( \tilde{B} = A(\lambda_0, \lambda_1, \ldots) \), the ring of polynomials in the commuting indeterminates \( \lambda_0, \lambda_1, \ldots \), with coefficients in \( A(x_1, x_2, \ldots) \). The new superderivation \( \tilde{D} : \tilde{B} \to \tilde{B} \) is \( \tilde{D} = D + \sum_{n=0}^{\infty} s_n \frac{\partial}{\partial \lambda_n} \).

Supposing \( \mu(\lambda_0, \lambda_1, \ldots) \) to be a polynomial with constant coefficients we ask for the first variation with respect to \( g \). When \( \mu = \lambda_n \) this is

\[
\begin{align*}
  \dot{\lambda}_n &= \frac{d}{dt} \bigg|_{t=0} D^{-1} (s_n \circ (a_1 + tg)) \\
  &= D^{-1} (s'_n g) \\
  &= (h \circ s)_n \in A(a_1, a_2, \ldots),
\end{align*}
\]

where \( (h \circ s)_n \) is the coefficient of \( \varepsilon^n \) in the power series \( h \circ s \).

This shows that \( \mu(\lambda_0, \lambda_1, \ldots) \) is a conserved quantity if

\[
\sum_{n=0}^{\infty} \lambda_n \frac{\partial \mu}{\partial \lambda_n} \in \tilde{D} \left( \tilde{B} \right).
\]
**Theorem 1** The coefficients of $e^{\varepsilon \lambda}$ are all nonlocal conserved quantities for the algebraic version of the SUSY KdV equation.

**Proof of Theorem 1** With

$$e^{\varepsilon \lambda} = 1 + \varepsilon \mu_1 + \varepsilon^2 \mu_2 + \cdots$$

$$\frac{\partial}{\partial \lambda} e^{\varepsilon \lambda} = \varepsilon^{n+1} e^{\varepsilon \lambda}$$

one has $\frac{\partial}{\partial \lambda} \mu_p = \mu_{p-n-1}$ and $\dot{\mu}_p = \sum_{n=0}^{p-1} \mu_{p-n-1} \lambda_n$.

This is the coefficient of $\varepsilon^{p-1}$ in the power series $e^{\varepsilon \lambda}(h \circ s)$.

Evidently $e^{\varepsilon \lambda}(h \circ s) \in \tilde{B}[\varepsilon]$.

The proof of the theorem is complete when we have shown that

$$e^{\varepsilon \lambda} (h \circ s) \in \tilde{D} \left( \tilde{B}[\varepsilon] \right).$$

However the substitution operation $\Phi(g) = g \circ \lambda$ gives a ring homomorphism

$$\Phi : A(a_0, a_1, \ldots)[\varepsilon] \rightarrow \tilde{B}[\varepsilon].$$

Obviously $\Phi(e^{\varepsilon a_0}) = e^{\varepsilon \lambda}$, while

$$\Phi(a_1) = a_1 \circ \lambda = D \lambda = s$$

$$\Phi(a_n) = D^n s \quad \text{for} \quad n \geq 1.$$

This shows that

$$\Phi(h) = h \circ s,$$

giving

$$\Phi \left( e^{\varepsilon a_0} h(a_0, a_1, \ldots, \varepsilon) \right) = e^{\varepsilon \lambda} \left( h \circ s \right).$$

The search for antiderivatives can therefore be done in the more accessible ring $A(a_0, a_1, \ldots)[\varepsilon]$.

Indeed

$$e^{\varepsilon a_0 h} = D l$$

with $l = e^{\varepsilon a_0}(F_0 + \varepsilon F_1 + \varepsilon^2 F_2)$ and $F_0, F_1, F_2$ certain fermionic elements of $A(a_1, a_2, \ldots)$.

The desired equation reduces to

$$DF_0 = a_6 + 3a_2a_2 - 3a_1a_3$$

$$DF_1 + a_1 F_0 = 0$$

$$DF_2 + a_1 F_1 = 3a_1a_2a_3 - 2a_2a_2a_2$$

$$a_1 F_2 = 0.$$
These equations are satisfied by

\[ F_0 = a_5 + 3a_1a_2, \quad F_1 = a_1a_4 - a_2a_3, \quad F_2 = -2a_1a_2a_2. \]

Because the ring homomorphism satisfies \( \Phi D = \tilde{D} \Phi \) we conclude that from

\[ \Phi(e^{s\alpha} h) = e^{\epsilon \Lambda}(h \circ s) \]

we may infer

\[ e^{\epsilon \Lambda}(h \circ s) = \tilde{D}(\Phi l). \]

This completes the proof of the theorem.

### 5 Conservation Laws for Superfields

The algebraic constructions done so far will now be applied to the SUSY KdV equation. This equation deals with superfields, which may be described as follows.

Suppose \( \Lambda \) is a finite dimensional Grassmann algebra generated by anticommuting elements \( \theta, \eta_1, \eta_2, \ldots \) which satisfy \( \theta^2 = \eta_1^2 = \eta_2^2 = \cdots = 0 \).

Any element of \( \Lambda \), after reorderings and sign changes, may be written uniquely as

\[ \phi = v(\eta_1, \eta_2, \ldots) + \theta u(\eta_1, \eta_2, \ldots). \]

Then the superderivation \( \frac{\partial}{\partial \eta} : \Lambda \to \Lambda \) is defined by \( \frac{\partial \phi}{\partial \eta} = u \).

A superfield is any infinitely differentiable function \( \phi : \mathbb{R} \to \Lambda \) and the ring of all superfields is denoted by \( C^\infty(\mathbb{R}, \Lambda) \). To avoid confusion with the algebraic case, the superderivation in this ring is written \( D_1 = \frac{\partial}{\partial \eta} + \theta \frac{\partial}{\partial x} \).

Thus \( \phi(x) = v(x) + \theta u(x) \) and \( D_1 \phi = u(x) + \theta v'(x) \).

Ring homomorphisms from algebra to analysis are given by substitution of elements of \( C^\infty(\mathbb{R}, \Lambda) \). For example if \( \phi = -\bar{\phi} \) in \( C^\infty(\mathbb{R}, \Lambda) \) one has the ring homomorphism

\[ \mathcal{A}(a_1, a_2, \ldots) \to C^\infty(\mathbb{R}, \Lambda) \]

which sends \( f(a_1, a_2, \ldots) \) to \( f \circ \phi = f(\phi, D_1 \phi, \ldots) \). This homomorphism interrelates the two superderivations, in the sense that

\[ D_1(f \circ \phi) = (Df) \circ \phi. \]

The associativity equation \( (g \circ f) \circ \phi = g \circ (f \circ \phi) \) continues to hold when \( f = -\bar{f} \) in \( \mathcal{A}(a_1, a_2, \ldots) \) and \( \phi = -\bar{\phi} \) in \( C^\infty(\mathbb{R}, \Lambda) \), while \( g \in \mathcal{A}(a_1, a_2, \ldots) \) is arbitrary. (The proof is the same as for Proposition 3).

For the convergence of integrals one must work in subrings of \( C^\infty(\mathbb{R}, \Lambda) \).
Let $C_\infty^\uparrow(\mathbb{R}, \Lambda)$ be the superfields that diminish rapidly at $x = \pm \infty$ together with all derivatives. When $\Phi$ satisfies $D_1 \Phi \in C_\infty^\uparrow$, $\Phi$ and all its derivatives are bounded functions, and in particular $\frac{\partial}{\partial \theta} \Phi \in C_\infty^\uparrow$. Thus $\psi \Phi \in C_\infty^\uparrow$ when $\Psi \in C_\infty^\uparrow$ and $D_1 \Phi \in C_\infty^\uparrow$.

The non-local extension of $C_\infty^\uparrow$ may be defined to be

$$C_\infty^\uparrow \mathbb{N}L(\mathbb{R}, \Lambda) = \{ \Phi \in C_\infty^\uparrow(\mathbb{R}, \Lambda) : D_1 \Phi \in C_\infty^\uparrow(\mathbb{R}, \Lambda) \}.$$  

Then $C_\infty^\uparrow \mathbb{N}L$ is again a derivation ring, and it contains $C_\infty^\uparrow$ as an ideal. The formulas

$$\phi(x) = v(x) + \theta u(x)$$
$$D^{-1} \phi(x) = \int_{-\infty}^{x} u(s) ds + \theta v(x)$$

give an explicit mapping $D^{-1}_1 : C_\infty^\uparrow \rightarrow C_\infty^\uparrow \mathbb{N}L$, with $D^{-1}_1 D_1 \phi = \phi$ as well as $D_1 D^{-1}_1 = \phi$ for all $\phi \in C_\infty^\uparrow$.

When $\Phi(x) = V(x) + \theta U(x) \in C_\infty^\uparrow \mathbb{N}L$ one can define the integral of $\Phi(x)$ to be

$$\int \Phi = \int_{-\infty}^{\infty} u(x) dx.$$  

Thus, integration is an additive mapping from $C_\infty^\uparrow \mathbb{N}L(\mathbb{R}, \Lambda)$ to the Grassmann algebra $\Lambda$.

And, when $\phi \in C_\infty^\uparrow \subset C_\infty^\uparrow \mathbb{N}L$ one has

$$\int D_1 \phi = 0.$$  

These preparations done we turn to the SUSY KdV equation, which is represented by $g = a_7 + 3a_1 a_4 + 3a_2 a_3$ : if $\phi$ is a time-dependent superfield then

$$\frac{\partial}{\partial t} \phi = \dot{\phi} = g \circ \phi.$$  

With $s = \sum_{n=0}^{\infty} \varepsilon^n s_n(a_1, a_2, \ldots)$ the inverse of Gardner transform, and any $\bar{\phi} = -\phi \in C_\infty^\uparrow$, we define

$$\Phi = D_1^{-1}(s \circ \phi),$$

a formal power series with $C_\infty^\uparrow \mathbb{N}L$ coefficients.

Then

$$J(\phi) = \int e^{\varepsilon \Phi}$$

will be shown to be a power series whose coefficients are nonlocal conservation laws for the SUSY KdV equation.

To compute $\dot{\Phi} = \frac{d}{dt}|_{t=0} \Phi(\phi + t \dot{\phi})$ we recall first that

$$\frac{d}{dt}|_{t=0} s \circ (a_1 + tg) = D(h \circ s)$$
in the ring of formal power series $A(a_1, a_2, \ldots)[\varepsilon]$, with $h = \sum_0^\infty h_n(a_1, a_2, \ldots)$ as computed before.

The substitution homomorphism given by $\phi$ then gives $\frac{d}{dt}|_{t=0} s \circ (\phi + t\dot{\phi}) = D_1(h \circ s \circ \phi)$. Consequently

$$
\Phi = \frac{d}{dt}|_{t=0} D_1^{-1} \left( s \circ \left( \phi + t\dot{\phi} \right) \right) = h \circ s \circ \phi.
$$

Since $\dot{J} = \varepsilon \int e^{\varepsilon \Phi} \dot{\Phi}$, the proof will be complete when it has been shown that

$$
\int e^{\varepsilon \Phi} (h \circ s \circ \phi) = 0.
$$

However, it was shown earlier that there exists $F = \sum_0^\infty \varepsilon^n F_n(a_1, a_2, \ldots)$ satisfying

$$
e^{\varepsilon a_0} h = D(e^{\varepsilon a_0} F)
$$

in the ring $A(a_0, a_1, \ldots)[\varepsilon]$ of formal power series with $A(a_0, a_1, \ldots)$ coefficients. Under the operation of substitution by $\Phi$ this equation becomes

$$
e^{\varepsilon \Phi} (h \circ s \circ \phi) = D_1 \left( e^{\varepsilon \Phi} (F \circ s \circ \phi) \right)
$$

in the ring $C_{\infty}^{NL}(\mathbb{R}, \Lambda)[\varepsilon]$, because $D_1 \Phi = s \circ \phi$ and $h$ and $F$ do not involve $a_0$.

The coefficients of $e^{\varepsilon \Phi}$ are in $C_{\infty}^{NL}$, while the coefficients of $F \circ s \circ \phi$ are in $C_{\infty}^{\infty}$.

Therefore their product is in $C_{\infty}^{\infty}$, where $\psi \in C_{\infty}^{\infty}$ implies $\int D_1 \psi = 0$.

This completes the proof that $\dot{J}(\phi) = \int e^{\varepsilon \Phi}$ is a conserved quantity for the SUSY KdV equation.

In closing we may compare $\int e^{\varepsilon \Phi}$ with some conserved quantities found in the literature. Starting with $\phi = -\phi \in C_{\infty}^{\infty}$, the first few coefficients of $\Phi = \sum_0^\infty \varepsilon^n \Phi_n = D^{-1}(s \circ \phi)$ can be found from the corresponding coefficients of the inverse Gardner transform $\sum_0^\infty \varepsilon^n s_n(a_1, a_2, \ldots)$. After replacing $D_1$ by the shorter notation $D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x}$ one finds that

$$
\Phi_0 = D^{-1}\phi \\
\Phi_1 = -D\phi \\
\Phi_2 = D^3\phi + D^{-1}(\phi D\phi) \\
\Phi_3 = -D^5\phi - 2(D\phi)^2 + 2\phi(D^2\phi).
$$

These elements of $C_{\infty}^{NL}$ are all bosonic, and the first few coefficients of $e^{\varepsilon \Phi} = 1 + \sum_{n=1}^\infty \Delta_n$ are

$$
\Delta_1 = \Phi_0
$$
\[ \Delta_2 = \frac{1}{2} \Phi_0^2 + \Phi_1 \]
\[ \Delta_3 = \frac{1}{6} \Phi_0^3 + \Phi_0 \Phi_1 + \Phi_2 \]
\[ \Delta_4 = \frac{1}{24} \Phi_0^4 + \frac{1}{2} \Phi_0^2 \Phi_1 + \Phi_0 \Phi_2 + \frac{1}{2} \Phi_1^2 + \Phi_3. \]

Because we are only interested in the integrals of \( \Delta_n(\phi) \), terms which fall into \( DC_i^{\infty} \) can be left out because they have identically zero integrals. For example

\[ \Phi_0 \Phi_1 = - (D^{-1}\phi) (D\phi) = -D ((D^{-1}\phi) \phi). \]

After rewriting the \( \Delta_n \) in terms of \( D^{-1}\phi, \phi, \ldots \) and simplifying in the manner just described we arrive at

\[ \Delta_1 = D^{-1}\phi \]
\[ \Delta_2 = \frac{1}{2} (D^{-1}\phi)^2 \]
\[ \Delta_3 = \frac{1}{6} (D^{-1}\phi)^3 + D^{-1}(\phi D\phi) \]
\[ \Delta_4 = \frac{1}{24} (D^{-1}\phi)^4 - \frac{1}{2} (D\Phi)^2 + (D^{-1}\phi) D^{-1}(\phi D\phi). \]

Replacing \( \phi \) by \( -\phi \) in these formulas we obtain constant multiples of the integrands which appear in the nonlocal conserved quantities \( J_{\frac{1}{2}}, J_{\frac{3}{2}}, J_{\frac{5}{2}}, J_{\frac{7}{2}} \) presented in reference [10].

The sign change comes from the ambiguity \( g = \pm a_7 + 3a_1a_4 + 3a_2a_3 \) in the definition of the SUSY KdV equation.

The two versions are interchanged by the transformation \( T : A(a_1, a_2, \ldots) \to A(a_1, a_2, \ldots) \) given by \( (Tf)(a_1, a_2, \ldots) = -f(-a_1, -a_2, \ldots) \).

This transformation is not a ring homomorphism but it satisfies \( DT = TD \). In terms of the substitution operation, \( Tg = -(g \circ (-a_1)) \) in general, with \( Tf = (-a_1) \circ f \circ (-a_1) \) when \( \bar{f} = -f \).

The associativity and the cancellation \( (-a_1)(-a_1) = a_1 \) then give

\[ T(g \circ f) = (Tg) \circ (Tf). \]

Therefore \( T \) also exchanges the respective Gardner transforms and conservation laws.

6 Conclusions

We introduced the fermionic substitution semigroup and the resulting Gardner category. We proved several propositions concerning their algebraic structure. This algebraic framework allows to define general Gardner transformations between different non-linear SUSY
differential equations. We then introduced a SUSY ring extension which allowed to consider in the same algebraic setting the construction of all the known non-local conserved quantities of $N = 1$ SKdV.

The algebraic version of the non-local conserved quantities was solved in terms of the exponential function applied to the $D^{-1}$ of the local conserved quantities of $N = 1$ SKdV. Finally the same formulas were shown to work for rapidly decreasing superfields.

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