Large-N scaling behavior of the ground-state energy, fidelity, and the order parameter in the Dicke Model

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Abstract

Within the numerically exact solution to the Dicke model proposed previously, we study the quantum criticality in terms of the ground-state (GS) energy, fidelity, and the order parameter. The finite size scaling analysis for the average fidelity susceptibility (FS) and second derivative of GS energy are performed. The correlation length exponent is obtained to be $\nu = 2/3$, which is the same as that in Lipkin-Meshkov-Glick model obtained previously, suggesting the same universality. It is observed that average FS and second derivative of GS energy show similar critical behavior, demonstrating the intrinsic relation in the Dicke model. The scaling behavior for the order parameter and the singular part of the GS energy at the critical point are also analyzed and the obtained exponents are consistent with the previous scaling hypothesis in $1/N$ expansion scheme.

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I. INTRODUCTION

The Dicke model\[1\] describes the interaction of N two-level atoms (qubits) with a single bosonic mode, which is a fundamental model in quantum optics. It exhibits a "superradiant" quantum phase transition (QPT)\[2\] in the thermodynamic limit. In recent years, the Dicke model has attracted considerable attentions\[3, 4, 5, 6, 7, 8, 9, 10, 11\]. On the one hand, the quantum entanglement\[12\] and Berry phase \[13\] have been used to characterize the QPTs. On the other hand, the Dicke model is closely related to many recent interesting fields in quantum optics and condensed matter physics, such as the superradiant behavior by an ensemble of quantum dots\[14\] and Bose-Einstein condensates\[15\], and coupled arrays of optical cavities\[16\].

The scaling exponents obtained at the critical points are of significance to distinguish the universality class of the QPTs among various models. By a modified Holstein-Primakoff approach, Vidal and Dusuel has predicted theoretically the nontrivial scaling exponent for several quantities in the Dicke model\[8\]. To our knowledge, the finite-size studies in the Dicke model were previously limited to numerical diagonalization in Bosonic Fock state\[3, 4, 5\] in small size system $N \leq 35$, the adiabatic approximation\[6\]. Recently, by using extended bosonic coherent states, a numerically exact technique to solve the Dicke model for large system size was developed by the present authors\[11\]. Therefore, numerical calculations of scaling exponents for some key quantities based on convincing treatment for large system size are clearly desirable for confirmation.

Recently, a concept in quantum information theory, i.e. the fidelity has been extensively used to identify the QPTs in various many-body systems from the perspective of the ground-state (GS) wave functions\[17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27\] (For more details, please refer to a review article\[28\]). In a mathematical sense, the fidelity is the overlap between two ground states where the transition parameters deviate slightly. However, the fidelity depends on an arbitrary small amount of the transition parameters, which in turn yields an artificial factor. Zanardi et al\[19\] introduced the Riemannian metric tensor and You et al\[20\] proposed the fidelity susceptibility (FS) to avoid this problem independently. The leading term of the fidelity are focused to account for the singularity of QPTs in both methods. It is thus implied that the fidelity have no singular behavior in the Kosterliz-Thouless phase transition, which can distinguish different transition types\[21, 22, 26\]. In addition, the
intrinsic relation between the GS fidelity and the derivative of GS energy has been studied and it is observed that they play an equivalent role in identifying the QPTs [22]. Since no \textit{a priori} knowledge of the order parameter is needed, it may be a great advantage to study the scaling behavior of FS to characterize the universality in quantum critical phenomena [21].

In this paper, we extended our previous numerically exact technique to calculate the GS energy, FS, and the order parameter in the Dicke model for finite N. The scaling behavior of FS, the second derivative of GS energy, and the order parameter of the QPT will be analyzed. We will also study the singular part of the GS energy, which is a further correction to the regular part [8].

II. MODEL HAMILTONIAN

Without the rotating-wave approximation, the Hamiltonian of $N$ identical atoms interacting with a single bosonic mode reads

$$H = \omega a^+ a + \omega_0 J_z + \frac{2\lambda}{\sqrt{N}}(a^+ + a) J_x,$$

(1)

where $a^+$ and $a$ are the bosonic annihilation and creation operators, $\omega_0$ and $\omega$ are the transition frequency of the qubit and the frequency of the single bosonic mode, $\lambda$ is the coupling constant. $J_x$ and $J_z$ are the usual angular momentum. There is a conserved parity operator $\Pi = e^{i\pi(J_z + N/2 + a^+ a)}$, which commutes with the Hamiltonian (1).

In our previous numerically exact approach [11], the wave function can be expressed in terms of the basis $\{|\varphi_n\rangle_b \otimes |j, n\rangle\}$ where $|j, n\rangle$ is the Dicke state with $j = N/2$ and $|\varphi_n\rangle_b$ is the bosonic state. The latter is given by

$$|\varphi_n\rangle_b = \sum_{k=0}^{N_{tr}} c_{n,k} \frac{1}{\sqrt{k!}}(a^+ + g_n)^k e^{-g_n a^+ - g_n^2/2} |0\rangle_a,$$

(2)

where $g_n = 2\lambda n/\omega\sqrt{N}$, $N_{tr}$ is the truncated bosonic number in the Fock space of the new operator $A = a + g_n$, the coefficient $c_{n,k}$ can be determined through the exact Lanczos diagonalization [29].

The driving Hamiltonian in the Dicke model is

$$H_1 = \frac{2}{\sqrt{N}}(a^+ + a) J_x,$$

(3)
and the transition parameter is $\lambda$. According to the definition, the FS is given by

$$S_F(\lambda) = \sum_{n \neq 0} \frac{\left| \langle \Psi_n | H_1 | \Psi_0 \rangle \right|^2}{[E_n - E_0]^2}$$

(4)

where $\Psi_n$ is the n-th eigen-states of the Hamiltonian (1). In terms of Eq. (2), we have

$$\langle \Psi_n | H_1 | \Psi_0 \rangle = 2 \sqrt{N} \sum_{m,i,j} \left( \frac{N}{2} - m \right) C_{m,i}^{n*} C_{m,j}^0 [\sqrt{j + 1} \delta_{i,j+1} + \sqrt{j} \delta_{i,j-1} - 2g_m \delta_{i,j}]$$

After the FS is well defined above, the fidelity is readily obtained through an artificially introduced small amount of the transition parameter $\delta \lambda$

$$F(\lambda, \delta \lambda) = \sqrt{1 - \delta \lambda^2 S_F(\lambda)}$$

(5)

Although the FS might not be a intensive quantity in some cases, for convenience, we will discuss the average FS: $\chi_F = S_F(\lambda)/N$ throughout the paper. It is interesting to note that the FS can be obtained through the wave function without the knowledge of the order parameter.

III. FINITE-SIZE SCALING ANALYSIS

We have studied the leading finite-size corrections to the GS energy in the Dicke model. For convenience, we have introduced one dimensionless parameter $D = \omega_0/\omega$. The QPT occurs at the critical point $\lambda_c = 0.5$ in the thermodynamic limit. The scaling exponent of the leading finite-size corrections is obtained to be $-1.0 \pm 0.02$ for three typical different values $D = 0.1, 1, 10$, consistent with that by a modified Holstein-Primakoff approach. In this paper, we first study the scaling behavior of the singular part of the GS energy, which can be obtained by subtracting the leading contribution $c_0 + c_1/N$, with $c_0 = -\omega_0/2$ and $c_1 = \frac{1}{2} \left[ -\omega - \omega_0 + (\omega^2 + \omega_0^2)^{1/2} \right]$ from the total GS energy. We exhibit the curve of $e_0 - (c_0 + c_1/N)$ as a function of $N$ at the critical point for different typical values of $D$ in Fig. 1. One can obviously observe that the scaling exponent estimated from the slope in all cases are $-1.325 \pm 0.005$, which is very close to $-4/3$, confirming the previous theoretical prediction in a modified Holstein-Primakoff approach.
FIG. 1: Scaling of the singular part of the GS energy $e_0 - \left(c_0 + \frac{c_1}{N}\right)$ as a function of $N$ at the critical point for $D = 0.1, 1, 5$. The inset shows the corresponding slope versus $1/N$.

Next, we illustrate the scaling behavior of the average FS. The finite-size scaling ansatz for the average FS to analyze the QPT take the form

$$\frac{\chi_{\text{max}}^F - \chi_F}{\chi_F} = f[N^\nu(\lambda - \lambda_{\text{max}})]$$

(6)

where $\chi_{\text{max}}^F$ is the value of average FS at the maximum point $\lambda_{\text{max}}$, $f$ is the scaling function and $\nu$ is the correlation length critical exponent. This function should be universal for large $N$ in the second-order QPTs, which is independent of the order parameter. As shown in Fig. 2, with $\nu = 2/3$, an excellent collapse in the critical regime is achieved according to Eq.(6) in the curve for different large size for three typical values of $D$. Beyond the critical regime, the collapse becomes poor. As $N$ increases, the critical regime become wider. It is demonstrated that $\nu$ is a universal constant and does not depended on the parameter $D$. It is also implied from the collapse that the correlation length behaves like $\xi \propto |\lambda - \lambda_c|^{-2/3}$.

It is very interesting to note that the value of $\nu$ is the same as that in the Lipkin-Meshkov-Glick (LMG) model obtained analytically [30] and numerically [27], suggesting the same universality in the Dicke and LMG model.

Fig. 3 shows the scaled average FS at the maximum point as a function of $N$ for different values of $D$ in log-log scale. A power law behavior $\chi_{\text{max}}^F \propto N^{\mu}$ exists in the large $N$. The finite-size exponents extracted from all curves tend to a converging value $\mu = 0.33 \pm 0.02$. This exponent is also independent of the value of $D$, and then is a universal constant. It is also very close to that in the LMG model, providing another evidence of the same universality class of these two models.

To exhibit the overall properties of the average FS in the whole coupling regime, we
FIG. 2: Finite-size scaling of the average FS according to Eq. (6) at the critical point for (a) $D = 0.1$, (b) $D = 1$, and (c) $D = 5$.

FIG. 3: Scaling of the maximum of the average FS as a function of $N$ at the critical point for $D = 0.1, 1, 5$. The inset shows the corresponding slope versus $1/N$.

calculate the average FS as a function of $\lambda$ for different size. From Fig. 4, we can see that $S_F = N\chi_F$ is a intensive quantity when $\lambda > \lambda_c$, while $\chi_F$ is a intensive quantity when $\lambda < \lambda_c$. It is pointed out in Ref. [21, 27] that the intensive average FS in the thermodynamic
limit scales generally like
\[ \chi_F \propto \frac{1}{|\lambda - \lambda_c|^\alpha} \]  
(7)
in the vicinity of the critical point. Within the similar analysis, we can get the exponent \( \alpha = 1/2 \) when \( \lambda > \lambda_c \) and \( \alpha = 2 \) when \( \lambda < \lambda_c \) through the relation \( \alpha = \mu/\nu \), which is readily derived from the above scaling ansatz. Also it is interesting to note that the average FS becomes divergent as the system size increases, demonstrating a Landau-type transitions in the Dicke model.

To illustrate the intrinsic relation between the ground-state fidelity and the ground state energy \( e_0 \) in the Dicke model, we calculate the second derivative of the GS energy for different lattice size at \( D = 1 \), which are shown in Fig. 5. Interestingly, similar to the average FS, the second derivative of the GS energy also shows divergent behavior around the critical point with increasing system size. This divergent behavior is also similar to that observed in 1D transverse field Ising model\[22\]. The only difference is that the maximum point \( \lambda_{\text{max}} \) approaches the critical point with the increasing system size in the Dicke model more fast than in the 1D transverse field Ising model, due to the smaller value of the correlation length exponent \( \nu \) in the former model.

Because the scaling \( \lambda - \lambda_{\text{max}} \propto N^{-2/3} \) holds, we try to plot the dimensionless quantities \( 1 - \vartheta/\vartheta_{\text{max}} \), where \( \vartheta \) is either the average FS or the second derivative of the GS energy for different size as a function of \( N^{2/3}(\lambda - \lambda_{\text{max}}) \) in the insets of Figs. 4 and 5. It is interesting to observe that the second derivative of the GS energy exhibits the same scaling behavior around the critical point as the average FS. It follows that these two quantities play the
same roles in the characterizing the QPTs in the Dicke model. In the 1D transverse field Ising model, the value of the critical exponent \( \nu = 1 \), which is equal to that in the classical two-dimensional Ising model, the scaling \( \lambda - \lambda_{\text{max}} \propto N^{-1} \) should be satisfied, which was just used in the Figs. 1 and 2 in Ref. [22].

We then perform the finite size scaling analysis on the order parameter of the QPT, i.e. the expectation value of the photon number per atom in the ground-state \( \langle a^\dagger a \rangle / N \). In the thermodynamic limit, this quantity changes from zero to finite value smoothly when crossing the critical point. In Fig. 6, we present this quantity as a function of \( N \) for different values of \( D \) in log-log scale. Derivative of these curves are plotted in the inset. The exponent of the order parameter is estimated to be \(-0.66 \pm 0.02\), which is consistent well with the predicted value \( 2/3 \) [8] by using the diagonalizing a expanded Hamiltonian at order \( 1/N \) based on Holstein-Primakoff representation.

Finally, to describe the new numerically exact approach with extended coherent states in more intuitive detail and see how it connects to results in the thermodynamic limit, we calculate the finite but large \( N \) wave functions, as done in Ref. [3]. We calculate the ground-state wave function \( \Psi(x, y) \) in the x-y representation for \( N = 1000 \) and \( D = 1 \) at various \( \lambda \) which cover \( \lambda_c(N = 1000) \). \( \lambda_c(N = 1000) \) falls in the range of \([0.5070, 0.5071]\), according to the position of the peak for FS. After tedious calculations, the numerical results for the wave function are shown in Fig. 7, where the displacement is not removed. One can observe that the wave packet becomes stretched in a direction with an angle around \( \pi/4 \) as \( \lambda \) increases. This stretching increases up to \( \lambda_c(N) \), where the wave function is not divergent.
due to the finite-size effect. Exceeding $\lambda_c$, the wave function splits into two peaked one. It is also a finite-size effect and not shown in the thermodynamic limit [cf. Fig. 6 in Ref. [3]]. With the further increase of $\lambda$, the two lobes start mixing, a single-peaked wave function is then formed like in the thermodynamic limit [3]. It follows that the essential feature in the thermodynamic limit already appears for large $N$, say $N = 1000$. Note that in Fig. 12 of Ref. [3], two lobes still separate even at $\lambda = 0.7$ for $j = 5$ (i.e. $N = 10$), different from the present observation for $N = 1000$.

IV. CONCLUSION

In summary, based on our previously proposed numerically exact technique in the finite-size the Dicke model, we study the quantum criticality in terms of the GS energy, fidelity, and the order parameter. By subtracting the regular part of the GS energy, we perform the analysis for the scaling behavior of the singular part of the GS energy at the critical point. The obtained exponent is very close to $4/3$, which agree well with the previous theoretical prediction in $1/N$ expansion scheme based on the Holstein-Primakoff transformation. Then, we perform the finite-size scaling analysis for the average FS. Several scaling exponents are obtained: $\nu = 2/3$, $\mu = 4/3$, and $\alpha = 1/2$ when $\lambda > \lambda_c$ and $\alpha = 2$ when $\lambda < \lambda_c$. All this exponents are the same as those in Lipkin-Meshkov-Glick model obtained both analytically and numerically, suggesting the same universality of these two models. We further study the scaling behavior of the average FS and the second derivative of GS energy, and observed that

FIG. 6: Scaling of the order parameter $\langle a^\dagger a \rangle / N$ as a function of $N$ at the critical point for $D = 0.1, 1, \text{ and } 5$. The inset shows the slope versus $1/N$. 
FIG. 7: The ground-state wave function \( \Psi(x, y) \) in the x-y representation for \( N = 1000 \) and \( D = 1 \) at \( \lambda = 0, 0.5, 0.507, 0.5071 \) (from left to right in the upper row), 0.525, 0.55, 0.555, 0.7 (the low row). Note that \( 0.5070 < \lambda_c(N = 1000) < 0.5071 \).

these two quantities play the same roles in the QPTs in the Dicke model. Brief comparisons with the 1D transverse field Ising model are also carried out. We also perform the analysis of the finite size scaling effect of the order parameter, and observe that the order parameter vanishes as \( N^{-2/3} \) at the critical point, consistent with the previous theoretical prediction. Finally, we calculate the finite but large N wave functions. It is observed that the essential feature in the thermodynamic limit has already shown up in large system size \( (N = 1000) \).

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