Probability Logic: 
A Model Theoretic Perspective

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Abstract

In this paper (propositional) probability logic (\(PL\)) is investigated from model theoretic point of view. First of all, the ultraproduct construction is adapted for \(\sigma\)-additive probability models, and subsequently when this class of models is considered it is shown that the compactness property holds with respect to a fragment of \(PL\) called basic probability logic (\(BPL\)). On the other hand, when dealing with finitely-additive probability models, one may extend the compactness property for a larger fragment of probability logic, namely positive probability logic (\(PPL\)). We finally prove that while the Löwenheim-Skolem number of the class of \(\sigma\)-additive probability models is uncountable, it is \(\aleph_0\) for the class of finitely additive probability models.

Keywords: Probability modal logic, Type spaces, Ultraproduct construction, Henkin method, Compactness property, Löwenheim-Skolem number.

1 Introduction and Preliminaries

Propositional probability logic (\(PL\)) is a framework for specifying and analyzing properties of structures involving probability, e.g. probability spaces or Markov processes. This logic provides rules of reasoning about these structures. This natural logic is a modal logic in which bounds on probability are treated as modal operators. So, for

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each $\alpha \in \mathbb{Q} \cap [0,1]$, the language of $PL$ includes a modal operator $L_\alpha$ interpreted as (an agent) assigns probability at least $\alpha$. This logic is shown to be useful in many research areas such as theoretical computer science, economics and philosophy. For example this logic might be used to reason about behavior of a program under probabilistic assumptions about inputs. Also one can highlight how probability systems and particular probability logic play a crucial role in game theory. A type space is an example of a probabilistic system, introduced by Harsanyi in [10], provides an implicit description of beliefs in games with incomplete information. So, in particular $PL$ is known to be useful for studying of type spaces.

This logic is studied from different perspectives. There is a rich source of papers involving full axiomatization of this logic, aiming to show different type of completeness results, [5] [11] [23, 24]. A coalgebraic point of view is another source of research in this area, [8] [15]. The aim of this paper is to study this logic from model theoretic perspectives.

To be able to state our results in technical terms we review basic concepts of $PL$.

Assume that $P$ is a countable set of propositional variables. The syntax of probability logic is obtained by adding countable probability modal operators $L_r, M_r$ for each $r \in \mathbb{Q} \cap [0,1]$ to propositional logic. When applying the operator $L_r$ to a formula $\varphi$, then $L_r \varphi$ is interpreted as “the formula $\varphi$ has probability at least $r$”. In the same way, the intended meaning of formula $M_r \varphi$ is “the formula $\varphi$ has probability at most $r$”.

**Definition 1.1.** Formulas of probability logic ($PL$) is defined by the following grammar:

$$PL := p \mid \neg \varphi \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid L_r \varphi \mid M_r \varphi,$$

where $p \in P$ and $r \in \mathbb{Q} \cap [0,1]$.

In the following we also consider two fragments of probability logic, namely, basic probability logic and positive probability logic.

**Definition 1.2.** The basic probability logic ($BPL$) and respectively positive probability logic ($PPL$) are defined by the following grammars:

$$BPL := p \mid \neg p \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid L_r \varphi.$$

$$PPL := BPL \mid M_r \varphi.$$

Note that in both $BPL$ and $PPL$ the negation only applies to elements of the set $P$. Furthermore, $BPL$ is a proper fragment of $PPL$ in which applying the modal
operators $L_r$ is only allowed. Note that $BPL \nsubseteq PPL \nsubseteq PL$. We also name these fragments as probability logics.

The other logical connectives $(\lor, \to, \leftrightarrow)$ have their standard definitions.

To interpret the formulas in $PL$ we have to define the notion of probability models.

For any measurable space $(\Omega, \mathcal{A})$ let $\Delta(\Omega, \mathcal{A})$ be the measurable space of all $\sigma$-additive probability measures on $\Omega$ whose $\sigma$-algebra generated by the sets

$$\{\mu \in \Delta(\Omega, \mathcal{A}) \mid \mu(E) \geq r\} \text{ for all } E \in \mathcal{A} \text{ and } r \in \mathbb{Q} \cap [0, 1].$$

**Definition 1.3.** A type space over a measurable space $(\Omega, \mathcal{A})$, is a triple $M = (\Omega, \mathcal{A}, T)$ where $T$ is a measurable function from $\Omega$ to $\Delta(\Omega, \mathcal{A})$.

$\Omega$ and $T$ are respectively called a set of states (or possible worlds) and a type function. It follows from the above definition, for $w \in \Omega$, $T(w)$ defines a probability measure on the $\sigma$-algebra $\mathcal{A}$. Furthermore, its measurability indicates that for each $E \in \mathcal{A}$ and $r \in \mathbb{Q} \cap [0, 1]$,

$$\{\omega \in \Omega \mid T(\omega, E) \geq r\} \in \mathcal{A}.$$

Type spaces are regarded as semantical devices for probability logics. We show that probability logic and its fragments introduced above have different model theoretic features.

**Definition 1.4.** A probability model is a tuple $M = (\Omega, \mathcal{A}, T, v)$ where the triple $(\Omega, \mathcal{A}, T)$ is a type space and $v : \mathcal{P} \to \mathcal{A}$ is a valuation function which assigns to each proposition $p \in \mathcal{P}$ the measurable set $v(p) \in \mathcal{A}$.

**Definition 1.5.** A finitely additive probability model is a tuple $M = (\Omega, \mathcal{A}, T : \Omega \times \mathcal{A} \to [0, 1], v : \mathcal{P} \to \mathcal{A})$ where

- $\mathcal{A}$ is an algebra over $\Omega \neq \emptyset$.
- For each $w \in \Omega$, $T(w)$ defines a finitely additive measure on $\mathcal{A}$.
- $T$ is a measurable function, i.e. for each $E \in \mathcal{A}$ and $r \in \mathbb{Q} \cap [0, 1]$, $\{w \in \Omega \mid T(w)(E) \geq r\} \in \mathcal{A}$.

A model $M$ with a distinguished point $w \in \Omega$ is called a pointed probability model and denoted by $(M, w)$.

Denote the class of pointed probability models by $\mathcal{PM}$ and respectively the class of finitely additive probability pointed models by $\mathcal{FPM}$. Notice that $\mathcal{PM} \nsubseteq \mathcal{FPM}$. 

3
Definition 1.6. For a pointed (finitely additive) probability model $(\mathcal{M}, w)$ and a formula $\varphi \in PL$ the satisfaction relation $\mathcal{M}, w \models \varphi$ is defined inductively in the usual way for propositional variables and boolean connectives. For $L_r, M_r$ operators, if we assume $[\varphi]_{\mathcal{M}} = \{w \in \Omega \mid \mathcal{M}, w \models \varphi\}$, then

\[
\mathcal{M}, w \models L_r \varphi \quad \text{if and only if} \quad T(w)([\varphi]_{\mathcal{M}}) \geq r
\]

and

\[
\mathcal{M}, w \models M_r \varphi \quad \text{if and only if} \quad T(w)([\varphi]_{\mathcal{M}}) \leq r.
\]

We often omit the subscript $\mathcal{M}$ from $[\varphi]_{\mathcal{M}}$ and write $[\varphi]$ when no confusion can arise. Note that, by definition of $v$ and measurability of $T$, it is easy to see that $[\varphi] \in A$ for any formula $\varphi$. Notice that for any formula $\varphi$ and $r \in \mathbb{Q} \cap [0,1]$, $\mathcal{M}, w \not\models L_r \varphi$ if and only if $\mathcal{M}, w \models M_s \varphi$, for some $s < r$. A similar statement holds for $M_r \varphi$. This means that the negation of positive formulas can be defined by an infinite disjunction of positive formulas.

The other syntactical and semantical components of probability logics can be defined in the usual way. In particular, if $\mathcal{L} \in \{PL, BPL, PPL\}$ then any set of $\mathcal{L}$-formulas is called an $\mathcal{L}$-theory. Let $\mathcal{K}$ be a subclass of $\mathcal{FPM}$. An $\mathcal{L}$-theory $T$ is satisfiable in $\mathcal{K}$ if there exists a pointed model $(\mathcal{M}, w) \in \mathcal{K}$ such that $\mathcal{M}, w \models \varphi$, for each $\varphi \in T$. Likewise, $T$ is finitely satisfiable in $\mathcal{K}$ if every finite subset of $T$ is satisfiable in $\mathcal{K}$. An $\mathcal{L}$-theory $T$ is maximally satisfiable (respectively maximally finitely satisfiable) if it is maximal in the poset of satisfiable $\mathcal{L}$-theories (respectively finitely satisfiable $\mathcal{L}$-theories) ordered by inclusion relation. We say that the logic $\mathcal{L}$ has the compactness property with respect to class $\mathcal{K}$ if an $\mathcal{L}$-theory $T$ is satisfiable in $\mathcal{K}$ if and only if $T$ is finitely satisfiable in $\mathcal{K}$. In this paper the following results are established.

1. (Theorem 2.5) $BPL$ has the compactness property with respect to $PM$.

2. (Theorem 2.13) $PPL$ has the compactness property with respect to $FPM$.

We also study the Löwenheim-Skolem number of the class of probability and finitely additive probability models. Recall that the Löwenheim-Skolem number of a class $\mathcal{K}$ of logic $\mathcal{L}$ is the least infinite cardinal $\kappa$ such that every satisfiable $\mathcal{L}$-theory has a model of size at most $\kappa$.

1. (Theorem 3.1) Let $\lambda$ be the Löwenheim-Skolem number of probability models with respect to probability logic. Then $\aleph_0 < \lambda \leq 2^{\aleph_0}$.

2. (Theorem 3.4) The Löwenheim-Skolem number of the class of finitely additive probability models with respect to probability logic is $\aleph_0$. 


2 Compactness Property

In this section we study the compactness property for \( BPL \) and \( PPL \).

2.1 Compactness for \( BPL \)

In this subsection we show that the basic probability logic, \( BPL \), satisfies the compactness property with respect to \( PM \). It is known that the compactness property does not hold for (full) probability logic. To see this consider the \( PL \)-theory

\[
T = \{L_{\frac{1}{2}, \frac{n}{2n+1}}p \mid n \in \mathbb{N}\} \cup \{\neg L_{\frac{1}{2}}p\}. \tag{1}
\]

It is easy to show that \( T \) is finitely satisfiable but not satisfiable.

The failure of compactness in \( PL \) is partly due to this fact that by using the negation one can express the strict inequality. However, as \( BPL \) only applies the negation on propositions the above example is not a \( BPL \)-theory. In fact then restrict ourselves to \( BPL \) we will see that the compactness holds in this logic. To achieve this, we adapt the ultraproduct construction for probability models and show that the Loś theorem holds for basic formulas, which means that \( BPL \) enjoys the compactness property. We recall first some primary notions related to the ultraproduct construction.

Let \( U \) be an ultrafilter over a non-empty set \( I \) and \( (a_i)_{i \in I} \) be a sequence of elements from \( \mathbb{R} \), the set of real numbers. The \( U \)-limit of this sequence, denoted by \( \lim_U a_i \), is an element \( r \in \mathbb{R} \) such that for every \( \epsilon > 0 \) we have \( \{i \in I \mid |a_i - r| < \epsilon\} \in U \). It is known that each bounded sequence of elements of \( \mathbb{R} \) has a unique limit over each ultrafilter. Furthermore, if \( (a_i)_{i \in I} \) is a bounded sequence of elements of \( \mathbb{R} \) and \( U \) is an ultrafilter over \( I \) then

- If \( \{i \in I \mid a_i \geq r\} \in U \), then \( \lim_U a_i \geq r \).
- If \( \lim_U a_i > r \), then \( \{i \in I \mid a_i > r\} \in U \).
- \( \lim_U a_i \geq r \) if and only if for every \( r' < r \) we have \( \{i \in I \mid a_i \geq r'\} \in U \).

For a family \( \langle \Omega_i : i \in I \rangle \) of sets indexed by \( I \), let \( \prod_{i \in I} \Omega_i \) be the Cartesian product of this family defined as the set \( \prod_{i \in I} \Omega_i = \{(w_i)_{i \in I} \mid w_i \in \Omega_i\} \). Two elements \( (w_i)_{i \in I} \) and \( (v_i)_{i \in I} \) in \( \prod_{i \in I} \Omega_i \) are \( U \)-equivalent, denoted by \( (w_i)_{i \in I} \sim_U (v_i)_{i \in I} \), if \( \{i \in I \mid w_i = v_i\} \in U \). Clearly \( \sim_U \) defines an equivalence relation on \( \prod_{i \in I} \Omega_i \). Let \( (w_i)_{U} \) be the equivalence class of \( (w_i)_{i \in I} \), and the resulting \( \prod_U \Omega_i \) be the set of all equivalence classes.
Now suppose $(\Omega_i, A_i, T_i)_{i \in I}$ is a family of type spaces. Then for each sequence $(A_i)_{i \in I}$ with $A_i \in A_i$ set

$$\textstyle (A_i)_U = \{(w_i)_U \in \prod U \Omega_i \mid \{i \in I \mid w_i \in A_i\} \in U\},$$

and let $\mathcal{A} = \{(A_i)_U \mid A_i \in A_i\}$. It is easy to see that $\mathcal{A}$ forms a boolean algebra over $\prod U \Omega_i$. Now, define the function $T^* : \prod U \Omega_i \times \mathcal{A} \to [0, 1]$ as follows:

$$\textstyle T^*((w_i)_U)((A_i)_U) = \lim_U T_i(w_i)(A_i). \quad (\ast)$$

Note that since for each $(w_i)_U$ the sequence $(T_i(w_i))_{i \in I}$ is a bounded sequence of real numbers, the $U$-limit $\lim_U T_i(w_i)(A_i)$ exists. Moreover, if $(w_i)_U \sim_U (v_i)_U$ then for each $(A_i)_U \in \mathcal{A}$ we have $\lim_U T_i(w_i)(A_i) = \lim_U T_i(v_i)(A_i)$. Also $(A_i)_U = (B_i)_U$ implies that $\{i \in I \mid A_i = B_i\} \in U$. Hence $(\ast)$ is well-defined.

**Lemma 2.1.** For each $(w_i)_U \in \prod U \Omega_i$, the function $T^*((w_i)_U)(.)$ is a premeasure on the boolean algebra $\mathcal{A}$.

**Proof.** Fix $(w_i)_U \in \prod U \Omega_i$. First note that

$$T^*((w_i)_U)(\varnothing) = \lim_U T_i(w_i)(\varnothing) = 0.$$

Now we have to show that whenever $\{(A_i^j)_U \mid j \in \mathbb{N}\}$ is a countable family of disjoint members of $\mathcal{A}$ if $\bigcup_{j \in \mathbb{N}} (A_i^j)_U \in \mathcal{A}$, then

$$T^*((w_i)_U)(\bigcup_{j \in \mathbb{N}} (A_i^j)_U) = \sum_{j \in \mathbb{N}} T^*((w_i)_U)((A_i^j)_U).$$

By Fact A.2 we prove that for each decreasing sequence $(A_i^0)_U \supseteq (A_i^1)_U \supseteq \ldots$ of elements of $\mathcal{A}$, if $\bigcap_j (A_i^j)_U = \varnothing$ then $\lim_{j \to \infty} T^*((w_i)_U)((A_i^j)_U) = 0$.

Now suppose on the contrary that $\lim_{j \to \infty} T^*((w_i)_U)((A_i^j)_U) > 0$. So, there exists $\epsilon > 0$ such that for all $j \in \mathbb{N}$ we have $\lim_U T_i(w_i)(A_i^j) > \epsilon$. So $I_j = \{i \in I \mid T_i(w_i)(A_i^j) \geq \epsilon\} \in U$, for all $j \in \mathbb{N}$.

Now we show that there is a decreasing sequence $((B_i^j)_{j \in \mathbb{N}})_U$ such that $(A_i^j)_U = (B_i^j)_U$ and for some $i \in I$, $\bigcap_j B_i^j = \varnothing$, while $T_i(w_i)(B_i^j) \geq \epsilon$, for each $j \in \mathbb{N}$.

Since $(A_i)_U \supseteq (A_i^{j+1})_U$, it follows that $\{i \in I \mid A_i^j \supseteq A_i^{j+1}\} \in U$, for each $j \in \mathbb{N}$.

Thus for each $j \in \mathbb{N}$ we have

$$S_j = \{i \in I \mid A_i^0 \supseteq A_i^j\} \cap \{i \in I \mid T_i(w_i)(A_i^j) \geq \epsilon\} \in U.$$

Now define the sets $B_i^j$ as follows:

$$B_i^j = \begin{cases} A_i^0 & \text{if } j = 0 \text{ or } i \notin S_j \text{ and } k = min_n(i \in S_n), \\ A_i^j & \text{if } j > 0 \text{ and } i \in S_j. \end{cases}$$
Hence, \( \{ i \in I \mid B_i^j \supseteq B_i^{j+1} \} \in U \) for each \( j \in \mathbb{N} \). Also, \( T_i(w_i)(B_i^j) \geq \epsilon \) for each \( j \in \mathbb{N} \) and each \( i \in I_0 \). Furthermore, by definition of \( B_i^j \)'s we have \( (B_i^j)_U = (A_i^j)_U \). So \( \cap_j (B_i^j)_U = \emptyset \). Since \( I_0 \in U \), there exists \( i \in I_0 \) such that \( \cap_j B_i^j = \emptyset \). But this contradicts \( T_i(w_i)(B_i^j) \geq \epsilon \).

Therefore, for each \( (w_i)_U \in \prod_U \Omega_i \), the function \( T'((w_i)_U)(.) \) is a premeasure on \( \mathcal{A} \). So by Fact A.5 it could be extended to the measure \( T((w_i)_U)(.) \) on \( \mathcal{A}_U = \sigma(\mathcal{A}) \).

Now to prove that \( T \) is a type function on \( (\prod_U \Omega_i, \mathcal{A}_U) \) we have to show that it is measurable, i.e.

\[
\{(w_i)_U \mid T((w_i)_U)(E) \geq \alpha \} \in \mathcal{A}_U,
\]

for each \( E \in \mathcal{A}_U \) and \( \alpha \in \mathbb{Q} \cap [0,1] \).

**Lemma 2.2.** \( T \) is a measurable function.

**Proof.** Let

\[
\mathcal{B} = \{ E \in \sigma(\mathcal{A}) \mid \text{for each } r \in \mathbb{Q} \cap [0,1], \{ w \mid T(w)(E) \geq r \} \in \sigma(\mathcal{A}) \}.
\]

First we show that if \( E \in \mathcal{A} \), then \( E \in \mathcal{B} \). In this situation, there are \( E_i \in \mathcal{A}_i \) such that \( E = (E_i)_U \). So we have

\[
\{(w_i)_U \mid T((w_i)_U)(E) \geq \alpha \} = \{(w_i)_U \mid \lim_U T_i(w_i)(E_i) \geq \alpha \} \\
= \{(w_i)_U \mid \forall \alpha' < \alpha, \alpha' \in \mathbb{Q} \cap [0,1], \{ i \in I \mid T_i(w_i)(E_i) \geq \alpha' \} \in U \} \\
= \bigcap_{\alpha' < \alpha} \{(w_i)_U \mid \{ i \in I \mid T_i(w_i)(E_i) \geq \alpha' \} \in U \}.
\]

On the other hand for each \( \alpha' \in \mathbb{Q} \cap [0,1] \) with \( \alpha' < \alpha \),

\[
\{(w_i)_U \mid \{ i \in I \mid T_i(w_i)(E_i) \geq \alpha' \} \in U \} = (A_i)_U
\]

where \( A_i = \{ w_i \mid T_i(w_i)(E_i) \geq \alpha' \} \), for each \( i \in I \). Hence \( (A_i)_U \in \mathcal{A} \) and therefore,

\[
\bigcap_{\alpha' < \alpha} \{(w_i)_U \mid \{ i \in I \mid T_i(w_i)(E_i) \geq \alpha' \} \in U \} \in \mathcal{A}_U.
\]

Next we show that if \( E \) is a union of an increasing sequence \( E_1 \subseteq E_2 \subseteq \ldots \) of elements of \( \mathcal{B} \) then \( E \in \mathcal{B} \).
Assume that $E_1 \subseteq E_2 \subseteq \ldots$ such that $E_j \in \mathcal{A}_U$ and $E = \bigcup_j E_j$ and the claim is true for each $E_j$.

$$H = \{(w_i)_U \mid T((w_i)_U)(E) \geq \alpha\} = \{(w_i)_U \mid T((w_i)_U)(\bigcup_j E_j) \geq \alpha\} = \{(w_i)_U \mid \lim_{j \to \infty} T((w_i)_U)(E_j) \geq \alpha\} = \{\forall \alpha' < \alpha \exists j \ T((w_i)_U)(E_j) \geq \alpha'\} = \bigcap_{\alpha' < \alpha} \bigcup_{j=1}^{\infty} \{((w_i)_U) \mid T((w_i)_U)(E_j) \geq \alpha'\}.$$  

By induction hypothesis for each $j$ and each $\alpha' < \alpha$ we have $\{(w_i)_U \mid T((w_i)_U)(E_j) > \alpha'\} \in \mathcal{A}_U$. Since $\mathcal{A}_U$ is a $\sigma$-algebra, it follows that $H \in \mathcal{A}_U$.

Therefore, $\mathcal{B}$ is a monotone class which includes the algebra $\mathcal{A}$. So by Fact A.1 $\mathcal{B} = \sigma(\mathcal{A})$ and the proof is complete.

Based on the above lemmas, we define ultraproduct of probability models.

**Definition 2.3.** Let $\langle \mathcal{M}_i = (\Omega_i, \mathcal{A}_i, T_i, v_i) : i \in I \rangle$ be a family of probability models and $U$ be a non-principal ultrafilter over $I$. The ultraproduct of the family of probability models $\langle \mathcal{M}_i : i \in I \rangle$ over $U$ is a model $\mathcal{M} = \prod_U \mathcal{M}_i = (\Omega_U, \mathcal{A}_U, T_U, v_U)$ where

- $\Omega_U$, $\mathcal{A}_U$ and $T_U$ are defined as above.
- $(w_i)_U \in v_U(p)$ if and only if $\{i \in I \mid w_i \in v_i(p)\} \in U$.

To ease the notation for each formula $\varphi$ we use $[\varphi]_U$ instead of $[\varphi]_{\prod_U \mathcal{M}_i}$.

The following theorem gives a weak version of the Loś theorem for basic probability logic.

**Theorem 2.4.** Let $\langle \mathcal{M}_i : i \in I \rangle$ be a family of probability models and $U$ be a non-principal ultrafilter over $I$. Suppose $\varphi$ is a basic formula. Then $\{i \in I \mid \mathcal{M}_i, w_i \models \varphi\} \in U$ implies $\prod_U \mathcal{M}_i, (w_i)_U \models \varphi$.

**Proof.** By induction on the complexity of basic formulas one can show that for each basic formula $\varphi$ we have

$$([\varphi]_{\mathcal{M}_i})_U \subseteq [\varphi]_U.$$

In fact, by definition, for the atomic formulas and their negations we have $( [\varphi]_{\mathcal{M}_i})_U = [\varphi]_U$. 

9
It is also easy to prove the induction step for the boolean connectives $\land$ and $\lor$. Now knowing the induction hypothesis for basic formula $\varphi$ we have the followings:

\[
(w_i)_U \in (\llbracket L_r \varphi \rrbracket_{\mathcal{M}_i})_U \Rightarrow \{ i \in I \mid w_i \in \llbracket L_r \varphi \rrbracket_{\mathcal{M}_i} \} \in U
\]
\[
\Rightarrow \{ i \in I \mid T_i(w_i)(\llbracket \varphi \rrbracket_{\mathcal{M}_i}) \geq r \} \in U
\]
\[
\Rightarrow \lim_U T_i(w_i)(\llbracket \varphi \rrbracket_{\mathcal{M}_i}) \geq r
\]
\[
\Rightarrow T((w_i)_U)(\llbracket \varphi \rrbracket_{\mathcal{M}_i})_U \geq r
\]
\[
\Rightarrow T((w_i)_U)(\llbracket \varphi \rrbracket_{U}) \geq r
\]
\[
\Rightarrow (w_i)_U \in \llbracket L_r \varphi \rrbracket_{U}.
\]

Note that the fifth line is obtained from the fourth line by the induction hypothesis. □

The above one directional statement is mainly due to the fundamental fact that $\lim_U (a_i) \geq r$ does not imply that $\{ i \in I : a_i \geq r \} \in U$. However the above theorem still enables us to prove the compactness theorem for basic probability logic.

**Theorem 2.5 (BPL-Compactness).** Suppose that $\Gamma$ is a $BPL$-theory. Then $\Gamma$ is satisfiable in $\mathcal{P}\mathcal{M}$ if and only if it is finitely satisfiable.

We conclude this subsection by giving an example which shows that the compactness fails even for positive probability logic. So, even by avoiding negation and mixing $L_r$ and $M_s$ operators we could find a theory $\Gamma$ which is finitely satisfiable but not satisfiable.

**Example 2.6.** Let

\[
\Sigma = \{ M_0(M_0p \lor L_1p) \} \cup \{ M_i(L_{\frac{1}{2^i}}p \land M_{1 - \frac{1}{2^i}}p) \mid i \in \mathbb{N} \}.
\]

We show that $\Sigma$ is finitely satisfiable but it is not satisfiable in any probability model. For each probability model $\mathfrak{M}$, Put

\[
A_0 = \{ w \in \Omega \mid T(w)(\llbracket p \rrbracket) = 0 \} \quad \text{and} \quad A'_0 = \{ w \in \Omega \mid T(w)(\llbracket p \rrbracket) = 1 \}.
\]

Also for each $i \in \mathbb{N}$, let

\[
A_i = \{ w \in \Omega \mid \frac{1}{2^i} \leq T(w)(\llbracket p \rrbracket) \leq 1 - \frac{1}{2^i} \}.
\]

For each finite subset $\Sigma' \subseteq \Sigma$, suppose $k$ is the greatest index $i$ such that $M_i(L_{\frac{1}{2^i}}p \land M_{1 - \frac{1}{2^i}}p) \in \Sigma'$. Let $\mathfrak{M}$ be a probability model $(\Omega = \{ w_1, w_2 \}, \mathcal{P}(\Omega), T, v)$ such that $v(p) = \{ w_1 \}$ and
• $T(w_1)(\{w_1\}) = T(w_1)(\{w_2\}) = \frac{1}{2}$,

• $0 < T(w_2)(\{w_1\}) < \frac{1}{2}$,

• $T(w_2)(\{w_2\}) = 1 - T(w_2)(\{w_1\})$.

Therefore, $\mathfrak{M}, w_1 \models \Sigma'$.

However, $\Sigma$ is not satisfiable, since otherwise, if $\mathfrak{M}, w \models \Sigma$ we have $T(w)(A_0) = T(w)(A'_0) = 0$ and $T(w)(A_i) \leq \frac{1}{2}$, for all $i \in \mathbb{N}$. As $T(w)$ is a $\sigma$-additive probability measure and $A_1 \subseteq A_2 \subseteq \ldots$, we have $T(w)(\bigcup_i A_i) = \lim_{i \to \infty} T(w)(A_i) \leq \frac{1}{2}$. On the other hand, $\Omega = A_0 \cup A'_0 \cup \bigcup_{i=1}^\infty A_i = A_0 \cup A'_0 \cup \{w' \in \Omega \mid 0 < T(w')(\llbracket p \rrbracket) < 1\}$. So, $T(w)(\Omega) = T(w)(A_0) + T(w)(A'_0) + T(w)(\bigcup_{i=1}^\infty A_i) < 1$, a contradiction.

In subsection 2.2, we show that in the above example $\Sigma$ has a finitely additive probability model. In fact when working with the finitely additive probability models the compactness property holds for positive probability logic $PPL$.

### 2.2 The Compactness for $PPL$

In this subsection we prove that $PPL$ has the compactness property with respect to the class of finitely additive probability models.

As we noted, Example 2.6 shows that when working with probability models the compactness property fails for $PPL$. So, it is not possible to adapt the ultraproduct construction for finitely additive probability models to prove the $PPL$-compactness. However, we will see that the Henkin method can be implemented to prove this property for finitely additive probability models.

In the following we show that the $PPL$-theory $\Sigma$ given in the Example 2.6 is satisfiable in $\mathcal{FPM}$, i.e. there exists a finitely additive pointed model $\mathfrak{M}, w \models \Sigma$.

**Example 2.7.** Let $\Sigma$ be a $PPL$-theory as given in Example 2.6. Define the model $\mathfrak{M} = (\mathbb{N}, \mathcal{P}(\mathbb{N}), T, v)$ which satisfies the following conditions:

• $v(p) = \{0\}$.

• For each $n \neq 0$, $T(n)$ is a $\sigma$-additive measure on $\mathcal{P}(\mathfrak{M})$ with the condition

\[
T(n)(\{x\}) = \begin{cases} 
\frac{1}{2^n} & \text{if } 0 \leq x \leq 2^n - 1, \\
0 & \text{if } x > 2^n - 1.
\end{cases}
\]
• For a non-principal ultrafilter $U$ over $\mathbb{N}$, we define $T(0)$ as:

$$T(0)(X) = \begin{cases} 1 & \text{if } X \in U, \\ 0 & \text{if } X \notin U. \end{cases}$$

It is easy to see that $T(0)$ is a finitely additive probability measure. Note that for every $i \neq 0$, \{w ∈ \mathbb{N} | \frac{1}{2^i} \leq T(w)[p] \leq 1 - \frac{1}{2^i} \} = \{1, \ldots, i\} \notin U$. So for the point $0$ we have

$$T(0)[p] = 0,$$

$$T(0)[M_0 p] = T(0)\{0\} = 0,$$

$$T(0)[L_1 p] = T(0)\emptyset = 0,$$

$$T(0)[L_{\frac{1}{2^i}}p \land M_{1 - \frac{1}{2^i}} p] = T(0)\{1, \ldots, i\} = 0,$$

$$T(0)\{n \mid 0 < T(n)[p] < 1\} = T(0)\{\mathbb{N} \setminus \{0\}\} = 1.$$

Therefore, $\mathcal{M}, 0 \models \Sigma$.

To prove the compactness property for $PPL$-theories, we construct a canonical finitely additive probability model $\mathcal{M}_C = (\Omega_C, P(\Omega_C), T_C, v_C)$ whose set of states $\Omega_C$ consists of all maximally finitely satisfiable positive theories such that its satisfaction relation is given by $\mathcal{M}_C, \Gamma \models \varphi$ if and only if $\varphi \in \Gamma$, for each $\Gamma \in \Omega_C$ and positive formula $\varphi$. Lemma [2.10] is the key ingredient in proving the truth lemma and is based on the following known fact which relates the satisfiability of a formula $\varphi$ with solvability of certain finite system of linear inequalities $S_\varphi$, [5, 25].

Before defining the canonical model we need to recall some basic notions. The (modal or) probability depth of a formula $\varphi$ is the maximum number of nesting probability operators used in $\varphi$. More formally,

**Definition 2.8.**  
- $\delta(p) = 0$, for each atomic formula $p$.

- $\delta(\neg \varphi) = \delta(\varphi)$.

- $\delta(\varphi \land \psi) = \delta(\varphi \lor \psi) = \max(\delta(\varphi), \delta(\psi))$.

- $\delta(L_r \varphi) = \delta(M_s \varphi) = \delta(\varphi) + 1$.

For a formula $\varphi$, let local language $L_\varphi$ be the largest set of formulas satisfying the following conditions:

- The propositional variables are those occur in $\varphi$. 
• Each \( r \in \mathbb{Q} \cap [0,1] \) appears in a probability operators of a formula in \( \mathcal{L}_\varphi \) is a multiple of \( \frac{1}{q_\varphi} \) where \( q_\varphi \) is the least common multiple of all denominators of the rational numbers appearing in probability operators in \( \varphi \).

• The depth of formulas in \( \mathcal{L}_\varphi \) is at most \( \delta(\varphi) \).

It is clear that up to logical equivalence the local language \( \mathcal{L}_\varphi \) consists of only finitely many formulas.

Now we state the following fact. We refer the reader to [25], Theorem 3, for the construction of \( S_\varphi \) as well as its complete proof. We should only point out whenever \( \varphi \) is a positive probability formula, the set \( S_\varphi \) consists of only closed inequalities, i.e. linear inequalities of form \( x_1 + \cdots + x_n \geq r \) or \( x_1 + \cdots + x_n \leq s \).

**Fact 2.9.** For any probability formula \( \varphi \) there is a system of linear inequalities \( S_\varphi \) such that \( \varphi \) is satisfiable if and only if \( S_\varphi \) is solvable.

The following crucial lemma follows from Fact 2.9 and is needed for characterizing the satisfaction relation of the canonical model.

**Lemma 2.10.** Let \( \Gamma \) be a finite set of positive formulas. If \( \Gamma \cup \{L_r \varphi\} \) is not satisfiable, then there is a rational number \( r' \) with \( 0 < r' < r \) such that \( \Gamma \cup \{L_r \varphi\} \) is not satisfiable. Similarly, if \( \Gamma \cup \{M_s \varphi\} \) is not satisfiable, then there is a rational number \( s' \) with \( s < s' < 1 \) such that \( \Gamma \cup \{M_s \varphi\} \) is not satisfiable.

**Proof.** First of all, without loss of generality we may assume that \( \Gamma \) is satisfiable. Put \( \psi = \land \Gamma \land \varphi \). Let \( \{H_1, \ldots, H_n\} \) be the set of all maximally satisfiable sets of formulas over \( \mathcal{L}_\psi \). Associate to each \( H_i \) a variable \( x_i \). Note that each formula in \( \mathcal{L}_\psi \) is logically equivalent to a disjunction of (conjunction of) \( H_i \)'s. So, for every formula \( \theta \) in this fragment let \( I_\theta \subseteq \{1, \ldots, n\} \), such that \( \theta = \lor_{i \in I_\theta} \land H_i \).

Since \( \Gamma \) is a finite set of positive formulas, \( \gamma = \land \Gamma \) is also a positive formula. On the other hand, the positive formula \( \gamma \) is equivalent to a disjunction of (satisfiable) formulas of the form

\[
\gamma_i = \bigwedge_j p_{ij} \land \bigwedge_j' \neg p_{ij'} \land \bigwedge_l L_{r_{il}} \varphi_{il} \land \bigwedge_{l'} M_{s_{il'}} \varphi_{il'}.
\]

\( \Gamma \cup \{L_r \varphi\} \) is not satisfiable, it follows that for each \( i \), \( \{\gamma_i\} \cup \{L_r \varphi\} \) is not satisfiable. So without loss of generality we may assume that \( \gamma \) is of the form

\[
\gamma = \bigwedge_j p_j \land \bigwedge_j' \neg p_{j'} \land \bigwedge_l L_{r_l} \varphi_l \land \bigwedge_{l'} M_{s_{l'}} \varphi_{l'}.
\]
Associate to $L_{r_l} \varphi_l$ a linear inequality of the form $\sum_{i \in I_{r_l}} x_i \geq r_l$. Likewise, for a formula $M_{r'_{r'}} \varphi_{r'}$, consider the linear inequality of the form $\sum_{i \in I_{r'_{r'}}} x_i \leq s'_{r'}$.

Let $S_T$ be the set of all above inequalities together with
\[
\begin{align*}
  x_i &\geq 0 & 1 \leq i \leq n \\
  x_i &\leq 1 & 1 \leq i \leq n \\
  x_1 + \cdots + x_n &\geq 1 \\
  x_1 + \cdots + x_n &\leq 1.
\end{align*}
\]

By Fact 2.9, $S_T$ is solvable, since $\Gamma$ is satisfiable. Now consider the following optimization problem:

\[
\begin{align*}
\text{Maximize} & \quad \sum_{i \in I_{r'}} x_i \\
\text{Subject to} & \quad S_T.
\end{align*}
\]

Since $S_T$ is solvable, it defines a non-empty closed and bounded set of $\mathbb{R}^n$. Hence by the Fundamental theorem of linear programming (see Theorem 3.4 in [21]) the above problem has a solution. Let $M$ be the maximum value of $\sum_{i \in I_{r'}} x_i$. Since $\Gamma \cup \{L_{r'} \varphi\}$ is not satisfiable, $S_T \cup \{\sum_{i \in I_{r'}} x_i \geq r\}$ is not solvable. Hence it follows that $M < r$. So for every rational number $r'$ with $M < r' < r$, one can see that $\Gamma \cup \{L_{r'} \varphi\}$ is not satisfiable.

The other assertion can be shown similarly.

**Proposition 2.11.** Let $\Gamma$ be a finitely satisfiable positive theory. Then $\Gamma$ can be extended to a maximally finitely satisfiable positive theory. Furthermore, if $\Gamma$ is maximally finitely satisfiable then for each PPL-formula $\varphi$ and $r \in \mathbb{Q} \cap [0,1]$, $\Gamma$ contains at least one of the formulas $L_{r} \varphi$ and $M_{r} \varphi$.

*Proof.* Suppose that $\psi_1, \psi_2, \ldots$ is an enumeration of positive formulas. Put $\Sigma_0 = \Gamma$. For each $n \in \mathbb{N}$ define
\[
\Sigma_{n+1} = \begin{cases} 
\Sigma_n \cup \{\psi_{n+1}\} & \text{if it is finitely satisfiable}, \\
\Sigma_n & \text{otherwise}.
\end{cases}
\]

Let $\Sigma = \bigcup \Sigma_n$. Then it is easy to see that $\Sigma$ is maximally finitely satisfiable positive theory. Now suppose that $\Gamma$ is maximally finitely satisfiable and $L_{r} \varphi \notin \Gamma$. So there is a finite subset $\Sigma' \subseteq \Gamma$ such that $\Sigma' \cup \{L_{r} \varphi\}$ is not satisfiable. But this implies that $\Sigma' = M_{r} \varphi$ and then each finite subset $\Sigma''$ of $\Gamma$ including $\Sigma'$ has a model satisfying $M_{r} \varphi$. Therefore $\Gamma \cup \{M_{r} \varphi\}$ is finitely satisfiable and as $\Gamma$ is maximally finitely satisfiable, it follows that $M_{r} \varphi \in \Gamma$. \qed
Next proposition introduces examples of maximally finitely satisfiable $PPL$-theories and is needed for Theorem 2.13.

**Proposition 2.12.** Let $(\mathcal{M}, w)$ be a finitely additive probability model. Then the positive theory of $(\mathcal{M}, w)$, i.e. $Th_+(\mathcal{M}, w) = \{ \varphi \in PPL \mid \mathcal{M}, w \models \varphi \}$, is a maximally finitely satisfiable $PPL$-theory.

**Proof.** Suppose that $\Sigma$ is a finitely satisfiable $PPL$-theory containing $Th_+(\mathcal{M}, w)$. By induction on the complexity of formulas we can show that if $\phi \in \Sigma$ then $\phi \in Th_+(\mathcal{M}, w)$, for each $PPL$-formula $\phi$.

**Theorem 2.13 (PPL-Compactness).** Let $\Gamma$ be a finitely satisfiable positive theory. Then $\Gamma$ has a finitely additive probability model.

**Proof.** In the following we define the model $(\mathcal{M}, w_0)$ in a way that $\mathcal{M}, w_0 \models \Sigma$.

Let $\Omega_C$ be the set of all maximally finitely satisfiable sets of positive formulas. Put

$$\Theta = \{ [\varphi] \mid \varphi \text{ is a positive formula} \}$$

where $[\varphi] = \{ w \in \Omega_C \mid \varphi \in w \}$. Note that the set $(\Theta, \cap, \cup, [\bot], [\top])$ forms a lattice and for every $\varphi, \psi \in PPL$, we have the followings

- $[\varphi] \cap [\psi] = [\varphi \land \psi]$.
- $[\varphi] \cup [\psi] = [\varphi \lor \psi]$.

Moreover, as $PPL$ is not closed under negation, $\Omega_C$ is not an algebra. Now we define the function $T' : \Omega_C \times \Theta \to [0, 1]$ as follows:

$$T'(w)([\varphi]) = \sup \{ r \in \mathbb{Q} \cap [0, 1] \mid L_r \varphi \in w \}.$$

**Claim.** For each $w \in \Omega_C$ the function $T'(w)$ is a valuation on the lattice $\Theta$.

**Proof of Claim.** Let $w \in \Omega_C$.

- $T'(w)(\emptyset) = 0$, since $L_r \bot$ is not satisfiable for any $r > 0$.
- Suppose that $[\varphi] \subseteq [\psi]$. Then we have to prove that $T'(w)([\varphi]) \leq T'(w)([\psi])$. We show that if $[\varphi] \subseteq [\psi]$, then we have $\varphi \models \psi$. Otherwise, there exists a model $\mathfrak{M}, v \models \varphi$ and $\mathfrak{M}, v \not\models \psi$. So, $\varphi \in Th_+(\mathfrak{M}, v)$ and $\psi \notin Th_+(\mathfrak{M}, v)$, which is a contradiction by Lemma 2.12.
• We have to show that for all $[\varphi_1], [\varphi_2] \in \Theta$,
\[ T'(w)([\varphi_1]) + T'(w)([\varphi_2]) = T'(w)([\varphi_1] \cup [\varphi_2]) + T'(w)([\varphi_1] \cap [\varphi_2]) \].

Suppose that $T'(w)([\varphi_i]) = \alpha_i$, for $i = 1, 2$ and $T'(w)([\varphi_1 \lor \varphi_2]) = \alpha_\lor$ and $T'(w)([\varphi_1 \land \varphi_2]) = \alpha_\land$.

If $\alpha_1 + \alpha_2 < \alpha_\lor + \alpha_\land$, then find $\epsilon_1, \epsilon_2, \epsilon_\lor, \epsilon_\land > 0$ such that $\alpha'_i = (\alpha_i + \epsilon_i) \in \mathbb{Q}$ for $i \in \{1, 2\}$, and $\alpha'_i = (\alpha_i - \epsilon_i) \in \mathbb{Q}$, for $i \in \{\land, \lor\}$, and $(\alpha'_i) + (\alpha'_i) < (\alpha'_i) + (\alpha'_i)$. But in this case \{\$M_{\alpha'_i} \varphi_1, M_{\alpha'_i} \varphi_2, \$L_{\alpha'_i} (\varphi_1 \land \varphi_2), L_{\alpha'_i} (\varphi_1 \lor \varphi_2)\} is a finite subset of $w$ and not satisfiable, a contradiction.

A similar argument shows that the inequality $\alpha_1 + \alpha_2 > \alpha_\lor + \alpha_\land$ leads to a contradiction.

\[ \Box \]

Now let $B(\Theta)$ be the boolean algebra generated by $\Theta$. By Fact A.3 for every $w \in \Omega_C$, one can extend the valuation $T'(w)$ to a finitely additive measure $T''(w)$ on $B(\Theta)$. Subsequently, by Fact A.4 we can extend each $T''(w)$ to a finitely additive measure $T_C(w)$ on $P(\Omega_C)$. Note that the measurability of $T_C : \Omega \times P(\Omega_C) \to [0, 1]$ comes for free. Now to define the valuation function $v_C$, for each proposition $p$, put $v_C(p) = \{w \in \Omega_C \mid p \in w\}$.

Having defined functions $T_C$ and $v_C$, we assume the model $\mathcal{M}_C = (\Omega_C, P(\Omega_C), T_C, v_C)$. The following claim characterizes the satisfaction relation of $\mathcal{M}_C$.

**Claim.** For every positive formula $\varphi$ and $w \in \Omega_C$,
\[ \mathcal{M}_C, w \models \varphi \quad \text{if and only if} \quad \varphi \in w. \]

The above claim states that inside $\mathcal{M}_C$, for each $\varphi$ we have $[\varphi]_{\mathcal{M}_C} = [\varphi]$.

**Proof of Claim.** The proof proceeds by induction on the complexity of positive formulas. The induction base for atomic formulas as well as the induction step for boolean operators $\land, \lor$ are clear.

Now consider the case where $\varphi = L_r \psi$, knowing that $[\psi]_{\mathcal{M}_C} = [\psi]$. Now suppose $w \in [L_r \psi]$. So, $L_r \psi \in w$ and $T_C(w)([\psi]) = \sup \{\alpha \mid L_\alpha \psi \in w\} \geq r$. So by induction hypothesis $T_C(w)([\psi]_{\mathcal{M}_C}) \geq r$. Conversely, suppose that $w \in [\varphi]_{\mathcal{M}_C}$. In this case by induction hypothesis we have $T_C(w)([\psi]) \geq r$. Therefore, $\sup \{\alpha \mid L_\alpha \psi \in w\} \geq r$. Now if $L_r \psi \notin w$, then, as $w$ is maximally finitely satisfiable, $w \cup \{L_r \psi\}$ is not finitely satisfiable. So there exists a finite subset $w'$ of $w$ such that $w' \cup \{L_r \varphi\}$ is not satisfiable. Thus,
by Lemma 2.10 there exists \( r' < r \) such that \( w' \cup \{L_{r'} \psi\} \) is not finitely satisfiable and \( L_{r'} \psi \notin w \). But this contradicts with \( \sup \{ \alpha \mid L_{\alpha} \psi \in w\} \geq r \). Hence \( [\varphi]_{M_C} = [\varphi] \) and the induction is proved for \( L_r \varphi \). Lemma 2.10 can be applied to show that the induction hypothesis holds for \( \varphi = M_s \psi \).

Now having proved the claim we can finish the proof by noticing that if \( \Gamma \) is a finitely satisfiable theory, then by Lemma 2.11 one can find a maximally finitely satisfiable \( w \) which includes \( \Gamma \). Hence we have that \( M_C, w \vDash \Gamma \).

\[ \square \]

3 The Löwenheim-Skolem Number of Probability Logics

In this section we study the Löwenheim-Skolem number of the class of probability and finitely additive probability models. The Löwenheim-Skolem number of a class of models \( C \) of a logic \( L \) is the least infinite cardinal \( \kappa \) such that every satisfiable \( L \)-theory has a model of size at most \( \kappa \). In this section we prove that this number is uncountable cardinal of at most \( 2^{\aleph_0} \) for the class of probability models, while it is \( \aleph_0 \) for the class of finitely additive models.

**Theorem 3.1.** Let \( \lambda \) be the Löwenheim-Skolem number of probability models with respect to probability logic. Then \( \aleph_0 < \lambda \leq 2^{\aleph_0} \).

**Proof.** If theory \( \Sigma \) is satisfiable then it is consistent. Hence by Theorem 3.2.13 of [22] there is a canonical model which models \( \Sigma \). But the size of this model is \( 2^{\aleph_0} \). So, \( \lambda \leq 2^{\aleph_0} \). Furthermore, the following example shows that there is a \( BPL \)-theory which does not have a countable model. Hence the proof is complete. \[ \square \]

**Example 3.2.** Let \[ \Gamma = \{L_{1/2} \neg (p_i \leftrightarrow p_j) \mid i < j, i, j \in \mathbb{N}\} \].

\( \Gamma \) is satisfiable, specially it has a model of size \( 2^{\aleph_0} \). To see this, define the model \( M \) as follows. Let \( \Omega_M = \{0, 1\}^{\mathbb{N}} \) and \( A_M \) be a product \( \sigma \)-algebra, i.e. a \( \sigma \)-algebra generated by direct product \( \prod_{i \in \mathbb{N}} A_i \) where except for a finite number of \( A_i \)s the rest of them are \( \{0, 1\} \). Suppose \( \mu(0) = \mu(1) = 1/2 \) and \( T((0_i)_{i \in \mathbb{N}}) \) is a product measure of \( \mu \). Moreover, for each proposition \( p_j \) put \( v(p_j) = \{(w_i)_{i \in \mathbb{N}} \mid w_j = 1\} \). Therefore,

\[
T((0_i)_{i \in \mathbb{N}})((w_i)_{i \in \mathbb{N}} \mid M, (w_i)_{i \in \mathbb{N}} \neq p_i \leftrightarrow p_j) = T((0_i)_{i \in \mathbb{N}})(\prod_{i \in \mathbb{N}} A_i) + T((0_i)_{i \in \mathbb{N}})(\prod_{i \in \mathbb{N}} B_i) = \frac{1}{2^2} + \frac{1}{2^2} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}
\]
where \( A_i = B_j = \{1\} \) and \( A_j = B_i = \{0\} \) and \( A_i = B_i = \{0, 1\} \) otherwise. So \( \mathfrak{M}, (0_i)_{i \in \mathbb{N}} \models \Gamma \).

Now we show that there is no countable model for \( \Gamma \). Suppose \( \mathfrak{M}, w \models \Gamma \) and \( \Omega_{2\mathfrak{M}} \) is a countable set. Specially we can assume that \( A_{2\mathfrak{M}} = \mathcal{P}(\Omega_{2\mathfrak{M}}) \). Hence we have \( \sum_{i=1}^{\infty} T(w)(\{w_i\}) = 1 \). Therefore, there is \( N \in \mathbb{N} \) such that \( \sum_{i=1}^{N} T(w)(\{w_i\}) > \frac{1}{2} \). On the other hand, for each finite number of worlds, say \( w_1, \ldots, w_n \), there are \( i, j \in \mathbb{N} \) such that \( \mathfrak{M}, w_k = (p_i \leftrightarrow p_j) \), for \( k = 1, \ldots, n \). Since \( T(w)(\{w'|\mathfrak{M}, w' \neq p_i \leftrightarrow p_j\}) \geq \frac{1}{2} \), we should have \( T(w)(\{w_1, \ldots, w_N\}) \leq T(w)(\{w'|\mathfrak{M}, w' = p_i \leftrightarrow p_j\}) \leq \frac{1}{2} \), a contradiction.

**Remark 3.3.** It is shown, for example in [13] any satisfiable PL-theory has an analytic probability model. On the other hand, any analytic space is either countable or has size of continuum. Hence the L"owenheim-Skolem number of class of analytic probability models is \( 2^{\aleph_0} \).

Now we turn to prove the L"owenheim-Skolem number of finitely additive models.

**Theorem 3.4.** The L"owenheim-Skolem number of the class of finitely additive probability models with respect to probability logic is \( \aleph_0 \).

**Proof.** Let \( \Gamma \) be a satisfiable theory and suppose a finitely additive probability model \( \mathfrak{M} = (M, \mathcal{B}, T : M \times \mathcal{B} \to [0, 1], v) \) models \( \Gamma \) at a point \( w_0 \in M \). We construct a countable finitely additive probability model \( \mathfrak{M}' \) which models \( \Gamma \). To this end, define a countable set \( \Omega \subseteq M \) which includes \( w_0 \) and has the property that for each probability formulas \( \varphi \), if \( \left[ \varphi \right]_{\mathfrak{M}} \neq \emptyset \), then \( \Omega \cap \left[ \varphi \right]_{\mathfrak{M}} \neq \emptyset \). For each formula \( \varphi \) let \( \left[ \varphi \right]_{\Omega} = \Omega \cap \left[ \varphi \right]_{\mathfrak{M}} \). Put \( \mathcal{B}_{\Omega} = \{\left[ \varphi \right]_{\Omega} \mid \varphi \in \text{PL} \} \). Note that, \( \mathcal{B}_{\Omega} \) forms an algebra. Furthermore, \( \text{Claim: If } \left[ \varphi \right]_{\Omega} = \left[ \psi \right]_{\Omega}, \text{ then } \left[ \varphi \right]_{\mathfrak{M}} = \left[ \psi \right]_{\mathfrak{M}}. \)

**Proof of Claim.** To see this, we may suppose that both sets \( \left[ \varphi \right]_{\Omega}, \left[ \psi \right]_{\Omega} \) are nonempty. Now if \( \left[ \varphi \right]_{\mathfrak{M}} \neq \left[ \psi \right]_{\mathfrak{M}} \), then either of the sets \( \left[ \varphi \land \neg \psi \right]_{\mathfrak{M}} \) and \( \left[ \psi \land \neg \varphi \right]_{\mathfrak{M}} \) are nonempty. Hence we have \( \left[ \varphi \land \neg \psi \right]_{\Omega} \cup \left[ \psi \land \neg \varphi \right]_{\Omega} \neq \emptyset \). But this implies that \( \left[ \varphi \right]_{\Omega} \neq \left[ \psi \right]_{\Omega} \).

Now define the function \( T_{\Omega} : \Omega \times \mathcal{B}_{\Omega} \to [0, 1] \) as follows:

\[
T_{\Omega}(v)(\left[ \varphi \right]_{\Omega}) = T_{\mathfrak{M}}(v)(\left[ \varphi \right]_{\mathfrak{M}}).
\]

By the above claim \( T_{\Omega} \) is a well-defined function. It is not hard to see that for each \( w \in \Omega, T_{\Omega}(w, -) \) defines a finitely additive probability measure. Moreover, for each formula \( \varphi \) and \( r \in \mathbb{Q} \cap [0, 1], \)

\[
\{w \in \Omega \mid T_{\Omega}(w, \left[ \varphi \right]_{\Omega}) \geq r\} = \left[ L_r \varphi \right]_{\Omega}.
\]

Hence \( T \) is a measurable function.
Finally, for each proposition $p$, put $v_\Omega(p) = v_M(p) \cap \Omega_\mathfrak{M}$ and set $\mathfrak{N} = (\Omega, B_\Omega, T_\Omega, v_\Omega)$.

By induction on the complexity of formulas one can prove that $[\varphi]_\mathfrak{M} = [\varphi]_\Omega$. Therefore, $\mathfrak{N}, w \models \Gamma$. \hfill \qed

4 Conclusion

In this paper we investigated probability logic from model theoretic point of view. Specifically we study the compactness property and the Löwenheim-Skolem number of probability logic with respect to both of the class of probability models and finitely additive models. We showed that, although probability logic does not have the compactness property the basic and positive fragments of that are compact respectively to the class of probability models and finitely additive probability models. Furthermore, we proved that the Löwenheim-Skolem number of probability logic is $2^{\aleph_0}$ while it is $\aleph_0$ when we consider finitely additive models.

One of the other interesting issue in model theory which is worthwhile to study for probability logic is the Lindström type theorem. The Lindström type theorems characterize logics in terms of model theoretic concepts. In 1969 Lindström proved that first-order logic has the maximal expressive power among the abstract logics containing it with the compactness and the Löwenheim-Skolem properties. This kind of characterization is widely studied for other logics specially for modal logics, for example [18, 20, 16, 4, 26]. In addition to the compactness, bisimulation invariance property used to prove a characterization theorem for modal logic. Bisimulation of Markov processes is widely studied in many literature and some kind of definitions are given for them, [2, 3, 1]. In all versions of Lindström’s style theorems the compactness property plays an essential rule. Kurz and Venema in [14] asked whether one can give a version of Lindström theorem for non-compact logic such as probability logic.

Since studying probability logic from coalgebraic perspective is significant in computer science, investigating the problems of this paper and finding an appropriate version of Lindström’s theorem for this logic can be a good guide for giving a general version of Lindström’s theorem for non-compact logic.

One of the other valuable issues is to study first-order probability modal logic. There are a few literature considering some versions of first-order probability logic, see [9, 19].

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A Appendix

In this part some basic notions and results from measure theory, used in this paper, are reviewed. For further reading on measure theory see [6].

Recall that a family $\mathcal{A}$ of subsets of a non-empty set $\Omega$ is called a lattice if $\emptyset, \Omega \in \mathcal{A}$ and it is closed under finite unions and intersections. If, furthermore, $\mathcal{A}$ is closed under complements then it is a boolean algebra (or simply an algebra). Call an algebra $\mathcal{A}$ a $\sigma$-algebra provided that if it is closed under countable unions. For a collection $\mathcal{A}$ of subsets of $\mathcal{P}(\Omega)$, there exists a $\sigma$-algebra $\sigma(\mathcal{A})$ generated by $\mathcal{A}$ which is the intersection of all $\sigma$-algebras containing $\mathcal{A}$. While $\mathcal{P}(\Omega)$ is an obvious example of a $\sigma$-algebra over set $\Omega$, for a topological space $(\Omega, \tau)$ the family $\mathcal{C}$ of closed subsets of $\Omega$ forms a lattice. Furthermore members of $\sigma(\mathcal{C})$ are called Borel subsets of $\Omega$.

Moreover, $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ is called a monotone class if it is closed under unions of countable increasing sequences and also intersections of countable decreasing sequences.

**Fact A.1** (Monotone class). (Lemma 2.35 in [6]) The monotone class generated by an algebra $\mathcal{A}$ is equal to $\sigma(\mathcal{A})$.

A measurable space is a pair $(\Omega, \mathcal{A})$ where $\mathcal{A}$ is a $\sigma$-algebra on the non-empty set $\Omega$. Each $A \in \mathcal{A}$ is named a measurable set. For two measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ the function $f : X \to Y$ is a measurable function if $f^{-1}(B) \in \mathcal{A}$ for each $B \in \mathcal{B}$.

Let $\mathcal{A}$ be a lattice over $\Omega$. Then a non-negative real-valued set function $\mu : \mathcal{A} \to \mathbb{R}$ is a valuation if it satisfies the following conditions:

- (Strictness) $\mu(\emptyset) = 0$,
- (Monotonicity) if $A \subseteq B$ is in $\mathcal{A}$, then $\mu(A) \leq \mu(B)$,
- (Modularity) $\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B)$, for all $A, B \in \mathcal{A}$.

In case $\mathcal{A}$ is an algebra then the function $\mu : \mathcal{A} \to \mathbb{R}$ is called a finitely additive measure. In this situation modularity implies monotonicity. Furthermore, $\mu$ is premeasure whenever for any $\{A_i\}_{i \in \mathbb{N}}$ of pairwise disjoint members of $\mathcal{A}$ if $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{A}$, then $\mu(\bigcup_{i \in \mathbb{N}} A_i) = \sum_{i \in \mathbb{N}} \mu(A_i)$.

Finally for a $\sigma$-algebra $\mathcal{A}$ a premeasure $\mu : \mathcal{A} \to \mathbb{R}$ is called a $\sigma$-additive measure.

A finitely or a $\sigma$-additive measure $\mu$ is a probability measure when $\mu(\Omega) = 1$. For brevity, a $\sigma$-additive measure is simply called a measure.
Fact A.2. Let $\mathcal{A}$ be an algebra over $\Omega$. A function $\mu : \mathcal{A} \to \mathbb{R}$ is a premeasure if for each decreasing sequence $A_0 \supseteq A_1 \supseteq \ldots$ of elements of $\mathcal{A}$, if $\bigcap_i A_i = \emptyset$ then $\lim_{i \to \infty} \mu(A_i) = 0$.

A measure space is a triple $(X, \mathcal{A}, \mu)$ where $\mu$ is a measure on the $\sigma$-algebra $\mathcal{A}$.

The following standard fact states how to extend a valuation over a lattice $\mathcal{L}$ to a finitely additive measure over algebra $\mathcal{B}(\mathcal{L})$ generated by $\mathcal{L}$.

Fact A.3 (Smiley–Horn–Tarski Theorem in [7]). Let $\mu$ be a valuation defined on a lattice $\mathcal{L}$ of subsets of $X$. Then $\mu$ can be uniquely extended to a finitely additive measure $\mu^*$ on the algebra $\mathcal{B}(\mathcal{L})$ generated by $\mathcal{L}$.

The following facts can also be shown using Carathéodory’s extension theorem.

Fact A.4. (Theorem 1.22 in [12]) Let $\mu$ be a finitely additive measure on a boolean algebra $\mathcal{A}$ of $X$. Then $\mu$ could be extended to a finitely additive measure on any boolean algebra $\mathcal{A}'$ containing $\mathcal{A}$.

Fact A.5. (Theorem 1.14 in [6]) Let $\mu$ be a finite premeasure on boolean algebra $\mathcal{A}$. Then $\mu$ has a unique extension to $\mu^*$ on $\sigma(\mathcal{A})$.

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