The Pauli Equation for Probability Distributions

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The “marginal” distribution for measurable coordinate and spin projection is introduced. Then, the analog of the Pauli equation for spin-$1/2$ particle is obtained for such probability distributions instead of the usual wave functions. That allows a classical-like approach to quantum mechanics. Some illuminating examples are presented.

\section{I. INTRODUCTION}

Since the early days of quantum mechanics, we have been forced to coexist with complex probability amplitudes without worrying about their lack of any reasonable physical meaning. One should not ignore, however, that the wave-like properties of quantum objects still raise conceptual problems on whose solutions, a general consensus is far from having been reached\cite{1,2}.

A possible way out of this difficulty has been implicitly suggested by Feynmann\cite{3}, who has shown that, by dropping the assumption that the probability for an event must always be nonnegative, one can avoid the use of probability amplitudes in quantum mechanics. This proposal goes back to the work by Wigner\cite{4}, who first introduced non-positive pseudoprobabilities to represent quantum mechanics in phase space, and to the Moyal approach to quantum mechanics\cite{5}.

From a conceptual point of view, the elimination of the waves from quantum theory is in line with the procedure inaugurated by Einstein with the elimination of the aether in the theory of electromagnetism.

The phase-space formulation of quantum mechanics\cite{4,6,7} provides a means of analyzing quantum-mechanical systems while still employing a classical framework. Moreover, a quantum mechanics without wave functions has been discussed\cite{8}.

Recently, the problem of quantum state measurement, initially posed by Pauli\cite{9}, received a lot of attention\cite{10}. The tomographic approach\cite{11,12} to the quantum state of a system has allowed to establish a map between the density operator (or any its representation) and a set of probability distributions, often called “marginals”. The latter have all the characteristics of classical probabilities; they are nonnegative, measurable, and normalized.

Based on this connection, a classical-like description of quantum dynamics by means of “symplectic tomography” has been formulated\cite{13}, providing a bridge between classical and quantum worlds. That is, the evolution of a quantum system with continuous observables (namely, quadrature components of a field mode) was described in terms of a classical-like equation for a marginal distribution. Different aspects of this classical-like description using tomographic probabilities were recently analyzed\cite{14,15}.

On the other hand, discrete observables, like spin or angular momentum, are so important in quantum mechanics as the continuous ones are. Hence, the tomography scheme for discrete variables was introduced\cite{16}, and the marginal distribution for rotated spin variables has been constructed\cite{17}, deriving an evolution equation for this function.

Here, we would extend the approach by considering a spin-$1/2$ particle moving in a potential, then constructing the marginal distributions for space coordinates and spin projections and finally deriving the evolution equation for such probabilities, which would be the analog of the Pauli equation. It would also be a generalization of approaches attempted by us in previous papers\cite{13}.

Essentially, our aim is to eliminate the hybrid procedure of describing the dynamical evolution of a system, which consists of a first stage, where the theory provides a deterministic evolution of the wave function, followed by a hand-made construction of the physically meaningful probability distributions. If the probabilistic nature of the microscopic phenomena is fundamental, and not simply due to our ignorance, as in classical statistical mechanics, why should it be impossible to describe them in probabilistic terms from the very beginning? On the other hand, the language of probability, suitably adapted to take into account all the relevant constraints, seems to be the only language capable of expressing the fundamental role of “chance” in nature\cite{18}.

The paper is organized as follow:
In Section 2, we review the general approach to construct known tomography schemes using density matrix in the specifically transformed reference frames. In Section 3, we derive the general evolution equation for tomographic probabilities (marginal distributions) which describe the quantum state instead of density matrix. In Section 4, the general scheme of tomography construction is used to re-derive the particular example of symplectic tomography, which is applied for measuring states depending on continuous quadrature. In Section 5, the general scheme is used to re-derive the construction of spin-state tomography. In Section 6, the general scheme of Section 2 is then applied to obtain tomographic probabilities in the combined situation described by spatial (multidimensional too) and spin variables. In Section 7, some examples are studied in the context of the probability representation of quantum mechanics. Section 8 concludes.

We are using the natural unit \( \hbar = c = 1 \).

II. GENERAL APPROACH TO QUANTUM TOMOGRAPHY

In this section, we give a short review of the general principles used to construct a tomography scheme for measuring quantum states. Recently, we established \([19]\) a quite general principle of constructing measurable probabilities, which determine completely the quantum state in the tomographic approach; more refined treatments then followed \([20,21]\). Here, we apply our general approach to derive the evolution equation for the tomographic probabilities that is alternative in some sense to the Schrödinger equation for the wave function (or the quantum Liouville equation for the density matrix).

Let us consider a quantum state described by the density operator \( \hat{\rho} \), which is a nonnegative hermitian operator, i.e.,

\[
\hat{\rho}^\dagger = \hat{\rho}, \quad \text{Tr} \hat{\rho} = 1,
\]

and

\[
\langle v | \hat{\rho} | v \rangle = \rho_{v,v} \geq 0.
\]

We label the vector basis \( | v \rangle \) in the space of pure quantum states by the multidimensional index \( v = (v_1, v_2, \ldots, v_N) \), where the number \( N \) shows the number of degrees of freedom of the system under consideration. Among indexes \( v_k \), \( k = 1, \ldots, N \), there are continuous ones like position (or momentum) and discrete ones like spin projections. In this sense, the wave function \( \psi(v) = \langle v | \psi \rangle \) of a pure state \( | \psi \rangle \) depends both on continuous and discrete observables. Formula (2) can be rewritten by using the hermitian projection operator

\[
\hat{\Pi}_v = | v \rangle \langle v |,
\]

in the following form

\[
\rho_{v,v} = \text{Tr} \left\{ \hat{\Pi}_v \hat{\rho} \right\}.
\]

The physical meaning of the projector \( \hat{\Pi}_v \) is that it extracts the state \( | v \rangle \) with given \( v \) (for example, with given position and spin projection), which is an eigenstate of the commuting-hermitian operators \( \hat{V} = (\hat{V}_1, \hat{V}_2, \ldots, \hat{V}_N) \)

\[
\hat{V}_k | v \rangle = v_k | v \rangle.
\]

In the space of states, there is a family of unitary transformation operators \( \hat{U}(\sigma) \) depending on the parameters \( \sigma = (\sigma_1, \ldots, \sigma_k \ldots) \), that can be sometimes identified with a group-representation operators. In these cases, the parameters \( \sigma \) describe the group element. It was shown \([20,21]\) that known tomography schemes can be considered from the viewpoint of the group theory by using appropriate groups. More recently this concept has been developed obtaining an elegant group theoretical approach to quantum state measurement \([21]\). Here, we formulate the tomographic approach in the following way. Let us introduce a “transformed density operator”

\[
\hat{\rho}_\sigma = \hat{U}^{-1}(\sigma) \hat{\rho} \hat{U}(\sigma).
\]

Its diagonal elements are still nonnegative probabilities

\[
\langle z | \hat{\rho}_\sigma | z \rangle = \langle \langle z | \hat{\rho} | z \rangle \rangle \equiv w(z, \sigma).
\]
Here, $|z\rangle$ is one of the possible vectors $|v\rangle$, while the symbol $|z\rangle\langle z|$ denotes the transformed vectors
\[ |z\rangle = \hat{U}(\sigma) |z\rangle, \tag{8} \]
which in turn are eigenstates of the transformed operators
\[ \hat{Z} = \hat{U}(\sigma)\hat{V}\hat{U}^{-1}(\sigma). \tag{9} \]
As a consequence of the unit trace of the density operator we also have the normalization condition
\[ \int dzw(z,\sigma) = 1. \tag{10} \]
Of course, in case of discrete indices, the integral in Eq. (10) is replaced by a sum over discrete variables.

Formula (7) can be interpreted as the probability density for the measurement of the observable $\hat{V}$ in an ensemble of transformed reference frames labeled by the index $\sigma$, if the state $\hat{\rho}$ is given. Along with this interpretation, one can also consider the transformed projector
\[ \hat{\Pi}_z(\sigma) = \hat{U}(\sigma)\hat{\Pi}_z\hat{U}^{-1}(\sigma) = |z\rangle\langle z|, \tag{11} \]
the explicit expression for the probability $w(z,\sigma)$ takes the form
\[ w(z,\sigma) = \text{Tr}\left\{ \hat{\rho}\hat{\Pi}_z(\sigma) \right\} = \text{Tr}\left\{ \hat{\rho} |z\rangle\langle z| \right\}. \tag{12} \]
These probability densities are also called “marginal” distributions as generalization of the concept introduced by Wigner [4]. The tomography schemes are based on the possibility to find the inverse of Eq. (12). If it is possible to solve Eq. (12), considering the probability $w(y,\sigma)$ as known function and the density matrix as unknown operator, the quantum state can be described by the positive probability instead of the density matrix. This property is the essence of state reconstruction techniques. In such cases, the inverse of Eq. (12) takes the form
\[ \hat{\rho} = \int w(z,\sigma) \hat{K}(z,\sigma) \, dz \, d\sigma. \tag{13} \]
Thus, there exist a family of operators $\hat{K}(z,\sigma)$ depending on both the variables $z$ and parameters $\sigma$ such that the density operator is reconstructed, if the probability $w(z,\sigma)$ is known. It is worth remarking that transformations $\hat{U}(\sigma)$ can form other algebraic constructions, which have no structure of groups [22]. The only condition for the existence of a tomography scheme is the possibility to invert Eq. (12). In the cases of optical tomography [12], symplectic tomography [13], and spin tomography [17,23], the sets of transformations $\hat{U}(\sigma)$ have the structure of corresponding Lie groups (i.e., rotation, symplectic and spin).

III. THE TIME EVOLUTION EQUATION

We are now interested in finding the evolution equation for the probability $w(z,\sigma,t)$, in which $t$ is the time parameter. Using Eq. (12) one has
\[ \partial_t w(z,\sigma,t) = \text{Tr}\left\{ \left[ \partial_t \hat{\rho}(t) \right] \hat{\Pi}_z(\sigma) \right\}. \tag{14} \]
On the other hand, the density operator satisfies the Liouville-Von Neumann equation
\[ \partial_t \hat{\rho}(t) = i \left[ \hat{\rho}(t), \hat{H} \right], \tag{15} \]
with $\hat{H}$ the system Hamiltonian. By inserting Eq.(13) in (14), and with the aid of Eq.(13), we find the evolution equation for the probability $w$ in a closed form
\[ \partial_t w(z,\sigma,t) = \int dz' \, d\sigma' \, w(z',\sigma',t) \text{Tr}\left\{ i \left[ \hat{K}(z',\sigma'), \hat{H} \right] \hat{\Pi}_z(\sigma) \right\}. \tag{16} \]
Equation (16) represents the classical-like version of the Liouville-Von Neumann equation, thus, it would be the analog of the Pauli equation for a system with space and spin degrees of freedom.
IV. QUANTUM TOMOGRAPHY WITH CONTINUOUS VARIABLES

Let us consider, in a one-dimensional system, an operator $\hat{X}$ as the linear combination of position $\hat{q}$ and momentum $\hat{p}$ [24,25]

$$\hat{X} = \mu \hat{q} + \nu \hat{p}, \quad (17)$$

which depends upon real parameters $\mu$, $\nu$ and, due to its hermiticity, is a measurable observable. Since the linear canonical transformation (17) belongs to the symplectic group $Sp(2, R)$, the tomography scheme under discussion was called “symplectic tomography” [25].

The probability (marginal) related to the observable (17) is given by

$$w(x, \mu, \nu) = \langle \langle x | \hat{\rho} | x \rangle \rangle, \quad (18)$$

where $\hat{\rho}$ is the system’s density operator, while the eigenstates $| x \rangle \rangle$ of the operator (17) can be written as

$$| x \rangle \rangle = \int dq \langle q | x \rangle \rangle | q \rangle, \quad (19)$$

with $| q \rangle$ the position eigenkets. The wave function $\langle q | x \rangle \rangle$ can be easily calculated by using the following equality

$$\langle q | \hat{X} | x \rangle \rangle = \langle q | \mu \hat{q} + \nu \hat{p} | x \rangle \rangle, \quad (20)$$

and then transforming it in a partial differential equation

$$x \langle q | x \rangle \rangle = \mu q \langle q | x \rangle \rangle - i\nu \frac{\partial}{\partial q} \langle q | x \rangle \rangle. \quad (21)$$

The solution is

$$\langle q | x \rangle \rangle = \left( \frac{1}{| \nu |} \right)^{1/2} \exp \left[ i x q - i \frac{\mu}{2} \nu q^2 \right]. \quad (22)$$

It is worth noting that as soon as $\mu \rightarrow 1$ and $\nu \rightarrow 0$, then $| x \rangle \rangle \rightarrow | x \rangle$ and the wavefunction (22) tends to $\delta(q - x)$.

Furthermore, Eq.(18) can be formally rewritten as

$$w(x, \mu, \nu) = Tr \left\{ \hat{\rho} \hat{\Pi}_x (\mu, \nu) \right\}, \quad (23)$$

where the transformed projector is given by

$$\hat{\Pi}_x (\mu, \nu) = \hat{U}(\mu, \nu) \hat{\Pi}_x \hat{U}^{-1}(\mu, \nu) = | x \rangle \langle x |. \quad (24)$$

Here, the transformation $\hat{U}(\sigma)$ is chosen to be the symplectic group representation [14]

$$\hat{U}(\mu, \nu) = \exp \left[ i \phi \left( \frac{\hat{p}^2}{2} + \frac{\hat{q}^2}{2} \right) \right] \exp \left[ i \lambda \left( \hat{q} \hat{p} + \hat{p} \hat{q} \right) \right]. \quad (25)$$

The rotation and scaling parameters $\phi$ and $\lambda$ are related to $\mu$ and $\nu$ by the following formulas

$$\mu = \lambda \cos \phi, \quad \nu = \lambda^{-1} \sin \phi, \quad (26)$$

$$\phi = \frac{1}{2} \arcsin(2\mu \nu), \quad \lambda = \pm \frac{1}{4} \sqrt{1 \pm \sqrt{1 - 4\mu^2 \nu^2}}. \quad (27)$$

This means that the marginal distribution $w(x, \mu, \nu)$ for this particular case of symplectic tomography is given by

$$w(x, \mu, \nu) = Tr \left\{ | x \rangle \langle x | \exp \left[ i \phi \left( \frac{\hat{p}^2}{2} + \frac{\hat{q}^2}{2} \right) \right] \exp \left[ i \lambda \left( \hat{q} \hat{p} + \hat{p} \hat{q} \right) \right] \hat{\rho} \exp \left[ -i \phi \left( \frac{\hat{p}^2}{2} + \frac{\hat{q}^2}{2} \right) \right] \exp \left[ -i \lambda \left( \hat{q} \hat{p} + \hat{p} \hat{q} \right) \right] \right\}. \quad (27)$$
Such measurable probability can be explicitly expressed as \[24\]
\[
    w(x, \mu, \nu) = \int dy \, dk \exp \left[ -ikx + i\mu \nu k^2/2 + iky \right] \rho(y + \nu k, y),
\]
(28)

where \(\rho(y + \nu k, y) = \langle y + \nu k | \rho | y \rangle\) is the representation of the density matrix over the position eigenkets. The marginals satisfy the following homogeneous property

\[
    w(x, \mu, \kappa \nu) = \frac{1}{\kappa} w(x/\kappa, \mu/\kappa, \nu),
\]
(29)

\[
    w(x, \kappa \mu, \nu) = \frac{1}{\kappa} w(x/\kappa, \mu, \nu/\kappa).
\]

The above relation (28) can be inverted \[25\] as

\[
    \hat{\rho} = \int dx \, d\mu \, d\nu \, w(x, \mu, \nu) \hat{K}(x, \mu, \nu),
\]
(30)

where the kernel operator takes the form

\[
    \hat{K}(x, \mu, \nu) = \frac{1}{2\pi \epsilon^2} \exp \left[ -i\epsilon X + \frac{i\epsilon^2 \mu \nu}{2} \right] e^{i\epsilon \mu \hat{q} + i\epsilon \nu \hat{p}}.
\]
(31)

Here, \(\epsilon\) can be set equal 1; this freedom reflects the overcompleteness of information obtainable by means of all possible marginals \[24, 25\].

The multi-mode generalization \[25\] is straightforward, and the analog of formula (27) holds with the following replacement

\[
    | x \rangle \rightarrow | \vec{x} \rangle, \quad \vec{x} = (x_1, x_2, \ldots),
\]

\[
    \phi \left( \frac{\hat{p}^2}{2} + \frac{\hat{q}^2}{2} \right) \rightarrow \phi_1 \left( \frac{\hat{p}_1^2}{2} + \frac{\hat{q}_1^2}{2} \right) + \phi_2 \left( \frac{\hat{p}_2^2}{2} + \frac{\hat{q}_2^2}{2} \right) + \ldots,
\]

\[
    \lambda (\hat{q} \hat{p} + \hat{p} \hat{q}) \rightarrow \lambda_1 (\hat{q}_1 \hat{p}_1 + \hat{p}_1 \hat{q}_1) + \lambda_2 (\hat{q}_2 \hat{p}_2 + \hat{p}_2 \hat{q}_2) + \ldots.
\]

Relations of the parameters \(\lambda_k, \phi_k\) to the parameters \(\mu_k, \nu_k\) are the same of Eq. (26).

\[\textbf{V. QUANTUM TOMOGRAPHY WITH DISCRETE VARIABLES}\]

Here, we consider a spin-\(j\) system. Following \[17, 23\] we will derive the expression for the density matrix of a spin state in terms of measurable probability distributions.

For arbitrary values of spin, let the spin state have the density matrix \(\rho^{(j)}_{mm'} = \langle jm \mid \hat{\rho}^{(j)} \mid jm' \rangle, \quad m = -j, -j + 1, \ldots, j - 1, j,\)

(33)

where

\[
    \hat{j}_3 \mid jm \rangle = m \mid jm \rangle, \quad \hat{j}^2 \mid jm \rangle = j(j + 1) \mid jm \rangle,
\]

(34)

and

\[
    \hat{\rho}^{(j)} = \sum_{m=-j}^{j} \sum_{m'=-j}^{j} \rho^{(j)}_{mm'} \langle jm \rangle \langle jm' \mid .
\]

(35)

The operator \(\hat{\rho}^{(j)}\) is the density operator of the state under consideration.

The general group construction of tomographic schemes \[19\] was also used for spin tomography \[17, 23\]. The idea is to consider the diagonal elements of the density matrix \(\rho\) in another reference frame, i.e. rotated one. To this end we introduce a rotated measureble spin projection

\[
    \hat{J}_3(\alpha, \beta, \gamma) = \hat{D}(\alpha, \beta, \gamma) \hat{j}_3 \hat{D}^{-1}(\alpha, \beta, \gamma),
\]

(36)
where the unitary rotation operator \( \hat{D} \) depends on the Euler angles \( \alpha, \beta, \gamma \). The role of the observable \( \hat{Z} \) is now played by the spin projection \( \hat{J}_3 \), while the rotation-transformation parameters are the Euler angles \( \sigma_1 = \alpha, \sigma_2 = \beta, \sigma_3 = \gamma \). The transformation \( \hat{U}(\sigma) \) is given by the matrix representation of the rotation group, i.e., the Wigner \( D \)-function [27].

The marginals are

\[
    w(s, \alpha, \beta, \gamma) = \langle (j s | \hat{\rho} | j s) \rangle ,
\]

where the rotated spin states becomes

\[
    | j s \rangle = \sum_{m=-j}^{j} D_{s m}(\alpha, \beta, \gamma) | j m \rangle .
\]

Here the matrix elements \( D_{m'm}^{(j)} (\alpha, \beta, \gamma) \) (Wigner \( D \)-functions) are the matrix elements of the rotation-group representation [27].

\[
    D_{m'm}^{(j)} (\alpha, \beta, \gamma) = e^{im'\gamma} d_{m'm}^{(j)} (\alpha, \beta) e^{im\alpha} ,
\]

where

\[
    d_{m'm}^{(j)} (\beta) = \binom{(j + m')!(j - m')!}{(j + m)!((j - m)!} \frac{1}{2} \left( \cos \frac{\beta}{2} \right)^{m'+m} \left( \sin \frac{\beta}{2} \right)^{m'-m} P_{m'm}(\cos \beta) ,
\]

with \( P_{n}^{(a,b)}(x) \) the Jacobi polynomials [27].

Moreover, the transformed spin projector will be

\[
    \hat{\Pi}_s (\alpha, \beta, \gamma) = \hat{D}(\alpha, \beta, \gamma) | j s \rangle \langle j s | \hat{D}^{-1}(\alpha, \beta, \gamma) = | j s \rangle \langle j s | ,
\]

then, we have

\[
    w(s, \alpha, \beta, \gamma) = \sum_{m_1=-j}^{j} \sum_{m_2=-j}^{j} D_{s m_1}^{(j)} (\alpha, \beta, \gamma) \rho_{m_1 m_2}^{(j)} D_{s m_2}^{(j)*} (\alpha, \beta, \gamma) .
\]

Since

\[
    D_{m'm}^{(j)*} (\alpha, \beta, \gamma) = (-1)^{m'-m} D_{-m'-m}^{(j)} (\alpha, \beta, \gamma) ,
\]

the marginal distribution really depends only on two angles, \( \alpha \) and \( \beta \). Hence

\[
    w(s, \alpha, \beta, \gamma) \rightarrow w(s, \alpha, \beta) ,
\]

which satisfies the normalization condition

\[
    \sum_{s=-j}^{j} w(s, \alpha, \beta) = 1 .
\]

As an example, for a spin-\( \frac{1}{2} \) state with spin projection +1/2, we have

\[
    \hat{\rho} = \begin{pmatrix}
        1 & 0 \\
        0 & 0
    \end{pmatrix} ,
\]

and the marginal distributions will be

\[
    w(s = \frac{1}{2}, \alpha, \beta) = \cos^2 \frac{\beta}{2} , \quad w(s = -\frac{1}{2}, \alpha, \beta) = \sin^{2} \frac{\beta}{2} .
\]

In Refs. [17,23], in view of the properties of the Wigner \( D \)-function and the Clebsch–Gordan coefficients, Eq. (42) was inverted and the density matrix was expressed in terms of the marginal distribution.
\[ \rho_{s, \alpha, \beta}^{(j)} = (-1)^{m_2} \sum_{j_3 = 0}^{2j} \sum_{m_3 = -j_3}^{j_3} (2j_3 + 1)^2 \sum_{s = -j}^{j} \int (-1)^s w(s, \alpha, \beta) \]
\[ \times D_{0, m_3}^{(j_3)}(\alpha, \beta, \gamma) W_{s - s_0}^{j_3} W_{m_1 - m_2}^{j_3} \frac{d\Omega}{8\pi^2} \]  
\tag{48}

where \( m_1, m_2 = -j, -j + 1, \ldots, j \) and \( W_{m_1, m_2}^{j_3} \) are the Wigner-3j symbols \[27\]. The integration is performed over the rotation parameters, i.e.
\[ \int d\Omega = \int_0^{2\pi} d\alpha \int_0^\pi \sin \beta d\beta \int_0^{2\pi} d\gamma. \]  
\tag{49}

Equation (48) can be presented in an invariant operator form \[23\]. We systematically introduce the following notation, first for the function on the unit sphere
\[ \Phi_j^{(j_3)}(s, \alpha, \beta) = (-1)^{m_2} \sum_{m_3 = -j_3}^{j_3} D_{0, m_3}^{(j_3)}(\alpha, \beta, \gamma) W_{m_1 - m_2}^{j_3} \]  
and then for the operator on the unit sphere
\[ \hat{A}_j^{(j_3)}(s, \alpha, \beta) = (2j_3 + 1)^2 \sum_{m_1 = -j}^{j} \sum_{m_2 = -j}^{j} \frac{j m_1}{j m_2} \Phi_j^{(j_3)}(s, \alpha, \beta) \langle j m_2 |. \]  
\tag{50}

In order to write a final expression for the density operator, we introduce an operator on the unit sphere which contains a dependence on the measurable projection of the spin
\[ \hat{K}^{(j)}(s, \alpha, \beta) = (-1)^s \sum_{j_3 = 0}^{2j} W_{s - s_0}^{j_3} \hat{A}_j^{(j_3)}(s, \alpha, \beta). \]  
\tag{52}

Finally, we obtain a compact expression for the density operator,
\[ \hat{\rho}^{(j)} = \sum_{s = -j}^{j} \int \frac{d\Omega}{8\pi^2} w(s, \alpha, \beta) \hat{K}^{(j)}(s, \alpha, \beta). \]  
\tag{53}

Formula (53) admits of the following interpretation. To determine the spin state for a spin \( j \), one has to experimentally measure the projection \( s \) of the spin for each direction specified by the angles \( \alpha \) and \( \beta \), obtaining a distribution function \( w(s, \alpha, \beta) \). The sum on the r.h.s. of Eq. (53) for a given point on the unit sphere represents the average operator \( \langle \hat{K}^{(j)}(s, \alpha, \beta) \rangle \). Then, the integral over the whole solid angle gives the desired density operator. Finally, we recognize that, for the spin case, the operator (52) plays the role of the operator \( \hat{K}(s, \sigma) \) of Eq. (13), employed in the general scheme of Section 2.

VI. THE GENERAL CASE

We are now able to consider the case of a particle with \( N - 1 \) spatial degrees of freedom, plus one spin-\( \frac{1}{2} \) degree. In this case, the state vector \( | \psi \rangle \) has the form
\[ | \vec{q}, m \rangle = | q_1, \ldots, q_{N-1} \rangle \otimes | \frac{1}{2}, m \rangle, \]  
\tag{54}

where \( \vec{q} \) is the eigenvalue of the position operator \( \hat{q} \) and the spin projection \( m = (-1/2, 1/2) \) is the eigenvalue of the Pauli matrix \( \hat{\sigma}_z \).

The transformation operator \( \hat{U}(\sigma) \) used to construct the tomography scheme, for this case, depends on \( 2(N - 1) \) parameters determining the symplectic transform, and on three Euler angles determining the spin rotation.

The transformation operator \( \hat{U}(\sigma) \) of Eq. (1) becomes the product of operators
\[ \hat{U}(\sigma) = \otimes_{k=1}^{N-1} \hat{U}(\mu_k, \nu_k) \otimes \hat{U}(\alpha, \beta, \gamma). \]  
\tag{55}
For the case of spin-$\frac{1}{2}$, the representation of the rotation group is given by

$$D(\alpha, \beta, \gamma) = \begin{pmatrix} e^{i\alpha/2} \cos (\beta/2) & e^{-i\gamma/2} \\ e^{i\gamma/2} \cos (\beta/2) & e^{-i\gamma/2} \sin (\beta/2) \end{pmatrix},$$

(56)

which determines the operator

$$\hat{U}(\alpha, \beta, \gamma) = \sum_{m_1=\pm 1/2}^{1/2} \sum_{m_2=\pm 1/2}^{1/2} D^{(1/2)}_{m_1, m_2}(\alpha, \beta, \gamma) |\frac{1}{2}, m_1 \rangle (\frac{1}{2}, m_2 |.$$

(57)

The marginal distribution $w(z, \sigma)$ (12) will depend on $N - 1$ continuous (noncompact) variables $z_1 = x_1$, ..., $z_{N-1} = x_{N-1}$, and one discrete spin projection $z_N = s$, as well as on parameters $\mu_k$, $\nu_k$ and on Euler angles $\alpha$, $\beta$. The dependence of the marginal distribution on the Euler angle $\gamma$ disappears, as it was shown in the previous section, due to the structure of Wigner $D$-functions.

In order to get an analog of the Pauli evolution equation for the marginal distribution, we consider the general equation (10) where the operator $K(z', \sigma')$ has the form

$$\hat{K}(z', \sigma') = \frac{1}{8 \pi^2} \otimes_{k=1}^{N-1} \hat{K}(x_k, \mu_k, \nu_k) \otimes \hat{K}^{(1/2)}(s, \alpha, \beta).$$

(58)

Here, the operator $\hat{K}(x_k, \mu_k, \nu_k)$ has the form of Eq. (31) with $\epsilon = 1$, and the operator $\hat{K}^{(1/2)}(s, \alpha, \beta)$ is given by formula (52) with $j = 1/2$. Moreover, we have to introduce the marginal distribution $w(\bar{x}, \bar{\mu}, \bar{\nu}, s, \alpha, \beta, t)$ describing a state of spin-$\frac{1}{2}$ particle which depends on the continuous variables $\bar{x}$, discrete spin projection $s$, symplectic reference frame’s labels $\bar{\mu}$ and $\bar{\nu}$, and Euler angles $\alpha$ and $\beta$. Then, for a given Hamiltonian $\hat{H}$ the general equation (10) takes the form of a Pauli-like equation

$$\partial_t w(\bar{x}, \bar{\mu}, \bar{\nu}, s, \alpha, \beta, t) = \frac{\partial}{\partial s'} \int d\bar{x}'d\bar{\mu}'d\bar{\nu}'d\gamma' w(\bar{x}', \bar{\mu}', \bar{\nu}', s', \alpha', \beta', t)$$

$$\times \Theta(\bar{x}, \bar{\mu}, \bar{\nu}, s, \alpha, \beta; \bar{x}', \bar{\mu}', \bar{\nu}', s', \alpha', \beta'),$$

(59)

where

$$\Theta = \frac{i}{8 \pi^2} \langle \bar{x}, s | \otimes_{k=1}^{N-1} \hat{K}(x_k', \mu_k', \nu_k) \otimes \hat{K}^{(1/2)}(s', \alpha', \beta'), \hat{H} \rangle | \bar{x}, s \rangle.$$  

(60)

The structure of the derived Pauli-like equation for probability distributions depends on the particular tomography schemes we have considered. Obviously, it would be useful to find the schemes which give the simplest form for such dynamical equation, nevertheless this is a nontrivial problem related to the possibility of finding properly transformed projector (11). The latter are investigated in Ref. [24], but for different purposes.

A. Limit cases

We want now to consider two limiting cases of the above equation (59).

First of all we consider the (one-dimensional) spatial case only with free motion

$$\hat{H} = \frac{p^2}{2}.$$

(61)

The spin part does not contribute since $\hat{H}$ does not contain the spin operators, that is

$$\int \frac{d\gamma'}{8 \pi^2} w(s', \alpha', \beta', -)(\langle s | \hat{K}^{(1)}(s', \alpha', \beta') | s \rangle) = \sum_{m_1, m_2 = -j}^{j} D^{(j)}_{m_1, m_2}(\alpha, \beta, \gamma) D^{(j)}_{m_1, m_2}(\alpha, \beta, \gamma)$$

$$\times \sum_{j_3=0}^{2j} \sum_{m_3=-j_3}^{j_3} \sum_{s'=s}^{j} (-)^{m_2-s'} (2j_3 + 1)^2 W^{j_3}_{s' s'} W^{j_3}_{s m_1 m_2 m_3}$$

$$\times \int \frac{d\gamma'}{8 \pi^2} w(s', \alpha', \beta', -) D^{(j_3)}_{m_3}(\alpha', \beta', \gamma') = w(s, \alpha, \beta, -),$$

(62)
where \( \) indicates other possible variables. Then, for what concerns the spatial part, it is important to calculate the commutator between the kernel and the Hamiltonian, given by

\[
\left[ e^{i\mu' \hat{q} e^{i\nu' \hat{p}}} \tilde{P} \right] = e^{i\mu' \hat{q} e^{i\nu' \hat{p}}} (-2\mu' \hat{p} - \mu'^2).
\]

(63)

Now, one can write

\[
\partial_t w(x, \mu, \nu, t) = \frac{i}{4\pi} \int dx' dy' dy'' \langle w(X', \mu', \nu', t) e^{-iX' + i\nu' / 2} \rangle \int dq \langle q | e^{i\nu' \hat{q} e^{i\nu'' \hat{p}}} \rangle \langle q | (-2\mu' \hat{p} - \mu'^2) | x \rangle
\]

(64)

By using the explicit form for the wave functions \( \langle q | x \rangle \) (22), together with the homogeneous property (29), it is possible to reduce the above equation to a very simple form

\[
\partial_t w = \mu \partial_{\nu} w
\]

(65)

which was derived in a different way in Ref. 13.

As a second case we study the dynamics of spin-\( \frac{1}{2} \) degree only. The Hamiltonian we wish to consider is

\[
\hat{H} = \begin{pmatrix}
a & 0 \\
0 & c
\end{pmatrix}.
\]

(66)

Of course, the spatial degree is not affected, so its variables can be disregarded; this also results from the fact that

\[
\frac{1}{2\pi} \int dx' dy' dy'' w(x', \mu', \nu', -) \langle x | e^{i\nu' \hat{q} e^{i\nu'' \hat{p}}} \rangle e^{-iX' + i\nu' / 2} = w(x, \mu, \nu, -)
\]

(67)

In this case the relation between the transformed spin state projection and the untransformed one is given by

\[
| s \rangle = \hat{D}_{s,1/2}^{(1/2)*}(\alpha, \beta) | \frac{1}{2} \rangle + \hat{D}_{s,-1/2}^{(1/2)*}(\alpha, \beta) | -\frac{1}{2} \rangle.
\]

(68)

Again, the central task is the calculation of the commutator between the kernel and the Hamiltonian. It is easy to see that

\[
\langle s | \left[ \hat{K}^{(1/2)}(s', \alpha', \beta'), \hat{H} \right] | s \rangle = (-1)^{-s'} \sum_{j_3=0}^{1/2} W_{s',-s'}^{1/2 1/2 j_3} W_{s,-s}^{1/2 1/2 j_3} (2j_3 + 1)^2 \sum_{m_1 \neq m_2, -1/2} (-)^{m_2} D_{m_3}^{(j_3)}(\alpha', \beta', \gamma) W_{m_1 - m_3}^{1/2 1/2 j_3} \times (-)^{1/2 - m_2} (a - c) D_{s_1 m_1}^{(1/2)}(\alpha, \beta, \gamma) D_{s_2 m_2}^{(1/2)}(\alpha, \beta, \gamma).
\]

(69)

Due to the properties of the Wigner-3j symbols we may see that the terms with \( j_3 = 0, 1 \), and \( m_3 = 0 \) do not give contributions; moreover, changing the value of \( s' \), it changes only the sign. Thus, we will get

\[
\partial_t w(\frac{1}{2}, \alpha, \beta, t) = \int \frac{d\Omega}{8\pi^2} \left[ w(\frac{1}{2}, \alpha', \beta', t) - w(-\frac{1}{2}, \alpha', \beta', t) \right] \times \frac{3}{2} (a - c) \sin \beta' \sin \beta \sin(\alpha - \alpha')
\]

(70)

and by using the normalization condition it can be rewritten as

\[
\partial_t w(s, \alpha, \beta, t) = 3(a - c) \sin \beta \int \frac{d\Omega}{8\pi^2} w(s, \alpha', \beta', t) \sin \beta' \sin(\alpha - \alpha')
\]

(71)

which is similar to that derived in Ref. 17 (the differences are due to the degeneracy of the spin-\( \frac{1}{2} \) systems). It should be noted in the above equation that the argument \( s \) is the same in both sides; this is consistent with the fact that \( \hat{H} \) in Eq. (66) does not mix states with different \( s \). On the other hand it can be easily checked that the sum over \( s \) at r.h.s. of Eq. (71) causes the integral to become zero; this is consistent with the fact that at l.h.s. we will obtain the time derivative of a constant. Also, if \( a = c \), the r.h.s. of Eq. (71) will be zero since the Hamiltonian (66) will be proportional to the identity and will not produce any evolution.
VII. EXAMPLES

In the previous section, we discussed the probability for the joint measurement of the spin and spatial variables. Therefore, here we would like to consider some examples involving both variables.

At first we consider a system with the following Hamiltonian

$$\hat{H} = \frac{1}{2} \left( \hat{p}^2 + \hat{q}^2 \right) + \left( | \frac{1}{2} \rangle \langle \frac{1}{2} | - | -\frac{1}{2} \rangle \langle -\frac{1}{2} | \right). \quad (72)$$

It could describe e.g. one vibrational degree of a trapped electron plus its spin [28]. The measurability of marginals in this system is investigated in Ref. [29]. Here, as a straightforward extension of the arguments of Sec. VI.1 we obtain

$$\partial_t w(x, \mu, \nu, s, \alpha, \beta) = \left( \mu \partial_\nu - \nu \partial_\mu \right) w(x, \mu, \nu, s, \alpha, \beta) \quad (73)$$

$$+ 6 \sin \beta \int \frac{d\Omega}{8\pi^2} w(x, \mu, \nu, s, \alpha', \beta', t) \sin \beta' \sin (\alpha - \alpha') \quad (73).$$

Let us now consider an initial entangled state like

$$\Psi(0) = \frac{1}{\sqrt{2}} \left( | 0 \rangle \otimes | -\frac{1}{2} \rangle + | 1 \rangle \otimes | \frac{1}{2} \rangle \right), \quad (74)$$

where \( | n \rangle \) represents the number eigenstate of a harmonic oscillator. At Eq.(74) corresponds the following marginal

$$w(x, \mu, \nu, s, \alpha, \beta, t = 0) = \frac{1}{2} \left[ w_{00\downarrow\downarrow} + w_{11\uparrow\uparrow} + w_{01\downarrow\uparrow} + w_{10\uparrow\downarrow} \right], \quad (75)$$

where

$$w_{00\downarrow\downarrow} = \frac{1}{\sqrt{\pi (\mu^2 + \nu^2)}} \exp \left[ -\frac{x^2}{\mu^2 + \nu^2} \right] D^{(1/2) \ast}_{s - \frac{1}{2}} (\alpha, \beta, \gamma) D^{(1/2)}_{s - \frac{1}{2}} (\alpha, \beta, \gamma), \quad (76)$$

$$w_{11\uparrow\uparrow} = \frac{2x^2}{\sqrt{\pi (\mu^2 + \nu^2)^3}} \exp \left[ -\frac{x^2}{\mu^2 + \nu^2} \right] D^{(1/2) \ast}_{s + \frac{1}{2}} (\alpha, \beta, \gamma) D^{(1/2)}_{s + \frac{1}{2}} (\alpha, \beta, \gamma), \quad (77)$$

$$w_{01\downarrow\uparrow} = i \frac{\sqrt{2x(\nu - i\mu)}}{\sqrt{\pi (\mu^2 + \nu^2)^3}} \exp \left[ -\frac{x^2}{\mu^2 + \nu^2} \right] D^{(1/2) \ast}_{s - \frac{1}{2}} (\alpha, \beta, \gamma) D^{(1/2)}_{s - \frac{1}{2}} (\alpha, \beta, \gamma), \quad (78)$$

$$w_{10\uparrow\downarrow} = w_{01\downarrow\uparrow}. \quad (79)$$

Then, the solution of the Pauli equation (73) is

$$w(x, \mu, \nu, s, \alpha, \beta, t) = \frac{1}{2} \left[ w_{00\downarrow\downarrow} + w_{11\uparrow\uparrow} + w_{01\downarrow\uparrow} e^{3it} + w_{10\uparrow\downarrow} e^{-3it} \right]. \quad (80)$$

As a second example, we want to consider the case of Landau levels [30], i.e. a charged particle moving in a classical magnetic field \( \vec{B} \) being time-independent and axial symmetric. The particle’s movement along the axis being free, instead the Hamiltonian of the transverse motion reads

$$\hat{H} = \frac{1}{2} \left[ (\hat{p}_1 - \hat{A}_1)^2 + (\hat{p}_2 - \hat{A}_2)^2 \right], \quad \hat{A} = \left[ \vec{B} \times \frac{\vec{r}}{2} \right], \quad (81)$$

where \( \vec{r} = (\hat{q}_1, \hat{q}_2) \) is the radius-vector of the particle’s center, \( \hat{p}_1 \) and \( \hat{p}_2 \) are the particle’s momentum components in the transverse plane. Having \( \vec{B} \) along the third axis and choosing \( | \vec{B} | = 2 \), we get

$$\hat{H} = \frac{1}{2} \left( \hat{p}_1^2 + \hat{p}_2^2 + \hat{q}_1^2 + \hat{q}_2^2 \right) + (\hat{p}_1 \hat{q}_2 - \hat{p}_2 \hat{q}_1) + \left( | \frac{1}{2} \rangle \langle \frac{1}{2} | - | -\frac{1}{2} \rangle \langle -\frac{1}{2} | \right). \quad (82)$$

In this case the kernel \( \Theta \) of Eq.(44) is given by
\[ \Theta = \frac{i}{4\pi^2} \int \frac{dq_1}{\nu_1} \int \frac{dq_2}{\nu_2} e^{\sum_{i=1}^2 [(\mu_i q_i/\nu_i) + (x_i q_i)] + \mu_i q_i + \mu_i q_i^2 - \mu_i q_i q_i} \]

\[ \times \left\{ \sum_{i=1}^2 (\mu_i q_i/\nu_i - \mu_i q_i + \mu_i^2/2 - \nu_i^2/2) - (x_1 - \mu_2 q_2)/\nu_2 \right\} \delta_{s,s'} \delta(\Omega - \Omega') + \frac{6}{8\pi^2} \sin \beta \sin \beta' \sin (\alpha - \alpha') \delta(\vec{x} - \vec{x}') \delta(\vec{\mu} - \vec{\mu}') \delta(\vec{v} - \vec{v}'). \] (83)

As nontrivial example we also consider here an initial state which is the entangled superposition

\[ \Psi(0) = \frac{1}{\sqrt{2}} \left[ |00\rangle \otimes |-\frac{1}{2}\rangle + |10\rangle \otimes |\frac{1}{2}\rangle \right]. \] (84)

It leads to nonfactorisable marginal

\[ w(x, \mu, \nu, s, \alpha, \beta, t = 0) = \frac{1}{2} \left[ w_{0000\downarrow\downarrow} + w_{1010\uparrow\uparrow} + w_{0010\downarrow\uparrow} + w_{1000\uparrow\downarrow} \right], \] (85)

where

\[ w_{n_1 n_2 n_1' n_2'} m_1 m_2 = w_{n_1 n_2 n_1' n_2'} \times D_{s m_1}^{(1/2)*}(\alpha, \beta, \gamma) D_{s m_2}^{(1/2)}(\alpha, \beta, \gamma). \] (86)

Here, \( m_1 = -\frac{1}{2} \) (\( m_1 = \frac{1}{2} \)) replaces downarrow (uparrow) while the spatial part \( w_{n_1 n_2 n_1' n_2'} \) is explicitely calculated in the Appendix.

It is now easy to see that the solution of the Eq. (59) with the kernel (83), subject to the above initial condition, is

\[ w(x, \mu, \nu, s, \alpha, \beta, t) = \frac{1}{2} \left[ w_{0000\downarrow\downarrow} + w_{1010\uparrow\uparrow} + w_{0010\downarrow\uparrow} e^{3it} + w_{1000\uparrow\downarrow} e^{-3it} \right]. \] (87)

**VIII. CONCLUSION**

We conclude that it is possible to obtain an evolution equation for the tomographic probabilities (marginal distributions) of an arbitrary tomography scheme. The main result of our paper is the analog of the Pauli equation for spin-\( \frac{1}{2} \) particle.

The explicit expression for the marginal distribution for a trapped particle as well as for Landau levels has been studied. It results that in the nonstationary case they obey the analog of the Pauli equation.

The examples considered demonstrate that the usual problems of conventional quantum mechanics can be cast into the form in which only positive probabilities are used to describe quantum states and their evolution. A possible disadvantage of the approach proposed is a complicated evolution equation (59), but, perhaps, this is the price one ought to pay for the possibility of describing quantum objects in terms of classical probabilities.

Anyway, our argumentations can constitute a step further from the Bohr position [31] about the inapplicability of classical modes of description in the quantum domain. In fact, while we believe that quantum mechanics is not classical physics in disguise, we retain (some) classical concepts still applicable against counterintuitive notions like complex statefunctions.

We also belive that the developed classical-like formalism could be applied to describe quantum mechanical paradoxes, because usually, if there is a paradox in quantum mechanics, there should also be a classical one, perhaps worse [32]. These aspects will be investigated in a forthcoming paper as well as the extension of the presented approach to the relativistic domain [33], in order to find an analog of the Dirac equation.

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11
The wave function of the particle’s coherent state in a magnetic field \( \vec{B} \) is

\[
\Psi_{\alpha, \beta}(q_1, q_2) = \frac{1}{\sqrt{\pi}} \exp \left\{ -\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} - i\alpha\beta + \beta (q_1 + iq_2) + i\alpha (q_1 - iq_2) \right\},
\]  

(88)

where \( q_1 \) and \( q_2 \) are the particle’s coordinates and \( \alpha \) and \( \beta \) complex numbers.

The coherent state (88) is the superposition of number states [34, 36], namely,

\[
\Psi_{\alpha, \beta}(q_1, q_2) = \exp \left( -\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} \right) \sum_{n=0}^{\infty} \frac{\alpha^n \beta^n \Psi_{n, n'}(q_1, q_2)}{\sqrt{n!n'n!}}.
\]

(89)

In view of the general relationship between the marginal distribution and wave function [37], we have

\[
w(x_1, x_2, \mu_1, \mu_2, \nu_2) = \frac{1}{4\pi^2 |\nu_1\nu_2|} \left| \int \int \exp \left( \frac{i\nu_1\mu_1}{\nu_1} - i\gamma x_1 + \frac{i\nu_2\mu_2}{\nu_2} - i\gamma x_2 \right) \Psi_{\alpha, \beta}(y_1, y_2) \, dy_1 \, dy_2 \right|^2,
\]

(90)

where parameters \( \mu_1, \nu_1, \mu_2, \nu_2 \), as usually, mark reference frames; then, one obtains for the marginal distribution of the particle’s coherent state without spin in a magnetic field the following expression

\[
w_{\alpha, \beta}(x_1, x_2, \mu_1, \mu_2, \nu_2) = \exp \left[ -\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} - i(\alpha\beta - \alpha^*\beta^*) \right]
\times \exp \left\{ \frac{(\nu_1 + i\mu_1)(i\alpha\nu_1 + \beta\nu_1 - ix_1)^2 + (\nu_1 - i\mu_1)(-i\alpha^*\nu_1\beta\nu_1 + ix_1)^2}{2\nu_1(\mu_1^2 + \nu_1^2)} 
+ \frac{(\nu_2 + i\mu_2)(\alpha\nu_2 + i\beta\nu_2 - ix_2)^2 + (\nu_2 - i\mu_2)(\alpha^*\nu_2 - i\beta^*\nu_2 + ix_2)^2}{2\nu_2(\mu_2^2 + \nu_2^2)} \right\}.
\]

(91)

Multiplying (91) by \( \exp(|\alpha|^2 + |\beta|^2) \) and expanding the expression obtained into the power series, we arrive at

\[
w_{\alpha, \beta}(x_1, x_2, \mu_1, \mu_2, \nu_2) e^{(|\alpha|^2 + |\beta|^2)} = \sum_{n_1, n_2, n_1', n_2'} \frac{\alpha^{n_1}(\alpha^*)^{n_2} \beta^{n_1'}(\beta^*)^{n_2'}}{\sqrt{n_1!n_2!n_1'!n_2'!}} w_{n_1, n_2, n_1', n_2'}.
\]

(92)

Taking into account the property of the generating function for multivariate Hermite polynomials [38], namely,

\[
\exp \left\{ -\frac{1}{2} \vec{u} \cdot \vec{M} \vec{u} + \vec{u} \cdot \vec{M} \vec{z} \right\} = \sum_{n_1, n_2, n_1', n_2'} \frac{\alpha^{n_1}(\alpha^*)^{n_2} \beta^{n_1'}(\beta^*)^{n_2'}}{n_1!n_2!n_1'!n_2'} H_{n_1, n_2, n_1', n_2'}^{(M)} \left( \vec{z} \right),
\]

(93)

where the vector \( \vec{u} \) has components \( u = (\alpha, \alpha^*, \beta, \beta^*) \), and comparing (92) with (93), we obtain

\[
w_{n_1, n_2, n_1', n_2'}(x_1, x_2, \mu_1, \mu_2, \nu_2) = \frac{1}{\pi \sqrt{(\nu_1^2 + \mu_1^2)(\nu_2^2 + \mu_2^2)}} 
\times \exp \left( -\frac{x_1^2}{\mu_1^2 + \nu_1^2} - \frac{x_2^2}{\mu_2^2 + \nu_2^2} \right) \frac{H_{n_1, n_2, n_1', n_2'}^{(M)} \left( \vec{z} \right)}{\sqrt{n_1!n_2!n_1'!n_2'!}},
\]

(94)

where the 4 \( \times \) 4 matrix \( M \) reads

\[
M = \left( \begin{array}{cccc}
M^{(1)} & M^{(2)} \\
M^{(4)} & M^{(3)}
\end{array} \right).
\]

(95)

The 2 \( \times \) 2 matrices \( M^{(r)} \) are given by
The argument of the multivariate Hermite polynomials $\vec{\zeta} = (\zeta_1, \zeta_2, \zeta_2^*)$ is expressed in terms of the parameters as follows

$$
\zeta_1 = \frac{i x_1}{\sqrt{\nu_1 + \nu_1^*}} \exp \left( i \tan^{-1} \frac{\nu_2}{\nu_2^*} \right) - \frac{x_2}{\sqrt{\nu_1^* + \nu_2^*}} \exp \left( i \tan^{-1} \frac{\nu_1}{\nu_1^*} \right),
$$

$$
\zeta_2 = \frac{i x_2}{\sqrt{\nu_1 + \nu_1^*}} \exp \left( i \tan^{-1} \frac{\nu_1}{\nu_1^*} \right) - \frac{x_1}{\sqrt{\nu_1^* + \nu_2^*}} \exp \left( i \tan^{-1} \frac{\nu_2}{\nu_2^*} \right).
$$

Taking $n_1 = n_2$ and $n_1' = n_2'$ we obtain the marginal distribution $w_{n,n'}(x_1, x_2, \mu_1, \nu_1, \mu_2, \nu_2)$ for the Landau level states $|nn')$

$$
w_{n,n'}(x_1, x_2, \mu_1, \nu_1, \mu_2, \nu_2) = \frac{1}{\pi \sqrt{(\nu_1^* + \nu_1^*)(\nu_2^* + \mu_2^*)}} \exp \left( -\frac{x_1^2}{\mu_1 + \nu_1} - \frac{x_2^2}{\mu_2 + \nu_2} \right) H_{n,n'}^{(M)}(\vec{\zeta}).
$$

where $n$ is the main quantum number and $n' - n = l$ is the angular momentum quantum number.
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