Note on Covariant Stückelberg Formalism and Absence of Boulware-Deser Ghost in Bi-gravity

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Abstract
The covariant Stückelberg formalism is applied to bi-gravity in order to revisit the issue of absence of the Boulware-Deser (BD) ghost. We first confirm that the leading order action in the decoupling limit for helicity-2 modes of metrics and helicity-0 mode of Stückelberg perturbations does not lead to higher time derivative in equations of motion, which suggests the absence of the BD ghost. By extending this method, we reconfirm that the BD ghost does not appear for arbitrary order of the perturbations at the decoupling limit in bi-gravity.

1. Introduction
One possible way of modification of general relativity is to introduce extra degrees of freedom to gravitation. In order for this kind of extension to be viable, these extra degrees of freedom must not destabilize the system, especially, must not be ghosts.

To give a mass to graviton is a potent way to add degrees of freedom to gravitation. On linear metric perturbations around the flat fiducial metric, Fierz and Pauli (FP) establish the theory which has five gravitational degrees of freedom without ghosts [1]. The theory consisting of the non-linear Einstein-Hilbert kinetic term and the FP mass potential, however, excites six degrees of freedom and the additional one is the Boulware-Deser (BD) ghost [2]. In the Stückelberg formalism with taking decoupling limit [3], the BD ghost instability can be regarded as the Ostrogradsky instability [4] associated with the dangerous higher time derivative of the helicity-0 mode of the Stückelberg fields. de Rham, Gabadadze and Tolley (dRGT) construct the mass potential where the self interactions of the helicity-0 mode take the Galileon form [5] and hence there is no Ostrogradsky instability at least in decoupling limit [6–8]. It is finally proven that dRGT theory is free from the BD ghost even without taking decoupling limit [9–11].

The theory of massive gravity is first constructed with the flat fiducial metric. dRGT massive gravity with the flat fiducial metric can be extend to that with a general fiducial metric and this theory is also proven to be BD ghost free [10, 12]. Since the full theory is BD ghost free, the equations of motion of the helicity-0 mode in the decoupling limit should not include higher time
derivatives. In Ref. [13], two of the present authors and their collaborators confirm this fact directly by formulating the covariant St"uckelberg analysis for general fiducial metric. There, non-trivial couplings between the curvature of fiducial metric and helicity-0 mode of St"uckelberg fields appear. Such coupling terms, however, do not produce higher time derivatives in the equations of motion because the fiducial metric is non-dynamical in massive gravity.

Massive gravity was further extended to bi-gravity, in which the Einstein-Hilbert kinetic term of a fiducial metric is added to dRGT massive gravity, and this bi-gravity theory is also shown to be BD ghost free [10, 14]. Then, the equations of motion in decoupling limit should not include higher time derivatives. On the other hand, by the analogy with the covariant St"uckelberg analysis of massive gravity, there should be non-trivial derivative couplings between the fiducial metric and the helicity-0 mode. The purpose of this short note is to clarify why such higher derivative interactions do not cause the Ostrogradsky instability.

In the next section, we will give a brief review of the covariant St"uckelberg analysis of dRGT massive gravity with a general fiducial metric established in Ref. [13]. Then, in Sec. 3, we will extend the results of dRGT theory to bi-gravity. Final section is devoted to conclusions.

2. Covariant St"uckelberg formalism of massive gravity

We begin with a brief review of the covariant St"uckelberg formalism of massive gravity with a general fiducial metric established in Ref. [13]. The action of dRGT massive gravity with a general fiducial metric $\bar{g}_{\mu\nu}$ is given as follows [7, 12, 15]:

$$S[g, \bar{g}] = \frac{M_{\text{pl}}^2}{2} \int d^4x \sqrt{-g} \left( R[g] + 2m^2 \sum_{i=0}^{4} \beta_i e_i(\gamma[g, \bar{g}]) \right),$$

where

$$\gamma[g, \bar{g}]^\mu_\nu = \sqrt{g^{-1} \bar{g}^\mu_\nu},$$

and

$$e_i[\gamma] = \gamma_i^{\mu_1 \mu_2 \cdots \mu_n}_{\mu_1 \cdots \mu_n}.$$ 

In this note, we focus on the following parameters used in the original dRGT theory [7]:

- $\beta_0 = 6 + 12\alpha_3 + 12\alpha_4$,
- $\beta_1 = -3 - 9\alpha_3 - 12\alpha_4$,
- $\beta_2 = 1 + 6\alpha_3 + 12\alpha_4$,
- $\beta_3 = -3\alpha_3 - 12\alpha_4$,
- $\beta_4 = 12\alpha_4$.

1In the original dRGT theory, the action is written in terms of $\alpha$ parameters as

$$S[g, \bar{g}] = \frac{M_{\text{pl}}^2}{2} \int d^4x \left[ \sqrt{-g} R[g] + \sqrt{-\bar{g}} m^2 \sum_{i=2}^{4} (i!) \alpha_i e_i(K) \right],$$

with $K^\mu_\nu = \delta^\mu_\nu - \gamma^\mu_\nu$ and $\alpha_2 = 1.$
Due to the presence of the mass term, the action (1) does not possess the gauge symmetry on diffeomorphism. However, we can rewrite the action (1) as gauge invariant one by introducing Stückelberg fields $\phi^a$ with $a = 0, 1, 2, 3$, and by replacing the original fiducial metric with covariantized one:

$$\bar{g}_{\mu\nu} \rightarrow f^\phi_{\mu\nu}(x) = \frac{\partial \phi^a(x)}{\partial x^\mu} \frac{\partial \phi^b(x)}{\partial x^\nu} \bar{g}_{ab}(\phi).$$  \hspace{1cm} (9)

The resultant action

$$S = \frac{M^2_{\text{pl}}}{2} \int d^4x \sqrt{-g} \left( R[g] + 2m^2 \sum_{i=0}^4 \beta_i \epsilon_i(\gamma[g, f^\phi]) \right)$$  \hspace{1cm} (10)

is invariant under the following gauge transformation,

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x) = g_{\alpha\beta}(y(x)) \frac{\partial y^\alpha(x)}{\partial x^\mu} \frac{\partial y^\beta(x)}{\partial x^\nu},$$  \hspace{1cm} (11)

$$\phi^a(x) \rightarrow \phi'^a(x) = \phi^a(y(x)).$$  \hspace{1cm} (12)

Here, the original action (1) can be recovered by fixing the gauge $\phi^a(x) = x^a$, which is called unitary gauge.

We consider perturbations of Stückelberg fields around the unitary gauge. When the fiducial metric is flat, $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$, the perturbations $\tilde{\pi}^a$ simply defined by $\phi^a(x) = x^a - \tilde{\pi}^a$ are well behaved because $\tilde{\pi}^a$ are covariant under the Lorentz transformation. In the non-linear theory consisting of the Einstein-Hilbert action and the linear FP mass potential, the BD ghost instability appears as the Ostrogradsky instability in the form of higher time derivative in the equation of motion of helicity-0 mode of $\tilde{\pi}^a$. Therefore, absence of such higher time derivatives is a necessary condition for being free from the BD ghost.

In the case of a general fiducial metric, since $\tilde{\pi}^a$ is not a covariant vector in curved spacetime, we need to refine the definition of Stückelberg perturbations. We have defined $\pi^a$ as a coordinate value of a point $\phi^a$ in the field space with the Riemann normal coordinate on the fiducial metric $\bar{g}_{ab}$ [13], that is,

$$\phi^a = x^a - \pi^a - \frac{1}{2} \bar{\Gamma}_{bc}^a \pi^b \pi^c + \frac{1}{6} \left( \partial_\lambda \bar{\Gamma}_{cd}^a - 2 \bar{\Gamma}_{bc}^a \bar{\Gamma}_{cd}^e \right) \pi^b \pi^c \pi^d + O(\epsilon^4),$$  \hspace{1cm} (13)

where $\epsilon$ represents the order of perturbations. From the definition of the Riemann normal coordinate, $\pi^a$ is a covariant vector. By this definition, the covariantized fiducial metric is expanded in a covariant way:

$$f^\phi_{\mu\nu} = \bar{g}_{\mu\nu} - 2 \nabla_{(\mu} \pi_{\nu)} + \nabla_{\mu} \pi_{\nu} \nabla_{\nu} \pi^\sigma - \bar{R}_{\mu\rho\sigma\pi} \pi^\rho \pi^\sigma$$

$$+ \frac{1}{3} \nabla_{\lambda} \bar{R}_{\mu\rho\sigma\pi} \pi^\rho \pi^\sigma + \frac{2}{3} \bar{R}_{\mu\rho\lambda\sigma} \nabla_{\nu} \pi^\lambda \pi^\rho \pi^\sigma + \frac{2}{3} \bar{R}_{\nu\rho\lambda\sigma} \nabla_{\mu} \pi^\lambda \pi^\rho \pi^\sigma + O(\epsilon^4),$$  \hspace{1cm} (14)

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2 The decoupling limit analysis based on the embedding method is investigated in [16] for de Sitter fiducial metric. This method is equivalent to our Riemann normal coordinate approach as proven in [17]. Another decoupling limit analysis based on the vielbein formalism is investigated in [18].
where $\pi_\mu = \bar{g}_{\mu\nu}\pi^\nu$. Another derivation of Eq. (14) is investigated in [17]. Hereafter, we concentrate only on the helicity-0 mode $\hat{\pi}$ defined by

$$\pi_\mu := \frac{\nabla_\mu \hat{\pi}}{m^2 M_{pl}}$$

(15)

to focus on the presence/absence of the BD ghost. We also consider the metric perturbations around the fiducial metric:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{\hat{h}_{\mu\nu}}{M_{pl}}.$$  

(16)

Here $\hat{\pi}$ and $\hat{h}_{\mu\nu}$ have canonically normalized dimensions.

After taking the extended $\Lambda_3$ decoupling limit [13],

$$M_{pl} \to \infty, \ m \to 0, \ \Lambda_3 := (M_{pl} m^2)^{1/3} \to \text{finite}, \ \frac{\bar{R}_{\mu\nu\rho\sigma}}{m^2} \to \text{finite},$$

(17)

the action reduces to

$$S = \int d^4 x \sqrt{-g} \left( -\frac{1}{4} \hat{h}_{\mu\nu} \bar{\epsilon}^{\mu\nu\rho\sigma} \hat{h}_{\rho\sigma} + \mathcal{L}^{\text{mass}}[\bar{g}, \hat{h}, \hat{\pi}] \right).$$

(18)

Here, the first term represents the contribution from the Einstein-Hilbert action, where the operators $\bar{\epsilon}^{\mu\nu\rho\sigma}$ are given by

$$\bar{\epsilon}^{\mu\nu\rho\sigma} \hat{h}_{\rho\sigma} = -\frac{1}{2} \Box \hat{h}^{\mu\nu} - \frac{1}{2} \nabla^\mu \nabla^\nu \hat{h} + \frac{1}{2} \hat{g}^{\mu\nu} \left( \Box \hat{h} - \nabla_\rho \nabla_\sigma \hat{h}^{\rho\sigma} \right) + \nabla_\rho \nabla^{(\mu} \hat{h}^{\nu)}.$$

(19)

On the other hand, $\mathcal{L}^{\text{mass}}$ represents the contribution from the dRGT mass potential and is concretely given as

$$\mathcal{L}^{\text{mass}}[\bar{g}, \hat{h}, \hat{\pi}] = \frac{1}{2} \hat{h}^{\mu\nu} X^{(1)}_\mu(\hat{\pi}) + \frac{1}{2} \frac{\bar{R}^{\mu\nu}}{m^2} \nabla_\mu \hat{\pi} \nabla_\nu \hat{\pi} + \frac{1}{2} \frac{3\alpha_3}{4\Lambda_3^3} \hat{h}^{\mu\nu} X^{(2)}_\mu(\hat{\pi}) + \frac{1}{2\Lambda_3^3} A_{\mu\nu\rho\sigma} \nabla^\mu \hat{\pi} \nabla^\nu \hat{\pi} \nabla^\rho \nabla^\sigma \hat{\pi}$$

$$+ \frac{\alpha_3 + 4\alpha_4}{4\Lambda_3^3} \hat{h}^{\mu\nu} X^{(3)}_\mu(\hat{\pi}) + \frac{1}{2\Lambda_3^3} \left( B_{\mu\nu\rho\sigma} \nabla^\rho \nabla^\sigma \hat{\pi} - \frac{5}{3} C_{\lambda\mu\nu\rho\sigma} \nabla^\lambda \hat{\pi} \right) \nabla^\mu \hat{\pi} \nabla^\nu \hat{\pi} \nabla^\rho \nabla^\sigma \hat{\pi} + O(\epsilon^5),$$

(20)

where $X^{(i)}_\mu(\hat{\pi})$ are defined by

$$X^{(1)}_\mu(\hat{\pi}) = \bar{g}_{\mu\nu} \Box \hat{\pi} - \nabla_\mu \nabla_\nu \hat{\pi},$$

(21)

$$X^{(2)}_\mu(\hat{\pi}) = \bar{g}_{\mu\nu} \left( \Box \hat{\pi}^2 - \nabla_\rho \nabla_\sigma \hat{\pi} \nabla^\rho \nabla^\sigma \hat{\pi} \right) + 2 \left( \nabla_\mu \nabla_\rho \hat{\pi} \nabla^\rho \nabla_\nu \hat{\pi} - \Box \hat{\pi} \nabla_\mu \nabla_\nu \hat{\pi} \right),$$

(22)

$$X^{(3)}_\mu(\hat{\pi}) = \bar{g}_{\mu\nu} \left( \Box \hat{\pi}^3 - 3\Box \hat{\pi} \nabla_\rho \nabla_\sigma \hat{\pi} \nabla^\rho \nabla^\sigma \hat{\pi} + 2 \nabla_\rho \nabla^\rho \hat{\pi} \nabla_\sigma \nabla_\lambda \hat{\pi} \nabla^\lambda \nabla_\rho \hat{\pi} \right)$$

$$+ 3 \nabla_\mu \nabla_\nu \hat{\pi} \left( \nabla_\rho \nabla_\sigma \hat{\pi} \nabla^\rho \nabla^\sigma \hat{\pi} - (\Box \hat{\pi})^2 \right)$$

$$+ 6 \nabla_\mu \nabla_\nu \hat{\pi} \left( \nabla_\rho \nabla_\sigma \hat{\pi} \nabla^\rho \hat{\pi} - \nabla_\mu \nabla^\sigma \hat{\pi} \nabla_\rho \nabla_\sigma \hat{\pi} \right),$$

(23)
and $A$, $B$ and $C$, which consist of the curvature of the fiducial metric, are defined by

$$A_{\mu\nu\rho\sigma} = \frac{1}{m^2} \left[ (1 + 2\alpha_3) \left( R_{\mu\nu} g_{\rho\sigma} + \bar{R}_{\rho(\mu\nu)} \right) - \alpha_3 \left( g_{\rho\mu} \bar{R}_\nu + g_{\sigma(\mu} \bar{R}_{\nu)\rho} \right) \right] ,$$

$$B_{\mu\nu\rho\sigma'} = \frac{1}{m^2} \left[ \frac{3}{2} (\alpha_3 + 2\alpha_4) \bar{R}_{\mu\nu} \left( 2g_{\rho\sigma'} g_{\sigma'\rho'} \right) + 12\alpha_4 \bar{R}_{\mu\rho} g_{\rho\nu} g_{\sigma'\rho'} + \frac{1}{3} \left( 1 + 9\alpha_3 + 18\alpha_4 \right) \left( \bar{R}_{\mu\nu\rho\sigma} - \bar{R}_{\mu\rho\nu\sigma} \right) - \frac{6\alpha_4 \bar{R}_{\mu\rho} g_{\rho\nu} g_{\sigma'\rho'} \right] ,$$

$$C_{\lambda\mu\nu\rho} = \frac{1}{m^2} \left[ \bar{g}_{\rho\sigma} \bar{\nabla}_{(\lambda} \bar{R}_{\mu\nu)} + \frac{1}{3} \left( \bar{\nabla}_{\lambda} \bar{R}_{\mu(\rho\nu)} + \bar{\nabla}_{\nu} \bar{R}_{\lambda(\rho\nu)} + \bar{\nabla}_{\mu} \bar{R}_{\lambda(\rho\sigma)} \right) \right] .$$

It should be noted that we implicitly assume $g_{\mu\nu} = \bar{g}_{\mu\nu}$ and $\phi^\mu = x^\mu$ is a solution of the background equations of motion in order for the linear order action to vanish. If this is not a vacuum solution, we should assume some matter fields to guarantee the absence of tadpole contributions.

The mixing terms $\hat{h}_{\mu\nu} X_{\mu\nu}^{(1)}$ and $\hat{h}_{\mu\nu} X_{\mu\nu}^{(2)}$ can be diagonalized by the field redefinition \(^3\)

$$\hat{h}_{\mu\nu} \rightarrow \hat{h}_{\mu\nu} + \hat{\pi} \bar{g}_{\mu\nu} - \frac{1 + 3\alpha_3}{A_3^2} \bar{\nabla}_\mu \hat{\pi} \bar{\nabla}_\nu \hat{\pi} .$$

The resultant action becomes

$$S = \int d^4x \sqrt{-g} \left( -\frac{1}{4} \bar{h}_{\mu\nu} \bar{\epsilon}^{\mu\nu\rho\sigma} \bar{h}_{\rho\sigma} + \mathcal{L}^G + \mathcal{L}^C + \frac{1}{4A_3^6} (\alpha_3 + 4\alpha_4) \bar{h}_{\mu\nu} X_{\mu\nu}^{(3)} (\hat{\pi}) + O(\epsilon^5) \right) ,$$

where

$$\mathcal{L}^G = -\frac{3}{4} \bar{g}_{\mu\nu} \bar{\nabla}^\mu \hat{\pi} \bar{\nabla}^\nu \hat{\pi} - \frac{3}{4A_3^2} (\bar{\nabla} \hat{\pi})^2 \bar{\nabla} \hat{\pi}$$

$$- \frac{1 + 8\alpha_3 + 9\alpha_3}{4A_3^2} (\bar{\nabla} \hat{\pi})^2 \left( (\bar{\nabla} \hat{\pi})^2 - \bar{\nabla}_\rho \hat{\pi} \bar{\nabla}_\sigma \hat{\pi} \bar{\nabla} \hat{\pi} \right) ,$$

$$\mathcal{L}^C = \frac{1}{m^2} \bar{R}_{\mu\nu} \bar{\nabla}^\mu \hat{\pi} \bar{\nabla}^\nu \hat{\pi} + \frac{1}{2A_3^2} A_{\mu\nu\rho\sigma} \bar{\nabla}^\mu \hat{\pi} \bar{\nabla}^\nu \hat{\pi} \bar{\nabla}^\rho \hat{\pi} \bar{\nabla}^\sigma \hat{\pi}$$

$$+ \frac{1}{2A_3^2} \left( B_{\mu\nu\rho\sigma'} \bar{\nabla}^\rho \hat{\pi} \bar{\nabla}^\sigma \hat{\pi} - \frac{1}{3} C_{\lambda\mu\nu\rho} \bar{\nabla}^\lambda \hat{\pi} \right) \bar{\nabla}^\mu \hat{\pi} \bar{\nabla}^\nu \hat{\pi} \bar{\nabla}^\rho \hat{\pi} \bar{\nabla}^\sigma \hat{\pi} .$$

Since the derivative operator $\bar{\nabla}_\mu$ is commutative in decoupling limit, the covariant Galileon term $\mathcal{L}^G$ does not lead to higher time derivative terms in the equation of motion of helicity-0 mode of $\hat{\pi}$. In addition, as found in Ref. [13], the curvature term $\mathcal{L}^C$ does not produce higher derivative terms either. Thus, no Ostrogradsky instability appears at least up to fourth order of perturbations.

\(^3\)The field redefinition (27) contains derivatives, so one may wonder if it changes the number of physical degrees of freedom. However, as we discuss in Sec. 3, the degrees of freedom in the perturbation theory can be determined only from the quadratic action. Since derivatives in (27) appear at the nonlinear level, it turns out that the number of degrees of freedom does not change at any order in the perturbation theory.
3. Covariant Stückelberg formalism of bi-gravity

The purpose of this note is to investigate the Ostrogradsky instability of helicity-0 mode of Stückelberg fields in bi-gravity by extending the discussions given in the previous section. The effect of the interactions between the curvature of the fiducial metric and \( \tilde{\pi} \) is not clear because the fiducial metric itself is dynamical in bi-gravity. While the covariant Galileon term \( \mathcal{L}^G \) does not lead to higher time derivative terms in the equation of motion of the fiducial metric as well as that of helicity-0 mode of \( \tilde{\pi}^a \), the curvature term \( \mathcal{L}^C \) naively generates higher time derivative terms in the equations of motion, which might be dangerous. We show, however, that there are no ghosts at the linear perturbation level, so that bi-gravity is free from the Ostrogradsky instability at any order in the perturbation theory.

Let us consider the theory of bi-gravity \([14]\), in which the action is given by

\[
S = \frac{M_{\text{pl}}^2}{2} \int d^4x \sqrt{-g} \left( R[g] + 2m^2 \sum_{i=0}^{4} \beta_i e_i(\gamma[g,f]) \right) + \frac{\kappa^2 M_{\text{pl}}^2}{2} \int d^4x \sqrt{-f} R[f].
\] (31)

First we introduce Stückelberg fields by analogy with massive gravity. The action of bi-gravity is invariant under the gauge transformation

\[
g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x) = g_{\alpha\beta}(y(x)) \frac{\partial y^\alpha(x)}{\partial x^\mu} \frac{\partial y^\beta(x)}{\partial x^\nu},
\] (32)

\[
f_{\mu\nu}(x) \rightarrow f'_{\mu\nu}(x) = f_{\alpha\beta}(y(x)) \frac{\partial y^\alpha(x)}{\partial x^\mu} \frac{\partial y^\beta(x)}{\partial x^\nu},
\] (33)

Stückelberg formalism of bi-gravity is obtained by the following replacement,

\[
\gamma[g,f] \rightarrow \gamma[g,f^\phi] \quad \text{with} \quad f_{\mu\nu}^\phi(x) = f_{\rho\sigma}(\phi(x)) \frac{\partial \phi^\rho}{\partial x^\mu} \frac{\partial \phi^\sigma}{\partial x^\nu}.
\] (34)

The resultant action,

\[
S = \frac{M_{\text{pl}}^2}{2} \int d^4x \sqrt{-g} \left( R[g] + 2m^2 \sum_{i=0}^{4} \beta_i e_i(\gamma[g,f^\phi]) \right) + \frac{\kappa^2 M_{\text{pl}}^2}{2} \int d^4x \sqrt{-f} R[f],
\] (35)

is invariant under the following two gauge symmetries. The first one is given by

\[
g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x) = g_{\alpha\beta}(y(x)) \frac{\partial y^\alpha(x)}{\partial x^\mu} \frac{\partial y^\beta(x)}{\partial x^\nu},
\] (36)

\[
f_{\mu\nu}(x) \rightarrow f'_{\mu\nu}(x) = f_{\gamma\delta}(z(x)) \frac{\partial z^\gamma(x)}{\partial x^\mu} \frac{\partial z^\delta(x)}{\partial x^\nu},
\] (37)

\[
\phi^\mu(x) \rightarrow \phi'^\mu(x) = \phi^\lambda(y(x)),
\] (38)

which leads to

\[
f_{\mu\nu}^\phi(x) \rightarrow f'_{\mu\nu}^\phi(x) = f_{\alpha\beta}^\phi(y(x)) \frac{\partial y^\alpha(x)}{\partial x^\mu} \frac{\partial y^\beta(x)}{\partial x^\nu}.
\] (39)

The second one is given by

\[
g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x) = g_{\mu\nu}(x),
\] (40)

\[
f_{\mu\nu}(x) \rightarrow f'_{\mu\nu}(x) = f_{\alpha\beta}^\phi(z(x)) \frac{\partial z^\alpha(x)}{\partial x^\mu} \frac{\partial z^\beta(x)}{\partial x^\nu},
\] (41)

\[
\phi^\mu(x) \rightarrow \phi'^\mu(x) = (z^{-1})^\mu(\phi(x)).
\] (42)
which leads to

\[ f^{\phi}_{\mu\nu}(x) \rightarrow f^{\phi'}_{\mu\nu}(x) = f^{\phi}_{\mu\nu}(x). \]  

(43)

The original action (35) can be recovered by fixing the unitary gauge $\phi^\mu = x^\mu$ and a combination of these gauge transformations with $y^\alpha(x) = z^\alpha(x)$ reproduces the gauge transformation with Eqs. (32) and (33).

In order to take the decoupling limit of the action in bi-gravity, we consider metric perturbations around $g_{\mu\nu} = \bar{g}_{\mu\nu}$ and $f_{\mu\nu} = \bar{g}_{\mu\nu}$,

\[ g_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{l_{\mu\nu}}{M_{pl}} + \frac{\hat{h}_{\mu\nu}}{M_{pl}} \]  

(44)

\[ f_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{l_{\mu\nu}}{M_{pl}} \]  

(45)

and covariant helicity-0 St"uckelberg perturbations around unitary gauge,

\[ \phi^\mu = x^\mu - \pi^\mu - \frac{1}{2\Lambda_3} \Gamma^{(f)}_{\mu\nu\rho} \pi^\nu \pi^\rho + \frac{1}{6} \left( \partial_b \Gamma^{(f)}_{\alpha\beta\gamma} e_i^\alpha \right) \pi^b \pi^c \pi^d + O(\epsilon^4), \]  

(47)

with

\[ \pi^\mu = \frac{f^{\mu\nu} \nabla_j^{(f)} \pi}{m^2 \Lambda_3}. \]  

(48)

The decoupling limit (17) of two Einstein-Hilbert actions reduces to

\[ \frac{M_2}{2} \int d^4 x \sqrt{-g} R[g] + \frac{\kappa^2 M_2^2}{2} \int d^4 x \sqrt{-f} R[f] \]

\[ = \int d^4 x \sqrt{-g} \left( - \frac{1}{4} (\hat{h}_{\mu\nu} + \kappa^{-1} l_{\mu\nu}) \bar{g}^{\mu\rho\nu\sigma} (\hat{h}_{\rho\sigma} + \kappa^{-1} l_{\rho\sigma}) - \frac{1}{4} l_{\mu\nu} \bar{g}^{\mu\rho\nu\sigma} l_{\rho\sigma} \right). \]  

(49)

Since Riemann tensor of $f_{\mu\nu}$ is related with that of $\bar{g}_{\mu\nu}$ as

\[ \frac{R^{(f)}_{\mu\nu\rho\sigma}}{m^2} = \frac{\bar{R}_{\mu\nu\rho\sigma}}{m^2} + \frac{1}{\Lambda_3^2} \frac{1}{2\kappa} \left( \nabla_\sigma \nabla_\mu \hat{l}_{\nu \rho} - \nabla_\rho \nabla_\mu \hat{l}_{\nu \sigma} - \nabla_\sigma \nabla_\nu \hat{l}_{\mu \rho} + \nabla_\rho \nabla_\nu \hat{l}_{\mu \sigma} \right) + O \left( \frac{1}{\Lambda_3^2 M_{pl}} \right), \]  

(50)

our decoupling limit (17) is equivalent to

\[ M_{pl} \rightarrow \infty, \ m \rightarrow 0, \ \Lambda_3 := (M_{pl} m^2)^{1/3} \rightarrow \text{finite}, \ \frac{R^{(f)}_{\mu\nu\rho\sigma}}{m^2} \rightarrow \text{finite}. \]  

(51)

Then, the decoupling limit of interaction terms is given by replacing $\bar{g}_{\mu\nu}$ in (20) with $f_{\mu\nu}$ as

\[ \frac{M_2}{2} \int d^4 x \sqrt{-g} \left( 2m^2 \sum_{i=0}^4 \beta_i e_i (\gamma [g, f^{(f)}]) \right) \]  

\[ = \int d^4 x \sqrt{-f} L^{\text{max}} [f, \hat{h}, \hat{\pi}]. \]  

(52)
Obviously, the interaction terms have non-trivial higher derivative couplings, for example,

$$\int d^4x \sqrt{-g} \frac{R_{\mu\nu}}{m^2} \nabla_{\mu}^{(f)} \nabla_{\nu}^{(f)} \nabla_{\rho}^{(f)} \nabla_{\sigma}^{(f)}$$

$$= \int d^4x \sqrt{-g} \left[ \frac{R_{\mu\nu}}{m^2} \nabla_{\mu} \nabla_{\nu} + \frac{1}{\Lambda_3^4} \frac{1}{2 \kappa} \left( 2 \nabla_{\sigma} \nabla_{\mu} \nabla_{\nu} - \nabla_{\sigma} \nabla_{\nu} \nabla_{\mu} + \nabla_{\nu} \nabla_{\sigma} \nabla_{\mu} \right) \nabla_{\rho} \nabla_{\sigma} \nabla_{\rho} \nabla_{\sigma} \nabla_{\rho} \right]. \quad (53)$$

The variation of (53) produces the terms with third order time derivatives. At a glance, this result seems to contradict the absence of BD ghost in the full non-perturbative theory. However, the key observation here is that such higher derivative interactions do not appear at the leading order of $\mathcal{L}_{mass}$, because

$$\int d^4x \sqrt{-\mathcal{L}_{mass}[f, \hat{h}, \hat{\pi}]} = \int d^4x \sqrt{-g} \mathcal{L}_{mass}[\hat{g}, \hat{h}, \hat{\pi}] + O(\epsilon^3). \quad (54)$$

Then the decoupling limit of the total action can be written as

$$S = \int d^4x \sqrt{-g} \left( -\frac{1}{4} (\hat{h}_{\mu\nu} + \kappa^{-1} \hat{i}_{\mu\nu}) \mathcal{E}_{\mu\nu\rho\sigma} \hat{h}_{\rho\sigma} + \kappa^{-1} \hat{i}_{\mu\nu} \right) \mathcal{L}_{mass}[\hat{g}, \hat{h}, \hat{\pi}] + O(\epsilon^3). \quad (55)$$

By introducing

$$L_{\mu\nu} = \frac{1}{\sqrt{1 + \kappa^2}} \left( \hat{h}_{\mu\nu} + \left( \kappa + \frac{1}{\kappa} \right) \hat{i}_{\mu\nu} \right), \quad (56)$$

the total action reduces to at leading order $O(\epsilon^2)$

$$S^{(2)}[\hat{h}, \hat{i}, \hat{\pi}] = \int d^4x \sqrt{-g} \left( -\frac{\kappa^2}{1 + \kappa^2} \frac{1}{4} \hat{h}_{\mu\nu} \mathcal{E}_{\mu\nu\rho\sigma} \hat{h}_{\rho\sigma} + \frac{\kappa^2}{2} \hat{h}_{\mu\nu} X_{\mu\nu}^{(1)}(\hat{\pi}) + \frac{1}{2} \mathcal{R}_{\mu\nu} \mathcal{E}_{\mu\nu\rho\sigma} \hat{L}_{\rho\sigma} - \frac{1}{4} L_{\mu\nu} \mathcal{E}_{\mu\nu\rho\sigma} \hat{L}_{\rho\sigma} \right). \quad (57)$$

It is manifest that $\hat{h}$ and $\hat{\pi}$ are decoupled from $L_{\mu\nu}$ at leading order. The first three terms in Eq. (57) coincide with the decoupling limit of the action in massive gravity when we expand $g_{\mu\nu} = \hat{g}_{\mu\nu} + \hat{h}_{\mu\nu}/M_{pl}$ and the Einstein-Hilbert term has additional coefficient $\kappa^2/(1 + \kappa^2)$. The last term coincides with the decoupling limit of the Einstein-Hilbert action. Thus, linear perturbations are free from the Ostrogradsky instability even in bi-gravity. This is one of the main conclusions of this note. It should be noticed that $R_{\mu\nu}$ in the third term comes from a non-dynamical field $\hat{g}_{\mu\nu}$.

The dynamical degree of freedom in the fiducial metric is encoded only in $\hat{h}_{\mu\nu}$.

We can extend this kind of discussion to the higher order perturbations and confirm the same result for them because the dynamics of the higher order perturbations is also determined by the functional form (structure) of the second order action, $S^{(2)}$. In order to verify this statement, first, we consider the second order perturbations,

$$\hat{h}_{\mu\nu} = \hat{h}_{\mu\nu}^{(1)} + \hat{h}_{\mu\nu}^{(2)}, \quad (58)$$

$$\hat{i}_{\mu\nu} = \hat{i}_{\mu\nu}^{(1)} + \hat{i}_{\mu\nu}^{(2)}, \quad (59)$$

$$\hat{\pi} = \hat{\pi}^{(1)} + \hat{\pi}^{(2)} \quad (60)$$
Here we regard only the second order perturbations as dynamical variables and the linear perturbations are just (given) solutions of the linear equations of motion. Since the full action can be written as

$$S = S^{(2)}[\hat{h}, \hat{i}, \hat{\pi}] + \text{(at least cubic order terms of } \hat{h}, \hat{i}, \hat{\pi})$$

the fourth order action can be schematically written as

$$S^{(4)} = S^{(2)}[\hat{h}^{(2)}, \hat{i}^{(2)}, \hat{\pi}^{(2)}] + \int d^4x \left( c^{ijk}_{nm} \Phi_i^{(2)} \partial^n \Phi_j^{(1)} \partial^m \Phi_k^{(1)} + \text{(terms without } \Phi^{(2)}) \right),$$

with some coefficients $c^{ijk}_{nm}$, where $\Phi_i^{(I)}$ represents $I$-th order perturbations $\hat{h}_{\mu\nu}^{(I)}, \hat{i}_{\mu\nu}^{(I)}, \hat{\pi}^{(I)}$. The equations of motion of the second order perturbations, which can be derived from fourth order action, are given as

$$\frac{\delta S^{(2)}[\Phi^{(2)}]}{\delta \Phi_i^{(2)}} = -c^{ijk}_{nm} \partial^n \Phi_j^{(1)} \partial^m \Phi_k^{(1)}.$$

Since the left hand side is the same form as the equation of motion of the linear perturbations, there is no higher time derivative term. Since the dynamics of linear perturbation has already been determined by the linear order equations of motion, the terms in right hand side of Eq. (63) are just source terms. Then the higher time derivative terms appearing in right hand side do not lead to the Ostrogradsky instability. To be more concrete, such higher time derivative terms can be reduced to lower derivatives ones by use of the linear order equations of motion. The extension to arbitrary higher order of perturbations is trivial. The equations of motion for $N$-th order perturbations can be derived from $2N$-th order action, and can be written as

$$\frac{\delta S^{(2)}[\Phi^{(N)}]}{\delta \Phi_i^{(N)}} = \text{source terms}.$$

The key observation is that there is no higher order derivative in the left hand side while the right hand side includes higher order derivatives but consists of up to the $(N - 1)$-th perturbations, whose dynamics has already been determined by the lower order equations of motion. Thus, by the same discussion on the linear order perturbations, the Ostrogradsky instability does not appear at any order of perturbations in the decoupling limit. This is none other than the main conclusions of this note.

It is worth noticing that our discussion here is similar to the one in the low-energy effective theory approach. In the low-energy effective action, there appear higher derivatives of low-energy degrees of freedom, e.g., as a consequence of integrating out massive modes. However, those higher derivatives do not imply the existence of ghosts, rather they provide the cutoff scale for the derivative expansion. More practically, such higher derivatives are eliminated by plugging the leading order equations of motion. See, e.g., Ref. [19] for more details. The main difference in our discussion is that we use the perturbative expansion based on the smallness of the perturbations around the fixed metric, rather than the derivative expansion. Just as the low-energy effective theory case, higher order derivatives can be eliminated order by order by using the lower order equations of motion, as long as the perturbations around the background are small. Let us then
close discussion by clarifying under which conditions such a perturbative expansion is justified. In \( \Lambda_3 \) decoupling limit, possible terms with \( n (\geq 2) \)-th order in perturbations are as follows:

\[
\frac{1}{\Lambda_3^{3(n-2)}} \hat{h} (\nabla^2 \hat{\pi})^{n-1}, \quad \frac{1}{\Lambda_3^{3(n-2)}} \nabla^2 (n-1-d) \nabla d \hat{\pi}^{n-1}, \quad \frac{1}{\Lambda_3^{3(n-2)}} \nabla^2 (n-1-d) \left( \frac{R}{m^2} \right) \nabla d \hat{\pi}^{n},
\]

where \( d \) is an integer which satisfies \( d \leq 2(n-1) \). If we denote the typical size of perturbations by \( \epsilon \) and assume that \( \bar{\nabla} \sim m^2 \) for simplicity, the interaction terms (65) do not dominate over the second order action (more precisely, the \( n (\geq 2) \)-th order term dominates the \( (n+1) \)-th order term) as long as

\[
\bar{\nabla}^2 \epsilon^2 \gg \frac{1}{\Lambda_3^{3(n-2)}} \nabla^2 (n-1) \epsilon^n \quad \leftrightarrow \quad \Lambda_3^3 \gg \bar{\nabla}^2 \epsilon.
\]

This is the condition for the validity of our perturbative expansion and we have shown that there are no BD ghosts in this regime. Notice that it is naturally satisfied in the \( \Lambda_3 \) decoupling regime because the scale of a derivative \( \bar{\nabla} \), denoted by \( \Lambda \), can be at most of the order of \( \Lambda_3 \) in the \( \Lambda_3 \) decoupling regime, which implies that \( \Lambda_3^3 \gg \Lambda^2 \epsilon \) for \( \epsilon \ll \Lambda_3 \).

4. Conclusion

We applied the St"{u}ckelberg formalism of dRGT massive gravity with a general fiducial metric established in Ref. [13] to bi-gravity. In the case of massive gravity, the decoupling limit of the action includes the non-trivial coupling between the curvature of the fiducial metric and the helicity-0 mode of St"{u}ckelberg fields, Eq.(30). However, since the fiducial metric is non-dynamical in massive gravity, such terms do not lead to the dangerous BD ghost.

In the case of bi-gravity, where the fiducial metric is dynamical, one may wonder if such terms would lead to the dangerous BD ghost. Then, we have revisited this question. First we derived the decoupling limit action for the linear perturbation and confirmed that higher time derivative terms do not appear, Eq.(57). Next, we confirm that the equations of motion of higher order perturbations are the same as those of the linear order perturbations except for the source term coming from the lower order perturbations, whose dynamics has already been determined by the lower order equations of motion. Then, by using this result, we reconfirm that the Ostrogradsky instability (BD ghost) does not appear for arbitrary order of the perturbations at the decoupling limit in bi-gravity as long as perturbative expansion is justified.

Acknowledgments

This work was in part supported by a grant from Research Grants Council of the Hong Kong Special Administrative Region [HKUST4/CRF/13G] (T.N.), the JSPS Grant-in-Aid for Scientific Research Nos. 25287054 (M.Y.), 26610062 (M.Y.), the JSPS Grant-in-Aid for Scientific Research on Innovative Areas No. 15H05888 (M.Y.), and the JSPS Research Fellowship for Young Scientists, No. 26-11495 (D.Y.).
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