Representation theory of order-related monoids of partial functions as locally trivial category algebras

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Abstract
In this paper we study the representation theory of three monoids of partial functions on an \( n \)-set. The monoid of all order-preserving functions (i.e., functions satisfying \( f(x) \leq f(y) \) if \( x \leq y \)) the monoid of all order-decreasing functions (i.e. functions satisfying \( f(x) \leq x \)) and their intersection (also known as the partial Catalan monoid). We use an isomorphism between the algebras of these monoids and the algebras of some corresponding locally trivial categories. We obtain an explicit description of a quiver presentation for each algebra. Moreover, we describe other invariants such as the Cartan matrix and the Loewy length.

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1 Introduction

Given a finite monoid \( M \), it is of interest to study its algebra \( kM \) over some field \( k \). Monoids with natural combinatorial structure are clearly of major interest. Denote by \( \mathcal{P}_n \) the monoid of all partial functions on an \( n \)-element set \( \{1, \ldots, n\} \). A partial function \( f \) is called order-preserving if \( x \leq y \) implies that \( f(x) \leq f(y) \) for all \( x, y \) in the domain of \( f \). \( f \) is called order-decreasing if \( f(x) \leq x \) for every \( x \) in the domain of \( f \). We denote by \( \mathcal{PO}_n \)
the submonoid of $\mathcal{P}_n$ consisting of all order-preserving partial functions and by $\mathcal{P}_n$ the submonoid of all order-decreasing partial functions. The intersection $\mathcal{P}_n = \mathcal{O}_n \cap \mathcal{F}_n$ is called the partial Catalan monoid. These monoids are well-studied. For instance, see [1, Chapter 14] and references therein. Denote by $\mathcal{E}_n$ the finite category whose objects are all subsets of $\{1, \ldots, n\}$ and given two subsets $A, B \subseteq \{1, \ldots, n\}$ the hom-set $\mathcal{E}_n(A, B)$ consists of all onto functions with domain $A$ and range $B$. In [19, Section 5] the author proved that each one of the monoid algebras $k\mathcal{O}_n$, $k\mathcal{F}_n$ and $k\mathcal{C}_n$ is isomorphic to a category algebra of some corresponding subcategory of $\mathcal{E}_n$. These are actually examples of an isomorphism that holds for a larger class of semigroups (for details, see [20, 18, 24]). In [19, Section 5] this isomorphism was used in order to describe the ordinary quiver of these algebras. In this paper we continue this study and obtain a description of other invariants of these algebras. Our main result is a description of a quiver presentation for these algebras. A quiver presentation of an algebra $A$ is a standard way to “present” an algebra in the theory of associative algebras. It consists of a (unique) “generating” graph $Q$ called the (ordinary) quiver of $A$ and a (non-unique) set $R$ of “relations” between paths in $Q$. A tuple $(Q, R)$ presents an algebra $B$ which does not need to be isomorphic to $A$ but it is Morita equivalent to $A$ (i.e., the module categories of $A$ and $B$ are equivalent). Therefore $A$ and $B$ share most of the important invariants of representation theory. Another important invariant that we will study is the Cartan matrix of an algebra, which gives the Jordan-Hölder decomposition of every (indecomposable) projective module into simple factors. We will also find the Loewy length of these algebras and their decomposition into a direct product of connected algebras.

The key fact in studying the algebras $k\mathcal{O}_n$, $k\mathcal{F}_n$ and $k\mathcal{C}_n$ is that each of their corresponding categories is locally trivial, i.e., the only endomorphisms are the identity maps of the category. We remark that locally trivial categories were used in monoid theory by Tilson in [22]. In Section 3 we study algebras of locally trivial categories in general. The main observation is that the description of many invariants of the category algebra can be reduced to some question about the category itself. Finding a quiver presentation for the algebra, can be reduced to finding a presentation for the category itself. Therefore we can reduce a representation theoretic problem into a combinatorial problem. Likewise the Cartan matrix has an immediate interpretation via hom-sets of the category. The connectedness of the algebra is reduced to connectedness of the category
as a graph. In Section 4 we use these observations to study $k\mathcal{PO}_n$, $k\mathcal{PF}_n$ and $k\mathcal{PC}_n$ and obtain an explicit descriptions of their invariants or at least an explicit way to compute them.

2 Preliminaries

2.1 Categories and $k$-linear categories

Graphs Let $Q$ be a (finite, directed) graph. We denote by $Q^0$ its set of vertices (or objects) and by $Q^1$ its set of edges (or morphisms). We allow $Q$ to have more than one morphisms between two objects. We denote by $d$ and $r$ the domain and range functions

$$d, r : Q^1 \rightarrow Q^0$$

which associate to every edge $m \in Q^1$ its domain $d(m)$ and range $r(m)$ respectively. The set of edges with domain $a$ and range $b$ (also called an hom-set) is denoted $Q(a, b)$.

Categories and their presentations Let $E$ be a finite category. Since any category has an underlying graph, the above notations hold also for $E$. A relation $R$ on a category $E$ is a relation on the set of morphisms $E^1$ such that $mRm'$ implies that $d(m) = d(m')$ and $r(m) = r(m')$. A relation $\theta$ on a category $E$ is called a (category) congruence if $\theta$ is an equivalence relation with the property that $m_1 \theta m_2$ and $m'_1 \theta m'_2$ implies $m'_1 m_1 \theta m'_2 m_2$ whenever $r(m_1) = d(m'_1)$ and $r(m_2) = d(m'_2)$. The quotient category $E/\theta$ is then defined in a natural way. The objects of $E$ and $E/\theta$ are identical and the morphisms of $E/\theta$ are the equivalence classes of $\theta$. Note that any relation $R$ on $E$ generates some congruence $\theta$, that is, there exists some congruence $\theta$ which is the minimal congruence on $E$ containing $R$. We will denote this congruence by $\theta_R$. For every graph $Q$, denote by $Q^*$ the free category generated by the graph $Q$. The object set of $Q^*$ and $Q$ are identical but the morphisms of $Q^*$ are the paths in $Q$ (including one empty path for every object). The composition in $Q^*$ is concatenation (from right to left) of paths and the empty paths are the identity elements. We say that a subgraph $Q$ of $E$ generates $E$ if $Q^0 = E^0$ and every non-identity morphism in $E$ can be written as a composition of morphisms from $Q$. In this case, one can define a congruence $\theta$ on $Q^*$ relating paths that represent the same morphism in $E$ and obviously $Q^*/\theta \simeq E$. Equivalently one can define
a projection functor $\pi : Q^* \to E$ which is identity on objects and sends every path to the morphism in $E$ if it represent. Then $\pi(p_1) = \pi(p_2)$ for two paths $p_1, p_2 \in (Q^*)$ if and only if $p_1 \theta p_2$. If $R$ is a relation on $Q^*$ that generates $\theta$ (i.e., $\theta = \theta_R$) the convention is to call the morphisms of $Q$ generators and the elements of $R$ relations. We also say that $E$ is presented by the generators $Q$ and relations $R$ and that $(Q, R)$ is a presentation of $E$. Note that the set of relations $R$ is not unique even if $Q$ is fixed. Note also that if $E$ is a monoid (viewed as a category with one object) then this definition reduces to the usual definition of a presentation of a monoid by generators and relations. It will be useful to view a category presentation also as a coequalizer. Consider a graph $Q$ and a relation $R = \{(m_i, m'_i) \mid i \in I\}$ on $Q^*$. Denote by $1$ the graph with two objects and one morphism connecting them, i.e, a graph that looks like $\ast \to \ast$. It is clear that any functor $F : 1 \ast \to Q^*$ corresponds to choosing a morphism of $Q^*$. Denote by $I$ a disjoint union of $|I|$ copies of $1$. A functor $F : I^* \to Q^*$ can be defined by associating each index $i \in I$ with a morphism $F(i)$ of $Q^*$. Now define two functors $M, M' : I^* \to Q^*$ by $M(i) = m_i$ and $M'(i) = m'_i$. The category $E$ which $(Q, R)$ presents is easily seen to be the coequalizer of the diagram

$$
\begin{array}{ccc}
I^* & \overset{M}{\rightarrow} & Q^* \\
\downarrow & & \downarrow \\
\rightarrow & \rightarrow & \\
\end{array}
$$

in the category of (small) categories.

$\mathbb{k}$-linear categories and their presentations A $\mathbb{k}$-linear category, is a category $L$ enriched over the category of $\mathbb{k}$-vector spaces $\text{VS}_\mathbb{k}$. This means that every hom-set of $L$ is a $\mathbb{k}$-vector space and the composition of morphisms is a bilinear map with respect to the vector space operations. A functor of $\mathbb{k}$-linear categories is a category functor which is also a linear transformation when restricted to any hom-set. For any category $E$, we associate a $\mathbb{k}$-linear category $L_\mathbb{k}[E]$, called the linearization of $E$, defined in the following way. The objects of $L_\mathbb{k}[E]$ and $E$ are identical, and every hom-set $L_\mathbb{k}[E](a, b)$ is a $\mathbb{k}$-vector space with basis $E(a, b)$. The composition of morphisms in $L_\mathbb{k}[E]$ is defined naturally in the only way that extends the composition of $E$ and forms a bilinear map. It is easy to see that $L_\mathbb{k}$ is actually a functor from the category of (small) categories to the category of (small) $\mathbb{k}$-linear categories. It is not difficult to check that it is the left adjoint of the natural forgetful functor from $\mathbb{k}$-linear categories to categories. Let $L$ be a $\mathbb{k}$-linear category. A relation $\rho$ on $L^1$ is called a ($\mathbb{k}$-linear
category) congruence if \( \rho \) is a category congruence and also a vector space congruence on every hom-set \( L(a,b) \). The quotient \( k \)-linear category \( L/\rho \) is then defined in the natural way. For every relation \( R \) on \( L \) there exists a unique minimal congruence on \( L \) that contains \( R \). We will denote this congruence by \( \rho_R \).

Now we can define a presentation of \( k \)-linear categories. Let \( Q \) be a subgraph of \( L \) such that \( Q^0 = L^0 \). The free \( k \)-linear category generated by \( Q \) is the category \( L_k[Q^*] \) of all linear combinations of paths. If \( R \) is a relation on \( L_k[Q^*] \) such that \( L_k[Q^*]/\rho_R \simeq L \), then we say that \((Q,R)\) is a presentation of \( L \). Again, \( L \) can be viewed as a coequalizer. Assume \( R = \{(m_i,m'_i) \mid i \in I \} \) and define \( \bf I \) and \( I \) as before. It is clear that a functor \( F : L_k[I^*] \to L_k[Q^*] \) can be defined by associating each index \( i \in I \) with a morphism \( F(i) \) of \( L_k[Q^*] \). Define two functors \( M, M' : L_k[I^*] \to L_k[Q^*] \) by \( M(i) = m_i \) and \( M'(i) = m'_i \). Then the category \( L \) is the coequalizer of the following diagram.

\[
\begin{array}{c}
L_k[I^*] \\
\downarrow M \quad \quad \quad \downarrow M'
\end{array} 
\xrightarrow{\quad} \quad \quad 
L_k[Q^*]
\]

Note that any category \( E \) can be naturally regarded as a subcategory of \( L_k[E] \). Therefore, if \( Q \) is a subgraph of \( E \) then it is clearly a subgraph of \( L_k[E] \) as well. If \( R \) is a relation on \( Q^* \) then it can also be regarded as a relation on \( L_k[Q^*] \). This allows us to state the following simple observation that will be useful in the sequel.

**Lemma 2.1.** Let \( Q \) be a subgraph of \( E \) and let \( R \) be a relation on \( Q^* \) such that \((Q,R)\) is a category presentation for \( E \). Then \((Q,R)\) is also a \( k \)-linear category presentation for \( L_k[E] \).

**Proof.** As described above, \( E \) is the coequalizer of the following diagram

\[
\begin{array}{c}
I^* \\
\downarrow M \quad \quad \quad \downarrow M'
\end{array} 
\xrightarrow{\quad} \quad \quad 
Q^*
\]

where \( I, M \) and \( M' \) are as defined above. Applying the functor \( L_k \), we obtain the diagram

\[
\begin{array}{c}
L_k[I^*] \\
\downarrow M \quad \quad \quad \downarrow M'
\end{array} 
\xrightarrow{\quad} \quad \quad 
L_k[Q^*].
\]

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\(^1\)The author would like to thank Prof. Stuart Margolis for pointing out that Lemma 2.1 can be proved by a categorical argument.
Since $L_k$ is a left adjoint it preserves coequalizers ([12, Chapter V, Section 5]). Hence $L_k[E]$ is the coequalizer of the second diagram which is precisely what we want to prove. □

More on categories and linear categories can be found in [12].

2.2 Algebras and representations

Recall that a representation of a $k$-linear category $L$ is a functor of $k$-linear categories from $L$ to the category of all $k$-vector spaces $\text{VS}_k$. Recall that a $k$-algebra is a $k$-linear category with one object. We will mainly be interested in category algebras. For some (finite) category $E$, the category algebra $kE$ is defined in the following way. It is a vector space over $k$ with basis the morphisms of $E$, that is, it consists of all formal linear combinations

$$\{k_1m_1 + \ldots + k_nm_n \mid k_i \in k, m_i \in E^1\}.$$

The multiplication in $kE$ is the linear extension of the following:

$$m' \cdot m = \begin{cases} m'm & \text{d}(m') = \text{r}(m) \\ 0 & \text{otherwise}. \end{cases}$$

Since a monoid $M$ is a category with one object, this definition also gives a definition for monoid algebras. In this case the monoid algebra contains linear combinations of elements of the monoid with the obvious multiplication. If $M$ has a zero element $0 \in M$ then $k\{0\}$ is an ideal of $kM$. In this special case we also define $k_0M = kM/k\{0\}$.

Let $A$ be some $k$-algebra. Recall that two idempotents $e, f \in A$ are called orthogonal if $ef = fe = 0$. A non-zero idempotent $e \in A$ is called primitive if it is not a sum of two non zero orthogonal idempotents. This is equivalent to $eAe$ being a local algebra (i.e., an algebra with no non-trivial idempotents). A complete set of primitive orthogonal idempotents is a set of primitive, mutually orthogonal idempotents $\{e_1, \ldots, e_r\}$ whose sum is 1. Recall that the radical of $A$ is the minimal ideal such that $A/\text{Rad}A$ is a semisimple algebra. $A$ is called split basic if $A/\text{Rad}A \simeq k^n$, i.e, the maximal semisimple quotient of $A$ is a direct product of the base field. We can associate to every algebra $A$ a linear category, denoted $L(A)$, in the following way. The objects are in one-to-one correspondence with a complete set of primitive idempotents. The hom-set
$\mathcal{L}(A)(e_i, e_j)$ is the set $e_j Ae_i$, i.e., all the elements $a \in A$ such that $e_j ae_i = a$. Composition of two morphisms is naturally defined as their product in the algebra $A$. The category of all $A$-representations is isomorphic to the category of all $\mathcal{L}(A)$-representations. There are some important invariants of the algebra $A$ that can be described by the category $\mathcal{L}(A)$ as we will see immediately. The (ordinary) quiver of $A$ is a directed graph $Q$ defined in the following way: The set of vertices of $Q$ is in one-to-one correspondence with $\{e_1, \ldots, e_n\}$ and the edges (more often called arrows) from $e_k$ to $e_r$ are in one to one correspondence with some basis of the vector space $(e_r + \text{Rad}^2 A)(\text{Rad} A/ \text{Rad}^2 A)(e_k + \text{Rad}^2 A)$.

It is well known that this definition does not depends on the exact choice of the primitive orthogonal idempotents. If $A$ is split basic then the quiver $Q$ of $A$ can be identified with a certain subgraph of $\mathcal{L}(A)$. Actually, there exists a $k$-linear relation $R$ on $L_k[Q^*]$ such that $(Q, R)$ is a presentation for $\mathcal{L}(A)$. Such a pair $(Q, R)$ is called a quiver presentation for the algebra $A$. We briefly recall some other definitions related to an algebra. An algebra $A$ is connected if 0, 1 are its only central idempotents. If $A$ is not connected then it is a direct product of connected algebras called the blocks of $A$. It is well known that the number of blocks of $A$ is the number of the connected components of its quiver $Q$ (as a graph). The Cartan matrix of $A$ is an $n \times n$ matrix whose $(i, j)$ entry is $\dim_k e_i Ae_j$. The descending Loewy series of an algebra $A$ is the decreasing sequence of ideals

$$0 \subset \cdots \subset \text{Rad}^2 A \subset \text{Rad} A \subset A$$

and the Loewy length of $A$ is the minimal integer $n$ such that $\text{Rad}^n A = 0$. More facts on representations of algebras and proofs can be found in [1, 2].

3 Algebras of locally trivial categories

Recall that an endomorphism in a category is a morphism whose domain and range are equal. A category $E$ is called locally trivial if the only endomorphisms of $E$ are the identity morphisms. In this section we will be interested in invariants related to the structure of the category algebra $kE$ such as the quiver presentation and the Cartan matrix. We show that the description of these invariants can be reduced quite easily to some questions about the structure of the category $E$ itself. All facts in this section appear in the literature or known as folklore. However the proofs are usually very simple and we will give some of them for the sake of completeness. Note that a partial order (considered as
a category) is a special case of a locally trivial category. Therefore, the results in this section generalize some well known-facts about incidence algebras (i.e. algebras of partial orders). Moreover, some of the results in this section were proved for the more general case of an EI-category, i.e., a category where every endomorphism is an isomorphism.

We start with describing the Jacobson radical of a locally trivial category algebra. The following proposition is a special case of [10, Proposition 4.6].

**Proposition 3.1.** Let $E$ be a finite locally trivial category. Then the Jacobson radical $\text{Rad}_k E$ is spanned by all the non-isomorphisms of $E$.

For simplicity of notation, we will write $R(E) = \text{Rad}_k E$.

**Corollary 3.2.** Let $E$ be a finite and locally trivial category. $R^k(E)$ is spanned by all the morphisms that can be written as a composition of $k$ non-isomorphisms.

Most of the invariants we will discuss are preserved under Morita equivalence. For studying invariants of this type one can use the skeleton of the category instead of the category itself. Recall that a category $E$ is called skeletal if it has no distinct isomorphic objects. The skeleton of a category $E$ is the full subcategory obtained from $E$ by choosing one object of every isomorphism class (recall that $D$ is a full subcategory of $E$ if $D^1(a, b) = E^1(a, b)$ for every $a, b \in D^0$). It is clear that the skeleton of any category is a skeletal category and it is unique up to isomorphism. Moreover, it is clear that a category and its skeleton are equivalent categories. Algebras of equivalent categories are Morita equivalent (see [25, Proposition 2.2]) and hence have the same quiver presentation, Cartan matrix etc. Note that if $E$ is locally trivial then its skeleton is locally trivial as well. Skeletal locally trivial categories were called deltas in [15, Section 22].

One important observation is that the objects of a skeletal locally trivial category are partially ordered in a natural way.

**Definition 3.3.** Let $E$ be a skeletal locally trivial category. Define a relation $\leq_E$ on $E^0$ by $a \leq_E b$ if $E(a, b) \neq \emptyset$.

**Lemma 3.4.** [11, Page 170] $\leq_E$ is a partial order.

**Remark 3.5.** Lemma 3.4 can also be described in the following neat way. Recall that a preorder (i.e., a reflexive and transitive relation) can be regarded as a category $R$ where every hom-set $R(a, b)$ can not contain more than one element [12, Page 11]. Likewise, a partial order can be regarded as a category where
every union of hom-sets of the form $R(a, b) \cup R(b, a)$ can not contain more than one element. For every category $E$ one can define the universal congruence $\theta_U$ defined by identifying any two morphisms with the same domain and range. In general, the quotient $E/\theta_U$ is a preorder. However, if $E$ is skeletal and locally trivial, Lemma 3.4 implies that $E/\theta_U = \leq_E$ is a partial order.

From now on let $E$ be a finite skeletal and locally trivial category. We denote the objects of $E$ by $\{e_1, \ldots, e_n\}$ and likewise their identity morphism by $\{id_1, \ldots, id_n\}$. We give some easy observations on the algebra $\mathbb{k}E$.

**Lemma 3.6.** The set $\{id_1, \ldots, id_n\}$ is a complete set of primitive orthogonal idempotents of $\mathbb{k}E$.

**Proof.** Let $m$ be some morphism of $E$. Clearly,

$$m id_k = \begin{cases} m & d(m) = e_k \\ 0 & \text{otherwise} \end{cases}, \quad id_k m = \begin{cases} m & r(m) = e_k \\ 0 & \text{otherwise}. \end{cases}$$

Hence it is clear that $\{id_1, \ldots, id_n\}$ are orthogonal idempotents and that

$$\sum_{k=1}^{n} id_k = 1_{\mathbb{k}E}.$$

Moreover, it also implies that for every $k$, the algebra $id_k \mathbb{k}E id_k$ is spanned by all morphisms $m$ with $d(m) = r(m) = e_k$. But $id_k$ is the unique such morphism in $E$ so $id_k \mathbb{k}E id_k \cong \mathbb{k}$ which implies that $id_k$ is primitive. \hfill $\Box$

Another good property of a skeletal locally trivial category $E$ is that $\mathbb{k}E$ is a split basic algebra.

**Corollary 3.7.** $\mathbb{k}E$ is a split basic algebra.

**Proof.** Proposition 3.1 implies that $\mathbb{k}E/\text{Rad} \mathbb{k}E$ is the algebra of the groupoid of isomorphisms of $E$. By *groupoid of isomorphisms* we mean the subcategory of $E$ with the same set of objects but whose morphisms are the isomorphisms of $E$. Since $E$ is skeletal and locally trivial, the only morphisms of this groupoid are the identity morphisms $\{id_1, \ldots, id_n\}$ so it is clear that

$$\mathbb{k}E/\text{Rad} \mathbb{k}E \cong \mathbb{k}^n$$

as required. \hfill $\Box$
Denote by $Q$ the quiver of $kE$. It will be sometimes convenient to call it simply the quiver of $E$ (and likewise for any other invariant of $kE$ we will discuss). The description of the quiver of a locally trivial category is given in [13, Theorem 6.14] as a special case of a formula for the quiver of an EI-category (see [10, Theorem 4.7] or [13, Theorem 6.13]). However, for a locally trivial category $E$ it is quite straightforward to obtain a description for the quiver so we will give it here for the sake of completeness. The vertices of the quiver of a split basic algebra are in one to one correspondence with a complete set of primitive orthogonal idempotents (because every idempotent corresponds to a different simple module). So the vertices in our case are corresponding to $\{id_1, \ldots, id_n\}$ by Lemma 3.6 and Corollary 3.7. Clearly we can associate the vertex corresponding to $id_k$ with $e_k$ hence we can consider $E_0$ as the vertex set of $Q$. In order to describe the arrows of the quiver we need some more notions.

**Definition 3.8.** A morphism $m$ of a locally trivial category $E$ is called *irreducible* if it is non-isomorphism but whenever $m = m'm''$, either $m'$ is an isomorphism or $m''$ is an isomorphism.

**Remark 3.9.** The definition of an irreducible morphism in a general category can be found in [13, subsection 6.1].

It is easy to see that a morphism $m$ is irreducible if and only if $m \in R(E)\setminus R^2(E)$. Therefore, the following is also an easy corollary of Proposition 3.1 and Corollary 3.2.

**Corollary 3.10.** Let $E$ be a finite, skeletal and locally trivial category. The set

$$\{m + R^2(E)\}$$

where $m$ ranges over all irreducible morphisms forms a basis for $R(E)/R^2(E)$.

We have seen that the number of arrows from $e_k$ to $e_r$ in $Q$ is the dimension of the vector space

$$(id_r + R^2(E))(R(E)/R^2(E))(id_k + R^2(E)).$$

However, by Corollary 3.10 this dimension is just the number of irreducible morphisms from $e_k$ to $e_r$. Concluding, we have the following observation:

**Lemma 3.11 ([13, Theorem 6.14]).** Let $E$ be a finite skeletal and locally trivial
category. The quiver of $E$ is the subgraph of $E$ having the same set of objects and the irreducible morphisms as arrows.

Lemma 3.11 has an immediate application regarding blocks.

**Lemma 3.12.** Let $E$ be a skeletal locally trivial category. Then the number of blocks of $\mathbb{k}E$ is the number of connected components of $E$.

**Proof.** Recall that the number of blocks of the algebra $\mathbb{k}E$ is the number of connected components of its quiver $Q$ (see [1, Lemma 2.5]). By Lemma 3.11 we can regard $Q$ as a subgraph of $E$. Let $e$ and $e'$ be two objects of $E$. Clearly if $e$ and $e'$ are in the same connected component of $Q$ then they are in the same connected component of $E$. All is left is to prove the converse. Clearly it is enough to prove that if there is a morphism $m \in E(e, e')$ from $e$ to $e'$ in $E$ then $e$ and $e'$ are in the same connected component of $Q$. Indeed, $m$ can be decomposed into a composition of irreducible morphisms

\[ m = m_k \cdots m_1. \]

where $d(m_1) = e$ and $r(m_k) = e'$. Since $m_i \in Q^1$ for every $1 \leq i \leq k$, we have that $e$ and $e'$ are in the same connected component of $Q$ as well. \qed

We now turn to describe a quiver presentation of $\mathbb{k}E$ where $E$ is a skeletal locally trivial category. As we have already seen, the primitive idempotents of $\mathbb{k}E$ correspond to the objects of $E$ and an element $x \in \mathbb{k}E$ will satisfy $i_j x i_i = x$ if and only if it is a linear combination of elements from $E(i, j)$. Therefore, in this special case we have that $\mathcal{L}[\mathbb{k}E] = L_\mathbb{k}[E]$. In other words, the $\mathbb{k}$-linear category corresponding to the algebra $\mathbb{k}E$ is just the linearization of the category $E$.

Therefore, Lemma 3.11 immediately implies the following result.

**Proposition 3.13.** Denote by $Q$ the quiver of $\mathbb{k}E$ regarded as a subgraph of $E$. Let $R$ be a relation on $Q^*$ such that $(Q, R)$ is a category presentation for $E$. Then $(Q, R)$ is also a quiver presentation for $\mathbb{k}E$.

It is easy to describe the Cartan matrix of $\mathbb{k}E$. It is clear that $i \mathbb{k} \mathbb{k}E i$ is the $\mathbb{k}$-vector space spanned by the hom-set $E(e_r, e_k)$. Therefore, we obtain the following result.
Lemma 3.14. The Cartan matrix of \( kE \) is the \( n \times n \) matrix whose \((i,j)\) entry is the number of morphisms in the hom-set \( E(e_j, e_i) \).

Remark 3.15. Note that if we order the columns in an ascending order with respect to the partial order defined in Definition 3.3 then the Cartan matrix of \( kE \) is an upper unitriangular matrix.

4 Representation theory of some order-related monoids of partial functions

Recall that \( \PO_n \) is the monoid of all order-preserving partial functions, \( \PF_n \) is the monoid of all order-decreasing partial functions and \( \PC_n \) is the monoid of all order-preserving and order-decreasing partial functions. The goal of this paper is to describe certain properties of the algebras \( k\PO_n, k\PF_n \) and \( k\PC_n \). For any \( n \in \mathbb{N} \) we define \([n] = \{1, \ldots, n\}\) and it will be convenient to set \([0] = \emptyset\).

Denote by \( \mathcal{E}_n \) the category defined as follows. The objects of \( \mathcal{E}_n \) are all subsets of \([n]\) and given two subsets \( A, B \subseteq [n] \) the hom-set \( \mathcal{E}_n(A,B) \) consists of all onto functions with domain \( A \) and range \( B \). We denote by \( \EO_n \) the subcategory of \( \mathcal{E}_n \) with the same set of objects but the hom-set \( \EO_n(A,B) \) consists only of order-preserving functions (with domain \( A \) and range \( B \)). Similarly, we denote by \( \EF_n \) (\( \EC_n \)) the subcategory whose morphisms are order-decreasing functions (respectively, order-preserving and order-decreasing functions). In [19, Section 5] the author have proved an isomorphism of algebras

\[ k\PO_n \simeq k\EO_n, \quad k\PF_n \simeq k\EF_n, \quad k\PC_n \simeq k\EC_n. \]

Using this isomorphism we were able to describe the quiver of these algebras [19, Propositions 5.2, 5.5 and 5.8]. In this section we will continue the study of these algebras. It is easy to see that \( \EO_n, \EF_n, \EC_n \) are all locally trivial so we can apply the results of Section 3.

4.1 Order-preserving partial functions

In this section we will study the representation theory of \( \PO_n \) using the category \( \EO_n \) defined above. Note that two objects \( A \) and \( B \) in \( \EO_n \) are isomorphic if and only if \( |A| = |B| \) and in this case there is only one order-preserving onto
function from $A$ to $B$. In particular, all the endomorphism monoids are trivial so $EO_n$ is indeed locally trivial.

**Loewy series and Loewy length** We define a “degree” function, 
\[ \deg : EO_n^1 \to \mathbb{N} \cup \{0\} \]
by 
\[ \deg(f) = |A| - |B| \]
for $f \in EO_n(A, B)$. Note that $EO_n$ is *graded* with respect to the function “deg”, that is, 
\[ \deg(gf) = \deg(g) + \deg(f) \]
wherever the composition $g \cdot f$ is defined. It is clear that $\deg(f) = 0$ if and only if $f$ is an isomorphism so Proposition 3.1 implies that 
\[ \text{Rad}_k EO_n = \text{span}\{ f \in (EO_n)^1 | \deg f \geq 1 \}. \]

The next proposition gives an explicit basis for all the terms of the descending Loewy series of $k EO_n$. A similar observation for the category of all epimorphisms is given in [19, Lemma 4.3].

**Lemma 4.1.** $\text{Rad}_k EO_n = \text{span}\{ f \in (EO_n)^1 | \deg f \geq k \}.$

**Proof.** Since $\text{Rad}_k EO_n$ is spanned by elements $f$ satisfying $\deg f \geq 1$ it is clear that 
\[ \text{Rad}_k EO_n \subseteq \text{span}\{ f \in (EO_n)^1 | \deg f \geq k \}. \]

The other inclusion is easily proved by induction. We have already seen that the case $k = 1$ is true. Assume that the statement is true for $k - 1$ and take a morphism $f \in EO_n(A, B)$ such that $\deg f = |A| - |B| \geq k$. Now take some $b \in B$ such that $f^{-1}(b)$ contains more than one element. Clearly, $B$ cannot be all $[n]$. Without loss of generality, take $r \in [n] \setminus B$ such that $b \leq l < r$ implies $l \in B$ (if no such $r$ exists, then there must be $r \in [n] \setminus B$ such that $r < l \leq b$ implies $l \in B$ and the proof is similar). Denote by $a$ the maximal element of $f^{-1}(b)$ and define two functions $g : A \to B \cup \{r\}$ and $h : B \cup \{r\} \to B$ by
\[
g(x) = \begin{cases} 
  f(x) + 1 & a \leq x \text{ and } f(x) < r \\
  f(x) & \text{otherwise}
\end{cases}, \quad h(x) = \begin{cases} 
  x - 1 & b + 1 \leq x \leq r \\
  x & \text{otherwise.}
\end{cases}
\]
It is easy to verify that \( g \) and \( h \) are well defined order-preserving onto functions and \( hg = f \). By the induction hypothesis \( g \in \text{Rad}^{k-1} k \mathcal{EO}_n \) and \( h \in \text{Rad} k \mathcal{EO}_n \), so we are done. \( \square \)

**Corollary 4.2.** The Loewy length of \( k \mathcal{EO}_n \) and hence of \( k \mathcal{PO}_n \) is \( n \).

We now give an explicit formula for the dimensions of the terms of the Loewy series. We state the following fact as a separate lemma for later use.

**Lemma 4.3** ([9, Part of Lemma 4.1]). Let \( A \) and \( B \) be subsets of \([n]\) such that \( m = |A| \geq |B| = l \). Then the number of onto order-preserving functions from \( A \) to \( B \) is \( \binom{m-1}{l-1} \).

**Proof.** If \( f : A \to B \) is an order-preserving function, \( \ker f \) divides \( A \) into \( l \) “convex” subsets and there are \( \binom{m-1}{l-1} \) ways to choose “barriers” between these subsets. \( \square \)

**Lemma 4.4.** \( \dim \text{Rad}^k k \mathcal{PO}_n = \sum_{m=k+1}^{n} \sum_{i=1}^{m-k} \binom{n}{m} \binom{n}{l} \binom{m-1}{l-1} \).

**Proof.** By Lemma 4.3 we need to count all onto order-preserving functions \( f : A \to B \) (where \( A, B \subseteq [n] \)) such that \( |A| - |B| \geq k \). There are \( \binom{n}{m} \) ways to choose a domain set \( A \) of size \( m \) and \( \binom{n}{l} \) ways to choose an image set \( B \) of size \( l \). By Lemma 4.3 there are \( \binom{m-1}{l-1} \) onto order-preserving functions from \( A \) to \( B \). Now all that is left is to sum up all the possible sizes for the domain and image. \( \square \)

We now want to discuss some invariants of \( k \mathcal{EO}_n \simeq k \mathcal{PO}_n \) which are preserved by Morita equivalence. As explained in Section 3, the algebra of the skeleton of \( \mathcal{EO}_n \) is Morita equivalent to \( k \mathcal{EO}_n \) so we can pass to the skeleton of \( \mathcal{EO}_n \), that is, the full subcategory obtained from \( \mathcal{EO}_n \) by choosing one object of every isomorphism class. We will denote this skeleton by \( \mathcal{SEO}_n \). As mentioned above, two objects \( A \) and \( B \) of \( \mathcal{EO}_n \) are isomorphic if and only if \( |A| = |B| \). So we can regard \( \mathcal{SEO}_n \) as the following category. The objects of \( \mathcal{SEO}_n \) are the sets \([k] \) for \( 0 \leq k \leq n \) and the morphisms are all total order-preserving epimorphisms between them. Note that by Corollary 3.7 \( k \mathcal{SEO}_n \) is a split basic algebra.
Blocks If $M$ is a monoid with zero, it is well known (see [21, Remark 5.3]) that its algebra can be decomposed into:

$$kM \simeq k\{0\} \times k_0 M \simeq k \times k_0 M.$$  

It is clear that $\mathcal{SEO}_n$ has two connected components (with $[0]$ being an isolated vertex). Therefore, Lemma 3.12 implies that the algebra $k\mathcal{PO}_n$ also has precisely two blocks. Therefore, we obtain the following corollary.

**Lemma 4.5.** The decomposition of $k\mathcal{PO}_n$ into a direct product of connected algebras is

$$k\mathcal{PO}_n \simeq k \times k_0 \mathcal{PO}_n.$$  

**Cartan matrix** The category $\mathcal{SEO}_n$ has $n + 1$ objects so by Lemma 3.14 the Cartan matrix is an $(n + 1) \times (n + 1)$ matrix. According to a natural ordering, the $(i, j)$ entry is the number of arrows from $[j]$ to $[i]$ which is $\binom{j - 1}{i - 1}$ for $j \geq i$ by Lemma 4.3. Therefore we obtain the following result.

**Lemma 4.6.** The Cartan matrix of $k\mathcal{PO}_n$ and $k\mathcal{EO}_n$ is an upper unitriangular $(n + 1) \times (n + 1)$ matrix whose $(i, j)$ entry is $\binom{j - 1}{i - 1}$ for $j \geq i$.

**Remark 4.7.** The upper triangular $n \times n$ matrix whose $(i, j)$ entry for $j \geq i$ is $\binom{j - 1}{i - 1}$ is called the upper-triangular Pascal matrix of size $n$. Therefore, Lemma 4.6 says that the Cartan matrix of $k\mathcal{PO}_n$ is the upper-triangular Pascal matrix of size $n + 1$.

**Quiver presentation** Now we will describe the quiver presentation of $k\mathcal{SEO}_n$.

By Lemma 3.11 in order to describe the quiver we need only to identify the irreducible morphisms.

**Lemma 4.8** ([19 Lemma 5.1]). A morphism $m \in \mathcal{SEO}_n([k], [r])$ is irreducible if and only if $k = r + 1$.

By Lemma 4.3 there are precisely $k$ order preserving onto functions from $[k + 1]$ to $[k]$ so we deduce the following corollary which is [19 Proposition 5.2].

**Corollary 4.9.** The vertex set of the quiver of $k\mathcal{PO}_n$ is $\{[0], \ldots, [n]\}$. There are $k$ arrows from $[k + 1]$ to $[k]$, for $k = 0, \ldots, n - 1$, and no other arrows.
By Proposition 3.13 we know that we just need to find a presentation for $\mathcal{SEO}_n$ (where the generators are the morphisms of the quiver $Q$). Denote by $\mathcal{SEO}_n^*$ the category obtained from $\mathcal{SEO}_n$ by removing the isolated vertex $\emptyset$. Clearly $\mathcal{SEO}_n$ and $\mathcal{SEO}_n^*$ have the same presentation relation. Instead of finding a presentation for $\mathcal{SEO}_n$ directly, we will show that $\mathcal{SEO}_n^*$ is isomorphic to another category whose presentation is well-known. Recall that a strict order-preserving function $g : X \to Y$ for some posets $(X, \leq)$ and $(Y, \leq)$ is a function for which $x_1 < x_2$ implies $g(x_1) < g(x_2)$. Clearly any strict order-preserving function is injective.

Denote be $\Delta_n$ the category defined as follows: The vertices are $[k]$ for $0 \leq k \leq n$ and the morphisms from $[r]$ to $[k]$ are all the strict order-preserving functions between those sets (including a unique empty function $f : [0] \to [k]$ for every $k \leq n$). Clearly, $\Delta_n([r],[k])$ is an empty set unless $r \leq k$. We claim:

**Proposition 4.10.** For every natural $n$, the categories $\mathcal{SEO}_n^{\bullet+1}$ and $\Delta_n^{\text{op}}$ are isomorphic.

**Proof.** We will prove that $(\mathcal{SEO}_n^{\bullet+1})^{\text{op}}$ is isomorphic to $\Delta_n$. Denote by $\Gamma_n$ the category defined as follows: The objects are $[k]$ for $1 \leq k \leq n$ and the morphisms from $[r]$ to $[k]$ are all the strict order-preserving functions $g : [r] \to [k]$ such that $g(1) = 1$. Note that one can think of any function $g : [r] \to [k]$ as a function from $\{2, \ldots, r+1\}$ to $\{2, \ldots, k+1\}$ and then add a fixed point $g(1) = 1$. So it should be clear that $\Gamma_n$ is isomorphic to $\Delta_n$. To be explicit, one can consider a functor $\mathcal{F} : \Delta_n \to \Gamma_{n+1}$ defined on objects by $\mathcal{F}([k]) = [k+1]$ and for every $g \in \Delta_n([r],[k])$ the morphism $\mathcal{F}(g) : [r+1] \to [k+1]$ is defined by

$$\mathcal{F}(g)(i) = \begin{cases} 1 & i = 1 \\ g(i-1) + 1 & \text{otherwise.} \end{cases}$$

It is easy to check that $\mathcal{F}$ is a functor and it has an inverse $\mathcal{F}^{-1} : \Gamma_{n+1} \to \Delta_n$ given on morphisms by

$$\mathcal{F}^{-1}(g)(i) = g(i+1) - 1$$

so $\Delta_n$ and $\Gamma_{n+1}$ are isomorphic categories. We will now prove that $(\mathcal{SEO}_n^{\bullet+1})^{\text{op}}$ is isomorphic to $\Gamma_{n+1}$. We can think of morphisms of $(\mathcal{SEO}_n^{\bullet+1})^{\text{op}}$ as being the inverses of the morphisms of $\mathcal{SEO}_n^{\bullet+1}$. The inverse $f^{-1} : [k] \to [r]$ of some function $f \in \mathcal{SEO}_n^{\bullet+1}([r],[k])$ is usually not a function, but we can work with it as a relation. Define a functor $\mathcal{G} : (\mathcal{SEO}_n^{\bullet+1})^{\text{op}} \to \Gamma_{n+1}$ in the following way.
On objects $\mathcal{G}$ is the identity function and on morphisms $\mathcal{G}$ is defined by

$$\mathcal{G}(f^{-1})(i) = \min f^{-1}(i).$$

It is easy to observe that $\min f^{-1}(i)$ is a strict order-preserving function and $\min f^{-1}(1) = 1$. $\mathcal{G}$ is indeed a functor. It is obvious that $\mathcal{G}$ sends identity morphisms to identity morphisms. Moreover, for every two morphisms $f_1 \in \mathcal{S}\mathcal{E}\mathcal{O}^*_{n+1}([r],[k])$ and $f_2 \in \mathcal{S}\mathcal{E}\mathcal{O}^*_{n+1}([k],[m])$ we have that $\mathcal{G}(f_2^{-1}f_1^{-1}) = \min f_2^{-1}(f_1^{-1}(i))$ and $\mathcal{G}(f_1^{-1})G(f_1^{-1}) = \min f_2^{-1}(\min f_1^{-1}(i))$. Since $f_2$ is order-preserving, it is clear that the minimal element of $f_2^{-1}(f_1^{-1}(i))$ will be in the set $f_2^{-1}(\min f_1^{-1}(i))$ so

$$\mathcal{G}(f_2^{-1}f_1^{-1}) = \min f_2^{-1}(f_1^{-1}(i)) = \min f_2^{-1}(\min f_1^{-1}(i)) = \mathcal{G}(f_2^{-1})\mathcal{G}(f_1^{-1})$$

which proves that $\mathcal{G}$ is a functor. It is also easy to see that $\mathcal{G}$ has an inverse. For a given $g \in \Gamma_{n+1}([k],[r])$ the inverse $\mathcal{G}^{-1}$ is given by

$$\mathcal{G}^{-1}(g) = f^{-1}$$

where $f : [r] \to [k]$ is the order-preserving onto function that sends $j \in [r]$ to the maximal $i$ such that $g(i) \leq j$. Again it is easy to check that this is indeed an inverse and establish that $\mathcal{G}$ is an isomorphism. So $(\mathcal{S}\mathcal{E}\mathcal{O}^*_{n+1})^{\text{op}}$ is isomorphic to $\Gamma_{n+1}$ and hence to $\Delta_n$ as required.

In order to give a presentation, we will need some notation for the arrows in the quiver. Clearly every $f \in \mathcal{S}\mathcal{E}\mathcal{O}_{n+1}([k+1],[k])$ (for $k \geq 1$) is determined by choosing one pair of successive numbers that will be sent to the same image. Hence, we can denote the arrows in the quiver by $d_i^k$ for $1 \leq k \leq n$ and $1 \leq i \leq k$. The arrow $d_i^k$ corresponds to the unique order preserving onto function $f : [k+1] \to [k]$ such that $f(i) = f(i + 1)$. By Proposition 4.10 $d_i^k$ corresponds to some morphism in $(\Delta_n)^{\text{op}}$. Following the explicit isomorphism given in Proposition 4.10 it is easy to see that $d_i^k$ corresponds to the inverse of the unique strict order preserving function from $[k-1]$ to $[k]$ such that $i$ is not in its image. The presentation for the category $(\Delta_n)^{\text{op}}$ according to this set of generators is well known in algebraic topology.

The category $\Delta_n^{\text{op}}$ captures in some sense the idea of face maps between $k$-simplices up to dimension $n$. Take some linearly ordered set $\{v_0, \ldots, v_k\}$ of $k + 1$ vectors in $\mathbb{R}^n$ such that $v_1 - v_0, \ldots, v_k - v_0$ are all linearly independent.
The convex set spanned by \{v_0, \ldots, v_k\} is called an \(k\)-simplex. A face of an \(k\)-simplex is a convex set spanned by some (ordered) subset of \{v_0, \ldots, v_k\} (where a \(r\)-dimensional face is a face spanned by \(r+1\) vectors). A face map is a function that sends the simplex to one of its faces. Note that a face map of a \(k\)-simplex to an \(r\)-dimensional face is determined by a choice of \(k-r\) elements to “delete”. One can view \(\Delta_n^{op}\) as the category of all face maps between \(k\)-simplices for \(0 \leq k \leq n\). The generators \(d^k_i\) correspond to \((k-1)\)-dimensional face maps of the \(k\)-simplex. The relations of this category with the generators \(d^k_i\) are well known. (However, the category \(\Delta_n\) has several other notations in the literature).

**Lemma 4.11** ([12 Section VII.5 exercise 2a] or [3 end of Section 2.3]). The category \(\Delta_n^{op}\) and hence \(\mathcal{SEO}_{n+1}\) is presented by the generating quiver \(Q\) with morphisms \(d^k_i\) as defined above and the relations

\[
d^{k-1}_i d_j^k = d^{k-1}_{j-1} d^k_i \quad (2 \leq k \leq n, \quad 1 \leq i < j \leq k).
\]

With Proposition 3.13 this immediately implies:

**Corollary 4.12.** Let \(Q\) be the quiver of \(k \mathcal{SEO}_n\) considered as a subgraph as given in Corollary 4.9 and denote the arrows of \(Q\) by \(d^k_i\) as above. A quiver presentation of \(k \mathcal{SEO}_n\) and hence of \(k \mathcal{PO}_n\) is given by

\[
d^{k-1}_i d_j^k = d^{k-1}_{j-1} d^k_i \quad (2 \leq k \leq n-1, \quad 1 \leq i < j \leq k)
\]

**Remark 4.13.** Since the range and domain objects can usually be understood from the context, the convention in the literature is to omit the superscripts. The above relations are then written as:

\[
d_id_j = d_{j-1}d_i \quad (i < j)
\]

### 4.2 The partial Catalan monoid

In this section we will study the representation theory of the partial Catalan monoid \(\mathcal{PC}_n\) using the category \(\mathcal{EC}_n\). Recall that the functions in \(\mathcal{PC}_n\) are both order-preserving and order-decreasing. For every set \(A \subseteq [n]\), it is clear that the identity function \(1_A\) is the only order-decreasing function with domain and image being \(A\) so \(\mathcal{EC}_n\) is indeed a locally trivial category. Moreover, if \(A, B \subseteq [n]\) for \(A \neq B\) then at least one of the hom-sets \(\mathcal{EC}_n(A,B)\) or \(\mathcal{EC}_n(B,A)\) is empty so the objects \(A\) and \(B\) are not isomorphic hence \(\mathcal{EC}_n\) is skeletal. We remark that
there is also a skeletal locally trivial category (in fact, a poset) whose algebra is isomorphic to the algebra of the catalan monoid $C_n$, i.e., the monoid of all total order-preserving and order-decreasing functions (see [14]).

**Blocks** For every non-empty $A \subseteq [n]$ there exists a morphism in $\mathcal{EC}_n$ with domain $A$ and image $\{1\}$. Therefore, it is clear that the category $\mathcal{EC}_n$ has two connected components (with $\emptyset$ being an isolated vertex). Since $\mathcal{PC}_n$ is a monoid with zero we can use the same argument as in Lemma 4.5 to obtain the following.

**Lemma 4.14.** The decomposition of $k\mathcal{PC}_n$ into a direct product of connected algebras is

$$k\mathcal{PC}_n \simeq k \times_{k_0} \mathcal{PC}_n.$$  

**Quiver presentation** Since $\mathcal{EC}_n$ is a skeletal locally trivial category, its quiver is the subgraph of all irreducible morphisms. So we just need to identify the irreducible morphisms in order to find the quiver. This was done in [19] with the following result.

**Lemma 4.15** ([19, Lemma 5.7]). A morphism $f \in \mathcal{EC}_n(A, B)$ is irreducible in $\mathcal{EC}_n$ if and only if there $j \in A$ such that $f(i) = i$ for any $i \in A \setminus \{j\}$ and $f(j) = j - 1$.

**Corollary 4.16.** The vertices in the quiver of $k\mathcal{PC}_n$ and $k\mathcal{EC}_n$ are in one-to-one correspondence with subsets of $[n]$. For $A, B \subseteq [n]$, the arrows from $A$ to $B$ are in one-to-one correspondence with onto functions $f : A \to B$ for which there exists $j \in A$ such that $f(i) = i$ for $i \in A \setminus \{j\}$ and $f(j) = j - 1$.

In this subsection we denote by $Q$ the quiver of $k\mathcal{EC}_n$. We now want to describe the quiver presentation of $k\mathcal{PC}_n$ using Proposition 3.13. We clearly need some way to index the morphisms of $Q$. We denote by $d_i^A$ the irreducible morphism whose domain is $A$ and $i \in A$ is its unique element such that $d_i^A(i) = i - 1$. Note that the range of $d_i^A$ is $(A \cup \{i-1\}) \setminus \{i\}$. For simplicity we denote this set by $A_i$.

**Lemma 4.17.** For every set $A \subseteq [n]$ the relations

$$(PC1) \ d_i^A d_j^A = d_j^A d_i^A \quad (j > i + 1, \ i, j \in A)$$
(PC2) $d^A_{i+1}d^A_i = d^{(A_i)_{i+1}}d^A_{i+1}d^A_i \ (i, i + 1 \in A)$

hold in $\mathcal{EC}_n$.

Remark 4.18. In order to simplify notation, we will drop the superscripts and remain with the "braid like" relations

(PC1) $d_id_j = d_jd_i \ (j > i + 1, \ i, j \in A)$

(PC2) $d_id_{i+1} = d_id_{i+1}d_i \ (i, i + 1 \in A)$

where the domain of every morphism should be understood from the context.

Proof of Lemma 4.17. This is a straightforward verification for every $k \in A$. For [PC1] we note that

$$d_id_j(k) = d_jd_i(k) = \begin{cases} k & k \neq i, j \\ i - 1 & k = i \\ j - 1 & k = j. \end{cases}$$

For [PC2] we have that

$$d_id_{i+1}(k) = d_id_{i+1}d_i(k) = \begin{cases} k & k \neq i, i + 1 \\ i - 1 & k = i, i + 1. \end{cases}$$

In this subsection we will denote the category relation defined in Lemma 4.17 by $R$. We will show that $(Q, R)$ is a quiver presentation for $k\mathcal{PC}_n$. Let $\theta_R$ is the category congruence generated by $R$.

Lemma 4.19. Let $f : A \to B$ be a non-identity morphism of $\mathcal{EC}_n$ and let

$$f = g_1 \cdots g_r$$

be some decomposition of $f$ into irreducible morphisms. Denote by $i \in A$ the minimal element $x \in A$ such that $f(x) < x$. Then $g_1 \cdots g_r$ is $\theta_R$ equivalent to

$$g'_1 \cdots g'_rd_i$$

for some irreducible morphisms $g'_1, \ldots, g'_r$.\]
Proof. We prove this by induction on the domain of \( f \) according to the partial order \( \leq_{\mathcal{EC}_n} \) defined on the objects of \( \mathcal{EC}_n \) (see Definition 3.3). If \( f \) is irreducible then there is nothing to prove. Now, consider a morphism \( f : A \rightarrow B \) and assume we have already proved the claim for every morphism with domain \( X \) for \( A <_{\mathcal{EC}_n} X \). If \( g_r = d_i \) then there is nothing to prove. Otherwise \( g_r = d_j \) for some \( i < j \). Define

\[
h = g_1 \cdots g_{r-1}
\]

Clearly the domain of \( h \) is \( A_j = (A \cup \{ j - 1 \}) \backslash \{ j \} \) and \( A < A_j \). Note that \( i \in A_j \) and it must be the minimal element \( x \in A_j \) such that \( h(x) < x \) so by the induction assumption this decomposition is \( \theta_R \) equivalent to

\[
g_1' \cdots g_l'd_i
\]

and therefore \( g_1 \cdots g_r \) is \( \theta_R \) equivalent to

\[
g_1' \cdots g_l'd_j.
\]

If \( j > i + 1 \) then \((PC1)\) implies that we can swap the two rightmost morphisms and obtain

\[
g_1' \cdots g_l'd_i d_j = g_1' \cdots g_l'd_j d_i.
\]

If \( j = i + 1 \) we can use \((PC2)\) and get

\[
g_1' \cdots g_l'd_i d_{i+1} = g_1' \cdots g_l'd_i d_{i+1} d_i.
\]

In any case we get a new decomposition (which might be of different length)

\[
g_1' \cdots g_l'd_i
\]

as required. \( \square \)

**Proposition 4.20.** The tuple \((Q, R)\) is a category presentation for \( \mathcal{EC}_n \).

**Proof.** In view of Lemma 4.17 it is left to show that these relations are enough. In other words, if \( f \) is a morphism of \( \mathcal{EC}_n \) with two different decompositions
into irreducible morphisms

\[ f = g_1 \cdots g_r \]
\[ f = h_1 \cdots h_l \]

we need to prove that these decompositions are \( \theta_R \) equivalent. We will prove this by induction on the domain of \( f \) according to the partial order \( \leq_{\mathcal{E}_\mathcal{C}_n} \). If \( f \) is irreducible there is nothing to prove. Now, consider a morphism \( f : A \to B \) and assume we have already proved the claim for every morphism with domain \( X \) for \( A <_{\mathcal{E}_\mathcal{C}_n} X \). Take \( i \) to be the minimal element \( x \in A \) such that \( f(x) < x \) (such an element exists if \( f \) is not an isomorphism). By Lemma 4.19, \( g_1 \cdots g_r \) and \( h_1 \cdots h_l \) are \( \theta_R \) equivalent to \( g'_1 \cdots g'_r, d_i \) and \( h'_1 \cdots h'_l, d_i \) respectively. Now, it is clear that the domain of both \( g'_1 \cdots g'_r, d_i \) and \( h'_1 \cdots h'_l, d_i \) is \( A_i = (A \cup \{ i - 1 \}) \setminus \{ i \} \). For every \( k \in A_i \), if \( k \neq i - 1 \) then

\[ g'_1 \cdots g'_r(k) = g'_1 \cdots g'_r, d_i(k) = f(k) = h'_1 \cdots h'_l, d_i(k) = h'_1 \cdots h'_l(k) \]

and if \( k = i - 1 \) then

\[ g'_1 \cdots g'_r(i - 1) = g'_1 \cdots g'_r, d_i(i) = f(i) = h'_1 \cdots h'_l, d_i(i) = h'_1 \cdots h'_l(i - 1). \]

Therefore, \( g'_1 \cdots g'_r \) and \( h'_1 \cdots h'_l \) present the same function. Note that \( A <_{\mathcal{E}_\mathcal{C}_n} A_i \) so by the inductive assumption, they are \( \theta_R \) equivalent. Hence \( g'_1 \cdots g'_r, d_i \) and \( h'_1 \cdots h'_l, d_i \) are also \( \theta_R \) equivalent and this finishes the proof.

In conclusion, we have the following.

**Theorem 4.21.** Let \( Q \) be the quiver of \( \mathbb{k} \mathcal{PC}_n \simeq \mathbb{k} \mathcal{E}_\mathcal{C}_n \) and denote the arrows of \( Q \) by \( d^A_i \) as above. A quiver presentation of these algebras is given by the relations

\[
d^A_i d^A_j = d^A_j d^A_i \quad (j > i + 1, \ i, j \in A)
\]
\[
d^A_{i+1} d^A_i = d^A_{i+1} d^A_{i+1} d^A_i \quad (i, i + 1 \in A)
\]

for every \( A \subseteq [n] \).

**Remark 4.22.** Note that there is some similarity between the quiver presentation of \( \mathbb{k} \mathcal{PC}_n \) and the monoid presentation of the Catalan monoid \( \mathcal{C}_n \) by “Kiselman relations” (see [3]).
Cartan matrix  The category $\mathcal{EC}_n$ has $2^n$ objects and therefore, $k\mathcal{EC}_n$ has $2^n$ irreducible representations, which are naturally indexed by subsets of $[n]$. Given $A, B \subseteq [n]$, Lemma 3.14 implies that the $(B, A)$ entry of the Cartan matrix is the number of (total) onto order-preserving and order-decreasing functions $f : A \to B$. We would like to give some method to enumerate this number. We denote by $C(A, B)$ the set of all order-preserving and order-decreasing functions $f : A \to B$ and by $\mathcal{EC}(A, B)$ the onto functions of $C(A, B)$. We will start by giving a way to count the elements of $\mathcal{EC}([n], B)$. By the inclusion exclusion principle on the poset of subsets of $[n]$ (see [17, Section 2.1]), it is clear that

$$|\mathcal{EC}([n], B)| = \sum_{X \subseteq B} (-1)^{|B|-|X|} |C([n], X)|.$$ 

Therefore, it is enough to count the elements of $C([n], B)$ in order to get an expression for $|\mathcal{EC}([n], B)|$. It is well known that elements of $C([n], [n])$ are in one-to-one correspondence with (North-East) lattice paths from $(1, 1)$ to $(n + 1, n + 1)$ that remain below the line $y = x$. For details see [6] or the introduction of [5] (a correspondence between $\mathcal{PC}_n$ and another type of lattice paths can be found in [8]). Order-preserving and order-decreasing functions with image contained in $B$ correspond to lattice paths whose horizontal steps, i.e. steps of the form $(i, j)$ to $(i + 1, j)$, satisfy $j \in B$. It will be convenient to use $n$-tuples instead of lattice paths. Every lattice path from $(1, 1)$ to $(n + 1, n + 1)$ can be identified with an $n$-tuple $(p_1, \ldots, p_n)$ where $p_i$ is the $y$ coordinate of the $(i, j) \to (i + 1, j)$ step. In the other direction any $n$-tuple $P = (p_1, \ldots, p_n)$ which is non decreasing and its elements satisfy $1 \leq p_i \leq n + 1$ corresponds to some lattice path from $(1, 1)$ to $(n + 1, n + 1)$. Therefore we can represent such paths with $n$-tuples. We say that a lattice path $P = (p_1, \ldots, p_n)$ is below a lattice path $T = (t_1, \ldots, t_n)$ and write $P \leq T$ if $p_i \leq t_i$ for every $i$. This clearly defines a partial order on lattice paths. It is clear that elements of $C([n], B)$ correspond to lattice paths $P = (p_1, \ldots, p_n)$ such that $p_i \in B$ for every $i$ and $P \leq (1, 2, \ldots, n)$ (since they are below $y = x$).

**Definition 4.23.** For every $B \subseteq [n]$, denote by $\overline{B} = \{\overline{b}_1, \ldots, \overline{b}_n\}$ the lattice path such that $\overline{b}_1 = 1$ and for $i > 1$ we have

$$\overline{b}_i = \begin{cases} 
\overline{b}_{i-1} & i \notin B \\
\overline{b}_{i-1} + 1 & i \in B.
\end{cases}$$
In other words, the ascends of $\bar{B}$ are in positions $i \in B$.

**Example 4.24.** If $n = 8$ and $B = \{1, 3, 4, 5, 8\}$ then

$$\bar{B} = (1, 1, 2, 3, 4, 4, 4, 5).$$

**Lemma 4.25.** There is a one-to-one correspondence between $C([n], B)$ and lattice paths (from $(1, 1)$ to $(n + 1, n + 1)$) $P$ such that $P \leq \bar{B}$.

**Proof.** We have already seen that there is a one-to-one correspondence between $C([n], B)$ and the set $\mathcal{P}$ of all lattice paths $P = (p_1, \ldots, p_n)$ such that $p_i \in B$ for every $i$ and $P \leq (1, 2, \ldots, n)$. Denote by $P_B$ a lattice path whose $i$-th element is the maximal $b \in B$ such that $b \leq i$. It is easy to see that $P_B \in \mathcal{P}$ is a maximum element. Therefore $\mathcal{P}$ is the set of lattice paths $P = (p_1, \ldots, p_n)$ such that $p_i \in B$ for every $i$ and $P \leq P_B$. Now the result follows by renaming the name of elements. More precisely, if $B = \{b_1, \ldots, b_k\}$ we can define a partial permutation $\sigma_B : B \rightarrow [k]$ by $\sigma_B(b_i) = i$. It is clear that $\sigma_B$ preserves order and that $\sigma_B(P_B) = \bar{B}$ (where $\sigma_B(P_B)$ means acting by $\sigma_B$ componentwise). Therefore there is a one-to-one correspondence between $\mathcal{P}$ and $\sigma_B(\mathcal{P})$ which is the set of lattice paths $P$ such that $P \leq \bar{B}$. 

**Example 4.26.** If $B = \{1, 3, 4, 5, 8\}$ as in Example 4.24 then

$$P_B = \left(\begin{array}{cccccc} 1 & 1 & 3 & 4 & 5 & 5 \\ 8 & 1 & 2 & 3 & 4 & 5 \end{array}\right)$$

and

$$\sigma_B = \left(\begin{array}{cccc} 1 & 3 & 4 & 5 \\ 8 & 1 & 2 & 3 \\ \end{array}\right).$$

so indeed $\sigma_B(P_B) = \bar{B}$.

**Theorem 4.27** (Part of [7] Theorem 10.7.1). Given a lattice path $X = (x_1, \ldots, x_n)$ from $(1, 1)$ to $(n + 1, n + 1)$, define a matrix $M_X$ by

$$[M_X]_{i,j} = \left(\begin{array}{c} x_i \\ j - i + 1 \end{array}\right).$$

The number of lattice paths $P = (p_1, \ldots, p_n)$ from $(1, 1)$ to $(n + 1, n + 1)$ which satisfy $P \leq X$ is the determinant of $M_X$.

**Corollary 4.28.** The size of $C([n], B)$ is the determinant of the matrix $M_{\bar{B}}$ where $\bar{B} = (\bar{b}_1, \ldots, \bar{b}_n)$ as defined above.
As mentioned above, the inclusion exclusion principle on the poset of subsets of \([n]\) immediately gives us the following corollary:

**Corollary 4.29.** The number of order-preserving and order-decreasing functions onto functions \(f : [n] \to B\) is given by

\[
\sum_{X \subseteq B} (-1)^{|B| - |X|} |M_X|.
\]

Now we want to count the elements in the set \(EC(A, B)\) where the domain \(A\) is not \([n]\) but some subset. If \(|A| = m\) we will show that the number of order-preserving and order-decreasing onto functions \(f : A \to B\) is the number of such functions \(f : [m] \to B'\) for some appropriate choice of \(B'\) so it can be also be enumerated by Corollary 4.29.

**Lemma 4.30.** Let \(A, B \subseteq [n]\) such that \(|A| = m\). There exists \(A'\) such that \(B \subseteq A'\) and \(|EC(A, B)| = |EC(A', B)|\).

**Proof.** Assume \(B = \{b_1, \ldots, b_k\}\), ordered by the standard order. We will build \(A'\) from \(A\) in \(k\) steps. At the first step we take the minimal element \(a \in A\) such that \(b_1 \leq a\) and define \(A_1 = (A \setminus \{a\}) \cup \{b_1\}\). It should be clear that \(|EC(A_1, B)| = |EC(A, B)|\). We repeat this process with the other elements of \(B\). In the \(i\)-th step we have already obtained a set \(A_{i-1}\) such \(|EC(A_{i-1}, B)| = |EC(A, B)|\) such that \(b_1, \ldots, b_{i-1} \in A_{i-1}\). Now take the minimal element \(a \in A\) such that \(b_i \leq a\) and define \(A_i = (A_{i-1} \setminus \{a\}) \cup \{b_i\}\). Now it is not difficult to see that \(|EC(A_i, B)| = |EC(A_{i-1}, B)|\). Formally we can define a bijection \(\tau_a : A_{i-1} \to A_i\) which is the identity on \(A_{i-1} \setminus \{a\}\) and \(\tau_a(a) = b_i\). Now the function \(\Phi : EC(A_i, B) \to EC(A_{i-1}, B)\) defined by \(\Phi(f) = f \tau_a\) is clearly a bijection between the two sets. Finally we define \(A' = A_k\) and it is clear that \(B \subseteq A'\) and \(|EC(A', B)| = |EC(A, B)|\). \(\square\)

It is now left to count the set \(EC(A', B)\). Assume \(A' = \{a_1, \ldots, a_m\}\) ordered by the standard order. Define \(\sigma_{A'}\) to be the partial bijection \(\sigma_{A'}(i) = a_i\) and denote \(\sigma_{A'}(B) = \{\sigma_{A'}(b_1), \ldots, \sigma_{A'}(b_k)\}\).

**Lemma 4.31.** The following equality holds

\[
|EC(A', B)| = |EC([m], \sigma_{A'}(B))|.
\]

25
Proof. Since $\sigma_{A'}$ is a (partial) permutation which preserves order, we can think of applying it as just renaming the elements of the sets so the claim is obvious.

In conclusion, we have displayed a method to enumerate the set $\mathcal{EC}(A, B)$ of all onto order-preserving and order-decreasing functions $f : A \to B$ which is the $(B, A)$ entry of the Cartan matrix.

**Loewy length** Recall that $\text{Rad}_k \mathcal{EC}_n$ is spanned by all the non-invertible morphisms of $\mathcal{EC}_n$, i.e. all the non identity morphisms.

**Lemma 4.32.** Let $f \in \mathcal{EC}_n(A, B)$ be a non-identity morphism. $f$ is the composition of at most $\binom{n}{2}$ non-identity morphisms.

**Proof.** In this proof it will be more convenient to assume that the objects of $\mathcal{EC}_n$ are all the subsets of $\{0, \ldots, n-1\}$. For every set $A \subseteq \{0, \ldots, n-1\}$ define $S(A)$ to be the sum of its elements

$$S(A) = \sum_{a \in A} a.$$ 

Since $f : A \to B$ is onto and order-decreasing, we must have that $S(B) < S(A)$. Since $S(\{0, \ldots, n-1\}) = \binom{n}{2}$ it is clear that a morphism cannot be written as a composition of more than $\binom{n}{2}$ non-identity morphism.

**Proposition 4.33.** The Loewy length of $\mathcal{EC}_n$ and hence of $\mathcal{PC}_n$ is $\binom{n}{2} + 1$.

**Proof.** By Lemma 4.32 it is clear that $\text{Rad}_k \mathcal{EC}_n = 0$ where $k = \binom{n}{2} + 1$. It is only left to prove that $\text{Rad}_k \mathcal{EC}_n \neq 0$ where $k = \binom{n}{2}$. Recall that we have denoted by $d_i^A$ the irreducible morphism whose domain is $A$ and $i \in A$ is its unique element such that $d_i^A(i) = i - 1$. For $2 \leq i \leq n$ define $f_i : \{1, i, i+1, \ldots, n\} \to \{1, i+1, \ldots, n\}$ to be the morphism which is the identity on all elements except $f_i(i) = 1$. It is clear that $f_i$ can be written as a composition of $i - 1$ morphisms

$$f_i = d_2 \cdots d_{i-1} d_i$$

where we have dropped the superscripts because they can be understood from the context. Now, it is easy to see that the constant function $1 : [n] \to \{1\}$ can
be written as the following composition

\[ 1 = f_n \cdots f_3 f_2 \]

and therefore we have found a morphism which can be written as a composition of \( \binom{n}{2} \) morphisms and we are done. \( \square \)

### 4.3 Order-decreasing partial functions

In this section we will study the representation theory of the monoid \( PF_n \) of all order-decreasing partial functions using the category \( EF_n \). We remark that \( PF_n \) is isomorphic to the monoid of all order-decreasing total functions on \( n+1 \) elements. This fact was first observed in \([23, \text{Corollary 2.4.3}]\). Another proof due to the referee of \([19]\) can be found in \([19, \text{Lemma 5.3}]\). We remark that a monoid is \( L \)-trivial (that is, any two distinct elements generate different left ideals) if and only if it is isomorphic to a submonoid of \( PF_n \) for some \( n \in \mathbb{N} \) \([13, \text{Chapter 4, Theorem 3.6}]\). For every set \( A \subseteq [n] \), it is clear that the identity function \( 1_A \) is the only order-decreasing function with domain and image being \( A \) so \( EF_n \) is indeed a locally trivial category. Moreover, if \( A, B \subseteq [n] \) for \( A \neq B \) then at least one of the hom-sets \( EF_n(A, B) \) or \( EF_n(B, A) \) is empty so the objects \( A \) and \( B \) are not isomorphic hence \( EF_n \) is skeletal.

**Blocks** It is clear that for every non-empty \( A \subseteq [n] \) there exists a constant order-decreasing function \( f : A \to \{1\} \). Therefore the category \( EF_n \) has precisely two connected components with the \( \emptyset \) object being isolated. Since \( PF_n \) is a monoid with zero we can use the same argument as in Lemma 4.5 to obtain the following.

**Lemma 4.34.** The decomposition of \( k\mathcal{P}F_n \) into a direct product of connected algebras is

\[ k\mathcal{P}F_n \cong k \times k_0 \mathcal{P}F_n. \]

**Cartan matrix** The category \( EF_n \) has \( 2^n \) objects and therefore, \( kEF_n \) has \( 2^n \) irreducible representations, which are naturally indexed by subsets of \( [n] \). Given \( B \subseteq [n] \) and \( i \in [n] \) we denote

\[ B_{\leq i} = \{ b \in B \mid b \leq i \} \]
Lemma 4.35. The number of order-decreasing (total, but not necessarily onto) functions \( f : A \rightarrow B \) is
\[
\prod_{i \in A} |B_{\leq i}|
\]

Proof. The image of every \( i \in A \) could be any element in \( B \) which is smaller than \( i \) hence there are \( |B_{\leq i}| \) options. The choice of image of any two elements of \( A \) is independent so we just take the product of the number of options.

Lemma 4.36. The Cartan matrix of \( k\mathcal{E}_n \) is a \( 2^n \times 2^n \) matrix. Given \( A, B \subseteq [n] \) the \((B, A)\) entry of the Cartan matrix is
\[
\sum_{X \subseteq B} (-1)^{|B| - |X|} \prod_{i \in A} |X_{\leq i}|
\]

Proof. The \((B, A)\) entry of the Cartan matrix is the number of (total) onto order-decreasing functions \( f : A \rightarrow B \) by Lemma 3.14. The claim follows immediately by the inclusion-exclusion principle on the poset of subsets of \([n]\) (see [17, Section 2.1]) and Lemma 4.35.

Quiver presentation Describing a quiver presentation for \( k\mathcal{P}_n \) is similar to the case of \( k\mathcal{P}_C \) but a bit more complicated. Again, \( \mathcal{E}_n \) is a skeletal locally trivial category, so its quiver is the subgraph of all irreducible morphisms. In order to describe the irreducible morphisms we will use the following notation. Let \( A \subseteq [n] \) and let \( i, j \in [n] \) be two distinct elements. We will write \( i \prec_A j \) if \( i < j \) and \( i < x \leq j \) implies that \( x \in A \). In other words, if all the elements between \( i \) and \( j \) (including \( j \)) are in \( A \).

Lemma 4.37 ([19, Lemma 5.4]). A morphism \( f \in \mathcal{E}_n(A, B) \) is irreducible if and only if there exists \( j \in A \) such that \( f(i) = i \) for any \( i \in A \setminus \{j\} \) and \( f(j) \prec_A j \).

Corollary 4.38. The vertices in the quiver of \( k\mathcal{P}_n \) and \( k\mathcal{E}_n \) are in one-to-one correspondence with subsets of [n]. For \( A, B \subseteq [n] \), the arrows from \( A \) to \( B \) are in one-to-one correspondence with onto functions \( f : A \rightarrow B \) for which there exists \( j \in A \) such that \( f(i) = i \) for any \( i \in A \setminus \{j\} \) and \( f(j) \prec_A j \).

Now denote by \( Q \) the quiver of \( k\mathcal{E}_n \). We now want to describe the quiver presentation of \( k\mathcal{P}_n \) using Proposition 3.13. We index the morphisms of \( Q \) in the following way. We denote by \( d_{i,j}^A \) the irreducible morphism whose domain
is $A$ and $j \in A$ is its unique element such that $d^A_{i,j}(j) = i \neq j$. Clearly, using this notation implies that $i <_A j$. Note that the range of $d^A_{i,j}$ is $(A \cup \{i\}) \setminus \{j\}$.

For simplicity we denote this set by $A_{i,j}$.

**Lemma 4.39.** Let $A \subseteq [n]$ and assume $j < t$. The relations

(PF1) $d^A_{i,j} \circ d^A_{s,t} = d^A_{s,t} \circ d^A_{i,j}$ ($s > j$, $t, j \in A$)

(PF2) $d^A_{i,j} \circ d^A_{s,t} = d^A_{s,t} \circ d^A_{i,j}$ ($s = j$, $t, j \in A$)

(PF3) $d^A_{i,j} \circ d^A_{s,t} = d^A_{s,j} \circ d^A_{i,j}$ ($i < s < j$, $s, t, j \in A$)

(PF4) $d^A_{i,j} \circ d^A_{s,t} = d^A_{s,j} \circ d^A_{i,j}$ ($i < s < j$, $s, t, j \in A$)

(PF5) $d^A_{i,j} \circ d^A_{s,t} = d^A_{s,j} \circ d^A_{i,j}$ ($i < s < j$, $t, j \in A$, $s \notin A$)

(PF6) $d^A_{i,j} \circ d^A_{s,i} = d^A_{i,j} \circ d^A_{s,j}$ ($s < i < j < t$, $t, j \in A$, $i \notin A$)

hold in $\mathcal{EF}_n$.

**Remark 4.40.** In order to simplify notation, we will drop the superscripts and remain with the relations

(PF1) $d_{i,j} \circ d_{s,t} = d_{s,t} \circ d_{i,j}$ ($s > j$, $t, j \in A$)

(PF2) $d_{i,j} \circ d_{s,t} = d_{s,t} \circ d_{i,j}$ ($s = j$, $t, j \in A$)

(PF3) $d_{i,j} \circ d_{s,t} = d_{s,j} \circ d_{i,j}$ ($i < s < j$, $s, t, j \in A$)

(PF4) $d_{i,j} \circ d_{s,t} = d_{s,j} \circ d_{i,j}$ ($i < s < j$, $s, t, j \in A$)

(PF5) $d_{i,j} \circ d_{s,t} = d_{s,j} \circ d_{i,j}$ ($i < s < j$, $t, j \in A$, $s \notin A$)

(PF6) $d_{i,j} \circ d_{s,i} = d_{s,j} \circ d_{i,j}$ ($s < i < j < t$, $t, j \in A$, $i \notin A$)

where the domain of every morphism should be understood from the context.

**Remark 4.41.** Note that relations (PF1)-(PF5) cover all the cases of a term $d_{i,j} \circ d_{s,t}$ satisfying $j < t$.

**Proof of Lemma 4.39.** This is a routine matter to check that all the compositions in Lemma 4.39 are well defined. Now, it is a straightforward verification to check equality. Choose some $k \in A$. For (PF1)(PF3)(PF4) and (PF5) we note that

29
\[ d_{i,j}d_{s,t}(k) = d_{s,t}d_{i,j}(k) = \begin{cases} 
  k & k \neq t, j \\
  i & k = j \\
  s & k = t 
\end{cases} \]

\[ d_{i,j}d_{s,t}(k) = d_{s,j}d_{j,i}(k) = \begin{cases} 
  k & k \neq t, j \\
  i & k = j \\
  s & k = t 
\end{cases} \]

\[ d_{i,j}d_{s,t}(k) = d_{s,j}d_{i,t}d_{i,s}d_{s,j}(k) = \begin{cases} 
  k & k \neq t, j \\
  i & k = j \\
  s & k = t. 
\end{cases} \]

For \([\text{PF2}]\) we have that

\[ d_{i,j}d_{s,t}(k) = d_{i,j}d_{j,i}(k) = \begin{cases} 
  k & k \neq t, j \\
  i & k = j, t 
\end{cases} \]

and for \([\text{PF6}]\) we obtain (note that \(k \neq i\) as \(i \notin A\))

\[ d_{i,j}d_{s,t}(k) = d_{s,j}d_{i,t}d_{i,s}d_{s,j}(k) = \begin{cases} 
  k & k \neq t, j \\
  i & k = j \\
  s & k = t. 
\end{cases} \]

For \([\text{PF2}]\) we have that

\[ d_{i,j}d_{s,t}(k) = d_{i,j}d_{j,i}(k) = \begin{cases} 
  k & k \neq t, j \\
  i & k = j, t 
\end{cases} \]

and for \([\text{PF6}]\) we obtain (note that \(k \neq i\) as \(i \notin A\))

\[ d_{i,j}d_{s,t}(k) = d_{s,j}d_{i,t}d_{i,s}d_{s,j}(k) = \begin{cases} 
  k & k \neq t, j \\
  i & k = j \\
  s & k = t. 
\end{cases} \]

In this subsection we will denote the category relation defined in Lemma 4.39 by \(R\). We will show that \((Q, R)\) is a quiver presentation for \(k\mathcal{PF}_n\).

**Lemma 4.42.** Let \(f : A \rightarrow B\) be a non-identity morphism of \(\mathcal{EF}_n\) and let

\[ f = g_1 \cdots g_r \]

be some decomposition of \(f\) into irreducible morphisms. Let \(j \in A\) be the minimal element \(x \in A\) such that \(f(x) < x\). Then \(g_1 \cdots g_r\) is \(\theta_R\) equivalent to

\[ g'_1 \cdots g'_r d_{i,j} \]

30
for some irreducible morphisms \( g'_1, \ldots, g'_r \), and some \( i <_A j \) (where \( \theta_R \) is the category congruence generated by \( R \)).

Proof. We prove this by induction on the domain of \( f \) according to the partial order \( \leq_{\mathcal{E}X_n} \) defined on the objects of \( \mathcal{E}X_n \) (see Definition 3.3). If \( f \) is irreducible then there is nothing to prove. Now, consider a morphism \( f : A \to B \) and assume we have already proved the claim for every morphism with domain \( X \) for \( A <_{\mathcal{E}X_n} X \). If \( g_r = d_{i,j} \) then we are done. Otherwise \( g_r = d_{i,t} \) for some \( t > j \). Define \( h_1 = g_1 \cdots g_{r-1} \). It is clear that the domain of \( h \) is \( A_{i_1,t} \) and that \( j \in A_{i_1,t} \) and \( A <_{\mathcal{E}X_n} A_{i_1,t} \).

Case 1. Assume that \( j \) is the minimal element \( x \in A_{i_1,t} \) such that \( h(x) < x \). In this case the induction assumption implies that \( g_1 \cdots g_{r-1} = \theta_R \) equivalent to \( g'_1 \cdots g'_d_{q,j} \) and therefore \( g_1 \cdots g_r = \theta_R \) equivalent to \( g'_1 \cdots g'_d_{q,j} d_{i,t} \). Recall that \( j < t \) so by one of the relations \([\text{PF1}],[\text{PF5}]\) we can “push” the \( d_{q,j} \) term to the rightmost position and obtain \( g'_1 \cdots g'_r d_{i,j} \) as required. This finishes this case.

Case 2. It might be the case that \( j \) is not the minimal element \( x \in \text{dom} h_1 \) such that \( h_1(x) < x \). The only other possibility for such minimal element is \( i_1 \). In this case, the induction assumption implies that \( g_1 \cdots g_{r-1} = \theta_R \) equivalent to \( g^{(1)}_1 \cdots g^{(1)}_l d_{i_2,i_1} \) so \( g_1 \cdots g_r = \theta_R \) equivalent to \( g^{(1)}_1 \cdots g^{(1)}_l d_{i_2,i_1} d_{i_1,t} \). Now denote \( h_2 = g^{(1)}_1 \cdots g^{(1)}_l d_{i_2,i_1} d_{i_1,t} \). By a similar argument, if \( j \) is not the minimal \( x \in \text{dom} h_2 \) such that \( h_2(x) < x \) then it must be \( i_2 \) so we obtain a \( \theta_R \) equivalence with \( g^{(2)}_1 \cdots g^{(2)}_l d_{i_2,i_2} d_{i_2,i_1} d_{i_1,t} \). This process must terminate at some point. Eventually we obtain a decomposition

\[
\prod_{i=1}^{n-1} g^{(n-1)}_i d_{i_{n-i},i_{n-i-1}} \cdots d_{i_2,i_1} d_{i_1,t}
\]

where \( j \) is the minimal element \( x \) in the domain of \( h_n = g^{(n-1)}_1 \cdots g^{(n-1)}_l \) such that \( h_n(x) < x \). Denote the domain of \( h_n \) by \( A_n \). By the induction assumption this decomposition is \( \theta_R \) equivalent to

\[
\prod_{i=1}^{n-1} g^{(n)}_i d_{i_{n-i},i_{n-i-1}} \cdots d_{i_2,i_1} d_{i_1,t}.
\]

Now, we know that \( i_1 < j \). However, it cannot be the case that \( q < i_1 < j \) because in this case \( q \notin A_n \) in contrary to the existence
of the $d_{q,j}$ term. Therefore, $i_1 \leq q$ and $i_n, \ldots, i_2 < q$ so we can use \([PF1]\) to swap terms and obtain $g_{i_1}^{(n)} \cdots g_{i_{n-1}}^{(n)} d_{q,j} d_{i_2,i_1} d_{i_1,t}$

Now, if $i_1 < q$ we can again swap terms with \([PF1]\) to get

$$g_{i_1}^{(n)} \cdots g_{i_{n-1}}^{(n)} d_{i_2,i_1} d_{q,j} d_{i_1,t}$$

and finish the proof with another use of \([PF1]\) \([PF5]\). If $i_1 = q$, the decomposition is of the form

$$g_{i_1}^{(n)} \cdots g_{i_{n-1}}^{(n)} d_{i_2,i_1} d_{i_1,t}.$$ 

By the minimality of $j$, we must have that $i_1 \notin A$ so we can use \([PF6]\) to obtain

$$g_{i_1}^{(n)} \cdots g_{i_{n-1}}^{(n)} d_{i_2,i_1} d_{j,t}$$

which finishes this case and the proof.

\[ \square \]

**Proposition 4.43.** The tuple $(Q, R)$ is a category presentation for $\mathcal{EF}_n$.

Proof. In view of Lemma 4.39 it is left to show that these relations are enough. In other words, if $f$ is a morphism of $\mathcal{EF}_n$ with two different decompositions into irreducible morphisms

$$f = g_1 \cdots g_r$$

$$f = h_1 \cdots h_l$$

we need to prove that these decompositions are $\theta_R$ equivalent. We will prove this by induction on the domain of $f$ according to the partial order $\leq_{\mathcal{EF}_n}$ defined on the objects of $\mathcal{EF}_n$. If $f$ is irreducible then there is nothing to prove. Now, consider a morphism $f : A \to B$ and assume we have already proved the claim for every morphism with domain $X$ for $A <_{\mathcal{EF}_n} X$. Take $j$ to be the minimal element $x \in A$ such that $f(x) < x$ (such an element exists if $f$ is not an isomorphism). By Lemma 4.32 $g_1 \cdots g_r$ and $h_1 \cdots h_l$ are $\theta_R$ equivalent to $g'_{i_1} \cdots g'_{i_r} d_{i_1,j}$ and $h'_{i_1} \cdots h'_{i_l} d_{i_1,j}$ respectively. We first claim that $i_1 = i_2$. Assume without loss of generality that $i_1 < i_2$. This implies that $f(j) < i_2$. Moreover, $i_1 \notin_A j$ so
\[ i_2 \in A \text{ and therefore} \]
\[ f(i_2) = h'_1 \cdots h'_r d_{i_2,j}(i_2) = h'_1 \cdots h'_r(i_2) = h'_1 \cdots h'_r d_{i_2,j}(j) = f(j) < i_2. \]

This contradicts the minimality of \( j \) and therefore \( i_1 = i_2 = i \). So \( g_1 \cdots g_r \) and \( h_1 \cdots h_l \) are \( \theta_R \) equivalent to \( g'_1 \cdots g'_{r'} d_{i,j} \) and \( h'_1 \cdots h'_{l'} d_{i,j} \) respectively. Now, it is clear that the domain of both \( g'_1 \cdots g'_{r'} \) and \( h'_1 \cdots h'_{l'} \) is \( A_{i,j} = (A \cup \{i\}) \setminus \{j\} \).

For every \( k \in A_{i,j} \), if \( k \neq i \) then
\[ g'_1 \cdots g'_{r'}(k) = g'_1 \cdots g'_{r'} d_{i,j}(k) = f(k) = h'_1 \cdots h'_{l'} d_{i,j}(k) = h'_1 \cdots h'_{l'}(k) \]
and if \( k = i \) then
\[ g'_1 \cdots g'_{r'}(i) = g'_1 \cdots g'_{r'} d_{i,j}(j) = f(j) = h'_1 \cdots h'_{l'} d_{i,j}(j) = h'_1 \cdots h'_{l'}(i). \]

Therefore, \( g'_1 \cdots g'_{r'} \) and \( h'_1 \cdots h'_{l'} \) present the same function. Note that \( A < A_{i,j} \) so by the inductive assumption, they are \( \theta_R \) equivalent. Hence \( g'_1 \cdots g'_{r'} d_{i,j} \) and \( h'_1 \cdots h'_{l'} d_{i,j} \) are also \( \theta_R \) equivalent and this finishes the proof. \( \square \)

In conclusion, we have the following.

**Theorem 4.44.** Let \( Q \) be the quiver of \( \mathbb{C} \mathcal{PF}_n \simeq \mathbb{C} \mathcal{EF}_n \). A quiver presentation of these algebras is given by the relations

\[
\begin{align*}
d^{A_{s,t}}_{i,j} d^A_{s,t} &= d^{A_{s,t}}_{i,j} d^A_{i,j} \quad (s > j, \quad t, j \in A) \\
d^{A_{s,t}}_{i,j} d^A_{s,t} &= d^{A_{s,t}}_{i,j} d^A_{i,j} \quad (s = j, \quad t, j \in A) \\
d^{A_{i,j}}_{s,t} d^A_{s,t} &= d^{A_{s,t}}_{i,j} d^A_{i,j} \quad (i < s < j, \quad s, t, j \in A) \\
d^{A_{i,j}}_{s,t} d^A_{s,t} &= d^{A_{s,t}}_{i,j} d^A_{i,j} \quad (s \leq i, \quad t, j \in A) \\
d^{(A_{i,j})\cdot}_{s,t} d^A_{s,t} &= d^{(A_{i,j})\cdot}_{s,t} d^A_{s,t} \quad (i < s < j, \quad t, j \in A, \quad s \notin A) \\
d^{(A_{i,j})\cdot}_{s,t} d^A_{s,t} &= d^{(A_{i,j})\cdot}_{s,t} d^A_{s,t} \quad (s < i < j, \quad t, j \in A, \quad i \notin A)
\end{align*}
\]

for \( j < t \) and every \( A \subseteq [n] \).

**Loewy length**

**Proposition 4.45.** The Loewy length of \( \mathbb{C} \mathcal{EF}_n \) and hence of \( \mathbb{C} \mathcal{PF}_n \) is \( \left(\begin{array}{c} n \\ 2 \end{array}\right) + 1 \).
Proof. The proof is similar to the case of $k\mathcal{E}C_n$. Note that $\mathcal{E}C_n$ is a subcategory of $\mathcal{E}F_n$. An identical argument of Lemma 4.32 proves that no morphism can be written as a composition of $\binom{n}{2} + 1$ non-identity elements and Proposition 4.33 proves that there exists a morphism which is a composition of $\binom{n}{2}$ non-identity morphisms.

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