Trapping of a model-system for a soliton in a well

G. Kälbermann
Soil and Water department
Faculty of Agriculture
Rehovot 76100, Israel

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Abstract

The nature of the interaction of a soliton with an attractive well is elucidated using a model of two interacting point particles. The system shows the existence of trapped states at positive kinetic energy, as well as reflection by an attractive impurity, as found when a topological soliton scatters off an attractive well.

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1 Introduction

Topological solitons arise as nontrivial solutions in field theories with nonlinear interactions. These solutions are stable against dispersion. Topology enters through the absolute conservation of a topological charge, or winding number.[1]

It is for this reason they become so important in the description of phenomena like, optical self-focusing, magnetic flux in Josephson junctions[2] or even the very existence of stable elementary particles, such as the skyrmion[3,4], as a model of hadrons.

Interactions of solitons with external agents become extremely important. These interactions allow us to test the validity of such models in real situations.

In a previous work[5] the interaction of a soliton in one space dimension with finite size impurities was investigated.

In the works of Kivshar et al.[5] (see also ref.[3,4]), it was found that the soliton displays unique phenomena when it interacts with an external impurity. The existence of trapped solutions for positive energy or, bound states in the continuum, is a very distinctive effect for the soliton in interaction with an attractive well.

We can understand the origin of impurity interactions of a soliton by looking at the impurity as a nontrivial medium in which the soliton propagates. An easy way to visualize these interactions consists in introducing a nontrivial metric for the relevant spacetime. The metric carries the information of the medium characteristics.

A 1+1 dimensional scalar field theory supporting topological solitons in flat space, immersed in a background determined by the metric $g_{\mu \nu}$ in a minimal coupling to the metric, is given by

$$\mathcal{L} = \sqrt{g} \left[ g^{\mu \nu} \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - U(\phi) \right]$$

(1)
where $g$ is the determinant of the metric, and $U$ is the self-interaction that enables the existence of the soliton. For a weak potential we have $\text{g}^{00} \approx 1 + V(x)$

$$g_{11} = -1$$

$$g_{-11} = g_{1-1} = 0$$

(2)

Where $V(x)$ is the external space dependent potential. The equation of motion of the soliton in the background space becomes

$$\frac{\partial^2 \phi}{\partial t^2} - \sqrt{g}^{-1} \frac{\partial}{\partial x} \left[ \sqrt{g} \frac{\partial \phi}{\partial x} \right] + g_{00} \frac{\partial U}{\partial \phi} = 0.$$  

(3)

This equation is identical, for slowly varying potentials, to the equation of motion of a soliton interacting with an impurity $V(x)$. Impurity interactions are therefore acceptable couplings of a soliton to an external potential. It is also the only way to couple the soliton without spoiling the topological boundary conditions. The source term generated by the metric is essentially a space dependent mass term.

The interaction of a soliton with an attractive impurity shows, however, some puzzling effects. A soliton can be trapped in it, when it impinges onto the well with positive kinetic energy. Energy conservation demands that the soliton fluctuates and distorts in trapped states inside the well. Even more counterintuitive is the fact that the soliton can be reflected by the well.

Neither of these effects are possible for classical point particles. The difference must obviously be due to the extended character of the soliton.

This was indeed put in evidence in the works of Kivshar et al. It was shown there that the bulk soliton behavior may be reproduced qualitatively, by taking the center of the soliton as a time dependent collective coordinate coupled to the major excitation modes of the soliton in the well, the impurity...
mode, and a shape distortion mode. The first mode is excited inside the well only and appears as an oscillating packet centered at the well. The shape mode is an excitation that accompanies the soliton with a time dependent amplitude along its scattering, and accounts for the distortions of the free soliton. The dynamics of the soliton was then replaced by the equations of motion of three classical particle-like excitations: the center of the soliton, the amplitude of the impurity mode and the amplitude of the shape mode. For a $\delta$ function type of well, it was found that the system behaves analogously to the soliton. Kinetic energy of the soliton center can be transferred to the impurity and the shape modes. The system of three effective degrees of freedom can resonate inside the well and be trapped. Reflection by the attractive well is also observed. The system captures the essential features of the behavior of the soliton.

However, both the interactions between the collective degrees of freedom and the dynamics of the system are derived from the soliton itself. Moreover, the behavior was shown to hold for $\delta$ type of wells only.

Later it was shown [8] that all the features of the scattering of a soliton show up in the case of a fine size well too.

In the present work we show that the effects are not specific to the soliton generated dynamics and interactions.

We are able to reproduce here all the abovementioned effects with a classical model for an extended object, regardless of the soliton dynamics. In the next section we show that a simple classical model exhibits the same behavior as the soliton. It can be trapped and reflected by an attractive well. The system will also show chaotic behavior.

The classical system can serve as a nice introductory example for the surprising behavior of solitons. It demands only basic lagrangian mechanics knowledge, but, it has many interesting features that are easily visualized by undergraduate students. It may serve also as an example of chaotic behavior in classical mechanics.
2 A classical model of trapping

Kinks and other topological defects are sometimes used to model classical systems of masses. We will here find, that the analogy is more than mathematical. The very behavior of the soliton is exactly the one observed on a two-body system interacting with an external potential.

In ref. [5], the dynamics of a soliton interacting with a well was also studied by selecting the collective coordinates of the soliton center, its shape-mode excitation amplitude and an impurity mode. Using this scheme, it was found that this classical system shows an analogous behavior to the soliton itself. Namely, trapped states and reflection by the attractive impurity. The specific dynamics of the soliton in terms of its collective coordinate, and its excitations including the shape mode were crucial for the obtention of the abovementioned results.

We show here that the effects are generic, any system resembling the behavior of the soliton, mainly, its extended character, will indeed yield similar results. This is an unexpected situation whose motivation arose entirely from the behavior of the soliton and its relevance goes beyond it, as will be explained below.

In order to justify the connection with the behavior of the model we appeal to a physical scenario in which both Sine-Gordon solitons and Kinks arise. Sine-Gordon solitons arise in large Josephson junctions and in the motion of dislocations in a one-dimensional crystal. Kinks arise in the latter also when the substrate potential is nonperiodic.

Consider the Hamiltonian of dislocations in a crystal with nearest neighbor interactions [10].

\[ H = \sum \left[ \frac{1}{2} m \left( \frac{du_n}{dt} \right)^2 + \frac{G}{2} \left( u_{n+1} - u_n \right)^2 + V(u_n) \right] \]  

(4)

Where \( u_n \) is the displacement of the dislocation at site \( n \), \( G \) is the spring
constant between particles and $V$ is a site potential generated by the substrate chain of fixed particles upon which the mobile dislocations move.

The above Hamiltonian supports solitons in the continuum, strong coupling limit. In particular for kinks, only a few dislocation centers are needed to generate the desired effect of the moving soliton. The minimal set would then be a couple of dislocation 'particles' moving along the substrate. Now suppose the above model is applied to a substrate for which the parameters of the substrate potential vary. This is analogous to the variation of the metric in the description of the previous section. In such a scenario we have to modify the substrate potential by adding a local interaction at fixed sites in the lattice. This is essentially the procedure in soliton-impurity scattering. If the effects found by Kivshar et al. are indeed based on the above simple dislocation model, then these should appear clearly when a couple of dislocation degrees of freedom scatter off an external potential. We will see below that this borne out in a simple two-particle model that imitates the soliton behavior.

Consider a system of two classical point particles connected by a massless spring, two of the dislocations above, and a repulsive force between them needed to prevent their collapse to zero size that subsums the behavior of the rest of the chain of dislocations. With only two degrees of freedom, we are eliminating the rigidity of the chain, thereby introducing the spurious possibility of complete overlap between the two sites, which does not occur when the chain is infinite. Hence the need for a repulsive interaction.

The above simplistic model is not directly related to the collective coordinate treatment of ref. deliberately. The aim is to show that the particular dynamics of the soliton is not the cause of the peculiar phenomena previously found.

When each particle in the system is allowed to interact with an external potential, we are imitating the impurity force or the local change in the dislocation potential.
The classical nonrelativistic one-dimensional lagrangian for the system of equal masses \( m_1 = m_2 = 1 \) becomes:

\[
\mathcal{L}_{\text{sys}} = \frac{\dot{x}_1^2}{2} + \frac{\dot{x}_2^2}{2} - k \frac{(x_1 - x_2)^2}{2} - \frac{\alpha}{|x_1 - x_2|^n} + V(x_1) + V(x_2) \tag{5}
\]

For the potential well we take

\[
V(x) = A e^{-\beta x^2} \tag{6}
\]

Although any finite size well may serve for this purpose. We prepare the two-particle system at rest at a large distance far away from the well with an initial speed \( v \). The equilibrium interparticle separation is \( r_0^{n+2} = \frac{n \alpha}{k} \). We here use \( n = 2 \).

The equations of motion are not solvable analytically. However we can show that for a well large compared to the equilibrium distance \( r_0 \) the system may be trapped and oscillate inside it. Transforming to relative and center of mass coordinates, \( r = \frac{x_2 - x_1}{2}, \ R = \frac{x_1 + x_2}{2} \) and using the ansatz \( r = r_0 + \delta(t) \), with \( \delta \) a small parameter, we find the equations of motion near the center of the well \( R = 0 \)

\[
\ddot{\delta} + 2k \delta + 2A e^{-r_0^2} \beta (r_0 + \delta) = 0 \\
\dot{R} + 2A \beta R e^{-r_0^2} \beta = 0 \tag{7}
\]

Where we have used \( \beta r_0^2 \ll 1 \), a wide well as compared to the equilibrium distance of the system. Passing to a new coordinate

\[
\delta(t) = \frac{r_0}{1 + \frac{k}{A \beta} e^{\beta r_0}} + \epsilon(t) \tag{8}
\]

the first of equations [7] becomes

\[
\ddot{\epsilon} + 2k \epsilon + 2A e^{-r_0^2} \beta \beta \epsilon = 0 \tag{9}
\]
It is clear from the above equations that the center of mass coordinate of the system oscillates around the center of the well, while the relative coordinate oscillates too with the small amplitude $\epsilon$. Moreover, the system shrinks inside the well. The oscillations of the center of mass coordinate $r$ compensate for the loss of kinetic energy of the system that impinged from infinity with a fixed relative separation. The above treatment demonstrates, that at least there is room for the trapping to occur.

We now proceed to show the numerical results of the exact calculation of the development of the model.

Using the numerical method used in ref. [8] we can find the outcome of the scattering events as a function of the initial speed. The system starts with a certain initial center of mass velocity $v$ far away from the well. The system is prepared with the relative separation of equilibrium $r_0$ and the outcome of the scattering is monitored as a function of the initial conditions.

Figure 1 exemplifies the results for the choice of parameters $k = 1$, $\alpha = 1$, $n = 2$, $A = 2$, $\beta = 1$.

Quite unexpectedly, it is found that the system behaves exactly like the soliton.

The system can be trapped $v_{final} = 0$, reflected, $v_{final} < 0$ or transmitted, $v_{final} > 0$ through the well by varying the initial speed.

When the system is trapped, it oscillates with a null average speed, the kinetic energy stored in the vibrational and deformation modes.

Minute changes of the initial speed around a value leading to a trapped state, may generate reflection or transmission events.

The effects are independent of the functional dependence of the interactions and external potential, as well as the values of the parameters.

In figure 1 we used a grid for $v$ of $dv = .001$. Using a finer grid, each region of reflection-transmission unfolds to more islands of trapping, reflection and transmission.

Finer and finer grids show more and more structure.
Figure 1: Final velocity of the two-particle system as a function of the initial velocity for the parameters $k = 1$, $\alpha = 1$, $n = 2$, $A = 2$, $\beta = 1$ with a velocity grid $dv = .001$

Figure 2 shows a detailed expansion of the velocity range around $v = .12$ with a grid $dv = .0002$. The system is chaotic, an infinitesimal change in the initial speed produces diverging results.

Many of the phenomena related to chaotic behavior may be identified in the system, namely scaling, bifurcation and perhaps even fractal structure.

It is now safer to claim that the unexpected behavior of a soliton interacting with an attractive well may be traced back to its extended nature. If we regard each $\phi(x)$ as a classical pointlike object we will find interactions between neighboring particles of attractive and repulsive character. The basic attractive interaction is provided by the space derivative of the soliton lagrangian and a piece of the self-interaction potential, while the repulsive
interaction is provided by the latter and the coupling to the remainder of the soliton or linear chain in a discrete model.

3 Final remarks

The simplest implementation of the system studied would be a toy-like system of two masses tied-up to a spring sliding on a frictionless table with a carved well on it. Two atoms in a molecule scattering off an external Van-der-Waals potential might show the same effects in a quasi-classical approximation. However, quantum effects can blur the picture due to interference.

Turning the process backwards: an extended object, may it be a soliton or a classical assembly of bound particles, in a trapped state, can suddenly be freed from it provided some random interaction causes the reversal of the
process of trapping, a process reminiscent of the decay of metastable states in quantum mechanics. The concept of trapping in general is not discussed in the classical mechanics literature. Soliton physics has taught us that an extended object made of particles in interaction has a much richer behavior than expected.

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References

[1] R. Rajaraman, *Solitons and instantons*, (North-Holland, Amsterdam, 1982).

[2] Samuel Shen, *A Course on Nonlinear waves*, Kluwer Academic Publishers, 1993.

[3] T.H.R. Skyrme, Proc. Roy. Soc. London, A260, 127 (1961); A262, 237 (1961) and Nucl. Phys. 31, 556 (1962).

[4] G.S. Adkins, C.R. Nappi, and E. Witten, Nucl. Phys. B228, 552 (1983).

[5] Y. S. Kivshar, Z. Fei and L. Vazquez, Phys. Rev. Lett. 67, 1177 (1991), Z. Fei, Y. S. Kivshar and L. Vazquez, Phys. Rev. A46, 5214 (1992).

[6] J. A. Gonzalez and B. de A. Mello, Phys. Scripta 54, 14 (1996),

[7] J. A. Gonzalez, B. de A. Mello, L. I. Reyes and L. E. Guerrero, Phys. Rev. Lett. 80 (1998) 1361.

[8] G. Kälbermann, Phys. Rev. E55, R6360 (1997).

[9] H. P. Robertson and T. W. Noonan, *General Relativity*, Saunders physics books, section 9.8, 1968.

[10] M. Remmoisenet, *Waves called solitons*, Springer Verlag, 1995.