An origin of spins of fields

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Abstract

Spins of fields are investigated in terms of the zero-energy eigenstates of 2-dimensional Schrödinger equations with central potentials \( V_\alpha(\rho) = -a^2 g_\alpha \rho^{2(n-1)} \) (\( a \neq 0, g_\alpha > 0 \) and \( \rho = \sqrt{x^2 + y^2} \)). We see that for \( a = N/2 \) (\( N \) = positive odd integers) one half spin states can naturally be understood as states with the angular momentum \( l = 1 \) in the \( \zeta_\alpha \) plane which is obtained by mapping the \( xy \) plane in terms of conformal transformations \( \zeta_\alpha = z^n \) with \( z = x + iy \). It is shown that the scalar and the 1/2-spin fields can obtain masses. Vortex structures and a supersymmetry for the zero-energy states are also pointed out.

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1. Introduction

Particles generally have definite spins. It is well-known that quantum mechanics allows the existence of half integer spins like \( J = 1/2, 3/2, 5/2, \cdots \). It is, however, also well-known that the angular momentum for Schrödinger equations with central potentials can take only integers in non-relativistic quantum mechanics. To understand 1/2 spin naturally we need to write the equation of motion in terms of Dirac’s relativistic equation. In this letter we shall see the fact that Schrödinger equations with some kind of two-dimensional central potentials can have the eigenstates that represent 1/2-spin states.

Let us start with 2-dimensional Schrödinger equations with the central potentials \( V_a(\rho) = -a^2 g_a \rho^{2(a-1)} \) \( (a \neq 0 \text{ and } \rho = \sqrt{x^2 + y^2}) \). It has been shown that the equations have zero-energy eigenstates which are infinitely degenerate \[1, 2, 3, 4\]. That is to say, the Schrödinger equations for the zero-energy eigenvalue are written as

\[
\left[ -\frac{\hbar^2}{2m} \triangle (x, y) + V_a(\rho) \right] \psi(x, y) = 0,
\]

where \( \triangle (x, y) = \partial^2 / \partial x^2 + \partial^2 / \partial y^2 \), and they are transformed into the following equation in terms of the conformal transformations \( \zeta_a = z^a \) with \( z = x + iy \) \[2, 3\]:

\[
\left[ -\frac{\hbar^2}{2m} \triangle_a - g_a \right] \psi_0(u_a, v_a) = 0,
\]

where \( \triangle_a = \partial^2 / \partial u_a^2 + \partial^2 / \partial v_a^2 \). Here the variables are defined by the relations \( \zeta_a = u_a + iv_a \) and

\[
u_a = \rho_a \cos(\varphi_a), \quad v_a = \rho_a \sin(\varphi_a),
\]

where \( \rho_a = \rho^a \) and \( \varphi_a = a\varphi \). We see that the zero-energy eigenstates for all the different numbers of \( a \) are described by the same plane-wave solutions in the \( \zeta_a \) space. Furthermore it is easily shown that the zero-energy states degenerate infinitely. Let us consider the case for \( a > 0 \) and \( g_a > 0 \). Putting the function \( f_n^\pm(u_a; v_a) e^{\pm ik_u u_a} \) with \( k_u = \sqrt{2mg_a / \hbar} \) into (2), where \( f_n^\pm(u_a; v_a) \) are polynomials of degree \( n \) \( (n = 0, 1, 2, \cdots) \), we obtain the equations for the polynomials

\[
[\triangle_a \pm 2ik_u \partial / \partial u_a] f_n^\pm(u_a; v_a) = 0.
\]

Note that from the above equations we can easily see the relation \((f_n^-(u_a; v_a))^* = f_n^+(u_a; v_a)\) for all \( a \) and \( n \). General forms of the polynomials have been obtained by using the solutions in the \( a = 2 \) case (2-dimensional parabolic potential barrier (2D PPB)) \[2, 3\]. Since all the solutions have the factors \( e^{\pm ik_u u_a} \) or \( e^{\pm ik_u v_a} \), we see that the zero-energy states describe stationary flows \[1, 2, 3\] and, of course, have no time factor. The general eigenfunctions with zero-energy are written as arbitrary linear combinations of the eigenfunctions included in two infinite series of \( \left\{ \psi_{0n}^{\pm(0)}(u_a; v_a) \right\} \) for \( n = 0, 1, 2, \cdots \), where

\[
\psi_{0n}^{\pm(0)}(u_a; v_a) = f_n^{\pm}(u_a; v_a) e^{\pm ik_a u_a}.
\]

(For the details, see the sections II and III of Ref. \[3\].) It has been also pointed out that the motions of the \( z \) direction perpendicular to the \( xy \) plane can be introduced as free motions represented by \( e^{\pm ik_z z} \). In this case the total energies \( E_T \) of the states are given by \( E_T = E_z \), where \( E_z \) are the energies of the free motions in the \( z \) direction. Note that the zero-energy eigenfunctions cannot be normalized as same as those in scattering processes \[3\]. Actually it has been shown that all the zero-energy states for \( g_a > 0 \) are eigenstates in the conjugate spaces of Gel’fand triplets \[1, 2, 3, 4\]. In the recent work \[4\] it has been pointed out that the freedom for the infinite degeneracy can be understood as a kind of interacting gauge fields. In this letter we shall study the infinite degeneracy in terms of the eigenstates with the definite angular momentum in the \( \zeta_a \) plane.
2. Boundary conditions

Let us start our discussion from a simple example for \( a = 1/2 \), because the simplest case for \( a = 1 \) is trivial such that \( u_1 = x \) and \( v_1 = y \). As shown in Eq. (4), the relation between the angle \( \varphi_{1/2} \) in the \( uv \) plane and the angle \( \varphi \) in the \( xy \) plane is given by \( \varphi_{1/2} = \varphi/2 \). This relation means that the whole of the \( uv \) plane must be represented by two sheets of the \( xy \) plane just as same as a Riemann surface [3, 7]. For instance, the flow for the wave function \( e^{-ik_1/2u_1(0)} \) has a cut from 0 to \( \infty \) on the \( x \) axis as shown in Fig. 1 [7]. Actually all the zero-energy eigenstates given in Eq. (5) are not rotationally symmetric. We easily see that the eigenfunctions are classified into two types satisfying the definite boundary conditions for the rotation in the \( xy \) plane such that

\[
\psi(\varphi + 2\pi) = \psi(\varphi), \quad \text{for bosonic boundary condition},
\]
\[
\psi(\varphi + 2\pi)_f = -\psi(\varphi), \quad \text{for fermionic boundary condition},
\]

where the eigenfunctions are written as

\[
\psi(\varphi)_b = \psi(\varphi) + \psi(\varphi + 2\pi),
\]
\[
\psi(\varphi)_f = \psi(\varphi) - \psi(\varphi + 2\pi).
\]

This means that we cannot decompose the eigenfunctions \( \psi(\varphi)_f \) in terms of the eigenfunctions of the angular momentum operator \(-i\hbar \partial/\partial \varphi\) in the \( xy \) plane, which are determined by the boundary condition \( \psi(\varphi + 2\pi) = \psi(\varphi) \). In the \( uv \) plane, however, from the relations \( u(\varphi_{1/2} + 2\pi) = u(\varphi_{1/2}) \) and \( v(\varphi_{1/2} + 2\pi) = v(\varphi_{1/2}) \) we see that all the eigenfunctions \( \psi \) satisfy the boundary condition

\[
\psi(\varphi_{1/2} + 2\pi) = \psi(\varphi_{1/2}).
\]

By considering the fact that the \( 2\pi \) rotation in the \( uv \) plane corresponds to the \( 4\pi \) rotation in the \( xy \) plane because of the relation \( \varphi_{1/2} = \varphi/2 \), we can write the Riemann sheet structure of the \( xy \) plane as shown in Fig. 1. The fact that all the eigenfunctions satisfy the boundary condition of Eq. (8) in the \( uv \)
plane indicates that the eigenfunctions can be written by those of the angular momentum with respect to the rotations for the angle $\varphi_{1/2}$ in the $uv$ plane.

3. The angular momentum eigenstates in the $uv$ plane

Let us here study the eigenstates for the angular momentum $\hat{L}_{uv}$ in the $uv$ plane, which are given by

$$\hat{L}_{uv} = -i\hbar \frac{\partial}{\partial \varphi_{1/2}}.$$  \hspace{1cm} (9)

Thus the Schrödinger equation (2) for the zero energy are written as

$$\left[ -\frac{1}{q} \frac{\partial}{\partial q} \left( q \frac{\partial}{\partial q} \right) + \frac{1}{q^2} \left( \frac{\hat{L}_{uv}}{\hbar} \right)^2 - g' \right] \psi_l(q, \varphi_{1/2}) = 0,$$

where $q = p^{1/2}$ and $g' = 2mg_1/\hbar^2$ are used. We can factorize the eigenfunctions as

$$\psi_l(q, \varphi_{1/2}) = h_l(q) \phi_l(\varphi_{1/2}).$$ \hspace{1cm} (11)

It is trivial that the eigenfunctions for the angular momentum are given as usual

$$\phi_l(\varphi_{1/2}) = e^{\pm i\varphi_{1/2}/\sqrt{2\pi}},$$ \hspace{1cm} (12)

which evidently satisfy the boundary condition (8) in the $uv$ plane only for $l =$ integers. Substituting Eq. (11) into Eq. (10), we obtain the equations for $h_l(q)$ as

$$\left[ -\frac{1}{q} \frac{\partial}{\partial q} \left( q \frac{\partial}{\partial q} \right) + \frac{l^2}{q^2} - g' \right] h_l(q) = 0.$$ \hspace{1cm} (13)

From the condition that $h_l(q)$ have no singularity at $q = 0$, we find out the solutions

$$h_l(q) = \sum_{n=0}^{\infty} c_{|l|+2n} q^{|l|+2n},$$

where $c_n$ are determined by the following recurrence formula;

$$c_{|l|+2n+2} = -\frac{g'}{4(n+1)(|l| + n + 1)} c_{|l|+2n} = \frac{(-g'/4)^{n+1}}{(n+1)(|l| + n + 1) \cdots (|l| + 1)} c_{|l|}. \hspace{1cm} (14)$$

Note that the eigenfunctions are unique for each $|l|$ except $c_{|l|}$. Since the eigenfunctions in Gel’fand triplets are not normalizable in general, the constants $c_{|l|}$ should not be determined by the normalizations of $h_l(q)$ \[8\]. Now we may conclude that the infinite degeneracy for the zero-energy eigenstates is expressed by the eigenstates of the angular momentum. It is strongly noticed that the infinite degeneracy can be represented as the infinite degeneracy of the states with arbitrary angular momenta in the $uv$ plane.

4. Spins in the $xy$ plane

Let us study the meaning of the angular momentum eigenstates $\phi_l(\varphi_{1/2})$ in the $xy$ plane. Since the case for $l = 0$ (scalar) is trivial, we study the case for $l = \pm 1$. The eigenfunctions written by $\phi_{\pm 1}(\varphi) \equiv \phi_{\pm 1}(\varphi_{1/2}) = e^{\pm i\varphi/\sqrt{2\pi}}$ in the $xy$ plane are, of course, degenerate as the solutions of the original Schrödinger equation with the zero-energy eigenvalue. It is important that both of them satisfy the fermionic boundary condition $\phi_{\pm 1}(\varphi + 2\pi) = -\phi_{\pm 1}(\varphi)$ in the $xy$ plane. Actually $\phi_{\pm 1}(\varphi)$ are the eigenfunctions of the angular momentum in the $xy$ plane, $\hat{L}_{xy} = -i\hbar \partial / \partial \varphi$, of which eigenvalues are, respectively, $(\pm 1/2)\hbar$. We may
say that \( \phi_{\pm 1} \) stand for the spin-up and -down states for the spin 1/2 states, where the direction of the spin is the \( z \) direction.

Let us examine the solutions in relativistic motions. The above argument can be extended to the relativistic equations for massless particles in 4-dimensional (4D) Lorentz spaces. In the 4D spaces the separation of the two directions perpendicular to a non-zero 3-momentum \( p \) can be carried out in terms of the 4-momentums defined by \( p^+ = (|p|, |p|) \) and \( p^- = (-|p|, |p|) \) such that \( \epsilon^{\mu
u\lambda\sigma}p^\mu p^\nu \), where \( \epsilon^{\mu
u\lambda\sigma} \) is the totally antisymmetric tensor defined by \( \epsilon_{0123} = 1 \). It is trivial to write \( \rho = x^2 + y^2 \) and \( \partial^z + \partial^t \) in the covariant expressions in terms of these tensors. The Klein-Gordon equation for massless fields with the momentum of the \( z \) direction can be written as

\[
\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta (x, y, z) + V_0(\rho) \psi_z(r) = 0,
\]

where \( \psi_z(r) = \psi(x, y) e^{i(p_z z - cp_z t)/\hbar} \) with \( p_0^2 = p_z^2 \). Note that the dimension of \( g_{1/2} \) in the 4D spaces is different from that in the 3D spaces. From the relation \( p_0^2 = p_z^2 \), we have no simple form corresponding to Dirac’s relativistic equations for massive 1/2-spin fields. The exception is the case of 2D PPB for the massless fields. We can write the equation of motion as

\[
\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \Delta (x, y) + V_{1/2}(\rho) \psi_{1/2}(r, t) = 0,
\]

where \( \partial_0 = (1/c) \partial_t \), and the transposed vector of \( \psi_{1/2} \) are given by

\[
\psi^t_{1/2}(r, t) = (\psi_{-1}(q, \varphi), \psi_1(q, \varphi)) e^{i(p_z z - cp_0 t)/\hbar}
\]

with \( p_0^2 = p_z^2 + m^2c^2 \). Note that for the case of \( a = 1/2 \) we have no simple form corresponding to Dirac’s relativistic equations for massive 1/2-spin fields. The exception is the case of 2D PPB for the massless fields. We can write the equation of motion as

\[
i\sigma^\mu (\partial_\mu - iA_\mu) \psi_2(r, t) = 0,
\]

where \( \sigma^0 \) is taken as the unite matrix, and

\[
A_\mu = e_0 \epsilon^{\mu\nu\lambda\sigma} r_{\mu} p_\lambda^+ p^- \sigma
\]
Fig. 2: Three corner flows for $a = 3/2$ in a $xy$ plane.

with $c_0 = \text{constant}$. By operating $-i\sigma^\mu(\partial_\mu + iA_\mu)$ to Eq. (18), we have a potential that is proportional to $\rho^2$. We see that only in the cases of the PPB the equation of motion for the massless fields can be written as the gauge interaction. It may be interesting to investigate the relations between the 2D structures composed of the zero-energy eigenstates for $a = 2$ and the membrane in string theory.

5. Spins for $a = N/2$

We can extend the above discussions to the cases for $a = N/2$ with $N =$ positive odd integers. The two boundary conditions for the angle $\varphi$ in the $xy$ plane given by Eq. (6) again appear and they are integrated into the same boundary condition for $\varphi_a$ as that given by Eq. (8) in the $u_a v_a$ plane. In these cases the Riemann surface composed of two sheets of the $xy$ plane correspond to $N$ sheets of the $u_a v_a$ plane, that is to say, the $4\pi$ rotation in the $xy$ plane corresponds to the $2N\pi$ rotation in the $u_a v_a$ plane. We should notice that the boundary condition (8) is defined by the $2\pi$ rotation in the $u_a v_a$ plane. In order to change the $2N\pi$ rotation into the $2\pi$ rotation, we have to consider the rotation in the $u_a v_a$ plane for $a = N/2$ by using the angle $\varphi_a/N$ instead of $\varphi_a$. Thus the angular-momentum eigenstates should be written by $e^{il\varphi_a/N} = e^{il\varphi/2}$ that are same for all of $N$ in the $xy$ plane. Let us examine this situation by writing the current corresponding to the lowest eigenstate of Eq. (5) in the $xy$ plane, which is expressed by the corner flow with the angle $\pi/a$. An example for $a = 3/2$ is shown in Fig. 2, where three corner flows with the angle $2\pi/3$ contained in a $xy$ plane are figured. In general for $a = N/2$ the $N$ number of corner flows are contained in a $xy$ plane. We see that one incoming (or outgoing) current composed of the two corner flows is contained in one sheet of the $N$ sheets of the $u_a v_a$ plane. The diagram composed of the $2N$ corner flows in one Riemann surface of the $xy$ representation (in the double sheets) is invariant with respect to the $4\pi/N$ rotation that exactly corresponds to the $2\pi$ rotation in the $u_a v_a$ plane. Note that in the consideration of the invariance the coincidence of the directions of the flows must be taken into account. For instance the $2\pi/N$ rotation change the directions of the flows.

We have seen that the half-integer spins originate from the 2-dimensional structure described by the zero-energy states of the potentials $V_a(\rho)$. It should strongly be noticed that, since the zero-energy states have no time development at all, the spin structures are absolutely stable in free motions. The change of
the structure will occur in the interaction with gauge fields induced by the zero-energy states [7].

6. Remarks

Finally we shall comment on two interesting properties of the zero-energy solutions.

(1) Vortices of the fields with spins
Let us investigate the vortex structure of the eigenfunctions \( \psi_l(q, \varphi) \) given in Eq. (11) for \( a = 1/2 \). It is known that the angular momentum eigenstates have a singularity in the velocity at the origin (\( \rho = 0 \)), and then a vortex appears at the origin [1, 2, 3]. Let us evaluate the circulation represented by the integration with respect to a closed circle \( C \) encircling the singularity as

\[
\Gamma_l = \oint_C \mathbf{v}_l \cdot ds,
\]

where the velocity is defined by \( \mathbf{v}_l = \Re \left[ \psi_l^*(\varphi) \frac{\partial}{\partial \varphi} \psi_l \right] / |\psi_l|^2 \) [4]. We obtain

\[
\Gamma_l = \frac{\pi \hbar}{m} l
\]

in the non-relativistic case. It is interesting that the circulation is quantized in the unite of \( \pi \hbar/m \), while it is quantized in the unite \( 2\pi \hbar/m \) for the eigenstates of the angular momentum for central potentials, which belong to Hilbert spaces. (For example, see Appendix A3 of Ref. [1].) This difference is easily understood from the fact that we obtain the same quantization with the unite \( 2\pi \hbar/m \) for the circulation in the \( u_a v_a \) plane, that is to say, since the \( 2\pi \) rotation in the \( u_a v_a \) plane corresponds to the \( 4\pi \) rotation in the \( xy \) plane, the circulation defined by the \( 2\pi \) rotation in the \( xy \) plane is quantized in the half of the unite \( 2\pi \hbar/m \) obtained in the \( u_a v_a \) plane. Similar arguments can be carried out in the cases of \( a > 1/2 \).

(2) Supersymmetry of the zero-energy states
Finally we would like briefly comment on supersymmetry. We have seen that in the cases of \( a = N/2 \) bosonic states and fermionic states appear alternately in increasing \( |l| \). This fact indicates that the total system composed of the zero-energy states have some kind of supersymmetry. We can understand the supersymmetric property of the zero-energy states as follows;

All the zero-energy eigenfunctions [5] can be written down by the two types corresponding to bosonic and fermionic states given in Eq. (7) such that

\[
\psi_{0n(b)}^\pm(\varphi) = \psi_{0n(\varphi)}^\pm(\varphi) + \psi_{0n(\varphi+2\pi)}^\pm(\varphi), \quad \text{for bosonic boundary condition}
\]

\[
\psi_{0n(f)}^\pm(\varphi) = \psi_{0n(\varphi)}^\pm(\varphi) - \psi_{0n(\varphi+2\pi)}^\pm(\varphi), \quad \text{for fermionic boundary condition}.
\]

This fact means that every zero-energy eigenstates can be written by the sum of the bosonic and fermionic states as

\[
\psi_{0n}(\varphi) = \frac{1}{2} [\psi_{0n(b)}^\pm(\varphi) + \psi_{0n(f)}^\pm(\varphi)].
\]

In these sum the weights for the bosonic state and the fermionic one are the same value \( 1/2 \). Except the states having definite angular momenta like \( \psi_l \) all stationary phenomena described by the zero-energy eigenstates [5] are expected to have a supersymmetry with respect to the exchange between the bosonic and fermionic states.
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