CONVERGENCE OF SOLUTIONS TO THE $p$-LAPLACE EVOLUTION EQUATION AS $p$ GOES TO 1

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Abstract. We prove that the set of solutions to the parabolic singular $p$-Laplace equation with Dirichlet boundary conditions on a bounded Lipschitz domain $\Omega$ for all space dimensions is continuous in the parameter $p \in [1, +\infty)$ and the initial data. The highly singular limit case $p = 1$ is included. In particular, we show that the solutions $u_p$ converge strongly in $L^2(\Omega)$, uniformly in time, to the solution $u_1$ of the parabolic 1-Laplace equation as $p \to 1$.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$ be an open bounded domain with Lipschitz boundary $\partial \Omega$. Consider the following family of singular diffusion equations indexed by $p \in (1, +\infty)$,

\[
\begin{aligned}
\frac{\partial u_p}{\partial t} - \text{div} \left[ |\nabla u_p|^{p-2} \nabla u_p \right] &= f_p \quad \text{in } (0, +\infty) \times \Omega, \\
u_p &= 0 \quad \text{on } (0, +\infty) \times \partial \Omega, \\
u_p(0) &= x_p,
\end{aligned}
\]

where $x_p \in L^2(\Omega)$ and $f_p \in L^2_{\text{loc}}((0, +\infty); L^2(\Omega))$. The notation $(pD)$ refers to Dirichlet boundary conditions.

Consider also

\[
\begin{aligned}
\frac{\partial u_1}{\partial t} - \text{div} \left[ \text{Sgn} \left( \nabla u_1 \right) \right] \ni f_1 \quad \text{in } (0, +\infty) \times \Omega, \\
u_1 &= 0 \quad \text{on } (0, +\infty) \times \partial \Omega, \\
u_1(0) &= x_1,
\end{aligned}
\]

where $x_1 \in L^2(\Omega)$ and $f_1 \in L^2_{\text{loc}}((0, +\infty); L^2(\Omega))$ and the vector-valued (and multi-valued) sign-operator $\text{Sgn} : \mathbb{R}^d \to 2^{\mathbb{R}^d}$ is defined by

\[
\text{Sgn}(x) := \begin{cases} 
\frac{x}{|x|}, & \text{if } x \in \mathbb{R}^d \setminus \{0\}, \\
\{y \in \mathbb{R}^d \mid |y| \leq 1\}, & \text{if } x = 0.
\end{cases}
\]

In fact, the expression “$\text{div} \left[ \text{Sgn} \left( \nabla \cdot \right) \right]$” is merely heuristic, we shall later replace it by the subdifferential $\partial \Phi_1$, where the convex potential $\Phi_1$ is defined in (2.6) below.

In this work, we consider weak variational solutions to equation $(pD)$, $p \in [1, +\infty)$, defined as limits of Yosida-regularized equations. We are interested in continuity properties of $(pD)$ in the initial data and, in particular, when the parameter $p$ varies. The $p$-Laplace evolution $(pD)$ belongs to the class of gradient flow-type equations, where the dynamics is generated by the (infinite dimensional) gradient of a differentiable functional.

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In order to prove the desired continuity of solutions in $p$, we shall apply a theorem by Attouch, stating that so-called Mosco continuity of convex, lower semicontinuous potentials implies the continuous dependence of the solutions to the related, subgradient-type evolution equations, see [6]. The classical theorem requires a Hilbert space framework, which explains our choice of $L^2$-data. For Banach spaces, variational convergence and viscosity solutions are brought together in [8].

Inclusion (1$_D$) requires some attention. It is called total variation flow, see [3,4]. We shall model it by the gradient of the total-variation functional on functions of bounded variation in $L^2(\Omega)$.

We shall also consider the $p$-Laplace evolution with Neumann boundary conditions,

\[
\left\{ \begin{array}{ll}
\frac{\partial u_p}{\partial t} - \text{div } [\nabla u_p|^{p-2}\nabla u_p] = f_p & \text{in } (0, +\infty) \times \Omega, \\
\frac{\partial u_p}{\partial \nu} = 0 & \text{on } (0, +\infty) \times \partial \Omega, \\
u_p(0) = x_p,
\end{array} \right.
\]

where $\nu$ is the generalized outward normal on $\partial \Omega$.

Also

\[
\left\{ \begin{array}{ll}
\frac{\partial u_1}{\partial t} - \text{div } [\text{Sgn } (\nabla u_1(t))] \ni f_1(t) & \text{in } (0, +\infty) \times \Omega, \\
\frac{\partial u_1}{\partial \nu} = 0 & \text{on } (0, +\infty) \times \partial \Omega, \\
u_1(0) = x_1,
\end{array} \right.
\]

where again the expression “$\text{div } [\text{Sgn } (\nabla \cdot )]$” is made rigorous as a subdifferential $\partial \Psi_1$ of the potential $\Psi_1$ defined in (2.7).

The singular diffusion operator in equations (p$_D$), (p$_N$) is called $p$-Laplacian. Both equations of evolution-type are covered by the theory of nonlinear semigroups associated to equations of subdifferential-type in Hilbert space as in the book of Brézis [17]. For the notion of solutions used in this paper, see Definition 3.2 below. Explicit representations for the subdifferential operators were given by Schuricht [39] for inclusion (1$_D$) and by Andreu, Ballester, Caselles and Mazón [3] for inclusion (1$_N$).

Classically, the $p$-Laplace equation appears in geometry, quasi-regular mappings, fluid dynamics and plasma physics, see [21, 22]. In [28], Ladyženskaja suggests the $p$-Laplace evolution as a model of motion of non-Newtonian fluids. A typical 2-dimensional application can be found in image restoration, see [13, Ch. 3] for a comprehensive treatment. General equations of $p$-type are studied in [41].

Eigenvalue problems for the 1-Laplacian have been studied e.g. by Fridman, Kawohl, Schuricht and Parini [25, 27, 37, 39].

The stochastic analog to (p$_D$) is being examined by Liu [32], respectively, to (1$_D$) by Barbu, Da Prato and Röckner [15]. Recently, in the case of stochastic equations, similar results to this paper have been proved by Ciotir and the author in [19]. The proof in this paper, however, originates from the author’s thesis [40, Ch. 8.3], where variable-space-approach, similar to that in [11], is used. Previous convergence results for the stationary problem have been obtained by Mercaldo, Rossi, Segura de León and Trombetti in [34–36]. Convergence of the evolution problem for local solutions has been previously investigated by Giga, Kashima and Yamazaki in [26].

Let us formulate the main result of this paper. Its proof can be found in section 4 below. The notion of solution is specified in section 3 below, see Definition 3.2.

**Theorem 1.1.** Let $p_0 \in [1, +\infty)$, $\{p_n\} \subset (1, +\infty)$ with $\lim_n p_n = p_0$. Let $T > 0$.

Suppose that $\lim_n \|x_{p_n} - x_{p_0}\|_{L^2(\Omega)} = 0$ and $\lim_n \int_0^T \|f_{p_n}(t) - f_{p_0}(t)\|_{L^2(\Omega)}^2 \, dt = 0$. 

Let us formulate the main result of this paper. Its proof can be found in section 4 below. The notion of solution is specified in section 3 below, see Definition 3.2.
Denote by $u_p$ the solution to equation $(p_D)$ in the sense of Definition 3.2 and by $v_p$ the solution to equation $(p_N)$ Definition 3.2.

Then

\begin{equation}
\lim_{n \to \infty} \sup_{t \in [0,T]} \|u_{p_n}(t) - u_p(t)\|_{L^2(\Omega)} = 0
\end{equation}

and

\begin{equation}
\lim_{n \to \infty} \sup_{t \in [0,T]} \|v_{p_n}(t) - v_p(t)\|_{L^2(\Omega)} = 0.
\end{equation}

In other words, if

$$F : [1, +\infty) \times L^2(\Omega) \times L^2_{\text{loc}}((0, +\infty); L^2(\Omega)) \to C([0, +\infty); L^2(\Omega))$$

denotes the map that assigns the solution $t \mapsto u_{p,f}^{-}(t)$ to equation $(p_D)$, our result states that $F$ is continuous. Certainly, a similar statement holds for $(p_N)$.

2. The related energies

We observe that, for $p \in (1, +\infty)$, the $p$-Laplace operator on $L^2(\Omega)$ is a gradient-type operator with the convex potential

$$\Phi_p(u) := \begin{cases} 
\frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx, & \text{if } u \in W^{1,p}_0(\Omega) \cap L^2(\Omega), \\
+\infty, & \text{if } u \in L^2(\Omega) \setminus W^{1,p}_0(\Omega).
\end{cases}$$

Note that the Dirichlet boundary conditions are encoded into the energy space $W^{1,p}_0(\Omega)$ which is the standard first order $p$-Sobolev space on $\Omega$ such that its elements have vanishing trace (see next paragraph).

In general, for an element $u \in W^{1,p}(\Omega)$, we define its trace $\gamma_p(u)$ to $\partial \Omega$ by

\begin{equation}
\gamma_p(u) \langle \eta, \nu \rangle \, dH^{d-1} = \int_{\partial \Omega} u \, \text{div} \, \eta \, dx + \int_{\Omega} \langle \eta, \nabla u \rangle \, dx \quad \forall \eta \in C^{1}(\overline{\Omega}; \mathbb{R}^d),
\end{equation}

where $\nu$ is the generalized outward normal on $\partial \Omega$ and $H^{d-1}$ is the $d-1$-dimensional Hausdorff measure on $\partial \Omega$. We have that $\gamma_p : W^{1,p}(\Omega) \to L^p(\partial \Omega, H^{d-1})$ is a continuous linear operator. Note that our definition of $W^{1,p}_0(\Omega)$ coincides with the standard one, where one takes $W^{1,p}_0(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$, see [23, §5.5, Theorem 2]. We denote by $\partial_i$, $\nabla$ resp. the weak or ordinary partial derivative in direction of the $i$-th coordinate, the weak or ordinary gradient resp. D_i and D refer to the corresponding objects in the sense of Schwartz distributions.

Sometimes we shall write $[Df]$ instead of $Df$ to indicate that $[Df]$ is a Radon measure. Of course, if $f \in L^1_{\text{loc}}(\Omega)$ and $Df \in L^1_{\text{loc}}(\Omega; \mathbb{R}^d)$ then $Df = \nabla f$ a.e.

Define also the energy of the $p$-Laplace operator in $L^2(\Omega)$ with Neumann boundary conditions by

$$\Psi_p(u) := \begin{cases} 
\frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx, & \text{if } u \in W^{1,p}(\Omega) \cap L^2(\Omega), \\
+\infty, & \text{if } u \in L^2(\Omega) \setminus W^{1,p}(\Omega).
\end{cases}$$

Let us continue investigating $\Phi_p$. It is easily seen that $\Phi_p$ is a proper convex function. It is well-known that $\Phi_p$ is lower semi-continuous, see e.g. [18].

The corresponding subdifferential $\partial \Phi_p$ is a realization of the $p$-Laplace operator on $\Omega$ with Dirichlet boundary conditions. Recall that, in general, for a convex, proper, lower semi-continuous function $\Psi : H \to (-\infty, +\infty]$ on a separable Hilbert space $H$, the subdifferential $\partial \Psi$, defined by $[x, y] \in \partial \Psi$ iff

$$(y, z - x)_H \leq \Psi(z) - \Psi(x) \quad \forall z \in H,$$
is a maximal monotone graph.

On smooth functions \( \varphi \in C_0^\infty(\Omega) \), \( \partial \Phi_p \) has the representation

\[
(\partial \Phi_p)(\varphi) = -\div [\left\{ |\nabla \varphi|^{p-1} \sgn(\nabla \varphi) \right\}],
\]

where \( \sgn \) is the selection of \( \text{Sgn} \) with minimal Euclidean norm.

Note that the extension of \( \Phi_p \) to \( W_0^{1,p}(\Omega) \) is Fréchet differentiable in \( W_0^{1,p}(\Omega) \)-norm, a fact which we, however, will not use. By the theory of subdifferential operators, \( \text{dom}(\partial \Phi_p) \) is dense in \( L^2(\Omega) \), see e.g. [14, Ch. 2, Proposition 2.6].

We shall continue investigating the limit case \( p = 1 \). We need some facts from the space \( BV \), which we shall collect here.

**Definition 2.1.** A function \( f \in L^1_{\text{loc}}(\Omega) \) is said to be of bounded variation if

\[
\|Df\|((\Omega)) := \sup \left\{ \int_\Omega f \div \eta \, dx \mid \eta \in C_0^\infty(\Omega; \mathbb{R}^d), \|\eta\|_\infty \leq 1 \right\} < +\infty.
\]

The value \( \|Df\|((\Omega)) \) is called the total variation of \( f \). The space of all classes of functions in \( L^1(\Omega) \) that are of bounded variation is denoted by \( BV(\Omega) \).

Let \( f \in BV(\Omega) \). Then there is a Radon measure \( \mu_f \) on \( \Omega \) and a measurable function \( \sigma_f : \Omega \to \mathbb{R}^d \) such that \( |\sigma_f| = 1 \) \( \mu_f \)-a.e. and

\[
(2.2) \quad \int_\Omega f \div \eta \, dx = -\int_\Omega \langle \eta, \sigma_f \rangle \, d\mu_f \quad \forall \eta \in C_0^1(\Omega; \mathbb{R}^d).
\]

By [24, §5.1], \( \|Df\|((\Omega)) = \mu_f(\Omega) \). Set \( d[Df] := \sigma_f d\mu_f(dx) \), which is a \( \mathbb{R}^d \)-valued Radon measure on \( \Omega \). Hence (2.2) becomes

\[
(2.3) \quad \int_\Omega f \div \eta \, dx = -\int_\Omega \langle \eta, d[Df] \rangle \quad \forall \eta \in C_0^1(\Omega; \mathbb{R}^d).
\]

For each \( f \in BV(\Omega) \) there is the trace \( \gamma_1(f) \) to \( \partial \Omega \) satisfying the following formula

\[
(2.4) \quad \int_\Omega f \div \eta \, dx = -\int_\Omega \langle \eta, d[Df] \rangle + \int_{\partial \Omega} \gamma_1(f) \langle \eta, \nu \rangle \, d\mathcal{H}^{d-1} \quad \forall \eta \in C^1(\overline{\Omega}; \mathbb{R}^d),
\]

where \( \nu \) is the outward normal and \( \mathcal{H}^{d-1} \) is the Hausdorff measure on \( \partial \Omega \). We have that \( \gamma_1(f) \in L^1(\partial \Omega, \mathcal{H}^{d-1}) \). We refer to [24, Ch. 5.3] and [2, Ch. 3] for details.

Also, we have that \( W^{1,p}(\Omega) \subset BV(\Omega) \), \( p \in [1, +\infty] \) and

\[
(2.5) \quad \int_\Omega |\nabla u| \, dx = \|Du\|((\Omega)) \quad \forall u \in W^{1,p}(\Omega),
\]

see [24, Ch. 5.1, Example 1].

**Remark 2.2.** Suppose that \( \partial \Omega \) is Lipschitz. Let \( u \in BV(\Omega) \) and extend \( u \) by zero outside \( \Omega \). Then \( u \in BV(\mathbb{R}^d) \) and

\[
\|Du\|((\mathbb{R}^d)) = \|Du\|((\Omega)) + \int_{\partial \Omega} |\gamma_1(u)| \, d\mathcal{H}^{d-1}
\]

where \( \gamma_1(u) \in L^1(\partial \Omega, \mathcal{H}^{d-1}) \) is the trace of \( u \). We refer to [2, Theorem 3.87].

We are ready to define the convex potential associated to the 1-Laplacian in \( L^2(\Omega) \) with Dirichlet boundary conditions. The Dirichlet problem was derived by Kawohl and Schuricht in [27, 39].

Define

\[
(2.6) \quad \Phi_1(u) := \begin{cases} \|Du\|((\mathbb{R}^d)), & \text{if } u \in BV(\Omega) \cap L^2(\Omega), \\ +\infty, & \text{if } u \in L^2(\Omega) \setminus BV(\Omega). \end{cases}
\]
\( \Phi_1 \) is obviously proper and convex. Lower semi-continuity follows by standard arguments via integration by parts since \( \|Du\| (\mathbb{R}^d) \) is lower semi-continuous with respect to weak convergence in \( L^1_{\text{loc}}(\mathbb{R}^d) \).

The 1-Laplacian is defined to be the subdifferential \( \partial \Phi_1 \) of \( \Phi_1 \). As above, \( \text{dom}(\partial \Phi_1) \) is dense in \( L^2(\Omega) \). A precise representation of \( \partial \Phi_1 \) is given in [39, Theorem 2.1] and [27, Theorem 4.23].

Define also

\[
\Psi_1(u) := \begin{cases} 
\|Du\| (\Omega), & \text{if } u \in BV(\Omega) \cap L^2(\Omega), \\
+\infty, & \text{if } u \in L^2(\Omega) \setminus BV(\Omega).
\end{cases}
\]

\( \Psi_1 \) is convex, proper and lower semi-continuous, see [4].

### 3. Mosco Convergence

Let \( H \) be a separable Hilbert space. Let \( \Phi_n : H \to (-\infty, +\infty], n \in \mathbb{N}, \Phi : H \to (-\infty, +\infty) \) be proper, convex, l.s.c. functionals.

Recall the following definition.

**Definition 3.1** (Mosco convergence). We say that \( \Phi_n \overset{M}{\rightharpoonup} \Phi \) in the Mosco sense if

1. \( \forall x \in H \forall x_n \in H, \ n \in \mathbb{N}, \ x_n \rightharpoonup x \text{ weakly in } H : \lim \inf_n \Phi_n(x_n) \geq \Phi(x) \).
2. \( \forall y \in H \exists y_n \in H, \ n \in \mathbb{N}, \ y_n \to y \text{ strongly in } H : \lim \sup_n \Phi_n(y_n) \leq \Phi(y) \).

Consider the following sequence of abstract gradient-type evolution equations in \( H \):

\[
\begin{cases}
\frac{d}{dt} u_n(t) + \partial \Phi_n(u_n(t)) \ni f_n(t), \ 0 < t < +\infty,
\end{cases}
\]

where \( x_n \in H \) and \( f_n \in L^2_{\text{loc}}((0, +\infty); H) \).

Also,

\[
\begin{cases}
\frac{d}{dt} u(t) + \partial \Phi(u(t)) \ni f(t), \ 0 < t < +\infty,
\end{cases}
\]

where \( x \in H \) and \( f \in L^2_{\text{loc}}((0, +\infty); H) \).

A solution to, say, equation (3.2) is defined as follows. Compare with [17, Définition 3.1].

**Definition 3.2.** We call \( u \in C([0, +\infty); H) \) a solution to equation (3.2) if \( u \) is locally absolutely continuous, \( u(0) = x, u(t) \in \text{dom}(\partial \Phi) \) for a.e. \( t > 0 \) and (3.2) holds for a.e. \( t > 0 \) or, equivalently, in the sense of distributions in \( L^2_{\text{loc}}((0, +\infty); H) \).

We shall need the following theorem.

**Theorem 3.3.** Suppose that for \( T > 0 \) we have that \( x_n \rightharpoonup x \text{ strongly in } H, f_n \rightharpoonup f \text{ strongly in } L^2((0, T); H) \).

Then \( \Phi_n \overset{M}{\rightharpoonup} \Phi \) implies that \( u_n \rightharpoonup u \text{ strongly in } H, \text{ uniformly on } [0, T] \).

**Proof.** See [7, Theorems 3.66 and 3.74].

See [5, 6, 9, 20] for related results.
4. Proof of Theorem 1.1

Let $p_0 \in [1, +\infty)$, $p_n \in (1, +\infty)$, $n \in \mathbb{N}$ such that $\lim_n p_n = p_0$. Write $q_n := \frac{p_n}{(p_n - 1)}$, $n \in \mathbb{N}$, $q_0 := \frac{p_0}{(p_0 - 1)}$ (with $1/0 := +\infty$). If we can prove that

\[
\Phi_n := \Phi_{p_n} \xrightarrow{n \to +\infty} \Phi_{p_0} =: \Phi,
\]

the assertion of Theorem 1.1 follows from Theorem 3.3.

Before we prove (4.1), we shall need the following approximation result.

**Lemma 4.1** (Littig–Parini–Schuricht). Let $\Omega \subset \mathbb{R}^d$ be open and bounded with Lipschitz boundary $\partial \Omega$ and let $u \in BV(\Omega) \cap L^p(\Omega)$ for some $p \in [1, +\infty)$. Then there is a sequence $\{u_k\}$ in $C_0^\infty(\Omega)$ such that, for any $\tilde{p} \in [1, p]$,

\[
\lim_{k \to \infty} \|u_k - u\|_{L^\tilde{p}(\Omega)} = 0 \quad \text{and} \quad \lim_{k \to \infty} \|D u_k\|_{(\mathbb{R}^d)} = \|D u\|_{(\mathbb{R}^d)}.
\]

The result was proved by Littig and Schuricht in [31, Theorem 3.2], see also [30, Satz 2.31]. Previously, if $\partial \Omega$ is $C^2$, the result was proved by Parini, see [37, Theorem 2.11] and also [38].

**Proof of (4.1).** We start with proving (M1) from Definition 3.1. Suppose that $u_n \in L^2(\Omega)$ with

\[
\liminf_n \Phi_n(u_n) < +\infty.
\]

Extract a subsequence (also denoted by $\{u_n\}$) such that

\[
\lim_n \Phi_n(u_n) = \liminf_n \Phi_n(u_n)
\]

and

\[
C := \sup_n \Phi_n(u_n) < +\infty.
\]

Since $u_n \in W^{1,p_n}_0(\Omega)$ and hence $\gamma_{p_n}(u_n) = 0$, we can extend $u_n$ to $\mathbb{R}^d$ by zero outside $\Omega$ (denoted also by $u_n$) and get that $u_n \in W^{1,p_n}(\mathbb{R}^d)$ (cf. [29, Exercise 15.26]).

Fix a ball $B \subset \mathbb{R}^d$. Suppose first that for a subsequence we have that $p_n \leq 2$. Then by Hölder inequality,

\[
\int_B |u_n| \, dx \leq \left( \int_B |u_n|^{p_n} \, dx \right)^{1/p_n} (\text{vol } B)^{1/q_n}
\]

\[
\leq \left( \text{vol } B + \int_B |u_n|^2 \, dx \right)^{1/p_n} (\text{vol } B)^{1/q_n}
\]

\[
\leq \left( 1 + \text{vol } B + \int_B |u_n|^2 \, dx \right) (1 + \text{vol } B)
\]

and

\[
\int_B |\nabla u_n| \, dx \leq (p_n \Phi_n(u_n))^{1/p_n} (\text{vol } B)^{1/q_n}
\]

\[
\leq (1 + C \sup_n p_n)(1 + \text{vol } B)
\]

Suppose for a while that

\[
\sup_n \|u_n\|_{L^2(\mathbb{R}^d)} < +\infty.
\]

Hence $\{u_n\}$ is bounded in $W^{1,1}(B)$. If for a subsequence $p_n > 2$, the $W^{1,1}$-bound can be established with the help of Jensen’s inequality.

By compactness of the embedding $W^{1,1}(B) \subset L^2(B)$, a subsequence of $\{u_n\}$ converges in $L^1(B)$ (see e.g. [33, §1.4.6, Lemma] or [1, Theorem 6.2]). By a diagonal argument, we can extract a subsequence such that $\{u_n\}$ converges strongly to some
$f \in L^1_{\text{loc}}(\mathbb{R}^d)$. W.l.o.g. $u_n \to f$ d-a.e., by extracting another subsequence, if necessary. Also, the measures $\{\partial_t u_n \, dx\}$ are vaguely bounded and hence vaguely relatively compact, see [16, Paragraphs 46.1, 46.2]. For each $1 \leq i \leq d$, we can extract a subsequence, such that $\{\partial_i u_n \, dx\}$ converges to some locally finite Radon measure $m_i$ on $\mathbb{R}^d$. By vague convergence and integration by parts,

$$
\int \varphi \, dm_i = \lim_n \int \varphi \partial_i u_n \, dx = - \lim_n \int \partial_i \varphi u_n \, dx = - \int \partial_i \varphi f \, dx,
$$

for every $\varphi \in C^\infty_0(\mathbb{R}^d)$ and every $1 \leq i \leq d$. Hence $m_i = D_i f$. Furthermore, for every $\varphi \in C^\infty_0(\mathbb{R}^d; \mathbb{R}^d)$,

$$
\frac{1}{p_n} \left| \int u_n \, \text{div} \varphi \, dx \right|^{p_n} = \frac{1}{p_n} \left| \int (\nabla u_n, \varphi) \, dx \right|^{p_n}
$$

$$
\leq \Phi_n(u_n) \times \begin{cases} \|\varphi\|_{L^{p_0}(\Omega)}^{p_0} \left( \text{vol(supp } \varphi) \right)^{p_0/p_n}, & \text{if } p_0 = 1, \\ \left( \int_{\Omega} |\varphi|^{q_n} \, dx \right)^{p_0/q_n}, & \text{if } p_0 > 1. \end{cases}
$$

Upon taking the limit, by Lebesgue’s dominated convergence theorem (since either for a subsequence $|\varphi^n| \leq 1 + |\cdot|^2$ or for a subsequence $|\varphi^n| \leq 1 + |\cdot|^{p p_0/q_0}$)

$$
\frac{1}{p_0} \left| \int f \, \text{div} \varphi \, dx \right|^{p_0} \leq \liminf_n \Phi_n(u_n) \times \begin{cases} \|\varphi\|_{L^{p_0}(\Omega)}^{p_0} \left( \text{vol(supp } \varphi) \right)^{p_0/p_0}, & \text{if } p_0 = 1, \\ \left( \int_{\Omega} |\varphi|^{p_0} \, dx \right)^{p_0/p_0}, & \text{if } p_0 > 1. \end{cases}
$$

Taking the supremum over all $\varphi$ with $\|\varphi\|_{L^{p_0}(\Omega)} \leq 1$ (if $p_0 = 1$) or with $\|\varphi\|_{L^{p_0}(\mathbb{R}^d)} \leq 1$ (if $p_0 > 1$) yields

$$
\|Df\|_{L^{p_0}(\mathbb{R}^d)} \quad \text{if } p_0 = 1,
$$

$$
\frac{1}{p_0} \left| \int |Df|^{p_0} \, dx \right| \text{ if } p_0 > 1 \leq \liminf_n \Phi_n(u_n).
$$

Suppose now that $u_n \rightharpoonup u$ weakly in $L^2(\Omega)$. This justifies (4.4). Clearly, for all $\varphi \in C_0(\mathbb{R}^d)$,

$$
\int u \varphi \, dx = \lim_n \int u_n \varphi \, dx = \int f \varphi \, dx,
$$

hence $u = f$ d-a.e. and $Du = Df$ in the sense of distributions. We are left to prove that, if $p_0 = 1$, then $u \in BV(\Omega)$, because then

$$
\|Du\|_{L^{p_0}(\mathbb{R}^d)} = \Phi(u) < +\infty.
$$

Similarly, if $p_0 > 1$ and if $u \in W^{1,p_0}(\Omega)$,

$$
\Phi(u) = \frac{1}{p_0} \int_{\Omega} |\nabla u|^{p_0} \, dx \leq \frac{1}{p_0} \int_{\mathbb{R}^d} |Du|^{p_0} \, dx < +\infty.
$$

We have that $u_n \to u$ in $L^1_{\text{loc}}(\mathbb{R}^d)$ and that $\partial_t u_n \, dx \to D_t u$ in the vague sense on $\mathbb{R}^d$. Hence by the definition of the trace (2.1), (2.4),

$$
0 = \lim_n \int_{\partial\Omega} \gamma_{p_n}(u_n) \, (\varphi, \nu) \, d\mathcal{H}^{d-1} = \int_{\partial\Omega} \gamma_{p_0}(u) \, (\varphi, \nu) \, d\mathcal{H}^{d-1}
$$

$$
\forall \varphi \in C^\infty_0(\mathbb{R}^d; \mathbb{R}^d).
$$

Let $\varphi \in C^\infty_0(\Omega; \mathbb{R}^d)$. By (2.4),

$$
\int_{\Omega} u \, \text{div} \varphi \, dx = - \int_{\Omega} \langle \varphi, d[Du] \rangle.
$$
Hence by definition, \( u \in BV(\Omega) \).

On the other hand, if \( p_0 > 1 \), we can take the the supremum over all \( \varphi \in C_0^\infty(\mathbb{R}^d, \mathbb{R}) \) such that \( \| \varphi \|_{L^{\infty}(\Omega)} \leq 1 \) to get that
\[
\int_{\partial \Omega} |\gamma_{p_0}(u)|^{p_0} \, d\mathcal{H}^{d-1} \leq \liminf_{n} \int_{\partial \Omega} |\gamma_n(u_n)|^{p_n} \, d\mathcal{H}^{d-1} = 0.
\]

Compare with [2, Lemma 3.90] and (4.6). We get that \( \gamma_{p_0}(u) = 0 \) \( \mathcal{H}^{d-1} \)-a.e. and hence \( u \in W_0^{1,p_0}(\Omega) \). Since we can repeat the steps for any subsequence of \( \{u_n\} \), we have proved (M1).

Let us prove (M2) from Definition 3.1. Let \( p_0 = 1 \) and \( u \in BV(\Omega) \). Then by Lemma 4.1 there is a sequence \( \{u_m\} \subset C_0^\infty(\Omega) \subset BV(\Omega) \) with
\[
u_m \to u \text{ in } L^2(\Omega) \text{ and } \|Du_m\|((\mathbb{R}^d) \to \|Du\|((\mathbb{R}^d) \text{ as } m \to +\infty.
\]

But by Lebesgue’s dominated convergence theorem (since for large \( n \) it holds that \( \frac{1}{p_n} |\cdot|^{p_n} \leq 1 + |\cdot|^2 \)) and (2.5) for each \( m \in \mathbb{N} \)
\[
\frac{1}{p_n} \int_{\Omega} |\nabla u_m|^{p_n} \, dx \to \int_{\Omega} |\nabla u_m| \, dx = \|D_{u_m}\|((\mathbb{R}^d) \text{ as } n \to +\infty.
\]

An application of Lemma A.1 shows that there exists a sequence \( \{m_n\}, m_n \uparrow +\infty, \) such that
\[
\lim_n \Phi_n(u_{m_n}) = \lim_n \frac{1}{p_n} \int_{\Omega} |\nabla u_n|^{p_n} \, dx = \|D_{u}\|((\mathbb{R}^d) = \Phi(u),
\]
which proves (M2). If \( p_0 > 1 \), we can repeat the above steps in the obvious manner.

Let us briefly treat the case of Neumann boundary conditions. The idea of proof is essentially the same with slightly different methods. Therefore, let \( p_0 \in [1, +\infty), \) \( p_n \in (1, +\infty), \) \( n \in \mathbb{N} \) such that \( \lim_n p_n = p_0 \). Write \( q_n := p_n/(p_n - 1), \) \( n \in \mathbb{N}, \) \( q_0 := p_0/(p_0 - 1) \) (with \( 1/0 := +\infty \)). In order to apply Theorem 3.3, we would like to prove that
\[
(4.7) \quad \Psi_n := \Psi_{p_n} \xrightarrow{M, n \to +\infty} \Psi_{p_0} =: \Psi.
\]

We need the following approximation result.

**Lemma 4.2.** Let \( u \in BV(\Omega) \cap L^2(\Omega) \). Then there exists a sequence \( \{u_k\} \subset C_0^\infty(\Omega) \cap BV(\Omega) \) such that
\[
\|u_k - u\|_{L^2(\Omega)} \to 0 \quad \text{and} \quad \|Du_k\|((\Omega) \to \|Du\|((\Omega).
\]

**Proof.** With \( L^2 \) replaced by \( L^1 \) this is well-known, see e.g. [10, Theorem 10.1.2]. Since the approximation is obtained by a mollifier, we get the strong convergence in \( L^2 \) by Hausdorff-Young inequality.

**Proof of (4.7).** We start with noting that by the Lipschitz assumption for \( \partial \Omega \), and the fact that \( W^{1,p}(\Omega) \subset BV(\Omega), \) \( p \in (1, +\infty) \), we can set any function \( u \in W^{1,p}(\Omega) \) a.e. zero outside \( \Omega \) (denoted also by \( u \)) and obtain a function \( u \in BV(\mathbb{R}^d) \), see [24, §5.4, Theorem 1].

Let us prove (M1) from Definition 3.1. Suppose that \( u_n \in L^2(\Omega) \) with
\[
\liminf_n \Phi_n(u_n) < +\infty.
\]
Suppose also that \( u_n \rightharpoonup u \) weakly in \( L^2(\Omega) \) for some \( u \). Extract a subsequence (also denoted by \( \{u_n\} \)) such that
\[
\lim_n \Phi_n(u_n) = \liminf_n \Phi_n(u_n).
\]
and

\[ C := \sup_n \Psi_n(u_n) < +\infty. \]

Hence \( u_n \in W^{1,p_n}(\Omega) \). Extend \( u_n \) by zero outside \( \Omega \), denoted also by \( u_n \). Note that \( u_n \in L^2(\mathbb{R}^d) \) and \( u_n \in BV(\mathbb{R}^d) \), but possibly \( u_n \notin W^{1,p}(\mathbb{R}^d) \).

As above we can extract a subsequence of \( \{u_n\} \) and some \( f \in BV(\Omega) \) such that \( u_n \to f \) in \( L^1_{\text{loc}}(\Omega) \) and the distributional gradient of \( u_n \) converges to some locally finite Radon measure \( m \) which is then found to be equal to \( |Df| \). In fact, \( u_n \to f \) in \( L^1(\Omega) \) by [24, Theorem 5.2.4].

The rest of the proof works exactly as above except that the test-functions in \( C_0^\infty(\mathbb{R}^d;\mathbb{R}^d) \) are replaced by test-functions in \( C_0^\infty(\Omega;\mathbb{R}^d) \) and in (4.6) we take the supremum over all \( \varphi \) such that \( \|\varphi\|_{L^q(\Omega)} \leq 1 \) instead.

The proof of (M2) from Definition 3.1 can easily be completed by the arguments of the above proof combined with Lemma 4.2 resp. the well-known approximation of \( W^{1,p}(\Omega) \)-functions by \( C_0^\infty(\Omega) \cap W^{1,p}(\Omega) \)-functions, see e.g. [24, §4.2, Theorem 3].

\[ \square \]

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Appendix A. A Diagonal Lemma

Lemma A.1. Let \( \{a_{n,m}\}_{n,m \in \mathbb{N}} \subset \mathbb{R} \) be a doubly indexed sequence of extended real numbers. Then there exists a map \( n \mapsto m(n) \) with \( m(n) \uparrow +\infty \) as \( n \to +\infty \) such that

(A.1) \[ \liminf_{n \to +\infty} a_{n,m(n)} \geq \liminf_{m \to +\infty} \left[ \liminf_{n \to +\infty} a_{n,m} \right], \]

or, equivalently

(A.2) \[ \limsup_{n \to +\infty} a_{n,m(n)} \leq \limsup_{m \to +\infty} \left[ \limsup_{n \to +\infty} a_{n,m} \right]. \]

Proof. See [12, Appendix] or [7, Lemma 1.15 et sqq.]. \[ \square \]

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