Survival of sharp $n = 0$ Landau levels in massive tilted Dirac fermions: Protection by generalized chiral symmetry

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Anomalously sharp (delta-function-like) $n = 0$ Landau level in the presence of disorder is usually considered to be a manifestation of the massless Dirac fermions in magnetic fields. This property persists even when the Dirac cone is tilted, which has been shown by Kawarabayashi et al. [Phys. Rev. B 83, 153414 (2011)] to be a consequence of a “generalized chiral symmetry”. Here we pose a question whether this property will be washed out when the tilted Dirac fermion becomes massive. Surprisingly, while the massive case with split $n = 0$ Landau levels may seem to degrade the anomalous sharpness, the levels do remain delta-function-like. This has been shown analytically in terms of the Aharonov-Casher argument extended to the massive tilted Dirac fermions. A key observation is that the conventional and generalized chiral operators are related with each other via a non-unitary transformation, with which the split, non-zero-energy $n = 0$ wave functions of the massive system can be identified as a gauge-transformed zero-mode wave functions of the massless system. This is confirmed from a numerical result for a model tight-binding system. A message is that the chiral symmetry, rather than a simpler notion of the sublattice symmetry, is essential for the robustness of the $n = 0$ Landau level, which is why the chiral symmetry remains applicable even to massive case.

I. INTRODUCTION

After the experimental discovery of graphene,1 fascination with the massless Dirac fermions has become one of the central interests in condensed matter physics.2 Physics of zero-gap semiconductors has actually a long history of studies, started by a theoretical work by Wallace and is now described in condensed-matter textbooks.2 There are various spin-offs, among which is the topological insulator with the quantized spin Hall effect, where the topological property of Dirac fermions plays a fundamental role.2–4 In the context of zero-gap semiconductors, the first topological insulator HgTe-CdTe can be made to have a small negative mass.5–7 Another important class of the Dirac fermions is an organic material, $\alpha$-(BEDT-TTF)$_2$I$_3$, where the Dirac cone dispersion is substantially tilted. In a broader context, anisotropic superconductors with d-wave symmetry has a Dirac cone in the dispersion for the Bogoliubov quasiparticle, which serves as another Dirac fermions in two dimensions.8,9,11 Quantum phase transitions of fermions associated with gap closing and opening can be described by a Dirac fermion in terms of reversing the sign of the mass.12 In this sense we have various realizations in diverse quantum phases, such as chiral spin states, flux phases, nodal fermions.

While the massless Dirac cone in graphene is related to the honeycomb lattice structure, the gap closing itself can be analyzed more generally in terms of the level crossing in quantum mechanics. According to the von-Neumann Wigner theorem, a degeneracy point has generically codimension three.15–17 This indicates that the existence of massless Dirac cones in three spatial dimensions is rather natural. Conversely, in two dimensions a Dirac cone is an accident unless some symmetry exists. The chiral symmetry is often evoked for graphene as represented by the honeycomb lattice, which is usually regarded as nothing but the sublattice symmetry against sign change of the wave function only one of the sublattices in a bipartite lattice structure. Hence it is usually considered that the reason why graphene realizes the massless Dirac fermions is because the material has a honeycomb structure. In two dimensional systems with a chiral symmetry, one can also prove the fermion doubling theorem as a two-dimensional analogue of the Nielsen-Ninomiya theorem in four dimensions,19–21 which guarantees the existence of an even number of massless Dirac fermions. It also brings a supersymmetric (SUSY) structure in the one-particle Hamiltonian.22,23 In the case of graphene this corresponds to the two Dirac cones at valleys K and K’. Thus in the physics of graphene the chiral symmetry is indeed important.15–22,25–28 For instance, in case of the d-wave superconductor, the chiral symmetry translates into the time-reversal symmetry in the Bogoliubov Hamiltonian,21,27 which protects the existence of nodes in the gap.

Now, in two dimensions the Dirac cone is in general tilted, as in the case of the organic material, where the conventional chiral symmetry is broken.28 One may then wonder if the existence of a Dirac cone itself suffices for the topological properties even when the chiral symmetry is apparently absent. The present authors have revealed that the notion of the chiral symmetry can actually be extended to accommodate the tilted cones so that the tilted cones has a symmetry against the “generalized chiral operator”, and demonstrated some of its consequences both analytically and numerically. Most importantly, if
we look at $n = 0$ Landau level (right at the Dirac point) in magnetic fields, its density of states remains delta-function-like even in the presence of disorder, while a usual wisdom was this property is a manifestation of the vertical Dirac cones.

In the present paper we pose a question whether this property will be washed out as soon as the tilted Dirac fermion becomes massive. While this question may seem too detailed, it is actually not so, since from this we can clarify an important query: are the existence of zero-modes and the chiral symmetry one and the same? While for a vertical cone, they are obviously the same, we have a splitting of the $n = 0$ Landau level with nonzero energies in a massive case, one might imagine that they are different in this case. Surprisingly, we shall find that the levels, now split, do remain delta-function-like. This has been shown analytically in terms of the Aharonov-Casher argument, which is known to construct wave functions in the zero-mode Landau level in vertical cones, and is here extended to the massive tilted Dirac fermions. A key observation is that the conventional and generalized chiral operators are related with each other via a non-unitary transformation, with which we can identify the split, nonzero-energy wave functions of the massive system as a gauge-transformed zero-mode wave functions of the massless ones. The anomalously sharp Landau level is confirmed from a numerical result for a model tight-binding system.

We can visualize the point as follows. While the conventional chiral symmetry dictates that each wave function in the $n = 0$ Landau level has nonzero amplitudes only on A sublattice (or B sublattice), the wave function for tilted Dirac fermions is not an eigenstate of the sublattice symmetry so that it has amplitudes on both of the two sublattices (or two components of the spinor). This may seem to suggest that the sharpness of the $n = 0$ Landau level is degraded for tilted Dirac fermions when we make the fermion massive by introducing a staggered potential over A and B sublattices. If this is the case, the tilted cone should differ from the vertical cone, and the sharpness of the $n = 0$ Landau level will be affected by the staggered potential. The present result shows that this is not the case. Thus a message of the present work is that (i) the (generalized) chiral symmetry rather than a simpler notion of the sublattice symmetry is essential for the robustness of the $n = 0$ Landau level, which is why (ii) the chiral symmetry applies even to massive case.

In this paper we first present in section II a numerical result for a lattice model that has tilted Dirac cones, where we find that the anomalous sharpness of the $n = 0$ Landau levels is surprisingly unaffected by the introduction of the mass term for the case of the spatially smooth (long-range) disorder. We further find numerically that, for spatially uncorrelated (short-range) disorder, the anomalous sharpness of the $n = 0$ Landau levels is unexpectedly recovered as the staggered potential is increased. This is just the opposite of the case of massless cones, where the sharpness of the $n = 0$ Landau level is degraded for spatially uncorrelated disorder. The recovery of the sharpness of the $n = 0$ Landau level for the uncorrelated disorder has also been reported for the shifted Dirac cones. In the present case, the energies of the two $n = 0$ Landau levels associated with the two valleys are split by the mass term (i.e. the staggered potential), although the Dirac cones themselves are not shifted in energy. The present recovery of the sharpness thus indicates that the disorder-induced mixing between the split $n = 0$ Landau levels is significantly suppressed by increasing the energy split introduced by the mass term.

We then present the main, analytic part, which provides a solution for the puzzling numerical result. Namely, in order to understand the origin of the anomalous sharpness of the $n = 0$ Landau level for a massive Dirac fermions, we develop a general effective theory, first for the massless case in section III and for the massive case in section IV, and find a simple algebraic relationship (which turns out to be non-unitary) between the generalized chiral operator and the conventional chiral operator. This enables us to discuss quantitatively the effect of the staggered potential on the anomalous sharpness of the $n = 0$ Landau levels for tilted Dirac fermions with a disorder that respects the generalized chiral symmetry. We then show that the $n = 0$ Landau levels of the massive Dirac fermions are still the eigenstate of the generalized chiral operator, where the robustness for the random gauge field is again shown analytically by the argument by Aharonov and Casher. Section V is devoted to summary.

II. NUMERICAL RESULT FOR MASSIVE TILTED DIRAC FERMIONS

To examine the $n = 0$ Landau level for massive and tilted Dirac fermions, we perform a numerical analysis based on the tight-binding lattice model on a two-dimensional square lattice having the next nearest-neighbor ($t'$) hopping as well as the nearest-neighbor one ($t$). The tight-binding Hamiltonian is given by

$$H_{TB} = \sum_{\mathbf{r}} \left[ -t c_{\mathbf{r}+\mathbf{y}}^\dagger c_{\mathbf{r}} + (-1)^{x+y} t c_{\mathbf{r}+\mathbf{x}}^\dagger c_{\mathbf{r}} + \text{h.c.} ight. $$

$$+ t' \left( c_{\mathbf{r}+\mathbf{x}+\mathbf{y}}^\dagger c_{\mathbf{r}} + c_{\mathbf{r}+\mathbf{x}}^\dagger c_{\mathbf{r}+\mathbf{y}} + \text{h.c.} \right),$$

where the lattice positions are denoted by $\mathbf{r} = x\mathbf{x} + y\mathbf{y}$ with the unit vector $\mathbf{x}(\mathbf{y})$ in the $(x(y))$-direction and the length in units of the lattice constant of the square lattice. This model has a pair of Dirac cones at $E = 0$ and $(k_x, k_y) = (0, \pm \pi/2)$ which are tilted when $t' \neq 0$. Focusing on the valence and conduction bands, we can consider a two-band Hamiltonian in a $2 \times 2$ matrix form. The effective low-energy Hamiltonian $H$ around the Dirac cones can then be expressed as

$$H = (X^0 \sigma_0 + X \cdot \sigma) \delta k_x + (Y^0 \sigma_0 + Y \cdot \sigma) \delta k_y.$$
Here, \( \delta k = k - k_0 \) is a deviation of the momentum from the gapless point \( k_0 \) and \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \equiv (\sigma_x, \sigma_y, \sigma_z) \) denotes the Pauli matrices and \( \sigma_0 \) a two-dimensional unit matrix. The parameters are \( 4X = (0, 2t, 0) \) and \( 4Y = (\mp 2t, 0, 0) \), and \( (X^0, Y^0) = (0, \pm 4t') \) that makes the Dirac cone tilted.

Now we make the fermions massive by introducing a mass term. This can be trivially done by introducing an additional staggered potential, and the massive Hamiltonian \( H_{TB}(m) \) is given by

\[
H_{TB}(m) = H_{TB} + mc^2 \sum_r (-1)^{x+y} c_r^\dagger c_r, \tag{1}
\]

where A (B) sublattice site energies are elevated (lowered). The effective low-energy Hamiltonian becomes

\[
H(m) = H + mc^2 \sigma_z.
\]

The term \( mc^2 \sigma_z \) that makes the Dirac fermions massive opens a gap at the Dirac point. In the case of usual vertical Dirac fermions, the mass term can be expressed in terms of the conventional chiral operator \( \Gamma \propto \sigma_z \). The chiral operator is defined as an operator that anti-commutes with the effective Hamiltonian \( H \), which, for the vertical Dirac cone with \( X^0 = Y^0 = 0 \), is given by \( \Gamma = \hat{n}_0 \cdot \sigma \) with \( \hat{n} \equiv X \times Y |X \times Y|^{28,30} \). For the present lattice model, the conventional chiral operator is simply \( \Gamma = \pm \sigma_z \), where the plus (minus) sign applies to the valley at \( (k_x, k_y) = (0, \pi/2) \) \((0, -\pi/2)\). The Hamiltonian \( H(m) \) can then be expressed with \( \Gamma \) as

\[
H(m) = H + mc^2 \Gamma \quad \text{for} \quad k = (0, \pi/2)
\]

\[
H - mc^2 \Gamma \quad \text{for} \quad k = (0, -\pi/2).
\]

We then apply an external magnetic field to carry out exact numerical diagonalization for a finite system in the presence of disorder. The disorder is introduced here as a random component, \( \delta \phi(r) \), in the magnetic flux \( \phi(r) = \phi + \delta \phi(r) \) piercing each square plaquette, where \( \phi \) denotes the uniform component. The random component \( \delta \phi(r) \) is assumed to obey a Gaussian distribution with the variance \( \sigma \) and with a spatial correlation as

\[
\langle \delta \phi(r) \delta \phi(r') \rangle = \sigma^2 \exp(-|r - r'|^2/4d^2),
\]

where \( d \) is the correlation length. Since the disorder in gauge degrees of freedom respects the generalized chiral symmetry, a randomness in the magnetic field generally respects the generalized chiral symmetry.

In Fig. 1 we show the density of states for the case of the spatially correlated disorder \( (d = 1.5) \) in units of the lattice constant for the system with tilted Dirac cones. For comparison, we also display the result for the vertical case. We can immediately notice that the introduction of the mass term \((mc^2 \sigma_z)\) does not affect the anomalous sharpness of the split \( n = 0 \) Landau levels even for the tilted cones as in the vertical cones. Since the other Landau levels (for e.g. \( n = \pm 1 \)) are broadened, the \( n = 0 \) levels do stand out.

A further surprise occurs when we examine the robustness of the split \( n = 0 \) Landau levels against the spatially uncorrelated disorder \((d/a = 0)\). For the massless \((m = 0)\) case, this degrades the sharp \( n = 0 \) Landau levels due to the inter-valley scattering. However, we can see in Fig. 2 that the anomalous sharpness is rather recovered as the mass is made heavier with the level splitting becoming wider. In the massive case, each \( n = 0 \) Landau level is associated with one of the two Dirac cones. The present result indicates that the mixing between the Dirac cones is effectively suppressed when the \( n = 0 \) Landau levels are split by the staggered potential. This reminds us of our previous work, where we have introduced a model in which the two Dirac cones remain massless but shifted in energy with a complex hopping. There, the robustness is recovered even for short-range disorder. The present result indicates that a similar suppression of the mixing is at work, where the energy offset comes not from the shifted cones but from a mass gap.

### III. TILTED MASSLESS DIRAC FERMIONS

#### A. A general formulation

To understand the robustness of the zero modes for massive and tilted Dirac fermions, let us first summarize a general effective theory for a massless and tilted Dirac fermions from the viewpoint of the generalized chiral symmetry. For this purpose we can introduce a compact four-dimensional notations, as employed by several authors. As in the previous section, a general form...
for the 2 $\times$ 2 effective Hamiltonian for the valence and conduction bands is $\sigma_0, \sigma_1, \sigma_2$ and $\sigma_3$ as

$$H_g(k) = \sigma_0 R_0(k) + \sigma \cdot R(k),$$

where $'R(k) = (R_1(k), R_2(k), R_3(k))$, $R_i \in \mathbb{R}$. The energy bands are given by

$$E_g(k) = R_0(k) \pm |R(k)|,$$

where $|R| = \sqrt{R_1^2 + R_2^2 + R_3^2}$, with the energy gap, $E_g(k)$, for each momentum $k$ being $E_g(k) = 2|R(k)|$. We have a semiconductor under a condition,

$$E_-(k_v) \leq E_+(k_c),$$

where $k_v(k_c)$ are the momenta in the valence (conduction) band.

In the case of a zero-gap semiconductor, the energy gap vanishes at some momentum $k_0$. Expanding the Hamiltonian around $k_0$, we have an effective Hamiltonian ($H_g \approx H$) as

$$H = (X^0 \sigma_0 + X \cdot \sigma) \delta k_x + (Y^0 \sigma_0 + Y \cdot \sigma) \delta k_y,$$

where $\delta k = k - k_0$, $X^0 = \partial_{k_x} R_0(k_0)$, $Y^0 = \partial_{k_x} R_0(k_0)$ and the three dimensional vectors $X = (X^1, X^2, X^3)$ and $Y = (Y^1, Y^2, Y^3)$ are defined by $X = \partial_{k_x} R|_{k_0}$, $Y = \partial_{k_y} R|_{k_0}$. Inclusion of $X_0 = -X^0$ and $Y_0 = -Y^0$ induce tilting of the Dirac cones. Note that for the case where $(X^0, Y^0) \neq 0$, the Dirac cones are tilted. When $(X^0, Y^0) = 0$, the Dirac cones can be anisotropic but they are vertical.

With an effective momentum around the gapless point as $p = \hbar \delta k$, it is expressed as

$$H = \hbar^{-1} (\sigma_{p} X^\mu \sigma_{p} Y^\nu) p,$$

where $p = (p_x, p_y)$, and a summation over repeated indices $\mu = 0, 1, 2, 3$ is assumed.

In the following, we introduced a four-dimensional notation to simplify the calculation. For this purpose, let us introduce, on top of the “contravariant” four-dimensional vectors $'X = (X^0, X^1, X^2, X^3)$ and $'Y = (Y^0, Y^1, Y^2, Y^3)$, the conjugated (or “covariant”) vectors $X$ and $Y$ defined as

$$X = (X_0, X_1, X_2, X_3) = i g = (-X^0, X^1, X^2, X^3),$$

where $g = \text{diag}(-1, 1, 1, 1)$ is a metric. Now we have a simple identity (see appendix $[3]$),

$$(\sigma_{\nu} X^\mu)(\sigma_{\nu} Y^\nu) = i Y X \sigma_0 - i \mu \sigma,$$

where $\eta = X^0 Y - Y^0 X$. Note that, while we have $XY = X^\nu Y_\mu = -X^0 Y_0 + X \cdot Y = Y X$, $n$ is anti-symmetric against $X \leftrightarrow Y$. Its norm becomes (see Appendix $[3]$)

$$n^2 = (\bar{X} X)(\bar{Y} Y) - (\bar{Y} X)(\bar{X} Y) \equiv (\eta c)^4,$$

where the Fermi velocity of the Dirac fermions, $c$, is defined. When $n^2 > 0$, the velocity $c$ is real.

By introducing the covariant notation, the discussion becomes transparent. For the usual (vertical) Dirac cones, it is known that considering a squared Hamiltonian, $H^2$, makes the analysis transparent. In the present case of tilted cones, this has to be modified. We can instead note that it is useful to define a Hamiltonian conjugate to Eq. (2) as

$$\bar{H} = \hbar^{-1} (\bar{X} \sigma_{\mu} \bar{Y} \sigma_{\mu}) \bar{p} = H - 2H_0,$$

where

$$H_0 = \hbar^{-1} \sigma_0(X^0, Y^0)p.$$ Now we can consider a product, $\bar{H} H$, as a “contraction” in the present four-dimensional representation. The expression can be put in a form,

$$\bar{H} H = \hbar^{-2} p^l G p,$$

where $G$ is a $4 \times 4$ matrix composed of $2 \times 2$ Pauli matrices and is evaluated, with the formula $[3]$, as the

$$G = \left( \begin{array}{cc} \bar{X} \sigma_{\mu} \bar{Y} \sigma_{\mu} & \bar{X} \sigma_{\nu} \bar{Y} \sigma_{\nu} \\ \bar{Y} \sigma_{\nu} - i \mu \sigma & \bar{Y} Y \sigma_0 \\ \end{array} \right).$$

The determinant of $G$ vanishes, since its rank is two, which can be confirmed directly by evaluating the determinant. We then have

$$\bar{H} H = c^2 p^l \Xi p \sigma_0,$$

where we have used $[p_x, p_y] = 0$ and we can note that $\det \Xi = 1$. 

![FIG. 2. For the spatially uncorrelated disorder ($d = 0$), the density of states for the lattice model having tilted Dirac cones is plotted against the mass (staggered potential) for the same magnetic field and the amplitude of disorder as in Fig. 1.](image)
From the Schrödinger equation, $H \Psi = E \Psi$, and Eq.\[4\], we have

$$
\hat{H} \Psi = E(H - 2H_0) \Psi = (E^2 - 2E H_0) \Psi,
$$
which reduces to

$$
\sigma_0 [c^2 \hbar^2 \hat{\mathbf{p}}^2 + 2(E/h)(X_0, Y_0) \mathbf{p}] \Psi = E^2 \Psi.
$$

By completing the square, we have (detail in the appendix) a simple, bilinear formula,

$$
c^2 \hbar \hat{\mathbf{p}}_E \cdot \mathbf{p}_E \sigma_0 \Psi = E^2 \Psi,
$$

where

$$
c_r = c \left[ \frac{\text{Re} \mathbf{n}^2}{(\text{Re} \mathbf{n})^2} \right]^{1/2} = \frac{c}{\cosh q},
$$

$$
\mathbf{p}_E = \mathbf{p} + \Delta \mathbf{p}_E,
$$

$$
\Delta \mathbf{p}_E = E \frac{1}{c^2 \hbar} \Xi^{-1} \left( \begin{array}{c} X_0 \\ Y_0 \end{array} \right).
$$

This implies the constant energy curve is an ellipse centered at $\Delta \mathbf{p}_E$. (See Fig.\[3\] and also Appendices.) The role of the parameter $q$ appearing in the renormalization factor for the velocity $c$ will become apparent when we discuss the relationship between the generalized chiral operator and the conventional chiral operator $\Gamma$ in section IV.

### B. Landau levels and the generalized chiral operator

Having formulated the case in zero magnetic field, let us move on to the Landau states when we apply an external magnetic field for the tilted Dirac fermion. In terms of the dynamical momentum $\pi_\mu$ with $\mu = x, y$,

$$
\pi_\mu = p_\mu - eA_\mu,
$$

$$
p_\mu = -i \hbar \partial_\mu,
$$

where $e$ is an elementary charge and $A_\mu$ a vector potential which describes a magnetic field perpendicular to the two-dimensional system as

$$
\mathbf{B} = \partial_x A_y - \partial_y A_x.
$$

The dynamical momentum satisfies

$$
[\pi_x, \pi_y] = i \hbar \mathbf{B} = i (\hbar/cB)^2,
$$

where $\ell_B = \sqrt{\hbar/eB}$ is the magnetic length. We may choose $eB > 0$ without loss of generality. With a substitution $\mathbf{p} \rightarrow \pi = \mathbf{p} - e\mathbf{A}$ we have a Hamiltonian,

$$
\hat{H} = \hbar^{-1} (\sigma_\mu X^\mu, \sigma_\nu Y^\nu) \pi,
$$

and its conjugate,

$$
\hat{\mathbf{H}} = \hbar^{-1} (\sigma_\mu X^\mu, \sigma_\nu Y^\nu) \pi = \hbar^{-1} (\sigma_0 X_0, \sigma_0 Y_0) \pi.
$$

Since $\pi_x$ and $\pi_y$ no longer commute in a magnetic field, we have an extra term proportional to $\mathbf{n} \cdot \sigma$ for $\hat{\mathbf{H}}$ as

$$
\hat{\mathbf{H}} = \hbar^{-2} \pi G \pi
$$

$$
= c^2 \pi \Xi \sigma_0 + i \hbar^{-2} \mathbf{n} \cdot \sigma \pi \pi
$$

$$
= c^2 \pi \Xi \sigma_0 - \ell_B^2 \mathbf{n} \cdot \sigma.
$$

From the Schrödinger equation, $H \Psi = E \Psi$, and the relation above, we get

$$
\begin{bmatrix} c^2 \pi \Xi \pi + \frac{2E}{\hbar} (X_0, Y_0) \end{bmatrix} \sigma_0 \Psi - \ell_B^2 \mathbf{n} \cdot \sigma \Psi = E^2 \Psi. \tag{8}
$$

We can readily complete the square to arrive at

$$
c^2 \pi \Xi \pi - \ell_B^2 \mathbf{n} \cdot \sigma \Psi = E^2 \Psi,
$$

where $\pi_\mathbf{E} = \pi + \Delta \mathbf{p}_E$. Here $c_\mathbf{r}$ is defined by Eq.\[4\] and we have introduced a generalized chiral operator $\gamma$ by

$$
\gamma = \frac{\mathbf{n} \cdot \sigma}{(\hbar c)^2},
$$

which has eigenvalues $\pm 1$ because $\text{Tr} \gamma = 0$, and

$$
\det \gamma = -n^2/(\hbar c)^4 = -1.
$$

However, it is not hermitian in general.

With the right-eigenstates $|\pm\rangle$ of $\gamma$,

$$
\gamma |\pm\rangle = \pm |\pm\rangle,
$$

the wave function is expressed as $\Psi_\pm = |\pm\rangle \psi_\pm$. Then the Schrödinger equation is reduced to a scalar equation,

$$
c^2 \pi \Xi \pi - (\hbar/eB)^2 \gamma |\pm\rangle = E^2 |\pm\rangle.
$$

If we note that the first term, $c^2 \pi \Xi \pi$, may be mathematically replaced by a Hamiltonian for anisotropic fermions with a parabolic dispersion having an effective mass $m^*$ in a magnetic field (Appendix B), we can introduce a single component Landau wave function $\psi_\mathbf{n}$, which satisfies

$$
\begin{bmatrix} 1/(2m^*) \Xi \pi \end{bmatrix} \mathbf{n} \mathbf{C} \left( n + \frac{1}{2} \right) \psi_\mathbf{n} = \hbar \omega_C \left( n + \frac{1}{2} \right) \psi_\mathbf{n},
$$

where the effective cyclotron frequency is

$$
\omega_C = \frac{eB}{m^*} = 2c^2 eB.
$$

We have then a spectrum, $E^2 = \hbar \omega_C \left[ (n + \frac{1}{2}) \mp \frac{1}{2} \right]$, i.e., the Dirac Landau levels,

$$
E_n = \pm c_\mathbf{r} \sqrt{2\hbar eB |n|}, \quad n = 0, 1, 2, \ldots.
$$

Note that the $n = 0$ Landau level is given by the eigenstate of $\gamma$ with the eigenvalue $+1$. 

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FIG. 3. A tilted Dirac dispersion and its cross section (an ellipse) with a constant-energy plane. The center of the ellipse is given by $\Delta \mathbf{p}_E$. 
C. Generalized chiral symmetry

Let us here discuss the generalized chiral operator \( \gamma = n \cdot \sigma / (hc)^2 \) defined in section [III B] assuming that \( c \) is real. Since \( \gamma \) is anti-symmetric against \( X \leftrightarrow Y \), we have from Eq. (3)

\[
\begin{align*}
2i(hc)^2 \gamma &= \bar{x}y - y\bar{x}, \\
2i(hc)^2 \gamma^\dagger &= x\bar{y} - \bar{y}x, \\
\end{align*}
\]

where \( x = x^\dagger = \sigma_\mu X^\mu, \bar{x} = \bar{x}^\dagger = \bar{\sigma}^\mu \sigma_\mu, \) etc. Since the Hamiltonian can be expressed as \( H = x\pi_x + y\pi_y \), we obtain

\[
\begin{align*}
2i(hc)^2 \gamma &= (x\bar{y} - y\bar{x})\pi_x + (y\bar{x} - \bar{y}x)\pi_y, \\
2i(hc)^2 \gamma^\dagger &= (x\bar{y} - y\bar{x})\pi_x + (y\bar{x} - \bar{y}x)\pi_y. \\
\end{align*}
\]

Since \( \bar{x}x = X^\mu X_\mu \sigma_0 \) commutes with \( y \) and \( \bar{y}y \) commutes with \( x, \gamma \) and \( H \) has an anti-commutation relation defined as

\[
\{H, \gamma\}_H = H\gamma + \gamma^\dagger H = 0,
\]

which we have called the generalized chiral symmetry.\(^{28}\)

Note again that

\[
\begin{align*}
\text{Tr} \gamma &= 0, \quad \det \gamma = -1, \\
\gamma^2 &= (\gamma^\dagger)^2 = \sigma_0, \quad \gamma^\dagger \neq \gamma.
\end{align*}
\]

The generalized chiral symmetry is essential to show that the zero modes are generally eigenstates of the generalized chiral operator.\(^{28}\)

D. Robust zero modes

Now let us focus on the zero modes (zero energy states). There is a long history of study of the zero modes in massless Dirac fermions, notably the well-known work of Aharonov and Casher.\(^{20,28,35}\) For \( E = 0 \) states, the Schrödinger equation \( H\Psi = 0 \) reduces to

\[
c^2 [\pi^\dagger \Xi \pi - (hcB)^2 \gamma] \Psi = 0.
\]

If we take the eigenstates, \( |\chi_+\rangle \), of the generalized chiral operator with the eigenvalue +1 with \( \Psi = |\chi_+\rangle \psi_+, \psi_+ \) satisfies

\[
\left[ \pi_x, \pi_y \right] \Xi \begin{pmatrix} \pi_x \\ \pi_y \end{pmatrix} + i[\pi_x, \pi_y] \psi_+ = 0,
\]

since \([\pi_x, \pi_y] = i(hcB)^2\). The matrix \( \Xi \), being real symmetric, can be diagonalized with an orthogonal matrix \( V_\Xi \) as

\[
\Xi = V_\Xi \text{diag} (\xi_1, \xi_2) V_\Xi^\dagger, \quad (10)
\]

where \( V_\Xi V_\Xi^\dagger = \sigma_0, \xi_1 > 0, \xi_2 > 0, \) and \( \xi_1 \xi_2 = \det \Xi = 1 \). Here we have assumed \( \det V_\Xi = 1 \) without loss of generality, since, if \( \det V_\Xi = -1, V_\Xi \sigma_x \) diagonalizes \( \Xi \) with \( \xi_1 \) and \( \xi_2 \) interchanged. Then we can define a new momentum,

\[
\Pi = \begin{pmatrix} \sqrt{\xi_1} & 0 \\ 0 & \sqrt{\xi_2} \end{pmatrix} V_\Xi \pi,
\]

which preserves the commutator,

\[
\left[ \Pi_1, \Pi_2 \right] = \sqrt{\xi_1 \xi_2} \sum_{i,j} V_{\Xi,11} V_{\Xi,2j}[\pi_i, \pi_j]
\]

\[
= (V_{\Xi,11} V_{\Xi,22} - V_{\Xi,12} V_{\Xi,21})[\pi_x, \pi_y]
\]

\[
= \det V_\Xi[\pi_x, \pi_y]
\]

\[
= [\pi_x, \pi_y].
\]

The zero-mode equation reads

\[
D^\dagger D \psi_+ = 0,
\]

where

\[
D = \Pi_1 + i\Pi_2.
\]

Since \( D^\dagger D \) is semi-positive definite, we have

\[
D \psi_+ = 0.
\]

Then noting that this is a first order differential equation, we have an explicit solution (which is given below) as the discussion by Aharonov-Casher.\(^{28,35}\) This guarantees the stability of the zero modes. Note that this argument is only possible for a real \( c^2 \), which is explicitly indicates that the index theorem for the elliptic operator is indeed relevant.\(^{28,31}\)

IV. MASSIVE AND TILTED DIRAC FERMIONS

A. General properties

Now we come to the massive case in question. Our motivation is to clarify the origin of the anomalous robustness of the split \( n = 0 \) Landau levels for the massive and tilted Dirac fermions. The generalized chiral operator \( \gamma \) introduced in section IIB can be expressed with the normalized vector \( \hat{n} = n/\Delta \), which puts Eq. (3) into

\[
\gamma = \hat{n} \cdot \sigma,
\]

where

\[
\Delta = \sqrt{n^2} = (\text{Re}n)^2 - (\text{Im}n)^2 = (hc)^2,
\]

is the norm of the vector \( n \). Recall in Eq. (3) that the real part and the imaginary part of \( n \) are given by \( \text{Re}n = X \times Y \) and \( \text{Im}n = \eta = (X_0 Y - Y_0 X) \), and therefore they are orthogonal with each other \( (\text{Re}n) \cdot (\text{Im}n) = 0 \). The conventional chiral operator \( \Gamma \) is expressed in a similar form in terms of a real vector \( \hat{n}_0 = \text{Re}n/\Delta_0 \) with \( \Delta_0 = |\text{Re}n| = |X \times Y| \) being the norm of \( \text{Re}n \) as

\[
\Gamma = \hat{n}_0 \cdot \sigma.
\]
We can then relate $\Gamma$ with the generalized chiral operator $\gamma$ as

$$
\gamma = (\hat{n}_0 \cdot \sigma)(\hat{n}_0 \cdot \sigma)(\hat{n} \cdot \sigma)
= \Gamma [\hat{n}_0 \cdot \hat{n}] \sigma_0 + i \sigma \cdot (\hat{n}_0 \times \hat{n})
= \Gamma [\hat{n}_0 \cdot \text{Re} \hat{n}] \sigma_0 - \sigma \cdot (\hat{n}_0 \times \text{Im} \hat{n})
= \Gamma \Delta_0 / \Delta - \sigma \cdot (\text{Re} \hat{n} \times \text{Im} \hat{n}) / (\Delta_0 \Delta),
$$

where we have inserted $\Gamma^2 = 1$ in the first line, used a formula above Eq. (A1) in the second line and the fact that $\hat{n}_0 \perp \text{Im} \hat{n}$ in the third line. Since $|\text{Re} \hat{n} \times \text{Im} \hat{n}| = |\text{Re} \hat{n}| |\text{Im} \hat{n}| = \Delta_0 \sqrt{\Delta^2 - \Delta^2}$, we end up with a compact expression,

$$
\gamma = \Gamma (\cosh q - \sigma \cdot \sigma \sinh q) = \Gamma e^{-q \tau \cdot \sigma},
$$

where the parameter $q$ is defined in Eq. (12) or equivalently $\tanh q = \sqrt{\Delta^2 - \Delta^2 / \Delta} = |\eta| / |X \times Y|$, and the unit vector $\tau$ is given by

$$
\tau = \frac{\text{Re} \hat{n} \times \text{Im} \hat{n}}{|\text{Re} \hat{n} \times \text{Im} \hat{n}|} = (X \times Y) \times \eta / |(X \times Y) \times \eta|.
$$

Note that the parameter $q$ is real as long as $\Delta^2 \geq 0$, which is equivalent to the ellipticity of the Hamiltonian (2) where the index theorem is relevant.

We can also note that $\{ \Gamma, \tau \cdot \sigma \} = 0$, since $\tau$ is normal to $X \times Y$, we have a suggestive representation,

$$
\gamma = \Gamma e^{-q \tau \cdot \sigma} = e^{q \tau \cdot \sigma} \Gamma = e^{q \tau \cdot \sigma^2 / 2} \Gamma e^{-q \tau \cdot \sigma^2 / 2}.
$$

This immediately implies that the eigenstates $|\pm\rangle$ of the conventional (hermitian) chiral operator $\Gamma$ (with $\Gamma|\pm\rangle = \pm|\pm\rangle$) can be related to the right-eigenstates $|\chi_{\pm}\rangle$ of the generalized (non-hermitian) chiral operator as

$$
|\chi_{\pm}\rangle = \frac{1}{\sqrt{\cosh q}} e^{q \tau \cdot \sigma^2 / 2}|\pm\rangle.
$$

The normalization factor $1/\sqrt{\cosh q}$ is introduced, since $\{ + \exp(q \tau \cdot \sigma) \} = -\{ - \exp(q \tau \cdot \sigma) \} = \cosh q$. On the other hand, we can readily verify a relation,

$$
\gamma \Gamma = \Gamma, \quad (11)
$$

which guarantees that

$$
\langle \chi_+ | \Gamma | \chi_- \rangle = \langle \chi_- | \Gamma | \chi_+ \rangle = 0.
$$

The diagonal matrix elements are evaluated as

$$
\langle \chi_{\pm} | \Gamma | \chi_{\pm} \rangle = \pm \frac{1}{\cosh q} \frac{\Delta}{|X \times Y|}.
$$

### B. Symmetry breaking and robust zero modes

The relations obtained above are useful in considering the effects of the mass term (i.e., staggered field proportional to $\Gamma$), which breaks the generalized chiral symmetry. For the vertical Dirac cones, the effect of the staggered potential is rather trivial, since the states in the $n = 0$ Landau level are also eigenstates of the chiral operator $\Gamma$ and their energies are simply shifted according to their eigenvalues of $\Gamma$. By sharp contrast, tilted Dirac cones have the states in the $n = 0$ Landau level that reside on both of the sub-lattices, and are not the eigenstates of $\Gamma$. This is why the effects of the staggered potential becomes nontrivial for tilted cone. We are now going to do is to employ the representation of $\Gamma$ in terms of the generalized chiral bases to explore the effects of the staggered potential on the $n = 0$ Landau level. Essentially, we shall show that the states in the $n = 0$ Landau level remain the eigenstates of $\gamma$ even in the presence of the staggered potential, which should exhibit an anomalous sharpness against the disorder.

For a typical source of mass gap, we can again introduce a chiral symmetry breaking term $mc^2 \Gamma$ in the Hamiltonian as

$$
H(m) = H + mc^2 \Gamma.
$$

For the massless, tilted cones, we have shown that it is usful to consider $\bar{\hat{H}} \hat{H}$. Let us extend this argument to the massive case by considering $\bar{\hat{H}}(m)H(m)$. Amazingly, we can simplify this into

$$
\bar{\hat{H}}(m)H(m) = (\bar{\hat{H}} + mc^2 \Gamma)(H + mc^2 \Gamma)
= \bar{\hat{H}}H + m^2 c^4,
$$

where cross terms between $\bar{\hat{H}}(m)$ and $H(m)$ vanish because the unperturbed Hamiltonian without tilting, $H_C = H - H_0$, is chiral symmetric with $\{ H_C, \Gamma \} = 0$. Now, following the case without tilting, let us assume that the $n = 0$ Landau state to be $\Psi^0 = |\chi_+\rangle\psi^m_+$. Then the Schrödinger equation, $H(m)\Psi^m = E\Psi^m$, implies

$$
\bar{\hat{H}}(m)H(m)\Psi^m = (E^2 - 2EH_0)\Psi^m
= c^2 \left[ \pi^T \Xi \pi - (h/\ell_B)^2 \chi + m^2 c^2 \right] \psi^m_+,
$$

which leads to

$$
c^2 \left[ \pi^T E \Xi \pi_\perp - (h/\ell_B)^2 + m^2 c^2 \right] \psi^m_+ = E^2 \psi^m_+.
$$

It is clear from this equation that the symmetry breaking term $mc^2 \Gamma$ indeed opens a gap $\pm mcc$, in the absence of a magnetic field. We can cast this into

$$
c^2 \left[ D_E^\dagger D_E + m^2 c^2 \right] \psi^m_+ = E^2 \psi^m_+,
$$

where

$$
D_E = \Xi_1, E + i \Xi_2, E,
\Xi_E = \left( \begin{array}{cc} \sqrt{\xi_1} & 0 \\ 0 & \sqrt{\xi_2} \end{array} \right) V_\Xi \pi_E,
$$

where $\xi_1, \xi_2$ are given in Eq. (10). Then $D_E^\dagger D_E$ is semi-positive definite, the wave function in the $n = 0$ Landau level is specified by

$$
D_E^\dagger \psi^m_+ = 0,
$$

and
which has an energy

\[ E = mc_r = mc^2 / \cosh q, \]

where we have used Eq. (11), and chosen the positive sign for the energy since it should tend to \( +mc^2 \) when the tilting becomes zero (See Fig. 4). As far as the velocity \( c \) is real, \( D \) and \( D_E \) (also \( \pi \) and \( \pi_E \) ) are simply related through a shift in the momentum by \( \Delta p_E \), which indicates \( \psi_+ \) and \( \psi^m_+ \) are also related via a gauge transformation,

\[ \psi^m_+ = e^{-i\Delta p_E \cdot r / \hbar} \psi_+. \]

To grasp the role of the generalized chiral symmetry and the relation (11) more explicitly, let us write the wave function \( \psi \) in the chiral basis as \( \Psi^m = |\chi_+\rangle \psi^m_+ + |\chi_-\rangle \psi^m_- \). Then the Schrödinger equation, \( H(m) \Psi^m = E \Psi^m \), becomes

\[
\begin{pmatrix}
\langle \chi_+ | mc^2 \Gamma | \chi_+ \rangle & \langle \chi_+ | H | \chi_- \rangle \\
\langle \chi_- | H | \chi_+ \rangle & \langle \chi_- | mc^2 \Gamma | \chi_- \rangle
\end{pmatrix}
\begin{pmatrix}
\psi^m_+ \\
\psi^m_-
\end{pmatrix}
= E
\begin{pmatrix}
1 & \beta^* \\
\beta & 1
\end{pmatrix}
\begin{pmatrix}
\psi^m_+ \\
\psi^m_-
\end{pmatrix},
\]

where \( \beta = \langle \chi_+ | \chi_- \rangle \). Due to the generalized chiral symmetry, \( H \) appears only in the off-diagonal elements, while the relation (11) guarantees that \( \Gamma \) appears only in the diagonal elements. From the explicit form of the matrix elements for \( \Gamma \), the equation is simplified to

\[
\begin{pmatrix}
mc_r & \alpha \cdot \pi_E \\
\alpha^* \cdot \pi_E & -mc_r
\end{pmatrix}
\begin{pmatrix}
\psi^m_+ \\
\psi^m_-
\end{pmatrix}
= E
\begin{pmatrix}
\psi^m_+ \\
\psi^m_-
\end{pmatrix},
\]

with

\[ \alpha \equiv (\alpha_{X}, \alpha_Y) = h^{-1}(\langle \chi_+ | X^\mu \sigma_\mu | \chi_- \rangle, \langle \chi_+ | Y^\mu \sigma_\mu | \chi_- \rangle) \]

(see Appendix F). Then we find that the normalizable wave functions for \( eB > 0 \) at energies \( E = \pm mc_r \) should have

\[ \psi^m_- = 0, \quad \alpha^* \cdot \pi_E \psi^m_+ = 0, \]

which indicates that the eigenstates at the bottom of the upper band \( E = mc_r \) are indeed the eigenstate of \( \gamma \) with the eigenvalue +1 \( (\Psi^m = |\chi_+\rangle \psi^m_+ \) because \( \psi^m_- = 0 \). In other words, the generalized chiral operator continues to commute with the Hamiltonian, within the \( n = 0 \) Landau subspace, even for massive Dirac fermions.

Namely, for tilted Dirac fermions, the wave functions of the \( n = 0 \) Landau levels for massless \( (m = 0) \) and those for massive \( (m \neq 0) \) fermions are thus related through the gauge transformation (13). We can therefore conclude that the robustness of the \( n = 0 \) Landau level at \( E = 0 \) against disorder that respects the generalized chiral symmetry persists to the cases where its energy is shifted to \( E = mc_r \) by the mass term \( mc^2 \Gamma \).

In the tight-binding lattice model discussed in section II, we have two valleys, for which the sign of the symmetry breaking term \( mc^2 \Gamma \) is opposite. The sign of the energy shift is therefore opposite for these two Dirac cones in the lattice model, which is actually seen as the split zero modes shown in Fig. 4. We show in Fig. 5 the energies of the split \( n = 0 \) Landau levels obtained for the tight-binding lattice model (11) as a function of \( mc^2 \). They are in a surprisingly good agreement with the analytical formula, \( \pm mc_r \), given by the effective theory.

V. SUMMARY

We have investigated the robustness of the zero modes for massive and tilted Dirac fermions in a magnetic field. It is clearly demonstrated numerically that the anomalous robustness of zero modes against disorder in gauge degrees of freedom is retained for a massive and tilted Dirac fermions. It is to be noted that for the massive fermions, the robustness shows up even in the case of the short-range disorder, while it is degraded for the massless Dirac fermions. We have also presented a general formu-
latter for the generic two-dimensional massless and massive Dirac fermions in which a simple algebraic transformation between the generalized chiral operator and the conventional chiral operator has been obtained. Based on the low-energy effective theory, we have explicitly discussed the applicability of the argument by Aharonov and Casher to show the robust zero modes of the massive and tilted Dirac fermions, where the wave function of the \( n = 0 \) Landau level for the massive case is related to that for the massless case through a gauge transformation. The present numerical and analytical results for tilted Dirac fermions, where the chiral symmetry and the sub-lattice symmetry are separated, clearly suggest that the generalized chiral symmetry, rather than the sub-lattice symmetry, is indeed a key ingredient for the robust zero modes for the generic Dirac fermions in two dimension.

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**Appendix A: Four dimensional notation**

Let us elaborate the general formulation for four-dimensional real vector, \( X = (X_0, X_1, X_2, X_3) \) with a metric \( g = \text{diag} (-1, 1, 1, 1) \), which defines

\[
\bar{X} = (X^0, X^1, X^2, X^3) \quad \text{and} \quad Xg = (-X_0, X_1, X_2, X_3)
\]

An inner product of the two 4-vectors \( \bar{X} \) and \( Y \) is expressed as

\[
\bar{X}Y = \bar{Y}X = X_\mu Y^\mu = -X^0Y^0 + X \cdot Y.
\]

For example, the norm of the four vector \( \bar{X}X \) is given as

\[
\bar{X}X = X^\mu X_\mu = |X|^2 = X_0^2.
\]

Noting that for three dimensional vectors \( X \) and \( Y \), one has \( (X \cdot \sigma)(Y \cdot \sigma) = (X \cdot Y)\sigma_0 + i(X \times Y) \cdot \sigma \), we have a simple formula,

\[
(\bar{X}^\mu \sigma_\mu)(Y^\nu \sigma_\nu) = -X^0\sigma_0 X_\mu Y^\mu \sigma_\mu + i(X \times Y) \cdot \sigma,
\]

where

\[
n(X, Y) = X \times Y + i\eta(X, Y),
\]

\[
\eta(X, Y) = X^0Y - X^0Y^0.
\]

Note that \( \bar{X}Y = \bar{X}Y \equiv X^\mu Y_\mu = -X^0Y^0 + X \cdot Y \) is symmetric, while \( n(X, Y) \) is anti-symmetric when one exchanges \( X \) and \( Y \).

Also, noting that

\[
\det A^\mu \sigma_\mu = \det \begin{pmatrix} A^0 + A^3 & A^1 - iA^2 \\ A^1 + iA^2 & A^0 - A^3 \end{pmatrix} = (A^0)^2 - |A|^2 = \bar{A}A,
\]

we have, by defining \( \sigma_A = A \cdot \sigma/|A| \),

\[
A^\mu \sigma_\mu = A^0\sigma_0 + A \cdot \sigma = \sqrt{AA}e^{\phi_A}A \sigma_A,
\]

where \( e^{\phi_A} = \sigma_0 \cosh \phi_A + \sigma_A \sinh \phi_A \) with \( \cosh \phi_A = |A|/\sqrt{AA} \) and \( \sinh \phi_A = A_0/\sqrt{AA} \).

**Appendix B: determinant of \( \Xi \)**

Let us here evaluate the determinant in the discussion as

\[
(ch)^4 = \frac{\bar{XX} \bar{XY}}{\bar{YX} \bar{YY}} \frac{X^0 Y^0}{X^0 Y^0} = \left| \begin{array}{ll} X^0 & Y^0 \\ X_1 & Y_1 \\ X_2 & Y_2 \\ X_3 & Y_3 \end{array} \right| = -(X^0 Y^0)^2 - (X \times Y)^2 (X^0 Y^0)^2 + X \times Y
\]

where \( \text{Im} (n \cdot n) = 2(X \times Y) \cdot (X^0 Y - X^0 Y^0) = 0 \).

It is also evaluated by the expansion of the minors as

\[
\begin{vmatrix} X^0 \\ X_1 \\ X_2 \\ X_3 \end{vmatrix} \begin{vmatrix} Y^0 \\ Y_1 \\ Y_2 \\ Y_3 \end{vmatrix} = \text{det} \left[ \begin{array}{cc} X^0 & Y^0 \\ X_1 & Y_1 \\ X_2 & Y_2 \\ X_3 & Y_3 \end{array} \right] = -(X^0 Y^0)^2 - (X \times Y)^2.
\]

\[
\begin{vmatrix} X^0 \\ X_1 \\ X_2 \\ X_3 \end{vmatrix} \begin{vmatrix} Y^0 \\ Y_1 \\ Y_2 \\ Y_3 \end{vmatrix} = 2(X \times Y) \cdot (X^0 Y - X^0 Y^0)
\]

\[
\frac{\bar{XX} \bar{XY}}{\bar{YX} \bar{YY}} = \text{det} \left[ \begin{array}{cc} X^0 & Y^0 \\ X_1 & Y_1 \\ X_2 & Y_2 \\ X_3 & Y_3 \end{array} \right] = -(X^0 Y^0)^2 - (X \times Y)^2.
\]
Appendix C: Completing the square

Here let us show details for deriving Eq.(6) by completing the square. We start with

\[
(X^0, Y^0) \Xi^{-1} \left( \begin{array}{c} X^0 \\ Y^0 \end{array} \right) (\hbar c)^2
= (X^0, Y^0) \Xi^{-1} \left( \begin{array}{c} X^0 \\ Y^0 \end{array} \right)
= |X^0 Y - Y^0 X|^2
= |\eta(X, Y)|^2
= (\text{Im } n)^2.
\]

Then we have

\[
c^2 p^\dagger \Xi \Xi^\dagger p
= \left[ c p^\dagger + \frac{E}{\hbar c} (X^0, Y^0) \Xi^{-1} \Xi \right] p
= \Xi^{-1} \left( \begin{array}{c} X^0 \\ Y^0 \end{array} \right)
= c^2 p^\dagger \Xi \Xi^\dagger p
= \frac{(\text{Im } n)^2}{(c \hbar)^4} E^2,
\]

where

\[
p_E = p + \Delta p_E,
\]

\[
\Delta p_E = E \frac{1}{c \hbar} \Xi^{-1} \left( \begin{array}{c} X^0 \\ Y^0 \end{array} \right).
\]

Also note that

\[
1 + \frac{(\text{Im } n)^2}{(c \hbar)^4} = \frac{n^2 + (\text{Im } n)^2}{n^2} = \frac{(\text{Re } n)^2}{(\text{Re } n)^2} = (\cosh q)^2.
\]

Appendix D: Three-dimensional representation

In the main text, we have given a compact treatment of the tilted Dirac cone physics with a four-dimensional representation. Here, let us show that how a three-dimensional representation. Here, let us show details for deriving Eq.(6) by completing the square. We start with

\[
(X^0, Y^0) \Xi^{-1} \left( \begin{array}{c} X^0 \\ Y^0 \end{array} \right) (\hbar c)^2
= (X^0, Y^0) \Xi^{-1} \left( \begin{array}{c} X^0 \\ Y^0 \end{array} \right)
= |X^0 Y - Y^0 X|^2
= |\eta(X, Y)|^2
= (\text{Im } n)^2.
\]

Then we have

\[
c^2 p^\dagger \Xi \Xi^\dagger p
= \left[ c p^\dagger + \frac{E}{\hbar c} (X^0, Y^0) \Xi^{-1} \Xi \right] p
= \Xi^{-1} \left( \begin{array}{c} X^0 \\ Y^0 \end{array} \right)
= c^2 p^\dagger \Xi \Xi^\dagger p
= \frac{(\text{Im } n)^2}{(c \hbar)^4} E^2,
\]

where

\[
p_E = p + \Delta p_E,
\]

\[
\Delta p_E = E \frac{1}{c \hbar} \Xi^{-1} \left( \begin{array}{c} X^0 \\ Y^0 \end{array} \right).
\]

Also note that

\[
1 + \frac{(\text{Im } n)^2}{(c \hbar)^4} = \frac{n^2 + (\text{Im } n)^2}{n^2} = \frac{(\text{Re } n)^2}{(\text{Re } n)^2} = (\cosh q)^2.
\]

FIG. 6. Geometrical meaning of the tilted Dirac cones in \((p_x, p_y, \xi)\) space.

Since \((H_0^0)^2 = [c^2 p^\dagger \Xi \Xi^\dagger p] \sigma_0 \propto \sigma_0^3\), we have a scalar equation for \(\Psi\),

\[
\hbar^{-2}(X, Y) \Xi \Xi^\dagger p = (E - z)^2,
\]

where

\[
\Xi_0 = \frac{1}{(\hbar c)^2} \left( \begin{array}{cc} X & X \cdot Y \\ Y & Y \cdot Y \end{array} \right),
\]

\[
c_0^2 = |X \times Y|/\hbar^2.
\]

The “light velocity” \(c_0\) is so chosen that \(\det \Xi_0 = 1\).

Geometrically (see Fig.7), a constant energy curve \(E(p_x, p_y) = \text{const.} \) in \((p_x, p_y, \xi)\) space is given by the intersection of the cone and the plane

\[
\xi^2 = c_0^2 p^\dagger \Xi \Xi^\dagger p
= (X^0/\hbar)p_x + (Y^0/\hbar)p_y + \xi = E,
\]

which can be a parabola, an ellipse, hyperbola or a point. Any intersection of the cone and the plane is an ellipse if the slope of the plane does not exceed that of the cone, which guarantees that the energy dispersion is given by the Dirac cone.

When the Dirac cone is not tilted, that is \(X^0 = Y^0 = 0\), the energy dispersion is given by \(E = z\). Since \(\Xi_0\) is a real symmetric matrix with \(\text{Tr } \Xi_0 > 0\), it is diagonalized by the orthogonal matrix \(V\) as

\[
\Xi_0 = V^\dagger \text{diag } (\xi_1^0, \xi_2^0) V,
\]

where \(\xi_1^0 > 0, \xi_2^0 > 0\) and \(\xi_1^0 \xi_2^0 = \det \Xi_0 = 1\). Now we have

\[
E = \pm c_0 \tilde{P},
\]

where \(\tilde{P} = \sqrt{\xi_1^0 P_x^2 + \xi_2^0 P_y^2} \), \(P = V \tilde{P}\) and \(c_0\) is the Dirac fermion velocity without tilting.

For the tilted case with finite \(X^0\) or/and \(Y^0 \neq 0\), we need to complete the square by rewriting Eq.(D1). Here let us complete the square in Eq.(D1). If we expand the
right-hand side as
\[
c^2 \xi_0 p^\dagger \Xi_0 p = h^{-2} p^\dagger \begin{pmatrix} X \\ Y \end{pmatrix} (X, Y)p
\]
\[
= \left[ E - \left( (X_0/h) p_x + (Y_0/h) p_y \right) \right]^2
\]
\[
= \left[ E^2 - 2E h^{-1} (X_0, Y_0) p + h^{-2} p^\dagger \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} (X_0, Y_0)p \right],
\]
we get
\[
[c^2 p^\dagger \Xi p + 2\frac{E}{h}(X_0, Y_0)p] \Psi = E^2 \Psi,
\]
where
\[
\Xi = \frac{1}{(\hbar c)^2} \begin{pmatrix} -X_0 X_0 + X \cdot X & -X_0 Y_0 + Y \cdot Y \\ -X_0 Y_0 + X \cdot Y & -Y_0 Y_0 + Y \cdot Y \end{pmatrix}
\]
\[
= \frac{1}{(\hbar c)^2} \begin{pmatrix} \hat{X} X & \hat{X} Y \\ \hat{X} Y & \hat{Y} Y \end{pmatrix}.
\]
The equation coincides with Eq.(5) in the four-dimensional notation in the text.

Although complicated, one can perform a similar process with a magnetic field as
\[
H = h^{-1} [\sigma, (X^\mu, Y^\mu) \pi], = H_0 + H_C,
\]
\[
H_0 = h^{-1} \sigma_0 (X_0, Y_0) \pi,
\]
\[
H_C = h^{-1} (X \cdot \sigma, Y \cdot \sigma) \pi,
\]
where \( \pi = p - eA \) is the dynamical momentum. The Schrödinger equation reads
\[
H_C \Psi = (E - Z) \Psi
\]
with \( Z = (X_0/h) \pi_x + (Y_0/h) \pi_y \). Using the relation
\[
[H_C, Z] = [h^{-1} (X \cdot \sigma) \pi_x + (Y \cdot \sigma) \pi_y], Z
\]
\[
= h^{-2} (Y_0 X - X_0 Y) \cdot \sigma |\pi_x, \pi_y\rangle
\]
\[
= -i \ell B^{-2} (\text{Im} \ n) \cdot \sigma
\]
and
\[
H_C^2 = h^{-2} [(X \cdot \sigma) \pi_x + (Y \cdot \sigma) \pi_y] \]
\[
= (\pi^\dagger \xi_0^\dagger \pi \sigma_0 + i h^{-2} (X \cdot Y) \cdot \sigma |\pi_x, \pi_y\rangle
\]
\[
= (\pi^\dagger \xi_0^\dagger \pi \sigma_0 - \ell B^{-2} (\text{Re} \ n) \cdot \sigma
\]
we have
\[
H_C^2 \Psi = H_C [(E - Z) \Psi
\]
\[
= \left[ (E - Z)H_C + [H_C, E - Z] \right] \Psi
\]
\[
= \left[ (E - Z)^2 + i \ell B^{-2} (\text{Im} \ n) \cdot \sigma \right] \Psi.
\]
This implies
\[
\left[ (\pi^\dagger \xi_0^\dagger \pi \sigma_0 - \ell B^{-2} n \cdot \sigma \right]\Psi = (E - Z)^2 \Psi.
\]
Similarly to the case without magnetic field, one has
\[
[c^2 \pi^\dagger \Xi \pi + 2E(h)(X_0, Y_0) \pi] \sigma^0 \Psi - \ell B^{-2} n \cdot \sigma \Psi = E^2 \Psi,
\]
which coincides with Eq.(5) in the text.

Appendix E: Landau levels for an anisotropic mass

Let us summarize the standard Landau quantization of electrons with parabolic dispersion with anisotropic masses (effective mass approximation) described by the following Hamiltonian
\[
H = \pi^\dagger \frac{1}{2m^*} \Xi_L \pi,
\]
with \( \pi = p - eA = \pi^\dagger \), \( \text{rot} \ A = B \hat{z} \), and
\[
\Xi_L = \begin{pmatrix} \xi_x & \xi_{xy} \\ \xi_{xy} & \xi_y \end{pmatrix},
\]
where
\[
\left( \frac{\ell_B}{\hbar} \right)^2 [\pi_x, \pi_y] = i, \quad \ell_B = \sqrt{\frac{\hbar}{eb}}.
\]
Here we have assumed \( eB > 0 \) without loss of generality. Since the matrix \( \Xi_L \) is real symmetric, it is diagonalized by the orthogonal matrix \( V \) as
\[
\Xi_L = V^\dagger \Xi_D V,
\]
\[
\Xi_D = \text{diag}(\xi_x, \xi_y), \quad \xi_x \xi_y = \det \Xi_L, \quad \xi_x + \xi_y = \text{Tr} \Xi_L.
\]
\[
V = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad 3 \theta \in \mathbb{R}
\]
Then we have
\[
H = \Pi^\dagger \Xi_D \Pi,
\]
\[
\Pi \equiv \Pi \pi,
\]
\[
\left( \frac{\ell_B}{\hbar} \right)^2 [\Pi_X, \Pi_Y] = i.
\]
Now defining a bosonic operator (with \( [a, a^\dagger] = 1 \),
\[
a = \frac{1}{\sqrt{2}} \frac{\ell_B}{\hbar} (\Pi_X + i \Pi_Y),
\]
the Hamiltonian is written as
\[
H = \frac{\hbar \omega}{4} \left[ \xi_x (a + a^\dagger)^2 - \xi_y (a - a^\dagger)^2 \right]
\]
with \( \omega = eB/m^* \).
Now we define a new bosonic operator \( [b, b^\dagger] = 1 \) as
\[
a = ub + v^* b^\dagger,
\]
\[
a^\dagger = u^* b^\dagger + vb
\]
with \( [a, a^\dagger] = [ub + v^* b^\dagger, u^* b^\dagger + vb] = |u|^2 - |v|^2 = 1 \). Here we choose
\[
\xi_x (u + v)^2 = \xi_y (u - v)^2,
\]
\[
u + v = C \sqrt{\xi_y},
\]
\[
u - v = -C \sqrt{\xi_x}.\]
Assuming $\xi_x, \xi_y > 0$ and imposing $|u|^2 - |v|^2 = 1$, we have $|C|^2 = 1/\sqrt{\xi_x \xi_y} = 1/(\det \Xi_L)^{1/2}$ and therefore arrive at

$$u = \frac{\sqrt{\xi_x} + \sqrt{\xi_y}}{2(\det \Xi_L)^{1/4}}, \quad v = \frac{-\sqrt{\xi_x} + \sqrt{\xi_y}}{2(\det \Xi_L)^{1/4}}.$$

Finally, the Hamiltonian is written as

$$H = \frac{1}{2} \hbar \omega (b^\dagger b + b^\dagger b) |C|^2 (\xi_x \xi_y) = \hbar \omega \left( \frac{b^\dagger b + 1}{2} \right),$$

$$\omega \Xi = \frac{\hbar}{m*} \sqrt{\det \Xi_L} = \frac{eB}{m*} \sqrt{\xi_x \xi_y}.$$

### Appendix F: Derivation of Eq. (14)

The equation above Eq. (14) can be expressed, by introducing a dynamical momentum $\pi'_E = \pi + \Delta p'_E$ in terms of a real vector $\Delta p'_E$ satisfying a relation $\alpha \cdot \Delta p'_E = -EB$, as

$$\begin{pmatrix} m_{cc}\tau \cdot \alpha \cdot \pi'_E \\ \alpha \cdot \pi'_E - m_{cc}\tau \end{pmatrix} \left( \begin{array}{c} \psi^m \\ \psi^m_\perp \end{array} \right) = E \left( \begin{array}{c} \psi^m \\ \psi^m_\perp \end{array} \right).$$

We can show that $\Delta p'_E = \Delta p_E$ by multiplying the matrix on the left-hand side of the equation once again to get

$$\left[ \text{Im}(\alpha \cdot \alpha^*_y) (\pi'_E \Xi', \pi'_E + h^2/\ell_B^2) + (m_{cc}\tau)^2 \right] \psi^m_\perp = E^2 \psi^m_\perp$$

with

$$\Xi' = \frac{1}{\text{Im}(\alpha \cdot \alpha^*_y)} \begin{pmatrix} |\alpha|^2 & \text{Re}(\alpha \cdot \alpha^*_y) \\ \text{Re}(\alpha \cdot \alpha^*_y) & |\alpha|^2 \end{pmatrix}.$$

Comparing this with Eq. (12), we can see that $\Xi = \Xi'$, $\Delta p_E = \Delta p'_E$, and $c_s^2 = \text{Im}(\alpha \cdot \alpha^*_y)$. 

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37. Using a well known formula, we have $(\Xi E)^2 = h^{-2} |(\sigma \cdot (X \cdot Y)) p|^2 = h^{-2} |(\sigma \cdot (X \cdot Y)) p + (\sigma \cdot Y) p_0|^2 = h^{-2} (X \cdot X p_0^2 + 2 X \cdot Y p_y + Y \cdot Y p_y)^2 \sigma_0 = |(X, Y) p|^2 \sigma_0 = \left( c_s^2 p^2 \Xi_0 p \right)^2 \sigma_0$. det $\Xi_0 = \frac{1}{2 (c_0 b)^2} [ (X | Y) ^2 - (X \cdot Y) ^2 ] = 1$. 
38. The equation above Eq. (14) can be expressed, by introducing a dynamical momentum $\pi'_E = \pi + \Delta p'_E$ in terms of a real vector $\Delta p'_E$ satisfying a relation $\alpha \cdot \Delta p'_E = -EB$, as