SOLVING A CONJECTURE ABOUT TESSELLATION GRAPHS OF $\mathbb{R}^2$

WALTER CARBALLOSA

ABSTRACT. In the paper Planarity and Hyperbolicity in Graphs, the authors present the following conjecture: every tessellation of the Euclidean plane with convex tiles induces a non-hyperbolic graph. It is natural to think that this statement holds since the Euclidean plane is non-hyperbolic. Furthermore, there are several results supporting this conjecture. However, this work shows that the conjecture is false.

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1. Introduction.

Hyperbolic spaces play an important role in geometric group theory and in the geometry of negatively curved spaces (see [1 12 13]). The concept of Gromov hyperbolicity grasps the essence of negatively curved spaces like the classical hyperbolic space, Riemannian manifolds of negative sectional curvature bounded away from 0, and of discrete spaces like trees and the Cayley graphs of many finitely generated groups. It is remarkable that a simple concept leads to such a rich general theory (see [1 12 13]).

The first works on Gromov hyperbolic spaces deal with finitely generated groups (see [13]). Initially, Gromov spaces were applied to the study of automatic groups in the science of computation (see, e.g., [21]); indeed, hyperbolic groups are strongly geodesically automatic, i.e., there is an automatic structure on the group [8].

The concept of hyperbolicity appears also in discrete mathematics, algorithms and networking. For example, it has been shown empirically in [25] that the internet topology embeds with better accuracy into a hyperbolic space than into an Euclidean space of comparable dimension; the same holds for many complex networks, see [19]. A few algorithmic problems in hyperbolic spaces and hyperbolic graphs have been considered in recent papers (see [9 10 11 18]). Another important application of these spaces is the study of the spread of viruses through the internet (see [16 17]). Furthermore, hyperbolic spaces are useful in secure transmission of information on the network (see [15 16 17 20]).

The study of Gromov hyperbolic graphs is a subject of increasing interest; see, e.g., [2 4 5 6 7 14 23 24] and the references therein.

There are several definitions of Gromov hyperbolicity. These different definitions are equivalent in the sense that if $X$ is $\delta$-hyperbolic with respect to the definition $A$, then it is $\delta'$-hyperbolic with respect to the definition $B$ for some $\delta'$ (see, e.g., [3 12]).

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Given a metric space $X$, we define the Gromov product of $x, y \in X$ with base point $w \in X$ by

$$(1) \quad (x|y)_w := \frac{1}{2} \left( d(x, w) + d(y, w) - d(x, y) \right).$$

We say that the Gromov product is $\delta$ hyperbolic if there is a constant $\delta \geq 0$ such that

$$(2) \quad (x|y)_w \geq \min \{ (x|z)_w, (z|y)_w \} - \delta$$

for every $x, y, z \in X$ and some $w \in X$. It is well known that this definition is independent of base point, see [1, Proposition 2.2] and [13, Lemma 1.1A]. In fact, if $X$ is a metric space, $w, w' \in X$ and the Gromov product based at $w$ is $\delta$-hyperbolic, then the Gromov product based at $w'$ is $2\delta$-hyperbolic. We say that $X$ is hyperbolic if there exists a constant $\delta \geq 0$ such that its Gromov product is $\delta$-hyperbolic for any base point, see, e.g., [12].

We say that the curve $\gamma$ in a metric space $X$ is a geodesic if we have $L(\gamma|[s,t]) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$ (then $\gamma$ is equipped with an arc-length parametrization). The metric space $X$ is said geodesic if for every couple of points in $X$ there exists a geodesic joining them; we denote by $[xy]$ any geodesic joining $x$ and $y$; this notation is ambiguous, since in general we do not have uniqueness of geodesics, but it is very convenient. Consequently, any geodesic metric space is connected. If the metric space $X$ is a graph, then the edge joining the vertices $u$ and $v$ will be denoted by $[u,v]$.

In order to consider a graph $G$ as a geodesic metric space, we must identify any edge $[u,v] \in E(G)$ with the real interval $[0,l]$ if $l := L([u,v])$; therefore, any point in the interior of any edge is a point of $G$. A connected graph $G$ is naturally equipped with a distance defined on its points, induced by taking shortest paths in $G$. Then, we see $G$ as a geodesic metric graph.

Throughout the paper we just deal with graphs which are connected and locally finite (i.e., each ball of finite radius contains just a finite number of edges); we also allow edges of arbitrary length. These conditions guarantee that the graph is a geodesic metric space. In particular, we pay attention to the planar graphs: the graphs which are the “boundary” (the 1-skeleton) of a tessellation of the Euclidean plane.

By a tessellation graph we mean the 1-skeleton (i.e., the set of 1-cells) of a CW 2-complex contained in a complete Riemannian surface $S$ (with or without boundary) such that every point in $S$ is contained in some face (2-cell) of the complex. We assume that every closed cell is embedded in the CW-complex, i.e., a face of the tessellation should not be glued to itself along a pair of edges. The edges of a tessellation graph are just rectifiable paths in $S$ and have the length induced by the metric in $S$ (they can be either geodesics or not in $S$). Note that this class of graphs contains as particular cases many planar graphs.

In [7] the authors present the following conjecture “every tessellation of the Euclidean plane with convex tiles induces a non-hyperbolic graph”. It is natural to think that this statement holds since the Euclidean plane is non-hyperbolic. Furthermore, there are several theorems in [7] supporting this conjecture. This work shows that the conjecture is false, see Section 3.
2. The conjecture and previous results

The conjecture was presented in [7] as follows.

**Conjecture 1.** Every tessellation graph of $\mathbb{R}^2 \cong \mathbb{C}$ with convex tiles is non-hyperbolic.

Note that many tessellation graphs of the Euclidean plane $\mathbb{R}^2$ are non-hyperbolic, and many tessellation graphs of hyperbolic space as the Poincaré disk $\mathbb{D}$ are hyperbolic. One can think that the tessellation graphs of the Euclidean plane $\mathbb{R}^2$ are always non-hyperbolic, since the plane is non-hyperbolic; however, in [22] the authors show a hyperbolic tessellation graph of $\mathbb{R}^2$. The main aim on it is enlarge the edges of a non-hyperbolic graph until to obtain a hyperbolic graph (similar to a tessellation graph of $\mathbb{D}$). However, this example is a tessellation graph of $\mathbb{R}^2$ with not convex tiles. Furthermore, it is easy to obtain a tessellation graph of $\mathbb{D}$ which is non-hyperbolic, taking huge tiles on $\mathbb{D}$.

The following results support the Conjecture 1.

**Theorem 1.** [7, Theorem 2.7] Given any tessellation graph $G$ of $\mathbb{R}^2$ with convex tiles $\{F_n\}$. If $\inf_n A(F_n) > 0$, then $G$ is not hyperbolic.

**Theorem 2.** [7, Theorem 2.9] Given any tessellation graph $G$ of $\mathbb{R}^2$ with convex tiles $\{F_n\}$. Let us assume that there exist balls $B_n \subset F_n$ with radius $r_n$ such that $L(\partial F_n) \leq c_1 r_n$ for some positive constant $c_1$ and for every $n$. Then $G$ is not hyperbolic.

In [5] the authors trying to solve the Conjecture 1 show that under appropriate assumptions adding or removing an infinite amount of edges to a given planar graph preserves its non-hyperbolicity. Also, a partial answer to Conjecture 1 was given. It is shown that in order to prove this conjecture it suffices to consider tessellations graphs of $\mathbb{R}^2$ such that every tile is a triangle.

**Theorem 3.** [5, Theorem 5.1] All tessellation graphs of $\mathbb{R}^2$ whose tiles are convex polygons are non-hyperbolic if and only if all tessellation graphs of $\mathbb{R}^2$ whose tiles are triangles are non-hyperbolic.

The following definitions and results will be useful in Section 3.

Let $(X, d_X)$ and $(Y, d_Y)$ be two (geodesic) metric spaces. A map $f : X \rightarrow Y$ is said to be an $(\alpha, \beta)$-quasi-isometric embedding, with constants $\alpha \geq 1$, $\beta \geq 0$ if for every $x, y \in X$:

$$\alpha^{-1} d_X(x, y) - \beta \leq d_Y(f(x), f(y)) \leq \alpha d_X(x, y) + \beta.$$  

The function $f$ is $\epsilon$-full if for each $y \in Y$ there exists $x \in X$ with $d_Y(f(x), y) \leq \epsilon$.

A map $f : X \rightarrow Y$ is said to be a quasi-isometry if there exist constants $\alpha \geq 1$, $\beta, \epsilon \geq 0$ such that $f$ is an $\epsilon$-full $(\alpha, \beta)$-quasi-isometric embedding.

A fundamental property of hyperbolic spaces is the following:

**Theorem 4** (Invariance of hyperbolicity). Let $f : X \rightarrow Y$ be an $(\alpha, \beta)$-quasi-isometric embedding between the geodesic metric spaces $X$ and $Y$. If $Y$ is hyperbolic, then $X$ is hyperbolic.

Furthermore, if $f$ is $\epsilon$-full for some $\epsilon \geq 0$ (a quasi-isometry), then $X$ is hyperbolic if and only if $Y$ is hyperbolic.

We say that a vertex $v$ of a graph $G$ is a cut-vertex if $G \setminus \{v\}$ is not connected. Given a graph $G$, a family of subgraphs $(G_n)_{n \in \Lambda}$ of $G$ is a $T$-decomposition of $G$ if $\bigcup_n G_n = G$ and $G_n \cap G_m$ is either a cut-vertex or the empty set for each $n \neq m$. We
will need the following result, which allows to obtain global information about the hyperbolicity of a graph from local information.

**Theorem 5.** [2, Theorem 5] Let $G$ be any graph and let $\{G_n\}_n$ be any $T$-decomposition of $G$. Then $\delta(G) = \sup_n \delta(G_n)$.

### 3. Solving the conjecture

In [4, 6] the authors obtain interesting results characterizing the hyperbolicity of many periodic graphs. Hence, we deal with periodic graphs.

A **geodesic line** is a geodesic with domain $\mathbb{R}$. We say that a graph $G$ is **periodic** if there exist a geodesic line $\gamma_0$ and an isometry $T$ of $G$ with the following properties:

1. $T\gamma_0 \cap \gamma_0 = \emptyset$,
2. $G \setminus \gamma_0$ has two connected components,
3. $G \setminus \{\gamma_0 \cup T\gamma_0\}$ has at least three connected components, two of them, $G_1$ and $G_2$, satisfy $\partial G_1 \subset \gamma_0$ and $\partial G_2 \subset T\gamma_0$, and the subgraph $G' := G \setminus \{G_1 \cup G_2\}$ is connected and $\bigcup_{n \in \mathbb{Z}} T^n(G') = G$.

Such subgraph $G'$ is called a **period graph** of $G$. In what follows, $G$ will denote a periodic graph and $G'$ a period graph of $G$.

The following result will be useful.

**Theorem 6.** [6, Theorem 1.1] Let $G$ be a periodic graph with $\inf_{z \in \gamma_0} d_G(z, Tz) > 0$. Then $G$ is hyperbolic if only if $G'$ is hyperbolic and $\lim_{|z| \to \infty, z \in \gamma_0} d_G(z, Tz) = \infty$.

**Figure 1.** Period graph $G'$ of a tessellation graph of $\mathbb{R}^2$ with convex tiles.

Let $G'$ be the graph showed in Figure 1, i.e., $G'$ is a planar graph contained in $\mathbb{R} \times [0, 1] \subset \mathbb{R}^2$ verifying the following properties:

1. $\mathbb{R} \times [0, 1] \subset G'$,
2. $G'$ is symmetric with respect to the vertical axis,
3. in $G \cap \{z \in \mathbb{C} : 2n - 2 \leq \text{Re } z \leq 2n\} (n \geq 1)$ there are $2^{n-1}$ graphs isomorphic to the subgraph in gray on Figure 1.

We say that a vertex $v$ of $G'$ is at level $i$ if $v$ belongs to $[i] \times [0, 1]$; besides, $G'$ just has $2^n$ vertices at level $2n$. Denote these vertices by $v_{2n,i}$ for $1 \leq i \leq 2^n$ with

$$\text{Im } v_{2n,i} < \text{Im } v_{2n,i+1}, \quad \text{for every } 1 \leq i \leq 2^n.$$

Consider now the tessellation graph $W$ of $\mathbb{R}^2$ with period graph $G'$.

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1. [Figure 1 shows a half (Re $z > 0$) of the graph $G'$.
In order to prove that $W$ is hyperbolic denote by
\[
G'' := G' \setminus \mathbb{R} \times [0, 1] \cap \{ z \in \mathbb{C} : \Re z \geq 0 \},
\]
and $G''$ a graph isomorphic to $G'$ with every edge of length 1. Note that $G''$ is not necessary embedded in $\mathbb{R}^2$; besides, every vertex of $G''$ at level $i$ is at distance $i$ to the vertex $v_{0,1}$.

**Lemma 1.** Let $k, m$ be positive integers with $1 \leq k \leq 2^n$. Then, $d_{G''}(v_{2n,k}, v_{2n+2m,k_n}) = 2m$ if $\max(2^m k - 2^{m+1} + 2, 1) \leq k_m \leq \min(2^m k + 2^m - 1, 2^{m+1})$, otherwise, the distance is greater than $2m + 1$.

**Proof.** Note that distance between two vertices at different levels is at least the difference of their levels. Clearly $d_{G''}(v_{2n,k}, v_{2n+2m,k_n})$ is an even number. Hence, it suffices to prove that $d_{G''}(v_{2n,k}, v_{2n+2m,k_2+2m-1}) = 2m$ but $d_{G''}(v_{2n,k}, v_{2n+2m,k_2+2m}) > 2m$ if $2^m k + 2^m - 1 \leq 2^{m+1}$ and $d_{G''}(v_{2n,k}, v_{2n+2m,k_2+2^{m+1}}) = 2m$ but $d_{G''}(v_{2n,k}, v_{2n+2m,k_2+2^{m+1}+1}) > 2m$ if $2^m k - 2^{m+1} + 1 \geq 1$. Note that $d_{G''}(v_{2n,k}, v_{2n+2m,k_2}) = 2$ if $\max(2k - 2, 1) \leq k_1 \leq \min(2k + 1, 2^{m+1})$; otherwise, the distance is greater than 3. Then, since $1 + 2 + \ldots + 2^{m-1} = 2^m - 1$, we have $d_{G''}(v_{2n,k}, v_{2n+2m,k_2+2^{m-1}}) = 2m$ if $k \leq 2^m - 1$ (i.e., $2^m k + 2^m - 1 < 2^{m+1}$); furthermore, it follows easily that $d_{G''}(v_{2n,k}, v_{2n+2m,k_2+2^n}) > 2m$ if $k \leq 2^{m-1}$. Similarly, since $2^m + \ldots + 2 = 2^{m+1} - 2$, we obtain $d_{G''}(v_{2n,k}, v_{2n+2m,k_2+2^{m+1}+2}) = 2m$ and $d_{G''}(v_{2n,k}, v_{2n+2m,k_2+2^{m+1}+1}) > 2m$ if $k \geq 2$. Thus, we have $d_{G''}(v_{2n,k}, v_{2n+2m,k_2}) = 2m$ for $\max(2^m k - 2^{m+1} + 2, 1) \leq k_m \leq \min(2^m k + 2^m - 1, 2^{m+1})$; otherwise, the distance is greater than $2m + 1$.

We will use the following facts.

**Remark 1.** For $1 \leq k \leq 2^n$ the vertex $v_{2n,k}$ is at distance $2m$ to at most $2^{m+1} + 2^m - 2$ vertices at level $2n + 2m$. Moreover, since $m \geq 1$ we have $2^{m+1} \leq 2^{m+1} + 2^m - 2 < 2^{m+2}$.

**Remark 2.** For $1 \leq l \leq 2^{m+1}$ the vertex $v_{2(n+e)m,l}$ is at distance $2m$ to at most three vertices at level $2n$, and furthermore these vertices at level $2n$ have consecutive indices.

**Lemma 2.** Let $v_{2n,i}, v_{2n,j}$ be two different vertices of $G''$ at level $2n$. Then, there is a fixed constant $\alpha$ such that

\[
\left| d_{G''}(v_{2n,i}, v_{2n,j}) - 4 \log_2 |i - j| \right| \leq \alpha.
\]

**Proof.** Without loss of generality we can assume that $i < j$. Note that if $j - i \leq 3$ then $d_{G''}(v_{2n,i}, v_{2n,j}) \leq 6$. Hence, (3) holds in this case for every $\alpha \geq 2$. Assume that $j - i \geq 4$.

Clearly, $d_{G''}(v_{2n,i}, v_{2n,j})$ is even. Note that every path joining $v_{2n,i}$ and $v_{2n,j}$ without vertices at levels lower than $2n - 1$ has length at least $2(j - i)$. Let $r$ be the positive integer such that $2^r \leq j - i < 2^{r+1}$, i.e., $r = \left\lfloor \log_2 (j - i) \right\rfloor$. On the one hand, since $j - i < 2^{r+1}$ by Remark 1 there is an integer $k$ such that

\[
2^{r+1} k - 2^{r+2} + 2 \leq i < 2^{r+1} k + 2^{r+1} - 1.
\]

Thus we have $d_{G''}(v_{2n,i}, v_{2(n-r-1)k}, v_{2n,j}) = 2r + 2$ and so $d_{G''}(v_{2n,i}, v_{2n,j}) \leq 4r + 4$. On the other hand, since $j - i \geq 2^r$ by Remark 2 there is not a vertex $v_{2(n-r+2)k}$ for some $1 \leq k \leq 2^{r+2}$ with

\[
d_{G''}(v_{2n,i}, v_{2(n-r+2)k}) = d_{G''}(v_{2(n-r+2)k}, v_{2n,j}) = 2r - 4.
\]
Lemma 3. For any positive integers \( n \) we have
\[
d_{G''}(v_{2n,i}, v_{2n,j}) > 4(r - 2)
\]
Assume that \( d_{G''}(v_{2n,i}, v_{2n,j}) \leq 4(r - 2) \). Hence, there is a geodesic \( \gamma := [v_{2n,i}, v_{2n,j}] \)
joining \( v_{2n,i} \) and \( v_{2n,j} \) which is contained between levels \( 2n - 2r + 2 \) and \( 2n \). Denote by \( 2m \) the lower even level with vertices in \( \gamma \). Let \( v_{2m,r}, v_{2m,j} \) be the vertices in \( \gamma \) at level \( 2m \) with the maximum distance between them. Thus, \( \gamma := [v_{2m,r}, v_{2m,j}] \cup [v_{2m,r}, v_{2m,j}] \cup [v_{2m,r}, v_{2m,j}] \). We have \( d_{G''}(v_{2n,i}, v_{2m,r}) = d_{G''}(v_{2n,j}, v_{2m,j}) \geq 2n - 2m \), and so,
\[
d_{G''}(v_{2m,r}, v_{2m,j}) = L(\gamma) - d_{G''}(v_{2n,i}, v_{2m,r}) - d_{G''}(v_{2n,j}, v_{2m,j})
\leq 4(r - 2) - 4(n - m)
\leq 2(2m - 2(n - r + 2)).
\]
But, since the subgeodesic \( [v_{2m,r}, v_{2m,j}] \subseteq \gamma \) joining \( v_{2m,r} \) and \( v_{2m,j} \) contains their vertices at level \( 2m - 1 \) or upper levels we have that \( |i' - j'| \leq 2m - 2(n - r + 2) \). Thus, by Lemma 1 there is a vertex \( v_{2(n-r+2),k} \) at level \( 2(n - r + 2) \) verifying \( d_{G''}(v_{2m,r}, v_{2(n-r+2),k}) = d_{G''}(v_{2(n-r+2),k}, v_{2m,j}) = 2m - 2(n - r + 2) \) and so, \( \ref{4} \) holds. This is the contradiction we were looking for, and then \( d_{G''}(v_{2n,i}, v_{2n,j}) > 4(r - 2) \). Therefore, we have \( 4r - 8 < d_{G''}(v_{2n,i}, v_{2n,j}) \leq 4r + 4 \). These inequalities finish the proof since \( r = \left\lfloor \log_2(j - i) \right\rfloor \). \( \square \)

Lemma 3. For any positive integers \( n, m \), consider two vertices \( v_{2n,i}, v_{2n,j} \) of \( G'' \) at level \( 2n \) and two vertices \( v_{2(n+m),i}, v_{2(n+m),j} \) at level \( 2(n + m) \) with \( d_{G''}(v_{2n,i}, v_{2(n+m),i}) = d_{G''}(v_{2n,j}, v_{2(n+m),j}) = 2m \). Then, there is a fixed constant \( \beta \) such that
\[
|d_{G''}(v_{2n,i}, v_{2(n+m),i}) - d_{G''}(v_{2n,j}, v_{2(n+m),j})| \leq \beta.
\]
Proof. Recall that
\[
d_{G''}(v_{0,s}, v_{2r,s}) = 2r \quad \text{for every} \quad r \geq 0, 1 \leq s \leq 2r.
\]
By Lemma 2 we have
\[
2n - 2 \log_2 |i - j| - \frac{\alpha}{2} \leq (v_{2n,i}, v_{2n,j})_{v_{0,1}} \leq 2n - 2 \log_2 |i - j| + \frac{\alpha}{2}
\]
and
\[
2(n + m) - 2 \log_2 |i_0 - j_0| - \frac{\alpha}{2} \leq (v_{2(n+m),i}, v_{2(n+m),j})_{v_{0,1}} \leq 2(n + m) - 2 \log_2 |i_0 - j_0| + \frac{\alpha}{2}.
\]
Hence, by Lemma 1 we have
\[
2^m i - 2^{m+1} + 2 \leq i_0 \leq 2^m i + 2^m - 1
\]
and
\[
2^m j - 2^{m+1} + 2 \leq j_0 \leq 2^m j + 2^m - 1.
\]
Therefore, we have
\[
2^m (|i - j| - 3) < |i_0 - j_0| < 2^m (|i - j| + 3)
\]
and the result follows. \( \square \)

Lemma 4. The graph \( G'' \) is hyperbolic.
Proof. In order to prove that the Gromov product based at \( w \) is hyperbolic it suffices to obtain
\[
\sup_{x, y, z} \left\{ \min \{ (x|z)_w - (x|y)_w, (z|y)_w - (x|y)_w \} \right\} < \infty.
\]
We will prove that the Gromov product based at $w := v_{0,1}$ is hyperbolic. Note that since every point in $G'_1$ is at distance at most 1 of the set $V_E$ of vertices at even levels, it suffices to prove (5) for $x, y, z \in V_E$.

Assume first that $x, y, z$ are at the same level, i.e., $x := v_{2n,j}, y := v_{2n,j}$ and $z := v_{2n,k}$. Clearly, we have $(x|z)_w - (x|y)_w = 1/2(d_{G'_1}(x, y) - d_{G'_1}(x, z))$ and $(z|y)_w - (x|y)_w = 1/2(d_{G'_1}(x, y) - d_{G'_1}(y, z))$. Note that if we do not have $i \leq k \leq j$ or $j \leq k \leq i$, then Lemma $\ref{lem:hyperbolicity}$ gives $\min\{(x|z)_w - (x|y)_w, (z|y)_w - (x|y)_w\} \leq 0$. Thus, we can assume that either $i \leq k \leq j$ or $j \leq k \leq i$. Hence, we have that one of $|i - k|, |j - k|$ is greater than or equal to $|i - j|/2$ and Lemma $\ref{lem:hyperbolicity}$ gives $\min\{(x|z)_w - (x|y)_w, (z|y)_w - (x|y)_w\} \leq \alpha + 2$.

Consider $(\alpha_1, \alpha_2, \alpha_3)$ a permutation of $(x, y, z)$. Assume that $x, y, z$ are at two different levels.

Suppose first that there are two vertices at the closest level to $w$. Without loss of generality we can assume that $n_2 := d_{G'_1}(w, \alpha_1) > n_1 := d_{G'_1}(w, \alpha_2) = d_{G'_1}(w, \alpha_3)$. Let $[\alpha_1, \alpha_2], [\alpha_2, \alpha_3]$ be geodesics joining $\alpha_1, \alpha_2$ and $\alpha_2, \alpha_3$, respectively, such that $\alpha_1', \alpha_2', \alpha_3' \in [\alpha_1, \alpha_2]$ and $\alpha_1'' \in [\alpha_2, \alpha_3]$ are vertices at level $n_1$ with $d_{G'_1}(\alpha_1', \alpha_1'') = n_2 - n_1 = d_{G'_1}(\alpha_1', \alpha_1'')$. So, by Remark $\ref{rem:geodesics}$ we have that $\alpha_1', \alpha_2''$ are either the same vertex or consecutive vertices or $d_{G'_1}(\alpha_1', \alpha_2'') = 4$, i.e., $d_{G'_1}(\alpha_1', \alpha_1'') = 4$. Hence, since $\alpha_1' \in [\alpha_1, \alpha_2]$ and $d_{G'_1}(\alpha_1', \alpha_1'') = n_2 - n_1$ we have $\min\{(\alpha_1|\alpha_1')_w, (\alpha_2|\alpha_1')_w\} = (\alpha_1'|\alpha_2)_w$. Similarly, we obtain $(\alpha_1|\alpha_3)_w = (\alpha_2'|\alpha_3)_w$. So, since $d_{G'_1}(\alpha_1', \alpha_2'') \leq 4$ we have that $|\alpha_1'|_w - (\alpha_1''|w) \leq 4$ for every $\alpha_1 \in G'_1$. Hence, in order to obtain (5) we can take $[\alpha_1', \alpha_2, \alpha_3]$ or $[\alpha_2', \alpha_2, \alpha_3]$ instead of $[\alpha_1, \alpha_2, \alpha_3]$. Thus, we can assume that $x, y, z$ are at the same level, and this is a case that has already been dealt with.

Suppose now that there is exactly one vertex at the closest level to $w$. Without loss of generality we can assume that $n_2 := d_{G'_1}(w, \alpha_1) = d_{G'_1}(w, \alpha_2) < n_1 := d_{G'_1}(w, \alpha_3)$. Using the previous arguments, there are $\alpha_1' \in [\alpha_1 \alpha_3]$ and $\alpha_2' \in [\alpha_2 \alpha_3]$ at level $n_1$ such that $d_{G'_1}(\alpha_1'|w) = (\alpha_1'|w)$ and $d_{G'_1}(\alpha_2'|w) = (\alpha_2'|w)$. So, in order to obtain (5) we can take $[\alpha_1', \alpha_2', \alpha_3]$ instead of $[\alpha_1, \alpha_2, \alpha_3]$. Thus, we can with this case assume that $x, y, z$ are at the same level, and this is a case of that has already been dealt with.

Assume now that $\alpha_1, \alpha_2, \alpha_3$ are at three different levels. Without loss of generality we can assume that $n_1 := d_{G'_1}(w, \alpha_1) < n_2 := d_{G'_1}(w, \alpha_2) < n_3 := d_{G'_1}(w, \alpha_3)$. Let $[\alpha_1, \alpha_2], [\alpha_3, \alpha_2]$ be geodesics joining $\alpha_1, \alpha_2$ and $\alpha_3, \alpha_2$, respectively, such that there are vertices at level $n_2$, denoted by $\alpha_1' \in [\alpha_1, \alpha_3]$ and $\alpha_2' \in [\alpha_3, \alpha_2]$ with $d_{G'_1}(\alpha_1', \alpha_2') = n_3 - n_2 = d_{G'_1}(\alpha_1', \alpha_2')$. So, by Remark $\ref{rem:geodesics}$ we have that $d_{G'_1}(\alpha_1', \alpha_2') \leq 4$. Indeed, since $\alpha_3' \in [\alpha_3, \alpha_2]$ and both are at level $n_2$ we have $d_{G'_1}(\alpha_3'|w) = (\alpha_3'|w)$ and $d_{G'_1}(\alpha_2'|w) = (\alpha_2'|w)$. Furthermore, it follows easily that $|\alpha_3'|_w - (\alpha_3''|w) \leq 2$ for $i \in \{1, 2\}$ and, consequently, we can take $[\alpha_1, \alpha_2, \alpha_3]$ or $[\alpha_1, \alpha_2, \alpha_3]$ instead of $[\alpha_1, \alpha_2, \alpha_3]$. Then, we can assume that $x, y, z$ are at two different levels, and this is a case that has already been dealt with.

Thus, we obtain
\[
\sup_{x,y,z \in V_E} \min\{(x|z)_w - (x|y)_w, (z|y)_w - (x|y)_w\} < \infty,
\]
and consequently $G'_1$ is hyperbolic. \qed
The following results provide a counterexample for Conjecture [1].

**Lemma 5.** The graph $G'$ is hyperbolic.

**Proof.** Since every edge $e \in G''$ verifies $1 \leq L(e) \leq \sqrt{5}/2$ and $G'', G''_1$ are isomorphic graphs, we have that there is a $0$-full $(\sqrt{5}/2, 0)$-quasi isometry from $G''_1$ to $G''$, since

$$d_{G''_1}(x, y) \leq d_{G''}(f(x), f(y)) \leq \sqrt{5} d_{G''_1}(x, y)/2.$$

Thus, by Theorem 4 and Lemma 4 we obtain that $G''_1$ is hyperbolic. Now, consider $G'_1 := G'/R \times \{0, 1\}$. Clearly, $v_{0,1}$ is a cut-vertex of $G'_1$, and there is a T-decomposition of $G'_1$ with two subgraphs isomorphic to $G''$. So, by Theorem 5 we obtain that $G'_1$ is hyperbolic. Finally, it follows easily that the natural inclusion $T : G'_1 \hookrightarrow G'$ is a $1$-full $(\sqrt{5}/2, 0)$-quasi isometry, and Theorem 4 gives the result. □

**Lemma 6.** We have

$$\lim_{|x| \to \infty} d_{G'}((x, 0), (x, 1)) = \infty.$$

**Proof.** This result is a direct consequence of Lemma 2 and 6, since we have

$$\lim_{n \to \infty} d_{G'}(v_{2n,1}, v_{2n,2^n}) = \infty.$$

□

**Theorem 7.** Conjecture [7] is false, i.e., there exists a hyperbolic tessellation graph of $\mathbb{R}^2$ with convex tiles.

**Proof.** It suffices to prove that $W$ is hyperbolic, and this is a consequence of Lemmas 5 and 6 and Theorem 6.

$W$ is a hyperbolic graph which is a tessellation graph of $\mathbb{R}^2$ with convex tiles (convex quadrilaterals and triangles). It follows easily that adding the largest diagonal of their quadrilaterals we obtain a hyperbolic triangulation of $\mathbb{R}^2$. This result is consistent with Theorem 5. However, adding to $W$ the smallest diagonal of their quadrilaterals we obtain a non-hyperbolic triangulation $T$ of $\mathbb{R}^2$: note that the period graph of $T$ is hyperbolic but it does not verify the condition $\lim_{|x| \to \infty} d_T((x, 0), (x, 1)) = \infty$, so Theorem 5 gives that $T$ is not hyperbolic.

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**E-mail address:** wcarballosato@conacyt.mx