Percolation, boundary, noise: an experiment

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Abstract
The scaling limit of the critical percolation, is it a black noise?
The answer depends on stability to perturbations concentrated along
a line. This text, containing no proofs, reports experimental results
that suggest the affirmative answer.

Introduction

By percolation I mean the two-dimensional critical site percolation on the
triangular lattice (see for instance [3]). Its (full) scaling limit exists, is con-
formally invariant and may be described by a random countable set of non-
crossing SLE\(_{6}\) curves (see [2]).

By a noise I mean a homogeneous continuous product of probability
spaces, as defined by [5, Def. 3d1]. The scaling limit of percolation could
lead to a noise (and the noise should be black, as defined by [5, Def. 7a1]);
this idea was discussed informally [5, Question 11b1] but encountered the
following difficulty.

Let a smooth domain \(D \subset \mathbb{R}^2\) be split by a smooth curve in two domains
\(D_1, D_2\). The random countable system of non-crossing SLE\(_{6}\) curves in \(D\), —
call it the random configuration in \(D\), — being restricted to \(D_1\) and \(D_2\) gives
two independent random configurations in \(D_1, D_2\). Sewing them together
appears to be a subtle matter. Does it admit any freedom? In other words,
is the configuration in \(D\) uniquely determined by its restrictions to \(D_1\) and
\(D_2\)?

More formally: the limiting model on the whole plane \(\mathbb{R}^2\) is described
by a probability space \((\Omega, \mathcal{F}, P)\) (of configurations). For every open interval
\((s, t) \subset \mathbb{R}\) we consider the domain \((s, t) \times \mathbb{R} \subset \mathbb{R}^2\) (strip) and the correspond-
ing ‘local’ sub-\(\sigma\)-field \(\mathcal{F}_{s,t} \subset \mathcal{F}\). Clearly, \(\mathcal{F}_{r,s}\) and \(\mathcal{F}_{s,t}\) are independent for
\(r < s < t\), and the upward continuity is satisfied:

\[
\mathcal{F}_{s,t} = \bigvee_{\varepsilon > 0} \mathcal{F}_{s+\varepsilon, t-\varepsilon}
\]
(that is, $\mathcal{F}_{s,t}$ is the least sub-$\sigma$-field containing all $\mathcal{F}_{s+\varepsilon,t-\varepsilon}$). The question is, whether

\[(1) \quad \mathcal{F}_{r,t} = \mathcal{F}_{r,s} \lor \mathcal{F}_{s,t}\]

or not; that is, whether the least sub-$\sigma$-field $\mathcal{F}_{r,s} \lor \mathcal{F}_{s,t}$ containing $\mathcal{F}_{r,s}$ and $\mathcal{F}_{s,t}$ is the whole $\mathcal{F}_{r,t}$, or its proper sub-$\sigma$-field. The latter case would mean that the line $\{s\} \times \mathbb{R} \subset \mathbb{R}^2$ cannot be neglected. Note that triviality of the $\sigma$-field $\mathcal{F}_{r,-s,+} = \cap_{\varepsilon>0} \mathcal{F}_{s-\varepsilon,s+\varepsilon}$ is only necessary, since in general $\cap_{\varepsilon>0} (\mathcal{F}_{r,s} \lor \mathcal{F}_{s-\varepsilon,s+\varepsilon} \lor \mathcal{F}_{s,t})$ may exceed $\mathcal{F}_{r,s} \lor \cap_{\varepsilon>0} (\mathcal{F}_{s-\varepsilon,s+\varepsilon} \lor \mathcal{F}_{s,t})$ (see [6] for detailed analysis). Note also that the difficulty cannot be avoided by waiving the upward continuity, since the latter holds for every noise [5] Prop. 3d3]. We may reformulate (1) as

\[(2) \quad \mathcal{F}_{r,t} = \lor_{\varepsilon>0} (\mathcal{F}_{r,s-\varepsilon} \lor \mathcal{F}_{s+\varepsilon,t}) .\]

The question is, whether percolation (or rather, its scaling limit) is stable under a strong perturbation in an infinitesimal strip, or not. This strong concentrated perturbation is quite different from the distributed weak perturbation examined in [1]. Sensitivity of percolation established there is micro-sensitivity in terms of [4 Sect. 5c/5.3], where it is opposed to block sensitivity. The latter also corresponds to a distributed weak perturbation, but this time the correlation length of the perturbation tends to 0 slower than the pitch of the lattice. Block sensitivity of percolation is noted in [4 8a2/Remark 8.2].

Two possibilities remain open. Maybe, percolation is strip stable, that is, satisfies (2). Then it leads to a noise, and the noise is black due to the block sensitivity. Or maybe percolation is strip unstable (that is, violates (2)). Then it does not lead to any noise. A trivial example of strip instability (and moreover, strip sensitivity) is given by the worst Boolean function, — the product of random signs (over all lattice points in the given domain). Here, trying scaling limit, we face an obstacle, — a continuum of independent random signs.

1 The idea of the experiment

Being unable to prove or disprove the equivalent properties (1), (2) for the scaling limit of percolation, I conducted an experiment as follows. First, the domain shown on Fig. (a) (the union of two equilateral triangles) is filled in with percolation data. That is, each hexagon gets black or white
with probabilities 0.5, 0.5 independently of others. However, the boundary is colored deterministically, its left part in white, right part in black. The corresponding exploration path is constructed. Second, the percolation data near the ‘equator’ are removed and generated anew. The second exploration path is constructed. Finally, the two exploration paths are compared. This is a single trial. Many such trials are conducted, independently of each other, for different values of parameters (see Sect. 2).

![Figure 1: the domain (a); percolation data and the first exploration path (b); the percolation data near the ‘equator’ are removed (c) and generated anew, giving rise to the second exploration path (d).](image)

The comparison of the two exploration paths $\gamma_1, \gamma_2$ is based on the following metric borrowed from [2, eq. (2)]:

\[
d(\gamma_1, \gamma_2) = \inf_{t \in [0,1]} \max \{|\gamma_1(t) - \gamma_2(t)|\};
\]

here $|\gamma_1(t) - \gamma_2(t)|$ is the usual Euclidean distance between the two points of the plane (the unit of length is such that the diameter of our domain equals 2, that is, each side of the rhombus equals $2/\sqrt{3}$), and the infimum is taken over all parameterizations of the paths. Similarly to [2], a path is treated as an equivalence class of continuous functions $[0,1] \to \mathbb{R}^2$ modulo monotonic (increasing) re-parameterizations.

## 2 Experimental results

Each trial is described by two parameters $n, k$ as follows. The number of hexagons on the ‘equator’ (including two boundary hexagons) equals $n + 2$;
that is, the (discrete) rhombus consists of \(2n + 1\) rows (of 2, 3, \ldots, \(n+1\), \(n+2, n+1, \ldots, 3, 2\) hexagons respectively). Out of these \(2n + 1\) rows, \(k\) rows are removed and generated anew. The set of \(k\) rows is either symmetric w.r.t. the ‘equator’ (if \(k\) is odd), or contains one additional row above the ‘equator’ (if \(k\) is even). For instance, \(n = 8\) and \(k = 2\) on Fig. 1(b,c,d).

Table 1 represents 27 samples, for 27 pairs \((n, k)\), of 250 trials each. For every sample, the median of 250 values of the distance is reported.

| \(k\) | \(n\) | 16 | 32 | 64 | 128 | 256 | 512 | 1024 |
|------|------|----|----|----|-----|-----|-----|------|
| 1    |      | 0.22 | 0.19 | 0.16 | 0.13 | 0.11 | 0.09 | 0.08 |
| 2    |      | 0.24 | 0.18 | 0.19 | 0.15 | 0.11 | 0.10 |      |
| 4    |      | 0.27 | 0.24 | 0.19 | 0.15 | 0.12 |      |      |
| 8    |      | 0.28 | 0.23 | 0.19 | 0.16 |      |      |      |
| 16   |      |      | 0.29 | 0.22 | 0.18 |      |      |      |
| 32   |      |      |      | 0.31 | 0.22 |      |      |      |

Table 1: Median values of the distance between the two exploration paths.

The distance is basically defined by (3). However, for speeding up the computation, I use an approximation to (3) (described in Sect. 4), which leads to a systematic error bounded by \(\pm 0.03\). Thus, each median in Table 1 has a systematic error bounded by \(\pm 0.03\), and a sampling error, the mean square deviation being about 7% of the median.

The results suggest that the typical distance tends to 0 as the strip width \(\varepsilon = \frac{k}{n}\) tends to 0. Maybe we observe a power law \(\varepsilon^\alpha\) for some exponent \(\alpha\) not far from \(1/3\).

### 3 Examples

The median (= 0.22) of the first sample \((n = 16, k = 1)\) resulted from a definite trial (no. 21, in fact), — call it the *median trial*, — presented on Fig. 2. Only the first exploration path is shown on Fig. 2(a) together with the hexagons relevant to it. On Fig. 2(b) the path is not shown explicitly, but you can restore it easily. On Fig. 2(c) the *second* exploration path is shown together with the hexagons relevant to the *first* path. Hexagons relevant to the second path are not shown explicitly, but you can restore them easily. Fig. 2(d) presents the most interesting, ‘equatorial’ part of Fig. 2(c).

More examples follow (Fig. 3, 4), in the same format as Fig. 2(d).
Figure 2: The median trial for $n = 16$, $k = 1$. The first exploration path (a) and the hexagons relevant to it (a,b); the same hexagons together with the second exploration path (c,d).

Figure 3: Median trials for $n = 32$. 
Figure 4: Median trials for $n = 64$. 

$k = 1$

$k = 2$

$k = 4$
4 Technicalities

First, about the metric (3) and its discrete approximation used. Treating the exploration path as a finite sequence of points, — vertices of the polygonal line (rather than the line itself), I define a distance \( d(a, b) \) between two such sequences \( a = (a_0, \ldots, a_M) \) and \( b = (b_0, \ldots, b_N) \) as follows:

\[
d(a, b) = \min_{(\mu, \nu)} \max_{k=0, \ldots, M+N} |a_{\mu_k} - b_{\nu_k}|;
\]

here \( |a_{\mu_k} - b_{\nu_k}| \) is the Euclidean distance between the two points, and \( (\mu, \nu) \) runs over all pairs of sequences \( \mu_0, \ldots, \mu_{M+N} \) and \( \nu_0, \ldots, \nu_{M+N} \) of integers satisfying

\[
0 = \mu_0 \leq \cdots \leq \mu_{M+N} = M, \quad 0 = \nu_0 \leq \cdots \leq \nu_{M+N} = N,
\]

\[
\mu_k + \nu_k = k \quad \text{for} \quad k = 0, \ldots, M + N.
\]

Clearly it is an approximation to (3). (Not a metric, since \( d(a, a) \neq 0 \), which is harmless.) Dynamic programming helps us to compute the minimum over \( (\mu, \nu) \). To this end, introduce

\[
d_{k,i} = \min_{(\mu, \nu); \mu_k=i} \max_{l=0, \ldots, k} |a_{\mu_l} - b_{\nu_l}|
\]

for \( i = 0, \ldots, \min(k, M) \) and consider the array \( D_k = (d_{k,i})_{i=0, \ldots, \min(k, M)} \). Having \( D_k \) we can compute \( D_{k+1} \) easily. Thus, we compute \( D_0, \) then \( D_1, \) and so on, up to \( D_{M+N}; \) the latter gives us \( d(a, b) \).

However, \( M + N \) is typically as large as \( 10^5 \) (for \( n = 1024, \) my worst case). For speeding up the computation, I replace the long sequence \( a = (a_0, \ldots, a_M) \) with a subsequence \( (a_{k_1}, \ldots, a_{k_p}) \) such that \( |a_{k_l} - a_l| \leq \varepsilon \) for \( k_l < l < k_{l+1}. \) (The same is done for the other sequence \( b. \)) The parameter \( \varepsilon \) was chosen to be 0.03.

Now, other technicalities. Random colors (of the hexagons) are produced by a Wichmann-Hill generator of pseudo-random numbers, initialized (before each trial) with a one-to-one function of \( \log_2 n, \log_2 k \) and the number of the trial (within the sample).

The program, in the ‘Python’ programming language, is available on my Web site.

The computer: a personal computer with matherboard Intel D815EGEW, processor Intel Pentium III (933 MHz), cache RAM 256 Kb and SDRAM 256 Mb (133 MHz).

The operating system: SuSE Linux 9.0.

Python: version 2.3+.

One trial takes typically from 0.5 min to 1 min for \( n = 1024, \) and 10–15 sec for \( n = 512. \)
Conclusion

The results (Table 1) suggest that the scaling limit of percolation is *strip stable*, and therefore may be treated as a black noise.

One could object that the strong concentrated perturbation is not enough; a distributed ‘microscopic’ perturbation should be added, such that exploration paths remain macroscopically close to the same $\text{SLE}_6$ curves, but are changed microscopically. That is correct, but I guess that such ‘micro’ perturbations may be absorbed somehow by the strong perturbation in the infinitesimal strip.

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