THE ASYMPTOTICAL BEHAVIOUR OF EMBEDDED EIGENVALUES FOR PERTURBED PERIODIC OPERATORS

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Abstract. Let $H$ be a periodic operator on $\mathbb{R}^+$ (or periodic Jacobi operator on $\mathbb{N}$). It is known that the absolutely continuous spectrum of $H$ is consisted of spectral bands $\cup [\alpha_l, \beta_l]$. Under the assumption that $\lim \sup_{x \to \infty} |V(x)| < \infty$ (or $\lim \sup_{n \to \infty} n|V(n)| < \infty$), the asymptotical behaviour of embedded eigenvalues approaching to the spectral boundary is established.

1. Introduction and main results

We consider the continuous Schrödinger equation on $\mathbb{R}$ ($\mathbb{R}^+$),

$$Hu = -u'' + (V(x) + V_0(x)) u = Eu,$$

where $V_0(x)$ is 1-periodic and $V(x)$ is a decaying perturbation.

In this paper, we always assume $V_0$ is 1-periodic and in $L^1[0,1]$. Without loss of generality, we only consider the half-line equation.

When $V \equiv 0$, we have a 1-periodic Schrödinger equation,

$$H_0 \varphi = -\varphi'' + V_0(x) \varphi = E\varphi.$$

It is known that the absolutely continuous spectrum (essential spectrum) of $H_0$ is consisted of a union of closed intervals (often referred to as spectral bands). We denote by

$$\sigma_{ac}(H_0) = \sigma_{ess}(H_0) = \bigcup_l [\alpha_l, \beta_l].$$

The bands may collapse at some boundaries. Putting the collapsed bands together, we denote by

$$\sigma_{ac}(H_0) = \sigma_{ess}(H_0) = \bigcup_l [\tilde{\alpha}_l, \tilde{\beta}_l].$$

The last band may have the form as $[c, \infty)$. In order to make the difference, we call $[\alpha_l, \beta_l]$ a standard spectral band and $[\tilde{\alpha}_l, \tilde{\beta}_l]$ a non-standard spectral band.

By Floquet theory, for any $E \in (\alpha_l, \beta_l)$, there exists $\varphi$ of (2), which has the following form

$$\varphi(x, E) = p(x, E)e^{ik(E)x}$$

where $k(E)$ is the quasimomentum, and $p(x, E)$ is 1-periodic. In each standard spectral band $[\alpha_l, \beta_l]$, $k(E)$ is monotonically increasing from 0 to $\pi$ or monotonically decreasing from $\pi$ to 0.

Similarly, we consider the periodic Jacobi equation,

$$(H_0 u)(n) := a_{n+1}u(n+1) + a_n u(n-1) + b_{n+1}u(n) = Eu(n), n \ge 0,$$

where the $\{a_j, b_j\}$ are real sequences with $q$-period, and $a_j$ is positive for all possible $j$.

Likewise, we consider a class of the perturbed periodic Jacobi equation,

$$(Hu)(n) = a_{n+1}u(n+1) + a_n u(n-1) + (b_{n+1} + V(n+1))u(n) = Eu(n), n \ge 0,$$
where $V(n)$ is an real sequences. Now we will extend the notations for the continuous case to the discrete one. For $E \in (\alpha, \beta)$, let $\varphi$ be the Floquet solution of (4), namely,

$$\varphi(n, E) = p(n, E) e^{i k(E) n},$$

where $p(n, E)$ is a $q$-periodic sequence and $k(E) \in (0, \pi)$ is the quasimomentum ($q$ is the period for $a_n$ and $b_n$). We also have the standard bands and non-standard bands notations,

$$\sigma_{ac}(H_0) = \sigma_{ess}(H_0) = \bigcup_l [\alpha_l, \beta_l] = \bigcup_l [\tilde{\alpha}_l, \tilde{\beta}_l].$$

It is a very interesting problem to study phenomenons of eigenvalues/singular continuous spectrum embedded into the absolutely continuous (or essential) spectrum based on the decaying perturbations, for example [4, 6, 7, 9–13, 18–22, 25, 26]. We care about the asymptotical behavior of eigenvalues approaching to the spectral boundaries. The asymptotical behavior of eigenvalues lying outside spectral bands has been well studied [1–3, 8, 24].

In order to state our main results, more notations are necessary. The Wronskian for the continuous case is given by

$$W(f, g)(x, E) = \tilde{\varphi}_1 f(x) - \tilde{\varphi}_2 g(x),$$

where $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are the Floquet solutions of (4), namely,

$$\tilde{\varphi}_1(x, E) = \tilde{\varphi}(x, E) - V(x),$$

and

$$\tilde{\varphi}_2(x, E) = \tilde{\varphi}(x, E) + V(x).$$

In the continuous case, define

$$\Gamma(E) = \int_0^1 \frac{4}{\omega^2} |\varphi(x, E)|^4 \, dx.$$

In the discrete case, define

$$\Gamma(E) = \frac{1}{q} \sum_{n=1}^q \frac{4}{\omega^2} |\varphi(n, E)|^4.$$

$\Gamma(E)$ is well defined for $E \in (\alpha, \beta)$ and blows up as $E \to \alpha$ or $E \to \beta$. In the following arguments, $H_0$ is always a continuous periodic operator or periodic Jacobi operator. $V(x)$ or $V(n)$ is the perturbed potential.

In the continuous case, define $P$ as

$$P = \{ E \in \mathbb{R} : -u'' + (V(x) + V_0(x))u = Eu \text{ has an } L^2(\mathbb{R}^+) \text{ solution } \},$$
and in the discrete case, define $P$ as
\begin{equation}
P = \{ E \in \mathbb{R} : \text{equation } 5 \text{ has an } l^2(\mathbb{N}) \text{ solution } \}.
\end{equation}

$P$ is the collection of eigenvalues for operators $H_0 + V$ with all the possible boundary conditions at 0.

**Theorem 1.1.** Suppose $\lim \sup_{x \to \infty} x|V(x)| = A$ ($\lim \sup_{n \to \infty} n|V(n)| = A$). For any standard spectral band $[\alpha_1, \beta_1]$, let $\delta_1$ be the unique point in $[\alpha_1, \beta_1]$ such that the quasimomentum $k(\delta_1) = \frac{\pi}{2}$. Then the set $P \cap (\alpha_1, \beta_1)$ is a countable set with two possible accumulation points $\alpha_1$ and $\beta_1$. Moreover,
\begin{equation}
\sum_{E_i \in P \cap (\alpha_1, \delta_1)} \frac{1}{\Gamma(E_i)} \leq \frac{A^2}{2},
\end{equation}
and
\begin{equation}
\sum_{E_i \in P \cap (\delta_1, \beta_1)} \frac{1}{\Gamma(E_i)} \leq \frac{A^2}{2}.
\end{equation}

For any $\lambda \in \cup_i \{\alpha_i, \beta_i\}$, we define $\kappa_\lambda = 1$ if $\lambda$ is non-collapsed. Otherwise, $\kappa_\lambda = 2$. See the formal description of “non-collapsed” in Theorem 2.1 and the Remark after.

Based on Theorems 1.1 we can get a lot of Corollaries.

**Corollary 1.2.** Suppose $\lim \sup_{x \to \infty} x|V(x)| = A$ ($\lim \sup_{n \to \infty} n|V(n)| = A$). Then for any standard spectral band $[\alpha_1, \beta_1]$, $P \cap [\alpha_1, \beta_1]$ is a countable set. Moreover, there is a constant $K$ (depends on $k$ uniformly in any bounded set ) such that
\begin{equation}
\sum_{E_i \in P \cap (\alpha_1, \delta_1)} \min\{|E_i - \alpha_1|^{\kappa_{\alpha_1}}, |E_i - \beta_1|^{\kappa_{\beta_1}}\} \leq KA^2.
\end{equation}

In particular,
\begin{equation}
\sum_{E_i \in P \cap (\alpha_1, \delta_1)} \min\{|E_i - \alpha_1|^2, |E_i - \beta_1|^2\} \leq KA^2.
\end{equation}

**Corollary 1.3.** Suppose $\lim \sup_{x \to \infty} x|V(x)| = A$ ($\lim \sup_{n \to \infty} n|V(n)| = A$). Then for any non-standard spectral band $[\tilde{\alpha}_1, \tilde{\beta}_1]$, there exists some $\epsilon$ (only depends on $H_0$ not on $l$) and a constant $K$ (depends on $l$ uniformly in any bounded set) such that
\begin{equation}
\sum_{E_i \in P \cap (\tilde{\alpha}_1, \tilde{\beta}_1)} \min\{|E_i - \tilde{\alpha}_1|, |E_i - \tilde{\beta}_1|\} \leq KA^2,
\end{equation}
and
\begin{equation}
\sum_{E_i \in P \cap (\tilde{\beta}_1 - \epsilon, \tilde{\beta}_1)} \min\{|E_i - \tilde{\alpha}_1|, |E_i - \tilde{\beta}_1|\} \leq KA^2.
\end{equation}

**Remark 1.4.** In the continuous case, $\beta_1 - \alpha_1 = 2\pi^2 l + O(1)$ as $k$ goes to infinity. Thus we can choose $\epsilon \approx \pi^2 l$ for large $l$.

**Corollary 1.5.** Suppose $V_0(x) = 0$ and $\lim \sup_{x \to \infty} x|V(x)| = A$. Then $P \cap (0, \infty)$ is a countable set. Moreover,
\begin{equation}
\sum_{E_i \in P \cap (0, \infty)} E_i \leq \frac{A^2}{2}.
\end{equation}
Corollary 1.6. Suppose \( V_0(n) = 0 \) and \( \limsup_{n \to \infty} n|V(n)| = A \). Then \( P \cap (-2, 2) \) is a countable set. Moreover,
\[
\sum_{E_i \in P \cap (-2, 2)} (4 - E_i^2) \leq 4A^2 + 4\min\{1, A\}.
\]

For the discrete case, \( H_0 \) has \( q \) standard spectral bands and the set of spectrum is bounded. Then the constant \( K(\epsilon) \) only depends on \( H_0 \).

Corollary 1.7. Suppose \( \limsup_{n \to \infty} n|V(n)| = A \). Then there is a constant \( K \) such that
\[
\sum_{E_i \in P \cap \sigma_{ess}(H_0)} d^2(E_i) \leq KA^2.
\]

Corollary 1.8. Suppose \( \limsup_{n \to \infty} n|V(n)| = A \). Then there exist constants \( \epsilon > 0 \) and \( K > 0 \) such that
\[
\sum_{E_i} \min\{|E_i - \tilde{\alpha}_i|, |E_i - \tilde{\beta}_i|\} \leq KA^2,
\]
where \( \{E_i\} \) goes over all the values in set \( P \cap \left( \cup_i (\tilde{\alpha}_i, \tilde{\alpha}_i + \epsilon) \cup (\tilde{\beta}_i - \epsilon, \tilde{\beta}_i) \right) \).

Remark 1.9.
- All the bounds go to 0 as \( A \) goes to zero. It implies that there is no \( L^2(\mathbb{R}^+) \) (or \( L^2(\mathbb{N}) \)) solution of \( (H_0 + V)u = Eu \) for \( E \in \cup_i (\alpha_i, \beta_i) \) if \( V(x) = \frac{a(1)}{1+x} \) \( (V(n) = \frac{a(1)}{1+n}) \). Thus under any fixed boundary condition at 0, there is no eigenvalues of the operator \( H_0 + V \) embedded into its bands if \( V(x) = \frac{a(1)}{1+x} \) \( (V(n) = \frac{a(1)}{1+n}) \).
- Corollary 1.5 (Corollary 1.6) with exactly the same bound has been proved in [10] (12). It means our bounds in Theorem 1.1 are effective.
- Fix \( l \). Corollary 1.2 implies the speed of \( E_i \in P \) going to the collapsed boundaries behaves like \( |E_i - \alpha_i| \approx \frac{1}{1+\sqrt{l}} \) \( (|E_i - \beta_i| \approx \frac{1}{1+\sqrt{l}}) \). The speed going to non-collapsed boundaries behaves like \( |E_i - \alpha_i| \approx \frac{1}{1+l} \) \( (|E_i - \beta_i| \approx \frac{1}{1+l}) \). Remling [23] shows that we can not improve it to \( |E_i - \alpha_i| \approx \frac{1}{1+l^{1/2}} \) in the continuous case. This means that our main results are optimal in some sense.

Our paper is organized in the following way. In Section 2 we will introduce the Floquet solution and generalized Prüfer transformation. In Section 3 we will give the proof of all the results in the continuous setting. In Section 4 we will give the proof of all the results in the discrete setting.

In the following, \( E \) is always in some spectral band \( (\alpha_i, \beta_i) \). The constant \( C \) depends on \( E \), and so does \( O(1) \). \( K \) depends on \( E \) (or say \( l \)) uniformly in any compact set.

2. Floquet solution and generalized Prüfer transformation for the continuous Schrödinger operator

Let \( T_0(E) \) be the transfer matrix of \( H_0 \) from 0 to 1, that is
\[
T_0(E) \left( \begin{array}{c} u(0) \\ u'(0) \end{array} \right) = \left( \begin{array}{c} u(1) \\ u'(1) \end{array} \right)
\]
for any solution \( u \) of \( H_0u = Eu \).

Let
\[
T_0(E) = \left( \begin{array}{cc} a(E) & b(E) \\ c(E) & d(E) \end{array} \right)
\]
and \( D(E) = a(E) + d(E) \)
Theorem 2.1. [3, Theorem 2.3.3] Let $\Lambda = \cup_t \{\alpha_t, \beta_t\}$. Then for any $\lambda \in \Lambda$, $D(\lambda)$ is either 2 or $-2$. Furthermore, one of $D'(\lambda)$ and $D''(\lambda)$ must be non-zero.

Remark 2.2. $D'(\lambda) \neq 0$ if and only if $\lambda$ is non-collapsed.

For $E \in (\alpha_l, \beta_l)$, let $k(E) \in (0, \pi)$ be the quasimomentum. Let $\varphi(x, E)$ be a solution of $H_0u = Eu$ with $E \in (\alpha_l, \beta_l)$ and satisfies boundary condition $\varphi(0) = -b(E)$ and $\varphi'(0) = a(E) - e^{ik(E)}$. It is easy to check that

$$T_0(E) \left( \begin{array}{c} \varphi(0) \\ \varphi'(0) \end{array} \right) = e^{ik(E)} \left( \begin{array}{c} \varphi(0) \\ \varphi'(0) \end{array} \right).$$

We get a Floquet solution $\varphi(x, E)$.

Since $\varphi(x, E)$ is a solution of $H_0u = Eu$, the Wronskian $W(\varphi, \varphi)$ is constant and

$$W(\varphi, \varphi) = 2i \text{ Im } (\varphi \varphi').$$

Let $\omega(E) \in \mathbb{R}$ be such that

$$2i \text{ Im } (\varphi(x, E) \varphi'(x, E)) = i\omega(E).$$

Thus

$$\omega(E) = 2b(E) \sin k(E).$$

Without loss of generality, assume $\omega(E) > 0$ (i.e., $b(E) > 0$).

Let $\gamma(x, E)$ be such that

$$\varphi(x, E) = |\varphi(x, E)|e^{i\gamma(x, E)}.$$

Then (See [11, Prop.2.1])

$$\gamma'(x, E) = \frac{\omega(E)}{2|\varphi(x, E)|^2}.$$ 

Proposition 2.3 (Proposition 2.2 and Theorem 2.3bc of [11]). Suppose $u$ is a real solution of (1). Then there exist real functions $R(x, E) > 0$ and $\theta(x, E)$ such that

$$|\ln R(x, E)|' = \frac{V(x)}{2\gamma'(x, E)} \sin 2\theta(x, E)$$

and

$$\theta(x, E)' = \gamma'(x, E) - \frac{V(x)}{2\gamma'(x, E)} \sin^2 \theta(x, E).$$

Moreover, there exists a constant $C$ (depending on $E$) such that

$$\frac{|u(x, E)|^2 + |u'(x, E)|^2}{C} \leq R(x, E)^2 \leq C(|u(x, E)|^2 + |u'(x, E)|^2).$$

Lemma 2.4. The following estimate holds,

$$\frac{1}{\gamma'(x, E)} \leq \frac{K(E)}{\sin k(E)},$$

and then (in the continuous case)

$$\Gamma(E) \leq \frac{K(E)}{|\sin k(E)|^2},$$

where $K(E)$ is uniformly bounded on any compact set of $E$. 

Thus it suffices to show
\begin{equation}
\frac{|\varphi(x,E)|^2}{\omega} \leq \frac{K(E)}{\sin k(E)}.
\end{equation}

Since $|\varphi(x,E)|$ is 1-periodic, it suffices to prove (15) for $0 \leq x \leq 1$. By the fact that $V_0 \in L^1[0,1]$ and Grönwall’s inequality, one has
\begin{equation}
|\varphi(x,E)|^2 \leq K(E)(|\varphi(0,E)|^2 + |\varphi'(0,E)|^2).
\end{equation}

Thus it suffices to show
\begin{equation}
\frac{|\varphi(0,E)|^2 + |\varphi'(0,E)|^2}{\omega} \leq \frac{K(E)}{\sin k(E)}.
\end{equation}

By the definition,
\begin{equation}
\frac{|\varphi(0,E)|^2 + |\varphi'(0,E)|^2}{\omega} = \frac{b^2(E) + (a(E) - \cos k(E))^2 + \sin^2 k(E)}{2b \sin k(E)}.
\end{equation}

Using $a(E) + d(E) = 2 \cos k(E)$ and $a(E)d(E) - b(E)c(E) = 1$, one has
\begin{align}
b^2(E) + (a(E) - \cos k(E))^2 + \sin^2 k(E) &= b^2(E) + a^2(E) - 2a(E) \cos k(E) + 1 \\
&= b^2(E) - a(E)d(E) + 1 \\
&= b^2(E) - b(E)c(E).
\end{align}

Since $||T_0|| \leq K(E)$, one has $|b(E)| \leq K, |c(E)| \leq K$. It implies
\begin{equation}
\frac{|\varphi(0,E)|^2 + |\varphi'(0,E)|^2}{\omega} = \frac{b(E) - c(E)}{2 \sin k(E)} \leq \frac{K(E)}{\sin k(E)}.
\end{equation}

We finish the proof. \hfill \Box

**Lemma 2.5.** [10, Lemma 4.4] Let $\{e_i\}_{i=1}^N$ be a set of unit vectors in a Hilbert space $\mathcal{H}$ so that
\begin{equation}
\alpha = N \sup_{i \neq j} |\langle e_i, e_j \rangle| < 1.
\end{equation}

Then
\begin{equation}
\sum_{i=1}^N |\langle g, e_i \rangle|^2 \leq (1 + \alpha) \|g\|^2.
\end{equation}

3. PROOF OF THE RESULTS IN CONTINUOUS SETTINGS

Although the following Lemma is proved for a special class of functions in [17], it works for all $L^2[0,1]$ functions.

**Lemma 3.1.** Suppose $V(x) = \frac{O(1)}{1+x}$ and $f \in L^2([0,1])$ is a 1-periodic function. Then
\begin{equation}
\left| \int_0^x f(t) \frac{\cos 4\theta(t,E)}{1+t} dt \right| = O(1),
\end{equation}

for $E \in \cup_i (\alpha_i, \beta_i)$ with $k(E) \neq \frac{\pi}{2}$. Suppose $E_1, E_2 \in \cup_i (\alpha_i, \beta_i)$ satisfy $k(E_1) \neq k(E_2)$ and $k(E_1) + k(E_2) \neq \pi$. Then we have
\begin{equation}
\left| \int_0^x f(t) \frac{\sin 2\theta(t,E_1) \sin 2\theta(t,E_2)}{1+t} dt \right| = O(1).
\end{equation}
Proof of Theorem 1.1 in the continuous setting. By the assumption of Theorem 1.1 for any $M > A$, we have
\[ |V(x)| \leq \frac{M}{1 + x} \]
for large $x$. By shifting the equation, we can assume
\[ (19) \quad |V(x)| \leq \frac{M}{1 + x} \]
for all $x > 0$.

Let us take any standard spectral band $(\alpha, \beta) = (\alpha_l, \beta_l)$ into consideration. Let $\delta \in (\alpha, \beta)$ be such that the quasimomentum $k(\delta) = \pi/2$.

Suppose $E_1, E_2, \ldots, E_N \in (\alpha, \delta) \cap P$. By (13) (also use $u \in L^2$ implies $u' \in L^2$), we have
\[ \sum_{i=1}^{N} R(x, E_i) \in L^2(\mathbb{R}^+), \]
and then there exists $B_j \to \infty$ such that
\[ (20) \quad R(B_j, E_i) \leq B_j^{-\frac{1}{2}}, \]
for all $i = 1, 2, \ldots, N$.

By (11), one has
\[ (21) \quad \ln R(x, E) - \ln R(0, E) = \int_0^x \frac{V(t)}{2\gamma'(t, E)} \sin 2\theta(t, E) dt. \]
By (20) and (21), we have
\[ (22) \quad \int_0^{B_j} \frac{V(t)}{\gamma'(t, E_i)} \sin 2\theta(t, E_i) dt \leq -B_j + O(1), \]
for all $i = 1, 2, \ldots, N$.

Now let us consider the Hilbert spaces
\[ \mathcal{H}_j = L^2((0, B_j), (1 + x)dx). \]
In $\mathcal{H}_j$, by (19) we have
\[ (23) \quad ||V||_{\mathcal{H}_j}^2 \leq M^2 \log(1 + B_j). \]
Let
\[ e_i^j(x) = \frac{1}{A_i^j} \sin 2\theta(x, E_i) \gamma'(x, E_i)(1 + x)^{1/2} \chi_{[0, B_j]}(x), \]
where $A_i^j$ is chosen so that $e_i^j$ is a unit vector in $\mathcal{H}_j$.

Obviously,
\[ A_i^j = \int_0^{B_j} \frac{\sin^2 2\theta(x, k_i)}{\gamma'(x, E_i)^2(1 + x)} dx \]
\[ (24) \quad = \int_0^{B_j} \frac{1}{2\gamma'(x, E_i)^2(1 + x)} dx - \int_0^{B_j} \frac{\cos 4\theta(x, k_i)}{\gamma'(x, E_i)^2(1 + x)} dx. \]
By (17), one has
\[ (25) \quad \left| \int_0^{B_j} \frac{\cos 4\theta(x, k_i)}{\gamma'(x, E_i)^2(1 + x)} dx \right| = O(1), \]
for all $i = 1, 2, \ldots, N$. 
Direct computation shows that
\[
\int_0^{B_j} \frac{1}{|\gamma'(x, E_i)|^2(1 + x)} \, dx = O(1) + \sum_{n=0}^{B_j-1} \int_n^{n+1} \frac{1}{|\gamma'(x, E_i)|^2(1 + n)} \, dx
\]
\begin{equation}
= O(1) + \Gamma(E_i) \log B_j.
\end{equation}

By (24), (25) and (26), we have
\begin{equation}
A_j^i = \frac{1}{2} \Gamma(E_i) \log B_j + O(1).
\end{equation}

Since \( E_i \in (\alpha, \delta) \) \((k(E_i) \in (0, \frac{1}{2}))\) for all \( i = 1, 2, \ldots, N \), one has \( k_i + k_g \neq \pi \),

for all \( 1 \leq i, g \leq N \).

By (18), we have
\begin{equation}
O(1) \log B_j \leq \langle V, e_j^i \rangle_{H_j} \leq \frac{O(1)}{\log B_j},
\end{equation}
for all \( 1 \leq i, g \leq N \) and \( i \neq g \).

By (27) and (22)
\begin{equation}
\langle V, e_j^i \rangle_{H_j} \leq -\frac{\sqrt{2}}{\sqrt{\Gamma(E_i)}} \sqrt{\log B_j} + O(1),
\end{equation}
for large \( j \). By (16) and (28), one has
\begin{equation}
\sum_{i=1}^{N} |\langle V, e_j^i \rangle_{H_j}|^2 \leq (1 + \frac{O(1)}{\log B_j}) \|V\|_{H_j}.
\end{equation}

By (29), (30) and (23), we have
\begin{equation}
\sum_{i=1}^{N} \frac{2}{\Gamma(E_i)} \log B_j \leq M^2 \log B_j + O(1).
\end{equation}

Let \( j \to \infty \), we get
\[
\sum_{i=1}^{N} \frac{1}{\Gamma(E_i)} \leq \frac{M^2}{2},
\]
for any \( M > A \). This implies
\begin{equation}
\sum_{i=1}^{N} \frac{1}{\Gamma(E_i)} \leq \frac{A^2}{2}.
\end{equation}

By (14), one has
\[
\Gamma(E) \leq \frac{K(E)}{|\sin k(E)|^2}.
\]
It implies
\begin{equation}
\sum_{i=1}^{N} |\sin k(E_i)|^2 \leq KA^2.
\end{equation}

\((\alpha, \delta) \cap P\) is a countable set with possible accumulation point \( \alpha \). Similarly, \((\delta, \beta) \cap P\) is a countable set with possible accumulation point \( \beta \), and the bounds as in (31) and (32) hold.

We finish the proof. \( \square \)
We need another lemma to treat the special situation $E = \delta$ separately.

**Lemma 3.2.** Fix any spectral band $[\alpha_l, \beta_l]$. Let $\delta_l \in [\alpha_l, \beta_l]$ be such that $k(\delta_l) = \frac{\pi}{2}$. Then there exists some $K > 0$ such that if $\limsup_{x \to \infty} x|V(x)| \leq \frac{1}{K}$, then $\delta_l \notin P$.

**Proof of Corollary 1.2 in the continuous setting.** By (32) and also taking $(\delta, \beta) \cap P$ into consideration, we have

\[(33) \sum_{E_i \in P \cap (\alpha, \beta), E_i \neq \delta} \sin k(E_i)^2 \leq KA^2.\]

Combining with Lemma 3.2, we have,

\[(34) \sum_{E_i \in P \cap (\alpha, \beta)} \sin k(E_i)^2 \leq KA^2.\]

By the fact that $2 \cos \pi k(E_i) = D(E_i)$, one has

\[(35) \sum_{E_i \in P \cap (\alpha, \beta)} (4 - D(E_i)^2) \leq KA^2.\]

Without loss of generality, assume $D(\alpha) = 2$ and $D(\beta) = -2$. If $\alpha$ is non-collapsed,

\[(36) 2 - D(E_i) \geq \frac{1}{K} |E_i - \alpha|.\]

If $\alpha$ is collapsed, by Theorem 2.1, one has

\[(37) 2 + D(E_i) \geq \frac{1}{K} |E_i - \alpha|^2.\]

Similarly, if $\beta$ is non-collapsed,

\[(38) 2 + D(E_i) \geq \frac{1}{K} |E_i - \beta|.\]

If $\beta$ is collapsed, by Theorem 2.1, one has

\[(39) 2 - D(E_i) \geq \frac{1}{K} |E_i - \beta|^2.\]

By (34), (36), (37), (38) and (39), we have

\[(40) \sum_{E_i \in P \cap (\alpha, \beta)} \min \{|E_i - \alpha|^\kappa_\alpha, |E_i - \beta|^\kappa_\beta\} \leq KA^2.\]

**Proof of Corollary 1.3 in the continuous setting.** Since $\tilde{\alpha}_l$ is non-collapsed, there exists some standard spectral band $[\alpha, \beta] \subset [\tilde{\alpha}_l, \beta_l]$ such that $\alpha = \tilde{\alpha}$ and $D'(\alpha) \neq 0$. Let $\epsilon = \frac{\beta - \alpha}{2}$. Obviously, for any $E \in (\alpha, \alpha + \epsilon) \cap P$,

\[|E - \beta| \geq \frac{1}{K}.\]

Then

\[(41) \min \{|E - \beta|, |E - \alpha|\} \geq \frac{|E - \alpha|}{K}.\]

Notice that $\kappa_\alpha = 1$. By (40) and (41), we have

\[\sum_{E_i \in P \cap (\tilde{\alpha}_l, \tilde{\alpha}_l + \epsilon)} |E_i - \tilde{\alpha}_l| \leq KA^2.\]
Similarly,
\[ \sum_{E_i \in P \cap (\tilde{\beta}_l - \epsilon, \tilde{\beta}_l]} \min\{|E_i - \tilde{\beta}_l|\} \leq KA^2. \]

**Proof of Corollary 1.5.** For the free Schrödinger equation, we can choose Floquet solution \( \varphi(x, E) = e^{i\sqrt{E}x} \) for \( E > 0 \). Direct computation and by (10), one has
\[ \gamma'(x, E) = \sqrt{E}. \]
and then
\[ (43) \Gamma(E) = \frac{1}{E}. \]
In this case, let \( \alpha = 0, \beta = \infty \) and \( \delta = \infty \). Following the proof of (31), we have
\[ \sum_{E_i \in P \cap (0, \infty]} E_i \leq A^2. \]
\[ \Box \]

4. **Proof of the results in discrete settings**

In this section, all the equations are discrete. Similarly, we can also introduce the Prüfer variables \( R(n, E) \) and \( \theta(n, E) \). See [11] for details.

\[ \frac{R(n+1, E)^2}{R(n, E)^2} = 1 - V(n+1) \frac{2}{\omega} \sin 2\theta(n, E)|\varphi(n, E)|^2 + \frac{4V(n+1)^2|\varphi(n, E)|^4}{\omega^2} \sin^2 \theta(n, E). \]

Let us take one standard spectral band \([\alpha, \beta] = [\alpha_l, \beta_l]\) into consideration. Let \( \delta \in (\alpha, \beta) \) be such that the quasimomentum \( k(\delta) = \frac{\pi}{2} \).

**Lemma 4.1.** [17] Suppose \( V(n) = \frac{O(1)}{1+n} \) and \( f(n) \) is \( q \) periodic. Let \( E_1, E_2 \in (\alpha, \beta) \) be such that \( k(E_1) \neq k(E_2) \) and \( k(E_1) + k(E_2) \neq \pi \). Assume \( k(E_1) \neq \frac{\pi}{2} \). Then for any \( \epsilon > 0 \), there exist \( D(E_1, E_2, \epsilon) \) and \( D(E_1, \epsilon) \) such that
\[ (44) \left| \sum_{t=1}^{n} f(t) \cos 4\theta(t, E_1) \right| \leq D(E_1, \epsilon) + \epsilon \ln n, \]
and
\[ (45) \left| \sum_{t=1}^{n} f(t) \sin 2\theta(t, E_1) \sin 2\theta(t, E_2) \right| \leq D(E_1, E_2, \epsilon) + \epsilon \ln n. \]

**Proof of Theorem 1.1 in the discrete setting.** Replacing Lemma 3.1 by Lemma 4.1, the proof of discrete case can be proceeded in a similar way as that of continuous case. We omit the details.

**Lemma 4.2.** [12] There exists some \( K > 0 \) such that if \( \limsup_{n \to \infty} n|V(n)| \leq \frac{1}{K} \), then \( \delta \notin P \).

**Proof of Corollaries 1.2 and 1.3 in the discrete setting.** The proof follows from the continuous case. We omit the details here.

In order to prove Corollary 1.6, we also need one Lemma.

**Lemma 4.3.** [12] Theorem 2.1 If \( V_0 = 0 \) and \( \limsup_{n \to \infty} n|V(n)| < 1 \), then 0 can not be in \( P \).
Proof of Corollary 1.6. In this case, we only have one spectral band $[-2, 2]$. Moreover, 
\[ \varphi(n, E) = e^{ik(E)n}, \quad 2\cos k(E) = E \text{ for } E \in (-2, 2) \text{ and } \Gamma(E) = \frac{1}{\sin^2 k(E)}. \]
By Theorem 1.1 ($\delta = 0$),
\[ \sum_{E_i \in P \cap (-2, 2); E_i \neq 0} \frac{1}{\Gamma(E_i)} \leq A^2. \]
Thus
\[ \sum_{E_i \in P \cap (-2, 2); E_i \neq 0} (4 - E_i^2) \leq 4A^2. \]
We may add 4 in the bound of (46) if 0 is in $P$. However, if $A < 1$, by Lemma 1.2, 0 cannot be in $P$. We finish the proof.

Proof of Corollaries 1.7 and 1.8. It follows from Corollaries 1.2 and 1.3 immediately since the discrete operator has finitely many spectral bands.

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