SYMmetry groups and lAgrangians associated to tîteica surfaces

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Abstract
Tîteica proved [23] that the surfaces for which the ratio $\frac{K}{d^4}$ is constant (where $K$ is the Gaussian curvature and $d$ is the distance from the origin to the tangent plane at an arbitrary point) are invariants under the group of centroaffine transformations. In this paper one applies the symmetry groups theory ([20], [21]) to study the PDEs which arise in Tîteica surfaces theory: the PDEs system (5) (equivalent to (15)+(16)) with the particular cases (7) and (9), the PDEs system (11) and the Liouville-Tîteica PDE (8), respectively the Tîteica PDE (10) (equivalent to (8') respectively (10')). In the case of the PDEs systems (15)+(16) and (11), the center of our attention is to find the symmetry subgroups $G_1$ of the full symmetry group $G$, respectively $\bar{G}_1$ of the full symmetry group $\bar{G}$, which act on the space of dependent variables, and also the symmetry subgroups $G_2$ of $G$ and $\bar{G}_2$ of $\bar{G}$, which act on the space of independent variables (Theorems 4, 5, 6 and 7). One proves (Theorem 4) that the subgroup $G_1$ is the unimodular subgroup of the group of the centroaffine transformations. One gives a new solution (21) for the Tîteica PDE for which it is a ruled Tîteica surface associated (Proposition 1). One finds the symmetry groups of Liouville-Tîteica PDE and Tîteica PDE (Theorems 9 and 10) and one proves that these are Euler-Lagrange equations with the Lagrangians (32) and (33) (Theorem 12). One gets the variational symmetry groups of the associated functionals (38) and (39) (Theorems 16 and 17) and conservation laws (Proposition 3). We make the remark that the Tîteica simple surfaces $z = f(x, y)$ was studied also in [28] and we proved that the Tîteica PDE is an Euler-Lagrange equation. All these results shows that Tîteica surfaces theory is strongly related to variational problems and hence it is a subject of global differential geometry.

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1 Introduction

Tîteica-the founder of the centroaffine geometry- introduced in 1907 a new class of surfaces, called the surfaces $S$, with the property that $\frac{K}{d^4}$=constant, where $K$
is the Gaussian curvature and \( d \) is the distance from the origin to the tangent plane at an arbitrary point [23]. These were called \( \dot{\tau} \) \( \epsilon \) \( \epsilon \) surfaces by Gheorghiu, or affine spheres by Blaschke and projectives spheres by Wilczynski. The most simple \( \dot{\tau} \) \( \epsilon \) \( \epsilon \) surfaces are the spheres and the quadrics. The extension of this class to hypersurfaces was considered by \( \dot{\tau} \) \( \epsilon \) \( \epsilon \) itself [24],[25]. Also, Mayer [19], Gheorghiu [12], Dobrescu [10] and Vr˘ anceanu [29],[30] studied the properties of these hypersurfaces. Gheorghiu made a remark on the hypersurfaces \( \dot{\tau} \) \( \epsilon \) \( \epsilon \): these can be considered as the affine spaces \( A^{n-1} \), embedded in a affine Euclidean space \( E^n \). Using this result, he introduced a new class of affine space \( A^{0,n} \) and new examples of these were considered by Udri¸ ste [26]. We start to make a short presentation of the \( \dot{\tau} \) \( \epsilon \) \( \epsilon \) surfaces.

Let \( D \subset \mathbb{R}^2 \) be an open set and let

\[ \Sigma : \quad r(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}, \quad (u, v) \in D, \]

be a surface in \( \mathbb{R}^3 \), different from a cone with the vertex at the origin of the system of coordinates. Thus, the position vector \( r \) of an arbitrary point of the surface satisfies the condition

\[ (r, r_u, r_v) \neq 0, \]

and this can be considered the solution of the second order PDEs system,

\[
\begin{align*}
    r_{uu} &= ar_u + br_v + cr \\
    r_{uv} &= a'r_u + b'r_v + c'r \\
    r_{vv} &= a''r_u + b''r_v + c''r,
\end{align*}
\]

which is completely integrable, i.e.,

\[ (r_{uu})_v = (r_{uv})_u, \quad (r_{uv})_v = (r_{vv})_u, \]

where \( a, a', a'', \ldots \) are nine functions of \( u \) and \( v \). The above system defines a surface, leaving a centroaffinity aside. The coefficients \( a, a', a'', \ldots \) are called the centroaffine invariants. If the surface \( \Sigma \) is related to the asymptotic lines (if a surface is not developable, then the two families of the asymptotic lines are distinct), then \( c = c'' = 0 \), and thus it is defined by the following completely integrable second order PDEs system

\[
\begin{align*}
    r_{uu} &= ar_u + br_v \\
    r_{uv} &= a'r_u + b'r_v + c'r \\
    r_{vv} &= a''r_u + b''r_v.
\end{align*}
\]

**Theorem 1 (\( \dot{\tau} \) \( \epsilon \) \( \epsilon \)).** Let \( \Sigma \) be a surface related to the asymptotic lines. The ratio \( I = \frac{K}{\dot{\tau}^2} \) is a constant if and only if \( a' = b' = 0 \).

Thus, the surfaces \( \dot{\tau} \) \( \epsilon \) \( \epsilon \) are defined by the PDEs system

\[
\begin{align*}
    r_{uu} &= ar_u + br_v \\
    r_{uv} &= h'r \\
    r_{vv} &= a''r_u + b''r_v,
\end{align*}
\]
where we denote \( c' = h \), and for which the integrability conditions (3) turn in

\[
\begin{align*}
ah &= h_u, & a_v &= ba'' + h, & b_v + bb'' &= 0, \\
h_v &= b'' h, & a_u'' + aa'' &= 0, & h &= b_u'' + a' b. \\
\end{align*}
\]

**Remark.** In the particular cases \( b = 0 \) or \( a'' = 0 \), \( \Sigma \) is a simply ruled surface. Thus, for \( b = 0 \), \( a'' \neq 0 \): the coordinates curves \( v = v_0 \) are straight lines, and for \( b \neq 0 \), \( a'' = 0 \): the coordinates curves \( u = u_0 \) are straight lines. If \( b = a'' = 0 \), then \( \Sigma \) is a double ruled surface (a quadric surface).

The ruled Titičca surfaces are given by the PDEs system

\[
\begin{align*}
\begin{cases}
  r_{uu} &= \frac{h}{r} r_u + \frac{\varphi(u)}{h} r_v, \\
  r_{uv} &= h r_u + h v, \\
  r_{vv} &= \frac{h}{h} r_v,
\end{cases}
\end{align*}
\]

where \( h \) is a solution of the Liouville-Titičca PDE

\[
(\ln h)_{uv} = h.
\]

The Titičca surfaces which are not ruled surfaces, are given by the PDEs system

\[
\begin{align*}
\begin{cases}
  r_{uu} &= \frac{h}{h} r_u + \frac{1}{h} r_v, \\
  r_{uv} &= h r_u + \frac{h}{h} r_v, \\
  r_{vv} &= \frac{1}{h} r_u + \frac{h}{h} r_v,
\end{cases}
\end{align*}
\]

where \( h \) is a solution of the Titičca PDE

\[
(\ln h)_{uv} = h - \frac{1}{h^2}.
\]

The PDEs system (5) can be identified with the completely integrable scalar PDEs system

\[
\begin{align*}
\begin{cases}
  \theta_{uu} &= a \theta_u + b \theta_v, \\
  \theta_{uv} &= h \theta, \\
  \theta_{vv} &= a' \theta_u + b' \theta_v,
\end{cases}
\end{align*}
\]

with the condition that three independent solutions of (11)+(6): \( x = x(u, v), y = y(u, v), z = z(u, v) \), define a Titičca surface. It is known that every linear combination of \( x, y, z \) is a solution of the system (11) also. Thus, a surface \( \Sigma \) is determined leaving a centroaffinity aside.

On the other hand, it is known that Sophus Lie is the founder of theory of symmetry groups. A modern presentation using the jets theory is introduced by Olver in his book [20]. A local group of transformations on the space of the independent and dependent variables of a studied PDEs system which transforms
the solutions of the system into its solutions, is called symmetry group or strong symmetry group of the system. The symmetry groups theory is very applied to study the ODEs, PDEs systems which appear in Geometry, Mechanics and Physics [2]-[6],[8],[9],[14],[17],[18],[20],[21],[27],[28],[31]. There are many computational programs for finding the defining system of infinitesimal symmetries, but in the ours cases of the PDEs systems (5) and (11) we cannot apply these. We make the remark that a other point of view in the study of the Liouville-Ţîţeica PDE and Ţîţeica PDE is contained the papers of Bobenko [7] and Wolf [31].

In this paper we shall apply this theory for finding infinitesimal symmetries of the PDEs systems which arise in Ţîţeica surfaces theory and we shall give a new point of view of Ţîţeica theory with the connection of the known results. We shall adopt the notation of the book of Olver [20].

2 Symmetry Group of PDEs System

Let consider the PDEs system

\[
\Delta_{\nu}(x, u^{(n)}) = 0, \quad \nu = 1, \ldots, l,
\]

with \( x = (x^1, \ldots, x^p) \), \( u = (u^1, \ldots, u^q) \) and \( \Delta(x, u^{(n)}) = (\Delta_1(x, u^{(n)}), \ldots, \Delta_l(x, u^{(n)})) \), is a differentiable function. We note \( u^{(n)} \) all the partial derivatives of the function \( u \) to 0 up \( n \). Any function \( u = h(x), h : D \subset \mathbb{R}^p \rightarrow U \subset \mathbb{R}^q \), \( h = (h^1, \ldots, h^q) \), induces the function \( u^{(n)} = \text{pr}^{(n)}h \) called the \( n \)-th prolongation of \( h \), which is defined by \( u_j^{(n)} = \partial_j h^\alpha \), \( \text{pr}^{(n)}h : D \rightarrow U^{(n)} \), and for each \( x \in D \), \( \text{pr}^{(n)}h \) is a vector whose \( qp^{(n)} = C_{p+n}^n \) entries represent the values of \( h \) and all its derivatives up to order \( n \) at the point \( x \).

The space \( D \times U^{(n)} \), whose coordinates represent the independent variables, the dependent variables and the derivatives of the dependent variables up to order \( n \), is called the \( n \)-th order jet space of the underlying space \( D \times U \). Thus \( \Delta \) is a map from the jet space \( D \times U^{(n)} \) to \( \mathbb{R}^l \). The PDEs system (12) determine the subvariety

\[
S = \{(x, u^{(n)}) | \Delta(x, u^{(n)}) = 0 \}
\]

of the total jet space \( D \times U^{(n)} \). One identifies the system of PDEs (12) with its corresponding subvariety \( S \).

Let \( M \subset D \times U \) be an open set. A symmetry group of the PDEs system (12) is a local group of transformations \( G \) acting on \( M \) with the property that whenever \( u = f(x) \) is a solution of (12) and whenever \( g \cdot f \) is defined for \( g \in G \), then \( u = g \cdot f(x) \) is also a solution of the system. The system (12) is called invariant with respect to \( G \).

Let us consider \( X \) a vector field on \( M \) with corresponding (local) 1-parameter group \( \exp(\varepsilon X) \) which is the infinitesimal generator of the symmetry group of the PDEs system (12). The infinitesimal generator of the corresponding prolonged
1-parameter group $pr^{(n)}[\exp(\varepsilon X)]$:

$$pr^{(n)}X|_{(x,u^{(n)})} = \frac{d}{d\varepsilon}pr^{(n)}[\exp(\varepsilon X)](x, u^{(n)})|_{\varepsilon=0}$$

for any $(x, u^{(n)}) \in M^{(n)}$, is a vector field on the $n$-jet space $M^{(n)}$ called the $n$-th prolongation of $X$ and denoted by $pr^{(n)}X$.

The PDEs system (12) is called to be of maximal rank if the Jacobi matrix

$$J_{\Delta}(x, u^{(n)}) = \left( \frac{\partial \Delta_{\nu}}{\partial x^i}, \frac{\partial \Delta_{\nu}}{\partial u_{J,i}^\alpha} \right)$$

of $\Delta$, with respect to all the variables $(x, u^{(n)})$, is of rank $l$ whenever $\Delta(x, u^{(n)}) = 0$.

**Theorem 2.** Let

$$X = \sum_{i=1}^{p} \zeta^i(x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_{\alpha}(x,u) \frac{\partial}{\partial u^\alpha}$$

be a vector field on open set $M \subset D \times U$. The $n$-th prolongation of $X$ is the vector field

$$pr^{(n)}X = X + \sum_{\alpha=1}^{q} \sum_{J} \phi_{\alpha}^J(x,u^{(n)}) \frac{\partial}{\partial u_{J,i}^\alpha},$$

defined on the corresponding jet space $M^{(n)} \subset D \times U^{(n)}$, the second summation being over all multi-indices $J = (j_1, \ldots, j_k)$ with $1 \leq j_k \leq p$, $1 \leq k \leq n$. The coefficient functions $\phi_{\alpha}^J$ of $pr^{(n)}X$ are given by the following formula

$$\phi_{\alpha}^J(x,u^{(n)}) = D_J \left( \phi_{\alpha} - \sum_{i=1}^{p} \zeta^i u_{i}^\alpha \right) + \sum_{i=1}^{p} \zeta^i u_{J,i}^\alpha,$$

where $u_{i}^\alpha = \frac{\partial u^\alpha}{\partial x^i}$, $u_{J,i}^\alpha = \frac{\partial u_{J,i}^\alpha}{\partial x^i}$.

**Theorem 3 (Criterion of infinitesimal invariance).** Let us consider the PDEs system (12) of maximal rank defined over $M \subset D \times U$. If $G$ is a local group of transformations acting on $M$ and

$$pr^{(n)}X[\Delta_{\nu}(x,u^{(n)})] = 0, \quad \nu = 1, \ldots, l,$$

whenever $\Delta_{\nu}(x,u^{(n)}) = 0$, for every infinitesimal generator $X$ of $G$, then $G$ is a symmetry group of the PDEs system (12).

**The algorithm for finding the symmetry group $G$ of the PDEs system (12):** One considers the vector field $X$ on $M$ and one writes the infinitesimal invariance condition (14); one eliminates any dependence between partial derivatives of the functions $u^\alpha$, determined by the PDEs system (12); one writes the condition (14) like polynomials in the partial derivatives of $u^\alpha$; one equates with zero the coefficients of partial derivatives of $u^\alpha$ in (14); it follows a PDEs system with respect to the unknown functions $\zeta^i$, $\phi_{\alpha}$ and this system defines the symmetry group $G$ of the studied PDEs system.
3 Symmetry Groups Associated to PDEs Systems of \( \mathcal{T} \)-\( \mathcal{L} \) Surfaces

3.1. In the first part of this section we shall study the symmetries of the PDEs system (5), which can be considered in the equivalent form

\[
\begin{align*}
  x_{uu} &= ax_u + bx_v \\
  x_{uv} &= hx \\
  x_{vv} &= a''x_u + b''x_v \\
  y_{uu} &= ay_u + by_v \\
  y_{uv} &= hy \\
  y_{vv} &= a''y_u + b''y_v \\
  z_{uu} &= az_u + bz_v \\
  z_{uv} &= hz \\
  z_{vv} &= a''z_u + b''z_v,
\end{align*}
\]

(15)

with the conditions (1) and (6). The condition (1) can be written as

\[
(y_u z_v - z_u y_v) x - (x_u z_v - x_v z_u) y + (x_u y_v - x_v y_u) z = f,
\]

where \( f = f(u, v) \) is a nonzero function. We consider the case of the real asymptotic lines. Let \( D \times U^{(2)} \) be the second order jet space associated to the PDEs system (15)+(16), whose coordinates represent the independent variables \( u, v \), the dependent variables \( x, y, z \) and the derivatives of the dependent variables till the order two. Denote \( x^1 = u, x^2 = v, u^1 = x, u^2 = y \) and \( u^3 = z \) (in the above section). Let \( M \subset D \times U \) be an open set and let

\[
X = \zeta \frac{\partial}{\partial u} + \eta \frac{\partial}{\partial v} + \phi \frac{\partial}{\partial x} + \lambda \frac{\partial}{\partial y} + \psi \frac{\partial}{\partial z}
\]

be the infinitesimal generator of the symmetry group \( G \) of the PDEs system (15)+(16), where \( \zeta, \eta, \phi, \lambda, \psi \) are functions of \( u, v, x, y \) and \( z \).

We shall study if there is a subgroup \( G_1 \) of the symmetry group \( G \), which acts on the space of the dependent variables \( x, y, z \) of the given system. Let suppose that it is, and let \( Y \) be its infinitesimal generator. In this case, we must have \( \zeta = 0, \eta = 0, \phi = \phi(x, y, z), \lambda = \lambda(x, y, z), \psi = \psi(x, y, z) \). By using the relations (13), we get the second prolongation of the vector field \( Y \), which is defined by the next functions

\[
\Phi^u = \phi_x x_u + \phi_y y_u + \phi_z z_u,
\]

\[
\Phi^{uu} = \phi_{xx} x_u^2 + \phi_{yy} y_u^2 + \phi_{zz} z_u^2 + 2\phi_{xy} x_u y_u + 2\phi_{xz} x_u z_u + 2\phi_{yz} y_u z_u + \phi_{xx} x_u x_{uu} + \phi_{yy} y_u y_{uu} + \phi_{zz} z_u z_{uu},
\]

\[
\Phi^{uv} = \phi_{xx} x_u x_v + \phi_{xy} x_v y_u + \phi_{xz} x_v z_u + \phi_{yy} y_u y_v + \phi_{yz} y_v y_u + \phi_{zz} z_u z_v + \phi_{xx} x_v x_{uv} + \phi_{yy} y_v y_{uv} + \phi_{zz} z_v z_{uv},
\]

\[
\Phi^{vv} = \phi_{yy} y_v^2 + \phi_{zz} z_v^2 + 2\phi_{xy} x_v y_v + 2\phi_{xz} x_v z_v + 2\phi_{yz} y_v z_v + \phi_{xx} x_v x_{vv} + \phi_{yy} y_v y_{vv} + \phi_{zz} z_v z_{vv}.
\]
\[ \Phi^{uv} = \phi_{xx}x^2 + 2\phi_{xy}x_uy + 2\phi_{xz}x_z + 2\phi_{yy}y^2 + 2\phi_{yz}y_z + 2\phi_{zz}z^2 + \]
\[ + \phi_{xuv} + \phi_{yuv} + \phi_{zuv}, \]
and also the functions \( \Lambda^u, \Lambda^v, \Lambda^{vu}, \Lambda^{uv}, \Phi^u, \Phi^v, \Phi^{uu}, \Phi^{uv}, \Phi^{vv} \) which are analogously written by substituting \( \phi \) with \( \lambda \), and respectively \( \psi \). The PDEs system (15)+(16) is of maximal rank. The infinitesimal invariance condition (14) for (15) turns in
\[
\begin{align*}
\left\{ \begin{array}{l}
a\Phi^u + b\Phi^v - \Phi^{uu} = 0 \\
h\Phi - \Phi^{uv} = 0 \\
a''\Phi^u + b''\Phi^v - \Phi^{vv} = 0
\end{array} \right. 
\end{align*}
\]
Let consider the first relation and let substitute the functions \( \Phi^u, \Phi^v \) and \( \Phi^{uu} \) gives by the above relations. We find
\[
a(\phi_x x_u + \phi_y y_u + \phi_z z_u) + b(\phi_x x_v + \phi_y y_v + \phi_z z_v) - \phi_{xx}x^2 - 2\phi_{xy}x_uy - 2\phi_{xz}x_z +
\]
\[-2\phi_{yy}y^2 - 2\phi_{yz}y_z + 2\phi_{zz}z^2 - \phi_{xu}x_u - \phi_{yu}y_u - \phi_{zu}z_u = 0.
\]
We eliminate any dependencies among the derivatives of the \( x, y, z \) by substituting
\[
x_{uu} = ax_u + bx_v, \quad y_{uu} = ay_u + by_v, \quad z_{uu} = az_u + bz_v.
\]
Then it results
\[
\phi_{xx}x^2 + \phi_{yy}y^2 + \phi_{zz}z^2 + 2\phi_{xy}x_uy + 2\phi_{xz}x_z + 2\phi_{yz}y_z = 0.
\]
Now we equate the coefficients of the remaining unconstrained partial derivatives of \( x, y, z \) to zero, and we get the PDEs system
\[
\phi_{xx} = 0, \quad \phi_{yy} = 0, \quad \phi_{zz} = 0, \quad \phi_{xy} = 0, \quad \phi_{xz} = 0, \quad \phi_{yz} = 0.
\]
It follows the solution \( \phi(x, y, z) = a_{11}x + a_{12}y + a_{13}z + k \), with \( a_{11}, a_{12}, a_{13}, k \in \mathbb{R} \). By substituting the function \( \phi \) in the next two relations of the system (17), we find \( k = 0 \) and thus \( \phi(x, y, z) = a_{11}x + a_{12}y + a_{13}z \), \( a_{11}, a_{12}, a_{13} \in \mathbb{R} \). Analogously, using the next six relations of the system (17) we get \( \lambda(x, y, z) = a_{21}x + a_{22}y + a_{23}z, \quad a_{21}, a_{22}, a_{23} \in \mathbb{R}, \quad \psi(x, y, z) = a_{31}x + a_{32}y + a_{33}z, \quad a_{31}, a_{32}, a_{33} \in \mathbb{R} \). Using the criterion of infinitesimal invariance (14), for (16), we get
\[
\phi(y_uz_v - z_uy_v) + \lambda(x_uz_v - x_uy_v) + \psi(x_uy_v - x_vy_u) + \Phi^u(zy_v - y_uz_v) + \Phi^v(yz_v - y_uz_v) +
\]
\[+ \Lambda^u(x_uz_v - x_uy_v) + \Lambda^v(x_uy_v - x_vy_u) + \Psi^u(xy_v - x_yu) + \Psi^v(xy_u - x_yv) = 0.
\]
If we substitute the functions \( \Phi^u, \Phi^v, \ldots \), then it results
\[
(x_uy_v - x_vy_u)(\psi - x\psi_x - y\psi_y - z\phi_x + z\lambda_y) + (x_vz_u - x_uz_v)(\lambda - x\lambda_x - z\lambda_z + y\phi_x +
\]
\[+ y\psi_z) + (y_uz_v - y_vz_u)(\phi - y\phi_y - z\phi_z + x\lambda_y + x\psi_z) = 0.
\]
We eliminate any dependencies among the derivatives of $x, y, z$ by using the relation (16) itself, and we find $\phi_x + \lambda_y + \psi_z = 0$ or equivalent $a_{33} + a_{11} + a_{22} = 0$. Thus, the next functions

$$
\begin{align*}
\phi(x, y, z) &= a_{11}x + a_{12}y + a_{13}z \\
\lambda(x, y, z) &= a_{21}x + a_{22}y + a_{23}z \\
\psi(x, y, z) &= a_{31}x + a_{32}y - (a_{11} + a_{22})z
\end{align*}
$$

define the infinitesimal generator $Y$ of the symmetry subgroup $G_1$:

$$
Y = a_{11} \left( x \frac{\partial}{\partial x} - z \frac{\partial}{\partial z} \right) + a_{22} \left( y \frac{\partial}{\partial y} - z \frac{\partial}{\partial z} \right) + a_{12}y \frac{\partial}{\partial x} + a_{13}z \frac{\partial}{\partial x} + a_{23}z \frac{\partial}{\partial y} + a_{31} \frac{\partial}{\partial y} + a_{32} \frac{\partial}{\partial z} + a_{33} \frac{\partial}{\partial z}.
$$

**Theorem 4.** The Lie algebra associated to the subgroup $G_1$ of the full symmetry group $G$ of the PDEs system (15)+(16) ($G_1$ acts on the space of the dependent variables) is generated by the vector fields

$$
(18) \quad Y_1 = x \frac{\partial}{\partial x} - z \frac{\partial}{\partial z}, \quad Y_2 = y \frac{\partial}{\partial y} - z \frac{\partial}{\partial z}, \quad Y_3 = y \frac{\partial}{\partial x}, \quad Y_4 = z \frac{\partial}{\partial x}
$$

$$
Y_5 = x \frac{\partial}{\partial y}, \quad Y_6 = z \frac{\partial}{\partial y}, \quad Y_7 = x \frac{\partial}{\partial z}, \quad Y_8 = y \frac{\partial}{\partial z},
$$

and, thus the Lie subgroup $G_1$ is the unimodular subgroup of the group of centroaffine transformations.

Using this result, we can find group-invariant solutions of the PDEs system (15)+(16). For example, if we consider the subalgebra described by the vector fields $Y_1$ and $Y_2$, then the function $F$ which is invariant under the associated group, satisfies $Y_1(F) = 0$, and $Y_2(F) = 0$. It results $F = \varphi(u, v, xyz)$ and the group-invariant solutions are defined by $u = C_1$, $v = C_2$ and $xyz = C_3$. Thus one gets the known Titeica surfaces

$$
(19) \quad z = \frac{C}{xy}, \quad C \in \mathbb{R}.
$$

**3.2.** We shall study if there is a subgroup $G_2$ of the symmetry group $G$ of the PDEs system (15)+(16) which acts on the space of the independent variables $u, v$ of the system. Let suppose that it is and let $Z$ be its infinitesimal generator. The vector field $Z$ is defined by the functions

$$
\zeta = \zeta(u, v), \quad \eta = \eta(u, v), \quad \phi = 0, \quad \lambda = 0, \quad \psi = 0.
$$

In this case, by using the above algorithm for finding the associated symmetries, we get
Theorem 5. The general vector field of the algebra of the infinitesimal symmetries associated to the subgroup \( G_2 \), where \( G_2 \) is the subgroup of the full symmetry group \( G \) of the PDEs system (15), which acts on the space of the independent variables, is

\[
Z = \zeta(u) \frac{\partial}{\partial u} + \eta(v) \frac{\partial}{\partial v},
\]

where the functions \( \zeta \) and \( \eta \) satisfy the next PDEs system:

\[
\begin{aligned}
&\zeta a_u + \eta a_v + a\zeta_u + \zeta_{uu} = 0 \\
&\zeta b_u + \eta b_v - b\eta_v + 2b\zeta_u = 0 \\
&\zeta h_u + \eta h_v + h(\zeta_u + \eta_v) = 0 \\
&\zeta a''_u + \eta a''_v - a''\zeta_u + 2a''\eta_v = 0 \\
&\zeta b''_u + \eta b''_v + b''\eta_v + \eta_{uv} = 0,
\end{aligned}
\]

and the functions \( a, b, h, a'', b'' \) satisfy the integrability conditions (6).

One considers the cases:

1. If \( \Sigma \) is a ruled TÎ­teica surface (7), then the completely integrability conditions (6) are

\[
a = \frac{h_u}{h}, \quad b = \frac{\varphi(u)}{h}, \quad a'' = 0, \quad b'' = \frac{h_v}{h},
\]

where \( h \) is a solution of the Liouville-TÎ­teica PDE (8). In this case, the relations (20) turn in

\[
\begin{aligned}
&\zeta h_u + \eta h_v + h(\zeta_u + \eta_v) = 0 \\
&\zeta^3 = \frac{k}{2}, \\
&hh_{uv} - h_uh_v = h^3.
\end{aligned}
\]

Let us consider the change of variables \( \zeta = \frac{1}{U} \) and \( \eta = -\frac{1}{V} \), where \( U = U(u) \) and \( V = V(v) \). Then the first PDE implies \( h = U'V'\mu(U + V) \) and by substituting this function in the last PDE of the above system (Liouville-TÎ­teica PDE), we find the following ODE

\[
\mu'' + \mu^2 = \mu^3,
\]

for which

\[
\mu(t) = \begin{cases} 
\frac{t^2}{(t+C)^2}, & k = 0 \\
\frac{t^2}{2\cos^2(\frac{\pi}{4}t+C)}, & k = -l^2 \\
\frac{t^2}{2\sin^2(\frac{\pi}{4}t+C)}, & k = l^2, \ l > 0.
\end{cases}
\]

is the general solution, with \( t = U + V \). One substitutes in \( h = U'V'\mu(U + V) \) and one consider the special change of the functions \( \bar{U} = F(U), \ \bar{V} = G(V), \ \bar{U} = U + C, \ \bar{V} = V \): for \( k = 0 \), \( \bar{U} = \text{th}_2(U + C), \ \bar{V} = \text{th}_2 V \), for \( k = l^2 \) and
\[ U = \cotg \left( \frac{1}{2} U + C \right), \quad V = \tan \frac{1}{2} V, \text{ for } k = -l^2, \] it results the general solution of the Liouville-\v{S}teica ([15],[24]) namely,

\[ h(u, v) = \frac{2 \tilde{U}' \tilde{V}'}{(\tilde{U} + \tilde{V})^2}. \]

2. If \( \Sigma \) is a \v{S}teica surface which are not ruled surface (9), then the completely integrability conditions (6) turn in

\[ a = \frac{h_u}{h}, \quad b = a'' = \frac{1}{h}, \quad b' = \frac{h_v}{h}, \]

where \( h \) is a solution of the \v{S}teica PDE (10). If we substitute these functions in the system (20), then we get

\[ \zeta_u = 0, \quad \eta_v = 0, \quad \zeta h_u + \eta h_v + h(\zeta_u + \eta_v) = 0. \]

It results \( \zeta = C_1, \eta = C_2 \) and \( h = \mu(C_1 v - C_2 u) \) and thus

\[ Z = C_1 \frac{\partial}{\partial u} + C_2 \frac{\partial}{\partial v}. \]

Let consider the \v{S}teica PDE (10) and let substitute the above function \( h = \mu(C_1 v - C_2 u) \). We get the following ODE:

\[ -C_1 C_2 (\mu \mu'' - \mu'^2) = \mu^3 - 1. \]

a. If \( C_1 C_2 = 0 \), then \( \mu = 1 \) and \( h = 1 \). This is the \v{S}teica solution [24].

b. If \( C_1 C_2 \neq 0 \), then denote \( k = -\frac{1}{C_1 C_2} \). The above ODE turns in

\[ \mu \mu'' - \mu'^2 = k(\mu^3 - 1). \]

We can consider \( k = 1 \). This ODE can be reduced to the following

\[ \mu'^2 = 2\mu^3 + C\mu^2 + 1, \quad C \in \mathbb{R}, \]

and using the change of function \( \mu = \frac{1}{2} g \), this becomes

\[ g'^2 = g^3 + C g^2 + 4. \]

Let \( \lambda \) the real solution of the right side polynom of the above ODE. It results that \( \lambda \neq 0 \), \( \lambda \) is not a triple solution of this and the ODE can writen as

\[ g'^2 = (g - \lambda) \left( g^2 - \frac{4}{\lambda^2} g - \frac{4}{\lambda} \right). \]

2.1. If \( \lambda = -1 \), then \( C = -3 \) and the ODE is

\[ g'^2 = (g + 1)(g - 2)^2. \]
If we consider \( g = \frac{1}{w^2} + 2 \) than we have the ODE

\[
w'^2 = \frac{1}{4}(3w^2 + 1),
\]

for which the solution \( w = w(t), \ t = u + v \) is

\[
w(t) = \frac{1}{\sqrt{3}} \sinh \left( \frac{t\sqrt{3}}{2} + C_1 \right), \quad C_1 \in \mathbb{R}.
\]

We get the next solution of the Tițeica PDE: \( h = \frac{1}{2w^2} + 1 \) which in the case \( C_1 = 0 \), it is

\[
(21) \quad h(t) = \frac{3}{2\sinh \left( \frac{t\sqrt{3}}{2} \right)} + 1, \quad t = u + v.
\]

2.2. If \( \lambda \neq -1 \), then the right side polynom has three distinct real solutions (\( \lambda > -1 \) or \( C < -3 \)) and repectively one is real and two complex (\( \lambda < -1 \) or \( C > -3 \)). In this case, the integral

\[
J = \int \frac{dg}{\sqrt{(g - \lambda)(g^2 - \frac{3}{4}g - \frac{1}{4})}}
\]

can be reduced to a first genus elliptical integral [11]

\[
J = \int \frac{d\varphi}{\sqrt{1 - k^2 \sin \varphi}}
\]

We get: for \( C \neq -3 \), the solutions of the PDE Tițeica with the form \( h = \mu(u + v) \) are gives in the terms of elliptical functions.

**Proposition 1.** The solution (21) gives a revolution Tițeica surface. Moreover, it is an associated ruled Tițeica surface.

**Proof.** In [24], 164-174, Tițeica studied the revolution surfaces defined by the system (11) and make the remark that, in this case, the function \( h \) must satisfy \( h_u = h_v \), and thus \( h = \mu(u + v) \). Tițeica obtained the above ODE

\[
\mu \mu'' - \mu'^2 = \mu^3 - 1,
\]

and not integrate it. He proved: by using

\[
\frac{\mu'^2 - 2\mu^3 - 1}{4\mu^2} = -k^2,
\]

one finds the solution of the studied system: for \( k \neq 0 \):

\[
(22) \quad \theta(u, v) = k_1 e^{\int \frac{b^2 - 1}{b^2} da} \cos k\beta + k_2 e^{\int \frac{b^2 - 1}{b^2} da} \sin k\beta + k_3 e^{\int \frac{b^2}{b^2 + 1} da},
\]

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and for $k = 0$:

$$\theta(u, v) = e^{\int \frac{k' - 1}{2} d\alpha} \left[ k_1 \left( \beta^2 + \int \frac{4\mu}{\mu' + 1} d\alpha \right) + k_2 \beta + k_3 \right],$$

where $\alpha = u + v$, $\beta = u - v \iff k_1, k_2, k_3 \in \mathbb{R}$.

On the other hand, our calculus imply that $k^2 = -\frac{\gamma}{\pi}$, and thus the function (21) defines a revolution surface (22). If we consider

$$\bar{U} = \text{th} \frac{\sqrt{3}}{2} (U + C_1), \quad \bar{V} = \text{th} \frac{\sqrt{3}}{2} V,$$

it results that the function $h$ can be written in the next form

$$h(u, v) = \frac{2\bar{U}'\bar{V}'}{\bar{U} + \bar{V}} + 1 = H(u, v) + 1.$$

But, the above calculus, implies that the function $H$ is a solution of Liouville-Titeica PDE (8) and this defines a ruled Titeica surface.

**Proposition 2.** The solution of Titeica (22) is invariant under the transformations subgroup of $G_2$, for which the Lie algebra is generated (in the case of not ruled Titeica surface) by

$$Z = C_1 \frac{\partial}{\partial u} + C_2 \frac{\partial}{\partial v}.$$

3.3. In this part, we shall apply the symmetry group theory for the PDEs system (11) with the integrability conditions (6). Let $D \times \bar{U}^{(2)}$ be the second order jet space associated to the PDEs system (11), whose coordinates are the independent variables $u, v$, the dependent variable $\theta$ and the derivatives of the dependent variable till the order two. Denote by $x^1 = u$, $x^2 = v$ and $u^1 = \theta$ (in the second section). Let consider $\bar{M} \subset D \times \bar{U}$ an open set and let

$$\bar{X} = \zeta \frac{\partial}{\partial u} + \eta \frac{\partial}{\partial v} + \alpha \frac{\partial}{\partial \theta}$$

be the infinitesimal generator of the symmetry group $\bar{G}$ of the PDEs system (11), where $\zeta$, $\eta$ and $\alpha$ are functions of $u$, $v$ and $\theta$. The relations (13) imply the first and the second prolongations of the vector field $\bar{X}$

$$pr^{(1)} \bar{X} = \bar{X} + \alpha^u \frac{\partial}{\partial \theta u} + \alpha^v \frac{\partial}{\partial \theta v},$$

$$pr^{(2)} \bar{X} = pr^{(1)} \bar{X} + \alpha^{uu} \frac{\partial}{\partial \theta uu} + \alpha^{uv} \frac{\partial}{\partial \theta uv} + \alpha^{vv} \frac{\partial}{\partial \theta vv},$$

where

$$\alpha^u = D_u (\alpha - \zeta \theta_u - \eta \theta_v) + \zeta \theta_{uu} + \eta \theta_{uv},$$

$$\alpha^v = D_v (\alpha - \zeta \theta_u - \eta \theta_v) + \zeta \theta_{uv} + \eta \theta_{vv},$$
\[
\begin{align*}
\alpha^{uu} &= D_{uu}(\alpha - \zeta\theta_u - \eta\theta_v) + \zeta\theta_{uuu} + \eta\theta_{uuv}, \\
\alpha^{uv} &= D_{uv}(\alpha - \zeta\theta_u - \eta\theta_v) + \zeta\theta_{uuv} + \eta\theta_{uvv}, \\
\alpha^{vv} &= D_{vv}(\alpha - \zeta\theta_u - \eta\theta_v) + \zeta\theta_{vvv} + \eta\theta_{vvv}.
\end{align*}
\]

We shall study if there is a subgroup $\bar{G}_1$ of the symmetry group $\bar{G}$, which acts on the space of the dependent variable $\theta$. Let suppose that it is and let
\[
\bar{Y} = \alpha \frac{\partial}{\partial \theta}, \quad \alpha = \alpha(\theta),
\]
be its infinitesimal generator. In this case, the above algorithm implies

**Theorem 6.** The Lie algebra of the infinitesimal symmetries associated to the subgroup $\bar{G}_1$ of the full symmetry group $\bar{G}$ of the PDEs system (11) ($\bar{G}_1$ acts on the space of the dependent variable $\theta$), is generated by the vector field
\[
(23) \quad \bar{Y}_1 = \theta \frac{\partial}{\partial \theta}.
\]

Analogously, we study the subgroup $\bar{G}_2$ of the symmetry group $\bar{G}$, which acts on the space of the independent variables $u, v$ of the PDEs system (11). Let
\[
\bar{Z} = \zeta \frac{\partial}{\partial u} + \eta \frac{\partial}{\partial v}, \quad \zeta = \zeta(u, v), \quad \eta = \eta(u, v),
\]
be the infinitesimal generator of it. It results

**Theorem 7.** The general vector field of the algebra of the infinitesimal symmetries associated to the subgroup $\bar{G}_2$ of the symmetry group $\bar{G}$ of the PDEs system (11) ($\bar{G}_2$ acts on the space of the independent variables $u, v$), is
\[
(24) \quad \bar{Z} = \zeta(u) \frac{\partial}{\partial u} + \eta(v) \frac{\partial}{\partial v},
\]
where $\zeta$ and $\eta$ satisfy the relations (20).

**Remark.** The subgroup $G_2$ and $\bar{G}_2$ are the same actions on the space of the independent variables $u, v$.

3.4. Now we shall study the symmetries of the Liouville-Titeica PDE (8) and Tîţeica PDE (10). Let consider the PDEs
\[
(8') \quad \omega_{uv} = e^\omega,
\]
and respectively
\[
(10') \quad \omega_{uv} = e^\omega - e^{-2\omega},
\]
which are equivalent to Liouville-Titeica PDE, and respectively Tîţeica PDE, where $\ln h = \omega$. We remark that these PDEs belong of the next class of second order PDE, of maximal rank,
\[
(25) \quad \omega_{uv} = H(\omega),
\]

13
which was studied by Sophus Lie himself. Also Pucci, Saccomandi, Mansfield have considered such equations. One proves [18] the next result

**Theorem 8.** If \( \zeta = \zeta(u, v, \omega) \), \( \eta = \eta(u, v, \omega) \) and \( \phi = \phi(u, v, \omega) \) are the solutions of the PDEs system

\[
\begin{align*}
\zeta_v &= 0 & \zeta_\omega &= 0 & \eta_u &= 0 & \eta_\omega &= 0 & \phi_{\omega \omega} &= 0 & \phi_{\omega v} &= 0 \\
\phi_{uv} + (\phi_\omega - \zeta_u - \eta_v - \phi)H - H'\phi &= 0,
\end{align*}
\]

where \( H = H(\omega) \), then

\[
X = \zeta \frac{\partial}{\partial u} + \eta \frac{\partial}{\partial v} + \phi \frac{\partial}{\partial \omega}
\]

is the infinitesimal generator of the symmetry group associated to one PDE of the form (25).

In the case of PDEs (8') and (10'), we get the results

**Theorem 9.** The general vector field which describes the algebra of infinitesimal symmetries associated to the Liouville-Titeica PDE (8') is the following

\[
W = f \frac{\partial}{\partial u} + g \frac{\partial}{\partial v} - (f' + g') \frac{\partial}{\partial \omega}
\]

where \( f = f(u) \) and \( g = g(v) \).

**Theorem 10.** The vector fields which generate the Lie algebra of infinitesimal symmetries associated with the PDE Titeica (10') are

\[
U_1 = u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v}, \quad U_2 = \frac{\partial}{\partial u}, \quad U_3 = \frac{\partial}{\partial v}.
\]

**Remark.** If \( \omega = f(u, v) \) is a solution of the Titeica PDE (8'), then the following functions

\[
\omega(1) = f(e^\varepsilon u, e^{-\varepsilon} v), \quad \omega(2) = f(u - \varepsilon, v), \quad \omega(3) = f(u, v - \varepsilon),
\]

where \( \varepsilon \) is a real number, are also solutions of the equation.

The adjoint representation of the symmetry group of the Titeica PDE (8'), is given by the next table

| Table 1 |
|---------|
| \( \text{Ad} \) | \( U_1 \) | \( U_2 \) | \( U_3 \) |
| \( U_1 \) | \( U_1 \) | \( e^\varepsilon U_2 \) | \( e^{-\varepsilon} U_3 \) |
| \( U_2 \) | \( U_1 - \varepsilon U_2 \) | \( U_2 \) | \( U_3 \) |
| \( U_3 \) | \( U_1 + \varepsilon U_2 \) | \( U_2 \) | \( U_3 \) |

The 1-dimensional subalgebras, described by \( U_2 \), \( U_3 \), \( U_2 - U_3 \), implies the finding the following group-invariant solutions:

1. For \( U_2 \), and respectively \( U_3 \), the solution is \( \omega = 0 \). Thus it results the Titeica solution \( h = 1 \) of the PDE (8).
2. In the case of the vector field \( U_2 - U_3 \), the group-invariant solutions have the form \( \omega = f(u + v) \) (respectively \( h = \mu(u + v) \) for PDE (10)). But this case was considered in the above section.

Now we look for the converse of the Theorem 10:

**Theorem 11.** The second order PDE invariant with respect to the symmetry group (28) of Titeica PDE, has the form

\[
H(\omega, \omega_u \omega_v, \omega_{uu}, \omega_{vv}) = 0.
\]

**Proof.** One considers the maximal chain of Lie subalgebras of the Lie algebra associated with the studied group:

\[
\{U_2\} \subset \{U_2, U_3\} \subset \{U_1, U_2, U_3\}.
\]

and a second order PDE: \( F(u,v,\omega^{(2)}) = 0 \) which is invariant with respect this group. So it satisfies the criterion of infinitesimal invariance for each second prolongation of these vectors fields.

1. For \( U_2 \), it results \( pr^{(2)} U_2(F) = 0 \), and thus \( F = F_1(v, \omega^{(2)}) \).
2. For \( U_3 \), the condition \( pr^{(2)} U_3(F) = 0 \) implies \( F = F_2(\omega^{(2)}) \).
3. For \( U_1 \), the relation \( pr^{(2)} U_1(F) = 0 \) is equivalent to

\[
U_1(F_2) = \omega_u \frac{\partial F_2}{\partial \omega_u} + \omega_v \frac{\partial F_2}{\partial \omega_v} - 2\omega_{uu} \frac{\partial F_2}{\partial F_2} \omega_{uu} + 2\omega_{vv} \frac{\partial F_2}{\partial \omega_{vv}} = 0,
\]

and finally, we get \( F_2 = H \) in (29).

4 **Lagrangians Associated to Titeica PDEs**

4.1. Our present study is for the inverse problem for the PDEs (8') and (10'). We recall that, the simple form of the inverse problem for a PDE in the calculus of variations is to determine if this is identically to an Euler-Lagrange PDE \([1],[8],[16],[20],[21],[28],[31]\).

Let us consider the second order PDE

\[
\Delta(u,v,\omega^{(2)}) = 0,
\]

where \( \omega^{(2)} \) is the second prolongation of the unknown function \( \omega = \omega(u,v) \). The PDE (30) is identically to an Euler-Lagrange equation if and only if the inegrability Helmholtz conditions

\[
\begin{aligned}
\frac{\partial \Delta}{\partial \omega_u} &= D_u \left( \frac{\partial \Delta}{\partial \omega_{uu}} + D_v \left( \frac{1}{2} \frac{\partial \Delta}{\partial \omega_{vv}} \right) \right) \\
\frac{\partial \Delta}{\partial \omega_v} &= D_u \left( \frac{1}{2} \frac{\partial \Delta}{\partial \omega_{uu}} \right) + D_v \left( \frac{\partial \Delta}{\partial \omega_{vv}} \right),
\end{aligned}
\]

are satisfied. In this case, there exists a function \( L \), called Lagrangian, such that the Euler-Lagrange PDE

\[
E(L) = \frac{\partial L}{\partial \omega} - D_u \left( \frac{\partial L}{\partial \omega_u} \right) - D_v \left( \frac{\partial L}{\partial \omega_v} \right) = 0
\]
is equivalent to the PDE (30), in the sense that every solution of the equation (30) is a solution of the Euler-Lagrange equation \( E(L) = 0 \) and conversely.

Also, the PDE (30) is called equivalent to an Euler-Lagrange equation if there exists a nonzero function \( f = f(u, v, \omega, \omega_u, \omega_v) \) such that \( f \cdot \Delta = E(L) \). The function \( f \) is called variational integrant factor.

**Theorem 12.** The Liouville-\( \tilde{T} \)iteica PDE (8') and \( \tilde{T} \)iteica PDE (10') are Euler-Lagrange equations with the next associated Lagrangians

\[
L_1(u, v, \omega^{(1)}) = -\frac{1}{2} \omega_u \omega_v - e^\omega,
\]

and

\[
L_2(u, v, \omega^{(1)}) = -\frac{1}{2} \omega_u \omega_v - e^\omega - \frac{1}{2} e^{-2\omega}.
\]

**Proof.** Indeed, one verifies that the Helmholtz integrability conditions (31) are satisfied and one verifies that \( L_1 \) and \( L_2 \) can be considered associated Lagrangians.

**Remark.** The PDEs (8) and (10) are equivalent to Euler-Lagrange PDEs, with the variational integrant factor \( \frac{1}{\lambda} \).

### 4.2.

We make a short presentation of the theory of variational symmetry groups for the functionals

\[
\mathcal{L}[\omega] = \int_{\Omega_0} L(u, v, \omega^{(1)})dudv,
\]

with \( \Omega_0 \) is a domain in \( \mathbb{R}^2 \) [20], [21].

Let \( D \subset \Omega_0 \) be a subdomain, \( U \) an open set in \( \mathbb{R} \) and \( M \subset D \times U \) an open set. We consider \( \omega \in C^2(D) \), \( \omega = f(u, v) \) such that \( \Gamma_\omega = \{(u, v, \omega(u, v)) | (u, v) \in D \} \subset M \). A local group \( G \) of transformations on \( M \) is called variational symmetry group for the functional (34), if \( g \in G \), \( g(u, v, \omega) = (\bar{u}, \bar{v}, \bar{\omega}) \), then the function \( \bar{\omega} = \bar{f}(\bar{u}, \bar{v}) = (g \cdot f)(\bar{u}, \bar{v}) \) is defined on \( \bar{\Omega} \subset \Omega_0 \) and

\[
\int_{\bar{D}} \int L(\bar{u}, \bar{v}, pr^{(1)} \bar{f}(\bar{u}, \bar{v}))d\bar{u}d\bar{v} = \int_{D} \int L(u, v, pr^{(1)} f(u, v))dudv.
\]

**Theorem 13 (Infinitesimal criterion for the variational problem).**

A connected group \( G \) of transformations acting on \( M \subset \Omega_0 \times U \) is a group of variational symmetries for the functional (34) if and only if

\[
pr^{(1)} X(L) + L \text{div} \xi = 0,
\]

is satisfied for \( \forall (u, v, \omega^{(2)}) \in M^{(2)} \subset D \times U^{(2)} \) and for any infinitesimal generator

\[
X = \zeta(u, v, \omega) \frac{\partial}{\partial u} + \eta(u, v, \omega) \frac{\partial}{\partial v} + \phi(u, v, \omega) \frac{\partial}{\partial \omega}
\]
of $G$, where $\xi = (\zeta, \eta)$ and $\text{Div}\xi = D_u\zeta + D_v\eta$.

**Theorem 14.** If $G$ is a variational symmetry group of the functional (34), then $G$ is a symmetry group of Euler-Lagrange equation $E(L) = 0$.

The converse of Theorem 14 is generally false. Let us consider the PDE (30). A conservation law is a divergence expression

$$\text{Div} P = 0$$

which vanishes for all solutions $u = f(x)$ of the given PDE. Here $P = (P^1, P^2)$ with $\text{Div} P = D_uP^1 + D_vP^2$, the total divergence. The function $P^1$ is called flow associated and $P^2$ is called conserved density to the conservation law. It results that there exists a function $Q$ such that

$$(36) \quad \text{Div} P = Q \cdot \Delta.$$  

This relation is called the characteristic form of the conservation law, and $Q$ is called the characteristic of the conservation law.

Let

$$X = \zeta(u, v, \omega) \frac{\partial}{\partial u} + \eta(u, v, \omega) \frac{\partial}{\partial v} + \phi(u, v, \omega) \frac{\partial}{\partial \omega}$$

be a vector field on $M$. The vector field

$$X_Q = Q \frac{\partial}{\partial u}, \quad Q = \phi - \zeta \omega_u - \eta \omega_v,$$

is called vector field of evolution associated to $X$, and $Q$ is called the characteristic associated to $X$.

**Theorem 15 (Noether Theorem).** Let $G$ be a local Lie group of transformations, which is a symmetry group of the variational problem (34) and let

$$X = \zeta(u, v, \omega) \frac{\partial}{\partial u} + \eta(u, v, \omega) \frac{\partial}{\partial v} + \phi(u, v, \omega) \frac{\partial}{\partial \omega}$$

be the infinitesimal generator of $G$. The characteristic $Q$ of the field $X$ is also a characteristic of the conservation law for the associated Euler-Lagrange equation $E(L) = 0$.

One proves ([20], 356) that for the Lagrangian $L = L(u, v, \omega^{(1)})$ we have

$$(37) \quad P = -(A + L\xi) = -(A^1 + L\zeta, A^2 + L\eta) = (P^1, P^2), \quad A = (A^1, A^2),$$

where $A^1 = Q \cdot E^{(u)}(L), \quad A^2 = Q \cdot E^{(v)}(L)$. The operators $E^{(u)}(L) = \frac{\partial L}{\partial \omega_u}$ and $E^{(v)}(L) = \frac{\partial L}{\partial \omega_v}$ are called first order Euler operators.

4.3. Let us consider the first order Lagrangians (32) and (33) and the associated functionals

$$(38) \quad \mathcal{L}[\omega] = \int \int_D L_1(u, v, \omega^{(1)})dudv$$

and

$$(39) \quad \tilde{\mathcal{L}}[\omega] = \int \int_D L_2(u, v, \omega^{(1)})dudv,$$
with $D$ is a domain in $\mathbb{R}^2$ and $\omega \in C^2(D)$.

**Theorem 16.** The Lie algebras of the variational symmetry groups of the functional (38) is described by the vector fields

$$W_1 = u \frac{\partial}{\partial u} - \frac{\partial}{\partial \omega}, \quad W_2 = v \frac{\partial}{\partial v} - \frac{\partial}{\partial \omega}, \quad W_3 = \frac{\partial}{\partial u}, \quad W_4 = \frac{\partial}{\partial v}.$$

**Proof.** According with Theorem 14, the vector fields which determine the Lie algebra of the variational symmetry group are founded between the vector fields of the Lie algebra of the symmetry group of the associated Euler-Lagrange equation. The condition (35) must be verified only for the vector fields in the algebra of the symmetry group of PDE (8'). Let us consider the vector field

$$W = f \frac{\partial}{\partial u} + g \frac{\partial}{\partial v} - (f' + g' \omega) \frac{\partial}{\partial \omega}, \quad f = f(u), \quad g = g(v),$$

which is given by the relation (27) and let

$$pr^{(2)}W = W - (f'' + f' \omega_u) \frac{\partial}{\partial \omega_u} - (g'' + g' \omega_v) \frac{\partial}{\partial \omega_v},$$

be the second prolongation. Introducing $\xi = (f, g)$ and $\text{Div}\xi = f' + g'$ in the relation (35), this turns in $f'' \omega_v + g'' \omega_u = 0$ and it implies $f'' = g'' = 0$. Thus, we get $f = C_1 u + C_3$, $g = C_2 v + C_4$ and also

$$W = (C_1 u + C_3) \frac{\partial}{\partial u} + (C_2 v + C_4) \frac{\partial}{\partial v} - (C_1 + C_2) \frac{\partial}{\partial \omega} =$$

$$= C_1 \left( u \frac{\partial}{\partial u} - \frac{\partial}{\partial \omega} \right) + C_2 \left( v \frac{\partial}{\partial v} - \frac{\partial}{\partial \omega} \right) + C_3 \frac{\partial}{\partial u} + C_4 \frac{\partial}{\partial v}.$$

**Theorem 17.** The Lie algebras of the variational symmetry groups of the functional (39) is described by the vector fields

$$U_1 = u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v}, \quad U_2 = \frac{\partial}{\partial u}, \quad U_3 = \frac{\partial}{\partial v}.$$

**Proposition 3.** The associated flows and respectively the conserved density in the case of Liouville-Ţîţeica PDE (8') and respectively for Ţîţeica PDE (10') are

| $W_i$ | $P^1$ | $P^2$ |
|-------|-------|-------|
| $W_1$ | $\frac{1}{2}\omega_u - uw u'$ | $\frac{1}{2}\omega_u(1 + uw u')$ |
| $W_2$ | $\frac{1}{2}\omega_v(1 + vw v')$ | $\frac{1}{2}\omega_v v' - v\omega'$ |
| $W_3$ | $\omega u'$ | $\frac{1}{2}\omega_u$ |
| $W_4$ | $\omega v'$ | $-\omega'$ |
Table 3

| $-U_1$ | $P^1$ | $P^2$ |
|--------|-------|-------|
| $-U_1$ | $-\frac{1}{2}ue^{-2\omega} - \frac{1}{2}u\omega_2^2 - ue\omega$ | $\frac{1}{2}u\omega_2^2 + ve\omega + \frac{1}{2}ve^{-2\omega}$ |
| $-U_2$ | $-e\omega - \frac{1}{2}e^{-2\omega}$ | $\frac{1}{2}e\omega^2$ |
| $-U_3$ | $\frac{1}{2}\omega_2^2$ | $-e\omega - \frac{1}{2}e^{-2\omega}$ |

**Proof.** For example, the characteristic associated to the vector field $-W_3$ is $Q = \omega_u$. Replacing in the relations (37), we obtain $A^1 = -\frac{1}{2}\omega_u\omega_v$, $A^2 = -\frac{1}{2}\omega_u^2$ and thus $P^1 = -e\omega$, $P^2 = \frac{1}{2}\omega_u^2$.

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