A note on dualities in Einstein’s gravity in the presence of a non-minimally coupled scalar field

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Abstract

We show that the action of Einstein’s gravity with a scalar field coupled in a generic way to spacetime curvature is invariant under a particular set of conformal transformations. These transformations relate dual theories for which the effective couplings of the theory are scaled uniformly. In the simplest case, this class of dualities reduce to the S-duality of low-energy effective action of string theory.

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I. INTRODUCTION

The theory we consider is Einstein’s gravity with the addition of a non-minimally coupled (NMC) real scalar field:

\[ \mathcal{L} = \sqrt{-g} \left\{ -\frac{1}{2} F(\psi) R + \frac{1}{2} g^{\mu \nu} \psi_{,\mu} \psi_{,\nu} - V(\psi) \right\}, \]  
\[ (1) \]

where \( F(\psi) \) is the coupling function of the scalar \( \psi \) to spacetime curvature, \( V(\psi) \) is an arbitrary scalar self-interaction and \( 8\pi G = 1 \) in our units.

It is convenient to change variables into the “Einstein frame”, in which the action (1) reduces to the Einstein-Hilbert term plus a minimally coupled scalar field [1,2]:

\[ \mathcal{L} = \sqrt{-\tilde{g}} \left\{ -\frac{1}{2} \tilde{R} + \frac{1}{2} \tilde{g}^{\mu \nu} \tilde{\psi}_{,\mu} \tilde{\psi}_{,\nu} - \tilde{V} \right\}. \]  
\[ (2) \]

We will show that classical solutions \([g_1, \psi_1, V_1]\) of the system (1) are dual to some other solutions \([g_2, \psi_2, V_2]\). These dualities follow because for each Einstein-frame solution \([\tilde{g}, \tilde{\psi}, \tilde{V}]\) of system (2), there are in general many corresponding physical solutions \([g_n, \psi_n, V_n]\).

II. THE EINSTEIN FRAME

Because the factor \( F(\psi) \) multiplies the scalar curvature in the Lagrangian (1), it is clear that the physical (helicity two) degrees of freedom of the gravitational field appear mixed with the scalar field. Diagonalization of the physical (or “Jordan-frame”) action (1) can be achieved by means of a conformal transformation of the metric [1]:

\[ \tilde{g}_{\mu \nu} \equiv \Omega^2 g_{\mu \nu}, \]  
\[ (3) \]

under which the Ricci scalar (in four dimensions) transforms as:

\[ \tilde{R} \equiv R[\tilde{g}] = \Omega^{-2} \left[ R - 6g^{\mu \nu} D_\mu D_\nu (\log \Omega) - 6g^{\mu \nu} D_\mu (\log \Omega) D_\nu (\log \Omega) \right], \]  
\[ (4) \]

where \( D_\mu[g] \) are covariant derivatives. Using this expression in Eq. (1) one obtains, after neglecting total derivatives:
\[
\sqrt{-\tilde{g}} \left\{ -\frac{F(\psi)}{2\Omega^2} \tilde{R} + \frac{1}{2\Omega^2} \tilde{g}^{\mu\nu} \left[ \psi_{,\mu} \psi_{,\nu} + 6F(\psi) (\log \Omega)_{,\mu} (\log \Omega)_{,\nu} \right] - \frac{V(\psi)}{\Omega^4} \right\} . \tag{5}
\]

We want to rewrite this Lagrangian in such a way that it resembles as much as possible the Einstein-Hilbert action plus a minimally coupled scalar field. Hence, we take \( \Omega^2 = \pm F(\psi) \), where the upper choice of sign applies when \( F > 0 \), while the minus sign applies in the case \( F < 0 \). This substitution leads to:

\[
\sqrt{-\tilde{g}} \left\{ \pm \frac{1}{2} \tilde{R} \pm \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{\psi}_{,\mu} \tilde{\psi}_{,\nu} \left( \frac{1}{F} + \frac{3 F^2}{2 F^2} \right) - \tilde{V}(\tilde{\psi}) \right\} . \tag{6}
\]

Introducing a conformally transformed field and potential:

\[
d\tilde{\psi}^2 \equiv \left( \frac{1}{F} + \frac{3 F^2}{2 F^2} \right) d\psi^2 , \tag{7}
\]

\[
\tilde{V} \equiv \pm \frac{V}{F^2} , \tag{8}
\]

and substituting into Eq. (6) we obtain:

\[
\mathcal{L} = \pm \sqrt{-\tilde{g}} \left\{ -\frac{1}{2} \tilde{R} + \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{\psi}_{,\mu} \tilde{\psi}_{,\nu} - \tilde{V}(\tilde{\psi}) \right\} , \tag{9}
\]

which is, up to the global sign, simply the action for gravity with a minimally coupled scalar field

On account of Eqs. (3), (7) and (8), the transformation is evidently singular if \( F = 0 \). At that point there is in fact a spacetime singularity which splits the theory into disconnected sectors [3,4]. For transformation (7) to make sense the terms inside the square root should be positive, that is:

\[
\frac{1}{F} + \frac{3 F^2}{2 F^2} \geq 0 . \tag{10}
\]

If this condition is violated, then the “effective” gravitons and scalars have energies with opposite signs and the theory is unstable [2,5].

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\(^1\)Since the equations of motion are insensitive to this overall sign, the two theories, with plus and minus signs, are identical. In any event, our main considerations will focus on cases where \( F > 0 \) and the choices of sign are the usual ones.
III. DUALITY

Suppose that $\tilde{\psi}$ defined in Eq. (7) is degenerate in $\psi$, that is, there are $n$ ($n \geq 2$) values $\psi_1, \cdots, \psi_n$ of the physical field $\psi$ which correspond to the same value of the Einstein-frame field $\tilde{\psi}$. If this is true, then there are at least two physical solutions which are related by the conformal transformation defined by Eqs. (3), (7) and (8):

$$(\psi_1, g_1, V_1) \uparrow (\tilde{\psi}, \tilde{g}, \tilde{V}) \downarrow (\psi_2, g_2, V_2).$$

The up-down arrow is a duality relation that is found by using the conformal transformation and its inverse. The group formed by these transformations can be discrete (e.g., parity), continuous (e.g., scaling), or a combination of the two.

A. Case 1: $F = \zeta \psi^2$

Consider the conformal function $F(\psi) = \zeta \psi^2$, where $\zeta > 0$. We can integrate Eq. (7) immediately, with the result:

$$\tilde{\psi}(\psi) = \pm \sqrt{6\zeta + 1} \zeta \log \left[ \frac{\psi}{q} \right],$$

where $q$ is a positive integration constant, and the choice of sign is arbitrary.
The solid (black) and dashed (blue) lines correspond respectively to the positive and the negative choices of the sign in Eq. (12), with $q = 1$ in both cases.

The simplest type of degeneracy is the one related to the choice of $q$, which shifts $\tilde{\psi}$ up or down by a constant factor. The dualities associated with this degeneracy are rescalings of the scalar field by a constant, $\psi \rightarrow \alpha \psi$.

A more profound type of duality arises from the arbitrary choice of the sign in Eq. (12). Let’s assume, without loss of generality, that $\psi > 0$. if $\psi_a/q_a = \psi_b/q_b$, then a differing choice of signs in each case means that the solutions $a$ and $b$ are dual to each other (see Fig. 1): for each $\tilde{\psi}$ there are two associated physical fields, $\psi_a$ and $\psi_b$. Given a solution of the Einstein and Klein-Gordon equations $\{\psi_a, g^{(a)}_{\mu\nu}\}$ for the theory with potential $V_a$, there is a dual solution to those equations, $\{\psi_b, g^{(b)}_{\mu\nu}\}$, but with a different potential $V_b$, which follows from Eqs. (3), (7) and (8). Defining $q_aq_b \equiv \psi_0^2$, the dual theory is expressed in terms of the original one by:

$$\psi_b = \frac{\psi_b^2}{\psi_a} ,$$

$$g^{(b)}_{\mu\nu} = \frac{F(\psi_a)}{F(\psi_b)}g^{(a)}_{\mu\nu} = \frac{\psi_a}{\psi_b^2}g^{(a)}_{\mu\nu} \quad (14)$$

$$V_b = \frac{F^2(\psi_b)}{F^2(\psi_a)}V_a = \frac{\psi_b^8}{\psi_a^8}V_a . \quad (15)$$

The relation between the “effective Newton’s constant” in the dual theory compared with the original one is:
\[ G_{\text{eff}}^{(a)} = \frac{G}{\zeta \psi_a^2}, \quad (16) \]
\[ G_{\text{eff}}^{(b)} = \frac{G}{\zeta \psi_b^2} = \frac{\psi_0^4}{\psi_b^4} G_{\text{eff}}^{(a)}, \quad (17) \]

while the effective Planck masses are related by:
\[ M_{P, \text{eff}}^{(b)} = \frac{\psi_0^2}{\psi_a^2} M_{P, \text{eff}}^{(a)} = \frac{\psi_0^2}{\psi_0^2} M_{P, \text{eff}}^{(a)} . \quad (18) \]

If \( \psi_a / \psi_0 \ll 1 \), then \( \psi_b / \psi_0 \gg 1 \), which means that while in the original theory theory \( M_{P, \text{eff}}^{(a)} \gg 1 \), in the dual theory \( M_{P, \text{eff}}^{(b)} \ll 1 \) — i.e., gravity is “weak” in the original, and “strong” in the dual theory.

In fact, under duality all effective masses scale in the same way as the effective Planck mass. Take the theory with scalar potential:
\[ V_a = \lambda_a M_a^4 \left( \frac{\psi_a}{M_a} \right)^{n_a} , \quad (19) \]

where the coupling \( \lambda_a \) is a dimensionless constant, \( M_a \) has dimensions of mass and \( n_a \) is some integer. The potential in a dual theory with \( \psi_b = \frac{\psi_0^2}{\psi_a} \) is given by Eq. (15):
\[ V_b = \frac{E_b^2}{E_a^2} V_a = \lambda_a \psi_0^4 \left( \frac{M_a}{\psi_0} \right)^{4-n_a} \left( \frac{\psi_b}{\psi_0} \right)^{8-n_a} \]
\[ = \lambda_a \left( M_a \frac{\psi_b}{\psi_0} \right)^4 \left( \frac{\psi_b}{M_a \frac{\psi_b}{\psi_0}} \right)^{n_a} . \quad (20) \]

Therefore, we can say that duality transforms all effective mass scales as:
\[ M \rightarrow M_b = \frac{\psi_0^2}{\psi_0^2} M_a . \quad (21) \]

However, notice that since \( \psi_b \) is in fact a dynamical variable, the dual theories are fundamentally different from each other. From inspection of Eqs. (19) and (20), the exact statement is that duality transforms the parameters of the potential as follows:
\[ n \rightarrow n_b = 8 - n_a , \]
\[ M \rightarrow \psi_0 , \]
\[ \lambda \rightarrow \lambda_b = \lambda_a \left( \frac{M_a}{\psi_0} \right)^{4-n_a} . \]
The dual of the case \( n = 2 \) with mass \( M \), for instance, is an \( n = 6 \) theory with coupling \( \lambda M^2 / \psi_0^2 \) and mass scale \( \psi_0 \).

In the present example with \( F = \zeta \psi^2 \), our duality takes \( \psi \rightarrow c/\psi \). We now show that this duality is nothing but “S-duality”, a symmetry of the low-energy effective action of string theory which switches the sign of the dilaton field, \( \phi \rightarrow -\phi \). With a conformal factor \( F(\psi) = \psi^2 / 4 \), consider the field redefinition:

\[
\psi = 2e^{-\phi/2},
\]

leading to the action:

\[
L = \sqrt{-g} e^{-\phi} \left\{ -\frac{1}{2} R + \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - e^{\phi} V(\phi) \right\},
\]

which is just the low-energy effective action of string theory restricted to gravity and the dilaton. It is well known that the transformation \( \phi \rightarrow -\phi \), together with an accompanying Weyl transformation of the metric, \( g_{\mu\nu} \rightarrow \exp(2\phi) g_{\mu\nu} \), leaves the action invariant — this is the so-called S-duality of low-energy string theory [6]. By Eq. (22), this duality takes \( \psi \rightarrow 4/\psi \), which is exactly the type of duality that we have discussed. Notice that choices of \( \zeta \) other than 1/4 are obtained simply by shifting the dilaton by a constant.

**B. Case 2: \( F = \zeta \psi^{2k} \)**

Generalizing the previous results, consider a conformal factor \( F(\psi) = \zeta \psi^{2k} \), where \( \zeta \) and \( k \neq 1 \) are positive real numbers. In this case the Einstein-frame field is:

\[
\tilde{\psi} = \psi_0 \pm \frac{1}{k - 1} \sqrt{\frac{6\zeta k^2 + \psi^{2-2k}}{\zeta}}
\]

\[
\times \left[ 1 - \sqrt{6\zeta k} \left( 6\zeta k^2 + \psi^{2-2k} \right)^{-1/2} \text{Arcsinh} \left( \sqrt{6\zeta k} \psi^{k-1} \right) \right].
\]

It is easy to show that this expression reduces to Eq. (12) when \( k \rightarrow 1 \). The qualitative behavior of \( \tilde{\psi} \) is the same as that of Fig. 1.

In the example with \( k = 1 \), because of the purely logarithmic dependence, the arbitrary integration constant of \( \tilde{\psi} \) translates into scaling invariance of the physical theory. In the
present case, the constant factor inside square brackets prevents the integration constant from being absorbed into the Arcsinh. Therefore, the case $k \neq 1$ does not exhibit scaling invariance — duality now involves some transcendental relation between different values of the scalar field. It might still be worth noting that for $k < 1$ the theory regains asymptotic scaling invariance in the limit $\psi \ll 1$.

C. Case 4: $F = 1 - \xi \psi^2$

This is the usual case when one considers NMC scalar fields. For $\xi = 0$ the coupling is said to be “minimal”, while $\xi = 1/6$ corresponds to the so-called “conformal coupling”. This theory admits three stable sectors [2]:

(a) $\xi \leq 0$ ,

(b) $\xi \geq 1/6$ with $\psi^2 < \xi^{-1}$ , and

(c) $\xi \geq 1/6$ with $\psi^2 > \xi^{-1}$.

Direct integration of Eq. (7) in this case yields the following expression:

$$
\tilde{\psi}(\psi) = \sqrt{3} \log \left[ \alpha \left( \sqrt{\xi (6\xi - 1) \psi^2} + \sqrt{1 + \xi (6\xi - 1) \psi^2} \right)^{-\frac{1}{2}} \right] \times \frac{1 + \psi \sqrt{\xi}}{1 - \psi \sqrt{\xi}} \times \frac{1 + \sqrt{6\xi} \sqrt{1 + \xi (6\xi - 1) \psi^2 + \sqrt{\xi (6\xi - 1) \psi}}}{1 + \sqrt{6\xi} \sqrt{1 + \xi (6\xi - 1) \psi^2 - \sqrt{\xi (6\xi - 1) \psi}}} , \tag{25}
$$

where $\alpha$ is an arbitrary integration constant. We have plotted $\tilde{\psi}(\psi)$ in Fig. 2 for the cases $\xi = -1/8$, $\xi = 1/6$ and $\xi = 1$, with the choice $\alpha = \pm 1$ to make the argument of the log positive (a different choice with the same sign would only shift $\tilde{\psi}$ by a constant.)
FIG. 2. Einstein-frame scalar field $\tilde{\psi}$ in the case $F = 1 - \xi \psi^2$, as a function of the physical scalar $\psi$, for three values of the curvature coupling $\xi$: $-1/8$, $1/6$ and $1$ (red, blue and purple lines, respectively.)

The case $\xi = -1/8$, representing sector $a$ ($\xi \leq 0$), is the simplest: $\tilde{\psi}$ is monotonic in $\psi$. In this case there is no degeneracy and, hence, no duality.

The case $\xi = 1/6$ (the middle line at the sides of Fig. 2) is a threshold case: here, the Einstein-frame field $\tilde{\psi}$ goes to zero when $|\psi| \rightarrow \infty$, so for each value of $\tilde{\psi}$ there are two associated values of the physical field $\psi$ — one in sector $b$, $|\psi_b| < \sqrt{6}$ and $F(\psi_b) > 0$, and one in sector $c$, $|\psi_c| > \sqrt{6}$ and $F(\psi_c) < 0$. In this case expression (25) simplifies considerably:

$$\tilde{\psi}(\psi) = \begin{cases} \sqrt{6} \tanh^{-1} \left[ \psi_b / \sqrt{6} \right] , & \psi_b^2 < 6 \ (\alpha = +1) , \\ \sqrt{6} \tanh^{-1} \left[ \sqrt{6} / \psi_c \right] , & \psi_c^2 > 6 \ (\alpha = -1) . \end{cases}$$

For $\xi = 1/6$ the explicit relation between the two physical solutions $(g_b, \psi_b, V_b)$ and $(g_c, \psi_c, V_c)$ corresponding to the same Einstein-frame solution $(\tilde{g}, \tilde{\psi}, \tilde{V})$ can be obtained by solving for the arguments of the $\tanh^{-1}$, and by using Eqs. (3) and (8):

$$\psi_b \psi_c = 6 ,$$

$$\psi_b g^b_{\mu\nu} = \psi_c g^c_{\mu\nu} ,$$

$$\psi_b^2 V_b = - \psi_c^2 V_c .$$

It can be easily seen from Eq. (27) that this particular duality relates physical solutions $\psi_b$ and $\psi_c$ such that, for $F_b \rightarrow 0^+$, $F_c \rightarrow 0^-$, and for $F_b \rightarrow 1$, $F_c \rightarrow -\infty$. 9
The case $\xi > 1/6$ is more interesting. In the example of Fig. 2, of $\xi = 1$, the region $|\psi| < 1$ (between the poles) corresponds to our sector $b$, while region $|\psi| > 1$ corresponds to our sector $c$. As before, the choice of the integration constant was $\alpha = \pm 1$ to make the argument of the log in Eq. (25) positive.

As can be easily seen from Fig. 2, when $\xi > 1/6$, for each value of $\tilde{\psi}$ there are now three associated values of $\psi$ — one in sector $b$ and one in each of the two independent branches of sector $c$.

Explicit relations between the dual solutions are found by solving for the threefold degeneracy of Eq. (25). Unfortunately, it has not been possible to solve these equations exactly, except in the case $\xi = 1/6$ explained above. Nevertheless, it suffices for our purposes to inspect the asymptotic limits $|\psi_1|\sqrt{\xi} \to 1$ and $|\psi_2| \to \infty$. We find that for each $\tilde{\psi}$ there is a threefold degeneracy:

$$1 - |\psi_{b1}|\sqrt{\xi} \approx |\psi_{c1}|\sqrt{\xi} - 1 \approx \beta [-\text{sign} (\psi_{c1}) \psi_{c2}]^{-\sqrt{\frac{2(6k-1)}{6k-\xi}}}$$

(30)

where $\beta$ is a $\xi$-dependent constant, $|\psi_{b1}|\sqrt{\xi} \to 1^-$, $|\psi_{c1}|\sqrt{\xi} \to 1^+$ and $|\psi_{c2}| \to \infty$. One of the degenerate solutions belongs to sector $b$, while the other two belong to sector $c$, as is also clear from Fig. 2. However, two of the dual solutions now have $|F(\psi_1)| \to 0$ in the limit $|\psi_1|\sqrt{\xi} \to 1$, while the third solution $|\psi_c| \to \infty$, which means that $F(\psi_c) \to -\infty$.

IV. CONCLUSION

We have found a class of dualities of Einstein’s gravity with a non-minimally coupled scalar field. The dualities relate inequivalent theories for which the effective couplings scale by the conformal factor $F(\psi)$. In particular, the dualities relate theories for which gravity is weak with theories for which it is strong. In the simplest case, $F(\psi) = \zeta \psi^2$, these dualities reduce to the S-duality of low-energy effective string theory. We have also studied the cases $F = \zeta \psi^{2k}$ and $F = 1 - \xi \psi^2$, and have shown how dualities are manifested in those theories.
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