A FAMILY OF SUPERCONGRUENCES INVOLVING MULTIPLE HARMONIC SUMS

MEGAN MCCOY*, KEVIN THIELEN*, LIUQUAN WANG†, AND JIANQIANG ZHAO*

Abstract. In recent years, the congruence
\[ \sum_{i+j+k=p \atop i,j,k>0} \frac{1}{ijk} \equiv -2B_{p-3} \pmod{p}, \]
first discovered by the last author have been generalized by either increasing the number of indices and considering the corresponding supercongruences, or by considering the alternating version of multiple harmonic sums. In this paper, we prove a family of similar supercongruences modulo prime powers \( p^r \) with the indexes summing up to \( mpr \) where \( m \) is coprime to \( p \), where all the indexes are also coprime to \( p \).

1. INTRODUCTION

Multiple harmonic sums are multiple variable generalization of harmonic numbers. Let \( \mathbb{N} \) be the set of natural numbers. For \( s = (s_1, \ldots, s_d) \in \mathbb{N}^d \) and any \( N \in \mathbb{N} \), we define the multiple harmonic sums (MHS) by
\[ H_N(s) := \sum_{N \geq k_1 > \cdots > k_d > 0} \prod_{i=1}^{d} \frac{1}{k_i^{s_i}}. \]
Since mid 1980s these sums have appeared in a few diverse areas of mathematics as well as theoretical physics such as multiple zeta values [4, 5, 7], Feynman integrals [1, 3], quantum electrodynamics and quantum chromodynamics [2, 10].

In [17] the last author started to investigate congruence properties of MHSs, which were also considered by Hoffman [5] independently. As a byproduct, the following intriguing congruence was noticed: for all primes \( p \geq 3 \)
\[ \sum_{i+j+k=p \atop i,j,k>0} \frac{1}{ijk} \equiv -2B_{p-3} \pmod{p}, \]
where \( B_k \) are Bernoulli numbers defined by the generating series
\[ \frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}. \]
This was proved by the last author using MHSs in [16], and by Ji using some combinatorial identities in [6]. Later on, a few generalizations and analogs were obtained by either increasing

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the number of indices and considering the corresponding supercongruences (see [11, 13, 15, 21]), or considering the alternating version of MHSs (see [9, 12]).

Let \( P_n \) be the set of positive integers not divisible by \( n \). To generalize the congruence in (1), we wonder if for every odd integer \( d \geq 3 \) there exists a rational number \( q_d \) such that

\[
\sum_{l_1, l_2, \ldots, l_d \in P_p} \frac{1}{l_1 l_2 \cdots l_d} \equiv q_d \cdot p^{r-1} B_{p-d} \pmod{p^r}
\]

for any prime \( p > d \) and integer \( r \geq 2 \). In [11, 13] it is shown that \( q_3 = -2 \) and \( q_5 = -5!/6 \).

We should point it out that when \( d \) is even, the congruence pattern is quite different, see [14, 19]. In this paper, we shall prove the following main result when \( d = 7 \).

**Theorem 1.1.** Let \( r \) and \( m \) be positive integers and \( p > 7 \) be a prime such that \( p \nmid m \).

(i) If \( r = 1 \), then

\[
\sum_{l_1, l_2, \ldots, l_7 \in P_p} \frac{1}{l_1 l_2 \cdots l_7} \equiv -(504m + 210m^3 + 6m^5)B_{p-7} \pmod{p^r}.
\]

(ii) If \( r \geq 2 \), then

\[
\sum_{l_1, l_2, \ldots, l_7 \in P_p} \frac{1}{l_1 l_2 \cdots l_7} \equiv -\frac{7!}{10} \cdot mp^{r-1} B_{p-7} \pmod{p^r}.
\]

To establish this result, for all positive integers \( n, m, r \) and primes \( p \), following the notation in [13], we define

\[
S^{(m)}_n(p^r) := \sum_{l_1, l_2, \ldots, l_n \in P_p} \frac{1}{l_1 l_2 \cdots l_n}.
\]

Notice that the sum in the theorem is not exactly the same type as that appearing in \( S^{(m)}_n \) since the condition \( p^r > l_i \) for all \( i \) is not present. The main idea of our proof is to show the special case when \( m = 1 \) first. In order to do this we will first prove the relation

\[
S^{(1)}_n(p^{r+1}) \equiv pS^{(1)}_n(p^r) \pmod{p^{r+1}}, \quad \forall r \geq 2,
\]

and then use induction. Notice that when \( r = 1 \) the above congruence usually does not hold anymore. So we will compute the congruence of \( S^{(1)}_n(p^2) \) and \( S^{(1)}_n(p) \) separately by relating them to the following quantities:

\[
R^{(m)}_n(p) := \sum_{l_1, l_2, \ldots, l_n \in P_p} \frac{1}{l_1 l_2 \cdots l_n}.
\]

To save space, throughout the paper when the prime \( p \) is fixed we often use the shorthand \( H(s) = H_{p-1}(s) \). Moreover, we shall also need the modified sum

\[
H^{(p)}(s) := \sum_{N \geq k_1 > \cdots > k_d > 0} \prod_{i=1}^d \frac{1}{k_i^{s_i}}.
\]
2. Preliminary lemmas

Let $C_{a,p}^{(m)}(n)$ denote the number of solutions $(x_1, \ldots, x_n)$ of the equation
$$x_1 + \cdots + x_n = mp - a, \quad 0 \leq x_i < p \ \forall i = 1, \ldots, n.$$ 
For all $b \geq 1$ set
$$\beta_n(a,b) := \begin{pmatrix} bp - a + n - 1 \\ n - 1 \end{pmatrix} \quad \text{and} \quad \gamma_n(a) := \frac{(-1)^a - 1}{a(\frac{n-1}{a})}.$$ 
It is not hard to see that
$$\beta_n(a,b) \equiv \frac{b(-1)^a - 1(n - a - 1)!}{(n - 1)!} \equiv \beta_n(a,b) \equiv b\gamma_n(a)p \pmod{p^2}. \quad (4)$$

Lemma 2.1. For all $m, n, a \in \mathbb{N}$ and primes $p$, we have
$$C_{a,p}^{(m)}(n) \equiv (-1)^{m-1}\begin{pmatrix} n - 2 \\ m - 1 \end{pmatrix}\gamma_n(a)p \equiv (-1)^{m-1}\begin{pmatrix} n - 2 \\ m - 1 \end{pmatrix}C_{a,p}^{(1)}(n) \pmod{p^2}.$$ 

Proof. The coefficient of $x^{mp-a}$ in the expansion of $(1 + x + \cdots + x^{p-1})^n = (x^p - 1)^n(x - 1)^{-n}$ is
$$C_{a,p}^{(m)}(n) = \sum_{i=0}^{m} \begin{pmatrix} n \\ i \end{pmatrix} \begin{pmatrix} -n \\ mp - ip - a \end{pmatrix}(-1)^{mp-a}$$
$$= \sum_{i=0}^{m} \begin{pmatrix} n \\ i \end{pmatrix}(-1)^{ip} \begin{pmatrix} n + mp - ip - a - 1 \\ n - 1 \end{pmatrix}$$
$$= \sum_{i=0}^{m} (-1)^{i}\begin{pmatrix} n \\ i \end{pmatrix} \begin{pmatrix} n + mp - ip - a - 1 \\ n - 1 \end{pmatrix}$$
$$\equiv \sum_{i=0}^{m} (-1)^{i}\begin{pmatrix} n \\ i \end{pmatrix}(m - i)\gamma_n(n - a)p \pmod{p^2}$$
by (4). Now we calculate the sum
$$A(m) = \sum_{i=0}^{m} (-1)^i\begin{pmatrix} n \\ i \end{pmatrix}(m - i).$$
It is easy to see that $A(m)$ is the coefficient of $x^m$ in the expansion of
$$(1 - x)^n \cdot \sum_{i=0}^{\infty} ix^i = (1 - x)^n \cdot \frac{x}{(1 - x)^2} = x(1 - x)^{n-2} = \sum_{m=1}^{n-1} (-1)^m \begin{pmatrix} n - 2 \\ m - 1 \end{pmatrix}x^m,$$
as desired. \hfill \Box

Corollary 2.2. When $n = 7$, we have
$$C_{1,p}^{(2)}(7) - C_{6,p}^{(2)}(7) \equiv -(5/3)p, \quad C_{1,p}^{(3)}(7) - C_{6,p}^{(3)}(7) \equiv (10/3)p \pmod{p^2},$$
$$C_{2,p}^{(3)}(7) - C_{5,p}^{(3)}(7) \equiv -(2/3)p, \quad C_{2,p}^{(2)}(7) - C_{5,p}^{(2)}(7) \equiv (1/3)p \pmod{p^2},$$
$$C_{3,p}^{(3)}(7) - C_{4,p}^{(3)}(7) \equiv (1/3)p, \quad C_{3,p}^{(2)}(7) - C_{4,p}^{(2)}(7) \equiv -(1/6)p \pmod{p^2}.$$ 
Part (ii) of the following lemma generalizes [13, Lemma 1(ii)].
Lemma 2.3. Let \( 1 \leq k \leq n - 1 \) and \( p > n \) a prime. For all \( r \geq 1 \), we have

(i) \( S_n^{(k)}(p^r) \equiv (-1)^n S_n^{(n-k)}(p^r) \pmod{p^r} \).

(ii) \( S_n^{(m)}(p^{r+1}) \equiv \sum_{a=1}^{n-1} C_{a,p}^{(m)}(n) S_n^{(a)}(p^r) \pmod{p^{r+1}} \).

Proof. (i) can be found in [13]. We now prove (ii). For any \( n \)-tuples \((l_1, \ldots, l_n)\) of integers satisfying \( l_1 + \cdots + l_n = mp^{r+1} \), \( p^r + 1 > l_1 \in P_p \), \( 1 \leq i \leq n \), we rewrite them as

\[
 l_i = x_i p^r + y_i, \quad 0 \leq x_i < p, \quad 1 \leq y_i < p^r, \quad y_i \in P_p, \quad 1 \leq i \leq n.
\]

Since

\[
 \left( \sum_{i=1}^{n} x_i \right) p^r + \sum_{i=1}^{n} y_i = mp^{r+1}
\]

and \( n < p \), we know there exists \( 1 \leq a < n \) such that

\[
 \begin{cases}
 x_1 + \cdots + x_n = mp - a, & 0 \leq x_i < p, \\
 y_1 + \cdots + y_n = ap^r.
\end{cases}
\]

For \( 1 \leq a < n \), the equation \( x_1 + \cdots + x_n = mp - a \) has \( C_{a,p}^{(m)}(n) \) integer solutions with \( 0 \leq x_i < p \). Hence by Lemma 2.1

\[
 S_n^{(m)}(p^{r+1}) = \sum_{l_1+\cdots+l_n=mp^{r+1}} \frac{1}{l_1 l_2 \cdots l_n}
 = \sum_{a=1}^{n-1} \sum_{0 \leq x_i < p} \sum_{y_i \in P_p, y_i < p^r} \frac{1}{(x_1 p^r + y_1) \cdots (x_n p^r + y_n)}
 \equiv \sum_{a=1}^{n-1} C_{a,p}^{(m)}(n) S_n^{(a)}(p^r) \pmod{p^{r+1}}
\]

since for each \( x_j \) \( (j = 1, \ldots, n) \), we have

\[
 \sum_{x_1+\cdots+x_n=mp-a \atop 0 \leq x_i < p} x_j = \frac{1}{n} \sum_{x_1+\cdots+x_n=mp-a} (x_1 + x_2 + \cdots + x_n) = \frac{mp-a}{n} C_{a,p}^{(m)}(n) \equiv 0 \pmod{p}
\]

by Lemma 2.1.

3. Congruences involving multiple harmonic sums

We first consider some unordered sums. Lemmas 3.1 and 3.3 were proved by Zhou and Cai [21].
Lemma 3.1. Let \( p \) be a prime and \( \alpha_1, \ldots, \alpha_n \) be positive integers, \( r = \alpha_1 + \cdots + \alpha_n \leq p - 3 \). Define the un-ordered sum
\[
U_b(\alpha_1, \ldots, \alpha_n) = \sum_{0 < l_1 < \cdots < l_n < b^p, l_i \neq l_j \forall i \neq j} \frac{1}{l_1 \cdots l_n}.
\]
Then
\[
U_1(\alpha_1, \ldots, \alpha_n) \equiv \begin{cases} (1) & \text{if } r \text{ is odd; } \\ (-1)^{n-1}(n-1)! \frac{1}{r+1} B_{p-r-1} \cdot p^2 & \text{if } r \text{ is even}. \end{cases}
\]
This easily leads to the following corollary (see also [17]).

Corollary 3.2. Let \( p \) be a prime and \( \alpha \) be a positive integer. Then
\[
H(\{\alpha\}) \equiv \begin{cases} (1) & \text{if } n\alpha \text{ is odd; } \\ (-1)^{n-1}(n-1)! \frac{1}{r+1} B_{p-n-1} \cdot p^2 & \text{if } n\alpha \text{ is even}. \end{cases}
\]

Lemma 3.3. Let \( n \) be a prime and \( \alpha \) be a positive integer and let \( \alpha \) be a positive integer and let \( p > n + 1 \) be a prime. Then
\[
B_n^{(1)}(p) = \sum_{l_1 + \cdots + l_n = p, l_1, \ldots, l_n > 0} \frac{1}{l_1 \cdots l_n} \equiv \begin{cases} (1) & \text{if } n \text{ is odd; } \\ (-1)^n(n-1)! B_{p-n} \cdot p & \text{if } n \text{ is even}. \end{cases}
\]

The next result generalizes Lemma 3.1.

Lemma 3.4. Let \( p \) be a prime and \( \alpha_1, \ldots, \alpha_n \) be positive integers, \( r = \alpha_1 + \cdots + \alpha_n \leq p - 3 \). Then
\[
U_b(\alpha_1, \ldots, \alpha_n) \equiv \begin{cases} (1) & \text{if } \alpha \text{ is odd; } \\ (-1)^{n-1}(n-1)! \frac{1}{r+1} B_{p-r-1} \cdot p^2 & \text{if } \alpha \text{ is even}. \end{cases}
\]

Proof. For all \( k \geq 1 \), we have
\[
\sum_{kp<l<(k+1)p} \frac{1}{l^\alpha} = \sum_{l_1} \frac{1}{(l+kp)^\alpha} = \sum_{l_1} \frac{1}{(1+kp/l)^\alpha} \frac{1}{l^\alpha}
\]
\[
\equiv \sum_{l_1} \left(1 - \frac{akp}{l} + \frac{\alpha(\alpha+1)k^2p^2}{2l^2} \right) \frac{1}{l^\alpha} \pmod{p^3}
\]
\[
\equiv \sum_{l_1} \frac{1}{l^\alpha} - \frac{akp}{\alpha} \sum_{l_1} \frac{1}{l^{\alpha+1}} \pmod{p^3}.
\]
By Lemma 3.1 we see that
\[
\sum_{kp<l<(k+1)p} \frac{1}{l^\alpha} \equiv \begin{cases} -\frac{\alpha(\alpha+1)}{\alpha+2} \left(\frac{1}{2} + k\right) B_{p-\alpha-2} p^2 & \text{if } \alpha \text{ is odd; } \\ -\frac{\alpha}{\alpha+1} B_{p-\alpha-1} p & \text{if } \alpha \text{ is even}. \end{cases}
\]
Therefore for any positive integer \( b \), we have
\[
\sum_{0 < i < bp, p \mid i} \frac{1}{l_i} \equiv \begin{cases} 
\frac{b^2 \alpha(\alpha + 1)}{2(\alpha + 2)} B_{p-\alpha-2p^2} \pmod{p^2}, & \text{if } \alpha \text{ is odd;} \\
\frac{b\alpha}{\alpha + 1} B_{p-\alpha-1p} \pmod{p^2}, & \text{if } \alpha \text{ is even.}
\end{cases}
\]

This proves the lemma in the case \( n = 1 \). Now assume the lemma holds when the number of variables is less than \( n \). Then
\[
U_b(\alpha_1, \ldots, \alpha_n) = \sum_{1 \leq l_1, \ldots, l_{n-1} < b, l_i \neq l_j, l_i \in \mathbb{P}_p} \frac{1}{l_1 n^{n-1}} \left( \sum_{1 \leq l_n < b, l_n \in \mathbb{P}_p} \frac{1}{l_n} - \sum_{i=1}^{n-1} \frac{1}{l_i n^{n-1}} \right)
\equiv U_b(\alpha_1, \ldots, \alpha_{n-1}) \left( \sum_{1 \leq l_n < b, l_n \in \mathbb{P}_p} \frac{1}{l_n} \right) - \sum_{i=1}^{n-1} U_b(\alpha_1, \ldots, \alpha_{i-1}, \alpha_i + \alpha_n, \alpha_{i+1}, \ldots, \alpha_{n-1}).
\]

By the induction assumption, we have
\[
U_b(\alpha_1, \ldots, \alpha_{n-1}) \sum_{1 \leq l_n < b, l_n \in \mathbb{P}_p} \frac{1}{l_n} \equiv \begin{cases} 
0 \pmod{p^2}, & \text{if } r \text{ is odd;} \\
0 \pmod{p^2}, & \text{if } r \text{ is even.}
\end{cases}
\]

Thus if \( r \) is odd, we have
\[
U_b(\alpha_1, \ldots, \alpha_n) \equiv -(n-1)U_b(\beta_1, \ldots, \beta_{n-1}) \quad \left( \text{here } \sum_{j=1}^{n-1} \beta_j = r \right)
\equiv -(n-1)(-1)^{n-1}(n-2)! \frac{b^2 r(r+1)}{2(r+2)} p^2 B_{p-r-2} \pmod{p^3}
\equiv (-1)^n(n-1)! \frac{b^2 r(r+1)}{2(r+2)} p^2 B_{p-r-2} \pmod{p^3}.
\]

Similarly, if \( r \) is even, we can derive
\[
U_b(\alpha_1, \ldots, \alpha_n) \equiv (-1)^{n-1}(n-1)! \frac{br}{r+1} p B_{p-r-1} \pmod{p^2}.
\]

\[\square\]

**Lemma 3.5.** Let \( n \) be an odd positive integer and \( p \) be a prime. Then
\[
R_n^{(2)}(p) = \sum_{l_1 + \cdots + l_n = 2p, l_i \neq l_j, l_i \in \mathbb{P}_p} \frac{1}{l_1 \cdots l_n} \equiv -\frac{n+1}{2} \cdot (n-1)! B_{p-n} \pmod{p}.
\]

**Proof.** We have
\[
\sum_{l_1 + \cdots + l_n = 2p, l_1, \ldots, l_n \in \mathbb{P}_p} \frac{1}{l_1 \cdots l_n}
= \sum_{l_1 + \cdots + l_n = 2p} \frac{1}{l_1 \cdots l_n} - \frac{n}{p} \sum_{l_1 + \cdots + l_{n-1} = p} \frac{1}{l_1 \cdots l_{n-1}}
\]
By Lemma 3.3, we have Corollary 3.6.

Let \( n \) be an odd positive integer with \( n \geq 5 \). Then for all prime \( p > n \), we have

\[
S_n^{(2)}(p) \equiv \frac{n-1}{2} \cdot (n-1)!B_{p-n} \pmod{p}.
\]

**Proof.** We observe that

\[
\sum_{l_1, \ldots, l_n} \frac{1}{l_1 \cdot \ldots \cdot l_n} \equiv \sum_{l_1, \ldots, l_n} \frac{1}{l_1 \cdot \ldots \cdot l_n} + \sum_{l_1, \ldots, l_n} \frac{1}{(l_1 + p)l_2 \cdot \ldots \cdot l_n} \pmod{p}.
\]

By Lemma 3.3 we have \( S_n^{(1)}(p) \equiv -(n-1)!B_{p-n} \pmod{p} \). So we deduce

\[
S_n^{(2)}(p) \equiv \sum_{l_1, \ldots, l_n} \frac{1}{l_1l_2 \cdot \ldots \cdot l_n} - nS_n^{(1)}(p) \equiv \frac{n-1}{2} \cdot (n-1)!B_{p-n} \pmod{p}.
\]

**Lemma 3.7.** Let \( n \geq 3 \) be an odd positive integer. Then for all prime \( p \geq \max\{n, 5\} \), we have

\[
R_n^{(3)}(p) = \sum_{l_1, \ldots, l_n} \frac{1}{l_1 \cdot \ldots \cdot l_n} \equiv -\frac{1}{n} \binom{n+2}{3} \cdot (n-1)!B_{p-n} - \frac{n!}{6} \sum_{a+b+c=n-3 \atop a, b, c \geq 1} \frac{B_{p-2a-1}B_{p-2b-2}B_{p-2c-1}}{(2a+1)(2b+1)(2c+1)} \pmod{p}.
\]
Proof. Let \( \nu = n - 1 \) throughout the proof. Let \( u_i = l_1 + \cdots + l_i, 1 \leq i \leq \nu \). We have
\[
\sum_{l_1 + \cdots + l_n = 3p \atop l_1, \ldots, l_n \in \mathcal{P}_p} \frac{1}{l_1 \cdots l_n} = \frac{n!}{3^p} \sum_{u_1, u_2 - u_1, \ldots, u_{n-2}, u_{n-2} \in \mathcal{P}_p} \frac{1}{u_1 \cdots u_{n-2}}. \tag{5}
\]
Evidently
\[
\sum_{u_1, u_2 - u_1, \ldots, u_{n-2}, u_{n-2} \in \mathcal{P}_p} \frac{1}{u_1 \cdots u_{n-2}} = \sum_{1 \leq u_1 \cdots < u_{n-2} < 3p} \frac{1}{u_1 \cdots u_{n-2}} + \sum_{i=2}^{n-4} \sum_{j=i+2}^{n-2} \sum_{1 \leq u_1 \cdots < u_{n-2} < 3p} \frac{1}{u_1 \cdots u_{n-2}} + \sum_{j=2}^{n-2} \sum_{\nu \leq j < u_{n-2} < 3p, u_j = p} \frac{1}{u_1 \cdots u_j \cdots u_{n-2}} + \sum_{j=2}^{n-2} \sum_{\nu \leq j < u_{n-2} < 3p, u_j = 2p} \frac{1}{u_1 \cdots u_j \cdots u_{n-2}}. \tag{6}
\]
Now we deal with the sums in (5) one by one. For \( 2 \leq j \leq n - 2 \),
\[
\sum_{1 \leq u_1 \cdots < u_{j-1} < 3p, u_j = p} \frac{1}{u_1 \cdots u_{j-1}} = \frac{H(\{1\}^{j-1})}{p} \sum_{0 < u_1 \cdots < u_{j-1} < 2p} \frac{1}{(u_1 + p) \cdots (u_{j-1} + p)} \]
\[
= \frac{1}{p} H(\{1\}^{j-1}) \left( H_{2p-1}^{(p)}(\{1\}^{n-j-1}) - p \sum_{i=0}^{n-j-2} H_{2p-1}^{(p)}(\{1\}^i, 2, \{1\}^{n-i-j-2}) \right)
- \sum_{i=1}^{\nu-j-2} \sum_{0 < u_1 \cdots < u_{j-1} < p} \frac{1}{(u_1 + p) \cdots (u_i + p)(u_{i+2p} + p) \cdots (u_{j-1} + 2p)} \pmod{p^2}.
\]
\[
= \frac{1}{p} H(\{1\}^{j-1}) \left( \frac{U_2(\{1\}^{n-j-1})}{(n-j-1)!} - \frac{U_2(2, \{1\}^{n-j-2})}{(n-j-2)!} \right) \pmod{p^2}
+ p \sum_{i=1}^{\nu-j-2} H(\{1\}^{i-1}, 3, \{1\}^{n-i-j-2}) + p \sum_{i=1}^{\nu-j-2} \sum_{k=0}^{i-1} H(\{1\}^k, 2, \{1\}^{i-k-2}, 2, \{1\}^{n-i-j-2})
+ 2p \sum_{i=1}^{\nu-j-2} H(\{1\}^{i-1}, 3, \{1\}^{n-i-j-2}) + 2p \sum_{k=0}^{\nu-j-2} \sum_{i=0}^{k} H(\{1\}^{i-1}, 2, \{1\}^{k}, 2, \{1\}^{n-i-j-k-3}) \equiv 0 \pmod{p^2} \tag{7}
\]
by Lemma 3.1 and Corollary 3.2 since one of \( j - 1 \) and \( n - j - 1 \) is even and the other is odd.
Similarly, for $2 \leq j \leq n - 2$,

\[
\sum_{1 \leq u_1 < \cdots < u_\nu < 3p, u_j = 2p, \forall k < j, u_k, u_2 - u_1, \ldots, u_\nu - u_{n - 2} \in \mathbb{P}_p} \frac{1}{u_1 \cdots u_\nu} \\
= \frac{1}{2p} \sum_{1 \leq u_1 < \cdots < u_{j - 1} < 2p, \forall k < j, u_k, u_2 - u_1, \ldots, u_{j - 1} - u_{j - 2} \in \mathbb{P}_p} \frac{1}{u_1 \cdots u_{j - 1}} \sum_{2p < u_{j + 1} < \cdots < u_\nu < 3p} \frac{1}{u_{j + 1} \cdots u_\nu} \\
= \frac{1}{2p} \left( H_{2p - 1}^p(\{1\}^{j - 1}) - \sum_{i=1}^{j - 2} \sum_{1 \leq u_1 < \cdots < u_{j - 2} < p} \frac{1}{u_1 \cdots u_i (u_i + p) \cdots (u_{j - 2} + p)} \right) \\
\times \sum_{0 < u_1 < \cdots < u_{n - j - 1} < p} \frac{1}{(u_1 + 2p) \cdots (u_{n - j - 1} + 2p)} \\
\equiv \frac{1}{2p} \left( H_{2p - 1}^p(\{1\}^{j - 1}) - \sum_{i=1}^{j - 2} H(\{1\}^{i - 1}, 2, \{1\}^{j - i - 2}) + p \sum_{i=1}^{j - 2} \sum_{k=0}^{j - i - 2} H(\{1\}^{i - 1}, 2, \{1\}^k, 2, \{1\}^{j - i - k - 3}) \right) \\
\times \left( H(\{1\}^{n - j - 1}) - 2p \sum_{i=0}^{n - j - 2} H(\{1\}^i, 2, \{1\}^{n - i - j - 2}) \right) \pmod{p^2} \\
\equiv \frac{1}{2p} \left( U_2(\{1\}^{j - 1}) - U_1(\{1\}^{j - 3}) + \frac{U_1(2, 2, \{1\}^{j - 4})}{2!(j - 4)!} p \right) \left( H(\{1\}^{n - j - 1}) - \frac{2U_1(2, \{1\}^{n - j - 2})}{(n - j - 2)!} p \right) \\
\equiv 0 \pmod{p^2}. \tag{8}
\]

Further, for all $2 \leq i \leq j - 2 \leq n - 4$, we obtain

\[
\sum_{1 \leq u_1 < \cdots < u_\nu < 3p, u_i = p, u_j = 2p, \forall k \neq i, k \neq j, u_k, u_2 - u_1, \ldots, u_\nu - u_{n - 2} \in \mathbb{P}_p} \frac{1}{u_1 \cdots u_\nu} \\
= \frac{1}{2p^2} \sum_{1 \leq u_1 < \cdots < u_{i - 1} < p} \frac{1}{u_1 \cdots u_{i - 1}} \sum_{p < u_{i + 1} < \cdots < u_{j - 1} < 2p} \frac{1}{u_{i + 1} \cdots u_{j - 1}} \sum_{2p < u_{j + 1} < \cdots < u_\nu < 3p} \frac{1}{u_{j + 1} \cdots u_\nu} \\
\equiv \frac{1}{2p^2} H(\{1\}^{i - 1})(H(\{1\}^{j - i - 1}) - p \sum_{\ell=1}^{j - i} H(\{1\}^\ell, 2, \{1\}^{j - i - \ell - 2})) \\
\times \left( H(\{1\}^{n - j - 1}) - 2p \sum_{\ell=1}^{n - j - 1} H(\{1\}^\ell, 2, \{1\}^{n - j - \ell - 2}) \right) \\
\equiv \frac{1}{2p^2} H(\{1\}^{i - 1})H(\{1\}^{j - i - 1})H(\{1\}^{n - j - 1}) \\
- \frac{1}{2p} H(\{1\}^{i - 1})H(\{1\}^{n - j - 1}) \frac{U_1(2, \{1\}^{j - i - 2})}{(j - i - 2)!} - \frac{1}{p} H(\{1\}^{i - 1})H(\{1\}^{j - i - 1}) \frac{U_1(2, \{1\}^{n - j - 2})}{(n - j - 2)!} \\
\equiv \frac{1}{2p^2} H(\{1\}^{i - 1})H(\{1\}^{j - i - 1})H(\{1\}^{n - j - 1}) \pmod{p^2}.
\]
Thus by Corollary 3.7, we deduce that

\[
\sum_{i=2}^{n-4} \sum_{j=i+2}^{n-2} \frac{1}{u_1 \ldots u_\nu}
\]

\[
\equiv \frac{1}{2p^2} \sum_{a+b+c=n-3}^{a, b, c \geq 1} H(\{1\}^a)H(\{1\}^b)H(\{1\}^c)
\]

\[
\equiv \frac{1}{2p^2} \sum_{a+b+c=2n-3}^{a, b, c \geq 1} H(\{1\}^{2a})H(\{1\}^{2b})H(\{1\}^{2c})
\]

\[
\equiv \frac{p}{2} \sum_{a+b+c=2n-3}^{a, b, c \geq 1} \frac{B_{p-2a-1}B_{p-2b-1}B_{p-2c-1}}{(2a+1)(2b+1)(2c+1)} \pmod{p^2}. \quad (9)
\]

For the first sum in (6), by the inclusion-exclusion principle,

\[
\sum_{1 \leq u_1 \leq \ldots \leq u_\nu < 3p} \frac{1}{u_1 \ldots u_\nu}
\]

\[
\equiv \sum_{1 \leq u_1 \leq \ldots \leq u_\nu < 3p} \frac{1}{u_1 \ldots u_\nu}
\]

\[
- \sum_{j=1}^{n-2} D_j - \sum_{j=1}^{n-2} T_j + \sum_{j=1}^{n-3} \sum_{k=j+2}^{n-2} T_{j,k} + \sum_{j=1}^{n-3} W_j
\]

\[
\equiv \frac{1}{(n-1)!} U_3(\{1\}^{n-1}) - \sum_{j=1}^{n-2} D_j - \sum_{j=1}^{n-2} T_j + \sum_{j=1}^{n-3} \sum_{k=j+2}^{n-2} T_{j,k} + \sum_{j=1}^{n-3} W_j \pmod{p^2}, \quad (10)
\]

where (setting \(v_{n-1} = 3p\))

\[
D_j = \sum_{1 \leq v_1 \leq \ldots < v_j < v_{j+1} + 2 \leq \ldots < v_{n-1} + 3p} \frac{1}{v_1 \ldots v_j (v_j + 2p) v_{j+1} \ldots v_{n-2}}.
\]

\[
T_j = \sum_{1 \leq v_1 \leq \ldots < v_j + p < v_{j+1} \leq \ldots < v_{n-2} + 3p} \frac{1}{v_1 \ldots v_j (v_j + p) v_{j+1} \ldots v_{n-2}}.
\]

\[
T_{j,k} = \sum_{1 \leq v_1 \leq \ldots < v_j < v_{j+1} + p < v_{j+1} + p < v_{k+1} + \ldots < v_{n-3} + 3p} \frac{1}{v_1 \ldots v_j (v_j + p) v_{j+1} \ldots v_{k+1} \ldots v_{n-3}}.
\]

\[
W_j = \sum_{1 \leq v_1 \leq \ldots < v_j + p < v_{j+1} + 2p < v_{j+1} + 3p} \frac{1}{v_1 \ldots v_j (v_j + p) (v_j + 2p) v_{j+1} \ldots v_{n-3}}.
\]
We have
\[
D_j = \sum_{1 \leq v_1 < \ldots < v_{n-2} < p \atop v_1, \ldots, v_{n-2} \in \mathcal{P}_p} \frac{1}{v_1 \ldots v_j(v_j + 2p)(v_{j+1} + 2p) \ldots (v_{n-2} + 2p)}
\]
\[= H(\{1\}^{j-1}, 2, \{1\}^{n-j-2}) - 2p \left( H(\{1\}^{j-1}, 3, \{1\}^{n-j-2}) + \sum_{i=0}^{n-j-3} H(\{1\}^{j-1}, 2, \{1\}^i, 2, \{1\}^{n-j-i-3}) \right) \pmod{p^2}. \]

So by Lemma 3.1 we have
\[
\sum_{j=1}^{n-2} D_j = \frac{U_1(2, \{1\}^{n-3})}{(n-3)!} - \frac{2U_1(3, \{1\}^{n-3})}{(n-3)!} p - \frac{2U_1(2, 2, \{1\}^{n-4})}{(n-4)!} p \equiv \frac{n-1}{n} B_{p-n} \cdot p \pmod{p^2}.
\]

Similarly,
\[
T_j = \sum_{1 \leq v_1 < \ldots < v_{n-2} < 2p \atop v_1, \ldots, v_{n-2} \in \mathcal{P}_p} \frac{1}{v_1 \ldots v_j(v_j + p)(v_{j+1} + p) \ldots (v_{n-2} + p)}
\]
\[= H_{2p-1}^{(p)}(\{1\}^{j-1}, 2, \{1\}^{n-j-2}) - p \left( H_{2p-1}^{(p)}(\{1\}^{j-1}, 3, \{1\}^{n-j-2}) + \sum_{i=0}^{n-j-3} H_{2p-1}^{(p)}(\{1\}^{j-1}, 2, \{1\}^i, 2, \{1\}^{n-j-i-3}) \right) \pmod{p^2}. \]

So by Lemma 3.1 we have
\[
\sum_{j=1}^{n-2} T_j = \frac{U_2(2, \{1\}^{n-3})}{(n-3)!} - \frac{U_2(3, \{1\}^{n-3})}{(n-3)!} p - \frac{U_2(2, 2, \{1\}^{n-4})}{(n-4)!} p \equiv \frac{2(n-1)}{n} B_{p-n} \cdot p \pmod{p^2}.
\]

Moreover,
\[
T_{j,k} = \sum_{1 \leq v_1 < \ldots < v_{n-3} < p \atop v_1, \ldots, v_{n-3} \in \mathcal{P}_p} \frac{1}{v_1 \ldots v_j(v_j + p)(v_{j+1} + p) \ldots (v_{k} + 2p) \ldots (v_{n-3} + 2p)}
\]
\[= H(\{1\}^{j-1}, 2, \{1\}^{k-j-1}, 2, \{1\}^{n-k-3}) - p \left( H(\{1\}^{j-1}, 3, \{1\}^{k-j-1}, 2, \{1\}^{n-k-3}) + \sum_{i=0}^{k-j-2} H(\{1\}^{j-1}, 2, \{1\}^i, 2, \{1\}^{k-i-j-2}, 2, \{1\}^{n-k-3}) + H(\{1\}^{j-1}, 2, \{1\}^{k-j-1}, 3, \{1\}^{n-k-3}) \right)
\]
\[- 2p \left( H(\{1\}^{j-1}, 2, \{1\}^{k-j-1}, 3, \{1\}^{n-k-3}) + \sum_{i=0}^{n-k-4} H(\{1\}^{j-1}, 2, \{1\}^{k-j-1}, 2, \{1\}^i, 2, \{1\}^{n-i-k-4}) \right) \pmod{p^2}. \]
Finally

\[
W_j = \sum_{1 \leq v_1 < \cdots < v_{n-3} < p \atop v_1, \ldots, v_{n-3} \in \mathbb{P}_p} \frac{1}{v_1 \cdots v_j (v_j + p)(v_j + 2p)(v_{j+1} + 2p) \cdots (v_{n-3} + 2p)}
\]

\[\equiv H(\{1\}^{j-1}, 3, \{1\}^{n-j-3}) - pH(\{1\}^{j-1}, 4, \{1\}^{n-j-3})
\]

\[- 2p \left( H(\{1\}^{j-1}, 4, \{1\}^{n-j-3}) + \sum_{i=0}^{n-k-4} H(\{1\}^{j-1}, 3, \{1\}^1, 2, \{1\}^{n-i-j-4}) \right) \pmod{p^2}.
\]

Thus

\[
\sum_{j=1}^{n-3} \sum_{k=j+1}^{n-3} T_{j,k} + \sum_{j=1}^{n-3} W_j \equiv \frac{U_1(2, 2, \{1\}^{n-5})}{2(n-5)!} + \frac{U_1(3, \{1\}^{n-4})}{(n-4)!}
\]

\[- 3p \left( \frac{U_1(4, \{1\}^{n-4})}{(n-4)!} + \frac{U_1(3, 2, \{1\}^{n-5})}{(n-5)!} + \frac{U_1(2 \{1\}^3, \{1\}^{n-6})}{3!(n-6)!} \right) \pmod{p^2}
\]

\[\equiv -(n-4)! \left( \frac{1}{2(n-5)!} + \frac{1}{(n-4)!} \right) B_{p-n} \equiv -\frac{(n-1)(n-2)}{2n} B_{p-n} \pmod{p^2}.
\]

Plugging this into (10), we have

\[
\sum_{1 \leq u_1 < \cdots < u_{3p} \atop u_1, \ldots, u_{3p} \in \mathbb{P}_p} \frac{1}{u_1 \cdots u_{3p}} \equiv -n! \frac{(n^2 + 3n + 2)}{3} \frac{2n}{p} B_{p-n}
\]

\[\equiv -\frac{(n+1)(n+2)}{6} (n-1)! B_{p-n} \pmod{p} \quad (11)
\]

by Lemma 3.1 again.

Now plugging (7), (8), (9) and (11) into (6), and then combining with (5), we get the desired result. \(\Box\)

**Corollary 3.8.** Let \(n \geq 3\) be an odd positive integer. Then for all prime \(p \geq \max\{n, 5\}\), we have

\[
S_n^{(3)}(p) \equiv \left( -\frac{1}{n} \binom{n}{3} \cdot (n-1)! B_{p-n} - \frac{n!}{6} \sum_{a+b+c = \frac{n}{3}} \frac{B_{p-2a-1} B_{p-2b-1} B_{p-2c-1}}{(2a+1)(2b+1)(2c+1)} \right) \pmod{p}.
\]

**Proof.** We observe that

\[
\sum_{l_1 + \cdots + l_n = 3p \atop l_j \in \mathbb{P}_p, \forall j} 1 \equiv \sum_{l_1 + \cdots + l_n = 3p \atop l_j < p, l_j \in \mathbb{P}_p, \forall j} \frac{1}{l_1 \cdots l_n} + \binom{n}{2} \sum_{l_1 + \cdots + l_n = p \atop l_1, \ldots, l_n < p} \frac{1}{(l_1 + p)(l_2 + p)l_3 \cdots l_n}
\]

\[+ n \sum_{l_1 + \cdots + l_n = p \atop l_1, \ldots, l_n < p} \frac{1}{(l_1 + 2p)l_2 \cdots l_n} + n \sum_{l_1 + \cdots + l_n = 2p \atop l_1, \ldots, l_n < p} \frac{1}{(l_1 + p)l_2 \cdots l_n} \pmod{p}.
\]
So we deduce
\[ S_n^{(3)}(p) \equiv \sum_{l_1, \ldots, l_n \equiv 3p \mod l_j} \frac{1}{l_1 \cdots l_n} - \binom{n+1}{2} S_n^{(1)}(p) - nS_n^{(2)}(p) \]
\[ \equiv -\frac{1}{n} \binom{n}{3} \cdot (n-1)! B_{p-n} - \frac{n!}{6} \sum_{a+b+c=n-3, a,b,c \geq 1} \frac{B_{p-2a-1}B_{p-2b-1}B_{p-2c-1}}{(2a+1)(2b+1)(2c+1)} \pmod{p} \]
by Lemma 3.7 since \( S_n^{(1)}(p) \equiv -(n-1)! B_{p-n} \pmod{p} \) by Lemma 3.3 and \( S_n^{(2)}(p) \equiv -\frac{n-1}{2}(n-1)! B_{p-n} \pmod{p} \) by Corollary 3.6.

4. Proof of the main theorem

First, we prove a special case of Theorem 1.1.

**Proposition 4.1.** For all \( r \geq 1 \) and prime \( p > 7 \) we have
\[ S_r^{(1)}(p^{r+1}) \equiv -\frac{7!}{10} B_{p-7} p^r \pmod{p^{r+1}}. \]

**Proof.** By Lemma 2.3 for all \( r \geq 1 \), we have
\[ S_n^{(m)}(p^{r+1}) \equiv \sum_{a=1}^{n-1} (-1)^{m-1} \binom{n-2}{m-1} \gamma_n(a)p + O(p^2) S_n^{(a)}(p^r) \pmod{p^{r+1}}. \]
Here the \( O(p^2) \) means a quantity which remains a \( p \)-adic integer after dividing by the \( p^2 \). By induction on \( r \) it is not hard to see that for all \( m = 1, \ldots, n-1 \), we have
\[ S_n^{(m)}(p^{r+1}) \equiv 0 \pmod{p^r}, \] for all \( r \geq 1 \).
Thus for all \( m = 1, \ldots, n-1 \), by Lemmas 2.1 and 2.3 we have
\[ S_n^{(m)}(p^{r+1}) \equiv (-1)^{m-1} \binom{n-2}{m-1} S_n^{(1)}(p^r) \pmod{p^{r+1}}. \]
Thus by Lemmas 2.1 and 2.3 for all \( r \geq 2 \)
\[ S_n^{(1)}(p^{r+1}) \equiv \sum_{m=1}^{n-1} C_{a,b}^m(n) S_n^{(m)}(p^r) \pmod{p^{r+1}} \]
\[ \equiv \sum_{m=1}^{n-1} (-1)^{m-1} \binom{n-2}{m-1} p^{\gamma_n(m)} S_n^{(1)}(p^r) \pmod{p^{r+1}} \]
\[ \equiv \sum_{m=1}^{n-1} \frac{(n-m-1)! (m-1)! p^{n-2}}{(n-1)!} \binom{n-2}{m-1} S_n^{(1)}(p^r) \pmod{p^{r+1}} \]
\[ \equiv p S_n^{(1)}(p^r) \pmod{p^{r+1}}, \]
which proves (3). Finally, by applying Lemma 2.3 when \( n = 7 \), we get

\[
S_7^{(1)}(p^2) \equiv \frac{p}{3}S_7^{(1)}(p) - \frac{p}{15}S_7^{(2)}(p) + \frac{p}{30}S_7^{(3)}(p^r) \pmod{p^2}
\]

\[
\equiv \left( \frac{p}{3} - \frac{3p}{15} - \frac{5p}{30} \right) 6!B_{p-7} \equiv -\frac{7}{10}B_{p-7}p \pmod{p^2}
\]

by Lemma 3.3 Corollary 3.6 and Corollary 3.8.

We are now ready to prove Theorem 1.1.

Let \( n = mp^r \), where \( p \) does not divide \( m \). For any 7-tuples \( (l_1, \ldots, l_7) \) of integers satisfying \( l_1 + \cdots + l_7 = n \), \( l_i \in \mathbb{P}_p \), \( 1 \leq i \leq 7 \), we rewrite them as

\[
l_i = x_ip^r + y_i, \quad x_i \geq 0, \quad 1 \leq y_i < p^r, \quad y_i \in \mathbb{P}_p, \quad 1 \leq i \leq 7.
\]

Since

\[
\left( \sum_{i=1}^{7} x_i \right)p^r + \sum_{i=1}^{7} y_i = mp^r,
\]

we know there exists \( 1 \leq a \leq 6 \) such that

\[
\begin{cases}
  x_1 + \cdots + x_7 = m - a, \\
y_1 + \cdots + y_7 = ap^r.
\end{cases}
\]

For \( 1 \leq a \leq 6 \), the equation \( x_1 + \cdots + x_7 = m - a \) has \( \binom{m+6-a}{6} \) nonnegative integer solutions. Hence

\[
\sum_{l_1 + \cdots + l_7 = mp^r \atop l_1, \ldots, l_7 \in \mathbb{P}_p} \frac{1}{l_1l_2 \cdots l_7} = \sum_{a=1}^{6} \sum_{x_1 + \cdots + x_7 = m-a \atop y_1 + \cdots + y_7 = ap^r \atop y_i \in \mathbb{P}_p, y_i < p^r} \frac{1}{(x_1p^r + y_1) \cdots (x_7p^r + y_7)} \equiv \sum_{a=1}^{6} \binom{m+6-a}{6} S_7^{(a)}(p^r) \pmod{p^r}. \tag{12}
\]

(i) If \( r = 1 \), then since \( S_7^{(1)}(p) \equiv -6!B_{p-7} \pmod{p} \). We also have \( S_7^{(2)}(p) \equiv 3 \cdot 6!B_{p-7} \pmod{p} \), \( S_7^{(3)}(p) \equiv -5 \cdot 6!B_{p-5} \pmod{p} \) and \( S_7^{(a)}(p) \equiv -S_7^{(7-a)}(p) \pmod{p} \) for \( 4 \leq a \leq 6 \). Hence from (12) we have

\[
\sum_{l_1 + \cdots + l_7 = n \atop l_1, \ldots, l_7 \in \mathbb{P}_p} \frac{1}{l_1l_2 \cdots l_7} \equiv \frac{1}{6!} \left( 504m + 210m^3 + 6m^5 \right) S_7^{(1)}(p) \pmod{p}.
\]

Since \( S_7^{(1)}(p) \equiv -6!B_{p-7} \pmod{p} \) we complete the proof of (i).

(ii) If \( r \geq 2 \), then we have \( S_7^{(2)}(p^r) \equiv -5S_7^{(1)}(p^r) \pmod{p^r} \) and \( S_7^{(3)}(p^r) \equiv 10S_7^{(1)}(p^r) \pmod{p^r} \). Meanwhile, we have \( S_7^{(a)}(p) \equiv -S_7^{(7-a)}(p) \pmod{p^r} \) for \( 4 \leq a \leq 6 \). Hence from (12) we obtain

\[
\sum_{l_1 + \cdots + l_7 = n \atop l_1, \ldots, l_7 \in \mathbb{P}_p} \frac{1}{l_1l_2 \cdots l_7} \equiv \sum_{a=0}^{5} (-1)^a \binom{5}{a} \binom{m+5-a}{6} S_7^{(1)}(p^r) \equiv mS_7^{(1)}(p^r) \pmod{p^r}.
\]

Since \( S_7^{(1)}(p^r) \equiv -\frac{7}{10}p^{r-1}B_{p-7} \pmod{p^r} \) by Proposition 4.1, we complete the proof of (ii).
5. Concluding remarks

Using similar ideas from [19] we find that it is unlikely to further generalize our main result to congruence (2) for \( r \geq 2 \), odd integer \( d \geq 9 \), and \( q_d \in \mathbb{Q} \) depending only on \( d \). By using PSLQ algorithm we find that both the numerator and the denominator of \( q_9 \) would have at least 60 digits if the congruence (2) holds for every prime \( p \geq 11 \). However, when \( r = 1 \) we have obtained a few general congruences in Lemma 3.3, Lemma 3.5 and Corollary 3.6, which can be rephrased as follows. Let \( m = 1, 2 \) and \( d \) be any odd integer greater than 2. Then for any prime \( p > d \), we have

\[
S_d^{(m)}(p) \equiv c_{d, m} \cdot (d - 1)! B_{p-d} \quad (\text{mod } p),
\]

where \( c_{d, 1} = -1 \) and \( c_{d, 2} = (d - 1)/2 \), and

\[
R_d^{(m)}(p) \equiv c'_{d, m} \cdot (d - 1)! B_{p-d} \quad (\text{mod } p),
\]

where \( c'_{d, 1} = -1 \) and \( c'_{d, 2} = -(d + 1)/2 \). Unfortunately, Lemma 3.7 and Corollary 3.8 imply that these do not generalize to \( m \geq 3 \). Computation with PSLQ algorithm suggests that if (13) and (14) hold for \( d = 9, 11, 13, 15 \), \( m = 3, 4 \) then both the numerators and the denominators of \( c_{d, m} \) and \( c'_{d, m} \) would have at least 60 digits. In fact, numerical evidence suggests the following conjecture.

**Conjecture 5.1.** For any prime \( p \geq 11 \), we have

\[
R_8^{(m)}(p) \equiv \frac{112}{5} m(m^2 + 16)(m^2 - 1) B_{p-3} B_{p-5} \quad (\text{mod } p),
\]

\[
R_9^{(m)}(p) \equiv - \frac{8!}{5} \binom{m + 2}{5} B_{p-3}^3 - 8m(m^6 + 126m^4 + 1869m^2 + 3044) B_{p-9} \quad (\text{mod } p),
\]

\[
R_{10}^{(m)}(p) \equiv - \frac{24}{35} m(m^4 + 71m^2 + 540)(m^2 - 1) (50B_{p-3} B_{p-7} + 21B_{p-5}^2) \quad (\text{mod } p).
\]

This conjecture is consistent with the general philosophy we have observed for the finite multiple zeta values (FMZVs). See, for example, [18, 20] for the definition of FMZVs and the relevant results. Note that according to the dimension conjecture of FMZVs discovered by Zagier and independently by the last author (see [20]) the weight 8 (resp. weight 10) piece of FMZVs has conjectural dimension 2 (resp. 3). Theorem 1.1 (i), Conjecture 5.1 and all the previous works in lower weights imply that \( R_d^{(m)}(p) \) (\( d \leq 10 \) and \( m \geq 2 \)) should lie in the proper subalgebra generated by the so-called \( A_1 \)-Bernoulli numbers defined in [20]. According to the analogy between FMZVs and MZVs, this subalgebra is the FMZV analog of the MZV subalgebra generated by the Riemann zeta values. It would be interesting to see if this phenomenon holds in every weight.

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Email address: wanglq@whu.edu.cn
Email address: zhaoj@ihes.fr

*Department of Mathematics, Eckerd College, St. Petersburg, FL 33711, USA
†Department of Mathematics, National University of Singapore, Singapore, 119076, Singapore

*Department of Mathematics, The Bishop’s School, San Diego, CA 92037