Downlink Coverage Analysis in a Heterogeneous Cellular Network

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Abstract—In this paper, we consider the downlink signal-to-interference-plus-noise ratio (SINR) analysis in a heterogeneous cellular network with $K$ tiers. Each tier is characterized by a base-station (BS) arrangement according to a homogeneous Poisson point process with certain BS density, transmission power, random shadow fading factors with arbitrary distribution, arbitrary path-loss exponent and a certain bias towards admitting the mobile-station (MS). The MS associates with the BS that has the maximum SINR under the open access cell association scheme. For such a general setting, we provide an analytical characterization of the coverage probability at the MS.

Index Terms—Multi-tier networks, Cellular Radio, Co-channel Interference, Fading channels, Poisson point process.

I. INTRODUCTION

The heterogeneous cellular network is a complex overlay of multiple cellular communication networks such as the macrocells, microcells, picocells, and femtocells. Research has shown heterogeneous networks support greater end-user data-rate and throughput as well as better indoor and cell-edge coverage. This has further led to its inclusion as an important feature for the 4G cellular networks [1]–[4].

In the conception of the heterogeneous cellular network, one is looking at an overlay of several dense, irregularly and often completely randomly deployed networks (namely microcells, picocells and femtocells) with a limited coverage area, all deployed on top of the conventional macrocell network. These networks consist of base-stations (BSs) with different transmission powers, different traffic-load carrying capability, and different radio environment which is based on the locations in which they are deployed. All these sum up to an extremely complicated network. As a result, the analysis of such a network through system simulations (which is largely the approach taken for studying the conventional macrocell network) is hampered by the curse of dimensionality due to the many parameters involved in designing and modeling each of the representative networks that make up the heterogeneous network. For this reason, we seek to develop an analytical model that captures all the design scenarios of interest.

The analysis in this paper applies to the downlink performance in terms of the signal-to-interference-plus-noise ratio (SINR) at the mobile-station (MS), where the MS associates itself with the BS that has the maximum SINR at the MS. The SINR at the MS is an important metric that determines the outage probability (or coverage probability), capacity and throughput of a cellular network in the downlink, and the characterization of the distribution of SINR aids in the complete understanding of the SINR metric.

The recent focus on stochastic geometry as a means to study and analyze large systems of essentially randomly deployed nodes, combined with the fact that certain representative networks (femtocell networks), that are a part of the heterogeneous network, are formed due to the end-user deployments, thus falling under the random deployment category, has naturally led to the application of stochastic geometry in modeling the heterogeneous network. While there is a vast literature corresponding to stochastic geometric modeling and analysis of systems with randomly deployed nodes, there have been three independent efforts to apply stochastic geometry to study heterogeneous networks [5]–[7]. All the 3 papers modeled the heterogeneous network as being composed of multiple tiers where the BSs in each tier are deployed according to a homogeneous Poisson point process, independent of the other tiers. While the first two references focused on the case where the fading coefficients were modeled as independent and identically distributed (i.i.d.) random variables with exponential distribution and obtained closed-form expressions for the distribution of SINR at the MS in a heterogeneous network for positive values of SINR (in dB), our work has provided semi-analytical expressions for the same quantities and holds for arbitrary fading distributions and for all values of SINR.

Jo et al. [8] have extended their results to obtain a complete characterization of the SINR in the heterogeneous network where the exponents for the power-law path-loss model were different for different tiers of the heterogeneous network, but for the case when the MS associates itself with the nearest BS rather than the BS that maximizes the SINR, and when an exponential distribution was assumed for fading. With this they could study the effect of varying cell-association biasing.

In this paper, we incorporate all the modeling details available in the literature for studying the heterogeneous cellular network, introduce a few additional important features, and obtain the complete characterization of the downlink SINR, and the coverage probability (i.e., outage probability) in a heterogeneous network, for the original case where the MS associates with the BS with the maximum SINR at the MS. In particular, the following paragraph lists our contributions.

From the modeling standpoint, the transmission and channel characteristics of a BS corresponding to a given tier include the transmission power, bias factor, path-loss exponent, distribution of fading coefficients, and the SINR threshold to be satisfied by the MS in order to communicate with the BS. These may be different for different tiers. This is more general than the setting in the previous heterogeneous network analysis literature. For this situation, we have obtained accurate characterization for the SINR distribution as well as the coverage probability for a given MS. Since we are able to handle arbitrary fading distributions, that are further different for different tiers, we are able to consider practical fading
models and obtain the coverage probabilities in those cases. Such a strong result is likely to be useful for studying and analyzing realistic scenarios. The following section introduces the system model.

II. SYSTEM MODEL

This section describes the various elements used to model the wireless network, namely, the BS layout, the radio environment, and the performance metrics of interest.

1) BS Layout: The BS layout for the $k^{th}$ tier, where $k \in \{1, \ldots, K\}$, is according to independent homogeneous Poisson point process with density, $\lambda_k$.

2) Cell-Association policy: The MS associates itself to the BS corresponding to the strongest instantaneous received power (the BS from which the MS has the maximum $\text{SINR}$). We focus on the open access scheme in this paper where the MS can freely communicate with any of the $K$ tiers.

3) Radio Environment: The received power at the MS from the $j^{th}$ BS belonging to the $k^{th}$ tier at a distance $D_{kj}$ ($> 0$) from the MS is given by $P = P_k \Psi_{kj} D_{kj}^{-\epsilon_k} B_k$, where $(P_k, \Psi_{kj}, B_k, \epsilon_k)$ corresponds to the constant transmission power, random channel gain coefficient, constant bias coefficient and the constant path-loss exponent ($> 2$) of the $k^{th}$ tier, respectively. Further, $\Psi_{kj}$ can assume any arbitrary distribution as long as $\mathbb{E}[\Psi_{kj}^{\frac{\epsilon_k}{\epsilon_k}}] < \infty$. It is independent and identically distributed (i.i.d.) across all the BSs of the $k^{th}$ tier, and independent of the other tiers and the underlying random process governing the BS arrangement. For this reason, we drop the subscript $j$ and for compactness denote the expectation by $\mathbb{E}[\Psi_{k}^{\frac{\epsilon_k}{\epsilon_k}}]$. In essence, the different tiers have their own arbitrary fading distributions. All BSs of the $k^{th}$ tier adopt an identical bias factor, $B_k$, which is some positive value. By manipulating the $B_k$’s corresponding to the different tiers, we may regulate the traffic from one tier to the other.

4) Performance Metric: In this paper, we are concerned with the SINR at a given MS. Without loss of generality, the MS is assumed to be located at the origin of the plane. SINR is defined as the ratio of the received signal from the desired BS to the sum of the interferences from all the BSs belonging to all the $K$ tiers and the background noise. As a result

\[
\text{SINR} = \frac{P_k \Psi_{kj} D_{kj}^{-\epsilon_k} B_k}{\sum_{m=1, (m,l) \neq (k,j)}^{K} \sum_{l=1}^{\infty} P_m \Psi_{ml} D_{ml}^{-\epsilon_m} B_m + \eta},
\]

where $(k, j)$ corresponds to the indices of the tier and the corresponding BS, which has the maximum received power at the MS, and $\eta$ is the power corresponding to the background noise. Further, each user can successfully communicate with its desired BS provided the $\text{SINR}$ is above the minimum threshold value, $\beta_k$, that is a characteristic of the tier. As a result, the coverage probability is defined as the probability that the MS is able to communicate with a BS using a specific BS association protocol.

III. USEFUL LEMMAS

We will first present some lemmas which will be useful in deriving the coverage probability.

**Lemma 1.** The SINR at the MS has the same distribution as that of a single-tier network where all BSs in the network have unity transmission power, channel gain and path-loss exponent, and are arranged according to a non-homogeneous 1-D Poisson point process with BS density function $\lambda(r) = \sum_{l=1}^{K} \lambda_l (P_l B_l)^{\frac{\epsilon}{\epsilon}} \mathbb{E} \Psi_l^{\frac{\epsilon}{\epsilon}} r^{\frac{\epsilon}{\epsilon}} - 1$, $r \geq 0$, as long as $\mathbb{E} \Psi_l^{\frac{\epsilon}{\epsilon}} < \infty$, $\forall l = 1, 2, \ldots, K$. That is

\[
\text{SINR} = \frac{\tilde{R}_l - 1}{\sum_{i=2}^{\infty} \tilde{R}_i + \eta} = \mathbb{E} \mathbb{E} \Psi_l^{\frac{\epsilon}{\epsilon}} r^{\frac{\epsilon}{\epsilon}} - 1, \lambda(r),
\]

where $=_{\text{st}}$ indicates the equivalence in distribution, and $\{\tilde{R}_i\}_{i=1}^{\infty}$ is the ascendingly ordered distances of the BSs from the origin, obtained from a non-homogeneous 1-D Poisson point process with BS density function $\lambda(r)$ defined above.

**Proof:** See Appendix A.

Now, for the equivalent single-tier network, we characterize the distance of the nearest $k^{th}$ tier BS from the MS, $\tilde{R}_1^{(k)}$.

**Lemma 2.** The tail probability of $\tilde{R}_1^{(k)}$ is

\[
\mathbb{P}\left(\left\{\tilde{R}_1^{(k)} > r\right\}\right) = \exp(-\lambda_k (P_k B_k)^{\frac{\epsilon}{\epsilon}} r^{\frac{\epsilon}{\epsilon}}), \forall r \geq 0.
\]

**Proof:** See Appendix B.

Using the above result, we will now characterize the random variable $I$, which represents the tier to which the desired BS (i.e., the BS nearest to the MS in the equivalent single-tier network mentioned in Lemma I) belongs.

**Lemma 3.** The desired BS belongs to the $k^{th}$ tier $(k = 1, 2, \ldots, K)$ with the probability

\[
\mathbb{P}\left(I = k\right) = \int_{0}^{\infty} \lambda_k (P_k B_k)^{\frac{\epsilon}{\epsilon}} 2\pi t r^{\frac{\epsilon}{\epsilon}} \exp\left(-\sum_{l=1}^{K} \lambda_l (P_l B_l)^{\frac{\epsilon}{\epsilon}} \mathbb{E} \Psi_l^{\frac{\epsilon}{\epsilon}} r^{\frac{\epsilon}{\epsilon}} - 1\right) dt.
\]

Further, in the special case $\{\epsilon_k\}_{k=1}^{K} = \epsilon,$

\[
\mathbb{P}\left(I = k\right) = \frac{\lambda_k (P_k B_k)^{\frac{\epsilon}{\epsilon}} \mathbb{E} \Psi_k^{\frac{\epsilon}{\epsilon}}}{\sum_{m=1}^{K} \lambda_m (P_m B_m)^{\frac{\epsilon}{\epsilon}} \mathbb{E} \Psi_m^{\frac{\epsilon}{\epsilon}}}.\]

**Proof:** See Appendix C.

Although we do not have a closed-form expression, (1) can be computed easily to any desired accuracy by numerical integration. Next, we obtain the p.d.f. of the distance of the serving BS from MS, given that it belongs to the $k^{th}$ tier, denoted by $f_{\tilde{R}_1^{(k)}}(r | k)$.

**Lemma 4.** The p.d.f. of the distance of the serving BS from the MS, given it belongs to the $k^{th}$ tier $(k = 1, 2, \ldots, K)$,
Theorem 1. The coverage probability of the MS is

$$P_{\text{open-access}} = \frac{\sum_{k=1}^{K} \lambda_k \mathbf{E}}{K} = \int_{0}^{\infty} \int_{-\infty}^{\infty} e^{-(\mu_1 + \mu_2) r^2} \frac{1}{\pi i \omega} \frac{1}{2 \pi i \omega} d\omega$$

where the denominator is obtained using Lemma 3.

Proof: See Appendix A.

Now, we are ready to obtain the expression for the coverage probability for a typical MS in this heterogeneous network.

IV. COVERAGE PROBABILITY

Recall that the MS is covered only if the SINR exceeds a certain threshold, $\beta_k$, where $k$ is the tier to which the desired BS belongs. Using the results from the previous section, we now present the coverage probability for an MS in a heterogeneous cellular network.

Theorem 1. The coverage probability of the MS is

$$P_{\text{open-access}} = \frac{\sum_{k=1}^{K} \lambda_k \mathbf{E}}{K} = \int_{0}^{\infty} \int_{-\infty}^{\infty} e^{-(\mu_1 + \mu_2) r^2} \frac{1}{\pi i \omega} \frac{1}{2 \pi i \omega} d\omega$$

where $\alpha_l = \frac{1}{2 \pi i \omega} F_1 \left( \frac{1}{\omega^2}, 1 - \frac{1}{\omega^2} = \frac{1}{i \omega} \right)$.

Proof: See Appendix A.

Though the above expression is not in closed form, this can also be computed to any desired accuracy using numerical integration methods to compute the double integral. Moreover, several insightful results arise for certain special cases of the above result. For example, in the interference-limited scenario, where the effect of the background noise may be ignored in the presence of the strong interferences from the BSs, the above result reduces to a simple form as shown below.

Corollary 1. In the interference-limited case, with $\{\varepsilon_k\}_{k=1}^{K}$ being independent and identically distributed random variables, the coverage probability is

$$P_{\text{open-access}} = \frac{\sum_{k=1}^{K} \lambda_k \mathbf{E}}{K} = \int_{0}^{\infty} \int_{-\infty}^{\infty} e^{-(\mu_1 + \mu_2) r^2} \frac{1}{\pi i \omega} \frac{1}{2 \pi i \omega} d\omega$$

where $\gamma_k = \frac{1}{2 \pi i \omega} F_1 \left( \frac{1}{\omega^2}, 1 - \frac{1}{\omega^2} = \frac{1}{i \omega} \right)$. Notice that the detailed proof will not be given. However, the corollary can be easily proved by noting that the integration w.r.t. $t$ in (7) evaluates to $\frac{1}{\pi} \int_{-\infty}^{\infty} e^{-i \omega t} dt$. Notice that $\gamma_k$ does not depend on any of the parameters that define the characteristics of the BSs of the various tiers, and only depends on the path-loss exponent, the SINR thresholds of the various tiers. Further, $\gamma_k$ is the coverage probability of a MS in a single-tier network. For the $\eta = 0$ case, Eq. 3 gives the coverage probability of a single-tier network with SINR threshold, $\gamma_{\eta}$, and channel gains that are i.i.d. exponentially distributed to be $\frac{\sin(2\pi \epsilon)}{2\pi \epsilon} - \gamma$. For the same case, in [9, Remark 4], we have shown that the single-tier network SINR distribution is the same irrespective of the transmission power of BSs and the distribution of the channel gains. As a result, the above expression that holds for exponential fading distribution also holds for any other fading distribution (even no fading).

The contributions of the BS density, transmission power, fading distribution, and the bias factor of the $k^{th}$ tier BS, are all captured as the multiplicative factors of the $k^{th}$ tier SINR threshold, $\gamma_k$. Further, the contribution of the fading coefficient is completely captured in terms of the $\frac{1}{\epsilon_k}$ moment of the random variable $\Psi_k$, used to represent the shadow fading factor of the $k^{th}$ tier. Next, having studied the coverage probability of the MS in the heterogeneous cellular network, we present some numerical examples to illustrate them.

V. NUMERICAL EXAMPLES AND DISCUSSION

In this section, we study various scenarios in order to clearly illustrate the results we have obtained. We restrict ourselves to a two-tier network consisting of a macrocell and picocell network for simplicity, and assume that the background noise is zero. We note that these studies can be extended to arbitrary number of tiers. Further, please refer Appendix E for the algorithm used to perform the Monte-Carlo simulations. For all the cases that will be considered next, we assume $\lambda_1 = 0.001$, $\lambda_2 = 0.002$, $P_1 = 53$ dBm, $P_2 = 33$ dBm, $B_1 = 1$, $B_2 = 1$, $\Psi_1$ and $\Psi_2$ are both exponential random variables with mean $1$, $\varepsilon_1 = 3.8$, $\varepsilon_2 = 3.5$, $\beta_1 = \beta_2 = 0$ dB, unless specified otherwise.

In our first example, we dwell in detail into our characterization of the SINR distribution as well as the coverage probability for arbitrary fading distribution that are different for different tiers. Figure 1 shows the plot of coverage probability versus the SINR threshold for various choices.
of fading distributions for each of the tiers. The first two curves (in the legend of Figure 1) show the two-tier network coverage probability when the channel gains at both the tiers are exponential random variables with mean 1. In the next two curves in Figure 1, we depict two scenarios where the channel gains are log-normal random variables with zero mean and standard deviations $\sigma_1 = 3.5$ dB for the first tier and $\sigma_2 = 4.65$ dB for the second tier. The characterization of the coverage probability if $\beta_1 \neq \beta_2$ and if $\beta_1, \beta_2 \geq 0$ dB were not known until now, for any chosen distribution of the channel gains, not even the exponential distribution. The results in this paper, in particular, Theorem 1 gives the coverage probability for all values of $\beta_1$ and $\beta_2$. The last two curves in Figure 1 consider the channel gain to have exponential distributions with means $\mu_1 = 46.5$ dB for the first tier and $\mu_2 = 50.3$ dB for the second tier. Notice that their coverage probability curves match exactly with those for the log-normal distributions. This is because, as illustrated in Theorem 1, the coverage probability only depends on the $2^{\text{nd}}$ moment of the random fading factor of the $k^{\text{th}}$ tier, i.e. $E[\Psi_k^2]$. If $\Psi_k$ is a log-normal random variable with standard deviation $\sigma_k$, then $E[\Psi_k^2] = \exp\left(2\sigma_k^2\right)$, and if $\Psi_k$ is an exponential random variable with mean $\mu_k$, the $E[\Psi_k^2] = \mu_k^2 (1 + 1/\sqrt{\mu_k})$. In this example, we have chosen $\sigma_1, \sigma_2, \mu_1,$ and $\mu_2$ in such a way that $E[\Psi_k^2]$’s are the same for $k = 1, 2$.

In the next example, we show that we are now able to study the heterogeneous network for different path-loss exponents at different tiers. Notice that there is an improvement when the path-loss exponents are large. This is because the signal power decays faster with the distance, thereby causing lesser intercell interference. Notice from Corollary 1 that, when the path-loss exponents and the SINR thresholds are identical across the tiers, then the coverage probability is the same as that of a single-tier network with the same path-loss exponent and the SINR threshold, and further, coverage probability varies log-linearly with the SINR threshold ($> 0$ dB). In Figure 2, notice that even when the path-loss exponents are not identical across the tiers, the coverage probability still has a log-linear behavior for SINR thresholds greater than 0 dB.

In the last example, we study the effect of varying the BS density ($\lambda_2$) and the bias factor ($B_2$) of the second tier on the coverage probability, where the second tier has a greater path-loss exponent, to mimic a typical indoor environment situation. While Corollary 1 shows that the coverage probability does not depend on the BS density of the tiers as well as the bias factors of the tiers when all the tiers have the same path-loss exponent and the SINR thresholds, when the path-loss exponents are different, Figure 3 shows that the coverage probability actually improves as we increase the bias factor of the second tier, and further it increases at a faster rate as the density of the second tier is increased. At the limits of the tier two bias factor, i.e. $B_2 \to 0$, and $B_2 \to \infty$, the two-tier network essentially collapses to a single-tier network consisting of only the macrocell network, and the femtocell network, respectively. As mentioned previously, the single tier network is invariant to changes in the BS density and the bias factor, and as a result the curves are straight lines at these limits.

The mathematical tools developed in this paper to study the heterogeneous network coverage probability are sufficient to characterize the average ergodic rate achieved at the MS, throughput and the per-tier traffic load, which are other important metrics of interest for understanding the heterogeneous network, and these will be considered in detail elsewhere.

VI. CONCLUSIONS

In this paper, we study a heterogeneous cellular network consisting of $K$ tiers, where each tier has its own BS density, BS transmission power and bias factor, path-loss exponent and channel gain with an arbitrary distribution, that is different for different tiers. For such a general model for the heterogeneous network, we develop mathematical tools based on stochastic geometry to characterize the distribution of the downlink SINR and the coverage probability at any given MS, where the MS
associates itself with the BS that has the SINR at the MS. Moreover, we have achieved a complete characterization of the SINR distribution and the coverage probability for all values of SINR thresholds, which has not been done before.

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APPENDIX

A. Proof for the Stochastic Equivalence Lemma

Given a BS belonging to the kth tier is at a distance \( R_k \) from the origin, then, \( \tilde{R} \mid k = (P_b B_k \Psi_{\kappa})^{-1} R_k \) represents the distance of the BS from the origin where the BS arrangement is according to a non-homogeneous 1-D Poisson point process with BS density function \( \lambda(k)(r) = \lambda_k 2 \Psi_{\kappa}(P_b B_k) \frac{2^\Psi_{\kappa}}{\Psi_{\kappa}} (P_b B_k \Psi_{\kappa})^{-1} r \), for each \( k = 1, 2, \cdots, K \). This is a consequence of the Mapping theorem [10] Page 18 and the Marking Theorem [10] Page 55 of the Poisson processes. Further, since the BS arrangements in the different tiers were originally independent of each other, the set of all the BSs in the equivalent 1-D non-homogeneous Poisson process is merely the union of all \( \tilde{R} \)’s \( k, \forall k = 1, 2, \cdots, K \). By the Superposition Theorem [10] Page 16 of Poisson process, \( \tilde{R} \) (notice that it is not conditioned on \( k \)) corresponds to the distance from origin of BS arrangement according to non-homogeneous Poisson point process with density function \( \lambda(r) = \sum_{k=1}^{K} \lambda(k)(r), r \geq 0 \).

In summary, we have converted the BS arrangement on a 2-D plane of a K-tier network to a BS arrangement of the equivalent single-tier network along 1-D (positive x-axis), and further the SINR distributions of both these networks are also equivalent. Further, by our construction, the BS corresponding to the strongest received power at the MS, in the K-tier network, corresponds to the BS that is nearest to the origin (MS) in the equivalent single tier network. As a result, SINR may be written in terms of the \( \tilde{R} \)’s indexed in the ascending order, and we get (2).

B. Tail Probability of Desired BS Distance from MS

From the proof of Lemma 1, the arrangement of the BSs of the kth tier in the equivalent single-tier network is according to a non-homogeneous Poisson point process with BS density function \( \lambda(k)(r) = \lambda_k 2 \Psi_{\kappa}(P_b B_k) \frac{2^\Psi_{\kappa}}{\Psi_{\kappa}} (P_b B_k \Psi_{\kappa})^{-1}, r \geq 0, \) and \( \tilde{R}_1 \) represents the distance of the nearest BS of this random process. Using the properties of the Poisson point process, \( P (\{ \tilde{R}_1 > r \}) = P (\{ N(\kappa) ([0, r]) = 0 \}) \), where \( N(\kappa) ([0, r]) \) represents the average number of BSs in the interval \([0, r]\) placed according to the Poisson point process with density \( \lambda(k)(r) \), and this is equal to (3).

C. Desired BS Tier Probability Lemma

The following sequence of equations provides the proof.

\[
\cos (\{ I = k \}) = \cos \left( \sum_{l=1}^{K} \cos \left( \tilde{R}_1 < \tilde{R}_1^{(l)} \right) \right)
\]

\[
= \mathbb{E}_{\tilde{R}_1} \left[ \sum_{l=1}^{K} \mathbb{E}_{\tilde{R}_1^{(l)}} \left( \tilde{R}_1^{(l)} > \tilde{R}_1 \right) \right]
\]

\[
= \int_{r=0}^{\infty} \lambda_k 2 \Psi_{\kappa}(P_b B_k) \frac{2^\Psi_{\kappa}}{\Psi_{\kappa}} (P_b B_k \Psi_{\kappa})^{-1} \times \exp \left( - \sum_{l=1}^{K} \lambda_l 2 \Psi_{\kappa}(P_l B_l) \frac{2^\Psi_{\kappa}}{\Psi_{\kappa}} (P_l B_l \Psi_{\kappa})^{-1} \right) dr,
\]

where \( \cos \) is obtained by noting that the desired BS belonging to the kth tier is closer to the origin than the nearest BS (to origin) of the rest of the tiers, \( \cos \) is due to the fact that \( \{ R_1^{(l)} \}_{l=1}^{K} \) are independent random variables, and \( \mathbb{E}_{\tilde{R}_1^{(l)}} \) represents the expectation with respect to the random variable \( \tilde{R}_1^{(l)} \); \( \cos \) is obtained by using Theorem 1 Equation 2 to compute the probability of the event \( \cos \) and to obtain the probability density function (p.d.f.) of \( \tilde{R}_1^{(l)} \) to evaluate the expectation, and finally, \( \cos \) is obtained by simplifying \( \cos \).

When \( \{ v \}_{k=1}^{K} = \varepsilon \), the integral in \( \cos \) simplifies to the form \( \int_{t=0}^{\infty} e^{-\alpha t} dt = \frac{1}{\alpha} \), which is rewritten in \( \cos \).

D. Proof for the p.d.f. of the Desired BS Distance given Tier

We first evaluate the probability of the event \( \tilde{R}_1 > r, I = k \). The steps are given in \( \cos \), where \( \cos \) is obtained by noting that the desired BS belonging to the kth tier is closer to the origin than the nearest BS (to origin) of the rest of the tiers, \( \cos \) rewrites joint probability in \( \cos \) in terms of the product of the marginal and the conditional probabilities, \( \cos \) is obtained by noting that the \( \tilde{R}_1^{(l)} \), \( \forall l = 1, 2, \cdots, K \) are independent random variables, \( \cos \) is obtained by substituting for the tail probability events in \( \cos \) using Theorem 1 Equation 3, and finally, \( \cos \) is
P \left( \left\{ \tilde{R}_1 > r, \ I = k \right\} \right) \quad \overset{(a)}{=} \quad P \left( \left\{ \tilde{R}_1^{(k)} > r \right\} \cap \bigcap_{l=1, \ l \neq k}^K \left\{ \tilde{R}_1^{(l)} > \tilde{R}_1^{(k)} \right\} \right)

\overset{(b)}{=} \quad \mathbb{E}_\tilde{R}_1^{(k)} \left[ I \left( \left\{ \tilde{R}_1^{(k)} > r \right\} \right) \cdot P \left( \bigcap_{l=1, \ l \neq k}^K \left\{ \tilde{R}_1^{(l)} > \tilde{R}_1^{(k)} \right\} \bigg| \tilde{R}_1^{(k)} \right) \right]

\overset{(c)}{=} \quad \mathbb{E}_\tilde{R}_1^{(k)} \left[ I \left( \left\{ \tilde{R}_1^{(k)} > r \right\} \right) \cdot \prod_{l=1, \ l \neq k}^K P \left( \left\{ \tilde{R}_1^{(l)} > \tilde{R}_1^{(k)} \right\} \bigg| \tilde{R}_1^{(k)} \right) \right]

\overset{(d)}{=} \quad \mathbb{E}_\tilde{R}_1^{(k)} \left[ I \left( \left\{ \tilde{R}_1^{(k)} > r \right\} \right) \cdot \exp \left( - \sum_{l=1, \ l \neq k}^K \lambda_l \pi \left( P_l B_l \tilde{R}_1^{(k)} \right)^\frac{2}{a} \Psi_i^\frac{2}{a} \right) \right]

\overset{(e)}{=} \quad \int_{r}^\infty \lambda_k \frac{2\pi}{\varepsilon_k} \left( P_k B_k \right)^\frac{2}{a} \Psi_k^\frac{2}{a} \int_{r}^{\infty} \exp \left( - \sum_{l=1}^K \lambda_l \pi \left( P_l B_l t \right)^\frac{2}{a} \Psi_i^\frac{2}{a} \right) dt, \quad (8)

\text{Probability of open-access coverage} \quad \overset{(a)}{=} \quad P \left( \left\{ \text{SINR} > \beta_T \right\} \bigg| \tilde{R}_1 \right) \overset{(b)}{=} \quad \mathbb{E}_{I, \tilde{R}_1} \left[ P \left( \left\{ \sum_{m=2}^{\infty} \tilde{R}_m^{-1} + \eta | \tilde{R}_1 \right\} \left( \omega \tilde{R}_1 \right) e^{-k\omega} \frac{2}{2\pi i\omega} d\omega dx \right) \right]

\overset{(c)}{=} \quad \mathbb{E}_I \left[ \int_{-\infty}^{\infty} \mathbb{E}_{\tilde{R}_1^{-1}} \left( \Phi \sum_{m=2}^{\infty} \tilde{R}_m^{-1} | \tilde{R}_1 \left( \omega \tilde{R}_1 \right) \right) e^{i\omega \eta \tilde{R}_1 \left( 1 - e^{-\frac{\omega}{\beta_T^1}} \right)} \frac{2}{2\pi i\omega} d\omega dx \right], \quad (9)

E. Proof for the Coverage Probability Theorem

The sequence of equations in (9), where (a) is obtained by using Lemma [1] and basic conditional probability properties, (b) is obtained by using [9] Theorem 1], using the BS density function specified in Lemma [1] for \( \lambda(r) \) and 1 for the path-loss exponent, and (c) is obtained by exchanging the order of integrations in (b), which is valid since the integrals are convergent. Upon simplifying, we get

\[ F_{\tilde{R}_1^{-1}}(\omega \tilde{R}_1) = \exp \left( \sum_{l=1}^K \lambda_l \pi \Psi_l^\frac{2}{a} \left( P_l B_l \tilde{R}_1 \right) \right) \times \left( 1 - F_1 \left( \frac{2}{\varepsilon_l} \left( 1, \ 1 \frac{2}{\varepsilon_l} \right) \right) \right). \]

Further, by evaluating the expectation in (c) by using Lemma [4] - Equation (8), and simplifying, we get (7).

F. Simulation Method

The \( k \)th tier of the heterogeneous network with \( K \) tiers is identified by the following set of system parameters: \( (\lambda_k, \ P_k, \ B_k, \ \Psi_k, \ \varepsilon_k, \ \beta_k) \), where the symbols have all been defined in Section [II] and \( k = 1, 2, \cdots, K \), where \( K \) is the total number of tiers. Now we illustrate the steps for simulating the heterogeneous network in order to obtain the SINR distribution and the coverage probability in the open-access cell association scheme. Assuming the MS to be at the origin, a single trial of heterogeneous cellular network arrangement in a cellular area with \( R_B \) as the boundary radius involves:

1) Generating the random numbers \( N_k \sim \text{Poisson} (\lambda_k \pi R_B^2) \), which is the number of BSs of the \( k \)th tier that will be deployed in the trial.

2) Generating \( N_k \) random variables according to a uniform distribution in the circular region of area \( \pi R_B^2 \), which represents the location of the \( k \)th tier BSs corresponding to the trial.

3) Computing the received power at the MS at the origin and computing the SINR as the ratio of the maximum of the received powers to the difference of the sum of all the received powers and the maximum received power.

4) Record the index \( I \) which corresponds to the tier to which the desired BS belongs, for the tier.

Repeat the same procedure \( T \) times. Typically, \( T \) is at least 50000. After this, we have an array containing the instantaneous SINRs and the tiers to which the desired BSs belonged, corresponding to the \( T \) trials. The tail probability of SINR at a certain point, say \( \eta \), is given by \( \left\{ \text{# of trials where } \text{SINR} > \eta \right\} \), and the coverage probability of the MS in the heterogeneous network is given by \( \sum_{k=1}^K \left\{ \text{# of trials where } I = k \text{ and } \text{SINR} > \beta_k \right\} \).