On the Uniqueness of the Non-Abelian Gauge Theories in Epstein-Glaser Approach to Renormalisation Theory

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Abstract
We generalise a result of Aste and Scharf saying that, under some reasonable assumptions, consistent with renormalisation theory, the non-Abelian gauge theories describes the only possibility of coupling the gluons. The proof is done using Epstein-Glaser approach to renormalisation theory.
1 Introduction

Renormalisation theory has a long history and among the many achievements one can count quantum electrodynamics (with the extremely good experimental verification) and, more recently, the non-Abelian gauge theories which are used to describe electro-weak theory and quantum chromodynamics. The major point is that these theories are renormalisable i.e. one can get rid of the so-called ultra-violet divergences in a consistent way and obtain a well-defined series for the $S$-matrix. Using the traditional approach based on Feynman diagrams, this assertion has been proved in second order of the perturbation theory by 't Hooft and Veltman and has been established in all orders of the perturbation theory in the extremely sophisticated approach of Becchi, Rouet and Stora.

However, it was proved by Epstein and Glaser [28], [30] that the most natural and straightforward way of dealing with perturbation theory and constructing a $S$-matrix fulfilling Bogoliubov axioms, is based on a direct exploitation of the causality axiom, and this can be done using a technical device, called the distribution splitting. This analysis was performed for a scalar field and extended for quantum electrodynamics in an external field in [26]. The full analysis of the interacting QED was done by G. Scharf and collaborators and is presented in a pedagogical manner in [13]. In recent years, the analysis was extended to the case of pure non-Abelian gauge symmetries [15], [16], [18], [19], [31]-[34], [37], [39] and of the electro-weak theory [27], [5]. Many other achievements of this approach are illustrated by the bibliography.

Recently Aste and Scharf [2] proved that the only possibility of non-trivial coupling between $r > 1$ zero-mass vector fields is through the usual Yang-Mills recipe. In other words, the existence of a compact semi-simple Lie group (to which the gauge principle is applied to obtain a local gauge theory) should not be assumed from the beginning: it simply follows from the consistency requirements of the theory. The main hypothesis leading to this result was a certain quantum form of the gauge principle invariance imposed to the $S$-matrix.

In this paper we will generalise this result and simplify somewhat they proof. Let us explain in what sense our approach is more general. The main obstacle in constructing the perturbation series for a zero-mass field is the fact that, as it happens for the electromagnetic field, one is forced to use non-physical degrees of freedom for the description of the free fields [50], [46], [36] in a Fock space formalism; one introduces in this Fock space a non-degenerate sesquilinear form and the true physical Hilbert space is obtained by a certain process of factorisation. More sophisticated, one can extend the Fock space to an auxiliary Hilbert space $H_{gh}$ including some ficticious fields, called ghosts, and construct an supercharge (i.e. an operator $Q$ verifying $Q^2 = 0$) such that the physical Hilbert space is $H_{phys} \equiv Ker(Q)/Im(Q)$ (see for instance [49] and references quoted there). We will present a careful analysis (which seems to be missing from the literature) of the Fock space representation of the electromagnetic field. The problem that one faces in the attempt to construct the perturbative $S$ matrix à la Epstein-Glaser is that one can define Wick monomials (from which the $S$-matrix is built) only on the auxiliary Fock space $H_{gh}$; so one must impose, beside the usual Bogoliubov axioms, the supplementary condition that the $S$ matrix factorizes to $H_{phys}$. We will prove that, the combination of these conditions leads uniquely to the pure Yang-Mills interaction. No other hypothesis are necessary to prove this statement. It is necessary to study rigorously the $S$ matrix only up to order 2.
in the perturbative sense. The gauge invariance condition is a consequence of our analysis and it is not necessary to impose it as an independent axiom. We also hope to present a simpler approach to the question of unitarity of the $S$-matrix.

The same analysis will be then applied to a theory of quarks and gluons. The result is that the Dirac Fermions should form a multiplet of the semi-simple compact group which appears naturally from the analysis of the pure Yang-Mills case.

We mention here that results of these type have been obtained quite a long time ago in [40] and [10] from some arguments concerning the high energy behaviour of the $S$ matrix elements in the tree approximation. However, a rigorous basis of this analysis seems to be lacking.

In Section 2 we will present the general scheme of construction of a perturbation theory in the causal approach of Epstein and Glaser giving all the relevant details about the Fock space construction of the theory. In Section 3 we will study in detail the quantization of the electromagnetic field, because some subtle points are simply overlooked in the literature or presented too summarily and not completely correct. We hope that we will be able to argue convincingly that a deep and clear understanding of these points is essential for the understanding of quantum gauge theories. In Section 4 we present our main result concerning the unicity of the Yang-Mills interaction and we generalise the analysis to the case when matter fields (quarks) are present. In the last Section we indicate some further possible developments.

Regarding the level of rigor, we make the following comments: (a) we will work with the formal notations for distributions for reasons of simplicity; (b) when working with Hilbert spaces of $L^2(X, d\alpha)$ one has to consider not functions but classes of function which are identically almost everywhere (a.e.); (c) domain problems for unbounded operators can be fixed in standard ways [50], [12].
2 Perturbation Theory in Fock Spaces

2.1 Second Quantization

Here we give the main concepts and formulae connected to the method of second quantization. We follow essentially [48] ch. VII; more details can be found in [9] and [7].

The idea of the method of second quantization is to provide a canonical framework for a multi-particle system in case one has a Hilbert space describing an “elementary” particle. (One usually takes the one-particle Hilbert space $\mathbb{H}$ to be some projective unitary irreducible representation of the Poincaré group, but this is not important for this subsection). Let $\mathbb{H}$ be a (complex) Hilbert space; the scalar product on $\mathbb{H}$ is denoted by $\langle \cdot, \cdot \rangle$. One first considers the tensor algebra $T(\mathbb{H}) \equiv \bigoplus_{n=0}^{\infty} \mathbb{H}^\otimes n$, \hfill (2.1.1)

where, by definition, the term corresponding to $n = 0$ is the division field $\mathbb{C}$. The generic element of $T(\mathbb{H})$ is of the type $(c, \Phi^{(1)}_1, \cdots, \Phi^{(n)}_n, \cdots)$, where $\Phi^{(n)}_n \in \mathbb{H}^\otimes n$; the element $\Phi^0_0 \equiv (1, 0, \cdots)$ is called the vacuum. Let us consider now the symmetrisation (resp. antisymmetrisation) operators $S^\pm$ defined by

$$S^\pm_n \equiv \bigoplus_{n=0}^{\infty} S^\pm_n,$$ \hfill (2.1.2)

where $S^\pm_0 = 1$ and $S^\pm_n$, $n \geq 1$ are defined on decomposable elements in the usual way

$$S^+_n \phi_1 \otimes \cdots \otimes \phi_n \equiv \frac{1}{n!} \sum_{P \in \mathcal{P}_n} \phi_{P(1)} \otimes \cdots \otimes \phi_{P(n)}$$ \hfill (2.1.3)

and

$$S^-_n \phi_1 \otimes \cdots \otimes \phi_n \equiv \frac{1}{n!} \sum_{P \in \mathcal{P}_n} (-1)^{|P|} \phi_{P(1)} \otimes \cdots \otimes \phi_{P(n)};$$ \hfill (2.1.4)

where $\mathcal{P}_n$ is the group of permutation of the numbers $1, 2, \ldots, n$ and $|P|$ is the sign of the permutation $P$. One extends the operators $S^\pm_n$ arbitrary elements of $T$ by linearity and continuity; it is convenient to denote the elements in defined by these relations by $\phi_1 \lor \cdots \lor \phi_n$ and respectively by $\phi_1 \land \cdots \land \phi_n$.

We now define the **Bosonic** (resp. **Fermionic**) Fock space according to:

$$\mathcal{F}^\pm(\mathbb{H}) \equiv S^\pm T(\mathbb{H});$$ \hfill (2.1.5)

obviously we have:

$$\mathcal{F}^\pm(\mathbb{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^\pm_n$$ \hfill (2.1.6)

where

$$\mathcal{H}^\pm_0 \equiv \mathbb{C}, \quad \mathcal{H}^\pm_n \equiv S^\pm_n \mathbb{H}^\otimes n \quad (n \geq 1)$$ \hfill (2.1.7)

are the so-called $n^{th}$-particle subspaces.

The operations $\lor$ (resp. $\land$) make $\mathcal{F}^\pm(\mathbb{H})$ into associative algebras. One defines in the Bosonic (resp. Fermionic) Fock space the **creation** and **annihilation** operators as follow: let $\phi \in \mathcal{H}$ be arbitrary. In the Bosonic case they are defined on elements from $\psi \in \mathcal{H}^+_n$ by

$$A(\phi)^\dagger \psi \equiv \sqrt{n+1} \phi \lor \psi$$ \hfill (2.1.8)
and respectively

\[ A(\phi)\psi \equiv \frac{1}{\sqrt{n}}i_\phi \psi \quad (2.1.9) \]

where \( i_\phi \) is the unique derivation of the algebra \( \mathcal{F}^+(H) \) verifying

\[ i_\phi 1 = 0; \quad i_\phi \psi = <\phi, \psi > 1. \quad (2.1.10) \]

**Remark 2.1** We note that the general idea is to associate to every element of the one-particle space \( \phi \in H \) a couple of operators \( A^\sharp(\phi) \) acting in the Fock space \( \mathcal{F}^+(H) \).

As usual, we have the canonical commutation relations (CCR):

\[ [A(\phi), A(\psi)] = 0, \quad [A(\phi)^\dagger, A(\psi)^\dagger] = 0, \quad [A(\phi), A(\psi)^\dagger] = <\phi, \psi > 1. \quad (2.1.11) \]

The operators \( A(\psi), \quad A(\psi)^\dagger \) are unbounded and adjoint one to the other.

In the Fermionic case we define these operators on elements from \( \psi \in H^-_n \) by

\[ A(\phi)^\dagger \psi \equiv \sqrt{n + 1}\phi \wedge \psi \quad (2.1.12) \]

and respectively

\[ A(\phi)\psi \equiv \frac{1}{\sqrt{n}}i_\phi \psi \quad (2.1.13) \]

where \( i_\phi \) is the unique graded derivation of the algebra \( \mathcal{F}^-(H) \) verifying

\[ i_\phi 1 = 0; \quad i_\phi \psi = <\phi, \psi > 1. \quad (2.1.14) \]

Now we have the canonical anticommutation relations (CAR):

\[ \{A(\phi), A(\psi)\} = 0, \quad \{A(\phi)^\dagger, A(\psi)^\dagger\} = 0, \quad \{A(\phi), A(\psi)^\dagger\} = <\phi, \psi > 1. \quad (2.1.15) \]

The operators \( A(\psi), \quad A(\psi)^\dagger \) are bounded and adjoint one to the other.

If \( U \) is a unitary (or antiunitary) operator on \( H \), it lifts naturally to an operator \( \Gamma(U) \) on the tensor algebra \( T(H) \), according to

\[ \Gamma(U) \equiv \bigoplus_0^\infty U^\otimes n \quad (2.1.16) \]

or, more explicitly, on decomposable elements

\[ \Gamma(U)\psi_1 \otimes \cdots \otimes \psi_n = U\psi_1 \otimes \cdots \otimes U\psi_n. \quad (2.1.17) \]

The operator \( \Gamma(U) \) leaves invariant the symmetric and resp. the antisymmetric algebras \( \mathcal{F}^\pm(H) \) and we have

\[ \Gamma(U_g)A(\phi)\Gamma(U_{g^{-1}}) = A(U_g\phi). \quad (2.1.18) \]
2.2 Elementary Relativistic Free Particles

As we have anticipated in the previous subsection, on usually takes $\mathcal{H}$ to be the Hilbert space of an unitary irreducible representation of the Poincaré group. We give below the relevant formulæ for the scalar particle of mass $m$ and for the photon. By comparison, one will be able to see the origin of the difficulties of the renormalisation theory for zero-mass particles.

According to [48], a scalar particle of mass $m$ can be described in the Hilbert space $\mathcal{H} \equiv L^2(X^+_m, \mathbb{C}, da^+_m)$ of Borel complex function $\phi$ defined on the upper hyperboloid of mass $m \geq 0$ $X^+_m \equiv \{ p \in \mathbb{R}^4 | \|p\|^2 = m^2 \}$ which are square integrable with respect to the Lorentz invariant measure $da^+_m \equiv \frac{dp}{\omega(p)}$. Here the conventions are the following: $\| \cdot \|$ is the Minkowski norm defined by $\|p\|^2 \equiv p_0^2 - p_1^2 - p_2^2 - p_3^2$ and $p_0 \cdot q \equiv p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3$.

If $p \in \mathbb{R}^3$ we define $\tau(p) \in X^+_m$ according to $\tau(p) \equiv (\omega(p), p)$, $\omega(p) \equiv \sqrt{p^2 + m^2}$.

The scalar product in $\mathcal{H}$ is:

$$<\phi, \psi> \equiv \int_{X^+_m} da^+_m \overline{\phi(p)} \psi(p). \quad (2.2.2)$$

The expression for the corresponding unitary irreducible representation of the Poincaré group is:

$$(U_{a,\Lambda} \phi)(p) \equiv e^{ia \cdot p} \phi(\Lambda^{-1} \cdot p) \quad \text{for} \quad \Lambda \in \mathcal{L}^\uparrow,$$

$$(U_{I,\phi})(p) \equiv \phi(I_s \cdot p); \quad (2.2.3)$$

here $I_s, I_t$ are the elements of the Lorentz group corresponding to the spatial and respectively temporal inversion. Also by $(\Lambda, p) \mapsto \Lambda \cdot p$ we denote the usual action of the Lorentz group on $\mathbb{R}^4$ and $\mathbb{C}^4$. The couple $(\mathcal{H}, U)$ is called scalar particle.

For the photon, such a simple description of the Hilbert space as a space of functions is no longer available. However, we can obtain a description of this type if one considers a factorisation procedure [48]. Let us consider the Hilbert space $\mathcal{H} \equiv L^2(X^+_0, \mathbb{C}, da^+_0)$ with the scalar product

$$<u, v>_{\mathbb{C}^4} \equiv \sum_{i=1}^4 \overline{u_i} v_i$$

where $<u, v>_{\mathbb{C}^4}$ is the usual scalar product from $\mathbb{C}^4$. In this Hilbert space we have the following (non-unitary) representation of the Poincaré group:

$$(U_{a,\Lambda} \phi)(p) \equiv e^{ia \cdot p} \phi(\Lambda^{-1} \cdot p) \quad \text{for} \quad \Lambda \in \mathcal{L}^\uparrow,$$

$$(U_{I,\phi})(p) \equiv \phi(I_s \cdot p). \quad (2.2.4)$$

Let us define on $\mathcal{H}$ the operator $g$ by

$$(g \cdot \phi)(p) \equiv g \cdot \phi(p) \quad (2.2.6)$$

and following non-degenerate sesquilinear form:

$$(\phi, \psi) \equiv - <\phi, g \cdot \psi>; \quad (2.2.7)$$
here \( g \in \mathcal{L}^\uparrow \) is the Minkowski matrix with diagonal elements \( 1, -1, -1, -1 \) and the operator \( g \) is appearing in (2.2.6) also called a *Krein operator*. Explicitly:

\[
(\phi, \psi) \equiv \int_{X_0^+} d\alpha_0^+ (\phi(p), \psi(p)) = \int_{X_0^+} d\alpha_0^+ g^{\mu\nu} \overline{\phi_\mu(p)} \psi_\nu(p);
\]

(2.2.8)

the indices \( \mu, \nu \) take the values 0, 1, 2, 3 and the summation convention over the dummy indices is used.

Then one easily establishes that we have

\[
(U_{a,\Lambda} \phi, U_{a,\Lambda} \psi) = (\phi, \psi), \quad \text{for} \quad \Lambda \in \mathcal{L}^\uparrow, \quad (U_I \phi, U_I \psi) = \overline{(\phi, \psi)}.
\]

(2.2.9)

We have now two elementary results:

**Lemma 2.2** Let us consider the following subspace of \( H \):

\[
H' \equiv \{ \phi \in H \mid \ p^\mu \phi_\mu(p) = 0 \}.
\]

Then the sesquilinear form \( (\cdot, \cdot)|_H \) is positively defined.

**Lemma 2.3** Let us consider the following subspace of \( H' \):

\[
H'' \equiv \{ \phi \in H' \mid \| \phi \| = 0 \}.
\]

Then

\[
H'' \equiv \{ \phi \in H \mid \text{there exists} \quad \lambda : X_0^+ \to \mathbb{C} \quad \text{s.t.} \quad \phi(p) = p\lambda(p) \}.
\]

(2.2.12)

Then we have the following result:

**Proposition 2.4** The representation (2.2.9) of the Poincaré group leaves invariant the subspaces \( H' \) and \( H'' \) and so, it induces an representation in the Hilbert space

\[
H_{\text{photon}} \equiv \overline{(H'/H'')} \quad (2.2.13)
\]

(here by the overline we understand completion). The factor representation, denoted also by \( U \) is unitary and irreducible. By restriction to the proper orthochronous Poincaré group it is equivalent to the representation \( H^{[0,1]} \oplus H^{[0,-1]} \).

By definition, the couple \((H_{\text{photon}}, U)\) is called *photon*. 

6
2.3 Free Fields and Wick products

Let us apply the second quantization procedure to the scalar particle, i.e. we consider that, in the general scheme from the first subsection, the one-particle subspace $H$ is the Hilbert space corresponding to the scalar particle and we consider Bose statistics i.e. the Bosonic Fock space. Then one can canonically identify the $n$th-particle subspace $F^+_n(H)$ with the set of Borel functions $\Phi_n: (X^+_m)^n \to \mathbb{C}$ which are square integrable with respect to the product measure $d(\alpha^+_m)^n$ and verify the symmetry property

$$\Phi_n(p_{P(1)}, \ldots, p_{P(n)}) = \Phi_n(p_1, \ldots, p_n), \quad \forall P \in \mathcal{P}_n. \quad (2.3.1)$$

On the dense domain of test functions one can define the annihilation operators:

$$(A(k)\Phi)^{(n)}(p_1, \ldots, p_n) \equiv \sqrt{n+1} \Phi^{(n+1)}(k, p_1, \ldots, p_n). \quad (2.3.2)$$

Working with the formalism of rigged Hilbert spaces one can also make sense of the creation operators:

$$(A(k)^\dagger \Phi)^{(n)}(p_1, \ldots, p_n) \equiv 2\omega(k) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \delta(k-p_i)\Phi^{(n-1)}(p_1, \ldots, \hat{p_i}, \ldots, p_n) \quad (2.3.3)$$

where the Bourbaki conventions $\sum_0 \equiv 0, \prod_0 \equiv 1$ are used.

These operators verify the canonical commutation relations (see (2.1.11)):

$$[A(k), A(k')] = 0, \quad [A(k)^\dagger, A(k')^\dagger] = 0, \quad [A(k), A(k')^\dagger] = 2\omega(k)\delta(k-k')1 \quad (2.3.4)$$

and can be used to express the creation and annihilation operators $A(\phi), \quad A(\phi)^\dagger$ defined in the preceding section; namely we have:

$$A(\phi) = \int_{X^+_m} d\alpha^+_m(k)\overline{\phi(k)}A(k), \quad A(\phi)^\dagger = \int_{X^+_m} d\alpha^+_m(k)\phi(k)A(k)^\dagger. \quad (2.3.5)$$

Now we define the scalar free field. Let $f \in \mathcal{S}(\mathbb{R}^4)$ be a test function. We define the Fourier transform with the convention:

$$\tilde{f}(p) \equiv \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} e^{-ip\cdot x} f(x). \quad (2.3.6)$$

We also define the functions

$$f_{\pm}(k) \equiv \tilde{f}(\pm k) \big|_{X^+_m}. \quad (2.3.7)$$

One can give now the expression of the free scalar field of mass $m$ on the dense domain of the test functions according to:

$$(\varphi(f)\Phi)^{(n)}(p_1, \ldots, p_n) = \sqrt{2\pi}[\sqrt{n} + 1] \int_{X^+_m} d\alpha^+_m(k)f_+(k)\Phi^{(n+1)}(k, p_1, \ldots, p_n) + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f_-(p_i)\Phi^{(n-1)}(p_1, \ldots, \hat{p_i}, \ldots, p_n). \quad (2.3.8)$$
Then one can extend it to a selfadjoint operator on the Fock space. One can define the scalar field in a fixed point as follows. First one defines the negative frequency part by:

\[
(\varphi_-(x)\Phi)^{(n)}(p_1, \ldots, p_n) \equiv (2\pi)^{-3/2} \sqrt{n+1} \int_{X^+_n} \alpha^+_m(k)e^{-ix\cdot k}\Phi^{(n+1)}(k, p_1, \ldots, p_n)
\]  
(2.3.9)

as a legitimate operator in the Fock space. Working with rigged Hilbert spaces one can define the positive frequency part:

\[
(\varphi_+(x)\Phi)^{(n)}(p_1, \ldots, p_n) \equiv (2\pi)^{-3/2} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} e^{ix\cdot p_i}\Phi^{(n-1)}(p_1, \ldots, \hat{p}_i, \ldots, p_n).
\]  
(2.3.10)

Then the expression

\[
\varphi(x) \equiv \varphi_+(x) + \varphi_-(x)
\]  
(2.3.11)

is the called real scalar field in the point \(x\). One can justify this definition by making sense of the formula:

\[
\varphi(f) = \int_{\mathbb{R}^4} dx f(x)\varphi(x) = \sqrt{2\pi} \int_{X^+_0} \alpha^+_m(k) \left[ f_+(k)A(k) + f_-(k)A^\dagger(k) \right] = \sqrt{2\pi} \left[ A(f_+) + A^\dagger(f_-) \right].
\]  
(2.3.12)

The attribute real is due to the equation:

\[
\varphi(x)^* = \varphi(x),
\]  
(2.3.13)

the attribute free to the equation:

\[
\Box \varphi(x) = 0
\]  
(2.3.14)

and the attribute scalar to the transformation properties with respect to the Poincaré group: if we define the natural extension of the representation (2.2.9) to the Fock space:

\[
\mathcal{U}_g \equiv \Gamma(U_g), \quad \forall g \in \mathcal{P}
\]  
(2.3.15)

then we have:

\[
\mathcal{U}_{a,\lambda}\varphi(x)\mathcal{U}_{a,\lambda}^{-1} = \varphi(\Lambda \cdot x + a), \quad \forall \Lambda \in \mathcal{L}^\dagger, \quad \mathcal{U}_{I_s}\varphi(x)\mathcal{U}_{I_s} = \varphi(I_s \cdot x)
\]  
(2.3.16)

Moreover, we have the important causality property:

\[
[\varphi(x), \varphi(y)] = 0, \quad \text{for} \quad (x - y)^2 \leq 0.
\]  
(2.3.17)

An important generalisation of the notion of free field is given by the concept of Wick monomials. According to the rigorous treatment of [50] one can make sense of the following expressions:

\[
W_{rs}(x) \equiv \varphi_+(x)^r\varphi_-(x)^s, \quad r, s \in \mathbb{N}.
\]  
(2.3.18)
Indeed, if one formally integrates with a test function \( f \in \mathcal{S}(\mathbb{R}^4) \) the expression defined above, one gets the following expression:

\[
W_{rs}(f) \equiv \int_{\mathbb{R}^4} f(x) W_{rs}(x)
\]

(2.3.19)

for which the following explicit formula are available. For \( n < r \):

\[
(W_{rs}(f)\Phi)^{(n)} = 0
\]

(2.3.20)

and for \( n \geq r \):

\[
(W_{rs}(f)\Phi)^{(n)} = (2\pi)^{2-\frac{d}{2}} \frac{1}{n!} \times
\]

\[
\mathcal{S}_n \int_{(X_m^+)^s} \prod_{j=1}^s da^+_m(k_j) \tilde{f} \left( \sum_{j=1}^s \tau(k_j) - \sum_{i=1}^r \tau(p_i) \right) \Phi^{(n-r+s)}(k_1, \ldots, k_s, p_{r+1}, \ldots, p_n);
\]

(2.3.21)

here the operator \( \mathcal{S}_n \) symmetrizes in the variables \( p_1, \ldots, p_n \).

The central result making the expressions above legitimate operators in the Fock space is the following lemma 16.

**Lemma 2.5** *In the conditions above the following function*

\[
F(p_1, \ldots, p_n) \equiv \int_{(X_m^+)^s} \prod_{j=1}^s da^+_m(k_j) \tilde{f} \left( \sum_{j=1}^s \tau(k_j) - \sum_{i=1}^r \tau(p_i) \right) \Phi^{(n-r+s)}(k_1, \ldots, k_s, p_{r+1}, \ldots, p_n)
\]

(2.3.22)

*is a test function.*

Now one defines Wick products to be the expressions

\[
: \varphi^l : (f) \equiv \sum_{r+s=l} W_{rs}(f) = \int_{\mathbb{R}^4} f(x) : \varphi(x)^l :
\]

(2.3.23)

with the usual definition for \( : \varphi^l : \) namely one puts all the creation operators at the left of the annihilation operators. In a similar way one can define Wick product with some derivative on the various factors from \( : \varphi^l : \).

Now we remind Wick theorem. Suppose that \( A_1(x), \ldots, A_n(x), B_1(x), \ldots, B_m(x) \) are the free scalar field or derivatives of it. Then we have the following formula:

\[
:A_1(x) \ldots A_n(x) \circ B_1(y) \ldots B_m(y) := A_1(x) \ldots A_n(x) B_1(y) \ldots B_m(y) + \]

\[
\sum_{p=1}^{\min(n,m)} \sum_{\text{partitions} t=1} \prod < \Phi_0, A_{i_1}(x) B_{j_1}(y) \Phi_0 > : A_{i_1'}(x) \ldots A_{i_{n-p}}(x) B_{j_1'}(y) \ldots B_{j_{m-p}}(y) :
\]

(2.3.24)

where \( \{i_1, \ldots, i_p\}, \{i_1', \ldots, i_{n-p}\} \) is a partition of \( 1, \ldots, n \), and \( \{j_1, \ldots, j_p\}, \{j_1', \ldots, j_{m-p}\} \) is a partition of \( 1, \ldots, m \); the sum runs over all such partitions.
We conclude with a remark concerning the choice of the statistics. One can also quantize the scalar field using Fermi statistics; in this way one will loose only the causality property (2.3.17). However, such unphysical fields, called ghosts will be essential as technical devices in the description of zero-mass fields as we will see in the next Section. We have to modify, however Wick theorem, in the sense that the sign of the permutation

\[(1, 2, \ldots, n, 1, 2, \ldots, m) \mapsto (i_t, \ldots, i_p, i'_1, \ldots, i'_{n-p}, j_1, \ldots, j_p, j'_1, \ldots, j'_{m-p})\] (2.3.25)

should appear.
2.4 Perturbation Theory in the Causal Approach

Perturbation theory, relies considerably on the axiom of causality, as shown by H. Epstein and V. Glaser \[28\]. According to Bogoliubov and Shirkov, the $S$-matrix is constructed inductively order by order as a formal series of operator valued distributions:

$$S(g) = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int_{\mathbb{R}^{4n}} dx_1 \cdots dx_n T_n(x_1, \ldots, x_n)g(x_1) \cdots g(x_n), \quad (2.4.1)$$

where $g(x)$ is a tempered test function that switches the interaction and $T_n$ are operator-valued distributions acting in the Fock space of some collection of free fields. For instance, in \[28\] (see also \[30\]) one considers a real free scalar field, i.e. one considers the Fock space is that one defined in the preceding subsection. These operator-valued distributions, which are called chronological products should verify some properties which can be argued starting from Bogoliubov axioms.

- First, it is clear that we can consider them completely symmetrical in all variables without loosing generality:
  $$T_n(x_{P(1)}, \cdots x_{P(n)}) = T_n(x_1, \cdots x_n), \quad \forall P \in \mathcal{P}_n. \quad (2.4.2)$$

- Next, we must have Poincaré invariance:
  $$\mathcal{U}_{a,\Lambda} T_n(x_1, \cdots, x_n) \mathcal{U}_{a,\Lambda}^{-1} = T_n(\Lambda \cdot x_1 + a, \cdots, \Lambda \cdot x_n + a), \quad \forall \Lambda \in \mathcal{L}^\dagger. \quad (2.4.3)$$

In particular, translation invariance is essential for implementing Epstein-Glaser scheme of renormalisation.

- The central axiom seems to be the requirement of causality which can be written compactly as follows. Let us firstly introduce some standard notations. Denote by $V^+ \equiv \{ x \in \mathbb{R}^4 \mid x^2 > 0, \quad x_0 > 0 \}$ and $V^- \equiv \{ x \in \mathbb{R}^4 \mid x^2 > 0, \quad x_0 < 0 \}$ the upper (lower) lightcones and by $\overline{V^\pm}$ their closures. If $X \equiv \{ x_1, \ldots, x_m \} \in \mathbb{R}^{4m}$ and $Y \equiv \{ y_1, \ldots, y_n \} \in \mathbb{R}^{4n}$ are such that $x_i - y_j \not\in \overline{V^-}, \quad \forall i = 1, \ldots, m, \quad j = 1, \ldots, n$ we use the notation $X \geq Y$. We use the compact notation $T_n(X) \equiv T_n(x_1, \cdots, x_n)$ and by $X \cup Y$ we mean the juxtaposition of the elements of $X$ and $Y$. In particular, the expression $T_{n+m}(X \cup Y)$ makes sense because of the symmetry property (2.4.2). Then the causality axiom writes as follows:
  $$T_{n+m}(X \cup Y) = T_m(X)T_n(Y), \quad \forall X \geq Y. \quad (2.4.4)$$

- The unitarity of the $S$-matrix can be most easily expressed (see \[28\]) if one introduces, the following formal series:
  $$\tilde{S}(g) = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{\mathbb{R}^{4n}} dx_1 \cdots dx_n \tilde{T}_n(x_1, \cdots, x_n)g(x_1) \cdots g(x_n), \quad (2.4.5)$$

where, by definition:
  $$(-1)^{|X|} \tilde{T}_n(X) \equiv \sum_{r=1}^{n} (-1)^r \sum_{n_1 + \cdots + n_r = n \text{ partitions}} T_{n_1}(X_1) \cdots T_{n_r}(X_r); \quad (2.4.6)$$
here $X_1, \ldots, X_r$ is a partition of $X$, $|X|$ is the cardinal of the set $X$ and the sum runs over all partitions. For instance, we have:

$$
\bar{T}_1(x) = T_1(x)
$$

(2.4.7)

and

$$
\bar{T}_2(x, y) = -T_2(x, y) + T_1(x)T_1(y) + T_1(y)T_1(x).
$$

(2.4.8)

One calls the operator-valued distributions $\bar{T}_n$ *anti-chronological products*. It is not very hard to prove that the series (2.4.5) is the inverse of the series (2.4.1) i.e. we have:

$$
\bar{S}(g) = S(g)^{-1}
$$

(2.4.9)

as formal series. Then the unitarity axiom is:

$$
\bar{T}_n(X) = T_n(X)\dagger, \quad \forall n \in \mathbb{N}, \quad \forall X.
$$

(2.4.10)

- The existence of the *adiabatic limit* can be formulated as follows. Let us take in (2.4.1) $g \rightarrow g_\epsilon$ where $\epsilon \in \mathbb{R}_+$ and

$$
g_\epsilon(x) \equiv g(\epsilon x).
$$

(2.4.11)

Then one requires that the limit

$$
S \equiv \lim_{\epsilon \rightarrow 0} S(g_\epsilon)
$$

exists, in the weak sense, and is independent of the the test function $g$. In other words, the operator $S$ should depend only on the *coupling constant* $g \equiv g(0)$. Equivalently, one requires that the limits

$$
T_n \equiv \lim_{\epsilon \rightarrow 0} T_n(g_\epsilon^\otimes n), \quad n \geq 1
$$

(2.4.13)

exists, in the weak sense, and are independent of the test function $g$. One also calls the limit performed above, the *infrared limit*.

- Finally, one demands the *stability of the vacuum* i.e.

$$
\lim_{\epsilon \searrow 0} < \Phi_0, S(g_\epsilon)\Phi_0 > = 1
$$

(2.4.14)

or,

$$
\lim_{\epsilon \searrow 0} < \Phi_0, T_n(g_\epsilon^\otimes n)\Phi_0 > = 0, \quad \forall n \in \mathbb{N}^*.
$$

(2.4.15)

A *renormalisation theory* is the possibility to construct such a $S$-matrix starting from the first order term:

$$
T_1(x) \equiv \mathcal{L}(x)
$$

(2.4.16)

where $\mathcal{L}$ is a Wick polynomial called *interaction Lagrangian* which should verify the following axioms:

$$
\mathcal{U}_{a,\Lambda}\mathcal{L}(x)\mathcal{U}^{-1}_{a,\Lambda} = \mathcal{L}(\Lambda \cdot x + a), \quad \forall \Lambda \in \mathcal{L}^\dagger, \quad (2.4.17)
$$

$$
[\mathcal{L}(x), \mathcal{L}(y)] = 0, \quad \forall x, y \in \mathbb{R}^4 \quad s.t. \quad (x - y)^2 < 0, \quad (2.4.18)
$$

$$
\mathcal{L}(x)^\dagger = \mathcal{L}(x)
$$

(2.4.19)
\[ L \equiv \lim_{\epsilon \to 0} \mathcal{L}(g_\epsilon) \quad (2.4.20) \]

should exists, in the weak sense, and should be independent of the test function \( g \). Moreover, we should have

\[ \langle \Phi_0, L\Phi_0 \rangle = 0. \quad (2.4.21) \]

In the analysis of Epstein and Glaser \cite{28,30} it is proved that if one considers a renormalisation theory for a real scalar free field of positive mass i.e. one works in the corresponding Fock space (see the first subsection) and takes \( \mathcal{L}(x) \) to be a Wick polynomial, then such a \( S \)-matrix exists. In the more familiar physicists language, \textit{no ultraviolet divergences} and \textit{no infrared divergences} appear, i.e. the \( T_n \)'s are finite and well defined and the adiabatic limit exists. The only remnant of the ordinary renormalisation theory is a non-uniqueness of the \( T_n \)'s due to \textit{finite} normalization terms which are distributions with the support \( \{ x_1 = \cdots = x_n = 0 \} \). The whole construction amounts to the operation of \textit{distribution splitting}. In this context appears what is sometimes called the \textit{normalisation} problem. First, we define the \textit{power} of a Wick monomial \( L \) as

\[ \omega(L) \equiv n_b + 3/2 n_f + n_d \quad (2.4.22) \]

where \( n_b(n_f) \) is the number of Bosonic (Fermionic) factors and \( n_d \) is the number of derivatives. Suppose that the interaction Lagrangian is a Wick polynomial of power \( l \). Then the finite renormalisations of the type described above are in fact Wick monomials. If we admit finite normalisation which are producing Wick expressions of power strictly greater than \( l \), then this would amount to corrections to the interaction Lagrangian which have been neglected from the very beginning. So, it is natural to impose that the arbitrariness involved in the process of distribution splitting, i.e. the finite normalisations should produce Wick monomials of power less or at most equal to the power of the interaction Lagrangian. We call this assumption the \textit{normalisation axiom}.

For the case of the real scalar field one can prove that a renormalisation theory do exist if the interaction Lagrangian verifies the properties (2.4.17), (2.4.18), (2.4.19) and (2.4.20). That’s it, one can use the normalisation arbitrariness to construct recurrently the chronological products verifying all the axioms. The main advantage of Epstein-Glaser approach is a direct control of the fulfilment of all axioms and the fact that one does not need a regularisation scheme.

It is instructive to show how the second order distribution \( T_2 \) can be constructed. First one constructs the distribution \( D_2(x, y) \)

\[ D_2(x, y) \equiv [T_1(x), T_1(y)]. \quad (2.4.23) \]

Form the axiom (2.4.18) one has:

\[ \supp(D_2) = \{ x - y \in V^+ \cup V^- \} \quad (2.4.24) \]

i.e. the distribution \( D_2 \) has causal support. Then \( D_2 \) is split into a retarded and an advanced part

\[ D_2 = R_2 - A_2 \quad (2.4.25) \]
with
\[ \text{supp}(R_2) = \{ x - y \in V^- \} \quad (2.4.26) \]
and
\[ \text{supp}(A_2) = \{ x - y \in V^+ \}. \quad (2.4.27) \]

Finally \( T_2 \) is given by
\[ T_2(x, y) = R_2(x, y) + T_1(y)T_1(x) = A_2(x, y) - T_1(x)T_1(y). \quad (2.4.28) \]

This solution is arbitrary up to an operator-valued distribution with support \( \{ x_1 = x_2 \} \). One can use this arbitrariness to satisfy the axioms of relativistic invariance and unitarity up to order 2 of the perturbation theory.

We mention now that if the interaction Lagrangian is a Wick polynomial, say \( \phi^l : (x) \), then the induction procedure leads to the following generic expression for the chronological products:
\[ T_n(g) = \int_{(\mathbb{R}^4)^n} dx_1 \cdots dx_n g(x_1, \ldots, x_n) \sum_{r_1, \ldots, r_n \leq l} F_{r_1, \ldots, r_n}(x_1, \ldots, x_n) : \phi^{r_1}(x_1) \cdots \phi^{r_n}(x_n) : \quad (2.4.29) \]
where the \( F \)'s are an ordinary distribution verifying translation invariance. The existence of such expressions as well defined operators is a consequence of the so-called zero theorem \[28\].

We close this Section with a comment about the possible choices of the interaction Lagrangian. Let us suppose that the interaction Lagrangian is of the form \( \phi^l :. \) with \( l = 2 \); then one can prove that the matrix element \( \langle p_1, \ldots, p_n | \int_{\mathbb{R}^4} dx : \phi^2 : (x) | q_1, \ldots, q_n \rangle \) is a sum of the type \( \delta(p_1 - q_{i_1}) \cdots \delta(p_n - q_{i_n}) \) which describes no interaction; a similar result is valid for the case \( l = 1 \).
3 The Quantisation of the Electromagnetic Field

3.1 Quantisation without Ghost Fields

We remind briefly the analysis from [50], [46], [36]. The idea is to apply the prescription from subsection 2.1 to the Hilbert space of the photon $H_{\text{photon}}$ given by (2.2.13). The idea is to express the (Bosonic) Fock space of the photon $F_{\text{photon}}$ as a factorization of the type (2.2.13). It is natural to start with the “bigger” Fock space

$$ H \equiv F^+(H) \equiv \bigoplus_{n \geq 0} H_n. \tag{3.1.2} $$

The first observation is that one can canonically identify the $n$th-particle subspace $H_n$ with the set of Borel functions $\Phi^{(n)}_{\mu_1, \ldots, \mu_n} : (X_0^+) \times \cdots \times (X_0^+) \to \mathbb{C}$ which are square summable:

$$ \int_{(X_0^+) \times \cdots \times (X_0^+)} \prod_{i=1}^{\infty} |\Phi^{(n)}_{\mu_1, \ldots, \mu_n}(k_1, \ldots, k_n)|^2 < \infty \tag{3.1.3} $$

and verify the symmetry property

$$ \Phi^{(n)}_{\mu_{P(1)}, \ldots, \mu_{P(n)}}(k_{P(1)}, \ldots, k_{P(n)}) = \Phi^{(n)}_{\mu_1, \ldots, \mu_n}(k_1, \ldots, k_n), \quad \forall P \in \mathcal{P}_n. \tag{3.1.4} $$

In $H$ the expression of the scalar product is:

$$ <\Psi, \Phi> \equiv <\Psi^{(0)}\Phi^{(0)} + \sum_{n=1}^{\infty} \int_{(X_0^+) \times \cdots \times (X_0^+)} \prod_{i=1}^{n} da_0^+(k_i) \sum_{\mu_1, \ldots, \mu_n=0}^{3} |\Phi^{(n)}_{\mu_1, \ldots, \mu_n}(k_1, \ldots, k_n)|^2, \Phi^{(n)}_{\mu_1, \ldots, \mu_n}(k_1, \ldots, k_n) > \tag{3.1.5} $$

and we have a (non-unitary) representation of the Poincaré group given by:

$$ U_g \equiv \Gamma(U_g), \quad \forall g \in \mathcal{P}; \tag{3.1.6} $$

Let us define the following Krein operator

$$ G \equiv \Gamma(-g) = \sum_{n \geq 0} (-g)^\otimes n \tag{3.1.7} $$

where the operator $g$ appears in (2.2.6). Then we can define the following non-degenerate sesquilinear form on $H$:

$$ (\Psi, \Phi) \equiv <\Psi, G\Phi> \tag{3.1.8} $$

or, explicitly:

$$ (\Psi, \Phi) \equiv <\Psi^{(0)}\Phi^{(0)} + \sum_{n=1}^{\infty} (-1)^n \int_{(X_0^+) \times \cdots \times (X_0^+)} \prod_{i=1}^{n} [da_0^+(k_i)g^{\mu_1\mu_1}] \Psi^{(n)}_{\mu_1, \ldots, \mu_n}(k_1, \ldots, k_n) \Phi^{(n)}_{\mu_1, \ldots, \mu_n}(k_1, \ldots, k_n) > \tag{3.1.9} $$
Then the sesquilinear form $(\cdot, \cdot)$ behaves naturally with respect to the action of the Poincaré group:

$$(U_g \Psi, U_g \Phi) = (\Psi, \Phi), \quad \forall g \in P^+, \quad (U_{t_i} \Psi, U_{t_i} \Phi) = (\Psi, \Phi).$$

We denote $|\phi|^2 = <\phi, \phi>$ and $\|\Phi\|^2 = (\Phi, \Phi)$.

Now one has from lemma 2.2:

**Lemma 3.1** Let us consider the following subspace of $\mathcal{H}$:

$$\mathcal{H}' \equiv \mathcal{F}^+(\mathcal{H}') = \oplus_{n \geq 0} \mathcal{H}'_n.$$ (3.1.11)

Then $\mathcal{H}'_n, \ n \geq 1$ is generated by elements of the form $\phi_1 \lor \cdots \lor \phi_n, \ \phi_1, \ldots, \phi_n \in \mathcal{H}'$ and, in the representation adopted previously for the Hilbert space $\mathcal{H}_n$ we can take

$$\mathcal{H}'_n = \{ \Phi^{(n)} \in \mathcal{H}_n | \ k_1^{\nu_1} \Phi^{(n)}_{\nu_1, \ldots, \nu_n}(k_1, \ldots, k_n) = 0 \}.$$ (3.1.12)

Moreover, the sesquilinear form $(\cdot, \cdot)|_{\mathcal{H}'}$ is positively defined.

Next, one has the analogue of lemma 2.3:

**Lemma 3.2** Let $\mathcal{H}'' \subset \mathcal{H}'$ given by

$$\mathcal{H}'' \equiv \{ \Phi \in \mathcal{H}' | \ \|\Phi\|^2 = 0 \} = \oplus_{n \geq 0} \mathcal{H}''_n.$$ (3.1.13)

Then, the subspace $\mathcal{H}''_n, \ n \geq 1$ is generated by elements of the type $\phi_1 \lor \cdots \lor \phi_n$ where at least one of the vectors $\phi_1, \ldots, \phi_n \in \mathcal{H}'$ belongs to $\mathcal{H}''$.

Moreover, in the representation adopted previously for the Hilbert space $\mathcal{H}_n$ the elements of $\mathcal{H}''_n$ are linearly generated by functions of the type:

$$\Phi^{(n)}_{\nu_1, \ldots, \nu_n}(k_1, \ldots, k_n) = \frac{1}{n} \sum_{i=1}^{n} (k_i)^{\nu_i} \lambda(k_i) \Psi^{(n-1)}_{\nu_1, \ldots, \nu_i, \ldots, \nu_n}(k_1, \ldots, \hat{k}_i, \ldots, k_n)$$ (3.1.14)

with $\Psi \in \mathcal{H}'$ and $\lambda : X^+_0 \to \mathbb{C}$ arbitrary.

Finally we have:

**Proposition 3.3** There exists an canonical isomorphism of Hilbert spaces

$$\mathcal{F}_{\text{photon}} \simeq \mathcal{H}'/\mathcal{H}''.$$ (3.1.15)

**Proof:** If $\psi \in \mathcal{H}'$ then we denote its class with respect to $\mathcal{H}''$ by $[\psi]$; similarly, if $\Phi \in \mathcal{H}'$ we denote its class with respect to $\mathcal{H}''$ by $[\Phi]$. Then the application $B : \mathcal{H}'/\mathcal{H}'' \to \mathcal{F}_{\text{photon}}$ is well defined by linearity, continuity and

$$B([\phi_1 \lor \cdots \lor \phi_n]) \equiv [\phi_1] \lor \cdots \lor [\phi_n]$$ (3.1.16)
and it is the desired isomorphism. Moreover, the sesquilinear form $(\cdot, \cdot)$ is strictly positive defined on the factor space, so it induces a scalar product. ■

Now we can define the electromagnetic field as an operator on the Hilbert space $\mathcal{H}$ in analogy to the construction from subsections 2.1 and 2.3 (see the relations (2.3.2) and (2.3.3)): we define for every $p \in X_0^+$ the annihilation and creation operators

\[
(A_\nu(p)\Phi)^{(n)}_{\mu_1, \ldots, \mu_n} (k_1, \ldots, k_n) \equiv \sqrt{n + 1} \Phi^{(n+1)}_{\nu, \mu_1, \ldots, \mu_n} (p, k_1, \ldots, k_n)
\] (3.1.17)

and

\[
(A_\nu^\dagger(p)\Phi)^{(n)}_{\mu_1, \ldots, \mu_n} (k_1, \ldots, k_n) \equiv -2\omega(p) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \delta(p - k_i) g_{\nu \mu_i} \Phi^{(n-1)}_{\mu_1, \ldots, \hat{\mu_i}, \ldots, \mu_n} (k_1, \ldots, \hat{k_i}, \ldots, k_n).
\] (3.1.18)

Then one has the canonical commutation relations (CAR)

\[
[A_\nu(p), A_\rho(p')] = 0, \quad [A_\nu^\dagger(p), A_\rho^\dagger(p')] = 0, \quad [A_\nu(p), A_\rho^\dagger(p')] = -2\omega(p) g_{\nu \rho} \delta(p - p') 1
\] (3.1.19)

and the relation

\[
(A_\nu^\dagger(p) \Psi, \Phi) = (\Psi, A_\nu(p) \Phi), \quad \forall \Psi, \Phi \in \mathcal{H}
\] (3.1.20)

which shows that $A_\nu^\dagger(p)$ is the adjoint of $A_\nu(p)$ with respect to the sesquilinear form $(\cdot, \cdot)$.

We also have a natural behaviour with respect to the action of the Poincaré group (see (2.1.18)):

\[
\mathcal{U}_{a, \Lambda} A_\nu(p) \mathcal{U}_{a, \Lambda}^{-1} = e^{i a \cdot p} (\Lambda^{-1})_\nu^\rho A_\rho(\Lambda \cdot p), \quad \forall \Lambda \in \mathcal{L}^+, \quad \mathcal{U}_{t, I} A_\nu(p) \mathcal{U}_{t, I}^{-1} = (I_t)_{\nu}^\rho A_\rho(I_s \cdot p)
\] (3.1.21)

and a similar relation for $A_\nu^\dagger(p)$.

Now we define the electromagnetic field in the point $x$ according to

\[
A_\nu(x) \equiv A_\nu^{(+)}(x) + A_\nu^{(-)}(x)
\] (3.1.22)

where the expressions appearing in the right hand side are the positive (negative) frequency parts and are defined by:

\[
A_\nu^{(+)}(x) \equiv \frac{1}{(2\pi)^{3/2}} \int_{X_0^+} d\alpha_0^+(p) e^{ip \cdot x} A_\nu^\dagger(p), \quad A_\nu^{(-)}(x) \equiv \frac{1}{(2\pi)^{3/2}} \int_{X_0^-} d\alpha_0^+(p) e^{-ip \cdot x} A_\nu(p).\n\] (3.1.23)

The explicit expressions are

\[
(A_\nu^{(+)}(x)\Phi)^{(n)}_{\mu_1, \ldots, \mu_n} (k_1, \ldots, k_n) = \sqrt{n + 1} \int_{X_0^+} d\alpha_0^+(p) e^{ip \cdot x} \Phi^{(n+1)}_{\nu, \mu_1, \ldots, \mu_n} (p, k_1, \ldots, k_n)
\] (3.1.24)

and

\[
(A_\nu^{(-)}(x)\Phi)^{(n)}_{\mu_1, \ldots, \mu_n} (k_1, \ldots, k_n) = -\frac{1}{(2\pi)^{3/2}} \sqrt{n} \sum_{i=1}^{n} e^{ik_i \cdot x} g_{\nu \mu_i} \Phi^{(n-1)}_{\mu_1, \ldots, \hat{\mu_i}, \ldots, \mu_n} (k_1, \ldots, \hat{k_i}, \ldots, k_n).
\] (3.1.25)
If $f \in S(\mathbb{R}^4, \mathbb{R}^4)$ we have the well-defined operators

$$A(f) \equiv \int_{\mathbb{R}^4} dx f^\mu(x) A_\mu(x)$$

(3.1.26)

or, more explicitly:

$$A(f) = \sqrt{2\pi} \int_{X_0^+} d\alpha_0^+ \left[ \tilde{f}^\mu(p) A_\mu(p) + \tilde{f}^\mu(-p) A_\mu^\dagger(p) \right].$$

(3.1.27)

The properties of the electromagnetic field operator $A_\nu(x)$ are summarised in the following elementary proposition:

**Proposition 3.4** The following relations are true:

$$\langle A_\nu(x) \Psi, \Phi \rangle = \langle \Psi, A_\nu(x) \Phi \rangle, \quad \forall \Psi, \Phi \in \mathcal{H},$$

(3.1.28)

and

$$D_0(x) = D_0^+(x) + D_0^-(x)$$

(3.1.32)

is the Pauli-Jordan distribution and $D^{(\pm)}(x)$ are given by:

$$D_0^{(\pm)}(x) \equiv \pm \frac{1}{(2\pi)^{3/2}} \int_{X_0^+} d\alpha_0^+ e^{\mp ip \cdot x}.$$

(3.1.33)

Let us note that we have:

$$\Box D_0^{(\pm)}(x) = 0, \quad \Box D_0(x) = 0.$$  

(3.1.34)

One can describe in very convenient way the subspaces $\mathcal{H}'$ and $\mathcal{H}''$ using the following operators

$$L(x) \equiv \partial^\mu A_\mu^-(x), \quad L^\dagger(x) \equiv \partial^\mu A_\mu^+(x).$$

(3.1.35)

Indeed, one has the following result:

**Proposition 3.5** The following relations are true:

$$\mathcal{H}' = \{ \Phi \in \mathcal{H} \mid L(x)\Phi = 0, \quad \forall x \in \mathbb{R}^4 \} = \cap_{x \in \mathbb{R}^4} \text{Ker}(L(x))$$

(3.1.36)

and

$$\mathcal{H}'' = \{ L(x)^\dagger \Phi \mid \forall \Phi \in \mathcal{H}, \quad \forall x \in \mathbb{R}^4 \} = \cup_{x \in \mathbb{R}^4} \text{Im}(L(x)^\dagger).$$

(3.1.37)

It follows that we have

$$\mathcal{F}_{\text{photon}} = \frac{\cap_{x \in \mathbb{R}^4} \text{Ker}(L(x))}{\cup_{x \in \mathbb{R}^4} \text{Im}(L(x)^\dagger)}.$$  

(3.1.38)
Let us note that we have:

\[ [L(x), L^\dagger(y)] = 0, \quad \forall x, y \in \mathbb{R}^4; \quad [L(f), L^\dagger(g)] = 0, \quad \forall f, g \in \mathcal{S}(\mathbb{R}^4). \quad (3.1.39) \]

We concentrate now on the constructions of observables on the Fock space of the photon \( \mathcal{F}_{\text{photon}} \). Because this space is rather hard to manipulate, we will distinguish a class of observables which are induced by self-adjoint operators on the Hilbert space \( \mathcal{H} \). Indeed, if \( O \) is such an operator and it leaves invariant the subspaces \( \mathcal{H}' \) and \( \mathcal{H}'' \) then it factorizes to an operator on \( \mathcal{F}_{\text{photon}} \) according to the formula

\[ [O][\Phi] \equiv [O\Phi]. \quad (3.1.40) \]

This type of observables are called *gauge invariant observables*. It is clear that not all observables on \( \mathcal{F}_{\text{photon}} \) are of this type.

Now we have the following result:

**Lemma 3.6** An operator \( O : \mathcal{H} \to \mathcal{H} \) induces a gauge invariant observable if and only if it verifies:

\[ [L(x), O]|_{\mathcal{H}'} = 0, \quad \forall x \in \mathbb{R}^4. \quad (3.1.41) \]

**Proof:** If the relation (3.1.41) is true, then one applies the proposition 3.3 and it is clear that the operator \( O \) leaves the subspaces \( \mathcal{H}' \) and \( \mathcal{H}'' \) invariant, so it induces the operator \([O]\). Conversely, if the operator \( O \) induces an operator \([O]\) then it should leave invariant the subspace \( \mathcal{H}' \) i.e. if \( \Phi \in \mathcal{H}' \) then we should also have \( O\Phi \in \mathcal{H}' \). Equivalently, we use proposition 3.3 and obtain that if \( L(x)\Phi = 0, \quad \forall x \in \mathbb{R}^4 \) then \( L(x)O\Phi = 0, \quad \forall x \in \mathbb{R}^4. \) But this implies that the relation (3.1.41) is true. \( \blacksquare \)

**Remark 3.7** It is clear that the same result is true if one replaces in (3.1.41) the commutator with the anticommutator.

In the end of this subsection we construct some typical gauge invariant observables. The verification of the condition from the preceding lemma is trivial. The first one is the so-called *strength of the electromagnetic field* defined by:

\[ F_{\mu\nu} \equiv \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x). \quad (3.1.42) \]

The second one is more subtle.

**Proposition 3.8** Let us consider a subset of the test functions space:

\[ \mathcal{S}_{\text{div}} \equiv \{ f \in \mathcal{S}(\mathbb{R}^4, \mathbb{R}^4) | \partial_\mu f^\mu = 0 \}. \quad (3.1.43) \]

For any \( f \in \mathcal{S}_{\text{div}} \) the operator \( A(f) \) (see (3.1.26) for the definition) induces a gauge invariant observable \([A(f)]\). Moreover, if we have \( f_\mu = \partial_\mu g \) for some test function verifying the equation \( \Box g = 0 \) then \([A(f)] = 0. \)
Now we can define, in analogy with the remark 2.1, annihilation and creation operators for the electromagnetic potentials as maps from $H'_{\text{photon}}$ into $F_{\text{photon}}$. We have:

**Proposition 3.9** Let $f \in H'$ and $[f]$ its equivalence class modulo $H''$. Then the operators $\mathcal{A}^{\pm}([f]): F_{\text{photon}} \to F_{\text{photon}}$ given by

$$\mathcal{A}([f])[\Phi] = \left[ \int_{X_0^+} d\alpha_0^+ f^\mu(p) A_\mu(p) \Phi \right], \quad \mathcal{A}^\dagger([f])[\Phi] = \left[ \int_{X_0^+} d\alpha_0^- f^\mu(p) A_\mu^\dagger(p) \Phi \right]$$

are well defined and one has the following formula

$$[\mathcal{A}(f)] = \sqrt{2\pi} \left( \mathcal{A}([f^+]) + \mathcal{A}^\dagger([f^-]) \right).$$

Let us note that the expression (3.1.45) is the perfect analogue of the generic formula (2.3.12). However, a major difference appears, namely one is not able to find analogues of the Wick monomials inducing well defined expressions on $F_{\text{photon}}$. More precisely, one can define without any problems expressions of the type

$$\mathcal{L}(g) = \int_{\mathbb{R}^4} dx g(x) \sum_{j} j^{(\nu_1,I_1),\ldots,(\nu_r,I_r)} (x) : \partial_{I_1} A_{\nu_1}(x) \cdots \partial_{I_r} A_{\nu_r}(x) :$$

for $r > 1$ on the Hilbert space $\mathcal{H}$, but it will be impossible to factorize such an expression into $F_{\text{photon}}$ even if one considers it in the adiabatic limit.

We continue this subsection with the analysis of possible interactions between the electromagnetic field and matter. Let us consider that the (Fock) space of the “matter” fields is denoted by $H_{\text{matter}}$. Then, in the hypothesis of weak coupling, one can argue that the Hilbert space of the combined system photons + matter is $H_{\text{total}} \equiv F_{\text{photon}} \otimes H_{\text{matter}}$. It is easy to see that, if we define, $\mathcal{H} \equiv H \otimes H_{\text{matter}}$, $\mathcal{H}' \equiv H' \otimes H_{\text{matter}}$ and $\mathcal{H}'' \equiv H'' \otimes H_{\text{matter}}$ we have as before:

$$H_{\text{total}} \simeq \mathcal{H}' / \mathcal{H}''.$$

In the Hilbert space $\mathcal{H}$ we can define as usual the expressions for the electromagnetic potentials and all properties listed previously stay true. In particular, there are no interactions of the type (3.1.46). So, we must try to construct the interaction in the form

$$T_1(g) \equiv \int_{\mathbb{R}^4} dx g(x) A_\nu(x) j''(x)$$

where $j''(x)$ are some Wick polynomials in the “matter” fields called currents.

**Remark 3.10** We note that it not necessary to consider more general expressions of the type $\mathcal{L}(x) = \sum j''(x) \partial A_\nu(x)$ because, in the adiabatic limit, by partial integration, one can exhibit the integrated expression $T_1(g) = \int_{\mathbb{R}^4} dx g(x) \mathcal{L}(x)$ into the form (3.1.48) from above.

Then we have a simple but very important result:
Theorem 3.11 The expression (3.1.48) induces, in the adiabatic limit, a well-defined expression on the Hilbert space $\mathcal{H}_{\text{total}}$ if and only if the current is conserved, i.e.

$$\partial_\mu j^\mu = 0.$$  

(3.1.49)

Proof: One applies the criterion from lemma 3.6. ■

Thus, we see that if we have a chiral interaction described by the axial current $j^\mu_A(x) = \bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)$ for some Dirac field of mass $m > 0$, then, because as it is well known, $\partial_\mu j^\mu_A(x) \neq 0$ this type of interaction does not produce a legitimate expression on the physical Hilbert space $\mathcal{H}_{\text{total}}$.

This result admits a generalisation. In studying higher orders of the perturbation theory, starting from an interaction Lagrangian of the type (3.1.48), one gets for the chronological products, expressions of the type (see (2.4.29)):

$$T_n(g) = \int_{\mathbb{R}^4 \times n} dx_1 \cdots dx_n g(x_1, \ldots, x_n)$$

$$\sum_{l=0}^{n} \frac{1}{l!} \sum_{k_1, \ldots, k_l} j^{\mu_1, \ldots, \mu_l}_{k_1, \ldots, k_l}(x_1, \ldots, x_n) : A_{\mu_1}(x_{k_1}) \cdots A_{\mu_l}(x_{k_l}) :$$

(3.1.50)

where $j^{\mu_1, \ldots, \mu_l}_{k_1, \ldots, k_l}(x_1, \ldots, x_n)$ are some distribution-valued operators build form the matter fields called multi-currents. The existence of such objects in the Hilbert space $\mathcal{H}$ is guaranteed by the zero theorem but we also have (see [43], formula (4.6.9)):

Theorem 3.12 The expressions (3.1.50) are inducing well-defined expression on $\mathcal{H}_{\text{total}}$, in the adiabatic limit, if and only if the following relations are verified:

$$\frac{\partial}{\partial x_{k_1}^{\nu}} j^{\mu_1, \ldots, \mu_l}_{k_1, \ldots, k_l}(x_1, \ldots, x_n) = 0$$

(3.1.51)

i.e. the multi-currents are also conserved.

Let us define now the so-called gauge transformation i.e. the operatorial transformation

$$\delta_\xi A_\nu(x) \equiv \partial_\nu \xi(x) \times 1$$

(3.1.52)

where $\xi$ is a test function. Then one can easily see that the identities (3.1.49) and (3.1.51) (see [43]) are equivalent to the following equations expressing the gauge invariance of the $S$-matrix:

$$\lim_{\epsilon \rightarrow 0} \delta_\xi T_n(g_{\epsilon} \otimes^n) = 0, \quad \forall n \in \mathbb{N}^* \quad \Leftrightarrow \quad \lim_{\epsilon \rightarrow 0} \delta_\xi S(g_{\epsilon}) = 0.$$  

(3.1.53)

So we see that the gauge invariance of the $S$-matrix should not be considered as an independent axiom, as it is usually done in the literature, but is a consequence of the requirement that the $S$-matrix factorizes, in the adiabatic limit, to the “physical” Hilbert space $\mathcal{H}_{\text{total}}$.

We close this subsection with some comments concerning other conditions which should be imposed to the currents. One should translate the requirements (2.4.17), (2.4.18) and (2.4.19) and obtains that the current $j^\mu(x)$ should be Poincaré covariant, should commute for spatially separated points and should be hermitian.
3.2 Quantisation with Ghost Fields

In this subsection we follow the analysis of the ghosts fields from [38] and then we show explicitly that one can obtain a new realisation of $F_{\text{photon}}$ which is essential for the construction of non-Abelian gauge fields. The details of this analysis seems to be missing from the literature (although some illuminating remarks do appear in [49]).

We consider the Hilbert space $H_{gh} \equiv L^2(X_m^+, \mathbb{C}^2; d\alpha_m^+)$ with the scalar product:

$$< \phi, \psi > \equiv \int_{X_m^+} d\alpha_m^+ < \phi(p), \psi(p) > |_{\mathbb{C}^2}. \quad (3.2.1)$$

In this space acts the following unitary representation of the Poincaré group:

$$(U_{a,\Lambda} \phi)(p) \equiv e^{ia \cdot p} \phi(\Lambda^{-1} \cdot p) \quad \text{for} \quad \Lambda \in \mathcal{L}^+, \quad (U_{I_S} \phi)(p) \equiv \phi(I_S \cdot p). \quad (3.2.2)$$

The Fock space $F^-(H_{gh})$ is called ghost particle Hilbert space. Remark the choice of the Fermi-Dirac statistics which seems to be essential for the whole analysis.

As in the subsection 2.3 one can canonically identify the $n$th-particle subspace $F^-(H_{gh})$ with the set of Borel functions $\Phi^{(n)}(X_m^+)^n \rightarrow \mathbb{C}^2$ which are square integrable with respect to the product measure $(\alpha_m^+)^n$ and verify the symmetry property

$$\Phi^{(n)}_{i_1, \ldots, i_n}(p_1, \ldots, p_n) = (-1)^{|P|} \Phi^{(n)}_{i_1, \ldots, i_n}(\hat{p}_1, \ldots, \hat{p}_n), \quad \forall P \in \mathcal{P}_n. \quad (3.2.3)$$

On the dense domain of test functions one can define the annihilation and creation operators for any $j = 1, 2$:

$$(d_j(k) \Phi)^{(n)}_{i_1, \ldots, i_n}(p_1, \ldots, p_n) \equiv \sqrt{n+1} \Phi^{(n+1)}_{j,i_1, \ldots, i_n}(k, p_1, \ldots, p_n) \quad (3.2.4)$$

and

$$(d_j^*(k) \Phi)^{(n)}_{i_1, \ldots, i_n}(p_1, \ldots, p_n) \equiv 2\omega(k) \frac{1}{\sqrt{n}} \sum_{s=1}^{n} (-1)^{s-1} \delta(k - p_s) \delta_{j_1 \ldots j_{s-1}}^s \Phi^{(n-1)}_{i_1, \ldots, i_{s-1}, p_{s+1}, \ldots, p_n}. \quad (3.2.5)$$

They verify canonical anticommutation relations

$$\{d_j(k), d_k(q)\} = 0, \quad \{d_j^*(k), d_k^*(q)\} = 0, \quad \{d_j(k), d_k^*(q)\} = \delta_{jk} 2\omega(q) \delta(k - q) 1 \quad (3.2.6)$$

and behave naturally with respect to Poincaré transform (see (2.1.18)). It is convenient to denote

$$b^\dagger(p) \equiv d_1^*(p), \quad c^\dagger(p) \equiv d_2^*(p). \quad (3.2.7)$$

Then the Fermionic fields

$$u(x) \equiv \frac{1}{(2\pi)^{3/2}} \int_{X_0^+} d\alpha_0^+ \left[ e^{-iq \cdot x} b(q) + e^{iq \cdot x} c^*(q) \right] \quad (3.2.8)$$
and
\[ \tilde{u}(x) \equiv \frac{1}{(2\pi)^{3/2}} \int_{X_0^+} d\alpha_0^+(q) \left[ -e^{-iq\cdot x} c(q) + e^{iq\cdot x} b^*(q) \right] \] (3.2.9)
are called ghost fields. They verify the wave equations:
\[ \Box u(x) = 0, \quad \Box \tilde{u}(x) = 0 \] (3.2.10)
and if we identify, as usual the positive (negative) frequency parts we have the canonical anticommutation relations:
\[
\begin{align*}
\{ u^{(\epsilon)}(x), u^{(\epsilon')} (y) \} &= 0, & \{ u(x), u(y) \} &= 0, & \{ \tilde{u}^{(\epsilon)}(x), \tilde{u}^{(\epsilon')} (y) \} &= 0, \\
\{ u^{(\epsilon)}(x), \tilde{u}^{(-\epsilon)} (y) \} &= D_0^{(-\epsilon)} (x-y), & \{ u(x), \tilde{u}(y) \} &= D_0 (x-y) \quad \forall \epsilon, \epsilon' = \pm 3.2.11
\end{align*}
\]

Now we consider that we are working into the Fock space
\[ H^g \equiv H \otimes F^{-}(H^g) \] (3.2.12)
i.e. we tensor the auxiliary Fock space $H$ used in the construction of the photon Fock space with the ghost Fock space (see (3.1.2)). In this Hilbert space, we can define without problems the electromagnetic potential and the ghosts. Now we can introduce an important operator:
\[ Q \equiv \int_{X_0^+} d\alpha_0^+(q) k^\mu \left[ A_\mu(k)c^*(k) + A_\mu^\dagger(k)b(k) \right] \] (3.2.13)
called supercharge. Its properties are summarised in the following proposition which can be proved by elementary computations:

**Proposition 3.13** The following relations are valid:
\[ Q \Phi_0 = 0; \] (3.2.14)
\[ [Q, A_\mu^\dagger(k)] = k_\mu c^*(k), \quad \{Q, b^*(k)\} = k^\mu A_\mu^\dagger(k), \quad \{Q, c^*(k)\} = 0; \] (3.2.15)
\[ [Q, A_\mu(k)] = k_\mu b(k), \quad \{Q, b(k)\} = 0, \quad \{Q, c(k)\} = k^\mu A_\mu(k); \] (3.2.16)
\[ Q^2 = 0; \] (3.2.17)
\[ \text{Im}(Q) \subset \text{Ker}(Q) \] (3.2.18)
and
\[ U_g Q = Q U_g, \quad \forall g \in \mathcal{P}. \] (3.2.19)

Moreover, one can express the supercharge in terms of the ghosts fields as follows:
\[ Q = \int_{\mathbb{R}^3} d^3 x \partial^\mu A_\mu(x) \tilde{u}(x). \] (3.2.20)
In particular (3.2.17) justify the terminology of supercharge and (3.2.18) indicates that it might be interesting to take the quotient. Indeed, we will rigorously prove that this quotient coincides with $F_{\text{photon}}$.

First, we give a more convenient representation for the Hilbert space $H^g$, namely

$$\mathcal{H}^g = \sum_{n,m,l=0}^\infty \mathcal{H}_{nml}$$

(3.2.21)

where $\mathcal{H}_{nml}$ consists of Borel functions $\Phi_{\mu_1,\ldots,\mu_n}^{(nm)} : (X_0^+)^{n+m+l} \to \mathbb{C}$ such that

$$\sum_{n,m,l=0}^\infty \int_0^{\infty} \frac{d\alpha_0^+}{\alpha_0^+} (K) d\alpha_0^+ (P) d\alpha_0^+ (Q) \sum_{\mu_1,\ldots,\mu_n=0}^3 |\Phi_{\mu_1,\ldots,\mu_n}^{(nm)} (K, P, Q)| \leq \infty$$

(3.2.22)

(here $K \equiv (k_1,\ldots,k_n)$, $P \equiv (p_1,\ldots,p_m)$ and $Q \equiv (q_1,\ldots,q_l)$) and verify the symmetry property

$$\Phi_{\mu_1,\ldots,\mu_n}^{(nm)} (k_{P(1)}^1,\ldots,k_{P(n)}^1; k_{Q(1)}^1,\ldots,k_{Q(m)}^1; q_{R(1)}^1,\ldots,q_{R(l)}^1) = (-1)^{|Q|+|R|} \Phi_{\nu_1,\ldots,\nu_m}^{(nml)} (k_1^1,\ldots,k_n^1; p_1^1,\ldots,p_m^1; q_1^1,\ldots,q_l^1), \quad \forall P \in P_n, Q \in P_m, R \in P_l$$

(3.2.23)

In this representation the annihilation operators have the following expressions:

$$A_r (\phi) \Phi_{\mu_1,\ldots,\mu_n}^{(nm)} (k_1^1,\ldots,k_n^1; p_1^1,\ldots,p_m^1; q_1^1,\ldots,q_l^1) = \sqrt{n + 1} \Phi_{\mu_1,\ldots,\mu_n}^{(n+1,m)} (r, k_1^1,\ldots,k_n^1; p_1^1,\ldots,p_m^1; q_1^1,\ldots,q_l^1)$$

(3.2.24)

$$b_r (\phi) \Phi_{\mu_1,\ldots,\mu_n}^{(nm)} (k_1^1,\ldots,k_n^1; p_1^1,\ldots,p_m^1; q_1^1,\ldots,q_l^1) = \sqrt{m + 1} \Phi_{\mu_1,\ldots,\mu_n}^{(n,m+1)} (k_1^1,\ldots,k_n^1; r, p_1^1,\ldots,p_m^1; q_1^1,\ldots,q_l^1)$$

(3.2.25)

and

$$(c_r (\phi) \Phi_{\mu_1,\ldots,\mu_n}^{(nm)} (k_1^1,\ldots,k_n^1; p_1^1,\ldots,p_m^1; q_1^1,\ldots,q_l^1) = (-1)^m \sqrt{l + 1} \Phi_{\mu_1,\ldots,\mu_n}^{(n,m,l+1)} (k_1^1,\ldots,k_n^1; q_1^1,\ldots,q_l^1)$$

(3.2.26)

similar expressions can be written for the creation operators.

**Remark 3.14** Let us note that the sign in the last expression is introduced such that we have “normal” anticommutation relations i.e. $b(p)$ anticommutes with $c(q)$. Without this sign we will have anomalous commutation relations, so the introduction of this sign is a Klein transform.

Now we can give the explicit expression of the supercharge in this representation; starting from the definition (3.2.13) we immediately get:

$$(Q \Phi_{\mu_1,\ldots,\mu_n}^{(nm)} (k_1^1,\ldots,k_n^1; p_1^1,\ldots,p_m^1; q_1^1,\ldots,q_l^1) = (-1)^m \sqrt{n + 1} \sum_{s=1}^l (-1)^{s-1} q_s \Phi_{\mu_1,\ldots,\mu_n}^{(n+1,m,l-1)} (q_s, k_1^1,\ldots,k_n^1; p_1^1,\ldots,p_m^1; q_1^1,\ldots,q_l^1)$$

$$- \sqrt{m + 1} \sum_{s=1}^n (k_s)_{\mu_1,\ldots,\mu_n} \Phi_{\mu_1,\ldots,\mu_n}^{(n-1,m+1,l)} (k_1^1,\ldots,k_s^1,\ldots,k_n^1; k_s^1,\ldots,k_n^1; p_1^1,\ldots,p_m^1; q_1^1,\ldots,q_l^1)$$

(3.2.27)

where, of course, we use Bourbaki convention $\sum_0 \equiv 0$. 

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Remark 3.15 Let us note that the relations (3.2.13) and (3.2.14) are uniquely determining the expression of the supercharge. Indeed, one can write any element of the Hilbert space $\mathcal{H}^{gh}$ in the form

$$\Phi = \sum_{n,m,l=0}^{\infty} \frac{(-1)^n}{\sqrt{n!m!l!}} \int (x_+^{a_i})^{n+m+l} \alpha^\dagger_0(K)\alpha^\dagger_0(P)\alpha^\dagger_0(Q) \Phi_{\mu_1,\ldots,\mu_n}(K,P,Q)$$

$$A^\dagger_{\mu_1}(k_1) \ldots A^\dagger_{\mu_m}(k_n) b^*_{p_1} \ldots b^*_{p_m} c^*(q_1) \ldots c^*(q_l) \Phi_0. \quad (3.2.28)$$

We apply the supercharge $Q$ and we commute it (using the relations (3.2.13)) till it gives zero on the vacuum. In this way the formula (3.2.27) is produced and formula (3.2.16) becomes a consequence.

Now we introduce on $\mathcal{H}^{gh}$ a Krein operator according to:

$$(J\Phi)^{(nml)}(K;P;Q) \equiv (-1)^{ml}(-g)^{\otimes n} \Phi^{(nml)}(K;Q;P). \quad (3.2.29)$$

In words, this operators acts on the “photon” variables as the Krein operator introduced in the previous subsection through the formula (3.1.7) and, moreover, inverts the rôles of the two types of ghosts (up to a sign). The properties of this operator are summarised in the following proposition which can be proved by elementary computations:

**Proposition 3.16** The following relations are verified:

$$J^* = J^{-1} = J \quad (3.2.30)$$

$$Jb(p)J = c(p), \quad Jc(p)J = b(p), \quad JA_{\mu}(p)J = A_{\mu}(p) \quad (3.2.31)$$

$$JQJ = Q^* \quad (3.2.32)$$

and

$$U_g J = J U_g, \quad \forall g \in \mathcal{P}. \quad (3.2.33)$$

Here $O^*$ is the adjoint of the operator $O$ with respect to the scalar product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}^{gh}$.

We can define now the sesquilinear form on $\mathcal{H}^{gh}$ according to

$$\langle \Psi, \Phi \rangle \equiv \langle \Psi, J\Phi \rangle; \quad (3.2.34)$$

then this form is non-degenerated. It is convenient to denote the conjugate of the arbitrary operator $O$ with respect to the sesquilinear form $(\cdot, \cdot)$ by $O^\dagger$ i.e.

$$(O^\dagger \Psi, \Phi) = \langle \Psi, O\Phi \rangle. \quad (3.2.35)$$

Then the following formula is available:

$$O^\dagger = JO^*J. \quad (3.2.36)$$
As a consequence, we have

\[ A_\mu(x)\dagger = A_\mu(x), \quad u(x)\dagger = u(x), \quad \bar{u}(x)\dagger = -\bar{u}(x). \tag{3.2.37} \]

From (3.2.33) it follows that we have:

\[ (U_\Psi, U_g \Phi) = (\Psi, \Phi), \forall g \in P^\dagger, \quad (U_I, U_{it} \Phi) = (\bar{\Psi}, \bar{\Phi}). \tag{3.2.38} \]

Now, we concentrate on the description of the factor space \( \text{Ker}(Q)/\text{Im}(Q) \). The following analysis, which is essential for the complete understanding of the photon description and of the Yang-Mills generalisation seems to be missing from the literature. We will construct a "homotopy" for the supercharge \( Q \).

Let us consider \( \epsilon : X_0^+ \to \mathbb{C}^4 \) a Borel bounded function such that

\[ \epsilon^\mu(k) k_\mu = -1, \quad \epsilon^\mu(k) \epsilon_\mu(k) = 0. \tag{3.2.39} \]

(One can construct such a function in the following way. If \( k_0 = (1, 0, 0, 1) \in X_0^+ \) then we define \( \epsilon(k_0) \equiv (-\frac{1}{2}, 0, 0, \frac{1}{2}) \). Now, let \( L_k \) be a Wigner rotation, i.e. a Borel map \( L \) from \( X_0^+ \) into the Lorentz group, such that \( L_k \cdot k_0 = k \). If we define \( \epsilon(k) \equiv L_k \cdot f(k_0) \) then we have the properties (3.2.39).)

Now we give the expression for the "homotopy" operator. We have the following result which follows by direct computations:

**Proposition 3.17** Let us define the operator

\[ \tilde{Q} \equiv \int_{X_0^+} d\alpha_0^+(q) \epsilon^\mu(k) \left[ A_\mu(k) b^*(k) + A_\mu(k) c(k) \right] \tag{3.2.40} \]

Then the following relation is valid:

\[ Y \equiv \{ Q, \tilde{Q} \} = N_b + N_c + X \tag{3.2.41} \]

where \( N_b \) (resp. \( N_c \)) are particle number operators for the ghosts of type \( b \) (resp. \( c \)) and

\[ X \equiv \int_{X_0^+} d\alpha_0^+(q) F^{\mu\nu}(k) A_\mu(k) A_\nu(k); \tag{3.2.42} \]

here we have introduced the notation:

\[ F^{\mu\nu}(k) \equiv k^\mu \epsilon^\nu(k) + k^\nu \epsilon^\mu(k). \tag{3.2.43} \]

Moreover the following relations are true:

\[ \tilde{Q}^2 = 0 \tag{3.2.44} \]

and

\[ [Y, Q] = 0, \quad [Y, \tilde{Q}] = 0. \tag{3.2.45} \]
We call the operator $\tilde{Q}$ the homotopy of $Q$. If the operator $Y$ would be invertible, then we could immediately conclude that the cohomology of the operator $Q$ is trivial. Fortunately this is not true. However, we have:

**Proposition 3.18** The operator $Y|_{\mathcal{H}_{nmL}}$ is invertible iff $m + l > 0$.

**Proof:** First we need another expression for the operator $X$ defined by (3.2.42). It is not very difficult to prove that one has:

$$X = A \otimes 1$$

(3.2.46)

where the operator $A$ acts only on the Bosonic variables and is given by the expression

$$A = d\Gamma(P);$$

(3.2.47)

here $d\Gamma$ is the familiar Cook functor \[9\] defined by;

$$d\Gamma(P)\psi_1 \otimes \cdots \otimes \psi_n \equiv P\psi_1 \otimes \psi_2 \cdots \otimes \psi_n + \cdots \psi_1 \otimes \psi_2 \cdots \otimes P\psi_n$$

(3.2.48)

and the operator $P$ is in our case given by:

$$(P\psi)_\mu(k) \equiv -F_{\mu\nu}(k)\psi^\nu(k).$$

(3.2.49)

In particular we immediately obtain that $P$ is a projector i.e. $P^2 = P$ and we have the direct sum decomposition of the one-particle Bosonic subspace into the direct sum of $\text{Ran}(P)$ and $\text{Ran}(1-P)$. Let us consider a basis in the one-particle Bosonic subspace formed by a basis $f_i, \ i \in \mathbb{N}$ of $\text{Ran}(P)$ and a basis $g_i, \ i \in \mathbb{N}$ of $\text{Ran}(1-P)$.

It is clear that a basis in the $n^\text{th}$-particle Bosonic subspace is of the form:

$$f_{i_1} \lor \cdots \lor f_{i_r} \lor g_{j_1} \lor \cdots \lor g_{j_s}, \ r, s \in \mathbb{N}, \ r + s = n.$$

Applying the operator $A$ to such a vector gives the same vector multiplied by $s$. So, in the basis chosen above, the operator $A$ is diagonal with diagonal elements from $\mathbb{N}$. It follows that the operator $Y|_{\mathcal{H}_{nmL}}$ can also be exhibited into a diagonal form with diagonal elements of the form $m + l + s, \ s \in \mathbb{N}$. It is obvious now that for $m + l > 0$ this is an invertible operator. ■

We have the following corollary:

**Corollary 3.19** Let us define $\mathcal{H}_0 \equiv \oplus_{n \geq 0}^\infty \mathcal{H}_{n00}$ and $\mathcal{H}_1 \equiv \oplus_{n \geq 0, m+l>0} \mathcal{H}_{nmL}$. Then the operator $Y$ has the block-diagonal form

$$Y = \begin{pmatrix} Y_1 & 0 \\ 0 & Y_0 \end{pmatrix}$$

(3.2.50)

with $Y_1$ an invertible operator.

Now we have the fundamental result
Proposition 3.20 There exists the following vector spaces isomorphism:

\[ \text{Ker}(Q)/\text{Im}(Q) \simeq \mathcal{H}'/\mathcal{H}'' \] (3.2.51)

where the subspaces \( \mathcal{H}' \) and \( \mathcal{H}'' \) have been defined in the previous subsection (see the lemmas 3.1 and 3.2 respectively).

Proof: (i) We note that the operators \( Q \) and \( \tilde{Q} \) have the block-diagonal form

\[ Q = \begin{pmatrix} Q_{11} & Q_{10} \\ Q_{01} & 0 \end{pmatrix}, \quad \tilde{Q} = \begin{pmatrix} \tilde{Q}_{11} & \tilde{Q}_{10} \\ \tilde{Q}_{01} & 0 \end{pmatrix} \] (3.2.52)

and from the relations (3.2.45) it easily follows that we have

\[ [Y_1, Q_{11}] = 0 \] (3.2.53)

\[ Y_1 Q_{10} = Q_{10} Y_0, \quad Y_0 Q_{01} = Q_{01} Y_1 \] (3.2.54)

and similar relations for the block-diagonal elements of the homotopy operator \( \tilde{Q} \). In particular we have

\[ [Y_1, Q_{10} \tilde{Q}_{01}] = 0. \] (3.2.55)

(ii) Let now \( \Phi \in \text{Ker}(Q) \). If we apply the relation to (3.2.41) the vector \( \Phi \) we obtain:

\[ Y \Phi = Q \Psi \] (3.2.56)

where we have defined

\[ \Psi \equiv \tilde{Q} \Phi. \] (3.2.57)

If we use the block-decomposition form for the vectors \( \Phi \) and \( \Psi \) we have in particular, from this relation that

\[ \Psi_0 = \tilde{Q}_{01} \Phi_1. \]

If we use this relation in (3.2.56) we obtain in particular that

\[ Y_1 \Phi_1 = Q_{11} \Psi_1 + Q_{10} \tilde{Q}_{01} \Phi_1. \] (3.2.58)

Because the operator \( Y_1 \) is invertible, we have from here

\[ \Phi_1 = Y_1^{-1} Q_{11} \Psi_1 + Y_1^{-1} Q_{10} \tilde{Q}_{01} \Phi_1. \] (3.2.59)

But from (3.2.53) and (3.2.55) we immediately obtain

\[ [Y_1^{-1}, Q_{11}] = 0, \quad [Y_1^{-1}, Q_{10} \tilde{Q}_{01}] = 0 \]

so the preceding relations becomes:

\[ \Phi_1 = Q_{11} Y_1^{-1} \Psi_1 + Q_{10} \tilde{Q}_{01} Y_1^{-1} \Phi_1. \] (3.2.60)
Now we define the vector $\psi$ by its components:

$$\psi_1 \equiv Y_1^{-1}\Psi_1, \quad \psi_0 \equiv \tilde{Q}_0 Y_1^{-1}\Phi_1$$  \hspace{1cm} (3.2.61)

and we get by a simple computation

$$\Phi - Q\psi = \begin{pmatrix} 0 \\ \phi_0 \end{pmatrix}$$  \hspace{1cm} (3.2.62)

where

$$\phi_0 \equiv \Phi_0 - Q_0 \psi_1.$$  

In other words, if $\Phi \in Ker(Q)$ then we have

$$\Phi = Q\psi + \tilde{\Phi}$$  \hspace{1cm} (3.2.63)

where

$$\tilde{\Phi}^{(nml)} = 0, \quad m + l > 0.$$  \hspace{1cm} (3.2.64)

(iii) The condition $Q\Phi = 0$ amounts now to $Q\tilde{\Phi} = 0$ or, with the explicit expression of the supercharge (3.2.27):

$$q^{\nu}_s \tilde{\Phi}^{(n+1,0,0)}_{s,\mu_1,\ldots,\mu_n} (q, k_1, \ldots, k_n; \theta; \theta) = 0, \quad \forall n \in \mathbb{N}$$  \hspace{1cm} (3.2.65)

i.e. the ensemble $\{\tilde{\Phi}^{(n00)}\}_{n \in \mathbb{N}}$ is an element from $\mathcal{H}'$ (see lemma 3.1).

It remains to see in what conditions such $\tilde{\Phi}$ is an element from $Im(Q)$ i.e. we have $\tilde{\Phi} = Q\chi$. It is clear that only the components $\chi^{(n10)}$ should be taken non-null. Then the expression of the supercharge (3.2.27) gives for any $n \in \mathbb{N}$ the following expressions:

$$(Q\chi)^{(n00)}_{\mu_1,\ldots,\mu_n} (k_1, \ldots, k_n; \theta; \theta) = -\frac{1}{\sqrt{n}} \sum_{s=1}^{n} (k_s)_{\mu_s} \chi^{(n-1,1,0)}_{\mu_1,\ldots,\mu_s,\ldots,\mu_n} (k_1, \ldots, \tilde{k}_s, \ldots, k_n; k_s; \emptyset)$$  \hspace{1cm} (3.2.66)

and

$$(Q\chi)^{(n11)}_{\mu_1,\ldots,\mu_n} (k_1, \ldots, k_n; p; q) = -\sqrt{n + 1} q^{\nu}_s \chi^{(n+1,1,0)}_{s,\mu_1,\ldots,\mu_n} (q, k_1, \ldots, k_n; p; \emptyset).$$  \hspace{1cm} (3.2.67)

But we must have $(Q\chi)^{(n11)} = 0$ because the vector $Q\chi = \tilde{\Phi}$ has only the projection on $\mathcal{H}_0$ non-null. This means that the expression (3.2.66) is an element from $\mathcal{H}''_n$ (see (3.1.14)). The isomorphism from the statement is now

$$[\Phi] \leftrightarrow [\tilde{\Phi}]$$

where in the left hand side we take classes modulo $Im(Q)$ and in the right hand side we take classes modulo $\mathcal{H}''$. ■

Remark 3.21 The homotopy formula derived above is considerably more complicated than the one appearing in [42].
We now have a standard result (see e.g. [12]):

Lemma 3.22 The sesquilinear form $(\cdot, \cdot)$ induces a strictly positive defined scalar product on the factor space $\overline{\text{Ker}(Q)/\text{Im}(Q)}$.

Proof: One must first show that the formula

$$( [\Psi], [\Phi] ) \equiv (\Psi, \Phi) \quad (3.2.68)$$

gives a well defined expression, i.e. the right hand side does not depend on the representatives $\Psi \in [\Psi]$ and $\Phi \in [\Phi]$; here $[\Psi], [\Phi]$ are arbitrary equivalence classes from $\text{Ker}(Q)/\text{Im}(Q)$. Next, one applies the preceding proposition and can take into the right hand side of (3.2.68) $\Psi, \Phi \in H'$. But in this case the sesquilinear form defined through (3.2.68) is positive defined because it coincides with the scalar product defined in proving proposition 3.3. ■

We also have:

Lemma 3.23 The representation of $U$ of the Poincaré group factors out at $\text{Ker}(Q)/\text{Im}(Q)$.

Proof: One combines the results (3.2.19) and (3.2.38). ■

The central result follows immediately:

Theorem 3.24 The isomorphism (3.2.51) extends to a Hilbert space isomorphism:

$$\overline{\text{Ker}(Q)/\text{Im}(Q)} \simeq \mathcal{F}_{\text{photon}}.$$

Remark 3.25 It is obvious that the whole construction of the photon Fock space as a factor space relies heavily on the property (3.2.14) of the supercharge. This property follows in turn from the choice of the “wrong” statistics for the ghosts fields i.e. Fermi-Dirac statistics. It is highly doubtful if a supercharge with all the required properties can be constructed using Bosonic ghosts. Nevertheless, one cannot claim that the construction above is unique. In fact, as it appears from the literature, there are many other “gauges” i.e. possibilities of obtaining a theorem of the type 3.24. It is an interesting, although not very well posed problem, to try to describe the most general quantization of the electromagnetic field of this kind.
3.3 Gauge-Invariant Observables

In this subsection we analyse in detail the same problem which was analysed at the end of the first subsection, namely the construction of observables on the factor space from the theorem 3.24. We hope to clarify some point not very well treated in the literature.

First we determine by direct calculus the following relations:

\[
\{Q, u^{(\pm)}(x)\} = 0, \quad \{Q, \bar{u}^{(\pm)}(x)\} = -i\partial^\mu A^{(\pm)}_\mu(x), \quad [Q, A^{(\pm)}_\mu(x)] = i\partial_\mu u^{(\pm)}(x);
\]

as a consequence:

\[
\{Q, u(x)\} = 0, \quad \{Q, \bar{u}(x)\} = -i\partial^\mu A_\mu(x), \quad [Q, A_\mu(x)] = i\partial_\mu u(x).
\]

Remark 3.26 The fact that the commutation with \(Q\) does not mix positive and negative parts of the various fields follows from the clever definition of \(u\) and \(\bar{u}\). If one would take, for instance, instead of \(\bar{u}\) the field \(u^*\) such nice relations would be lost.

Next, we denote by \(\mathcal{W}\) the linear space of all Wick monomials on the Fock space \(\mathcal{H}^g\) i.e. containing the fields \(A_\mu(x), u(x)\) and \(\bar{u}(x)\). If \(M\) is such a Wick monomial, we define by \(gh_\pm(M)\) the degree in \(\bar{u}\) (resp. in \(u\)). The total degree of \(M\) is obviously

\[
\text{deg}(M) \equiv gh_+(M) + gh_-(M).
\]

The ghost number is, by definition, the expression:

\[
gh(M) \equiv gh_+(M) - gh_-(M).
\]

Then we have the following well-known result:

Lemma 3.27 If \(M \in \mathcal{W}\) let us define the operator:

\[
d_QM \equiv: QM : -(-1)^{gh(M)} : MQ :
\]

on monomials \(M\) and extend it by linearity to the whole \(\mathcal{W}\). Then \(d_QM \in \mathcal{W}\) and

\[
gh(d_QM) = gh(M) - 1.
\]

Proof: It is done by induction on \(\text{deg}(M)\). First, we consider the case when \(M\) contains only the ghost field \(u\) and prove easily by induction on \(gh_-(M)\) that \(d_QM = 0\). Indeed, if \(\text{deg}(M) = 1\) we obviously have from the first relation (3.3.1) this equality. If we assume that the equality is true for \(M\) of degree \(n\) in \(u\), let \(M\) be of degree \(n + 1\). We can write it as follows: \(M =: BC:\) where \(B\) (resp. \(C\)) are Wick monomials of degree 1 (resp. \(n\)) in \(u\). We have from here:

\[
M = B^+C + (-1)^nCB^-
\]

and the commutator of \(Q\) with this expression can be computed without any problem, producing the desired result if one uses the induction hypothesis.
Next, we take $M$ to be a monomial in $u$ and $A_\mu$ and prove the assertion from the statement by induction on the degree in $A_\mu$ in the same way. Finally, one considers the general case applying the same tricks. ■

The operator $d_Q : \mathcal{W} \to \mathcal{W}$ is called the BRST operator; other properties of this object are summarized in the following elementary:

**Proposition 3.28** The following relations are verified:

\[
d^2_Q = 0, \\
d_Qu = 0, \quad d_Q\tilde{u} = -i\partial^\mu A_\mu, \quad d_QA_\mu = i\partial_\mu u; \\
d_Q(MN) = (d_QM)N + (-1)^{gh(M)}M(d_QN), \quad \forall M, N \in \mathcal{W}.\]

Now we can distinguish a class of observables on the factor space from theorem 3.24; we have in complete analogy to the construction (3.1.40) the following result:

**Lemma 3.29** If $O : \mathcal{H}^{gh} \to \mathcal{H}^{gh}$ verifies the condition

\[
d_QO = 0 \quad (3.3.10)
\]

then it induces a well defined operator $[O]$ on the factor space $\overline{\text{Ker}(Q)/\text{Im}(Q)} \simeq \mathcal{F}_{\text{photon}}$.

Moreover, in this case the following formula is true for the matrix elements of the factorized operator $[O]$:

\[
([\Psi], [O][\Phi]) = (\Psi, O\Phi). \quad (3.3.11)
\]

This kind of observables on the physical space will also be called gauge invariant observables. Next, we have in analogy to lemma 3.6:

**Lemma 3.30** An operator $O : \mathcal{H}^{gh} \to \mathcal{H}^{gh}$ induces a gauge invariant observables if and only if it verifies:

\[
d_QO|_{\text{Ker}(Q)} = 0. \quad (3.3.12)
\]

The criteria from lemma 3.30 can be applied, as in the first subsection, to the electromagnetic field strength (3.1.42) and to the electromagnetic potentials (3.1.45) proving that they are gauge-invariant observables.

Not all operators verifying the condition (3.3.10) are interesting. In fact, we have from (3.3.7):

**Lemma 3.31** The operators of the type $d_QO$ are inducing a null operator on the factor space; explicitly, we have:

\[
[d_QO] = 0. \quad (3.3.13)
\]

Moreover, we have:
Theorem 3.32 Let the interaction Lagrangian be a Wick monomial $T_1 \in \mathcal{W}$ with $gh(T_1) \neq 0$. Then the chronological product are null, i.e. there is no non-trivial $S$-matrix.

Proof: The generic form of $T_1$ is

$$T_1(x) =: u_1^x(x) \cdots u_n^x(x) M(x):$$

where $M$ is a Wick monomial in $A_{\mu}(x)$. One must determine the corresponding chronological products $T_n$, $n \geq 2$. It is not very hard to prove, by induction, that we also have:

$$gh(T_n) \neq 0 \quad (3.3.14)$$

Indeed, to compute $T_2$ one splits causally the commutator $D_2(x, y) = [T_1(x), T_1(y)]$. But one can compute this expression using Wick theorem. From (3.2.11) one can see that the only non-null pairing are between a ghost of type $u$ and a ghost of type $\tilde{u}$. So, we have $gh(D_2) \neq 0$. It is clear that this property is preserved by the process of distribution splitting, so we also have $gh(T_2) \neq 0$. The argument goes on now to an arbitrary order producing chronological product verifying (3.3.14).

We compute now the matrix elements of $T_n$ using (3.3.11); according to proposition 3.20 we can take into the right hand side, $\Psi, \Phi$ of the form $\Psi = \Psi' \otimes \Psi_0^{gh}$ and $\Phi = \Phi' \otimes \Phi_0^{gh}$ where $\Psi_0^{gh}$ is the vacuum state in the ghost Hilbert space and $\Psi', \Phi'$ are arbitrary vectors from $\mathcal{H}$. It follows that we have:

$$([\Psi], [T_n(x_1, \ldots, x_n)] [\Phi]) = \sum_\alpha (\Phi_0^{gh}, G_\alpha^{gh} \Phi_0^{gh})(\Psi', M_\alpha^{gh} \Phi')$$

where $G_\alpha^{gh}$ are Wick monomials in the ghost fields and $M_\alpha^{gh}$ are Wick monomials in the electromagnetic potentials. According to (3.3.14) we have $gh(G_\alpha^{gh}) \neq 0$. But, in this case one can compute $(\Phi_0^{gh}, G_\alpha^{gh} \Phi_0^{gh})$ using Wick theorem (see the end of subsection 2.3) and (3.2.11); because only the pairing of $u$ and $\tilde{u}$ can give non-null contributions, it follows that, in the end we get $([\Psi], [T_n(x_1, \ldots, x_n)] [\Phi]) = 0$. ■

As in the end of subsection 3.1 one can see that interaction Lagrangians of the type (3.1.46) do not factorise to the “physical” space $\text{Ker}(Q)/\text{Im}(Q)$. Finally, we want to analyse possible interactions between the electromagnetic field and “matter” in this new representation of the electromagnetic field. Presumably, we will obtain the same result as in the end of subsection 3.1. Indeed, if $\mathcal{H}_\text{matter}$ is the corresponding Hilbert space of the matter fields, it is elementary to see that we can realise the total Hilbert space $\mathcal{H}_\text{total} \equiv \mathcal{F}_\text{photon} \otimes \mathcal{H}_\text{matter}$ as the factor space $\text{Ker}(Q)/\text{Im}(Q)$ where the supercharge $Q$ is defined on $\mathcal{H}_\text{gh} \equiv \mathcal{H}_\text{gh} \otimes \mathcal{H}_\text{matter}$ by the obvious substitution $Q \rightarrow Q \otimes 1$.

Now we have:

Theorem 3.33 Let us define on $\mathcal{H}_\text{gh}$ the interaction Lagrangian of the form (3.1.48) where the current $j^{\mu}(x)$ is a Wick monomial in the matter and ghost fields. The this expression factorises, in the adiabatic limit, to the physical space $\mathcal{H}_\text{total}$ and gives a non-null $S$-matrix if and only if it does not depend on the ghost fields and it is conserved in the sense (3.1.49).
Proof: We apply the criterion from lemma 3.30 to the interaction Lagrangian (3.1.48) and easily obtain the conservation law (3.1.49) and the condition $d Q^\mu(x) = 0$. Using the relations (3.3.8) it is not hard to prove that the current $j^\mu(x)$ should not contain ghosts of the type $\tilde{u}$. But in this case one can apply proposition 3.32 and obtain that, in fact, the current does not depend on $u$ either. \[\blacksquare\]
4 Pure Yang-Mills Fields

4.1 The General Framework

In this section, we derive the following result: the only possible coupling between \( r > 1 \) electromagnetic-type fields of power not greater than 4 is through an Yang-Mills Lagrangian. This type of result has recently appeared in [2]. The main differences in our approach is the systematic utilisation of the criterion from lemma 3.30; we also hope to streamline the arguments.

First, we have to define in an unambiguous way what we mean by Yang-Mills fields. We make in subsection 3.2 the following modifications.

- Instead of \( H \) (see subsection 2.3) we consider Hilbert space \( H^r \equiv L^2(X_0^+, \mathbb{C}^4)^r, d\alpha_0^+ \) with the scalar product

\[
< \phi, \psi > \equiv \sum_{a=1}^r \int_{X_0^+} d\alpha_0^+ < \phi_a(p), \psi_a(p) >_{\mathbb{C}^4}; \tag{4.1.1}
\]

The subspaces \( H_{Y,M}^r \equiv \{ \phi \in H^r | p^\mu \phi_{a\mu} = 0 \} \) and \( H_{Y,M}^r \equiv \{ \phi \in H_{Y,M}^r | ||\phi|| = 0 \} \) are introduced by complete analogy and we call the factor space \( H_{Y,M}^r \equiv H_{Y,M}^r/H_{Y,M}^r \) the Hilbert space of the Yang-Mills particles (or gluons). The description of the corresponding Fock space can be done as in subsection 3.1.

- The construction of the ghosts fields starts from \( H^{gh,r} \equiv L^2(X_m^+, \mathbb{C}^2)^r, d\alpha_m^+ \) with the scalar product

\[
< \phi, \psi > \equiv \sum_{a=1}^r \int_{X_m^+} d\alpha_m^+ < \phi_a(p), \psi_a(p) >_{\mathbb{C}^2}; \tag{4.1.2}
\]

the corresponding Fock space is called the space of ghosts particles.

- The construction of the Yang-Mills interaction will be done in the Hilbert space \( H_{Y,M}^{gh,r} \equiv H_{Y,M}^r \otimes F^{-}(H^{gh,r}) \). One can decompose this Fock space into subspaces with fixed number of gluons and ghosts.

- One can define now the Yang-Mills and the ghosts fields \( A^a_{\mu}, u_a, \tilde{u}_a, a = 1, \ldots, r \) generalising in an obvious way the formulæ (3.1.22) + (3.1.23):

\[
A_{a\mu}(x) \equiv \frac{1}{(2\pi)^{3/2}} \int_{X_0^+} d\alpha_0^+(p) \left[ e^{-ip^x} A_{a\mu}(p) + e^{ip^x} A^\dagger_{a\mu}(p) \right] \tag{4.1.3}
\]

and respectively (3.2.8) + (3.2.9):

\[
u_a(x) \equiv \frac{1}{(2\pi)^{3/2}} \int_{X_0^+} d\alpha_0^+(q) \left[ e^{-iq^x} b_a(q) + e^{iq^x} c^\dagger_a(q) \right] \tag{4.1.4}
\]

and

\[
\tilde{u}_a(x) \equiv \frac{1}{(2\pi)^{3/2}} \int_{X_0^+} d\alpha_0^+(q) \left[ -e^{-iq^x} c_a(q) + e^{iq^x} b_a^\dagger(q) \right]. \tag{4.1.5}
\]

- The only significant modification appears in the canonical (anti)commutation relations (3.1.30) + (2.3.4) and (3.2.11), namely we have:

\[
\left[ A^{(\mp)}_{a\mu}(x), A^{(\mp)}_{b\nu}(y) \right] = -\delta_{ab}g_{\mu\nu} D_0^{(\mp)}(x-y) \times 1, \quad [A_{a\mu}(x), A_{b\nu}(y)] = -\delta_{ab}g_{\mu\nu} D_0(x-y) \times 1 \tag{4.1.6}
\]
\{u^{(e)}_a(x), \tilde{u}^{(-e)}_b(y)\} = \delta_{ab}D_0^{(-e)}(x-y), \quad \{u_a(x), \tilde{u}_b(y)\} = \delta_{ab}D_0(x-y). \quad \text{(4.1.7)}

- The supercharge is given by (see (3.2.13)):

\[
Q \equiv \sum_{a=1}^{r} \int_{X_0}^{} d\alpha^\mu_0(q)k^\mu [A_{a\mu}(k)c^\dagger_a(k) + A^\dagger_{a\mu}(k)b_a(k)]
\]

\[
\text{(4.1.8)}
\]

and verifies all the expected properties.

- The Krein operator has an expression similar to (3.2.29) and can be used to construct a sesquilinear form like in (3.2.34). Then relations of the type (3.2.37) are still true;

\[
A_{a\mu}(x)\dagger = A_{a\mu}(x), \quad u_a(x)\dagger = u_a(x), \quad \tilde{u}_a(x)\dagger = -\tilde{u}_a(x).
\]

\[
\text{(4.1.9)}
\]

- As a consequence, proposition 3.20, and the main theorem 3.24 stay true.

- The ghost degree is defined in an obvious way and the expression of the BRST operator (3.3.5) is the same in this more general framework and the corresponding properties are easy to obtain. In particular we have (see (3.3.8)):

\[
d_Q u_a = 0, \quad d_Q \tilde{u}_a = -i\partial^\mu A_\mu, \quad d_Q A_{a\mu} = i\partial_\mu u_a.
\]

\[
\text{(4.1.10)}
\]

- Finally, the characterisation of gauge-invariant observables is done in the same way as in subsection 3.3. In particular we have the theorem 3.32.

As we see, the Yang-Mills fields are nothing but a \(r\)-th component electromagnetic-type free field. This assumption has its limitation, but it is one of the main ingredients of the theory, although not always admitted explicitly in the literature.

We will try to construct a \(S\)-matrix on the auxiliary Fock space \(H_{YM}^{gh,r}\) having all the properties from the section 2.4. In particular we should be able to construct the first-order term, i.e. the interaction Lagrangian \(T_1(x)\) verifying the properties (2.4.17), (2.4.18), (2.4.19) and (2.4.20). We emphasise that in the condition (2.4.19) we will consider the adjoint with respect to the sesquilinear form \((\cdot, \cdot)\) defined with the help of the Krein operator. Moreover, we will impose that the chronological products constructed in this way should have a well defined adiabatic limit. According to lemma 3.30, this condition writes as:

\[
\lim_{\epsilon \searrow 0} d_Q T_n(x_1, \ldots, x_n)|_{Ker(Q)} = 0, \quad \forall n \geq 1.
\]

\[
\text{(4.1.11)}
\]

If these two requirements are fulfilled, then we get an unitary \(S\)-matrix on the factor space \(Ker(Q)/Im(Q)\); indeed, if the chronological products are factorizing to this quotient, the theorem must be true for the antichronological products, because they are expressed by (2.4.6) in terms of the chronological ones. In this way, the whole unitarity argument of Epstein and Glaser [28] works in our case also: we have from (2.4.19), by induction, that (2.4.10) can be fulfilled and by factorization we get an unitary \(S\)-matrix on the quotient (“physical”) space. This is due to the fact that the sesquilinear form induces on this quotient space a true scalar product according to lemma 3.22.
4.2 The Derivation of the Yang-Mills Lagrangian

In this subsection we prove two theorems which characterise completely the Yang-Mills interaction of gluons. We assume the summation convention of the dummy indices \(a, b, \ldots\).

**Theorem 4.1** Let us consider the operator

\[
T_1(g) = \int_{\mathbb{R}^4} dx \; g(x) T_1(x)
\]  

(4.2.1)

defined on \(H_{YM}^{\text{gr}}\) with \(T_1\) a Lorentz-invariant Wick polynomial in \(A_\mu\), \(u\) and \(\bar{u}\) verifying also \(\omega(T_1) \leq 4\). Then \(T_1(g)\) can induce an well defined non-trivial \(S\)-matrix, in the adiabatic limit, if and only if it has the following form:

\[
T_1(g) = \int_{\mathbb{R}^4} dx \; g(x) \{ f_{abc} : A_{a\mu}(x) A_{b\nu}(x) \partial^\nu A_\mu^b(x) : - : A_\mu^a(x) u_b(x) \partial_\mu \bar{u}_c(x) :, - 2 : \bar{u}_a(x) u_b(x) : \}.
\]  

(4.2.2)

Here \(f_{abc}\) \((h_{ab})\) are completely antisymmetric (symmetric) constants.

**Proof:** (i) If we take into account Lorentz invariance, the power counting condition from the statement and the restriction of non-triviality of theorem 3.32 we end up with the following linear independent possibilities:

- of degree 2:
  \[T^{(1)} = h_{ab}^{(1)} : A_{a\mu}(x) A_b^\mu(x) :, \quad T^{(2)} = h_{ab}^{(2)} : \bar{u}_a(x) u_b(x) :\]

- of degree 3: none

- of degree 4:

\[
\begin{align*}
T^{(1)} &= f_{abc}^{(1)} : A_{a\mu}(x) A_{b\nu}(x) \partial^\nu A_\mu^c(x) : \quad T^{(2)} = f_{abc}^{(2)} : A_\mu^a(x) u_b(x) \partial_\mu \bar{u}_c(x) : \\
T^{(3)} &= f_{abc}^{(3)} : A_\mu^a(x) \partial_\mu u_b(x) \bar{u}_c(x) : \\
T^{(4)} &= f_{abc}^{(4)} : \partial_\mu A_\mu^a(x) u_b(x) \bar{u}_c(x) : \\
T^{(5)} &= f_{abc}^{(5)} : A_{a\mu}(x) A_b^\mu(x) \partial_\nu A_\nu^c(x) : \\
T^{(6)} &= g_{abcd}^{(1)} : A_{a\mu}(x) A_b^\mu(x) A_c^\nu(x) A_d^\sigma(x) : \\
T^{(7)} &= g_{abcd}^{(2)} : u_a(x) A_b^\mu(x) \bar{u}_c(x) \bar{u}_d(x) : \\
T^{(8)} &= g_{abcd}^{(3)} : u_a(x) u_b(x) \bar{u}_c(x) \bar{u}_d(x) : \\
T^{(9)} &= g_{abcd}^{(4)} : \epsilon_{\mu\nu\rho\sigma} A_\mu^a(x) A_\nu^b(x) A_\rho^c(x) A_\sigma^d(x) : \\
T^{(10)} &= g_{ab}^{(1)} : \partial_\mu A_\mu^a(x) \partial_\nu A_\nu^b(x) : \\
T^{(11)} &= g_{ab}^{(2)} : \partial_\mu A_\mu^a(x) \partial_\nu A_\nu^b(x) : \\
T^{(12)} &= g_{ab}^{(3)} : \partial_\mu A_\mu^a(x) \partial_\nu A_\nu^b(x) : \\
T^{(13)} &= g_{ab}^{(4)} : A_\mu^a(x) \partial_\mu A_\nu^b(x) : \\
T^{(14)} &= g_{ab}^{(5)} : \epsilon_{\mu\nu\rho\sigma} F_\mu^a(x) F_\nu^b(x) : \\
T^{(15)} &= g_{ab}^{(6)} : \partial_\mu u_a(x) \partial_\nu \bar{u}_b(x) :
\end{align*}
\]

(4.2.4)

Without losing generality we can impose the following symmetry restrictions on the constants from the preceding list:

\[
\begin{align*}
h_{ab}^{(1)} &= h_{ba}^{(1)} \quad g_{ab}^{(1)} = g_{ba}^{(1)} \quad g_{abcd}^{(1)} = g_{cdab}^{(1)} \quad g_{abcd}^{(2)} = g_{cdab}^{(2)} \\
g_{abcd}^{(3)} &= -g_{bacd}^{(3)} \quad g_{abcd}^{(3)} = -g_{bacd}^{(3)} \quad g_{ab}^{(i)} = g_{ba}^{(i)} \quad i = 1, 2, 3, 5
\end{align*}
\]  

(4.2.5)
and one can suppose that \( g_{abcd}^{(4)} \) are completely antisymmetric in all indices.

(ii) By integration over \( x \) some of the linear independence is lost in the adiabatic limit. Namely:
- One can eliminate \( T^{(3)} \) by redefining the constants \( f_{abc}^{(2)} \) and \( f_{abc}^{(4)} \);
- One can eliminate \( T^{(5)} \) by redefining the constants \( f_{abc}^{(1)} \);
- One can eliminate \( T^{(12)} \) and \( T^{(13)} \) by redefining the constants \( g_{ab}^{(2)} \);
- One can eliminate \( T^{(10)} \) and \( T^{(15)} \) using the equation of motion \((3.2.9)\) and \((3.2.10)\);
- Finally \( T^{(14)} \) is null in the adiabatic limit.

(iii) Some of the remaining expressions are of the form \( dQ O \) so they do not count, according to lemma \((3.31)\). Namely,
\[
T^{(11)} = id_Q \left( g_{ab}^{(2)} : \partial_\mu A_\mu \tilde{u} \right)
\]
and if we define
\[
O \equiv g_{abc} : \tilde{u}_a u_b \tilde{u}^c :
\]
with the constants \( g_{abc} \) antisymmetric in the indices \( a \) and \( c \), then we have
\[
d_Q O = 2i g_{abc} : \partial_\mu A_\mu u_b \tilde{u}^c :
\]
so, it follows that we can choose
\[
f_{abc}^{(4)} = f_{cba}^{(4)}.
\] (4.2.6)

(iv) As a conclusion, we can keep in \( T_1 \) only the expressions \( T^{(1')} \), \( T^{(2')} \), \( T^{(1)} \), \( T^{(2)} \), \( T^{(4)} \) and \( T^{(6)} - T^{(9)} \) with the symmetry properties \((1.2.5)\) and \((4.2.6)\).

We compute now the expression \( d_Q T_1 \); we find:
\[
d_Q T_1 = i \partial_\mu \left[ 2h_{ab}^{(1)} : A_\mu u_b : \right.
\]
\[
+ \left( f_{abc}^{(1)} - f_{cba}^{(1)} \right) : \partial_\mu A_\mu \partial_\nu A_\nu : + f_{bac}^{(1)} : u_a A_\nu \partial_\mu A_\nu :
\]
\[
+ f_{ca}^{(1)} : \partial_\mu u_a A_\mu A_\nu : - f_{c}^{(1)} : u_a \partial_\nu A_\nu A_\nu : ]
\]
\[
- i f_{abc}^{(1)} : \partial_\mu A_\mu u_b : + i f_{cba}^{(1)} : u_a \partial_\mu A_\nu A_\nu :
\]
\[
+ i f_{abc}^{(1)} : u_a A_\nu \partial_\mu A_\nu : + i f_{cba}^{(1)} : \partial_\mu A_\nu A_\mu u_b : + i f_{abc}^{(1)} : \partial_\mu A_\nu A_\nu u_b :
\]
\[
- i f_{abc}^{(1)} : \partial_\mu A_\mu A_\mu u_c : - i f_{cba}^{(1)} : \partial_\nu A_\mu A_\mu u_c : + 4i g_{abcd}^{(1)} : \partial_\mu u_a A_\mu A_\nu A_\nu :
\]
\[
+ i g_{abcd}^{(2)} : \partial_\mu A_\mu \partial_\nu A_\nu u_c : + i g_{abcd}^{(3)} : A_\mu A_\nu A_\mu u_c : - 2i g_{abcd}^{(3)} : u_a u_b \partial_\mu A_\mu \tilde{u}^c :
\]
\[
- 4i g_{abcd}^{(4)} : \partial_\mu u_a A_\mu \tilde{u}^c A_\nu A_\nu :
\] (4.2.7)

Using lemma \((3.30)\) we impose the condition from the statement:
\[
\lim_{\epsilon \to 0} d_Q \int_{\mathbb{R}^4} dx \ g_c(x) \left. T_1(x) \right|_{Ker(Q)} = 0.
\] (4.2.8)

The divergence gives no contribution and the other terms can be computed on vectors from \( \mathcal{H}' \) according to the proposition \((3.20)\). In this way we see that we get independent conditions from each term i.e. we have:
\[
g_{abcd}^{(i)} = 0, \quad i = 1, 2, 3, 4
\] (4.2.9)
\[ 2h^{(1)}_{ab} + h^{(2)}_{ab} = 0 \]  

(4.2.10)

and

\[
\begin{align*}
&f^{(1)}_{cba} - f^{(1)}_{abc} = -(b \leftrightarrow c) \\
f^{(1)}_{abc} - f^{(1)}_{bac} + f^{(2)}_{cba} = 0 \\
f^{(1)}_{abc} + f^{(4)}_{acb} = -(a \leftrightarrow b) \\
f^{(1)}_{acb} = -(a \leftrightarrow b).
\end{align*}
\]  

(4.2.11)

We exploit completely the last system of equations. From the last relation of the system we see that the expression \( f^{(1)}_{abc} \) is antisymmetric in \( a \) and \( c \). In this case, the first relation of the system gives us the antisymmetry in \( b \) and \( c \); so the expressions \( f^{(1)}_{abc} \) are completely antisymmetric in all indices. Then the second relation of the system leads to

\[-f^{(1)}_{cba} + f^{(2)}_{cba} = 0.\]

At last, we use (4.2.6) in the third relation of the system and get that

\[ f^{(4)}_{abc} = 0. \]

The expression from the statement have been obtained.

(vi) It remains to prove that the expression from the statement cannot be of the type \( d_Q O \) and this can be done without problems. 

\[ \blacksquare \]

**Remark 4.2** The second contribution in the generic expression for \( T_1 \) just obtained could be left out according to the argument presented at the end of subsection 2.3. We will prefer to keep this term in the following and show that it must be null using only arguments of gauge invariance.

**Corollary 4.3** In the condition of the preceding theorem, one has:

\[ d_Q T_1 = i\partial_{\mu} K^\mu_1 \]  

(4.2.12)

where:

\[ K^\mu_1 = 2h_{ab} : A^\mu_a u_b : + f_{abc} : u_a A_{b} \Gamma^\nu_c \mu : - \frac{1}{2} : u_a u_b \partial^\mu \bar{u}_c : \]  

(4.2.13)

Moreover, we have:

**Proposition 4.4** The expression \( T_1 \) from the preceding theorem verifies the unitarity condition

\[ T_1(x)^\dagger = T_1(x) \]

if and only if the constants \( f_{abc} \) and \( h_{ab} \) have real values.
The proof is very simple and relies on the relations \((4.1.9)\).
Finally we have

**Proposition 4.5** The expression \(T_1\) determined in the preceding theorem verifies the causality condition:

\[ [T_1(x), T_1(y)] = 0, \quad \forall x, y \in \mathbb{R}^4 \quad \text{s.t.} \quad (x - y)^2 < 0. \]

**Proof:** One must determine the commutator appearing in the lefthand side. After a tedious computation one gets:

\[
D_2(x, y) = -f_{\mu\nu} f_{\sigma\tau} \partial_{\sigma\tau} D_0(x - y) \cdot A_{\alpha\beta}(x) F_{\mu\nu}(y) D_0(x - y) + \frac{1}{2} f_{\alpha\beta\gamma} f_{\mu\nu} \partial_{\alpha\beta} D_0(x - y) \cdot A_{\mu\nu}(x) F_{\alpha\beta}(y) D_0(x - y) = -f_{\mu\nu} f_{\sigma\tau} \partial_{\sigma\tau} D_0(x - y) \cdot A_{\alpha\beta}(x) F_{\mu\nu}(y) D_0(x - y)
\]

\[
-2 f_{\mu\nu} f_{\alpha\beta} h_{abc} h_{\mu\nu} \partial_{\alpha\beta} D_0(x - y) \cdot A_{\mu\nu}(x) F_{\alpha\beta}(y) D_0(x - y) = -f_{\mu\nu} f_{\sigma\tau} \partial_{\sigma\tau} D_0(x - y) \cdot A_{\alpha\beta}(x) F_{\mu\nu}(y) D_0(x - y)
\]

where

\[
D_0(x - y) \equiv D_0^{(+)}(x - y)^2 - D_0^{(-)}(x - y)^2, \quad D_1(x - y) \equiv D_1^{(+)}(x - y)^2 + D_1^{(-)}(x - y)^2\]

are well defined distributions with causal support and

\[
D_{0,\mu\nu}(x - y) \equiv \frac{1}{6} (\partial_{\mu} \partial_{\nu} - \square g_{\mu\nu}) D_0^2(x - y).
\]

The preceding expression show that we have causality in the first order. ■
We note that the following identity is valid:

$$\partial^\mu D_{0,\mu\nu}(x) = 0.$$  \hspace{1cm} (4.2.17)

We go now to the second order of the perturbation theory following closely [15]. We split causally the commutator (4.2.14) according to the prescription given (2.4.25) and (2.4.28) and include the most general finite arbitrariness of the decomposition. We can prove that a causal splitting which preserves Lorentz covariance of the distributions $D_0(x), D_0^2(x), D_0^3(x), D_{0,\mu\nu}$ exists. Moreover, the splitting can be chosen such that it will preserve the property (4.2.17). The result is contained in:

**Proposition 4.6** The generic for of the distribution $T_2$ is

$$T_2(x, y) = - f_{cab} f_{cde} D_{0,F}(x - y) : A_{\alpha\nu}(x) F_{\mu}^{\nu}(x) A_{\beta\nu}(y) F_{\epsilon\mu}(y) : + : u_a(x) \partial_\mu \tilde{u}_b(x) u_d(y) \partial^\mu \tilde{u}_e(y) : - A_{\alpha\nu}(x) F_{\mu}^{\nu}(x) u_d(y) \partial_\mu \tilde{u}_e(y) :
$$

$$- : u_d(x) \partial_\mu \tilde{u}_e(y) A_{\alpha\nu}(y) F_{\mu}^{\nu}(y) : + \frac{1}{2} f_{abc} f_{dce} D_{0,F}^2(x - y) : F_{d}^{\mu\nu}(x) F_{d\nu\mu}(y) :
$$

$$+ f_{cab} f_{cde} \frac{\partial}{\partial x_\rho} D_{0,F}(x - y) : A_{\alpha\nu}(x) F_{\mu}^{\nu}(x) A_{\beta\rho}(y) A_{\gamma\nu}(y) : - : A_{\beta\rho}(x) A_{\gamma\nu}(x) A_{\alpha\nu}(y) F_{\mu}^{\nu}(y) :
$$

$$+ : A_{\alpha\nu}(x) A_{\beta\rho}(x) u_d(y) \partial^\mu \tilde{u}_e(y) : - : \partial^\mu \tilde{u}_e(y) A_{\alpha\nu}(y) A_{\beta\rho}(y) :
$$

$$+ : A_{\alpha\nu}(x) \partial_\mu \tilde{u}_b(x) A_{\beta\rho}(y) u_e(y) : - : A_{\beta\rho}(x) u_e(y) A_{\alpha\nu}(y) \partial_\mu \tilde{u}_b(y) :
$$

$$- f_{cab} f_{cde} \frac{\partial^2}{\partial x_\nu \partial x_\rho} D_{0,F}(x - y) + c_1 g^{\rho\sigma} \delta(x - y) : A_{\alpha\nu}(x) A_{\beta\nu}(x) A_{\gamma\rho}(y) A_{\delta\sigma}(y) :
$$

$$- 2 f_{abc} f_{dce} [D_{0,F}^2(x - y) + c_2 g_{\mu\nu} \delta(x - y) ] : A_{\alpha\nu}(x) A_{\beta\nu}(x) A_{\gamma\rho}(y) A_{\delta\sigma}(y) :
$$

$$+ \frac{1}{2} f_{abc} f_{dce} \frac{\partial}{\partial x_\rho} D_{0,F}^2(x - y) : A_{\alpha\nu}(x) F_{d}^{\mu\nu}(y) :
$$

$$- : F_{d}^{\mu\nu}(x) A_{\alpha\nu}(y) : - : u_a(x) \partial_\rho \tilde{u}_d(y) : - : \partial_\rho \tilde{u}_d(y) u_a(y) : - \frac{1}{3} f_{abc} f_{abe} D_{0,F}^3(x - y) 1
$$

$$- 4 h_{ab} h_{cd} [D_{0,F}(x - y) + c_3 \delta(x - y)] : A_{\alpha\nu}(x) A_{\beta\nu}(y) : + : u_a(x) \tilde{u}_c(y) : - : \tilde{u}_a(x) u_c(y) : + 4 h_{ab} h_{cd} D_{0,F}^3(x - y) 1 + c_3 \delta(x - y) 1
$$

$$- 2 f_{cab} h_{cd} [D_0(x - y) + c_4 \delta(x - y)] : A_{\alpha\nu}(x) F_{\beta}^{\nu\mu}(x) A_{\delta\mu}(y) : + : A_{\delta\mu}(x) A_{\alpha\nu}(y) F_{\beta}^{\nu\mu}(y) :
$$

$$- : u_a(x) \partial_\mu \tilde{u}_b(x) A_{\beta\nu}(y) : - : A_{\beta\nu}(x) u_a(y) \partial_\mu \tilde{u}_b(y) :
$$

$$- : A_{\beta\nu}(x) \partial_\mu \tilde{u}_b(y) u_d(y) : - : u_d(x) A_{\beta\nu}(y) \partial_\mu \tilde{u}_b(y) :
$$

$$- 2 f_{cab} h_{cd} \frac{\partial}{\partial x_\nu} [D_0(x - y) + c_5 \delta(x - y)] : A_{\alpha\nu}(x) A_{\beta\nu}(x) A_{\delta\mu}(y) : + : A_{\beta\nu}(x) A_{\delta\mu}(y) A_{\alpha\nu}(y) :
$$

$$+ : A_{\delta\mu}(x) u_b(x) \tilde{u}_d(y) : - : \tilde{u}_d(x) A_{\delta\mu}(y) u_b(y) :
$$

(4.2.18)

Here

$$D_{0,F}(x) \equiv D_0^{ret}(x) + D_0^{(+)}(x),$$

(4.2.19)

is the usual Feynman propagator and

$$D_{0,F}^2(x) \equiv D_0^{ret,2}(x) + D_0^{(+)}(x)^2, \quad D_{0,F}^3(x) \equiv D_0^{ret,3}(x) + D_0^{(+)}(x)^3$$

(4.2.20)

are some generalisation of it.
Proof: The proof is elementary. The only things to be noticed is that if one includes all the finite renormalisations compatible with power counting and Lorentz covariance, one will get beside the contributions already appearing into the formula above, some additional terms which however, can be proven to be null in the adiabatic limit. More precisely:

• in the second term one could also include the expression: \( \delta(x - y) : F^\nu_\mu(x)F_{d\nu\mu}(y) : \) but this will produce, in the adiabatic limit, terms of the type \( T^{(10)} \) and \( T^{(12)} \) from (4.2.4) which can be discarded on the same grounds as there;

• in the fifth term one could include \( c_6g_{\mu\nu}\Box\delta(x - y) + c_7\partial_\mu\partial_\nu\delta(x - y) \right] : A^\nu_\delta(x)A^\mu_\alpha(y) : \) but, in the adiabatic limit, they will lead to expressions of the type \( T^{(10-12)} \) which can be discarded;

• in the sixth term one could include the finite renormalisation \( \partial_x^\mu \rho \delta(x - y) \right] \) but in the adiabatic limit we get contributions of the type \( T^{(10-13)} \) which can be discarded;

• finally, one could add at will local terms of the type \( P(\Box)\delta(x - y) \times 1 \) with \( P \) a polynomial of degree \( > 1 \), but these terms are null in the adiabatic limit. ■

Of course, we do not have the guarantee that the expression (4.2.18) leads to a well-defined operator on the factor space \( H^r_{YM} \); in fact, one can show that this can happen if and only if some severe restrictions are placed on the constants appearing in the expression of the interaction Lagrangian [2]. For the sake of completeness we give below this result.

Theorem 4.7 The expression \( T_2 \) appearing in the preceding proposition leads, in the adiabatic limit, to an well defined operator on \( H^r_{YM} \) if and only if: (a) the constants \( f_{abc} \) verify the Jacobi identities:

\[
f_{abc}f_{dec} + f_{bdc}f_{aec} + f_{dac}f_{bec} = 0;
\]

(b) the second order terms are not present i.e.

\[
h_{ab} = 0;
\]

(c) the constants \( c_i, \quad i = 1, \ldots, 4 \) are given by

\[
c_1 = \frac{i}{2}, \quad c_2 = 0.
\]

Proof: (i) The “brute force” computation can be avoided using the following trick [13]. We have from the definition (2.4.23) of the distribution \( D_2 \), the Leibnitz formula (3.3.9) and (1.2.12):

\[
d_QD_2(x, y) = [d_QT_1(x), T_1(y)] + [T_1(x), d_QT_1(y)] = \\
i\frac{\partial}{\partial x^\mu}[K_1^\mu(x), T_1(y)] - i\frac{\partial}{\partial y^\mu}[K_1^\mu(y), T_1(x)];
\]

this formula will be used to compute the left hand side. It is elementary to see that the distribution \( d_QD_2(x, y) \) still has a causal support so it can be split causally.
On the other hand, the causal splitting (2.4.25) implies
\[
d_QD_2(x, y) = d QR_2(x, y) - d QA_2(x, y).
\] (4.2.25)

So, if we split causally the right hand side of the formula (4.2.24) without ruining Lorentz covariance and power counting, we get a posteriori valid expressions for the distributions \(d QR_2(x, y)\) and \(d QA_2(x, y)\). Of course, in this way we do not get the most general expression for these distribution because we have the possibility of finite normalisations. But the arbitrariness for \(d QR_2(x, y)\) is exactly the same as the arbitrariness for \(d QT_2(x, y)\) which can be read from (4.2.18). So, in this way, we get in a roundabout way, the most general expression for the distributions \(d QR_2(x, y)\) and \(d QA_2(x, y)\).

Finally, we see that (4.1.11) for \(n = 2\) is equivalent to:
\[
\lim_{\epsilon \to 0} dQR_2(x, y)|_{\ker(Q)} = 0;
\] (4.2.26)

(one must use, of course the fact that we already have (4.1.11) for \(n = 1\)). Imposing this condition on the expression determined in the way outlined above will lead to the conditions from the statement.

(ii) A long, but straightforward computation gives us the following expression for the first commutator appearing in (4.2.24):
\[
[K^\nu_1(x, T_1(y)] = D_0(x - y) T^\mu(x, y) + \frac{\partial}{\partial x^\nu} D_0(x - y) T^{\mu\nu}(x, y)
+ \frac{\partial}{\partial x^\nu} D_{0,2}(x - y) \tilde{T}^{\mu\nu}(x, y) + \frac{\partial}{\partial x^\nu} \frac{\partial}{\partial x^\rho} D_0(x - y) T^{\mu\nu\rho}(x, y) + D_0^{\mu\nu}(x - y) \tilde{T}_\nu(x, y)
+ \frac{\partial}{\partial x^\mu} D_0(x - y) T(x, y) + \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\rho} D_0(x - y) \tilde{T}_\rho(x, y)
\] (4.2.27)

where we have:
\[
T^\mu(x, y) \equiv f_{cab} f_{cd} : u_a(x) F_b^{\mu\nu}(x) A_c^\nu(y) F_{e\nu}(y) : + : u_a(x) F_b^{\mu\nu}(x) u_d(y) \partial_{\nu} \tilde{u}_e(y) : [4.2.28]
\]
\[
T^{\mu\nu}(x, y) \equiv - f_{cab} f_{cd} : u_a(x) A_b^\mu(x) A_{d\nu}(y) F_{e\nu}(y) : - : A_{a\nu}(x) F_b^{\mu\nu}(x) A^\nu_c(y) u_e(y) :
- : u_a(x) A_{a\nu}(y) F_b^{\mu\nu}(y) : + f_{cab} h_{cd} : u_d(x) u_a(y) \partial^{\nu} \tilde{u}_b(y) :
- 2(h^2)_{ab} [2 : u_a(x) A_b^\mu(y) : + : A_c^\mu(x) u_b(y) : ]
\] (4.2.29)
\[
\tilde{T}^{\mu\nu}(x, y) \equiv \frac{1}{2} f_{cab} f_{cde} : u_a(x) F_{b\mu\nu}(y) : + : F_{d\mu\nu}(x) u_a(y) :
\] (4.2.30)
\[
T^{\mu\nu\rho}(x, y) \equiv f_{cab} f_{cde} : u_a(x) A_b^\mu(x) A_c^\nu(y) A_d^\rho(y) :
\] (4.2.31)
\[
\tilde{T}_\nu(x, y) \equiv f_{cab} f_{cde} : u_a(x) A_{d\nu}(y) :
\] (4.2.32)
that a possible splitting is appearing in the formula (4.2.27) leads to
\[
\partial \quad \text{which one integrates by parts. Finally, we obtain (after some combin atorial rearrangements)}
\]
\[
\delta \quad \text{difference appears in the last two contributions. We have, for insta nce for the distribution:}
\]
\[
\partial x \quad \text{works for the next three contributions appearing into the right ha nd side of (4.2.27). A major}
\]
\[
\partial x \quad \text{and}
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where the explicit expression of the distribution $L_\mu(x,y)$ is not important.

(iv) The decomposition of the distribution $\frac{\partial}{\partial y^\mu}[K_1^\mu(y), T_1(x)]$ can be obtained from the previous expression if we notice that in the causal splitting $D(y-x) = -D^{adv}(y-x) + D^{ret}(y-x)$ the first term plays the role of the retarded part. So, one should make the substitution $D^{ret}(x-y) \rightarrow -D^{adv}(y-x)$ in the previous expression. Finally, one gets a possible causal splitting of the distribution $d_Q R_2(x,y)$ from (I.2.24):

$$d_Q R_2(x,y) = i \frac{\partial}{\partial x_\mu} R_\mu(x,y) - \delta(x-y) \left[ (2 f_{cab} f_{cd} - f_{cab} f_{cd}) : u_a(x) F_{b\nu}(x) A^\nu_d(x) A^\rho_e(x) : 
- f_{cab} f_{cd} : \partial_\rho u_a(x) A_{b\nu}(x) A^\nu_d(x) A^\rho_e(x) : + (2 f_{cad} f_{be} - f_{cab} f_{cd}) : u_a(x) u_b(x) A^\nu_d(x) \partial_\nu \tilde{u}_e(x) : 
- 4 f_{cab} h_{cd} : u_a(x) A_{b\nu}(x) A^\nu_d(x) : - 2 f_{cab} h_{cd} : u_a(x) u_b(x) \tilde{u}_d(x) : \right]$$

The arbitrariness of the decomposition can be read from (I.2.18), so the previous formula gets corrected to

$$d_Q R_2(x,y) = i \frac{\partial}{\partial x_\mu} R_\mu(x,y) - \delta(x-y) \left[ (2 f_{cab} f_{cd} - f_{cab} f_{cd}) : u_a(x) F_{b\nu}(x) A^\nu_d(x) A^\rho_e(x) : 
- f_{cab} f_{cd} (1 - 2i c_1) : \partial_\rho u_a(x) A_{b\nu}(x) A^\nu_d(x) A^\rho_e(x) : + (2 f_{cad} f_{be} - f_{cab} f_{cd}) : u_a(x) u_b(x) A^\nu_d(x) \partial_\nu \tilde{u}_e(x) : 
- 4 f_{cab} h_{cd} : u_a(x) A_{b\nu}(x) A^\nu_d(x) : - 2 f_{cab} h_{cd} : u_a(x) u_b(x) \tilde{u}_d(x) : + c_{ad} : \partial_\mu u_a(x) A^\mu_d(x) : + c_4 P_4(x) + c_5 P_5(x) \right]$$

(4.2.36)

where $c_{ad} \equiv c_2 f_{cab} f_{cde} + c_3 (h^2)_{ad}$ and the polynomial $P_4(x)$ (resp. $P_5(x)$) contain Wick monomials of the type $A\partial A \partial u :$, $AA \partial \partial u :$ (resp. $u A \partial \partial A :$, $u \partial \partial \partial u :$) which do not appear in the other contributions.

Now we impose the condition (I.2.20). It is not very hard to see that one obtains:

$$\lim_{\epsilon \searrow 0} \int_{\mathbb{R}^4} dx g_\epsilon(x)^2 \left[ (2 f_{cab} f_{cde} - f_{cab} f_{cde}) : u_a(x) F_{b\nu}(x) A^\nu_d(x) A^\rho_e(x) : 
- f_{cab} f_{cde} (1 - 2i c_1) : \partial_\rho u_a(x) A_{b\nu}(x) A^\nu_d(x) A^\rho_e(x) : 
+ (2 f_{cad} f_{be} - f_{cab} f_{cd}) : u_a(x) u_b(x) A^\nu_d(x) \partial_\nu \tilde{u}_e(x) : 
- 4 f_{cab} h_{cd} : u_a(x) A_{b\nu}(x) A^\nu_d(x) : - 2 f_{cab} h_{cd} : u_a(x) u_b(x) \tilde{u}_d(x) : 
+ c_{ad} : \partial_\mu u_a(x) A^\mu_d(x) : + c_4 P_4(x) + c_5 P_5(x) \right] \vert_{Ker(Q)} = 0.$$ 

From the third term we get

$$2 f_{cad} f_{cbe} - f_{cab} f_{cde} = (a \leftrightarrow b)$$

which is equivalent to the Jacobi identity (I.2.21) from the statement. But, in this case the first terms is also null. We get from the rest $c_1 = \frac{i}{2}$, $c_{ad} = 0$ and $f_{cab} h_{cd} = 0$. Because the constants $f_{abc}$ correspond to a semi-simple Lie algebra, the Killing-Cartan form is strictly positively defined and we get from the last condition that in fact we have $h_{ab} = 0$. In this case the constants $c_3, c_4, c_5$ should not be included in the expression of $T_2$ and we also have $c_2 = 0$. 

■
Corollary 4.8 The chronological product $T_2$ is given by the following expression:

$$T_2(x, y) = \frac{i}{2} f_{abc} f_{cde} D_0(x - y) : A_{\alpha\nu} F_{\alpha\beta}(x) A_{\beta\gamma}(x) A_{\gamma\delta}(x) :$$

$$- f_{abc} f_{cde} D_0(x - y) [: A_{\alpha\nu} F_{\alpha\beta}(x) A_{\beta\gamma}(y) F_{\gamma\mu}(y) + : u_{\alpha}(x) \partial_{\nu} u_{\mu}(x) + : u_{\beta}(y) \partial_{\mu} u_{\nu}(y) - : A_{\alpha\nu} F_{\alpha\beta}(x) u_{\beta}(y) \partial_{\mu} u_{\nu}(y) - : u_{\beta}(y) \partial_{\nu} u_{\alpha}(x) A_{\alpha\gamma}(y) F_{\gamma\mu}(y) : + : A_{\gamma\nu}(x) F_{\gamma\mu}(x) u_{\gamma}(y) \partial_{\mu} u_{\nu}(y) - : A_{\gamma\nu}(x) \partial_{\mu} u_{\gamma}(y) A_{\mu\beta}(y) A_{\beta\nu}(y) : + : A_{\alpha\nu}(x) A_{\beta\mu}(y) u_{\mu}(y) \partial_{\nu} u_{\beta}(y) + : \partial_{\nu} u_{\alpha}(x) A_{\alpha\mu}(y) \partial_{\mu} u_{\beta}(y) : - f_{abc} f_{cde} \frac{\partial}{\partial x_{\rho}} D_0(x - y) : A_{\alpha\nu} A_{\beta\gamma}(x) A_{\gamma\delta}(y) A_{\delta\mu}(y) :$$

$$- 2 f_{abc} f_{cde} D_0(x - y) : A_{\alpha\nu}(x) A_{\beta\gamma}(y) : + \frac{i}{2} f_{abc} f_{cde} \frac{\partial}{\partial x_{\rho}} D_0(x - y) [: A_{\alpha\nu}(x) F_{\beta\gamma}(y) : - : F_{\beta\gamma}(x) A_{\alpha\nu}(y) : - : u_{\alpha}(x) \partial_{\nu} u_{\gamma}(y) - : \partial_{\gamma} u_{\alpha}(x) u_{\nu}(y) : - \frac{1}{3} f_{abc} f_{cde} \Box D_0(x - y) 1 + c_0 \delta(x - y) 1. \quad (4.2.37)$$

Here the constants $f_{abc}$ are completely antisymmetric and verify the Jacobi identities and $c_0$ is arbitrary. The condition of unitarity can be satisfied if and only if $c_0$ is purely imaginary.

**Proof:** The first statement follows without any problems from the expression (4.2.18) if we take into account the preceding theorem. The second statement amounts to:

$$T_2^+(x, y) + T_2(x, y) = T_1(x) T_1(y) + T_1(y) T_1(x)$$

which easily follows from:

$$D_0(x) + D_0(x) = D_0^+(x) + D_0^-(x) \quad (4.2.38)$$

and similar relations for $D_{0,F}, D_{0,F}^\alpha$ and $D_{0,F,\mu\nu}$. ■
4.3 Yang-Mills Fields coupled to Matter

In this subsection we study the possibility of coupling Yang-Mills fields to “matter”. As previously, we suppose that we are given the Hilbert space of “matter” \( \mathcal{H}_{\text{matter}} \) which should also be a Fock space. Then the coupled system is described in the tensor product Hilbert space \( \mathcal{F}_{\text{YM}} \otimes \mathcal{H}_{\text{matter}} \). One can describe this Fock space, as in the end of subsection 3.1, by considering \( \mathcal{H}_{\text{YM}}^{gh,r} = \mathcal{H}_{\text{YM}} \otimes \mathcal{H}_{\text{matter}} \) with the corresponding supercharge operator and forming the quotient \( \text{Ker}(Q)/\text{Im}(Q) \).

First, we have to generalise theorem 4.1:

**Theorem 4.9** Let us consider the operator

\[
T_1(g) = \int_{\mathbb{R}^4} dx \ g(x) T_1(x)
\]

defined on \( \mathcal{H}_{\text{YM}}^{gh,r} \) with \( T_1 \) a Lorentz-invariant Wick polynomial in \( A_\mu \), \( u \), \( \bar{u} \) and the matter fields, verifying also \( \omega(T_1) \leq 4 \). Then \( T_1(g) \) can induce an well defined non-trivial S-matrix, in the adiabatic limit, if and only if it has the following form:

\[
T_1(g) = \int_{\mathbb{R}^4} dx \ g(x) \{ f_{abc} [ A_{a\mu}(x) A_{b\nu}(x) \partial^\rho A^{\mu}_{c}(x) : - : A^\mu_{a}(x) u_b(x) \partial_\mu \bar{u}_c(x) : ] + h_{ab} [ A_{a\mu}(x) A^{\mu}_{b}(x) : - 2 : \bar{u}_a(x) u_b(x) : ] + A^\mu_{a}(x) j_{a\mu}(x) : + T_{1,\text{matter}}(x) \}. \tag{4.3.2}
\]

Here \( f_{abc} \) (\( h_{ab} \)) are completely antisymmetric (symmetric) constants, \( j_{a\mu} \) is a conserved current build only from the matter fields with \( \omega(j_{a\mu}) = 1, 2, 3 \) and \( T_{1,\text{matter}} \) contains only the matter fields.

The proof follows the lines of theorem 4.1 only we have to add to the list of possible building blocks of \( T_1 \), beside the list \( T^{(1)} - T^{(15)} \), and \( T_{1,\text{matter}} \), combinations of the type

\[
T^{(16)} = A^\mu_a A^\nu_b A^\rho_c j^{abc}_{\mu\nu\rho} : \quad T^{(17)} = A^\mu_a A^\nu_j j^{ab}_{\mu\nu} : \quad T^{(18)} = A^\mu_a \partial^\rho A^\nu_{bj\rho} : \quad T^{(19)} = A^\mu_a j^{a}_{\mu} : \quad T^{(20)} = : u_a \bar{u}_b j^{ab} :
\]

with \( \omega(j^{abc}_{\mu\nu\rho}) = 1, \quad \omega(j^{ab}_{\mu\nu}) = 1, 2 \quad \omega(j^{ab}_{\mu\nu}) = 1, \quad \omega(j^a_{\mu}) = 1, 2, 3 \quad \omega(j^{ab}) = 1, 2 \). Moreover, we can impose the symmetry properties:

\[
j^{ab}_{\mu\nu} = j^{ba}_{\nu\mu}, \quad j^{ab}_{\mu\nu} = j^{ba}_{\nu\mu}.
\]

Proceeding as before we end up with the expression from the statement. Moreover, we have as before:

**Proposition 4.10** The expression for \( T_1 \) verifies the unitarity requirement if and only if we have:

\[
j^{a}_{\mu}(x)^\dagger = j^a_{\mu}(x) \tag{4.3.3}
\]

and verifies the causality condition if and only if:

\[
[j^{a}_{\mu}(x), j^b_{\nu}(x)] = 0, \quad (x - y)^2 < 0. \tag{4.3.4}
\]
Now we consider a special case of matter fields, namely a collection of free Dirac fields. This situation it is supposed to described quantum chromodynamics. First we have

**Proposition 4.11** Suppose that $\mathcal{H}_{\text{matter}}$ is the Fock space of an ensemble of Dirac fields $\psi_A$ of masses $m_A \geq 0, \ A = 1, \ldots, N$. Then the generic form of the current, verifying power counting, Lorentz covariance, unitarity and conservation is:

$$j_i^\mu(x) = :\bar{\psi}_A(x)(t_a)_{AB} \gamma^\mu \psi_B(x) : + :\bar{\psi}_A(x)(t'_a)_{AB} \gamma^\mu \gamma_5 \psi_B(x) :$$

(4.3.5)

where the numerical matrices $t_a, \ t'_a, \ a = 1, \ldots, r$ must be hermitian. Moreover, the first set of matrices can be exhibited into a block diagonal structure (eventually after a relabelling of the Dirac fields) and the masses corresponding to the same block must be equal. The second term is admissible if and only if all the masses are null. Finally, the expression of this current satisfy the causality condition from the preceding proposition.

**Proof:** The generic form for the current from the statement follows from power counting and Lorentz covariance. The hermiticity of the matrices is a consequence of the unitarity condition from the preceding proposition. Finally, imposing the conservation of this current we end up with the conditions:

$$(m_A - m_B)(t_a)_{AB} = 0, \ (m_A + m_B)(t'_a)_{AB} = 0, \ \forall A, B = 1, \ldots, N, \ \forall a = 1, \ldots, r$$

which lead to the conclusions from the statement. The verification of the causality property is by straightforward computation. ■

We will restrict ourselves in the following to the case of non-zero masses. It is clear from the block structure of the matrices $t_a$ that, in fact, we do not lose the generality of the analysis if we take only one block corresponding to a single mass $m$. Let us define some distributions with causal support which will be needed in the next proposition and which do appear in spinorial QED [43]:

$$S^{(\pm)}(x) \equiv (i\gamma \cdot \partial + m)D^{(\pm)}_m(x), \ S(x) \equiv S^{(+)}(x) + S^{(-)}(x) = (i\gamma \cdot \partial + m)D_m(x),$$

(4.3.6)

where $D_m(x)$ is the Pauli-Jordan function for arbitrary mass $m$ defined similarly to $D_0(x)$ in (3.1.3) but the integral is done over the hyperboloid of mass $X_m^\pm$. The causal splitting

$$D_m(x) = D^{\text{ret}}_m(x) - D^{\text{adv}}_m(x)$$

induces a similar splitting for the distribution $S(x)$ and in this way Feynman propagator can be obtained. We also have the distribution with causal support:

$$\Sigma(x) \equiv \gamma^\mu \left[ D^{(+)}(x)S^{(+)}(x) + D^{(+)}(-x)S^{(-)}(x) \right] \gamma^\mu$$

(4.3.7)

$$P^{(\pm)}_{\mu\nu}(x) \equiv \pm Tr \left[ \gamma_\mu S^{(-)}(\mp x)\gamma_\nu S^{(+)}(\pm x) \right], \ P_{\mu\nu}(x) \equiv P^{(+)}_{\mu\nu}(x) + P^{(-)}_{\mu\nu}(x)$$

(4.3.8)

and

$$V^{(\pm)}(x) \equiv \pm g^{\mu\nu}D_0^{(\pm)}(x)P^{(\pm)}_{\mu\nu}(x), \ V(x) \equiv V^{(+)}(x) + V^{(-)}(x).$$

(4.3.9)

All these distribution can be split causally and preserving Lorentz covariance and we will denote the corresponding retarded, advanced and Feynman distributions in an obvious way.

Then we have the following generalisation of the formula (4.2.18):
Proposition 4.12 Suppose that $\psi_A, A = 1, \ldots, N$ are Dirac fields of mass $m > 0$ such that the current is vectorial. We also suppose that there is no contribution $T_{1,matter}$ in the first order chronological product. Then, the generic form of the second order chronological product is:

$$T_2(x, y) = T_2^{YM}(x, y)$$

$$-f_{abc}D_{0,F}(x - y): A_{a\nu}(x)F_{b\mu}^{\nu}(x)j_{c\mu}(y) : - : u_a(x)\partial^\mu \bar{u}_b j_{c\mu}(y) : + (x \leftrightarrow y)$$

$$-f_{abc}\frac{\partial}{\partial x^\nu}D_{0,F}(x - y): A_{a\nu}^\mu(x)A_{b\mu}^\nu(x)j_{c\mu}(y) : - (x \leftrightarrow y)$$

$$-2h_{ab}D_{0,F}(x - y): A_{a\mu}(x)j_{b\mu}(y) : + (x \leftrightarrow y)$$

$$-D_{0,F}(x - y): \overline{\psi}_A(x)(t_a)_{AB}\gamma^\mu \psi_B(x)\overline{\psi}_C(y)(t_a)_{CD}\gamma^\mu \psi_D(y) :$$

$$\pm : \overline{\psi}_A(x)(t_a^2)_{AB}\Sigma_F(x - y)\psi_B(y) : + : \overline{\psi}_A(y)(t_a^2)_{AB}\Sigma_F(y - x)\psi_B(x) :$$

$$+ \delta(x - y): \overline{\psi}_A(x)M_{AB}\psi_B(x) : + : \overline{\psi}_A(x)\gamma_5 M_{AB}\psi_B(x) : + Tr(t_a^2)_{V}(x - y)1$$

$$+ : A_{a}^\mu(x)A_{b}^\nu(y) : \{ \overline{\psi}_A(x)(t_a t_b)_{AB}\gamma_\mu S_F(x - y)\gamma_\nu \psi_B(y) :$$

$$\pm : \overline{\psi}_A(y)(t_a t_b)_{AB}\gamma_\nu S_F(y - x)\gamma_\mu \psi_B(x) :$$

$$+ Tr(t_a t_b) [P_{F,\mu\nu}(x - y) + f_1 g_{\mu\nu} \delta(x - y) + f_2 \partial_\mu \partial_\nu \delta(x - y)1]. \quad (4.3.10)$$

Proof: The first step is, as before, to compute explicitly the commutator $D_2$. If we denote the pure Yang-Mills contribution (4.2.14) by $D_2^{YM}(x, y)$ then we have by elementary computations:

$$D_2(x, y) = D_2^{YM}(x, y)$$

$$-f_{abc}D_0(x - y): A_{a\nu}(x)F_{b\mu}^{\nu}(x)j_{c\mu}(y) : - : u_a(x)\partial^\mu \bar{u}_b j_{c\mu}(y) : + (x \leftrightarrow y)$$

$$-f_{abc}\frac{\partial}{\partial x^\nu}D_0(x - y): A_{a\nu}^\mu(x)A_{b\mu}^\nu(x)j_{c\mu}(y) : - (x \leftrightarrow y)$$

$$-2h_{ab}D_0(x - y): A_{a\mu}(x)j_{b\mu}(y) : + (x \leftrightarrow y)$$

$$-D_0(x - y): \overline{\psi}_A(x)(t_a)_{AB}\gamma^\mu \psi_B(x)\overline{\psi}_C(y)(t_a)_{CD}\gamma^\mu \psi_D(y) :$$

$$\pm : \overline{\psi}_A(x)(t_a^2)_{AB}\Sigma_F(x - y)\psi_B(y) : + : \overline{\psi}_A(y)(t_a^2)_{AB}\Sigma_F(y - x)\psi_B(x) : + Tr(t_a^2)_{V}(x - y)1$$

$$+ : A_{a}^\mu(x)A_{b}^\nu(y) : \{ \overline{\psi}_A(x)(t_a t_b)_{AB}\gamma_\mu S_F(x - y)\gamma_\nu \psi_B(y) :$$

$$\pm : \overline{\psi}_A(y)(t_a t_b)_{AB}\gamma_\nu S_F(y - x)\gamma_\mu \psi_B(x) :$$

$$+ Tr(t_a t_b) [P_{F,\mu\nu}(x - y) + f_1 g_{\mu\nu} \delta(x - y) + f_2 \partial_\mu \partial_\nu \delta(x - y)1].$$

Now one must proceed to the causal splitting of this distribution. The finite renormalisations can affect the sixth, the seventh and the last term of the expression from above. Lorentz covariance and power counting must be used now to fix the generic form of the finite renormalisation. For the last term, the structure of these contributions is clear. For the sixth term, we can add to $\Sigma_F$ the finite renormalisation $\delta(x - y): \overline{\psi}_A(x)[M_{AB}\gamma \cdot \partial + \gamma_5 \gamma \cdot \partial] \psi_B(x) :$ but if we use Dirac equation for the free Dirac fields $\psi_B(x)$ we end up with the expression from the statement. ■

Finally, we check if the expression just derived induces an well-defined operator on the physical space.

Theorem 4.13 In the conditions of the preceding proposition, the second order chronological product $T_2$ induces, in the adiabatic limit, an well-defined operator on the physical space $\mathcal{H}_{total}$.
if and only if, beside the conditions from theorem 4.7 we also have:

\[ [t_a, t_b] = i f_{abc} t_c, \quad \forall a, b = 1, \cdots, r \]  

(4.3.11)

and

\[ f_1 = f_2 = 0. \]  

(4.3.12)

**Proof:** As in the proof of theorem 4.7, we compute the commutator \([K_1^\mu(x), T_1(y)]\); for simplicity, we denote the pure Yang-Mills contribution given by the formula (4.2.27) in a suggestive way by \([K_1^\mu(x), T_1(y)]^{YM}\). Then we get by direct computation:

\[ [K_1^\mu(x), T_1(y)] = [K_1^\mu(x), T_1(y)]^{YM} \]

\[ -f_{abc} \frac{\partial}{\partial x^\nu} D_0(x-y) : j_a^\mu(x) A_0^\nu(x) u_c(y) : + : A_0^\nu(x) u_c(x) j_a^\mu(y) : \]

\[ -2h_{ab} D_0(x-y) : j_a^\mu(x) u_b(y) : - : u_b(x) j_a^\mu(y) : + f_{abc} D_0(x-y) : u_a(x) F_b^{\nu \mu}(x) j_c(y) : + \]

\[ f_{abc} \frac{\partial}{\partial x^\mu} D_0(x-y) : u_a(x) A_0^\mu(x) j_c(y) : - \cdot u_a(x) \overline{\psi}_A (t_b t_a)_{AB} \gamma_\rho S(y-x) \gamma^\mu \psi_B (x) A_0^\rho (x) : \]

\[ + : u_a(x) \overline{\psi}_A (t_a t_b)_{AB} \gamma_\mu S(y-x) \gamma_\rho \psi_B (y) A_0^\rho (y) : + Tr(t_a t_b) P^{\mu \rho} (x-y) : u_a(x) A_0^\rho (y) : \]

So, beside the causal decompositions analysed previously in theorem 4.7, we have to decompose distributions of the type:

\[ \frac{\partial}{\partial x^\mu} : T_\alpha(x) \gamma^\mu S(x-y) T_\beta(y) := \frac{\partial T_\alpha}{\partial x^\mu} (x) \gamma^\mu S(x-y) T_\beta(y) : + : T_\alpha(x) \gamma \cdot \partial S(x-y) T_\beta(y) : \]

\[ : \frac{\partial T_\alpha}{\partial x^\mu} (x) \gamma^\mu S(x-y) T_\beta(y) : - i m : T_\alpha(x) S(x-y) T_\beta(y) : \]

where in the last line we have used Dirac equation for the Pauli-Jordan distribution. A possible splitting is given by

\[ : \frac{\partial T_\alpha}{\partial x^\mu} (x) \gamma^\mu S^{ret}(x-y) T_\beta(y) : - i m : T_\alpha(x) S^{ret}(x-y) T_\beta(y) : \]

\[ \frac{\partial}{\partial x^\mu} [ T_\alpha(x) \gamma^\mu S^{ret}(x-y) T_\beta(y) : ] + i : T_\alpha(x) (i \gamma \cdot \partial - m) S^{ret}(x-y) T_\beta(y) : \]

\[ \frac{\partial}{\partial x^\mu} [ T_\alpha(x) \gamma^\mu S^{ret}(x-y) T_\beta(y) : ] + \delta(x-y) : T_\alpha(x) T_\beta(y) : \]  

(4.3.13)

and a similar analysis is valid for the term \(\frac{\partial}{\partial x^\mu} : T_\alpha(x) \gamma^\mu S(y-x) T_\beta(y) : \) with the result that only the \(\delta\)-term has here a minus sign.

As a consequence, we have the following causal splitting

\[ \frac{\partial}{\partial x^\mu} [K_1^\mu(x), T_1(y)] = \frac{\partial}{\partial x^\mu} L_\mu + \delta(x-y)^{YM} - i \delta(x-y) f_{abc} : u_a(x) A_0^\nu(x) j_c(x) : \]

\[ \delta(x-y) : u_a(x) \overline{\psi}_A (t_b t_a)_{AB} \gamma_\rho \psi_B (x) A_0^\rho (x) : + : u_a(x) \overline{\psi}_A (t_a t_b)_{AB} \gamma_\rho \psi_B (x) A_0^\rho (x) : \]  

(4.3.14)

where by \(\delta(x-y)^{YM}\) we mean the \(\delta\)-terms appearing in formula (4.2.35). The treatment of the contribution \(\frac{\partial}{\partial x^\mu} [K_1^\mu(y), T_1(x)]\) follows the lines described in the derivation of the theorem.
and we obtain in the end the following modification of the possible choice for the retarded component of the commutator $D_2$ (see (4.2.36)):

$$d_Q R_2(x, y) = d_Q R_2(x, y)^{YM} + 2i\delta(x - y) : u_a(x)\overline{\psi}_A(x) ([t_a, t_b] - if_{abc}t_c)_{AB} \gamma_\rho \psi_B(x) A_\rho^b (x) (4.3.15)$$

where $d_Q R_2(x, y)^{YM}$ is the expression given by (4.2.36). The arbitrariness of this decomposition becomes augmented (with respect to the similar one from the pure Yang-Mills case) by the contribution:

$$2i Tr(t_a t_b)\delta(x - y) [f_1 : \partial_\mu u_a A_\mu^b : + f_2 (\partial_\rho \partial^\rho u_a \partial_\mu A_\mu^b : + \partial^\rho u_a \partial_\mu \partial_\nu A_\nu^b :) \div (4.3.16)$$

where by $\div$ we mean a total divergence.

The assertion from the statement follows now immediately from the condition (4.2.26). ■

The meaning of the theorem is that the Dirac “matter” fields should form a multiplet for the Lie algebra with structure constants $f_{abc}$ i.e. they should transform according to some (finite dimensional) representation of this algebra.

**Corollary 4.14** In the preceding conditions the expression of the second order chronological product is:

$$T_2(x, y) = T_2^{YM}(x, y)$$

$$-f_{abc}D_{0,F}(x - y) [A_{\mu}(x) F_\mu^\nu(x) j_{\nu}(y) : - \partial_\nu u_a \partial^\nu u_a \partial_\mu A_\mu^b(y) : + (x \leftrightarrow y)]$$

$$-f_{abc} \partial_{\partial x^\nu} D_{0,F}(x - y) [A_\mu^b(x) A_\mu^c(y) j_{\nu}(y) : - (x \leftrightarrow y)]$$

$$-D_{0,F}(x - y) [\overline{\psi}_A(x)(t_a)_{AB} \gamma^\mu \psi_B(x) \overline{\psi}_C(y)(t_a)_{CD} \gamma_\mu \psi_D(y) :$$

$$+ \partial_{\partial x^\nu} \overline{\psi}_A(x)(t_a)_{AB} \Sigma_F(x - y) \psi_B(x) :$$

$$+ \delta(x - y) [\overline{\psi}_A(x) M_{AB} \psi_B(x) : + \overline{\psi}_A(x) \gamma_\mu M'_{AB} \psi_B(x) :] + Tr(t_a^2) V_F(x - y) \mathbf{1}$$

$$+ A_\mu^b(x) A_\mu^c(y) : [\overline{\psi}_A(x)(t_a t_b)_{AB} \gamma_\mu S_F(x - y) \gamma_\mu \psi_B(y) :$$

$$+ \overline{\psi}_A(y)(t_b t_a)_{AB} \gamma_\mu S_F(y - x) \gamma_\mu \psi_B(x) :$$

$$+ Tr(t_a t_b) F_{F, \mu\nu}(x - y)]. (4.3.17)$$

**Remark 4.15** One should expect that the arbitrariness of the expression written above included in the “mass” terms $M_{AB}$, $M'_{AB}$ will drop out, like in QED, if one goes to the third order of the perturbation theory.
The expressions (4.2.37) and (4.3.17) are remarkable in the sense that combining the general principles of perturbation theory with the factorisation requirement (i.e. “gauge invariance”) we obtain fewer free parameters that we would expect from the similar analysis performed on a model without gauge invariance, like for instance, a scalar field theory with a $: \phi^4:$ interaction; however, these expressions have a common “disease”, namely they do not have, in fact, adiabatic limit. Indeed, the constant term from (4.2.37) exhibits the usual logarithmic divergence and cannot be compensated like in the usual treatment of infra-red divergences. So, we do not have the stability of the vacuum and rigorously speaking, these theories do not exist!

This is the usual problem with zero-mass theories and one can deal with it in two ways. One can argue that from the physical point of view, the quarks are never free and as a consequence a perturbation theory which describes asymptotic free particles has no reason to exist. A more profound answer would be that the usual treatments of the infra-red divergences are not sufficiently general and one should proceed to a modification of the free Fock space taking into account the long range nature of the interaction mediated by zero-mass particles. Thus one should look for some generalisation, in the Fock space formalism, of the Dollard formalism of modified “free” evolution.

There some further developments which will also be considered in future publications, namely the same analysis in $2+1$ dimensions. In this way Chern-Simons Lagrangian could be derived from general principles of perturbation theory. A generalisation of the analysis from section 3 to massive spin one particles is also necessary to complete the study of the electroweak interaction [27], [3]. Finally, the same arguments should be also implemented for the case of spin 2 particles into an attempt to construct a perturbative theory of gravity.
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