DEGREE 4 COVERINGS OF ELLIPTIC CURVES BY GENUS 2 CURVES

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Abstract. Genus two curves covering elliptic curves have been the object of study of many articles. For a fixed degree $n$ the subloci of the moduli space $M_2$ of curves having a degree $n$ elliptic subcover has been computed for $n = 3, 5$ and discussed in detail for $n$ odd; see [17, 22, 3, 4]. When the degree of the cover is even the case in general has been treated in [16]. In this paper we compute the sublocus of $M_2$ of curves having a degree 4 elliptic subcover.

1. Introduction

Let $\psi : C \rightarrow E$ be a degree $n$ covering of an elliptic curve $E$ by a genus two curve $C$. Let $\pi_C : C \rightarrow \mathbb{P}^1$ and $\pi_E : E \rightarrow \mathbb{P}^1$ be the natural degree 2 projections. There is $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ such that the diagram commutes.

$\begin{array}{ccc}
C & \xrightarrow{\pi_C} & \mathbb{P}^1 \\
\psi \downarrow & & \downarrow \phi \\
E & \xrightarrow{\pi_E} & \mathbb{P}^1 
\end{array}$

(1)

The ramification of induced coverings $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ can be determined in detail; see [16] for details. Let $\sigma$ denote the fixed ramification of $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$. The Hurwitz space of such covers is denoted by $\mathcal{H}(\sigma)$. For each covering $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ (up to equivalence) there is a unique genus two curve $C$ (up to isomorphism). Hence, we
We denote by $L_2$ points paper is to study $L \in \mathcal{M}_2$ have a map

$$\Phi : \mathcal{H}(\sigma) \to \mathcal{M}_2$$

$$[\phi] \to [C].$$

We denote by $L_n(\sigma)$ the image of $\mathcal{H}(\sigma)$ under this map. The main goal of this paper is to study $L_4(\sigma)$.

2. Preliminaries

Most of the material of this section can be found in [23]. Let $C$ and $E$ be curves of genus 2 and 1, respectively. Both are smooth, projective curves defined over $k$, $\text{char}(k) = 0$. Let $\psi : C \to E$ be a covering of degree $n$. From the Riemann-Hurwitz formula, $\sum_{P \in C} (e_{\psi}(P) - 1) = 2$ where $e_{\psi}(P)$ is the ramification index of points $P \in C$, under $\psi$. Thus, we have two points of ramification index 2 or one point of ramification index 3. The two points of ramification index 2 can be in the same fiber or in different fibers. Therefore, we have the following cases of the covering $\psi$:

**Case I:** There are $P_1$, $P_2 \in C$, such that $e_{\psi}(P_1) = e_{\psi}(P_2) = 2$, $\psi(P_1) \neq \psi(P_2)$, and $\forall P \in C \setminus \{P_1, P_2\}$, $e_{\psi}(P) = 1$.

**Case II:** There are $P_1$, $P_2 \in C$, such that $e_{\psi}(P_1) = e_{\psi}(P_2) = 2$, $\psi(P_1) = \psi(P_2)$, and $\forall P \in C \setminus \{P_1, P_2\}$, $e_{\psi}(P) = 1$.

**Case III:** There is $P_1 \in C$ such that $e_{\psi}(P_1) = 3$, and $\forall P \in C \setminus \{P_1\}$, $e_{\psi}(P) = 1$.

In case I (resp. II, III) the cover $\psi$ has 2 (resp. 1) branch points in $E$.

Denote the hyperelliptic involution of $C$ by $w$. We choose $O$ in $E$ such that $w$ restricted to $E$ is the hyperelliptic involution on $E$. We denote the restriction of $w$ on $E$ by $v$, $v(P) = -P$. Thus, $\psi \circ w = v \circ \psi$. $E[2]$ denotes the group of 2-torsion points of the elliptic curve $E$, which are the points fixed by $v$. The proof of the following two lemmas is straightforward and will be omitted.

**Lemma 1.** a) If $Q \in E$, then $\forall P \in \psi^{-1}(Q)$, $w(P) \in \psi^{-1}(-Q)$.

b) For all $P \in C$, $e_{\psi}(P) = e_{\psi}(w(P))$.

Let $W$ be the set of points in $C$ fixed by $w$. Every curve of genus 2 is given, up to isomorphism, by a binary sextic, so there are 6 points fixed by the hyperelliptic involution $w$, namely the Weierstrass points of $C$. The following lemma determines the distribution of the Weierstrass points in fibers of 2-torsion points.

**Lemma 2.** The following hold:

1. $\psi(W) \subset E[2]$
2. If $n$ is an even number then for all $Q \in E[2]$, $\#(\psi^{-1}(Q) \cap W) = 0 \mod 2$

Let $\pi_C : C \to \mathbb{P}^1$ and $\pi_E : E \to \mathbb{P}^1$ be the natural degree 2 projections. The hyperelliptic involution permutes the points in the fibers of $\pi_C$ and $\pi_E$. The ramified points of $\pi_C$, $\pi_E$ are respectively points in $W$ and $E[2]$ and their ramification index is 2. There is $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ such that the diagram commutes.

$$\begin{array}{ccc}
C & \xrightarrow{\pi_C} & \mathbb{P}^1 \\
\psi \downarrow & & \downarrow \phi \\
E & \xrightarrow{\pi_E} & \mathbb{P}^1
\end{array}$$

(3)
Next, we will determine the ramification of induced coverings \( \phi : \mathbb{P}^1 \to \mathbb{P}^1 \). First we fix some notation. For a given branch point we will denote the ramification of points in its fiber as follows. Any point \( P \) of ramification index \( m \) is denoted by \((m)\). If there are \( k \) such points then we write \((m)^k\). We omit writing symbols for unramified points, in other words \((1)^k \) will not be written. Ramification data between two branch points will be separated by commas. We denote by \( \pi_E(E[2]) = \{ q_1, \ldots, q_4 \} \) and \( \pi_C(W) = \{ w_1, \ldots, w_6 \} \).

Let us assume now that \( \deg(\psi) = n \) is an even number. Then the generic case for \( \psi : C \to E \) induce the following three cases for \( \phi : \mathbb{P}^1 \to \mathbb{P}^1 \):

**I:** \( \left\{ (2)^{\frac{n}{2}}, (2)^{\frac{n}{4}}, (2)^{\frac{n}{4}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}} \right\} \)

**II:** \( \left\{ (2)^{\frac{n}{2}}, (2)^{\frac{n}{4}}, (2)^{\frac{n}{4}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}} \right\} \)

**III:** \( \left\{ (2)^{\frac{n}{2}}, (2)^{\frac{n}{4}}, (2)^{\frac{n}{4}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}} \right\} \)

Each of the above cases has the following degenerations (two of the branch points collapse to one)

**I:**
1. \( \left\{ (2)^{\frac{n}{2}}, (2)^{\frac{n}{4}}, (2)^{\frac{n}{4}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}} \right\} \)
2. \( \left\{ (2)^{\frac{n}{2}}, (2)^{\frac{n}{4}}, (2)^{\frac{n}{4}}, (4)(2)^{\frac{n}{2}}, (2)^{\frac{n}{2}} \right\} \)
3. \( \left\{ (2)^{\frac{n}{2}}, (2)^{\frac{n}{4}}, (2)^{\frac{n}{4}}, (4)(2)^{\frac{n}{2}}, (2)^{\frac{n}{2}} \right\} \)
4. \( \left\{ (3)(2)^{\frac{n}{4}}, (2)^{\frac{n}{4}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}} \right\} \)

**II:**
1. \( \left\{ (2)^{\frac{n}{2}}, (2)^{\frac{n}{4}}, (2)^{\frac{n}{4}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}} \right\} \)
2. \( \left\{ (2)^{\frac{n}{2}}, (2)^{\frac{n}{4}}, (2)^{\frac{n}{4}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}} \right\} \)
3. \( \left\{ (4)(2)^{\frac{n}{2}}, (2)^{\frac{n}{4}}, (2)^{\frac{n}{4}}, (2)^{\frac{n}{2}} \right\} \)
4. \( \left\{ (2)^{\frac{n}{2}}, (4)(2)^{\frac{n}{2}}, (2)^{\frac{n}{4}}, (2)^{\frac{n}{4}} \right\} \)
5. \( \left\{ (2)^{\frac{n}{2}}, (4)(2)^{\frac{n}{2}}, (2)^{\frac{n}{4}}, (2)^{\frac{n}{4}} \right\} \)
6. \( \left\{ (3)(2)^{\frac{n}{4}}, (2)^{\frac{n}{4}}, (4)(2)^{\frac{n}{2}}, (2)^{\frac{n}{2}} \right\} \)
7. \( \left\{ (2)^{\frac{n}{2}}, (3)(2)^{\frac{n}{4}}, (2)^{\frac{n}{4}}, (2)^{\frac{n}{2}} \right\} \)

**III:**
1. \( \left\{ (2)^{\frac{n}{2}}, (2)^{\frac{n}{4}}, (2)^{\frac{n}{4}}, (4)(2)^{\frac{n}{2}} \right\} \)
2. \( \left\{ (2)^{\frac{n}{2}}, (2)^{\frac{n}{4}}, (2)^{\frac{n}{4}}, (4)(2)^{\frac{n}{2}} \right\} \)
3. \( \left\{ (2)^{\frac{n}{2}}, (2)^{\frac{n}{4}}, (2)^{\frac{n}{4}}, (4)(2)^{\frac{n}{2}} \right\} \)
4. \( \left\{ (3)(2)^{\frac{n}{4}}, (2)^{\frac{n}{4}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}} \right\} \)

For details see [16].

### 3. Degree 4 Case

In this section we focus on the case \( \deg(\phi) = 4 \). The goal is to determine all ramifications \( \sigma \) and explicitly compute \( \mathcal{L}_4(\sigma) \).

There is one generic case and one degenerate case in which the ramification of \( \deg(\phi) = 4 \) applies, as given by the above possible ramification structures:
i) \((2, 2, 2, 2^2, 2)\) (generic)

ii) \((2, 2, 2, 4)\) (degenerate)

4. **Computing the locus \(L_4\) in \(\mathcal{M}_2\)**

4.1. **Non-degenerate case.** Let \(\psi : C \rightarrow E\) be a covering of degree 4, where \(C\) is a genus 2 curve and \(E\) is an elliptic curve. Let \(\phi\) be the Frey-Kani covering with \(\text{deg}(\phi) = 4\) such that \(\phi(1) = 0\), \(\phi(\infty) = \infty\), \(\phi(p) = \infty\) and the roots of \(f(x) = x^2 + ax + b\) be in the fiber of 0. In the following figure, bullets (resp., circles) represent places of ramification index 2 (resp., 1).

\[
\begin{align*}
\bullet & \quad \bullet & \quad \bullet & \quad \bullet & \quad \infty & \quad \bullet & \quad \bullet \\
\bullet & \quad \bullet & \quad \bullet & \quad \bullet & \quad \infty & \quad \bullet & \quad \bullet \\
p^1 & \quad \bullet & \quad \bullet & \quad \bullet & \quad p & \quad \infty & \quad 0
\end{align*}
\]

**Figure 1.** Degree 4 covering for generic case

Then the cover can be given by
\[
\phi(x) = \frac{k(x-1)^2(x^2+b)}{(x-p)^2}.
\]

Let \(\lambda\) be a 2-torsion point of \(E\). To find \(\lambda\), we solve
\[
(4) \quad \phi(x) - \lambda = 0.
\]

According to this ramification we should have 3 solutions for \(\lambda\), say \(\lambda_1, \lambda_2, \lambda_3\). The discriminant of the Eq. (4) gives branch points for the points with ramification index 2. So we have the following relation for \(\lambda\), with \(p \neq 1\).

\[
\begin{align*}
(b-p^2 - k^2 b^2 - 36 k^2 b^2 p - 3 k^2 b^3 - 20 k^2 b p^2 + 8 k^3 b^2 p + 18 k^4 b p - k^2 p^2)\lambda^2 & = 0.
\end{align*}
\]

Using Eq.(4) and Eq.(5) we find the degree 12 equation with 2 factors. One of them with degree 6 corresponds to the equation of genus 2 curve and the other corresponds to the double roots in the fiber of \(\lambda_1, \lambda_2\) and \(\lambda_3\).
The equation of genus 2 curve can be written as follows:

\[ C : y^2 = a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \]

where

- \( a_6 = p^2 + b \)
- \( a_5 = 4p^3 - 6p^2 + 4pb - 6b \)
- \( a_4 = -4p^4 - 10p^3 + (-5b + 13)p^2 - 8pb + 12b \)
- \( a_3 = 12p^4 + (4 + 6b)p^3 + (12b) + 8b^2 - 6b) p - 8b - 8b^2 \)
- \( a_2 = (-11 - 4b)p^4 + (20b + 6)p^3 + (4 + 13b - 12b^2)p^2 + 10pb + 12b^2 \)
- \( a_1 = (14b + 2)p^4 + (6b^2 - 4 + 4b)p^3 + (-24b + 6b^2)p^2 + (-6b^2 + 4b)p - 6b^2 \)
- \( a_0 = (-b^2 + 1 - 11b)p^4 + (14b - 2b^2)p^3 - 2bp^2 + 2b^2p + b^2. \)

Notice that we write the equation of genus 2 curve in terms of only 2 unknowns. We denote the Igusa invariants of \( C \) by \( J_2, J_4, J_6, \) and \( J_{10}. \) The absolute invariants of \( C \) are given in terms of these classical invariants:

\[ i_1 = 144J_4/J_2^2, \quad i_2 = -1728J_2J_4 - 3J_6, \quad i_3 = 486J_{10}/J_2^5. \]

Two genus 2 curves with \( J_2 \neq 0 \) are isomorphic if and only if they have the same absolute invariants. Notice that these invariants of our genus 2 curve are polynomials in \( p \) and \( b. \) By using a computational symbolic package (as Maple) we eliminate \( p \) and \( b \) to determine the equation for the non-degenerate locus \( L_4. \) The result is very long. We don’t display it here.

5. Degenerate Case

Notice that only one degenerate case can occur when \( n = 4 : (2, 2, 2, 4). \) In this case one of the Weierstrass points has ramification index 3, so the cover is totally ramified at this point.

Let the branch points be 0, 1, \( \lambda, \) and \( \infty, \) where \( \infty \) corresponds to the element of index 4. Then, above the fibers of 0, 1, \( \lambda \) lie two Weierstrass points. The two Weierstrass points above 0 can be written as the roots of a quadratic polynomial \( x^2 + ax + b; \) above 1, they are the roots of \( x^2 + px + q; \) and above \( \lambda, \) they are the roots of \( x^2 + sx + t. \) This gives us an equation for the genus 2 curve \( C: \)

\[ C : y^2 = (x^2 + ax + b)(x^2 + px + q)(x^2 + sx + t). \]

The four branch points of the cover \( \phi \) are the 2-torsion points \( E[2] \) of the elliptic curve \( E, \) allowing us to write the elliptic subcover as

\[ E : y^2 = x(x - 1)(x - \lambda). \]

The cover \( \phi : \mathbb{P}^1 \to \mathbb{P}^1 \) is Frey-Kani covering and is given by

\[ \phi(x) = cx^2(x^2 + ax + b). \]

Using \( \phi(1) = 1, \) we get \( c = \frac{1}{1+a+b}. \) Then,

\[ \phi(x) - 1 = c(x - 1)^2(x^2 + px + q). \]
This implies that $\phi'(1) = 0$, so we get $c(4 + 3a + 2b) = 0$. Since $c$ cannot be 0, we must have $4 + 3a + 2b = 0$, which implies $a = \frac{-2(b+2)}{3}$. Combining this with our equation for $c$, we get $c = \frac{3}{b+1}$.

Now, since $\phi(x) - 1 - c(x-1)^2(x^2 + px + q) = 0$, we want all of the coefficients of this polynomial to be identically 0; thus
\[ p = \frac{2(1-b)}{3}, q = \frac{1-b}{3}. \]

Finally, we consider the fiber above $\lambda$. We write
\[ \phi(x) - \lambda = c(x-r)^2(x^2 + sx + t). \]

Similar to above, we set the coefficients of the polynomial to 0 to get:
\[ \lambda = \frac{b^3(4-b)}{16(b-1)}, \quad r = \frac{b}{2}, \quad s = \frac{b-4}{3}, \quad t = \frac{b(b-4)}{12}. \]

Hence we have $C$ and $E$ with equations:
\[ C : y^2 = \left( \frac{1-b}{3} + \frac{2}{3}(1-b)x + x^2 \right) \left( \frac{1}{12}(b-4)b + \frac{1}{3}(b-4)x + x^2 \right) \]
\[ E : v^2 = u(u-1) \left( u - \frac{b^3(4-b)}{16(b-1)} \right) \]

where the corresponding discriminants of the right sides must be non-zero. Hence,
\[ \Delta_C := b(b-4)(b-2)(b-1)(2+b) \neq 0 \]
\[ \Delta_E := \frac{(b-4)^2(b-2)^6b^6(b+2)^2}{65536(b-1)^4} \neq 0. \]

From here on, we consider the additional restriction on $b$ that it does not solve $J_2 = 0$, that is,
\[ J_2 = \frac{5}{486}(256 - 384b - 4908b^2 + 5068b^3 - 1227b^4 - 24b^5 + 4b^6) \neq 0. \]

The case when $J_2 = 0$ is considered separately. We can eliminate $b$ from this system of equations by taking the numerators of $i_j - i_j(b)$ and setting them equal to 0, where $i_j$ are absolute invariants of genus 2 curve.

Thus, we have 3 polynomials in $b, i_1, i_2, i_3$. We eliminate $b$ using the method of resultants and get the following:
\[ 3652054494822999 - 312800728170302145i_1 - 24778254774362875i_1^2 = 0 \]
\[ +3039113062253125i_1^3 - 522534367747902600i_2 - 2801773453711500i_1i_2 - 238234372300000i_3^2 = 0 \]
\[ 1158391804615233525i_1 - 17653298856896250i_1^2 + 100894442906250i_1^3 + 4256292578125i_1^4 - 4441460625i_1^5 - 3238901679891027326688000000i_3 = 0 \]
\[ -1487967225288904960000000i_1i_3 - 4060943110225800000000000i_1^2i_3 - 16677181699665669 + 3474053619183583968614400000000000i_3^2 = 0 \]

These equations determine the degenerate locus $L'_4$ when $J_2 \neq 0$.\[ \]
When \( J_2 = 0 \), we must resort to the \( a \)-invariants of the genus 2 curve. These invariants are defined as
\[
a_1 = \frac{J_4 J_6}{J_{10}}, \quad a_2 = \frac{J_4 J_6}{J_4^2}.
\]

Two genus 2 curves with \( J_2 = 0 \) are isomorphic iff their \( a \)-invariants are equal. For our genus 2 curve,
\[
J_4 = \frac{1}{5184} \left( 65536 - 196608b - 307200b^2 + 1218560b^3 - 834288b^4 - 294432b^5 \\
+456600b^6 - 73608b^7 - 52143b^8 + 19040b^9 - 1200b^{10} - 192b^{11} + 16b^{12} \right)
\]

It can be guaranteed that \( J_4 \) and \( J_2 \) are not simultaneously 0 because the resultant of these two polynomials in \( b \) is
\[
\frac{117849780515223957076466728960000000000000000}{42391158275216203514294433201},
\]
so there are no more subcases. We want to eliminate \( b \) from the set of equations:
\[
\begin{align*}
J_2 &= 0 \\
a_1 - a_1(b) &= 0 \\
a_2 - a_2(b) &= 0.
\end{align*}
\]

Similar to what we did above with the \( i \)-invariants, we take resultants of combinations of these and set them equal to 0. Doing so tells us
\[
20a_1 - 55476394831 = 0 \\
1022825924657928a_2 - 522665 = 0.
\]

So in other words, if \( C \) is a genus 2 curve with a degree 4 elliptic subcover with \( J_2 = 0 \), then
\[
a_1 = \frac{55476394831}{20}, \quad a_2 = \frac{522665}{1022825924657928}.
\]

So up to isomorphism, this is the only genus 2 curve with degree 4 elliptic subcover with \( J_2 = 0 \). In this case the equation of the genus 2 curve is given by Eq.(6), where \( b \) is given by the following:
\[
b = \frac{2\alpha + \sqrt{429\alpha^2 + 60123\alpha + \beta}}{2\alpha}
\]
with \( \alpha = \sqrt[3]{2837051 + 9408\sqrt[5]{5}} \) and \( \beta = 8511153 + 28224i\sqrt[5]{5} \). We summarize the above results in the following theorem.

**Theorem 1.** Let \( C \) be a genus 2 curve with a degree 4 degenerate elliptic subcover. Then \( C \) is isomorphic to the curve given by Eq.(6) where \( b \) satisfies Eq.(12) or its absolute invariants satisfy Eq. (10) and Eq. (11).

**Remark 1.** The genus 2 curve, when \( J_2 = 0 \), is not defined over the rational.
Remark 2. When the genus 2 curve has non zero $J_2$ invariant the $j$ invariant of the elliptic curve satisfies the following equation:

\[ 0 = (262144000000000000 J_1^4 - 143329853440000000 J_3^2 J_4^4 - 15871355382432000 J_6^6 J_4 + 1586874322944 J_2^8 + 26122821304320000 J_2^4 J_4^2)^2 + (-258941458659766450406400000000 J_4^4 - 203482361042468209670400000000 J_2^2 J_4^3 + 39862710766802552045625 J_8^6 - 19433806326190741141800000 J_6^6 J_4 + 3259543004362746907416000000 J_4^4 J_4^2). \]

5.1. Genus 2 curves with degree 4 elliptic subcovers and extra automorphisms in the degenerate locus of $L_4$. In any characteristic different from 2, the automorphism group Aut(C) is isomorphic to one of the groups: $C_2, C_10, V_4, D_8, D_{12}, C_3 \rtimes D_8, GF(2^3)$, or $2^+ S_5$; See [21] for the description of each group. We have the following lemma.

Lemma 3. (a) The locus $L_2$ of genus 2 curves $C$ which have a degree 2 elliptic subcover is a closed subvariety of $M_2$. The equation of $L_2$ is given by

\[
0 = 8748 J_{10} J_2^2 - 507384000 J_1^5 J_2^2 - 19245600 J_1^2 J_2 J_6^2 - 592272 J_1 J_2^4 J_6^2 + 77436 J_1 J_2 J_6^2 J_6^2 + 4743360 J_1^2 J_2 J_6^3 - 870912 J_2 J_6^4 J_6 + 3090960 J_1^3 J_2^3 J_6^3 - 1259712000000 J_1 J_2^3 J_6^3 - 1332 J_2^4 J_6^4 + 384 J_2^6 J_6 + 41472 J_2^6 J_6^2 + 159 J_2^4 J_2 J_6^2 - 80 J_2^4 J_2 - 47952 J_2 J_6^4 + 104976000 J_1 J_2^5 J_6^3 - 1728 J_1^5 J_2 J_6^3 - 31104 J_6^6 + 6912 J_1^3 J_2^3 J_6 - 5832 J_1^3 J_4 J_6 - 54 J_1^2 J_2 J_6^2 J_6 + 108 J_1^2 J_2 J_6^3 + 972 J_1 J_6^5 J_6.
\]

(b) The locus $M_2(D_8)$ of genus 2 curves $C$ with Aut(C) $\equiv D_8$ is given by the equation of $L_2$ and

\[
0 = 1706 J_1^2 J_2^2 + 2650 J_4^3 + 27 J_2 J_6^2 - 81 J_2^2 J_6 - 14880 J_2 J_4 J_6 + 28800 J_6^2.
\]

(c) The locus $M_2(D_{12})$ of genus 2 curves $C$ with Aut(C) $\equiv D_{12}$ is

\[
0 = -J_4 J_2 J_6 - 12 J_2^3 J_6 - 52 J_2^4 J_6 + 80 J_4^3 + 960 J_2 J_4 J_6 - 3600 J_6^2.
\]

\[
0 = -864 J_1 J_2 J_4 J_6 + 3456000 J_1 J_2 J_4 J_6 - 432000 J_1 J_2 J_4 J_6 - 232280000 J_1 J_2 J_4 J_6 - 768 J_2 J_5 J_6 + 48 J_2^2 J_4 J_6 + 4096 J_5 J_6.
\]

We will refer to the locus of genus 2 curves $C$ with Aut(C) $\equiv D_{12}$ (resp., Aut(C) $\equiv D_8$) as the $D_{12}$-locus (resp., $D_8$-locus).

Equations (10), (11), and (13) determine a system of 3 equations in the 3 $i$-invariants. The set of possible solutions to this system contains 20 rational points and 8 irrational or complex points (there may be more possible solutions, but finding them involves the difficult task of solving a degree 15 or higher polynomial).
Among the 20 rational solutions, there are four rational points which actually solve the system.

\[(i_1, i_2, i_3) = \left( \frac{102789}{12005}, \frac{-73594737}{2847524900000}, \frac{531441}{2847524900000} \right)\]

\[(i_1, i_2, i_3) = \left( \frac{66357}{9245}, \frac{-892323}{46225}, \frac{7776}{459401384375} \right)\]

\[(i_1, i_2, i_3) = \left( \frac{235629}{1156805}, \frac{-28488591}{214008925}, \frac{53747712}{531441} \right)\]

\[(i_1, i_2, i_3) = \left( \frac{1078818669}{383775605}, \frac{-77466710644803}{16811290377025}, \frac{1356226634181762}{161294078381836186878125} \right)\]

Of these four points, only the first one lies on the $D_{12}$-locus, and none lie on the $D_8$-locus, so the other three curves have automorphism groups isomorphic to $V_4$ (See Remark 3 for their equations). We have the following proposition.

**Proposition 1.** There is exactly one genus 2 curve $C$ defined over $\mathbb{Q}$ (up to $\mathbb{Q}$-isomorphism) with a degree 4 elliptic subcover which has an automorphism group $D_{12}$ namely the curve

\[C = 100X^6 + 100X^3 + 27\]

and no such curves with automorphism group $D_8$.

**Proof.** From above discussion there is exactly one rational point which lies on the $D_{12}$-locus and three rational points which lies on the $V_4$-locus. Furthermore we have the fact that $\text{Aut}(C) \cong D_{12}$ if and only if $C$ is isomorphic to the curve given by $Y^2 = X^6 + X^3 + t$ for some $t \in k$; see [19] for more details.

Suppose the equation of the $D_{12}$ case is $Y^2 = X^6 + X^3 + t$. We want to find $t$. We can calculate the $i$-invariants in terms of $t$ accordingly, so we get a system of equations, $i_j - i_j(t) = 0$ for $j \in \{1, 2, 3\}$. Those equations simplify to the following:

\[0 = 1600i_1t^2 - 80i_1 + i_1 - 6480t^2 - 1296t\]
\[0 = 64000i_2t^3 - 4800i_2t^2 + 120i_2t - i_2 + 23280t^3 + 303264t^2 - 11664t\]
\[0 = 16384000000i_3t^5 - 2048000000i_3t^4 + 10240000000i_3t^3 - 25600000i_3t^2 + 3200i_3t - 16i_3 + 729t^2 + 34992t^2 - 46656t^5 - 8748t^3\]

Replacing our i-invariants into the above system of equations we get:

\[0 = 866700000t^2 - 237816000t + 102789\]
\[0 = -40239342000000t^3 + 1245223960000t^2 - 43137816840t + 73594737\]
\[0 = -8231536350000000t^5 + 61776053451150000t^4 - 15443994116835000t^3 + 1287019350200250t^2 + 1062882000t - 531441\]

There is only root those three polynomials share: $t = \frac{279}{199}$. Thus, there is exactly one genus 2 curve $C$ defined over $\mathbb{Q}$ (up to $\mathbb{Q}$-isomorphism) with a degree 4 elliptic subcover which has an automorphism group $D_{12}$

\[C : \quad y^2 = 100X^6 + 100X^3 + 27\]

Similarly, we show that there are no such curves with automorphism group $D_8$. □
Remark 3. There are at least three genus 2 curves defined over $\mathbb{Q}$ with automorphism group $V_4$. The equations of these curves are given by the following:

**Case 1:** $(i_1, i_2, i_3) = \left(\frac{66357}{9245}, \frac{-892324}{4625}, \frac{7776}{25901384375}\right)$

$$\begin{align*}
C : y^2 &= 1432139730944x^6 + 34271993769359360x^5 + 267643983706245216000x^4 \\
&\quad + 1267919172426862313120000x^3 + 2394555897022486213835350000x^2 \\
&\quad + 27433066616264915379359980475000x + 1025623291911204380755800513010015625.
\end{align*}$$

**Case 2:** $(i_1, i_2, i_3) = \left(\frac{235629}{1156805}, \frac{-28488591}{214008925}, \frac{53747712}{80459143207503125}\right)$

$$\begin{align*}
C : y^2 &= 41871441565158964373437321767075023159296x^6 \\
&\quad + 156000358914872008908017177004915818496000x^5 \\
&\quad + 8994429753268252328699175313122263040000000x^4 \\
&\quad + 1785753740382156157948005357453312000000000x^3 \\
&\quad + 7750181515625167813522653681664000000000x^2 \\
&\quad + 1158249382368910116792368993736000000000000x \\
&\quad + 2678752767946851427317565520959888458251953125.
\end{align*}$$

**Case 3:** $(i_1, i_2, i_3) = \left(\frac{1078818669}{383775605}, \frac{-77466710644803}{16312907838136186878125}\right)$

$$\begin{align*}
C : y^2 &= 9224408124038149308993379217084884661375653227720704x^6 \\
&\quad + 373075876766898487772512960488815232203536482648192000x^5 \\
&\quad + 113852328380343991240386194142199809225534591301754000000x^4 \\
&\quad + 189425049047623621895235859065864784120488845750000000x^3 \\
&\quad + 76212520567614919095032412154382214439329394838317128906250000x^2 \\
&\quad + 16717294192730754705692151510108692898208834624189098203125000x \\
&\quad + 276688898904544873606744431686092459654926161559210811614990234375.
\end{align*}$$

We summarize by the following:

**Theorem 2.** Let $\psi : C \to E$ be a degree 4 covering of an elliptic curve by a genus 2 curve. Then the following hold:

i) In the generic case the equation of $C$ can be written as follows:

$$C : y^2 = a_6x^6 + a_5x^5 + \cdots + a_1x + a_0$$

where

$$\begin{align*}
a_0 &= p^2 + b \\
a_1 &= 4p^3 - 6p^2 + 4pb - 6b \\
a_2 &= -11 - 4b + p^4 + (4b + 6)p^3 + (4 + 13b - 12b^2)p^2 + 10pb + 12b^2 \\
a_3 &= 14b + 2b + (6b^2 - 4 + 4b)p^3 - (24b + 6b)p^2 + (2b + 4b)p - 6b^2 \\
a_4 &= -b^2 + 1 - 11b + 14b - 2b^3 - 2bp^2 + 2b^2p + b^2.
\end{align*}$$
ii) In the degenerate case the equation of $L'_4$ is given by
\[
154108612576000 J_2^2 J_4^2 - 228353122360690000 J_2 J_4 J_6 + 509067694763 J_2^6
-87822719004672000 J_4^2 + 11768712184572476480 J_4^4 J_4 J_4 + 1244820710298880000 J_4^3
-371579945829526000 J_2^3 J_6 = 0
\]
\[
18662560000000 J_2^2 J_4^4 + 13886214476734358745760000000000 J_4^2 + 28242953648000000 J_2 J_6^2
+619923800736000000 J_4 J_6^2 - 256000000000000 J_2^5 J_10 - 5102020224000000 J_2^3 J_4 J_10
+693067624145820000000000 J_2 J_4 J_4 J_10 + 176351670812388376000000 J_2^2 J_4 J_10 = 0
\]

iii) The intersection $L'_4 \cap M_2(D_8) = \emptyset$ and the intersection $L'_4 \cap M_2(D_{12})$ contains a single point, namely the curve
\[ C : \quad y^2 = 100X^6 + 100X^3 + 27 \]

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