MINIMIZATION OF ARAKELOV K-ENERGY FOR MANY CASES

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Abstract. We prove that for various polarized varieties over \( \overline{\mathbb{Q}} \), which broadly includes K-trivial case, K-ample case, Fano case, minimal models, certain classes of fibrations, certain metrized “minimal-like” models minimizes the Arakelov theoretic analogue of the Mabuchi K-energy, as conjectured in [Od15]. This is an Arakelov theoretic analogue of [Hat22b].

1. Introduction

The K-stability of polarized varieties was originally designed to give an algebro-geometric counterpart of the existence of canonical Kähler metrics [Tia97, Don02] (see §2.1 for more details). The second author introduced arithmetic framework for K-stability in [Od15], which discusses certain modular heights of polarized varieties \((X, L)\) over \( \overline{\mathbb{Q}} \), which for instance conjecturally allows generalization of Faltings heights of abelian varieties [Fal83]. The plan is to achieve it as the infimum or minimum of what [Od15] calls Arakelov K-energy or K-modular height which depends on metrized models. [Od15, Conjecture 3.12, 3.13] (see our Conjecture 1.2) means to characterize the models which attain such minimum, whose partial resolution is the aim of this paper. It is done by fitting the theory of “special K-stability” by the first author [Hat22b] in usual algebraic geometry (cf., §2.1.4), to the arithmetic framework [Od15], with some differential geometric inputs as [CSI17, Che21, Zha21a].

Notation 1 (Arithmetic setup). We slightly change notation from [Od15] to fit more to [Hat22b]. Let \( F \) be a number field, \( X_\eta \) a \( n \)-dimensional smooth projective variety over \( F \) and \( L_\eta \) an ample line bundle (polarization) on it. We consider an ample-polarized normal projective model \((X, L)\) over \( \mathcal{O}_F \), the ring of integers in \( F \), with the generic fiber \((X_\eta, L_\eta)\) possibly after the extention of scalars i.e., replacing \( F \) by its finite extension. \((X_C, L_C)\) denotes the base change \((X_\eta, L_\eta) \times_F \mathbb{C} \) and the reduction of \((X, L)\) over a prime ideal \( p \) of \( \mathcal{O}_F \) as \((X_p, L_p)\).

We write a hermitian metric of \( L_C \) of real type, as \( h_{L} \) and its corresponding 1-st Chern form as \( \omega_{h_L} \) which we assume to be positive definite. The pair \((L, h_L)\) is often denoted as \( \overline{L} \) or \( \overline{L}^{h_L} \). The dual of a line bundle is denoted by \( \check{} \).

When we focus on the complex place or (positive characteristic) reduction, we use different notations to be set as Notation 2 and 3 later.

Date: May 6, 2024.
2020 Mathematics Subject Classification. 14E30, 14G40, 32Q26.
Key words and phrases. Arakelov geometry, K-stability, Kähler geometry.
The following is our main object to study.

**Definition 1.1** ([Od15, §2]). We define the Arakelov-K-energy (or K-modular height) as

\[ h_{K}^{\text{Ar}} (X, L, h_{L}) \]

\[ := \frac{1}{[F : \mathbb{Q}]} \left\{ -\frac{n(L_{n}^{-1}.K_{X_{\eta}})}{(n + 1)(L_{n})^{n+1}} + \frac{((L^{h_{L}})_{n}.K_{X}/\mathcal{O}_{F_{\ell}}^{Ric(\omega_{h_{L}})})}{(L_{n})^{n+1}} \right\}. \]

In the above, the superscript \( Ric(\omega_{h_{L}}) \) means the metrization of \( K_{X(\mathbb{C})} \) which corresponds to the Monge-Ampere measure \( \omega_{h_{L}} \). The above is slightly different from [Od15, Definition 2.4] by a normalizing constant \((n+1)(L_{n})^{n}\).

We excerpt a part of the series of conjectures in [Od15] as follows, which is what we partially prove in this paper. This is somewhat analogous to the CM minimization conjecture, introduced by the second author (cf., [Od13b], [Od20]), though implicitly also combined a little with usual Yau-Tian-Donaldson conjecture (Conjecture 2.3).

**Conjecture 1.2** (Arithmetic Yau-Tian-Donaldson conjecture [Od15]). We fix a normal polarized projective variety \((X_{\eta}, L_{\eta})\) over a number field \( F \).

Then, we consider all the metrized polarized normal models \((X, L, h_{L})\) (in the sense of above Notation 1) over \( \mathcal{O}_{F_{\ell}} \) where \( F_{\ell} \) also runs over all finite extensions of \( F \). Then, \( h_{K}^{\text{Ar}} (X, L, h_{L}) \) attains their minimum if and only if

(i) all the reductions \((X_{p}, L_{p})\) are K-semistable,
(ii) \( \omega_{h_{L}} \) is a Kähler form with constant scalar curvature.

Recall that the attained minimum above for abelian varieties case is essentially the Faltings height [Fal83], modulo some simple constants, as confirmed in [Od15].

Recently, as we briefly review at the subsection §2.1.4, the first author [Hat22b] introduces the notion of “special K-stability” which, nevertheless of its name, include many cases. The notion is defined by using J-stability (cf., §2.1.2 [Hat21]) and the \( \delta \)-invariant (cf., §2.1.3 [FO18, BLJ20]) in the field of K-stability. Then, the first author showed the special K-(semi)stability implies the usual K-(semi)stability [Hat22a] (see §2.1 for more details).

Our main theorem 3.1 is roughly as follows, which partially confirms the “if” direction of the above Conjecture 1.2.

**Theorem 1.3** (Main Theorem (=Theorem 3.1)). If a metrized polarized model \((X, L, h_{L})/\mathcal{O}_{F} \) satisfies analogues of special K-stability over any place of \( F \), \( h_{K} (X, L, h_{L}) \) attains the minimum for the fixed geometric generic fiber \((X, L) \times_{F} \mathbb{Q} \) over \( \mathbb{Q} \).

There are many classes of polarized varieties which have a model which satisfies the above-mentioned condition, such as K-trivial case, K-ample case, K-stable Fano varieties case, minimal models and some fibrations for instance. Thus, the above theorem at least broadly generalizes [Od15, Theorem 3.14]. For instance, the cases of minimal models and certain algebraic fibrations are newly included (compare [Hat21]). Large
parts of this paper are devoted to preparations to rigorously formulate (and prove) Theorem 1.3, finally resulting to the main theorem 3.1. Some review of the background is also contained in the next section §2 for the readers convenience.

Our discussion of the K-modular height (Definition 1.1) and its variant is based on the framework of [Od15], which uses the Gillet-Soulé intersection theory [GS90] and its developments. The main discussion of the proof of Theorem 3.1 closely follow [Hat22b]. Thus, we also refer to them for more details of the background.

2. Preliminaries

This section consists of three subsections. The first §2.1 briefly reviews some basics of K-stability for convenience of readers, introducing the J-stability and the $\delta$-invariant as well. They are both recent useful tools to study K-stability. The latter two subsections §2.3, §2.2 are technical results prepared for our main theorem in the next section §3. The materials from the section 2.2 and later are new.

2.1. Review of K-stability. In this subsection, we review the usual K-stability in the complex geometric setup. For that, we first re-set the notation for this subsection §2.1:

Notation 2 (Complex geometric setup). We consider a polarized smooth projective variety $(X_\mathbb{C}, L_\mathbb{C})$ over $\mathbb{C}$ which, in this subsection, does not necessary descends over $\mathbb{Q}$ as Notation 1. As in Notation 1, we denote a hermitian metric of $L_\mathbb{C}$, as $h_{L_\mathbb{C}}$ and its corresponding 1-st Chern form as $\omega_{h_{L_\mathbb{C}}}$ which we assume to be positive definite i.e., a Kähler form. For a smooth real function $\varphi$ on $X(\mathbb{C})$, we set $\omega_{\varphi} := \omega_{h_{L_\mathbb{C}}} + \sqrt{-1} \partial \bar{\partial} \varphi$.

2.1.1. K-stability ([Tia97, Don02]). Now, we review the definition of the K-stability.

Definition 2.1. A test configuration of $(X_\mathbb{C}, L_\mathbb{C})$ of the exponent $r (\in \mathbb{Z}_{>0})$ means a projective scheme $X_\mathbb{C}$ flat over $\mathbb{P}^1_\mathbb{C}$, a relatively ample line bundle $L_\mathbb{C}$ on $X_\mathbb{C}$, $\mathbb{G}_m$-action on $(X_\mathbb{C}, L_\mathbb{C})$ together with a $\mathbb{G}_m$-equivariant isomorphism $(X_\mathbb{C}, L_\mathbb{C})|_{(\mathbb{P}^1_\mathbb{C}\setminus \{0\})} \simeq (X_\mathbb{C}, L_\mathbb{C}^{\otimes r}) \times (\mathbb{P}^1_\mathbb{C}\setminus \{0\})$.

We simply denote this set of data, forming a test configuration, as $(X_\mathbb{C}, L_\mathbb{C})$.

Definition 2.2 ([Don02, Wan12, Od13a]). The Donaldson-Futaki invariant $\text{DF}(X_\mathbb{C}, L_\mathbb{C})$ of a test configuration $(X_\mathbb{C}, L_\mathbb{C})$, where $X_\mathbb{C}$ is normal, is defined as

$$\frac{-n(L_\mathbb{C}^{n-1}.K_{X_\mathbb{C}})}{(n+1)(L_\mathbb{C})^n}(L_\mathbb{C})^{n+1} + r(L_\mathbb{C}^n.K_{X_\mathbb{C}/\mathbb{P}^1_\mathbb{C}}).$$

Note that, thanks to the homogeneity of the above, it is convenient to replace $L$ by $L/r$ as a $\mathbb{Q}$-line bundle (of exponent 1). We say $(X_\mathbb{C}, L_\mathbb{C})$ is $K$-stable (resp., $K$-semistable) if they are positive unless $X_\mathbb{C}$ is $X_\mathbb{C} \times \mathbb{P}^1$ (resp., they are always non-negative). We also say $(X_\mathbb{C}, L_\mathbb{C})$ is $K$-polystable if they are positive unless $X_\mathbb{C}$ is a $X_\mathbb{C}$-fiber bundle over $\mathbb{P}^1$.

The Donaldson-Futaki invariant is recently also called non-archimedean Mabuchi energy (cf., [BHJ17]) modulo a technical slight difference. Also, note that our definition of the K-modular height (Definition 1.1) is designed after the above intersection number formula. The original motivation for K-stability is the following well-known conjecture in complex geometry.
Conjecture 2.3 (The Yau-Tian-Donaldson conjecture [Don02]). For any polarized smooth complex projective variety \((X_C, L_C)\), it is K-polystable if and only if \(X_C\) admits a constant scalar curvature Kähler metric of the Kähler class \(c_1(L_C)\).

2.1.2. J-stability. The J-stability of polarized variety is a certain toy-model analogue of the K-stability, originally named after the J-flow of Donaldson [Don99] (cf., also [Che00]). After Notation 2, we further consider another auxiliary ample \(R\)-line bundle \(H\) of \(X_C\).

The differential geometric counterpart of the J-stability (Definition 2.4) is the following so-called \(J\)-equation

\[
\text{tr}_\omega(\chi) = \text{constant},
\]

where \(\text{tr}_\omega\) means the trace with respect to \(\omega\). See e.g., [LS15, Che21, DP21, Son20, Hat21] for more detailed context. Here, we only briefly review it at the level we use in this paper.

Definition 2.4 (J-stability). For a test configuration \((X_C, L_C)\) of a polarized variety \((X_C, L_C)\), we take a resolution of indeterminancy of birational map \(X_C \times \mathbb{P}^1 \to X_C\) as

\[
X_C \times \mathbb{P}^1 \xrightarrow{\xi} Y \xrightarrow{\varphi} X_C,
\]

so that \(p\) and \(q\) are morphisms. We also denote the first projection \(X_C \times \mathbb{P}^1 \to X_C\) as \(p_1\). Then we define

\[
J^{H, \text{NA}}(X_C, L_C) := -n(L_C^{n-1}.H)/(n+1)(L_C^n)^{n+1} + r(L_C^n.(p_1 \circ p)^*H).
\]

A polarized variety \((X_C, L_C)\) is \(J^H\)-semistable if \(J^{H, \text{NA}}(X_C, L_C) \geq 0\) for any test configuration, \((X_C, L_C)\) is called uniformly \(J^H\)-stable if there exists \(\epsilon > 0\) such that \((X_C, L_C)\) is \(J^{H-\epsilon L_C}\)-semistable.

Remark 2.5. Note that the above Definition 2.4 does not particularly use that the base field is \(\mathbb{C}\). Hence, we can also define J-stability of polarized varieties over any field, including positive characteristic, in the same way.

The analogue of Yau-Tian-Donaldson conjecture for the J-stability is now a theorem, as conjectured by Lejmi-Szekelyhidi [LS15].

Theorem 2.6 ([Che21, DP21, Son20]). Fix a Kähler form \(\chi\) such that \([\chi] = c_1(H_C)\).

Then, the following are equivalent:

(i) There is a (unique) Kähler form \(\omega\) such that \([\omega] = c_1(L_C)\) which satisfies the J-equation \((1)\) above.

(ii) \((X_C, L_C)\) is uniformly \(J^H\)-stable.

Here, uniform J-stability above is a slight strengthening of the J-stability i.e., it implies J-stability, after the idea of [BHJ17, Der16].

Next, we recall the definition of filtrations.

Definition 2.7. Let \(X\) be a proper reduced scheme over a field \(k\) with an ample line bundle \(L\). Suppose that \(H^0(X, L^\otimes m)\) is generated by \(H^0(X, L)\) for any \(m \in \mathbb{Z}_{>0}\). We
call a set of subspaces \( \mathcal{F} := \{ \mathcal{F}^\lambda H^0(X, L^{\otimes m}) \}_{m \in \mathbb{Z}_{>0}, \lambda \in \mathbb{Z}} \) of \( H^0(X, L^{\otimes m}) \) a filtration if \( \mathcal{F} \) satisfies the following:

(i) \( \mathcal{F}^{\lambda} H^0(X, L^{\otimes m}) \cdot \mathcal{F}^{\lambda'} H^0(X, L^{\otimes m'}) \subset \mathcal{F}^{\lambda + \lambda'} H^0(X, L^{\otimes m + m'}) \),

(ii) \( \mathcal{F}^{\lambda} H^0(X, L^{\otimes m}) \subset \mathcal{F}^{\lambda} H^0(X, L^{\otimes m}) \) for any \( \lambda > \lambda' \),

(iii) there exists \( N > 0 \) such that if \( \lambda > Nm \) then \( \mathcal{F}^{\lambda} H^0(X, L^{\otimes m}) = 0 \) and if \( \lambda \leq Nm \) then \( \mathcal{F}^{\lambda} H^0(X, L^{\otimes m}) = H^0(X, L^{\otimes m}) \).

For any \( \mathcal{F} \), we fix \( N \) as above. Then we take an approximation to \( \mathcal{F} \) as follows. For any \( l \in \mathbb{Z}_{>0} \), let \( \mathcal{F}(l) \) be a filtration generated by \( \mathcal{F}^t H^0(X, L^{\otimes m}) \) and \( \mathcal{F}^{\lambda} H^0(X, L^{\otimes l}) \) for any \( t \leq -Nm, \lambda \) and \( m \) as [Hat22b, Definition 2.16]. We call a sequence \( \{ \mathcal{F}(l) \}_{l \in \mathbb{Z}_{>0}} \) an approximation to \( \mathcal{F} \). We take the normal test configuration \((X^{(l)}, L^{(l)})\) for \((X, L)\) induced by \( \mathcal{F}(l) \) as [Hat22b, Definition 2.19], which is defined as follows. Let \( a(l) \) be the image of the following \( \mathcal{O}_X[t] \)-homomorphism

\[
\bigoplus_{\lambda \in \mathbb{Z}} t^{-\lambda} \mathcal{F}^{\lambda} H^0(X, L^{\otimes l}) \otimes \mathcal{O}_{X \times \mathbb{A}^1}(-l(L \times \mathbb{A}^1)) \rightarrow \mathcal{O}_X[t, t^{-1}],
\]

where \( t \) is the canonical coordinate of \( \mathbb{A}^1 \). Then, \( \mu_t: X^{(l)} \rightarrow X \times \mathbb{P}^1 \) be the blow up along \( a(l) \) and

\[
L^{(l)} := \mu_t^* (L \times \mathbb{P}^1) - \frac{1}{l} \mu_t^{-1}(a(l)).
\]

Let \( H \) be an ample divisor on \( X \) and take \( D \in |mH| \) for any \( m \in \mathbb{Z}_{>0} \). We say that \( D \) is compatible with \( \{ \mathcal{F}(l) \} \) if the support of \( \mu_t^* D \times \mathbb{P}^1 \) contains no \( \mu_t \)-exceptional divisor for any \( l \). Finally, we close this subsection with the following lemma.

**Lemma 2.8** ([Hat22b, Lemma 2.20]). *In the above situation, we have that*

\[
\mathcal{J}^{H,NA}(\mathcal{F}) := \lim_{l \to \infty} \mathcal{J}^{H,NA}(X^{(l)}, L^{(l)}). \]

*If \((X, L)\) is further \( J^H \)-semistable, then*

\[
\mathcal{J}^{H,NA}(\mathcal{F}) \geq 0.
\]

**Proof.** If \( \mathbb{k} \) is uncountable, then we can choose a compatible divisor \( D \in |mH| \) for some \( m \) with \( \{ \mathcal{F}(l) \} \). Thus, \( \lim_{l \to \infty} \mathcal{J}^{H,NA}(X^{(l)}, L^{(l)}) \) exists and coincides with the value [Hat22b, (5)] by [Hat22b, Lemma 2.20] (whose proof also works for the positive characteristic case). For the general case, we reduce to the previous case by changing the base field \( \mathbb{k} \) to some uncountable field (cf. [Hat22b, Remark 2.21]). \( \square \)

### 2.1.3. Delta invariant ([FO18, BLJ20]).

First, we recall the log canonical threshold.

**Definition 2.9.** Let \( X_{\mathbb{C}} \) be a normal variety over \( \mathbb{C} \) such that \( K_{X_{\mathbb{C}}} \) is \( \mathbb{Q} \)-Cartier with an effective \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-Weil divisor \( D \) on \( X_{\mathbb{C}} \). For any prime divisor \( E \) over \( X_{\mathbb{C}} \), we set the log discrepancy

\[
A_{X_{\mathbb{C}}}(E) = 1 + \text{ord}_E(K_Y - \pi^*K_X),
\]

where \( \pi: Y \rightarrow X \) is a log resolution such that \( E \) is a divisor on \( Y \). Then, we set the log canonical threshold of \( X_{\mathbb{C}} \) with respect to \( D \) as

\[
\inf_E \frac{A_{X_{\mathbb{C}}}(E)}{\text{ord}_E(D)}.
\]
where $E$ runs over all prime divisors over $X_C$.

The *delta invariant* $\delta(X_C, L_C)$ (a.k.a. *stability thresholds*) introduced by [FO18, BJ20] is a real invariant of a Fano variety or a general polarized variety $(X_C, L_C)$ (see Notation 2) which is now known to give an effective criterion for the K-stability. For its definition, a special type of $\mathbb{Q}$-divisors as follows is introduced ([FO18]).

**Definition 2.10 ([FO18]).** Let $k$ be a natural number. For any basis $s_1, \ldots, s_{h^0(L_C^\otimes k)}$ of $H^0(L_C^\otimes k)$, taking the corresponding divisors $D_1, \ldots, D_{h^0(L_C^\otimes k)}$ (note $D_i \sim L_C^\otimes k$), we obtain a $\mathbb{Q}$-divisor

\[ D := \frac{D_1 + \cdots + D_{h^0(L_C^\otimes k)}}{k \cdot h^0(L_C^\otimes k)}. \]

This kind of effective $\mathbb{Q}$-divisor is called an ($\mathbb{Q}$-)divisor of $k$-basis type.

**Definition 2.11 ([FO18]).** For $k \in \mathbb{Z}_{>0}$, we define

\[ \delta_k(X_C, L_C) := \inf_{(L_C \sim_D D; D: k\text{-basis type})} \text{lct}(X_C; D), \]

where lct stands for log canonical thresholds. It is easy to see that there exist a prime divisor $E$ over $X_C$ and a divisor $D$ of $k$-basis type such that

\[ \delta_k(X_C, L_C) = \frac{A_{X_C}(E)}{\text{ord}_E(D)}. \]

Then, we say that $E$ *computes* $\delta_k(X_C, L_C)$. On the other hand, we set

\[ \delta(X_C, L_C) := \lim_{k \to \infty} \delta_k(X_C, L_C). \]

The above limit is known to exists by [BJ20]. Its original motivation is the following criterion.

**Theorem 2.12 ([FO18, BJ20]).** For any Fano manifold $X_C$, $\delta(X_C, -K_{X_C}) > 1$ (resp., $\geq 1$) then $(X_C, -K_{X_C})$ is uniformly K-stable (resp., K-semistable).

Here, again, the uniform K-stability above is an a priori strengthening of the K-stability due to [BHJ17, Der16] but more recently they are confirmed to be equivalent again for anticanonically polarized $\mathbb{Q}$-Fano varieties ([LXZ22]).

**Remark 2.13.** We note that $\delta_k(X_K, L_K)$ for any $k \in \mathbb{Z}_{>0}$ and $\delta(X_K, L_K)$ for any polarized klt pair over any field in the same way as the complex field. See also [Zhu21, Definition 2.3].

2.1.4. *Special K-stability* ([Hat22a, Hat22b]). Recently the delta invariant turned out to be also efficient for studying K-stability of more general varieties (cf., e.g., [Zha21a, Hat22a, Hat22b]). In particular, [Hat22b] introduces the following notion which also forms a key idea of the current paper.
Definition 2.14 ([Hat22b, Definition 3.10]). We call a polarized complex variety $(X_C, L_C)$ is specially $K$-stable (resp. specially $K$-semistable) if $X_C$ is semi-log-canonical and both of the following hold:

(i) $K_{X_C} + \delta(X_C, L_C)L_C$ is ample (resp. nef),
(ii) $(X_C, L_C)$ is uniformly $J_{K_{X_C} + \delta(X_C, L_C)L_C}$-stable (resp. $J_{K_{X_C} + \delta(X_C, L_C)L_C}$-semistable).

The main point of this notion is:

Theorem 2.15 ([Hat22b, Corollary 3.21]). If $(X_C, L_C)$ is specially $K$-stable (resp. specially $K$-semistable), it is also uniformly $K$-stable (resp. specially $K$-semistable).

As [Hat22b, Theorem 3.12] reviews (cf., also e.g., [Hat21]), there are many classes of polarized varieties which satisfy special K-(semi)stability.

2.2. Positive characteristic analogue of $\delta$-invariant. Now we turn to a preparation for the reductions at non-archimedean places, which is to introduce a positive characteristic analogue of the $\delta$-invariant ([FO18, BlJ20]). In the next section, we use it to formulate a positive characteristics analogue of special $K$-semistability (Definition 2.14).

Since the following arguments work more generally, i.e., not only for reductions of arithmetic models, we use the following (compatible) notation in this subsection §3.

Notation 3 (Positive characteristic setup). $X_p$ is a projective scheme over a field of positive characteristic, and $L_p$ is an ample line bundle on it. Unlike Notation 1, $(X_p, L_p)$ does not necessarily lift to $O_{\overline{Q}}$, the ring of integers in $\overline{Q}$.

Definition 2.16 (Frobenius $\delta$-invariant). For a triple $(X_p, \Delta, L_p)$ of geometrically normal projective variety $X_p$ over a field of characteristic $p > 0$, its effective $\mathbb{Q}$-Weil divisor such that $K_{X_p} + \Delta$ is $\mathbb{Q}$-Cartier, $(X_p, \Delta)$ being locally F-pure ([HW02, Definition 2.1]) an ample line bundle $L_p$ over $X_p$, we consider the following invariants.

(i) For a positive integer $k$, we set the $k$-(quantized) Frobenius $\delta$-invariant

$$\delta^F_{(X_p, \Delta), k}(L_p) := \inf_{\substack{D: \text{k-basis type} \atop (L_p \sim \mathcal{O}D)}} \text{Fpt}((X_p, \Delta); D),$$

where Fpt denotes the F-pure threshold

$$\sup\{c \mid (X_p, \Delta + cD) \text{ is locally F-pure}\}$$

as originally introduced in [TW04, §2] for the affine setup. Here, $D$ runs over all $k$-basis type divisors for $L_p$ in the sense of [FO18, Definition 0.1], [BlJ20, Introduction].

(ii) Then we define the Frobenius $\delta$-invariant as

$$\delta^F_{(X_p, \Delta)}(L_p) := \lim_{k \to \infty} \inf \delta^F_{(X_p, \Delta), k}(L_p).$$

Recall that if we replace Fpt by lct, the above is nothing but $\delta((X_p, \Delta)(L_p)$ in the original form [FO18] (see also [BlJ20]). We sometimes omit $(X_p, \Delta)$ from the subscripts in the
above and simply write \( \delta^F_p(L_p) \) and \( \delta^F(L_p) \) respectively. On the other hand, we can define \( \delta^{F}_{(X_p,\Delta),k}(\overline{L}_p) \) and \( \delta^{F}_{(X_p,\Delta)}(\overline{L}_p) \) in the same way, i.e.
\[
\delta^{F}_{(X_p,\Delta),k}(\overline{L}_p) := \inf_{D: k\text{-basis type}} \text{Fpt}((X_p,\Delta); D),
\]
\[
\delta^{F}_{(X_p,\Delta)}(\overline{L}_p) := \lim_{k \to \infty} \inf_{D: k\text{-basis type}} \delta^{F}_{(X_p,\Delta),k}(\overline{L}_p),
\]
where \( (X_p,\Delta,\overline{L}_p) = (X_p,\Delta, L_p) \times_{\mathcal{O}_F/p} \overline{\mathcal{O}_F/p} \) and \( \overline{\mathcal{O}_F/p} \) denotes the algebraic closure of \( \mathcal{O}_F/p \). We sometimes simply write them as \( \delta^F_k(L_p) \) and \( \delta^F(L_p) \) respectively. We note that
\[
\delta^F_k(L_p) \leq \delta^F_k(L_p) \\
\delta^F(L_p) \leq \delta^F(L_p).
\]

We remark that there are some examples where the above inequality can be strict (indeed, similar arguments to Remark \[3.5\] applies to nontrivial twists of elliptic curves). Also note that by the simple combination of [HW02, 3.3], [TW04, 2.2(5)] and [BLJ20, A], we have \( \delta_{(X_p,\Delta)}(L) \geq \delta^F_{(X_p,\Delta)}(L_p) \). Note also that the above definition naturally extends to \( cL \) with a line bundle \( L \) and \( c \in \mathbb{R}_{>0} \) as \( \delta^F_{(X_p,\Delta),k}(cL_p) = \frac{1}{c} \delta^F_{(X_p,\Delta),k}(L_p) \).

2.3. Twisted analogue of [Od15]. We now go back to the Arakelov geometric setup, and discuss after Notation \[\square\] henceforth. The following are auxiliary “twisted” analogues of the original Arakelov K-energy (Definition \[\square\] of [Od15]) and its variants. The “twist” here refers to consideration of (again) an additional hermitian-metrized line bundle \( (H,h) \) so that the original untwisted setup means the case when \( H = \mathcal{O}_X \) and \( h \) is trivial metric over any infinite place. See e.g., [Der16] for more background.

We refrain from considering any boundary, i.e., “logarithmic” extension with mild singularities, to avoid non-substantial technical complications. We use these to partially prove Conjecture \[\square\] resulting to our main Theorem \[3.1\].

**Definition 2.17.** Fix an ample-polarized normal projective model \((X,L)\) over \( \mathcal{O}_F \) with \( h_L \) as Notation \[\square\]

(i) For a metrized line bundle \( \overline{H} = (H,h) \) on \( X \), we define the \((H,h)\)-twisted Arakelov-K-energy as
\[
h^{Av}_{K,H}(X,L,h_L).
\]

\[
:= \int \left\{ n(L_\eta^{n-1}.K_{X_\eta} \otimes H|_{X_\eta})(\overline{L}_n)^{n+1} + (\overline{L}^{h_L}/(n+1)(L_\eta)^n \right\}.
\]

In the above, the intersection numbers of the metrized line bundles on the total spaces are that of [GS90] and the superscript \( \text{Ric}(\omega_{h_L}) \) means the metrization of \( K_{X(\mathbb{C})} \) which corresponds to the Monge-Ampere measure \( \omega_{h_L}^{n} \). Note that if \((H,h)\) is trivial, the above quantity is nothing but Definition \[\square\] [Od15, Definition 2.4] modulo a normalizing constant \((n+1)(L_\eta)^n\).
(ii) For a line bundle $H$ on $X$, with a real type hermitian metric $h$ on $H|_{X_{\eta}}(\mathbb{C})$, we suppose $L_{\eta} = (K_{X_{\eta}} \otimes H|_{X_{\eta}})^{\nu}$. Then, we define the $(H, h)$-twisted Arakelov-Ding functional as

$$D_{H}^{\text{Ar}}(X, L, h_{L}) := \frac{1}{[F : \mathbb{Q}]} \left\{ - \frac{(L)^{n+1}}{(n+1)(L_{\eta})^{n}} + \deg H^{0}(K_{X/\mathcal{O}_{F}} \otimes \overline{H}^{h} \otimes \mathcal{L}^{h_{L}}) \right\},$$

where $H^{0}(K_{X/\mathcal{O}_{F}} \otimes \overline{H}^{h} \otimes \mathcal{L}^{h_{L}})$ is associated with the $L^{2}$-metric.

(iii) For a metrized line bundle $\overline{H} = (H, h)$ on $X$, we set the Arakelov-$J^{H}$-energy as

$$J_{\text{Ar}, H}(X, L, h_{L}) := \frac{1}{[F : \mathbb{Q}]} \left\{ - \frac{n(L_{\eta}^{n-1}H|_{X_{\eta}})}{(n+1)(L_{\eta})^{n}} \right\}.$$

We note that $h_{K,\overline{H}}^{\text{Ar}}(X, L, h_{L}) = h_{K}^{\text{Ar}}(X, L, h_{L}) + J_{\text{Ar}, \overline{H}}(X, L, h_{L})$ holds.

**Lemma 2.18.** If $L_{\eta} = (K_{X_{\eta}} \otimes H|_{X_{\eta}})^{\nu}$, then

$$h_{K,\overline{H}}^{\text{Ar}}(X, L, h_{L}) \geq D_{H}^{\text{Ar}}(X, L, h_{L}) - (L_{\eta}^{n}) \log(L_{\eta}^{n}),$$

so that

$$h_{K,\overline{H}}^{\text{Ar}}(X, L, h_{L}) \geq J_{\text{Ar}, \overline{H}}(X, L, h_{L}) + D_{H}^{\text{Ar}}(X, L, h_{L}) - (L_{\eta}^{n}) \log(L_{\eta}^{n}).$$

Furthermore, equality holds if $L = (K_{X} \otimes H)^{\nu}$ and $\omega_{h_{L}}$ is the $\omega_{h}$-twisted Kähler-Einstein metric, where $\omega_{h}$ is the curvature form of $h$.

The non-twisted version is discussed in [AB22, Prop 7.3], which we generalize here.

**Proof.** Since $(L \otimes H)|_{X_{\eta}} = -K_{X_{\eta}}$, its hermitian metric $h_{L} \cdot h$ determines a (non-holomorphic) volume form $\nu$ on $X(\mathbb{C})$. Then,

$$h_{K,\overline{H}}^{\text{Ar}}(X, L, h_{L}) - D_{H}^{\text{Ar}}(X, L, h_{L})$$

$$= \frac{1}{(n+1)(L_{\eta})^{n}}((\mathcal{L}^{h_{L}})^{n} \mathcal{L}^{h_{L}} \otimes K_{X/\mathcal{O}_{F}}^{\text{Ric}(\omega_{h_{L}})})$$

$$- \deg H^{0}(X, \mathcal{L}^{h_{L}} \otimes K_{X/\mathcal{O}_{F}}^{\text{Ric}(\omega_{h_{L}})} \otimes \overline{H}^{h}),$$

where $H^{0}(X, \mathcal{L}^{h_{L}} \otimes K_{X/\mathcal{O}_{F}}^{\text{Ric}(\omega_{h_{L}})} \otimes \overline{H}^{h})$ is regarded as a $\mathcal{O}_{F}$-module with the $L^{2}$-metric. If we take a section $s$ of $L \otimes H \otimes K_{X/\mathcal{O}_{F}}$ which is non-vanishing at the generic fiber, it decides an effective vertical divisor $D = \text{div}(s)$, which we further decompose as $\text{div}(D_{F}) + D'$ where $D_{F}$ is a divisor of $\mathcal{O}_{F}$ and $D'$ is still effective which does not contain any non-trivial (scheme-theoretic) fiber. Note that the weight of our metric on $L(\mathbb{C}) \otimes H(\mathbb{C}) \otimes K_{X(\mathbb{C})}$ is $\log \frac{\omega_{h_{L}}}{\nu}$. Hence, we continue the standard calculation as

$$\geq \int_{X(\mathbb{C})} \log(L_{\eta}^{n}) \omega_{h_{L}}^{n} = (L_{\eta}^{n}) \log(L_{\eta}^{n}) + \int_{X(\mathbb{C})} \log \frac{\omega_{h_{L}}}{\nu} \omega_{h_{L}}^{n}. $$
We finally apply the Jensen’s inequality for the logarithmic function to the last relative entropy term to finish the proof. □

3. Main theorem and the proof

Now, we are ready to state and prove our main theorem as follows. It partially proves “if direction” of Conjecture 1.2 for the case of specially K-stable varieties in the sense of [Hat22b]. A point is that, nevertheless of the adjective “special”, it broadly includes many cases, hence in particular generalizing the results of [Od15, Theorem 3.14].

**Theorem 3.1** (Main theorem). Suppose that there exists an ample-polarized (metrized) normal projective model \((X, L) = (X, L, h_{\text{cscK}})\), whose \(\omega_{X/\mathcal{O}_F}\) is \(\mathbb{Q}\)-Cartier, satisfying the following:

(i) (at complex place) \((X_\mathbb{C}, L_\mathbb{C})\) has a constant scalar curvature Kähler metric \(\omega_{h_{\text{cscK}}} \) with the Kähler class \(2\pi c_1(L_\mathbb{C})\).

(ii) (on reductions) For each prime ideal \(\mathfrak{p}\) of \(\mathcal{O}_F\), \(X_\mathfrak{p} := X_\mathfrak{p} \times_{\mathcal{O}_F/\mathfrak{p}} \overline{\mathcal{O}_F/\mathfrak{p}}\) is locally F-pure, \((X_\mathfrak{p}, L_\mathfrak{p})\) is \(J_{K_{X_\mathfrak{p}}+\delta F(L_\mathfrak{p})L_\mathfrak{p}}\)-semistable (cf. Remark 2.5), and \(K_{X_\mathfrak{p}}+\delta F(L_\mathfrak{p})L_\mathfrak{p}\) is nef.

Then, \(h_K(X, L, h_{\text{cscK}})\) attains the minimum among \(h_K(X', L', h'_L)\) for all metrized ample polarized models \((X', L', h'_L)\) with the same generic fibers \((X_\eta, L_\eta)\).

**Remark 3.2.** For the complex place, we remark that if \((X_\mathbb{C}, L_\mathbb{C})\) is specially K-stable, then it implies the condition (i) i.e., there exists a unique metric \(h_L\) such that \(\omega_{h_L}\) has a constant scalar curvature by [Zha21b, Corollary 5.2] and [CC21, Theorem 4.1].

The assumption (ii) is clearly a positive characteristic analogue of the special K-semistability (cf., Definition 2.14). Therefore, roughly speaking, the above three assumptions are analogues of special K-(semi)stabilities for each place.

By [Od15, §2], the obtained minimum gives a generalization of the Faltings height for abelian varieties ([Fal83]). Although the “special K-stability” type assumptions in Theorem 3.1 on \((X, L)\) may look quite technical, many examples (compared with [Od15, 3.14]) should satisfy as [Hat22b, Theorem 3.12] summarizes (also cf., §2.1.4, [Hat21]).

**Example 3.3.**

(i) Either if \(X_\eta\) is smooth proper curve of genus \(g \geq 2\) and all reductions are stable curves, or if \(X_\eta\) is smooth elliptic curve and all reductions are \(I_m\)-type reductions for \(m \geq 1\), then these classical examples of curves satisfy (3.1).

(ii) In the case when \(\dim X_\eta = 2\), if \(X_\eta\) (resp., \(X_\mathfrak{p}\)) is a smooth (resp., F-pure) minimal model, whose \(L_\eta\) (resp., \(L_\mathfrak{p}\)) is close enough to \(K_{X_\eta}\) (resp., \(K_{X_\mathfrak{p}}\)), the assumption (ii) is satisfied by [Hat21, §8] and the assumption (i) also holds by loc.cit. [Zha21b] (cf., also the earlier references therein. In [Hat21], we do not deal with the positive characteristic case, but we can also show the special K-stability of klt minimal models of dimension two).

(iii) For any projective module \(E\) over \(\mathcal{O}_F\), \((\mathbb{P}(\mathcal{E}), \mathcal{O}(1))\) satisfies the above conditions of Theorem 3.1 and hence satisfy the arithmetic Yau-Tian-Donaldson conjecture 1.2 which is not confirmed in [Od15].
(iv) More generally, we expect that most of K-stable Fano varieties over \( \mathbb{Q} \) have some polarized models which satisfies the conditions of Theorem 3.1. For instance, recently the first author, S. Pandey, T. Takamatsu confirmed that general del Pezzo surfaces \( S \) of degree 1, whose \( \vert -K_S \vert \) contains only elliptic curves or rational curves with only one nodal singularity, have \( \delta^F(S, -K_S) > 1 \).

To show Theorem 3.1, we prepare a lemma below, which is a mixed characteristic analogue of [BL22, Theorem 6.6] and [Xu24, Theorem 7.27].

**Lemma 3.4.** Let \( (X, L) \) be an ample-polarized (metrized) normal projective model. Let \( p \) be a prime ideal of \( \mathcal{O}_F \) such that \( X_p \) is locally \( F \)-pure. Then \( \delta(L_p) \leq \delta(X, L) \).

**Proof.** This lemma follows from [ST21, Corollary 3.9] and a similar argument of [Xu24, Theorem 7.27], but we give a proof for the reader’s convenience here.

For any \( \epsilon > 0 \) and sufficiently divisible integer \( k \in \mathbb{Z}_{>0} \), we have \( \delta_k(X, L) \leq \delta(X, L) + \epsilon \). Let \( F \) be the algebraic closure of \( F \) and \( X \eta := X \eta \times \text{spec} F \text{ spec } F \). We note that \( \delta_k(X, L) = \delta_k(X, L) \). It is well-known to experts but we write the proof of this fact. First, it is easy to see that \( \delta_k(X, L) \geq \delta_k(X, L) \) since the log canonicity is stable under changes of algebraically closed base fields. Next, we argue the converse inequality. Recall that \( \mathbb{P}^{h_0(X, L_F \oplus_k)}-\text{1} \text{\ parametrizes effective divisors linearly equivalent to } L_F \text{ and let } D \subset X \times \mathbb{P}^{h_0(X, L_F \oplus_k)}-\text{1} \text{ be the universal divisor. Then, we set } D' \subset X \times \mathbb{P}^{h_0(X, L_F \oplus_k)}-\text{1} \text{ as}
\[
D' := \frac{1}{kh_0(X, L_F \oplus_k)} \sum_{i=1}^{h_0(X, L_F \oplus_k)} \text{pr}_1^* D,
\]
where \( \text{pr}_1 : X \times \mathbb{P}^{h_0(X, L_F \oplus_k)}-\text{1} \times h_0(X, L_F \oplus_k) \rightarrow X \times \mathbb{P}^{h_0(X, L_F \oplus_k)}-\text{1} \text{ is induced by the } i\text{-th projection of } \mathbb{P}^{h_0(X, L_F \oplus_k)}-\text{1} \times h_0(X, L_F \oplus_k) \text{, and}
\]
\[
U := \{ s \in \mathbb{P}^{h_0(X, L_F \oplus_k)}-\text{1} \times h_0(X, L_F \oplus_k) | D'_s \text{ is } k\text{-basis type}\}.
\]
It is easy to see that the fiber of \( (X \times \mathbb{P}^{h_0(X, L_F \oplus_k)}-\text{1} \times h_0(X, L_F \oplus_k), \delta_k(X, L)D') \) over any geometric point \( \bar{s} \in U \) is log canonical. For any \( k\text{-basis type divisor } D_{\mathcal{C}} \text{ in } L_{\mathcal{C}} \text{, we see that } (X_{\mathcal{C}}, D_{\mathcal{C}}) \text{ is the base change of some geometric fiber of } (X \times \mathbb{P}^{h_0(X, L_F \oplus_k)}-\text{1} \times h_0(X, L_F \oplus_k), \delta_k(X, L)D') \text{ over } \mathbb{P}^{h_0(X, L_F \oplus_k)}-\text{1} \times h_0(X, L_F \oplus_k) \text{ and hence log canonical. Thus, we conclude that } \delta_k(X, L) \leq \delta_k(X, L) \).

Let \( E_F \) be a prime divisor over \( X \) such that \( E_F \) computes \( \delta_k(X, L) \), i.e., there exists a \( k\text{-basis type divisor } D_{k,F} \sim q L_F \text{ such that } (X \times \mathbb{P}^{h_0(X, L_F \oplus_k)}-\text{1} \times h_0(X, L_F \oplus_k), \delta_k(X, L)D') \text{ is log canonical and } E_F \text{ is an lc place. Then, we can take a finite field extension } K \text{ of } F \text{ such that}
\]
\[
\delta_k(X, L) = \delta_k(X, L) = \delta_k(X, L)
\]
and there exists a prime divisor \( E_K \) over \( X_K := X \eta \times \text{spec } F \text{ spec } K \) that computes \( \delta_k(X_K, L_K) \), where \( L_K \) is the pullback of \( L \). Let \( \mathcal{O}_K \) be the integral closure of \( \mathcal{O}_F \) in \( K \) and \( \eta' \) the generic point of \( \text{spec } (\mathcal{O}_K) \). Let \( \mathcal{F}_{E_K} \) be the filtration of \( H_0(X_K, L_K \oplus_k) \) which is defined by \( E_K \), that is
\[
\mathcal{F}_{E_K} H_0(X_K, L_K \oplus_k) := H_0(X_K, L_K \oplus_k (-\lambda E_K))
\]
for any \( \lambda \in \mathbb{Z} \). Let \( \pi_K: X \times \text{Spec}(\mathcal{O}_P) \to \text{Spec}(\mathcal{O}_K) \) and \( \mu: X \times \text{Spec}(\mathcal{O}_P) \to X \) be the canonical morphisms. By the properness of the flag varieties and the fact that \( \mathcal{O}_K \) is a Dedekind domain, we see that there exists a filtration \( \mathcal{F} \) of the sheaf \( \pi_*(\mathcal{O}_{X \times \text{Spec}(\mathcal{O}_P)}|_{\text{Spec}(\mathcal{O}_K)}) \) such that

\[
\mathcal{F}^\lambda \pi_*(\mathcal{O}_{X \times \text{Spec}(\mathcal{O}_P)}|_{\text{Spec}(\mathcal{O}_K)}) \big|_{\eta'} = \mathcal{F}^\lambda_{E_K} H^0(X_K, L^\otimes_k)
\]

is flat over \( \text{Spec}(\mathcal{O}_K) \) for any \( \lambda \in \mathbb{Z} \). Take a prime ideal \( \mathfrak{p}' \) of \( \mathcal{O}_K \) that is mapped to \( \mathfrak{p} \in \text{Spec}(\mathcal{O}_F) \). Then we can choose a free basis \( \{s_1, \ldots, s_{h^0(X_K, L^\otimes_k)}\} \) of \( \pi_*(\mathcal{O}_{X \times \text{Spec}(\mathcal{O}_P)}|_{\text{Spec}(\mathcal{O}_K)}) \otimes_{\mathcal{O}_K} \mathcal{O}_{K, \mathfrak{p}}' \) such that for any \( \lambda \), we can choose a subset of \( \{s_1, \ldots, s_{h^0(X_K, L^\otimes_k)}\} \) that is a free basis of \( \mathcal{F}^\lambda \pi_*(\mathcal{O}_{X \times \text{Spec}(\mathcal{O}_P)}|_{\text{Spec}(\mathcal{O}_K)}) \otimes_{\mathcal{O}_K} \mathcal{O}_{K, \mathfrak{p}}' \).

Let \( D = \frac{1}{kh^0(X_K, L^\otimes_k)} \sum_{j=1}^{h^0(X_K, L^\otimes_k)} \text{div}(s_j) \) on \( X \times \text{Spec}(\mathcal{O}_P) \to \text{Spec}(\mathcal{O}_K) \). By taking \( k \) large enough, we may assume that \( H^1(X_p, L^\otimes_k) = 0 \). Then, \( D_{\mathfrak{p}}' \) and \( D_K \) are \( k \)-basis type divisors. By the choice of \( D \) and the proof of [FO13, Lemma 2.2], we see that \( D_K \) attains \( \max_{D_{\mathfrak{p}}'} \ord_{E_K}(D_{\mathfrak{p}}') \), where \( D_{\mathfrak{p}}' \) runs over all \( k \)-basis type divisors, and hence

\[
\delta_k(X_K, L_K) = \text{lct}(X_K; D_K) = \frac{A_{X_K}(E_K)}{\ord_{E_K}(D_K)}.
\]

Therefore, it follows from [ST21, Corollary 3.9] that \( \text{Fpt}(X_{\mathfrak{p}}', D_{\mathfrak{p}}') \leq \delta_k(X_K, L_K) \), which means that

\[
\delta^F_{X_{\mathfrak{p}}', k}(L_{\mathfrak{p}}') \leq \delta_k(X_K, L_K) = \delta_k(X_C, L_C).
\]

By the definition of F-purity (cf. [ST21, Definition 2.7]) and Definition 2.16, we have

\[
\delta^F_{\overline{\mathcal{L}_{\mathfrak{p}}}} \leq \delta^F_{X_{\mathfrak{p}}', k}(L_{\mathfrak{p}}').
\]

This shows

\[
\delta^F_k(\overline{\mathcal{L}_{\mathfrak{p}}}) \leq \delta_k(X_C, L_C) \leq \delta(X_C, L_C) + \epsilon
\]

for any \( \epsilon > 0 \) and sufficiently large \( k \). Therefore, we have

\[
\delta^F(\overline{\mathcal{L}_{\mathfrak{p}}}) = \liminf_{k \to \infty} \delta^F_k(\overline{\mathcal{L}_{\mathfrak{p}}}) \leq \delta(X_C, L_C) + \epsilon.
\]

Thus, we have \( \delta^F(\overline{\mathcal{L}_{\mathfrak{p}}}) \leq \delta(X_C, L_C) \) and complete the proof. \( \square \)

Remark 3.5. By the same argument as the above proof, we see that \( \delta^F(\mathcal{L}_{\mathfrak{p}}) \leq \delta(X_{\mathfrak{p}}, L_{\mathfrak{p}}) \). However, we cannot replace \( \delta^F(\mathcal{L}_{\mathfrak{p}}) \) with \( \delta^F(\mathcal{L}_{\mathfrak{p}}) \) in the statement of Lemma 3.2 since the inequality \( \delta(X_{\mathfrak{p}}, L_{\mathfrak{p}}) \geq \delta(X_C, L_C) \) could be strict in general. Indeed, we have the following example. Let \( K \) be a finitely generated field over \( \mathbb{C} \) whose transcendence degree is two. It is well-known (cf. [Har77 III, Exercise 9.10 (b)]) that there exists a proper smooth variety \( X_K \) over \( K \) with ample \(-K_{X_K} \) such that \( X_K \times \text{Spec}(K) \cong \mathbb{P}^1_K \) but \( X_K \not\cong \mathbb{P}^1_K \). Even though the K-semistability of \((X_K, -K_{X_K})\) and \((\mathbb{P}^1_K, -K_{\mathbb{P}^1_K})\) are equivalent by [Zhu21, Theorem 1.1], we have \( \delta(X_K, -K_{X_K}) \neq \delta(\mathbb{P}^1_K, -K_{\mathbb{P}^1_K}) \) in this case. We see this fact as follows. It is not hard to see that \( E \times \text{Spec}(K) \) is a union of distinct \( n_E \)-points on \( \mathbb{P}^1_K \) for every prime divisor \( E \) over \( X_K \). Since \( X_K \not\cong \mathbb{P}^1_K \), we have that \( n_E > 1 \). This means that for any nonzero section \( s \in H^0(X_K, -mK_{X_K}) \), the
pullback of $s$ to $\mathbb{P}^1_K$ has the same vanishing order on each point of $E \times_{\text{Spec}(K)} \text{Spec}(\mathbb{K})$. Therefore,
\[
\delta(X_K, -K_{X_K}) = \inf_E \frac{\deg(-K_{X_K}) \cdot \text{A}_X(E)}{\int_0^\infty \text{vol}(-K_{X_K} - xE)dx} = \inf_E \frac{2}{\int_0^{n_E} (2 - n_E x)dx} = \inf_E n_E > 1,
\]
where $E$ runs over all prime divisors over $X_K$ and note that $\text{vol}(-K_{X_K} - xE) = \max\{0, \deg(-K_{X_K} - xE)\}$. Here, we used the fact that $\text{A}_X(E) = 1$ and [BLJ20 Corollary 3.9, Theorem 4.4]. On the other hand, it is well-known that $\delta(\mathbb{P}^1_K, -K_{\mathbb{P}^1_K}) = 1$. Thus, $\delta(\mathbb{P}^1_K, -K_{\mathbb{P}^1_K}) < \delta(X_K, -K_{X_K})$.

We prepare the following application of the Fujita vanishing theorem (cf. [Fuj17 Theorem 3.8.1]).

Lemma 3.6. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_A$, where $A$ is an Artinian local ring. Let $H$ be an ample line bundle on $\mathbb{P}^n_A$. Then we obtain the following.

(i) There exists $m \in \mathbb{Z}_{>0}$ depending only on $\mathcal{F}$ such that $H^j(\mathbb{P}^n_A, \mathcal{F}(mH + D)) = 0$ for any $j > 0$ and nef Cartier divisor $D$, and

(ii) for any nef Cartier divisor $D$, we have
\[
\text{length}_A(H^j(\mathbb{P}^n_A, \mathcal{F}(mD))) = O(m^{l-j})
\]
for any $j$ and sufficiently large $m$, where $l = \dim \text{Supp}(\mathcal{F})$. Here, $\text{length}_A$ denotes the length of an $A$-module.

Proof. First, we note that if $A$ is a field, then both (i) and (ii) hold. Indeed, (i) and (ii) are shown (cf. Fuj17 3.8.1, 3.9.1) when $A$ is an algebraically closed field. If $A$ is not algebraically closed, we conclude that (i) and (ii) also hold in this case by changing the base field $A$ to an algebraically closed field.

For general case, consider the following short exact sequence
\[
0 \to m^n\mathcal{F} \to m^{n-1}\mathcal{F} \to m^{n-1}\mathcal{F}/m^n\mathcal{F} \to 0
\]
for $n \in \mathbb{Z}_{>0}$, where $m$ is the maximal ideal of $A$. Since $A$ is Artinian, $m^n\mathcal{F} = 0$ for some $n$ and then $m^{n-1}\mathcal{F}$ is a sheaf on $\mathbb{P}^n_{A/m}$. It is easy to see the following for any $n$:

- if (i) and (ii) hold for $m^n\mathcal{F}$ and $m^{n-1}\mathcal{F}/m^n\mathcal{F}$, then (i) and (ii) also hold for $m^{n-1}\mathcal{F}$.

Therefore, (i) and (ii) hold for $\mathcal{F}$ by the induction on $n$. \hfill \Box

Proof of Theorem 3.1. Take any other positively metrized ample polarized model $(X', L', h')$ whose generic fiber is the same i.e., $(X_n, L_n)$. Then we can take a finite sequence of metrized polarized models $(X(k), L(k), h(k))$ for $k = 0, \cdots , m$ such that

(i) $(X(0), L(0), h(0)) = (X, L, h_{\text{cscK}}),$

(ii) $(X(1), L(1), h(1)) = (X, L, h'),$

(iii) $(X(m), L(m), h(m)) = (X', L', h'),$

(iv) For each $k \geq 1$, $(X(k), L(k), h(k))$ and $(X(k + 1), L(k + 1), h(k + 1))$ differs exactly at one non-archimedean place of $F$, which corresponds to $p_k \subset \mathcal{O}_F$ and are all distinct.
Note that \( L(i) \) is independent of \( i \).

Firstly, we have
\[
(4) \quad h_K(X(0), L(0), h(0)) = h_{cscK} \leq h_K(X(1), L(1), h(1) = h'),
\]
because of the change of metric formula (cf., e.g., [Od15, 2.2]) and the assumption that \( \omega_{h_{cscK}} \) is a cscK metric which minimizes the (complex) Mabuchi’s K-energy.

Next, we deal with the inequality
\[
(5) \quad h_K(X(k+1), L(k+1), h(k+1)) \geq h_K(X(k), L(k), h(k))
\]
for any \( k \geq 1 \), which completes the proof of Theorem 3.1. Indeed, combining (5) with (4), we have that
\[
\begin{align*}
& h_K(X(m), L(m), h(m)) - h_K(X(0), L(0), h(0)) \\
& = \sum_{k=0}^{m-1} h_K(X(k+1), L(k+1), h(k+1)) - h_K(X(k), L(k), h(k)) \geq 0.
\end{align*}
\]
First, we deal with (5) in the case when \( \delta^F(\overline{L}_{p_k}) > 0 \).

Case 1. \( \delta^F(\overline{L}_{p_k}) > 0 \). In this section, to show (5) for any \( k \geq 1 \), we change the models and reduce the argument to comparing Arakelov-Ding functionals and J-energies in the case when \( \delta^F(\overline{L}_{p_k}) > 0 \). Here, we note that \( h(k+1) = h(k) \). Take a normalized blow up \( \nu: X'(k+1) \to X(k+1) \) along some closed subschemes supported on \( X(k+1)_p \) such that there exists a proper birational morphism \( \mu: X'(k+1) \to X(k) \). We construct a model \( (X''(k+1), L''(k+1), h(k)) \) (resp. \( (X''(k), L''(k), h(k)) \)), where there exists a canonical projective birational morphism \( \mu'': X''(k+1) \to X''(k) \), by patching \( (X(0), L(0), h(k)) \times_{SpecO_F} (SpecO_F \setminus \{p_k\}) \) and \( (X'(k+1), \nu'^*L(k+1), h(k)) \times_{SpecO_F} (SpecO_F \setminus \{p_j\}_{j \neq k}) \) together over Spec\(O_F \setminus \{p_1, \ldots, p_m\} \). Then, it is easy to see that
\[
(6) \quad h_K(X(k+1), L(k+1), h(k)) - h_K(X(k), L(k), h(k))
= h_K(X''(k+1), L''(k+1), h(k)) - h_K(X''(k), L''(k), h(k))
= h_K(X''(k+1), L''(k+1), h(k)) - h_K(X''(k), L''(k), h(k)).
\]
On the other hand, for any sufficiently small \( \gamma > 0 \) such that \( \delta^F(\overline{L}_{p_k}) - \gamma \in \mathbb{Q}_{>0} \), by replacing \( \overline{L} \) with \( (\delta^F(\overline{L}_{p_k}) - \gamma)\overline{L} \) and setting \( \epsilon := \frac{2\gamma}{\delta^F(\overline{L}_{p_k}) - \gamma} \), we may assume that
\[
\delta^F(\overline{L}_{p_k}) > 1 \text{ and } H(k) := K_{X''(k)} + (1 + \epsilon)L''(k) \text{ is ample on } X''_{p_k}.
\]
Then we also have \( \delta(X_C, L_C) > 1 \) by Lemma 3.4. Take an arbitrary hermitian metric \( h_{H(k)} \) on \( H(k)C \) and set \( H(k) = (H(k), h_{H(k)}) \). By [Zha21b, Theorem 2.3], we have a unique \( -\omega_{H(k)} + \epsilon \omega_{H(k)} \)-twisted Kähler-Einstein metric \( \omega_{H_L(k)} \) in \( 2\pi C_1(L_C) \). Now, we claim the following.

Claim 1. Suppose that
\[
(7) \quad h_{K, e(\mu''*L''(k), h_{L}(k))}(X''(k+1), L''(k+1), h_L(k)) - h_{K, e(L''(k), h_{L}(k))}(X''(k), L''(k), h_L(k)) \geq 0
\]
is nonnegative for any sufficiently small \( \gamma \) and \( \epsilon > 0 \). Then, the inequality (5) holds.

Proof. Assume that Claim 1 fails. Then
\[
h_K(X(k+1), L(k+1), h(k)) - h_K(X(k), L(k), h(k)) < 0.
\]
Note that
\[
\lim_{\epsilon \to 0} \left( h_{K, \epsilon^{\mu''} L''(k), h(k)}(X''(k + 1), L''(k + 1), h(k))
\right.
\]
\[- h_{K, \epsilon^{\mu''} L''(k), h(k)}(X''(k), L''(k), h(k)) \right)
= h_K(X''(k + 1), L''(k + 1), h(k)) - h_K(X''(k), L''(k), h(k)).
\]
By the above equation and (8), we can take sufficiently small \(\epsilon\) such that (8)
\[
h_{K, \epsilon^{\mu''} L''(k), h(k)}(X''(k + 1), L''(k + 1), h(k)) - h_{K, \epsilon^{\mu''} L''(k), h(k)}(X''(k), L''(k), h(k)) < 0.
\]
By the change of metric formula (cf., e.g., [Od15 2.2]), we have that
\[
h_{K, \epsilon^{\mu''} L''(k), h_L(k)}(X''(k + 1), L''(k + 1), h_L(k)) - h_{K, \epsilon^{\mu''} L''(k), h_L(k)}(X''(k), L''(k), h_L(k))
= h_{K, \epsilon^{\mu''} L''(k), h_L(k)}(X''(k + 1), L''(k + 1), h_L(k)) - h_{K, \epsilon^{\mu''} L''(k), h_L(k)}(X''(k), L''(k), h_L(k))
\]
for any \(\epsilon \geq 0\). By the above equation and (8), we have that
\[
h_{K, \epsilon^{\mu''} L''(k), h_L(k)}(X''(k + 1), L''(k + 1), h_L(k)) - h_{K, \epsilon^{\mu''} L''(k), h_L(k)}(X''(k), L''(k), h_L(k)) < 0.
\]
This contradicts to the assumption that (7) is nonnegative for any sufficiently small \(\epsilon > 0\). We complete the proof of Claim 1.

From now, we fix a sufficiently small \(\epsilon > 0\) and deal with (7). By Lemma 2.18, \(K_{X''(k)} + \epsilon L''(k) - H(k) = -L''(k)\) and the property of \(\omega_{h_L(k)}\), we have that
\[
h_{K, \epsilon^{\mu''} L''(k), h_L(k)}(X''(k + 1), L''(k + 1), h_L(k))
\geq \mathcal{J}_L^{A_{\mu''} H(k)}(X''(k + 1), L''(k + 1), h_L(k))
+ \mathcal{D}_L^{A_{\mu''} \epsilon \{} L''(k) - H(k)\}(X''(k + 1), L''(k + 1), h_L(k))
- (L''_\eta) \log (L''_\eta), \quad \text{and}
\]
\[
h_{K, \epsilon^{\mu''} L''(k), h_L(k)}(X''(k), L''(k), h_L(k)) = \mathcal{J}_L^{A_{\mu''} H(k)}(X''(k), L''(k), h_L(k))
+ \mathcal{D}_L^{A_{\mu''} \epsilon \{} L''(k) - H(k)\}(X''(k), L''(k), h_L(k))
- (L''_\eta) \log (L''_\eta).
\]
Therefore,
\[
(7) \geq \mathcal{J}_L^{A_{\mu''} H(k)}(X''(k + 1), L''(k + 1), h_L(k)) - \mathcal{J}_L^{A_{\mu''} H(k)}(X''(k), L''(k), h_L(k))
+ \mathcal{D}_L^{A_{\mu''} \epsilon \{} L''(k) - H(k)\}(X''(k + 1), L''(k + 1), h_L(k)) - \mathcal{D}_L^{A_{\mu''} \epsilon \{} L''(k) - H(k)\}(X''(k), L''(k), h_L(k)).
\]
To show (7) is nonnegative, it suffices to show the following values are nonnegative:
(9) \(\mathcal{J}_L^{A_{\mu''} H(k)}(X''(k + 1), L''(k + 1), h_L(k)) - \mathcal{J}_L^{A_{\mu''} H(k)}(X''(k), L''(k), h_L(k))\)
(10) \(\mathcal{D}_L^{A_{\mu''} \epsilon \{} L''(k) - H(k)\}(X''(k + 1), L''(k + 1), h_L(k)) - \mathcal{D}_L^{A_{\mu''} \epsilon \{} L''(k) - H(k)\}(X''(k), L''(k), h_L(k)).\)

Next, we apply the same arguments as [Hat22b 3.15] (comparing twisted Arakelov J-energy) to show (9) is nonnegative as we recap as follows. First, let \(f(k) : X''(k) \to \text{Spec}(\mathcal{O}_F)\) be the canonical morphism. As [Hat22b 3.15], we may assume that \(E := \mu'' L(k) - L''(k + 1)\) is an effective divisor supported on \(X''(k + 1)\). We note that
(X_{p_k}, L_{p_k}) is \( J^{H_{p_k}} \)-semistable and \( H_{p_k} \) is ample by the choice of \( H_{p_k} \). After [Hat22b], we construct a filtration \( \mathcal{F} \) for \((X''(k)_{p}, L''(k)_{p}) = (X(k), L(k))\) from \((X''(k), L''(k))\) and \((X''(k+1), L''(k+1))\). For each \( m \geq 0 \), we first take the filtration of \( f(k)_*(L''(k)^{\otimes m}) \)

\[
\mathcal{F}^i f(k)_*(L''(k)^{\otimes m}) := \begin{cases} 
\mathcal{P}_k^i ((f(k) \circ \mu''_*) L''(k+1)^{\otimes m}) & \text{for } i \leq 0 \\
0 & \text{for } i > 0,
\end{cases}
\]

and then we set

\[
\mathcal{F}^i H^0(X(k)_{p_k}, L(k)^{\otimes m})
\]

as the images of \( \mathcal{F}^i f(k)_*(L''(k)^{\otimes m}) \to H^0(X(k)_{p_k}, L(k)^{\otimes m}) \). It is easy to check that \( \mathcal{F} \) satisfies Definition [2.7]. We set as in [Hat22b, Theorem 3.5] the following value

\[
w_{\mathcal{F}}(m) := \sum_{i=0}^{\infty} i \cdot \text{length}_{\mathcal{O}_F}(\mathcal{F}^i H^0(X(k)_{p_k}, L(k)^{\otimes m})/\mathcal{F}^{i+1} H^0(X(k)_{p_k}, L(k)^{\otimes m})).
\]

We note that all but finitely many terms in the above sum are zero. On the other hand, the value (11) equals to

\[
\frac{1}{[F: \mathbb{Q}] L^m_{\eta}} \left( -E \cdot \sum_{j=0}^{n-1} L''(k+1)^j \cdot \mu''_* L''(k)^{n-1-j} \right) + \frac{n H_{\eta} \cdot L_{\eta}^{n-1}}{(n+1)L^m_{\eta}} E \cdot \left( \sum_{j=0}^{n} L''(k+1)^j \cdot \mu''_* L''(k)^{n-j} \right).
\]

Since the support of \( E \) is proper, the above intersection numbers are well-defined. To show this value is nonnegative, we may assume that \( L''(k+1) \) is relatively ample by perturbing the coefficients of \( E \). Then, we note that the following claim holds as [Hat22b, Theorem 3.5]. We remark that we cannot directly apply [Hat22b, Theorem 3.5] to obtain the following claim since we assumed there that the base curve \( C \) is proper.

**Claim 2.**

\[
\lim_{m \to \infty} \frac{(n+1)!}{m^{n+1}} w_{\mathcal{F}}(m) = -E \cdot \left( \sum_{j=0}^{n} L''(k+1)^j \cdot \mu''_* L''(k)^{n-j} \right).
\]

**Proof.** As the proof of [Hat22b, Theorem 3.5], we see that

\[
w_{\mathcal{F}}(m) = -\text{length}_{\mathcal{O}_F}(f(k)_*(L''(k)^{\otimes m})/(f(k) \circ \mu''_*) (L''(k+1)^{\otimes m}))
\]

for any sufficiently large and divisible \( m \in \mathbb{Z}_{\geq 0} \). Thus, it suffices to show that

\[
\lim_{m \to \infty} \frac{(n+1)!}{m^{n+1}} \text{length}_{\mathcal{O}_F}(f(k)_*(L''(k)^{\otimes m})/(f(k) \circ \mu''_*) (L''(k+1)^{\otimes m}))
\]

\[
= -E \cdot \left( \sum_{j=0}^{n} L''(k+1)^j \cdot \mu''_* L''(k)^{n-j} \right).
\]
Note that there exists the following exact sequence of coherent sheaves on $X''(k+1)$ for any $i$ and $m$,

$$0 \to \mu^{\prime\prime} L''(k)^{\otimes m}(-i+1)E \to \mu^{\prime\prime} L''(k)^{\otimes m}(-iE) \to \mu^{\prime\prime} L''(k)^{\otimes m}(-iE)|_E \to 0.$$ 

Note also that the schematic image structure of $f(k)(E)$ is an Artinian scheme. By Lemma 3.6 applied to $(\mu^{\prime\prime} L''(k)^{\otimes m}(-iE))|_{p_k}$ and $E$, there exists $N > 0$ such that $R^j(f(k) \circ \mu^{\prime\prime})(\mu^{\prime\prime} L''(k)^{\otimes m}(-iE)) = 0$ around $p_k$ and $H^i(E, \mu^{\prime\prime} L''(k)^{\otimes m}(-iE)|_E) = 0$ for any sufficiently large $m$, $j > 0$ and $N \leq i \leq m$. Thus, the following injective homomorphism

$$(f(k) \circ \mu^{\prime\prime})_* (\mu^{\prime\prime} L''(k)^{\otimes m}(-iE))/(f(k) \circ \mu^{\prime\prime})_* (\mu^{\prime\prime} L''(k)^{\otimes m}(-i+1)E)$$

$$\to H^0(E, \mu^{\prime\prime} L''(k)^{\otimes m}(-iE)|_E)$$

is bijective and

$$\text{length}_{\mathcal{O}_E}(H^0(E, \mu^{\prime\prime} L''(k)^{\otimes m}(-iE)|_E)) = \chi(E, \mu^{\prime\prime} L''(k)^{\otimes m}(-iE)|_E)$$

for any $N \leq i \leq m−1$ and sufficiently large $m$. Here, we set

$$\chi(E, \mu^{\prime\prime} L''(k)^{\otimes m}(-iE)|_E) := \sum_{j=0}^n (-1)^j \text{length}_{\mathcal{O}_E}(H^j(E, \mu^{\prime\prime} L''(k)^{\otimes m}(-iE)|_E)).$$

On the other hand, we apply Lemma 3.6 to $H^j(E, \mu^{\prime\prime} L''(k)^{\otimes m}(-iE)|_E)$ and obtain that

$$\text{length}_{\mathcal{O}_E}(H^0(E, \mu^{\prime\prime} L''(k)^{\otimes m}(-iE)|_E)) = \chi(E, \mu^{\prime\prime} L''(k)^{\otimes m}(-iE)|_E) + O(m^{n−1})$$

for any $i$ and sufficiently large $m$. Note that $\chi(E, \mu^{\prime\prime} L''(k)^{\otimes m}(-iE)|_E)$ is a polynomial of $m$ and $i$ of degree $n$ with the leading term $\frac{m^n}{n!} (\mu^{\prime\prime} L''(k) − \frac{i}{m} E)^n \cdot E$ (cf. [F+05, Appendix B]). It means that

$$\text{length}_{\mathcal{O}_E}(f(k)_*(L''(k)^{\otimes m})/(f(k) \circ \mu^{\prime\prime})_* (L''(k+1)^{\otimes m}))$$

$$= \sum_{i=0}^{m-1} \text{length}_{\mathcal{O}_E}(f(k) \circ \mu^{\prime\prime})_* (\mu^{\prime\prime} L''(k)^{\otimes m}(-iE))/(f(k) \circ \mu^{\prime\prime})_* (\mu^{\prime\prime} L''(k)^{\otimes m}(-(i+1)E))$$

$$= \sum_{i=0}^{m-1} \frac{m^n}{n!} \left( \mu^{\prime\prime} L''(k) − \frac{i}{m} E \right)^n \cdot E + O(m^n).$$

By the dominated convergence theorem, we obtain

$$\lim_{m \to \infty} \frac{(n+1)!}{m^{n+1}} \left( \sum_{i=0}^{m-1} \frac{m^n}{n!} \left( \mu^{\prime\prime} L''(k) − \frac{i}{m} E \right)^n \cdot E + O(m^n) \right)$$

$$=(n+1) \int_0^1 (\mu^{\prime\prime} L''(k) − xE)^n \cdot Edx$$

$$=E \cdot \left( \sum_{j=0}^{n} L''(k+1)^j \cdot \mu^{\prime\prime} L''(k)^{n-j} \right),$$

which shows [12]. □
Take a sufficiently large integer $a > 0$ such that $a H(k)_{p_k}$ is very ample. Take a discrete valuation ring $R$ dominating $\mathcal{O}_{F, p_k}$ such that the residue field $R/\mathfrak{m}_R$ of $R$ is an uncountable algebraically closed field by \cite[Theorem 83]{Mat80}. Here, $\mathfrak{m}_R$ is the maximal ideal of $R$. Let $X''(k)_R := X''(k) \times_{\Spec(\mathcal{O}_F)} \Spec(R)$, $X''(k+1)_R := X''(k+1) \times_{\Spec(\mathcal{O}_F)} \Spec(R)$ and $g_R: X''(k)_R \to X''(k)$ be the canonical morphism. Then, we can take a general section $s \in H^0(X''(k)_R, \mathcal{O}_{X''(k)_R}(ag_R H(k)))$ such that $\div(s)$ satisfying the following (by the Bertini theorem for very ample divisors and the fact that $R/\mathfrak{m}_R$ is uncountable):

- $\div(s)|_{\mathfrak{m}_R}$ is reduced,
- $\div(s)|_{\mathfrak{m}_R}$ is compatible with an approximation $\{\mathcal{F}(t)\}_{t > 0}$ of the filtration $\mathcal{F}_R$, which is defined by

$$\mathcal{F}_R^\lambda H^0((X''(k)_R)_{\mathfrak{m}_R}, (L''(k)_R)_{\mathfrak{m}_R}^\otimes m) := \mathcal{F}_R^\lambda H^0(X(k)_{p_k}, L(k)_{p_k}^\otimes m) \otimes_{\mathcal{O}_F/p_k} (R/\mathfrak{m}_R)$$

(for the definition of the approximation $\{\mathcal{F}(t)\}_{t > 0}$, we refer to Definition 2.7), and
- the support of $\mu'' \div(s)$ contains no $\mu''$-exceptional divisor, where $\mu'' = X''(k+1)_{\mathfrak{m}_R} \to X''(k)_{\mathfrak{m}_R}$ is the morphism induced by $\mu''$.

The last condition implies that $\mu'' \div(s)$ is reduced at all points of $(X''(k+1)_{\mathfrak{m}_R})_{\mathfrak{m}_R}$ of codimension one. Thus, the reduced structure $\mu'' \div(s)_{\text{red}}$ of $\mu'' \div(s)$ is flat over $R$ and isomorphic to $\div(s)$ over the generic point of $\Spec(R)$. We construct a filtration $\mathcal{F}_{\div(s)}^\lambda H^0(\div(s)_{\mathfrak{m}_R}, (L''(k)_R)_{\mathfrak{m}_R}^\otimes m|_{\div(s)_{\mathfrak{m}_R}})$ for $\div(s)$ and $\mu'' \div(s)_{\text{red}}$ as (11). By Claim 2 applied to $\mathcal{F}_{\div(s)}$, we obtain that

$$J = \lim_{m \to \infty} \frac{1}{[F : \mathbb{Q}] L_\eta^m} \left( \frac{n! w_{\mathcal{F}(s)}(m)}{am^n} - \frac{n H_\eta \cdot L_\eta^{n-1} (n + 1)! w_{\mathcal{F}}(m)}{(n + 1) L_\eta^n m^{n+1}} \right).$$

Then, the same discussion as \cite[3.8, 3.15]{Hat22b} shows that (for the definition of $J^{F, NA}(\mathcal{F}_R)$, we refer to Lemma 2.8)

$$\lim_{m \to \infty} \left( \frac{n! w_{\mathcal{F}(s)}(m)}{am^n} - \frac{n H_\eta \cdot L_\eta^{n-1} (n + 1)! w_{\mathcal{F}}(m)}{(n + 1) L_\eta^n m^{n+1}} \right) \geq J^{F, NA}(\mathcal{F}_R).$$

By the construction of $\mathcal{F}_R$, we have $J^{F, NA}(\mathcal{F}_R) = J^{F, NA}(\mathcal{F})$. On the other hand, $J^{F, NA}(\mathcal{F}) \geq 0$ by Lemma 2.8. Summarizing them, we obtain

$$J \geq \frac{1}{[F : \mathbb{Q}] L_\eta^m} J^{F, NA}(\mathcal{F}_R) \geq 0.$$

Finally, we apply the same arguments as \cite[3.19]{Hat22b} to show (10) is nonnegative by using a recent variant of inversion of adjunction due to \cite[Theorem 3.8]{ST21} via the theory of F-singularities. More precisely speaking, we discuss as follows. By making use of Claim 2 instead of \cite[Theorem 3.5]{Hat22b}, we apply the same argument as the proof of \cite[3.19]{Hat22b} and obtain the following estimate:

$$10 \geq \liminf_{l \to \infty} \inf_D \operatorname{lct}(X, X_{p_k} + D; X_{p_k}),$$

where $D$ runs over all effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors such that the support of $D$ does not contain $X_{p_k}$ and $D_{p_k}$ is an $l$-basis type divisor with respect to $L''(k)_{p_k}$ where $l$ is
sufficiently large. Since we assumed that \( \delta^F(L_{p_k}) > 1 \) in the fourth paragraph of this proof, \( \delta^F(L_{p_k}) > 1 \) also holds. Therefore, it follows from [ST21, Theorem 3.8] that \( (X''(k), X''(k)_p + D) \) is log canonical for any effective \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor \( D \) whose support does not contain \( X_{p_k} \) and whose restriction \( D_{p_k} \) to \( X''(k)_p \) is an \( l \)-basis type divisor with respect to \( L''(k)_{p_k} \) where \( l \) is sufficiently large. Thus, we have

\[
\liminf_{l \to \infty} \inf_{D} \leq \lim(X, X_{p_k} + D; X_{p_k}) \geq 0,
\]

which shows that \( (10) \) is nonnegative. Since \( (9) \) and \( (10) \) are nonnegative, so is \( (7) \). Therefore, we complete the proof that \( (5) \) holds in Case 1.

**Case 2.** We deal with the case when \( \delta^F(L_{p_k}) = 0 \). By \( (5) \), we may replace the models \( (X(k), L(k), h(k)) \) and \( (X(k + 1), L(k + 1), h(k)) \) with \( (X''(k), L''(k), h(k)) \) and \( (X''(k + 1), L''(k + 1), h(k)) \) respectively. We have that

\[
\lim_{\epsilon \to 0} (J_{X''(k)})_{K_{X''(k)}} + L''(k) (X''(k) + 1, L''(k) + 1, h(k))
\]

\[
- J_{X''(k)} (X''(k), L''(k), h(k))
\]

\[
= J_{X''(k)} (X''(k + 1), L''(k + 1), h_L(k)) - J_{X''(k)} (X''(k), L''(k), h_L(k)).
\]

By \( (6) \) and the above equation, we have that

\[
J_{X''(k)} (X''(k + 1), L''(k + 1), h_L(k)) - J_{X''(k)} (X''(k), L''(k), h_L(k)) \geq 0.
\]

Since \( X_{p_k} \) is locally F-pure, so is \( X_{p_k} \). By the same argument of the proof of Case 2 in [Hat22b, Theorem 3.20] and the local F-purity of \( X_{p_k} \), we have that

\[
h_K(X''(k + 1), L''(k + 1), h(k)) - h_K(X''(k), L''(k), h(k))
\]

\[
\geq J_{X''(k)} (X''(k + 1), L''(k + 1), h_L(k)) - J_{X''(k)} (X''(k), L''(k), h_L(k)) \geq 0.
\]

This shows that \( (5) \) also holds in this case. We complete the proof of Theorem 3.1 by the argument of the third paragraph of this proof. \( \square \)

**Acknowledgements.** This work is partially supported by JSPS KAKENHI 22J20059 (Grant-in-Aid for JSPS Fellows DC1) for M.H. and KAKENHI 16H06335 (Grant-in-Aid for Scientific Research (S)), 20H00112 (Grant-in-Aid for Scientific Research (A)), 21H00973 (Grant-in-Aid for Scientific Research (B)) and KAKENHI 18K13385 (Grant-in-Aid for Early-Career Scientists) for Y.O. We thank Shunsuke Takagi and Shou Yoshikawa for useful comments. We also thank Teppei Takamatsu and Swaraj Pande for allowing us to write Example 3.3 (iv). We would like to thank the referee for the careful reading and suggestions.

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