A FIRST GUIDE TO THE CHARACTER THEORY OF FINITE GROUPS OF LIE TYPE

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Abstract. This survey article is an introduction to some of Lusztig’s work on the character theory of a finite group of Lie type \( G(\mathbb{F}_q) \), where \( q \) is a power of a prime \( p \). It is partly based on two series of lectures given at the Centre Bernoulli (EPFL) in July 2016 and at a summer school in Les Diablerets in August 2015. Our focus here is on questions related to the parametrization of the irreducible characters and on results which hold without any assumption on \( p \) or \( q \).

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1. Introduction

According to Aschbacher [2], when faced with a problem about finite groups, it seems best to attempt to reduce the problem or a related problem to a question about simple groups or groups closely related to simple groups. The classification of finite simple groups then supplies an explicit list of groups which can be studied in detail using the effective description of the groups. In recent years, this program has led to substantial advances on various long-standing open problems in the representation theory of finite groups; see, for example, Malle’s survey [61]. The classification of finite simple groups itself highlights the importance of studying “finite groups of Lie type”, which are the subject of our survey.

So let \( p \) be a prime and \( k = \mathbb{F}_p \) be an algebraic closure of the field with \( p \) elements. Let \( G \) be a connected reductive algebraic group over \( k \) and assume that \( G \) is defined over the finite subfield \( \mathbb{F}_q \subseteq k \), where \( q \) is a power of \( p \). Let \( F: G \to G \) be the corresponding Frobenius map. Then the group of rational points \( G^F = G(\mathbb{F}_q) \) is called a “finite group of Lie type”. (For the basic theory of these groups, see [9], [22], [62], [74].) We are interested in finding out as much information as possible about the complex irreducible characters of \( G^F \).

In Section 2 we begin by recalling some basic results about the virtual characters \( R_{T,\theta} \) of Deligne-Lusztig [12]. In Section 3 we explain the fact that the order of \( G^F \) and the degrees of the irreducible characters of \( G^F \) can be obtained by evaluating...
certain polynomials at \( q \). The “unipotent” characters of \( G^F \) form a distinguished subset of the set of irreducible characters of \( G^F \). In Section 4 we describe, following Lusztig [54], a canonical bijection between the unipotent characters and a certain combinatorially defined set which only depends on the Weyl group \( W \) of \( G \) and the action of \( F \) on \( W \). In Section 5 we expose some basic results from Lusztig’s book [41], assuming that the centre \( Z(G) \) is connected. The “regular embeddings” in Section 6 provide a technique to transfer results from the connected centre case to the general case; see [45], [50]. Taken together, one obtains a full classification of the irreducible characters of \( G^F \) (without any condition on \( Z(G) \)) including, for example, explicit formulae for the character degrees. Finally, in Section 7 we discuss some basic features of Lusztig’s theory of character sheaves [43]. In [53], Lusztig closed a gap in this theory which now makes it possible to state a number of results without any condition on \( p \) or \( q \).

As an application, and in response to a question from Pham Huu Tiep, we will then show that the results in [26, §6] on the number of unipotent \( \ell \)-modular Brauer characters of \( G^F \) (for primes \( \ell \neq p \)) hold unconditionally. In an appendix we show that, with all the methods available nowadays, it is relatively straightforward to settle an old conjecture of Lusztig [39] on “uniform” functions.

Our main references for this survey are, first of all, Lusztig’s book [11], and then the monographs by Carter [9], Digne–Michel [15], Cabanes–Enguehard [8] and Bonnafe [5] (in chronological order). These five volumes contain a wealth of ideas, theoretical results and concrete data about characters of finite groups of Lie type. As far as character sheaves are concerned, we mostly rely on the original source [43]. We also recommend Lusztig’s lecture [55] for an overview of the subject, as well as Shoji’s older survey [69]. The aim of the ongoing book project [30] is to provide a guided tour to this vast territory, where new areas and directions keep emerging; see, e.g., Lusztig’s recent papers [50], [58]. It is planned that a substantially expanded version of this survey will appear as Chapter 2 of [30].

1.1. Notation. We denote by \( \text{CF}(G^F) \) the vector space of complex valued functions on \( G^F \) which are constant on the conjugacy classes of \( G^F \). Given \( f, f' \in \text{CF}(G^F) \), we denote by \( \langle f, f' \rangle = |G^F|^{-1} \sum_{g \in G^F} f(g)f'(g) \) the standard scalar product of \( f, f' \) (where the bar denotes complex conjugation). Let \( \text{Irr}(G^F) \) be the set of complex irreducible characters of \( G^F \); these form an orthonormal basis of \( \text{CF}(G^F) \) with respect to the above scalar product. In the framework of [12], [43], class functions are constructed whose values are algebraic numbers in \( \bar{\mathbb{Q}}_\ell \) where \( \ell \neq p \) is a prime. By choosing an embedding of the algebraic closure of \( \mathbb{Q} \) in \( \bar{\mathbb{Q}}_\ell \) into \( \mathbb{C} \), we will tacitly regard these class functions as elements of \( \text{CF}(G^F) \).

Also note that we do assume that \( G \) is defined over \( \mathbb{F}_q \) and so we exclude the Suzuki and Ree groups from the discussion. This certainly saves us from additional technical complications in the formulation of some results; it also makes it easier to give precise references. Note that, in most applications to finite group theory (as mentioned above), the Suzuki and Ree groups can be regarded as “sporadic groups” and be dealt with separately.

2. THE VIRTUAL CHARACTERS OF DELIGNE AND LUSZTIG

Let \( G, F, q \) be as in the introduction. The framework for dealing with questions about the irreducible characters of \( G^F \) is provided by the theory originally
developed by Deligne and Lusztig [12]. In this set-up, one associates a virtual character \( R_{T,\theta} \) of \( G^F \) to any pair \((T,\theta)\) where \( T \subseteq G \) is an \( F \)-stable maximal torus and \( \theta \in \operatorname{Irr}(T^F) \). An excellent reference for the definition and basic properties is Carter’s book [14]: orthogonality relations, dimension formulae and further character relations can all be found in [9, Chap. 7]. We shall work here with a slightly different (but equivalent) model of \( R_{T,\theta} \). (There is nothing new about this: it is already contained in [12, 1.9]; see also [39, 3.3].)

### 2.1. Throughout, we fix an \( F \)-stable Borel subgroup \( B_0 \subseteq G \) and write \( B_0 = U_0 \rtimes T_0 \) (semidirect product) where \( U_0 \) is the unipotent radical of \( B_0 \) and \( T_0 \) is an \( F \)-stable maximal torus. Let \( N_0 := N_G(T_0) \) and \( W := N_0/T_0 \) be the corresponding Weyl group, with set \( S \) of simple reflections determined by \( B_0 \), let \( I: W \to \mathbb{Z}_{\geq 0} \) be the corresponding length function. Then \((B_0, N_0)\) is a split \( BN \)-pair in \( G \). Now \( F \) induces an automorphism \( \sigma: W \to W \) such that \( \sigma(S) = S \). By taking fixed points under \( F \), we also obtain a split \( BN \)-pair \((B_0^F, N_0^F)\) in the finite group \( G^F \), with corresponding Weyl group \( W^\sigma = \{w \in W \mid \sigma(w) = w\} \). (See [22, §4.2].) If \( w \in W \), then \( \hat{w} \) always denotes a representative of \( w \) in \( N_G(T_0) \).

One advantage of the model of \( R_{T,\theta} \) that we will now introduce is that everything is defined with respect to the fixed pair \((B_0, T_0)\).

### 2.2. For \( w \in W \) we set \( Y_{\hat{w}} := \{x \in G \mid x^{-1}F(x) \in \hat{w}U_0\} \subseteq G \). Then \( Y_{\hat{w}} \) is a closed subvariety which is stable under left multiplication by elements of \( G^F \). Now consider the subgroup

\[
T_0[w] := \{t \in T_0 \mid F(t) = \hat{w}^{-1}t\hat{w}\} \subseteq T_0.
\]

(Note that this does not depend on the choice of the representative \( \hat{w} \).) One easily sees that \( T_0[w] \) is a finite group; see, e.g., Remark 2.3 below. We check that \( Y_{\hat{w}} \) is also stable under right multiplication by elements of \( T_0[w] \). Indeed, let \( t \in T_0[w] \) and \( x \in Y_{\hat{w}} \). Then \( F(t) = \hat{w}^{-1}t\hat{w} \) and so

\[
(\hat{x})^{-1}F(\hat{x}) = t^{-1}x^{-1}F(x)F(t) = t^{-1}\hat{w}U_0F(t) = \hat{w}(\hat{w}^{-1}t^{-1}\hat{w}U_0\hat{w}^{-1}t) = \hat{w}U_0
\]

since \( T_0 \) normalizes \( U_0 \). Consequently, by the same argument as in [9, §7.2], a pair \((g, t) \in G^F \times T_0[w]\) induces linear maps on the \( \ell \)-adic cohomology spaces with compact support \( H^i_c(Y_{\hat{w}}) \) \((i \in \mathbb{Z})\); furthermore, if \( \theta \in \operatorname{Irr}(T_0[w]) \), then we obtain a virtual character \( R_{\theta}^\theta \) of \( G^F \) by setting

\[
R_{\theta}^\theta(g) := \frac{1}{|T_0[w]|} \sum_{t \in T_0[w]} \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{Trace}((g, t), H^i_c(Y_{\hat{w}}))\theta(t) \quad (g \in G^F).
\]

**Remark 2.3.** Let \( w \in W \). By Lang’s Theorem (see, e.g., [9, §1.17]), we can write \( \hat{w} = g^{-1}F(g) \) for some \( g \in G \). Then \( T := gT_0g^{-1} \) is an \( F \)-stable maximal torus and \( T_0[w] = g^{-1}T^Fg \). (In particular, \( T_0[w] \) is finite.) Thus, \( T \) is a torus “of type \( w \)” and may be denoted by \( T_w \). We define \( \theta \in \operatorname{Irr}(T_w^F) \) by \( \theta(t) := \theta(g^{-1}tg) \) for \( t \in T_w^F \). Then one checks that \( R_{\theta}^\theta \) equals \( R_{T_w,\theta} \) (as defined in [9, §7.2, 39, 2.2]) and \( R_{\theta}^\theta \) does not depend on the choice of the representative \( \hat{w} \). (See, e.g., [22, 4.5.6].)

**Example 2.4.** Let \( w = 1 \). Then we can take \( \hat{w} = g = 1 \) and so \( T_0[1] = T_0^F \).

Since \( F(U_0) = U_0 \), Lang’s Theorem implies that \( Y_1 = \{yu \mid y \in G^F, u \in U_0\} \);

furthermore, we obtain a surjective morphism \( Y_1 \to (G/B_0)^F, x \mapsto xB_0 \), which is compatible with the actions of \( G^F \) by left multiplication on \( Y_1 \) and on \( (G/B_0)^F \).

Let \( Z_1, \ldots, Z_n \) be the fibres of this morphism, where \( Z_1 \) is the fibre of \( B_0 \) and
n = |(G/B_0)_F|. Then $G^F$ permutes these fibres transitively and the stabiliser of $Z_1$ is $B'_0$. By one of the basic properties of $\ell$-adic cohomology listed in [9, §7.1], this already shows that $R^\theta_1$ is obtained by usual induction from a virtual character of $B'_0$. Now $B'_0 = U'_0 \times T'_0$ and so there is a canonical homomorphism $\pi : B'_0 \to T'_0$ with kernel $U'_0$. Hence, if $\theta \in \text{Irr}(T'_0)$, then $\theta \circ \pi \in \text{Irr}(B'_0)$ and one further shows that (see the proof of [9 7.2.4] for details):

$$R^\theta_1 = \text{Ind}_{B'_0}^{G^F}(\theta \circ \pi).$$

So, in this case ($w = 1$), $R^\theta_1$ is just ordinary induction of characters. Thus, in general, one may call $R^\theta_w$, “cohomological induction”.

**Proposition 2.5.** We have $R^\theta_w(1) = (-1)^{(a)} q^{-N} |G^F : T_0[w]|$, where $N$ denotes the number of reflections in $W$.

**Proof.** Write $R^\theta_w = R_{T_w, g}$ as in Remark 2.3. By [9 7.5.1 and 7.5.2], we have $R_{T_w, g}(1) = (-1)^{(a)} |G^F : T'_w|$. It remains to note that $q^N$ is the $p$-part of $|G^F|$. (See also the formula for $|G^F|$ in 3.1 below.)

Given $w, w' \in W$, we denote $N_{W, \sigma}(w, w') := \{ x \in W \mid xw\sigma(x)^{-1} = w' \}$. Then a simple calculation shows that $xT_0[w]x^{-1} = T_0[w']$ if $x \in N_{W, \sigma}(w, w')$.

**Proposition 2.6.** Let $w, w' \in W$, $\theta \in \text{Irr}(T_0[w])$ and $\theta' \in \text{Irr}(T_0[w'])$. Then

$$\langle R^\theta_w, R^\theta_{w'} \rangle = |\{ x \in N_{W, \sigma}(w, w') \mid \theta(t) = \theta'(xTx^{-1}) \text{ for all } t \in T_0[w] \}|.$$  

In particular, $R^\theta_w$ and $R^\theta_{w'}$ are either equal or orthogonal to each other.

**Proof.** Write $R^\theta_w = R_{T_w, g}$ where $T_w = gT_0g^{-1}$ and $g^{-1}F(g) = \hat{w}$, as in Remark 2.3. Similarly, we write $R^\theta_{w'} = R_{T_{w'}, g'}$ where $T_{w'} = g'T_0g'^{-1}$ and $g'^{-1}F(g') = \hat{w}'$. By [9 Theorem 7.3.4], the above scalar product is given by

$$(*) \quad |\{ x \in G^F \mid xT_0x^{-1} = T_{w'} \text{ and } \theta'(xTx^{-1}) = \theta(t) \text{ for all } t \in T'_w \}| / |T'_w|.$$

Hence, if $T_w, T_{w'}$ are not $G^F$-conjugate, then $N_{W, \sigma}(w, w') = \varnothing$ (see [9 3.3.3]) and both sides of the desired identity are 0. Now assume that $T_w, T_{w'}$ are $G^F$-conjugate. Then we may as well assume that $w = w'$, $T_w = T_{w'}, g = g'$. Then $(*)$ equals

$$|\{ nT'_w \in N_G(T'_w)^F / T'_w \mid \theta'(ntn^{-1}) = \theta(t) \text{ for all } t \in T'_w \}|.$$

By [9 3.3.6], $N_G(T'_w)^F / T'_w$ is isomorphic to the group $N_{W, \sigma}(w, w)$, where the isomorphism is given by sending $nT'_w$ to the coset of $x_n := g^{-1}ng \in N_G(T_0)$ mod $T_0$. Now, for any $t \in T'_w$, we have $\theta(t) = \theta(g^{-1}tg)$ and $\theta'(ntn^{-1}) = \theta'(x_ng^{-1}tgx_n^{-1})$. This yields the desired formula. Finally note that, if $(*)$ is non-zero, then the pairs $(T_w, \hat{g})$ and $(T_{w'}, \hat{g}')$ are $G^F$-conjugate, which implies that $R_{T_w, g} = R_{T_{w'}, g'}$. □

**Definition 2.7.** We say that $f \in \text{CF}(G^F)$ is uniform if $f$ can be written as a linear combination of $R^\theta_w$ for various $w, \theta$. Let

$$\text{CF}_{\text{unif}}(G^F) := \{ f \in \text{CF}(G^F) \mid f \text{ is uniform} \}.$$  

For example, by [12 7.5], [15 12.14], the character of the regular representation of $G^F$ is uniform; more precisely, that character can be written as

$$\frac{1}{|W|} \sum_{w \in W} \sum_{\theta \in \text{Irr}(T_0[w])} R^\theta_w(1) R^\theta_w.$$
Proposition 2.8. Let $\rho \in \text{Irr}(G^F)$. Then
\[
\rho(1) = \frac{1}{|W|} \sum_{w \in W} \sum_{\theta \in \text{Irr}(T_0[w])} \langle R^\theta_w, \rho \rangle R^\theta_w(1).
\]

Proof. Just take the scalar product of $\rho$ with the above expression for the character of the regular representation of $G^F$. \hfill □

Remark 2.9. Let $G_{\text{uni}}$ be the set of unipotent elements of $G$. Let $w \in W$ and $\theta \in \text{Irr}(T_0[w])$. If $u \in G_{\text{uni}}$, then $R^\theta_w(u) \in \mathbb{Z}$ and this value is independent of $\theta$; see [9, 7.2.9] and the remarks in [9, §7.6]. Hence, we obtain a well-defined function $Q_w : G^F_{\text{uni}} \to \mathbb{Z}$ such that $Q_w(u) = R^\theta_w(u)$ for $u \in G^F_{\text{uni}}$. This is called the Green function corresponding to $w$; see [12, 4.1]. There is a character formula (see [9, 7.2.8]) which expresses the values of $R^\theta_w$ in terms of $Q_w$, the values of $\theta$ and Green functions for various smaller groups. That formula immediately implies that
\[
\sum_{\theta \in \text{Irr}(T_0[w])} R^\theta_w(g) = \begin{cases} |T_0[w]|Q_w(g) & \text{if } g \in G^F \text{ is unipotent,} \\ 0 & \text{otherwise.} \end{cases}
\]

(This is also the special case $s_0 = 1$ in Lemma 8.3 further below.) Using also Proposition 2.5, we can re-write the identity in Proposition 2.8 as follows:
\[
\rho(1) = \frac{1}{|W|} \sum_{w \in W} |T_0[w]| \langle Q_w, \rho \rangle Q_w(1) = \frac{1}{|W|} q^{-N |G^F|} \sum_{w \in W} (-1)^{l(w)} \langle Q_w, \rho \rangle,
\]
where we regard $Q_w$ as a function on all of $G^F$, with value 0 for $g \in G^F \setminus G^F_{\text{uni}}$.

2.10. By Proposition 2.8, every irreducible character of $G^F$ occurs with non-zero multiplicity in some $R^\theta_w$; furthermore, by Proposition 2.6, $R^\theta_w$ can have more than one irreducible constituent in general. This suggests to define a graph $\mathcal{G}(G^F)$ as follows. It has vertices in bijection with $\text{Irr}(G^F)$. Two characters $\rho \neq \rho'$ in $\text{Irr}(G^F)$ are joined by an edge if there exists some pair $(w, \theta)$ such that $\langle R^\theta_w, \rho \rangle \neq 0$ and $\langle R^\theta_w, \rho' \rangle \neq 0$. Thus, the connected components of this graph define a partition of $\text{Irr}(G^F)$. There are related partitions of $\text{Irr}(G^F)$ into so-called “geometric conjugacy classes” and into so-called “rational series”, but the definitions are more complicated; see [9, §12.1], [12, §10], [15, Chap. 13], [38, §7]. We just note at this point that the definitions immediately show the following implications:
\[
\rho, \rho' \text{ belong to the same connected component of } \mathcal{G}(G^F) \quad \Rightarrow \quad \rho, \rho' \text{ belong to the same rational series}
\]
\[
\rho, \rho' \text{ belong to the same connected component of } \mathcal{G}(G^F) \quad \Rightarrow \quad \rho, \rho' \text{ belong to the same geometric conjugacy class.}
\]

We will clarify the relations between these notions in Theorem 5.2 and Remark 6.14 below. (See also Bonnafé [5, §11] for a further, detailed discussion).

Example 2.11. Let $G = \text{SL}_2(\mathbb{F}_p)$, where $p \neq 2$, and $F$ be the Frobenius map which raises each matrix entry to its $q$th power; then $G^F = \text{SL}_2(\mathbb{F}_q)$. The character table of $G^F$ is, of course, well-known. (See, e.g., Fulton–Harris [17, §5.2].) The set
\( \text{Irr}(G^F) \) consists of the following irreducible characters:

\[
\begin{align*}
1_G & \quad \text{(trivial character)}; \\
\text{St}_G & \quad \text{with } \text{St}_G(1) = q \text{ (Steinberg character)}; \\
\rho_i & \quad \text{with } \rho_i(1) = q + 1, \text{ for } 1 \leq i \leq (q - 3)/2; \\
\pi_j & \quad \text{with } \pi_j(1) = q - 1, \text{ for } 1 \leq j \leq (q - 1)/2; \\
\rho'_0, \rho''_0 & \quad \text{each of degree } (q + 1)/2; \\
\pi'_0, \pi''_0 & \quad \text{each of degree } (q - 1)/2.
\end{align*}
\]

Thus, in total, we have \(|\text{Irr}(G^F)| = q + 4\). Let us re-interpret this in terms of the characters \( R^\theta_w \); cf. Bonnafé [6 §5.3]. For \( \xi \in k^\times \), denote by \( S(\xi) \) the diagonal matrix with diagonal entries \( \xi, \xi^{-1} \). Then an \( F \)-stable maximal torus \( T_0 \subseteq G \) as above and the corresponding Weyl group are given by

\[
T_0 = \{ S(\xi) \mid \xi \in k^\times \} \quad \text{and} \quad W = \langle s \rangle \quad \text{where} \quad s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

Let \( \xi_0 \in k^\times \) be a fixed element of order \( q - 1 \) and \( \xi'_0 \in k^\times \) be a fixed element of order \( q + 1 \). Then

\[
\begin{align*}
T_0[1] = T_0^F = \{ S(\xi_0^a) \mid 0 \leq a < q - 1 \} & \quad \text{is cyclic of order } q - 1; \\
T_0[s] = \{ S(\xi_0^b) \mid 0 \leq b < q + 1 \} & \quad \text{is cyclic of order } q + 1.
\end{align*}
\]

Let \( \varepsilon \in \mathbb{C} \) be a primitive root of unity of order \( q - 1 \); let \( \eta \in \mathbb{C} \) be a primitive root of unity of order \( q + 1 \). For \( i \in \mathbb{Z} \) let \( \theta_i \in \text{Irr}(T_0[1]) \) be the character which sends \( S(\xi_0^a) \) to \( \varepsilon^{ai} \); for \( j \in \mathbb{Z} \) let \( \theta'_j \in \text{Irr}(T_0[s]) \) be the character which sends \( S(\xi_0^b) \) to \( \eta^{bj} \).

Then one finds that:

\[
R^\theta_1 = \left\{ \begin{array}{ll}
1_G + \text{St}_G & \text{if } i = 0, \\
\rho_i & \text{if } 1 \leq i \leq \frac{q - 3}{2}, \\
\rho'_0 + \rho''_0 & \text{if } i = \frac{q - 1}{2},
\end{array} \right. \quad R^\theta_j = \left\{ \begin{array}{ll}
1_G - \text{St}_G & \text{if } j = 0, \\
-\pi_j & \text{if } 1 \leq j \leq \frac{q - 1}{2}, \\
-\pi'_0 - \pi''_0 & \text{if } j = \frac{q + 1}{2};
\end{array} \right.
\]

furthermore, \( R'^{\theta}_{j-1} = R^\theta_j \) and \( R''^\theta_j = R'^\theta_{j+1} \) for \( i, j \in \mathbb{Z} \). Hence, the graph \( \mathcal{G}(G^F) \) has \( q + 1 \) connected components, which partition \( \text{Irr}(G^F) \) into the following subsets:

\[
\{1_G, \text{St}_G\}, \quad \{\rho_i\} \quad (1 \leq i \leq \frac{q - 3}{2}), \quad \{\pi_j\} \quad (1 \leq j \leq \frac{q - 1}{2}), \quad \{\rho'_0, \rho''_0\}, \quad \{\pi'_0, \pi''_0\}.
\]

The pairs \((1, \theta_i)\) and \((s, \theta'_j)\), where \( i = \frac{q - 1}{2} \) and \( j = \frac{q + 1}{2} \), play a special role in this context; see Example 5.8 below.

It is a good exercise to re-interpret similarly Srinivasan’s character table [72] of \( G^F = \text{Sp}_4(\mathbb{F}_q) \). (See also [73].)

**Remark 2.12.** In the setting of Example 2.11 we see that \( \dim \text{CF}_{\text{unif}}(G^F) = q + 2 \) and so \( \text{CF}_{\text{unif}}(G^F) \subseteq \text{CF}(G^F) \). Furthermore, we see that the two class functions

\[
\frac{1}{2}(\rho'_0 - \rho''_0 + \pi'_0 - \pi''_0) \quad \text{and} \quad \frac{1}{2}(\rho'_0 - \rho''_0 - \pi'_0 + \pi''_0)
\]

form an orthonormal basis of the orthogonal complement of \( \text{CF}_{\text{unif}}(G^F) \) in \( \text{CF}(G^F) \). Using the character table of \( G^F \) [74 §5.2], the values of these class functions are given as follows, where \( \delta = (-1)^{(q-1)/2} \).

| \( J \) | \( J' \) | \(-J\) | \(-J'\) | Otherwise |
|---|---|---|---|---|
| \( \frac{1}{2}(\rho'_0 - \rho''_0 + \pi'_0 - \pi''_0) \) | 0 | 0 | \( \delta \sqrt{\delta q} \) | \( -\delta \sqrt{\delta q} \) | 0 |
| \( \frac{1}{2}(\rho'_0 - \rho''_0 - \pi'_0 + \pi''_0) \) | \( \sqrt{\delta q} \) | \( -\sqrt{\delta q} \) | 0 | 0 | 0 |
Here, $J$ is $G^F$-conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $J'$ is $G^F$-conjugate to $\begin{pmatrix} 1 & \xi_0 \\ 0 & 1 \end{pmatrix}$. We will encounter these two functions again at the end of this paper, in Example 7.8.

Example 2.13. Assume that $G$ has a connected centre and that $W$ is either $\{1\}$ or a direct product of Weyl groups of type $A_n$ for various $n \geq 1$. Then it follows from Lusztig–Srinivasan [59] that every class function on $G^F$ is uniform. In all other cases, we have $\text{CF}_{\text{unif}}(G^F) \subsetneq \text{CF}(G^F)$, and this is one reason why the character theory of $G^F$ is so much more complicated in general.

In general, it seems difficult to say precisely how big $\text{CF}_{\text{unif}}(G^F)$ is inside $\text{CF}(G^F)$. To close this section, we recall a conjecture of Lusztig concerning this question, and indicate how a proof can be obtained by the methods that are available now. (One can probably formulate a more complete answer in the framework of the results in Section 7, but we will not pursue this here.) Let $\mathcal{C}$ be an $F$-stable conjugacy class of $G$; then $\mathcal{C}^F$ is a union of conjugacy classes of $G^F$. We denote by $f_{\mathcal{C}}^G \in \text{CF}(G^F)$ the characteristic function of $\mathcal{C}^F$; thus, $f_{\mathcal{C}}^G$ takes the value 1 on $\mathcal{C}^F$ and the value 0 on the complement of $\mathcal{C}^F$.

Theorem 2.14 (Lusztig [39, Conjecture 2.16]). The function $f_{\mathcal{C}}^G$ is uniform.

If $\mathcal{C}$ is a unipotent class and $p$ is a “good” prime for $G$, then this easily follows from the known results on Green functions; see Shoji [65]. It is shown in [21 Prop. 1.3] that the condition on $p$ can be removed, using results from [43] and [67, Theorem 5.5]. So Theorem 2.14 holds when $\mathcal{C}$ is a unipotent class. By an argument analogous to that in [43, 25.5] (at the very end of Lusztig’s character sheaves papers) one can deduce from this that Theorem 2.14 holds in complete generality; details are given in an appendix at the end of this paper (Section 8).

3. Order and degree polynomials

A series of finite groups of Lie type (of a fixed “type”) is an infinite family of groups like $\{\text{SL}_n(F_q) \mid \text{any } q\}$ (where $n$ is fixed) or $\{E_8(F_q) \mid \text{any } q\}$. It then becomes meaningful to say that the orders of the groups in the family are given by a polynomial in $q$, or that the degrees of the characters of these groups are given by polynomials in $q$. In this section we explain how these polynomials can be formally defined. This will allow us to attach some useful numerical invariants to the irreducible characters of $G^F$.

3.1. Let $X = X(T_0)$ be the character group of $T_0$, that is, the abelian group of all algebraic homomorphisms $\lambda: T_0 \to k^\times$; this group is free abelian of rank equal to $\dim T_0$. There is an embedding $W \hookrightarrow \text{Aut}(X)$, $w \mapsto \underline{w}$, such that $\underline{w}(\lambda)(t) = \lambda(\underline{w}^{-1}t\underline{w})$ for all $\lambda \in X$, $w \in W$, $t \in T_0$.

Furthermore, $F$ induces a homomorphism $X \to X$, $\lambda \mapsto \lambda \circ F$, which we denote by the same symbol. There exists $\varphi_0 \in \text{Aut}(X)$ of finite order such that $\lambda(F(t)) = q^{\varphi_0}(\lambda)(t)$ for all $t \in T_0$.

(See [30, 1.4.16].) Then the automorphism $\sigma: W \to W$ in [2.1] is determined by $\underline{w} \circ \varphi_0 = \varphi_0 \circ \underline{w}$ for all $w \in W$ (see, e.g., [30, 6.1.1]). Now, the triple $G := (X, \varphi_0, W \hookrightarrow \text{Aut}(X))$
may be regarded as the combinatorial skeleton of $G$, $F$; note that the prime power $q$ does not occur here. Let $q$ be an indeterminate over $\mathbb{Q}$. For any $w \in W$, let us set
\[ |T_w| := \det(q \text{id}_X - \varphi_0^{-1} \circ w) \in \mathbb{Z}[q], \]
Note that $|T_w|$ is monic of degree $\dim T_0$. By [9, 3.3.5], the order of the finite subgroup $T_0[w] \subseteq G$ introduced in Section 2 is given by evaluating $|T_w|$ at $q$. In particular, the order of $T_0^F$ is given by $|T_1|(q)$. Let us now define
\[ |G| := q^N |T_1| \sum_{w \in W^*} q^{l(w)} \in \mathbb{Z}[q], \]
where $N \geq 0$ denotes the number of reflections in $W$. Then the order of $G^F$ is given by evaluating $|G|$ at $q$; see, e.g., [9, §2.9], [22, 4.2.5]. We call $|G| \in \mathbb{Z}[q]$ the “order polynomial” of $G^F$. Note that $|G|$ has degree $\dim G = 2N + \dim T_0$ and that $q^N$ is the largest power of $q$ which divides $|G|$. Since the order of $T_0[w]$ divides the order of $G^F$, one deduces that $|T_w|$ divides $|G|$ in $\mathbb{Q}[q]$. An alternative expression for $|G|$ is given as follows. Steinberg [75, Theorem 14.14] (see also [9, 3.4.1]) proves a formula for the total number of $F$-stable maximal tori in $G$. This formula yields the identity
\[ |G| = q^{2N} \left( \frac{1}{|W|} \sum_{w \in W} \frac{1}{|T_w|} \right)^{-1}. \]
(For further details, see [7, §1], [30, §1.6], [62, 24.6, 25.5].)

**Remark 3.2.** Let $Z = Z(G)$ be the centre of $G$ and $|Z^0| \in \mathbb{Q}[q]$ be the order polynomial of the torus $Z^0$. Then the discussion in [9, §2.9] also shows that
\[ |T_1| = |Z^0| \prod_{J \in S_\sigma} (q^{\lceil J \rceil} - 1), \]
where $S_\sigma$ denotes the set of orbits of $\sigma$ on $S$. We call $|S_\sigma|$ the semisimple $\mathbb{F}_q$-rank of $G$. Thus, $(q-1)^{|S_\sigma|}$ is the exact power of $q-1$ which divides the order polynomial of $G_{\text{der}}$, where $G_{\text{der}}$ is the derived subgroup of $G$. Note that $G = Z^0 G_{\text{der}}$ and $Z^0 \cap G_{\text{der}}$ is finite; then $|G^F| = |(Z^0)^F| |G_{\text{der}}^F|$ and $|G| = |Z^0| |G_{\text{der}}|$ (see [9, §2.9]).

### 3.3.

Now let us turn to the irreducible characters of $G^F$. In order to define “degree polynomials”, we consider the virtual characters $R_w^\theta$ from the previous section. By Proposition 2.5 and the above formalism, we see that $R_w^\theta(1)$ is given by evaluating the polynomial $(-1)^{l(w)} q^{-N} |G|/|T_w| \in \mathbb{Q}[q]$ at $q$. Hence, setting
\[ \mathbb{D} \rho := \frac{1}{|W|} \sum_{w \in W} \sum_{\theta \in \text{Irr}(T_0[w])} (-1)^{l(w)} \langle R_w^\theta, \rho \rangle q^{-N} |G|/|T_w| \in \mathbb{Q}[q], \]
we deduce from Proposition 2.8 that $\rho(1)$ is obtained by evaluating the polynomial $\mathbb{D} \rho$ at $q$; in particular, $\mathbb{D} \rho \neq 0$. Having $\mathbb{D} \rho \in \mathbb{Q}[q]$ at our disposal, we obtain numerical invariants of $\rho$ as follows.

\[ A_\rho := \text{degree of } \mathbb{D} \rho, \]
\[ a_\rho := \text{largest non-negative integer such that } q^{a_\rho} \text{ divides } \mathbb{D} \rho, \]
\[ n_\rho := \text{smallest positive integer such that } n_\rho \mathbb{D} \rho \in \mathbb{Z}[q]. \]

All we can say at this stage is that $0 \leq a_\rho \leq A_\rho \leq N$ (since $q^{-N} |G|/|T_w| \in \mathbb{Q}[q]$ has degree $N$); furthermore, $n_\rho$ divides $|W|$, since $|T_w| \in \mathbb{Z}[q]$ is monic, $|G| \in \mathbb{Z}[q]$.
and, hence, $|W|D_{\rho} \in \mathbb{Z}[q]$. In fact, it is known — but this requires much more work — that $D_{\rho}$ always has the following form:

\[ D_{\rho} = \frac{1}{n_{\rho}}(q^{A_{\rho}} + \ldots \pm q^{a_{\rho}}), \]

where the coefficients of all intermediate powers $q^i$ ($a_{\rho} < i < A_{\rho}$) are integers and the number $n_{\rho}$ is typically much smaller than the order of $W$. Indeed, if $Z(G)$ is connected, then this is contained in [11, 4.26] and [11, §3], [11, §71B]; for the general case, one uses an embedding of $G$ into a group with a connected centre and the techniques described in Section 6 below (see Remark 6.5). We also mention that a formula like (♣) already appeared early in Lusztig’s work [10, §8].

**Example.** Let $G^F = SL_2(\mathbb{F}_q)$. Then $|G| = q(q^2 - 1)$, $|T_1| = q - 1$ and $|T_s| = q + 1$, where we use the notation in Example 2.11. For $1 \leq i \leq (q - 3)/2$, the character $\pi_i$ occurs with multiplicity 1 in $R^i_{\rho_1}$ and in $R^i_{\rho_{q_i}}$. So $D_{\rho_1} = \frac{1}{2}((q - 1) + (q - 1)) = q - 1$, as expected. Similarly, one finds that $D_{\rho_2} = D_{\rho_q} = q$, $D_{\pi_j} = q + 1$ (for $1 \leq j \leq (q - 1)/2$), $D_{\rho'_{\rho}} = D_{\rho''_{\rho}} = \frac{1}{2}(q - 1)$ and $D_{\pi'_{\rho}} = D_{\pi''_{\rho}} = \frac{1}{2}(q + 1)$.

### 3.4. The integers $n_{\rho}$ and $a_{\rho}$

Attached to $\rho$ can also be characterized more directly in terms of the character values of $\rho$, as follows. Let $\Theta$ be an $F$-stable unipotent conjugacy class of $G$. Then $\Theta^F$ is a union of conjugacy classes of $G^F$. Let $u_1, \ldots, u_r \in \Theta^F$ be representatives of the classes of $G^F$ contained in $\Theta^F$. For $1 \leq i \leq r$ we set $A(u_i) := C_G(u_i)/C_{G^F}(u_i)$. Since $F(u_i) = u_i$, the Frobenius map $F$ induces an automorphism of $A(u_i)$ which we denote by the same symbol. Let $A(u_i)^F$ be the group of fixed points under $F$. Then we set

\[ AV(f, \Theta) := \sum_{1 \leq i \leq r} |A(u_i) : A(u_i)^F| f(u_i) \quad \text{for any } f \in CF(G^F). \]

Note that this does not depend on the choice of the representatives $u_i$; furthermore, the map $CF(G^F) \to \mathbb{C}$, $f \mapsto AV(f, \Theta)$, is linear.

Now let $\rho \in \text{Irr}(G^F)$ and set $d_{\rho} := \max\{\dim \Theta : AV(\rho, \Theta) \neq 0\}$. By the main results of [28, 47], there is a unique $\Theta$ such that $\dim \Theta = d_{\rho}$ and $AV(\rho, \Theta) \neq 0$. This $\Theta$ will be denoted by $\Theta_{\rho}$ and called the unipotent support of $\rho$. Let $u \in \Theta_{\rho}$. Then we have

\[ a_{\rho} = (\dim C_G(u) - \dim T_0)/2 \quad \text{and} \quad AV(\rho, \Theta_{\rho}) = \pm \frac{1}{n_{\rho}} q^{a_{\rho}}|A(u)|. \]

(See [28, Theorem 3.7].) Thus, from $\Theta_{\rho}$ and $AV(\rho, \Theta_{\rho})$, we obtain $a_{\rho}$, $A(u)$ and, hence, also $n_{\rho}$. For further characterisations of these integers, see Lusztig [52].

**Remark 3.5.** The above results on the unipotent support of the irreducible characters of $G^F$ essentially rely on Kawanaka’s theory [32, 33] of “generalized Gelfand–Graev representations”, which are defined only if $p$ is a “good” prime for $G$. Lusztig [47] showed how the characters of these representations can be determined assuming that $p, q$ are sufficiently large. The latter restrictions on $p, q$ have been for a long time a drawback for applications of this theory. Recently, Taylor [76] has shown that Lusztig’s results hold under the single assumption that $p$ is good.

**Example 3.6.** For any unipotent $u \in G$, we have $\dim C_G(u) \geq \dim T_0$. Furthermore, there is a unique unipotent class $\Theta$ such that $\dim C_G(u) = \dim T_0$ for $u \in \Theta$; this is called the regular unipotent class and denoted by $\Theta_{\text{reg}}$. (See [9, §5.1].) In
particular, $\mathcal{O}_{\text{reg}}$ is $F$-stable. Let $\rho \in \text{Irr}(G^F)$. We say that $\rho$ is a “semisimple character” if $\text{AV}(\rho, \mathcal{O}_{\text{reg}}) \neq 0$. These characters were originally singled out in the work of Deligne–Lusztig [12, 10.8] (see also [9, §8.4]). Clearly, if $\rho$ is semisimple, then $\mathcal{O}_{\text{reg}}$ is the unipotent support of $\rho$ in the sense of [3, 4] furthermore, $\rho$ is semisimple if and only if $a_\rho = 0$.

Semisimple characters appear “frequently” in the virtual characters $R_\theta^G$. Indeed, let $w \in W$ and $\theta \in \text{Irr}(T_0[w])$. By [12, Theorem 9.16], we have $R_\theta^G(u) = 1$ for any $u \in \mathcal{O}_{\text{reg}}^F$. This certainly implies that $\text{AV}(R_\theta^G, \mathcal{O}_{\text{reg}}) \neq 0$ and so there exists some $\rho \in \text{Irr}(G^F)$ such that $\langle R_\theta^G, \rho \rangle \neq 0$ and $\text{AV}(\rho, \mathcal{O}_{\text{reg}}) \neq 0$. Thus, every connected component of the graph $\mathcal{G}(G^F)$ in [2, 10] has at least one vertex labeled by a semisimple character.

**Remark 3.7.** Assume that $Z(G)$ is connected. Then note that our definition of a “semisimple character” looks slightly different from that in [9, p. 280], [12, 10.8], where the global average value on $\mathcal{O}_{\text{reg}}$ is used instead of our $\text{AV}(\rho, \mathcal{O}_{\text{reg}})$. Let us check that the two definitions do agree. First note that $\mathcal{O}_{\text{reg}} \cap U_0^F \neq \emptyset$; furthermore, if $u \in \mathcal{O}_{\text{reg}} \cap U_0$, then $C_G(u) = Z(G), C_{U_0}(u) \subseteq T_0, U_0 = B_0$ (see [15, 14.15]).

Hence, a standard application of Lang’s Theorem shows that we can find elements $u_1, \ldots, u_r \in U_0^F$ which form a set of representatives of the classes of $G^F$ inside $\mathcal{O}_{\text{reg}}$. By [15, 14.15, 14.18], the group $A(u_i)$ is generated by the image of $u_i$ in $A(u_i)$; hence, we have $A(u_i)^F = A(u_i)$ and so

$$\text{AV}(\rho, \mathcal{O}_{\text{reg}}) = \rho(u_1) + \ldots + \rho(u_r).$$

On the other hand, as above, we have $C_G(u_i) = Z(G), C_{U_0}(u_i)$ and so $|C_G(u_i)^F| = |Z(G)^F||C_{U_0}(u_i)^F|\text{Av}(u_i)$. Now $C_{U_0}(u_i)$ is a connected unipotent group and so $|C_{U_0}(u_i)^F| = q^{d_i}$ where $d_i := \text{dim} C_{U_0}(u_i)$; see, e.g., [22, 4.2.4]. Also note that all $d_i$ are equal and that all the groups $A(u_i)$ have the same order. Hence, $|C_G(u_i)^F|$ does not depend on $i$. So the global average value is given by

$$\sum_{g \in \mathcal{O}_{\text{reg}}^F} \rho(g) = \sum_{1 \leq i \leq r} |G^F : C_G(u_i)^F| \rho(u_i) = c(\rho(u_1) + \ldots + \rho(u_r)) \quad \text{where } c \neq 0.$$

Thus, indeed, the global average value is non-zero if and only if $\text{AV}(\rho, \mathcal{O}_{\text{reg}}) \neq 0$.

(Note that, in general, the global average value of $\rho$ on an $F$-stable unipotent class $\mathcal{O}$ will not be proportional to $\text{AV}(\rho, \mathcal{O})$.)

**3.8.** Let $\mathcal{P}(S)$ be the set of all subsets of $S$. Then $\sigma : W \to W$ (see [2, 1]) acts on $\mathcal{P}(S)$ and we denote by $\mathcal{P}(S)^\sigma$ the $\sigma$-stable subsets. For $J \in \mathcal{P}(S)^\sigma$ we have a corresponding $F$-stable parabolic subgroup $P_J = \langle B_0, s \mid s \in J \rangle \subseteq G$. This has a Levi decomposition $P_J = U_J \rtimes L_J$ where $U_J$ is the unipotent radical of $P_J$ and $L_J$ is an $F$-stable closed subgroup such that $T_0 \subseteq L_J$; furthermore, $L_J$ is connected reductive, $B_0 \cap L_J$ is an $F$-stable Borel subgroup of $L_J$, and the Weyl group $N_{L_J}(T_0)/T_0$ of $L_J$ is isomorphic to the parabolic subgroup $W_J := \langle J \rangle$ of $W$.

We have $P_J^F = U_J^F \rtimes L_J^F$ and so there is a canonical homomorphism $\pi_J : P_J^F \to L_J^F$ with kernel $U_J^F$. Hence, if $\phi \in \text{Irr}(L_J^F)$, then $\phi \circ \pi_J \in \text{Irr}(P_J^F)$ and we define

$$R_J^S(\phi) := \text{Ind}_{P_J^F}^{G_F}(\phi \circ \pi_J) \quad \text{“Harish-Chandra induction”}.$$ We say that $\rho \in \text{Irr}(G^F)$ is cuspidal if $\langle R_J^S(\phi), \rho \rangle = 0$ for any $J \in \mathcal{P}(S)^\sigma \setminus \{S\}$ and any $\phi \in \text{Irr}(L_J^F)$. If $J \in \mathcal{P}(S)^\sigma$ and $\phi \in \text{Irr}(L_J^F)$ is cuspidal, then let

$$\text{Irr}(G^F | J), \phi := \{ \rho \in \text{Irr}(G^F) \mid \langle R_J^S(\phi), \rho \rangle \neq 0 \}.$$
With this notation, \( \text{Irr}(G^F) \) is the union of the sets \( \text{Irr}(G^F|J, \phi) \) for various \((J, \phi)\). (See \[9, \text{Chap. 9}], \[11, \S 7.6.5], \[12, \text{Chap. 6}]\.) The following result strengthens earlier results of Howlett–Lehrer which are described in detail in \[9, \text{Chap. 10}]\).

**Proposition 3.9** (Lusztig \[11, 8.7\]). Assume that \( Z(G) \) is connected. Let \( J \in \mathcal{P}(S)^a \) and \( \phi \in \text{Irr}(L^F_J) \) be cuspidal. Let \( W_J(\phi) \) denote the stabilizer of \( \phi \) in \( N_G(L_J)^F/L^F_J \). Then \( W_J(\phi) \) is a finite Weyl group and there is a natural bijection

\[
\text{Irr}(W_J(\phi)) \xrightarrow{\sim} \text{Irr}(G^F|J, \phi), \quad \epsilon \mapsto \phi[\epsilon],
\]

(transition on the choice of a square root of \( q \) in \( \mathbb{C} \)).

The characters of finite Weyl groups are well-understood; see, e.g., \[31\]. So the above result provides an effective parametrization of the characters in \( \text{Irr}(G^F|J, \phi) \).

The situation is technically more complicated when the centre \( Z(G) \) is not connected; see, e.g., \[9, \text{Chap. 10}]\). (Note that one complication has disappeared in the meantime: the 2-cocyle appearing in \[9, 10.8.5\] is always trivial; see \[20\].)

**Remark 3.10.** Let \( \rho \in \text{Irr}(G^F) \) and \( J \in \mathcal{P}(S)^a \) be such that \( \langle R^S_J(\phi), \rho \rangle \neq 0 \) for some cuspidal \( \phi \in \text{Irr}(L^F_S) \). As in Remark 3.2, let \( S_\sigma \) be the set of orbits of \( \sigma \) on \( S \); similarly, let \( J_\sigma \) be the set of orbits of \( \sigma \) on \( J \). Then the number \( t := |S_\sigma| - |J_\sigma| \) may be called the “depth” of \( \rho \), as in \[33, 4.1\]. It is known (see, e.g., \[9, 9.2.3\]) that \( t \) is well-defined. Thus, the “depth” provides a further numerical invariant attached to \( \rho \). For example, \( \rho \) has depth 0 if and only if \( \rho \) is cuspidal.

**Remark 3.11.** One can show that the degree polynomials \( D_\rho \) in \[3.3\] behave in many ways like true character degrees. We leave it as a challenge to the interested reader to prove the following statements for \( \rho \in \text{Irr}(G^F) \):

\begin{itemize}
  \item[(a)] \( D_\rho \) divides \(|G|\).
  \item[(b)] If \( \langle R^\theta_R, \rho \rangle \neq 0 \), then \( D_\rho \) divides \(|G|/|T_w|\).
  \item[(c)] If \( d = \max\{i \geq 0 \mid (q - 1)^i \text{ divides } D_\rho\} \), then \( \rho \) has depth \( |S_\sigma| - d \geq 0 \); in particular, \( \rho \) is cuspidal if and only if \((q - 1)^{|S_\sigma|} \text{ divides } D_\rho\).
\end{itemize}

(For hints, see \[13, \S 2\], \[14, \S 2\]. If \( q, p \) are very large, then the above properties are equivalent to analogous properties of actual character degrees. In order to reduce to this case, one can use an argument as in the proof of \[29, \text{Theorem 3.7}]\.)

### 4. Parametrization of unipotent characters

We say that \( \rho \in \text{Irr}(G^F) \) is unipotent if \( \langle R^1_{w_1}, \rho \rangle \neq 0 \) for some \( w \in W \), where \( 1 \) stands for the trivial character of \( T_0[w] \). We set

\[
\mathcal{U}(G^F) = \{ \rho \in \text{Irr}(G^F) \mid \rho \text{ unipotent} \}.
\]

We will see that these play a distinguished role in the theory.

**Remark 4.1.** For any \( w \in W \), we have \( \langle R^1_{w_1}, 1_G \rangle = 1 \) and \( \langle R^1_{w_1}, \text{St}_G \rangle = (-1)^{l(w)} \), where \( 1_G \) denotes the trivial character of \( G^F \) and \( \text{St}_G \) denotes the Steinberg character of \( G^F \) (see \[9, 7.6.5, 7.6.6\]). Thus, we have \( 1_G \in \mathcal{U}(G^F) \) and \( \text{St}_G \in \mathcal{U}(G^F) \). In fact, it is even true that \( 1_G \) and \( \text{St}_G \) are uniform:

\[
1_G = \frac{1}{|W|} \sum_{w \in W} R^1_w \quad \text{and} \quad \text{St}_G = \frac{1}{|W|} \sum_{w \in W} (-1)^{l(w)} R^1_w;
\]

see \[15, 12.13, 12.14\]. We also have \( \langle R^\theta_R, \rho \rangle = 0 \) if \( \rho \in \mathcal{U}(G^F) \) and \( \theta \neq 1 \); see \[9, \S 12.1\]. Hence, \( \mathcal{U}(G^F) \) defines a connected component of the graph \( \mathcal{G}(G^F) \) in \[2.10\].
By Lusztig’s Main Theorem 4.23 in [41], the classification of $\mathfrak{U}(G^F)$ only depends on the pair $(W, \sigma)$. We shall now describe this classification, where we do not follow the scheme in [41] but that in [54, §3]. This will be done in several steps.

4.2. Let us begin by explaining how the classification of $\mathfrak{U}(G^F)$ is reduced to the case where $G$ is a simple algebraic group of adjoint type. (See [36, 1.18] and [39, 3.15]). First, since $G/Z(G)$ is semisimple, there exists a surjective homomorphism of algebraic groups $\pi: G \to G_{\text{ad}}$ which factors through $G/Z(G)$ and where $G_{\text{ad}}$ is a semisimple group of adjoint type (see [30, 1.5.8] and [74, p. 45/64]). Furthermore, there exists a Frobenius map $F: G_{\text{ad}} \to G_{\text{ad}}$ (relative to an $\mathbb{F}_q$-rational structure on $G_{\text{ad}}$) such that $F \circ \pi = \pi \circ F$; thus, $\pi$ is defined over $\mathbb{F}_q$. (See [30, 1.5.9(b)] and [75, 9.16].) Hence, we obtain a group homomorphism $\pi: G^F \to G_{\text{ad}}$ (but note that this is not necessarily surjective). By [12, 7.10], this induces a bijection

$$\mathfrak{U}(G^F) \sim \sim \mathfrak{U}(G^F), \quad \rho \mapsto \rho \circ \pi.$$ 

Now we can write $G_{\text{ad}} = G_1 \times \ldots \times G_r$ where each $G_i$ is semisimple of adjoint type, $F$-stable and $F$-simple, that is, $G_i$ is a direct product of simple algebraic groups which are cyclically permuted by $F$. Let $h_i \geq 1$ be the number of simple factors in $G_i$, and let $H_i$ be one of these. Then $F^{h_i}(H_i) = H_i$ and

$$\iota_i: H_i \to G_i, \quad g \mapsto gF(g) \ldots F^{h_i-1}(g),$$

is an injective homomorphism of algebraic groups which restricts to an isomorphism $\iota_i: H_i^1 \sim \sim G_i^1$ where we denote $F_i := F^{h_i}|_{H_i}: H_i \to H_i$. (See [30, 1.5.15].) Let $f: G_1 \times \ldots \times G_r \to G_{\text{ad}}$ be the product map. Then, finally, it is shown in [36, 1.18] that $f$ and the homomorphisms $\iota_1, \ldots, \iota_r$ induce bijections

$$\mathfrak{U}(G_{\text{ad}}^F) \sim \sim \mathfrak{U}(G_1^F) \times \ldots \times \mathfrak{U}(G_r^F) \sim \sim \mathfrak{U}(H_1^F) \times \ldots \times \mathfrak{U}(H_r^F).$$

Thus, the classification of $\mathfrak{U}(G^F)$ is reduced to the case where $G$ is simple of adjoint type.

4.3. In order to parametrize the set $\mathfrak{U}(G^F)$, we need some further invariants attached to the unipotent characters of $G^F$. (For example, the invariants $A_\rho, a_\rho, n_\rho$ in [33] are not sufficient.) For this purpose, we use an alternative characterization of the unipotent characters of $G^F$. This is based on the varieties (see [12, §1])

$$X_w := \{gB_0 \in G/B_0 \mid g^{-1}F(g) \in B_0wB_0\} \quad (w \in W).$$

Note that $X_w$ is stable under left multiplication by elements of $G^F$. Hence, any $g \in G^F$ induces a linear map of $H_c^i(X_w)$ ($i \in \mathbb{Z}$). We have a morphism of varieties $Y_w \to X_w, x \mapsto xB_0$, which turns out to be surjective. By studying the fibres of this morphism and by using some basic properties of $\ell$-adic cohomology with compact support, one shows that

$$R^i_{\ell}(g) = \sum_i (-1)^i \text{Trace}(g, H^i_c(X_w)) \quad \text{for all } g \in G^F;$$

see [9, 7.7.8, 7.7.11]. Now let $\delta \geq 1$ be the order of $\sigma \in \text{Aut}(W)$. Then $F^\delta(X_w) = X_w$ and, hence, $F^\delta$ induces a linear map of $H^i_c(X_w)$ which commutes with the linear maps induced by the elements of $G^F$. Consequently, if $\mu \in \mathcal{O}_\ell$ is an eigenvalue of $F^\delta$ on $H^i_c(X_w)$, then the corresponding generalized eigenspace $H^i_c(X_w)_\mu$ is a $G^F$-module. Now every $\rho \in \mathfrak{U}(G^F)$ occurs as a constituent of $H^i_c(X_w)_\mu$ for some $w$, some $i$ and some $\mu$. By Digne–Michel [13, III.2.3] and Lusztig [39, 3.9], there is a well-defined root of unity $\omega_\rho$ with the following property. If $\rho$ occurs in $H^i_c(X_w)_\mu$.
for some $w, i, \mu$, then $\mu = \omega_{\rho} q^{m^2/2}$ where $m \in \mathbb{Z}$. (Here, we assume that a square root $q^{1/2} \in \overline{\mathbb{Q}}_\ell$ has been fixed.) We call $\omega_{\rho}$ the Frobenius eigenvalue of $\rho$. We shall regard $\omega_{\rho}$ as an element of $\mathbb{C}$ (via our chosen embedding of the algebraic numbers in $\overline{\mathbb{Q}}_\ell$ into $\mathbb{C}$; see [11]).

4.4. Assume that $G/Z(G)$ is simple or $\{1\}$. Then $W = \{1\}$ or $W \neq \{1\}$ is an irreducible Weyl group. We now define a subset $\mathfrak{X}^\circ(W, \sigma) \subseteq \mathbb{C}^\times \times \mathbb{Z}$, which only depends on the pair $(W, \sigma)$. Assume first that $\sigma$ is the identity; then we write $\mathfrak{X}(W) = \mathfrak{X}(W, \text{id})$. If $W = \{1\}$, then $\mathfrak{X}(W) = \{(1, 1)\}$. Now let $W \neq \{1\}$. Then the sets $\mathfrak{X}(W)$ are given as follows.

- **Type $A_n$** ($n \geq 1$): $\mathfrak{X}(W) = \emptyset$.
- **Type $B_n$ or $C_n$** ($n \geq 2$): $\mathfrak{X}(W) = \{((-1)^{n/2}, 2\ell)\}$ if $n = \ell^2 + \ell$ for some integer $\ell \geq 1$, and $\mathfrak{X}(W) = \emptyset$ otherwise.
- **Type $D_n$** ($n \geq 4$): $\mathfrak{X}(W) = \{((-1)^{n/4}, 2^{\ell-1})\}$ if $n = 4\ell^2$ for some integer $\ell \geq 1$, and $\mathfrak{X}(W) = \emptyset$ otherwise.
- **Type $G_2$:** $\mathfrak{X}(W) = \{(1, 6), (-1, 2), (\theta, 3), (\theta^2, 3)\}$.
- **Type $F_4$:** $\mathfrak{X}(W) = \{(1, 8), (1, 24), (-1, 4), (\pm i, 4), (\theta, 3), (\theta^2, 3)\}$.
- **Type $E_6$:** $\mathfrak{X}(W) = \{(\theta, 3), (\theta^2, 3)\}$.
- **Type $E_7$:** $\mathfrak{X}(W) = \{\pm i, 2\}$.
- **Type $E_8$:** $\mathfrak{X}(W) = \{(1, 8), (1, 120), (-1, 12), (\pm i, 4), (\pm \theta, 6), (\pm \theta^2, 6), (\zeta, 5), (\zeta^2, 5), (\zeta^3, 5), (\zeta^4, 5)\}$.

Here, $\theta, i, \zeta \in \mathbb{C}$ denote fixed primitive roots of unity of order 3, 4, 5, respectively.

Now assume that $\sigma$ is not the identity; let $\delta \geq 2$ be the order of $\sigma$. Then the sets $\mathfrak{X}^\circ(W, \sigma)$ are given as follows.

- **Type $A_n$** ($n \geq 2$) and $\delta = 2$: $\mathfrak{X}(W, \sigma) = \{((-1)^{[n+1]/2}, 1)\}$ if $n + 1 = \ell(\ell - 1)/2$ for some integer $\ell \geq 1$, and $\mathfrak{X}(W, \sigma) = \emptyset$ otherwise.
- **Type $D_n$** ($n \geq 4$) and $\delta = 2$: $\mathfrak{X}(W, \sigma) = \{(1, 2^{\ell+1})\}$ if $n = (2\ell + 1)^2$ for some integer $\ell \geq 1$, and $\mathfrak{X}(W, \sigma) = \emptyset$ otherwise.
- **Type $D_4$ and $\delta = 3$:** $\mathfrak{X}(W, \sigma) = \{\pm 1, 2\}$.
- **Type $E_6$ and $\delta = 2$:** $\mathfrak{X}(W, \sigma) = \{(1, 6), (\theta, 3), (\theta^2, 3)\}$.

Let us denote $\mathfrak{U}(G^F) := \{\rho \in \mathfrak{U}(G^F) \mid \rho$ cuspidal}. Now we can state:

**Theorem 4.5** (Lusztig [34]). Assume that $G/Z(G)$ is simple or $\{1\}$. There exists a unique bijection $\mathfrak{X}(W(S), \sigma) \sim \mathfrak{U}(G^F)$ with the following property. If $\rho \in \mathfrak{U}(G^F)$ corresponds to $x = (\omega, m) \in \mathfrak{X}(W(S), \sigma)$, then $\omega = \omega_{\rho}$ (see 4.3), $m = n_{\rho}$ (see 3.3).

**Remark 4.6.** (a) The second component of the pairs in $\mathfrak{X}(W, \sigma)$ is only needed to distinguish the two cuspidal unipotent characters in types $F_4, E_8$ which both have Frobenius eigenvalue 1. Lusztig [34 3.3], [35 3.3]) uses slightly different methods to achieve this distinction. A similar unicity statement is contained in [11 11.2].

(b) The proof of Theorem 4.5 relies on the explicit knowledge of the degree polynomials $\mathfrak{D}_\rho$ and of the Frobenius eigenvalues $\omega_{\rho}$ in all cases. Tables with the degrees of the cuspidal unipotent characters can be found in [11 13.7]. If $\sigma =$ id, then $\omega_{\rho}$ is determined in [11 11.2]. If $\sigma \neq$ id, then this can be extracted from [35 7.3], [33 3.3], [29 4.4]. (The case where $W$ is of type $A_n$ and $\delta = 2$ can also be dealt with by a similar argument as in the proof of [29 4.11].) The explicit knowledge of the degree polynomials also shows that the function $\rho \mapsto a_{\rho}$ is constant on $\mathfrak{U}(G^F)$. 


4.7. Next we use the fact that the classification of $\mathfrak{U}(G^F)$ can be reduced to the classification of $\mathfrak{U}_c(G^F)$, using the concept of Harish-Chandra induction as in 3.8 (See Lusztig [39, 3.25] for a detailed explanation of this reduction.) Thus, if $G/Z(G)$ is simple, then we have a partition

$$\mathfrak{U}(G^F) = \bigsqcup_{(J,\phi)} \text{Irr}(G^F|J,\phi);$$

here, the union runs over all pairs $(J,\phi)$ such that $J \in \mathcal{P}(S)^\sigma$ and $\phi \in \mathfrak{U}_c(L^F_J)$; also note that, if $\mathfrak{U}_c(L^F_J) \neq \emptyset$, then $L_J/Z(L_J)$ is simple or $\{1\}$. Furthermore, the stabiliser $W_J(\phi)$ in Proposition 3.9 now has a more explicit description:

$$W_J(\phi) = \mathcal{W}^{S/J} := \{ w \in W \mid \sigma(w) = w \text{ and } wJw^{-1} = J \};$$

this is a Weyl group with simple reflections in bijection with the orbits of $\sigma$ on $S \setminus J$ (see [II, 8.2, 8.5]). As before, we have a natural bijection

$$\text{Irr}(\mathcal{W}^{S/J}) \sim \text{Irr}(G^F|J,\phi), \quad \epsilon \mapsto \phi[\epsilon],$$

(depending on the choice of a square root of $q$ in $C$). Now let

$$\mathfrak{X}(W,\sigma) := \{(J,\epsilon,x) \mid J \in \mathcal{P}(S)^\sigma, \epsilon \in \text{Irr}(\mathcal{W}^{S/J}), x \in \mathfrak{X}(\mathcal{W}_J,\sigma)\}.$$

We have an embedding $\mathfrak{X}^o(W,\sigma) \hookrightarrow \mathfrak{X}(W,\sigma), x \mapsto (S,1,x)$.

**Corollary 4.8** (Lusztig [54]). Assume that $G/Z(G)$ is simple or $\{1\}$. There exists a unique bijection $\mathfrak{X}(W,\sigma) \sim \mathfrak{U}(G^F)$ with the following property. If $\rho \in \mathfrak{U}(G^F)$ corresponds to $(J,\epsilon,x) \in \mathfrak{X}(W,\sigma)$, then $\rho = \phi[\epsilon]$ where $\phi \in \mathfrak{U}_c(L^F_J)$ corresponds to $x$ under the bijection in Theorem 4.3.

In this picture, those $\rho \in \mathfrak{U}(G^F)$ which occur in $R_1^1$ (the character of the permutation module $C[G^F/B^F_0]$) correspond to triples $(J,\epsilon,x)$ where $J = \emptyset$, $x = (1,1)$ and $\epsilon \in \text{Irr}(W^\sigma)$. (Note that $\mathcal{W}^{S/\emptyset} = W^\sigma$.) If $\epsilon$ is the trivial character, then $\rho = 1_G$ is the trivial character of $G^F$; if $\epsilon$ is the sign character, then $\rho = \text{St}_G$ is the Steinberg character of $G^F$. (See, e.g., [II, 68B].) At the other extreme, the cuspidal unipotent characters of $G^F$ correspond to triples $(J,\epsilon,x)$ where $J = S$, $\epsilon = 1$ and $x \in \mathfrak{X}^o(\mathcal{W},\sigma)$. (Note that $\mathcal{W}^{S/S} = \{1\}$.)

4.9. If $G/Z(G)$ is not simple or $\{1\}$, then consider the reduction arguments in 4.2. Thus, we obtain a natural bijection

$$\mathfrak{U}(G^F) \sim \mathfrak{U}(H^F_1) \times \ldots \times \mathfrak{U}(H^F_r),$$

where each $H_i \subseteq G_{\text{ad}}$ is a simple algebraic group and $F_i = F^{h_i}|_{H_i} : H_i \to H_i$ for some $h_i \geq 1$. Let $W_i$ be the Weyl group of $H_i$ and $\sigma_i$ be the automorphism of $W_i$ induced by $F_i$. Then Corollary 4.8 yields a bijection

(a) $$\mathfrak{U}(G^F) \sim \mathfrak{X}(W,\sigma) := \mathfrak{X}(W_1,\sigma_1) \times \ldots \times \mathfrak{X}(W_r,\sigma_r).$$

By [II, Main Theorem 4.23], there exist integers $m_{\hat{x},w} \in \mathbb{Z}$ (for $\hat{x} \in \mathfrak{X}(W,\sigma)$ and $w \in W$), which only depend on the pair $(W,\sigma)$, with the following property. If $\rho \in \mathfrak{U}(G^F)$ corresponds to $\hat{x} \in \mathfrak{X}(W,\sigma)$ under the bijection in (a), then

(b) $$\langle R^1_w, \rho \rangle = m_{\hat{x},w} \quad \text{for all } w \in W.$$

Furthermore, there are explicit formulae for $m_{\hat{x},w}$ in terms of Lusztig’s “non-abelian Fourier matrices” (which first appeared in [10, 4]), the function $\Delta : \mathfrak{X}(W,\sigma) \to$
{±1} in \cite[4.21]{41}, and the ("σ-twisted") character table of \( W \). Finally, let \( \hat{x}_0 \in \mathfrak{X}(W, \sigma) \) correspond to the trivial character \( 1_G \in \mathfrak{U}(G^F) \). Then

\[(c) \quad \hat{x}_0 \text{ is uniquely determined by the condition that } m_{\hat{x}_0, w} = 1 \text{ for all } w \in W.\]

Indeed, \( \langle R_w^1, 1_G \rangle = 1 \) for all \( w \in W \); see Remark \ref{remark4.11}. On the other hand, if we also have \( m_{\hat{x}, w} = 1 \) for some \( \hat{x} \in \mathfrak{X}(W, \sigma) \), then \( \langle R_w^1, \rho \rangle = 1 \) for the corresponding \( \rho \in \mathfrak{U}(G^F) \). But then \( \langle 1_G, \rho \rangle = \frac{1}{|W|} \sum_{w \in W} \langle R_w^1, \rho \rangle = 1 \) and so \( \rho = 1_G \), hence \( \hat{x} = \hat{x}_0 \).

**Remark 4.10.** For \( \rho \in \mathfrak{U}(G^F) \), we denote by \( \mathbb{Q}(\rho) = \{ \rho(g) \mid g \in G^F \} \subseteq \mathbb{C} \) the character field of \( \rho \). Then \( \mathbb{Q}(\rho) \) is explicitly known in all cases; see \cite{23, 49}. Assume that \( G/Z(G) \) is simple. Let \( J \in \mathcal{P}(S)^F \) and \( \phi \in \mathfrak{U}^c(L_J^F) \) be cuspidal such that \( \rho \in \mathfrak{U}(G^F|J, \phi) \). Then \( \mathbb{Q}(\rho) \subseteq \mathbb{Q}(q^{1/2}, \omega_0) \), where \( q^{1/2} \) is only needed for certain \( \rho \) for types \( E_7, E_8 \); see \cite[5.4, 5.6]{23} for further details.

**Remark 4.11.** One may wonder what general statements about the multiplicities \( m_{\hat{x}, w} \) could be made. For example, is it true that, for any \( \rho \in \mathfrak{U}(G^F) \), there exists some \( w \in W \) such that \( \langle R_w^1, \rho \rangle = \pm 1 \)? In \cite[p. 356]{41}, there is an example of a cuspidal unipotent character (for the Ree group of type \( ^2F_4 \)) which has even multiplicity in all \( R_w^1 \). There are also examples (e.g., in type \( C_4 \)) of non-cuspidal unipotent characters which have even multiplicity in all \( R_w^1 \); see Lusztig \cite[2.21]{49}.

## 5. Jordan decomposition (connected centre)

The Jordan decomposition reduces the problem of classifying the irreducible characters of \( G^F \) to the classification of unipotent characters for certain “smaller” groups associated with \( G \). If the centre \( Z(G) \) is connected, then this is achieved by Lusztig’s Main Theorem 4.23 in \cite{41}; in \cite{45, 50} this is extended to the general case. This section is meant to give a first introduction into the formalism of Lusztig’s book \cite{41}. We try to present this here in a way which avoids the close discussion of the underlying technical apparatus, which is quite elaborate.

Recall from \cite[10]{23} the definition of the graph \( \mathcal{G}(G^F) \). We have seen in Example 3.6 that every connected component of this graph contains at least one vertex which is labeled by a semisimple character of \( G^F \). The first step is to clarify this situation.

**Definition 5.1.** We set \( \mathcal{S}(G^F) := \{ \rho_0 \in \text{Irr}(G^F) \mid \rho_0 \text{ semisimple} \} \). For \( \rho_0 \in \mathcal{S}(G^F) \), we define

\[\mathcal{E}(\rho_0) := \{ \rho \in \text{Irr}(G^F) \mid \langle R_w^\theta, \rho \rangle \neq 0 \text{ and } \langle R_w^\theta, \rho_0 \rangle \neq 0 \text{ for some } w, \theta \}.\]

The following result is already contained in the original article of Deligne–Lusztig \cite{12} (see Proposition 6.13 below for the case where \( Z(G) \) is not connected).

**Theorem 5.2** (Deligne–Lusztig \cite[§10]{12}). Assume that \( Z(G) \) is connected.

(a) Every connected component of \( \mathcal{G}(G^F) \) contains a unique semisimple character. If \( \rho_0 \in \mathcal{S}(G^F) \), then \( \mathcal{E}(\rho_0) \) is a connected component of \( \mathcal{G}(G^F) \).

(b) We have a partition \( \text{Irr}(G^F) = \bigsqcup_{\rho_0 \in \mathcal{S}(G^F)} \mathcal{E}(\rho_0) \).

(c) The partition of \( \text{Irr}(G^F) \) in (b) corresponds precisely to the partition into “geometric conjugacy classes” (as defined in \cite[§10]{12}, \cite[§12.1]{9}).

**Proof.** All that we need to know about “geometric conjugacy classes” (in addition to the implications in \cite[10]{23}) is the following crucial fact: if \( Z(G) \) is connected, then every geometric conjugacy class of characters contains precisely one semisimple
character; see [12, 10.7], [9, 8.4.6]. Now we can argue as follows. We have already remarked in [2.4.10] that every geometric conjugacy class of characters is a union of connected components of \( G \). We also know that every connected component of \( G \) has at least one vertex labeled by a semisimple character. So the above crucial fact shows that the partition of \( \text{Irr}(G) \) defined by the graph \( G \) corresponds precisely to the partition into geometric conjugacy classes.

In order to complete the proof, it now remains to show that \( \mathcal{E}(\rho_0) \), for \( \rho_0 \in \mathcal{S}(G) \), is a connected component of \( G \). This is seen as follows. Clearly, \( \mathcal{E}(\rho_0) \) is contained in a connected component. Conversely, consider any \( \rho \in \text{Irr}(G) \) such that \( \rho, \rho_0 \) belong to the same connected component. Let \( \psi, \theta \) be such that \( \langle R^0_\psi, \rho \rangle \neq 0 \). By Example 3.6, there exists some \( \rho_0 \in \mathcal{S}(G) \) such that \( \langle R^0_\psi, \rho_0 \rangle \neq 0 \). Then \( \rho_0, \rho_0 \) belong to the same connected component, hence to the same geometric conjugacy class and, hence, \( \rho_0 = \rho_0 \), again by the crucial fact above. □

The next step consists of investigating a piece \( \mathcal{E}(\rho_0) \) in the partition in Theorem 5.2(b). The basic idea is as follows. We wish to associate with \( \rho_0 \) a subgroup \( W' \subseteq W \) which should be itself a Weyl group (i.e., generated by reflections of \( W \)); furthermore, there should be an induced automorphism \( \gamma : W' \rightarrow W' \) such that we can apply the procedure in the previous section to form the set \( \mathfrak{X}(W', \gamma) \). Then, finally, there should be a bijection \( \mathcal{E}(\rho_0) \leftrightarrow \mathfrak{X}(W', \gamma) \) satisfying some further conditions. Now, in order to associate \( W' \) with \( \rho_0 \), we have to use in some way the underlying algebraic group \( G \).

5.3. For the discussion to follow, we need to fix an embedding \( \psi : k^\times \hookrightarrow \mathbb{C}^\times \).

This can be obtained as follows. Recall that \( k = \mathbb{F}_p \). Let \( \mathbb{A} \) be the ring of algebraic integers in \( \mathbb{C} \). Then one can find a surjective homomorphism of rings \( \kappa : \mathbb{A} \rightarrow k \), which we will fix from now on. (A definite choice of \( \kappa \) could be made, for example, using a construction of \( k \) via Conway polynomials; see [60, §4.2].) Another way to make this canonical is described by Lusztig [51, §16].) Let \( \mu_{p'} \) be the group of all roots of unity in \( \mathbb{C} \) of order prime to \( p \). Then \( \kappa \) restricts to an isomorphism \( \mu_{p'} \xrightarrow{\sim} k^\times \), and we let \( \psi : k^\times \xrightarrow{\sim} \mu_{p'} \subseteq \mathbb{C}^\times \) be the inverse isomorphism.

5.4. We can now relate \( \text{Irr}(T_0[w]) \) (for \( w \in W \)) to elements in the character group \( X = X(T_0) \) (see 3.1). The map \( F' : T_0 \rightarrow T_0, t \mapsto \bar{w}F(t)\bar{w}^{-1} \), can also be regarded as a Frobenius map on \( T_0 \), and we have \( T_0[w] = T_0^{F'} \); see [75, 10.9] or [30, 1.4.13]. There is an induced map \( X \rightarrow X, \lambda \mapsto \lambda \circ F', \) as in 3.1. Then, by [12, (5.2.2)*] (see also [9, 3.2.3] or [15, 13.7]), we have an exact sequence

\[
\{0\} \hookrightarrow X \xrightarrow{F'-\text{id}_X} X \twoheadrightarrow \text{Irr}(T_0[w]) \twoheadrightarrow \{1\},
\]

where the map \( X \rightarrow \text{Irr}(T_0[w]) \) is given by restriction, followed by our embedding \( \psi : k^\times \leftrightarrow \mathbb{C}^\times \). Now we proceed as follows, see Lusztig [11, 6.2]. Let \( \theta \in \text{Irr}(T_0[w]) \) and \( n \geq 1 \) be the smallest integer such that \( \theta(t)^n = 1 \) for all \( t \in T_0[w] \). Then \( p \nmid n \) and the values of \( \theta \) lie in the image of our embedding \( \psi : k^\times \leftrightarrow \mathbb{C}^\times \). So we can write \( \theta = \psi \circ \hat{\theta} \) where \( \hat{\theta} : T_0[w] \rightarrow k^\times \) is a group homomorphism. By (a), there exists some \( \lambda_1 \in X \) such that \( \hat{\theta} \) is the restriction of \( \lambda_1 \). Now \( \theta^n = 1 \) and so \( n\lambda_1 \) is in the image of the map \( F' - \text{id}_X : X \rightarrow X \). So there exists some \( \lambda' \in X \) such that \( \lambda' \circ F' - \lambda' = n\lambda_1 \). Setting \( \lambda := \overline{w^{-1}}(\lambda') \in X \), we obtain

\[
\lambda_1(t^n) = (n\lambda_1)(t) = \lambda'(F'(t)t^{-1}) = \lambda(F(t)\bar{w}^{-1}t^{-1}\bar{w}) \quad \text{for all } t \in T_0.
\]
Thus, we have associated with \((w, \theta)\) a pair \((\lambda, n)\) where \(\lambda \in X\) and \(n \geq 1\) is an integer such that \(p \nmid n\) and (b) holds for some \(\lambda_1 \in X\) such that \(\theta = \psi \circ \lambda_1|_{T_0[w]}\). (Note that Lusztig actually works with line bundles \(L\) over \(G/B_0\) instead of characters \(\lambda \in X\), but one can pass from one to the other by [11, 1.3.2].)

**Remark 5.5.** In the setting of [5.4] the integer \(n\) is uniquely determined by \((w, \theta)\), but \(\lambda \in X\) depends on the choice of \(\lambda_1 \in X\) such that \(\theta = \psi \circ \lambda_1|_{T_0[w]}\). Note that, once \(\lambda_1\) is chosen, then \(\lambda\) is uniquely determined since the map \(T_0 \to T_0, t \mapsto F(t)w^{-1}t^{-1}w\), is surjective (Lang's Theorem). Now, let \(\mu_1 \in X\) also be such that \(\theta\) is the restriction of \(\mu_1\). Then \(\mu_1 - \lambda_1\) is trivial on \(T_0[w]\) and so, by [5.4(a)], we have \(\mu_1 - \lambda_1 = \nu \circ F' - \nu\) for some \(\nu \in X\). But then \(\mu := \lambda + nw^{-1}(\nu)\) is the unique element of \(X\) such that [5.4(b)] holds with \(\lambda_1\) replaced by \(\mu_1\). Thus, we can associate with \((w, \theta)\) the well-defined element

\[
\left(\frac{1}{n} + Z\right) \otimes \lambda \in \left(\mathbb{Q}/\mathbb{Z}\right)_{p'} \otimes X,
\]

where \(\left(\mathbb{Q}/\mathbb{Z}\right)_{p'}\) is the group of elements of \(\mathbb{Q}/\mathbb{Z}\) of order prime to \(p\) (see [9, §4.1], [12, §5], [15, Chap. 13], [11, 8.4] for a further discussion of this correspondence).

**5.6.** Conversely, let us begin with a pair \((\lambda, n)\) where \(\lambda \in X\) and \(n \geq 1\) is an integer such that \(p \nmid n\). Following Lusztig [11, 2.1], we define \(Z_{\lambda, n}\) to be the set of all \(w \in W\) for which there exists some \(\lambda_w \in X(T_0)\) such that [5.4(b)] holds, that is,

\[
\lambda(F(t)) = \lambda(\tilde{w}^{-1}tw)\lambda_w(t^n) \quad \text{for all } t \in T_0.
\]

Note that \(\lambda_w\), if it exists, is uniquely determined by \(w\) (since \(T_0 = \{t^n \mid t \in T_0\}\)). Assume now that \(Z_{\lambda, n} \neq \emptyset\). Then, for any \(w \in Z_{\lambda, n}\), the restriction of \(\lambda_w\) to \(T_0[w]\) is a group homomorphism

\[
\tilde{\lambda}_w: T_0[w] \to k^\times \quad \text{such that} \quad \tilde{\lambda}_w^n = 1.
\]

Using our embedding \(\psi: k^\times \hookrightarrow \mathbb{C}^\times\), we obtain a character \(\theta_w := \psi \circ \tilde{\lambda}_w \in \text{Irr}(T_0[w])\), such that \(\theta_w^n = 1\). So each \(w \in Z_{\lambda, n}\) gives rise to a virtual character \(R_{\lambda, n}^\theta\).

**Definition 5.7.** Assume that \(Z_{\lambda, n} \neq \emptyset\). Following Lusztig [11, 2.19, 6.5], we set

\[
\mathcal{E}_{\lambda, n} := \{\rho \in \text{Irr}(G^F) \mid \langle R_{\lambda, n}^\theta, \rho \rangle \neq 0 \text{ for some } w \in Z_{\lambda, n}\}.
\]

By [5.4] any \(\rho \in \text{Irr}(G^F)\) belongs to \(\mathcal{E}_{\lambda, n}\) for some \((\lambda, n)\) as above. (Note that Lusztig assumes that \(Z(G)\) is connected, but the definition can be given in general.)

**Example 5.8.**

(a) Let \(n = 1\) and \(\lambda: T_0 \to k^\times, t \mapsto 1\), be the neutral element of \(X\). Let \(w \in W\) and set \(\lambda_w := \lambda\). Then the condition in [5.6] trivially holds. So, in this case, we have \(Z_{\lambda, 1} = W\) and \(R_{w, \lambda} = R_{w}^1\) for all \(w \in W\). Consequently, \(\mathcal{E}_{\lambda, 1}\) is precisely the set of unipotent characters of \(G^F\).

(b) Let \(G^F = SL_2(\mathbb{F}_q)\) where \(q\) is odd. As in Example [2.11] we write \(T_0 = \{S(\xi) \mid \xi \in k^\times\}\) and \(W = \{1, s\}\). Let \(n = 2\) and \(\lambda \in X\) be defined by \(\lambda(S(\xi)) = \xi\) for all \(\xi \in k^\times\). Now note that, if \(t \in T_0\), then \(t^{q-1} = F(t)t^{-1}\) and \(t^{q+1} = F(t)s^{-1}t^{-1}s\). Thus, if we define \(\lambda_1, \lambda_s \in X\) by

\[
\lambda_1(S(\xi)) = \xi^{(q-1)/2} \quad \text{and} \quad \lambda_s(S(\xi)) = \xi^{(q+1)/2} \quad \text{for all } \xi \in k^\times,
\]

then the condition in [5.6] holds. So \(Z_{\lambda, 2} = W\). The corresponding characters of \(T_0[1]\) and \(T_0[s]\) are the unique non-trivial characters of order 2. So we obtain

\[
\mathcal{E}_{\lambda, 2} = \{\rho'_0, \rho''_0, \pi'_0, \pi''_0\}.
\]
Thus, while \( \{ \rho'_0, \rho''_0 \} \) and \( \{ \pi'_0, \pi''_0 \} \) form different connected components of the graph \( \mathcal{G}(G^F) \), the above constructions reveal a hidden relation among these four characters. (This hidden relation is precisely the relation of “geometric conjugacy”; see the references in Remark 5.3.)

5.9. Now the situation simplifies when the centre \( Z(G) \) is connected. Assume that this is the case. Let \( \lambda, n \) be as in 5.6 such that \( Z_{\lambda,n} \neq \emptyset \). Then, by [41, 1.8, 2.19], there exists a unique element \( w_1 \in W \) of minimal length in \( Z_{\lambda,n} \) and we have

\[
Z_{\lambda,n} = w_1 W_{\lambda,n}
\]

where \( W_{\lambda,n} \) is a reflection subgroup with a canonically defined set of simple reflections \( S_{\lambda,n} \). Furthermore, \( F \) induces an automorphism \( \gamma : W_{\lambda,n} \to W_{\lambda,n} \) such that \( \gamma(S_{\lambda,n}) = S_{\lambda,n} \) and \( \gamma(y) = \sigma(w_1 y w_1^{-1}) \) for \( y \in W_{\lambda,n} \); see [41, 2.15]. By [41, 3.4.1], \( \gamma \) induces an automorphism of the underlying root system. So we can apply the procedure in Section 4 to \( (W_{\lambda,n}, \gamma) \), and obtain a corresponding set \( \mathfrak{X}(W_{\lambda,n}, \gamma) \).

**Theorem 5.10** (Lusztig [41, Main Theorem 4.23]). Assume that \( Z(G) \) is connected. Let \( \rho_0 \in \mathcal{S}(G^F) \) and \( (\lambda, n) \) be a pair as above such that \( \rho_0 \in \mathcal{E}_{\lambda,n} \) (cf. Definition 5.7). Then we have \( \mathcal{E}(\rho_0) = \mathcal{E}_{\lambda,n} \) and there exists a bijection

\[
\mathcal{E}(\rho_0) \sim \mathfrak{X}(W_{\lambda,n}, \gamma), \quad \rho \mapsto \hat{x}_\rho,
\]

such that \( \langle R_{w_1 y_1}^{\theta_1}, \rho \rangle = (-1)^{(w_1)} m_{\hat{x}_\rho,y} \) for \( \rho \in \mathcal{E}_{\lambda,n} \) and \( y \in W_{\lambda,n} \), where \( Z_{\lambda,n} = w_1 W_{\lambda,n} \) (see 5.9) and \( m_{\hat{x}_\rho,y} \) are the numbers in 4.9(b) (with respect to \( (W_{\lambda,n}, \gamma) \)).

**Remark 5.11.** As explained in [41, 8.4], a pair \( (\lambda, n) \) as above can be interpreted as a semisimple element \( s \in G^* \) where \( G^* \) is a group “dual” to \( G \). There is a corresponding Frobenius map \( F^* : G^* \to G^* \) and then the conjugacy class of \( s \) is \( F^* \)-stable. This actually gives rise to a bijection (see also [12, §10]):

\[
\mathcal{S}(G^F) \sim \{ F^* \text{-stable semisimple conjugacy classes of } G^* \}.
\]

Furthermore, the set \( \mathfrak{X}(W_{\lambda,n}, \gamma) \) parametrizes the unipotent characters of \( C_{G^*}(s)^{F^*} \); note that \( C_{G^*}(s) \) is connected since \( Z(G) \) is connected (by a result of Steinberg; see [9, 4.5.9]). For further details about “dual” groups, see [9, Chap. 4].

**Remark 5.12.** Let \( \mathcal{E}(\rho_0) \sim \mathfrak{X}(W_{\lambda,n}, \gamma) \) as in Theorem 5.10. Then the semisimple character \( \rho_0 \in \mathcal{E}(\rho_0) \) corresponds to the unique \( \hat{x}_0 \in \mathfrak{X}(W_{\lambda,n}, \gamma) \) such that \( m_{\hat{x}_0,y} = 1 \) for all \( y \in W_{\lambda,n} \) (see 4.9(c)). Indeed, by [9, 8.4.6], [12, 10.7], \( \rho_0 \) is uniform and it has an explicit expression as a linear combination of virtual characters \( R_{T,\theta} \). As in [15, 14.47], this can be rewritten in the form

\[
\rho_0 = (-1)^{(w_1)} |W_{\lambda,n}|^{-1} \sum_{w \in Z_{\lambda,n}} R_{\theta}^w.
\]

Now let \( \rho \in \mathcal{E}(\rho_0) \) correspond to \( \hat{x}_0 \). By Theorem 5.10, we have \( \langle R_{w}^{\theta}, \rho \rangle = (-1)^{(w_1)} \) for all \( w \in Z_{\lambda,n} \). As in 4.9(c), we find that \( \langle \rho_0, \rho \rangle = 1 \) and, hence, \( \rho = \rho_0 \).

**Remark 5.13.** For any \( w' \in W \) and any \( \theta' \in \text{Irr}(T_0[w']) \), the virtual character \( R_{w'}^{\theta'} \) is equal to one of the virtual characters in the set

\[
\{ R_{w}^{\theta} \mid w \in Z_{\lambda,n} \text{ for some } (\lambda, n) \text{ as above} \}.
\]

Indeed, let \( \rho_0 \in \mathcal{S}(G^F) \) be such that \( \langle R_{w}^{\theta}, \rho_0 \rangle \neq 0 \). Then choose \( (\lambda, n) \) such that \( Z_{\lambda,n} \neq \emptyset \) and \( \rho_0 \in \mathcal{E}_{\lambda,n} \). Using Remark 5.12, we have \( \langle R_{w}^{\theta}, R_{w'}^{\theta'} \rangle \neq 0 \) for some \( w \in Z_{\lambda,n} \). So, finally, \( R_{w'}^{\theta} = R_{w}^{\theta} \) by Proposition 2.6.
5.14. The above results lead to a general plan for classifying the irreducible characters of $G^F$, assuming that $Z(G)$ is connected. In a first step, one considers the dual group $G^*$ and determines the $F$-stable semisimple conjugacy classes of $G^*$. As in Remark 5.11, this gives a parametrization of the set $\mathcal{S}(G^F)$. Since each $\rho_0 \in \mathcal{S}(G^F)$ is uniform, one can even compute — at least in principle — the character values of $\rho_0$ by using the character formula for $R_w^0$ (mentioned in Remark 2.9) and known results on Green functions [85], [95]. For a given $\rho_0 \in \mathcal{S}(G^F)$, one then determines a corresponding pair $(W_{\lambda,n}, \gamma)$ as above. The characters in $\mathcal{S}(\rho_0)$ are parametrized by $\mathfrak{X}(W_{\lambda,n}, \gamma)$ and we know the multiplicities of these characters in $R_w^0$ for all $w, \theta$; hence, we can also work out the corresponding degree polynomials. A large portion of this whole procedure (and also the one in 6.12 below) can even be put on a computer; see [10] and Lübeck [35] where explicit data are made available for various series of groups.

6. Regular embeddings

Let us drop the assumption that $Z(G)$ is connected. Then we can find an embedding $G \subseteq \tilde{G}$, where $\tilde{G}$ is a connected reductive group over $k$ such that

- $G$ is a closed subgroup of $\tilde{G}$ and $G, \tilde{G}$ have the same derived subgroup;
- $Z(\tilde{G})$ is connected;
- there is a Frobenius map $\tilde{F} : \tilde{G} \to \tilde{G}$ whose restriction to $G$ equals $F$.

Such an embedding is called a “regular embedding”. (See [12, 5.18], [45].) This is the key tool to transfer results from the connected centre case to the general case.

Example 6.1. (a) Assume that $Z(G)$ is connected and let $G_{\text{der}} \subseteq G$ be the derived subgroup of $G$. Then, clearly, $G_{\text{der}} \subseteq G$ is a regular embedding. The standard example is given by $G = \text{GL}_n(k)$ where $G_{\text{der}} = \text{SL}_n(k)$.

(b) Let $G = \text{SL}_n(k)$ and suppose we did not know yet of the existence of $\text{GL}_n(k)$. Then we can construct a regular embedding $G \subseteq \tilde{G}$ as follows. Let

$$\tilde{G} := \{(A, \xi) \in M_n(k) \times k^\times \mid \xi \det(A) = 1\};$$

then $Z(\tilde{G}) = \{((\xi I_n, \xi^{-n}) \mid \xi \in k^\times\}$ is connected and $\dim Z(\tilde{G}) = 1$. Identifying $A \in G$ with $(A, 1) \in \tilde{G}$, we obtain a regular embedding. A corresponding Frobenius map $\tilde{F} : \tilde{G} \to \tilde{G}$ is defined by $\tilde{F}(A, \xi) = (F(A), \xi^q)$ if $G^F = \text{SL}_n(q)$, and by $\tilde{F}(A, \xi) = (F(A), \xi^{-q})$ if $G^F = \text{SU}_n(q)$. This is just an example of the general construction mentioned above. For further examples see [30, §1.7].

We now fix a regular embedding $G \subseteq \tilde{G}$.

6.2. Here are some purely group-theoretical properties; see Lehrer [34, §1]. First of all, one easily sees that $G^F$ is a normal subgroup of $\tilde{G}^F$ such that $\tilde{G}^F/G^F$ is abelian. Let $Z = Z(G)$ and $\tilde{Z} = Z(\tilde{G})$. If $T \subseteq G$ is an $F$-stable maximal torus, then $\tilde{T} := \tilde{Z}.T \subseteq \tilde{G}$ is an $\tilde{F}$-stable maximal torus; a similar statement holds for Borel subgroups. In particular, we obtain an $F$-stable split $BN$-pair $(\tilde{T}_0, \tilde{B}_0)$ in $\tilde{G}$ and the inclusion $N_G(T_0) \subseteq N_{\tilde{G}}(\tilde{T}_0)$ induces a canonical isomorphism between the Weyl group $W = N_G(T_0)/T_0$ of $G$ and the Weyl group $N_G(\tilde{T}_0)/\tilde{T}_0$ of $\tilde{G}$. We have $\tilde{G} = G.\tilde{Z}$ but $G^F, \tilde{Z}^F \not\subseteq \tilde{G}^F$, in general. The gap is measured by the group $(Z/Z^o)_F$ (the largest quotient of $Z/Z^o$ on which $F$ acts trivially). By [34, 1.2], [30]...
Note that every irreducible character of $G^F . \tilde{Z}^F$ restricts irreducibly to $G^F$, so if some irreducible character of $\tilde{G}^F$ becomes reducible upon restriction to $G^F$, then the splitting must happen between $\tilde{G}^F$ and $G^F . \tilde{Z}^F$.

**Theorem 6.3** (Lusztig). Let $\tilde{\rho} \in \Irr(\tilde{G}^F)$. Then the restriction of $\tilde{\rho}$ to $G^F$ is multiplicity-free, that is, we have $\tilde{\rho}|_{G^F} = \rho_1 + \ldots + \rho_r$ where $\rho_1, \ldots, \rho_r$ are distinct irreducible characters of $G^F$.

This statement appeared in [45, Prop. 10] (see also [42]), with an outline of the strategy of the proof; the details of this proof, which are surprisingly complicated, were provided much later in [50]. In the meantime, Cabanes and Enguehard also gave a proof in [8 Chap. 16].

Let us now consider the virtual characters $R^\theta_w$ of $G^F$. As discussed in [6.2] we can identify $W$ with the Weyl group $N_G(\tilde{T}_0)/\tilde{T}_0$ of $\tilde{G}$. So it makes sense to define

$$\tilde{T}_0[w] := \{ t \in \tilde{T}_0 \mid \tilde{F}(t) = \tilde{w}^{-1}t\tilde{w} \} \subseteq \tilde{T}_0 \quad \text{for } w \in W,$$

where $\tilde{w} \in N_G(\tilde{T}_0)$ also is a representative of $w$ in the Weyl group of $\tilde{G}$. We have $T_0[w] \subseteq \tilde{T}_0[w]$ and so we can restrict characters from $\tilde{T}_0[w]$ to $T_0[w]$.

**Lemma 6.4.** Let $w \in W$ and $\tilde{\theta} \in \Irr(\tilde{T}_0[w])$. Then $R^\theta_w|_{G^F} = R^\tilde{\theta}_w$, where $\theta \in \Irr(T_0[w])$ is the restriction of $\tilde{\theta}$ to $T_0[w]$.

**Proof.** As in Remark 2.3 we write $R^\theta_w = R_{T_w, \theta}$ and, similarly, $R^\tilde{\theta}_w = R_{\tilde{T}_w, \tilde{\theta}}$. Under these identifications, $\tilde{\theta} \in \Irr(\tilde{T}_0[w])$ is the restriction of $\tilde{\theta} \in \Irr(\tilde{T}_0^F)$ to $T_0[w]$. Then

$$R^\tilde{\theta}_w|_{G^F} = R_{\tilde{T}_w, \tilde{\theta}}|_{G^F} = R_{\tilde{T}_w, \tilde{\theta}} = R^\theta_w$$

where the second equality holds by [15, 13.22] (see also [19, Lemma 1.4]). \hfill $\Box$

Before we continue, we settle a point that was left open in 3.3; this will also be an illustration of how regular embeddings can be used to transfer results on characters from the connected centre case to the general case.

**Remark 6.5.** Let $\tilde{\rho} \in \Irr(\tilde{G}^F)$ and write $\tilde{\rho}|_{G^F} = \rho_1 + \ldots + \rho_r$ as in Theorem 6.3.

(a) The degree polynomials (see 3.3) are related by $D_{\rho_i} = \frac{1}{r}D_{\tilde{\rho}}$ (as it should be).

(b) All $\rho_i$ have the same unipotent support, which is the unipotent support of $\tilde{\rho}$. In particular, $\tilde{\rho} \in S(\tilde{G}^F)$ if and only if $\rho_i \in S(G^F)$ for all $i$.

**Proof.** (a) For $w \in W$ let $Q_w$ and $\tilde{Q}_w$ be the corresponding Green functions (see Remark 2.9) for $G^F$ and $\tilde{G}^F$, respectively. Then we can write $D_{\rho_i}$ as

$$D_{\rho_i} = \frac{1}{|W|} \sum_{w \in W} (-1)^{l(w)}|T_0[w]|\langle Q_w, \rho_i \rangle q^{-|G|/|T_w|},$$
and we have a similar formula for $D_{\tilde{\rho}}$. By Lemma 6.4 $Q_w$ is the restriction of $\tilde{Q}_w$. Since the $\rho_i$ are conjugate under $\tilde{G}^F$, this implies that $\langle Q_w, \rho_i \rangle$ does not depend on $i$ and so all $D_{\rho_i}$ are equal. Thus,

$$rD_{\rho_i} = D_{\rho_1} + \ldots + D_{\rho_r} = \frac{1}{|W|} \sum_{w \in W} (-1)^{l(w)}|T_0[w]|\langle Q_w, \tilde{\rho}|_{G^F} \rangle q^{-|G|/|T_w|},$$
for any fixed \( i \in \{1, \ldots, r\} \). Next note that \( G_{\text{uni}} = \tilde{G}_{\text{uni}} \) and so
\[
\text{Ind}_{\tilde{G}^F}^{G^F}(\tilde{\rho}|_{G^F})(u) = |\tilde{G}^F : G^F|\tilde{\rho}(u) \quad \text{for all unipotent } u \in \tilde{G}^F.
\]
Hence, using Frobenius reciprocity, we obtain 
\[
\langle Q_w, \tilde{\rho}|_{G^F} \rangle = |\tilde{G}^F : G^F|\langle \tilde{Q}_w, \tilde{\rho} \rangle.
\]
Finally, the formulae in Remark 3.2 imply that 
\[
|G|/|G_w| = |G|/|T_w| \text{ for any } w \in W.
\]
This immediately yields that \( r\mathbb{P}_{p_i} = \mathbb{D}_{\tilde{p}} \) for all \( i \).

(b) Let \( \Theta \) be an \( F \)-stable unipotent class of \( G \); then \( \Theta \) also is an \( \tilde{F} \)-stable unipotent class in \( \tilde{G} \). Since the \( \rho_i \) are \( G^F \)-conjugate, we conclude that \( \text{AV}(\rho_i, \Theta) \) does not depend on \( i \). As in the proof of [28, 3.7], we then have
\[
r[A(u)|\text{AV}(\rho_i, \Theta)] = |A(u)|\text{AV}(\tilde{\rho}, \Theta) \quad \text{for any fixed } i,
\]
where \( A(u) = C_G(u)/C_{\tilde{G}}(u) \) and \( \tilde{A}(u) = C_{\tilde{G}}(u)/C_{\tilde{G}}(u) \) for \( u \in \Theta \). This clearly yields the statement about the unipotent supports of \( \rho_i \).

\section{6.6.}
We now discuss some purely Clifford-theoretic aspects (cf. [34, §2]). Let \( \Theta \) denote the group of all linear characters \( \tilde{\eta} : \tilde{G}^\tilde{F} \to \mathbb{C}^\times \) with \( G^F \subseteq \text{ker}(\tilde{\eta}) \). Then \( \Theta \) acts on \( \text{Irr}(\tilde{G}^\tilde{F}) \) via \( \tilde{\rho} \mapsto \tilde{\eta} \cdot \tilde{\rho} \) (usual pointwise multiplication of class functions). If \( \tilde{\rho}, \tilde{\rho}' \in \text{Irr}(\tilde{G}^\tilde{F}) \), then either \( \tilde{\rho}|_{G^F}, \tilde{\rho}'|_{G^F} \) do not have any irreducible constituent in common, or we have \( \tilde{\rho}|_{G^F} = \tilde{\rho}'|_{G^F} \); the latter case happens precisely when \( \tilde{\rho}' = \tilde{\eta} \cdot \tilde{\rho} \) for some \( \tilde{\eta} \in \Theta \). Given \( \tilde{\rho} \in \text{Irr}(\tilde{G}^\tilde{F}) \), we denote the stabilizer of \( \tilde{\rho} \) by
\[
\Theta(\tilde{\rho}) := \{ \tilde{\eta} \in \Theta \mid \tilde{\eta} \cdot \tilde{\rho} = \tilde{\rho} \}.
\]
Now write \( \tilde{\rho}|_{G^F} = \rho_1 + \ldots + \rho_r \), as in Theorem 6.3. Then
\[
r = \langle \tilde{\rho}|_{G^F}, \tilde{\rho}|_{G^F} \rangle = \langle \tilde{\rho}, \text{Ind}_{G^F}^{\tilde{G}^F}(\tilde{\rho}|_{G^F}) \rangle = \sum_{\tilde{\eta} \in \Theta} \langle \tilde{\rho}, \tilde{\eta} \cdot \tilde{\rho} \rangle = |\Theta(\tilde{\rho})|.
\]
Furthermore, let again \( Z = Z(G) \) and \( \tilde{Z} = Z(\tilde{G}) \) as in 6.2. Then
\[
\Theta(\tilde{\rho}) \subseteq \{ \tilde{\eta} \in \Theta \mid \tilde{Z}^\tilde{F} \subseteq \text{ker}(\tilde{\eta}) \},
\]
since any element of \( \tilde{Z}^\tilde{F} \) acts as a scalar in a representation affording \( \tilde{\rho} \). Consequently, \( r = |\Theta(\tilde{\rho})| \) divides \( |\tilde{G}^F : G^F.\tilde{Z}^\tilde{F}| \) and, hence, the order of \( (Z/Z^\ast)^F \) (where we use the exact sequence in 6.2).

\textbf{Remark 6.7.} The set \( \mathcal{S}(\tilde{G}^\tilde{F}) \) is invariant under the action of \( \Theta \). This is clear by the definition in Example 3.6. Just note that \( \tilde{G}_{\text{uni}} = G_{\text{uni}} \) and \( \tilde{\eta}(u) = 1 \) for all \( \tilde{\eta} \in \Theta \) and all unipotent elements \( u \in G^F \).

\textbf{Lemma 6.8.} Let \( \tilde{\rho}_1, \tilde{\rho}_2 \in \text{Irr}(\tilde{G}^\tilde{F}) \) and \( \tilde{\eta} \in \Theta \). If \( \tilde{\rho}_1 \) and \( \tilde{\rho}_2 \) belong to the same connected component of \( \mathcal{S}(G^F) \), then so do \( \tilde{\eta} \cdot \tilde{\rho}_1 \) and \( \tilde{\eta} \cdot \tilde{\rho}_2 \). Thus, the action of \( \Theta \) on \( \text{Irr}(\tilde{G}^\tilde{F}) \) induces a permutation of the connected components of \( \mathcal{S}(\tilde{G}^\tilde{F}) \) satisfying
\[
\tilde{\eta} \cdot \mathcal{S}(\tilde{\rho}_0) = \mathcal{S}(\tilde{\rho}_0) \quad \text{for } \tilde{\rho}_0 \in \mathcal{S}(\tilde{G}^\tilde{F}) \text{ (cf. Definition 5.4)}.
\]

\textbf{Proof.} By Theorem 5.2 we have \( \tilde{\rho}_1 \in \mathcal{S}(\tilde{\rho}_0) \) for some \( \tilde{\rho}_0 \in \mathcal{S}(G^F) \). By the definition of \( \mathcal{S}(\tilde{\rho}_0) \), we have \( \langle R_\theta^\tilde{\rho}_0, \tilde{\rho}_1 \rangle \neq 0 \) and \( \langle R_\theta^\tilde{\rho}_0, \tilde{\rho}_0 \rangle \neq 0 \) for some \( w \in W \) and \( \tilde{\theta} \in \text{Irr}(\tilde{T}_0[w]) \). We write \( R_\theta^\tilde{\rho}_0 = R_{\tilde{T}_w, \tilde{\theta}}^\tilde{\rho}_0 \) as in Remark 2.3. Now \( \tilde{\eta} \) is a “p-constant function” on \( \tilde{G}^\tilde{F} \) since \( G^F \subseteq \text{ker}(\tilde{\eta}) \); see [15, 7.2]. So, by [15, 12.6], we have
\[
\tilde{\eta} \cdot R_{\tilde{T}_w, \tilde{\theta}}^\tilde{\rho}_0 = R_{\tilde{T}_w, \tilde{\eta} \tilde{\theta}}^\tilde{\rho}_0
\]
where, on the right hand side, \( \tilde{\eta} \) also denotes the restriction of \( \tilde{\eta} \) to \( \tilde{T}_0^F \). Also note that \( R_{\tilde{T}_0, \tilde{\eta}} \) for a unique \( \tilde{\theta}' \in \text{Irr}(\tilde{T}_0[w]) \) (again, by Remark 2.3). Hence,

\[
\langle R_{\tilde{T}_0, \tilde{\eta}}^{\tilde{\theta}'}, \tilde{\eta} \cdot \tilde{\rho}_1 \rangle = \langle \tilde{\eta} \cdot R_{\tilde{T}_0, \tilde{\eta}}^{\tilde{\theta}'}, \tilde{\eta} \cdot \tilde{\rho}_1 \rangle = \langle R_{\tilde{T}_0, \tilde{\eta}}^{\tilde{\theta}'}, \tilde{\eta} \cdot \tilde{\rho}_1 \rangle \neq 0
\]

and, similarly, \( \langle R_{\tilde{T}_0, \tilde{\eta}}^{\tilde{\theta}'}, \tilde{\eta} \cdot \tilde{\rho}_0 \rangle = \langle R_{\tilde{T}_0, \tilde{\eta}}^{\tilde{\theta}'}, \tilde{\rho}_0 \rangle \neq 0 \). By Remark 6.7, we have \( \tilde{\eta} \cdot \tilde{\rho}_0 \in \mathcal{S}(\tilde{G}_F) \) and so \( \tilde{\eta} \cdot \tilde{\rho}_1 \in \mathcal{S}(\tilde{G}_F) \), by the definition of \( \mathcal{S}(\tilde{G}_F) \). We conclude that, if \( \tilde{\rho}_1 \) and \( \tilde{\rho}_2 \) belong to \( \mathcal{S}(\tilde{G}_F) \), then \( \tilde{\eta} \cdot \tilde{\rho}_1 \) and \( \tilde{\eta} \cdot \tilde{\rho}_2 \) belong to \( \mathcal{S}(\tilde{G}_F) \). It remains to use the characterisation of connected components in Theorem 5.2.

Let \( \tilde{\rho}_0 \in \mathcal{S}(\tilde{G}_F) \). Then Lemma 6.8 shows that, in particular, the set \( \mathcal{S}(\tilde{G}_F) \) is preserved under multiplication with any \( \tilde{\eta} \in \Theta(\tilde{\rho}_0) \).

**Proposition 6.9** (Lusztig [11.14.1], [15.11]). Let \( \tilde{\rho}_0 \in \mathcal{S}(\tilde{G}_F) \) and \( \tilde{\rho} \in \mathcal{S}(\tilde{\rho}_0) \). Let \( O \subseteq \mathcal{S}(\tilde{\rho}_0) \) be the orbit of \( \tilde{\rho} \) under the action of \( \Theta(\tilde{\rho}_0) \). Write

\[
\tilde{\rho}|_{G_F} = \rho_1 + \ldots + \rho_r \quad \text{where} \quad \rho_1, \ldots, \rho_r \in \text{Irr}(G_F) \quad \text{(see Theorem 6.3)}.
\]

Then \( r = |\Theta(\tilde{\rho})| \) and \( \Theta(\tilde{\rho}) \subseteq \Theta(\tilde{\rho}_0) \). Let \( w \in W \) and \( \theta \in \text{Irr}(T_0[w]) \). If \( \theta \) is the restriction of some \( \tilde{\theta} \in \text{Irr}(T_0[w]) \) such that \( \langle R_{\tilde{T}_0, \tilde{\eta}}^{\tilde{\theta}}, \tilde{\rho}_0 \rangle \neq 0 \), then

\[
\langle R_{\tilde{T}_0, \tilde{\eta}}^{\tilde{\theta}}, \rho_i \rangle = \sum_{\tilde{\rho}' \in O} \langle R_{\tilde{T}_0, \tilde{\eta}}^{\tilde{\theta}}, \tilde{\rho}' \rangle \quad \text{for} \quad 1 \leq i \leq r;
\]

otherwise, we have \( \langle R_{\tilde{T}_0, \tilde{\eta}}^{\tilde{\theta}}, \rho_i \rangle = 0 \) for \( 1 \leq i \leq r \).

**Proof.** By Lemma 6.6, we have \( r = |\Theta(\tilde{\rho})| \). If \( \tilde{\eta} \in \Theta(\tilde{\rho}) \), then \( \tilde{\rho} = \tilde{\eta} \cdot \tilde{\rho} \) and \( \tilde{\eta} \cdot \tilde{\rho}_0 \in \mathcal{S}(\tilde{G}_F) \) belong to the same connected component of \( \mathcal{S}(G_F) \) by Lemma 6.8. Hence, we must have \( \tilde{\rho}_0 = \tilde{\eta} \cdot \tilde{\rho}_0 \) by Theorem 5.2. This shows that \( \Theta(\tilde{\rho}) \subseteq \Theta(\tilde{\rho}_0) \). It remains to prove the multiplicity formula. Let \( w \in W \) and \( \theta \in \text{Irr}(T_0[w]) \) be the restriction of some \( \tilde{\theta} \in \text{Irr}(T_0[w]) \). By Lemma 6.4 and Frobenius reciprocity, we have

\[
(\ast) \quad \langle R_{\tilde{T}_0, \tilde{\eta}}^{\tilde{\theta}}, \rho_i \rangle = \langle R_{\tilde{T}_0, \tilde{\eta}}^{\tilde{\theta}}, \rho_i \rangle = \langle R_{\tilde{T}_0, \tilde{\eta}}^{\tilde{\theta}}, \text{Ind}^{\tilde{G}_F}_{G_F}(\rho_i) \rangle = \sum_{\tilde{\rho}' \in O'} \langle R_{\tilde{T}_0, \tilde{\eta}}^{\tilde{\theta}}, \tilde{\rho}' \rangle
\]

where \( O' \subseteq \text{Irr}(G_F) \) is the full orbit of \( \tilde{\rho} \) under the action of \( \Theta \). Now assume that \( \langle R_{\tilde{T}_0, \tilde{\eta}}^{\tilde{\theta}}, \rho_i \rangle \neq 0 \). Let \( \tilde{\rho}' \in O' \) be such that \( \langle R_{\tilde{T}_0, \tilde{\eta}}^{\tilde{\theta}}, \tilde{\rho}' \rangle \neq 0 \); then \( \tilde{\rho}' \in \mathcal{S}(\tilde{\rho}_0) \). If we write \( \tilde{\rho}' = \tilde{\eta} \cdot \tilde{\rho} \) where \( \tilde{\eta} \in \Theta \), then Lemma 6.8 shows that \( \tilde{\rho}' \) and \( \tilde{\eta} \cdot \tilde{\rho}_0 \) belong to the same connected component of \( \mathcal{S}(\tilde{G}_F) \) and so \( \tilde{\eta} \cdot \tilde{\rho}_0 = \tilde{\rho}_0 \), that is, \( \tilde{\rho}' \in O \). Thus, the desired formula holds in this case.

Finally, assume that \( \langle R_{\tilde{T}_0, \tilde{\eta}}^{\tilde{\theta}}, \rho_i \rangle \neq 0 \). Then we must show that \( \tilde{\theta} \) can be chosen such that \( \langle R_{\tilde{T}_0, \tilde{\eta}}^{\tilde{\theta}}, \rho_i \rangle \neq 0 \). Now, \( (\ast) \) shows that \( \langle R_{\tilde{T}_0, \tilde{\eta}}^{\tilde{\theta}}, \tilde{\rho}' \rangle \neq 0 \) for some \( \tilde{\rho}' \in O' \). Let \( \tilde{\eta} \in \Theta \) be such that \( \tilde{\rho}' = \tilde{\eta} \cdot \tilde{\rho} \), and let \( \tilde{\eta}^* \) be the complex conjugate of \( \tilde{\eta} \). Now write again \( R_{\tilde{T}_0, \tilde{\eta}}^{\tilde{\theta}} = R_{\tilde{T}_0, \tilde{\eta}}^{\tilde{\theta}} \) and \( R_{\tilde{T}_0, \tilde{\eta}}^{\tilde{\theta}} = R_{\tilde{T}_0, \tilde{\eta}}^{\tilde{\theta}} \) as in the proof of Lemma 6.8. Then we have:

\[
\langle R_{\tilde{T}_0, \tilde{\eta}}^{\tilde{\theta}}, \tilde{\rho} \rangle = \langle \tilde{\rho}^*, R_{\tilde{T}_0, \tilde{\eta}}^{\tilde{\theta}}, \tilde{\rho} \rangle = \langle R_{\tilde{T}_0, \tilde{\eta}}^{\tilde{\theta}}, \tilde{\rho} \rangle = \langle R_{\tilde{T}_0, \tilde{\eta}}^{\tilde{\theta}}, \tilde{\rho} \rangle \neq 0.
\]

Now define \( \tilde{\theta}' \in \text{Irr}(\tilde{T}_0[w]) \) by the condition that \( \tilde{\theta}' = \tilde{\rho}^* \cdot \tilde{\theta} \); then \( \langle R_{\tilde{T}_0, \tilde{\eta}}^{\tilde{\theta}'}, \tilde{\rho} \rangle \neq 0 \).

Since \( \tilde{\rho} \in \mathcal{S}(\tilde{\rho}_0) \), we also have \( \langle R_{\tilde{T}_0, \tilde{\eta}}^{\tilde{\theta}'}, \tilde{\rho}_0 \rangle \neq 0 \). Finally, since \( \tilde{\theta} \) is the restriction of \( \tilde{\theta} \) to \( T_0^F \), we also have that \( \tilde{\theta} \) is the restriction of \( \tilde{\theta}' \) to \( T_0^F \). So the restriction of \( \tilde{\theta}' \) to \( T_0^F \) equals \( \tilde{\theta} \), as required. \( \square \)
The crucial ingredient in the above result is the action of $\Theta$ on $\text{Irr}(\tilde{G}^F)$, and it would be very useful to describe this action in terms of the bijections in Theorem 5.10. This is done in the further parts of [45], leading to the final statement of a “Jordan decomposition” in [45] §12. In this respect, the following example plays a special role; see part (a) of the proof of [45] Prop. 8.1.

Example 6.10 (Lusztig [41 p. 353]). Assume that $p > 2$ and $G$ is simple of simply-connected type $E_7$. Then $Z(G)$ has order 2 and so an irreducible character of $\tilde{G}^F$ either remains irreducible upon restriction to $G^F$, or the restriction splits up as a sum of two distinct irreducible characters. Now, there is a semisimple character $\tilde{\rho}_0 \in \mathcal{I}(\tilde{G}^F)$ such that, in the setting of Theorem 5.10 the group $W_{\lambda,n}$ is of type $E_6$ and $\gamma = \text{id}$. By [45], the set $\mathcal{X}(W_{\lambda,n}, \gamma) \hookrightarrow \mathcal{X}(W_{\lambda,n})$ contains two elements of the form $(\theta, 3)$, $(\theta^2, 3)$. Let $\tilde{\rho}_1, \tilde{\rho}_2$ be the corresponding irreducible characters in $\mathcal{E}(\tilde{\rho}_0)$. The question is whether these two characters are fixed by $\Theta(\tilde{\rho}_0)$ or not, and this turns out to be difficult to tell without knowing some extra information, e.g., sufficiently many character values. In [41] p. 353, it is shown by a separate argument that the restrictions of $\tilde{\rho}_1, \tilde{\rho}_2$ to $G^F$ are reducible; so the conclusion is that these two characters must be fixed by $\Theta(\tilde{\rho}_0)$.

Here is an example where $O$ in Proposition 6.9 has more than one element.

Example 6.11. Let $G^F = \text{Sp}_4(\mathbb{F}_q)$, where $q$ is odd. Then we have a regular embedding $G \subset \tilde{G}$ where $\tilde{G}^F = \text{CSp}_4(\mathbb{F}_q)$ is the conformal symplectic group. Here, $\tilde{G}^F/G^F \cong \mathbb{F}_q^*$ and $Z(G)$ has order 2. The character tables of $G^F$ and $\tilde{G}^F$ are known by Srinivasan [72] and Shinoda [64], respectively. Let $\tilde{\rho}_0 \in \mathcal{I}(\tilde{G}^F)$ be one of the $\frac{1}{2}(q-1)$ semisimple characters of $\tilde{G}^F$ denoted by $\tau_1(\lambda)$ in [64] 5.1. We have

$$\tilde{\rho}_0(1) = q^2 + 1 \quad \text{and} \quad \Theta(\tilde{\rho}_0) = \langle \tilde{\eta} \rangle \cong \mathbb{Z}/2\mathbb{Z},$$

where $\tilde{\eta} \in \Theta$ is the unique character of order 2. (In the setting of Remark 5.11, $\tilde{\rho}_0$ corresponds to a semisimple $s \in \tilde{G}^*$ such that the Weyl group of $C_{\tilde{G}^*}(s)$ is of type $A_1 \times A_1$.) By [64] 5.3, the corresponding connected component of $\mathcal{I}(\tilde{G}^F)$ contains exactly four irreducible characters. Using the notation in [64] 5.1, we have

$$\mathcal{E}(\tilde{\rho}_0) = \{ \tau_1(\lambda), \tau_2(\lambda), \tau_2(\lambda^*), \tau_3(\lambda) \} \quad (\tilde{\rho}_0 = \tau_1(\lambda)),$$

where $\tau_2(\lambda)(1) = \tau_2(\lambda^*)(1) = q(q^2 + 1)$ and $\tau_3(\lambda)(1) = q^2(q^2 + 1)$. Now we restrict these characters to $G^F$. Using the notation of [72], we find that

$$\tau_1(\lambda)_{|G^F} = \theta_3 + \theta_4, \quad \tau_3(\lambda)_{|G^F} = \theta_1 + \theta_2, \quad \tau_2(\lambda)_{|G^F} = \tau_2(\lambda^*)_{|G^F} = \Phi_9,$$

where $\theta_1, \theta_2, \theta_3, \theta_4, \Phi_9 \in \text{Irr}(G^F)$. We conclude that $\tau_2(\lambda^*) = \tilde{\eta} \cdot \tau_2(\lambda)$ and so

$$O = \{ \tau_2(\lambda), \tau_2(\lambda^*) \} \quad \text{is the } \Theta(\tilde{\rho}_0)-\text{orbit of } \tau_2(\lambda).$$

Let us evaluate the formula in Proposition 6.9 in some particular cases. First, by [64] 5.3, there is some $\tilde{\theta} \in \text{Irr}(\tilde{T}_0[1])$ such that

$$R_{\tilde{\theta}}^1 = \tau_1(\lambda) + \tau_2(\lambda) + \tau_2(\lambda^*) + \tau_3(\lambda).$$

Let $\theta \in \text{Irr}(T_0[1])$ be the restriction of $\tilde{\theta}$. Then, by Proposition 6.9 we have

$$\langle R_{\tilde{\theta}}^1, \Phi_9 \rangle = \langle R_{\tilde{\theta}}^1, \tau_2(\lambda) \rangle + \langle R_{\tilde{\theta}}^1, \tau_2(\lambda^*) \rangle = 1 + 1 = 2,$$
Proof. Let \( \theta' \in \text{Irr}(\tilde{T}_0[w]) \) be the restriction of \( \tilde{\theta}' \). Then, by Proposition 6.9 we have

\[
\langle R_{w\alpha}^{\theta'}, \Phi_\theta \rangle = \langle R_{w\alpha}^{\theta'}, \tau_2(\lambda) \rangle + \langle R_{w\alpha}^{\theta'}, \tau_2(\lambda^*) \rangle = -1 + 1 = 0,
\]

(which is consistent with [73, p. 191]). Thus, the terms in the sum \( \sum_{\tilde{\theta}' \in \Theta} \langle R_{w\alpha}^{\tilde{\theta}'}, \rho' \rangle \) in Proposition 6.9 can actually take different values. (Similar examples also exist for the regular embedding \( SL_n(k) \subseteq GL_n(k) \), as was pointed out by C. Bonnafé.)

6.12. The above results lead to a general plan for classifying the irreducible characters of \( G^F \), assuming that the analogous problem for \( \tilde{G}^F \) has been solved (see 5.14). There is one further step to do: determine the action of \( \Theta \) on \( \text{Irr}(\tilde{G}^F) \). Recall from Remark 6.7 that \( \mathcal{J}(\tilde{G}^F) \) is invariant under this action. Then we have a partition

\[
\text{Irr}(G^F) = \bigsqcup_{\tilde{\rho}_0} \mathcal{E}(G^F, \tilde{\rho}_0)
\]

where \( \tilde{\rho}_0 \) runs over a set of representatives of the \( \Theta \)-orbits on \( \mathcal{J}(\tilde{G}^F) \) and

\[
\mathcal{E}(G^F, \tilde{\rho}_0) := \{ \rho \in \text{Irr}(G^F) \mid \langle \tilde{\rho}_0|_{G^F}, \rho \rangle \neq 0 \text{ for some } \tilde{\rho} \in \mathcal{E}(\tilde{\rho}_0) \}
\]

for \( \tilde{\rho}_0 \in \mathcal{J}(\tilde{G}^F) \). Using Proposition 6.9 we can work out the number of irreducible constituents \( \rho_i \) of any \( \tilde{\rho} \in \mathcal{E}(\tilde{\rho}_0) \) and the multiplicities \( \langle R_{w\alpha}^\theta, \rho_i \rangle \); the degree polynomials of \( \rho_i \) are determined by Remark 6.4. Note that the above union is indeed disjoint: if \( \rho \in \text{Irr}(G^F) \) and \( \langle \tilde{\rho}_0|_{G^F}, \rho \rangle \neq 0 \), \( \langle \rho'|_{G^F}, \rho \rangle \neq 0 \), where \( \tilde{\rho} \in \mathcal{E}(\tilde{\rho}_0) \) and \( \rho' \in \mathcal{E}(\tilde{\rho}'_0) \), then there exists some \( \tilde{\eta} \in \Theta \) such that \( \tilde{\rho}' = \tilde{\eta} \cdot \tilde{\rho} \) (see 6.6) and so \( \tilde{\rho}'_0 = \tilde{\eta} \cdot \tilde{\rho}_0 \) (see Lemma 6.8 and Theorem 5.2).

In a somewhat different setting and formulation, part (b) of the following result appears in Bonnafé [5, Prop. 11.7].

**Proposition 6.13.** Let \( \tilde{\rho}_0 \in \mathcal{J}(\tilde{G}^F) \) and \( \rho_0 \in \text{Irr}(G^F) \) be such that \( \langle \tilde{\rho}_0|_{G^F}, \rho_0 \rangle \neq 0 \).

(a) We have \( \rho_0 \in \mathcal{J}(G^F) \) (see Remark 6.3 (b)).

(b) The set \( \mathcal{E}(G^F, \tilde{\rho}_0) \) (see 6.12) equals the set \( \mathcal{E}(\rho_0) \) (see Definition 5.1).

(c) The partition of \( \text{Irr}(G^F) \) in 6.12 corresponds precisely to the partition given by the connected components of \( \mathcal{J}(G^F) \).

**Proof.** First we show that the characters in a connected component of \( \mathcal{J}(G^F) \) are all contained in a set \( \mathcal{E}(G^F, \tilde{\rho}_0) \) as above. Indeed, let \( \rho, \rho' \in \text{Irr}(G^F) \) and assume that \( \langle R_{w\alpha}^\theta, \rho \rangle \neq 0 \), \( \langle R_{w\alpha}^\theta, \rho' \rangle \neq 0 \) for some \( w, \theta \). Choose any \( \tilde{\theta} \in \text{Irr}(\tilde{T}_0[w]) \) such that \( \theta \) is the restriction of \( \tilde{\theta} \). By Lemma 6.4 and Frobenius reciprocity, there exist \( \tilde{\rho}, \tilde{\rho}' \in \text{Irr}(\tilde{G}^F) \) such that the scalar products \( \langle R_{w\alpha}^\theta, \tilde{\rho} \rangle, \langle R_{w\alpha}^\theta, \tilde{\rho}' \rangle, \langle \tilde{\rho}|_{G^F}, \rho \rangle, \langle \tilde{\rho}'|_{G^F}, \rho \rangle \) are all non-zero. Then \( \tilde{\rho}, \tilde{\rho}' \) belong to the same connected component of \( \mathcal{J}(G^F) \) and so \( \tilde{\rho}, \tilde{\rho}' \in \mathcal{E}(\tilde{\rho}_0) \) for some \( \tilde{\rho}_0 \in \mathcal{J}(\tilde{G}^F) \) (see Theorem 5.2). But then, by definition, \( \rho, \rho' \in \mathcal{E}(G^F, \tilde{\rho}_0) \), as claimed. It now remains to show that \( \mathcal{E}(G^F, \tilde{\rho}_0) \subseteq \mathcal{E}(\rho_0) \). So let \( \rho \in \mathcal{E}(G^F, \tilde{\rho}_0) \). We must show that \( \langle R_{w\alpha}^\theta, \rho \rangle \neq 0 \) and \( \langle R_{w\alpha}^\theta, \rho_0 \rangle \neq 0 \) for some pair \( (w, \theta) \). For this purpose, as in the proof of [5, Prop. 11.7], we consider the linear combination \( \sum_{\tilde{\rho} \in \mathcal{E}(\tilde{\rho}_0)} \tilde{\rho}(1) \tilde{\rho} \). By [38, 7.7], that linear combination is a uniform
function. So there exist \( w_i \in W \) and \( \tilde{\theta}_i \in \text{Irr}(\tilde{T}_0[w_i]) \) (where \( 1 \leq i \leq n \) for some \( n \)), such that

\[
\sum_{\tilde{\rho} \in \mathcal{E}(\tilde{\rho}_0)} \tilde{\rho}(1)\tilde{\rho} = \sum_{1 \leq i \leq n} c_i \tilde{R}_{\tilde{\theta}_i}^{\tilde{\rho}_0} \quad \text{where} \quad c_i \in \mathbb{C}, \ c_i \neq 0 \text{ for all } i.
\]

Here, we can assume that \( \tilde{R}_{w_i}^{\tilde{\theta}_i} \) and \( \tilde{R}_{w_j}^{\tilde{\theta}_j} \) are orthogonal if \( i \neq j \) (see Proposition 2.7). Let \( \theta_i \) denote the restriction of \( \tilde{\theta}_i \) to \( T_0[w_i] \). Now \( \rho \in \mathcal{E}(G^F, \tilde{\rho}_0) \) has a non-zero scalar product with the restriction of the left hand side of the above identity to \( G^F \).

Hence, using Lemma 6.4 there exists some \( i \) such that \( \langle \tilde{R}_{w_i}^{\tilde{\theta}_i}, \tilde{\rho} \rangle \neq 0 \). On the other hand, since \( c_i \neq 0 \), the above identity shows that \( \langle \tilde{R}_{w_i}^{\tilde{\theta}_i}, \tilde{\rho} \rangle \neq 0 \). By Example 3.6 there exists some \( \tilde{\rho}_0' \in \mathcal{E}(\tilde{G}^F) \) such that \( \langle \tilde{R}_{w_i}^{\tilde{\theta}_i}, \tilde{\rho}_0' \rangle \neq 0 \). Then \( \tilde{\rho} \in \mathcal{E}(\tilde{\rho}_0') \) and so Theorem 5.2 shows that \( \tilde{\rho}_0' = \tilde{\rho}_0 \). Thus, we have \( \langle R_{w_i}^{\theta_i}, \rho_0 \rangle \neq 0 \).

Now Proposition 6.9 yields that \( \langle R_{w_i}^{\theta_i}, \rho_0 \rangle = \langle R_{w_i}^{\tilde{\theta}_i}, \tilde{\rho}_0 \rangle \neq 0 \); note that the orbit of \( \rho_0 \) under the action of \( \Theta(\tilde{\rho}_0) \) is just \( \{ \tilde{\rho}_0 \} \). Thus, \( \rho \in \mathcal{E}(\rho_0) \) as claimed. In particular, \( \rho, \rho_0 \) belong to the same connected component of \( \mathcal{G}(G^F) \).

\[ \square \]

Remark 6.14. As promised in 2.10, we can now finally clarify the relations between the various partitions of \( \text{Irr}(G^F) \) given in terms of

- the connected components of \( \mathcal{G}(G^F) \),
- “rational series” and
- “geometric conjugacy classes”.

If \( Z(G) \) is connected, then Theorem 5.2(c) shows that the implications in 2.10 are, in fact, equivalences; so the three partitions agree in this case.

Now assume that \( Z(G) \) is not connected. Then Examples 2.11 and 5.8 already show that “geometric conjugacy classes” may be strictly larger than the connected components of \( \mathcal{G}(G^F) \). By Proposition 6.13, the latter are given by the sets \( \mathcal{E}(G^F, \tilde{\rho}_0) \). In turn, the sets \( \mathcal{E}(G^F, \tilde{\rho}_0) \) are known to be equal to the “rational series” of characters of \( G^F \). (This follows from Theorem 5.2(c) and [5 Prop. 11.7], [38, 7.5].) Thus, in general, “rational series” are just given by the connected components of \( \mathcal{G}(G^F) \); in other words, the first implication in 2.10 is always an equivalence. (This also follows from Digne–Michel [15 Theorem 14.51].)

7. Character sheaves

Finally, we turn to the problem of computing the values of the irreducible characters of \( G^F \). As in [38], it will be convenient to express this in terms of finding the base changes between various vector space bases of \( \text{CF}(G^F) \).

7.1. The first basis to consider will be denoted by \( B_0 \); it consists of the characteristic functions \( f_C : G^F \rightarrow \mathbb{C} \) of the various conjugacy classes \( C \) of \( G^F \). (Here, \( f_C \) takes the value 1 on \( C \) and the value 0 on the complement of \( C \).) This basis is well-understood; see, e.g., the chapters on conjugacy classes in Carter’s book [9] (and the references there). As a model example, see Mizuno’s [63] computation of all the conjugacy classes of \( G^F = E_6(\mathbb{F}_q) \). The second basis is, of course, \( A_0 = \text{Irr}(G^F) \), the set of irreducible characters of \( G^F \). Thus, the character table of \( G^F \) is the matrix which expresses the base change between \( A_0 \) and \( B_0 \). Note that the results discussed in the previous sections provide a parametrization of \( A_0 = \text{Irr}(G^F) \) in a way which is almost totally unrelated with the basis \( B_0 \).
7.2. The third basis to consider will be denoted by $A_1$; it consists of Lusztig’s “almost characters”. These are defined as certain explicit linear combinations of the irreducible characters of $G^F$. (The definition first appeared in [41, 4.25], assuming that $Z(G)$ is connected; see [57] for the general case.) Almost characters are only well-defined up to multiplication by a root of unity, but we can choose a set $A_1$ of almost characters which form an orthonormal basis of $CF(G^F)$. The matrix which expresses the base change between $A_0 = \text{Irr}(G^F)$ and $A_1$ is explicitly known and given in terms of Lusztig’s “non-abelian Fourier matrices”.

7.3. Already in [41, 13.7], Lusztig conjectured that there should be a fourth basis, providing a geometric interpretation of the almost characters. The theory of character sheaves [43] gives a positive answer to this conjecture. Character sheaves are certain simple perverse sheaves in the bounded derived category $\mathcal{D}G$ of constructible $\mathbb{Q}_\ell$-sheaves (in the sense of Beilinson, Bernstein, Deligne [3]) on the algebraic group $G$, which are equivariant for the action of $G$ on itself by conjugation. If $A$ is such a character sheaf, we consider its inverse image $F^*A$ under the Frobenius map. If $F^*A \cong A$ in $\mathcal{D}G$, we choose an isomorphism $\phi: F^*A \cong A$ and then define a class function $\chi_A \in CF(G^F)$, called “characteristic function”, by $\chi_A(g) = \sum_i (-1)^i \text{Trace}(\phi, H^i_g(A))$ for $g \in G^F$, where $H^i_g(A)$ are the stalks at $g$ of the cohomology sheaves of $A$ (see [43, 8.4]). Such a function is only well-defined up to multiplication with a non-zero scalar. Let $\hat{G}^F$ be the set of character sheaves (up to isomorphism) which are isomorphic to their inverse image under $F$, and set

$$B_1 := \{ \chi_A \mid A \in \hat{G}^F \} \subseteq CF(G^F),$$

where, for each $A \in \hat{G}^F$, the characteristic function $\chi_A$ is defined with respect to a fixed choice of $\phi: F^*A \cong A$. In [43] §17.8, Lusztig states a number of properties of character sheaves. These include a “multiplicity formula” [43, 17.8.3] (rather analogous to the Main Theorem 4.23 of [41]), a “cleanness condition” and a “parity condition” [43, 17.8.4], and a characterisation of arbitrary irreducible “cuspidal perverse sheaves” on $G$ [43, 17.8.5]. In [43, Theorem 23.1], these properties were proved under a mild condition on $p$; the conditions on $p$ were later completely removed by Lusztig [53]. (As a side remark we mention that a portion of the proof in [53] relies on computer calculations, as discussed in [24, 5.12].) As pointed out in [53, 3.10], this allows us to state without any assumption on $G$, $p$ or $q$:

**Theorem 7.4** (Lusztig [43, 53]). For $A \in \hat{G}^F$, an isomorphism $\phi: F^*A \cong A$ can be chosen such that the values of $\chi_A$ belong to a cyclotomic field and $\langle \chi_A, \chi_A \rangle = 1$. The corresponding set $B_1 = \{ \chi_A \mid A \in \hat{G}^F \}$ is an orthonormal basis of $CF(G^F)$.

7.5. Consider the bases $A_1$ (almost characters) and $B_1$ (character sheaves) of $CF(G^F)$. Lusztig [41, 13.7] (see also p. 226 in part II and p. 103 in part V of [43]) conjectured that the matrix which expresses the base change between $A_1$ and $B_1$ should be diagonal. For “cuspidal” objects in $B_1$ this is proved in [43, Theorem 0.8], assuming that $p, q$ are sufficiently large. If $Z(G)$ is connected, then this conjecture was proved by Shoji [66], [67], under some mild conditions on $p$. These conditions can now be removed since the properties in [43, §17.8] mentioned in 7.3 are known to hold in complete generality. Thus, we can state:

**Theorem 7.6** (Shoji [66, 67]). Assume that $Z(G)$ is connected. Then the matrix which expresses the base change between the bases $A_1$ (see 7.2) and $B_1$ (see 7.3)
is diagonal. Thus, there is a bijection $A_1 \leftrightarrow B_1$ which is also explicitly determined in terms of the parametrizations of $A_1$ (see [41, 4.23]) and $B_1$ (see [43, 23.1]).

Remark 7.7. There is a notion of “cuspidal” perverse sheaves in $\mathcal{D}G$; see [43, 3.10, 7.1]. If $A \in \hat{G}^F$ is cuspidal, then the “cleanness condition” [43, 17.8.4] implies that $\chi_A$ has the following property. The support $\{g \in G^F \mid \chi_A(g) \neq 0\}$ is contained in $\Sigma^F$ where $\Sigma \subseteq G$ is an $F$-stable subset which is the inverse image of a single conjugacy class in $G/Z(G)^o$ under the natural map $G \to G/Z(G)^o$. In particular, if $G$ is semisimple, then $\Sigma$ is just a conjugacy class in $G$. Explicit information about the sets $\Sigma$ can be extracted, for example, from the proofs in [43, §19–§21].

Note that, in general, a class function on a finite group which has non-zero values on very few conjugacy classes only, is typically a linear combination of many irreducible characters of the group. In contrast, it is actually quite impressive to see how the characteristic functions of $F$-stable cuspidal character sheaves are expressed as linear combinations of very few irreducible characters of $G^F$.

Example 7.8. (a) Let $G^F = \text{SL}_2(\mathbb{F}_q)$ where $q$ is odd, as in Example 2.11. By [43, §18], there are two cuspidal character sheaves in $\hat{G}^F$. Their characteristic functions are equal (up to multiplication by a scalar of absolute value 1) to the two class functions in Remark 2.12. The corresponding sets $\Sigma$ are the $G$-conjugacy classes of the elements $J$ and $-J$, respectively.

(b) Let $G^F = \text{Sp}_4(\mathbb{F}_q)$ where $q$ is odd, as in Example 6.11. In Srinivasan’s work [72], apart from constructing characters by induction from various subgroups, an extra function $\Gamma_1$ was constructed in [72, (7.3)] by ad hoc methods in order to complete the character table. This function $\Gamma_1$ takes the values $q, -q, \sqrt{q}, q$ on the classes denoted $D_{31}, D_{32}, D_{33}, D_{34}$, respectively, and vanishes on all other classes of $G^F$. We have

$$\Gamma_1 = \frac{1}{2}(\theta_9 + \theta_{10} - \theta_{11} - \theta_{12}) \quad \text{where} \quad \theta_9, \theta_{10}, \theta_{11}, \theta_{12} \in \mathcal{U}(G^F).$$

On the other hand, by (a') in the proof of [43, 19.3], there is a unique cuspidal character sheaf $A_0 \in \hat{G}^F$. In [73, p. 192], it is pointed out that $\Gamma_1$ is a characteristic function associated with $A_0$.

(c) Let $G$ be simple of adjoint type $D_4$ and $F$ be such that $G^F = \mathfrak{g}D_4(\mathbb{F}_q)$, where $q$ is odd. By part (d) in the proof of [43, 19.3], there are four cuspidal character sheaves, but only one of them is isomorphic to its inverse image under $F$ (this is seen by an argument similar to that in the proof of [43, 20.4]); let us denote the latter one by $A_0$. The supporting set $\Sigma$ is the conjugacy class of an element $g = su = us \in G^F$ where $s$ has order 2 and $u$ is regular unipotent in $C_G(s)^o$. The set $\Sigma^F$ splits into two classes in $G^F$, with representatives $su', su''$ as described in [71, 0.8]. The characteristic function $\chi_{A_0}$ can be normalized such that it takes values $q^2$ and $-q^2$ on $su'$ and $su''$, respectively (and vanishes on all other classes of $G^F$). Using Spaltenstein’s table [71] of the unipotent characters of $G^F$, we see that (cf. [71, p. 681]):

$$\chi_{A_0} = \frac{1}{2}([\rho_1] - [\rho_2] + 3D_4[1] - 3D_4[-1]).$$

(d) Further examples for $G$ of exceptional type are given by Kawanaka [32, §4.2]. These examples re-appear in the more general framework of Lusztig [17, §7].

7.9. Finally, there is an inductive description of $\text{CF}(G^F)$ which highlights the relevance of the characteristic functions of cuspidal characters sheaves. First, some definitions. As in [38, 7.2], a closed subgroup $L \subseteq G$ is called a regular subgroup
if $L$ is $F$-stable and there exists a parabolic subgroup $P \subseteq G$ (not necessarily $F$-stable) such that $L$ is a Levi subgroup of $P$, that is, $P = U_P \times L$ where $U_P$ is the unipotent radical of $P$. By generalizing the construction of the virtual characters $R_{T, \theta}$, Lusztig [37] (see also [46, 1.7]) defines a “twisted” induction

$$R^G_{L \subseteq P} : \text{CF}(L^F) \to \text{CF}(G^F),$$

which sends virtual characters of $L^F$ to virtual characters of $G^F$. (A model of $R^G_{L \subseteq P}$ analogous to the model $R^2_\theta$ in Section 2 is described in [43, 6.21].)

On the other hand, let $A_0 \in \hat{L}^F$ be cuspidal and $\phi : F^*A_0 \to A_0$ be an isomorphism. To $(L, A_0, \phi)$ one can associate a pair $(K, \tau)$ where $K$ is an object in $\mathcal{O}_G$ (obtained by a geometric induction process from $A_0$) and $\tau : F^*K \to K$ is an isomorphism; see [43, 8.1], [46, 1.8]. We have corresponding characteristic functions $\chi_{A_0} \in \text{CF}(L^F)$ and $\chi_K \in \text{CF}(G^F)$.

**Theorem 7.10** (Lusztig [46, 8.13, 9.2] + Shoji [68, §4]). Assume that $Z(G)$ is connected. Then, with the above notation, the map $R^G_{L \subseteq P} : \text{CF}(L^F) \to \text{CF}(G^F)$ does not depend on $P$ and, hence, can be denoted by $R^G_L$. If $(K, \tau)$ is associated with $(L, A_0, \phi)$ as above (where $A_0$ is cuspidal), then $\chi_K = (-1)^{\dim \Sigma} R^G_L(\chi_{A_0})$.

In [46, 8.13, 9.2], these statements are proved under a mild condition on $p$ and for $q$ a sufficiently large power of $p$. Again, the condition on $p$ can now be removed since the properties in [43, §17.8] mentioned in [73] (especially, the “cleanness condition”) are known to hold in complete generality [43]; this even works when $Z(G)$ is not connected. Shoji [68, §4] showed that one can also remove the assumption on $q$, when $Z(G)$ is connected. Then we also have:

**Corollary 7.11.** Assume that $Z(G)$ is connected. Then

$$\text{CF}(G^F) = \left\langle R^G_L(\chi_{A_0}) \mid L \subseteq G \text{ regular and } A_0 \in \hat{L}^F \text{ cuspidal} \right\rangle_C.$$

**Proof.** Since the properties in [43, §17.8] hold in complete generality, the set $\hat{G}$ coincides with the set of “admissible complexes” on $G$ in [43, 7.1.10]. Then the above statement is contained in [43, §10.4]; see also the discussion in [69, 4.2]. □

**Remark 7.12.** In this picture, the general strategy for computing the character values of $\rho \in \text{Irr}(G^F)$ is as follows (see [69, §4] for further details).

If $A \in \hat{G}$, then $\chi_A$ can be written as an explicit linear combination of induced class functions $R^G_L(\chi_{A_0})$ as in Corollary 7.11 (see [43, §10.4]). Now the values of $R^G_L(\chi_{A_0})$ can be determined by the character formula [43, 8.5], which involves certain “generalized Green functions”. The latter are computable by an algorithm which is described in [43, §24]; see also Shoji [65], [69, §4.3]. Thus, the values of the basis elements in $B_1$ can, at least in principle, be computed. Then one uses the transition from $B_1$ to $A_1$ in Theorem 7.6 and, finally, the transition from $A_1$ to $A_0 = \text{Irr}(G^F)$ in 7.2. The remaining issue in this program is the determination of the scalars in the diagonal base change in Theorem 7.6 This problem appears to be very hard; it is not yet completely solved. But for many applications, one can already draw strong conclusions about character values without knowing these scalars precisely (an example will be given below).

The first successful realization of this whole program was carried out by Lusztig [44], where character values on unipotent elements are determined. See also Bonnafé [5], Shoji [68], [69], [70], Waldspurger [77] and the further references there.
To close this section, we briefly mention an application of the above results to a concrete problem on the modular representation theory of $G^F$ in the non-defining characteristic case (see [60] for general background on modular representations).

7.13. Let $\ell \neq p$ be a prime and $\text{CF}_\ell(G^F)$ be the space of all class functions $G^F \to \mathbb{C}$, where $G^F$ denotes the set of elements $g \in G^F$ whose order is not divisible by $\ell$. For any $f \in \text{CF}(G^F)$, we denote by $\bar{f}$ the restriction of $f$ to $G^F$. Let $\text{IBr}_\ell(G^F) \subseteq \text{CF}_\ell(G^F)$ be the set of irreducible Brauer characters; these form a basis of $\text{CF}_\ell(G^F)$. (The set $\text{IBr}_\ell(G^F)$ is in bijection with the isomorphism classes of irreducible representations of $G^F$ over an algebraically closed field of characteristic $\ell$.) For $\rho \in \text{Irr}(G^F)$, we have

$$\hat{\rho} = \sum_{\beta \in \text{IBr}_\ell(G^F)} d_{\rho\beta} \beta \quad \text{where} \quad d_{\rho\beta} \in \mathbb{Z}_{\geq 0}.$$ 

The matrix $(d_{\rho\beta})_{\rho,\beta}$ is Brauer’s “$\ell$-modular decomposition matrix” of $G^F$. We say that $\beta \in \text{IBr}_\ell(G^F)$ is unipotent if $d_{\rho\beta} \neq 0$ for some $\rho \in \mathfrak{U}(G^F)$. (See also the general discussion in [26, §3] for further comments and references.)

| Type | $\ell = 2$ | $\ell = 3$ | $\ell = 5$ | $\ell$ good |
|------|------------|------------|------------|-------------|
| $G_2$ | 9          | 8          | 10         |             |
| $F_4$ | 28         | 35         | 37         |             |
| $E_6, ^2E_6$ | 27       | 28         | 30         |             |
| $E_7$ | 64         | 72         | 76         |             |
| $E_8$ | 131        | 150        | 162        | 166         |

(No entry means: same number as for $\ell$ good)

Assume now that $Z(G)$ is connected. In [26, §6], we found the number of unipotent $\beta \in \text{IBr}_\ell(G^F)$ for $G$ of exceptional type and “bad” primes $\ell$, under some mild restrictions on $p$. (The table in [26] contained an error which was corrected in [16, §4.1].) For “good” $\ell$, those numbers were already known by [25] to be equal to $|\mathfrak{U}(G^F)|$. For $G$ of classical type, see [26, 6.6] and the references there.

**Proposition 7.14** (Cf. [26, §6]). Assume that $Z(G)$ is connected and $G/Z(G)$ is simple of type $G_2$, $F_4$, $E_6$, $E_7$ or $E_8$. If $\ell \neq p$, then the number of unipotent $\beta \in \text{IBr}_\ell(G^F)$ is given by Table 7. Furthermore, we have the equality

$$\langle \beta \mid \beta \in \text{IBr}_\ell(G^F) \ \text{unipotent} \rangle_{\mathbb{C}} = \langle \hat{\rho} \mid \rho \in \mathfrak{U}(G^F) \rangle_{\mathbb{C}} \subseteq \text{CF}_\ell(G^F).$$

**Proof.** If we knew the character tables of the groups in question, then this would be a matter of a purely mechanical computation. Since those tables are not known, the argument in [26, §6] uses results on character sheaves but it requires three assumptions A, B, C, as formulated in [26, 5.2]. At the time of writing [26], these assumptions were known to hold under some mild conditions on $p$. These conditions can now be completely removed thanks to the facts that the “cleanness condition” holds unconditionally [53] and that Theorem 7.4, 7.6, 7.10 are valid as stated above. Otherwise, the argument remains the same as in [26, §6]. □
8. Appendix: On uniform functions

The main purpose of this appendix is to provide a proof of Theorem 2.14 (Lusztig’s conjecture on uniform functions). This may also serve as another illustration of the methods that are available in order to deal with a concrete problem. Recall from Section 2 the definition of \(R_w^\theta\). It will now be convenient to work with the model \(R_{T,\theta}\) defined in [3, §7.2], [39, 2.2] (cf. Remark 2.3). We will now further write \(R_{T,\theta}\) as \(R_T^G(\theta)\). Thus, for fixed \(T \subseteq G\), we have a map \(\theta \mapsto R_T^G(\theta)\). Extending this linearly to all class functions, we obtain a linear map

\[ R_T^G : CF(T^F) \rightarrow CF(G^F). \]

Let \(G_{uni}\) be the set of unipotent elements of \(G\). Then, as in Remark 2.9 we obtain the Green function \(Q_T^G : G_{uni}^F \rightarrow \mathbb{Z}\), \(u \mapsto R_T^G(\theta)(u)\).

By adjunction, there is a unique linear map \(^*R_T^G : CF(G^F) \rightarrow CF(T^F)\) such that

\[ \langle R_T^G(f'), f \rangle = \langle f', ^*R_T^G(f) \rangle \quad \text{for all } f' \in CF(T^F) \text{ and } f \in CF(G^F). \]

Then we have the following elegant characterisation of uniform functions.

**Proposition 8.1** (Digne–Michel [14, 12.12]). A class function \(f \in CF(G^F)\) is uniform if and only if

\[ f = \frac{1}{|G^F|} \sum_{T \in \mathcal{T}(G)} |T^F|(R_T^G \circ ^*R_T^G)(f), \]

where \(\mathcal{T}(G)\) denotes the set of all \(F\)-stable maximal tori \(T \subseteq G\).

**8.2.** Let us fix a semisimple element \(s_0 \in G^F\) and assume that \(H := C_G(s_0)\) is connected. Then \(H\) is \(F\)-stable, closed, connected and reductive (see [3, 3.5.4]). Thus, \(H\) itself is a connected reductive algebraic group and \(F: H \rightarrow H\) is a Frobenius map. Let \(T \subseteq G\) be an \(F\)-stable maximal torus such that \(s_0 \in T\). Then \(T \subseteq H = C_G(s_0)\) and so we can form the Green function \(Q_T^H : H_{uni}^F \rightarrow \mathbb{Z}\), \(u \mapsto R_T^H(\theta)(u)\) (where \(\theta\) is any irreducible character of \(T^F\)).

**Lemma 8.3** (Cf. Lusztig [43, 25.5]). Let \(g \in G^F\) and write \(g = su = us\) where \(s \in G^F\) is semisimple, \(u \in G^F\) is unipotent. Then, in the setting of 8.2, we have

\[ \frac{1}{|T^F|} \sum_{\theta \in \text{Irr}(T^F)} \theta(s_0)^{-1} R_T^G(\theta)(g) = \begin{cases} Q_T^H(u) & \text{if } s = s_0, \\ 0 & \text{if } s \text{ is not } G^F\text{-conjugate to } s_0. \end{cases} \]

**Proof.** Consider the character formula for \(R_T^G(\theta)\) in [3, 7.2.8]; we have

\[ R_T^G(\theta)(g) = \frac{1}{|H_s^F|} \sum_{x} \theta(x^{-1}sx) Q_{\mathbb{Z}xT^{-1}}^H(u) \]

where \(H_s := C_G^0(s)\) is connected reductive (see again [3, 3.5.6]) and the sum runs over all \(x \in G^F\) such that \(x^{-1}sx \in T\) (and hence, \(xTx^{-1} \subseteq H_s\)). This yields that

\[ \sum_{\theta \in \text{Irr}(T^F)} \theta(s_0)^{-1} R_T^G(\theta)(g) = \frac{1}{|H_s^F|} \sum_{x} \left( \sum_{\theta \in \text{Irr}(T^F)} \theta(s_0^{-1}x^{-1}sx) \right) Q_{\mathbb{Z}xT^{-1}}^H(u). \]

Now the sum \(\sum_{\theta \in \text{Irr}(T^F)} \theta\) is the character of the regular representation of \(T^F\). Hence, for \(x\) as above, we have

\[ \sum_{\theta \in \text{Irr}(T^F)} \theta(s_0^{-1}x^{-1}sx) = \begin{cases} |T^F| & \text{if } s_0 = x^{-1}sx, \\ 0 & \text{otherwise}. \end{cases} \]
So, if $s$ is not $G^F$-conjugate to $s_0$, then
\[ \sum_{\theta \in \text{Irr}(T^F)} \theta(s_0)^{-1} R_T^G(\theta)(g) = 0, \]
as desired. On the other hand, if $s = s_0$, then $H = H_s$ and we obtain
\[ \sum_{\theta \in \text{Irr}(T^F)} \theta(s_0)^{-1} R_T^G(\theta)(g) = \frac{1}{|T^F|} \sum_{x \in H^F} |T^F| Q_{xT^F}^H(u). \]
Since $Q_{xT^F}^H = Q_T^H$ for all $x \in H^F$, this yields the desired formula. \hfill \square

**8.4.** Let $s_0 \in G^F$ and $H = C_G(s_0)$ be as in 8.2. For any $f \in \text{CF}(H^F)$ such that \( \{ h \in H^F \mid f(h) \neq 0 \} \subseteq H^F_{\text{uni}} \), we can uniquely define a class function $\hat{f} \in \text{CF}(G^F)$ by the requirement that
\[
\begin{cases}
\hat{f}(g) = f(u) & \text{if } s = s_0, \\
0 & \text{if } s \text{ is not } G^F\text{-conjugate to } s_0,
\end{cases}
\]
where $g \in G^F$ and $g = su = us$ with $s \in G^F$ semisimple, $u \in G^F$ unipotent (see Lusztig [43, p. 151]). Thus, Lemma 8.3 means that
\[
\hat{Q}_T^H = \frac{1}{|T^F|} \sum_{\theta \in \text{Irr}(T^F)} \theta(s_0)^{-1} R_T^G(\theta),
\]
for any $F$-stable maximal torus $T \subseteq H$. Hence, we deduce that
\[
f \text{ uniform } \Rightarrow \hat{f} \text{ uniform.}
\]
Indeed, if $f$ is uniform, then $f$ can be written as a linear combination of $R_T^H(\theta)$ for various $T, \theta$. Since the restriction of $R_T^H(\theta)$ to $H^F_{\text{uni}}$ is the Green function $Q_T^H$, we can write $f$ as a linear combination of Green functions $Q_T^H$ for various $T$. Clearly, the map $f \mapsto \hat{f}$ is linear. Hence, (b) implies that $\hat{f}$ is uniform; so (c) holds.

**Corollary 8.5.** Assume that the derived subgroup $G_{\text{der}}$ of $G$ is simply-connected. Then Theorem 2.14 holds.

**Proof.** The assumption implies that the centralizer of any semisimple element is connected (see, e.g., [9, 3.5.6]). Hence, we can apply the above discussion. Let $\mathcal{C}$ be an $F$-stable conjugacy class of $G$ and $f_\mathcal{C}^G \in \text{CF}(G^F)$ be the characteristic function of $\mathcal{C}^F$. Let $g_0 \in \mathcal{C}^F$ and write $g_0 = s_0 u_0 = u_0 s_0$ where $s_0 \in G^F$ is semisimple and $u_0 \in G^F$ is unipotent. Let $H := C_G(s_0)$ and set
\[ \mathcal{C}' := \{ u \in H_{\text{uni}} \mid s_0 u \in \mathcal{C} \}. \]
Then $\mathcal{C}'$ is an $F$-stable unipotent class of $H$ and we denote by $f_{\mathcal{C}'}^H \in \text{CF}(H^F)$ the characteristic function of $\mathcal{C}'^F$. Let $f = f_{\mathcal{C}}^H$ and consider $\hat{f}$ as defined in 8.4(a). One immediately checks that $\hat{f} = f_{\mathcal{C}'}^G$. Now, by [21 Prop. 1.3], $f = f_{\mathcal{C}'}^H$ is uniform. Hence, 8.3(c) shows that $\hat{f} = f_{\mathcal{C}'}^G$ is uniform. \hfill \square

The final step in the proof of Theorem 2.14 is to show that we can reduce it to the case where $G_{\text{der}}$ is simply-connected. So let us now drop the assumption that $G_{\text{der}}$ is simply-connected. Then we can find a surjective homomorphism of algebraic groups $\iota: G' \rightarrow G$ where
\begin{itemize}
\item $G'$ is connected reductive and $G'_{\text{der}}$ is simply-connected,
\item the kernel $\ker(\iota)$ is contained in $Z(G')$ and is connected,
\end{itemize}
• there is a Frobenius map $F': G' \to G'$ such that $\iota \circ F' = F \circ \iota$.

(See [13, p. 152] and [30, 1.7.13].) Since $\ker(\iota)$ is connected, a standard application of Lang’s Theorem shows that $\iota$ restricts to a surjective map $G^F' \to G^F$ which we denote again by $\iota$. In this setting, we have:

**Proposition 8.6** (Digne–Michel). Let $\iota: G' \to G$ be as above. Let $T \in \mathcal{T}(G)$. Then $T' = \iota^{-1}(T) \in \mathcal{T}(G')$ and we have:

(a) $R^G_T(f') \circ \iota = R^G_{T'}(f' \circ \iota)$ for any $f' \in CF(T')$.

(b) $^*R^G_T(f) \circ \iota = ^*R^G_{T'}(f \circ \iota)$ for any $f \in CF(G^F)$.

**Proof.** (a) This is a special case of a much more general result; see [15, 13.22] and also Bonnafé [4, §2]. (I thank Jean Michel for pointing out the latter reference.)

(b) We have the following character formula (see [15, 12.2]):

$$^*R^G_T(f)(t) = |T^F||H^F_t|^{-1} \sum_{u \in H^F_{t,\text{uni}}} Q^H_{T^F}(u)f(tu) \quad \text{for any } f \in CF(G^F)$$

where $t \in T^F$ and $H_t := C^G_{G^F}(t)$. Now, applying (a) with the trivial character of $T^F$, we obtain $Q^G_{T} \circ \iota = Q^G_{T'}$. Let $t' \in T'^F$ and $t = \iota(t') \in T^F$. Let $H_{t'} := C^G_{G^F}(t')$. Then $\iota$ induces a bijection $H^F_{t',\text{uni}} \sim \to H^F_{t,\text{uni}}$. Since also $|T^F||H^F_t|^{-1} = |T'^F||H^F_{t'}|^{-1}$, we obtain (b) using the character formulae for $^*R^G_T$ and for $^*R^G_{T'}$. (See also [4, §2].) \( \square \)

Now we can complete the proof of Theorem 2.14 as follows. Let $\mathcal{C}$ be an $F$-stable conjugacy class of $G$. Then $\Sigma := \iota^{-1}({\mathcal{C}})$ is a union of $F$-stable conjugacy classes of $G'$. Let $f^G_{\Sigma} \in CF(G^{F'})$ be the characteristic function of $\Sigma^F$. Then $f^G_{\Sigma} \circ \iota = f^G_{\Sigma'}$ is uniform by Corollary 8.5. So, using Propositions 8.1 and 8.6 we obtain:

$$f^G_{\iota} = \frac{1}{|G^F|} \sum_{T' \in \mathcal{T}(G')} |T'^{F'}|(R^G_{T'} \circ \iota)(f^G_{\Sigma'} \circ \iota)$$

$$= \frac{1}{|G^F|} \sum_{T \in \mathcal{T}(G)} |T^F|(R^G_T \circ ^*R^G_T)(f^G_{\iota}) \quad \text{for any } f^G_{\iota} \in CF(G^F)$$

where we used that the map $\mathcal{T}(G) \to \mathcal{T}(G')$, $T \mapsto T' := \iota^{-1}(T)$, is a bijection, such that $|G^{F'}|/|T^{F'}| = |G^F|/|T^F|$ for all $T \in \mathcal{T}(G)$. Hence, we conclude that

$$f^G_{\iota} = \frac{1}{|G^F|} \sum_{T \in \mathcal{T}(G)} |T^F|(R^G_T \circ ^*R^G_T)(f^G_{\iota})$$

and so $f^G_{\iota}$ is uniform, by Proposition 8.1. Thus, Theorem 2.14 is proved. \( \square \)

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