Convergence of fundamental solutions of linear parabolic equations under Cheeger-Gromov convergence

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May 03, 2010

1 Introduction

Heat kernel is an important tool and has many applications in differential geometry (see, for example, [LY]) and physics. Heat kernel and related ideas play an important role in Perelman’s work on Ricci flow, one particular example is the so-called Perelman’s Li-Yau type differential Harnack inequality ([Pe02I, Corollary 9.3], see also [Ni] and [CTY] for more details of the proof).

In the proof of the pseudo locality theorem (§10.1, in particular, p.25 in [Pe02I]) Perelman used the following statement about the convergence of heat kernels. Let \( (M^n_k, g_k(\tau), x_{0k}) \), \( \tau \in [0, T] \), be a pointed sequence of \( n \)-dimensional complete solutions of the backward Ricci flow. Suppose that each solution has bounded curvature and suppose that the sequence converges to \( (M^n, g_\infty(\tau), x_{0\infty}) \) in the pointed Cheeger-Gromov sense, then the heat kernels \( H_k \) for the adjoint heat equations associated with \( (M^n_k, g_k(\tau), x_{0k}) \) sub-converge to a positive limit \( H_\infty \) defined on \( M_\infty \times (0, T] \). Motivated by this, the proof of the statement is studied by several groups of authors, Chau, Tam and Yu ([CTY]), Hsu ([Hs]), Kleiner and Lott ([KL]), S. Zhang ([ZhS]), and Chow (§22.2 in [CCIII]).

In this note by combining the techniques from the literature mentioned above we give a proof of the convergence of fundamental solutions for general linear parabolic equations (see Theorem 2.1 below). More precisely our proof of the existence of the limit is similar to the one given by Chow in [CCIII]. Our proof of the \( \delta \)-function property of the limit is very close to the proofs given in [CTY] and [Hs]. Another feature worth mention is that we do not assume that the sequence has uniformly bounded curvature (compare to [ZhS, Theorem 0.1]). This is made possible by avoiding the use of decay estimates of heat kernels in our proof.

The above mentioned proof of Theorem 2.1 will be given in §2. In §3 we will discuss when the assumption, about the upper bound of \( L^1 \)-norms made in Theorem 2.1 could be satisfied. In §4 we consider the uniqueness of the limit given by Theorem 2.1. In §5 we construct a cut-off function following Perelman ([Pe02I §8.3]) and use it to prove a local integral estimate of fundamental solutions (see Corollary 5.4), essentially this is the only novel part in this note.

2 The convergence of fundamental solutions

2.1 The fundamental solutions. Let \( M^n \) be a smooth connected manifold and let \( \Omega \subset M \) be a smooth connected domain with (possibly empty) smooth boundary \( \partial \Omega \). Fix a \( T > 0 \),
Let \( g(\tau), \tau \in [0, T] \), be a smooth family of smooth metrics on \( \Omega \). Define the symmetric 2-tensor \( R_{ij} \) and the scalar function \( R \) by

\[
R_{ij} = \frac{1}{2} \frac{\partial g_{ij}}{\partial \tau}, \\
R(x, \tau) \doteq g^{ij}(x, \tau) R_{ij}(x, \tau).
\]  

(1)

(2)

Let \( Q : \Omega \times [0, T] \to \mathbb{R} \) be a smooth function. Let \( X(\tau), \tau \in [0, T] \), be a smooth family of smooth vector fields on \( \Omega \). We will consider the following linear parabolic equation on \( \Omega \times [0, T] \) with respect to the evolving metrics,

\[
\square^* u \doteq \frac{\partial u}{\partial \tau} - \Delta_{g(\tau)} u + \nabla_X u + Qu = 0
\]

(3)

where \( \nabla_X u \doteq X(\tau) u \).

Let \( \text{Int}(\Omega) \) denote the set of interior points of \( \Omega \). Fix a base point \( x_0 \in \text{Int}(\Omega) \). Let \( H(x, \tau) \) be a function belong to \( C^0(\Omega \times (0, T], \mathbb{R}) \cap C^\infty(\text{Int}(\Omega) \times (0, T], \mathbb{R}) \).

\( H \) is called a fundamental solution of (3) centered at \( x_0 \) if it satisfies the following. \( H \) is positive on \( \text{Int}(\Omega) \times (0, T] \) and

\[
\square^* H = 0,
\]

(4)

\[
\lim_{\tau \to 0^+} H(\cdot, \tau) = \delta_{x_0}.
\]

\( H \) is called the heat kernel of (3) centered at \( x_0 \) if \( H \) is the minimal fundamental solution of (3) centered at \( x_0 \).

**Remark.** Fundamental solutions as defined above are not necessarily unique. In particular when \( \Omega \) is a compact manifold with nonempty boundary. Let \( h \) be a solution of \( \square^* h = 0 \) and which satisfies boundary conditions \( h|_{\Omega \times (0]} = 0 \) and \( h|_{\partial \Omega \times [0, T]} \geq 0 \). If \( H \) is a fundamental solution, then \( H + h \) is also a fundamental solution.

**Remark.** In this note we do not address the issue of the existence of \( H_k \). It is well-known that convergence theorems, such as Theorem 2.1, can be used to prove the existence of heat kernels on complete noncompact manifold assuming the existence of Dirichlet heat kernels on compact manifolds with boundary.

**2.2 The main theorem.** Let \( \{M_k\}_{k=1}^\infty \) be a sequence of smooth manifolds, let \( \Omega_k \subset M_k \) be a smooth connected domain with (possibly empty) smooth boundary \( \partial \Omega_k \) for each \( k \), and let \( g_k(\tau), \tau \in [0, T] \), be a smooth family of smooth metrics on \( \Omega_k \) for each \( k \). Let \( x_{0k} \in \text{Int}(\Omega_k) \). We assume that the sequence \( \{(\Omega_k, g_k(\tau), (x_{0k}, 0))\} \) converges, in the \( C^\infty \) pointed Cheeger-Gromov sense, to a smooth limit \( (M^\infty, g_\infty(\cdot), (x_{0\infty}, 0)), \tau \in [0, T] \). Here \( (M^\infty, g_\infty(\cdot)) \) may have boundary and may not be complete. By definition the convergence means that there exist an exhaustion \( \{U_k\}_{k=1}^\infty \) of \( M_\infty \) by pre-compact open sets and a sequence of diffeomorphisms \( \Phi_k : U_k \to V_k \doteq \Phi_k(U_k) \subset \Omega_k \) such that

\( \text{CG1} \) \( x_{0\infty} \in U_k \) and \( \Phi_k(x_{0\infty}) = x_{0k} \) for each \( k \), and

\( \text{CG2} \) \( \left(U_k^0, \Phi^k_{x_{0\infty}}[g_k(\tau)]_{V_k^0}\right) \) converges uniformly in \( C^\infty \)-topology to \( g_\infty(\cdot) \) on any compact subset of \( M^\infty \times [0, T] \).

From the definition it is easy to see that \( x_{0\infty} \in \text{Int}(M^\infty) \).

Let \( Q_k : \Omega_k \times [0, T] \to \mathbb{R} \) be a smooth function for each \( k \). Let \( X_k(\tau), \tau \in [0, T] \), be a smooth family of smooth vector fields on \( \Omega_k \) for each \( k \). Suppose that under the convergence
of \{ (\Omega_k, g_k(\tau), (x_{0k}, 0)) \} the sequence \{ Q_k \} converges to a smooth limit \( Q_\infty : M_\infty \times [0, T] \rightarrow \mathbb{R} \), i.e., the sequence of functions \{ Q_k(\Phi_k(\cdot), \tau) \} converges uniformly in \( C^\infty \)-topology to \( Q_\infty(\cdot, \tau) \) on any compact subset of \( M_\infty \times [0, T] \). We also assume that the sequence of vector fields \{ X_k(\tau) \} converges to a smooth vector field limit \( X_\infty(\tau) \) on \( M_\infty \times [0, T] \), i.e., the sequence of push-forward vector fields \{ (\Phi_k^{-1})_* X_k(\tau) \} converges uniformly in \( C^\infty \)-topology to \( X_\infty(\tau) \) on any compact subset of \( M_\infty \times [0, T] \).

We will consider the following equation on \( \Omega_k \times [0, T] \).

\[
\square_k u \triangleq \frac{\partial u}{\partial \tau} - \Delta_{g_k(\tau)} u + \nabla_{X_k} u + Q_k u = 0.
\] (5)

Let \( H_k : \Omega_k \times (0, T] \rightarrow (0, \infty) \) be a fundamental solution of \( (\square_k u) \) centered at \( x_{0k} \), i.e., \( \square_k H_k = 0 \) and

\[
\lim_{\tau \to 0} H_k(\cdot, \tau) = \delta_{x_{0k}}.
\] (6)

We define function \( f_k : \Omega_k \times (0, T] \rightarrow \mathbb{R} \) by

\[
f_k(x, \tau) \triangleq (4\pi\tau)^{-n/2} e^{-f_k(x, \tau)}.
\] (7)

The following is the convergence property of fundamental solutions, under the pointed \( C^\infty \) Cheeger-Gromov convergence of the underlying manifolds, the convergence of potential functions \( Q_k \), and the convergence of vector fields \( X_k \).

**Theorem 2.1** Under the setup and notations above, suppose there is a constant \( C_* \) independent of \( k \) such that

\[
\int_{\Omega_k} H_k(x, \tau) \, d\mu_{g_k(\cdot)}(x) \leq C_*
\] (8)

for each \( k \) and \( \tau \in (0, T] \), then there is a subsequence (still indexed by \( k \)) such that the following holds. There are smooth nonnegative function \( H_\infty \) defined on \( \text{Int}(M_\infty) \times (0, T] \) and smooth function \( f_\infty \) defined on \( \text{Int}(M_\infty) \times (0, T] \) satisfying

\[
H_\infty(x, \tau) = (4\pi\tau)^{-n/2} e^{-f_\infty(x, \tau)},
\] (9)

such that

\[
\tilde{H}_k(\cdot, \cdot) \triangleq H_k(\Phi_k(\cdot), \cdot) \rightarrow H_\infty(\cdot, \cdot)
\] (10)

uniformly in \( C^\infty \)-topology on any compact subset of \( \text{Int}(M_\infty) \times (0, T] \), and

\[
\tilde{f}_k(\cdot, \cdot) \triangleq f_k(\Phi_k(\cdot), \cdot) \rightarrow f_\infty(\cdot, \cdot)
\] (11)

uniformly in \( C^\infty \)-topology on any compact subset of \( \text{Int}(M_\infty) \times (0, T] \). Moreover, \( H_\infty \) is a fundamental solution to the parabolic equation on \( \text{Int}(M_\infty) \times [0, T] \)

\[
\square^* H_\infty \triangleq \left( \frac{\partial}{\partial \tau} - \Delta_{g_\infty(\cdot)} + \nabla_{X_\infty} + Q_\infty \right) H_\infty = 0
\] (12)

\[
\lim_{\tau \to 0^+} H_\infty(\cdot, \tau) = \delta_{x_{0\infty}},
\] (13)

and

\[
\int_{M_\infty} H_\infty(x, \tau) \, d\mu_{g_\infty(\cdot)}(x) \leq C_*
\] (14)

for each \( \tau \in (0, T] \).
Remark. From the proof (see (32) and (10)), the assumption (8) can be replaced by the following condition and we still have the convergence to a fundamental solution. For any compact domain \( D \subset \text{Int}(M_\infty) \) there is a constant \( C_D \) independent of \( k \) such that when \( k \) is large enough
\[
\int_{\Phi_k(D)} H_k(x, \tau) \, d\mu_{g_k(\tau)}(x) \leq C_D
\]  
for any \( \tau \in (0, |) \). However under assumption (15) the limit \( H_\infty \) may not satisfy (14).

Remark. When \( M_\infty \) in Theorem 2.1 is a compact manifold with nonempty boundary, the theorem does not address the issue about whether the convergence can be extended to the boundary.

The remaining of this section is devoted to give a proof of Theorem 2.1.

2.3 The mean value inequality. The following parabolic mean value inequality will be used in the proof of Theorem 2.1.

Let \( \Omega \subset M^n \) be a smooth connected domain in a smooth manifold. Let \( g(\tau), \tau \in [0, T] \), be a smooth family of smooth metrics on \( \Omega \). We adopt the notations \( R_{ij}, R, X, \) and \( Q \) from the beginning of \( \S 2.1 \).

Let \( \tilde{g} \) be a smooth metric on \( \Omega \) satisfying \( Rc(\tilde{g}) \geq -K \) on \( \Omega \) for some \( K \geq 0 \). We assume that
\[
C_0^{-1} \tilde{g} \leq g(0) \leq C_0 \tilde{g}
\]
in \( \Omega \) for some constant \( C_0 \).

**Theorem 2.2** Under the setup and the assumption above. Let \( u : \Omega \times [0, T] \to \mathbb{R} \) be a nonnegative sub-solution to (3), i.e.,
\[
\frac{\partial u}{\partial \tau} - \Delta g(\tau)u + \nabla_X u + Q u \leq 0.
\]
If \( (x_0, \tau_0) \in \Omega \times (0, T] \) and \( r_0 > 0 \) are such that the closed ball \( B_{\tilde{g}}(x_0, 2r_0) \) is contained in \( \text{Int}(\Omega) \) and is compact. We assume \( \tau_0 \geq (2r_0)^2 \) so that the parabolic cylinder
\[
B_{\tilde{g}}(x_0, 2r_0) \times [\tau_0 - (2r_0)^2, \tau_0] \subset \Omega \times [0, T] .
\]
Then for any \( (x, \tau) \in B_{\tilde{g}}(x_0, r_0) \times [\tau_0 - r_0^2, \tau_0] \) we have
\[
u(x, \tau) \leq \frac{C_1 e^{C_2 \tau_0 + C_3 \sqrt{\tau_0} R_{ij}}}{r_0^3 \text{Vol}_{\tilde{g}} (B_{\tilde{g}}(x_0, r_0))} \int_{B_{\tilde{g}}(x_0, 2r_0) \times [\tau_0 - (2r_0)^2, \tau_0]} u(y, \sigma) \, d\mu_{\tilde{g}}(y) \, d\sigma,
\]
where constant \( C_1 \) depends only on \( n, T, C_0, \) and \( \text{sup}_{\Omega \times [0, T]} |R_{ij}|, \) constant \( C_2 \) depends only on \( \text{sup}_{\Omega \times [0, T]} \max\{ -Q, 0 \} \), \( \text{sup}_{\Omega \times [0, T]} |X|_{g(\tau)} \) and \( \text{sup}_{R \times [0, T]} |R| \), and constant \( C_3 \) depends only on \( n \).

Remark When \( X = 0 \) Theorem 2.2 is Lemma 3.1 in [CTY], which is based on [ZhiQ, §5]. The following proof of the mean value inequality is a slight modification of the proof given in [CCHI1] [25.1] which also considers the case \( X = 0 \).

**Proof** Given a nonnegative sub-solution \( u : \Omega \times [0, T] \to \mathbb{R} \) to (3), define
\[
v \triangleq e^{-At} u,
\]
where $A \geq 0$ to be chosen later (see (28) below). We have
\[
\frac{\partial v}{\partial \tau} - \Delta v + \nabla_X v + (Q + A) v \leq 0
\]
where we have dropped $g(\tau)$ from our notation, e.g., $\Delta = \Delta_{g(\tau)}$. Hence for any real number $p \in [1, \infty)$
\[
\frac{\partial}{\partial \tau} (v^p) - \Delta (v^p) + \nabla_X v^p + p(Q + A) v^p \leq -p(p-1) v^{p-2} |\nabla v|^2 \leq 0 \tag{20}
\]
on $\Omega \times [0, T]$. To prove the so-called reverse Poincare-type inequality (see (29) below), we will first localize this inequality and then integrate it.

Let $0 \leq \tau_1 < \tau_2 \leq T$ and let
\[
\psi : \Omega \times [\tau_1, \tau_2] \to [0, 1] \tag{21}
\]
be a cutoff function with support contained in $D \times [\tau_1, \tau_2]$, where $D \subset \Omega$ is a compact domain with $C^1$-boundary. Assume that
\[
\psi(x, \tau_1) \equiv 0 \quad \text{for } x \in \Omega. \tag{22}
\]
Furthermore we assume $\frac{\partial \psi}{\partial \tau} \geq 0$ and
\[
\psi \frac{\partial \psi}{\partial \tau} + |\nabla \psi|_p^2 \leq L \quad \text{on } D \times [\tau_1, \tau_2] \tag{23}
\]
for some constant $L \in [0, \infty)$.

Multiplying the inequality (20) by $\psi^2 v^p$ and integrating by parts in space and time, we have
\[
0 \geq \int_{\tau_1}^{\tau_2} \int_D \psi^2 \left( \frac{1}{2} \frac{\partial}{\partial \tau} (v^p) - v^p \Delta (v^p) + v^p \nabla_X v^p + p(Q + A) v^{2p} \right) d\mu d\tau
\]
\[
= \int_{\tau_1}^{\tau_2} \int_D \left( -\psi \frac{\partial \psi}{\partial \tau} v^{2p} - \frac{1}{2} \psi^2 v^{2p} R \right) d\mu d\tau + \frac{1}{2} \left( \int_D \psi^2 v^{2p} d\mu \right) (\tau_2)
\]
\[
+ \int_{\tau_1}^{\tau_2} \int_D \left( |\nabla (\psi v^p)|^2 - v^{2p} |\nabla \psi|^2 + p(Q + A) \psi^2 v^{2p} \right) d\mu d\tau
\]
\[
+ \int_{\tau_1}^{\tau_2} \int_D (\psi v^p \nabla_X (\psi v^p) - \psi v^{2p} \nabla_X \psi) d\mu d\tau \tag{24}
\]
where we have used $\frac{\partial}{\partial \tau} d\mu = R d\mu$ and
\[
- \int_D \psi^2 v^p \Delta (v^p) d\mu = \int_D |\nabla (\psi v^p)|^2 d\mu - \int_D v^{2p} |\nabla \psi|^2 d\mu
\]
to derive the equality.

Let $A_1 \equiv 1 + \sup_{D \times [0, T]} |X|_{g(\tau)}$. We compute
\[
\left| \int_{\tau_1}^{\tau_2} \int_D \psi v^p \nabla_X (\psi v^p) d\mu d\tau \right|
\]
\[
\leq A_1^2 \int_{\tau_1}^{\tau_2} \int_D \psi^2 v^{2p} d\mu d\tau + \frac{1}{4A_1^2} \int_{\tau_1}^{\tau_2} \int_D |\nabla_X (\psi v^p)|^2 d\mu d\tau
\]
\[
\leq A_1^2 \int_{\tau_1}^{\tau_2} \int_D \psi^2 v^{2p} d\mu d\tau + \frac{1}{4} \int_{\tau_1}^{\tau_2} \int_D |\nabla (\psi v^p)|^2 d\mu d\tau. \tag{25}
\]
We estimate
\[
\left| \int_{\tau_1}^{\tau_2} \int_D \psi v^{2p} \nabla_X \psi d\mu d\tau \right| \\
\leq \frac{1}{4} A_1^2 \int_{\tau_1}^{\tau_2} \int_D \psi^2 v^{2p} d\mu d\tau + \frac{1}{A_1^2} \int_{\tau_1}^{\tau_2} \int_D v^{2p} |\nabla_X \psi|^2 d\mu d\tau \\
\leq \frac{1}{4} A_1^2 \int_{\tau_1}^{\tau_2} \int_D \psi^2 v^{2p} d\mu d\tau + \int_{\tau_1}^{\tau_2} \int_D v^{2p} |\nabla \psi|^2 d\mu d\tau. \quad (26)
\]

Hence by combining (24), (25), and (26) we get
\[
0 \geq \frac{3}{4} \int_{\tau_1}^{\tau_2} \int_D |\nabla (\psi v^p)|^2 d\mu d\tau + \frac{1}{2} \left( \int_D \psi^2 v^{2p} d\mu \right) (\tau_2) \\
- \int_{\tau_1}^{\tau_2} \int_D \left( \psi \frac{\partial \psi}{\partial \tau} + 2 |\nabla \psi|^2 \right) v^{2p} d\mu d\tau \\
+ \int_{\tau_1}^{\tau_2} \int_D \left( p(Q + A) - \frac{1}{2} R - \frac{5}{4} A_1^2 \right) \psi^2 v^{2p} d\mu d\tau. \quad (27)
\]

Now choose \( A \in [0, \infty) \), depending only on \( \sup_{\Omega \times [0,T]} \max \{-Q(x, \tau), 0\} \), \( \sup_{\Omega \times [0,T]} R \), and \( \sup_{\Omega \times [0,T]} |X|_{g(\tau)} \) (in particular, \( A \) is independent of \( p \) and \( D \)), so that
\[
Q + A - \frac{1}{2p} R - \frac{5}{4p} A_1^2 \geq 0 \quad (28)
\]
on \( D \times [0,T] \) for all \( p \geq 1 \). Hence from (24) and (27) we have the following.

**Lemma 2.3** For the choice of \( A \) as in (28) the function \( v \) defined in (19) satisfies
\[
\int_{\tau_1}^{\tau_2} \int_D |\nabla (\psi v^p)|^2 d\mu d\tau \leq \frac{8}{3} L \int_{\tau_1}^{\tau_2} \int_D \psi^2 v^{2p} d\mu d\tau
\]
and
\[
\left( \int_D \psi^2 v^{2p} d\mu \right) (\tau_2) \leq 4L \int_{\tau_1}^{\tau_2} \int_D v^{2p} d\mu d\tau, \quad (29)
\]
where \( D \subset \Omega \) is a compact domain with \( C^1 \)-boundary and where \( \psi \) satisfies (23) and \( \text{supp}(\psi) \subset D \times [\tau_1, \tau_2] \).

The remaining proof of Theorem 2.2 is the same as the proof given in §25.1 of [CCIII], we omit it. Now we turn to the proof of Theorem 2.1.

**2.4 Step 1 The existence of \( H_\infty \).** First we apply Theorem 2.2 to \( H_k \) in Theorem 2.1 to get some local \( C^0 \) estimate of \( H_k \) uniform in \( k \). In this subsection we adopt the notation of §2.2. Fix an arbitrary interval \([\tau_1, \tau_2] \subset (0, T)\) and fix a compact domain set \( D \subset \text{Int}(M_\infty) \) which contains \( x_{0_\infty} \). Let \( \tilde{g} = g_{\infty}(0) \). We define for any \( r > 0 \)
\[
D(r) = \bigcup_{x \in D} B_{\tilde{g}}(x, r) \subset M_\infty.
\]
Below we fix a constant \( r_1 > 0 \) such that the closure \( \overline{D(r_1)} \) is compact in \( \text{Int}(M_\infty) \).

Choose \( k_0 \) large enough such that \( D(r_1) \subset U_k \) for all \( k \geq k_0 \). Since \( g_k(\tau), X_k, \) and \( Q_k \), after adjustments by \( \Phi_k \), converge to \( g_{\infty}(\tau), X_\infty, \) and \( Q_\infty \) respectively and uniformly in
where $k \geq k_0$.

$C_0^{-1}\hat{g} \leq \Phi^*_{g_k}(\tau) \leq C_0\hat{g}$ on $D(r_1)$ for all $\tau \in [0, T]$, \(\leq 2.2\.

\[ \sup_{\Phi_k(D(r_1)) \times [0, T]} |(R_k)_{ij}|_{g_k(\tau)} \leq C_0 \left( \sup_{\Phi_k(D(r_1)) \times [0, T]} |(R_\infty)_{ij}|_{g_\infty(\tau)} + 1 \right), \]

\[ \sup_{\Phi_k(D(r_1)) \times [0, T]} |R_k| \leq C_0 \left( \sup_{\Phi_k(D(r_1)) \times [0, T]} |R_\infty| + 1 \right), \]

\[ \sup_{\Phi_k(D(r_1)) \times [0, T]} |X_k|_{g_k(\tau)} \leq C_0 \left( \sup_{\Phi_k(D(r_1)) \times [0, T]} |X_\infty|_{g_\infty(\tau)} + 1 \right), \]

\[ \sup_{\Phi_k(D(r_1)) \times [0, T]} \max \{-Q_k, 0\} \leq C_0 \left( \sup_{\Phi_k(D(r_1)) \times [0, T]} \max \{-Q_\infty, 0\} + 1 \right). \]

Note that there is a constant $K \geq 0$ such that

\[ Rc(\hat{g}) \geq -K \mbox{ in } D(r_1). \]

To apply Theorem 2.2 to $u = H_k$ we choose $\Omega$ in Theorem 2.2 to be $D(2r_0)$ where

\[ r_0 \in (0, \min \left\{ \sqrt{\frac{1}{2}}, \frac{1}{2} r_1, \frac{1}{2} \right\}). \]

we take $g(\tau) = g_k(\tau)$, $X = X_k$, and $Q = Q_k$. Now we verify the assumption of Theorem 2.2. For any $(x_*, \tau_*) \in D \times [\tau_1, \tau_2]$, by the choice of $r_1$ and $r_0$ we conclude that the closed ball $B_{\hat{g}}(x_*, 2r_0)$ is compact in $D(r_1)$ and that $\tau_* \geq (2r_0)^2$. Also for any $x_* \in D$ we have

\[ \text{Vol}_{\hat{g}}(B_{\hat{g}}(x_*, r_0)) \geq c_1 \text{Vol}_{\hat{g}}(B_{\hat{g}}(x_{\infty}, r_0)) \]

where $c_1$ is a constant depending on $n$, $K$, $r_0$ and $\text{diam}_{\hat{g}}(D)$, this follows from a standard argument using Bishop-Gromov volume comparison theorem.

By the local mean value property (18) and bounds in (20) we have the following. When $k \geq k_0$, for any $(x_*, \tau_*) \in D \times [\tau_1, \tau_2]$ and for any $x_* \in D$ we have

\[ 0 \leq H_k(\Phi_k(x), \tau) \leq \sup_{B_{\hat{g}}(x_*, r_0) \times [\tau_1 - (2r_0)^2, \tau]} H_k(\Phi_k(x), \tau) \]

\[ \leq C_4 \int_{B_{\hat{g}}(x_*, 2r_0) \times [\tau_1 - (2r_0)^2, \tau]} H_k(\Phi_k(y), \tau) d\mu_{\hat{g}}(y) d\tau \]

\[ \leq C_4 \int_{B_{\hat{g}}(x_*, 2r_0) \times [\tau_1 - (2r_0)^2, \tau]} H_k(\Phi_k(y), \tau) \left( C_0^* d\mu_{\Phi^*_{g_k}(\tau)}(y) \right) d\tau \]

\[ \leq C_4 C_0^* \int_{\tau_1 - (2r_0)^2}^{\tau} \int_{B_{\hat{g}}(x_*, 2r_0))} H_k(z, \tau) d\mu_{g_k(\tau)}(z) d\tau, \]

where

\[ C_4 \leq \frac{C_1 e^{C_2 T + C_3 \sqrt{r_0}}}{r_0^2 \text{Vol}_{\hat{g}}(B_{\hat{g}}(x_*, r_0))} \leq \frac{C_1 e^{C_2 T + C_3 \sqrt{r_0}}}{r_0^2 c_1 \text{Vol}_{\hat{g}}(B_{\hat{g}}(x_{\infty}, r_0))}. \]
Here constant $C_1$ depends only on $n$, $T$, $C_0$, and $\sup_{D(r_1)\times[0,T]} |(\mathcal{R}_\infty)_{ij}|_{g_\infty(\tau)}$. Constant $C_2$ depends only on $C_0$, $\sup_{D(r_1)\times[0,T]} \max \{-Q_\infty,0\}$, $\sup_{D(r_1)\times[0,T]} |X_\infty|_{g_\infty(\tau)}$, and $\sup_{D(r_1)\times[0,T]} |\mathcal{R}_\infty|$. Constant $C_3$ depends only on $n$.

In summary so far we have proved that when $k \geq k_0$, for any $(x,\tau) \in D(r_0) \times [\tau_1 - r_0^2, \tau_2]$,

$$|H_k(\Phi_k(x),\tau)| \leq C_5 \int_{\tau_1 -(2r_0)^2}^{\tau_2} \int_{\Phi_k(B_\delta(x,2r_0))} H_k(z,\tau) \, d\mu_{g_k(\tau)}(z) \, d\tau,$$

(32)

where $C_5$ is a constant independent of $k$ and $x_*$ is a point in $D$ such that $x \in B_\delta(x_*,r_0)$. It follows from the assumption (33) that

$$\sup_{D(r_0) \times [\tau_1 - r_0^2, \tau_2]} |H_k(\Phi_k(x),\tau)| \leq C_6$$

(33)

where $C_6$ is a constant independent of $k$.

Next we want to get local high derivative estimates of $H_k(\Phi_k(\cdot),\cdot)$. Below we assume $k \geq k_0$. From the compactness of $\overline{D(r_1)}$ there is a $K_1 > 0$ such that $|\text{Rm}_\bar{g}\bar{g}| \leq K_1$ for all $x \in D(r_1)$. Let $\tilde{r}_0 \overset{\triangle}{=} \min \left\{ r_0, \frac{\sqrt{\text{Rm}(x_0)} \cdot \bar{g}(x_0)}{\bar{g}(x_0)} \right\}$. Fix an arbitrary $x_* \in D$, let $\exp_{x_*} : B(0,\tilde{r}_0) \to B_\delta(x_*,\tilde{r}_0) \subset M_\infty$ be the exponential map of metric $\bar{g}$ and let $\bar{x} = (x_i)$ be an associated normal coordinates on $B(0,\tilde{r}_0)$. We will consider the pull back functions

$$\hat{H}_k(\bar{x},\tau) \overset{\triangle}{=} H_k(\Phi_k(\exp_{x_*}\bar{x},\tau))$$

(34)

defined on $B(0,\tilde{r}_0) \times (0,T]$.

Let $\hat{X}_k(\tau)$ be the vector fields on $B(0,\tilde{r}_0)$ such that the push-forward vector field $(\Phi_k \circ \exp_{x_*})_* \hat{X}_k(\tau) = X_k(\tau)$. Since $H_k$ satisfies (35), $\hat{H}_k$ satisfies the following

$$\frac{\partial \hat{H}_k}{\partial \tau} - \Delta_{(\Phi_k \circ \exp_{x_*})^* g_k(\tau)} \hat{H}_k + \nabla_{\hat{X}_k} \hat{H}_k + Q_k(\Phi_k(\exp_{x_*}\bar{x},\tau)) \hat{H}_k = 0$$

(35)

on $B(0,\tilde{r}_0) \times (0,T]$.

Because of the $C^\infty$ Cheeger-Gromov convergence of $(\Omega_k, g_k(\tau), (x_{0k},0))$ to $(M_\infty, g_\infty(\tau), (x_{0\infty},0))$ for $\tau \in [0,T]$, the structure coefficient functions of the parabolic equation (35) converge in $C^\infty$-norm on $B(0,\tilde{r}_0) \times [0,T]$ to the corresponding coefficients of the following parabolic equation

$$\frac{\partial \hat{H}_\infty}{\partial \tau} - \Delta_{\exp_{x_*}^* g_\infty(\tau)} \hat{H}_\infty + \nabla_{\hat{X}_\infty} \hat{H}_\infty + Q_\infty(\exp_{x_*}^* \bar{x},\tau) \hat{H}_\infty = 0,$$

(36)

where $\hat{X}_\infty(\cdot,\tau)$ is the vector fields on $B(0,\tilde{r}_0)$ such that $(\exp_{x_*})_* \hat{X}_\infty(\cdot,\tau) = X_\infty(\cdot,\tau)$. In particular for any fixed $l \in \mathbb{N}$ and $\alpha \in (0,1)$ the parabolic Hölder norms

$$\|\|_{C^{2l+\alpha,l+\alpha/2}(B(0,\tilde{r}_0) \times [0,T])}$$

of these structure coefficient functions are bounded by a constant independent of $k$. Note that from (33) $\hat{H}_k$ is uniform bounded on $B(0,\tilde{r}_0) \times [\tau_1 - r_0^2, \tau_2]$.

We can apply the interior Schauder estimates for linear parabolic equation (see Theorem 4.9 and Exercise 4.5 in [Lie], for example) to conclude that for any integer $l \geq 0$ and $\alpha \in (0,1)$ we have

$$\|\hat{H}_k\|_{C^{2l+2,\alpha,l+1/2}(B(0,\tilde{r}_0/2) \times [\tau_1,\tau_2])} \leq C_{l,\alpha}$$

(37)
where $C_{t, \alpha}$ is a constant independent of $k$.

Finally we can show the existence of $H_\infty$ in Theorem 2.1. By estimates (37) and Arzela and Ascoli theorem there is a subsequence (still indexed by $k$) such that $\hat{H}_k \to H_\infty$ in $C^\infty$-norm on $\tilde{B}(0, \hat{r}_0/2) \times [\tau_1, \tau_2]$ and $H_\infty$ is a solution of (38). Note that $\hat{H}_k(x, \tau) = H_k(\Phi_k(x), \tau)$ satisfies the following equation

$$\frac{\partial \hat{H}_k}{\partial \tau} - \Delta_{\tilde{x}^*g_k(\tau)} \hat{H}_k + \nabla \tilde{x}_k \cdot \hat{H}_k + Q_k(\Phi_k(x), \tau) \hat{H}_k = 0 \quad (38)$$

where $\tilde{x}_k(\tau)$ is a family of vector field such that the push-forward vector field $(\Phi_k)_* \tilde{x}_k(\tau) = X_k(\tau)$. Define function $H_\infty$ on $\exp_{x_*} \tilde{B}(0, \hat{r}_0/2) \times [\tau_1, \tau_2]$ by $H_\infty(\exp_{x_*}(\tilde{x}), \tau) \equiv H_\infty(\tilde{x}, \tau)$. Though $\exp_{x_*}$ is not necessarily a diffeomorphism, it is clear that $H_\infty$ is well defined and that $\hat{H}_k$ converges to $H_\infty$ in $C^\infty$-norm. Furthermore $H_\infty$ is a solution of (12) on $\exp_{x_*} \tilde{B}(0, \hat{r}_0/2) \times [\tau_1, \tau_2]$.

Note that our definition of $\hat{H}_k(\tilde{x}, \tau)$ in (31) depends on $x_*$. Since $x_*$ is an arbitrary point in $D$, we conclude by a diagonalization argument that there is a subsequence of $\hat{H}_k$ which converges in $C^\infty$-norm to some function $H_{\infty}$ defined on $D \times [\tau_1, \tau_2]$. $H_\infty$ is a solution of (12) on $D \times [\tau_1, \tau_2]$.

Let $\tau_2 = T$ and let $\tau_i$ be a sequence approaching to $0^+$. Let $D_i$ be a sequence of compact domains which exhaust $\Int(M_\infty)$. We can apply the above construction of $H_\infty$ on $D \times [\tau_1, \tau_2]$ repeatedly to each $D_i \times [\tau_{i-1}, T]$. Using a diagonalization argument we can find a subsequence such that $\hat{H}_k$ converges in $C^\infty$-norm on $D_i \times [\tau_{i-1}, T]$ for each $i$. This implies that $\hat{H}_k$ converges on $\Int(M_\infty) \times (0, T]$ to a function $H_\infty$, the convergence is uniform in $C^\infty$-norm on any compact subset of $\Int(M_\infty) \times (0, T]$.

In short we have defined a function $H_\infty \geq 0$ which satisfies (12) on $\Int(M_\infty) \times (0, T]$.

2.5 Step 2 Proof of (13). We will show $\lim_{\tau \to 0^+} H_\infty(x, \tau) = \delta_{x_0\infty}$. Let $F : M_\infty \to \mathbb{R}$ be a nonnegative $C^2$ function such that support $\text{supp}(F)$ is a compact subset of $\Int(M_\infty)$. We compute using (38)

$$\frac{d}{d\tau} \int_{M_\infty} \hat{H}_k F d\mu_{\tilde{x}^*g_k(\tau)}$$

$$= \int_{M_\infty} \left( \frac{\partial \hat{H}_k}{\partial \tau} + \nabla \tilde{x}_k \cdot \hat{H}_k + Q_k(\Phi_k(x), \tau) \hat{H}_k \right) F d\mu_{\tilde{x}^*g_k(\tau)}$$

$$= \int_{M_\infty} \left( \Delta_{\tilde{x}^*g_k(\tau)} \hat{H}_k - 
abla \tilde{x}_k \cdot \hat{H}_k + (\nabla \tilde{x}_k \cdot \hat{H}_k) \hat{H}_k \right) F d\mu_{\tilde{x}^*g_k(\tau)}$$

$$= \int_{M_\infty} \left( \frac{\Delta_{\tilde{x}^*g_k(\tau)} F}{\nabla \tilde{x}_k \cdot \hat{H}_k} + (\nabla \tilde{x}_k \cdot \hat{H}_k) \right) \hat{H}_k F d\mu_{\tilde{x}^*g_k(\tau)}$$

where in the last equality we have used the divergence theorem

$$\int_{M_\infty} \left( -\nabla \tilde{x}_k \cdot \hat{H}_k \right) F d\mu_{\tilde{x}^*g_k(\tau)} = \int_{M_\infty} \left( -\left( \frac{\nabla \tilde{x}_k \cdot \hat{H}_k}{\Phi_k(x), \tau} \right) F \right) d\mu_{\tilde{x}^*g_k(\tau)}$$

$$= \int_{M_\infty} \left( \text{div}_{\tilde{x}^*g_k(\tau)}(F \tilde{x}_k) \hat{H}_k d\mu_{\tilde{x}^*g_k(\tau)} \right).$$
Because of the following uniform convergence on \( \text{supp}(F) \times [0, T] \)

\[
\Delta \phi_{\tau, g_k}(\tau) F \rightarrow \Delta_{g_\infty}(\tau) F \\
\text{div} \phi_{\tau, g_k}(\tau) F \Phi_k \rightarrow \text{div}_{g_\infty}(\tau)(FX_\infty) \\
R_k(\Phi_k, \tau) \rightarrow R_\infty(x, \tau) \\
Q_k(\Phi_k, \tau) \rightarrow Q_\infty(x, \tau),
\]

there is a constant \( C_7 \) independent of \( k \) such that

\[
\sup_{\text{supp}(F) \times [0, T]} |\Delta \phi_{\tau, g_k}(\tau) F| \leq C_7 \\
\sup_{\text{supp}(F) \times [0, T]} |\text{div} \phi_{\tau, g_k}(\tau) F \phi_k| \leq C_7 \\
\sup_{\text{supp}(F) \times [0, T]} |R_k(\Phi_k, \tau)| \leq C_7 \\
\sup_{\text{supp}(F) \times [0, T]} |Q_k(\Phi_k, \tau)| \leq C_7.
\]

Here and below we assume \( k \) are large enough. Hence it follows from (39) and (8)

\[
\left| \frac{d}{d\tau} \int_{M^\infty} \hat{H}_k F d\mu_{\phi_{\tau, g_k}}(\tau) \right| \leq 2C_7 \int_{M^\infty} \hat{H}_k F d\mu_{\phi_{\tau, g_k}}(\tau) + 2C_8 \tag{40}
\]

for \( \tau \in (0, T] \) where \( C_8 \equiv C_7 C_* \).

Let

\[
U_k(\tau) \equiv \int_{M^\infty} \hat{H}_k(x, \tau) F(x) d\mu_{\phi_{\tau, g_k}}(\tau). 
\]

By a simple integration of (40) we get

\[
U_k(0) e^{-2C_7 \tau} - \frac{C_8}{C_7} (1 - e^{-2C_7 \tau}) \leq U_k(\tau) \leq U_k(0) e^{2C_7 \tau} + \frac{C_8}{C_7} (e^{2C_7 \tau} - 1). \tag{41}
\]

By the definition of \( \hat{H}_k \) we have

\[
U_k(0) = \int_{M^\infty} H_k(y, 0) F(\Phi_k(y)) d\mu_{g_k}(\tau) = F(x_{0\infty}).
\]

By the convergence of \( \hat{H}_k \) proved in §2.4 and the compactness of \( \text{supp}(F) \) we have that for any \( \tau > 0 \)

\[
\lim_{k \rightarrow \infty} U_k(\tau) = \int_{M^\infty} \lim_{k \rightarrow \infty} \left( \hat{H}_k(x, \tau) F(x) d\mu_{\phi_{\tau, g_k}}(\tau) \right) \\
= \int_{M^\infty} H_\infty(x, \tau) F(x) d\mu_{g_\infty}(\tau).
\]

By taking \( k \rightarrow \infty \) limit of (41) we get

\[
e^{-2C_7 \tau} F(x_{0\infty}) - \frac{C_8}{C_7} (1 - e^{-2C_7 \tau}) \leq \int_{M^\infty} H_\infty F d\mu_{g_\infty}(\tau) \leq e^{2C_7 \tau} F(x_{0\infty}) + \frac{C_8}{C_7} (e^{2C_7 \tau} - 1)
\]
for \( \tau \in (0, T] \). Hence
\[
\lim_{\tau \to 0^+} \int_{M_\infty} H_\infty F d\mu_{g_\infty}(\tau) = F(x_0) \quad (42)
\]
for any nonnegative \( C^2 \) function with compact support in \( \text{Int}(M_\infty) \). This implies \( \lim_{\tau \to 0^+} H_\infty(x, \tau) = \delta_{x_0} \).

### 2.6 Step 3 Finishing the proof of Theorem 2.1

By (42) we conclude that there is a \( T_1 \in (0, T) \) such that for any \( \tau \in (0, T_1] \) there is a \( x_\tau \in B_{\tilde{g}}(x_0, \tilde{r}_0) \) such that \( H_\infty(x_\tau, \tau) > 0 \). Since \( H_\infty \geq 0 \) and \( H_\infty \) satisfies the equation (12) on \( M_\infty \times (0, T] \), it follows from the strong maximum principle that \( H_\infty(x, \tau) > 0 \) on \( M_\infty \times (0, T] \). Hence \( \ln H_k(x, \tau) \to \ln H_\infty(x, \tau) \) uniformly in \( C_\infty \)-norm on any compact subset of \( \text{Int}(M_\infty) \times (0, T] \), and the claimed convergence of \( f_k \) to \( f_\infty \) in Theorem 2.1 follows easily. Now Theorem 2.1 is proved.

### 3 The upper bound of integrals of fundamental solutions

In this section we discuss how to bound the integrals of fundamental solutions, this is related to the assumption (8). Here we use a special case of the setup described at the beginning of §2.1, we assume \( \Omega = M \). We also adopt the notations used at the beginning of §2.1, in particular, \( H \) is a fundamental solution of (4) on \( M \times [0, T] \) centered at \( x_0 \in \text{Int}(M) \). It is clear
\[
\lim_{\tau \to 0^+} \int_M H(x, \tau) d\mu_{g(\tau)}(x) = 1. \quad (43)
\]

The results in this section and the next section are well-known, we include them here because of their relation with Theorem 2.1. Below we divide our discussion into three cases.

**Case 1** \( M \) is a closed manifold. Let \( C_0 \) be a constant such that
\[
\sup_{M \times [0, T]} (\text{div}_{g(\tau)} X + R - Q) \leq C_0.
\]

We compute
\[
\frac{d}{dt} \int_M H d\mu_{g(\tau)} = \int_M \left( \frac{\partial H}{\partial \tau} + RH \right) d\mu_{g(\tau)}
\]
\[
= \int_M (\Delta H - \nabla_X H + (R - Q)H) d\mu_{g(\tau)}
\]
\[
= \int_M (\text{div}_{g(\tau)} X + R - Q) H d\mu_{g(\tau)} \quad (44)
\]
\[
\leq C_0 \int_M H d\mu_{g(\tau)}.
\]

Hence it follows from (43) that
\[
\int_M H d\mu_{g(\tau)} \leq e^{C_0 \tau}
\]
for \( \tau \in [0, T] \).
In the special case when $X = 0$ and $Q = \mathcal{R}$ it follows from (44) that
\[ \int_M H \, d\mu_g(\tau) = 1 \]
for $\tau \in [0, T].$

**Case 2** *$M$ is a compact manifold with nonempty boundary.* Let $C_0$ be a constant such that
\[ \sup_{M \times [0, T]} (\text{div}_g(\tau) X + \mathcal{R} - Q) \leq C_0. \]

Let $\nu = \nu(x, \tau)$ be the outward unit normal direction on $\partial M$ with respect to the metric $g(\tau)$. We assume $\langle X, \nu \rangle g(\tau) \geq 0$ on the boundary $\partial M \times [0, T]$. We compute
\[
\begin{align*}
\frac{d}{d\tau} & \int_M H \, d\mu_g(\tau) \\
& = \int_M (\Delta H - \nabla X H + (\mathcal{R} - Q)H) \, d\mu_g(\tau) \\
& = \int_{\partial M} \left( \frac{\partial H}{\partial \nu} - \langle X, \nu \rangle g(\tau)H \right) \, d\mu_g(\tau) + \int_M \left( \text{div}_g(\tau) X + \mathcal{R} - Q \right) H \, d\mu_g(\tau) \\
& \leq \int_{\partial M} \frac{\partial H}{\partial \nu} \, d\mu_g(\tau) + \int_M \left( \text{div}_g(\tau) X + \mathcal{R} - Q \right) H \, d\mu_g(\tau). \quad (45)
\end{align*}
\]

If $H$ satisfies the Dirichlet boundary condition $H|_{\partial M \times (0, T]} = 0$, then $\frac{\partial H}{\partial \nu} \leq 0$ on $\partial M \times (0, T]$. It follows from (45) that
\[ \int_M H \, d\mu_g(\tau) \leq e^{C_0 \tau} \]
for $\tau \in (0, T]$. In the special case when $X = 0$ and $Q = \mathcal{R}$ it follows from (45) that
\[ \int_M H \, d\mu_g(\tau) \leq 1 \quad \text{for any } \tau \in (0, T]. \]

If $H$ satisfies the Neumann boundary condition $\frac{\partial H}{\partial \nu}|_{\partial M \times (0, T]} = 0$, then (45) becomes
\[ \frac{d}{d\tau} \int_M H \, d\mu_g(\tau) \leq \int_{\partial M} \left( \text{div}_g(\tau) X + \mathcal{R} - Q \right) H \, d\mu_g(\tau). \]

Hence
\[ \int_M H \, d\mu_g(\tau) \leq e^{C_0 \tau} \quad \text{for any } \tau \in (0, T]. \]

In the special case when $X = 0$ and $Q = \mathcal{R}$ it follows that
\[ \int_M H \, d\mu_g(\tau) = 1 \quad \text{for any } \tau \in (0, T]. \]

**Case 3** $(M, g(\tau))$, $\tau \in [0, T]$, are complete noncompact manifolds. Because of the potential non-uniqueness of the fundamental solutions, it is desirable to consider the integrals of the minimal fundamental solutions. We refer the reader to the literature (see, for example, Lemma 5.1 in [CTY], or Lemma 26.14 and Corollary 26.15 in [CCIII]).
4 Uniqueness of fundamental solutions

In this section we consider the uniqueness of fundamental solutions. Note that it is pointed out by Hsu in [Hs] that when the limit fundamental solution in Theorem 2.1 is unique, then the sub-convergence in the theorem can be improved to be convergence for the whole sequence. Here we use the same setup as the one used §3, in particular, $H$ is a fundamental solution of (1) on $M^n \times [0, T]$ centered at $x_0 \in \text{Int}(M)$.

**Proposition 4.1** (i) When $M$ is a closed manifold, the fundamental solution is unique.

(ii) When $M$ is a compact manifold with nonempty boundary, the fundamental solution with any Dirichlet boundary condition is unique.

(iii) When $M$ is a compact manifold with nonempty boundary, the fundamental solution with any Neumann boundary condition is unique.

(iv) When $(M, g(\tau)), \tau \in [0, T]$, are complete and noncompact manifold, the fundamental solution satisfying the assumption (A1), (A2), and (A3) below is unique.

**Remark** The proof is based on an idea of Brett Kotschwar (see Footnote 14 on p.345 of [CCIII]).

**Proof.** (i) Let $H_1$ and $H_2$ be two fundamental solutions centered at $x_0$. Define $F(x, \tau) = H_1(x, \tau) - H_2(x, \tau)$. Then $F$ satisfies

$$\frac{\partial F}{\partial \tau} - \Delta_{g(\tau)} F + \nabla_X F + QF = 0.$$  \hfill (46)

Let $\varphi : M \to \mathbb{R}$ be an arbitrary $C^2$ function. Fix a $\bar{\tau} \in (0, T]$. Let $\Phi : M \times [0, \bar{\tau}] \to \mathbb{R}$ be the solution to the initial value problem

$$\left(\frac{\partial}{\partial \tau} + \Delta_{g(\tau)} + \nabla_X + (\text{div}_{g(\tau)} X + R - Q)\right) \Phi = 0,$$  \hfill (47)

The solution always exists. For $\varepsilon \in (0, \bar{\tau})$ we have

$$0 = \int_{\varepsilon}^{\bar{\tau}} \int_M \Phi \left(\frac{\partial}{\partial \tau} - \Delta_{g(\tau)} + \nabla_X + Q\right) F d\mu_{g(\tau)} d\tau$$

$$= \int_{\varepsilon}^{\bar{\tau}} \left(\frac{d}{d\tau} \int_M \Phi F d\mu_{g(\tau)}\right) d\tau$$

$$- \int_{\varepsilon}^{\bar{\tau}} \int_M F \left(\frac{\partial}{\partial \tau} + \Delta_{g(\tau)} + \nabla_X + (\text{div}_{g(\tau)} X + R - Q)\right) \Phi d\mu_{g(\tau)} d\tau$$

$$= \int_M \Phi F d\mu_{g(\varepsilon)} - \int_M \Phi F d\mu_{g(\varepsilon)}.$$

Since

$$\lim_{\varepsilon \to 0} \int_M \Phi F d\mu_{g(\varepsilon)} = \lim_{\varepsilon \to 0} \int_M \Phi H_1 d\mu_{g(\varepsilon)} - \lim_{\varepsilon \to 0} \int_M \Phi H_2 d\mu_{g(\varepsilon)}$$

$$= \Phi(x_0, 0) - \Phi(x_0, 0) = 0,$$
we have proved
\[ \int_M \varphi F d\mu_g(\tau) = \int_M \Phi F d\mu_g(\tau) = 0. \]
Since both \( \varphi \) and \( \tau \) are arbitrary, we concluded \( F = 0 \) and hence \( H_1 = H_2 \).

(ii) Let \( \psi: M \times [0, T] \to \mathbb{R} \) be a continuous function. Let \( H_1 \) and \( H_2 \) be two fundamental solution of (41) on \( M \times (0, T] \) centered at \( x_0 \) which satisfy the following Dirichlet boundary condition
\[ H_1|_{\partial M \times (0, T]} = H_2|_{\partial M \times (0, T]} = \psi|_{\partial M \times (0, T]}. \]
Define \( F(x, \tau) \div H_1(x, \tau) \div H_2(x, \tau) \). Then \( F \) satisfies (46) and the boundary condition \( F|_{\partial M \times (0, T]} = 0 \). Let \( \varphi: M \to \mathbb{R} \) be an arbitrary \( C^2 \) function which vanishes on \( \partial M \). Fix a \( \bar{\tau} \in (0, T) \). Let \( \Phi: M \times [0, \bar{\tau}] \to \mathbb{R} \) be the solution to the initial-boundary value problem (47) with Dirichlet boundary condition \( \Phi|_{\partial M \times [0, T]} = 0 \). The solution always exists.

By the divergence theorem we have
\[ \begin{align*}
\int_M \Phi \Delta_g(\tau) F d\mu_g(\tau) &= \int_M F \Delta_g(\tau) \Phi d\mu_g(\tau) + \int_{\partial M} \left( \frac{\partial F}{\partial \nu} \Phi - F \frac{\partial \Phi}{\partial \nu} \right) d\sigma_g(\tau) \\
\int_M \Phi \nabla_X F d\mu_g(\tau) &= \int_M \left( -F \nabla_X \Phi - F \Phi \Delta_g(\tau) X \right) d\mu_g(\tau) \\
&\quad + \int_{\partial M} \langle X, \nu \rangle F \Phi d\sigma_g(\tau)
\end{align*} \]
where \( d\sigma_g(\tau) \) is the volume form on \( \partial M \) defined by the metric \( g(\tau)|_{\partial M} \). Because \( F = \Phi = 0 \) on \( \partial M \times (0, T] \), it follows from similar calculations and arguments as in (i) that for \( \varepsilon \in (0, \bar{\tau}) \)
\[ \int_M \Phi F d\mu_g(\tau) = \int_M \Phi F d\mu_g(\tau) \]
and hence \( F = 0 \). This proves \( H_1 = H_2 \).

(iii) Let \( \psi: M \times [0, T] \to \mathbb{R} \) be a continuous function. Let \( H_1 \) and \( H_2 \) be two fundamental solution of (41) on \( M \times (0, T] \) centered at \( x_0 \in \text{Int}(M) \) which satisfy the following Neumann boundary condition
\[ \frac{\partial H_1}{\partial \nu}|_{\partial M \times (0, T]} = \frac{\partial H_2}{\partial \nu}|_{\partial M \times (0, T]} = \psi|_{\partial M \times (0, T]}. \]
Define \( F(x, \tau) \div H_1(x, \tau) \div H_2(x, \tau) \). Then \( F \) satisfies (46) and the boundary condition \( \frac{\partial F}{\partial \nu}|_{\partial M \times (0, T]} = 0 \). Let \( \varphi: M \to \mathbb{R} \) be an arbitrary \( C^2 \) function which vanishes on \( \partial M \). Fix a \( \bar{\tau} \in (0, T) \). Let \( \Phi: M \times [0, \bar{\tau}] \to \mathbb{R} \) be the solution to the initial-boundary value problem (47) with oblique derivative boundary condition
\[ \left( \frac{\partial \Phi}{\partial \nu} - \langle X, \nu \rangle \Phi \right)|_{\partial M \times [0, T]} = 0. \]
The solution always exists. From (48) and similar calculation and argument as in (i) we conclude that for \( \varepsilon \in (0, \bar{\tau}) \)
\[ \int_M \Phi F d\mu_g(\tau) = \int_M \Phi F d\mu_g(\tau). \]
and hence $F = 0$. This proves $H_1 = H_2$.

(iv) Let $\varphi : M \to \mathbb{R}$ be an arbitrary $C^2$ function with compact support. Fix a $\tilde{\tau} \in (0, T]$. Let $\Phi : M \times [0, \tilde{\tau}] \to \mathbb{R}$ be the bounded solution to the initial value problem \[17\]. The solution always exists.

Let $H_i, i = 1, 2$, be two fundamental solution of \[14\] on $M \times (0, T]$ centered at $x_0 \in \text{Int}(M)$. Assume that each $H_i$ satisfies the following equalities: for any $\Phi$ defined above
\begin{align*}
(A1) \quad & \frac{d}{dt} \int_M H_i \Phi d\mu_g(\tau) = \int_M \frac{\partial}{\partial \tau} (H_i \Phi d\mu_g(\tau)), \\
(A2) \quad & \int_M \Phi \Delta_g(\tau) H_i d\mu_g(\tau) = \int_M H_i \Delta_g(\tau) \Phi d\mu_g(\tau), \\
(A3) \quad & \int_M \Phi \nabla_X H_i d\mu_g(\tau) = \int_M (\Phi \nabla_X H_i - H_i \Phi \text{div}_g(\tau) X) d\mu_g(\tau).
\end{align*}

Define $F(x, \tau) \equiv H_1(x, \tau) - H_2(x, \tau)$. Then $F$ satisfies \[16\]. The above assumptions enable us to perform the similar calculation and argument as in (i), we conclude that for $\varepsilon \in (0, \tilde{\tau})$
\begin{equation}
\int_M \Phi F d\mu_g(\tau) = \int_M \Phi F d\mu_g(\varepsilon)
\end{equation}
and hence $F = 0$. This proves $H_1 = H_2$. $\blacksquare$

**Remark** Here we will not discuss when the assumptions (A1), (A2), and (A3) will be satisfied. The interested readers may find various estimates of $\Phi$ and $H_i$ from literature which guarantee that the assumptions hold (see [KQ], for example).

## 5 A local integral estimate of fundamental solutions

### 5.1 The cut-off function $h$

The following construction of $h$ is adapted from the cut-off function used by Perelman in his proof of pseudo-locality theorem ([Pe02I, p.25]), a similar function is also used by Perelman in his proof of the localized no local collapsing theorem ([Pe02I, p.21]). Let $M^n$, $\Omega$, $g(\tau)$ with $\tau \in [0, T]$, $Q$, and $x_0 \in \text{Int}(M)$, be defined as at the beginning of §2.1. We also adopt the notations used there. Let $\tilde{g}$ be a smooth metric on $\Omega$ satisfying \[16\] and $\text{Rc}(\tilde{g}) \geq -K$ on $\Omega$. Fix a $r_\ast > 0$ such that $B_{\tilde{g}}(x_0, r_\ast) \subset \Omega$ is compact. We assume
\begin{align}
\sup_{B_{\tilde{g}}(x_0, r_\ast) \times [0,T]} |R_{ij}(x, \tau)|_{g(\tau)} & \leq K_\ast, \\
\sup_{B_{\tilde{g}}(x_0, r_\ast) \times [0,T]} |\text{Rc}(x, \tau)|_{g(\tau)} & \leq K_\ast.
\end{align}

By a simple calculation we have
\begin{equation}
e^{-2K_\ast T} C_0^{-1} \tilde{g}(x) \leq g(x, \tau) \leq e^{2K_\ast T} C_0 \tilde{g}(x)
\end{equation}
for all $x \in B_{\tilde{g}}(x_0, r_\ast)$ and $\tau \in [0, T]$. Let
\begin{equation}
\tilde{r} \equiv \frac{1}{2} e^{-2K_\ast T} C_0^{-1} r_\ast.
\end{equation}

Then closed ball
\begin{equation*}
B_{\tilde{g}(\tau)}(x_0, \tilde{r}) \subset B_{\tilde{g}}(x_0, r_\ast) \quad \text{for all } \tau \in [0, T].
\end{equation*}

For any $x \in B_{\tilde{g}(\tau)}(x_0, \tilde{r})$, we have (see Lemma 18.1 in [CCIII], for example)
\begin{equation}
\frac{\partial}{\partial \tau} d_{g(\tau)}(x, x_0) = \int_0^{d_{g(\tau)}(x, x_0)} \mathcal{R}_{ij}(\gamma(s), \tau) \frac{d\gamma_i}{ds} \frac{d\gamma_j}{ds} ds.
\end{equation}
where \( \gamma(s) \) is some unit speed minimal geodesic with respect to metric \( g(\tau) \) between \( x \) and \( x_0 \). Hence
\[
\frac{\partial}{\partial \tau} d_{g(\tau)}(x, x_0) \leq K_\ast d_{g(\tau)}(x, x_0) \leq K_\ast \hat{r}.
\] (52)

Let \( \xi : [0, d_{g(\tau)}(x, x_0)] \to [0, 1] \) be a continuous piecewise smooth function with \( \xi(0) = 0 \) and \( \xi(d_{g(\tau)}(x, x_0)) = 1 \). We have (see Lemma 18.6 in [CCIII], for example)
\[
\Delta_{g(\tau)} d_{g(\tau)}(x, x_0) \leq \int_0^{d_{g(\tau)}(x, x_0)} (n - 1) (\xi'(s))^2 - \xi^2(s) \text{Rc}(\gamma'(s), \gamma'(s)) ds
\]

Let \( x \in B_{g(\tau)}(x_0, \hat{r}) \setminus B_{g(\tau)}(x_0, \frac{1}{10} \hat{r}) \).

We choose
\[
\xi(s) = \begin{cases} 
\frac{10}{\hat{r}} & \text{if } s \in [0, \frac{\hat{r}}{10}], \\
1 & \text{if } s \in (\frac{\hat{r}}{10}, d_{g(\tau)}(x, x_0)].
\end{cases}
\]

By a simple calculation and (50) we have
\[
\Delta_{g(\tau)} d_{g(\tau)}(x, x_0) \leq \frac{10(n - 1)}{\hat{r}} + K_\ast \hat{r}.
\] (53)

Combining (52) and (53) we have proved

**Lemma 5.1** Under the assumption given at the beginning of this subsection, we have
\[
\left( \frac{\partial}{\partial \tau} + \Delta_{g(\tau)} \right) d_{g(\tau)}(x, x_0) \leq \frac{10(n - 1)}{\hat{r}} + 2K_\ast \hat{r}
\] (54)

for any \( x \in B_{g(\tau)}(x_0, \hat{r}) \setminus B_{g(\tau)}(x_0, \frac{1}{10} \hat{r}) \).

Let \( \phi : \mathbb{R} \to [0, 1] \) be a smooth function which is strictly decreasing on the interval \([1, 2]\) and which satisfies
\[
\phi(s) = \begin{cases} 
1 & \text{if } s \in (-\infty, 1] \\
0 & \text{if } s \in [2, \infty)
\end{cases}
\] (55)

and
\[
(\phi'(s))^2 \leq 10\phi(s)
\] (56)
\[
\phi''(s) \geq -10\phi(s)
\] (57)

for \( s \in \mathbb{R} \).

Let \( T_1 \in (0, T] \) be a constant to be chosen later (see (59)). We define a function \( h : \Omega \times [0, T_1] \to [0, 1] \) by
\[
h(x, \tau) = \phi \left( \frac{d_{g(\tau)}(x, x_0) + a(T_1 - \tau)}{b} \right)
\] (58)

where \( a \) is a positive constant to be chosen later (see (59)) and \( b = \frac{1}{2} \hat{r} \). Note that \( \text{supp } h(\cdot, \tau) \subset B_{g(\tau)}(x_0, \hat{r}) \). Let \( X(\tau), \tau \in [0, T] \), be a smooth family of vector fields on \( B_g(x_0, r_\ast) \), and let
\[
w(x, \tau) = \frac{d_{g(\tau)}(x, x_0) + a(T_1 - \tau)}{b}.
\]
We compute
\[ \frac{\partial h}{\partial \tau} + \Delta_{g(\tau)} h - \nabla_X h = \frac{\phi'(w)}{b} \left( \left( \frac{\partial}{\partial \tau} + \Delta_{g(\tau)} - \nabla_X \right) d_{g(\tau)}(x,x_0) - a \right) + \frac{\phi''}{b^2} \left| \nabla_{g(\tau)} d_{g(\tau)}(x,x_0) \right|^2_{g(\tau)}. \] (59)

First we choose
\[ a \geq \frac{10(a - 1)}{\hat{r}} + 2K_1 \hat{r} + K_1 \] (60)
where \( K_1 \doteq \sup_{B_{\hat{r}}(x_0, r_*) \times [0,T]} |X|_{g(\tau)}. \) Next we choose \( T_1 \) such that
\[ \frac{1}{10} \hat{r} + aT_1 \leq b = \frac{1}{2} \hat{r}. \] (61)

Then \( w(x, \tau) \leq 1 \) and \( \phi'(w) = 0 \) for any \( x \in B_{\hat{r}}(x_0, \frac{1}{10} \hat{r}) \) and \( \tau \in [0, T_1]. \) We have proved that either \( \phi'(w(x, \tau)) = 0 \) or
\[ \left( \frac{\partial}{\partial \tau} + \Delta_{g(\tau)} - \nabla_X \right) d_{g(\tau)}(x,x_0) - a \leq 0, \]
i.e.,
\[ \frac{\phi'(w)}{b} \left( \left( \frac{\partial}{\partial \tau} + \Delta_{g(\tau)} - \nabla_X \right) d_{g(\tau)}(x,x_0) - a \right) \geq 0 \] (62)
for \( x \in M \) and \( \tau \in [0, T_1]. \)

Combining (59, 62) and \( \left| \nabla_{g(\tau)} d_{g(\tau)}(x,x_0) \right|_{g(\tau)} = 1 \) we have proved

**Lemma 5.2** Under the assumption given at the beginning of this subsection, the function \( h \) defined in (53) with the choice of \( a, b \) and \( T_1 \) given by (60) and (61) satisfies
\[ \frac{\partial h}{\partial \tau} + \Delta_{g(\tau)} h - \nabla_X h \geq -\frac{10}{b^2} h. \]

### 5.2 Lower bound of integrals of fundamental solutions on balls.

In this subsection we use the setup described at the beginning of §3. Let \( r_* \) in §5.1 be a positive constant such that \( \overline{B}_{\hat{r}}(x_0, r_*) \) is compact subset of \( M \) and such that (10) and (50) hold. Let \( h(x, \tau) = \phi(w(x, \tau)) \) be the function defined by (53) with structure constants \( \hat{r} \) defined by (51), \( a \) satisfying (50), \( b = \frac{1}{2} \hat{r}, \) and \( T_1 \) satisfying (61). Note that the support \( \text{supp} h(\cdot, \tau) \subset B_{\hat{r}}(x_0, r_*) \) for each \( \tau \in [0, T_1]. \)

Define \( K_2 \) by
\[ K_2 \doteq \sup_{B_{\hat{r}}(x_0, r_*) \times [0,T]} \left\{ Q, \text{div}_{g(\tau)} X \right\}. \] (63)

Let \( u \) be a nonnegative solution of (3) on \( M \times (0,T] \). We compute
\[
\frac{d}{d\tau} \int_M uhd\mu_{g(\tau)} = \int_M h \left( \frac{\partial u}{\partial \tau} - \Delta_{g(\tau)} u + \nabla_X u + Qu \right) d\mu_{g(\tau)}
+ \int_M u \left( \frac{\partial h}{\partial \tau} + \Delta_{g(\tau)} h - \nabla_X h \right) d\mu_{g(\tau)}
\geq - \left( \frac{10}{b^2} + nK_* + K_2 \right) \int_M uhd\mu_{g(\tau)},
\]

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where we have used \( \frac{d}{d\tau} \ln \int_M uhd\mu_{g(\tau)} \geq - \left( \frac{10}{b^2} + nK_* + K_2 \right) \). Hence for any \( 0 < \tau_1 < \tau \leq T_1 \) we have

\[
\int_M uhd\mu_{g(\tau)} \geq e^{-\left( \frac{10}{b^2} + nK_* + K_2 \right) (\tau - \tau_1)} \int_M uhd\mu_{g(\tau_1)}. \tag{64}
\]

Suppose that \( u \) can be continuously extended to a function defined on \( M \times [0, T] \), then we have

\[
\int_M uhd\mu_{g(\tau)} \geq e^{-\left( \frac{10}{b^2} + nK_* + K_2 \right) T_1} \int_M uhd\mu_{g(0)} \quad \text{for} \quad \tau \in (0, T_1].
\]

Since \( \text{supp} \ h(\cdot, \tau) \subset B_{\tilde{g}}(x_0, r_*) \) for any \( \tau \in [0, T_1] \), using (61) we have proved the following.

**Proposition 5.3** Let \( g(\tau), \tau \in [0, T], \) be a smooth family of smooth metrics on \( M^n \) and let \( \tilde{g} \) be a smooth metric on \( M \). We assume that \( B_{\tilde{g}}(x_0, r_*) \) is compact subset of \( M \). Let \( u \) be a nonnegative solution of \( (4) \) on \( M \times [0, T] \). Then

\[
\int_{B_{\tilde{g}}(x_0, r_*)} uhd\mu_{g(\tau)} \geq e^{-\left( \frac{10}{b^2} + nK_* + K_2 \right) T_1} \int_{B_{\tilde{g}}(x_0, r_*)} uhd\mu_{g(0)} \quad \text{for} \quad \tau \in (0, T_1].
\]

Here \( b = \frac{1}{2} \tilde{r} \) is defined by (51), \( K_* \) is defined by (49) and (50), \( K_2 \) is defined by (52), and \( T_1 \) is defined by (60) and (61).

When \( H \) is a fundamental solution of \( (4) \) on \( M \times (0, T] \) centered at \( x_0 \in \text{Int}(M) \), we have

\[
\lim_{\tau_1 \to 0} \int_M H(x, \tau_1) h(x, \tau_1) d\mu_{g(\tau_1)}(x) = h(x_0, 0) = 1.
\]

Then it follows from (64) that

\[
\int_M Hdhd\mu_{g(\tau)} \geq e^{-\left( \frac{10}{b^2} + nK_* + K_2 \right) T_1} \quad \text{for} \quad \tau \in (0, T_1].
\]

Hence we have proved the following.

**Corollary 5.4** Let \( g(\tau), \tau \in [0, T], \) be a smooth family of smooth metrics on \( M^n \) and let \( \tilde{g} \) be a smooth metric on \( M \). We assume that \( B_{\tilde{g}}(x_0, r_*) \) is compact subset in \( M \). Let \( H \) be a fundamental solution of \( (4) \) on \( M \times (0, T] \) centered at \( x_0 \in \text{Int}(M) \). Then

\[
\int_{B_{\tilde{g}}(x_0, r_*)} Hhd\mu_{g(\tau)} \geq e^{-\left( \frac{10}{b^2} + nK_* + K_2 \right) T_1} \quad \text{for} \quad \tau \in (0, T_1].
\]

Here \( b = \frac{1}{2} \tilde{r} \) is defined by (51), \( K_* \) is defined by (49) and (50), \( K_2 \) is defined by (52), and \( T_1 \) is defined by (60) and (61).

**Acknowledgments**

Part of this work was done while the author was visiting Department of Mathematics, University of California at San Diego in early 2009. The author thanks Bennett Chow and Lei Ni for their invitation and hospitality and thanks Bennett Chow for the helpful discussion.
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