Joint Chance Constraints in AC Optimal Power Flow: Improving Bounds through Learning

Kyri Baker, Member, IEEE and Andrey Bernstein, Member, IEEE

Abstract—This paper considers distribution systems with a high penetration of distributed, renewable generation and addresses the problem of incorporating the associated uncertainty into the optimal operation of these networks. Joint chance constraints, which satisfy multiple constraints simultaneously with a prescribed probability, are one way to incorporate uncertainty across sets of constraints, leading to a chance-constrained optimal power flow problem. Departing from the computationally-heavy scenario-based approaches or approximations that transform the joint constraint into conservative deterministic constraints, this paper develops a scalable, data-driven approach which learns operational trends in a power network, eliminates zero-probability events (e.g., inactive constraints), and accurately and efficiently approximates bounds on the joint chance constraint iteratively. In particular, the proposed framework improves upon the classic methods based on the union bound (or Boole’s inequality) by generating a much less conservative set of single chance constraints that also guarantees the satisfaction of the original joint constraint. The proposed framework is evaluated numerically using the IEEE 37-node test feeder, focusing on the problem of voltage regulation in distribution grids.

I. INTRODUCTION

The AC optimal power flow (OPF) problem is one of the fundamental problems in power system operation and analysis; see, e.g., [1] for an overview. Chance-constrained AC OPF (CC-AC-OPF) is one way to deal with the uncertainties associated with high penetration of distributed and variable generation in the distribution level [2], [3]. A prototypical CC-AC-OPF is given by

\[
\begin{align*}
(P0) \min_{x \in X} & \quad \mathbb{E}_y f(x, y) \\
\text{subject to: } & \quad y = h(x, \xi) \\
& \quad \mathbb{P}\{y \in Y\} \geq 1 - \epsilon,
\end{align*}
\]

where \(x\) is a vector that collects all the controllable inputs to the system, typically active and reactive power injections of the controllable distributed energy resources (DERs); \(\xi\) is a random vector representing the uncertainty in the system (e.g., power injections of the uncontrollable assets and solar irradiance); \(y\) is the vector of state variables, such as voltage phasors across the buses of the network; \(\{1\}\) are the power-flow constraints; \(\{1\}\) is an operational constraint formulated as a chance constraint on the state vector \(y\); and we use the notation \(\mathbb{E}_y\) to denote the expected value with respect to the distribution of \(y\). In particular, \(\{1\}\) ensures that the state vector lies in some prescribed operational set \(Y\) with probability at least \(1 - \epsilon\) for some (small) probability \(\epsilon > 0\).

In many applications, the constraint \(y \in Y\) is composed of several individual constraints \(y \in Y_i, i = 1, \ldots, n\), that have to be satisfied simultaneously; therefore chance constraint \(\{1\}\) is of the form

\[
\mathbb{P}(\cap_{i=1}^{n}\{y \in Y_i\}) \geq 1 - \epsilon.
\]  

Examples of individual constraints that have to be satisfied simultaneously with high probability include joint constraints over different buses in the network, constraints that link timesteps (e.g., ensuring that power delivered to a sensitive resource is satisfied with high probability across the timesteps after a contingency), or even simply two-sided constraints (e.g., constraining the upper and lower limits on uncertain line flows or voltage magnitudes).

Considering simultaneous probabilistic constraints generally requires either computationally heavy sampling-based approaches which are limited by problem size [4], or assumptions about the random parameters [5]; or the use of the union bound, or Boole’s inequality [6], to create conservative upper bounds on the single chance constraints [7], [8]. In [4], a Monte Carlo method is proposed to solve a sequence of convex optimization problems, avoiding the use of Boole’s inequality, with a guarantee that the algorithm converges to a KKT point. However, it is limited by problem size to small or medium size problems with less than 100 dimensions. Scenario approaches can be used to simplify joint constraints into deterministic single constraints; however, these approaches do not offer any guarantees, can be overly conservative, and can actually perform worse as the number of samples increases [9].

Using the union bound (or, Boole’s inequality) is the most popular way to relax \(\{1\}\) that boils down to replacing it with \(n\) chance constraints

\[
\begin{align*}
\mathbb{P}\{y \in Y_i\} & \geq 1 - \epsilon_i, \quad i = 1, \ldots, n, \quad (3)
\end{align*}
\]

It is easy to see that if \(\sum_{i=1}^{n} \epsilon_i = \epsilon\), \(\{3\}\) implies \(\{2\}\); particularly, if no additional information is used regarding the individual constraints, the typical choice is \(\epsilon_i \equiv \frac{\epsilon}{n}\). However, this choice may result in highly conservative solution to \((P0)\).

To illustrate this fact, consider two constraints; \(y \in Y_1\) and \(y \in Y_2\). Suppose that the events \(A_i := \{y \notin Y_i\}\) are highly correlated, in the sense that with very high probability, whenever \(A_1\) happens, \(A_2\) happens as well (and vice versa). For example, \(A_i\) can represent a violation of voltage upper bound at bus \(i\) equipped with a photovoltaic (PV) panel, and both buses are geographically close to one another. In this case,

\[
\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2) \approx \mathbb{P}(A_1) \approx \mathbb{P}(A_2)
\]

because \(\mathbb{P}(A_1 \cap A_2) \approx \mathbb{P}(A_1) \approx \mathbb{P}(A_2)\). Therefore, the joint chance constraint \(\{2\}\) would boil down to a single constraint

\[
\begin{align*}
\mathbb{P}(\{y \in Y_1\} \cap \{y \in Y_2\}) & = 1 - \mathbb{P}(A_1 \cup A_2) \\
& \approx 1 - \mathbb{P}(A_1) \geq 1 - \epsilon,
\end{align*}
\]
or equivalently, \( \mathbb{P}(A_i) \leq \epsilon \). However, the union bound approximation (3) will impose a pair of constraints \( \mathbb{P}(A_i) \leq \frac{1}{2} \), \( i = 1, 2, \) therefore unnecessarily restricting the constraint set.

In this paper, we leverage statistical learning tools to address the problem of computationally burdensome joint chance constraints in AC OPF problems, with the following key ingredients:

- We present a framework for reducing a joint chance constraint into a series of single chance constraints in a method that drastically reduces the conservativeness compared to using Boole’s inequality [10] by using support vector classifiers to classify events \( A_i := \{ \mathbf{y} \notin \mathcal{Y}_i \} \) as having either zero or non-zero probabilities (e.g., voltage constraints are classified as active or inactive).
- An estimation method is presented which iteratively provides a tighter upper bound on the joint chance constraint and can be terminated before the estimation is finished in computationally restrictive or high dimensional settings where the entire joint constraint cannot be estimated. Unlike classic Monte-Carlo-based approaches, the proposed framework is scalable to high-dimensional constraints. Moreover, the reduction of the joint chance constraint into single chance constraints allows for the use of many of the distributionally-robust single chance constraint reformulations in the literature [11], [12]. In addition, the proposed method can also reduce computation time in non-stochastic settings by removing non-binding constraints from the deterministic optimization problem.

Simulation results are presented for the IEEE 37-node test system with a high penetration of distributed solar in an active distribution network. While the results presented here are focused on voltage regulation in distribution networks, the method proposed in this paper can be applied in general CC-AC-OPF settings for any type of joint chance constraints.

The remainder of the paper is structured as follows. Section II discusses the joint chance constraints formulation and outlines our approach. Section III presents a method to classify inactive constraints and to estimate the remaining joint constraints. Section IV outlines the distribution system model and related notation. Section V discusses the application of the proposed method to voltage regulation problem in active distribution networks. Section VI presents the numerical results. Finally, Section VII concludes the paper.

II. OUTLINE OF THE APPROACH

To explain how we will use statistical learning to reduce the complexity of the joint chance constraint in power network optimization, consider (3) and let \( A_i := \{ \mathbf{y} \notin \mathcal{Y}_i \} \). Then, \( \mathbb{P}(\bigcap_{i=1}^n \{ \mathbf{y} \in \mathcal{Y}_i \}) = 1 - \mathbb{P}(\bigcup_{i=1}^n A_i) \), and from the probabilistic version of the inclusion-exclusion principle

![Fig. 1. An outline of the general procedure for solving the joint chance constrained problem.](image-url)

III. CONSTRAINT CLASSIFICATION AND ESTIMATION

In an optimization problem, inactive constraints are those which, if removed from the problem, would not change the optimal solution. Active constraints, on the other hand, are essential in determining the optimal solution and would change the optimal solution if removed. Machine learning approaches to solve OPF problems have recently been realized as a powerful tool [13], [14]; here, we leverage machine learning for identifying active constraints in AC OPF problems with joint chance constraints. This section discusses how we can learn which constraints are likely to be inactive in power system optimization given certain system conditions, reducing the computational burden of calculating each term in (5).
A. A Simple Example - Two Sided Constraint

To illustrate the overall idea of the framework, consider the two-sided joint chance constraint which constrains the state of charge $E^{(t)}$ of an energy storage system (ESS) to be within desired bounds $\underline{E}$ and $\overline{E}$ with probability at least $1-\epsilon$:

$$\mathbb{P}(\underline{E} \leq E^{(t)} \leq \overline{E}) \geq 1 - \epsilon,$$

(6)

While maintaining ESS state of charge within certain bounds can extend the lifetime of the ESS, under certain situations it may be more beneficial or unavoidable to violate these limits. Intuitively, in certain situations it can be obvious if $\underline{E} \leq E^{(t)}$ or $E^{(t)} \leq \overline{E}$ is an inactive constraint; for example, if the ESS is currently at its maximum charge ($E^{(t)} = \overline{E}$), and the maximum discharge rate makes it impossible for the ESS to reach $\underline{E}$ in the next time step, we know with certainty that $\underline{E} < E^{(t+1)}$ and thus $\mathbb{P}(E^{(t)} > E^{(t+1)}) = 0$. So, from the inclusion-exclusion principle, $\mathbb{P}(\underline{E} \leq E^{(t)} \leq \overline{E}) = 1 - \mathbb{P}(E^{(t)} > E^{(t+1)}) + \mathbb{P}(E^{(t)} > \overline{E}) + \mathbb{P}(E^{(t)} < \underline{E}) = \mathbb{P}(E^{(t)} \leq \overline{E})$, reducing the joint chance constraint (6) to the single chance constraint $\mathbb{P}(E^{(t)} \leq \overline{E}) \geq 1 - \epsilon$. However, when dealing with multi-time step problems, assuming one of these events has a zero probability may not be trivial; it may also not be trivial depending on maximum charge/discharge rates, time in between control decisions, the level of uncertainty, or distance between $\underline{E}$ and $\overline{E}$. In addition, while we may have physical intuition as to when a constraint is likely to be relevant or not, there can be many factors influencing the outcome of an optimization problem, and we would like to have an automated way of reducing the complexity of joint chance constraints. Thus, it is desirable to develop a rule that may allow us to exploit these patterns by learning them over time and having the optimization problem automatically decompose the joint chance constraints into single chance constraints depending on the outcome of these rules.

In general, recall that if $\mathbb{P}(A_i) = 0$, $\mathbb{P}(A_i \cap A_j) = 0$ for all $A_j$; if even a single constraint is classified as inactive, a significant number of terms in the joint chance constraint expansion are eliminated from the calculations and do not have to be estimated further. As a larger example, consider a four-event union $\mathbb{P}(A_1 \cup A_2 \cup A_3 \cup A_4)$ and its expansion via (5):

$$\begin{align*}
\mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) + \mathbb{P}(A_4) \\
- \mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1 \cap A_3) - \mathbb{P}(A_1 \cap A_4) \\
- \mathbb{P}(A_2 \cap A_3) - \mathbb{P}(A_2 \cap A_4) - \mathbb{P}(A_3 \cap A_4) \\
+ \mathbb{P}(A_1 \cap A_2 \cap A_3) + \mathbb{P}(A_1 \cap A_2 \cap A_4) \\
+ \mathbb{P}(A_1 \cap A_3 \cap A_4) + \mathbb{P}(A_2 \cap A_3 \cap A_4) \\
- \mathbb{P}(A_1 \cap A_2 \cap A_3 \cap A_4)
\end{align*}$$

(7)

If constraint $A_1$ is classified as inactive, the above reduces to

$$\begin{align*}
\mathbb{P}(A_2) + \mathbb{P}(A_3) + \mathbb{P}(A_4) \\
- \mathbb{P}(A_2 \cap A_3) - \mathbb{P}(A_2 \cap A_4) - \mathbb{P}(A_3 \cap A_4) \\
+ \mathbb{P}(A_2 \cap A_3 \cap A_4)
\end{align*}$$

(8)

dramatically reducing the number of intersections we must estimate. For sizable joint chance constraints, identifying zero probability events can potentially make an otherwise intractable problem possible to solve via sampling approaches.

B. Support Vector Classification (SVC)

We will use a popular machine learning technique for classification called Support Vector Classification (SVC). With two classes of interest for classifying a constraint, namely active ($\ell_i = -1$) and inactive ($\ell_i = +1$), we wish to create a rule which uses selected inputs to determine whether or not we include that constraint in the optimization problem. Here, we seek to form an affine classifier of the form $w^T \phi + b$ with weights $w \in \mathbb{R}^n$ and bias $b$ that classifies constraints as active ($w^T \phi + b \geq 0$) or inactive ($w^T \phi + b < 0$) based on input features $\phi \in \mathbb{R}^n$ (e.g., load and available solar at a node in the distribution network as considered in the example in Section [15] below). In our formulation, called “weighted SVC”, we heavily penalize misclassifications of active constraints as inactive, while maximizing the separation between classes [15]. In the training stage, we build the classifier by using $m$ samples of labeled training data $\ell$, by solving the following optimization problem:

$$\begin{align*}
(P0) \min_{w,z,b} & \quad \frac{1}{2} w^T b + c^T z \\
\text{s.t.} & \quad \ell_i (w^T \phi_i + b) \geq 1 - z_i \\
& \quad z_i \geq 0
\end{align*}$$

(9a)

(9b)

(9c)

where $c \in \mathbb{R}^m$ is a penalty parameter and $c_i = 1$ if $\ell_i = +1$ and $c_i = a$ if $\ell_i = -1$, $a \gg 1$. Slack variables $z_i$ are nonzero if $\phi_i$ is classified incorrectly, and zero otherwise.

C. Classifying Power System Constraints

We next illustrate the classifier chosen with the traditional SVC versus the conservative weighted SVC (wSVC) by considering the example of distribution grid voltage regulation under high PV production (see Section [VII] for the detailed description of this application). Low-voltage traditional distribution networks, historically constructed assuming one-directional flows of power from distribution substation to loads, are now experiencing many operational challenges due to the increase in distributed generation (namely, increased solar production). Overvoltage conditions ($V > V_l$) are primarily caused when solar generation exceeds consumption (load), and it is reasonable to exploit this relationship to use the current and forecasted levels of solar and load to determine which voltage constraints will be relevant or binding. In Fig [2] this relationship is demonstrated; at each node in the considered distribution network, we construct an SVC/wSVC at each timestep in the optimization with features consisting of the net solar and load at that node. The training data consists from previous optimization runs, with voltages at their maximum limit, $V \geq V_l$ labeled as in the “active” class and voltages with $V < V_l$ labeled as in the “inactive” class. Traditional SVC and wSVC may not perfectly separate the two classes, but the conservative wSVC ensures that all training points labeled as active are correctly classified.

While in this paper we focus on using these classifiers for probabilistic constraints, the approach would also provide computational benefits for constraint removal from deterministic programs as well. In particular with the voltage regulation
pairs probabilities for not be preserved. We therefore only estimate intersection if certain intersections are included in the expansion but not in which these intersections are computed is very important; constraint, allowing for the termination of the algorithm before way that maintains an upper bound on the original joint chance estimation algorithm is performed). In the bottom subfigure, an improved upper bound on \( P \) data availability, and problem size allow, maintaining an upper bound on the original constraint by performing pairwise inter-

B_K := \sum_{m \in M} P(A_m)
- \sum_{k=1}^{K} \left[ \sum_{I \subseteq \{1, ..., |M|\}} |I| = k \sum_{I \subseteq \{1, ..., |M|\}} |I| = k+1 \right] P(A_I)
\]

for \( K = 1, \ldots, \left\lfloor \frac{|M|+1}{2} \right\rfloor \), where \( I \subseteq \{1, ..., |M|\} \), \(|I| = k\) denotes all subsets \( I \) of indices \( 1, ..., |M| \) which contain exactly \( k \) elements, and \( A_I := \bigcap_{i \in I} A_i \).

**Observation 1** We have that
\[
P\left( \bigcup_{i=1}^{n} A_i \right) \leq B_K \leq \sum_{m \in M} P(A_m)
\]
for all \( K \in \{1, \ldots, \left\lfloor \frac{|M|+1}{2} \right\rfloor \} \).

**Proof.** The proof follows by the inclusion-exclusion principle, the monotonicity of \( B_K \) in \( K \), and the fact that for \( K = \left\lfloor \frac{|M|+1}{2} \right\rfloor \), \( P\left( \bigcup_{i=1}^{n} A_i \right) = B_K \).

Note that Observation 1 allows us to terminate the estimation process of the joint probabilities before \( K = \left\lfloor \frac{|M|+1}{2} \right\rfloor \). A four-event example shown in Fig. 3 to illustrate this: in the top subfigure, an improved upper bound on \( P(A_1 \cup A_2 \cup A_3 \cup A_4) \) is sought by removing redundant intersections (right) as time, data availability, and problem size allow, maintaining an upper bound on the original constraint by performing pairwise intersection estimations (here, only one iteration of the intersection estimation algorithm is performed). In the bottom subfigure, the event intersection probabilities are iteratively removed in the order of \([\Pi]\), no longer maintaining an upper bound on the union of events. This provides a benefit over the convenient but extremely conservative Boole’s inequality as well as a more reliable and robust alternative to scenario-based approaches, which may require more time than available in between control actions in large networks. In the worst case (no computation...
time is allowed to estimate intersections), the algorithm is equivalent to using Boole’s inequality to create tractable single chance constraints.

In this paper, we estimate the remaining joint probabilities in (5) under conservative control policies by using a sampling approach, and represent these probabilities with their relative frequencies; i.e., if an event \(A_1 \cap A_2\) occurred 3,000 times out of 10,000, we would assign \(P(A_1 \cap A_2) = \frac{3,000}{10,000} = 0.3\). This process is discussed in more detail in Sec. [V-B]

IV. DISTRIBUTION NETWORK AND SYSTEM MODELS

Consider a distributionfeeder comprising \(N\) PQ nodes and a single slack node. Let \(V_n \in \mathbb{R}\) denote the line-to-ground voltage magnitude at node \(n\), and define the \(N\)-dimensional vector \(v := [V_1, \ldots, V_N]^T \in \mathbb{R}^N\). Constants \(P_{\ell,n}\) and \(Q_{\ell,n}\) denote the real and reactive demands at node \(n\), and we can define the vectors \(p_\ell := [P_{\ell,1}, \ldots, P_{\ell,N}]^T\) and \(q_\ell := [Q_{\ell,1}, \ldots, Q_{\ell,N}]^T\); if no load is present at node \(n\), then \(P_{\ell,n} = Q_{\ell,n} = 0\).

Here, we use a linearization of the AC power-flow equations [16], [17] which linearly relates the voltage magnitudes \(v\) to the injected real and reactive powers \(p \in \mathbb{R}^N\) and \(q \in \mathbb{R}^N\) in the form

\[
v \approx Rp + Bq + a, \tag{13}
\]

where \(R, B, a\) and \(\alpha\) are parameters that are dependent on the system model [16], [17]. While the proposed methodology does not require problem convexity, we leverage a linearization in order to provide a clear exposition of the joint chance constraint reformulation in Section [V].

**Photovoltaic (PV) Systems**

Random quantity \(P_{av,n}\) denotes the maximum renewable-based generation at node \(n\) — hereafter referred to as the available solar power. Particularly, \(P_{av,n}\) coincides with the maximum power point at the AC side of the inverter. When RESs operate at unity power factor and inject the available solar power \(P_{av,n}\), issues related to power quality and reliability in distribution systems may be encountered. For example, voltages exceeding prescribed limits at a particular node may be experienced when RES generation exceeds the load of that consumer [15].

Efforts to ensure reliable operation of existing distribution systems with increased behind-the-meter renewable generation are focus on the possibility of inverter providing reactive power compensation and/or curtailing real power. To account for the ability of the RES inverters to adjust the output of real power, let \(\alpha_n \in [0,1]\) denote the fraction of available solar power curtailed by RES-inverter \(n\). If no PV system/inverter is at a particular node, \(P_{av,i} = \alpha_i = 0\). For convenience, define the vectors \(\alpha := [\alpha_1, \ldots, \alpha_N]^T\) and \(p_{av} := [P_{av,1}, \ldots, P_{av,N}]^T\).

The available active power generation from solar is modeled as \(p_{av} = \bar{p}_{av} + \delta_{av}\), where \(\bar{p}_{av} \in \mathbb{R}^N\) is a vector of the forecasted values and \(\delta_{av} \in \mathbb{R}_{av} \subseteq \mathbb{R}^N\) is a random vector whose distribution captures spatial dependencies among forecasting errors. We assume that the distribution system operator has a certain amount of information about the probability distributions of the forecasting errors \(\delta_{av}\). This information can come in the form of either knowledge of the probability density functions, or a model of \(\delta_{av}\) from which one can draw samples. In this paper, we make the assumption that these errors are normally distributed; however, distributionally robust formulations of single chance constraints [11], [12] can easily be incorporated into the framework here.

V. JOINT CHANCE CONstrained FORMULATION

A. Optimization problem reformulation

The joint chance constraint optimization for voltage regulation in distribution systems is shown below:

\[
\begin{align}
(P1) \min_{v,\alpha} & \quad \mathbb{E}(f(v, \alpha, p_\ell, q_\ell)) \\
\text{subject to} & \\
& \quad v = R(I - \text{diag}(\alpha))^T \mathbb{E}(p_{av} - p_\ell) \\
& \quad - Bq_\ell + a \\
& \quad P\{V_i \leq V_{\max}; \ldots; V_N \leq V_{\max}\} \geq 1 - \epsilon \\
& \quad 0 \leq \alpha_i \leq 1, \ i = 1, \ldots, N; \\
\end{align}
\]

note that it is in the general form of (P0). Constraint (14b) represents a surrogate for the power balance equation; constraint (14c) is the joint chance constraint that requires voltage
magnitudes to be within $V_{\text{max}}$ with at least $1 - \epsilon$ probability; and constraint (14d) limits the curtailment percentage from 0–100%. The cost function $f(v, \alpha, p_c, q_c)$ is convex and can consider a sum of penalties on curtailment, penalties on power drawn from the substation, penalties on voltage violations, etc.

By reformulating the joint constraint (14c) as a series of single chance constraints, we can write

$$\mathbb{P}(V_i \leq V_{\text{max}}) \geq 1 - \epsilon_i$$

(15)

for all $i \in 1, \ldots, N$, and each $\epsilon_i$ is chosen such that $\sum_{i=1}^N \epsilon_i = \epsilon$. In the case of Boole’s inequality, we choose $\epsilon_i = \frac{\epsilon}{N}$; in the case of what we call the improved Boole’s inequality, using the method proposed in this paper, we choose $\epsilon_i = \frac{\epsilon}{N} + \frac{P_c}{N}$, where $P_c$ is our estimation of the non-zero probabilities in (4).

If $P_c > 0$ (all events are not mutually exclusive), it is clear that the improved Boole’s inequality provides a less conservative upper bound on the chance constraints.

B. Estimating Event Intersections

We use a conservative relative frequency sampling approach to estimate the event intersections that have not been classified as zero. If event $A_i$ represents an overvoltage at node $i$, and $I$ is a subset of nodes, we can estimate the probability of the intersection of overvoltages at the nodes included in $I$ as

$$\mathbb{P}\left(\bigcap_{i \in I} A_i\right) \approx \frac{\sum_{s=1}^{N_s} \{v_I(\delta_s) > V_{\text{max}}\}}{N_s}$$

(16)

for $N_s$ random draws of the uncertainty distribution, where draw $s$ is denoted $\delta_s$. Vector $v_I$ contains the voltage magnitudes at each of the nodes in $I$, and $\{v_I(\delta_s) > V_{\text{max}}\}$ is one if all of the elements in $v_I(\delta_s)$ are greater than $V_{\text{max}}$ and zero otherwise. To represent the most conservative case, for each sample $\delta_s$, the voltage vector $v(\delta_s) = R(I(\bar{p}_{av} + \delta_s - p_I)) - B q_c + a$; i.e., the curtailment variables $\alpha$ are chosen to be zero to represent no curtailment and thus the most conservative case for the control policy. The impact of different sample sizes $N_s$ and computational burden of the estimation process is discussed in further detail in the next section.

C. Analytical Reformulation of Single Chance Constraints

The single chance constraints can be reformulated as exact, tractable constraints $[19]$, assuming $\epsilon \leq 0.5$. Assuming the joint distribution of the random variables is a multivariate Gaussian with mean $\mu$ and covariance matrix $\Sigma$, define $\mu_i$ as the $i$-th value in $\mu$ and $\sigma_i$ as the $(i, i)$-th entry in $\Sigma$. Then, define the following function at each node $i$:

$$h(p_{av,i}) = \sum_j (R_{ij}((1 - \alpha_j)p_{av,j} - p_{\epsilon,j}))$$

$$- \sum_j (B_{ij}q_i,j) + a_i - V_{\text{max}}$$

(17)

where $R_{ij}$ is the $(i, j)$-th entry of $R$, $B_{ij}$ is the $(i, j)$-th entry of $B$, and $a_i$ is the $i$th element of $a$. Then $h(p_{av,i})$ is also normally distributed with the following mean $\mu'_i$ and variance $\sigma'_i$:

$$\mu'_i = \sum_j (R_{ij}((1 - \alpha_j)p_{\epsilon,j}) - \sum_j (B_{ij}q_i,j) + a_i - V_{\text{max}}$$

$$\sigma'_i = \sum_j R_{ij}(1 - \alpha_j)\sigma_j$$

Thus, the constraints (15) can be reformulated using the Gaussian cumulative distribution function (CDF) $\Phi$:

$$\mathbb{P}\{h(p_{av,i}) \leq 0\} = \Phi\left(\frac{0 - \mu'_i}{\sigma'_i}\right) \geq 1 - \epsilon_i$$

With the final analytical constraint written using the quantile function (the inverse of the Gaussian CDF):

$$R_i((1 - \alpha_i)p_{\epsilon,i} - B_iq_i,i) + a_i - V_{\text{max}}$$

$$\leq -R_i\alpha_i\sigma_i\Phi^{-1}(1 - \epsilon_i)$$

(18)

Which can be explicitly included into problem (P1) for each $i$ in place of the joint constraint (14c).

Remark 1 In these results, the individual solar forecasting errors are modeled as Gaussian. Because of this, the single chance constraints can be exactly analytically reformulated. Without loss of generality, other distributionally robust methods for single chance constraints can also be used here [12], [20], but as the contribution of this paper is in the decomposition of the joint chance constraint, not in addressing the tractability of single chance constraints, we have kept the marginal distributions Gaussian for simplicity of exposition. The method proposed in this paper is not distribution-specific.

Remark 2 Note that the original use of Boole’s inequality ensures the satisfaction of the original constraint by choosing $\epsilon_i$ such that $\sum_{i=1}^n \epsilon_i = \epsilon$; however, without optimizing this parameter, suboptimal performance of this reformulation is possible [21]. We leave the optimal choice of $\epsilon_i$ as a direction for future work.

VI. Numerical Results

The IEEE-37 node test feeder [22] was used for the following simulations. Five-minute load and solar irradiance data from weekdays in August 2012 was obtained from [23] for the simulations, and in order to emulate a situation with high-PV penetration and risks of overvoltage, 8 200-kW rated PV systems were placed at nodes 29-36. The considered cost function seeks to minimize renewable curtailment; specifically,

$$f(v, \alpha, p_c, q_c) = \sum_{i \in N} d_i\alpha_i^2$$

(18)

where the cost of curtailing power at each node is set to be $d_i = 80.10$. The number of samples used to calculate each intersection was $N_s = 10,000$. The considered joint chance
constraint considers maintaining voltages at nodes 29–36. Each \( \mu_i, i = 1...N \) was chosen to be the power generated from the forecasted PV at that node, based on the shape of the aggregate solar irradiance from [23] and shifted using samples from a uniform distribution from \(+/-1 \) kW across each node. The covariance matrix \( \Sigma \) was formed by setting each entry \((i,j)\) to \( \Sigma_{ij} = \mathbb{E}[(P_{av,i} - \mu_i)(P_{av,j} - \mu_j)^T] \). Three cases were considered in the following numerical results. First, a deterministic case was considered, which uses a certainty equivalence formulation and uses the mean of each of the uncertain parameters in place of each random variable in the optimization problem. Second, Boole’s inequality was used to separate the joint chance constraint into a series of conservative single constraints, each with \( \sum_{i=1}^{N}c_i = \epsilon \). Third, an Improved Boole’s inequality is considered, where the proposed methodology is implemented to approximate each of the intersections in (11), and \( \epsilon_i = \frac{c_i}{N} + \frac{P_{\mu_i}}{N} \).

### A. Training, Testing, and Choosing the Number of Samples

Each of the classifiers (one per constraint; 8 classifiers total) were trained using 1152 samples (4 training days), and tested using 864 samples (3 testing days), using \( c_i = 1 \) when \( \ell_i = +1 \) and \( c_i = 10 \) when \( \ell_i = -1 \). The overall classification error was 0.19% for false classification of binding events and 4.73% for false classification of non-binding events. A larger classification error for the \( \ell_i = +1 \) is to be expected from our formulation in (P0); it is more detrimental to the performance of the algorithm if we exclude constraints by mistake rather than include non-binding constraints unnecessarily.

In Fig. 4, the number of active constraints (out of 8 total) is indicated with red dots for each time instance. The total computation time (s) required for calculating the corresponding intersections is shown in blue (for \( N_s = 1,000 \) samples) and purple (for \( N_s = 10,000 \) samples). Computational time can be reduced by potentially sacrificing accuracy of estimating event intersections; in Fig. 5 the value of increasing the number of samples wanes around \( N_s = 1,000 \). Thus, in the following simulations, the conservative choice of \( N_s = 10,000 \) was made to estimate each event intersection; however, in a general setting this number is dependent on the underlying distribution.

### B. Voltage Regulation Results

In the following results, the maximum joint constraint violation probability was set to \( \epsilon = 0.02 \) for the Boole’s and Improved Boole’s cases, and all of the terms in (11) were estimated. In Fig. 6 the maximum voltage magnitudes from the resulting control policies are shown for each of the three cases. The deterministic case does not take forecast uncertainty into account, and as a result, the voltages are pushed to the maximum voltage of 1.05 pu. The Boole’s case curtails enough solar generation to ensure that overvoltages will not occur with a high probability; the Improved Boole’s case reduces this probability and results in less curtailment.

A Monte Carlo validation procedure was implemented to demonstrate the behavior of the control policies for \( N_m = 10,000 \) random draws of the uncertainty distributions at each timestep. In Fig. 7 these resulting probabilities are shown. The deterministic case, which only considers the mean of the random variables, violates the desired chance constraint bound of 0.02 when compared with the chance constrained methods, because that method offers no guarantee that the voltages will be within limits. Boole’s method is generally more conservative than the Improved Boole’s method and results in lower violation probabilities, with both methods resulting in satisfaction of the original joint chance constraint.
Improved Boole’s solutions ensure that the constraint is satisfied with probability $\geq 98\%$, while the stochastic times of high solar irradiance, the deterministic control policy does not allocate the estimated intersection probabilities constraint violation probability than a deterministic certainty to result in a lower cost than Boole’s inequality and lower PV penetration, and the proposed method was demonstrated addressed voltage regulation in distribution networks with high by Boole’s inequality. Simulation results were shown which resulted in a slightly higher violation probability than the highest probability of voltage violations); the Boole’s case is overly conservative, resulting in a higher level of curtailment than that provided an iterative approach appropriate for fast timescale chance constraints via sampling methods. In addition, we can increase the computational efficiency of computing joint uncertainty when identifying and estimating the underlying probability distributions.

Table I shows the total objective function value and voltage violation probability across the three-day test period for the deterministic, Boole’s, and Improved Boole’s cases. As expected, the deterministic case results in the lowest cost (but highest probability of voltage violations); the Boole’s case is overly conservative, resulting in a higher level of curtailment and thus cost, but lowest probability of voltage violations. The Improved Boole’s case strikes a balance between the two, resulting in a slightly higher violation probability than the original Boole’s case but with a lowered objective value.

### VII. Conclusion

In this paper, we demonstrated how identifying zero-probability events using conservative support vector classifiers can increase the computational efficiency of computing joint chance constraints via sampling methods. In addition, we provided an iterative approach appropriate for fast timescale optimization, ensuring that if the entire joint chance constraint cannot be computed, that the resulting approximation of the constraint always provides an upper bound of the original constraint at every iteration which is tighter than that provided by Boole’s inequality. Simulation results were shown which addressed voltage regulation in distribution networks with high PV penetration, and the proposed method was demonstrated to result in a lower cost than Boole’s inequality and lower constraint violation probability than a deterministic certainty equivalence formulation.

Future work will address the question of how to optimally allocate the estimated intersection probabilities $P_c$ to the individual chance constraints (rather than allocating them equally across single constraints as in this paper), determining how many samples are adequate for estimating event intersections, and identifying which statistical learning techniques are best suited for identifying active constraints in power systems optimization problems. In addition, an important question for future work is how to incorporate sampling error and uncertainty when identifying and estimating the underlying probability distributions.

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