CATEGORIFICATIONS OF $QSym$ USING SUPERCHARACTER THEORIES AND A NEW BASIS FOR $NSym_{\mathbb{C}(q,t)}$

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Abstract. Let us fix a positive integer $\nu > 1$. For each positive integer $n > 1$, we consider a normal supercharacter theory $S_n$ of $G_n$, where $G_n$ is the direct-product of $n-1$ copies of the cyclic group of order $\nu$. Then we endow $\bigoplus_{n \geq 0} \text{scf}(S_n)$, the direct-product of supercharacter function spaces, with the Hopf algebra structure that is isomorphic to the Hopf algebra $QSym$ of quasisymmetric functions. Furthermore, we compute the structure constants of the Hopf algebra thus obtained for the basis consisting of superclass identifier functions. Using our categorifications, we study a new basis for the Hopf algebra $NSym_{\mathbb{C}(q,t)}$ of noncommutative symmetric functions over the rational function field $\mathbb{C}(q,t)$ in commuting variables $q$ and $t$, with an emphasis on the structure constants of $NSym_{\mathbb{C}(q,t)}$ for this basis. Some interesting applications are also obtained via the specializations of $q$ and $t$.

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1. Introduction

Categorifying an algebraic object allows us to look at the hidden side of the object. Through this process, we can derive information that is difficult to see from the object itself. Perhaps the most foundational example is the categorification of the Hopf algebra \( \text{Sym} \) of symmetric functions via the tower of the symmetric group algebras \( \mathbb{C}S_n \), where \( n \) ranges over the set of nonnegative integers. More precisely, equipped with induction product and restriction, the Grothendieck group associated with \( \bigoplus_{n \geq 0} \mathbb{C}S_n \) turns out to be isomorphic to \( \text{Sym} \) as Hopf algebras ([10, 18]). In fact, this categorification has a tremendous influence on the theory of symmetric functions (see [15] and [9]).

In this paper, we focus on the Hopf algebra \( \text{QSym} \) of quasisymmetric functions. In [8], Duchamp, Krob, Leclerc, and Thibon categorified \( \text{QSym} \) via the tower of the 0-Hecke algebras \( H_n(0) \). Similar to the above, equipped with induction product and restriction, the Grothendieck group associated with \( \bigoplus_{n \geq 0} H_n(0) \) turns out to be isomorphic to \( \text{QSym} \) as Hopf algebras. However, since the 0-Hecke algebras are not group algebras, one can naturally ask whether \( \text{QSym} \) can be categorified via the tower of certain group algebras as in \( \text{Sym} \). Here we give a positive answer to this question. In more detail, we successfully categorify \( \text{QSym} \) via supercharacter theories of finite abelian groups. The results thus obtained will be used in studying a new basis for the Hopf algebra \( \text{NSym} \) of noncommutative symmetric functions.

Let us briefly review supercharacter theories of finite groups. Classifying the conjugacy classes and the irreducible characters of the unipotent upper triangular matrix group \( UT_n(q) \) over a finite field \( \mathbb{F}_q \), where \( n \) ranges over the set of nonnegative integers, has been regarded as a difficult problem. Supercharacter theory was initiated by André([6]) and Yan([17]) as a tractable method for studying the representation theory of \( UT_n(q) \). Inspired by André’s and Yan’s work, Diaconis and Isaacs [7] extended the concept of supercharacter theories to arbitrary finite groups in an axiomatic method.

Given a finite group \( G \), a supercharacter theory \( S \) of \( G \) is defined as a pair of a set partition of \{conjugacy classes of \( G \)\} and a set partition of \{irreducible characters of \( G \)\} that satisfies appropriate axioms (Definition 2.1). The following table illustrates key analogies between the character theory and a supercharacter theory well (for more information, see Subsection 2.1).

| The character theory of \( G \)          | A supercharacter theory of \( G \) |
|----------------------------------------|----------------------------------|
| conjugacy classes                      | superclasses                     |
| irreducible characters                 | supercharacters                  |
| class identifier functions             | superclass identifier functions   |
| class functions                        | supercharacter functions         |

Supercharacter theories have been used extensively in the problems of categorifying combinatorial Hopf algebras. In 2013, using André’s supercharacter theory of \( UT_n(q) \),
Aguiar et al. [1] succeeded in categorifying the Hopf algebra NCSym of symmetric functions in noncommuting variables. Very recently, using normal supercharacter theories, Aliniaeifard and Thiem succeeded in categorifying the Hopf algebra FQSym of free quasisymmetric functions in [4] and the Hopf algebra NSym of noncommutative symmetric functions in [5]. Since FQSym, QSym, NSym, and Sym are the four most classical and important combinatorial Hopf algebras, it would be very natural to deal with QSym as a follow-up study of the above works.

The first objective of the present paper is to categorify QSym using normal supercharacter theories of finite abelian groups. Our setup is as follows. Let $n$ and $\nu$ be any positive integers $> 1$, $C_\nu$ the additive cyclic group of order $\nu$, and $G$ the direct product of $n - 1$ copies of $C_\nu$. Then we let

$$\mathcal{N}_n(\nu) := \{ Q_I(\nu) \mid I \subseteq [n - 1] \},$$

where $Q_I(\nu)$ is the subgroup of $G$ whose $j$th component is $C_\nu$ if $j \in I$ and $\{0\}$ otherwise. According to Aliniaeifard’s result [2, Theorem 3.4], it gives rise to a normal supercharacter theory $S(\mathcal{N}_n(\nu))$. Denote by $\{ \chi^I(\nu) \mid I \subseteq S \}$ and $\{ \kappa_I(\nu) \mid I \subseteq S \}$ the set of supercharacters and the set of superclass identifier functions of $S(\mathcal{N}_n(\nu))$, respectively. For full information, see Subsection 3.1. It can be easily checked that the supercharacter function space of $S(\mathcal{N}_n(\nu))$ has the same dimension as $\text{QSym}_n$, the $n$th homogeneous component of QSym.

We elaborately define a product $\mathbf{m}$ and a coproduct $\mathbf{\triangle}$ on the $\mathbb{C}$-vector space

$$\bigoplus_{n \geq 0} \text{scf}(S(\mathcal{N}_n(\nu)))$$

as compositions of certain linear operators. Here $\text{scf}(S(\mathcal{N}_n(\nu)))$ is the supercharacter function of $\mathcal{N}_n(\nu)$ and $\text{scf}(S(\mathcal{N}_n(\nu))) \cong \mathbb{C}$ for $n = 0, 1$. Equipped with these operations, it is proven that $\bigoplus_{n \geq 0} \text{scf}(\mathcal{N}_n(\nu))$ has a Hopf algebra structure and the characteristic map

$$\text{ch}_\nu : \bigoplus_{n \geq 0} \text{scf}(\mathcal{N}_n(\nu)) \to \text{QSym}, \quad \chi^I(\nu) \mapsto L_{\text{comp}(I)} \quad (I \subseteq [n - 1])$$

is an isomorphism of Hopf algebras, where $\chi^I(\nu) = \chi^I(\nu)/\chi^I(\nu)(0)$, $\text{comp}(I)$ is the composition of $n$ corresponding to $I$, and $L_{\text{comp}(I)}$ is the fundamental quasisymmetric function attached to $\text{comp}(I)$ (Theorem 3.13).

The second objective of the present paper is to study the basis $\{ B(q, t)_{\alpha} \mid \alpha \in \text{Comp} \}$ for the Hopf algebra $\text{NSym}_{\mathbb{C}(q, t)}$ of noncommutative symmetric function over $\mathbb{C}(q, t)$ ($=$ the rational function field in commuting variables $q$ and $t$), where

$$B(q, t)_{\text{comp}(I)} := \sum_{J : I \cup J = [n - 1]} q^{|I \setminus J|} t^{|I \cap J|} H_{\text{comp}(J)} \in \text{NSym}_{\mathbb{C}(q, t)}$$

for $I \subseteq [n - 1]$ and $H_{\text{comp}(J)}$ is the complete homogeneous noncommutative symmetric function attached to $\text{comp}(J)$. This basis has some noteworthy properties. For example, if
q and t are specialized suitably, it interpolates some well known bases for $\text{NSym}$ as follows:

$$\mathcal{B}(1,0)_\alpha = H_{\alpha^c}, \quad \mathcal{B}(-1,1)_\alpha = \Lambda_{\alpha^c}, \quad \mathcal{B}(1,-1)_\alpha = \mathcal{E}^*_\alpha,$$

where $H_\alpha$, $\Lambda_\alpha$, and $\mathcal{E}^*_\alpha$ are the complete homogeneous, the elementary, the dual essential noncommutative symmetric function in $\text{NSym}$, respectively, and $\alpha^c$ denotes the complement of $\alpha$ (see 4.1). The main result here is the calculation of the structure constants of $\text{NSym}_{\mathbb{C}(q,t)}$ for this basis. In other words, we provide explicit expansions of $\mathcal{B}(q,t)_\alpha \mathcal{B}(q,t)_\beta$ and $\Delta \mathcal{B}(q,t)_\alpha$ in this basis. Let us briefly sketch the calculation process. We first derive formulas on the structure constants of $\text{NSym}$. For instance, for each $I \subseteq [n-1]$, we let

$$\Pi(\nu)_{\text{comp}(I)} := \text{ch}_\nu \left( \frac{\kappa_I(\nu)}{(\nu-1)!} \right)$$

and consider the basis $\{\Pi(\nu)_\alpha \mid \alpha \in \text{Comp}\}$ for $\text{QSym}$. Third, via the duality of $\text{QSym}$ and $\text{NSym}$, we obtain formulas on the structure constants of $\text{NSym}$ for the dual basis $\{\Pi(\nu)^*_\alpha \mid \alpha \in \text{Comp}\}$ of $\{\Pi(\nu)_\alpha \mid \alpha \in \text{Comp}\}$ (Lemma 4.6). And we show that

$$\Pi(\nu)^*_\alpha = \mathcal{B}(-\nu,\nu-1)_\alpha$$

for every $\alpha \in \text{Comp}$ and $\nu > 1$. Finally, putting these together, we derive formulas on the structure constants of $\text{NSym}_{\mathbb{C}(q,t)}$ for the basis $\{\mathcal{B}(q,t)_1^n \mid n \geq 0\}$, a generating set of $\text{NSym}_{\mathbb{C}(q,t)}$ (Theorem 4.9).

As applications, we obtain generating sets for $\text{NSym}$ and $\text{Sym}$ from the description of the structure constants associated with the product. For instance,

$$\left\{ \sum_{\lambda: \text{partitions of } n} a^{n-\ell(\lambda)} b^{\ell(\lambda)-1} C_\lambda h_\lambda \mid n \geq 0 \right\}$$

is a generating set of $\text{Sym}$ for each $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$, where $C_\lambda = \ell(\lambda)! / \prod_i m_i(\lambda)!$ for $\lambda = 1^{m_1(\lambda)} 2^{m_2(\lambda)} \cdots$ and $h_\lambda$ is the complete homogeneous symmetric function attached to $\lambda$ (Corollary 4.11). Also, using the description of the structure constants associated with the coproduct, we redescribe the number of overlapping shuffles of two compositions $\alpha$ and $\beta$ with weight $\gamma$ (Corollary 4.14).

As seen above, our categorifications provide many interesting applications related to quasisymmetric functions and noncommutative symmetric functions. We expect our approach to be useful in studying the structure constants of various combinatorial Hopf algebras.

The paper is organized as follows. In Section 2, we briefly review normal supercharacter theories and the Hopf algebras $\text{QSym}$, $\text{NSym}$, and $\text{FQSym}$. In Section 3, for each integer $\nu > 1$, we consider the normal supercharacter theories $\mathcal{N}_n(\nu)$ for $n \geq 0$ and then endow the
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The $\mathbb{C}$-space $\bigoplus_{n \geq 0} \text{scf}(S, \mathcal{N}_n(\nu))$ with the Hopf algebra structure that is isomorphic to $QSym$. Furthermore, we compute the structure constants of the Hopf algebra thus obtained for the basis consisting of superclass identifier functions. Section 4 is devoted to the study of a new basis $\{B(q, t)_\alpha\}$ for $\text{NSym}_{\mathbb{C}(q, t)}$, with an emphasis on the structure constants of $\text{NSym}_{\mathbb{C}(q, t)}$ for this basis. Some interesting applications are also obtained via the specializations of $q$ and $t$.

2. Preliminaries

Given any integers $m$ and $n$, define $[m, n]$ to be the interval $\{t \in \mathbb{Z} \mid m \leq t \leq n\}$ whenever $m \leq n$ and the empty set $\emptyset$ else. For simplicity, we set $[n] := [1, n]$ and therefore $[n] = \emptyset$ if $n < 1$. Unless otherwise stated, $n$ will denote a nonnegative integer throughout this paper.

2.1. Supercharacter theories of a finite group. In this subsection, $G$ denotes a finite group. A supercharacter theory of $G$ is a variation of the character theory of $G$. Given a set partition $\mathcal{K}$ of $G$, let $f(G; \mathcal{K})$ be the $\mathbb{C}$-vector space consisting of functions which are constant on the blocks in $\mathcal{K}$, that is,

$$f(G; \mathcal{K}) := \{\psi \mid G \to \mathbb{C} \mid \psi(g) = \psi(k) \text{ if } g \text{ and } h \text{ are in the same block in } \mathcal{K}\}.$$ 

And, we let $\text{Irr}(G)$ be the set of irreducible characters of $G$.

Definition 2.1. ([7]) A supercharacter theory $\mathcal{S}$ of $G$ is a pair $(\text{cl}(\mathcal{S}), \text{ch}(\mathcal{S}))$, where $\text{cl}(\mathcal{S})$ is a set partition of $G$ and $\text{ch}(\mathcal{S})$ is a set partition of $\text{Irr}(G)$, such that

C1. $\{e\} \in \text{cl}(\mathcal{S})$,
C2. $|\text{cl}(\mathcal{S})| = |\text{ch}(\mathcal{S})|$, 
C3. For each block $X$ in $\text{ch}(\mathcal{S})$,

$$\chi^X := \sum_{\psi \in X} \psi(e) \psi \in f(G; \text{cl}(\mathcal{S})),$$

where $e$ denotes the identity of $G$.

Each block in $\text{cl}(\mathcal{S})$ is called a superclass of $\text{cl}(\mathcal{S})$ and, for each block $X$ of $\text{ch}(\mathcal{S})$, $\chi^X$ is called a supercharacter of $\mathcal{S}$. And, $f(G; \text{cl}(\mathcal{S}))$ is called the supercharacter function space of $\mathcal{S}$, denoted by $\text{scf}(\mathcal{S})$.

Perhaps the most familiar examples of supercharacter theories are

- $\langle\{\text{conjugacy classes of } G\}, \{\psi\mid \psi \in \text{Irr}(G)\}\rangle$ and
- $\langle\{e\}, G \setminus \{e\}, \{\{1\}, \text{Irr}(G) \setminus \{1\}\}\rangle$,

where $1$ is the trivial character of $G$.

For each $I \in \text{cl}(\mathcal{S})$, consider the function $\kappa_I \in f(G; \text{cl}(\mathcal{S}))$ defined by

$$\kappa_I(g) = \begin{cases} 1 & \text{if } g \in I, \\ 0 & \text{otherwise}, \end{cases}$$
which is called the \textit{superclass identifier function} attached to \( I \). Combining the orthogonality of irreducible characters with the conditions \( \text{C2} \) and \( \text{C3} \), one can easily see that 
\[ \{ \kappa_I \mid I \in \text{cl}(S) \} \] 
and 
\[ \{ \chi^X \mid X \in \text{ch}(S) \} \]
are \( \mathbb{C} \)-bases of \( \text{scf}(S) \).

Supercharacter theories can be generated in many ways. Here we introduce the normal supercharacter theory introduced by Aliniaeifard \cite{2}. Let \( \ker(G) := \{ N \unlhd G \} \) be the lattice of normal subgroups of \( G \) ordered by inclusion. For \( M, N \in \ker(G) \), the meet and join of \( M \) and \( N \) are given by 
\[ M \lor N = MN, \quad M \land N = M \cap N. \]

Define a \textit{sublattice} \( \mathcal{N} \) of \( \ker(G) \) by a subset of \( \ker(G) \) such that
1. \( \{ e \}, G \in \mathcal{N} \)
2. \( \mathcal{N} \) is closed under meet and join operations.

Obviously every sublattice also forms a lattice under \( \lor, \land \). Given \( N \in \mathcal{N} \), let 
\[ C(L) := \{ O \in \mathcal{N} \mid O \text{ covers } L \} \]
and 
\[ N_\circ := \{ g \in N \mid g \notin M \text{ for all } M \in \mathcal{N} \text{ with } N \in C(M) \} \]
\[ X^{N^\bullet} := \{ \psi \in \text{Irr}(G) \mid N \subseteq \ker(\psi), \text{ but } O \notin \ker(\psi) \text{ for all } O \in C(N) \}. \]

With this notation, the following theorem is proved in \cite{2}.

\textbf{Theorem 2.2.} (\cite[Theorem 3.4]{2}) \textit{Given a sublattice} \( \mathcal{N} \) \textit{of} \( \ker(G) \), \textit{let}
\[ \text{cl}(S(\mathcal{N})) := \{ N_\circ \mid N \in \mathcal{N} \text{ and } N_\circ \neq \emptyset \} \]
\[ \text{ch}(S(\mathcal{N})) := \{ X^{N^\bullet} \mid N \in \mathcal{N} \text{ and } X^{N^\bullet} \neq \emptyset \}. \]

\textit{Then} \( (\text{cl}(S(\mathcal{N})), \text{ch}(S(\mathcal{N}))) \) \textit{defines a supercharacter theory} \( S(\mathcal{N}) \) \textit{of} \( G \).

For each sublattice \( \mathcal{N} \subseteq \ker(G) \), the supercharacter theory \( S(\mathcal{N}) \) is called a \textit{normal supercharacter theory} of \( G \). If there is no danger of confusion, we simply write \( \mathcal{N} \) for \( S(\mathcal{N}) \). Using normal supercharacter theories, Aliniaeifard and Thiem \cite{4, 5} successfully categorify the Hopf algebra \( \text{FQSym} \) and the Hopf algebra \( \text{NSym} \). In Section 3, we present a categorification of the Hopf algebra \( \text{QSym} \) using normal supercharacter theories of certain finite abelian groups.

\textbf{2.2. The Hopf algebras in our consideration.} We start by recalling the definitions and properties of Hopf algebras. All of these are borrowed unchanged from \cite{13}.

A Hopf algebra is a bialgebra \( H \) over \( \mathbb{C} \) together with a \( \mathbb{C} \)-linear map \( S : H \to H \), called the antipode, which satisfy certain compatible relations. If \( A = \bigoplus_{n \geq 0} A_n \) is a \( \mathbb{Z}_{\geq 0} \)-graded \( \mathbb{C} \)-algebra with \( A_0 \cong \mathbb{C} \), then \( A \) is said to be \textit{connected}. It is well known that every connected graded bialgebra \( H \) has a unique antipode \( S \) endowing it with a Hopf structure (for instance, see \cite[Proposition 1.4.14]{13}).
In the present paper, we deal with Hopf algebras $\text{QSym}$, $\text{NSym}$, and $\text{FQSym}$, which were firstly introduced in [12], [11], and [16], respectively. Since all of them are connected $\mathbb{Z}_{\geq 0}$-graded bialgebras, we usually do not mention their antipodes unless otherwise stated. Before introducing these algebras, let us collect the necessary basic definitions and notation.

A composition is a finite tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ of positive integers. Its length is defined to be $l$ and denoted by $\ell(\alpha)$ and its size is defined to be $\alpha_1 + \alpha_2 + \cdots + \alpha_l$ and denoted by $|\alpha|$. Denote the set of composition of $n$ by $\text{Comp}_n$. A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is a composition with additional condition that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$. Denote the set of partition of $n$ by $\text{Par}_n$. A composition $\alpha$ of $n$ determines a partition by rearranging in order of $|\alpha_i|$. Denote it by $\lambda(\alpha)$.

For each positive integer $n$, there is a 1-1 correspondence between $\text{Comp}_n$ and subsets of $[n-1]$ given by

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l) \mapsto \text{set}(\alpha) := \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{l-1}\}$$

$$S = \{s_1 < s_2 < \cdots < s_i\} \mapsto \text{comp}(S) := (s_1, s_2 - s_1, \ldots, s_i - s_{i-1}, n - s_i).$$

2.2.1. The Hopf algebra of quasisymmetric functions. Let $x = (x_1, x_2, \ldots)$ be the infinite totally ordered set of commuting variables, and let $\mathbb{C}[[x_1, x_2, \ldots]]$ be the algebra of formal power series of bounded degree. For each composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$, define the monomial quasisymmetric function to be

$$M_\alpha = \sum_{j_1 < j_2 < \cdots < j_l} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_l^{\alpha_l}.$$ 

The algebra $\text{QSym}$ of quasisymmetric functions over $\mathbb{C}$ is defined by

$$\text{QSym} := \bigoplus_{n \geq 0} \text{QSym}_n \subseteq \mathbb{C}[[x_1, x_2, \ldots]],$$

where $\text{QSym}_n := \text{span}_\mathbb{C}\{M_\alpha | \alpha \in \text{Comp}_n\}$.

For $\alpha, \beta \in \text{Comp}_n$, we say that $\alpha$ refines $\beta$ if one can obtain $\beta$ from $\alpha$ by combining some of its adjacent parts. Alternatively, this means that $\text{set}(\beta) \subseteq \text{set}(\alpha)$. Denote this by $\alpha \preceq \beta$. With this ordering, we have two other bases $\{F_\alpha\}$ and $\{E_\alpha\}$, where

$$F_\alpha = \sum_{\beta \succeq \alpha} M_\beta \quad (\text{the fundamental quasisymmetric function})$$

$$E_\alpha = \sum_{\alpha \succeq \beta} M_\beta \quad (\text{the essential quasisymmetric function}).$$

The latter basis was investigated intensively by Hoffman in [14].

The algebra $\text{QSym}$ has a natural coproduct structure. In particular, in the basis of monomial quasisymmetric functions, the coproduct formula can be expressed in the following...
form:
\[ \Delta M_\alpha = \sum_{\beta \cdot \gamma = \alpha} M_\beta \otimes M_\alpha, \]
where \( \beta \cdot \gamma \) is the concatenation of compositions \( \beta \) and \( \gamma \). For reference, in this case, the antipode is given as follows:
\[ S(M_\alpha) = (-1)^{\ell(\alpha)} \sum_{\gamma \geq \alpha^r} M_\gamma, \]
where \( \alpha^r = (\alpha_l, \alpha_{l-1}, \ldots, \alpha_1) \) is the reverse composition of \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l) \).

2.2.2. The Hopf algebra of noncommutative symmetric functions. The Hopf algebra \( \text{NSym} \) of noncommutative symmetric functions over \( \mathbb{C} \) is defined by the graded dual of \( \text{QSym} \), i.e., \( \text{NSym} = \text{QSym}^\circ \). Let \( (\cdot, \cdot) : \text{NSym} \otimes \text{QSym} \to \mathbb{C} \) be the dual pairing and \( H_\alpha \) be the dual basis of \( M_\alpha \) so that \( (H_\alpha, M_\beta) = \delta_{\alpha, \beta} \). Letting \( H_n := H_{(n)} \) for \( n = 1, 2, \ldots, \) one can see that \( \text{NSym} \simeq \mathbb{C}\langle H_1, H_2, \ldots \rangle \) is a free associative algebra on generators \( \{H_1, H_2, \ldots \} \) with the coproduct determined by
\[ \Delta H_n = \sum_{i+j=n} H_i \otimes H_j \]
(see [13, Thm5.4.2]).

The noncommutative symmetric function \( H_\alpha \) \( (\alpha \in \text{Comp}) \) are called the noncommutative complete homogeneous symmetric functions. There are many well known bases for \( \text{NSym} \) other than \( \{H_\alpha\} \). We are particularly interested in the bases \( \{\Lambda_\alpha\}, \{R_\alpha\}, \{E_\alpha^*\} \), where
\[ \Lambda_\alpha := \sum_{\beta \leq \alpha} (-1)^{n-\ell(\beta)} H_\beta \quad \text{(the noncommutative elementary symmetric function)} \]
\[ R_\alpha := \sum_{\alpha \leq \beta} (-1)^{\ell(\beta) - \ell(\alpha)} H_\beta \quad \text{(the noncommutative ribbon Schur function)} \]
\[ E_\alpha^* := \sum_{\beta \leq \alpha} (-1)^{\ell(\beta) - \ell(\alpha)} H_\beta \quad \text{(the dual essential noncommutative symmetric function)} \]
In fact, \( \{R_\alpha\} \) is the dual basis of \( \{F_\alpha\} \) and \( \{E_\alpha^*\} \) is the dual basis of \( \{E_\alpha\} \) with respect to the dual pairing \( (\cdot, \cdot) \).

**Theorem 2.3.** ([13, Corollary 5.4.3]) Let \( \text{Sym} \) be the Hopf algebra of symmetric functions over \( \mathbb{C} \) and \( h_n \) be the \( n \)th complete homogeneous symmetric function. The algebra homomorphism
\[ \text{comm} : \text{NSym} \to \text{Sym}, \quad H_n \mapsto h_n \]
(2.1)
is a surjective Hopf algebra homomorphism. And, \( \text{comm}(\Lambda_n) = e_n \) where \( e_n \) is the \( n \)th elementary symmetric function.
2.2.3. The Hopf algebra of free quasisymmetric functions. We begin by introducing the necessary notation. For positive integers $m$ and $n$, let

$$\text{Sh}_{m,n} := \left\{ \sigma \in S_{m+n} \mid \sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(m) \quad \text{and} \quad \sigma^{-1}(m+1) < \sigma^{-1}(m+2) < \cdots < \sigma^{-1}(m+n) \right\}. \tag{2.2}$$

For words $u = u_1 u_2 \ldots u_m$, $v = v_1 v_2 \ldots v_n$, and a permutation $\sigma \in \text{Sh}_{m,n}$, set

$$u \uplus_{\sigma} v := w_1 w_2 \ldots w_{m+n},$$

where $w_{\sigma^{-1}(i)} = u_i$ for $1 \leq i \leq m$ and $w_{\sigma^{-1}(m+j)} = v_j$ for $1 \leq j \leq n$. Then we define

$$u \uplus v := \{ u \uplus_{\sigma} v \mid \sigma \in \text{Sh}_{m,n} \} \quad \text{(as a multiset)}.$$

Using one line notation, let us identify each permutation $\pi \in S_n$ with the word $\pi_1 \pi_2 \ldots \pi_n$, where $\pi_i = \pi(i)$ for $i \in [n]$. The $m$-shifted permutation of $\pi$ is defined by the permutation $\pi[m] \in S_{m+1,m+n}$ corresponding to the word $(\pi_1 + m)(\pi_2 + m)\ldots(\pi_n + m)$. For instance, $21[2] = 43$ and therefore

$$12 \uplus 21[2] = \{1243, 1423, 1432, 4123, 4132, 4312\}.$$

**Definition 2.4.** ([16, Theorem 3.3]) The Hopf algebra $\mathbb{F}Q\text{Sym}$ of free quasisymmetric functions is defined to be the graded Hopf algebra over $\mathbb{C}$

$$\mathbb{F}Q\text{Sym} := \bigoplus_{n \geq 0} \mathbb{F}Q\text{Sym}_n,$$

where $\mathbb{F}Q\text{Sym}_n := \text{span}_{\mathbb{C}} \{ F_w \mid w \in S_n \}$, with the following product and coproduct:

- For each $u \in S_m$ and $v \in S_n$, the product is defined by

$$F_u F_v := \sum_{w \in u \uplus v[m]} F_w,$$

- and, for each $w \in S_n$, the coproduct is defined by

$$\Delta F_w := \sum_{k=0}^{n} F_{\text{std}(w_{k+1}w_{k+2}\ldots w_n)} \otimes F_{\text{std}(w_1w_2\ldots w_k)},$$

where $\text{std}(w)$ stands for the standardization of $w$.

**Example 2.5.** $\Delta(F_{132}) = 1 \otimes F_{132} + F_1 \otimes F_{21} + F_{12} \otimes F_1 + F_{132} \otimes 1$.

**Theorem 2.6.** ([13, Corollary 8.1.14]) The $\mathbb{C}$-linear map

$$\pi : \mathbb{F}Q\text{Sym} \to \mathbb{Q}\text{Sym}, \quad F_w \mapsto L_{\text{comp}(\text{Des}(w))} \quad (w \in S_n, n \geq 0)$$

is a surjective Hopf algebra homomorphism, where $\text{Des}(w)$ is the descent set of $w$, i.e.,

$$\text{Des}(w) = \{ i \mid w(i) > w(i + 1) \} \subseteq [n - 1].$$
The diagram below shows the relationship between the Hope algebras discussed so far. The morphisms $\pi^*$ and $\text{comm}^*$ represent the dual maps of $\pi$ and $\text{comm}$, respectively, and the unlabelled arrow in the middle represents the duality of the Hopf algebras.

3. **Categorifications of $\text{QSym}$ using supercharacter theories**

We assume that $\nu$ is a positive integer $> 1$, which will be fixed throughout this section.

### 3.1. Normal supercharacter theories of $\bigoplus C_\nu$.

In [4], Aliniaeifard and Thiem categorified $\text{FQSym}$ via towers of groups and their supercharacter theories, and our approach here is basically based on this paper.

For a cyclic group $C_\nu$ of order $\nu$ and a finite set $S$, let

$$Q_S(\nu) := \bigoplus_{s \in S} C_{\nu,s},$$

where $C_{\nu,s} = C_\nu$ for all $s \in S$. For clarity, we use 0 to represent the additive identity of $C_\nu$ and 0 to represent the identity of $Q_S(\nu)$. For each subset $I$ of $S$, we identify $Q_I(\nu)$ with the subgroup of $Q_S(\nu)$ whose $j$th component is $C_\nu$ if $j \in I$ and $\{0\}$ else. Let

$$N_S(\nu) := \{Q_I(\nu) \mid I \subseteq S\}.$$

Under this identification, it can be easily seen that $N_S(\nu)$ is a sublattice of $Q_S(\nu)$ and thus gives rise to a normal supercharacter theory $S(N_S(\nu))$ of $Q_S(\nu)$ (see Theorem 2.2). For simplicity, we write $Q_n(\nu)$ and $N_n(\nu)$ for $Q_{[n-1]}(\nu)$ and $N_{[n-1]}(\nu)$, respectively.

The sublattice $N_S(\nu)$ form a distributive lattice which implies that $N_0 \neq 0$ and $X^{N^*} \neq \emptyset$ by [3, Corollary 3.11]. Therefore, all blocks of $\text{cl}(S(N_n(\nu)))$ and $\text{ch}(S(N_n(\nu)))$ are parametrized by subsets of $[n-1]$. This implies that the superclass function space, denoted by $\text{scf}(S(N_n(\nu)))$, is of dimension $|\{I \subseteq [n-1]\}| = 2^{n-1}$, so it has the same dimension as $\text{QSym}_n$.

For each $I \subseteq S$, set

$$\text{cl}_I(\nu) := Q_I(\nu)_o \quad \text{and} \quad \text{ch}_I(\nu) := X^{Q_I(\nu)^*},$$

$$\kappa_I(\nu) := \kappa_{\text{cl}_I(\nu)} \quad \text{and} \quad \chi^I(\nu) := \chi^{\text{ch}_I(\nu)}.$$  \hfill (3.1)

**Remark 3.1.** It should be noted that the notation in (3.1) depends $S$ as well as $I$. If necessary, we will clarify $S$ as in Definition (3.4).
From the definition of $Q_I(\nu)$, it follows that

$$\text{cl}_I(\nu) = \{(g_i)_{i \in S} \in Q_n(\nu) \mid g_i \text{ is nonzero if } i \in I \text{ and zero else}\}.$$ 

In the following, we provide a formula for $\chi^I(\nu)$. Let $1$ be the trivial character of $C_\nu$ and $\text{reg}$ be the character of the regular representation of $C_\nu$.

**Proposition 3.2.** Let $I \subseteq S$. For $g = (g_i)_{i \in S} \in Q_S(\nu)$, we have

$$\chi^I(\nu)(g) = \prod_{i \in I} 1(g_i) \prod_{j \in I^c} (\text{reg} - 1)(g_j).$$

**Proof.** By [3, Corollary 3.4, 3.5], for $g \in \text{cl}_J(\nu)$, we have

$$\chi^I(\nu)(g) = \begin{cases} \frac{\nu^{n-1}}{\nu^{|I|}} \left(\frac{\nu - 1}{\nu}\right)^{|I \cap J|} \frac{1}{1 - \nu} & \text{if } I \prec_c J, \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} (\nu - 1)^{|I|} \left(\frac{-1}{\nu - 1}\right)^{|J \setminus I|} & \text{if } I \prec_c J, \\ 0 & \text{otherwise}, \end{cases}$$

where $\prec_c$ represents the covering relation of the poset $\{I : I \subseteq S\}$ ordered by inclusion. Now, the assertion follows from the fact that $1(g) = 1$ for all $g \in G$ and

$$(\text{reg} - 1)(g) = \begin{cases} \nu - 1 & \text{if } g = 0, \\ -1 & \text{otherwise.} \end{cases}$$

□

It is convenient to express $\chi^I(\nu)$ in the form of coordinates as in [4]. For a finite group $G$, let $\text{cf}(G)$ denote the $\mathbb{C}$-vector space of the class functions on $G$. Consider the natural isomorphism of vector spaces

$$[\cdot] : \bigotimes_{s \in S} \text{cf}(C_{\nu,s}) \to \text{cf}(Q_S(\nu))$$

defined by $\left[\bigotimes_{i \in S} \phi_i\right](g) = \prod_{i \in S} \phi_i(g_i)$ for $\phi_i \in \text{cf}(C_{\nu,i})$ and $g = (g_i)_{i \in S} \in Q_S(\nu)$.

**Convention.** When $S = \{s_1 < s_2 < \cdots < s_t\}$, we write $\bigotimes_{i \in S} \phi_i$ and $\left[\bigotimes_{i \in S} \phi_i\right]$ as $(\phi_{s_1}, \phi_{s_2}, \ldots, \phi_{s_t})$ and $[\phi_{s_1}, \phi_{s_2}, \ldots, \phi_{s_t}]$, respectively.

For each $i \in S$, let

$$\chi^I(\nu)_i := \begin{cases} 1 & \text{if } i \in I, \\ \text{reg} - 1 & \text{if } i \in S \setminus I. \end{cases}$$
It follows from Proposition 3.2 that
\[
\chi^I(\nu) = \left[ (\chi^I(\nu)_i)_{i \in S} \right].
\]

For example, if \( I = \{1, 6\} \subseteq S = \{1, 2, 4, 6, 7\} \), then
\[
\chi^I(\nu) = \left[ 1, \overline{\text{reg} - 1}, \overline{\text{reg} - 1}, 1, \text{reg} - 1 \right].
\]

### 3.2. A Hopf algebra structure of \( \bigoplus_{n \geq 0} \text{scf}(\mathcal{N}_n(\nu)) \).

Hereafter we will simply write \( \text{scf}(\mathcal{N}_n(\nu)) \) for \( \text{scf}(\mathcal{S}(\mathcal{N}_n(\nu))) \), the supercharacter function space of \( \mathcal{S}(\mathcal{N}_n(\nu)) \).

Given a subset \( S = \{s_1 < s_2 < \cdots < s_t\} \) of \([n-1]\) with \( t := |S| \), consider the standardization, i.e., the group isomorphism
\[
\iota_S : Q_S(\nu) \to Q_{t+1}(\nu), \quad (g_s)_{s \in S} \mapsto (g_s)_{1 \leq i \leq t}.
\]

This induces a \( \mathbb{C} \)-linear isomorphism
\[
\iota_S^* : \text{cf}(Q_{t+1}(\nu)) \to \text{cf}(Q_S(\nu)), \quad \phi \mapsto \phi \circ \text{stan}.
\]

For simplicity, we set \( \iota_S := \text{stan} \circ \iota_S^*. \)

Given a subset \( S = \{s_1 < s_2 < \cdots < s_t\} \) of \([n-1]\) with \( t := |S| \), consider the group isomorphism
\[
\iota_S : Q_S(\nu) \to Q_{t+1}(\nu), \quad (g_s)_{s \in S} \mapsto (g_s)_{1 \leq i \leq t},
\]

which is obtained by the standization of indices. This induces a \( \mathbb{C} \)-linear isomorphism
\[
\iota_S^* : \text{cf}(Q_{t+1}(\nu)) \to \text{cf}(Q_S(\nu)), \quad \phi \mapsto \phi \circ \iota_S.
\]

For \( I \subseteq [t] \), let \( S_I := \{s_i \mid i \in I\} \). Since \( \iota_S^*(\chi^I(\nu)) = \chi^{S_I}(\nu) \), this isomorphism restricts to the following isomorphism of the supercharacter function spaces:
\[
\iota_{S_I}^* : \text{scf}(\mathcal{N}_{t+1}(\nu)) \to \text{scf}(\mathcal{N}_S(\nu)), \quad \chi^I(\nu) \mapsto \chi^{S_I}(\nu).
\]

Let us identify \( Q_S(\nu) \times Q_{S^c}(\nu) \) with \( Q_{[n-1]}(\nu) \) in the natural way. For \( \phi \in \text{cf}(Q_S(\nu)) \) and \( \psi \in \text{cf}(Q_{S^c}(\nu)) \), let \( \phi \otimes_S \psi \) be the class function of \( Q_{[n-1]}(\nu) \) defined by
\[
(\phi \otimes_S \psi)(a, b) = \phi(a)\psi(b),
\]

where \((a, b) \in Q_S(\nu) \times Q_{S^c}(\nu) = Q_{[n-1]}(\nu)\). It is easy to see that if \( \phi \in \text{scf}(\mathcal{N}_S(\nu)) \) and \( \psi \in \text{scf}(\mathcal{N}_{S^c}(\nu)) \) then \( \phi \otimes_S \psi \in \text{scf}(\mathcal{N}_n(\nu)) \).

In case where \( S = [n-2] \), or equivalently \( S^c = \{n-1\} \), we simply write \( \otimes \) for \( \otimes_S \).

Let \( A \) be a subset of \([n]\). A subset of \( A \) is said to be connected (in \( A \)) if it consists of consecutive integers or a single element, and maximally connected (in \( A \)) if it is maximal among connected subsets with respect to inclusion order. Let
\[
\text{conn}(A) := \{\text{maximally connected subsets of } A\}.
\]
Label the subsets in $\text{conn}(A)$ as $A_1, A_2, A_3, \ldots$, where $\min A_i < \min A_j$ if $i < j$. Let
\[
\begin{align*}
  c_1(A) & := \{\max B \mid B \in \text{conn}(A) \} \setminus \{n\}, \\
  c_2(A) & := \{\max B \mid B \in \text{conn}(A^c) \} \setminus \{n\}, \\
  c(A) & := c_1(A) \cup c_2(A).
\end{align*}
\]

**Example 3.3.** Let $n = 9$ and $A = \{1, 2, 5, 7, 8, 9\} \subseteq [9]$. Then $A_1 = \{1, 2\}$, $A_2 = \{5\}$, $A_3 = \{7, 8, 9\}$ are all maximally connected subsets of $A$. Moreover, $c_1(A) = \{2, 5\}$, $c_2(A) = \{4, 6\}$, and thus $c(A) = \{2, 4, 5, 6\}$.

In what follows, we will define a product and a coproduct on $\bigoplus_{n \geq 0} \text{scf}(\mathcal{N}_n(\nu))$. To do this, we need the following notations.

- Let $G$ be a group and $H$ a subgroup of $G$. For $\phi \in \text{cf}(G)$, denote by $\phi \downarrow_H^G$ the restriction of $\phi$ from $G$ to $H$.
- For positive integers $a$ and $b$, denote by $\binom{a}{b}$ the collection of subsets of $[a]$ with size $b$.
- For $I \subseteq S$, set $\chi^I(\nu) := \frac{\chi^I(\nu)}{\chi^I(\nu)(0)}$.
- Let $m, n \geq 0$ and $A \in \binom{[m+n]}{n}$. For $\phi \in \text{cf}(Q_m(\nu))$ and $\psi \in \text{cf}(Q_n(\nu))$, set
  \[
  s_A(\phi, \psi) := \left(\tau_A^{-1} \left( \phi \otimes \frac{\nu - 1}{\nu - 1} \right) \right) \otimes_{A^c} \left( \tau_A^{-1} \left( \psi \otimes \frac{\nu - 1}{\nu - 1} \right) \right). \tag{3.2}
  \]

Here, we are viewing $\otimes_{A^c}$ as a map
\[
\otimes_{A^c} : \text{cf}(Q_{A^c}(\nu)) \times \text{cf}(Q_{[m+n]\setminus A^c}(\nu)) \to \text{cf}(Q_{m+n+1}(\nu)).
\]

**Definition 3.4.** Let $m$ and $n$ be nonnegative integers.

(a) For $A \in \binom{[m+n]}{n}$, let
\[
\mathbf{m}_A : \text{cf}(Q_m(\nu)) \times \text{cf}(Q_n(\nu)) \to \text{cf}(Q_{m+n}(\nu))
\]
be the $\mathbb{C}$-bilinear map given by
\[
\mathbf{m}_A(\phi, \psi) = \begin{cases}
  \phi \psi & \text{if } m = 0 \text{ or } n = 0, \\
  \chi^{c_1(A)}(\nu) \otimes_{c(A)} \left( s_A(\phi, \psi) \downarrow_{Q_{[m+n+1]\setminus c(A)}(\nu)} \right) & \text{otherwise}
\end{cases}
\]
for $\phi \in \text{cf}(Q_m(\nu))$ and $\psi \in \text{cf}(Q_n(\nu))$. Here, we are viewing $\chi^{c_1(A)}(\nu)$ as an element of $\text{cf}(Q_A(\nu))$ and $\otimes_{c(A)}$ as a map
\[
\otimes_{c(A)} : \text{cf}(Q_{c(A)}(\nu)) \times \text{cf}(Q_{[m+n-1]\setminus c(A)}(\nu)) \to \text{cf}(Q_{m+n}(\nu)).
\]

(b) We define
\[
\mathbf{m} : \text{cf}(Q_m(\nu)) \times \text{cf}(Q_n(\nu)) \to \text{cf}(Q_{m+n}(\nu))
\]
by

\[ \mathbf{m} := \sum_{A \in \binom{[m+n]}{m}} \mathbf{m}_A. \]

For \( n \geq 2 \) and \( \phi \in \text{cf}(Q_n(\nu)) \) and \( 1 \geq k \geq n-1 \), let \( (\phi_1^k, \phi_2^k) \) be a pair satisfying that

\[ \phi_1^k \in \text{cf}(Q_{[1,k-1]}(\nu)), \quad \phi_2^k \in \text{cf}(Q_{[k+1,n-1]}(\nu)), \text{ and} \]

\[ \phi \downarrow Q_{[1,k-1] \sqcup [k+1,n-1]}(\nu) = \phi_1^k \otimes \phi_2^k. \]

**Definition 3.5.** Let \( n \) be a nonnegative integer.

(a) For \( 0 \leq k \leq n \), let

\[ \blacktriangle_k : \text{cf}(Q_n(\nu)) \to \text{cf}(Q_k(\nu)) \otimes \text{cf}(Q_{n-k}(\nu)) \]

be the \( \mathbb{C} \)-linear map given by

\[ \blacktriangle_k(\phi) = \begin{cases} \mathbf{1}_0 \otimes \phi & \text{if } k = 0, \\ \phi_1^k \otimes (\iota_{[k+1,n-1]}^*)^{-1}(\phi_2^k) & \text{if } 1 \leq k < n, \\ \phi \otimes \mathbf{1}_0 & \text{if } k = n, \end{cases} \]

where \( \phi \in \text{cf}(Q_n(\nu)) \).

(b) We define

\[ \blacktriangle : \text{cf}(Q_n(\nu)) \to \bigoplus_{k \in [n-1]} \text{cf}(Q_k(\nu)) \otimes \text{cf}(Q_{n-k}(\nu)) \]

by

\[ \blacktriangle := \sum_{k=0}^{n} \blacktriangle_k. \]

**Remark 3.6.** It should be remarked that the pair \( (\phi_1^k, \phi_2^k) \) is not unique, but the value of \( \blacktriangle_k(\phi) \) does not depend on the choices of this pair.

Extending \( \mathbf{m} \) bi-additively and \( \blacktriangle \) additively, we obtain a \( \mathbb{C} \)-bilinear map

\[ \mathbf{m} : \bigoplus_{n \geq 0} \text{cf}(Q_n(\nu)) \otimes \bigoplus_{n \geq 0} \text{cf}(Q_n(\nu)) \to \bigoplus_{n \geq 0} \text{cf}(Q_n(\nu)) \]

and a \( \mathbb{C} \)-linear map

\[ \blacktriangle : \bigoplus_{n \geq 0} \text{cf}(Q_n(\nu)) \to \bigoplus_{n \geq 0} \text{cf}(Q_n(\nu)) \otimes \bigoplus_{n \geq 0} \text{cf}(Q_n(\nu)). \]

In the following, we will show that \( \mathbf{m} \) and \( \blacktriangle \) restrict to the supercharacter function spaces. For this purpose, we first introduce necessary notation.
Definition 3.7. Let \( m \) and \( n \) be nonnegative integers and let \( A \in {m+n \choose n} \), \( I \subseteq [m-1] \), and \( J \subseteq [n-1] \).

(a) Let 
\[
(I\#_A J)' := (A^c)_I' \sqcup A_{J'} \quad (\subseteq [m+n]).
\]
Here \( I' = I, J' = J \), but we are viewing them as \( I' \subseteq [m] \) and \( J' \subseteq [n] \).

(b) The \( A \)-preshuffle \( I\#_A J \) of \( I \) and \( J \) is defined by the subset of \([m+n-1]\) with the same elements as \( (I\#_A J)' \).

(c) Let 
\[
I \sqcup_A J := c_1(A) \sqcup ((I\#_A J) \setminus c(A)).
\]

For \( S = \{s_1, s_2, \ldots, s_t\} \subseteq [k+1, n-1] \), let 
\[
S - k := \{s_1 - k, s_2 - k, \ldots, s_t - k\} \quad (\subseteq [n-k+1]).
\]

Lemma 3.8. We have the following.

(a) Let \( m \) and \( n \) be nonnegative integers and \( A \in {m+n \choose n} \). For \( I \subseteq [m-1] \) and \( J \subseteq [n-1] \), we have 
\[
M_A (\hat{\chi}^I(\nu), \hat{\chi}^J(\nu)) = \hat{\chi}^{I\#_A J}(\nu).
\]

(b) Let \( n \geq 2 \) and \( 1 \leq k \leq n-1 \). For \( I \subseteq [n-1] \), we have 
\[
\triangle_k(\hat{\chi}^I(\nu)) = \hat{\chi}^{I\cap [k-1]}(\nu) \otimes \hat{\chi}^{(I\cap [k+1,n-1]) - k}(\nu).
\]

Proof. (a) Let 
\[
\hat{\chi}^I(\nu)_i := \begin{cases} \frac{1}{\nu - 1} & \text{if } i \in I, \\ \frac{\nu + 1}{\nu - 1} & \text{if } i \in [n-1] \setminus I. \end{cases}
\]
Since \( \chi^I(\nu)(0) = (\nu - 1)^{|I|} \), it follows from Proposition 3.2 that 
\[
\left( (\hat{\chi}^I(\nu)_i)_{i \in [n-1]} \right) = \hat{\chi}^I(\nu).
\]

Let 
\[
s_A(\phi, \psi)_i := \begin{cases} \frac{1}{\nu - 1} & \text{if } i \in (I\#_A J)', \\ \frac{\nu + 1}{\nu - 1} & \text{otherwise} \end{cases}
\]
for all \( i \in [m+n] \). Combining Definition 3.7 with (3.2), we derive that 
\[
\left( (s_A(\phi, \psi)_i)_{i \in [m+n]} \right) = s_A(\phi, \psi).
\]
Next, for each $i \in [m + n - 1] \setminus c(A)$, we let

$$\left( s_A(\phi, \psi) \downarrow_{Q[m+n-1]\setminus c(A)} \right)_i := \begin{cases} 1 & \text{if } i \in (I\#_A J) \setminus c(A), \\ \frac{\text{reg} - 1}{\nu - 1} & \text{otherwise}. \end{cases}$$

In view of

$$\frac{\text{reg} - 1}{\nu - 1}(0) = \frac{\nu - 1}{\nu - 1} = 1 \quad \text{and} \quad \mathbb{1}(0) = 1,$$

we have

$$\left( \left( (s_A(\phi, \psi)) \downarrow_{Q[m+n-1]\setminus c(A)} \right)_i \right)_{i \in [m+n-1]\setminus c(A)} = s_A(\phi, \psi) \downarrow_{Q[m+n-1]\setminus c(A)}^{Q_{m+n+1}}.$$

Finally, let

$$m_A(\hat{x}^I(\nu), \hat{x}^J(\nu))_i := \begin{cases} 1 & \text{if } i \in c_1(A) \cup ((I\#_A J) \setminus c(A)), \\ \frac{\text{reg} - 1}{\nu - 1} & \text{otherwise} \end{cases}$$

for $i \in [m + n - 1]$. It holds that

$$\left[ \left( m_A(\hat{x}^I(\nu), \hat{x}^J(\nu))_i \right)_{i \in [m+n-1]} \right] = \hat{x}^{c_1(A)}(\nu) \otimes_{c(A)} \left( s_A(\phi, \psi) \downarrow_{Q[m+n-1]\setminus c(A)}^{Q_{m+n+1}(\nu)} \right).$$

Now the desired result follows from Definition 3.4 (a) and 3.7 (c).

(b) Let $\phi = \hat{x}^I(\nu)$, and take $\phi^k_1 = ((\phi^k_{1i})_{i \in [k-1]})$, $\phi^k_2 = ((\phi^k_{2i})_{i \in [k+1, n-1]})$ as follows:

$$\phi^k_{1i} = \begin{cases} \frac{1}{\nu - 1} & \text{if } i \in I \cap [k - 1], \\ \frac{\text{reg} - 1}{\nu - 1} & \text{otherwise}, \end{cases}$$

$$\phi^k_{2i} = \begin{cases} 1 & \text{if } i \in I \cap [k + 1, n - 1], \\ \frac{\text{reg} - 1}{\nu - 1} & \text{otherwise}. \end{cases}$$

It is easy to show that

$$\hat{x}^I(\nu) \downarrow_{Q_{[1,k-1] \cup [k+1,n-1]}(\nu)} = \phi^k_1 \otimes_{[1,k-1]} \phi^k_2$$

and

$$\phi^k_1 = \hat{x}^{I \cap [k-1]}(\nu) \quad \text{and} \quad \phi^k_2 = \hat{x}^{(I \cap [k+1,n-1]) \setminus k}(\nu),$$

which proves the assertion. \hfill \Box

**Example 3.9.** Let $m = 4$ and $n = 3$, and consider the case where $I = \{2, 3\} \subseteq [m - 1]$, $J = \{2\} \subseteq [n - 1]$, and $A = \{1, 3, 4\}$. Then we have

$$\hat{x}^I(\nu) = \left[ \frac{\text{reg} - 1}{\nu - 1}, 1, 1 \right], \quad \text{and} \quad \hat{x}^J(\nu) = \left[ \frac{\text{reg} - 1}{\nu - 1}, 1 \right].$$
This implies
\[ \dot{\chi}^I(\nu) \otimes_1 \frac{\text{reg} - 1}{\nu - 1} = \left[ \frac{\text{reg} - 1}{\nu - 1}, 1, \frac{\text{reg} - 1}{\nu - 1} \right] \] and
\[ \dot{\chi}^J(\nu) \otimes_1 \frac{\text{reg} - 1}{\nu - 1} = \left[ \frac{\text{reg} - 1}{\nu - 1}, 1, \frac{\text{reg} - 1}{\nu - 1} \right]. \]

Hence
\[
s_A(\dot{\chi}^I(\nu), \dot{\chi}^J(\nu)) = \tau_A^* \left( \dot{\chi}^I(\nu) \otimes \frac{\text{reg} - 1}{\nu - 1} \right) \otimes_A \left( \tau_A^* \left( \dot{\chi}^J(\nu) \otimes \frac{\text{reg} - 1}{\nu - 1} \right) \right)
\]
\[ = \left[ \frac{\text{reg} - 1}{\nu - 1}, 1, \frac{\text{reg} - 1}{\nu - 1}, 1, \frac{\text{reg} - 1}{\nu - 1}, 1 \right]. \]

and therefore
\[
s_A(\dot{\chi}^I(\nu), \dot{\chi}^J(\nu)) \downarrow_{Q_{[m+n-1]}(\nu)}^{Q_{m+n+1}(\nu)} = \left[ \underbrace{1, \frac{\text{reg} - 1}{\nu - 1}}_{\text{reg}}, 1, 1, 1, 1 \right] \in \text{cf}(Q_{m+n-1}(\nu)). \]

Combining the above calculation with the equality
\[ \dot{\chi}^{c_1(A)}(\nu) = \left[ \underbrace{1, \frac{\text{reg} - 1}{\nu - 1}}_{\text{reg}}, 1, 1, 1, 1 \right] \in \text{cf}(Q_{c(A)}(\nu)), \]
we see that \( m_A(\dot{\chi}^I(\nu), \dot{\chi}^J(\nu)) \) is equal to
\[
\dot{\chi}^{c_1(A)}(\nu) \otimes_{c(A)} (s_A(\dot{\chi}^I(\nu), \dot{\chi}^J(\nu)) \downarrow_{Q_{[m+n-1]}(\nu)}^{Q_{m+n+1}(\nu)})
\]
\[ = \left[ \underbrace{1, \frac{\text{reg} - 1}{\nu - 1}}_{\text{reg}}, 1, 1, 1, 1 \right] \]
\[ = \dot{\chi}^{c_1(A)}(\nu). \]

**Example 3.10.** Let \( n = 5, I = \{1, 3, 4\} \subseteq [4] \). Then
\[ \dot{\chi}^I(\nu) = \left[ 1, \frac{\text{reg} - 1}{\nu - 1}, 1, 1 \right], \]
thus
\[ \triangledown_0(\dot{\chi}^I(\nu)) = 1_0 \otimes \left[ 1, \frac{\text{reg} - 1}{\nu - 1}, 1, 1 \right], \]
\[ \triangledown_1(\dot{\chi}^I(\nu)) = 1_1 \otimes \left[ \frac{\text{reg} - 1}{\nu - 1}, 1, 1 \right], \]
\[ \triangledown_2(\dot{\chi}^I(\nu)) = \left[ 1 \right] \otimes \left[ 1, 1 \right], \]
\[ \triangledown_3(\dot{\chi}^I(\nu)) = \left[ 1, \frac{\text{reg} - 1}{\nu - 1} \right] \otimes \left[ 1 \right], \]
\[ \triangledown_4(\dot{\chi}^I(\nu)) = \left[ 1, \frac{\text{reg} - 1}{\nu - 1} \right] \otimes 1_1, \]
\[ \triangledown_5(\dot{\chi}^I(\nu)) = \left[ 1, \frac{\text{reg} - 1}{\nu - 1}, 1, 1 \right] \otimes 1_0. \]
Lemma 3.8 implies that the maps \( m \) and \( \triangle \) restrict to the supercharacter function spaces. Thus we have
\[
\begin{align*}
m &: \bigoplus_{n \geq 0} \text{scf}(N_n(\nu)) \otimes \bigoplus_{n \geq 0} \text{scf}(N_n(\nu)) \to \bigoplus_{n \geq 0} \text{scf}(N_n(\nu)), \\
\triangle &: \bigoplus_{n \geq 0} \text{scf}(N_n(\nu)) \to \bigoplus_{n \geq 0} \text{scf}(N_n(\nu)) \otimes \bigoplus_{n \geq 0} \text{scf}(N_n(\nu)).
\end{align*}
\]

(3.4)

For each nonnegative integer \( n \), define
\[
\text{ch}_n^\nu : \text{scf}(N_n(\nu)) \to \text{QSym}_n
\]
by the \( \mathbb{C} \)-vector space isomorphism given by
\[
\hat{\chi}^I(\nu) \mapsto L_{\text{comp}(I)} \quad (I \subset [n - 1]).
\]

This induces a \( \mathbb{C} \)-vector space isomorphism:
\[
\text{ch}_\nu := \bigoplus_{n \geq 0} \text{ch}_n^\nu : \bigoplus_{n \geq 0} \text{scf}(N_n(\nu)) \to \text{QSym}
\]

**Lemma 3.11.** For nonnegative \( m \) and \( n \), the following hold.

(a) For \( I \subseteq [m - 1] \) and \( J \subseteq [n - 1] \), we have
\[
\text{ch}_\nu(m(\hat{\chi}^I(\nu), \hat{\chi}^J(\nu))) = L_{\text{comp}(I)}L_{\text{comp}(J)}.
\]

(b) For \( I \subseteq [n - 1] \), we have
\[
\text{ch}_\nu \otimes \text{ch}_\nu(\triangle(\hat{\chi}^I(\nu))) = \triangle(L_{\text{comp}(I)}),
\]

where \( \triangle \) denotes the coproduct of \( \text{QSym} \).

**Proof.** By Lemma (3.8) (a), it suffices to show that
\[
L_{\text{comp}(I)}L_{\text{comp}(J)} = \sum_{A \in \binom{[m+n]}{n}} L_{\text{comp}(I \cup A J)}.
\]

(3.5)

Let \( w_I \in S_m \) be a permutation with \( \text{Des}(w_I) = I \) and \( w_J \in S_n \) be a permutation with \( \text{Des}(w_J) = J \). Theorem 2.6 implies that
\[
L_{\text{comp}(I)}L_{\text{comp}(J)} = \sum_{w \in w_I \cup w_J[m]} L_{\text{comp}(\text{Des}(w))}.
\]

(3.6)

Note that \( \text{Sh}_{m,n} \) in (2.2) is in bijection with \( \binom{[m+n]}{n} \) under the following assignment:
\[
\sigma \mapsto \{ \sigma^{-1}(m + 1), \sigma^{-1}(m + 2), \ldots, \sigma^{-1}(m + n) \}.
\]
For $A \in \binom{[m+n]}{n}$, we write $\sqcup A$ for $\sqcup \sigma$, where $\sigma$ is the permutation corresponding to $A$ under this bijection. With this notation, for any word $v$ of length $m$ and $w$ of length $n$, we have the equality:

$$v \sqcup w = \left\{ v \sqcup_A w \mid A \in \binom{[m+n]}{n} \right\}.$$ 

The word $w_I \sqcup_A w_J[m]$ can be obtained from $w_I$ and $w_J[m]$ in the following manner:

- Place the alphabets of $w_J[m]$ in order in the positions occupied by $A$.
- Place the alphabets of $w_I$ in order in the positions occupied by $A^c$.

Since any alphabet in $w_J[m]$ is larger than the greatest alphabet in $w_I$, it follows that $c_1(A) \subseteq \text{Des}(w_I \sqcup_A w_J[m])$.

Similarly, since any alphabet in $w_I$ is smaller than the smallest alphabet in $w_J[m]$, it follows that $c_2(A) \subseteq \text{Des}(w_I \sqcup_A w_J[m])^c$.

The positions which are not occupied by $c_1(A)$ and $c_2(A)$ depends only on the descent pattern of $w_I \sqcup_A w_J$, which implies

$$\text{Des}(w_I \sqcup_A w_J[m]) \setminus c(A) = \text{Des}(w_I \sqcup_A w_J) \setminus c(A).$$

Since

$$(I\#A)J \setminus c(A) = \text{Des}(w_I \sqcup_A w_J) \setminus c(A),$$

we have

$$\text{Des}(w_I \sqcup_A w_J[m]) = c_1(A) \sqcup ((I\#A)J \setminus c(A)).$$

It follows that (3.6) is equal to (3.5).

(b) By Lemma (3.8) (b), it suffices to show that

$$\triangle L_{\text{comp}(I)} = L_\emptyset \otimes L_{\text{comp}(I)} + \sum_{k=1}^{n-1} L_{\text{comp}(I \cap [k-1])} \otimes L_{\text{comp}((I \cap [k+1,n-1]) \setminus k)} + L_{\text{comp}(I)} \otimes L_\emptyset.$$

On the other hand, Theorem 2.6 implies that

$$\triangle L_{\text{comp}(\text{Des}(w))} = \sum_{k=0}^{n} L_{\text{comp}(\text{Des}(\text{std}(w_{1\ldots k})))} \otimes L_{\text{comp}(\text{Des}(\text{std}(w_{k+1\ldots n})))}.$$ 

Let $w_I \in \mathfrak{S}_n$ be a permutation with $\text{Des}(w_I) = I$. It is easy to see that

- $\text{Des}(\text{std}((w_I)_1 \ldots (w_I)_k)) = I \cap [k-1] \subseteq [k-1]$,
- $\text{Des}(\text{std}((w_I)_{k+1} \ldots (w_I)_n)) = (I \cap [k+1,n-1]) - k \subseteq [n-k-1]$,

for $k = 1, \ldots, n - 1$, as required. \qed
Example 3.12. Let us revisit Example 3.9. Take $w_I = 1432$, $w_J = 132$. Then $c_1(A) = \{1, 4\}$, $c_2(A) = \{2\}$, and $I \# A J = \{5, 6\} \sqcup \{3\}$. As a consequence, we have $w_I \sqcup_A w_J[m] = \{1, 4\} \sqcup (\{3, 5, 6\} \setminus \{1, 2, 4\}) = \{1, 3, 4, 5, 6\}$.

By Lemma 3.11, we have the following commuting diagrams:

$$\bigoplus_{n \geq 0} \text{scf}(N_n(\nu)) \otimes \bigoplus_{n \geq 0} \text{scf}(N_n(\nu)) \xrightarrow{m} \bigoplus_{n \geq 0} \text{scf}(N_n(\nu))$$

$$\downarrow \chi_{\nu} \otimes \chi_{\nu} \quad \downarrow \chi_{\nu}$$

$$\text{QSym} \otimes \text{QSym} \quad \text{QSym}$$

$$\bigoplus_{n \geq 0} \text{scf}(N_n(\nu)) \otimes \bigoplus_{n \geq 0} \text{scf}(N_n(\nu)) \xleftarrow{\triangle} \bigoplus_{n \geq 0} \text{scf}(N_n(\nu))$$

$$\downarrow \chi_{\nu} \otimes \chi_{\nu} \quad \downarrow \chi_{\nu}$$

$$\text{QSym} \otimes \text{QSym} \quad \text{QSym}$$

Now we are ready to state the main result of this section.

**Theorem 3.13.** We have the following.

(a) $(\bigoplus_{n \geq 0} \text{scf}(N_n(\nu)), m, \triangle)$ has a Hopf algebra structure.

(b) The characteristic map

$$\chi_{\nu} : \bigoplus_{n \geq 0} \text{scf}(N_n(\nu)) \to \text{QSym}$$

is an isomorphism of Hopf algebras.

**Proof.** By Lemma 3.11, $\chi_{\nu}$ is a bialgebra isomorphism. As QSym is a connected graded Hopf algebra, this proves our assertions. \(\square\)

**Remark 3.14.** It should be pointed out that even if $C_\nu$ is replaced by any finite group $G$ of order $q$, all results in this section are still valid. In this case, $Q_S(\nu) = \bigoplus_{s \in S} G_s$, where $G_s = G$ for all $s \in S$.

3.3. **The superclass identifiers of $\bigoplus_{n \geq 0} \text{scf}(N_n(\nu))$.** The purpose of this subsection is to study the superclass identifiers of $\bigoplus_{n \geq 0} \text{scf}(N_n(\nu))$. To be precise, for $I \subseteq [n - 1]$ and $J \subseteq [m - 1]$, we expand $m(\kappa_I(\nu), \kappa_J(\nu))$ in the basis $\{\kappa_I(\nu) : I \subseteq [m + n - 1]\}$ for $\text{scf}(N_{m+n}(\nu))$ and $\triangle(\kappa_I(\nu))$ in the basis for $\bigoplus_{k \in [n-1]} \text{scf}(N_k(\nu)) \otimes \text{scf}(N_{n-k}(\nu))$ consisting of tensor products of the superclass identifiers. This results obtained here will play a crucial role in Section 4.

**Lemma 3.15.** For $I \subseteq [n - 1]$, let

$$\kappa_I(\nu)_i := \begin{cases} 1 - \frac{1}{\nu} \text{reg} & \text{if } i \in I, \\ \frac{1}{\nu} \text{reg} & \text{if } i \in [n - 1] \setminus I \end{cases}$$
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for $i \in [n-1]$. Then $[(\kappa_I(\nu))_{i \in [n-1]}] = \kappa_I(\nu)$.

Proof. The assertion follows from

$$\frac{1}{\nu} \text{reg}(g) = \begin{cases} 1 & \text{if } g = 0, \\ 0 & \text{otherwise}. \end{cases}$$

\[\square\]

Let us first introduce the expansion of $m(\kappa_I(\nu), \kappa_J(\nu))$ in the basis consisting of the superclass identifiers.

**Proposition 3.16.** Let $I \subseteq [m-1], J \subseteq [n-1]$. Then

$$m(\kappa_I(\nu), \kappa_J(\nu)) = \sum_{K \subseteq [m+n-1]} d_K \kappa_K(\nu),$$

where

$$d_K = \sum_{A \subseteq \binom{[m+n]}{n}} \left(\frac{1}{1-\nu}\right)^{|K \cap c_2(A)|}.$$}

Proof. Observe the following equalities:

$$1 = \left(1 - \frac{1}{\nu} \text{reg}\right) + \frac{1}{\nu} \text{reg}, \quad (3.7)$$

$$\frac{\text{reg} - 1}{\nu - 1} = \left(\frac{-1}{\nu - 1}\right) \left(1 - \frac{1}{\nu} \text{reg}\right) + \frac{1}{\nu} \text{reg}. \quad (3.8)$$

For $A \subseteq [m+n]$, set

$$z_A := \begin{cases} \max A & \text{if } m+n \notin A, \\ \max [m+n] \setminus A & \text{if } m+n \in A. \end{cases}$$

If we let

$$\phi_i := \begin{cases} 1 - \frac{1}{\nu} \text{reg} & \text{if } i \in (I \# A)\setminus J, \\ \frac{\text{reg} - 3}{\nu - 1} & \text{if } i = z_A \text{ or } i = m+n, \\ \frac{1}{\nu} \text{reg} & \text{otherwise}, \end{cases}$$

then

$$\left[(\phi_i)_{i \in [m+n]}\right] = s_A(\kappa_I(\nu), \kappa_J(\nu)).$$

Applying (3.8) to the $\phi_{z_A}$ in this vector notation and then using Lemma 3.15, one can simplify

$$s_A(\kappa_I(\nu), \kappa_J(\nu)) \downarrow_{Q_{m+n+1}(\nu)}^{Q_{m+n+1}(\nu)} \downarrow_{Q_{[m+n-1]\setminus c(A)}(\nu)}^{Q_{m+n-1}(\nu)}$$
in the following simple form:

\[
\left(\left(\frac{-1}{\nu - 1}\right)^{K(I_{A}^\#J)\cup z_{A}}(\nu) + K_{I_{A}^\#J}(\nu)\right)\bigg|_{Q_{m+n+1}(\nu) + Q_{m+n-1}\setminus c(A)(\nu)}^{Q_{m+n}(\nu)}
\]  

(3.9)

On the other hand, one can easily see that for any subsets \(S, T \subseteq [n - 1]\),

\[
\kappa_{S}(\nu)\bigg|_{Q_{n}(\nu)}^{Q_{T}(\nu)} = \begin{cases} 
\kappa_{S}(\nu) & \text{if } S \subseteq T, \\
0 & \text{otherwise}.
\end{cases}
\]  

(3.10)

Since \(z_{A} \in c(A)\), from (3.10) it follows that

\[
(3.9) = \begin{cases} 
K_{I_{A}^\#J}(\nu) & \text{if } I_{A}^\#J \subseteq [m + n - 1] \setminus c(A), \\
0 & \text{otherwise}.
\end{cases}
\]

Finally, using (3.7) and (3.8), we derive that

\[
\mathbf{m}_{A}(\kappa_{I}(\nu), \kappa_{J}(\nu))
= \begin{cases} 
\chi^{c_{1}(A)}(\nu) \otimes_{c(A)} K_{I_{A}^\#J}(\nu) & \text{if } I_{A}^\#J \subseteq [m + n - 1] \setminus c(A), \\
0 & \text{otherwise}
\end{cases}
\]

\[
= \begin{cases} 
\sum_{I_{A}^\#J \subseteq K_{c_{1}(A)}(\nu)}(\frac{1}{1 - \nu})^{K_{c_{2}(A)}(\nu)} & \text{if } I_{A}^\#J \subseteq [m + n - 1] \setminus c(A), \\
0 & \text{otherwise}.
\end{cases}
\]

This completes the proof. \(\square\)

**Example 3.17.** Let \(m = 2, n = 3\) and \(I = \{1\}, J = \{2\}\). The following table shows all the statistics required to calculate \(\mathbf{m}(\kappa_{I}(\nu), \kappa_{J}(\nu))\).

| \(A\) | \(c_{1}(A)\) | \(c_{2}(A)\) | \(c(A)\) | \(I_{A}^\#J\) | \((I_{A}^\#J) \cap c(A)\) | \((I_{A}^\#J) \cup c(A)\) |
|---|---|---|---|---|---|---|
| \(\{1,2,3\}\) | \(\{3\}\) | \(\emptyset\) | \(\{3\}\) | \(\{2,4\}\) | \(\emptyset\) | \(\{2,3,4\}\) |
| \(\{1,2,4\}\) | \(\{2,4\}\) | \(\{3\}\) | \(\{2,3\}\) | \(\{2,3\}\) | \(\{2,3\}\) | \(\text{unnecessary}\) |
| \(\{1,2,5\}\) | \(\{2\}\) | \(\{4\}\) | \(\{2,4\}\) | \(\{2,3\}\) | \(\{2\}\) | \(\text{unnecessary}\) |
| \(\{1,3,4\}\) | \(\{1,4\}\) | \(\{2\}\) | \(\{1,2,4\}\) | \(\{2,3\}\) | \(\{2\}\) | \(\text{unnecessary}\) |
| \(\{1,3,5\}\) | \(\{1,3\}\) | \(\{4\}\) | \(\{1,3,4\}\) | \(\{2,3\}\) | \(\{3\}\) | \(\text{unnecessary}\) |
| \(\{1,4,5\}\) | \(\{1\}\) | \(\{3\}\) | \(\{1,3\}\) | \(\{2,4\}\) | \(\emptyset\) | \(\{1,2,3,4\}\) |
| \(\{2,3,4\}\) | \(\{4\}\) | \(\{1\}\) | \(\{1,4\}\) | \(\{1,3\}\) | \(\{1\}\) | \(\text{unnecessary}\) |
| \(\{2,3,5\}\) | \(\{3\}\) | \(\{1,4\}\) | \(\{1,3,4\}\) | \(\{1,3\}\) | \(\{1,3\}\) | \(\text{unnecessary}\) |
| \(\{2,4,5\}\) | \(\{2\}\) | \(\{1,3\}\) | \(\{1,2,3\}\) | \(\{1,4\}\) | \(\{1\}\) | \(\text{unnecessary}\) |
| \(\{3,4,5\}\) | \(\emptyset\) | \(\{2\}\) | \(\{1,4\}\) | \(\emptyset\) | \(\emptyset\) | \(\{1,2,4\}\) |
From this, we have
\[
\begin{align*}
    m(k_f(\nu), \kappa_f(\nu)) &= \left( \frac{1}{1 - \nu} \right)^0 \kappa_{\{1,4\}}(\nu) + \left\{ \left( \frac{1}{1 - \nu} \right)^0 + \left( \frac{1}{1 - \nu} \right)^1 \right\} \kappa_{\{2,4\}}(\nu) \\
    &+ \left\{ \left( \frac{1}{1 - \nu} \right)^0 + \left( \frac{1}{1 - \nu} \right)^1 \right\} \kappa_{\{1,2,4\}}(\nu) \\
    &+ \left\{ \left( \frac{1}{1 - \nu} \right)^0 + \left( \frac{1}{1 - \nu} \right)^1 \right\} \kappa_{\{2,3,4\}}(\nu) + \left( \frac{2 - \nu}{1 - \nu} \right) \kappa_{\{1,2,4\}}(\nu) \\
    &+ \left( \frac{2 - \nu}{1 - \nu} \right) \kappa_{\{1,3,4\}}(\nu).
\end{align*}
\]

For two compositions \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l) \) and \( \beta = (\beta_1, \beta_2, \ldots, \beta_k) \), the near-concatenation \( \alpha \odot \beta \) of \( \alpha \) and \( \beta \) is defined by the composition
\[
\alpha \odot \beta := (\alpha_1, \alpha_2, \ldots, \alpha_l + \beta_1, \beta_2, \ldots, \beta_k).
\]

The next result concerns the expansion of \( \mathbf{\check{A}}(k_f(\nu)) \).

**Proposition 3.18.** Let \( n \) be a nonnegative integer. For \( \gamma \in \text{Comp}_n \), we have
\[
\mathbf{\check{A}}(k_{\text{set}}(\gamma)(\nu)) = \sum_{\alpha \odot \beta = \gamma} k_{\text{set}}(\alpha)(\nu) \otimes k_{\text{set}}(\beta)(\nu).
\]

**Proof.** Let \( A := \{(\alpha, \beta) \mid \alpha \cdot \beta = \gamma\} \) and \( B := \{(\alpha, \beta) \mid \alpha \odot \beta = \gamma\} \). Then \( A \cap B = \{(\emptyset, \gamma), (\gamma, \emptyset)\} \), where \( \emptyset \) denotes the empty composition in \( \text{Comp}_0 \). It is well known that \( \Delta(L_{\gamma}) = \sum_{\alpha \beta = \gamma} L_\alpha \otimes L_\beta \) (see [13, Proposition 5.2.15]). Therefore, by Theorem 3.13, we have
\[
\mathbf{\check{A}}(\check{\chi}_{\text{set}}(\gamma)(\nu)) = \sum_{\alpha \beta = \gamma} \check{\chi}_{\text{set}}(\alpha)(\nu) \otimes \check{\chi}_{\text{set}}(\beta)(\nu)
\]
\[
= \sum_{(\alpha, \beta) \in A \cup B} \check{\chi}_{\text{set}}(\alpha)(\nu) \otimes \check{\chi}_{\text{set}}(\beta)(\nu). \tag{3.11}
\]

For simplicity, set \( I := \text{set}(\gamma) \). Combining Lemma 3.8 (b) with (3.11), it can be easily seen that \( A \cup B \) is equal to
\[
\{(\text{comp}(I \cap [1, k - 1]), \text{comp}((I \cap [k + 1, n - 1]) - k)) \mid 1 \leq k \leq n - 1\} \cup \{(\emptyset, \gamma), (\gamma, \emptyset)\}.
\]
For $1 \leq k \leq n - 1$, it holds that
\[
\operatorname{comp}(I) = \begin{cases} \operatorname{comp}(I \cap [1, k-1]) \otimes \operatorname{comp}((I \cap [k+1, n-1]) - k) & \text{if } k \notin I, \\ \operatorname{comp}(I \cap [1, k-1]) \cdot \operatorname{comp}((I \cap [k+1, n-1]) - k) & \text{if } k \in I, \end{cases} \quad (3.12)
\]
and which implies
\[
B = \{(\operatorname{comp}(I \cap [1, k-1]), \operatorname{comp}((I \cap [k+1, n-1]) - k)) \mid k \notin I\} \sqcup \{() \}, \quad (\emptyset, \gamma), (\gamma, \emptyset)\}.
\]
On the other hand, by (3.10), we have that
\[
\Delta_k(\kappa_I(\nu)) = \begin{cases} \kappa_{I \cap [1,k-1]}(\nu) \otimes (\nu_{[k+1,n-1]})^{-1}(\kappa_{I \cap [k+1,n-1]}(\nu)) & \text{if } k \notin I, \\ 0 & \text{if } k \in I \end{cases} \quad (3.13)
\]
for $1 \leq k \leq n - 1$. If $k = 0$ or $n$, then $\Delta_0(\phi) = \mathbb{1}_0 \otimes \phi$ and $\Delta_n(\phi) = \phi \otimes \mathbb{1}_0$, so
\[
\Delta_0(\kappa_{\set(\gamma)}(\nu)) = \kappa_{\set(\emptyset)}(\nu) \otimes \kappa_{\set(\emptyset)}(\nu) \text{ and } \Delta_n(\kappa_{\set(\gamma)}(\nu)) = \kappa_{\set(\gamma)}(\nu) \otimes \kappa_{\set(\emptyset)}(\nu).
\]
Putting these together, we conclude that
\[
\Delta(\kappa_I(\nu)) = \sum_{k=0}^n \Delta_k(\kappa_I(\nu)) = \sum_{(\alpha, \beta) \in B} \kappa_{\set(\alpha)}(\nu) \otimes \kappa_{\set(\beta)}(\nu).
\]
\[\square\]

**Example 3.19.** Let $\gamma = (1, 3, 2) = \operatorname{comp}([1,4])$. All the possible ways to write $\gamma$ as $\alpha \otimes \beta$ are
\[
(1, 3, 2) \otimes (1, 1) = (1, 1) \otimes (2, 2) = (1, 2) \otimes (1, 2) = (1, 3, 1) \otimes (1) = (1, 3, 2) \otimes \emptyset.
\]
Therefore $\Delta(\kappa_{[1,4]}(\nu))$ is equal to
\[
\mathbb{1}_0 \otimes \kappa_{[1,4]}(\nu) + \kappa_{[1]}(\nu) \otimes \kappa_{[2]}(\nu) + \kappa_{[1]}(\nu) \otimes \kappa_{[2]}(\nu) + \kappa_{[1,4]}(\nu) \otimes \mathbb{1}_1 + \kappa_{[1,4]}(\nu) \otimes \mathbb{1}_0,
\]
where
\[
\mathbb{1}_0 \otimes \kappa_{[1,4]}(\nu) \in \operatorname{scf}(\mathcal{N}_0(\nu)) \otimes \operatorname{scf}(\mathcal{N}_0(\nu)), \\
\kappa_{[1]}(\nu) \otimes \kappa_{[2]}(\nu) \in \operatorname{scf}(\mathcal{N}_2(\nu)) \otimes \operatorname{scf}(\mathcal{N}_4(\nu)), \\
\kappa_{[1]}(\nu) \otimes \kappa_{[2]}(\nu) \in \operatorname{scf}(\mathcal{N}_3(\nu)) \otimes \operatorname{scf}(\mathcal{N}_5(\nu)), \\
\kappa_{[1,4]}(\nu) \otimes \mathbb{1}_1 \in \operatorname{scf}(\mathcal{N}_5(\nu)) \otimes \operatorname{scf}(\mathcal{N}_1(\nu)), \\
\kappa_{[1,4]}(\nu) \otimes \mathbb{1}_0 \in \operatorname{scf}(\mathcal{N}_6(\nu)) \otimes \operatorname{scf}(\mathcal{N}_0(\nu)).
\]

4. A new basis for $\operatorname{NSym}_{\mathbb{C}(q,t)}$ and the structure constants

Although defined over the base field $\mathbb{C}$, the Hopf algebras in Section 2.2 can be defined over any nonzero field (see [13]). The main object of the present section is $\operatorname{NSym}_{\mathbb{C}(q,t)}$, the Hopf algebra of noncommutative symmetric functions defined over $\mathbb{C}(q,t)$, where $q$ and $t$ are commuting variables. We introduce and investigate a new basis $\{B(q,t)_\alpha \mid \alpha \in \operatorname{Comp}\}$ for $\operatorname{NSym}_{\mathbb{C}(q,t)}$. We also consider its variant $\{\tilde{B}(q,t)_\alpha \mid \alpha \in \operatorname{Comp}\}$ obtained by reparametrizing the indices. Particular emphasis will be placed on the structure constants
for these bases and Theorem 3.13 will play an important role in performing this. As before, 
\(n\) is assumed to be any nonnegative integer throughout this section.

4.1. A new basis \(\{B(q,t)_{\alpha} \mid \alpha \in \text{Comp}\}\) for \(\text{NSym}_{\mathbb{C}(q,t)}\). We begin with the definition of \(B(q,t)_{\alpha}\).

**Definition 4.1.** For \(I \subseteq [n-1]\), define

\[
B(q,t)_{\text{comp}(I)} := \sum_{J: I \cup J = [n-1]} q^{|J\setminus I|} t^{|J\cap I|} \, H_{\text{comp}(J)} \in \text{NSym}_{\mathbb{C}(q,t)}.
\]

Set

\[
\alpha^c := \text{comp}([n-1] \setminus \text{set}(\alpha))
\]

for all \(\alpha \in \text{Comp}_n\). Then, choose a linear extension \(\preceq^c_{\text{lin.ext.}}\) of the partial order \(\preceq^c\) on \(\{B(q,t)_{\alpha} \mid \alpha \in \text{Comp}_n\}\), where

\[
B(q,t)_{\alpha} \preceq^c B(q,t)_{\beta} \text{ if and only if } B(q,t)_{\alpha^c} \preceq B(q,t)_{\beta^c}.
\]

And, we also choose a linear extension \(\preceq^c_{\text{lin.ext.}}\) of the partial order \(\preceq\) on \(H_{\alpha}\). Since \(I \cup J = [n-1]\) if and only if \(I^c \subseteq J\) if and only if \(\text{comp}(J) \preceq \text{comp}(J)^c\), the transition matrix from the ordered basis \(\{B(q,t)_{\alpha}\}, \preceq^c_{\text{lin.ext.}}\) to the ordered basis \(\{H_{\alpha}\}, \preceq_{\text{lin.ext.}}\) is triangular with non-zero diagonals, Therefore \(\{B(q,t)_{\alpha} \mid \alpha \in \text{Comp}\}\) is a basis for \(\text{NSym}_{\mathbb{C}(q,t)}\). It is quite interesting to note that this basis recovers well known bases for \(\text{NSym}\) such as \(\{\Lambda_{\alpha}\}, \{H_{\alpha}\}, \{E^*_\alpha\}\) via suitable specializations of \(q\) and \(t\).

**Proposition 4.2.** For each \(\alpha \in \text{Comp}\), we have

(a) \(B(1,0)_{\alpha} = H_{\alpha^c}\),
(b) \(B(-1,1)_{\alpha} = \Lambda_{\alpha^c}\),
(c) \(B(1,-1)_{\alpha} = E^*_\alpha\).

**Proof.** (a) The assertion follows from the identity

\[
\Lambda_{\alpha} = \sum_{\beta \leq \alpha} (-1)^{n-\ell(\beta)} \, H_{\beta}.
\]

(b) Recall that the noncommutative elementary symmetric function \(\Lambda_{\alpha}\) is given by

\[
\Lambda_{\alpha} = \sum_{\beta \leq \alpha} (-1)^{n-\ell(\beta)} \, H_{\beta}.
\]

Therefore the assertion follows from the calculation below:

\[
\Lambda_{\text{comp}(I^c)} = \sum_{I^c \subseteq J} (-1)^{n-|J|+1} \, H_{\text{comp}(J)}
\]

\[
= \sum_{I^c \subseteq J} (-1)^{|J^c|} \, H_{\text{comp}(J)}
\]

\[
= \sum_{J: I \cup J = [n-1]} (-1)^{|J\setminus I|} \, H_{\text{comp}(J)}.
\]
(c) Recall that the dual essential quasisymmetric function $E^*_\alpha$ is given by
\[
E^*_\alpha = \sum_{\beta \leq \alpha} (-1)^{\ell(\beta) - \ell(\alpha)} H_\beta.
\]
Therefore,
\[
E^*_{\text{comp}(I^c)} = \sum_{I^c \subseteq J} (-1)^{|J|+1-(|I^c|+1)} H_{\text{comp}(J)} = \sum_{I^c \subseteq J} (-1)^{|J|-|I^c|} H_{\text{comp}(J)}.
\]
This completes the proof since $I^c \subseteq J$ implies $|J| - |I^c| = |I \cap J|$.

4.2. The structure constants of $\text{NSym}_{\mathbb{C}(q,t)}$ for $\{B(q,t)_{\alpha} \mid \alpha \in \text{Comp}\}$. As in Section 3, $\nu$ denotes any positive integer $> 1$. For each $I \subseteq [n - 1]$, let
\[
\Pi(\nu)_{\text{comp}(I)} := \text{ch}_\nu \left( \frac{\kappa_I(\nu)}{(\nu - 1)|I|} \right).
\]
Since $\{\kappa_I(\nu) : I \subseteq [n - 1]\}$ is a basis for $\text{scf}(\mathcal{N}_n(\nu))$ and $\text{ch}_\nu$ is an isomorphism, $\{\Pi(\nu)_{\alpha} : \alpha \in \text{Comp}_n\}$ is a basis for $\text{QSym}_n$. The following lemma shows the changes of basis between $\{L_\alpha : \alpha \in \text{Comp}_n\}$ and $\{\Pi(\nu)_{\alpha} : \alpha \in \text{Comp}_n\}$.

Lemma 4.3. Let $I, J \subseteq [n - 1]$. Then
\begin{enumerate}
  \item $L_{\text{comp}(I)} = \sum_J (-1)^{|J|}(\nu - 1)^{|I \cap J|} \Pi(\nu)_{\text{comp}(J)}$ and
  \item $\Pi(\nu)_{\text{comp}(J)} = \sum_I \frac{1}{\nu^{n-1}} (-1)^{|J|}(\nu - 1)^{|I \cup J|} L_{\text{comp}(I)}$.
\end{enumerate}

Proof. For $I, J \subseteq [n - 1]$, let $\chi^I_J(\nu)$ be the coefficient of $\kappa_J(\nu)$ in $\chi^I(\nu)$, that is,
\[
\chi^I_J(\nu) = \sum_I \chi^I_J(\nu) \kappa_J(\nu).
\]
Proposition 3.2 implies that $\chi^I_J(\nu) = (-1)^{|I \cap J|}(\nu - 1)^{|I \cup J|}$. Let $\langle \cdot, \cdot \rangle$ be the Hall-inner product on $\text{cf}(Q_n(\nu))$, that is,
\[
\langle \phi, \psi \rangle = \frac{1}{|Q_n(\nu)|} \sum_{g \in Q_n(\nu)} \phi(g) \psi(g^{-1}) \quad (\phi, \psi \in \text{cf}(Q_n(\nu))).
\]
By Proposition 3.2 and Lemma 3.15, we see that
\[
\langle \chi^I(\nu), \chi^I(\nu) \rangle = (\nu - 1)^{|I^c|} \quad \text{and} \quad \langle \kappa_I(\nu), \kappa_I(\nu) \rangle = \frac{\nu^{n-1}}{(\nu - 1)|I|}.
\]
Letting
\[
\chi^I(\nu) := \sqrt{\frac{1}{(\nu - 1)^{|I^c|}}} \chi^I(\nu) \quad \text{and} \quad \kappa_I(\nu) := \sqrt{\frac{(\nu - 1)|I|}{\nu^{n-1}}} \kappa_I(\nu),
\]
one can see that \( \{ \chi^I(\nu) : I \subseteq [n-1] \} \) and \( \{ \kappa^I(\nu) : I \subseteq [n-1] \} \) are orthonormal bases for \( \text{scf}(\mathcal{N}_n(\nu)) \). Using this notation, we can rewrite (4.1) as

\[ \chi^I(\nu) = \sum_J \chi^I_J(\nu) \kappa_J(\nu) \]  

(4.2)

with

\[ \chi^I_J(\nu) := \frac{\chi^I_J(\nu)}{\sqrt{(\nu - 1)|J|}} \sqrt{\frac{\nu - 1}{\nu^{n-1}}}. \]

Since the matrix \( (\chi^I_J(\nu))_{I,J} \) is unitary, we have

\[ \kappa^I(\nu) = \sum_I \chi^I_J(\nu) \chi_I(\nu). \]  

(4.3)

Now, our assertions can be obtained by taking \( \text{ch}_\nu \) on both sides (4.2) and (4.3).

The following lemma shows the change of basis for \( \{ M_\alpha \} \) and \( \{ \Pi(\nu)_\alpha \} \), where \( \{ M_\alpha \} \) is the basis of the monomial quasisymmetric functions.

**Lemma 4.4.** Let \( I, J \subseteq [n-1] \). Then

(a) \( M_{\text{comp}(I)} = \sum_{J : I \cup J = [n-1]} (-\nu)^{|J\setminus I|}(\nu - 1)^{|I \cap J|}\Pi(\nu)_{\text{comp}(J)} \) and

(b) \( \Pi(\nu)_{\text{comp}(J)} = \left( \frac{1}{1 - q} \right)^{|J|} \sum_{I : I \cap J = \emptyset} \left( \frac{q - 1}{q} \right)^{(n-1) - |I|} M_{\text{comp}(I)}. \)

**Proof.** (a) In view of Lemma 4.3 (a), we have that

\[ M_{\text{comp}(I)} = \sum_{K: I \subseteq K} (-1)^{|K\setminus I|} L_{\text{comp}(K)} \]

\[ = \sum_{K: I \subseteq K} (-1)^{|K\setminus I|} \left( \sum_J (-1)^{|J\setminus K|}(\nu - 1)^{|K \cap J|}\Pi(\nu)_{\text{comp}(J)} \right) \]

\[ = \sum_J \left( \sum_{K: J \subseteq K} (-1)^{|K\setminus J|}(\nu - 1)^{|K \cap J|} \right) \Pi(\nu)_{\text{comp}(J)}. \]

For simplicity, we use the following abbreviations:

\[ V := ([n - 1] \setminus (I \cup J)) \cap K \quad \text{and} \quad W := (J \setminus I) \cap K. \]
Then $K = I \sqcup V \sqcup W$, and therefore our assertion follows from the calculation below:

$$
\sum_{K: I \subseteq K} (-1)^{|K \setminus I|} (-1)^{|I \cap K|} (\nu - 1)^{|K \cap J|} = \\
\sum_{V \subseteq [n-1] \setminus (I \cup J)} (-1)^{|V|} (-1)^{|I \cap V|} (\nu - 1)^{|I \cap J| + |V|} = \\
(-1)^{|I \cap J|} (\nu - 1)^{|I \cap J|} \sum_{V \subseteq [n-1] \setminus (I \cup J)} (-1)^{|V|} (\nu - 1)^{|V|} = \\
(-1)^{|I \cap J|} (\nu - 1)^{|I \cap J|} 0^{[n-1] \setminus (I \cup J)} \nu^{|J \cap I|}.
$$

The last equality can be derived by applying the binomial expansion formula.

(b) Combining Lemma 4.3 (b) with $L_{\text{comp}}(K) = \sum_{I: K \subseteq I} M_{\text{comp}(I)}$, we derive that

$$
\Pi(\nu)_{\text{comp}(J)} = \sum_{K} \frac{1}{\nu^{n-1}} (-1)^{|K \setminus I|} (\nu - 1)^{|I \cup J|} \left( \sum_{I: K \subseteq I} M_{\text{comp}(I)} \right) = \\
\frac{1}{\nu^{n-1}} \sum_{I} \left( \sum_{K: K \subseteq I} (-1)^{|K \setminus I|} (\nu - 1)^{|I \cup J|} \right) M_{\text{comp}(I)}.
$$

We use the following abbreviations:

$$X := (I \cap J) \cap K \quad \text{and} \quad Y := (I \setminus J) \cap K.$$

Then it holds that $K = V \sqcup Y$ whenever $K \subseteq I$. Now, our assertion follows from the calculation below:

$$
\sum_{K: K \subseteq I} (-1)^{|K \setminus I|} (\nu - 1)^{|I \cap J|} = \\
\sum_{X \subseteq I \cap J} (-1)^{|I \cap J| - |X|} (\nu - 1)^{(n-1) - |I \cap J| - |Y|} = \\
(-1)^{|J|} 0^{|I \cap J|} (\nu - 1)^{(n-1) - |J|} \left( 1 + \frac{1}{\nu - 1} \right)^{|I \setminus J|}.
$$

□

Remark 4.5. Let $M$ be the $2^{n-1} \times 2^{n-1}$ matrix whose columns and rows are indexed by subsets of $[n-1]$ and with entries

$$M_{I,J} = \begin{cases} 
q^{|J \setminus I|} & \text{if } I \cup J = [n-1], \\
0 & \text{otherwise}.
\end{cases}$$
In fact, it is the transition matrix from $\mathcal{B}(q,t)_{\text{comp}(I)}$ to $H_{\text{comp}(J)}$. Lemma 4.4 (a) implies that $\Pi(\nu)_{\text{comp}(I)} = \mathcal{B}(-\nu,\nu - 1)_{\text{comp}(I)}^*$ since $\{M_\alpha\}$ is the dual basis of $\{H_\alpha\}$. Hence Lemma 4.4 (b) implicitly implies that the inverse matrix $N$ of $M$ is given by

$$N_{I,J} = \begin{cases} q^{[|J|-(n-1)(-t)^{(n-1)-|I|-|J|}]} & \text{if } I \cap J = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

This can be verified by a direct calculation. Consequently, letting $\{\mathcal{B}(q,t)_\alpha^* \mid \alpha \in \text{Comp}\}$ be the dual basis of $\{\mathcal{B}(q,t)_\alpha \mid \alpha \in \text{Comp}\}$, we have the following expansion:

$$\mathcal{B}(q,t)^*_{\text{comp}(I)} = \sum_{J: I \cap J = \emptyset} q^{[|J|-(n-1)(-t)^{(n-1)-|I|-|J|}]} M_{\text{comp}(J)}.$$

Next, we study $\Pi(q)^*$, the dual basis of $\Pi(q)$, which plays an important role in proving Theorem 4.7.

**Lemma 4.6.** Let $m$, $n$, and $k$ be nonnegative integers.

(a) Let $K \subseteq [k-1]$. Then

$$\triangle \Pi(\nu)^*_{\text{comp}(K)} = \sum_{\substack{m+n=k \\ I \subseteq [m-1] \\ J \subseteq [n-1]}} C^K_{I,J}(\nu)(\Pi(\nu)^*_{\text{comp}(I)} \otimes \Pi(\nu)^*_{\text{comp}(J)}),$$

where

$$C^K_{I,J}(\nu) = \frac{1}{(\nu - 1)^{|I|+|J|}} \sum_{A \in \binom{\binom{m+n}{n}}{I \cap A \subseteq [n-1]; I \not\subseteq [I \cap A]; J \cup A \subseteq [n-1]}} (-1)^{|K \cap c_2(A)}|\nu - 1|^{K \setminus c_2(A)}.$$

(b) For $\alpha \in \text{Comp}_m$ and $\beta \in \text{Comp}_n$,

$$\Pi(\nu)^*_\alpha \Pi(\nu)^*_\beta = \Pi(\nu)^*_{\alpha \odot \beta}.$$

**Proof.** Recall that the structure constant for the product (coproduct) of $\Pi^*$ is equal to the structure constant for the coproduct (product) of $\Pi$, that is,

$$\Pi(\nu)^*_\alpha \Pi(\nu)^*_\beta = \sum C^\gamma_{\alpha,\beta} \Pi(\nu)^*_\gamma \iff \triangle \Pi(\nu)^*_\gamma = \sum C^\gamma_{\alpha,\beta}(\Pi(\nu)^*_\alpha \otimes \Pi(\nu)^*_\beta).$$

Also, we recall that $\text{ch}_\nu(\kappa_I(\nu)) = (\nu - 1)^{|I|}\Pi(\nu)_{\text{comp}(I)}$

(a) The assertion follows from Proposition 3.16 and the equality

$$\left(\frac{1}{1-\nu}\right)^{|K\cap c_2(A)}(\nu - 1)^{|K|} = (-1)^{|K\cap c_2(A)}(\nu - 1)^{|K\setminus c_2(A)|}.$$

(b) The assertion follows from Proposition 3.18. □
We are now ready to state the main result of this subsection.

**Theorem 4.7.** Let $m$, $n$, and $k$ be nonnegative integers.

(a) Let $K \subseteq [k-1]$. Then

$$\Delta B(q,t)_{\text{comp}(K)} = \sum_{m+n=k \atop I \subseteq [m-1] \atop J \subseteq [n-1]} C^K_{I,J}(q,t) \left( B(q,t)_{\text{comp}(I)} \otimes B(q,t)_{\text{comp}(J)} \right),$$

where

$$C^K_{I,J}(q,t) = t^{-|I|-|J|} \sum_{A \in \binom{[m+n]}{n}} (-1)^{|K \cap c_2(A)| (q-t)^{|I \cap c_2(A)|}}.$$

(b) For $\alpha \in \text{Comp}_m$, $\beta \in \text{Comp}_n$,

$$B(q,t)_\alpha B(q,t)_\beta = B(q,t)_{\alpha \otimes \beta}.$$

**Proof.** (a) For each integer $\nu > 1$ we have that $B(-\nu, \nu-1)_{\text{comp}(t)} = \Pi(\nu)^*_{\text{comp}(t)}$ by Lemma 4.4 (a). Let

$$\Delta B(-\nu, \nu-1)_{\text{comp}(K)} = \sum_{I,J} C^K_{I,J}(\nu) \left( B(-\nu, \nu-1)_{\text{comp}(I)} \otimes B(-\nu, \nu-1)_{\text{comp}(J)} \right).$$

By Lemma 4.6, we have

$$C^K_{I,J}(\nu) = \frac{1}{(\nu-1)^{|I|+|J|}} \sum_{A \in \binom{[m+n]}{n}} (-1)^{|K \cap c_2(A)| (\nu-1)^{|I \cap c_2(A)|}}.$$

Since $\nu$ ranges over the infinite set $\{2, 3, \ldots\}$,

$$C^K_{I,J}(q) = \frac{1}{(q-1)^{|I|+|J|}} \sum_{A \in \binom{[m+n]}{n}} (-1)^{|K \cap c_2(A)| (q-1)^{|I \cap c_2(A)|}}$$

as rational functions in $q$. On the other hand, one can easily see that

$$B(q,t)_{\text{comp}(K)} = (-q-t)^{|K|} B(-Q, Q-1)_{\text{comp}(K)},$$

where $Q = \frac{q}{q+t}$. Taking $\Delta$ on both sides, we obtain the desired result.

(b) The assertions can be obtained by using similar arguments as in (a). \qed
In the rest of this subsection, we extend Theorem 4.7 to $\text{NSym}_k(q,t)$, where $k$ is an arbitrary nonzero field with identity $1$. Let us explain the problem in more detail. Irregardless of the field $k$, the coproduct of $\text{NSym}_k(q,t)$ is universally expressed by the rule:

$$\triangle H_n = \sum_{i+j=n} H_i \otimes H_j$$

(for more discussion, we refer the readers to [13, Thm5.4.2]). However, it is not obvious if this phenomenon happens for other bases, particularly for $\{\hat{B}(q,t)_{\alpha}\}$.

For our purpose, we first observe that Definition 4.1 still works for $\text{NSym}_k(q,t)$. This is because the diagonal entries in the transition matrix from $\{\hat{B}(q,t)_{\alpha}\}, \preceq_{\text{lin.ext.}}$ to $(\{H_{\alpha}\}, \preceq_{\text{lin.ext.}})$ are all units in $k(q,t)$. Next, let us consider the ring homomorphism

$$- : \mathbb{Z} \to k, \quad a \mapsto \overline{a} := a \cdot 1.$$  

Clearly it induces the ring homomorphism

$$- : \mathbb{Z}[q,t] \to k[q,t], \quad \sum_{i,j} a_{ij} q^i t^j \mapsto \sum_{i,j} \overline{a_{ij}} q^i t^j.$$  

The following corollary shows that the coproduct rule of $\text{NSym}_k(q,t)$ is universally described not only for $\{H_{\alpha}\}$ but also for $\{\hat{B}(q,t)_{\alpha}\}$.

**Corollary 4.8.** Let $k$ be an arbitrary nonzero field.

(a) The polynomials $C^K_{I,J}(q,t)$ in Theorem 4.7(a) belong to $\mathbb{Z}[q,t]$.

(b) Let $k \geq 0$ and $K \subseteq [k - 1]$. Then

$$\triangle \hat{B}(q,t)_{\text{comp}(K)} = \sum_{\substack{m+n=k \\ I \subseteq [m-1] \\ J \subseteq [n-1]}} C^K_{I,J}(q,t)(B(q,t)_{\text{comp}(I)} \otimes B(q,t)_{\text{comp}(J)}).$$

(c) For $\alpha \in \text{Comp}_m, \beta \in \text{Comp}_n$,

$$B(q,t)_{\alpha}B(q,t)_{\beta} = B(q,t)_{\alpha \circ \beta}.$$  

**Proof.** (a) Note that $|I \# A J| = |I| + |J|$. Using this, one can see that $|I| + |J| \leq |K \setminus c_2(A)|$ if $(I \# A J) \cap c(A) = \emptyset$ and $I \# A J \subseteq K$.

(b) The assertion follows from (a).

(c) The assertion can be obtained in the same way as (b) is obtained from (a).

\[\square\]

4.3. A variant of $\{\hat{B}(q,t)_{\alpha} \mid \alpha \in \text{Comp}\}$. In this subsection, we consider a variant $\{\hat{B}(q,t)_{\alpha} \mid \alpha \in \text{Comp}\}$ of $\{B(q,t)_{\alpha} \mid \alpha \in \text{Comp}\}$, where $\hat{B}(q,t)_{\alpha} := B(q,t)_{\alpha \cdot c}$. The main reason we are considering it is because most formulas can be expressed in simpler forms on this basis compared to $\{B(q,t)_{\alpha} \mid \alpha \in \text{Comp}\}$. 

For $k \geq 0$, we set $\widehat{B}(q, t)_k := \widehat{B}(q, t)_{(k)}$. Generally, we can call a $k$-basis $\{B_\alpha \mid \alpha \in \text{Comp}\}$ of $Q\text{Sym}_{k}$ or $N\text{Sym}_{k}$ multiplicative if $B_\alpha B_\beta = B_{\alpha \beta}$ for $\alpha, \beta \in \text{Comp}$. Similarly, we can call a $k$-basis $\{B_\lambda \mid \lambda \in \text{Par}\}$ of $\text{Sym}_{k}$ multiplicative if $B_\lambda B_\mu = B_{\lambda \mu}$ for $\lambda, \mu \in \text{Par}$. And, we denote by $\{\Pi(q)_\alpha \mid \alpha \in \text{Comp}_n\}$ the dual basis of $\{\Pi(q)_\alpha \mid \alpha \in \text{Comp}_n\}$. In the following, we will investigate how Theorem 4.7 (a) is written for $\widehat{B}(q, t)_k$. In fact, it can be written in a much simpler form than it is now. To see this, for $A \subseteq [k]$, we set $\text{conn}(A) := (A_1, A_2, \ldots, A_l)$ and $\text{conn}(A^c) := ((A^c)_1, (A^c)_2, \ldots, (A^c)_l)$, i.e., the collections of maximally connected subsets of $A$ and $A^c$, respectively. Then we define two compositions $\alpha_A$ and $\beta_A$ as follows:

$$\alpha_A := (|A_1|, |A_2|, \ldots, |A_l|) \quad \text{and} \quad \beta_A := (|(A^c)_1|, |(A^c)_2|, \ldots, |(A^c)_l|).$$

The main result of this subsection is stated in the following form.

**Theorem 4.9.** Let $k$ be a nonnegative integer. Then we have

(a) The basis $\{\widehat{B}(q, t)_\alpha\}$ is multiplicative.

(b) $\{\widehat{B}(q, t)_k \mid k \geq 0\}$ is a generating set of $N\text{Sym}_{C(q, t)}$.

(c) For the generators $\widehat{B}(q, t)_k$, the coproduct formula in Theorem 4.7 (a) reads as follows:

$$\Delta \widehat{B}(q, t)_k = \sum_{A \subseteq [k]} (q + t)^{|c_2(A)|}t^{|c_1(A)|} \left(\widehat{B}(q, t)_{\alpha_A} \otimes \widehat{B}(q, t)_{\beta_A}\right).$$

In particular, by letting $(q, t) = (1, 0)$ and $(q, t) = (-1, 1)$, we can recover the formulas:

$$\Delta H_k = \sum_{i+j=k} H_i \otimes H_j \quad (\text{in } N\text{Sym}),$$

$$\Delta \Lambda_k = \sum_{i+j=k} \Lambda_i \otimes \Lambda_j \quad (\text{in } N\text{Sym}).$$

**Proof.** (a) Let $\widehat{\Pi}(q)_\alpha := \Pi(q)_\alpha^c$. One can show

$$(\alpha \otimes \beta)^c = \alpha^c \cdot \beta^c,$$

which follows from (3.12). Thus, by Lemma 4.6 (b), $\{\widehat{\Pi}(q)_\alpha : \alpha \in \text{Comp}_n\}$ is multiplicative. Now, the assertions can be obtained by using similar arguments as in the proof of Theorem 4.7.

(b) By (a), we have

$$\widehat{B}(q, t)_\alpha = \widehat{B}(q, t)_{\alpha_1} \widehat{B}(q, t)_{\alpha_2} \cdots \widehat{B}(q, t)_{\alpha_l}$$

for every composition $\alpha = (\alpha_1, \ldots, \alpha_l)$, which proves the assertion.
(c) Fix an integer \(0 \leq m \leq k\) and let \(n := k - m\) and \(K := [k - 1]\). For \(I \subseteq [m - 1]\), and \(J \subseteq [n - 1]\), we showed in Theorem 4.7 (a) that
\[
C^K_{I,J}(q, t) = \sum_{A \in \binom{[m+n]}{n}} (q + t)^{|c_2(A)|} t^{(k-1)-|I|-|J|-|c_2(A)|}.
\]

Choose a subset \(A \in \binom{[m+n]}{n}\). Let \((I_A, J_A)\) be a pair satisfying the conditions:
- \(I_A \# A J_A \subseteq [m + n - 1] \setminus c(A)\), and
- \(I_A \# A J_A \subseteq K \subseteq (I_A \# A J_A) \cup c(A)\).

Then
\[(I_A \# A J_A) \cup c(A) = [m + n - 1],
\]
which implies that \((I_A, J_A)\) is uniquely determined by \(A\). To be precise,
\[
I_A = [m - 1] \setminus \{(A^c)_1, (A^c)_1 + (A^c)_2, \ldots, (A^c)_1 + (A^c)_2 + \ldots + (A^c)_{t-1}\},
\]
\[J_A = [n - 1] \setminus \{|A_1|, |A_1| + |A_2|, \ldots, (|A_1| + |A_2| + \ldots + |A_{t-1}|)\}.
\]
Therefore,
\[
C^K_{I,J}(q, t) = \begin{cases} (q + t)^{|c_2(A)|} t^{(k-1)-|I|-|J|-|c_2(A)|} & \text{if } I = I_A, J = J_A \text{ for some } A, \\ 0 & \text{otherwise}. \end{cases}
\]

Note that \(|I_A| = (m - 1) - |\text{conn}(A^c)| - 1\), \(|J_A| = (n - 1) - |\text{conn}(A)| - 1\), and
\[
|\text{conn}(A)| = \begin{cases} |c_1(A)| & \text{if } k \in A_t, \\ |c_1(A)| + 1 & \text{if } k \notin A_t \end{cases}
\]
\[
|\text{conn}(A^c)| = \begin{cases} |c_2(A)| & \text{if } k \in (A^c)_t, \\ |c_2(A)| + 1 & \text{if } k \notin (A^c)_t. \end{cases}
\]

Furthermore, we see that \(k \in (A^c)_i\) if and only if \(k \notin A_t\), thus
\[(k - 1) - |I_A| - |J_A| - |c_2(A)| = |\text{conn}(A)| + |\text{conn}(A^c)| - |c_2(A)| - 1 = |c_1(A)|.
\]

Now the first assertion follows from the observation that \(\alpha_A = \text{comp}((I_A)^c)\) and \(\beta_A = \text{comp}((J_A)^c)\).

Since
\[
|c_1(A)| = 0 \text{ if and only if } A = [k] \setminus [i] \text{ for some } 0 \leq i \leq k,
\]
\[
|c_2(A)| = 0 \text{ if and only if } A^c = [k] \setminus [i] \text{ for some } 0 \leq i \leq k,
\]
we have
\[
C^K_{I,J}(1, 0) = \begin{cases} 1 & \text{if } I = \{1, 2, \ldots, i - 1\}, J = \{i + 1, i + 2, \ldots, k - 1\} \text{ for some } 0 \leq i \leq k, \\ 0 & \text{otherwise}, \end{cases}
\]
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\( C^K_{I,J}(−1, 1) = \begin{cases} 1 & \text{if } J = \{1, 2, \ldots, i − 1\}, I = \{i + 1, i + 2, \ldots, k − 1\} \text{ for some } 0 ≤ i ≤ k, \\ 0 & \text{otherwise.} \end{cases} \)

This justifies the second assertion. \( \square \)

**Example 4.10.** The following table shows the calculations of \( c_1(A), c_2(A), α_A, \) and \( β_A \) for each \( A ∈ [3] \).

| \( A \) | \( c_1(A) \) | \( c_2(A) \) | \( α_A \) | \( β_A \) |
|---|---|---|---|---|
| \( ∅ \) | \( ∅ \) | \( ∅ \) | \( 0 \) | \( (3) \) |
| \( \{1\} \) | \( \{1\} \) | \( 0 \) | \( 0 \) | \( (1) \)
| \( \{2\} \) | \( \{2\} \) | \( \{1\} \) | \( 0 \) | \( (1, 1) \)
| \( \{3\} \) | \( ∅ \) | \( \{2\} \) | \( 0 \) | \( (2) \)
| \( \{1,2\} \) | \( \{2\} \) | \( ∅ \) | \( (2) \) | \( (1) \)
| \( \{1,3\} \) | \( \{2\} \) | \( \{1\} \) | \( (1) \) | \( (1) \)
| \( \{2,3\} \) | \( ∅ \) | \( \{2\} \) | \( (2) \) | \( (1) \)
| \( \{1,2,3\} \) | \( ∅ \) | \( ∅ \) | \( (3) \) | \( ∅ \) |

Thus, Theorem 4.9 (c) says that

\[
\triangle \tilde{B}(q,t)_3 = (1 \otimes \tilde{B}(q,t)_3) + (q + 2t) \left( \tilde{B}(q,t)_1 \otimes \tilde{B}(q,t)_2 \right) + (q + t)t \left( \tilde{B}(q,t)_1 \otimes \tilde{B}(q,t)_1 \right) \\
+ t \left( \tilde{B}(q,t)_2 \otimes \tilde{B}(q,t)_1 \right) + (q + t)t \left( \tilde{B}(q,t)_1 \otimes \tilde{B}(q,t)_1 \right) \\
+ (q + t) \left( \tilde{B}(q,t)_2 \otimes \tilde{B}(q,t)_1 \right) + \left( \tilde{B}(q,t)_3 \otimes 1 \right).
\]

In the rest of this subsection, we provide some interesting consequences obtained by specializations of \( q \) and \( t \). The first consequence can be obtained by combining Theorem 4.7 and the surjective Hopf algebra homomorphism in (2.1)

\[ \text{comm} : \text{NSym} \rightarrow \text{Sym}, \quad H_n \mapsto h_n. \]

**Corollary 4.11.** Let \( a ∈ \mathbb{C} \setminus \{0\} \) and \( b ∈ \mathbb{C} \). Then the following hold.

(a) \( \{\tilde{B}(a,b)_n \mid n ≥ 0\} \) is a generating set of \( \text{NSym} \).

(b) \( \{\text{comm}(\tilde{B}(a,b)_n) \mid n ≥ 0\} \) is a generating set of \( \text{Sym} \).

**Proof.** (a) The diagonal entries of the transition matrix from \( \tilde{B}(a,b)_{\text{comp}(I)} \) to \( H_{\text{comp}(J)} \) are non-zero if \( a ≠ 0 \), and therefore \( \{\tilde{B}(a,b)_α \mid α ∈ \text{Comp}\} \) is a basis for \( \text{NSym} \) if \( a ≠ 0 \). On the other hand, from Theorem 4.9 (a) it follows that all \( \tilde{B}(a,b)_α \)'s are generated by \( \tilde{B}(a,b)_n (n ≥ 0) \).

(b) Since \( \text{comm} \) is an algebra homomorphism, the assertion follows from (a). \( \square \)
It should be remarked that \( \widehat{B}(a, b)_n \) and \( \text{comm}(\widehat{B}(a, b)_n) \) in Corollary 4.11 can be written in a more concrete form as follows:

\[
\widehat{B}(a, b)_n = \sum_{J \subseteq [n-1]} a^{(n-1)-|J|} b^{|J|} H_{\text{comp}(J)} = \sum_{\beta \in \text{Comp}_n} a^{n-\ell(\beta)} b^{\ell(\beta)-1} H_\beta,
\]

and therefore

\[
\text{comm}(\widehat{B}(a, b)_n) = \sum_{\beta \in \text{Comp}_n} a^{n-\ell(\beta)} b^{\ell(\beta)-1} h_\lambda(\beta) = \sum_{\lambda \in \text{Par}_n} a^{n-\ell(\lambda)} b^{\ell(\lambda)-1} C_\lambda h_\lambda,
\]

where \( C_\lambda = \ell(\lambda)! / \prod_i m_i(\lambda)! \) for \( \lambda = 1^{m_1(\lambda)} 2^{m_2(\lambda)} \ldots \).

The second consequence concerns a symmetry on the set \( \{ \widehat{B}(a, b)_\alpha | a, b \in \mathbb{C}, \alpha \in \text{Comp} \} \). Recall that \( \text{NSym} \) has the involutive anti-homomorphism

\[
\omega : \text{NSym} \to \text{NSym}, \quad H_\alpha \mapsto \Lambda_\alpha (\alpha \in \text{Comp})
\]

(see [11, Chapter 4]). Proposition 4.2 (a) and (b) tell us that \( \omega(\widehat{B}(1, 0)_\alpha) = \widehat{B}(-1, 1)_\alpha \).

In fact, this is a special case of the following proposition.

**Proposition 4.12.** For any \( \alpha \in \text{Comp} \) and \( a, b \in \mathbb{C} \),

\[
\omega(\widehat{B}(a, b)_\alpha) = \widehat{B}(-a, a + b)_\alpha.
\]

**Proof.** It is straightforward to check the equality when \( a = 0 \), so we assume \( a \neq 0 \). Since \( \omega \) is an involutive anti-homomorphism and \( \{ \widehat{B}(a, b)_\alpha \} \) is multiplicative by Theorem 4.9 (a), we only have to show that \( \omega(\mathcal{B}(-a, a + b)_{\text{comp}([n-1])}) = \mathcal{B}(a, b)_{\text{comp}([n-1])} \) for each \( n \geq 1 \). Note that we have

\[
\omega(\mathcal{B}(-a, a + b)_{\text{comp}([n-1])}) = \sum_{J \subseteq [n-1]} (-a)^{(n-1)-|J|} (a + b)^{|J|} \Lambda_{\text{comp}(J)}
\]

\[
= \sum_{J \subseteq [n-1]} (-a)^{(n-1)-|J|} (a + b)^{|J|} \Lambda_{\text{comp}(J)},
\]

since \( |\text{set}(\text{comp}(J)^r)| = |J| \). By [11, Proposition 4.3], we have

\[
\Lambda_{\text{comp}(J)} = \sum_{J \subseteq K} (-1)^{(n-1)-|K|} H_{\text{comp}(K)}
\]
for \( J \subseteq [n-1] \). Therefore,

\[
\omega(B(-a, a + b)_{\text{comp}(\alpha)}) = \sum_{J \subseteq [n-1]} (-a)^{(n-1)-|J|} (a + b)^{|J|} \left( \sum_{K \subseteq J} (-1)^{(n-1)-|K|} H_{\text{comp}(K)} \right)
\]

\[
= \sum_{K \subseteq [n-1]} (-1)^{(n-1)-|K|} \left( \sum_{J \subseteq K} (-a)^{(n-1)-|J|} (a + b)^{|J|} \right) H_{\text{comp}(K)}
\]

\[
= \sum_{K \subseteq [n-1]} (-1)^{(n-1)-|K|} (-a)^n (\frac{a + b}{a} + 1)^{|K|} H_{\text{comp}(K)}
\]

\[
= \sum_{K \subseteq [n-1]} a^{(n-1)-|K|} b^{|K|} H_{\text{comp}(K)}
\]

\[
= B(a, b)_{\text{comp}(\alpha)}.
\]

□

Let \( \text{Comp}_{n,l} \) (resp. \( \text{Par}_{n,l} \)) be the set of compositions (resp. partitions) of \( n \) with length \( l \). Then \( \mathcal{S}_l \) acts on \( \text{Comp}_{n,l} \) on the right by place permutations, i.e.,

\[
(\alpha_1, \alpha_2, \ldots, \alpha_l) \cdot \sigma = (\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \ldots, \alpha_{\sigma(l)}).
\]

Clearly \( \text{Par}_{n,l} \) is a complete set of orbit-representatives. With this preparation, we state the third consequence.

**Proposition 4.13.** Let \( a \in \mathbb{C} \setminus \{0\} \) and \( b \in \mathbb{C} \). Then \( \{\text{comm}(\hat{B}(a, b)_\lambda) \mid \lambda \in \text{Par}\} \) is a multiplicative basis for \( \text{Sym} \).

**Proof.** Since \( \{\hat{B}(a, b)_\alpha \mid \alpha \in \text{Comp}_n\} \) is a multiplicative basis for \( \text{NSym} \) by Theorem 4.9 (a) and \( \text{comm} : \text{NSym} \rightarrow \text{Sym} \) is a surjective algebra homomorphism, we have only to show that \( \{\text{comm}(\hat{B}(a, b)_\lambda) \mid \lambda \in \text{Par}\} \) is a basis for \( \text{Sym} \). Note that \( \{\text{comm}(\hat{B}(a, b)_\alpha) \mid \alpha \in \text{Comp}_n\} \) is a spanning set of \( \text{Sym}_n \) for each \( n \geq 0 \). Hence, for our purpose, it suffices to show that \( \text{comm}(\hat{B}(a, b)_\alpha) = \text{comm}(\hat{B}(a, b)_\beta) \) whenever \( \alpha, \beta \in \text{Comp}_{n,l} \) are in the same orbit. To see this, we first recall that

\[
\hat{B}(q, t)_\alpha = \sum_{\gamma \preceq \alpha} q^{|\text{set}(\alpha) \setminus \text{set}(\gamma)|} t^{|\text{set}(\alpha) \cap \text{set}(\gamma)|} H_{\gamma}.
\]

Let \( \Gamma_\alpha := \{\gamma \mid \gamma \preceq \alpha\} \) and \( \Gamma_\beta := \{\gamma \mid \gamma \preceq \beta\} \). Since \( \text{comm}(H_\alpha) = \text{comm}(H_\beta) = h_{\lambda(\alpha)} \), we have only to see that there is a bijection \( \Phi : \Gamma_\alpha \rightarrow \Gamma_\beta \) such that \( \lambda(\gamma) = \lambda(\Phi(\gamma)) \) and \( |\text{set}(\alpha) \cap \text{set}(\gamma)| = |\text{set}(\beta) \cap \text{set}(\Phi(\gamma))| \). For \( \gamma \preceq \alpha \), we write it as \( (\gamma^{(1)}; \gamma^{(2)}; \ldots; \gamma^{(l)}) \) where \( \gamma^{(i)} \in \text{Comp}_{\alpha_i} \).

Then we define \( \Phi : \Gamma_\alpha \rightarrow \Gamma_\beta \) by

\[
\Phi(\gamma) = (\gamma^{(\sigma(1))}; \gamma^{(\sigma(2))}; \ldots; \gamma^{(\sigma(l))}).
\]
It can be easily seen that $\Phi$ satisfies the desired properties, so we are done.

The final consequence to be introduced concerns the structure constants of $\mathbb{QSym}$ for the basis $\{M_\alpha\}$ of the monomial quasisymmetric functions. It is well known that

$$M_\alpha M_\beta = \sum_{\gamma} e_{\alpha,\beta}^\gamma M_\gamma,$$

where $e_{\alpha,\beta}^\gamma$ counts the overlapping shuffles of $\alpha$ and $\beta$ with weight $\gamma$. For more information, we refer the readers to [13, Proposition 5.1.3]. Putting Proposition 4.2 and Theorem 4.7 together, one can derive the following combinatorial descriptions of the overlapping shuffles.

**Corollary 4.14.** Let $I \subseteq [m-1], J \subseteq [n-1], \text{ and } K \subseteq [m+n-1]$. The following three sets have the same cardinality.

(a) The set of overlapping shuffles of $\text{comp}(I^c)$ and $\text{comp}(J^c)$ with weight $\text{comp}(K^c)$

(b) \[
A \in \binom{[m+n]}{n} \mid \begin{cases} (I \#_A J) \cap c(A) = \emptyset, \\
(\#_A I \subseteq K \subseteq (I \#_A J) \cup c(A), \\
|K \backslash c_2(A)| = |I| + |J| 
\end{cases}
\]

(c) \[
A \in \binom{[m+n]}{n} \mid \begin{cases} (I \#_A J) \cap c(A) = \emptyset, \\
(\#_A I \subseteq K \subseteq (I \#_A J) \cup c(A), \\
K \cap c_2(A) = \emptyset 
\end{cases}
\]

**Proof.** Proposition 4.2 (a) says that $\mathcal{B}(1,0)_{\text{comp}(K)} = \mathcal{H}_{\text{comp}(K^c)}$. Therefore, from the fact that $\{H_\alpha\}$ is the dual basis of $\{M_\alpha\}$ it follows that the cardinality of (a) is given by $C_{I,J}^K(1,0)$ (=the cardinality of (b)).

On the other hand, Proposition 4.2 (a) says that $\mathcal{B}(-1,1)_{\text{comp}(K)} = \Lambda_{\text{comp}(K^c)}$. It was shown in [11, Proposition 3.8] that $\Lambda_\alpha \Lambda_\beta$ and $H_\alpha H_\beta$ have the same structure constants, thus $C_{I,J}^K(1,0)$ equals $C_{I,J}^K(-1,1)$ (=the cardinality of (c)).

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