Graph Powers and Graph Homomorphisms

Hossein Hajiabolhassan and Ali Taherkhani

Department of Mathematical Sciences
Shahid Beheshti University, G.C.,
P.O. Box 19834, Tehran, Iran
hhaji@sbu.ac.ir
a_taherkhani@sbu.ac.ir

Abstract
In this paper we investigate some basic properties of fractional powers. In this regard, we show that for any rational number $1 \leq \frac{2r+1}{2s+1} < \phi(G)$, $G^{\frac{2r+1}{2s+1}} \rightarrow H$ if and only if $G \rightarrow H^{\frac{2s+1}{2r+1}}$. Also, for two rational numbers $\frac{2r+1}{2s+1} < \frac{2p+1}{2q+1}$ and a non-bipartite graph $G$, we show that $G^{\frac{2r+1}{2s+1}} < G^{\frac{2p+1}{2q+1}}$. In the sequel, we introduce an equivalent definition for circular chromatic number of graphs in terms of fractional powers. We also present a sufficient condition for equality of chromatic number and circular chromatic number.

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1 Introduction

Throughout this paper we only consider finite graphs. For a graph $G$, let $V(G)$ and $E(G)$ denote its vertex and edge sets, respectively. Given two graphs $G$ and $H$, a homomorphism from $G$ to $H$ is a map $f : V(G) \rightarrow V(H)$ such that adjacent vertices in $G$ are mapped into adjacent vertices in $H$, i.e., $uv \in E(G)$ implies $f(u)f(v) \in E(H)$. For simplicity, the existence of a homomorphism is indicated by the symbol $G \rightarrow H$. Two graphs $G$ and $H$ are homomorphically equivalent, denoted by $G \leftrightarrow H$, if $G \rightarrow H$ and $H \rightarrow G$. Also, $G < H$ means that $G \rightarrow H$ and no homomorphism exists from $H$ to $G$. In this terminology, we say that $H$ is a bound for a class $\mathcal{C}$ of graphs, if $G \rightarrow H$ for all $G \in \mathcal{C}$. The problem of the existence of a bound with some special properties, for a given class of graphs, has been a subject of study in graph homomorphism. A retract of a graph $G$ is a subgraph $H$ of $G$ such that there exists a homomorphism $r : G \rightarrow H$, called retraction with $r(u) = u$ for any vertex $u$ of $H$. A core is a graph which does not retract to a proper subgraph. Any graph is homomorphically equivalent to a unique core. Also, the symbol $\text{Hom}(G,H)$ is used to denote the set of all homomorphisms from $G$ to $H$ (for more on graph homomorphisms see [2, 3, 8, 11]).

Circular coloring, introduced by Vince [22], is a model for coloring the vertices of graphs that provides a more refined measure of coloring difficulty than the ordinary chromatic number. If $n$ and $d$ are positive integers with $n \geq 2d$, then the circular complete graph $K_n^d$ is the graph with vertex set $\{v_0, v_1, \ldots, v_{n-1}\}$ in which $v_i$ is

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2Correspondence should be addressed to hhaji@sbu.ac.ir.
connected to \( v \) if and only if \( d \leq |i - j| \leq n - d \). A graph \( G \) is said to be \((n, d)\)-
colorable if \( G \) admits a homomorphism to \( K_{2, d} \). The circular chromatic number (also
known as the star chromatic number [22]) \( \chi_c(G) \) of a graph \( G \) is the minimum of
those ratios \( \frac{n}{d} \) for which \( \text{gcd}(n, d) = 1 \) and such that \( G \) admits a homomorphism to
\( K_{2, d} \). It can be shown that one may only consider onto-vertex homomorphisms [23].

A \((n, d)\)-coloring is circular in the sense that we may view the colors as points on a
circle, and the requirement for \((n, d)\)-coloring is that the colors on adjacent vertices
must be at least \( d \) positions apart on the circle. Zhu [23] provides a thorough survey
of results on circular chromatic number.

As usual, we denote by \([m]\) the set \( \{1, 2, \ldots, m\} \), and denote by \( \binom{[m]}{n} \) the
collection of all \( n \)-subsets of \([m]\). The Kneser graph \( KG(m, n) \) is the graph with
vertex set \( \binom{[m]}{n} \), in which \( A \) is connected to \( B \) if and only if \( A \cap B = \emptyset \). It
was conjectured by Kneser [13] in 1955, and proved by Lovász [15] in 1978, that
\( \chi(KG(m, n)) = m - 2n + 2 \). The Schrijver graph \( SG(m, n) \) is the subgraph of
\( KG(m, n) \) induced by all 2-stable \( n \)-subsets of \([m]\). It was proved by Schrijver [18]
that \( \chi(SG(m, n)) = \chi(KG(m, n)) \) and that every proper subgraph of \( SG(m, n) \) has
a chromatic number smaller than that of \( SG(m, n) \). Also, for a given graph \( G \), the
notation \( og(G) \) stands for the odd girth of graph \( G \).

For a graph \( G \), let \( G^k \) be the \( k \)th power of \( G \), which is obtained on the vertex
set \( V(G) \), by connecting any two vertices \( u \) and \( v \) for which there exists a walk of
length \( k \) between \( u \) and \( v \) in \( G \). Note that the \( k \)th power of a simple graph is not
necessarily a simple graph itself. For instance, the \( k \)th power may have loops on its
vertices provided that \( k \) is an even integer. The chromatic number of graph powers
has been studied in the literature (see [11 5 7 9 19 21]).

The following simple and useful lemma was proved and used independently in
[5 17 21].

**Lemma A.** Let \( G \) and \( H \) be two simple graphs such that \( \text{Hom}(G,H) \neq \emptyset \). Then,
for any positive integer \( k \), \( \text{Hom}(G^k,H^k) \neq \emptyset \).

Note that Lemma A trivially holds whenever \( H^k \) contains a loop, e.g., when
\( k = 2 \). As immediate consequences of Lemma A we obtain \( \chi_c(P) = \chi(P) \) and
\( \text{Hom}(C, C_s) = \emptyset \), where \( P \) and \( C \) are the Petersen and the Coxeter graphs, respectively, see [5].

The local chromatic number of a graph is defined in [6] as the minimum number of
colors that must appear within distance 1 of a vertex. For a given graph \( G \) with
odd \( (G) \geq 7 \), the chromatic number of \( G^5 \) provides an upper bound for local
chromatic number of \( G \). In [19], it was proved if \( \chi(G^5) \leq m \) then \( \psi(G) \leq \left\lfloor \frac{m}{2} \right\rfloor + 2 \).

Now, we recall a definition from [9].

**Definition 1.** Let \( m, n, \) and \( k \) be positive integers with \( m \geq 2n \). Set \( H(m, n, k) \) to
be the helical graph whose vertex set contains all \( k \)-tuples \( (A_1, \ldots, A_k) \) such that for
any \( 1 \leq r \leq k \), \( A_r \subseteq [m], |A_1| = n, |A_r| \geq n \) and for any \( s \leq k - 1 \) and \( t \leq k - 2 \),
\( A_s \cap A_{s+1} = \emptyset, A_t \subseteq A_{t+2} \). Also, two vertices \( (A_1, \ldots, A_k) \) and \( (B_1, \ldots, B_k) \) of
\( H(m, n, k) \) are adjacent if for any \( 1 \leq i, j \leq k \), \( A_i \cap B_i = \emptyset, A_j \subseteq B_{j+1}, \) and
\( B_j \subseteq A_{j+1} \). \( \blacklozenge \)
Note that $H(m,1,1)$ is the complete graph $K_m$ and $H(m,n,1)$ is the Kneser graph $KG(m,n)$. It is easy to verify that if $m > 2n$, then the odd girth of $H(m,n,k)$ is greater than or equal to $2k+1$.

The following theorem shows that the helical graphs are bound of high odd girth graphs.

**Theorem A.** [9] Let $G$ be a non-empty graph with odd girth at least $2k+1$. Then, we have $\text{Hom}(G^{2k-1}, KG(m,n)) \neq \emptyset$ if and only if $\text{Hom}(G, H(m,n,k)) \neq \emptyset$.

Chromatic number of helical graphs has been characterized as follows.

**Theorem B.** [9] Let $m, n,$ and $k$ be positive integers with $m \geq 2n$. The chromatic number of the helical graph $H(m,n,k)$ is equal to $m - 2n + 2$.

A graph $H$ is said to be a subdivision of a graph $G$ if $H$ is obtained from $G$ by subdividing some of the edges. The graph $S_t(G)$ is said to be the $t$-subdivision of a graph $G$ if $S_t(G)$ is obtained from $G$ by replacing each edge by a path with exactly $t-1$ inner vertices. Note that $S_1(G)$ is isomorphic to $G$.

Hereafter, for a given graph $G$, we will use the following notation for convenience.

$$G^{2+1}_{2n+1} \overset{\text{def}}{=} (S_{2n+1}(G))^{2+1}.$$ 

For instance, if $n \geq 3$ is a positive integer, then $K_{6n+1} \simeq K_{2n^2-6n+3}$. It was proved in [9], if $G$ is a graph with odd girth at least $2k+1$, then a homomorphism from graph $G$ to $(2k+1)$-cycle exists if and only if the chromatic number of $G^{2k+1}_{2n+1}$ is less than or equal to 3.

**Theorem C.** [9] Let $G$ be a graph with odd girth at least $2k+1$. Then, $\chi(G^{2k+1}_{2n+1}) \leq 3$ if and only if $\text{Hom}(G, C_{2k+1}) \neq \emptyset$.

In what follows we are concerned with fractional powers. The paper is organized as follows. In second section, we study some basic properties of fractional power. In this regard, we show that for any rational number $1 \leq \frac{2r+1}{2s+1} < \text{og}(G)$, $G^{2r+1}_{2s+1} \rightarrow H$ if and only if $G \rightarrow H^{2s+1}_{2r+1}$. Also, for two rational numbers $\frac{2p+1}{2q+1} < \frac{2r+1}{2s+1}$ and a non-bipartite graph $G$, we show that $G^{2r+1}_{2s+1} < G^{2p+1}_{2q+1}$. In third section, we investigate some basic properties of power thickness. In fourth section, we introduce an equivalent definition for circular chromatic number of graphs in terms of fractional powers. We also present a sufficient condition for equality of chromatic number and circular chromatic number in terms of power thickness. Finally, in Section five, we make some concluding remarks about open problems and natural directions of generalization.

### 2 Fractional Powers

In this section we investigate the basic properties of graph powers. The following simple lemma can easily be proved by constructing graph homomorphisms and its proofs is omitted for the sake of brevity.
Lemma 1. Let $G$ be a graph.

a) If $s$ is a non-negative integer, then $G^{2s+1} \twoheadrightarrow G$.

b) If $s$ is a non-negative integer where $2s + 1 < \text{og}(G)$, then $(G^{2s+1})^{-\frac{1}{2s+1}} \rightarrow G$.

The next lemma will be useful throughout the paper.

Lemma 2. Let $G$ and $H$ be two graphs where $2s + 1 < \text{og}(H)$. Then, $G^{2s+1} \twoheadrightarrow H$ if and only if $G \rightarrow H^{2s+1}$.

Proof. Let $G^{\frac{1}{2s+1}} \rightarrow H$; then $(G^{\frac{1}{2s+1}})^{2s+1} \rightarrow H^{2s+1}$. In view of Lemma 1(a), we have $G \rightarrow (G^{\frac{1}{2s+1}})^{2s+1} \rightarrow H^{2s+1}$. Conversely, assume that $G \rightarrow H^{2s+1}$. Hence, $(G^{\frac{1}{2s+1}}) \rightarrow (H^{2s+1})^{-\frac{1}{2s+1}}$. On the other hand, Lemma 1(b) shows that $(H^{2s+1})^{-\frac{1}{2s+1}} \rightarrow H$, as desired. ■

For given graphs $G$ and $H$ with $v \in V(G)$, set

$$N_i(v) \overset{\text{def}}{=} \{u|\text{there is a walk of length } i \text{ joining } u \text{ and } v\}.$$  

Also, for a graph homomorphism $f : G \rightarrow H$, define

$$f(N_i(v)) \overset{\text{def}}{=} \bigcup_{u \in N_i(v)} f(u).$$

Also, for two subsets $A$ and $B$ of the vertex set of a graph $G$, we write $A \bowtie B$ if every vertex of $A$ is joined to every vertex of $B$. Also, for any non-negative integer $s$, define the graph $G^{\frac{1}{2s+1}}$ as follows

$$V(G^{\frac{1}{2s+1}}) \overset{\text{def}}{=} \{ (A_1, \ldots, A_{s+1}) | A_i \subseteq V(G), |A_1| = 1, \emptyset \neq A_i \subseteq N_{i-1}(A_1), i \leq s+1 \}.$$  

Two vertices $(A_1, \ldots, A_{s+1})$ and $(B_1, \ldots, B_{s+1})$ are adjacent in $G^{\frac{1}{2s+1}}$ if for any $1 \leq i \leq s$ and $1 \leq j \leq s + 1$, $A_i \subseteq B_{i+1}$, $B_i \subseteq A_{i+1}$, and $A_j \bowtie B_j$. Also, for any graph $G$ and $\frac{2s+1}{2r+1} \leq 1$, define the graph $G^{-\frac{2s+1}{2r+1}}$ as follows

$$G^{-\frac{2s+1}{2r+1}} \overset{\text{def}}{=} (G^{-\frac{1}{2r+1}})^{2s+1}.$$  

It is easy to verify that if $r$ is a non-negative integer, then the odd girth of $G^{-\frac{1}{2r+1}}$ is greater than or equal to $2r + 3$. The following theorem is a generalization of Theorem A and Lemma 3(ii) of [21].

Theorem 1. Let $G$ and $H$ be two graphs and $1 \leq \frac{2s+1}{2r+1} < \text{og}(G)$. We have $G^{\frac{2s+1}{2r+1}} \rightarrow H$ if and only if $G \rightarrow H^{-\frac{2s+1}{2r+1}}$. 

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**Proof.** First, we show that

\[ G^{2r+1} \longrightarrow H \text{ if and only if } G \longrightarrow H^{-\frac{1}{2r+1}}. \]  

(1)

Assume that \( g \in \text{Hom}(G^{2r+1}, H) \). Now, we present a graph homomorphism, say \( f \), from \( G \) to \( H^{-\frac{1}{2r+1}} \). If \( v \) is an isolated vertex of \( G \), then consider an arbitrary vertex, say \( f(v) \), of \( H \) as image of \( f \). For any non-isolated vertex \( v \in V(G) \), define

\[ f(v) := (g(v), g(N_1(v)), g(N_2(v)), \ldots, g(N_r(v))). \]

Since \( g \) is a graph homomorphism from \( G^{2r+1} \) to \( H \), one can verify that for any vertex \( v \in V(G) \), \( f(v) \in H^{-\frac{1}{2r+1}} \). Also, for any \( 0 \leq i, j \leq r \), we have \( g(N_i(v)) \nRightarrow g(N_j(u)) \), \( g(N_j(v)) \subseteq g(N_{j+1}(u)) \), and \( g(N_j(u)) \subseteq g(N_{j+1}(v)) \) provided that \( u \) is adjacent to \( v \). Hence, \( f \) is a graph homomorphism from \( G \) to \( H^{-\frac{1}{2r+1}} \).

Next, let \( \text{Hom}(G, H^{-\frac{1}{2r+1}}) \neq \emptyset \) and \( f : G \longrightarrow H^{-\frac{1}{2r+1}} \). Assume \( v \in V(G) \) and \( f(v) = (A_1, A_2, \ldots, A_{r+1}) \). Define, \( g(v) := A_1 \). We show that \( g \in \text{Hom}(G^{2r+1}, H) \).

Assume further that \( u, v \in V(G) \) such that there is a walk of length \( 2t+1 \) \((t \leq r)\) between \( u \) and \( v \) in \( G \), i.e., \( uv \in E(G^{2r+1}) \). Consider adjacent vertices \( u' \) and \( v' \) such that \( u' \in N_i(u) \) and \( v' \in N_i(v) \). Also, let \( f(v) = (A_1, A_2, \ldots, A_{r+1}) \), \( f(u) = (B_1, B_2, \ldots, B_{r+1}) \), \( f(v') = (A'_1, A'_2, \ldots, A'_{r+1}) \), and \( f(u') = (B'_1, B'_2, \ldots, B'_{r+1}) \). In view of the definition of \( H^{-\frac{1}{2r+1}} \), we obtain \( A_1 \subseteq A'_{t+1} \) and \( B_1 \subseteq B'_{t+1} \). On the other hand, \( A'_{t+1} \nRightarrow B'_{t+1} \), which yields \( g(v) \) is adjacent to \( g(u) \). Thus, \( \text{Hom}(G^{2r+1}, H) \neq \emptyset \).

Now, assume that \( G^{2r+1} \longrightarrow H \). In view of (1), one has \( G^{-\frac{1}{2r+1}} \longrightarrow H^{-\frac{1}{2r+1}} \); consequently, \( G \longrightarrow (G^{-\frac{1}{2r+1}})^{2s+1} \longrightarrow (H^{-\frac{1}{2r+1}})^{2s+1} \). Conversely, suppose \( G \longrightarrow H^{-\frac{1}{2r+1}} \). Considering Lemma 2, we have \( G^{-\frac{1}{2r+1}} \longrightarrow H^{-\frac{1}{2r+1}} \). Now, in view of (1), one can conclude that \( G^{2r+1} \longrightarrow H \). 

Note that in Theorem 1 we assume that \( 1 \leq \frac{2r+1}{2s+1} \), since \( 2s+1 \) should be less than the odd girth of \( H^{-\frac{1}{2r+1}} \). In fact, we don’t know the exact value of \( \text{og}(H^{-\frac{1}{2r+1}}) \). Though, we specify the odd girth of \( H(m, 1, k) \) in Lemma 7. This can be used to generalize Theorem 1. Also, it should be noted that the above theorem, for the case \( s = 0 \), was obtained by C. Tardif (personal communication).

**Corollary 1.** Let \( G \) be a non-bipartite graph. If \( 1 \leq \frac{2s+1}{2r+1} < \text{og}(G) \), then \( G^{2r+1} \longrightarrow K_m \) if and only if \( G \longrightarrow H(m, 1, r+1)^{2s+1} \).

**Lemma 3.** Let \( G \) be a non-bipartite graph. For any non-negative integer \( r \) we have

\[ (G^{-\frac{1}{2r+1}})^{2r+1} \longleftrightarrow G. \]

**Proof.** First, note that \( G^{-\frac{1}{2r+1}} \longrightarrow G^{-\frac{1}{2r+1}} \). Hence, in view of Theorem 1 we have \( (G^{-\frac{1}{2r+1}})^{2r+1} \longrightarrow G \). Next, \( G^{2r+1} \longrightarrow G \). Considering Theorem 1 we have \( G^{2r+1} \longrightarrow G^{2r+1} \). Thus, \( G \longrightarrow (G^{2r+1})^{2r+1} \longrightarrow (G^{-\frac{1}{2r+1}})^{2r+1} \), as required.
However, in general, \((G^{2r+1})^{\frac{1}{2x+1}}\) and \(G\) are quite different. For example, 
\((C_5^2)^{-\frac{1}{5}} = K_5^{-\frac{1}{3}}\) is not homomorphically equivalent to \(C_5\). In fact, 
\(\chi(K_5^{-\frac{1}{3}}) = \chi(H(5,1,2)) = 5\), while \(\chi(C_5) = 3\). Also, it should be noted that for given positive integers \(k, m,\) and \(n\) where \(m > 2n\), the helical graph \(H(m, n, k)\) and the graph \(KG(m, n)^{\frac{1}{2k-1}}\) are homomorphically equivalent. Although, if \(k \geq 2\) and \(n \geq 2\), then the number of vertices of \(H(m, n, k)\) is less than that of \(KG(m, n)^{\frac{1}{2k-1}}\).

We introduce some notation used for the remainder of the paper. Let \(G\) be a graph which does not contain isolated vertices. Set the vertex set of \(G\) the aforementioned notation for the vertex set of \(G\).

**Lemma 4.** Let \(G\) be a non-bipartite graph.

a) If \(\frac{(2r+1)(2p+1)}{2x+1} < og(G)\), then \(G = (G^{2r+1})^{2p+1}\).

b) If \(\frac{2r+1}{2x+1} < \frac{(2r+1)(2p+1)}{2x+1} < og(G)\), then \(G^{2r+1} = G^{2p+1}\).

**Proof.** Part (a) follows by a simple argument. To prove part (b), note that

\[
G^{2r+1} \xrightarrow{2x+1} G \xrightarrow{(2r+1)(2p+1)} G^{2p+1}
\]

by Lemma 4(a)).

One important property of the family of circular complete graph is that \(K_2^x < K_2^y\) if and only if \(\frac{x}{2} < \frac{y}{q}\). Fortunately, for a given non-bipartite graph \(G\), we have a similar property for the family of fractional powers of \(G\).

**Theorem 2.** Let \(G\) be a non-bipartite graph. If \(\frac{2r+1}{2x+1} < \frac{2p+1}{2q+1} < \og(G)\), then 
\(G^{2r+1} < G^{2p+1}\).
Proof. First, we show that if $1 < \frac{2r+1}{2s+1} < o_2(G)$, then $G < G^{\frac{2r+1}{2s+1}}$. We know that $G \rightarrow G^{\frac{2r+1}{2s+1}}$. Hence, it is sufficient to show that there is no homomorphism from $G^{\frac{2r+1}{2s+1}}$ to $G$. First, we prove that if $G$ is a core, then the statement is true. On the contrary, suppose that there is no homomorphism from $G^{\frac{2r+1}{2s+1}}$ to $G$. Since, $G$ is a core and induced subgraph of $G^{\frac{2r+1}{2s+1}}$, this homomorphism provides an isomorphism between two copies of $G$. For any edge of $G$, say $e = uv$, The vertex $(uv)_1$ (resp. $(vu)_1$) of $G^{\frac{2r+1}{2s+1}}$ is adjacent to all neighborhood of the vertex $u$ (resp. $v$). $G$ is a core; therefore, the image of $(uv)_1$ (resp. $(vu)_1$) should be the same as $u$ (resp. $v$). By induction, one can show that image of $(uv)_k$ (resp. $(vu)_k$) should be the same as $u$ (resp. $v$) whenever $1 \leq k \leq s$. Now, note that since $G$ is a non-bipartite graph; hence, it contains a triangle or an induced path of length three. Assume that $G$ contains a triangle with vertex set $\{u, v, w\}$. Consider two vertices $(uv)_s$ and $(uw)_s$. It was shown that images of $(uv)_s$ and $(uw)_s$ should be $u$. Also, $1 < \frac{2r+1}{2s+1}$; consequently, $(uv)_s$ and $(uw)_s$ are adjacent which is a contradiction. Similarly, if $G$ contains an induced path of length three, we get a contradiction.

Now, suppose that $G$ is an arbitrary non-bipartite graph. It is well-know that $G$ contains a core, say $H$, as induced subgraph. On the contrary, suppose that $G^{\frac{2r+1}{2s+1}} \rightarrow G$. Then, we have $H^{\frac{2r+1}{2s+1}} \rightarrow G^{\frac{2r+1}{2s+1}} \rightarrow G \rightarrow H$, which is a contradiction. Consequently, if $1 < \frac{2r+1}{2s+1} < o_2(G)$, then $G < G^{\frac{2r+1}{2s+1}}$.

It is readily to check that

$$G^{\frac{2r+1}{2s+1}} \rightarrow G^{\frac{2r+1}{2s+1}}(2r+1)(2s+1) \quad \text{and} \quad G^{\frac{2r+1}{2s+1}} \rightarrow G^{\frac{2r+1}{2s+1}}(2r+1)(2s+1).$$

On the other hand, we have $\frac{2r+1}{2s+1} < \frac{2p+1}{2q+1}$; hence, $G^{\frac{2r+1}{2s+1}} \rightarrow G^{\frac{2r+1}{2s+1}}(2r+1)(2s+1)$. It remains to show that the inequality is strict. On the contrary, assume that $G^{\frac{2r+1}{2s+1}} \rightarrow G^{\frac{2r+1}{2s+1}}$. Then, in view of Lemma 4(b) we have

$$G^{\frac{2p+1}{2q+1}}(2r+1)(2s+1) \rightarrow G^{\frac{2p+1}{2q+1}}(2r+1)(2s+1) \rightarrow G^{\frac{2p+1}{2q+1}}(2r+1)(2s+1).$$

Note that $\frac{2r+1}{2s+1} > 1$ which is a contradiction, as desired. 

3 Power Thickness

Considering Theorem C, it is worth studying the following definition which has been introduced in [9].

Definition 2. Assume that $G$ is a non-bipartite graph. Also, let $i \geq -\chi(G) + 3$ be an integer. $i$th power thickness of $G$ is defined as follows.

$$\theta_i(G) \stackrel{\text{def}}{=} \sup \left\{ \frac{2r+1}{2s+1} \chi(G^{\frac{2r+1}{2s+1}}) \right\} = \chi(G) + i, \quad \frac{2r+1}{2s+1} < o_2(G).$$

For simplicity, when $i = 0$, the 0th power thickness of $G$ is called power thickness of $G$ and it is denoted by $\theta(G)$. 

The importance of power thickness is that it allows us to obtain necessary condition for the existence of graph homomorphisms.
**Lemma 5.** Let \( G \) and \( H \) be two non-bipartite graphs with \( \chi(G) = \chi(H) - j \), \( j \geq 0 \). If \( G \xrightarrow{} H \) and \( i + j \geq -\chi(G) + 3 \), then

\[
\theta_{i+j}(G) \geq \theta_j(H).
\]

**Proof.** Consider a rational number \( \frac{2r+1}{2s+1} < \text{og}(H) \) for which \( \chi(H \uparrow_{2r+1}^2) \leq \chi(H) + i \). We know that \( \text{og}(G) \geq \text{og}(H) \) since \( G \xrightarrow{} H \). Hence, \( \frac{2r+1}{2s+1} < \text{og}(G) \) and \( G \uparrow_{2r+1}^2 \xrightarrow{} H \uparrow_{2r+1}^2 \) which implies that \( \chi(G \uparrow_{2r+1}^2) \leq \chi(H) + i = \chi(G) + i + j \). \( \blacksquare \)

In view of Theorem C, it is a hard task to compute the power thickness of arbitrary graphs. Hereafter, we will introduce some results in this regard. Finding graphs with high power thickness arises naturally in the mind. In this direction, we compute the power thickness of some helical graphs.

**Theorem 3.** Let \( k, l, \) and \( m \) be positive integers where \( m \geq 3 \) and \( \frac{2l-1}{2k-1} \leq 1 \). Then,

\[
\theta(H(m, 1, k)^{2l-1}) = \frac{2k-1}{2l-1}.
\]

**Proof.** In view of Lemma 4(b) and Theorem A, we have \( (H(m, 1, k)^{2l-1}) \uparrow_{2r+1} \xrightarrow{} H(m, 1, k)^{2k-1} \xrightarrow{} K_m \); therefore, \( \theta(H(m, 1, k)^{2l-1}) \geq \frac{2k-1}{2l-1} \). Suppose, on the contrary, that \( \theta(H(m, 1, k)^{2l-1}) = t > \frac{2k-1}{2l-1} \). Choose a rational number \( 1 < \frac{2r+1}{2s+1} \) such that \( 1 < \frac{(2r+1)(2k-1)}{(2s+1)(2l-1)} < t \). Set \( G \overset{\text{def}}{=} (H(m, 1, k)^{2l-1}) \uparrow_{2r+1} \). In view of Lemma 4(b) and definition of power thickness, one has \( \chi(G \uparrow_{2r+1}^2) \leq m \). By Theorem 1, one has \( G \xrightarrow{} H(m, 1, k)^{2l-1} \). Thus, \( (H(m, 1, k)^{2l-1}) \uparrow_{2r+1} \xrightarrow{} H(m, 1, k)^{2l-1} \) which contradicts Theorem 2 as claimed. \( \blacksquare \)

The next definition provides a sufficient condition for the graphs with \( \theta(G) = 1 \).

**Definition 3.** Let \( G \) be a graph with chromatic number \( k \). \( G \) is called a colorful graph if for any \( k \)-coloring \( c \) of \( G \), there exists an induced subgraph \( H \) of \( G \) such that for any vertex \( v \) of \( H \), all colors appear in closed neighborhood of \( v \), i.e., \( c(N[v]) = \{1, 2, \ldots, k\} \).

**Theorem 4.** For any non-bipartite colorful graph \( G \), we have \( \theta(G) = 1 \).

**Proof.** On the contrary, suppose that \( \theta(G) > 1 \). Choose a rational number \( 1 < \frac{2r+1}{2s+1} < \theta(G) \). By definition, \( \chi(G \uparrow_{2r+1}^2) = \chi(G) = k \). Consider a \( k \)-coloring of the graph \( G \uparrow_{2r+1}^2 \). Since, \( G \) is a colorful graph and an induced subgraph of \( G \uparrow_{2r+1}^2 \), there exists an induce subgraph of \( G \uparrow_{2r+1}^2 \), denoted by \( H \), such that for any vertex \( v \) of \( H \), all colors appear in closed neighborhood of \( v \). For any edge of \( H \), say \( e = uv \), the vertex \((uv)_1\) (resp. \((vu)_1\)) of \( G \uparrow_{2r+1}^2 \) is adjacent to all neighborhood of the vertex \( u \) (resp. \( v \)). Therefore, the color of \((uv)_1\) (resp. \((vu)_1\)) should be the same as \( u \) (resp. \( v \)). By induction, one can show that the color of \((uv)_k\) (resp. \((vu)_k\)) should
be the same as $u$ (resp. $v$) provided that $uv \in E(H)$. In view of coloring property of $H$, it should contain a triangle or an induced path of length three whose end vertices have the same color. Assume that $H$ contains an induced path with vertex set $\{u, v, w, x\}$ and edge set $\{uv, vw, wx\}$ such that $u$ and $x$ have the same color. Consider two vertices $(uv)_s$ and $(xw)_s$. It was shown that colors of $(uv)_s$ and $(xw)_s$ should be the same as $u$ and $x$, i.e., they have the same color. On the other hand, $1 < \frac{2r+1}{2s+1}$; consequently, $(uv)_s$ and $(xw)_s$ are adjacent which is a contradiction. Similarly, if $H$ contains a triangle, we get a contradiction. \[\blacksquare\]

We know that any uniquely colorable graph is a colorful graph. Hence, the power thickness of non-bipartite uniquely colorable graphs is one.

**Corollary 2.** Let $K_n$ be complete graph with $n \geq 3$ vertices. Then, $\theta(K_n) = 1$

A less ambitious objective is to find all graphs with power thickness one. Also, we don’t know whether any graph with power thickness one is colorful.

Of particular interest is the conclusion that circular complete graph $K_{2n+1}^{2t+1}$ is isomorphic to $C_{2n+1}^{2t+1}$. This allows us to investigate some coloring properties of circular complete graph powers.

**Lemma 6.** Given non-negative integers $n$ and $t$ where $n > t$. We have

a) $C_{2n+1}^{2t+1} \simeq K_{\frac{2n+1}{n-t}}$

b) $\theta(C_{2n+1}) = \frac{2n+1}{3}$.

**Proof.** Part (a) follows by a simple discussion. Note that $C_{2n+1}^{2t+1}$ and $C_{(2n+1)(2s+1)}^{2r+1}$ are isomorphic. Also, in view of part (a), $\chi(C_{2n+1}) = \lceil \frac{2n+1}{n-t} \rceil$. Now, part (b) follows by part (a). \[\blacksquare\]

Now, we are ready to specify the odd girth of $H(m, 1, k)$.

**Lemma 7.** Let $m \geq 3$ and $k$ be positive integers. The odd girth of the helical graph $H(m, 1, k)$ is equal to $2k + 2\lceil \frac{2k-1}{m-2} \rceil - 1$.

**Proof.** Let $C_{2n+1} \rightarrow H(m, 1, k)$; then $C_{2n+1}^{2k-1} \rightarrow H(m, 1, k)^{2k-1}$. On the other hand, we know that $\chi(H(m, 1, k)^{2k-1}) = m$; consequently, in view of Lemma 6(a) we have $\lceil \frac{2n+1}{n-k+1} \rceil \leq m$. Also, if $\lceil \frac{2n+1}{n-k+1} \rceil \leq m$, then by using Lemma 6(a) we have $C_{2n+1}^{2k-1} \rightarrow K_m$ which this implies that $C_{2n+1} \rightarrow H(m, 1, k)$. Therefore, the odd girth of the helical graph $H(m, 1, k)$ is the smallest value of $2n + 1$ for which $\lceil \frac{2n+1}{n-k+1} \rceil \leq m$. It is easy to check that the odd girth of $H(m, 1, k)$ should be $2k + 2\lceil \frac{2k-1}{m-2} \rceil - 1$. \[\blacksquare\]
4 Circular Coloring

The remainder of this paper is devoted to connection between chromatic number of graph powers and circular coloring. In the next theorem we introduce an equivalent definition for circular chromatic number of graphs.

**Theorem 6.** Let $G$ be a non-bipartite graph with chromatic number $\chi(G)$. Then, $\chi(G) \neq \chi_c(G)$ if and only if there exists a rational number $\frac{2r+1}{2s+1} > \frac{\chi(G)}{3(\chi(G)-2)}$ for which $\chi(G^{\frac{2r+1}{2s+1}}) = 3$. Moreover, $\chi_c(G) = \inf\{\frac{2n+1}{n-t}|\chi(G^{\frac{2n+1}{2n+1}}) = 3, n > t > 0\}$. Also, if $\frac{2r+1}{2s+1} \leq \frac{\chi(G)}{3(\chi(G)-2)}$, then $\chi(G^{\frac{2r+1}{2s+1}}) = 3$.

**Proof.** First, assume that $\chi(G) \neq \chi_c(G)$ and $\chi_c(G) < \frac{2d\chi(G)-1}{2d} < \chi(G)$ where $d$ is sufficiently large. By Lemma 6(a), $C_n^{2r+1} \simeq K_\frac{2r+1}{2s+1}$ whenever $n = dx - 1$ and $t = d(x - 1) - 1$. Therefore, $G \rightarrow K_{\frac{2n+1}{n-t}}$. On the other hand,

$$G \rightarrow K_{\frac{2n+1}{n-t}} \iff G \rightarrow K_{\frac{2t+1}{2s+1}} \text{ (by Lemma 6(a))}$$

$$\iff G^{\frac{2t+1}{2s+1}} \rightarrow K_{\frac{2n+1}{n-t}} \text{ (by Lemma 2)}$$

$$\iff G^{\frac{2t+1}{2s+1}} \rightarrow C_{\frac{2n+1}{n-t}} \text{ (by Theorem 3)}$$

It is readily seen that $\frac{2n+1}{3(2t+1)} > \frac{\chi(G)}{3(\chi(G)-2)}$ whenever $n = dx - 1$ and $t = d(x - 1) - 1$. Hence, it suffices to set $\frac{2r+1}{2s+1} = \frac{2n+1}{3(2t+1)}$; consequently, $\chi(G^{\frac{2n+1}{2n+1}}) = 3$. Conversely, let $\chi(G^{\frac{2n+1}{2n+1}}) = 3$ for a rational number $\frac{2r+1}{2s+1} > \frac{\chi(G)}{3(\chi(G)-2)}$. Choose positive integers $n$ and $t$ which satisfy $\frac{\chi(G)}{3(\chi(G)-2)} < \frac{2n+1}{3(2t+1)} \leq \frac{2r+1}{2s+1}$. By Theorem 2, $G^{\frac{2n+1}{3(2t+1)}} \rightarrow G^{\frac{2r+1}{2s+1}}$ and so $3 \leq \chi(G^{\frac{2n+1}{3(2t+1)}}) \leq \chi(G^{\frac{2r+1}{2s+1}})$. In view of (2) we have $G \rightarrow K_{\frac{2n+1}{n-t}}$ provided that $\chi(G^{\frac{2n+1}{3(2t+1)}}) = 3$. Also, it is straightforward to verify $\frac{2n+1}{n-t} < \chi(G)$ whenever $\frac{\chi(G)}{3(\chi(G)-2)} < \frac{2n+1}{3(2t+1)}$. Thus, $\chi_c(G) < \chi(G)$.

Also, the (2) shows that $\chi_c(G) = \inf\{\frac{2n+1}{n-t}|\chi(G^{\frac{2n+1}{3(2t+1)}}) = 3\}$. Finally, suppose that $\frac{2n+1}{n-t} \leq \frac{\chi(G)}{3(\chi(G)-2)}$. As before, choose positive integers $n$ and $t$ which satisfy $\frac{2n+1}{n-t} \leq \frac{\chi(G)}{3(\chi(G)-2)}$. One can see that $\frac{2n+1}{n-t} \geq \chi(G)$, and therefore $G \rightarrow K_{\frac{2n+1}{n-t}}$. Now, (2) implies that $\chi(G^{\frac{2n+1}{3(2t+1)}}) = 3$. By Theorem 2 we have $3 \leq \chi(G^{\frac{2n+1}{3(2t+1)}}) \leq \chi(G^{\frac{2n+1}{3(2t+1)}}) = 3$, as claimed.

We show that the power thickness of circular complete graphs $K_{\frac{q}{p}}$ is greater than one provided that $q \nmid p$.

**Theorem 6.** For any rational number $\frac{q}{p} > 2$ where $q \nmid p$ we have

$$\theta(K_{\frac{q}{p}}) > 1.$$
Proof. Set \(m \overset{\text{def}}{=} \lceil \frac{p}{q} \rceil\). Choose a positive integer \(d\) such that \(\frac{2d}{q} < \frac{2dm - 1}{2d} < m\). We know that \(K_{\frac{p}{q}} \rightarrow K_{\frac{2dm - 1}{2d}}\); hence, it is sufficient to show that there exists a positive integer \(s\) such that \((K_{\frac{2dm - 1}{2d}})^{\frac{2s + 1}{2s - 1}} \rightarrow K_m\).

Set \(n \overset{\text{def}}{=} dm - 1\), \(t \overset{\text{def}}{=} d(m - 2) - 1\). In view of Lemma 6(a) and Lemma 4(b), we have

\[
(K_{\frac{2dm - 1}{2d}})^{\frac{2s + 1}{2s - 1}} \simeq (C_{2n+1})^{\frac{2s + 1}{2s - 1}} \rightarrow (C_{2n+1})^{\frac{(2t+1)(2s+1)}{2s - 1}} \simeq (C_{(2n+1)(2s-1)})^{(2t+1)(2s+1)}.
\]

On the other hand, Lemma 6(b) confirms that \(
\chi((C_{(2n+1)(2s-1)})^{(2t+1)(2s+1)}) = \lceil \frac{(2n + 1)(2s - 1)}{(n - t)(2s + 1) - 2n - 1} \rceil.
\)

Therefore,

\[
\chi((K_{\frac{2dm - 1}{2d}})^{\frac{2s + 1}{2s - 1}}) \leq \lceil \frac{(2dm - 1)(2s - 1)}{2d(2s + 1) - 2dm + 1} \rceil.
\]

It is easily to see that if \(s\) is sufficiently large, then \(\chi((K_{\frac{2dm - 1}{2d}})^{\frac{2s + 1}{2s - 1}}) = m\). In other words, \(\theta(K_{\frac{p}{q}}) \geq \frac{2s + 1}{2s - 1} > 1\).

The aforementioned theorem provides a sufficient condition for equality of chromatic number and circular chromatic number of graphs. In fact, if we show that power thickness of a graph \(G\) is equal to one, then \(\chi(G) = \chi_c(G)\).

In case \(\chi(G) = 3\), it is well-known that \(\chi_c(G) = 3\) if and only if \(G\) is a colorful graph.

**Theorem 7.** Let \(G\) be a graph with chromatic number 3. Then, \(\theta(G) = 1\) if and only if \(\chi_c(G) = 3\).

The problem whether the circular chromatic number and the chromatic number of the Kneser graphs and the Schrijver graphs are equal has received attention and has been studied in several papers [4, 10, 12, 14, 16, 19]. Johnson, Holroyd, and Stahl [12] proved that \(\chi_c(KG(m, n)) = \chi(KG(m, n))\) if \(m \leq 2n + 2\) or \(n = 2\). This shows \(KG(2n + 1, n)\) is a colorful graph.

**Corollary 3.** Let \(n\) be a positive integer. Then, \(\theta(KG(2n + 1, n)) = 1\)

They also conjectured that the equality holds for all Kneser graphs.

**Conjecture 1.** \(\chi_c(KG(m, n)) = \chi(KG(m, n))\) for all \(m \geq 2n + 1\).

**Question 1.** Given positive integers \(m\) and \(n\) where \(m \geq 2n\), is the Kneser graph \(KG(m, n)\) a colorful graph? Is it true that \(\theta(KG(m, n)) = 1\)?

Theorem A shows that \(\theta(H(m, n, k)) \geq 2k - 1\) whenever \(m \geq 2n + 1\). Another problem which may be of interest is the following.
Question 2. Given positive integers \( m \) and \( n \) where \( m \geq 2n + 1 \), is it true that 
\[
\theta(H(m, n, k)) = 2k - 1
\]

Odd cycles are symmetric and they have sparse structure. Hence, it can be useful if circular chromatic number can be expressed as homomorphism to odd cycles. Now, let \( G \) be a non-bipartite graph and \( t \) be a positive integer. Define,

\[
f(G, 2t + 1) \overset{\text{def}}{=} \max\{2n + 1|G^{\frac{1}{2t+1}} \rightarrow C_{2n+1}\}.
\]

One can see that \( 3 \leq f(G, 2t + 1) \leq (2t + 1) \times \text{og}(G) \). In view of proof of Theorem 5 one can compute \( f(G, 2t + 1) \) in terms of circular chromatic number of graph \( G \) and vice versa. In fact, we have

\[
\chi_c(G) = \inf\{\frac{2n + 1}{n - t}|G^{\frac{1}{2t+1}} \rightarrow C_{2n+1}, n > t > 0\}.
\]

Moreover,

\[
f(G, 2t + 1) = 2\left\lfloor \frac{1}{\chi_c(G)} \frac{1 + t\chi_c(G)}{\chi_c(G) - 2} \right\rfloor + 1.
\]

Also, note that there exists an necessary condition for the existence of homomorphism to symmetric graphs in terms of eigenvalue of Laplaican matrix. The next theorem can be useful in studying circular chromatic number of graphs.

Theorem D. [2, 3, 4] Let \( G \) be a graph with \( |V(G)| = m \). If \( \sigma \in \text{Hom}(G, C_{2n+1}) \), then,

\[
\lambda_m^G \geq \frac{2|E(G)|}{2m} \lambda_{C_{2n+1}}^{C_{2n+1}},
\]

where \( \lambda_m^G \) and \( \lambda_{C_{2n+1}}^{C_{2n+1}} \) stand for the largest eigenvalues of Laplacian matrices of \( G \) and \( C_{2n+1} \), respectively.

5 Concluding Remarks

It is instructive to add some notes on the whole setup we have introduced so far. It is evident from our approach that any kind of information about power thickness of a graph has important consequences on graph homomorphism problem. There are several questions about power thickness which remain open. In fact, we don’t know whether the power thickness is always a rational number.

Question 3. Let \( G \) be a non-bipartite graph and \( i \geq -\chi(G) + 3 \) be an integer. Is \( \theta_i(G) \) a rational number? Also, for which real number \( r > 1 \) there exists a graph \( G \) with \( \theta_i(G) = r \).

Finally, we consider the following parameter as a natural generalization of power thickness and as a measure for graph homomorphism problem.

Definition 4. Let \( G \) and \( H \) be two graphs. Set

\[
\theta_H(G) \overset{\text{def}}{=} \sup\{\frac{2r + 1}{2s + 1}|G^{\frac{2r+1}{2s+1}} \rightarrow H, \frac{2r + 1}{2s + 1} < \text{og}(G)\}.
\]

\[\blacktriangle\]
It is easy to show that for any non-bipartite graphs \( G \) and \( H \), \( \theta_H(G) \) is a real number. Also, it is obvious to see that there is a homomorphism from \( G \) to \( H \) if and only if \( \theta_H(G) \geq 1 \).

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