New method of computing prolate spheroidal wavefunctions and bandlimited extrapolation: A general approach to bandlimited Fredholm kernels

Vishal Vaibhav

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The paper deals with numerical solution of the Fredholm integral equation associated with the classical problem of extrapolating bandlimited functions known on (−1, 1) to the entire real line. The approach presented can be characterized as the degenerate kernel method using the spherical Bessel functions as basis functions. This discretization also facilitates the solution of the associated eigenvalue problem whose eigenfunctions are the prolate spheroidal wave functions of order zero, thus yielding a new method of computing these functions on the entire real line. These ideas are then extended to Fredholm integral equations whose kernel belongs to a class of bandlimited functions that are square integrable. The proposed discretization scheme is used to solve the associated eigenvalue problem as well as the inverse problem that arises in the estimation of object function from its image function.

I. INTRODUCTION

It was demonstrated in the pioneering work of Pollak, Landau and Slepian [1–3] that the prolate spheroidal functions (PSWFs) are the eigenfunctions of the finite Fourier transform operator, which made it an appropriate tool for the study of the relationship between signals and their Fourier transforms. These functions form a natural basis for the representation of bandlimited signals and have found diverse applications such as in the determination of modes of a laser resonator with finite apertures [4, 5], in the analysis of diffraction-limited imaging systems [5–7] and, more recently, in the numerical solution of PDEs and ODEs [8–10].

On the interval (−1, 1), the standard method of computing the PSWFs is that of Bouwkamp [11] which involves solving a Sturm-Liouville eigenvalue problem using the Legendre-Galerkin method. In this paper, we revisit the problem of computation of PSWFs using the discretization of the associated Fredholm integral equation in the spirit of the work of Khare and George [12], and, Walter and Solesky [12]. The key difference here lies in the fact that instead of translates of the sinc function, we propose to use the spherical Bessel functions for the discretization of the integral equation. We further analyze how these representations can be used to solve the classical problem of extrapolation of bandlimited signals known on (−1, 1) to the entire real line [15–17]. It is well known that bandlimited extrapolation is inherently an ill-posed problem; therefore, we must turn to the standard techniques of solving ill-posed problems such as Tikhonov regularization [18–20]. In contrast to the approach adopted in [21], the regularization is applied to the discrete system.

At the heart of the method described in [12, 14] and in this paper is the fact that degenerate approximation of the sinc-kernel (either in terms of the translates of sinc function as in [12, 14] or the spherical Bessel functions proposed in this paper) can be obtained without any computational effort. As it turns out, this result can be extended to a class of real-valued σ-bandlimited kernel functions in $L^2(\mathbb{R})$ denoted by $\text{BL}(\sigma; L^2)$. For any $K \in \text{BL}(\sigma; L^2)$, the corresponding Fredholm integral operator is defined by

\begin{equation}
(\mathcal{K} \phi)(t) = \int_{-1}^{1} K(t - s)\phi(s)ds.
\end{equation}

The eigenvalue problem $\lambda \phi(t) = (\mathcal{K} \phi)(t)$ and the Fredholm integral equation of the first kind, $y(t) = (\mathcal{K} \phi)(t)$, appear in many problems of physical significance. Examples from optics include the analysis of imaging systems where $K(t)$ becomes the point spread function (PSF) which turns out to be the Fourier transform of the aperture function in case of coherent illumination or its autocorrelation function in case of incoherent illumination [6, 7, 22–24]. Note that, in the presence of aberrations, the PSF can take very general form while still being bandlimited [25]. The aforementioned problems are also the starting point in the Kac–Siegent analysis where $K(t)$ is identified as an autocorrelation function in detection problems (see, for example, [26] where this analysis was used to study the laser speckle pattern). In contrast to the PSWFs, the eigenfunctions of $\mathcal{K}$ are not in general known to be related to a Sturm-Liouville problem; therefore, numerical solution of the integral equation is the only available method

The method presented in this paper can be classified as the degenerate kernel method which is one of the standard methods for solving Fredholm equations [27]. Among the other methods are the Nyström method and the projection method (see [27] for a comprehensive treatment); however, our method has an intuitive appeal on account of the fact that the solutions are bandlimited functions, and, it is natural to think of their representation in terms of bandlimited functions that form an orthonormal basis. In particular, our objective is

\footnote{With regard to the computation of PSWFs, as noted in [13], the discrete system obtained via the associated Fredholm equation is ill-conditioned and provides poor accuracy compared to the method of Bouwkamp [11]. Therefore, the significance of the work [12, 14] and the present paper must be viewed in the light of the fact that they provide a discrete framework for the solution of the eigenvalue problem for the class of kernel functions $\text{BL}(\sigma; L^2)$ as well as for the solution of certain inverse problems such as the bandlimited extrapolation problem, or, determination of the object function from its image function.}

* vishal.vaibhav@gmail.com
to compare the performance of the spherical Bessel functions with that of translates of sinc function. The tests reveal that spherical Bessel functions exhibit better performance for the class of functions in $\mathcal{BL}(\sigma; L^2)$ whose spectrum is $\mathbb{C}(-\sigma, \sigma)$ where $p$ is large or infinite. Finally, let us mention that, in treating the inverse problems, we have assumed that the input to the algorithm is not contaminated with noise. Note that the inverse problems in question are ill-posed regardless of the presence of noise. Therefore, we assume that it is possible to make precise measurements of the input or make a sufficiently accurate estimate of the input through multiple measurements.

The paper is organized as follows: Sec. II deals with the computation of PSWFs and Sec. III deals with the extrapolation of bandlimited functions. In Sec. IV, we consider the eigenvalue problem, $\Lambda \phi(t) = (\mathcal{H} \phi)(t)$, and the Fredholm equation, $y(t) = (\mathcal{H} x)(t)$, where the kernel function belongs to $\mathcal{BL}(\sigma; L^2)$. Sec V concludes the paper.

II. ANGULAR PROLATE SPHEROIDAL WAVE FUNCTIONS OF ORDER ZERO

For a given $\sigma \in \mathbb{R}_+$, referred to as the bandlimiting parameter, the prolate spheroidal wave functions (PSWFs) [1, 28] are defined as eigenfunctions of the eigenvalue problem given by

$$\int_{-1}^{1} \frac{\sin(\sigma(t-s))}{\pi(t-s)} \phi(s) ds = \lambda \phi(t), \quad t \in (-1, 1). \quad (2)$$

The equation (2) is also valid for all $t \in \mathbb{R}$ and it defines the values of the PSWFs on the real line. The eigenvalues are all positive real numbers indexed in the decreasing order of their magnitude, $1 > \lambda_0 > \lambda_1 > \ldots > \lambda_n > \ldots$, such that $\lim_{n \to \infty} \lambda_n = 0$ and the corresponding eigenfunctions are denoted by $\phi_n(t)$, $n = 0, 1, \ldots$, respectively. The eigenfunctions form a complete orthogonal basis in $L^2((-1, 1))$. On the real line, they form a complete orthonormal basis spanning the class of bandlimited functions in $L^2(\mathbb{R})$. The double orthogonality property of PSWFs is characterized by the following relations:

$$\langle \phi_n, \phi_m \rangle_{(-1, 1)} = \int_{-1}^{1} \phi_n(t) \phi_m(t) dt = \lambda_n \delta_{nm},$$

$$\langle \phi_n, \phi_m \rangle_{\mathbb{R}} = \int_{-\infty}^{\infty} \phi_n(t) \phi_m(t) dt = \delta_{nm},$$

where $\langle \cdot, \cdot \rangle$ stands for the inner product and $\delta_{nm}$ denotes the Kronecker delta. We enumerate some of the important properties of PSWFs for ready reference [28]:

- The eigenfunctions $\phi_n(t)$ satisfy the parity relation
  $$\phi_n(-t) = (-1)^n \phi_n(t),$$
  and have exactly $n$ zeros in $(-1, 1)$.

- The PSWFs also satisfy the following eigenvalue problem
  $$\int_{-1}^{1} e^{i\sigma t} \phi(s) ds = \nu_n \phi(t), \quad t \in (-1, 1), \quad (5)$$
  where $\nu_n = i^n \sqrt{2\pi \lambda_n \sigma}$. The equation (5) is also valid for $t \in \mathbb{R}$ so that either of the relationships (2) or (5) can be used for computing the values of the PSWFs outside $(-1, 1)$.

- The Fourier Transform of $\phi_n(t)$, denoted by $\Phi_n(\xi)$, is given by
  $$\Phi_n(\xi) = \int_{-\infty}^{\infty} \phi_n(t) e^{-i\xi t} dt = (-i)^n \sqrt{\frac{2\pi}{\sigma \lambda_n}} \phi_n(\xi) \Pi(\frac{\xi}{\sigma}),$$

where $\Pi(\xi)$ is the rectangle function which is defined to be unity for $\xi \in (-1, 1)$ and zero otherwise.

A. The Legendre-Galerkin method

PSWFs can be computed on $(-1, 1)$ by the solution of the Sturm-Liouville problem [11, 29]

$$\frac{d}{dt} \left[(1-t^2) \frac{d}{dt} \psi - \sigma^2 \psi + \chi \psi \right] = 0, \quad t \in (-1, 1),$$

(7)

(where $\chi$ is the eigenvalue) using any of the spectral methods such as the Legendre-Galerkin method [30]. Using the expansion in terms of normalized Legendre polynomials, $\bar{P}_n(t) = \sqrt{n+1/2}P_n(t)$,

$$\psi(t) = \sum_{n=0}^{\infty} a_n \bar{P}_n(t) \approx \sum_{n=0}^{N-1} a_n \bar{P}_n(t),$$

(8)

we obtain the discrete eigenvalue problem given by

$$[A - \chi I] a = 0,$$

(9)

where $A = [A_{jk}]_{N \times N}$ is a symmetric tri-diagonal matrix with the non-zero diagonal elements given by [31]

$$A_{kk} = k(k+1) + \frac{2(k+1) - 1}{(2k-1)(2k+3)} \sigma^2,$$

$$A_{k-1,k} = \frac{k(k-1)}{(2k-1)(2k+3)} \sigma^2,$$

$$A_{k+2,k} = \frac{(k+1)(k+2)}{(2k+3)(2k+5)} \sigma^2.$$

(10)

The even and odd eigenfunctions can be computed separately. The indexing is done in the ascending order of the eigenvalues. For normalization, we use the relations in (3) which yields $\phi_n(t) = \sqrt{\lambda_n} \psi_n(t)/||\psi_n||_{L^2((-1, 1))}$. The normalization evidently requires the computation of the eigenvalues $\lambda_n$ which is carried out by means of the following well-conditioned numerical procedure [31]: Taking the unnormalized eigenfunctions, $\psi_n$, find $\nu_n$ from the general formula

$$\nu_n = [\psi_{2n}(0)]^{-1} \int_{-1}^{1} \psi_{2n}(t) dt = \frac{\sqrt{\lambda_n} \psi_n(2n)}{\psi_{2n}(0)}.$$
The other eigenvalues are then obtained using the connection formula
\[
\frac{v_m}{v_n} = |\langle \phi'_m, \phi_n \rangle|^{1/2} = |\langle \phi'_n, \phi_m \rangle|^{1/2}, \quad m \neq n \mod 2, \tag{11}
\]
where the superscript in \(a_n^{(2n)}\) corresponds to the index of the eigenfunction. Finally, \(a_0 = \sqrt{v_0^2} \sigma^2/2\pi\). The values of the derivative at Legendre-Gauss-Lobatto (LGL) points can be computed using the differentiation matrix given by [30]
\[
(D_N)_{j,l} = \begin{cases} 
\frac{P_N(x_j)}{x_j - x_l}, & j \neq l, \\
\frac{P_N(x_l)}{x_j - x_l} - \frac{N(N + 1)}{4}, & j = l = 0, \\
\frac{N(N + 1)}{4}, & j = l = N, \\
0, & \text{otherwise},
\end{cases}
\tag{12}
\]
so that \(\psi'(x_j) = \sum_{l} (D_N)_{j,l} \psi(x_l)\). The inner products are evaluated using Gauss quadrature with LGL nodes.

In most of the extrapolation problems, the value of \(\phi_n(t)\) is needed in the region \(t \in \mathbb{R} \setminus (-1, 1)\). One of the standard methods to compute these values is to apply numerical quadrature techniques to the oscillatory integral in (5) which reads as
\[
\phi_n(t) = \frac{(-i)^n}{\sqrt{2\pi A_n/\sigma}} \int_{-1}^{1} \phi_n(s)e^{i\sigma s} ds. \tag{13}
\]
Gauss-type quadrature schemes tend to perform poorly in computing these integrals on account of the oscillatory nature of the integrand which deviates considerably from polynomials, specially for larger values of \(\sigma\). There is a vast amount of literature devoted to treating such problems, for instance, see [32, Section 2.10] and the references therein. The most convenient method that fits naturally into our setting is the method of Bakhvalov and Vasil’eva [33] which, employing the series expansion (8), yields the representation
\[
\phi_n(t) = \sum_{m=0}^{\infty} \sigma^{-m} \alpha_m n^{m/2} j_m(\sigma t), \tag{14}
\]
using the identity
\[
\int_{-1}^{1} e^{i\sigma s} P_n(s) ds = 2\pi j_n(t), \tag{15}
\]
where \(j_n(t)\) denotes the spherical Bessel function of the first kind defined as \(j_n(t) = \sqrt{\pi/2} J_{n+1/2}(t)\). On account of the presence of the factor \(J_{(n+1)/2}\), the numerical conditioning of the expression (14) is extremely poor for \(n > 2\sigma/\pi\) [31]. We therefore turn to other techniques for computing the PSWFs outside \((-1, 1)\).

B. The approach based on spherical Bessel functions

The method outlined in this section is motivated by the following representation of the sinc-kernel in (2):
\[
\frac{\sin \sigma(t - s)}{\pi(t - s)} = \frac{\sigma}{\pi} \sum_{n=0}^{\infty} (2n + 1) j_n(\sigma t) j_n(\sigma s). \tag{16}
\]
Observing the orthogonality relation
\[
\int_{-\infty}^{\infty} dt \ j_m(\sigma t) j_n(\sigma t) = \frac{\pi}{\sigma} \left( \delta_{mn} + \frac{1}{2} \right), \tag{17}
\]
let us introduce the normalized basis functions \(\bar{j}_m(t) = \sqrt{\rho(2n+1)/\pi} j_n(\sigma t)\) for convenience. The eigenvalue problem in (2) can be discretized by writing
\[
\phi(t) = \sum_{n=0}^{\infty} \beta_n \bar{j}_n(t), \tag{18}
\]
so that
\[
\lambda \sum_{n=0}^{\infty} \beta_m j_m(t) = \sum_{m=0}^{\infty} \bar{j}_m(t) \sum_{n=0}^{\infty} \left( \int_{-1}^{1} \bar{j}_m(s) \bar{j}_n(s) \right) \beta_n. \tag{19}
\]
Equating the coefficient of \(\bar{j}_m(t)\) on both sides of for every \(m\), we have
\[
\lambda \beta_m = \sum_{n=0}^{\infty} J_{mn} \beta_n, \tag{20}
\]
where the matrix elements \(J_{mn}\) are given by
\[
J_{mn} = \int_{-1}^{1} ds \bar{j}_m(s) \bar{j}_n(s). \tag{21}
\]
For \(\sigma < \infty\), using Cauchy-Schwartz inequality, we have
\[
|J_{mn}| \leq ||\bar{j}_m||_{L^2(-1,1)} ||\bar{j}_n||_{L^2(-1,1)} < 1,
\]
where we used the fact that \(||\bar{j}_m||_{L^2(-1,1)} < ||\bar{j}_m||_{L^2(\mathbb{R})} = 1\). It must be noted that \(J_{nn} = 0\) for \(m \neq n \mod 2\); therefore, the even system of equations can be decoupled with that of the odd. In order to study the decay property of the overlap integral in (21), we start with the integral representation
\[
\bar{j}_n(x) = \frac{x^n}{2\pi^{n+1} n!} \int_{0}^{\pi} \cos(x \cos \theta) \sin^{2n+1} \theta d\theta \tag{22}
\]
so that
\[
J_{mn} = C_{mn} \int_{0}^{\pi} d\theta d\theta' \int_{-1}^{1} ds \ s^{n+m} \times \cos(\sigma s \cos \theta) \cos(\sigma s \cos \theta') \sin^{2n+1} \theta \sin^{2n+1} \theta',\]
where
\[
C_{mn} = \frac{\sigma^{m+n+1} \sqrt{(2m+1)(2n+1)} 2 \cdot (2m)!! 2 \cdot (2n)!!}{2^{m+n+1} m! (m + n + 1) \pi (2m + 1)!! (2n + 1)!!}.
\]
This leads to the following estimate
\[
|J_{mn}| \leq \frac{\sigma^{m+n+1} \sqrt{(2m+1)(2n+1)} 2 \cdot (2m)!! 2 \cdot (2n)!!}{2^{m+n+1} m! (m + n + 1) \pi (2m + 1)!! (2n + 1)!!} \leq \frac{4\sigma^{m+n+1}}{\pi (m + n + 1) (2m + 1)! (2n + 1)!}, \tag{23}
\]
where we have used the result
\[ \int_0^\infty \sin^{2n+1} \theta d\theta = 2 \frac{(2m)!}{(2m+1)!} \]  
(24)
Applying the Stirling’s formula to (23), we obtain
\[ |J_{mn}| < \frac{c e^2}{4\pi^2} \frac{e \sigma}{2m+2} \left( \frac{e \sigma}{2n+2} \right)^n. \]  
(25)
This inequality allows us to conclude that \( J \) is a Hilbert-Schmidt operator on \( \ell^2 \):
\[ \|J\|^2_\ell = \sum_{m,n=0}^\infty |J_{mn}|^2 \leq \left( \frac{c e^2}{4\pi^2} \right)^2 \sum_{n=0}^\infty \left( \frac{e \sigma}{2n+2} \right)^{2n}, \]
where ‘F’ stands for the Frobenius norm. It is also possible to estimate the spectral norm of \( J \) as follows:
\[ \sum_{m,n \geq 0} J_{mn} \beta_m \beta_n = \int_{-1}^1 \left( \sum_{n \geq 0} \beta_n J_n(s) \right)^2 ds \]
\[ < \int_{-\infty}^\infty \left( \sum_{n \geq 0} \beta_n J_n(s) \right)^2 ds = \|\beta\|^2_\ell, \]
so that \( \|J\|_\ell < 1 \) provided \( \sigma < \infty \). Therefore, it is straightforward to conclude that \( J \) is a self-adjoint, positive definite and compact operator on \( \ell^2 \); consequently, it admits of an orthonormal sequence of eigenvectors which correspond to positive eigenvalues less that unity. Finally, the estimate in (25) allows us to truncate the infinite matrix \( J \) to a \( N \times N \) square matrix by choosing \( N \gg e\sigma/2 \).

Let \( \beta_n = (\beta_0^{(n)}, \beta_1^{(n)}, \ldots)^t \) define \( \phi_n(t) \). We assume that the eigenvectors \( \beta_n \) are normalized such that \( \|\beta_n\|_\ell^2 = 1 \). The inner-product on \( \mathbb{R} \) is given by
\[ \int_{-\infty}^\infty \phi_m(t) \phi_n^*(t) dt = \sum_{k=0}^\infty \beta_k^{(m)} \beta_k^{(n)*} \equiv \beta_k^* \beta_m = \delta_{mn}, \]  
(26)
using the orthogonality of the spherical Bessel functions. The inner-product on \((-1,1)\) is given by
\[ \int_{-1}^1 \phi_m(t) \phi_n^*(t) dt = \sum_{k=0}^\infty \beta_k^{(m)} \beta_k^{(n)*} J_{k\eta} \equiv \beta_k^* J \beta_m = \lambda_n \delta_{mn}, \]  
which follows from the symmetric nature of the matrix \( J \) in (20).

The eigenfunctions \( \phi_n(t) \) have definite parity which implies that the Fourier transform of \( \phi_n(t) \) must be either purely real or imaginary. Using the identity
\[ \int_{\mathbb{R}} dt \tilde{J}_n(t) e^{-i\xi t} = \sqrt{\frac{2\pi}{\sigma}} (-i)^n \tilde{P}_n \left( \frac{\xi}{\sigma} \right), \quad \xi \in (-\sigma, \sigma), \]  
(27)
we have
\[ \Phi_n(\xi) = \sqrt{\frac{2\pi}{\sigma}} \sum_{k=0}^\infty \beta_k^{(n)} (-i)^k \tilde{P}_k \left( \frac{\xi}{\sigma} \right) \Pi \left( \frac{\xi}{\sigma} \right), \]  
(28)
Noting that, for non-zero value of the coefficients, \( k \) either runs through all even or all odd values; it is evident that \( \beta_k^{(n)} \in \mathbb{R} \) (or identically zero) which confirms that the eigenfunctions are real valued.

Next, we prove that the Fourier transform \( \Phi_n(\sigma s), s \in (-1,1) \), also satisfies the eigenvalue problem in (2). Consider
\[ I = \int_{-1}^1 \sin[\sigma(t-s)] \Phi_n(\sigma s) ds \]
\[ = \sigma \int_{-1}^1 \int_{-1}^1 e^{i\sigma(\eta-s)} d\eta \Phi_n(\sigma s) ds \]  
(29)
Observing that [29]
\[ e^{-i\sigma \eta} = \sqrt{\frac{2\pi}{\sigma}} \sum_{k=0}^\infty (-i)^k \tilde{J}_k(\eta), \quad t \in (-1,1), \]  
(30)
we have
\[ I = \sqrt{\frac{2\pi}{\sigma}} \sum_{k=0}^\infty \beta_k^{(n)} \int_{-1}^1 \int_{-1}^1 d\eta \tilde{J}_k(\eta) \tilde{J}_n(\eta) \]
\[ = \sqrt{\frac{2\pi}{\sigma}} \sum_{k=0}^\infty \beta_k^{(n)} J_k(\sigma s) \]
\[ = \lambda_n \sqrt{\frac{2\pi}{\sigma}} \sum_{k=0}^\infty \beta_k^{(n)} (-i)^k \tilde{P}_k(\eta) = \lambda_n \Phi_n(\sigma t), \]
for \( t \in (-1,1) \). This shows that we can write \( \Phi_n(\sigma t) = \gamma_n \phi_n(t) \) on account of the fact that the eigenfunctions are unique up to scalar multiplier. Noting that
\[ \gamma_n^{-2} \int_{-1}^1 \Phi_n^2(\sigma t) dt = \gamma_n^{-2} (2\pi/\sigma) \sum_k \beta_k^{(n)} \beta_k^{(n)*} (-i)^{2k} = \lambda_n, \]
it follows that for the even case \( \gamma_n = \pm \sqrt{2\pi/\sigma} \lambda_n \) and for the odd case \( \gamma_n = \pm i \sqrt{2\pi/\sigma} \lambda_n \). The ambiguity in sign is resolved by requiring \( \phi_n(t) \to P_n(t) \) as \( \sigma \to 0 \). This leads to \( \gamma_n = (-1)^{n} \sqrt{2\pi/\sigma} \lambda_n \). From (29) it also becomes evident that
\[ (-1)^n \lambda_n \Phi_n(\sigma t) = \int_{-1}^1 d\eta e^{i\eta \phi_n(\eta)} \]
\[ = \int_{-1}^1 d\eta e^{i\eta \phi_n(\eta)}. \]
Therefore, \( \phi_n(t) \) and, equivalently, \( \Phi_n(\sigma t) \) satisfy the eigenvalue problem in (5) for \( t \in (-1,1) \). The eigenvalues \( \nu_n \) works out to be \( \nu_n = (-1)^n \lambda_n \gamma_n = (-1)^n \sqrt{2\pi/\sigma} \lambda_n \).

Let us conclude this section with a discussion of the differentiation matrix. The first order derivative of the spherical Bessel functions obey the recurrence relation
\[ 2\sigma \tilde{J}'_n(\sigma t) = \tilde{J}_{n-1}(\sigma t) - \tilde{J}_{n+1}(\sigma t). \]
For the sake of brevity, let $\phi(t)$ represent any of the PSWFs; then

$$
\phi'(t) = \sqrt{\frac{\sigma}{\pi}} \sum_{n=0}^{\infty} \beta_n \sqrt{2n + 1} \sigma j_n(\sigma t)
$$

Equating the coefficient of $j_{m}(\sigma t)$ both sides, we have

$$
2 \sqrt{2m + 1} \beta_m' = \sqrt{2m + 3} \beta_{m+1}' - \sqrt{2m - 1} \beta_{m-1}'.
$$

which yields the First order differentiation matrix $D^{(1)}$ which is a symmetric banded matrix with zeros on the diagonal and the non-zero off-diagonal elements are given by

$$
D^{(1)}_{m,m+1} = \frac{\sqrt{2m + 3}}{2 \sqrt{2m + 1}}, \quad D^{(1)}_{m-1,m} = -\frac{\sqrt{2m - 1}}{2 \sqrt{2m + 1}}.
$$

Similarly, the second order differentiation matrix $D^{(2)}$ which is also a symmetric banded matrix with diagonal elements given by $D^{(2)}_{m,m} = -1/2$ and the nonzero off-diagonal elements are given by

$$
D^{(2)}_{m,m+2} = \frac{\sqrt{2m + 5}}{4 \sqrt{2m + 1}}, \quad D^{(2)}_{m-2,m} = -\frac{\sqrt{2m - 3}}{4 \sqrt{2m + 1}}.
$$

C. The approach based on sampling theory

The sampling theory approach being discussed in this section is based on [12, 14]. Let the PSWFs be expanded as

$$
\phi(t) = \sum_{n \in \mathbb{Z}} \beta_n \psi_n(t),
$$

where $\psi_n(t)$ denotes the normalized translates of the sinc function defined as

$$
\psi_n(t) = \sqrt{\frac{\sigma}{\pi}} \text{sinc}[\sigma(t - t_n)] = \sqrt{\frac{\sigma}{\pi}} \sin(\sigma t - n\pi). \quad (35)
$$

These functions form an orthonormal system on $\mathbb{R}$:

$$
\sum_{n = -\infty}^{\infty} \psi_n(t)\psi_n(t)dt = \delta_{mn}.
$$

The coefficients $\beta_n$ can be worked out using the orthonormality property or by using direct sampling at $t_n = n\pi/\sigma$ so that $\beta_n = \sqrt{\pi/\sigma \psi(t_n)}$ which is just another way of stating the Shannon-Whittaker sampling theorem for $\sigma$-bandlimited functions [34]. Consider the sinc-kernel in (2) which can be written as

$$
\frac{\sin[\sigma(t - s)]}{\pi(t - s)} = \sum_{n \in \mathbb{Z}} \psi_n(t)\psi_n(s). \quad (37)
$$

The eigenvalue problem in (2) can now be discretized in the following manner:

$$
\lambda \sum_{m \in \mathbb{Z}} \beta_m \psi_m(t) = \sum_{m \in \mathbb{Z}} \psi_m(t) \sum_{n \in \mathbb{Z}} \beta_n \int_{-1}^{1} ds \psi_m(s)\psi_n(s). \quad (38)
$$

Equating the coefficient of $\psi_m(t)$ both sides, we have

$$
\lambda \beta_m = \sum_{n \in \mathbb{Z}} A_{mn} \beta_n
$$

where

$$
A_{mn} = \int_{-1}^{1} \psi_m(t)\psi_n(t)dt.
$$

Using Cauchy-Schwartz inequality, we have

$$
|A_{mn}|^2 \leq \frac{2\sigma^2}{\pi(m^2\pi^2 - \sigma^2)} \frac{2\sigma^2}{\pi(n^2\pi^2 - \sigma^2)}, \quad |m|, |n| > \frac{\sigma}{\pi}. \quad (41)
$$

The operator $A$ can be shown to be a self-adjoint, positive definite and compact operator on $L^2$ so that it admits of an orthonormal sequence of eigenvectors which correspond to positive eigenvalues [14]. Various properties of the PSWFs can also be deduced from the fact that they are eigenfunctions of the operator $A$ as was done in [12, 14]. We conclude this section with the remark that the linear system in (39) can be split into two parts where one corresponds to the odd and the other to the even parity PSWFs by introducing the odd and even parity basis functions as follows: For $n \in \mathbb{Z}$, define

$$
\psi_n^{(+)}(t) = \frac{1}{\sqrt{2}} [\psi_n(t) + \psi_{-n}(t)], \quad \psi_n^{(-)}(t) = \frac{1}{\sqrt{2}} [\psi_n(t) - \psi_{-n}(t)],
$$

with $\psi_0^{(+)}(t) = \psi_0(t)$. The sinc-kernel in (37) can be rewritten as

$$
\frac{\sin[\sigma(t - s)]}{\pi(t - s)} = \sum_{n \geq 0} \psi_n^{(+)}(t)\psi_n^{(+)}(s) + \sum_{n < 0} \psi_n^{(-)}(t)\psi_n^{(-)}(s). \quad (43)
$$

Observing that the odd basis functions do not “interact” with the even ones, it is straightforward to decouple the two systems. Rest of the details are entirely similar to what has been carried out above which can be applied individually to the odd and the even parity PSWFs. In our tests, the numerical conditioning improves as a result of this splitting.

D. Numerical results

In this section we compare the results of the three algorithms discussed so far. The number of Legendre polynomials used in the Legendre-Galerkin (LG) method is taken to be $N = 1000$. The number of basis functions used in the integral equation (IE) method (using either Bessel or sinc functions) is $N = 1000$. The number of nodes used for the Gauss quadrature in the IE method is $N_{\text{quad}} = 4000$. The LG method is used to compute the PSWFs only on the domain $(-1, 1)$ which serves as a reference solution for the IE method for $t \in (-1, 1)$. The results for $r \in [8, 12]$ are plotted in Fig. 1 and Fig. 1 with their eigenvalues listed in Table I and Table II, respectively. As evident from these tables, the accuracy of the IE methods...
falls as the eigenvalues, $\lambda_n$, get closer to each other. The IE method based on the spherical Bessel functions shows better agreement with the LG method even with fewer number of basis functions on account of the fact that the elements of the infinite matrix involved has sharp decay with respect to the indices.

### III. Extrapolation of Bandlimited Signals

The extrapolation problem for bandlimited signals known on $(-1, 1)$ requires the solution of the Fredholm equation [16]

$$x(t) = \int_{-1}^{1} \frac{\sin \sigma (t - s)}{\pi (t - s)} y(s) ds, \quad t \in (-1, 1),$$  \hspace{1cm} (44)

where $x(t)$ is known in the interval $(-1, 1)$ and $y(t)$ is an unknown signal. The extrapolation of $x(t)$ for $t \in \mathbb{R} \setminus (-1, 1)$ is carried out using the same equation once $y(t)$ is determined on $(-1, 1)$.

The discretization of this equation can be accomplished by using the representation in (16) so that

$$x(t) = \sum_{n=0}^{\infty} \hat{J}_n(t) \int_{-1}^{1} \hat{J}_n(s) y(s) ds,$$

where we have used the fact that $\hat{J}_n(t)$ are precisely the PSWFs discussed in the last section. In the discrete form, the solution can be stated by writing the expansion of $J^{-1}$ using its eigenvectors

$$\hat{y} = \sum_{n} \left( \frac{\hat{x} \cdot \beta_n}{\lambda_n} \right) \beta_n,$$  \hspace{1cm} (47)

provided

$$\sum_{n} \left| \frac{\hat{x} \cdot \beta_n}{\lambda_n} \right|^2 < \infty.$$  \hspace{1cm} (48)

This is a special case of Picard’s theorem [35, Chap. VI]. For $n > 2\sigma/\pi$, the eigenvalues $\lambda_n$ show a sharp decrease rendering the sum (48) extremely sensitive to the errors in $[\hat{x} \cdot \beta_n]$ making this problem inherently ill-posed. Thus, in order to obtain any meaningful solution of the problem, one must turn to regularization techniques for solving the linear system in (46).

#### A. Tikhonov Regularization

The Tikhonov regularization [18] for the Fredholm equation (44) in the discretized form [36, 37] can be stated as the minimization of the following function of $\hat{y}$:

$$H(\hat{y}) = ||J y - \hat{x}||_2^2 + \mu^2 ||y||_2^2,$$

where we have used the fact that $J$ is symmetric. This is known as the standard form of Tikhonov regularization. The minimization problem for $\hat{y}$ boils down to the solution of the linear system given by [19, Chap. 5]

$$(J^2 + \mu^2 )\hat{y} = J\hat{x}.$$  \hspace{1cm} (50)

The solution can be stated in terms of the eigenvectors of $J$ as

$$\hat{y}_\mu = \sum_{n} \frac{\lambda_n}{\lambda_n^2 + \mu^2} (\hat{x} \cdot \beta_n) \beta_n.$$  \hspace{1cm} (51)

Given that the bandlimited signals are smooth functions, we may introduce extra “penalty” for the lack of smoothness by considering a minimization problem of the form

$$H(\hat{y}) = ||J \hat{y} - \hat{x}||_2^2 + \mu^2 \left( ||\hat{y}||_2^2 + ||J^{(1)} \hat{y}||_2^2 + ||J^{(2)} \hat{y}||_2^2 \right).$$  \hspace{1cm} (52)
The quantity in parenthesis above represents a Sobolev norm. As before, this minimization problem is equivalent to solving the linear system given by
\[
[J^2 + \mu^2(I + D^{(1)}D^{(1)} + D^{(2)}D^{(2)})]\hat{y} = J\hat{x}.
\] (53)

This equation can be solved using the generalized SVD (see [19, 37]).

Finally, let us remark that the Tikhonov regularization can also be discussed in the discrete framework based on the translates of the sinc function. These details are being omitted here because the line of reasoning is entirely similar. However, with regard to the translates of the sinc function, let us note that it is considerably harder to implement the Sobolev norm in this basis; therefore, we restrict ourselves to the standard form of the Tikhonov regularization, i.e. (49), in this case.

### B. Numerical examples

For the purpose of numerical tests, we first label the algorithms being tested as follows:

- **T₁-Bessel**: Standard Tikhonov regularization with spherical Bessel functions as basis functions.
- **T₂-Bessel**: Tikhonov regularization where regularity is enforced via a Sobolev norm with spherical Bessel functions as basis functions.
- **T₁-sinc**: Standard Tikhonov regularization with translates of sinc function as basis functions.

The number of basis functions taken is \(N = 400\). The number of quadrature nodes is chosen to be \(N_{\text{quad.}} = 1600\). The
FIG. 2. The figure shows some of the PSWFs computed using the Legendre-Galerkin method for the ODEs and the integral equation method with spherical Bessel and translates of sinc function as basis functions. The bandlimiting parameter is given by $\sigma = 12$.

The bandlimiting parameter $\sigma$ is chosen from $\{10, 20\}$. The regularization parameter $\mu$ is allowed to vary in a range for which we look at the error in the extrapolated signal. The error is quantified by a relative $L^2(\Omega)$-norm where $\Omega = (-5, 5)$:

$$e_{rel.} = \frac{\|x_{\text{exact}} - x_{\text{extrap}}\|_{L^2(\Omega)}}{\|x_{\text{exact}}\|_{L^2(\Omega)}}$$

where $x_{\text{exact}}$ denotes the exact signal and $x_{\text{extrap.}}$ denotes the extrapolated signal. Note that the signal is assumed to be known in $(-1, 1)$. In this paper, we do not present a study of the effectiveness of different methods of finding the optimal regularization parameter $\mu$; however, let us mention that the L-curve method seems to perform satisfactorily.

Following are the two favorable examples where the Fourier spectrum belongs to $C^\infty(-1, 1)$:

$$x_1(t) = \sum_{j=0}^4 \text{sinc} \left[ \frac{\sigma}{\kappa_j} \sqrt{1 + \kappa_j^2 (t - \tau_j)^2} \right],$$

$$x_2(t) = \sum_{j=0}^4 \frac{J_\nu \left[ \frac{\sigma}{\kappa_j} (t - \tau_j) \right]}{\left[ \sigma(t - \tau_j)^2 \right]},$$

where $\kappa = (5, 8, 8, 10, 10)$ and $\tau = (0, -0.1, +0.2, -0.3, +0.4)$. The error for various values of the regularization parameter are plotted in Fig. 3 and Fig. 4. For smaller values of the regularization parameter, the Bessel function based methods seem to outperform that based on the sinc function in terms of accuracy.

The final test we consider is a signal whose Fourier spec-
The result for different values of the regularization parameter is plotted in Fig. 5. It turns out all methods perform equally worse in this case.

For specific values of the regularization parameter which are listed in Table III, a comparison of the extrapolated and the exact signal for $x(t) = x_j(t)$, $j = 1, 2, 3$, is presented in figures 6, 7 and 8, respectively. The plots show how dramatically the point-wise error, defined by $|x(t) - x_{\text{extrap.}}(t)|$, increases outside the interval $(-1, 1)$. Note that we have not conducted a thorough study of how to choose the regularization parameter here and the values in the Table III are read off Fig. 3–5. Let us however remark that our experiments with the L-curve method often yielded values that were close to that listed in Table III. It is well known that any methodology of choosing the regularization parameter in the absence additional information about the solution can be defeated by suitably constructed counter examples (see Neumaier [20]). Since the present paper largely focuses on the method of discretization, we have omitted this analysis.
TABLE III. The table shows the regularization parameter corresponding to the bandlimiting parameter chosen for various extrapolation algorithms for the signals $x_j(t)$ as defined in (57). The signal is assumed to be known in $(-1, 1)$ and the extrapolation error is computed over $\Omega = (-5, 5)$.

| Method       | Signal: $x_1(t)$ | Signal: $x_2(t)$ | Signal: $x_3(t)$ |
|--------------|------------------|------------------|------------------|
| $\sigma = 10$ | $4 \times 10^{-14}$ | $10^{-12}$       | $10^{-12}$       |
| $\sigma = 20$ | $10^{-11}$       | $10^{-12}$       | $10^{-12}$       |

IV. FREDHOLM INTEGRAL EQUATIONS WITH BANDLIMITED KERNEL

Let us consider the integral operator of the form

$$(\mathcal{K}x)(t) = \int_{-1}^{1} K(t - s)x(s)ds,$$  \hspace{1cm} (58)

where $K \in \text{BL}(\sigma; L^2)$, i.e., a real-valued $\sigma$-bandlimited function such that

$$\int_{-\infty}^{\infty} |K(t)|^2 dt < \infty,$$  \hspace{1cm} (59)

which guarantees that $\mathcal{K}$ is a Hilbert-Schmidt operator and therefore compact. Inversion of such an operator is required in reconstruction of any object from its diffraction-limited image in the limit of fully coherent or fully incoherent illumination [6, 7, 22, 25]. Here, the imaging system is modeled as a linear-shift invariant system which is completely characterized by the impulse response function, $K(t)$, referred to as the point spread function (PSF) in this context. The Fourier transform of $K(t)$ denoted by $\tilde{K}(\xi)$.

In general, the operator $\mathcal{K}$ is not self-adjoint and a system of eigenfunctions can be computed for $\mathcal{K}$ and its adjoint, respectively, which would together form a biorthogonal system on $(-1, 1)$. The general properties of this system have already been studied in [38]. Here our goal is to provide an alternative basis expansion to translates of the sinc function; therefore, it suffices to consider the simple case of a self-adjoint operator $\mathcal{K}$. This restriction implies that

$$K(t - s) = K(s - t),$$  \hspace{1cm} (60)

so that $\tilde{K}(\xi)$ is real and even. The eigenvalue problem $\lambda \phi(t) = (\mathcal{K} \phi)(t)$ is therefore central to the analysis of the such systems which is taken up in the next section. In the last part of this section, we consider the inverse problem of estimating $\gamma(t)$ for $t \in (-1, 1)$ from the observation $x(t)$ on the interval $(-1, 1)$ such that $x(t) = (\mathcal{K} \gamma)(t)$.

A. The eigenvalue problem

1. The approach based on spherical Bessel functions

The fact that the sinc kernel in (44) can be expressed as a linear combination of spherical Bessel functions may seem to be a special case but it turns out that such a result also exists for the class of bandlimited kernels $\text{BL}(\sigma; L^2)$.

Recalling the identity

$$\int_{\mathbb{R}} e^{-i\xi t} f(t)dt = \frac{2\pi}{\sigma} (-i)^n \tilde{P}_n \left( \frac{\xi}{\sigma} \right), \quad \xi \in (-\sigma, \sigma),$$  \hspace{1cm} (61)

for the normalized spherical Bessel functions and the normal-
FIG. 6. The figure shows a comparison of the extrapolated and the exact signal $x(t) = x_1(t)$ for $\sigma = 10$ (first row) and $\sigma = 20$ (third row). The point-wise error, $\vert x(t) - x_{\text{extrap}}(t) \vert$, for $\sigma = 10$ and $\sigma = 20$ is displayed the second and the third row, respectively.

Using Legendre polynomials, we may write

\[
K_{mn} = \int_{\mathbb{R}} ds \, \tilde{j}_m(s) \int_{\mathbb{R}} dt \, \tilde{j}_n(t) K(t-s)
\]

\[
= \int_{-1}^{1} P_n(\xi) K(\sigma \xi) P_m(\xi) d\xi.
\]  

(63)

The last step follows from the Plancherel’s theorem [39]. Let $K$ denote the matrix with elements $\tilde{j}^m n K_{mn}$. The discrete
FIG. 7. The figure shows a comparison of the extrapolated and the exact signal $x(t) = x_2(t)$ for $\sigma = 10$ (first row) and $\sigma = 20$ (third row). The point-wise error, $|x(t) - x_{\text{extrap}}(t)|$, for $\sigma = 10$ and $\sigma = 20$ is displayed in the second and the third row, respectively.

Form of the eigenvalue problem $\lambda \phi(t) = (\mathcal{K} \phi)(t)$ then works out to be

$$\lambda \beta = J'K\beta,$$

where $\beta_n = \langle \phi, J_n \rangle_{(-1,1)}$.

Now, on account of the completeness of the spherical Bessel functions for the class of bandlimited function $BL(\sigma; L^2)$, we may write

$$\int_R [K(t-s)]^2 dt = \sum_{m \geq 0} \sum_{n \geq 0} ^{m+n} |K_{mn} J_n(s)|^2,$$

where each of the sums on the right hand side is bounded. The
FIG. 8. The figure shows a comparison of the extrapolated and the exact signal \( x(t) = x_1(t) \) for \( \sigma = 10 \) (first row) and \( \sigma = 20 \) (third row). The point-wise error, \(|x(t) - x_{\text{extrap}}(t)|\), for \( \sigma = 10 \) and \( \sigma = 20 \) is displayed the second and the third row, respectively.

boundedness of the matrix elements \([JK]_{ml}\) now follows from where we have used the Cauchy-Schwartz inequality. Now, the Frobenius norm of \( JK \) can be estimated as

\[
||JK||^2_F = \sum_{m,l \geq 0} ||JK_{ml}||^2,
\]

\[
\leq 2||J||^2_{L^2([-1,1])} \sum_{m \geq 0} ||J_m||^2_{L^2([-1,1])} = \frac{2\sigma}{\pi} ||K||^2_{L^2([-1,1])}.
\]
FIG. 9. The figure shows some of the eigenfunctions of the Fredholm operator (58) with the kernel $K_1(t)$ defined in (78) and the bandlimiting parameter given by $\sigma = 8$. The method labeled with ‘Bessel’ uses the spherical Bessel as basis functions while the method labeled with ‘sinc’ uses translates of sinc function.

It follows that $J^*K$ is a Hilbert-Schmidt operator over $\ell^2$. From the overlap integrals in (63), noting that $\widetilde{K}(\xi)$ is even, it follows that $K_{mn} = 0$ for $m = n \mod 2$ (a result that allows us to split the linear system in (64) according to odd and even parity) so that $K$ turns out to be a real symmetric matrix. Next, on account of $\|J\|_s < 1$, it follows that there exists an infinite orthogonal matrix $Q$ such that $J = QDQ^*$ where $D$ is a diagonal matrix with positive entries less than unity so that the eigenvalue problem (64) can be written as $\lambda \alpha = M\alpha$ where $\alpha = D^{-1/2}Q^* \beta$ and $M = D^{1/2}Q^* K D^{1/2}$ which is manifestly symmetric and $\|M\|_s \leq \|K\|_s$. Therefore, its eigenvalues are positive and the eigenfunctions corresponding to distinct eigenvalues are orthogonal. This verifies the basic spectral properties of the self-adjoint operator $\mathcal{K}$ in the discrete framework. Finally, the eigenfunction $\phi(t)$ is recovered as

$$\phi(t) = \sum_{m \geq 0} \left( \sum_{n \geq 0} K_{mn}\beta_n \right) j_m(t),$$

with the normalization such that $\|K\beta\|_{\ell^2} = 1$ and it is straightforward to see that the eigenfunction have a definite parity in this representation.
FIG. 10. The figure shows some of the eigenfunctions of the Fredholm operator (58) with the kernel $K_2(t)$ defined in (79) and the bandlimiting parameter given by $\sigma = 8$. The method labeled with ‘Bessel’ uses the spherical Bessel as basis functions while the method labeled with ‘sinc’ uses translates of sinc function.

2. The approach based on sampling theory

For any $K \in \mathcal{B}(\sigma; L^2)$, we can write [25, 38]

$$K(t - s) = \sum_{m} \sqrt{\frac{\pi}{\sigma}} K(t_m - s)\phi_m(t) = \frac{\pi}{\sigma} \sum_{m,n} K\left(\frac{(m-n)\pi}{\sigma}\right)\psi_m(t)\psi_n(s).$$

(67)

Let $K_{mn} = (\pi/\sigma)K((m-n)\pi/\sigma)$ and denote this matrix by $\mathcal{K}$. Following the procedure employed in the earlier sections, the discrete form of the eigenvalue problem $\lambda\phi(t) = (\mathcal{K}\phi)(t)$ can be stated as

$$\lambda\beta_m = \sum_{n \in \mathbb{Z}} A_{mn}K_{nm}\beta_n,$$

(68)

where $\beta_l = \langle \phi, \psi_l \rangle_{(-1, 1)}$ and $A_{mn}$ is given by (40). If $\mathcal{K}$ is self-adjoint, then $K_{mn} = K_{nm}$. Let us now show that $\sum_m |K_{mn}|^2 = (\pi/\sigma)\|K\|^2_{L^2(\mathbb{R})}$; observing that

$$\int_{\mathbb{R}} |K(t - s)|^2 dt = \frac{\pi}{\sigma} \int_{\mathbb{R}} \left( \sum_{m \in \mathbb{Z}} K(t_m - s)\psi_m(t) \right)^2 dt$$

$$= \frac{\pi}{\sigma} \sum_{m \in \mathbb{Z}} |K(t_m - s)|^2,$$
the result follows by setting $s = n\pi/r$. The boundedness of the matrix elements $[AK_m]_{nl}$ now follows from

$$
[AK_m]_{nl}^2 = \left| \int_{-1}^{1} ds \psi_m(s) \left( \sum_{n \in \mathbb{Z}} \psi_n(s) K_{nl} \right) \right|^2 \\
\leq \left( \frac{\pi}{\sigma} \int_{-1}^{1} |K(s-t)|^2 ds \right) \|\psi_m\|_{L^2(-1,1)}^2 \|\psi_n\|_{L^2(-1,1)}^2,
$$

where we have used the Cauchy-Schwartz inequality. Now, the Frobenius norm of $AK$ can be estimated as

$$
\|AK\|_F^2 = \sum_{m,n \in \mathbb{Z}} [AK_m]_{nl}^2 \\
\leq 2\|K\|_F^2 \sum_{m \in \mathbb{Z}} \|\psi_m\|_{L^2(-1,1)}^2 = \frac{2\sigma}{\pi} \|K\|_F^2.
$$

(69)

It follows that $AK$ is a Hilbert-Schmidt operator over $\ell^2$. Noting that $\|A\| < 1$, there exists an infinite orthogonal matrix $Q$ such that $A = QDQ^\top$ where $D$ is a diagonal matrix with positive entries less than unity so that the eigenvalue problem (68) can be written as $\lambda \mathbf{\alpha} = D^{1/2} Q^\top \mathbf{K} Q D^{1/2} \mathbf{\alpha}$ where $\mathbf{\alpha} = D^{-1/2} Q^\top \mathbf{\beta}$. Define $M = (D^{1/2} Q^\top) K (Q D^{1/2})$ which is manifestly symmetric and $\|M\|_a \leq \|K\|_a$. Finally, the eigenfunction $\phi(t)$ is recovered as

$$
\phi(t) = \sum_{m \in \mathbb{Z}} \left( \sum_{n \in \mathbb{Z}} K_{mn} \beta_n \right) \psi_m(t),
$$

with the normalization such that $\|K\beta\|_c = 1$.

### B. The inverse problem

In this section, we consider the inverse problem of estimating $y(t)$ from the observed signal $x(t)$ in $(-1, 1)$ such that

$$
x(t) = (x,y)(t) = \int_{-1}^{1} K(t-s)y(s)ds,
$$

(71)

where $K \in L^2(\mathbb{R})$ is a real-valued $\sigma$-bandlimited even function of $t \in \mathbb{R}$. In the context of a 4f-imaging system [22, 25], $K(t)$ is the impulse response (IR) of the linear shift-invariant system. The support of the PSF is given by supp $K = [-\sigma, \sigma]$. The PSF and IR of this system is assumed to be known. Note that the range over which the object $x(t)$ can be measured is scaled to $(-1, 1)$ and supp $y \subset (-1, 1).

The representation of $K(t-s)$ stated in (62) allows us to write

$$
x(t) = \sum_{m \geq 0} \left( \sum_{n \geq 0} (-i)^{m-n} K_{mn} \hat{y}_n \right) \tilde{\psi}_m(t),
$$

(72)

where $\hat{y}_n = \int_{-1}^{1} y(t) \tilde{\psi}_m(t) dt$. Putting $\hat{x}_m = \int_{-1}^{1} x(t) \tilde{\psi}_m(t) dt$, we have

$$
\hat{x}_m = \sum_{l \geq 0} \left( \sum_{n \geq 0} J_{mn} (-i)^{m-l} K_{nl} \right) \hat{y}_l.
$$

(73)

In order to compute $y(t)$ for $t \in (-1, 1)$, we proceed by setting

$$
y(t) = \sum_{n \geq 0} \hat{y}_n \tilde{\psi}_n(t), \quad t \in (-1, 1),
$$

(74)

so that $\hat{x} = M\hat{y}$, where $M = JKJ$. The linear system obtained above is highly ill-conditioned which is a manifestation of the ill-posed nature of the Fredholm equation (71). In order to solve this system, we adopt the method of regularization as discussed in the Sec. III A. Here, we consider two forms of Tikhonov regularization where one minimizes a function $H(\hat{y})$ of the form

$$
H(\hat{y}) = \|M\hat{y} - \hat{x}\|^2 + \mu^2 \|\hat{y}\|^2_2.
$$

These minimization problems can be solved using the generalized SVD (see Hansen [19]).

Finally, let us observe that the procedure carried out above can also be repeated with translates of the sinc function. Let us briefly outline these steps below. Starting from the representation of $K(t-s)$ stated in (67), we have

$$
x(t) = \sum_{m \in \mathbb{Z}} \left( \sum_{n \in \mathbb{Z}} K_{mn} \hat{y}_n \right) \psi_m(t),
$$

(75)

where $\hat{y}_n = \langle y, \psi_n \rangle_{(-1,1)}$. Putting $\hat{x}_n = \langle x, \psi_n \rangle_{(-1,1)}$, we have

$$
\hat{x}_m = \sum_{n \in \mathbb{Z}} \left( \sum_{n \in \mathbb{Z}} A_{mn} K_{nl} \right) \hat{y}_l.
$$

(76)

In order to recover $y(t)$ for $t \in (-1, 1)$, we use

$$
y(t) = \sum_{n \in \mathbb{Z}} \hat{y}_n \psi_n(t),
$$

(77)

so that $\hat{y}_m = \sum_{n \in \mathbb{Z}} A_{mn} \hat{y}_n$ which yields $\hat{x} = M\hat{y}$ where $M = AK_0$. The standard form of Tikhonov regularization is implemented through the minimization of $H(\hat{y})$ given by

$$
H(\hat{y}) = \|M\hat{y} - \hat{x}\|^2 + \mu^2 \|\hat{y}\|^2_2.
$$

C. Numerical examples

In order to demonstrate the effectiveness of the methods introduced in Sec. IV A and Sec. IV B, we consider the following two examples for the Kernel function:

- Example of a non-smooth PSF is

$$
K_1(t) = \frac{\sigma}{2\pi} \left( \int \frac{\sin(\sigma t/2)}{\sigma t/2} \right)^2,
$$

$$
\bar{K}_1(\sigma \xi) = (1 - |\xi|) \Pi(\xi).
$$

- Example of a smooth PSF is

$$
K_2(t) = \frac{2\sigma}{\pi} \left[ \int \frac{\sin(\sigma t) - \cos(\sigma t)}{\sigma^2 t^2} \right],
$$

$$
\bar{K}_2(\sigma \xi) = (1 - \xi^2) \Pi(\xi).
$$
TABLE IV. The table shows some of the computed eigenvalues of the Fredholm operator (58) with kernel $K_1(t)$ defined in (78) and bandlimiting parameter $\sigma = 8$. Here $N$ is the number of basis functions used.

| Basis: spherical Bessel function |
|----------------------------------|
| Index  | $N = 200$ | $N = 400$ | $N = 600$ | $N = 800$ | $N = 1000$ |
| 0      | 8.6123075491 × 10^{-4} | 8.6122622670 × 10^{-4} | 8.6122538760 × 10^{-4} | 8.6122509385 × 10^{-4} | 8.6122495787 × 10^{-4} |
| 1      | 6.7173526812 × 10^{-4} | 6.7173526829 × 10^{-4} | 6.7173526830 × 10^{-4} | 6.7173526831 × 10^{-4} | 6.7173526830 × 10^{-4} |
| 2      | 4.8980990431 × 10^{-4} | 4.89890961765 × 10^{-4} | 4.8980956454 × 10^{-4} | 4.8980954594 × 10^{-4} | 4.8989095373 × 10^{-4} |
| 3      | 3.1377573188 × 10^{-4} | 3.1377573191 × 10^{-4} | 3.1377573191 × 10^{-4} | 3.1377573191 × 10^{-4} | 3.1377573191 × 10^{-4} |
| 4      | 1.5436810526 × 10^{-4} | 1.5436802488 × 10^{-4} | 1.5436800999 × 10^{-4} | 1.5436800477 × 10^{-4} | 1.5436800236 × 10^{-4} |
| 5      | 4.6752039706 × 10^{-2} | 4.6752039713 × 10^{-2} | 4.6752039713 × 10^{-2} | 4.6752039713 × 10^{-2} | 4.6752039713 × 10^{-2} |
| 6      | 7.9355199572 × 10^{-3} | 7.9355134714 × 10^{-3} | 7.9355122697 × 10^{-3} | 7.9355118490 × 10^{-3} | 7.9355116543 × 10^{-2} |
| 7      | 8.1676648852 × 10^{-4} | 8.1676648871 × 10^{-4} | 8.1676648872 × 10^{-4} | 8.1676648872 × 10^{-4} | 8.1676648872 × 10^{-4} |

| Basis: translates of sinc function |
|-----------------------------------|
| Index  | $N = 200$ | $N = 400$ | $N = 600$ | $N = 800$ | $N = 1000$ |
| 0      | 8.6122471636 × 10^{-4} | 8.6122471611 × 10^{-4} | 8.6122471609 × 10^{-4} | 8.6122471608 × 10^{-4} | 8.6122471608 × 10^{-4} |
| 1      | 6.7173535961 × 10^{-4} | 6.7173529047 × 10^{-4} | 6.7173527862 × 10^{-4} | 6.7173527430 × 10^{-4} | 6.7173527224 × 10^{-4} |
| 2      | 4.8980952933 × 10^{-4} | 4.8980952227 × 10^{-4} | 4.8980952206 × 10^{-4} | 4.8980952206 × 10^{-4} | 4.8980952205 × 10^{-4} |
| 3      | 3.1377601251 × 10^{-4} | 3.1377580645 × 10^{-4} | 3.1377576645 × 10^{-4} | 3.1377575195 × 10^{-4} | 3.1377574505 × 10^{-4} |
| 4      | 1.5436791381 × 10^{-4} | 1.5436798732 × 10^{-4} | 1.5436799486 × 10^{-4} | 1.5436799671 × 10^{-4} | 1.5436799737 × 10^{-4} |
| 5      | 4.6783408028 × 10^{-2} | 4.6760705739 × 10^{-2} | 4.6756106030 × 10^{-2} | 4.6754412778 × 10^{-2} | 4.6753601059 × 10^{-2} |
| 6      | 7.9355781257 × 10^{-3} | 7.9355195332 × 10^{-3} | 7.9355137313 × 10^{-3} | 7.9355123273 × 10^{-3} | 7.9355118289 × 10^{-3} |
| 7      | 8.1994908742 × 10^{-4} | 8.1765489432 × 10^{-4} | 8.1718460959 × 10^{-4} | 8.1701084294 × 10^{-4} | 8.1692738958 × 10^{-4} |

FIG. 11. The figure shows the input signal and the corresponding output signal for the Fredholm equation discussed in Sec. IV B. The signals $y_1(t)$ and $x_1(t)$ are plotted on the left ($\sigma = 4\pi$), and, the signals $y_2(t)$ and $x_2(t)$ are plotted on the right ($\sigma = 8\pi$).

Let us first consider the eigenvalue problem discussed in Sec. IV A. The number of basis functions chosen (using either Bessel or sinc functions) is $N = 1000$. The number of nodes used in the Gauss quadrature for computing the overlap integrals involved is $N_{quad} = 4000$. The results for $\sigma = 8$ is plotted in Fig. 9 (corresponding to the kernel $K_1(t)$) and Fig. 10 (corresponding to the kernel $K_2(t)$). The method based on spherical Bessel functions is labeled as ‘Bessel’ and that using translates of sinc function is labeled as ‘sinc’. The two methods seem to be in agreement with each other as evidence by these plots. In the absence of any reference to compare results, we devise the next test where we look at the first eight eigenvalues for $N \in \{200, 400, \ldots, 1000\}$. While this does not constitute a test for convergence, it is interesting to see how
TABLE V. The table shows some of the computed eigenvalues of the Fredholm operator (58) with kernel $K_2(t)$ defined in (79) and bandlimiting parameter $\sigma = 8$. Here $N$ is the number of basis functions used.

| Basis: spherical Bessel function | Index | $N = 200$ | $N = 400$ | $N = 600$ | $N = 800$ | $N = 1000$ |
|----------------------------------|-------|-----------|-----------|-----------|-----------|-----------|
| 0                                | 0.9699333440 × 10^{-1} | 0.9699333440 × 10^{-1} | 0.9699333440 × 10^{-1} | 0.9699333440 × 10^{-1} | 0.9699333440 × 10^{-1} |
| 1                                | 6.893197918 × 10^{-1} | 6.893197918 × 10^{-1} | 6.893197918 × 10^{-1} | 6.893197918 × 10^{-1} | 6.893197918 × 10^{-1} |
| 2                                | 7.0927453157 × 10^{-1} | 7.0927453157 × 10^{-1} | 7.0927453157 × 10^{-1} | 7.0927453157 × 10^{-1} | 7.0927453157 × 10^{-1} |
| 3                                | 4.9749372701 × 10^{-1} | 4.9749372701 × 10^{-1} | 4.9749372701 × 10^{-1} | 4.9749372701 × 10^{-1} | 4.9749372701 × 10^{-1} |
| 4                                | 2.5819098828 × 10^{-1} | 2.5819098828 × 10^{-1} | 2.5819098828 × 10^{-1} | 2.5819098828 × 10^{-1} | 2.5819098828 × 10^{-1} |
| 5                                | 7.9576215670 × 10^{-1} | 7.9576215670 × 10^{-1} | 7.9576215670 × 10^{-1} | 7.9576215670 × 10^{-1} | 7.9576215670 × 10^{-1} |
| 6                                | 1.3380226454 × 10^{-2} | 1.3380226454 × 10^{-2} | 1.3380226454 × 10^{-2} | 1.3380226454 × 10^{-2} | 1.3380226454 × 10^{-2} |
| 7                                | 1.3630727514 × 10^{-3} | 1.3630727514 × 10^{-3} | 1.3630727514 × 10^{-3} | 1.3630727514 × 10^{-3} | 1.3630727514 × 10^{-3} |

| Basis: translates of sinc function | Index | $N = 200$ | $N = 400$ | $N = 600$ | $N = 800$ | $N = 1000$ |
|-----------------------------------|-------|-----------|-----------|-----------|-----------|-----------|
| 0                                | 0.96993334492 × 10^{-1} | 0.96993334492 × 10^{-1} | 0.96993334492 × 10^{-1} | 0.96993334492 × 10^{-1} | 0.96993334492 × 10^{-1} |
| 1                                | 6.893199921 × 10^{-1} | 6.893199921 × 10^{-1} | 6.893199921 × 10^{-1} | 6.893199921 × 10^{-1} | 6.893199921 × 10^{-1} |
| 2                                | 7.0927453678 × 10^{-1} | 7.0927453678 × 10^{-1} | 7.0927453678 × 10^{-1} | 7.0927453678 × 10^{-1} | 7.0927453678 × 10^{-1} |
| 3                                | 4.9749481012 × 10^{-1} | 4.9749481012 × 10^{-1} | 4.9749481012 × 10^{-1} | 4.9749481012 × 10^{-1} | 4.9749481012 × 10^{-1} |
| 4                                | 2.5819081601 × 10^{-1} | 2.5819081601 × 10^{-1} | 2.5819081601 × 10^{-1} | 2.5819081601 × 10^{-1} | 2.5819081601 × 10^{-1} |
| 5                                | 7.9632172709 × 10^{-1} | 7.9632172709 × 10^{-1} | 7.9632172709 × 10^{-1} | 7.9632172709 × 10^{-1} | 7.9632172709 × 10^{-1} |
| 6                                | 1.3380360856 × 10^{-2} | 1.3380360856 × 10^{-2} | 1.3380360856 × 10^{-2} | 1.3380360856 × 10^{-2} | 1.3380360856 × 10^{-2} |
| 7                                | 1.3682496634 × 10^{-3} | 1.3682496634 × 10^{-3} | 1.3682496634 × 10^{-3} | 1.3682496634 × 10^{-3} | 1.3682496634 × 10^{-3} |

FIG. 12. The figure shows the error (quantified by (85)) as a function of the regularization parameter ($\mu$) for various inversion algorithms where the observed signal is $x_1(t)$ as defined in (82).

soon the corresponding eigenvalues computed with increasing $N$ start agreeing with each other. Table IV and Table V
The methods are labeled according the basis function and the type of Tikhonov regularization used as in Sec. III B: T₁-Bessel, T₂-Bessel and T₁-sinc. The error is quantified by a relative \( L^2 \) norm:

\[
    e_{\text{rel.}} = \frac{\| y_{\text{exact}} - y_{\text{num.}} \|_{L^2(-1,1)}}{\|y_{\text{exact}}\|_{L^2(-1,1)}}.
\]

The error in the estimation of \( y(t) \) as a function of the regularization parameter are displayed in Fig. 12 and Fig. 13. The methods T₁-Bessel and T₂-Bessel appear to be equally accurate and significantly superior to T₁-sinc. The search for the optimal regularization parameter can be performed by means of L-curve [40].

V. CONCLUSION

To conclude, we have devised a new degenerate kernel method for solving Fredholm integral equations whose kernel belongs to a class of bandlimited functions that are square integrable. The choice of using spherical Bessel functions as the basis functions improves the performance of the numerical algorithms when compared to a similar method using the translates of the sinc function. The idea was first applied to the case of sinc-kernel which yields a new method of computation of the PSWFs on the entire real line and extrapolation of bandlimited signals known on \((-1,1)\) to the entire real line. Notable among other examples is the case of sinc\(^2\)-kernel where we also considered the inverse problem of estimating the object function from the image function. The solution of the ill-conditioned discrete linear system was carried out using the Tikhonov regularization method.

\[\text{FIG. 13. The figure shows the error (quantified by (85)) as a function of the regularization parameter (\( \mu \)) for various inversion algorithms where the observed signal is } x_2(t) \text{ as defined in (83).}\]
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