Dirac or Inverse Seesaw Neutrino Masses with $B-L$ Gauge Symmetry and $S_3$ Flavour Symmetry

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Abstract

Many studies have been made on extensions of the standard model with $B-L$ gauge symmetry. The addition of three singlet (right-handed) neutrinos renders it anomaly-free. It has always been assumed that the spontaneous breaking of $B-L$ is accomplished by a singlet scalar field carrying two units of $B-L$ charge. This results in a very natural implementation of the Majorana seesaw mechanism for neutrinos. However, there exists in fact another simple anomaly-free solution which allows Dirac or inverse seesaw neutrino masses. We show for the first time these new possibilities and discuss an application to neutrino mixing with $S_3$ flavour symmetry.

I. INTRODUCTION

The standard model (SM) of quarks and leptons is free of gauge anomalies. If it is extended to include an extra $U(1)$ gauge group, the conditions for the absence of gauge anomalies impose nontrivial constraints. Nevertheless, interesting discoveries of such possibilities have been made. One example [1] adds a lepton triplet per family to the SM. Another [2] adds a number of new superfields to the minimal supersymmetric standard model (MSSM) of three families. On the other hand, the most studied extension is that of

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Using the particle content of the standard model, all triangle gauge anomalies are zero except for
\[
\sum U(1)_{B-L}^3 = -3
\] (1)
This is easily solved with the addition of three (right-handed) singlet neutrinos, which contribute \(-3(-1)^3 = +3\). Note that the gauge-gravitational anomaly is also zero because \(-3(-1) = +3\). Numerous studies have been made regarding this model of \(U(1)_{B-L}\).

In this paper, we point out that there is another simple choice for three families. Let \(\nu_{Ri} \sim n_i\) under \(B - L\), then \(n_{1,2,3} = (+5, -4, -4)\) yields
\[
-(+5)^3 - (-4)^3 - (-4)^3 = +3
\] (2)
as well. In fact, since \(-5 - (-4) - (-4) = +3\), the mixed gauge-gravitational anomaly is also zero. Now the standard-model Higgs doublet \((\phi^+, \phi^0)^T\) does not connect \(\nu_L\) with \(\nu_R\) and there is no neutrino mass. Consider then three heavy Dirac singlet fermions \(N_{L,R}\), all transforming as \(-1\) under \(B - L\). They do not change the anomaly conditions, but now in the case of \(\nu_{R2}\) and \(\nu_{R3}\), \((\Bar{\nu}_L, \Bar{N}_L)\) is linked to \((\nu_R, N_R)\) through the \(2 \times 2\) mass matrix as follows
\[
M_{\nu,N} = \begin{pmatrix} 0 & m_0 \\ m_3 & M \end{pmatrix}
\] (3)
where \(m_0\) comes from \(\langle \phi^0 \rangle\) and \(m_3\) comes from \(\langle \chi_3 \rangle\), assuming the Yukawa coupling \(\bar{N}_L \nu_R \chi_3\), i.e. \(\chi_3 \sim +3\) under \(B - L\). Now, the invariant mass \(M\) is naturally large, so the Dirac seesaw \([3]\) yields a small neutrino mass \(m_3 m_0 / M\). In the conventional \(U(1)_{B-L}\) model, \(\chi_2 \sim +2\) under \(B - L\), is chosen to break the gauge symmetry, so that \(\nu_R\) gets a Majorana mass and lepton number \(L\) is broken to \((-1)^L\). Here, \(\chi_3 \sim +3\) means that \(L\) remains a conserved global symmetry, with \(\nu_{L,R}\) and \(N_{L,R}\) all having \(L = 1\). Since \(\nu_{R1} \sim +5\) does not connect with \(\nu_L\) or \(N_L\) directly, there is one massless neutrino in this case. However, the dimension-five operator \(\bar{N}_L \nu_R \chi_3^* \chi_3^*/\Lambda\) is allowed by \(U(1)_{B-L}\) and would give it a very small Dirac mass. Alternatively, a second scalar \(\chi_6 \sim 6\) may be added.
Another possible outcome of the above scenario is obtained with the choice of two complex scalar fields $\chi_2 \sim 2$ and $\chi_6 \sim 6$ under $U(1)_{B-L}$. In this case, $\nu_L$ is not connected to $\nu_{R1,R2} \sim -4$. It is connected however to $N_{L,R}$ through the mass matrix spanning $(\bar{\nu}_L, N_R, \bar{N}_L)$ as follows:

$$M_{\nu,N} = \begin{pmatrix} 0 & m_0 & 0 \\ m_0 & m'_2 & M \\ 0 & M & m_2 \end{pmatrix}$$

where $m_2$ and $m'_2$ come from $\chi_2$. This is then an inverse seesaw \[4-6\], i.e. $m_\nu \simeq m^2_2 m_2 / M^2$.

In the case of $\nu_{R3} \sim +5$, the corresponding mass matrix spanning $(\bar{\nu}_L, N_R, \bar{N}_L, \nu_{R3})$ is given by

$$M_{\nu,N} = \begin{pmatrix} 0 & m_0 & 0 & 0 \\ m_0 & m'_2 & M & 0 \\ 0 & M & m_2 & m_6 \\ 0 & 0 & m_6 & 0 \end{pmatrix}$$

where $m_6$ comes from $\chi_6$. Thus $\nu_{R3}$ also gets an inverse seesaw mass $\simeq m^2_6 m'_2 / M^2$. Note that Eq. (5) is the $4 \times 4$ analog of the $3 \times 3$ lopsided seesaw discussed in \[7\].

In this scheme, lepton number $L$ is broken to $(-1)^L$ for $\nu_L$ and $\nu_{R3}$, and $N_{1,2,3}$ are heavy pseudo-Dirac fermions. However, $\nu_{R1,R2}$ remain massless and may only be produced in pairs. They are thus good dark-matter candidates if they become massive. This may be accomplished with a third scalar $\chi_8 \sim 8$.

II. THE $S_3$ FLAVOUR SYMMETRY: $\mu - \tau$ SECTOR

In this section we expand upon the first scenario discussed in the previous section, using two singlet scalar fields transforming as $+3$ under the $B-L$ symmetry, in analogy to having $\nu_{R2}$ and $\nu_{R3}$ transforming as $-4$. This allows us to use the doublet representations of the non-Abelian discrete symmetry group $S_3$ to understand the leptonic family structure as indicated by present neutrino oscillation data and other experimental constraints. We will first work out the details of our model for the simpler case of $\mu - \tau$ sector and show how
one can use $S_3$ symmetry to obtain maximal mixing. Then we will discuss the implications of imposing $S_3$ symmetry to the full $3 \times 3$ mixing matrix. However, before presenting our model let us briefly discuss the $S_3$ symmetry group.

The $S_3$ group is the smallest non-Abelian discreet symmetry group and is the group of the permutation of three objects. It consists of six elements and is also isomorphic to the symmetry group of the equilateral triangle. It admits three irreducible representations $1$, $1'$ and $2$ with the tensor product rules

\[ 1 \times 1' = 1', \quad 1' \times 1' = 1, \quad 2 \times 1 = 2, \quad 2 \times 1' = 2, \quad 2 \times 2 = 1 + 1' + 2 \]  

(6)

In this work, following the earlier works [8, 9], we choose to work with the complex representation of the $S_3$ group. In the complex representation, if

\[
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\psi_1 \\
\psi_2
\end{pmatrix}, \quad \begin{pmatrix}
\phi_1 \\
\phi_2 \\
\psi_1 \\
\psi_2
\end{pmatrix} \in 2 \Rightarrow \begin{pmatrix}
\phi_1^T \\
\phi_2^T \\
\psi_1^T \\
\psi_2^T
\end{pmatrix}, \quad \begin{pmatrix}
\phi_1^T \\
\phi_2^T \\
\psi_1^T \\
\psi_2^T
\end{pmatrix} \in 2
\]

(7)

such that

\[
\phi_1 \psi_2 + \phi_2 \psi_1, \quad \phi_1^T \psi_2 + \phi_2^T \psi_1 \in 1
\]

\[
\phi_1 \psi_2 - \phi_2 \psi_1, \quad \phi_1^T \psi_2 - \phi_2^T \psi_1 \in 1'
\]

\[
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\psi_1
\end{pmatrix}, \quad \begin{pmatrix}
\phi_1^T \\
\phi_2^T \\
\psi_1^T
\end{pmatrix} \in 2
\]

(8)

With this brief summary of $S_3$ group and its irreducible representations, we now move on to constructing an $S_3$ invariant $\mu - \tau$ sector. In later section we will generalize the discussion of this section to the full $3 \times 3$ mixing matrix.

**A. The $S_3$ invariant $\mu - \tau$ sector**

We denote the left handed lepton doublets by $L^\alpha = (\nu_L^\alpha, l_L^\alpha)^T$ where $\alpha = \mu, \tau$; the right handed charged leptons are denoted as $\mu_R, \tau_R$ and the right handed neutrinos as $\nu_R^\mu, \nu_R^\tau$. Also, let us denote the heavy singlet fermions as $N_{L,R}^i; i = 2, 3$. The “Standard Model like”
scalar doublet is denoted by $\Phi = (\phi^+, \phi^0)^T$ and the singlet scalars are denoted by $\chi_{2,3}$.

Let the $B - L$ charge and $S_3$ assignment of the above fields be as shown in Table I.

| Fields | $B - L$ | $S_3$ | Fields | $B - L$ | $S_3$ |
|--------|---------|-------|--------|---------|-------|
| $L^\mu$ | -1 | 1' | $L^\tau$ | -1 | 1 |
| $\mu_R$ | -1 | 1' | $\tau_R$ | -1 | 1 |
| $N^2_L$ | -1 | 1' | $N^3_L$ | -1 | 1 |
| $N^2_R$ | -1 | 1' | $N^3_R$ | -1 | 1 |
| $\Phi$ | 0 | 1 | | | |
| $(\nu^\mu_R \nu^\tau_R)$ | -4 | 2 | $(\chi^2 \chi^3)$ | 3 | 2 |

**TABLE I:** The $B - L$ and $S_3$ charge assignment for the fields.

The $S_3$ and $B - L$ invariant Yukawa $\mathcal{L}^{(2)}_Y$ interaction can be written as

$$\mathcal{L}^{(2)}_Y = \mathcal{L}^{(2)}_{L^\mu L^\tau} + \mathcal{L}^{(2)}_{L^\mu N_R} + \mathcal{L}^{(2)}_{N_L N_R} + \mathcal{L}^{(2)}_{N_L \nu_R}$$ (9)

where

$$\mathcal{L}^{(2)}_{L^\mu L^\tau} = y_{\mu} \bar{L}^\mu \Phi \mu_R + y_{\tau} \bar{L}^\tau \Phi \tau_R$$
$$\mathcal{L}^{(2)}_{L^\mu N_R} = g_{2} \bar{L}^\mu \hat{\Phi}^* N^2_R + g_{3} \bar{L}^\tau \hat{\Phi}^* N^3_R$$
$$\mathcal{L}^{(2)}_{N_L N_R} = M_{2} \bar{N}_L^2 N^2_R + M_{3} \bar{N}_L^3 N^3_R$$
$$\mathcal{L}^{(2)}_{N_L \nu_R} = f_{2} \bar{N}_L^2 \otimes \left[ \begin{pmatrix} \nu^\mu_R \\ \nu^\tau_R \end{pmatrix} \otimes \begin{pmatrix} \chi^2 \\ \chi^3 \end{pmatrix} \right]_{\nu} + f_{3} \bar{N}_L^3 \otimes \left[ \begin{pmatrix} \nu^\mu_R \\ \nu^\tau_R \end{pmatrix} \otimes \begin{pmatrix} \chi^2 \\ \chi^3 \end{pmatrix} \right]_{\nu}$$ (10)

In writing (10), we used the notation $\hat{\Phi}^* = i\tau_2 \Phi^* = (\phi^0, \phi^-)^T$. Moreover, $y_{\mu}, y_{\tau}$ are the Yukawa couplings of the charged lepton sector whereas $f_i, g_i$ and $M_i$ denote the dimensionless coupling constants between the leptons and the heavy fermions.

After symmetry breaking the scalar fields get vacuum expectation values (VEVs) $\langle \phi^0 \rangle = v$, $\langle \chi_i \rangle = u_i$; $i = 2, 3$. Then the mass matrix relevant to charged leptons is given by
\[ M_i = v \begin{pmatrix} y_\mu & 0 \\ 0 & y_\tau \end{pmatrix} \]  

(11)

Also, the $4 \times 4$ mass matrix spanning $(\bar{\nu}_L^\mu, \bar{\nu}_L^\tau, \bar{N}_L^2, \bar{N}_L^3)$ and $(\nu_R^\mu, \nu_R^\tau, N_R^2, N_R^3)^T$ of neutrinos and the heavy fermions is given by

\[
M_{\nu,N} = \begin{pmatrix} 0 & 0 & g_2 v^* & 0 \\ 0 & 0 & 0 & g_3 v^* \\ f_2 u_3 & -f_2 u_2 & M_2 & 0 \\ f_3 u_3 & f_3 u_2 & 0 & M_3 \end{pmatrix} \]  

(12)

As remarked earlier, the mass terms $M_i$ between the heavy fermions can be naturally large, so we can block diagonalize (12) assuming that $f_i, g_i << M_i$. The block diagonalized mass matrix of light neutrinos is given by

\[
M_{\nu} = \frac{m^{(2)}_{N_L\nu_R}}{M^{(2)}_{N_L, N_R}} m^{(2)}_{L, N_R} \begin{pmatrix} f_2 u_3 & -f_2 u_2 & M_2 & 0 \\ f_3 u_3 & f_3 u_2 & 0 & M_3 \end{pmatrix} \]  

(13)

where $m^{(2)}_{L, N_R}$, $M^{(2)}_{N_L, N_R}$ and $m^{(2)}_{N_L, \nu_R}$ are the $2 \times 2$ mass matrices obtained respectively from $L^{(2)}_{L, N_R}$, $L^{(2)}_{N_L, N_R}$ and $L^{(2)}_{N_L, \nu_R}$ terms of (10). This light neutrino mass matrix can be further diagonalized by the bi-unitary transformation. The neutrino masses and the mixing angles obtained from (13) will be dependent on the specific values of the coupling constants $f_i, g_i, M_i$ as well as the VEVs $v, u_i; i = 2, 3$. For sake of illustration we explicitly compute them for two simplifying scenario leading to maximal mixing.

**Case I:** $f_2 = f_3 = f$, $g_2 = g_3 = g$, $M_2 = M_3 = M$, $u_2 \neq u_3$.

In this case the neutrino mass matrix becomes

\[
M_{\nu}^I = \frac{fgv^*}{M} \begin{pmatrix} u_3 & -u_2 \\ u_3 & u_2 \end{pmatrix} \]  

(14)
Case II: \( f_2 = f_3 = f, \ u_2 = u_3 = u, \ \frac{q_2}{M_2} \neq \frac{q_3}{M_3}. \)

In this case the neutrino mass matrix becomes

\[
\mathcal{M}^{II}_\nu = f u v^* \begin{pmatrix}
\frac{q_2}{M_2} & \frac{q_3}{M_3} \\
\frac{q_3}{M_3} & \frac{q_2}{M_2}
\end{pmatrix}
\]  

(15)

Both the mass matrices in (14) and (15) can be written as

\[
\mathcal{M}_\nu = \kappa \begin{pmatrix}
a & -b \\
b & a
\end{pmatrix}
\]  

(16)

where

\[
\kappa = \begin{cases}
f v v^* & \text{In Case I} \\
f u v^* & \text{In Case II}
\end{cases}, \quad a = \begin{cases}
u_3 & \text{In Case I} \\
\frac{q_2}{M_2} & \text{In Case II}
\end{cases}, \quad b = \begin{cases}
u_2 & \text{In Case I} \\
\frac{q_3}{M_3} & \text{In Case II}
\end{cases}
\]  

(17)

The mass matrix in (16) can be easily diagonalized by a bi-unitary transformation leading to the neutrino masses \( \nu_2 = \sqrt{2}ka \) and \( \nu_3 = \sqrt{2}kb \) with mixing angle \( \theta_{23} = \frac{\pi}{4} \) where \( \kappa, a, b \) for both cases are given in (17). Therefore, \( S_3 \) allows us to understand maximal mixing in the \( \mu - \tau \) sector.

### III. THE COMPLETE \( S_3 \) INvariant Lepton Sector

The \( B - L \) charge and \( S_3 \) assignment of the fields for the full lepton sector is as shown in Table II.

The \( S_3 \) and \( B - L \) invariant Yukawa \( \mathcal{L}_Y \) interaction can be written as

\[
\mathcal{L}_Y = \mathcal{L}_{L^\alpha l_R} + \mathcal{L}_{L^\alpha N_R} + \mathcal{L}_{N_L N_R} + \mathcal{L}_{N_L \nu_R}
\]  

(18)

where
Table II: The $B - L$ and $S_3$ charge assignment for the fields.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\text{Fields} & B - L & S_3 & \text{Fields} & B - L & S_3 \\
\hline
L^e & -1 & 1' & \nu^e_R & -1 & 1' \\
L^\mu & -1 & 1' & \nu^\mu_R & -1 & 1' \\
L^\tau & -1 & 1 & \tau_R & -1 & 1 \\
N^L_1 & -1 & 1' & N^1_R & -1 & 1' \\
N^L_2 & -1 & 1' & N^2_R & -1 & 1' \\
N^L_3 & -1 & 1 & N^3_R & -1 & 1 \\
\Phi & 0 & 1 & \nu^c_R & 5 & 1' \\
\left(\begin{array}{c}
\nu^\mu_R \\
\nu^\tau_R
\end{array}\right) & -4 & 2 & \left(\begin{array}{c}
\chi_2 \\
\chi_3
\end{array}\right) & 3 & 2 \\
\hline
\end{array}
\]

\[\mathcal{L}_{L^a} = y'_e L^e \Phi e_R + y'_{12} L^e \Phi \mu_R + y'_{21} \bar{L}^\mu \Phi e_R + y'_{\mu} \bar{L}^\mu \Phi \mu_R + y_{\tau} \bar{L}^\tau \Phi \tau_R\]
\[\mathcal{L}_{L^a N^i} = g'_{11} L^e \hat{\Phi}^* N^1_R + g'_{12} L^e \hat{\Phi}^* N^2_R + g'_{21} L^\mu \hat{\Phi}^* N^1_R + g'_{22} L^\mu \hat{\Phi}^* N^2_R + g_{33} L^\tau \hat{\Phi}^* N^3_R\]
\[\mathcal{L}_{N^i N^j} = M'_{11} \bar{N}^1_L N^1_R + M'_{12} \bar{N}^1_L N^2_R + M'_{21} \bar{N}^2_L N^1_R + M'_{22} \bar{N}^2_L N^2_R + M'_{33} \bar{N}^3_L N^3_R\]
\[\mathcal{L}_{N^i \nu^j} = \frac{f'_{11}}{\Lambda} (\bar{N}^1_L \nu^c_R) \otimes \left[\begin{array}{c}
\chi_2 \\
\chi_3
\end{array}\right] \otimes \left[\begin{array}{c}
\chi_2 \\
\chi_3
\end{array}\right]_1 + \frac{f'_{21}}{\Lambda} (\bar{N}^2_L \nu^c_R) \otimes \left[\begin{array}{c}
\chi_2 \\
\chi_3
\end{array}\right] \otimes \left[\begin{array}{c}
\chi_2 \\
\chi_3
\end{array}\right]_1 \\
+ f'_{12} \bar{N}^1_L \otimes \left[\begin{array}{c}
\nu^\mu_R \\
\nu^\tau_R
\end{array}\right] \otimes \left[\begin{array}{c}
\chi_2 \\
\chi_3
\end{array}\right]_1' + f'_{22} \bar{N}^2_L \otimes \left[\begin{array}{c}
\nu^\mu_R \\
\nu^\tau_R
\end{array}\right] \otimes \left[\begin{array}{c}
\chi_2 \\
\chi_3
\end{array}\right]_1' \\
+ f'_{33} \bar{N}^3_L \otimes \left[\begin{array}{c}
\nu^\mu_R \\
\nu^\tau_R
\end{array}\right] \otimes \left[\begin{array}{c}
\chi_2 \\
\chi_3
\end{array}\right]_1' \right]
\] (19)

As before, in writing (19), we used the notation $\hat{\Phi}^* = i\tau_2 \Phi^* = (\phi^0, \phi^-)^T$. Moreover, $y_a$ are the Yukawa couplings of the charged leptons whereas $f_{ij}$, $g_{ij}$ and $M_{ij}$ denote the dimensionless coupling constants between the leptons and the heavy fermions.

At this point we like to remark that in (19) there is still a freedom to redefine a few fields (i.e. the pairs $N^1_L - N^2_L$, $N^1_R - N^2_R$ and $e_R - \mu_R$) in a way that certain couplings can
be made equal to zero. For sake of later convenience we choose to use this freedom of field redefinition to make \( f'_{11} = M'_{12} = y'_{21} = 0 \). Moreover, we relabel the remaining non-zero couplings of these redefined fields as \( f'_{ij} \rightarrow f_{ij}, g'_{ij} \rightarrow g_{ij}, M'_{ij} \rightarrow M_{ij} \).

Now, after symmetry breaking the scalar fields get VEVs \( \langle \phi^0 \rangle = v, \langle \chi_i \rangle = u_i; \ i = 2, 3 \). Then the mass matrices relevant to charged lepton is given by

\[
M_l = v \begin{pmatrix} y_e & y_{12} & 0 \\ 0 & y_\mu & 0 \\ 0 & 0 & y_\tau \end{pmatrix}
\]  

(20)

The mass matrix (20) can be readily diagonalized by bi-unitary transformation. In the limit of \( y_e << y_\mu \) we get

\[
\theta_{12}^l \approx \tan^{-1} \left( \frac{-y_{12}}{y_\mu} \right); \quad m_e \approx v y_e \cos \theta_{12}^l \\
m_\mu \approx v (y_\mu \cos \theta_{12}^l - y_{12} \sin \theta_{12}^l); \quad m_\tau \approx v y_\tau
\]  

(21)

If \( y_{12} = y_\mu \) then maximal mixing is achieved i.e. \( \theta_{12}^l = -\frac{\pi}{4} \), with \( m_\mu = \sqrt{2} vy_\mu \).

Also, the 6 \times 6 mass matrix spanning \( (\bar{\nu}_L, \bar{\nu}_L^\mu, \bar{\nu}_L^\tau, \bar{N}_L^1, \bar{N}_L^2, \bar{N}_L^3) \) and \( (\nu_R, \nu_R^\mu, \nu_R^\tau, N_R^1, N_R^2, N_R^3) \) of neutrinos and the heavy fermions is given by

\[
M_{\nu,N} = \begin{pmatrix}
0 & 0 & 0 & g_{11} v^* & g_{12} v^* & 0 \\
0 & 0 & 0 & g_{21} v^* & g_{22} v^* & 0 \\
0 & 0 & 0 & 0 & 0 & g_{33} v^* \\
0 & f_{12} u_3 - f_{12} u_2 & M_{11} & 0 & 0 \\
f_{22} u_3^* u_3^* & f_{22} u_3 - f_{22} u_2 & M_{21} & M_{22} & 0 \\
0 & f_{33} u_3 & f_{33} u_2 & 0 & 0 & M_3
\end{pmatrix}
\]  

(22)

As remarked earlier, the mass terms \( M_{ij} \) between the heavy fermions can be naturally large, so we can block diagonalize (22) assuming that \( f_{ij}, g_{ij} << M_{ij} \). The block diagonalized mass matrix of light neutrinos is given by
\[
M_\nu = m_{N_L N_R} \frac{1}{M_{N_L N_R}} m_{L^\nu N_R}.
\]

\[
= v^* \begin{pmatrix}
\frac{(g_{21} M_{11} - g_{11} M_{21}) f_{12} u_2}{M_{11} M_{22}} & \frac{(g_{22} M_{11} - g_{12} M_{21}) f_{12} u_2}{M_{11} M_{22}} & -f_{12} g_{33} u_3 \\
\frac{(g_{21} M_{11} - g_{11} M_{21}) f_{33} u_2}{M_{11} M_{22}} & \frac{(g_{22} M_{11} - g_{12} M_{21}) f_{33} u_2}{M_{11} M_{22}} & f_{33} g_{33} u_3 \\
M_{11} M_{22} & M_{11} M_{22} & M_{33}
\end{pmatrix}
\]

(23)

where we have written \(u_6 = \frac{u^*_2 u^*_3}{\Lambda}\). Also, the 3 \times 3 mass matrices \(m_{L^\nu N_R}, M_{N_L N_R}\) and \(m_{N_L N_R}\) are obtained from the terms \(\mathcal{L}_{L^\nu N_R}, \mathcal{L}_{N_L N_R}\) and \(\mathcal{L}_{N_L N_R}\) of (19), respectively. This light neutrino mass matrix can be further diagonalized by the bi-unitary transformation. The neutrino masses and the mixing angles obtained from (23) will be dependent on the specific values of the coupling constants \(f_{ij}, g_{ij}, M_{ij}\) as well as the VEVs \(v, u_i; i = 2, 3\).

In the simplifying case of \(g_{ij} = g\) and \(M_{ij} = M\) we get

\[
M_\nu = \frac{g v^*}{M} \begin{pmatrix}
0 & 0 & -f_{12} u_3 \\
f_{21} u_6 & f_{21} u_6 & -f_{22} u_3 \\
0 & 0 & f_{33} u_3
\end{pmatrix}
\]

(24)

Diagonalizing the mass matrix (24) we have

\[
\theta_{12}^\nu \approx 0; \quad \theta_{13}^\nu \approx \tan^{-1}\left(\frac{f_{12}}{f_{33}}\right); \quad \theta_{23}^\nu \approx \tan^{-1}\left(\frac{f_{22}}{\sqrt{f_{12}^2 + f_{33}^2}}\right)
\]

\[
m_{1}^\nu \approx 0; \quad m_{2}^\nu \approx \sqrt{2(f_{12}^2 + f_{33}^2) f_{21} g |v|} |u_6|; \quad m_{3}^\nu \approx \sqrt{f_{12}^2 + f_{22}^2 + f_{33}^2 g |v|} |u_3|
\]

(25)

Since, \(u_6 << u_3\), we have a normal hierarchy pattern with two nearly massless neutrinos and one relatively heavy neutrino. Moreover, the massless neutrino will also gain small mass, if any of the \(M_{ij}\)’s or \(g_{ij}\)’s are not equal to \(M\) or \(g\) respectively. Also, if they deviate significantly from these values then one can possibly recover degenerate or inverted hierarchy patterns also.

Now, if \(U_l\) and \(U_\nu\) are the mixing matrices of the charged leptons and neutrinos respectively, then the PMNS mixing matrix is given by
\[ U_{\text{PMNS}} = U_1^\dagger U_\nu \]  

Taking \( y_{12} = y_{1\mu} \), \( f_{12} = -\frac{f_{1\nu}}{2} \) and \( f_{22} = \sqrt{f_{12}^2 + f_{33}^2} \) in (21) and (25) we get \( \theta_{23}^\nu = -\theta_{12}^l = \frac{\pi}{4} \) and \( \theta_{13}^\nu = \tan^{-1}(\frac{1}{2}) \) which gives PMNS mixing angles consistent\(^1\) with present 3\( - \sigma \) limits of global fits obtained from experiments [10].

The Quark Sector

In our minimal model with only one doublet scalar, the quark sector can be accommodated in a simple way if both the left handed quark doublets \( Q_i^L = (u_i^L, d_i^L)^T \), \( i = 1, 2, 3 \) and the right handed quark singlets \( u_i^R, d_i^R; i = 1, 2, 3 \) transform as 1 of \( S_3 \).

A better understanding of the quark sector can be obtained if, to our minimal model, we add more doublet scalars transforming non-trivially under \( S_3 \). One such example for quark sector, albeit in context of a different model for lepton sector, has already been worked out in [8, 9]. We plan to present a similar extension of our minimal model in a future work.

IV. CONCLUSIONS

The idea that \( B - L \) should be a gauge symmetry has been around for a long time. The minimal version of three \( \nu_R \) transforming as \( -1 \) is very well-known. Other exotic variants are possible with extra particles, such as the two models recently proposed [11, 12]. Here we point out the simple anomaly-free solution of three \( \nu_R \)'s transforming as \( +5, -4, -4 \). We show how these assignments may be used to obtain seesaw Dirac neutrino masses, as well as inverse seesaw Majorana neutrino masses. We then apply \( S_3 \) symmetry to the first case, and obtain realistic neutrino and charged-lepton mass matrices with a mixing pattern consistent with experiments.

\(^1\) With \( \theta_{23}^\nu = -\theta_{12}^l = \frac{\pi}{4} \) and \( \theta_{13}^\nu = \tan^{-1}(\frac{1}{2}) \) we get the value of mixing angles as \( \theta_{12} = 30.29^\circ \), \( \theta_{13} = 7.53^\circ \) and \( \theta_{23} = 50.36^\circ \). The values of \( \theta_{12} \) and \( \theta_{13} \) are slightly below the lower limits (30.59\(^\circ\), 7.62\(^\circ\) respectively) quoted in [10]. A much better fit can be obtained if we take slightly different values, for instance taking \( \theta_{12} = -50^\circ \), \( \theta_{13} = -29.39^\circ \) and \( \theta_{23} = \frac{\pi}{4} \) gives us \( \theta_{12} = 33.26^\circ \), \( \theta_{13} = 9.00^\circ \) and \( \theta_{23} = 51.41^\circ \).
Acknowledgments

RS will like to thank G. Rajasekaran for valuable comments and suggestions. EM is supported in part by the U. S. Department of Energy under Grant No. de-sc0008541.

[1] E. Ma, Mod. Phys. Lett. A 17, 535 (2002) [hep-ph/0112232].
[2] E. Ma, Phys. Rev. Lett. 89, 041801 (2002) [hep-ph/0201083].
[3] P. Roy and O. U. Shanker, Phys. Rev. Lett. 52, 713 (1984) [Erratum-ibid. 52, 2190 (1984)].
[4] D. Wyler and L. Wolfenstein, Nucl. Phys. B 218, 205 (1983).
[5] R. N. Mohapatra and J. W. F. Valle, Phys. Rev. D 34, 1642 (1986).
[6] E. Ma, Phys. Lett. B 191, 287 (1987).
[7] E. Ma, Mod. Phys. Lett. A 24, 2161 (2009) [arXiv:0904.1580 [hep-ph]].
[8] S. L. Chen, M. Frigerio and E. Ma, Phys. Rev. D 70, 073008 (2004) [Erratum-ibid. D 70, 079905 (2004)] [hep-ph/0404084].
[9] E. Ma and B. Melic, Phys. Lett. B 725, 402 (2013) [arXiv:1303.6928 [hep-ph]].
[10] F. Capozzi, G. L. Fogli, E. Lisi, A. Marrone, D. Montanino and A. Palazzo, Phys. Rev. D 89, 093018 (2014) [arXiv:1312.2878 [hep-ph]].
[11] S. Kanemura, T. Nabeshima and H. Sugiyama, Phys. Rev. D 85, 033004 (2012) [arXiv:1111.0599 [hep-ph]].
[12] S. Kanemura, T. Matsui and H. Sugiyama, Phys. Rev. D 90, 013001 (2014) [arXiv:1405.1935 [hep-ph]].