Fatigue Scatter Factor Analysis of Airplane Structures Based on Zero-failure Data

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Abstract. In the fatigue life test of high-reliability airplane structures, a large number of zero-failure data can be obtained, in which case the fatigue life scatter factor is significantly different from that of complete data. This paper conducted a study to define a scatter factor correction coefficient for normal distribution and two-parameter Weibull distribution under the condition of zero-failure data and derived the difference of the scatter factor under these two cases theoretically. At the same time, in order to analyze the difference, a comparative analysis of numerical simulation results under a certain reliability index is presented. The results show that this method makes full use of the product information under the conditions of zero-failure data to enhance the structures’ fatigue safe life and to meet the practical engineering requirements.

1. Introduction
In the aerospace field, specimen numbers of large-scale component fatigue life tests tend to be small, or even only one or two. Therefore, how to determine product safety life based on limited test data becomes a very important issue. Nowadays, the scatter factor method is widely used. However, as the improvement of product reliability and the appearance of abundant zero-failure data, new problem on scatter factor determination raised [1]. In this case, the product safety life will be far less than the physical safe life still using the life scatter of complete data, which affects the normal use of the product. Therefore, it’s highly recommended to establish a fatigue life scatter factor calculation method under the condition of zero-failure data. Extensive research has shown that the fatigue life of fighter aircrafts obeyed log-normal distribution while the fatigue life of the passenger aircrafts obeyed two-parameter Weibull distribution [2]. This paper is proposed to give the scatter factor correction coefficient of normal distribution and two-parameter Weibull distribution under the condition of zero-failure data, and to comparatively analyze the scatter factor of complete data and zero-failure data theoretically and numerically. Compared with the existing extensive test results, this method could make a more accurate prediction and is easy to be used in practical engineering applications.

2. Methods and results
For the fatigue tests of full-scale aircrafts or some large components, specimens are limited, sometimes only one available. Under this situation, the scatter factor method is widely applied to evaluate the safe life.

According to the reference [3], the fatigue life scatter factor $L_f$ is defined as
\[
L_f = \frac{N_{50}}{N_p}
\]

where \( N_{50} \) is the measured mean life, \( N_p \) is an estimated safe life under a certain reliability index \( P \). However, the traditional scatter factor should be revised to approach the actual safe life under the condition of zero-failure data. The new form of fatigue life scatter factor can be defined by the following formula:

\[
L_f' = L_f \cdot k
\]

where \( k \) is the correction coefficient, \( L_f \) is the scatter factor calculated under the condition of complete data. The derivation process and the analysis of \( k \) are given below.

2.1. Normal distribution situation

2.1.1. The fatigue life scatter factor under condition of complete data. In order to calculate the fatigue life scatter factor under the condition that the fatigue life is lognormal distributed with complete data. The fatigue life scatter factor considering the reliability only is defined as follows.

Assuming that the fatigue life \( N \) follows lognormal distribution:

\[
X = \log N = N(\mu, \sigma^2)
\]

where \( X \) is the logarithm of \( N \), \( \mu \) is the mathematical expectation, \( \sigma \) is the standard deviation. Thus the \( P \)th percentile is established as

\[
x_p = \mu + u_p \sigma
\]

where \( u_p \) is the standard normal deviation of probability \( P \). Thus the safe life \( N_p \) can be expressed as

\[
N_p = 10^{\gamma} = 10^{\mu + u_p \sigma}
\]

Since the theoretical median life \( N_{50} = 10^\mu \), the theoretical fatigue life scatter factor under condition of lognormal distribution can be calculated by

\[
L_f = \frac{10^\sigma}{10^{\mu + u_p \sigma}} = 10^{\gamma - \sigma}
\]

The confidence level is introduced to prevent that the estimate is larger than the actual value when the specimens are limited. In order to make the estimated parent mean \( \hat{\mu}_{1-\gamma} \) less than the true value \( \mu \) in most cases, \( \hat{\mu}_{1-\gamma} \) can be expressed as:

\[
\hat{\mu}_{1-\gamma} = \bar{X} - C
\]

where \( C \) is a constant, and to the desired confidence, equation (8) should be satisfied.

\[
P(\mu \geq \hat{\mu}_{1-\gamma} = \bar{X} - C) = \gamma
\]

When \( \sigma \) is known, for a given confidence \( \gamma \),

\[
P\left( \frac{\bar{X} - \mu}{\sigma} \leq u_p \right) = \gamma
\]

Namely,

\[
\hat{\mu}_{1-\gamma} = \bar{X} - u_p \frac{\sigma}{\sqrt{n}}
\]

Therefore, by substituting \( \mu \) with \( \hat{\mu}_{1-\gamma} \), the log safety life \( x_p \) of desired probability \( P \) and confidence \( \gamma \) can be obtained.

\[
x_p = \bar{X} - u_p \frac{\sigma}{\sqrt{n}} + u_p \sigma
\]

Namely,
\[ N_p = 10^{\frac{u_p - \mu}{\sigma}} \]  

By the symmetry character of normal distribution, the parent mean equals the parent median, therefore the median life \( N_{50} = 10^\sigma \) can be estimated by \( \tilde{N}_{50} = 10^\sigma \).

Therefore the fatigue life scatter factor \( L_f \) under complete data can be expressed as

\[
L_f = \frac{10^\sigma}{10^{\frac{u_p - \mu}{\sigma}}} = 10^{\left(\frac{u_p - \mu}{\sigma}\right)}
\]  

(13)

2.1.2. The fatigue life scatter factor under condition of zero-failure data. Large components or the full-scale of aircrafts in fatigue tests are all not destroyed, so the life data is zero-failure data. In order to obtain the scatter factor when the fatigue life follows lognormal distribution with zero-failure data, the parent standard deviation needs to be determined first.

Given that the parent standard deviation is known, the derivation process of the fatigue life scatter factor with zero-failure data is as follows:

Let \( x_i = \log N_i, \ i = 1,2,\ldots,n \) is a set of zero-failure data coming from normal distribution \( N(\mu, \sigma) \), and the standard deviation \( \sigma \) is known. As reference [4] addressed, to a certain probability \( P_0 = 1 - (1 - \gamma)^2 \), the one-sided lower confidence limits of log life with the confidence level \( \gamma \) is

\[
P(x_p \geq \bar{x}) = \gamma
\]  

(14)

For a given probability \( P_0 \), according to equation (4),

\[
x_p = x_p + (u_p - u_p)\sigma
\]  

(15)

Substitute equation (15) into equation (14), thus

\[
P(x_p \geq \bar{x} + (u_p - u_p)\sigma) = \gamma
\]  

(16)

Therefore the log safety life of a certain probability \( P \) and confidence \( \gamma \) can be expressed as

\[
x_p = \bar{x} + (u_p - u_p)\sigma
\]  

(17)

Since \( \tilde{N}_{50} = 10^\sigma \), the fatigue life scatter factor \( L_f \) of zero-failure data is

\[
L_f = \frac{10^\sigma}{10^{\frac{u_p - \mu}{\sigma}}} = 10^{\left(\frac{u_p - \mu}{\sigma}\right)}
\]  

(18)

2.1.3. The correction coefficient of fatigue life scatter factor of normal distribution. The derivation illuminated that the scatter factor under the condition of zero failure data is different from that of complete data. Therefore, the traditional fatigue life scatter factor needs correction to apply for the incomplete data case.

According to the definition of scatter factor, the correction coefficient \( k \) of fatigue life scatter factor under the condition of normal distribution can be expressed as

\[
k = \frac{10^{\left(\frac{u_p - \mu}{\sigma}\right)}}{10^{\left(\frac{u_p - \mu}{\sigma}\right)}} = 10^{\left(\frac{u_p - \mu}{\sigma}\right)}
\]  

(19)

Figure 1 shows the value of \( k \) varies with the standard deviation \( \sigma \) and sample size \( n \) in a certain range when the confidence \( \gamma = 0.9 \). Equation (19) and figure 1 denote information as follows:

1. When the sample size \( n = 1 \), the correction factor \( k = 1 \), that is, the scatter factor of zero-failure data equals the scatter factor of complete data when the test sample is single; when the sample size \( n > 1 \), the correction factor \( k < 1 \), that is, the scatter factor under the condition of zero-failure data is less than that under the conditions of complete data.
2. The scatter factor correction coefficient is inversely proportional to the standard deviation \( \sigma \), that is, when the other parameters keep constant, the larger the standard deviation \( \sigma \), the smaller the coefficient \( k \), which indicates that a smaller disparity will lay a greater influence on the scatter factor.

3. The correction factor of scatter factor has no relation with the given reliability index, but do be influenced by the confidence level, and become more sensitive when the confidence is higher. The correction factor is also sensitive to sample size and will decrease sharply when \( n \) becomes larger, but the impact will tend to be stabilized when \( n \) is up to 5.

![Figure 1. The correction coefficient of fatigue life scatter factor of normal distribution, zero-failure data.](image1.png)

![Figure 2. The correction coefficient of fatigue life scatter factor of Weibull distribution, zero-failure data.](image2.png)

2.2. Weibull distribution situation

2.2.1. The fatigue life scatter factor under condition of complete data. When assessing the aircraft life, the fatigue life of civil aircraft structure is generally believed to obey two-parameter Weibull distribution [2]. The study on fatigue life scatter factor of two-parameter Weibull distribution is relatively complex, because under the condition of incomplete data, the scatter factor has different forms in accordance with the parameters which based on. Reference [5] gives the scatter factor based on the median life expectancy, which is widely used in the aviation sector of China. Therefore, the correction of scatter factor based on the median life expectancy is discussed in the following part.

Assuming that the fatigue life \( N \) follows two parameters Weibull distribution, then

\[
F(N) = 1 - \exp \left( - \left( \frac{N}{\beta} \right)^\alpha \right)
\]

where \( \alpha \) is the shape parameter, \( \beta \) is the scale parameter. Thus the \( P \) th lower percentile \( N_p \) in accordance with probability \( P \) can be defined as follows

\[
N_p = \beta(\ln P)^{\frac{1}{\alpha}}
\]

Therefore, the theoretical fatigue life scatter factor of Weibull distribution can be expressed as

\[
L_f = \frac{\beta(-\ln 0.5)^{\frac{1}{\alpha}}}{\beta(-\ln P)^{\frac{1}{\alpha}}} = \left(\frac{\ln 0.5}{\ln P}\right)^{\frac{1}{\alpha}}
\]

The median value of two-parameter Weibull distribution doesn’t equal the mean with an asymmetry distribution curve, which is different from normal distribution. The estimate of the median life can be expressed as

\[
\hat{N}_{\text{med}} = \hat{\beta}(-\ln 0.5)^{\frac{1}{\alpha}}
\]
where $\hat{\beta}$ is the estimate of characteristic life, which can be obtained by maximum likelihood method.

\[
\hat{\beta} = \left( \frac{1}{n} \sum_{i=1}^{n} N_i^\alpha \right)^{\frac{1}{\alpha}}
\]  

(24)

The confidence level should be introduced in the calculation of the safe life $N_p$ to prevent estimate is larger than the actual value. After the introduction of the confidence level $\gamma$, the safe life $N_{p\gamma}$ can be calculated by

\[
N_{p\gamma} = \hat{\beta}_\gamma \left( -\ln P \right)^{\frac{1}{\alpha}}
\]  

(25)

where $\hat{\beta}_\gamma$ is the estimate of characteristic life after the introduction of confidence level, and

\[
\hat{\beta}_\gamma = \frac{\hat{\beta}}{S_\gamma}
\]  

(26)

where $S_\gamma$ is confidence coefficient satisfied equation (27)

\[
P\left( \frac{\hat{\beta}}{\beta} < S_\gamma \right) = \gamma
\]  

(27)

Therefore, $S_\gamma$ can be obtained after the probability distribution density of $X = \frac{\hat{\beta}}{\beta}$ is known, reference [2] addressed the equation which satisfied

\[
\int_{0}^{S_\gamma} \frac{\alpha n^\alpha}{\Gamma(n)} x^{\alpha-1} e^{-n x} dx = \gamma
\]  

(28)

where $\gamma$ is the confidence level, $\alpha$ is the shape parameter, $n$ is the sample size. Therefore, according to the scatter factor calculation equation (1), the scatter factor of two-parameter Weibull distribution considering confidence level can be expressed as

\[
L_{\gamma} = \frac{\hat{\beta}(\ln 0.5)^{\frac{1}{\alpha}}}{\hat{\beta}_\gamma \left( -\ln P \right)^{\frac{1}{\alpha}}} = S_\gamma \left( \ln 0.5 \right)^{\frac{1}{\alpha}}
\]  

(29)

### 2.2.2. The fatigue life scatter factor under condition of zero-failure data.

As mentioned above, due to the zero-failure data cannot offer any concrete information of population disparity, the previous accumulated test data have to be relied on to determine the shape parameter of two-parameter Weibull distribution, and then to amend the fatigue life scatter factor and estimate the safe life.

The following analysis on safe life and fatigue life scatter factor is under the condition of zero-failure data and that the shape parameter of two-parameter Weibull is known. Assuming that $N_i, i = 1, 2, \cdots, n$ is a set of zero-failure data from exponential distribution.

\[
F(N) = 1 - \exp \left\{ \frac{-N}{\theta} \right\}
\]  

(30)

The one-sided lower confidence limit of reliable life $N_p, \gamma$ with confidence $\gamma$ is [1]

\[
N_p = \frac{\ln P}{\ln(1-\gamma)} \sum_{i=1}^{n} N_i
\]  

(31)

For two-parameter Weibull distribution, if the shape parameter is known, there is theorem 1 [6].

Theorem 1: if $N$ follows two-parameter Weibull distribution as equation (30), then $Z = N^\alpha$ follows Weibull distribution with the only parameter $\theta = \beta^\alpha$.

Therefore, according to theorem 1 and equation (31), the estimated safety life $\hat{N}_{p\gamma}$ under the condition that the zero-failure data and shape parameter $\alpha$ is known can be expressed as
\[ N_{s0} = \left( -\ln 0.5 - \sum_{i=1}^{n} N_i^\alpha \right)^{1/\alpha} \]

The scatter factor based on median life in zero-failure data case can be expressed as

\[ L_y = \left( \frac{-\ln 0.5 - \sum_{i=1}^{n} N_i^\alpha}{n \ln P} \right)^{1/\alpha} = \left( \frac{-\ln(1-\gamma) \ln 0.5}{n \ln P} \right)^{1/\alpha} \]

2.2.3. The correction coefficient of fatigue life scatter factor of Weibull distribution. The above discussion presented the difference of the scatter factor of fatigue median life obeyed two-parameter Weibull under the conditions of zero-failure data and complete data. Therefore, the scatter factor needs correction to meet the need of zero-failure data case. According to the definition of scatter factor, we can obtain the correction coefficient \( k \) of fatigue life scatter factor under the Weibull distribution case.

\[ k = \frac{\ln(1-\gamma) \ln 0.5^{1/\alpha}}{n \ln P} = \frac{-\ln(1-\gamma) \ln 0.5^{1/\alpha}}{n \ln P} \]

where \( \alpha \) is the shape parameter, \( \beta \) is the scale parameter, \( S_y \) can be obtained by equation (28). Figure 2 shows the value of \( k \) varies with the shape parameter \( \alpha \) and sample size \( n \) in a certain range when the confidence \( \gamma = 0.95 \). Equation (35) and figure 2 denote information as follows:

1. The characteristics of the correction coefficient of fatigue life scatter factor of zero-failure data under Weibull distribution is the same as that of the normal distribution case. That means, when the sample size \( n = 1 \), the correction factor \( k = 1 \), that is, the scatter factor of zero-failure data equals the scatter factor of complete data when the test sample is single; when the sample size \( n > 1 \), the correction factor \( k < 1 \), that is, the scatter factor under the condition of zero-failure data is less than that under the conditions of complete data.

2. The scatter factor correction coefficient is inversely proportional to the shape parameter \( \alpha \), that is when the other parameters are constant, the larger the \( \alpha \), the smaller the \( k \). Similar to the result of log-normal distribution, more disparity indicates less influence on the scatter factor brought by zero-failure data.

3. The correction coefficient of scatter factor has no relation with the reliability index, but do be influenced by the confidence level, and become more sensitive when the confidence is higher. The coefficient factor is also sensitive to sample size and will decrease sharply when \( n \) becomes larger, but the impact will tend to be stabilized when \( n \) up to 5.

3. Comparison

A concrete numerical example is given to discuss the difference among these cases here. Table 1 and table 2 list the fatigue life scatter factor following normal distribution of complete data and the non-failure data, respectively. Table 3 and table 4 list that of two-parameter Weibull distribution case.

Table 1 and table 2 denote:

1. When the sample size \( n = 1 \), the scatter factor of zero-failure data equals the scatter factor of complete data, when the sample size \( n > 1 \), when the reliability, confidence and standard deviation are the same, the scatter factor under the condition of zero-failure data is less than that under the condition
of complete data, which is more distinct with a larger sample size $n$ and standard deviation $\sigma$. These properties denote that the zero-failure data lay a great impact on the fatigue life scatter factor in normal distribution case.

2. Reliability, confidence and standard deviation are proportional to fatigue life scatter factor.

Table 3 and table 4 denote:

1. When the sample size $n = 1$, the scatter factor of zero-failure data equals the scatter factor of complete data, but when the sample size $n > 1$, when the reliability, confidence and standard deviation are the same, the scatter factor under the condition of zero-failure data is less than that under the condition of complete data, which is more distinct with a larger sample size $n$ and standard deviation $\sigma$. These properties denote that the zero-failure data lay a great impact on the fatigue life scatter factor in Weibull distribution case.

2. Reliability, confidence is proportional to fatigue life scatter factor, whereas the shape parameter is inversely proportional to fatigue life scatter factor.

| Simple Size $n$ | Standard Deviation $\sigma$ | Reliability/Confidence ($P/\gamma$) |
|-----------------|-----------------------------|----------------------------------|
|                 |                             | 99.9/95 99.9/90 99.9/70 95/95 95/90 95/70 |
| 1               | 0.176                       | 6.8 5.9 4.3 3.8 3.3 2.4 |
|                 | 0.2                         | 8.9 7.5 5.3 4.5 3.8 2.7 |
| 2               | 0.176                       | 5.6 5.1 4.1 3.1 2.8 2.3 |
|                 | 0.2                         | 7.1 6.3 4.9 3.6 3.2 2.5 |

| Simple Size $n$ | Standard Deviation $\sigma$ | Reliability/Confidence ($P/\gamma$) |
|-----------------|-----------------------------|----------------------------------|
|                 |                             | 99.9/95 99.9/90 99.9/70 95/95 95/90 95/70 |
| 1               | 0.176                       | 6.8 5.9 4.3 3.8 3.3 2.4 |
|                 | 0.2                         | 8.9 7.5 5.3 4.5 3.8 2.7 |
| 2               | 0.176                       | 4.8 4.2 3.3 2.7 2.4 1.9 |
|                 | 0.2                         | 5.9 5.2 3.9 3.0 2.7 2.0 |

| Simple Size $n$ | Shape Parameter $\alpha_0$ | Reliability/Confidence ($P/\gamma$) |
|-----------------|-----------------------------|----------------------------------|
|                 |                             | 99.9/95 99.9/90 99.9/70 95/95 95/90 95/70 |
| 1               | 4                           | 6.8 6.3 5.4 2.5 2.4 2.0 |
|                 | 3                           | 12.7 11.7 9.4 3.4 3.1 2.5 |
|                 | 2.2                         | 32.1 28.5 21.3 5.4 4.8 3.6 |
| 2               | 4                           | 6.4 6.1 5.4 2.4 2.3 2.0 |
|                 | 3                           | 11.8 11.1 9.5 3.2 3.0 2.5 |
|                 | 2.2                         | 29 26.4 21.3 4.8 4.4 3.6 |

| Simple Size $n$ | Shape Parameter $\alpha_0$ | Reliability/Confidence ($P/\gamma$) |
|-----------------|-----------------------------|----------------------------------|
|                 |                             | 99.9/95 99.9/90 99.9/70 95/95 95/90 95/70 |
4. Conclusion
1. The fatigue life scatter factor under the condition of zero-failure data is distinctly different from that of complete data.
2. The fatigue life scatter factor correction coefficients are defined in the cases of normal distribution and two-parameter Weibull distribution respectively. When the sample size $n = 1$, the correction coefficient equals 1, and when the sample size $n > 1$, the correction coefficient is less than 1.
3. The fatigue life scatter factor is less than that of complete data when the sample size is larger than 1, which is in line with the engineering practice.
4. The difference of scatter factor between complete data and zero-failure data is influenced by sample size, reliability, confidence and dispersivity. The larger the sample size, the higher the confidence and the more dispersive, the more distinction between them.
5. The data analysis of scatter factor of two kinds of distribution and under complete and incomplete data condition denotes that: the method addressed in this paper can make full use of the newly obtained zero-failure test data and the existed test results, raise the product safe fatigue life and meet the engineering requirements better.

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