Quantum Time Crystal By Decoherence:
Proposal With Incommensurate Charge Density Wave Ring

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We show that time translation symmetry of a ring system with a macroscopic quantum ground state is broken by decoherence. In particular, we consider a ring-shaped incommensurate charge density wave (ICDW ring) threaded by a fluctuating magnetic flux: the Caldeira-Leggett model is used to model the fluctuating flux as a bath of harmonic oscillators. We show that the charge density expectation value of a quantized ICDW ring coupled to its environment oscillates periodically. The Hamiltonians considered in this model are time independent unlike “Floquet time crystals” considered recently. Our model forms a metastable quantum time crystal with a finite length in space and in time.

I. INTRODUCTION

The original proposal of a quantum time crystal (QTC) given by Wilczek [1] and Li et al. [2] is a quantum mechanical ground state which breaks time translation symmetry. In this QTC ground state there exists an operator $\hat{Q}$ whose expectation value oscillates permanently with a well-defined “lattice constant” $P$, that is, with a well-defined period.

Volovik [3] relaxed the condition of permanent oscillation and proposed the possibility of effective QTC, that is, a periodic oscillation in a metastable state such that the oscillation will persist for a finite duration $\tau_Q \gg P$ in the time domain and will eventually decay. In this paper we promote Volovik’s line and consider the possibility of a metastable QTC state without spontaneous symmetry breaking: We consider symmetry breaking by decoherence of a macroscopic quantum ground state (FIG. 1). Decoherence is defined as the loss of quantum coherence of a system coupled to its environment. Coupling to environment will inevitably introduce friction to the system such that the oscillation will eventually decay at $t = \tau_{damp}$. However, for $t < \tau_Q \ll \tau_{damp}$ the oscillation period $P$ is well defined. If friction is sufficiently weak such that $P \ll \tau_Q$, then we have a model of effective QTC with life time $\tau_Q$.

Our model consists of a ring-shaped incommensurate charge density wave (ICDW ring) threaded by a fluctuating magnetic flux (FIG. 2(a)). A charge density wave (CDW) is a periodic (spatial) modulation of electric charge density which occurs in quasi-one-dimensional crystals [5]: the periodic modulation of the electric charge density occurs due to electron-phonon interaction. If the ratio between the CDW wavelength $\lambda$ and the lattice constant $a$ of the crystal is a simple fraction like 2, 5/2 etc, then the CDW is commensurate with the underlying lattice. A commensurate CDW cannot move freely because of commensurability pinning, i.e. the CDW phase is pinned by ions position in the crystal. On the other hand, if $\lambda/a$ is effectively an irrational number [5], then the CDW is incommensurate with the underlying lattice. An ICDW ring with a radius of 10$\mu$m, for instance, contains approximately 10$^5$ wavelengths [6], so it is possible that $a/\lambda$ is very close to an irrational number. See the discussion section for an elaboration of this assumption. The sliding of an ICDW without pinning is described by
oscillates periodically (FIG. 2 (c)). If the ICDW ring weakly couples to its environment then this state is a metastable ground state. Therefore, our model forms an effective QTC with a finite length in space and in time.

Before developing our main arguments, we compare our work to recent developments of QTC. In analogy with spatial crystal, the original proposal of QTC is based on the spontaneous breaking of time translation symmetry. However, Bruno [12] and Watanabe and Oshikawa [13] theoretically proved that spontaneous breaking of time translation symmetry cannot occur at ground state. Recently, it was shown that there is a notion of spontaneous breaking of time translation symmetry in periodically driven (Floquet) states [14] and this idea was proved experimentally [15]. On the other hand, the periodic oscillation we consider in this paper is inherent to ring systems with a macroscopic wave function.

II. GROUND STATE OF AN ISOLATED ICDW RING

A. Classical Theory of ICDW Ring

It is well known that the electric charge density of a quasi-one-dimensional crystal becomes periodic by opening a gap at the Fermi wave number \( k_F \) and form a charge density wave (CDW) ground state with a wavelength \( \lambda = \pi/k_F \). Consider a CDW formed on a ring-shaped quasi-one-dimensional crystal with radius \( R \). The order parameter of this CDW ring is a complex scalar order parameter of this CDW ring is a complex scalar

\[
\psi(x,t) = |\psi(x,t)| \exp[i\theta(x,t)]
\]

where \(|\psi(x,t)|\) is the amplitude of the wave. Bogachek et al. [16] derived the following Lagrangian density of the phase of a ring-shaped incommensurate CDW (ICDW ring) threaded by a magnetic flux

\[
\mathcal{L}_0 \left( \frac{\partial \theta}{\partial t}, \frac{\partial \theta}{\partial x} \right) = \frac{N_0}{2} \left[ \left( \frac{\partial \theta}{\partial t} \right)^2 - c_0^2 \left( \frac{\partial \theta}{\partial x} \right)^2 \right] + \frac{eA}{\pi} \frac{\partial \theta}{\partial t}
\]

where \( A \) is the magnetic vector potential, \( N_0 = \frac{v_F^2}{\hbar^2} N(\varepsilon_F)/(2c_0^2) \), \( N(\varepsilon_F) \) is the density of states of electrons at the Fermi level per unit length and per spin direction, \( v_F \) is the Fermi velocity of the crystal, \( c_0 = \sqrt{m^*/m} v_F \) is the phason velocity, \( m^* \) is the effective mass of electrons and \( \hbar \) is the reduced Planck constant.

We first consider an isolated CDW ring with \( A = 0 \). Assuming that \( N(\varepsilon_F) \) is equivalent to the density of states of electrons on a one dimensional line, that is, \( N(\varepsilon_F) = 1/(\pi\hbar v_F) \), we have \( N_0 = \hbar v_F/(2\pi c_0^2) \). An incommensurate CDW (ICDW) can slide freely because of spatial translation symmetry, so the dynamics of an

We consider an incommensurate CDW ring threaded by a fluctuating magnetic flux. This figure shows a commensurate CDW with \( \lambda/a = 2 \) because it is easier to visualise. The wave (typically \( \sim 10^5 \) wavelengths) represents the charge density and the dots represent the atoms of a quasi-one dimensional crystal. (b) ICDW ring crystals such as monoclinic TaS\(_3\) ring crystals and NbSe\(_3\) ring crystals can be produced experimentally [5]. Our model can be tested provided clean ring crystals with almost no defects and impurities can be produced. (c) The ground state of an isolated quantized ICDW ring is a superposition of periodically oscillating ICDWs, hence the oscillation is unobservable. Coupling to environment (fluctuating magnetic flux) will break the superposition and the oscillation becomes apparent.

We show in section II that the charge density expectation value of an isolated CDW ring with moment of inertia \( I \) is periodic in time with period \( P = 4\pi I/\hbar \). This periodicity is a consequence of the uncertainty relation on \( S^1 \) (ring). However, this oscillation becomes unobservable at ground state because the ground state of an isolated CDW ring is a plane wave state, i.e. a coherent superposition of ICDWs with different phases. Therefore, in section III we use the Caldeira-Leggett model [10, 11] to show that time translation symmetry is broken by decoherence. More precisely, the superposition is broken by decoherence and the amplitude of the ICDW

\[ a \text{ gapless Nambu-Goldstone (phason) mode} \]

and the energy of an ICDW is independent of its phase (i.e. position), which implies that the expected ground state of an ICDW ring is a superposition of ICDWs with different phases. Ring-shaped crystals and ring-shaped ICDWs have been produced experimentally [5] (FIG. 2 (b)). The presence of circulating CDW current [8] and Aharonov-Bohm oscillation (evidence of macroscopic wave function) [8] are verified experimentally.

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ICDW is understood by its phase $\theta$. We further assume the rigid-body model of ICDW, i.e. the ICDW ring does not deform locally and the phase $\theta(x, t) = \theta(t)$ is independent of position. Then, the Lagrangian, the canonical angular momentum and the Hamiltonian of the ICDW ring are, respectively

$$L_0(\dot{\theta}) = \int_0^{2\pi R} dx L_0(\dot{\theta}) = \frac{I}{2} \dot{\theta}^2,$$

$$\pi_0(\theta) = \frac{\partial L_0(\dot{\theta})}{\partial \ddot{\theta}} = I \ddot{\theta},$$

$$H_0(\pi_0) = \pi_0 \theta - L_0(\theta) = \frac{\pi^2}{2I},$$

where $\dot{\theta} = d\theta/dt$ and $I = hRv/c_0^2$ is the moment of inertia. We note that $\theta$, $\pi_0$ and $I$ are time independent.

B. Quantization of an Isolated ICDW Ring

Next, we quantize the ICDW ring system. We show that a quantized ICDW ring possesses an inherent oscillation which originates from the uncertainty principle. Let $\hat{H}_0 = \hat{\pi}_0^2/(2I)$ and $\hat{\pi}_0$ be the Hamiltonian operator and angular momentum operator of the ICDW ring, respectively. The macroscopic quantum state $\psi \in \mathcal{H}$ is defined in the Hilbert space $\mathcal{H}$ of positive square-integrable functions with the periodic boundary condition $\psi(\theta + 2\pi) = \psi(\theta)$. The canonical commutation relation $[\hat{\pi}_0, \hat{\theta}] = i\hbar$ is not satisfactory because $\hat{\theta}$ is a multi-valued operator and is not well-defined. Ohnuki and Kitakado resolved this difficulty by using the unitary operator $\hat{W}$ and the self-adjoint angular momentum operator $\hat{\pi}_0$ defined by

$$\langle \theta | \hat{W} | \psi \rangle = e^{i\theta} \psi(\theta),$$

which satisfy the commutation relation on $\mathcal{H}$

$$[\hat{\pi}_0, \hat{W}] = i\hbar \hat{W}.$$  (5)

$\hat{H}_0$ is a function of $\hat{\pi}_0$ only, hence the complete orthonormal set $\{\psi_l(I)\} = e^{i\theta} / \sqrt{2\pi}$ of momentum eigenstates spans $\mathcal{H}$ and satisfy $\psi_l(\theta) = e^{i\theta} / \sqrt{2\pi}$. The eigenvalues of $\hat{\pi}_0$ are quantized with $\langle \psi_l | \hat{\pi}_0 | \psi_l \rangle = l\hbar, l \in \mathbb{Z}$. $\hat{W}$ and $\hat{W}^\dagger$ are ladder operators which satisfy $\hat{W} \psi_l = \psi_{l+1}$ and $\hat{W} \psi_l = \psi_{l-1}$. Therefore, (5) is the one dimensional realization of the well known angular momentum algebra [18]. Time evolution is introduced via the Heisenberg picture: $\hat{\pi}_0(t) = e^{i\hat{H}_0 t/\hbar} \hat{\pi}_0 e^{-i\hat{H}_0 t/\hbar}$ and $\hat{W}(t) = e^{i\hat{H}_0 t/\hbar} \hat{W} e^{-i\hat{H}_0 t/\hbar}$. $\hat{\pi}_0$ commutes with $\hat{H}_0$, so $\hat{\pi}_0(t) = \hat{\pi}_0$. From the commutation relation (5) we obtain the following solutions of $\hat{W}(t)$:

$$\hat{W}(t) = e^{i\hat{\pi}_0 t/\hbar} \hat{W} e^{-\hat{\pi}_0 t/\hbar} = \hat{W} e^{i\hat{\pi}_0 t/\hbar} e^{i\pi_0 \hat{\theta}_t}/2.$$  (6)

where $\mu = I/h$. The two different expressions in (6) arise from the noncommutativity between $\hat{W}$ and $e^{i\hat{\pi}_0 t/\hbar}$.

For a QTC we need a periodic expectation value at the ground state. So, we define the time dependent charge density operator

$$\hat{n}(x, t) = n_0 + \frac{\mu_1}{2} \left( e^{2ikRv} \hat{W}(t) + \text{h.c.} \right)$$  (7)

and replace the classical charge density expression (1) by the expectation value

$$n(x, t) = \langle \hat{n}(x, t) \rangle.$$  (8)

Any states in $\mathcal{H}$ must be a linear superposition of $\{|\psi_l\rangle\}$, that is $\psi = \sum_{l \in \mathbb{Z}} c_l \psi_l$ provided $\sum |c_l|^2 = 1$. Therefore, the expectation values $\langle \hat{W}(t) \rangle$ and $\langle \hat{n}(x, t) \rangle$ are periodic with period $P = 4\pi \mu$ for any state $\psi$:

$$\langle \hat{W}(t) \rangle = \frac{1}{2} \text{tr} \left[ \hat{W}(t) \dot{\rho} + \rho \dot{\hat{W}}(t) \right] = \frac{1}{2} \int_{-\pi}^{\pi} d\theta e^{i\theta} \left[ \exp \left( i \theta \hat{\pi}_0 \right) \rho(\theta + t/\mu, \theta) + \rho(\theta, \theta - t/\mu) e^{-i \theta} \right].$$  (9)

From the two different expressions of $\hat{W}(t)$ in (6) we can define the Weyl form of the commutation relation [19]

$$\hat{W} e^{i\pi_0 t/\hbar} = e^{i\pi_0 t/\hbar} \hat{W} e^{-i\pi_0 t/\hbar}$$

hence the phase $\pi_0 t/\hbar$ and the periodic oscillation with period $4\pi \mu$ is a manifestation of the uncertainty principle. For an alternative explanation, let us consider an electron with effective mass $m^*$ confined in a finite space with volume $L \sim 2R$. From the uncertainty principle, the uncertainty momenta of this particle is $\Delta p \sim \hbar/(4R)$. This means that the particle's wave packet expands with velocity $v = \Delta p/m^* \sim \hbar/(4m^* R)$. Then, because of the periodic boundary condition, the physical quantity $W = e^{ix/\lambda}$ is periodic with period $P = \lambda/v \sim 4\pi m^* R/\hbar k_F \sim 4\pi m^* R/m v_F \sim 4\pi R v_F/c_0^2 = 4\pi \mu$. So, the origin of the periodicity is (i) the macroscopic wave function of the ICDW ring diffuses due to the uncertainty principle then (ii) $W = e^{i\theta}$ oscillates periodically. However, the oscillation in (9) is not observable at the ground state $\rho_0 = |\psi_0 \rangle \langle \psi_0 |$ because $\langle \theta | \rho_0 | \phi \rangle = \frac{\pi}{2}$ and the $\theta$ integral vanishes. Therefore, the ground state of an isolated ICDW ring is not yet a time crystal because of superposition.

III. COUPLING TO ENVIRONMENT

Now, suppose that the ICDW ring starts to interact with its surrounding environment at $t = 0$. Then, we expect decoherence of the phase $\theta$. This interaction is modeled using the Caldeira-Leggett model [10] which is a model quantum Brownian motion. It describes a particle coupled to its environment. This environment is described as a set of non-interacting harmonic oscillators. First, the classical solution $\theta(t)$ is calculated to study the dynamics of the ICDW ring. Next, this system is quantized to calculate the amplitude of the charge density expectation value $\langle \hat{n}(x, t) \rangle$. 
A. Classical Theory of ICDW Ring With Environment

Let us consider the following Lagrangian of an ICDW ring threaded by a fluctuating magnetic flux

$$\tilde{L}(\dot{\theta}, q, \dot{q}) = \frac{1}{2} I \ddot{\theta}^2 + A(q) \dot{\theta} + \sum_{j=1}^{N} \left( \frac{1}{2} m q_j^2 - \frac{1}{2} m \omega_j^2 q_j^2 \right)$$

(10)

where $q_j$ are the normal coordinates of the fluctuation and $\tilde{g}_\theta = \partial L / \partial \dot{\theta}$ and $p_j = \partial L / \partial q_j$ are the canonical momenta of the ICDW ring and the environment, respectively. The fluctuating magnetic flux is given by

$$A(q) = \sum_{j=1}^{N} c q_j.$$  

Classically, this magnetic flux will randomly change the phase and the mechanical angular momentum $I \dot{\theta}$ of the ICDW ring due to electromotive force. An equivalent Lagrangian obtained by a Canonical transformation is

$$L(\dot{\theta}, R, \dot{R}) = \frac{1}{2} I \ddot{\theta}^2 + \frac{m}{2} \sum_{j=1}^{N} \left[ \ddot{R}_j^2(\theta) - \omega_j^2 \left( R_j(\theta) - \frac{C_j}{m \omega_j^2} \right)^2 \right]$$

(11)

which is the Lagrangian of a bath of field particles $R_j$ coupled to the phase $\theta$ by springs. $R_j(p_j, \theta) = -\frac{p_j}{m \omega_j}$, $P_j(q_j) = m \omega_j q_j$ and $C_j = c \omega_j$.  (10) and (11) are precisely the kinds of Lagrangian considered by Caldeira and Leggett, so we can use the results in [10] but with slight modifications due to the periodicity of the ring.

B. Classical Solution

The equation of motion of $\theta$ obtained from the Lagrangian (11) is the generalized Langevin equation [20]

$$I \ddot{\theta}(t) + 2 \int_{0}^{t} d\tau \alpha(t - \tau) \dot{\theta}(\tau) = \xi(t)$$

(12)

with the dissipation kernel $\alpha(t - \tau)$, the memory function $\gamma(t - \tau)$ and the classical fluctuating force $\xi(t)$ defined by

$$\alpha(t - \tau) = I \gamma(0) \delta(t - \tau) + \frac{1}{2} \frac{d}{dt} \gamma(t - \tau),$$

$$\gamma(t - \tau) = \sum_{j=1}^{N} \frac{C_j^2}{I m \omega_j^2} \cos \omega_j(t - \tau),$$

$$\xi(t) = \sum_{j=1}^{N} C_j \left[ R_j(0) \cos \omega_j t + \frac{P_j(0)}{m \omega_j} \sin \omega_j t \right].$$

The correlation function of the classical force is given by the noise kernel

$$\langle \xi(t) \xi(\tau) \rangle_{\text{env}} = \hbar o_{R}(t - \tau)$$

$$\alpha_R(t - \tau) = \sum_{j=1}^{N} \frac{C_j^2}{2 m \omega_j} \coth \left( \frac{\hbar \omega_j}{2 k_B T} \right) \cos \omega_j(t - \tau)$$

where the average $\langle \cdot \rangle_{\text{env}}$ is taken with respect to the environment coordinate at equilibrium. It is convenient to define the spectral density function

$$J(\omega) = \frac{\pi}{2} \sum_{j=1}^{N} \frac{C_j^2}{m \omega_j} \delta(\omega - \omega_j)$$

(13)

and assume the power law spectrum $J(\omega) = I g_s \omega^s$ [21] with a cutoff frequency $\Omega$ and $0 < s < 2$. Then, $\alpha_R(t - \tau)$ can be written

$$\alpha_R(t - \tau) = \frac{I g_s}{\pi} \int_{0}^{\Omega} \omega^s \coth \left( \frac{\hbar \omega}{2 k_B T} \right) \cos \omega(t - \tau) d\omega.$$  

The classical solution of $\theta$ is

$$\theta(t) = G(t) \dot{\theta}(0) + \dot{G}(t) \theta(0) + \frac{1}{T} \int_{0}^{t} d\tau G(t - \tau) \xi(t)$$

(14)

with the fundamental solution

$$G(t) = L^{-1} \left[ \frac{1}{z^2 + z \gamma(z)} \right](t),$$

where $L^{-1}$ is the inverse Laplace transform. The Laplace transform of the memory function $\gamma(t)$ can be written [11]

$$\hat{\gamma}(z) = \omega^2 z^{-s-1}, \quad \omega_s = \left( \frac{g_s}{\sin \frac{\gamma}{2}} \right)^{1/(2-s)}$$

and $G(t)$ takes the form of a generalized Mittag-Leffler function $E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha \beta + n)}$:

$$G(t) = t E_{2-s,2}[-(\omega_s t)^{2-s}].$$

For ohmic damping with $s = 1$ and $\gamma(z) = g_1 \equiv 2 \gamma$, we obtain

$$G(t) = \frac{1 - e^{-2 \gamma t}}{2 \gamma}.$$  

(15)

$G(t)$ and $\dot{G}(t)$ are shown in FIG. 3. We note that $\dot{G}(t) \approx 1$ for $t$ less than some damping time scale $\tau_{\text{damp,s}}$. In other words, the fluctuating magnetic flux does not affect the dynamics of an ICDW ring for $t < \tau_{\text{damp,s}}$ and $e^{i\theta(t)}$ oscillates periodically with period $P \approx 4 \pi \mu$. Next, we quantize the ICDW ring + environment system to show that this oscillation is observable for a finite time $\tau_Q$ and form an effective QTC as a metastable state.
FIG. 3. These plots are shown with \( g_s = 1 \text{ Hz}^{-2/3} \) and \( \mu = 10^{-8} \text{ sec} \). (a) The fundamental solution \( G(t) \) of a classical ICDW ring coupled to its environment is shown for sub-ohmic \((s < 1)\), ohmic \((s = 1)\) and super-ohmic \((s > 1)\) damping. Note that \( G(t) \approx t \) for \( t \) less than some time scale \( \tau_{\text{damp},s} \). The time \( t \) is normalized by \( 4\pi\mu \) so the horizontal axis gives the number of “lattice points”. (b) The derivative of the fundamental solution, \( G'(t) \), heuristically describes the velocity of the ICDW ring. Note that \( G'(t) \approx 1 \) for \( t < \tau_{\text{damp},s} \).

### C. Quantization of ICDW Ring Coupled to Environment

The ICDW ring + environment system is quantized using the commutation relations [\( \hat{x}_\theta, \hat{W} \) = \( i\hbar \hat{W} \) and \( \hat{q}_j, \hat{p}_k = i\hbar \delta_{jk} \)]. Define the orthonormal position state \( |q\rangle = \prod_{j=1}^N |q_j\rangle \) and the orthonormal momentum state \( |p\rangle = \prod_{p=1}^N |p_i\rangle \) such that

\[
(q|\hat{q}_j|\psi) = q_j (q|\psi), \quad (q|\hat{p}_j|\psi) = -i\hbar \frac{\partial}{\partial q_j} (q|\psi),
\]

and the inner product of \( |q\rangle \) and \( |p\rangle \) is defined as \( (q|p) = \frac{1}{\sqrt{2\pi\hbar}} \exp \left( i\hat{q} \cdot \hat{p} \right) \). The periodic boundary condition of the ring implies that \( (\theta + 2\pi n|\tilde{\pi}_\theta\rangle = (\theta|\tilde{\pi}_\theta\rangle \) for some integer \( n \), hence the angular momentum eigenstates are quantized: \( (\psi_l|\tilde{\pi}_\theta|\psi_l) = \hbar t = 0, \pm, \pm 2, \ldots, |\tilde{\pi}_\theta\rangle = \hbar^{-1/2} |\psi_l\rangle \), \( (\theta|\tilde{\pi}_\theta) = \frac{1}{\sqrt{2\pi}} e^{i\theta} \). Moreover, one can easily show that

\[
(\theta, p|\tilde{\pi}_\theta, q) = (\theta, R(\theta)|\pi_\theta, P) .
\]

\( \hat{W} \) is independent of the environmental coordinate. So, the expectation value of \( \hat{W} \) is

\[
\langle \hat{W}(t) \rangle = \frac{1}{2} \int_{-\pi}^{\pi} d\theta_1 \int_{-\pi}^{\pi} d\phi_1 \rho(\theta_1, \phi_1, t) \langle \phi_1|\hat{W}|\theta_1\rangle + \frac{1}{2} \int_{-\pi}^{\pi} d\theta_1 \int_{-\pi}^{\pi} d\phi_1 \langle \theta_1|\hat{W}|\phi_1\rangle \rho(\phi_1, \theta_1, t).
\]

where the reduced density matrix of the ICDW ring is (see Appendix A)

\[
\rho(\theta_1, \phi_1, t; \theta_1, \phi_1, 0) = F^2(t) \exp \left( \frac{i}{\hbar} S[\varphi^+, \varphi^-] - \Gamma[\varphi^-] \right) .
\]

The exact form of \( J(\theta_1, \phi_1, t; \theta_1, \phi_1, 0) \) for ohmic dissipation \( s = 1 \) was calculated in [10]. For general damping with arbitrary \( s \) the computation of the reduced density matrix is essentially equivalent to [10], and we obtain

\[
\rho(\theta_1, \phi_1, t; \theta_1, \phi_1, 0) = F^2(t) \exp \left( \frac{i}{\hbar} S[\varphi^+, \varphi^-] - \Gamma[\varphi^-] \right) .
\]

The classical action and the noise action are given by

\[
I_{\varphi^-}(u) + 2 \int_{-\pi}^{\pi} d\tau \varphi_{\varphi^-}(\tau) \alpha_1(\tau - u) = 0,
\]

\[
I_{\varphi^+}(u) + 2 \int_{-\pi}^{\pi} d\tau \varphi_{\varphi^+}(\tau) \alpha_2(u - \tau) = 0
\]

whose solution are given in terms of boundary conditions

\[
\varphi^+_1 = \varphi^+_1(0), \varphi^-_1 = \varphi^-_1(\tau) .
\]

The classical action and the noise action are given by

\[
S[\varphi^+, \varphi^-] = S_\theta(\varphi^+, \varphi^-, \varphi^-_1, 0)
\]

\[
= -I[\varphi^+_1(\tau) \varphi^- - \varphi^+_1(0) \varphi^-_1],
\]

\[
\Gamma[\varphi^-] = \Gamma_\varphi(\varphi^-_1, \varphi^-_1, 0)
\]

\[
= \frac{1}{2\hbar} \int_{-\pi}^{\pi} d\tau \int_{-\pi}^{\pi} d\tau' \varphi^- (\tau) \alpha_\varphi(\tau - \tau') \varphi^- (s).
\]

\( F^2(t) \) is a normalization function such that \( \text{tr}[\rho(t)] = (1) = 1 \). The winding numbers \( l_1 \) and \( l_2 \) can be absorbed into the \( \theta_1 \) and \( \phi_1 \) integrals, respectively, by changing the domain of \( \theta_1 \) and \( \phi_1 \) from \( S^1 \) to \( R^1 \). Then, taking care of
the non-Hermiticity of \( \hat{W} \), we obtain

\[
\langle \hat{W}(t) \rangle = \frac{r_1^+(t) + r_1^-(t)}{r_2^+(t) + r_2^-(t)}.
\]

\[
r_1^+ = \int_0^\pi d\theta_1 \sum_{n \in S_1(\theta_1,t)} \rho(\theta_1 - f_1(t), \theta_1, 0)
\]
\[
\times e^{-i\theta_1 - \pi \mu f_1(t) + \frac{\hbar}{2} f_1(t) f_1(t) - \Gamma_{\text{cl}}(2\pi n; f_1(t), 0)},
\]
\[
r_1^- = \int_0^\pi d\theta_1 \sum_{n \in S_1(\theta_1,t)} \rho(\theta_1 + f_1(t), \theta_1, 0)
\]
\[
\times e^{-i\theta_1 - \pi \mu f_1(t) - \frac{\hbar}{2} f_1(t) f_1(t) - \Gamma_{\text{cl}}(2\pi n; f_1(t), 0)},
\]
\[
r_2^- = \int_0^\pi d\theta_1 \sum_{n \in S_2(\theta_1,t)} \rho(\theta_1 + f_2(t), \theta_1, 0)
\]
\[
\times e^{-i\theta_1 - \pi \mu f_2(t) - \frac{\hbar}{2} f_2(t) f_2(t) - \Gamma_{\text{cl}}(2\pi n; f_2(t), 0)},
\]
\[
r_2^+ = \int_0^\pi d\theta_1 \sum_{n \in S_2(\theta_1,t)} \rho(\theta_1 - f_2(t), \theta_1, 0)
\]
\[
\times e^{-i\theta_1 - \pi \mu f_2(t) + \frac{\hbar}{2} f_2(t) f_2(t) - \Gamma_{\text{cl}}(2\pi n; f_2(t), 0)},
\]

with \( f_1(t) = 2\pi n \hat{G}(t) - G(t)/\mu, f_2(t) = 2\pi n \hat{G}(t), S_1(\theta_1, t) = \{ n \in \mathbb{Z} | -\pi < \theta + f_1(t) < \pi \}, \) and \( S_2(\theta_1, t) = \{ n \in \mathbb{Z} | -\pi < \theta + f_2(t) < \pi \}. \) This is the most general form of the expectation value of \( \hat{W} \) for an ICDW ring coupled to its environment. Although the derivation of \( \langle \hat{W}(t) \rangle \) is exact, it is not very insightful, so we make some approximations.

### D. Early Time Approximation

Let us consider the classical solution (22) with \( \varphi_{\text{cl}} = \theta_1 \) and ohmic damping [15]. Then, we see immediately that \( \theta(t) \approx \theta(0)e^{-2\gamma t} \). Therefore, we are interested in the range \( t \ll 1/(2\gamma) \approx \tau_{\text{damp,1}}. \) For general dissipation, we can see from Fig. 3 that there exist a time scale \( \tau_{\text{damp,s}} \) such that \( G(t) \approx t, \hat{G}(t) \approx 1 \) for \( t \ll \tau_{\text{damp,s}}. \) Then, writing \( t = 2\pi I(m + a)/\hbar \) for an integer \( m \) and \( 0 < a < 1, \) we have \( S_1(\theta < -2a\pi, t) = \{ m + 1 \}, S_1(\theta > 2a\pi, t) = \{ m \}, \) and \( S_2(\theta, t) = \{ 0 \}. \) Therefore, we conclude that \( S_1(\theta, t) \) is approximately the \( (m/2) \) th lattice point. Then, using \( f_1(t) \approx -\hat{G}(t) \hbar/\Gamma \) we obtain the approximate form

\[
\langle \hat{W}(t) \rangle \approx \int_0^\pi d\theta_1 \sum_{n \in S_1(\theta_1,t)} \rho(\theta_1 - G(t)/\mu, 0)
\]
\[
\times \exp \left\{ i\theta_1 \hat{G}(t) - \frac{G(t)\hat{G}(t)}{2\mu} - \Gamma_{\text{cl}}(2\pi n; f_1(t), 0) \right\}
\]
\[
+ \int_0^\pi d\theta_1 \sum_{n \in S_1(\theta_1,t)} \rho(\theta_1 + G(t)/\mu, \theta_1, 0)
\]
\[
\times \exp \left\{ i\theta_1 \hat{G}(t) + \frac{G(t)\hat{G}(t)}{2\mu} - \Gamma_{\text{cl}}(2\pi n; f_1(t), 0) \right\}.
\]

For \( t \ll \tau_{\text{damp,s}} \) we have \( \varphi_{\text{cl}}(\tau) \approx \tau/\mu \) and the noise action can be written

\[
\Gamma_{\text{cl}}(2\pi n; f_1(t), 0) \approx \Gamma_{T,s}(t)
\]
\[
= \frac{g_s}{2\pi \mu} \int_0^\Omega d\omega \coth \left( \frac{\hbar \omega}{2k_B T} \right) T(\omega),
\]
\[
T(\omega) = \omega^{-a}(2 + \omega^2 t^2 - 2 \cos \omega t - 2 \omega \sin \omega t)
\]

Taking the low temperature limit \( \frac{\hbar \Omega}{2k_B T} \to \infty \) such that \( \coth \frac{\hbar \Omega}{2k_B T} \to 1 \) we obtain

\[
\Gamma_{T,s}(t) = -\frac{g_s}{2\pi \mu} \frac{\Omega^{s-1} - \Omega^{s+1}}{s - 3} \left( I_F^2 \left( \frac{1}{s-1}; \frac{s+1}{2} \right) - \frac{1}{4} \Omega^2 \right)
\]
\[
= \frac{g_s^2 t^2 \Omega^{s+1}}{2\pi \mu} \frac{\Omega^{s-1} - \Omega^{s+1}}{s - 1} \left( I_F^2 \left( \frac{1}{s-1}; \frac{s+1}{2} \right) - \frac{1}{4} \Omega^2 \right)
\]

where \( I_F^2(a_1; a_2) \) is the generalized hypergeometric function. Define the decoherence time \( \tau_{\text{decoh,s}} \) such that \( \Gamma_{T,s}(\tau_{\text{decoh,s}}) = 1. \) Numerical analysis shows that the order of \( \tau_{\text{decoh,s}} \) does not change with \( \Omega < 1/(2\mu) \) and decreases rapidly for \( \Omega > 1/(2\mu). \) If we set \( \Omega \sim 1/\mu \) and use the ground state \( \rho_0, \) then for \( t \ll \tau_Q \equiv \tau_{\text{damp,s}} \)
min\{\tau_{\text{damp}, s}, \tau_{\text{decoh}, s}\} \text{ we have}
\begin{align*}
n(x, t) & \approx n_0 + n_1^{\text{osc}}(t) \cos(2k_F x), \\
n_1^{\text{osc}}(t) & = n_1 \text{sinc}[\pi \hat{G}(t)] \cos \left[ \frac{\hat{G}(t) G(t)}{2\mu} \right] e^{-\Gamma t} e^{i \phi(t)}. \quad (27)
\end{align*}

Equation (24) is the main result of this paper. It shows that the amplitude of an ICDW ring threaded by a (time independent) fluctuating magnetic flux oscillates for a finite time \( \tau_Q \) and form an effective QTC. In the no-damping limit \( \gamma(\omega) \to 0 \) we have \( G(t) \to t, G(t) \to 1 \) and recover \( \langle W(t) \rangle = 0 \). This charge density oscillation is shown in FIG. 4.

**IV. DISCUSSION**

First, we elaborate our assumption of ICDW ring. A mathematical definition of ICDW is that \( \lambda/a \) is an irrational number. We note that a CDW formed on a macroscopic crystal is basically incommensurate because the wavelength of a CDW is given by \( \lambda = \pi/k_F \), where the Fermi wave number \( k_F \) is usually an irrational number for an arbitrary band filling. However, strictly speaking, \( \lambda/a \) of a finite-size system can never be irrational. A physical condition is that the commensurability pinning energy is negligible, which is possible if \( \lambda/a \) cannot be expressed as a simple fraction like 2, 5/2, etc. In order to explain this, suppose that for some integer \( M \geq 2 \), \( Ma/\lambda \) is an integer. In other words, the same atom-electron configuration is obtained if we move the CDW by \( M \) wavelengths and \( \epsilon_{k+2Mk_F} = \epsilon_k \) (\( \epsilon_k \) is the energy of an electron with momentum \( hk \)). Then, the energy required to move a CDW by a small phase \( \phi \) from its equilibrium is

\[ \epsilon(\phi; M) = \frac{|\Delta|^2}{\epsilon_F} \left( \frac{\epsilon |\Delta|}{W} \right)^{M-2} \frac{M \sigma^2}{2} \]

where \( \epsilon_F \) is the Fermi energy, \( W \) is the band width, \( |\Delta| \) is the CDW gap width and \( \epsilon \) is the elementary charge in natural units. We see that \( \epsilon(\phi; M) \) approaches zero rapidly for large \( M \) as the distinction between rational and irrational numbers becomes academic. For example, for a ring crystal with \( N_a \) atoms and \( N_N \) CDW wavelengths, we obtain \( \lambda/a = N_a/N_N \). Therefore, for a large \( N_a \) and a large \( N_N \), \( M \) can always be arbitrary large (order of \( N_a \)), hence the commensurability energy is completely negligible. In fact, we can experimentally make sub-micrometer scale ICDW rings such that, we expect, \( N_a \) and \( N_N \) are not so large but \( M \) is very large such that commensurability energy is negligible.

Usually, superposition of ICDWs with different phases is not observed because of impurity pinning. However, the probability of impurity decreases with decreasing radius. Commensurability effect may become significant for small radius (more precisely, for some small \( M \)). But, the origin of the time crystal periodicity in our model is the uncertainty principle, which appears as a collective fluctuation of the ICDW phase. If commensurability effect becomes significant, then the ICDW phase is expected to fluctuate periodically around some phase \( \theta = \theta_0 \) determined by the ion's position. This fluctuation is expected to become apparent and oscillate periodically by coupling to environment (fluctuating magnetic flux).

Next, we discuss the presence of an upper bound and a lower bound for the radius of the ICDW ring in our model. Decoherence induced breaking of time translation symmetry occurs, in principle, only in mesoscopic systems: There is an upper bound for the CDW radius determined by the coupling strength \( \gamma = \omega_s/2 \) and a lower bound given by the CDW wavelength \( \lambda \). We will first calculate the upper bound by replacing the environment with an equivalent LC circuit. Here, we focus on Ohmic damping because an approximate form of \( \gamma \) can be calculated explicitly. In general, the coupling strength \( \gamma \) depends on the CDW radius \( R \). In order to explicitly see the \( R \) dependence, suppose that the fluctuating magnetic flux in our model comes from an external coil connected to a series of parallel capacitors. Then, the Lagrangian of the CDW + environment system is

\[ L_{\text{circuit}} = L_0 + MI_{\text{CDW}} \sum_{j=1}^{N} I_j + \sum_{j=1}^{N} \frac{m}{2} (I_j^2 - \omega_j^2 Q_j^2). \]

\( L_0 \) is the Lagrangian of an isolated ICDW ring, \( I_{\text{CDW}} = e\theta/\pi \) is the CDW current induced by the fluctuating flux, \( Q_j \) is the net charge on the capacitor \( j \) with capacitance \( \mathcal{C}_j \), \( m \) is the inductance of the coil, \( \omega_j = 1/\sqrt{m_0 \mathcal{C}_j} \), and \( M \) is the mutual inductance between the coil and the CDW. We immediately see that \( L_{\text{circuit}} \) is equivalent to the Lagrangian in equation (10) but with the interaction Lagrangian replaced by \( \theta \sum_{j=1}^{N} \frac{d}{dt} \mathcal{C}_j \). Therefore, define \( p_j = m I_j \), \( C_j = M e \omega_j^2 / \pi \) and we obtain the Lagrangian in equation (11) after the canonical transformation

\[ L = L_{\text{circuit}} - \frac{d}{dt} \sum_{j=1}^{N} (MI_{\text{CDW}} + p_j) Q_j \]

with \( P_j = m \dot{R}_j = m \omega_j Q_j \) and \( R_j = -(p_j - M I_{\text{CDW}})/(m \omega_j) \). Assume that the radius of the coil \( r_{\text{coil}} \) is much larger than the CDW radius \( R \) and that the coil and the CDW are concentric. Then, we obtain \( M = (\mu_0 \pi R^2)/(2r_{\text{coil}}) \). Next, integrate (13) with respect to \( \omega \) from \( \omega = 0 \) to \( \omega = \Omega = 1/\mu \) and obtain \( \gamma = \beta R \), \( \beta = (\pi \mu_0^2 e^2 \rho_{\text{coil}}^5)/(32 \pi h r_{\text{coil}}^2 m_0^3 \omega_j^2) \). Here, \( \mu_0 \) is the permeability of free space and \( \rho \) is the density of states defined by \( \sum_{j} \to \int d\omega \). If we assume that \( \rho \sim \Omega \) like in the Caldeira-Leggett model, then the order of \( \beta \) may change depending on the parameters of the CDW and the parameters of the coil, but \( \gamma = \beta R \) is usually smaller than 1Hz. Now, if the radius of the CDW ring is too large, then periodicity does not appear because the oscillation period exceeds the lifetime \( \tau_Q \) of the time crystal. The upper bound of the CDW radius \( R \) is given by the condition
Finally, we discuss how our model can be tested experimentally and discuss how our results may be applicable to other physical systems. Ring-shaped crystals and ICDW ring crystals (such as monoclinic TaS$_3$ ring crystals and NbSe$_3$ ring crystals) have been produced and studied by the Hokkaido group [7–9]. Therefore, our model can be tested provided ring crystals with almost no defects and impurities can be produced. The oscillation in [26] implies that the local charge density of the ICDW ring oscillates with frequency $\omega = h/2I$. For a ring with diameter $2R = 1\mu m$, $v_F/c_0 = 10^5$ and $v_F = 10^3 m/s$, we have $\omega = 10^8 Hz$. The time dependence of the charge density modulation can be measured using scanning tunneling microscopy (STM) [23] and/or using narrow-band noise with vanishing threshold voltage [24].

We recall that the origin of the quantum oscillation in our model is the uncertainty principle. If we were to consider a single particle with mass $m^*$ confined on a ring with radius $R$, then the particle’s wave packet expands with a velocity $v \sim h/(4m^*R)$ and $e^{i\theta}$ oscillates with period $P = 2\pi R/v$. However, a charge density wave has an internal periodicity given by the wavelength $\lambda$. Then the period of oscillation is $P \sim \lambda/v$. Therefore, because an ICDW ring is described by a macroscopic wave function with internal periodicity, the number of lattice points $N$ in our model is numerous. And yet, the periodicity of $W(t)$ seems to be universal for any wave functions on $S^1$ (ring system). Therefore, our results may be applicable to earlier models of QTC such as [11, 24] and annular Josephson junctions [26]. Moreover, it was shown very recently in [27] that the ground state of a $^{40}$Ca$^+$ ring trap possesses rotational symmetry as the number of ions is decreased. Our results predict that quantum oscillations may appear in such ring traps with the appropriate set up. We also recall that Volovik’s proposal of metastable effective QTC [3] is not restricted to ring systems. Therefore, time translation symmetry breaking by decoherence may occur in other systems coupled to time-independent environment, and without a periodic driving field. We also expect that many other incommensurate systems such as incommensurate spin density waves [5], incommensurate mass density waves [28–30], or possibly some dielectrics that exhibit incommensurate phases [31], may be used to test our results and to model QTC without spontaneous symmetry breaking.

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Appendix A: QUANTUM BROWNIAN MOTION ON \( S^1 \)

1. Reduced Density Matrix On \( S^1 \)

Consider an ICDW ring \( A \) coupled to its environment \( B \). Let \( \rho_A \) and \( \rho_B \) denote the density operators of \( A \) and \( B \), respectively. Let \( \theta \) and \( \phi \) be angular coordinates on the ring system \( A \) and let \( p = \{ p_k : k = 1, N \} \) and \( s = \{ s_k : k = 1, N \} \) be momentum coordinates of the bath \( B \). Then, the density matrix of the coupled system can be written

\[
\rho(\theta, p, \phi, s) = \langle \theta, p | \rho_{AB}(t) | \phi, s \rangle = \int_0^{2\pi} d\theta' d\phi' \int_{-\infty}^{\infty} dp' ds' K(\theta, p, t; \theta', p', 0) K^*(\phi, s; t; \phi', s', 0) \]

where

\[
K(\theta, p, t; \theta', p', 0) = \langle \theta, p | e^{-\frac{it\hat{H}_t}{\hbar}} | \theta', p' \rangle \quad (A1)
\]

and

\[
K^*(\phi, s; t; \phi', s', 0) = \langle \phi', s' | e^{\frac{it\hat{H}_t}{\hbar}} | \phi, s \rangle \quad (A2)
\]

are recognized as Feynman propagators if we notice that \( |p\rangle \) and \( |s\rangle \) are actually position states after canonical transformation. The propagators \( (A1) \) and \( (A2) \) can be written using path integrals by dividing the time \( t \) into \( N \) time steps of length \( \epsilon = t/(N + 1) \), \( p = p_n, p' = p_0, \theta = \theta_N \) and \( \theta' = \theta_0 \). For \( N \to \infty \) the propagator becomes

\[
K(\theta, p, t; \theta', p', 0) = \left\langle \theta, p \right| \lim_{\epsilon \to 0} \left( \exp \left[ \frac{-it\hat{H}}{\hbar} \right] \right)_{N}^{N} \left\rangle \theta', p' \right. 
\]

\[
= \lim_{\epsilon \to 0} \int_0^{2\pi} \left( \prod_{n=1}^{N-1} d\theta_n \right) \int_{-\infty}^{\infty} \left( \prod_{n=1}^{N-1} dp_n \right) \prod_{n=1}^{N} K_n(\theta_n, p_n, \epsilon; \theta_{n-1}, p_{n-1}, 0). 
\]

The Hamiltonian operator \( \hat{H} \) can be decomposed into the kinetic part \( \hat{K} \) and the potential part \( \hat{V} \), i.e. \( \hat{H} = \hat{K} + \hat{V} \), which satisfy the eigenvalue equations

\[
\hat{K} |\psi_t, q\rangle = \mathcal{K}(t, P) |\psi_t, q\rangle, \quad \hat{V} |\theta, p\rangle = \mathcal{V}(\theta, R(\theta)) |\theta, p\rangle. 
\]

(A3)
Then we obtain
\[
K(\theta_n, \mathbf{p}_n, \epsilon; \theta_{n-1}, \mathbf{p}_{n-1}, 0)
\]
\[
= \sum_{l_n} \int_{-\infty}^{\infty} d\mathbf{q}_n \left\langle \theta_n, \mathbf{p}_n \left| \exp \left( -\frac{ie}{\hbar} \hat{K} \right) \psi_{l_n}, \mathbf{q}_n \right. \right| \psi_{l_n}, \mathbf{q}_n \left| \exp \left( -\frac{ie}{\hbar} \hat{V} \right) \theta_{n-1}, \mathbf{p}_{n-1} \right\rangle
\]
\[
= \sum_{l_n=\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\mathbf{q}_n \frac{d\mathbf{q}_n}{(2\pi\hbar)^N} \exp \left\{ -\frac{ie}{\hbar} \mathcal{K}(l_n, \mathbf{p}_n) - \frac{ie}{\hbar} \mathcal{V}(\theta_{n-1}, \mathbf{R}_{n-1}(\theta_{n-1})) \right\}
\]
\[
\times \exp \left\{ il_n(\theta_n - \theta_{n-1}) - \frac{i}{\hbar} \mathbf{q}_n \cdot (\mathbf{p}_n - \mathbf{p}_{n-1}) \right\}.
\]

Next, define \( A_n = \sum_j c_{q_j,n} = \sum_j \frac{C_j}{m\omega_j^2} P_{j,n} \) and obtain
\[
K(\theta_n, \mathbf{p}_n, \epsilon; \theta_{n-1}, \mathbf{p}_{n-1}, 0)
\]
\[
= \sum_{l_n=\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\mathbf{q}_n \frac{d\mathbf{q}_n}{(2\pi\hbar)^N} \exp \left\{ -\frac{ie}{\hbar} \mathcal{K}(l_n, \mathbf{p}_n) - \frac{ie}{\hbar} \mathcal{V}(\theta_{n-1}, \mathbf{R}_{n-1}(\theta_{n-1})) \right\}
\]
\[
\times \exp \left\{ i(l_n - A_n/\hbar)(\theta_n - \theta_{n-1}) + \frac{i}{\hbar} \mathbf{p}_n \cdot (\mathbf{R}_n(\theta_n) - \mathbf{R}_{n-1}(\theta_{n-1})) \right\}.
\]

The sum over \( l_n \) can be replaced by a sum of integrals using the Poisson resummation formula
\[
\sum_{l \in \mathbb{Z}} f(l) = \sum_{l \in \mathbb{Z}} \int_{-\infty}^{\infty} f(\zeta) \exp(2\pi il\zeta) d\zeta.
\]

Then,
\[
K(\theta_n, \mathbf{p}_n, \epsilon; \theta_{n-1}, \mathbf{p}_{n-1}, 0)
\]
\[
= \sum_{l_n=\infty}^{\infty} \int_{-\infty}^{\infty} d\zeta_n \int_{-\infty}^{\infty} d\mathbf{q}_n \frac{d\mathbf{q}_n}{(2\pi\hbar)^N} \exp \left\{ -\frac{ie}{\hbar} \mathcal{K}(\zeta_n, \mathbf{p}_n) - \frac{ie}{\hbar} \mathcal{V}(\theta_{n-1}, \mathbf{R}_{n-1}(\theta_{n-1})) \right\}
\]
\[
\times \exp \left\{ i((\zeta_n - A_n/\hbar)(\theta_n - \theta_{n-1} + 2\pi l_n) + \frac{i}{\hbar} \mathbf{p}_n \cdot (\mathbf{R}_n(\theta_n + 2\pi l_n) - \mathbf{R}_{n-1}(\theta_{n-1})) \right\}.
\]

For our Hamiltonian of an ICDW ring threaded by a fluctuating magnetic flux we have
\[
\mathcal{K}(\zeta_n, \mathbf{p}_n) = \frac{(\zeta_n - A_n)^2}{2l} + \sum_j \frac{1}{2m} P_{j,n}^2
\]
\[
\mathcal{V}(\theta_{n-1}, \mathbf{R}_{n-1}(\theta_{n-1})) = \sum_j \frac{1}{2m} \omega_j^2 \left( R_{j,n-1}(\theta_{n-1}) - \frac{C_j}{m\omega_j} \right)^2.
\]

We note that that the potential \( \mathcal{V}(\theta, \mathbf{R}(\theta)) \) is rotationally invariant. That is, for an arbitrary rotation \( \theta \rightarrow \theta + \delta \), the potential is \( \mathcal{V}(\theta + \delta, \mathbf{R}(\theta + \delta)) = \mathcal{V}(\theta, \mathbf{R}(\theta)) \). Let \( \tilde{\zeta}_n = \zeta_n - A_n/\hbar \). We use the fact that the phase space volume is conserved under canonical transformation. Then,
\[
K(\theta, \mathbf{p}; t; \theta', \mathbf{p}', 0)
\]
\[
= \left( \prod_{j=1}^{N} \frac{1}{m\omega_j} \right) \lim_{\epsilon \to 0} \left( \prod_{n=1}^{N-1} \int_0^{2\pi} \frac{d\theta_n}{2\pi} \int_{-\infty}^{\infty} d\mathbf{R}_n(\theta_n) \right) \left( \prod_{n=1}^{N} \sum_{l_n=\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\mathbf{p}_n}{(2\pi\hbar)^N} \frac{d\tilde{\zeta}_n}{2\pi} \right)
\]
\[
\times \exp \left\{ -\frac{ie}{\hbar} \left( \frac{\hbar^2}{2l} + \sum_j \frac{1}{2m} P_{j,n}^2 \right) - \frac{ie}{\hbar} \mathcal{V}(\theta_{n-1}, \mathbf{R}_{n-1}(\theta_{n-1})) \right\}
\]
\[
\times \exp \left\{ i\tilde{\zeta}_n(\theta_n - \theta_{n-1} + 2\pi l_n) + \frac{i}{\hbar} \mathbf{p}_n \cdot (\mathbf{R}_n(\theta_n + 2\pi l_n) - \mathbf{R}_{n-1}(\theta_{n-1})) \right\}.
\]
Solve the $\zeta_n$ and $P_{j,n}$ integrals to obtain

$$K(\theta, p, t; \theta', p', 0)$$

$$= \left( \prod_{j=1}^{N} \frac{1}{m \omega_j} \right) \lim_{\epsilon \to 0} \left( \prod_{n=1}^{N-1} \int_{0}^{2\pi} d\theta_n \int_{-\infty}^{\infty} dR_n(\theta_n) \right) \left( \prod_{n=1}^{N} \sum_{l_n=-\infty}^{\infty} \sqrt{\frac{I}{2\pi i \hbar}} \sqrt{\frac{m}{2\pi i \hbar}} \right)$$

$$\times \exp \sum_{n=1}^{N} \left\{ \frac{i I}{\hbar} (\theta_n - \theta_{n-1} + 2\pi l_n)^2 + \frac{i m}{\hbar} (R_n(\theta_n + 2\pi l_n) - R_{n-1}(\theta_{n-1}))^2 \right\}$$

$$\times \exp \sum_{n=1}^{N} \left\{ -\frac{i \epsilon}{\hbar} \nu(\theta_{n-1}, R_{n-1}(\theta_{n-1})) \right\} .$$

The integral including $\theta_{N-1}$ is

$$I_{N-1} = \sum_{l_{N-1}=-\infty}^{\infty} \sum_{l_{N-1}=-\infty}^{\infty} \sqrt{\frac{I}{2\pi i \hbar}} \sqrt{\frac{m}{2\pi i \hbar}} \int_{0}^{2\pi} d\theta_{N-1} \int_{-\infty}^{\infty} dR_{N-1}(\theta_{N-1})$$

$$\times \exp \left\{ \frac{i I}{\hbar} (\theta_N - \theta_{N-1} + 2\pi l_N)^2 + \frac{i I}{\hbar} (\theta_{N-1} - \theta_{N-2} + 2\pi l_{N-1})^2 \right\}$$

$$\times \exp \left\{ \frac{i m}{\hbar} (R_N(\theta_N + 2\pi l_N) - R_{N-1}(\theta_{N-1}))^2 \right\}$$

$$\times \exp \left\{ \frac{i m}{\hbar} (R_{N-1}(\theta_{N-1} + 2\pi l_{N-1}) - R_{N-2}(\theta_{N-2}))^2 \right\}$$

$$\times \exp \left\{ -\frac{i \epsilon}{\hbar} \nu(\theta_{N-1}, R_{N-1}(\theta_{N-1})) \right\} .$$

The sum over $l_{N-1}$ can be absorbed into the $\theta_{N-1}$ integral by changing the domain of $\theta_{N-1}$ integration from $[0, 2\pi)$ to $(-\infty, \infty)$. This is done by the following procedure:

1. transform $l_N$ to $\tilde{l}_N = l_N + l_{N-1}$ and write $\tilde{\theta}_{N-1} = \theta_{N-1} + 2\pi l_{N-1}$ to obtain

$$I_{N-1} = \sum_{l_{N-1}=-\infty}^{\infty} \sum_{l_{N-1}=-\infty}^{\infty} \sqrt{\frac{I}{2\pi i \hbar}} \sqrt{\frac{m}{2\pi i \hbar}} \int_{0}^{2\pi} d\theta_{N-1} \int_{-\infty}^{\infty} dR_{N-1}(\theta_{N-1})$$

$$\times \exp \left\{ \frac{i I}{\hbar} (\tilde{\theta}_N - \tilde{\theta}_{N-1} + 2\pi \tilde{l}_N)^2 + \frac{i I}{\hbar} (\tilde{\theta}_{N-1} - \tilde{\theta}_{N-2})^2 \right\}$$

$$\times \exp \left\{ \frac{i m}{\hbar} (R_N(\theta_N + 2\pi \tilde{l}_N) - R_{N-1}(\tilde{\theta}_{N-1}))^2 \right\}$$

$$\times \exp \left\{ \frac{i m}{\hbar} (R_{N-1}(\tilde{\theta}_{N-1} + 2\pi \tilde{l}_{N-1}) - R_{N-2}(\theta_{N-2}))^2 \right\}$$

$$\times \exp \left\{ -\frac{i \epsilon}{\hbar} \nu(\tilde{\theta}_{N-1}, R_{N-1}(\tilde{\theta}_{N-1})) \right\} .$$

2. Change the variable of integration $\int_{0}^{2\pi} d\theta_n \to \int_{2\pi n}^{2\pi(n+1)} d\tilde{\theta}_n$ and change the domain of $\theta_{N-1}$ integration from...
[0, 2\pi) to (−∞, ∞) and use the periodicity of \(\mathcal{V}(\theta_{N-1}, R_{N-1}(\theta_{N-1}))\) to obtain

\[
I_{N-1} = \sum_{l_N=-\infty}^{\infty} \sqrt{\frac{I}{2\pi i\hbar}} \sqrt{\frac{m}{2\pi i\hbar}} \int_{-\infty}^{\infty} d\theta_{N-1} \int_{-\infty}^{\infty} dR_{N-1}(\theta_{N-1})
\]

\[
\times \exp \left\{ \frac{iI}{\hbar 2\epsilon} (\theta_N - \theta_{N-1} + 2\pi l_N)^2 + \frac{iI}{\hbar 2\epsilon} (\theta_N - \theta_{N-2})^2 \right\}
\]

\[
\times \exp \left\{ \frac{im}{\hbar 2\epsilon} [R_N(\theta_N + 2\pi l_N) - R_{N-1}(\theta_{N-1})]^2 \right\}
\]

\[
\times \exp \left\{ \frac{im}{\hbar 2\epsilon} [R_{N-1}(\theta_{N-1}) - R_{N-2}(\theta_{N-2})]^2 \right\}
\]

\[
\times \exp \left\{ -\frac{i\epsilon}{\hbar} \mathcal{V}(\theta_{N-1}, R_{N-1}(\theta_{N-1})) \right\}.
\]

Repeat this procedure for the integrals involving \(\theta_n\) from \(n = N - 2\) to \(n = 1\), and obtain

\[
K(\theta, q, t; \theta', q', 0)
\]

\[
= \sum_{l=-\infty}^{\infty} \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \left( \prod_{n=1}^{N-1} \frac{d\theta_n}{A_n} \right) \int_{-\infty}^{\infty} \left( \prod_{n=1}^{N-1} \frac{dR_{n-1}(\theta_{n-1})}{A_{B_n}} \right) \sqrt{\frac{I}{2\pi i\hbar}} \sqrt{\frac{m}{2\pi i\hbar}}
\]

\[
\times \exp \left\{ \frac{iI}{\hbar 2\epsilon} (\theta_n - \theta_{n-1} + 2\pi l_\delta n)\right)^2 \right\}
\]

\[
\times \exp \left\{ \frac{im}{\hbar 2\epsilon} [R_n(\theta_n + 2\pi l_\delta n) - R_{n-1}(\theta_{n-1})]^2 - \frac{i\epsilon}{\hbar} \mathcal{V}(\theta_{n-1}, R_{n-1}(\theta_{n-1})) \right\}.
\]

Define

\[
\mathcal{A}_A = \sqrt{\frac{2\pi i\hbar}{I}}, \quad \mathcal{A}_B = \sqrt{\frac{2\pi i\hbar}{m}},
\]

\[
\int \mathcal{D}\theta = \frac{1}{\mathcal{A}_A} \int_{-\infty}^{\infty} \left( \prod_{n=1}^{N-1} \frac{d\theta_n}{A_n} \right), \quad \int \mathcal{D}R(\theta) = \frac{1}{\mathcal{A}_B} \int_{-\infty}^{\infty} \left( \prod_{n=1}^{N-1} \frac{dR_n(\theta_n)}{A_{B_n}} \right),
\]

\[
L_n(\theta_n, R_n(\theta_n)) = \frac{I}{2} \left( \frac{\theta_n - \theta_{n-1} + 2\pi l_\delta n_N}{\epsilon} \right)^2
\]

\[
+ \frac{m}{2} \left( \frac{R_n(\theta_n + 2\pi l_\delta n_N) - R_{n-1}(\theta_{n-1})}{\epsilon} \right)^2 - \mathcal{V}(\theta_{n-1}, R_{n-1}(\theta_{n-1}))
\]

where

\[
L[\theta, R(\theta)] = \lim_{\epsilon \to 0} L_n[\theta_n, R_n(\theta_n)] = L_0[\theta] + L_B[\theta, R(\theta)]
\]

\[
L_0[\theta] = \frac{I}{2} \theta^2
\]

\[
L_B[\theta, R(\theta)] = \sum_{j=1}^{N} \frac{m}{2} \dot{R}_j(\theta)^2 - \sum_{j=1}^{N} \frac{1}{2} m \omega_j^2 \left( \frac{R_j(\theta) - C_j\theta}{m \omega_j^2} \right)^2
\]

is the Lagrangian of the coupled system. The action \(S[\theta, R(\theta)]\) is given by

\[
S[\theta, R(\theta)] = \int_0^t L[\theta, R(\theta)] ds = S_0[\theta] + S_B[\theta, R(\theta)]
\]

\[
S_0[\theta] = \int_0^t d\tau L[\theta]
\]

\[
S_B[\theta, R(\theta)] = \int_0^t d\tau L_B[\theta, R(\theta)].
\]
Now, suppose that we initially have the total density operator given by
\[ \hat{\rho}_{AB}(0) = \hat{\rho}_A(0)\hat{\rho}_B(0). \] (A14)

Then, the reduced density matrix is
\[ \rho(\theta, \phi, t) = \int_{-\infty}^{\infty} d\mathbf{R}(\theta) d\mathbf{Q}(\phi) \delta(\mathbf{R}(\theta) - \mathbf{Q}(\phi)) \rho(\theta, \mathbf{p}, \phi, s, t) \]
\[ = \int_0^{2\pi} d\theta' d\phi' J(\theta, \phi, t; \theta', \phi', 0) \rho_A(\theta', \phi', 0) \] (A15)

where
\[ J(\theta, \phi, t; \theta', \phi', 0) = \sum_{i=-\infty}^{\infty} \sum_{i'=-\infty}^{\infty} \int_{\theta'}^{\theta+2\pi i} D\theta \int_{\phi'}^{\phi+2\pi i} D\phi' \exp \left( \frac{i}{\hbar} (S_0[\theta] - S_0[\phi]) \right) \mathcal{F}[\theta, \phi] \] (A16)
is the propagator of the density matrix,
\[ \mathcal{F}[\theta, \phi] = \int_{-\infty}^{\infty} d\mathbf{R}(\theta) d\mathbf{Q}(\phi) \delta(\mathbf{R}(\theta) - \mathbf{Q}(\phi)) \rho_B(\mathbf{p}', \mathbf{s}', 0) \]
\[ \times \int_{\mathbf{R}(\theta')}^{\mathbf{R}(\phi')} D\mathbf{R}(\theta) D\mathbf{Q}(\phi) \exp \left( \frac{i}{\hbar} (S_B[\mathbf{R}(\theta)] - S_B[\mathbf{Q}(\phi)]) \right) \] is the influence functional and \( Q(\phi) = -(s_j - c\phi)/m\omega_j \). Supposed that the density operator \( \hat{\rho}_B \) can be written as a canonical ensemble at \( t = 0 \). That is,
\[ \rho_B = \frac{\exp(-\beta \hat{H}_B)}{\text{tr}_B[\exp(-\beta \hat{H}_B)]} \] (A17)

Introduce the imaginary time \( \tau = -i\hbar\beta \), then the density matrix of the environment at \( t = 0 \) is
\[ \rho_B(\mathbf{p}', \mathbf{s}', 0) = \frac{\left. \langle \mathbf{p}' \left| \exp \left( \frac{-i\beta \hat{H}_B}{\hbar} \right) \right| \mathbf{s}' \rangle \right|_{\text{tr}_B[\exp(-\beta \hat{H}_B)]}}{\int d\mathbf{q} K_B(\mathbf{q}, \tau; \mathbf{q}, 0)} = K_B(\mathbf{p}', \tau; \mathbf{s}', 0) \]

2. Momentum Representation of the Density matrix of a Harmonic Oscillator

The propagator of a harmonic oscillator with the Lagrangian
\[ L = \frac{m}{2} x^2 - \frac{1}{2} m\omega x^2 \] (A18)
has a well known solution
\[ K(x, t; x', 0) = \int dx \exp \left( \frac{i}{\hbar} \int_0^t Lds \right) \]
\[ = \sqrt{\frac{m\omega}{2\pi i\hbar \sin \omega t}} \exp \left( \frac{i}{\hbar} \frac{m\omega}{2\sin \omega t} \left[ (x^2 + x'^2) \cos \omega t - 2xx' \right] \right) \] (A20)

\( K(p, t; p', 0) \) is obtained by a Fourier transformation
\[ K(p, t; p', 0) = \int_{-\infty}^{\infty} \frac{dx dx'}{2\pi} K(x, t; x', 0) \exp \left( \frac{i}{\hbar} (px - p'x') \right). \] (A21)

Let \( A = \frac{m\omega \cos \omega t}{2\sin \omega t} \) and \( B = \frac{m\omega}{2\sin \omega t} \), then
\[ K(p, t; p', 0) = \frac{F(t)}{\sqrt{4B^2 - 4A^2}} \exp \left( \frac{i}{\hbar} \frac{Ap^2 + Ap'^2 - 2Bpp'}{4B^2 - 4A^2} \right) \]
where

$$4B^2 - 4A^2 = \frac{m^2 \omega^2(1 - \cos^2 \omega t)}{\sin^2 \omega t} = m^2 \omega^2,$$

so,

$$K(p, t; p', 0) = \frac{F(t)}{m \omega} \exp \frac{i}{\hbar} S[p(t)/m \omega].$$

Therefore, using $\tau = -i \hbar \beta, \beta = 1/k_B T, \nu_j = h \omega_j \beta$ and $i \sinh x = \sin i x$ we have

$$\langle p' | \exp(-i \tau \hat{H}_B/\hbar) | s' \rangle = \prod_{j=1}^{N} \frac{1}{m \omega_j} \sqrt{\frac{m \omega_j}{2 \pi \hbar \sinh \nu_j}} \exp - \frac{m \omega_j}{2 \hbar \sinh \nu_j} \left[ \frac{p_j^2 + s_j^2}{m^2 \omega_j^2} \cosh \nu - \frac{2 p_j s_j}{m^2 \omega_j^2} \right],$$

and

$$\text{tr}_B[\exp(-i \tau \hat{H}_B/\hbar)] = \int_{-\infty}^{\infty} dq \langle q | \exp(-i \tau \hat{H}_B/\hbar) | q \rangle = \prod_{j=1}^{N} \frac{1}{2 \sinh \nu_j/2}.$$

We are assuming that the environment is not coupled to the system at $t = 0$, so $R'_j = p'_j/m \omega_j$ and $Q'_j = s'_j/m \omega_j$. Therefore,

$$\rho_B(p', s', 0) = \left( \prod_{j=1}^{N} \frac{1}{m \omega_j} \right) \rho_B(R', Q', 0)$$

where

$$\rho_B(R', Q', 0) = \prod_{j=1}^{N} 2 \sinh \frac{\nu_j}{2} \sqrt{\frac{m \omega_j}{2 \pi \hbar \sinh \nu_j}} \exp - \frac{m \omega_j}{2 \hbar \sinh \nu_j} \left[ (R'_j^2 + Q'_j^2) \cosh \nu_j - 2R'_j Q'_j \right].$$

Let us rewrite this density matrix into a simpler form. Define $x_{0j} = R'_j + Q'_j$ and $x'_{0j} = R'_j - Q'_j$. Then, using

$$(R'_j^2 + Q'_j^2) \cosh \nu_j - 2R'_j Q'_j = \frac{1}{2} (\cosh \nu_j - 1)x_{0j}^2 + \frac{1}{2} (\cosh \nu_j + 1)x'_{0j}^2$$

and the identity

$$\frac{\cosh \nu_j - 1}{\sinh \nu_j} = \frac{\sinh \nu_j}{\cosh \nu_j + 1} = \tanh \frac{\nu_j}{2},$$

we obtain

$$\rho_B(R', Q', 0) = \prod_{j=1}^{N} \sqrt{\frac{m \omega_j}{\pi \hbar \mu_j}} \exp - \frac{m \omega_j}{4 \hbar} \left[ \frac{x_{0j}^2}{\mu_j} + \frac{x'_{0j}^2}{2 \mu_j} \right].$$

$$\mu_j = \coth \frac{\nu_j}{2}.$$  \hspace{1cm} (A30)

3. The Influence Functional $\mathcal{F}[\theta, \phi]$

Now, we calculate the density functional $\mathcal{F}[\theta, \phi]$ explicitly. We will write $R$ and $Q$ instead of $R[\theta]$ and $Q[\phi]$ for simplicity, but keep in mind that they depend on $\theta$ and $\phi$, respectively. Then, the influence functional can be written as

$$\mathcal{F}[\theta, \phi] = \int_{-\infty}^{\infty} dx_r dx'_r dx_0 dx'_0 (1/2)^N \delta(x'_0) \rho_B(R', Q', 0) \int_{R_r}^{R} D\mathbf{R} \int_{Q_r}^{Q} D\mathbf{Q}^* \times \exp \left\{ \frac{i}{\hbar} \int_{0}^{t'} ds \sum_{j} \left[ \frac{m}{2} (R_j^2 - \dot{Q}_j^2) - \frac{1}{2} m \omega_j^2(R_j^2 - Q_j^2) + C_j(\theta R_j - \phi Q_j) - \frac{C_j^2}{2 m \omega_j^2}(\theta^2 - \phi^2) \right] \right\}.$$
To evaluate this we introduce the variables
\[ \varphi = \theta + \phi, \quad \varphi' = \theta - \phi, \]
\[ x_j = R_j + Q_j, \quad x'_j = R_j - Q_j. \] (A32)

Then, the influence functional becomes
\[ F[\theta, \phi] = \int_{-\infty}^{\infty} dx_i dx'_i dx_0 dx'_0 (1/2)^N \delta(x'_i) \rho_B(R', Q', 0) \int DxDx' \]
\[ \times \exp \left\{ \frac{i}{\hbar} \int_0^t ds \sum_j \left[ \frac{m}{2} \dot{x}_j \dot{x}'_j - \frac{1}{2} m \omega_j^2 x_j x'_j + \frac{C_j}{2} (\varphi x'_j + \varphi' x_j) - \frac{C_j^2}{2 m \omega_j^2} \varphi' \varphi \right] \right\} \]
\[ = \int_{-\infty}^{\infty} dx_i dx'_i dx_0 dx'_0 (1/2)^N \delta(x'_i) \rho_B(R', Q', 0) \]
\[ \times \exp \left\{ \frac{i m}{2 \hbar} \sum_j (x_{1j} \dot{x}'_{1j} - x_{0j} \dot{x}'_{0j}) - \frac{i}{\hbar} \int_0^t ds \sum_j \frac{C_j^2}{2 m \omega_j^2} \varphi' \varphi \right\} \]
\[ \times \int DxDx' \exp \left\{ \frac{i}{\hbar} \int_0^t ds \sum_j \left[ -g(x'_j) x_j + \frac{C_j}{2} \varphi x'_j \right] \right\} \]

where
\[ g(x'_j) = \frac{m}{2} x'_j + \frac{1}{2} m \omega_j^2 x'_j - \frac{C_j}{2} \varphi'. \] (A33)

We note that the classical solution \( x'_{j, \text{cl}} \) satisfies the Euler-Lagrange equation
\[ g(x'_{j, \text{cl}}) = 0, \quad g(x'_{j}(0)) = 0, \quad g(x'_{j}(t)) = 0. \] (A34)

The path integral over \( x \) can be done first. Calling this integral \( I_{x, x'} \) we obtain
\[ I_{x, x'} = \int DxDx' \exp \left\{ \frac{i}{\hbar} \int_0^t ds \sum_j \left[ -g(x'_j) x_j + \frac{C_j}{2} \varphi x'_j \right] \right\} \] (A35)
\[ = \frac{1}{|A|} \int_{-\infty}^{\infty} \left( \prod_{n=1}^{N-1} \int dx\! dx' \right) \prod_{n=1}^{N} \prod_{j} \exp \frac{i e}{\hbar} \left[ -g(x'_{n,j}) x_{n,j} + \frac{C_j}{2} \varphi x'_j \right] \] (A36)
\[ = \frac{1}{|A|} \int_{-\infty}^{\infty} \left( \prod_{n=1}^{N-1} \int dx\! dx' \right) \prod_{j} \left( \prod_{n=1}^{N-1} \delta \left[ \frac{e}{2 \pi \hbar} g(x'_{n,j}) \right] \right) \prod_{n=1}^{N} \exp \frac{i e}{\hbar} \frac{C_j}{2} \varphi x'_{n,j}. \] (A37)

\(|A|^{-2}\) is absorbed into the delta functions. The interpretation of the delta function is to insert the classical solution \( x'_{j, \text{cl}} \) into \( x'_j \) which satisfy \( g(x'_{j, \text{cl}}) = 0 \). The outcome is,
\[ I_{x, x'} = \frac{1}{|A|} \exp \left( \frac{i}{\hbar} \int_0^t ds \sum_j \frac{C_j}{2} \varphi x'_{j, \text{cl}} \right). \] (A38)

Therefore, the influence functional is
\[ F[\theta, \phi] = \int_{-\infty}^{\infty} dx_i dx'_i dx_0 dx'_0 (1/2)^N \delta(x'_i) \rho_B(R', Q', 0) \exp \frac{i m}{2 \hbar} \sum_j (x_{0j} \dot{x}'_{0j} - x_{0j} x'_{0j}) \]
\[ \times \frac{1}{|A|} \exp \left( \frac{i}{\hbar} \int_0^t ds \sum_j \left[ -\frac{C_j^2}{2 m \omega_j^2} \varphi' \varphi + \frac{C_j}{2} \varphi x'_{j, \text{cl}} \right] \right). \]
The $x$, integral gives a delta function of $x'_j$

$$
\mathcal{F}[\theta, \phi] = \int_{-\infty}^{\infty} d\mathbf{x}'_t d\mathbf{x}_0 d\mathbf{x}'_0 (1/2)^N \delta(\mathbf{x}') \delta(\mathbf{x}'_t) \rho_B(R', Q', 0)
$$

$$
\times \exp \left( \frac{i}{\hbar} \int_0^t ds \sum_j \left[ -\frac{C_j^2}{2m \omega_j^2} \varphi' \phi + \frac{C_j}{2} \varphi x'_d \right] + \frac{i}{\hbar} \sum_j x_{0j} x'_{0j} \right).
$$

Insert equation (A29) for the density matrix and do the $x_0$ integral, then we obtain

$$
\mathcal{F}[\theta, \phi] = \int_{-\infty}^{\infty} d\mathbf{x}'_t d\mathbf{x}'_0 \delta(\mathbf{x}') \delta(\mathbf{x}'_t)
$$

$$
\times \prod_j \exp \left( \frac{i}{\hbar} \int_0^t ds \left[ -\frac{C_j^2}{2m \omega_j^2} \varphi' \phi + \frac{C_j}{2} \varphi x'_d \right] - \frac{m \mu_j \omega_j}{4 \hbar} \left( \frac{x'_{0j}^2}{\omega_j^2} + x_{0j}^2 \right) \right).
$$

The two delta functions $\delta(\mathbf{x}')$ and $\delta(\mathbf{x}'_t)$ can be interpreted as boundary conditions on the classical solution of $g(\mathbf{x}(s)) = 0$. The result of doing the remaining integrals would be to substitute the classical solution of $x'_d$, $x'_{0j}$ and $x_{0j}$ which satisfy these boundary conditions. The solution to the classical solution of $g(x) = 0$, that is

$$
\ddot{x}'_j + \omega_j^2 x'_j - \frac{C_j}{m} \phi' = 0.
$$

(A39)

is

$$
x_j(\tau) = - \int_\tau^t \frac{C_j^2 \varphi'(s)}{m \omega_j} \sin \omega_j(\tau - s) ds + x'_{0j} \cos \omega_j(t - \tau) - \frac{\dot{x}'_{0j}}{\omega_j} \sin \omega_j(t - \tau).
$$

(A40)

For our boundary condition this solution reduces to

$$
x'_j(\tau) = - \int_\tau^t \frac{C_j \varphi'(s)}{m \omega_j} \sin \omega_j(\tau - s) ds
$$

(A41)

$$
\dot{x}'_j(\tau) = - \int_\tau^t \frac{C_j \varphi'(s)}{m} \cos \omega_j(\tau - s) ds
$$

(A42)

Therefore, the result of integration is

$$
\mathcal{F}[\theta, \phi] = \exp \left( - \sum_j \frac{i C_j^2}{\hbar 2m \omega_j} \int_\tau^t \frac{\varphi(\tau)}{\omega_j} \delta(\tau - s) \sin \omega_j(\tau - s) \varphi'(s) d\tau ds \right)
$$

$$
\times \exp \left( - \sum_j \frac{\mu_j}{4 \hbar} \int_0^t \int_0^t \frac{C_j^2}{m \omega_j^2} \varphi'(\tau) \cos \omega_j(\tau - s) \varphi'(s) ds d\tau \right).
$$

Since $s$ and $\tau$ enter into the second integral symmetrically it can be rewritten

$$
\int_0^t \int_0^t E(\tau, s) ds d\tau = 2 \int_0^t \int_0^\tau E(\tau, s) ds d\tau.
$$

(A43)

The first integral can be written

$$
\int_0^t \int_0^t E(\tau, s) ds d\tau = \int_0^t \int_0^\tau E(s, \tau) ds d\tau.
$$

(A44)

Then, the influence functional becomes

$$
\mathcal{F}[\theta, \phi] = \exp i \Phi[\theta, \phi]
$$

(A45)
where the influence phase \( \Phi[\theta, \phi] \) can be written as

\[
i\Phi[\theta, \phi] = -\frac{i}{\hbar} \int_0^t \int_0^\tau d\tau ds \varphi(s) \alpha_I(\tau - s) \varphi'(\tau) + \frac{1}{\hbar} \int_0^t \int_0^\tau ds d\tau \varphi'(\tau) \alpha_R(\tau - s) \varphi'(s)
\] (A46)

\[
\alpha_R(\tau - s) = \sum_j \frac{C_j^2}{2\omega_j} \coth \frac{\hbar \omega_j}{2k_B T} \cos \omega_j(\tau - s)
\] (A47)

\[
\alpha_I(\tau - s) = \sum_j \frac{C_j^2}{2\omega_j} \left( \frac{2}{\omega_j} \delta(\tau - s) + \sin \omega_j(\tau - s) \right).
\] (A48)

Therefore, we obtain the density matrix propagator \( J(\theta, \phi; t; \theta', \phi', 0) \)

\[
J(\theta, \phi; t; \theta', \phi', 0) = \int_{\theta_i}^{\theta_f} D\theta \int_{\phi_i}^{\phi_f} D\phi^* \exp \left\{ \frac{i}{\hbar} S[\varphi^+, \varphi^-] - \Gamma[\varphi^-] \right\}
\] (A49)

\[
\frac{\varphi^+}{\varphi^-} = \frac{\varphi^+(0)}{\varphi^-(0)}
\] (A50)

the classical action \( S[\varphi^+, \varphi^-] \) is

\[
S[\varphi, \varphi'] = -\int_0^t d\tau I\dot{\varphi}^+(\tau) \dot{\varphi}^-(\tau)
\]

\[
+ 2\int_0^t d\tau \int_0^\tau ds \varphi_-(\tau) \alpha_I(\tau - s) \varphi^+(s)
\]

and the \( \Gamma[\varphi^-] \) is given by

\[
\Gamma[\varphi^-] = \frac{1}{2\hbar} \int_0^t d\tau \int_0^\tau ds \varphi_-(\tau) \alpha_R(\tau - s) \varphi^-(s).
\]

The path integral in (A50) can be done exactly and we obtain

\[
J(\theta, \phi; t; \theta_i, \phi_i, 0) = F^2(t) \exp \left( \frac{i}{\hbar} S[\varphi^+, \varphi^-] - \Gamma[\varphi^-] \right)
\] (A51)

\( \varphi^\pm \) are the classical coordinates obtained from the Euler-Lagrange equation

\[
I\ddot{\varphi}^\pm(u) + 2\int_u^t d\tau \varphi^\pm(\tau) \alpha_I(\tau - u) = 0
\] (A52)

\[
I\ddot{\varphi}^\pm(u) + 2\int_0^u d\tau \varphi^\pm(u) \alpha_I(u - \tau) = 0
\] (A53)

whose solution are given in terms of boundary conditions \( \varphi^\pm = \varphi^\pm(0), \varphi^\pm = \varphi^\pm(t) \):

\[
\varphi^+_\varphi^+(u) = \kappa_i(u; t) \varphi^+_\varphi^+(u) + \kappa_f(u; t) \varphi^+_\varphi^+(t)
\] (A54)

\[
\varphi^+_\varphi^-(u) = \kappa_i(t - u; t) \varphi^+_\varphi^-(u) + \kappa_f(t - u; t) \varphi^+_\varphi^-(t)
\] (A55)

\[
\kappa_i(u; t) = \frac{\dot{G}(u)}{G(t)} G(u), \quad \kappa_f(u; t) = \frac{G(u)}{G(t)}.
\] (A56)

Then, the classical action reduces to

\[
S[\varphi^+, \varphi^-] = -I[\varphi^+_\varphi^+(t) \varphi^+_\varphi^-(0) \varphi^-_\varphi^-].
\] (A57)