Nonexistence of global solutions of a class of coupled nonlinear Klein-Gordon equations with nonnegative potentials and arbitrary initial energy

Yanjin Wang *

Graduate School of Mathematical Sciences, University of Tokyo
3-8-1 Komaba, Meguro, Tokyo, 153-8914, Japan

Abstract

In the paper we consider the nonexistence of global solutions of the Cauchy problem for coupled Klein-Gordon equations of the form

\[
\begin{aligned}
  u_{tt} - \Delta u + m_1^2 u + K_1(x)u &= a_1 |v|^{p+1} |u|^{p-1} u \\
  v_{tt} - \Delta v + m_2^2 v + K_2(x)v &= a_2 |u|^{p+1} |v|^{q-1} v \\
  u(0, x) &= u_0; u_t(0, x) = u_1(x) \\
  v(0, x) &= v_0; v_t(0, x) = v_1(x)
\end{aligned}
\]

on \( \mathbb{R} \times \mathbb{R}^n \).

Firstly for some special cases of \( n = 2, 3 \), we prove the existence of ground state of the corresponding Lagrange-Euler equations of the above equations. Then we establish a blow up result with low initial energy, which leads to instability of standing waves of the system above. Moreover as a byproduct we also discuss the global existence. Next based on concavity method we prove the blow up result for the system with non-positive initial energy in the general case: \( n \geq 1 \). Finally when the initial energy is given arbitrarily positive, we show that if the initial datum satisfies some conditions, the corresponding solution blows up in a finite time.

**Keywords:** Coupled Klein-Gordon equations; variational calculus; Blow up; Arbitrarily initial energy; Non-negative potential.

**AMS subject classification:** 34A34, 35G25, 35L70, 35J60

1 Introduction

The motion of charged mesons in an electromagnetic field can be described by the following coupled Klein-Gordon equations:

\[
\begin{aligned}
  u_{tt} - \Delta u + \alpha^2 u + g^2 v^2 u &= 0 \\
  v_{tt} - \Delta v + \beta^2 v + h^2 u^2 v &= 0
\end{aligned}
\]

(1.1)

where \( \Delta \) is Laplacian operator on \( \mathbb{R}^n \), \( \alpha \) and \( \beta \) are non zero real constants. The system was firstly introduced by I. Segal [23]. A lot of authors have discussed this mixed system; see for example [8], [11], [15], [16]. And a sharp condition for global existence and blowing up has been given by Zhang [29] for the mixed problem (1.1), where the blow up result is given under the condition that the initial energy is below the energy wall. The system was generalized by Miranda and Medeiros [17].

*Corresponding email: wangyj@ms.u-tokyo.ac.jp
for the case where the nonlinear terms are of the form $|v|^{p+1} |u|^{p-1}u$ and $|u|^{p+1} |v|^{p-1}v$ with some $p$. Li and Tsai [12] recently considered a class of nonlinear term which includes the above generalized nonlinear terms on a bounded domain of $\mathbb{R}^n$. And some other nonlinear term was considered by Delort, Fang and Xue [3] on $\mathbb{R}^2$.

In this paper we are interested in the initial boundary value problem for the coupled Klein-Gordon equations with nonnegative potentials of the form

$$
\begin{cases}
  u_{tt} - \Delta u + m_1^2 u + K_1(x) u = a_1 |v|^{q+1} |u|^{p-1} u \\
  v_{tt} - \Delta v + m_2^2 v + K_2(x) v = a_2 |u|^{q+1} |v|^{p-1} v \\
  u(0,x) = u_0; u_t(0,x) = u_1(x), x \in \mathbb{R}^n \\
  v(0,x) = v_0; v_t(0,x) = v_1(x), x \in \mathbb{R}^n
\end{cases}
$$

(1.2)

where the parameters $a_1$ and $a_2$ are positive constants, the masses are nonzero, $m_1 \neq 0$ and $m_2 \neq 0$.

In the paper we make the following restriction on the real numbers $p > 1$ and $q > 1$:

If $n = 1, 2$,

$$1 < p, q < \infty;$$

(1.3)

And if $n \geq 3$, then

$$q < p + 1 < \frac{n + 2}{n - 2} \text{ or } p + 1 < q < \frac{n + 2}{n - 2},$$

(1.4)

and

$$p < q + 1 < \frac{n + 2}{n - 2} \text{ or } q + 1 < p < \frac{n + 2}{n - 2}.$$

(1.5)

Throughout the paper we will assume that $K_i(x)$ satisfies

$$K_i(x) \geq 0 \ (\forall x \in \mathbb{R}^n)$$

(1.6)

for $i = 1, 2$.

Before describing our results, we first recall the existing results about the Cauchy problem for the single Klein-Gordon equation

$$u_{tt} - \Delta u + m^2 u + K(x) u = f(u) \text{ on } [0, \infty) \times \mathbb{R}^n,$$

(1.7)

where $K(x) \geq 0$.

When $K(x) \equiv 0$, there are numerous results about the existence and blow up of solutions for the equation (1.7). It is well known that the solution blows up in a finite time when the initial energy is negative. Here we refer to [5], [6], [9], [25]. The instability of standing wave of the equation (1.7) was studied in [1], [2], [14], [19]. Based on the results in [1], [2], Zhang [28] established a sharp condition of global existence and blow up for the equation (1.7). Recently the author [26] has proposed a sufficient condition of the initial datum with arbitrarily positive initial energy such that the corresponding solution of the equation (1.7) blows up in a finite time.

As for the case $K(x) \neq 0$, when the nonlinear term vanishes it has been shown that the equation (1.7) has time periodic and spatially localized solutions in [22]. And Soffer and Weinstein [24] considered a class of nonlinear Klein-Gordon equations (1.7) with nonlinear term $f(u)$, which is real-valued, smooth in a neighborhood of $u = 0$ and has an expansion $f(u) = u^3 + O(u^4)$ on $\mathbb{R}^3$. Recently Gan and Zhang [4] considered standing waves for the equation (1.7) with $f(u) = |u|^{p-1}u$.

Now we return to the coupled Klein-Gordon equations (1.2). As we know, until now there is no result for the system (1.2). In the paper we are concerned with the nonexistence of global solutions.
of the system (1.2). We first establish the existence of a unique local weak solution of the equations (1.2) by applying the Banach contraction mapping principle.

Based on the local existence theorem, our first purpose of the present paper is to establish a blowing up result by using the ground state solution. This blowing up result leads to instability of standing wave for the equation (1.2). The proof is done by first showing the existence of the ground state solution of the corresponding Lagrange-Euler equations by variational method which was firstly introduced in [1], [2], and then by discussing a blow up result with low initial energy based on a potential well argument and concavity method, which is originated by Payne and Sattinger [21] and Levine [9], [10], respectively. Because of the restriction of the embedding theorem of $H^1_{K_i} \hookrightarrow L^r_{\sigma}$ $(2 < r < \frac{2n}{n-2})$, the blow up result will be established only on $\mathbb{R}^n$ $(n = 2, 3)$. As a byproduct we, however, also establish the global existence of solutions of the system (1.2) when $n = 2, 3$.

Our next purpose is to show the blow up result when the initial energy is non-positive by a concavity argument.

The final purpose is to construct sufficient conditions of the initial datum such that the corresponding solution blows up in a finite time with arbitrarily positive initial energy, that is, we show that there exists a finite time $T$ such that $\lim_{t \to T^-} (\|u(t)\|^2 + \|v(t)\|^2) = \infty$.

To the best of our knowledge, this is the first blowing up result for the coupled Klein-Gordon equations with arbitrarily positive initial energy.

The paper is organized as follows. In Section 2 we state the local existence of solutions by a fixed point argument. In Section 3 we study the existence of the standing wave of the system (1.2) with the ground state by using variational method. In section 4, based on the result obtained in Section 3 we establish blowing up result for the system (1.2) on $\mathbb{R}^n$ (n=2,3), which will lead to the instability of standing waves. In Section 5, using concavity argument we show a blowing up result when the initial energy is negative. In the last section, we establish some sufficient conditions of initial datum with arbitrarily initial energy such that the corresponding solution blows up in a finite time.

2 Local existence

In the paper we will work in the energy space:

$$H^1_{0,K_i} = \left\{ u \in H^1_0(\mathbb{R}^n) : \int K_i(x)|u(x)|^2 dx < \infty \right\}$$

with the following norm:

$$\|u\|_{H^1_{0,K_i}} = \|\nabla u\|^2 + m^2_2 |u|^2 + \int_{\mathbb{R}^n} K_i(x) |u(x)|^2 dx$$

where $K_i(x) \geq 0 \ (\forall x \in \mathbb{R}^n)$ for $i = 1, 2$.

And for simplicity we denote $\int_{\mathbb{R}^n} dx$ by $\int dx$. The notation $t \to T^-$ means that $t \to T$ and $t < T$.

Firstly we rewrite the coupled Klein-Gordon equations (1.2) in the following equivalent form

$$\begin{cases}
\alpha u_{tt} - \Delta u + \alpha m^2_2 u + \alpha K_1(x) u = \alpha_2'(p+1)|v|^{q+1} |u|^{p-1} u \\
v_{tt} - \Delta v + m^2_2 v + K_2(x) v = \alpha_2'(q+1)|u|^{p+1} |v|^{q-1} v \\
u(0,x) = u_0; u_t(0,x) = u_1(x) \\
v(0,x) = v_0; v_t(0,x) = v_1(x)
\end{cases} \quad (2.1)$$
on $\mathbb{R} \times \mathbb{R}^n$, where $\alpha = \frac{a_2(p + 1)}{a_1(q + 1)}$ and $a'_2 = \frac{a_2}{q + 1}$.

**Definition 2.1** A function $(u, v)$ is said to be a solution of the system (2.1), if it satisfies that $u \in C^0([0, T), H^1_{0, K_1}(\mathbb{R}^n)) \cap C^1([0, T), L^2(\mathbb{R}^n))$ and $v \in C^0([0, T), H^1_{0, K_2}(\mathbb{R}^n)) \cap C^1([0, T), L^2(\mathbb{R}^n))$, where

$$
\alpha \left[ \int u_t v_1 dx + \int \nabla u(t) \nabla w_1 dx + m_1^2 \int u(t) w_1 dx + \int K_1(x) u(t) w_1 dx \right] = a'_2 \int |v(t)|^{q+1} |u(t)|^{p-1} u(t) v_1 dx,
$$

and

$$
\left\{ \begin{array}{l}
u_t v_2 dx + \int \nabla v(t) \nabla w_2 dx + m_2^2 \int v(t) w_2 dx + \int K_2(x) v(t) w_2 dx \end{array} \right. = a'_2 \int |u(t)|^{p+1} |v(t)|^{q-1} v(t) w_2 dx,
$$

for $(w_1, w_2) \in H^1_{0, K_1} \times H^1_{0, K_2}, x \in \mathbb{R}^n$ and $t \in [0, T)$.

Now we state the local existence theorem of the initial boundary value problem for the equivalent system (2.1).

**Theorem 2.1** Assume that $p$ and $q$ satisfy the conditions (1.3), (1.4) and (1.5). Let $(u_0, v_0) \in H^1_{0, K_1} \times H^1_{0, K_2}$ and $(u_1, v_1) \in L^2 \times L^2$. Then there exists a unique solution $(u(t), v(t), x)$ of the equations (2.1) on a maximal time interval $[0, T_{\text{max}})$ for some $T_{\text{max}} \in (0, \infty)$ such that $(u, v) \in C^0([0, T_{\text{max}}); H^1_{0, K_1}(\mathbb{R}^n)) \times C^0([0, T_{\text{max}}); H^1_{0, K_2}(\mathbb{R}^n))$.

Furthermore, we have the following alternatives:

$$
T_{\text{max}} = \infty; \quad (2.2)
$$

or

$$
T_{\text{max}} \neq \infty \quad \text{and} \quad \lim_{t \to T_{\text{max}}} \left( \alpha \|u(t)\|^2_2 + \|v(t)\|^2_2 \right) = \infty. \quad (2.3)
$$

Moreover, the local solution $(u, v)$ satisfies the following conservation law of energy:

$$
E(t) = E(0) \quad (2.4)
$$

for every $t \in [0, T_{\text{max}})$, where

$$
E(t) = \frac{1}{2} \int |u(t)|^2 + |v(t)|^2 dx + \frac{1}{2} \left[ \int \alpha \left( |\nabla u(t)|^2 + m_1^2 |u(t)|^2 + K_1(x) |u(t)|^2 \right) dx \right. 
$$

$$
+ \left. \int \left( |\nabla v(t)|^2 + m_2^2 |v(t)|^2 + K_2(x) |v(t)|^2 \right) dx \right] - a'_2 \int |u(t)|^{p+1} |v(t)|^{q-1} dx. \quad (2.5)
$$

In order to prove the above theorem, we consider the following scalar equation

$$
\left\{ \begin{array}{l}
w_{tt} - \Delta w + m^2 w + K(x) w = f(t, x) \\
w(0, x) = w_0, w_t(0, x) = w_1(x)
\end{array} \right. \quad (2.6)
$$

where $K(x) \geq 0$ and $m \neq 0$. 
Theorem 2.2 Assume that $f(t,x)$ is a Lipschitz function with respect to $x$. If $(w_0, w_1) \in H^{1}_{0,K}(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, then there exists a unique function $w \in C^1([0,T],H^{1}_{0,K})$ satisfying the equation (2.6) for $T > 0$.

The proof of this theorem follows the argument by Haraux [7], Lions and Magens [13]. We here omit it.

We next give two estimates on the nonlinear term of the system (2.1).

Lemma 2.1 Assume that $p$ and $q$ satisfy the conditions (1.3), (1.4) and (1.5). Then we have the following estimates:

\[
\|v^{q+1}|u|^{p-1}uw\|_1 \leq \|\nabla v\|_r \|v\|_{r_1} \|u\|_{r_2} \|w\|_{r_3},
\]

(2.7)

\[
\|u^{p+1}|v|^{q-1}uw\|_1 \leq \|\nabla u\|_r \|u\|_{r_1} \|v\|_{r_2} \|w\|_{r_3}.
\]

(2.8)

Proof. Firstly we consider the case $n \geq 3$ and $q + 1 > p$. By Hölder inequality and Sobolev inequality we have

\[
\|v^{q+1}|u|^{p-1}uw\|_1 \leq \|\nabla v\|_r \|v\|_{r_1} \|u\|_{r_2} \|w\|_{r_3},
\]

where $r_1$, $r_2$ and $r_3$ satisfy the following conditions:

\[
\frac{p}{r_1} + \frac{q + 1 - p}{r_2} + \frac{1}{r_3} = 1,
\]

\[2 < r_1 < \frac{2n}{n - 2},\]

\[2 < r_2 < \frac{2n}{n - 2},\]

\[2 < r_3 < \frac{2n}{n - 2}.
\]

From the inequalities above on $r_1$, $r_2$ and $r_3$, we get that $q < \frac{4}{n - 2}$.

For the other cases, using the same argument as above we can obtain the desired result.

\[\square\]

Proof of Theorem 2.1. The proof relies on the Banach contraction mapping principle. For $T > 0$ and $R > 0$ we define the following space

\[X_{T,R} \equiv \{(u(t), v(t)); u(t) \in C^0([0,T]; H^{1}_{0,K_1}(\mathbb{R}^n)) \cap L^2([0,T]; H^{1}_{0,K_1}), v(t) \in C^0([0,T]; H^{1}_{0,K_2}(\mathbb{R}^n)) \cap L^2([0,T]; H^{1}_{0,K_2}), (u_t, v_t) \in C^1([0,T]; L^2(\mathbb{R}^n)) \times C^1([0,T]; L^2(\mathbb{R}^n)), e(u, v) \leq R^2 \text{ for all } t \in [0,T], (u(0), v(0)) = (u_0, v_0) \text{ and } (u_t(0), v_t(0)) = (u_1, v_1)\}

where

\[e(u, v) = \max_{t \in [0,T]} \{\alpha(\|u(t)\|_{H^{1}_{0,K_1}}^2 + \|u_t(t)\|_{2}^2) + \|v(t)\|_{H^{1}_{0,K_2}}^2 + \|v_t(t)\|_{2}^2\}.
\]
Thus taking the maximum on \([0,T]\), we define a nonlinear mapping \(S\) in the following way: for any \((u,v) \in X_{T,R}\), \((\bar{u}, \bar{v}) = S(u,v)\) is the unique solution of the following linear wave equation with nonnegative potential

\[
\begin{align*}
\alpha (u_{tt} - \Delta u + m_1^2 \bar{u} + K_1(x) \bar{u}) &= a'_2(p + 1)|v|^{q + 1}|u|^{p - 1}u \\
\bar{v}_{tt} - \Delta \bar{v} + m_2^2 \bar{v} + K_2(x) \bar{v} &= a'_2(q + 1)|u|^{p + 1}|v|^{q - 1}v \\
\bar{u}(0, x) &= u_0, \bar{v}(0, x) = v_0 \\
\bar{u}_t(0, x) &= u_1, \bar{v}_t(0, x) = v_1
\end{align*}
\]

(2.9)

Obviously, by Theorem 2.2 the existence and uniqueness of the solution \((\bar{u}(t), \bar{v}(t))\) can be obtained for \((u, v) \in H_{0,K_1}^1 \times H_{0,K_2}^1\).

We next claim that, for suitable \(R\) and \(T\), \(S\) is a contraction mapping satisfying \(S(X_{T,R}) \subseteq X_{T,R}\). Indeed, in the following part of the proof, we will take \(R^2 = e(u_0, v_0)\). Given \((u, v) \in X_{T,R}\), for every \(t \in (0,T]\) the corresponding solution \((\bar{u}, \bar{v}) = S(u,v)\) satisfies the energy identity:

\[
\begin{align*}
\frac{\alpha}{2} \left( \|\bar{u}_t(t)\|^2 + \|\nabla \bar{u}(t)\|^2 + m_1^2 \|\bar{u}(t)\|^2 + \int K_1(x)|\bar{u}(t)|^2 dx \right) \\
&= \frac{\alpha}{2} \left( \|u_0\|^2_{H_{0,K_1}} + \|u_1\|^2_2 + a'_2(p + 1) \int_0^t \int |v|^{q + 1}|u|^{p - 1}u \bar{u} d\tau, \\
\frac{1}{2} \left( \|\bar{v}_t(t)\|^2 + \|\nabla \bar{v}(t)\|^2 + m_2^2 \|\bar{v}(t)\|^2 + \int K_2(x)|\bar{v}(t)|^2 dx \right) \\
&= \frac{1}{2} \left( \|v_0\|^2_{H_{0,K_2}} + \|v_1\|^2_2 + a'_2(q + 1) \int_0^t \int |u|^{p + 1}|v|^{q - 1}v \bar{v} d\tau. \\
\right)
\end{align*}
\]

For the last terms of the right hand side above, we have by Lemma 2.1

\[
\begin{align*}
\int_0^T \int |v|^{q + 1}|u|^{p - 1}u \bar{u} d\tau &\leq cTR^{2(p + q + 1)} + 2 \int_0^T \|\nabla \bar{u}\|^2 d\tau, \\
\int_0^T \int |u|^{p + 1}|v|^{q - 1}v \bar{v} d\tau &\leq cTR^{2(p + q + 1)} + 2 \int_0^T \|\nabla \bar{v}\|^2 d\tau.
\end{align*}
\]

Thus taking the maximum on \([0,T]\), we have

\[
e(u, v) \leq \frac{1}{2} R^2 + cTR^{2(p + q + 1)}.
\]

Obviously, taking \(T > 0\) sufficient small, we have \(e(\bar{u}, \bar{v}) \leq R^2\), which means \(S(X_{T,R}) \subseteq X_{T,R}\).

We next show that \(S\) is a contraction in \(X_{T,R}\) with the distance \(e(u_1 - u_2, v_1 - v_2)\). Take \((u_1, v_1)\) and \((u_2, v_2)\) from \(X_{T,R}\), and denote the corresponding solution of (2.9) by \((\bar{u}_1, \bar{v}_1)\) and \((\bar{u}_2, \bar{v}_2)\), respectively. Then by mean value theorem we have

\[
\begin{align*}
||v_1|^{q + 1}|u_1|^{p - 1}u_1 - |v_2|^{q + 1}|u_2|^{p - 1}u_2| &\leq |v_1 - v_2| |\omega_1(x,t)| |u_1|^p + |u_1 - u_2| |\omega_2(t, x)||v_2|^{q + 1}, \\
||u_1|^{p + 1}|v_1|^{q - 1}v_1 - |u_2|^{p + 1}|v_2|^{q - 1}v_2| &\leq |u_1 - u_2| |\omega_3(x, t)| |v_1|^q + |v_1 - v_2| |\omega_4(t, x)||u_2|^{p + 1}.
\end{align*}
\]

where

\[
\begin{align*}
\omega_1(t) &\leq (|v_1| + |v_2|)^q, \\
\omega_2(t) &\leq (|u_1| + |u_2|)^{p - 1}, \\
\omega_3(t) &\leq (|u_1| + |u_2|)^p, \\
\omega_4(t) &\leq (|v_1| + |v_2|)^{q - 1}.
\end{align*}
\]
Thus by H"older inequality and Sobolev inequality we have
\[ e(\bar{u}_1 - \bar{u}_2, \bar{v}_1 - \bar{v}_2) \leq c R^{2(p+q)} T e(u_1 - u_2, v_2 - v_2). \]
If we let \( T \) sufficiently small, then \( c R^{2(p+q)} T < 1 \). This implies that \( S \) is a contraction in \( X_{T,R} \).

From the above argument, by applying Banach fixed point theorem, we obtain the existence of a unique solution of the system (2.1) on a maximum time interval.

For the last statement, let the interval \([0, T_{\text{max}}]\) is maximal interval where the solution of (2.1) exists. We assume that \( T_{\text{max}} < \infty \) and \( K = \lim_{t \to T_{\text{max}}} (\alpha \|u(t)\|_2^2 + \|v(t)\|_2^2) < \infty \). Then there exists a sequence \( \{t_j\}_{j=1}^\infty \) such that
\[ t_j \to T_{\text{max}}, \]
\[ \alpha \|u(t_j)\|_2^2 + \|v(t_j)\|_2^2 \leq K. \]
Using the same argument as above with the initial data at \( t_j \), we see that there exists a unique solution of (2.1) on \([t_j, t_j + T_{\text{max}}]\). Thus we can get \( T_{\text{max}} < t_j + T_{\text{max}},j \) for some \( j \) large enough. Obviously this contradicts the definition of \( T_{\text{max}} \). Thus we have completed the proof of the local existence theorem.

\[ \Box \]

3 Standing wave with ground state

If a real function \((\phi, \psi)\) verifies the following system
\[
\begin{cases}
- \alpha \Delta \phi + am_1^2 \phi + \alpha K_1(x) \phi = a_2'(p + 1)|\psi|^{q+1}|\phi|^{p-1}\phi \\
- \Delta \psi + m_2^2 \psi + K_2(x) \psi = a_2'(q + 1)|\phi|^{p+1}|\psi|^{q-1}\psi
\end{cases}
\tag{3.1}
\]
and
\[(\phi, \psi) \in H^1_0 K_1(\mathbb{R}^n) \times H^1_0 K_2(\mathbb{R}^n),\]
then \((u, v) = (\phi, \psi)\) verifies the system (2.1) for \( t \geq 0, x \in \mathbb{R}^n \).

We now define the action \( J(\phi, \psi) \) of the solution \((\phi, \psi)\) of the system (3.1) as follows
\[
J(\phi, \psi) = \frac{1}{2} \int (|\nabla \phi(x)|^2 + |\nabla \psi(x)|^2 + \alpha m_1^2 |\phi(x)|^2 + m_2^2 |\psi(x)|^2)
+ \alpha K_1(x)|\phi(x)|^2 + K_2(x)|\psi(x)|^2 \) dx - a_2'(p + 1)|\psi(x)|^{q+1}dx.
\tag{3.2}
\]
In addition, we let
\[
I(\phi, \psi) = \int (|\nabla \phi(x)|^2 + |\nabla \psi(x)|^2 + \alpha m_1^2 |\phi(x)|^2 + m_2^2 |\psi(x)|^2)
+ \alpha K_1(x)|\phi(x)|^2 + K_2(x)|\psi(x)|^2 \) dx - a_2'(p + q + 2)|\phi(x)|^{p+1}dx.
\tag{3.3}
\]

We now state a proposition, which describes the relation between \( J(\phi, \psi) \) and \( I(\phi, \psi) \).

**Proposition 3.1** Let \((\phi, \psi) \in H^1_0 K_1 \times H^1_0 K_2\) satisfy \((\phi, \psi) \neq 0\) and set \((\phi^\lambda(x), \psi^\lambda(x)) = (\lambda \phi(x), \lambda \psi(x))\) for \( \lambda > 0 \). Then for \( p > 0 \) and \( q > 0 \) there exists a unique \( \lambda_1 \) such that
\[
I(\phi^\lambda, \psi^\lambda) =
\begin{cases}
> 0 \text{ when } \lambda \in (0, \lambda_1), \\
= 0 \text{ when } \lambda = \lambda_1, \\
< 0 \text{ when } \lambda \in (\lambda_1, \infty).
\end{cases}
\tag{3.4}
\]
and
\[ J(\phi^\lambda, \psi^\lambda) \leq J(\phi^{\lambda_1}, \psi^{\lambda_1}) \] (3.5)
for all \( \lambda > 0 \).

**Proof.** By (3.3) and \((\phi^\lambda, \psi^\lambda) = (\lambda \phi, \lambda \psi)\), we easily see that \( I(\phi^\lambda, \psi^\lambda) \) is continuous in \( \lambda \). Thus it is obvious that there exists a unique \( \lambda_1 \) such that \( I(\phi^\lambda, \psi^\lambda) \) satisfies the relation (3.4).

Furthermore, by a direct computation we see that for \( \lambda > 0 \)
\[ \lambda \frac{d}{d\lambda} J(\phi^\lambda, \psi^\lambda) = I(\phi^\lambda, \psi^\lambda), \]
which implies the property (3.5) by (3.4).
\[ \square \]

Next define the set \( M \) by
\[ M = \{(\phi, \psi) \in H_1(\mathbb{R}^n) \times H_2(\mathbb{R}^n); I(\phi, \psi) = 0, (\phi, \psi) \neq 0\}. \] (3.6)

We then set the following constrained variational problem
\[ d = \min_{(\phi, \psi) \in M} J(\phi, \psi). \]

Now we are in a position to state the theorem about the ground state of (3.1).

**Theorem 3.1** Assume that \( 1 < p, q < \infty \) when \( n = 2 \) and \( 1 < p, q < \frac{2}{n-2} \) when \( n = 3 \). Then there exists \((\Phi, \Psi) \in M\) such that

1. \( J(\Phi, \Psi) = \inf_{(\phi, \psi) \in M} J(\phi, \psi) = d; \)
2. \((\Phi, \Psi)\) is a ground state solution of (3.1).

To prove the above theorem we first introduce a compactness lemma, whose proof can be found in [20], [27].

**Lemma 3.1** Let \( 1 \leq r < \frac{n + 2}{n - 2} \) when \( n \geq 3 \) and \( 1 \leq r < \infty \) when \( n = 1, 2 \). Then for \( K(x) \geq 0 \) \( (\forall x \in \mathbb{R}^n) \) the embedding \( H_{1,K} \hookrightarrow L^{r+1} \) is compact.

**Proof of Theorem 3.1.** Firstly we claim that \( J(\phi, \psi) \) is bounded below on \( M \). Indeed, By (3.2) and (3.6) we see that
\[ J(\phi, \psi) = \frac{p+q}{2(p+q+2)} \int \left( \alpha |\nabla \phi|^2 + |\nabla \psi|^2 + \alpha m_1^2 |\phi|^2 + m_2^2 |\psi|^2 + \alpha K_1 |\phi|^2 + K_2 |\psi|^2 \right) dx \geq 0. \] (3.7)

Then there exists a minimizing sequence \( \{(\phi_j, \psi_j)\}_{j=1}^\infty \) satisfying
\[ I(\phi_j, \psi_j) = 0 \text{ for every } j \in \mathbb{N}, \] (3.8)
\[ J(\phi_j, \psi_j) \to d \text{ as } j \to \infty. \] (3.9)

By (3.7) and (3.9) there exists a subsequence of \( \{(\phi_j, \psi_j)\}_{j=1}^\infty \), which we still denote by \( \{(\phi_j, \psi_j)\}_{j=1}^\infty \), such that
\[ (\phi_j, \psi_j) \rightharpoonup (\phi_\infty, \psi_\infty) \text{ weakly in } H_{0,K_1}^1 \times H_{0,K_2}^1. \]
By Lemma 3.1 we have
\[
(\phi_j, \psi_j) \to (\phi_\infty, \psi_\infty) \text{ strongly in } L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n); \tag{3.10}
\]
\[
(\phi_j, \psi_j) \to (\phi_\infty, \psi_\infty) \text{ strongly in } L^{2(p+1)}(\mathbb{R}^n) \times L^{2(q+1)}(\mathbb{R}^n). \tag{3.11}
\]

Next by a contradiction argument we prove that
\[
(\phi_\infty, \psi_\infty) \neq 0. \tag{3.12}
\]

We assume that \((\phi_\infty, \psi_\infty) \equiv 0\). Then by (3.10) and (3.11) we obtain
\[
(\phi_j, \psi_j) \to 0 \text{ in } L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2); \tag{3.13}
\]
\[
(\phi_j, \psi_j) \to 0 \text{ in } L^{2(p+1)}(\mathbb{R}^2) \times L^{2(q+1)}(\mathbb{R}^2). \tag{3.14}
\]

Noting the fact \((\phi_j, \psi_j) \in M\), we see that \(I(\phi_j, \psi_j) = 0\) implies that
\[
\int (\alpha(|\nabla \phi_j|^2 + m_1^2|\phi_j|^2 + K_1(x)|\phi_j|^2) + |\nabla \psi_j|^2 + m_2^2|\psi_j|^2 + K_2(x)|\psi_j|^2) \, dx \to 0,
\]
which means
\[
\int (\alpha(|\nabla \phi_j|^2 + K_1(x)|\phi_j|^2) + |\nabla \psi_j|^2 + K_2(x)|\psi_j|^2) \, dx \to 0 \tag{3.15}
\]
as \(j \to \infty\).

On the other hand, from (3.2) and (3.9) we see
\[
\int (\alpha(|\nabla \phi_j| + K_1(x)|\phi_j|^2) + |\nabla \psi_j| + K_2(x)|\psi_j|^2) \, dx \to \frac{2(p + q + 2)}{p + q} \tag{3.16}
\]
as \(j \to \infty\).

Obviously if we can show \(d > 0\), then there exists a contradiction between (3.15) and (3.16), which implies that it is impossible that \((\phi_\infty, \psi_\infty) \equiv 0\). We next show it.

Indeed, by (3.2), (3.3) and (3.6) we have
\[
J(\phi_j, \psi_j) = \frac{p + q}{2(p + q + 2)} \int (\alpha(|\nabla \phi_j| + m_1^2|\phi_j|^2 + K_1(x)|\phi_j|^2) + |\nabla \psi_j| + m_2^2|\psi_j|^2 + K_2(x)|\psi_j|^2) \, dx. \tag{3.17}
\]

Using Hölder inequality, the Sobolev’s embedding theorem and \(I(\phi_j, \psi_j) = 0\) we obtain
\[
\int |\phi_j|^{p+1}|\psi_j|^{q+1} \, dx \leq \left( \int |\phi_j|^{2(p+1)} \, dx \right)^{1/2} \left( \int |\psi_j|^{2(q+1)} \, dx \right)^{1/2}
\leq c \left( \alpha \int (|\nabla \phi_j|^2 + m_1^2|\phi_j|^2) \, dx \right)^{(p+1)/2} \left( \int (|\nabla \psi_j|^2 + m_2^2|\psi_j|^2) \, dx \right)^{(q+1)/2}
\leq c \left( \int (\alpha(|\nabla \phi_j|^2 + m_1^2|\phi_j|^2) + |\nabla \psi_j|^2 + m_2^2|\psi_j|^2) \, dx \right)^{(p+q+2)/2}
\]
for some constant \(c > 0\), and
\[
a'_2(p + q + 2) \int |\phi_j|^{p+1}|\psi_j|^{q+1} \, dx
\leq \int (\alpha(|\nabla \phi_j|^2 + m_1^2|\phi_j|^2 + K_1(x)|\phi_j|^2) + |\nabla \psi_j|^2 + m_2^2|\psi_j|^2 + K_2(x)|\psi_j|^2) \, dx
\]
9
for every \( j \in \mathbb{N} \).

Thus we see that

\[
\int (\alpha(|\nabla \phi_j|^2 + m_1^2|\phi_j|^2 + K_1(x)|\phi_j|^2) + |\nabla \psi_j|^2 + m_2^2|\psi_j|^2 + K_2(x)|\psi_j|^2)dx
\leq c' \left( \int (|\nabla \phi_j|^2 + m_1^2|\phi_j|^2) + |\nabla \psi_j|^2 + m_2^2|\psi_j|^2)dx \right)^{(p+q+2)/2}
\leq \left( \int (|\nabla \phi_j|^2 + m_1^2|\phi_j|^2 + K_1|\phi_j|^2) + |\nabla \psi_j|^2 + m_2^2|\psi_j|^2 + K_2|\psi_j|^2)dx \right)^{(p+q+2)/2},
\]

for some constant \( c' > 0 \), which implies that for some constant \( c'' > 0 \)

\[
\int (\alpha(|\nabla \phi_j|^2 + m_1^2|\phi_j|^2 + K_1|\phi_j|^2) + |\nabla \psi_j|^2 + m_2^2|\psi_j|^2 + K_2|\psi_j|^2)dx \geq c'' > 0
\]

for every \( j \in \mathbb{N} \).

Therefore from (3.17) it follows that

\[ d = \lim_{j \to \infty} J(\phi_j, \psi_j) > 0. \]

Thus we have completed the proof of \( (\varphi, \psi) \neq 0 \).

By Proposition 3.1 there exists a unique \( \lambda_0 \in (0, \infty) \) such that

\[ I(\varphi_{\lambda_0}^0, \psi_{\lambda_0}^0) = 0, \]

where

\[ (\varphi^\lambda(x), \psi^\lambda(x)) = (\lambda \phi(x), \lambda \psi(x)). \]

Then it follows from Proposition 3.1 and \( I(\phi_j, \psi_j) = 0 \) that

\[ J(\phi_{\lambda_0}^0, \psi_{\lambda_0}^0) \leq J(\phi_j, \psi_j) \to d \]
as \( j \to \infty \).

Noting the fact \( I(\varphi_{\lambda_0}^0, \psi_{\lambda_0}^0) = 0 \), we have \( I(\Phi, \Psi) = 0 \) and \( J(\Phi, \Psi) = d \) with \( \Phi = \varphi_{\lambda_0}^0 \) and \( \Psi = \psi_{\lambda_0}^0 \).

Since \((\Phi, \Psi)\) is a solution of Lagrange-Euler equation (3.1), there exists a Lagrange multiplier \( \Theta \) such that

\[
\begin{align*}
J_0(\Phi, \Psi) + \Theta J_0(\Phi, \Psi) &= 0, \quad (3.18) \\
J_0(\Phi, \Psi) + \Theta J_0(\Phi, \Psi) &= 0, \quad (3.19)
\end{align*}
\]

and

\[
\begin{align*}
\langle J_0(\Phi, \Psi) + \Theta J_0(\Phi, \Psi), \Phi \rangle &= 0, \quad (3.20) \\
\langle J_0(\Phi, \Psi) + \Theta J_0(\Phi, \Psi), \Psi \rangle &= 0. \quad (3.21)
\end{align*}
\]

Since \( I(\Phi, \Psi) = 0 \), it follows from (3.18) - (3.21) that

\[ \Theta \int |\Phi|^q + |\Psi|^{q+1}dx = 0 \]

which implies \( \Theta \equiv 0 \). Therefore \((\Phi, \Psi)\) solves the equation (3.1) in \( H^1_{0, K_1}(\mathbb{R}^n) \times H^1_{0, K_2}(\mathbb{R}^n) \). This completes the proof of Theorem 3.1. \( \square \)

**Remark 3.2** The standing wave of the system (2.1) is the form of \( (\exp(i \omega_1 t) \phi_{\omega_1}(x), \exp(i \omega_2 t) \psi_{\omega_1}(x)) \), where \((\phi_{\omega_1}, \psi_{\omega_1})\) is the ground state solution of the corresponding Lagrange-Euler equations. Indeed, in the above argument we just consider the case \( (\omega_1, \omega_2) = 0 \). Recently, for the single Klein-Gordon equation (4.7) with \( K(x) \equiv 0 \), the strong instability of the standing waves \( \exp(i \omega t) \varphi_{\omega}(x) \), where \( \varphi_{\omega}(x) \) is a ground state solution for the corresponding single Lagrange-Euler equation, has been considered in [14], [15] with \( |\omega| < 1 \). But for \( (\omega_1, \omega_2) \neq 0 \), the strong instability of the standing waves of the system (1.2) is unknown.

10
4 Blow up with low initial energy when $n = 2, 3$

In the section, based on the result obtained in Section 3 we prove the blow up result by potential well argument and concavity method, which leads to the instability of standing waves of the system (2.1).

Firstly define two sets $\Gamma_1$ and $\Gamma_2$ as

$$\Gamma_1 = \{(u, v) \in H_1 \times H_2; J(u, v) < d, I(u, v) < 0\},$$
$$\Gamma_2 = \{(u, v) \in H_1 \times H_2; J(u, v) < d, I(u, v) > 0\},$$

where $d$ is defined in theorem 3.1.

**Lemma 4.1** Assume $1 < p, q < \infty$ when $n = 2$ and $1 < p, q < \frac{2}{n-2}$ when $n = 3$. Let the initial energy satisfying $E(0) < d$. Then the set $\Gamma_1$ is invariant under the flow generated by (2.1) in the sense that: If nonzero $(u_0, v_0) \in \Gamma_1$, then the unique solution $(u(t), v(t))$ of the equations (2.1), $0 \leq t < T_{\text{max}}$, with the initial data $(u_0, v_0)$ satisfies

$$(u(t), v(t)) \in \Gamma_1, \text{ for } t \in [0, T_{\text{max}}),$$

where $T_{\text{max}} > 0$ is the maximum existing time of the solution $(u(t), v(t))$.

**Proof.** It is observed by the conservation law of energy (2.1) that

$$J(u(t), v(t)) \leq E(t) = E(0) < d$$

for $0 \leq t < T_{\text{max}}$.

To prove $(u(t), v(t)) \in \Gamma_1$, there is only one thing left to be checked: $I(u(t), v(t)) < 0$ for $0 \leq t < T_{\text{max}}$. In the following we show it by a contradiction argument.

We assume that it is wrong that $I(u(t), v(t)) < 0$ for $0 \leq t < T_{\text{max}}$. Then by continuity, we see there exists a time $T > 0$ such that

$$T = \min \{0 < t < T_{\text{max}}; I(u(t), v(t)) = 0\}.$$ 

It is natural that $I(u(T), v(T)) = 0$ by the continuity of $I(u(t), v(t))$ in $t$. Then $(u(T), v(T)) \in M$. By Theorem 3.1 we see that it is impossible that $J(u(T), v(T)) < d$ and $(u(T), v(T)) \in M$. Thus, we have obtained that $I(u(t), v(t)) < 0$ for $0 \leq t < T_{\text{max}}$. So $\Gamma_1$ is invariant under the flow generated by (2.1).

$\square$

**Theorem 4.1** Assume that $1 < p, q < \infty$ when $n = 2$ and $1 < p, q < \frac{2}{n-2}$ when $n = 3$. If the initial datum $(u_0, v_0)$ and $(u_1, v_1)$ satisfy $E(0) < d$ and $I(u_0, v_0) < 0$, then the solution $(u(t), v(t))$ of the Cauchy problem (2.1) blows up in a finite time, that is,

$$\lim_{t \to T_{\text{max}}} (\|u(t)\|^2 + \|v(t)\|^2) = \infty.$$ 

**Proof.** By Lemma 4.1 we see that

$$I(u(t), v(t)) < 0$$

for every $t \in [0, T_{\text{max}})$. 

11
We first define the following auxiliary function
\[ G(t) = \int (\alpha|u(t,x)|^2 + |v(t,x)|^2)dx + b(t + T_1)^2. \]  
(4.2)
where \( b \) and \( T_1 \) is two positive parameters, which will be determined later.
By simple calculation we see that
\[
G'(t) = \frac{d}{dt} G(t)
= 2 \int (\alpha u(t, x) u_t(t, x) + v(t, x) v_t(t, x))dx + 2b(t + T_1),
\]
(4.3)
and
\[
\frac{1}{2} G''(t) = \int (\alpha|u_t(t,x)|^2 + |v_t(t,x)|^2)dx + \int \alpha u(t, x) u_{tt}(t, x)dx + \int v(t, x) v_{tt}(t, x)dx + b
= \int (\alpha|u_t(t,x)|^2 + |v_t(t,x)|^2)dx + \int \alpha u(t, x) (\Delta u - m_1^2 u - K_1(x)u + a_1|v|^{q+1}|u|^{p-1}u)dx
+ \int v(t, x) (\Delta v - m_2^2 v - K_2(x)v + a_2|u|^{q+1}|v|^{q-1}v)dx + b
= \int (\alpha|u_t(t,x)|^2 + |v_t(t,x)|^2)dx + a'_2(p + q + 2) \int |v|^{q+1}|u|^{p+1}dx
- \alpha \int (|\nabla u|^2 + m_1^2 |u|^2 + K_1(x)|u|^2)dx - \int (|\nabla v|^2 + m_2^2 |v|^2 + K_2(x)|v|^2)dx + b
\]
(4.4)
Thus it is obvious by (4.1) that
\[
G''(t) > 0
\]
(4.5)
on \([0, T_{\text{max}}])\), which implies that the function \( G(t) \) is convex in \( t \).
From (2.4), (2.5) and (4.4) it follows that
\[
G''(t) = (p + q + 4) \int (\alpha|u_t|^2 + |v_t|^2)dx + (p + q) \left[ \alpha \int (|\nabla u|^2 + m_1^2 |u|^2 + K_1(x)|u|^2)dx
+ \int (|\nabla v|^2 + m_2^2 |v|^2 + K_2(x)|v|^2)dx \right] - 2(p + q + 2) E(t) + 2b
\geq (p + q + 4) \int (\alpha|u_t|^2 + |v_t|^2)dx + (p + q) \min\{m_1, m_2\} G(t) - 2(p + q + 2) E(t) + 2b.
\]
We now choose a sufficiently large time \( T_1 \) and suitable \( b \) satisfying
\[
G'(0) > 0
\]
(4.6)
and
\[
(p + q) \min\{m_1, m_2\} G(0) - 2(p + q + 2) E(0) > (p + q + 2)b.
\]
(4.7)
Thus we see that \( G(t) \) is strictly increasing on \([0, T_{\text{max}}])\) by (4.5) and (4.6). So we have
\[
(p + q) \min\{m_1, m_2\} G(t) - 2(p + q + 2) E(0) > (p + q + 2)b
\]
Theorem 4.2
Assume that for any \( \epsilon > 0 \) with the property that for every \( t \)
and the corresponding solution of the system (1.2) blows up in a finite time with the initial data \( \delta \).

Next we will consider the instability of standing waves of the equation (2.1). Before we give the

Definition 4.1 The standing wave \((u(t, x), v(t, x)) = (\phi(x), \psi(x))\) of the system (2.1) is instable by
blow up if for any \( \epsilon > 0 \) there exists \( (u_0, v_0) \in H^1_{0,K_1} \times H^1_{0,K_2} \) such that \( \|(u_0, v_0) - (\phi, \psi)\|_{H^1_{0,K_1} \times H^1_{0,K_2}} < \delta \) and the corresponding solution of the system (1.2) blows up in a finite time with the initial data

\[
\begin{align*}
  u(0, x) &= u_0, u_t(0, x) = 0, \\
  v(0, x) &= v_0, v_t(0, x) = 0.
\end{align*}
\]

Now we are in a position to state the instability theorem.

Theorem 4.2 Assume that 1 < \( p, q < \infty \) when \( n = 2 \) and 1 < \( p, q < \frac{2}{n-2} \) when \( n = 3 \). Let \((\phi, \psi)\) be a ground state solution of (3.1). Then for any \( \epsilon > 0 \) there exists \( T_{\text{max}} < \infty \) and \( (u_0, v_0) \in H^1_1 \times H^1_2 \) with the property

\[
\begin{align*}
  &\|u_0 - \phi\|_{H^1_{0,K_1}} < \epsilon, \\
  &\|v_0 - \psi\|_{H^1_{0,K_2}} < \epsilon.
\end{align*}
\]
such that the solution \((u,v)\) of the system (1.2) blows up in a finite time \(T_{\text{max}}\) with the initial data
\[
\begin{align*}
  u(0,x) &= u_0, u_t(0,x) = 0 \\
  v(0,x) &= v_0, v_t(0,x) = 0
\end{align*}
\] (4.12)

**Proof.** By (2.5) and (4.12) we easily see that
\[
E(0) = J(u_0, v_0).
\] (4.13)

We next let
\[
u_0(x) = \gamma \phi(x), v_0(x) = \gamma \psi(x)
\] (4.14)
where \(\gamma > 1\).

Obviously, for any \(\epsilon > 0\) we can take a suitable \(\gamma > 1\) such that
\[
\|u_0 - \phi\|_{H^1_0} = (\gamma - 1)\|\phi\|_{H^1_0} < \epsilon,
\]
\[
\|v_0 - \psi\|_{H^1_0} = (\gamma - 1)\|\psi\|_{H^1_0} < \epsilon.
\]
Noting the fact \(\gamma > 1\), we see by (4.14) and Proposition 3.1 that
\[
I(u_0, v_0) < I(\phi, \psi) = 0,
\]
\[
J(u_0, v_0) < J(\phi, \psi) = d
\]
and by (4.13) we have \(E(0) < d\).

Thus by Theorem 3.1 we have completed the proof of Theorem 3.2.

\(\square\)

As a byproduct we have the following global existence theorem for the system (1.2).

**Theorem 4.3** Assume that \(1 < p,q < \infty\) when \(n = 2\) and \(1 < p,q < \frac{2}{n-2}\) when \(n = 3\). If the initial data satisfy that \(E(0) < d\) and \(I(u_0, v_0) > 0\) then the corresponding solution \((u(t,x), v(t,x))\) of the system (2.1) exists globally.

**Proof.** As in the proof of Lemma 3.1 we can prove that \(\Gamma_2\) is invariant under the flow generated by the system (2.1). Thus we see that
\[
J(u(t), v(t)) < d,
\]
\[
I(u(t), v(t)) > 0
\]
for each \(t \in [0, T_{\text{max}}]\).

Thus by (3.2) we get
\[
J(u(t), v(t)) > \frac{p+q}{2(p+q+2)} \left[ \int \alpha (|\nabla u(t)|^2 + m_1^2 |u(t)| + K_1(x)|u(t)|^2) dx \\
+ \int (|\nabla v(t)|^2 + m_2^2 |v(t)| + K_2(x)|v(t)|^2) dx \right]
\]
\[
\geq 0.
\] (4.17)

From (4.15) and (4.17) it follows that
\[
\int (\alpha |u_t(t)|^2 + |v_t(t)|^2) dx < \frac{2(p+q+2)}{p+q} d
\]
for each \( t \in [0, T_{\text{max}}) \).

By (4.11), (4.12) and (4.13), we see that
\[
\int (\alpha |\nabla u|^2 + |\nabla v|) dx < \frac{2(p + q + 2)}{p + q}d,
\]
\[
\int (\alpha m_1^2 |u|^2 + m_2^2 |v|^2) dx < \frac{2(p + q + 2)}{p + q}d,
\]
\[
\int (\alpha K_1(x)|u|^2 + K_2|v|^2) dx < \frac{2(p + q + 2)}{p + q}d
\]
for each \( t \in [0, T_{\text{max}}) \).

Thus we see that the solution is uniformly bounded on \([0, T_{\text{max}})\). The proof of the theorem is finished.

\[\square\]

5 Blow up with non-positive initial energy

Because of the embedding theorem (Lemma 3.1) we cannot claim the blowing up result as in Section 4 for other case. So in this section we will prove the blow up result for the system (2.1) by the concavity method when the initial energy is not positive. We state our theorem.

**Theorem 5.1** Assume that \( p \) and \( q \) satisfy the conditions (1.3), (1.4) and (1.5). If the nonzero datum \((u_0, v_0) \in H_{0,K_1}^1 \times H_{0,K_2}^1\) and \((u_1,v_1) \in L^2 \times L^2\) satisfy
\[E(0) < 0,\]
or
\[\int \alpha u_0 u_1 + v_0 v_1 \geq 0 \text{ when } E(0) = 0.\]
then the corresponding local solution of the system (2.1) blows up in a finite time \(T_{\text{max}} < \infty\), that is,
\[\lim_{t \to T_{\text{max}}} (\alpha \|u(t)\|^2 + \|v(t)\|^2) = \infty.\]

**Proof.** We first consider the case \( E(0) < 0 \). The auxiliary function \( G(t) \) in (4.2) will still be used here. Naturally by (2.3), (2.5) and (4.3) we see that
\[
G''(t) = (p + q + 4) \int (\alpha |u_i|^2 + |v_i|^2) dx + (p + q) \left[ \alpha \int (|\nabla u_i|^2 + m_1^2|u_i|^2 + K_1(x)|u_i|^2) dx \\
+ \int (|\nabla v_i|^2 + m_2^2|v_i|^2 + K_2(x)|v_i|^2) dx \right] - 2(p + q + 2)E(t) + 2b.
\]

Since \( E(0) < 0 \), we now let the constant \( b \) satisfy
\[0 < b \leq -2E(0).\]
Then it follows that
\[-2(p + q + 2)E(t) + 2b \geq (p + q + 4)b,\]
which implies that
\[ G''(t) \geq (p + q + 4) \left[ \int (\alpha |u|^2 + |v|^2)dx + b \right]. \]

Obviously
\[ G''(t) > 0 \] (5.1)
on \([0, T_{\text{max}}]).\)

Moreover we can take a sufficiently large \(T_1 > 0\) and a suitable \(b\) such that
\[ G'(0) = 2 \int (\alpha u_t u + v_t v)dx + bT_1 > 0 \] (5.2)

Thus by (5.1) and (5.2) we obtain that \(G(t) > 0\) and \(G'(t) > 0\) for every \(t \in [0, T_{\text{max}}]\). That is, \(G(t)\) and \(G'(t)\) are strictly increasing on \([0, T_{\text{max}}]).\) Then as in (4.9) we see that
\[ G''(t)G(t) - \frac{p + q + 4}{4}(G'(t)) \geq 0 \]

Thus we have
\[ \frac{d}{dt}G^{-\frac{p+q}{4}}(t) = -\frac{p + q}{4} G^{-\frac{p+q+4}{4}}(t)G'(t) < 0, \] (5.3)

\[ \frac{d^2}{dt^2}G^{-\frac{p+q}{4}}(t) = -\frac{p + q}{4} G^{-\frac{p+q+8}{4}} \left( G''(t)G(t) - \frac{p + q + 4}{4}(G'(t)) \right) \leq 0 \] (5.4)

for every \(t \in [0, T_{\text{max}}]\), which implies that \(G^{-\frac{p+q}{4}}(t)\) is concave on \([0, T_{\text{max}}]).\) From (5.3) and (5.4) it follows that the function \(G^{-\frac{p+q}{4}} \to 0\) when \(t < T_{\text{max}}\) and \(t \to T_{\text{max}}\) \(T_{\text{max}} \leq \frac{4G(0)}{(p + q)G'(0)}.\)

Thus we see that there exists a finite time \(T_{\text{max}} > 0\) such that
\[ \lim_{t \to T_{\text{max}}} (\alpha \|u(t)\|_2^2 + \|v(t)\|_2^2) = \infty. \]

We next deal with the case \(E(0) = 0\) with \(\int (\alpha u_0u_1 + v_0v_1)dx \geq 0.\) Here we define
\[ G(t) = \int (\alpha |u(t)|^2 + |v(t)|^2)dx. \]

By direct calculation we have
\[ G'(t) = 2 \int (\alpha u_t u + v_t v)dx \] (5.5)

and
\[ G''(t) = 2 \int (\alpha |u_t|^2 + |v_t|^2)dx - I(u(t), v(t)). \] (5.6)
By (2.4) and (2.5) we see that
\[
\int (\alpha(|\nabla u|^2 + m_1^2|u|^2 + K_1(x)|u|^2) + (|\nabla v|^2 + m_2^2|v|^2 + K_2(x)|v|^2))\,dx
\]
for every \( t \in [0, T_{\max}) \). Thus we easily have
\[
I(u(t, x), v(x)) = \int (\alpha(|\nabla u|^2 + m_1^2|u|^2 + K_1(x)|u|^2) + (|\nabla v|^2 + m_2^2|v|^2 + K_2(x)|v|^2))\,dx
\]
for every \( t \in [0, T_{\max}) \).

By (5.6) we then see that
\[
G''(t) > 0 \quad (5.7)
\]
on \([0, T_{\max})\). And noting the fact \( \int (\alpha u_t u + v_t v)\,dx \geq 0 \), we have
\[
G'(t) > 0 \quad (5.8)
\]
for every \( t \in (0, T_{\max}) \).

Thus, by (5.7) and (5.8) we see that \( G(t) \) and \( G'(t) \) are strictly increasing on \([0, T_{\max})\).

Moreover,
\[
G''(t) = (p + q + 4) \int (\alpha |u_t|^2 + |v_t|^2)\,dx + (p + q) \left[ \alpha \int (|\nabla u|^2 + m_1^2|u|^2 + K_1(x)|u|^2)\,dx \right.
\]
\[
+ \left. \int (|\nabla v|^2 + m_2^2|v|^2 + K_2(x)|v|^2)\,dx \right] - 2(p + q + 2)E(0) \]
Noting here \( E(0) = 0 \), we then have
\[
G''(t) \geq (p + q + 4) \int (\alpha |u_t|^2 + |v_t|^2)\,dx.
\]
Since \( G(t) > 0 \) for every \( t \in [0, T_{\max}) \), we obtain by Cauchy-Schwartz inequality
\[
G''(t)G(t) - \frac{p + q + 4}{4}(G'(t))^2 \geq 0
\]
for every \( t \in [0, T_{\max}) \).

Then by a concavity argument as in Theorem 4.1, we can claim that there exists a finite time \( T_{\max} \) such that
\[
\lim_{t \to T_{\max}} (\alpha \|u(t)\|^2 + \|v(t)\|^2) = \infty.
\]

\[\square\]

Remark 5.2 For \( 1 < p, q < \infty \) when \( n = 2 \) and for \( 1 < p, q < \frac{2}{n-2} \) when \( n = 3 \), Theorem 5.1 reproduces the blowing up result in Section 4. But because of the restriction of the embedding theorem (Lemma 3.1), we cannot apply the method of Section 4 to get the blow up result in this section.
6 Blow up with arbitrarily positive initial energy

To the best of our acknowledgement, there is no result for a system of Klein-Gordon equations when the initial energy is given arbitrarily positive. In the section we will prove a blow up result for the system (1.2) with arbitrarily positive initial energy. Indeed, we give the sufficient conditions for the initial datum with positive initial energy such that the corresponding solution blows up in a finite time.

Theorem 6.1 Assume that $p$ and $q$ satisfy the conditions (1.3), (1.4) and (1.5). If the initial data $(u_0, v_0) \in H_0^{1,K_1} \times H_0^{1,K_2}$ and $(u_1, v_1) \in L^2 \times L^2$ satisfy

\begin{align*}
E(0) &> 0; \quad (6.1) \\
I(u_0, v_0) &< 0; \quad (6.2) \\
\int (\alpha u_0 u_1 + v_0 v_1) dx &\geq 0; \quad (6.3) \\
\alpha \|u_0\|^2 + \|v_0\|^2 &> \frac{2(p+q+2)}{\min\{m_1^2, m_2^2\}} (p+q) E(0). \quad (6.4)
\end{align*}

Then the corresponding solution $(u(t), v(t))$ of the system (2.1) blows up in a finite time $T_{\text{max}} < \infty$, that is,

$$\lim_{t \to T_{\text{max}}} (\alpha \|u(t)\|^2 + \|v(t)\|^2) = \infty.$$ 

Proof. We will prove the result in two steps.

Firstly we show that

$$I(u(t), v(t)) < 0 \quad (6.5)$$

and

$$\alpha \|u(t)\|^2 + \|v(t)\|^2 > \frac{2(p+q+2)}{\min\{m_1^2, m_2^2\}} (p+q) E(0) \quad (6.6)$$

for every $t \in [0, T_{\text{max}})$.

We prove (6.5) by a contradiction argument. Assume that (6.5) is wrong at some $t \in (0, T_{\text{max}})$, that is to say, there exists $T > 0$ such that

$$T = \min\{t \in [0, T_{\text{max}}); I(u(t), v(t)) \geq 0\}. \quad (6.7)$$

Then by the continuity of $I(u(t), v(t))$ in $t$ we see that

$$I(u(T), v(T)) = 0. \quad (6.8)$$

Now letting

$$G(t) = \int (\alpha |u(t,x)|^2 + |v(t,x)|^2) dx,$$

we have

$$G'(t) = 2 \int (\alpha u_t u + v_t v) dx \quad (6.9)$$
and
\[ G''(t) = 2 \int (\alpha |u_t|^2 + |v_t|^2) dx - I(u(t), v(t)). \] (6.10)

Noting the definition (6.7) we see that
\[ I(u(t), v(t)) < 0 \] (6.11)
for every \( t \in [0, T) \). Thus it follows that \( G''(t) > 0 \) on \([0, T)\). And by (6.3) we have \( G'(t) > 0 \) for \( t \in (0, T) \). In other words, \( G(t) \) and \( G'(t) \) are strictly increasing on \([0, T)\). So by (6.3)
\[ G(t) > \frac{2(p + q + 2)}{\min\{m_1^2, m_2^2\}} E(0). \] (6.12)
for every \( t \in [0, T) \).

Furthermore, since \( u(t) \) and \( v(t) \) are continuous in \( t \) we get by (6.12)
\[ G(T) > \frac{2(p + q + 2)}{\min\{m_1^2, m_2^2\}} E(0). \] (6.13)

On the other hand, by (6.4) and (6.5) we have
\[ \int (\alpha (|\nabla u|^2 + m_1^2 |u|^2 + K_1 |u|^2) + (|\nabla v|^2 + m_2^2 |v|^2 + K_2 |v|^2)) dx - 2a' \int |u|^{p+1} |v|^{q+1} dx \leq 2E(0). \]

By (6.8) we then have
\[ G(T) = \int (\alpha |u(T)|^2 + |v(T)|^2) dx \]
\[ \leq \frac{2(p + q + 2)}{\min\{m_1^2, m_2^2\}} E(0). \] (6.14)

Obviously there is a contradiction between (6.13) and (6.14). Thus we have proved that
\[ I(u(t), v(t)) < 0 \] (6.15)
for every \( t \in [0, T_{\text{max}}) \).

By the argument above we see that \( G(t) \) is strictly increasing on \([0, T_{\text{max}})\) if \( I(u(t), v(t)) < 0 \) for every \( t \in [0, T_{\text{max}}) \) and (6.3) holds. Namely (6.15) implies that
\[ G(t) > \frac{2(p + q + 2)}{\min\{m_1^2, m_2^2\}} E(0). \] (6.16)
for every \( t \in [0, T_{\text{max}}) \).

Now we are going to show the blow up result. By a simple computation we have
\[ G''(t) = (p + q + 4) \int (\alpha |u_t|^2 + |v_t|^2) dx + (p + q) \left[ \alpha \int (|\nabla u|^2 + m_1^2 |u|^2 + K_1(x) |u|^2) dx + \int (|\nabla v|^2 + m_2^2 |v|^2 + K_2(x) |v|^2) dx \right] - 2(p + q + 2) E(0) \]
\[ \geq (p + q + 4) \int (\alpha |u_t|^2 + |v_t|^2) dx + (p + q) \min\{m_1^2, m_2^2\} \int (|\nabla u|^2 + |v|^2) dx - 2(p + q + 2) E(0) \]
\[ \geq (p + q + 4) \int (\alpha |u_t|^2 + |v_t|^2) dx. \]
for every $t \in [0, T_{\text{max}})$.

Thus, by Cauchy-Schwartz inequality we get

$$G''(t)G(t) - \frac{p + q + 4}{4}(G'(t))^2 = (p + q + 4)\int (\alpha|u|^2 + |v|^2)dx \int (\alpha|u|^2 + |v|^2)dx$$

$$- \int (\alpha u_t u + v_t v)dx \geq 0.$$ 

So we have

$$\frac{d}{dt}G^{-(p+q)/4}(t) = -\frac{p + q}{4}G^{-(p+q+4)/4}(t)G'(t) < 0 \quad (6.17)$$

$$\frac{d^2}{dt^2}G^{-(p+q)/4}(t) \leq 0$$

for every $t \in [0, T_{\text{max}})$, which implies that $G^{-(p+q)/4}(t)$ is concave on $[0, T_{\text{max}})$. From (6.17) and (6.18) it follows that the function $G^{-(p+q)/4} \to 0$ when $t < T_{\text{max}}$ and $t \to T_{\text{max}}$ ($T_{\text{max}} \leq 4G(0)/(p + q)G'(0)$). Thus we see that there exists a finite time $T_{\text{max}} > 0$ such that

$$\lim_{t \to T^-} (\alpha\|u(t)\|^2_2 + \|v(t)\|^2_2) = \infty.$$ 

(6.19)

\[\square\]

Acknowledgments. The author wishes to express his deep gratitude to Prof. Hitoshi Kitada for his constant encouragement and careful reading the manuscript. The study is supported by Japanese Government Scholarship.

References

[1] H. Berestycki and T. Cazenave, Instabilité des états stationnaires dans les équations de Schrödinger et de Klein-Gordon non linéaires. C. R. Acad. Sci. Paris 293 (1981), pp. 489-492.

[2] H. Berestycki, T. Gallouët and O. Kavian, équations de champs scalaires euclidiens nonlinéaires dans le plan. C. R. Acad. Sci. Paris. Serie I 297 (1983), pp. 307-310.

[3] J.M. Delort, D. Fang and R. Xue, Global existence of small solutions for quadratic quasilinear Klein-Gordon systems in two space dimensions $\mathbb{R}^2$, Journal of Functional Analysis, 211(2004) 288-323.

[4] Z. Gan and J. Zhang, Standing waves for nonlinear Klein-Gordon equations with nonnegative potentials. Appl. Math. Comput. 166 (2005), 551–570.

[5] J. Ginibre and G. Velo, The global Cauchy problem for the nonlinear Klein-Gordon equation. Math. Z. 189 (1985), no. 4, 487–505.

[6] J. Ginibre and G. Velo, The global Cauchy problem for the nonlinear Klein-Gordon equation. II. Ann. Inst. H. Poincaré Anal. Non Linéaire 6 (1989), no. 1, 15–35.
[7] A. Haraux, Nonlinear evolution equations: Global behavior of solutions, in; Lecture Notes in Mathematics, Vol. 841, Springer, Berlin, 1987.

[8] K. Jörgens, Nonlinear wave equations, University of Colorado, Department of Mathematics, 1970.

[9] H. A. Levine, Instability and nonexistence of global solutions to nonlinear wave equations of the form $P_{u_{tt}} = -Au + f(u)$, Trans. Amer. Math. Soc. 192 (1974), 1-21.

[10] H.A. Levine, Some additional remarks on the nonexistence of global solutions to nonlinear wave equations. SIAM J. Math. Anal. 5 (1974), 138–146.

[11] M.R. Li, L.Y. Tsai, On a system of nonlinear wave equations, Proceedings of the Workshop on DiHerential Equations, Vol. V, Tsinghua University, Hsinchu, 1997, pp. 1-5.

[12] M.R. Li and L.Y. Tsai, Existence and nonexistence of global solutions of some system of semilinear wave equations, Nonlinear analysis, 54(2003) 1397-1415.

[13] J.L. Lions and e. Magenes, Nonhomogeneous boundary value problem, Vol. 2, Springer, Berlin, 1975.

[14] Y. Liu, M. Ohta and G. Todorova, Strong instability of solitary waves for nonlinear Klein-Gordon equations and generalized Boussinesq equations, Ann. Inst. H. Poincaré Anal. Non Linéaire (2006), doi:10.1016/j.anihpc.2006.03.005.

[15] V. Makhandov, Dynamics of classical solutions in integrable system, Physics Reports, (Sect C of Physics Letters) 35(1978) 1-128.

[16] L.A. Medeiros and G. Perla Menzala, On a mixed problem for a class of nonlinear Klein-Gordon equations, Atas 21º Seminário Brasileiro de Análise, Brasilia, May 1985.

[17] Medeiros L A, Miranda M M. Weak solutions for a system of nonlinear Klein-Gordon equations[J]. Annali di Math Pura Appl, 4(1987) 173-183.

[18] M. Miranda and L. Medeiros, On the existence of global solutions of a coupled nonlinear Klein-Gordon equations, Funkcialaj Ekvacjoj, 30(1987) 147-161.

[19] M. Ohta and G. Todorova, Strong instability of standing waves for nonlinear Klein-Gordon equations. Discrete Contin. Dyn. Syst. 12 (2005), no. 2, 315–322.

[20] W. Omana and W. Willem, Homoclinic orbit for a class of Hamiltonian systems, Diff. Int. Eq. 5(1992) 1115-1120.

[21] I.E. Payne and D.H. Sattinger, Saddle points and instability of nonlinear hyperbolic equations. Israel Journal of Mathematics 22(1975) 273-303.

[22] H.A. Rose and M.I. Weinstein, On the bound states of the nonlinear Schrödinger equation with a linear potential, Physica D 30(1988)207-218.

[23] I. Segal, Nonlinear partial differential equations in Quantum Field Theory, Proc. Symp. Appl. Math. A.M.S. 17(1965) 210-226.

[24] H.A. Soffer and M.I. Weinstein, Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations, Invent. Math. 136(1999)9-74.
[25] W.A. Strauss, Nonlinear invariant Wave Equations, Lecture Notes in Physics, Springer-Verlag, New York/Berlin, 1978.

[26] Y. Wang, A sufficient condition for finite time blow up of a nonlinear Klein-Gordon equation with arbitrary positive initial energy, 2006. Submitted.

[27] J. Zhang, Stability of standing waves for nonlinear Schrödinger equations with unbounded potentials, Z. Angew. Math. Phys. 51(2000)498-503.

[28] J. Zhang, Sharp conditions of global existence for nonlinear Schrödinger and Klein-Gordon equations. Nonlinear Anal. 48 (2002), no. 2, Ser. A: Theory Methods, 191–207.

[29] J. Zhang, On the standing wave in coupled non-linear Klein-Gordon equations. Math. Methods Appl. Sci. 26 (2003), no. 1, 11–25.