Super–Schwarzians via nonlinear realizations

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Abstract

The $\mathcal{N} = 1$ and $\mathcal{N} = 2$ super–Schwarzian derivatives were originally introduced by physicists when computing a finite superconformal transformation of the super stress–energy tensor underlying a superconformal field theory. Mathematicians like to think of them as the cocycles describing central extensions of Lie superalgebras. In this work, a third possibility is discussed which consists in applying the method of nonlinear realizations to $\mathfrak{osp}(1|2)$ and $\mathfrak{su}(1,1|1)$ superconformal algebras. It is demonstrated that the super–Schwarzians arise quite naturally, if one decides to keep the number of independent Goldstone superfields to a minimum.

Keywords: super–Schwarzian, superconformal algebra, the method of nonlinear realizations
1. Introduction

A recent study of supersymmetric extensions of the Sachdev–Ye–Kitaev model\(^1\) generated renewed interest in the \(\mathcal{N} = 1\) and \(\mathcal{N} = 2\) super–Schwarzian derivatives.\(^2\) Such derivatives were originally introduced by physicists when computing a (finite) superconformal transformation of the super stress–energy tensor underlying a superconformal field theory\(^3\)\(^4\)\(^5\). Mathematicians used to regard them as the cocycles describing central extensions of Lie superalgebras (see, e.g.,\(^6\) and references therein).

A remarkable property of the \(\mathcal{N} = 1\) and \(\mathcal{N} = 2\) super–Schwarzian derivatives is that they hold invariant under (finite) transformations forming \(OSp(1|2)\) and \(SU(1,1|1)\) superconformal groups, respectively.\(^2\) It is then natural to wonder whether the super–Schwarziants can be obtained within the conventional group–theoretic construction\(^7\), thus providing an alternative to the approaches mentioned above.

The goal of this paper is to apply the method of nonlinear realizations to \(osp(1|2)\) and \(su(1,1|1)\) superconformal algebras and to demonstrate that the \(\mathcal{N} = 1\) and \(\mathcal{N} = 2\) super–Schwarzian derivatives arise quite naturally, if one decides to keep the number of independent Goldstone superfields to a minimum. Similar group–theoretic derivation of the ordinary Schwarzian derivative was recently reported in\(^8\).

The idea behind this work is rather straightforward. First, each generator in the superalgebra is accompanied by a superfield of the same Grassmann parity. Then the coset space element \(\hat{g}\) is introduced on which a supergroup representative \(g\) acts by the left multiplication, \(\hat{g}' = g \cdot \hat{g}\). After that the invariant \(\hat{g}^{-1}D\hat{g}\) is considered, which involves the covariant derivative \(D\). This can be decomposed into a linear combination of the generators belonging to the superalgebra each of which is multiplied by a specific invariant built from the superfields and their covariant derivatives. The latter can be used to impose constraints eliminating some of the superfields as well as reproducing the expression for the super–Schwarzian derivative.

The work is organized as follows. In the next section, the \(\mathcal{N} = 1\) super–Schwarzian derivative is obtained by applying the method of nonlinear realizations to \(osp(1|2)\) superconformal algebra. Five superfield invariants are constructed which fit the number of generators of the superalgebra. Imposing four constraints so as to express some of the Goldstone superfields in terms of the other and substituting the result into the remaining fifth invariant, one reproduces the \(\mathcal{N} = 1\) super–Schwarzian. Similar group–theoretic derivation of the \(\mathcal{N} = 2\) super–Schwarzian derivative based upon \(su(1,1|1)\) superalgebra is given in Sec. 3. In contrast to the previous case, the \(\mathcal{N} = 2\) super–Schwarzian comes about when one analyses the reality condition for the superfield associated with the generator of special conformal transformations. We summarize our results and discuss possible further developments in the

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\(^1\)The literature on the subject is rather extensive. For a good recent account and further references see Ref.\(^2\).

\(^2\)In modern literature, the superconformal groups are sometimes designated by the number of spacetime dimensions in which they are realized and the number of real supersymmetry charges at hand. In this nomenclature \(OSp(1|2)\) and \(SU(1,1|1)\) are identified with the \(d = 1, \mathcal{N} = 1\) and \(d = 1, \mathcal{N} = 2\) superconformal groups, respectively.
concluding Sect. 4. Some useful identities relevant for computation of the superconformal invariants in Sect. 2 and Sect. 3 are gathered in Appendix.

2. $\mathcal{N} = 1$ super–Schwarzian derivative via nonlinear realizations

The $\mathcal{N} = 1$ super–Schwarzian derivative

$$S[\psi(t, \theta); t, \theta] = \frac{D^4 \psi}{D\psi} - 2\frac{D^3 \psi}{D\psi} D^2 \psi$$

(1)

where $\psi(t, \theta)$ is a real fermionic superfield and $D$ is the covariant derivative, was first introduced in [3] by computing a finite superconformal transformation of the super stress–energy tensor underlying an $\mathcal{N} = 1$ superconformal field theory. Our objective in this section is to demonstrate that (1) comes about naturally if one applies the method of nonlinear realizations [7] to $OSp(1|2)$ supergroup and keeps the number of independent Goldstone superfields to a minimum.

Consider a real superspace $\mathcal{R}^{1|1}$ parametrized by a bosonic coordinate $t$ and a fermionic coordinate $\theta$, $\theta^2 = 0$. The supersymmetry transformations

$$t' = t + a; \quad t' = t + i\epsilon \theta, \quad \theta' = \theta + \epsilon,$$

(2)

where $a$ and $\epsilon$ are even and odd real supernumbers, respectively, realise the action of the $d = 1, \mathcal{N} = 1$ supersymmetry algebra

$$\{q, q\} = 2h$$

(3)

in the superspace.

Within the method of nonlinear realizations, $\mathcal{R}^{1|1}$ is represented by the supergroup element

$$\tilde{g} = e^{i\theta b} e^{q},$$

(4)

while the left action of the supergroup on itself, $\tilde{g}' = e^{i\theta b} e^{q} \cdot \tilde{g}$, reproduces (2). The covariant derivative, which anticommutes with the supersymmetry generator, reads

$$D = \partial_\theta - i\theta \partial_t, \quad D^2 = -i\partial_t,$$

(5)

where $\partial_t = \frac{\partial}{\partial t}$, $\partial_\theta = \frac{\partial}{\partial \theta}$.

Real bosonic and fermionic superfields are power series in $\theta$

$$\rho(t, \theta) = b(t) + i\theta f(t), \quad \psi(t, \theta) = F(t) + \theta B(t), \quad (D\rho)^* = -D\rho, \quad (D\psi)^* = D\psi,$$

(6)

which involve the bosonic components $(b(t), B(t))$ and the fermionic partners $(f(t), F(t))$. The covariant derivative (5) and a real fermionic superfield $\psi(t, \theta)$ are the building blocks entering Eq. (1) above.

3To be more precise, in Ref. [3] a complexified version of (1) was considered.
In order to derive the $\mathcal{N} = 1$ super-Schwarzian derivative within the method of nonlinear realizations, let us consider $osp(1|2)$ superconformal algebra

\[
[P, D] = iP, \quad [P, K] = 2iD, \\
[D, K] = iK, \quad [D, Q] = -\frac{i}{2}Q, \\
[D, S] = \frac{i}{2}S, \quad [P, S] = -iQ, \\
[K, Q] = iS, \quad \{Q, S\} = -2D, \\
\{Q, Q\} = 2P, \quad \{S, S\} = 2K, \\
\{Q, K\} = iS,
\]

(7)

where $(P, D, K)$ are the bosonic generators of translations, dilatations and special conformal transformations, respectively. $Q$ and $S$ are the fermionic generators of supersymmetry transformations and superconformal boosts.

As the next step, each generator in the superalgebra (7) is accompanied by a real Goldstone superfield of the same Grassmann parity and both $R_{1|1}$ and the superfields on it are represented by the element

\[
\tilde{g} = e^{i\theta h} e^{i\rho Q} e^{i\rho(t,\theta)P} e^{\psi(t,\theta)Q} e^{\phi(t,\theta)S} e^{i\mu(t,\theta)K} e^{i\nu(t,\theta)D}.
\]

(8)

It is assumed that $(h, q)$ (anti)commute with $(P, D, K, Q, S)$.

Left multiplication by a group element $\tilde{g}' = g \cdot \tilde{g}$, where $g = e^{iaP} e^{iQ} e^{iS} e^{icK} e^{ibD}$ involves real bosonic parameters $(a, b, c)$ and real fermionic parameters $(\epsilon, \sigma)$, determines the action of the superconformal group $OSp(1|2)$ on the superfields. Focusing on the infinitesimal transformations and making use of the Baker–Campbell–Hausdorff formula

\[
e^{iA} T e^{-iA} = T + \sum_{n=1}^{\infty} \frac{i^n}{n!} [A, [A, [A, \ldots [A, T] \ldots]], \text{ n times}],
\]

(9)

one gets

\[
\rho' = \rho + a; \\
\rho' = \rho + b\rho, \quad \nu' = \nu + b, \\
\mu' = \mu - b\mu, \quad \psi' = \psi + \frac{1}{2}b\psi, \quad \phi' = \phi - \frac{1}{2}b\phi; \\
\rho' = \rho + c\rho^2, \quad \nu' = \nu + 2c\rho, \\
\mu' = \mu + c - 2c\rho\mu - ic\psi\phi, \quad \psi' = \psi + c\rho\psi, \quad \phi' = \phi - c\psi - c\rho\phi; \\
\rho' = \rho + i\epsilon\psi, \quad \psi' = \psi + \epsilon; \\
\rho' = \rho - i\sigma\rho\psi, \quad \nu' = \nu - 2i\sigma\psi, \\
\mu' = \mu + i\sigma\phi + 2i\sigma\mu\psi, \quad \psi' = \psi - \sigma\rho, \quad \phi' = \phi + \sigma + i\sigma\psi\phi.
\]

(10)
Note that both the original and transformed superfields depend on the same arguments \((t, \theta)\) such that the transformations affect the form of the superfields only, e.g., 
\[ \delta \rho = \rho'(t, \theta) - \rho(t, \theta). \]
Computing the algebra of the infinitesimal transformations \((10)\), one can verify that it does reproduce the structure relations \((11)\). 

As the next step, one computes the invariant superfield combinations 
\[
\begin{align*}
\omega_P &= (D\rho + i\psi D\psi) e^{-\nu}, \\
\omega_D &= D\nu - 2i\phi D\psi - 2\mu (D\rho + i\psi D\psi), \\
\omega_K &= (D\mu + i\phi D\phi + 2i\mu \phi D\psi + \mu^2 (D\rho + i\psi D\psi)) e^\nu, \\
\omega_Q &= (D\psi - \phi (D\rho + i\psi D\psi)) e^{\frac{-\tau}{2}}, \\
\omega_S &= (D\phi + \mu (D\psi - \phi (D\rho + i\psi D\psi))) e^{\frac{\tau}{2}},
\end{align*}
\]
which originate from
\[
\tilde{g}^{-1} D \tilde{g} = i \omega_P P + i \omega_D D + i \omega_K K + \omega_Q Q + \omega_S S + q. \tag{12}
\]
Note that the Grassmann parities of the superfields entering \((11)\) are opposite to those of the conventional Maurer–Cartan one–forms \(\tilde{g}^{-1} d \tilde{g}\) because \(D\) is an odd operator. When obtaining \((11)\), the identities exposed in Appendix were heavily used.

At this stage, one can use the invariants \((11)\) so as to impose constraints allowing one to eliminate some of the Goldstone superfields. By analogy with our study of the Schwarzian derivative in \([8]\), let us decide to keep the number of independent Goldstone superfields to a minimum and impose four conditions 
\[
\omega_P = 0, \quad \omega_D = 0, \quad \omega_Q = g^{-1}, \quad \omega_S = p, \tag{13}
\]
where \(g\) and \(p\) are even supernumbers. It seems quite natural to set the fermionic invariants to vanish and the bosonic invariants to take constant values as only the \(c\)-numbers are observable.

The leftmost equation in \((13)\) gives 
\[
(D\rho + i\psi D\psi) = 0, \tag{14}
\]
while the rest links \((\nu, \mu, \phi)\) to \(\psi\)
\[
e^{\frac{-\tau}{2}} = gD\psi, \quad \phi = -\frac{\partial \psi}{(D\psi)^2}, \quad \mu = \frac{p}{g(D\psi)^2} + \frac{1}{D\psi} D \left( \frac{\partial \psi}{(D\psi)^2} \right). \tag{15}
\]

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4In order to verify the structure relations \((7)\), one first computes the commutators \([\delta_1, \delta_2] = \delta_3\) acting upon \((\rho, \mu, \nu, \psi, \phi)\) for all the transformations entering \((10)\). Then one represents each \(\delta\) as the product of a parameter and the corresponding generator, e.g. \(\delta_a = a \cdot P\). Finally, one substitutes these into \([\delta_1, \delta_2] = \delta_3\) and discards the parameters on both the right and left hand sides of the equality. For the case at hand, this yields \((7)\) after the rescaling \((P, K, D) \rightarrow (iP, iK, iD)\).
Substituting these relations into the remaining invariant \( \omega_K \), one gets
\[
\omega_K = ig^2 \left( \frac{D^4 \psi}{D \psi} - 2 \frac{D^3 \psi}{D \psi} \frac{D^2 \psi}{D \psi} \right).
\] (16)

Up to an irrelevant constant factor, this coincides with (1).

Thus, the \( \mathcal{N} = 1 \) super-Schwarzian derivative arises quite naturally if one applies the method of nonlinear realizations to \( OSp(1|2) \) superconformal group and decides to keep the number of independent Goldstone superfields to a minimum.

We conclude this section with a discussion of symmetries of (1). If one is interested in infinitesimal transformations, it suffices to consider (10) and focus on the transformation laws of \( \rho \) and \( \psi \). A straightforward computation shows that both (11) and the supplementary condition (14) hold invariant.

If one is concerned with finite transformations, then, following Ref. [3], one has to consider a generic super-diffeomorphism of \( \mathcal{R}^{1|1} \)
\[
t' = \rho(t, \theta), \quad \theta' = \psi(t, \theta),
\] (17)
under which the covariant derivative transforms homogeneously. The condition
\[
\mathcal{D} = (\mathcal{D} \theta') \mathcal{D}'.
\] (18)
yields the restriction on the bosonic superfield \( \rho \)
\[
\mathcal{D} \rho + i \psi \mathcal{D} \psi = 0,
\] (19)
yet leaves the fermionic superfield \( \psi \) unconstrained. Acting on (19) by the covariant derivative \( \mathcal{D} \), one gets the equation
\[
\partial_t \rho - i \psi \partial_t \psi - (\mathcal{D} \psi)^2 = 0,
\] (20)
which can be used to fix \( \rho \) provided \( \psi \) is known. Note that within the method of nonlinear realizations the supplementary condition (11) comes about as the constraint \( \omega_P = 0 \).

Given \( \rho(t, \theta) \) and \( \psi(t, \theta) \) obeying (19), consider a coordinate transformation (17) and a new real fermionic superfield \( \psi'(t', \theta') = \psi(\rho, \psi) \). Taking into account Eq. (20), one gets the formula
\[
\partial_{t'} = (\partial_{t'} \theta') \mathcal{D}' + (\mathcal{D} \theta')^2 \partial_{t'},
\] (21)
which then leads to the transformation law of the \( \mathcal{N} = 1 \) super-Schwarzian
\[
S[\psi'(\rho, \psi); t, \theta] = S[\psi(t, \theta); t, \theta] + (\mathcal{D} \theta')^3 S[\psi'(t', \theta'); t', \theta'].
\] (22)

\[5\text{It proves helpful to keep in mind the identities: } \frac{D^4 \psi}{D \psi} - 2 \frac{D^3 \psi}{D \psi} \frac{D^2 \psi}{D \psi} = \mathcal{D} \psi \mathcal{D} \left( \frac{D^3 \psi}{D \psi} \right) = \mathcal{D} \psi \mathcal{D}^2 \left( \frac{D^2 \psi}{D \psi} \right) = -\mathcal{D} \psi \mathcal{D}^3 \left( \frac{1}{D \psi} \right).\]
Thus, if \( S[\psi(t', \theta'); t', \theta'] = 0 \), the \( N = 1 \) super–Schwarzian derivative holds invariant under the change of the argument \( \psi(t, \theta) \rightarrow \psi'(\rho(t, \theta), \psi(t, \theta)) \). Solving \( S[\psi'(t', \theta'); t', \theta'] = 0 \) and integrating the analogue of Eq. (20), one finally gets

\[
\psi' = \alpha + \frac{\beta + \psi}{c \rho + d}, \quad \rho' = \frac{a \rho + b - i (\psi \psi' - \alpha \beta)}{c \rho + d},
\]

(23)

where \((a, b, c, d)\) are real even supernumbers obeying \(ad - cb = 1\) and \((\alpha, \beta)\) are real odd supernumbers.

Eq. (23) describes the finite form of \( OSp(1|2) \) transformations acting upon the form of the superfields \( \rho \) and \( \psi \) which leave the \( N = 1 \) super–Schwarzian and the supplementary condition (19) invariant (cf. (10)).

3. \( N = 2 \) super–Schwarzian derivative via nonlinear realizations

The \( N = 2 \) super–Schwarzian derivative [4]

\[
S[\psi(t, \theta, \bar{\theta}); t, \theta, \bar{\theta}] = \frac{\partial_t \mathcal{D} \psi - \partial_{\bar{\theta}} \mathcal{D} \bar{\psi}}{\mathcal{D} \psi} - 2i \partial_t \psi \partial_t \bar{\psi} \frac{\mathcal{D} \psi}{\mathcal{D} \psi \mathcal{D} \psi}
\]

(24)

involves a complex chiral fermionic superfield \( \psi \) defined on \( \mathcal{R}^{1|2} \) superspace and its complex conjugate partner \( \bar{\psi} = \psi^* \)

\[
\mathcal{D} \psi = 0, \quad \mathcal{D} \bar{\psi} = 0.
\]

(25)

\( \mathcal{R}^{1|2} \) is parametrized by a bosonic coordinate \( t \) and a pair of complex conjugate fermionic coordinates \((\theta, \bar{\theta})\), \( \theta \bar{\theta} = -\bar{\theta} \theta, \theta^2 = \bar{\theta}^2 = 0 \). The \( d = 1, N = 2 \) supersymmetry transformations read

\[
t' = t + a; \quad \theta' = \theta + \epsilon, \quad \bar{\theta}' = \bar{\theta} + \bar{\epsilon}, \quad t' = t + i(\epsilon \bar{\theta} + \bar{\epsilon} \theta),
\]

(26)

and the covariant derivatives, which anticommute with the supersymmetry generators, are realised as follows:

\[
\mathcal{D} = \partial_\theta - i \bar{\theta} \partial_t, \quad \bar{\mathcal{D}} = \partial_{\bar{\theta}} - i \theta \partial_t, \quad \{\mathcal{D}, \bar{\mathcal{D}}\} = -2i \partial_t, \quad \mathcal{D}^2 = \bar{\mathcal{D}}^2 = 0,
\]

(27)

where \( \partial_t = \frac{\partial}{\partial t}, \partial_\theta = \frac{\partial}{\partial \theta}, \partial_{\bar{\theta}} = \frac{\partial}{\partial \bar{\theta}} \).

Within the method of nonlinear realizations \( \mathcal{R}^{1|2} \) is identified with the supergroup element

\[
\tilde{g} = e^{i h} e^{q + \bar{q}}, \quad \{q, \bar{q}\} = 2h,
\]

(28)

while (26) follow from \( \tilde{g}' = e^{i h} e^{q + \bar{q}} \cdot \tilde{g} \).

\( \mathcal{N} = 2 \) superfields are power series in \( \theta \) and \( \bar{\theta} \) which involve component fields dependent on \( t \). If \( \rho \) is a real bosonic superfield and \( \psi \) is a complex fermionic superfield, the complex conjugation rules for their covariant derivatives read

\[
(\mathcal{D} \rho)^* = -\bar{\mathcal{D}} \rho, \quad (\mathcal{D} \psi)^* = \bar{\mathcal{D}} \bar{\psi}, \quad (\bar{\mathcal{D}} \psi)^* = \mathcal{D} \bar{\psi}.
\]

(29)
In order to obtain the $\mathcal{N} = 2$ super–Schwarzian derivative within the method of nonlinear realizations, let us consider $su(1, 1|1)$ superconformal algebra

\[
\begin{align*}
[P, D] &= iP, & [P, K] &= 2iD, \\
[D, K] &= iK, & [D, Q] &= -\frac{i}{2}Q, \\
[D, \bar{Q}] &= -\frac{i}{2}\bar{Q}, & [D, S] &= \frac{i}{2}S, \\
[D, \bar{S}] &= \frac{i}{2}\bar{S}, & [P, S] &= -iQ, \\
[P, \bar{S}] &= -i\bar{Q}, & [K, Q] &= iS, \\
[K, \bar{Q}] &= i\bar{S}, & [J, Q] &= \frac{1}{2}Q, \\
[J, \bar{Q}] &= -\frac{1}{2}\bar{Q}, & [J, S] &= \frac{1}{2}S, \\
[J, \bar{S}] &= -\frac{1}{2}\bar{S}, & \{Q, \bar{Q}\} &= 2P, \\
\{Q, \bar{S}\} &= -2(D + iJ), & \{S, \bar{S}\} &= 2K, \\
\{Q, S\} &= -2(D - iJ),
\end{align*}
\]

where $(P, D, K, J)$ are bosonic generators and $(Q, \bar{Q}, S, \bar{S})$ are their fermionic partners. As compared to the $\mathcal{N} = 1$ case, there appears a new $U(1)$–symmetry generator $J$, while $Q$ and $S$ become complex.

Before constructing a group–theoretic element similar to (8), it proves instructive to make recourse to a generic super–diffeomorphism of $\mathcal{R}^{1|2}$

\[
t' = \rho(t, \theta, \bar{\theta}), \quad \theta' = \psi(t, \theta, \bar{\theta}), \quad \bar{\theta}' = \bar{\psi}(t, \theta, \bar{\theta}),
\]

where $\rho$ is a real bosonic superfield and $\psi$ is a complex fermionic superfield, and find conditions which follow from the requirement that the covariant derivatives transform homogeneously. The elementary computation

\[
\mathcal{D} = (\mathcal{D}t' + \bar{\theta}'\mathcal{D}\theta') \partial_{t'} + (\mathcal{D}\theta') \mathcal{D}' + (\mathcal{D}\bar{\theta}') \partial_{\bar{\psi}} = (\mathcal{D}\theta')\mathcal{D}'
\]

(32) gives

\[
\mathcal{D}\bar{\psi} = 0, \quad \mathcal{D}\rho + i\bar{\psi}\mathcal{D}\psi = 0,
\]

(33) while $\bar{\mathcal{D}} = (\bar{\mathcal{D}}\bar{\theta}') \bar{\mathcal{D}}'$ yields the complex conjugate restrictions

\[
\bar{\mathcal{D}}\psi = 0, \quad \bar{\mathcal{D}}\rho + \psi\mathcal{D}\bar{\psi} = 0.
\]

(34)

These are the $\mathcal{N} = 2$ analogues of Eq. (19) above. Note the simple corollary of Eqs. (33) and (34)

\[
\partial_t \rho = (\mathcal{D}\psi)(\bar{\mathcal{D}}\bar{\psi}) + i\psi\partial_t \bar{\psi} - i\partial_t \bar{\psi}\bar{\psi},
\]

(35)
which may be used to fix $\rho$ provided $\psi$ is known.

In what follows, we shall assume that the conditions (33) and (34) hold. As a matter of fact, the method of nonlinear realizations allows one to reproduce the equation for $\rho$, which comes about as the constraint $\omega_P=0$, but not the chirality condition for the fermionic superfield $\psi$.

In view of all the foregoing, consider the group-theoretic element

$$\tilde{g} = e^{ih} e^{\theta e + \bar{\theta} e} e^{i\rho P} e^{\psi Q + \bar{\psi} Q} e^{\phi S + \bar{\phi} S} e^{i\mu K} e^{i\nu D} e^{i\lambda J}, \quad (36)$$

where $(\rho, \mu, \nu, \lambda)$ are real bosonic superfields and $(\psi, \phi)$ are complex fermionic superfields.

Note that such a choice of $\tilde{g}$ is suggested by the previous study of $d = 1, \mathcal{N} = 2$ superconformal mechanics within the method of nonlinear realizations [9]. The superconformal invariants, which derive from $\tilde{g}^{-1} D \tilde{g}$, read

$$\omega_D = D\nu - 2i\bar{\phi}D\psi,$$
$$\omega_K = (D\mu + i\phi D\bar{\phi} + i\bar{\phi} D\phi + 2i\mu \bar{\phi} D\psi) e^\nu,$$
$$\omega_J = D\lambda + 2\bar{\phi} D\psi,$$
$$\omega_Q = D\psi e^{-\frac{\nu}{2}} e^{-\frac{i\lambda}{2}},$$
$$\omega_S = (D\phi + \mu D\psi - i\phi \bar{\phi} D\psi) e^\nu e^{-\frac{i\lambda}{2}},$$
$$\omega_{\bar{S}} = D\phi e^\nu e^{\frac{i\lambda}{2}}. \quad (37)$$

When obtaining Eqs. (37), the identities gathered in Appendix were heavily used.

As the next step, let us impose constraints similar to those in the preceding section

$$\omega_D = 0, \quad \omega_Q = g^{-1}, \quad \omega_S = p, \quad (38)$$

where $g$ and $p$ are complex even supernumbers. They allow one to express $(\mu, \nu, \lambda, \phi)$ in terms of $\psi$

$$\mu = \frac{pg}{e^\nu} + i\phi \bar{\phi} - D\phi \frac{D\psi}{D\bar{\psi}}, \quad e^\nu = g\bar{g}D\psi \bar{D}\bar{\psi}, \quad e^{i\lambda} = gD\psi \frac{D\psi}{g\psi}, \quad \phi = -\frac{\partial_{\psi} \psi}{D\bar{\psi} \bar{D}\bar{\psi}}, \quad (39)$$

which, in their turn, ensure $\phi$ to be a chiral superfield, $\bar{D}\phi = 0$, and force the remaining invariants to vanish, $\omega_K = \omega_J = \omega_{\bar{S}} = 0$. Finally, taking into account that $\mu$ is a real superfield, $\bar{\mu} = \mu$, one gets the equation

$$\frac{\partial_{\psi} \psi}{D\psi} - \frac{\partial_{\bar{\psi}} \bar{D}\bar{\psi}}{D\bar{\psi}} - 2i \frac{\partial_{\psi} \psi \partial_{\bar{\psi}} \bar{\psi}}{D\psi \bar{D}\psi} = \frac{\bar{g}g - pg}{g\bar{g}}, \quad (40)$$

the right hand side of which reproduces (24).

As compared to the $\mathcal{N} = 1$ case, the constraints (38) turn out to be more stringent and result in a variant of $\mathcal{N} = 2$ super–Schwarzain mechanics in which the super–Schwarzian
derivative is equal to a (coupling) constant $\frac{\partial^2 \phi}{\partial \bar{z} \partial z}$. The latter is an $N = 2$ analogue of the model studied recently in [10]. As was mentioned in the Introduction, our primarily concern in this work is to understand how the super–Schwarzian derivatives may be obtained within the method of nonlinear realizations. The dynamics of the specific model (40) will not be studied any further.

Concluding this section, let us discuss symmetries of Eqs. (24) and (33), (34). The infinitesimal form of such transformations follows from

$$\tilde{g}' = e^{i a P} e^{i Q + i \sigma S + i \delta S} e^{i K} e^{i \bar{\psi} D} e^{i \bar{\xi} J} \cdot \tilde{g},$$

(41)

where $\tilde{g}$ is given in (36), while $(a, b, c, \xi)$ and $(\epsilon, \sigma)$ are bosonic and fermionic infinitesimal parameters, respectively. Implementing the Baker–Campbell–Hausdorff formula (9) and discarding the transformation laws of $(\mu, \nu, \lambda, \phi, \bar{\phi})$, one gets

$$\rho' = \rho + a, \quad \psi' = \psi, \quad \bar{\psi}' = \bar{\psi};$$

$$\rho' = \rho + b \rho, \quad \psi' = \psi + \frac{1}{2} b \psi, \quad \bar{\psi}' = \bar{\psi} + \frac{1}{2} b \bar{\psi};$$

$$\rho' = \rho + c \rho^2, \quad \psi' = \psi + c \rho \psi, \quad \bar{\psi}' = \bar{\psi} + c \rho \bar{\psi};$$

$$\rho' = \rho, \quad \psi' = \psi + \frac{i}{2} \xi \psi, \quad \bar{\psi}' = \bar{\psi} - \frac{i}{2} \xi \bar{\psi};$$

$$\rho' = \rho + i \left( \epsilon \bar{\psi} + \bar{\epsilon} \psi \right), \quad \psi' = \psi + \epsilon, \quad \bar{\psi}' = \bar{\psi} + \bar{\epsilon};$$

$$\rho' = \rho - i \rho \left( \sigma \bar{\psi} + \bar{\sigma} \psi \right), \quad \psi' = \psi - \rho \sigma + i \sigma \psi \bar{\psi}, \quad \bar{\psi}' = \bar{\psi} - \rho \bar{\sigma} - i \bar{\sigma} \psi \bar{\psi}. \quad (42)$$

As in the $N = 1$ case, the transformations act upon the form of the superfields only and do not affect the arguments $(t, \theta, \bar{\theta})$. It is easy to compute the commutators $[\delta_1, \delta_2]$ and verify that they do reproduce the structure relations (30). A straightforward calculation then shows that both (24) and the supplementary conditions (33), (34) hold invariant under the infinitesimal transformations.

In order to determine a finite form of symmetries of the $N = 2$ super–Schwarzian, one considers the coordinate transformation (31), which obeys the subsidiary conditions (33), (34), and a new complex chiral fermionic superfield $\psi'(t', \theta', \bar{\theta}') = \psi'(\rho, \psi, \bar{\psi})$. Taking into account the relations

$$\mathcal{D} = (D \theta') \mathcal{D}', \quad \bar{\mathcal{D}} = (\bar{D} \bar{\theta}') \bar{\mathcal{D}}', \quad \partial_i = (\partial_i \theta') \mathcal{D}' + (\partial_i \bar{\theta}') \bar{\mathcal{D}}' + (D \theta' \bar{D} \bar{\theta}') \partial_i, \quad (43)$$

one gets

$$S[\psi'(\rho, \psi, \bar{\psi}); t, \theta, \bar{\theta}] = S[\psi(t, \theta, \bar{\theta}); t, \theta, \bar{\theta}] + (D \theta' \bar{D} \bar{\theta}') S[\psi'(t', \theta', \bar{\theta}'); t', \theta', \bar{\theta}]. \quad (44)$$

As in the $N = 1$, after computing the algebra one has to redefine the bosonic generators $(P, D, K, J) \rightarrow (iP, iD, iK, iJ)$ so as to fit the notation in (30).
Thus, the $\mathcal{N} = 2$ super–Schwarzian derivative remains intact under the change of the argument $\psi(t, \theta, \bar{\theta}) \rightarrow \psi'(\rho(t, \theta, \bar{\theta}), \psi(t, \theta, \bar{\theta}), \bar{\psi}(t, \theta, \bar{\theta}))$ provided $S[\psi'(t', \theta', \bar{\theta}'); t', \theta', \bar{\theta}'] = 0$. Solving the latter equation and integrating the analogue of (35) for $\rho'$, one finally gets

$$
\psi' = \alpha - \frac{\beta}{cp + d} + \psi' \frac{e^{-iv}}{cp + d} \left( c - \frac{i\beta \bar{\alpha}}{cp + d} \right) - \frac{i\bar{\psi}c\beta}{(cp + d)^2},
$$

$$
\rho' = \frac{\alpha p + b}{cp + d} - \frac{i(\alpha\bar{\beta} - \bar{\beta}\alpha)}{cp + d} - \frac{i\bar{\psi}e^{-iv}}{cp + d} \left( c\bar{\alpha} - \frac{\beta(c + i\beta\bar{\alpha})}{cp + d} \right)
$$

$$
- \frac{i\bar{\psi}e^{iv}}{cp + d} \left( \frac{\beta(c + i\beta\bar{\alpha})}{cp + d} \right) - \frac{\psi\bar{\psi}c\beta}{(cp + d)^2} \left( \frac{\alpha\bar{\beta} + \beta\bar{\alpha} - \frac{2\beta\bar{\beta}}{cp + d} \right),
$$

where $(a, b, c, d, v)$ are real even supernumbers obeying $ad - cb = c^2$ and $(\alpha, \beta)$ are complex odd supernumbers.

It is straightforward to verify that the supplementary conditions (33), (34) hold invariant under the transformation (45) as well.

4. Discussion

To summarize, in this work we have demonstrated how the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ super–Schwarzian derivatives can be obtained within the method of nonlinear realizations applied to $OSp(1|2)$ and $SU(1, 1|1)$ superconformal groups, thus providing an alternative to the existing approaches.

Turning to possible further developments, the most urgent question is to generalise the analysis to the $\mathcal{N} = 4$ case, i.e. to treat $SU(1, 1|2)$ superconformal group in a similar fashion. Note that there is some controversy on this issue in the literature. In Ref. [5] it is stated that an $\mathcal{N} = 4$ super–Schwarzian is a non–local expression and only the covariant derivative of it is given in explicit form. Mathematicians report on obstruction to obtain a projective cocycle for $\mathcal{N} \geq 3$ (see, e.g., the discussion in [6]). An $\mathcal{N} = 4$ super–Schwarzian proposed in [3] does not seem to be invariant under finite $SU(1, 1|2)$ transformations (any super–Schwarzian should be a homogeneous function of degree zero under the rescaling $\psi \rightarrow b\psi$).

A preliminary consideration shows that an $\mathcal{N} = 4$ super–Schwarzian generated by the method of nonlinear realizations might read

$$
\mathcal{D}^\alpha \psi_{\beta} \bar{D}_\alpha \left( \frac{\partial_t \bar{\psi}^\beta}{\bar{D}_t \bar{\psi}} \right) - \bar{D}_\alpha \bar{\psi}^\beta \mathcal{D}^\alpha \left( \frac{\partial_t \psi_\beta}{D_t \psi} \right) + \frac{4i\partial_t \psi_\alpha \partial_t \bar{\psi}^\alpha}{\bar{D}_t \psi^\alpha},
$$

where $\psi_\alpha$ is a fermionic chiral superfield on $\mathcal{R}^{1|4}$ superspace carrying an $SU(2)$ spinor index $\alpha = 1, 2$, $\bar{\psi}^\alpha$ is its complex conjugate $(\psi_\alpha)^* = \bar{\psi}^\alpha$, and $\mathcal{D}^\alpha, \bar{D}_\alpha$ are the covariant derivatives. Above we abbreviated $(\bar{D}_t \psi^\alpha) = \mathcal{D}^\alpha \psi_\beta \bar{D}_\alpha \bar{\psi}^\beta$. However, it turns out that along with (46) the methods yields an extra nonlinear constraint on $\psi_\alpha$ which still needs to be understood.

---

7 The standard form of $SL(2, R)$ transformations with $ad - cb = 1$ is recovered by rescaling $\psi' \rightarrow \frac{1}{x} \psi'$, $\rho' \rightarrow \frac{1}{x} \rho'$, $\bar{\psi} \rightarrow a$, $\bar{\psi} \rightarrow b$. 1
We hope to report on the progress as well as to describe a more general case of the \( D(2, 1; \alpha) \) super–Schwarzian elsewhere.

An elegant derivation of the \( \mathcal{N} = 1 \) and \( \mathcal{N} = 2 \) super–Schwarzian derivatives within the context of a one-dimensional \( Osp(N|2M) \) pseudoparticle mechanics was proposed in \[12\]. It would be interesting to see if the analysis in \[12\] can be generalised to the \( \mathcal{N} = 4 \) case.

A link between the conventional second order conformal mechanics and the Schwarzian mechanics was discussed in a very recent work \[13\]. It would be interesting to explore whether the analysis in \[13\] can be extended to produce super–Schwarzians. The construction of higher derivative superconformal mechanics of the Schwarzian type along the lines in \[14\] is of interest as well.

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**Appendix**

In this Appendix we gather some identities which were used in the main text when computing the superconformal invariants \( (11) \) and \( (37) \).

The identities which facilitate the derivation of Eq. \((11)\) read
\[
\begin{align*}
e^{-i\nu D} Pe^{i\nu D} &= e^{-\nu} P, \\
e^{-i\nu D} Qe^{i\nu D} &= e^{-\frac{\nu}{2}} Q, \\
e^{-i\mu K} Pe^{i\mu K} &= P - 2\mu D + \mu^2 K, \\
e^{-i\mu K} Qe^{i\mu K} &= Q + \mu S, \\
e^{-i\rho P} Ke^{i\rho P} &= K + 2\rho D + \rho^2 P, \\
e^{-i\rho P} De^{i\rho P} &= D - \rho P, \\
e^{-i\rho P} Se^{i\rho P} &= S - \rho Q.
\end{align*}
\]

Note that these relations are also valid for the \( su(1,1|1) \) superconformal algebra, in which case \( Q \) and \( S \) are regarded complex. In that case the identities involving \( \bar{Q}, \bar{S} \) follow by the Hermitian conjugation.

When computing the superconformal invariants \( (37) \), the following identities:
\[
\begin{align*}
e^{-(\phi S + \bar{\phi} \bar{S})} Q e^{\phi S + \bar{\phi} \bar{S}} &= Q + 2\tilde{\phi}(D + iJ) - i\phi \tilde{\phi} S, \\
e^{-(\psi Q + \bar{\psi} \bar{Q})} \left( De^{\psi Q + \bar{\psi} \bar{Q}} \right) &= D\psi \left( Q - \bar{\psi} P \right), \\
e^{-(\phi S + \bar{\phi} \bar{S})} P e^{\phi S + \bar{\phi} \bar{S}} &= P - i\phi Q - i\bar{\phi} \bar{Q} + 2\phi \bar{\phi} J, \\
e^{-(\psi Q + \bar{\psi} \bar{Q})} \left( De^{\psi Q + \bar{\psi} \bar{Q}} \right) &= D\bar{\psi} \left( \bar{Q} - \psi P \right), \\
e^{-(\phi S + \bar{\phi} \bar{S})} \bar{Q} e^{\phi S + \bar{\phi} \bar{S}} &= \bar{Q} + 2\phi(D - iJ) + i\phi \bar{\phi} \bar{S}, \\
e^{-(\phi S + \bar{\phi} \bar{S})} \left( De^{\phi S + \bar{\phi} \bar{S}} \right) &= D\phi \left( S - \phi K \right) + D\hat{\phi} \left( \bar{S} - \phi K \right),
\end{align*}
\]
\[ e^{-(\phi S + \bar{\phi} \bar{S})} \left( \bar{D} e^{\phi S + \bar{\phi} \bar{S}} \right) = \bar{D} \bar{\phi} \left( \bar{S} - \phi K \right) + \bar{D} \phi \left( S - \bar{\phi} K \right), \]

\[ e^{-i\lambda J} Q e^{i\lambda J} = e^{-\frac{\lambda}{2} Q}, \quad e^{-i\lambda J} \bar{Q} e^{i\lambda J} = e^{\frac{\lambda}{2} \bar{Q}}, \]

\[ e^{-i\lambda J} S e^{i\lambda J} = e^{-\frac{\lambda}{2} S}, \quad e^{-i\lambda J} \bar{S} e^{i\lambda J} = e^{\frac{\lambda}{2} \bar{S}}, \]

proved helpful. Recall that \( \psi \) is a chiral fermionic superfield.

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