Energy Method on Dynamic Buckling of Elastic Bar with Clamped-Free boundary Condition subjected to Impact of Rigid Mass

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Abstract. Considering the effect of stress wave, the dynamic buckling problem of elastic bar with clamped-free boundary condition subjected to impact of rigid mass is studied using the energy method. The Lagrange function is built. Both of the trial function that satisfy the boundary conditions and the Lagrange function are substituted into the second Lagrange equation. After that an analytical expression of critical impact velocity can be deduced. By the analytical expression, it is obtained that the critical impact velocity is related to the impact mass of rigid block, the critical length and the inertia radius of the bar. The result show that the critical velocity decreases by increasing the impact mass and critical length, while increases by the increasing of inertia radius.

1. Introduction
Straight rods are widely used in many industrial fields such as aerospace, atomic energy, chemicals, ships, construction and machinery. Therefore, the buckling failure of the bar under the impact of strong dynamic load has greatly aroused interest and concern of many scholars, and it has become one of the most active research fields in solid mechanics. Lindberg H E[1] used the ultra-high-speed photography technology to reveal the effect of stress wave effects for the first time on the buckling of thin rods. D. Karagiozova[2] and Ü. Lepik[3] studied the buckling of elastoplastic bars under stress waves. Chen Guo-sheng[4] studied the buckling of composite laminated beams with delamination damage under uniform speed loading. Phani Motamarri[5] studied the buckling of elastic rods during low and uniform speed loading, and ignored the effect of stress wave to derive the relationship between axial stress and loading speed. Wang Yayun[6] studied the buckling problem of elastic beams under the step load stress waves by differential quadrature method. Hu Zhelin[7] used the difference method to study the problem of uniform speed loading of a straight rod, obtained the relationship between the loading speed and the critical stress, and carried out finite element simulation verification. Pengcheng Jiao[8] studied the static and dynamic buckling of slender rods by using the total potential energy of the system instead of the total energy, the relationship between deflection and axial load was obtained without considering the stress wave effect, and carried out experimental verification. To sum up, one kind of the study of dynamic buckling problem is that considers the stress wave effect, usually using step load loading; the other is that the stress wave effect is not considered, and the loading form is full load. In theoretical analysis, it is generally difficult to obtain an analytical solution, and there is
less research on dynamic buckling problems of mass impact rods with engineering application background. Zhang xu\cite{9} studied the dynamic response of c-section steel columns under the impact of falling weights by simulation and experiment, and established a numerical analysis model. J. Deng\cite{10} studied the rockburst problem in underground construction engineering and established a dynamic buckling model of the mass block impact the structure. The above description shows that the dynamic buckling problem of bar subjected to impact of rigid mass is complicated due to the stress wave propagation, which makes the theoretical research difficult, and the results obtained are different due to different angles of consideration and the methods used.

Based on this, this paper considers the effect of stress waves, studies the impact buckling of rigid mass, and uses the energy method to establish the Lagrangian function of the system. Both of the trial function that satisfy the boundary conditions and the function are substituted into the second Lagrange equation, and a second-order linear ordinary differential equation on the time function is obtained. According to the stability of the equation solution, the determination condition of buckling is determined, and the analytical expression of the critical impact velocity of buckling is derived. Through the example analysis, it is discussed that influence of impact mass, critical length and radius of inertia on the critical impact velocity of the bar under the impact of mass block.

2. Derivation of critical conditions

As shown in figure 1. The rod is free on the left edge, clamped on the right edge. $L$ stands for the length of bar, $A$ denotes area of cross profile, $\rho$ is density, and corresponding displacement of $z$ direction is $w$. The bar is impacted by the rigid block of mass $M$ with velocity $v_0$. When the stress wave propagates to a distance of $l$ from the clamping edge, sudden buckling occurs in the stress section of the member.

![Figure 1](image1.png)

Figure 1 Free-clamped bar subjected to impact of rigid mass

It can be deduced from the characteristic line compatibility condition that the stress wave propagates in the rod before reflection\cite{11}:

\[
\sigma = \begin{cases} 
\rho c v_0 \frac{\rho A x^2}{M} & 0 \leq x \leq ct \\
0 & ct < x \leq L 
\end{cases}
\]  

(1)

The stress wave propagation is shown as figure 2, where $c = \frac{E}{\sqrt{\rho}}$ is the velocity of stress wave:

![Figure 2](image2.png)

Figure 2 Propagation of stress wave

Thus, the axial force of the bar can be expressed as:

\[
P = \begin{cases} 
\rho c v_0 A c^2 \frac{\rho A x^2}{M} & 0 \leq x \leq ct \\
0 & ct < x \leq L
\end{cases}
\]  

(2)

Ignoring the axial inertia, the system kinetic energy $T$, the system strain energy $U$ and the work done by the external force $W$ are respectively:
Lagrange function of the system:

\[ L = T - U + W \]  

(6)

Substituting eq. (1)-(5) into eq. (6), it can be sorted into:

\[ L = \frac{1}{2} \rho A \int_0^l \left( \frac{\partial w}{\partial t} \right)^2 dx - \frac{1}{2} EI \int_0^l \left( \frac{\partial^2 w}{\partial x^2} \right)^2 dx + \frac{1}{2} \int_0^l P \left( \frac{\partial w}{\partial x} \right)^2 dx \]  

(7)

Introduce trial function:

\[ w = X(x)T(t) \]  

(8)

Suppose:

\[ X(x) = k_1 \cos \frac{n\pi}{l} (l - x) + k_2 \cos \frac{n\pi}{l} (l - x) + k_3 \]  

(9)

Substitute that into the boundary condition:

\[ \begin{cases} X(l) = 0 & X'(l) = 0 \\ X''(0) = 0 & X''(0) = 0 \end{cases} \]  

(10)

It can be obtain:

\[ \begin{cases} k_1 + k_2 + k_3 = 0 \\ k_1 \frac{n^2 \pi^2}{l^2} \cos n\pi - k_2 \frac{n^2 \pi^2}{l^2} \cos n\pi = 0 \end{cases} \]  

(11)

Take \( n_1 = n \), \( n_2 = n + 1 \), if there is solution to eq. (11), \( k_1 = 1 \), \( k_2 = \frac{n^2}{(n+1)^2} \), \( k_3 = \frac{2n^2 + 2(n+1)}{(n+1)^2} \), then the solution is:

\[ X(x) = \cos \frac{n\pi}{l} (l - x) + \frac{n^2}{(n+1)^2} \cos \frac{n\pi}{l} (l - x) - \frac{2n^2 + 2(n+1)}{(n+1)^2} \cos \frac{n\pi}{l} (l - x) \]  

(12)

Where \( n \) is the number of buckling modes.

Substitute eq. (8) into eq. (11) to obtain:

\[ L = \frac{1}{2} \rho A \int_0^l X^2 dx \cdot \left( \frac{d^r}{dt^r} \right)^2 - \frac{1}{2} EI \int_0^l \left( \frac{d^2 x}{dx^2} \right)^2 dx \cdot T + \frac{1}{2} \int_0^l P \left( \frac{\partial w}{\partial x} \right)^2 dx \cdot T^2 \]  

(13)

Substitute eq. (13) into the second Lagrange equation:

\[ \frac{\partial L}{\partial T} - \frac{d}{dt} \frac{\partial L}{\partial T} = 0 \]  

(14)

Sorted out:

\[ c_1 \ddot{T} + c_2 T - c_3 T = 0 \]  

(15)

Where:

\[ c_1 = \rho A \int_0^l X^2 dx \quad c_2 = EI \int_0^l \left( \frac{d^2 x}{dx^2} \right)^2 dx \quad c_3 = \rho c v_o A e^{-\frac{\rho l}{M}} i \int_0^l e^{-\frac{\rho l}{M}} \left( \frac{dx}{dx} \right)^2 dx \]  

(16)

Cite:

\[ l_0 = \frac{M}{\rho A} \]  

(17)

Substitute eq. (12) into eq. (16), the three coefficients of eq. (15) can be obtained as follows:

\[ c_1 = \frac{1}{2} \rho A l \quad c_2 = EI \frac{n^2 \pi^2}{l^2} \quad c_3 = \rho c v_o A \frac{n^2 \pi^2}{l^2} l_0 \]  

(18)

Where:

\[ \gamma = n^2 \left( 1 - \frac{1}{2} \left[ \frac{\rho n^2 \pi^2}{M^2} + \frac{\rho n^2 \pi^2}{M^2} + \frac{n^2 \pi^2}{M^2} \right] \right) \]  

(19)

Substitute eq. (18) into eq. (15) to obtain:

\[ \dot{T} + \frac{\alpha + \beta}{l_0} T = 0 \]  

(20)

Where \( \dot{T} = \frac{1}{A} \) cite \( \alpha = \frac{2c^2 v_o^2 n^2 \pi^2}{l_0^2} \), \( \beta = \frac{2c v_o n^2 \pi^2}{l_0^2} l_0 \), eq. (20) can be sort as:

\[ \ddot{T} + (\alpha - \beta) T = 0 \]  

(21)

The properties of the solution of analytic eq. (21) are as follows:
When $\alpha - \beta > 0$, the solution of the equation is a steady trigonometric function with time, and the bar does not buckle.

When $\alpha - \beta < 0$, the solution is exponential diffusion with time, indicating that the bar has buckled.

When $\alpha - \beta = 0$, the bar is in the critical state of buckling, then the critical buckling speed of the bar is:

$$V_{cr} = \frac{\pi^2 n^4 \rho}{2(l_0)^2}$$  \hspace{1cm} (n=0, 1, 2, \ldots) \hspace{1cm} (22)$$

Eq. (22) is buckling critical velocity of clamped-free bar impacted by the rigid block.

3. Examples analysis

For example analysis, the steel rod with rectangular section was taken, the elastic modulus $E=200\text{GPa}$, the density $\rho=7.8\times10^3\text{kg/m}^3$, the length of the rod $L=1\text{m}$, width $b=0.011\text{m}$, height $h=0.005\text{m}$, so that the section area of the rod $A=55\times10^{-6}\text{m}^2$, and the radius of inertia $i=1.443\times10^{-3}\text{m}$.

![Figure 3 Buckling modal of free-clamped bar](image1)

![Figure 4 Relationship between $V_{cr}$ and $l_0$](image2)

Figure 3 is a schematic diagram of the buckling modes when $n$ is 0, 1 and 2. The peak and trough of the buckling mode of the rod are in the same direction. The maximum deformation of the first order mode occurs at $x=0$. The maximum deformation of second order mode is about $x=0.6$. The maximum deformation of the third order mode occurs about $x=0.7$.

![Figure 5 Relationship between $V_{cr}$ and $l$](image3)

![Figure 6 Relationship between $V_{cr}$ and $i$](image4)

Figure 4 is the relation curve of the critical buckling velocity $V_{cr}$ with the mass ratio $l_0$ of the impact mass block, where $l=1\text{m}$, $n=1, 2$ and 3. The $V_{cr}$ decreases with the increase of $l_0$. With the constant $l_0$, the $V_{cr}$ increases with the increase of the number of modes, indicating that the higher modes of buckling are easily inspired with the increase of the initial kinetic energy of impact.

Figure 5 shows the relationship between the buckling critical velocity $V_{cr}$ and the critical length $l$ where $M=0.429\text{kg}$, $l_0=1$ and $n=0, 1$ and 2. The $V_{cr}$ decreases with the increase of $l$, indicating that the
stress wave effect has a significant influence on the dynamic buckling. The $V_c$ increases with the increase of mode number, indicating that the impact velocity increases and the higher mode of buckling is easy to be excited.

Figure 6 shows the relationship curve of the critical buckling velocity $V_c$ with the radius of inertia $i$ where $l=1m$, $M=0.429kg$, $h_0=1$, $n=0, 1, 2$, the section area of the rod is kept unchanged, and the radius of inertia is changed by changing the section shape of the rod. When the area of the bar is constant, the $V_c$ increases with the increase of $i$. That mean, the resistance to impact buckling can be effectively improved by increasing the inertia radius through changing the geometry of the section.

4. Conclusions
The following conclusions are obtained through theoretical analysis and example analysis:

(1) This paper uses the energy method to obtain a second-order linear ordinary differential equation about the time function, determines the buckling determination conditions according to the stability of the solution, and derives the analytical expression of the critical impact velocity of buckling;

(2) An example analysis discusses the effects of modal, impact mass, critical length, and inertia radius on the critical buckling speed of the rod under free-clamped boundary conditions:

The larger the impact mass, the smaller the critical impact velocity, indicating that the initial impact kinetic energy has an important influence on buckling. The increase of the initial impact kinetic energy can easily induce higher-order modes of buckling. The larger the critical length, the smaller the critical impact velocity, indicating the stress wave effect has an important influence on buckling. By changing the cross-section geometry and increasing the cross-section inertia radius, the ability of the member to resist buckling can be effectively improved.

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