Bifunctor Theorem and Gray monoidal structure for double categories with lax double functors

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Abstract

We introduce inner hom for $\text{Dbl}^{st}_{\text{lx}}$, the category of strict double categories and lax double functors, and characterize a lax double functor into it obtaining a lax double quasi-functor. The latter consists of a pair of lax double functors with a 2-cell reminiscent of a distributive law. We extend this characterization to a 2-category isomorphism $q\text{-Lax}_{\text{hop}}(A \times B, C) \cong \text{Lax}_{\text{hop}}(A, [B, C])$. We prove a bifunctor theorem by which certain type of lax double quasi-functors give rise to lax double functors on the Cartesian product, extend it to a 2-functor $q\text{-Lax}_{\text{hop}}^{\text{ns}}(A \times B, C) \rightarrow \text{Lax}_{\text{hop}}(A \times B, C)$ and show when it is a biequivalence. The (un)currying 2-functors are studied. We construct a Gray monoidal product in $\text{Dbl}^{st}_{\text{lx}}$. We prove that a lax double functor from the trivial double category is a monad in the codomain double category, and show that our above 2-functor in the form $q\text{-Lax}_{\text{hop}}^{\text{ns}}(\ast \times \ast, D) \rightarrow \text{Lax}_{\text{hop}}^{\text{ns}}(\ast, D)$ recovers the specification $\text{Comp}(\mathcal{H}(D)) : \text{Mnd}(\text{Mnd}(\mathcal{H}(D))) \rightarrow \text{Mnd}(\mathcal{H}(D))$ of the natural transformation $\text{Comp}$ introduced by Street, where $\mathcal{H}(D)$ is the horizontal 2-category of $D$.

Keywords: bicategories, double categories, Gray monoidal product.

1 Introduction

In recent years the importance of double categories, and more generally of internal categories, has been increasingly recognized in the literature. It was observed by various authors (e.g. [3][4][6]) that it is often more convenient to work in the internal i.e. double category setting, than in the bicategorical one. In Mod-type bicategories the 1-cells are (also) “objects” but of different nature than the 0-cells, and they do not present real maps between 0-cells, so the latter are missing in the picture. This also happens in the 2-category $\text{Mnd}(\mathcal{K})$ of monads in a 2-category $\mathcal{K}$, a fact which gave rise to the introduction of the (pseudo) double category of monads in a (pseudo) double category in [6].
For pseudo double categories the Strictification Theorem is proved in [9, Section 7.5]. One has that the category of pseudo double categories and pseudo double functors is equivalent to the category $Dbl_{st}^{lt}$ of (strict) double categories and (strict) double functors. However, the lax double functors can not be “strictified”, so the category of double categories and lax double functors $Dbl_{st}^{lt}$ is properly more general than $Dbl_{st}^{lt}$. Apart from the Cartesian monoidal product known in the literature for both categories, a Gray type monoidal product $\otimes$ was introduced in [2] for $Dbl_{st}^{lt}$. In [3] we have described the monoidal category structure of $A \otimes B$ for double categories $A$ and $B$. One of the things that we accomplish in the present paper is the construction and description of a Gray type monoidal product for the category $Dbl_{lt}^{lt}$.

Recent results on 2-categories of [4] naturally inspired us to study the analogous properties in double categories. Namely, in loc. cit. conditions were studied for two families of pseudofunctors with a common codomain 2-category to collate into a bifunctor, i.e. a pseudofunctor on the Cartesian product 2-category. (This question corresponds to a 2-category analogue of the first Proposition in [11], page 37.) More generally, the authors proved a version of a bifunctor theorem for lax functors, which even extends to a 2-functor $K : \text{Dist}(A, B, C) \to \text{Lax}_{op}(A \times B, C)$ into the corresponding 2-category of bifunctors. The 2-functor $K$ is proved to restrict to a biequivalence on certain sub-2-categories.

In the case of pseudofunctors above, we noticed that the conditions found by the authors to fulfill the above-mentioned goal are the weak (pseudo) version of the 2-categorical part of the data of a cubical double functor, that we introduced in [5, Definition 2.2]. Namely, starting from the Gray type closed monoidal structure on the category $(Dbl_{st}^{lt}, \otimes)$ constructed in [2], we characterized in [5, Proposition 2.1] a (strict) double functor $F : A \to [[B, C]]$ with the codomain the inner hom object. We obtained that $F$ corresponds to two families of double functors with codomain $C$, satisfying a longer list of conditions. The latter pair of families we called a cubical double functor, in analogy to [8, Section 4.2].

Our above-mentioned observation led us to conjecture that by weakening our characterization in [5, Proposition 2.1] to strict double categories and pseudo double functors, the corresponding “cubical pseudo double functors” would contain - at the level of 2-categories - the exact conditions from [4] needed for collation. Considering this more generally, we pondered and this became one of our main goals in this paper, to study if characterizing lax double functors of strict double categories analogously to [5, Proposition 2.1] would lead to a 2-functor into the corresponding 2-category of lax double bifunctors, generalizing the above 2-functor $K$ to a double category setting. We attained this goal in Section 4 (concretely, we constructed the 2-functor $F$ in Proposition 4.2), where we also identify a biequivalence functor which is a restriction of $F$ (Theorem 4.3). We present this and the rest of our results in more details in the continuation.

We start by introducing a strict double category $[[A, B]]$, for strict double categories $A$ and $B$, by adapting the construction of inner-homs in $Dbl_{st}^{lt}$ from [2, Section 2.2] to our case of $Dbl_{lt}^{lt}$. The horizontal 1-cells in our $[[A, B]]$ are horizontal oplax transformations of lax double functors. We then characterize a lax double functor $F : A \to [[B, C]]$ as a pair of two families of lax double functors into $C$, satisfying a list of properties, that we call a lax double quasi-functor. We use this characterization to describe a double category $A \otimes B$ at the end of Section 2, where $\otimes$ will show in Subsection 5.2 to be a Gray type
monoidal product for $Dbl_{tx}$. In Section 3 we introduce two 2-categories: $\text{Lax}_{hop}(\mathcal{A}, [\mathcal{B}, \mathcal{C}])$, of lax double functors $\mathcal{A} \to [\mathcal{B}, \mathcal{C}]$, horizontal oplax transformations and their modifications, and $q$-$\text{Lax}_{hop}(\mathcal{A} \times \mathcal{B}, \mathcal{C})$, the 2-category consisting of lax double quasi-functors, horizontal oplax transformations of the latter and their modifications. In this double categorical context, the 1-cells in $q$-$\text{Lax}_{hop}(\mathcal{A} \times \mathcal{B}, \mathcal{C})$ have four defining axioms $\text{HOT}_1^q$ – $\text{HOT}_4^q$, whereas in the analogous 2-categorical situation, the 1-cells of the 2-category $\text{Dist}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ from [4] have a single axiom, corresponding to our $\text{HOT}_1^q$, called a Yang-Baxter equation therein. (The 2-category $\text{Dist}(\mathcal{A}, \mathcal{B}, \mathcal{C})$, in turn, is a lax version on 0-cells and an oplax version on 1-cells of the 2-category $q$-$\text{Fun}(\mathcal{A} \times \mathcal{B}, \mathcal{C})$ from [10, Section I.4].) In Subsection 3.4 we prove that the 2-categories $q$-$\text{Lax}_{hop}(\mathcal{A} \times \mathcal{B}, \mathcal{C})$ and $\text{Lax}_{hop}(\mathcal{A}, [\mathcal{B}, \mathcal{C}])$ are isomorphic. Here we find particularly interesting the 1-1 correspondence on 1-cells, for which Proposition 3.5 is very insightful.

The objective in Section 4 is to find a description of lax double quasi-functors in terms of ordinary lax double functors on the Cartesian product $\mathcal{A} \times \mathcal{B}$. The horizontal and vertical 1-cells in $[\mathcal{A}, \mathcal{B}]$ are strong horizontal, respectively vertical (op)lax transformations in the sense of [2, Subsection 7.4], meaning that a part of their data is a non-trivial globular 2-cell. In order to obtain the mentioned description of lax double quasi-functors, we find that it is necessary to quit the strongness of horizontal oplax transformations, in which case we obtain a strict double category $[\mathcal{A}, \mathcal{B}]^{ns}$. Concretely, it is on the isomorphic counterpart $q$-$\text{Lax}_{hop}^{ns}(\mathcal{A} \times \mathcal{B}, \mathcal{C})$ of $\text{Lax}_{hop}(\mathcal{A}, [\mathcal{B}, \mathcal{C}]^{ns})$ (in the isomorphism from the previous paragraph) that we managed to construct the 2-functor $\mathcal{F} : q$-$\text{Lax}_{hop}^{ns}(\mathcal{A} \times \mathcal{B}, \mathcal{C}) \to \text{Lax}_{hop}(\mathcal{A} \times \mathcal{B}, \mathcal{C})$ in Proposition 4.2. Restricting to certain sub-2-categories we obtain a 2-equivalence $\mathcal{F}' : q$-$\text{Lax}_{hop}^{ns-\sigma}(\mathcal{A} \times \mathcal{B}, \mathcal{C}) \to \text{Lax}_{hop}^{ud}(\mathcal{A} \times \mathcal{B}, \mathcal{C})$ in Theorem 4.3. In terms of pseudo double functors it comes down to a 2-equivalence $\mathcal{F}'' : q$-$\text{Ps}_{hop}^{ns}(\mathcal{A} \times \mathcal{B}, \mathcal{C}) \to \text{Ps}_{hop}(\mathcal{A} \times \mathcal{B}, \mathcal{C})$.

The 2-category isomorphism from Section 3, the 2-functor $\mathcal{F}$ above and the 2-equivalences $\mathcal{F}'$ and $\mathcal{F}''$ are generalizations to double categories of the corresponding results in [4]. In Section 5 we show applications of these results of ours in three different contexts. In [3] we obtain a double category version $\text{Lax}_{hop}(\mathcal{A}, [\mathcal{B}, \mathcal{C}]^{ns}) \to \text{Lax}_{hop}(\mathcal{A} \times \mathcal{B}, \mathcal{C})$ of the “uncurrying” 2-functor $j$ from [4, Section 4] and establish a “currying” functor, i.e. a 2-equivalence $\text{Lax}_{hop}^{id}(\mathcal{A} \times \mathcal{B}, \mathcal{C}) \cong \text{Lax}_{hop}^{a}(\mathcal{A}, [\mathcal{B}, \mathcal{C}]^{ns-\sigma})$ in a double category setting.

In Subsection 5.2 we prove a 2-category isomorphism $q$-$\text{Lax}_{hop}^{ns}(\mathcal{A} \times \mathcal{B}, \mathcal{C}) \cong \text{Lax}_{hop}(\mathcal{A} \times \mathcal{B}, \mathcal{C})$, yielding further $\text{Lax}_{hop}(\mathcal{A} \times \mathcal{B}, \mathcal{C}) \cong \text{Lax}_{hop}(\mathcal{A}, [\mathcal{B}, \mathcal{C}]^{ns})$, and we finally conclude that $\otimes$ is a Gray type monoidal product on the category $(Dbl_{tx}^{st}, \otimes)$. The final Subsection is devoted to applications to monads in double categories. In it we show that a monad in a double category $\mathcal{D}$, as defined in [6, Definition 2.4], is a lax double functor $* \to \mathcal{D}$ from the trivial double category. Moreover, we obtain isomorphisms of 2-categories $\text{Lax}_{hop}(\mathbb{1}, \mathcal{D}) \cong \text{Mnd}(\mathcal{H}([\mathcal{D}])))$ and $q$-$\text{Lax}_{hop}(\mathcal{D} \times \mathcal{D}) \cong \text{Mnd}(\mathcal{H}([\mathcal{D}])))$, where $\mathcal{H}([\mathcal{D}])$ is the horizontal 2-category of $\mathcal{D}$. We argue that a version of our 2-functor $\mathcal{F}$ from above, $q$-$\text{Lax}_{hop}(\mathcal{D} \times \mathcal{D}) \to \text{Lax}_{hop}(\mathcal{D} \times \mathcal{D})$, corresponds via the above isomorphisms to the natural transformation $\text{Comp} : \text{MndMnd} \to \text{Mnd}$ evaluated at the 2-category $\mathcal{H}([\mathcal{D}])$. 





2 Towards a Gray monoidal product in \((\text{Dbl}^\text{st}_\text{lx}, \otimes)\)

Let \(\text{Dbl}^\text{st}_\text{lx}\) denote the category of (strict) double categories and lax double functors. For two (strict) double categories \(A, B\) we will construct a (strict) double category \([A, B]\) analogous to one in \([2\text{ Section 2.2}]\), which will induce a functor \([-, -] : (\text{Dbl}^\text{st}_\text{lx})^\text{op} \times \text{Dbl}^\text{st}_\text{lx} \to \text{Dbl}^\text{st}_\text{lx}\). Then we will characterize a lax functor \(F : A \to [B, C]\) for another (strict) double category \(C\) in terms of a bifunctor from the Cartesian product \(A \times B \to C\) of (strict) double categories. Setting \(C = A \times B\) and reading off the structure of the image double category \(F(A)(B)\), we will obtain a full description of a new structure on the underlying (strict) double category \(A \times B\). Thus obtained structure we will denote by \(A \otimes B\). In Subsection 5.2 we will see how this provides a Gray type monoidal product on the category \(\text{Dbl}^\text{st}_\text{lx}\).

Let us fix the notation in a (strict) double category \(D\). Objects we denote by \(A, B, \ldots\), horizontal 1-cells we will call for brevity 1h-cells and denote them by \(f, f', g, F, \ldots\), vertical 1-cells we will call 1v-cells and denote by \(u, v, U, \ldots\), and squares we will call just 2-cells and denote them by \(\alpha, \zeta, \ldots\). We denote the horizontal identity 1-cell by \(1_A\), vertical identity 1-cell by \(1^A\) for an object \(A \in D\), horizontal identity 2-cell on a 1v-cell \(u\) by \(\text{Id}_u\), and vertical identity 2-cell on a 1h-cell \(f\) by \(\text{Id}_f\) (with subindexes we denote those identity 1- and 2-cells which come from the horizontal 2-category lying in \(D\)). For a (vertically) globular 2-cell \(\alpha\), that is, a one whose 1v-cells are identities, we will write \(\alpha : f \Rightarrow g\) as in bicategories. The composition of 1h-cells as well as the horizontal composition of 2-cells we will denote by juxtaposition, while the composition of 1v-cells as well the vertical composition of 2-cells we will denote by fractions \(\frac{\alpha}{\beta}\). When combining horizontal and vertical composition of 2-cells we will also use the notation: \([\alpha, \beta] := \beta\alpha\) for the horizontal composition.

We start by noticing that a lax double functor \(F : C \to D\) is given by: 1) the data: images on objects, 1h-, 1v- and 2-cells of \(C\), globular 2-cells \(F(g)F(f) \Rightarrow F(gf), \quad 1_{F(A)} \Rightarrow F(1_A)\) in \(D\), and 2) rules (in \(D\)):

\[
\begin{align*}
\frac{F(a)}{F(a')}, \quad 1_{F(A)} = F(1^A), \quad \frac{F(\omega)}{F(\zeta)} = \frac{F(\omega)}{F(\zeta)}, \quad \frac{F(\lambda|\text{Id}_{\phi(r)})}{F_{\beta|\text{Id}_{\phi(s)}}} = \frac{F(\lambda|\text{Id}_{\phi(r)})}{F_{\beta|\text{Id}_{\phi(s)}}},
\end{align*}
\]

for composable 1h-cells \(f, g, h\) and

\[
\begin{align*}
\frac{F(A)}{F(A)} \Rightarrow \frac{F(A)}{F(u)} \Rightarrow \frac{F(u)}{F(1_{A'})} \Rightarrow \frac{F(1_{A'})}{F(A')} \Rightarrow \frac{F(A')}{F(A')},
\end{align*}
\]

\[
\begin{align*}
\frac{F(A)}{F(A)} \Rightarrow \frac{F(A)}{F(1_A)} \Rightarrow \frac{F(1_A)}{F(A)} \Rightarrow \frac{F(A)}{F(u)} \Rightarrow \frac{F(u)}{F(1_{A'})} \Rightarrow \frac{F(1_{A'})}{F(A')},
\end{align*}
\]
In order to avoid a lengthy definition, we give the following one indicating only the differences with the definition from [2 Section 2.2].

**Definition 2.1** We define a (strict) double category \([\mathcal{A}, \mathcal{B}]\) as done in [2 Section 2.2], with the following differences:

1. the 0-cells are lax double functors;

2. in 1h-cells horizontal functoriality holds like this:

   ![Diagram of Functors]

   and

   ![Diagram of Functors]

3. in 1v-cells horizontal functoriality holds like this:

   ![Diagram of Functors]

for composable 1h-cells \(h, k\).

**Remark 2.2** In this definition we kept the notation \(x, y\) from [2 Section 2.2]. There the double functors are strict, whereas in our case here they are lax, hence only these three items in the two definitions differ, so that now the non-trivial 2-cells for coherence of double functors with horizontal compositions appear. In our notation in [3], \(x_A : F(A) \to G(A)\) corresponds to \(\alpha_1(A)\), \(y_A\) corresponds to \(\delta_{\alpha_0, h}\), \(y_A : F(A) \to H(A)\) corresponds to \(\alpha_0(A)\) and \(y^h\) corresponds to \((\alpha_0)_h\).
The 1h-cells in $[A, B]$ we will call \textit{horizontal oplax transformations between lax double functors}, and the 1v-cells: \textit{vertical lax transformations between lax double functors}. The 2-cells are the modifications. The full definition of 1h-cells, \textit{i.e.} of the horizontal oplax transformations between lax double functors we will give in Definition 3.1. (Observe that the modifications in Definition 3.2 being modifications between the horizontal oplax transformations, correspond to the \textit{“globular”} type of modifications which are 2-cells in $[A, B]$.)

Since in this paper we will deal only with horizontal oplax transformations (and not with vertical lax ones), the above-mentioned notation $\alpha_1(A)$ and $\delta_{n,j}\alpha_1$, we will use here without index 1 (the subindex 1 refers to horizontal, while the subindex 0 refers to vertical transformations).

Thus defined double category is indeed strict: this relies on the facts that the double categories $A$ and $B$ are strict, and that the associatively and unitality of 2-cells in both horizontal and vertical direction hold strictly.

Having in mind the characterization of a lax double functor before Definition 2.1 and the definition of $[[B, C]]$, writing out the list of the data and relations that determine a lax double functor $\mathcal{F} : A \to [[B, C]]$, one gets the following characterization of it:

**Proposition 2.3** A lax double functor $\mathcal{F} : A \to [[B, C]]$ of (strict) double categories consists of the following:

1. lax double functors $(-, A) : B \to C$ and $(B, -) : A \to C$

such that $(-, A)|_B = (B, -)|_A = (B, A)$, for objects $A \in A, B \in B$,

2. given 1h-cells $A \xrightarrow{f} A'$ and $B \xrightarrow{g} B'$ and 1v-cells $A \xrightarrow{U} \tilde{A}$ and $B \xrightarrow{\tilde{U}} \tilde{B}$ there are 2-cells

\[
\begin{array}{c}
(B, A) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
(B, B') \quad (B', A') \quad (B, A') \quad (B, A') \quad (B', A') \quad (B', A')
\end{array}
\]

\[
\begin{array}{c}
(u, A) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
(B, A) \quad (B, A') \quad (B, A') \quad (B, A') \quad (B, A') \quad (B, A')
\end{array}
\]

\[
\begin{array}{c}
(u, A) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
(B, A) \quad (B, A') \quad (B, A') \quad (B, A') \quad (B, A') \quad (B, A')
\end{array}
\]

which satisfy:
a) (11)
\[
(B, A) \xrightarrow{(f, A)} (B', A) \xrightarrow{(f', F)} (B'', A)
\]
and
\[
(B, A) \xrightarrow{(-, A)_A} (B', A) \xrightarrow{(f, A)} (B'', A)
\]

where the 2-cells \((-, A)_A\) and \((B, -)_A\) come from laxity of the lax double functors \((-, A)\) and \((B, -)\);

(21) \(1^B,F = \text{Id}_{(B,F)}\) and \((u, 1_A) = \text{Id}^{(u,A)}\)

(12) \(1^B,U = \text{Id}^{(B,U)}\) and \((f, 1^A) = \text{Id}^{(f,A)}\)

(22) \((1^B, U) = \text{Id}^{(B,U)}\) and \((u, 1^A) = \text{Id}^{(u,A)}\);

b) (11)
\[
(B', A) \xrightarrow{(f', A)} (B'', A) \xrightarrow{(B'', F)} (B'', A')
\]

where \((-, A)_{f,f}\) is the 2-cell from the laxity of \((-, A)\), and

\[
(B', A) \xrightarrow{(f', A)} (B'', A') \xrightarrow{(B'', F') \cdot (B'', F)} (B'', A'')
\]
where \((B, -)_{F, F'}\) is the 2-cell from the laxity of \((B, -)\), and

\[
(\mu, F) = \left(\frac{u, F}{u, F}\right) \text{ and }
\]

\[
\begin{align*}
(B, A) & \xrightarrow{(f, A)} (B', A) \\
(B, A) & \xrightarrow{(f', A)} (B''', A) \\
\end{align*}
\]

\[
\begin{align*}
(B, A) & \xrightarrow{(f, A)} (B, A') \\
(B, A) & \xrightarrow{(B', F')} (B, A'') \\
\end{align*}
\]

\[
\begin{align*}
(B, A) & \xrightarrow{(f, A)} (B', A') \\
(B, A) & \xrightarrow{(B', F')} (B, A'') \\
\end{align*}
\]

\[
\begin{align*}
(B, A) & \xrightarrow{(f, A)} (B, A) \\
(B, A) & \xrightarrow{(f, A')} (B, A') \\
\end{align*}
\]

\[
\begin{align*}
(B, A) & \xrightarrow{(f, A')} (B, A'') \\
(B, A) & \xrightarrow{(u, F')} (B, A'') \\
\end{align*}
\]

\[
\begin{align*}
(B, A) & \xrightarrow{(f, A)} (B', A) \\
(B, A) & \xrightarrow{(f', A)} (B'', A) \\
\end{align*}
\]

\[
\begin{align*}
(B, A) & \xrightarrow{(f, A)} (B, A) \\
(B, A) & \xrightarrow{(f, A')} (B, A') \\
\end{align*}
\]

\[
\begin{align*}
(B, A) & \xrightarrow{(f, A)} (B, A'') \\
(B, A) & \xrightarrow{(f, A')} (B, A'') \\
\end{align*}
\]

\[
\begin{align*}
(B, A) & \xrightarrow{(f, A)} (B, A) \\
(B, A) & \xrightarrow{(f, A')} (B, A') \\
\end{align*}
\]

\[
\begin{align*}
(B, A) & \xrightarrow{(f, A)} (B, A'') \\
(B, A) & \xrightarrow{(u, F')} (B, A'') \\
\end{align*}
\]

and \( (f, \frac{U}{U'}) = \left(\frac{f, U}{f, U'}\right) \)
c) (11)

\[
\begin{align*}
(B, A) & \xrightarrow{(f, A)} (B', A) \xrightarrow{(B', F)} (B', A') \\
= & \xrightarrow{(f, F)} \xrightarrow{(B', A')} (B', A') \\
(u, A) & \xrightarrow{(u, F)} (B', A') \xrightarrow{(g, A')} \xrightarrow{(g, F)} (v, A') \\
(B, A) & \xrightarrow{(B, F)} (B', A) \xrightarrow{(B', F)} (B', A') \\
= & \xrightarrow{(B, U)} (B', U) \xrightarrow{(B', \tilde{C})} (B', V) \\
(u, \tilde{A}) & \xrightarrow{(u, \tilde{A})} (B', U) \xrightarrow{(g, \tilde{A})} (B', \tilde{A}) \xrightarrow{(g, \tilde{A})} (B', \tilde{A}) \\
(B, \tilde{A}) & \xrightarrow{(B, \tilde{A})} (B, \tilde{A}) \xrightarrow{(B', A')} \xrightarrow{(B', A')} (B', A') \\
(B, A) & \xrightarrow{(B, F)} (B, A') \xrightarrow{(u, A')} \xrightarrow{(u, \tilde{A})} (B', \tilde{A}) \\
(u, \tilde{A}) & \xrightarrow{(u, \tilde{A})} (B, \tilde{A}) \xrightarrow{(B, \tilde{A})} (B, \tilde{A}) \xrightarrow{(B', A')} \xrightarrow{(B', \tilde{A})} (B', \tilde{A}) \\
(B, A) & \xrightarrow{(B, A)} (B, A') \xrightarrow{(B, A')} \xrightarrow{(B, A')} (B, A') \\
\end{align*}
\]

and

\[
\begin{align*}
(B, \tilde{A}) & \xrightarrow{(B, \tilde{A})} (B', \tilde{A}) \xrightarrow{(B', \tilde{A})} (B', \tilde{A}) \\
(u, \tilde{A}) & \xrightarrow{(u, \tilde{A})} (B', \tilde{A}) \xrightarrow{(B', \tilde{A})} (B', \tilde{A}) \\
(B, \tilde{A}) & \xrightarrow{(B, \tilde{A})} (B, \tilde{A}) \xrightarrow{(B', \tilde{A})} (B', \tilde{A}) \xrightarrow{(B', \tilde{A})} (B', \tilde{A}) \\
(u, \tilde{A}) & \xrightarrow{(u, \tilde{A})} (B, \tilde{A}) \xrightarrow{(B, \tilde{A})} (B, \tilde{A}) \xrightarrow{(B, \tilde{A})} (B, \tilde{A}) \\
\end{align*}
\]

for any 2-cells

\[
\begin{align*}
& B \xrightarrow{f} B' \quad \text{and} \quad A \xrightarrow{F} A' \\
& u \xrightarrow{\omega} v \quad \text{in} \ B, \text{respectively} \ A.
\end{align*}
\]
Remark 2.4 That \((-, A) : B \to C\) is a lax double functor and 14 of the 20 rules in item 2. of the above Proposition cover the “data” that \(\mathcal{F} : A \to \llbracket B, C \rrbracket\) generates by mapping the four types of cells. The resting 6 rules are: b) (11)/2 and b) (21)/2 determine how \(\mathcal{F}(F'F)\) is defined for 1h-cells \(F, F'\) in \(A\); b) (12)/2 and b) (22)/1 determine how \(\mathcal{F}(U_{\mathcal{F}})\) is defined for 1v-cells \(U, U'\) in \(A\); a) (11)/2 and a) (21)/2 determine the (globular) modification \(\text{Id}_{(-, A)} = (\cdot, A)\), which corresponds to the 2-cell \(1_F(A) \Rightarrow \mathcal{F}(1_A)\) in \(\llbracket B, C \rrbracket\) from the characterization of a lax double functor before Definition 2.1. On the other hand, by this characterization \(\mathcal{F}\) is equipped with two 2-cells \(\mathcal{F}_{PF} \) and \(\mathcal{F}_{B}\) that satisfy the six listed rules. If we think of \(\mathcal{F}(\bullet)(\cdot) = (\cdot, \cdot)\), then when plugging in \(B\) in the left slot in the image these 2-cells and six rules mean that \((B, \cdot) : A \to C\) is a lax double functor. Plugging in the rest of cells in \(B\) yields identities already listed in the above 20 rules (plugging in a 2-cell \(\omega\) only makes sense in the case \((\omega, A)\)).

In analogy to \([8, \text{Section 4.2}]\) we set:

**Definition 2.5** The characterization in the above Proposition gives rise to an application \(H : A \times B \to C\) such that \(H(A, -) = (-, A)\) and \(H(-, B) = (B, -)\). Such an application of double categories we will call lax double quasi-functor.

From the data in the above Proposition we may draw several consequences.

**Corollary 2.6** For 1h-cells \(F : A \to A'\) in \(A\) and \(f : B \to B'\) in \(B\) one has that \((-, F) : (-, A) \to (-, A)\) and \((f, -) : (B, -) \to (B', -)\) are horizontal oplax transformations of lax double functors (Remark 2.2).

**Proof.** Name \(\alpha := (-, F)\), then set \(\alpha(B) = (B, F), \delta_{\alpha, f} = (f, F)\) for a 1h-cell \(f : B \to B'\) in \(B\) and \(\alpha_u = (u, F)\) for a 1v-cell \(u : B \to B\), and similarly for \((f, -)\). Then the claim follows from the properties c) (11), first one in b) (21) and second one in b) (12), first one in a) (21) and second one in a) (12), b) (11) and a) (11) of Proposition 2.3. \(\Box\)

We may now describe a double category \(A \otimes B\) by reading off the structure of the image double category \(F(A)(B)\) for any double functor \(F : A \to \llbracket B, A \times B \rrbracket\) using the characterization of a double functor before Proposition 2.3. Substituting \(F(x)(y)\) by \(x \otimes y\) for any 0-, 1h-, 1v- or 2-cells \(x\) of \(A\) and \(y\) of \(B\) we obtain that a double category \(A \otimes B\) is generated by the following data:

**objects:** \(A \otimes B\) for objects \(A \in A, B \in B\);

**1h-cells:** \(A \otimes f, F \otimes B\) and horizontal compositions of such (modulo associativity and unity constraints) obeying the following rules:

\[(A \otimes f')(A \otimes f) = A \otimes (f' f), \quad (F' \otimes B)(F \otimes B) = (F' F) \otimes B, \quad A \otimes 1_B = 1_{A \otimes B} = 1_A \otimes B\]

where \(f, f'\) are 1h-cells of \(B\) and \(F, F'\) 1h-cells of \(A\);

**1v-cells:** \(A \otimes u, U \otimes B\) and vertical compositions of such obeying the following rules:

\[A \otimes u = A \otimes \frac{u}{u'}, \quad U \otimes B = \frac{U}{U'} \otimes B, \quad A \otimes 1_B = 1_{A \otimes B} = 1_A \otimes B\]
where \( u, u' \) are \( 1 \)-cells of \( B \) and \( U, U' \) \( 1 \)-cells of \( A \);

2-cells: \( A \otimes \omega, \zeta \otimes B \):

\[
\begin{array}{ccc}
A \otimes B & A \otimes f & A \otimes B' \\
A \otimes u & A \otimes \omega & A \otimes u' \\
A \otimes \bar{B} & A \otimes u & A \otimes \bar{B}'
\end{array}
\]

\[
\begin{array}{ccc}
A \otimes B & F \otimes B & A' \otimes B \\
U \otimes B & C \otimes B & V \otimes B \\
\bar{A} \otimes B & \bar{A} \otimes C & \bar{A} \otimes B'
\end{array}
\]

where \( \omega \) and \( \zeta \) are as in (1), and four types of 2-cells coming from the 2-cells of point 2. in Proposition 2.3: vertically invertible globular 2-cell \( F \otimes f : (A' \otimes f)(F \otimes B) \Rightarrow (F \otimes B')(A \otimes f) \), horizontally invertible globular 2-cell \( U \otimes u : F \otimes u \Rightarrow F \otimes \bar{B} \), 2-cells \( F \otimes u \) and \( U \otimes f \), and the horizontal and vertical compositions of these (modulo associativity and unity constraints in the horizontal direction and the interchange law) subject to the rules induced by a), b) and c) of point 2. in Proposition 2.3 and the following ones:

\[
\begin{align*}
A \otimes (\omega' \omega) &= (A \otimes \omega')(A \otimes \omega), & (\zeta' \zeta) \otimes B &= (\zeta' \otimes B)(\zeta \otimes B), \\
A \otimes \omega &= A \otimes \omega' & \zeta \otimes B &= \zeta' \otimes B, \\
A \otimes \text{Id}_f &= \text{Id}_{A \otimes f}, & \text{Id}_f \otimes B &= \text{Id}_{F \otimes B}, & A \otimes \text{Id}_u &= \text{Id}_{A \otimes u}, & \text{Id}_U \otimes B &= \text{Id}_{U \otimes B}.
\end{align*}
\]

3 The 2-categories \( \text{Lax}_{hop}(\mathcal{A}, [\mathcal{B}, \mathcal{C}]) \) and \( q\text{-Lax}_{hop}(A \times \mathcal{B}, \mathcal{C}) \) are isomorphic

Observe that distributive laws of lax functors of 2-categories defined in [4, Definition 3.1] are lax version of the “quasi-functors of two variables” of [10, Definition I, 4.1]. The single condition QF\(_2\) of the latter is equivalent to the two conditions (D5) and (D6) of the former. In [7,8] the 2-cells of a quasi-functor of two variables, corresponding to \( \sigma_{f,g} \) of [4] and \( \gamma_{f,g} \) of [10], were considered to be invertible. Such a quasi-functor of two variables in these references was called “cubical functor”. In Proposition 2.1 and Definition 2.2 of [5] we generalized cubical functors to strict double categories and called them \textit{cubical double functors}.

In Proposition 2.3 and Definition 2.5 above we generalized cubical double functors to the lax case. (Observe that the corresponding 2-cell mentioned in the above paragraph is not invertible, so we do not work here in a cubical setting, and follow Gray’s terminology.) Thus our Proposition 2.3 is a generalization to the double category setting of [4, Lemma 4.1].

Morphisms of distributive laws of lax functors from [4, Definition 4.3] are the oplax version of quasi-natural transformations from [10, Definition I, 4.1]. In this Section we will first give the definitions of horizontal oplax transformations between lax double functors and modifications among them, then we will introduce the corresponding notions in the lax double quasi-functor setting, and finally prove that the latter are in 1-1 correspondence with the horizontal oplax transformations between lax double functors of the form \( A \rightarrow [\mathcal{B}, \mathcal{C}] \) and their modifications.
3.1 The 2-category $\text{Lax}_{hop}(\mathcal{A}, [\mathcal{B}, \mathcal{C}])$

This 2-category consists of lax double functors $\mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$ (with $[\mathcal{B}, \mathcal{C}]$ from Definition 2.1), horizontal oplax transformations of the latter lax double functors and modifications of the latter. For reader’s convenience we give the explicit definitions of horizontal oplax transformations of lax double functors and their modifications.

**Definition 3.1** A horizontal oplax transformation between lax double functors $F, G : \mathcal{A} \rightarrow \mathcal{B}$ consists of the following:

- for every 0-cell $A$ in $\mathcal{A}$ a 1h-cell $\alpha(A) : F(A) \rightarrow G(A)$ in $\mathcal{B}$,
- for every 1v-cell $u : A \rightarrow A'$ in $\mathcal{A}$ a 2-cell in $\mathcal{B}$:

$$
\begin{array}{ccc}
F(A) & \xrightarrow{\alpha(A)} & G(A) \\
F(u) & \xrightarrow{\alpha_u} & G(u) \\
F(A') & \xrightarrow{\alpha(A')} & G(A')
\end{array}
$$
- for every 1h-cell $f : A \rightarrow B$ in $\mathcal{A}$ there is a 2-cell in $\mathcal{B}$:

$$
\begin{array}{ccc}
F(A) & \xrightarrow{F(f)} & F(B) \\
= & \xrightarrow{\delta_{\alpha,f}} & = \\
F(A) & \xrightarrow{\alpha(A)} & G(A) \\
F(A') & \xrightarrow{\alpha(A')} & G(A')
\end{array}
$$

so that the following are satisfied:

1. **Pseudonaturality of 2-cells:** for every 2-cell in $\mathcal{A}$ the following identity in $\mathcal{B}$ must hold:

$$
\begin{array}{ccc}
F(A) & \xrightarrow{F(f)} & F(B) \\
F(u) & \xrightarrow{\alpha_u} & G(u) \\
F(A') & \xrightarrow{\alpha(A')} & G(A')
\end{array}
\quad=
\begin{array}{ccc}
F(A) & \xrightarrow{F(f)} & F(B) \\
= & \xrightarrow{\delta_{\alpha,f}} & = \\
F(A) & \xrightarrow{\alpha(A)} & G(A) \\
F(A') & \xrightarrow{\alpha(A')} & G(A')
\end{array}
$$

2. **Vertical functoriality:** for any composable 1v-cells $u$ and $v$ in $\mathcal{A}$:

$$
\alpha_u^v = \frac{\alpha_u}{\alpha_v} \quad \text{and} \quad \alpha_{1^A} = \text{Id}_{\alpha(A)};
$$
3. horizontal functoriality for $\delta_{a,-}$: for any composable 1h-cells $f$ and $g$ in $\mathbf{A}$ the 2-cell $\delta_{a,gf}$ satisfies:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
F(A) F(f) \xrightarrow{\delta_{a,f}} F(g) \\
\xrightarrow{\delta_{a,gf}} F(gf) = \\
\xrightarrow{\alpha(A)} G(A) \\
\end{array}
\end{array}
\end{array}
\]

and $\delta_{a,id_A}$:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
F(A) = \\
\xrightarrow{\delta_{a,id_A}} F(id_A) \\
\xrightarrow{\alpha(A)} G(A(A)) \\
\xrightarrow{\delta_{a,id_A}} G(id_A) \\
\end{array}
\end{array}
\end{array}
\]

The above Definition is an “oplax and horizontal” version of a “strong vertical transformation” from [9, Section 7.4] for strict (rather then pseudo) double categories. Similarly, the following is a horizontal version of a “strong modification” from loc. cit. with $\mathcal{H}$ and $\mathcal{K}$ being identities.

**Definition 3.2** A modification between two horizontal oplax transformations $\alpha$ and $\beta$ which act between lax double functors $F \Rightarrow G$ (acting between $\mathbf{A} \rightarrow \mathbf{B}$) is an application $\alpha : \alpha \Rightarrow \beta$ such that for each 0-cell $A$ in $\mathbf{A}$ there is a vertically globular 2-cell $a(A) : \alpha(A) \Rightarrow \beta(A)$ in $\mathbf{B}$ which for each 1h-cell $f : A \rightarrow B$ and 1v-cell $u : A \rightarrow A'$ satisfies:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
F(A) F(f) \xrightarrow{\delta_{a,f}} G(B) \\
\xrightarrow{\alpha(A)} G(f) = \\
\xrightarrow{\beta(A)} G(A) \\
\end{array}
\end{array}
\end{array}
\]

and

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
F(A) \xrightarrow{\alpha(A)} G(A) \\
\xrightarrow{\delta_{a,u}} F(u) = \\
\xrightarrow{\beta(A)} G(A') \\
\end{array}
\end{array}
\end{array}
\]
3.2 The 2-category $q$-Lax$_{hop}(A \times B, C)$

By $q$-Lax$_{hop}(A \times B, C)$ we will denote the 2-category consisting of lax double quasi-functors, horizontal oplax transformations of lax double quasi-functors and modifications among the latter. In this Subsection we define its 1- and 2-cells.

**Definition 3.3** A horizontal oplax transformation $\Theta : (-, -)_1 \Rightarrow (-, -)_2$ between lax double quasi-functors $(-, -)_1, (-, -)_2 : A \times B \rightarrow C$ is given by collections: for each $A \in A$ a horizontal oplax transformation $\theta^A : (-, A)_1 \rightarrow (-, A)_2$ and for each $B \in B$ an oplax transformation $\theta^B : (B, -)_1 \rightarrow (B, -)_2$, both of lax double functors, such that $\theta^A = \theta^B$ and such that

$$\text{(HOT)}_q^A$$

\[
\begin{array}{c}
(B', A)_1 \xrightarrow{(B', f)_1} (B', A')_1 \\
\downarrow \quad \downarrow \quad \downarrow \\
(B, A)_1 \xrightarrow{(B, f)_1} (B, A')_1
\end{array}
\]

$$\text{(HOT)}_q^B$$

\[
\begin{array}{c}
(B, A)_1 \xrightarrow{(B, f)_1} (B, A')_1 \\
\downarrow \quad \downarrow \quad \downarrow \\
(B, A)_2 \xrightarrow{(B, f)_2} (B, A')_2
\end{array}
\]

for every 1h-cells $f : A \rightarrow A'$ and $g : B \rightarrow B'$,

$$\text{(HOT)}_q^A$$

\[
\begin{array}{c}
(u, A)_1 \xrightarrow{(u, f)_1} (u, A')_1 \\
\downarrow \quad \downarrow \quad \downarrow \\
(\bar{B}, A)_1 \xrightarrow{(\bar{B}, f)_1} (\bar{B}, A')_1
\end{array}
\]

$$\text{(HOT)}_q^B$$

\[
\begin{array}{c}
(u, A)_1 \xrightarrow{(u, f)_1} (u, A')_1 \\
\downarrow \quad \downarrow \quad \downarrow \\
(\bar{B}, A)_2 \xrightarrow{(\bar{B}, f)_2} (\bar{B}, A')_2
\end{array}
\]

for every 1h-cell $f : A \rightarrow A'$ and 1v-cell $u : B \rightarrow \bar{B}$,

$$\text{(HOT)}_q^B$$

\[
\begin{array}{c}
(B, A)_1 \xrightarrow{(B, f)_1} (B, A')_1 \\
\downarrow \quad \downarrow \quad \downarrow \\
(\bar{B}, A)_1 \xrightarrow{(\bar{B}, f)_1} (\bar{B}, A')_1
\end{array}
\]

$$\text{(HOT)}_q^A$$

\[
\begin{array}{c}
(B, A)_1 \xrightarrow{(B, f)_1} (B, A')_1 \\
\downarrow \quad \downarrow \quad \downarrow \\
(\bar{B}, A)_2 \xrightarrow{(\bar{B}, f)_2} (\bar{B}, A')_2
\end{array}
\]

for every 1v-cell $U : A \rightarrow \tilde{A}$ and 1h-cell $g : B \rightarrow B'$, and

$$\text{(HOT)}_q^A$$

\[
\begin{array}{c}
(B, A)_1 \xrightarrow{(B, f)_1} (B, A')_1 \\
\downarrow \quad \downarrow \quad \downarrow \\
(\bar{B}, A)_1 \xrightarrow{(\bar{B}, f)_1} (\bar{B}, A')_1
\end{array}
\]

\[
\begin{array}{c}
(B, A)_1 \xrightarrow{(B, f)_1} (B, A')_1 \\
\downarrow \quad \downarrow \quad \downarrow \\
(\bar{B}, A)_2 \xrightarrow{(\bar{B}, f)_2} (\bar{B}, A')_2
\end{array}
\]

for every 1v-cell $U : A \rightarrow \tilde{A}$ and 1h-cell $g : B \rightarrow B'$, and
for every 1v-cells $U : A \rightarrow \tilde{A}$ and $u : B \rightarrow \tilde{B}$.

**Definition 3.4** A modification $a : \Theta \Rightarrow \Theta'$ of horizontal oplax transformations $\Theta, \Theta'$ between lax double quasi-functors $F, G : A \times B \rightarrow C$ is given by modifications of horizontal oplax transformations of lax double functors: $\tau^A : \Theta^A \rightarrow (\Theta')^A$ for each $A \in A$ and $\tau^B : \Theta^B \rightarrow (\Theta')^B$ for each $B \in B$, such that $\tau^A_B = \tau^B_A$ (globular 2-cells) for every $A \in A, B \in B$.

### 3.3 The 1-1 correspondence between 1-cells

We proceed to show that the 2-categories $\text{Lax}_{\text{ho}}(A, [B, C])$ and $\text{q-Lax}_{\text{ho}}(A \times B, C)$ are isomorphic. From Proposition 2.3 and Definition 2.5 we know that we have a 1-1 correspondence between their corresponding 0-cells.

Let $F, G : A \rightarrow [B, C]$ be two lax double functors and take a horizontal oplax transformation $\alpha : F \Rightarrow G$. Set $(-, -)_1$ and $(-, -)_2$ for the two lax double quasi-functors obtained from $F$ and $G$, respectively. Evaluating at a 0-cell $A \in A$ we get $\alpha(A) : F(A) \rightarrow G(A)$ a 1h-cell in $[B, C]$ of the form $(-, -)_1 \rightarrow (-, -)_2$. Being this 1h-cell a horizontal oplax transformation between lax double functors, it can be further evaluated at a 0-cell $B \in B$ (yielding a 1h-cell $\alpha(A)_B : (B, A)_1 \rightarrow (B, A)_2$ in $C$), and also at a 1h-cell $g : B \rightarrow B'$ (yielding a globular 2-cell $\alpha(A)_g = \delta_{\alpha(A)_g}$ in $C$), and at a 1v-cell $u : B \rightarrow \tilde{B}$ (yielding a 2-cell $\alpha(A)_u$ in $C$). The 2-cells $\alpha(A)_g$ and $\alpha(A)_u$ in $C$ satisfy five axioms from Definition 3.1.

On the other hand, evaluating the horizontal oplax transformation $\alpha : F \Rightarrow G$ at a 1h-cell $f : A \rightarrow A'$ in $A$, one obtains a globular 2-cell $\alpha_f = \delta_{\alpha(f)} : \alpha(A) \Rightarrow \alpha(A')$ in $[B, C]$, which, by Definition 2.1, is a modification between (the vertical composition of) horizontal oplax transformations of lax double functors. (As such it is a modification in the sense of Definition 3.4.) It has a free slot for 0-cells in $B$, so that after evaluation at some $B \in B$ it yields a globular 2-cell $\alpha_f(B)$ in $C$. Finally, evaluating $\alpha$ at a 1v-cell $U : A \rightarrow \tilde{A}$ in $A$, one obtains a 2-cell $\alpha_U$:

$$
\begin{equation}
\begin{array}{c}
(B, A)_1 \xrightarrow{\alpha(A)} (B, A)_2 \\
(B, U)_1 \xrightarrow{(\cdot)} (B, \tilde{A})_1 \xrightarrow{\alpha(\tilde{A})} (B, \tilde{A})_2 \\
\end{array}
\end{equation}
$$

in $[B, C]$ (thus a modification in the sense of Remark 2.2) with a free slot for 0-cells in $B$ (after evaluation at $B \in B$ it yields a 2-cell $\alpha_U(B)$ in $C$).

The families of 2-cells $\alpha_f$ and $\alpha_U$ in $[B, C]$ from the horizontal oplax transformation $\alpha : F \Rightarrow G$ satisfy five axioms from Definition 3.1. Evaluating these five axioms at $B \in B$ one obtains five axioms for families of 2-cell $\alpha_f(B)$ and $\alpha_U(B)$ in $C$. The latter
axioms mean that $\alpha(-)_B : F(-)(B) \Rightarrow G(-)(B)$, obtained by reading $\alpha : F \Rightarrow G$ described above the other way around, that is, evaluating at a 0-cell $B \in \mathcal{B}$ and leaving a free slot for cells from $\mathcal{A}$, is a horizontal oplax transformation between lax double functors $(B,-)_1 \rightarrow (B,-)_2$ which act between $\mathcal{A} \rightarrow \mathcal{C}$. Namely, set $\alpha(f)_B = \delta_{\alpha(-)_B, f} := \alpha_f(B)$ and $\alpha(U)_B := \alpha_U(B)$, for a 1h-cell $f : A \rightarrow A'$ and 1v-cell $U : A \rightarrow A'$ in $\mathcal{A}$.

Now, we may set $\delta^A := \alpha(A)$ and $\delta^B := \alpha(-)_B$ for two horizontal oplax transformations between lax double functors. We do have that $\delta^A = \delta^B$, it remains to check the other four conditions in order for the pairs $(\delta^A, \delta^B)$ for $A \in \mathcal{A}, B \in \mathcal{B}$ to make a horizontal oplax transformation $\Theta : (-,-)_1 \Rightarrow (-,-)_2$ between lax double quasi-functors.

**Proposition 3.5** Let $F, G : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$ be two lax double functors with the corresponding lax double quasi-functors $(-,-)_1, (-,-)_2 : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$. For every $A \in \mathcal{A}$ and $B \in \mathcal{B}$ let $\alpha(A) : F(A) \Rightarrow G(A)$ and $\alpha(-)_B : F(-)(B) \Rightarrow G(-)(B)$ be horizontal oplax transformations between lax double functors. The following are equivalent:

1. $\alpha_f : (\cdot, f)_1 \Rightarrow \alpha(A)_2 \Rightarrow \alpha(A)_1$ is a modification on the vertical composition of horizontal oplax transformations of lax double functors (with components $(\alpha_f)_B = \delta_{\alpha(-)_B, f} : (B, \cdot, f)_2 : \alpha(A)_1 \Rightarrow \alpha(A)_2$ for every 1h-cell $f : A \rightarrow A'$ in $\mathcal{A}$, and $\alpha_U$ of the form (2) is a modification in the sense of Remark 2.2 for every 1v-cell $U : A \rightarrow A'$ in $\mathcal{A}$;

2. The pairs $(\delta^A, \delta^B) := (\alpha(A), \alpha(-)_B)$ for $A \in \mathcal{A}, B \in \mathcal{B}$ form a horizontal oplax transformation $\Theta : (-,-)_1 \Rightarrow (-,-)_2$ between lax double quasi-functors.

**Proof.** In Corollary 2.6 we saw that $(-, f)_i, i = 1, 2$ are horizontal oplax transformations. In [5, Lemma 3.7] we proved that vertical composition of horizontal pseudonatural transformations of two double pseudo functors is again such a horizontal transformation. One similarly shows that the analogous claim with the same structures holds in our current case: for oplax (instead of pseudo) and lax double functors (instead of double pseudo functors). From there we have that the composite transformations evaluated at a 1h-cell $g : B \rightarrow B'$: $[\alpha(A), (\cdot, f)_2]_g$ and $[\alpha(A), (\cdot, f)_2]_g$, have the following form:

\[
\delta_{\alpha(-)_B, f} = \begin{array}{c}
(B,A)_1 \\
(B,A)_2
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
(B,A) \\
(B,A)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
(B,A) \xrightarrow{\alpha(A)_1} (B,A') \\
(B,A') \xrightarrow{\alpha(A)_1} (B,A')
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
(B,A) \xrightarrow{\alpha(A)_1} (B,A') \\
(B,A') \xrightarrow{\alpha(A)_1} (B,A')
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
(B,A) \xrightarrow{\alpha(A)_1} (B,A') \\
(B,A') \xrightarrow{\alpha(A)_1} (B,A')
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

and

\[
\delta_{\alpha(A)_B, f} = \begin{array}{c}
(B,A)_1 \\
(B,A)_2
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
(B,A) \\
(B,A)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
(B,A) \xrightarrow{\alpha(A)_1} (B,A') \\
(B,A') \xrightarrow{\alpha(A)_1} (B,A')
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
(B,A) \xrightarrow{\alpha(A)_1} (B,A') \\
(B,A') \xrightarrow{\alpha(A)_1} (B,A')
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
(B,A) \xrightarrow{\alpha(A)_1} (B,A') \\
(B,A') \xrightarrow{\alpha(A)_1} (B,A')
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

16
Now the first modification condition for $\alpha_f$ reads:

\[
\begin{align*}
(B, A)_1 &\xrightarrow{(g, A)_1} (B', A')_1 \xleftarrow{(B', f)_1} (B', A')_2 \\
(B, A)_1 &\xrightarrow{(B, f)_1} (B', A')_2 \xrightarrow{\delta_{(-, f)_B}} (B', A')_2 \\
(B, A)_1 &\xrightarrow{\alpha(A)_{B'}} (B', A')_2 \xrightarrow{\delta_{(-, -)_B}} (B', A')_2
\end{align*}
\]

Recall from Corollary 2.6 that $\delta_{(-, f)_B} = (g, f)_i$ for $i = 1, 2$ and that we are setting $\theta^A := \alpha(A)$ and $\theta^B := \alpha(-)_B$, thus $\delta_{\alpha(A)_B} = \theta^A_\delta$ and $\delta_{\alpha(-)_B} = \theta^B_\delta$. We have that the above modification condition is precisely the first condition in Definition 3.3.

By [5 Lemma 3.7] adapted to our setting we have:

\[
\begin{align*}
\frac{(-, f)_1}{\alpha(A')_2} &\big|_u = (u, A)_1 \xrightarrow{(u, f)_1} (u, A')_1 \xrightarrow{\alpha(A')_B} (u, A')_2 \\
&\quad \xrightarrow{(\tilde{B}, A)_1} (\tilde{B}, A')_2 \xrightarrow{\alpha(A')_B} (\tilde{B}, A')_2
\end{align*}
\]

and

\[
\begin{align*}
\frac{\alpha(A)}{(-, f)_2} &\big|_u = (u, A)_1 \xrightarrow{\alpha(A)_B} (u, A)_2 \xrightarrow{(u, f)_2} (u, A')_2 \\
&\quad \xrightarrow{(\tilde{B}, A)_1} (\tilde{B}, A)_2 \xrightarrow{\alpha(A)_B} (\tilde{B}, A)_2
\end{align*}
\]

for a 1v-cell $u : B \rightarrow \tilde{B}$. Now the second modification condition for $\alpha_f$ reads:

\[
\begin{align*}
(B, A)_1 &\xrightarrow{(B, f)_1} (B, A')_2 \xrightarrow{\alpha(A')_B} (B, A')_2 \\
&\quad \xrightarrow{(\tilde{B}, A)_1} (\tilde{B}, A')_2 \xrightarrow{\alpha(A')_B} (\tilde{B}, A')_2 \\
&\quad = (B, A)_1 \xrightarrow{\alpha(A)_B} (B, A)_2 \xrightarrow{(B, f)_2} (B, A')_2 \\
&\quad = (B, A)_1 \xrightarrow{\alpha(A)_B} (B, A)_2 \xrightarrow{(B, f)_2} (B, A')_2
\end{align*}
\]

Setting $\theta^A_u = \alpha(A)_u$ and $\theta^B_f = \delta_{\alpha(-)_B}$ this is precisely the second condition in Definition 3.3.

The two modification conditions for $\alpha_{U}$ are:

\[
\begin{align*}
(B, A)_1 &\xrightarrow{(g, A)_1} (B', A')_1 \xrightarrow{\alpha(A)_{B'}} (B', A')_2 \\
(B, U)_1 &\xrightarrow{(g, U)_1} (B', U')_1 \xrightarrow{\alpha(U)_{B'}} (B', U')_2 \\
(B, \tilde{A})_1 &\xrightarrow{(\tilde{g}, \tilde{A})_1} (B', \tilde{A}')_1 \xrightarrow{\alpha(\tilde{A})_{B'}} (B', \tilde{A}')_2 \\
(B, \tilde{A})_1 &\xrightarrow{\alpha(\tilde{A})_{B'}} (B', \tilde{A}')_2 \xrightarrow{(g, \tilde{A})_2} (B', \tilde{A}')_2
\end{align*}
\]

\[
\begin{align*}
(B, A)_1 &\xrightarrow{(g, A)_1} (B', A')_1 \xrightarrow{\alpha(A)_{B'}} (B', A')_2 \\
(B, U)_1 &\xrightarrow{(g, U)_1} (B', U')_1 \xrightarrow{\alpha(U)_{B'}} (B', U')_2 \\
(B, \tilde{A})_1 &\xrightarrow{(\tilde{g}, \tilde{A})_1} (B', \tilde{A}')_1 \xrightarrow{\alpha(\tilde{A})_{B'}} (B', \tilde{A}')_2 \\
(B, \tilde{A})_1 &\xrightarrow{\alpha(\tilde{A})_{B'}} (B', \tilde{A}')_2 \xrightarrow{(g, \tilde{A})_2} (B', \tilde{A}')_2
\end{align*}
\]

\[
\begin{align*}
(B, U)_1 &\xrightarrow{(g, U)_1} (B', U')_1 \xrightarrow{\alpha(U)_{B'}} (B', U')_2 \\
(B, \tilde{A})_1 &\xrightarrow{(\tilde{g}, \tilde{A})_1} (B', \tilde{A}')_1 \xrightarrow{\alpha(\tilde{A})_{B'}} (B', \tilde{A}')_2 \\
(B, \tilde{A})_1 &\xrightarrow{\alpha(\tilde{A})_{B'}} (B', \tilde{A}')_2 \xrightarrow{(g, \tilde{A})_2} (B', \tilde{A}')_2
\end{align*}
\]
and

\[
\begin{align*}
(B, A) & \xrightarrow{(B, U)} (B, A) & (B', A) \\
(B, U_1) & \xrightarrow{(u, A)_1} (B, A) & (B', A) \\
(B, A) & \xrightarrow{(u, U, I)} (B, A) & (B', A) \\
(B, U_1) & \xrightarrow{(u, A)_2} (B, A) & (B', A) \\
(u, A) & \xrightarrow{(b, U, A)} (B', A) & (B', A) \\
(u, A_1) & \xrightarrow{(u, A)_1} (B, A) & (B', A) \\
(u, A_2) & \xrightarrow{(u, A)_2} (B, A) & (B', A) \\
(u, A) & \xrightarrow{(u, A)} (B, A) & (B', A)
\end{align*}
\]

which by additional identifications \(\theta^A_u = \alpha(A)_u\) and \(\theta^B_u = \alpha_U(B)\) are the last two conditions in Definition 3.3.

Now we have that \(\alpha : F \Rightarrow G\) yields \(\Theta : (-, -)_1 \Rightarrow (-, -)_2\). Before seeing the converse, let us summarize our above findings:

**Proposition 3.6** A horizontal oplax transformation \(\alpha : F \Rightarrow G\) between lax double functors \(F, G : A \rightarrow [B, C]\) consists of the following data:

- \(\alpha(A) : F(A) \rightarrow G(A)\) is a horizontal oplax transformation between lax double functors for every \(A \in A\);
- \(\alpha_f\) is a (globular) modification for every 1h-cell \(f : A \rightarrow A'\);
- \(\alpha_U\) is a modification for every 1v-cell \(U : A \rightarrow \tilde{A}\);

so that \(\alpha_f\) and \(\alpha_U\) obey five axioms, which (after evaluation at \(B \in B\)) yield that \(\alpha(-)_B : F(-)(B) \rightarrow G(-)(B)\) is a horizontal oplax transformation between lax double functors for every \(B \in B\) (by setting \(\alpha(f)_B := \alpha_f(B)\) and \(\alpha(U)_B := \alpha_U(B)\)). (Both modifications above are meant in the sense of Definition 2.1.)

Now, assuming that \(\Theta : (-, -)_1 \Rightarrow (-, -)_2\) given by pairs \((\theta^A, \theta^B)\) for \(A \in A, B \in B\) is a horizontal oplax transformation between lax double quasi-functors, whose corresponding lax double functors are \(F, G : A \rightarrow [B, C]\), we define a horizontal oplax transformation \(\alpha : F \Rightarrow G\) as follows. For \(A \in A\) set \(\alpha(A) := \theta^A\), for a 1h-cell \(f : A \rightarrow A'\) set for the desired modification \(\alpha_f\) to be given by components \(\alpha_f(B) := \theta^B_f\), and for a 1v-cell \(U : A \rightarrow \tilde{A}\) set for the desired modification \(\alpha_U\) to be given by components \(\alpha_U(B) := \theta^B_U\). Since moreover \(\alpha(A)_B = \theta^A_B = \theta^B_{A'}\) we have that \(\alpha(-)_B = \theta^B\) is a horizontal oplax transformation of lax double functors with \(\alpha(f)_B = \theta^B = \alpha_f(B)\) and \(\alpha(U)_B = \theta^B_U = \alpha_U(B)\). Now by Proposition 3.3, \(\alpha_f\) and \(\alpha_U\) are modifications. By Proposition 3.6 we have that \(\alpha : F \Rightarrow G\) is indeed a horizontal oplax transformation between lax double functors.

The two assignments of horizontal oplax transformations are clearly inverse to each other.

### 3.4 The 1-1 correspondence among 2-cells

A modification \(\alpha : \alpha \Rightarrow \beta\) between horizontal oplax transformations \(\alpha, \beta : F \Rightarrow G\) between lax double functors \(F, G : A \rightarrow [B, C]\) consists of a globular 2-cell \(\alpha(A) : \)
$\alpha(A) \Rightarrow \beta(A)$ in $[[B, C]]$ (which is a modification in the sense of Definition 2.1) and being globular it is a modification in the sense of Definition 3.2 which satisfies two axioms with free slots in $B \in B$. Evaluating the latter two axioms in a fixed $B \in B$, and considering the slot occupied by 0-, 1h and 1v-cells in $A$ as variable, these two axioms mean that one has a modification $a(-)_{B} : \alpha(-)_{B} \Rightarrow \beta(-)_{B}$ in the sense of Definition 3.2.

With the identifications $\tau^{B} = a(\alpha)$ and $\tau^{B} = a(-)_{B}$ and recalling the identifications from Subsection 3.3 $\theta^{A} = \alpha(A), \theta^{B} = \alpha(-)_{B}$, and analogously we have $\theta^{A} = \beta(A), \theta^{B} = \beta(-)_{B}$, we clearly have: $\tau^{A} = \tau^{B}$ and hence that $\tau^{A} : \theta^{A} \Rightarrow \theta^{A}$ and $\tau^{B} : \theta^{B} \Rightarrow \theta^{B}$ make a modification $\tau$ of horizontal oplax transformations $\Theta = (\theta^{A}, \theta^{B})_{A \in A}$ and $\Theta' = (\theta^{A}, \theta^{B})_{A \in A}$ between lax double quasi-functors.

Reading the above characterization of a modification $a : \alpha \Rightarrow \beta$ and how we obtained the modification $\tau$ in the reversed order, one finds the converse assignment, and it is clear that these two assignments are inverse to each other.

It is directly seen that the assignments that we defined in Subsection 3.3 and this Subsection determine a strict 2-functor between 2-categories $Lax_{hop}(A, [[B, C]])$ and $q\cdot Lax_{hop}(A \times B, C)$. To see that it is compatible with compositions viewing Proposition 3.6 may be helpful. We conclude that there is an isomorphism of 2-categories

$$q\cdot Lax_{hop}(A \times B, C) \cong Lax_{hop}(A, [[B, C]]). \quad (3)$$

### 4. A 2-functor from $q\cdot Lax_{hop}^{ns}(A \times B, C)$ to $Lax_{hop}(A \times B, C)$

Horizontal and vertical lax transformations between lax double functors in Definition 2.1 (recall Remark 2.2) are called strong in [9, Subsection 7.4]. It is the non-triviality of the globular 2-cells $x^{k}$ and $y^{k}$ in Definition 2.1 (2, Section 2.2) that makes these transformations strong. When the 2-cells $y^{k}$, i.e. the 2-cells $(u, U)$ in Proposition 2.3 are identities, we have stricter version of vertical lax transformations. In these conditions we may consider the full sub-2-category $q\cdot Lax_{hop}^{ns}(A \times B, C)$ of $q\cdot Lax_{hop}(A \times B, C)$ differing only in 0-cells, so that the 2-cells $(u, U)$ of its lax double quasi-functors are trivial. The super-index “$ns$” alludes to non-strong. (The corresponding full sub-2-category of $Lax_{hop}(A, [[B, C]])$ isomorphic to $q\cdot Lax_{hop}(A \times B, C)$ in (3) we denote by $Lax_{hop}(A, [[B, C]]^{ns})$.

Here $[[B, C]]^{ns}$ stands for the variation of $[[B, C]]$ in Definition 2.1 in which the vertical lax transformations are not strong.)

We can prove that there is a 2-functor from $q\cdot Lax_{hop}^{ns}(A \times B, C)$ (and thus also from $Lax_{hop}(A, [[B, C]]^{ns})$) to the 2-category $Lax_{hop}(A \times B, C)$, consisting of lax double functors on the Cartesian product of double categories, and their corresponding horizontal oplax transformations and modifications. We will denote it by

$$\mathcal{F} : q\cdot Lax_{hop}^{ns}(A \times B, C) \rightarrow Lax_{hop}(A \times B, C). \quad (4)$$

Moreover, restricting to certain sub-2-categories of $q\cdot Lax_{hop}^{ns}(A \times B, C)$ and $Lax_{hop}(A \times B, C)$ we obtain a biequivalence of 2-categories. We provide the proofs for these facts in this Section. This will generalize to the context of double categories Theorems 4.10 and 5.3 of [4].
4.1 The 2-functor $\mathcal{F}$ on 0-cells

Let us show that a lax double quasi-functor $H : A \times B \to C$, with lax double functors $H(A, -) = (\cdot, A)$ and $H(-, B) = (B, \cdot)$, determines a lax double functor $P : A \times B \to C$ on the Cartesian product. We highlight, as indicated in the introduction of the Section, that we will assume the 2-cells $(u, U)$ of $H$ to be identities.

Instead of typing the whole proof, we will indicate the list of its steps. For that purpose recall the notation for the formulaic computations $[\alpha|\beta] = \beta\alpha$ for the horizontal composition of 2-cells $\alpha$ (first) and $\beta$ (second) from the end of the second paragraph of Section 2.

We set:

$P(A, B) = H(A, B),$

$P(f, g) = H(A', g)H(f, B) = (g, A')(B, f)$, for $f : A \to A'$ and $g : B \to B'$,

$P(U, u) = \langle \tilde{B}, \tilde{A} \rangle (u, \tilde{A})$ for 1v-cells $U : A \to \tilde{A}, u : B \to \tilde{B}$, and

$$P(\alpha, \beta) := \begin{pmatrix}
(B, A) & (B, A') & (B', A') \\
(B, U) & (u, \tilde{A}) & (u', \tilde{A}') \\
(\tilde{B}, A) & (\tilde{B}, A') & (\tilde{B}', A')
\end{pmatrix},$$

for 2-cells $\alpha$ in $A$ and $\beta$ in $B$ as in (6);

for the lax structure $\gamma_{(f', g')}(f, g) : P(f', g')P(f, g) \to P(f', f g')$ and $\iota^P : \text{Id}_{P(A, B)} \Rightarrow P(\text{Id}_{(A, B)})$ of $P$ we set: $\gamma_{(f', g')}(f, g) = \langle \text{Id}_{B'} | \langle g' f'\text{Id}_{A'}, \text{Id}_{A'} \rangle \rangle : (g', A')(B', f')(g, A')(B, f) \Rightarrow (g' g, A'')(B, f f')$;

a globular 2-cell, and

$\iota^P_{(A, B)} = [\iota^B_{A(B)}] : \text{Id}_{P(A, B)} \Rightarrow (\text{Id}_{B}, A)(\text{Id}_{B}, A)$, where $\iota^B_{A} = (B, -)_{B}$ and $\iota^A_{B} = (-, A)_{B}$ (from a) (11) of Proposition 3.

Since the hexagonal law for $\gamma$ and the unit laws rely only on 1h-cells, these laws hold true as in the 2-categorical setting of [4, Theorem 3.2]. The same holds for horizontal naturality of $\iota^P$ (with respect to 1h-cells). We will discuss its vertical naturality (with respect to 1v-cells), the (only) naturality of $\gamma$ (with respect to 2-cells), and the vertical functoriality of $P$ (with respect to 2-cells).

For the naturality of $\gamma$ we take two 2-cells $\alpha, \alpha'$ in $A$ and two 2-cells $\beta, \beta'$ in $B$:

$$\begin{align*}
\begin{array}{c}
\begin{array}{ccc}
A & \xrightarrow{f} & A' \\
\downarrow \alpha & & \downarrow \alpha' \\
\tilde{A} & \xrightarrow{\tilde{f}} & \tilde{A}'
\end{array}
\end{array}
\quad \begin{array}{ccc}
\begin{array}{ccc}
\begin{array}{ccc}
A & \xrightarrow{f} & A' & \xrightarrow{f'} & A'' \\
\downarrow \alpha & & \downarrow \alpha' & & \downarrow \alpha''
\end{array}
\end{array}
\end{array}
\quad \begin{array}{ccc}
\begin{array}{ccc}
\begin{array}{ccc}
A & \xrightarrow{f} & \tilde{A}' & \xrightarrow{\tilde{f}} & \tilde{A}
\end{array}
\end{array}
\end{array}
\end{align*}$$

and

$$\begin{align*}
\begin{array}{c}
\begin{array}{ccc}
B & \xrightarrow{g} & B' \\
\downarrow \beta & & \downarrow \beta' \\
\tilde{B} & \xrightarrow{\tilde{g}} & \tilde{B}'
\end{array}
\end{array}
\quad \begin{array}{ccc}
\begin{array}{ccc}
\begin{array}{ccc}
B & \xrightarrow{g} & B' & \xrightarrow{g'} & B'' \\
\downarrow \beta & & \downarrow \beta' & & \downarrow \beta''
\end{array}
\end{array}
\end{array}
\quad \begin{array}{ccc}
\begin{array}{ccc}
\begin{array}{ccc}
B & \xrightarrow{g} & \tilde{B}' & \xrightarrow{\tilde{g}} & \tilde{B}
\end{array}
\end{array}
\end{array}
\end{align*}$$

(6)
Recall that here we assume that the 2-cells of the form $(\gamma, A')$. b) $(21)$

For the vertical naturality of $u$,

In order to show that the left-hand side equals the right one, apply the following rules: 1) naturality of the laxity of $(B, -)$ in $\alpha'' \alpha$ and of $(-, A'')$ in $\beta' \beta$; 2) b) $(21)/2$, b) $(12)/1$ and c) $(22)/2$; 3) c) $(11)/1$ followed by c) $(11)/2$, and 4) c) $(22)/2$.

Recall that here we assume that the 2-cells of the form $(u', U'')$ are identities, thus taking inverse of $(u', U'')$ above makes sense. As a matter of fact, to prove the naturality of $\gamma$, it is sufficient to assume merely invertibility of such globular 2-cells.

For the vertical naturality of $t^P$ we should prove the equality: $\frac{t^P_{\alpha' \alpha}}{t^P_{(1, \beta)}} = \frac{t^P_{(1, \gamma)}}{t^P_{(1, \beta)}}$, which
translates into:

\[
\begin{align*}
(B, A) & \cong (B, A) \\
(B, U) & \cong (B, U) \\
(u, \tilde{A}) & \cong (u, \tilde{A})
\end{align*}
\]

\[
\begin{align*}
(B, A) & \cong (B, A) \\
(B, U) & \cong (B, U) \\
(u, \tilde{A}) & \cong (u, \tilde{A})
\end{align*}
\]

which is true by vertical naturality of \((B, -)_A\) and \((- , A)_B\) and c) \((22)/2\).

For the vertical functoriality of \(P\), i.e. its compatibility with the vertical composition of 2-cells, take two vertically composable pairs of 2-cells:

\[
\begin{align*}
A & \xrightarrow{f} \tilde{A} \\
U & \xrightarrow{\delta} \tilde{V} \\
A' & \xrightarrow{f'} \tilde{A}' \\
U' & \xrightarrow{\delta'} \tilde{V}' \\
A'' & \xrightarrow{f''} \tilde{A}''
\end{align*}
\]

and

\[
\begin{align*}
B & \xrightarrow{g} \tilde{B} \\
U & \xrightarrow{\eta} \tilde{V} \\
B' & \xrightarrow{g'} \tilde{B}' \\
U' & \xrightarrow{\eta'} \tilde{V}' \\
B'' & \xrightarrow{g''} \tilde{B}''
\end{align*}
\]

To write out \(P(\delta, \delta', \eta, \eta')\) one uses the following rules in b): both in \((22), (21)/1\) and \((12)/2\). Then apply to it the both rules in c) \((22)\). One obtains an expression consisting of an appropriate concatenation of \(P(\xi, \xi', \gamma, \gamma')\) and the 2-cells \((u, U')\) and \((v', V)\). By the hypothesis, the latter 2-cells are identities, so we have functoriality of \(P\).

**Remark 4.1** In the setting of strong vertical lax transformations this is the only obstacle in proving that a lax double quasi-functor \(H\) induces a lax double functor \(P\), and thus the reason why we require the 2-cells \((u, U)\) to be identities at the beginning of the Section.

This finishes the proof that we have a lax double functor \(P : A \times B \rightarrow C\).

Observe that if \(i^A\) and \(i^B\) are invertible, then so is \(i^P\) and also \(\gamma(1_A, \xi)(f, 1_B)\). This is exactly the same as in [4, Lemma 5.2]. When \(i\)'s are invertible the lax double functor in question is called *unital*, whereas the lax double functor \(P : A \times B \rightarrow C\) is called *decomposable* when \(\gamma(1_A, \delta)(f, 1_B)\) is invertible in *loc.cit.*.

We will prove later that the full sub-2-category \(\text{q-Lax}^{\text{cons}}(A \times B, C)\) of \(\text{q-Lax}^{\text{cons}}(A \times B, C)\) consisting of *unital* lax double quasi-functors (in the sense that both \((- A\) and \((B, -)\) are unital) is biequivalent to the full sub-2-category \(\text{Lax}^{\text{unital}}(A \times B, C)\) of \(\text{Lax}^{\text{unital}}(A \times B, C)\) consisting of the *unital and decomposable* lax double functors. Let

\[
\mathcal{F}' : \text{q-Lax}^{\text{cons}}(A \times B, C) \rightarrow \text{Lax}^{\text{unital}}(A \times B, C).
\]

(7) denote the corresponding restriction of \(\mathcal{F}\) from [4, and let \(\mathcal{G}\) denote a to-be-defined quasi-inverse 2-functor for \(\mathcal{F}'\).

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4.2 A quasi-inverse $G$ of $F'$ on 0-cells

We will first show that a unital and decomposable lax double functor $P : A \times B \rightarrow C$ with the structures $\gamma$ and $i^p$ determines a lax double quasi-functor $H : A \times B \rightarrow C$. Let $P(A, g) := P(1_A, g)$, $P(A, u) := P(1_A, u)$, $P(1_A, \beta) := P(P(1_A, \beta)$, where $\mathbb{I}_A$ is the identity 2-cell for the object $A$, and $g, u, \beta$ as usual, and similarly for $P(\sim, B)$. It follows: $P(1_A, 1_B) = P(1_A, 1_B) = P(1_A, B)$.

Now set $(-, A) = P(A, -)$ and $(B, -) = P(\sim, B)$ and $i^p_A = \iota_A^p := i^p_{(A,B)}$. Then $(g', A)(g, A) = P(1_A, g')P(1_A, g)$ and similarly for $(B, -)$, and we may set:

$$(-, A)_{g,g'} := \gamma_{(1_A, g'), (1_A, g)} \quad \text{and} \quad (B, -)_{f,f'} := \gamma_{(f', 1_B),(f, 1_B)}$$

and we get that $(-, A)$ and $(B, -)$ are lax double functors.

Observe the form of the 2-cell:

$$P(U, u) \xrightarrow{P(\alpha, \beta)} P(V, v)$$

and notice then whenever either of the two 1h-cells or either of the two 1v-cells in $P(\sim, -)$ above is identity, the form of $P(\sim, -)$ becomes $(B, x)$ i.e. $(x, A)$ for the corresponding 1h- or 1v-cell $x$. Then we may further define:

$$(g, f) := \frac{(g, A) \rightarrow (B', f)}{(B, f) \rightarrow (g, A')} \quad \text{and} \quad (u, f) := \frac{(u, A) \rightarrow (g, f)}{(B, f) \rightarrow (u, A')}$$

Since $P$ is strict in the vertical direction, we have: $\frac{(B, 1_U)}{(u, A)} = \frac{(u, A)}{(B, 1_U)}$, so we may define a 2-cell $(u, U)$ (in the desired form) to be identity. For the same reason the rules b) (21)/1, (12)/2 and c) (22) hold. The rules b) (21)/2 and (12)/1 hold by laxity of $P$.

Since $P$ as a lax functor obeys that evaluated at an identity 2-cell equals identity, the following rules in a) hold true: both in (21) and both in (12). The two rules in (22) hold since we defined $(u, U)$ to be identity.

Observe that the naturality of $\gamma$ with respect to 2-cells $Id_f$ from $A$ and $\beta$ from $B$ reads:

$$P(1_A, B) \xrightarrow{P(Id_f, \beta)} P(A', B') = P(1^A, u) \xrightarrow{P(Id_f, \beta)} P(1^A, v)$$

and

$$P(1_A, u) \xrightarrow{P(Id_f, \beta)} P(A', B') = P(1^A, u) \xrightarrow{P(Id_f, \beta)} P(1^A, v)$$

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Then applying this naturality of \( \chi(\gamma, 1_B)(1_A, \delta) \) and analogously of \( \chi^{-1}(1_A, \delta')(\gamma, 1_B) \), one obtains that the rule c) (11)/1 holds. Analogously, the naturality of \( \gamma \) with respect to 2-cells \( \alpha \) from \( A \) and \( \text{Id}_x \) from \( B \) are used to prove the rule c) (11)/2.

All the resting rules from Proposition 2.3 concern only the horizontal structures and are already shown to hold in [4, Theorem 5.3]. We conclude that the unital and decomposable lax double functor \( P : A \times B \to C \) indeed determines a unital lax double quasi-functor \( H : A \times B \to C \). Thus, \( G \), with \( G(P) = H \), is well-defined on 0-cells.

Let us now start to define two 2-natural transformations: \( \kappa : \text{Id} \Rightarrow GF' \) and \( \lambda : F'G \Rightarrow \text{Id} \). They will be isomorphisms if we restrict to unital double lax quasi-functors in \( q\text{-Lax}_{\text{hop}}(A \times B, C) \) and to unital and decomposable lax double functors in \( \text{Lax}_{\text{hop}}^\text{ns}(A \times B, C) \). This will yield that \( F' \) in (7) is a biequivalence, as announced.

Observe that \( GF'H(f, B) = P(f, B) = (B, f)(1_B, A') \), and similarly \( GF'H(A, g) = P(A, g) = (B, 1_A)(g, A) \). Then we set \( \alpha(B) = 1_{(B, A')} \), for a globular 2-cell \( \alpha \) which is actually \( \alpha_\gamma : (g, A) \Rightarrow (B, 1_A)(g, A) \), we set \( \alpha_\gamma = [\text{Id}_{(g, A)}]_{(1_B, 1_A)} \), and \( \alpha_\gamma = (u, 1_A') = \text{Id}_{(u, A')}(g, A) \), with notations as usual. We obtain that \((-, A)\) is naturally isomorphic to \( P(A, -) \) (where \( P = F'H \)) if \( t_B \) is invertible, through the horizontal opolax transformation \( \alpha = : \chi^A \). To prove the property 1) of Definition 3.1 in this double category setting the interchange law is used. Analogously, one proves a natural isomorphism \((B, -) \Rightarrow P(-, B) \) through \( \chi^B \), if \( t_B \) is invertible. Thus when \( H \) is unital (i.e. \( t_A \) and \( t_B \) are invertible), \( \chi^A \) and \( \chi^B \) are isomorphisms. It is easily seen that \( \chi^A_B = \chi^B \).

Observe further that \( GF'(g, f) \) is a 2-cell whose both source and target 1h-cells are compositions of four 1h-cells, and not of two 1h-cells as in the case of \((g, f)\). To express \( GF'(g, f) \), one uses that \( \gamma^{-1}(1_{(1_A, \gamma)}(1_A, 1_B)) = [\text{Id}_{(B, f)}]_{(1_A, 1_B)}[\text{Id}_{(g, A)}][\text{Id}_{(g, A')}] \) by [4, Lemma 5.2]. Moreover, \( GF'G'(g, f) \) and \( GF'(g, U) \) are 2-cell whose source and target 1h-cells are compositions of two 1h-cells, and not a single 1h-cell as in the case of \((u, f)\) and \((g, U)\), respectively. It is easily seen and it is proved in [4, Theorem 5.3] that \( \chi^A \) and \( \chi^B \) obey \( \text{HOT}_1^2 \). The axioms \( \text{HOT}_1^1 \) for \( \chi^A \) and \( \chi^B \) hold almost trivially: \( \chi^A \), \( \chi^B \) are identities and in properties \( \text{HOT}_1^2 \) and \( \text{HOT}_1^3 \) use the interchange law to move the 2-cells \( \odot \).

In this way we have defined a 0-component \( \kappa \) at a 0-cell \( H \) in \( q\text{-Lax}_{\text{hop}}^\text{ns}(A \times B, C) \) of \( \kappa : \text{Id} \Rightarrow GF' \), which will be a 2-natural transformation.

To define 0-component of a 2-natural transformation \( \lambda : F'G \Rightarrow \text{Id} \), we see that \( F'G(P)(A, B) = P(A, B) \) for a unital decomposable double lax functor \( P : A \times B \to C \), so we may set \( \lambda^P \) to be identity between 0-cells. Similarly, as \( F'G(U, u) = (B, U) \) we may set \( \lambda^U \) to be identity also between 1v-cells. Though, \( F'G(f, g) = (g, A')(B, f) = P(1_A, g)P(f, 1_B) \), then we set \( \lambda^P \) on a 1h-cell \((f, g)\) to be \( \gamma(1_A, g)(1_B, f) : P(1_A, g)P(f, 1_B) \Rightarrow P(f, g) \). Such defined \( \lambda^P \) is indeed a horizontal opolax transformation of double lax functors: property 3) is proved in [4, Theorem 5.3], property 2) holds since 1v-components of \( \lambda^P \) are identities, and the first property holds by naturality of \( \gamma \).

In the next two Subsections we will finalize the proof that \( \kappa \) and \( \lambda \) are 2-natural transformations. Observe from above that restricting to the full sub-categories \( q\text{-Lax}_{\text{hop}}^\text{ns}(A \times B, C) \) and \( \text{Lax}_{\text{hop}}^\text{ad}(A \times B, C) \) we indeed obtain a 2-equivalence. Namely, in these sub-2-categories \( \kappa \) is an isomorphism since so are \( \chi^A \)'s and \( \chi^B \)'s, and \( \lambda \) is an isomorphism since the 0-component of \( \lambda \) is defined to be \( \gamma(1_A, g)(1_B, f) \) on 1h-cells \((f, g)\).
4.3 \( \mathcal{F} \) and a quasi-inverse \( \mathcal{G} \) of \( \mathcal{F}' \) on 1-cells

We first define \( \mathcal{F} \) on 1-cells. Let a horizontal oplax transformation between lax double quasi-functors \((-,-)_1,(-,-)_2\) with images \(P, P'\) be given via a family \(\Theta^A, \Theta^B, A \in \mathcal{A}, B \in \mathcal{B}\) of horizontal oplax transformations. We define \(\Theta : P \Rightarrow P'\) by \(\Theta(A, B) := \Theta^A_B, \Theta(U_{1u}) := \Theta^{B}_{1}\) and \(\delta_{\Theta(f,g)} := \Theta(f,g) \equiv \frac{[\text{Id}_{\mathcal{B}}(f), \Theta^{B}_{1}]}{[\text{Id}_{\mathcal{A}}(g), \Theta^{A}_{1}]}\). The property 3) of Definition 3.1 for \(\Theta\) is proved in [4, Theorem 5.3]. The property 2) follows by 2) for \(\Theta^B\) and \(\Theta^A\) (\(\text{HOT}_1\)) and since by assumption the vertically globular 2-cells \((u, V)_1, (u, V)_2\) are identities. (This includes the proof for \(\Theta(U_{1u}) = \text{Id}_{\Theta(A,B)}\).)

To prove the property 1) of Definition 3.1 for \(\Theta\) one uses: property c) (22)/1 of \((-,-)_1\) and property 1) for \(\Theta^A\), then simultaneously (\(\text{HOT}_2\)) and (\(\text{HOT}_3\)), and finally property 1) for \(\Theta^B\) and c) (22)/1 of \((-,-)_2\). Then \(\Theta : P \Rightarrow P'\) is indeed a horizontal oplax transformation of lax double functors.

For \(\mathcal{G}\), let \(P, P' : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}\) be unital and decomposable lax double functors and \(\Theta : P \Rightarrow P'\) a horizontal oplax transformation between them. Then \(\mathcal{G}(\Theta)\) is a horizontal oplax transformation between lax double quasi-functors given by the families of \(\Theta^A := \Theta(A,-), \Theta^B := \Theta(-,B)\) for \(A \in \mathcal{A}, B \in \mathcal{B}\). It’s clearly \(\Theta^B_A = \Theta^A_B\) and the condition (\(\text{HOT}_4\)) is proved to hold in [4, Theorem 5.3]. Conditions (\(\text{HOT}_2\)) and (\(\text{HOT}_3\)) hold by the property 1) of \(\Theta\) from Definition 3.1 with \(a = (\text{Id}_f, \text{Id}^u)\) and \(a = (\text{Id}_{\mathcal{U}}, \text{Id}_{\mathcal{L}})\), respectively, while (\(\text{HOT}_4\)) holds by the property 2) thereof: both sides in (\(\text{HOT}_4\)) equal \(\Theta(U_{1u})\).

Let us now study \(\kappa : \text{Id} \Rightarrow \mathcal{G}\mathcal{F}'\) at a 1-cell component, a horizontal oplax transformation between lax double quasi-functors given by a family \((\Theta^A, \Theta^B), A \in \mathcal{A}, B \in \mathcal{B}\). To prove that \(\kappa\) is a 2-natural transformation we should show the identity \(\kappa^{H} \circ (\Theta^A, \Theta^B) = \mathcal{F}' \mathcal{G}(\Theta^A, \Theta^B) \circ \kappa^{H}\). This means that both \(\chi^A \circ \Theta^A = \mathcal{F}' \mathcal{G}(\Theta^A) \circ \chi^A\) (with abuse of notation by writing \(\mathcal{F}' \mathcal{G}(\Theta^A)\) which is easily understood) and the analogous identity for \(B\) must hold. We check only the first identity. At a 0-component \(B\) we have that \(\chi^A_B\) is identity and the equality is easily seen to hold. At a 1-component \(g\) of \(\mathcal{F}' \mathcal{G}(\Theta^A_B) = \frac{[\text{Id}_{\mathcal{B}}(g), \Theta^B_{1}]}{[\text{Id}_{\mathcal{A}}(g), \Theta^{A}_{1}]}\) and observe that by the property 2) of Definition 3.1 we have that \(\Theta^B_{1}\) is identity. Recall that the \(\chi^A_{\mathcal{B}}\)'s are identities by construction, so the compositions in the identity \(\chi^A_B \circ \Theta^B_A = \mathcal{F}' \mathcal{G}(\Theta^A) \circ \chi^A_B\) make sense, and the identity is shown to hold by the interchange law. Finally, at a 1v-component \(u\) we have that \(\chi^A_{u}\) is identity, on one hand, and observe that \(\mathcal{F}' \mathcal{G}(\Theta^A_u) = \frac{[\text{Id}_{\mathcal{B}}(u), \Theta^B_{1}]}{[\text{Id}_{\mathcal{A}}(u), \Theta^{A}_{1}]}\), on the other. But \(\Theta^B_{1}\) is identity, so we get indeed \(\chi^A_{u} \circ \Theta^A_{u} = \mathcal{F}' \mathcal{G}(\Theta^A) \circ \chi^A_{u}\), as desired.

To prove that \(\lambda\) is a 2-natural transformation, to the proof in [4, Theorem 5.3] we only need to add the check of if \(\lambda^{P}_{(U_{1u})} \circ \mathcal{F}' \mathcal{G}(\Theta(U_{1u})) = \Theta(U_{1u}) \circ \lambda^{P}_{(U_{1u})}\) for an oplax transformation of double lax functors \(\Theta : P \Rightarrow P'\). But the 1v-components of \(\lambda^P\) are identities, so it remains to check if \(\mathcal{F}' \mathcal{G}(\Theta(U_{1u})) = \Theta(U_{1u})\) holds. We find: \(\mathcal{F}' \mathcal{G}(\Theta(U_{1u})) = \frac{\Theta(U_{1u})}{\Theta(U_{1u})} = \Theta(U_{1u})\), the latter identity holds by the property 2) of \(\Theta\) being an oplax transformation of double lax functors. Thus we proved the desired equality.
4.4 \( \mathcal{F} \) and \( \mathcal{G} \) on 2-cells

We start by defining \( \mathcal{F} \) on modifications. Let \( a : \Theta \Rightarrow \Theta' \) be a modification \( a = (\tau^A, \tau^B)_{A \in A, B \in B} \) between horizontal oplax transformations of lax double quasi-functors \( H = (-, -)_1 \) and \( H' = (-, -)_2 \). We define \( \mathcal{F}(a) : \mathcal{F}(\Theta) \Rightarrow \mathcal{F}(\Theta') \) by \( \mathcal{F}(a)_{(A,B)} := \tau^A_B = \tau^B_A \). It is directly and easily checked that this is a modification in the sense of Definition 3.2 by using the interchange law.

Conversely, given a modification \( a \) between horizontal oplax transformations of lax double functors, define \( a^A := a(A, -) \) and \( a^B := a(-, B) \). It is directly seen that they give modifications in the sense of Definition 3.2, and it is clearly \( a^A_B = a^B_A \), so we obtain a modification \( \mathcal{G}(a) = (a^A, a^B)_{A \in A, B \in B} \) of horizontal oplax transformations of lax double functors.

The naturality of \( \kappa \) and \( \lambda \) with respect to 2-cells is already proved to hold in [4, Theorem 5.3], as modifications are given by (vertically) globular 2-cells.

To summarize, in this Section we have proved the following results:

**Proposition 4.2** With notations as at the beginning of Section 4 there is a functor

\[ \mathcal{F} : q^{-} \text{Lax}_{\text{hop}}(A \times B, C) \rightarrow \text{Lax}_{\text{hop}}(A \times B, C). \]

**Theorem 4.3** With notations as explained above (7) the 2-functor \( \mathcal{F} \) restricts to an equivalence 2-functor

\[ \mathcal{F}' : q^{-} \text{Lax}_{\text{hop}}^{\text{ns-u}}(A \times B, C) \rightarrow \text{Lax}_{\text{hop}}^{\text{ud}}(A \times B, C) \]

with quasi-inverse \( \mathcal{G} \).

This Theorem is a double category version of [4, Theorem 5.3]. We can straighten a bit its formulation by passing to pseudo (quasi-) functors, as they are lax unital (quasi-) functors. The notation \( \text{Lax} \) changes then to \( \text{Ps} \), the supraindex \( u \) becomes superfluous, but also \( d \) in the right hand-side. Moreover observe that in the 0-cells of \( q^{-} \text{Ps}_{\text{hop}}^{\text{ns}}(A \times B, C) \) in the left, the 2-cells \((g, f)\) of quasi pseudofunctors are bijective. Then the 2-equivalence functor \( \mathcal{F}' \) restricts to a 2-equivalence

\[ \mathcal{F}'' : q^{-} \text{Ps}_{\text{hop}}^{\text{ns}}(A \times B, C) \rightarrow \text{Ps}_{\text{hop}}(A \times B, C). \]

Observe that by [4, Proposition 6.2] if the 2-cells \((g, f)\) of a lax unital quasi-functor are bijective, as it is the case in pseudo quasi-functors, then the property a) (11) in Proposition 2.3 is redundant.

5 Applications

After proving our main results in Sections 3 and 4 we dedicate this last Section to some specific cases. We also address the question on the Gray type monoidal product for double categories and lax double functors, and finally we discuss monads in double categories.
5.1 "(Un)currying" functor

At the beginning of Section 4 we commented that the 2-category isomorphism \( (3) \) restricts to a 2-category isomorphism \( q: \text{Lax}_{\text{hop}}^{\text{cs}}(A \times B, C) \cong \text{Lax}_{\text{hop}}(A, [B, C]^{\text{ns}}) \). Composing this with \( \mathcal{F} \) we obtain a 2-functor:

\[
\text{Lax}_{\text{hop}}(A, [B, C]^{\text{ns}}) \rightarrow \text{Lax}_{\text{hop}}(A \times B, C)
\]

(8)

that is a double category version of the "uncurrying" 2-functor \( J \) at the end of Section 4 of [4]. (\( J \) was implicitly constructed in [12].)

In (7) we moreover restricted to \( \text{unital} \) lax double (quasi) functors. On the left-hand-side therein (and in the last Theorem above) unitality of a lax double quasi-functor \( H \) refers to the unitality of both \( (-, A) \) and \( (B, -) \) lax double functors comprising \( H \). In the isomorphism \( q: \text{Lax}_{\text{hop}}^{\text{cs}}(A \times B, C) \cong \text{Lax}_{\text{hop}}(A, [B, C]^{\text{ns}}) \) unitality of \( (-, A) \) corresponds to the unitality of 0-cells in \( [B, C]^{\text{ns}} \), while unitality of \( (B, -) \) corresponds to the unitality of 0-cells in \( \text{Lax}_{\text{hop}}(A, [B, C]^{\text{ns}}) \). Then the isomorphism \( q: \text{Lax}_{\text{hop}}^{\text{cs}}(A \times B, C) \cong \text{Lax}_{\text{hop}}(A, [B, C]^{\text{ns}}) \) restricts further to a 2-category isomorphism

\[
q: \text{Lax}_{\text{hop}}^{\text{cs}}(A \times B, C) \cong \text{Lax}_{\text{hop}}^{\text{ns}}(A, [B, C]^{\text{ns}})
\]

(9)

where \( [B, C]^{\text{ns}} \) denotes the (strict) double category of 0: unital lax double functors \( B \rightarrow C \), 1h: horizontal opolax natural transformations, 1v: non-strong vertical lax natural transformations, and 2: modifications, and \( \text{Lax}_{\text{hop}}^{\text{ns}}(A, [B, C]^{\text{ns}}) \) is the 2-category of 0: unital lax double functors \( A \rightarrow [B, C]^{\text{ns}} \), 1: horizontal opolax transformations between them and modifications between the latter. Joining [9] and Theorem [13] yields

\[
\text{Lax}_{\text{hop}}^{\text{ns}}(A \times B, C) \cong \text{Lax}_{\text{hop}}^{\text{ns}}(A, [B, C]^{\text{ns}})
\]

which presents a setting in which the uncurrying 2-functor \( \Theta \) restricts to a 2-category equivalence, \( i.e. \) in which a "currying" functor exists.

5.2 The Gray monoidal product in \( (\text{Db}_{\text{lx}}^{\text{st}}, \otimes) \)

At the beginning of Section 2 we announced a construction of a Gray monoidal product in the category of (strict) double categories and lax double functors \( (\text{Db}_{\text{lx}}^{\text{st}}, \otimes) \). We complete the proof that we indeed have a Gray monoidal product in this Subsection.

There is clearly a lax double quasi-functor \( J: A \times B \rightarrow A \otimes B \) given by \( J(-, B)(x) = x \otimes_B y \) for cells \( x \) in \( A \) and \( y \) in \( B \) and with unique 2-cells \( F \otimes f, F \otimes u, U \otimes f \) and \( U \otimes u \) in \( A \otimes B \), where the usual notation is used.

**Proposition 5.1** There is an isomorphism of 2-categories

\[
q: \text{Lax}_{\text{hop}}^{\text{ns}}(A \times B, C) \cong \text{Lax}_{\text{hop}}(A \otimes B, C).
\]

**Proof.** It is clear that for every \( H \in q: \text{Lax}_{\text{hop}}^{\text{ns}}(A \times B, C) \) there is a unique lax double functor \( \overline{H}: A \otimes B \rightarrow C \) such that \( H = \overline{H} J \). Given a horizontal opolax transformation \( \theta = (\theta_A, \theta_B)_{A \in A, B \in B} \) between lax double quasi-functors \( H \Rightarrow H' \) we define a horizontal opolax transformation \( \Theta: \overline{H} \Rightarrow \overline{H}' \) by setting: \( \Theta(A \otimes B) = \theta_A, \theta_B \), \( \Theta_{(A, B)} = \theta_A, \Theta_{(f, B)} = \theta_B \).
and $\Theta_{(A,a)} = \theta^A_a, \Theta_{(U,B)} = \theta^B_U$. To check the property 1) of Definition 3.1 for $\Theta$ one should check it for six types of 2-cells in $A \otimes B$: for $a$ being $(A, \omega)$ or $(\zeta, B)$ the property 1) for $\Theta$ holds since $\theta^A$ respectively $\theta^B$ is a horizontal oplax transformation, while for $a$ being $f \otimes g, f \otimes u, U \otimes g$ and $U \otimes u$ the property 1) for $\Theta$ holds by the properties $HOT^1_1 - HOT^q_{4'}$, respectively. The properties 2) and 3) of Definition 3.1 for $\Theta$ hold by the same properties for $\theta^A$ and $\theta^B$.

Given a modification $a = (\tau, \omega)_{A \in A, B \in B}$ we define a modification $a : \Theta \Rightarrow \Theta'$ by $a(A \otimes B) = \tau^A B = \tau^B A$, then it is immediate to see that $a$ is well-defined.

On all the three levels of cells it is clear that one has a 1-1 correspondence, so that one obtains an isomorphism of 2-categories, as claimed.

Joining the isomorphism from the above Proposition and (3) we obtain that there is an isomorphism of 2-categories:

$$Lax_{hop}(A \otimes B, C) \cong Lax_{hop}(A, [B, C]^{ns})$$

(recall the 2-category $Lax_{hop}(A, [B, C]^{ns})$ from the beginning of Section 4). This is a double category version of [10, Theorem I.4.9]. The proof that $A \otimes B$ is indeed a monoidal product is analogous to that of [10, Theorem I.4.9]. By the 1-1 correspondence at the level of 0-cells in the above 2-category isomorphism we conclude that $\otimes$ indeed presents a Gray type monoidal product on the category $Dbl_{ls}^{st} \otimes$. Hence, there is a natural isomorphism of sets

$$Dbl_{ls}^{st}(A \otimes B, C) \cong Dbl_{ls}^{st}(A, [B, C]^{ns}).$$

5.3 Monads in double categories

In [6, Definition 2.4] the authors defined a monad in a double category $\mathcal{D}$ as a 2-monad in the horizontal 2-category $\mathcal{H}(\mathcal{D})$ of $\mathcal{D}$. Bénabou observed in [11] that a lax functor $* \rightarrow K$ from the trivial 2-category to a 2-category $K$ is nothing but a 2-monad in $K$. It is straightforwardly seen that the analogous holds for monads in a (strict) double category $\mathcal{D}$. Let now $*$ denote the trivial double category, then we have:

**Proposition 5.2** A lax double functor $* \rightarrow \mathcal{D}$ is a monad in $\mathcal{D}$.

Moreover, a 0-cell in $q^* Lax_{hop}(*, \mathcal{D})$ is then given by two monads in $\mathcal{D}$, and the only surviving 2-cell (and laws) in the characterization Proposition 2.3 is the one of type $(f, \bar{F}) = (id_*, id_*)$ and the rules a) (11) and b) (11), which correspond to monad-monad distributive laws. Doing the computations further one directly verifies the following isomorphisms of 2-categories, where $\text{Mnd}(\mathcal{K})$ denotes the 2-category of monads in a 2-category $\mathcal{K}$:

**Proposition 5.3** The following two pairs of 2-categories are isomorphic:

$$Lax_{hop}(*, \mathcal{D}) \cong \text{Mnd}(\mathcal{H}(\mathcal{D}))$$

and

$$q^* Lax_{hop}(*, \mathcal{D}) \cong \text{Mnd}(\text{Mnd}(\mathcal{H}(\mathcal{D}))).$$
Remark 5.4 One can consider two distinct versions of the 2-category $\text{Mnd}(\mathcal{K})$ depending on the side of the monad “distributive law” in the 1-cells. More precisely, for a 1-cell $\overline{F} : (\mathcal{A}, T) \to (\mathcal{A}, S)$ one can consider its 2-cell part of the data in the form $\psi : SF \Rightarrow FT$ or $\phi : FT \Rightarrow SF$, where $T$ is a monad on a 0-cell $\mathcal{A}$ in $\mathcal{K}$ and $S$ on $\mathcal{A}'$. This choice is correlated with the choice to consider “lax” or “oplax”, respectively, horizontal transformations in the 1-cells in the left hand-sid ed 2-categories in the above isomorphisms.

In this setting, being $\mathcal{A} = \mathcal{B} = \ast$, note that we can write the 2-functor $\mathcal{F}$ from (4) as $\mathcal{F} : q\text{-}\text{Lax}_{\text{hop}}(* \times *, \mathcal{D}) \to \text{Lax}_{\text{hop}}(*, \mathcal{D})$. On the other hand, recall that $\text{Mnd}$ is an endofunctor on the category $\text{Cat}_{1}$ of categories (it sends a 2-category $\mathcal{K}$ to the 2-category $\text{Mnd}(\mathcal{K})$, as we used above) and that there is a natural transformation $\text{Comp} : \text{Mnd}\text{Mnd} \to \text{Mnd}$, which evaluated at $\mathcal{K}$ sends a distributive law in $\mathcal{K}$ to the induced composite monad, [15]. One has that the following diagram commutes

$$
\begin{array}{ccc}
q\text{-}\text{Lax}_{\text{hop}}(* \times *, \mathcal{D}) & \xrightarrow{\mathcal{F}} & \text{Lax}_{\text{hop}}(*, \mathcal{D}) \\
\cong \downarrow & & \cong \downarrow \\
\text{Mnd}(\text{Mnd}(\mathcal{H}(\mathcal{D}))) & \xrightarrow{\text{Comp}(\mathcal{H}(\mathcal{D}))} & \text{Mnd}(\mathcal{H}(\mathcal{D}))
\end{array}
$$

and it moreover indicates that the general 2-functor $\mathcal{F}$ can be seen as a sort of generalization of the functor $\text{Comp}(\mathcal{K})$. (Although our formulations here include double categories, the ideas are basically 2-categorical, by the nature of the definition of monads in double categories.) Either of the two horizontal arrows in this diagram corresponds to the 2-categorification of Beck’s result, that given a monad-monad distributive law between monads $T$ and $S$ (given by the 2-cell $(\text{id}, \text{id})$, i.e. $\phi : ST \Rightarrow TS$ in $\mathcal{K}$), then $ST$ is again a monad in $\mathcal{K}$. In terms of our 2-functor $\mathcal{F}$, the 2-cell $\gamma : ST \Rightarrow TS$ corresponds to the 2-cell $\gamma'((\text{id}, \text{id}),(\text{id}, \text{id}))$ from Subsection 4.1.

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