Two-Party Quantum Protocols Do Not Compose Securely Against Honest-But-Curious Adversaries

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Abstract. In this paper, we build upon the model of two-party quantum computation introduced by Salvail et al. [SSS09] and show that in this model, only trivial correct two-party quantum protocols are weakly self-composable. We do so by defining a protocol $\Pi$ calling any non-trivial sub-protocol $\pi$ $N$ times and showing that there is a quantum honest-but-curious strategy that cannot be modeled by acting locally in every single copy of $\pi$. In order to achieve this, we assign a real value called payoff to any strategy for $\Pi$ and show that there is a gap between the highest payoff achievable by coherent and local strategies.

1 Introduction

The most striking result in quantum cryptography is certainly the capacity to perform secret-key distribution [BB84] securely by a universally composable quantum protocol [RK05,BHL+05]. This is in sharp contrast with what is achievable using classical communication alone. A different class of cryptographic primitives, called two-party computation, is not as easy to solve using quantum communication. In fact, some two-party primitives are as impossible to achieve using quantum communication as they are based solely on classical communication. In particular, well-known two-party primitives like oblivious transfer [Lo97], bit commitment [May97,LC97], and fair coin-tossing [Kit03] have neither classical nor quantum secure implementations. However, there exists weaker primitives achievable by quantum protocols but impossible in the classical world. For instance, sharing an EPR pair allows for two players to implement a noisy version of a two-party primitive called non-local box (NLB)\footnote{NLB : $(x^A,y^B) \mapsto (a^A,(a \oplus xy)^B)$, where $x$ and $y$ are Alice’s and Bob’s respective input bits, $a$ is a uniformly random output bit for Alice, and $a \oplus xy$ is the output bit for Bob.} with noise rate $\sin^2 \frac{\pi}{8}$ [PR94,BLM+05], which is a task impossible to achieve classically. Due to the local equivalence between randomized NLB and randomized one-out-of-two oblivious transfer (1-2-OT)\footnote{1-2-OT : $((x_0,x_1)^A,c^B) \mapsto x^B_c$, where $x_0$ and $x_1$ denote Alice’s input bits, $c$ denotes Bob’s input bits, and $x_c$ denotes Bob’s output bit.} [WW05b], a noisy version of randomized 1-2-OT with noise rate $\sin^2 \frac{\pi}{8}$ can also be obtained from one shared EPR-pair while no such classical protocol exists.

The cryptographic power of quantum protocols for two-party computations have been investigated in [SSS09]. Let Alice and Bob be the two parties involved in a two-party computation. In this model, a primitive is modelled by a joint probability distribution $P_{X,Y}$ where Alice outputs $x$ and Bob $y$ with probability $P_{X,Y}(x,y)$. Any two-party primitive can be randomized (the input to the functionality are picked at random) so that its functionality is captured by an appropriate choice of $P_{X,Y}$. We say that $P_{X,Y}$ is trivial if it can be implemented by a correct classical protocol against honest-but-curious (HBC) adversaries. Intuitively, a quantum protocol for primitive $P_{X,Y}$ is correct if once Alice and Bob get their
respective outputs with joint probability $P_{X,Y}$ then nothing else is available to each party about the other party’s output. Such a protocol can be purified and the measurements yielding the outcomes $X$ and $Y$ can be postponed to the end of the protocol’s execution. The state of the protocol just before the final measurements take place, is then called quantum embedding of the implemented primitive. In addition, regular embedding of a primitive is defined to be an embedding where Alice and Bob do not possess any other (auxiliary) registers than the ones used to measure their respective outputs. In [SSS09], it is shown that although quantum protocols can implement correctly non-trivial functionalities they will always leak extra information even against the weak class of honest-but-curious quantum adversaries. While classical protocols can only implement trivial primitives, quantum protocols necessarily leak when they correctly implement something non-trivial.

In this paper, we look at another aspect of two-party quantum protocols: their ability to compose against quantum honest-but-curious adversaries (QHBC). In order to guarantee composability, the functionality of a quantum protocol should be modeled by some classical ideal functionality. An ideal functionality is a classical description of what the protocol achieves independently of the environment in which it is executed. If a protocol does not admit such a description then it can clearly not be used in any environment while keeping its functionality, and such a protocol would not compose securely in all applications. In particular, in this thesis we investigate composability of non-trivial quantum protocols. An embedding of $P_{X,Y}$ is called trivial if both parties can access at least the same amount of information about the functionality as it is possible in some classical protocol for $P_{X,Y}$ in the HBC model. Otherwise, it is said to be non-trivial. A quantum protocol is non-trivial if its bipartite purification results in a non-trivial embedding. We show that no non-trivial quantum protocol composes freely even if the adversary is restricted to be honest-but-curious. No ideal functionality, even with an uncountable set of rules, can fully characterize the behavior of a quantum protocol in all environments. This is clearly another severe limit to the cryptographic power of two-party quantum protocols.

It is not too difficult to show that any trivial embedding can be implemented by a quantum protocol that composes against QHBC adversaries. In the other direction, let $|\psi(\pi)\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ be a non-trivial embedding of $P_{X,Y}$ corresponding to the bipartite purification of quantum protocol $\pi$. We know that $|\psi(\pi)\rangle$ necessarily leaks information towards a QHBC adversary. Any ideal functionality $\text{ID}_\pi$ for protocol $\pi$ trying to account for honest-but-curious behaviors should allow to simulate all measurements applied either in $\mathcal{H}_A$ or $\mathcal{H}_B$ through an appropriate call to $\text{ID}_\pi$. One way to do this is to define $\text{ID}_\pi$ by a function $[0..1] \times [0..1] \mapsto [0..1] \times [0..1]$ where $\text{ID}_\pi(0,0)$ corresponds to the honest behavior on both sides: $\text{ID}_\pi(0,0) = (x,y)$ with probability $P_{X,Y}(x,y)$ where $(x,y)$ is encoded as a pair of real numbers. Other inputs to the ideal functionality allow for the simulation of different strategies mounted by the QHBC adversary. In its most general form, an ideal functionality could have an uncountable set of possible inputs in order to allow the simulation of all QHBC adversaries. We show that even allowing for these general ideal functionalities, composed non-trivial protocols cannot be modeled by one single ideal functionality. It means that for a protocol $\Pi$ calling $N$ times any non-trivial sub-protocol $\pi$, there is a QHBC strategy that cannot be modeled by arbitrarily many calls of $\text{ID}_\pi$, each of them acting locally on a single copy of $\pi$.

In order to achieve this, we provide a generic example of such a protocol. Protocol $\Pi$ produces, as output, a real-value $p$ that we call payoff. The payoff $p$ represents how well the adversary can compare, without error, two factors of product states extracted from the $N$ executions of protocol $\pi$. From a result of [KKB05], the product states are constructed in such a way that no individual measurement can do as well as the best coherent measurement. It follows that the payoff corresponding to any adversary
restricted to deal with $\pi$ through any ideal functionality would necessarily be lower than the one an adversary applying coherent strategies on both parts of the product state could get. This implies that no ideal functionality for $\pi$ would ever account for all QHBC strategies in $\Pi$. Moreover, the advantage of coherent strategies over individual ones can be made constant. The result follows.

2 Preliminaries

CLASSICAL INFORMATION THEORY – DEPENDENT PART The following definition introduces a random variable describing the correlation between two random variables $X$ and $Y$.

Definition 2.1 (Dependent part [WW04]). For two random variables $X, Y$, let $f_X(x) := P_{Y|X=x}$. Then the dependent part of $X$ with respect to $Y$ is defined as $X \leftrightarrow Y := f_X(X)$.

The dependent part $X \leftrightarrow Y$ is the minimum random variable from the random variables computable from $X$ such that $X \leftrightarrow X \leftrightarrow Y \leftrightarrow Y$ is a Markov chain [WW04]. It means that for any random variable $K = f(X)$ such that $X \leftrightarrow K \leftrightarrow Y$ is a Markov chain, there exists a function $g$ such that $g(K) = X \leftrightarrow Y$. Immediately from the definition we get several other properties of $X \leftrightarrow Y$ [WW04]:

- $H(Y|X \leftrightarrow Y) = H(Y|X)$,
- $I(X;Y) = I(X \leftrightarrow Y;Y)$, and
- $X \leftrightarrow Y = X \leftrightarrow (Y \leftrightarrow X)$. The second and the third formula yield $I(X;Y) = I(X \leftrightarrow Y;Y \leftrightarrow X)$.

The notion of dependent part has been further investigated in [FWW04,IMNW04,WW05a]. Wullschleger and Wolf have shown that quantities $H(X \leftrightarrow Y|Y)$ and $H(Y \leftrightarrow X|X)$ are monotones for two-party protocols [WW05a]. That is, none of these values can increase during classical two-party protocols. In particular, if Alice and Bob start without sharing any non-trivial cryptographic resource then classical two-party protocols can only produce $(X,Y)$ such that: $H(X \leftrightarrow Y|Y) = H(Y \leftrightarrow X|X) = 0$, since $H(X \leftrightarrow Y|Y) > 0$ if and only if $H(Y \leftrightarrow X|X) > 0$ [WW05a]. Conversely, any primitive satisfying $H(X \leftrightarrow Y|Y) = H(Y \leftrightarrow X|X) = 0$ can be implemented securely in the honest-but-curious (HBC) model. We call such primitives trivial.

QUANTUM INFORMATION THEORY AND STATE DISTINGUISHABILITY Let $|\psi\rangle_{AB} \in \mathcal{H}_{AB}$ be an arbitrary pure state of the joint systems $A$ and $B$. The states of these subsystems are $\rho_A = \text{tr}_B |\psi\rangle\langle\psi|$ and $\rho_B = \text{tr}_A |\psi\rangle\langle\psi|$, respectively. We denote by $S(A) := S(\rho_A)$ and $S(B) := S(\rho_B)$ the von Neumann entropy (defined as the Shannon entropy of the eigenvalues of the density matrix) of subsystem $A$ and $B$ respectively. Since the joint system is in a pure state, it follows easily from the Schmidt decomposition that $S(A) = S(B)$ (see e.g. [NC00]). Analogously to their classical counterparts, we can define quantum conditional entropy $S(A|B) := S(AB) - S(B)$, and quantum mutual information $S(A;B) := S(A) + S(B) - S(AB) = S(A) - S(A|B)$. Even though in general, $S(A|B)$ can be negative, $S(A|B) \geq 0$ is always true if $A$ is a classical random variable.

The following lemma gives a relation between the probability of error and the probability of conclusive answer of a POVM used for discriminating two pure state.

Lemma 2.2 ([CB98]). Let the probability of a conclusive outcome and the error-probability of some POVM applied to a state, sampled uniformly at random from a pair of pure states $(|\psi_0\rangle,|\psi_1\rangle)$, be denoted by $q_c$ and $q_{err}$, respectively. Then

$$q_{err} \geq \frac{1}{2} \left( q_c - \sqrt{q_c^2 - (q_c - 1 + |\langle\psi_0|\psi_1\rangle|)^2} \right).$$
Notice that for the marginal case where $q_{err} = 0$ we get that $q_c \leq 1 - |\langle \psi_0 | \psi_1 \rangle|$ [Iva87,Die88,Per88], and for the marginal case where $q_c = 1$ (no inconclusive answer is allowed) we get $q_{err} \geq \frac{1}{2} - \sqrt{1 - |\langle \psi_0 | \psi_1 \rangle|^2}$ [Hel76].

Purification All security questions we ask are with respect to (quantum) honest-but-curious adversaries. In the classical honest-but-curious adversary model (HBC), the parties follow the instructions of a protocol but store all information available to them. Quantum honest-but-curious adversaries (QHBC), on the other hand, are allowed to behave in an arbitrary way that cannot be distinguished from their honest behavior by the other player.

Almost all impossibility results in quantum cryptography rely upon a quantum honest-but-curious behavior of the adversary. This behavior consists in purifying all actions of the honest players. Purifying means that instead of invoking classical randomness from a random tape, for instance, the adversary relies upon quantum registers holding all random bits needed. The operations to be executed from the random outcome are then performed quantumly without fixing the random outcomes. For example, suppose a protocol instructs a party to pick with probability $p$ state $|\phi_0 \rangle_C$ and with probability $1 - p$ state $|\phi^1 \rangle_C$ before sending it to the other party through the quantum channel $C$. The purified version of this instruction looks as follows: Prepare a quantum register in state $\sqrt{p}|0 \rangle_R + \sqrt{1 - p}|1 \rangle_R$ holding the random process. Add a new register initially in state $|0 \rangle_C$, before applying the unitary transform $U : |r \rangle_R|0 \rangle_C \mapsto |r \rangle_R|\phi^r \rangle_C$ for $r \in \{0,1\}$ and send register $C$ through the quantum channel and keep register $R$.

From the receiver’s point of view, the purified behavior is indistinguishable from the one relying upon a classical source of randomness because in both cases, the state of register $C$ is $\rho = p|\phi_0 \rangle \langle \phi_0 | + (1 - p)|\phi^1 \rangle \langle \phi^1 |$. All operations invoking classical randomness can be purified similarly[LC97,May97]. The result is that measurements are postponed as much as possible and only extract information required to run the protocol in the sense that only when both players need to know a random outcome, the corresponding quantum register holding the random coin will be measured. If both players purify their actions then the joint state at any point during the execution will remain in pure state, until the very last step of the protocol when the outcomes are measured.

Correct Two-Party Quantum Protocols and Their Embeddings In this section we define when a protocol correctly implements a joint distribution $P_{X,Y}$ which may correspond to some standard cryptographic task with uniformly random inputs. We call such a probability distribution primitive. As an example of a primitive, we can take e.g. $P_{X,Y}$ such that for all $x_0, x_1, y, c \in \{0,1\}$, $P_{X,Y}(x_0, x_1, c, y) = 1/8$ if and only if $y = x_c$. $P_{X,Y}$ then corresponds to a cryptographic task known as one-out-of-two oblivious transfer (1-2-OT), first introduced by Wiesner [Wie83]. It lets Alice send two bits $(x_0, x_1)$ to Bob, of which he selects one ($x_c$) to receive. In the randomized version, we assume the inputs $x_0, x_1$, and $c$ to be chosen uniformly at random. For standard cryptographic primitives such as 1-2-OT, the version with inputs can be securely implemented from the randomized version [WW05b]. It follows that for such primitives, considering the randomized version is without loss of generality.

As a result of purification of a protocol implementing primitive $P_{X,Y}$, up to the point when the final measurements take place, Alice and Bob obtain a shared pure state $|\psi \rangle$. Without loss of generality, we may assume that the final measurements yielding the implemented probability distribution are in the standard (computational) basis. Besides the registers $A$ and $B$ needed to compute $X$ and $Y$, the players could use auxiliary registers $A'$ and $B'$, yielding the final state $ket\psi$ to be in $\mathcal{H}_{AA'} \otimes \mathcal{H}_{BB'}$, where $\mathcal{H}_{AA'}$ and $\mathcal{H}_{BB'}$ denote the subsystems controlled by Alice and Bob, respectively. Informally, we call $|\psi \rangle$ an
embedding of $P_{X,Y}$, if the extra working registers $A'$ and $B'$ do not provide any extra information to the honest players, measuring their respective registers $A$ and $B$ in the computational bases. By “extra information” we mean additional information about the other party’s output, not available to a player from the ideal functionality for $P_{X,Y}$. A protocol whose purification produces an embedding of $P_{X,Y}$ as the final state is then called correct protocol for $P_{X,Y}$. Formally, we define an embedding of and a correct protocol for a given primitive as follows:

**Definition 2.3 ([SSS09]).** A protocol $\pi$ for $P_{X,Y}$ is correct if its final state satisfies $S(X;YB') = S(XA';Y) = I(X;Y)$ where $X$ and $Y$ are Alice’s and Bob’s honest measurement outcomes in the computational basis and $A'$ and $B'$ denote the extra working registers of Alice and Bob. The state $|\psi\rangle \in \mathcal{H}_{AB} \otimes \mathcal{H}_{A'B'}$ is called an embedding of $P_{X,Y}$ if it can be produced by the purification of a correct protocol for $P_{X,Y}$.

Correctness is a natural restriction imposed on two-party quantum protocols, since nothing can prevent honest players to perform any measurement they wish in the systems which are not needed to compute their desired outputs. In the following, we also use the notion of regular embedding which, as it turns out, simplifies the analysis of two-party quantum protocols.

**Definition 2.4 ([SSS09]).** Regular embedding of $P_{X,Y}$ is an embedding where the auxiliary registers $A'$ and $B'$ are trivial.

[SSS09] shows that any embedding of $P_{X,Y}$ can be easily converted into its regular embedding by a measurement performed on either side.

**Lemma 2.5 ([SSS09]).** Let $|\psi\rangle_{A'B'B'}$ be an embedding of $P_{X,Y}$. Then $|\psi\rangle$ is locally equivalent to a state $|\psi^*\rangle$ in the form:

$$|\psi^*\rangle = \sum_k \lambda_k |k, k\rangle_{A'B'} |\psi_k\rangle_{AB},$$

where $\lambda_k$ are all nonnegative real numbers and for each $k$, $|\psi_k\rangle$ is a regular embedding of $P_{X,Y}$.

It follows easily from the lemma above that Alice can convert $|\psi\rangle$ into a product state $|\psi_k\rangle_{AB} \otimes |\varphi\rangle_{B'}$ by a proper measurement in register $A'$. An analogous statement holds for Bob.

Informally, an embedding $|\psi\rangle_{A'B'B'}$ of $P_{X,Y}$ is called trivial, if it allows a dishonest player to access at least the same amount of information as he/she is allowed in some classical implementation of $P_{X,Y}$. Formally, we define trivial and non-trivial embeddings of a given primitive as follows:

**Definition 2.6 ([SSS09]).** Let $|\psi\rangle_{A'B'B'}$ be an embedding of $P_{X,Y}$. We call $|\psi\rangle$ a trivial embedding of $P_{X,Y}$ if it satisfies $S(Y \triangleright X|AA') = 0$ or $S(X \triangleright Y|BB') = 0$. Otherwise, we call it non-trivial.

Notice that $P_{X,Y}$ can be implemented by the following classical protocol:

1. Bob samples $x' = P_{Y|X \triangleright Y = x'}$ from the distribution $P_{X \triangleright Y}$ and sends it to Alice. He samples $y$ from the distribution $x'$.
2. Alice samples $x$ from the distribution $P_{X|X \triangleright Y = x'}$.

Clearly, in the case where $S(X \triangleright Y|BB') = 0$, $|\psi\rangle$ allows dishonest Bob and Alice to access at least as much information about the other party’s outputs, as they can in the classical implementation above.
3 Non-Trivial Protocols and Composability

In the following we show that quantum protocols even characterized only by the embeddings of the corresponding primitives (i.e. without considering whether or not that state can be distributed fairly) do not compose without allowing the adversary to mount joint attacks that cannot be simulated by attacks applied to individual copies. We are allowed to make this simplification because any attack of an embedding of a primitive can be modeled by an at least equally efficient (in terms of the amount of extra information accessible by a cheater) attack of the associated protocol. We define trivial protocols to be such that produce trivial embeddings.

Definition 3.1. A correct protocol for a primitive $P_{X,Y}$ is trivial, if the embedding produced by such a protocol is trivial. Otherwise, it is called non-trivial.

In order to show non-composability of a non-trivial embedding $|\psi\rangle \in \mathcal{H}_{ABA'B}$ of a primitive $P_{X,Y}$, satisfying $t S_\psi(X \rightarrow Y|BB') > 0$ and $S_\psi(Y \rightarrow X|AA') > 0$, it is sufficient to show that no non-trivial regular embedding of $P_{X,Y}$ can be composed, for the following reason: Lemma 2.5 shows that by measuring register $A'$ of $|\psi\rangle$, Alice converts $|\psi\rangle$ into $|\psi_k\rangle$ for some $k \in \{1, \ldots, K\}$, which is a regular embedding of $P_{X,Y}$. If she performs such a measurement on many copies of $|\psi\rangle$, with high probability at least some constant fraction of them collapses into the same non-trivial regular embedding of $P_{X,Y}$. Non-composability of such a regular embedding then implies non-composability of embedding $|\psi\rangle$ of $P_{X,Y}$. The protocol composability questions can therefore be reduced to investigating composability of regular embeddings.

In the following, we formalize the weakness of non-composability inherent to any two-party quantum protocol, preventing us from building strong cryptographic primitives even from non-trivial weak ones. This is in a sharp contrast with quantum key distribution - a three-party game that can be shown to be universally composable [BHL05].

Composability of quantum protocols has been studied by Ben-Or and Mayers [BM02,BM04] and by Unruh [Unr04]. The former approach is an extension of Canetti’s framework [Can01] to the quantum case while the latter is an extension of Backes, Pfitzmann, and Waidner [BPW04]. We are going to consider a weaker version of composability called weak composability and show that almost no quantum protocol satisfies it. Informally, we call a quantum two-party protocol weakly self-composable if any adversarial strategy acting, possibly coherently, upon $n$ independent copies of the protocol is equivalent to a strategy which acts individually upon each copy of the protocol.

4 Ideal Functionalities

In order to guarantee composability, the functionality of a quantum protocol should be modeled by some classical ideal functionality. An ideal functionality is a classical description of what the protocol achieves independently of the environment in which it is executed. If a protocol does not admit such a description then it can clearly not be used in any environment while keeping its functionality, and such a protocol would not compose securely in all applications.

In the following, let $H_A$ and $H_B$ denote Alice’s and Bob’s quantum systems, respectively, and let $X'$ and $Y'$ denote the set of classical outcomes of Alice’s and Bob’s final measurements.

Intuitively, a pure state $|\psi\rangle \in H_A \otimes H_B$ implements the ideal functionality $ID_\psi$ if whatever the adversary does on his/her part of $|\psi\rangle$, there exists a classical input to $ID_\psi$ for the adversary that produces
the same view. The ideal functionality $\text{ID}_\psi$ accepts inputs for Alice and for Bob in $[0..1]$, where the elements of $[0..1]$ encode all possible strategies for both parties. When a party inputs $0$ to $\text{ID}_\psi$, the outcome of measuring this party’s part of $|\psi\rangle$ in the computational basis, encoded by a number in $[0..1]$ is returned to the party. This corresponds to the honest behavior. When $m \in [0..1]$ is input to $\text{ID}_\psi$, a measurement depending upon $m$ is applied to register $\mathcal{H}_A$ (resp. $\mathcal{H}_B$) of $|\psi\rangle$ and the classical outcome is returned to Alice (resp. Bob). Such a measurement acts only locally on the specified system. Clearly, for $\text{ID}_\psi$ to be of any cryptographic value, the set of possible strategies should be small, otherwise it would be very difficult to characterize exactly what $\text{ID}_\psi$ achieves. As we are going to show next, even if $|\psi\rangle$ implements such an $\text{ID}_\psi$ where $[0..1]$ is used to encode all possible POVMs in $\mathcal{H}_A$ and $\mathcal{H}_B$ then all adversarial strategies against $|\psi\rangle^\otimes n$ cannot be modeled by calls to $n$ copies of $\text{ID}_\psi$.

We write $\text{ID}_\psi(m, 0) = (\bar{w}, z)$ for the ideal functionality corresponding to pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ with honest Bob and dishonest Alice using strategy $m \in (0..1]$. The output $\bar{w}$ is provided to Alice and $z \in [0..1]$ encoding an event in $\mathcal{Y}$ to Bob. Similarly, we write $\text{ID}_\psi(0, m) = (z, \bar{w})$ when Alice is honest and Bob is dishonest and is using strategy $m \in (0..1]$. Notice that an ideal functionality for state $|\psi\rangle$ is easy to implement by letting $\text{ID}_\psi$ simulate Alice’s and Bob’s strategies through a classical interface.

In general, $\text{ID}_\psi$ returns one party’s output as soon as its strategy has been specified. The ideal functionality never waits for both parties before returning the outcomes. This models the fact that shared pure states never signal from one party to the other. The ideal functionality $\text{ID}_\psi$ can be queried by one party more than once with different strategies. The ideal functionality keeps track of the residual state after one strategy is applied. If a new strategy is applied then it is applied to the residual state. This feature captures the fact that the first measurement can be applied before knowing how to refine it, which may happen when Alice and Bob are involved in an interactive protocol using only classical communication from shared state $|\psi\rangle$. Dishonest Alice may measure partially her part of $|\psi\rangle$ before announcing the outcome to Bob. Bob could then send information to Alice allowing her to refine her measurement of $|\psi\rangle$ dependently of what she received from him. This procedure can be simulated using $\text{ID}_\psi$ after specifying a partial POVM for Alice’s first measurement among the set of POVMs encoded by the elements of $[0..1]$. Then, Alice refines her first measurement by specifying a new POVM represented by an element of $[0..1]$ to the ideal functionality $\text{ID}_\psi$.

5 Simulation

A pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ implements the ideal functionality $\text{ID}_\psi$ if any attack implemented via POVM $\mathcal{M}$ by adversary Alice (resp. adversary Bob) can be simulated by calling the ideal functionality with some $m \in [0..1]$. The attack in the simulated world calls $\text{ID}_\psi$ only once as it is in the real case. The ideal functionality $\text{ID}_\psi$ therefore refuses to answer more than one query per party. Remember also that $\text{ID}_\psi$ returns the outcome to one party as soon as its strategy has been specified irrespectively of whether the other party has specified its own.

First, let us show on an example what do we mean by simulation of an attack using the calls to the ideal functionality.

Example 5.1. Consider that Alice and Bob are sharing $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ which is an embedding of the joint probability distribution $P_{X,Y}$ with $P_{X,Y}(0,0) = P_{X,Y}(1,1) = 1/2$. Alice’s and Bob’s honest measurement happen to be in the Schmidt basis. We can define the ideal functionality $\text{ID}_{\text{EPR}}$ as follows:

$$\text{ID}_{\text{EPR}}(0,0) = (x, x) \text{ with prob. } \frac{1}{2}.$$
Since both players are measuring in the Schmidt basis, it follows that \( \text{ID}_{\text{EPR}} \) models any adversarial behavior. \( \text{ID}_{\text{EPR}} \) is an ideal functionality for \( |\Psi^+\rangle \) even in a context where it is a part of a larger system. However, \( |\Psi^+\rangle \) is a trivial embedding!

Notice that any strategy against \( |\Psi^+\rangle^\otimes m \) can be simulated by appropriate calls to \( m \) copies of \( \text{ID}_{\text{EPR}} \). In other words, \( |\psi^+\rangle \) is self-composable in a weak sense. In the following section we show that in fact, all weakly self-composable regular embeddings of joint probability distributions are trivial.

6 Self-Composability of Embeddings

We define the **classical weak self-composability** of a regular embedding \( |\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \) of a joint probability distribution \( P_{X,Y} \) as its ability to be composed with itself without allowing the adversary to get information about \( X \) resp. \( Y \) that is not available through calls to independent copies of \( \text{ID}_{\psi} \).

**Definition 6.1.** Embedding \( |\psi\rangle \) of \( P_{X,Y} \) is weakly self-composable if there exists an ideal functionality \( \text{ID}_{\psi} \) such that all attacks against \( |\psi\rangle^\otimes m \) for any \( m > 0 \) can be simulated by appropriate calls to \( m \) ideal functionalities \( \text{ID}_{\psi} \).

Next, we show that only (not necessarily all) trivial regular embeddings can be weakly self-composed. The idea behind this result is the definition of a protocol computing a function, between Alice and Bob sharing \( |\psi\rangle^\otimes m \) such that Bob can make the expected value of the function strictly larger provided he has the capabilities to measure his part of \( |\psi\rangle^\otimes m \) coherently rather than individually. Only individual measurements can be performed by Bob if \( \text{ID}_{\psi} \) is modelling the behavior of \( |\psi\rangle \) in any situation. Consider that Alice and Bob are sharing a non-trivial regular embedding \( |\psi\rangle \) of \( P_{X,Y} \) that can be written as:

\[
|\psi\rangle = \sum_{x \in \mathcal{X}} \sqrt{P_X(x)} |x\rangle^A |\psi_x\rangle^B.
\]

We show in Lemma 6.2 that \( |\psi\rangle \) being non-trivial (i.e. \( S(X \perp Y | \rho_B) > 0 \)) implies existence of \( x_0 \neq x_1 \in \mathcal{X} \) such that

\[
0 < |\langle \psi_{x_0} | \psi_{x_1} \rangle|^2 < 1.
\]

Protocol 1 challenges Bob to **identify** in some sense the state of two positions chosen uniformly and at random among the following possibilities:

\[
\{|\psi_{x_0} \rangle | \psi_{x_0} \rangle, |\psi_{x_0} \rangle | \psi_{x_1} \rangle, |\psi_{x_1} \rangle | \psi_{x_0} \rangle, |\psi_{x_1} \rangle | \psi_{x_1} \rangle\}.
\]

We will show that Bob, restricted to interact with his subsystem through the ideal functionality \( \text{ID}_{\psi} \), cannot make the expected value of a certain function as large as when it is allowed to interact unconditionally (i.e. coherently) with his subsystem. We now prove that such \( x_0, x_1 \in \mathcal{X} \) exist for any non-trivial regular embedding.

**Lemma 6.2.** If \( |\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \) is a non-trivial regular embedding of \( P_{X,Y} \) then there exist \( x_0, x_1 \in \mathcal{X} \) such that \( |\psi_{x_0}\rangle \) and \( |\psi_{x_1}\rangle \) satisfy

\[
0 < |\langle \psi_{x_0} | \psi_{x_1} \rangle| < 1.
\]

**Proof.** Let us write \( |\psi\rangle \) as,

\[
|\psi\rangle = \sum_{x \in \mathcal{X}} \sqrt{P_X(x)} |x\rangle^A |\psi_x\rangle^B.
\]

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Let \( \{ |\psi^*_x \rangle, \ldots, |\psi^*_m \rangle \} \subseteq \{ |\psi_x \rangle \} \) be the set of different states \( |\psi_x \rangle \) available to Bob when Alice measures \( X \). Equation (3) can be re-written as,

\[
|\psi \rangle = \sum_{j=1}^{\ell} \left( \sum_{x \in X} e^{i\theta(x)} \sqrt{P_x(x)} |x \rangle \right) \otimes |\psi^*_j \rangle,
\]

for some \( \theta(x) \in [0 \ldots 2\pi) \).

If \( \{ |\psi^*_j \rangle \}^{\ell}_{j=1} \) are mutually orthogonal then if Bob measures in this basis no uncertainty about \( X \setminus Y \) is left contradicting the fact that \( S(X \setminus Y | \rho_B) > 0 \).

In Protocol 1 Alice asks Bob to compare the two pure states on his side. In the next section we define a game related to the state comparison problem and show that there is a coherent strategy which in this game can succeed strictly better than any separable one, and therefore also LOCC strategy on Bob’s registers.

**CHALLENGE:**

1. Let \( p := 0 \) and let Alice and Bob both know \( x_0, x_1 \in X \) such that \( 0 < |\langle \psi_{x_0} | \psi_{x_1} \rangle| = \tau < 1 \) is satisfied.
2. Alice gets \( X^m = X_1, \ldots, X_m \) by measuring her part in all \( m \) copies of \( |\psi \rangle \) in the computational basis. She identifies 4 positions \( 1 \leq i \neq i', j \neq j' \leq m \) such that \( X_i = X_{i'} = x_0 \) and \( X_j = X_{j'} = x_1 \). If such four positions do not exist then Alice announces to Bob that \( p = 0 \) and aborts.
3. Alice picks \( (h, h') \in \{ i, i', j, j' \} \) with \( h \neq h' \) such that \( (X_h, X_{h'}) = (\alpha, \beta) \) with probability \( 1/4 \) for any choice of \( \alpha, \beta \in \{ x_0, x_1 \} \) and announces \( (h, h') \) to Bob.
4. Bob sends \( b \in \{ 0, 1, ? \} \) to Alice, guessing whether the pair of pure states on the positions \( h, h' \) is one of \( A_0 := \{ |\psi_{x_0} \rangle |\psi_{x_0} \rangle, |\psi_{x_1} \rangle |\psi_{x_1} \rangle \}, A_1 := \{ |\psi_{x_0} \rangle |\psi_{x_1} \rangle, |\psi_{x_1} \rangle |\psi_{x_0} \rangle \} \), or responds by “don’t know”.
5. Alice sets the payoff value \( p : p := -c \) if Bob responded incorrectly, \( p := 0 \) if he answered “don’t know”, and \( p := 1 \) if he answered the challenge correctly.

**Fig. 1.** A state comparison challenge to Bob.

### 7 State-Comparison Game with a Separably Inapproximable Coherent Strategy

Consider the challenge from Protocol 1. In the game defined by this protocol, Alice lets Bob compare two states defined by a non-trivial regular embedding of a given primitive, which are either identical or different, but not orthogonal. Bob is allowed to respond inconclusively however, for such an answer he obtains 0 points. On the other hand, if his guess is right, he obtains 1 point and if it is wrong, he obtains \(-c\) points for some positive number \( c \) which we determine later. We call his score payoff. With respect to the game defined by Protocol 1, let the maximal achievable expected payoff over the set of all measurement strategies be denoted by \( p_{\text{max}} \). In this section we show that there exists \( c \) such that the maximal average payoff \( p_{\text{max}} \) can be only achieved with a strategy coherent on the registers corresponding to the two factors of Bob’s product state. Furthermore, we show that for such a \( c \) there is a constant gap between the maximal payoff achievable with a separable strategy and \( p_{\text{max}} \). Separable measurements on a quantum system consisting of two subsystems are such that any of their elements \( M \)
is in the form $M = \sum_{i,j} F_{i}^{0} \otimes F_{j}^{1}$, where $F_{i}^{0}, F_{j}^{1}$ are the operators acting on the respective subsystems of the given system. According to [BDF+99], separable measurements form a strict superset of all LOCC measurements.

It is shown in [KKB05] that for $0 < \tau < 1$, the optimal no-error measurement is always coherent. Furthermore, they prove that the highest success rate achievable by a separable unambiguous measurement is $(1 - \tau)^2$ whereas the optimal measurement has a success rate $(1 - \tau)$.

Fix the value of $0 < \tau < 1$. For sufficiently large the best coherent strategy is to apply the best unambiguous measurement with the correct-answer rate $1 - \tau$, and to output don’t know for an uncertain result. Therefore, for some $c$ we have $p_{\text{max}} = 1 - \tau$. Let $p_{s}$ denote the supremum of average payoffs in the game from Protocol 1 achievable by separable strategies.

**Theorem 7.1.** In the game from Protocol 1 there exists $c > 0$ such that $p_{s} \leq p_{\text{max}} - f(\tau)$, where $f(\tau) > 0$ whenever $0 < \tau < 1$.

Before proving the actual theorem, we introduce a useful lemma.

**Lemma 7.2.** Let $|\varphi_0\rangle, |\varphi_1\rangle \in \mathcal{H}$ be pure states such that $|\langle \varphi_0 | \varphi_1 \rangle| = \tau$. For a discrimination strategy $S$ with three possible outcomes 0, 1, and “don’t know”, let $q_{c}$ denote the probability of a conclusive answer and $q_{\text{err}}$ the probability of a wrong answer. Then,

$$q_{c} \leq 2q_{\text{err}} + 1 - \tau + 2\sqrt{q_{\text{err}}(1 - \tau)}.$$

**Proof.** According to Lemma 2.2,

$$q_{\text{err}} \geq \frac{1}{2} \left(q_{c} - \sqrt{q_{c}^{2} - (q_{c} - (1 - \cos \theta))^{2}}\right).$$

Equivalently, we get:

$$\sqrt{q_{c}^{2} - (q_{c} - (1 - \cos \theta))^{2}} \geq q_{c} - 2q_{\text{err}}.$$

By squaring both sides of the inequality we obtain:

$$2q_{c}(1 - \cos \theta) - (1 - \cos \theta)^{2} \geq q_{c}^{2} + 4q_{\text{err}}^{2} - 4q_{c}q_{\text{err}}$$

$$q_{c}^{2} - q_{c}(4q_{\text{err}} + 2(1 - \tau)) + (1 - \tau)^{2} + 2q_{\text{err}}^{2} \leq 0. \quad (5)$$

By solving the quadratic equation

$$q_{c}^{2} - q_{c}(4q_{\text{err}} + 2(1 - \tau)) + (1 - \tau)^{2} + 2q_{\text{err}}^{2} = 0,$$

we get the solutions $2q_{\text{err}} + 1 - \tau \pm 2\sqrt{q_{\text{err}}(1 - \tau)}$, implying the solutions of (5) to be

$$q_{c} \leq 2q_{\text{err}} + 1 - \tau + 2\sqrt{q_{\text{err}}(1 - \tau)}.$$

**Proof (Theorem 7.1).** The method we use is the following: For given parameters $\tau, c \in \mathbb{R}$ such that $0 < \tau < 1$ and $c > 0$, and an additional parameter $k > 0$, we divide the set of all separable measurements into three subsets according to the probability $q_{\text{err}}$ of Bob’s incorrect (conclusive) answer in the state-comparison, expressed as a function of $c, k,$ and $\tau$. We construct an upper bound on $p_{s}$ in each of the...
three sets separately and dependently on \( c, k, \) and \( \tau \). Finally, we find the conditions for \( c \) and \( k \) such that in all three sets we get \( q_s \leq p_{\text{max}} - f(\tau) \) for some \( f(\tau) > 0 \).

[KKB05] shows that the best separable unambiguous strategy for solving the 2-out-of-2 state comparison problem is applying the best unambiguous measurements on each part of Bob’s register independently. Lemma A.2 (see Appendix A) says that the payoff achieved by such a strategy in the case where probability \( q_{\text{err}} \) is small, is close to the optimal payoff. The analysis of such a situation is captured in the first of the three cases, where we consider the separable measurements with \( q_{\text{err}} \leq \frac{1}{2k(c+1)^2} \).

1. \((q_{\text{err}} \leq \frac{1}{2k(c+1)^2})\) Lemma A.2 shows that to any separable measurement \( \mathcal{M} = (E_0, E_1, E_2) \) with probability of error \( q_{\text{err}} \leq \frac{1}{2k(c+1)^2} \) and the expected payoff \( p \), there exists a separable measurement \( \mathcal{M}' = (E'_0, E'_1, E'_2) \) with the expected payoff \( p' \), satisfying \( p \leq p' + \frac{1}{k} + O(1/\sqrt{c}) \), such that its elements can be written in the form:

\[
E'_0 = G^0_0 \otimes G^0_0 + G^0_1 \otimes G^1_1, \quad E'_1 = G^0_0 \otimes G^1_0 + G^0_1 \otimes G^0_1, \quad E'_2 = 1 - E'_0 - E'_1,
\]

where the upper index of \( G^0_0 \) refers to the subsystem and the lower index determines the guess of the state of the corresponding subsystem.

The upper bound on the value of \( p' \) which we compute next, can then be used to upper bound \( p \). Consider an extended problem where Bob is supposed to identify each factor of his product state (in contrast to just comparing the factors in the game). Let \( q^0_{\text{err}}, q^1_{\text{err}}, \) and \( q^0_c, q^1_c \) denote the probabilities of Bob’s incorrect resp. conclusive answers in each of his subsystems. Then the probability of comparing the states incorrectly can be expressed as follows:

\[
q_{\text{err}} = q^0_{\text{err}}(q^1_c - q^1_{\text{err}}) + q^1_{\text{err}}(q^0_c - q^0_{\text{err}}) = q^0_c q^0_{\text{err}} + q^1_c q^1_{\text{err}} - 2q^0_{\text{err}} q^1_{\text{err}}.
\]

For separable strategies for which \( q^1_c < 1 - \tau - 2/k \) or \( q^0_c < 1 - \tau - 2/k \), we obtain \( p' < 1 - \tau - 2/k \) and hence, \( p < 1 - \tau - 1/k + O(1/\sqrt{c}) \) due to Lemma A.2. For \( c \) sufficiently large we then get:

\[
\frac{1}{p_{\text{max}} - p} \geq \frac{1}{2k}. \tag{6}
\]

Next, we discuss the case (not disjoint with the previous one) where both \( q^0_c, q^1_c \geq 1 - \tau - 1/k \) =: \( \gamma \), which implies that

\[
q_{\text{err}} \geq \gamma(q^0_{\text{err}} + q^1_{\text{err}}) - 2q^0_{\text{err}} q^1_{\text{err}}. \tag{7}
\]

For upper bounding the probability \( q^0_c \) of a conclusive answer of the measurement \( \mathcal{M}' \) we use Lemma 7.2 (an analogous formula holds for \( q^1_c \)):

\[
q^0_c \leq 2q^0_{\text{err}} + 1 - \tau + 2\sqrt{q^0_{\text{err}}(1 - \tau)}. \tag{8}
\]

The probability of correct state-identification in the first of Bob’s subsystems then satisfies:

\[
q^0_c - q^0_{\text{err}} \leq q^0_{\text{err}} + 1 - \tau + 2\sqrt{q^0_{\text{err}}(1 - \tau)}. \tag{8}
\]

Inequalities (7) and (8) give us an upper bound on \( p' \) for \( c > 9 \):

\[
p' \leq -cq_{\text{err}} + q^0_{\text{err}} q^1_{\text{err}} + (q^0_{\text{err}} + 1 - \tau + 2\sqrt{q^0_{\text{err}}(1 - \tau)})(q^1_{\text{err}} + 1 - \tau + 2\sqrt{q^1_{\text{err}}(1 - \tau)})
\leq -cq_{\text{err}} + (1 - \tau)^2 + 2(\sqrt{q^0_{\text{err}} + \sqrt{q^1_{\text{err}}}}) + 9q_{\text{err}}
\leq (1 - \tau)^2 + \frac{4\sqrt{q_{\text{err}}}}{\sqrt{\gamma}} \leq (1 - \tau)^2 + \frac{4}{\sqrt{2\gamma k(c + 1)}}
\]

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hence by Lemma A.2, \( p \leq (1 - \tau)^2 + \frac{4}{\sqrt{2}k(c+1)} + \frac{1}{k} + O(1/\sqrt{c}) \). For \( c \) sufficiently large we get:

\[
p \leq (1 - \tau)^2 + \frac{2}{k}.
\] (9)

2. Second, we assume that \( \frac{1}{2k(c+1)} < q_{\text{err}} \leq \frac{1}{256(1 - \tau)} \). To upper bound the probability of comparing the states correctly, we use the same argument as in (8) and get that:

\[
q_c - q_{\text{err}} \leq q_{\text{err}} + 1 - \tau + 2\sqrt{q_{\text{err}}(1 - \tau)},
\]

where \( q_c \) denotes the probability of a conclusive outcome. This inequality implies the upper bound on \( p \):

\[
p \leq -cq_{\text{err}} + (q_c - q_{\text{err}}) \leq -c - \frac{1}{c+1} \cdot \frac{1}{2k} + 1 - \tau + 2\sqrt{q_{\text{err}}(1 - \tau)},
\]

yielding that for \( c \) sufficiently large,

\[
p \leq -\frac{1}{2k} + 1 - \tau + 2\sqrt{q_{\text{err}}(1 - \tau)}.
\] (10)

Consequently, we have three upper bounds on the value of \( p \), given by (6), (9), and (10): \( B_0 := 1 - \tau - \frac{1}{2k} \), \( B_1 := (1 - \tau)^2 + \frac{2}{k} \), and \( B_2 := 1 - \tau + 2\sqrt{q_{\text{err}}(1 - \tau)} - \frac{1}{2k} \). Since \( B_2 \geq B_0 \), we only have to find \( f(\tau) \) and \( k \) such that \( B_1, B_2 \leq (1 - \tau) - f(\tau) \), or equivalently:

\[
2f(\tau) + 4\sqrt{q_{\text{err}}(1 - \tau)} \leq \frac{1}{k} \leq \frac{\tau(1 - \tau) - f(\tau)}{2}.
\]

It is easy to verify that for \( d \leq \frac{1}{256(1 - \tau)} \), the two inequalities are satisfied for \( k := \frac{20}{\sqrt{\tau}(1 - \tau)} \) and \( f(\tau) := \frac{\tau(1 - \tau)}{10} \). Thus, there exists \( c > 0 \) such that in any separable strategy with the probability of error \( q_{\text{err}} \leq \frac{1}{256(1 - \tau)} \) and the expected payoff \( p \):

\[
p \leq p_{\text{max}} - \frac{\tau(1 - \tau)}{10}.
\]

3. For separable strategies with the probability of error \( q_{\text{err}} > \frac{1}{256(1 - \tau)} \), we can simply set \( c > 256(1 - \tau) \) which ensures that the payoff \( p \leq 0 \).

Set \( c \) to be the maximum over the values required by the discussed subcases. For such a \( c \) and any separable strategy, the corresponding expected payoff \( p \) satisfies \( p \leq p_{\text{max}} - \frac{\tau(1 - \tau)}{10} \), yielding that

\[
p_{\text{max}} - p_s \geq \frac{\tau(1 - \tau)}{10}.
\]
8 Only Trivial Embeddings Can Be Composed

As a straightforward corollary of Theorem 7.1, we now get that there exists a constant $c$ such that any Bob restricted to interact with his system through the ideal functionality $ID_\psi^{\otimes m}$ can never get the expected value of $p$ as large and not even close as with the best coherent strategy. This remains true for any possible description of the ideal functionality since even if $ID_\psi$ allowed to specify an arbitrary POVM then the ideal functionality would not be as good as the best coherent strategy.

Notice that any strategy Bob may use for querying the ideal functionality $ID_\psi$ for both systems involved in order to pass the challenge with success, can also be carried by two parties restricted to local quantum operation and classical communication (LOCC). This is because $ID_\psi$ only returns classical information. Local quantum operations can be performed by asking $ID_\psi$ to apply a POVM to a local part of $|\psi\rangle$.

We now formally prove that non-trivial regular embeddings do not compose since Bob can always succeed better in Protocol 1 if he could measure all his registers coherently.

**Theorem 8.1.** Only trivial regular embeddings of a primitive $P_{X,Y}$ are weakly self-composable.

**Proof.** Let $|\psi\rangle = \sum_{x\in X} \sqrt{P_X(x)} |x\rangle |\psi_x\rangle$ be a non-trivial regular embedding of $P_{X,Y}$. According to Lemma 6.2 there exist $x_0, x_1 \in X$ such that $0 < |\langle \psi_{x_0} | \psi_{x_1}\rangle| < 1$. Theorem 7.1 then implies that there is $c \in \mathbb{R}^+$ such that in Protocol 1 played with $|\psi_{x_0}\rangle$ and $|\psi_{x_1}\rangle$ satisfying the condition above, the expected payoff achievable by the best coherent strategy is strictly better than what can be achieved by separable i.e. also LOCC strategies. By definition of weak self-composability it means that non-trivial regular embedding $|\psi\rangle$ of $P_{X,Y}$ is not weakly self-composable.

**Corollary 8.2.** Only trivial (correct) two-party quantum protocols are weakly self-composable.

**Proof.** The statement follows from the fact that any quantum honest-but-curious attack of an embedding can be modeled by an attack of the corresponding two-party protocol. Lemma 2.5 shows that for any party there is a measurement converting a regular embedding $|\psi\rangle \in \mathcal{H}_{ABA'B'}$ of a primitive $P_{X,Y}$ into an embedding $|\psi_k\rangle$ of $P_{X,Y}$ for some $k \in \{1, \ldots, K\}$. The other party can also learn the index $k$ by measuring his/her additional register. Non-composability of non-trivial quantum two-party protocols for $P_{X,Y}$ then follows from non-composability of non-trivial regular embeddings of $P_{X,Y}$ by including a pre-stage into the game from Protocol 1. In this stage, Alice and Bob convert each of the many embeddings of $P_{X,Y}$ corresponding to the protocol copies into a regular embedding of $P_{X,Y}$ known to both parties. This conversion results into a non-trivial regular embedding of $P_{X,Y}$ with constant probability. This is because if all regular embeddings in the conversion-range were trivial, then the measurement converting the embedding into regular embeddings could be used as a part of a measurement revealing $X \ \downarrow \ Y$ completely to Bob, or revealing $Y \ \downarrow \ X$ completely to Alice. Hence, such an embedding and the corresponding protocol would then be trivial. Due to the law of large numbers, from several copies of an embedding Alice obtains at least some constant fraction of the same non-trivial regular embeddings except of probability negligible in the number of copies. Alice and Bob then play the game from Protocol 1, using the subset of copies where Alice obtained the same non-trivial regular embedding.

Finally, let us mention several facts related particularly to (non-)composability of trivial two-party quantum protocols implementing trivial primitives. Clearly, every trivial primitive has a protocol which is composable against quantum honest-but-curious adversaries, namely the classical one implementing...
the primitive securely in the HBC model. Formally, for a trivial \( P_{X,Y} \) we show composability of quantum protocols implementing only \( P_{X \rightarrow Y, Y \rightarrow X} \) (which corresponds to secure implementation in the HBC model) instead of \( P_{X,Y} \), where the desired distribution \( P_{X,Y} \) is obtained from the implementation of \( P_{X \rightarrow Y, Y \rightarrow X} \) by local randomization. Since a trivial primitive satisfies \( H(X \rightarrow Y | Y \rightarrow X) = H(Y \rightarrow X | X \rightarrow Y) = 0 \) or in other words, the implemented dependent parts are accessible to both parties already in one protocol copy, coherent attacks do not help in getting any more information. Because the rest of \( X \) and \( Y \) is computed purely locally, there is no attack, individual or coherent, revealing any information about the result of this operation.

On the other hand, not all protocols for trivial primitives are composable. As an example let us take a protocol for a primitive \( P_{X,Y} \) defined by \( P_{X,Y}(0,0) = P_{X,Y}(1,0) = 3/8, P_{X,Y}(0,1) = P_{X,Y}(1,1) = 1/8 \), represented by the following regular embedding:

\[
|\psi\rangle = \frac{1}{\sqrt{2}} |0\rangle \otimes \left( \frac{\sqrt{3}}{2} |0\rangle + \frac{1}{2} |1\rangle \right) + \frac{1}{\sqrt{2}} |1\rangle \otimes \left( \frac{\sqrt{3}}{2} |0\rangle - \frac{1}{2} |1\rangle \right)
\]

Such an embedding (and therefore, the corresponding protocol) is trivial because it implements a trivial primitive. Formally, \( 0 = H(X \rightarrow Y | Y) \) and \( H(X \rightarrow Y | Y) \geq S(X \rightarrow Y | B) \) imply that \( S(X \rightarrow Y | B) = 0 \). On the other hand, the states

\[
|\psi_0\rangle := \frac{\sqrt{3}}{2} |0\rangle + \frac{1}{2} |1\rangle, \quad |\psi_1\rangle := \frac{\sqrt{3}}{2} |0\rangle - \frac{1}{2} |1\rangle
\]

that Bob gets for Alice’s respective outcomes 0 and 1 of the measurement in the canonical basis, satisfy the condition \( 0 < |\langle \psi_0 | \psi_1 \rangle| < 1 \) from Protocol 1. Hence, the arguments from the proof of Theorem 7.1 apply, yielding that \( |\psi\rangle \) cannot be composed.

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A Lemma A.2 from the proof of Theorem 7.1

Before starting with the actual Lemma A.2, we formulate and prove an auxiliary lemma, needed for the main proof. In the following, \( \|T\|_\infty \) denotes the norm of an operator \( T \in \mathbb{C}^{n \times n} \), which equals the operator’s largest singular value.

**Lemma A.1.** Let \( f : \mathbb{R}^+ \to \mathbb{C}^{2 \times 2} \) be a function mapping \( c \) positive into a positive-semidefinite operator \( F_c \in \mathbb{C}^{2 \times 2} \) such that \( \|F_c\|_\infty = 1 \) and for some unit vector \( |v_0\rangle \in \mathbb{C}^2 \), \( \langle v_0|F_c|v_0\rangle \in O(1/c) \). Then the dominant eigenvector of \( F_c \) is of the form \( \gamma_0^c |v_0\rangle + \gamma_1^c |v_1\rangle \), where \( \langle v_0|v_1\rangle = 0 \), \( \|\gamma_0^c\|^2 + |\gamma_1^c|^2 = 1 \), and \( |\gamma_0^c|^2 \in O(1/c) \). Furthermore, the second largest eigenvalue \( \lambda_c \) of \( F_c \) satisfies \( \lambda_c \in O(1/c) \).
Proof. Let us write $F_c$ in the form: $F_c = M_c^\dagger M_c$, for a matrix $M_c \in \mathbb{C}^{2 \times 2}$. This is possible due to the fact that $F_c$ is positive-semidefinite. We define $|u_0\rangle := M_c|v_0\rangle$ and $|u_1\rangle := M_c|v_1\rangle$. According to the assumption, $\langle u_0|u_0\rangle \in O(1/c)$. Let us write the (unit) dominant eigenvector of $F_c$ in the basis $\{|v_0\rangle, |v_1\rangle\}$ as:

$$\langle w\rangle = \gamma_0|v_0\rangle + \gamma_1|v_1\rangle.$$ 

It follows that

$$1 = \langle w|F_c|w\rangle = |\gamma_0|^2 \langle u_0|u_0\rangle + |\gamma_1|^2 \langle u_1|u_1\rangle + 2\text{Re}(\gamma_0\gamma_1^\ast \langle u_0|u_1\rangle).$$

Assume that there exists an unbounded increasing sequence of positive numbers such that for its elements $c$, we get $|\gamma_1|^2 = 1 - \Theta(1/c^\delta)$ for $1/2 \leq \delta < 1$. From $\langle u_0|u_0\rangle \in O(1/c)$ we get that $|\gamma_0\gamma_1^\ast \langle u_0|u_1\rangle| \in \Theta(1/c^\delta)$ and $|\langle u_0|u_1\rangle| \in O(1/\sqrt{c})$, yielding that

$$|\gamma_0|^2 \in \omega(1/c^{\delta-1/2}).$$

Since $|w\rangle$ is a unit vector, for some $k$ positive, we get

$$1 = |\gamma_1|^2 + |\gamma_0|^2 \geq 1 - \frac{k}{c^\delta} + |\gamma_0|^2,$$

and thus, $|\gamma_0|^2 \in O(1/c^\delta)$. From the two conditions we conclude that

$$|\gamma_0|^2 \in \omega(1/c^{2\delta-1}) \cap O(1/c^\delta),$$

yielding that $\delta = 1$, since the intersection of the two sets has to be non-empty. Therefore, the function $f$ satisfies

$$|\gamma_0|^2 \in O(1/c)$$

on the entire domain.

Now we upper bound the second largest eigenvalue of $F_c$. Since the second eigenvector $|w^\perp\rangle$ of $F_c$ is orthogonal to its dominant eigenvector, it can be written in the form:

$$|w^\perp\rangle = \tilde{\gamma}_1|v_0\rangle + \tilde{\gamma}_0|v_1\rangle,$$

where $|\tilde{\gamma}_1| = |\gamma_1|$ and $|\tilde{\gamma}_0| = |\gamma_0|$. We get that

$$\lambda_c = \langle w^\perp|F_c|w^\perp\rangle = |\tilde{\gamma}_1|^2 \langle u_0|u_0\rangle + |\tilde{\gamma}_0|^2 \langle u_1|u_1\rangle + 2\text{Re}(\tilde{\gamma}_1^\ast \tilde{\gamma}_0^\ast \langle u_0|u_1\rangle).$$

From the assumption $\langle u_0|u_0\rangle \in O(1/c)$ and (11) we conclude that

$$\lambda_c \in O(1/c).$$

Lemma A.2. Let $c, k > 0$. Consider the game from Prot. 1 and let $X$ and $Y$ denote the respective registers of Bob, corresponding to Alice’s choices of $h$ and $h'$. To any strategy based on the outcomes of a separable measurement $M = (E_0, E_1, E_2)$ on $\mathcal{H}_X \otimes \mathcal{H}_Y$ with probability of error $q_{\text{err}} \leq \frac{1}{2(c+1)k}$ and the expected payoff $p$, there exists a strategy using a separable measurement $M' = (E_0', E_1', E_2')$ in the form:

$$E_0' = G_0^0 \otimes G_1^0 + G_1^0 \otimes G_1^1, \quad E_1' = G_0^0 \otimes G_1^1 + G_1^0 \otimes G_0^0, \quad E_2' = 1 - E_0' - E_1'$$

with the expected payoff $p'$, satisfying:

$$|p - p'| \leq \frac{1}{k} + O(1/\sqrt{c}).$$
**Proof.** For simplicity of the notation, let us define $|\psi'_0\rangle := |\psi_{x_0}\rangle$ and $|\psi'_1\rangle := |\psi_{x_1}\rangle$, where $|\psi_{x_0}\rangle$ and $|\psi_{x_1}\rangle$ come from Prot. 1.

Every element of a separable measurement on $\mathcal{H}_X \otimes \mathcal{H}_Y$ can be written as a sum of tensor products of positive semi-definite operators. In particular, the elements of $\mathcal{M}$ can be written in the form:

$$E_{b(x,y)} := \sum_{x,y} F^0_{b(x,y),x} \otimes F^1_{b(x,y),y}. $$

Operators $F^0_{b(x,y),x} \otimes F^1_{b(x,y),y}$ can be viewed as the elements of a new measurement $\mathcal{N}$, refining $\mathcal{M}$. Since the states $|\psi'_0\rangle$ and $|\psi'_1\rangle$ span a 2-dimensional Hilbert space, all operators $F^0_{b(x,y),x}$ and $F^1_{b(x,y),y}$ can be restricted to correspond to $2 \times 2$ matrices in some basis of this space.

The function $b : (x, y) \rightarrow \{0, 1, ?\}$ is a post-processing function of the outcomes of $\mathcal{N}$, determining the outcome of $\mathcal{M}$ (0 corresponds to the states being equal, 1 to them being different, and $?$ denotes an inconclusive answer). Let $A$ denote the sets of all pairs $(x, y)$ of outcomes of $\mathcal{N}$. To every pair $(x, y) \in A$ we assign $(q^0_x, q^1_y) \in [0, 1/2]^2$ – the probabilities of error in guessing the factor states of $\mathcal{H}_X$ and $\mathcal{H}_Y$, conditioned on measuring $x$ and $y$, respectively. Let $W_0$ and $W_1$ denote the random variables assigned to the states of $\mathcal{H}_X$ and $\mathcal{H}_Y$, respectively. The probability space of both $W_0$ and $W_1$ is $\{0, 1\}$, since the state of either of the subsystems is $|\psi'_0\rangle$ or $|\psi'_1\rangle$. For $\zeta \in \{0, 1\}$, let $x \rightarrow \zeta, y \rightarrow \zeta$ stand for $\Pr[W_0 = 1 - \zeta|x], \Pr[W_1 = 1 - \zeta|y] \leq \frac{1}{2(c+1)}$, respectively, where the probabilities are conditioned on the outcomes of $\mathcal{N}$ in the respective subsystems. Consider measurement $\mathcal{M}^* := (E^*_0, E^*_1, E^*_?)$ with the same refined set of outputs $A$ as $\mathcal{M}$ (which now will be indexed differently) in the following form:

$$E^*_0 = E^*_{0,0} + E^*_{1,1}, \quad E^*_1 = E^*_{0,1} + E^*_{1,0}, \quad E^*_? = \mathbb{I} - E^*_0 - E^*_1,$$

where

$$E^*_{\alpha,\beta} := \sum_{x \rightarrow \alpha, y \rightarrow \beta} F^0_{\alpha,x} \otimes F^1_{\beta,y}. $$

We show that the difference of the expected payoff $p$ of $\mathcal{M}$ and the expected payoff $p^*$ of $\mathcal{M}^*$ satisfies:

$$|p - p^*| \leq \frac{1}{k}. \tag{13}$$

Since the refined sets of possible outcomes of both $\mathcal{M}^*$ and $\mathcal{M}$ are the same, the two measurements only differ in the post-processing functions denoted by $b$ and $b^*$, respectively. In other words, $\mathcal{M}^*$ differs from $\mathcal{M}$ in the arrangement of the same set of summands in the three sums defining measurement elements $(E_0, E_1, E_?)$ and $(E^*_0, E^*_1, E^*_?)$.

Consider any strategy which upon measuring $(x, y)$ yields a conclusive answer. For the corresponding expected payoff $p_{x,y}$ conditioned on measuring $(x, y)$ we then get:

$$p_{x,y} = (1 - q^0_x)(1 - q^1_y) + q^0_x q^1_y - c(q^0_x(1 - q^1_y) + (1 - q^0_x)q^1_y) = 1 - (c + 1)(q^0_x + q^1_y - 2q^0_x q^1_y). \tag{14}$$

If on the other hand, measuring $(x, y)$ implies the answer of $\mathcal{M}$ to be inconclusive, the expected payoff conditioned on measuring $(x, y)$ will be 0. Consequently, the optimal post-processing strategy (with the maximum payoff) should output $b(x, y) = ?$ for every $(x, y)$ satisfying $q^0_x + q^1_y - 2q^0_x q^1_y > \frac{1}{c+1}$. 

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otherwise it outputs a conclusive answer. In particular, the output should be inconclusive for all pairs \((x, y)\) such that \(q_x^0 > \frac{1}{c+1}\) or \(q_y^1 > \frac{1}{c+1}\), and conclusive if both \(q_x^0, q_y^1 \leq \frac{1}{2(c+1)}\).

However, only the knowledge that \((q_x^0, q_y^1) \in [0, \frac{1}{c+1}]^2 \\setminus [0, \frac{1}{2(c+1)}]^2\) does not allow us to determine what is the best output in order to maximize the payoff. We analyze this problem with respect to the probability of error allowed for the post-processing function.

We assume that the answer of \(\mathcal{M}\) with the post-processing function \(b\) can be false with probability at most \(q_{err} \leq \frac{1}{2k(c+1)}\). According to Markov’s inequality, measuring \((x, y)\) such that either \(q_x^0 > kq_{err}\) or \(q_y^1 > kq_{err}\) does not allow to output a conclusive answer with probability larger than \(1/k\). Thus, for either \(q_x^0 > \frac{1}{2(c+1)}\) or \(q_y^1 > \frac{1}{2(c+1)}\), the answer cannot be conclusive with probability larger than \(1/k\).

In the latter we analyze the difference of the expected payoffs for the post-processing function \(b\) and for a newly defined \(b^*\) such that for any \((x, y)\) satisfying \(q_x^0 > \frac{1}{2(c+1)}\) or \(q_y^1 > \frac{1}{2(c+1)}\), the output is \(b^*(x, y) = ?\).

Consider every pair \((x, y)\) such that by modifying \(b(x, y)\) into \(b^*(x, y)\), \(p_{x,y}\) decreases and compute the difference of \(p_{x,y}\) and \(p^*_{x,y}\) in this case. We have that either \(q_x^0 \in (\frac{1}{2(c+1)}, \frac{1}{c+1}]\) or \(q_y^1 \in (\frac{1}{2(c+1)}, \frac{1}{c+1}]\), yielding that

\[
p_{x,y} = 1 - (c+1)(q_x^0 + q_y^1 - 2q_x^0 q_y^1) < \frac{1}{2}.
\]

It means that for every pair \((x, y)\) for which the value of the post-processing function was modified, \(p_{x,y}\) decreased by at most \(1/2\). However, since the answer of \(\mathcal{M}\) is false with probability at most \(q_{err}\), the functions \(b\) and \(b^*\) cannot differ anywhere except for a set of \((x, y)\) measured with probability at most \(1/k\), concerning that \(q_{err} \leq \frac{1}{2k(c+1)}\). This gives us

\[
|p - p^*| \leq \frac{1}{k}.
\] (15)

We have shown that a separable measurement \(\mathcal{M}\) can be approximated by a separable measurement \(\mathcal{M}^*\) in the special form. In the following we show that \(\mathcal{M}^*\) can be approximated by a measurement in the form from the statement up to a difference in payoffs which is in \(O(1/\sqrt{c})\). The statement of the lemma then follows from the triangle inequality.

Our next goal is to construct a measurement \(\mathcal{M}' = (E_0^0, E_1^1, E_β)\) in the form:

\[
E_0^0 = C^0_{00} \otimes G^1_{00} + G^0_{11} \otimes G^1_{11}, \quad E_1^1 = C^0_{01} \otimes G^1_{01} + G^0_{10} \otimes G^1_{10}, \quad E_β = I - E_0^0 - E_1^1,
\]

approximating the measurement \(\mathcal{M}^*\) with respect to the expected payoff. In the definition of the elements of \(\mathcal{M}'\), the upper index of \(G^k\) specifies the subsystem, the first bit of the lower index determines the outcome in the first subsystem, and the second bit of the lower index determines the outcome in the second subsystem.

Consider the previously constructed measurement \(\mathcal{M}^*\). Fix \(\alpha, \beta \in \{0, 1\}\) and define \(F_x^0 := \frac{F_{x,y}^1}{\|F_{x,y}^1\|_∞}\), \(F_y^1 := \|F_{α,x}^0\|_∞ \cdot \|F_{β,y}^1\|_∞\). First, we construct positive-semidefinite operators \(\tilde{G}^0_{α,β} \otimes \tilde{G}^1_{α,β}\) approximating

\[
E_{α,β}^n = \sum_{x,α,α,β} μ_{x,y} F_x^0 \otimes F_y^1
\]

(defined by (12)), where the guesses of \(α\) and \(β\) conditioned on measuring \(F_x^0\) and \(F_y^1\) are incorrect with probability at most \(\frac{1}{2(c+1)}\). We require these operators to satisfy:
1. \( p_{E^*_{\alpha,\beta}} = p_{\tilde{G}^0_{\alpha,\beta} \otimes \tilde{G}^1_{\alpha,\beta}} \), where \( p_{E^*_{\alpha,\beta}} \) and \( p_{\tilde{G}^0_{\alpha,\beta} \otimes \tilde{G}^1_{\alpha,\beta}} \) denote the expected payoffs conditioned on measuring \( E^*_{\alpha,\beta} \) and \( \tilde{G}^0_{\alpha,\beta} \otimes \tilde{G}^1_{\alpha,\beta} \), respectively.

2. For all \( \zeta_0, \zeta_1, \alpha, \beta \in \{0, 1\} : \)
\[
\left| \langle \psi_{\zeta_0}, \psi_{\zeta_1} | \tilde{G}^0_{\alpha,\beta} \otimes \tilde{G}^1_{\alpha,\beta} | \psi_{\zeta_0}, \psi_{\zeta_1} \rangle - \langle \psi_{\zeta_0}, \psi_{\zeta_1} | E^*_{\alpha,\beta} | \psi_{\zeta_0}, \psi_{\zeta_1} \rangle \right| \in O(1/\sqrt{c}),
\]

3. \( \| \sum_{\alpha,\beta} \tilde{G}^0_{\alpha,\beta} \otimes \tilde{G}^1_{\alpha,\beta} \|_\infty \in 1 + O(1/c) \).

We now describe the construction of operators \( \tilde{G}^0_{\alpha,\beta} \) and \( \tilde{G}^1_{\alpha,\beta} \). The respective dominant eigenvectors of \( F^0_x \) and \( F^1_y \) can be written as
\[
|w_0\rangle = \gamma^0_{0,x} |\psi_{1-a}\rangle + \gamma^0_{1,x} |\psi_{1-a}^\perp\rangle,
|w_1\rangle = \gamma^1_{0,y} |\psi_{1-b}\rangle + \gamma^1_{1,y} |\psi_{1-b}^\perp\rangle,
\]
where for each \( \zeta \in \{0, 1\} \), \( |\psi_{\zeta}^\perp\rangle \) denotes the unit vector spanned by \( |\psi_0\rangle \) and \( |\psi_1\rangle \), orthogonal to \( |\psi_{\zeta}\rangle \).

According to Lemma A.1, there exists \( \kappa \) positive such that for each \( x \) and \( y \), \( |\gamma^0_{0,x}|^2 \leq \frac{\kappa}{c} \) and \( |\gamma^1_{0,y}|^2 \leq \frac{\kappa}{c} \).

We define operators \( \tilde{G}^0_{\alpha,\beta} \) and \( \tilde{G}^1_{\alpha,\beta} \) by
\[
\tilde{G}^0_{\alpha,\beta} := \left(1 - \frac{\kappa}{c}\right) \cdot \sqrt{\sum_{x,y} \mu_{x,y} |\psi_{1-a}^\perp\rangle \langle \psi_{1-a} | + \nu_0(c) |\psi_{1-a}\rangle \langle \psi_{1-a}|},
\tilde{G}^1_{\alpha,\beta} := \left(1 - \frac{\kappa}{c}\right) \cdot \sqrt{\sum_{x,y} \mu_{x,y} |\psi_{1-b}^\perp\rangle \langle \psi_{1-b} | + \nu_1(c) |\psi_{1-b}\rangle \langle \psi_{1-b}|},
\]
for non-negative functions \( \nu_0, \nu_1 \in O(1/c) \) chosen to be such that
\[
p_{E^*_{\alpha,\beta}} = p_{\tilde{G}^0_{\alpha,\beta} \otimes \tilde{G}^1_{\alpha,\beta}}.
\]

Such a choice of parameters is possible, due to the fact the the probability of a wrong guess, conditioned on the outcome \( E^*_{\alpha,\beta} \) is in \( O(1/c) \). Since operators \( \{E^*_{\alpha,\beta}\}_{\alpha,\beta} \) form a valid POVM, after projecting them by a projector \( P := |\psi_{1-a}^\perp\rangle \langle \psi_{1-a} | \otimes |\psi_{1-b}^\perp\rangle \langle \psi_{1-b}| \), we get a valid POVM on the support of \( P \). In other words, \( \{P E^*_{\alpha,\beta} P\}_{\alpha,\beta} \) form a POVM and therefore, also the operators
\[
J_{\alpha,\beta} := \left(1 - \frac{\kappa}{c}\right) \cdot \left(\sum_{x \rightarrow a, y \rightarrow b} \mu_{x,y} \right) |\psi_{1-a}^\perp\rangle \langle \psi_{1-a} | \otimes |\psi_{1-b}^\perp\rangle \langle \psi_{1-b}|,
\]
lower-bounding \( P E^*_{\alpha,\beta} P \), form valid POVMs. From the condition
\[
\| \sum_{\alpha,\beta} J_{\alpha,\beta} \|_\infty \leq 1,
\]
we conclude that
\[
\| \sum_{\alpha,\beta} \tilde{G}^0_{\alpha,\beta} \otimes \tilde{G}^1_{\alpha,\beta} \|_\infty \in 1 + O(1/c).
\]
It remains to show that

\[ \forall \zeta_0, \zeta_1, \alpha, \beta \in \{0, 1\} : \left| \langle \psi_{\zeta_0}, \psi_{\zeta_1} | \tilde{G}_{\alpha,\beta}^0 \otimes \tilde{G}_{\alpha,\beta}^1 | \psi_{\zeta_0}, \psi_{\zeta_1} \rangle - \langle \psi_{\zeta_0}, \psi_{\zeta_1} | E_{\alpha,\beta}^* | \psi_{\zeta_0}, \psi_{\zeta_1} \rangle \right| \in O(1/\sqrt{c}). \]

By definition of \( \tilde{G}_{\alpha,\beta}^0 \) and \( \tilde{G}_{\alpha,\beta}^1 \), this is true if \( \zeta_0 \neq \alpha \) or \( \zeta_1 \neq \beta \). We now discuss the remaining case. It follows from Lemma A.1, applied to each \( F_x^0 \otimes F_y^1 \) and the construction of \( \tilde{G}_{\alpha,\beta}^0 \otimes \tilde{G}_{\alpha,\beta}^1 \) that

\[ \left\| \sum_{x, y} \mu_{x,y} F_x^0 \otimes F_y^1 - \tilde{G}_{\alpha,\beta}^0 \otimes \tilde{G}_{\alpha,\beta}^1 \right\|_\infty \in O(1/\sqrt{c}). \]

Hence, also

\[ \left| \langle \psi_\alpha, \psi_\beta | \tilde{G}_{\alpha,\beta}^0 \otimes \tilde{G}_{\alpha,\beta}^1 | \psi_\alpha, \psi_\beta \rangle - \langle \psi_\alpha, \psi_\beta | E_{\alpha,\beta}^* | \psi_\alpha, \psi_\beta \rangle \right| \in O(1/\sqrt{c}). \]

We have defined a set of operators \( \{ \tilde{G}_{\alpha,\beta}^0 \otimes \tilde{G}_{\alpha,\beta}^1 \}_{\alpha,\beta} \), almost forming a POVM due to the condition (iii). Therefore, we can re-scale the elements of the set by a factor in \( 1 - O(1/c) \), and thereby create a POVM \( \{ \tilde{G}_{\alpha,\beta}^0 \otimes \tilde{G}_{\alpha,\beta}^1 \}_{\alpha,\beta} \). Due to the condition (i), the expected payoffs conditioned on measuring either \( E_{\alpha,\beta}^* \) or \( \tilde{G}_{\alpha,\beta}^0 \otimes \tilde{G}_{\alpha,\beta}^1 \) are the same. Finally, due to the condition (ii), the probabilities of measuring an outcome from \( \{ E_{\alpha,\beta}^* \}_{\alpha,\beta} \) and an outcome from \( \{ \tilde{G}_{\alpha,\beta}^0 \otimes \tilde{G}_{\alpha,\beta}^1 \}_{\alpha,\beta} \) differ by a value in \( O(1/\sqrt{c}) \). Hence, if the probability of a conclusive answer of \( M^* \) is constant then the measurement with elements

\[ E_0'' := \tilde{G}_{0,0}^0 \otimes \tilde{G}_{0,0}^1 + \tilde{G}_{1,1}^0 \otimes \tilde{G}_{1,1}^1, \quad E_1'' := \tilde{G}_{0,1}^0 \otimes \tilde{G}_{0,1}^1 + \tilde{G}_{1,0}^0 \otimes \tilde{G}_{1,0}^1, \quad E_?'' := \mathbb{I} - E_0'' - E_1'' \]

gives a conclusive answer with probability lower by at most a value in \( O(1/\sqrt{c}) \), and differs from \( M^* \) in its payoff by a value in \( O(1/\sqrt{c}) \). According to \( [KKB05] \), the state of each of the two subsystems after applying the measurement given above is independent of the outcome in the other one. Therefore, in order to achieve certain expected payoff, the local measurements can be optimized separately. It follows that the payoff of measurement \( (E_0'', E_1'', E_?'') \) can be matched by the payoff \( p' \) of some measurement \( M' \) in the form:

\[ E_0' = \tilde{G}_{0,0}^0 \otimes \tilde{G}_{0,0}^1 + \tilde{G}_{1,1}^0 \otimes \tilde{G}_{1,1}^1, \quad E_1' = \tilde{G}_{0,1}^0 \otimes \tilde{G}_{0,1}^1 + \tilde{G}_{1,0}^0 \otimes \tilde{G}_{1,0}^1, \quad E_? = \mathbb{I} - E_0' - E_1'. \]

By applying (13) and the triangle inequality, we finally get that

\[ |p - p'| \in O(1/k) + O(1/\sqrt{c}). \]