An Invariant Set Approach for Optimization on Integrable Manifolds

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Abstract—Recent results in control systems and numerical integration literature ([1]) utilize invariant set theory to lift dynamical systems evolving on nonlinear manifolds to those evolving on vector spaces. We leverage this technique to propose an algorithm to solve a class of constrained optimization problems as unconstrained problems.

I. PROBLEM FORMULATION
Consider the following optimization problem

\[ \begin{align*}
\text{minimize } & f(x) \\
\text{subject to } & h(x) = 0
\end{align*} \] (1)

where the function \( h : \mathbb{R}^n \to \mathbb{R}^k \) is a submersion ([2]) on an open set \( S \subset \mathbb{R}^n \) and the cost function \( f \) is \( C^1 \). We would like to design an iterative algorithm

\[ x_{k+1} = F(x_k) \]

whose iterates converge to the optimizer of (1), thus solving the constrained optimization problem. Towards this end, we interpret iterative algorithms as discrete-time dynamical systems and use tools from invariant set theory. Exploiting the inherent geometry in the considered class of problems, we provide an algorithm that possesses a local minimum of (1) as an attractive fixed point.

II. EXTENSION OF DYNAMICAL SYSTEMS ON EMBEDDED MANIFOLDS
Consider the following continuous-time dynamical system with states \( x \) evolving on a compact manifold \( Q \subset \mathbb{R}^n \), driven by a vector field \( X \)

\[ \dot{x} = X(x) \] (2)

Suppose that there exists a \( C^1 \) function \( V : \mathbb{R}^n \to \mathbb{R}^+ \) such that \( Q \) is its 0-level set, i.e.,

\[ Q = V^{-1}(0) = \{ x \in \mathbb{R}^n | V(x) = 0 \} \]

and

\[ \nabla V(x) \cdot X(x) \leq 0 \quad \forall x \in \mathbb{R}^n \]

By differentiability of \( V \), we also have that

\[ \nabla V(x) = 0 \quad \forall x \in Q \] (3)

by the first order condition of optimality. Now consider the following extended system

\[ \dot{x} = X(x) - \nabla V(x) \quad \forall x \in \mathbb{R}^n \] (4)

Observe that system (4) is defined on the entire space \( \mathbb{R}^n \) within which the state space \( Q \) is embedded. Moreover, it reduces to system (2) on the manifold \( Q \), implying that \( Q \) is an invariant set for the flow of (4).

Lemma 1: All trajectories of the extended system converge to the manifold \( Q \) if the following assumptions hold

A1 \( Q = V^{-1}(0) \) is the only set of critical points of \( V \), i.e.,

\[ \nabla V(x) = 0 \iff x \in Q \]

A2 \( V \) is non-increasing along the flow of the system, i.e.,

\[ \nabla V(x) \cdot X(x) \leq 0 \quad \forall x \in \mathbb{R}^n \]

A3 The set

\[ V^{-1}[0,V(x(0))] = \{ x \in \mathbb{R}^n | 0 \leq V(x) \leq V(x(0)) \} \]

is compact.

Proof
On examining the evolution of \( V \) along the flow of system (4), we obtain

\[ \dot{V}(x) = \nabla V(x) \cdot (X(x) - \nabla V(x)) \leq -||\nabla V(x)||^2 \leq 0 \]

By the first assumption, \( Q \) is the largest invariant for which \( \nabla V(x) = 0 \) under the flow of (4). Along with the third assumption, the hypotheses of La Salle’s invariance theorem are satisfied and thus all trajectories of the extended system asymptotically converge to the manifold \( Q \).

III. OPTIMIZATION OVER COMPACT, INTEGRABLE MANIFOLDS
In this section, we set up the geometric framework for our problem and motivate our design for an algorithm that solves (1).

Lemma 2: The distribution given by \( Ker(\nabla h(x)) \) is involutive on \( S \)

Proof
Since \( h(x) \) is a submersion on \( S \), we have that \( \nabla h : S \to \mathbb{R}^{n \times k} \) is well defined and has (full) rank \( k \) on \( S \). The set \( Ker(\nabla h(x)) \) is given by

\[ Ker(\nabla h(x)) = \{ z \in \mathbb{R}^n | \nabla h^T(x)z = 0 \iff L_2 h = 0 \} \]

and is always \( n-k \) dimensional on \( S \). \( L_2 f(x) \) is the Lie derivative of a function \( f \) along vector field \( g \).
Considering any \( z_1, z_2 \in Ker(\nabla h(x)) \), we see that their Lie bracket satisfies
\[
\nabla h^T(x)[z_1, z_2] = \mathcal{L}_{[z_1, z_2]}h(x) \\
= \mathcal{L}_{z_1}z_2 - \mathcal{L}_{z_2}z_1h(x) \\
= \mathcal{L}_{z_1}\mathcal{L}_{z_2}h(x) - \mathcal{L}_{z_2}\mathcal{L}_{z_1}h(x) \\
= 0 \quad (\because \text{by definition of } z_1 \text{ and } z_2)
\]
\[
\Rightarrow [z_1, z_2] \in Ker(\nabla h(x))
\]
The distribution \( Ker(\nabla h(x)) \) has shown to be closed under the Lie bracket and is hence, involutive.

Using the Frobenius theorem ([3]) for the distribution \( Ker(\nabla h(x)) \) further gives us that \( h(x) = 0 \) describes an integrable smooth submanifold \( Q \subset S \), i.e.,
\[
Q = \{ x \in \mathbb{R}^n \mid h(x) = 0 \}
\]
with tangent spaces given by \( T_xQ = Ker(\nabla h(x)) \).

In view of A3, we assume to work with functions \( h \) with compact sub-level sets to yield compact submanifolds \( (x_1^2 + x_2^2 = 1 \text{ yielding the unit circle } S^1 \subset \mathbb{R}^2 \text{, for instance}) \). The compactness assumption of the manifold and continuity of \( f \) ensures that the problem is well-posed in the sense that a solution to problem (1) exists by Weierstrass’ theorem ([5]). Moreover, the differentiability of \( f \) and the assumption that \( h \) is a submersion ensure that a regular first-order KKT point \( x^* \in Q \) exists ([4]) and satisfies
\[
-\nabla f(x^*) = \nabla h(x^*)A^*
\]
The above equation gives us a characterization of the local optima of the considered class of optimization problems. Intuitively, the equation says that at a point of optimality, any direction that leads to a decrease in cost must necessarily also cause the point to violate the constraint equation.

\section*{IV. DESCENT ALGORITHM}
\subsection*{A. Continuous-time Formulation}
Sections II and III now provide the necessary ingredients for designing our algorithm. Consider the differentiable function \( V : \mathbb{R}^n \rightarrow R^+ \) and its gradient as given by
\[
V(x) = \frac{1}{2}h^T(x)h(x)
\]
\[
\nabla V(x) = \nabla h(x)h(x)
\]
Clearly, the function attains the value 0 only on the manifold \( Q \) and is positive everywhere else. Since \( h \) is a submersion, the gradient \( \nabla V \) is 0 only when \( h(x) = 0 \), i.e., on the manifold \( Q \). The function \( V \) thus meets the requirements of A1. We now proceed towards designing the “on-manifold” dynamics so that the state \( x \) remains on the manifold.

\textbf{Proposition 1:} The manifold \( Q \) is invariant for the flow of the following continuous-time system
\[
\dot{x} = -\nabla \tilde{f}(x)
\]
where \( \nabla \tilde{f}(x) \) is the projection of \( \nabla f(x) \) onto the space \( Ker(\nabla h(x)) \).
\textbf{Proof}
If \( x(0) = x_0 \in Q \), we have that \( T_{x_0}Q = Ker(\nabla h(x_0)) \) and thus \( \nabla \tilde{f}(x_0) \in T_{x_0}Q \). Examining the derivative of \( h(x) \) along \( \dot{\tilde{x}} \) yields
\[
\dot{h}(x) = \nabla h^T(x)\dot{x} = \nabla h^T(x)\nabla \tilde{f}(x) = 0
\]
So the solutions to (6) are integral curves ([2]) of \( Q \) and \( x(t) \in Q \forall t \geq 0 \).

Observe that since \( \nabla \tilde{f}(x) \in Ker(\nabla h(x)) \), we have
\[
\nabla V(x) \cdot (-\nabla \tilde{f}(x)) = -h^T(x)\nabla h^T(x)\nabla \tilde{f}(x) = 0
\]
We have so shown that assumption A2 is met. The following proposition extends the system (6) such that the manifold \( Q \) becomes an invariant and attractive set for its flow.

\textbf{Proposition 2:} The manifold \( Q \) is invariant and attractive for the flow of the following continuous-time system
\[
\dot{x} = -\nabla \tilde{f}(x) - \nabla V(x)
\]
\textbf{Proof}
On examining the evolution of \( V \) along the flow of system (7), we obtain
\[
\dot{V}(x) = \nabla V(x) \cdot (-\nabla \tilde{f}(x) - \nabla V(x))
\]
\[
= -h^T(x)\nabla h^T(x)\nabla \tilde{f}(x) - ||\nabla V(x)||^2
\]
\[
= 0 - ||\nabla V(x)||^2
\]
\[
\leq 0
\]
Since the requirements of the three assumptions of lemma 1 have been met, the hypotheses of La Salle’s invariance theorem are satisfied.
\[
\Rightarrow \text{all trajectories of system (7) asymptotically converge to the manifold } Q.
\]

We can also consider the following algorithm which operates on two time-scales.

\textbf{Proposition 3:} The manifold \( Q \) is invariant and attractive for the flow of the following continuous-time system
\[
\dot{x} = -\nabla \tilde{f}(x) - \nabla h(x)(\nabla h^T(x)\nabla h(x))^{-1}z
\]
\[
\epsilon\dot{z} = -z
\]
\[
z = \frac{1}{\epsilon}h(x)
\]
where \( \epsilon > 0 \).
\textbf{Proof}
We have \( z = \frac{1}{\epsilon}h(x) \) and its time evolution is given by
\[
\dot{z} = \frac{1}{\epsilon}\nabla h^T(x)\dot{x}
\]
\[
\Rightarrow \dot{z} = -\frac{1}{\epsilon}z
\]
For $\epsilon > 0$, the origin $z = \frac{1}{\epsilon} h(x) = 0 \in \mathbb{R}^k$ is exponentially stable for the above dynamics and thus, $Q$ is invariant and attractive.

Observe further that continuity of $h(x)$ implies that at the singularity $\epsilon = 0$, we necessarily have $h(x) = 0$. The following analysis follows the spirit of Tikhonov’s theorems to show that the solutions to (8) converge to local optima.

**Theorem 1:** Let $f^*$ be a local minimum for problem (11). Solutions to the system (8) asymptotically converge to a local optimum as $\epsilon \to 0$.

**Proof**
Consider the following the positive, differentiable function

$$V = (1 - d)(f(x) - f^*) + \frac{d}{2}||z||^2$$

$$\Rightarrow \dot{V} = (1 - d)\nabla f^T(x)\dot{x} - \frac{d}{\epsilon}||z||^2$$

$$\leq -(1 - d)||\nabla \tilde{f}(x)||^2 - \frac{d}{\epsilon}||z||^2 + (1 - d)||\lambda(x)||||z||$$

where $\nabla f(x) = \nabla \tilde{f}(x) + \nabla h(x)\lambda(x)$. Since $h(x)$ yields compact level-sets, the proof of proposition 3 implies that $x$ lies in a closed and bounded set. Using the continuity of the functions $\nabla f(x)$ and $\nabla h(x)$ allows us to write the above inequality as follows for some finite $M > 0$

$$\dot{V} \leq -(1 - d)||\nabla \tilde{f}(x)||^2 - \frac{d}{\epsilon}||z||^2(1 - \frac{1}{d}||z||M)$$

As $\epsilon \to 0$, $\dot{V}$ is negative except at the local optima. Application of La Salle’s invariance theorem then gives us the desired conclusion.

**B. Discrete-time formulation**

For our descent algorithm, we use Euler discretization on the continuous-time system (7) to obtain the following discrete-time system

$$x_{k+1} = x_k - t\nabla \tilde{f}(x_k) - t\nabla V(x_k)$$

(11)

where $t$ is the step size. Observe that at a fixed point $x^*$ of algorithm (11), we necessarily have

$$\nabla \tilde{f}(x^*) = 0 \quad \nabla V(x^*) = 0$$

(12)

because of the orthogonality of $\nabla \tilde{f}(x)$ and $\nabla V(x)$, which successfully characterizes $x^*$ as a local optimum in view of (5). Convexity of $f$ then subsequently implies that in fact a local minimum. The proposed algorithm (11) bears resemblance to the Augmented Lagrangian method (4) wherein the constrained optimization problem is converted to an unconstrained problem by minimizing the linear approximations of Lagrangian $L_k = f(x) + \lambda_k h(x) + c_k||h(x)||^2$. Our method exploits the additional structure in the problem to explicitly obtain $\lambda_k$ as the projection operator via $\nabla \tilde{f}(x_k) + \lambda_k \nabla h(x_k) = \nabla \tilde{f}(x_k)$. This also alleviates the need for the additional parameter $c_k$.

We show that the iterates generated by (11) converges to a local optimum value under some mild assumptions. We proceed to do so in steps via ancillary lemmas which culminate in our main result.

**Lemma 3 (Descent Lemma (4))** If the gradient of a $C^1$ function $g$ is $L$–Lipschitz on $\mathbb{R}^n$, then the following holds

$$g(y) \leq g(x) + \nabla g^T(x)(y - x) + \frac{L}{2}||y - x||^2$$

In the following analyses, we assume that the gradient of $f$ is $L_f$–Lipschitz. While this assumption can be relaxed (slightly) to hold on a sublevel set of $f$, in this work, we assume the former for simplicity.

**Lemma 4:** Trajectories given by the discrete flow (11) converge to the manifold $Q$ if the step size $t$ is chosen via Armijo’s rule:

$$t_a = \beta^m s$$

for some $\beta, s \in (0, 1)$ and $m$ is the smallest positive integer such that the following holds for some $\sigma > 0$

$$V(x_{k+1}) - V(x_k) \leq \sigma t_a \nabla V^T(x_k)(x_{k+1} - x_k)$$

**Proof**
Expressing $V(x_{k+1})$ using the Taylor expansion around $x_k$, we have

$$V(x_{k+1}) - V(x_k) = \nabla V^T(x_k)(x_{k+1} - x_k) + o(||x_{k+1} - x_k||^2)$$

$$= - t||\nabla V(x_k)||^2 - t\nabla V^T(x_k)\nabla \tilde{f}(x_k) + o(t^2)$$

$$= - t||\nabla V(x_k)||^2 + o(t^2)$$

For small enough $t$, the linear term in $t$ dominates the $o(t^2)$ term and for $t = t_a$, we have

$$V(x_{k+1}) - V(x_k) \leq - t_a \nabla V(x_k)||^2 \leq 0$$

(13)

The sequence $\{V(x_k)\}$ is thus monotonically decreasing and is lower bounded by 0. By the Monotone Convergence theorem (5), we thus have that $\lim_{k \to \infty} V = c$ where $c \geq 0$. Taking the limits on both sides of the above inequality, we further get

$$\lim_{k \to \infty} ||\nabla V(x_k)||^2 \leq 0$$

$$\Rightarrow \lim_{k \to \infty} ||\nabla V(x_k)||^2 = 0$$

$$\Rightarrow \lim_{k \to \infty} ||\nabla h(x)h(x)||^2 = 0$$

As noted earlier, since $h$ is a submersion, we have $A1$ and so, $\lim_{k \to \infty} x_k \in Q$ and $\lim_{k \to \infty} V = c = 0$.

**Corollary 1:** Using equation (13), it can be shown that for any $\delta > 0$, the set $\{x \in \mathbb{R}^n | V(x) \leq \delta\}$ is compact and invariant under the discrete flow (11).

The main mechanism of convergence of the iterates of (11) to a local optimum can be described in two steps:

1) Get arbitrarily close to $Q$, based on the step length $t$
2) Convergence to a local optimum via dynamics approximately given by $\mathbf{[9]}$.

This is formalized in the next theorem and its proof as follows.

**Theorem 2:** If the step size is chosen such that

$$t < \bar{t} = \min(t_{a}, \frac{2}{L_{f}}),$$

then the sequence $\{x_{k}\}$ generated by the proposed descent algorithm $\mathbf{[1]}$ converges to a local optimum of the constrained optimization problem $\mathbf{[1]}$ asymptotically, i.e.,

$$\lim_{k \to \infty} x_{k} = x^{*}$$

where $x^{*}$ satisfies $\mathbf{[5]}$.

**Proof**

Using the Descent Lemma, we have for $f(x)$,

$$f(x_{k+1}) - f(x_{k}) \leq \nabla^{T}(x_{k})\nabla(x_{k+1} - x_{k}) + \frac{L_{f}}{2}||x_{k+1} - x_{k}||^{2} = -t\nabla^{T}(x_{k})\nabla V(x_{k}) + \nabla \tilde{f}(x_{k}) + \frac{L_{f}}{2}||\nabla V(x_{k})||^{2} + ||\nabla \tilde{f}(x_{k})||^{2}$$

We write $\nabla \tilde{f}(x_{k}) = \nabla \tilde{f}(x_{k}) + \lambda_{k}\nabla h(x_{k})$ for some $\lambda_{k} \in \mathbb{R}^{k}$ and $\nabla V(x_{k}) = \nabla h(x_{k})h(x_{k})$ to express the above inequality as

$$f(x_{k+1}) - f(x_{k}) \leq -t||\nabla \tilde{f}(x_{k})||^{2} - t\lambda_{k}^{T}\nabla h^{T}(x_{k})\nabla h(x_{k})h(x_{k}) + t^{2}\frac{L_{f}}{2}||\nabla V(x_{k})||^{2} + t^{2}\frac{L_{f}}{2}||\nabla \tilde{f}(x_{k})||^{2}$$

$$\leq -t(1 - t\frac{L_{f}}{2})||\nabla \tilde{f}(x_{k})||^{2} + tM_{k}||\lambda_{k}||||h(x_{k})||$$

$$+ t^{2}\frac{L_{f}}{2}||h(x_{k})||^{2}$$

where $M_{k}$ is the largest eigenvalue of the positive definite matrix $\nabla h^{T}(x_{k})\nabla h(x_{k})$.

From lemma 4 and corollary 1, we have that $x_{k}$ always lies in a compact subset of $\mathbb{R}^{n}$, which furthermore, is closed and bounded (Heine-Borel theorem [5]). Since $M_{k}$ is a continuous function of the entries of matrix $\nabla h^{T}(x_{k})\nabla h(x_{k})$ which is a continuous of $x$ itself, we can conclude using Weierstrass’ theorem that $M_{k}$ is finite because $x_{k}$ lies in a closed and bounded set. By a similar line of reasoning, $\lambda_{k}$ is finite as well because $\nabla \tilde{f}(x_{k})$ is continuous.

Now consider a sequence $\{\delta_{k}\}$ given by $\delta_{k} = ||h(x_{k})||$. The above arguments help us rewrite inequality (11) as

$$f(x_{k+1}) - f(x_{k}) \leq -t(1 - t\frac{L_{f}}{2})||\nabla \tilde{f}(x_{k})||^{2} + o(t\delta_{k})$$

Note that $t < \frac{2}{L_{f}} \Rightarrow 1 - t\frac{L_{f}}{2} > 0$. From lemma 4, we have

$$\lim_{k \to \infty} V = 0 \Rightarrow \lim_{k \to \infty} \delta_{k} = 0$$

(16)

We arrive at the desired conclusion by analyzing the sign of the right hand side of inequality (12).

**Case 1:** $\exists \bar{k}$ such that

$$t(1 - t\frac{L_{f}}{2})||\nabla \tilde{f}(x_{k})||^{2} > o(t\delta_{k}) \ \forall k \geq \bar{k}$$

Then there is an increasing sequence $k_{1} < k_{2} < k_{3}$, such that

$$0 \leq t(1 - t\frac{L_{f}}{2})||\nabla \tilde{f}(x_{k_{i}})||^{2} \leq o(t\delta_{k_{i}})$$

Taking limits on both sides and using the fact that $||\nabla \tilde{f}(x_{k})||$ is continuous, yields

$$\lim_{k \to \infty} ||\nabla \tilde{f}(x_{k})|| = 0$$

(17)

**Case 2:** $\exists \bar{k}$ such that

$$t(1 - t\frac{L_{f}}{2})||\nabla \tilde{f}(x_{k})||^{2} > o(t\delta_{k}) \ \forall k \geq \bar{k}$$

Then for $k \geq \bar{k}$, the sequence $\{f(x_{k})\}$ is decreasing as is evident from inequality (12). Moreover, the continuity of $f$ and corollary 1 further imply that the sequence is bounded. We appeal to the Monotone Convergence theorem again to conclude that the sequence $\{f(x_{k})\}$ has a limit and we take limits on both sides of inequality (12) to yield

$$\lim_{k \to \infty} ||\nabla \tilde{f}(x_{k})|| = 0$$

(18)

Equation (16) coupled with either (17) or (18) gives us the necessary condition of optimality of the limit point via ([12]). Convexity of $f$ then further establishes that this limit point is a local minimum.

**Fortification against maxima and saddle points:**

**Lemma 5:** Saddle points and maxima are unstable equilibria for $\mathbf{[7]}$.

**Proof**

Let $x^{*}$ be a saddle point or maximum of the system. This is a KKT point that satisfies $\mathbf{[5]}$ with Lagrange multiplier $\Lambda^{*}$. We analyze the dynamics of $\mathbf{[7]}$ when perturbed on the tangent space $T_{x}Q = Ker(\nabla h(x))$ to get

$$\delta \dot{x} = -\nabla^{2}f(x^{*}) + \sum_{i=1}^{k} \Lambda_{i}^{*}\nabla^{2}h_{i}(x^{*})\delta x$$

(19)

This is nothing but the negative hessian of the Lagrangian $L(x, \Lambda^{*}) = f(x) + (\Lambda^{*})^{T}h(x)$, i.e.,

$$\delta \dot{x} = -\nabla^{2}L(x^{*}, \Lambda^{*})\delta x$$

Because $x^{*}$ is not a local minimum, the second order conditions of optimality tell us that $\exists \varepsilon \in Ker(\nabla h(x^{*}))$ such that

$$z^{T}\nabla^{2}L(x^{*}, \Lambda^{*})z < 0 \Rightarrow -z^{T}\nabla^{2}L(x^{*}, \Lambda^{*})z > 0$$

Since, $-\nabla^{2}L(x^{*}, \Lambda^{*})$ is a symmetric matrix, the above inequality implies that it has a positive eigenvalue, and consequently, that $x^{*}$ is unstable (by Lyapunov’s indirect method [3]).

While in practice, we almost surely converge to a local minimum, we use the following heuristic to escape fixed points that aren’t local minima.

**Proposition 4:** Suppose that the iterates of $\mathbf{[11]}$ converge to $\hat{x}$. Define the projection operator onto $Ker(\nabla h(x^{*}))$ as

$$P = I - \nabla h(x)(\nabla h^{T}(x)\nabla h(x))^{-1}\nabla h(x).$$

If the matrix $-\nabla^{2}L(x^{*}, \Lambda^{*})$ has a positive eigenvalue with corresponding eigenvector $z$, then restart the algorithm with $x_{0} = x^{*} + \varepsilon z$ for some $\varepsilon > 0$. 

V. Examples

The algorithm is implemented for minimizing the convex function

\[ f(x) = (x_1 + 3)^2 + (x_2 + 2)^2 + (x_3 + 2)^2 \]

defined on \( \mathbb{R}^3 \), over two different constraint manifolds:

Sphere: \( h(x) = x_1^2 + x_2^2 + x_3^2 - 1 = 0 \)

Parabola: \( h(x) = x_1^2 + x_2^2 - x_3 = 0 \)

The results are illustrated in the following figures for various initial conditions. The colour on the surface indicates the cost at the point— the cooler the colour, lower is the cost at that point.

Fig. 1. Convergence to local minimum on the sphere

Fig. 2. Convergence to local minimum on the parabola

References

[1] Chang, D., E., Controller design for systems on manifolds in Euclidean space. 2017 IEEE 56th Annual Conference on Decision and Control (CDC)

[2] Holm, D., Schmah, T. and Stoica, C., Geometric Mechanics and Symmetry: From Finite to Infinite Dimensions, Oxford University Press

[3] Sastry, S., Nonlinear systems: analysis, stability, and control, Vol. 10. Springer Science & Business Media, 2013.

[4] Bertsekas, D., Nonlinear Programming, Athena Scientific

[5] Rudin, W., Principles of Mathematical Analysis, Vol. 3. New York: McGraw-hill, 1964.

[6] Lang, S., Algebra, Springer Graduate Texts in Mathematics, 2004