Structural parts as quadrics: Elasticity ellipses revisited

Tamás Baranyai

Abstract
Elasticity ellipses or central ellipses have been long used in graphic statics to capture the elastic behaviour of structural elements. The paper gives a generalisation the concept both in dimensions and in the possibility of degenerate conics/quadrics. The effect of projective transformations of these quadrics is also given, such that the entire mechanical system can be transformed preserving equilibrium and compatibility between its elements.

Keywords
Euler-Bernoulli beam, elasticity ellipse, quadric surface, projective transformations

Introduction
The idea of graphically representing the elastic behaviour of a structural element can be found in many classical works of graphic statics. Possibly the best known examples are the ellipse of inertia used for graphically constructing the core (Kern) of a cross-section, and the ellipse of elasticity used to graphically construct the force in a rod, given the centre of relative rotation of its ends. In ‘Die Graphische Statik’ of Culmann one finds these as central ellipses, along with a central ellipsoid in 3D containing these ellipses. While in these cases the area moments of inertia are used, the ellipsoidal representation of the mass moments of inertia is even older, introduced by Poinsot and further investigated by Clebsch.

This three dimensional treatment seems to be missing from later interpretations graphic statics, as the focus was on planar ruler and compass constructions. Eventually the use of these ellipses became sparse even in the planar case, as the graphic analysis of indeterminate structures evolved into the fixed-point method of Suter that later took the algebraic form of the Cross-method.

Although not as wide-spread these graphic tools are still being used today, for instance in seismic analysis, the examination of historic structures or even dental prostheses.

Furthermore, graphic statics is currently undergoing a renaissance partly due to computerization as it allows efficient creation of constructions visually representing the forces inside a structure. Apart from this it is used for structural optimization through application of reciprocal diagrams or through projective transformations.

This paper revisits elasticity ellipses in a more contemporary way. From the engineering standpoint it will follow the logic of numerical methods, building up a structure from a set of members with different supporting conditions represented by different stiffness matrices. Linear members are examined first, then (sub-)structures as their sums. It will be shown how in a global coordinate system the known stiffness matrices can readily be interpreted as conics and quadrics similar to the elasticity ellipses, representing the geometrical relations the stiffness of the members entail.

Engineering motivation
The study of conics is often done via projective geometry as this way intersections of the conics with lines and (hyper) planes at infinity can be treated naturally. This is not the

1Budapest University of Technology and Economics, Budapest, Hungary

Corresponding author:
Tamás Baranyai, Budapest University of Technology and Economics, Műegyetem Rakpart 1-3. K.II.61, Budapest 1111, Hungary.
Email: baranyai.tamas@epk.bme.hu
only advantage of projective geometry: as there is an associated projective space to any (at least two dimensional) vector-space, one can interpret any problem admitting a vector-space description geometrically. The insights gained from this interpretation depends heavily on the algebraic description chosen. The following example will serve both as a demonstration of this and an engineering introduction, since the subject is not part of typical engineering curricula. A more formal treatment will follow in the bulk of the paper.

Consider the planar structure in Figure 1! One widespread method of solving would describe the motion of the nodes (vertices) with their translations and their rotations around their positions. The projective space associated to this six dimensional vector-space is hard to relate to the geometry of the problem. Instead, one could describe the motion of each node with two translational components \( \Delta x \), \( \Delta y \) and a rotational component around the origin of the coordinate system \( \Phi_z \) (instead of around the node) and keep track of everything in this three dimensional space. This space is relatble to the geometry of the planar problem by projecting the nodal displacement vectors \( d^i = (\Delta x^i, \Delta y^i, \Phi_z^i) \) from the origin to the image plane \( \Phi_z = 1 \). If one chooses the \( x \) axis to point in the \(-\Delta y\) direction and the \( y \) axis to point in the \( \Delta x \) direction, the projections will give the centres of the rotation around which the nodes move in the planar problem (see Figure 2).

Similarly, any planar force can be described with two force and a moment component as \( f^i = (F_x^i, F_y^i, M_z^i) \). Identifying the plane of the 2D problem with the \( M_z = 1 \) plane such that the \( x \) axis points in the \(-F_y\) direction and the \( y \) axis in the \( F_x \) direction the intersection of the normal hyperplane of \( f^i \) with the \( M_z = 1 \) plane will give the line of action of the force \( f^i \) represents (see Figure 3).

**Mathematical description**

Due to the mechanical motivation, we will present concepts for real (finite dimensional) projective spaces. The reader may find details in the books of Richter-Gebert\(^\text{15}\) or Pottmann and Wallner.\(^\text{16}\)

**Projective space associated to a vector-space**

Consider \( \mathbb{R}^{n+1} \) with the equivalence relation

\[
\mathbf{u} \sim \mathbf{v} \iff \mathbf{u} = \lambda \mathbf{v}, \quad \mathbf{u}, \mathbf{v} \in \mathbb{R}^{n+1}, \lambda \in \mathbb{R} \setminus \{0\}
\]

(1)

(we will treat all vectors as column vectors if the distinction is necessary). Factorizing \( \mathbb{R}^{n+1} \setminus \{0\} \) with this relation leads to equivalence classes \( \mathbf{u}_\lambda \) that can be considered points of an \( n \) dimensional real projective space \( \text{PG}(n) \). We will use the fact that \( n - 1 \) dimensional projective
subspaces (hyperplanes) can be represented similarly through the scalar product: point $p_\perp$ lies in hyperplane $h_\perp$ if and only if $\langle p, h \rangle = 0$ holds (it can be seen that the choice of vectors from the equivalence class is irrelevant.)

In general a $k$ dimensional projective subspace in $PG(n)$ is identified with a $k+1$ dimensional subspace minus the origin of $\mathbb{R}^{n+1}$.

**Collineations and correlations**

We will be looking at two types of transformations: collineations and correlations. In both cases there is a bijection between all such transformations and all invertible matrices. As such the algebraic description on homogeneous coordinates can be given with matrix multiplication. (In the descriptions below points are 0 dimensional projective subspaces.)

Collineations map $k$ dimensional subspaces of $PG(n)$ to $k$ dimensional subspaces of $PG(n)$ such that all incidences are preserved. Given matrix $P$, the transformation corresponding to its equivalence class be described with

$$\text{points to points } p_\perp \mapsto Pp_\perp.$$  \hspace{1cm} (2)

hyperplanes to hyperplanes $h_\perp \mapsto P^T h_\perp$. \hspace{1cm} (3)

Correlations map $k$ dimensional subspaces of $PG(n)$ to $n-k-1$ dimensional subspaces of $PG(n)$ such that all incidences are preserved (point $p$ incident with hyperplane $h$ is mapped into hyperplane $p'$ incident with point $h'$). Given matrix $P$, the transformation corresponding to its equivalence class be described with

$$\text{points to hyperplanes } p_\perp \mapsto Pp_\perp.$$  \hspace{1cm} (4)

$$\text{hyperplanes to points } h_\perp \mapsto P^T h_\perp.$$  \hspace{1cm} (5)

Correlations of period 2 (where $p \sim P^T P p$) are called polarities, and it can be seen they correspond to invertible symmetric and anti-metric matrices. Points that are mapped to hyperplanes incident with them are called self conjugate and satisfy $\langle p, pP \rangle = 0$. For anti-metric matrices (null-polarities) all points are such, while for symmetric matrices if they exist they correspond to conics and quadrics, as follows.

**Conics and quadrics**

The classical mechanical subject at hand seems to require the more general approach to conics (and quadrics), which goes beyond ellipses parabolas and hyperbolas, even in the plane. With the help of bilinear forms we can have a bijection between conics (2D) and quadrics (higher dimensions) and equivalence classes of symmetric matrices. The conic $C$ as a set of points is given as

$$C := \{p_\perp \mid \langle p, C p \rangle = 0\}$$

we prescribe $C = C^T$.  \hspace{1cm} (6)

Given a collineation acting on points with matrix $P$, conics transform as

$$C \mapsto P^T C P^{-1}.$$  \hspace{1cm} (7)

and conics that can be transformed into each other are called projectively equivalent. In this setting not all conics are projectively equivalent, for instance to a positive definite $C$ the bilinear form in (6) has no real solutions and the empty set can not be projectively mapped into a circle. We will call such ‘invisible’ conics complex conics and we will see how the energy principles of classical mechanics often lead to them. Another way to conics not equivalent with the unit circle is if $C$ degenerates, the different number of zero eigenvalues of $C$ will also have a mechanical interpretation.

**Lines in 3D**

Beyond points and hyperplanes we will need the Plücker coordinates of lines in $PG(3)$. They are homogeneous coordinates, we will have equivalence classes of sextuples $(l_1...l_6)_\perp$ that represent a line if and only if they satisfy

$$l_1 l_4 + l_2 l_5 + l_3 l_6 = 0$$  \hspace{1cm} (8)

which will have a mechanical motivation below. Since such equivalence classes can be considered points in $PG(5)$, we can imagine all lines as a subset in $PG(5)$. It can be seen that Equation (8) is in fact an equation of a quadric, meaning the set of all lines forms a quadric in $PG(5)$ called Klein quadric (we will denote it with $Q$).
One can compute the effect of three dimensional correlations and collineations on Plücker coordinates, leading to a linear map $\mathbb{R}^6 \to \mathbb{R}^5$, which can be thought of as a correlation or collineation in $PG(5)$. We will rely on the following theorem linking the two:

**Theorem 1.** Projective collineations and correlations of $PG(3)$ induce projective automorphisms of the Klein quadric, and the Klein quadric does not admit any other projective automorphisms.

**Posing the mechanical problem as a vector space**

We will look at structures with the usual assumptions of the Euler-Bernoulli beam theory. We can pose the arising mechanical problems such that both static and kinematic dynames can be elements of a respective vector space, we can associate a projective space to. The idea started from Sir Robert Ball’s Screw Theory, the reader may find more in the books of Pottmann and Wallner or Davidson.

After a choice of coordinate system the effect of any force system can be given with a force vector $F \in \mathbb{R}^3$ and a moment vector $M \in \mathbb{R}^3$ with respect to the origin. We can combine them into a single vector $f = (M, F) \in \mathbb{R}^6$.

For the kinematic dyname it may be useful to start from the better known instantaneous kinematics: the velocity state of a rigid body can be described with the vector pair $(\Omega, V)$ where $\Omega \in \mathbb{R}^3$ is the angular velocity of the body as it rotated around an axis passing through the origin and $V \in \mathbb{R}^3$ is the translational velocity of the origin. We can get the small displacement approximation we are going to use by letting the velocity state act for a small time, displacing each point in the direction of its velocity. The effect of this can be captured in the kinematic dyname $d = (\Phi, \Delta) \in \mathbb{R}^6$, where $\Phi \in \mathbb{R}^3$ describes rotation about an axis passing through the origin while $\Delta \in \mathbb{R}^3$ describes the displacement of the origin.

The effect of these dynames can be represented by a single force or rotation if and only if the sextuple satisfies (8). In these cases we can think of these mechanical quantities as line representants, and the line they represent is the line of action of the force, or the axis of rotation. (In both cases it might be an ideal line at infinity, corresponding to moments in the static and translations in the kinematic case.) We may interpret (8) as the property that a force induced moment vector is orthogonal to the force vector inducing it, and given a rotating rigid body the velocity vector of its points is orthogonal to the axis of the rotation.

Displacement $d$ will be treated as a point in $PG(5)$, given by the equivalence class $d_\cdot$. This is nothing else then the Klein-embedding of lines into five dimensions, extended to kinematic dynames not reducible to a single rotation.

Static dyname $f$ will be identified with a four dimensional hyperplane of $PG(5)$ given by equivalence class $f_\cdot$. This is dual to the usual Klein embedding, dyname $f$ is reducible to a single force if the corresponding hyperplane is a tangent hyperplane of the Klein quadric.

In what follows we will give the relation of the mechanical properties and will treat them directly as points or hyperplanes with the equivalence signs neglected.

**Structural parts as conics**

In numerical analysis it is typical to decompose complex structures into pieces with known behaviours. In case of frames these known elements are usually linear, rods having a defined axis and connecting joints or vertices of the structure. To each known element or in certain cases sets of elements corresponds a stiffness matrix. Here we will show how these matrices can be considered conics and how these conics contain geometrical information relevant to the forces and displacements involved. The stiffness matrix $K$ connects displacements and forces as

$$f = Kd.$$  \hfill (9)

Since displacements are identified with points of $PG(5)$ while forces with four dimensional subspaces, if $K$ is invertible we will interpret this as a correlation and denote it with $\kappa$. In case of a degenerate matrix the map from points to hyperplanes is meaningful, but the entire incidence structure of $PG(5)$ is not preserved. The set

$$\mathcal{K} := \{d \mid \langle d, Kd \rangle = 0\}$$  \hfill (10)

will be called the corresponding stiffness conic or stiffness quadric. In contrast the name elasticity ellipse/ellipsoid/quadric will be used for shapes corresponding to a correlation different then $\kappa$. When using the elasticity ellipse the difference is corrected by adding a geometric operation (typically reflection) before or after the correlation given by the elasticity quadric.

We will examine a structural member first, then describe the effect of different support conditions and how structures can be built from members. This will be followed by visualization methods and a few results on projective transformations of the mechanical systems.

**Stiffness conics of a structural member**

Consider a rod joining vertices $i$ and $j$! If vertex $j$ is displaced relatively to vertex $i$ with $d_{ij}$, force $f_j$ will act on the $j$ end of the rod and on vertex $i$ while force $f_i = -f_j$ will act on vertex $j$ and on the $i$ end of the rod. We can describe the stiffness of the rod with a stiffness matrix as
\[ f_{ij} = K d_{ij}. \]  

We know, that \( K = K^T \) due to Betti’s theorem, implying that all it’s eigenvalues are real and there is at least one eigenvector to each. Furthermore, no negative eigenvalue is possible: if \( K v = \lambda v \) existed with \( \lambda < 0 \), we could consider end \( i \) clamped and end \( j \) free for the moment and apply force \( f_{ij} = \lambda v \) on end \( j \), resulting in displacement \( d_{ij} = v \). The own work of the force on the displacement it caused would be \( \frac{1}{2} \langle f_j, d_j \rangle = \frac{1}{2} \langle v, K v \rangle = \frac{\lambda}{2} \| v \|^2 \), which can not be negative. This implies that equation \( \langle d_j, K d_j \rangle = 0 \) is either never satisfied in a real vector-space (\( K \) is positive definite) or all solutions are inside the kernel of \( K \) (\( \ker(K) \)). Geometrically speaking \( K \) is either a complex conic not appearing in real projective space, or a degenerate conic corresponding to a projective subspace. The number of zero eigenvalues and the dimensionality of \( \ker(K) \) depends on the supporting conditions on the ends of the rod. On the displacement side \( K \) is precisely the set of displacements that can happen with no arising forces. On the side of forces any non-zero force \( f_{ij} \) must be in the image space of \( K \) (\( \text{im}(K) \)). Since for symmetric matrices \( \text{im}(K) = \ker(K)^\perp \) holds, we have

\[ f_{ij} \neq 0 \Rightarrow \langle f_j, d \rangle = 0 \ \forall d \in K, \]

that is any force arising from relative displacement \( d_{ij} \) must be incident with all points of the stiffness conic. It is not hard to see, that as the supports become less strict \( \ker(K) \) grows in dimension, less types of forces are possible leading to stricter incidence conditions given by \( K \). This is more pronounced in the case of planar problems, where degenerate conics directly appear in relation to the geometry of the structures. A few examples illustrating this are presented in the Appendix.

**Combined effect of members**

One use of tying our structural elements to coordinate-free matrices forming a vector-space is that we may use their linear combinations, representing the combined effect of these members. The idea of an elasticity ellipse of a set of elements is not new, we may find it in the works of Culmann and Richter. They give elasticity ellipses for cells of trusses (Fach) considering different geometries. One could generalize their method of testing the structure to appropriately chosen displacements, but we are in a better position thanks to the linear algebraical treatment of conics.

Consider two rigid bodies \( a \) and \( b \), connected by a set of elements numbered \( i \in \{1 \ldots n\} \), with corresponding stiffness conics \( K_i \). Given relative displacement \( d_{ab} \), the force from the displacement in each element is \( f_i = K_i d_{ab} \) (acting on body \( a \)). As the total force acting on body \( a \) is

\[ \sum_i f_i = \sum_i (K_i d_{ab}) = \left( \sum_i K_i \right) d_{ab} \]

we have deduced that the stiffness conic of the combined elements is the sum of the stiffness conics of the parts.

All the things stated for the stiffness conics of members can be stated for stiffness conics of their sums. The sum of positive (semi-) definite matrices will be positive (semi-) definite and the types of conics corresponding to sums of parts will be the same as in the case of the members. Adding more members to a structure will decrease the dimensionality of \( \ker(K) \) and thus \( K \), implying a looser incidence condition on the forces.

**Visualization**

In a lot of cases we build structures that resist all types of motion and their stiffness conics are complex – invisible in real projective space. A way around this is given by the idea of elasticity ellipses, giving a graphical way to construct points of displacements and lines of forces from each other. We will extend this idea to the case of spatial forces and displacements, resulting in a five dimensional elasticity quadric. We will again consider the case of a single member to have a concrete example, the arguments except for the 3D visualization part generalize to structures from multiple members as provided in the previous subsection.

Although the example in the following will be a ‘straight, uniform’ rod for simplicity, such restriction is not necessary and it is known how to create the stiffness matrix of a non uniform rod from infinitesimally short uniform members.

**The five dimensional elasticity quadric**

Consider a rod of length \( L \) joining vertices \( i \) and \( j \). Let us pick the coordinate system such that the origin is in the midpoint of the rod, let the rod be parallel with the \( x \) axis and let \( y \) and \( z \) be the principal directions of its cross-section. We will denote the area of the cross section with \( A \), the principal inertia moments with \( I_y \) and \( I_z \) and the polar inertia of the cross-section with \( I_x \). The Young-modulus will be denoted with \( E \), the shear modulus with \( G \). It is known how in this coordinate system the mechanical behaviour of a ‘straight, uniform’ member gives the map:

\[ f_{ij} = \text{diag}(GI_x / L, EI_y / L, EI_z / L, EA / L, 12EI_y / L, 12EI_z / L) d_{ij} \]

where \( \text{diag}( ) \) is shorthand for diagonal matrix. In engineering books there are a number of tacit or explicit assumptions involved, when ‘straight, uniform’ or ‘homogeneous’ members are used. To avoid confusion a formal
definition is presented what the paper will mean under straight uniform rods:

**Definition 1.** straight uniform rod. A line segment with cross-sections as fictitious rigid objects corresponding to each point on the line segment, connected with neighbouring cross-sections elastically. In the stress-free case the location of the centroids of the cross sections as well as their sizes and orientations have to change in a continuous way, when considered as functions over the line segment. The relative motion of cross sections at one endpoint and an internal point of the axis (the line of the line-segment)

(i) under axial force is pure translation in the axial direction with magnitude proportional to the length of line segment between the points.

(ii) under torsion is rotation around the axis with magnitude proportional to the length of the line segment between the points.

(iii) under pure bending moment is rotation around an axis the direction of which is independent of the length of the line segment between the points. The magnitude of the rotation is proportional to the length of the line segment between the points.

**Remark 1.** Shear forces and deformations perpendicular to the axis of the rod are missing as the Euler-Bernoulli beam theory neglects shear deformations. The effect of shear forces is captured in the fact they cause bending moments. The displacements orthogonal to the axis arise from the relative rotations of the cross-sections of the rod.

**Remark 2.** This definition is stricter than the rod having the same cross-section everywhere, as the centroid and shear-center of the cross-sections have to coincide.

In order to visualise the mechanical behaviour given in (14), let us introduce the collineation \( \tau \) represented by matrix \( T \) and the correlation \( \kappa \) represented by \( K \), where:

\[
T := \text{diag}(-1,-1,-1,1,1,1) \quad (15)
\]

\[
K := KT. \quad (16)
\]

We can see, that \( T = T^T \), and \( K = TKT \), meaning we have a construction similar to the elasticity ellipse of Culmann. As such, we will call the set the elasticity quadric of the member.

\[
\kappa := \{d \mid \langle d, Kd \rangle = 0\} \quad (17)
\]

**Remark 3.** This is not the only way to visualize \( K \), even in the planar case if we reflected with respect to a line we would have an elasticity hyperbola and not an ellipse. The map \( \tau \) has been selected because it seems the most consistent with earlier works of Culmann and Ritter, while having the property that the Klein quadric is invariant under it. In fact it can be interpreted as an action on three dimensional lines, reflecting them with respect to the centre of the coordinate system.

Notable sections of the elasticity quadric of the member. Recall the radii of inertia being defined as

\[
i_y := \sqrt{\frac{I_y}{A}} \quad \text{and} \quad i_z := \sqrt{\frac{I_z}{A}} \quad (18)
\]

Let us restrict ourself to the \( \Phi_x = \Phi_y = \Delta_x = 0 \) subspace (which is \( PG(2) \)) and adopt the drawing convention that the Euclidean (finite) points are represented with vectors satisfying \( \Phi_z = 1 \). (Other displacements that are a scalar multiple of this appear on the same place on the projective plane.) We can use the radii of inertia and multiply (17) with \( \frac{L}{EAI_z^2} \) giving an equivalent equation of \( K \) (restricted to this subspace) as

\[
-\Phi_x^2 + \frac{\Delta_x^2}{i_y^2} + \frac{12 \Delta_x^2}{L^2} = 0 \quad (19)
\]

which is nothing else then the elasticity ellipse of classical planar graphic statics. A similar observation can be made in the \( \Phi_x = \Delta_y = \Delta_z = 0 \) subspace. The mechanical problem appears rotated with \( \pi/2 \) in these planes, due to the way we represent cross products with scalar products.

Furthermore, in the \( \Phi_x = \Delta_x = \Delta_z = 0 \) subspace drawn with finite points corresponding to \( \pi = 1 \) the section is a dual ellipse of the usual ellipse used for graphically constructing the core of a cross-section. This can be seen by multiplying (17) with \( \frac{L}{EA} \) giving

\[
-\Phi_{ix}^2 - \Phi_{iz}^2 + 1 = 0. \quad (20)
\]

The classic graphic construction connects points of attack of forces to the neutral axes, where the stresses are zero. It is not hard to see if we consider an infinitely short rod the relative motion of the endpoints will describe the relative motion of two neighbouring rigid plates of the rod model and the neutral axis of stresses is the axis of rotation. In this setting forces are identified with points and displacements with lines, which is the dual of our setting, hence the dual ellipse\(^{15} \) in the \( \Phi_x = \Delta_y = \Delta_z = 0 \) subspace. The primal ellipse of Culmann in this case would correspond to \( K^{-1} \), and the coordinate system is again rotated with \( \pi/2 \).
Three dimensional graphic representation

We typically see forces and rotations having lines of action in $PG(3)$ and would like to interpret the mechanical behaviour in three dimensions instead of 5. Due to the nature of line geometry and statics we cannot possibly handle everything this way, we only give a few cases where this visualization is reasonable.

**Fully supported member**

Given a structural member with a non-degenerate five dimensional elasticity quadric two natural questions arise:

(i) Can we represent the behaviour of the member with a three dimensional quadric?

(ii) Is it true that kinematic dynames reducible to a rotation around an axis in $PG(3)$ are mapped to static dyname reducible to a force having a line of action in $PG(3)$?

We know from Theorem 1 that geometrically speaking these two questions are equivalent if we consider all possible lines with corresponding static and kinematic dynames. We will show that there are rods for which the answer to these two questions is unconditionally 'yes'. We will also show that for all rods the answer 'it depends' is also applicable, implying a condition on the geometry of lines involved.

Let us embed the Euclidean space into $PG(3)$ such that finite points have a representant of shape $(x, y, z, 1)$, while ideal points of shape $(x, y, z, 0)$! To each rod we can create a polarity $\kappa_3$ represented by

$$K_3 := \text{diag} \left( \frac{12}{E}, \frac{1}{i_z^2}, \frac{1}{i_y^2}, 1 \right)$$

acting on homogeneous coordinates of $PG(3)$. The corresponding three dimensional quadric $K_3$ is again complex.

We can do the same as we did in five dimensions and introduce $r_3$ to be a reflection with respect to the centre of the coordinate system, and another polarity defined as $\kappa_3 := K_3 \circ r_3$ (reflection first, but in this particular case the order is irrelevant). Note, how the effect of $r_3$ on lines is the restriction of $r$ to the Klein quadric.

It is easy to see that the self conjugate points of $K_3$ form an ellipsoid $K_3$ with equation

$$\frac{x^2}{L} + \frac{y^2}{i_z^2} + \frac{z^2}{i_y^2} + 1 = 0$$

which can be used to visualize and graphically construct the effect of $K_1$. With this, we can more formally give answers to the aforementioned questions, as:

**Theorem 2.** For static and kinematic dynames reducible to lines correlation $\kappa_3$ gives correct lines of action

(i) for all lines if and only if the mechanical properties of the rod satisfy

$$Gl_x = 12 \frac{E \lambda^2_i z^2}{L^2}$$

(ii) for all rods, if and only if the dynames satisfy

$$\langle d, l \rangle = 0 \iff \langle f, l \rangle = 0 \text{ or } d = \lambda_3 l \iff f = \lambda_3 l$$

with $l = (1, 0, 0, 0, 0)$ and some $\lambda_3 \in \mathbb{R} \setminus \{0\}$.

**Proof.** In order to show (i) we have to show that the correlation induced by $\kappa_3$ in $PG(5)$ is $\kappa$, that is the linear maps describing the two five dimensional correlations are scalar multiples of each other. Given points $(p_1, p_2, p_3, 1) = (p, 1)$ and $(q_1, q_2, q_3, 1) = (q, 1)$, Plücker coordinates $\gamma$ of lines passing through them can be calculated as

$$(1 p - lq, q \times p).$$

Polarity $\kappa_3$ maps points $(p, 1)$ and $(q, 1)$ into planes $\text{diag}(12 / L^2, i_z^2, i_y^2) p, 1)$ and $\text{diag}(12 / L^2, i_z^2, i_y^2) q, 1)$. The intersection line of these planes has Plücker coordinates $\gamma$ of lines passing through them can be calculated as

$$I = \text{diag}(12 / L^2, i_z^2, i_y^2) q \times \text{diag}(12 / L^2, i_z^2, i_y^2) p$$

$$\tilde{I} = \text{diag}(12 / L^2, i_z^2, i_y^2) (1 p - lq).$$

Using that for any invertible $3 \times 3$ matrix $A$ and vectors $a$ and $b$

$$(Aa) \times (Ab) = \text{det}(A)A^{-T}(a \times b)$$

holds the expression in (24) turns into:

$$I = \text{diag}(i_z^2, i_z^2, i_y^2, i_z^2, i_z^2, i_y^2) (q \times p)$$

$$\tilde{I} = \text{diag}(12 / L^2, i_z^2, i_y^2) (1 p - lq).$$

As such, the correlation induced by $\kappa_3$ in $PG(5)$ can be described by the linear map

$$\kappa := \gamma \leftrightarrow f$$

$$f = \text{diag} \left( \frac{12}{L^2}, \frac{i_y^2}{i_z^2}, \frac{i_y^2}{i_z^2}, \frac{i_y^2}{i_z^2}, \frac{i_y^2}{i_z^2}, \frac{i_y^2}{i_z^2} \right) d.$$

Comparing this diagonal matrix to the one in expression (14), we see that they are scalar multiples of each other if and only if

$$\frac{Gl_x}{L} = 12 \frac{E \lambda^2_i z^2}{L^2}$$

holds.

The validity of (ii) can also be seen by comparing equatons (14) and (27).
Assuming the material properties $E$ and $G$ as given, equation (28) can be interpreted as a condition on the torsion stiffness of the rod; while other properties are governed by the length of the rod and the radii of inertia, determining the axes of the three dimensional ellipsoid. The mechanical behaviour has four free parameters while the ellipsoid only three, hence the constraint. If it is not satisfied the restriction of the loading condition may still make the graphic approach usable, as in a large number of cases the built structures are torsion free. The torsion stiffness of the rods are usually much smaller than the bending stiffness and hence get neglected, while engineers are taught to avoid torsion while designing structures.

The three dimensional elasticity ellipsoid does directly contain both ellipses of classical graphic statics, and the planar constructions can be interpreted as the intersection of spatial constructions and an image plane, as illustrated in Figure 5. The three dimensional ellipsoid itself was known in the late 1800’s but the spatial use of it does not seems widespread. This may be due to the technical restrictions of that time, in our days computers will solve the required operations on homogeneous coordinates in fractions of a second.

We can also interpret Theorem 2 in five dimensions, relating the elasticity quadric $\mathcal{K}$ to the Klein-quadric $\mathcal{Q}$. According to the $i)$ part and the fact that $\mathcal{Q}$ is invariant under $\tau$ iff condition $GI EA_i i = 12 \frac{I_2}{L^2}$ is satisfied, $\mathcal{K}$ is such that its tangent hyperplanes at the intersection $\mathcal{K} \cap \mathcal{Q}$ also touch $\mathcal{Q}$, although not necessarily at the same point. The set $\mathcal{L} := \{ d | d \in \mathcal{Q} \text{and} (d, I_i) = 0 \}$ is a special linear complex, the intersection of the Klein-quadric with one of its tangent hyperplanes. According to the $ii)$ part regardless of the torsion stiffness the tangent hyperplanes of $\mathcal{K}$ at the intersection $\mathcal{K} \cap \mathcal{L}$ also touch $\mathcal{Q}$. (Point $I_i$ is not on $\mathcal{K}$.)

Three degenerate cases

There are a number of ways the five dimensional stiffness conic can degenerate, depending on the supporting conditions. We will take a look at a few of them, that correspond to more often used supporting conditions. The geometrical principle guiding our selection is the known type of projective subspaces the Klein quadric (with the usual $PG(3)$ - $PG(5)$ pairing) may contain:

(i) a single point in $PG(5)$ representing a line in $PG(3)$
(ii) a line in $PG(5)$ representing a pencil of lines in $PG(3)$
(iii) a plane in $PG(5)$ representing either all lines through a point in $PG(3)$ (Latin plane in $PG(5)$) or all the lines in a plane (Greek plane in $PG(5)$).

Figure 5. Spatial interpretation of previous planer ellipses of graphic statics. Top: choosing the $x = 0$ plane as an image plane the ellipse of the cross section appears connecting (intersection) points of attack of forces to (intersection) lines of attack of rotations – the neutral axes in case of cross-sections. Bottom: choosing the $z = 0$ plane as image plane the elasticity ellipse of planar graphic statics appears, connecting (intersection) points of rotation to (intersection) lines of action of forces.

The simplest case is the dual of case $(i)$ when the support is such that the line of action of the force is known, for instance in case of a rod with ball-joints on the two ends, or a rod with a fix support on one end and a roller on the other. In our settings forces correspond to hyperplanes of $PG(5)$ so the five dimensional stiffness quadric is a tangent hyperplane of the Klein quadric.

The second case would be the dual of case $(ii)$, which arises for instance when the rod is supported by a fix
support on one end and a pair of rollers on the other. The arising force must pass through the non-fixed end and must lie in the plane determined by the rollers. The five dimensional stiffness quadric is a three dimensional subspace touching the Klein quadric in a line.

The third case dual to case (iii) arises when the line of action of the force is known to be incident with a given point or a plane. An example for the given point is when the member is supported by a fixed support on one end and a ball-joint on the other and the force has to pass through the ball-joint. In this case the stiffness quadric is the Latin plane corresponding to the point of the ball-joint, as rotation of the other end around the joint can happen without internal forces. The case of a given plane would give a Greek plane as stiffness quadric.

**Graphic example, relation with Rankine-reciprocals**

As an example we look at the Latin plane case with a three dimensional focus. The example considered will be a frame built from three rods supported with ball-joints on one end and connected to each other on the other end in a moment-bearing way (see Figure 6).

Let us consider now a single rod! Taking the ball-joint as the origin of the coordinate system (and having the x axis be parallel with the rod) the corresponding stiffness matrix will be

\[
K = \text{diag}(0, 0, 0, EA/L, 3EI_z/L^3, 3EI_y/L^3) \quad (29)
\]

which can be decomposed similarly to the elasticity ellipsoid’s case as \( K = DKK' = KDD' \) with

\[
D = \text{diag}(0, 0, 0, 1, 1, 1) \quad (30)
\]

\[
K = \text{diag}(GI_{1z}/L, EI_z/L, EI_y/L, EA/L, 3EI_z/L^3, 3EI_y/L^3). \quad (31)
\]

Similarly to the non-degenerate case \( D \) can be interpreted as a translation on lines translating them to the prescribed point of attack, while \( K \) can be interpreted as a map induced by a three dimensional ellipsoid centred at the ball-joint with \( x, y \) and \( z \) directional axes \( \sqrt{\frac{L^2}{3}} \), \( i_z \) and \( i_y \), respectively. The rod directional axis of the ellipsoid is different, due to the different supporting conditions and \( GI \), need not be the actual torsional stiffness of the rod, as it cancels out any-ways.

As in this example each member-force must pass through the centre of the corresponding ellipsoid, one can conveniently represent the displacement of the internal node with a line at infinity (other lines behave similarly, with the additional need for translation). The line of action of the force \( f_i \) will be the dual of the line at infinity \( d \), with respect to the ellipsoid \( K' \), see Figure 7. This is also shown in a side-view of the construction presented in Figure 8.

We may note that since in this example the location of the forces is known, determining the geometry of the forces requires only determining directions, which can be treated both as ideal points or planes containing Euclidean points. In the construction above the direction of force \( f_i \) is represented with ideal point \( \psi_i \) (drawn twice since the ideal plane is represented as a sphere around the problem with antipodal points identified – see half-sphere model of the projective plane). The direction of the displacement is represented with any plane orthogonal to the direction of
the displacement, in the figure the one through the centre of the ellipsoid is drawn.

This orthogonal representation of directions is similar to the Rankine-reciprocals currently investigated in graphic statics,\textsuperscript{12,21} where forces are represented by planes orthogonal to them. In order to represent forces this way, we have to dualize the above described construction. Forces will be represented by planes $\Psi_i$, dual to points $\psi_i$ of the previous construction, while the displacement will be represented by ideal point $\delta$, dual to plane $\Delta$ of the previous construction (under the duality represented by the four dimensional identity matrix). Correspondingly, instead of the elasticity ellipsoid one has to use the dual of it, with the lengths of the axes inverted. The polyhedron from planes $\Psi_i$ of the member-forces and $\Psi$ of the loads will give a representation of the static equilibrium, where the magnitude of the forces is proportional to the area of the corresponding polyhedral face. This construction is shown in Figures 9 and 10.

**Transformations**

As the mechanical properties of structures are now are tied to the geometry we can take the projective transform of an entire mechanical system, with linear maps (2), (3) and (7). In practice we are interested in three or two dimensional collineations but they can be analysed through the five dimensional collineations they induce. While these relations

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**Figure 8.** Side view of the elasticity ellipsoid. The tangent planes at the intersection points of the line of the force and the ellipsoid intersect at the line of the displacement. The sphere drawn around the problem represents the plane at infinity, with antipodal points identified.

**Figure 9.** Point $\delta$ representing the displacement direction is dual to plane $\Psi_i$ representing the member force, with respect to the dual of the elasticity ellipsoid of the member. The axis-lengths of the dual ellipsoid are the inverse of those of the elasticity ellipsoid. The sphere drawn around the problem represents the plane at infinity, with antipodal points identified.

**Figure 10.** Side view of the dual of the elasticity ellipsoid. Point $\delta$ is the tip of the tangent cone touching the ellipsoid in a planar curve lying in the plane $\Psi_i$ representing the member force. The sphere drawn around the problem represents the plane at infinity, with antipodal points identified.
define a matrix equivalence class $P_\pi$ for collineation $\pi$, we have additional requirements that will narrow down the possibilities. Yet, we will see that there is no unique way to transform the mechanical system corresponding to a projective change in geometry. To show this let us chose a fixed representant $P \in P_\pi$ to describe the effect of $\pi$ on the mechanical system. The transformations preserves compatibility of displacements with each other if for all $d_1$ and $d_2$

$$\pi(d_1 + d_2) = \pi(d_1) + \pi(d_2)$$ (32)

holds, implying

$$\pi(d) = \lambda_d Pd \quad \text{for a fixed } \lambda_d \neq 0.$$ (33)

Similarly, in order to preserve static equilibrium, the transformations need to satisfy:

$$\pi(f_1 + f_2) = \pi(f_1) + \pi(f_2) \quad \forall f_1, f_2$$ (34)

implying

$$\pi(f) = \lambda f P^{-T} f \quad \text{for a fixed } \lambda_f \neq 0.$$ (35)

In order to be able to take sums of stiffness conics we need

$$\pi(K_1 + K_2) = \pi(K_1) + \pi(K_2) \quad \forall K_1, K_2$$ (36)

implying

$$\pi(K) = \lambda_K P^{-T} KP^{-1} \quad \text{for a fixed } \lambda_K \neq 0.$$ (37)

Finally, in order to preserve the compatibility of forces and displacements we need

$$\pi(Kd) = \pi(K)\pi(d) \quad \forall K, d \Rightarrow \lambda_K = \frac{\lambda_f}{\lambda_d}.$$ (38)

The effect of $\lambda_K$ can be considered as scaling the Young moduli $E$ of the materials involved, as it is the only linear term present in all the stiffness matrices ($G$ can be expressed from $E$ using Poisson’s ratio). In the end we can choose two of the three $\lambda$ values freely.

According to Sylvester’s law of inertia the eigenvalue signature of $\pi(K)$ is the same as that of $K$ implying that projective transformations preserve the property that no force can have negative own work. Furthermore, the dimensionality of $\ker(K)$ is also preserved, which is tied to the projective invariance of the degree of static indeterminacy.

Whether the given transformation corresponding to collineation $\pi$ makes sense or not may depend on the problem at hand. A deep categorization of problems in this respect is left for another occasion, we only present the following theorem providing a safely usable subset of transformations (the paper uses the usual categorization of transformations: Euclidean $\subset$ similarity $\subset$ affine $\subset$ projective).

**Theorem 3.** Exactly similarity transformations preserve the straight uniform property of fully supported rods.

**Proof.** We will check properties (i), (ii), and (iii) of Definition 1. In each case we will split the rod (with endpoints $i$ and $j$) in two, at point $k$ in-between. The proportionality of the magnitudes of the displacements will be checked by comparing them on the rod-segments. Let us denote the lengths of the two segments with $L_{ik}$ and $L_{kj}$, the three dimensional similarity transformation with $\sigma$ and the matrix describing effect on displacements with $S$. As similarity transformations are a subset of affine transformations (preserving ratios of parallel line segments), we have

$$\frac{L_{ik}}{L_{kj}} = \frac{L_{i\sigma(k)}(j)}{L_{k\sigma(j)}(i)}$$ (39)

where $L_{i\sigma(k)}(j)$ denotes the distance between the images of $i$ and $k$ under $\sigma$.

(i) Let us apply an axial (normal) force $f_A$ with magnitude $N$ on the endpoints of the rod, implying $f_{ik} = f_A = f_{kj}$. The own works of force $f_A$ on the two axial displacements $\delta_{ik} = \delta_{kj}$ are

$$\frac{1}{2} N \delta_{ik} = \frac{1}{2} (f_A, K_{ik}^{-1} f_A)$$ (40)

$$\frac{1}{2} N \delta_{kj} = \frac{1}{2} (f_A, K_{kj}^{-1} f_A)$$ (41)

such that

$$\frac{L_{ik}}{L_{kj}} = \frac{\delta_{ik}}{\delta_{kj}} = \frac{\langle f_A, K_{ik}^{-1} f_A \rangle}{\langle f_A, K_{kj}^{-1} f_A \rangle}$$ (42)

holds due to the homogeneity of the rod. The proportion of the transformed axial displacements can be calculated to be

$$\frac{\langle S^{-T} f_A, (S^{-T} K_{ik} S^{-1})^{-1} S^{-T} f_A \rangle}{\langle S^{-T} f_A, (S^{-T} K_{kj} S^{-1})^{-1} S^{-T} f_A \rangle}$$

$$= \frac{\langle f_A, K_{ik}^{-1} f_A \rangle}{\langle f_A, K_{kj}^{-1} f_A \rangle} \frac{L_{ik}}{L_{kj}}$$ (43)

implying the transformed displacements are proportional to the lengths of the transformed rod segments.

The fact that $Sd_{ik}$ and $Sd_{kj}$ represent pure axial displacements can be seen through the angle preserving nature of the similarity transformations. In the 3D setting the axial translational displacements are represented by an ideal line, the intersection line of the planes orthogonal to the axis of the rod. As this orthogonality is preserved, the transformed displacement will be represented by an ideal line lying in the planes orthogonal to the transform of the axis. Note how more general affine transformations would not preserve the axial direction of the displacements.

(ii) If we replace $f_A$ in $(a)$ with a torsional moment we can repeat the shown calculation leading to an equation similar to (43), implying the proportionality of axial
rotations. The axis of the transformed rotation being the rod axis follows trivially.

(iii) If we replace \( f_i \) in \( a \) with a pure bending moment \( f_M \) and repeat the calculation leading to (43) we get the proportionality of the rotations from bending. As the starting rod was straight uniform the starting displacements \( d_{ik} = K_{ij} f_M \) and \( d_{ij} = K_{ij} f_M \) have parallel lines as axes. The similarity transformation preserves this parallelism, which completes the proof.

Summary

Structural members have been identified with conics and quadrics whose points, if they exist in real projective space express an incidence condition on the deformational forces that can arise. If no such points exist the elasticity conics and quadrics can be used as detailed. The latter exists in three and five dimensions as well, the applicability of the three dimensional ellipsoid depends on the properties of the member as well as the loads, as detailed in Theorem 2.

The description of projective transformations of the entire system of forces, displacements and stiffness relations has been presented, along with a basic result showing straight uniform rods stay straight uniform under similarity transformations (Theorem 3). Notably, as Fivet\(^1\) pointed out affine transformations preserve the applicability of the gravity field while general projective collineations do not. If one wishes to preserve the straight uniform property of the rods fully supported on both ends (along with the compatibility of forces with displacements), further restriction to a subset of affine transformations is necessary.

The description provided in the paper has multiple properties that make it suitable for computerised use. The treatment of dynamics as vectors and the vector-space nature of symmetric matrices makes calculation with them fast and efficient. The construction of structures and substructures from previously given known elements is methodical and can easily be automated.

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ORCID iD

Tamás Baranyai https://orcid.org/0000-0002-0512-7161

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Appendix

Planar examples

A few planar examples are provided, where the degenerating conics can readily be seen in relation to the mechanical problem at hand. We will consider the Euclidean $xy$ plane, and embed it into $\mathbb{P}G(2)$ with $(x, y, 1)$. If we represent forces with $(-F_y, F_x, M_z)$ as homogeneous line coordinates and displacements as $(-\Delta_x, \Delta_y, \Phi_z)$ as homogeneous point coordinates, we get correct lines and points of action. The three examples covered are presented in Figure A1.

![Figure A1. Degenerate conics K appearing as incidence conditions on forces. Top: single point (the origin), where the force must pass through. Middle and bottom: lines of action the force must have.](image)

The first rod on the top has stiffness matrix

$$K = \begin{bmatrix} 3EI_z / L^3 \\ EA / L \\ 0 \end{bmatrix}$$

with the corresponding stiffness conic

$$3x^2 / L^2 + y^2 / L^2 = 0,$$

satisfied by $(0, 0)$, the point all forces from the kinematic load must go through. The second rod in the middle of Figure 11 has stiffness matrix

$$K = \begin{bmatrix} 3EI_z / L^3 \\ 0 \\ 0 \end{bmatrix},$$

which represents a conic degenerated to the line

$$3x^2 / L^2 = 0 \rightarrow x = 0$$

which is the line of action of the arising force from kinematic loads. The third rod to the bottom in Figure 11 has stiffness matrix

$$K = \begin{bmatrix} 0 \\ EA / L \\ 0 \end{bmatrix}$$

which represents line $y = 0$, again the line of action of the arising force from kinematic loads.

Dual examples

We will consider elements of a grillage in the $xy$ plane, with a three dimensional subset of forces and displacements. If we want to see the $xy$ plane as the euclidean part of a projective plane corresponding to the factorization of a relevant vector-space of a mechanical problem, we should think of forces as triplets $f = (-M_y, M_x, F_z)$ and displacements as triplets $d = (-\Phi_y, \Phi_x, \Delta_z)$. This way point $(p_x, p_y, 1) \sim F_z(p_x, p_y, 1) = (-M_y, M_x, F_z)$ represents the point of attack of the force, while triplets $(-\Phi_y, \Phi_x, \Delta_z)$ represent lines around which the rotations happen. In accordance with engineering practice, we assume the members have negligible rotational stiffness. The stiffness matrix of a member with fixed supports on both ends (Figure 2A, left) turns into:

$$K = \begin{bmatrix} EI_y / L \\ 0 \\ 12EI_y / L^2 \end{bmatrix}$$

which needs to be interpreted as a dual conic, since we see forces as points and displacements as lines. The kernel of $K$ is spanned by $(0, 1, 0)$, which represents the line $y = 0$. This is nothing else than the axis of the rod, with a mechanical interpretation similar to the ‘primal’ case. The only displacement not causing stresses is rotation around this axis (as we neglected rotational stiffness),
and possible deformational forces from relative motion of the endpoints of the member must lie on this line.

![Figure A2. Grillage members, with corresponding degenerate stiffness conics. The conics are points and lines in the x, y plane, the z direction is drawn for convenience.]

A member supported with a fixed support on one end and a ball-joint on the other in an appropriate coordinate system (Figure A2, right) has stiffness matrix

\[
K = \begin{bmatrix}
0 & 0 \\
0 & 3EI_y / L^3
\end{bmatrix}
\]  

(50)

The kernel of this is spanned by \{(1,0,0),(0,1,0)\} and this two dimensional linear subspace corresponds to a one dimensional pencil of projective lines: all lines passing through the origin. Mechanically speaking the force from the displacement has to act at the ball-joint and any rotation with axis passing through this point will not induce stresses and forces.