QUASI-STATE RIGIDITY FOR FINITE-DIMENSIONAL LIE ALGEBRAS

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ABSTRACT. We say that a Lie algebra \( g \) quasi-state rigid if every Ad-invariant Lie quasi-state on it is the directional derivative of a homogeneous quasimorphism. Extending work of Entov and Polterovich, we show that every reductive Lie algebra, as well as the algebras \( \mathbb{C}^n \times u(n) \), \( n \geq 2 \), are rigid. For solvable Lie algebras which split over a codimension one abelian ideal, we characterize rigidity in terms of spectral data. In particular we show that the Lie algebra of the \((ax+b)\)-group is rigid, but the Lie algebras of the SOL group and the three-dimensional Heisenberg group are not.

1. INTRODUCTION

1.1. Background and history. Let \( g \) be a real Lie algebra. A function \( \zeta: g \rightarrow \mathbb{R} \) is called a Lie quasi-state if

\[
\zeta(aX + bY) = a\zeta(X) + b\zeta(Y),
\]

for all \( a, b \in \mathbb{R} \) and every pair \((X, Y)\) of commuting elements in \( g \). In other words, \( \zeta \) is linear on abelian subalgebras of \( g \).

We refer the reader to [11, 14] for an overview of the history of the notion of Lie quasi-states. Roughly speaking, Lie quasi-states (or closely related notions) arose more or less independently in three different contexts: In connection with the foundations of quantum mechanics (see e.g. [18, 10, 6]), in symplectic topology (see the recent survey [11] for references) and in the study of quasimorphisms on finite (see e.g. [19, 7, 21, 8, 2, 3, 9]) and infinite-dimensional (see e.g. [12, 4, 1, 22]) Lie groups.

One of the basic theorems in the mathematical foundations of quantum mechanics is Gleason’s theorem on rigidity of frame functions [18, 10]. Although Lie quasi-states do not feature explicitly in Gleason’s work, his result is essentially equivalent to the statement that every locally bounded Lie quasi-state on \( u(n) \) is linear, provided \( n \geq 3 \). (See the introductions of [16] and [14] for a discussion of this equivalence.) This can be seen as the first major non-trivial result concerning Lie quasi-states.

In symplectic topology, Lie quasi-states constructed from Floer homology and spectral invariants (as in [12, 4, 22]) have recently become an important tool in studying the displacability of subsets of symplectic manifolds under Hamiltonian diffeomorphisms (see [13]). Entov’s recent ICM address [11] gives an overview of these developments and an extensive list of references. In most of these symplectic applications, the Lie quasi-states considered arise as directional derivatives of continuous quasimorphisms on the corresponding infinite-dimensional Lie groups.

Both in the quantum-mechanical and the symplectic setting the focus is naturally on Lie quasi-states on infinite-dimensional Lie algebras. A systematic analysis of Lie quasi-states on finite-dimensional Lie algebras was initiated only recently by Entov and Polterovich in [14]. Even if one is primarily interested in the infinite-dimensional case, such an analysis is relevant in order to understand the behaviour of Lie quasi-states along finite-dimensional subalgebras. However, to

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the best of our knowledge [14] remains the only paper so far which concerns Lie quasi-states on finite-dimensional Lie algebras.

The purpose of the present paper is to extend some of the results from [14] to larger classes of finite-dimensional Lie algebras and to obtain a clearer picture about Lie quasi-states on general finite-dimensional Lie algebras through some key examples. Our main focus will be on Ad-invariant Lie quasi-states, since these are comparatively easy to handle and at the same time the most relevant ones in applications. One particular goal of this article is to understand their connection with homogeneous quasimorphisms on finite-dimensional Lie groups.

1.2. Integrable Lie quasi-states and homogeneous quasimorphisms. From now on we assume that all Lie algebras are real and finite-dimensional. We denote by $Q(g)$ the space of Lie quasi-states on $g$ and by $g^* \subset \mathcal{D}(g) \subset Q(g)$ the subspaces of linear and continuous Lie quasi-states on $g$ respectively. Note that the adjoint group associated to the Lie algebra $g$ acts on Lie quasi-states by $g.\zeta(X) = \zeta(\text{Ad}(g)^{-1}(X))$ preserving these subspaces.

The notion of a Lie quasi-state on a Lie algebra $g$ has a global counterpart on the level of Lie groups. Indeed, given a Lie group $G$ (not necessarily connected) we can consider integrated Lie quasi-states, i.e. Borel measurable functions $f : G \to \mathbb{R}$ such that

$$f(gh) = f(g) + f(h),$$

for all pairs $(g, h)$ of commuting elements in $G$. Any integrated Lie quasi-state $f$ on a Lie group $G$ induces a Lie quasi-state $\zeta$ on the Lie algebra $g$ on $G$, given by the directional derivative

$$\zeta(X) = f(\exp_G(X)), \quad \text{for all } X \in g,$$

where $\exp_G$ denotes the exponential map from $g$ to $G$. We note that $\zeta$ is continuous provided $f$ is. If $\zeta$ is the directional derivative of an integrated Lie quasi-state $f$, then we say that $\zeta$ is integrable and we say that $\zeta$ integrates to $f$. Typical examples of integrated Lie quasi-states are given by homogeneous (continuous) quasimorphisms, i.e. continuous functions $f : G \to \mathbb{R}$ which satisfy

$$D(f) : = \sup_{g, h \in G} |f(gh) - f(g) - f(h)| < \infty,$$

and $f(g^n) = n \cdot f(g)$ for all $n \in \mathbb{Z}$. We note that if $g$ and $h$ in $G$ commute, then

$$n \cdot f(gh) = f((gh)^n) = f(g^n h^n) - f(g^n) - f(h^n) + n \cdot (f(g) + f(h))$$

for all $n$, which readily implies that $f(gh) = f(g) + f(h)$ upon dividing with $n$ and letting $n$ tend to infinity. In particular, every homogeneous (continuous) quasimorphism is a (conjugation-invariant) integrated Lie quasi-state on $G$, and thus its directional derivative gives rise to an Ad-invariant (continuous) Lie quasi-states on $g$.

It turns out that homogeneous quasimorphisms on connected Lie groups are rare. On a solvable Lie group every homogeneous quasimorphism is in fact a homomorphism, and a simple Lie group admits a (non-trivial) homogeneous quasimorphism if and only if it has infinite center, in which case there is a unique (non-trivial) homogeneous quasimorphism up to multiples (see e.g. [3]).

1.3. Lie quasi-state rigidity. In the sequel we shall denote by $Q_{\text{Ad}}(g)$ and $Q_{\text{int}}(g)$ the classes of Ad-invariant and integrable Lie quasi-states respectively. We also denote by $Q_{\text{qm}}(g)$ the space of Lie quasi-states which are directional derivatives of homogeneous quasimorphisms on some Lie group $G$ with Lie algebra $g$. We then use the notations $\mathcal{D}_{\text{Ad}}(g)$, $\mathcal{D}_{\text{int}}(g)$, $\mathcal{D}_{\text{qm}}(g)$ for the corresponding subspaces of continuous quasi-states. Note that

$$g^* \subset Q_{\text{qm}}(g) = \mathcal{D}_{\text{qm}}(g) \subset \mathcal{D}_{\text{Ad}}(g) \cap \mathcal{D}_{\text{int}}(g) \subset \mathcal{D}_{\text{Ad}}(g) \subset \mathcal{D}(g).$$
The first inclusion is an equality for all Lie algebras not containing a simple Hermitian Lie algebra, and it is of codimension one for simple Hermitian Lie algebras. In particular, $Q_{qm}(g)$ is always finite-dimensional, whereas all of the larger spaces listed above can be infinite-dimensional. We say that a Lie algebra $g$ is quasi-state rigid, or just rigid for short, if

$$Q_{qm}(g) = \mathcal{Q}_{Ad}(g).$$

This implies in particular, that every Ad-invariant Lie quasi-state is integrable and that $\mathcal{Q}_{Ad}(g)$ is finite-dimensional. For Lie algebras not containing a simple Hermitian Lie algebra it is moreover equivalent to showing that every Ad-invariant Lie quasi-state is linear. In the sequel we will refer to the problem of classifying all rigid Lie algebras as the (quasi-state) rigidity problem, and will be the main concern of the present paper.

1.4. Statements of main results. We recall that every finite-dimensional Lie algebra $g$ is the semidirect product of a semisimple Lie subalgebra and a maximal solvable ideal, called the solvable radical of $g$. It is therefore a common strategy in the study of Lie algebras to separately analyze a problem for semisimple (or, slightly more generally, reductive) and solvable Lie algebras and then to attack the general 'mixed' case by means of semi-direct products. This is also the approach which we will take towards the quasi-state rigidity problem here.

The reductive case is by far the simplest one, since the fine structure of reductive Lie algebras is very well understood. It was already established in [14, Theorem 4.2] that for every Hermitian simple Lie algebra $g$ we have

$$Q_{Ad}(g) = \mathcal{Q}_{Ad}(g) = Q_{qm}(g).$$

By [14, Theorem 4.1] the same holds for any compact Lie algebra $g$ (which is automatically reductive). In this paper, we strengthen this result as follows:

**Theorem 1.1.** For every reductive Lie algebra $g$ we have

$$Q_{Ad}(g) = \mathcal{Q}_{Ad}(g) = Q_{qm}(g).$$

In particular, every reductive Lie algebra is rigid and every Ad-invariant Lie quasi-state on a non-Hermitian simple Lie algebra (such as $sl_n(\mathbb{R})$, $n \geq 3$) is trivial.

This settles the quasi-state rigidity problem in the reductive setting completely. However, we stress that the classification of non Ad-invariant quasi-states on reductive Lie algebras remains open. There are some important partial results towards such a classification. For instance, as remarked in the introduction of [14], the classical theorem of Gleason [18] can be re-interpreted as the statement that for $n \geq 3$ the Lie algebra $g = u(n)$ satisfies

$$\mathcal{Q}(g)/g^* = Q_{qm}(g)/g^*. $$

The main theorem of [14] extends this result to the Lie algebras $g = sp(2n)$ for $n \geq 3$. We do not know how to establish such a result for general compact Lie algebras, let alone reductive Lie algebras. If such a general result exists, then it has to evoke some kind of higher rank assumption, since the property fails for $sp(2)$ and $u(2)$.

Let us mention in passing that there is also a global version of Theorem 1.1 which can be stated as follows:

**Theorem 1.2.** Let $G$ be a connected reductive Lie group. Then every conjugation-invariant integrated Lie quasi-state on $G$ is a homogeneous quasimorphism.

We now turn to the solvable case. Here the rigidity problem becomes much more subtle than in the reductive case. In the simplest special case of an abelian Lie algebra, rigidity always hold by
definition. Arguably the next simple case arises when \( g \) admits a codimension one abelian ideal \( V \) such that the extension

\[
0 \to V \to g \to \mathbb{R} \to 0
\]
splits as a semidirect product. More explicitly this means that, as a real vector space, \( g = V \oplus \mathbb{R} \)
with a Lie bracket given by

\[
[(v, s), (w, t)] = (s\varphi(w) - t\varphi(v), 0)
\]
for all \( v, w \in V \) and \( s, t \in \mathbb{R} \), and for some \( \varphi \in \text{End}(V) \) called the underlying endomorphism of \( g \). If we wish to emphasize the dependence on \( \varphi \) we write \( g_\varphi \) for the Lie algebra \( g \), and we shall always assume that \( \varphi \neq 0 \), that is to say, \( g_\varphi \) is non-abelian.

In this case we can solve the rigidity problem completely and it turns out that the outcome heavily depends on subtle spectral properties of the endomorphism \( \varphi \).

In order to state our results precisely, we denote by

\[
H_\pm := \{ z \in \mathbb{C} \mid \pm \Re(z) > 0 \}
\]
the right-/left-halfplane of the complex plane, excluding the axis \( \Re(z) = 0 \). We note that if \( W \) denotes the image of \( \varphi \), then \( \varphi \) induces an endomorphism \( \varphi_W : W \to W \) whose spectrum shall be denoted by \( \sigma(\varphi_W) \subset \mathbb{C} \). Using this notation, we have:

**Theorem 1.3.** The Lie algebra \( g_\varphi \) is rigid if and only if \( \sigma(\varphi_W) \subset H_+ \) or \( \sigma(\varphi_W) \subset H_- \). Otherwise the inclusion \( \mathcal{Z}_{qm}(g) \subsetneq \mathcal{Z}_{Ad}(g) \) is of infinite codimension. Moreover, \( \dim \mathcal{Z}(g) \) is always infinite-dimensional independently of \( \varphi \).

For example, the Lie algebra of the \((ax + b)\)-group is rigid, while the 3-dimensional Heisenberg algebra is not. Indeed the corresponding endomorphisms \( \varphi \) can be given in matrix form as

\[
\varphi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \varphi_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},
\]
and \( \sigma((\varphi_1)_W) = \{ 1 \} \subset H_+ \), whereas \( \sigma((\varphi_2)_W) = \{ 0 \} \not\subset H_\pm \). Similarly, the Lie algebras associated with the three-dimensional SOL group and the group \( C^1 \times U(1) \) of one-dimensional unitary motions are non-rigid. The example of the 3-dimensional Heisenberg Lie algebra is particularly interesting given Proposition 2.13 in [14] which asserts that the higher-dimensional Heisenberg Lie algebras are always rigid. In fact, on these algebras every Lie quasi-state (not even assumed continuous) is linear.

These examples already indicate that the rigidity question for solvable Lie algebras does not have a simple answer, but should depend on very specific properties of the Lie algebra under consideration. We would like to single out one question: Consider the moduli space of solvable Lie algebras of a fixed dimension (parameterized by the structure constants). Is it true that the subset of quasi-state rigid solvable Lie algebras is open? If one restricts to Lie algebras of the form \( g_\varphi \) as above, then this is indeed the case by Theorem 1.3. However, at least among the algebras \( g_\varphi \) quasi-state rigidity is not a generic (i.e. open and dense) property.

The example of the 3-dimensional Heisenberg Lie algebra also shows that the semidirect product between a rigid solvable Lie algebra \( s \) and a reductive Lie algebra \( g_0 \) need not be rigid again. In order to understand the general ‘mixed’ case it would be crucial to understand under which additional assumptions on \( g_0 \) (e.g. semi-simplicity, higher rank, spectral properties of the representation on \( s \) etc.) this is the case. At the moment we are far from understanding this problem in a systematic way. However, we do understand some specific, but important, examples:
Theorem 1.4. Let \( g \) be the \( n \)-dimensional unitary motion algebra, i.e. the semidirect product of the Lie algebras \( u(n) \) and \( \mathbb{C}^n \) with respect to the standard representation of \( u(n) \) on \( \mathbb{C}^n \). Then \( g \) is rigid if \( n \geq 2 \).

In fact, every locally bounded (but not necessarily continuous) Ad-invariant quasi-state on \( g \) is linear for \( n \geq 2 \). For \( n \geq 3 \) (but not for \( n = 2 \)), every locally bounded quasi-state on \( g \) is linear.

The proof of Theorem 1.4 is close in spirit to the proofs of Gleason and of Entov-Polterovich of the rigidity of the Lie algebras \( u(n) \) and \( sp(n) \) respectively in that it depends on the analysis of the values of a given Lie quasi-state at certain elements related to rank one projections. Extending Theorem 1.4 even to the Euclidean motion algebras \( \mathbb{R}^n \rtimes o(n) \) would require a deeper understanding of (non-Ad-invariant) Lie quasi-states on \( o(n) \).

1.5. Organization of the paper. The paper consists of four main sections and one appendix. In Section 2 we discuss additive Jordan decompositions of simple Lie algebras and we show how these decompositions can be used to prove Theorem 1.1. In Section 3 we describe the structure of centralizers of solvable Lie algebras with an abelian ideal of codimension one, and give an explicit description of Lie quasi-states in this setting, which in particular yields Theorem 1.3. In Section 4 we extend some arguments of Gleason and Entov-Polterovich to the setting of unitary motion Lie algebras and establish Theorem 1.4. Finally, in the appendix we collect some general remarks about maximal abelian subalgebras in Lie algebras which are used in the proof of Theorem 1.3.

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2. The reductive case

Recall that a Lie algebra \( g \) is reductive if it decomposes as a direct sum of abelian and simple Lie algebras. The study of (Ad-invariant) Lie quasi-states on such Lie algebras can be immediately reduced to the case of simple Lie algebras by means of the following observation:

Lemma 2.1. If \( g = g_1 \oplus g_2 \) is a direct sum of Lie algebras, then \( Q(g) \) decomposes as \( Q(g) = Q(g_1) \oplus Q(g_2) \), and the subspaces \( Q_{qm}(g), Q_{Ad}(g), Q(g), L_{qm}(g), L_{Ad}(g), L(g) \) decompose accordingly. If \( g \) is abelian then all these spaces coincide with \( g^* \).

2.1. Additive Jordan decompositions of simple Lie algebras. Thus in order to establish Theorem 1.1 it suffices to consider the case of a finite-dimensional simple real Lie algebra \( g \). Such Lie algebras come in two different flavors, Hermitian and non-Hermitian. More precisely, let \( \theta \) be a Cartan involution on \( g \) and let \( \mathfrak{t} < g \) be the fixed point algebra of \( \theta \). Then \( \mathfrak{t} \) is reductive, i.e. can be written as \( \mathfrak{t} = z(\mathfrak{t}) \oplus \mathfrak{t}_{ss} \) where \( \mathfrak{t}_{ss} \) is semisimple and where \( z(\mathfrak{t}) \) denotes the centre of \( \mathfrak{t} \), and moreover \( \dim z(\mathfrak{t}) \leq 1 \). Now \( g \) is Hermitian if \( \dim z(\mathfrak{t}) = 1 \) and non-Hermitian otherwise.

A key feature of simple Lie algebras is that they are always algebraic, i.e. their associated adjoint groups are not only Lie groups, but in fact algebraic groups. Now simple algebraic groups
admit a (multiplicative) Jordan decomposition, and this induces an additive Jordan decomposition of the corresponding Lie algebras. We will now describe this decomposition explicitly in our setting.

We first recall that given the pair \((\mathfrak{g}, \mathfrak{t})\), there exists an Iwasawa decomposition of the form
\[
\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n} = \mathfrak{z}(\mathfrak{t}) \oplus \mathfrak{t}_{ss} \oplus \mathfrak{a} \oplus \mathfrak{n},
\]
such that \(a < g\) is an abelian subalgebra with \(\theta|_a \equiv -1\) and maximal with respect to these two properties and \(n\) consists of ad-nilpotent elements [20, Prop. 6.43]. Now the additive refined Jordan decomposition of \(g\) can be stated as follows [5, Cor. 2.5]:

**Lemma 2.2.** Let \(g\) be a simple Lie algebra with Iwasawa decomposition \(g = \mathfrak{z}(\mathfrak{t}) \oplus \mathfrak{t}_{ss} \oplus \mathfrak{a} \oplus \mathfrak{n}\) and let \(G\) be the adjoint connected Lie group with Lie algebra \(g\). Then for every \(X \in \mathfrak{g}\) there exist unique elements \(X_c, X_k, X_a, X_n \in \mathfrak{g}\) and (in general non-unique) elements \(Y_c \in \mathfrak{z}(\mathfrak{t}), Y_k \in \mathfrak{t}_{ss}, Y_a \in \mathfrak{a}, Y_n \in \mathfrak{n}\) such that the following hold:

(i) \(X = X_c + X_k + X_a + X_n\).
(ii) \(X_c, X_k, X_a, X_n\) commute pairwise.
(iii) \(X_j \in \text{Ad}(G)(Y_j)\) for \(j \in \{c, k, a, n\}\).

Note that the elements \(Y_c, Y_k, Y_a, Y_n\) are determined up to the action of \(\text{Ad}(G)\). In fact, the element \(Y_c\) is uniquely determined, as can be seen as follows: Assume \(Y'_c, Y''_c\) were elements of \(\mathfrak{z}(\mathfrak{t})\) and \(\text{Ad}(g)(Y'_c) = Y''_c\). Since \(\dim \mathfrak{z}(\mathfrak{t}) \leq 1\) we have \(Y'_c = \lambda Y''_c\) for some \(\lambda \in \mathbb{R}\). In particular, if \(f : G \to \mathbb{R}\) is conjugation-invariant then \(f(\exp(Y'_c)) = f(\exp(Y''_c))\). If \(\lambda \neq 1\) we would deduce that every continuous conjugation-invariant homogeneous function on \(G\) would have to vanish on \(Z(K)\); however, a non-trivial such function is given by the Guichardet-Wigner quasimorphism [19]. We deduce that \(Y_c\) is uniquely determined by \(X\) and refer to \(Y_c\) as the **central elliptic part** of \(X\).

We need one other basic structural property of semisimple Lie algebras. Recall from [17, Prop. 14.31] that given an irreducible abstract root system \(\Sigma\) spanning a vector space \(V\), the action of the Weyl group \(W\) of \(\Sigma\) on \(V\) is irreducible. It follows that the space \(V^W\) of \(W\)-invariants is trivial, and this conclusion extends to reducible root systems. In particular, if \(a\) is the Cartan subalgebra of a semisimple Lie algebra and \(W\) the corresponding Weyl group, then \((a^*)^W = \{0\}\).

### 2.2. Ad-invariant Lie quasi-states on simple Lie algebras

We can now prove the following strengthening of Theorem [1.1]

**Theorem 2.3.** Let \(g\) be a simple real Lie algebra and \(\zeta \in Q_{\text{Ad}}(g)\). Then there exists a linear functional \(\alpha \in \mathfrak{z}(\mathfrak{t})^*\) such that for every \(X \in \mathfrak{g}\) with central elliptic part \(Y_c \in \mathfrak{z}(\mathfrak{t})\) we have
\[
\zeta(X) = \alpha(Y_c),
\]
In particular,
\[
\dim Q_{\text{Ad}}(g) = \begin{cases} 1, & g \text{ Hermitian,} \\ 0, & g \text{ non-Hermitian,} \end{cases}
\]
and thus \(Q_{\text{Ad}}(g) = D_{\text{Ad}}(g) = D_{\text{qm}}(g)\).

**Proof.** Let \(\zeta\) be an Ad- invariant quasi-state on \(g\). By the first two parts of Lemma [2.2] we have
\[
\zeta(X) = \zeta(X_c) + \zeta(X_k) + \zeta(X_a) + \zeta(X_n).
\]
Since \(\zeta\) is Ad- invariant the last part of that lemma yields \(\zeta(X_j) = \zeta(Y_j)\) for \(j \in \{k, a, n\}\), whence
\[
\zeta(X) = \zeta(Y_c) + \zeta(Y_k) + \zeta(Y_a) + \zeta(Y_n).
\]
Since \(\mathfrak{z}(\mathfrak{t})\) is abelian the restriction \(\alpha := \left.\zeta\right|_{\mathfrak{z}(\mathfrak{t})}\) is a linear functional. It thus remains to show only that \(\zeta\) vanishes on \(a, n\) and \(\mathfrak{t}_{ss}\).
Since \( a \) is abelian, \( \zeta|_a \in a^* \) is linear, and since \( \zeta \) is Ad-invariant, \( \zeta|_a \) is invariant under the adjoint action of the Weyl group \( W := N_K(a)/Z_K(a) \). Now \( (a^*)_W = \{0\} \) by the remark at the end of the last subsection, and hence \( \zeta|_a = 0 \).

Now let \( X \in \mathfrak{g} \) be an arbitrary nilpotent element. By the Jacobson-Morozov theorem, there exists a subalgebra \( \mathfrak{g}_0 < \mathfrak{g} \) which is isomorphic to \( \mathfrak{sl}_2(\mathbb{R}) \) and which contains \( X \) and hence also \( -X \). Now by the Jordan canonical form, every non-zero nilpotent element in \( \mathfrak{sl}_2(\mathbb{R}) \) is conjugate to the matrix

\[
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}.
\]

This shows in particular that \( X \) and \( -X \) are conjugate in \( \mathfrak{g}_0 \), whence \( \zeta(X) = \zeta(-X) = -\zeta(X) \) by Ad-invariance. This shows that \( \zeta \) vanishes on all nilpotent elements of \( \mathfrak{g} \), and in particular on \( n \).

The fact that \( \zeta \) vanishes on \( \mathfrak{t}_{ss} \) was already observed in [14]. We repeat the argument here for completeness: Since \( \mathfrak{t}_{ss} \) is compact, every Ad-orbit intersects any given maximal total subalgebra \( t < \mathfrak{t}_{ss} \), whence \( \zeta|_{\mathfrak{t}_{ss}} \) is determined by \( \zeta|_{t} \). Since \( t \) is abelian we have \( \zeta|_t = \mathfrak{t}^* \) and moreover \( \zeta|_{t} \) is invariant under the adjoint action of \( \mathfrak{w}_t := N_K(t)/Z_K(t) \). Since \( \mathfrak{t}_{ss} \) is semisimple we have \( (\mathfrak{t}^*)_W = \{0\} \) (again by the remark at the end of the last subsection) and thus \( \zeta|_{t} = 0 \) and consequently also \( \zeta|_{\mathfrak{t}_{ss}} = 0 \). The theorem follows.

\[\square\]

Remark 2.4. (i) The most substantial ingredient in the proof of Theorem 2.3 is the vanishing of \( \zeta \) along nilpotent elements, which we deduced from the Jacobson-Morozov theorem. If one is willing to assume continuity of \( \zeta \) then one can give a more elementary proof of this result which does not invoke the Jacobson-Morozov theorem. Instead one uses the fact that for every \( X \in n \) there exists a sequence \( (g_n) \) in \( G \) such that

\[
\lim_{n \to \infty} \text{Ad}(g_n)(X) = 0.
\]

Assuming continuity of \( \zeta \) this is enough to deduce that

\[
0 = \zeta(\lim_{n \to \infty} \text{Ad}(g_n)(X)) = \lim_{n \to \infty} \zeta(\text{Ad}(g_n)(X)) = \lim_{n \to \infty} \zeta(X) = \zeta(X).
\]

(ii) In the case of a simple Hermitian Lie algebra, Theorem 2.3 gives an explicit formula for all Ad-invariant Lie quasi-states on \( \mathfrak{g} \). Now let \( G \) denote the simply-connected Lie group associated with \( \mathfrak{g} \) and \( Z(K) \cong \mathbb{R} \) denote the analytic subgroup of \( G \) with Lie algebra \( \mathfrak{z}(\mathfrak{g}) \). Then for every measurable homomorphism \( \chi : Z(K) \to \mathbb{R} \) there is a unique homogeneous quasimorphism \( f \) on \( G \) subject to the normalisation \( f|_{Z(K)} = \chi \), and by Theorem 2.3 we have

\[
f(\exp(X)) = \chi(\exp(Y_c)).
\]

A global version of this formula was first pointed out in [8].

(iii) The classification of Lie quasi-states on \( \mathfrak{g} \) does not by itself provide a classification of integrated Lie quasi-states on \( G \), since there is no general argument which would ensure that two integrated Lie quasi-states with the same directional derivatives coincide. (Such an argument would work e.g. if the Lie quasi-states in question were of class \( C^1 \), but in that case they are necessarily linear anyway.) It therefore requires additional effort to prove the following global version of Theorem 2.3.

\textbf{Theorem 2.5.} Let \( G \) be a connected reductive Lie group and \( f : G \to \mathbb{R} \) a conjugation-invariant integrated Lie quasi-state. Then \( f \) is a homogeneous quasimorphism.

\textit{Proof.} We first observe that we may assume that \( G \) is simply-connected. Indeed, if \( G \) is arbitrary and \( \tilde{G} \) is its universal covering group, then every conjugation-invariant integrated Lie quasi-state on \( G \) lifts to \( \tilde{G} \), and if this can be shown to be a homogeneous quasimorphism, then it descends
to a homogeneous quasimorphism on $G$, see [3]. Now a simply-connected reductive Lie group $G$ is a product of abelian and simple factors. Since every integrated Lie quasi-state on an abelian group is a homomorphism, it suffices to prove the theorem for a general simply-connected simple Lie group $G$.

In this case, we consider the Lie algebra $g$ of $G$ and and fix an Iwasawa decomposition $g = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Let $K$ be a maximal Ad-compact subgroup of $G$ with Lie algebra $\mathfrak{k}$ and let $A$ and $N$ be the analytic subgroups associated with $\mathfrak{a}$ and $\mathfrak{n}$. Since $G$ is connected and $K$ is a deformation retract of $G$, also $K$ is connected. Since $K$ is compact-times-abelian and $A$ and $N$ are nilpotent and all three are connected the restricted exponential functions $\mathfrak{k} \to K$, $a \to A$ and $n \to N$ are all onto. It thus follows from Theorem 2.3 that there exists a homogeneous quasimorphism $\varphi : G \to \mathbb{R}$ such that $f_0 := f - \varphi$ vanishes on $K$, $A$ and $N$. It suffices to show that $f_0 \equiv 0$.

To this end we first observe that since $Z(G) \subset K$ we have $f_0|_{Z(G)} \equiv 0$, and hence $f$ factors through a quasimorphism on $G_0 := \text{Ad}(G) = G/Z(G)$ which vanishes on the images $K_0$, $A_0$ and $N_0$ of $K$, $A$ and $N$ in $G_0$. The group $G_0$ (unlike $G$) is automatically algebraic, whence admits a multiplicative Jordan decomposition: Every $g \in G$ can be written as a product of elements $g_k$, $g_a$ and $g_n$ which pairwise commute and are conjugate to elements in $K_0$, $A_0$ and $N_0$ respectively. It follows that $f_0(g) = f_0(g_k) + f_0(g_a) + f_0(g_n) = 0$, which finishes the proof. 

3. SEMIDIRECT PRODUCTS AND SOLVABLE EXAMPLES

The goal of this section is to establish Theorem 1.3 concerning Ad-invariant Lie quasi-states on certain solvable Lie algebras. Before we turn to the specific setting of that theorem we collect a few general facts about Lie algebras with an abelian ideal, which will also be useful for our study of unitary motion algebras in the next section. Thus let $g = V \times h$, where $V < g$ is an abelian ideal and $h < g$ is a complementary subalgebra. We denote by $\rho : h \to \text{gl}(V)$ the restriction of the adjoint action of $h$ to $V$; then $\rho$ is a representation of $h$ and $g$ is uniquely determined by $\rho$.

In considering Lie quasi-states on $g$ we may restrict our attention to the case where $\rho$ is a faithful representation. Indeed, if $\rho$ is not faithful, then $g$ is the Lie algebra direct sum of $V \oplus \rho(h)$ and $\ker(\rho)$, and all quasi-states decompose accordingly. To simplify notation we will thus assume without loss of generality that $h < \text{gl}(V)$ is a subalgebra and that $\rho$ is given by the inclusion map. In particular, given $X \in h$ and $v \in V$ we simply write $X.v := \rho(X)(v)$. Given $(v,X) \in g$ we denote by $\text{Cent}(v,X)$ the centralizer of $(v,X)$ in $g$.

**Lemma 3.1.** Let $g = V \times h$ as above. Then,

(i) For every $(v,X) \in g$ we have

$$\text{Cent}(v,X) = \{(w,Y) \in g \mid Y.v = X.w, [X,Y] = 0\}.$$  

In particular, $\text{Cent}(v,0) = V$ and $\text{Cent}(0,X) = \text{Cent}_h(X)$.

(ii) $Q(g)$ can be written as the internal direct sum

$$Q(g) = Q_0(g) \oplus V^* = Q_{00}(g) \oplus V^* \oplus Q(h),$$  

where

$$Q_0(g) = \{\zeta \in Q(g) \mid \forall v \in V : \zeta(v,0) = 0\},$$

and

$$Q_{00}(g) = \{\zeta \in Q(g) \mid \forall v \in V, X \in h : \zeta(v,0) = \zeta(0,X) = 0\}.$$
Moreover, we denote by $\phi$. In the introduction we then write $g \in X$, where $X$ is the restriction of the adjoint action of the generator $1$. We call elements $X, Y \in g$ equivalent if $\text{Cent}(X) = \text{Cent}(Y)$. The equivalence class of $X$ will be denoted by $[X]$ and we write $\text{Cent}([X]) := \text{Cent}(X)$. The following lemma collects some basic structural properties of $g$ and $Q(g)$.

**Lemma 3.2.** Let $X = (v, t) \in g \setminus U$. Then the following hold:

(i) If $X \in V$ (i.e., $t = 0$), then $\text{Cent}(X) = V$ is the unique maximal abelian subalgebra containing $X$.

(ii) If $X \not\in V$, then $\text{Cent}(X)$ is the unique maximal abelian subalgebra containing $X$ and contains $U$ as a codimension one subspace.

(iii) If $Y \in g \setminus U$ is not equivalent to $X$ then $\text{Cent}(X) \cap \text{Cent}(Y) = U = \mathfrak{z}(g)$.

(iv) $Q(g) = Q_0(g) \oplus V^*$ with $Q_0(g)$ given as in Lemma 3.1.

(v) We have the identification

$Q_0(g) \cong \left\{ (\zeta|_X) \in \prod_{[X] \in (g \setminus V)^\sim} \text{Cent}([X])^* \mid \forall [X] : \zeta|_X|_U \equiv 0 \right\}$

via the isomorphism

$\zeta \mapsto (\zeta|_{\text{Cent}(X)})$.

(vi) If $X \in g \setminus V$ and $\alpha \in \text{Cent}(X)^*$ vanishes on $U$, then there exists $c \in \mathbb{R}$ such that

$\alpha(w, s) = c \cdot s$.

For the proof of the lemma we shall make use of various results concerning centralizers and Lie quasi-states, which can be found in the appendix of this paper.

**Proof.** (i) If $X \in V$ then $\text{Cent}(X) = V$ by Lemma 3.1(i). In particular, $\text{Cent}(X)$ is abelian, whence the unique maximal abelian subalgebra containing $X$ by Lemma A.2.

(ii) If $X \not\in V$ the the formula for $\text{Cent}(X)$ from Lemma 3.1(i) reads

$\text{Cent}(v, t) = \{ (v', t') \in g \mid t'\varphi(v) = t\varphi(v') \}
= \{ (v', t') \in g \mid \varphi(t'v/t) = \varphi(v') \}
= \{ (v', t') \in g \mid t'v/t - v' \in U \}$. 

In particular, \( U < \text{Cent}(v, t) \). Moreover, for every \( t \in \mathbb{R} \) there exists \( v' \in V \) with \( t'v/t - v' \in U \) which is unique up to an element of \( U \). This shows that \( \text{dim Cent}(v, t)/U = 1 \).

Finally, if \((v', t'), (v'', t'') \in \text{Cent}(v, t)\), then \( t'\varphi(v') = t'\varphi(v) \) and \( t\varphi(v'') = t''\varphi(v) \) and hence
\[
\alpha t'\varphi(v'') = t'/t \cdot t\varphi(v') = t'/t \cdot t'\varphi(v) = t''/t \cdot t\varphi(v) = t''/t \cdot t\varphi(v'),
\]
which implies that \((v', t')\) and \((v'', t'')\) commute. Thus \( \text{Cent}(v, t) \) is abelian and (ii) follows from Lemma A.2.

(iii) It follows from (i) and (ii) that \( U < \text{Cent}(X) \cap \text{Cent}(Y) \) for all \( X, Y \in g \). Conversely, if \( X \) and \( Y \) are not equivalent then at most one of them is contained in \( V \), thus without loss of generality we may assume that \( X \not\subseteq V \). Since \([X] \neq [Y]\), the inclusion \( \text{Cent}(X) \cap \text{Cent}(Y) \subset \text{Cent}(X) \) is proper. We conclude that
\[
\text{dim } U \leq \text{dim } \text{Cent}(X) \cap \text{Cent}(Y) < \text{dim } \text{Cent}(X) = \text{dim } U + 1,
\]
and thus \( U = \text{Cent}(X) \cap \text{Cent}(Y) \).

(iv) is just a special case of Lemma 3.1(ii), and (v) is a direct consequence of (i), (ii), (iii) and Corollary A.3. Finally, (vi) is an immediate consequence of (ii). \( \square \)

3.2. **Classification of Lie quasi-states when \( h \) is one-dimensional.** In the sequel, given a Lie quasi-state \( \zeta \in Q(g) \) we refer to the linear functional \( \zeta_V \in V^* \) given by
\[
\zeta_V(v) := \zeta(v, 0), \quad \text{for all } v \in V,
\]
as the **canonical character** of \( \zeta \). In this language, the elements of \( Q_0(g) \) are precisely the Lie quasi-states of vanishing canonical character. The following theorem provides a full classification of Lie quasi-states on \( g \). We recall our convention that \( W < V \) denotes the image of the endomorphism \( \varphi \).

**Theorem 3.3.** A function \( \zeta : g \to \mathbb{R} \) is a Lie quasi-state of canonical character \( \alpha \in V^* \) if and only if the following hold: There exists a function \( c : W \to \mathbb{R} \) such that
\[
\zeta(v, t) = \begin{cases} 
c(\varphi(v)/t) \cdot t + \alpha(v) & t \neq 0 \\
\alpha(v) & t = 0
\end{cases}
\]
The quasi-state given by (3.1) is continuous if and only if \( c \) is continuous and sublinear in the sense that for every norm \( \| \cdot \| \) on \( W \) we have
\[
\lim_{w \to \infty} \frac{c(w)}{\|w\|} = 0.
\]

**Proof.** Given a function \( \zeta : g \to \mathbb{R} \) let \( \zeta_0(v, t) := \zeta(v, t) - \alpha(v) \). By Parts (iv) and (v) of Lemma 3.2, the function \( \zeta \) is a Lie quasi-state of canonical character \( \alpha \in V^* \) if and only if for every \( X \in g \setminus V \) the restriction \( \zeta|_X := \zeta_0|_{\text{Cent}(X)} \) is a linear functional vanishing on \( U \). By Part (vi) of that lemma the latter condition is equivalent to
\[
\zeta|_X(w, s) = c \cdot s
\]
for some constant \( c \) depending on \( \text{Cent}(X) \). Now two elements \( X = (v, t) \) and \( Y = (w, s) \) in \( g \setminus V \) are equivalent if and only if \( \varphi(v)/t = \varphi(w)/s \), and thus the constant \( c \) depends only on \( \varphi(v)/t \). This shows that Lie quasi-states on \( g \) are precisely the functions of the form (3.1). Now continuity of \( \zeta \) as given by (3.1) is equivalent to continuity of \( c \) and the additional condition that for all \( w \in W \) we have
\[
\lim_{t \to 0} c(w/t) \cdot t = 0.
\]
This latter condition amounts to
\[
0 = \lim_{t \to 0} \frac{|c(w/t) | t|}{|R|} = \lim_{R \to \infty} \frac{|c(R \cdot w)|}{|R|}
= \lim_{R \to \infty} \frac{\|w\| \cdot |c(R \cdot w)|}{\|R\| \cdot \|w\|} = \|w\| \cdot \lim_{R \to \infty} \frac{|c(R \cdot w)|}{\|R \cdot w\|},
\]
which is equivalent to (3.2). □

In the sequel we write \( \zeta_{\alpha,c} \) for the Lie quasi-state on \( \mathfrak{g} \) given by equation (3.1). If we denote by \( C_{sl}(W) < C(W) \) the space of sub-linear continuous functions on \( W \), that is to say, those continuous function on \( W \) which satisfy Condition (3.2), then we have:

**Corollary 3.4.** If \( \mathfrak{g} = \mathfrak{g}_\varphi \) for some \( \varphi \neq 0 \) and \( W = \text{Im}(\varphi) \) then the map
\[
V^* \oplus C_{sl}(W) \to \mathcal{D}(\mathfrak{g}), \quad (\alpha,c) \mapsto \zeta_{\alpha,c}.
\]
is an isomorphism and thus \( \mathcal{D}(\mathfrak{g}) \) is infinite-dimensional.

While \( \mathcal{D}(\mathfrak{g}) \) is always infinite-dimensional, the dimension of \( \mathcal{D}_{\text{Ad}}(\mathfrak{g}) \) depends on subtle spectral properties of the underlying endomorphism. The Ad-invariance condition spells out as follows:

**Proposition 3.5.** The Lie quasi-state \( \zeta = \zeta_{\alpha,c} \) on \( \mathfrak{g}_\varphi \) is Ad-invariant if and only if the following two conditions hold:

(i) \( \alpha(\exp(s\varphi)v) = \alpha(v) \) for all \( v \in V \) and \( s \in \mathbb{R} \).

(ii) \( c(\exp(s\varphi)w) = c(w) \) for all \( w \in W \) and \( s \in \mathbb{R} \).

**Proof.** First observe that
\[
\text{Ad}(\exp(w,s))(v,t) = \exp(\text{ad}(w,s))(v,t) = (\exp(s\varphi)v, t).
\]
We note that the condition \( \zeta(\text{Ad}(\exp(w,s))(v,0)) = \zeta(v,0) \) amounts to (i), and for \( t \neq 0 \) the invariance condition \( \zeta(\text{Ad}(\exp(w,s))(v,t)) = \zeta(v,t) \) simply means that
\[
c \left( \frac{\varphi(\exp(s\varphi)v)}{t} \right) \cdot t + \alpha(\exp(s\varphi)v) = c(\varphi(v)/t) \cdot t + \alpha(v).
\]
In view of (i) and the condition \( t \neq 0 \) this can be rewritten as
\[
c(\varphi(v)/t) = c \left( \frac{\varphi(\exp(s\varphi)v)}{t} \right) = c \left( \frac{\exp(s\varphi)\varphi(v)}{t} \right).
\]
Now as \( v \) runs through \( V \) the expression \( w := \varphi(v)/t \) runs through all of \( W \), and thus we obtain condition (ii). □

### 3.3. A spectral criterion for quasi-state rigidity.
In the sequel we denote by \( \varphi_W : W \to V \) the restriction of \( \varphi \) to \( W = \text{Im}(\varphi) \). Note that \( \varphi_W(W) \subset W \) so that we can consider \( \varphi_W \) as an endomorphism of \( W \). We denote by \( \sigma(\varphi_W) \subset \mathbb{C} \) the spectrum of \( \varphi_W \). Also recall from the introduction the notation
\[
H_\pm := \{ z \in \mathbb{C} | \pm \Re(z) > 0 \}
\]
for the right-/left-halfplane of the complex plane.

**Theorem 3.6.** With notation as above the following conditions are equivalent:

(i) \( \sigma(\varphi_W) \subset H_+ \) or \( \sigma(\varphi_W) \subset H_- \).

(ii) \( \mathfrak{g}_\varphi \) is rigid.

(iii) \( \mathcal{D}_{\text{Ad}}(\mathfrak{g}_\varphi) \) is finite-dimensional.
Proof. Assume first that (i) holds. If \( \sigma(\varphi_W) \subset H_+ \), then for every \( w \in W \) we have
\[
\lim_{s \to \infty} \exp(-s\varphi)w = 0.
\]
Thus if \( \zeta = \zeta_{a,c} \in \mathcal{D}_{\text{Ad}}(g) \), then by Proposition 3.5 we have for every \( w \in W \),
\[
c(w) = \lim_{s \to \infty} c(\exp(-s\varphi)w) = c(0).
\]
If instead \( \sigma(\varphi_0) \subset H_- \), then we argue similarly to obtain
\[
c(w) = \lim_{s \to \infty} c(\exp(s\varphi)w) = c(0),
\]
Either way, the function c is constant, and thus \( \zeta \) is linear. Now every continuous Ad-invariant linear map on \( g \) is a homomorphism and thus contained in \( \mathcal{D}_{Qm}(g) \). This shows (ii), and since \( \mathcal{D}_{Qm}(g) \subset g^* \) is finite-dimensional, (ii) implies (iii).

It thus remains to show only that (iii) implies (i). We will argue by contraposition and assume that \( \sigma(\varphi_W) \) is not contained in \( H_+ \) nor in \( H_- \). We then have to show that \( \mathcal{D}_{\text{Ad}}(g) \) is infinite-dimensional. Now the assumption that \( \sigma(\varphi_0) \) is not contained in \( H_\pm \) implies that one of the following cases happens:

1. There exists real eigenvalues \( \lambda_i, \lambda_j \) with \( \lambda_1 > 0 > \lambda_j \).
2. \( \lambda_j = 0 \) for some \( j \).
3. There exists a complex pair of eigenvalue \( (\lambda_j, \overline{\lambda}_j) \in i\mathbb{R} \) for some \( j \).
4. There exist two complex pairs of eigenvalues \( (\lambda_i, \overline{\lambda}_j), (\lambda_j, \overline{\lambda}_j) \) such that \( \Re(\lambda_i) > 0 > \Re(\lambda_j) \).
5. There exists a real eigenvalue \( \lambda_i \) and a complex pair of eigenvalues \( (\lambda_j, \overline{\lambda}_j) \in i\mathbb{R} \) with \( \Re(\lambda_i) > 0 > \lambda_j \) or vice versa.

In each of these cases we will now construct an infinite-dimensional space of functions \( c \in C_c(W) \) satisfying the condition in Proposition 3.5 (ii). In fact, it suffices to find a \( \varphi_W \)-invariant subspace \( W_0 \subset W \) and a single non-constant continuous function \( c_0 : W_0 \to \mathbb{R} \) satisfying the condition in Proposition 3.5 (ii), for then any function of the form \( f \circ c_0 \circ \pi_{W_0} \) with \( f \in C_c(\mathbb{R}) \) and \( \pi_{W_0} \), a suitable projection onto \( W_0 \) will be of the desired form.

1. Here we find a 2-dimensional subspace of \( W_0 \) on which \( \varphi_W \) (in a suitable basis) acts by
\[
\varphi_W(x, y) = \begin{bmatrix} \lambda_i & 0 \\ 0 & \lambda_j \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]
Then we can choose \( c_0(x, y) = |x|^{-\lambda_i}|y|^\lambda_j \) for \( x, y \neq 0 \). Note that the assumption \( \lambda_j < 0 < \lambda_i \) is used in order to guarantee that \( c_0 \) is well-defined and continuous at 0.

2. Here we can choose \( W_0 := \ker(\varphi_w) \) and \( c_0 = \Id_{W_0} \).

3. Here we find a 2-dimensional subspace \( W_0 \) on which \( \varphi_W \) acts by rotations, so any radial function \( c_0 \) will do.

4. Here we find a 4-dimensional subspace \( W_0 \) on which \( \varphi_W \) acts by
\[
\varphi_W(w, x, y, z) = \begin{bmatrix} a_i & b_i & 0 & 0 \\ -b_i & a_i & 0 & 0 \\ 0 & 0 & a_j & b_j \\ 0 & 0 & -b_j & -a_j \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix},
\]
where \( a_i = \Re(\lambda_i) > 0 > a_j = \Re(\lambda_j) \). In this case we can choose
\[
c_0(w, x, y, z) = (\sqrt{w^2 + x^2})^{-a_i} \cdot (\sqrt{y^2 + z^2})^{a_j}.
\]
Here we find a 3-dimensional subspace \( W_0 \) on which \( \varphi_W \) acts by
\[
\varphi_0(x, y, z) = \begin{pmatrix} a_i & b_i & 0 \\ -b_i & a_i & 0 \\ 0 & 0 & \lambda_j \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},
\]
where \( a_i = \Re(\lambda_i) > 0 > \lambda_j \), and we can choose
\[
c_0(x, y, z) = \left( \sqrt{x^2 + y^2} \right)^{-\lambda_j} \cdot z^a_i.
\]
Thus in any case we have constructed an infinite-dimensional subspace of \( \mathcal{Q}_{Ad}(g) \). This finishes the proof. \( \square \)

**Remark 3.7.** The proof of Theorem 3.6 actually yields the stronger conclusion that \( \dim \mathcal{Q}_{Ad}(g_\varphi) \) is either finite-dimensional or has the dimension of the continuum. This conclusion actually holds in any example we know. It is well-known that the space of homogeneous quasimorphisms on any given group satisfies the same dichotomy. In the latter case this dichotomy is explained by the fact that the second bounded cohomology of a group is always a Banach space (and thus any Hamel basis has either finite or uncountable cardinality). We do not know whether there is a similar explanation in the Lie quasi-state case.

### 3.4. Examples in low dimensions.

Up to isomorphism there are seven non-abelian Lie algebras of the form \( g_\varphi \) in dimension \( \leq 3 \) (i.e. \( \dim(V) \leq 2 \)) whose quasi-state rigidity properties are given as follows (using Theorem 3.6):

(i) For every vector space \( V \) there is the group \( V \rtimes \mathbb{R}^\times \) of affine homotheties of \( V \), whose Lie algebra is of the form \( g = g_{\text{Id}_V} \). These Lie algebras are always rigid since \( \sigma(\text{Id}_V) \subset H_+ \). For \( \dim V = 1 \) we obtain the Lie algebra of the \((ax + b)\)-group, which is the unique solvable Lie algebra of dimension 2, and for \( \dim V = 2 \) we obtain the Lie algebra of the group of affine homotheties of \( \mathbb{R}^2 \). Similarly, if \( G \) denotes the product of the \((ax + b)\)-group with the abelian group \( \mathbb{R} \), then the Lie algebra \( g \) of \( G \) equals \( g_\varphi \) with
\[
\varphi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\]
which is also rigid.

(ii) The 3-dimensional Heisenberg algebra \( g \) can be written as \( g = g_\varphi \) with
\[
\varphi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]
Since \( \varphi_W = 0 \) it is non-rigid.

(iii) The Lie algebra \( so(3) \) of the 3-dimensional SOL-group is of the form \( g_\varphi \) with
\[
\varphi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
and is non-rigid.

(iv) The one-dimensional unitary motion algebra \( g = C^1 \rtimes u(1) \) can be written as \( g = g_\varphi \) with
\[
\varphi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]
and is non-rigid. However, the closely related Lie algebra \( g_\varphi \) with
\[
\varphi = \begin{pmatrix} \epsilon & 1 \\ -1 & \epsilon \end{pmatrix},
\]
for some \( \epsilon > 0 \) is rigid.

4. Unitary Motion Algebras

In this section we shall consider the family of unitary motion algebras \( g_n = \mathbb{C}^n \oplus u(n) \). The case \( n = 1 \) was already dealt with in Subsection 3.4, where we have seen that \( g_1 \) is non-rigid. On the contrary we will show here that \( g_n \) is rigid for all \( n \geq 2 \). In the sequel we shall always assume that \( n \geq 2 \).

We start with a description of general (i.e. not necessarily Ad-invariant) Lie quasi-states on \( g_n \). We first recall from Lemma 3.1 that
\[
\mathcal{Q}(g_n) = \mathcal{Q}_{00}(g_n) \oplus (\mathbb{C}^n)^* \oplus \mathcal{Q}(u(n)),
\]
where
\[
\mathcal{Q}_{00}(g_n) = \{ \zeta \in \mathcal{Q}(g_n) \mid \zeta(v,0) = \zeta(0,X) = 0, \text{ for all } v \in \mathbb{C}^n \text{ and } X \in u(n) \}.
\]
We refer to elements of \( \mathcal{Q}_{00}(g_n) \) as normalized Lie quasi-states on \( g_n \). We shall now parameterize normalized Lie quasi-states by functions which are reminiscent of the frame functions which appear in Gleason’s work [18]. To this end we introduce the following notation. We denote by \( \langle \cdot, \cdot \rangle \) an inner product on \( \mathbb{C}^n \) with respect to which elements of \( u(n) \) are skew-Hermitian. We use the convention that \( \langle \cdot, \cdot \rangle \) is linear in the first argument and anti-linear in the second argument. Given a vector \( v \in \mathbb{C}^n \setminus \{0\} \) we define \( P_v \in u(n) \) by
\[
P_v(w) = i \frac{\langle w, v \rangle}{\langle v, v \rangle} v,
\]
so that \( -iP_v \) is the orthogonal projection onto the line spanned by \( v \). Note that \( P_v = P_w \) if and only if \( v = \lambda w \) for some \( \lambda \in \mathbb{C}^* \) and that for every orthogonal basis \( (v_1, \ldots, v_n) \) we have
\[
\sum_{j=1}^n P_{v_j} = i \cdot \text{Id}.
\]

**Definition 4.1.** A function \( f : \mathbb{C}^n \to \mathbb{R} \) is called a **generalized frame function** if it satisfies the following conditions:

(i) \( f(0) = 0 \).
(ii) If \( (v_1, \ldots, v_n) \) is an orthogonal basis, then
\[
\sum_{j=1}^n f(P_{v_j}(w)) = f(iw), \quad \text{for all } w \in \mathbb{C}^n \setminus \{0\}.
\]

A typical example of a generalized frame function is given by \( f(v) = \|v\|^2 \). Indeed, this is the content of the Pythagoras theorem.

We now turn to a description of quasi-states in terms of generalized frame functions.

**Proposition 4.2.** Let \( \zeta \in \mathcal{Q}_{00}(g_n) \). Then,

(i) The function \( f = f_\zeta : \mathbb{C}^n \to \mathbb{R} \) given by \( f(0) = 0 \) and
\[
f(w) := \zeta(-iw, P_w) \quad \text{for all } w \in \mathbb{C}^n \setminus \{0\}
\]
is a generalized frame function.
(ii) \( \zeta \) is uniquely determined by \( f_\zeta \). In particular, \( f_\zeta = 0 \) implies \( \zeta = 0 \).

(iii) If \( \zeta \) is continuous, then \( f_\zeta \) is continuous and sublinear in the sense that

\[
\lim_{v \to \infty} \frac{f_\zeta(v)}{\|v\|} = 0
\]

for any norm \( \| \cdot \| \) on \( \mathbb{C}^n \).

**Proof.** (i) Let \( (v_1, \ldots, v_n) \) is an orthogonal basis and \( w \in \mathbb{C}^n \setminus \{0\} \). If we define index sets

\[
J = \{ j \in \{1, \ldots, n\} \mid \langle w, v_j \rangle \neq 0 \}, \quad \text{and} \quad J^c := \{1, \ldots, n\} \setminus J,
\]

then we have

\[
(w, i \cdot \text{Id}) = \sum_{j \in J} (-iP_{v_j}(w), P_{v_j}(w)) + \sum_{j \in J^c} (0, P_{v_j}).
\]

By Lemma 3.1, all summands in this decomposition commute. More precisely this follows from the fact that orthogonal projections onto different orthogonal lines commute together with the fact that for all \( j \neq k \) we have

\[
P_{v_j}(w)(-iP_{v_k}(w)) = 0.
\]

We deduce that

\[
\zeta(w, i \cdot \text{Id}) = \sum_{j \in J} \zeta(-iP_{v_j}(w), P_{v_j}(w)) + \sum_{j \in J^c} \zeta(0, P_{v_j})
\]

\[
= \sum_{j \in J} f(P_{v_j}(w)),
\]

where we used the definition of \( f \) and the fact that \( \zeta \) is normalized. On the other hand, if \( j \in J^c \), then

\[
f(P_{v_j}(w)) = f(0) = 0,
\]

hence

\[
\sum_{j=1}^n f(P_{v_j}(w)) = \zeta(w, i \cdot \text{Id}),
\]

which is independent of the choice of orthogonal basis \( (v_1, \ldots, v_n) \). Now let \( (v_1', \ldots, v_n') \) be a different orthogonal basis with \( v_1' = w \) and thus \( P_{v_j'}(0) = 0 \) for all \( j \geq 2 \). Then we obtain

\[
\sum_{j=1}^n f(P_{v_j}(w)) = \zeta(w, i \cdot \text{Id}) = \sum_{j=1}^n f(P_{v_j'}(w)) = f(iw) + \sum_{j=2}^n f(0) = f(iw).
\]

This proves that \( f \) is a generalized frame function.

(ii) Let \( X \in \mathfrak{u}(n) \) and let \( (v_1, \ldots, v_n) \) be an orthonormal basis of eigenvectors of \( X \) with corresponding (purely imaginary) eigenvalues \( (i\lambda_1, \ldots, i\lambda_n) \). Note that \( X \in \mathfrak{u}(n) \) implies that \( \lambda_j \in \mathbb{R} \), so we can order them by decreasing absolute value and assume that \( |\lambda_1| \geq \cdots \geq |\lambda_l| > \lambda_{l+1} = \cdots = \lambda_n = 0 \), for some index \( l = l(X) \). If we wish to further emphasize the dependence on \( X \) we write \( \lambda_j(X) \) and \( v_j(X) \) instead of \( \lambda_j \) and \( v_j \).

Now let \( w \in \mathbb{C}^n \) and set \( w_j := P_{v_j}(w) \). We observe that by (4.2),

\[
X.w = X \left( -i \cdot \sum_{j=1}^n P_{v_j}(w) \right) = \sum_{j=1}^n \lambda_j P_{v_j}(w).
\]

and

\[
w = -i \sum_{j=1}^n w_j.
\]
We thus obtain
\[ (w, X) = \sum_{j=1}^{n} (-iw_j, \lambda_j P_{v_j}). \]

This is again a sum of commuting elements since rank one projectors in different directions commute and for \( j \neq k \) we have \( \lambda_j P_{v_j}(-iw_k) = 0 = \lambda_k P_{v_k}(-iw_j) \). We thus deduce that
\[
\zeta(w, X) = \sum_{j=1}^{n} \zeta(-iw_j, \lambda_j P_{v_j})
\]

We are now going to express the right hand side in terms of \( f := f_{\zeta} \). Since \( \zeta \) is normalized we have \( \zeta(-iw_j, \lambda_j P_{v_j}) = 0 \) whenever \( \lambda_j = 0 \), i.e. \( j > l \). Now assume that \( j \leq l \) and \( w_j \neq 0 \). Then \( w_j/\lambda_j \) is proportional to \( v_j \), whence \( P_{w_j/\lambda_j} = P_{v_j} \). In this case we thus have
\[
\zeta(-iw_j, \lambda_j P_{v_j}) = \lambda_j \cdot \zeta(-iw_j/\lambda_j, P_{v_j}) = \lambda_j \cdot \zeta(-iw_j/\lambda_j, P_{w_j/\lambda_j}) = \lambda_j \cdot f(w_j/\lambda_j) = \lambda_j \cdot f(P_{v_j}(w)/\lambda_j).
\]

If \( w_j = 0 \) for some \( j \leq l \) then we can use again that \( \zeta \) is normalized to deduce
\[
\zeta(-iw_j, \lambda_j P_{v_j}) = 0 = \lambda_j \cdot f(0) = \lambda_j \cdot f(w_j/\lambda_j) = \lambda_j \cdot f(P_{v_j}(w)/\lambda_j).
\]

We have thus established the formula
\[
\zeta(w, X) = \sum_{j=1}^{l(X)} \lambda_j(X) \cdot f_{\zeta}(P_{v_j}(X)(w)/\lambda_j(X)).
\]

This shows in particular that \( \zeta \) is uniquely determined by \( f_{\zeta} \).

(iii) It is not hard to see from (4.4) that the continuity of \( \zeta \) implies the continuity of \( f_{\zeta} \). To prove sublinearity of \( f_{\zeta} \), we fix \( w \in C^n \) with \( \|w\| = 1 \). By the definition of a quasi-state,
\[
f_{\zeta}(\lambda w) = \frac{1}{\lambda} \cdot \zeta(-iw, P_{\lambda w}/\lambda) = \zeta(-iw, P_{w}/\lambda).
\]

Hence, by continuity of \( \zeta \), we have
\[
\lim_{\lambda \to \infty} \frac{f_{\zeta}(\lambda w)}{\lambda} = \lim_{\lambda \to \infty} \zeta(-iw, \frac{P_w}{\lambda}) = \zeta(-iw, 0) = 0,
\]
since \( \zeta \in Q_{00}(g_n) \) and \( P_{\lambda w} = P_w \) for every non-zero real \( \lambda \).

4.1. The Ad-invariant case. We now restrict our attention to continuous Ad-invariant Lie quasi-states.

Theorem 4.3. Every continuous Ad-invariant Lie quasi-state on \( g_n \) is of the form
\[
\zeta(v, X) = i\lambda \cdot \text{tr}(X)
\]
for some \( \lambda \in \mathbb{R} \). In particular, \( \text{dim} \mathcal{Q}_{\text{Ad}}(g_n) = 1 \) and
\[
\mathcal{Q}_{\text{Ad}}(g_n) = \mathcal{Q}_{\text{qm}}(g_n).
\]

Proof. By (4.1) every Lie quasi-state on \( g_n \) is of the form
\[
\zeta(v, X) = \psi(v) + \zeta_{\text{ut}(n)}(X) + \zeta_0(v, X)
\]
for some \( \psi \in V^*, \alpha \in Q(\text{ut}(n)) \) and \( \zeta_0 \in Q_{00}(g_n) \). Now \( \zeta \) is Ad-invariant if and only if for all \( (v, X) \in g_n \) and all \( k \in U(n) \) we have
\[
\psi(k \cdot v) + \zeta_{\text{ut}(n)}(\text{Ad}(k)(X)) + \zeta_0(k \cdot v, \text{Ad}(k)(X)) = \psi(v) + \zeta_{\text{ut}(n)}(X) + \zeta_0(v, X).
\]
Setting $X = 0$, respectively $v = 0$, we deduce that $\psi$ and $\zeta_{u(n)}$ have to be $U(n)$-invariant. It follows that
\begin{equation}
\zeta_0(k.u, \text{Ad}(k)(X)) = \zeta_0(v, X).
\end{equation}
Since $-\text{Id} \in U(n)$ we deduce from the $U(n)$-invariance of $\psi$ that $\psi(v) = \psi(-v) = -\psi(v)$, showing that $\psi = 0$. Since $u(n)$ is reductive, we deduce from Theorem [1.1] that every Ad-invariant Lie quasi-state on $u(n)$ is linear. It follows that $\zeta_{u(n)} \in \text{Hom}(u(n), \mathbb{R})$, and thus $\zeta_{u(n)}(X) = i\lambda \cdot \text{tr}(X)$ for some $\lambda \in \mathbb{R}$. It remains to show only that if $\zeta_0 \in \mathcal{D}_{00}(g)$ satisfies (4.5), then $\zeta_0 \equiv 0$.

For this, let $f = f_{\zeta_0}$ denote the frame function associated with $\zeta_0$. From (4.5) we see that
\[
 f(w) = \zeta_0(-iw, P_w) = \zeta_0(k \cdot (-iw), \text{Ad}(k)(P_w)) = \zeta_0(-i(k \cdot w), P_{k \cdot w}) = f(k \cdot w),
\]
for all $k \in U(n)$ and thus $f$ is a radial (continuous) function on $\mathbb{C}^n$, i.e. there exists a (continuous) function $f_0 : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ such that $f(w) = f_0(\|w\|)$.

Given $t_1, \ldots, t_n \in \mathbb{R}_{\geq 0}$ and any orthonormal basis $(v_1, \ldots, v_n)$ we can use the defining property of a frame function to compute
\[
 f_0(t_1) + \cdots + f_0(t_n) = f(t_1 \cdot v_1) + \cdots + f(t_n \cdot v_n)
 = f\left(\sum_{j=1}^{n} t_j \cdot v_j\right) = f_0\left(\left\| \sum_{j=1}^{n} t_j \cdot v_j \right\|\right)
 = f_0(\sqrt{t_1^2 + \cdots + t_n^2}).
\]
Now we use our assumption $n \geq 2$ in order to deduce that for all $s, t \in \mathbb{R}_{\geq 0}$ we have
\[
f_0(s) + f_0(t) = f_0(\sqrt{s^2 + t^2}).
\]
From this we show by induction that for all $n \in \mathbb{N}$ and $s \geq 0$,
\[
n \cdot f_0(s) = f_0(\sqrt{n} \cdot s).
\]
In particular, for $s > 0$ we have
\[
\sqrt{n} \cdot \frac{f_0(s)}{s} = \frac{f_0(\sqrt{n} \cdot s)}{\sqrt{n} \cdot s}.
\]
Now if $\zeta$ is a continuous Lie quasi-state, then $f$, and hence $f_0$, are sublinear by Theorem [4.2] We conclude that for $s > 0$ we have
\[
\lim_{n \to \infty} \sqrt{n} \cdot \frac{f_0(s)}{s} = \lim_{n \to \infty} \frac{f_0(\sqrt{n} \cdot s)}{\sqrt{n} \cdot s} = 0.
\]
Thus $f_0(s) = 0$ for all $s > 0$ and also $f_0(0) = 0$ since $f(0) = 0$. Thus $f = f_{\zeta_0} = 0$ and hence $\zeta_0 = 0$ by Theorem [4.2] The theorem follows.

4.2. The general case. We stress that Theorem [4.3] is not true without the assumption of Ad-invariance. Indeed, for $n = 2$ there are non-linear continuous quasi-states on $u(2)$, and these give rise to non-linear quasi-states on $C^2 \ltimes u(2)$. However for $n \geq 3$ the theorem remains true. Moreover we can replace the assumption of continuity by the weaker assumption of local boundedness. Here a quasi-state $\alpha : g \to \mathbb{R}$ is called locally bounded if there exists a neighbourhood $U$ of 0 such that $\alpha|U$ is bounded. This is (equivalent to) the assumption used in Gleason’s original work on quasi-states on $u(n)$ (see the introduction of [14]).

Theorem 4.4. Let $g = \mathbb{C}^n \ltimes u(n)$, $n \geq 2$ and let $\zeta : g \to \mathbb{R}$ be a locally bounded Lie quasi-state. If either $n \geq 3$, or $n \geq 2$ and $\zeta$ is Ad-invariant, then $\zeta$ is linear.
In preparation for the proof of the theorem we observe that a positive definite bilinear form on the Lie algebra \( u(n) \) is given by
\[
\langle X, Y \rangle := -\text{tr}(XY).
\]

**Lemma 4.5.** For every \( v \in \mathbb{C}^n \setminus \{0\} \) there exist \( v_1, \ldots, v_{N-1} \in \mathbb{C}^n \), where \( N = \dim u(n) \), such that the corresponding skew-projections \( (P_v, P_{v_1}, \ldots, P_{v_{N-1}}) \) form an orthonormal basis of \( u(n) \) with respect to the bilinear form \( \langle \cdot, \cdot \rangle \).

**Proof.** By the spectral theorem, the elements \( P_w \) with \( w \in \mathbb{C}^n \) generate \( u(n) \). It follows that there exists a basis of \( u(n) \) of the form \( (P_v, P_{v_1}, \ldots, P_{v_{N-1}}) \). Now observe that \( -\text{tr}(P_w^2) = 1 \) for every \( w \in \mathbb{C}^n \) and that \( \text{tr}(P_{w_1}P_{w_2}) = 0 \) provided \( P_{w_1} \) and \( P_{w_2} \) (and hence \( w_1 \) and \( w_2 \)) are linearly independent.

**Proof of Theorem 4.4.** As in the proof of Theorem 4.3 every Lie quasi-state \( \zeta \) on \( g \) can be written as
\[
\zeta(v, X) = \psi(v) + \zeta_{u(n)}(X) + \zeta_0(v, X)
\]
for some \( \psi \in V^* \), \( a \in Q(u(n)) \) and \( \zeta_0 \in Q_{00}(g_0) \). If \( \zeta \) is locally bounded, then so is \( \zeta_{u(n)} \). If \( n \geq 3 \) it then follows from Gleason's theorem that \( \zeta_{u(n)} \) is linear. For \( n = 2 \) and under the additional assumption of Ad-invariance, this follows from Theorem 4.1 in [14] (or Theorem 1.1 in this paper). It thus only remains to show that \( \zeta_0 \) is linear. We will in fact show that \( \zeta_0 = 0 \).

For every \( v \in \mathbb{C}^n \) consider the subalgebra \( h_v \subset g \) given by
\[
h_v = \{ (Xv, X) \in g \mid X \in u(n) \}.
\]
We observe that \( h_v \) is isomorphic to \( u(n) \) via the projection onto the second coordinate. In particular, the restriction \( \zeta_v := \zeta_0|_{h_v} \) is linear by Gleason's theorem (or by Theorem 1.1 if \( n = 2 \) and the quasi-state \( \zeta_0 \) is Ad-invariant). There thus exists \( A_v \in u(n) \) such that \( \zeta_v(Xv, X) = \langle A_v, X \rangle \). Since we have assumed that \( \zeta_0 \in Q_{00}(g) \) we conclude that \( A_0 = 0 \).

Next we observe that for \( v, w \in \mathbb{C}^n \) we have \( \zeta_v|_{h_v \cap h_w} = \zeta_w|_{h_v \cap h_w} \). Since
\[
h_v \cap h_w = \{ (Xv, X) \in g \mid v - w \in \ker(X) \},
\]
this implies that
\[
v - w \in \ker(X) \Rightarrow \langle A_v - A_w, X \rangle = 0.
\]
Now assume \( v - w \neq 0 \). According to Lemma 4.5 we can then find an orthonormal basis of \( u(n) \) of the form \( (P_{v-w}, P_{v_1}, \ldots, P_{v_{N-1}}) \). Now for every index \( j = 1, \ldots, N - 1 \) we have \( v - w \in \ker(P_{v_j}) \) and thus \( \langle A_v - A_{w}, P_{v_j} \rangle = 0 \). We deduce that there exist \( \lambda_v, w \in \mathbb{R} \) such that
\[
A_v - A_w = \lambda_v, w \cdot P_{v-w}.
\]
If we set \( \lambda_v := \lambda_v, 0 \) then in particular \( A_v = \lambda_v \cdot P_v \). Now if \( v \) and \( w \) are linearly independent, then the only possibility for the equation
\[
\lambda_v \cdot P_v - \lambda_w \cdot P_w = \lambda_{v, w} \cdot P_{v-w}
\]
to hold is that \( \lambda_v = \lambda_w = \lambda_{v, w} = 0 \). This shows that \( A_v = 0 \) for all \( v \in \mathbb{C}^n \) and consequently \( \zeta_0 \) vanishes on each of the subalgebras \( h_v \). This in turn implies that \( f_{\zeta_0} = 0 \) and thus \( \zeta_0 = 0 \). Indeed, given any \( (w, X) \in g \), we can find \( u \in \mathbb{C}^n \) and \( v \in \ker(X) \) such that \( w = Xu + v \), and thus
\[
\zeta_0(w, X) = \zeta_0(Xu, X) + \zeta_0(v, 0) = \zeta_0(Xu, X) + \zeta_0(v, 0) = \zeta_0(X) + \zeta_0(v, 0) = 0,
\]
since \( (Xu, X) \) and \( (v, 0) \) commute in \( g \) and \( \zeta_0 \in Q_{00} \).
Appendix A. Centralizers and Lie quasi-states

In this appendix we record a few general facts concerning the relation between centralizers of elements in a Lie algebra $\mathfrak{g}$ and Lie quasi-states on $\mathfrak{g}$. Throughout this appendix we assume that $\mathfrak{g}$ is a finite-dimensional real Lie algebra.

The collection $\mathcal{A} = \mathcal{A}(\mathfrak{g})$ is of abelian subalgebras of $\mathfrak{g}$ is partially ordered with respect to inclusion and maximal elements are called maximal abelian subalgebras. By finite-dimensionality every chain in $(\mathcal{A}, \subseteq)$ is finite, and thus every abelian subalgebra is contained in a (in general non-unique) maximal one.

Denote by $A_0 \subset \mathcal{A}$ the collection of maximal abelian subalgebras and by $A_0 \subset \mathcal{A}$ the closure of $A_0$. Given $a, b \in A$ with $a \subset b$ there is a canonical (dual) restriction map $b^* \to a^*$. We denote by $A^* = A^*(\mathfrak{g})$ the diagram whose objects are vector spaces $a^*$ with $a \in A$ and whose maps are restriction maps.

Proposition A.1. For every finite-dimensional Lie algebra $\mathfrak{g}$ we have

$$Q(\mathfrak{g}) = \lim_{\leftarrow} A^*(\mathfrak{g}).$$

Proof. If we replace $A$ by $\mathcal{A}$ then this is just the definition of a Lie quasi-state. Since every abelian subalgebra is contained in a maximal one we can restrict ourselves to the sub-poset $A \subset \mathcal{A}$. □

Thus the space of Lie quasi-states depends only on the maximal abelian subalgebras of $\mathfrak{g}$ and their intersections. We now relate maximal abelian subalgebras to centralizers. Given an element $X \in \mathfrak{g}$ we denote by $\text{Cent}(X)$ the centraliser of $X$ in $\mathfrak{g}$.

Lemma A.2. (i) A subalgebra $a \subset \mathfrak{g}$ is abelian if and only if

$$a \subset \bigcap_{X \in a} \text{Cent}(X).$$

(ii) If $\text{Cent}(X)$ is abelian for some $X \in \mathfrak{g}$ then it is the unique maximal abelian subalgebra containing $X$.

(iii) A subalgebra $a \subset \mathfrak{g}$ is maximal abelian if and only if

$$a = \bigcap_{X \in a} \text{Cent}(X).$$

(A.1)

Proof. (i) A subalgebra is abelian iff it centralizes each of its elements. (ii) By (i) every abelian subalgebra containing $X$ is necessarily contained in $\text{Cent}(X)$, hence the latter is uniquely maximal abelian if it is abelian. (iii) If $a$ satisfies (A.1) and $a' \supset a$ is abelian then by (i) we have

$$a' \subset \bigcap_{X \in a'} \text{Cent}(X) \subset \bigcap_{X \in a} \text{Cent}(X) = a,$$

showing that $a$ is maximal abelian. Conversely, if the inclusion is proper and

$$Y \in \bigcap_{X \in a} \text{Cent}(X) \setminus a,$$

then $Y$ commutes with every $X \in a$, hence $\text{span}(a \cup Y) \supset a$ is abelian. □

Thus, at least theoretically maximal abelian subalgebras can be constructed iteratively by intersecting centralizers. However, in practice determining centralizers and maximal abelian subalgebras for arbitrary elements is a hard problem. A particularly nice situation is given if all or at least many centralizers are abelian. In such a situation one can often read off the space of Lie quasi-states directly.
We illustrate this principle with one specific example, which is used in Section 3. As there, let us call elements $X, Y \in g$ equivalent provided $\text{Cent}(X) = \text{Cent}(Y)$. We also denote the equivalence class of $X$ by $[X]$ and set $\text{Cent}([X]) := \text{Cent}(X)$.

**Corollary A.3.** Let $g$ be a Lie algebra with center $\mathfrak{z}(g)$. Assume that for every $X \in g \setminus \mathfrak{z}(g)$ the centralizer $\text{Cent}(X)$ is abelian, and that distinct centralizers intersect in $\mathfrak{z}(g)$. Moreover, let $[X_0]$ denote a distinguished equivalence class in $g \setminus \mathfrak{z}(g)$. Then the space of Lie quasi-states on $g$ vanishing on $[X_0]$ can be identified with the set

$$\left\{ (\zeta_{[X]} \mid \prod_{[X] \in (g \setminus [X_0])} \text{Cent}([X])^*) \mid \forall [X] : \zeta_{[X]} |_{\mathfrak{z}(g)} \equiv 0 \right\}$$

via the map

$$\zeta \mapsto (\zeta |_{\text{Cent}([X])}).$$

**Proof.** In view of Proposition A.1 we consider the diagram $A^* (g)$. In view of Lemma 1.(ii) the centralizers of elements $X \in g \setminus \mathfrak{z}(g)$ are the only maximal abelian subalgebras. Since they pairwise intersect in the centre, the diagram $A^*(g)$ looks schematically as follows:

$$\begin{array}{ccc}
\text{Cent}([X_0])^* & \text{Cent}([X_1])^* & \text{Cent}([X_\lambda])^* \\
& \downarrow & \downarrow \\
& \mathfrak{z}(g)^* & \\
\end{array}$$

Now an element of the inverse limit of this diagram which vanishes along $\text{Cent}([X_0])$ and hence along $\mathfrak{z}(g)$ is precisely of the form described, so the corollary follows from Proposition A.1.

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