Fibre product approach to index pairings for the
generic Hopf fibration of $SU_q(2)$

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Abstract
A fibre product construction is used to give a description of quantum line bundles over the generic Podleś spheres by gluing two quantum discs along their boundaries. Representatives of the corresponding $K_0$-classes are given in terms of 1-dimensional projections belonging to the C*-algebra, and in terms of analogues of the classical Bott projections. The $K_0$-classes of quantum line bundles derived from the generic Hopf fibration of quantum $SU(2)$ are determined and the index pairing is computed. It is argued that taking the projections obtained from the fibre product construction yields a significant simplification of earlier index computations.

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1 Introduction

The main goal of the paper is to show that the fibre product approach provides an effective tool for simplifying index computations. Our example is one of the most extensively studied quantum spaces, namely the Podleś spheres [20]. These two parameter deformations of the classical 2-sphere served as a guiding example in the development of the theory of coalgebra bundles (quantum principal bundles) [3, 4]. In this setting, the Podleś spheres and the quantum $SU(2)$ play the roles of base and total space of a noncommutative Hopf fibration, respectively. A projective module description of all associated quantum line bundles was given in [11] for the standard Podleś sphere, and in [12, 21] for general parameters. Since the cyclic cohomology of the Podleś spheres was known [17], the projective module description made it possible to compute the index pairing (Chern-Connes pairing) between cyclic cohomology and the $K_0$-classes of the quantum line bundles. This was done first in [10] for the standard Podleś sphere, and then extended to the general case in [12]. Their proofs relied heavily on the index theorem, arguing that the index is an integer and taking
a suitable limit of the parameters. The main obstacle for a direct computation was the growing size of the projection matrices, leading to complicated expressions which are difficult to handle. It is therefore desirable to find a tool for obtaining simpler representatives of $K$-theory classes.

The computations of the $K$-groups of quantum 2-spheres in [13] and of quantum 3-spheres in [2] strongly encourage to use a fibre product approach to index pairings. Furthermore, an explicit description of projective modules and a significant simplification of the index problem were obtained in [8] by using Bass’ construction of the Mayer-Vietoris boundary map in (algebraic) $K$-theory [1]. In the present paper, we shall see another example where the fibre product approach simplifies the index computation considerably.

The fact that the C*-algebra of generic Podleś spheres is given by a fibre product of two Toeplitz algebras follows from [22, Proposition 1.2]. In Section 3 we explain how this description can be interpreted as gluing two quantum discs along the boundary (see also [5, 6]). Our first result concerns the construction of quantum line bundles. Since the boundaries of the quantum discs are identified with the classical circle $S^1$, we can glue two trivial bundles over the quantum disc by using the same transition function as in the commutative case. The gluing procedure is described by a fibre product and the resulting projective modules are considered as quantum line bundles of winding number $N$, where $N \in \mathbb{Z}$ is the degree of the transition function. We then show that the quantum line bundles are isomorphic to projective modules given by elementary 1-dimensional projections. In a sense, the quantum case is simpler than its classical counterpart since there are no non-trivial 1-dimensional projections in $C(S^2)$.

The link between the fibre product approach of quantum line bundles and the Hopf fibration of $SU_q(2)$ will be established in Section 4. We prove that, for each winding number, the projection derived from the noncommutative Hopf fibration is Murray-von Neumann equivalent to the 1-dimensional projection of the fibre product construction. As an intermediate step, we introduce Murray-von Neumann equivalent $2 \times 2$-projections whose entries are rational functions of the generators of the Podleś spheres. These $2 \times 2$-projections can be viewed as analogues of the classical Bott projections. Moreover, their explicit description enables us to give a short direct proof of the statement that the direct sum of quantum line bundles with winding number $N$ and $-N$ is equivalent to a free rank 2 module.

In the final section, we carry out the index computations. Because of the previous results, we are free to choose the most convenient representatives of the $K_0$-classes. By using the 1-dimensional projections from the fibre product approach, the index pairing reduces to its simplest possible form; it remains to calculate a trace of a projection onto a finite dimensional subspace. In analogy to the classical case, one generator of $K$-homology computes the winding number, and the other detects the rank of the projective module.
2 Preliminaries

2.1 Fibre products

Let \( \pi_0 : A_0 \to A_{01} \) and \( \pi_1 : A_1 \to A_{01} \) be morphisms of C*-algebras. The fibre product \( A := A_0 \times_{(\pi_0, \pi_1)} A_1 \) is defined by the pull-back diagram

\[
\begin{array}{c}
A_0 \xrightarrow{\pi_0} A_0 \times_{(\pi_0, \pi_1)} A_1 \xrightarrow{\pi_1} A_1 \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
A_0 \quad \quad A_0 \end{array}
\]

Up to a unique isomorphism, \( A \) is given by

\[
A = \{(ao, a_1) \in A_0 \times A_1 : \pi_0(a_0) = \pi_1(a_1)\},
\]

where \( A_0 \times A_1 \) denotes the outer direct sum of C*-algebras with multiplication \((a_0, a_1)(b_0, b_1) = (a_0b_0, a_1b_1)\) and involution \((a_0, a_1)^* = (a_0^*, a_1^*)\). The morphisms \( \text{pr}_0 : A \to A_0 \) and \( \text{pr}_1 : A \to A_1 \) are then the left and right projections, respectively.

Similarly, if \( B \) is a C*-algebra, and \( \pi_0 : A_0 \to A_{01} \) and \( \pi_1 : A_1 \to A_{01} \) are morphisms of left \( B \)-modules, then the fibre product of vector spaces \( A := A_0 \times_{(\pi_0, \pi_1)} A_1 \) is a left \( B \)-module with left action given by \( b.(ao, a_1) = (b,a_0, b.a_1) \), where the dot denotes the left action.

Now let \( A_i \) be left \( B_j \)-modules, \( j = 0, 1, 01 \), and let \( \rho_0 : B_0 \to B_{01} \) and \( \rho_1 : B_1 \to B_{01} \) be morphisms of C*-algebras. Suppose that the vector space morphisms \( \pi_0 : A_0 \to A_{01} \) and \( \pi_1 : A_1 \to A_{01} \) intertwine the actions, that is, \( \pi_i(b_i, a_i) = \rho_i(b_i).\pi_i(a_i) \) for all \( b_i \in B_i \) and \( a_i \in A_i \), \( i = 0, 1 \). Then \( A \) is a left \( B = B_0 \times_{(\pi_0, \rho_0)} B_1 \)-module. This follows from the preceding with the left \( B \)-actions given by \( b.a_i := \text{pr}_i(b).a_i \) for \( a_i \in A_i \), \( i = 0, 1 \), and \( b.a_{01} := \rho_1 \circ \text{pr}_0(b).a_{01} = \rho_1 \circ \text{pr}_1(b).a_{01} \) for \( a_{01} \in A_{01} \).

2.2 The generic Podleś spheres

Let \( q \in (0, 1) \) and \( s \in [0, \infty) \). The *-algebra \( \mathcal{O}(S^2_{qs}) \) of polynomial functions on the quantum 2-sphere \( S^2_{qs} \) is generated by \( \eta_s, \eta_s^* \) and an hermitian element \( \zeta_s \) satisfying the relations

\[
\zeta_s \eta_s = q^2 \eta_s \zeta_s, \quad \zeta_s^* \eta_s = (1 - \zeta_s)(s^2 + \zeta_s), \quad \eta_s \eta_s^* = (1 - q^{-2} \zeta_s)(s^2 + q^{-2} \zeta_s).
\]

The case \( s = 0 \) corresponds to the standard Podleś sphere. Its algebraic and C*-algebraic properties differ considerably from the other Podleś spheres which we call “generic”. The standard Podleś sphere is treated in [14] and will be excluded here. Note that, for \( s > 0 \), \( \mathcal{O}(S^2_{qs}) \) and \( \mathcal{O}(S^2_{q, s^{-1}}) \) are isomorphic, where an isomorphism is given by \( \zeta_s \to -s^2 \zeta_{s^{-1}} \) and \( \eta_s \to s \eta_{s^{-1}} \). For practical reasons, we shall consider both versions of the same *-algebra.

Recall that the universal C*-algebra of some *-algebra is defined as the C*-completion with respect to the universal C*-norm given by the supremum of
the operator norms over all bounded \(*\)-representations (if the supremum exists).

The universal \(\mathbb{C}\)-algebra of \(\mathcal{O}(S_{qs}^2)\) will be denoted by \(\mathcal{C}(S_{qs}^2)\).

The \(*\)-representations of \(\mathcal{O}(S_{qs}^2)\) were described in [17, 20] and are classified as follows. Given a Hilbert space \(\mathcal{H}_0\), set \(\mathcal{H} := \bigoplus_{n=0}^{\infty} \mathcal{H}_n\), where \(\mathcal{H}_n = \mathcal{H}_0\). For \(e \in \mathcal{H}_0 \setminus \{0\}\) and \(n \in \mathbb{N}_0\), we denote by \(e_n\) the vector in \(\mathcal{H}\) which has \(e\) as its \(n\)-th component and zero otherwise. On \(\mathcal{H}\), the shift operator \(S\) is given by

\[
Se_n = e_{n+1}.
\]

Now let \(\mathcal{H}_0^+, \mathcal{H}_0^-, \mathcal{H}_0^0\) be Hilbert spaces and let \(U\) be a unitary operator on \(\mathcal{H}_0^\perp\). Set \(\mathcal{H}^+ := \bigoplus_{n=0}^{\infty} \mathcal{H}_n^+\) and \(\mathcal{H}^- := \bigoplus_{n=0}^{\infty} \mathcal{H}_n^-\). For \(s \neq 0\), a \(*\)-representation \(\rho\) of \(\mathcal{O}(S_{qs}^2)\) on a Hilbert space \(\mathcal{H}\) has the direct sum decomposition \(\rho = \rho_- \oplus \rho_0 \oplus \rho_+\) on \(\mathcal{H} = \mathcal{H}^- \oplus \mathcal{H}_0 \oplus \mathcal{H}^+\) and is given by

\[
\rho_- (\zeta_s) e_n^- = -s^2 q^{2(n+1)} e_n^-,
\rho_- (\eta_s) e_n^- = s \sqrt{(1 + s^2 q^{2(n+1)})(1 - q^{2(n+1)})} Se_n^-,
\rho_0 (\zeta_s) e^0 = 0,
\rho_0 (\eta_s) e^0 = sU,
\rho_+ (\zeta_s) e_n^+ = q^{2(n+1)} e_n^+,
\rho_+ (\eta_s) e_n^+ = \sqrt{(1 - q^{2(n+1)})(s^2 + q^{2(n+1)})} Se_n^+.
\]

The representations \(\rho_-\), \(\rho_0\) and \(\rho_+\) are irreducible if and only if the Hilbert spaces \(\mathcal{H}_0^+, \mathcal{H}_0^-, \mathcal{H}_0^0\) are 1-dimensional.

It was argued in [17, 22] that the direct sum \(\rho_- \oplus \rho_+\) of the irreducible representations \(\rho_-\) and \(\rho_+\) on \(\ell^2(\mathbb{N}_0)\) yields a faithful representation of \(\mathcal{C}(S_{qs}^2)\). Therefore we can consider \(\mathcal{C}(S_{qs}^2)\) as the concrete \(\mathbb{C}\)-algebra given by the representation \(\rho_- \oplus \rho_+\) on \(\ell^2(\mathbb{N}_0) \oplus \ell^2(\mathbb{N}_0)\). By a slight abuse of notation, we shall frequently identify elements of \(\mathcal{C}(S_{qs}^2)\) with its image under the mapping \(\rho_- \oplus \rho_+\).

For later use, let us also mention that

\[
\text{spec}(\zeta_s) = \{-s^2 q^2, -s^2 q^4, \ldots\} \cup \{0\} \cup \{q^2, q^4, \ldots\}.
\]

To give an explicit description of \(\mathcal{C}(S_{qs}^2)\), we make use of the quantum disc. The \(*\)-algebra \(\mathcal{O}(D_2^s)\) of polynomial functions on the quantum disc is generated by two generators \(z\) and \(z^*\) with relation

\[
z^*z - q^2 zz^* = 1 - q^2.
\]

The universal \(\mathbb{C}\)-algebra of \(\mathcal{O}(D_2^s)\) is well known. It has been discussed by several authors (see, e.g., [13, 17, 22]) that it is isomorphic to the Toeplitz algebra \(\mathcal{T}\), where \(\mathcal{T}\) can be characterized as the universal \(\mathbb{C}\)-algebra generated by the unilateral shift \(S\) on \(\ell^2(\mathbb{N}_0)\).

Let \(U\) denote the unitary generator of \(\mathcal{C}(S^1)\). The so-called symbol map is the \(*\)-homomorphism \(\sigma: \mathcal{T} \to \mathcal{C}(S^1)\) defined by \(\sigma(S) = U\). In [22], it has been shown that (for \(s > 0\))

\[
\mathcal{C}(S_{qs}^2) \cong \{(a_0, a_1) \in \mathcal{T} \oplus \mathcal{T} : \sigma(a_0) = \sigma(a_1)\}.
\]

The \(*\)-representations \(\rho_-\) and \(\rho_+\) of \(\mathcal{C}(S_{qs}^2)\) on \(\ell^2(\mathbb{N}_0)\) are then given by

\[
\rho_- ((a_0, a_1)) = a_0, \quad \rho_+ ((a_0, a_1)) = a_1, \quad (a_0, a_1) \in \mathcal{C}(S_{qs}^2).
\]
The K-theory and K-homology of $\mathcal{C}(S^2_{qs})$ has been computed in [17]. There it is shown that $K_0(\mathcal{C}(S^2_{qs})) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $K^0(\mathcal{C}(S^2_{qs})) \cong \mathbb{Z} \oplus \mathbb{Z}$. For $s > 0$, the two generators of the $K_0$-group are given by the class $[1]$ of the unit element $1 \in \mathcal{C}(S^2_{qs})$, and by the class $[(0, 1 - SS^*)]$ of the 1-dimensional projection onto $\mathbb{C}(0, e_0) \subseteq \ell^2(\mathbb{N}_0) \oplus \ell^2(\mathbb{N}_0)$.

Describing an even Fredholm module by a pair of representations on the same Hilbert space, one generator of $K^0(\mathcal{C}(S^2_{qs}))$ is given by $[(\rho_+, \rho_-)]$ on $\ell^2(\mathbb{N}_0)$. The second generator is obtained by a pull-back of an even Fredholm module on $\mathcal{C}(S^1)$ via the symbol map $\sigma : \mathcal{C}(S^2_{qs}) \to \mathcal{C}(S^1)$, $\sigma((a_0, a_1)) := \sigma(a_0) = \sigma(a_1)$. The even Fredholm module on $\mathcal{C}(S^1)$ was described in [16] by the following pair of representations $\pi_{\pm} : \mathcal{C}(S^1) \to \ell^2(\mathbb{Z})$:

$$
\pi_+(U)e_n = e_{n+1}, \quad n \in \mathbb{Z}, \\
\pi_-(U)e_n = e_{n+1}, \quad n \in \mathbb{Z} \setminus \{-1, 0\}, \quad \pi_-(U)e_{-1} = e_1, \quad \pi_-(U)e_0 = 0.
$$

Note that the representation $\pi_-$ is non-unital. More precisely, $\pi_-(1)$ is the projection onto $\text{span}\{e_n : n \in \mathbb{Z} \setminus \{0\}\}$. Composing $\pi_{\pm}$ with the symbol map $\sigma$ yields the second generator of $K^0(\mathcal{C}(S^2_{qs}))$, say $[(\varepsilon_+, \varepsilon_-)]$, where $\varepsilon_{\pm} = \pi_{\pm} \circ \sigma$.

### 2.3 Hopf fibrations of quantum SU(2) over the generic Podleś spheres

Let again $q \in (0, 1)$. The Hopf *-algebra $\mathcal{O}(SU_q(2))$ of polynomial functions on the quantum group $SU_q(2)$ is generated by $\alpha, \beta, \gamma, \delta$ with relations

$$
\alpha \beta = q \beta \alpha, \quad \alpha \gamma = q \gamma \alpha, \quad \beta \delta = q \delta \beta, \quad \gamma \delta = q \delta \gamma, \quad \beta \gamma = \gamma \beta,
\alpha \delta - q \beta \gamma = 1, \quad \delta \alpha - q^{-1} \beta \gamma = 1,
$$

and involution $\alpha^* = \delta$, $\beta^* = -q \gamma$. The Hopf algebra structure is given by

$$
\Delta(\alpha) = \alpha \otimes \alpha + \beta \otimes \gamma, \quad \Delta(\beta) = \alpha \otimes \beta + \beta \otimes \delta, \quad \\
\Delta(\gamma) = \gamma \otimes \alpha + \delta \otimes \gamma, \quad \Delta(\delta) = \gamma \otimes \beta + \delta \otimes \delta, \\
\varepsilon(\alpha) = \varepsilon(\delta) = 1, \quad \varepsilon(\beta) = \varepsilon(\gamma) = 0, \\
\kappa(\alpha) = \delta, \quad \kappa(\beta) = -q^{-1} \beta, \quad \kappa(\gamma) = -q \gamma, \quad \kappa(\delta) = \alpha.
$$

The polynomial *-algebra $\mathcal{O}(S^2_{qs})$ can be embedded into $\mathcal{O}(SU_q(2))$ by setting

$$
\eta_s := (\delta + q^{-1} s \beta)(\beta - s \delta), \quad \zeta_s := 1 - (\alpha - qs \gamma)(\delta + s \beta).
$$

By restricting the comultiplication $\Delta$ of $\mathcal{O}(SU_q(2))$ to $\mathcal{O}(S^2_{qs})$, the latter becomes a right $\mathcal{O}(SU_q(2))$-comodule *-algebra.

Let $J_s$ denote the coalgebra $\mathcal{O}(SU_q(2))/\mathcal{O}(S^2_{qs})^+ \mathcal{O}(SU_q(2))$ with quotient map $\text{pr}_s : \mathcal{O}(SU_q(2)) \to J_s$, where $\mathcal{O}(S^2_{qs})^+ = \{x \in \mathcal{O}(S^2_{qs}) : \varepsilon(x) = 0\}$. It has been shown in [15] that $J_s$ is spanned by group-like elements $g_N, N \in \mathbb{Z}$. Set

$$
M_N := \{x \in \mathcal{O}(SU_q(2)) : \text{pr}_s(x_{(1)}) \otimes x_{(2)} = g_N \otimes x\}.
$$
Then the algebra of co-invariants $M_0$ is isomorphic to $\mathcal{O}(S^2_{qs})$ (via the embedding $\mathfrak{M}$), $M_N$ is a finitely generated projective $\mathcal{O}(S^2_{qs})$-module, and $\mathcal{O}(SU_q(2))$ is the direct sum of all $M_N$ [18]. Analogous to the classical Hopf fibration over the 2-sphere, the projective modules $M_N$ are considered as line bundles over the quantum 2-sphere $S^2_{qs}$ with winding number $N$.

Descriptions of idempotents representing $M_N$ for all $N$ and all parameters $s$ have been given in [12, 21]. For the convenience of the reader, we review briefly the construction of idempotents from [21].

For $N \in \mathbb{Z}$, set
\[
\begin{align*}
u_N &= q^{-N}(\beta - q^s \delta)(\beta - q^{2s} \delta) \cdots (\beta - q^{Ns} \delta), \quad N > 0, \\
u_N &= (\delta + q^{-1}s\beta)(\delta + q^{-2}2\beta) \cdots (\delta + q^{N}s\beta), \quad N < 0,
\end{align*}
\]
and $\nu_0 = 1$. From [4], we conclude that $\mathfrak{pr}_s(\nu_N) = q \nu_N$ and thus $\nu_N \in M_N$. (In order to apply the results of [4], one has to switch from right coalgebras to left coalgebras by using the $*$-algebra automorphism and coalgebra antihomomorphism $\theta : \mathcal{O}(SU_q(2)) \to \mathcal{O}(SU_q(2))$ determined by $\theta(\alpha) = \alpha$, $\theta(\delta) = \delta$, $\theta(\beta) = -q^{-1} \beta$.)

For $l \in \frac{1}{2} \mathbb{N}_0$ and $j, k = -l, -l + 1, \ldots, l$, let $t_{j,k}$ denote the matrix elements of finite dimensional unitary corepresentations of $\mathcal{O}(SU_q(2))$. By the explicit description of the matrix elements (see, e.g., [19, Section 4.2.4]), one sees that $u_{\pm 2l} \in \text{span}\{t_{j,k}^l : j = -l, -l + 1, \ldots, l\}$. In particular, $u_{\pm 2l}$ is a highest weight vector of an irreducible spin-$l$ corepresentation. That is, there are elements $v_{k,\pm t}^l$, $k = -l, -l + 1, \ldots, l$, such that $\Delta(v_{k,\pm t}^l) = \sum_j v_{j,\pm t}^l \otimes t_{j,k}^l$ and $v_{k,\pm t}^l = c_{\pm t} u_{\pm 2l}$ with positive constants $c_{\pm t}$ determined as follows: Since $t_{j,k}^l$ are matrix elements of unitary corepresentations, we have $\Delta(\sum_k v_{k,\pm t}^{l*} v_{k,\pm t}^l) = (\sum_k v_{k,\pm t}^{l*} v_{k,\pm t}^l) \otimes 1$, so the sum belongs to the trivial corepresentation. Thus we find $c_{\pm t} \in \mathbb{R}_+$ such that $\sum_k v_{k,\pm t}^{l*} v_{k,\pm t}^l = 1$.

Next, define
\[
V_{\pm 2l} := (v_{-l,\pm t}^l, \ldots, v_{l,\pm t}^l)^t \quad \text{and} \quad P_{\pm 2l} = V_{\pm 2l} V_{\pm 2l}^*,
\]
where the superscript $t$ denote the transpose. As $V_{\pm 2l} P_{\pm 2l}$ is a self-adjoint idempotent. It has been shown in [21] that the entries of $P_{\pm 2l}$ belong to $\mathcal{O}(S^2_{qs})$ and that $M_N \cong \mathcal{O}(S^2_{qs})^{\otimes |N|+1} P_N$, where $N = \pm 2l \in \mathbb{Z}$.

Let $\mathcal{C}(SU_q(2))$ denote the C*-completion of $\mathcal{O}(SU_q(2))$ with respect to the universal C*-norm. Clearly, each *-representation of $\mathcal{O}(SU_q(2))$ restricts to one of $\mathcal{O}(S^2_{qs})$. On the other hand, the faithful representation $\rho_- \oplus \rho_+ \circ \varphi$ of $\mathcal{O}(S^2_{qs})$ on $L^2(N_0) \oplus L^2(N_0)$ can be realized as a subrepresentation of a *-representation of $\mathcal{O}(SU_q(2))$, for instance, of the GNS-representation associated with the Haar state on $\mathcal{O}(SU_q(2))$ (see, e.g., [21, Section 6]). From the definition of the universal C*-norm and the isomorphism $\mathcal{C}(S^2_{qs}) \ni \xi \mapsto (\rho_- \oplus \rho_+)(\xi) \in (\rho_- \oplus \rho_+)(\mathcal{C}(S^2_{qs}))$, it follows that the embedding is an isometry. Thus $\mathcal{C}(S^2_{qs})$ can be viewed as a subalgebra of $\mathcal{C}(SU_q(2))$. 

6
2.4 Two auxiliary lemmas

The first lemma is used to handle expressions like \( f(\zeta_s)\eta_s = \eta_s f(q^2 \zeta_s) \) and \( f(\zeta_s)S = SF(q^2 \zeta_s) \).

**Lemma 2.1.** Given a unital \( C^* \)-algebra \( A \), let \( A, B, C \in A \) such that \( A^* = A \), \( B^* = B \) and \( AC = CB \). Then \( f(A)C = Cf(B) \) for any continuous function \( f : \text{spec}(A) \cup \text{spec}(B) \to \mathbb{C} \).

**Proof.** From \( AC = CB \), it follows that \( p(A)C = Cp(B) \) for any polynomial \( p \) in one indeterminate. Approximating \( f \) uniformly by polynomials on the compact set \( \text{spec}(A) \cup \text{spec}(B) \) gives the result since the multiplication on \( A \) is continuous.

The second lemma summarizes some crucial relations between the generators of \( \mathcal{O}(S_u^2) \) and highest/lowest weight vectors of \( M_N \). Before stating the lemma, recall the definition of the highest weight vectors \( u_N \in M_N \) from the previous subsection. Now, for \( N \in \mathbb{Z} \), set

\[
\begin{align*}
  w_N & := (\alpha - qN \gamma)(\alpha - q^2 \gamma) \cdots (\alpha - q^N \gamma), \quad N > 0, \\
  w_N & := (-q)^{-N}(\gamma + q^{-1} \gamma)(\gamma + q^{-2} \gamma) \cdots (\gamma + q^{-N} \gamma), \quad N < 0,
\end{align*}
\]

and \( w_0 = 1 \). Applying the coaction \( \Delta \) on \( u_N \) and taking only the elements with \( \alpha \) and \( \gamma \) in the left tensor factor, one sees that \( w_N \) is a lowest weight vector belonging to the same corepresentation as \( u_N \). In particular, \( w_{\pm 2l} = d_{\pm 2l} v_{-l, \pm l} \) for some non-zero real constant \( d_{\pm 2l} \), where \( l \in \frac{1}{2} \mathbb{N}_0 \).

**Lemma 2.2.** For \( n \in \mathbb{N}_0 \),

\[
\begin{align*}
  u_n \zeta q^n & = \zeta u_n, \quad w_n \zeta q^n & = q^{2n} \zeta w_n, \\
  u_{-n} \zeta q^n & = q^{-2n} \zeta u_{-n}, \quad w_{-n} \zeta q^n & = \zeta w_{-n}, \\
  u_n \eta q^n & = q^n \eta u_n, \quad u_{-n} \eta q^n & = q^{-n} \eta u_{-n};
\end{align*}
\]

\[
\begin{align*}
  u_n u_n^* & = q^{-2n}(q^2 s^2 + \zeta)(q^4 s^2 + \zeta) \cdots (q^{2n} s^2 + \zeta), \\
  w_n w_n^* & = (1 - \zeta)(1 - q^2 \zeta) \cdots (1 - q^{2(n-1)} \zeta), \\
  u_{-n} u_{-n}^* & = (1 - q^{-2} \zeta)(1 - q^{-4} \zeta) \cdots (1 - q^{-2n} \zeta), \\
  w_{-n} w_{-n}^* & = (q^2 s^2 + \zeta)(q^{-2} s^2 + \zeta) \cdots (q^{-2(n-1)} s^2 + \zeta), \\
  u_n w_n^* & = q^{n(n-1)/2} \eta^n, \quad u_{-n} w_{-n}^* = q^{-n(n-1)/2} \eta_{-n}^n;
\end{align*}
\]

\[
\begin{align*}
  u_n^* u_n & = q^{-2n}(q^2 s^2 + \zeta q^n)(q^4 s^2 + \zeta q^n) \cdots (q^{2n} s^2 + \zeta q^n), \\
  w_n^* w_n & = (1 - q^{-2} \zeta q^n)(1 - q^{-4} \zeta q^n) \cdots (1 - q^{-2n} \zeta q^n), \\
  u_{-n}^* u_{-n} & = (1 - \zeta q^{-n})(1 - q^2 \zeta q^{-n}) \cdots (1 - q^{2(n-1)} \zeta q^{-n}), \\
  w_{-n}^* w_{-n} & = (s^2 + \zeta q^{-n})(q^{-2} s^2 + \zeta q^{-n}) \cdots (q^{-2(n-1)} s^2 + \zeta q^{-n}).
\end{align*}
\]

**Proof.** The lemma is proved by direct computations using the embedding \( q \), the commutation relations in \( \mathcal{O}(SU_q(2)) \), and induction on \( n \).
3 Fibre product approach to line bundles over the generic Podleś spheres

The isomorphism (7) admits a geometric interpretation as describing the generic Podleś spheres by gluing two quantum discs along their boundaries. To see this, note that \( \text{spec}(\rho_- (\zeta_s)) \subseteq [-s^2 q^2, 0] \) and \( \text{spec}(\rho_+ (\zeta_s)) \subseteq [0, q^2] \). As a consequence, \((1 - \rho_- (\zeta_s))\) and \((s^2 + \rho_+ (\zeta_s))\) are strictly positive operators with bounded inverses. Let \( z_- \) and \( z_+ \) be defined by “stereographic projection”:

\[
    z_- := s^{-1} \rho_- (\eta_s) (1 - \rho_- (\zeta_s))^{-1/2}, \quad z_+ := \rho_+ (\eta_s) (s^2 + \rho_+ (\zeta_s))^{-1/2}.
\]  

(25)

Then one readily checks that \( z_- \) and \( z_+ \) satisfy the defining relation (4) of the quantum disc and therefore generate \( T \), the \( C^* \)-algebra of \( O(D^2_{q}) \). The generators \( \rho_{\pm} (\zeta_s) \) and \( \rho_{\pm} (\eta_s) \) can be recovered from \( z_{\pm} \) since

\[
    \rho_- (\zeta_s) = (q^{-2} - 1)^{-1} s^2 (z_- z_-^* - z_+ z_-), \quad \rho_+ (\zeta_s) = (q^{-2} - 1)^{-1} (z_+ z_-^* - z_+ z_+).
\]

Thus we can view the two copies of \( T \) in (7) as algebras of continuous functions on the quantum disc derived from continuous functions on the northern hemisphere \( \zeta_s \geq 0 \) and the southern hemisphere \( \zeta_s \leq 0 \) by “stereographic projection”.

Furthermore, the isomorphism (7) shows that \( C(S^2_{q^2}) \) is obtained as the fibre product of two Toeplitz algebras by the following pull-back diagram:

\[
    \begin{array}{ccc}
    C(S^2_{q^2}) & \longrightarrow & T \\
    \downarrow^{\text{pr}_1} & & \downarrow^{\sigma} \\
    T & \longrightarrow & C(S^1). \\
    \end{array}
\]  

(26)

In the classical case \( q = 1 \), the surjection \( \sigma : T \rightarrow C(S^1) \) corresponds to an embedding of the circle \( S^1 \) into the closed disc \( D^2 \), and the relation \( \sigma(a_0) = \sigma(a_1) \) in (7) means that a pair of continuous functions on the northern and southern hemispheres is identified with a continuous function on the 2-sphere iff their restrictions to \( S^1 \) coincide. This identification captures precisely the meaning of gluing two (quantum) discs along their boundaries.

Recall that complex line bundles with winding number \( N \in \mathbb{Z} \) over the classical 2-sphere can be constructed by taking trivial bundles over the northern and southern hemispheres and gluing them together along the boundary via the map \( U^N : S^1 \rightarrow S^1, \ U^N (e^{i\phi}) = e^{iN\phi} \).

Our aim is to give a non-commutative analogue of this construction. The fibre product (26) tells us that the trivial bundles over the discs \( D^2 \) should be replaced by \( T \), whereas the transition map \( U^N \) remains the same. This leads
to the following pull-back diagram:

\[
\begin{array}{ccc}
T \times (U^N_\sigma, \sigma) & \overset{\text{pr}_1}{\rightarrow} & T \\
\sigma & \downarrow & \sigma \\
C(S^1) & \overset{f \rightarrow U^N f}{\rightarrow} & C(S^1),
\end{array}
\]

where

\[
T \times (U^N_\sigma, \sigma) \cong \{(a_0, a_1) \in T \oplus T : U^N \sigma(a_0) = \sigma(a_1)\}.
\]  

(27)

Thinking of \(T\) and \(C(S^1)\) as left modules over themselves, it follows from the discussion in the end of Section 2.1 that \(T \times (U^N_\sigma, \sigma)\) is a left \(C(S^2_{qs})\)-module. This can also be seen directly from Equations (7) and (27).

The next proposition gives a projective module description of \(T \times (U^N_\sigma, \sigma)\).

**Proposition 3.1.** For \(N \in \mathbb{Z}\), let \(L_N := T \times (U^N_\sigma, \sigma)\), and define

\[
E_N := (S^N S^* S^N, 1), \quad N \geq 0, \quad E_N := (1, S^{|N|} S^{|N|}), \quad N < 0.
\]  

(28)

Then the left \(C(S^2_{qs})\)-modules \(L_N\) and \(C(S^2_{qs})E_N\) are isomorphic.

**Proof.** Clearly, \(E_N\) is a projection in \(C(S^2_{qs})\) since \(\sigma(S^n S^{*n}) = 1 = \sigma(1)\) and \(S^n S^n = 1\) for all \(n \in \mathbb{N}_0\).

Let \(N \geq 0\). From \(\sigma(S^*) = U^*, \) (7) and (27), it follows that \((a_0 S^N, a_1)\) belongs to \(C(S^2_{qs})\) for all \((a_0, a_1) \in L_N\). Consider now the \(C(S^2_{qs})\)-linear map

\[
\Psi_N : L_N \rightarrow C(S^2_{qs})E_N, \quad \Psi_N((a_0, a_1)) := (a_0 S^N, a_1) E_N.
\]

We claim that \(\Psi_N\) is an isomorphism. Note that \(\Psi_N((a_0, a_1)) = (a_0 S^N, a_1)\) since, as above, \(S^N S^N = 1\). Assume that \((a_0, a_1) \in \text{Ker}(\Psi_N)\). Then \(0 = (a_0 S^N, a_1)\). Hence \(a_1 = 0\) and, as \(S^N\) is right invertible, \(a_0 = 0\). Therefore \(\text{Ker}(\Psi_N) = \{0\}\), so \(\Psi_N\) is injective.

Next, let \((a_0, a_1) \in C(S^2_{qs})\). Then \((a_0 S^N, a_1) \in L_N\) by (7) and (27), and \(\Psi_N((a_0 S^N, a_1)) = (a_0 S^N S^N, a_1) = (a_0, a_1) E_N\). Therefore \(\Psi_N\) is also surjective.

For \(N < 0\), one proves analogously that

\[
\Psi_N : L_N \rightarrow C(S^2_{qs})E_N, \quad \Psi_N((a_0, a_1)) := (a_0, a_1 S^{|N|}) E_N,
\]

is an isomorphism. \(\square\)
4 Projective module descriptions of line bundles associated to the generic Hopf fibration

In Section 2.2 we explained that the quantum line bundles $M_N$ from the generic Hopf fibration of $SU_q(2)$ are, as left $\mathcal{O}(S^2_q)$-modules, isomorphic to $\mathcal{O}(S^2_q)^{|N|+1}P_N$, where $P_N$ was given in (11). As the entries of $P_N$ belong to $\mathcal{O}(S^2_q)$, these projections represent $K_0$-classes of $\mathcal{C}(S^2_q)$. The aim of this section is to prove that $P_N$ and the 1-dimensional projection $E_N$ from Equation (28) define the same class in $K_0$-theory. As an intermediate step, we first reduce the $|N|+1 \times |N|+1$-projections $P_N$ to Murray-von Neumann equivalent $2 \times 2$-projections by assembling the corners of $P_N$. These $2 \times 2$-projections are quantum analogues of the classical Bott projections. After that, we prove the Murray-von Neumann equivalence of the “Bott projections” and $E_N$. Note that the 1-dimensional projections $E_N$, $N \neq 0$, do not have classical counterparts since $S^2$ is connected and therefore all projections in $\mathcal{C}(S^2)$ are trivial.

Let $l \in \frac{1}{2}\mathbb{N}$. Recall from the Preliminaries that $P_{\pm 2l} = V_{\pm 2l}V_{\mp 2l}^*$, where $V_{\pm 2l} = (v_{l, \pm 1}^t, \ldots, v_{l, \pm l}^t)^t$ and that $v_{l, \pm 1}^t \sim u_{\pm 2l}$ and $v_{l, \pm l}^t \sim w_{\pm 2l}$ with $u_{\pm 2l}$ and $w_{\pm 2l}$ given in Equations (10) and (12), respectively. By Lemma 2.2 we have, for $n \in \mathbb{N}$,

$$u_{-n}^*u_{-n} + q^{n(n-1)}w_{-n}^*w_{-n} = \Pi_{k=0}^{n-1} (1 - q^{2k}\zeta_{q^{-n}}) + \Pi_{k=0}^{n-1} (s^2 + q^{2k}\zeta_{q^{-n}}),$$

$$q^{-n(n-1)}u_n^*u_n + w_n^*w_n = \Pi_{k=0}^{n-1} (s^2 + q^{2k-2n}\zeta_{q^{-n}}) + \Pi_{k=0}^{n-1} (1 - q^{2k-2n}\zeta_{q^{-n}}).$$

(29)

Here, the constants $q^{\pm n(n-1)}$ were inserted in order to obtain more symmetric formulas. Using (3), one easily shows that $u_{-n}^*u_{-n} + q^{n(n-1)}w_{-n}^*w_{-n}$ and $q^{-n(n-1)}u_n^*u_n + w_n^*w_n$ are strictly positive operators with bounded inverses in $\mathcal{C}(SU_q(2))$. Thus we can define the following 2-vectors with entries in $\mathcal{C}(SU_q(2))$:

$$W_{-n} := (u_{-n}, -q^{n(n-1)/2}w_{-n})^t(u_{-n}^*u_{-n} + q^{n(n-1)}w_{-n}^*w_{-n})^{-1/2},$$

$$W_n := (q^{-n(n-1)/2}u_n, w_n)^t(q^{-n(n-1)}u_{-n}^*u_{-n} + w_n^*w_n)^{-1/2}. $$

(30)

Obviously, $W_{\pm n}^*W_{\pm n} = 1$, so that

$$Q_{\pm n} := W_{\pm n}^*W_{\pm n}$$

(31)

is a self-adjoint idempotent.

**Lemma 4.1.** Let $n \in \mathbb{N}$. With $t$ an indeterminate, define

$$f_n(t) := \Pi_{k=0}^{n-1} (1 - q^{2k}t), \quad g_n(t) := \Pi_{k=0}^{n-1} (s^2 + q^{2k}t).$$

(32)

Set $\mathcal{S} := (S, S) \in \mathcal{C}(S^2_q)$, where $S$ denotes the shift operator on $\ell^2(\mathbb{N}_0)$. Then

$$Q_{-n} = \begin{pmatrix}
\frac{f_n(q^{-2n}\zeta)}{f_n(q^{-2n}\zeta) + g_n(q^{-2n}\zeta)} & -\sqrt{f_n(q^{-2n}\zeta)g_n(q^{-2n}\zeta)} \\
\sqrt{f_n(q^{-2n}\zeta)g_n(q^{-2n}\zeta)} & \frac{g_n(q^{-2n}\zeta)}{f_n(q^{-2n}\zeta) + g_n(q^{-2n}\zeta)}
\end{pmatrix}^n,$$

where $\zeta$ is an indeterminate, define
\[
Q_n = \begin{pmatrix}
\frac{g_n(q^{-2n} \zeta_r)}{f_n(q^{-2n} \zeta_r) + g_n(q^{-2n} \zeta_r)} & \sqrt{f_n(q^{-2n} \zeta_r) g_n(q^{-2n} \zeta_r)} S_n^* \\
\sqrt{f_n(\zeta_r) g_n(\zeta_r)} S_n & \frac{f_n(\zeta_r)}{f_n(\zeta_r) + g_n(\zeta_r)}
\end{pmatrix}.
\]

In particular, the entries of \(Q_{\pm n}\) belong to \(C(S^2_{qs})\).

**Proof.** The entries \((Q_{\pm n})_{ij}\), \(i, j = 1, 2\), of the \(2 \times 2\) matrices \(Q_{\pm n}\) are easily computed by using Lemmas 2.1 and 2.2. For instance,

\[
(Q_n)_{12} = q^{-n(n-1)/2} u_n(q^n(n-1) u_n^* u_n + w_n^* w_n)^{-1} w_n^* \\
= q^{-n(n-1)/2} u_n(\Pi_{k=0}^{n-1} (s^2 + q^{2k-2n} \zeta_n) + \Pi_{k=0}^{n-1} (1 - q^{2k-2n} \zeta_n))^{-1} w_n^* \\
= q^{-n(n-1)/2} (\Pi_{k=0}^{n-1} (s^2 + q^{2k-2n} \zeta_n) + \Pi_{k=0}^{n-1} (1 - q^{2k-2n} \zeta_n))^{-1} u_n w_n^* \\
= (f_n(q^{-2n} \zeta_n) + g_n(q^{-2n} \zeta_n))^{-1} \eta_n^*,
\]

where we used (29) in the second equality, (13) in the third, and (20) and (32) in the fourth. From the second relation in (3) and the identification \(\eta_n \mapsto (\rho_-(\eta_n), \rho_+(\eta_n)) \in C(S^2_{qs})\), it follows that the polar decomposition of \(\eta_n\) is given by \(\eta_n = \mathcal{S}(1 - \zeta_n)(s^2 + \zeta_n)\). Since \(\mathcal{S}\zeta_n = q^{-2} \zeta_n \mathcal{S}\), Lemma 2.1 implies that \(\eta_n^* = \sqrt{f_n(q^{-2n} \zeta_n) g_n(q^{-2n} \zeta_n)} \mathcal{S}^n\). Inserting the last equation into the above expression for \((Q_n)_{12}\) gives the result.

Clearly, \(\mathcal{S}^n\) and \(\mathcal{S}^n\) belong to \(C(S^2_{qs})\). Thus, to prove the last statement of the lemma, it suffices to show that the functions in \(\zeta_n\) are continuous on \(\text{spec}(\zeta_n)\). This can easily be verified by observing that the denominators are non-zero. \(\square\)

**Remark 4.2.** Since \(u_{\pm n} w_{\pm n}^* \sim \eta_n^*\), the entries of \(Q_{\pm n}\) are actually rational functions in the generators of \(\mathcal{O}(S^2_{qs})\). This is in analogy to the classical Bott projections (see, e.g., [2, Section 2.6]).

Recall the definitions of \(P_N\) and \(E_N\) in Equations (11) and (28), respectively. For convenience of notation, set \(Q_0 := (1, 0)^t(1, 0)\). In the next proposition, we shall show that, for fixed \(N \in \mathbb{Z}\), the projections \(P_N\), \(E_N\) and \(Q_N\) are Murray-von Neumann equivalent.

**Proposition 4.3.** For all \(N \in \mathbb{Z}\), the projections \(P_N\), \(E_N\) and \(Q_N\) belong to the same class in \(K_0(C(S^2_{qs}))\).

**Proof.** The case \(N = 0\) is trivial. For \(N \neq 0\), set

\[
X_N := V_N W_N^*.
\]

where \(V_N\) and \(W_N\) are defined in Equations (11) and (30), respectively. Since \(V_N^* V_N = 1\) and \(W_N^* W_N = 1\), we have \(X_N X_N^* = P_N\) by (11) and \(X_N^* X_N = Q_N\) by (31). To prove the Murray-von Neumann equivalence of \(P_N\) and \(Q_N\), it remains to verify that the entries of \(X_N\) belong to \(C(S^2_{qs})\). To this end, recall that \(u_{2l} \sim v_{l,l}^I\) and \(w_{2l} \sim v_{-l,-l}^I\) for \(l \in \frac{1}{2} \mathbb{N}\), and that \(v_{l,k,l}^I v_{k,l}^I \in \mathcal{O}(S^2_{qs})\) since \(P_{2l}\)
belongs to Mat$_{2n+1,2n+1}(\mathcal{O}(S^2_{qs}))$. Let $r_{2l}((\xi_1)_{2l}) := (q^{-l}(2l-1)u_{2l}^*w_{2l} + w_{2l}^*u_{2l})^{-1}$. Then, by Lemma 2.1, Equation (13) and the foregoing,

\[ (X_{2l})_{j,l} \sim v_{j,l}^* r_{2l}(\xi_{2l}^*) u_{2l}^* \sim v_{j,l}^* v_{j,l}^* r_{2l}(\xi_{2l}^*) \in \mathcal{C}(S^2_{qs}), \]

\[ (X_{2l})_{j,2l} \sim v_{j,l}^* r_{2l}(\xi_{2l}^*) w_{2l}^* \sim v_{j,l}^* v_{j,l}^* r_{2l}(q^2 \xi_{2l}^*) \in \mathcal{C}(S^2_{qs}), \]

hence $X_N \in \text{Mat}_{n+1,2n}(\mathcal{C}(S^2_{qs}))$ for all $N > 0$. A similar argumentation yields $X_N \in \text{Mat}_{N+1,2n}(\mathcal{C}(S^2_{qs}))$ for $N < 0$.

Next, for $n \in \mathbb{N}$, define

\[ Y_n := \left( \frac{\sqrt{g_n(q^{-2n}\xi_n)}}{f_n(q^{-2n}\xi_n) + g_n(q^{-2n}\xi_n)} \mathbb{S}^n, \frac{\sqrt{f_n(q^{-2n}\xi_n)}}{f_n(q^{-2n}\xi_n) + g_n(q^{-2n}\xi_n)} \mathbb{S}^n \right), \]

\[ Y_{-n} := \left( \frac{\sqrt{g_n(q^{-2n}\xi_n)}}{f_n(q^{-2n}\xi_n) + g_n(q^{-2n}\xi_n)} \mathbb{S}^n, \frac{\sqrt{f_n(q^{-2n}\xi_n)}}{f_n(q^{-2n}\xi_n) + g_n(q^{-2n}\xi_n)} \mathbb{S}^n \right). \]

By the same arguments as in the proof of Lemma 4.1, it follows that the entries of $Y_{\pm n}$ belong to $\mathcal{C}(S^2_{qs})$. Since, by Lemma 2.1, $\mathbb{S}^n f(\xi_n) = f(q^2 \xi_n) \mathbb{S}^n$ for all continuous functions $f$ on $\text{spec}(\xi_n)$, and since $\mathbb{S}^n \mathbb{S}^n = 1$, one sees immediately that $Y_{\pm n}^* Y_{\pm n} = Q_{\pm n}$. Thus, the proposition will be proved, if we show that $Y_{\pm n}^* Y_{\pm n} = E_{\pm n}$.

Observe that $\rho_-(g_n(q^{-2n}\xi_n))e_k = 0$ and $\rho_+(f_n(q^{-2n}\xi_n))e_k = 0$ for all $k = 0, \ldots, n-1$, and that

\[ \rho_-(g_n(q^{-2n}\xi_n)) e_j = f_n(-s^2 q^{2(j+1-n)}) e_j \neq 0, \]

\[ \rho_+(f_n(q^{-2n}\xi_n)) e_j = g_n(q^{2(j+1-n)}) e_j \neq 0, \quad j \in \mathbb{N}_0. \]

Using the identification $h \mapsto (\rho_-(h), \rho_+(h))$ for elements $h \in \mathcal{C}(S^2_{qs})$, one concludes that

\[ \frac{g_n(q^{-2n}\xi_n)}{f_n(q^{-2n}\xi_n) + g_n(q^{-2n}\xi_n)}(1 - S^n S^{*n}, 1 - S^n S^{*n}) = (0, 1 - S^n S^{*n}), \]

\[ \frac{f_n(q^{-2n}\xi_n)}{f_n(q^{-2n}\xi_n) + g_n(q^{-2n}\xi_n)}(1 - S^n S^{*n}, 1 - S^n S^{*n}) = (1 - S^n S^{*n}, 0). \]

Since $\mathbb{S}^n \mathbb{S}^n = (S^n S^{*n}, S^n S^{*n})$, we get

\[ Y_n Y_n^* = \left( \frac{g_n(q^{-2n}\xi_n)}{f_n(q^{-2n}\xi_n) + g_n(q^{-2n}\xi_n)} \mathbb{S}^n \mathbb{S}^n + \frac{f_n(q^{-2n}\xi_n)}{f_n(q^{-2n}\xi_n) + g_n(q^{-2n}\xi_n)}(1 - \mathbb{S}^n \mathbb{S}^n) \right) \mathbb{S}^n \mathbb{S}^n \]

and similarly $Y_{-n}^* Y_{-n} = E_{-n}$. This concludes the proof. \hfill \Box

From Proposition 3.11 and the proof of Proposition 4.2, we obtain the following isomorphisms of left $\mathcal{C}(S^2_{qs})$-modules:

\[ L_N := T \times_{(\mathbb{N} \sigma, T)} T \cong \mathcal{C}(S^2_{qs}) \mathcal{E}_N \cong \mathcal{C}(S^2_{qs})^2 Q_N \cong \mathcal{C}(S^2_{qs})^{\lfloor N \rfloor + 1} P_N, \quad N \in \mathbb{Z}. \]
The common $K_0$-class will be denoted by $[L_N]$. 

Classically, the direct sum of two line bundles over the $2$-sphere with winding number $N$ and $-N$ is isomorphic to a trivial rank $2$ bundle. An analogous result holds in the quantum case:

**Corollary 4.4.** The direct sum of the line bundles $L_N$ and $L_{-N}$ is isomorphic to a free rank $2$ module.

**Proof.** For $N \neq 0$, we have obviously $Q_N + Q_{-N} = 1 \in \text{Mat}_{2,2}(C(S^2_{qs}))$, so $Q_N$ and $Q_{-N} = 1 - Q_N$ are complementary projections in $\text{Mat}_{2,2}(C(S^2_{qs}))$. Therefore $L_N \oplus L_{-N} \cong C(S^2_{qs})^2 Q_N \oplus C(S^2_{qs})^2 Q_{-N} \cong C(S^2_{qs})^2$. The case $N = 0$ is trivial. \(\square\)

5  Index computation for quantum line bundles

Let $A$ be a C*-algebra, $p \in \text{Mat}_{n,n}(A)$ a projection, and $\varrho_+$ and $\varrho_-$ *-representations of $A$ on a Hilbert space $\mathcal{H}$ such that $[(\varrho_+, \varrho_-)] \in K^0(A)$. If the following traces exist, then the formula

$$\langle [(\varrho_+, \varrho_-)], [p] \rangle = \text{tr}_\mathcal{H}(\text{tr}_{\text{Mat}_{n,n}}(\varrho_+ - \varrho_-)(p))$$

(35)

computes the index of the Fredholm operator $\varrho_+(p)\varrho_-(p) : \varrho_- (p)\mathcal{H}^n \to \varrho_+(p)\mathcal{H}^n$ and therefore yields a pairing between $K^0(A)$ and $K_0(A)$ \(\square\) Section IV.1.

In general, the difficulty of computing the traces increases with the growing size of the matrices. However, the fibre product approach to quantum line bundles provided us with $1$-dimensional projections as representatives of $K_0$-classes. Their simple form makes them very suitable for the calculation of the index pairing.

To compute the Fredholm indices (Chern numbers) for quantum line bundles, we shall use the generators $[(\varepsilon_+, \varepsilon_-)]$ and $[(\rho_+, \rho_-)]$ of $K^0(C(S^2_{qs}))$ defined in the end of Section 2.2.

**Proposition 5.1.** The pairing between the generators of $K^0(C(S^2_{qs}))$ and the $K_0$-class of the quantum line bundle $L_N$, $N \in \mathbb{Z}$, yields

$$\langle [(\varepsilon_+, \varepsilon_-)], [L_N] \rangle = 1, \quad \langle [(\rho_+, \rho_-)], [L_N] \rangle = N.$$

**Proof.** As already mentioned, we choose the $1$-dimensional projection $E_N$ as a representative of the $K_0$-class $[L_N]$. Recall from Section 2.2 that $\pi_+(1) = 1$ and $\pi_-(1)$ is the projection onto $\text{span}\{e_k : k \in \mathbb{Z} \setminus \{0\}\}$. Thus, $(\pi_+ - \pi_-)(1)$ is the projection onto $\mathbb{C}e_0$. Moreover, $\sigma((S^nS^{*n}, 1)) = \sigma((1, S^nS^{*n})) = 1$ for all $n \in \mathbb{N}_0$, therefore $\sigma(E_N) = 1$. Since $\varepsilon_\pm = \pi_\pm \circ \sigma$, we get

$$\langle [(\varepsilon_+, \varepsilon_-)], [L_N] \rangle = \text{tr}_{\mathcal{H}(\mathbb{Z})}(\pi_+ - \pi_-)(\sigma(E_N)) = \text{tr}_{\mathcal{H}(\mathbb{Z})}(\pi_+ - \pi_-)(1) = 1.$$

Let $N \geq 0$. By (8), $(\rho_+ - \rho_-)(E_N) = (\rho_+ - \rho_-)(S^NS^{*N}, 1) = 1 - S^NS^{*N}$ is the projection onto $\text{span}\{e_0, \ldots, e_{N-1}\}$. Thus

$$\langle [(\rho_+, \rho_-)], [L_N] \rangle = \text{tr}_{\mathcal{H}(\mathbb{N}_0)}(\rho_+ - \rho_-)(E_N) = \text{tr}_{\mathcal{H}(\mathbb{N}_0)}(1 - S^NS^{*N}) = N.$$
If $N < 0$, we get $(\rho_+ - \rho_-)(E_N) = (\rho_+ - \rho_-)(1, S^{|N|}S^*|N|) = S^{|N|}S^*|N| - 1$, again by (8). Hence
\[
\langle ([\rho_+, \rho_-]), [L_N] \rangle = \text{tr}_{\ell^2(N_0)}(\rho_+ - \rho_-)(E_N) = -\text{tr}_{\ell^2(N_0)}(1 - S^{|N|}S^*|N|) = -|N|.
\]
Therefore $\langle ([\rho_+, \rho_-]), [L_N] \rangle = N$ for all $N \in \mathbb{Z}$.

We remark that the additive map $\langle ([\varepsilon_+, \varepsilon_-]), [\cdot] \rangle : K_0(C(S^2_{q_s})) \to \mathbb{Z}$ can be used to detect the rank of the bundle (equal to 1 for line bundles). A similar statement was made in [11, Remark 3.4] for the standard Podleś sphere.

The Chern number $\langle ([\rho_+, \rho_-]), [L_N] \rangle$ coincides with the power of $U$ in (27) and thus computes the “winding number”, i.e., the number of rotations along the equator of the two glued quantum discs. That the pairing between $([\rho_+, \rho_-])$ and the $K_0$-class of $P_N$ gives the winding number $N$ has been shown before in [12], relying heavily on the index theorem. The advantage of our fibre product approach is that we are able to compute the index pairing directly by finding projections of small matrix size.

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