An overdetermined problem in Riesz-potential and fractional Laplacian

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Abstract. The main purpose of this paper is to address two open questions raised by W. Reichel in \([R2]\) on characterizations of balls in terms of the Riesz potential and fractional Laplacian. For a bounded \(C^1\) domain \(\Omega \subset \mathbb{R}^N\), we consider the Riesz-potential

\[
u(x) = \int_{\Omega} \frac{1}{|x-y|^{N-\alpha}} dy
\]

for \(2 \leq \alpha \neq N\). We show that \(u = \text{constant on } \partial \Omega\) if and only if \(\Omega\) is a ball. In the case of \(\alpha = N\), the similar characterization is established for the logarithmic potential \(u(x) = \int_{\Omega} \log \frac{1}{|x-y|} dy\). We also prove that such a characterization holds for the logarithmic Riesz potential

\[
u(x) = \int_{\Omega} |x-y|^\alpha \log \frac{1}{|x-y|} dy
\]

when the diameter of the domain \(\Omega\) is less than \(e^{\frac{\alpha}{N}}\) in the case when \(\alpha - N\) is a non-negative even integer. This provides a characterization for the overdetermined problem of the fractional Laplacian. These results answer two open questions in \([R2]\) to some extent.

1. Introduction

It is well-known that the gravitational potential of a ball of constant mass density is constant on the surface of the ball. It is shown by Fraenkel \([Fr]\) that this property indeed provides a characterization of balls. In fact, Fraenkel proves the following

**Theorem A \([Fr]\):** Let \(\Omega \subset \mathbb{R}^N\) be a bounded domain and \(\omega_N\) be the surface measure of the unit sphere in \(\mathbb{R}^N\). Consider

\[
u(x) = \begin{cases} \frac{1}{2\pi} \int_{\Omega} \log \frac{1}{|x-y|} dy, & N = 2, \\ \frac{1}{(N-2)\omega_N} \int_{\Omega} \frac{1}{|x-y|^N} \log \frac{1}{|x-y|} dy, & N \geq 3. \end{cases}
\]

If \(u(x)\) is constant on \(\partial \Omega\), then \(\Omega\) is a ball.

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This result has been extended by Reichel [R2] to more general Riesz potential, but under a more restrictive assumption on the domain Ω, i.e., Ω is assumed to be convex. In [R2], Reichel considers the integral equation

\[
    u(x) = \begin{cases} 
        \int_{\Omega} \log \frac{1}{|x-y|} \, dy, & N = \alpha, \\
        \int_{\Omega} \frac{1}{|x-y|^{N-\alpha}} \, dy, & N \neq \alpha,
    \end{cases}
\]

and proves the following theorem.

**Theorem B [R2]**: Let \( \Omega \subset \mathbb{R}^N \) be a bounded convex domain and \( \alpha > 2 \), if \( u(x) \) is constant on \( \partial \Omega \), then \( \Omega \) is a ball.

This more general Riesz potential is actually closely related to the fractional Laplacian \( (-\Delta)^{\frac{\alpha}{2}} \) in \( \mathbb{R}^N \). Let \( N_0 \) be the collection of nonnegative integers. It is known that the fundamental solution \( G(x,y) \) for pseudo-differential operator \( (-\Delta)^{\frac{\alpha}{2}} \) in \( \mathbb{R}^N \) has the following representation

\[
    G(x,y) = \begin{cases} 
        \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{2^\alpha \pi^{\frac{N}{2}} \Gamma\left(\frac{\alpha}{2}\right)} |x-y|^{\alpha-N}, & \text{if } \frac{\alpha-N}{2} \notin N_0, \\
        \frac{(-1)^k}{2^{\alpha-1} \pi^{\frac{N}{2}} \Gamma\left(\frac{\alpha}{2}\right)} |x-y|^{\alpha-N} \log \frac{1}{|x-y|}, & \text{if } \frac{\alpha-N}{2} \in N_0.
    \end{cases}
\]

We note that for the case of \( \alpha = 2 \), Fraenkel’s result is under weaker assumption on the domain \( \Omega \), namely, \( \Omega \) only needs to be bounded and open in \( \mathbb{R}^N \). The surprising part for \( \alpha = 2 \) is that there is neither regularity nor convexity requirement for \( \Omega \). Thus, two open problems were raised by Reichel in [R2].

**Question 1.** Is Theorem B true if we remove the convexity assumption of \( \Omega \)?

**Question 2.** Is there an analogous result as Theorem B for Riesz-Potential of the form

\[
    u(x) = \int_{\Omega} |x-y|^{\alpha-N} \log \frac{1}{|x-y|} \, dy?
\]

It is meaningful to study (1.4) because in the case of \( \frac{\alpha-N}{2} \in N_0 \), up to some rescaling, the kernel function in above integral is the fundamental solution of the fractional Laplacian \( (-\Delta)^{\frac{\alpha}{2}} \).

Our goal is to address the above two open questions.

The first result we establish does remove the convexity assumption in Theorem B.

**Theorem 1.** Let \( \Omega \) be a \( C^1 \) bounded domain. If \( u \) in (1.2) is constant on \( \partial \Omega \), then \( \Omega \) is a ball.

As far as Question 2 is concerned, we partially solve it under some additional assumption on the diameter of the domain \( \Omega \). Since we are only interested in the case when \( \alpha > N \), we will assume this when we address Question 2.

**Theorem 2.** Assume \( \alpha > N \). Let \( \Omega \) be a \( C^1 \) bounded domain with \( \text{diam } \Omega < e^{\frac{N}{N-\alpha}} \). Thus, \( \Omega \) is a ball if \( u(x) \) in (1.4) is constant on \( \partial \Omega \).
Remark 1.1. In the above two theorems, if the conclusion that $\Omega$ is a ball is verified, then we can easily deduce that $u(x)$ is radially symmetric with respect to the center of the ball.

There has been extensive study in the literature about overdetermined problems in elliptic differential equations and integral equations. In his seminal paper [Se], Serrin showed that the overdetermined boundary value determines the geometry of the underlying set. This is, if $\Omega$ is a bounded $C^2$ domain and $u \in C^2(\overline{\Omega})$ satisfies the following

\[
\begin{aligned}
\triangle u &= -1 \quad \text{in } \Omega, \\
u &= 0, \quad \frac{\partial u}{\partial n} = \text{constant} \quad \text{on } \partial \Omega,
\end{aligned}
\]

then $\Omega$ is a ball and $u$ is radially symmetric with respect to its center of the ball. Serrin’s proof is based on what is nowadays called the moving planes method relying on the maximum principle of solutions to the differential equations, which is originally due to Alexandrov, and has been later used to derive further symmetry results for more general elliptic equations. Important progress as for the moving plane methods since then are the works of Gidas-Ni-Nirenberg [GNN], Caffarelli-Gidas-Spruck [CGS], to just name some of the early works in this direction.

Immediately after Serrin’s paper, Weinberger [W] obtained a very short proof of the same result, using the maximum principle applied to an auxiliary function. However, compared to Serrin’s approach, Weinberger’s proof relies crucially on the linearity of the Laplace operator.

Since the work of [Se], many results are obtained about overdetermined problems. The interested reader may refer to [AB], [B], [BK], [BNST], [BNST1], [CS], [EP], [FG], [FGK], [FK], [FV], [G], [GL], [HPP], [Lim], [Liu], [M], [MR], [PP], [PS], [P], [Sh], [Si], [WX] and references therein, for more general elliptic equations. See also [R1] and reference therein for overdetermined problems in an exterior domain or general domain. In [BNST], an alternative shorter proof of Serrin’s result, not relying explicitly on the maximum principle has been given, where they deduce some global information concerning the geometry of the solution.

Overdetermined problems are important from the point of view of mathematical physics. Many models in fluid mechanics, solid mechanics, thermodynamics, and electrostatics are relevant to the overdetermined Dirichlet or Neumann boundary problems of elliptic partial differential equations. We refer the reader to the article [FG] for a nice introduction in that aspect.

Instead of a volume potential, single layer potential is also considered in overdetermined problems. A single layer potential is given by

\[
u(x) = \begin{cases} 
A \int_{\partial \Omega} \frac{-1}{2\pi} \log \frac{1}{|x-y|} \, d\sigma_y, & N = 2, \\
A \int_{\partial \Omega} \frac{1}{(N-2)\omega_N |x-y|^{N-2}} \, d\sigma_y, & N \geq 3,
\end{cases}
\]

where $A > 0$ is the constant source density on the boundary of the domain $\Omega$. If $u$ is constant in $\overline{\Omega}$, then $\Omega$ can be proved to be a ball under different smoothness assumption on the domain $\Omega$. See [M] for the case of $n = 2$ and [R1] for the case of $n \geq 3$, and also
some related works in [Lim] and [Sh]. We also refer the reader to the book of C. Kenig [K] on this subject of layer potential.

Generally speaking, two approaches are widely applied in dealing with overdetermined problems. One is the classical moving plane method. In [Se], the moving plane method with a sophisticated version of Hopf boundary maximum principle plays a very important role in the proof. The other way is based on an equality of Rellich type, as well as an interior maximum principle, see [W]. Our approach is a new variant of moving plane method - Moving plane in integral forms. It is much different from the traditional methods of moving planes used for partial differential equations. Instead of relying on the differentiability and maximum principles of the structure, a global integral norm is estimated. The method of moving planes in integral forms can be adapted to obtain symmetry and monotonicity for solutions. The method of moving planes on integral equations was developed in the work of W. Chen, C. Li and B. Ou [CLO], see also Y.Y. Li [Li], the book by W. Chen and C. Li [CLI] and an exhaustive list of references therein, where the symmetry of solutions in the entire space was proved. Moving plane method in integral form over bounded domains requires some additional efforts and has been carried out recently in symmetry problems arising from the integral equations over bounded domains, see the work of D. Li, G. Strohmer and L. Wang [LSW].

We end this introduction with the following remark concerning the characterization of balls by using the Bessel potential. The Bessel kernel $g_\alpha$ in $\mathbb{R}^N$ with $\alpha \geq 0$ is defined by

$$g_\alpha(x) = \frac{1}{r(\alpha)} \int_0^\infty \exp\left(-\frac{\pi}{\delta} |x|^2\right) \exp\left(-\frac{\delta}{4\pi}\right)^{\frac{\alpha-N-2}{2}} d\delta,$$

where $r(\alpha) = (4\pi)^{\frac{\alpha}{2}} \Gamma\left(\frac{\alpha}{2}\right)$.

In the paper [HLZ], we consider the Bessel potential type equation:

$$u(x) = \int g_\alpha(x - y) dy.$$  

(1.8)

Overdetermined problems for Bessel potential over a bounded domain in $\mathbb{R}^N$ have been recently studied in [HLZ]. For instance, the following theorem is proved in [HLZ], among some other results:

**Theorem 3.** Let $\Omega$ be a $C^1$ bounded domain in $\mathbb{R}^N$. If $u$ in (1.8) is constant on $\partial \Omega$, then $\Omega$ is a ball.

It is well-known that (1.8) is closely related to the following fractional equation

$$(I - \triangle)_{\frac{N}{2}} u = \chi_\Omega.$$

In the case of $\alpha = 2$, it turns out to be the ground state of the Schrödinger equation.

The paper is organized as follows. In Section 2, we show Theorem 1. In Section 3, we carry out the proof of Theorem 2. Throughout this paper, the positive constant $C$ is frequently used in the paper. It may differ from line to line, even within the same line. It also may depends on $u$ in some cases.

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2. Proof of Theorem 1

In this section, we will prove Theorem 1 by adapting the moving plane method in integral forms, see [CLO]. Since we are dealing with the case of bounded domains, we modify the method accordingly (see also [LSW], [CZ]).

We first introduce some notations. Choose any direction and, rotate coordinate system if it is necessary such that $x_1$-axis is parallel to it. For any $\lambda \in \mathbb{R}$, define

$$T_\lambda = \{(x_1, \ldots, x_n) \in \Omega | x_1 = \lambda\}.$$  

Since $\Omega$ is bounded, if $\lambda$ is sufficiently negative, the intersection of $T_\lambda$ and $\Omega$ is empty. Then, we move the plane $T_\lambda$ all the way to the right until it intersects $\Omega$. Let

$$\lambda_0 = \min\{\lambda : T_\lambda \cap \overline{\Omega} \neq \emptyset\}.$$  

For $\lambda > \lambda_0$, $T_\lambda$ cuts off $\Omega$. We define

$$\Sigma_\lambda = \{x \in \Omega | x_1 < \lambda\}.$$  

Set

$$x_\lambda = \{2\lambda - x_1, \ldots, x_n\}$$  

and

$$\Sigma'_\lambda = \{x_\lambda \in \Omega | x \in \Sigma_\lambda\}.$$  

At the beginning of $\lambda > \lambda_0$, $\Sigma'_\lambda$ remains within $\Omega$. As the plane keeps moving to the right, $\Sigma'_\lambda$ will still stay in $\Omega$ until at least one of the following events occurs:

(i) $\Sigma'_\lambda$ is internally tangent to the boundary of $\Omega$ at some point $P_\lambda$ not on $T_\lambda$.

(ii) $T_\lambda$ reaches a position where it is orthogonal to the boundary of $\Omega$ at some point $Q$.

Let $\bar{\lambda}$ be the first value such that at least one of the above positions is reached.

We assert that $\Omega$ must be symmetric about $T_{\bar{\lambda}}$; i.e.,

$$\Sigma_\lambda \cup T_{\bar{\lambda}} \cup \Sigma'_\lambda = \Omega.$$  

If this assertion is verified, for any given direction in $\mathbb{R}^N$, there also exists a plane $T_{\bar{\lambda}}$ such that $\Omega$ is symmetric about $T_{\bar{\lambda}}$. Moreover, $\Omega$ is connected. Then the only domain with those properties is a ball, see [AI].

In order to assert (2.1), we introduce

$$u_\lambda(x) = u(x_\lambda),$$  

$$\Omega_\lambda = \Omega \setminus (\Sigma_\lambda \cup \Sigma'_\lambda).$$  

We first establish some lemmas. Throughout the paper we assume $\alpha \geq 2$.

**Lemma 2.1.** Let $l \in \mathbb{N}$ with $1 \leq l < \alpha$. Then for any solution in (1.2), $u \in C^l(\mathbb{R}^N)$ and differentiation of order $l$ can be taken under the integral.

**Proof.** The proof is standard. We refer the reader to [R2].

**Lemma 2.2.** For $\lambda_0 < \lambda < \bar{\lambda}$ and $u(x)$ satisfying (1.2), we have

(i) If $N \geq \alpha$, $u_\lambda(x) > u(x)$ for any $x \in \Sigma_\lambda$.

(ii) If $N < \alpha$, $u_\lambda(x) < u(x)$ for any $x \in \Sigma_\lambda$.  


**Proof.** For $x \in \Sigma_\lambda$, in the case of $N = \alpha$, we rewrite $u(x)$ and $u_\lambda(x)$ as

$$u(x) = \int_{\Sigma_\lambda} \log \frac{1}{|x - y|} \, dy + \int_{\Sigma_\lambda} \log \frac{1}{|x_\lambda - y|} \, dy + \int_{\Omega_\lambda} \log \frac{1}{|x - y|} \, dy,$$

and

$$u_\lambda(x) = \int_{\Sigma_\lambda} \log \frac{1}{|x_\lambda - y|} \, dy + \int_{\Sigma_\lambda} \log \frac{1}{|x - y|} \, dy + \int_{\Omega_\lambda} \log \frac{1}{|x_\lambda - y|} \, dy.$$

Then

$$u_\lambda(x) - u(x) = \int_{\Omega_\lambda} \log \frac{|x - y|}{|x_\lambda - y|} \, dy.$$  

Since $|x - y| > |x_\lambda - y|$ for $x \in \Sigma_\lambda$ and $y \in \Omega_\lambda$, then

$$u_\lambda(x) > u(x).$$

While in the case of $N \neq \alpha$, $u_\lambda(x)$ and $u(x)$ have the following representations respectively:

$$u(x) = \int_{\Sigma_\lambda} |x - y|^\alpha - N \, dy + \int_{\Sigma_\lambda} |x_\lambda - y|^\alpha - N \, dy + \int_{\Omega_\lambda} |x - y|^\alpha - N \, dy,$$

and

$$u_\lambda(x) = \int_{\Sigma_\lambda} |x_\lambda - y|^\alpha - N \, dy + \int_{\Sigma_\lambda} |x - y|^\alpha - N \, dy + \int_{\Omega_\lambda} |x_\lambda - y|^\alpha - N \, dy.$$

Thus,

$$u_\lambda(x) - u(x) = \int_{\Omega_\lambda} (|x_\lambda - y|^\alpha - N - |x - y|^\alpha - N) \, dy,$$

Note that $|x - y| > |x_\lambda - y|$ for $x \in \Sigma_\lambda$ and $y \in \Omega_\lambda$. Thus, (i) and (ii) are concluded.

\[\square\]

**Lemma 2.3.** Assume that $u(x)$ satisfies (1.2) and suppose $\lambda = \bar{\lambda}$ in the first case; i.e. $\Sigma_\lambda$ is internally tangent to the boundary of $\Omega$ at some point $P_\lambda$ not on $T_\lambda$, then $\Sigma_{\bar{\lambda}} \cup T_\lambda \cup \Sigma_\lambda = \Omega$.

**Proof.** When $N \geq \alpha$, thanks to Lemma 2.1, $u_\lambda(x) \geq u(x)$ for $x \in \Sigma_\lambda$. While $N < \alpha$, $u_\lambda(x) \leq u(x)$ for $x \in \Sigma_\lambda$. We argue by contradiction. Suppose $\Sigma_\lambda \cup T_\lambda \cup \Sigma' \subseteq \bar{\Omega}$; that is, $\Omega_{\bar{\lambda}} \neq \emptyset$. At $P_\lambda$, from (2.2) and (2.3), $u(P_\lambda) > u(P)$ in the case of $N \geq \alpha$. It is a contradiction since $P_\lambda, P \in \partial \Omega$ and $u(P_\lambda) = u(P) = \text{constant}$. From the same reason, $u(P_\lambda) < u(P)$ when $N < \alpha$. It also contradicts the fact that $u$ is constant on the boundary. Therefore, the lemma is completed.

\[\square\]

**Lemma 2.4.** Assume that $u(x)$ satisfies (1.2) and suppose that the second case occurs; i.e. $T_\lambda$ reaches a position where is orthogonal to the boundary of $\Omega$ at some point $Q$, then, $\Sigma_\lambda \cup T_\lambda \cup \Sigma_\lambda = \Omega$.
PROOF. Since $u(x)$ is constant on the boundary and $\Omega \in C^1$, $\nabla u$ is parallel to the normal at $Q$. As implied in the second case, $\frac{\partial u}{\partial x_1}|Q = 0$. We denote the coordinate of $Q$ by $z$. Suppose $\Omega_\lambda \neq \emptyset$, there exists a ball $B \subset \Omega_\lambda$. Choose a sequence $\{x^i\}_1^\infty \in \Sigma_\lambda \setminus T_\lambda$ such that $x^i \to z$ as $i \to \infty$. It is easy to see that $x^i_\lambda \to z$ as $i \to \infty$. Since $B \subset \Omega_\lambda$, we can also find a $\delta$ such that $diam \Omega > |x^i_\lambda - y| > \delta$ for any $y \in B$ and any $x^i_\lambda$.

If $N = \alpha$, by (2.2),

$$u(x^i_\lambda) - u(x^i) = \int_{\Omega_\lambda} \log \frac{|x^i - y|}{|x^i_\lambda - y|} dy.$$ 

Let $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^N$, then $(x^i_\lambda - x^i) \cdot e_1$ is the first component of $(x^i_\lambda - x^i)$. By the Mean Value theorem,

$$\frac{u(x^i_\lambda) - u(x)}{(x^i_\lambda - x^i) \cdot e_1} = \int_{\Omega_\lambda} \frac{\log |x^i - y| - \log |x^i_\lambda - y|}{(x^i_\lambda - x^i) \cdot e_1} dy$$

$$= \int_{\Omega_\lambda} \frac{(y - \bar{x}^i_\lambda) \cdot e_1}{|y - \bar{x}^i_\lambda|^2} dy$$

$$> C \int_B \frac{1}{|diam \Omega|^2} dy$$

$$> C,$$

(2.4)

where $\bar{x}^i_\lambda$ is some point between $x^i_\lambda$ and $x^i$. Nevertheless,

$$\lim_{i \to \infty} \frac{u(x^i_\lambda) - u(x^i)}{(x^i_\lambda - x^i) \cdot e_1} = \frac{\partial u}{\partial x_1}|Q = 0,$$

which contradicts (2.4). Therefore, $\Omega_\lambda = \emptyset$.

In the case of $N > \alpha$, similarly we have

$$\frac{u(x^i_\lambda) - u(x^i)}{(x^i_\lambda - x^i) \cdot e_1} = \int_{\Omega_\lambda} \frac{|x^i_\lambda - y|^\alpha - N - |x^i - y|^\alpha - N}{(x^i_\lambda - x^i) \cdot e_1} dy$$

$$= \int_{\Omega_\lambda} \frac{(\alpha - N)|\bar{x}^i_\lambda - y|^\alpha - 2((x^i_\lambda - y) \cdot e_1)}{(x^i_\lambda - x^i) \cdot e_1} dy$$

$$> \int_B (\alpha - N)|\bar{x}^i_\lambda - y|^\alpha - 2((x^i_\lambda - y) \cdot e_1) dy$$

$$> C.$$

(2.5)

It also contradicts $\frac{\partial u}{\partial x_1}|Q = 0$, thus $\Omega_\lambda = \emptyset$.

The same idea can be applied to the case of $N < \alpha$ with minor modification. In conclusion, $\Sigma_\lambda \cup T_\lambda \cup \Sigma_\lambda^* = \Omega$ when the second case occurs. 

Combining Lemma (2.3) and Lemma (2.4), Theorem 1 is implied.
3. Proof of Theorem 2

In this section, we will prove theorem 2 under some restriction on the diameter of $\Omega$. Since we are mainly interested in the case of $\frac{\alpha - N}{2} \in \mathbb{N}_0$, this is the case when the fundamental solution of $(-\Delta)^{\frac{\alpha}{2}}$ has the representation (1.3). Therefore, we will assume $\alpha > N$ in this section. Obviously, $u \in C^1(\mathbb{R}^N)$ in (1.4). We begin with establishing several lemmas.

Lemma 3.1. For $\lambda_0 < \lambda < \bar{\lambda}$, assume $u(x)$ satisfies (1.4) with $\text{diam} \Omega < e^{\frac{1}{\alpha}}$, then $u_\lambda(x) < u(x)$ for any $x \in \Sigma_\lambda$.

**Proof.** Since $|x_\lambda - y_\lambda| = |x - y|$, and $|x_\lambda - y| = |x - y_\lambda|$, we write $u(x)$ and $u_\lambda(x)$ in the following forms:

$$ u(x) = \int_{\Sigma_\lambda} |x - y|^{\alpha - N} \log \frac{1}{|x - y|} dy + \int_{\Sigma_\lambda} |x_\lambda - y|^{\alpha - N} \log \frac{1}{|x_\lambda - y|} dy $$

$$ + \int_{\Omega_\lambda} |x - y|^{\alpha - N} \log \frac{1}{|x - y|} dy, $$

and

$$ u_\lambda(x) = \int_{\Sigma_\lambda} |x_\lambda - y|^{\alpha - N} \log \frac{1}{|x_\lambda - y|} dy + \int_{\Sigma_\lambda} |x - y|^{\alpha - N} \log \frac{1}{|x - y|} dy $$

$$ + \int_{\Omega_\lambda} |x_\lambda - y|^{\alpha - N} \log \frac{1}{|x_\lambda - y|} dy. $$

Then,

$$ u_\lambda(x) - u(x) = \int_{\Omega_\lambda} |x - y|^{\alpha - N} \log |x - y| dy - \int_{\Omega_\lambda} |x_\lambda - y|^{\alpha - N} \log |x_\lambda - y| dy. $$

We consider the function $s^{\alpha - N} \log s$. Note $\alpha > N$, thus

$$(s^{\alpha - N} \log s)' = s^{\alpha - N - 1}[(\alpha - N) \log s + 1] < 0,$$

whenever $s < e^{\frac{1}{\alpha - N}}$. Since $|x - y| > |x_\lambda - y|$ for $x \in \Sigma_\lambda$, $y \in \Omega_\lambda$, and $\text{diam} \Omega < e^{\frac{1}{\alpha - N}}$, we easily infer that $u_\lambda(x) < u(x)$ for any $x \in \Sigma_\lambda$.

Lemma 3.2. $u(x)$ satisfies (1.4) and suppose $\lambda = \bar{\lambda}$ in the first case; i.e. $\Sigma'_\lambda$ is internally tangent to the boundary of $\Omega$ at some point $P_{\bar{\lambda}}$ not on $T_{\bar{\lambda}}$, then $\Sigma_\lambda \cup T_{\bar{\lambda}} \cup \Sigma'_\lambda = \Omega$.

**Proof.** The proof is essentially the same as that of Lemma 2.3.

Lemma 3.3. Suppose that $u(x)$ satisfies (1.4) with $\text{diam} \Omega < e^{\frac{1}{\alpha - N}}$ and that the second case occurs; i.e. $T_{\bar{\lambda}}$ reaches a position where is orthogonal to the boundary of $\Omega$ at some point $Q$, then $\Sigma_\lambda \cup T_{\bar{\lambda}} \cup \Sigma'_\lambda = \Omega$.

**Proof.** The argument follows that of the proof of Lemma 2.4. Since $u(x)$ is constant on $\partial \Omega$ and $\Omega \in C^1$, $\frac{\partial u}{\partial n}|_Q = 0$. We denote the coordinate of $Q$ by $z$. Suppose $\Omega_{\bar{\lambda}} \neq \emptyset$, there exits a ball $B \subset \subset \Omega_{\bar{\lambda}}$. Choosing a sequence $\{x^i\}_{i=1}^\infty \in \Sigma_\lambda \setminus T_{\bar{\lambda}}$ such that $x^i \to z$ as $i \to \infty$, we have
then $x^i_\lambda \to z$ as $i \to \infty$. Since $B \subset \subset \Omega_\lambda$, we find a $\delta$ such that $\text{diam} \Omega > |x^i_\lambda - y| > \delta$ for any $y \in B$ and any $x^i_\lambda$.

From (3.1), by Mean Value Theorem,

$$
\frac{u(x^i_\lambda) - u(x^i)}{(x^i_\lambda - x^i) \cdot e_1} = \int_{\Omega_\lambda} \frac{|x^i - y|^{\alpha-N} \log |x^i - y| - |x^i_\lambda - y|^{\alpha-N} \log |x^i_\lambda - y|}{(x^i_\lambda - x^i) \cdot e_1} dy \\
= \int_{\Omega_\lambda} -|\bar{x}^i_\lambda - y|^{\alpha-N-2}((x^i_\lambda - y) \cdot e_1)((\alpha - N) \log |\bar{x}^i_\lambda - y| + 1) dy \\
< \int_{B} -|\bar{x}^i_\lambda - y|^{\alpha-N-2}((x^i - y) \cdot e_1)((\alpha - N) \log |\bar{x}^i_\lambda - y| + 1) dy \\
< -C.
$$

(3.2)

Where $\bar{x}^i_\lambda$ is some point between $x^i_\lambda$ and $x^i$. The assumption $\text{diam} \Omega < e^{N-\alpha}$ is applied in the last inequalities. Consequently, (3.2) contradicts $\frac{\partial u}{\partial x_1}(Q) = 0$ as $i \to \infty$. Therefore, the lemma is verified. \qed

With the help of the above two lemmas, Theorem 2 is confirmed.

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