A NOTE ON LATTICE POINTS IN CERTAIN FINITE TYPE DOMAINS IN $\mathbb{R}^d$

JINGWEI GUO

Abstract. We study the Fourier transforms of indicator functions of some special high-dimensional finite type domains and obtain estimates of the associated lattice point discrepancy.

1. Introduction

Let $B \subset \mathbb{R}^d$ be a compact convex domain, which contains the origin in its interior and has a smooth boundary $\partial B$. The number of lattice points $\mathbb{Z}^d$ in the dilated domain $tB$ is approximately $\text{vol}(B)t^d$ and the lattice point problem is to study the remainder, $P_B(t)$, in the equation

$$P_B(t) = \#(tB \cap \mathbb{Z}^d) - \text{vol}(B)t^d \quad \text{for } t \geq 1.$$ 

For a rich theory of this problem see Kratzel’s monographs [9, 10], Huxley’s [5], as well as a survey article [7] by Ivić, Krätzel, Kühleitner, and Nowak. The study of non-convex domains is also interesting. For example see Nowak [18] and Chamizo and Raboso [2] for study of a torus in $\mathbb{R}^3$.

If the boundary $\partial B$ has points of vanishing curvature, the problem is hard and our knowledge (especially of high dimensions) is fragmentary. We assume $d \geq 3$ below and refer interested readers to [10], [7], and the author [3] for the planar case which is comparatively well understood.

Randol [21] considered super spheres

$$B = \{ x \in \mathbb{R}^d : |x_1|^{\omega} + |x_2|^{\omega} + \cdots + |x_d|^{\omega} \leq 1 \}$$

for even integer $\omega \geq 3$, and proved that

$$P_B(t) = \begin{cases} O(t^{d-2+2/(d+1)}) & \text{for } \omega < d+1, \\ O(t^{d-1}(1-1/\omega)) & \text{for } \omega \geq d+1, \end{cases}$$

and this estimate is the best possible when $\omega \geq d+1$. Krätzel [8] extended this result to odd $\omega \geq 3$ and actually gave an asymptotic formula of the remainder. See [9] for further results and also Müller [16] for related (sharp) results when $d$ is much larger than $\omega$.

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Krätzel \[13\] and Krätzel and Nowak \[14, 15\] study a specific class of convex domains in \(\mathbb{R}^3\), the so-called bodies of pseudo revolution. They evaluate the contribution of flat points precisely and also of other boundary points and obtain asymptotic formulas of \(P_B(t)\).

Many authors tried to study more general domains in \(\mathbb{R}^3\). Partial results are available in Krätzel \[11, 12\], Popov \[20\], Peter \[19\], and Nowak \[17\], in which a variety of (somewhat complicated) curvature assumptions are made. Concerning curvature assumptions, Krätzel and Nowak write in \[14, 17\]: “In general, in dimension 3 it is not at all clear how ‘natural’ assumptions should look like, concerning the subset of \(\partial B\) where the curvature vanishes.” The geometry is even more complicated in high dimensions.

One concise and possibly proper curvature assumption for every dimension \(d \geq 3\) may be the “of finite type” condition (in the sense of Bruna, Nagel, and Wainger \[1\]). A few results are known for convex domains of finite type. The classical method (see for example Randol \[21\]) readily yields

\[
P_B(t) = O\left(t^{(d-1)(1-\frac{1}{-d+1})}\right)
\]

(\(\omega\) is the type of the boundary) as a consequence of the following bound (due to \[1\] Theorem B) of the Fourier transform of the indicator function of \(B\):

\[
|\widehat{\chi}_B(\xi)| \lesssim |\xi|^{-1-(d-1)/\omega}.
\]

The bound \[1.2\] is crude. Iosevich, Sawyer, and Seeger \[6\] Theorem 1.3] gives a better one, which shows its dependence on the multitype of \(\partial B\) and follows from a finer version of \[1.3\] (6 Proposition 1.2)). It is unknown to us whether or not their bound can be improved in general since in high dimensions the set of boundary points with vanishing curvature may be complicated and the related problem of estimating \(\widehat{\chi}_B\) is not easy.

For specific finite type domains, however, Iosevich, Sawyer, and Seeger’s bound may be improved since the two difficulties mentioned above may be overcome. For example Randol’s bound \[1.1\] for super spheres is better when \(\omega\) is not large (say, of size \(O(d)\)). Motivated by examples studied by Randol, Krätzel, and Nowak, we study a class of high-dimensional domains and prove the following result.

**Theorem 1.1.** Let \(\omega_1, \ldots, \omega_d \in \mathbb{N}\) be even and

\[
D = \{x \in \mathbb{R}^d : x_1^{\omega_1} + \cdots + x_d^{\omega_d} \leq 1\}.
\]
Then

\[ P_D(t) = \sum_{S \in \mathcal{P}_1(N_d)} O, \Omega \left( \frac{1}{t} \sum_{1 \leq l \leq d} \frac{1}{\omega_l} \right) + \]
\[ \sum_{S \in \mathcal{P}_j(N_d), 2 \leq j \leq d} O \left( \frac{1}{t} \sum_{1 \leq l \leq d} \frac{1}{\omega_l} \right). \]

(1.4)

where \( N_d = \{1, 2, \ldots, d\} \) and \( \mathcal{P}_j(N_d) \) is the collection of all subsets of \( N_d \) having \( j \) elements. If \( \omega = \max\{\omega_1, \ldots, \omega_d\} \) then

\[ P_D(t) \lesssim t^{d-1(1-1/\omega)} + t^{d-2+2/(d+1)}. \]

Remarks: In (1.4), the first sum is the contribution of boundary points which lie on coordinate axes; the terms for \( 2 \leq j \leq d - 1 \) come from boundary points lying on coordinate planes but not on axes; the term for \( j = d \) is \( O(t^{d-2+2/(d+1)}) \), due to boundary points that are not on any coordinate plane.

Similar examples can be studied in the same way. For interesting results on rotations of convex domains in \( \mathbb{R}^d \) see [6], the author [4], etc.

Notations: We set \( \mathbb{Z}_d = \mathbb{Z} \setminus \{0\} \), and \( \mathbb{R}_d = \mathbb{R} \setminus \{0\} \). The Fourier transform of any function \( f \in L^1(\mathbb{R}^d) \) is \( \hat{f}(\xi) = \int f(x) \exp(-2\pi x \cdot \xi) \, dx \).

For functions \( f \) and \( g \) with \( g \) taking nonnegative real values, \( f \lesssim g \) means \( |f| \leq Cg \) for some constant \( C \). If \( f \) is nonnegative, \( f \gtrsim g \) means \( g \lesssim f \). The Landau notation \( f = O(g) \) is equivalent to \( f \lesssim g \). The notation \( f \asymp g \) means that \( f \lesssim g \) and \( g \lesssim f \).

2. Main Estimates

Let \( D \) be as defined in Theorem [14]. If \( x \in \partial D \) let \( T_x \) be the affine tangent plane to \( \partial D \) at \( x \). Bruna, Nagel, and Wainger [1] defines a "ball"

\[ \tilde{B}(x, \delta) = \{ y \in \partial D : \text{dist}(y, T_x) < \delta \} \]
to be a cap near \( x \) cut off from \( \partial D \) by a plane parallel to \( T_x \) at distance \( \delta \) from it.

For nonzero \( \xi \in \mathbb{R}^d \) let \( x(\xi) \) be the unique point on \( \partial D \) where the unit exterior normal is \( \xi/|\xi| \). We first prove a bound for the surface measure of \( \tilde{B}(x(\xi), |\xi|^{-1}) \):

Lemma 2.1. Let \( 0 < c \leq 1 \) be a constant. For any nonzero \( \xi \in \mathbb{R}^d \) with \( |\xi_d|/|\xi| \geq c \) we have

\[ \sigma(\tilde{B}(x(\xi), |\xi|^{-1})) \lesssim \prod_{l=1}^{d-1} \min \left( |\xi|^{-\frac{1}{2l}}, (|\xi_l|/|\xi|)^{-\frac{w_l}{2\omega_l}} |\xi|^{-\frac{1}{2}} \right), \]
where the implicit constant only depends on $c$ and $D$, and the minimum takes the value $|\xi|^{-1/\omega_l}$ if $\xi_l$ vanishes for some $l$.

**Proof.** For any fixed $\xi$ denote $x(\xi) = (a_1, \ldots, a_d) \in \partial D$. By definition $\tilde{B}(x(\xi), |\xi|^{-1})$ is the cap near $x(\xi)$ cut off from $\partial D$

\[ x_1^{\omega_1} + \cdots + x_d^{\omega_d} = 1 \]

by the plane

\[ \langle x - (x(\xi) - |\xi|^{-2}\xi), \xi \rangle = 0. \]

Changing variables $y_i = x_i - a_i$, combining the two equations above, and eliminating $y_d$ yield the equation

\[ \sum_{i=1}^{d-1} (a_i + y_i)^{\omega_i} + \left( a_d - \xi_d^{-1} - \xi_d^{-1} \sum_{i=1}^{d-1} \xi_i y_i \right)^{\omega_d} = 1. \]

To estimate $\sigma(\tilde{B}(x(\xi), |\xi|^{-1}))$ it suffices to estimate the size of the $(d-1)$-dimensional domain bounded by (2.2) (namely, the projection of the cap onto $\mathbb{R}^{d-1}$). Hence it suffices to show that if $(y_1, \ldots, y_{d-1})$ satisfies (2.2) then for each $1 \leq l \leq d-1$

\[ \max |y_l| \lesssim \min \left( |\xi|^{-\frac{1}{\omega_l}}, (|\xi_l|/|\xi|)^{-\frac{\omega_l-2}{\omega_l-1}} |\xi|^{-\frac{1}{2}} \right). \]

Without loss of generality we only prove the case $l = 1$ since other cases are the same.

**Case 1:** $a_1 = 0$. Then (2.2) implies that

\[ \max |y_1| \lesssim |\xi|^{-1/\omega_1}. \]

Indeed, Taylor’s expansion gives

\[ (a_i + y_i)^{\omega_i} \geq a_i^{\omega_i} + \omega_i a_i^{\omega_i-1} y_i \quad \text{if } a_i \neq 0, \]

and

\[ \left( a_d - \xi_d^{-1} - \xi_d^{-1} \sum_{i=1}^{d-1} \xi_i y_i \right)^{\omega_d} \geq a_d^{\omega_d} - \omega_d a_d^{\omega_d-1} \xi_d^{-1} \sum_{i=1}^{d-1} \xi_i y_i + O(|\xi|^{-1}), \]

since $\omega_i$ is even and $|\xi_d| \approx |\xi|$. Recall that $x(\xi)$ satisfies (2.1) and note that $\xi_1 = 0$ since

\[ (\omega_1 a_1^{\omega_1-1}, \ldots, \omega_d a_d^{\omega_d-1}) \parallel \xi. \]

Applying these facts and the two inequalities (2.5) and (2.6) to (2.2) yields (2.4), hence (2.3).

**Case 2:** $a_1 > 0$ (the negative case is similar). We assert that: if $\max |y_1| \leq c_1 a_1$ for a sufficiently small constant $c_1$ then (2.3) holds with $l = 1$; otherwise it still holds.
If \( \max |y_1| \leq c_1 a_1 \), applying (2.5), (2.6), and
\[(a_1 + y_1)^{\omega_1} = a_1^{\omega_1} + \omega_1 a_1^{\omega_1-1} y_1 + a_1^{\omega_1-2} y_1^2 (\omega_1 (\omega_1 - 1)/2 + O(c_1)) + y_1^{\omega_1}\]
to (2.2) (like what we did in Case 1) yields
\[(2.8) \quad |y_1| \lesssim \min \left( |\xi|^{-\frac{1}{\omega_1}}, |a_1|^{-\frac{\omega_1-2}{\omega_1}} |\xi|^{-\frac{1}{2}} \right)\]
if \( c_1 \) is sufficiently small. Note that (2.7) implies
\[|a_1| \asymp \left( |\xi_1| / |\xi| \right)^{1/(\omega_1 - 1)}.
\]
Hence the first assertion follows immediately.

If \( y_1 > 0 \) or \( y_1 < 0 \) but \( \max_{y_1 < 0} |y_1| \leq c_1 a_1 \), a similar argument as above proves the desired bound.

Hence, to prove the second assertion, we may assume \( y_1 < 0 \) and \( \max_{y_1 < 0} |y_1| \leq c_1 a_1 \). By a compactness argument there exists a constant \( C_1 \) (depending only on \( c_1 \) and \( D \)) such that \( \tilde{B}(x(\xi), C_1 |\xi|^{-1}) \) intersects the coordinate plane \( x = 0 \). Let \( P \) be a point of the intersection. Then the cap \( \tilde{B}(P, C_1 |\xi|^{-1}) \) intersects \( \tilde{B}(x(\xi), C_1 |\xi|^{-1}) \). By [1, Theorem A] there exists a constant \( C_2 = C_2(D) \) such that
\[\tilde{B}(x(\xi), C_1 |\xi|^{-1}) \subset \tilde{B}(P, C_2 C_1 |\xi|^{-1}).\]
By this inclusion and the result of Case 1 (applying to \( \tilde{B}(P, C_2 C_1 |\xi|^{-1}) \)), we get
\[(2.9) \quad \max |y_1| \lesssim |\xi|^{-1/\omega_1}.\]
Hence \( a_1 \lesssim |\xi|^{-1/\omega_1} \), which implies
\[(2.10) \quad |\xi|^{-\frac{1}{\omega_1}} \lesssim (|\xi_1| / |\xi|)^{-\frac{\omega_1-2}{\omega_1-1}} |\xi|^{-\frac{1}{2}}.\]
The inequalities (2.9) and (2.10) gives (2.3) with \( l = 1 \). This finishes the proof. \( \square \)

By the Gauss-Green formula, [1, Theorem B], and Lemma 2.1, we immediately get the following bound of the Fourier transform of the indicator function of \( \mathcal{D} \):

**Theorem 2.2.** Let \( 0 < c \leq 1 \) be a constant. For any \( \xi \in S^{d-1} \) with \( |\xi_d| \geq c \) and \( \lambda > 0 \) we have
\[(2.11) \quad |\widehat{\chi_\delta}(\lambda \xi)| \lesssim \lambda^{-1} \prod_{l=1}^{d-1} \min \left( \lambda^{-\frac{1}{\omega_l}}, |\xi_l|^{-\frac{\omega_l-2}{\omega_l-1}} \lambda^{-\frac{1}{2}} \right),\]
where the implicit constant only depends on \( c \) and \( \mathcal{D} \), and the minimum takes the value \( \lambda^{-1/\omega_l} \) if \( \xi_l \) vanishes for some \( l \).

**Remark:** This Lemma is a generalization of [21, II, Theorem 2]. Our proof is simpler due to an application of the results from [1]. The result
shows that the size of $|\hat{\chi}_D(\lambda \xi)|$ may depend on both the size of $\lambda$ and the direction of $\xi$.

3. Proof of Theorem 1.1

Let $0 \leq \rho \in C_0^\infty(\mathbb{R}^d)$ be such that $\int_{\mathbb{R}^d} \rho(y) \, dy = 1$, $\epsilon > 0$, $\rho_\epsilon(y) = \epsilon^{-d} \rho(\epsilon^{-1} y)$, and

$$N_\epsilon(t) = \sum_{k \in \mathbb{Z}^d} \chi_{tD} \ast \rho_\epsilon(k),$$

where $\chi_{tD}$ denotes the characteristic function of $tD$. By the Poisson summation formula we have

$$(3.1) \quad N_\epsilon(t) = t^d \sum_{k \in \mathbb{Z}^d} \hat{\chi}_D(tk) \hat{\rho}(\epsilon k) = \text{vol}(D) t^d + R_\epsilon(t),$$

where

$$R_\epsilon(t) = t^d \sum_{k \in \mathbb{Z}^d} \hat{\chi}_D(tk) \hat{\rho}(\epsilon k).$$

To estimate $R_\epsilon(t)$, by using a partition of unity, we decompose it as the sum of $S_j$, $1 \leq j \leq d$, where

$$S_j = t^d \sum_{k \in \mathbb{Z}^d} \Omega_j(k) \hat{\chi}_D(tk) \hat{\rho}(\epsilon k)$$

with each $\Omega_j$ supported in $\Gamma_j = \{ x \in \mathbb{R}^d : |x_j| \geq (2d)^{-1/2} |x| \}$ and smooth away from the origin. We then split $S_j$ (as follows) depending on the number of nonzero components of $k$:

$$S_j = \sum_{i=1}^d S_{i,j}$$

with

$$S_{i,j} = t^d \sum_{(i)} \Omega_j(k) \hat{\chi}_D(tk) \hat{\rho}(\epsilon k),$$

where the summation is over all $k \in \mathbb{Z}_*^d \cap \text{supp}(\Omega_j)$ having exactly $i$ nonzero components.

Without loss of generality we may assume $j = d$, namely, restrict the $k \in \mathbb{Z}_*^d$ to a cone about $x_d$-axis. By Theorem 2.2 we have

$$(3.2) \quad |S_{1,d}| \lesssim t^{d-1-\sum_{i=1}^{d-1} 1/\omega_i} \sum_{k_d \in \mathbb{Z}_*^d} |k_d|^{-1-\sum_{i=1}^{d-1} 1/\omega_i} \lesssim t^{d-1-\sum_{i=1}^{d-1} 1/\omega_i}.$$
For $2 \leq i \leq d$ we apply Theorem 2.2, compare the sums with integrals in polar coordinates, and get

$$|S_{i,d}| \lesssim \sum_{S \in \mathcal{P}((Nd):d \in S} t^{d-\frac{i}{\omega_i} - \frac{1}{\omega_i}} \left(1 + \varepsilon^{\frac{-i}{\omega_i} + \frac{1}{\omega_i}} \right).$$

Note that the first term of the summand above is not larger than the right side of (3.2).

By using the two bounds above and similar results for other $j$’s we obtain a bound of $R_\varepsilon(t)$. Note that (3.3)

$$N_\varepsilon(t - C\varepsilon) \leq \#(t\mathcal{D} \cap \mathbb{Z}^d) = \sum_{k \in \mathbb{Z}^d} \chi_{t\mathcal{D}}(k) \leq N_\varepsilon(t + C\varepsilon),$$

where $C$ is a positive constant depending on $\mathcal{D}$. Thus by using (3.1) and $\varepsilon = t^{-1/(d+1)}$ we get the desired upper bound in (1.4), from which it is simple to derive (1.5).

To prove the lower bound in (1.4) (see also [6, P.167-168]), we first apply the asymptotic expansion in Schulz [22] to get

$$\widehat{n}_d d\sigma(tk) = C_1 \sin(2\pi tk_d - \pi\nu/2)(tk_d)^{-\nu} + O((tk_d)^{-\nu-1/\eta}),$$

where $n_d$ is the $d$th component of the Gauss map of $\partial\mathcal{D}$, $d\sigma$ is the induced Lebesgue measure on $\partial\mathcal{D}$, $k = (0, \ldots, 0, k_d)$, $k_d \in \mathbb{N}$, $\nu = \sum_{l=1}^{d-1} 1/\omega_l$, $C_1$ is a constant (depending on $\omega_1, \ldots, \omega_{d-1}$), and $\eta$ is the least common multiple of $\omega_1, \ldots, \omega_{d-1}$.

Hence by the Gauss–Green formula we can readily get an expansion of $\widehat{\chi}_D(tk)$. We then split the sum $S_{1,d}$ as follows

$$S_{1,d} = t^d \sum_{k=(0,\ldots,0,k_d) \atop k_d \in \mathbb{Z}_1^d} \widehat{\chi}_D(tk) + t^d \sum_{k=(0,\ldots,0,k_d) \atop k_d \in \mathbb{Z}_1^d} \widehat{\chi}_D(tk)(\widehat{\rho}(\varepsilon k) - 1) =: I + II$$

and apply the expansion. Therefore

$$I = t^{d-1-\nu}g(t) + O(t^{d-1-\nu-1/\eta}),$$

where

$$g(t) = C_2 \sum_{k_d \in \mathbb{Z}_1^d} |k_d|^{-\nu-1} \sin(2\pi t |k_d| - \pi\nu/2)$$

with a positive constant $C_2$ (depending on $\omega_1, \ldots, \omega_{d-1}$), and

$$II = O(t^{d-1-\nu}(\varepsilon^{\nu} + \varepsilon)).$$

Note that the function $g$ is periodic and not identically zero, hence

$$\lim_{t \to \infty} |g(t)| > 0.$$
symmetry, (3.1), and (3.3), we get the desired lower bound in (1.4). This finishes the proof.

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JINGWEI GUO, SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA, HEFEI, ANHUI PROVINCE 230026, PEOPLE’S REPUBLIC OF CHINA  
E-mail address: jwguo@ustc.edu.cn