Dynamics of a Bose–Einstein condensate in a symmetric triple-well trap

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Abstract

We present a complete analysis of the dynamics of a Bose–Einstein condensate trapped in a symmetric triple-well potential. Our classical analogue treatment, based on a time-dependent variational method using SU(3) coherent states, includes the parameter dependence analysis of the equilibrium points and their local stability, which is closely related to the condensate collective behaviour. We also consider the effects of off-site interactions, and how these ‘cross-collisions’ may become relevant for a large number of trapped bosons. Even in the presence of cross-collisional terms, the model still features an integrable sub-regime, known as the twin-condensate dynamics, which corresponds to invariant surfaces in the classical phase space. However, the quantum dynamics preserves the twin-condensate defining characteristics only partially, thus breaking the invariance of the associated quantum subspace. Moreover, the periodic geometry of the trapping potential allowed us to investigate the dynamics of finite angular momentum collective excitations, which can be suppressed by the emergence of chaos. Finally, using the generalized purity associated with the su(3) algebra, we were able to quantify the dynamical classicality of a quantum evolved system, as compared to the corresponding classical trajectory.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The experimental observation of the Bose–Einstein condensation in the systems of ultracold dilute alkali atomic clouds [1] has led to an active development of various studies concerning collective quantum phenomena. The special property of coherence presented in the...
condensates of a large number of atoms allowed classical and semiclassical approaches to discover the fascinating dynamics of such a large quantum system.

The prospect of using magneto-optical traps with diverse confining geometries allowed the study of various quantum dynamical regimes, also including the possibility of testing the frontier between the quantum and classical mechanics through the increase of the total particle number to the macroscopic scale.

Several surveys have been performed in the case of a Bose–Einstein condensate trapped in a double-well potential under the two-mode approximation. Various subjects were considered in these studies, such as tunneling dynamics [2], entanglement of modes [3] and quantum phase transition [4]. More recently the effects of the so-called cross-collisions have also been considered, and it was shown that these usually neglected higher order approximation terms can become quite important for large numbers of trapped bosons [5].

The dynamics of condensates in a triple-well potential has also been investigated several times in the literature [6–8]. However, little attention has been paid to the specific case in which all three wells are identically coupled in pairs [9–11]. This arrangement is very interesting because it represents the simplest trapping potential where we can observe the rotation of the condensate.

In this work, we examine the dynamics of a condensate in a symmetric triple-well potential, under the three-mode approximation and considering the cross-collision terms. First, we investigate the classical analogue of the model and its phase space structure, by determining its fixed points and the Hamiltonian flow in the vicinity of these special solutions, which is closely related to the condensate collective excitations. In this way we specify how the cross-collisions modify the position and stability of equilibrium points, though preserving all the classical invariant spaces. Then, we choose the coherent states located in the proximity of the classical stationary states as initial quantum states. This enables us to capture in the quantum dynamics the characteristics associated with each of the classical fixed points. This strategy has been used before to observe the effect of chaos in the entanglement dynamics [12].

In section 2, we derive the model in the three-mode approximation, including the cross-collisional terms, and the corresponding Hamiltonian in terms of the SU(3) group generators. We also present the classical approximation based on the time-dependent variational principle (TDVP) [13] and the SU(3) coherent states [14–16]. The equilibrium points of the classical analogue equations of motion are divided into three classes: twin-condensate states, single depleted well states and vortex states. Here, we show in detail the stability diagrams of the fixed points as functions of two collision parameters. The behaviour of the lowest energy fixed point and the occurrence of a quantum phase transition has been reported recently [17].

In section 3, we relate a simple permutation symmetry of the Hamiltonian to a transformation of the single-particle basis that leads to the twin-condensate subspace. Then, we demonstrate that the classically invariant subspace of twin condensates is not fully preserved quantum mechanically. In addition, we compare the quantum population dynamics in the local modes with the corresponding classical evolution. An appropriate choice of initial conditions allowed us to better characterize the dynamics in the vicinity of each fixed point, in comparison with previous publications. Particularly interesting is the behaviour of the vortex states [11], which are associated with the condensate angular momentum along the symmetry axis of the trapping potential. The variation of the collision parameters can lead the vortex solutions to the chaotic dynamics, and consequently inhibiting the condensate rotation in a preferred sense. Moreover, similar to the previous results for the double-well condensate model [4] with the measure known as generalized purity [18], we make a quantitative comparison between the...
quantum evolution and the classical approximation based on the coherent states. Section 4 is reserved for our conclusions and final considerations.

2. Model

Suppose a spinless boson of mass $m$ trapped in a potential $V(\vec{r})$, whose equivalent minima are located at $\vec{r}_j = (x_j, y_j, 0)$, $j = 1, 2, 3$:

$$V(\vec{r}) = \frac{m \omega^2}{18q_0^2} \prod_{j=1}^{3} [((x - x_j)^2 + (y - y_j)^2] + \frac{m \omega^2}{2} \vec{z}^2. \quad (2.1)$$

Note that the potential is harmonic in the $z$-direction with angular frequency $\omega$ and we choose $V(\vec{r}_j) = 0$. The parameter $q_0$ represents the distance between the potential minima and the origin of the coordinate system. For simplicity, we also choose $V(\vec{r})$ so that the harmonic approximation $V_j^{(2)}(\vec{r})$ around each minimum is isotropic with angular frequency $\omega$. We can assign to each potential $V_j^{(2)}(\vec{r})$, taken independently, a localized ground state:

$$\langle \vec{r} | u_j \rangle = u_j(\vec{r}) = \frac{1}{\pi^{d/2} |d|^2} \exp \left[ - \frac{(x - x_j)^2 + (y - y_j)^2 + \vec{z}^2}{2d^2} \right]. \quad (2.2)$$

The Gaussian width is described by the parameter $d = \sqrt{\hbar/m \omega}$. Considering that $q_0 \gg d$, the states $|u_j\rangle$ are almost orthogonal since $\epsilon = \langle u_j | u_k \rangle = \exp \left[ -3q_0^2/4d^2 \right] \ll 1$ for $j \neq k$.

The states $|u_j\rangle$ are not the eigenstates of the full single-particle Hamiltonian $H = \vec{p}^2/2m + V(\vec{r})$, but these localized states generate the same eigenspace of the three lowest energy states of $H$ at first order in $\epsilon$. That is, we have the following single-particle lowest energy states at $O(\epsilon)$:

$$\begin{align*}
|e_1\rangle &= \frac{1}{\sqrt{3}} (|u_1\rangle + |u_2\rangle + |u_3\rangle) \\
|e_2\rangle &= \frac{1}{\sqrt{6}} (|u_1\rangle + |u_2\rangle - 2|u_3\rangle) \\
|e_3\rangle &= \frac{1}{\sqrt{2}} (|u_1\rangle - |u_2\rangle) \quad .
\end{align*} \quad (2.3)$$

The states $|e_2\rangle$ and $|e_3\rangle$ are degenerated and the energy of these states differs from the energy of $|e_1\rangle$ by $3\Omega$, with the tunneling rate defined by

$$\Omega \equiv \langle u_j \vert V(\vec{r}) - V_j^{(2)}(\vec{r}) \vert u_k \rangle + \epsilon \frac{3m \omega}{2}; \quad j \neq k. \quad (2.4)$$

Generally, for $q_0 \gg d$, $\Omega$ is negative and $|e_1\rangle$ is the single-particle ground state, whose energy difference for the next two states is proportional to $\epsilon$.

So far we studied only the single-particle potential. Now, considering a system of $N$ condensates bosons, we introduce the many-particle Hamiltonian

$$\hat{H} = \int d^3r \hat{\psi}^\dagger(\vec{r}) \hat{H}(\vec{r}) \hat{\psi}(\vec{r}) + \frac{1}{2} \int d^3r d^3r' \hat{\psi}^\dagger(\vec{r}) \hat{\psi}(\vec{r}') \mathcal{V}(\vec{r}, \vec{r}') \hat{\psi}(\vec{r}') \hat{\psi}(\vec{r}), \quad (2.5)$$

where $H$ is the single-particle Hamiltonian and $\mathcal{V}(\vec{r}, \vec{r}')$ is the pair interaction potential. Note that only the interactions between pairs of particles are significant, since we assume a dilute condensate gas, characterized by a large mean free path compared to the size of a boson, which makes higher order collisions unlikely. For particles of momentum approaching zero, as in the case of a condensate, a good approximation for the interaction potential is the following effective potential [19]:

$$\mathcal{V}(\vec{r}, \vec{r}') = \frac{4\pi \hbar^2 a}{m} \delta(\vec{r} - \vec{r}') = V_0 \delta(\vec{r} - \vec{r}'). \quad (2.6)$$
In the previous equation, we defined the s-wave scattering length, denoted by \( a \). Now, supposing that the states of (2.3) are only significantly populated, we obtain our second approximation in (2.5), namely, the three-mode approximation:

\[
\hat{\psi}(\vec{r}) \approx \langle \hat{r}|u_1\rangle a_1 + \langle \hat{r}|u_2\rangle a_2 + \langle \hat{r}|u_3\rangle a_3;
\]

(2.7)

where we defined \( a_j \) as the annihilation operator for the state \( |u_j\rangle \). Therefore, the field operator \( \hat{\psi}(\vec{r}) \) may be expanded in the local modes described in (2.2), which generate the same single-particle space as the energy states of (2.3). Substituting (2.6) and (2.7) into (2.5), we obtain the following Hamiltonian, retaining only the terms up to order \( \epsilon^2 \):

\[
\hat{H} = \Omega' \sum_{j \neq k} a_j^\dagger a_k + \kappa \sum_j a_j^2 a_j^\dagger - 2\Lambda \sum_{j,k,m} a_j^\dagger a_j a_k^\dagger a_m^\dagger.
\]

(2.8)

The tilde over the third summation symbol in (2.8) indicates the sum over three different indices. In the previous equation, we defined the effective tunneling rate \( \Omega' = \Omega + 2\Lambda(N - 1) \) and the following collision parameters:

\[
\kappa = \frac{V_0}{2} \int d^3r \; u_j^\dagger(\vec{r}) \frac{V_0}{2\pi \pi^2 d^3};
\]

\[
\Lambda = \frac{V_0}{2} \int d^3r \; u_j^\dagger(\vec{r}) u_k(\vec{r}) = \kappa \epsilon^2; \quad j \neq k.
\]

(2.9)

The self-collision rate \( \kappa \) is proportional to the collision frequency between bosons in the same potential well, while the cross-collision rate \( \Lambda \) is proportional to the collision frequency between bosons coming from different local modes. Note that the effective tunneling rate \( \Omega' \) depends on the cross-collision parameter and the total number of trapped bosons. Also, observe that we have \( |\Lambda| \ll |\kappa| \) for \( \epsilon \ll 1 \). Although the cross-collision rate has a lower order of magnitude than \( \Omega \), the product \( N\Lambda \) in \( \Omega' \) may be relevant for large numbers of trapped bosons. Therefore, the cross-collisions become important in comparing theoretical and experimental results, because they include significant effects of the number of particles in the effective tunneling rate.

In order to evidence the contribution of the cross-interactions to the tunneling term in (2.8), we employed the following identity to recast the cross-collision terms arising from equation (2.5):

\[
\sum_{j \neq k} a_j^\dagger a_j a_k + \sum_{j \neq k} a_j^\dagger a_k + a_k^\dagger a_j^\dagger = (\hat{N} - 1) \sum_{j \neq k} a_j^\dagger a_k - \sum_{j,k,m} a_j^\dagger a_j a_k^\dagger a_m^\dagger.
\]

(2.10)

This expression follows directly from the bosonic canonical commutation relations. In equation (2.10) we also defined the total particle number operator \( \hat{N} = a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3 \). Note that in (2.8) we replace the operator \( \hat{N} \) by one of its eigenvalues. This procedure relies on the particle number conservation under unitary evolution, which allows us to consider a fixed value for \( N \).

Also as a consequence of \( [\hat{H}, \hat{N}] = 0 \), we can use the following homomorphism between the bilinear bosonic operators and the generators of the SU(3) group:

\[
Q_1 = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2), \quad Q_2 = \frac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2 - 2a_3^\dagger a_3),
\]

\[
J_k = i(a_k^\dagger a_j - a_j^\dagger a_k), \quad P_k = a_k^\dagger a_j + a_j^\dagger a_k
\]

(2.11)

for \( k = 1, 2, 3 \) and \( j = (k + 1) \mod 3 + 1 \). The Hamiltonian (2.8) can be rewritten in terms of the operators (2.11), characterizing SU(3) as the dynamical group of the system

\[
\hat{H} = \Omega' - 2\Lambda \frac{N}{3} (P_1 + P_2 + P_3) + \frac{\kappa}{2} (4Q_1^2 + 3Q_2^2)
\]

\[
+ \Lambda [2Q_1(P_1 - P_3) + Q_2(2P_2 - P_1 - P_3)].
\]

(2.12)
In the previous equation, we discarded some terms dependent only on the number operator $\hat{N}$, as such terms do not change the system dynamics for a fixed value of $N$, except by the introduction of a global phase. Note that the tunneling term is linear in the SU(3) generators, while the collision terms are quadratic.

Within the three-mode approximation, a single-particle operator $A$ and its corresponding many-particle operator $\hat{A}$ are related by

$$\hat{A} = \int d^3r d^3r' \hat{\psi}^\dagger(\vec{r}) \langle \vec{r}|A|\vec{r}' \rangle \hat{\psi}(\vec{r}') \approx \sum_{j,k=1}^3 \langle u_j|A|u_k \rangle a_j^\dagger a_k. \quad (2.13)$$

Therefore, considering terms up to first order in $\epsilon$, we can find relations between the generators of SU(3) and basic system observables. For instance, the condensate angular momentum operator along the $z$-axis is proportional to the sum $J_S = J_1 + J_2 + J_3$:

$$\hat{L}_z = \sqrt{3} \epsilon q_0 4 d^2 J_S. \quad (2.14)$$

### 2.1. Classical approximation

Classical trajectories are used widely in the literature for the treatment of the condensate dynamics. Here we introduce the classical approach to our problem based on the well-known scheme combining the TDVP [13] with the coherent-state picture [14]. Considering independent variations in an arbitrary state $|\psi\rangle$ of the Hilbert space and its conjugate $\langle \psi|$, one can show that the extremization of the following action functional is equivalent to the Schrödinger equation:

$$S = \int_{t_i}^{t_f} dt \left[ \frac{i}{2} \langle \psi|\dot{\psi}\rangle - \frac{\langle \dot{\psi}|\psi\rangle}{\langle \psi|\psi\rangle} - \langle \psi|\hat{H}|\psi\rangle \right]. \quad (2.15)$$

We can make approximations to the Schrödinger equation by a suitable parametrization of the time dependence of the state $|\psi\rangle$, restricting the state evolution to a subspace of the Hilbert space. The classical approximation is achieved when the state is restricted to the nonlinear subspace constituted only by the coherent states associated with the appropriate dynamical group [20]. Specifically for the SU(3) bosonic representations, we have the following coherent states [15, 16]:

$$|N; w_1, w_2\rangle = \sum_{n_1+n_2+n_3=N} \frac{N!}{n_1!n_2!n_3!} \left( w_1^{n_1} w_2^{n_2} |n_1, n_2, n_3\rangle \right) \frac{1}{(|w_1|^2 + |w_2|^2 + 1)^{\frac{N}{2}}}$$

$$= \frac{1}{\sqrt{N!}} \left[ w_1 a_1^\dagger + w_2 a_2^\dagger + a_3^\dagger \right]^N |0\rangle. \quad (2.16)$$

The variables $w_1, w_2 \in \mathbb{C}$ parametrize the subspace composed of coherent states for a fixed number particles $N$.

1. In what follows we make $\hbar = 1$.
2. Except for the normalization factor, the coherent states are the analytic functions of the variables $w_1$ and $w_2$, which have unrestricted domain in the complex space. In fact, this property makes the adopted choice of parametrization, corresponding to spherical projective coordinates [15], very convenient in classical [21] and semiclassical [22] applications.
the variational principle to the action functional (2.15) with \( |\psi\rangle = |N; w_1, w_2\rangle \), we obtain the classical equations of motion for the complex variables \( w_j \):

\[
iw_j = \frac{|w_j|^2 + |w_k|^2 + 1}{N} \left[ (|w_j|^2 + 1) \frac{\partial \mathcal{H}}{\partial w_j^*} + w_j w_k^* \frac{\partial \mathcal{H}}{\partial w_k^*} \right]
\] (2.17)

for \( j, k = 1, 2 \) and \( k \neq j \). The quantity \( \mathcal{H}(w_j^*, w_j^*, w_k, w_k) \) is the effective classical Hamiltonian given by the average of \( \hat{H} \) in the coherent states

\[
\frac{\mathcal{H}}{N} = \frac{\langle N; w_1, w_2|\hat{H}|N; w_1, w_2 \rangle}{N} = \Omega \left( (1 + 2\mu) \frac{w_1 w_2 + w_1 w_1^* + w_1 + w_2^* + w_2 + w_2^*}{1 + |w_1|^2 + |w_2|^2} + \chi \left( \frac{|w_1|^4 + |w_2|^4 + 1}{(|w_1|^2 + |w_2|^2 + 1)^2} \right) - 2\mu \left( \frac{|w_1|^2 (w_2^* + w_2) + |w_2|^2 (w_1^* + w_1) + w_1^* w_2 + w_2^* w_1}{(|w_1|^2 + |w_2|^2 + 1)^2} \right) \right).
\] (2.18)

In the previous equation, we introduced new collision parameters which allowed us to eliminate the dependence on \( N \) from the average energy per particle and also prevented the problems of divergence in the macroscopic limit \( N \to \infty \). The quantities \( \chi \) and \( \mu \) as functions of the old collision parameters are given by

\[
\chi = \frac{\kappa (N - 1)}{\Omega}, \quad \mu = \frac{\Lambda (N - 1)}{\Omega}.
\] (2.19)

Note that \( |\chi| \gg |\mu| \), because supposedly we have \( |\kappa| \gg |\Lambda| \). Substituting the Hamiltonian (2.18) into (2.17), we obtain the classical equations of motion for the three-mode condensate:

\[
iw_j = \Omega \left( (1 + 2\mu)(1 - w_j)(w_j + w_k + 1) + 2\chi \frac{w_j (|w_j|^2 - 1)}{|w_j|^2 + |w_k|^2 + 1} \right.
\]

\[
- 2\mu \left( \frac{(1 - w_j) w_k^*(w_j + w_k + w_j w_k) - (1 + w_j) w_k (|w_j|^2 - 1)}{|w_j|^2 + |w_k|^2 + 1} \right) \right) \},
\] (2.20)

again for \( j, k = 1, 2 \) and \( k \neq j \). The TDVP-based classical approximation with coherent states is exact for any linear Hamiltonian in the dynamical group generators, i.e. the classical equations of motion (2.20) reproduce the correct quantum evolution of a coherent state in the case of non-interacting bosons, for \( \chi = \mu = 0 \). The approximation also becomes exact in the classical limit\(^3\), which coincides with the macroscopic limit \( N \to \infty \), since the total number of trapped bosons plays a role equivalent to the reciprocal of \( \hbar \) [23]. However, considering \( \chi \) and \( \mu \) as independent parameters, we see that the equations of motion (2.20) and the Hamiltonian per particle (2.18) are independent of \( N \), and therefore represent the exact results in the classical–macroscopic limit.

The equations of motion (2.17) assume the canonical form under the change of dynamical variables:

\[
w_j = \left( \frac{K_j}{N - K_j - K_2} \right)^{\frac{1}{2}} e^{-i\phi_j} \] (2.21)

for \( j = 1, 2 \). The canonical variables \( K_j \) and \( \phi_j \) have a useful and immediate physical interpretation. The variable \( K_j \) is the mean bosonic occupation in the \( j \)th potential well, and

\(^3\) In this paper, the term ‘classical’ generally refers to the trajectory approach defined by the combination of the TDVP and the SU(3) coherent states. However, note that the composite term ‘classical limit’ indicates the limit at which the quantum evolution coincides with the classical approximation.
(N − K1 − K2) is the mean population in the third well due to the conservation of the total number of particles. The angular variable \( \phi_j \) represents the phase difference between the portions of the condensate located at the \( j \)th and third wells.

2.2. Equilibrium points

The equations of motion (2.20) lead to the same equilibrium points of the case \( \mu = 0 \) [11] since we are considering the hypothetical restriction \( \mu \ll \chi \). However, the number, position and stability of fixed points can change significantly with the presence of cross-collisions. In this subsection, we describe how the cross-collisional terms affect the classical dynamics in phase space.

2.2.1. Twin condensates. According to the equations of motion (2.20), the classical phase space of the condensate has three invariant subspaces arising from the discrete rotational symmetry of the trapping potential, which are described by the following conditions:

\[
    w_1 = w_2, \quad w_1 = 1, \quad w_2 = 1. \quad (2.22)
\]

Each of these equivalent invariant subspaces follows from the equality in the phase and mean occupation of a pair of local modes, according to transformation (2.21). Therefore, if two localized condensates have the same initial classical state, they remain identical during their classical evolution. This integrable sub-regime of the model is known as the twin-condensate dynamics [7, 9].

The majority of the equilibrium points of the equations of motion (2.20) are located in the twin-condensate subspaces under the additional constraint \( w_1, w_2 \in \mathbb{R} \). Therefore, by making \( w_1 = w_2 \in \mathbb{R} \) and \( w_j = 0 \) in (2.20), we obtain the following polynomial equation for the location of the twin-condensate fixed points:

\[
    4(1 + \mu)w_1^4 - 2(1 + \chi + 4\mu)w_1^3 + 6\mu w_1^2 + (2\chi - 1)w_1 - 1 - 2\mu = 0. \quad (2.23)
\]

The above equation clearly has the solution \( w_1 = w_2 = 1 \), regardless of the choice of the collision parameter values. This equilibrium point, according to the interpretation of transformation (2.21), corresponds to the state with an identical phase and mean occupation in all three local modes. We denote this solution by \( s_1 \), which is the intersection of the three invariant surfaces in (2.22). Excluding the solution \( s_1 \) from (2.23), we obtain a cubic equation for the remaining real fixed points:

\[
    w_1^3 + \frac{(1 - \chi - 2\mu)}{2(1 + \mu)}w_1^2 + \frac{(1 - \chi + \mu)}{2(1 + \mu)}w_1 + \frac{(1 + 2\mu)}{4(1 + \mu)} = 0. \quad (2.24)
\]

The number of real solutions of equation (2.24) depends on the sign of the following discriminant:

\[
    \Delta = -\chi^2 - 2(3 + 7\mu)\chi^3 + 3(2 - 11\mu^2)\chi^2 + 2(5 + 12\mu - 18\mu^2 - 52\mu^3)\chi
    + 2(9 + 76\mu + 228\mu^2 + 264\mu^3 + 76\mu^4). \quad (2.25)
\]

This discriminant has two real roots in \( \chi \); the positive (negative) root is denoted by \( \chi_+(\mu) \) (\( \chi_-(\mu) \)). If \( \chi_-(\mu) < \chi < \chi_+(\mu) \), equation (2.24) has only one real solution that we denote by \( s_2(\chi, \mu) \). However, for \( \chi = \chi_\pm(\mu) \) we have a new equilibrium point, which undergoes a bifurcation for \( \chi > \chi_+(\mu) \) and \( \chi < \chi_-(\mu) \), giving rise to the solutions denoted by \( s_3(\chi, \mu) \) and \( s_4(\chi, \mu) \). Figure 1 shows the number of real solutions of equation (2.23) as a function of the collision parameters. Note that for \( \chi = \frac{5}{3} + \mu \) we have only three real solutions to (2.23), because on this curve the solutions \( s_3 \) and \( s_4(\chi, \mu) \) become degenerate.
Figure 1. The number of real solutions of equation (2.23) as a function of the collision parameters. The light grey curves represent the transitional situation of the existence of only three real solutions. Note that only the quadrants where $\chi \mu > 0$ are physically accessible since the two collision parameters must have the same sign.

Figure 2. (a) Stability analysis of the equilibrium point $s_2(\chi, \mu)$ as a function of collision parameters. (b) Stability analysis of the single depleted well states. Note that these are unstable equilibrium points on the curve $\chi = \mu$.

By the linearization of the equations of motion (2.20), we can analyse the stability of the dynamics in the vicinity of each equilibrium point. Although the position of the equilibrium point $s_1$ does not depend on the collision parameters, we can show the following condition of stability for this equilibrium point

$$\lambda_{s_1}^2 = \frac{|\Omega|^2}{3} (3 + 4\mu)(4\chi - 9 - 4\mu) < 0.$$ (2.26)

Therefore, considering $\mu > -\frac{3}{4}$, $s_1$ is stable for $\chi < \frac{9}{4} + \mu$. Note that this critical curve of stability change coincides with the curve of degeneracy of $s_1$ and $s_3(\chi, \mu)$.

The behaviour of the fixed point $s_2(\chi, \mu)$ as a function of the collision parameters is shown in figure 2(a). Therefore, for $|\chi| \gg |\mu|$, $s_2$ is generally stable (unstable) for $\chi < 0$ ($\chi > 0$). Also for $|\chi| \gg |\mu|$, we can demonstrate that the equilibrium point $s_3$ ($s_4$) is unstable
(stable) in its region of existence in the parameter space, thus characterizing a *saddle-node* bifurcation\(^4\).

### 2.2.2. Single depleted well states.

Returning to the equations of motion, we can find two more real equilibrium points with coordinates independent of the collision parameters:

\[
  w_j = -1, \quad w_k = 0
\]

for \( j, k = 1, 2 \) and \( j \neq k \). The fixed points above are known as *single depleted well states* (SDWS) [11], and according to transformation (2.21) they always possess one completely empty local mode, while the two other modes have opposite phases and the same average population\(^5\). The SDWS have the following stability conditions, also shown in figure 2 (b):

\[
  \lambda_{\text{SDW}, \pm}^2 = \frac{\Omega_1^2}{2} \left[ -9 + 4\mu(10 + 11\mu) + \chi(2 + \chi) \right] \pm \left[ (9 + 16\mu - \chi) \right. \\
  \left. \times (9 + 128\mu^2 - 8\mu^2(\chi - 19) + \chi(5 + 3\chi - \chi^2) + 16\mu(4\chi + \chi^2)) \right]^{1/2} < 0.
\]

(2.28)

### 2.2.3. Vortex states.

Within the classical approximation, the angular momentum of the condensate along the symmetry axis of the trapping potential is proportional to

\[
  \langle N; w_1, w_2 | J_3 | N; w_1, w_2 \rangle = \frac{2N}{|w_1|^2 + |w_2|^2 + 1} \text{Im}(w_1 - w_2 + w_1^* w_2). \quad (2.29)
\]

Therefore, the classical angular momentum is directly related to the imaginary parts of the variables \( w_1 \) and \( w_2 \). Since all equilibrium points found previously have real coordinates, they represent irrotational condensate configurations. However, equations (2.20) have a further pair of fixed points

\[
  w_1 = e^{i\frac{\pi}{3}}, \quad w_2 = e^{i\frac{2\pi}{3}}.
\]

(2.30)

According to (2.21), these latter equilibrium points have the same mean occupation in all three local modes, since \( |w_1| = |w_2| = 1 \), but the phase difference between each pair of local condensates is \( \pm \frac{\pi}{2} \), which is the angle of rotational symmetry of the trapping potential. The states corresponding to (2.30) are known as *vortex states* [11], due to their nonzero angular momentum, which is proportional to the total number of condensate bosons:

\[
  \langle N; w_1 = w_2^* = e^{i\frac{\pi}{3}} | J_3 | N; w_1 = w_2^* = e^{i\frac{2\pi}{3}} \rangle = \pm \sqrt{3}N.
\]

(2.31)

Note that the two vortex states are equivalent, because they differ only in their sense of rotation. The stability condition for the vortex states is given by

\[
  \lambda_{\text{V}, \pm}^2 = \frac{\Omega_1^2}{6} \left[ -104\mu^2 - 16\mu(6 + \chi) - 3(9 + 4\chi) \right. \\
  \left. \pm \sqrt{3}(3 + 4\mu)(3 + 8\mu)^2(9 + 4\mu + 8\chi) \right] < 0.
\]

(2.32)

Therefore, concerning \( \mu > -\frac{1}{4} \), the vortex states are stable for \( \chi \geq -\frac{4\mu + 9}{8} \), except on the curve \( \chi = 2\mu(6 + 11\mu)/(3 + 4\mu) \).

---

\(^4\) We do not show the algebraic conditions of stability for \( s_2, s_1 \) and \( s_4 \) because they are exceedingly long and complicated, due mostly to the position dependence of these equilibrium points on \( \chi \) and \( \mu \).

\(^5\) The rotational symmetry of the trapping potential indicates the presence of a third single depleted well state, corresponding to the complete depletion of the third well. However, according to transformations (2.21), this third equilibrium point would receive the coordinates \( w_1, w_2 \to \infty \).
3. Condensate dynamics

3.1. Twin-condensate dynamics

The integrable sub-regime of twin condensates is a direct consequence of the symmetry of $\hat{H}$ under the permutation of indices of the three local modes. The equivalence of the three local modes remains in the Hamiltonian (2.18), which has the permutation symmetry of the quantities $\alpha w_1, \alpha w_2$ and $\alpha$, where $\alpha = (|w_1|^2 + |w_2|^2 + 1)^{-\frac{1}{2}}$.

Without loss of generality, we study the twin-condensate sub-regime under the restriction $w_1 = w_2$, because the invariant surfaces in (2.22) are dynamically equivalent. Applying this condition to the coherent states in (2.16), we obtain

$$|N; w_1 = w_2\rangle = \sum_{m_1 + m_2 = N} \frac{\sqrt{N!}}{m_1! m_2!} \left( \frac{\sqrt{2} w_1}{2|w_1|^2 + 1} \right)^{m_1 + m_2} |0\rangle.$$  \hspace{1cm} (3.1)

Now it is convenient to change the basis of the single-particle Hilbert space. Such transformation may be described by the following linear combinations of the bosonic operators:

$$\begin{align*}
b_1^\dagger &= a_1^\dagger + a_2^\dagger \sqrt{2} \\
b_2^\dagger &= a_3^\dagger \\
b_3^\dagger &= a_1^\dagger - a_2^\dagger \sqrt{2}.
\end{align*}$$ \hspace{1cm} (3.2)

The set of states $\{|m_1, m_2, m_3\rangle\}$ constitutes a new basis for the bosonic Fock space, where $m_j$ represents the eigenvalue of the number operator $b_j^\dagger b_j$. Note that the SU(3) coherent state restricted to the subspace of twin condensates has no occupation in the mode related to the operator $b_3^\dagger b_3$, also called the opposite phase mode, evidencing the phase equality between the twin condensates in the classical dynamics. Besides, the coherent state (3.3) is also an SU(2) coherent state with complex coordinate $\sqrt{2} w_1 = e^{-i\theta} \sec \frac{\phi}{2}$, indicating that the invariant surface of twin condensates is topologically isomorphic to the unit sphere ($S^2$) [24].
Figure 3. Classical trajectories in the invariant subspace of twin condensates \( w_1 = w_2 = \frac{c}{\lambda} \sin \varphi \) for \( N = 30 \), \( |\Omega| = 1 \), \( \mu = 0 \) and self-collision parameter values: (a) \( \chi = 1.5 \), (b) \( \chi = 1.98 \) and (c) \( \chi = 3 \). The circles (crosses) indicate the location of stable (unstable) fixed points with respect to the complete four-dimensional phase space of the three-mode condensate. The angles \( \phi \) and \( \theta \) are similar to the usual spherical coordinate angles, except that, for convenience, we take the origin of \( \theta \) along the negative \( z \)-axis.

The states with a zero mean occupation in the opposite phase mode, given by 
\[
\langle \psi | b_1^\dagger b_3 | \psi \rangle = 0 \\
\text{or} \\
|\psi\rangle = \sum_{m_1,m_2} c_{m_1,m_2} |m_1, m_2, 0\rangle \\
\text{with } c_{m_1,m_2} \in \mathbb{C},
\]
satisfy the following identities:
\[
\langle \psi | a_1^\dagger a_1 + a_2^\dagger a_2 | \psi \rangle = \langle \psi | a_1^\dagger a_2 + a_2^\dagger a_1 | \psi \rangle; \\
\langle \psi | a_1^\dagger a_1 | \psi \rangle = \langle \psi | a_2^\dagger a_2 | \psi \rangle; \\
\frac{d}{dt} \langle \psi | Q_1 | \psi \rangle = 0. \tag{3.4}
\]

Therefore, the quantum dynamics preserves the population equality between the twin condensates, because the average of the population imbalance operator \( Q_1 \) remains zero during the quantum evolution for all the initial states in the subspace generated by the basis \( \{|m_1, m_2, 0\rangle\} \). However, the quantum dynamics does not preserve the identical phases of twin condensates, unlike the classical dynamics, because the average of \( b_1^\dagger b_3 \) does not remain constant, in spite of its initial zero value.

At this point it is interesting to define the generators of a subalgebra of \( \text{su}(3) \) isomorphic to \( \text{su}(2) \), which are related to the modes present in the twin condensate sub-regime:
\[
S_x = \frac{b_1^\dagger b_2 + b_2^\dagger b_1}{2}, \\
S_y = \frac{b_1^\dagger b_2 - b_2^\dagger b_1}{2}, \\
S_z = \frac{b_3^\dagger b_1 - b_1^\dagger b_3}{2}. \tag{3.5}
\]

Now, employing identities (3.4), we can show that
\[
\langle \psi | S_z | \psi \rangle = \frac{1}{2} \langle \psi | a_1^\dagger a_1 + a_2^\dagger a_2 - a_3^\dagger a_3 | \psi \rangle. \tag{3.6}
\]

Thus, \( S_z \) is the population imbalance operator between the twin condensates and the solitary mode, but only if the average occupation of the opposite phase mode is zero.

Figure 3 displays the classical dynamics of the three-mode model in the invariant subspace of twin condensates, which is represented by the unit sphere, for various values of the
self-collision rate $\chi$, but neglecting the presence of cross-collisions. Observe that the Cartesian coordinates directly represent the normalized classical averages of the SU(2) generators:

$$
\begin{align*}
I_x &= \frac{2}{N} \langle N; w_1 = w_2 | S_x | N; w_1 = w_2 \rangle = \sin \theta \cos \phi \\
I_y &= \frac{2}{N} \langle N; w_1 = w_2 | S_y | N; w_1 = w_2 \rangle = \sin \theta \sin \phi \\
I_z &= \frac{2}{N} \langle N; w_1 = w_2 | S_z | N; w_1 = w_2 \rangle = -\cos \theta.
\end{align*}
$$

Figure 3(a), for $\chi = 1.5 < \chi_*$ ($\mu = 0$) $\approx 1.971$, shows the classical dynamics in the absence of the equilibrium points $s_3$ and $s_4$. Note that all orbits surround the stable equilibrium point $s_1$ and, therefore, the population imbalance varies around the value $I_z(w_1 = w_2 = 1) = \frac{1}{4}$, which represents an identical average population in all three local modes. Thus, there is no preferential occupation of any local mode, and this behaviour characterizes the dynamical regime known as Josephson oscillation (JO). In figure 3(b) we show the classical dynamics for $\chi = 1.98$, i.e. a self-collision rate value slightly higher than the critical bifurcation parameter. The bifurcation is accompanied by the appearance of a separatrix, which crosses the unstable equilibrium point $s_3$. The separatrix is the boundary between the orbits of the JO regime and the trajectories around the new stable equilibrium point $s_4$. The trajectories near $s_4$ vary around the negative values of $I_z$, indicating a preferential occupation of the third well. Therefore, the tunneling between twin condensates and solitary mode is suppressed, giving rise to the effect known as macroscopic self-trapping (MST). The JO and MST sub-regimes are also present in the double-well model, whose classical approximation with SU(2) coherent states is quite similar to the dynamics of twin condensates [25].

Figure 3(c) displays the dynamics of twin condensates for $\chi = 3$, a parameter value significantly higher than $\chi_*$. For $\chi > \frac{9}{2}$, the fixed point $s_1$ becomes unstable and takes the place of $s_3$ in the separatrix. Although the orbits around $s_3$ in the subspace of twin condensates are regular, we should note that this equilibrium point is not a stable centre in the complete phase space since $s_3$ behaves like a saddle point in the directions orthogonal to the surface $w_1 = w_2$. Therefore, the self-trapping dynamics which favors the occupation of the twin condensates, characterized by the regular trajectories with $I_z > \frac{1}{4}$ in the vicinity of $s_3$, is restricted to the invariant surface. Besides, we observe that an increase in the self-collision rate is accompanied by an expansion of the region on $S^2$ occupied by MST orbits.

Figure 4 shows the effect of cross-collisions on the dynamics of the mean population imbalance $I_z$ for the initial coherent state $|N, w_1 = w_2 = 0 \rangle = |0, 0, N \rangle$, corresponding to the south pole in figure 3. The solid (dashed) curve in figure 4(a) displays the classical time evolution of $I_z$ for $N = 30$, $|\Omega| = 1$, $\chi = 4$ and $\mu = \frac{1}{10} (\mu = \frac{1}{8})$. Note that the greater value of the cross-collision parameter inhibits the MST, due to its correspondent increase of the effective tunneling rate $\Omega^2 = \Omega (1 + 2\mu)$. Therefore, the condensate dynamics can be completely changed by the cross-collisions, even when the collision parameters $\mu$ and $\chi$ have different magnitudes.

Figure 4(b) exhibits the quantum time evolution of $I_z$ for a direct comparison with the classical results. Note that, unlike previous classical results, the amplitude of quantum
oscillations does not remain constant during the evolution of the condensate, evidencing a quantitative disagreement between the two approaches. However, the solid (dashed) curve properly characterizes the MST (JO) regime, because the fluctuations in the population imbalance remain around $I_z < 0$ ($I_z = 1/3$). Consequently, the qualitative agreement between the classical and quantum approaches persists, even for a number of particles as small as $N = 30$.

3.1.1. Generalized purity associated with the su(3) algebra. The classical equations of motion (2.20) do not depend on the total number of trapped bosons since they represent the dynamics in the macroscopic limit $N \to \infty$. Therefore, the qualitative agreement between the two approaches would not be expected for a small number of particles such as $N = 30$, unlike observed in figure 4. Although classical theories are quite common in the treatment of the condensate dynamics, little is known about the quality of these approximations with respect to the exact quantum calculation for a microscopic or mesoscopic condensate. The classical dynamics is almost exact for a macroscopic number of particles, but for a mesoscopic or microscopic number of bosons we must be able to evaluate quantitatively the quality of our approximations as a function of $N$ and the propagation time, because in an usual experiment the number of condensate particles can range from a few hundred to the order of $10^{10}$ bosons [26].

Our classical approximation consists of restricting the system time evolution to the nonlinear subspace of coherent states. However, the exact quantum dynamics for few particles can promote the departure of an initial coherent state from the classical subspace, thus introducing quantitative errors in our approximation. The generalized purity associated with the su(3) algebra, which is a measure capable of quantifying the proximity of a given $N$-particle state $|\psi\rangle$ to the classical subspace, is given by [18]

$$P_{su(3)}(|\psi\rangle) = \frac{9}{N^2} \left( \frac{\langle \psi | Q_1 | \psi \rangle^2}{3} + \frac{\langle \psi | Q_2 | \psi \rangle^2}{4} + \sum_{j=1}^{3} \frac{\langle \psi | P_j | \psi \rangle^2}{12} + \sum_{j=1}^{3} \frac{\langle \psi | J_j | \psi \rangle^2}{12} \right). \quad (3.8)$$

The generalized purity is limited to the interval $0 \leq P_{su(3)}(|\psi\rangle) \leq 1$, but we have $P_{su(3)}(|\psi\rangle) = 1$ if and only if $|\psi\rangle$ is a coherent state. On the other hand, the value of $P_{su(3)}(|\psi\rangle)$ is decreasing with the ‘distance’ of $|\psi\rangle$ to the subspace of coherent states. Therefore, $P_{su(3)}(|\psi\rangle)$ is a classicality measure, because the purity value is increasing with
the classical character of $|\psi\rangle$, taking the coherent states as the most classical pure states, due to their minimum uncertainty on the phase space.

Figure 5(a) displays the evolution of $\mathcal{P}_{su(3)}$ for the same initial state and parameters in figure 4. Note that the state initially suffers a sudden purity loss, which is responsible for the quantitative disagreement between the classical and quantum results. Therefore, only for a very short time interval the classical approximation roughly coincides with the quantum dynamics, when considering a small number of particles. However, after a brief initial period of purity loss, $\mathcal{P}_{su(3)}$ only exhibits fluctuations around a stable value. The purity loss in the MST regime is inferior to the JO regime, indicating a better quality of the classical approximation in the self-trapping dynamics. The orbits associated with the JO regime traverse a larger region of the phase space compared with the localized MST trajectories, as shown in figure 3. Therefore, the quantum states related to the JO regime have greater delocalization (uncertainty) on the phase space, which is responsible for their accentuated classicality loss.

Within the classical approximation, the invariant subspaces of twin condensates are defined by the equality of the phase and mean occupation between two local modes. Although the quantum dynamics preserves the population equality between identical local condensates, according to equation (3.4), the phase difference does not remain zero. In general, using transformation (3.2) in the Hamiltonian (2.8), we can show that

$$\frac{d}{dt} \langle N; w_1 = w_2 | b_3^\dagger b_3 | N; w_1 = w_2 \rangle \neq 0, \quad (3.9)$$

except for $\chi = \mu = 0$ or $N \to \infty$, when the classical approximation is exact. Therefore, the subspaces with no phase difference between two local modes are not quantum invariant. The population dynamics of the opposite phase mode for the states $|N, w_1 = w_2 \rangle$ cannot be described classically, and therefore it represents a process of purity (classicality) loss. Figure 5(b) shows the behaviour of $\langle b_3^\dagger b_3 \rangle$ for the same initial state and parameters as of the previous figure. In comparison with figure 5(a), we observe that a higher occupation of the opposite phase mode is accompanied by a greater purity loss. Therefore, the nonzero occupation of the opposite phase mode is partially responsible for breaking the quantum–classical correspondence in the sub-regime of twin condensates. Moreover, the quantity $\langle b_3^\dagger b_3 \rangle$ can be used as a measure of validity (quality) of the classical approximation in the invariant subspaces.
3.2. Single depleted well dynamics

It is well known that the sub-regime of twin condensates is equivalent to the dynamics of a condensate in a double-well potential [7, 8]. Therefore, the SDWS represent the first example of an entirely new population dynamics for the three-mode model.

According to transformation (2.21), the SDWS have a completely empty local mode, while the other two modes remain with the same mean occupation and opposite phases. Similar to the self-trapping regime, the SDWS represent a dynamical effect of tunneling suppression, but the opposite phases between the preferentially occupied local modes distinguish the latter regime.

Due to the rotational symmetry of the trapping potential, all the SDWS are equivalent. Thus, without loss of generality, we restrict our results to the region of phase space in the vicinity of \((w_1 = -1, w_2 = 0)\), where the second local mode is empty.

The SDWS have fixed positions in the phase space, but their stability varies as a complicated function of the collision parameters \(\chi\) and \(\mu\), according to condition (2.28) and figure 2(b). Figure 6(a) shows the Poincaré section at \(\phi_2 = 0\) for \(\chi = 5\) and \(\mu = \frac{\chi}{100}\), considering only the trajectories with the same energy as the SDWS. Note that the section presents the stable SDWS located in the centre of a bundle of regular orbits. The energetically accessible region of phase space is filled by orbits with a behaviour similar to the SDWS since they exhibit only small fluctuations in the average occupation of the second local mode, while the other two modes oscillate in opposite phase around \(N_2\).

Figure 6(b) shows the Poincaré section in the vicinity of the SDWS for \(\chi = -5\) and \(\mu = \frac{\chi}{100}\). Here we see an unstable SDWS inserted in a chaotic region of the phase space. We also observe that the SDWS are the boundary between two distinct sets of chaotic orbits, because the trajectories above (below) the equilibrium point in this section preserve the condition \(K_1 > \frac{\chi^2}{2} (K_1 < \frac{\chi^2}{2})\) during their time evolution. Unlike the stable case, the energetically accessible region of phase space in the vicinity of the fixed point does not show a behaviour similar to the SDWS since there is no persistent depletion of the second local mode.
Figure 7. Left (right) : the classical population dynamics for an initial condition near one of the SDWS, considering $N = 30, |\Omega| = 1, \chi = 5 (\chi = -5), \mu = \frac{\chi}{100}$.

Figure 8. Left (centre): the quantum evolution of the average occupations in the three local modes for $N = 30, |\Omega| = 1, \chi = 5 (\chi = -5), \mu = \frac{\chi}{100}$, considering as the initial state the coherent state centred at the initial condition of the trajectory in figure 7(a) (figure 7(b)). Note in (b) that the curves for $\langle a_1^\dagger a_1 \rangle$ and $\langle a_3^\dagger a_3 \rangle$ are almost superimposed. The solid and dashed curves in (c) represent the dynamics of $P_{su(3)}$ corresponding to the results displayed in (a) and (b), respectively.

The regular population dynamics in the vicinity of the SDWS for $\chi = 5$ and $\mu = \frac{\chi}{100}$ is illustrated in figure 7(a), employing the initial condition indicated by the square marker in figure 6(a). As expected, the second local mode remains almost empty, while the two remaining modes exhibit periodic population inversions around $\frac{N}{2}$. Figure 7(b) shows the chaotic population dynamics for $\chi = -5$ and $\mu = \frac{\chi}{100}$, given an initial condition very close to the SDWS with $K_1 > \frac{N}{2}$. Note that the second local mode does not remain empty, and it presents successive population inversions with the third mode, while the first mode holds more than half of the trapped bosons.

For direct comparison with the previous classical results, the exact quantum population dynamics is shown in figure 8, where we used as initial states the coherent states centred at the initial conditions in figure 7. Figure 8(a) displays the quantum population dynamics corresponding to the regular trajectory near the SDWS. The mean occupation in the second local mode is slightly higher than in the classical approximation, but the preferential occupation in the other two modes still prevails. Therefore, there is fair agreement between the exact
quantum result and the classical approximation when we consider a regular orbit close to the stable SDWS. However, the quantum dynamics corresponding to the chaotic trajectory, shown in figure 8(b), has little resemblance to its classical analogue. Unlike the classical approximation, there is no preferential occupation of the first mode, which displays values below \( \sqrt{N/2} \). On the other hand, the second well does not remain empty, as expected for the irregular dynamics near the SDWS.

The dynamics of \( P_{\mu(3)} \) confirms that the classical approximation is more accurate for the regular trajectory, as we see in figure 8(c). The state associated with the chaotic dynamics has a large purity loss, which corresponds to a large delocalization in the phase space. Conversely, the time evolution of the state under the regular dynamics preserves its similarity to the coherent states, i.e. the state remains well localized in the phase space and exhibits relatively high values of purity during its evolution.

### 3.3. Vortex dynamics

The vortex states are the only equilibrium points of the classical approximation with nonzero imaginary parts. Thus, they are the only ones with nonzero angular momentum along the symmetry axis of the trapping potential. To emphasize the imaginary parts of the complex variables \( w_j \) during their time evolution, we introduce a new set of canonical variables:

\[
    w_j = \frac{q_j + ip_j}{\sqrt{2N - q_1^2 - p_1^2 - q_2^2 - p_2^2}}.
\]

(3.10)

The real variables \( q_j \) and \( p_j \) are directly related to the real and imaginary parts of \( w_j \), respectively. Note also that these new dynamical variables must satisfy the condition \( 2N \geq q_1^2 + p_1^2 + q_2^2 + p_2^2 \). We show below only the results for the vortex state parametrized by \( w_1 = e^{i\pi/3} = w_2^* \) since the two vortex states differ only by the sense of rotation, as previously discussed.

Figure 9 shows the dynamics in the vicinity of the vortex state for two completely different situations. For \( \chi = -1 \) and \( \mu = -100 \), the Poincaré section at \( p_2 = -\sqrt{N/2} \) displays the stable vortex state in a regular region of phase space. However, for \( \chi = -5 \) and \( \mu = 1000 \), the vortex state is unstable and the dynamics of the system is almost completely chaotic.
Figure 10. Left (middle): the classical (quantum) dynamics of $\langle J_3 \rangle$, quantity directly proportional to the angular momentum of the condensate along the symmetry axis of the trapping potential. The solid (dashed) curve shows the time evolution for $N = 30$, $|\Omega| = 1$, $\chi = -1$ ($\chi = -5$), $\mu = \frac{\hbar}{2} \Omega$ and the initial condition indicated by the star marker in figure 9(a) (figure 9(b)). Right: the dynamics of $P_{\text{su}(3)}$ for the same initial conditions.

According to equation (2.14), the angular momentum of the condensate along the symmetry axis of the trapping potential is directly proportional to the su(3) operator $J_3 = J_1 + J_2 + J_3$. Therefore, the rotational dynamics of the condensate is illustrated in figure 10, where we consider the initial conditions indicated by the star markers in figure 9. The solid (dashed) curve in figure 10(a) shows the classical dynamics of a regular (chaotic) trajectory near the vortex state. Note that only the regular trajectory represents a persistent collective rotation of the condensate. Therefore, the rotation of the condensate in a preferential sense is present only when the vortex state is stable. The chaotic trajectory does not exhibit preferential sense of rotation, but the presents oscillations of angular momentum with large amplitudes. However, the maximum value of $\langle J_3(t) \rangle$ in both trajectories remains bounded by $\sqrt{3}N$, which corresponds to the mean value of $J_3$ for the vortex state, according to (2.31).

Figure 10(b) shows the quantum dynamics of rotation, where we consider the initial coherent states centred on the initial conditions in figure 10(a). Note that even for $N = 30$, a relatively small number of trapped bosons, the quantum results of $\langle J_3(t) \rangle$ associated with the regular trajectory demonstrate good agreement with the classical approximation. However, in the chaotic case, the amplitude of the oscillations are strongly attenuated. Thus, as expected, the approximation is not quantitatively satisfactory in the chaotic regime, although the absence of a preferential sense of rotation is evidenced in both approaches. Figure 10(c) displays the dynamics of $P_{\text{su}(3)}$ for the two orbits previously selected. Note that the purity loss in the chaotic regime is much faster and more intense when compared to the regular system, confirming the differences found in the precision of the classical approximation.

4. Conclusion

In this work, we have investigated the quantum dynamics of a Bose–Einstein condensate confined in a symmetric triple-well potential employing an approximation based on the TDVP and the SU(3) coherent states. We have also considered the effects of the cross-collisional terms, which do not introduce new dynamical regimes, but their contribution to the tunneling term can significantly modify the structure of the classical phase space.

We have exploited maximally the use of coherent states as initial states chosen to be centred in the appropriate points of the classical manifold, particularly near the equilibrium points of the classical equations of motion. This scheme allowed us to capture in the complete
quantum calculations some of the particularities of the classical solutions. Moreover, in order to evaluate the quality of the classical Hamiltonian evolution, as compared to the quantum results, we employed the purity associated with the su(3) algebra. The generalized purity measures the departure from the subspace of coherent states, in which our classical approximation is restricted, thus enabling us to quantify the quality of the approximation. The classical approximation quality depends not only on the total number of particles, but also on the location in the phase space of the initial coherent state and the regularity of the Hamiltonian flow in its vicinity. We concluded that even for a relatively small number of particles such as \( N = 30 \), which is very far from the classical–macroscopic limit, the classical results obtained for the regular dynamical regime display fair agreement with the quantum time evolution, unlike the chaotic regime.

Considering the sub-regime of twin condensates, we have shown that the self-trapping dynamics may be suppressed by the cross-collisions effects on the effective tunneling rate, which can become quite important for a large number of trapped bosons. Since this sub-regime is classically integrable, its classical approximation exhibits excellent results. Whereas, the single depleted well regime, which is characterized by the complete depletion of a local mode, can be effectively eliminated in phase space when its corresponding fixed points become unstable, giving way to chaos. In the chaotic regime, a single Hamiltonian trajectory clearly does not account for the quantum dynamics of an initial coherent state at long times. This fact is precisely indicated by the generalized purity. In the same line of reasoning, the persistent collective rotation in a preferential sense, presented in the vicinity of the vortex states in phase space, can only be properly observed when the corresponding fixed points are stable.

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References

[1] Parkins A S and Walls D F 1998 Phys. Rep. 303 1–80
Dalfovo F, Giorgini S, Pitaevskii L and Stringari S 1999 Rev. Mod. Phys. 71 463–512
Cornell E A and Wieman C E 2002 Rev. Mod. Phys. 74 875–93
[2] Milburn G J, Corney J, Wright E M and Walls D F 1997 Phys. Rev. A 55 4318–24
Smerzi A, Fantoni S, Giovanazzi S and Shenoy S R 1997 Phys. Rev. Lett. 79 4950–53
[3] Hines A P, McKenzie R H and Milburn G J 2003 Phys. Rev. A 67 013609
Sanz L, Angelo R M and Furuya K 2003 J. Phys. A: Math. Gen. 36 9737–54
[4] Viscondi T F, Furuya K and de Oliveira M C 2009 Phys. Rev. A 80 013610
[5] de Oliveira M C and da Cunha B R 2009 Int. J. Mod. Phys. B 23 S867–80
[6] Buonsante P, Franzosi R and Penna V 2003 Phys. Rev. Lett. 90 050404
[7] Trimborn F, Witthaut D and Korsch H J 2009 Phys. Rev. A 79 013608
Buonsante P, Franzosi R and Penna V 2009 J. Phys. A: Math. Theor. 42 285307
[8] Buonsante P, Franzosi R and Penna V 2004 Laser Phys. 14 556–64
Buonsante P, Franzosi R and Penna V 2004 J. Phys. B: At. Mol. Opt. Phys. 37 5229–38
[9] Nemoto K, Holmes C A, Milburn G J and Munro W J 2000 Phys. Rev. A 63 013604
Franzosi R and Penna V 2001 Phys. Rev. A 65 013601
[10] Buonsante P, Franzosi R and Penna V 2003 Laser Phys. 13 537–42
  Pando L C L 2005 Laser Phys. 15 643–9
  Pando L C L and Doedel E J 2007 Phys. Rev. E 75 016213
  Kolovsky A R 2007 Phys. Rev. Lett. 99 020401
[11] Franzosi R and Penna V 2003 Phys. Rev. E 67 046227
[12] Furuya K, Nemes M C and Pellegrino G Q 1998 Phys. Rev. Lett. 80 5524
[13] Kramer P and Saraceno M 1980 Geometry of the Time-Dependent Variational Principle in Quantum Mechanics (Lecture Notes Physics vol 140) (New York: Springer)
[14] Perelomov A M 1986 Generalized Coherent States and Their Applications (Berlin: Springer)
  Gilmore R 1972 Ann. Phys., NY 74 391–463
  Zhang W-M, Feng D H and Gilmore R 1990 Rev. Mod. Phys. 62 867–927
[15] Gilmore R, Bowden C M and Narducci L M 1975 Phys. Rev. A 12 1019–31
[16] Nemoto K 2000 J. Phys. A: Math. Gen. 33 3493–506
  Nemoto K and Sanders B C 2001 J. Phys. A: Math. Gen. 34 2051–62
  Trimborn F, Witthaut D and Korsch H J 2008 Phys. Rev. A 77 043631
  Mathur M and Sen D 2001 J. Math. Phys. 42 4181
  Raghu Nath K, Seetharaman M and Vasan S S 1989 J. Phys. A: Math. Gen. 22 L1089–92
[17] Viscondi T F, Furuya K and de Oliveira M C 2010 Eur. Phys. Lett. 90 10014
[18] Klyachko A A 2002 Coherent states, entanglement, and geometric invariant theory arXiv:quant-ph/0206012v1
  Somma R, Ortiz G, Barnum H, Knill E and Viola L 2004 Phys. Rev. A 70 042311
  Barnum H, Knill E, Ortiz G and Viola L 2003 Phys. Rev. A 68 032308
  Barnum H, Knill E, Ortiz G, Somma R and Viola L 2004 Phys. Rev. Lett. 92 107902
  Delbourgo R and Fox J R 1977 J. Phys. A: Math. Gen. 10 L233
  Delbourgo R 1977 J. Phys. A: Math. Gen. 10 1837
[19] Pethick C J and Smith H 2002 Bose–Einstein Condensation in Dilute Gases (Cambridge: Cambridge University Press)
[20] Zhang W-M, Feng D H, Yuan J-M and Wang S-J 1989 Phys. Rev. A 40 438–47
  Zhang W-M, Feng D H and Yuan J-M 1991 Phys. Rev. A 42 7125–50
[21] de Aguiar M A M, Furuya K, Lewenkopf C H and Nemes M C 1992 Ann. Phys., NY 216 291–312
[22] Ribeiro A D, de Aguiar M A M and de Toledo Piza A F R 2006 J. Phys. A: Math. Gen. 39 3085–97
[23] Yaffe L G 1982 Rev. Mod. Phys. 54 407–35
[24] Arecchi F T, Courtens E, Gilmore R and Thomas H 1972 Phys. Rev. A 6 2211–37
[25] Viscondi T F, Furuya K and de Oliveira M C 2009 Ann. Phys., NY 324 1837–46
[26] Leggett A J 2001 Rev. Mod. Phys. 73 307–56