Eikonal Methods in AdS/CFT: Regge Theory and Multi–Reggeon Exchange

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Abstract

We analyze conformal field theory 4–point functions of the form $A \sim \langle O_1(x_1) O_2(x_2) O_1(x_3) O_2(x_4) \rangle$, where the operators $O_i$ are scalar primaries. We show that, in the Lorentzian regime, the limit $x_1 \rightarrow x_3$ is dominated by the exchange of conformal partial waves of highest spin. When partial waves of arbitrary spin contribute to $A$, the behavior of the Lorentzian amplitude for $x_1 \rightarrow x_3$ must be analyzed using complex–spin techniques, leading to a generalized Regge theory for CFT’s. Whenever the CFT is dual to a string theory, the string tree–level contribution $A_{\text{tree}}$ to the amplitude $A$ presents a Regge pole corresponding to a gravir–reggeon exchange. In this case, we apply the impact parameter representation for CFT amplitudes, previously developed, to analyze multiple reggeon exchanges in the eikonal limit. As an example, we apply these general techniques to $\mathcal{N} = 4$ super–Yang–Mills theory in $d = 4$ in the limit of large ’t Hooft coupling, including the leading string corrections to pure graviton exchange.

1 Introduction

In this work, we wish to reconsider the theory of Regge poles \cite{1, 2} in the context of the AdS/CFT duality \cite{3}. Most of the discussion will be phrased uniquely in terms of CFT data, and is therefore independent of the existence of a gravity dual. On the other hand, whenever a string dual is present, we expect that tree–level string interactions will be dominated at high energy by a gravi–reggeon pole in AdS. In this case, we shall show how to resum multi–reggeon interactions in the eikonal limit, using an impact parameter representation for CFT amplitudes developed in \cite{4, 5, 6}. We take as an example $\mathcal{N} = 4$ $SU(N)$ supersymmetric Yang–Mills theory in $d = 4$ at large ’t Hooft coupling, where the gravi–reggeon pole corresponds dually to the pomeron trajectory \cite{7}. We analyze the leading string corrections to pure graviton exchange, following \cite{0, 7, 8, 9}.

Recall that, due to the presence of a time–like boundary, the AdS analogue of the flat space scattering matrix computes correlators $\langle O_1(x_1) \cdots \rangle$ in the dual CFT. Moreover, the role played in flat space scattering by external particle

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momenta is now played by the boundary points $x_i$ in the dual correlator. This interplay of AdS “momenta” and CFT positions is at the heart of the simplicity of Regge theory for CFT’s in position space. In particular, we shall focus our attention on the AdS analogue of $2 \rightarrow 2$ scattering in flat space, which computes the conformal field theory correlator

$$A \sim \langle O_1(x_1) O_2(x_2) O_1(x_3) O_2(x_4) \rangle_{\text{CFT}},$$

where the operators $O_i$ are scalar primaries. In analogy with a flat space scattering process, we shall consider the operators $O_1(x_1)$ and $O_2(x_2)$ to be dual to incoming states, whereas $O_1(x_3)$ and $O_2(x_4)$ will be dual to outgoing ones. Accordingly, we shall denote with $S$– and $T$–channel the limits where $x_1 \rightarrow x_2$ and $x_1 \rightarrow x_3$, respectively.

Let us first recall the situation in flat space scattering. At high energy, interactions are dominated by the partial wave of highest spin $J$ exchanged in the $T$–channel, with a characteristic growth of the amplitude given by $s^{J}$. Whenever partial waves of all spins are present, the high energy limit of the amplitude must be determined with the aid of a Sommerfeld–Watson transform in the complex $J$–plane, which leads directly to Regge theory. Tree–level string theory in flat space is a paradigmatic example of this behavior, where massive higher spin interactions resum to an effective grav–reggeon pole with high energy behavior $s^{2+\frac{2}{d-4}}$. Moreover, in this case, higher loop string interactions can be resummed in the eikonal limit [10] at large transverse distances. This is achieved by determining the phase shift using the impact parameter representation of the leading tree level grav–reggeon amplitude.

Consider now the situation in conformal field theory. In the Euclidean regime of the CFT, the correlator (1) is determined, in the $T$–channel $x_1 \rightarrow x_3$, by the usual operator product expansion and is therefore dominated by conformal partial waves of lowest energy. The situation is therefore rather different from the analogous flat space $2 \rightarrow 2$ S–matrix amplitude, where highest exchanged spin dominates. On the other hand, if we consider the CFT amplitude in a specific Lorentzian kinematical regime, the situation is analogous to a flat space $S$–matrix $S$–matrix [6]. More specifically, we shall consider the situation where $x_4$ is in the future of $x_1$ and $x_2$ is in the future of $x_3$, with $x_1, x_4$ spacelike related to $x_2, x_3$ (see figure [4]). This kinematics corresponds, dually, to a true $2 \rightarrow 2$ scattering process in AdS and the limit $x_1 \rightarrow x_3$ of a $T$–channel partial wave of spin $J$ now behaves as $|x_3 - x_1|^{-J}$. As in flat space, highest spins now dominate. When the amplitude contains intermediate conformal partial waves of all spins $J$, its limiting form for $x_1 \rightarrow x_3$ must be analyzed by resumming all contributions in the complex $J$–plane. In this paper, we show how to perform the relevant analytic continuation in the complex $J$–plane and how to extract the leading behavior of the Lorentzian amplitude in the limit $x_1 \rightarrow x_3$. In section [3] we shall describe, in particular, the behavior of the amplitude $A$ when it is dominated by a leading Regge pole in the $J$–plane. The pole is described by a trajectory $j(\nu)$, where $\nu$ is the AdS analogue of transverse momentum transfer. More precisely, $\nu$ parameterizes the spectral decomposition of the Laplacian operator on the relevant transverse space, which turns out to be the hyperbolic space $H_{d-1}$, independently of the existence of a gravity AdS dual [5].

Finally, in section [7] we consider theories which admit a string dual description. The string tree level contribution $A_{\text{tree}}$ to the full amplitude will then
be dominated, in the T-channel, by a leading Regge pole, which corresponds to gravi-reggeon exchange in AdS. We then analyze multi-reggeon exchanges in the limit $x_1 \to x_3$ using eikonal techniques. This is easily achieved using the relevant impact parameter representation for the tree-level amplitude $A_{\text{tree}}$, which was developed in \cite{5} and which determines the leading contribution to the phase shift. Under the assumption that the phase shift is determined only by the leading tree contribution for large impact parameters, we are able to resum the contributions of multi-reggeon exchanges, neglecting self-interactions, as in flat space \cite{10}.

As an example of the general formalism, we conclude, in section \cite{5} with the discussion of the gravi-reggeon trajectory $j(\nu, g)$ dual to the pomeron trajectory of $\mathcal{N} = 4 \, SU(N)$ supersymmetric Yang-Mills theory in $d = 4$. We analyze the leading string corrections to the trajectory at large 't Hooft coupling, quickly rederiving the results of \cite{7}. Our general formalism then allows to resum multipomeron exchanges at large 't Hooft coupling in the large impact parameter limit. We will extend this analysis to the weak 't Hooft coupling regime in \cite{11}.

NOTE ADDED: As this work was being completed, a paper appeared \cite{9} which contains some overlapping results. The results in this paper have been presented at the GGI Workshop "String and M theory approaches to particle physics and cosmology", May 2007, at the 3rd RTN Workshop, Valencia 1–5 October 2007 and at Centro E. Fermi on September 5th, 2007.

2 Basic Kinematics

2.1 General Setting

In this first section, we shall review the results obtained in \cite{4,5,6} in a language which will be useful for the analysis of the rest of the paper. Subject of our analysis will be a $d$-dimensional conformal field theory $\text{CFT}_d$ defined on Minkowski space $M^d$. We parameterize points $x \in M^d$ with two lightcone coordinates $x^+, x^-$ and with a point $x$ in the Euclidean transverse space $\mathbb{E}^{d-2}$, and we choose metric

$$-dx^+ dx^- + dx \cdot dx .$$

We will be interested in the analysis of the correlator

$$A(x_1, x_2, x_3, x_4) = \langle O_1(x_1) O_2(x_2) O_1(x_3) O_2(x_4) \rangle_{\text{CFT}_d} ,$$

where $O_1$ and $O_2$ are scalar primary operators of dimension $\Delta_1$ and $\Delta_2$ respectively. As discussed in the introduction, the above correlator is the CFT analogue of $2 \rightarrow 2$ scattering in a quantum field theory with a conventional scattering matrix. Conformal invariance implies that $A$ can be written in terms of a reduced amplitude $\mathcal{A}$ as

$$A(x_1, \cdots, x_4) = \frac{1}{(x_{13}^2 + i\epsilon)^{\Delta_1} (x_{24}^2 + i\epsilon)^{\Delta_2}} A(z, \bar{z}) ,$$

\hspace{1cm} (2)
Figure 1: Basic Lorentzian kinematics for the correlator $A(x_1, x_2, x_3, x_4)$. The relevant light cones are shown in (a) and (b). The gray region in (b) with $x^+ > 0$ and $x^- < 0$ is mapped, under the conformal maps $\pi, \bar{\pi}$, to the future Milne cone $M$.

where $x_{ij} = x_i - x_j$ and where we have defined \[12\] the cross-ratios $z, \bar{z}$ by

\[
zz = \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2}, \quad (1 - z)(1 - \bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2}.
\]

We are working in a Minkowskian setting, and therefore the causal relations between the four points $x_i$ is of primary importance. In particular, we shall be interested in two basic configurations. Firstly, we will consider the Euclidean configuration, where all points $x_i$ are space-like related so that $x_{ij}^2 > 0$. This first case is directly related to the Euclidean version of the CFT, since the reduced amplitude $A$ is identical to the one of the Euclidean theory, whenever $\bar{z} = z^*$. We shall reserve from now on the name $A$ for the Euclidean amplitude.

Of basic importance for our discussion will be, on the other hand, configurations with a different causal relation between the points $x_i$, which really probe the Lorentzian nature of the theory at hand and which are more closely related to standard scattering theory. More precisely, we shall be interested in configurations, called Lorentzian from now on, where

\[
x_4 \text{ in the future of } x_1, \\
x_3 \text{ in the future of } x_2, \\
x_{12}^2, x_{13}^2, x_{24}^2, x_{34}^2 > 0,
\]

as shown in figure 1a. As discussed extensively in [6], in this regime the amplitude $A$ takes the same form [2], with a reduced amplitude

\[
\hat{A}(z, \bar{z}) = [A(z, \bar{z})]^{\circ}
\]

related to the Euclidean amplitude $A$ by analytic continuation. More precisely, the relevant continuation $\circ$ obtained by Wick rotation is shown in figure 2a, and

\footnote{Throughout the paper, we shall consider barred and unbarred variables as independent, with complex conjugation denoted by $\star$.}
Figure 2: Relevant analytic continuation in $z, \bar{z}$ for the Lorentzian amplitude $\hat{A}$ starting from the Euclidean amplitude $A$ with $\bar{z} = z^*$. In (b) we show the same paths under the map $z \rightarrow z/(z-1)$ which exchanges the roles of particles 2 and 4.

is given explicitly by

$$\bar{z} \text{ fixed },$$

$$z \text{ counter–clockwise around } 0, 1 .$$

For the purposes of this paper $z, \bar{z}$ are real and positive in $\hat{A}(z, \bar{z})$, with an infinitesimal positive imaginary part $+i\epsilon$.

In what follows we shall restrict, without loss of generality due to conformal invariance, to the following simpler configurations, as shown in figure 1b. We shall fix points $x_2, x_3$ and send $x_1, x_4$ to infinity by setting $x_1 = x_4 = 0$ and $x_1^- = x_4^+ = 0$, and by considering the limit

$$A(x_2, x_3) = (x_3^+)^{\Delta_1} (-x_2^-)^{\Delta_2} \lim_{x_1^- \rightarrow -\infty, x_1^+ \rightarrow +\infty} (-x_2^+)^{\Delta_1} (x_4^-)^{\Delta_2} A(x_1, \cdots, x_4) .$$

The Lorentzian conditions (3) now simply imply that

$$x_2^+, x_3^+ > 0 ,$$

$$x_2^-, x_3^- < 0 ,$$

$$x_3 \text{ in the future of } x_2 .$$

The cross–ratios are then determined by

$$z \bar{z} = \frac{x_3^- x_2^+}{x_3^+ x_2^-} ,$$

$$(1 - z)(1 - \bar{z}) = \frac{x_2^+ x_3^-}{x_2^- x_3^+} ,$$

and the amplitude is explicitly given by

$$A(x_2, x_3) = \left( \frac{x_2^+}{-x_3^-} \right)^{\Delta_1} \left( \frac{-x_2^-}{x_3^+} \right)^{\Delta_2} \hat{A}(z, \bar{z}) .$$
2.2 Transverse Space

In order to best understand the geometric nature of the correlator $A$, it is often useful to use $[4,5,6]$, instead of the coordinates $x_3, x_2$, the related coordinates

$$x = \pi (x_3), \quad \bar{x} = \bar{\pi} (x_2),$$

where the two conformal transformations $\pi, \bar{\pi} : \mathbb{M}^d \to \mathbb{M}^d$ are defined by

\[
\begin{align*}
  x^+ &= \frac{1}{x_3^+}, \\
  x^- &= -x_3^- + \frac{x_3 \cdot x_3}{x_3^+}, \\
  x &= \frac{x_3}{x_3^+},
\end{align*}
\]

and

\[
\begin{align*}
  \bar{x}^+ &= -\frac{1}{x_2^+}, \\
  \bar{x}^- &= x_2^+ - \frac{x_2 \cdot x_2}{x_2^+}, \\
  \bar{x} &= -\frac{x_2}{x_2^+}.
\end{align*}
\]

To understand the geometric meaning of the transformations $\pi, \bar{\pi}$, it is best to think of the CFT as being defined on the boundary of global AdS $[4,5]$, given by the light–cone

$$uv = w^2 = -w^+ w^- + w \cdot w$$

up to positive rescalings of the vector $(u, v, w)$. The CFT correlator $A$ is originally defined on the Poincaré patch corresponding to the choice $u = 1$. On the other hand, as shown in $[4,5,6]$ and in figure 3, it is more convenient geometrically to parameterize the four points of the amplitude $A$ using left and right Poincaré patches, corresponding to insertions of the operators $O_1$ and $O_2$ respectively. These patches are defined by the choice $w^+ = 1$ and $w^- = -1$ respectively, inverting the role of the light–cone coordinates $u, v$ and $w^+, w^-$ in $[4]$. For instance, the point $x_3$ relative to operator $O_1$ corresponds to the vector $(u, v, w) = (1, x_3^+, x_3)$. Rescaling $x_3^+$ and inverting the role of the light–cone coordinates we obtain the map $\pi$. Similarly, the map $\bar{\pi}$ is obtained by rescaling $(1, x_2^+, x_2)$ by $-x_2^-$. 

Note that

$$x^2 = \frac{x_3^2}{x_3^+}, \quad \bar{x}^2 = \frac{x_2^2}{x_2^+},$$

and that the regions $x_3^+, -x_3^- > 0$ and $x_2^+, -x_2^- > 0$ are mapped, by the transformations $\pi, \bar{\pi}$, into the future Milne cone $M \subset \mathbb{M}^d$ of Minkowski space. Therefore, the Lorentzian conditions $[4]$ now imply that

$$x, \bar{x} \in M,$$

$$-x / x^2 \text{ in the future of } \bar{x}.$$
The cross-ratios are given by
\[ z \bar{z} = x^2 \bar{x}^2 , \quad z + \bar{z} = -2x \cdot \bar{x}, \]
and the amplitude \( A \) now reads
\[ A(x, \bar{x}) = \frac{1}{(-x^2)^{\Delta_1} (-\bar{x}^2)^{\Delta_2}} \hat{A}(z, \bar{z}). \quad (6) \]

In what follows, it will be often convenient to denote the cross-ratios as
\[ z = \frac{\sigma}{4} e^\rho , \]
\[ \bar{z} = \frac{\sigma}{4} e^{-\rho} . \]

In this notation, \( \sigma = 4 |x| |\bar{x}| \) and \( \rho \) is the geodesic distance between the points
\[ \frac{x}{|x|} , \frac{\bar{x}}{|\bar{x}|} \in H_{d-1} , \]
where \( H_{d-1} \subset M \subset M^d \) represents the transverse hyperboloid of points \( w \in M \) with \( w^2 = -1 \).

In some situations, and again without loss of generality due to conformal invariance, we shall find it convenient to restrict the kinematics so that \( x_2, x_3, \) and therefore \( x, \bar{x}, \) have vanishing transverse components \( x_2 = x_3 = x = \bar{x} = 0. \)

Then we have that
\[ z = \frac{x^-}{x^2}, \quad \bar{z} = \frac{x^+}{x^2} = x^+ \bar{x}^- , \]
and
\[
A = \left( \frac{x_3^+}{-x_3^-} \right)^{\Delta_1} \left( \frac{-x_2^-}{x_2^+} \right)^{\Delta_2} \tilde{A} \left( \frac{x_3^-}{x_2^-}, \frac{x_3^+}{x_2^+} \right)
\]
\[
= \frac{1}{(x^+ x^-)^{\Delta_1} (\bar{x}^+ \bar{x}^-)^{\Delta_2}} \tilde{A} \left( x^+ \bar{x}^+, x^- \bar{x}^- \right).
\]

2.3 The Impact Parameter Representation

To conclude this introductory section, we briefly recall the results of [5, 6] on the impact parameter representation of $A$ in the Lorentzian regime. This is obtained by considering the Fourier transform $B$ of the amplitude
\[
A(x, \bar{x}) = (-)^{-\Delta_1} (-)^{-\Delta_2} \int_M d\mathbf{y} d\bar{\mathbf{y}} e^{-2i\mathbf{y} \cdot \mathbf{x} - 2i\bar{\mathbf{y}} \cdot \bar{\mathbf{x}}} B(y, \bar{y}),
\]
which has support uniquely in the future Milne cone $M$. The amplitude $B$ has a representation similar to (6)
\[
B(y, \bar{y}) = \frac{N_{\Delta_1}}{(-y^2)^{\frac{\Delta_1}{2}}} \frac{N_{\Delta_2}}{(-\bar{y}^2)^{\frac{\Delta_2}{2}}} B(h, \bar{h}),
\]
where
\[
N_{\Delta} = \frac{2\pi \pi^{-\frac{d}{2}}}{\Gamma(\Delta) \Gamma(1 + \Delta - \frac{d}{2})}
\]
and where the reduced amplitude $\tilde{B}$ depends on $y, \bar{y}$ through the effective cross-ratios $h, \bar{h}$ defined by
\[
h^2 \bar{h}^2 = y^2 \bar{y}^2, \quad h^2 + \bar{h}^2 = -2y \cdot \bar{y}.
\]

We shall also use, instead of $h, \bar{h}$, the equivalent ratios $s, r$, defined by
\[
h^2 = \frac{s}{4} e^r, \quad \bar{h}^2 = \frac{s}{4} e^{-r},
\]
where again $r$ is the geodesic distance between the points $y/|y|$ and $\bar{y}/|\bar{y}|$ in $H_{d-1}$. Recall [6] that it is often convenient to express $B$ as
\[
\tilde{B}(h, \bar{h}) = e^{-2\pi i \Gamma(h, \bar{h})},
\]
where $\Gamma(h, \bar{h})$ plays the dual role of phase shift $-\pi \Gamma$ in the AdS dual of the CFT under consideration, and of anomalous dimension $2\Gamma$ of the composites $\mathcal{O}_1 \partial \cdots \partial \mathcal{O}_2$ at large values of the energy $h + \bar{h}$ and spin $h - \bar{h}$.

3 Regge Theory in CFT

In this section we shall use the results of [5] to develop a Regge theory suitable for conformal field theories. From now on, we shall continue the discussion assuming $d = 4$. This is done uniquely for notational simplicity, since most formulæ generalize to general $d$, as shown in Appendix A.

\footnote{Throughout this paper we shall use the notation $(-)^\alpha \equiv e^{i\pi \alpha}$.}
3.1 T–Channel Partial Waves

We start by considering the contribution to the amplitude \( A \) coming from the exchange of a primary operator of \( 3 \) energy \( 2 + E \), spin \( J \), in the T–channel \( x_1 \rightarrow x_3, x_2 \rightarrow x_4 \). This corresponds to the limit \( z, \bar{z} \rightarrow 0 \) or to \( \sigma \rightarrow 0 \).

We shall denote the corresponding contribution to the reduced Euclidean amplitude \( A \) by \( T_{E,J}(z, \bar{z}) \).

The expression for \( T_{E,J} \) can be written, in \( d = 4 \), in terms of the chiral expression \( T_k(z) = (-z)^k F(k, k, 2k, z) \),

\[
T_{E,J}(z, \bar{z}) = -\frac{z \bar{z}}{z - \bar{z}} \left[ T_k(z) T_{k-1}(\bar{z}) - T_k(\bar{z}) T_{k-1}(z) \right],
\]

where

\[
k + \bar{k} = 2 + E,
\]

\[
k - \bar{k} = J.
\]

The limit \( \sigma \rightarrow 0 \) of \( T_{E,J} \) is determined by the usual operator product expansion, and we have chosen the normalization of \( T_{E,J} \) so that, for integer spin \( J \), we have

\[
T_{E,J} \sim \sum_{m=0}^{J} (-z)^{k-m} (-\bar{z})^{k+m} \approx (-1)^J \left( \frac{\sigma^4}{4} \right)^{2+E} \frac{\sinh((J+1)\rho)}{\sinh(\rho)}.\]

Clearly, the leading behavior of the amplitude \( A \) for \( \sigma \rightarrow 0 \) is determined by the operators of lowest energy. As shown in [5], the \( \sigma \rightarrow 0 \) limit of the Lorentzian partial wave \( \tilde{T}_{E,J} \) is, on the other hand, rather different. It is given explicitly by [6]

\[
\tilde{T}_{E,J} \sim 2\pi i \left( -1 \right)^J \left( \frac{\sigma^4}{4} \right)^{1-J} \frac{\Pi E (\rho)}{\pi (J + E)}.
\]

\(^3\)We shall always write dimensions \( \Delta \) of operators as \( \frac{d}{2} + E \), so that the corresponding scalar mass \( \Delta (\Delta - d) \) is real either for \( E \) purely real, or for \( E = i\nu \) purely imaginary. This convention is useful also when considering high energy interactions. As shown in [3, 5, 6], eikonal interactions mediated by a particle of dimension \( \Delta \) are effectively described by a scalar particle in the transverse \( d - 2 \) dimensions. The conformal dimension of this effective particle is \( \Delta - 1 \), and therefore the energy \( E \) is invariant.

\(^4\)The expressions which follow are valid for general complex \( J \). The leading behavior \( \tilde{T}_{E,J} \sim \sigma^{1-J} \) is valid for \( \text{Re } J > -1 \), and always dominates \( \sigma^{2+E} \) for \( \text{Re } E > 0 \). For \( \text{Re } J < -1 \) we must replace below \( J \rightarrow -2 - J \), since \( T_{E,J} = -T_{E,-2-J} \).
where the coefficient $\tau(x)$ is given by

$$\tau(x) = -\frac{1}{2\pi} \frac{\Gamma(1 + \frac{x}{2})^4}{\Gamma(2 + x)\Gamma(1 + x)},$$

and where

$$\Pi_E(\rho) = \frac{1}{4\pi} \frac{e^{-E\rho}}{\sinh \rho}$$

is the scalar propagator in the transverse hyperbolic space $\mathbb{H}_3$ of dimension $1 + E$, normalized as

$$[\Box_{\mathbb{H}_3} - E^2 + 1] \Pi_E = -\delta_{\mathbb{H}_3}.$$ 

We see that the behavior for $\sigma \to 0$ is now controlled by the operators of highest spin which are exchanged in the $T$–channel. The $\sigma \to 0$ limit of the Lorentzian amplitude is then much richer in structure than the corresponding limit of the Euclidean amplitude. In fact, at a fixed energy, we can have only a finite number of partial waves due to the unitarity bounds of the conformal field theory and to the discreteness of the spin variable. On the other hand, at fixed spin, we have a continuous of possible energy contributions.

### 3.2 Regge Poles in CFT

The situation in the Lorentzian regime is rather similar to the usual partial wave decomposition in flat space. In that case, one decompose amplitudes as $\sum_J T_J (-s/t) a_J(t)$, where $s,t$ are the Mandelstam invariants and where the partial waves $T_J$ are now polynomials of degree $J$. In the high energy limit $s \to \infty$ with $t$ fixed, a single partial wave behaves as $s^J$ and thus highest spin waves dominate. On the other hand, if there are contributions from partial waves of all spins, one must resum all contributions in order to analyze the high $s$ behavior of the amplitude. In the flat space case, this resummation is well understood in terms of the Sommerfeld–Watson transform, and leads to the theory of Regge poles \[1, 2\]. We shall see that the extension of these usual techniques to conformal field theories is rather straight–forward.

Before starting the discussion, we need some basic results regarding harmonic analysis of functions on the transverse space $\mathbb{H}_3$. More precisely, we shall consider functions $G(w, w')$, with $w, w' \in \mathbb{H}_3$, which depend only on the geodesic distance $\rho$ between $w, w'$, given by $\cosh \rho = -w \cdot w'$. We shall call those functions radial for short, with an example given by the propagator $\Pi_E(w, w')$ in (11). In order to analyze radial functions, we first define $\Omega_E(w, w')$ as

$$\Omega_E = \frac{E}{2\pi} (\Pi_E - \Pi_{-E}) = -\frac{E}{4\pi^2} \frac{\sinh E\rho}{\sinh \rho},$$

which satisfies the homogeneous equation

$$[\Box_{\mathbb{H}_3} - E^2 + 1] \Omega_E = 0,$$

and which is even in $E$, since $\Omega_E = \Omega_{-E}$. It should be clear to the reader that the functions $\Omega_{i\nu}$ with $E = i\nu$ pure imaginary are the analogues, in $\mathbb{H}_3$, of the
usual radial Bessel functions which give a Fourier basis for radial functions in $\mathbb{E}^3$. We then expect that a general radial function $G(w, w')$ can be expressed as

$$G(w, w') = \int d\nu \; g(\nu) \; \Omega_{\nu w}(w, w'),$$

(15)

with $g(\nu) = g(-\nu)$. For example, we have that

$$\Pi_E = \int d\nu \; \frac{\Omega_{\nu w}}{\nu^2 + E^2}.$$  

(16)

This can be shown by first using the definition of $\Omega_E$ to rewrite the right hand side of the above equation as

$$-2 \int \frac{d\nu}{2\pi i} \frac{\nu}{\nu^2 + E^2} \Pi_{\nu w},$$

and then by closing the $\nu$–contour in the $\text{Im} \; \nu \leq 0$ region, noting that $\Pi_{\nu w}$ is free of singularities and decays exponentially at infinity. The transform (15) can be inverted by first noting that

$$2 \int_{H_3} dw' \; \Omega_{\nu w}(w, w') \; \Omega_{\nu w'}(w, w'') = [\delta(\nu - \bar{\nu}) + \delta(\nu + \bar{\nu})] \; \Omega_{\nu w}(w, w'').$$

(17)

The normalization in the above expression can be fixed by integrating in $\nu$ and by using the fact that

$$\delta_{H_3}(w, w') = \int d\nu \; \Omega_{\nu w}(w, w'),$$

which immediately follows from (16), (12) and (14). We then have that

$$g(\nu) \; \Omega_{\nu w}(w, w') = \int_{H_3} d\nu \; \Omega_{\nu w}(w, w') \; G(w', w''),$$

and, choosing $w = w''$, that

$$g(\nu) = \frac{1}{\Omega_{\nu w}(w, w)} \int_{H_3} d\nu \; \Omega_{\nu w}(w, w') \; G(w', w) ,$$

$$= \frac{4\pi}{\nu} \int_0^\infty d\rho \; \sin \nu \rho \; \sinh \rho \; G(\rho).$$

To conclude this preliminary discussion, we note that (17) implies that convolution $\int_{H_3} dw' \; G(w, w') F(w', w'')$ in $H_3$ goes to the product $g(\nu) f(\nu)$ in Fourier space, as usual.

We may now start the discussion of Regge theory by recalling that, in the previous section, we had shown the limit, in the Lorentzian regime and for $\sigma \to 0$, of

$$\tau (J + E) \; T_{E, J} \to 2\pi i \; (-)^J \left(\frac{\sigma}{4}\right)^{1-J} \Pi_E(\rho).$$

In analogy with (13), we then introduce the composite partial waves

$$G_{E, J} = \frac{E}{2\pi} \left[ \tau (J + E) \; T_{E, J} - \tau (J - E) \; T_{-E, J}\right],$$

11
so that we have the correspondence
\[ G_{E,J} \rightarrow 2\pi i (-)^J \left( \frac{\sigma}{4} \right)^{1-J} \Omega_E (\rho) . \]

Equation (15) then suggests to consider the following generalized integrals
\[ \int d\nu \ g_J (\nu) \ G_{i\nu,J} , \] (18)

with \( g_J (\nu) = g_J (-\nu) \) and with \( \nu g_J (\nu) \) square–integrable, in order to have well defined transforms. To compute the integral above, we first write it as
\[ -2 \int \frac{d\nu}{2\pi i} \ g_J (\nu) \ \nu \ \tau (J + i\nu) \ T_{i\nu,J} \\
\] and we then close the contour in the region \( \text{Im} \nu \leq 0 \) to get
\[ -2i \sum_E \ \text{Res} \ g_J (-iE) \cdot E \ \tau (J + E) \ T_{E,J} + \cdots . \] (19)

We have only shown the contributions coming from the poles of \( g_J (\nu) \), with the omitted \( \cdots \) indicating contributions from the poles of the partial waves \( T_{i\nu,J} \), since \( \nu \ \tau (J + i\nu) \) is regular in the region of interest. These extra contributions are given by partial waves \( T_{J+1,\nu} \) of lower spin \( J' < J \), fixed energy \( J+1 \), whose precise form will not be fundamental for the main discussion of this paper. We shall therefore describe them in detail in a separate section in equation (27), mostly for completeness. In any case, since the omitted contribution is built explicitly out of partial waves of lower spin \( J' < J \), the leading contribution to the Lorentzian amplitude in the \( \sigma \rightarrow 0 \) limit coming from (18) is given by
\[ \int d\nu \ g_J (\nu) \ \hat{G}_{i\nu,J} \rightarrow_{\sigma \rightarrow 0} 2\pi i (-)^J \left( \frac{\sigma}{4} \right)^{1-J} \ G_J (\rho) , \] (20)

with
\[ G_J (\rho) = \int d\nu \ g_J (\nu) \ \Omega_{i\nu} (\rho) . \]

The full amplitude \( \mathcal{A} \) will have then a decomposition in terms of T–channel conformal partial waves given by
\[ \mathcal{A} = \sum_{J \geq 0} \int d\nu \ g_J (\nu) \ G_{i\nu,J} . \]

In order to understand the behavior of the Lorentzian amplitude \( \hat{\mathcal{A}} \) in the limit \( \sigma \rightarrow 0 \), we clearly cannot use (20) directly whenever the sum over \( J \) in the above decomposition is unbounded. We must therefore use, as in flat space, a Sommerfeld–Watson resummation, which starts by separating the contributions of even and odd spins as \( \mathcal{A} = \mathcal{A}_+ + \mathcal{A}_- \), with
\[ \mathcal{A}_\pm = \frac{1}{2} \sum_{J \geq 0} \int d\nu \ g_J (\nu) \left[ G_{i\nu,J} (z, \bar{z}) \pm G_{i\nu,J} \left( \frac{z}{z-1}, \frac{\bar{z}}{\bar{z}-1} \right) \right] . \]
Figure 4: The path $\Gamma$ in the complex $J$–plane, which gives the usual sum over integral spins, is deformed to analyze the $\sigma \to 0$ limit of the Lorentzian amplitude. Contributions come from Regge poles and cuts (a pole is shown at $J = j(\nu)$), together with subdominant shadow contributions coming from the dashed contours. This includes the usual path at $\text{Re} J = -1$ together with the contributions of the spurious poles at $J = \pm i\nu + m$ with $m$ a positive odd integer, which have an effective spin $-1 \pm i\nu$.

Note that the transformation $z \to z/ (z - 1)$ and $\bar{z} \to \bar{z}/ (\bar{z} - 1)$ interchanges the roles of the operators at $x_2$ and $x_4$, and in flat space corresponds to the exchange of the roles of the Mandelstam variables $s, u$. Moreover it is easy to show that, for $J$ integral,

$$ T_{E,J} \left( \frac{z}{z-1}, \frac{\bar{z}}{\bar{z}-1} \right) = (-)^J T_{E,J} (z, \bar{z}), \quad (J \in \mathbb{Z}) $$

with a similar equation for $G_{E,J}$. Extending $g_J(\nu)$ in the complex $J$–plane to $g_{\pm}(\nu, J)$, we may write an explicit Sommerfeld–Watson representation for $A_\pm$

$$ A_\pm = \frac{i}{4} \int d\nu \frac{dJ}{\sin \pi J} g_{\pm}(\nu, J) \left[ G_{iv,J} \left( \frac{z}{z-1}, \frac{\bar{z}}{\bar{z}-1} \right) \pm G_{iv,j(\nu)} (z, \bar{z}) \right], \quad (21) $$

where the $J$–contour of integration picks up the zeros of $\sin \pi J$ for $J \geq 0$. As shown in figure 4 we shall deform the $J$ integration contour, assuming that the functions $g_{\pm}(\nu, J)$ have a leading pole

$$ g_{\pm}(\nu, J) \sim \frac{r_{\pm}(\nu)}{J - j(\nu)} $$

along the Regge trajectory $J = j(\nu)$, with residue $r_{\pm}(\nu)$. The contribution to the amplitude $A$ from this pole is then given by

$$ A \simeq -\frac{\pi}{2} \int d\nu \frac{r_{\pm}(\nu)}{\sin \pi j(\nu)} \left[ G_{iv,j(\nu)} \left( \frac{z}{z-1}, \frac{\bar{z}}{\bar{z}-1} \right) \pm G_{iv,j(\nu)} (z, \bar{z}) \right]. $$

We are now interested in the $\sigma \to 0$ limit in the Lorentzian regime. In this
regime we have that
\[
G_{i\nu,j} (\nu) \sim 2\pi i (-)^{j(\nu)} \left(\frac{\sigma}{4}\right)^{1-j(\nu)} \Omega_{i\nu} (\rho) ,
\]
\[
\tilde{G}_{i\nu,j} (\nu) \left(\frac{z}{z-1}, \frac{\bar{z}}{\bar{z}-1}\right) \sim 2\pi i \left(\frac{\sigma}{4}\right)^{1-j(\nu)} \Omega_{i\nu} (\rho) ,
\]
where the relevant analytic continuations are shown in figure 2a,b. We then obtain
\[
\hat{A} \sim 2\pi i \int d\nu (-)^{j(\nu)} \alpha (\nu) \sigma^{1-j(\nu)} \Omega_{i\nu} (\rho) ,
\] (22)
where \(\alpha (\nu)\) is given, for poles of positive and negative signature, by
\[
\alpha (\nu) = \left(\begin{array}{c}
-2
\end{array}\right) \frac{\pi}{2} \left(\begin{array}{c}
4j(\nu)-1
\end{array}\right) \left(\begin{array}{c}
r_+ (\nu) \\
\sin \frac{\pi j(\nu)}{2}
\end{array}\right) ,
\]
\[
\alpha (\nu) = \left(\begin{array}{c}
1
\end{array}\right) \frac{\pi}{2} \left(\begin{array}{c}
4j(\nu)-1
\end{array}\right) \left(\begin{array}{c}
r_- (\nu) \\
\cos \frac{\pi j(\nu)}{2}
\end{array}\right) .
\]
To conclude this section, we consider the contribution from a Regge pole directly in the impact parameter representation. This can be easily achieved by writing
\[
\mathcal{B} \sim -2\pi i \int d\nu \beta (\nu) \sigma^{j(\nu)-1} \Omega_{i\nu} (r) .
\] (23)
It is then easy to show, using (7) and (8), that the contribution to the amplitude \(A\) is given
\[
A (x, \bar{x}) \simeq 2\pi i N_{\Delta_1} N_{\Delta_2} \int d\nu (-)^{j(\nu)} 4^{j(\nu)-1} \Gamma (\eta_1 (\nu)) \Gamma (\eta_2 (\nu)) \beta (\nu) \times
\]
\[
\times \int_{\mathbb{H}_3} dy d\bar{y} \frac{\Omega_{i\nu} (y, \bar{y})}{(-2x \cdot y)^{\eta_1 (\nu)} (-2\bar{x} \cdot \bar{y})^{\eta_2 (\nu)}} ,
\]
where
\[
\eta_i (\nu) = 2\Delta_i - 1 + j (\nu) . \quad (i = 1, 2)
\]
In order to write the reduced amplitude \(\hat{A} = (-x^2)^{\Delta_1} (-\bar{x}^2)^{\Delta_2} A\) in the Regge form (22), we must consider the Fourier transform of the radial functions \((-2x \cdot y)^{-\eta}\), which in general is given by the integral representation.
\[
\frac{1}{(-2x \cdot y)^\eta} = \int d\nu V (\nu, \eta) \Omega_{i\nu} (x, y) ,
\] (24)
with
\[
V (\nu, \eta) = \frac{4\pi}{\nu} \int_0^\infty d\rho \sin \nu \rho \frac{\sinh \rho}{(2 \cosh \rho)^\eta} = \frac{\pi}{2} \frac{\Gamma \left(\frac{\eta+1+i\nu}{2}\right) \Gamma \left(\frac{\eta-1-i\nu}{2}\right)}{\Gamma (\eta)} .
\]
It is then easy to show that the reduced amplitude is of the form (22), with
\[
\alpha (\nu) = V_1 (\nu) \beta (\nu) V_2 (\nu) ,
\] (25)
Figure 5: Spurious poles of $T_{E,J}$ in the complex $E$-plane for $\text{Re} E > 0$. They are located at $E = J - m$, with $m$ a positive odd integer, and have residues proportional to $T_{1+J,J-1-m}$.

where we have defined the generalized couplings $V_i(\nu)$ as

\[
V_i(\nu) = N_{\Delta_i} \frac{4^{j(\nu)}-1}{\Gamma(\Delta_i-1)} \frac{\Gamma(\Delta_i-1+\frac{1}{2}j(\nu))}{\Gamma(\Delta_i)} \Gamma(\Delta_i-1) \cdot \Gamma(\Delta_i) \Gamma(\Delta_i-1) .
\]

3.3 Spurious Poles

In this section, we discuss the extra contributions in (19), hidden in the omitted $\cdots$, which come from the poles of the partial waves $T_{E,J}$ in the physical region $\text{Re} E \geq 0$. These contributions are subleading in the $\sigma \to 0$ limit. Therefore, the discussion which follows is included for completeness and can be skipped at a first reading.

We first note that the chiral partial wave (10) has poles at

\[
\text{Res}_{k \to 1 - \frac{m}{2}} T_{k-1} = -\frac{1}{2\pi^2} \frac{\Gamma(2-k)^4}{\Gamma(4-2k)\Gamma(3-2k)} T_{2-k} 
\]

\[(m = 1, 3, 5, \cdots)\]

Using the exact expression for $T_{E,J}$ it is then immediate to prove, as shown in figure 5, that

\[
\text{Res}_{E \to 1+J',J} T_{E,J} = \frac{2}{\pi} \tau (J - J' - 1) \ T_{1+J,J'} ,
\]

\[(26)\]

where

\[J' = J - 2, J - 4, \cdots \quad (\text{Re} J' \geq -1)\]

Note that, at the residue point, the conformal Casimir for energy and spin $(E, J)$ given by (12)

\[E^2 + (J + 1)^2 - 6 \]

is identical for the pairs $(1 + J', J)$ and $(1 + J, J')$. Equation (26) then shows that the omitted terms in (19) are given by

\[-\frac{4}{\pi} \sum_{J'} g_{J'} (-i (1 + J')) (1 + J') \tau (J + J' + 1) \tau (J - J' - 1) \ T_{1+J,J'} .
\]

(27)

As anticipated in the previous section, the extra contributions have spins $J' < J$ and therefore do not effect the $\sigma \to 0$ behavior in (20). When one considers the
full partial wave decomposition for the reduced amplitude

\[ A = \sum_{J \geq 0} \int d\nu \ g_J(\nu) \ G_{i\nu,J}, \]

one usually uses a representation in which the extra contributions from the spurious poles in (27) are cancelled by contributions from poles of \( g_{J'}(\nu) \). This is achieved by demanding that

\[ 4g_J(-i(1+J')) = -2\pi i \frac{1+J}{1+J'} \frac{\text{Res} g_{J'}(-i(1+J))}{\tau(J-J'-1)}. \]

Let us note that the above relation is satisfied if

\[ g_J(\nu) = (k - \bar{k} + 1) \frac{\Gamma(2k) \Gamma(4 - 2\bar{k})}{\Gamma(k)^2 \Gamma(2 - k)^2} f(k, \bar{k}), \]

where

\[ k + \bar{k} = 2 + i\nu, \quad k - \bar{k} = J, \]

and where the function \( f(k, \bar{k}) \) satisfies, for \( k - \bar{k} \in \mathbb{N}_0, \)

\[ f(k, \bar{k}) = f(2 - \bar{k}, 2 - k), \]
\[ f(k, \bar{k}) = f(k, 3 - \bar{k}). \]

\( \bar{k} = 5/2 + \mathbb{N}_0 \)

We conclude this section by discussing the role of these spurious poles in the Sommerfeld–Watson resummation procedure. First we note that immediately implies that

\[ \text{Res} \ G_{i\nu,J} = \pm \frac{iv}{\pi^2} \tau(m) \tau(m \pm 2iv) \ T_{i\nu+m+1, \pm i\nu-1} \]

where \( m \) is an odd positive integer. We then see that, when performing the Sommerfeld–Watson transform, we get contributions from the poles in (28), of the form

\[ -\frac{1}{\pi} \sum_{m \in 1 + 2\mathbb{N}_0} \int d\nu \ \frac{\nu}{\sinh(\pi \nu)} \ \tau(m) \tau(2iv + m) \ \ g_{\pm}(\nu, iv + m) \times \]
\[ \times \left[ T_{i\nu+m+1, \nu-1} \left( \frac{z}{z - 1}, \frac{\bar{z}}{\bar{z} - 1} \right) \pm T_{i\nu+m+1, \nu-1} \left( z, \bar{z} \right) \right]. \]

We note that, although the above contributions come from poles at \( J = \pm iv + m \), the residues are given by partial waves of spin \(-1 \pm iv\), which are subordinate compared with the leading Regge pole in the Lorentzian regime in the \( \sigma \to 0 \) limit. As shown in figure 4 the above contributions should therefore be considered as shadow contributions, and should be added to the usual shadow contribution given by (21), where the \( J \) integral runs from \(-1 - i\infty\) to \(-1 + i\infty\).

4 Multi–Reggeon Exchange in the Eikonal Limit

In this section, we consider conformal field theories which admit a string dual description. The amplitude \( A \) then has a perturbative expansion in the string coupling

\[ A = 1 + A_{\text{tree}} + \cdots. \]
In particular, we expect the tree level contribution $\mathcal{A}_{\text{tree}}$ to be dominated, in the Lorentzian regime, by an even Regge pole at $J = j(\nu)$ corresponding to the graviton trajectory in AdS, so that

$$\hat{\mathcal{A}}_{\text{tree}} \simeq 2\pi i \int d\nu \ (-)^{j(\nu)} \alpha(\nu) \sigma^{1-j(\nu)} \Omega_{i\nu}(\rho) \ .$$

Using (9) and (23) we then quickly deduce that the tree–level contribution to the phase shift $\Gamma$ is given by

$$\Gamma(s, r) \simeq \int d\nu \beta(\nu) \ s^{j(\nu)-1} \Omega_{i\nu}(r) \ , \quad (29)$$

where $\beta(\nu)$ is related to $\alpha(\nu)$ by the kinematical relation (25). As in flat space [10], we expect that, for large impact parameter $r$, the phase shift $\Gamma$ is dominated by the leading contribution (29). In this limit, the impact parameter representation

$$A(x, \bar{x}) = (-)^{-\Delta_1} \cdot (-)^{-\Delta_2} N_{\Delta_1} N_{\Delta_2} \times$$

$$\times \int_M \frac{dy d\bar{y}}{|y|^{d-2\Delta_1} |\bar{y}|^{d-2\Delta_2}} e^{-2i\bar{\kappa} \cdot \bar{y}} e^{-2\pi i \Gamma(y, \bar{y})}$$

resums multi–Reggeon interactions in the eikonal limit, exactly as in flat.

The procedure we have followed to eikonalize Reggeon exchanges follows closely the analogous technique in flat space, restoring unitarity at high energy. Our formulae therefore implicitly incorporate unitarity requirements in AdS. The relation to unitarity in the dual CFT has been partly explored in [9, 13, 14], but surely deserves a more thorough investigation.

5 Example: $\mathcal{N} = 4$ SYM at Large ’t Hooft Coupling

In the remainder of the paper, we shall focus our attention on the canonical example of SU ($N$), $\mathcal{N} = 4$ super Yang–Mills in $d = 4$, which is dual [3] to Type IIB strings on AdS$_5 \times S^5$. Denoting with $\ell$ the AdS radius and with $\sqrt{\alpha'}$ the string length, the ’t Hooft coupling

$$g^2 \equiv g_{YM}^2 N$$

of the dual YM theory is given by

$$g = \frac{\ell^2}{\alpha'} .$$

Moreover, the number of colors $N$ is given, in terms of the AdS$_5$ gravitational constant $G$, by

$$\frac{1}{N^2} = \frac{2}{\pi} \frac{G}{\ell^3} .$$

One of the basic results of [6, 8] is the fact that, in the supergravity limit of large $g$, which neglects string corrections, the leading eikonal gravitational interaction
in AdS can be resummed. More precisely, the phase shift \( \Gamma(y, \bar{y}) \), which gives \( B = e^{-2\pi i \Gamma} \), is determined by the tree–level interaction and reads
\[
\Gamma = -\frac{2G}{\ell^3} \Pi_2(r) = -\frac{2G}{\ell^3} s \int \frac{d\nu}{\nu^2 + 4} \Omega_{\nu \nu}(r),
\]
which corresponds to the exchange of the graviton dual to the dimension 4 stress–energy tensor. As in flat space, in the full string theory this tree level result is actually determined by the contribution from the full Regge trajectory in the limit in which uniquely the graviton dominates. This corresponds to a positive signature pole, and in general we shall have that
\[
\Gamma = -\frac{2G}{\ell^3} \int d\nu \beta(\nu, g) s^{j(\nu, g)-1} \Omega_{\nu \nu}(r),
\]
where we have shown the explicit dependence of \( \beta \) and \( j \) on the \'t Hooft coupling \( g \). In particular, the Regge trajectory \( j(\nu, g) \) determines the spectrum of lowest twist operators in the dual YM theory. More precisely, in terms of the inverse \( E(j, g) \) defined by \( j(\pm iE(j, g), g) = j \), the dimensions of the lowest twist operators of increasing spin \( J = 2, 4, 6, \ldots \) is given by \( 2 + E(J, g) \). We recall some known facts about \( j(\nu, g) \) in Appendix B. In order to derive the leading string corrections in AdS to \( j \) and \( \beta \), we consider the flat space limit \( \ell \to \infty \), where we know the result from the exact tree level string computation in the Regge regime. More precisely, we must consider the limit
\[
\nu = \ell q, \quad g = \frac{\ell^2}{\alpha'}, \quad (\ell \to \infty) \quad (30)
\]
with \( q, \alpha' \) fixed. Note that, in this limit, the variable \( q \) measures momentum transfer, since \( \ell^{-2}(1 + \nu^2) \) measures the value of the Laplacian \( -\ell^{-2}\Box_{H^3} \) on the transverse space \( H^3 \to \mathbb{R}^3 \). Therefore, matching the linear graviton Regge trajectory in flat space we must have that
\[
\lim_{\ell \to \infty} j(\ell q, \ell^2/\alpha') = 2 - \frac{\alpha'}{2} q^2. \quad (31)
\]
Moreover, we know that the graviton remains massless in AdS at any value of the \'t Hooft coupling \( g \) due to the vanishing of the anomalous dimension of the boundary stress energy tensor. Therefore we must also have that
\[
j(\pm 2i, g) = 2. \quad (32)
\]
At strong coupling, we expand the general expression for the spin \( j(\nu, g) \) in inverse powers of \( g \) for small \( \nu \) as
\[
j(\nu, g) = \sum_{n \geq 0} \frac{j_n(\nu)}{g^n} = \sum_{0 \leq m \leq n} j_{n,m} \frac{\nu^{2m}}{g^n},
\]
where we have assumed that \( j(\nu, g) \) is regular at \( \nu = 0 \). In the above sum \( m \leq n \) since the general term
\[
\frac{\nu^{2m}}{g^n} = \ell^{2(m-n)}(\alpha')^n q^{2m}
\]
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has a well-behaved $\ell \to \infty$ limit only in this case. Moreover, the coefficients $j_{n,m}$ for $n = m$ are determined by the flat space limit (31) to be $j_{0,0} = 2$, $j_{1,1} = -1/2$ and $j_{n,n} = 0$ for $n \geq 2$. Equation (32) then implies that $j_n (\pm 2i) = 0$ for $n \geq 1$. Since $j_1 (\nu) = j_{1,0} + j_{1,1}\nu^2$, we then deduce that $j_{1,0} = -2$ and finally

$$j (\nu, g) = 2 - \frac{4 + \nu^2}{2g} - \left(4 + \nu^2\right) \sum_{n \geq 2} \frac{\tilde{j}_n (\nu)}{g^n}$$  \hspace{1cm} (33)$$

with

$$\tilde{j}_n (\nu) \text{ even polynomial of order } 2n - 4.$$  

In particular, $\tilde{j}_2$ is just a constant and $j_2 \propto j_1$.

A similar argument can be followed to deduce the leading string corrections to $\beta (\nu, g)$. We have seen that, in the limit $g \to \infty$ with $\nu$ fixed we must have that

$$\beta \sim \frac{1}{4 + \nu^2}.$$  

To recover the leading string corrections we must consider, on the other hand, the limit (30) so that $-\pi \Gamma$ converges to the full phase shift in flat space. In the limit we note that

$$\lim_{\ell \to \infty} \frac{1}{\ell^2} \Omega_{q \epsilon t} \left(\frac{R}{\ell}\right) = \frac{q}{4\pi^2 R} \sin qR = \frac{q^2}{4\pi} j_0 (qR),$$

$$\lim_{\ell \to \infty} \frac{s}{\ell^2} = S,$$

where $R$ is the physical impact parameter in transverse Euclidean space $E^3$, $S$ is the physical Mandelstam energy-squared and $q^2 j_0 (qR)$ computes Fourier transforms of radial functions in $E^3$. Then $\beta$ should converge to $(\ell q)^{-2}$ in the absence of string corrections, to reproduce the usual gravitational interaction, and to the usual factor

$$\frac{\alpha'}{4\ell^2} (-)^{\alpha' q^2} \frac{\Gamma \left(\frac{\alpha'}{4} q^2\right)}{\Gamma \left(1 - \frac{\alpha'}{4} q^2\right)}$$

to reproduce the string corrections. Therefore we deduce that

$$\beta (\nu, g) = -\frac{1}{4g} (-)^{-j(\nu,g)/2} \frac{\Gamma \left(1 - \frac{j(\nu,g)}{2}\right)}{\Gamma \left(\frac{j(\nu,g)}{2}\right)} + \cdots,$$

with $\cdots$ vanishing in the limit (30).

We conclude by noting that the results derived in this section confirm and extend the results derived in [7].

6 Conclusions

In this paper we have discussed the extension of Regge theory to conformal field theories. Although inspired by the AdS/CFT correspondence, most of the discussion is independent of the existence of a gravity dual description of
the CFT. On the other hand, in the presence of a dual string formulation, we show how to resum, in the eikonal limit of large impact parameter, multiple reggeon interactions, following the analogous prescription used in flat space. As an example, we have considered $\mathcal{N} = 4 \ SU(N)$ supersymmetric Yang–Mills theory in $d = 4$ at large ’t Hooft coupling, where the gravi–reggeon pole corresponds dually to the pomeron. We have in particular analyzed the leading string corrections to pure graviton exchange. In a subsequent work [11], we plan to extend some of the techniques of this paper to the weak coupling regime, connecting to the formalism of BFKL [15] for hard pomeron exchange.

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A Regge Theory in General Dimension $d$

In this appendix, we wish to briefly outline the results of section 3 for general dimension $d$. The exchange of a primary of energy and spin

\[ \frac{d}{2} + E = k + \bar{k}, \]
\[ J = k - \bar{k}, \]

in the $T$-channel contributes to the reduced amplitude $A$ as $T_{E,J}(z,\bar{z})$, we normalize partial waves so that

\[ \lim_{z\to 0} \lim_{\bar{z}\to 0} T_{E,J} \sim (-z)^k (-\bar{z})^{\bar{k}}. \]

The $\sigma \to 0$ limit of the Lorentzian partial wave $\hat{T}_{E,J}$ is then given by

\[ \hat{T}_{E,J} \sim 2\pi i (-)^J \left(\frac{\sigma}{4}\right)^{1-J} \frac{\Pi_E(\rho)}{\tilde{\tau}(E)\tau\left(\frac{d-2}{2} + J - E\right)}, \]

where we have defined

\[ \tilde{\tau}(E) = \pi^{\frac{1+d}{2}} \frac{\Gamma(E + \frac{d-2}{2})}{\Gamma(E + 1)} \]

and where

\[ \Pi_E(\rho) = \frac{\tilde{\tau}(E)}{2\pi} e^{-\rho(E+\frac{d-2}{2})} F\left(\frac{d}{2} - 1, E + \frac{d-2}{2}, E + 1, e^{-2\rho}\right) \]

is the scalar propagator in the transverse hyperbolic space $H_{d-1}$ of dimension $\frac{d-2}{2} + E$, canonically normalized. In the important cases $d = 2, 4, 6$ the propagator reads

\[ \frac{1}{2} e^{-E\rho}, \quad (d = 2) \]
\[ \frac{1}{4\pi} e^{-E\rho}, \quad (d = 4) \]
\[ \frac{1}{8\pi^2} \frac{e^{-E\rho}(\cosh\rho + E\sinh\rho)}{\sinh^3\rho}. \quad (d = 6) \]

Harmonic analysis of radial functions in $H_{d-1}$ is performed using

\[ \Omega_E = \frac{E}{2\pi} (\Pi_E - \Pi_{-E}) \]
\[ = -\frac{1}{2^{d-1}\pi^{d-1}} E \sin \pi E \frac{\Gamma\left(\frac{d-2}{2} + E\right) \Gamma\left(\frac{d-2}{2} - E\right)}{\Gamma\left(\frac{d-2}{2}\right)} \times \]
\[ \times F\left(\frac{d-2}{2} + E, \frac{d-2}{2} - E, \frac{d-1}{2}, \frac{1 - \cosh\rho}{2}\right), \]

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which is given, in \( d = 2, 4, 6 \), by

\[
\begin{align*}
\frac{1}{2\pi} \cosh E\rho , & \quad (d = 2) \\
- \frac{E}{4\pi^2} \sinh E\rho \, \frac{1}{\sinh \rho} , & \quad (d = 4) \\
- \frac{E}{8\pi^3} \frac{\cosh \rho \sinh E\rho}{\sinh^3 \rho} + \frac{E^2 \cosh E\rho}{8\pi^3 \sinh^2 \rho} & \quad (d = 6)
\end{align*}
\]

General radial functions \( G(w, w') \) can then be expressed by

\[
G(w, w') = \int d\nu \, g(\nu) \Omega_{i\nu}(w, w')
\]

\[
g(\nu) = \frac{1}{\Omega_{i\nu}(w, w)} \int_{t_{d-1}} \Omega_{i\nu}(w, w') \, G(w', w) .
\]

Since, for general \( d \), we have the limit for \( \sigma \to 0 \)

\[
\tilde\tau (E) \tau \left( \frac{d-4}{2} + J + E \right) T_{E, J} \to 2\pi i \left( - \right)^J \left( \frac{\sigma}{4} \right)^{1-J} \Pi_E (\rho) ,
\]

we can introduce the composite partial waves

\[
G_{E, J} = \frac{E}{2\pi} \left[ \tilde\tau (E) \tau \left( \frac{d-4}{2} + J + E \right) T_{E, J} - \tilde\tau (-E) \tau \left( \frac{d-4}{2} + J - E \right) T_{-E, J} \right] ,
\]

so that we have

\[
G_{E, J} \to 2\pi i \left( - \right)^J \left( \frac{\sigma}{4} \right)^{1-J} \Omega_E (\rho) .
\]

In particular, for \( d = 2 \) we have that

\[
G_{E, J} = \frac{1}{2} \left[ \tau (J + E - 1) T_{E, J} + \tau (J - E - 1) T_{-E, J} \right] . \quad (d = 2)
\]

The full amplitude \( A \) will have then a decomposition in T–channel partial waves given by

\[
A = \sum_{J \geq 0} \int d\nu \, g_J(\nu) \, G_{i\nu, J} .
\]

In the presence of a Regge pole, equations (22) and (23) are still valid, with the relation between \( \alpha(\nu) \) and \( \beta(\nu) \) still given by equation (25). The only equation which must be modified involves \( V(\nu, \eta) \), which has different expressions in diverse dimensions. The function \( V(\nu, \eta) \), still defined by (24), is given in general by

\[
V(\nu, \eta) = \frac{2\pi}{\Gamma \left( \frac{d-1}{2} \right)} \int d\rho \, \sinh^{d-2} \rho \, \frac{\Omega_{i\nu}(\rho)}{\Omega_{i\nu}(0)} (2 \cosh \rho)^{-\eta} .
\]

The integral can be explicitly done and gives

\[
V(\nu, \eta) = \frac{\pi^{d/2 - 1}}{2} \frac{\Gamma \left( \frac{\eta - 1/2 + \nu}{2} \right) \Gamma \left( \frac{\eta - 1/2 - \nu}{2} \right)}{\Gamma(\eta)} .
\]
Therefore, the generalized couplings $V_i(\nu)$ in (25) are given in general by

$$V_i(\nu) = N_{\Delta_i} 4^{j(\nu)-1} \Gamma (\eta_i(\nu)) \frac{V(\nu, \eta_i(\nu))}{\Gamma (\Delta_i) \Gamma (\Delta_i + 1 - \frac{d}{2})}.$$ 

B Known Facts About $j(\nu, g)$

In this appendix, we recall some basic known facts about the Regge trajectory $j(\nu, g)$ which determines the spectrum of lowest twist operators in the dual YM theory. The strong and weak coupling limit of $j$ are given respectively by

$$\lim_{g \to \infty} j(\pm iE, g) = 2, \quad \lim_{g \to 0} j(\pm iE, g) = \max (|E|, 1).$$

The inverse function $E(j, g)$ has a universal large $j$ behavior

$$\lim_{j \to \infty} E(j, g) = j + f(g) \ln j + \ldots$$

determined by the so-called cusp anomalous dimension $f(g)$. The function $f(g)$ is known to all orders [14], with well-defined strong and weak coupling expansions

$$f(g) = \frac{g}{\pi} - \frac{3 \ln 2}{\pi} + \ldots,$$

$$f(g) = \frac{1}{2\pi^2} \left( g^2 - \frac{1}{48} g^4 + \frac{11}{11520} g^6 + \ldots \right).$$
Finally the functions $E(j, g)$ and $j(\pm iE, g)$ have weak coupling expansion given by

$$E(j, g) = j + \frac{g^2}{2\pi^2} (\Psi(j - 1) - \Psi(1)) + \cdots,$$

$$j(\pm iE, g) = 1 + \frac{g^2}{4\pi^2} \left(2\Psi(1) - \Psi\left(\frac{1 + E}{2}\right) - \Psi\left(\frac{1 - E}{2}\right)\right) + \cdots,$$

where the second result is the celebrated BFKL spin [15] for the hard perturbative QCD pomeron. Some of this information is graphically summarized in figure [15] following [7].

References

[1] R. J. Eden, P. V. Landshoff, D. I. Olive and J. C. Polkinghorne, *The Analytic S–Matrix*, Cambridge Univ. Pr. (2002), 295 p.

[2] V. N. Gribov, *The theory of complex angular momenta: Gribov lectures on theoretical physics*, Cambridge Univ. Pr. (2003), 297 p.

[3] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, *Large N Field Theories, String Theory and Gravity*, Phys. Rept. 323 (2000) 183, [arXiv:hep-th/9905111].

[4] L. Cornalba, M. S. Costa, J. Penedones and R. Schiappa, *Eikonal approximation in AdS/CFT: From shock waves to four-pointfunctions*, JHEP 0708, 019 (2007), [arXiv:hep-th/061122].

[5] L. Cornalba, M. S. Costa, J. Penedones and R. Schiappa, *Eikonal Approximation in AdS/CFT: Conformal Partial–Waves and Finite N Four–Point Functions*, Nucl. Phys. B 767, 327 (2007) [arXiv:hep-th/0611123].

[6] L. Cornalba, M. S. Costa, J. Penedones, *Eikonal Approximation in AdS/CFT: Resumming the Gravitational Loop Expansion*, JHEP 0709, 037 (2007), [arXiv:0707.0120 [hep-th]].

[7] R. C. Brower, J. Polchinski, M. J. Strassler and C. I. Tan, *The Pomeron and Gauge/String Duality*, [arXiv:hep-th/0603115].

[8] R. C. Brower, J. Polchinski, M. J. Strassler and C. I. Tan, *The Pomeron and Gauge/String Duality*, Phys. Rev. Lett. 88 (2002) 031601, [arXiv:hep-th/0109174].

[9] R. C. Brower, M. J. Strassler and C. I. Tan, *On the Eikonal Approximation in AdS Space*, arXiv:0707.2408 [hep-th].

[10] D. Amati, M. Ciafaloni and G. Veneziano, *Superstring Collisions at Planckian Energies*, Phys. Lett. B 197, 81 (1987).

[11] L. Cornalba, M. S. Costa, J. Penedones, *To appear*
[12] F. A. Dolan and H. Osborn, *Conformal Partial Waves and the Operator Product Expansion*, Nucl. Phys. **B678** (2004) 491, [arXiv:hep-th/0309180].

[13] Y. Hatta, E. Iancu and A. H. Mueller, *Deep inelastic scattering at strong coupling from gauge/string duality: the saturation line*, arXiv:0710.2148 [hep-th].

[14] L. Alvarez-Gaume, C. Gomez and M. A. Vazquez-Mozo, *Scaling Phenomena in Gravity from QCD*, Phys. Lett. B **649**, 478 (2007) [arXiv:hep-th/0611312].
L. Alvarez-Gaume, C. Gomez, A. Sabio Vera, A. Tavanfar and M. A. Vazquez-Mozo, *Gluon Saturation and Black Hole Criticality*, arXiv:0710.2517 [hep-th].

[15] V. S. Fadin, E. A. Kuraev and L. N. Lipatov, *On The Pomeranchuk Singularity In Asymptotically Free Theories*, Phys. Lett. B **660** (1975) 50–52
E. A. Kuraev, L. N. Lipatov and V. S. Fadin, *The Pomeranchuk Singularity in Nonabelian Gauge Theories*, Sov. Phys. JETP **45** (1977) 199–204
Ya. Ya. Balitsky and L. N. Lipatov, *The Pomeranchuk Singularity in Quantum Chromodynamics*, Sov. J. Nucl. Phys. **28** (1978) 822–829
L. N. Lipatov, *The Bare Pomeron In Quantum Chromodynamics*, Sov. Phys. JETP **63**, 904 (1986) [Zh. Eksp. Teor. Fiz. **90**, 1536 (1986)].
L. N. Lipatov, *Small–x physics in perturbative QCD*, Phys. Rept. **286**, 131 (1997) [arXiv:hep-ph/9610276].

[16] N. Beisert, B. Eden and M. Staudacher, *Transcendentality and crossing*, J. Stat. Mech. **0701**, P021 (2007) [arXiv:hep-th/0610251].