FUNCTIONS OF NEARLY MAXIMAL GOWERS-HOST-KRA NORMS ON EUCLIDEAN SPACES

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Abstract. Let \( k \geq 2, n \geq 1 \) be integers. Let \( f : \mathbb{R}^n \to \mathbb{C} \). The \( k \)th Gowers-Host-Kra norm of \( f \) is defined recursively by

\[
\|f\|_{U^k}^2 = \int_{\mathbb{R}^n} |T^h f \cdot f|_{U^{k-1}}^2 \, dh
\]

with \( T^h f(x) = f(x + h) \) and \( \|f\|_{U^1} = |\int_{\mathbb{R}^n} f(x) \, dx| \). These norms were introduced by Gowers in his work on Szemerédi’s theorem, and by Host-Kra in ergodic setting. It’s shown by Eisner and Tao that for every \( k \geq 2 \) there exist \( A(p, k, n) \) and \( p_k = 2^k/(k + 1) \) such that \( \|f\|_{U^k} \leq A(k, n) \|f\|_{L^{p_k}} \), for all \( f \in L^{p_k}(\mathbb{R}^n) \). The optimal constant \( A(k, n) \) and the extremizers for this inequality are known. In this exposition, it is shown that if the ratio \( \|f\|_{U^k}/\|f\|_{L^{p_k}} \) is nearly maximal, then \( f \) is close in \( L^{p_k} \) norm to an extremizer.

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1. Introduction

Let \( k \geq 2, n \geq 1 \) be integers and \( f : \mathbb{R}^n \to \mathbb{C} \) be a measurable function. The \( k \)th Gowers-Host-Kra norm of \( f \) is defined recursively by:

\[
\|f\|_{U^k}^2 = \int_{\mathbb{R}^n} |T^h f \cdot f|_{U^{k-1}}^2 \, dh.
\]

Here, \( T^h f(x) = f(x + h) \), and \( \|f\|_{U^1} = |\int_{\mathbb{R}^n} f(x) \, dx| \). These norms were introduced by Gowers [11] in his work on Szemerédi’s theorem, and by Host-Kra [12] in ergodic setting. They are also discussed in [17]. There is an alternative expression to (1.1) [17]:

\[
\|f\|_{U^k}^2 = \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} \mathcal{C}^{\omega_n} f(x + \alpha \cdot \vec{h}) \, dxd\vec{h}.
\]
Here $\alpha = (\alpha_1, \ldots, \alpha_k) \in \{0,1\}^k$, $\vec{h} = (h_1, \ldots, h_k) \in \mathbb{R}^k$ and $\alpha \cdot \vec{h} = \sum_{i=1}^k \alpha_i h_i \in \mathbb{R}^n$. $Cf = \vec{f}$ is the conjugation operator and $\omega_\alpha$ is the parity of the number of ones in $\alpha \in \{0,1\}^k$. For instance, if $k = 2$, then (1.2) becomes
\[
\|f\|_{U^2}^4 = \int_{\mathbb{R}^{3n}} f(x)f(x+h_1)f(x+h_2)f(x+h_1+h_2)
\]
\[\times dh_1 dh_2.
\]
It follows that, after some changes in variables,
\[
\|f\|_{U^2}^4 = \int_{\mathbb{R}^{3n}} f(x)f(y)f(z)f(x+y-z)
\]
\[\times dxdydz = \int_{\mathbb{R}^{2n}} f(x)f(y)(f*\overline{f})(x+y)
\]
\[\times dxdy + \int_{\mathbb{R}^n} (f*\overline{f})^2(y)
\]
\[\times dy = \int_{\mathbb{R}^n} |\hat{f}|^4(x)
\]
(1.3) or $\|f\|_{U^2} = \|\hat{f}\|_4$. Hence the second Gowers-Host-Kra norm is an actual norm. The same has been shown for higher order Gowers-Host-Kra norms [17]. It’s also shown in [10] that for every $k \geq 2$, $n \geq 1$ there exist $A(k,n) = A(k,1)^n$ with $A(k,1) = 2^{k/2^k}/(k+1)^{(k+1)/2^k+1}$ and $p_k = 2^k/(k+1)$ such that, for every $f \in L^{p_k}(\mathbb{R}^n)$,
\[
\|f\|_{U^k} \leq A(k,n)\|f\|_{p_k}.
\]
We call $f$ an extremizer of (1.4) if equality is attained with this choice of $f$. See Section 2 below for terminology used in this exposition. It’s shown in [10] that this constant $A(k,n)$ is best possible and that equality is attained iff $f(x) = C \exp(-(x-c) \cdot M (x-c)) \exp(2 \pi i P(x))$, with $C \in \mathbb{C}$, $c \in \mathbb{R}^n$, $M$ is a positive-definite $n \times n$ matrix and $P : \mathbb{R}^n \to \mathbb{R}$ is a polynomial of degree at most $k-1$. In this exposition, we seek to answer the question, of what happens if the ratio $\|f\|_{U^k}/\|f\|_{p_k}$ is nearly maximized on $\mathbb{R}^n$. This question was not addressed in [10]. We set out to show that $f$ must then be close in $L^{p_k}$ norm to an extremizer.

**Theorem 1.1.** Let $k \geq 2$, $n \geq 1$ be integers. For every $\epsilon > 0$ there exists $\delta > 0$ with the following property. Suppose $f \in L^{p_k}(\mathbb{R}^n)$ and
\[
\|f\|_{U^k} \geq (1 - \delta)A(k,n)\|f\|_{p_k}.
\]
Then there exists an extremizer $F$ of (1.4) on $\mathbb{R}^n$ such that $\|F - f\|_{p_k} \leq \epsilon\|f\|_{p_k}$. Theorem 1.1 can also be equivalently stated in terms of sequences.

**Theorem 1.2.** Let $k \geq 2$, $n \geq 1$ be integers. Suppose $\{f_j\}_j$ is a sequence of functions on $\mathbb{R}^n$ such that $\|f_j\|_{p_k} = 1$ and
\[
\|f_j\|_{U^k} \geq (1 - \delta_j)A(k,n)\|f_j\|_{p_k} = (1 - \delta_j)A(k,n)
\]
with $\delta_j \to 0$ as $j \to \infty$. Then there exists an extremizer $F$ of (1.4) on $\mathbb{R}^n$, $\lambda_j \in \mathbb{R}_{\geq 0}, c_j \in \mathbb{R}^n$, real-valued polynomial of degree at most $k-1$, $P_j$, and positive definite $M_j \in M^{n \times n}(\mathbb{R})$ such that
\[
\|F - \lambda_j^{p_k}f_{j} e^{iP_j}(\lambda_j(M_j \cdot c_j))\|_{p_k} \leq o(\delta_j)\|f_j\|_{p_k} = o(\delta_j).
\]

Theorem 1.1 and Theorem 1.2 are equivalent to each other for the following reasons. Note that (1.5) is invariant under an orthogonal action on the underlying Euclidean space $\mathbb{R}^n$ as well as translation and dilation. Now (1.5) is also unchanged if $f$ is replaced by $f e^{iQ}$, with $Q$ being a real-valued function, which will be shown in Section 10 to be "nearly" polynomial, in the sense defined there. Conversely, if $\{f_j\}_j$ is a sequence as in Theorem 1.2, then $\{|f_j|\}_j$ must first be appropriately translated, dilated and transformed so that all $|f_j|$ can be close to the same nonnegative extremizer of (1.4) in $L^{p_k}$ norm. If the $f_j$ are complex-valued then their complex phases of $f_j$ must also be adjusted, by multiplying with appropriate polynomial phases. These modifications are done in order to introduce compactness to such sequence $\{f_j\}_j$ and these are the only modifications required, based on the description of extremizers.
for (1.4). Hence, we will not distinguish the two theorems in this discussion and simply refer to both as Theorem 1.1. Via the relation between the second Gowers-Host-Kra norm of \( f \) and the \( L^4 \) norm of \( \hat{f} \) shown in (1.3), this near extremization question has been resolved in [6], as \( p_2 = 4/3 \), and hence \( \|f\|_{L^2} \geq (1-\delta)A(2,n)\|f\|_{4/3} \) becomes \( \|f\|_4 \geq (1-\delta)A(2,n)\|f\|_{4/3} \).

The alternative expression (1.2) of the Gowers-Host-Kra norm leads to an inner product inequality. The Gowers inner product [17] of degree \( 2^k \), \( \mathcal{T}_k \), is defined as follows. For each \( \alpha \in \{0,1\}^k \), let \( f_\alpha : \mathbb{R}^n \to \mathbb{C} \) be a measurable function. Then

\[
\mathcal{T}_k(f_\alpha : \alpha \in \{0,1\}^k) = \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} C^{\omega_\alpha} f_\alpha(x + \alpha \cdot \tilde{h}) \, dx d\tilde{h}.
\]

The mentioned inner product inequality is the following

(1.6)

\[
|\mathcal{T}_k(f_\alpha : \alpha \in \{0,1\}^k)| \leq A(k,n)^{2^k} \prod_{\alpha \in \{0,1\}^k} \|f_\alpha\|_{pk};
\]

see Section 2 below. Observe that (1.4) is a consequence of (1.6) if \( f_\alpha = f \) for all \( \alpha \in \{0,1\}^k \). A more general version of (1.5) is then

\[
|\mathcal{T}_k(f_\alpha : \alpha \in \{0,1\}^k)| \geq (1-\delta)A(k,n)^{2^k} \prod_{\alpha \in \{0,1\}^k} \|f_\alpha\|_{pk}.
\]

We will not be investigating the near extremization question in this full generality; however, the arguments presented in the second part of this exposition, which spans from Section 4 to Section 10, can be extended for the said case. The conclusion of Theorem 1.1 is qualitative; the dependence of \( \epsilon \) on \( \delta \) is not made explicit in this exposition. This non-quantitative dependence \( \epsilon = \epsilon(\delta) \) is a result of implicitness arisen in Proposition 6.1 and Subsection 7.4; other steps could be made explicit without this implicitness. There will be two parts in this exposition. The first part, Section 3, is a very short proof of Theorem 1.1 in the case of nonnegative functions, which utilizes a stability result of Young’s convolution inequality in [7]. The second part, spanning from Section 4 to Section 10, concludes the theorem for general complex-valued functions and does not rely on the said stability result. The framework used in the second part parallels that laid out in [7]. Another such inequality that is closely related to the Gowers product inequality is the following:

\[
|\mathcal{T}_k(f_\alpha : \alpha \in \{0,1\}^k)| \leq C(k,n,\vec{p}) \prod_{\alpha \in \{0,1\}^k} \|f_\alpha\|_{p_\alpha}
\]

with \( \vec{p} = (p_\alpha : \alpha \in \{0,1\}^k) \). The \( p_\alpha \) satisfy the necessary scaling condition \( \sum_{\alpha \in \{0,1\}^k} p_\alpha^{-1} = k+1 \). In Section 4 below, it’s shown that the said inequality holds (in other words, \( C(k,n,\vec{p}) < \infty \)) if \( \vec{p} \) lies in a small neighborhood of the point \( \vec{P} = (2^k/(k+1), \ldots, 2^k/(k+1)) \) on the hyperplane \( \sum_{\alpha \in \{0,1\}^k} p_\alpha^{-1} = k + 1 \). No general sufficient algebraic conditions on \( p_\alpha \) are known as of now; however, see [1] for a geometric description. This discovery at least confirms that the characteristic polytope, as discussed in [1], of the Gowers inner product structure - which is a Brascamp-Lieb structure - is not a degenerate one-point set. The Gowers-Host-Kra norm inequality is an important step toward the study of the near extremization problems of these more general cases.

2. Basics

The notation \( \mathbb{N} \) denotes the set of nonnegative integers. The notation \( x, y \) or \( h \) indicates an element in a Euclidean space while the notation \( \vec{x}, \vec{y} \) or \( \vec{h} \) indicates a tuple of such elements. In particular, the notation \( (x,\vec{h}) \) always means \( (x,\vec{h}) \in \mathbb{R}^{(k+1)n} \), with \( x \in \mathbb{R}^n, \vec{h} = (h_1, \ldots, h_k) \).
and $h_i \in \mathbb{R}^n$, $i \in \{1, \ldots, k\}$. The notation $\text{crd}(S)$ denotes the cardinality of a finite set $S$. The notation $\mathcal{L}(S)$ denotes the $n$-dimensional Lebesgue measure of a Euclidean set $S \subseteq \mathbb{R}^n$. If $S$ is a ball with center $c$ and $r > 0$, then $rS$ denotes the dilated set $\{r(x - c) \mid x \in S\}$. The notation $\text{spt}(f)$ denotes the essential support of a measurable function $f$ on a Euclidean space. If $f$ is continuous then this support is taken to be the closure of the set of points at which $f$ does not vanish. The symbol $\cdot$ denotes several meanings of multiplication, which will be clear from context. The notation $\langle a, b \rangle_{\mathbb{R}^n}$ denotes the usual real inner product between two Euclidean elements $a, b$.

Let $a, b > 0$. The symbol $\asymp$ in $a \asymp b$ means the following: There exist $C > c > 0$ such that $ca \leq b \leq Ca$. In our context, $c, C$ will be integral powers of 2, say $2^ja \leq b \leq 2^j a$; in some cases, $j$ and $l$ are chosen so that $2^j a$ and $2^l a$ are among the closest numbers of this type to $b$. Let $S \subseteq \mathbb{R}^n$. Then

$$
\tilde{S}^{k+1} = \{(x, \vec{h}) \in \mathbb{R}^{(k+1)n} \mid x, h_i, x + \alpha \cdot \vec{h} \in S, \forall i \in \{1, \ldots, k\}, \forall \alpha \in \{0,1\}^k\}.
$$

By "$f$ is of unit norm" we mean $\|f\|_p = 1$, if the $L^p$ norm is understood. By "a Gaussian on $\mathbb{R}^n$" we mean the function, $m \exp(-a|x-c|^2)$, with $m, a \in \mathbb{R}_{>0}$ and $c \in \mathbb{R}^n$. By "centered Gaussian" we mean $m \exp(-|x|^2)$, with $m > 0$ chosen so that $\|G\|_p = 1$, for some appropriate $L^p$ norm. By "measurable" we mean "Lebesgue measurable." By "a centered ball" we mean "a ball that is centered at the origin." Finally, we employ some short-hand notations - for instance, if $f : \mathbb{R}^n \to \mathbb{R}$ then $\{f > \lambda\}$ means the super-level set $\{x \in \mathbb{R}^n : f(x) > \lambda\}$ - and we make an effort to indicate parametric dependence of our constants whenever it’s important for our calculations. When the dimension $n = 1$, we simply write $C$ (other parameters, 1) = $C$ (other parameters).

We call (1.4) "the $k$th Gowers-Host-Kra norm inequality" and (1.6) "the $k$th Gowers inner product inequality." Let $f_\alpha, f : \mathbb{R}^n \to \mathbb{C}$ be measurable functions, $\alpha \in \{0,1\}^k$, and $\vec{f} = (f_\alpha : \alpha \in \{0,1\}^k)$. We say $f$ is an extremizer of the $k$th Gowers-Host-Kra norm inequality if:

$$
\|f\|_{U^k} = A(k, n)\|f\|_{p_k}.
$$

We say $f$ is a $(1 - \delta)$ near extremizer of the $k$th Gowers-Host-Kra norm inequality if:

$$
\|f\|_{U^k} \geq (1 - \delta)A(k, n)\|f\|_{p_k}
$$

and we say $\{f_i\}_i$ is an extremizing sequence of the $k$th Gowers-Host-Kra norm inequality if $\|f_i\|_{p_k} = 1$ and

$$
\|f_i\|_{U^k} \geq (1 - \delta_i)A(k, n)\|f_i\|_{p_k} = (1 - \delta_i)A(k, n)
$$

with $\delta_i \to 0$ as $i \to \infty$. We say $\vec{f}$ is an extremizing tuple of the $k$th Gowers inner product inequality if:

$$
\mathcal{T}_k(\vec{f}) = \mathcal{T}_k(f_\alpha : \alpha \in \{0,1\}^k) = A(k, n)^{2k} \prod_{\alpha \in \{0,1\}^k} \|f_\alpha\|_{p_k}.
$$

We say $\vec{f}$ is a $(1 - \delta)$ near extremizing tuple of the $k$th Gowers inner product inequality if:

$$
\mathcal{T}_k(\vec{f}) = \mathcal{T}_k(f_\alpha : \alpha \in \{0,1\}^k) \geq (1 - \delta)A(k, n)^{2k} \prod_{\alpha \in \{0,1\}^k} \|f_\alpha\|_{p_k}.
$$

Gowers-Cauchy-Schwarz inequality implies Gowers product inequality:

$$
\mathcal{T}_k(f_\alpha : \alpha \in \{0,1\}^k) \leq \prod_{\alpha \in \{0,1\}^k} \|f_\alpha\|_{U^k} \leq A(k, n)^{2k} \prod_{\alpha \in \{0,1\}^k} \|f_\alpha\|_{p_k}.
$$
The first inequality is the Gowers-Cauchy-Schwarz inequality [17]; the second is the Gowers-Host-Kra norm inequality. We called the resulting inequality, the Gowers product inequality.

Relation of Gowers-Host-Kra norm inequality to sharp Young’s inequality:
Specializing the version of the sharp Young’s inequality given in [10] to the exponents \( s = 2^{k-1} \), \( r = p_{k-1} \) and \( t = q = p_k \), we have:

\[
\left( \int_{\mathbb{R}^n} \| (T^h f) \|^{2^{k-1}}_{L^{p_{k-1}}(\mathbb{R}^n)} \, dh \right)^{1/2^{k-1}} \leq \left( \frac{C_{2^k/(k+1)}^2}{C_k} \right)^{nk/2^{k-1}} \| f \|_{L^{p_k}(\mathbb{R}^n)} \| g \|_{L^{p_k}(\mathbb{R}^n)}
\]

with \( C_t = \left( \frac{t^{1/2}}{1/t + 1/t'} \right)^{1/2} \), \( 1/t + 1/t' = 1 \). It’s shown in [10] that \( A(k) \leq 1 = A(1) \) and that \( A(k) = \left( \frac{C_{2^k/(k+1)}^2}{C_k} \right)^{k/2^{k-1}} \) for all \( k \geq 1 \). Hence:

\[
\left( \int_{\mathbb{R}^n} \| (T^h f) \|^{2^{k-1}}_{L^{p_{k-1}}(\mathbb{R}^n)} \, dh \right)^{1/2^{k-1}} \leq \left( \frac{A(k, n)}{A(k-1, n)^{1/2}} \right)^2 \| f \|_{L^{p_k}(\mathbb{R}^n)}^2 \| g \|_{L^{p_k}(\mathbb{R}^n)}.
\]

Now letting \( f = g \), incorporating this with the \((k-1)\)th Gowers-Host-Kra norm inequality and using the definition of the Gowers-Host-Kra norms, we obtain:

\[
\| f \|_{U^k(\mathbb{R}^n)}^2 = \left( \int_{\mathbb{R}^n} \| (T^h f) \|^{2^{k-1}}_{U^{k_{k-1}}(\mathbb{R}^n)} \, dh \right)^{1/2^{k-1}} \leq A(k-1, n) \left( \frac{A(k, n)}{A(k-1, n)^{1/2}} \right)^2 \| f \|_{L^{p_k}(\mathbb{R}^n)}^2 = A(k, n)^2 \| f \|_{L^{p_k}(\mathbb{R}^n)}^2
\]

which is the \( k \)th Gowers-Host-Kra norm inequality. Due to this relationship between the two inequalities, the optimal constants \( A(k, n) \) satisfy: \( A(k, m \cdot n) = (A(k, m))^n \); in particular, \( A(k, m) = (A(k))^m \).

Facts about Lorentz semi-norms: [16]

Let \( f : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) be a measurable function. There exists a unique decomposition \( f = \sum_{j \in \mathbb{Z}} 2^j f_j \) such that \( 1_{F_j} \leq f_j < 2 \cdot 1_{F_j} \), with the measurable sets \( F_j \) being pairwise disjoint up to null sets. This decomposition comes from the layer cake representation: \( f(x) = \int_0^\infty 1_{\{f > t\}}(x) \, dt = \sum_{j \in \mathbb{Z}} 2^{j+1} 1_{\{f > t\}}(x) \, dt = \sum_{j \in \mathbb{Z}} 2^j f_j(x) \), with \( f_j(x) = 2^{-j} \int_{2^{j+1}}^{2^j} 1_{\{f > t\}}(x) \, dt \).

It’s readily checked that \( 1_{F_j} \leq f_j < 2 \cdot 1_{F_j} \), with \( F_j = \{ 2^j \leq f < 2^{j+1} \} \). Let \( f_* \) be the nonincreasing rearrangement of \( f \) [16], defined as follows. If \( \lambda \) denotes the distribution function of \( f \), then \( f_*(t) = \inf \{ s : \lambda(s) \leq t \} \). We claim that:

Lemma 2.1. Let \( 1 \leq q < \infty, 1 \leq \hat{q} \leq \infty \). Then:

\[
\| f \|_{(q, \hat{q})} = \left( \frac{\hat{q}}{q} \int_0^\infty \left( \frac{1}{t^{1/q}} f_*(t) \right)^{\hat{q}} \, dt \right)^{1/\hat{q}} \propto \left( \sum_{j \in \mathbb{Z}} 2^j (\mathcal{L}(F_j))^{\hat{q}/q} \right)^{1/\hat{q}}
\]

if \( \hat{q} < \infty \), and,

\[
\| f \|_{(q, \infty)} = \sup_{t > 0} t^{1/q} f_*(t) = \sup_{j ; F_j \not= \emptyset} 2^j (\mathcal{L}(F_j))^{1/q}.
\]

Proof. First note that if \( f = 1_E \), \( E \) being a measurable set of \( \mathbb{R}^n \), then \( \| f \|_{(q, \hat{q})} = (\mathcal{L}(E))^{1/q} \), \( 1 \leq \hat{q} \leq \infty \). Since \( 1_{F_j} \leq f_j < 2 \cdot 1_{F_j} \), we first consider the case \( f = \sum_{j = -1}^\infty 2^j 1_{E_j} \) with pairwise
disjoint measurable sets \( E_j \) of \( \mathbb{R}^n \) and \( l \in \mathbb{Z}_{>0} \). Then:

\[
\|f\|_{(q,q)} = \left( \sum_{j=-l}^{l} 2^{jq}(B_j^{q} - B_{j+1}^{q}) \right)^{1/q} \leq \left( \sum_{j=-l}^{l} 2^{jq}(L(E_j))^q \right)^{1/q}.
\]

Here \( B_j = \sum_{i=j}^{l}(L(E_i)) \) and \( B_{l+1} = 0 \). The first equality is easily deduced from the case of a single set indicator function - see also [16]. The approximation that follows is an application of the algebraic fact: \((a^t + b^t) \leq (a+b)^t \leq 2^t(a^t + b^t)\), for \( t \geq 1 \) and \( a, b > 0 \). That means, the constants in the approximation depend only on \( q, \tilde{q} \) and not on \( l \). Then for the general case:

\[
\|f\|_{(q,q)} = \|\sum_{j \in \mathbb{Z}} 2^{j} F_j \|_{(q,q)} = \lim_{l \to \infty} \left( \sum_{j=-l}^{l} 2^{jq}(L(F_j))^q \right)^{1/q}.
\]

for all sufficiently positively large \( l \) depending on \( f \). Hence this ultimately allows us to write:

\[
\|f\|_{(q,q)} = \lim_{l \to \infty} \left( \sum_{j=-l}^{l} 2^{jq}(L(F_j))^q \right)^{1/q} = \left( \sum_{j \in \mathbb{Z}} 2^{jq}(L(F_j))^q \right)^{1/q}.
\]

Similarly, if \( \tilde{q} = \infty \) and if \( f = \sum_{j=-l}^{l} 2^{j} 1_{E_j} \), then \( \|f\|_{(q,\infty)} = \sup_{-l \leq j \leq l} 2^{j} B_j^{1/q} \leq \sup_{-l \leq j \leq l} 2^{j}(L(E_j))^{1/q} \).

Then for the general case:

\[
\|f\|_{(q,\infty)} = \|\sum_{j \in \mathbb{Z}} 2^{j} F_j \|_{(q,\infty)} = \lim_{l \to \infty} \left( \sup_{-l \leq j \leq l} 2^{j}(L(F_j))^q \right)^{1/q} = \sup_{j:F_j \neq \emptyset} 2^{j}(L(F_j))^{1/q}.
\]

The last approximation follows because the first approximation holds for all sufficiently positively large \( l \).

3. A SHORT PROOF OF THEOREM 1.1 FOR NONNEGATIVE FUNCTIONS

Let \( \beta \in \{0,1\}^{k+1} \), then \( \beta = (\alpha, 0) \) or \( \beta = (\alpha, 1) \) for some \( \alpha \in \{0,1\}^k \). Let \( f = (f_{\beta}: \beta \in \{0,1\}^{k+1}) \), and suppose \( f_{\beta} \geq 0 \) for all \( \beta \in \{0,1\}^{k+1} \). Then

\[
\mathcal{T}_{k+1}(f_{\beta}: \beta \in \{0,1\}^{k+1}) = \int_{\mathbb{R}^n} \mathcal{T}_{k}(T^h f_{(\alpha,0)} : f_{(\alpha,1)} : \alpha \in \{0,1\}^k) \, dh
\]

\[
\leq \int_{\mathbb{R}^n} \prod_{\alpha \in \{0,1\}^k} \|T^h f_{(\alpha,0)} : f_{(\alpha,1)}\|_{U^k} \, dh
\]

\[
\leq A(k,n)^{2^k} \int_{\mathbb{R}^n} \prod_{\alpha \in \{0,1\}^k} \|T^h f_{(\alpha,0)} : f_{(\alpha,1)}\|_{p_k} \, dh
\]

\[
\leq A(k,n)^{2^k} \prod_{\alpha \in \{0,1\}^k} \left( \int_{\mathbb{R}^n} \|T^h f_{(\alpha,0)} : f_{(\alpha,1)}\|_{p_k}^2 \, dh \right) \left( \int_{\mathbb{R}^n} \|T^h f_{(\alpha,0)} : f_{(\alpha,1)}\|_{p_k}^2 \, dh \right)^{1/2k}.
\]

The first inequality is due to the Gowers-Cauchy-Schwarz inequality, the second to the Gowers-Host-Kra norm inequality and the last to Hölder’s inequality. Substituting \( p_k = 2^k/(k+1) \) in
The sole inequality in the display (3.2) above follows from Young’s convolution inequality. Here \( \tilde{f}(y) = f(-y) \), \( 2q^{-1} = (k + 1)^{-1} + 1 \), or equivalently, \( q = p_{k+1}/p_k \) and \( Y(k + 1, n) \) is the optimal constant in Young’s convolution inequality for the involved exponents. Combining (3.1) and (3.2), we obtain the following majorization:

\[
\mathcal{T}_{k+1}(f_\beta : \beta \in \{0, 1\}^{k+1}) \leq A(k, n)^{2k} Y(k + 1, n)^{k+1} \prod_{\beta \in \{0, 1\}^{k+1}} \|f_\beta\|_{p_{k+1}} = A(k + 1, n)^{2k+1} \prod_{\beta \in \{0, 1\}^{k+1}} \|f_\beta\|_{p_{k+1}}
\]

as it happens that \( A(k, n)^{2k} Y(k + 1, n)^{k+1} = A(k + 1, n)^{2k+1} \) - see [10] and Section 2. Now suppose that there exists \( \delta > 0 \) such that

\[
\mathcal{T}_{k+1}(f_\beta : \beta \in \{0, 1\}^{k+1}) \geq (1 - \delta) A(k + 1, n)^{2k+1} \prod_{\beta \in \{0, 1\}^{k+1}} \|f_\beta\|_{p_{k+1}}.
\]

Then each inequality in (3.1) and (3.2) must hold in reverse, up to a factor of \( 1 - c(k)\delta \), for some small \( c(k) > 0 \). In particular, for each \( \alpha \in \{0, 1\}^k \),

\[
\|f_{p_{k}\alpha}^{\beta} * f_{p_{k}\alpha}^{\beta}\|_{k+1}^{1/k+1} \geq (1 - c(k)\delta) Y(k + 1, n)^{k+1} \|f_{p_{k}\alpha}^{\beta}\|_{k+1}^{1/k+1} \|f_{p_{k}\alpha}^{\beta}\|_{k+1}^{1/k+1}.
\]

Since \( (\alpha, 0), (\alpha, 1) \in \{0, 1\}^{k+1} \) if \( \alpha \in \{0, 1\}^k \), a stability result for Young’s convolution inequality in [7] implies from (3.3) that, for each \( \beta \in \{0, 1\}^{k+1} \), there exists a Gaussian function \( G_\beta \) such that:

\[
\|G_\beta - f_\beta\|_q \leq o(\delta) \|f_\beta\|_q.
\]

Since \( \|G_\beta^{1/p_k} - f_\beta\|_{p_{k+1}} \leq C(k) \|G_\beta - f_\beta\|_q^{1/p_k} \), by a simple calculation, (3.4) leads to \( \|G_\beta^{1/p_k} - f_\beta\|_{p_{k+1}} \leq o(\delta) \|f_\beta\|_{p_{k+1}} \).

This discussion so far applies to \( k + 1 \geq 3 \), since \( k \geq 2 \). For the characterization of a near extremizing tuple \( \tilde{f} = (f_1, f_2, f_3, f_4) \) of the second Gowers inner product inequality, note that

\[
\mathcal{T}_2(f_1, f_2, f_3, f_4) = \langle \tilde{f}_1 * f_2, \tilde{f}_3 * f_4 \rangle \leq \|\tilde{f}_1 * f_2\|_2 \|\tilde{f}_3 * f_4\|_2 \leq A(2, n)^4 \prod_{i=1}^4 \|f_i\|_{4/3}.
\]

The first inequality is an application of the Cauchy-Schwarz inequality while the second follows from Young’s convolution inequality. The appearance of the constant in (3.5) is due to the facts that \( A(j, n)^{2j} B(j + 1, n)^{j+1} = A(j + 1, n)^{2j+1} \) actually holds for all \( j \geq 1 \) and that \( A(1, n) = A(1)^n = 1 \). If for some \( \delta > 0 \),

\[
\mathcal{T}_2(f_1, f_2, f_3, f_4) \geq (1 - \delta) A(2, n)^4 \prod_{i=1}^4 \|f_i\|_{4/3}
\]

then with a similar argument as above, we trade each inequality in (3.5) for a reverse inequality with a factor of \( 1 - c\delta \) and use the said stability result for Young’s convolution inequality to conclude that, for each \( i \in \{1, \ldots, 4\} \), there exists a Gaussian \( G_i \) such that \( \|G_i - f_i\|_{4/3} \leq o(\delta) \|f_i\|_{4/3} \). As Gaussians are the only nonnegative maximizers of the Gowers-Host-Kra norm.
inequality; hence this concludes Theorem 1.1 for the case of nonnegative near extremizers.

\[ \square \]

4. Localization around a single scale

4.1. Normalization. Let \( \Theta : \mathbb{R}_{>0} \to \mathbb{R}_{>0} \) be such that \( \Theta(t) \to 0 \) if \( t \to \infty \). Let \( f : \mathbb{R}^n \to \mathbb{C} \) and \( f \in L^q, \quad q \in [1, \infty) \). We say that \( f \) is normalized (with respect to \( \Theta \)) if the following two conditions occur:

\[
\int_{|f|>\rho} |f|^q \, dx \leq \Theta(\rho) \\
\int_{|f|<\rho} |f|^q \, dx \leq \Theta(\rho^{-1}).
\]

We often say "\( f \) is normalized" for short as the presence of such a function \( \Theta \) will always be implied, and the selection of \( \Theta \) will precede the selection of the involved near extremizers, so as to create no ambiguity. We will also allow the following relaxed version of normalization. Let \( \nu > 0 \) be a small number and \( \Theta \) be as above. Then \( f \) is said to be \( \nu \)-normalized, with respect to \( \Theta \), if \( f = g + h, \) \( g \) is normalized with respect to \( \Theta \) and \( \|h\|_q \leq \nu \). In our application, \( q = p_k = \frac{2^k}{p_{k+1}} \). From now on, whenever we say a "growth function" we mean a function \( \Theta : \mathbb{R}_{>0} \to \mathbb{R}_{>0} \) satisfying \( \Theta(\rho) \to 0 \) as \( \rho \to \infty \). Our claim is that, there exist a growth function \( \Theta \) and a \( \delta_0 > 0 \), so that the following property occurs. If \( 0 < \delta \leq \delta_0 \) and \( f \) is a \((1-\delta)\) near extremizer of the Gowers-Host-Kra norm inequality, then there exists \( \lambda > 0 \), such that, \( \tilde{f}(x) = \lambda^{n/p_k} f(\lambda x) \) is normalized, with respect to \( \Theta \). toward this end, we will prove an equivalent result below.

Since the normalization condition concerns the absolute value of \( f \) and since \( |f| \) is a \((1-\delta)\) near extremizer if \( f \) is, as \( \|f\|_{U^k} \leq \|f\|_{U^k} \), we consider only nonnegative functions. Let \( f : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) be a measurable function and \( f = \sum_{j \in \mathbb{Z}} 2^j F_j \) be the unique decomposition of \( f \) mentioned in Section 2. Recall here that \( 1_{F_j} \leq \tilde{F}_j < 2 \cdot 1_{F_j} \), and the measurable sets \( F_j \) are pairwise disjoint up to null sets. This decomposition will set up the stage for the use of Lorentz semi-norms which will be needed. In particular, we want to show:

**Proposition 4.1.** There exist positive functions \( \phi, \Phi \) satisfying \( \lim_{t \to \infty} \phi(t) = 0 \) and \( \lim_{t \to 0} \Phi(t) = 0 \), and positive constants \( \delta_0, c_0 \) such that the following holds.

Let \( 0 \leq f \in L^p(\mathbb{R}^n) \). Let \( f = \sum_{j \in \mathbb{Z}} 2^j F_j \) with the \( F_j \) related to \( f \) as indicated above. Suppose that \( \|f\|_{p_k} = 1 \), and for some \( 0 < \delta \leq \delta_0 \), we have

\[
\|f\|_{U^k} \geq (1-\delta)A(k,n)\|f\|_{p_k} = (1-\delta)A(k,n).
\]

Then there exists \( l \in \mathbb{Z} \) such that:

\[
2^l (\mathcal{L}(F_j))^{1/p_k} \geq c_0 \|f\|_{p_k} = c_0
\]

\[
\sum_{|j-l| \geq m} 2^{jp} \mathcal{L}(F_j) \leq (\phi(m) + \Phi(\delta)) \cdot \|f\|_{p_k}^{p_k} = \phi(m) + \Phi(\delta).
\]

**Remark 4.1.** Statement (4.1) ensures that there exists a scale \( 2^l \) so that the contribution of the quantity \( 2^l (\mathcal{L}(F_j))^{1/p_k} \) to the total norm \( \|f\|_{p_k} \) is non-negligible. Choose \( \lambda \) to be an integral power of 2 to dilate the set \( F_j \), so that its dilated version has measure as close to one as possible. Statement (4.2) is equivalent to saying, this dilated version of \( f \) is then \( \nu \)-normalized with respect to an appropriate growth function \( \Theta \) and a positive \( \nu = \nu(\delta) \) that tends to zero as \( \delta \to 0 \).

Let \( 1 \leq q, \tilde{q} < \infty \) and \( f = \sum_{j \in \mathbb{Z}} 2^j F_j \) be as above. Recall from Section 2 that \( \|f\|_{(q,\tilde{q})} \simeq (\sum_{j \in \mathbb{Z}} 2^{j\tilde{q}} (\mathcal{L}(F_j))^{\tilde{q}})^{1/q} \) if \( \tilde{q} < \infty \), and \( \|f\|_{(q,\infty)} \simeq \sup_{j \in \mathbb{Z}} 2^{j\tilde{q}} (\mathcal{L}(F_j))^{1/q} \).
A proof of (4.4), then if \( \tilde{q} \in \Omega \),

\[
\|F\|_{q_\alpha} \leq C(k, n) \prod_{\alpha \in \{0,1\}^k : \alpha \neq \vec{0}} \|f_\alpha\|_{q_\alpha}.
\]

If we vary one \( q_\alpha, \alpha \neq \vec{0} \), while keeping other values \( q_\beta, \beta \neq \vec{0}, \alpha \), fixed and apply interpolation for the Lorentz norms, we obtain from (4.6):

\[
\|F\|_{(r', \infty)} \leq C(k, n) \|f_\alpha\|_{(p_k, \infty)} \prod_{\beta \in \{0,1\}^k : \beta \neq \vec{0}, \alpha} \|f_\beta\|_{q_\beta}
\]

whenever \( p_k^{-1} + r_\alpha^{-1} + \sum q_\beta^{-1} = k + 1 \) and \( r_\alpha, q_\beta \in \left[ p_k - 2^{-k}, p_k + 2^k \right] \). We continue applying interpolation on other exponents \( q_\beta \), one by one. Suppose that \( q_\gamma \) is the last exponent to be interpolated, then:

\[
\|F\|_{(r', \infty)} \leq C(k, n) \|f_\gamma\|_{q_\gamma} \prod_{\beta \in \{0,1\}^k : \beta \neq \vec{0}, \gamma} \|f_\beta\|_{(p_k, \infty)}
\]

whenever \( r_\gamma^{-1} + q_\gamma^{-1} + (2^k - 2)p_k^{-1} = k + 1 \) and \( r_\gamma, q_\gamma \in \left[ p_k - 2^{-k}, p_k + 2^k \right] \). Applying interpolation one last time, we have:

\[
\|F\|_{(p_k', 1)} \leq C(k, n) \|f_\gamma\|_{(p_k, \infty)} \prod_{\alpha \in \{0,1\}^k : \alpha \neq \vec{0}, \gamma} \|f_\alpha\|_{(p_k, \infty)}
\]

\[
\|F\|_{(p_k', \infty)} \leq C(k, n) \|f_\gamma\|_{(p_k, \infty)} \prod_{\alpha \in \{0,1\}^k : \alpha \neq \vec{0}, \gamma} \|f_\alpha\|_{(p_k, \infty)}.
\]

Since \( \gamma \) can be any of the values \( \alpha \in \{0,1\}^k \), these calculations yield,

\[
\|F\|_{(p_k', 1)} \leq C(k, n) \prod_{\alpha \in \{0,1\}^k : \alpha \neq \vec{0}} \|f_\alpha\|_{(p_k, r_\alpha)}
\]

\[
\|F\|_{(p_k', \infty)} \leq C(k, n) \prod_{\alpha \in \{0,1\}^k : \alpha \neq \vec{0}} \|f_\alpha\|_{(p_k, \infty)}
\]

whenever \( \sum r_\alpha^{-1} = 1 \) and \( r_\alpha \in \{1, \infty\} \). Then (4.4), (4.7) and (4.8) yield:

\[
\|T_k(\vec{f})\| \leq C \|f_0\|_{(p_k, r_{\vec{0}})} \|F\|_{(p_k', \infty)} \leq C(k, n) \prod_{\alpha \in \{0,1\}^k} \|f_\alpha\|_{(p_k, r_\alpha)}
\]
whenever $\sum r^{-1}_\alpha = 1$ and $r_\alpha \in \{1, \infty\}$. To complete the proof of (4.3), we need the following short lemma:

**Lemma 4.2.** [5] Let $T$ be a scalar-valued $m$-linear form on $\mathbb{R}^n$ and $q \in [1, \infty)$. Suppose that

$$|T(\vec{f})| \leq C(m) \prod_{1 \leq i \leq m} \|f_i\|_{(q, r_i)}$$

whenever $r_i \in \{1, \infty\}$ and $\sum r_i^{-1} = 1$. Suppose furthermore that $\vec{q} = (q_i)_{i=1}^m$ with $\sum q_i^{-1} = 1$ and $q_i \in (1, \infty)$. Then

$$|T(\vec{f})| \leq C(m, \vec{q}) \prod_{1 \leq i \leq m} \|f_i\|_{(q, q_i)}.$$  

**Proof.** We use the described decomposition $f_i = \sum_{j \in \mathbb{Z}} 2^j F_{i,j}$, with $1_{F_{i,j}} \leq 2^{-1} \cdot 1_{F_{i,j}}$ and the measurable sets $F_{i,j}$ pairwise disjoint. Let $S_{i,\lambda} = \{ j : 2^\lambda \leq 2^j (L(F_{i,j}))^{1/p_k} < 2^{\lambda+1} \}$ for each $\lambda \in \mathbb{Z}$. Define $f_{i,\lambda} = \sum_{j \in S_{i,\lambda}} 2^j F_{i,j}$. Note that $f_i = \sum_{\lambda \in \mathbb{Z}} 2^\lambda f_{i,\lambda}$ and for $s \in [1, \infty)$:

$$\|f_i\|_{(q,s)} = \left( \sum_{\lambda \in \mathbb{Z}} 2^{2\lambda \chi}(\text{crd}(S_{i,\lambda}))^{1/\alpha} \right)^{1/s}.$$  

If $s = \infty$:

$$\|f_i\|_{(q, \infty)} \approx \lim_{t \to \infty} \left( \sum_{\lambda \in \mathbb{Z}} 2^{\lambda \chi}(\text{crd}(S_{i,\lambda}))^{1/t} \right)^{1/t} = \sup_{\lambda:S_{i,\lambda} \neq \emptyset} 2^\lambda.$$

Let $\vec{q} = (\lambda_i)_{i=1}^m \in \mathbb{Z}^m$. Then from (4.10), (4.12) and (4.13):

$$|T(\vec{f})| \leq C(m, \vec{q}) \sum_{\lambda} 2^{2\lambda \chi} \min_{1 \leq i \leq m} (\text{crd}(S_{i,\lambda})) \leq C(m) \sum_{\lambda} \theta((\text{crd}(S_{i,\lambda}))_{i=1}^m) \prod_{1 \leq i \leq m} 2^\lambda \text{crd}(S_{i,\lambda})^{1/q_i}$$

with $\sum q_i^{-1} = 1$ and $\theta((t_i)_{i=1}^m) = (\min t_i) \cdot \prod_{1 \leq i \leq m} t_i^{-1/q_i}$. Consider all the vectors $\vec{q}$ for which:

$$\text{crd}(S_{i,\lambda_i}) = \min_i \text{crd}(S_{i,\lambda_i})$$

$$\text{crd}(S_{m,\lambda_m}) = \max_i \text{crd}(S_{i,\lambda_i}).$$

The following analysis applies the same way for other vectors $\vec{q}$ with only minor changes in notation. For each $i \in \{1, \ldots, m\}$, define a nonnegative integer $n_i$ so that

$$2^{-n_i - 1} \text{crd}(S_{i,\lambda_i}) \leq \text{crd}(S_{i,\lambda_i}) \leq 2^{-n_i} \text{crd}(S_{i,\lambda_i}).$$

Fix a tuple $\vec{n} = (n_i)_{i=1}^m \in \mathbb{N}^m$; we sum the right hand side of (4.14) over $\vec{q}$ for which all three (4.15), (4.16) and (4.17) are satisfied with this fixed $\vec{n}$. Define $I_1 = 1$ and $I_i, i \neq 0$, is the largest index $l$ such that $2^{-n_i - 1} \text{crd}(S_{i,\lambda_i}) \leq \text{crd}(S_{i,l}) \leq 2^{-n_i} \text{crd}(S_{i,\lambda_i})$. With these choices of $I_i$, it’s clear that $I_1 \to I_i$ are bounded-to-one functions, uniformly in $\vec{n}$. Then for this fixed $\vec{n}$, the sum on the right hand side of (4.14) will be majorized by:

$$C(m) \sum_{I_m} 2^{-n_m} \prod_{1 \leq i < m} 2^{n_i/q_i} \prod_{1 \leq i < m} 2^{l_{i,q_i} \text{crd}(S_{i,l_i})^{1/q_i}} \leq C(m) 2^{-r} \sum_{I_m} \prod_{1 \leq i < m} 2^{l_{i,q_i} \text{crd}(S_{i,l_i})^{1/q_i}}$$

with $u_m = n_m$ and $u_i = n_m - n_i \geq 0$ and $0 < r = \min_i 1/q_i$. Once again, with these choices of $I_i$, the constant $C(m)$ in (4.18) doesn’t depend on $\vec{n}$. Applying Hölder’s inequality to the right hand side sum in (4.18) to have it further majorized by:

$$C(m) 2^{-r} \sum_{I_m} \prod_{1 \leq i < m} \left( \sum_{I_m} 2^{l_{i,q_i} \text{crd}(S_{i,l_i})^{1/q_i}} \right)^{1/q_i} \leq C(m) 2^{-r} \sum_{I_m} \prod_{1 \leq i < m} \|f_i\|_{(q, q_i)}.$$  

Summing the sum in the right hand side of (4.14) over $\vec{n} \in \mathbb{N}^m$ and utilizing the bound in (4.19) and the convergence of geometric series, we conclude that (4.10) indeed implies (4.11). □
Applying this Lemma 4.2 to (4.9), with \( q = p_k \) and \( m = 2^k \) with indices \( \alpha \in \{0,1\}^k \), we obtain (4.3) with the desired exponents \( q_i = 2^k \).

**Remark 4.2.** The interpolation result of this discussion is a very restricted result that is suited to the task at hand. A more general interpolation result can be found in [13].

**Proof of (4.5).** [5] We prove the following by induction. For every \( k \geq 2 \), there exists \( r_k \) such that, if \( \vec{p} = (p_\alpha : \alpha \in \{0,1\}^k) \) is such that \( \sum_\alpha p_\alpha^{-1} = k + 1 \) and \( |p_\alpha - 2^i/(k+1)| \leq r_k \), for every \( \alpha \in \{0,1\}^k \), then

\[
T_k(\vec{f}) \leq \prod_{\alpha \in \{0,1\}^k} \|f_\alpha\|_{p_\alpha}
\]

for every \( \vec{f} = (f_\alpha : \alpha \in \{0,1\}^k) \) with each \( f_\alpha \) nonnegative and measurable.

If \( k = 2 \) then the conclusion is a consequence of Young’s and Hölder’s inequalities:

\[
T_k(\vec{f}) = \langle f_1 \ast \tilde{f}_2, f_3 \ast \tilde{f}_4 \rangle_R \leq \|f_1 \ast \tilde{f}_2\|_{p_1,2} \|f_3 \ast \tilde{f}_4\|_{p_3,4} \leq \prod_{i=1}^4 \|f_i\|_{p_i}.
\]

Here, \( \tilde{f}(x) = f(-x) \). The condition on \( p_i, i \in \{1, \ldots, 4\} \) is simply, \( p_{1,2}^{-1} + p_{3,4}^{-1} = 1 \), or equivalently, \( \sum_{i=1}^4 p_i^{-1} = 2 + 1 \).

Assume the conclusion holds for the case \( k - 1 \). Let \( \vec{f} = (f_\alpha \geq 0 : \alpha \in \{0,1\}^k) \). For each \( \alpha \in \{0,1\}^k \), there is a unique \( \beta \in \{0,1\}^{k-1} \) such that \( \alpha = (\beta,0) \) or \( \alpha = (\beta,1) \). Then for each \( \beta \in \{0,1\}^k \), let \( g_\beta^0(x) = f_{(\beta,0)}(x) f_{(\beta,1)}(x + t) \). Write \( \vec{g} = (g_\beta^0 : \beta \in \{0,1\}^{k-1}) \), then

\[
T_k(\vec{f}) = \int_{\mathbb{R}^n} T_{k-1}(\vec{g}^0) \, dt \leq \int_{\mathbb{R}^n} \prod_{\beta \in \{0,1\}^{k-1}} \|g_\beta^0\|_{p_\beta} \, dt \leq \prod_{\beta \in \{0,1\}^{k-1}} \left( \int_{\mathbb{R}^n} \|g_\beta^0\|_{p_\beta}^{2^{k-1}} \, dt \right)^{1/2^{k-1}} \leq \prod_{\beta \in \{0,1\}^{k-1}} \|f_{(\beta,0)}\|_{p_{(\beta,0)}} \|f_{(\beta,1)}\|_{p_{(\beta,1)}}.
\]

The first inequality follows from the induction hypothesis for some \( r_{k-1} > 0 \) such that, \( \sum_{\beta \in \{0,1\}^{k-1}} p_\beta^{-1} = k \) and \( |p_\beta - 2^{k-1}/k| \leq r_{k-1} \), for every \( \beta \in \{0,1\}^{k-1} \). The second inequality follows from Minkowski’s inequality, and the last from Young’s inequality, with the following relations between the exponents: \( 1/2^{k-1} + 1/p_\beta = 1/p_{(\beta,0)} + 1/p_{(\beta,1)} \). Now it’s easily observed that if \( p_{(\beta,0)} = 2^k/(k+1) = p_{(\beta,1)} \) then \( p_\beta = 2^{k-1}/k \); if \( \sum_{\beta \in \{0,1\}^{k-1}} p_\beta^{-1} = k \) then \( \sum_{\beta \in \{0,1\}^k} p_{(\beta,0)}^{-1} + p_{(\beta,1)}^{-1} = k + 1 \), and for each \( \beta \in \{0,1\}^{k-1} \), \( p_\beta \) is a continuous function of \( p_{(\beta,0)} \) and \( p_{(\beta,1)} \). Hence if for each \( \beta \in \{0,1\}^{k-1} \), \( p_{(\beta,0)} \) and \( p_{(\beta,1)} \) are taken sufficiently close to \( 2^k/(k+1) \), so that \( p_\beta \) is within an \( r_{k-1} \) distance to \( 2^{k-1}/k \), then the conclusion for the case \( k \) follows from the induction hypothesis for the case \( k - 1 \). The proof of (4.5) is now complete.

**4.3. Proof of Proposition 4.1.** Let \( \eta \equiv \eta(\delta) > 0 \) be a small number to be chosen later. Define \( S = S(\eta) = \{j \in \mathbb{Z} : 2^j (\mathcal{L}(F_j))^{1/p_k} > \eta\} \). Statement (4.1) will follow if this set \( S \) is nonempty, which is indeed true if \( \eta \) is sufficiently small. To see this, consider \( f_S = \sum_{j \in S} 2^j F_j \) and \( f_S^c = \sum_{j \notin S} 2^j F_j \). Then:

\[
\|f_S\|_{p_k,2^k} = \|\sum_{j \notin S} 2^j F_j\|_{p_k,2^k} = \sum_{j \notin S} 2^{j2^k}(\mathcal{L}(F_j))^{2^k/p_k} \leq \max_{j \notin S}(2^j (\mathcal{L}(F_j))^{1/p_k})^{2^k-p_k} \sum_{j \notin S} (2^j (\mathcal{L}(F_j))^{1/p_k})^{p_k} \leq \eta^{2^k-p_k} \|f_{S^c}\|_{p_k}.
\]
or
\[(4.20) \quad \|f_{S^c}\|_{(p_k,2^k)} \leq C \eta^{-p_k/2^k} \|f_{S^c}\|_{p_k/2^k}.\]

As above, for a set $A \subset \mathbb{Z}$, we denote $f_A = \sum_{j \in A} 2^j F_j$ and define a $2^k$ tuple-valued function on $\mathbb{R}^n$, $\tilde{f}_A = (f_A, f, \ldots, f)$. With this notation:

\[(4.21) \quad |\mathcal{T}_k(\tilde{f}_{S^c})| = \left| \int_{\mathbb{R}^{(k+1)n}} f_{S^c}(x) \prod_{\alpha \in \{0,1\}^k: \alpha \neq 0} f(x + \alpha \cdot \tilde{h}) \, dx \, d\tilde{h} \right| \leq C(k,n)\|f_{S^c}\|_{(p_k,2^k)} \|f\|_{p_k}^{2^k-1} \leq C(k,n) \eta^{-p_k/2^k} \|f_{S^c}\|_{p_k/2^k} \|f\|_{p_k}^{2^k-1}.
\]

The first inequality follows from (4.3) and the fact that $\|f\|_{(p_k,2^k)} \leq \|f\|_{p_k}$ and the last inequality from (4.20). Recall that $\|f\|_{p_k} = 1$. Hence $S = \emptyset$ in the context of (4.21) would imply that:

\[(4.22) \quad \|f\|_{U^k}^{2^k} \leq C(k,n) \eta^{-p_k/2^k} \|f\|_{p_k}^{2^k} = C(k,n) \eta^{-p_k}.
\]

Since $1 - p_k/2^k > 0$, the right hand side of (4.22) is small if $\eta$ is small. From the near extremizing hypothesis, $\|f\|_{U^k} \geq A(k,n)(1 - \delta)$. Hence, if $\eta$ is sufficiently small then $S = S(\eta)$ must be nonempty; otherwise (4.22) implies $\|f\|_{U^k} = 0$, a contradiction. Note that the argument also shows, for given $\eta$, if $\delta$ is sufficiently small then $S = S(\eta) \neq \emptyset$. In addition, from the definition of $S$ and Chebyshev’s inequality,

\[(4.23) \quad \text{crd}(S) \leq C \eta^{-p_k} \|f\|_{p_k}^{2^k} = C \eta^{-p_k}.
\]

We now prove statement (4.2). Let $\eta > 0$ be a small number, and consequently let $\delta > 0$ be small enough if needed, so that $S = S(\eta) \neq \emptyset$. Let $M = \max_{i,j \in S} |i - j|$. In order to prove (4.2), it suffices to show that $M$ is bounded by a finite upper bound that does not depend on a particular $(1 - \delta)$ near extremizer $f$. Assume that $S$ has more than one element. Note that since $S$ is a set of integers, $M \geq \text{crd}(S)$. Let $N > 0$ be a large integer to be chosen below, which will depend only on $n, k$ and $\eta$ - and consequently on $\delta$. Suppose that with this choice of $N$ we have $M \leq 10N\text{crd}(S)$, then we obtain the desired bound, since from (4.23):

\[M \leq C N \text{crd}(S) \leq C(k,n,\delta) \eta(\delta)^{-p_k}.
\]

If $M > 10N\text{crd}(S)$, then there must exist integers $J > I$ such that: $S \cap (-\infty, I] \neq \emptyset$, $S \cap [J, \infty) \neq \emptyset$, $S \cap (I, J) = \emptyset$ and $J - I \geq M/(N\text{crd}(S))$. Indeed, choose an integer $L$ such that $M/(2N\text{crd}(S)) \leq L \leq M/(N\text{crd}(S))$. Consider the intervals $I_j = [jL, (j+1)L) \subset (\inf(S), \sup(S))$ and let $K$ be the set of all these indices $j$. Then $\text{crd}(K) \approx M/L \approx N\text{crd}(S)$. Hence $I_j \cap S = \emptyset$ for at least one of these intervals. Since

\[\|f_{S^c}\|_{p_k/2^k}^{2^k} = \sum_{l \in S} \|2^l F_l\|_{p_k/2^k}^{2^k} \geq \sum_{j \in K: I_j \subseteq S = \emptyset, l \not\in I_j} \|2^l F_l\|_{p_k/2^k}^{2^k},
\]

and $\text{crd}(K) - \text{crd}(S) \geq c(N-1)\text{crd}(S) \geq cN - 1$, there exists one such interval $I_j$ such that,

\[\|f_{S^c}\|_{p_k/2^k}^{2^k} \geq C N^{-1} \sum_{l \in I_j} \|2^l F_l\|_{p_k/2^k}^{2^k}.
\]

Take such an interval, then $I = jL$ and $J = (j+1)L$ are our desired integers. Moreover, with this pair of $(I, J)$:

\[(4.24) \quad \sum_{l < l < J} \|2^l F_l\|_{p_k/2^k}^{2^k} \leq C N^{-1} \|f_{S^c}\|_{p_k/2^k}^{2^k} \leq C N^{-1} \eta^{2^k-1}.
\]
The second inequality follows from (4.20). Define $f_u = \sum_{i \in I} 2^i F_i$, $f_d = \sum_{i \in I} 2^i F_i$ and $f_b = f - f_u - f_d$. As (4.24) states, $\|f_\Omega\|_{p_k} \leq C N^{\alpha} \gamma^c$ for some $c > 0$. Now in terms of these $f_u, f_d, f_b$:

$$ \|f_k\|_{p_k} = |T_k(f_u + f_d + f_b, ..., f_u + f_d + f_b)|. $$

Expanding the above sum, we will have $C(k)$ terms in which at least one of the components is $f_b$, $C(k)$ mixed terms in which the components consist of only $f_u$ and $f_d$ but no $f_b$, and two pure terms in which the components are either $f_u$ or $f_d$, which we denote $T_k(f_u')$ and $T_k(f_d')$, respectively. We claim that the contribution of all the mixed terms is majorized by $C(k, n)2^{-\eta p_k M/N}$. We will first need the following fact about the $2^k$ linear form $T_k$ on indicators of sets. Recall that $\Omega = \{q = (q_\alpha : \alpha \in \{0, 1\}) \in Q_r(\Omega) : \sum_{\alpha \in \{0, 1\}} q_\alpha^{-1} = k + 1\}$ and $Q_r(\Omega)$ denotes a closed cube of sufficiently small size $r$ centered at $\Omega = (p_1, ..., p_k) \in \mathbb{R}^k$. Then:

**Lemma 4.3.** Suppose $\bar{\Omega} = (p_\alpha : \alpha \in \{0, 1\}) \in \Omega$. Then there exist $\tau > 0$ and $C$ such that, if $E_\alpha \subset \mathbb{R}^n$ are sets with finite measures, $\alpha \in \{0, 1\}$, then:

$$ T_k(1_{E_\alpha} : \alpha \in \{0, 1\}) \leq C(k, n) \rho(\mathcal{L}(E_\alpha) : \alpha \in \{0, 1\})^\tau \prod_{\alpha \in \{0, 1\}} (\mathcal{L}(E_\alpha))^{1/P_{\alpha}} \tag{4.25} $$

with $\rho(c_\alpha : \alpha \in \{0, 1\}) = \min_{\beta + \gamma \in \{0, 1\}} \frac{c_\beta}{c_\gamma}$.

**Proof.** Select $\bar{q} = (q_\alpha : \alpha \in \{0, 1\}) \in \overline{\Omega}$ that differs from $\bar{\Omega} = (p_\alpha : \alpha \in \{0, 1\})$ only in two components, for convenience, say $\alpha = 0$ and $\alpha = (1, ..., 1) = 1$, so that $q_0^{-1} = p_0^{-1} + \tau$ and $q_1^{-1} = p_1^{-1} - \tau$ for some $\tau > 0$ small enough. Then from (4.5), for this pair $(0, 1)$ we obtain:

$$ T_k(1_{E_\alpha} : \alpha \in \{0, 1\}) \leq C(k, n) \prod_{\alpha \in \{0, 1\}} |E_\alpha|^{1/q_\alpha} \leq C(k, n) \left(\frac{\mathcal{L}(E_0)}{\mathcal{L}(E_1)}\right)^\tau \prod_{\alpha \in \{0, 1\}} (\mathcal{L}(E_\alpha))^{1/P_{\alpha}}. $$

Repeating this process to the other pairs $(\beta, \gamma)$, $\beta, \gamma \in \{0, 1\}$, we obtain the conclusion (4.25). \qed

**Lemma 4.4.** Using the terminology above, then the contribution of all the quantities $T_k$ with mixed-term inputs is majorized by $C(k, n)2^{-\eta p_k M/N}$ in absolute value.

**Remark 4.3.** This lemma confirms the intuition that if the components are not "compatible" then their multilinear product doesn't contribute much.

**Proof of Lemma 4.4.** Take a typical mixed term $T_k(f_u, ..., f_d)$ in the expansion of $T_k(f_u + f_d + f_b, ..., f_u + f_d + f_b)$, whose at least two components are assumed to be different. The notation used here is only for convenience; the argument will be independent of which two components are different. Let $\mathbb{Z}^{2^k} \supset R = \{i = (i_\alpha : \alpha \in \{0, 1\}) : i_\alpha \geq 0, i_\alpha \leq 1\}$. Let $\epsilon > 0$ be a small number and let $R_\epsilon = \{i = (i_\alpha : \alpha \in \{0, 1\}) : 2^i \mathcal{L}(F_{i_\alpha})^{1/p_k} \geq \epsilon, \forall \alpha \in \{0, 1\}\}$. We take $\epsilon$ sufficiently small so that $R_\epsilon \not\subset \mathcal{O}$. By Chebyshev’s inequality,

$$ \text{crd}(R_\epsilon) \leq C \epsilon^{-2^k}. \tag{4.26} $$

By the definition of $R_\epsilon$ and the Gowers product inequality:

$$ \sum_{i \in R_\epsilon} 2^{\sum_{\alpha \in \{0, 1\}} i_\alpha} T_k(1_{F_{i_\alpha}} : \alpha \in \{0, 1\}) \leq C. \tag{4.27} $$

Next we need to find an upper bound for the corresponding sum over $\bar{i} \in R_\epsilon$. Note that if $\bar{i} = (i_\alpha : \alpha \in \{0, 1\}) \in R_\epsilon$, then $2^i \mathcal{L}(F_{i_\bar{\alpha}})^{1/p_k} \leq C$, since $\|f\|_{p_k} = 1$, and that $2^i \mathcal{L}(F_{i_\bar{r}})^{1/p_k} \geq \epsilon$, which then implies

$$ \mathcal{L}(F_{i_\bar{r}}) \geq \epsilon p_k 2^{-i_\bar{r} p_k} \geq \epsilon p_k 2^{-i_\bar{r} p_k}. \tag{4.28} $$
Furthermore, since \( i_\delta \geq J \geq I + M/(N \operatorname{crd}(S)) \geq I + c\eta^{pk} M/N \),

\[
\mathcal{L}(\mathcal{F}_{i_\delta}) \leq C 2^{-i_\delta p_k} \leq C 2^{-Ip_k - c\eta^{pk} M/N}.
\]  

(4.28) and (4.29) conclude:

\[
\frac{\mathcal{L}(\mathcal{F}_{i_\delta})}{\mathcal{L}(\mathcal{F}_{i_\delta}^s)} \leq C \epsilon^{-pk} 2^{-c\eta^{pk} M/N}.
\]  

(4.30) Applying Lemma 4.3 to the sets \( \mathcal{L}(\mathcal{F}_{i_\delta}^s) \) and utilizing (4.26) and (4.30), we obtain a bound on the sum over \( \vec{v} \in R_e \):

\[
\sum_{\vec{v} \in R_e} 2\sum_{i=0}^{k} i \mathcal{T} \left( (1_{\mathcal{F}_{i_\delta}} : \alpha \in \{0, 1\}^k) \right) \leq C(k, n)\epsilon^{-pk} 2^{-c\eta^{pk} M/N} \operatorname{crd}(R_e)
\]

(4.31)

\[
\leq C(k, n)\epsilon^{-C(k)2^{-c\eta^{pk} M/N}}.
\]  

(4.27) and (4.31) give:

\[
|\mathcal{T}(\vec{f}_u, \ldots, \vec{f}_d)| \leq C \epsilon + C(k, n)\epsilon^{-C(k)2^{-c\eta^{pk} M/N}}.
\]  

Choose \( \epsilon = e^{-c\eta^{pk} M/N} \) for a sufficiently small \( \tau = \tau(k) > 0 \); we have from (4.32):

\[
|\mathcal{T}(\vec{f}_u, \ldots, \vec{f}_d)| \leq C(k, n)2^{-c\eta^{pk} M/N}
\]

which is the conclusion of Lemma 4.4. \( \square \)

Recall that (4.24) implies \( \|\vec{f}_0\|_{pk} \leq CN^{-1}\eta^p \). Then (4.32), the definition of the Gowers-Host-Kra norms and the Gowers-Host-Kra norm inequality give

\[
\|f\|_{L^2}^k \leq |\mathcal{T}(\vec{f}_u)| + |\mathcal{T}(\vec{f}_d)| + C(k, n)2^{-c\eta^{pk} M/N} + CN^{-1}\eta^p
\]

(4.33)

\[
\leq A(k, n)2^k (\|f_u\|_{pk}^2 + \|f_d\|_{pk}^2) + C(k, n)2^{-c\eta^{pk} M/N} + CN^{-1}\eta^p.
\]

From the definition of \( f_u \) and \( f_d \), we have \( \min(\|f_u\|_{pk}, \|f_d\|_{pk}) \geq \eta \). Since \( (\|f_u\|_{pk} + \|f_d\|_{pk})1/2k \leq \|f\|_{pk} = 1 \), \( \max(\|f_u\|_{pk}, \|f_d\|_{pk}) \leq (1 - c\eta^{pk})\|f\|_{pk} = 1 - c\eta^{pk} \). Hence:

\[
\|f_u\|_{pk}^2 + \|f_d\|_{pk}^2 \leq \max(\|f_u\|_{pk}, \|f_d\|_{pk})2^k - c\eta^{pk} (\|f_u\|_{pk} + \|f_d\|_{pk}) \leq 1 - c\eta^{pk}.
\]  

(4.34)

Inserting the bound in (4.34) into (4.33) and utilizing the fact that \( f \) is a \((1 - \delta)\) near extremizer:

\[
A(k, n)2^k (1 - \delta) \leq A(k, n)2^k (1 - c\eta^{pk}) + C(k, n)2^{-c\eta^{pk} M/N} + CN^{-1}\eta^p
\]

or,

\[
2^{-c\eta^{pk} M/N} \geq c(k, n)\eta^{pk} - c(k, n)N^{-1} - C(k, n)\delta.
\]  

(4.35)

Now choose \( N \) large enough to be the nearest strictly positive integer \( \approx c\eta^{-pk} \) for some small \( c \), so that from (4.35):

\[
2^{-c\eta^{pk} M/N} \geq c(k, n)\eta^{pk} - C(k, n)\delta.
\]  

(4.36)

Choose \( \delta_0 \) sufficiently small so that \( c(k, n)\eta^{pk} - C(k, n)\delta_0 > 0 \), which yields, \( \eta \geq C_0(k, n)\delta_0^{-1/2k} \) for some \( C_0(k, n) > 0 \). We replace the first term in the right hand side of (4.36) with this new lower bound of \( \eta \); the inequality then yields:

\[
M \leq C\eta^{-2p_k}
\]

with \( C = C(k, n, \delta) \), which is a desired bound on \( M \). This completes the proof of statement (4.2) and consequently, the proof of Proposition 4.1. \( \square \)
5. Control of distribution functions

We wish to obtain two conclusions in this chapter; one is that, for a near extremizer, the contribution in $L^p$ norm of the superposition of its super-level sets associated with values outside a large compact range is negligible, and the other is, the measure of a super-level set with value in the said compact range closely approximates the measure of a super-level set of an extremizer of the same value.

5.1. Precompactness of symmetric rearrangements. Note that if $f \geq 0$ is a $(1 - \delta)$ near extremizer then so is its symmetric rearrangement, $f^*$ [14]:

$$
\|f^*\|_{L^k}^{2k} = \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} f^*(x + \alpha \cdot \vec{h}) \, dx \, d\vec{h} \geq \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} f(x + \alpha \cdot \vec{h}) \, dx \, d\vec{h}
$$

$$
= \|f\|_{L^k}^{2k} \geq (1 - \delta) A(k, n)^{n2k} \|f\|_{L^k}^{2k} = (1 - \delta) A(k, n)^{n2k} \|f^*\|_{L^k}^{2k}.
$$

The first inequality is due to the general rearrangement inequality and the second to the $L^p$-norm preservation property of symmetric rearrangements [14]. By definition, the normalization condition preserves measure. That means if $\Theta$ is a growth function and if $f \geq 0$ is normalized with respect to $\Theta$ then so is $f^*$:

$$
\int_{\{|f^*| \geq \rho\}} f^*(x) \, dx = \int_{\{|f| \geq \rho\}} |f|(x) \, dx \leq \Theta(\rho)
$$

$$
\int_{\{|f^*| < \rho\}} f^*(x) \, dx = \int_{\{|f| < \rho\}} |f|(x) \, dx \leq \Theta(\rho^{-1}).
$$

Let $\{f_i\}$ be a sequence of measurable functions on $\mathbb{R}^n$. We say $\{f_i\}$ is a normalized extremizing sequence if $\|f_i\|_{L^k} = 1$, $f_i$ is normalized with respect to $\Theta$ for all $i$ and if there exists a sequence $\{\delta_i\}$ satisfying $\delta_i \to 0$ as $i \to \infty$, such that:

$$
\|f_i\|_{L^k} \geq (1 - \delta_i) A(k, n) \|f_i\|_{L^k}.
$$

Suppose furthermore that $f_i \geq 0$ for all $i$ and let $\{f_i^*\}$ be the corresponding symmetric rearrangement sequence of $\{f_i\}$. Then as noted, $\{f_i^*\}$ is also a normalized extremizing sequence.

Lemma 5.1. Let $\{f_i^*\}$ be as above. Let $\epsilon > 0$. Then there exist $r > 0, R > 0$ such that, for all sufficiently large $i$:

$$
\int_{\{|x| \leq r\}} (f_i^*)^{p_k} \, dx \leq \epsilon
$$

(5.1)

$$
\int_{\{|x| \geq R\}} (f_i^*)^{p_k} \, dx \leq \epsilon.
$$

Proof. Indeed, by normalization,

$$
\int_{\{|x| \leq r\}} (f_i^*)^{p_k} \, dx \leq \int_{\{|x| \leq r\} \cap \{f_i^* \leq \alpha\}} (f_i^*)^{p_k} \, dx + \int_{\{|x| \geq \alpha\}} (f_i^*)^{p_k} \, dx
$$

$$
\leq c(n) \alpha^{p_k} r^n + \int_{\{|x| \geq \alpha\}} (f_i^*)^{p_k} \, dx \leq c(n) \alpha^{p_k} r^n + \Theta(\alpha).
$$

(5.3)

The third inequality follows from the fact that $f_i^*$ is normalized. Choose $\alpha$ sufficiently large so that $\Theta(\alpha) \leq (1/2)\epsilon$ and with this $\alpha$, $r$ small enough so that $c(n) \alpha^{p_k} r^n \leq (1/2)\epsilon$. Putting all these upper bounds in (5.3), we obtain (5.1). The fact that $f_i^*$ is radially symmetric and nonincreasing gives us,

$$
c(n)|s|^n (f_i^*)^{p_k}(s) \leq \int_{\{|x| \leq s\}} (f_i^*)^{p_k} \, dx \leq 1.
$$
Hence, if \( R \leq |s| \leq R_0 \):

\[
(f^*_i)^{pk}(s) \leq C(n)|s|^{-n} \leq C(n)R^{-n}.
\]

(5.4)

In other words, \( f^*_i \) is bounded above by \( C(n)R^{-n/pk} \) in an annulus \( \{ R \leq |x| \leq R_0 \} \), with \( R_0 > R \), uniformly for all \( i \), and hence, \( \{ |x| \geq R \} \subseteq \{ (f^*_i)^{pk} \leq C(n)R^{-n} \} \). Thus:

\[
\int_{\{|x| \geq R\}} (f^*_i)^{pk} \, dx \leq \int \|(f^*_i)^{pk} \leq C(n)R^{-n}\) \( (f^*_i)^{pk} \, dx \leq \Theta((c(n)R^n)^{1/pk}).
\]

(5.5)

The last inequality in (5.5) follows from normalization. Now choose \( R \) sufficiently large so that \( \Theta((c(n)R^n)^{1/pk}) \leq \epsilon \). Then (5.5) implies (5.2). This completes the proof of the lemma. \( \square \)

Let \( r > 0, R > 0 \) and \( C_{r,R} = \{ r \leq |x| \leq R \} \). The proof of Lemma 5.1, particularly (5.4), shows that the sequence \( \{ f^*_i \} \) restricted to a ray cutting the annulus \( C_{r,R} \), produces a uniformly bounded, nonincreasing sequence of functions on an interval \( [r,R] \). It follows from Helly’s selection principle [15] that there exists a subsequence of \( \{ f^*_i \} \) that converges pointwise on every such ray. Since \( f^*_i \) is radial for all \( i \), such a subsequence also converges pointwise on the annulus \( C_{r,R} \). We denote such a subsequence by \( \{ f^*_{i;r,R} \} \) and the corresponding pointwise limit by \( F_{r,R} \). It’s clear that \( F_{r,R} \) is also radial. We select \( r \) to be an increasing function of \( \epsilon \) and \( R \) a decreasing function of \( \epsilon \). Choose a sequence \( \{ \epsilon_i \} \) that decreases to zero and denote \( r(\epsilon_i) = r_i, R(\epsilon_i) = R_i \). Then by a Cantor diagonal argument and passing to a subsequence of \( \{ f^*_{i;r_i,R_i} \} \), we obtain a sequence of nonincreasing sequence of radial functions \( \{ F_{r_i,R_i} \} \) such that \( F_{r_i,R_i} \) is supported on the annulus \( C_{r_i,R_i} \), for all \( i \), and \( F_{r_i,R_i} = F_{r_i,R_i} \) on \( C_{r_i,R_i} \) if \( i \geq n \). Let \( F \) be the pointwise limit of \( \{ F_{r_i,R_i} \} \). Note that \( F_{r_i,R_i} \leq F_{r_i,R_i} \leq F \), if \( i \geq n \). For convenience, we denote the subsequence of \( \{ f^*_{i;r_i,R_i} \} \) in the construction of \( F_{r_i,R_i} \) at every index \( i \) as simply \( \{ f^*_i \} \). We argue that \( F \) must be a centered Gaussian function on \( \mathbb{R}^n \). Indeed, by Lemma 5.1, for all sufficiently large \( i \),

\[
\int_{\{|x| \leq r_i\}} (f^*_i)^{pk} \, dx + \int_{\{|x| \geq R_i\}} (f^*_i)^{pk} \, dx \leq 2\epsilon_i.
\]

(5.6)

Take \( \eta > 0 \). Then take \( I \) sufficiently large so that \( 2\epsilon_i < \eta, \) if \( i \geq I \). By letting \( i \to \infty \), it follows from (5.6) and the fact that \( F_{r_i,R_i} = F_{r_i,R_i} \) on \( C_{r_i,R_i} \) if \( i \geq n \geq I \),

\[
\int_{\mathbb{R}^n} |F_{r_i,R_i} - F_{r_i,R_i}|^{pk} \, dx \leq \int_{\{|x| \leq r_j\}} |F_{r_i,R_i} - F_{r_i,R_i}|^{pk} + \int_{\{|x| \geq R_j\}} |F_{r_i,R_i} - F_{r_i,R_i}|^{pk} \, dx \leq \eta.
\]

Hence \( \{ F_{r_i,R_i} \} \) is a Cauchy sequence in \( L^{pk} \) and thus \( F \) is also its \( L^{pk} \)-limit. On the other hand, by the Dominated Convergence Theorem, \( F_{r_i,R_i} \) is the pointwise limit and thus the \( L^{pk} \)-limit of \( \{ f^*_i \} \) on \( C_{r_i,R_i} \). Hence \( F \) is the \( L^{pk} \)-limit of \( \{ f^*_i \} \) on \( \mathbb{R}^n \). By the Gowers-Host-Kra norm inequality, \( \| F - f^*_i \|_{L^{pk}} \leq A(k,n)\| F - f^*_i \|_{pk} \), and hence, as \( i \to \infty \), \( \| F \|_{L^{pk}} \to \| F \|_{pk} \). But since \( \| f^*_i \|_{L^{pk}} \geq (1 - \delta)A(k,n)\| f^*_i \|_{pk} \), these imply that \( \| F \|_{L^{pk}} \to A(k,n)\| F \|_{pk} \). By the characterization of extremizers of the Gowers-Host-Kra norm inequality, \( F \) must be a Gaussian, which then is a centered Gaussian, as it is the pointwise limit of radial functions.

5.2. Control of distribution functions. Recall that if \( f : \mathbb{R}^n \to \mathbb{R} \) is a \( (1 - \delta) \) near extremizer then so is \( |f| \). Hence we only consider nonnegative near extremizers for now. Suppose \( \| f \|_{L^{pk}} \geq (1 - \delta)A(k,n)\| f \|_{pk} \) and \( \| F - f^* \|_{pk} \leq \delta \) with \( F \) being a centered Gaussian on \( \mathbb{R}^n \). Denote, for \( s > 0, F_s \{ f > s \}, F^*_s \{ f^* > s \} \) and \( F_s \{ F > s \} \).
The proof is complete.

Likewise:

\[ \delta \]

\[ \eta \]

Remark 5.1. We can summarize the findings above in the following language. Given a \( \delta \) small, there exist \( \eta(\delta) \) and \( \epsilon(\eta, \delta) \) with the following properties. If \( \delta \to 0 \) then \( \eta(\delta) \to 0 \), and if \( \eta \) is fixed, \( \delta \to 0 \) then \( \epsilon(\eta, \delta) \to 0 \). If \( F_s, F_s \) are as above, then \( \sup_{s \in [\eta, \|F\|_{\infty} - \eta]} |\mathcal{L}(F_s) - \mathcal{L}(F_s)| \leq \epsilon(\eta, \delta) \). Moreover, by the description of \( \epsilon(\eta, \delta) \), we have that if \( \xi(\eta) \) is a positive continuous function satisfying \( \xi(\eta) \to 0 \) as \( \eta \to 0 \), there exists a function \( \eta \to \delta_0(\eta) > 0 \) such that \( \sup_{0 < \delta \leq \delta_0(\eta)} \epsilon(\eta, \delta) \leq \xi(\eta) \). In particular, \( \lim_{\eta \to 0} \sup_{0 < \delta \leq \delta_0(\eta)} \epsilon(\eta, \delta) = 0 \), for some positive function \( \delta_0 \). These properties of \( \epsilon(\eta, \delta) \) will be needed below in Section 6.

Lemma 5.2. Let \( f, f^*, F_s, F_s^s, F_s^\star \) be as above. Given \( \delta > 0 \). There exists \( \eta = \eta(\delta) > 0 \) satisfying \( \eta(\delta) \to 0 \) as \( \delta \to 0 \), such that if \( s \in [\eta, \|F\|_{\infty} - \eta] \) then

\[
\mathcal{L}(F_s \Delta F_s^\star) = o(\delta)
\]

\[
|\mathcal{L}(F_s) - \mathcal{L}(F_s)| = o(\delta).
\]

Proof. Let \( \eta \in (0, 1) \) be a small number to be chosen below. Consider \( F_s^s \setminus F_s - \eta \). If \( x \in F_s^s \setminus F_s - \eta \) then \( |(F - f^*)(x)| \geq \eta \). Chebyshev’s inequality gives, \( \mathcal{L}(F_s^s \setminus F_s - \eta) \leq \eta^{-p_k} \|F - f^*\|_{p_k} \leq \eta^{-p_k} \delta^{p_k} \), and, \( \mathcal{L}(F_s^s \setminus F_s^\star) \leq \eta^{-p_k} \delta^{p_k} \). Hence:

\[
\mathcal{L}(F_s \Delta F_s^\star) \leq c \eta^{-p_k} \delta^{p_k}.
\]

On the other hand, since \( F \) is a Gaussian and all the super-level sets of \( F \) are nested, centered ellipsoids, we also have

\[
\mathcal{L}(F_s \setminus F_s^\star) \leq C \eta^{1/2}.
\]

Optimizing (5.9) and (5.10) to have:

\[
\mathcal{L}(F_s \setminus F_s^\star) \leq C \delta^{p_k/(2p_k+1)}.
\]

Since the super-level sets of \( f^\star \) are also centered ellipsoids, if \( \eta = \delta^{2p_k/(2p_k+1)} \) then

\[
\mathcal{L}(F_s^s \setminus F_s) \leq c \delta^{p_k/(2p_k+1)}
\]

(5.11), (5.12), (5.13) then imply \( \mathcal{L}(F_s^s \setminus F_s^\star) \leq C \delta^{p_k/(2p_k+1)} \), and consequently, \( |\mathcal{L}(F_s^s) - \mathcal{L}(F_s)| \leq C \delta^{p_k/(2p_k+1)} \), which are the promises (5.7) and (5.8), respectively.

Remark 5.1. We can summarize the findings above in the following language. Given a \( \delta \) small, there exist \( \eta(\delta) \) and \( \epsilon(\eta, \delta) \) with the following properties. If \( \delta \to 0 \) then \( \eta(\delta) \to 0 \), and if \( \eta \) is fixed, \( \delta \to 0 \) then \( \epsilon(\eta, \delta) \to 0 \). If \( F_s, F_s \) are as above, then \( \sup_{s \in [\eta(\delta), \|F\|_{\infty} - \eta(\delta)]} |\mathcal{L}(F_s) - \mathcal{L}(F_s)| \leq \epsilon(\eta, \delta) \). Moreover, by the description of \( \epsilon(\eta, \delta) \), we have that if \( \xi(\eta) \) is a positive continuous function satisfying \( \xi(\eta) \to 0 \) as \( \eta \to 0 \), there exists a function \( \eta \to \delta_0(\eta) > 0 \) such that \( \sup_{0 < \delta \leq \delta_0(\eta)} \epsilon(\eta, \delta) \leq \xi(\eta) \). In particular, \( \lim_{\eta \to 0} \sup_{0 < \delta \leq \delta_0(\eta)} \epsilon(\eta, \delta) = 0 \), for some positive function \( \delta_0 \). These properties of \( \epsilon(\eta, \delta) \) will be needed below in Section 6.

Lemma 5.3. Let \( f^\star, F_s, F_s \) and \( \eta \) be as above. There exists \( C > 0 \) so that,

\[
\left\| \int_0^\eta 1_{F_s} \, ds \right\|_{p_k} \leq C \|F - f^\star\|_{p_k} + C \eta (\log(1/\eta))^C
\]

and,

\[
\left\| \int_{|F|_{\infty} - \eta}^\infty 1_{F_s} \, ds \right\|_{p_k} \leq C \|F - f^\star\|_{p_k} + O(\eta).
\]

Proof. Indeed,

\[
\left\| \int_0^\eta 1_{F_s} \, ds \right\|_{p_k} = \left\| \int_0^\eta 1_{F^\star} \, ds \right\|_{p_k} = \|\min(f, \eta)\|_{p_k} \leq \|F - f^\star\|_{p_k} + \|\min(F, \eta)\|_{p_k}
\]

\[
\leq \|F - f^\star\|_{p_k} + \eta (\log(1/\eta))^C.
\]

Likewise:

\[
\left\| \int_{|F|_{\infty} - \eta}^\infty 1_{F_s} \, ds \right\|_{p_k} = \left\| \int_{|F|_{\infty} - \eta}^\infty 1_{F^\star} \, ds \right\|_{p_k} = \|\max(0, f^\star - (|F|_{\infty} - \eta))\|_{p_k}
\]

\[
\leq \|F - f^\star\|_{p_k} + \|\max(0, F - (|F|_{\infty} - \eta))\|_{p_k} \leq \|F - f^\star\|_{p_k} + O(\eta).
\]

The proof is complete.

Remark 5.2. One can appreciate the usefulness of the normalization condition in obtaining the decay properties (5.1) and (5.2) and consequently the precompactness of \( \{ f_i^* \} \) in \( L^{p_k} \), which then allows us to extract an extremizer \( F \). Moreover, one observes that the only needed hypotheses for Lemma 5.2 and Lemma 5.3 are that \( f^* \) is radially symmetric and decreasing, \( F \) is a centered Gaussian, and \( \| \mathcal{F} - f \|_{p_k} \leq \delta, \| \mathcal{F} - f^* \|_{p_k} \leq \delta \); the specific fact that \( f \) is a near extremizer for the Gowers-Host-Kra norm inequality did not enter. This observation was first used by Christ to obtain similar conclusions about near extremizing triple \((f, g, h)\) of Young’s inequality on Euclidean spaces [7].

6. Control of super-level sets

Let \( n = 1 \). Let \( f \) be a \((1 - \delta)\) near extremizer of the Gowers-Host-Kra norm inequality and \( \mathcal{F} \) a centered Gaussian such that \( \| f \|_{p_k} = 1 \) and \( \| \mathcal{F} - f^* \|_{p_k} \leq \delta \). In this chapter, we only consider \( f \geq 0 \). Since \( f \) is of unit norm, so are \( f^* \) and \( \mathcal{F} \). For \( t > 0 \), let \( \Omega(t) = [t, \| \mathcal{F} \|_{\infty} - t] \).

Recall that for \( s > 0 \), \( F_s = \{ f > s \}, F_s^* = \{ f^* > s \}, \mathcal{F}_s = \{ \mathcal{F} > s \} \).

Proposition 6.1. For every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for every \((1 - \delta)\) near extremizer \( f \) and every \( \nu_\alpha \in \Omega(\epsilon), \alpha \in \{0, 1\}^k \),

\[
T_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) \geq (1 - \epsilon)T_k(1_{F^*_{\nu_\alpha}} : \alpha \in \{0, 1\}^k).
\]

Remark 6.1. We will prove the statement in the following form:

For every \( \epsilon > 0 \) there exists \( \zeta > 0 \) and \( \delta > 0 \) such that if \( f \) is a \((1 - \delta)\) near extremizer of the Gowers-Host-Kra norm inequality and if \( \nu_\alpha \in \Omega(\zeta), \alpha \in \{0, 1\}^k \) then,

\[
T_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) \geq (1 - \epsilon)T_k(1_{F^*_{\nu_\alpha}} : \alpha \in \{0, 1\}^k).
\]

(6.2) is equivalent to (6.1) if \( \zeta = o(\epsilon) \) and \( \epsilon = o(\zeta) \), which will be the case.

Assume Proposition 6.1 for a moment. Assume \( \nu_\alpha = \nu \) for all \( \alpha \in \{0, 1\}^k \). Then by definition, \( T_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) = T_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) = \| \mathcal{F}_{\nu_\alpha} \|_{p_k}^2 \) and \( T_k(1_{F^*_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) = \| \mathcal{F}_{\nu_\alpha}^* \|_{p_k}^2 \). Then (6.2) becomes \( \| \mathcal{F}_{\nu_\alpha} \|_{U^k} \geq (1 - \epsilon) \| \mathcal{F}_{\nu_\alpha}^* \|_{U^k} \), which entails that the sets \( F_{\nu_\alpha} \) will then be nearly intervals, by the following result:

Proposition 6.2. [8] Let \( n \geq 1 \). For every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that if \( E \subset \mathbb{R}^n \) is a measurable set with finite measure and \( \| 1_E \|_{U^k} \geq (1 - \delta) \| 1_{E^*} \|_{U^k} \) then there exists an ellipsoid \( \mathcal{E} \subset \mathbb{R}^n \) such that \( \mathcal{L}(E \Delta \mathcal{E}) < \epsilon \mathcal{L}(E) \).

In what follows, \( \bar{\nu} = (\nu_\alpha : \alpha \in \{0, 1\}^k) \).

6.1. Proof of Proposition 6.1.

6.1.1. Set-up: Let \( t \in (0, 1) \). By the Gowers product inequality

\[
(6.3) \quad \int_{\mathbb{R}^{0, 1}^k \setminus \Omega(t) (0, 1)^k} T_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) \ d\bar{\nu} \leq A(k) 2^k \int_{\mathbb{R}^{0, 1}^k \setminus \Omega(t) (0, 1)^k} \prod_{\alpha \in \{0, 1\}^k} \| 1_{F_{\nu_\alpha}} \|_{p_k} \ d\bar{\nu}.
\]

Moreover \( \mathbb{R}^{0, 1}^k \setminus \Omega(t) (0, 1)^k \subset \cup_{\alpha \in \{0, 1\}^k} (\mathbb{R} \setminus \Omega(t)) \times \mathbb{R}^{0, 1}^{k-1} \). Hence an immediate consequence of Lemma 5.3, Fubini’s theorem and (6.3) is that there exists a universal \( C(k) > 0 \) such that for all small \( t > 0 \):

\[
(6.4) \quad \int_{\mathbb{R}^{0, 1}^k \setminus \Omega(t) (0, 1)^k} T_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) \ d\bar{\nu} \leq A(k) 2^k \prod_{\alpha \in \{0, 1\}^k} \left\| \int_{\mathbb{R} \setminus \Omega(t)} 1_{F_{\nu_\alpha}} \ d\nu_\alpha \right\|_{p_k} \leq C(k) \delta + C(k) t (\log(1/t))^C.
\]
Define:

\[ H(\nu_0) = \mathcal{T}_k((1_F, 1_{F^n}) : \alpha \in \{0, 1\}^k; \alpha \neq 0) \]

\[ \mathcal{H}(\nu_0) = \mathcal{T}_k((1_F, 1_{F^n}) : \alpha \in \{0, 1\}^k; \alpha \neq 0). \]

In other words, we fix all the values \( \nu_0, \alpha \neq 0 \) and consider \( H, \mathcal{H} \) as nonnegative functions of only \( \nu_0 \). Since level sets are nested, both \( H \) and \( \mathcal{H} \) are non-increasing. Moreover, since \( \mathcal{F} \) is a centered Gaussian of unit norm, if \( \nu_0 \in \Omega(t) \), for all \( \alpha \in \{0, 1\}^k \), then \( \mathcal{H} \) is bounded below by a strictly positive function of \( t \) as long as \( 0 < t < \|\mathcal{F}\|_\infty \). Similarly, \( \mathcal{T}_k((1_F, 1_{F^n}) : \alpha \in \{0, 1\}^k) \geq \phi(t) \), with \( \phi \) being a strictly positive continuous function satisfying \( \phi \to 0 \) as \( t \to 0 \); this function \( \phi \) will be needed below. Moreover, \( \mathcal{H} \) is also Lipschitz continuous with a Lipschitz constant \( L(t) \) that is independent of a specific \( \bar{\nu} \), as long as \( \nu_0 \in \Omega(t) \), for all \( \alpha \in \{0, 1\}^k \).

Now:

\[
\int_{\Omega(t)^{0,1}^k} \mathcal{T}_k(1_{F^n}) : \alpha \in \{0, 1\}^k \, d\bar{\nu} = \left\| \mathcal{F}_{\bar{\nu}} \right\|_{U^k}^2 - \int_{\mathbb{R}^{0,1}_m \setminus \Omega(t)^{0,1}^k} \mathcal{T}_k(1_{F^n}) : \alpha \in \{0, 1\}^k \, d\bar{\nu} \\
\geq (1 - \delta) \left( \|\mathcal{F}\|_{U^k} - C(k)\delta - C(k)(\log(1/t))^{C} \right) \\
\geq (1 - \delta) \int_{\Omega(t)^{0,1}^k} \mathcal{T}_k(1_{F^n}) : \alpha \in \{0, 1\}^k \, d\bar{\nu} - C(k)\delta - C(k)(\log(1/t))^{C}. \tag{6.5}
\]

The first inequality follows from (6.4) and the near extremizing hypothesis. The second inequality follows from the definition of the Gowers-Host-Kra norms. Recall from Remark 5.1, that if \( s \in \Omega(t) \), then:

\[
\text{sup}_{s \in \Omega(t)} |\mathcal{L}(\mathcal{F}_s) - \mathcal{L}(F_s)| \leq \epsilon(t, \delta) \tag{6.6}
\]

and \( \epsilon(t, \delta) \) has the following properties:

1. \( \epsilon(t, \delta) \to 0 \) if \( \delta \to 0 \) and \( t \) is fixed.
2. For every positive continuous function \( \xi(t) \) satisfying \( \xi(t) \to 0 \) as \( t \to 0 \), there exists \( \delta_0(t) > 0 \) such that \( \epsilon(t, \delta) \leq \xi(t) \) for all \( 0 < \delta \leq \delta_0(t) \); hence \( \lim_{t \to 0} \sup_{0 < \delta \leq \delta_0(t)} \epsilon(t, \delta) = 0 \).

It follows from (6.6) and the Gowers product inequality, that if \( \bar{\nu} \in \Omega(t)^{0,1}^k \) then:

\[
|\mathcal{T}_k(1_{F^n}) : \alpha \in \{0, 1\}^k| - \mathcal{T}_k(1_{F^n}) : \alpha \in \{0, 1\}^k| \\
\leq C(k) \sup_{s \in \Omega(t)} |(\mathcal{L}(\mathcal{F}_s))^{1/p_k} - (\mathcal{L}(F_s))^{1/p_k}| \cdot ((\mathcal{L}(\mathcal{F}_t))^{1/p_k} + \epsilon(t, \delta))^{2k-1} \leq C(k, t)\epsilon(t, \delta). \tag{6.7}
\]

The constant \( C(k, t) \) depends on \( \mathcal{L}(\mathcal{F}_t) \) and satisfies \( C(k, t) \to \infty \) if \( t \to 0 \). Combining (6.7) and the general rearrangement inequality, we obtain:

\[
\mathcal{T}_k(1_{F^n}) : \alpha \in \{0, 1\}^k \leq \mathcal{T}_k(1_{F^n}) : \alpha \in \{0, 1\}^k \leq \mathcal{T}_k(1_{F^n}) : \alpha \in \{0, 1\}^k + C(k, t)\epsilon(t, \delta). \tag{6.8}
\]

Take two numbers \( \rho, \eta \in (0, 1) \) satisfying \( \rho \leq \eta \), then \( \Omega(\eta) \subset \Omega(\rho) \). Integrating \( \bar{\nu} \) in (6.8) over \( \Omega(\rho)^{0,1}^k \setminus \Omega(\eta)^{0,1}^k \) gives:

\[
\int_{\Omega(\rho)^{0,1}^k \setminus \Omega(\eta)^{0,1}^k} \mathcal{T}_k(1_{F^n}) : \alpha \in \{0, 1\}^k \, d\bar{\nu} \\
\leq \int_{\Omega(\rho)^{0,1}^k \setminus \Omega(\eta)^{0,1}^k} \mathcal{T}_k(1_{F^n}) : \alpha \in \{0, 1\}^k \, d\bar{\nu} + C(k, \rho)\epsilon(\rho, \delta). \tag{6.9}
\]
Here again \( C(k, \rho) \to \infty \) if \( \rho \to 0 \). Substituting \( t = \rho \) in (6.5) to have:

\[
(6.10) \quad \int_{\Omega(\rho)(0,1)k} T_k(1_{F_{\nu_\alpha}} : \alpha \in \{0,1\}^k) \, d\tilde{\nu} \geq (1 - \delta) \int_{\Omega(\rho)(0,1)k} T_k(1_{F_{\nu_\alpha}} : \alpha \in \{0,1\}^k) \, d\tilde{\nu} - C(k)\delta - C(k)\rho(\log(1/\rho))^C.
\]

Subtracting (6.9) from (6.10):

\[
(6.11) \quad \int_{\Omega(\eta)(0,1)k} T_k(1_{F_{\nu_\alpha}} : \alpha \in \{0,1\}^k) \, d\tilde{\nu} \geq \int_{\Omega(\eta)(0,1)k} T_k(1_{F_{\nu_\alpha}} : \alpha \in \{0,1\}^k) \, d\tilde{\nu} - C(k)\delta\phi(\eta) - C(k)\delta - C(k)\rho(\log(1/\rho))^C - C(k, \rho)\epsilon(\rho, \delta).
\]

Let \( \xi(t) \) be a positive continuous function satisfying \( \xi(t) \to 0 \) as \( t \to 0 \). We now require in our selection of \( \rho, \eta, \) that \( C(k)\rho(\log(1/\rho))^C + C(k)\phi(\eta) \leq \xi(\eta) \). By the second property of \( \epsilon(t, \delta) \), there exists a positive function \( \delta_0(t) \) such that for all \( 0 < \delta \leq \delta_0(t), \ t \in (0,1) \), the two following conditions are satisfied:

\[
\epsilon(t, \delta) \leq \xi(t) \leq C(k, t)^{-1}\xi(t)
\]

with \( C(k, \cdot) \) having the same meaning as in (6.11). Since \( \xi(t) \to 0 \) as \( t \to 0 \), \( c_1\xi(\rho) \leq \xi(\eta) \leq c_2\xi(\rho) \), if \( \rho \leq \eta \) are both sufficiently small. These observations together with the conditions allow us to rewrite (6.11) as:

\[
(6.12) \quad \int_{\Omega(\eta)(0,1)k} (T_k(1_{F_{\nu_\alpha}} : \alpha \in \{0,1\}^k) - T_k(1_{F_{\nu_\alpha}} : \alpha \in \{0,1\}^k)) \, d\tilde{\nu} \leq C(k)\delta + C(k)\xi(\eta).
\]

Note that if we re-define \( \epsilon(s, \delta) = C(k)\delta + C(k)\xi(s) \), then this new \( \epsilon(s, \delta) \) still satisfies the two properties of the error function \( \epsilon \) mentioned above. We rewrite (6.8) and (6.12) respectively in this new language, with \( \nu_\alpha \in \Omega(t), \ \alpha \in \{0,1\}^k \):

\[
(6.13) \quad T_k(1_{F_{\nu_\alpha}} : \alpha \in \{0,1\}^k) \leq T_k(1_{F_{\nu_\alpha}} : \alpha \in \{0,1\}^k) \leq T_k(1_{F_{\nu_\alpha}} : \alpha \in \{0,1\}^k) + C(k)\epsilon(t, \delta)
\]

\[
(6.14) \quad \int_{\Omega(t)(0,1)k} (T_k(1_{F_{\nu_\alpha}} : \alpha \in \{0,1\}^k) - T_k(1_{F_{\nu_\alpha}} : \alpha \in \{0,1\}^k)) \, d\tilde{\nu} \leq \epsilon(t, \delta).
\]

By Fubini’s theorem, (6.14) is simply,

\[
(6.15) \quad \int_{\Omega(t)(0,1)k} (T_k(1_{F_{\nu_\alpha}} : \alpha \in \{0,1\}^k) - T_k(1_{F_{\nu_\alpha}} : \alpha \in \{0,1\}^k)) \, d\tilde{\nu} = \int_{\Omega(t)(0,1)k} (H(\nu_\alpha) - H(\nu_\alpha)) \, d\tilde{\nu} \leq \epsilon(t, \delta).
\]

(6.13) implies, if \( \nu_\alpha \in \Omega(t), \) for all \( \alpha \in \{0,1\}^k \):

\[
(6.16) \quad H(\nu_\alpha) - H(\nu_\alpha) \geq -C(k)\epsilon(t, \delta).
\]

Integrating (6.16) over \( \Omega(t) \), we have uniformly for \( (\nu_\alpha : \alpha \in \{0,1\}^k ; \alpha \neq \overline{0}) \in \Omega(t)^{0,1,k\setminus\overline{0}} \):

\[
(6.17) \quad \int_{t}^{|F|_{x-t}} (H(\nu_\alpha) - H(\nu_\alpha)) \, d\nu_\alpha \geq -C(k)\epsilon(t, \delta).
\]
6.1.2. A process: Recall that $\mathcal{H}$ is Lipschitz continuous with a Lipschitz constant majorized by a quantity $L(t)$ that is independent of $t$, as long as $\nu_\alpha \in \Omega(t)$, for all $\alpha \in \{0,1\}^k$. Let $K = K(t) = \max(1, L(t))$. Note that $K(t) \to \infty$ as $t \to 0$, since $\mathcal{L}(\mathcal{F}||x_0 - t) \to 0$ and $\mathcal{L}(\mathcal{F}) \to \infty$ as $t \to 0$. So far we have two parameters $\rho \leq \eta$. The parameter $\rho$ is an auxiliary one whose ultimate use was to define the new error function $\epsilon$; our main parameter is the parameter $\eta$. We now will define new parameters in terms of $\eta$. Let $r = r(\eta)$ be a small quantity to be chosen below, and suppose that there exists $\nu'_0 \in [\eta, \|\mathcal{F}\|_{L^\infty} - \eta - r]$ such that:

$$H(\nu'_0) \leq H(\nu'_{00}) - r. \quad (6.18)$$

Then, we claim, a similar property will hold for a sub-range of $\nu_0 \in [\nu'_0, \nu'_0 + cK^{-1}r]$:

$$H(\nu_0) \leq H(\nu'_{00}) - r \leq H(\nu'_0) - r. \quad (6.19)$$

Indeed, the non-increasing property of $H$ gives the first inequality in (6.19); the second is just (6.18), and the Lipschitz continuity of $H$ over the selected interval gives the last. Furthermore, if (6.18) happens, we can increase the lower bound in (6.17):

$$\int_{\eta}^{\|\mathcal{F}\|_{L^\infty} - \eta} (H(\nu_0) - H(\nu'_0)) \, d\nu_0 = \int_{[\nu'_0, \nu'_0 + cK^{-1}r]} (H(\nu_0) - H(\nu'_0)) \, d\nu_0 \geq cr^2K^{-1} - C(k)\epsilon(\eta, \delta). \quad (6.20)$$

The inequality in (6.20) follows from applying the respective lower bounds as in (6.17), with $t = \eta$, and (6.19). It will be in our desire that $r(\eta) \to 0$ as $\eta \to 0$ but at a rate $o(\eta)$. To this end, take $q \in (0, 1)$. We consider only sufficiently small values of $\delta$ so that $qcK^{-1}r^2 \geq C\epsilon(\eta, \delta)$. This is possible by the first property of $\epsilon$ described above. This allows us to rewrite (6.20) as:

$$\int_{\eta}^{\|\mathcal{F}\|_{L^\infty} - \eta} (H(\nu_0) - H(\nu'_0)) \, d\nu_0 \geq c(k)K^{-1}r^2. \quad (6.21)$$

Let $S$ be the set of $(\nu_\alpha : \alpha \in \{0,1\}^k; \alpha \neq \overline{0}) \in \Omega(\eta)^{(0,1)^k\setminus\{0\}}$ for which there exists at least one $\nu'_0$ such that (6.18) is satisfied. By (6.15), with $t = \eta$, (6.21) and Markov’s inequality:

$$\mathcal{L}(S) \leq C(k)K^{-2}\epsilon(\eta, \delta). \quad (6.22)$$

Select the parameter $\delta = \delta(\eta)$ so that $\delta \to 0$ as $\eta \to 0$, with a rate sufficiently rapid so that $K(\eta)\epsilon(\eta, \delta(\eta)) \to 0$ as $\eta \to 0$; this is possible due to the first property of $\epsilon(\eta, \delta(\eta))$ mentioned above. Then choose $r = r(\eta) \to 0$ satisfying $r \to 0$ as $\eta \to 0$, with a rate sufficiently slow so that $K^2(\eta)(r^{-2}(\eta)\epsilon(\eta, \delta(\eta)))^{1/2k} \to 0$ as $\eta \to 0$. Since $K(\eta) \to \infty$ as $\eta \to 0$ and $K^2(\eta)(r^{-2}(\eta)\epsilon(\eta, \delta(\eta)))^{1/2k} \to 0$ (6.22) implies $\mathcal{L}(S) \to 0$ as $\eta \to 0$. Introduce another parameter $\zeta = \zeta(\eta)$ satisfying $\zeta \to 0$ as $\eta \to 0$ and $\zeta \geq \eta$, so that $\Omega(\zeta) \subset \Omega(\eta)$. Let $S = S(\eta)$ to be the set of all $\nu = (\nu_\alpha : \alpha \in \{0,1\}^k; \alpha \neq \overline{0}) \in \Omega(\zeta)^{(0,1)^k}$ so that $(\nu_\alpha : \alpha \in \{0,1\}^k; \alpha \neq \overline{0}) \in S$. It’s clear from definition and (6.22) that $\mathcal{L}(S) \to 0$ as $\eta \to 0$. By the definition of $S$, if $\nu \in \Omega(\zeta)^{(0,1)^k\setminus S}$ then,

$$T_k(1_{F_{\nu_\alpha}} : \alpha \in \{0,1\}^k) \geq T_k(1_{F_{\nu_\alpha}} : \alpha \in \{0,1\}^k) - r. \quad (6.23)$$

Recall that $T_k(1_{F_{\nu_\alpha}} : \alpha \in \{0,1\}^k) \geq \phi(\zeta)$ if $\nu_\alpha \in \Omega(\zeta)$ and $\phi$ is a positive continuous function. Then (6.23) and the general rearrangement inequality imply:

$$T_k(1_{F_{\nu_\alpha}} : \alpha \in \{0,1\}^k) \geq \phi(\zeta) - r. \quad (6.24)$$

(6.13), with $t = \zeta$, and (6.23) furthermore imply:

$$T_k(1_{F_{\nu_\alpha}} : \alpha \in \{0,1\}^k) \geq T_k(1_{F_{\nu_\alpha}} : \alpha \in \{0,1\}^k) - r \geq T_k(1_{F_{\nu_\alpha}} : \alpha \in \{0,1\}^k) - r - C(k)\epsilon(\zeta, \delta). \quad (6.25)$$
We now further require \( \zeta = \zeta(\eta) \) tending to zero as \( \eta \to 0 \) with a rate sufficiently slow so that \( \phi(\zeta) \) also tends to zero slowly and,

\[
(6.26) \quad \frac{r + C(k)\epsilon(\eta, \delta)}{\phi(\zeta) - r} \to 0,
\]

\[
(6.27) \quad \frac{K(\eta)^2 r^{-2} \epsilon(\eta, \delta)}{\phi(\zeta) - r} \to 0.
\]

(6.24), (6.25) and (6.26) imply:

\[
(6.28) \quad \mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) \geq (1 - o(\eta)) \mathcal{T}_k(1_{F'_{\nu_\alpha}} : \alpha \in \{0, 1\}^k).
\]

Note that (6.28) is precisely (6.2) for the case \( \tilde{\nu} \in \Omega(\zeta)^{[0,1]^k}\setminus S \). We now investigate the exceptional set \( S \). Hence, in addition to the above requirements on \( \zeta \), we also enforce \( \zeta(\eta) \to 0 \) as \( \eta \to 0 \) with a rate sufficiently slow so that \( (L(S))^{1/2k}/\zeta(\eta) \to 0 \) as \( \eta \to 0 \). This allows us to find, if \( \eta \) is sufficiently small, for every \( \tilde{\nu} \in S \cap \Omega(2\zeta)^{[0,1]^k} \) two vectors \( \tilde{\nu}' \neq \tilde{\nu}'' \in \Omega(\zeta)^{[0,1]^k} \) such that \( \nu_\alpha - 2(L(S))^{1/2k} \leq \nu_\alpha'' \leq \nu_\alpha' \leq \nu_\alpha + 2(L(S))^{1/2k} \), for all \( \alpha \in \{0, 1\}^k \). Then for these vectors \( \tilde{\nu} \in \Omega(2\zeta)^{[0,1]^k} \subset \Omega(\eta)^{[0,1]^k} \):

\[
(6.29) \quad \mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) \geq (1 - o(\eta)) \mathcal{T}_k(1_{F'_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) - C(\zeta)(L(S))^{1/2k} - r
\]

Since level sets are nested, \( \mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) \) is non-increasing in terms of each variable \( \nu_\alpha \); that explains the first inequality in (6.29). The second inequality comes from the definition of the set \( S \). Since \( \Omega(2\zeta) \) is a compact set and \( F \) is a centered Gaussian, \( \mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) \) is Lipschitz continuous with a Lipschitz constant \( C(\zeta) \) in each variable \( \nu_\alpha \in \Omega(2\zeta) \); hence the third inequality follows. Finally the last comes from (6.13). It remains to show:

\[
(6.30) \quad C(\zeta)(L(S))^{1/2k} + r + C(k)\epsilon(\eta, \delta) \leq o(\eta) \mathcal{T}_k(1_{F'_{\nu_\alpha}} : \alpha \in \{0, 1\}^k).
\]

Indeed, from the definition of \( C(\zeta) \) we have \( C(\zeta) \leq CK(\zeta) \leq CK(\eta) \), and from the definition of \( S \), we have,

\[
C(\zeta)(L(S))^{1/2k} \leq C(k)K(\eta)(K(\eta)r^{-2}\epsilon(\eta, \delta))^{1/2k} \leq C(k)K^2(\eta)(r^{-2}\epsilon(\eta, \delta))^{1/2k}.
\]

Hence (6.24), (6.26), (6.27) and (6.29) then imply (6.30):

\[
C(\zeta)(L(S))^{1/2k} + r + C(k)\epsilon(\eta, \delta) \leq \frac{C(k)K^2(\eta)(r^{-2}\epsilon(\eta, \delta))^{1/2k}}{\phi(\zeta) - r} \cdot (\phi(\zeta) - r) + \frac{r + C(k)\epsilon(\eta, \delta)}{\phi(\zeta) - r} \cdot (\phi(\zeta) - r) \leq o(\eta) \mathcal{T}_k(1_{F'_{\nu_\alpha}} : \alpha \in \{0, 1\}^k).
\]

Note that \( \mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) \) has a non-increasing property in terms of each variable \( \nu_\alpha \), similarly to that of \( \mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) \). This fact, (6.29) and (6.30) together give us the desired conclusion for \( \tilde{\nu} \in S \):

\[
(6.31) \quad \mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) \geq (1 - o(\eta)) \mathcal{T}_k(1_{F'_{\nu_\alpha}} : \alpha \in \{0, 1\}^k).
\]

Finally (6.28) and (6.31) together give the desired conclusion of (6.2). \( \square \)
Remark 6.2. It’s important for us to analyze the subset $S$, even when its measure is at most $o(\eta)$, in order to prepare for our next discussions. As noted above, the conclusion of Proposition 6.2 is only applicable when we have the conclusion of Proposition 6.1 for the diagonal case $\nu_\alpha = \nu$, for all $\alpha \in \{0, 1\}^k$. Hence we can’t afford to bypass even a subset of measure zero.

7. Conclusion for one dimension

7.1. Preparation. Let $n = 1$. Recall that in the proof of Proposition 6.1, the parameter $\delta = \delta(\eta)$ satisfies $\delta \to 0$ as $\eta \to 0$. We can choose $\delta(\eta)$ to be a one-to-one function, in which case it allows us to rephrase the findings of the previous two sections as follows:

Let $\mathcal{F}$ be a centered Gaussian. For every $\delta$ there exists $\eta = \eta(\delta)$ satisfying $\eta \to 0$ as $\delta \to 0$ such that the following occurs. Suppose $f \in L^{p_k}(\mathbb{R})$ is a nonnegative $(1 - \delta)$ near extremizer with $\|\mathcal{F} - f^*\|_{p_k} \leq \delta \|f\|_{p_k}$. Then for every $s \in \Omega(\eta) = \{\eta \in \mathcal{F} \to \mathcal{F} - \eta\}$, there exists an interval $I_s$ such that $\mathcal{L}(I_s \Delta F_s) = o(\delta) \mathcal{L}(F_s), \mathcal{L}(I_s^* \Delta F^*_s) = o(\delta)$ and consequently $|\mathcal{L}(F_s) - \mathcal{L}(F_s)| = o(\delta)$.

As hinted, to further analyze the distribution of $f$, we replace the measurable set $F_s$ with one such corresponding interval $I_s$. One conclusion of this chapter is that, if $I_s$ and $I_s'$ are two intervals such that $s, s' \in \Omega(\eta)$ and $c_s, c_{s'}$ are centers of $I_s, I_{s'}$ respectively, then $c_s$ must be close to $c_{s'}$ in an appropriate sense that will be made clear. To this end, we first argue that a selected tuple $(1_{I_{\nu_\alpha}} : \alpha \in \{0, 1\}^k)$ nearly achieves equality in the rearrangement inequality:

Lemma 7.1. Let $\delta, \eta$ be as above. Let $\tilde{\nu} = (\nu_\alpha : \alpha \in \{0, 1\}^k)$ with $\nu_\alpha \in \Omega(\eta)$. There exists $\delta_0 > 0$ such that if $\delta \leq \delta_0$ then:

$$\mathcal{T}_k(1_{I_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) \geq (1 - o(\delta)) \mathcal{T}_k(1_{I_{\nu_\alpha}^*} : \alpha \in \{0, 1\}^k).$$

Proof. Since $\mathcal{L}(I_{\nu_\alpha}^* \Delta F_{\nu_\alpha}^*) \leq \mathcal{L}(I_s \Delta F_s), \mathcal{L}(I_{\nu_\alpha}^* \Delta F_{\nu_\alpha}^*) = o(\delta) \mathcal{L}(F_s)$ if $s \in \Omega(\eta)$. Proposition 6.1 and the Gowers product inequality then imply the following three inequalities:

$$|\mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) - \mathcal{T}_k(1_{F_{\nu_\alpha}^*} : \alpha \in \{0, 1\}^k)| = o(\delta) \prod_{\alpha \in \{0, 1\}^k} (\mathcal{L}(F_{\nu_\alpha}))^{1/p_k}$$

$$|\mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) - \mathcal{T}_k(1_{I_{\nu_\alpha}} : \alpha \in \{0, 1\}^k)| = o(\delta) \prod_{\alpha \in \{0, 1\}^k} (\mathcal{L}(F_{\nu_\alpha}))^{1/p_k}$$

$$|\mathcal{T}_k(1_{F_{\nu_\alpha}^*} : \alpha \in \{0, 1\}^k) - \mathcal{T}_k(1_{I_{\nu_\alpha}^*} : \alpha \in \{0, 1\}^k)| = o(\delta) \prod_{\alpha \in \{0, 1\}^k} (\mathcal{L}(F_{\nu_\alpha}))^{1/p_k}$$

which then give us:

$$\mathcal{T}_k(1_{I_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) \geq \mathcal{T}_k(1_{I_{\nu_\alpha}^*} : \alpha \in \{0, 1\}^k) - o(\delta) \prod_{\alpha \in \{0, 1\}^k} (\mathcal{L}(F_{\nu_\alpha}))^{1/p_k}. \tag{7.1}$$

Since $\mathcal{L}(I_{\nu_\alpha}) \geq (1 - o(\delta)) \mathcal{L}(F_{\nu_\alpha}) \geq (1 - o(\delta)) \mathcal{L}(|\mathcal{F}_{\nu_\alpha} - \eta|) \geq \mathcal{L}(|\mathcal{F}_{\nu_\alpha} - \eta|) \geq \mathcal{L}(I_{\nu_\alpha}^* \subseteq I_{\nu_\alpha} - \eta)$, for $\nu_\alpha \in \Omega(\eta)$ and $\alpha \in \{0, 1\}^k$. Define:

$$\psi(\eta) = \mathcal{T}_k(1_{I_{\nu_\alpha}^* \subseteq \mathcal{F}_{\nu_\alpha} - \eta} \ldots, 1_{I_{\nu_\alpha}^* \subseteq \mathcal{F}_{\nu_\alpha} - \eta}).$$

Then the said set inclusion implies:

$$\mathcal{T}_k(1_{I_{\nu_\alpha}^*} : \alpha \in \{0, 1\}^k) \geq \mathcal{T}_k(1_{I_{\nu_\alpha}^* \subseteq \mathcal{F}_{\nu_\alpha} - \eta} \ldots, 1_{I_{\nu_\alpha}^* \subseteq \mathcal{F}_{\nu_\alpha} - \eta}) = \psi(\eta). \tag{7.2}$$

Note that $\mathcal{L}(\mathcal{F}_{\eta}) = C(\log(1/\eta))^C$ and that $\psi(\eta)$ stays strictly bounded below as long as $\eta > 0$. We select $\eta = \eta(\delta) \to 0$ sufficiently slowly as $\delta \to 0$ and $\delta_0 > 0$ so that if $\delta \leq \delta_0$ then
implies the centers below that Lemma 7.2. By Lemma 7.2.

Lemma 7.2. Let \( f_i \) and \( h_i \) be as above. Then \( \| f_i - h_i \|_{L^p} \to 0 \) as \( i \to \infty \).

Proof. By Lemma 5.3 and Minkowski’s integral inequality,

\[
\| f_i - h_i \|_{L^p} \leq \left\| \int_{\eta_i}^{\bar{\eta}_i} 1_{F_i,s} ds \right\|_{L^p} + C\eta_i(\log(1/\eta_i))^C
\]

By Lemma 7.2, if we can establish that \( \{h_i\}_i \) is precompact in \( L^p \), then we have the same result for \( \{f_i\}_i \). We show here a related compactness result which will be needed.

Lemma 7.3. Let \( a < b \in \mathbb{R} \) and \( 1 \leq q < \infty \). Let \( [-B, B] \) be a closed interval. For each \( i \in \mathbb{Z}_{\geq 0} \) and \( s \in \mathbb{R} \) let \( I_i,s \subset [-B, B] \) be an interval. Suppose further that the function \( (x, s) \mapsto 1_{I_i,s}(x) \) is measurable. Then \( \{\int_0^b 1_{I_i,s}(x) ds\}_i \) is precompact in \( L^q \).

Proof. Let \( g_i(x) = \int_0^b 1_{I_i,s}(x) ds \). It’s clear that, there exists \( C > 0 \) such that \( \|g_i\|_q \leq C \). Moreover, \( \int_{|x| > r} |g_i(x)|^q dx = 0 \), for every \( r > B \), and \( \lim_{r \to 0} \|T^k g_i - g_i\| = 0 \) uniformly in \( i \). Hence by the Fréchet-Kolmogorov theorem [3], \( \{g_i\}_i \) is precompact in \( L^q \).

7.2. A monotonicity result. Let \( I = [-1, 1] \) and \( J = [-\eta - 1, 1 + \eta] \) for some \( \eta \in [0, \frac{2}{k-1}] \). Define:

\[
\phi(t) = \int_{\mathbb{R}^{k+1}} 1_{J + \bar{\eta}^t} \prod_{\alpha \in \{0,1\}^k; \alpha \neq \bar{\eta}} 1_f(x + \alpha \cdot \bar{\eta}) d\bar{\eta} dx.
\]

\( \phi(t) \) is a continuous, nonnegative even function of \( t \) and has a compact support. Furthermore, let:

\[
H(x) = \int_{\mathbb{R}^k} \prod_{\alpha \in \{0,1\}^k; \alpha \neq \bar{\eta}} 1_f(x + \alpha \cdot \bar{\eta}) d\bar{\eta}.
\]
Then \( \phi(t) = \int_{\mathbb{R}} 1_{J+t} \cdot H(x) \, dx \). \( H \) is also a continuous, nonnegative even function whose support is the interval \([-\frac{k+1}{k}, \frac{k-1}{k}]\). Indeed, the interval \([-1, 1]\) is clearly contained in the support of \( H \). Suppose \( x > 1 \) and \( x \in \text{spt}(H) \). Then there exists \( \tilde{h} = (h_i)_i \in \mathbb{R}^k \) such that \( |x + \alpha \cdot \tilde{h}| \leq 1 \) for all \( \alpha \in \{0, 1\}^k \) (there exists, in fact, a set of such \( \tilde{h} \) of positive measure). Let \( j \in \{1, \ldots, k\} \) and define \( \beta \in \{0, 1\}^k \) by \( \beta_j = 1 \) and \( \beta_i = 0 \) if \( i \neq j \). Then \( |x + \beta \cdot \tilde{h}| = |x + h_j| \leq 1 \) implies \( h_j \leq 1 - x \). Since this holds for every \( j \in \{1, \ldots, k\} \), \( k^{-1} \sum_{i=1}^{k} h_i \leq 1 - x \). Now consider \( \alpha = (1, \ldots, 1) \). Then \( |x + \tilde{h} \cdot \tilde{h}| = |x + \sum_{i=1}^{k} h_i| \leq 1 \) implies \( k^{-1} \sum_{i=1}^{k} h_i \geq -k^{-1}(1 + x) \). By transitivity, \( k^{-1}(1 + x) \leq 1 - x \), or equivalently \( x \leq \frac{k+1}{k} \). Similarly, if \( x < -1 \) and \( x \in \text{spt}(H) \) then \( x \geq -\frac{k+1}{k} \).

Now suppose \( x \in \left[-\frac{k+1}{k}, \frac{k+1}{k}\right] \). We define \( \tilde{h} = (h_i)_i \in \mathbb{R}^k \) such that \( h_i \leq \frac{1-x}{k} \) if \( 1 < x \leq \frac{k+1}{k} \) and \( h_i \leq \frac{x}{k} \) if \( -\frac{k+1}{k} \leq x < -1 \). For the former case, \( -1 \leq x(1-k) + k \leq x + \alpha \cdot \tilde{h} \leq x+1-x = 1 \), for all \( \alpha \in \{0, 1\}^k \); hence \( x \in \text{spt}(H) \). We also obtain the same conclusion for the latter case.

We claim that \( \phi(t) \) is a strictly decreasing function of \( t \geq 0 \) in its support. We make a quick remark that this claim and the fact that \( \text{spt}(H) = \left[-\frac{k+1}{k}, \frac{k+1}{k}\right] \) are why we take \( \eta \in \left[0, \frac{2}{k-1}\right] \).

Since \( \eta > \frac{2}{k-1} \), then there exist \( t_1 > t_0 \) sufficiently small, so that they are both in the support of \( \phi \), say, \( t_1 = (\eta - \frac{2}{k-1})/10 \) and \( t_0 = (\eta - \frac{2}{k-1})/20 \), such that,

\[
\phi(t_1) = \int_{\mathbb{R}^{k+1}} 1_{J+t_1}(x) \prod_{\alpha \in \{0,1\}^k \setminus \{0\}} 1_f(x + \alpha \cdot \tilde{h}) \, d\tilde{h} dx
\]

\[
= \int_{\mathbb{R}^{k+1}} 1_{J+t_0}(x) \prod_{\alpha \in \{0,1\}^k \setminus \{0\}} 1_f(x + \alpha \cdot \tilde{h}) \, d\tilde{h} dx = \phi(t_0).
\]

With our choice of \( \eta \), \( \text{spt}(\phi) \subset \left[-\frac{2(k+1)}{k-1}, \frac{2(k+1)}{k-1}\right] \). Note that:

\[
(d/dt)\phi(t) = \int_{\mathbb{R}} (\delta_{t+1+\eta} - \delta_{t-1-\eta}) \cdot H(x) \, dx = H(t + 1 + \eta) - H(t - 1 - \eta).
\]

Furthermore, with our choice of \( \eta \), \( t - 1 - \eta \) always lies in the support of \( H \). Hence our claim on the monotonicity of \( \phi \) will follow if we can show \( H(t) \) is strictly decreasing for \( t \geq 0 \) in its support. Let \( \mathcal{C} = \{(x, \tilde{h}) \in \mathbb{R}^{k+1} : x + \alpha \cdot \tilde{h} \in I, \forall \alpha \in \{0,1\}^k \} \). Then \( \mathcal{C} \) is a compact, convex, balanced subset of \( \mathbb{R}^{k+1} \). Let \( \mathcal{C}_x = \{ \tilde{h} \in \mathbb{R}^k : (x, \tilde{h}) \in \mathcal{C} \} \). We observe that \( H(x) = \mathcal{L}(\mathcal{C}_x) \).

**Lemma 7.4.** Let \( H \) be as above. Then \( H(0) > H(x) \), for all \( x > 0 \).

Lemma 7.4, if assumed true, will imply that \( H(x) > H(y) \) if \( 0 \leq x < y \) and \( x, y \) both lie in the support of \( K \). Indeed, let \( 0 < x < y \) be in the interior of the support of \( H \), then \( H(x) = \mathcal{L}(\mathcal{C}_x) > 0 \) and \( H(y) = \mathcal{L}(\mathcal{C}_y) > 0 \). Let \( t \in (0,1) \) be such that \( x = (1-t) \cdot 0 + ty \). By the convexity of \( \mathcal{C} \), \( \mathcal{C}_x \supset (1-t)\mathcal{C}_0 + t\mathcal{C}_y \). Then by the Brunn-Minkowski inequality, \( \mathcal{L}(\mathcal{C}_x) \geq (\mathcal{L}(\mathcal{C}_0))^{1-t}(\mathcal{L}(\mathcal{C}_y))^t \), and by Lemma 7.4:

\[
H(x) \geq H(0)^{1-t}H(y)^t > H(y)^{1-t}H(y)^t = H(y).
\]

7.3. **Proof of Lemma 7.4.** We will deduce Lemma 7.4 from the following more general result. The setup is as follows:

Let \( M, N \in \mathbb{Z}_{\geq 0} \). For \( i \in \{1, \ldots, M\} \), let \( J_i \) be a closed interval centered at 0, \( \mathcal{L}(J_i) = l_i \in \mathbb{R}_{>0} \), and \( L_i : \mathbb{R}^N \to \mathbb{R} \) be surjective linear mappings. For \( \bar{t} = (t_i)_i \in \mathbb{R}^M \), we let:

\[
\Psi(\bar{t}) = \int_{\mathbb{R}^N} \prod_{i=1}^{M} 1_{J_i+t_i}(L_i(\bar{y})) \, d\bar{y}.
\]
We make two observations. First, if $\vec{v} \in \mathbb{R}^N$:

$$\int_{\mathbb{R}^N} \prod_{i=1}^M 1_{J_i}(L_i(\vec{y} + \vec{v})) \, d\vec{y} = \int_{\mathbb{R}^N} \prod_{i=1}^M 1_{J_i}(L_i(\vec{y})) \, d\vec{y}.$$ 

Secondly, by the general rearrangement inequality, for $\vec{t} = (t_i)_i \in \mathbb{R}^M$:

$$\int_{\mathbb{R}^N} \prod_{i=1}^M 1_{J_i+t_i}(L_i(\vec{y})) \, d\vec{y} \leq \int_{\mathbb{R}^N} \prod_{i=1}^M 1_{J_i}(L_i(\vec{y})) \, d\vec{y}$$

or equivalently, $\Psi(\vec{t}) \preceq \Psi(\vec{0})$. It then follows that, for $\vec{t} \in \mathbb{R}^M$, $\Psi(\vec{t}) = \Psi(\vec{0})$ if there exists $\vec{v} \in \mathbb{R}^N$ such that $L_i(\vec{v}) = t_i$, $i \in \{1, \ldots, M\}$. It will be shown that this is also a necessary condition, provided that $(L_i, l_i)_{i=1}^M$ is an admissible tuple.

**Admissibility:** For $i \in \{1, \ldots, M\}$, let $J_i$ be a closed interval centered at 0, $\mathcal{L}(J_i) = l_i \in \mathbb{R}_{>0}$ and $L_i : \mathbb{R}^N \to \mathbb{R}$ be surjective linear mappings. Let $\vec{t} = (l_i)_i \in (\mathbb{R}_{>0})^M$. Define $K_{\vec{t}} = \{ \vec{x} \in \mathbb{R}^N : |L_i(\vec{x})| \leq l_i, \forall i \in \{1, \ldots, M\} \}$. We said that $(L_i, l_i)_i$ is an admissible tuple if for every $m \in \{1, \ldots, M\}$ there exists $\vec{x}_m \in K_{\vec{t}}$ such that $|L_m(\vec{x}_m)| = l_m$.

**Lemma 7.5.** [5] Let $L_i, J_i, l_i$ and $\Psi$ be as above. Suppose $(L_i, l_i)_{i=1}^M$ is an admissible tuple. Let $\vec{t} \in \mathbb{R}^M$. Then $\Psi(\vec{0}) \geq \Psi(\vec{t})$ and equality holds iff there exists $\vec{v} \in \mathbb{R}^N$ such that $L_i(\vec{v}) = t_i$ for all $i \in \{1, \ldots, M\}$.

**Proof.** Let $K = \{ (\vec{x}, \vec{t}) \in \mathbb{R}^N \times \mathbb{R}^M : L_i(\vec{x}) \in J_i + t_i, \forall i \in \{1, \ldots, M\} \}$. For each $\vec{t} \in \mathbb{R}^M$, let $K(\vec{t}) = \{ \vec{x} \in \mathbb{R}^N : (\vec{x}, \vec{t}) \in K \}$. It’s clear that $K$ is convex, and if $(\vec{x}, \vec{t}) \in K$ then $|L_i(\vec{x}) - t_i| \leq l_i$. Since $\Psi(\vec{t})$ represents the $N$-dimensional volume of $K(\vec{t})$ and $J_i$ are centered at 0, $\Psi(\vec{t}) = \Psi(-\vec{t})$ or $\mathcal{L}(K(\vec{t})) = \mathcal{L}(K(-\vec{t}))$. Moreover,

$$K(\vec{0}) \supseteq (1/2)K(\vec{t}) + (1/2)K(-\vec{t}).$$

Suppose that $\Psi(\vec{t}) = \Psi(\vec{0})$ for some $\vec{t} \in \mathbb{R}^M$, which implies $\Psi(\vec{0}) = \mathcal{L}(K(\vec{0})) = \Psi(\vec{t}) = (\mathcal{L}(K(\vec{t})))^{1/2}(\mathcal{L}(K(-\vec{t})))^{1/2}$. Then (7.4) and the Brunn-Minkowski inequality imply:

$$\mathcal{L}(K(\vec{0})) \geq (1/2)\mathcal{L}(K(\vec{t})) + (1/2)\mathcal{L}(K(-\vec{t})) \geq (\mathcal{L}(K(\vec{t})))^{1/2}(\mathcal{L}(K(-\vec{t})))^{1/2} = \mathcal{L}(K(\vec{0}))$$

which yields $\mathcal{L}((1/2)K(\vec{t}) + (1/2)K(-\vec{t})) = (\mathcal{L}(K(\vec{t})))^{1/2}(\mathcal{L}(K(-\vec{t})))^{1/2}$. Hence by the characterization of the equality case of the Brunn-Minkowski inequality, there exists $\vec{v} \in \mathbb{R}^N$ such that $K(\vec{0}) + \vec{v} = K(\vec{0})$. By definition, this means, if $\vec{x} \in \mathbb{R}^N$ and $i \in \{1, \ldots, M\}$:

$$L_i(\vec{x}) \in J_i + t_i \iff L_i(\vec{x} + \vec{v}) \in J_i.$$

Let $\vec{z} = \vec{x} + \vec{v}$, then by (7.5), $|L_i(\vec{z})| \leq l_i$ implies,

$$|L_i(\vec{z}) - t_i - L_i(\vec{v})| \leq l_i.$$ 

Fix $m \in \{1, \ldots, M\}$. Admissibility assumption implies that there exist $\vec{x}_{m,\pm} \in \mathbb{R}^N$ such that $|L_i(\vec{x}_{m,\pm})| \leq l_i$ and $L_m(\vec{x}_{m,\pm}) = \pm l_m$. In particular, (7.6) implies

$$|L_m(\vec{x}_{m,\pm}) - t_m - L_m(\vec{v})| \leq l_m.$$ 

Suppose $t_m + L_m(\vec{v}) < 0$. Then $L_m(\vec{x}_{m,\pm}) - t_m - L_m(\vec{v}) = l_m - (t_m + L_m(\vec{v})) > l_m$, which poses a contradiction to (7.7). Similarly, $t_m + L_m(\vec{v}) > 0$ will also imply the contradiction to (7.7) since $L_m(\vec{x}_{m,\pm}) - (t_m + L_m(\vec{v})) = -l_m - (t_m + L_m(\vec{v})) < -l_m$. Hence $t_m = L_m(-\vec{v})$ for every $m \in \{1, \ldots, M\}$. We’ve completed the proof of the lemma as the "if" direction is apparent by the discussion at the beginning of this section. □
To finish the proof of Lemma 7.4, let $M = 2^k - 1$, $N = k$. We note that for each $\vec{0} \neq \alpha \in \{0, 1\}^k$, $L_\alpha : \mathbb{R}^k \to \mathbb{R}$ defined by $L_\alpha(\vec{h}) = \alpha \cdot \vec{h}$ is a surjective linear mapping. Moreover, for each $\vec{0} \neq \alpha \in \{0, 1\}^k$, let $|\alpha| = \sum_{i=1}^k \alpha_i$ and $\vec{h}_\alpha \in \mathbb{R}^k$ be such that $(\vec{h}_\alpha)_i = 1/|\alpha|$ if $\alpha_i = 1$ and $(\vec{h}_\alpha)_i = 0$ otherwise. Then it’s easily checked that $L_\alpha(\pm \vec{h}_\alpha) = \pm 1$ and $|L_\beta(\vec{h}_\alpha)| \leq 1$ if $\alpha \neq \beta \in \{0, 1\}^k \setminus \{\vec{0}\}$. Hence the tuple $((L_\alpha, 1)_\alpha : \alpha \in \{0, 1\}^k \setminus \{\vec{0}\})$ is admissible. Note that with $\vec{x} = (x)_i \in \mathbb{R}^{\{0, 1\}^k \setminus \{\vec{0}\}}$:

$$H(x) = \int_{\mathbb{R}^k} \prod_{\alpha \in \{0, 1\}^k : \alpha \neq \vec{0}} 1_I(x + \alpha \cdot \vec{h}) \, d\vec{h} = \int_{\mathbb{R}^k} \prod_{\alpha \in \{0, 1\}^k : \alpha \neq \vec{0}} 1_{I - x}(\alpha \cdot \vec{h}) \, d\vec{h} = \Psi(-\vec{x}).$$

Then from Lemma 7.5, $H(0) = \Psi(\vec{0}) \geq \Psi(-\vec{x}) = H(x)$. If $H(0) = H(x)$ for some $x > 0$ then there must exist $\vec{v} \in \mathbb{R}^k$ such that $L_\alpha(\vec{v}) = \alpha \cdot \vec{v} = -x$, for all $\vec{0} \neq \alpha \in \{0, 1\}^k$, which is clearly impossible. Hence $H(0) > H(x)$ for all $x > 0$, as we wish to conclude. $\square$

7.4. Alignment of super-level sets. Now Subsection 7.2 concludes that, if $I$ and $J$ are two intervals "compatible" in size, and if

\[ T_k(J, I, \ldots, I) = \int_{\mathbb{R}^{k+1}} 1_I(x) \prod_{\alpha \in \{0, 1\}^k : \alpha \neq \vec{0}} 1_I(x + \alpha \cdot \vec{h}) \, d\vec{h} \, dx \]

is nearly maximized over all tuples of intervals of the same sizes, then the centers of $I$ and $J$ must be "close" to each other. The compatibility condition is given by allowing $L(J) = L(I) + \eta$ with $\eta \in [0, \frac{2}{k-1}L(I)]$.

Let $0 < \epsilon \ll \frac{1}{k-1}$. Then the discussion in the beginning of Subsection 7.1 and Lemma 7.1 conclude that there exist $\delta > 0$ and $\eta = \eta(\delta) > 0$ satisfying the following properties:

1. If $f$ is a nonnegative $(1 - \delta)$ near extremizer and $\|F - f\|_{P_k} \leq \Delta$ for a centered Gaussian $\mathcal{F}$, then for every $s \in \Omega(\eta)$, $\|\langle F \rangle \|_\infty \in [\eta, \|\langle F \rangle \|_\infty - \eta]$ there exists an interval $I_s$ such that $L_i(\mathcal{F}_s \Delta \mathcal{F}_s) < c\epsilon L(\mathcal{F})$ and $|L(\mathcal{F}) - L(\mathcal{F}_s)| < c\epsilon$.

2. Furthermore, $T_k(1_{I_{\nu}} : \alpha \in \{0, 1\}^k) \geq (1 - \epsilon)T_k(1_{I_{\nu}} : \alpha \in \{0, 1\}^k)$, if $\nu \in \Omega(\eta)$, $\alpha \in \{0, 1\}^k$.

Let $\mathcal{F}_s$ denote the super-level set of the Gaussian $\mathcal{F}$ associated with the value $s$. There exist $s_1 = \eta < s_2 < \ldots < s_N = \|\langle F \rangle \|_\infty - \eta$ in $[\eta, \|\langle F \rangle \|_\infty - \eta]$ with $N = N(\eta)$ such that $L(\mathcal{F}_{s_i} \Delta \mathcal{F}_{s_{i+1}}) \leq \frac{1}{k-1}L(\mathcal{F}_{s_i})$ for all $i \in \{1, \ldots, N - 1\}$. Since super-level sets are nested, $L(\mathcal{F}_{s_i} \Delta \mathcal{F}_{s_j}) \leq \frac{c}{k-1}L(\mathcal{F}_{s_j})$, where $c$ being some sufficiently small constant, if $s_i \leq s \leq s_{i+1}$, and consequently, $|L(\mathcal{I}_{i}) - L(\mathcal{I}_{s})| \leq \frac{c}{k-1}L(\mathcal{I}_{s})$, $i \in \{1, \ldots, N - 1\}$. That is to say, the size compatibility condition is also satisfied by the sub-intervals with $s \in [s_i, s_{i+1}] \subset \Omega(\eta)$. Recall that $\mathcal{F}_s$ denotes the center of an interval $I_s$. If $\epsilon$ is sufficiently small, then by the second property, the previous paragraph and the satisfaction of the compatibility condition, $|c_s - c_{s_i}| = o(\epsilon)L(I_{s_i})$ if $s_i \leq s \leq s_{i+1}$, $i \in \{1, \ldots, N - 1\}$. If $s \in \Omega(\eta)$, then $s$ must lie in one such interval $[s_i, s_{i+1}]$; hence $|c_s - c_{s_i}| = o(\epsilon)L(\mathcal{F}) = o(\epsilon)(\log(1/\eta))^C$ for $s \in \Omega(\eta)$ since $L(\mathcal{F}_s) = C(\log(1/\eta))^C$. Note that if $\epsilon \to 0$ then $\delta \to 0$, and if $\eta > \eta'$ then $\Omega(\eta) \subset \Omega(\eta')$. We now require $\eta = \eta(\delta) \to 0$ sufficiently slow so that $|c_s - c_{s_i}| = o(\epsilon)L(\mathcal{F}) = o(\epsilon)(\log(1/\eta))^C = o(\delta)$ as $\delta \to 0$. The intervals $I_s$ might change as the parameters change, but the size estimates still hold. We now obtain the following lemma:

**Lemma 7.6.** Let $\delta, \eta = \eta(\delta)$ and $c_s$ be as above. Then $|c_s - c_{s_i}| = o(\delta)$ if $s \in \Omega(\eta)$. 


7.5. A compactness result. Suppose we have a nonnegative extremizing sequence \( \{ f_i \}_i \) such that \( \| f_i \|_{L^k} = 1 \) and 
\[
\| f_i \|_{L^k} \geq (1 - \delta_i)A(k) \| f_i \|_{L^k} = (1 - \delta_i)A(k)
\]
and that \( \| F - f_i \|_{L^k} \leq \delta_i \) for some centered Gaussian \( F \) with \( \delta_i \to 0 \) as \( i \to \infty \). Recall that in Subsection 7.1 we defined for such a sequence \( \{ f_i \}_i \) an associate superposition sequence \( h_i(x) = \int_{\eta_i}^{\| F \|_x - \eta_i} 1_{I_{i,s}}(x) \, ds \). We select \( \eta_i = \eta(\delta_i) \) satisfying \( \eta_i \to 0 \) as \( i \to \infty \) such that \( |c_s - c_{s'}| = o(\delta_i) \) if \( s \in \Omega(\eta_i) \), as in Subsection 7.4. The conclusions of Lemma 7.2 and Lemma 7.3 stay unchanged with this selection of \( \{ \eta_i \}_i \).

Remark 7.1. It’s possible to select \( I_{i,s_i} \) so that \( (x, s_i) \mapsto 1_{I_{i,s_i}}(x) \) is a measurable function. Fix \( i \) and denote \( \eta_i = \eta, f_i \) as \( f, F_{i,s_i} \) as \( F_s \) and \( I_{i,s_i} \) as \( I_s \). The set \( E = \bigcup_{s \in \Omega(\eta)} \{ s \} \times F_s \) is a measurable subset of \( \Omega(\eta) \times \mathbb{R} \). Let \( a_s < b_s \) denote the endpoints of \( I_s \), \( s \in \Omega(\eta) \), then it comes down to the ability to select the endpoints \( a_s, b_s \) of the intervals \( I_s \) in a measurable manner.

We propose for a moment that \( f \) is a continuous function. Decompose the range of values \( \Omega(\eta) \) into a finite number \( C_f(\eta) \) of smaller sub-ranges of values \( \Omega'(\eta) \) such that \( \mathcal{L}(F_s \Delta F_{s'}) = o(\eta) \) for each pair \( s, s' \in \Omega'(\eta) \). Fix one such sub-range \( \Omega'(\eta) \) and let \( F_{s_1}, F_{s_2} \) denote the super-level sets of \( f \) associated with the largest and smallest values of the range, respectively. By replacing \( F_{s_1} \) with \( I_{i,s_1} \) and \( F_{s_2} \) with \( I_{i,s_2} \), we assume \( F_{s_1} \) and \( F_{s_2} \) are both intervals. Let \( a_{s_1} \) denote the left endpoint of \( F_{s_1} \) and \( a_{s_2} \) the left endpoint of \( F_{s_2} \); note that \( |a_{s_1} - a_{s_2}| = C_f o(\eta) \). Consider the part of the graph of \( f \) inside the rectangle box \( \Omega'(\eta) \times [a_{s_2}, a_{s_1}] \); call this set \( R_{\eta} \).

By the Measurable Choice Theorem [2], there exists a measurable function \( s \mapsto a(s) \), so that \( (s, a(s)) \in R_{\eta} \), for every \( s \in \Omega'(\eta) \). Take \( a(s) \) to be the left endpoint of our interval \( I_s \). Proceed similarly to obtain a measurable function \( s \mapsto b(s) \) for the right endpoint of \( I_s \). Note that by construction \( \mathcal{L}(I_s \Delta F_s) = o(\eta) \mathcal{L}(F_s) \), if \( s \in \Omega'(\eta) \). Continue this procedure for each of these sub-ranges \( \Omega'(\eta) \) and concatenate the obtained left endpoint and right endpoint functions to obtain a measurable function \( (s, x) \mapsto 1_{I_s}(x) \), \( s \in \Omega(\eta) \). For the general case we approximate \( f \) with a positive continuous function \( g \) so that \( \| f - g \|_{L^k} = \rho(\eta) \) with \( \rho \ll \eta \) in order for us to have \( \mathcal{L}(G_s \Delta F_s) = o(\eta) \mathcal{L}(F_s) \), for a.e \( s \in \Omega(\eta) \). Then apply the described procedure with \( G_s \) in place of \( F_s \).

Another and easier way is to construct a piecewise constant function \( (x, s) \mapsto I_s(x) \); i.e., \( I_s(x) = I_{s'}(x) \) if \( s, s' \in \Omega'(\eta) \). This ensures measurability but might come at the expense of increasing the (still finite) number of sub-ranges so that \( \mathcal{L}(I_s \Delta F_s) = o(\eta) \mathcal{L}(F_s) \) is still guaranteed.

Proposition 7.7. Let \( \{ h_i \}_i \) be as above. There exists \( \{ a_i \}_i \) such that \( \{ h_i(-a_i) \}_i \) is precompact in \( L^p \).

Proof. Let \( a_i = c_{\eta_i} \) with \( c_{\eta_i} \) being the center of the interval \( I_{\eta_i} \). Define:
\[
g_i(x) = h_i(x - a_i) = \int_{\eta_i}^{\| F \|_x - \eta_i} 1_{I_{i,s}}(x - a_i) \, ds.
\]
Let \( \epsilon > 0 \) be a small number and select \( \eta > 0 \) such that:
\[
\| \int_0^\eta 1_{F_s} \, ds \|_{L^k} + \| \int_{\| F \|_x - \eta}^{\| F \|_x} 1_{F_s} \, ds \|_{L^k} \leq \epsilon.
\]
For simplicity, we assume that \( \eta_i \leq \eta \) for all \( i \). Recall that \( |\mathcal{L}(F_s) - \mathcal{L}(I_{i,s})| = o(\delta_i) \mathcal{L}(F_s) \) if \( s \in \Omega(\eta_i) \). Then (7.8) entails:
\[
\| \int_{\eta_i}^\eta 1_{I_{i,s}}(-a_i) \, ds \|_{L^k} + \| \int_{\| F \|_x - \eta}^{\| F \|_x - \eta_i} 1_{I_{i,s}}(-a_i) \, ds \|_{L^k} \leq C \epsilon.
\]
The fact that \( |\mathcal{L}(F_s) - \mathcal{L}(I_{i,s})| = o(\delta_i) \mathcal{L}(F_s) \) also implies that the intervals \( I_{i,s}(x - a_i) \) are contained within some compact interval \([-B, B]\) if \( s \in \Omega(\eta) = [\eta_i, \| F \| - \eta_i] \subset \Omega(\eta_i) = \).
and Proposition 7.2. For nonnegative extremizing tuple \( \vec{a}, \vec{p} \) of the Brascamp-Lieb datum \( \vec{F}, \vec{B} \), we recover the classic Riesz-Sobolev inequality [14]. In this case, strict admissibility is equivalent to the strict admissibility polygon condition given by Burchard [4], which is: \( l_i < l_j + l_k \) for any permutation \( (i, j, k) \) of \( (1, 2, 3) \). Indeed, if the set \( \{ (x_1, x_2) : |x_1| \leq l_1, |x_2| \leq l_2, |x_1 + x_2| \leq l_3 \} \) is a nonempty proper subset of \( \{ (x_1, x_2) : |x_1| \leq l_1, |x_2| \leq l_2 \} \), i.e. the condition \( |L_3((x_1, x_2))| \leq l_3 \) is not redundant, then \( l_3 > l_2 + l_1 \), and conversely.

8. Nonnegative extremizers of Gowers product inequality

Our induction step to higher dimensions will need a complete characterization of an arbitrary nonnegative extremizing tuple \( \vec{f} = (f_\alpha : \alpha \in \{0, 1\}^k) \) of the Gowers product inequality in one dimension. In fact, we obtain a characterization for all dimensions.

Let \( m, n \geq 1 \) be integers. For \( 1 \leq i \leq m \), let \( H, H_i \) be vector spaces, \( B_i : H \to H_i \) be a surjective linear mapping, with \( \ker(B_i) \cap \ker(B_j) = \{0\} \), \( 1 \leq i \neq j \leq m \), and \( 1 \leq p_i \leq \infty \). Then \( (\vec{B}, \vec{p}) = ((B_1, \ldots, B_m), (p_1, \ldots, p_m)) \) is called a Brascamp-Lieb datum [1]. A Brascamp-Lieb inequality is an inequality of the form:

\[
\int_H \prod_{1 \leq i \leq m} f_i \circ B_i(x) \, dx \leq B L(\vec{B}, \vec{p}) \| f_i \|_{L^{p_i}(H_i)}.
\]
BL(\(\tilde{B}, \tilde{p}\)) is the smallest constant such that the above inequality is satisfied for all input tuples \(\tilde{f} = (f_1, \ldots, f_m)\) with measurable \(f_i : H_i \to \mathbb{R}_{\geq 0}\). An extremizing tuple \(\tilde{f}\) is an input tuple with which the equal sign happens. We quote the following result:

**Theorem 8.1.** [1] Let \((\tilde{B}, \tilde{p})\) be an extremizable Brascamp-Lieb datum with \(1 \leq p_i < \infty\) for all \(i\). Suppose also that \(B_i^* \cap B_j^* = \{0\}\) whenever \(1 \leq i < j \leq m\). Then if \(\tilde{f} = (f_i)\) is an extremizing input, then all the \(f_i\) are Gaussians, thus there exist real numbers \(C, c_i > 0\), positive definite transformations \(M_i : H_i \to H_i\), and points \(x_i \in H_i\) such that \(f_i(x) = c_i \exp(-C \langle A_i(x-x_i), (x-x_i) \rangle_{H_i})\). Moreover, \(x_i = B_i w, \) for some \(w \in H\).

Our example of \((\tilde{B}, \tilde{p})\) is \(((B_\alpha : \alpha \in \{0,1\}^k), (p_k, \ldots, p_k))\) with \(B_\alpha : \mathbb{R}^{(k+1)n} \to \mathbb{R}^n\) defined by \(B_\alpha(x, \tilde{h}) = x + \alpha \cdot \tilde{h}\) is a Brascamp-Lieb datum. Hence the quoted result allows us to say, if \(\tilde{f} = (f_\alpha : \alpha \in \{0,1\}^k)\), \(f_\alpha : \mathbb{R}^n \to \mathbb{R}_{\geq 0}\), is a nonnegative extremizing tuple for the inequality,

\[
|T_k(\tilde{f})| = |T_k(f_\alpha : \alpha \in \{0,1\}^k)| \leq A(k, n)^{2k} \prod_{\alpha \in \{0,1\}^k} \|f_\alpha\|_{p_k} = A(k, n)^{2k}
\]

then each \(f_\alpha\) is a Gaussian of the described form, provided that the hypotheses are met. Indeed, for each \(\alpha \in \{0,1\}^k\), define \(B_\alpha^* : \mathbb{R}^n \to \mathbb{R}^{(k+1)n}\) by \(B_\alpha^*(x) = (x, \bar{X}_\alpha)\) with \(\bar{X}_\alpha = (X_1^\alpha, \ldots, X_k^\alpha)\) and \(\mathbb{R}^n \ni X_\alpha^\alpha = x\) if \(\alpha = 1\) and \(X_\alpha^\alpha = 0\) otherwise. It’s easy to check that \(B_\alpha^*\) is indeed the adjoint of \(B_\alpha\) and that \(B_\alpha^* \cap B_\beta^* = \{0\}\), \(\alpha \neq \beta \in \{0,1\}^k\). As indicated by the Gowers-Host-Kra norm inequality, the datum \(((B_\alpha : \alpha \in \{0,1\}^k), (p_k, \ldots, p_k))\) is extremizable. Hence \(f_\alpha(x) = m_\alpha \exp(-C \langle M_\alpha(x-c_\alpha), (x-c_\alpha) \rangle_{\mathbb{R}^n})\), with \(C, m_\alpha > 0\), \(M_\alpha : \mathbb{R}^n \to \mathbb{R}^n\) a positive definite transformation, and \(c_\alpha = c_0 + \alpha \cdot \bar{c}\), for some \((c_0, \bar{c}) \in \mathbb{R}^{(k+1)n}\).

Furthermore, in our case, we claim that there exists a positive definite transformation \(M\) such that \(M_\alpha = M\) for all \(\alpha \in \{0,1\}^k\). To see this, note that if \(\tilde{f} = (f_\alpha : \alpha \in \{0,1\}^k)\) is an extremizing tuple then so is its symmetric rearrangement tuple, \(\tilde{f}^* = (f_\alpha^* : \alpha \in \{0,1\}^k)\); this is simply a consequence of the general rearrangement inequality, \(T_k(\tilde{f}) = T_k(f_\alpha : \alpha \in \{0,1\}^k) \leq T_k(f_\alpha^* : \alpha \in \{0,1\}^k) = T_k(\tilde{f}^*)\) and the fact that \(\|f_\alpha\|_{p_k} = \|f_\alpha^*\|_{p_k}\). It then suffices to assume \(f_\alpha(x) = m_\alpha \exp(-C \langle M_\alpha x, x \rangle_{\mathbb{R}^n})\). For \(k = 2\), the Cauchy-Schwarz inequality, Young’s convolution inequality and the fact that \(A(2, n) = ((C_{4/3}/C_2)^{1/2})^n A(1, n) = ((C_{4/3}/C_2)^{1/2})^n\) - see again Section 2 - give us:

\[
\mathcal{T}_2(f_1, f_2, f_3, f_4) = \int_{\mathbb{R}^n} (f_1 \ast f_2) \cdot (f_3 * f_4)(x) \, dx \leq \|f_1 \ast f_2\|_2 \|f_3 \ast f_4\|_2 \leq A(2)^4 \prod_{i=1}^4 \|f_i\|_{4/3}.
\]

The equal sign is a simple consequence of change of variables. The conclusion for the case \(k = 2\) then follows from the characterization of extremizers of the Cauchy-Schwarz inequality.
and of Young’s convolution inequality. Assume the claim is true for index \( k \) or lower. Then:

\[
T_{k+1}(f_\gamma : \gamma \in \{0,1\}^{k+1}) = \int_{\mathbb{R}^n} T_k(T^h f_{(\alpha,1)} \cdot f_{(\alpha,0)} : \alpha \in \{0,1\}^k) \, dh \\
\leq \int_{\mathbb{R}^n} \prod_{\alpha \in \{0,1\}^k} \|T^h f_{(\alpha,1)} \cdot f_{(\alpha,0)}\|_{U^k} \, dh \\
= A(k,n)^{2k} \int_{\mathbb{R}^n} \prod_{\alpha \in \{0,1\}^k} \|T^h f_{(\alpha,1)} \cdot f_{(\alpha,0)}\|_{p_k} \, dh \\
\leq A(k,n)^{2k} \prod_{\alpha \in \{0,1\}^k} \left( \int_{\mathbb{R}^n} \|T^h f_{(\alpha,1)} \cdot f_{(\alpha,0)}\|_{p_k}^2 \, dh \right)^{1/2k} \\
\leq A(k+1,n)^{2k+1} \prod_{\alpha \in \{0,1\}^k} \|f_{(\alpha,0)}\|_{p_{k+1}} \|f_{(\alpha,1)}\|_{p_{k+1}} \\
= A(k+1,n)^{2k+1} \prod_{\gamma \in \{0,1\}^{k+1}} \|f_{\gamma}\|_{p_{k+1}}.
\]

The first inequality is the Gowers product inequality. The second equality is due to the fact that Gaussians are extremizers of the Gowers-Host-Kra norm inequality. The third inequality follows from Hölder’s inequality and the fourth from the sharp Young’s inequality, as discussed in Section 2. If \((f_\gamma : \gamma \in \{0,1\}^{k+1})\) is an extremizing tuple with \(f_\gamma(x) = m_\gamma \exp(-C\langle M_\gamma x, x \rangle_{\mathbb{R}^n})\), this forces the first and third inequalities in the display to become equalities. Due to the Gowers-Host-Kra norm inequality and the fact that all the integrands involved are continuous, equal sign in the first inequality happens only when for all \( h \in \mathbb{R}^n \) and \( \alpha \in \{0,1\}^k \),

\[
(8.1) \quad T_k(T^h f_{(\alpha,1)} \cdot f_{(\alpha,0)}) = A(k,n)^{2k} \prod_{\alpha \in \{0,1\}^k} \|T^h f_{(\alpha,1)} \cdot f_{(\alpha,0)}\|_{p_k}.
\]

\(T^h f_{(\alpha,1)} \cdot f_{(\alpha,0)}\) is still a Gaussian; hence by the induction hypothesis, (8.1) implies in particular for \( h = 0 \):

\[
(8.2) \quad M_{(\alpha,0)} + M_{(\alpha,1)} = M_{(\beta,0)} + M_{(\beta,1)}
\]

for all \( \alpha, \beta \in \{0,1\}^k \). On the other hand, equal sign in the third inequality gives, for each \( \alpha \in \{0,1\}^k \):

\[
(8.3) \quad \left( \int_{\mathbb{R}^n} \|T^h f_{(\alpha,1)} \cdot f_{(\alpha,0)}\|_{p_k}^2 \, dh \right)^{1/2k} = A(k,n)^{2k} \|f_{(\alpha,0)}\|_{p_k} \|f_{(\alpha,1)}\|_{p_k}.
\]

The characterization of extremizers of the sharp Young’s inequality [10] and (8.3) implies there exist \( m_0, m_1 \in \mathbb{R}_{>0} \), a positive definite transformation \( M : \mathbb{R}^n \to \mathbb{R}^n \) and \( c \in \mathbb{R}^n \) such that \( \exp(-p_k \langle M(x-c), (x-c) \rangle_{\mathbb{R}^n}) = m_0 f_{(\alpha,0)}(x) = m_1 f_{(\alpha,1)}(x) \), which in turns implies

\[
(8.4) \quad M_{(\alpha,0)} = M_{(\alpha,1)} = M
\]

for all \( \alpha \in \{0,1\}^k \). Then (8.2) and (8.4) conclude \( M_\gamma = M_{\gamma'} \) for all \( \gamma, \gamma' \in \{0,1\}^{k+1} \) and hence the induction step. In particular, we obtain:

**Corollary 8.2.** Let \( k \geq 2, n \geq 1 \) be integers and \( \vec{f} = (f_\alpha : \alpha \in \{0,1\}^k) \) with measurable \( f_\alpha : \mathbb{R}^n \to \mathbb{R}_{\geq 0}, \|f_\alpha\|_{p_k} = 1 \). If

\[
T_k(\vec{f}) = \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} f_\alpha(x + \alpha \cdot \vec{h}) \, dx d\vec{h} = A(k,n)^{2k} \prod_{\alpha \in \{0,1\}^k} \|f_\alpha\|_{p_k}
\]
then \( f_\alpha = m \exp(-\langle M(x - c_\alpha), (x - c_\alpha) \rangle R^n) \), with \( m > 0 \), \( M : R^n \to R^n \) a positive definite transformation, and \( c_\alpha = c_0 + \alpha \cdot \vec{c} \), for some \((c_0, \vec{c}) \in R^{(k+1)n}\).

**Remark 8.1.** Another proof for the fact that \( c_\alpha = c_0 + \alpha \cdot \vec{c} \), \( \alpha \in \{0, 1\}^k \), for some \((c_0, \vec{c}) \in R^{k+1}\) (hence only applicable for \( n = 1 \)) is as follows. If \( f_\alpha(x) = m \exp(-a(x - c_\alpha)^2) \) and

\[
T_k(\vec{f}) = \int T_k(1_{F_{\alpha,v}} : \alpha \in \{0, 1\}^k) \, dv = A(k)^2k = T_k(\vec{f}^*) = \int T_k(1_{F_{\alpha,v}} : \alpha \in \{0, 1\}^k) \, dv
\]

then this entails \( T_k(1_{F_{\alpha,v}} : \alpha \in \{0, 1\}^k) = T_k(1_{F_0,v} : \alpha \in \{0, 1\}^k) \) for all \( v \in (0, m) \). Note that \((F_{\alpha,v})^* = F_0^v\). Fix \( v \) and let \( L(F_{\alpha,v}) = L(F_0^v) = l_\alpha \). Then apply Lemma 7.5 to the admissible tuple \((B_\alpha, l_\alpha)_{\alpha \in \{0, 1\}^k}\) to obtain the desired conclusion.

### 8.1. Gaussian near extremizers in one dimension.

We now characterize Gaussian near extremizers of the Gowers product inequality in one dimension. Given a Gaussian tuple \( \vec{f} = (f_\alpha = c_\alpha^{1/2p_k} \exp(-a_\alpha(x - c_\alpha)^2) : \alpha \in \{0, 1\}^k) \), so that \( \|f_\alpha\|_{p_k} = 1 \). Suppose that for some \( \delta > 0 \) small,

\[
T_k(\vec{f}) = T_k(f_\alpha : \alpha \in \{0, 1\}^k) \geq (1 - \delta)A(k)^2k \prod_{\alpha \in \{0, 1\}^k} \|f_\alpha\|_{p_k} = (1 - \delta)A(k)^2k.
\]

We claim that there exist \( a, \Gamma = \Gamma(k) > 0 \) and a nonnegative function \( \eta = \eta_k(\delta) \) that is increasing for small values of \( \delta \) such that \( |a_\alpha/a - \Gamma| \leq \eta_\alpha \) for all \( \alpha \in \{0, 1\}^k \). As before, we can first assume that \( c_\alpha = 0 \) for all \( \alpha \in \{0, 1\}^k \). We start with the induction step for the case \( k + 1 \) and assume the claim is true for index \( k \) or lower. If:

\[
T_{k+1}(f_\beta : \beta \in \{0, 1\}^{k+1}) \geq (1 - \delta)A(k + 1)^2k + 1.
\]

It then follows from the calculations in Section 3 that there exists \( c(k) > 0 \) such that for each \( \alpha \in \{0, 1\}^k \),

\[
\|f_{(\alpha,0)}^{p_k} \ast f_{(\alpha,1)}^{p_k}\|_{k+1}^{k+1} \geq (1 - c(k)\delta)Y(k + 1)^{k+1} \|f_{(\alpha,0)}^{p_k}\|_q \|f_{(\alpha,1)}^{p_k}\|_q^{k+1}.
\]

Here, \( q = p_{k+1}/p_k \) and \( Y(k + 1) \) is the optimal constant of Young's convolution inequality for the involved exponents. Since all the functions involved are centered Gaussians in one dimension and \( p_k = 2k/(k + 1) \), it's a simple calculation to show that there exist \( \delta_0 > 0 \) sufficiently small, a function \( \eta = \eta_k(\delta) \) that is increasing for \( 0 < \delta \leq \delta_0 \) and \( \Gamma = \Gamma(k) \), such that \( |a_{(\alpha,0)}/a_{(\alpha,1)} - \Gamma| \leq \eta \), for all \( \alpha \in \{0, 1\}^k \). By symmetry, this means that if \( \beta, \beta' \in \{0, 1\}^{k+1} \) are such that \( \beta_i = \beta'_i \) for all but one single index \( i \in \{1, \ldots, k + 1\} \), then \( |a_{\beta}/a_{\beta'} - \Gamma| \leq \eta \); in other words, there exists \( \alpha > 0 \) such that \( |a_{\beta}/a - \Gamma| \leq \eta \) for all \( \beta \in \{0, 1\}^{k+1} \). The case \( k = 2 \) is proved using similar arguments.

We now claim that if \( \delta \) is small enough then there exist \( c_0 \in R \) and \( \vec{c} \in R^k \) such that \( |c_\alpha - (c_0 + \alpha \cdot \vec{c})| = o(\delta) \). To see this, we now assume \( a_\alpha = a > 0 \) for all \( \alpha \in \{0, 1\}^k \) as permitted by the above reasoning. For each \( \alpha \in \{0, 1\}^k \), let \( \xi_\alpha \in R^{k+1} \) be such that \( B_\alpha(\xi_\alpha) = c_\alpha \). Define \( R^{k+1} \ni \vec{T} \) so that \( \sum_{\alpha \in \{0, 1\}^k} B_\alpha B_\alpha \vec{T} = \sum_{\alpha \in \{0, 1\}^k} B_\alpha B_\alpha \vec{c}_\alpha \) (it's an easy calculation that \( |\text{det} \sum_{\alpha \in \{0, 1\}^k} B_\alpha B_\alpha| > 0 \)). Here, \( B_\alpha \) is defined as above with \( n = 1 \). Then by a change of
functions of nearl y maximal Gowers-Host-Kra norms on Euclidean spaces

variables:
\[
\int_{\mathbb{R}^{k+1}} \prod_{\alpha \in \{0,1\}^k} f_{\alpha} \circ B_\alpha(\vec{x}) \, d\vec{x} = C \int_{\mathbb{R}^{k+1}} \exp \left\{ -a \sum_{\alpha \in \{0,1\}^k} (B_\alpha(\vec{x} - \vec{\xi}_\alpha))^2 \right\} d\vec{x}
\]

\[
= C \int_{\mathbb{R}^{k+1}} \exp \left\{ -a \sum_{\alpha \in \{0,1\}^k} (B_\alpha(\vec{x} - (\vec{\xi}_\alpha - \vec{T}))^2 \right\} d\vec{x}
\]

\[
= C \exp \left\{ -a \sum_{\alpha \in \{0,1\}^k} (B_\alpha(\vec{\xi} - \vec{T}))^2 \right\} \int_{\mathbb{R}^{k+1}} \exp \left\{ -a \sum_{\alpha \in \{0,1\}^k} (B_\alpha(\vec{x})^2 \right\} d\vec{x}
\]

(8.5)

\[
\leq C \int_{\mathbb{R}^{k+1}} \exp \left\{ -a \sum_{\alpha \in \{0,1\}^k} (B_\alpha(\vec{x})^2 \right\} d\vec{x}.
\]

Note that by definition of \( \vec{T} \), \( \int_{\mathbb{R}^{k+1}} \exp \left\{ 2a \sum_{\alpha \in \{0,1\}^k} B_\alpha(\vec{x} \cdot B_\alpha(\vec{\xi} - \vec{T}) \right\} d\vec{x} = 1 \), hence the last equality in (8.5) follows. It’s also now clear from (8.5) and the fact that its last expression is \( \|c \exp(-a \vec{x}^2)\|_{L^k} = \|A(\vec{k})\|^{2k} \), that if \( T_k(\vec{f}) = \int_{\mathbb{R}^{k+1}} \prod_{\alpha \in \{0,1\}^k} f_{\alpha} \circ B_\alpha(\vec{x}) \, d\vec{x} \) is nearing its optimal value then \( |c_\alpha - B_\alpha(\vec{T})| = o(\delta) \) for all \( \alpha \in \{0,1\}^k \). Hence the claim follows.

**Remark 8.2.** If \( \vec{f} = (f_\alpha : \alpha \in \{0,1\}^k) \) is such that
\[
T_k(\vec{f}) \geq (1 - \delta) A(\vec{k})^2 \prod_{\alpha \in \{0,1\}^k} \|f_\alpha\|_{P_k},
\]

then it follows from the Gowers-Cauchy-Schwarz inequality that for each \( \alpha \in \{0,1\}^k \), \( \|f_\alpha\|_{L^k} \geq (1 - o(\delta)) A(\vec{k}) \|f_\alpha\|_{P_k} \). If furthermore, \( f_\alpha \geq 0 \) then by the result for dimension one, there exists a Gaussian \( g_\alpha \) such that \( \|g_\alpha - f_\alpha\|_{P_k} = o(\delta) \|f_\alpha\|_{P_k} \). Then from the Gowers product inequality, \( \vec{g} = (g_\alpha : \alpha \in \{0,1\}^k) \) is also a near extremizing tuple: \( T_k(\vec{g}) \geq (1 - o(\delta)) A(\vec{k})^2 \prod_{\alpha \in \{0,1\}^k} \|g_\alpha\|_{P_k}, \)

and the analytic descriptions of the \( g_\alpha \) are given above. We note that all of these arguments can be generalized to higher dimensions. For now, a characterization of nonnegative near extremizing tuples for the Gowers product inequality is sufficient for an induction process in Section 9 below. We also note that the arguments given in this section are stronger in the sense that they establish analytic properties of near extremizers, not just extremizers.

9. Extension to higher dimensions

9.1. An additive relation. We first quote a result in [7]:

**Proposition 9.1.** [7] Let \( n \geq 1 \). There exists a positive constant \( K = K(n) > 0 \) with the following property. Let \( B \) be a ball of positive, finite radius. Let \( \alpha, \beta, \gamma : \mathbb{R}^n \to \mathbb{C} \) be measurable functions. Let \( \tau \in (0, \infty) \) and \( \delta \in (0, 1] \). Suppose that,
\[
\mathcal{L} \left( \{ (x, y) \in B^2 : |\alpha(x) + \beta(y) - \gamma(x + y)| > \tau \} \right) < \delta(\mathcal{L}(B))^2.
\]

Then there exists an affine function \( L : \mathbb{R}^n \to \mathbb{C} \) such that
\[
\mathcal{L} \left( \{ x \in B : |\alpha(x) - L(x)| > K\tau \} \right) < K\delta\mathcal{L}(B).
\]

Using this proposition, we prove the following:

**Proposition 9.2.** Let \( n \geq 1 \) and \( l \geq 2 \) be an integer. Let \( C_i > 0 \) be such that \( C_i \sim C_j \) for \( i, j \in \{1, \ldots, l\} \). There exists \( K > 0 \) with the following property. Let \( B_i \) be a ball in \( \mathbb{R}^n \) such
that $\mathcal{L}(B_i) = C_i$ and let $f_i : \mathbb{R}^n \to \mathbb{C}$ be a measurable function, $i \in \{1, \ldots, l\}$. Let $\tau \in (0, \infty)$ and $\delta \in (0, 1]$. Suppose that,

$$
|f_{i+1}(x_1 + \ldots + x_l) - \sum_{i=1}^l f_i(x_i)| \leq \tau
$$

(9.1)

for all $(x_1, \ldots, x_l) \in B_1 \times \ldots \times B_l$ outside a subset of measure $\delta \prod_{i=1}^l \mathcal{L}(B_i)$. Then there exist an affine function $a : \mathbb{R}^n \to \mathbb{C}$ and a positive function $\eta$ satisfying $\lim_{\delta \to 0} \eta(\delta) = 0$, such that

$$
|f_1(x) - a(x)| \leq K\tau
$$

for all $x \in B_1$ outside a subset of measure $K\eta(\delta)\mathcal{L}(B_1)$.

Proof. Since the conclusion doesn’t change after applying a finite number of translations and dilations, we assume that $B_1 = \ldots = B_l = B$ - here $B$ is a ball of positive radius. Then (9.1) gives that, for all $(x_1, \ldots, x_l) \in \times_{i=1}^l B$ outside a subset of measure $\delta(\mathcal{L}(B))^l$, the following holds:

$$
f_{i+1}(x_1 + x_2 + \sum_{i=3}^l x_i) - \sum_{i=3}^l f_i(x_i) = f_1(x_1) + f_2(x_2) + O(\tau).
$$

(9.2)

That means there exists a positive function $\eta$ satisfying $\lim_{\delta \to 0} \eta(\delta) = 0$ such that the following holds. For each $(x_3, \ldots, x_l) \in \times_{i=3}^l B$ outside a subset of measure at most $\eta(\delta)(\mathcal{L}(B))^l - 2$, (9.2) is true for all $(x_1, x_2) \in B \times B$ except for a subset of measure at most $\eta(\delta)(\mathcal{L}(B))^2$. Take such a point $(x_3, \ldots, x_l) \in \times_{i=3}^l B$. For this point, define $\tilde{f} : \mathbb{R}^n \to \mathbb{C}$ by $\tilde{f}_{i+1}(u) = f_{i+1}(u + \sum_{i=3}^l x_i) - \sum_{i=3}^l f_i(x_i)$. Then (9.2) becomes:

$$
\tilde{f}_{i+1}(x_1 + x_2) = f_1(x_1) + f_2(x_2) + O(\tau)
$$

for all $(x_1, x_2) \in B \times B$ outside a subset of measure at least $l(1 - \eta(\delta))(\mathcal{L}(B))^2$. Now apply Proposition 9.1 to $\tilde{f}_{i+1}, f_1, f_2$ - and an appropriate translation and dilation if necessary - to obtain an affine function $\eta : \mathbb{R}^n \to \mathbb{C}$ such that $f_1(x) = a(x) + O(\tau)$ for all $x \in B_1$ outside a subset of measure $K\eta(\delta)\mathcal{L}(B_1)$.

9.2. Extension to higher dimensions. Let $n \geq 1$. Assume Theorem 1.1 is true for non-negative functions and for dimensions $n$ and lower. Let $f(x, s) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a $(1 - \delta)$ near extremizer of the Gowers-Host-Kra norm inequality. Assume that $\|f\|_{p_k} = 1$. Define $F : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ by $F(x) = \|f(x, \cdot)\|_{p_k}$. Define $\tilde{f}_x : \mathbb{R} \to \mathbb{R}_{\geq 0}$ by $\tilde{f}_x(s) = \frac{f(x, s)}{F(x)}$ if $F(x) \neq 0$, and $\tilde{f}_x(s) = 0$ if $F(x) \in \{0, \infty\}$ - which happens only for a null subset of $\text{spt}(F) \subset \mathbb{R}^n$, outside of which, $\|\tilde{f}_x\|_{p_k} = \|f\|_{p_k} = 1$. Thus, $\|F\|_{p_k} = \|f\|_{p_k} = 1$. From the definition of the Gowers-Host-Kra norms and Fubini’s theorem,

$$
\|f\|^2_{U_k} = \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0, 1\}^k} F(x + \alpha \cdot \vec{h}) \mathcal{T}_k f_{x+\alpha \vec{h}} : \alpha \in \{0, 1\}^k \, dx dh.
$$

The Gowers-Cauchy-Schwarz inequality then gives,

$$
(1 - \delta)A(k, n + 1)^{2k} \leq \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0, 1\}^k} F(x + \alpha \cdot \vec{h}) \mathcal{T}_k f_{x+\alpha \vec{h}} : \alpha \in \{0, 1\}^k \, dx dh
$$

(9.3)

$$
\leq A(k)^{2k} \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0, 1\}^k} F(x + \alpha \cdot \vec{h}) \, dx dh = A(k)^{2k} \|F\|^2_{U_k}.
$$

Since $A(k, m) = A(k)^m$, (9.3) implies $\|F\|^2_{U_k} \geq (1 - \delta)A(k, n)^{2k}$. Then by the inductive assumption, there exists a Gaussian $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that $\|\mathcal{F}\|_{p_k} = 1$ and $\|\mathcal{F} - F\|_{p_k} = o(\delta)$. 


Let $B_R \subset \mathbb{R}^n$ denote a centered ball of radius $R$. Take $\eta > 0$ small. There exist $\delta = \delta(\eta) > 0$ small and $B_R$ with $R = R(\eta) > 0$ large such that $\delta \to 0$ and $R \to \infty$ as $\eta \to 0$ with the following extra property. Let $\mathcal{F}^\downarrow$ denote the standard centered Gaussian on $\mathbb{R}^n$. Suppose $\mathcal{F}^\downarrow$ is such that $\|\mathcal{F}^\downarrow\|_{p_k} = 1$ and $\|\mathcal{F}^\downarrow - \mathcal{F}\|_{p_k} \leq \delta$. Then:

$$\int_{\mathbb{R}^{(k+1)n} \setminus B_k^{R+1}} \prod_{\alpha \in \{0,1\}^k} \mathcal{F}^\downarrow(x + \alpha \cdot \vec{h}) \, dx \, d\vec{h} = \int_{\mathbb{R}^{(k+1)n} \setminus B_k^{R+1}} \prod_{\alpha \in \{0,1\}^k} \mathcal{F}^\downarrow(x + \alpha \cdot \vec{h}) \, dx \, d\vec{h} < \eta.$$  

(9.4)

Indeed, since $\mathcal{F}^\downarrow$ is the standard centered Gaussian, for every $\eta > 0$, there exists $B_R$ with $R$ sufficiently large so that:

$$\int_{\mathbb{R}^{(k+1)n} \setminus B_k^{R+1}} \prod_{\alpha \in \{0,1\}^k} \mathcal{F}^\downarrow(x + \alpha \cdot \vec{h}) \, dx \, d\vec{h} = \int_{\mathbb{R}^{(k+1)n} \setminus B_k^{R+1}} \prod_{\alpha \in \{0,1\}^k} \mathcal{F}^\downarrow(x + \alpha \cdot \vec{h}) \, dx \, d\vec{h} < \eta/2.$$  

(9.5)

Applying the Gowers product inequality to have:

$$\int_{\mathbb{R}^{(k+1)n} \setminus B_k^{R+1}} \prod_{\alpha \in \{0,1\}^k} |\mathcal{F}^\downarrow - \mathcal{F}| \chi_{\mathbb{R}^n \setminus B_R}(x + \alpha \cdot \vec{h}) \, dx \, d\vec{h} \leq A(k,n)^{2k} \prod_{\alpha \in \{0,1\}^k} \|\mathcal{F}^\downarrow - \mathcal{F}\|_{p_k} < A(k,n)^{2k} \delta.$$  

(9.6)

Simply take $\delta$ small enough so that $A(k,n)^{2k} \delta < \eta/2$. Then (9.4) follows from (9.5) and (9.6).

Suppose now that our Gaussian $\mathcal{F}$ is indeed the standard centered one; this can be done by applying an affine transformation and scaling by a suitable constant. Then there exists a sufficiently large $R$ so that:

$$\int_{\mathbb{R}^{(k+1)n} \setminus B_k^{R+1}} \prod_{\alpha \in \{0,1\}^k} F(x + \alpha \cdot \vec{h}) \, dx \, d\vec{h} < \eta.$$  

(9.7)

Now by the Gowers product inequality:

$$\mathcal{T}_k(f_{x + \alpha \cdot \vec{h}} : \alpha \in \{0,1\}^k) \leq A(k)^{2k} \prod_{\alpha \in \{0,1\}^k} \|f_{x + \alpha \cdot \vec{h}}\|_{p_k} = A(k)^{2k}$$  

which then with (9.7), implies

$$\int_{\mathbb{R}^{(k+1)n} \setminus B_k^{R+1}} \prod_{\alpha \in \{0,1\}^k} F(x + \alpha \cdot \vec{h}) \mathcal{T}_k(f_{x + \alpha \cdot \vec{h}} : \alpha \in \{0,1\}^k) \, dx \, d\vec{h} < \eta A(k)^{2k}.$$  

(9.8)

Note that if $R \to \infty$ and $\delta \to 0$ then $\eta = \eta(R,\delta) \to 0$ in (9.7) and (9.8). We’ve set up the case for restricting our analysis in $B_k^{R+1}$ with $R$ big enough.

**Lemma 9.3.** Let $f, F, f_x, R$ be as above. Then there exist $\Omega \subset B_k^{R+1}$ and $\omega \subset B_R$ such that $\mathcal{L}(\Omega) + \mathcal{L}(\omega) = o(\delta)$ satisfying:

For $(x, \vec{h}) \in B_k^{R+1} \setminus \Omega$ and $\alpha \in \{0,1\}^k$,

$$\|\phi^\omega_{x + \alpha \cdot \vec{h}} - f_{x + \alpha \cdot \vec{h}}\|_{p_k} = o(\delta)$$  

with $\phi^\omega_{x + \alpha \cdot \vec{h}}(s) = ca(x + \alpha \cdot \vec{h})^{1/2p} \exp(-a(x + \alpha \cdot \vec{h})(s - d(x + \alpha \cdot \vec{h}))^2)$. The functions $a : \mathbb{R}^n \to \mathbb{R}_{>0}$ and $d : \mathbb{R}^n \to \mathbb{R}$ are measurable and satisfy the following properties:

There exists a scalar $a > 0$ such that for $x \in B_R \setminus \omega$,

$$|a(x) - a| = o(\delta).$$  

(9.10)

For $(x, \vec{h}) \in B_k^{R+1} \setminus \Omega$,

$$|d(x + \alpha \cdot \vec{h}) - (1, \alpha) \cdot (d(x), d(h_1), \ldots, d(h_k))| = o(\delta).$$  

(9.11)
Lastly,

\[(9.12) \quad \int_{\Omega} \prod_{\alpha \in \{0,1\}^k} F(x + \alpha \cdot \vec{h}) \, dx d\vec{h} = o(\delta).\]

**Proof.** From the near extremizing hypothesis and the Gowers-Host-Kra norm inequality,

\[
\|f\|_{U^k}^{2k} = \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} F(x + \alpha \cdot \vec{h}) \mathcal{T}_k(f_{x+\alpha \cdot \vec{h}} : \alpha \in \{0,1\}^k) \, dx d\vec{h}
\geq (1 - \delta) A(k, n + 1)^{2k} = (1 - \delta) A(k)^{2k} A(k, n)^{2k} \|F\|_{p_k}^{2k}
\geq (1 - \delta) A(k)^{2k} \|F\|_{U^k}^{2k} = (1 - \delta) A(k)^{2k} \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} F(x + \alpha \cdot \vec{h}) \, dx d\vec{h}.
\]

(9.13)

To take \(R\) appropriately so that, as in (9.7) and (9.8),

\[(9.14) \quad \int_{\mathbb{R}^{(k+1)n} \backslash \tilde{B}_R^{k+1}} \prod_{\alpha \in \{0,1\}^k} F(x + \alpha \cdot \vec{h}) \, dx d\vec{h} = o(\delta)\]

and (9.15)

\[(9.15) \quad \int_{\mathbb{R}^{(k+1)n} \backslash \tilde{B}_R^{k+1}} \prod_{\alpha \in \{0,1\}^k} F(x + \alpha \cdot \vec{h}) \mathcal{T}_k(f_{x+\alpha \cdot \vec{h}} : \alpha \in \{0,1\}^k) \, dx d\vec{h} = o(\delta).\]

There might still exist \(\Omega_0 \subset \tilde{B}_R^{k+1}\) such that,

\[(9.16) \quad \int_{\Omega_0} \prod_{\alpha \in \{0,1\}^k} F(x + \alpha \cdot \vec{h}) \, dx d\vec{h} = o(\delta)\]

and hence, due to the Gowers product inequality,

\[(9.17) \quad \int_{\Omega_0} \prod_{\alpha \in \{0,1\}^k} F(x + \alpha \cdot \vec{h}) \mathcal{T}_k(f_{x+\alpha \cdot \vec{h}} : \alpha \in \{0,1\}^k) \, dx d\vec{h} = o(\delta) A(k)^{2k}.
\]

Since \(F\) is \(o(\delta)\) close in \(L^{p_k}\) norm to the standard centered Gaussian on \(\mathbb{R}^n\), \(\mathcal{F}\), (9.16) implies a similar inequality for \(\mathcal{F}\):

\[\int_{\Omega_0} \prod_{\alpha \in \{0,1\}^k} \mathcal{F}(x + \alpha \cdot \vec{h}) \, dx d\vec{h} = o(\delta)\]

which then implies,

\[\mathcal{L}(\Omega_0) = o(\delta).\]

Now (9.13), (9.14), (9.15) and (9.17) imply that \(\Omega_0\) also has the following properties:

\[(9.18) \quad \int_{\tilde{B}_R^{k+1} \backslash \Omega_0} \prod_{\alpha \in \{0,1\}^k} F(x + \alpha \cdot \vec{h}) \mathcal{T}_k(f_{x+\alpha \cdot \vec{h}} : \alpha \in \{0,1\}^k) \, dx d\vec{h}
\geq (1 - o(\delta)) A(k)^{2k} \int_{\tilde{B}_R^{k+1} \backslash \Omega_0} \prod_{\alpha \in \{0,1\}^k} F(x + \alpha \cdot \vec{h}) \, dx d\vec{h}.
\]

Since \(\mathcal{T}_k(f_{x+\alpha \cdot \vec{h}} : \alpha \in \{0,1\}^k) \leq A(k)^{2k}\) for all \((x, \vec{h}) \in \mathbb{R}^{(k+1)n}\), by the Gowers inner product inequality, (9.17) and (9.18) imply that for a.e \((x, \vec{h}) \in \tilde{B}_R^{k+1} \backslash \Omega_0,\)

\[\mathcal{T}_k(f_{x+\alpha \cdot \vec{h}} : \alpha \in \{0,1\}^k) \geq (1 - o(\delta)) A(k)^{2k}\]

which, by the Gowers-Cauchy-Schwarz inequality, entails

\[\prod_{\alpha \in \{0,1\}^k} \|f_{x+\alpha \cdot \vec{h}}\|_{U^k} \geq \mathcal{T}_k(f_{x+\alpha \cdot \vec{h}} : \alpha \in \{0,1\}^k) \geq (1 - o(\delta)) A(k)^{2k} \prod_{\alpha \in \{0,1\}^k} \|f_{x+\alpha \cdot \vec{h}}\|_{p_k}\]
which then gives that, for each $\alpha \in \{0,1\}^k$ and a.e. $(x, \vec{h}) \in \tilde{B}_R^{k+1} \setminus \Omega_0$:

$$
\|f_{x+\alpha \vec{h}}\|_{L^p} \geq (1 - o(\delta)) A(k) \|f_{x+\alpha \vec{h}}\|_{p_k}.
$$

(9.19)

Excluding a null subset if necessary, then the inductive hypothesis for dimension $n = 1$ and (9.19) imply that, if $(x, \vec{h}) \in \tilde{B}_R^{k+1} \setminus \Omega_0$ then $f_{x+\alpha \vec{h}}$ is $o(\delta)$ close in $L^{p_k}$ norm to a Gaussian

$$
\phi_{x+\alpha \vec{h}}(s) = c a(x + \alpha \cdot \vec{h})^{1/2p} \exp(-a(x + \alpha \cdot \vec{h})(s - d(x + \alpha \cdot \vec{h}))^2).
$$

For $x \in B_R$, let $E^x = \{ \vec{h} \in \tilde{B}_R^{k+1} : (x, \vec{h}) \in \tilde{B}_R^{k+1} \setminus \Omega_0 \}$ and $\omega_0 = \{ x \in B_R : \mathcal{L}(E^x) \leq (1 - \delta') (\mathcal{L}(B_R))^k \}$, with $\delta' = \delta'(\delta)$ satisfying $\delta' \to 0$ sufficiently slow compared to $\delta \to 0$ so that $\mathcal{L}(\omega_0) = o(\delta) \mathcal{L}(B_R)$ and for every $x \in B_R \setminus \omega_0$, $\mathcal{L}(E^x) \geq (1 - o(\delta))(\mathcal{L}(B_R))^k$. From the definitions of $\omega_0$ and $\Omega_0$, if $x \in B_R \setminus \omega_0$ and $\vec{h} \in E^x$, we have a decomposition,

$$
f_{x+\alpha \vec{h}} = \phi_{x+\alpha \vec{h}} + \rho_{x+\alpha \vec{h}},
$$

(9.20)

for some Gaussian $\phi_{x+\alpha \vec{h}}(s) = c a(x + \alpha \cdot \vec{h})^{1/2p} \exp(-a(x + \alpha \cdot \vec{h})(s - d(x + \alpha \cdot \vec{h}))^2)$, and $\|\rho_{x+\alpha \vec{h}}\|_{p_k} = o(\delta)$. This decomposition can be done so that $(x, \vec{h}) \mapsto a(x + \alpha \cdot \vec{h})$, $(x, \vec{h}) \mapsto d(x + \alpha \cdot \vec{h})$, $(x, \vec{h}, s) \mapsto \rho_{x+\alpha \vec{h}}(s)$ are all measurable. Take $x \in B_R \setminus \omega_0$ and $\vec{h} \in E^x$. By (9.19), (9.20) and the Gowers product inequality:

$$
\mathcal{T}_k(\phi_{x+\alpha \vec{h}} : \alpha \in \{0,1\}^k) \geq (1 - o(\delta)) A(k)^{2k}.
$$

(9.21)

By the result in Subsection 8.1, it follows from (9.21) that for every $\alpha \in \{0,1\}^k$, $x \in B_R \setminus \omega_0$ and $\vec{h} \in E^x$,

$$
a(x) = a(x + \alpha \cdot \vec{h}) + o(\delta)
$$

(9.22)

$$
|d(x + \alpha \cdot \vec{h}) - (1, \alpha) \cdot (d(x), d(h_1), \ldots, d(h_k))| = o(\delta).
$$

(9.23)

Since $\mathcal{L}(B_R \setminus \omega_0) = (1 - o(\delta)) \mathcal{L}(B_R)$, there exists $x_0 \in B_R \setminus \omega_0$ such that $|x_0| = o(\delta)$. Moreover, $\mathcal{L}(E^{x_0}) \geq (1 - o(\delta))(\mathcal{L}(B_R))^k+1$ implies that the set $V$ of values $x_0 + \alpha \cdot \vec{h}$ for some $\alpha \in \{0,1\}^k$ and $\vec{h} \in E^{x_0}$ must take up a measure of $(1 - o(\delta))(\mathcal{L}(B_R))$ in $B_R$. We define $\omega$ to be the complement of the intersection of $B_R \setminus \omega_0$ and $V$. Then by (9.22), for every $x \in B_R \setminus \omega$,

$$
a(x) = a(x_0) + o(\delta) = a + o(\delta)
$$

(9.24)

for some $a > 0$. Define $\Omega$ by letting $\tilde{B}_R^{k+1} \setminus \Omega$ to be the set $(x, \vec{h}) \in \tilde{B}_R^{k+1}$ with $x \in \omega$ and $\vec{h} \in E^x$, then $\mathcal{L}(\Omega) + \mathcal{L}(\omega) = o(\delta)$. Hence (9.16) is still retained with $\Omega$ in place of $\Omega_0$:

$$
\int \Omega \prod_{\alpha \in \{0,1\}^k} F(x + \alpha \cdot \vec{h}) dx d\vec{h} = o(\delta)
$$

which is (9.12). Moreover, if $\alpha \in \{0,1\}^k$ and $(x, \vec{h}) \in \tilde{B}_R^{k+1} \setminus \Omega$,

$$
\|\rho_{x+\alpha \vec{h}}\|_{p_k} = o(\delta)
$$

which is (9.9). Finally, (9.23) and (9.24) are (9.11) and (9.10), respectively. Hence this completes the proof of Lemma 9.3.

Remark 9.1. It’s possible to select a decomposition in (9.20) so that $(x, \vec{h}) \mapsto a(x + \alpha \cdot \vec{h})$, $(x, \vec{h}) \mapsto d(x + \alpha \cdot \vec{h})$, $(x, \vec{h}, s) \mapsto \rho_{x+\alpha \vec{h}}(s)$ are all measurable. Let $\delta > 0$ be as above, then since $\mathcal{L}(\omega) = o(\delta)$, at each $x \in B_R \setminus \omega$, we can define $\phi_x = ca(x)^{1/2p} \exp(-a(x)(s - d(x))^2)$ in a way that both $a(x)$ and $d(x)$ are locally piecewise constant in a sufficiently small neighborhood $N_x$ of $x$, so that $\|\phi_y - f_y\|_{p_k} = o(\delta)$ for all $y \in N_x$. Then it’s clear that the decomposition $f_x = \phi_x + \rho_x$ satisfies the conditions $\|\rho_x\|_{p_k} = o(\delta)$ and $x \mapsto \rho$ is measurable, if $x \in B_R \setminus \omega$. □
Theorem 9.4. Let $f(x, s) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}_{>0}$ be a $(1 - \delta)$ near extremizer of the Gowers-Host-Kra norm inequality. Then there exists a Gaussian $G : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}_{>0}$ such that $\|G\|_{p_k} = 1$ and $\|G - f\|_{p_k} = o(\delta)$.

Proof. Let $\psi_x(s) = ca^{1/2p}\exp(-a(s - d(x))^2)$, with $a > 0$ and the function $d$ be as above; $x \in B_R$. By (9.21), (9.24) and the continuity of exponential functions, for $x \in B_R \setminus \omega$,

$$
(9.25) \quad \|\psi_x - f_x\|_{p_k} = o(\delta).
$$

An application of Proposition 9.2 to (9.11) gives for every $x \in B_R$ except for a subset of measure $o(\delta)\mathcal{L}(B_R)$:

$$
(9.26) \quad d(x) = L(x) + o(\delta).
$$

Here $L$ is an affine function on $\mathbb{R}^n$, which we can take to be real-valued, as is the center function $d$. Let $\omega'$ be the union of $\omega$ and this new subset of measure $o(\delta)\mathcal{L}(B_R)$; it’s clear $\mathcal{L}(\omega') = o(\delta)$. Let $z_x(s) = ca^{1/2p}\exp(-a(s - L(x))^2)$. Then (9.26) implies for $x \in B_R \setminus \omega'$:

$$
(9.27) \quad \|\psi_x - z_x\|_{p_k} = o(\delta).
$$

(9.25) and (9.27) then give $\|f_x - \psi_x\|_{p_k} = o(\delta)$, if $x \in B_R \setminus \omega$. Let $G(x, s) = F(x)z_x(s)$. It’s clear that $\|G\|_{p_k} = 1$. Recall that $f(x, s) = F(x)f_x(s)$. Then:

$$
(9.28) \quad \|G - f\|_{p_k} \leq C \int_{\mathbb{R}^n} F^{pk}(x)\|f_x - \psi_x\|_{p_k} dx + C \int_{\mathbb{R}^n} F^{pk}(x)\|\psi_x - z_x\|_{p_k} dx + C\|\mathcal{F} - F\|_{p_k}.
$$

The presence of the last term in (9.28) is due to the fact that $z_x(s)$ is a Schwartz function in terms of $s$ and of $L^\infty$ in terms of $x$. Moreover, since $\|\mathcal{F} - F\|_{p_k} = o(\delta)$, the contribution of this last term is that of size $o(\delta)$ in absolute value. For the first term and the second term, we split them as follows, respectively:

$$
(9.29) \quad \int_{\mathbb{R}^n} F^{pk}(x)\|f_x - \psi_x\|_{p_k} dx \leq \int_{B_R \setminus \omega'} |F^{pk}(x) - F^{pk}(x)|\|f_x - \psi_x\|_{p_k} dx
$$

$$
+ \int_{B_R \setminus \omega'} \mathcal{F}^{pk}(x)\|f_x - \psi_x\|_{p_k} dx + C \int_{(\mathbb{R}^n \setminus B_R) \cup \omega'} \mathcal{F}^{pk}(x) dx
$$

and,

$$
(9.30) \quad \int_{\mathbb{R}^n} F^{pk}(x)\|\psi_x - z_x\|_{p_k} dx \leq \int_{B_R \setminus \omega'} |F^{pk}(x) - F^{pk}(x)|\|\psi_x - z_x\|_{p_k} dx
$$

$$
+ \int_{B_R \setminus \omega'} \mathcal{F}^{pk}(x)\|\psi_x - z_x\|_{p_k} dx + C \int_{(\mathbb{R}^n \setminus B_R) \cup \omega'} \mathcal{F}^{pk}(x) dx.
$$

The first and second terms in (9.29) and (9.30) has the size of $o(\delta)$ in absolute value due to (9.25), (9.27) and the fact that $\|\mathcal{F} - F\|_{p_k} = o(\delta)$. The third terms can be further dominated by the sum $C \int_{(\mathbb{R}^n \setminus B_R) \cup \omega'} \mathcal{F}^{pk}(x) dx + \|\mathcal{F} - F\|_{p_k}$, the first term of which is of size $o(\delta)$ in absolute value by the choice of $R$ and the fact that $\mathcal{L}(\omega') = o(\delta)$. All of these yield $\|G - f\|_{p_k} = o(\delta)$. \qed

With this theorem, the extension step to higher dimensions for nonnegative near extremizers is now complete.

10. Complex-valued case

10.1. Preparation. Let $f : \mathbb{R}^n \to \mathbb{C}$ be a $(1 - \delta)$ near extremizer of Gowers-Host-Kra norm inequality. We write $f(x) = |f(x)a(x)|$, with $a(x) = e^{i2\pi q(x)}$ and $q : \mathbb{R}^n \to \mathbb{R}/\mathbb{Z}$. Assume $\|f\|_{p_k} = 1$. We first make a few observations. Recall that if $f$ is a $(1 - \delta)$ near extremizer then so is $|f|:

$$
(10.1) \quad \|f\|_{L^k} \geq \|f\|_{L^k} \geq (1 - \delta)A(k, n)\|f\|_{p_k} = (1 - \delta)A(k, n)\|f\|_{p_k} = (1 - \delta)A(k, n).
$$
Then by the previous sections, \( |f| \) is \( o(\delta) \) close in \( L^p k \) to a Gaussian extremizer, which by an affine change of variables, we can assume to be the standard centered Gaussian \( F^\dagger \) on \( \mathbb{R}^n \): \( \| F^\dagger - |f| \|_{p k} = o(\delta) \) and \( \| F^\dagger \|_{p k} = 1 \). Now (10.1) and Gowers-Host-Kra norm inequality imply,

\[
A(k, n) \| f \|_{p k} = A(k, n) \geq \| f \|_{U^k} \geq \| f \|_{U^k} \geq (1 - \delta) A(k, n)
\]

which entails \( \| f \|_{U^k} \geq (1 - o(\delta)) \| f \|_{U^k} \). Then since \( \| f \|_{U^k} > 0 \):

\[
\| f \|_{U^k}^{2k} = \text{Re}(\| f \|_{U^k}^{2k}) \\
\quad = \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} |f|(x + \alpha \cdot \tilde{h}) \text{Re} \left( \prod_{\alpha \in \{0,1\}^k} C^\omega a(x + \alpha \cdot \tilde{h}) \right) dx \tilde{h}
\]

(10.2)

\[
\geq (1 - o(\delta)) \| f \|_{U^k}^{2k} = (1 - o(\delta)) \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} |f|(x + \alpha \cdot \tilde{h}) dx \tilde{h}.
\]

On the other hand, from the Gowers product inequality,

(10.3) \[
\left| \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} F^\dagger(x + \alpha \cdot \tilde{h}) dx \tilde{h} \right| - \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} |f|(x + \alpha \cdot \tilde{h}) dx \tilde{h} \right| \leq A(k, n)^2 \| F^\dagger - |f| \|_{p k}^{2k} = o(\delta)
\]

and similarly, since \( |a| = 1 \) on \( \mathbb{R}^n \),

(10.4) \[
\left| \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} F^\dagger(x + \alpha \cdot \tilde{h}) \text{Re} \left( \prod_{\alpha \in \{0,1\}^k} C^\omega a(x + \alpha \cdot \tilde{h}) \right) dx \tilde{h} \\
- \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} |f|(x + \alpha \cdot \tilde{h}) \text{Re} \left( \prod_{\alpha \in \{0,1\}^k} C^\omega a(x + \alpha \cdot \tilde{h}) \right) dx \tilde{h} \right| = o(\delta).
\]

Then (10.3) and (10.4) allow us to replace \( |f| \) with \( F^\dagger \) in (10.2):

(10.5) \[
\int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} F^\dagger(x + \alpha \cdot \tilde{h}) \text{Re} \left( \prod_{\alpha \in \{0,1\}^k} C^\omega a(x + \alpha \cdot \tilde{h}) \right) dx \tilde{h}
\]

\[
\geq (1 - o(\delta)) \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} F^\dagger(x + \alpha \cdot \tilde{h}) dx \tilde{h}.
\]

Let \( B_R \subset \mathbb{R}^n \) denote a centered ball of radius \( R \). We can find a sufficiently large positive \( R \), so that \( \int_{\mathbb{R}^{(k+1)n} \setminus B_R} (F^\dagger)^p k = o(\delta) \), and \( \int_{\mathbb{R}^{(k+1)n} \setminus B_R} |f|^p k = o(\delta) \), since \( \| F^\dagger - |f| \|_{p k} = o(\delta) \). Moreover, we can select this \( R \) so that the following properties are also satisfied:

(10.6) \[
\int_{\mathbb{R}^{(k+1)n} \setminus B_R^{k+1}} \prod_{\alpha \in \{0,1\}^k} F^\dagger(x + \alpha \cdot \tilde{h}) dx \tilde{h} = o(\delta)
\]

(10.7) \[
\int_{\mathbb{R}^{(k+1)n} \setminus B_R^{k+1}} \prod_{\alpha \in \{0,1\}^k} F^\dagger(x + \alpha \cdot \tilde{h}) \text{Re} \left( \prod_{\alpha \in \{0,1\}^k} C^\omega a(x + \alpha \cdot \tilde{h}) \right) dx \tilde{h} = o(\delta).
\]

Then (10.5), (10.6) and (10.7) allow us to reduce our analysis within a bounded region:

(10.8) \[
\int_{B_R^{k+1}} \prod_{\alpha \in \{0,1\}^k} F^\dagger(x + \alpha \cdot \tilde{h}) \text{Re} \left( \prod_{\alpha \in \{0,1\}^k} C^\omega a(x + \alpha \cdot \tilde{h}) \right) dx \tilde{h}
\]

\[
\geq (1 - o(\delta)) \int_{B_R^{k+1}} \prod_{\alpha \in \{0,1\}^k} F^\dagger(x + \alpha \cdot \tilde{h}) dx \tilde{h}.
\]
Since $|a| = 1$ on $\mathbb{R}^n$, it follows from (10.8) that:

$$\left| \text{Re} \left( \prod_{\alpha \in \{0,1\}^k} C^{\omega_{\alpha}} a(x + \alpha \cdot \vec{h}) \right) - 1 \right| = o(\delta)$$

or,

$$\prod_{\alpha \in \{0,1\}^k} C^{\omega_{\alpha}} a(x + \alpha \cdot \vec{h}) = 1 + o(\delta)$$

for $(x, \vec{h})$ in a subset of $\tilde{B}_R^{k+1}$ of measure $(1 - o(\delta))\mathcal{L}(\tilde{B}_R^{k+1})$. Since $a(x) = e^{i2\pi q(x)}$, this last display is simply

$$e^{i2\pi \Delta_{h_1} \cdots \Delta_{h_k} q(x)} = 1 + o(\delta).$$

Here $\Delta_h f(x) = f(x + h) - f(x)$.

**Theorem 10.1.** Let $n, k \geq 1$. There exists $K > 0$ with the following property. Let $B \subset \mathbb{R}^n$ be a centered ball of positive radius and $\psi : B \to \mathbb{R}/\mathbb{Z}$ be a measurable function. Let $\eta, \tau > 0$ be small numbers. Suppose

$$\mathcal{L}(\{(x, \vec{h}) \in \tilde{B}_R^{k+1} : |e^{i2\pi \Delta_{h_1} \cdots \Delta_{h_k} \psi(x)} - 1| > \tau\}) < \eta \mathcal{L}(\tilde{B}_R^{k+1}).$$

Then there exists a polynomial $P : \mathbb{R}^n \to \mathbb{R}$ of degree at most $k - 1$ and a positive function $\rho$ satisfying $\lim_{\tau \to 0} \rho(\tau) = 0$ such that

$$\mathcal{L}(\{x \in B : |e^{i2\pi \psi(x)} e^{-i2\pi P(x)} - 1| > \rho(\tau)\}) < K \eta \mathcal{L}(B).$$

10.2. **Proof of Theorem 10.1.** Let $B^2 = \{(x, h_1) \in \mathbb{R}^{2n} : x \in B, h_1 \in B \text{ and } x + h_1 \in B\}$. When $k = 1$, (10.10) becomes

$$e^{i2\pi (\psi(x + h_1) - \psi(x))} = 1 + O(\tau)$$

which happens for a subset of $B^2$ of measure $(1 - \eta)\mathcal{L}(B^2)$. Then there exists $c \in B$ such that $|c| = o(\eta)$ and that $c$ satisfies (10.11) for $h_1 \in B$ outside a subset of measure at most $O(\eta)\mathcal{L}(B)$. That means

$$e^{i2\pi \psi(x)} = (1 + O(\tau))e^{i2\pi \psi(c)}$$

for $x \in B$ outside a subset of measure at most $O(\eta)\mathcal{L}(B)$. Hence we can take the constant polynomial $P \equiv \psi(c)$. Assuming the conclusion is true for the case $k - 1$, we now prove it for the case of index $k$.

Rewrite (10.10) as

$$1 + O(\tau) = e^{i2\pi \Delta_{h_1} \cdots \Delta_{h_k} \psi(x)} = e^{i2\pi \Delta_{h_1} \cdots \Delta_{h_{k-1}} (\Delta_{h_k} \psi(x))}.$$

This holds for $(x, \vec{h}) \in \tilde{B}_R^{k+1}$ outside a subset of measure at most $\eta \mathcal{L}(B)$. Applying the induction hypothesis to $\Delta_{h_k} \psi(x)$ in (10.12), we conclude that there exists a polynomial in $x$, $P_{h_k}(x) = \sum_{j=0}^{k-2} \sum_{|\gamma| = j} a_{\gamma}(h_k)x^j$ satisfying, for $(x, h_k) \in B^2$ outside a subset of measure at most $O(\eta)\mathcal{L}(B^2)$,

$$e^{i2\pi \Delta_{h_k} \psi(x)}e^{-i2\pi P_{h_k}(x)} = 1 + o(\tau).$$

The coefficient functions $a_{\gamma}$ can be selected to be measurable functions in terms of $h_k$ (in fact, these coefficient functions can be selected to be locally piecewise constant, in a manner that is described in Remark 9.1). To resolve (10.13), we prove a sub-claim:
Claim 10.1. Let \( \psi, B, \eta, \tau \) be as above. Let \( P_t(x) = \sum_{j=0}^{k-1} \sum_{|\gamma|=j} a_\gamma(t)x^\gamma \), with \( a_\gamma : \mathbb{R}^n \to \mathbb{R} \) being measurable functions. Suppose for \((x,t) \in \tilde{B}^2 \) outside a set of measure at most \( O(\eta)\mathcal{L}(\tilde{B}^2) \) the following occurs:

\[
e^{i2\pi \Delta \psi(x)} e^{-i2\pi P_t(x)} = 1 + O(\tau).
\]

Then there exists a polynomial \( Q \) of degree at most \( k \) such that, for \( x \in B \) outside a subset of measure at most \( O(\eta)\mathcal{L}(B) \),

\[
e^{i2\pi \psi(x)} e^{-i2\pi Q(x)} = 1 + o(\tau).
\]

Proof of Claim 10.1. We again use induction. If \( k = 1 \) then (10.14) becomes

\[
e^{i2\pi \psi(x)} e^{-i2\pi a_0(t)} = 1 + O(\tau).
\]

Suppose (10.16) holds for \((x,t) \in \tilde{B}^2 \) outside a subset of measure at most \( \eta\mathcal{L}(\tilde{B}^2) \), we borrow the following result:

Proposition 10.2. [7] Let \( n \geq 1 \). There exists a positive constant \( K = K(n) > 0 \) with the following property. Let \( B \) be a centered ball of positive, finite radius. Let \( \delta > 0 \) and \( \eta \in (0,1/2] \). Let \( f_1, f_2, f_3 : 2B \to \mathbb{C} \) be measurable functions that vanish only on sets of Lebesgue measure zero. Suppose that

\[
\mathcal{L}\left( \{(x,y) \in B^2 : |f_1(x)f_2(y)f_3^{-1}(x+y) - 1| > \eta \} \right) < \delta (\mathcal{L}(B))^2.
\]

Then for each index \( j \) there exists a real-linear function \( L_j : \mathbb{R}^n \to \mathbb{C} \) such that

\[
\mathcal{L}\left( \{x \in B : |f_j(x)e^{-L_j(x)}| > K\eta^{1/K} \} \right) < K\delta \mathcal{L}(B).
\]

Applying Proposition 10.2 verbatim, there exist a constant \( K = K(n) \), an affine function \( L : \mathbb{R}^n \to \mathbb{R} \) and a subset of \( B \) of measure at most \( K\eta \mathcal{L}(B) \) outside of which, \( |e^{i2\pi \psi(x)} e^{-i2\pi L(x)}| < K\eta^{1/K} \). Hence the claim holds for the base case \( k = 1 \). Assume the claim is true for the case \( k - 1 \), that is, if (10.14) holds for a polynomial \( P_t \) of degree at most \( k - 1 \) in \( x \), for \((x,t) \in \tilde{B}^2 \) outside a subset of measure at most \( O(\eta)\mathcal{L}(\tilde{B}^2) \), then (10.15) follows, with a polynomial \( Q \) of degree at most \( k \), in terms of \( x \).

See also [9] for the following argument. Suppose now that \( P_t \) in (10.14) is a polynomial of degree at most \( k \) in \( x \). The leading terms in \( x \) in \( P_t \) are \( \sum_{|\gamma|=k} a_\gamma(t)x^\gamma \). Suppose also for the moment that the leading coefficient functions, for \( |\gamma| = k \), are, \( a_\gamma(t) = \langle A_\gamma, t \rangle_{\mathbb{R}^n} \), for some elements \( A_\gamma \in \mathbb{R}^n \):

\[
\sum_{|\gamma|=k} a_\gamma(t)x^\gamma = \sum_{|\gamma|=k} \langle A_\gamma, t \rangle_{\mathbb{R}^n} x^\gamma.
\]

Define:

\[
q(x) = \sum_{|\beta|=k+1} \left( \sum_{i=1}^{n} \beta_i^{-1}(A_\beta)_i \right) x^\beta.
\]

The relations between indices \( \gamma \) in (10.17) and \( \beta \) in (10.18), and correspondingly, \( A_\gamma \) and \( A_\beta \), are as follows. Each \( \beta \) has the form \( \beta = (\beta_1, \ldots, \beta_i, \ldots, \beta_n) = (\gamma_1, \ldots, \gamma_i + 1, \ldots, \gamma_n) \), for some \( i \in \{1, \ldots, n\} \), and for each \( \beta \) that is arisen from a \( \gamma \) in this way, \( A_\beta = A_\gamma \). Note that \( q \) is a polynomial of degree at most \( k + 1 \), and the leading \( x \)-terms of \( \Delta q \) are,

\[
\sum_{|\beta|=k+1} \left( \sum_{i=1}^{n} (A_\beta)_i t_i x_i^{\beta_i-1} \prod_{j \neq i} x_j^{\beta_j} \right) = \sum_{|\gamma|=k} \langle A_\gamma, t \rangle_{\mathbb{R}^n} x^\gamma.
\]
which is to say, $\Delta_t q$ is a polynomial of degree at most $k$ in terms of $x$. Consider $\Psi(x) = \psi(x) - q(x)$. Then (10.14) yields, for $(x, t) \in \tilde{B}^2$ outside a subset of measure at most $O(\eta)\mathcal{L}(\tilde{B}^2)$,

$$e^{i2\pi \Delta_t \Psi(x)} = e^{i2\pi \Delta_t (\psi - q)(x)} = (1 + O(\tau))e^{i2\pi P_t(x)}e^{-i2\pi \Delta_t q(x)} = (1 + O(\tau))e^{i2\pi S_t(x)}.$$  

Note that (10.17) and (10.19) imply that in $S_t$ in (10.20) takes the following form:

$$S_t(x) = \sum_{k=1}^{N-1} \sum_{j=0}^{N} c_{\mu}(t)x^\mu.$$  

Hence by the induction hypothesis, we conclude from (10.20) that there exists a polynomial $Q$ of degree at most $k$ such that $e^{i2\pi \Psi(x)} = (1 + o(\tau))e^{i2\pi Q(x)}$, and in turn, $e^{i2\pi \psi(x)} = (1 + o(\tau))e^{i2\pi (Q(x) + q(x))}$, for $x \in B$ outside a subset of measure at most $O(\eta)\mathcal{L}(B)$. It’s obvious that $Q + q$ is a polynomial of degree at most $k + 1$; we now close the induction loop.  

In the above proof of Claim 10.1, we need to make an assumption that if $|\gamma| = k$, $a_\gamma(t) = \langle A_\gamma, t \rangle_{\mathbb{R}^n}$ for some $A_\gamma \in \mathbb{R}^n$, for $t \in B$ outside a subset of measure at most $O(\eta)\mathcal{L}(B)$. Now we show that such condition must indeed occur:

**Claim 10.2.** Let $\psi, \eta, \tau$ be as above. Given $P_t(x) = \sum_{j=0}^{k} \sum_{|\gamma| = j} a_\gamma(t)x^\gamma$, with $a_\gamma : \mathbb{R}^n \to \mathbb{R}$ being measurable functions. Suppose for $(x, t) \in \tilde{B}^2$ outside a subset of measure at most $O(\eta)\mathcal{L}(\tilde{B}^2)$, the following happens:

$$e^{i2\pi \Delta_t \psi(x)} = (1 + O(\tau))e^{i2\pi P_t(x)}.$$  

Then, for $|\gamma| = k$, there exist $A_\gamma \in \mathbb{R}^n$ such that $a_\gamma(t) = \langle A_\gamma, t \rangle_{\mathbb{R}^n}$, for $t \in B$ outside of a subset of measure at most $O(\eta)\mathcal{L}(B)$.

**Proof of Claim 10.2.** (10.21) gives us,

$$\psi(x + t) - \psi(x) \equiv \sum_{j=0}^{k} \sum_{|\gamma| = j} a_\gamma(t)x^\gamma + o(\tau).$$  

By "$u \equiv v$", we mean $u - v \in \mathbb{Z}$. Let $A^x = \{ t \in B : (x, t) \in \tilde{B}^2 \}$ and (10.22) apply, for each $x \in B$. Let $A$ denote the set of all $x \in (1/2)B$ such that $\mathcal{L}(A^x) > (1 - O(\eta))\mathcal{L}(B)$. Then there exists $c > 0$ such that, for all sufficiently small $\eta$, $\mathcal{L}(A) > c$. Then uniformly for $x \in A$, as $t$ varies in $A^x$, these values $x + t \in B$ occupy a subset of $B$ of measure at least $(1 - O(\eta))\mathcal{L}(B)$.

Fix $x \in A$, this set of values $x + t$ has a nonempty intersection with $A$. Fix one such value $x + t \in A$. Then $\mathcal{L}(A^{x+t}) > (1 - O(\eta))\mathcal{L}(B)$. That means, the set of values $t + s$, as $s$ varies in $A^{x+t}$, occupy a subset of $B$ of measure at least $(1 - O(\eta))\mathcal{L}(B)$. This set of values $t + s$ will then have a nonempty intersection with $A^x$. In other words, we can select $x, t, s$ so that the following argument is applicable:

Consider $\psi(x + t + s) - \psi(x + t)$. From (10.22) we can write the said difference in two ways. Firstly:

$$\psi((x + t) + s) - \psi(x + t) \equiv \sum_{j=0}^{k} \sum_{|\gamma| = j} a_\gamma(s)(x + t)^\gamma + o(\tau)$$  

$$= \sum_{|\gamma| = k} a_\gamma(s)(x + t)^\gamma + O(x^{k-1}) + o(\tau) = \sum_{|\gamma| = k} a_\gamma(s)x^\gamma + O(x^{k-1}) + o(\tau).$$
By "$O(x^{k-1})$" we mean a linear combination of monomials in terms of $x$ of degree at most $k - 1$. Secondly:

\begin{equation}
\psi(x + t + s) - \psi(x + t) = [\psi(x + (t + s)) - \psi(x)] - [\psi(x + t) - \psi(x)] = \sum_{|\gamma| = k} [a_\gamma(t + s) - a_\gamma(t)] x^\gamma + O(x^{k-1}) + o(\tau).
\end{equation}

Since $\mathcal{L}(A) > c$, a comparison between (10.23) and (10.24) shows that, for $|\gamma| = k$:

\begin{equation}
\alpha_\gamma(t + s) = \alpha_\gamma(t) + \alpha_\gamma(s).
\end{equation}

By the argument presented above, (10.25) is satisfied for $t, s \in B$ except for a subset of measure at most $O(\eta)\mathcal{L}(B)$. Let $\{\rho_i\}_i$ be a sequence tending to zero. Then from (10.25), for a.e. $(t, s) \in B \times B$ outside a subset of measure at most $O(\eta)\mathcal{L}(B \times B)$ and for every $\rho_i$, we have, $|\alpha_\gamma(t + s) - \alpha_\gamma(t) - \alpha_\gamma(s)| < \rho_i$. By Proposition 9.1 and its proof given in [7], there exist affine functions $L_{\rho_i}(t) = \langle A_{\rho_i} t, t \rangle_{\mathbb{R}^n} + b_{\rho_i}$, $A_{\rho_i} \in \mathbb{R}^n$, $b_{\rho_i} \in \mathbb{R}$, and a subset $U \subset B$, such that $\alpha_\gamma(t) - L_{\rho_i}(t) = O(\rho_i)$ for a.e. $t \in U$. The relative complement of $U$ has measure at most $O(\eta)\mathcal{L}(B)$, and the implicit constants in the notations $O(\rho_i), O(\eta)\mathcal{L}(B)$ are independent of $i$. This implies in particular that the $L_{\rho_i}$ are uniformly bounded and converges pointwise in $U$.

Since $L_{\rho_i}$ are affine functions on $\mathbb{R}^n$, this in turn implies, by passing to a subsequence, $A_{\rho_i} \rightarrow A \in \mathbb{R}^n$ and $b_{\rho_i} \rightarrow b \in \mathbb{R}$ as $i \rightarrow \infty$ [15]. Hence $\alpha_\gamma(t) = \langle A, t \rangle_{\mathbb{R}^n} + b$ for $t \in B$ outside of a subset of measure at most $O(\eta)\mathcal{L}(B)$. But then (10.25) gives $b = 0$, hence the proof of Claim 10.2 is now complete. \(\square\)

We have also finished the proof of Theorem 10.1. \(\square\)

### 10.3. Production of an extremizer

We combine the results of the previous sections, (10.9) and Theorem 10.1 to conclude that there exist a centered ball $B_R$, with $R$ sufficiently large, so that

\begin{equation}
\int_{\mathbb{R}^n \setminus B_R} |f|^p \, d\mu = o(\delta)
\end{equation}

and a polynomial $P$ of degree at most $k - 1$ so that,

\begin{equation}
\|e^{2\pi P} - a\|_{L^p} = o(\delta).
\end{equation}

Let $\mathcal{G} = \mathcal{F}^\dagger e^{2\pi P}$, with $\mathcal{F}^\dagger$ as in Section 10.1. Then $\mathcal{G}$ is an extremizer of (1.4). Observe:

\begin{equation}
\|\mathcal{G} - f\|_{L^p}^p = \int_{\mathbb{R}^n \setminus B_R} |\mathcal{G} - f|^p + \int_{B_R} |\mathcal{G} - f|^p.
\end{equation}

By (10.26), the facts that $f(x) = |f(x) a(x)$ and $|e^{2\pi P} - a| \leq C$ uniformly on $\mathbb{R}^n$:

\begin{equation}
\int_{\mathbb{R}^n \setminus B_R} \|f| e^{2\pi P} - |f| a\|^p \, d\mu = o(\delta),
\end{equation}

which together with the fact that $\|\mathcal{F}^\dagger - f\|^p_{L^p} = o(\delta)$ allows us to conclude the following for the first term in (10.28):

\begin{equation}
\int_{\mathbb{R}^n \setminus B_R} \mathcal{F}^\dagger e^{2\pi P} - |f|^a|^p \leq C(k) \int_{\mathbb{R}^n \setminus B_R} |\mathcal{F}^\dagger e^{2\pi P} - |f|^a|^p \leq C(k)\mathcal{F}^\dagger - |f|^a|^p + o(\delta) = o(\delta).
\end{equation}

In the same spirit, we split the second term in (10.28):

\begin{equation}
\int_{B_R} \mathcal{G} - f|^p \leq C(k) \int_{B_R} \mathcal{F}^\dagger e^{2\pi P} - \mathcal{F}^\dagger a|^p + C(k) \int_{B_R} \mathcal{F}^\dagger a - |f|^a|^p.
\end{equation}
The second term in (10.29) is at most $o(\delta)$ in absolute value, again by $\|\mathcal{F}^\dagger - |f|\|^p_k = o(\delta)$ and $|a| = 1$ on $\mathbb{R}^n$. It follows from (10.27) that the contribution of first term in (10.29) is also at most $o(\delta)$ in absolute value:

$$\int_{BR} |\mathcal{F}^\dagger e^{i2\pi P} - \mathcal{F}^\dagger a|^p_k \leq C(k) \int_{BR} |e^{i2\pi P} - a|^p_k = o(\delta).$$

Hence we conclude that $\|g - f\|^p_k = o(\delta)$.

We have now obtained the conclusion for Theorem 1.1 for a general measurable near extremizer $f : \mathbb{R}^n \to \mathbb{C}$, $n \geq 1$, of the $k$th Gowers-Host-Kra norm inequality for $k \geq 2$. □

REFERENCES

[1] Jonathan Bennett, Anthony Carbery, Michael Christ, and Terence Tao. The brascamp–lieb inequalities: finiteness, structure and extremals. *Geometric and Functional Analysis*, 17(5):1343–1415, 2008.
[2] Vladimir Igorevich Bogachev and Maria Aparecida Soares Ruas. *Measure theory*, volume 1. Springer, 2007.
[3] Haim Brezis and Haim Brézis. *Functional analysis, Sobolev spaces and partial differential equations*, volume 2. Springer, 2011.
[4] Almut Burchard. Cases of equality in the riesz rearrangement inequality. *Annals of mathematics*, 143(3):499–527, 1996.
[5] Michael Christ. personal communication.
[6] Michael Christ. A sharpened haussdorff-young inequality. *arXiv preprint arXiv:1406.1210*, 2014.
[7] Michael Christ. Near-extremizers of young’s inequality for euclidean groups. *Revista Matemática Iberoamericana*, 35(7):1925–1972, 2019.
[8] Michael Christ. Subsets of euclidean space with nearly maximal gowers norms. *Acta Mathematica Sinica, English Series*, 35(6):771–782, 2019.
[9] Michael Christ. On young’s convolution inequality for heisenberg groups. In *Geometric aspects of harmonic analysis*, pages 223–260. Springer, 2021.
[10] Tanja Eisner and Terence Tao. Large values of the gowers-host-kra seminorms. *Journal d’analyse mathématique*, 117(1):133–186, 2012.
[11] William T Gowers. A new proof of Szemerédi’s theorem. *Geometric & Functional Analysis GAFA*, 11(3):465–588, 2001.
[12] Bernard Host and Bryna Kra. Nonconventional ergodic averages and nilmanifolds. *Annals of Mathematics*, pages 397–488, 2005.
[13] Svante Janson. On interpolation of multi-linear operators. In *Function Spaces and Applications: Proceedings of the US-Swedish Seminar held in Lund, Sweden, June 15–21, 1986*, pages 290–302. Springer, 2006.
[14] Elliott H Lieb and Michael Loss. *Analysis*, volume 14. American Mathematical Soc., 2001.
[15] Walter Rudin et al. *Principles of mathematical analysis*, volume 3. McGraw-hill New York, 1976.
[16] Elias M Stein and Guido Weiss. *Introduction to Fourier analysis on Euclidean spaces*, volume 1. Princeton university press, 1971.
[17] Terence Tao and Van H Vu. *Additive combinatorics*, volume 105. Cambridge University Press, 2006.