Generalized Mean Estimation in Monte-Carlo Tree Search

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Abstract

We consider Monte-Carlo Tree Search (MCTS) applied to Markov Decision Processes (MDPs) and Partially Observable MDPs (POMDPs), and the well-known Upper Confidence bound for Trees (UCT) algorithm. In UCT, a tree with nodes (states) and edges (actions) is incrementally built by the expansion of nodes, and the values of nodes are updated through a backup strategy based on the average value of child nodes. However, it has been shown that with enough samples the maximum operator yields more accurate node value estimates than averaging. Instead of settling for one of these value estimates, we go a step further proposing a novel backup strategy which uses the power mean operator, which computes a value between the average and maximum value. We call our new approach Power-UCT, and argue how the use of the power mean operator helps to speed up the learning in MCTS.

We theoretically analyze our method providing guarantees of convergence to the optimum. Moreover, we discuss a heuristic approach to balance the greediness of backups by tuning the power mean operator according to the number of visits to each node. Finally, we empirically demonstrate the effectiveness of our method in well-known MDP and POMDP benchmarks, showing significant improvement in performance and convergence speed w.r.t. UCT.

Introduction

Monte-Carlo Tree Search (MCTS) (Coulom 2006) is an effective strategy for combining Monte-Carlo search with an incremental tree structure. MCTS is becoming increasingly popular in the community, especially after the outstanding results recently achieved in the game of Go (Silver et al. 2016). In the last years, the research about MCTS mainly focused on the study of effective ways of expanding the tree, performing rollouts, and backing up the average reward computed from rollouts to the parent nodes. We consider the Upper Confidence bound applied to Trees (UCT) algorithm (Kocsis, Szepesvári, and Willemsen 2006), which combines tree search with the well-known UCB1 sampling policy (Auer, Cesa-Bianchi, and Fischer 2002), as an effective way of dealing with the action selection to expand the tree. In UCT, the estimate of the value of each node is computed performing multiple rollouts starting from it, and updating its value as the average of the collected rewards; then, its value is backed up to its parent nodes that are updated with the average of their children nodes. Considering that the action selection policy tends to favor the best actions in the long run, UCT has theoretical guarantees about the convergence of the estimates to the optimal values. However, it has been shown that using the average reward for backup leads to an underestimation of the optimal value, slowing down the learning; on the other hand, using the maximum reward leads to an overestimation causing the same learning problems, especially in stochastic settings (Coulom 2006). This problem is also evinced in the well-known Q-Learning algorithm (Watkins 1989), where the maximum operator leads to overestimation of the optimal values because of the positively biased estimate of the maximum expected value computed by the maximum operator (Smith and Winkler 2006). Some variants of Q-Learning based on (weighted) mean operators have been successfully proposed to address this issue (Hasselt 2010, D’Eramo, Restelli, and Nuara 2016).

In this paper, we introduce a novel backup operator based on a power mean (Bullen 2013) that, through the tuning of a single coefficient, computes a value between the average reward and the maximum one. This allows to balance between the negatively biased estimate of the average reward, and the positively biased estimate of the maximum reward; in practice, this translates in balancing between a safe but slow update, and a greedy but misleading one. In the following, we propose a variant of UCT based on the power mean operator, which we then call Power-UCT. We theoretically prove the convergence of Power-UCT, based on the consideration that the algorithm converges for all values between the range computed by the power mean. Moreover, we describe an intuitive heuristic, based on the number of visits to each node, to tune the greediness of the power mean operator, and show their effect on the learning process. Eventually, we empirically evaluate Power-UCT w.r.t. UCT in classic MDPs, and we consider POMCP (Silver and Veness 2010), a well-known algorithm based on UCT to solve POMDPs, to compare Power-UCT w.r.t. UCT under partial observability. Remarkably, we show how Power-UCT outperforms the baselines both in terms of quality and speed of learning. Thus, our contribution is threefold:

1. We propose a new backup operator for UCT based on a...
power mean, and prove the convergence to the optimal values;

2. We discuss a heuristic to tune the greediness of the power mean operator each node, according to the number of visits to them;

3. We empirically evaluate the effectiveness of our approach comparing it with UCT in well-known MDPs and POMDPs, showing significantly better performance.

The rest of this paper is organized as follows: we first introduce related works in MCTS. Next, we discuss background knowledge of MCTS method, UCB algorithm and its extension to UCT, and introduce Power-UCT Operator. We then describe in detail our algorithm and show Theoretical Analysis of Power-UCT with convergence guarantee. Finally, we present evaluations results in both MDP and POMDP problems and show that it outperforms UCT in all scenarios and conclude the paper.

Related Work

Several works focus on the proposal of methods to apply UCB1  (Auer, Cesa-Bianchi, and Fischer 2002) to MCTS. For this purpose UCB1-tuned (Auer, Cesa-Bianchi, and Fischer 2002) modifies the upper confidence bound of UCB1 to account for variance in order to improve exploration. Bayesian UCT (Tesauro, Rajan, and Segal 2012) propose a Bayesian version of UCT, which obtains better estimates of node values and uncertainties given limited experience, but is more computation-intensive. While most work on bandits in MCTS focuses on discrete actions, work on continuous action MCTS also exists (Mansley, Weinstein, and Littman 2011). Since our MCTS algorithm is based on the UCT algorithm, which is an extension of UCB1, our method could be applied to all of these MCTS algorithms. Many heuristic approaches based on specific domain knowledge have been proposed, such as adding a bonus term to value estimates based on domain knowledge (Gelly and Wang 2006; Teytaud and Teytaud 2010; Childs, Brodeur, and Kocsis 2008; Kozelek et al. 2009; Chaslot et al. 2008) or prior knowledge collected during policy search (Gelly and Silver 2007; Helmbold and Parker-Wood 2009; Lorentz 2010; Tom 2010; Hoock et al. 2010). We point out that we provide a novel node value backup approach that could be applied in combination with all of these methods.

To improve upon UCT algorithm in MCTS, [Khandelwal et al. 2016] formalizes and analyzes different on-policy/off-policy complex backup approaches from Reinforcement Learning literature for MCTS planning, and propose four complex backup strategies: MCTS(λ), MaxMCTS(λ), MCTSv, MaxMCTSv. It also reports that MaxMCTS(λ) and MaxMCTSv perform better than UCT for certain setup of parameter. Vodopivec, Samothrakis, and Stier(2017) proposed an approach called SARSA-UCT, which performs the dynamic programming backups using SARSA (Rumery 1995). Both Khandelwal et al. (2016) and Vodopivec, Samothrakis, and Stier (2017) directly borrow value backup ideas from Reinforcement Learning in order to estimate the value at each tree node. However, they do not provide any proof of convergence. Instead, our method provides a completely novel way of backing up values in each MCTS node using a power mean operator, for which we prove the convergence to the optimal policy in the limit.

Background

In this section, we first discuss an overview of Monte Carlo Tree Search method. Next, we discuss UCB algorithm and subsequently an extension of UCB to UCT algorithm. Finally, we discuss the definition of Power Mean operator and its properties.

Monte-Carlo Tree Search

MCTS combines tree search with Monte-Carlo sampling in order to build a tree, where states and actions are respectively modeled as nodes and edges, to compute optimal decisions. MCTS requires a generative black box simulator for generating a new state based on the current state and chosen action. The MCTS algorithm consists of a loop of four steps:

- Selection: start from the root node, interleave action selection and sampling the next state (tree node) until a leaf node is reached
- Expansion: expand the tree by adding a new edge (action) to the leaf node and sample a next state (new leaf node)
- Simulation: rollout from the reached state to the end of the episode using random actions or a heuristic
- Backup: update the nodes backwards along the trajectory starting from the end of the episode until the root node according to the rewards collected

In the next subsection, we discuss UCB algorithm and its extension to UCT.

Upper Confidence bound for Trees

In this work, we consider the UCT (Upper Confidence bounds for Trees) (Kocsis, Szepesvári, and Willemson 2006) algorithm, an extension of the well-known UCB1 (Auer, Cesa-Bianchi, and Fischer 2002) multi-armed bandit algorithm, used in MCTS. UCB1 chooses the arm (action a) using

\[
a = \arg \max_{i \in \{1, \ldots, K\}} \bar{X}_{i,T_i(n-1)} + C \sqrt{\frac{\log n}{T_i(n-1)}}.
\]

where \(T_i(n) = \sum_{t=1}^{n} 1\{t = i\}\) is the number of times arm \(i\) is played up to time \(n\). \(\bar{X}_{i,T_i(n-1)}\) denotes the average reward of arm \(i\) up to time \(n - 1\) and \(C = \sqrt{2}\) is an exploration constant. In UCT, each node is different bandit problem, where the arms correspond to the actions, and the payoff is the reward of the episodes starting from them. In the backup phase, value is backed up recursively from the leaf node to the root as

\[
\bar{X}_n = \sum_{i=1}^{K} \left( \frac{T_i(n)}{n} \right) \bar{X}_{i,T_i(n)}.
\]
At each step corresponding to the selection step in MCTS, each arm in the tree is chosen as the maximum value of nodes in the current non-stationary multi-armed bandits setup.

\[ a = \arg \max_{s,a} \sum_{i=1}^{K} \frac{T_i(n-1)}{n-1} X_i, T_i(n-1) + C \sqrt{\frac{\log n}{T_i(n-1)}}. \]

\[ (3) \]

Kocsis, Szepesvári, and Willemson (2006) proved that UCT converges in the limit to the optimal policy.

**Power Mean**

In this paper, we introduce a novel way of estimating the expected value of a bandit arm \((X_i, T_i(n-1))\) in MCTS that can adapt according to the number of samples. For this purpose, we will use the power mean (Bullen 2013), an operator belonging to the family of functions for aggregating sets of numbers, that includes as special cases the Pythagorean means (arithmetic, geometric, and harmonic means):

**Definition 1.** For a sequence of positive numbers \(X = (X_1, ..., X_n)\) and positive weights \(w = (w_1, ..., w_n)\), the power mean of order \(p\) \((p\) is an extended real number\) is defined as

\[ M_n^{[p]}(X, w) = \left( \frac{\sum_{i=1}^{n} w_i X_i^p}{\sum_{i=1}^{n} w_i} \right)^{\frac{1}{p}}. \]

\[ (4) \]

\(p = 1\) we obtain the weighted arithmetic mean. With \(p \to 0\) we have the geometric mean, and with \(p = -1\) we have the harmonic mean (Bullen 2013)

\[ M_n^{[0]}(X, w) = \lim_{p \to 0} M_n^{[p]}(X, w) = \sqrt[n]{X_1 \cdots X_n}, \]

\[ (5) \]

\[ M_n^{[-1]}(X, w) = \frac{n}{1/X_1 + \cdots + 1/X_n}. \]

\[ (6) \]

Furthermore, we get (Bullen 2013)

\[ M_n^{[\infty]}(X, w) = \lim_{p \to \infty} M_n^{[p]}(X, w) = \operatorname{Min}(X_1, ..., X_n), \]

\[ (7) \]

\[ M_n^{[-\infty]}(X, w) = \lim_{p \to -\infty} M_n^{[p]}(X, w) = \operatorname{Max}(X_1, ..., X_n), \]

\[ (8) \]

The weighted arithmetic mean lies between \(\operatorname{Min}(X_1, ..., X_n)\) and \(\operatorname{Max}(X_1, ..., X_n)\). Moreover, the following lemma shows that \(M_n^{[p]}(X, w)\) is an increasing function.

**Lemma 1.** \(M_n^{[p]}(X, w)\) is an increasing function meaning that

\[ M_n^{[1]}(X, w) \leq M_n^{[p]}(X, w) \leq M_n^{[q]}(X, w), \forall p \geq q \geq 1 \]

\[ (9) \]

For the proof, see (Bullen 2013).

**Power Mean Backup**

As previously described, it is well known that performing backups using the average of the rewards results in an underestimate of the true value of the node, while using the maximum results in an overestimate of it (Coulom 2006). Usually, the average backup is used when the number of simulations is low, for a conservative update of the nodes due to the lack of samples; on the other hand, the maximum operator is favoured when the number of simulations is high. We address this problem proposing a novel backup operator for UCT based on the power mean (Equation 4):

\[ \bar{X}_n(p) = \left( \frac{\sum_{i=1}^{K} T_i(n)}{n} X_i^{p} \right)^{1/p}. \]

\[ (10) \]

This way, we bridge the gap between the average and maximum estimators with the purpose of getting the advantages of both. We call our approach Power-UCT and describe it in more detail in the following.

**Power-UCT**

The introduction of our novel backup operator in UCT does not require major changes to the algorithm. Indeed, the Power-UCT pseudocode shown in Algorithm 1 is almost identical to the UCT one, with the only few differences highlighted for clarity. MCTS has two type of nodes: \(V\)Nodes corresponding to the state-value, and \(Q\) Nodes corresponding to state-action values. An action is taken from the \(V\)Node of the current state leading to the respective \(Q\) Node, then it leads to the \(V\) Node of the reached state. We skip the description of all the procedures since they are well-known components of MCTS, and we focus only on the ones involved in the use of the power mean backup operator. In SIMULATE\(_V\), Power-UCT updates the value of each \(V\) Node using the power mean of its children \(Q\) Nodes, that are computed in SIMULATE\(_Q\). Note that our algorithm could be applied to several bandit based enhancements of UCT, but for simplicity we only focus on UCT.

**Adaptive \(p\) coefficient**

To take advantage of Power-UCT, it is desirable to have a different \(p\) coefficient for each node and adapt its value during the training according to the number of visits. In particular, we decrease \(p\) when the number of visits is small since the average operator performs better in this case; on the contrary, we increase it when the number of visits is large, which is the case where the maximum operator is more effective. We propose a heuristic adaptive procedure to tune \(p\), that considers \(n_0, p_{\text{init}}, p_{\text{max}}, p_{\text{stepsize}}\) as hyper-parameters. When the number of simulations is less than \(n_0\), and the variance of the current node is large, we use \(p = p_{\text{init}}\). Then, when the number of simulations is larger than a threshold, we slowly increase the value of \(p\) as follows

\[ p = p_{\text{max}} - p_{\text{stepsize}} \log \left( \frac{n_{s,a} + 1}{n_{s,a}} \right), \]

\[ (11) \]

where \(n_{s,a}\) is the number of simulations in the current node. The intuitive motivation of this approach is supported by its theoretical guarantees of convergence for any value of \(p\), that
Algorithm 1 Power-UCT

1: $s$: state
2: $a$: action
3: $N(s)$: number of simulations of $V$-Node of state $s$
4: $n(s, a)$: number of simulations of $Q$-Node of state $s$ and action $a$
5: $V(s)$: Value of $V$-Node at state $s$. Default is 0
6: $Q(s, a)$: Value of $Q$-Node at state $s$, action $a$. Default is 0
7: $\tau(s, a)$: transition function
8: $\gamma$: discount factor
9:
10: procedure SELECT_ACTION(s)
11: \hspace{1em} return $\arg \max_a Q(s, a) + C \frac{\log N(s)}{n(s, a)}$
12: end procedure
13:
14: procedure SEARCH(s)
15: repeat
16: \hspace{1em} SIMULATE $V(s, 0)$
17: until TIMEOUT()
18: return $\arg \max_a Q(s, a)$
19: end procedure
20:
21: procedure ROLLOUT(s, depth)
22: if $\gamma^{\text{depth}} < \epsilon$ then
23: \hspace{1em} return 0
24: end if
25: \hspace{1em} $a \sim \pi_{\text{rollout}}(\cdot)$
26: \hspace{1em} $(s', r) \sim \tau(s, a)$
27: return $r + \gamma^{\text{depth} + 1}$
28: end procedure
29:
30: procedure SIMULATE_V(s, depth)
31: \hspace{1em} $a \leftarrow$ SELECT_ACTION(s)
32: \hspace{1em} SIMULATE_Q(s, a, depth)
33: \hspace{1em} $N(s) \leftarrow N(s) + 1$
34: \hspace{1em} Estimate $p$ according to $N(s)$
35: \hspace{1em} $V(s) \leftarrow (\sum_a \frac{n(s, a)}{N(s)} Q(s, a)^p)^{1/p}$
36: end procedure
37:
38: procedure SIMULATE_Q(s, a, depth)
39: \hspace{1em} $(s', r) \sim \tau(s, a)$
40: if $s' \not\in \text{Terminal}$ then
41: \hspace{1em} if $V(s')$ not expanded then
42: \hspace{2em} ROLLOUT($s'$, depth)
43: \hspace{2em} else
44: \hspace{3em} SIMULATE_V($s'$, depth + 1)
45: \hspace{1em} end if
46: \hspace{1em} end if
47: \hspace{1em} $n(s, a) \leftarrow n(s, a) + 1$
48: \hspace{1em} $Q(s, a) \leftarrow (\sum_a \frac{n(s, a)}{N(s)} Q(s, a)^p + \gamma \sum_t N(s') V(s')) / n(s, a)$
49: end procedure

we prove in the following section. To the best of our knowledge, this is the first work to show an adaptive method of backup value in MCTS that still ensures the convergence to the optimal value.

Theoretical analysis

In this section, we prove the convergence of Power-UCT following the same steps of the proof of convergence of UCT described in Kocsis, Szepesvári, and Willemsen (2006), but using the power mean instead of the average. Given i.i.d. random variables $X_{it}$, according to Hoeffding’s inequality (Hoeffding 1994) it holds that

$$P(X_{is} \geq \mu_i + \epsilon_{t,s}) \leq 2 \cdot e^{-\frac{2 \epsilon^2}{\mu_i}}$$

$$P(X_{is} \leq \mu_i - \epsilon_{t,s}) \leq 2 \cdot e^{-\frac{2 \epsilon^2}{\mu_i}}$$

where $\epsilon_{t,s} = \sqrt{\frac{2 \log(t)}{s}}$ is the bias exploration constant. Let $\mu_{in} = E[X_{in}]$. According to the Hoeffding’s inequalities, the expected value of the average reward of an arm converges to the mean value of that arm:

$$\mu_i = \lim_{n \to \infty} \mu_{in}.$$  

Let $\delta_{in} = \mu_i - \mu_{in}$. We assume the average reward to drift as function of time and converges only in the limit, which means that

$$\lim_{n \to \infty} \delta_{in} = 0.$$  

From now on, let $*$ be the upper index for all quantities related to optimal arm. By assumption, the rewards lie between 0 and 1.

Let start with Assumption 1 (Kocsis, Szepesvári, and Willemsen 2006)

Assumption 1. Fix $1 \leq i \leq K$. Let $\{F_t\}$ be a filtration such that $\{X_{it}\}$ is $\{F_t\}$-adapted and $X_{it}$ is conditionally independent of $F_{t+1}, F_{t+2}, \ldots$, given $F_{t-1}$. Then $0 \leq X_{it} \leq 1$ and the limit of $\mu_{in} = E[X_{in}]$ exists. Further, we assume that there exists a constant $C > 0$ and an integer $N_p$ such that for $n > N_p$, for any $\delta > 0$, $\Delta_n(\delta) = C \sqrt{n \log(1/\delta)}$, the following bounds hold:

$$P(nX_n \geq n E[X_n] + \Delta_n(\delta)) \leq \delta$$

$$P(nX_n \leq n E[X_n] - \Delta_n(\delta)) \leq \delta$$

From Assumption 1 we could derive the upper bound for the regret of choosing a sub-optimal arm.

Theorem 1. (Theorem 2 in Kocsis, Szepesvári, and Willemsen 2006) Consider UCB1 applied to a non-stationary problem where the pay-off sequence satisfies Assumption 1 and where the bias sequence, $\epsilon_{t,s}$, used by UCB1 is given by $c_{t,s} = \sqrt{\frac{2 \log(t)}{s}}$. Fix $\epsilon \geq 0$. Let $T_i(n)$ denote the number of plays of arm $i$. Then if $i$ is the index of a sub-optimal arm then

$$E[T_i(n)] \leq \frac{16 C^2 \log n}{(1 - \epsilon)^2 \Delta_i^2} + N_0(\epsilon) + N + 1 + \frac{\pi^2}{3}. \quad (14)$$
Theorem 1 derives an upper bound on the error for the expected number of times suboptimal arms are played. As in the convergence proof for UCT, we refer to this theorem, to derive our version of Theorem 3 in (Kocsis, Szepesvári, and Willemsen 2006), which computes the upper bound of the difference between the value backup of an arm with \( \mu^* \) up to time \( n \).

**Theorem 2.** Under the assumptions of Theorem 1,

\[
|E[\overline{X}_n(p)] - \mu^*| \leq |\delta^*_n| + O\left(\frac{K(C^2 \log n + N_0)}{n}\right)^{1/p}.
\]

**Proof.** In UCT, the value of each node is used for backup as \( \overline{X}_n = \sum_{i=1}^{K} \left( \frac{T_i(n)}{n} \right) \overline{X}_{i,T_i(n)} \), and the authors show that

\[
|E[\overline{X}_n] - \mu^*| \leq |\delta^*_n| + |\mu^*_n - \mu^*| = |\delta^*_n| + |E[\overline{X}_n] - \mu^*_n| \\
\leq |\delta^*_n| + O\left(\frac{K(C^2 \log n + N_0)}{n}\right) \tag{15}
\]

We derive the same results replacing the average with the power mean. First, we have

\[
E[\overline{X}_n(p)] - \mu^*_n = E\left[\left( \sum_{i=1}^{K} \frac{T_i(n)}{n} \overline{X}_{i,T_i(n)} \right)^{1/p} \right] - \mu^*_n. \tag{16}
\]

In the proof, we will make use of the following inequalities:

\[
0 \leq X_i \leq 1, \tag{17}
\]

\[
x^{1/p} \leq y^{1/p} \text{ when } 0 \leq x \leq y, \tag{18}
\]

\[
(x+y)^m \leq x^m + y^m (0 \leq m \leq 1), \tag{19}
\]

\[
E[f(X)] \leq f(E[X]) \quad (f(X) \text{ is concave}). \tag{20}
\]

With \( i^* \) being the index of the optimal arm, we can derive an upper bound on the difference between the value backup and the true average reward:

\[
E\left[\left( \sum_{i=1}^{K} \frac{T_i(n)}{n} \overline{X}_{i,T_i(n)} \right)^{1/p} \right] - \mu^*_n \\
\leq E\left[\left( \sum_{i=1,i\neq i^*}^{K} \frac{T_i(n)}{n} + \overline{X}_{i*,T_{i^*}(n)} \right)^{1/p} \right] - \mu^*_n \text{ (see (17))} \\
\leq E\left( \sum_{i=1,i\neq i^*}^{K} \frac{T_i(n)}{n} \right)^{1/p} + E[\overline{X}_{i*,T_{i^*}(n)}] - \mu^*_n \\
= E\left( \sum_{i=1,i\neq i^*}^{K} \frac{T_i(n)}{n} \right)^{1/p} + \mu^*_n - \mu^*_n \\
\leq \left( \sum_{i=1,i\neq i^*}^{K} E[T_i(n)] \right)^{1/p} \text{ (see (20))} \\
\leq (K-1)O\left(\frac{K(C^2 \log n + N_0)}{n}\right)^{1/p} \text{ (Theorem 1 & (18))} \tag{21}
\]

According to Lemma 1, it holds that

\[
E[\overline{X}_n(p)] \geq E[\overline{X}_n]
\]

for \( p \geq 1 \). Because of this, we can reuse the lower bound given by (15):

\[
- O\left(\frac{K(C^2 \log n + N_0)}{n}\right) \leq E[\overline{X}_n] - \mu^*_n,
\]

so that:

\[
- O\left(\frac{K(C^2 \log n + N_0)}{n}\right) \leq E[\overline{X}_n] - \mu^*_n \\
\leq E[\overline{X}_n(p)] - \mu^*_n. \tag{22}
\]

Combining (21) and (22) concludes our prove:

\[
|E[\overline{X}_n(p)] - \mu^*_n| \leq |\delta^*_n| + O\left(\frac{K(C^2 \log n + N_0)}{n}\right)^{1/p}.
\]

\( \square \)

We refer to the other theorems in (Kocsis, Szepesvári, and Willemsen 2006), which we do not need to modify for our approach. In particular, Theorem 4 derives a lower bound of the number of time an arm is pulled by a factor of (15); Theorem 5 deals with the concentration of estimated payoff around its mean; Theorem 6 studies the convergence of failure probability.
Theorem 3. Consider the Power-UCT algorithm running on a game tree of depth D, branching factor K with stochastic payoff at the leaves. Assume that the payoffs lie in the interval [0, 1]. Then the bias of the estimated expected payoff, $\overline{X}_n$, is $O(KD(\log(n)/n)^{1/p} + K^{D-1}(1/n)^{1/p})$. Further, the failure probability at the root convergences to zero as the number of samples grows to infinity.

Proof. The proof is done by induction on D. When D = 1, Power-UCT becomes UCB1 problem and the convergence is guaranteed directly from Theorem 2 and Theorem 6.

Now we assume that the result holds up to depth $D-1$ and consider the tree of Depth D. Running Power-UCT on root node is equivalence as UCB1 on non-stationary bandit settings. The error bound of running Power-UCT for the whole tree is the sum of payoff at root node with payoff starting from any node i after the first action chosen from root node until the end. This payoff by induction at depth $(D-1)$ is

$$O(K(D-1)(\log(n)/n)^{1/p} + K^{D-1}(1/n)^{1/p}).$$

According to the Theorem 2, the payoff at the root node is

$$|\delta^*_n| + O\left(\frac{K(\log n + N_0)}{n}\right)^{1/p}.$$

The payoff of the whole tree with depth D:

$$|\delta^*_n| + O\left(\frac{K(\log n + N_0)}{n}\right)^{1/p} = O(K(D-1)(\log(n)/n)^{1/p} + K^{D-1}(1/n)^{1/p}) + O\left(\frac{K(\log n + N_0)}{n}\right)^{1/p} \leq O(K(D-1)(\log(n)/n)^{1/p} + K^{D-1}(1/n)^{1/p}) + O\left(\frac{K}{n}\right)^{1/p} + KN_0 \left(\frac{1}{n}\right)^{1/p} = O(KD(\log(n)/n)^{1/p} + K^{D}(1/n)^{1/p})$$

with $N_0 = O((K - 1)K^{D-1})$, which completes our proof of the convergence of Power-UCT. Interestingly, the proof guarantees the convergence for any finite value of $p$. □

Experiments

In the experiments, we ask the question: does using the power mean instead of the mean in MCTS offer higher performance in practice? To answer this question we compare the Power-UCT algorithm to standard UCT in both an MDP problem and in POMDP problems.

FrozenLake

For MDPs, we consider the well-known FrozenLake problem as implemented in OpenAI Gym [Brockman et al. 2016]. In this problem, the agent needs to reach a goal position in an 8x8 grid ice world while avoiding falling into the water by stepping onto unstable spots. The challenge of this task arises from the high-level of stochasticity, which makes the agent only move towards the intended direction one-third of the time, and into one of the two tangential directions the rest of it. Reaching the goal position yields a reward of 1, while all other outcomes (reaching the time limit or falling into the water) yield a reward of zero.

As can be seen in Figure 1, Power-UCT improves the performance compared to UCT. We adapt the value of $p$ to the number of visits according to equation 11. We ran Bayesian Optimization to find the hyper-parameters of this function and set the UCT-Constant to $C = \sqrt{2}$ for both UCT and Power-UCT and finds it works for $p_{\text{init}} = 1.852$, $n_0 = 80$, $p_{\text{max}} = 2.295$, $p_{\text{stepsize}} = 38.292$.

We also investigated the influence of the tree depth in dependence of the value of $p$. For this, we computed the average number of steps until the end of the search tree for different values of constant $p$ in the FrozenLake environment. The results in Table 1 indicate that a higher value of $p$ tends to lead to a deeper search-tree. When reviewing Table 1, it is important to note that due to its stochasticity, the FrozenLake environment has an effective branching factor of 12, leading to a strong growth of the number of nodes in the tree with increasing depth.

Rocksample and PocMan

In POMDP problems, we compare Power-UCT against the POMCP algorithm [Silver and Veness 2010] which is a standard UCT algorithm for POMDPs. Since the state is not fully observable in POMDPs, POMCP assigns a unique action-observation history, which is a sufficient statistic for optimal decision making in POMDPs, instead of the state, to each tree node. Similarly to fully observable UCT, POMCP chooses actions using the UCB1 bandit. Therefore, modifying POMCP to use the power mean to get a POMDP version of Power-UCT can be done identically to modifying stan-
We further evaluate our algorithm in the pocman environment. In pocman, an agent called PocMan must travel in a maze of size (17x19) by only observing the local neighborhood in the maze. PocMan tries to eat as many as possible food pellets. Four ghosts try to kill PocMan. After moving initially randomly the ghosts start to follow directions with a high number of food pellets more likely. If PocMan eats a power pill, he is able to eat ghosts for 15 time steps. PocMan receives a reward of $-1$ at each step he travels, $+10$ for eating each food pellet, $+25$ for eating a ghost and $-100$ for dying. The pocman problem has 4 actions, 1024 observations, and approximately $10^{96}$ states.

Fig. 2 shows that Power-UCT converges faster to the optimum compared to POMCP. We reuse the hyper-parameters for computing $p$ from the rocksample problem and only change $n_o$ from 1500 to 1000.

Table 1: Average search tree depth of the Power-UCT algorithm for different constant values of $p$ and number of rollouts. Mean ± two times standard deviation are computed from 100 trials.

| $p$ | 1024 | 4996 | 10384 | 131072 |
|-----|------|------|-------|--------|
| $p = 1$ | 3.06 ± 0.17 | 3.55 ± 0.28 | 4.04 ± 0.35 | 4.87 ± 0.56 |
| $p = 4$ | 3.09 ± 0.16 | 3.6 ± 0.26 | 4.13 ± 0.24 | 5.08 ± 0.41 |
| $p = 10$ | 3.1 ± 0.18 | 3.64 ± 0.21 | 4.15 ± 0.32 | 5.2 ± 0.41 |

**Conclusion**

We have proposed the use of power mean as a novel backup operator in MCTS, and derived a variant of UCT based on this operator, which we call Power-UCT. We theoretically prove the convergence of Power-UCT to the optimal value, given that the value computed by the power mean lies between the average and the maximum value of children nodes. Moreover, we discuss a heuristic approach to adaptively tune the greediness of power mean, according to the number of visits to each node. The empirical evaluation on stochastic MDPs and POMDPs, evinces the advantages of Power-UCT w.r.t. UCT.

Even if the heuristic approach to adapt the greediness of power mean has an intuitive explanation and good empirical performance, we plan to study a more rigorous and theoretically justified way of doing it. Future work could be analyse the Bias and Variance of Power Mean estimator or analyse the regret bound of Power-UCT in MCTS. Moreover, we are interested to test our methodology in more challenging Reinforcement Learning problems through the use of parametric function approximator, e.g. neural networks.
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