VERSAL DG DEFORMATION OF CALABI–YAU MANIFOLDS

HAYATO MORIMURA

ABSTRACT. We prove the equivalence of the deformation theory for a higher dimensional Calabi–Yau manifold and that for its dg category of perfect complexes by giving a natural isomorphism of the deformation functors. As a consequence, the dg category of perfect complexes on a versal deformation of the original manifold provides a versal Morita deformation of its dg category of perfect complexes. Besides the classical uniqueness up to étale neighborhood of the base, we prove another sort of uniqueness of versal Morita deformations. Namely, given a pair of derived-equivalent higher dimensional Calabi–Yau manifolds, the dg categories of perfect complexes of their algebraic deformations over a common base, which always exist, become quasi-equivalent close to effectivizations. Then the base change along the corresponding first order approximation yields quasi-equivalent versal Morita deformations. We introduce the generic fiber of the versal Morita deformation as a Drinfeld quotient, which is quasi-equivalent to the dg category of perfect complexes on the generic fiber of the versal deformation.

1. Introduction

The derived category of coherent sheaves on an algebraic variety is an intensively studied invariant which carries rich information about geometric properties of the variety. For instance, given a smooth projective variety either of whose canonical or anticanonical bundle is ample, one can reconstruct the variety from its derived category [BO01]. The condition guarantees the absence of nontrivial autoequivalences of the derived category. Such autoequivalences often stem from the derived equivalence of nonisomorphic, sometimes even nonbirational Calabi–Yau manifolds. According to the homological mirror symmetry conjecture by Kontsevich, derived-equivalent Calabi–Yau manifolds should share their mirror partner. Usually, the homological mirror symmetry is considered for families of Kähler manifolds.

A goal of this paper is to study the relationship between deformations and the derived category of a higher dimensional Calabi–Yau manifold. There seems to be a consensus among some experts that deforming an algebraic variety and its derived category are essentially the same. Philosophically, it is reasonable since their Hochschild cohomology, which in general is known to control deformations of a mathematical object, are isomorphic. However, before [LV06b] we were not given the correct framework to study deformations of even linear nor abelian categories. To every second Hochschild cocycle on a smooth projective variety, Toda associated the category of twisted coherent sheaves on the corresponding noncommutative scheme over the ring of dual numbers [Tod09]. In [DLL17] Dinh–Liu–Lowen showed that Toda’s construction indeed yields flat abelian first order deformations of the category of coherent sheaves on the variety in the sense of [LV06b]. We fill the gap between this point and the conclusion stated below more precisely.

Let $X_0$ be a Calabi–Yau manifold of dimension more than two in the strict sense, i.e., a smooth projective $k$-variety with $\omega_{X_0} \cong \mathcal{O}_{X_0}$ and $H^i(\mathcal{O}_{X_0}) = 0$ for $0 < i < \dim X_0$. We denote by $\text{Perf}_{dg}(X_0)$ the dg category of perfect complexes on $X_0$. The deformation functor

$$\text{Def}_{X_0} : \text{Art}_k \to \text{Set}$$

sends each local artinian $k$-algebra $A \in \text{Art}_k$ with residue field $k$ to the set of equivalence classes of $A$-deformations of $X_0$ and each morphism $B \to A$ in $\text{Art}_k$ to the map $\text{Def}_{X_0}(B) \to \text{Def}_{X_0}(A)$.
induced by the base change. Consider another deformation functor
\[ \text{Def}_{\text{Perf}_d}(X_0)^{\text{embr}}: \text{Art}_k \to \text{Set} \]
which sends each \( A \in \text{Art}_k \) to the set of isomorphism classes of Morita \( A \)-deformations of \( \text{Perf}_d(X_0) \) and each morphism \( B \to A \) in \( \text{Art}_k \) to the map \( \text{Def}_{\text{Perf}_d}(X_0)^{\text{embr}}(B) \to \text{Def}_{\text{Perf}_d}(X_0)^{\text{embr}}(A) \) induced by the derived dg functor \( - \otimes B \). Our first main result claims that the deformation theory for \( X_0 \) is equivalent to that for \( \text{Perf}_d(X_0) \) in the following sense.

**Theorem 1.1.** (Theorem 7.1) There is a natural isomorphism
\[ \zeta: \text{Def}_{X_0} \to \text{Def}_{\text{Perf}_d}(X_0)^{\text{embr}} \]
of deformation functors.

In particular, Morita deformations of \( \text{Perf}_d(X_0) \) is controlled by the Kodaira–Spencer differential graded Lie algebra. To obtain \( \zeta \) we need to consider certain maximal partial curved dg deformations of \( \text{Perf}_d(X_0) \). Curved dg deformations of a dg category is a special case of curved \( A_{\infty} \)-deformations of an \( A_{\infty} \)-category. Let \((a, \mu)\) be a dg category over \( R \in \text{Art}_k \) with a square zero extension
\[ 0 \to I \to S \to R \to 0. \]
Choose generators \( \epsilon = (\epsilon_1, \ldots, \epsilon_l) \) of \( I \) regarded as a free \( R \)-module of rank \( l \). By [Low08, Theorem 4.11] there is a bijection
\[ (1.1) \quad H^2 C(a)^{\text{ad}} \to \text{Def}_{\text{ad}}^{\text{dg}}(S), \quad \phi \mapsto a_\phi = (a[\epsilon], \mu + \phi \epsilon) \]
where \( \phi \) is a Hochschild cocycle. In other words, curved dg \( S \)-deformations of \( a \) are classified by the direct sum of the second Hochschild cohomology.

Assume that \( a \) is an \( R \)-linear category. We denote by \( \text{Com}^+(a) \) the dg category of bounded below complexes of \( a \)-objects. Then by [Low08, Theorem 4.8] the characteristic morphism
\[ \chi_{a}^{\text{ad}}: H^* C(a)^{\text{ad}} \to \mathfrak{z}^* K^+(a)^{\text{ad}} \]
maps \( \phi \in Z^2 C(a)^{\text{ad}} \) to obstructions against deforming objects of \( K^+(a) \) to objects of \( K^+(a_\phi) \). In particular, for each \( C \in K^+(a) \) there exists a lift to \( K^+(a_\phi) \) if and only if \( \chi_{a}^{\text{ad}}(\phi)_C = 0 \). The characteristic morphism \( \chi_{a}^{\text{ad}} \) is induced by a \( B_{\infty} \)-section
\[ \text{embr}_{\beta}: C(C(\text{Com}^+(a))^{\text{ad}} \to C(\text{Com}^+(a))^{\text{ad}} \]
of the canonical projection, which is a quasi-isomorphism of \( B_{\infty} \)-algebras [Low08, Theorem 3.22]. Hence \((1.1) \) induces another bijection
\[ (1.2) \quad H^2 C(a)^{\text{ad}} \to \text{Def}_{\text{Com}^+(a)}^{\text{dg}}(S), \quad \phi \mapsto \text{Com}^+(a)_{\text{embr}_{\beta}(\phi)} = (\text{Com}^+(a)[\epsilon], \text{embr}_{\beta}(\mu + \phi \epsilon)). \]
From the proof of [Low08, Theorem 4.8] it follows that \( \chi_{a}^{\text{ad}}(\phi)_C = 0 \) if and only if the curvature element \( \text{embr}_{\beta}(\mu + \phi \epsilon))_{0,C} \) vanishes for each \( C \in \text{Com}^+(a) \). Hence any full dg subcategory of \( \text{Com}^+(a) \) spanned by object \( C \) with \( \chi_{a}^{\text{ad}}(\phi)_C = 0 \) dg deforms along the restriction of \( \text{embr}_{\beta}(\phi) \).

Let \( X \) be an \( R \)-deformation of \( X_0 \) and \( X_\phi \) its deformation along a cocycle \( \phi \in HH^2(X)^{\text{ad}} = H^1(\mathcal{J}_{X/R}^{X/R})^{\text{ad}} \). The above argument can be adapted to our setting so that for each \( E \in \text{Perf}_d(X) \) the curvature element vanishes if and only if there exists a lift of \( E \in \text{Perf}(X) \) to \( \text{Perf}(X_\phi) \). With a little more effort one can apply [KL09, Proposition 3.12] to obtain

**Theorem 1.2.** (Theorem 6.8) There is a bijection
\[ \text{Def}_{\text{Perf}_d}(X)(S) \to \text{Def}_{\text{Perf}_d}(X)^{\text{embr}}(S) \]
between the set of isomorphism classes of Morita \( S \)-deformations and the set of isomorphism classes of curved \( dg \) \( S \)-deformations of \( \text{Perf}_d(X) \).
In particular, giving curved dg $S$-deformations of $\text{Perf}_{dg}(X)$ is equivalent to giving its Morita $S$-deformations. Consider the dg category $\text{Perf}_{dg}(X_\phi)$ of perfect complexes on $X_\phi$. It defines a Morita $S$-deformation of $\text{Perf}_{dg}(X)$. Let 

$$m(\phi) = \text{Perf}_{dg}(X_\phi) \otimes_R S$$

be the image of the derived base change. Then any $h$-flat resolution $\text{Perf}_{dg}(X_\phi)$ defines a dg deformation $\text{Perf}_{dg,\Gamma}(X_\phi)$ of $m(\phi)$, where $\text{Perf}_{dg,\Gamma}(X_\phi)$ is the full dg subcategory of $\text{Perf}_{dg}(X_\phi)$ consisting the collection of one chosen lift of each object in $m(\phi)$. There is an isomorphism

$$HH^2(X)^{\text{eff}} \cong H^2C(\text{Perf}_{dg}(X))^{\text{eff}}$$

induced by the $B_{\infty}$-section

$$\text{embr}_\phi : C(\text{Inj}(Qch(X)))^{\text{eff}} \to C(\text{Com}^+(\text{Inj}(Qch(X))))^{\text{eff}},$$

where $\text{Inj}(Qch(X)) \subset Qch(X)$ is the full $R$-linear subcategory of injective objects. We denote by $\text{embr}_\phi(\phi)$ the image of $\phi$ under the isomorphism, which defines another dg $S$-deformation $m(\phi)_{\text{embr}_\phi(\phi)}$ of $m(\phi)$ along $\text{embr}_\phi(\phi)$. Deformations and taking the dg category of perfect complexes intertwine in the following sense.

**Theorem 1.3.** (Theorem [6,9]) There is an isomorphism

$$\text{Perf}_{dg,\Gamma}(X_\phi) \cong m(\phi)_{\text{embr}_\phi(\phi)}$$

of dg $S$-deformations of $m(\phi)$. In particular, the Morita $S$-deformation $\text{Perf}_{dg}(X_\phi)$ defines a maximal partial dg $S$-deformation of $\text{Perf}_{dg}(X)$ along $\text{embr}_\phi(\phi)$.

This is the key to prove Theorem [11]. Unwinding Toda’s construction, from [DLL17, Theorem 5.12] we obtain an equivalence $Qch(X)_\phi \cong Qch(X_\phi)$ of Grothendieck abelian categories, where $Qch(X)_\phi$ is the flat abelian $S$-deformation of $Qch(X)$ along $\phi \in Z^2C_{ab}(Qch(X))^{\text{eff}}$. Here, we use the same symbol $\phi$ to denote the image under the isomorphism

$$HH^2(X)^{\text{eff}} \cong H^2C_{ab}(Qch(X))^{\text{eff}}.$$

Via the induced equivalence

$$D_{dg}(Qch(X_\phi)) \cong D_{dg}(Qch(X_\phi))$$

we regard $\text{Perf}_{dg}(X_\phi)$ as the full dg subcategory of compact objects of $D_{dg}(Qch(X_\phi))$. Based on the idea in the proof of [Low08, Theorem 4.15], we compare the dg structure on $\text{Perf}_{dg,\Gamma}(X_\phi)$ with that on $m(\phi)_{\text{embr}_\phi(\phi)}$.

Working with Morita deformations of $\text{Perf}_{dg}(X_0)$, by [Coh] Corollary 5.7 we may apply [BFN10, Theorem 1.2] to obtain reductions. In particular, given a deformation $(X_B, i_B) \in \text{Def}_{X_0}(B)$ and a morphism $B \to A$ in $\text{Art}_k$, there is a Morita equivalence

$$\text{Perf}_{dg}(X_B) \otimes^L_B A \cong_{\text{mo}} \text{Perf}_{dg}(X_B) \otimes^L_B \text{Perf}_{dg}(A) \cong_{\text{mo}} \text{Perf}_{dg}(X_A)$$

of $A$-linear dg categories, where $- \otimes^L_B -$ is the derived pointwise tensor product of dg categories. Further application of [BFN10, Theorem 1.2] shows that any universal formal family for $\text{Def}^{\text{mor}}_{\text{Perf}_{dg}(X_0)}$ is effective. If we ignore the set theoretical issues, the deformation functor $\text{Def}^{\text{mor}}_{\text{Perf}_{dg}(X_0)}$ can naturally be extended to a functor defined on the category $\text{Alg}^{\text{mor}}(k)$ of augmented noetherian $k$-algebras. Although we do not know whether it would be locally of finite presentation (colimit preserving), one can always construct a versal Morita deformation via geometric realization in the following sense.

**Corollary 1.4.** (Corollary [7,4]) Any effective universal formal family for $\text{Def}^{\text{mor}}_{\text{Perf}_{dg}(X_0)}$ is algebraizable. In particular, an algebraization is given by $\text{Perf}_{dg}(X_S)$ where $(\text{Spec}S, s, X_S)$ is a versal deformation of $X_0$. 

3
The versal Morita deformation $\text{Perf}_{dg}(X_S)$ may be regarded as a family of Morita deformations of $\text{Perf}_{dg}(X_0)$. More generally, for such a family determined by an enough nice $S$-scheme $X_S$ we introduce its generic fiber as follows.

**Definition 1.5.** Let $X_S$ be a smooth separated scheme over a noetherian connected regular affine $k$-scheme $\text{Spec } S$ whose closed points are $k$-rational. Then the dg categorical generic fiber of $\text{Perf}_{dg}(X_S)$ is the Drinfeld quotient

$$\text{Perf}_{dg}(X_S)/\text{Perf}_{dg}(X_S)_0,$$

where $\text{Perf}_{dg}(X_S)_0 \subset \text{Perf}_{dg}(X_S)$ is the full dg subcategory of perfect complexes with $S$-torsion cohomology.

We impose a technical assumption on $S$ to include also the case where $S$ is a formal power series ring. The Drinfeld quotient is a natural dg enhancement of the categorical generic fiber introduced in [Morb], which is in turn based on the categorical general fiber by Huybrechts–Macrì–Stellari [HMS11]. Taking the generic fiber and the dg category of perfect complexes intertwine in the following sense.

**Proposition 1.6.** (Proposition 7.5) Let $X_S$ be a smooth separated scheme over a noetherian connected regular affine $k$-scheme $\text{Spec } S$ whose closed points are $k$-rational. Then there is a quasi-equivalence

$$\text{Perf}_{dg}(X_S)/\text{Perf}_{dg}(X_S)_0 \cong_{\text{qeq}} \text{Perf}_{dg}(X_{Q(S)}),$$

where $Q(S)$ is the quotient field of $S$ and $X_{Q(S)}$ is the generic fiber of $X_S$.

Another goal of this paper is to show the uniqueness of versal Morita deformations with respect to geometric realizations. Recall that up to étale neighborhood of the base versal deformations of $X_0$ are unique. Namely, if $(\text{Spec } S, s, X_S) (\text{Spec } S', s', X_S')$ are two versal deformations of $X_0$, then there is another versal deformation $(\text{Spec } S'', s'', X_S'')$ such that $(\text{Spec } S'', s'')$ is an étale neighborhood of $s, s'$ in $\text{Spec } S$. The versal deformation $\text{Perf}_{dg}(X_S)$ may be regarded only by quasi-equivalent universal formal families and the same sufficiently large index $j \in I$. Hence the ambiguity of $X_S$ stems from the choice of $j \in I$, besides the choice of étale neighborhoods.

In [Mora] the author constructed smooth projective versal deformations $X_S, X_S'$ of $X_0, X_0'$ over a common nonsingular affine variety $\text{Spec } S$, while deforming simultaneously the Fourier–Mukai kernel connecting deformations of $X_0, X_0'$. By Corollary 1.4 we have two versal Morita deformations $\text{Perf}_{dg}(X_S), \text{Perf}_{dg}(X_S')$ of $\text{Perf}_{dg}(X_0)$. Theorem 1.1 together with the construction of versal deformations suggests that $\text{Perf}_{dg}(X_S), \text{Perf}_{dg}(X_S')$ should be determined only by quasi-equivalent universal formal families and the same sufficiently large index $j \in I$. From this observation we arrive at our second main result.

**Theorem 1.7.** (Theorem 8.3) Let $X_0, X_0'$ be derived-equivalent Calabi–Yau manifolds of dimension more than two and $P_0 \in D^b(X_0 \times_k X_0')$ the Fourier–Mukai kernels. Then there exists an index $j \in I$ such that for all $k \geq j$ the integral functors

$$\Phi_{P_k}: \text{Perf}(X_{R_j}) \to \text{Perf}(X'_{R_j})$$
defined by deformations \( \mathcal{P}_k \) of \( \mathcal{P}_0 \) are equivalences of triangulated categories of perfect complexes. In particular, the dg categories \( \text{Perf}_d(X_{R_k}), \text{Perf}_d(X'_{R_k}) \) of perfect complexes are quasi-equivalent.

Theorem \[1.7\] tells us that, given two algebraic Morita deformations \( \text{Perf}_d(X_{R_k}), \text{Perf}_d(X'_{R_k}) \) geometrically realized by algebraic deformations \( X_{R_k}, X'_{R_k} \) of two derived-equivalent higher dimensional Calabi–Yau manifolds \( X_0, X'_0 \), if \( X_{R_k}, X'_R \) are enough close to effectivizations \( X_{R_k}, X'_R \) then \( \text{Perf}_d(X_{R_k}), \text{Perf}_d(X'_{R_k}) \) are Morita equivalent. The base change along the homomorphism \( R_k \rightarrow S \) yields Morita equivalent versal Morita deformations \( \text{Perf}_d(X_S), \text{Perf}_d(X'_S) \). In other words, up to Morita equivalence the versal Morita deformation \( \text{Perf}_d(X_S) \) does not depend on the choice of geometric realizations in the following sense.

**Corollary 1.8.** (Corollary \[8.4\]) Let \( X_0, X'_0 \) be derived-equivalent Calabi–Yau manifolds of dimension more than two and \( X_S, X'_S \) their smooth projective versal deformations over a common nonsingular affine \( k \)-variety \( \text{Spec} \, S \). Let \( (R_i)_{i \in I} \) be a filtered inductive system of finitely generated \( T \)-subalgebras \( R_i \subset R \) whose colimit is \( R \) with

\[
T = k[t_1, \ldots, t_d], \quad R \cong k[[t_1, \ldots, t_d]], \quad d = \dim_k H^1(\mathcal{F}_{X_0}).
\]

Assume that \( X_S, X'_S \) correspond to a first order approximation \( R_j \rightarrow S \) of \( R_j \leftarrow R \) for sufficiently large \( j \in I \). Then \( X_S, X'_S \) are derived-equivalent. In particular, the dg categories \( \text{Perf}_d(X_S), \text{Perf}_d(X'_S) \) of perfect complexes are quasi-equivalent.

The uniqueness result also holds for the dg categorical generic fiber. Corollary \[1.8\] slightly improves [Mora, Theorem 1.1], which extends the derived equivalence from special to general fibers. Here, the advantage is that we do not have to shrink the base \( \text{Spec} \, S \) as long as the construction passes enough close to effectivizations. In particular, beginning with a pair of general fibers, one obtains the derived equivalence of special fibers contained in the versal deformations. Hence the above corollary partially provides a method for the opposite direction, i.e., how to extend the derived equivalence from general to special fibers.

**Notations and conventions.** We work over an algebraically closed field \( k \) of characteristic 0 throughout this paper. For an augmented \( k \)-algebra \( A \) by \( \mathfrak{m}_A \) we denote its augmentation ideal. All higher dimensional Calabi–Yau manifolds we treat are smooth projective \( k \)-varieties \( X_0 \) of dimension more than two with \( \omega_{X_0} \cong \mathcal{O}_{X_0} \) and \( H^i(\mathcal{O}_{X_0}) = 0 \) for \( 0 < i < \dim X_0 \).

**Acknowledgements.** The author is supported by SISSA PhD scholarships in Mathematics. The author would like to thank Yukinobu Toda for pointing out mistakes in earlier version.

### 2. Hochschild cohomology of relatively smooth proper schemes

In this section, we review various kinds of complexes whose cohomology controls deformations of associated mathematical objects, mainly following the exposition from [DLL17] Section 2, 3. We always assume that all algebras have units, morphism of algebras preserve units, and modules are unital. In the sequel, we fix a local artinian \( k \)-algebra \( R \) with residue field \( k \) and its square zero extension

\[
0 \rightarrow I \rightarrow S \rightarrow R \rightarrow 0,
\]

and choose generators \( \epsilon = (\epsilon_1, \ldots, \epsilon_l) \) of \( I \) regarded as a free \( R \)-module of rank \( l \). For smooth proper \( R \)-schemes, we explain the correspondence between its relative Hochschild cohomology and cohomology of the Gerstenhaber–Shack complex associated with its restricted structure sheaf.
2.1. **Relative Hochschild cohomology of schemes.** Let $X$ be a smooth proper $R$-scheme. We denote by $\Delta_R : X_R \hookrightarrow X \times_R X$ the relative diagonal embedding. The *relative Hochschild cohomology* is defined as the graded $R$-algebra

$$HH^\bullet(X/R) = \text{Ext}_X^\bullet(\mathcal{O}_{\Delta R}, \mathcal{O}_{\Delta R}) \cong \text{Ext}_X^\bullet(\Delta_R^* \mathcal{O}_{\Delta R}, \mathcal{O}_X).$$

Here, the multiplication in $HH^\bullet(X/R)$ is given by the composition in $D(X \times_R X)$. Then the natural map $R \to \text{End}_{X \times_R X}(\mathcal{O}_{\Delta R})$ induces the $R$-algebra structure. There is a quasi-isomorphism

$$\Delta_R^* \mathcal{O}_{\Delta R} \cong \bigoplus_i \Omega^i_{X/R}[i]$$

called the *relative Hochschild–Kostant–Rosenberg isomorphism*, which induces an isomorphism

$$\text{Ext}_X^\bullet(\Delta_R^* \mathcal{O}_{\Delta R}, \mathcal{O}_X) \to \text{Ext}_X^\bullet\left(\bigoplus_i \Omega^i_{X/R}[i], \mathcal{O}_X\right) \to \bigoplus_i H^{i-\bullet}(X, \wedge^i \mathcal{T}_{X/R})$$

where $\mathcal{T}_{X/R}$ is the relative tangent sheaf and $\Omega^i_{X/R}$ is its dual. We also call the compositions

$$H^{n}_{\text{HKR}} : \text{Ext}_X^n(\mathcal{O}_{\Delta R}, \mathcal{O}_{\Delta R}) = HH^n(X/R) \to HT^n(X/R) = \bigoplus_{p+q=n} H^p(X, \wedge^q \mathcal{T}_{X/R})$$

the relative Hochschild–Kostant–Rosenberg isomorphisms.

2.2. **Hochschild cohomology of algebras.** Let $A = (A, m)$ be an $R$-algebra and $M$ an $A$-bimodule. The Hochschild complex $C(A, M)$ has $C^n(A, M) = \text{Hom}_R(A^n, M)$ as its $n$-th term and $d^n_{\text{Hoch}} : C^n(A, M) \to C^{n+1}(A, M)$, called the *Hochschild differential*, as its differential which is given by

$$d^n_{\text{Hoch}}(\phi)(a_n, a_{n-1}, \ldots, a_0) = a_n \phi(a_{n-1}, \ldots, a_0) + \sum_{i=0}^{n-1} (-1)^i \phi(a_n, \ldots, a_{n-i}a_{n-i-1}, \ldots, a_0) + (-1)^{n+1} \phi(a_n, \ldots, a_1)a_0.$$

A cochain $\phi \in C^n(A, M)$ is *normalized* if $\phi(a_{n-1}, \ldots, a_0) = 0$ whenever $a_i = 1$ for some $0 \leq i \leq n-1$. The normalized cochains form a subcomplex $\tilde{C}(A, M)$ quasi-isomorphic to $C(A, M)$ via the inclusion. When $M = A$, we call $C(A) = C(A, A)$ the Hochschild complex and $H^nC(A)$ the $n$-th Hochschild cohomology of $A$. Note that the multiplication $m$ on $A$ belongs to $C^2(A)$.

The direct sum of the second normalized Hochschild cohomology of $A$ classifies $S$-deformations of $A$ up to equivalence. Recall that an $S$-*deformation* of $A$ is an $S$-algebra $(\tilde{A}, \tilde{m}) = (A[\epsilon] = A \otimes_R S, m + m\epsilon)$ with $m \in C^2(A)^{gl}$ such that the unit of $\tilde{A}$ is the same as that of $A$. Two deformations $(\tilde{A}, \tilde{m}), (\tilde{A}', \tilde{m}')$ are *equivalent* if there is an isomorphism of the form $1 + g\epsilon : \tilde{A} \to \tilde{A}'$ with $g \in C^1(A)^{gl}$. We denote by $\text{Def}_A^{alg}(S)$ the set of equivalence classes of $S$-deformations of $A$. It is known that there is a bijection

$$H^2\tilde{C}(A)^{gl} \to \text{Def}_A^{alg}(S), \ m \mapsto (A[\epsilon], m + m\epsilon), \ m \in Z^2\tilde{C}(A)^{gl}.$$

2.3. **Simplicial cohomology of presheaves.** Let $\mathcal{U}$ be a small category and $N(\mathcal{U})$ its simplicial nerve. We write

$$\sigma = (d\sigma = U_0 \xrightarrow{u_1} U_1 \xrightarrow{u_2} \cdots \xrightarrow{u_p} U_p \xrightarrow{u_{p+1}} U_{p+1} = c\sigma)$$
for a \((p+1)\)-simplex \(\sigma \in N_{p+1}(\mathcal{U})\). Let \((\mathcal{F}, f), (\mathcal{G}, g)\) be presheaves of \(\mathbf{R}\)-modules with restriction maps \(f^u: \mathcal{F}(U) \to \mathcal{F}(V), g^u: \mathcal{G}(U) \to \mathcal{G}(V)\) for \(u: V \to U\) in \(\mathcal{U}\). We write \(f^\tau\) for the map \(f_{u_{p+1} \cdots u_1}: \mathcal{F}(U_{p+1}) \to \mathcal{F}(U_0)\). Consider a complex whose \(p\)-th term is

\[
C^p_{\text{simp}}(\mathcal{G}, \mathcal{F}) = \prod_{\tau \in N_p(\mathcal{U})} \text{Hom}_R(\mathcal{G}(c\tau), \mathcal{F}(d\tau)).
\]

and whose differential \(d^p_{\text{simp}}\) is defined as follows. Recall that we have the maps

\[
\partial_i: N_{p+1}(\mathcal{U}) \to N_p(\mathcal{U}), \quad \sigma \mapsto \partial_i \sigma,
\]

for \(i = 0, 1, \ldots, p+1\) given by

\[
\partial_i \sigma = (U_0 \xrightarrow{u_1} \cdots U_{i-1} \xrightarrow{u_i} U_{i+1} \xrightarrow{u_{i+1}} \cdots U_p \xrightarrow{u_{p+1}} U_{p+1}), \quad i \neq 0, p+1,
\]

\[
\partial_0 \sigma = (U_1 \xrightarrow{u_2} U_2 \xrightarrow{u_3} \cdots U_p \xrightarrow{u_{p+1}} U_{p+1}),
\]

\[
\partial_{p+1} \sigma = (U_0 \xrightarrow{u_1} U_1 \xrightarrow{u_2} \cdots U_p).
\]

Each \(\partial_i\) induces a map

\[
d_i: C^p_{\text{simp}}(\mathcal{G}, \mathcal{F}) \to C^{p+1}_{\text{simp}}(\mathcal{G}, \mathcal{F}), \quad \phi = (\phi^\tau)_\tau \mapsto d_i \phi = ((d_i \phi)^\tau)_\tau
\]

given by

\[
(d_i \phi)^\tau = \phi^{0, \tau}, \quad i \neq 0, p+1,
\]

\[
(d_0 \phi)^\tau = f^\tau \circ \phi^{0, \tau},
\]

\[
(d_{p+1} \phi)^\tau = \phi^{p+1, \tau} \circ g^{u_{p+1}}.
\]

Then one defines

\[
d_{\text{simp}} = \sum_{i=0}^{p+1} (-1)^i d_i: C^p_{\text{simp}}(\mathcal{G}, \mathcal{F}) \to C^{p+1}_{\text{simp}}(\mathcal{G}, \mathcal{F}).
\]

When \(\mathcal{G}\) is the constant presheaf \(\mathbf{R}\), we call \(H^p(\mathcal{U}, \mathcal{F}) = H^p C^\text{simp}_{\text{simp}}(\mathcal{G}, \mathcal{F}) = H^p C^\text{simp}_{\text{simp}}(\mathbf{R}, \mathcal{F})\) the simplicial presheaf cohomology of \(\mathcal{F}\). A \((p+1)\)-simplex \(\sigma \in N_{p+1}(\mathcal{U})\) is degenerate if \(u_i = 1_{U_i}\) for some \(1 \leq i \leq p+1\). A \(p\)-cochain \(\phi = (\phi^\tau)_\tau \in C^p(\mathcal{G}, \mathcal{F})\) is reduced if \(\phi^\tau = 0\) whenever \(\tau\) is degenerate. All 0-cochains are reduced by convention. The reduced cochains are preserved by \(d_{\text{simp}}\) and form a subcomplex \(C^\prime_{\text{simp}}(\mathcal{G}, \mathcal{F})\), which is quasi-isomorphic to \(C_{\text{simp}}(\mathcal{G}, \mathcal{F})\) by \(\text{[DLL17]}\) Proposition 2.9.

The direct sum of the first reduced simplicial presheaf cohomology of \(\mathcal{F}\) classifies \(\mathfrak{S}\)-deformations of \(\mathcal{F}\) up to equivalence. Recall that an \(\mathfrak{S}\)-deformation of \(\mathcal{F}\) is a presheaf of \(\mathfrak{S}\)-modules \((\mathcal{F}, f) = (\mathcal{F}[\epsilon], f + f\epsilon)\) with \(f \in C^1_{\text{simp}}(\mathcal{F}, \mathcal{F})\). Two deformations \((\mathcal{F}, f), (\mathcal{F}', f')\) are equivalent if there is an isomorphism of the form \(1 + g\epsilon\) with \(g \in C^0_{\text{simp}}(\mathcal{F}, \mathcal{F})\). We denote by \(\text{Def}^\text{sh}_{\mathcal{F}}(\mathfrak{S})\) the set of equivalence classes of \(\mathfrak{S}\)-deformations of \(\mathcal{F}\).

**Lemma 2.1.** \(\text{[DLL17]}\) Proposition 2.11) Let \((\mathcal{F}, f)\) be a presheaf of \(\mathbf{R}\)-modules. Then there is a bijection

\[
H^1 C_{\text{simp}}'(\mathcal{F}, \mathcal{F}) = \text{Def}^\text{sh}_{\mathcal{F}}(\mathfrak{S}), \quad \phi \mapsto (\mathcal{F}[\epsilon], f + f\epsilon), \quad f \in Z^1 C_{\text{simp}}'(\mathcal{F}, \mathcal{F}).
\]

Another cocycle \(f' \in Z^1 C_{\text{simp}}'(\mathcal{F}, \mathcal{F})\) maps to an equivalent deformation if and only if there is an element \(g \in C^0_{\text{simp}}(\mathcal{F}, \mathcal{F})\) satisfying \(f' - f = d_{\text{simp}}(g)\).
2.4. Gerstenhaber–Schack complexes. Let \( \mathcal{A} \) be a small category and \((\mathcal{A}, m, f)\) a presheaf of \( R \)-algebras on \( \mathcal{U} \). The Gerstenhaber–Schack complex \( C_{GS}^{\bullet}(\mathcal{A}) \) introduced in [GS88] is the total complex of the double complex whose \((p, q)\)-term for \( p, q \geq 0 \) is
\[
C_{GS}^{p,q}(\mathcal{A}) = \prod_{\tau \in N_\tau(\mathcal{U})} \text{Hom}_R(\mathcal{A}(c\tau)^{0q}, \mathcal{A}(d\tau)),
\]
where we regard \( \mathcal{A}(d\tau) \) as an \( \mathcal{A}(c\tau) \)-bimodule via \( f^*: \mathcal{A}(c\tau) \to \mathcal{A}(d\tau) \). When \( q \) is fixed, we have
\[
C_{GS}^{\bullet,q}(\mathcal{A}) = C_{simp}(\mathcal{A}^{eq}, \mathcal{A})
\]
endowed with the simplicial differential \( d_{simp} \) horizontally. When \( p \) is fixed, we have
\[
C_{GS}^{p,\bullet}(\mathcal{A}) = C(\mathcal{A}(c\tau), \mathcal{A}(d\tau))
\]
endowed with the product Hochschild differential \( d_{Hoch} \) vertically. The differential
\[
d_{GS}^m: C_{GS}^{m}(\mathcal{A}) \to C_{GS}^{m+1}(\mathcal{A})
\]
is defined as \( d_{GS}^m = (-1)^{m+1} d_{simp} + d_{Hoch} \).

A cochain \( \phi = (\phi^\tau)_\tau \in C_{GS}^{p,q}(\mathcal{A}) \) is normalized if \( \phi^\tau \) is normalized for each \( p \)-simplex \( \tau \), and it is reduced if \( \phi^\tau = 0 \) whenever \( \tau \) is degenerate. The normalized cochains form a subcomplex \( \tilde{C}_{GS}^{\bullet}(\mathcal{A}) \) of \( C_{GS}^{\bullet,\bullet}(\mathcal{A}) \) called the normalized Hochschild complex of \( \mathcal{A} \), and the normalized reduced cochains form a subcomplex \( \tilde{C}_{GS}^{\bullet}(\mathcal{A}) \) of \( C_{GS}^{\bullet,\bullet}(\mathcal{A}) \) called the normalized reduced Hochschild complex of \( \mathcal{A} \). These three complexes are quasi-isomorphic via the inclusions. Eliminating the bottom row from \( C_{GS}^{\bullet}(\mathcal{A}) \), one obtains a subcomplex \( C_{GS}^{\bullet}(\mathcal{A}) \) called the truncated Hochschild complex. There is a short exact sequence
\[
0 \to C_{IGS}(\mathcal{A}) \to C_{GS}(\mathcal{A}) \to C_{simp}(\mathcal{A}) \to 0.
\]
Since \( R \) is commutative, one can apply [DLL17, Proposition 2.14] to see that the sequence splits and we have
\[
C_{GS}(\mathcal{A}) = C_{IGS}(\mathcal{A}) \oplus C_{simp}(\mathcal{A}).
\]

Similarly, we have
\[
\tilde{C}_{GS}(\mathcal{A}) = \tilde{C}_{IGS}(\mathcal{A}) \oplus C_{simp}^{\bullet}(\mathcal{A}).
\]

The direct sum of the second normalized reduced Gerstenhaber–Schack cohomology of \( \mathcal{A} \) classifies twisted \( S \)-deformations of \( \mathcal{A} \) up to equivalence. Recall that a twisted presheaf \( \mathcal{A} = (\mathcal{A}, m, f, c, z) \) of \( R \)-algebras on \( \mathcal{U} \) consists of the following data:

- for each \( U \in \mathcal{U} \) an \( R \)-algebra \( (\mathcal{A}(U), m^U) \),
- for each \( u: V \to U \) in \( \mathcal{U} \) a homomorphism of \( R \)-algebras \( f^u: \mathcal{A}(U) \to \mathcal{A}(V) \),
- for each pair \( u: V \to U \), \( v: W \to V \) in \( \mathcal{U} \) an invertible element \( e^{uv} \in \mathcal{A}(W) \) satisfying for any \( a \in \mathcal{A}(U) \)
  \[
e^{uv} f^u(f^v(a)) = f^{uv}(a) e^{uv}.
\]
- for each \( U \in \mathcal{U} \) an invertible element \( e^U \in \mathcal{A}(U) \) satisfying for any \( a \in \mathcal{A}(U) \)
  \[
e^U a = f^U(a) e^U.
\]
Moreover, these data must satisfy
\[
e^{uvw} e^{vw} = e^{uv} f^w(e^{uv}),
e^{uv} e^U = 1, \quad e^{U} f^u(e^U) = 1
\]
for each triple \( u: V \to U, v: W \to V, w: T \to W \) in \( \mathcal{U} \). When \( c^{uv}, z^U \) are central for all \( u, v \) and \( U \), we call \( \mathcal{A} \) a twisted presheaf with central twists and denote by \( |\mathcal{A}| = (\mathcal{A}, m) \) the underlying presheaf.

For twisted sheaves \( \mathcal{A} = (\mathcal{A}, m, f, c, z), \mathcal{A'} = (\mathcal{A'}, m', f', c', z') \) of \( \mathbf{R} \)-algebras on \( \mathcal{U} \), a morphism \((g, h): \mathcal{A} \to \mathcal{A'}\) consists of the following data:

- for each \( U \in \mathcal{U} \) a homomorphism of \( \mathbf{R} \)-algebras \( g^U: \mathcal{A}(U) \to \mathcal{A'}(U) \),
- for each \( u: V \to U \) in \( \mathcal{U} \) an invertible element \( h^u \in \mathcal{A}'(V) \).

Moreover, these data must satisfy

\[
m^U(g^V f^u(a), h^u) = m^U(h^v, f^v(g^U(a))),
\]

\[
m^W(h^v, c^m,v) = m^W(g^W(c^m,v), h^v, f^v(h^u)),
\]

\[
m^U(h^v, z^U) = g^U(z^U)
\]

for all \( u, v \) and \( a \in \mathcal{A}(U) \). Morphisms can be composed and the identity \( 1_\mathcal{A} \) is given by \( g^U = 1_{\mathcal{A}(U)} \) and \( h^u = 1 \in \mathcal{A}'(V) \). When \( g^U \) are isomorphisms of \( \mathbf{R} \)-algebras for all \( U \), we call \((g, h)\) an isomorphism. Any twisted presheaf \( (\mathcal{A}, m, f, c) \) is isomorphic to the one of the form \((\mathcal{A}', m', f', c', 1)\).

Let \( \mathcal{A} = (\mathcal{A}, m, f, c) \) be a twisted presheaf of \( \mathbf{R} \)-algebras on \( \mathcal{U} \). A twisted \( \mathbf{S} \)-deformation of \( \mathcal{A} \) is a twisted presheaf

\[
\mathcal{A} = (\mathcal{A}, \bar{m}, \bar{f}, \bar{c}) = (\mathcal{A}[\epsilon], m + \epsilon m, f + \epsilon f, c + \epsilon c)
\]

of \( \mathbf{S} \)-algebras such that \((\mathcal{A}(U), \bar{m}^U)\) is an \( \mathbf{S} \)-deformation of \((\mathcal{A}(U), m^U)\) for each \( U \in \mathcal{U} \) with

\[
(m, f, c) \in C^2_{GS}(\mathcal{A})^{\mathbf{S}\mathbf{d}} = C^0_{GS}(\mathcal{A})^{\mathbf{d}} \oplus C^1_{GS}(\mathcal{A})^{\mathbf{d}} \oplus C^2_{GS}(\mathcal{A})^{\mathbf{d}}.
\]

Two twisted deformations \((\mathcal{A}, \bar{m}, \bar{f}, \bar{c}), (\mathcal{A'}, \bar{m'}, \bar{f'}, \bar{c'})\) are equivalent if there is an isomorphism of the form \((1 + \epsilon g, 1 + \epsilon h)\) with

\[
(g, h) \in C^1_{GS}(\mathcal{A})^{\mathbf{d}} = C^0_{GS}(\mathcal{A})^{\mathbf{d}} \oplus C^1_{GS}(\mathcal{A})^{\mathbf{d}}.
\]

We denote by \( \text{Def}_{\mathcal{A}}(\mathbf{S}) \) the set of equivalence classes of twisted \( \mathbf{S} \)-deformations of \( \mathcal{A} \).

When \( c = 1 \), a presheaf \( \mathbf{S} \)-deformation of \( \mathcal{A} \) is a twisted \( \mathbf{S} \)-deformation with \( c = 0 \). Two presheaf deformations \((\mathcal{A}, \bar{m}, \bar{f}), (\mathcal{A'}, \bar{m'}, \bar{f'})\) are equivalent if there is an isomorphism of the form \( 1 + \epsilon g \) with \( g \in C^0_{GS}(\mathcal{A})^{\mathbf{d}} \). We denote by \( \text{Def}_{\mathcal{A}}(\mathbf{S}) \) the set of equivalence classes of presheaf \( \mathbf{S} \)-deformations of \( \mathcal{A} \).

**Lemma 2.2.** ([Delli 2017] Theorem 2.21) Let \( (\mathcal{A}, m, f) \) be a presheaf of \( \mathbf{R} \)-algebras on \( \mathcal{U} \). Then there is a bijection

\[
H^2 \tilde{C}_{GS}(\mathcal{A})^{\mathbf{d}} \to \text{Def}_{\mathcal{A}}^{w}(\mathbf{S}), (m, f, c) \mapsto (\mathcal{A}[\epsilon], m + \epsilon m, f + \epsilon f, c + \epsilon c), (m, f, c) \in \mathbb{Z}^2 \tilde{C}_{GS}(\mathcal{A})^{\mathbf{d}}.
\]

Another cocycle \((m', f', c') \in \mathbb{Z}^2 \tilde{C}_{GS}(\mathcal{A})^{\mathbf{d}} \) maps to an equivalent deformation if and only if there is an element \((g, h) \in C^1_{GS}(\mathcal{A})^{\mathbf{d}} \) satisfying \((m', f', c') - (m, f, c) = d_{GS}(g, h) \). In particular, there is a bijection

\[
H^2 \tilde{C}_{GS}(\mathcal{A})^{\mathbf{d}} \to \text{Def}_{\mathcal{A}}^{w}(\mathbf{S}), (m, f) \mapsto (\mathcal{A}[\epsilon], m + \epsilon m, f + \epsilon f).
\]

2.5. **Hodge decomposition.** In the group algebra \( \mathbb{Q} S_n \) of the \( n \)-th symmetric group \( S_n \), there is a collection of pairwise orthogonal idempotents \( e_r \) for \( 1 \leq r \leq n \) such that \( \sum_{r=1}^n e_r(r) = 1 \) ([G-S 1987], Theorem 1.2). Put \( e_r(0) = 0, e_0(0) = 1 \in \mathbb{Q}, \) and \( e_r(r) = 0 \) for \( r > n \). Let \( A \) be a commutative \( \mathbf{R} \)-algebra and \( M \) a symmetric \( A \)-bimodule. The subcomplex \( C(A, M)_r \subset C(A, M) \) whose \( n \)-th term is \( C(A, M)e_n(r) \) gives rise to a Hodge decomposition

\[
C(A, M) = \bigoplus_{r \in \mathbb{N}} C(A, M)_r.
\]
Assume that \( \mathcal{A} \) is a presheaf of commutative \( \mathbb{R} \)-algebras. Then the Hodge decomposition

\[
\text{Hom}(\mathcal{A}(ct)^{0q}, \mathcal{A}(d\tau)) = \bigoplus_{r=0}^{q} \text{Hom}(\mathcal{A}(ct)^{0q}, \mathcal{A}(d\tau))_r
\]

induces a decomposition of the double complex \( C^{n,q}_{\mathcal{G}S}(\mathcal{A}) \) preserved by \( d_{\text{Hoch}} \) and \( d_{\text{simp}} \). Hence one obtains a Hodge decomposition

\[
C^{n,q}_{\mathcal{G}S}(\mathcal{A}) = \bigoplus_{r \in \mathbb{N}} C^{n,q}_{\mathcal{G}S}(\mathcal{A})_r.
\]

Taking cohomology yields a decomposition for \( H^\bullet C^{n,q}_{\mathcal{G}S}(\mathcal{A}) \).

Assume further the following.

- The restriction map \( f^u : \mathcal{A}(U) \to \mathcal{A}(V) \) is a flat epimorphism of rings for each \( u : V \to U \).
- The algebra \( \mathcal{A}(U) \) is essentially of finite type and smooth \( \mathbb{R} \)-algebra for each \( U \).

Recall that a homomorphism of rings is called an epimorphism if it is an epimorphism in the category of noncommutative rings. For instance, every surjective homomorphism of commutative rings is an epimorphism. Then one obtains the presheaf of differential \( \Omega_{\mathcal{A}} : \mathcal{U}^{op} \to \text{Mod}(\mathcal{A}) \) with \( \Omega_{\mathcal{A}}(U) = \Omega_{\mathcal{A}(U)}/\mathbb{R} \). Since we have a canonical isomorphism \( \mathcal{A}(V) \otimes_{\mathcal{A}(U)} \Omega_{\mathcal{A}(U)} \cong \Omega_{\mathcal{A}(V)} \) by the first additional assumption, the induced restriction maps \( \mathcal{T}_{\mathcal{A}(U)/\mathbb{R}} \to \mathcal{T}_{\mathcal{A}(V)/\mathbb{R}} \) yield the tangent presheaf \( \mathcal{T}_{\mathcal{A}} : \mathcal{U}^{op} \to \text{Mod}(\mathcal{A}) \) with \( \mathcal{T}_{\mathcal{A}}(U) = \mathcal{T}_{\mathcal{A}(U)/\mathbb{R}} \). From the second additional assumption it follows that antisymmetrizations \( \Lambda^\bullet \mathcal{T}_{\mathcal{A}(U)} \to H^\bullet \mathcal{C}(\mathcal{A}(U)) \) are isomorphisms.

Lemma 2.3. ([DDL77 Theorem 3.3]) Let \( \mathcal{U} \) be a small category and \( \mathcal{A} : \mathcal{U}^{op} \to \text{CAlg}(\mathbb{R}) \) a presheaf of commutative algebras. Assume that the algebra \( \mathcal{A}(U) \) is essentially of finite type and smooth \( \mathbb{R} \)-algebra for each \( U \). Assume further that the restriction map \( f^u : \mathcal{A}(U) \to \mathcal{A}(V) \) is a flat epimorphism of rings for each \( u : V \to U \). Then there is a canonical bijection

\[
H^p C^{n,q}_{\mathcal{G}S}(\mathcal{A}) = \bigoplus_{r=0}^{n} H^p C^{n,q}_{\mathcal{G}S}(\mathcal{A})_r \cong \bigoplus_{p+q=n} H^p(\mathcal{U}, \Lambda^q \mathcal{T}_{\mathcal{A}}).
\]

From the proof, one sees that any Gerstenhaber–Shack cohomology class \( \zeta_{\mathcal{G}S} \) is represented by a normalized reduced decomposable cocycle \( \theta_{n,0}, \theta_{n-1,0}, \ldots, \theta_{0,0} \) in the sense that \( \theta_{n-r, r} \) are reduced and belong to \( \tilde{C}^{n-r, r}(\mathcal{A}) \). Each \( \theta_{n-r, r} = (\theta_{n-r, r})_{r \in N_{\mathcal{U}(\mathcal{U})}} \) lifts to a unique simplicial cocycle \( \Theta_{n-r, r} = (\Theta_{n-r, r})_{r \in N_{\mathcal{U}(\mathcal{U})}} \in C^{n-r, r}_{\text{simp}}(\Lambda^r \mathcal{T}_{\mathcal{A}}) \). The image of \( \zeta_{\mathcal{G}S} \) under the bijection is the cohomology class \( \zeta_{\text{simp}} \) represented by \( \Theta_{0,0}, \Theta_{1,1}, \ldots, \Theta_{n,0} \).

2.6. Comparison with relative Hochschild cohomology. We describe the relation between simplicial cohomology and Čech cohomology for a presheaf \( \mathcal{F} : \mathcal{U}^{op} \to \text{Mod}(\mathbb{R}) \) in the case where \( \mathcal{U} \) is a poset with binary meets. We use the symbol \( \cap \) to denote meets in \( \mathcal{U} \). For a \( p \)-sequence \( \tau = (U_{0}^1, U_{0}^2, \ldots, U_{p}^1) \in \mathcal{U}^{p+1} \) we denote by \( \cap \tau \) the meet of all coordinates of \( \tau \). The Čech complex \( \bar{C}(\mathcal{F}) \) of \( \mathcal{F} \) has

\[
\bar{C}(\mathcal{F}) = \prod_{\tau \in \mathcal{U}^{p+1}} \mathcal{F}(\cap \tau)
\]
as the \( p \)-th term with the usual differentials. A Čech cochain \( \psi = (\psi^s)_{\tau} \) is alternating if \( \psi^s = 0 \) whenever two coordinates of \( \tau \) are equal, and \( \psi^{ps} = (-1)^{\text{sign}(s)} \psi^s \) for any permutation \( s \) of the set \( \{0, 1, \ldots, p\} \). Here, we regard \( \tau \) as a set theoretic map \( \{0, 1, \ldots, p\} \to \mathcal{U} \). The alternating Čech cochains form a subcomplex \( \bar{C}'(\mathcal{F}) \) which is quasi-isomorphic to \( \bar{C}(\mathcal{F}) \) via the inclusion.

To a \( p \)-sequence \( \tau \), one associates a \( p \)-simplex

\[
\bar{\tau} = (d\bar{\tau} = \cap_{j=0}^{p} U_{j}^s \to \cap_{j=1}^{p} U_{j}^s \to \cdots \to \cap_{j=p-1}^{p} U_{j}^s \to U_{p}^s = c\bar{\tau}).
\]
Conversely, any \( p \)-simplex \( \mu \) can be regarded as a \( p \)-sequence \( \tilde{\mu} \) by forgetting the inclusions. Define a map \( \delta_i : \mathcal{U}^{p+1} \to \mathcal{U}^p \) for \( i = 1, \ldots, p \) as
\[
\delta_i \tau = (U^i_0, \ldots, U^i_{i-2}, U^i_{i-1} \cap U^i_{i+1}, U^i_{i+1}, \ldots, U^i_p).
\]
There are morphisms \( \iota : C'_{\text{simp}} \to \tilde{C}'(\mathcal{F}), \pi : \tilde{C}'(\mathcal{F}) \to C'_{\text{simp}} \) of complexes defined as
\[
\iota(\phi)^\tau = \sum_{s \in \mathbb{Z}_{p+1}} (-1)^{\text{sign}(s)} \phi^{\tau^s}, \phi \in C'_{\text{simp}}(\mathcal{F}), \tau \in \mathcal{U}^{p+1},
\]
\[
\pi(\psi)^\mu = \psi^\mu, \psi \in \tilde{C}'(\mathcal{F}), \mu \in N_p(\mathcal{U}),
\]
which induce mutually inverse isomorphisms between \( H^* (\mathcal{U}, \mathcal{F}) \) and \( \tilde{H}^* (\mathcal{U}, \mathcal{F}) \) \cite{DLL17} Lemma 3.9.

Now, for a smooth proper \( \mathbf{R} \)-scheme \( X \) we give an alternative description of the relative Hochschild cohomology. As explained above, we have \( HH^*(X/\mathbf{R}) \cong H^* (X/\mathbf{R}) \). Choose a finite affine open cover \( \mathcal{U} \) closed under intersections. By definition \( \mathcal{U} \) is semi-separating, i.e., \( \mathcal{U} \) is closed under finite intersections. For every quasi-coherent sheaf \( \mathcal{F} \) on \( X \), one can apply \cite{DLL17} Lemma 3.9 and Leray’s theorem \cite{Har77} Theorem 4.5] to obtain
(2.2) \[
H^* (\mathcal{U}, \mathcal{F}|_{\mathcal{U}}) \cong \tilde{H}^* (\mathcal{U}, \mathcal{F}|_{\mathcal{U}}) \cong \tilde{H}^* (\mathcal{U}, \mathcal{F}) \cong H^* (X, \mathcal{F})
\]
with \( \mathcal{F}|_{\mathcal{U}} \) regarded as a presheaf on \( \mathcal{U} \). Since \( X \) is smooth over \( \mathbf{R} \) and open immersions \( V \hookrightarrow U \) in \( \mathcal{U} \) define flat epimorphisms \( \mathcal{O}_X(U) \to \mathcal{O}_X(V) \), combining (2.1) with (2.2), we obtain

Lemma 2.4. (\cite{DLL17} Corollary 3.4) Let \( X \) be a smooth proper \( \mathbf{R} \)-scheme with a finite affine open cover \( \mathcal{U} \) closed under intersections. Let \( \mathcal{O}_X|_{\mathcal{U}}, \mathcal{F}_X|_{\mathcal{U}} \) be the restrictions of \( \mathcal{O}_X, \mathcal{F}_X/\mathbf{R} \) to \( \mathcal{U} \) respectively. Then there are canonical isomorphisms
(2.3) \[
H^n C_{GS} (\mathcal{O}_X|_{\mathcal{U}}) = \bigoplus_{r=0}^n H^n C_{GS} (\mathcal{O}_X|_{\mathcal{U}}), \cong \bigoplus_{p+q=n} H^p (\mathcal{U}, : \mathcal{F}_X/\mathbf{R}|_{\mathcal{U}}) \cong HH^n (X/\mathbf{R}),
\]
where the first isomorphism respects the Hodge decomposition.

3. Deformations of relatively smooth proper schemes

In this section, we review the classical deformation theory of schemes. The main reference is \cite{Har10}. We explain how deformations of smooth proper \( k \)-varieties extend to Toda’s construction \cite{Tod09}, which can be adapted to deformations of relatively smooth proper schemes along square zero extensions in a straightforward way. When the original scheme is a deformation of a higher dimensional Calabi–Yau manifold, Toda’s construction gives the category of quasi-coherent sheaves on deformations of the Calabi–Yau manifold.

3.1. Deformations of schemes. Let \( X \) be a \( k \)-scheme and \( A \) a local artinian \( k \)-algebra with residue field \( k \). An \( A \)-deformation of \( X \) is a pair \( (X_A, i_A) \), where \( X_A \) is a scheme flat over \( A \) and \( i_A : X \hookrightarrow X_A \) is a closed immersion such that the induced map \( X \to X_A \times_A k \) is an isomorphism. Two deformations \( (X_A, i_A), (X'_A, i'_A) \) are equivalent if there is an \( A \)-isomorphism \( X_A \to X'_A \) compatible with \( i_A, i'_A \). The deformation functor
\[
\text{Def}_X : \text{Art}_k \to \text{Set}
\]
sends each \( A \in \text{Art}_k \) to the set of equivalence classes of \( A \)-deformations of \( X \).

Assume that \( X \) is projective over \( k \). Then \( \text{Def}_X \) satisfies Schlessinger’s criterion and there exists a miniversal formal family \( (R, \xi) \) for \( \text{Def}_X \), where \( R \) is a complete local noetherian \( k \)-algebra with residue field \( k \), and \( \xi = [\xi_n]_n \) belongs to the limit
\[
\text{Def}_X(R) = \lim \text{Def}_X(R/m^n_k)
\]
of the inverse system
\[ \cdots \to \text{Def}_X(R/m_R^{n+2}) \to \text{Def}_X(R/m_R^{n+1}) \to \text{Def}_X(R/m_R^n) \to \cdots \]
induced by the natural quotient maps \( R/m_R^{n+1} \to R/m_R^n \). The formal family \( \xi \) corresponds to a natural transformation
\[ h_R = \text{Hom}_{\text{alg}}(R, -) \to \text{Def}_X, \]
which sends each \( g \in h_R(A) \) factorizing through \( R \to R/m_R^{n+1} g_n \to A \) to \( \text{Def}_X(g_n)(\xi_n) \).

Let \( X_0 \) be the schemes which define \( \xi_n \). There is a noetherian formal scheme \( \mathcal{X} \) over \( R \) such that \( X_n \cong \mathcal{X} \times_R R/m_R^{n+1} \) for each \( n \). By abuse of notation, we use the same symbol \( \xi \) to denote \( \mathcal{X} \). Thus any scheme which defines an equivalence class \([X_A, i_A]\) can be obtained as the pullback of \( \xi \) along some morphism of noetherian formal schemes \( \text{Spec} A \to \text{Spf} R \). If \( X \) has no infinitesimal automorphisms which restrict to the identity of \( X \), then every equivalence class \([X_A, i_A]\) becomes just a deformation \((X_A, i_A)\) and we have a natural isomorphism \( h_R \cong \text{Def}_X \). In this case, we call \( \text{Def}_X \) prorepresentable and \((R, \xi)\) a universal formal family for \( \text{Def}_X \).

3.2. Algebraization. Let \( X \) be a projective \( k \)-variety. We call a miniversal formal family \((R, \xi)\) for \( \text{Def}_X \) effective when there exists a scheme \( X_R \) flat and of finite type over \( R \) whose formal completion along the closed fiber \( X \) is isomorphic to \( \xi \). By [GD61, Theorem III5.4.5] the family \((R, \xi)\) is effective if deformations of any invertible sheaf on \( X \) are unobstructed. Note that this is the case, for instance, if we have \( H^2(\mathcal{O}_X) = 0 \). From the proof, one sees that \( X_R \) is projective over \( R \). We will call such \( X_R \) an effectivization of \( \xi \).

The deformation functor \( \text{Def}_X \) can naturally be extended to a functor defined on the category \( \text{Alg}_{\text{aug}}(k) \) of augmented noetherian \( k \)-algebras. By abuse of notation, we use the same symbol \( \text{Def}_X \) to denote the extended functor, which sends each \((P, m_P) \in \text{Alg}_{\text{aug}}(k)\) to the set of equivalence classes of deformations over \((P, m_P)\). Since the functor \( \text{Def}_X \) is locally of finite presentation, by [Art69, Theorem 1.6] the miniversal formal family is algebraizable, i.e., there exists a triple \((S, s, X_S)\) where \( S \) is an algebraic \( k \)-scheme with a distinguished closed point \( s \in S \), and \( X_S \) is a flat and of finite type \( S \)-scheme whose formal completion along the closed fiber \( X \) over \( s \) is isomorphic to \( \xi \). We call the scheme \( X_S \) a versal deformation over \( S \). When there exists a versal deformation, we say that the miniversal formal family \((R, \xi)\) is algebraizable.

3.3. Deformations of higher dimensional Calabi–Yau manifolds. Here, we focus on a special case where several interesting results hold. Let \( X_0 \) be a Calabi–Yau manifold of dimension more than two. Then the deformation functor \( \text{Def}_{X_0} \) has an effective universal formal family \((R, \xi)\). Since deformations of Calabi–Yau manifolds are unobstructed, the complete local noetherian ring \( R \) is regular and we have
\[ R \cong k[[t_1, \ldots, t_d]] \]
with \( d = \dim_k H^1(\mathcal{O}_{X_0}) \). Every \( A \)-deformation of \( X_0 \) is smooth projective over \( A \), as we have

**Lemma 3.1.** ([Mora, Lemma 2.4]) The effectivization \( X_R \) for \((R, \xi)\) is regular and the morphism \( \pi_R : X_R \to \text{Spec} R \) is smooth of relative dimension \( \dim X_0 \).

Now, we briefly recall the construction of \( X_S \). Consider the extended functor
\[ \text{Def}_{X_0} : \text{Alg}_{\text{aug}}(k) \to \text{Set}. \]
Fix an isomorphism \( R \cong k[[t_1, \ldots, t_d]] \). Let \( T = k[t_1, \ldots, t_d] \) and \( t \in \text{Spec} T \) be the closed point corresponding to maximal ideal \((t_1, \ldots, t_d)\). There is a filtered inductive system \( \{ R_i \}_{i \in I} \) of finitely generated \( T \)-subalgebras of \( R \) whose colimit is \( R \). Since \( \text{Def}_{X_0} \) is locally of finite presentation, \([X_R, t_R]\) is the image of some element \( \xi_i \in \text{Def}_{X_0}((R_i, m_{R_i})) \) by the canonical map
Lemma 3.3. (Har10, Theorem 3.4) Let $X_0$ be a Calabi–Yau manifold of dimension more than two. Then there exists a nonsingular affine $k$-variety $Spec S$ with a versal deformation $X_S$ which is smooth projective of relative dimension $dim X_0$ over $S$.

3.4. $T^i$ functors. Let $A \to B$ be a ring homomorphism and $M$ a $B$-module. Define the groups $T^i(B/A, M)$ for $i = 0, 1, 2$ as the $i$-th cohomology of the complex $\text{Hom}_B(L_\bullet, M)$, where

$$L_\bullet = L_2 \xrightarrow{d_2} L_1 \xrightarrow{d_1} L_0$$

is the cotangent complex. When the ring homomorphism $A \to B$ is a surjection with kernel $J$, $L_\bullet$ is given as follows. Choose a free $A$-module $P$ and a surjection $j: P \to J$ with kernel $Q$. We have two exact sequences

$$0 \to J \to A \to B \to 0, \quad 0 \to Q \to P \xrightarrow{j} J \to 0.$$ 

Let $P_0$ be the submodule of $P$ generated by all relations of the form $j(a)b - j(b)a$ for $a, b \in P$. From $j(P_0) = 0$ it follows $P_0 \subset Q$. Take $L_2 = Q/P_0$, $L_1 = P \otimes_A B$, and $L_0 = 0$. Note that $L_2$ is a $B$-module. Indeed, for $a \in J$ there is an element $a' \in P$ such that $a = j(a')$. Then we have $ax \equiv j(x)a' \equiv 0$ modulo $P_0$ for $x \in Q$. The differential $d_2: L_2 \to L_1$ is the map induced by the inclusion $Q \to P$ and $d_1 = 0$. By [Har10] Lemma 3.2 the $B$-modules $T^i(B/A, M)$ do not depend on the choice of $P$ up to isomorphism.

Lemma 3.3. (Har10 Theorem 3.4) Let $A \to B$ be a homomorphism of rings. Then

$$T^i(B/A, -) : \text{Mod}(B) \to \text{Mod}(B), \quad i = 0, 1, 2$$

define covariant additive functor.

The construction of $T^i$ functors is compatible with localization and one obtains sheaves $\mathcal{T}^i(X/Y, \mathcal{F})$, $i = 0, 1, 2$ for any morphism of $k$-schemes $f: X \to Y$ and any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ [Har10] Exercise 3.5]. The sections of $\mathcal{T}^i(X/Y, \mathcal{F})$ over $U = \text{Spec} B \subset f^{-1}(V)$ give $T^i(B/A, M)$, where $V = \text{Spec} A \subset Y$ and $\mathcal{F}|_U = M$ for some $B$-module $M$.

3.5. Infinitesimal extension of schemes. Let $X$ be a scheme of finite type over $\mathbb{R}$ and $\mathcal{F}$ a coherent sheaf on $X$. An infinitesimal extension of $X$ by $\mathcal{F}$ is a pair $(Y, \mathcal{Z})$, where $Y$ is a scheme of finite type over $\mathbb{S}$ and $\mathcal{Z} \subset \mathcal{O}_Y$ is an ideal sheaf such that $\mathcal{Z}^2 = 0$, $(\mathcal{O}_Y/\mathcal{Z}) \cong (X, \mathcal{O}_X)$, and $\mathcal{Z} \cong \mathcal{F}$ as an $\mathcal{O}_X$-module. Two infinitesimal extensions $(Y, \mathcal{Z}), (Y', \mathcal{Z}')$ are equivalent if there is an isomorphism $\mathcal{O}_Y \to \mathcal{O}_Y'$ which makes the diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{O}_Y & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
0 & \uparrow{id} & \downarrow{id} & & & & & & \\
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{O}_Y' & \longrightarrow & \mathcal{O}_X' & \longrightarrow & 0
\end{array}
$$

commute. The trivial extension is a sheaf $\mathcal{O}_X \oplus \mathcal{F}$ of abelian group endowed with the ring structure by

$$(a, f) \cdot (a', f') = (aa', af' + a'f).$$
Assume that $X$ is smooth proper over $\mathbb{R}$. Recall that for the square zero extension
\begin{equation}
0 \to I \to S \to R \to 0
\end{equation}
we have $I \cong R^{\text{ad}}$ as an $R$-module. Note that given an infinitesimal extension $(Y, \mathcal{I})$ of $X$ by $\mathcal{O}_X^{\text{ad}}$, $Y$ is flat over $S$ since $\mathcal{O}_X$ is flat over $R$ and $\mathcal{O}_X^{\text{ad}} \to \mathcal{O}_Y$ is injective [Har10, Proposition 2.2]. Below, we collect fundamental results necessary to describe the relation between deformations and extensions of schemes.

**Lemma 3.4.** ([Har10, Exercise 4.7]) Let $X$ be a smooth $\mathbb{R}$-scheme and $g: Y \to X$ a morphism from an affine $\mathbb{R}$-scheme $Y$ to $X$, and $i_S: Y \hookrightarrow Y'$ an $S$-deformation of $Y$. Then $g$ lifts to a morphism $h: Y' \to X$ such that $h \circ i_S = g$.

**Lemma 3.5.** ([Har10, Proposition 3.6, Exercise 5.2]) Let $A \to B$ be a homomorphism of rings, $M$ a $B$-module, and $B'$ an extension of $B$ by $M$. Then the automorphism group of $B'$ is given by $T^0(B/A, M) = \text{Hom}_B(\Omega^1_{B/A}, M) = \text{Der}_A(B, M)$.

**Lemma 3.6.** ([Har10, Theorem 5.1]) Let $A \to B$ be a homomorphism of rings and $M$ a $B$-module. Then there is a bijection between the set of equivalence classes of extensions of $B$ by $M$ and the group $T^1(B/A, M)$. The trivial extension corresponds to the zero element.

**Lemma 3.7.** ([Har10, Theorem 4.11]) Let $f: X \to Y$ be an of finite type morphism of noetherian $k$-schemes. Then $f$ is smooth if and only if it is flat and $\mathcal{T}^1(X/Y, \mathcal{F}) = 0$ for every coherent $\mathcal{O}_X$-module $\mathcal{F}$.

Now, we are ready to show relevant results to our setting.

**Lemma 3.8.** Let $X$ be a smooth separated $\mathbb{R}$-scheme. Then every $S$-deformation $(Y, j)$ of $X$ is locally trivial.

**Proof.** Since $Y$ is flat over $S$, (3.1) induces a short exact sequence
\[0 \to \mathcal{O}^{\text{ad}}_X \to \mathcal{O}_Y \to \mathcal{O}_X \to 0,
\]
which defines equivalence classes of infinitesimal extensions of coordinate rings on affine open subschemes of $X$. Let $i_S: \text{Spec } B \hookrightarrow \text{Spec } A$ be the induced deformation of any affine open subscheme. Since $\text{Spec } B$ is smooth over $\mathbb{R}$, by Lemma 3.4, the identity $\text{Spec } B \to \text{Spec } B$ lifts to a morphism $h: \text{Spec } A \to \text{Spec } B$ such that $h \circ i_S = \text{id}$. The lift $h$ induces a morphism $\text{Spec } A \to \text{Spec } B \times_{\mathbb{R}} S$ of schemes flat of finite type over $S$. Now, one can apply [Har10, Exercise 4.2] to see that the induced morphism is an isomorphism. \hfill \square

**Proposition 3.9.** Let $X$ be a smooth separated $\mathbb{R}$-scheme. Then there is a bijection
\[\text{Def}_X(S) \cong H^1(X, \mathcal{T}_{X/R}^{\text{ad}}),\]
where $\mathcal{T}_{X/R}$ is the relative tangent sheaf on $X$.

**Proof.** Let $(Y, j)$ be an $S$-deformation of $X$. Take an affine open cover $U = \{U_i\}_{i \in I}$ of $X$. By Lemma 3.8 we may assume that the induced deformations $U_i \hookrightarrow V_i \subset Y$ are trivial. Choose isomorphisms $\varphi_i: U_i \times_{\mathbb{R}} S \to V_i$ and write $\varphi_{ij}$ for the composition $\varphi_i^{-1} \circ \varphi_j$ on $U_i \times_{\mathbb{R}} S$, where the intersections $U_{ij} = U_i \cap U_j$ are again affine as $X$ is separated over $\mathbb{R}$. Let $\text{Spec } B = U_{ij}$ and $\text{Spec } A = U_{ij} \times_{\mathbb{R}} S$. According to Lemma 3.5, the set of automorphisms of extensions $\hat{A}$ of $\hat{B}$ by $B^{\text{ad}}$ bijectively corresponds to $T^0(B/\mathbb{R}, B^{\text{ad}}) \cong \text{Hom}(\Omega_{B/\mathbb{R}}^{\text{ad}}, B^{\text{ad}})$. Then $\{\varphi_{ij}\}_{i, j \in I}$ define a collection $\{\vartheta_{ij}\}_{i, j \in I}$ of sections $\vartheta_{ij} \in H^0(U_{ij}, \mathcal{T}_{X/R}^{\text{ad}})$ on $U_{ij}$. One checks $\vartheta_{ij} + \vartheta_{jk} + \vartheta_{kl} = 0$ and $\{\vartheta_{ij}\}_{i, j \in I}$ is a $\text{Cech}$ 1-cocycle with respect to $U$. Another choice of isomorphisms $\varphi'_i: U_i \times_{\mathbb{R}} S \to V_i$ yields a collection $\{\varphi'_i\}_{i \in I}$ of automorphisms such that $\varphi'_{ij} = (\varphi'_j \circ \varphi'_i)^{-1} \circ \varphi_{ij} \circ (\varphi'_i \circ \varphi'_j)$. It...
follows $\theta'_{ij} = \theta_{ij} + \alpha_i - \alpha_j$ for some sections $\alpha_i \in H^0(U_i, \mathcal{T}_X/R)^{\text{ad}}$. Thus we obtain a well defined assignment

\[ \text{Def}_X(S) \to H^1(X, \mathcal{T}_X/R)^{\text{ad}}, \quad [Y, j] \mapsto \{ \theta_{ij}\}_{i,j\in I}, \]

as $\{\theta_{ij}\}_{i,j\in I}$ does not depend on $\mathcal{U}$.

Conversely, an element of $H^1(X, \mathcal{T}_X/R)^{\text{ad}}$ can be represented by Čech 2-cocycle $\{\theta_{ij}\}_{i,j\in I}$ with respect to $\mathcal{U}$. As explained above, the cocycle define automorphisms of the trivial deformations $U_{ij} \times_R S$, which glue to yield a global deformation $(Y', j')$ of $X$. Clearly, this construction gives the inverse assignment. \hfill \Box

**Corollary 3.10.** There is a canonical bijection between $\text{Def}_X(S)$ and the set of equivalence classes of infinitesimal extensions of $X$ by $\mathcal{O}_X^{\text{ad}}$.

**Proof.** By Lemma 3.6 and Lemma 3.7 any extension of $X$ by $\mathcal{O}_X^{\text{ad}}$ is locally trivial. Then due to Lemma 3.5 the claim follows from the same argument as in the proof of Proposition 3.9. \hfill \Box

3.6. Toda’s construction. Let $X_0$ be a smooth projective $k$-variety. In [Tod09] Toda constructed the category of $\tilde{a}$-twisted sheaves on the noncommutative scheme $(X_0, \mathcal{O}_{X_0}^{\beta, \gamma})$ over the ring of dual numbers for each $[\phi_0] \in HT^2(X_0)$ represented by a cocycle

\[ (\alpha_0, \beta_0, \gamma_0) \in H^2(\mathcal{O}_{X_0}) \oplus H^1(\mathcal{T}_{X_0}) \oplus H^0(\wedge^2 \mathcal{T}_{X_0}). \]

Here, we apply his idea to a smooth proper $k$-scheme $X$ and $[\phi] \in HT^2(X/R)^{\text{ad}}$ represented by

\[ (\alpha, \beta, \gamma) = ((\alpha^1, \ldots, \alpha^l), (\beta^1, \ldots, \beta^l), (\gamma^1, \ldots, \gamma^l)) \in H^2(\mathcal{O}_X)^{\text{ad}} \oplus H^1(\mathcal{T}_{X/R})^{\text{ad}} \oplus H^0(\wedge^2 \mathcal{T}_{X/R})^{\text{ad}}. \]

Take a finite affine open cover $\mathcal{U} = \{ U_i \}_{i=1}^N$ of $X$ and let $\mathcal{U} \times_R S = \{ U_i \times_R S \}_{i=1}^N$. Consider the extension of $X$ by $\mathcal{O}_X^{\text{ad}}$ whose equivalence class corresponds to $\beta$, giving rise to an classical $S$-deformation $X_\beta$ of $X$ by Corollary 3.10. We modify the multiplication on $\mathcal{O}_X \oplus C(\mathcal{U}, \mathcal{O}_X^{\text{ad}})$ as

\[ (a, \{ b_i^1 \}, \ldots, \{ b_i^l \}) \ast (c, \{ d_i^1 \}, \ldots, \{ d_i^l \}) = (ac, \{ ad_i^1 + b_i^1 c + \gamma_i(a, c), \ldots, ad_i^l + b_i^l c + \gamma_i(a, c) \}), \]

where $\gamma_i : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$ are regarded as bidifferential operators. We denote by $X_{(\beta, \gamma)} = (X_\beta, \mathcal{O}_X^{\beta, \gamma})$ the resulting noncommutative $S$-scheme. By the standard argument, one sees that up to isomorphism the scheme does not depend on the choice of $\mathcal{U}$ and Čech representative of $\gamma$. From $\alpha$ one obtains an element

\[ \tilde{a} = \{ 1 - \alpha^1_{i_0i_1} e_{i_1} - \cdots - \alpha^l_{i_0i_l} e_{i_l} \}_{i_0i_1i_2} \in C^2(X_\beta, Z(\mathcal{O}_X^{\beta, \gamma})) \]

which is a cocycle. Then $\tilde{a}$-twisted sheaves on $X_{(\beta, \gamma)}$ form a category $\text{Mod}(X_{(\beta, \gamma)}, \tilde{a})$. By the similar argument to [Cal00, Lemma 1.2.3, 1.2.8], one sees that up to equivalence the category does not depend on the choice of $\mathcal{U}$ and Čech representative of $\alpha$. We denote by $\text{Qch}(X, \phi)$ the full abelian subcategory spanned by $\tilde{a}$-twisted quasi-coherent sheaves.

Assume that $X$ is an $R$-deformation of a higher dimensional Calabi–Yau manifold. Then we have

\[ HT^2(X/R) = H^1(\mathcal{T}_{X/R}). \]

In this case, Toda’s construction yields nothing but the category of quasi-coherent sheaves on the $S$-deformation of $X$ along $\phi$.

**Proposition 3.11.** Let $X_0$ be a Calabi–Yau manifold with $\dim X_0 > 2$ and $X$ an $R$-deformation of $X_0$. Then for every cocycle $\phi \in HT^2(X/R)^{\text{ad}} = H^1(\mathcal{T}_{X/R})^{\text{ad}}$ we have

\[ \text{Qch}(X, \phi) = \text{Qch}(X_\phi), \]

where $X_\phi$ is the $S$-deformation of $X$ along $\phi$. 

15
In this section, we review the deformation theory of linear and abelian categories developed by Lowen and van den Bergh in [LV06b], introducing the fundamental notion of flatness. As explained there, when considering only flat nilpotent deformations over a certain class of rings, one avoids any set-theoretic issue by choosing sufficiently large universes. Moreover, both linear and abelian deformations reduce to strict linear deformations without affecting the deformation theory up to equivalence. Along square zero extensions, flat deformations of linear and abelian categories are controlled by the second Hochschild cohomology of the corresponding linear categories.

4.1. Universes. First, we need to extend the Zermelo–Fraenkel axioms of the set theory to avoid foundational issues in the deformation theory of categories. One solution is the theory of universes introduced by Grothendieck with the axiom of choice and the universe axiom. A universe $\mathcal{U}$ is a set with the following properties:

- if $x \in \mathcal{U}$ and $y \in x$ then $y \in \mathcal{U}$,
- if $x, y \in \mathcal{U}$ then $\{x, y\} \in \mathcal{U}$,
- if $x \in \mathcal{U}$ then the powerset $\mathcal{P}(x)$ of $x$ is in $\mathcal{U}$,
- if $(x_i)_{i \in I}$ is a family of objects of $\mathcal{U}$ indexed by an element of $\mathcal{U}$ then $\bigcup_{i \in I} x_i \in \mathcal{U}$,
- if $U \in \mathcal{U}$ and $f : U \to \mathcal{U}$ is a function then $\{f(x) \mid x \in U\} \in \mathcal{U}$.

A universe $\mathcal{U}$ containing $\mathbb{N}$ is a model for the Zermelo–Fraenkel axioms of the set theory with the axiom of choice. Since the known nonempty universe only contains finite sets, the universe axiom is added, which imposes every set to be an element of a universe.

Consider the category $\mathcal{U} - \text{Set}$ whose objects are elements of $\mathcal{U}$ and whose morphisms are ordinary maps between sets in $\mathcal{U}$. The category $\mathcal{U} - \text{Cat}$ consists of categories whose objects and morphisms respectively form sets being an element of $\mathcal{U}$. Similarly, by requiring the underlying sets to belong to $\mathcal{U}$, we obtain categories with a structure such as abelian groups and rings. We call a category $\mathcal{U}$-small when its objects and morphisms respectively form sets with the same cardinality as an element of $\mathcal{U}$, and essentially $\mathcal{U}$-small when it is equivalent to a $\mathcal{U}$-small category. A $\mathcal{U}$-category is a category whose Hom-sets have the the same cardinality as an element of $\mathcal{U}$. The axiom of choice allows us to replace a $\mathcal{U}$-category $\mathcal{C}$ by an equivalent category $\mathcal{C}'$ with $\text{Ob}(\mathcal{C}) = \text{Ob}(\mathcal{C}')$ and $\mathcal{C}'(C, D) \in \mathcal{U}$ for all $C, D \in \text{Ob}(\mathcal{C}')$. When $\mathcal{C}$ is abelian with a generator, we call $\mathcal{C}$ $\mathcal{U}$-Grothendieck. Every $\mathcal{U}$-Grothendieck category $\mathcal{C}$ admits $\mathcal{U}$-small colimits and $\mathcal{U}$-small filtered colimits are exact in $\mathcal{C}$.

Throughout the paper, we work with a fixed universe $\mathcal{U}$ containing $\mathbb{N}$. All the notion based on universes will be with respect to $\mathcal{U}$ and all the related symbols will be tacitly prefixed by $\mathcal{U}$. By taking $\mathcal{U}$ sufficiently large, we may assume all categories to be small. Unless otherwise specified, we will be free from any issue caused by the choice of universes.

4.2. Flatness. The notion of flatness for abelian categories was introduced in [LV06b]. For a while, we temporarily drop the assumption on $\mathcal{R}$ and $\mathcal{S}$ imposed at the beginning of Section 2. Let $\mathcal{R}$ be a commutative ring. An $\mathcal{R}$-linear category is a category $\mathcal{A}$ enriched over the abelian category $\text{Mod}(\mathcal{R})$ of $\mathcal{R}$-modules. Namely, $\mathcal{A}$ is a pre-additive category together with a ring map $\rho : \mathcal{R} \to \text{Nat}(1_{\mathcal{A}}, 1_{\mathcal{A}})$ inducing a ring map $\rho_A : \mathcal{R} \to \mathcal{A}(A, A)$ for each $A \in \mathcal{A}$ and an action of $\mathcal{R}$ on each Hom-set.

Assume that $\mathcal{R}$ is coherent, i.e., any finitely generated ideal is finitely presented as an $\mathcal{R}$-module. Typical examples are given by noetherian rings. We denote by $\text{mod}(\mathcal{R})$ the full abelian subcategory of finitely presented $\mathcal{R}$-modules. Let $\mathcal{C}$ be an $\mathcal{R}$-linear abelian category. We call an object $\mathcal{C} \in \mathcal{C}$ flat if the natural finite colimit preserving functor $(-) \otimes_{\mathcal{R}} \mathcal{C} : \text{mod}(\mathcal{R}) \to \mathcal{C}$
is exact, and coflat if the natural finite limit preserving functor \( \text{Hom}_R(\cdot, C) : \text{mod}(R) \to \mathcal{C} \) is exact.

An \( R \)-linear category \( a \) is flat if its Hom-sets are flat \( R \)-modules. Namely, the functors \( - \otimes_R a(a, A') : \text{mod}(R) \to \text{Mod}(R) \) are exact for all \( A, A' \in a \). An \( R \)-linear abelian category \( \mathcal{C} \) is flat if for each \( Y \in \text{mod}(R) \) the functor \( \text{Tor}^R(Y, -) : \mathcal{C} \to \mathcal{C} \) is co-effeable, i.e., for each \( C \in \mathcal{C} \) there is an epimorphism \( f : C' \to C \) with \( \text{Tor}^R(Y, f) = 0 \) \[LV06b\] Proposition 3.1]. Here, \( \text{Tor}^R(Y, -) \) is the left derived functor of the finite colimit preserving functor \( Y \otimes_R (-) : \mathcal{C} \to \mathcal{C} \). The flatness has the following characterizations \[LV06b\] Proposition 3.3, 3.4, 3.6, 3.7).

- \( \mathcal{C} \) is flat if and only if \( \mathcal{C}^{\text{op}} \) is flat.
- \( \mathcal{C} \) is flat if and only if all injectives in \( \mathcal{C} \) are coflat.
- \( \mathcal{C} \) is flat if and only if \( \text{Ind}(\mathcal{C}) \) is flat.
- \( a \) is flat if and only if the abelian category \( \text{Mod}(a) \) is flat.

Here, \( \text{Ind}(\mathcal{C}) \) is the category of ind-objects, i.e., the full subcategory of \( \text{Mod}(\mathcal{C}) \) consisting of left exact functors, where \( \text{Mod}(\mathcal{C}) \) is the category of covariant additive functors from \( \mathcal{C} \) to the category \( \text{Ab} \) of abelian groups. Note that we are assuming all categories to be small in our fixed universe \( \mathcal{U} \).

4.3. Base change. We fix a homomorphism \( \theta : S \to R \) of commutative rings. For an \( R \)-module \( M \), by \( \bar{M} \) we denote \( M \) regarded as an \( S \)-module via \( \theta \). Let \( a \) be an \( R \)-linear category. We have the category \( \bar{a} \) with \( \text{Ob}(\bar{a}) = \text{Ob}(a) \) and \( \bar{a}(A, A') = a(A, A') \). For an \( S \)-linear category \( b \), we denote by \( b \otimes_S R \) the \( R \)-linear category with \( \text{Ob}(b \otimes_S R) = \text{Ob}(b) \) and \( (b \otimes_S R)(B, B') = b(B, B') \otimes_S R \). The functor \( (-) \otimes_S R \) is left adjoint to \( (\cdot) \) in the sense that there is a natural isomorphism

\[
\text{Add}(R)(b \otimes_S R, a) \cong \text{Add}(S)(b, \bar{a})
\]

of \( S \)-linear categories, where \( \text{Add}(S) \) is the category of \( S \)-linear functors.

Let \( (b, \rho) \) be an \( S \)-linear category. We have the category \( b_R \) of \( R \)-linear objects whose objects are pairs \( (B, \varphi) \) where \( B \in b \) and \( \varphi : R \to b(B, B) \) is a ring map with \( \varphi \circ \theta = \rho_B \), and whose morphisms are those of \( b \) compatible with the ring maps. An object \( B \in b \) belongs to \( b_R \) if and only if \( 1_B \) is annihilated by the kernel of \( \theta \). Taking \( R \)-linear objects defines a functor \( (-)_R : b \to b_R \), which is right adjoint to \( (-) \) in the sense that there is a natural isomorphism

\[
\text{Add}(R)(a, b_R) \cong \text{Add}(S)(\bar{a}, b)
\]

of \( S \)-linear categories. If \( \mathcal{D} \) is an \( S \)-linear abelian category, then \( \mathcal{D}_R \) is also abelian and by \[LV06b\] Proposition 4.2] the forgetful functor \( \mathcal{D}_R \to \mathcal{D} \) is exact. From \( (\text{Mod}(S))_R \cong \text{Mod}(R) \), it follows

\[
\text{Add}(R)(a, \text{Mod}(R)) \cong \text{Mod}(a)
\]

for any \( R \)-linear category \( a \).

**Lemma 4.1.** \([LV06b\) Proposition 4.4(1)] Let \( b \) be an \( S \)-linear category. Then there is an equivalence \( \text{Mod}(b \otimes_S R) \to \text{Mod}(b)_R \) of \( R \)-linear categories which makes the diagram

\[
\begin{array}{ccc}
\text{Mod}(b \otimes_S R) & \xrightarrow{\cong} & \text{Mod}(b)_R \\
\downarrow & & \downarrow \\
\text{Mod}(b) & \xrightarrow{\text{id}} & \text{Mod}(b)
\end{array}
\]

commutes, where the left vertical arrow is the dual to \( b \to b \otimes_S R \) and the right vertical arrow is the forgetful functor.

17
4.4. **Deformations of linear categories.** Let \( a \) be an \( R \)-linear category. A **linear \( S \)-deformation** of \( a \) is an \( S \)-linear category \( b \) together with an \( S \)-linear functor \( b \to a \) inducing an equivalence \( b \otimes_S R \to a \). Two deformations \( f : b \to a, f' : b' \to a \) are **equivalent** if there is an equivalence \( \Phi : b \to b' \) of \( S \)-linear categories such that \( f' \circ \Phi = f \). When \( b \) is flat over \( R \), we call the deformation \( b \) **flat**. We denote by \( \text{Def}_{\text{lin}}(S) \) the set of equivalence classes of flat linear \( S \)-deformations of \( a \). The notation will be justified below with respect to the choice of universe. When \( b \otimes_S R \to a \) is an isomorphism, we call the deformation \( b \) **strict**. Two strict linear deformations \( f : b \to a, f' : b' \to a \) are equivalent if there is an isomorphism \( \Phi : b \to b' \) of \( S \)-linear categories such that \( f' \circ \Phi = f \). We denote by \( \text{Def}_{\text{lin}}(S) \) the set of equivalence classes of strict flat linear \( S \)-deformations of \( a \). Also this notation will be justified below.

4.5. **Deformations of abelian categories.** Let \( C \) be an \( R \)-linear abelian category. An **abelian \( S \)-deformation** of \( C \) is an \( S \)-linear abelian category \( D \) together with an \( S \)-linear functor \( \overline{C} \to D \) inducing an equivalence \( C \to DR \). When \( D \) is flat over \( R \), we call the deformation \( D \) **flat**. Two deformations \( g : \overline{C} \to D, g' : \overline{C} \to D' \) are equivalent if there is an equivalence \( \Psi : D \to D' \) of \( S \)-linear abelian categories such that \( \Psi \circ g' = g \). We denote by \( \text{Def}^S_{\text{ab}}(S) \) the set of equivalence classes of flat abelian \( S \)-deformations of \( C \). The notation will be justified below with respect to the choice of universe. When \( C \to DR \) is an isomorphism, we call the deformation \( D \) **strict**. Two strict abelian deformations \( g : \overline{C} \to D, g' : \overline{C} \to D' \) are equivalent if there is an isomorphism \( \Psi : D \to D' \) of \( S \)-linear abelian categories such that \( \Psi \circ g' = g \). We denote by \( \text{Def}^S_{\text{ab}}(S) \) the set of equivalence classes of strict flat abelian \( S \)-deformations of \( C \). Also this notation will be justified below.

Assume that \( \theta : S \to R \) is a homomorphism of coherent commutative rings with \( R \) being finitely presented as an \( S \)-module. Then the bifunctors

\[
(-) \otimes_S (-) : \mathcal{D} \times \text{mod}(S) \to \mathcal{D}, \quad \text{Hom}_S(-, -) : \text{mod}(S) \times \mathcal{D} \to \mathcal{D}
\]

yield respectively left and right adjoint

\[
(-) \otimes_S R : \mathcal{D} \to \mathcal{D}_R \simeq \mathcal{C}, \quad \text{Hom}_S(R, -) : \mathcal{D} \to \mathcal{D}_R \simeq \mathcal{C}
\]

to the natural inclusion functor \( \mathcal{C} \simeq \mathcal{D}_R \hookrightarrow \mathcal{D} \) [LV06b, Proposition 4.3]. They agree with the adjoints in the Section 4.3.

4.6. **Flat nilpotent deformations of categories.** Assume further that \( \theta \) is surjective. Then for an \( S \)-linear abelian category \( D \) the forgetful functor \( D_R \to D \) is fully faithful. When the kernel \( I = \ker \theta \) is nilpotent, we call both linear and abelian \( S \)-deformations **nilpotent**. From now on, we restrict our attention to flat nilpotent deformations. The following properties of \( R \)-linear category \( a \) and \( R \)-linear abelian category \( C \) are respectively preserved under flat nilpotent linear and abelian deformations [LV06b, Proposition 6.7, 6.9, Theorem 6.16, 6.29, 6.36].

- \( a, C \) are essentially small.
- \( C \) has enough injectives.
- \( C \) is a Grothendieck category.
- \( C \) is a locally coherent Grothendieck category.

Here, we call \( C \) **locally coherent Grothendieck** when it is Grothendieck and generated by a small abelian subcategory of finitely presented objects.

4.7. **Deformation pseudofunctors.** In order to be careful about our choices of universes, we temporarily make them explicit in the notation. Let \( \mathcal{U} \) be a universe containing the field \( k \). We denote by \( \mathcal{U} - \text{Rng}^0 \) the category whose objects are coherent commutative \( \mathcal{U} \)-rings and whose morphisms are surjective ring maps with finitely generated nilpotent kernels. We are interested in the category \( \mathcal{U} - \text{Rng}^0 / k \). Fix some other universe \( \mathcal{W} \). A deformation pseudofunctor
is a pseudofunctor $D: \mathcal{U} \to \text{Rng}^0 / k \to \mathcal{W} - \text{Gd}$. Two deformation pseudofunctors $D, D'$ are equivalent if there is a pseudonatural transformation $\mu: D \to D'$ such that for each $R \in \mathcal{U} - \text{Rng}^0 / k$ we have an equivalence $D(R) \to D'(R)$ of categories. For any enlargement $\mathcal{U}'$ of $\mathcal{U}$, the canonical functor

$$\mathcal{U} - \text{Rng}^0 / k \to \mathcal{U}' - \text{Rng}^0 / k$$

is an equivalence of categories [LV06b, Proposition 8.1]. Thus the deformation pseudofunctor is independent of the choice of $\mathcal{U}$ up to equivalence.

Let $a$ be a flat $k$-linear $\mathcal{U}$-category and $\mathcal{G}$ a flat $k$-linear abelian $\mathcal{U}$-category. Fix a universe $\mathcal{V}$ such that $a, \mathcal{G}$ are essentially $\mathcal{V}$-small and $\mathcal{U} \in \mathcal{V}$. For $R \in \mathcal{U} - \text{Rng}^0 / k$ we consider the groupoid $\mathcal{V} - \text{def}_a^\text{lin}(R)$ whose objects are flat linear $R$-deformations of $a$ belonging to $\mathcal{V}$, and whose morphisms are equivalences of deformations up to natural isomorphism of functors. Also we consider the groupoid $\mathcal{V} - \text{def}_\mathcal{G}^{\text{ab}}(R)$ whose objects are flat abelian $R$-deformations of $\mathcal{G}$ belonging to $\mathcal{V}$, and whose morphisms are equivalences of deformations up to natural isomorphism of functors. Enlarging the universe $\mathcal{W}$ if necessary, we may assume that $\mathcal{V} \in \mathcal{W}$ and we obtain deformation pseudofunctors

$$\mathcal{V} - \text{def}_a^\text{lin}, \mathcal{V} - \text{def}_\mathcal{G}^{\text{ab}}: \mathcal{U} - \text{Rng}^0 / k \to \mathcal{W} - \text{Gd}.$$  

The universe $\mathcal{W}$ is a purely technical device which guarantees $\mathcal{V} - \text{def}_a^\text{lin}, \mathcal{V} - \text{def}_\mathcal{G}^{\text{ab}}$ taking values in categories. Moreover, whether two deformation pseudofunctors are equivalent is preserved under enlargement of $\mathcal{W}$. On the other hand, by [LV06b, Proposition 8.3] for any enlargement $\mathcal{V}' \in \mathcal{W}$ of $\mathcal{V}$, the canonical pseudonatural transformations

$$\mathcal{V} - \text{def}_a^\text{lin} \to \mathcal{V}' - \text{def}_a^\text{lin}, \mathcal{V} - \text{def}_\mathcal{G}^{\text{ab}} \to \mathcal{V}' - \text{def}_\mathcal{G}^{\text{ab}}$$

define equivalences of deformation pseudofunctors.

In summary, as long as we consider flat nilpotent deformations, the choice of universe does not affect deformation pseudofunctors up to equivalence. Thus we simply write $\text{def}_a^\text{lin}, \text{def}_\mathcal{G}^{\text{ab}}$ for deformation pseudofunctors. Since they have small skeletons [LV06b, Theorem 8.4, 8.5], we also write $\text{Def}_a^\text{lin}, \text{Def}_\mathcal{G}^{\text{ab}}$ for deformation functors

$$\mathcal{V} - \text{Def}_a^\text{lin}, \mathcal{V} - \text{Def}_\mathcal{G}^{\text{ab}}: \mathcal{U} - \text{Rng}^0 / k \to \mathcal{W} - \text{Set}$$

which take values in sets.

Finally, we collect relevant results on deformations of linear and abelian categories.

**Lemma 4.2.** ([LV06b, Theorem 8.16]) Let $S \to R$ be a morphism in $\mathcal{U} - \text{Rng}^0 / k$ and $a$ an essentially small flat $R$-linear category. Then there is a bijection

$$\text{Def}_a^\text{lin}(S) \to \text{Def}_a^{\text{Mod}(a)}(S), \ b \mapsto \text{Mod}(b).$$

In particular, deformations of a module category are module categories.

**Lemma 4.3.** ([LV06b, Theorem 8.17]) Let $S \to R$ be a morphism in $\mathcal{U} - \text{Rng}^0 / k$ and $\mathcal{G}$ an essentially small flat $R$-linear abelian category with enough injectives. Then there is a bijection

$$\text{Def}_a^\text{lin}(\mathcal{G}) \to \text{Def}_a^{\text{Mod}(\mathcal{G})}(S), \ j \mapsto (\text{mod}(j))^{op}.$$
Now, let again $R$ be the fixed local artinian $k$-algebra with residue field $k$, the square zero extension

$$0 \to I \to S \to R \to 0,$$

and the chosen generators $\epsilon = (\epsilon_1, \ldots, \epsilon_l)$ of $I$ regarded as a free $R$-module of rank $l$.

**Lemma 4.5.** ([Low08, Proposition 4.2]) Let $(a, m)$ be a flat $R$-linear category with compositions $m$. Then there is a bijection

$$(4.1) \quad H^2 C(a)^{gl} \to \text{Def}_a^{lin}(S), \quad \phi \mapsto (a[\epsilon], m + \phi \epsilon), \quad \phi \in Z^2 C(a)^{gl}.$$

Another cocycle $\phi' \in Z^2 C(a)^{gl}$ maps to an isomorphic linear deformation if and only if there is an element $h \in C^1(a)$ satisfying $\phi' - \phi = d_a(h)$.

**Lemma 4.6.** ([LV06a Theorem 3.1]) Let $\mathcal{C}$ be a flat $R$-linear abelian category. Then there is a bijection

$$(4.2) \quad H^2 C_{ab}(\mathcal{C})^{gl} \to \text{Def}_{\phi}^{ab}(S).$$

Here, $C(a)$ is the Hochschild object associated with $a$. The compositions $m$ is an element of

$$\prod_{A_0, A_1, A_2 \in a} [a(A_1, A_2) \otimes_R a(A_0, A_1), a(A_0, A_2)]^0$$

rather than the ring map $\rho$ defining the $R$-linear structure. For an $R$-linear abelian category $\mathcal{C}$, the associated Hochschild object is defined as $C_{ab}(\mathcal{C}) = C_{ab}(\text{Ind}(\text{Inj}(\mathcal{C})))$, where $C_{ab}(\text{Ind}(\text{Inj}(\mathcal{C})))$ is the Shukla complex associated with $\text{Ind}(\text{Inj}(\mathcal{C}))$. Note that we have $H^2 C_{ab}(a) = H^2 C(a)$. We will review the definitions in Section 6.

4.8. **Examples.** Let $X$ be a smooth proper $R$-scheme. Since it is noetherian, the category $\text{Qch}(X)$ of quasi-coherent sheaves on $X$ has enough injectives. We denote $i = \text{Inj}(\text{Qch}(X))$ the full $R$-linear subcategory of injective objects. Since $X$ is flat separated, by [DLL17 Proposition 4.28, 4.30(2)] the $R$-linear abelian category $\text{Qch}(X)$ is flat. From [LV06b Proposition 2.9(6)] it follows that the $R$-linear category $i$ is flat. Then by Lemma 4.5 and Lemma 4.6 or Lemma 4.3 both flat linear $S$-deformations of $i$ and flat abelian $S$-deformations of $\text{Qch}(X)$ are classified by $H^2 C(i)^{gl}$.

5. THE CATEGORY OF QUASI-COHERENT SHEAVES

In this section, we review an alternative description of Toda’s construction in terms of the descent category of the category of twisted quasi-coherent presheaves over the restricted structure sheaf, following the exposition from [DLL17 Section 4.5]. It follows that, for square zero extension of relatively smooth $R$-schemes, Toda’s construction coincides with the deformation of the category of quasi-coherent sheaves along the corresponding Hochschild cocycle. As a consequence, deforming the category of the quasi-coherent sheaves is equivalent to deforming the complex structure for higher dimensional Calabi–Yau manifolds. In particular, deformations of the category of quasi-coherent sheaves are given by the category of quasi-coherent sheaves on deformations.

5.1. **Descent categories.** Let $\mathcal{U}$ be a small category and $\text{Cat}(R)$ the category of small $R$-linear categories and $R$-linear functors. A prestack $\mathcal{A}$ is a pseudofunctor $\mathcal{U}^{op} \to \text{Cat}(R)$ consists of the following data:

- for each $U \in \mathcal{U}$ an $R$-linear category $\mathcal{A}(U)$,
- for each $u: V \to U$ in $\mathcal{U}$ an $R$-linear functor $f^u: \mathcal{A}(U) \to \mathcal{A}(V)$,
- for each pair $u: V \to U$, $v: W \to V$ in $\mathcal{U}$ a natural isomorphism $c^{u,v}: f^v f^u \to f^{uv}$,
- for each $U \in \mathcal{U}$ a natural isomorphism $z^U: 1_{\mathcal{A}(U)} \to f^{1_U}$.
Moreover, these data must satisfy
\[ c_{w,v}^{u,w}(c^{v,w} \circ f^w) = c^{w,v}(f^w \circ c^{u,v}), \]
\[ c_{u,v}^{u,v}(z^v \circ f^w) = 1, \quad c^{v,u}(f^w \circ z^u) = 1 \]
for each triple \( u: V \to U, \ v: W \to V, \ w: T \to W \) in \( \mathfrak{U} \). With \( \mathscr{A}(U) \) regarded as one-objected categories, a twisted presheaf \( \mathscr{A} \) of \( \mathbb{R} \)-algebras provides an example of a prestack of \( \mathbb{R} \)-linear category.

For prestacks \( \mathscr{A} = (\mathscr{A}, m, f, c, z), \mathscr{A}' = (\mathscr{A}', m', f', c', z') \) of \( \mathbb{R} \)-linear category on \( \mathfrak{U} \), a morphism \((g, h): \mathscr{A} \to \mathscr{A}'\) is a pseudonatural transformation which consists of the following data:

- for each \( U \in \mathfrak{U} \) an \( \mathbb{R} \)-linear functor \( g^U: \mathscr{A}(U) \to \mathscr{A}'(U) \),
- for each \( u: V \to U \) in \( \mathfrak{U} \) a natural isomorphism \( h^u: f^u g^U \to g^V f^u \).

Moreover, these data must satisfy
\[ h^u(c^{u,v} \circ g^U) = (g^W \circ c^{u,v})(h^v \circ f^u)(f^w \circ h^u), \]
\[ h^v(z^u \circ g^U) = g^U \circ z^u. \]
for each pair \( u: V \to U, \ v: W \to V \) in \( \mathfrak{U} \). When \( \mathscr{A} \) is a twisted presheaf of \( \mathbb{R} \)-algebras, morphisms of twisted presheaves of \( \mathbb{R} \)-algebras coincide with morphisms of prestacks.

A **pre-descent datum** in a prestack \( \mathscr{A} \) is a collection \((A_U)_U \) of objects \( A_U \in \mathscr{A}(U) \) with a morphism \( \varphi_u: f^u A_U \to A_V \) in \( \mathscr{A}(V) \) for each \( u: V \to U \) in \( \mathfrak{U} \), which satisfies
\[ \varphi_v f^u \varphi_u = \varphi_{uv} c^{u,v} A_U; \]
given an additional \( v: W \to V \) in \( \mathfrak{U} \). A *morphism* of pre-descent data \( g: (A_U)_U \to (A'_U)_U \) is a collection \((g_U)_U \) of compatible morphisms \( g_U: A_U \to A'_U \). Pre-descent data and their morphisms form a category \( \text{PDes}(\mathscr{A}) \) equipped with a canonical functor
\[ \pi_V: \text{PDes}(\mathscr{A}) \to \mathscr{A}(V), \quad (A_U)_U \mapsto A_V. \]
When all \( \varphi_u \) are isomorphisms, \((A_U)_U \) is called a *descent datum* and we denote by \( \text{Des}(\mathscr{A}) \) the full subcategory of descent data. Given limits and colimits in each \( \mathscr{A}(U) \) preserved by all \( f^u: \mathscr{A}(U) \to \mathscr{A}(V) \), there exist ones in \( \text{Des}(\mathscr{A}) \) preserved by all \( \pi_V: \text{Des}(\mathscr{A}) \to \mathscr{A}(V) \) \([\text{DLL17]} \) Proposition 4.5(3)]. In particular, if each category \( \mathscr{A}(U) \) is abelian and all \( f^u \) are exact, then \( \text{Des}(\mathscr{A}) \) is abelian and \( \pi_V \) are exact.

### 5.2. The category of quasi-coherent modules over a prestack.

The category of right quasi-coherent modules over a prestack \( \mathscr{A} \) is defined as
\[ \text{Qch}(\mathscr{A}) = \text{Qch}'(\mathscr{A}) = \text{Des}(\text{Mod}_{\mathscr{A}}), \]
where \( \text{Mod}_{\mathscr{A}} \) is the *associated prestack* with a prestack \( \mathscr{A} \) given by
\[ \text{Mod}_{\mathscr{A}} = \text{Mod}'_{\mathscr{A}}: \mathfrak{U}^{\text{op}} \to \text{Cat}(\mathbb{R}), \quad U \mapsto \text{Mod}_{\mathscr{A}}(U) = \text{Mod}(\mathscr{A}(U)), \]
whose restriction functor
\[ \otimes_u \mathscr{A}(V): \text{Mod}(\mathscr{A}(U)) \to \text{Mod}(\mathscr{A}(V)) \]
is the unique colimit preserving functor extending \( f^u: \mathscr{A}(U) \to \mathscr{A}(V) \). The functor sends each \( F \in \text{Mod}(\mathscr{A}(U)) \) to an \( \mathbb{R} \)-linear functor \( F \otimes_u \mathscr{A}(V): \mathscr{A}(V)^{\text{op}} \to \text{Mod}(\mathbb{R}) \) such that
\[ F \otimes_u \mathscr{A}(V)(B) = \bigoplus_{A \in \mathscr{A}(U)} F(A) \otimes_{\mathbb{R}} \mathscr{A}(V)(B, f^u A) / \sim, \]
for each \( B \in \mathscr{A}(V) \). Here, \( \sim \) denotes the equivalence relation defined as
\[ F(a)(x) \otimes y \sim x \otimes f^u(a)y \]
for $x \in F(A')$, $y \in \mathcal{A}(V)(B, f^u A)$, and $a: A \to A'$ in $\mathcal{A}(U)$.

In the case where $F = \mathcal{A}(U)(-, A')$ for some $A' \in \mathcal{A}(U)$, $f^u$ induces an isomorphism
\[
\theta_a': \mathcal{A}(U)(-, A') \otimes_a \mathcal{A}(V) \to \mathcal{A}(V)(-, f^u A').
\]
If $u = 1_U$, then $\mathcal{A}(U) \to f^1_U$ induces an isomorphism
\[
\text{Mod}(z)^U: 1_{\text{Mod}(\mathcal{A}(U))} \to - \otimes_{1_U} \mathcal{A}(U).
\]
Since we have
\[
F \otimes_a \mathcal{A}(V) \otimes_v \mathcal{A}(W)(C) = \bigoplus_{A \in \mathcal{A}(U), B \in \mathcal{A}(V)} F(A) \otimes_R \mathcal{A}(V)(B, f^u A) \otimes_R \mathcal{A}(W)(C, f^v B)/_,
\]
$\theta^{f^a u}_v$ and $c^{u^v}$ induce another isomorphism
\[
\text{Mod}(c^{u^v}_u): - \otimes_u \mathcal{A}(V) \otimes_v \mathcal{A}(W) \to - \otimes_{uv} \mathcal{A}(U).
\]
When $\mathcal{A}$ is a twisted presheaf of $R$-algebras, $\text{Mod}(\mathcal{A}(U))$ coincides with the category of right $\mathcal{A}(U)$-modules whose restriction functor is the ordinary tensor product and $\text{Mod}(c^{u^v}_u)$, $\text{Mod}(z)^U$ are respectively given by
\[
\text{Mod}(c^{u^v}_u)_M: M \otimes_u \mathcal{A}(V) \otimes_v \mathcal{A}(W) \to M \otimes_{uv} \mathcal{A}(U), m \otimes a \otimes b \mapsto m \otimes c^{u^v}(a)b,
\]
\[
\text{Mod}(z)^U_M: M \to M \otimes_{1_U} \mathcal{A}(U), m \mapsto m \otimes z^U
\]
for any right $\mathcal{A}(U)$-module $M$.

5.3. The category of twisted quasi-coherent presheaves over a twisted presheaf. Let $\mathcal{A}$ be a presheaf of $R$-algebras on $\mathcal{U}$. We denote by $\text{Pr}(\mathcal{A}|_U)$ the category of presheaves of right $\mathcal{A}|_U$-modules on $\mathcal{U}/U$, where $\mathcal{A}|_U$ is the induced presheaf on $\mathcal{U}/U$ with $\mathcal{A}|_U(V \to U) = \mathcal{A}(V)$ for $U \in \mathcal{U}$ and $u: V \to U$ in $\mathcal{U}$. Each $u: V \to U$ in $\mathcal{U}$ induces a functor $u^*_V: \text{Pr}(\mathcal{A}|_U) \to \text{Pr}(\mathcal{A}|_V)$.

Since we have $v^*_u: \mathcal{A}(V) \to \mathcal{A}(U)$, we can adapt $\text{Mod}(\mathcal{A}(U))$, $\text{Mod}(\mathcal{A}(U))$, $\text{Mod}(z)^U$, $\text{Mod}(c^{u^v}_u)$, $\text{Mod}(z)^U$ as follows.

Let $M$ be a right $\mathcal{A}(U)$-module. Then $\tilde{M}(u) := M \otimes_u \mathcal{A}(V) = M \otimes_{\mathcal{A}(U)} \mathcal{A}(V)$ is a right $\mathcal{A}(V)$-module with $\mathcal{A}(V)$ regarded as a left $\mathcal{A}(U)$-module via $f^u$. Suppose that $u': V' \to U$ satisfies $u\cdot v = u'$ for $v: V \to V$. We have the right $\mathcal{A}(U)$-module homomorphism $1_M \otimes f^v': \tilde{M}(u) \to \tilde{M}(u')$. The assignments $u \mapsto \tilde{M}(u)$ and $f^v' \mapsto 1_M \otimes f^v'$ define a presheaf $\tilde{M}$ of right $\mathcal{A}(U)$-modules on $\mathcal{U}/U$. Any $\mathcal{A}(U)$-module homomorphism $g: M \to N$ induces a natural transform $\tilde{g} = [\tilde{g}^u := g \circ 1_{\mathcal{A}(V)}]_u$. Thus the assignments $M \mapsto \tilde{M}$ and $g \mapsto \tilde{g}$ define a functor
\[
\text{Q}^U: \text{Mod}(\mathcal{A}(U)) \to \text{Pr}(\mathcal{A}|_U).
\]

We have the canonical isomorphism
\[
\text{can}_{M}^{u^v}: M \otimes_u \mathcal{A}(V) \otimes_v \mathcal{A}(W) \to M \otimes_{uv} \mathcal{A}(U), m \otimes a \otimes b \mapsto m \otimes f^v(a)b.
\]
By [DLL17] Lemma 4.10 the functor $\text{Q}^U$ is fully faithful and there is a natural isomorphism
\[
\tau^u: u^*_V \text{Q}^U \to \text{Q}^U(- \otimes_u \mathcal{A}(V))
\]
induced by $(\text{can}_{M}^{u^v})^{-1}$. A quasi-coherent presheaf over $\mathcal{A}|_U$ is defined as the essential image of some $\mathcal{A}(U)$-module $M$ by $\text{Q}^U$. We denote by $\text{QPr}(\mathcal{A}|_U)$ the category of quasi-coherent presheaves over $\mathcal{A}|_U$.

When $\mathcal{A}$ is a twisted presheaf with central twists $c$, one can adapt $\text{Mod}(c)^{u^v}_u$ to $\text{Pr}(c)^{u^v}_u$ as follows. For $\mathcal{F} \in \text{Pr}(\mathcal{A}|_U)$ and $w: T \to W$ in $\mathcal{U}/W$ the central invertible element $f^{uw}(c^{u^v})$ in $\mathcal{A}(T)$ gives an automorphism
\[
f^{uw}(c^{u^v}): \mathcal{F}(uw) \to \mathcal{F}(uw), m \mapsto mf^{uw}(c^{u^v})
\]
inducing an isomorphism
\[ \Pr(c)^{u,v}_x : v^*_x u^*_y (\mathcal{F}) \to (uv)^*_z (\mathcal{F}) \]
in \( \Pr(\mathcal{A}||U) \). Since we have
\[ \Pr(c)^{u,v}_x \Pr(c)^{v,w} = \Pr(c)^{w,u}_z \Pr(c)^{u,v}_x, \]
the assignments \( U \mapsto \Pr(\mathcal{A}||U) \) and \( u \mapsto u^*_y \) define a prestack
\[ \Pr_{\mathcal{A}} : \mathcal{U}^{op} \to \text{Cat}(\mathcal{R}) \]
whose twist functor is given by \( \Pr(c) \) and \( z \) is given by the identity. Restricting to the essential images \( \text{QPr}(\mathcal{A}||U) \), we obtain another prestack \( \text{QPr}_{\mathcal{A}} \). The category of right twisted quasi-coherent presheaves over a twisted presheaf \( \mathcal{A} \) is defined as
\[ \text{QPr}(\mathcal{A}) = \text{Des(\text{QPr}_{\mathcal{A}})}. \]

**Lemma 5.1.** ([DLL17, Theorem 4.12]) Let \( \mathcal{A} : \mathcal{U}^{op} \to \text{Alg}(\mathcal{R}) \) be a twisted presheaf with central twists. Then \( Q = (Q^U, \tau^u)_{U,\mathfrak{m}} \) defines an equivalence
\[ \text{Qch}(\mathcal{A}) = \text{QPr}(\mathcal{A}) \]
of \( \mathcal{R} \)-linear categories, where \( Q^U, \tau^u \) are given by (5.1), (5.2) respectively.

5.4. **Deformations of the restricted structure sheaves.** Let \( \mathcal{A} : \mathcal{U}^{op} \to \text{Cat}(\mathcal{R}) \) be a flat prestack. Recall that \( \mathcal{A} \) is flat if \( \mathcal{R} \)-modules \( \mathcal{A}(U)(A, A') \) are flat for all \( U \in \mathfrak{U} \) and \( A, A' \in \mathcal{A}(U) \). An \( S \)-deformation of \( \mathcal{A} \) is a flat prestack \( \mathcal{B} : \mathcal{U}^{op} \to \text{Cat}(\mathcal{S}) \) together with an equivalence of prestacks \( \mathcal{B} \otimes_{\mathcal{S}} \mathcal{R} \to \mathcal{A} \), i.e., for each \( U \) there is a morphism of prestacks inducing an equivalence \( \mathcal{B}(U) \otimes_{\mathcal{S}} \mathcal{R} \to \mathcal{A}(U) \) of \( \mathcal{R} \)-linear categories ([DLL17] Proposition 4.7). Two deformations \( \mathcal{B}, \mathcal{B}' \) are equivalent if there is an equivalence \( \mathcal{B} \to \mathcal{B}' \) of prestacks compatible with equivalences \( \mathcal{B} \otimes_{\mathcal{S}} \mathcal{R} \to \mathcal{A}, \mathcal{B}' \otimes_{\mathcal{S}} \mathcal{R} \to \mathcal{A} \). We denote by \( \text{Def}^{\text{tw}}_{\mathcal{A}}(\mathcal{S}) \) the set of equivalence classes of \( \mathcal{S} \)-deformations of \( \mathcal{A} \). When \( \mathcal{B} \otimes_{\mathcal{S}} \mathcal{R} \to \mathcal{A} \) is an isomorphism of prestacks, i.e., for each \( U \) there is a morphism of prestacks inducing an isomorphism \( \mathcal{B}(U) \otimes_{\mathcal{S}} \mathcal{R} \to \mathcal{A}(U) \) of \( \mathcal{R} \)-linear categories, we call the deformation \( \mathcal{B} \) strict. Two strict deformations \( \mathcal{B}, \mathcal{B}' \) are equivalent if there is an isomorphism \( \mathcal{B} \to \mathcal{B}' \) of prestacks inducing the identity on \( \mathcal{A} \). We denote by \( \text{Def}^{\text{tw}}_{\mathcal{A}}(\mathcal{S}) \) the set of equivalence classes of strict \( \mathcal{S} \)-deformations of \( \mathcal{A} \). Recall that twisted presheaves of \( \mathcal{R} \)-algebras can be regarded as a prestack. Due to the lemma below, as long as we consider equivalence classes of twisted deformations of flat presheaves, we may restrict our attention to strict twisted deformations.

**Lemma 5.2.** ([DLL17, Proposition 5.9]) Let \( (\mathcal{A}, m, f, c, z) \) be a flat prestack of \( \mathcal{R} \)-linear categories on \( \mathfrak{U} \). Then the canonical map
\[ \text{Def}^{\text{tw}}_{\mathcal{A}}(\mathcal{S}) \to \text{Def}^{\text{tw}}_{\mathcal{A}}(\mathcal{S}) \]
is bijective.

Let \( \mathfrak{U} \) be a finite poset with binary meets. Then any prestack on \( \mathfrak{U} \) is quasi-compact since \( \mathfrak{U} \) is finite. A prestack \( \mathcal{A} : \mathcal{U}^{op} \to \text{Cat}(\mathcal{R}) \) is right semi-separated if the associated prestack \( \text{Mod}_{\mathcal{A}} \) is of affine localizations. Namely, for all \( U, V, W \in \mathfrak{U} \) with \( v : V \to U, w : W \to U \) in \( \mathfrak{U} \) and the pullback diagram
\[ V \cap W \xrightarrow{\tilde{v}} V \]
\[ W \xrightarrow{w} U \]
the following conditions are satisfied.
• The category $\mathsf{Mod}_\mathscr{A}(U)$ is Grothendieck abelian.
• The functor $\nu^* : \mathsf{Mod}_\mathscr{A}(U) \to \mathsf{Mod}_\mathscr{A}(V)$ is exact.
• The functor $\nu^*$ admits a fully faithful exact right adjoint $\nu_* : \mathsf{Mod}_\mathscr{A}(V) \to \mathsf{Mod}_\mathscr{A}(U)$.
• There are natural isomorphisms
  $$(\nu_* \nu^*)(w \times w) \cong (\nu \nu^*)(\nu \nu^*) \cong (w \times w)(\nu \nu^*).$$

A presheaf $\mathscr{A} : \mathcal{U}^{op} \to \mathbf{Alg}(\mathbf{R})$ is right semi-separated if so is $\mathscr{A}$ with $\mathcal{A}(U)$ regarded as one-objected categories. Every right semi-separated prestack is geometric, i.e., the restriction functor
$$- \otimes_\mathbb{R} \mathscr{A}(V) : \mathsf{Mod}(\mathscr{A}(U)) \to \mathsf{Mod}(\mathscr{A}(V))$$
is exact. Note that for any geometric prestack $\mathscr{A} : \mathcal{U}^{op} \to \mathbf{Cat}(\mathbf{R})$ on a small category $\mathsf{Qch}(\mathscr{A})$ is a Grothendieck abelian category [DLL17, Theorem 4.14].

Let $X$ be a smooth proper $\mathbf{R}$-scheme. Choose a finite affine open cover $\mathcal{U}$ closed under intersections. We denote by $\mathscr{O}_{X|\mathcal{U}}$ the restricted structure sheaf to $\mathcal{U}$. Since $U \cap V$ is affine as $X$ is separated, we have isomorphisms of $\mathscr{O}_X(U)$-modules
$$\mathscr{O}_X(V) \otimes_\mathscr{O}_X(U) \mathscr{O}_X(W) \cong \mathscr{O}_X(U \cap V) \cong \mathscr{O}_X(W) \otimes_\mathscr{O}_X(U) \mathscr{O}_X(V)$$
for all $U, V, W \in \mathcal{U}$ with $V, W \subset U$. Since pushforwards along open immersions $V \hookrightarrow U$ of affine schemes are fully faithful, by [DLL17] Lemma 3.1 the restriction maps $\mathscr{O}_X(U) \to \mathscr{O}_X(V)$ are flat epimorphisms of rings. Then one can apply [DLL17] Proposition 4.28 to see that the presheaf $\mathscr{O}_{X|\mathcal{U}} : \mathcal{U}^{op} \to \mathbf{Alg}(\mathbf{R})$ is right semi-separated. Since $\mathscr{O}_X(U)$ are flat $\mathbf{R}$-modules, the category
$$\mathsf{Qch}(\mathscr{O}_{X|\mathcal{U}}) \cong \mathsf{QPr}(\mathscr{O}_{X|\mathcal{U}}) \cong \mathsf{Qch}(\mathscr{O}_X)$$
is flat over $\mathbf{R}$ and Grothendieck abelian [DLL17, Proposition 4.30].

**Lemma 5.3.** ([DLL17] Theorem 5.10) Let $X$ be a smooth proper $\mathbf{R}$-scheme with a finite affine open cover $\mathcal{U}$ closed under intersections. Then every twisted $\mathbf{S}$-deformations of the restricted structure sheaf $\mathscr{O}_{X|\mathcal{U}}$ is a quasi-compact semi-separated presheaf on $\mathcal{U}$ and there is a bijection
$$\text{Def}_{\mathcal{U}}^{\mathbf{S}}(\mathbf{S}) \to \text{Def}_{\mathsf{Qch}(X)}^{\mathbf{S}}(\mathbf{S}), \quad (\mathscr{O}_{X|\mathcal{U}})^{\phi} \mapsto \mathsf{Qch}((\mathscr{O}_{X|\mathcal{U}})^{\phi}),$$
where $\phi \in H^2_{\mathbf{C}(\mathbf{S})}(\mathcal{U})^{\otimes}$ is a cocycle and $(\mathscr{O}_{X|\mathcal{U}})^{\phi}$ is the twisted $\mathbf{S}$-deformation of $\mathscr{O}_{X|\mathcal{U}}$ along $\phi$. In particular, the category of right quasi-coherent modules over a twisted deformation of $\mathscr{O}_{X|\mathcal{U}}$ is given by an abelian deformation of the category $\mathsf{Qch}(X)$ of quasi-coherent sheaves.

**5.5. Toda’s construction revisited.** Let $\mathcal{U}$ be a small category and $(\mathscr{A}, m, f)$ a presheaf of $\mathbf{R}$-algebras on $\mathcal{U}$. The simplicial complex of presheaves associated with $\mathscr{A}$ is the complex $(\mathscr{A}^\bullet, \mathbf{d}^\bullet)$ defined as follows. Consider the presheaf of algebras $\mathscr{A}^n = (\mathscr{A}^n, m^n, f^n)$ for $n \geq 0$ given by
$$\mathscr{A}^n(U) = \prod_{\tau \in \mathcal{N}_n(\mathcal{U}/U)} \mathscr{A}|_{\mathcal{U}}(\tau)$$
endowed with the product algebra structure $m^n|_U$. Here, $\tau \in \mathcal{N}_n(\mathcal{U}/U)$ is identified with the object $d\tau \to U \in \mathcal{U}/U$ by composing all morphisms of $\tau$, and the restriction map
$$f^n|_U : \mathscr{A}^n(U) \to \mathscr{A}^n(V), \quad (a^\tau) \mapsto (a^\partial^\tau)_\tau$$
is induced by the natural map $\mathcal{N}_n(\mathcal{U}/V) \to \mathcal{N}_n(\mathcal{U}/U)$, $\sigma \mapsto \iota(\sigma)$. Define morphisms of presheaves $\varphi^n : \mathscr{A}^n \to \mathscr{A}^{n+1}$ as
$$\varphi^n|_U : \prod_{\tau \in \mathcal{N}_n(\mathcal{U}/U)} \mathscr{A}|_{\mathcal{U}}(\tau) \to \prod_{\sigma \in \mathcal{N}_{n+1}(\mathcal{U}/U)} \mathscr{A}|_{\mathcal{U}}(\sigma), \quad (a^\tau) \mapsto \left( f^n(a^\partial^\sigma) + \sum_{i=1}^{n+1} (-1)^i (a_{^\partial^\sigma}) \right)_\sigma$$
which specialize to
\[ \varphi^{0, U} : \prod_{u : V \to U} \mathcal{A}(V) \to \prod_{u : V \to U, \, W \to V} \mathcal{A}(W), \quad (a^u)_u \mapsto f^u(a^u) - a^{w}. \]

Then one obtains the complex \((\mathcal{A}, \varphi^*\varphi)\) with \(\text{ker}(\varphi^0) \cong \mathcal{A}\) [DLL17] Lemma 2.12).

Using a part of the simplicial complex of presheaves, one can give an alternative description of Toda’s construction. Since \(\mathcal{R}\) is commutative, by [DLL17] Proposition 2.14] every normalized reduced cocycle
\[ \phi = (m, f, c) = (m_1, \ldots, m_l, f_1, \ldots, f_l; c_1, \ldots, c_l) \in \tilde{C}_{\mathcal{R}}\text{simp}(\mathcal{A})^{\otimes} \oplus \tilde{C}_{\mathcal{R}}^{1,1}(\mathcal{A})^{\otimes} \oplus \tilde{C}_{\mathcal{R}}^{2,0}(\mathcal{A})^{\otimes}. \]

admits a weak decomposition
\[ (m, f, c) = (m, f, 0) + (0, 0, c) \in \tilde{C}_{\mathcal{R}}^{2,0}(\mathcal{A})^{\otimes} \oplus \tilde{C}_{\mathcal{R}}^{2,0}(\mathcal{A})^{\otimes}. \]

From [DLL17] Proposition 2.24] it follows that the twisted \(S\)-deformation \(\mathcal{A}\phi\) of \(\mathcal{A}\) along \(\phi\) has central twists and the underlying presheaf \(|\mathcal{A}\phi|\) is the presheaf \(S\)-deformation of \(\mathcal{A}\) along \(|\phi| = (m, f, 0)\).

Consider the morphism \(F : \mathcal{A} \oplus (\mathcal{A}^0)^{\otimes} \to (\mathcal{A}^1)^{\otimes}\) of presheaves defined as
\[ F^U : \mathcal{A}(U) \oplus \prod_{u : V \to U} \mathcal{A}^0(V)^{\otimes} \to \prod_{u : V \to U, \, W \to V} \mathcal{A}^1(W)^{\otimes}, \]
\[ (a, (b^u_1, \ldots, b^u_l)_u) \mapsto (f^u_a(a) + v^u(b^u_1), \ldots, f^u_a(a) + v^u(b^u_l)) \]

where we denote \(f^u_a\) by \(u^*\) for clarity. Define the multiplication on \(\mathcal{A} \oplus (\mathcal{A}^0)^{\otimes}\) as
\[ (a, (b^u_1, \ldots, b^u_l)_u) \cdot (a', (b'^u_1, \ldots, b'^u_l)_u) \]
\[ = (a'a', (u^*(a)b'^u_1 + b'^u_1u'(a')) + m_1(u^*(a), u'(a'))), \ldots, \]

With the scalar given by
\[ (\lambda + \kappa_1 e_1 + \ldots + \kappa_l e_l)(a, (b^u_1, \ldots, b^u_l)_u) = (\lambda a, (\kappa_1 u^*(a) + \lambda b^u_1, \ldots, \kappa_l u^*(a) + \lambda b^u_l)_u), \]

\(\mathcal{A} \oplus (\mathcal{A}^0)^{\otimes}\) becomes an \(S\)-algebra. Then the morphism \(G : |\mathcal{A}\phi| \to \mathcal{A} \oplus (\mathcal{A}^0)^{\otimes}\) of presheaves of \(S\)-algebras defined as
\[ G^U : |\mathcal{A}\phi|(U) \to \mathcal{A}(U) \oplus \mathcal{A}^0(U)^{\otimes}, \]
\[ a + b^u_1 e_1 + \ldots + b^u_l e_l \mapsto (a, (f^u_a(a) + u^*(b_1), \ldots, f^u_a(a) + u^*(b_l)))_u \]

yields an exact sequence
\[ 0 \to |\mathcal{A}\phi| \xrightarrow{G} \mathcal{A} \oplus (\mathcal{A}^0)^{\otimes} \xrightarrow{F} (\mathcal{A}^1)^{\otimes}. \]

Consider the case where \(\mathcal{A}\) is the restricted structure sheaf \(\mathcal{O}_X|_U\) of a smooth proper \(R\)-scheme \(X\). Fix a finite affine open cover \(U\) of \(X\) closed under intersections. As explained above, \(\mathcal{O}_X|_U\) gives a quasi-compact right semi-separated presheaf of \(R\)-algebras. Since \(\mathcal{O}_X(U)\) is smooth \(R\)-algebra for each \(U \in \mathcal{U}\), we may assume further that \(\phi = (m, f, c)\) is decomposable. We use the same symbol \(\phi\) to denote the cocycle
\[ (\alpha, \beta, \gamma) \in H^2(\mathcal{O}_X) \oplus H^1(\mathcal{V}_X/R) \oplus H^0(\mathcal{A}_X/R) \]
which is the image of \((m, f, c)\) under the bijection (2.3). Then \(\phi\) defines the \(S\)-linear abelian category \(\text{Qch}(X, \phi)\) obtained by Toda’s construction.

**Lemma 5.4.** ([DLL17] Theorem 5.12) For a smooth proper \(R\)-scheme \(X\) with a finite affine open cover \(U\) closed under intersections, let \((\mathcal{O}_X|_U)_\phi\) be the twisted \(S\)-deformation of the restricted structure sheaf \(\mathcal{O}_X|_U\) along a normalized reduced decomposable cocycle
\[ \phi = (m, f, c) \in \tilde{C}_{\mathcal{R}}^{0,2}(\mathcal{O}_X|_U)^{\otimes} \oplus \tilde{C}_{\mathcal{R}}^{1,1}(\mathcal{O}_X|_U)^{\otimes} \oplus \tilde{C}_{\mathcal{R}}^{2,0}(\mathcal{O}_X|_U)^{\otimes}, \]

25
which maps to a cocycle

$$(\alpha, \beta, \gamma) \in H^2(\mathcal{O}_X) \oplus H^1(\mathcal{T}_X/\mathcal{R}) \oplus H^0(\wedge^2\mathcal{T}_X/\mathcal{R})$$

under the bijection (2.3). Then there is an equivalence

$$\text{Qch}((\mathcal{O}_X|_{\mathcal{U}})_\phi) \approx \text{Qch}(X, \phi)$$

of $S$-linear Grothendieck abelian categories, where $\text{Qch}(X, \phi)$ is the abelian category obtained by Toda’s construction from $\text{Qch}(X)$ along $(\alpha, \beta, \gamma)$.

By (2.3), Lemma 2.2, and Lemma 5.3 we obtain a bijection

$$(5.3) \quad HT^2(X/\mathcal{R})^{ab} \cong HH^2(X/\mathcal{R})^{ab} \cong HH^2_{ab}(\text{Qch}(X))^{ab}.$$  

Let $Q(X)_\phi$ the flat abelian $S$-deformation of $Q(X)$ along the image of $(\alpha, \beta, \gamma)$ under (5.3). Combining Lemma 5.3 and Lemma 5.4, we obtain

**Proposition 5.5.** For a smooth proper $R$-scheme $X$, let $Q(X)_\phi$ be the flat abelian $S$-deformation of $Q(X)$ and $Q(X, \phi)$ the abelian category obtained by Toda’s construction from $Q(X)$ corresponding to $[\phi] \in HH^2(X/\mathcal{R})^{ab}$ via the isomorphism (5.3). Then there is an equivalence

$$Q(X)_\phi \approx Q(X, \phi)$$

of $S$-linear Grothendieck abelian categories.

Now, we return to our setting. Let $X_0$ be a Calabi–Yau manifold with $\dim X_0 > 2$ and $(X, i_R)$ an $R$-deformation of $X_0$. Since we have

$$HT^2(X/R) = H^2(\mathcal{O}_X/R) \oplus H^1(\mathcal{T}_X/R) \oplus H^0(\wedge^2\mathcal{T}_X/R) \cong H^1(\mathcal{T}_X/R),$$

every cocycle $\phi \in HH^2(X/R)^{ab}$ defines an $S$-deformation $(X_\phi, i_S)$ of $(X, i_R)$. By Proposition 3.11, we have $Q(X, \phi) \approx Q(X_\phi)$. Along square zero extensions, deforming Calabi–Yau manifolds and taking the category of quasi-coherent sheaves are compatible in the following sense.

**Corollary 5.6.** Let $X_0$ be a Calabi–Yau manifold with $\dim X_0 > 2$, $(X, i_R)$ an $R$-deformation of $X_0$, and $(X_\phi, i_S)$ the $S$-deformation of $(X, i_R)$ corresponding to $[\phi] \in HH^2(X/R)^{ab}$. Then there is an equivalence

$$Q(X)_\phi \approx Q(X_\phi)$$

of $S$-linear Grothendieck abelian categories, where $Q(X)_\phi$ is the flat abelian $S$-deformation of $Q(X)$ corresponding to $[\phi]$ via the isomorphism (5.3).

**Remark 5.1.** Since we have $HT^2(X_\phi/S) \cong H^1(\mathcal{T}_{X_\phi}/S)$ by Calabi–Yau condition and the finite affine open cover $\mathcal{U} = \{U_i\}_{i=1}^N$ of $X$ closed under intersections canonically lifts to the locally trivial deformation $\mathcal{U} \times_R S = \{U_i \times_R S\}_{i=1}^N$, one may iteratively use Corollary 5.6 along a sequence of square zero extensions.

6. **DEFORMATIONS OF THE DG CATEGORY OF PERFECT COMPLEXES**

In this section, we review the deformation theory of dg category following the exposition from [Low08] and [KL09]. Based on the ideas thereof, for a higher dimensional Calabi–Yau manifold we prove the compatibility of deformations with taking the dg category of perfect complexes. Namely, the dg category of perfect complexes on a deformation is Morita equivalent to the corresponding dg deformation of a certain full dg category determined by the direction of the deformation.
6.1. **Curved $A_{\infty}$-categories.** In the sequel, by a *quiver* we will mean a $\mathbb{Z}$-graded quiver. We choose shift functors $\Sigma^k$ on the category $G(\mathbb{R})$ of $\mathbb{Z}$-graded $\mathbb{R}$-modules. Let $a$ be an $\mathbb{R}$-linear quiver. Namely, a consists of a set $\text{Ob}(a)$ of objects and a $\mathbb{Z}$-graded $\mathbb{R}$-module $a(A, A')$ for each pair $A, A' \in \text{Ob}(a)$. The category of quivers with a fixed set of objects admits a tensor product
\[
a \otimes b(A, A') = \bigoplus_{A''} a(A'', A') \otimes_\mathbb{R} b(A, A'')
\]
and an internal hom
\[
[a, b](A, A') = [a(A, A'), b(A, A')].
\]
Morphisms of degree $k$ are elements of $[a, b]^k = \prod_{A, A'} [a, b](A, A')^k$.

The *tensor cocategory* $T(a)$ of $a$ is the quiver
\[
T(a) = \bigoplus_{n \geq 0} a^\otimes_n
\]
equipped with the comultiplication which separates tensors. There is a natural brace algebra structure on $[T(a), a] = \prod_{n \geq 0}[T(a), a]_n$, where
\[
[T(a), a]_n = [a^\otimes_n, a] = \prod_{A_0, \ldots, A_n \in a} [a(A_{n-1}, A_n) \otimes_\mathbb{R} \cdots \otimes_\mathbb{R} a(A_0, A_1), a(A_0, A_n)].
\]
It is given by the operations
\[
[T(a), a]_n \otimes_\mathbb{R} [T(a), a]_n \otimes_\mathbb{R} \cdots \otimes_\mathbb{R} [T(a), a]_n \to [T(a), a]_{n-i+n_1+\cdots+n_i},
\]
\[
(\phi, \phi_1, \ldots, \phi_i) \mapsto \phi[\phi_1, \ldots, \phi_i]
\]
with
\[
\phi[\phi_1, \ldots, \phi_i] = \sum \phi(1 \otimes \cdots \otimes \phi_1 \otimes 1 \otimes \cdots \phi_i \otimes 1 \otimes \cdots 1)
\]
satisfying
\[
\phi[\phi_1, \ldots, \phi_i][\psi_1, \ldots, \psi_j] = \sum (-1)^{\alpha} \phi[\psi_1, \ldots, \phi_1[\psi_{m_1}, \ldots], \psi_{n_1}, \ldots, \phi_1[\psi_{m_i}, \ldots], \psi_{n_i}, \ldots, \psi_j],
\]
where $\alpha = \sum_{k=1}^i |\phi_k| \sum_{l=1}^{n_k-1} |\psi_l|$. We denote by $B\mathfrak{a}$ the Bar cocategory $T(\Sigma a)$ and by $C_{br}(a)$ the brace algebra $[B\mathfrak{a}, \Sigma a]$. The associated Hochschild object is defined as $C(a) = \Sigma^{-1} C_{br}(a)$. By [Low08 Proposition 2.2] the tensor coalgebra $T(C_{br}(a)) = B\mathcal{C}(a)$ becomes a graded bialgebra with the associative multiplication defined by the composition.

A *curved $A_{\infty}$-structure* on $a$ is an element $b \in C_{br}(a)$ satisfying $b(b) = 0$. The pair $(a, b)$ is called a curved $A_{\infty}$-category. When the defining morphisms $b_n : \Sigma^{\otimes n} \to \Sigma a$ vanish for $n \geq 3$, we call $(a, b)$ a cdg category. The curvature elements of $(a, b)$ is the morphism $b_0$. When it vanishes, we drop “curved” and “c” from the notation.

**Definition 6.1.** ([Low08 Definition 2.5]) For curved $A_{\infty}$-categories $(a, b), (a', b')$ with $\text{Ob}(a) = \text{Ob}(a')$ a morphism is a fixed object morphism of quivers $f : B\mathfrak{a} \to B\mathfrak{a}'$, which is determined by morphisms $f_n : (\Sigma a)^{\otimes n} \to \Sigma a'$ and respects the comultiplications and the curved $A_{\infty}$-structures.

6.2. **Hochschild complexes of curved $A_{\infty}$-categories.** The associated Lie bracket with the brace algebra $C_{br}(a)$ is defined as
\[
\langle \phi, \psi \rangle = \phi[\psi] - (-1)^{|\phi||\psi|} \psi[\phi].
\]
Via an isomorphism
\[
C_{br}(a) \cong \text{Coder}(B\mathfrak{a}, B\mathfrak{a})
\]
of \( \mathbb{Z} \)-graded \( \mathbf{R} \)-modules to coderivations between cocategories, it corresponds to the commutator of coderivations. For a curved \( A_\infty \)-structure \( b \) on \( a \) the Hochschild differential on \( C_{br}(a) \) is defined as

\[
d_b = \langle b, - \rangle \in [C_{br}(a), C_{br}(a)]^{1}, \quad \phi \mapsto \langle b, \phi \rangle.
\]

In particular, \( C_{br}(a) \) can be regarded as a dg Lie algebra. Then \( C(a) \) is known to be isomorphic to the classical Hochschild complex of \( a \), whose definition we will review later. Since \( b \) naturally belongs to \( BC(a)^{1} \), it induces a differential

\[
D_b = [b, -] \in [BC(a), BC(a)]^{1}, \quad \phi \mapsto [b, \phi],
\]

where \([-, -]\) is the commutator of the multiplication given by [Low08 Proposition 2.2]. As \( D_b \) belongs to \( \text{Coder}(BC(a), BC(a)) \), it defines a curved \( A_\infty \)-structure on \( C(a) \). The differential \( D_b \) gives a dg bialgebra structure on \( BC(a) \) and \( C(a) \), \( C_{br}(a) \) become \( B_\infty \)-algebras [Gl Definition 5.2].

We will use the same symbol \( C(a) \) to denote the bigraded object with

\[
C^{p,q}(a) = \prod_{A_0,...,A_q \in \mathbb{N}} [a(A_{q-1}, A_q) \otimes_{\mathbf{R}} \cdots \otimes_{\mathbf{R}} a(A_0, A_1), a(A_0, A_q)]^{p}.
\]

An element \( \phi \in C^{p,q}(a) \) is said to have the degree \( |\phi| = p \), the arity \( \text{ar}(\phi) = q \), and the Hochschild degree \( \text{deg}(\phi) = n = p + q \). The total complex of Hochschild degree \( n \) is defined as \( C^n(a) = \prod_{p+q=n} C^{p,q}(a) \). Via the canonical isomorphisms

\[
\Sigma^{1-q}[a(A_{q-1}, A_q) \otimes_{\mathbf{R}} \cdots \otimes_{\mathbf{R}} a(A_0, A_1), a(A_0, A_q)] \\
\rightarrow [\Sigma a(A_{q-1}, A_q) \otimes_{\mathbf{R}} \cdots \otimes_{\mathbf{R}} \Sigma a(A_0, A_1), \Sigma a(A_0, A_q)],
\]

the \( B_\infty \)-structure on \( C_{br}(a) \) is translated in terms of \( a \). For instance, the operation

\[
\text{dot}: C_{br}(a)_q \otimes C_{br}(a)_r \rightarrow C_{br}(a)_{q+r-1}, \quad (\phi, \psi) \mapsto \phi(\psi)
\]

induces the classical “dot product”

\[
\bullet: C^{p,q}(a) \otimes C^{0,s}(a) \rightarrow C^{p+r,q+s-1}(a)
\]

on \( C(a) \) given by

\[
\phi \bullet \psi = \sum_{k=0}^{q-1} (-1)^{\ell} \phi(1^{\otimes q-k-1} \otimes \psi \otimes 1^{\otimes k}),
\]

where \( \beta = (\text{deg}(\phi) + k + 1)(\text{ar}(\psi) + 1) \). We also call the bigraded object \( C(a) \) the Hochschild complex of \( a \) and its elements Hochschild cochains. In the sequel, curved \( A_\infty \)-structure on \( a \) will often be translated into an element of \( C^2(a) \) without further comments.

6.3. Curved dg category of precomplexes. Let \( a \) be an \( \mathbf{R} \)-linear category. Consider the category \( \text{PCom}(a) \) of precomplexes of \( a \)-objects. A precomplex of \( a \)-objects is a \( \mathbb{Z} \)-graded \( a \)-objects \( C \) with \( C^i \in a \) together with a predifferential, a \( \mathbb{Z} \)-graded \( a \)-morphism \( \delta_C : C \rightarrow C \) of degree 1. For any \( C, D \in \text{PCom}(a) \) the Hom-set \( \text{PCom}(a)(C, D) \) is a \( \mathbb{Z} \)-graded \( \mathbf{R} \)-module with

\[
\text{PCom}(a)(C, D)^k = \prod_{i \in \mathbb{Z}} a(C^i, D^{i+k}).
\]
The cdg structure $\mu \in C(a)^3$ on $\text{PCom}(a)$ consists of compositions $m$, differentials $d$, and curvature elements $c$, where

$$m = \mu_2 \in \prod_{C_0, C_1, C_2 \in \text{PCom}(a)} [\text{PCom}(a)(C_1, C_2) \otimes_R \text{PCom}(a)(C_0, C_1), \text{PCom}(a)(C_0, C_2)]^\mu,$$

$$d = \mu_1 \in \prod_{C_0, C_1 \in \text{PCom}(a)} [\text{PCom}(a)(C_0, C_1), \text{PCom}(a)(C_0, C_1)]^1,$$

$$c = \mu_0 \in \prod_{C \in \text{PCom}(a)} \text{PCom}(a)(C, C)^2$$

are given by

$$m(g, f)_i = (gf)_i = g_{i+1}f_i : C_0^i \to C_2^{i+f_0+|f|},$$

$$d(f) = \delta C_1 f - (-1)^{|f|}\delta C_0,$$

$$c_c = -\delta^2 C_c$$

for morphisms $f : C_0 \to C_1, g : C_1 \to C_2$ in $\text{PCom}(a)$. One can check that $m$, $d$, and $c$ satisfy

$$d(c) = 0,$$

$$d^2 = -m(c \otimes 1 - 1 \otimes c),$$

$$dm = m(d \otimes 1 + 1 \otimes d),$$

$$m(m \otimes 1) = m(1 \otimes m).$$

We denote by $\text{Com}(a)$ the full dg subcategory of complexes of $a$-objects, where $\delta_C$ become differentials.

Here, we demonstrate how the cdg structure is translated. The differential

$$d_b = \langle b, - \rangle = \langle \Sigma c + d + \Sigma^{-1} m, - \rangle \in [C_{br}(\text{PCom}(a)), C_{br}(\text{PCom}(a))]^1$$

on $C_{br}(\text{PCom}(a))$ sends $\Sigma^{-q}\phi \in C_{br}(\text{PCom}(a))$ with $\phi \in C^{p,q}(\text{PCom}(a))$ to

$$\text{dot}(\Sigma c + d + \Sigma^{-1} m, \Sigma^{-q}\phi) - (-1)^{1-q+q} \text{dot}(\Sigma^{-q+q} \phi, \Sigma c + d + \Sigma^{-1} m).$$

In terms of $C(a)$ the image corresponds to $[c + d + m, \phi]$, where

$$[c, \phi] = \sum_{k=0}^{q-1} (-1)^{k+1} \phi(1^{\otimes q-k-1} \otimes c \otimes 1^{\otimes k}),$$

$$[d, \phi] = (-1)^{ar(\phi)+1} d(\phi) + \sum_{k=0}^{q-1} (-1)^{\text{deg}(\phi)} \phi(1^{\otimes q-k-1} \otimes c \otimes 1^{\otimes k}),$$

$$[m, \phi] = m(\phi \otimes 1) + (-1)^{ar(\phi)+1} m(1 \otimes \phi) + \sum_{k=0}^{q-1} (-1)^{k+1} \phi(1^{\otimes q-k-1} \otimes c \otimes 1^{\otimes k}).$$

### 6.4. Curved dg deformations of dg categories.

A cdg $S$-deformation of $a$ is an $S$-linear cdg structure on an $S$-linear quiver $b$ together with an isomorphism $b \to a[e] = a \otimes_R S$ of $S$-linear quivers whose reduction $b \otimes_S R \to a$ induces an isomorphism of cdg categories. Two cdg deformations $b, b'$ are isomorphic if there is an isomorphism $b \to b'$ of cdg categories inducing the identity on $a$. We denote by $\text{Def}_{a}^{cdg}(S)$ the set of isomorphism classes of cdg $S$-deformations of $a$.

**Theorem 6.2.** ([Low08, Theorem 4.11]) Let $(a, \mu)$ be an $R$-linear cdg category. Then there is a bijection

$$H^2 C(a)^{\oplus} \to \text{Def}_{a}^{cdg}(S), \phi \mapsto (a[e], \mu + \phi e), \phi \in Z^2 C(a)^{\oplus}.$$
Another cocycle \( \phi' \in Z^2(C(a)\oplus) \) maps to an isomorphic cdg deformation if and only if there is an element \( h \in C^1(a) \) satisfying \( \phi' - \phi = d_\mu(h) \).

A partial cdg \( S \)-deformation of \( a \) is a cdg \( S \)-deformation of some full cdg subcategory \( a' \). Two partial cdg deformations \( b, b' \) are isomorphic if there is an isomorphism \( b \to b' \) of cdg categories inducing the identity on \( a' \). A morphism of partial cdg deformations \( b, b' \) is an isomorphism of cdg deformations between \( b \) and a full cdg subcategory of \( b' \). When every morphism of \( b \to b' \) of partial cdg deformations is an isomorphism, we call \( b \) maximal. We denote by \( \text{Def}^{\text{mp}}_{a}(S) \) the set of morphism classes of partial cdg \( S \)-deformations of \( a \) and by \( \text{Def}^{\text{mp}}_{a}(S) \) the set of isomorphism classes of maximal partial cdg \( S \)-deformations of \( a \).

Assume further that \( a \) is a dg category. For \( \phi \in Z^2(C(a) \) the \( [\phi] - \infty \) part of \( a \) is the full dg subcategory \( a_{[\phi] - \infty} \) spanned by objects \( A \in a \) satisfying

\[
H^2(\pi_0)([\phi])_A = 0 \in H^2(a(A, A))
\]

where \( \pi_0 : C(a) \to C(a)_0 \) is the projection onto the zero part

\[
C(a)_0 = \Sigma^{-1}[T(\Sigma a), \Sigma a]_0 = \prod_{A \in a} a(A, A).
\]

For a cdg \( S \)-deformation \( b = (\tilde{a}, \mu + \phi \epsilon) \) of \( a \), the \( -\infty \) part \( b_{-\infty} \) is the full cdg subcategory spanned by objects \( B \in b \) satisfying

\[
(\mu + \phi \epsilon)_{0,B} = 0 \in b(B, B)^2.
\]

It is a partial dg deformation of \( a \) and a dg deformation of \( a_{[\phi] - \infty} \). More explicitly, if we restrict \( \phi \) to \( a_{[\phi] - \infty} \), then \( \phi_0 \) becomes a coboundary and there is an element

\[
h \in \prod_{A \in a_{[\phi] - \infty}} a(A, A)^1 \subset C^1(a)
\]

with \( d_\mu(h) = (\phi|_{a_{[\phi] - \infty}})_0 \). Thus the cocycle \( \phi|_{a_{[\phi] - \infty}} - d_\mu(h) \) has trivial curvature elements.

**Proposition 6.3.** ([Low08, Proposition 4.14]) Let \( (a, \mu) \) be an \( R \)-linear dg category. Then there is a map

\[
(6.2) \quad H^2C(a)\oplus \to \text{Def}_{a}^{\text{mp}}(S), \phi \mapsto (a_{[\phi] - \infty}[\epsilon], \mu + (\phi|_{a_{[\phi] - \infty}} - d_\mu(h))\epsilon).
\]

6.5. The characteristic morphism. Let \( i \) be an \( R \)-linear category. The canonical projection

\[
\pi : C(P\text{Com}(i)) \to C(i)
\]

has a \( B_{-\infty} \)-section

\[
(6.3) \quad \text{embr}_{\delta} : C(i) \to C(P\text{Com}(i)),
\]

whose restriction to the full dg subcategory \( \text{Com}^*(i) \) of bounded below complexes of \( i \)-objects is an inverse in the homotopy category \( \text{Ho}(B_{-\infty}) \) of \( B_{-\infty} \)-algebras [Low08, Theorem 3.21]. Consider the projection onto the zero part

\[
(6.4) \quad \pi_0 : C(\text{Com}(i)) \to C(\text{Com}(i))_0 = \prod_{C \in \text{Com}(i)} \text{Com}(i)(C, C).
\]

Since \( \text{Com}(i) \) is uncurved, \( \pi_0 \) induces a morphism of dg algebras [Low08, Proposition 2.7]. Composing (6.3) and (6.4), one obtains the characteristic dg morphism

\[
C(i) \to \prod_{C \in \text{Com}(i)} \text{Com}(i)(C, C).
\]

On cohomology it induces the characteristic morphism

\[
\chi_{i} : H^*C(i) \to \mathcal{Z}^*K(i)
\]

30
for a linear category $i$. Here, $\mathfrak{Z}K(i)$ is the graded center of $K(i)$, i.e., the center

$$\mathfrak{Z}(\text{Com}(i)) = \text{Hom}(1_{\text{Com}(i)}, 1_{\text{Com}(i)})$$

of the graded category $\text{Com}(i)$, where $\text{Hom}$ denotes the graded $R$-module of graded natural transformations [Low08, Remark 4.6]. The characteristic morphism can be interpreted in terms of deformations of categories.

**Theorem 6.4.** ([Low08, Theorem 4.8]) Let $\mathcal{C}$ be an $R$-linear category and $i_{\mathcal{C}}$ its $S$-deformation along $\phi \in \mathbb{Z}^2C(i)^{\oplus}$. Then for each $C \in K(i)$ the element $\chi^\phi_C \in K(i)(C, C)[2]^{\oplus}$ is the obstruction against deforming $C$ to an object of $K(i)_{\phi}$.

Let $\mathcal{C}$ be an $R$-linear abelian category with enough injectives. Assume that $i$ is the subcategory $\text{Inj}(\mathcal{C})$ of injective objects. Taking cohomology of (6.4) restricted to $\text{Com}^+(i)$ and composing with the isomorphism $HH^*_ab(\mathcal{C}) \cong HC^*(\text{Com}^+(i))$ induced by (6.3), one obtains the characteristic morphism

$$\chi_{\mathcal{C}} : HH^*_ab(\mathcal{C}) \to \mathfrak{Z}(D^+(\mathcal{C}))$$

for an abelian category $\mathcal{C}$. Here, we use the isomorphism $HH^*_ab(\mathcal{C}) \cong HC^*(\text{Com}^+(i))$ obtained from [LV06b, Theorem 6.6]. Note that the graded center $\mathfrak{Z}(D^+(\mathcal{C}))$ of $D^+(\mathcal{C}) \cong K^+(i)$ is given by the center $\mathfrak{Z}(\text{Com}^+(i))$.

**Corollary 6.5.** ([Low08, Corollary 4.9]) Let $\mathcal{C}$ be an $R$-linear abelian category with enough injectives and $\mathcal{C}_\phi$ its abelian $S$-deformation along a cocycle $\phi \in HH^2_{ab}(\mathcal{C})^{\oplus}$. Then for each $C \in D^+(\mathcal{C})$ the element $\chi^\phi_C \in \text{Ext}^2_{C}(C, C)^{\oplus}$ is the obstruction against deforming $C$ to an object of $D^+(\mathcal{C}_\phi)$.

6.6. Maximal partial dg deformations of the dg category of bounded below complexes. Consider the map

$$\rho' : H^2C(i)^{\oplus} \to \text{Def}_{\text{Com}^+(i)}^{p=\infty}(S), \ \phi \mapsto (\text{Com}^+(i)_{\text{embr}^\phi})_{\infty}$$

obtained from (6.2) and (6.3). The partial dg deformation $(\text{Com}^+(i)_{\text{embr}^\phi})_{\infty}$ of $\text{Com}^+(i)$ coincides with a dg deformation $(\text{Com}^+(i)_{\text{embr}^\phi})_{\infty}$ of $(\text{Com}^+(i)_{\text{embr}^\phi})_{\infty}$, where the cdg structure $\text{embr}^\phi$ restricted to the $[\text{embr}^\phi]_{\infty}$ part.

**Theorem 6.6.** ([Low08, Theorem 4.15(iii)]) Let $\mathcal{C}$ be an $R$-linear category and $i_{\mathcal{C}}$ its $S$-deformation along $\phi \in \mathbb{Z}^2C(i)^{\oplus}$. Then, for every collection of complexes $\Gamma \in \mathcal{C}$ with $\mathcal{C} \otimes_R C = C$, the full dg subcategory $\text{Com}^+_i(i_{\mathcal{C}}) \subset \text{Com}^+(i_{\mathcal{C}})$ spanned by $\Gamma$ is a maximal partial dg $S$-deformation of $\text{Com}^+(i)$ representing $\rho'(\phi)$.

From the proof, one sees that $\text{Com}^+_i(i_{\mathcal{C}})$ is a dg deformation of $\text{Com}^+(i)_{\text{embr}^\phi}_{\infty}$. According to [Low08, Example 4.13], an object $C \in \text{Com}^+(i)$ belongs to $\text{Com}^+(i)_{\text{embr}^\phi}_{\infty}$ if and only if

$$\chi^\phi_C = 0 \in K(i)(C, C[2])^{\oplus}.$$ 

Since $\chi^\phi_C$ is the obstruction against deforming $C$ to an object of $\text{Com}^+(i)$, we may take a collection of one chosen lift of each unobstructed complex in $\text{Com}^+(i)$ as $\Gamma$. Clearly, any dg subcategory of $\text{Com}^+(i)_{\text{embr}^\phi}_{\infty}$ dg deforms along the restriction of $\text{embr}^\phi$. 

6.7. Hochschild cohomology of the dg category of perfect complexes. We review the definition of the classical Hochschild complex of dg categories. Assume that $\mathcal{A}$ is a small $R$-cofibrant dg category, i.e., all Hom-sets are cofibrant in the dg category $\text{Mod}_{d}(\mathcal{A}) = \text{Com}(\mathcal{Mod}(\mathcal{A}))$ of complexes of $R$-modules. Recall that $N \in \text{Mod}_d(R)$ is cofibrant if its terms are projective. For
an a-bimodule \(M : a^{op} \otimes a \to \text{Mod}_{dg}(R)\), the Hochschild complex \(C(a, M)\) of a with coefficients in \(M\) is the total complex of the double complex whose \(q\)-th columns are given by

\[
\prod_{A_0, \ldots, A_q \in a} \text{Hom}(a(A_{q-1}, A_q) \otimes_R \cdots \otimes_R a(A_0, A_0), M(A_0, A_q))
\]

with horizontal differentials \(d^H\). When \(M = a\) we call \(C(a) = C(a, a)\) the Hochschild complex of a. The Hochschild complex satisfies a “limited functoriality” property. Namely, if \(f : a \hookrightarrow b\) is a fully faithful dg functor to a small \(R\)-cofibrant dg category \(b\), then there is an associated map between Hochschild complexes

\[
j^* : C(b) \to C(a)
\]

given by restriction. As mentioned above, the Hochschild complex is isomorphic to the associated Hochschild object \(C(a) = \Sigma^{-1} C_{sh}(a)\). In particular, the Hochschild complex \(C(a)\) has a \(B_{\infty}\)-algebra structure compatible with the map \(j^*\).

The definition of the Hochschild complex was modified by Shukla and Quillen [Shu61, Qui70] to general dg categories. Now, we drop the assumption on \(a\) to be \(R\)-cofibrant and fix a good \(R\)-cofibrant resolution \(\tilde{a} \to a\), which is a quasi-equivalence with an \(R\)-cofibrant dg category \(\tilde{a}\) inducing surjection of Hom-sets in the graded category [LV06a, Proposition-Definition 2.3.2]. The Shukla complex of \(a\) with coefficient \(M\) is defined as

\[
C_{sh}(a, M) = C(\tilde{a}, M).
\]

According to [LV06a, Section 4.2], which in turn is attributed to [Kel], the assignment

\[
C_{sh} : a \mapsto C_{sh}(a)
\]

defines up to canonical natural isomorphism a contravariant functor on a suitable category of small dg categories with values in \(\text{Ho}(B_{\infty})\). In particular, \(C_{sh}(a)\) does not depend on the choice of good \(R\)-cofibrant resolutions of \(a\) up to canonical isomorphism. The functor \(C_{sh}\) satisfies some extended “limited functoriality”. Let \(j : a \hookrightarrow b\) be a fully faithful dg functor to a small dg category \(b\) with a good \(R\)-cofibrant resolution \(\tilde{b} \to b\). One may restrict the resolution to a good \(R\)-cofibrant resolution \(\tilde{a} \to a\) of \(a\). Then the restriction along the extended fully faithful dg functor \(\tilde{a} \hookrightarrow \tilde{b}\) defines a morphism of Shukla complexes

\[
C_{sh}(b) \to C_{sh}(a)
\]

still denoted by \(j^*\). In the sequel, we write \(C(a)\) for \(C_{sh}(a)\).

Now, we return to our setting. Let \(X_0\) be a Calabi–Yau manifold with \(\text{dim} X_0 > 2\). We denote by \(D_{dg}(\text{Qch}(X_0))\) the dg category of unbounded complexes of quasi-coherent sheaves on \(X_0\). In our setting, the full dg subcategory \(\text{Perf}_{dg}(X_0)\) of compact objects consists of perfect complexes on \(X_0\). The canonical embedding \(\text{Perf}_{dg}(X_0) \hookrightarrow D_{dg}(\text{Qch}(X_0))\) factorizes through the dg category \(D^+_{dg}(\text{Qch}(X_0))\) of bounded below complexes of quasi-coherent sheaves on \(X_0\). Let \((X, i_R)\) be an \(R\)-deformation of \(X_0\) and \(i = \text{Inj}(\text{Qch}(X))\). As explained above, \(X\) is smooth projective over \(R\). The Hochschild cohomology of \(\text{Perf}_{dg}(X)\) can be expressed in terms of \(i\).

**Lemma 6.7.** There is an isomorphism

\[
\text{HC}^*(\text{Com}^+(i)) \to \text{HC}^*(\text{Perf}_{dg}(X)).
\]

**Proof.** Consider the quasi-fully faithful functor

\[
\text{Perf}_{dg}(X) \hookrightarrow D^+_{dg}(\text{Qch}(X)) \to \text{Com}^+(i),
\]

where the first functor is the canonical embedding and the second functor is a quasi-equivalence induced by the canonical equivalence

\[
D^+(\text{Qch}(X)) \to K^+(i)
\]

32
of their homotopy categories [CNS, Theorem A]. The functor (6.7) induces a morphism
\[ C(\text{Com}^+ (i)) \to C(D_{dg}^+ (\text{Qch}(X))) \to C(\text{Perf}_{dg}(X)) \]
of $B_\infty$-algebra, which in turn induces a morphism
\[ H^*C(\text{Com}^+ (i)) \to H^*C(D_{dg}^+ (\text{Qch}(X))) \to H^*C(\text{Perf}_{dg}(X)) \]
of Hochschild cohomology. Since quasi-equivalences preserve Hochschild cohomology, the first arrow in (6.9) is an isomorphism.

It remains to show that the second arrow in (6.9) is an isomorphism. We claim that the restriction
\[ C(D_{dg} (\text{Qch}(X))) \to C(\text{Perf}_{dg}(X)) \]
is an isomorphism in $\text{Ho}(B_\infty)$. Fix a good $R$-cofibrant resolution
\[ D_{dg}(\text{Qch}(X)) \to D_{dg}(\text{Qch}(X)). \]
Via the canonical embedding $\text{Perf}_{dg}(X) \hookrightarrow D_{dg}(\text{Qch}(X))$ it induces a good $R$-cofibrant resolution $\text{Perf}_{dg}(X) \to \text{Perf}_{dg}(X)$ and the fully faithful embedding
\[ \overline{j} : \text{Perf}_{dg}(X) \hookrightarrow D_{dg}(\text{Qch}(X)). \]
We denote by $j$ the induced fully faithful functor on the homotopy categories. One can apply [Por10, Theorem 1.2] to see that the functor
\[ D(\text{Qch}(X)) \to \text{Mod}(\text{Perf}(X)), F \mapsto \text{Hom}_{D(\text{Qch}(X))}(j(-), F) \]
lifts to a localization $D(\text{Qch}(X)) \to D(\text{Mod}_{dg}(\text{Perf}_{dg}(X)))$, where $D(\text{Mod}_{dg}(\text{Perf}_{dg}(X)))$ is the derived category of right dg modules over the dg category $\text{Mod}_{dg}(\text{Perf}_{dg}(X))$ of right dg modules over $\text{Perf}_{dg}(X)$. In particular, the lift is fully faithful. Then the claim follows from [DL13, Proposition 5.1]. Similarly, one can show that the restriction
\[ C(D_{dg}(\text{Qch}(X))) \to C(D_{dg}^+ (\text{Qch}(X))) \]
is an isomorphism in $\text{Ho}(B_\infty)$. Hence the restriction
\[ C(D_{dg}(\text{Qch}(X))) \to C(D_{dg}^+ (\text{Qch}(X))) \]
is an isomorphism in $\text{Ho}(B_\infty)$, which induces the desired isomorphism on Hochschild cohomology.  

6.8. **Morita deformations of the dg category of perfect complexes.** Let $\text{dgCat}_R$ be the category of small $R$-linear dg categories and dg functors. The category $\text{dgCat}_R$ has two model structures, so called the Dwyer–Kan model structure and the Morita model structure, constructed by Tabuada respectively in [Tab05a] and [Tab05b]. On the Dwyer–Kan model structure, weak equivalences are given by quasi-equivalences of dg categories. On the Morita model structure, weak equivalences are given by Morita morphisms. Recall that a dg functor in $\text{dgCat}_R$ is a **Morita morphism** if it induces an derived equivalent. Also recall that for each object $a \in \text{dgCat}_R$ the derived category $D(a)$ of right dg modules over $a$ is defined as the Verdier quotient
\[ [\text{Mod}_{dg}(a)]/[\text{Acycl}(a)] \]
of the homotopy category of the dg category $\text{Mod}_{dg}(a)$ of right dg modules over $a$ by the homotopy category of the full dg subcategory $\text{Acycl}(a)$ of acyclic right dg modules.

We denote by $\text{Ho}_R$ the localization of $\text{dgCat}_R$ by weak equivalences in the Dwyer–Kan model structure and by $\text{Hmo}_R$ the localization of $\text{dgCat}_R$ by weak equivalences in the Morita model structure. Passing to $\text{Ho}_R$, two quasi-equivalent $R$-linear dg categories $a, b$ get identified. Passing to $\text{Hmo}_R$, two Morita equivalent $R$-linear dg categories $a, b$ get identified. Recall that two
\(\mathbb{R}\)-linear dg categories \(a, b\) are said to be **Morita equivalent** if they are connected by a Morita morphism. By [Töe Proposition 7] or [Töe Exercise 28] for \(\mathbb{R}\)-linear triangulated dg categories Morita equivalences coincide with quasi-equivalences. The dg category \(\text{Perf}_{dg}(X)\) is triangulated, as it is pretriangulated and closed under homotopy direct summands.

Let \(a\) be a small \(\mathbb{R}\)-linear dg category. A **Morita \(S\)-deformation** of \(a\) is an \(S\)-linear dg category \(b\) together with a Morita equivalence \(b \otimes_S^{\mathbb{L}} \mathbb{R} \to a\), where \(\otimes_S^{\mathbb{L}}: \text{Hom}_S \to \text{Hom}_\mathbb{R}\) is the derived functor of the base change \(\otimes_S: \text{Cat}_R \to \text{Cat}_R\). Two Morita deformations \(b, b'\) are **isomorphic** if there is a Morita equivalence \(b \to b'\) inducing the identity on \(a\). We denote by \(\text{Def}_a^{\text{mor}}(S)\) the set of isomorphism classes of Morita \(S\)-deformations of \(a\). By [KL09] Proposition 3.3] there is a canonical map

\[
(6.10) \quad \text{Def}_a^{\text{mor}}(S) \to H^2C(a)^{\mathbb{L}L}
\]

defined as follows. Any Morita \(S\)-deformation \(b\) of a small \(\mathbb{R}\)-linear dg category \(a\) can be represented by a \(h\)-flat resolution \(\tilde{b} \to b\), which defines a \(\mathbb{S}\)-deformation of \(\tilde{b} \otimes_S \mathbb{R}\). Let \(\phi_\tau \in Z^2C(\tilde{b} \otimes_S \mathbb{R})^{\mathbb{L}L}\) be a Hochschild cocycle representing \(\tilde{b}\) via the bijection \((6.1)\). The map \((6.10)\) sends \(b\) to the image \(\phi_\tau\) of \(\phi_\tau\) under the isomorphism \(H^2C(\tilde{b} \otimes_S \mathbb{R})^{\mathbb{L}L} \to H^2C(a)^{\mathbb{L}L}\) induced by the Morita equivalence \(\tilde{b} \otimes_S \mathbb{R} \to a\).

**Theorem 6.8.** The composition

\[
(6.11) \quad \text{Def}_a^{\text{mor}}(S) \to \text{Def}_a^{\text{cdg}}(S)
\]

of \((6.10)\) with the inverse of \((6.1)\) is bijective.

**Proof.** To show the injectivity, by [KL09 Proposition 3.7] it suffices to check that \(\text{Perf}_{dg}(X)\) has bounded above cohomology, i.e., the dg module \(\text{Perf}_{dg}(X)(E, F)\) has bounded above cohomology for each \(E, F \in \text{Perf}_{dg}(X)\). Consider the spectral sequence

\[
E_2^{p, q} = H^p(X, \text{Ext}_X^q(E, F)) \Rightarrow \text{Ext}_X^{p+q}(E, F).
\]

Since we have \(\text{Ext}_X^q(E, F) \cong \mathcal{H}_q(\text{Ext}_X^* \otimes_{\mathcal{E}} F)\), the cohomology

\[
\mathcal{H}^{p+q}(\text{Perf}_{dg}(X)(E, F)) \cong \text{Ext}_X^{p+q}(E, F)
\]

vanishes whenever \(p, q\) are sufficiently large.

To show the surjectivity, consider the characteristic dg morphism

\[
(6.12) \quad \mathbf{C}(\text{Perf}_{dg}(X)) \to \mathbf{C}(\text{Tw}(\text{Perf}_{dg}(X))) \to \prod_{E \in \text{Tw}(\text{Perf}_{dg}(X))} \text{Tw}(\text{Perf}_{dg}(X))(E, E)
\]

for the dg category \(\text{Perf}_{dg}(X)\) [Low08 Theorem 3.19], where the first arrow is given by the \(B_{\infty}\)-section of the canonical projection \(\mathbf{C}(\text{Tw}(\text{Perf}_{dg}(X))) \to \mathbf{C}(\text{Perf}_{dg}(X))\). Here,

\[
\text{Tw}(\text{Perf}_{dg}(X)) = \text{Tw}_{\text{init}}(\text{Perf}_{dg}(X))_{\infty}
\]

is the \(\infty\)-part of the dg category of locally nilpotent twisted object over \(\text{Perf}_{dg}(X)\). It is a dg enhancement of the derived category \(\mathbf{D}(\text{Perf}_{dg}(X))\) of right dg modules over \(\text{Perf}_{dg}(X)\). We denote by \(\text{Tw}(\text{Perf}_{dg}(X))^c\) the full dg subcategory of \(\text{Tw}(\text{Perf}_{dg}(X))\) spanned by compact objects. Its homotopy category \(\mathbf{D}(\text{Perf}_{dg}(X))^c\), the full triangulated subcategory of \(\mathbf{D}(\text{Perf}_{dg}(X))\) spanned by compact objects, get identified with \(\text{Perf}_{dg}(X)\) via the Yoneda embedding as \(\text{Perf}_{dg}(X)\) is pretriangulated and closed under homotopy direct summands. When restricted to \(\text{Tw}(\text{Perf}_{dg}(X))^c\), on cohomology \((6.12)\) induces the characteristic morphism

\[
\chi_{\text{Perf}_{dg}(X), E}: H^*\mathbf{C}(\text{Perf}_{dg}(X)) \to \text{Ext}_X^*(E, E).
\]
Consider the quasi-fully faithful functor (6.7). One can apply [KL09] Proposition 2.3 to obtain a commutative diagram

\[
\begin{array}{ccc}
H^\bullet C(\text{Perf}_{dg}(X)) & \rightarrow & \text{Ext}^\bullet_X(E, E) \\
\downarrow & & \downarrow \\
H^\bullet C(D_{dg}^+(\text{Qch}(X))) & \rightarrow & \text{Ext}^\bullet_X(E, E) \\
\downarrow & & \downarrow \\
H^\bullet C(\text{Com}^+(i)) & \rightarrow & \text{Ext}^\bullet_X(E, E),
\end{array}
\]

whose horizontal arrows are the characteristic morphisms and whose vertical arrows are the canonical isomorphisms induced by quasi-equivalences. Applying [KL09] Proposition 2.3 to the quasi-fully faithful functor,

\[
i \rightarrow \text{Com}^+(i)
\]

we obtain another commutative diagram

\[
\begin{array}{ccc}
H^\bullet C(\text{Com}^+(i)) & \rightarrow & \text{Ext}^\bullet_X(E, E) \\
\downarrow & & \downarrow \\
H^\bullet C(i) & \rightarrow & \text{Ext}^\bullet_X(E, E)
\end{array}
\]

whose horizontal arrows are the characteristic morphisms and whose vertical arrows are the canonical isomorphisms, as the morphism $C(\text{Com}^+(i)) \rightarrow C(i)$ induced by the quasi-fully faithful functor (6.14) coincides with the canonical projection and by [Low08] Proposition 3.20 its inverse in $\text{Ho}(B_\infty)$ is the $B_\infty$-section (6.3), which induces an isomorphism on cohomology. Note that $\text{Tw}(i)$ is precisely $\text{Com}(i)$.

By Lemma 6.7 any element of $H^2 C(\text{Perf}_{dg}(X))^{\otimes l}$ can be represented by the image $\text{embr}_l(\phi)$ of $\phi \in Z^2 C(i)^{\otimes l}$ under the $B_\infty$-section (6.3). We use the same symbol to denote the image under the direct sum of (6.6). Since $\text{Perf}_{dg}(X)$ has bounded above cohomology, by [KL09] Proposition 3.12 the map (6.11) is surjective if there exists a full dg subcategory $m(\phi) \subset \text{Perf}_{dg}(X)$ which is Morita equivalent to $\text{Perf}_{dg}(X)$ such that $\chi_{\text{Perf}_{dg}(X)}^{\otimes l}(\text{embr}_l(\phi))_E = 0$ for any $E \in m(\phi)$ and cocycle $\text{embr}_l(\phi) \in Z^2 C(\text{Perf}_{dg}(X))^{\otimes l}$. The composition of vertical arrows in (6.13) and (6.15) maps $\text{embr}_l(\phi)$ to $\phi$ up to coboundary. By Corollary 6.5 the image of $\phi$ under $\chi_{i, E}^{\otimes l}$ is the obstruction against deforming $E$ to an object of $D^+(\text{Qch}(X))$. Here, for a cocycle $\phi \in HH_{ab}^2(\text{Qch}(X))^{\otimes l}$ we denote by $(X_\phi, i_S)$ the $S$-deformation of $(X, i_R)$ along $\phi$ via the bijection (5.3). Let $\text{Perf}_{dg}(X_\phi)$ be an $h$-flat resolution of $\text{Perf}_{dg}(X_\phi)$. Then the dg category $m(\phi) = \text{Perf}_{dg}(X_\phi) \otimes_S \mathbb{R}$ is a full dg subcategory of $\text{Perf}_{dg}(X)$ with a Morita equivalence

\[
m(\phi) \simeq_{mo} \text{Perf}_{dg}(X_\phi) \otimes_S \mathbb{R} \simeq_{mo} \text{Perf}_{dg}(X)
\]

from [BFI10] Theorem 1.2. As each object $E \in m(\phi)$ lifts to an object of $\text{Perf}(X_\phi)$, we obtain $\chi_{\text{Perf}_{dg}(X)}^{\otimes l}(\text{embr}_l(\phi))_E = 0$. Chasing the diagrams (6.13) and (6.15), we obtain $\chi_{\text{Perf}_{dg}(X)}^{\otimes l}(\text{embr}_l(\phi))_E = 0$ which completes the proof.

\[\square\]

\textbf{Remark 6.1.} The canonical equivalence

\[
\text{Perf}_{\infty}(X_\phi) \otimes_{\text{Perf}_{\infty}(S)} \text{Perf}_{\infty}(\mathbb{R}) \simeq_{\infty} \text{Perf}_{\infty}(X)
\]
of corresponding ∞-categories from [BFN10] Theorem 1.2 translates via [Coh Corollary 5.7] into a Morita equivalence

\[
\text{Perf}_{dg}(X_\phi) \otimes_{\text{Perf}_{dg}(S)} \text{Perf}_{dg}(R) \cong_{mo} \text{Perf}_{dg}(X)
\]

of dg categories, where \(-\otimes_-:\text{Hm}_Q \times \text{Hm}_Q \to \text{Hm}_Q\) is the derived pointwise tensor product of dg categories. The left hand side of (6.16) is a triangulated dg category split-generated by objects of the form \(E_\phi \otimes M\) for \(E_\phi \in \text{Perf}_{dg}(X_\phi)\) and \(M \in \text{Perf}_{dg}(R)\), which maps to \(E \otimes_{\mathcal{O}_X} \pi_R^* M \in \text{Perf}_{dg}(X)\) via the Morita equivalence. Here, \(E = E_\phi \otimes_S R\) and \(\pi_R : X \to \text{Spec} R\) is the structure morphism. Hence we obtain a Morita equivalence

\[
\text{Perf}_{dg}(X_\phi) \otimes_S R \cong_{mo} \text{Perf}_{dg}(X_\phi) \otimes_{\text{Perf}_{dg}(S)} \text{Perf}_{dg}(R) \cong_{mo} \text{Perf}_{dg}(X)
\]

used in the above proof.

6.9. Maximal partial dg deformations of the dg category of perfect complexes. As explained above, the category \(\text{Perf}_{dg}(X_\phi)\) defines a Morita \(S\)-deformation of \(\text{Perf}_{dg}(X)\). Let \(\text{Perf}_{dg}(X_\phi)\) be the \(h\)-flat resolution of \(\text{Perf}_{dg}(X_\phi)\) from [CNS Proposition 3.10]. Choose one lift for each object of \(m(\phi) = \text{Perf}_{dg}(X_\phi) \otimes_S R\) to \(\text{Perf}_{dg}(X_\phi)\). Let \(\Gamma\) be the collection of the choices and \(\text{Perf}_{dg,\Gamma}(X_\phi)\) the full dg subcategory spanned by objects belonging to \(\Gamma\). It gives a dg \(S\)-deformation of \(m(\phi)\) representing the Morita \(S\)-deformation \(\text{Perf}_{dg}(X_\phi)\). On the other hand, \(m(\phi)\) admits a dg deformation

\[
m(\phi)_{\text{embr}_s(\phi)} = (m(\phi)[\epsilon]) = m(\phi) \otimes_S S, \text{embr}_s(m) + \text{embr}_s(\phi)\epsilon,
\]

where \(\text{embr}_s(m) = Z^2C(\text{Perf}_{dg}(X))\) and \(\text{embr}_s(\phi) \in Z^2C(\text{Perf}_{dg}(X))^{br}\) are respectively the images under the \(B_{\infty}\)-section (6.3) and its direct sum of the compositions \(m\) in \(s\) and the cocycle \(\phi \in ZC^2(i)^{br}\) corresponding to \(\phi \in HH^2_{st}(\text{Qch}(X))^{br}\) via the isomorphism obtained from [LV06b Theorem 6.6]. Here, as above we use the same symbol to denote the images under the compositions of the bijections induced by the quasi-equivalences with (6.6) and its direct sum respectively. Also, we use the same symbol to denote the images under the morphism \(C(\text{Perf}_{dg}(X)) \to C(m(\phi))\) of \(B_{\infty}\)-algebras induced by the Morita equivalence \(m(\phi) \to \text{Perf}_{dg}(X)\).

**Theorem 6.9.** There is an isomorphism

\[
\text{Perf}_{dg,\Gamma}(X_\phi) \cong m(\phi)_{\text{embr}_s(\phi)}
\]

of dg \(S\)-deformations of \(m(\phi)\). In particular, the Morita \(S\)-deformations \(\text{Perf}_{dg}(X_\phi)\) defines a maximal partial dg \(S\)-deformation of \(\text{Perf}_{dg}(X)\) along \(\text{embr}_s(\phi)\).

**Proof.** Since both dg deformations share their underlying quiver \(m(\phi)[\epsilon]\), it suffices to show the coincidence of their dg structures up to coboundary. The dg structure on \(m(\phi)_{\text{embr}_s(\phi)}\) is \(\text{embr}_s(m) + \text{embr}_s(\phi)\epsilon\). Let \(D^+_{dg}(\text{Qch}(X_\phi))\) be the \(h\)-flat resolution of \(D^+_{dg}(\text{Qch}(X_\phi))\) from [CNS Proposition 3.10]. There is a canonical dg functor

\[
\text{Perf}_{dg}(X_\phi) \hookrightarrow D^+_{dg}(\text{Qch}(X_\phi))
\]

extending the canonical embedding \(\text{Perf}_{dg}(X_\phi) \hookrightarrow D^+_{dg}(\text{Qch}(X_\phi))\). The dg structure on \(\text{Perf}_{dg,\Gamma}(X_\phi)\) is the restriction of that on \(D^+_{dg}(\text{Qch}(X_\phi))\) by [Low08 Proposition 2.6].

We compute the dg structure on \(\text{Perf}_{dg}(X_\phi)\). Consider the quasi-fully faithful functor

\[
\text{Perf}_{dg}(X_\phi) \hookrightarrow D^+_{dg}(\text{Qch}(X_\phi)) \to \text{Com}^+(i_\phi)
\]

where the first functor is the canonical embedding and the second functor is a quasi-equivalence induced by the canonical equivalence

\[
D^+(\text{Qch}(X_\phi)) \to K^+(i_\phi)
\]
Remark (6.10) is represented by embr. This example was informed to the author by Yukinobu Toda.

Theorem 4.15] to obtain

\[ \text{embr}_{\delta + \delta' \epsilon}(m + \phi \epsilon) = (m + \phi \epsilon)[\delta + \delta' \epsilon, \delta + \delta' \epsilon] + (m + \phi \epsilon)\{\delta + \delta' \epsilon\} + (m + \phi \epsilon). \]

We use the same symbol to denote the images under the composition of (6.20) with the morphism \( C(\text{Com}^+(i_0)) \to C(\text{Com}^+(i_0)) \) induced by the h-flat resolution. Note that \( \delta \in C^1(\text{Perf}_{dg}(X)) \) is the differentials of objects in \( \text{Perf}_{dg}(X) \) and \( \delta + \delta' \epsilon = C^1(\text{Perf}_{dg}(X_0)) \) is the differentials of objects in \( \text{Perf}_{dg}(X_0) \) with \( \delta' = (\delta'_1, \ldots, \delta'_l) \in C^l(\text{Perf}_{dg}(X_0))^{\text{gl}} \), as (6.20) is a morphism of \( B_{\infty} \)-algebras induced by the canonical equivalence (6.18). One can apply the same argument as in the proof of [Low08, Theorem 4.15] to obtain

\[ \text{embr}_{\delta + \delta' \epsilon}(m + \phi \epsilon) = \text{embr}_{\delta}(m) + \text{embr}_{\delta}(\phi) \epsilon + d_{\text{embr}_{\delta}(m)}(\delta'_1) \epsilon_1 + \cdots + d_{\text{embr}_{\delta}(m)}(\delta'_l) \epsilon_l \]

on \( \text{Perf}_{dg,l}(X_0) \). Note that up to coboundary the image of

\[ d_{\text{embr}_{\delta}(m)}(\delta'_1) \epsilon_1 + \cdots + d_{\text{embr}_{\delta}(m)}(\delta'_l) \epsilon_l \in C(\text{Com}^+(i_0))^{\text{gl}} \]

under (6.20) coincide with the images of

\[ \delta'_1 \epsilon_1 + \cdots + \delta'_l \epsilon_l \in C(\text{Perf}_{dg}(X_0))^{\text{gl}} \]

under Hochschild differential \( d_{\text{embr}_{\delta}(m)} \) on \( C(\text{Perf}_{dg}(X_0)) \). Hence we obtain an isomorphism

\[ \text{Perf}_{dg,l}(X_0) = (m(\phi)[\epsilon], \text{embr}_{\delta + \delta' \epsilon}(m + \phi \epsilon)) \]

\[ \cong (m(\phi)[\epsilon], \text{embr}_{\delta}(m) + \text{embr}_{\delta}(\phi) \epsilon) \]

\[ = m(\phi) \text{embr}_{\delta}(\phi). \]

Remark 6.2. From the above theorem it follows that the image of \( \text{Perf}_{dg}(X_0) \) under the map (6.10) is represented by \( \text{embr}_{\delta}(\phi) \).

Remark 6.3. In general, \( m(\phi) \) is strictly smaller than \( \text{Perf}_{dg}(X) \). For instance, let \( X_0 \) be a quintic 3-fold of Fermat type. By [AK91, Proposition 2.1] for any general first order deformation \( X_1 \) of \( X_0 \), there is no line in \( X_0 \) which lifts to a closed subvariety of \( X_1 \). Hence deformations of any perfect complex quasi-isomorphic to the pushforward of the structure sheaf of a line in \( X_0 \) is obstructed. This example was informed to the author by Yukinobu Toda.
7. Deformations of higher dimensional Calabi–Yau manifolds revisited

Now, we are ready to prove our first main result. Consider the functor
\[ \text{Def}^{\text{mo}}_{\text{Perf}_{dg}(X_0)} : \text{Art}_k \to \text{Set} \]
which sends each \( A \in \text{Art}_k \) to the set of isomorphism classes of Morita \( A \)-deformations of \( \text{Perf}_{dg}(X_0) \) and each morphism \( B \to A \) in \( \text{Art}_k \) to the map \( \text{Def}^{\text{mo}}_{\text{Perf}_{dg}(X_0)}(B) \to \text{Def}^{\text{mo}}_{\text{Perf}_{dg}(X_0)}(A) \) induced by \( - \otimes_k B \). The deformation theory for \( X_0 \) is equivalent to that for \( \text{Perf}_{dg}(X_0) \) in the following sense.

**Theorem 7.1.** There is a natural isomorphism
\[ (7.1) \quad \zeta : \text{Def}_{X_0} \to \text{Def}^{\text{mo}}_{\text{Perf}_{dg}(X_0)} \]
of deformation functors.

**Proof.** We show that the assignment
\[ (X_A, i_A) \mapsto (\text{Perf}_{dg}(X_A), i_A^*) \]
for each \( A \in \text{Art}_k \) defines a natural transformation. Here, we use the same symbol \( i_A^* \) to denote both the derived pullback functor
\[ i_A^* : \text{Perf}_{dg}(X_A) \to \text{Perf}_{dg}(X_0) \]
and the induced Morita equivalence
\[ \text{Perf}_{dg}(X_A) \otimes_k k \simeq_{\text{mo}} \text{Perf}_{dg}(X_0). \]
The surjection \( A \to k \) factorizes through a sequence
\[ A = A_m \to A_{m-1} \to \cdots \to A_1 \to k \]
of small extensions. Pullback of \( X_A \) yields a sequence
\[ (X_{A_m}, i_{A_m}) \mapsto (X_{A_{m-1}}, i_{A_{m-1}}) \mapsto \cdots \mapsto (X_{A_1}, i_{A_1}) \mapsto X_0 \]
of deformations of \( X_0 \). Let \( \phi_{A_1} \in \text{HH}^2(X_0) = H^1(\mathcal{F}_{X_0}) \) be the cocycle representing \( (X_{A_1}, i_{A_1}). \)

By Theorem 6.9, the Morita deformation \( \text{Perf}_{dg}(X_{A_1}) \) of \( \text{Perf}_{dg}(X) \) corresponds to \( \text{embr}_{\theta_0}(\phi_{A_1}) \) via the bijection \( (6.11) \). Here, \( \text{embr}_{\theta_0}(\phi_{A_1}) \) denotes the image under the composition of
\[ H^2(C(\text{Com}^+(\text{Inj}(Qch(X_0)))) \cong H^2(C(\text{Perf}_{dg}(X_0))) \]
with the induced isomorphism
\[ HH^2(X_0) \cong H^2(C(\text{Inj}(Qch(X_0)))) \cong H^2(C(\text{Com}^+(\text{Inj}(Qch(X_0))))). \]
Induction yields a sequence
\[ [\text{Perf}_{dg}(X_{A_m}), i_{A_m}^*] \mapsto [\text{Perf}_{dg}(X_{A_{m-1}}), i_{A_{m-1}}^*] \mapsto \cdots \mapsto [\text{Perf}_{dg}(X_{A_1}), i_{A_1}^*] \mapsto \text{Perf}_{dg}(X_0) \]
of Morita deformations of \( \text{Perf}_{dg}(X_0) \). In particular, \( (\text{Perf}_{dg}(X_{A_n}), i_{A_n}^*) \) is a Morita deformation of \( \text{Perf}_{dg}(X_0) \) corresponding to the collection \( \{\text{embr}_{\theta_{n-1}}(\phi_{A_n})\}_{n=1}^m \). Here, \( \text{embr}_{\theta_{n-1}}(\phi_{A_n}) \) denotes the image of \( \phi_{A_n} \in H^1(\mathcal{F}_{X_{A_{n-1}}}/A_{n-1})^{\oplus l_{n-1}} \), where \( l_{n-1} \) is the rank of the kernel of square zero extension \( A_n \to A_{n-1} \) as a free \( A_{n-1} \)-module, under the composition of
\[ H^2(C(\text{Com}^+(\text{Inj}(Qch(X_{A_{n-1}})))))^{\oplus l_{n-1}} \cong H^2(C(\text{Perf}_{dg}(X_{A_{n-1}})))^{\oplus l_{n-1}} \]
with the induced isomorphism
\[ HH^2(X_{A_{n-1}}, A_{n-1})^{\oplus l_{n-1}} \cong H^2(C(\text{Inj}(Qch(X_{A_{n-1}}))))^{\oplus l_{n-1}} \cong H^2(C(\text{Com}^+(\text{Inj}(Qch(X_{A_{n-1}}))))^{\oplus l_{n-1}} \]
by the \( B_{\infty} \)-section of the canonical projection \( C(\text{Inj}(Qch(X_{A_{n-1}}))) \to C(\text{Com}^+(\text{Inj}(Qch(X_{A_{n-1}}))))). \)
It follows that the assignment defines a map

\[ \zeta_A : \text{Def}_{X_0}(A) \to \text{Def}_{\text{Perf}_{dg}(X_0)}^{mo}(A), \quad (X_A, i_A) \mapsto [\text{Perf}_{dg}(X_A), i_A^*]. \]

For each morphism \( f : B \to A \) in \( \text{Art}_k \) the diagram

\[
\begin{array}{ccc}
\text{Def}_{X_0}(B) & \longrightarrow & \text{Def}_{\text{Perf}_{dg}(X_0)}^{mo}(B) \\
\text{Def}_{X_0}(f) \downarrow & & \downarrow \text{Def}_{\text{Perf}_{dg}(X_0)}^{mo}(f) \\
\text{Def}_{X_0}(A) & \longrightarrow & \text{Def}_{\text{Perf}_{dg}(X_0)}^{mo}(A)
\end{array}
\]

(7.2)

commutes. To see this, we may assume that \( f \) is a square zero extension \( B = S \) of \( A = R \). Then for any \((X_S, i_S) \in \text{Def}_{X_0}(S)\) with \( X_S \times_S R \cong X_R \) we have \( X_S \cong (X_R)_B \) for some cocycle \( \phi \in H^1(\mathcal{F}_{X_R}/R)^{\text{fd}} \). We already know that \( (\text{Perf}_{dg}(X_S), i_S^*) \) is the Morita deformation of \( \text{Perf}_{dg}(X_R) \). Since \((X_S, i_S)\) maps to \((X_R, i_R)\), the derived pullback functor \( i_S^* \) factorizes through \( i_R^* \). Thus the assignments \( \{\zeta_A\}_{A \in \text{Art}_k} \) defines a natural transformation \( \zeta : \text{Def}_{X_0} \to \text{Def}_{\text{Perf}_{dg}(X_0)}^{mo} \).

It remains to show that \( \zeta_A \) is bijective for each \( A \in \text{Art}_k \). We will proceed by induction. Now, assume that \( \zeta_A \) are bijective for all \( 1 \leq i \leq n \). In order to show the surjectivity of \( \zeta_{A_{n+1}} \), take any element \([a_{A_{n+1}}, u_{A_{n+1}}^*] \in \text{Def}_{\text{Perf}_{dg}(X_0)}^{mo}(A_{n+1})\). By the assumption of induction, the reduction \([a_A, u_A^*] \in \text{Def}_{\text{Perf}_{dg}(X_0)}^{mo}(A_n)\) is equal to \([\text{Perf}_{dg}(Y_{A_n}), j_{A_n}^*]\) for some \((Y_{A_n}, j_{A_n}) \in \text{Def}_{X_0}(A_n)\). Combining Theorem 6.8 with Theorem 6.9 one sees that the Morita \( A_{n+1}\)-deformation \( (a_{A_{n+1}}, u_{A_{n+1}}^*) \) of \( \text{Perf}_{dg}(Y_{A_n}) \) is represented by \( \text{embr}_{A(n)}(\phi_{A_{n+1}}) \) for some cocycle \( \phi_{A_{n+1}} \in H^1(\mathcal{F}_{X_{A_n}/A_n})^{\text{fd}} \). Then we have

\[ [a_{A_{n+1}}, u_{A_{n+1}}^*] = [\text{Perf}_{dg}(Y_{A_n}, \phi_{A_{n+1}}), j_{A_n}^*]. \]

In order to show the injectivity, suppose that we have

\[ [\text{Perf}_{dg}(X_{A_{n+1}}), i_{A_{n+1}}^*] = [\text{Perf}_{dg}(Y_{A_n}), j_{A_n}^*], \]

i.e., there is a Morita equivalence \( \text{Perf}_{dg}(X_{A_{n+1}}) \cong_{mo} \text{Perf}_{dg}(Y_{A_n}) \) reducing to the identity on \( \text{Perf}_{dg}(X_0) \). Combining the above argument with the commutative diagram (7.2), we have

\[ [\text{Perf}_{dg}(X_{A_{n+1}}), i_{A_{n+1}}^*] = [\text{Perf}_{dg}(X_{A_n}, \phi_a), i_{A_n}^*], \quad [\text{Perf}_{dg}(Y_{A_n}), j_{A_n}^*] = [\text{Perf}_{dg}(Y_{A_n}, \phi_a), j_{A_n}^*] \]

for some elements

\[ (X_{A_n}, i_{A_n}), (Y_{A_n}, j_{A_n}) \in \text{Def}_{X_0}(A_n) \]

and cocycles

\[ \phi_a \in H^1(\mathcal{F}_{X_{A_n}/A_n})^{\text{fd}}, \quad \psi_{A_n} \in H^1(\mathcal{F}_{X_{A_n}/A_n})^{\text{fd}}. \]

Applying \(- \otimes_{A_{n+1}} A_{n}\), we obtain a Morita equivalence \( \text{Perf}_{dg}(X_{A_n}) \cong_{mo} \text{Perf}_{dg}(Y_{A_n}) \) reducing to the identity on \( \text{Perf}_{dg}(X_0) \). By the assumption of induction there is an isomorphism \( X_{A_n} \cong Y_{A_n} \) reducing to the identity on \( X_0 \). Then (7.3) implies \( [\text{embr}_{A_{n+1}}(\phi_{A_{n+1}})] = [\text{embr}_{A_{n+1}}(\psi_{A_{n+1}})] \), which in turn implies \([\phi_{A_n}] = [\psi_{A_n}]\). Thus we obtain an isomorphism \( X_{A_{n+1}} \cong Y_{A_{n+1}} \) reducing to the identity on \( X_0 \). \( \square \)

**Remark 7.1.** Consider the functor

\[ \overline{\text{Def}}_{\text{Perf}_{dg}(X_0)}^{mo} : \text{Art}_k \to \text{Set} \]

which sends each \( A \in \text{Art}_k \) to the set of isomorphism classes of Morita \( A\)-deformations of \( \text{Perf}_{dg}(X_0) \) and each morphism \( B \to A \) in \( \text{Art}_k \) to the map \( \text{Def}_{\text{Perf}_{dg}(X_0)}^{mo}(B) \to \text{Def}_{\text{Perf}_{dg}(X_0)}^{mo}(A) \).
induced by the derived pointwise tensor product with $\text{Perf}_{dg}(A)$ over $\text{Perf}_{dg}(B)$. Based on Remark 6.1, one can rewrite the proof of Theorem 7.1 in terms of $\widetilde{\text{Def}}^{\text{mo}}_{\text{Perf}_{dg}(X_0)}$ to obtain a natural isomorphism

$$\tilde{\xi} : \text{Def}_{X_0} \to \widetilde{\text{Def}}^{\text{mo}}_{\text{Perf}_{dg}(X_0)}$$

of deformation functors. In the sequel, we will identify the deformation functors

$$\text{Def}^{\text{mo}}_{\text{Perf}_{dg}(X_0)}, \widetilde{\text{Def}}^{\text{mo}}_{\text{Perf}_{dg}(X_0)} : \text{Art}_k \to \text{Set}$$

without further comments.

**Remark 7.2.** Theorem 7.1 tells us that infinitesimal deformations of $\text{Perf}_{dg}(X_0)$ is controlled by the Kodaira–Spencer differential graded Lie algebra $\KS_{X_0}$ of $X_0$. Consider the functor $\text{Def}_{\KS_{X_0}} : \text{Art}_k \to \text{Set}$ defined as

$$\text{Def}_{\KS_{X_0}}(A) = \frac{\text{MC}_{\KS_{X_0}}(A)}{\text{gauge equivalence}}$$

for each $A \in \text{Art}_k$, where

$$\text{MC}_{\KS_{X_0}}(A) = \left\{ x \in K_{X_0}^i \otimes_k m_A \mid dx + \frac{1}{2}[x, x] = 0 \right\}.$$

Recall that a differential graded Lie algebra $L$ and a commutative $k$-algebra $m$ there exists a natural structure of a differential graded Lie algebra on the tensor product $L \otimes_k m$ given by

$$d(x \otimes_k r) = dx \otimes_k r, \quad [x \otimes_k r, y \otimes_k s] = [x, y] \otimes_k rs, \quad x, y \in L, \quad r, s \in m.$$

For every surjection $A \to k[t]/t^2$ in Art$\mathfrak{k}$ the set $\text{Def}_{\KS_{X_0}}(A)$ consists of solutions of the extended Maurer–Cartan equation to $m_A$. Giving higher order deformations of $X_0$ is equivalent to giving solutions of the extended Maurer–Cartan equation. Indeed, we have $\text{Def}_{\KS_{X_0}} \simeq \text{Def}_{X_0}$ by [Man09, Example 2.3].

**Corollary 7.2.** The functor $\text{Def}^{\text{mo}}_{\text{Perf}_{dg}(X_0)}$ is pro-represented by $R$.

**Proof.** This follows immediately as $\text{Def}_{X_0}$ is pro-represented by $R$. \hfill \Box

**Corollary 7.3.** The functor $\text{Def}^{\text{mo}}_{\text{Perf}_{dg}(X_0)}$ has an effective universal formal family.

**Proof.** Let $(R, \tilde{\xi})$ be a universal formal family for $\text{Def}^{\text{mo}}_{\text{Perf}_{dg}(X_0)}$, where $\tilde{\xi} = \{\tilde{\xi}_n\}_n$ belongs to the limit

$$\widetilde{\text{Def}}^{\text{mo}}_{\text{Perf}_{dg}(X_0)}(R) = \lim_{\leftarrow} \text{Def}^{\text{mo}}_{\text{Perf}_{dg}(X_0)}(R/m^n_R)$$

of the inverse system

$$\cdots \to \text{Def}^{\text{mo}}_{\text{Perf}_{dg}(X_0)}(R/m^n_R) \to \text{Def}^{\text{mo}}_{\text{Perf}_{dg}(X_0)}(R/m^{n+1}_R) \to \text{Def}^{\text{mo}}_{\text{Perf}_{dg}(X_0)}(R/m^{n+2}_R) \to \cdots$$

induced by the natural quotient maps $R/m^{n+1}_R \to R/m^n_R$. Recall that for the universal formal family $(R, \xi)$ there is a noetherian formal scheme $\mathcal{X}$ over $R$ such that $X_n \cong \mathcal{X} \times_R R/m^{n+1}_R$ for each $n$, where $(X_n, i_n)$ are $R_n$-deformations of $X_0$ defining $\xi_n$. By [GD61, Theorem III5.4.5] there exists a scheme $X_R$ flat projective over $R$ whose formal completion along the closed fiber $X_0$ is isomorphic to $\mathcal{X}$. From the proof of Theorem 7.1 it follows that $(\text{Perf}_{dg}(X_0), i_n)$ defines $\tilde{\xi}_n$. Then by [BFN10, Theorem 1.2] the $R$-linear dg category $\text{Perf}_{dg}(X_R)$ yields the compatible system $\{\tilde{\xi}_n\}_n$ via reduction along the natural quotient maps $R/m^{n+1}_R \to R/m^n_R$, which means $\tilde{\xi}$ is effective. \hfill \Box
Remark 7.3. Recall that the Dwyer–Kan model structure on dgCat_k has a natural simplicial enrichment [Toc, Section 5]. We denote by dgCat^∞_k the underlying ∞-category. There is a notion of limits in ∞-categories that behaves similarly to the classical one [Lur09, Chapter 4]. As dgCat^∞_k is the underlying ∞-category of a simplicial model category, it admits limits [Lur09, Corollary 4.2.4.8]. Hence we obtain a limit

\[
\widehat{\text{Perf}}_{dg}(X_R) = \lim_{\leftarrow} \text{Perf}_{dg}(X_n)
\]

of the inverse system

\[
\cdots \to \text{Perf}_{dg}(X_{n+2}) \to \text{Perf}_{dg}(X_{n+1}) \to \text{Perf}_{dg}(X_n) \to \cdots
\]

of small k-linear dg categories induced by the natural quotient maps \(R/m^{n+1}_R \to R/m^n_R\).

We claim that the limit is quasi-equivalent to \(\text{Perf}_{dg}(X_s)\). By [GD61, Corollary 5.1.3] the canonical map

\[
\text{Hom}_{X_R}(E, F) \to \text{Hom}_X(\hat{E}, \hat{F})
\]

defined by taking the formal completion of each morphism along the closed fiber is an isomorphism for all coherent sheaves \(E, F\) on \(X_R\). In particular, we may write

\[
\text{Hom}_{X_R}(\hat{E}, \hat{F}) = \text{Hom}_X(\hat{E}, \hat{F}).
\]

Since \(X_R\) is projective over a complete local noetherian ring \(R\), by [GD61, Corollary III5.1.6] the functor

\[
\text{coh}(X_R) \to \text{coh}(X),
\]

which sends each coherent sheaf \(F\) on \(X_R\) to its formal completion \(\hat{F}\) along the closed fiber is an equivalence of abelian categories. We obtain the induced derived equivalence

\[
\text{Perf}(X_R) \simeq D^b(X_R) \simeq D^b(X) \simeq \text{Perf}(X).
\]

Hence for \(E, F \in \text{Perf}_{dg}(X_R)\) with formal completions \(\hat{E}, \hat{F} \in \text{Perf}_{dg}(X)\) we may write

\[
\text{Ext}^*_X(\hat{E}, \hat{F}) = \text{Ext}^*_X(\hat{E}, \hat{F}).
\]

Now, one sees that the objects and morphisms in \(\text{Perf}(X)\) satisfy universality with respect to the induced inverse system on homotopy categories. Thus the dg functor

\[
\text{Perf}_{dg}(X) \to \text{Perf}_{dg}(X_R)
\]

uniquely determined by universality of the limit is a quasi-equivalence. Namely, the formal completion of \(\text{Perf}_{dg}(X_R)\) is quasi-equivalent to \(\text{Perf}_{dg}(X)\).

Corollary 7.4. Any effective universal formal family for \(\text{Def}_{\text{Perf}}^{\text{mo}}(X_0)\) is algebraizable. In particular, an algebraization is given by \(\text{Perf}_{dg}(X_S)\) where \(X_S\) is a versal deformation of \(X_0\).

Proof. Consider the triple \((\text{Spec } S, s, \text{Perf}_{dg}(X_S))\). Since the reduction of \(X_S\) along the natural quotient maps \(S/m_S^{n+1} \to S/m_S^n\) yields a compatible system isomorphic to \(\xi\), the reduction of \(\text{Perf}_{dg}(X_S)\) yields a compatible system isomorphic to \(\hat{\xi}\). Thus \((\text{Spec } S, s, \text{Perf}_{dg}(X_S))\) gives a versal Morita deformation of \(\text{Perf}_{dg}(X_0)\).

Proposition 7.5. There is a quasi-equivalence

\[
\text{Perf}_{dg}(X_S)/\text{Perf}_{dg}(X_S)_0 \simeq_{qeq} \text{Perf}_{dg}(X_{Q(S)})
\]

where \(Q(S)\) is the quotient field of \(S\) and \(X_{Q(S)}\) is the generic fiber of \(X_S\).
Proof. By [Dri04, Theorem 3.4] and [Morb Theorem 1.1] we have an equivalence

\[ \text{Perf}_{dg}(X_\mathcal{S}) / \text{Perf}_{dg}(X_\mathcal{S})_0 \simeq \text{Perf}(X_\mathcal{S}) / \text{Perf}(X_\mathcal{S})_0 \simeq \text{Perf}(X_\mathcal{Q}(S)) \]

of idempotent complete triangulated categories, where the middle category is the Verdier quotient by the full triangulated subcategory Perf(X_\mathcal{S})_0 \subset Perf(X_\mathcal{S}) of perfect complexes with S-torsion cohomology. Then the claim follows from [CNS, Theorem B]. \qed

Remark 7.4. From the proof one sees that the dg categorical generic fiber is a natural dg enhancement of the categorical generic fiber introduced in [Morb], which is in turn based on the categorical general fiber by Huybrechts–Macrì–Stellari [HMS11].

8. Independence from geometric realizations

Due to Corollary 7.4, a versal Morita deformation of Perf_{dg}(X_0) is given by Perf_{dg}(X_\mathcal{S}) where X_\mathcal{S} is a versal deformation of X_0. Suppose that there is another Calabi–Yau manifold X'_\mathcal{S} derived-equivalent to X_0. Since by [CNS Theorem B] dg enhancements of

\[ \text{Perf}(X_0) \simeq D^b(X_0) \simeq D^b(X'_0) \simeq \text{Perf}(X'_0) \]

are unique, we obtain a quasi-equivalence

\[ \text{Perf}_{dg}(X_0) \rightarrow \text{Perf}_{dg}(X'_0). \]

Hence Perf_{dg}(X_\mathcal{S}) gives also a versal Morita deformation of Perf_{dg}(X'_\mathcal{S}).

By Lemma 3.2 we may assume X_\mathcal{S} to be smooth projective over S. Then one finds a smooth projective versal deformation X'_\mathcal{S} over the same base. The construction requires the deformation theory of Fourier–Mukai kernels, which we briefly review below. It passes through effectivizations, i.e., there are effectivizations X_R, X'_R of X_0, X'_0 over the same regular affine scheme Spec R. Applying [Mora, Corollary 4.2] and [CNS, Theorem B], we obtain a quasi-equivalence

\[ \text{Perf}_{dg}(X_R) \simeq qeq \text{Perf}_{dg}(X'_R). \]

Unwinding the construction of versal deformations recalled in Section 3.3 one sees that, up to equivalence of deformations, the ambiguity of X_\mathcal{S} essentially stems from the choice of indices \( i \in I \) of the filtered inductive system \( \{ R_i \}_{i \in I} \), where \( R_i \) are finitely generated T-subalgebras of \( R \) whose colimit is \( R \). The versal deformations X_\mathcal{S}, X'_\mathcal{S} over the same base are obtained by choosing the same sufficiently large index. From this observation combined with Theorem 7.1 and the quasi-equivalence (8.1), it is natural to expect that the versal Morita deformations Perf_{dg}(X_\mathcal{S}), Perf_{dg}(X'_\mathcal{S}) become quasi-equivalent close to effectivizations. In this section, we prove our second main result which yields the quasi-equivalence as a corollary.

8.1. Deformations of Fourier–Mukai kernels. Suppose that the derived equivalence of X_0, X'_0 is given by a Fourier–Mukai kernel \( \mathcal{P}_0 \in D^b(X_0 \times X'_0) \). In order to define a relative integral functor from \( D^b(X_\mathcal{S}) \) to \( D^b(X'_\mathcal{S}) \), we deform \( \mathcal{P}_0 \) to a perfect complex \( \mathcal{P}_\mathcal{S} \) on \( X_\mathcal{S} \times_{\mathcal{S}} X'_\mathcal{S} \). Here, for a deformation \( [X_p, \nu_p] \in \text{Def}_{\mathcal{S}}((\mathcal{P}, m_p)) \) of a \( \text{k} \)-scheme \( X \), by a deformation of \( E \in \text{Perf}(X) \) over \( (\mathcal{P}, m_p) \) we mean a pair \( (E_p, \nu_p) \), where \( E_p \in \text{Perf}(X_p) \) and \( \nu_p : E_p \otimes_{m_p} \text{k} \rightarrow E \) is an isomorphism. Two deformations \( (E_p, \nu_p) \) and \( (F_p, \nu_p) \) are equivalent if there is an isomorphism \( E_p \rightarrow F_p \) reducing to an isomorphism of \( E \).

The R_n-deformations \( X_n, X'_n \) of \( X_0, X'_0 \) and their fiber product \( X_n \times_{R_n} X'_n \) form the diagram

\[
\begin{array}{ccc}
X_n & \xrightarrow{p_n} & X'_n \\
\downarrow q_n & & \downarrow \\
X_n & & X'_n \\
\end{array}
\]
with the natural projections $q_n$ and $p_n$. For any perfect complex $\mathcal{P}_n$ on $X_n \times_{R_n} X'_n$, the relative integral functor

$$\Phi_{\mathcal{P}_n}(-) = R\mathcal{P}_n \left( \mathcal{P}_n \otimes^L q_n^*(-) \right)$$

sends each object of $D^b(X_n)$ to $D^b(X'_n)$. Due to the Grothendieck–Verdier duality the functor $\Phi_{\mathcal{P}_n}$ admits the right adjoint $\Phi_{\mathcal{P}_n}^*$ with kernel $(\mathcal{P}_n)_R = \mathcal{P}_n \otimes p_n^* \omega_{\pi_n^*}[\dim X_0]$, where $\omega_{\pi_n^*}$ is the determinant of the relative cotangent sheaf associated with the natural projection $\pi_n^*: X'_n \to \Spec R_n$.

**Lemma 8.1.** ([Mora, Lemma 3.1, 3.2]) Assume that $\Phi_{\mathcal{P}_n}$ is an equivalence. Then for any thickening $X_n \hookrightarrow X_{n+1}$ there exist a thickening $X'_n \hookrightarrow X'_{n+1}$ and a perfect complex $\mathcal{P}_{n+1}$ on $X_{n+1} \times_{R_{n+1}} X'_{n+1}$ with an isomorphism $\mathcal{P}_{n+1} \otimes_{R_{n+1}}^L R_n \cong \mathcal{P}_n$ such that the integral functor $\Phi_{\mathcal{P}_{n+1}}: D^b(X_{n+1}) \to D^b(X'_n)$ is an equivalence.

Iterative application of Lemma 8.1 allows us to deform the Fourier–Mukai kernel $\mathcal{P}_0 \in D^b(X_0 \times X'_0)$ to some Fourier–Mukai kernel $\mathcal{P}_n \in \Perf(X_n \times_{R_n} X'_n)$ for arbitrary order $n$. We obtain a system of deformations $\mathcal{P}_n \in \Perf(X_n \times_{R_n} X'_n)$ of $\mathcal{P}_0$ with compatible isomorphisms $\mathcal{P}_{n+1} \otimes_{R_{n+1}}^L R_n \cong \mathcal{P}_n$. According to [Lie06, Proposition 3.6.1] there exists an effectivization, i.e., a perfect complex $\mathcal{P}_R$ on $X_R \times X'_R$ with compatible isomorphisms $\mathcal{P}_R \otimes_{R}^L R \cong \mathcal{P}_n$. Recall that to algebrize $\mathcal{Z}$ we used a filtered inductive system $\{R_i\}_{i \in \mathbb{Z}}$ of finitely generated $T$-subalgebras of $R$ whose colimit is $R$. Taking an index $i$ sufficiently large, one finds smooth projective $R_i$-deformations $X_{R_i}, X'_{R_i}$ of $X_0, X'_0$ whose pullback along the canonical homomorphism $R_i \hookrightarrow R$ are $X_{R_i}, X'_{R_i}$. Since we have $X_R \times \times X'_R \oplus \left( X_{R_i} \times_{R_i} X'_{R_i} \right)$, by [Lie06, Proposition 2.2.1] there exists a perfect complex $\mathcal{P}_S$ on $X_S \times X'_S$ with an isomorphism $\mathcal{P}_S \otimes_{R}^L R \cong \mathcal{P}_R$. Finally, the derived pullback $\mathcal{P}_S \in \Perf(X_S \times X'_S)$ along $R_i \to S$ yields a deformation of $\mathcal{P}_R$.

**Lemma 8.2.** ([Mora, Proposition 3.3]) Let $\mathcal{P}_0$ be a Fourier–Mukai kernel defining the derived equivalence of Calabi–Yau manifolds $X_0, X'_0$ of dimension more than two. Then there exists a perfect complex $\mathcal{P}_S$ on the fiber product $X_S \times X'_S$ of smooth projective versal deformations with an isomorphism $\mathcal{P}_S \otimes_{R}^L k \cong \mathcal{P}_0$.

### 8.2. Inherited equivalences.

The schemes $X_{R_i}, X'_{R_i}$ and their fiber product $X_R \times_{R_i} X'_R$, together with the pullbacks along $T$-algebra homomorphisms $R_i \to R_j \to R$ for $i \leq j$ form the commutative diagram

\[
\begin{array}{ccc}
X_R \times_{R_i} X'_R & \xrightarrow{q_i} & X_R \\
\downarrow f_i & & \downarrow f_i \\
X_{R_j} \times_{R_i} X'_{R_j} & \xrightarrow{q_j} & X_{R_j} \\
\downarrow f_j & & \downarrow f_j \\
X_{R_i} \times_{R_i} X'_{R_i} & \xrightarrow{p_i} & X'_{R_i} \\
\downarrow f_i & & \downarrow f_i \\
X_{R_i} \times_{R_i} X'_{R_i} & \xrightarrow{p_i} & X'_{R_i} \\
\downarrow q_i & & \downarrow q_i \\
X_{R_i} \times_{R_i} X'_{R_i} & \xrightarrow{p_i} & X'_{R_i} \\
\end{array}
\]

where $q_i, p_i$ are smooth projective of relative dimension $\dim X_0$. Given a collection $\{\mathcal{P}_i\}_{i \in \mathbb{Z}}$ with $\mathcal{P}_i \in \Perf(X_R \times_{R_i} X'_R)$ satisfying $\mathcal{P}_j \cong \mathcal{P}_{R_j} \oplus_{R_j} R_j$ and $\mathcal{P}_R \cong \mathcal{P}_{R_i} \oplus_{R_i} R$ for all $i \leq j$, consider the relative integral functors

$$\Phi_{\mathcal{P}_i} = R\mathcal{P}_i \left( \mathcal{P}_i \otimes^L q_i^*(-) \right): D^b(X_R) \to D^b(X'_R).$$
Since $p_i$ is projective and $\mathcal{P}_i$ is of finite homological dimension, i.e., $\mathcal{P}_i \otimes^L q_i^* F_R$ are bounded for each object $F_R \in D^b(X_R)$, one can apply [LST13, Lemma 1.8] to see that $\Phi_{P_i}$ send perfect complexes to perfect complexes. We use the same symbol to denote the restricted functor.

**Theorem 8.3.** There exists an index $j \in I$ such that for all $k \geq j$ the functors

$$\Phi_{P_j} : \text{Perf}(X_{R_k}) \rightarrow \text{Perf}(X'_{R_k})$$

are equivalences of triangulated categories of perfect complexes. In particular, the dg categories $\text{Perf}_{dg}(X_{R_k}), \text{Perf}_{dg}(X'_{R_k})$ of perfect complexes are quasi-equivalent.

**Proof.** Under the assumption one always finds deformations $[X_{R_j}, t_{R_j}], [X'_{R_j}, t'_{R_j}]$ smooth projective over $(R_j, \mathfrak{m}_{R_j})$ for sufficiently large index $j \in I$. Moreover, the pullbacks along $R_j \rightarrow R$ and $R_j \rightarrow S$ yield respectively effectivizations $X_{R_k}, X'_{R_k}$ of universal formal families $\xi, \xi'$ and versal deformations $(\text{Spec } S, s, X_S), (\text{Spec } S, s, X'_S)$ of $X_0, X'_0$. Recall that $(\text{Spec } S, s)$ is an étale neighborhood of $t$ in $\text{Spec } T$ with $t$ corresponding to the maximal ideal $(t_1, \ldots, t_d) \subset T$, and the formal completions of $X_S, X'_S$ along the closed fibers over $s$ are isomorphic to $\hat{X}_R, \hat{X}'_R$. In summary, we have the pullback diagrams

$$
\begin{align*}
X_0 \overset{\eta}{\longrightarrow} X_S \overset{f_{S,k}}{\longrightarrow} X_{R,j} \overset{f_j}{\longrightarrow} X_R \\
\text{Spec } \mathbb{k} \overset{\pi}{\longleftarrow} \text{Spec } S \overset{\pi_{R,j}}{\longleftarrow} \text{Spec } R_j \overset{\pi_s}{\longleftarrow} \text{Spec } R
\end{align*}
$$

\begin{align*}
X'_0 \overset{\eta'}{\longrightarrow} X'_S \overset{f'_{S,k}}{\longrightarrow} X'_{R,j} \overset{f'_j}{\longrightarrow} X'_R.
\end{align*}

Let $\mathcal{P}_0 \in D^b(X_0 \times_k X'_0)$ be a Fourier–Mukai kernel defining the derived equivalence. As explained above, one can deform $\mathcal{P}_0$ to a perfect complex $\mathcal{P}_j \in D^b(X_{R_j} \times_{R_j} X'_{R_j})$. Due to the Grothendieck–Verdier duality the functor $\Phi_{P_j}$ admits a left adjoint $\Phi_{P_j}^L = \Phi_{P_jL}$ with kernel $(\mathcal{P}_j)_L = \mathcal{P}_j^\vee \otimes^L \omega_{P,j}^\vee [\dim X_0]$. By [BV03, Corollary 3.1.2] the category $\text{Perf}(X_{R,j})$ is generated by some single object $E_{R,j}$. Namely, each object $F_{R,j} \in \text{Perf}(X_{R,j})$ can be obtained from $E_{R,j}$ by taking isomorphisms, finite direct sums, direct summands, shifts, and bounded number of cones. The counit morphism $\eta_j : \Phi_{P,j}^L \circ \Phi_{P,j}(E_{R,j}) \rightarrow \text{id}_{\text{Perf}(X_{R,j})}$ gives the distinguished triangle

$$\Phi_{P,j}^L \circ \Phi_{P,j}(E_{R,j}) \overset{\eta_j(E_{R,j})}{\longrightarrow} E_{R,j} \rightarrow C(E_{R,j}) := \text{Cone}(\eta_j(E_{R,j})).$$

For sufficiently large $k \geq j$ we will show that $\eta_k(E_{R,k})$ is an isomorphism and then $\Phi_{P_k}$ is fully faithful. Similarly, one can show that $\Phi_{P_j}^L$ is also fully faithful. Thus $\Phi_{P_k}$ is an equivalence, as it is a fully faithful functor admitting a fully faithful left adjoint.

Pullback along $R_j \subset R_k$ yields

$$\Phi_{P,j}^L \circ \Phi_{P,j}(E_{R,j}) \overset{f_{j,k}^* \eta_j(E_{R,j})}{\longrightarrow} E_{R,j} \rightarrow f_{j,k}^* C(E_{R,j})$$

with $E_k = f_{j,k}^* E_j$ and $\mathcal{P}_k = (f_{j,k} \times f_{j,k})^* \mathcal{P}_j$. Further pullback along $R_k \subset R$ yields

$$\Phi_{P_k}^L \circ \Phi_{P,k}(E_R) \overset{f_{j,k}^* \eta_j(E_{R,j})}{\longrightarrow} E_R \rightarrow f_{j,k}^* C(E_{R,j}),$$

where $f_j : X_R \rightarrow X_{R,j}$ satisfies $f_{j,k} \circ f_k = f_j$. Restriction to the closed fiber $X_0$ yields

$$\Phi_{P_0}^L \circ \Phi_{P_0}(E_{R|X_0}) \overset{\eta_j(E_{R,j}|X_0)}{\longrightarrow} E_{R|X_0} \rightarrow (f_j^* C(E_{R,j}))(X_0).$$

44
Note that since $f_j^{-1}(X_0) = X_0$ and the restriction of the counit morphism is the counit morphism, we have $(f_j^* \eta(E_R))|_{X_0} = \eta(E_R|_{X_0})$. Each term in the above distinguished triangle is perfect so that we may consider the restriction to the closed fiber. Since $\Phi_{R_0}$ is an equivalence, $\eta_j(E_R|_{X_0})$ is an isomorphism and we obtain a quasi-isomorphism $f_j^* C(E_R)|_{X_0} \cong 0$. Then the support of $f_j^* C(E_R)$ is a proper closed subscheme of $X_R$ which does not contain any closed point of $X_R$. Thus the quasi-isomorphism extends to $f_j^* C(E_R) \cong 0$. From [Lie06, Proposition 2.2.1] it follows $f_j^* C(E_R) \cong 0$ when $k \in I$ is sufficiently large.

Take any closed point $u \in \text{Spec} R_j$ whose inverse image by $g_{jk} : \text{Spec} R_k \to \text{Spec} R_j$ is not empty. We have the pullback diagrams

$$
\begin{array}{ccc}
X_R & \xrightarrow{\eta_j} & f_{jk}^{-1}(X_u) \\ \downarrow \eta_j & & \downarrow f_{jk} \\
\text{Spec} R_j & \xleftarrow{g_{jk}^{-1}(u)} & \text{Spec} k
\end{array}
$$

Note that $f_{u,jk}$ is surjective by construction and flat as $g_{u,jk}$ is flat. The restriction of (8.2) to $f_{jk}^{-1}(X_u)$ yields

$$
\Phi_{P_{u,jk}} \circ \Phi_{P_{u,jk}}(E_R|_{f_{jk}^{-1}(X_u)}) \xrightarrow{f_{jk}^{-1}(\eta_j)|_{X_u}} E_R|_{f_{jk}^{-1}(X_u)} \xrightarrow{f_{u,jk}^*} f_{u,jk}^* C(E_R|_{X_0}) \cong 0,
$$

where $P_{u,jk} = P_k|_{f_{jk}^{-1}(X_u)}$. It follows $C(E_R|_{X_0}) \cong 0$ and $\eta_j(E_R|_{X_0})$ is an isomorphism.

By [BV03, Lemma 3.4.1] the restriction $E_R|_{X_0}$ is a generator of Perf($X_u$). Then each object $F_u \in \text{Perf}(X_u)$ can be obtained from $E_R|_{X_0}$ by taking isomorphisms, finite direct sums, direct summands, shifts, and bounded number of cones. We may assume that $E_R|_{X_0}$ has no nontrivial direct summands, as $\Phi_{P_{u,j}}^L$ and $\Phi_{P_{u,j}}^L$ commute with direct sums on Perf($X'_u$) and Perf($X_u$) respectively with $P_{u,j} = P_k|_{X_u \times X'_u}$ [BV03, Corollary 3.3.4]. One inductively sees that the counit morphism $\Phi_{P_{u,j}}^L \circ \Phi_{P_{u,j}}^L(F_u) \to F_u$ is an isomorphism. In other words, the restriction $\Phi_{P_{u,j}}^L$ of $\Phi_{P_{u,j}}^L$ to $X_u$ is fully faithful. Similarly, the restriction $\Phi_{P_{u,j}}^L$ of $\Phi_{P_{u,j}}^L$ to $X'_u$ is also fully faithful. Thus $\Phi_{P_{u,j}}^L$ is an equivalence.

Since $X_u$ is a smooth projective $k$-variety,

$$
\Phi_{P_{u,j}}^L \circ \Phi_{P_{u,j}}^L \cong \text{id}_{\text{Perf}(X_u)}, \quad \Phi_{P_{u,j}}^L \circ \Phi_{P_{u,j}}^L \cong \text{id}_{\text{Perf}(X'_u)}
$$

imply

$$
P_{u,j} * (P_{u,j})_L \cong O_{\Delta_{u,j}}, \quad (P_{u,j})_L * P_{u,j} \cong O_{X'_u}\n$$

where

$$
\Delta_{u,j} : X_u \dashv X_u \times X_u, \quad \Delta'_{u,j} : X'_u \dashv X'_u \times X'_u
$$

are the diagonal embeddings. Pullback by $f_{u,jk}$ yields

$$
P_{u,jk} * (P_{u,jk})_L \cong O_{\Delta_{u,j}}, \quad (P_{u,jk})_L * P_{u,jk} \cong O_{\Delta'_{u,jk}}\n$$

where

$$
\Delta_{u,jk} : f_{jk}^{-1}(X_u) \dashv f_{jk}^{-1}(X_u) \times g_{jk}^{-1}(u) f_{jk}^{-1}(X_u), \quad \Delta'_{u,jk} : (f_{jk}^{-1}(X'_u)) \dashv (f_{jk}^{-1}(X'_u)) \times g_{jk}^{-1}(u) (f_{jk}^{-1}(X'_u))
$$

are the relative diagonal embeddings. Thus $\Phi_{P_{u,j}}$ is an equivalence. Since $X_{R_k}$ is covered by the collection $(f_{jk}^{-1}(X_u))_u$ with $u$ running through all the closed points of $\text{Spec} R_j$, from [LST13, Proposition 1.3] it follows that $\Phi_{P_k}$ is an equivalence. By the same argument, we conclude that $\Phi_{P_k}$ are equivalences for all $l \geq k$. Applying [CNS, Theorem B], we obtain a quasi-equivalence $\text{Perf}_{d_k}(X_{R_k}) \simeq_{\text{eq}} \text{Perf}_{d_k}(X'_{R_k})$ for all $l \geq k$. \qed

45
Corollary 8.4. Let $X_0, X'_0$ be derived-equivalent Calabi–Yau manifolds of dimension more than two and $X_S, X'_S$ their smooth projective versal deformations over a common nonsingular affine $k$-variety $\text{Spec S}$. Assume that $X_S, X'_S$ correspond to a first order approximation $R_1 \to S$ of $R_j \to R$ for sufficiently large $j \in I$. Then $X_S, X'_S$ are derived-equivalent. In particular, the dg categories $\text{Perf}_{dg}(X_S), \text{Perf}_{dg}(X'_S)$ of perfect complexes are quasi-equivalent.

Proof. By assumption one can apply Theorem 8.3 to find an index $j \in I$ such that $X_S, X'_S$ are the pullbacks of smooth projective families $X_{R_j}, X'_{R_j}$ over $R_j$ satisfying $\text{Perf}_{dg}(X_{R_j}) \simeq_{\text{eq}} \text{Perf}_{dg}(X'_{R_j})$. Consider the distinguished triangle

$$\Phi_{\mathcal{P}_j} \circ \Phi_{\mathcal{P}_j}(E_{R_j}) \xrightarrow{\eta(E_{R_j})} E_{R_j} \to C(\eta_j(E_{R_j})) \cong 0.$$ 

Applying the same argument in the above proof to $X_S$ instead of $X_{R_k}$, one sees that $\Phi_{\mathcal{P}_j} : D^b(X_S) \to D^b(X'_S)$ is an equivalence with $\mathcal{P}_S = (\mathcal{P}_j \times f'_{S_j})^* \mathcal{P}_j$. □

Proposition 8.5. Let $X_0, X'_0$ be derived-equivalent Calabi–Yau manifolds of dimension more than two and $X_S, X'_S$ smooth projective versal deformations over a common nonsingular affine variety $\text{Spec S}$. Then the dg categorical generic fibers are quasi-equivalent.

Proof. We have

$$\text{Perf}_{dg}(X_S)/ \text{Perf}_{dg}(X_S)_0 \simeq_{\text{eq}} \text{Perf}_{dg}(X_{Q(S)}) \simeq_{\text{eq}} \text{Perf}_{dg}(X'_S)/ \text{Perf}_{dg}(X'_S)_0,$$

where the first and the third quasi-equivalences follow from Proposition 7.5. The second quasi-equivalence follows from the above corollary, [Mora, Theorem 1.1], [Morb, Corollary 4.2], and [CNS, Theorem B]. □

References

[AK91] A. Albano and S. Katz, *Lines on the Fermat quintic threefold and the infinitesimal generalized Hodge conjecture*, Trans. Amer. Math. Soc. 324(1), 353-368 (1991).

[Art69a] M. Artin, *Algebraic approximation of structures over complete local rings*, Publ. Math-Paris. 36, 23-58 (1969).

[Art69b] M. Artin, *Algebraization of formal moduli: I*, Global Analysis (Papers in Honor of K. Kodaira), University of Tokyo Press, 21-71 1969, ISBN:978-1-4008-7123-0.

[BO01] A. Bondal and D. Orlov, *Reconstruction of a variety from the derived category and groups of autoequivalences*, Compos. Math. J. 125, 327-344 (2001).

[BV03] A. Bondal and M. Van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Mosc. Math. J. 3(1), 1-36 (2003).

[BFN10] D. Ben-zvi, J. Francis, and D. Nadler, *Integral transforms and Drinfeld centers in derived algebraic geometry*, J. Amer. Math. Soc. 23(4), 909-966 (2010).

[Coh] L. Cohn, *Differential graded categories are k-linear stable infinity categories*, arXiv:1308.2587

[Câl00] A. Căldăraru, *Derived categories of twisted sheaves on Calabi–Yau manifolds*, Ph.D. thesis, Cornell University (2000).

[CNS] A. Canonaco, A. Neeman and P. Stellari, *Uniqueness of enhancements for derived and geometric categories*, arXiv:2101.04404

[DL13] O. D. Dekken and W. Lowen, *On deformations of triangulated models*, Adv. Math 243, 330-374 (2013).

[DLL17] H. Dinh Van, L. Liu and W. Lowen, *Non-commutative deformations and quasi-coherent modules*, Sel. Math. New Ser. 23, 1061–1119 (2017).

[Dri04] V. Drinfeld, *DG quotients of DG categories*, J. Algebra 272(2), 643-691 (2004).

[GD61] A. Grothendieck and J. Dieudonné, *Elements de géométrie algébrique: III. Étude cohomologique des faisceaux cohérents, première partie.*, Publ. Math-Paris. 11, 5-167 (1961).

[GJ] E. Getzler and J.D.S. Jones, *Operads, homotopy algebra and iterated integrals for double loop spaces*, arXiv:hep-th/9403055

[GS87] M. Gerstenhaber and S. D. Schack, *A Hodge-type decomposition for commutative algebra cohomology*, J. Pure. Appl. Algebra. 48(3) 229-247 (1987).
[GS88] M. Gerstenhaber and S. D. Schack, Algebraic cohomology and Deformation theory, Deformation Theory of Algebras and Structures and Applications. NATO ASI Series (Series C: Mathematical and Physical Sciences), 247, Springer. https://doi.org/10.1007/978-94-009-3057-5_2

[Har77] R. Hartshorne, Algebraic geometry, Graduate Texts in Mathematics, 52, Springer-Verlag, 1977, ISBN: 0-387-90244-9.

[Har10] R. Hartshorne, Deformation theory, Graduate Texts in Mathematics, 257, Springer-Verlag, 2010, ISBN: 978-1-4419-1596-2.

[HMS11] D. Huybrechts, E. Macrì, and P. Stellari, Formal deformations and their categorical general fibre, Comment. Math. Helv. 86(1), 41-71 (2011).

[Kel] B. Keller, Derived invariance of higher structures on the Hochschild complexes, http://www.math.jussieu.fr/~keller/publ/dih.dvi

[KL09] B. Keller and W. Lowen, On hochschild cohomology and morita deformations, Int. Math. Res. Not. 17, 3221–3235 (2009)

[Lie06] M. Lieblich, Moduli of complexes on a proper morphism, J. Algebraic Geom. 15, 175-206 (2006).

[Low08] W. Lowen, Hochschild cohomology, the characteristic morphism and derived deformations, Compos. Math. 144(6), 1557-1580 (2008).

[LV06a] W. Lowen and M. Van den Bergh, Hochschild cohomology of abelian categories and ringed spaces, Adv. Math. 198(1), 172-221 (2006).

[LV06b] W. Lowen and M. Van den Bergh, Deformation theory of abelian categories, Trans. Amer. Math. Soc. 358(12), 5441-5483 (2006).

[LST13] A. C. López Martín, D. Sánchez Gómez, and C. Tejero Prieto, Relative Fourier–Mukai transforms for Weierstrass fibrations, abelian schemes and Fano fibrations, Math. Proc. Cambridge. 155(1), 129-153 (2013).

[Lur09] J. Lurie, Higher topos theory, Annals of Mathematics Studies, 170, Princeton University Press, 2009, ISBN: 978-0-691-14049-0

[Man09] M. Manetti, Differential graded Lie algebras and formal deformation theory, Proceedings of Symposia in Pure Mathematics, 80(2), American Mathematical Society, Providence, RI, 2009. ISBN: 978-0-8218-4703-9

[Mora] H. Morimura, Algebraic deformations and Fourier–Mukai transforms for Calabi–Yau manifolds, to appear in Proc. Amer. Math. Soc.

[Morb] H. Morimura, Categorical generic fiber, arXiv:2111.00239

[Por10] M. Porta, The Popescu–Gabriel theorem for triangulated categories, Adv. Math. 225(3), 1669-1715 (2010).

[Qui70] D. Quillen, On the (co-)homology of commutative rings, Applications of categorical algebra (Proc. Sympos. Pure. Math., Vol. XVII, New York, 1968), Amer. Math. Soc., Providence, R.I., 65-87 (1970)

[Shu61] U. Shukla, Cohomologie des algèbres associatives, Ann. Sci. École Norm. Sup. 78(3), 163-209 (1961).

[STZ14] N. Sibilla, D. Treumann, and E. Zaslow Ribbon Graphs and Mirror Symmetry I, Selecta Mathematica (N.S.) 20(4), 979-1002 (2014).

[Tab05a] G. Tabuada, Une structure de catégorie de modèles de Quillen sur la catégorie des dg-catégories, Comptes Rendus de lAcadadémie de Sciences de Paris. 340, 15–19 (2005).

[Tab05b] G. Tabuada, Invariants additifs de DG-catégories, Int. Math. Res. Not. 53, 3309-3339 (2005).

[Tod09] Y. Toda, Deformations and Fourier–Mukai transforms, J. Differ. Geom. 81(1), 197-224 (2009).

[Töe] B. Töen, Lectures on DG-categories, http://www.math.univ-toulouse.fr/toen/