Some remarks on the maximally modulated
Calderón-Zygmund operator satisfying
$L^r$-Hörmander condition

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Abstract

In this work, by recent work of Lerner and Ombrasi (J. Geom. Anal. 30(1): 1011-1027, 2020), we show a maximally modulated singular integral operator which its kernel satisfying $L^r$-Hörmander condition can be dominated by sparse operators. Also, the local exponential decay estimates for these operators are obtained.

Key Words: Maximally modulated singular integrals, Sparse domination theorem, $L^r$-Hörmander condition, Local decay estimate.

2000 Mathematics Subject Classification: primary 42B20; secondary 42B15.

1 Introduction and Preliminaries

After work of T. Hytönen [11], about the full proof of the $A_2$ conjecture, there were a number of breakthroughs results that have delineated a new theory that we may now call "sparse domination technique". A decisive step toward a modern development of domination by sparse operators was carried out by Andrei Lerner [18], which gave an alternative simple proof of the $A_2$ theorem. He showed that Calderón–Zygmund operators can be controlled in norm from above by a very special dyadic type operators. Later the pointwise (dual) dominating of an operator is obtained. Since then, sparse bounds for different operators have been a very attractive realm in harmonic analysis. These type dominating of an operator which is typically signed and non-local, by a positive and localized expression of the form (1.1) has seen an explosion of applications in dyadic harmonic Analysis. The sparse operators are defined in the form

\[ A_{p,S}(f)(x) := \sum_{Q \in S} \langle f \rangle_{Q,p} 1_Q(x), \]  

where $p \geq 1$, and for any cube $Q$,

\[ \langle f \rangle_{Q,p} := \left( \frac{1}{|Q|} \int_Q |f(x)|^p \, dx \right)^{\frac{1}{p}}. \]

The purpose of this paper is to present some remarks concerning maximally modulated singular integrals, which it’s kernel satisfy $L^r$-Hörmander condition. These operators are investigated inspire of Carleson’s operator. Carleson’s operator is the modulated Hilbert transform
define by

\[ C f(x) = \sup_{\xi \in \mathbb{R}} |H(M^{\xi} f)(x)|, \]

where \( M^{\xi} f(x) = e^{2\pi i \xi x} f(x) \).

The organization of this paper is as follows. The first section of the paper is devoted to obtaining a sparse bound to maximally modulated singular integrals and (maximally) modulated maximal singular integral, using known methods especially the works of [21] and its improved version [25]. In the non-modulated case, here are some works, for example [13, 17, 21, 25], which authors were considering weaker regularity conditions on the kernels of Calderón–Zygmund operators. In the second section, we will consider local exponential decay estimates for the maximally modulated Calderón-Zygmund operator satisfying \( L^r \)-Hörmander condition.

We will work with an improved version of inequality (1.4) due to Karagulyan [16]:

\[ |\{ x \in \mathbb{R}^n : T^* f(x) > t M(f) \} | \leq c e^{-ct |Q|}, \quad t > 0. \]

We note that the analogue of (1.2) for the maximally modulated Calderón-Zygmund operator where \( T \) is classical was obtained in [9]. So far, there are several methods for obtaining local decay in references. We will consider the works [7, 26]. The advantage of the method used in [7], is it can be applied for every operator which has sparse bound.

### 1.1 Notations and basic definitions

In this article we will be concerned with sparse domination bound for the maximally modulated Calderón-Zygmund operator and then the local exponential decay estimates for such operators will be established. We say that \( T \) is a Calderón–Zygmund operator if \( T \) is a linear operator of weak type \((1, 1)\) such that

\[ T f(x) = \int_{\mathbb{R}^n} k(x, y) f(y) dy \quad x \notin \text{supp} f. \]

with kernel \( K \) satisfying \( L^r \)-Hörmander condition (see definition 1.1).
Definition 1.1. The kernel $K$ satisfies the $L^r$-Hörmander condition, $1 \leq r \leq \infty$, if

$$
\kappa_r := \sup_{Q} \sup_{x',x'' \in \frac{1}{2}Q} \sum_{k=1}^{\infty} 2^k Q \left| K(x',\cdot) - K(x'',\cdot) \right|_{L^r(2^k Q \setminus 2^{k-1} Q)} < \infty.
$$

where $r'$ is conjugate exponent of $r$. Denote by $\mathcal{H}_r$ the class of all kernel satisfying the $L^r$-Hörmander condition.

Let $\mathcal{H}_r$ denote the class of kernels satisfying the $L^r$-Hörmander condition. One can see that for any $r > s$, $\mathcal{H}_r$ contained in $\mathcal{H}_s$ i.e. $\mathcal{H}_r \subseteq \mathcal{H}_s$.

Remark 1.2. We remark that $L^r$-Hörmander condition is weaker than assumption $H_2$ used in [3 Proposition 3.2]. Then sparse bound obtained in section 2, is valid for modulated of nonstandard kernel operators such as Fourier multipliers and the Riesz transforms associated to Schrödinger operators [28, Remark 3.4]. Also $L^r$-Hörmander condition is weaker than the Dini condition [28 Proposition 3.1]. As a result, sparse bound obtained in section 2, is valid for modulated singular integrals which satisfy the Dini’s condition.

In [25], Lerner and Ombersi give a method to find sparse bound for Calderón-Zygmund operators which it is an improved version of [21]. Authors obtain nearly minimal assumptions on a singular integral operator $T$ for which it admits a sparse domination. Based on them ideas, we are going to give sparse bound for maximally modulated Calderón-Zygmund operators. Our next definition is the definition of $W_q$ property of $T$, which is defined in [25].

Definition 1.3. We call a sub-linear $T$ satisfy the property of $W_q$, if there exist a non-increasing function $\psi_{T,q}(\lambda)$ such that for every $Q$ and for every $f \in L^q(Q)$,

$$
|\{ x \in Q : |T(f \chi_Q)(x)| > \psi_{T,q}(\lambda)\langle f\rangle_{q,Q}\}| \leq \lambda|Q| \quad (0 < \lambda < 1). \quad (1.7)
$$

For definition of maximal singular operator of $T$, we recall associated with $T$ there is a truncated operator $T_\epsilon$ which is defined as follows:

$$
T_\epsilon f(x) = \int_{|x-y|<\epsilon} k(x,y)f(y)dy, \quad T_\epsilon f(x) = \sup_{\epsilon > 0} |T_\epsilon f(x)|. \quad (1.8)
$$

Definition of Maximally modulated operators $T^F$ inspire of the Carleson operator, is defined as following: let $\mathcal{F} = \{\phi_{\alpha}\}_{\alpha \in A}$ be a family of real-valued measurable functions indexed by some set $A$, and let $T$ be above mentioned operator.

$$
T^F f(x) := \sup_{\alpha \in A} |T\left(\mathcal{M}^{\phi_{\alpha}} f\right)(x)|,
$$

where $\mathcal{M}^{\phi_{\alpha}} f(x) = e^{2\pi i \phi_{\alpha}(x)} f(x)$. We recall that the Carleson operator is the modulated of Hilbert transform and the family $F$ consists of the linear functions $\phi_{\alpha}(y) = \alpha y$ with $\alpha \in \mathbb{R}$. We also define the (maximally) modulated maximal singular integral associated with $T$ and $\phi$ via

$$
T^F_\epsilon f(x) = \sup_{\epsilon > 0} \sup_{\alpha \in A} |T_\epsilon(\mathcal{M}^{\phi_{\alpha}} f(x))|. \quad (1.9)
$$

We will consider these operators under a priori assumption

$$
\|T^F(f)\|_{L^p,\infty} \lesssim \psi(p') \|f\|_{L^p}, \quad 1 < p \leq 2 \quad (1.10)
$$
where $\psi$ is a non-decreasing function on $[1, \infty)$ and $f \in L^1(\mathbb{R}^n)$ with compact support. We remark that inspired by recent approach to the study of the $p=1$ end-point behavior of the Calerson operator via weak $L^p$ bound, we choose to work under a priori assumption (1.10). For more detail information corresponding to make sense of this assumption see [27]. Also, we note that the condition (1.10), means that this operator satisfy $W_p$ property, i.e. for any $0 < \lambda < 1$

$$\left| \left\{ x \in Q : \left| T^F(f\chi_Q)(x) \right| > \xi_{T^F,q}(\lambda)\langle f \rangle_{q,Q} \right\} \right| \leq \lambda |Q| \quad (0 < \lambda < 1) \quad (1.11)$$

where $\xi_{T^F,p}(\lambda) := \frac{1}{\lambda^{\Phi \left( \frac{p}{p'} \right)}} < f > Q$.

Now we define following quantity

$$M^\Phi_{T^F,\alpha} f(x) := \sup_{Q \ni x} \text{ess sup}_{x',x'' \in Q} |T^F(f\chi_{\mathbb{R}^n \setminus \alpha Q})(x) - T^F(f\chi_{\mathbb{R}^n \setminus \alpha Q})(x')|$$

where $x, x' \in P$ and $\alpha \geq 3$. This quantity is of weak type $(r, r)$ which is proved in Lemma 2.1

Now we recall Orlicz space and some notion related this space which we will use them.

Let $\Phi$ be a Young function, that is, $\Phi : [0, \infty) \to [0, \infty)$, $\Phi$ is continuous, convex, increasing, $\Phi(0) = 0$ and $\Phi(t) \to \infty$ as $t \to \infty$.

Let $f$ be a measurable function defined on a set $Q \subset \mathbb{R}^n$ with finite Lebesgue measure. The $\Phi$-norm of $f$ over $Q$ is defined by

$$\|f\|_{\Phi,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \Phi \left( \frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\}. \quad (1.12)$$

The Orlicz maximal operator $M_{\Phi}$ is defined by

$$M_{\Phi} f(x) := \sup_{Q \ni x} \|f\|_{\Phi,Q}.$$

Similar to proof of Proposition 6.1 in [27], we can see that for each cube $Q \subset \mathbb{R}^n$ and for $T^F$ maximally modulated Calderón-Zygmund operator mentioned above, the condition

$$\|T^F(f\chi_Q)\|_{L^1 \to L^{\gamma_{\Phi}(p)}}(Q) \leq |Q|\|f\|_{\Phi,Q}, \quad (1.13)$$

is sufficient condition for validity of the priori assumption (1.10), where the where the Young function $\Phi$ is such that

$$\gamma_{\Phi}(p) = \sup_{t \geq 1} \frac{\Phi(t)}{t^{p'}} < \infty, \quad \forall p > 1.$$

### 1.2 Local Mean Oscillation estimate.

Let $f : Q \to R$ be a measurable function. Here $Q$ could be any set of finite positive measure, but later on it will mostly be a cube; hence the choice of the letter. The median of $f$ on $Q$ is any real number $m_f(Q)$ with the following two properties:

$$|Q \cap \{ f > m_f(Q) \}| \leq \frac{1}{2}|Q|$$

$$|Q \cap \{ f < m_f(Q) \}| \leq \frac{1}{2}|Q|$$
The mean local oscillation of a measurable function $f$ on a cube $Q$ is defined by the following expression

$$
\omega_{\lambda}(f; Q) = \inf_{c \in \mathbb{R}} ((f - c)\chi_Q)^*(\lambda|Q|),
$$

for all $0 < \lambda < 1$, and the local sharp maximal function on a fixed cube $Q_0$ is defined as

$$
M^\#_{\lambda; Q_0} f(x) = \sup_{x \in Q \subset Q_0} \omega_{\lambda}(f; Q),
$$

where the supremum is taken over all cubes $Q$ contained in $Q_0$ and such that $x \in Q$.

The decreasing rearrangement concept can be defined for any measurable function $f$. We denote

$$
f^*(t) := \inf\{\alpha \geq 0 : |\{|f| > \alpha\}| \leq t\}
$$

We make the following observations (see [12]):

- $f^*$ is non-increasing.
- The set inside the infimum is of the form $[a_0, \infty)$ (or $\emptyset$). Hence the infimum is reached as a minimum; in particular, $f^*$ itself is an admissible value of $\alpha$, so that

$$
|\{|f| > f^*\}| \leq t
$$

- We have $(f\chi_Q)^*(t) = \inf\{\alpha \geq 0 : |Q \cap \{|f| > \alpha\}| \leq t\}$.

We will use several times that for any $\delta > 0$, and $0 < \lambda < 1$,

$$
(f\chi_Q)^*(\lambda|Q|) \leq \left(\frac{1}{\lambda|Q|} \int_Q |f|^\delta \, dx\right)^{\frac{1}{\delta}}
$$

(1.14)

this is true since

$$
(f\chi_Q)^*(t) \leq \frac{1}{t^\delta} \|f\chi_Q\|_{\delta, \infty}.
$$

(1.15)

On the other hand, the following lemma is valid which is proved in [12].

**Lemma 1.4.** The following estimate holds for all $\lambda \in (0, \frac{1}{2})$ and all medians $m_f(Q)$:

$$
|m_f(Q)| \leq (f\chi_Q)^*(\lambda|Q|)
$$

(1.16)

Recall that, for a fixed cube $Q_0$, $D(Q_0)$ denotes all the dyadic subcubes with respect to the cube $Q_0$. As before, if $Q \in D(Q_0)$ and $Q \neq Q_0$, $\hat{Q}$ will be the ancestor dyadic cube of $Q$, i.e., the only cube in $D(Q_0)$ that contains $Q$ and such that $|\hat{Q}| = 2^n|Q|$.

The following theorem was proved by Hytönen [14, Theorem 2.3] which is improved version of Lerner’s formula. The original version of Lerner’s formula is given in [22, 23].

**Theorem 1.5.** For any measurable function $f$ on a cube $Q^0 \subset \mathbb{R}^n$, we have

$$
|f(x) - m_f(Q^0)| \leq 2 \sum_{L \in \mathcal{L}} \frac{\omega_{\frac{1}{|Q^0|}}(f; L)\chi_L(x)}{\omega_{\frac{1}{|Q^0|}}(f; L)\chi_L(x)},
$$

(1.17)

where $\mathcal{L} \subset D(Q^0)$ is sparse: the collection $\mathcal{L} \subset D(Q^0)$ is called a $\eta = \frac{1}{2}$-sparse family of cubes if there exist pairwise disjoint subsets $E_L \subset L$ with $\eta|L| \leq |E_L|$ for each $L \in \mathcal{L}$. 

2 Main Results

The following lemma shows that $M_{T,F}^2$ is of weak type $(r,r)$, where $1 \leq r \leq \infty$.

**Lemma 2.1.** Let $T^F$ be maximally modulated operator defined above. For every $1 \leq r \leq \infty$, the following inequality is satisfied:

$$M_{T,F}^2 f(x) \leq CM_r f(x).$$

**Proof.** Let $x_p$ be the center of cube $P$ and $P^* = \alpha P$, where $\alpha \geq 3$ then

$$|T^F (f\chi_{\mathbb{R}^n\setminus P^*}) (x) - T^F (f\chi_{\mathbb{R}^n\setminus P^*}) (x')|$$

$$\leq |T^F (f\chi_{\mathbb{R}^n\setminus P^*}) (x) - T^F (f\chi_{\mathbb{R}^n\setminus P^*}) (x_p)| + |T^F (f\chi_{\mathbb{R}^n\setminus P^*}) (x') - T^F (f\chi_{\mathbb{R}^n\setminus P^*}) (x_p)|.$$

It is sufficient we estimate one of two terms. Let us start with first

$$|T^F (f\chi_{\mathbb{R}^n\setminus P^*}) (x) - T^F (f\chi_{\mathbb{R}^n\setminus P^*}) (x_p)|$$

$$\leq \left| \sup_{\alpha \in A} |T (\mathcal{M}^{\phi_\alpha} f\chi_{\mathbb{R}^n\setminus P^*}) (x) - \sup_{\alpha \in A} |T (\mathcal{M}^{\phi_\alpha} f\chi_{\mathbb{R}^n\setminus P^*}) (x_p)| \right|$$

$$\leq \sum_{j=1}^{\infty} \int_{j2^kQ \setminus (j-1)2^kQ} |K(x,y) - K(x_p,y)| |f(y)| dy$$

$$\leq \sum_{j=1}^{\infty} \left( \int_{j2^kQ \setminus (j-1)2^kQ} |K(x,y) - K(x_p,y)| |f(y)| dy \right)^{\frac{1}{r}} \left( \int_{(j-1)2^kQ \setminus j2^kQ} |f(y)|^{r'} dy \right)^{\frac{1}{r'}}$$

$$\leq \sum_{j=1}^{\infty} \|K(x,.) - K(x_p,.)\|_{L^r(2^kQ \setminus (2^kQ \setminus 2^{k-1}Q))} \left( \frac{|2^kQ|}{|2^kQ|} \int_{2^kQ} |f(y)|^{r'} dy \right)^{\frac{1}{r'}}$$

$$\leq \sup_{Q \ni x} \sup_{x' \in Q} \sum_{j=1}^{\infty} |2^kQ|^{\frac{1}{r'}} \|K(x',.) - K(x''.,.)\|_{L^r(2^kQ \setminus (2^kQ \setminus 2^{k-1}Q))} \times M_r f(x)$$

$$\leq \kappa_r \ M_r f(x).$$

Therefore we have

$$|T^F (f\chi_{\mathbb{R}^n\setminus P^*}) (x) - T^F (f\chi_{\mathbb{R}^n\setminus P^*}) (x')| \lesssim \kappa_r \ M_r f(x),$$

This argument follows $M_{T,F}^2 f(x)$ is weak type $(r', r')$. □

Following theorem give sparse bound for maximally modulated singular integral.
Theorem 2.2. Let $T^F$ be a sub-linear operator satisfying the $W_q$ condition and such that $M^#_{T^F,\alpha}$ is of weak type $(r, r)$ for some $\alpha \geq 3$, where $1 \leq q, r < \infty$. Let $s = \max(q, r)$. Then, for every compactly supported $f \in L^s(\mathbb{R}^n)$, there exists $\frac{1}{2\alpha}$-sparse family $S$ such that

$$|T^F f(x)| \leq C \sum_{Q \in S} \langle f \rangle_{s, Q} \chi_Q(x)$$

for a.e $x \in \mathbb{R}^n$, where $C = c_{n, r, s, \alpha} \left( \psi_{T^F, \alpha}( \frac{1}{12(2\alpha)^n}) + \|M^#_{T^F, \alpha}\|_{L^r \rightarrow L^{r, \infty}} \right)$.

Proof. The proof is almost identical to the proof of \cite{25} Theorem 2.2]. For the sake of completeness, we give the proof here. For any arbitrary cube $Q$, set $Q^* = \alpha Q$ where $\alpha \geq 3$. In point view of the definition of operator $M^#_{T,\alpha}$, for every cube $P$ and for all $x', x'' \in P$, we have the following inequality

$$|T^F (f\chi_{\mathbb{R}^n \setminus P^*}) (x') - T^F (f\chi_{\mathbb{R}^n \setminus P^*}) (x'')| \leq \inf_P \|M^#_{T^F,\alpha}(f)\|. \tag{2.1}$$

Set

$$\tilde{M}_{T^F} f = \max \left( |T^F f|, M^#_{T^F,\alpha} f \right).$$

In point view of the theorem assumptions, the weak type $(1, 1)$ of $M$ and Hölder’s inequality, the set

$$\Omega = \left\{ x \in Q : \max \left( \frac{M_s(f\chi_{Q^*})(x)}{c}, \frac{|\tilde{M}_{T^F}(f\chi_{Q^*})(x)|}{A} \right) > \langle f \rangle_{s, Q^*} \right\}$$

satisfies $|\Omega| \leq \frac{1}{2\alpha} |Q|$, where

$$A = 2\psi_{T^F, \alpha}(\frac{1}{12(2\alpha)^n}) + c_{n, r, s, \alpha} \|M^#_{T^F, \alpha}\|_{L^r \rightarrow L^{r, \infty}}.$$

The local Calderón-Zygmund decomposition guarantees that there exists a family of disjoint cubes $\{P_j\}_{j=1}^{\infty} \subset Q$ such that

$$\frac{1}{2^{n+1}} |P_j| < |P_j \cap \Omega| \leq \frac{1}{2} |P_j|, \quad |\Omega \setminus \bigcup_{j=1}^{\infty} P_j| = 0. \tag{2.2}$$

consequently,

$$|T(f\chi_{Q^*})(x)| \leq A(f)_{s, Q^*} \quad \text{for a.e } x \in Q \setminus \bigcup_{j=1}^{\infty} P_j. \tag{2.3}$$

On the other hands, for almost all $x \in P_j$ and $x' \in P_j \setminus \Omega,$

$$|T^F (f\chi_{Q^* \setminus P_j^*})(x)| \leq \inf_{P_j} \frac{M_{T^F,\alpha}^#(f \chi_{P_j}) + |T^F (f\chi_{Q^* \setminus P_j^*})(x')|}{\inf_{P_j} M_{T^F,\alpha}^#(f \chi_{P_j}) + |T^F (f\chi_{Q^* \setminus P_j^*})(x')|}$$

$$\leq \frac{M_{T^F,\alpha}^#(f \chi_{P_j}) + |T^F (f\chi_{Q^*})(x')| + |T^F (f\chi_{P_j})(x')|}{\inf_{P_j} M_{T^F,\alpha}^#(f \chi_{P_j}) + A(f)_{s, Q^*} + |T^F (f\chi_{P_j})(x')|}$$

$$\leq \frac{M_{T^F,\alpha}^#(f \chi_{P_j}) + A(f)_{s, Q^*} + |T^F (f\chi_{P_j})(x')|}{\inf_{P_j} M_{T^F,\alpha}^#(f \chi_{P_j}) + A(f)_{s, Q^*} + |T^F (f\chi_{P_j})(x')|}$$

$$= 2A(f)_{s, Q^*} + |T^F (f\chi_{P_j})(x')|. \tag{2.4}$$
Thanks to $\text{(2.2)}$, one can obtain $|P_j \setminus \Omega| \geq \frac{1}{2}|P_j|$. On the other hand,

$$|\Omega'| = \left| \left\{ x \in P_j : |T^F(f \chi_{P_j^*})(x)| > A(f)_{s,P_j^*} \right\} \right| \leq \frac{1}{2^{n+2}} |P_j|.$$

As a result, we have

$$\inf_{P_j \setminus \Omega} |T^F(f \chi_{P_j^*})| \leq A(f)_{s,P_j^*} \leq A \inf_{P_j^*} M_s f \leq A \inf_{P_j} M_s f \leq A \inf_{\Omega} M_s f \leq c A(f)_{s,\Omega^*},$$

which, combined with $\text{(2.1)}$, implies that for all $x \in P_j$,

$$|T^F(f \chi_{\Omega^* \setminus P_j^*})(x)| \leq (2 + c) A(f)_{s,\Omega^*}. \quad (2.5)$$

Form this and from $\text{(2.3)}$, for a.e $x \in Q$

$$|T^F(f \chi_{\Omega^*})(x)| \chi_Q(x) = |T^F(f \chi_{\Omega^*})(x)| \chi_Q \cup_{j=1}^{\infty} P_j(x) + |T^F(f \chi_{\Omega^*})(x)| \chi_{\bigcup_{j=1}^{\infty} P_j}(x)
= (3 + c) A(f)_{s,\Omega^*} + \sum_{j=1}^{\infty} |T^F(f \chi_{P_j^*})(x)| \chi_{P_j}(x). \quad (2.6)$$

By $\text{(2.2)}$, $\sum_{j=1}^{\infty} |P_j| \leq \frac{1}{2}|Q|$. Therefore, iterating $\text{(2.6)}$, we obtain a $\frac{1}{2}$-sparse family $F_Q$ of sub cubes of $Q$ such that

$$|T^F(f \chi_{Q^*})(x)| \leq (3 + c) A \sum_{R \in F_Q} \langle f \rangle_{s,\Omega^*} \chi_R(x). \quad (2.7)$$

So, the proof is completed with $\text{(2.1)}$, Lemma 2.1].

\[\square\]

**Remark 2.3.** The cubes of the resulting sparse family $S$ are not dyadic. But there is a well known result which says for an arbitrary cube $Q$, there are $n+1$ general dyadic grids $D^a$ such that every cube $Q \subset \mathbb{R}^n$ is contained in some cube $Q' \in D^n$ such that $|Q| \leq c_n |Q'|$ (see [6]). So, in Theorem 2.2 one can write

$$|T^F f(x)| \leq C_{n,s} \sum_{j=1}^{n+1} \sum_{Q \in S_j} \langle f \rangle_{s,Q} \chi_Q(x), \quad (2.8)$$

where $S_j$ is a sparse family from a dyadic grid $D^j$. We note that in most of papers number of dyadic grids assumed $3^n$ or $2^n$, for example see the papers [21] [15] [10], but the number $n + 1$ is optimal.

Readily we have following theorem regarding sparse bound to $T^*_s f(x)$ which defined in [10].

**Proposition 2.4.** Let $T^F$ be a sub-linear operator satisfying the $W_q$ condition and such that $M^F_{T^F, \alpha}$ is of weak type $(r,r)$ for some $\alpha \geq 3$, where $1 \leq q, r < \infty$. Let $s = \max(q,r)$. Then, for every compactly supported $f \in L^s(\mathbb{R}^n)$, there exists a $\frac{1}{2^{n+2}}$-sparse family $S$ such that

$$|T^*_s f(x)| \leq C_{n,s} \sum_{j=1}^{n+1} A^s_{S_j} f(x), \quad (2.9)$$

where $A^s_{S_j} f(x) = \sum_{Q \in S_j} \langle f \rangle_{s,Q} \chi_Q(x)$. 

\[\square\]
Theorem 2.5. Let $T^F$ be a maximally modulated singular integral operator with the (maximally) modulated maximal singular integral $T^F$. Let $Q$ be a cube and let $f \in L^\infty_c(\mathbb{R}^n)$ such that $\text{supp}(f) \subseteq Q$. Then there are constants $\alpha, c > 0$ such that
\[
|\{x \in Q : |T^F f(x)| > t M_r f(x)\}| \leq c e^{-\alpha t} |Q|, \quad t > 0. \tag{2.10}
\]

Proof. It follows from Lemma 1.4, 1.11, and Kolmogorov's inequality
\[
|m_{T^F}(Q)| \leq \left( \frac{4}{|Q|} \int_Q |T^F|^\delta dx \right)^\frac{1}{\delta},
\]
\[
\leq c_\delta \|T^F(f)\|_{L^1,\infty(Q, \frac{dx}{|x|^r})}
\]
\[
\leq c_\delta \|f(x)\|_{\mathcal{F}, Q}
\]
\[
\leq c_\delta M_\Phi(f)(x)
\]
\[
\leq c_\delta \gamma_\Phi(r)^\frac{\delta}{2} M_r(f)(x)
\]
where $r > 1$ and for any $0 < \delta$. In last inequality we use the [27, Lemma 6.3]. By consequence of the definitions of following notions, for given a cube $Q$, $\delta > 0$ and $0 < \lambda < \frac{1}{2}$, there exists a constant $c = c_\lambda$ such that
\[
M_{\lambda, Q}^2(f \chi_Q)(x) \leq c M_{\lambda}^2(f \chi_Q)(x), \quad x \in Q. \tag{2.11}
\]
where $M_{\delta}^2(f \chi_Q)(x) = \sup_{Q \ni x} \left( \frac{1}{|Q|} \int_Q |f(y) - c^\delta|^\frac{1}{\delta} \right)^\frac{1}{\delta}$. This inequality, readily valid as following:
\[
M_{\lambda, Q}^2 f(x) = \sup_{Q \ni x} \omega_\lambda(f; Q)
\]
\[
= \sup_{Q \ni x} \{((f - c) \chi_Q)^*(\lambda |Q|),
\]
\[
\leq c_\lambda \sup_{Q \ni x} \left( \frac{1}{|Q|} \int_Q |f(y) - c|^\delta dy \right)^\frac{1}{\delta}
\]
\[
\leq c_\lambda M_{\delta}^2(f \chi_Q)(x)
\]
the last inequality comes from $(f \chi_Q)^*(t) \leq \frac{1}{t} \|f \chi_Q\|_{2,\infty}$.

Also, based on arguments in [27, proposition 6.1] and [27, Lemma 6.3], one can observe
\[
M_{\delta}^2(T^F f)(x) \lesssim M_{\delta} f(x) \lesssim M_r f(x).
\]
Moreover, the following Feierman-Stein inequality was obtained in [20]:
\[
|\{x \in Q : |f(x) - m_Q(f)| > t M_{\lambda, Q}^2(f)(x)\}| \leq c_3 e^{\lambda t} |Q|, \quad \lambda = \frac{1}{2n+2}.
\]

Consequently, we deduce that for $t > c_1$
\[
|\{x \in Q : |T^F f(x)| > t M_r(f)(x)\}| \leq |\{x \in Q : |T^F f(x)| > t M_r(f)(x)\}|
\]
\[
\leq |\{x \in Q : |T^F f(x) - m_{T^F f}(Q)| > (t - c_1)M_r(f)(x)\}|
\]
\[
\leq |\{x \in Q : |T^F f(x)| > c_2^{-1}(t - c_1)M_{\delta,2^{-2}, -n, Q}^2(T^F f)(x)\}|
\]
\[
\leq c_3 e^{-\frac{\beta(t - c_1)}{2}} |Q|.
\]
So, taking $c = \max\{1, c_3\} e^{\frac{\beta c_1}{2}}$ and $\alpha = \frac{\beta c_1}{2}$, we obtain the desired result. \qed
Remark 2.6. We remark that local decay estimate can be derive by the combination of the sparse domination results obtained in Theorem 2.2 and Proposition 2.4 and the estimate

\[ |\{ x \in Q : A_S^r f > tM_r(f) \} | \leq c_1 e^{c_2 r^2} |Q|, \quad r > 0 \]

which is proved in [7].

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