On analytical construction of observable functions in extended dynamic mode decomposition for nonlinear estimation and prediction

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Abstract—We propose an analytical construction of observable functions in the extended dynamic mode decomposition (EDMD) algorithm. EDMD is a numerical method for approximating the spectral properties of the Koopman operator. The choice of observable functions is fundamental for the application of EDMD to nonlinear problems arising in systems and control. Existing methods either start from a set of dictionary functions and look for the subset that best fits, in a certain sense, the underlying nonlinear dynamics; or they rely on machine learning algorithms, e.g., neural networks, to “learn” observable functions that are not explicitly available. Conversely, we start from the dynamical system model and lift it through the Lie derivatives, rendering it into a polynomial form. This transformation into a polynomial form is exact, although not unique, and it provides an adequate set of observable functions. The strength of the proposed approach is its applicability to a broader class of nonlinear dynamical systems, particularly those with nonpolynomial functions and compositions thereof. Moreover, it retains the physical interpretability of the underlying dynamical system and can be readily integrated into existing numerical libraries. The proposed approach is illustrated with an application to electric power systems. The modeled system consists of a single generator connected to an infinite bus, in which case nonlinear terms include sine and cosine functions. The results demonstrate the effectiveness of the proposed procedure in off-attractor nonlinear dynamics for estimation and prediction; the observable functions obtained from the proposed construction outperformed existing methods that use dictionary functions comprising monomials or radial basis functions.

Index Terms—Extended dynamic mode decomposition, Koopman spectral analysis, Lie derivative, nonlinear estimation and prediction, observable function, polynomialization.

I. INTRODUCTION

KOOPMAN operator theory (KOT) and associated numerical methods [1]–[3] are promising for system identification [4], state estimation [5], stability assessment [6], and control [7] of nonlinear dynamical systems. The increasing interest in the applications of KOT to systems and control is primarily due to two of its characteristics: i) it does not rely on any model—from beginning to end, numerical methods based on KOT are truly data driven, yet they are supported by a mathematical foundation anchored on the spectral theory of dynamical systems [1], [2]; and ii) linear and nonlinear modes are captured, although the numerical methods rely exclusively on linear algebra. In simple words, these outstanding characteristics can be explained by the fact that the Koopman operator is a linear, infinite-dimensional operator that acts on functions; in principle, any measured quantity of a dynamical system can be expressed as a function of its state variables, \( x \), hereafter referred to as an observable function, \( g(x) \). See, e.g., [1], [2] for a formal exposition of this topic.

A great deal of progress has been made in devising numerical methods that provide a finite-dimensional approximation to the infinite-dimensional Koopman operator. Extended dynamic mode decomposition (EDMD) [8] is an example of a powerful numerical method tailored to this purpose; see [9] for a study on the convergence of EDMD to the Koopman operator. EDMD is sensitive to the set of observable functions provided as input, however [10]; finding the right set of observable functions, i.e., a set of observable functions that yields a Koopman invariant subspace, is nontrivial [11] and an unsolved problem. Although there is an increase in the number of applications based on EDMD, very few researchers have tackled the fundamental challenge of choosing the right set of observable functions. One strategy is to start from a large set of dictionary functions and apply a sparse regression penalty on the number of functions selected to approximate the dynamics of the underlying system [12]. Another strategy is to use neural networks to “learn” the observable functions [10], [13], [14]. These strategies have found success in a broad range of complex problems where data are abundant and state-space models are scarce or nonexistent. A question that always plagues these strategies is how well the discovered mapping describes the underlying system dynamics beyond sampled trajectories. This is a challenging question to answer with limited to no access to a model.

On the other hand, state-space models are often available in systems and control. Although in certain cases the uncertainty in the parameters is large, the structure of the model is known. In this context, for a given set of observable functions, one can optimize the approximated spectral properties of the Koopman operator, in particular the Koopman eigenfunctions [15]. But the selection of observable functions is particularly challenging if the state-space model contains nonlinear terms given by non-polynomial functions and the underlying dynamical system has multiple fixed points [11]. Note that the Carleman linearization [16] is limited to polynomial vector fields. This problem has not been addressed before, and it is the main contribution of this letter.

Our investigations reveal that a broad class of dynamical systems comprise elementary nonlinear functions, such as \( \sin x, \cos x, e^x, \frac{1}{\sqrt{1+x^2}}, \) and compositions of these elementary
functions; and that dynamical systems that fall into this class can be put into a polynomial form by lifting the original system to a space of higher dimension. The lifting procedure, originally proposed in [17], relies on Lie derivatives. We show that this embedding, referred to as polynomialization, makes EDMD more effective for the aforementioned class of dynamical systems. This letter is intended to set the direction for others working on the problem of selecting observable functions. Additionally, we envision the numerical illustration in Section IV to serve as a benchmark problem and solution that researchers could use to compare their choice of observable functions against.

II. Preliminaries

A. Koopman Operator Theory

Let an autonomous dynamical system evolving on a finite, n-dimensional manifold \( \mathbb{X} \) be:

\[
\dot{x}(t) = f(x(t)),
\]

for continuous time \( t \in \mathbb{R} \), where \( x \in \mathbb{X} \) is the state, and \( f : \mathbb{X} \to \mathbb{X} \) (tangent bundle of \( \mathbb{X} \)) is a nonlinear vector-valued function. In what follows, we introduce the Koopman operator for continuous time systems.

Let \( g(x) \) be a scalar-valued function defined in \( \mathbb{X} \), such that \( g : \mathbb{X} \to \mathbb{C} \). The function \( g \) is referred to as an observable function. The space of observable functions is \( \mathcal{F} \subseteq C^0 \), where \( C^0 \) denotes all continuous functions. Note that the choice of \( \mathcal{F} \) is discussed in [2].

The Koopman operator, denoted by \( \mathcal{K}_t \), is a linear, infinite-dimensional operator [18] that acts on \( g \) in the following manner:

\[
\mathcal{K}_t g := g \circ S_t,
\]

where

\[
S_t : \mathbb{X} \to \mathbb{X} ; \ x(0) \to x(t) = x(0) + \int_0^t f(x(\tau)) d\tau
\]

is called the flow.

The Koopman eigenvalues, \( \lambda \), and Koopman eigenfunctions, \( \phi(x) \), of the continuous time system (1) are such that:

\[
\mathcal{K}\phi_i = e^{\lambda_i t} \phi_i, \quad i = 1, \ldots, \infty,
\]

where \( \lambda_i \in \mathbb{C} \), and \( \phi_i \in \mathcal{F} \) is nonzero. Now consider a vector-valued function, \( g : \mathbb{X} \to \mathbb{C}^q \). If all elements of \( g \) lie within the span of the Koopman eigenfunctions, then

\[
g(x(t)) \approx \sum_{i=1}^{\infty} \phi_i(x(t)) \upsilon_i = \sum_{i=1}^{\infty} \phi_i(x(0)) \upsilon_i e^{\lambda_i t},
\]

where \( \upsilon_i \in \mathbb{C}^q \), \( i = 1, \ldots, q \), are referred to as Koopman modes [19].

B. Extended Dynamic Mode Decomposition

Consider pairs of snapshots of the system state variables, \( \{x_{k-1}, x_k\}, \quad k = 1, \ldots, m \), as sampled data of continuous flows, i.e., under a sampling period \( \Delta t \), we have \( x_k = x(k \Delta t) \).

The data matrices are defined as

\[
X = [x_0 \ldots x_{m-1}], \quad X^+ = [x_1 \ldots x_m],
\]

where \( X, X^+ \in \mathbb{R}^{n \times m} \). The vector of observable functions is defined as

\[
g(x_k) := \begin{bmatrix} g_1(x_k) & \cdots & g_q(x_k) \end{bmatrix}^T,
\]

where \( g : \mathbb{R}^n \to \mathbb{R}^q, \quad q > n \). Also, the matrices of observables are defined as

\[
G = [g(x_0) \ldots g(x_{m-1})], \quad G^+ = [g(x_1) \ldots g(x_m)],
\]

where \( G, G^+ \in \mathbb{R}^{q \times m} \).

A finite-dimensional approximation to the Koopman operator is estimated as follows:

\[
K = G^+ G^\dagger,
\]

where \( G^\dagger \) denotes the Moore-Penrose pseudoinverse of \( G \), and \( K \in \mathbb{R}^{q \times q} \). The eigenvalues of \( K \) are approximations to the Koopman eigenvalues, \( \lambda \), whereas an approximation to the Koopman eigenfunctions is given by

\[
\phi(x_k) \approx L g(x_k),
\]

where the matrix \( L \) contains the left eigenvectors of \( K \), and \( \phi(x_k) = [\phi_1(x_k) \ldots \phi_q(x_k)]^T \). Finally, to recover the Koopman modes for the full set of state variables \( g(x_k) = x_k \), let the projection matrix, \( P \in \mathbb{R}^{q \times q} \), be a matrix defined such that

\[
x_k = Pg(x_k).
\]

From (10), we have that \( g(x_k) = L^{-1} \phi(x_k) \), and thus

\[
x_k = Pg(x_k) = PL^{-1} \phi(x_k).
\]

Hence, an approximation to the Koopman modes is provided by the column vectors of \( U = PL^{-1}, \quad U \in \mathbb{C}^{q \times q} \), and

\[
x_k = \sum_{i=1}^{q} \phi_i(x_k) \upsilon_i = \sum_{i=1}^{q} \phi_i(x_0) \upsilon_i \lambda_i^k.
\]

Note that (13) is a finite truncation of (5) under the sampling.

III. Analytical Construction of Observable Functions

This section contains the main contribution of this letter. Let \( \mathbb{X} \subseteq \mathbb{R}^n \) in the continuous time system (1). In what follows, we drop the time index, \( t \), for simplicity. We are interested in the case where the nonlinear functions, \( f(x) \), can be written as a linear combination of elementary functions, \( h(x) \in \mathcal{F} \)—that is, if we consider the \( i \)-th state variable,

\[
\dot{x}_i = k_i^T x + k_1 h_1(x) + \cdots + k_m h_m(x),
\]

where \( k_i^T \) denotes the transpose of \( k_i \).

The elementary functions include \( \sin x, \cos x, e^x, \) and \( \frac{x}{1+x} \), as well as compositions of these elementary functions. Note that because of the composition of functions, these elementary functions encompass a broad class of models encountered in engineering, making this approach well-suited to applications of KOT. Indeed, mathematical models of many engineering systems can be written in the form of (14)—including models of ion channels [20], semiconductor devices [21], and power systems [22], [23]—thereby motivating the search for a state-inclusive Koopman observable space [24].

The following procedure was proposed in [17] as part of a model order reduction method, and it was recently applied in the context of system identification on a lifted space [25]. For each elementary function, \( h_i(x) \), proceed as follows:

1) Introduce a new variable \( z_i = h_i(x) \).
2) Replace \( h_i(x) \) by \( z_i \) in the original equations.
3) Add \( \dot{z}_i = \frac{\partial h_i(x)}{\partial x} f \) in the set of original equations.

The equation added in Step 3 is the Lie derivative of \( z_i \) with respect to \( f \). Note that Lie derivative is a Koopman generator in terms of the vector field \( f \). The resulting lifted system is as follows:

\[
\dot{x}_i = k_i^T x + k_1 z_1 + \cdots + k_m z_m,
\]

\[
\dot{z}_i = L f h_i(x),
\]

where

\[
L f h_i(x) = \frac{\partial h_i(x)}{\partial x_1} \dot{x}_1 + \cdots + \frac{\partial h_i(x)}{\partial x_n} \dot{x}_n.
\]
Table I shows examples of transformations for univariate elementary functions, and the following remarks are in order:

- $x^{-1}$ can be removed from the new differential equations by introducing another new variable, $y = x^{-1}$.
- There are elementary functions that need to be handled by adding two new variables, e.g., $\sin x$.

Now, for compositions of elementary functions, i.e., $h(x) = (h_1 \circ h_2)(x) = h_2(h_1(x))$, proceed as follows:

1. Introduce new variables $z_1 = h_1(x)$ and $z_2 = h_2(z_1)$.
2. Replace $h_2(h_1(x))$ by $z_2$ in the original equations.
3. Add $\dot{z}_1 = \frac{\partial h_1(x)}{\partial x}f$ and $\dot{z}_2 = \frac{\partial h_2(z_1)}{\partial z_1} \dot{z}_1$ in the set of original equations.

Two examples of polynomialization involving the composition of elementary functions are given in Table II, from which two remarks are in order:

- Elementary functions that need to be handled by adding two new variables must be considered.
- Polynomialization is not a unique transformation, as is clear from the second example.

Note that polynomial systems obtained from the previous procedure can be put into a quadratic-linear form by applying another lifting. Moreover, the previous procedures also apply for control systems of the form:

$$\dot{x}_i = k_i^T x + k_1 h_1(x) + \cdots + k_m h_m(x) + Bu,$$  \hfill (18)

where $u \in \mathbb{R}$ is the input. The interested reader is referred to [17]. Upon completion of the polynomialization, we select the state variables, along with the obtained new variables, as observable functions in the EDMD, i.e., $\{x_1, \ldots, x_n, z_1, z_2, \ldots\}$. This choice of observable functions is justified by the fact that polynomialization is an exact transformation, from the original state space to a higher dimension space; therefore, the lifted representation serves as a weak canonical form of the original system, given its nonuniqueness.

### TABLE I

| Elementary function | New variable(s) | New differential equation(s) |
|---------------------|----------------|-----------------------------|
| $h(x) = e^x$        | $z = e^x$     | $\dot{z} = e^x = z$         |
| $h(x) = \frac{x}{x+y}$ | $z = \frac{x}{x+y}$ | $\dot{z} = -\frac{y}{(x+y)^2} = -z^2$ |
| $h(x) = x^k$        | $z = x^k$     | $\dot{z} = kx^{k-1} = k\dot{x}$ |
| $h(x) = \ln x$      | $z = \ln x$   | $\dot{z} = x^{-1}$          |
| $h(x) = \sin x$     | $z_1 = \sin x$ | $\dot{z}_1 = \cos x = z_2$  |
|                     | $z_2 = \cos x$ | $\dot{z}_2 = -\sin x = -z_1$ |

### TABLE II

| Original system | New variables | Lifted system |
|----------------|--------------|--------------|
| $\dot{x} = \frac{1}{1+x^2}$ | $z_1 = \frac{1}{1+x^2}$ | $\dot{z}_1 = \frac{1}{1+x^2}$ |
| $z_2 = \frac{1}{1+\tan^2 x}$ | $\dot{z}_2 = \frac{1}{1+\tan^2 x}z_1 = \frac{z_1}{z_1^2 + 1}$ | $\dot{z}_2 = -\frac{z_1}{z_1^2 + 1}$ |
| $\dot{x} = x \cos x$ | $z_1 = \cos x$ | $\dot{z}_1 = \dot{x}_2$ |
| $z_2 = x z_1$ | $z_1 = -\sin x (x \cos x) = -z_2 z_3 = -z_1 z_4$ | $\dot{z}_2 = -z_2 z_3$ |
| $z_3 = \sin x$ | $\dot{z}_3 = -x z_2 z_3 = -z_2 z_4$ | $\dot{z}_3 = -z_4$ |
| $z_4 = x z_3$ | $\dot{z}_4 = \cos x (x \cos x) = z_1 z_2$ | $\dot{z}_4 = z_1 z_2 + z_4^2$ |

### Fig. 1

Phase portrait of the dynamical system in (19)–(20). The symbol $\ast$ denotes fixed points, and $\bullet$ denotes the initial states, $(\delta_0, \omega_0)$, in cases 1 to 4 in Fig. 3. The rectangle centered at the origin indicates the region where trajectories were sampled to compute EDMD. The dotted line $\cdots$ delineates the attractor.

$$\dot{\delta} = \omega,$$

$$\dot{\omega} = \frac{1}{M} \left( k_1 + k_2 \cos \delta + k_3 \sin \delta - \frac{D}{\omega_s} \omega \right).$$  \hfill (20)

Details of the power system model (19)–(20) are provided in Appendix A, and its phase portrait is shown in Fig. 1. Trajectories with starting point $(\delta_0, \omega_0)$ in the lattice $\delta = (-0.50 : 0.25 : 0.50), \omega = (-1.00 : 0.25 : 1.00)$, are sampled at each $\Delta t = 0.005$ second. This sampling rate is consistent with the available technology for power system measurement devices, namely phasor measurement units. The lattice is indicated in Fig. 1 by the blue rectangle centered at the origin, and it contains 45 starting points, thereby leading to 45 sampled trajectories. Note that all the trajectories in the lattice are in a linear region of the state space. We record the initial 0.8 second of each trajectory, thereby leading to 160 samples per trajectory. These trajectories are used to compute EDMD.

Now, define $z_1 := \delta, z_2 := \omega, z_3 := \sin \delta$ and $z_4 := \cos \delta$. By applying the procedure outlined in Section III, we have

$$\dot{z}_1 = z_2,$$

$$\dot{z}_2 = \frac{1}{M} \left( k_1 + k_2 z_4 + k_3 z_3 - \frac{D}{\omega_s} z_2 \right),$$

$$\dot{z}_3 = L_f \sin \delta = \frac{\partial \sin \delta}{\partial \delta} \dot{\delta} + \frac{\partial \sin \delta}{\partial \omega} \dot{\omega} = z_2 z_4,$$

$$\dot{z}_4 = L_f \cos \delta = \frac{\partial \cos \delta}{\partial \delta} \dot{\delta} + \frac{\partial \cos \delta}{\partial \omega} \dot{\omega} = -z_2 z_3,$$

and (21)–(24) contain only monomials in $z$. The lifted dynamical system in $z$ suggests the use of the following observable functions, $[z_1 \ z_2 \ z_3 \ z_4 \ z_2 z_3 \ z_2 z_4]^T$, which yields to:

$$g = [\delta \omega \sin \delta \cos \delta \sin \delta \cos \delta]^T.$$  \hfill (25)

We compute EDMD using the observable functions given by (25), which we refer to as EDMD-Lie. Additionally, for comparison, we compute EDMD with other sets of observable functions that have been widely used in the literature. In what follows, EDMD-pN denotes the case where all monomials of the state variables up to degree $N$ are used. For example, in the case of EDMD-p3:

$$g = [\delta \omega \delta \omega^2 \delta^2 \omega \delta \omega^2 \delta^3 \omega^3]^T.$$  \hfill (26)
Fig. 2. Example of a trajectory starting at $(\delta_0, \omega_0) = (-0.50, -0.75)$, i.e.,
within the sampled region surrounding the fixed point $(0, 0)$ where trajectories were sampled to compute EDM. The system mode is mildly damped, $D = \frac{76}{3}$, $\lambda_{true} = -0.5000 \pm j8.8503$, $f_{true} = 1.4086$Hz, and $\xi_{true} = 5.6406\%$.

We consider using all monomials of the state variables up to degree 2, 3, and 4, respectively denoted by EDM-p2, EDM-p3, and EDM-p4. Further, the use of radial basis functions is also common in the literature. EDM-rbfN denotes the case where the state variables, $(\delta, \omega)$, plus $N^{-2}$ thin-plate spline radial basis functions with center at $x_c$:

$$g(x) = \|x - x_c\|^2 \log(\|x - x_c\|), \quad (27)$$

are used [15].

We consider two cases with radial basis functions. The first case, EDM-rbf6, is designed to have the same size of the Lie lifted system. This will allow us to make a fair comparison between EDM-Lie and the case where radial basis functions is used. The second case, EDM-rbf19, is designed to exploit the maximum possible number of radial basis functions before the Koopman Kalman filter in Section IV-B becomes unobservable.

By linearizing (19)–(20) around the fixed point, $(0, 0)$, and by computing the eigenvalues of the obtained Jacobian matrix, one finds a pair of complex-conjugate eigenvalues, $\lambda_{true} = -0.5000 \pm j8.8503$, associated with the linear mode of frequency $f_{true} = 1.4086$ Hz and damping ratio $\xi_{true} = 5.6406\%$. For comparison, the eigenvalues estimated through EDM with different sets of observable functions are shown in Table III. The pairs of complex-conjugate eigenvalues associated with the linear mode, referred to as principal eigenvalues, are shaded in blue. A comparison between the dimension of the lifted model and the accuracy of the estimated principal eigenvalues is provided in Table IV. Note that the principal eigenvalues are well approximated in all cases; however, although estimating the linear mode with good numerical accuracy is important, this is only part of the information needed to represent the entire domain of attraction through the approximated Koopman tuples, $\{\lambda, \phi, \nu\}$.

A. Reconstruction and Prediction of Known Trajectories

To further assess the performance of EDM with different sets of observables functions, we use the approximated Koopman tuples to reconstruct the trajectories given as inputs to EDM. In Fig. 2, we show the results obtained with EDM-Lie and EDM-p2. We omit the results obtained with other sets of observable functions because they are quantitatively and qualitatively similar to EDM-Lie. We observe that these trajectories do not pose any challenge to EDM, independent of the choice of observable functions, because they are in a linear region of the state space. In this case, however, it is difficult to assess whether EDM is performing well or simply overfitting the input data.

B. Prediction of Strongly Nonlinear Trajectories by Applying the Koopman Kalman Filter

In theory, the Koopman operator is valid in the entire domain of attraction [27]; hence, the approximation of the Koopman operator via EDM should provide the means to predict, with good numerical accuracy, any trajectory in the same domain of attraction for which EDM was computed. This is when the choice of observable functions plays a key role—for example, it will directly affect the transient stability analysis [28] of electric power grids.

We use the approximated Koopman tuples to predict trajectories that start far from the fixed point and are strongly nonlinear. Further, these trajectories are not used as inputs to EDM in first place. This test will reveal how well the Koopman tuples approximated by EDM are representative of the entire domain of attraction. Four trajectories are selected, with initial states $(\delta_0, \omega_0)$ indicated by $\bullet$ in Fig. 1.

The Koopman tuples estimated through EDM are used to design a robust Koopman Kalman filter (KKF) [22], [29], with real and reactive power measured at the generator terminal. Figs. 3a–3d compare the trajectory obtained with the KKF design with EDM-Lie versus the full nonlinear model, (19)–(20), for each of the selected starting points. To avoid making the plot too crowded, Figs. 3a–3d do not show the trajectories obtained with the KKF design using other sets of observable functions; instead, we calculate the absolute error in $\delta$ and $\omega$ for each case and present their statistics in Table V. Also, the absolute error in $\delta$ is shown in Figs. 3e–3h.

Table V and Fig. 3 show that EDM-Lie has overall better performance, indicating that the proposed analytical approach to ascertain observable functions captures the system dynamics well enough to predict unforeseen scenarios. In this illustration, the availability of the system model enabled us to make an informed decision on the selection of observable

| $\lambda$ | $\phi$ | $\nu$ |
|----------|-------|-------|
| Real part | Imaginary part | $\%$ |
| 5 | 0.68 | 0.99 | 3.70 |
| 6 | 0.22 | 1.13 | 1.36 |
| 7 | 1.14 | 0.01 | 1.15 |
| 8 | 1.0 | 0.02 | 1.00 |
| 9 | 1.18 | 1.22 | 2.42 |
| 10 | 0.48 | 0.91 | 0.54 |

| $\lambda_{true}$ | $f_{true}$ | $\xi_{true}$ |
|------------------|-------------|--------------|
| $-0.5000 \pm j8.8503$ | $1.4086$Hz | $5.6406\%$ |
functions; however, the procedure suggested here based on Lie derivatives can be applied even if the system model is not known a priori. For example, a system identification method based on KOT, e.g., the sparse identification of nonlinear dynamics [12], can be used to identify the elementary functions, \( h(x) \). Then, in a second step, the system with elementary functions can be lifted to improve the performance of EDMD.

V. CONCLUSIONS AND FUTURE WORK

We provide an analytical method to select the observable functions to perform extended dynamic mode decomposition of nonlinear dynamics. This method is particularly attractive for dynamical systems where elementary nonlinear functions are beyond polynomial nonlinearities. The method can be applied to a broad class of nonlinear dynamical systems encountered in many engineering fields.

In future research, we will investigate the application of this method to power system models of increased complexity and size, including wind turbine models and their data-driven control.

APPENDIX A

SINGLE-MACHINE INFINITE BUS SYSTEM

Consider a synchronous generator represented by model 0.0 [30], also referred to as the classical model:

\[
\dot{\delta} = \omega - \omega_s, \\
M \dot{\omega} + \frac{D}{\omega_s} (\omega - \omega_s) = P_m - P_r, 
\]

connected to an infinite bus, as shown in Fig. 4. In (28)–(29), \( \delta \) is the electrical angle related to the rotor mechanical angle, \( \omega \) is the angular velocity of the revolving magnetic field, \( \omega_s = 2\pi f \) is the synchronous angular velocity of the revolving magnetic field, \( f \) is the system frequency, \( M = 2H/\omega_s \) is the inertia constant, \( D \) is the damping constant, \( P_m \) is the mechanical power, and \( P_r \) is the electrical power.

\[
\text{TABLE VI} \quad \text{SYSTEM DATA}^1
\]

| \( R \) | \( X \) | \( V_1 \) | \( V_2 \) | \( P \) | \( f \) |
|---|---|---|---|---|---|
| 0.05 | 0.30 | 1.00 | 0.60 | 0.80 | 0.20 |
| 10 | 5 | 60 |

Fixed point:

Let \( S = P + jQ \) and \( Y = Ye^{j\gamma} = G + jB = \frac{1}{\pi + jX} \). Then:

1Adapted from Example 2.3 in [31]. All values are given in per unit, except \( H \), which is given in MJ/MVA, and \( f \), which is given in Hz.
\[ \bar{S} = \bar{V}_1 \bar{T} = V_1 e^{i \omega_1} (V_2 e^{-i \omega_1} - V_2 e^{i \omega_1}) Y e^{i \gamma} = V_1^2 Y e^{-i \gamma} - V_1 V_2 Y e^{i(\theta - \gamma)} = V_1^2 Y (\cos \gamma - j \sin \gamma) - V_1 V_2 Y (\cos (\theta - \gamma) + j \sin (\theta - \gamma)), \quad (30) \]

\bar{T} \] is the complex-conjugate of the current phasor injected into node 1. Then:

\[ P = \text{Re} \{ \bar{S} \} = \bar{V}_1^2 Y \cos \gamma - V_1 V_2 Y \cos (\theta - \gamma) = V_1^2 Y \cos \gamma - V_1 V_2 Y \cos \gamma \sin \theta + V_1 V_2 Y \sin \gamma \sin \theta = V_1^2 Y - V_1 V_2 Y \cos \theta + V_1 V_2 B \sin \theta. \quad (31) \]

By substituting values into (31) and solving, \( \theta = 0.2243 \). It follows that:

\[ \bar{T} = (\bar{V}_1 - \bar{V}_2) \bar{Y} = 0.7718 e^{j 0.0640}. \]

\[ \bar{E} = E e^{j \delta} = \bar{V}_1 + j X'_1 \bar{Y} = 1.0854 e^{j 0.3651}. \]

Let \( \bar{S}_e = P_e + jQ_e \) and \( \bar{Y}_{eq} = G_{eq} + jB_{eq} = \frac{1}{R + j(X + Y)} \).

Following (30)-(31), we obtain \( P_e = E^2 G_{eq} - E V_2 G_{eq} \cos \delta - E V_2 B_{eq} \sin \delta \). Then:

\[ M \dot{\omega} + \frac{D}{\omega_s} (\omega - \omega_s) = k_1 + c_2 \cos \delta + c_3 \sin \delta, \quad (32) \]

where \( k_1 = P_m - E^2 G_{eq}, c_2 = EV_2 G_{eq}, c_3 = EV_2 B_{eq}, \) and the fixed point \( (\delta_0, \omega_0) = (0.3651, 2\pi f) \).

Shift fixed point to the origin:

Let \( \delta = x_1 + \delta_0 \) and \( \omega = x_2 + \omega_0 \). Then:

\[ \dot{x}_1 = x_2 + \omega_0 - \omega_0 = \dot{x}_2, \]

\[ M \dot{x}_2 + \frac{D}{\omega_s} x_2 = k_1 + c_2 \cos (x_1 + \delta_0) + c_3 \sin (x_1 + \delta_0) = k_1 + c_2 \cos x_1 \cos \delta_0 - c_2 \sin x_1 \sin \delta_0 + c_3 \sin x_1 \cos \delta_0 + c_3 \sin \delta_0 \cos x_1. \quad (33) \]

Finally:

\[ \dot{x}_1 = x_2, \]

\[ M \dot{x}_2 + \frac{D}{\omega_s} x_2 = k_1 + k_2 \cos x_1 + k_3 \sin x_1, \quad (34) \]

where \( k_2 = c_2 \cos \delta_0 + c_3 \sin \delta_0 \) and \( k_3 = c_3 \cos \delta_0 - c_2 \sin \delta_0 \), and the fixed point \( (x_{1f}, x_{2f}) = (0, 0) \). By substituting numerical values, \( k_1 = 0.5667, k_2 = -0.5667, \) and \( k_3 = -2.0843 \).

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