Euclidean Freedman–Schwarz model

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The N=4 gauged SU(2) × SU(1,1) supergravity in four-dimensional Euclidean space is obtained via a consistent dimensional reduction of the N=1, D=10 supergravity on $S^3 \times AdS_3$. The dilaton potential in the theory is proportional to the difference of the two gauge coupling constants, which is due to the opposite signs of the curvatures of $S^3$ and $AdS_3$. As a result, the potential can be positive, negative, or zero – depending on the values of the constants. A consistent reduction of the fermion supersymmetry transformations is performed at the linearized level, and special attention is paid to the Euclidean Majorana condition. A further reduction of the D=4 theory is considered to the static, purely magnetic sector, where the vacuum solutions are studied. The Bogomol’nyi equations are derived and their essentially non-Abelian monopole-type and sphaleron-type solutions are presented. Any solution in the theory can be uplifted to become a vacuum of string or M-theory.

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1 Introduction

Supergravity backgrounds play an important role in the analysis of string theory. Besides genuine fully supersymmetric string vacua, also solutions with partial supersymmetry (p-branes, monopoles etc. [1]) are presently obtaining much consideration, in particular in view of their role in verifying various duality conjectures. Unfortunately, apart from stringy monopoles [2] and related solutions [1] obtained via the heterotic five-brane construction [3] (see also [4, 5]), most of the literature is devoted to solutions with Abelian gauge fields. This is easily understood, since such configurations can often be obtained from the known solutions of the Einstein-Maxwell system, while the non-Abelian sector is much more difficult to study. On the other hand, it is to be expected that also configurations with non-Abelian gauge fields will eventually play an important role. Apart from this, gauged supergravity models obtainable from string or M-theory via the Kaluza-Klein reduction have recently regained considerable interest in view of the AdS/CFT correspondence; see for example [6, 7, 8]. This also suggests studying classical solutions of supergravities with non-Abelian gauge fields. Finally, systems with gravitating Yang-Mills fields can be studied in the context of General Relativity, where they have recently attracted a lot of attention in view of the unusual properties of their solutions [9]. Unfortunately, due to the high complexity of the equations, our knowledge is largely based on numerical analysis. At the same time, gauged supergravities provide the rare opportunity to obtain analytical solutions via solving the Bogomol’nyi equations.

The present work was motivated by the desire to analytically obtain certain particle-like solutions for gravitating Yang-Mills fields, which requires to identify the corresponding gauged supergravity model. In the recent work [10, 11] non-Abelian partially supersymmetric vacua were obtained within the N=4 gauged SU(2)×SU(2) supergravity, also known as Freedman–Schwarz (FS) model [12]. These solutions are globally regular, but not asymptotically flat – due to the presence of the dilaton potential,

\[ U(\phi) = -\frac{1}{8} \left( g_{(1)}^2 + g_{(2)}^2 \right) e^{-2\phi}, \]

where \( g_{(1)} \) and \( g_{(2)} \) are the gauge coupling constants. It is well-known that a dilaton potential unbounded from below is generically present in gauged supergravities (see, however, Ref.[13]). The potential renders solutions non-
asymptotically flat, and it is therefore necessary to get rid of it if one wants to obtain particle-like solutions. For this purpose the following trick was employed in [14]: to truncate the FS model to the purely magnetic sector and then to pass to imaginary values of the gauge coupling constant \( g(2) \):

\[
g(2) \rightarrow ig(2).
\]

(1.2)

For \(|g(2)| = g(1)|\) the potential vanishes. Surprisingly, such a formal trick does not destroy supersymmetry – the replacement (1.2) in the FS fermion supersymmetry transformations leads to non-trivial Bogomol’nyi equations. These admit asymptotically flat solutions. As the existence of Bogomol’nyi equations is usually related to supersymmetry, it was conjectured in [14] that there is another, hitherto unknown consistent gauged supergravity that can be formally related to the FS model via the replacement (1.2). The justification of this conjecture is the main subject of the present paper.

To understand what the new supergravity is, let us remember that the FS model can be obtained via dimensional reduction of the N=1, D=10 supergravity on the group manifold SU(2) \(\times\) SU(2) [11]. Now, the replacement (1.2) suggests considering another reduction of the same theory: on the group manifold SU(2) \(\times\) SU(1,1). Since SU(1,1) is non-compact and its invariant metric is non-positive definite, the timelike coordinate of the ten-dimensional space should be viewed as one of the internal coordinates. Specifically, in order to match the metric signature in ten dimensions, one chooses a positive Cartan metric for the SU(2) factor and a negative one for the SU(1,1) factor. The geometry on the internal six-space is then described by the standard metric on \( S^3 \times \text{AdS}_3 \) with the signature \((+++---)\), while the remaining four-space becomes Euclidean.

The FS dilaton potential (1.1) arises upon reduction as the contribution of the scalar curvatures of the internal manifolds (and also due to the D=10 three-form). Since the scalar curvatures for the two \( S^3 \) factors are positive, the radii of the spheres being \( 1/g(1) \) and \( 1/g(2) \), the result is proportional to the sum \( g^2(1) + g^2(2) \). Now, in the case of the reduction on \( S^3 \times \text{AdS}_3 \) the scalar curvatures for these two factors have different signs. As a result, choosing again \( 1/g(1) \) and \( 1/g(2) \) to be the (real) radii of the internal manifolds, the dilaton potential of the resulting Euclidean theory is proportional to the difference \( g^2(1) - g^2(2) \).

To summarize, the new theory appears to be an N=4 gauged supergravity in four-dimensional Euclidean space. We call it Euclidean Freedman-Schwarz...
(EFS) model. Its matter content is similar to the one of the FS model, but the gauge group is now $SU(2) \times SU(1,1)$. The dilaton potential can be positive, negative or zero – depending on the values of the two gauge coupling constants. This allows one to study various supersymmetric solutions for gravitating Yang-Mills fields by integrating the first order Bogomol’nyi equations.

The above qualitative considerations will be confirmed by detailed calculations. In Sec.2 we describe the dimensional reduction procedure in the bosonic sector. We use the general recipes for the reduction on group manifolds given in [15] and, in order to keep control over the calculations at every step, consider the reductions on $S^3 \times AdS_3$ and $S^3 \times S^3$ simultaneously. Specifically, our dimensional reduction ansatz and most of other formulas contain a selective parameter, $s$. The value $s = 1$ corresponds to the reduction on $S^3 \times AdS_3$, while for $s = \sqrt{-1}$ we recover the results of the analysis for the $S^3 \times S^3$ case described in [11]. Since we are interested in obtaining a specific rather than the most general four-dimensional model, we truncate many of the degrees of freedom and always work at the level of equations of motion in order to maintain consistency. The main result of Sec.2 is that a consistent reduction on $S^3 \times AdS_3$ is possible and the equations of motion of the resulting theory in D=4 can be obtained by varying the Lagrangian in Eq. (2.30). If a four-dimensional configuration fulfills these equations then its uplifted ten-dimensional version will be on shell.

In Sec.3 we consider the reduction of the D=10 linearized fermion supersymmetry transformations down to D=4. After discussing the Euclidean Majorana condition we derive the four-dimensional supersymmetry variations in Eqs. (3.27), (3.36). If these variations vanish for a given four-dimensional configuration, then its uplifted version will be supersymmetric in the ten-dimensional sense. This completes the dimensional reduction procedure, as we obtain the bosonic Lagrangian and the fermion supersymmetry transformations, which is sufficient for deriving the Bogomol’nyi equations.

In Sec.4 we apply our results in order to obtain supersymmetric vacua with gravitating Yang-Mills fields by deriving and integrating the Bogomol’nyi equations. First, we consider a further truncation of the theory to the static, purely magnetic sector. It turns out that, in this sector, the field equations and supersymmetry transformations for the FS and EFS models are formally related via the ‘analytic continuation’ (1.2). This provides a complete explanation of the conjecture of Ref.[14]. We then impose a sphe-
ical symmetry and derive the supersymmetry constraints, whose consistency conditions give us a system of first order non-linear Bogomol’nyi equations. Essentially non-Abelian solutions of these equations are known in two special cases \[10, 11, 14\]. For \( g(1) \neq 0, g(2) = 0 \) the solution preserves 1/4 of the supersymmetries and turns out to be of regular monopole type. For \( g(1) = g(2) \neq 0 \) the solution is of sphaleron type and has only 1/8 of the supersymmetries unbroken.

The last section contains some concluding remarks. We use units where \( \hbar = c = 4\pi G = 1 \).

2 Bosons

We start from the bosonic part of the action of D=10, N=1 supergravity

\[
S_{10} = \int \left( \frac{1}{4} \hat{R} - \frac{1}{2} \partial_M \hat{\phi} \partial^M \hat{\phi} - \frac{1}{12} e^{-2\hat{\phi}} \hat{H}_{MNP} \hat{H}^{MNP} \right) \sqrt{-\hat{g}} \, d^{10}\hat{x},
\]

whose equations of motion are

\[
\hat{\nabla}_M \hat{\nabla}^M \hat{\phi} = \frac{1}{6} e^{-2\hat{\phi}} \hat{H}_{MNP} \hat{H}^{MNP},
\]

\[
\hat{\nabla}_M \left( e^{-2\hat{\phi}} \hat{H}^{MNP} \right) = 0,
\]

\[
\hat{R}_{MN} = 2 \partial_M \hat{\phi} \partial_N \hat{\phi} + e^{-2\hat{\phi}} \hat{H}_{MPQ} \hat{H}^{PQ} - \frac{1}{12} e^{-2\hat{\phi}} \hat{g}_{MN} \hat{H}_{PQS} \hat{H}^{PQS}.
\]

Our notation is as follows. The hatted symbols are reserved for D=10 quantities. We shall always use late letters for base space indices and early ones for tangent space indices. Indices in ten, four, and six dimensions are denoted by capital Latin, small Greek, and small Latin letters, respectively, such that \( M \equiv (\mu = 0, \ldots, 3; m = 1, \ldots, 6) \) is the base space index, and \( A \equiv (\alpha = 0, \ldots, 3; a = 1, \ldots, 6) \) is the tangent space index. The spacetime coordinates are \( \hat{x}^M \equiv (x^\mu, z^m) \). We shall sometimes be considering the further split of the four-indices into 3+1 as \( \mu \equiv (0, k) \) and \( \alpha \equiv (0, a) \) with the three-indices denoted by bold-faced letters. The 6-space will be assumed to be a direct product of two three-dimensional group spaces labeled by \( (\sigma) = 1, 2 \) with the indices for each of the internal three-spaces denoted by italic letters. As a result, every 6-index will be replaced by a pair of indices
as \( m \equiv ((\sigma), i) \) and \( a \equiv ((\sigma), a) \), such that \( z^m \equiv z^{(\sigma)i} \), say. Unless explicitly stated, we do not assume summation over repeated indices \((\sigma)\).

The D=10 metric is related to the vielbein, \( \hat{g}_{MN} = \hat{\eta}_{AB} \hat{\Theta}^A_M \hat{\Theta}^B_N \), where

\[
\hat{\eta}_{AB} = \begin{pmatrix}
\eta_{\alpha\beta} & \eta^{(1)}_{\alpha\beta} & \eta^{(2)}_{\alpha\beta} \\
4\text{-space} & 6\text{-space}
\end{pmatrix}
\]  

(2.5)

Here the parameter \( s \) assumes two values: \( s = 1 \) or \( s = i = \sqrt{-1} \). The two options correspond to the same theory in D=10 – up to a renumbering of coordinates – but to two different choices of the four-space. For \( s = i \) the time coordinate in D=10 is \( \hat{x}^0 \) and the four-metric is Lorentzian. For \( s = 1 \) the time is \( \hat{x}^9 \), which is regarded as one of the internal coordinates, and the four-space is Euclidean.

The invariant 1-forms on the six-space are denoted by \( \tilde{\theta}^a \), the invariant vectors being \( \tilde{e}_b \), one has \( \langle \tilde{\theta}^a, \tilde{e}_b \rangle = \delta^a_b \). The tetrads vectors and dual 1-forms for the four-space are \( e_\alpha \) and \( \theta^\beta \), the four-metric being \( g_{\mu\nu} = \eta_{\alpha\beta} \theta^\alpha_\mu \theta^\beta_\nu \) with \( \eta_{\alpha\beta} \) defined in (2.5). The D=10 vielbein vectors are \( \hat{E}_A \), they are dual to the 1-forms \( \hat{\Theta}^B \). Our sign conventions for the Riemann and Ricci tensors are \( \hat{R}^P_{QMN} = \partial_M \hat{\Gamma}^P_Q - \ldots \) and \( \hat{R}_{MN} = \hat{R}^Q_{MQN} \).

2.1 The dimensional reduction ansatz

Our goal is to find a parameterization of \( \hat{g}_{MN}, \hat{H}_{MNP} \) and \( \hat{\phi} \) in terms of four-dimensional variables which reduces Eqs. (2.2)–(2.4) to a consistent system of four-dimensional equations. As a first step, we choose \( \hat{g}_{MN} \) as

\[
ds^2 = e^{-3\phi/2} g_{\mu\nu} dx^\mu dx^\nu + e^{\phi/2} \gamma_{ab} (A^a_\mu dx^\mu - \tilde{\theta}^a_m dz^m)(A^b_\nu dx^\nu - \tilde{\theta}^b_n dz^n). \]  

(2.6)

Here the four-metric \( g_{\mu\nu} \), the four-dilaton \( \phi = -\frac{1}{2} \hat{\phi} \), and the fields \( A^a_\mu \) depend only on \( x^\mu \), while the one-forms \( \tilde{\theta}^a \equiv \tilde{\theta}^a_m dz^m \) are functions of only the internal coordinates \( z^m \). The matrix \( \gamma_{ab} \) is assumed to be constant and diagonal. We assume also that \( z^m \) span a semi-simple (and not necessarily compact) group space \( \mathcal{G} \), and that \( \tilde{\theta}^a \) are the invariant forms. The vectors \( \tilde{e}_a \) dual to \( \tilde{\theta}^a \) satisfy the commutation relations

\[
[\tilde{e}_a, \tilde{e}_b] = f^c_{ab} \tilde{e}_c, \]  

(2.7)
with $f_{ab}^c$ being the structure constants of $\mathcal{G}$. As a result, we can view the metric coefficients $A_{\mu}^a$ as a four-dimensional Yang-Mills field for the gauge group $\mathcal{G}$. The gauge field tensor is

$$F_{\mu\nu}^a = \partial_{\mu}A_{\nu}^a - \partial_{\nu}A_{\mu}^a + f_{abc}^a A_{\mu}^b A_{\nu}^c.$$  

(2.8)

We shall assume the group $\mathcal{G}$ to be the direct product:

$$\mathcal{G} = \mathcal{G}^{(1)} \otimes \mathcal{G}^{(2)}. \quad (2.9)$$

Here $\mathcal{G}^{(1)}=SU(2)$, while there are two options for the second factor in the product, $\mathcal{G}^{(2)} \equiv \mathcal{G}^{(2)}_s$, where $\mathcal{G}^{(2)}_s=SU(1,1)$ for $s = 1$ and $\mathcal{G}^{(2)}_s=SU(2)$ for $s = i$. The direct product structure implies that the invariant group metric, structure constants, etc., decompose into direct sums. For example,

$$f_{ab}^c = f_{ab}^{(1)c} \oplus f_{ab}^{(2)c}, \quad (2.10)$$

where

$$f_{ab}^{(\sigma)c} = \eta^{(\sigma)cd} \varepsilon_{dab}, \quad (2.11)$$

with $\eta_{\sigma}^{(\sigma)}$ ($\sigma = 1, 2$) defined in (2.5) and $\varepsilon_{abc}$ being the antisymmetric tensor ($\varepsilon_{123} = 1$). In addition, we assume that all other quantities that carry internal indices also split into direct sums. For example,

$$A_{\mu}^a = A_{\mu}^{(1)a} \oplus A_{\mu}^{(2)a}, \quad F_{\mu\nu}^a = F_{\mu\nu}^{(1)a} \oplus F_{\mu\nu}^{(2)a}, \quad (2.12)$$

where

$$F_{\mu\nu}^{(\sigma)a} = \partial_{\mu}A_{\nu}^{(\sigma)a} - \partial_{\nu}A_{\mu}^{(\sigma)a} + f_{bc}^{(\sigma)a} A_{\mu}^{(\sigma)b} A_{\nu}^{(\sigma)c}. \quad (2.13)$$

In particular, we choose

$$\gamma_{ab} = \frac{2}{g_{(1)}^2} \eta_{ab}^{(1)} \oplus \frac{2}{g_{(2)}^2} \eta_{ab}^{(2)}, \quad (2.14)$$

with $g_{(1)}$ and $g_{(2)}$ being real constants. As a result, we can replace the six-dimensional indices in all formulas by the three-dimensional ones at the expense of adding the index $(\sigma) = 1, 2$. For example, we shall often write $\gamma_{ab}^{(\sigma)}$ instead of $\gamma_{ab}$.

Each of the two factors in (2.14) is proportional to the Cartan metric for the corresponding group space. We note that for $s = 1$ the proportionality
coefficients have different signs. Specifically, normalizing the Cartan metric as
\[ K_{ab}^{(\sigma)} = -\frac{1}{2} f^{(\sigma)c}_{\ ab} f^{(\sigma)d}_{\ cb}, \] (2.15)
we have
\[ K_{ab}^{(1)} = \eta_{ab}^{(1)}, \quad K_{ab}^{(2)} = -s^{2} \eta_{ab}^{(2)}. \] (2.16)
As a result, the metric for the SU(2) part of the internal space is (proportional to) the corresponding Cartan metric, while the one for the SU(1,1) factor is the negative Cartan metric.

Let us now describe the structure of the vielbein in D=10. We have
\[ \hat{g}_{MN} \, dx^{M} \, dx^{N} = \hat{\eta}^{AB} \hat{\Theta}^{A} \hat{\Theta}^{B} = \eta_{\alpha\beta} \hat{\Theta}^{\alpha} \hat{\Theta}^{\beta} + \sum_{(\sigma)=1,2} \eta_{ab}^{(\sigma)} \hat{\Theta}^{(\sigma)a} \hat{\Theta}^{(\sigma)b}, \] (2.17)
where
\[ \hat{\Theta}^{\alpha} = e^{-3\phi/4} \theta^{\alpha}_{\mu} \, dx^{\mu}, \quad \hat{\Theta}^{(\sigma)a} = \sqrt{2} g^{(\sigma)} \left( A^{(\sigma)a}_{\mu} dx^{\mu} - \tilde{\theta}^{(\sigma)a} \right). \] (2.18)
The dual basis \( \hat{E}_{B} \) is specified by
\[ \hat{g}^{MN} \frac{\partial}{\partial \hat{x}^{M}} \frac{\partial}{\partial \hat{x}^{N}} = \hat{\eta}^{AB} \hat{E}_{A} \hat{E}_{B} = \eta_{\alpha\beta} \hat{E}_{\alpha} \hat{E}_{\beta} + \sum_{(\sigma)=1,2} \eta_{ab}^{(\sigma)} \hat{E}_{a}^{(\sigma)} \hat{E}_{b}^{(\sigma)}, \] (2.19)
where
\[ \hat{E}_{\alpha} = e^{3\phi/4} \left( e_{\alpha} + \sum_{(\sigma)=1,2} e^{(\sigma)}_{\mu} A^{(\sigma)a}_{\mu} \tilde{c}^{(\sigma)}_{a} \right), \quad \hat{E}_{a}^{(\sigma)} \equiv \hat{E}_{a} = -\frac{g^{(\sigma)} e^{-3\phi/4} \tilde{c}^{(\sigma)}_{a}}{\sqrt{2}}. \] (2.20)
Here \( e_{\alpha} \equiv e^{\mu}_{\alpha} \partial / \partial x^{\mu} \) are the basis 4-vectors dual to the \( \theta^{\alpha}_{\mu} \)s, and \( \tilde{c}^{(\sigma)}_{a} \equiv \tilde{c}^{(\sigma)}_{a} \partial / \partial z^{(\sigma)i} \) are the invariant vectors on the group spaces \( G^{(\sigma)} \).

We shall also need explicit expressions for \( \hat{R}_{MN} \). These can be obtained in the standard way from the vielbein connection \( \hat{\omega}_{AB,C} \), which is computed in Eq. (3.31) below. This yields components \( \hat{R}_{AB} = (\hat{R}_{\alpha\beta}, \hat{R}_{\alpha a}, \hat{R}_{ab}) \) in the basis (2.20):
\[ e^{-3\phi/2} \hat{R}_{\alpha\beta} \theta^{\alpha}_{\mu} \theta^{\beta}_{\nu} = R_{\mu\nu} - \frac{3}{2} \partial_{\mu} \phi \partial_{\nu} \phi + \frac{3}{4} g_{\mu\nu} \nabla_{\rho} \nabla^{\rho} \phi - \frac{1}{2} e^{2\phi} \gamma_{ab} F_{\mu\rho}^{a} F_{\nu}^{b \rho}, \]
\[
\frac{\sqrt{2}}{g_{(a)}^2} e^{-\phi/2} \hat{R}_{\alpha\mu} = -\frac{1}{2} \gamma_{ab} \left( \nabla^b \left( e^{2\phi} F^{b\rho}_\mu \right) + e^{2\phi} f^{b}_{cde} A^c_{\rho} F^{d\mu}_\rho \right), \quad (2.21)
\]
\[
8 \frac{\sqrt{2}}{g_{(a)}^2} e^{2\phi/2} \hat{R}_{ab} = -e^{2\phi} \gamma_{ab} \nabla^c \nabla^d \phi + e^{4\phi} \gamma_{ab} \gamma_{cd} F^c_{\mu\nu} F^{d\mu\nu} + f_{abc} f_{da},
\]
where \( \nabla_\rho \) and \( R_{\mu\nu} \) are the covariant derivative and Ricci tensor for \( g_{\mu\nu} \), and \( g_{(a)} \equiv g(\sigma) a = g(\sigma) ).

So far we have expressed the ten-dimensional \( \hat{g}_{MN} \) and \( \hat{\phi} \) in terms of the four-metric \( g_{\mu\nu} \), the dilaton \( \phi \), and the gauge fields \( A_{\mu}^{(\sigma) a} \). It remains to specify the ten-dimensional antisymmetric tensor \( \hat{H}_{MNP} \). We choose its non-vanishing components in the basis \( (2.20) \) to be 

\[
\hat{H}^{(\sigma)}_{abc} = g(\sigma) e^{-3\phi/4} \varepsilon_{abc},
\]
\[
\hat{H}^{(\sigma)}_{a\alpha\beta} = -\frac{1}{\sqrt{2}} g(\sigma) e^{5\phi/4} \eta^{(\sigma)}_{ab} F^{(\sigma)b}_{a\alpha\beta},
\]
\[
\hat{H}_{\alpha\beta\gamma} = e^{-7\phi/4} \varepsilon_{\alpha\beta\gamma} \delta e^\mu_\delta \partial_\mu a. \quad (2.22)
\]

Here the axion \( a \) depends only on \( x^\mu \), we choose \( \varepsilon^{0123} = 1 \), and \( F^{(\sigma)b}_{a\alpha\beta} \) are components of the gauge field strength with respect to the tetrad \( e_\alpha \).

### 2.2 Four-dimensional theory

We now have the complete ansatz for \( \hat{g}_{MN} \), \( \hat{H}_{MNP} \) and \( \hat{\phi} \), and this we insert into the supergravity equations \( (2.2) \)–\( (2.4) \). Let us consider first Eq. \( (2.2) \) for the dilaton \( \hat{\phi} \). Using the above definitions, it is not difficult to see that this equation assumes the following four-dimensional form:

\[
\nabla_\rho \nabla^\rho \phi = \frac{1}{2} e^{2\phi} \sum_{(\sigma)=1,2} \frac{1}{g_{(\sigma)}^2} \eta_{ab}^{(\sigma)} F^{(\sigma)a}_{\mu\nu} F^{(\sigma)b\mu\nu} + 2s^2 e^{-4\phi} \partial_\mu a \partial^\mu a - 2U(\phi), \quad (2.23)
\]

with the dilaton potential

\[
U(\phi) = -\frac{1}{8} \left( g_{(1)}^2 - s^2 g_{(2)}^2 \right) e^{-2\phi}. \quad (2.24)
\]

The next step is to check Eq. \( (2.3) \) for \( \hat{H}_{MNP} \). This is in fact a system of 49 equations labeled by pairs of indices \((M,N)\), and it can be split into three
groups labeled by \((\mu, \nu), (a, b),\) and \((\mu, a)\), respectively. A direct computation then reveals that the \((\mu, \nu)\) and \((a, b)\) equations are identically fulfilled, while the \((\mu, a)\) ones reduce to

\[
\nabla_\mu (e^{2\phi} F^{(\sigma)a\rho\mu}) + e^{2\phi} f^{(\sigma)a}_{bc} A_\rho^{(\sigma)b} F^{(\sigma)c\rho\mu} = 2 * F^{(\sigma)a\mu\rho} \partial_\mu a. \tag{2.25}
\]

These are the four-dimensional Yang-Mills equations; the dual tensor is defined as

\[
*F^{(\sigma)a}_{\mu\nu} = \frac{1}{2} \sqrt{|g|} \varepsilon_{\mu\nu\lambda\rho} F^{(\sigma)a}_{\lambda\rho}.
\]

Let us now turn to the Einstein equations in (2.4). Splitting these into the \((\mu, \nu), (a, b)\) and \((\mu, a)\) groups and using the expressions for \(\hat{R}_{MN}\) in (2.21) together with all the above definitions, we find that the \((a, b)\) equations are identically fulfilled, while the \((\mu, a)\) ones again reduce to the Yang-Mills equations (2.25). The \((\mu, \nu)\) group gives the four-dimensional Einstein equations:

\[
R_{\mu\nu} = 2 \partial_\mu \phi \partial_\nu \phi - 2 s^2 e^{-4\phi} \partial_\mu a \partial_\nu a + 2 T_{\mu\nu} + 2 U(\phi) g_{\mu\nu}, \tag{2.26}
\]

with the Yang-Mills energy-momentum tensor

\[
T_{\mu\nu} = e^{2\phi} \sum_{(\sigma)=1,2} \frac{1}{g^{(\sigma)}} \eta^{(\sigma)}_{ab} \left( F^{(\sigma)a}_{\mu\rho} F^{(\sigma)b\rho}_{\nu} - \frac{1}{4} g_{\mu\nu} F^{(\sigma)a}_{\lambda\rho} F^{(\sigma)b\lambda\rho} \right). \tag{2.27}
\]

At this point our procedure successfully terminates, since all ten-dimensional equations are now fulfilled, provided that the four-dimensional dilaton, Yang-Mills, and Einstein equations in (2.23)–(2.27) hold. We note, however, that so far we have not obtained the equation for the axion \(a\). This arises as the consistency condition for the Einstein equations (2.26). Specifically, the Bianchi identities for (2.26) are fulfilled by virtue of the dilaton and Yang-Mills equations together with

\[
\nabla_\rho \nabla^\rho (e^{-4\phi} a) = \frac{s^2}{2} \sum_{(\sigma)=1,2} \frac{1}{g^{(\sigma)}} \eta^{(\sigma)}_{ab} * F^{(\sigma)a}_{\mu\nu} F^{(\sigma)b\mu
u}. \tag{2.28}
\]

This completes the system of four-dimensional equations. It is not difficult to see that all equations in (2.23)–(2.28) can be obtained by varying the four-dimensional action

\[
S_4 = \int L_4 \sqrt{|g|} \, d^4x, \tag{2.29}
\]
where

\[ L_4 = \frac{R}{4} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{s^2}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{-4\phi} \partial_\mu \mathbf{a} \partial^\mu \mathbf{a} - \frac{1}{4} e^{2\phi} \sum_{(\sigma)=1,2} \frac{1}{g_{(\sigma)}^2} \eta_{ab}^{(\sigma)} F^{(\sigma)\mu a} F^{(\sigma)\mu b} \]

\[ - \mathbf{a} \sum_{(\sigma)=1,2} \frac{1}{g_{(\sigma)}^2} \eta_{ab}^{(\sigma)} \ast F^{(\sigma)\mu a} F^{(\sigma)\mu b} + \frac{1}{8} \left( g_{(1)}^2 - s^2 g_{(2)}^2 \right) e^{-2\phi}. \]  

(2.30)

We have therefore obtained a four-dimensional theory via the consistent dimensional reduction of ten-dimensional supergravity. In fact, we have obtained two different theories distinguished by the values of the parameter \( s = 1, i \). Let us recall that the signature of the spacetime metric is \((s^2, +1, +1, +1)\), the internal metrics are \( \eta_{ab}^{(2)} = \text{diag}(+1, +1, -s^2) \) and \( \eta_{ab}^{(1)} = \delta_{ab} \), and the field tensors \( F^{(\sigma)\mu a} \) are specified by Eqs. (2.11), (2.13).

The model (2.30) describes interacting gravitational, axion, dilaton, and two non-Abelian gauge fields with gauge group \( G \) and two independent gauge coupling constants \( g_{(1)} \) and \( g_{(2)} \). For \( s = i \) the gauge group is \( SU(2) \times SU(2) \), and the theory coincides with the bosonic sector of the gauged supergravity of Freedman and Schwarz [12]. We have therefore reproduced the result of Ref. [11] that the Freedman-Schwarz model can be obtained via dimensional reduction of D=10, N=1 supergravity on \( S^3 \times S^3 \). The inverse radii of the spheres determine the gauge coupling constants \( g_{(\sigma)} \).

The principal new result that we obtain here is the model determined by (2.30) for \( s = 1 \). This has been hitherto unknown. It is somewhat similar to the Freedman-Schwarz model, apart from the fact that it lives in Euclidean and not Lorentzian space, also the gauge group is now \( SU(2) \times SU(1,1) \), and the dilaton potential is proportional to the difference and not to the sum of the coupling constants. The latter is due to the opposite signs of the scalar curvatures of the group manifolds used for the dimensional reduction: \( SU(2) \) with the positive metric and \( SU(1,1) \) with the negative metric.

We call the new theory Euclidean Freedman-Schwarz (EFS) model. Some of its features are as follows. We notice that the kinetic terms for the dilaton and axion in (2.30) have opposite signs, which is typical for Euclidean theories. We notice also that the non-compact Lie-algebra components of \( F^{(2)\mu a} \) \( (a = 1, 2) \) give positive contributions to the energy density, while the compact one \( (a = 3) \) makes a negative contribution. This is because the metric for bilinear combinations of the gauge field strength is not the Cartan metric for \( SU(1,1) \) but its negative, \( \eta_{ab}^{(2)} \). So far the new supergravity is not yet
complete, since we have described only its bosonic sector. We shall now pass to considering the fermions, in which we shall restrict ourselves to deriving the linearized fermion supersymmetry transformations.

3 Fermions

The fermion fields of D=10, N=1 supergravity, the gravitino $\hat{\psi}_M$ and the gaugino $\hat{\chi}$, can be consistently set to zero, which leads to the action (2.1). However, their supersymmetry variations do not necessarily vanish and are given by

$$\delta \hat{\psi}_M = \hat{D}_M \hat{\epsilon} - \frac{1}{48} e^{-\hat{\phi}} \left( \hat{\Gamma}^{S\hat{P}\hat{Q}}_M + 9 \delta^S_M \hat{\Gamma}^{P\hat{Q}} \right) \hat{H}_{S\hat{P}\hat{Q}} \hat{\epsilon},$$

$$\delta \hat{\chi} = -\frac{1}{\sqrt{2}} (\hat{\Gamma}^M \partial_M \hat{\phi}) \hat{\epsilon} - \frac{1}{12\sqrt{2}} e^{-\hat{\phi}} \hat{\Gamma}^{S\hat{P}\hat{Q}} \hat{H}_{S\hat{P}\hat{Q}} \hat{\epsilon}. \quad (3.2)$$

Here $\hat{\epsilon}$ is the Majorana-Weyl spinor parameter of supersymmetry transformations, and its covariant derivative is

$$\hat{D}_M \hat{\epsilon} = \left( \partial_M + \frac{1}{4} \hat{\omega}_{AB,M} \hat{\Gamma}^{AB} \right) \hat{\epsilon}, \quad (3.3)$$

with $\partial_M \equiv \partial / \partial \hat{x}^M$ and $\hat{\omega}_{AB,M}$ being the spin connection for the vielbein $\hat{E}_A$.

The D=10 gamma matrices span the Clifford algebra $\hat{\Gamma}^A \hat{\Gamma}^B + \hat{\Gamma}^A \hat{\Gamma}^B = 2 \hat{\eta}^{AB}$; one has $\hat{\Gamma}^M \equiv \hat{\Gamma}^A \hat{E}_A^M$ and $\hat{\Gamma}^{M\ldots N} \equiv \hat{\Gamma}^{[M \ldots} \hat{\Gamma}^{N]}$.

We shall now proceed as in the bosonic case to express the ten-dimensional quantities in terms of the four-dimensional ones. Our aim is to consistently derive the four-dimensional supersymmetry transformations from the ten-dimensional rules (3.1),(3.2). The vanishing of the four-dimensional SUSY variations will then imply that the ten-dimensional variations $\delta \hat{\psi}_M$ and $\delta \hat{\chi}$ vanish.

3.1 The D=10 gamma matrices

We parameterize the $32 \times 32$ gamma matrices $\hat{\Gamma}^A \equiv (\hat{\Gamma}^\alpha, \hat{\Gamma}^{(1)a}, \hat{\Gamma}^{(2)a})$ as

$$\begin{align*}
\hat{\Gamma}^\alpha &= \mathbb{1}_2 \otimes \gamma^\alpha \otimes \mathbb{1}_4, \\
\hat{\Gamma}^{(1)a} &= i \tau^1 \otimes \gamma_5 \otimes \alpha^{(1)a}, \\
\hat{\Gamma}^{(2)a} &= \frac{1}{s} \tau^3 \otimes \gamma_5 \otimes \alpha^{(2)a}.
\end{align*} \quad (3.4)$$
Here as usual $s = 1, i$; $\tau^a$ are the Pauli matrices, $\gamma^\alpha$ are the D=4 gamma matrices, and the $4 \times 4$ matrices $\alpha^{(\sigma)a}$ generate the Lie algebra of the group $G$. One has

$$\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2 \eta^{\alpha \beta} \equiv 2 \text{diag}(s^2, +1, +1, +1).$$  \hspace{1cm} (3.5)$$

Since $\varepsilon^{0123} = 1$ and $\varepsilon_{0123} = s^2$, and also $\sqrt{-\eta} = is$, we have

$$\gamma_5 = \frac{i}{4!} \sqrt{-\eta} \varepsilon_{\alpha \beta \gamma \delta} \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta = -\frac{1}{s} \gamma^0 \gamma^1 \gamma^2 \gamma^3,$$  \hspace{1cm} (3.6)$$

such that $\gamma_5^2 = 1$ and $\{\gamma_5, \gamma^\alpha\} = 0$. The matrices $\alpha^{(\sigma)a}$ in (3.4) are specified by the relations

$$\alpha^{(\sigma)a} \alpha^{(\sigma)b} = -\varepsilon^{abc} \eta_{cd}^{(\sigma)} \alpha^{(\sigma)d} - K^{(\sigma)ab},$$

$$(\alpha^{(\sigma)a})^\dagger = -K^{(\sigma)ab} \delta_{bc} \alpha^{(\sigma)c},$$

$$[\alpha^{(1)a}, \alpha^{(2)b}] = 0,$$  \hspace{1cm} (3.7)$$

where $\varepsilon^{abc} = \varepsilon_{abc}$ and the Cartan metric $K^{(\sigma)ab}$ is defined in Eq. (2.15). For $T^{(\sigma)}_a = -\frac{1}{2} \delta_{ab} \alpha^{(\sigma)b}$ we have

$$[T^{(\sigma)}_a, T^{(\sigma)}_b] = \varepsilon_{abc} \eta^{(\sigma)cd} T^{(\sigma)}_d,$$  \hspace{1cm} (3.8)$$

which are the commutation relations for the Lie algebra of $G$.

It is not difficult to see that Eqs. (3.4)–(3.7) imply the correct Clifford algebra relations for the $\hat{\Gamma}^A$’s. Although the actual choice of $\gamma^\alpha$ and $\alpha^{(\sigma)a}$ is not important, it is sometimes convenient to have an explicit representation. One can choose

$$\gamma^0 = s \sigma^1 \otimes \mathbb{I}_2, \quad \gamma^\alpha = \sigma^2 \otimes \sigma^\alpha, \quad \gamma_5 = \sigma^3 \otimes \mathbb{I}_2,$$  \hspace{1cm} (3.9)$$

and also

$$\alpha^{(1)a} = i \tau^a \otimes \mathbb{I}_2; \quad \alpha^{(2)b} = s \mathbb{I}_2 \otimes \tau^b \quad (b = 1, 2), \quad \alpha^{(2)3} = i \mathbb{I}_2 \otimes \tau^3,$$  \hspace{1cm} (3.10)$$

with $\sigma^a$ and $\tau^a$ being Pauli matrices.
3.2 The Majorana-Weyl condition

Having expressed the $\hat{\Gamma}^A$’s in terms of four-dimensional quantities, we need a similar reduction also for the spinors. It is important that the spinors in Eqs. (1.1), (1.2) are Majorana-Weyl with $\delta \hat{\psi}_S$ and $\hat{\epsilon}$ being right handed, while $\delta \hat{\chi}$ is left-handed [16]. Upon dimensional reduction the $D=10$ Majorana-Weyl condition will reduce to Majorana-type constraints for the $D=4$ spinors, which we shall discuss in some detail, especially in the Euclidean case.

Let us first consider the Weyl condition. As is well-known, this can be imposed in any even-dimensional space. In the particular case of $D=10$ one defines the chirality matrix as

$$\hat{\Gamma}_{11} = -\hat{\Gamma}^0 \hat{\Gamma}^1 \ldots \hat{\Gamma}^9 = \tau^2 \otimes \gamma_5 \otimes \mathbb{1}_4,$$

(3.11)

the Weyl spinors $\hat{\psi}_\pm$ being solutions of

$$\hat{\Gamma}_{11} \hat{\psi}_\pm = \pm \hat{\psi}_\pm.$$  

(3.12)

Let us now consider the Majorana condition. This is defined by the matrix $\hat{B}$ subject to

$$\hat{B} \hat{\Gamma}^A \hat{B}^{-1} = (\hat{\Gamma}^A)^*,$$

(3.13)

where the asterisk denotes complex conjugation. Such a matrix always exists, since $\hat{\Gamma}^A \rightarrow (\hat{\Gamma}^A)^*$ is a symmetry of the Clifford algebra. This implies that for any spinor $\hat{\psi}$ in any dimension $D$ and for an arbitrary spacetime signature one can define the conjugated spinor $\hat{\psi}_M \equiv \hat{B}^{-1} \hat{\psi}^*$, which transforms as $\hat{\psi}$. What is not always possible is to choose $\hat{B}$ such that

$$\hat{B} \hat{B}^* = 1.$$  

(3.14)

As a result, the Majorana condition, $\hat{\psi}_M = \hat{\psi}$, or explicitly

$$\hat{\psi}^* = \hat{B} \hat{\psi},$$

(3.15)

is not always consistent, since it requires that (3.14) must hold. For example, for $D=4$ the solution to (3.13), (3.14) exists in Minkowski space but not in Euclidean space. In those cases where the Majorana condition can be imposed, it is not always compatible with the Weyl condition, since chirality may change under the Majorana conjugation.
As is well-known, in ten-dimensional space with the signature (1,9) the
Majorana condition can be imposed and is compatible with the Weyl con-
dition. Let us see this explicitly. In the representation (3.4) one has
\[ \hat{B} = \Omega \otimes A \otimes B, \]  
(3.16)
where
\[ \Omega = \begin{pmatrix} 1 & 0 \\ 0 & -\gamma_5 \end{pmatrix}, \]  
(3.17)
and the 4 \times 4 matrices \( A \) and \( B \) are such that
\[ A\gamma^\alpha A^{-1} = (\gamma^\alpha)^*, \quad B\alpha^{(\sigma)a} B^{-1} = (\alpha^{(\sigma)a})^*, \]  
(3.18)
and also
\[ AA^* \otimes BB^* = \mathbb{1}. \]  
(3.19)
It follows that \( A\gamma_5 A^{-1} = \gamma_5 \gamma_5 \). The explicit form of \( A \) and \( B \) depends on the
representation of \( \gamma^\alpha \) and \( \alpha^{(\sigma)a} \). In the representation (3.9), (3.10) one has
\[ A = \mathbb{1}_2 \otimes \sigma_2, \quad B = \tau^2 \otimes \tau^1 \]  
(3.20)
for \( s = 1 \), while for \( s = i \) one finds
\[ A = \sigma_2 \otimes \sigma_2, \quad B = \tau^2 \otimes \tau^2. \]  
(3.21)
Using these definitions one can see that the Weyl condition in (3.11), (3.12)
and the Majorana condition in (3.15) can be solved simultaneously as
\[ \hat{\psi}_\pm = \begin{pmatrix} \psi \\ \pm i\gamma_5 \psi \end{pmatrix}, \]  
(3.22)
with
\[ \psi^* = A \otimes B \psi. \]  
(3.23)
Here \( \hat{\psi} \) is written in the form of a two-component spinor which is acted
upon by the 2 \times 2 bold-faced matrices like \( \mathbb{1}_2, \tau^a \) and \( \Omega \). The spinor \( \psi \)
has 16 components, which are acted upon by the 4 \times 4 gamma-matrices \( \gamma^\alpha \)
and group-generators \( T^{(\sigma)}_a \). One can write \( \hat{\psi} \equiv \psi^I_\kappa \), where \( I = 1, \ldots, 4 \) is the
group index and \( \kappa = 1, \ldots, 4 \) is the spinor index. In view of (3.23) only
8 components of \( \psi \) are independent. We notice that the condition (3.23) is
invariant under 4-rotations generated by $\frac{1}{4}[\gamma^{\alpha}, \gamma^{\beta}]$ and gauge transformations generated by $T_{a}^{(\sigma)}$. As a result, $\psi$ can be viewed as a $G$-group multiplet of Majorana spinors in D=4 with the Majorana condition given by (3.23). Eqs. (3.22), (3.23) therefore provide the sought expression for the ten-dimensional spinors in terms of the four-dimensional ones.

The Majorana condition (3.23) is obtained for both values of the parameter $s$. One might think that for $s = 1$, when the space is Euclidean, this contradicts the well-known fact that there are no Majorana spinors in four-dimensional Euclidean space. However, there is no contradiction, since the spinors have internal degrees of freedom. Specifically, the normalization (3.13), which is the analog of (3.14), holds because there are two factors on the left hand side of (3.19), which satisfy

$$AA^{\ast} = -s^{2}, \quad BB^{\ast} = -s^{2}.$$  (3.24)

For $s = 1$ each of the two terms here has the wrong sign, but two wrongs make a right – their product in (3.19) has the correct sign. Without the matrix $B$ one would be left with just one wrong sign, and this implies that singlet fermions cannot be Majorana. To recapitulate, the Euclidean Majorana condition (3.23) is consistent due to the group degrees of freedom. We also note that, since $A\gamma_{5}A^{-1} = + (\gamma_{5})^{\ast}$ for $s = 1$, the Majorana spinors can be at the same time Weyl.

For $s = i$ each of the two factors in (3.19) has the correct sign on its own. In particular, for $s = i$ one can choose all the $\alpha^{(\sigma)a}$’s to be real [12], in which case $B = 1$. As a result, the Majorana condition can be imposed both for group singlets and multiplets. However, since $A\gamma_{5}A^{-1} = - (\gamma_{5})^{\ast}$, spinors cannot at the same time be Majorana and Weyl.

### 3.3 Four-dimensional SUSY variations

We now have all necessary tools in order to reduce the D=10 SUSY transformations in (3.1), (3.2) to four dimensions. Let us first consider Eq. (3.2) for $\delta \hat{\chi}$. The spinors $\delta \hat{\chi}$ and $\hat{\epsilon}$ have left and right chiralities, respectively, and we therefore choose according to (3.22)

$$\delta \hat{\chi} = -\frac{1}{2}e^{5\phi/8}\begin{pmatrix} \delta \chi \\ -i\gamma_{5} \delta \chi \end{pmatrix}, \quad \hat{\epsilon} = e^{-\phi/8}\begin{pmatrix} \epsilon \\ i\gamma_{5} \epsilon \end{pmatrix}.$$  (3.25)
Inserting this into (3.2), using the $\Gamma^A$'s from (3.4) and $\hat{H}_{ABC}$ from (2.22), and also utilizing the identity
\[
\gamma_\alpha \gamma_5 = \frac{i}{6} \sqrt{-\eta} \varepsilon_{\alpha\beta\gamma} \gamma^\beta \gamma^\gamma \gamma^\delta ,
\]  
(3.26)
Eq. (3.2) reduces to the following relation between $\delta \chi$ and $\epsilon$:
\[
\delta \chi = \left( \frac{1}{\sqrt{2}} \gamma^\mu \partial_\mu \phi - \frac{1}{\sqrt{2}s} e^{-2\phi} \gamma_5 \gamma^\mu \partial_\mu a \right) \epsilon + \frac{1}{2s} e^\phi \left( s \mathcal{F}^{(1)} - \gamma_5 \mathcal{F}^{(2)} \right) \epsilon + \frac{1}{4s} e^{-\phi} \left( s g^{(1)} - g^{(2)} \gamma_5 \right) \epsilon ,
\]  
(3.27)
with
\[
\mathcal{F}^{(\sigma)} = \frac{1}{2g^{(\sigma)}} \eta^{(\sigma)}_{ab} \gamma^a \gamma^b \mathcal{F}^{(\sigma)a}_{ab} \alpha^{(\sigma)b} .
\]  
(3.28)
We note that this relation is compatible with the Majorana condition (3.23) for the spinors $\delta \chi$ and $\epsilon$. Specifically, taking the Majorana conjugate of (3.27) and using (3.18), the whole expression reproduces itself.

Consider now the equation for $\delta \hat{\psi}_M$ in (3.1). The procedure in this case is somewhat more involved. The first step is to compute the spinor covariant derivatives in (3.3), and for this we need the spin-connection $\hat{\omega}_{AB,M}$. This can be obtained as
\[
\hat{\omega}_{AB,C} = \frac{1}{2} \left( C_{B,AC} + C_{C,AB} - C_{A,BC} \right) ,
\]  
(3.29)
where $C_{A,BC} = \hat{\eta}_{ABD} C^{D}_{BC}$ are determined by the commutation relations for the basis vectors of the vielbein (2.20),
\[
[\hat{E}_A, \hat{E}_B] = C^{C}_{AB} \hat{E}_C .
\]  
(3.30)
The result is
\[
\hat{\omega}^{\alpha\beta\gamma} = e^{3\phi/4} \left( \omega^{\alpha\beta\gamma} + \frac{3}{4} (\eta_{\beta\gamma} e^\mu_\alpha - \eta_{\alpha\gamma} e^\mu_\beta) \partial_\mu \phi \right) ,
\]
\[
\hat{\omega}^{(\sigma)}_{\alpha\beta,a} = \hat{\omega}^{(\sigma)}_{\alpha a,\beta} = -\frac{1}{\sqrt{2}g^{(\sigma)}} e^{7\phi/4} \eta^{(\sigma)}_{ab} F^{(\sigma)b}_{a\beta} ,
\]
\[
\hat{\omega}^{(\sigma)}_{\alpha\beta,b} = -\frac{1}{4} e^{3\phi/4} \eta_{ab} e^\mu_\alpha \partial_\mu \phi , \quad \hat{\omega}^{(\sigma)}_{ab,\alpha} = -e^{3\phi/4} \varepsilon_{abc} A^{(\sigma)c} ,
\]
\[
\hat{\omega}^{(\sigma)}_{ab,c} = \frac{g^{(\sigma)}_{2\sqrt{2}} e^{-\phi/4} \varepsilon_{abc} ,}
\]  
(3.31)
where $\omega_{\alpha\beta,\gamma}$ is the spin-connection for the tetrad $e_\alpha$.

Using these expressions together with (3.4) and (2.22) we first consider that part of the SUSY variation $\delta \hat{\psi}_M$ for which the index runs over the internal coordinates, $M=m$. Utilizing the identity
\[
\hat{\Gamma}^{ABC}_D \hat{H}_{ABC} = (-\hat{\Gamma}^D \hat{\Gamma}^{ABC} + 3\delta_D^A \hat{\Gamma}^{BC}) \hat{H}_{ABC},
\]
a straightforward computation gives the relation for the D=10 spinors,
\[
\delta \hat{\psi}_a + \frac{1}{2\sqrt{2}} \hat{\Gamma}_a \delta \hat{\chi} = -\frac{g(a)}{\sqrt{2}} \hat{e}_m \frac{\partial}{\partial z_m} \hat{\epsilon},
\]
where $g(a)$ is defined after Eqs. (2.21) (one has $\delta \hat{\psi}_m = \hat{\Theta}^A_m \delta \hat{\psi}_A = \hat{\Theta}^a_m \delta \hat{\psi}_a$).

Notice the following important fact: the expression on the right contains the partial and not the covariant derivative. Specifically, when simplifying the expression on the left the spin connection arises twice, and a careful examination reveals that the two terms cancel each other. As a result, the expression in (3.33) appears to be not covariant under local rotations of the internal basis $\hat{e}_a$. This, however, is simply a consequence of the fact that the whole theory under consideration does not allow for such rotations. Indeed, the crucial assumption is that the basis $\hat{e}_a$ consists of invariant vector fields, for which only global rotations are allowed. Under these, obviously, Eq. (3.33) is covariant.

Consider now the four-dimensional part of the gravitino, $\delta \hat{\psi}_\mu$. Since $\delta \hat{\psi}_M$ has positive chirality, we consider the linear combination
\[
\delta \hat{\psi}_\mu - \frac{3}{2\sqrt{2}} \gamma_\mu \delta \hat{\chi} \equiv e^{\phi/8} \left( \begin{array}{c} \delta \hat{\psi}_\mu \\ i \gamma_5 \delta \hat{\psi}_\mu \end{array} \right).
\]

Taking into account all the definitions above and making use of the identity
\[
\gamma_\alpha \gamma_\beta \gamma_5 = -\frac{i}{2} \sqrt{-\eta} \epsilon_{\alpha\beta\gamma\delta} \gamma^\gamma \gamma^\delta + \eta_{\alpha\beta} \gamma_5,
\]
the remaining part of the SUSY variation in (3.1) reduces to the following relation:
\[
\delta \hat{\psi}_\mu = \left( \partial_\mu + \frac{1}{4} \omega_{\alpha\beta,\mu} \gamma^\alpha \gamma^\beta - \frac{1}{2} \sum_{(\sigma)=1,2} K^\sigma_{ab} \alpha^{(\sigma)a} A_\mu^{(\sigma)b} + \frac{1}{2s} e^{-\phi} \gamma_5 \partial_\mu a \right) \epsilon \\
+ \frac{1}{2\sqrt{2s}} e^{\phi} \left( s F^{(1)} + \gamma_5 F^{(2)} \right) \gamma_\mu \epsilon + \frac{1}{4\sqrt{2s}} e^{\phi} \left( s g^{(1)} + g^{(2)} \gamma_5 \right) \gamma_\mu \epsilon,
\]
with the Cartan metric $K^{(\sigma)}_{ab}$ from (2.13).

To recapitulate, Eqs. (3.25), (3.27), (3.33), (3.34), and (3.36) provide an equivalent representation of the ten-dimensional SUSY variations $\delta \hat{\chi}$ and $\delta \hat{\psi}_M$, as no truncation of the fermionic degrees of freedom has been done so far. Let us now assume that the parameter $\epsilon$ does not depend on the internal coordinates

\[ \frac{\partial}{\partial z^m} \epsilon = 0. \] 

(3.37)

This is consistent due to the appearance of the partial derivative in (3.33) discussed above, and eventually due to the fact that the internal space is a group manifold. In the case of dimensional reduction on general homogeneous spaces the dependence of spinors on internal coordinates is usually more complicated and is given in terms of Killing spinors on the internal space \[17\].

Let us now suppose that $\delta \chi = \delta \psi_\mu = 0$. Eq. (3.25) then implies that $\delta \hat{\chi} = 0$, Eq. (3.34) shows that $\delta \hat{\psi}_\mu = 0$, while Eq. (3.33) ensures in view of (3.37) that $\delta \hat{\psi}_m = 0$. As a result, all components of the ten-dimensional SUSY variations vanish, $\delta \hat{\chi} = \delta \hat{\psi}_M = 0$. This shows that we can restrict our considerations to the four-dimensional SUSY variations $\delta \chi$ and $\delta \psi_\mu$ given by (3.27) and (3.36). The vanishing of these implies that the background bosonic configuration is supersymmetric when lifted to ten dimensions.

We have completed our program of deriving the four-dimensional theory from the ten-dimensional one. Summarizing, in addition to the bosonic Lagrangian (2.30) we now have also the four-dimensional supersymmetry transformations (3.27), (3.36). For $s = i$ these exactly coincide with the Lagrangian and linearized SUSY variations of the Freedman-Schwarz model described in \[12\], up to a change of the overall sign of the metric:

\[ g_{\mu\nu} \rightarrow -g_{\mu\nu}, \quad \gamma^\mu \rightarrow i\gamma^\mu, \quad \gamma_\mu \rightarrow -i\gamma_\mu, \quad \gamma_5 \rightarrow \gamma_5. \] 

(3.38)

For $s = 1$ Eqs. (2.30), (3.27), (3.36) give us the Lagrangian and SUSY variations of the Euclidean Freedman-Schwarz model. This appears to be the N=4 gauged SU(2)\(\times\)SU(1,1) supergravity in four-dimensional Euclidean space. Here N=4 is due to the fact that the spinor supersymmetry parameter $\epsilon$ in (3.27), (3.36) is a multiplet of four Majorana spinors.

Having obtained the theory, we shall now proceed with studying its vacuum structure. Since we have a gauged supergravity, we shall mainly be
interested in solutions with non-Abelian gauge fields. In particular, we still need to explain the relation $g(2) \rightarrow ig(2)$ between the two models, as this apparently does not hold at the level of the full D=4 theories, which are rather related via $s \rightarrow is$.

4 Vacua

A supersymmetric vacuum is an on-shell bosonic configuration which is invariant under some or all of the SUSY transformations. This manifests in the existence of non-trivial spinor parameters $\epsilon$ for which the fermion SUSY variations vanish. Such $\epsilon$’s are called supersymmetry Killing spinors. In an N=4 supergravity a vacuum can have at most 16 Killing spinors, which is the number of the real components of $\epsilon$, and such a vacuum is called maximally supersymmetric.

The Freedman-Schwarz model has no maximally supersymmetric vacua. This is because the latter are expected to respect the maximal number of spacetime isometries, while the model does not admit solutions with maximal symmetry. In view of the relation to ten dimensional supergravity, this is guaranteed by the “ten into four won’t go” theorem: N=1, D=10 supergravity does not admit solutions of the form $M \times S^3 \times S^3$, where $M$ is a maximal symmetry space (Minkowski, de Sitter, or AdS) \cite{18}. One can also see directly that Eqs. (2.23)–(2.28) for $s = i$ do not admit maximally symmetric solutions, since the dilaton potential in (2.24) has no stationary points. However, there are vacua in the model which are of the type of a direct product of the two maximally symmetric spaces, $AdS_2 \times E^2$, and these solutions preserve half of the supersymmetries \cite{19}. The model also admits domain-wall-type vacua with half of the supersymmetries preserved \cite{21, 22}, as well as other vacuum solutions which can be of various types and typically preserve less than half of the supersymmetries \cite{21, 22}. When lifted to ten dimensions, some of the known FS vacua can be interpreted as the near-horizon geometries of certain intersecting brane solutions \cite{23, 24}. Almost all known FS vacua are characterized by the gauge fields belonging to the Cartan subalgebra of the Lie algebra of SU(2)×SU(2). Only one solution is known whose gauge field is truly non-Abelian, this is of regular monopole type \cite{10, 11}, it preserves 1/4 of the supersymmetries and will be briefly discussed below.
Let us now turn to the Euclidean Freedman-Schwarz model. In this case we can get rid of the dilaton potential by choosing $g^{(1)} = g^{(2)}$. This allows us to set in Eqs. (2.23)–(2.28) the vector fields to zero and scalar fields to constant values. The non-trivial bosonic equations then reduce to

$$R_{\mu\nu} = 0,$$

(4.1)

whose solution can be any gravitational instanton $\mathcal{M}_4$. The conditions $\delta \chi = \delta \psi_\mu = 0$ read

$$\nabla_\mu \epsilon = 0,$$

$$g^{(1)} (1 - \gamma_5) \epsilon = 0,$$

(4.2)

with $\nabla_\mu$ being the geometrical covariant derivative, which requires that $\mathcal{M}_4$ should admit chiral geometrical Killing spinors (remember that the Euclidean Majorana condition for $\epsilon$ is compatible with the Weyl condition). This gives the simplest vacua in the EFS model, and these can be uplifted to D=10 with the use of Eqs. (2.6) and (2.22) leading to solutions of the form $\mathcal{M}_4 \times S^3 \times AdS_3$, the radii of the internal manifolds being $1/g^{(1)}$. The D=10 dilaton is constant, while the antisymmetric tensor field coincides up to $g^{(1)}_2$ with the direct sum of the volume forms on the internal three-spaces.

The EFS vacua described above include the flat instanton $E^4$, which is maximally symmetric. In view of the chirality condition in (4.2) this has only eight Killing spinors and thus is not maximally supersymmetric. The reason for this is clear from the ten-dimensional point of view, since $E^4 \times S^3 \times AdS_3$ is not maximally symmetric. For $g^{(1)} \to 0$ the second condition in (4.2) disappears and the number of supersymmetries doubles, while the ten-dimensional configuration reduces to the flat metric.

One can also study more complex EFS vacua, in particular those with non-trivial scalars and gauge fields. Such solutions are probably relatively easy to obtain in the case where the gauge fields belong to the Cartan subalgebra. However, since we have a gauged supergravity, our primary interest will be in configurations with essentially non-Abelian structures, and we shall explicitly present such solutions below. Our strategy will be as follows. First, we shall further reduce the theory from D=4 to D=3 and recover in this case the relation between the FS and EFS models via $g^{(2)} \to ig^{(2)}$, which has been the main motivation for the present work. Next, we shall impose spherical symmetry and give a complete derivation of the supersymmetry constraints.
and the Bogomol'nyi equations. Finally we shall describe the known non-Abelian solutions of the Bogomol'nyi equations and their interpretation.

4.1 Reduction to D=3

We wish to further reduce the four-dimensional theory specified by (2.30), (3.27), (3.36) to D=3. In order to maintain consistency of the procedure we shall again work at the level of equations of motion and, as usual, keep both values of the parameter $s$. In brief, our dimensional reduction ansatz is

$$\frac{\partial}{\partial x^0} = 0, \quad A^{(2)}_\mu = 0, \quad A^{(1)}_0 = 0, \quad a = 0. \quad (4.3)$$

The first condition here means that $\partial/\partial x^0$ is a Killing vector, and thus there is a gauge where all variables depend only on the spatial coordinates $x^i$ (we shall denote the spatial base space and tangent space indices by bold-faced letters $i, k = 1, 2, 3$ and $a, b = 1, 2, 3$, respectively). Next, the bosonic field equations (2.23)–(2.28) show that one can consistently set the second gauge field to zero, $A^{(2)}_\mu = 0$. However, we keep at the same time $g^{(2)} \neq 0$. This leaves us with only one gauge field, $A^a_\mu \equiv A^{(1)}_\mu$, whose gauge group is SU(2). Without loss of generality we can assume that $g^{(1)} = 1$. Next, we require that $A^a_0 = 0$, and this implies that the $*FF$ invariant of the gauge field vanishes. It follows then from Eq. (2.28) that one can consistently set the axion to zero, $a = 0$. Finally, we assume that the Killing vector $\partial/\partial x^0$ is hypersurface orthogonal, in which case the metric can be chosen as

$$ds^2 = s^2 e^{2V} dt^2 + h_{ik} dx^i dx^k. \quad (4.4)$$

As a result, we are now left with only the metric amplitude $V$, the three-metric $h_{ik}$, the dilaton $\phi$, and the three-dimensional gauge field $A^a_i$. In the Lorentzian case ($s = i$) our truncation corresponds to the static and purely magnetic sector of the FS model with the second gauge field set to zero. In the Euclidean ($s = 1$) domain the notions ‘static’ and ‘purely magnetic’ do not have an invariant meaning. It will be convenient to use together with the three-metric $h_{ik}$ also its conformally rescaled version, $g_{ik}$, the ‘spatial’ line element being

$$dt^2 = h_{ik} dx^i dx^k = e^{-2V} g_{ik} dx^i dx^k = \delta_{ab} \theta^a \theta^b, \quad (4.5)$$
where $\theta^a$ are the basis one-forms, which are dual to the triad $e_a$.

One can verify that under the conditions in (4.3) and (4.4) the D=4 field equations (2.23)–(2.28) consistently reduce to the Lagrangian equations for the action

$$S_3 = \int L_3 \sqrt{g_3} d^3 x,$$

with $g_3 \equiv \det(g_{ik})$ and

$$L_3 = \frac{1}{4} \left( R^{(3)} - \frac{1}{2} \partial_i \phi \partial^i \phi - \frac{1}{2} \partial_i V \partial^i V - \frac{1}{4} e^{2\phi + 2V} F_{ik}^a F^{aik} + \frac{1}{8} (1 - \xi^2) e^{-2\phi - 2V} \right).$$

Here $F_{ik} = \partial_i A_k - \partial_k A_i + \varepsilon_{abc} A^b_i A^c_k$ is the gauge field tensor, $R^{(3)}$ is the Ricci-scalar for $g_{ik}$, and $\xi \equiv g^{(2)}/s$. Notice the following important fact: the reduction above has been done for both values of $s$. At the same time, the $s$-dependence is now almost completely gone, and if we ignored the dilaton potential then the field equations would be exactly the same in the Euclidean and Lorentzian cases. This is because the system is ‘static’ and ‘purely magnetic’. It is only the explicit dependence of the dilaton potential on $s$ that breaks the complete symmetry between the two cases.

The action (4.6) admits the global symmetry

$$\phi \rightarrow \phi + a, \quad V \rightarrow V - a,$$

and this implies that there is the Noether current, whose conservation law reads

$$\nabla^{(3)}_i \nabla^1 (\phi - V) = 0,$$

where $\nabla$ is the covariant derivative with respect to $g_{ik}$. As a result, we can consistently set

$$V = \phi - \phi_\infty,$$

where $\phi_\infty$ is the value of the dilaton at ‘spatial’ infinity.

Let us now consider the reduction of the D=4 SUSY variations in (3.27) and (3.36) to D=3. Inserting (4.3), (4.4) into (3.27) and (3.36) the result is

$$\delta \chi = \left( \frac{1}{\sqrt{2}} \gamma^i \partial_i \phi - \frac{1}{2} e^\phi T_a F_{ik}^a \gamma^k + \frac{1}{4} e^{-\phi} (1 - \gamma_5 \xi) \right) \epsilon,$$

$$\delta \psi_j = D_j \epsilon + \frac{1}{2 \sqrt{2}} \left( -e^\phi T_a F_{ik}^a \gamma^k + \frac{1}{2} e^{-\phi} (1 + \gamma_5 \xi) \right) \gamma_j \epsilon.$$
Here the covariant derivative $D_j \equiv \partial_j + \frac{1}{4} \omega_{ab,j} \gamma^a \gamma^a + T^a A^a_j$, where $\omega_{ab,j}$ is the spin-connection for $\theta^a$, and $[T_a, T_b] = \varepsilon_{abc} T_c$ are the SU(2) generators. The $\gamma^a$'s are the four-dimensional gamma matrices for the spatial values of the index, $\gamma^a = (\gamma^0, \gamma^a)$, one has $\gamma^a \gamma^b + \gamma^b \gamma^a = 2 \delta^{ab}$ and $\gamma^i = e^i_a \gamma^a$.

The temporal component $\delta \psi_0$ obeys

$$\delta \psi_0 - \frac{1}{\sqrt{2}} \gamma_0 \delta \chi = \frac{1}{2} \left( \gamma_0 \gamma^k \partial_k (V - \phi) \right) \epsilon = 0,$$

where the last equality on the right is due to (4.10). This shows that $\delta \psi_0$ is not independent and vanishes whenever other SUSY variations vanish, provided that the condition (4.10) holds, the latter thus being one of the supersymmetry conditions.

We have completed the reduction to D=3. The bosonic sector of the resulting theory is described by Eqs. (4.6), (4.7) together with the constraint in (4.10), while the fermion SUSY transformations are given by (4.11). We now make the following observation. All expressions above depend on $s$ only via the ratio $\xi = g(2)/s$. For real values of $\xi$ we obtain the EFS model, while choosing $\xi$ imaginary gives the FS model. This shows that starting from the static, purely magnetic sector of the Freedman-Schwarz supergravity and making the formal replacement $g(2) \rightarrow i g(2)$ gives the ‘static’ and ‘purely magnetic’ sector of the Euclidean Freedman-Schwarz theory. This explains the empirical observation made in Ref.[14] that a formal analytic continuation in the supergravity equations gives a meaningful result – because we obtain in this way another consistent supergravity model.

The D=3 field equations in (4.6), (4.7), (4.10) allow one to study the static solutions. As there is little hope to directly solve the equations for the bosonic action (4.6), one can start from the equations for the supersymmetry Killing spinors $\epsilon$ obtained from (4.10) by setting $\delta \chi = \delta \psi_j = 0$. The consistency conditions for these equations can be formulated as a set of first order Bogomol’nyi equations for the underlying bosonic configuration. The Bogomol’nyi equations are compatible with the second order field equations and their solutions automatically give supersymmetric vacua. So far, however, the corresponding construction has been carried out only for the spherically symmetric fields. These will be considered below.
4.2 Spherical symmetry

Let us consider the reduction of the D=3 theory described by Eqs. (4.6), (4.7), and (4.10) to the spherically symmetric sector. The most general spherically symmetric 3-metric is

\[ dl^2 = e^{2\lambda} dr^2 + e^{2\tau} d\Omega^2, \]  

(4.13)

where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \) is the line element of the unit sphere. The spherically symmetric and purely magnetic Yang-Mills field is given by

\[ A = w ( -T_2 d\theta + T_1 \sin \theta d\phi ) + T_3 \cos \theta d\phi. \]  

(4.14)

Here \( \lambda, \tau, w \) as well as the dilaton \( \phi \) depend only on the radial coordinate \( r \), and the reparameterization invariance \( r \rightarrow \tilde{r}(r) \) implies that one coordinate condition can be imposed on the four amplitudes. Taking the condition in (4.10) into account, the 4-metric reads

\[ ds^2 = s^2 e^{2(\phi - \phi_\infty)} dt^2 + dl^2. \]  

(4.15)

The complete set of the field equations for the bosonic action (4.6) is

\[ e^{2\tau} (\tau'^2 + 2\tau' \phi') = e^{2\tau} \phi'^2 + 2e^{2\phi} w'^2 + e^{2\lambda} \left(1 - e^{2\phi - 2\tau} (w^2 - 1)^2\right) + \frac{1}{4} (1 - \xi^2) e^{2\lambda - 2\phi + 2\tau}, \]

\[ (e^{\phi - \lambda + 2\tau} \tau')' = e^{\phi + \lambda} \left(1 - e^{2\phi - 2\tau} (w^2 - 1)^2\right) + \frac{1}{4} (1 - \xi^2) e^{\lambda - \phi + 2\tau}, \]

\[ (e^{\phi - \lambda + 2\tau} \phi')' = 2e^{3\phi - \lambda} w'^2 + e^{3\phi + \lambda - 2\tau} (w^2 - 1)^2 + \frac{1}{4} (1 - \xi^2) e^{\lambda - \phi + 2\tau}, \]

\[ (e^{3\phi - \lambda} w')' = e^{3\phi + \lambda - 2\tau} w (w^2 - 1), \]  

(4.16)

with \( ' := \frac{d}{dr} \). The same equations can be obtained by varying the four dimensional action (2.30) and using Eq. (1.10).

The supersymmetric vacua that we shall be considering are solutions to these equations for which there are non-trivial \( \epsilon \)'s such that \( \delta \chi = \delta \psi_j = 0 \). Of course, it is very difficult to directly solve the non-linear equations in (4.10) (apart from some trivial cases). For this reason we shall start from the equations \( \delta \chi = \delta \psi_j = 0 \) for the Killing spinors \( \epsilon \). These equations are sometimes called supersymmetry constraints, and they are generically inconsistent. One can analyze the consistency conditions under which non-trivial solutions for
the $\epsilon$'s exist. These conditions can be given in the form of a set of nonlinear first order differential equations for the underlying bosonic configuration – usually called Bogomol’nyi equations. The Bogomol’nyi equations are compatible with the second order field equations, and their solutions therefore describe supersymmetric vacua.

Following this strategy, our procedure will be to analyze the supersymmetry constraints $\delta \chi = \delta \psi_j = 0$ obtained from Eqs. (4.11) in the case of spherical symmetry.

4.2.1 The supersymmetry constraints

It is convenient to choose the isotropic gauge in the line element (4.15), $re^\lambda = e^r$, and then to pass to the Cartesian coordinates $x^i$ with $r = \sqrt{\delta_{ik}x^ix^k}$. The three-metric reads

$$dl^2 = e^{2\lambda} (dr^2 + r^2 d\Omega^2) = e^{2\lambda} \delta_{ik} dx^i dx^k. \quad (4.17)$$

The triad vectors and one-forms are $e_a = e^{-\lambda} \partial_a$ and $\theta^a = e^{\lambda} dx^a$, respectively. The spin-connection is

$$\omega_{ab, c} = \lambda' e^{-\lambda} (n_b \delta_{ac} - n_a \delta_{bc}). \quad (4.18)$$

Here and below $x^a \equiv \delta^a_{kb}, \partial_a \equiv \partial/\partial x^a$, and $n^a \equiv x^a/r$. The triad components of the gauge field (4.14) read

$$A_a = e^{-\lambda} \frac{1 - w}{r} \varepsilon_{aab} n^b, \quad (4.19)$$

and the gauge field strength is

$$F^a_{ab} = e^{-2\lambda} \varepsilon_{abc} (f_1 \delta^{ac} + f_2 n^a n^c), \quad (4.20)$$

with

$$f_1 = \frac{w'}{r}, \quad f_2 = \frac{w^2 - 1}{r^2} - \frac{w'}{r}. \quad (4.21)$$

Here we handle the triad and internal indices with the metric tensors $\delta_{ab}$ and $\delta_{ab}$ and allow for objects with mixed indices like $\varepsilon_{ab}$ and $\delta_{ab}$.

Let us now introduce four different spinor two-spaces and denote the corresponding Pauli matrices by $\sigma^a, \sigma^a, \tau^a, \tau^a$, respectively. We choose the gamma matrices and group generators as

$$\gamma^a = \sigma^2 \otimes \sigma^a, \quad \gamma_5 = \sigma^3 \otimes \mathbb{I}_2, \quad T^a = \frac{1}{2i} \tau^a \otimes \mathbb{I}_2. \quad (4.22)$$
Here $\mathbb{L}_a$ acts in the $\tau^a$ space. In what follows we shall not write down explicitly the direct product sign and the unit operators. Taking into account all the definitions above, the SUSY variations in (4.11) assume the form

$$\delta \chi = \frac{1}{\sqrt{2}} e^{-\phi} \sigma^2 (\bar{n} \bar{\sigma}) \epsilon - \frac{1}{2} e^{\phi - 2\lambda} \left( f_1 (\bar{\sigma} \bar{\tau}) + f_2 (\bar{n} \bar{\sigma}) (\bar{n} \bar{\tau}) \right) \epsilon$$

$$+ \frac{1}{4} e^{-\phi} (1 - \xi \sigma^3) \epsilon,$$

$$\delta \psi_a = e^{-\lambda} \left( \partial_a + \frac{i}{2} \lambda' \varepsilon_{abc} n^b \sigma^c + \frac{i w - 1}{2} \varepsilon_{abc} n^b \tau^c \right) \epsilon$$

$$- \frac{1}{2\sqrt{2}} e^{\phi - 2\lambda} \sigma^2 \left( f_1 (\bar{\sigma} \bar{\tau}) + f_2 (\bar{n} \bar{\sigma}) (\bar{n} \bar{\tau}) \right) \sigma_a \epsilon$$

$$+ \frac{1}{4\sqrt{2}} e^{-\phi} (\sigma^2 - i \xi \sigma^1) \sigma_a \epsilon .$$

(4.23)

Here $(\bar{n} \bar{\sigma}) = \delta_{ab} n^a \sigma^b$, also $(\bar{n} \bar{\tau}) = \delta_{ab} \bar{n}^a \tau^b$, and $(\bar{\sigma} \bar{\tau}) = \delta_{ab} \bar{\sigma}^a \tau^b$.

Let us recall that $\epsilon$ is a 16-component spinor subject to the Majorana condition

$$(\epsilon)^* = \sigma^2 \tau^2 \tau^1 \epsilon$$

(4.24)

for real $\xi$ (Euclidean theory), and

$$(\epsilon)^* = \sigma^2 \sigma^2 \tau^2 \tau^2 \epsilon$$

(4.25)

for imaginary $\xi$ (Lorentzian theory). Now, since we are considering spherically symmetric backgrounds, it is natural to choose $\epsilon$ to be an eigenstate of the total angular momentum. Let us first study the sector with zero angular momentum, $J = 0$; the case of $J > 0$ will be discussed later. We choose

$$\epsilon = \epsilon_q \equiv \left( \Psi^{(+)}_q \psi_q + \Psi^{(-)}_q \sigma^2 \psi_q (\bar{n} \bar{\sigma}) \right) \left( \psi_+ \chi_- - \psi_- \chi_+ \right) \chi.$$  

(4.26)

Here $q = \pm 1$, $\Psi^{(\pm)}_q$ are functions of $r$, while $\psi_q$, $\psi_\pm$, $\chi_\pm$, and $\chi$ are constant two-component spinors from the four different spinor spaces in which the operators $\sigma^a$, $\sigma^a$, $\tau^a$, $\tau^a$, respectively, act. One has $\sigma^3 \psi_q = q \psi_q$, $\sigma^3 \psi_\pm = \pm \psi_\pm$, and $\tau^3 \chi_\pm = \pm \chi_\pm$. Notice that the ansatz for $\epsilon$ in (4.26) is the most
general expression annihilated by the total (orbital plus spin plus isospin) angular momentum operator,

\[ (-i\varepsilon^{abc}x_b\partial_c + \frac{1}{2}\sigma^a + \frac{1}{2}\tau^a)\epsilon = 0. \]  

(4.27)

Let us insert the ansatz (4.26) into (4.23) and set the left-hand sides to zero. After some spinor algebra the angular dependence decouples and we obtain a system of equations for \(\Psi^{(\pm)}_q\):

\[
\left(\frac{d}{dr} - \frac{\phi'}{2}\right)\Psi^{(\pm)}_q \pm \sqrt{2}e^{\phi-\lambda}w'_{q}\Psi^{(\mp)}_q = 0, \\
\left(\frac{\lambda'}{2} + \frac{1 \mp w}{2r}\right)\Psi^{(\pm)}_q + \left(\frac{e^{\phi-\lambda} - 1 - w^2}{2\sqrt{2}} \frac{1 - w^2}{r^2} + \frac{e^{\lambda-\phi}}{4\sqrt{2}}(1 \pm q\xi)\right)\Psi^{(\mp)}_q = 0, \\
\frac{\phi'}{2}\Psi^{(\pm)}_q + \left(\frac{e^{\phi-\lambda}(w^2 - 1)}{2r^2} + \frac{w'_{q}}{r}\right) + \frac{e^{\lambda-\phi}}{4}(1 \pm q\xi)\Psi^{(\mp)}_q = 0.
\]  

(4.28)

These are the supersymmetry constraints we are interested in.

4.2.2 The Bogomol’nyi equations

It is not difficult to find the consistency conditions for Eqs. (4.28)–(4.30). Consider, for example, Eq. (4.30). This is in fact a system of two homogeneous algebraic equations, which has a non-trivial solution only if the determinant of the coefficient matrix vanishes. The same is true with (4.29). In addition, the solutions obtained from (4.29) and (4.30) should agree. As a result, we obtain three algebraic conditions for the coefficients in Eqs. (4.29) and (4.30), which can be expressed in the form

\[
1 + r\frac{d\lambda}{dr} = \sqrt{u^2 + \frac{1}{8}e^{2(\phi-\lambda-ln r-\phi)}((B-1)^2 - q^2\xi^2)} \equiv \nu, \\
Ar\frac{dw}{dr} = 2q\xi w + q^2\xi^2(w^2 - 1) - 2w^2(B + 1), \\
Ar\frac{d\phi}{dr} = -(B + 1)(q\xi w + (B - 1)),
\]

(4.31)

with \(A \equiv 8w\nu e^{2(\phi-\lambda-ln r)} + q\xi (B - 1)\) and \(B \equiv 2e^{2(\phi-\lambda-ln r)}(w^2 - 1)\). Under these conditions Eqs. (4.29) and (4.30) specify the algebraic relation between
Ψ(+) and Ψ(−). Next, this relation should be consistent with the remaining
differential constraints in (4.28). A direct calculation shows that the required
consistency holds by virtue of Eqs. (4.31). The differential constraints (4.28)
then fix the solution for Ψ_q(±) uniquely – up to a normalization constant,

$$
Ψ_q^+ + iΨ_q^- = \exp \left( i \sqrt{2} \int_{r_0}^r e^{\phi - \lambda} \frac{w'}{r} dr \right). 
$$

As a result, the equations in (4.31) constitute a complete set of consistency
conditions under which the supersymmetry Killing spinors exist. These first
order Bogomol’nyi equations are compatible (for q = ±1) with the second
order field equations in (4.16). In other words, they are first integrals for
the field equations. Although this fact is expected, its direct verification is
not at all trivial and provides a very good check of the consistency of the
whole procedure. Any solution of the Bogomol’nyi equations hence fulfills
the field equations and admits non-trivial supersymmetry Killing spinors,
thus describing a supersymmetric vacuum.

Before we pass to studying the Bogomol’nyi equations, let us count the
number of supersymmetry Killing spinors obtained from (4.26), (4.28)–(4.30).
Consider first the EFS case, where ξ is real. Then all coefficients in Eqs.
(4.28)–(4.30) are real and the solution for Ψ_q(±) in (4.32) can be chosen to be
real as well. As was mentioned above, this solution is specified uniquely. We
do not obtain solutions for both values of q, since the Bogomol’nyi equations
for ξ ̸= 0 are not invariant under q → −q, but only under

$$
q \to -q, \quad w \to -w. 
$$

Thus, unless w ≡ 0, the value of q is fixed by the background configuration
and there is only one solution for the Ψ_q(±). Notice however that Eq. (4.26)
contains an additional degeneracy due to the arbitrary constant spinor χ.
The Majorana condition imposes the restrictions

$$
\bar{ψ}_q = (ψ_q)^*, \quad \Sigma^1 \chi = - (χ)^*. 
$$

There are two independent solutions to these conditions for a given q. This
finally gives two supersymmetry Killing spinors. For ξ = 0 the number of
supersymmetries doubles, since the Bogomol’nyi equations are then invariant
under q → −q and both values of q are allowed in (4.26) and (4.34).
Consider now the FS case, where \( \xi \) is imaginary. Then \( \xi = 0 \) is the only allowed value, since otherwise the Bogomol’nyi equations contain imaginary coefficients and their solutions are not real. The solution for \( \epsilon \) is given by (4.26), where one can take both values of \( q \), and again there is an additional degeneracy due to \( \chi \). In order to fulfill the Majorana condition in this case one should take linear combinations of solutions with different \( q \). Omitting the index \( q \) of \( \psi_q \), the Majorana condition in (4.25) reduces to

\[
(\sigma^2 \psi) (\tau^2 \chi) = - (\psi \chi)^*,
\]

and this has four independent solutions.

To recapitulate, supersymmetric vacua exist in the EFS model for arbitrary real \( \xi \), and in the FS case for \( \xi = 0 \) only. For \( \xi \neq 0 \) the vacua preserve two supersymmetries. For \( \xi = 0 \), when one of the two gauge coupling constants vanishes and the theory is ‘half-gauged’, the vacuum admits four supersymmetry Killing spinors and fulfills the equations of both FS and EFS models.

It is worth noting that the above analysis uses only the Killing spinors from the sector with zero angular momentum, \( J = 0 \). At the same time, there could be additional Killing spinors for \( J > 0 \), even though the background is spherically symmetric. Indeed, as was mentioned above, the \( E^4 \) instanton solution, which is given by \( \lambda = \phi = 0 \) and \( w = \xi = 1 \), has eight Killing spinors. Of these only two are recovered in the system (4.28)–(4.30), and the remaining six must therefore reside in sectors with \( J > 0 \). These sectors therefore should also be taken into consideration. Now, one can show that any additional Killing spinors can only exist for \( J = 1 \). In this case the whole procedure described above can be repeated, which leads to a system of seven non-linear Bogomol’nyi equations for the three amplitudes \( w, \phi, \) and \( \lambda \). The \( E^4 \) instanton fulfills these new equations, which accounts for its additional six Killing spinors. However, since the equations are overdetermined, it is unclear whether they admit any other solutions. We shall not concentrate on this here but rather pass to considering the solutions of the Bogomol’nyi equations (4.31). For these solutions one can show that all their Killing spinors live in the \( J = 0 \) sector.
4.2.3 Vacuum solutions for $\xi = 0$. Supersymmetric monopoles

It is not difficult to see that the Bogomol’nyi equations in (4.31) can be made autonomous, in which case the system essentially reduces to one first order differential equation. This equation turns out to be rather complicated, but it can be analyzed for special values of $\xi$. Let us consider the value $\xi = 0$, in which case the solutions will have N=1 supersymmetry. Denoting $x \equiv w^2$ and $R \equiv \frac{1}{2} e^{2(\lambda+\ln r-\phi)}$, Eqs. (4.31) give [10, 11]

$$x(x+1)\frac{dR}{dx} + (x+1)R + (x-1)^2 = 0. \quad (4.36)$$

The two remaining Bogomol’nyi equations are solved by quadratures as soon as the solution $R(x)$ is obtained. After the substitution [10]

$$x = \rho^2 e^{g(\rho)}, \quad R = -\rho \frac{d\rho}{d\rho} - \rho^2 e^{g(\rho)} - 1, \quad (4.37)$$

the Abel equation (4.36) reduces to the Liouville equation

$$\frac{d^2 y}{d\rho^2} = 2 e^y, \quad (4.38)$$

which is integrable. This gives a globally regular solution described by

$$ds^2 = 2 e^{2\phi} \left( s^2 dt^2 + d\rho^2 + R d\Omega^2 \right), \quad (4.39)$$

$$R = 2\rho \coth \rho - \frac{\rho^2}{\sinh^2 \rho} - 1, \quad w = \pm \frac{\rho}{\sinh \rho}, \quad e^{2\phi} = \frac{\sinh \rho}{\sqrt{R}}, \quad (4.40)$$

where the scaling symmetry (4.8) has been used to set $\phi(0) = 0$. The new result we obtain here as compared to that of [10] is that this solution turns out to be the vacuum of both the FS and EFS models, preserving in each case 1/4 of the supersymmetries. Corresponding to this there is the parameter $s$ in the metric in (4.39).

For $s = 1$ the solution describes a globally regular Riemannian manifold with an essentially non-Abelian gauge field. Since the configuration does not depend on $t$, the action is infinite. Passing to the Lorentzian sector via choosing $s = i$, the solution describes a globally regular magnetic monopole with unit charge. This is geodesically complete and globally hyperbolic [10, 11].
Unfortunately, the ADM mass is infinite and the solution is not asymptotically flat – due to the dilaton potential. In view of its supersymmetry, it is very plausible that the Lorentzian solution is stable, while for its Euclidean counterpart the notion of dynamical stability makes no sense.

4.2.4 Vacuum solutions for $\xi = 1$. Supersymmetric sphalerons

For any $\xi \neq 0$ solutions of the Bogomol’nyi equations (4.31) will preserve only 1/8 of the supersymmetries. Let us consider the value $\xi = 1$, in which case the dilaton potential vanishes. After some transformations described in [14] the Bogomol’nyi equations can be reduced to

$$\frac{1}{2r} \frac{dw}{dr} = \frac{1 - w^2}{4r^2} - \frac{(w + 1)^3}{8} + \frac{(w - 1)^3}{8r^4},$$

(4.41)

which is invariant under $r \rightarrow 1/r$, $w \rightarrow -w$. When the solution $w(r)$ is found, the whole configuration is reconstructed as follows. Computing the combination

$$U = \frac{r^2(1+w)^2 + (1-w)^2}{r^2(1+w)^2 - (1-w)^2},$$

(4.42)

the metric function $\lambda$ is obtained from

$$\lambda = \ln(2) + \int_0^r \left( \frac{U + w}{2} - 1 \right) \frac{dr}{r},$$

(4.43)
while the dilaton is given by

$$
\phi = \lambda + \ln(r) + \frac{1}{2} \ln \left( \frac{(U + w)^2 - 2w^2 - 2}{2(w^2 - 1)^2} \right),
$$

(4.44)

which is normalized such that \( \phi(0) = 0 \). The metric is

$$
ds^2 = e^{2(\phi - \Phi_\infty)} dt^2 + e^{2\lambda}(dr^2 + r^2d\Omega^2).
$$

(4.45)

Unfortunately, analytical solutions to Eq. (4.41) are unknown (apart from singular ones [14]). The numerical integration (see Fig.1) reveals the existence of a globally regular solution in the interval \( r \in [0, \infty) \) which monotonically interpolates between the values specified by the local asymptotic solutions: \( w = 1 - \frac{2}{3}r^2 + O(r^4) \) for \( r \to 0 \) and \( w = -1 + 2\sqrt{2}r^{-1} + O(r^{-2}) \) as \( r \to \infty \). This gives a globally regular supersymmetric Euclidean solution with non-trivial Yang-Mills field and infinite action.

One can pass to the Lorentzian sector by changing the sign of \( dt^2 \) in the metric (4.45). The resulting configuration fulfills the equations of motion of the Einstein-Yang-Mills-Dilaton (EYMD) model with the action

$$
S_{\text{EYMD}} = \int \left( \frac{1}{4} R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{2\phi} F^a_{\mu \nu} F^{a \mu \nu} \right) \sqrt{-g} \, d^4x.
$$

(4.46)

The static and purely magnetic sector of this model can be embedded into the heterotic string theory. However, the solution is not supersymmetric in this model, since it is not self-dual [3]. Nevertheless, this EYMD solution is interesting as it describes a regular particle-like object with finite ADM mass \( M \) determined by the asymptotic behaviour of the metric in (4.43), \( e^{2(\phi - \Phi_\infty)} = 1 - 2Me^{-\lambda(\infty)} r^{-1} + O(r^{-2}) \) [14]. This EYMD particle is static, spherically symmetric and neutral, but it has a non-trivial purely magnetic gauge field that asymptotically vanishes like \( 1/r^3 \). It turns out that this solution resembles the well-known sphaleron solution of the Weinberg-Salam theory [25]. Specifically, one can show that the solution relates to the top of the potential barrier between the topological vacua of the EYMD theory, which implies in particular that it is unstable. This suggests the name ‘EYMD sphaleron’. One can argue that this solution is responsible for fermion number non-conserving processes in heterotic string theory. Despite its instability, it bears an imprint of supersymmetry as it fulfills the first order

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Bogomol’nyi equations. Passing back to the Euclidean theory, the counterpart of the EYMD sphaleron is a genuinely supersymmetric configuration, and we call it ‘supersymmetric sphaleron’.

Historically, it was the EYMD sphaleron solution which was first obtained by numerical integration of the second order field equations for the action \( (4.46) \) \( [26, 27, 28] \). Only later it was discovered in \( [14] \) that the solution fulfills the first order Bogomol’nyi equation \( (4.41) \), and it was conjectured that the configuration becomes supersymmetric upon continuation to the Euclidean sector. The justification of this conjecture has been the main subject of the present paper.

5 Conclusion

In this paper we have studied the dimensional reduction of the \( N=1, D=10 \) supergravity on \( S^3 \times AdS_3 \). The resulting four-dimensional theory is Euclidean \( N=4 \) gauged \( SU(2) \times SU(1,1) \) supergravity with the bosonic Lagrangian \( (2.30) \) and the fermion supersymmetry transformations specified by \( (3.27), (3.36) \). An interesting feature of this model is that its dilaton potential \( U(\phi) \) can be positive, negative, or zero, depending on the values of the two gauge coupling constants. This allows one to apply the model for generating various solutions with gravitating Yang-Mills fields via solving the Bogomol’nyi equations, which sometimes even gives solutions in a closed analytical form. The two examples – monopole-type and sphaleron-type non-Abelian vacua – were described above. In view of the relation to \( N=1, D=10 \) supergravity, which is in turn related to \( D=11 \) supergravity \( [16] \), any solution of the theory can be uplifted to become a vacuum of string or M theory.

It is worth noting that reductions on \( AdS \times \text{Sphere} \) are often considered; see for example \( [29] \). In particular, gravity and string theory on \( AdS_3 \times S^3 \) have been studied in detail \( [31, 30] \). In our analysis, however, the emphasis is quite different, since we are interested not in an effective Lorentzian theory with an \( AdS \) ground state, but rather in a Euclidean theory admitting \( E^4 \) as a vacuum. Let us also note that in most cases the dimensional reduction is performed only at the level of the Lagrangian, which raises the issue of consistency; see \( [4, 8] \) for a recent discussion. Our results on the other hand provide an example of a consistent reduction carried out at the level of equations of motion.
One can expect that the model described above admits interesting solutions also beyond the static, spherically symmetric and purely magnetic sector with \( U(\phi) = 0 \). For \( U(\phi) > 0 \) these could be, for example, compact instantons, and possibly also non-compact, asymptotically flat configurations. For \( U(\phi) = 0 \) the theory probably admits non-compact instantons with finite action. One can also study static multi-sphaleron solutions by deriving the Bogomol’nyi equations from Eqs. (4.11). An interpretation of the Euclidean solutions can sometimes be obtained by continuation to the Lorentzian sector – negative energy states will not arise if the SU(1,1) gauge field vanishes. All solutions for gravitating Yang-Mills fields of this type are expected to be relevant for string/M-theory and in the context of the general study of non-linear phenomena in field theory.

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