PROPERLY TWISTED GROUPS AND THEIR ALGEBRAS

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ABSTRACT. A proper twist on a group $G$ is a function $\alpha : G \times G \mapsto \{-1, 1\}$ with the property that, if $p, q \in G$ then $\alpha(p, q) \alpha(q, q^{-1}) = \alpha(pq, q^{-1})$ and $\alpha(p^{-1}, p) \alpha(p, q) = \alpha(p^{-1}, pq)$. The span $V$ of a set of unit vectors $B = \{i_p \mid p \in G\}$ over a ring $\mathbb{K}$ with product $xy = \sum_{p,q} \alpha(p, q)xpyq$ is a twisted group algebra. If the twist $\alpha$ is proper, then the conjugate defined by $x^* = \sum \alpha(p, p^{-1})x^*_p i_{p^{-1}}$ and inner product $\langle x, y \rangle = \sum x_pi_py^*_p$, satisfy the adjoint properties $\langle xy, z \rangle = \langle y, x^*z \rangle$ and $\langle x, yz \rangle = \langle xz^*, y \rangle$ for all $x, y, z \in V$. Proper twists on $\mathbb{Z}_N$ over the reals produce the complex numbers, quaternions, octonions and all higher order Cayley-Dickson and Clifford algebras.

1. Introduction

A vector space $V$ with orthonormal basis $B$ can be transformed into an algebra by defining a product for the elements of $B$ and extending that product to $V$. An obvious way to define a product on $B$ corresponding element of a group product to $V$ has the product $i \in \mathbb{N}$ integer $N$ of elements $\alpha$ the group algebra. Either a Cayley-Dickson or Clifford algebra by an appropriate twist on the direct $Z$ product $B$ by defining a product for the elements of $V$. A vector space $V$ denote a group having only $n$ elements. Let $G$ denote a Cayley-Dickson algebra, Clifford algebra, quaternions, octonions, sedenions, geometric algebra.

Given a twisted group algebra $V$ with element $x$, the transformation $L_x : V \mapsto V$ defined by $L_x(y) = xy$ is a linear transformation with standard matrix $[x]$. The product of elements $x, y \in V$ is $xy = [x]y$. A result of Wedderburn guarantees that $\|xy\| \leq \sqrt{n} \|x\| \|y\|$ where $n$ is the order of $[x]$. If the twist is “associative” then $V$ is associative and $[xy] = [x][y]$ making $V$ isomorphic to a ring of square matrices. The Clifford algebras fall into this category, as well as the reals, complex numbers and quaternions. Cayley-Dickson algebras from the octonions upwards are non-associative.

2. Twisted Group Algebras

Let $V$ denote an $n$-dimensional inner product space over the ring $\mathbb{K}$. Let $G$ denote a group having only $n$ elements. Let $B = \{i_p \mid p \in G\}$ denote a set of unit basis vectors for $V$. Then, for each $x \in V$, there exist elements $\{x_p \mid x_p \in \mathbb{K}, p \in G\}$ such that $x = \sum x_p i_p$.  

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Define a product on the elements of $B$ and their negatives in the following manner.

Let $\alpha : G \times G \mapsto \{-1, 1\}$ denote a sign function or 'twist' on $G$. Then for $p, q \in G$ define the product of $i_p$ and $i_q$ as follows.

**Definition 2.1.**

$$i_p i_q = \alpha(p, q) i_{pq}$$

Extend this product to $V$ in the natural way. That is,

**Definition 2.2.**

$$xy = \left( \sum_p x_p i_p \right) \left( \sum_q y_q i_q \right)$$

$$= \sum_{p,q} x_p y_q i_{pq}$$

$$= \sum_{p,q} \alpha(p, q) x_p y_q i_{pq}$$

In defining the product this way, one gets the closure and distributive properties "for free", as well as $(cx) y = x(cy) = c(xy)$

This product transforms the vector space $V$ into a twisted group algebra. The properties of the algebra depend upon the properties of the twist and the properties of $G$.

**Notation 2.3.** Given a group $G$, twist $\alpha$ on $G$ and ring $K$, let $[G, \alpha, K]$ denote the corresponding twisted group algebra. If $K = \mathbb{R}$, abbreviate this notation $[G, \alpha]$.

**Definition 2.4.** Let $x \in V = [G, \alpha, K]$. Define the matrix $[x] : G \times G \mapsto K$ such that for $r, s \in G$, $[x](r, s) = \alpha(rs^{-1}, s)x_{rs^{-1}}$.

**Theorem 2.5.** For $x, y \in V = [G, \alpha, K]$, $xy = [x] y$.

**Proof.** Given $xy = \sum_{p,q} \alpha(p, q) x_p y_q i_{pq}$ let $pq = r$. Then $p = rq^{-1}$, so

$$xy = \sum_{r,q} \alpha(rq^{-1}, q)x_{rq^{-1}}y_q i_r = [x] y.$$  

□

**Theorem 2.6.** $[i_p](r, s) = \begin{cases} \alpha(p, s) & \text{if } ps = r \\ 0 & \text{otherwise} \end{cases}$

**Corollary 2.7.** $[\alpha(p, q)i_{pq}](r, s) = \begin{cases} \alpha(p, q) \alpha(pq, s) & \text{if } pqs = r \\ 0 & \text{otherwise} \end{cases}$

**Theorem 2.8.** $([i_p][i_q])(r, s) = \begin{cases} \alpha(p, qs) \alpha(q, s) & \text{if } pqs = r \\ 0 & \text{otherwise} \end{cases}$

**Corollary 2.9.** $[i_p][i_q] = \alpha(p, q)[i_{pq}]$ provided $\alpha(p, q) \alpha(pq, s) = \alpha(p, qs) \alpha(q, s)$ for $p, q, s \in G$. 
3. Symmetric and anti-symmetric products

Definition 3.1. For group $G$, the interior of $G \times G$, written $G^2_0$, is the set of all ordered pairs $(p, q) \in G \times G$ such that $p \neq e$, $q \neq e$ and $pq \neq e$ or equivalently, $e \neq p \neq q^{-1} \neq e$.

Definition 3.2. For group $G$ and twist $\alpha$, the symmetric part of $G \times G$ with respect to $\alpha$, written $G^2 + \alpha$, is the set of all ordered pairs in $G \times G$ such that $\alpha(p, q) = \alpha(q, p)$. The anti-symmetric part of $G \times G$ with respect to $\alpha$, written $G^2 - \alpha$, is the set of all ordered pairs in $G \times G$ such that $\alpha(p, q) \neq \alpha(q, p)$.

Every twisted product can be decomposed into its symmetric and anti-symmetric parts.

Definition 3.3. For $x, y \in [G, \alpha, K]$, define the symmetric product $x \lor y$ as

$$x \lor y = \sum_{p, q} \alpha(p, q)x_py_q i_{pq}$$

and define the anti-symmetric product $x \land y$ as

$$x \land y = \sum_{p, q} \alpha(p, q)x_py_q i_{pq}$$

The following results are immediate.

Theorem 3.4. For $x, y \in [G, \alpha, K]$,

1. $xy = x \lor y + x \land y$
2. $x \lor y = y \lor x$
3. $x \land y = -(y \land x)$
4. $x \lor y = \frac{xy + yx}{2}$
5. $x \land y = \frac{xy - yx}{2}$

Definition 3.5. For group $G$ and twist $\alpha$, the symmetric interior of $G \times G$ with respect to $\alpha$, written $G^2_0 + \alpha$, is the intersection of the symmetric part of $G \times G$ with the interior of $G \times G$. The anti-symmetric interior written $G^2_0 - \alpha$, is the intersection of the anti-symmetric part of $G \times G$ with the interior of $G \times G$.

Theorem 3.6. If $\alpha(p, p^{-1}) = \alpha(p^{-1}, p)$ for $p \in G$ then

$$x \lor y = x_ey + x_ey + \sum_p \alpha(p, p^{-1})x_py_p^{-1} - 2x_ey + \sum_{p, q} \alpha(p, q) \left( \frac{x_py_q + x_qy_p}{2} \right) i_{pq}$$

and

$$x \land y = \sum_{p, q} \alpha(p, q) \left( \frac{x_py_q - x_qy_p}{2} \right) i_{pq}$$

Proof. Separating out the terms of $x \lor y$ for which $p = e$, $q = e$, $p = q^{-1}$ and $e \neq p \neq q^{-1} \neq e$ yields

$$x \lor y = x_ey + x_ey + \sum_p \alpha(p, p^{-1})x_py_p^{-1} - 2x_ey + \sum_{p, q} \alpha(p, q) \left( \frac{x_py_q + x_qy_p}{2} \right) i_{pq}$$
The set of all \((p, q) \in G \times G\) for which \(\alpha(p, q) \neq \alpha(q, p)\) is precisely the anti-symmetric interior of \(G \times G\), since \(\alpha(e, q) = \alpha(p, e) = 1\) and \(\alpha(p, p^{-1}) = \alpha(p^{-1}, p)\). Thus

\[
x \wedge y = \sum_{p,q} \alpha(p, q) \left( \frac{x_p y_q - x_q y_p}{2} \right) i_{pq}
\]

4. Twists and Ring Properties

Let \(G\) denote a finite group with identity \(e\), \(K\) a ring and \(\alpha\) a twist on \(G\). Let \(V = [G, \alpha, K]\).

In order for \(i_e\) to be the identity element 1 of the group algebra \(V\), we require for all \(p \in G\)

\[(4.1) \quad \alpha(e, p) = \alpha(p, e) = 1\]

**Definition 4.1.** If \(\alpha(p, q) = \alpha(q, p)\) for all \(p, q \in G\), then \(\alpha\) is a commutative twist.

**Theorem 4.2.** If \(V = [G, \alpha, K]\) and \(\alpha\) is commutative, then the resulting product on \(V\) is commutative.

**Proof.** For \(p, q \in G\), \(i_p i_q = \alpha(p, q) i_{pq} = \alpha(q, p) i_{qp}\). Thus, for \(x, y \in V\),

\[
xy = \sum_{p,q} x_p y_q \alpha(p, q) i_{pq} = \sum_{q,p} x_q y_p \alpha(q, p) i_{qp} = yx
\]

Corollary 2.9 establishes that a twisted group algebra is isomorphic to a ring of square matrices provided that a particular condition on the twist is satisfied, motivating the following definition.

**Definition 4.3.** If \(\alpha(p, q) \alpha(pq, r) = \alpha(p, qr) \alpha(q, r)\) for \(p, q, r \in G\), then \(\alpha\) is an associative twist on \(G\).

**Theorem 4.4.** If \(p, q, r \in G\) and \(\alpha\) is associative, then \(i_p (i_q i_r) = (i_p i_q) i_r\).

**Proof.**

\[
i_p (i_q i_r) = i_p (\alpha(q, r) i_{qr}) = \alpha(q, r) i_p i_{qr} = \alpha(p, qr) \alpha(q, r) i_{pq} i_{qr} = \alpha(p, qr) i_{pq} i_{qr} = \alpha(p, qr) i_{pq} i_r = (i_p i_q) i_r.
\]

**Theorem 4.5.** If \(x, y, z \in V = [G, \alpha, K]\) and if \(\alpha\) is associative, then \(x(yz) = (xy) z\).
Proof. $yz = \sum_{q,r} (yq z_r) i_q i_r$, so
\[
x(yz) = \left( \sum_p x_p i_p \right) \left( \sum_{q,r} (yq z_r) i_q i_r \right)
\]
\[
= \sum_{p,q,r} x_p (yq z_r) i_p (i_q i_r)
\]
\[
= \sum_{p,q,r} (x_p y_q) z_r (i_p i_q) i_r
\]
\[
= \left( \sum_{p,q} (x_p y_q) i_p i_q \right) \sum_r z_r i_r
\]
\[
= (xy) z.
\]

\[\square\]

**Corollary 4.6.** If $\alpha$ is associative, then $< [G, \alpha, \mathbb{K}], +, \cdot >$ is a ring with unity.

**Theorem 4.7.** For each $p \in G$, $\alpha \left( p, p^{-1} \right) i_{p^{-1}}$ and $\alpha \left( p^{-1}, p \right) i_p$ are right and left inverses, respectively, of $i_p$.

**Proof.**
\[
i_p \left( \alpha \left( p, p^{-1} \right) i_{p^{-1}} \right) = \alpha \left( p, p^{-1} \right) i_p i_{p^{-1}}
\]
\[
= \alpha \left( p, p^{-1} \right) \left( \alpha \left( p, p^{-1} \right) i_{pp^{-1}} \right)
\]
\[
= i_e = 1
\]
\[
(\alpha \left( p^{-1}, p \right) i_p) i_p = \alpha \left( p^{-1}, p \right) i_{p^{-1}} i_p
\]
\[
= \alpha \left( p^{-1}, p \right) \left( \alpha \left( p^{-1}, p \right) i_{pp^{-1}} \right)
\]
\[
= i_e = 1.
\]

\[\square\]

**Definition 4.8.** If $\alpha \left( p, p^{-1} \right) = \alpha \left( p^{-1}, p \right)$ for $p \in G$, then $\alpha$ is an **invertive** twist on $G$.

**Theorem 4.9.** If $p \in G$ and if $\alpha$ is invertive, then $i_p$ has an inverse $i_{p^{-1}} = \alpha \left( p, p^{-1} \right) i_{p^{-1}}$.

**Proof.** Follows immediately from Theorem 4.7 and Definition 4.8.

\[\square\]

**Definition 4.10.** If $\alpha$ is invertive, and $x \in V = [G, \alpha, \mathbb{K}]$, then let $x^* = \sum_p x_p i_p^{-1}$ denote the **conjugate** of $x$.

**Theorem 4.11.** If $\alpha$ is an invertive twist on $G$, and if $x, y \in V = [G, \alpha, \mathbb{K}]$, then
\begin{enumerate}
  \item \(x^* = \sum_p \alpha \left( p^{-1}, p \right) x_p^* i_p\)
  \item \(x^{**} = x\)
  \item \((x + y)^* = x^* + y^*\)
  \item \((cx)^* = c^* x^* \) for all $c \in \mathbb{K}$.
\end{enumerate}

**Proof.**
(i) Let \( q^{-1} = p \). Then

\[
x^* = \sum_q x_q^* i_q^{-1} = \sum_q x_q^* \alpha(q, q^{-1}) i^{-1}_{q-1} = \sum_p \alpha(p^{-1}, p) x_{p^{-1}}^* i_p
\]

(ii) Let \( z = x^* \). Then

\[
z = \sum_p \alpha(p^{-1}, p) x_{p^{-1}}^* i_p = \sum_p z_{p} i_p
\]

where \( z_p = \alpha(p^{-1}, p) x_{p^{-1}}^* \). Then \( z_{p^{-1}} = \alpha(p, p^{-1}) x_p \), and

\[
z_{p^{-1}} = \alpha(p, p^{-1}) x_p. \text{So } x^{**} = z^* = \sum_p \alpha(p^{-1}, p) z_{p^{-1}} i_p = \sum_p \alpha(p^{-1}, p) \alpha(p, p^{-1}) x_p i_p = \sum_p x_p i_p = x
\]

(iii)

\[
(x + y)^* = \sum_p (x_p + y_p)^* i_p^{-1} = \sum_p (x_p^* + y_p^*) i_p^{-1} = \sum_p x_p^* i_p^{-1} + \sum_p y_p^* i_p^{-1} = x^* + y^*.
\]

(iv)

\[
(cx)^* = \sum_p (cx_p)^* i_p^{-1} = \sum_p c x_p^* i_p^{-1} = c^* \sum_p x_p^* i_p^{-1} = c^* x^*
\]
The invertive property is not sufficient for establishing the algebraic property \((xy)^* = y^*x^*\). For that, we need the *proper* property.

5. Proper Twists

**Definition 5.1.** The statement that the twist \(\alpha\) on \(G\) is *proper* means that if \(p, q \in G\), then
\[
\begin{align*}
(1) & \quad \alpha(p, q) \alpha(q, q^{-1}) = \alpha(pq, q^{-1}) \\
(2) & \quad \alpha(p^{-1}, p) \alpha(p, q) = \alpha(p^{-1}, pq).
\end{align*}
\]

**Theorem 5.2.** Every associative twist is proper.

*Proof.*

\[
\begin{align*}
(1) & \quad \alpha(p, q) \alpha(q, q^{-1}) = \alpha(pq, q^{-1}) \alpha(p, qq^{-1}) \\
& \quad = \alpha(pq, q^{-1}) \alpha(p, e) = \alpha(pq, q^{-1}) \alpha(e, e) = \alpha(pq, q^{-1}) \\
(2) & \quad \alpha(p^{-1}, p) \alpha(p, q) = \alpha(p^{-1}, p) \alpha(p^{-1}, pq) \\
& \quad = \alpha(e, q) \alpha(p^{-1}, pq) = \alpha(e, e) \alpha(p^{-1}, pq) = \text{sgn}(p^{-1}, pq)
\end{align*}
\]

\[
\square
\]

**Theorem 5.3.** Every proper twist is invertive.

*Proof.* Suppose \(\alpha\) is proper. Then \(\alpha(p, p^{-1}) \alpha(p^{-1}, p) = \alpha(pp^{-1}, p) = \alpha(e, p) = 1\), thus \(\alpha(p, p^{-1}) = \alpha(p^{-1}, p)\). So \(\alpha\) is invertive.

\[
\square
\]

**Theorem 5.4.** If \(\alpha\) is a proper twist on \(G\), then \((i_p i_q)^* = i_q^* i_p^*\) for all \(p, q \in G\).

*Proof.* Since \(i_p i_q = \alpha(p, q)i_{pq}\), \((i_p i_q)^* = (\alpha(p, q)i_{pq})^* = \alpha(p, q)(i_{pq})^* = \alpha(p, q)i_{pq\alpha(q^{-1}, p)}\).

On the other hand,
\[
\begin{align*}
i_{q^{-1}p}^* & \quad = i_q^{-1}i_{p^{-1}} = (\alpha(q^{-1}, q) i_{q^{-1}})(\alpha(p^{-1}, p) i_{p^{-1}}) \\
& \quad = \alpha(q^{-1}, q) \alpha(p^{-1}, p) \alpha(q^{-1}, p^{-1}) i_{q^{-1}p^{-1}} \\
& \quad = \alpha(q^{-1}, q) \alpha(p, p^{-1}) \alpha(q^{-1}, p^{-1}) i_{pq^{-1}}
\end{align*}
\]

Therefore, in order to show that \((i_p i_q)^* = i_q^* i_p^*\), it is sufficient to show that \(\alpha(p, q) \alpha((pq)\alpha(q^{-1}, p)) = \alpha(q^{-1}, q) \alpha(p^{-1}, p) \alpha(q^{-1}, p^{-1})\).

Beginning with the expression on the left,
\[
\begin{align*}
\alpha(p, q) \alpha((pq)^{-1}, pq) & \quad = \alpha(p, q) \alpha(q, q^{-1}) \alpha(q^{-1}, pq) \\
& \quad = \alpha(pq, q^{-1}) \alpha(q, q^{-1}) \alpha((pq)^{-1}, pq) \\
& \quad = \alpha((pq)^{-1}, pq) \alpha(pq, q^{-1}) \alpha(q, q^{-1}) \\
& \quad = \alpha((pq)^{-1}, p) \alpha(q, q^{-1}) \\
& \quad = \alpha((pq)^{-1}, p) \alpha(p, p^{-1}) \alpha(p, p^{-1}) \alpha(q, q^{-1}) \\
& \quad = \alpha(q^{-1}, p^{-1}) \alpha(p, p^{-1}) \alpha(q, q^{-1}) \\
& \quad = \alpha(q^{-1}, p^{-1}) \alpha(p, p^{-1}) \alpha(q, q^{-1}) \\
& \quad = \alpha(q^{-1}, p^{-1}) \alpha(p, p^{-1}) \alpha(q, q^{-1})
\end{align*}
\]

\[
\square
\]
Theorem 5.5. If $\alpha$ is a proper twist on $G$ and if $x, y \in V = [G, \alpha, K]$, then $(xy)^* = y^*x^*$.

Proof.

$$(xy)^* = \left( \left( \sum_p x_p i_p \right) \left( \sum_q y_q i_q \right) \right)^*$$

$$= \left( \sum_{p,q} x_p y_q i_p i_q \right)^*$$

$$= \sum_{p,q} (x_p y_q i_p i_q)^*$$

$$= \sum_{q,p} y^{*^1}_{i_q^*} x^{*^1}_{i_p^*}$$

$$= \left( \sum_q y^{*^1}_{i_q^*} \right) \left( \sum_p x^{*^1}_{i_p^*} \right)$$

$$= y^*x^*$$

\[\square\]

Definition 5.6. The inner product of elements $x$ and $y$ in $V = [G, \alpha, K]$ is $\langle x, y \rangle = \sum_p x_p y^*_p$.

Theorem 5.7. If $\alpha$ is a proper twist on $G$, and if $x, y \in V = [G, \alpha, K]$, then the product $xy$ has the Fourier expansion

$$xy = \sum_r (x, i_r y^*) i_r = \sum_r (y, x^* i_r) i_r$$

Proof.

(i) $y^* = \sum_s \alpha (s, s^{-1}) y^*_{s^{-1}i_s}$. Let $p = rs$. Then

$$i_r y^* = \sum_s \alpha (s, s^{-1}) y^*_{s^{-1}i_s i_r}$$

$$= \sum_s \alpha (r, s) \alpha (s, s^{-1}) y^*_{s^{-1}i_r}$$

$$= \sum_s \alpha (rs, s^{-1}) y^*_{s^{-1}i_r}$$

$$= \sum_p \alpha (p, p^{-1}r) y^*_{p^{-1}i_p}$$

Thus,

$$\langle x, i_r y^* \rangle = \sum_p \alpha (p, p^{-1}r) x_p y^*_{p^{-1}i_r}$$

$$= \sum_p \alpha (p, p^{-1}r) x_p y_{p^{-1}i_r}$$
(ii) \( x^* = \sum_s \alpha(s^{-1}, s) x^*_{s^{-1}i_s} \). Let \( q = sr \). Then
\[
x^* i_r = \sum_s \alpha(s^{-1}, s) x^*_{s^{-1}i_s} i_r
= \sum_s \alpha(s^{-1}, s) \alpha(s, r) x^*_{s^{-1}i_{sr}}
= \sum_s \alpha(s^{-1}, sr) x^*_{s^{-1}i_{sr}}
= \sum_q \alpha(rq^{-1}, q) x^*_{rq^{-1}i_q}
\]

So \( \langle y, x^* i_r \rangle = \sum_q \alpha(rq^{-1}, q) x_{rq^{-1}} y_q \).

(iii) \( xy = \sum_{p,q} x_p y_q i_p i_q \)
\[
= \sum_{p,q} \alpha(p, q) x_p y_q i_{pq}
= \sum_{q,p} \alpha(p, q) x_p y_q i_{pq}
\]

Let \( pq = r \). Then \( p = rq^{-1} \) and \( q = p^{-1}r \).
Thus,
\[
xy = \sum_{r,q} \alpha(rq^{-1}, q) x_{rq^{-1}} y_q i_r = \sum_r \langle y, x^* i_r \rangle i_r.
\]
And
\[
xy = \sum_{r,p} \alpha(p, p^{-1}r) x_p y_{p^{-1}} i_r = \sum_r \langle x, i_r y^* \rangle i_r. \quad \Box
\]

**Corollary 5.8.** If \( \alpha \) is proper, then \( \langle xy, i_r \rangle = \langle x, i_r y^* \rangle = \langle y, x^* i_r \rangle \).

**Definition 5.9.** A proper \(*\)-algebra is an inner product space with an involution \((*)\) satisfying the adjoint properties:

\[
\langle xy, z \rangle = \langle y, x^* z \rangle \quad (5.1)
\]
\[
\langle x, yz \rangle = \langle xz^*, y \rangle \quad (5.2)
\]

for all \( x, y, z \) in the algebra.

Note: The Cayley-Dickson algebras are known to be proper \(*\)-algebras [4].

**Theorem 5.10.** If \( V = [G, \alpha, \mathbb{K}] \) and if \( \alpha \) is proper, then \( V \) is a proper \(*\)-algebra.
Proof. Let \( x, y, z \in G \). Then

\[
\langle xy, z \rangle = \langle xy, \sum_r z_r i_r \rangle \\
= \sum_r \langle xy, z_r i_r \rangle \\
= \sum_r z_r^* \langle xy, i_r \rangle \\
= \sum_r z_r^* \langle y, x^* i_r \rangle \\
= \sum_r \langle y, x^* (z_r i_r) \rangle \\
= \langle y, x^* \sum_r z_r i_r \rangle \\
= \langle y, x^* z \rangle
\]

and

\[
\langle x, yz \rangle = \left\langle \sum_r x_r i_r, yz \right\rangle \\
= \sum_r \langle x_r i_r, yz \rangle \\
= \sum_r x_r \langle i_r, yz \rangle \\
= \sum_r x_r \langle yz, i_r \rangle^* \\
= \sum_r x_r \langle y, i_r z^* \rangle^* \\
= \sum_r x_r \langle i_r z^*, y \rangle \\
= \sum_r \langle (x_r i_r) z^*, y \rangle \\
= \left\langle \sum_r (x_r i_r) z^*, y \right\rangle \\
= \langle x z^*, y \rangle
\]

\[\square\]

The next theorem and its corollaries apply specifically to real Cayley-Dickson and Clifford algebras.

**Theorem 5.11.** Suppose \( V = [G, \alpha] \), \( \alpha \) is proper and \( p = p^{-1} \) for all \( p \in G \). Then for all \( p \in G \), \( [1 - \alpha(p, p)] \langle x, i_p x \rangle = 0 \).

**Proof.** \( i_p x = \sum_q x_q i_p i_q = \sum_q \alpha(p, q)x_q i_p = \sum_r \alpha(p, pr)x_{pr} i_r \) where \( q = pr \).
\[ 2 \langle x, i_p x \rangle = 2 \sum_r \alpha(p, pr)x_r x_{pr} \]
\[ = 2 \alpha(p, p) \sum_r \alpha(p, r)x_r x_{pr} \]
\[ = \alpha(p, p) \sum_r \alpha(p, r)x_r x_{pr} + \alpha(p, p) \sum_q \alpha(p, q)x_q x_{pq} \]
\[ = \alpha(p, p) \sum_r \alpha(p, r)x_r x_{pr} + \alpha(p, p) \sum_r \alpha(p, pr)x_{pr} x_r \]
\[ = \alpha(p, p) \sum_r \alpha(p, r)x_r x_{pr} + \alpha(p, p) \alpha(p, p) \sum_r \alpha(p, r)x_r x_{pr} \]
\[ = [\alpha(p, p) + 1] \sum_r \alpha(p, r)x_r x_{pr} \]
\[ = [\alpha(p, p) + 1] \alpha(p, p) \sum_r \alpha(p, pr)x_r x_{pr} \]
\[ = [1 + \alpha(p, p)] \langle x, i_p x \rangle \]

Thus \( [1 - \alpha(p, p)] \langle x, i_p x \rangle = 0 \). □

**Corollary 5.12.** Suppose \( V = [G, \alpha] \), \( \alpha \) is proper and \( p = p^{-1} \) for all \( p \in G \). Then \( \langle x, i_p x \rangle = 0 \) provided \( \alpha(p, p) = -1 \).

**Corollary 5.13.** Suppose \( V = [G, \alpha] \), \( \alpha \) is proper and \( p = p^{-1} \) for all \( p \in G \). Then \( \langle xx^*, i_p \rangle = 0 \) provided \( \alpha(p, p) = -1 \).

6. The Cayley-Dickson Algebras

The complex numbers can be constructed as ordered pairs of real numbers, and the quaternions as ordered pairs of complex numbers. The Cayley-Dickson process continues this development. Ordered pairs of quaternions are octonions and ordered pairs of octonions are sedenions. For each non-negative integer \( N \) the Cayley-Dickson algebra \( S_N \) is a properly twisted group algebra on \( \mathbb{R}^{2^N} \), with \( S_0 = \mathbb{R} \) denoting the reals, \( S_1 = \mathbb{C} \) the complex numbers, \( S_2 = \mathbb{H} \) the quaternions and \( S_3 = \mathbb{O} \) the octonions.

In the remainder of the paper we will develop the real Cayley-Dickson and Clifford algebras as proper subsets of the Hilbert space \( \ell^2 \) of square summable sequences. The canonical basis for \( \ell^2 \) will be indexed by the group \( \mathbb{Z}^+ = \{0, 1, 2, 3, \cdots\} \) of non-negative integers with group operation the 'bit-wise exclusive or' of the binary representations of the elements.

Equate a real number \( r \) with the sequence \( r, 0, 0, 0, \cdots \). Given two real number sequences \( x = x_0, x_1, x_2, \cdots \) and \( y = y_0, y_1, y_2, \cdots \), equate the ordered pair \( (x, y) \) with the 'shuffled' sequence

\[ (x, y) = x_0, y_0, x_1, y_1, x_2, y_2, \cdots \]
The sequences

\[ i_0 = 1, 0, 0, 0, \cdots \]

\[ i_1 = 0, 1, 0, 0, \cdots \]

\[ i_2 = 0, 0, 1, 0, \cdots \]

\[ i_3 = 0, 0, 0, 1, \cdots \]

\[ \vdots \]

\[ i_{n-1} \]

form the canonical basis for \( \mathbb{R}^n \) and satisfy the identities

\[ i_{2n} = (i_n, 0) \]  
(6.1)

\[ i_{2n+1} = (0, i_n) \]  
(6.2)

This produces a numbering of the unit basis vectors for the Cayley-Dickson algebras which differs from other numberings, yet it arises naturally from equating ordered pairs with shuffled sequences. It also produces, quite naturally, the Cayley-Dickson conjugate identity.

In order to define a conjugate satisfying \( x + x^* \in \mathbb{R} \), we must have \( i_0^* = i_0 \) (else 1 will not be the identity) and \( i_n^* = -i_n \) for \( n > 0 \). This leads to the result

\[ (x, y)^* = (x^*, -y) \]  
(6.3)

The usual way of multiplying ordered pairs of real numbers \( (a, b) \) and \( (c, d) \) regarded as complex numbers is

\[ (a, b)(c, d) = (ac - bd, ad + bc) \]

This method of multiplying ordered pairs, if repeated for ordered pairs of ordered pairs ad infinitum, produces a sequences of algebras of dimension \( 2^N \). However, the four dimensional algebra produced when \( N = 2 \) is not the quaternions but a four dimensional algebra with zero divisors. The twist on \( S_n = \mathbb{R}^{2^n} \) produced by this product is \( \eta(p, q) = (-1)^{p \wedge q} \) with \( p \wedge q \) the bitwise ‘and’ of the binary representations of \( p \) and \( q \) and \( \langle p \rangle \) the ‘sum of the bits’ of the binary representation of \( p \). \( i_p i_q = (-1)^{p \wedge q} i_{pq} \) where \( pq = p \wedge q \): the ‘bit-wise’ “exclusive or” of \( p \) and \( q \).

(This group product is equivalent to addition in \( \mathbb{Z}_2^N \).) Since \( \eta \), considered as a matrix, is a Hadamaard matrix, \( \eta(p, q) = (-1)^{p \wedge q} \) may be termed the ‘Hadamaard twist’. It is a simple exercise to show that the Hadamaard twist is associative.

The product which produces the quaternions from the complex numbers is the Cayley-Dickson product

\[ (a, b)(c, d) = (ac - bd^*, a^*d + cb) \]  
(6.4)

For real numbers \( a, b, c, d \) this is the complex number product. For complex numbers \( a, b, c, d \) this is the quaternion product. For quaternions, it is the octonion product, etc.

One may establish immediately that \( i_0 = (1, 0) \) is both the left and the right identity. Furthermore, applying this product to all the unit basis vectors yields the following identities:
This product on the unit basis vectors recursively defines a product and a twist \( \gamma \) on the indexing sets \( \mathbb{Z}_N^2 \) for each Cayley-Dickson space \( S_N \) in such a way that \( S_N \subset S_{N+1} \subset \ell^2 \) for all \( N \). The product \( pq \) of elements of \( \mathbb{Z}_N^2 \) implied by these identities is the bit-wise ‘exclusive or’ \( p \oplus q \) of the binary representations of \( p \) and \( q \), which is equivalent to addition in \( \mathbb{Z}_2^N \). For this product, \( 0 \) is the identity and \( p^{-1} = p \) for all \( p \in \mathbb{Z}_2^N \).

These identities imply the following defining properties of the Cayley-Dickson twist \( \gamma \):

\[
\begin{align*}
\gamma(0, 0) &= \gamma(p, 0) = \gamma(0, q) = 1 \\
\gamma(2p, 2q) &= \gamma(p, q) \\
\gamma(2p + 1, 2q) &= \gamma(q, p) \\
\gamma(2p, 2q + 1) &= \begin{cases} -\gamma(p, q) & \text{if } p \neq 0 \\ 1 & \text{otherwise} \end{cases} \\
\gamma(2p + 1, 2q + 1) &= \begin{cases} \gamma(q, p) & \text{if } p \neq 0 \\ -1 & \text{otherwise} \end{cases}
\end{align*}
\]

These properties, in turn, imply the quaternion properties:

For \( 0 \neq p \neq q \neq 0 \),

\[
\begin{align*}
\gamma(p, p) &= -1 \\
\gamma(p, q) &= -\gamma(q, p) \\
\gamma(p, q) &= \gamma(pq, q) = \gamma(p, pq)
\end{align*}
\]

The quaternion properties imply that if \( 0 \neq p \neq q \neq 0 \) and if \( i_p i_q = i_r \) then \( i_p i_q = i_q i_r = i_r i_p \) and \( i_p i_q = -i_q i_p \). Thus elements of \( \mathbb{Z}_2^k \) for \( k \geq 2 \) can be arranged into triplets of the form \( (p, q, pq) \) for which \( i_p i_q = i_{pq} \). For \( \mathbb{Z}_2^3 \) these are \( (1, 2, 3), (1, 4, 5), (1, 6, 7), (2, 4, 6), (2, 7, 5), (3, 6, 5) \) and \( (3, 7, 4) \).

The quaternion properties also imply that \( \gamma \) is a proper sign function. That is, for all \( p, q \),

\[
\begin{align*}
\gamma(p, q) \gamma(q, q) &= \gamma(pq, q) \\
\gamma(p, p) \gamma(p, q) &= \gamma(p, pq)
\end{align*}
\]

Thus Cayley-Dickson algebras are proper \(*\)-algebras satisfying the adjoint properties: For all \( x, y, z \),

\[
\begin{align*}
\langle xy, z \rangle &= \langle y, x^* z \rangle \\
\langle x, yz \rangle &= \langle xz^*, y \rangle
\end{align*}
\]
Left open is the question whether $\ell^2$ is the completion of the sequence of Cayley-Dickson algebras. It is clear that if $x \in \ell^2$ then $xy \in \ell^2$ provided $y \in S_N$. It is not clear, however, that $xy \in \ell^2$ for all $x, y \in \ell^2$.

Since the interior of $Z_2^N \times Z_2^N$ is anti-symmetric, by theorem 3.6 we have

**Theorem 6.1.**

\[
x \vee y = x_0y + y_0x + \langle x, y^* \rangle - 2x_0y_0
\]
\[
= x_0y + y_0x - \langle x, y \rangle
\]

7. **The Cayley-Dickson Twist on $Z^+$**

The recursive definition of the Cayley-Dickson twist on $Z^+$ may be restated as follows: For $p, q \in Z^+$ and $r, s \in \{0, 1\}$,

\[
(7.1) \quad \gamma(0, 0) = 1
\]
\[
(7.2) \quad \gamma(2p + r, 2q + s) = \gamma(p, q)A_{pq}(r, s)
\]

where

\[
(7.3) \quad A_{pq} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{if } p = 0
\]
\[
(7.4) \quad = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{if } 0 \neq p = q \text{ or } p \neq q = 0
\]
\[
(7.5) \quad = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \quad \text{if } 0 \neq p \neq q \neq 0.
\]

Define $\gamma_0 = (1)$. Then, for each non-negative integer $N$, $\gamma_{N+1}$ is a partitioned matrix defined by

\[
(7.6) \quad \gamma_{N+1} = \langle \gamma_N(p, q)A_{pq} \rangle
\]

The twist for the sedenions, octonions, quaternions, complex numbers and reals is given by the matrix $\gamma_4$:

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
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1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1
\end{pmatrix}
\]
8. Clifford Algebra

In Clifford algebra [2], the same unit basis vectors \( B = \{ i_p \mid p \in \mathbb{Z}_N^2 \} \) and the same bit-wise ‘exclusive or’ group operation on \( \mathbb{Z}_N^2 \) may be used. Only the twist \( \phi \) will differ.

In Clifford algebra, the unit basis vectors are called ‘blades’. Each blade has a numerical ‘grade’. The grade of the blade \( i_p \) is the sum of the bits of the binary representation of \( p \) denoted here by \( \langle p \rangle \).

\( i_0 = 1 \) is the unit scalar, and is a 0-blade.

\( i_1, i_2, i_4, \cdots, i_{2^n} \) are 1-blades, or ‘vectors’ in Clifford algebra parlance.

\( i_3, i_5, i_6, \cdots \) are 2-blades or ‘bi-vectors’.

\( i_7, i_{11}, i_{13}, i_{14}, \cdots \) are 3-blades or ‘tri-vectors’, etc.

As was the case with Cayley-Dickson algebras, this is not the standard notation for the basis vectors. However, it has the advantage that the product of unit basis vectors satisfies \( i_p i_q = \phi(p, q) i_{pq} \) for a suitably defined Clifford twist \( \phi \) on \( \mathbb{Z}_N^2 \).

In the standard notation, 1-blades or ‘vectors’ are denoted \( e_1, e_2, e_3, \cdots \), whereas 2-blades or ‘bivectors’ are denoted \( e_{12}, e_{13}, e_{23}, \cdots \) etc.

Translating from the \( e \)-notation to the \( i \)-notation is straightforward. For example, the 3-blade \( e_{134} \) translates as \( i_{13} \) since the binary representation of 13 is 1101 with bits 1, 3 and 4 set. The 2-blade \( e_{23} = i_6 \) since the binary representation of 6 is 110, with bits 2 and 3 set.

All the properties of \( n \)-blades can be deduced from five fundamental properties:

1. The square of 1-blades is 1.
2. The product of 1-blades is anticommutative.
3. The product of 1-blades is associative.
4. The conjugate of a 1-blade is its negative.
5. Every \( n \)-blade can be factored into the product of \( n \) distinct 1-blades.

The convention is that, if \( j < k \), then \( e_j e_k = e_{jk} \), thus \( e_k e_j = -e_{jk} \).

Any two \( n \)-blades may be multiplied by first factoring them into 1-blades. For example, the product of \( e_{134} \) and \( e_{23} \), is computed as follows:

\[
e_{134} e_{23} = e_1 e_3 e_4 e_2 e_3 \\
= -e_1 e_4 e_3 e_2 e_3 \\
= e_1 e_4 e_2 e_3 \\
= -e_1 e_2 e_4 \\
= -e_{124}
\]

Since \( e_{134} = i_{13} \) and \( e_{23} = i_6 \), and the bit-wise ‘exclusive or’ of 13 and 6 is 11, the same product using the ‘\( i \)’ notation is

\( i_{13} i_6 = \phi(13, 6) i_{11} \)

so evidently, \( \phi(13, 6) = -1 \).

The Clifford twist \( \phi \) can be defined recursively as follows:

\[
\phi(0, 0) = 1 \\
\phi(2p + 1, 2q) = \phi(2p, 2q) = \phi(p, q) \\
\phi(2p + 1, 2q + 1) = \phi(2p, 2q + 1) = (-1)^p \phi(p, q)
\]

For Clifford algebra, \( x + x^* \) is not generally a real number.
Since

\[(8.4) \quad i_p^* = \phi(p, p)i_p\]

and since it can be shown that

\[(8.5) \quad \phi(p, p) = (-1)^s \text{ where } s = \frac{(p)}{(p) + 1)}\]

it follows that, if \(i_p\) is an \(n\)-blade, then

\[(8.6) \quad i_p^* = \begin{cases} -i_p & \text{if } n = 4k + 1 \text{ or } n = 4k + 2 \\ i_p & \text{if } n = 4k \text{ or } n = 4k + 3 \end{cases}\]

This divides the \(n\)-blades into the ‘real’ blades and the ‘imaginary’ blades, where

1. \(0\)-blades are real
2. \(1\)-blades are imaginary
3. \((n + 2)\)-blades have the opposite ‘parity’ of \(n\)-blades.

The Clifford twist \(\phi\) can be shown to be associative:

\[(8.7) \quad \phi(p, q)\phi(pq, r) = \phi(p, qr)\phi(q, r)\]

Since associative twists are proper, Clifford algebras are proper \(*\)-algebras. That is, for multivectors \(x, y\) and \(z\),

\[(8.8) \quad \langle xy, z \rangle = \langle y, x^*z \rangle\]

\[(8.9) \quad \langle x, yz \rangle = \langle xz^*, y \rangle\]

9. Conclusion

There are a number of interesting questions about properly twisted group algebras.

The union \(S_\infty = \bigcup_{N=0}^{\infty} S_N\) of all the finite dimensional Cayley-Dickson algebras is itself a Cayley-Dickson algebra. However, it is incomplete. The Hilbert space \(\ell^2 \supset S_\infty\) of square-summable sequences is a natural completion for \(S_\infty\). Furthermore, for \(x, y \in \ell^2\) the product \(xy\) is a well-defined sequence. However, the product \(xy\) is not obviously square-summable so it is not clear that \(\ell^2\) is closed under the Cayley-Dickson product. It is easily shown, though, that if \(x \in \ell^2\) and \(y \in S_N\), then \(xy \in \ell^2\).

**Question 9.1.** Is the Hilbert space \(\ell^2\) of square-summable sequences a Cayley-Dickson algebra?

It follows from a result of Wedderburn [5] that for \(x, y \in S_N\) \(\| xy \| \leq \sqrt{2^N} \| x \| \| y \|\). Computer generation of random products \(xy\) in \(S_9\) by the author has never produced a value of \(\| xy \|\) larger than \(\sqrt{2^9}\). It is tempting to conjecture that \(\| xy \| \leq \sqrt{2^N} \| x \| \| y \|\) for all Cayley-Dickson algebras. If the conjecture is true, then the preceding question may be answered in the affirmative.

**Question 9.2.** Is there a Cayley-Dickson algebra and elements \(x\) and \(y\) such that \(\| xy \| > \sqrt{2^N} \| x \| \| y \|\)?
In the Clifford algebras, the $n$-blades are distinct products of $n$ 1-blades and the properties of the basis vectors derive from the properties of 1-blades. A similar blade construction can be specified for the basis vectors of Cayley-Dickson algebras.

Cayley-Dickson basis vectors, or $n$-blades should be deducible from the properties of Cayley-Dickson 1-blades:

1. The square of 1-blades is $-1$.
2. The product of 1-blades is anticommutative.
3. The conjugate of a 1-blade is its negative.
4. Every $n$-blade can be factored into the ‘left’-product of $n$ distinct 1-blades.

An example of this last is $i_{51} = e_{1245} = e_1(e_2e_4e_5)$. This works since, by induction, $\gamma(2^n, k2^{n+1}) = 1$ for all non-negative integers $n$ and positive integers $k$. The Cayley-Dickson product is non-associative past the quaternions, so working out the correct sign for the product of Cayley-Dickson blades from their factors would be a challenge.

**Question 9.3.** Can any insights into Cayley-Dickson algebras be gleaned from representing the basis vectors as products of 1-blades?

The Cayley-Dickson spaces are implicit in equations 6.3 and 6.4.

**Question 9.4.** Do there exist equations corresponding to equations 6.3 and 6.4 for the Clifford algebras?

The existence of the Fourier expansion (Theorem 5.7) and the adjoint properties of a twisted group product depend upon the twist being proper. This makes propriety an interesting twist property.

**Question 9.5.** How many proper twists exist for a given group? Which of these have interesting algebras?

The proper twists on the groups $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$ and $\mathbb{Z}_5$ can be found by ‘back of the envelope’ investigation, and they all are commutative and associative.

**Question 9.6.** Are all proper twists on $\mathbb{Z}_N$ commutative and associative?

If $G$ is a group and if $\mathcal{P}(G)$ is the set of all proper twists on $G$, then $\mathcal{P}(G)$ is itself an abelian group with identity the trivial twist $\iota(p, q) = 1$ for all $p, q \in G$ and product $\alpha\beta(p, q) = \alpha(p, q)\beta(p, q)$ for $\alpha, \beta \in \mathcal{P}(G)$.

**Question 9.7.** How is the abelian group $\mathcal{P}(G)$ related to $G$? Is there an interesting theory here?

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