SHIFTED CONVOLUTION SUMS FOR $GL(3) \times GL(2)$ AVERAGED OVER WEIGHTED SETS

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ABSTRACT. Let $A(1, m)$ be the Fourier coefficients of a $SL(3, \mathbb{Z})$ Hecke-Maass cusp form $\pi_1$ and $\lambda(m)$ be those of a $SL(2, \mathbb{Z})$ Hecke holomorphic or Hecke-Mass cusp form $\pi_2$. Let $\mathcal{C} \subset [-X^{1-\epsilon}, X^{1+\epsilon}]$ and $|a(h)|_{h \in \mathcal{C}} < C$ be a sequence. We show that if $\mathcal{C} \subset \ell + [0, X^{1/2+\epsilon}]$ for some $\ell \geq 0$, 

$$D_{a, \mathcal{C}}(X) := \frac{1}{|\mathcal{C}|} \sum_{h \in \mathcal{C}} a(h) \sum_{m=1}^{\infty} A(1, m) \lambda(m + h) V \left( \frac{m}{X} \right) \ll \varepsilon_{\pi_1, \pi_2, \epsilon} X^{1+\epsilon}$$

for any $\epsilon > 0$, and a similar bound when $|\mathcal{C}|$ is big. This improves Sun’s bound and generalizes it to an average with arbitrary weights. Moreover, we demonstrate how one can recover the factorizable moduli structure given by the Jutila’s circle method via studying a shifted sum with weighted average. This allows us to recover Munshi’s bound on the shifted sum with a fixed shift without using the Jutila’s circle method.

1. INTRODUCTION

The shifted convolution sum problem concerns sums of the form

$$\sum_{m=1}^{\infty} a(m) b(m + \ell) V \left( \frac{m}{X} \right).$$

Such a sum is related to many different problems by specializing the sequences $a$ and $b$, and many cases have been extensively studied. Obtaining a non-trivial bound often yields deep applications. For example, if $a$ and $b$ are specialized to be combinations of von Mangoldt function, Mobius function and/or the divisor function, these shifted sums are related to classical problems including twin prime conjecture, Chowla conjecture, moments of zeta function and many others. When $a$ and $b$ are specialized to be coefficients of GL(2) coefficients, these shifted sums are related to subconvexity and Quantum Unique Ergodicity, see for example [2], [5], [7], [9], [10], [13], [15], [18], [19], [24], [25].

We now consider a higher rank analogue, when $a$ is a GL(3) coefficient and $b$ is a GL(2) coefficient. Pitt [23] studied the shifted sum between the ternary divisor function and a GL(2) coefficient. Let $A(1, m)$ be the Fourier coefficients of a $SL(3, \mathbb{Z})$ Hecke-Maass cusp form $\pi_1$ and $\lambda(m)$ be those of a $SL(2, \mathbb{Z})$ Hecke holomorphic of Hecke-Mass cusp form $\pi_2$. For a $a$ coming from a $SL(3, \mathbb{Z})$ cusp form, the first breakthrough is done by Munshi in [22], who obtained (after a minor correction in his Lemma 11)

$$D_h(X) = \sum_{m=1}^{\infty} A(1, m) \lambda(m + h) V \left( \frac{m}{X} \right) \ll \varepsilon_{\pi_1, \pi_2, \epsilon} X^{1-1/26+\epsilon}.$$ 

He used the Jutila’s circle method in order to have factorizable moduli, which provides him a way to better balance the mass between the diagonal and offdiagonal contribution and obtained the first non-trivial bound. He also showed a similar result for the ternary divisor function case in [21]. This result was later improved by Xi in [27], who showed that

$$D_h(X) \ll \varepsilon_{\pi_1, \pi_2, \epsilon} X^{1-1/22+\epsilon}.$$ 

He used a similar starting point as Munshi’s idea, exploiting the factorizable moduli structure given by the Jutila’s circle method. Additionally, he used an exponent pair argument that works on moduli that are smooth numbers to obtain the extra saving.

In the mean time, inspired by a comment made by Munshi in [22], Sun studied the shifted sum with an average over the shifts in [26]. For any integer $r \geq 1$, she showed that

$$\frac{1}{H} \sum_{H=1}^{\infty} W \left( \frac{h}{H} \right) \sum_{m=1}^{\infty} A(1, m) \lambda(m + h) V \left( \frac{m}{X} \right) \ll \varepsilon_{\pi_1, \pi_2, \epsilon} \begin{cases} 
X^{-A} & \text{for any } A > 0 \\
X^{1-\delta+\epsilon} & \text{if } (rX)^{1/2+\epsilon} \leq H \leq X \\
X^{3/2}X^{1/4+7\delta/2} & \text{if } r \leq (rX)^{1/2+\epsilon}. 
\end{cases}$$
1.1. **Main results.** Let \( r > 0 \) be a fixed positive integer. Let \( \varepsilon > 0 \), \( \mathcal{H} \subset [-X^{1-\varepsilon}, X^{1+\varepsilon}] \) and \( \{a(h)\}_{h \in \mathcal{H}} \) be a sequence of complex numbers. Here \([a, b] \) denotes the set \( \{x \in \mathbb{Z} : a \leq x < b\} \). Write \( \|a\|_2^2 = \sum_{h \in \mathcal{H}} |a(h)|^2 \). In this paper, we study the \( GL(3) \times GL(2) \) shifted convolution sum averaged over weighted sets,

\[
D_{a, \mathcal{H}}(X) := \frac{1}{|\mathcal{H}|} \sum_{h \in \mathcal{H}} a(h) \sum_{m=1}^{\infty} A(1, m) \lambda(m + h) V \left( \frac{m}{X} \right). 
\] (1.1)

**Remark.** Fixing \( r \) here is purely due to aesthetic reasons. One can easily generalize the arguments here to varying \( r > 0 \), but the statements and proofs will be messier.

The main goal of this paper is twofold. The **first goal** is to prove the following two main theorems, which improves Sun’s bound in [26] and generalize it to weighted shifts averaged over an arbitrary set.

**Theorem 1.1.** Let \( \varepsilon > 0 \) and let \( \mathcal{H} \subset [-X^{1-\varepsilon}, X^{1+\varepsilon}] \). We have

\[
D_{a, \mathcal{H}}(X) \ll_{\eta_1, \eta_2, \varepsilon} X^{3/4 + \varepsilon} \frac{1}{|\mathcal{H}|} \|a\|_2 \left( \sum_{h \in X^{1/2 + \varepsilon} \cap \mathcal{H}} \sup \left| \{h' \in \mathcal{H} : h' \equiv h \pmod{b} \} \right| \right)^{1/2}. 
\]

In particular, if \( \mathcal{H} \subset \ell + [0, X^{1/2 + \varepsilon}] \) for some \( \ell \geq 0 \), then we have

\[
D_{a, \mathcal{H}}(X) \ll_{\eta_1, \eta_2, \varepsilon} X^{1/2 + \varepsilon} \frac{1}{|\mathcal{H}|} \|a\|_2; 
\]

and if \( \mathcal{H} \supset \ell + [0, H] \) for some \( H \gg X^{1/2 + \varepsilon} \), then we have

\[
D_{a, \mathcal{H}}(X) \ll_{\eta_1, \eta_2, \varepsilon} X^{3/4 + \varepsilon} \frac{1}{\sqrt{|\mathcal{H}|}} \|a\|_2. 
\]

Compared to the current best bound \( X^{1-1/22 + \varepsilon} \) for a fixed shift obtained by Xi in [27], this shows that having an average of shifts is beneficial when \( |\mathcal{H}| \gg X^{1/11} \).

If \( \mathcal{H} \) and \( \{a(h)\} \) are ‘factorizable’ in the following manner: Let \( D, Q_1, Q_2 > 0 \) such that \( DQ_1Q_2 < X^{1-\varepsilon} \) and let \( \mathcal{O}_1 \subset [1, Q_1] \). Let \( \{a'(q_1)\}_{q_1 \in \mathcal{O}_1} \) be a sequence of complex numbers. Let \( V_1, V_2 \in C_0^\infty(\mathbb{R}^*) \) be fixed functions. Let \( 0 \leq \ell \ll X^{1+\varepsilon} \) and suppose \( a(h) \) is of the form

\[
a_\ell(h) = \sum_{\pm dq_1, q_2 = h-\ell} V_1 \left( \frac{d}{D} \right) V_2 \left( \frac{q_2}{Q_2} \right) a'(q_1) 
\]

and

\[
\mathcal{H} \supset \{h \in \mathbb{Z} : a_\ell(h) \neq 0\},
\]

then we have the following theorem.

**Theorem 1.2.** Let \( \varepsilon > 0 \). Suppose \( \mathcal{H} \) and \( \{a_\ell(h)\} \) satisfy (*). If \( D + Q_1 \gg X^{1/2 + \varepsilon} \), we have

\[
D_{a_\ell, \mathcal{H}}(X) \ll_{\eta_1, \eta_2, \varepsilon} X^{1/2 - A} 
\]

for any \( A > 0 \). Otherwise, if \( \mathcal{O}_1 \subset [1 \text{ or Primes in } [Q_1, 2Q_1]] \) with \( Q_1Q_2 \gg X^{1/2 + \varepsilon} \), we have

\[
D_{a_\ell, \mathcal{H}}(X) \ll_{\eta_1, \eta_2, \varepsilon} X^{1/2 - A} \left( 1 + \frac{\sqrt{XQ_1}}{DQ_2} \right)^{1/2} \|a\|_\infty
\]

The special case (*) is of particular interest to us. Not only do we improve and generalize Sun’s result in [26] by considering \( D = 1 \) and \( \mathcal{O}_1 = [1] \) with \( a'(1) = 1 \), such factorization is related to our **second goal** of this paper: To recover the factorizable moduli structure of the Jutila’s circle method, which we now explain and exemplify with the fixed shift problem.

Let \( 0 \leq \ell \ll X^{1+\varepsilon} \), write \( Y = X + \ell \) and consider

\[
D_\ell(X) = \sum_{m=1}^{\infty} A(1, m) \lambda(m + \ell) V \left( \frac{m}{X} \right)
\]
as before. Let \( \varphi \in C_c^\infty([1/2, 5/2]) \) such that \( \varphi(x) = 1 \) for \( 1 \leq x \leq 2 \). Applying a divisor or hyperbola trick similar to the initial steps on how the reformulation of the Delta method is proved (see Lemma A.1 and appendix A), we obtain the following simple Lemma.

**Lemma 1.3.** Let \( q \) be a positive integer and \( D > 0 \) be a parameter. Let \( F \) be any function such that \( F(0) = 1 \). Then we have

\[
\delta(n - 0) = \delta(n \equiv 0 \pmod{q}) F\left( \frac{n}{Dq} \right) - \sum_{0 \neq d \in \mathbb{Z}} \delta(n = d q) F\left( \frac{d}{D} \right).
\]

In particular, for any set of positive integers \( \mathcal{D} \) with a sequence \( \{b(q)\}_{q \in \mathcal{D}} \), we have

\[
D_{\mathcal{D}}(X) = M.T. - A.S^+. - A.S^-,
\]

where

\[
M.T. := \frac{1}{|\mathcal{D}|} \sum_{q \in \mathcal{D}} b(q) \sum_{m=1}^{\infty} A(1, m) V\left( \frac{m}{X} \right) \sum_{n=1}^{\infty} \lambda(n) \varphi\left( \frac{n}{Y} \right) \delta(m + \ell \equiv n \pmod{q}) F\left( \frac{m + \ell - n}{Dq} \right)
\]

and

\[
A.S^\pm := \frac{1}{|\mathcal{D}|} \sum_{q \in \mathcal{D}} b(q) \sum_{d=1}^{\infty} \sum_{m=1}^{\infty} A(1, m) V\left( \frac{m}{X} \right) \lambda(m + \ell \pm d q) \varphi\left( \frac{m + \ell \pm d q}{Y} \right) F\left( \frac{\pm d}{D} \right).
\]

**Proof.** Using \( F(0) = 1 \), we have

\[
\sum_{0 \neq d \in \mathbb{Z}} \delta(n = d q) F\left( \frac{d}{D} \right) = \sum_{0 \neq d \in \mathbb{Z}} \delta(n = d q) F\left( \frac{n}{Dq} \right)
\]

\[
= \sum_{d \in \mathbb{Z}} \delta(n = d q) F\left( \frac{n}{Dq} \right) - \delta(n = 0)
\]

\[
= \delta(n \equiv 0 \pmod{q}) F\left( \frac{n}{Dq} \right) - \delta(n = 0).
\]

\[\square\]

With

\[
\delta(m + \ell \equiv n \pmod{q}) = \frac{1}{q} \sum_{a (\mod{q})} e\left( \frac{a(m + \ell - n)}{q} \right) = \sum_{0 \neq q' \equiv q \pmod{q'}} e\left( \frac{a(m + \ell - n)}{q'} \right),
\]

\( M.T. \) is almost the main term when one applies the Jutila’s circle method, with the only difference being the presence of the \( q_0 \)-sum. In application, such a difference is easy to handle by choosing \( \mathcal{D} \) carefully. On the other hand, \( A.S^\pm \) is precisely a shifted sum with the sums summing over the set \( \mathcal{H} = \{ \ell \pm d q : F(d) \neq 0, q \in \mathcal{D} \} \) with weights \( a(h) = \sum_{d q = h} F\left( \frac{\pm d}{D} \right) \), with the sum going over \( \{d : F(d) \neq 0\} \), \( q \in \mathcal{D} \). Applying a dyadic subdivision on the \( d \)-sum followed by Theorem 1.2, we recover Munshi’s result on the \( GL(3) \times GL(2) \) shifted convolution sum for a fixed shift by analysing the average version of the shifted sum, instead of invoking the Jutila’s circle method.

**Theorem 1.4.** ([22, Thm 1]) For \( 0 < h \ll X^{1+\varepsilon} \),

\[
\mathcal{D}_h(X) \ll X^{1-1/26+\varepsilon}.
\]

Looking at the shifted convolution sum problem without an average over shifts, we have demonstrated how one can obtain an arbitrary moduli structure allowed in the Jutila’s circle method by studying an average shifted sum instead of the implicit error term that comes with the Jutila’s circle method. This framework works in general, as depicted by the simplicity of Lemma 1.3. Surprisingly in the case of \( GL(3) \times GL(2) \) shifted sum, the bound coming from the average shifted sum we obtain in Theorem 1.2 matches with the error term treatment in the Jutila’s circle method, and hence giving the exact same bound Munshi obtained in [22] (after fixing the error in his Lemma 11).\(^1\) (See Section 6 for details.) However, the method to bound \( A.S^\pm \) is quite different from how the error term in the Jutila’s circle method is treated, which has the following two implications.

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\(^1\)The moduli structure we have here is essentially of the form \( |Q_1, 2Q_1| \times |\text{Primes in } [Q_2, 2Q_2]| \) for some \( Q_1, Q_2 > 0 \). This is technically different from Munshi’s choice of the form \( |\text{Primes in } [Q_1, 2Q_1]| \times |\text{Primes in } [Q_2, 2Q_2]| \). However, the difference is technical but not essential.
• One can improve the resulting bound by improving the bound on the average shifted sums.

• It may lead to different estimates if one applies this framework in other problems when the Jutila’s circle method is helpful.

On the other hand, recovering a special case of the Jutila’s circle method using our Delta method (see Section 3) is interesting on its own. The author showed in [15, Ch 3] that appropriate choices of parameters in our Delta method recovers most Delta methods together with a simple case of the Jutila’s circle method. It is then natural to ask if we can recover other special cases of the Jutila’s circle method. Since our Delta method is used in proving Theorem 1.1 and 1.2, the above discussions provide a partial affirmative answer.

Unfortunately, we have not been able to recover Xi’s $x^{1-1/22+\varepsilon}$ bound for the fixed shift problem in [27]. This is due to the fact that his method seems to crucially rely on the choice of moduli being square-free smooth numbers. In such a case, Theorem 1.1 and the methods used in proving Theorem 1.2 are not sufficient to yield a good enough bound for these kind of shifts.

1.2. **Two crucial tricks.** To prove Theorem 1.1 and 1.2, we used the Delta method (Lemma 3.1) to separate the oscillations. In the process, we used two simple, yet crucial, tricks to obtain the main results.

1. At some Cauchy Schwarz inequality step, we apply a lengthening trick to the outer sum, i.e.

$$\sum_{m \leq M} |F(m)|^2 \ll \sum_{m \leq M L} |F(m)|^2$$

for any $L \geq 1$. This lengthening trick can be seen as a simpler analogue of Xiannan Li’s trick on his recent paper [16]. By adding more summation terms outside an absolute value square, we boost the diagonal contribution while lowering the offdiagonal contribution. See Section 4.3 and 5.3.2 for details.

2. To prove Theorem 1.2, we add a small oscillation right before we replace the Delta symbol by our Delta method, i.e. if $b > 0$, we have

$$\delta(a+b) = \delta(a) b^2 e^{2\pi i t}$$

for any $t \in \mathbb{R}$. This allows us to remove certain zero frequency terms coming from Poisson summations that would otherwise be difficult to deal with. See Lemma 5.1 and its proof in Section 5.4 for details.

**Notation.** Throughout the paper, $\varepsilon > 0$ is a very small number that may vary depending on context. We denote $e(x) = e^{2\pi i x}$. We write $[a, b]$ as the set $\{x \in \mathbb{Z} : a \leq x < b\}$. We say a function $F$ is $Z$-inert if it satisfies the bound $x^{-\frac{1}{2}} \frac{d^i}{dx^i} f(x) \ll Z^j$ for any $j \geq 0$.

2. **Preliminaries**

2.1. **$SL(3, \mathbb{Z})$ Maass forms.** Let $\pi$ be a Maass form of type $(\nu_1, \nu_2)$ for $SL_3(\mathbb{Z})$, which is an eigenfunction for all the Hecke operators. Let the Fourier coefficients be $A(n_1, n_2)$, normalized so that $A(1, 1) = 1$. The Langlands parameter $(\alpha_1, \alpha_2, \alpha_3)$ associated with $\pi$ are $\alpha_1 = -\nu_1 - 2\nu_2 + 1$, $\alpha_2 = -\nu_1 + \nu_2$, $\alpha_3 = 2\nu_1 + \nu_2 - 1$. We refer the reader to Goldfeld’s book [6] for more details.

The Fourier coefficients satisfy the following bound.

**Lemma 2.1.** We have

$$\sum_{m \leq M} |A(n, m)|^2, \sum_{m \leq M} |A(n, m)|^2 \ll M^{1+\varepsilon}.$$ 

Together with the Hecke relations [6, Thm 6.4.11] and Mobius inversion, we have the following Corollary.

**Corollary 2.1.1.** We have

$$\sum_{n_1 \leq N_1} \sum_{n_2 \leq N_2} |A(n_1, n_2)|^2 \ll (N_1 N_2)^{1+\varepsilon}.$$

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2They include the trivial Delta method [1], Kloosterman’s refinement of the circle method [12], DFI Delta method [4] and some special cases of the GL(2) Delta method (using Petersson trace formula as a Delta symbol).
Proof. Applying the Hecke multiplicativity, we have
\[
\sum_{n_1 \leq N_1} \sum_{n_2 \leq N_2} |A(n_1, n_2)|^2 = \sum_{n_1 \leq N_1} \sum_{n_2 \leq N_2} \sum_{d|\text{lcm}(n_1, n_2)} \mu(d) A(n_1/d, 1) A(1, n_2/d) |^2.
\]
Applying Cauchy-Schwarz inequality to take out the \(d\)-sum, the above is bounded by
\[
\ll X^2 \sum_{n_1 \leq N_1} \sum_{n_2 \leq N_2} \sum_d |A(n_1/d, 1) A(1, n_2/d)|^2
\]
\[
= X^2 \sum_d \sum_{n_1 \leq N_1/d} \sum_{n_2 \leq N_2/d} |A(n_1, 1) A(1, n_2)|^2.
\]
Then Lemma 2.1 yields the desired bound. \(\square\)

We will also need the following Voronoi summation formula. Let \(g\) be a compactly supported smooth function on \(\mathbb{R}^+\), and let \(\tilde{g}(s) = \int_0^\infty g(x) x^{s-1} \, dx\) be its Mellin transform. For \(s > 1 + \max\{\text{Re}(\alpha_1), \text{Re}(\alpha_2), \text{Re}(\alpha_3)\}\) and \(a = 0, 1\), define
\[
\gamma_a(s) = \frac{\pi^{-3s/2}}{2} \prod_{i=1}^3 \Gamma\left(\frac{s}{2} - \frac{a + \alpha_i}{2}\right).
\]
Further set \(\gamma_{\pm}(s) = \gamma_0(s) \pm i \gamma_1(s)\) and let
\[
G_{\pm}(y) = \frac{1}{2\pi i} \int_{(u)} y^{-s} \gamma_{\pm}(s) \tilde{g}(-s) \, ds.
\]
Then we have the following lemma. (See [3], [17], [20]).

Lemma 2.2. Let \(h\) be a compactly supported smooth function on \((0, \infty)\). We have
\[
\sum_{n=1}^\infty \lambda(n) e\left(\frac{an}{c}\right) g\left(\frac{n}{X}\right) = \sum_{n_0 | c} \lambda(n, n_0) \sum_{n|n_0} S\left(\frac{a}{c}, \pm; n_0; n_0\right) G_{\pm}\left(\frac{n_0^2 n X}{c^3}\right),
\]
where \((a, c) = 1\) and \(a c^{-1} \equiv 1 \mod c\).

Suppose \(g\) is supported on a fixed compact set in \(\mathbb{R}^+\) and it is \(X^\epsilon\)-inert, i.e. \(x^{-j} \frac{d^j}{dx^j} g(x) \ll X^{1-\epsilon}\) for any \(j \geq 0\). Then a standard integral analysis on \(G_{\pm}(n_0^2 n X/c^3)\) implies that we get arbitrary saving unless \(n_0^2 n \ll c^3 X^{-1+\epsilon}\). Moreover, by [3, Lemma 7], we have
\[
x^{-j} \frac{d^j}{dx^j} G_{\pm}(x) \ll j (X^\epsilon + x^{1/3})^j x^{2i/3} \|g\|_\infty.
\]
(2.1)

2.2. \(SL(2, \mathbb{Z})\) cusp forms. Let \(f\) be a primitive holomorphic Hecke eigen cusp form or a primitive \(GL(2)\) Hecke-Maass cusp form of full level. Let \(\lambda_f(n)\) be its Fourier coefficients. Then \(f\) has the Fourier expansion
\[
f(z) = \sum_{n \geq 1} \lambda_f(n) n^{k-1} e(nz)
\]
if \(f\) is holomorphic of weight \(k\); and
\[
f(z) = \sqrt{y} \sum_{n \neq 0} \lambda_f(n) K_{1/2}(2\pi|n|y) e(nx)
\]
if \(f\) is a Maass form with Laplace eigenvalue \(\frac{1}{4} + t^2 \geq 0\).

These Fourier coefficients satisfy the following bound.

Lemma 2.3. We have
\[
\sum_{n \ll N} |\lambda(n)| \ll N^{1+\epsilon}.
\]
Moreover, we have the following Voronoi summation formula.

Lemma 2.4. [13, Thm A.4] Let \(a, c \neq 0\) be integers such that \((a, c) = 1\) and let \(F \in C_c^\infty(\mathbb{R}^+)\). Then we have
\[
\sum_{n \geq 1} \lambda_f(n) e\left(\frac{an}{c}\right) F\left(\frac{n}{X}\right) = X \sum_{n \geq 1} \lambda_f(\pm n) e\left(\pm \frac{an_2 n}{c}\right) \int_0^\infty F(y) J_{\pm,f} \left(\frac{4\pi \sqrt{ny}}{c}\right) \, dy.
\]
where
\[ J_{+}f(x) = 2\pi i^k J_{k-1}(x) \]
and \( J_{-}f = 0 \) if \( f \) is holomorphic of weight \( k \), and
\[ J_{+}f(x) = -\frac{\pi}{\sin(\pi i r)} (J_{2i r}(x) - J_{-2i r}(x)) \]
and
\[ J_{-}f(x) = \epsilon_f 4 \cosh(\pi r) K_{2i r}(x) \]
if \( f \) is Hecke-Maass with Laplace eigenvalue \( \frac{1}{4} + i^2 \), and \( \epsilon_f = \begin{cases} 1 & \text{if } f \text{ is even} \\ -1 & \text{if } f \text{ is odd} \end{cases} \).

If \( F \) is a \( Z \)-inert function, by the derivative properties of the Bessel functions, repeated integration by parts on the \( y \)-integral gives us arbitrary saving unless \( n \ll c^2(1 + Z^2)X^{-1+e} \).

### 2.3. Integral analysis

We need the following lemma for a simple integral analysis.

**Lemma 2.5.** [11, Lemma 3.1] Suppose that \( w \) is a \( Z \)-inert function with a fixed compact support on \( \mathbb{R}^+ \). Let \( \phi \) be a real-valued, smooth function. Let
\[ I = \int_{-\infty}^{\infty} w(t)e^{i\phi(t)} \, dt. \]
If \( |\phi'(t)| \gg Z X^\epsilon \) for all \( t \) in the support of \( w \), then \( I \ll AZ^{-A} \) for any \( A > 0 \).

### 3. Reformulation of the Delta method

To prove Theorem 1.1 and Theorem 1.2, we apply our reformulation of the (DFI) Delta method. The following version can be viewed as a simplified version of the DFI delta method (see [4], [8]), and it is introduced by the author in [14] and [15, Ch 3]. For a more general statement with its proof, see Lemma A.1 in Appendix A.

**Lemma 3.1.** Let \( \epsilon > 0 \), \( n, q \) be integers such that \( q > 0 \), \( |n| \ll N \to \infty \) and let \( C > N^\epsilon \) be a parameter. Let \( U \in C_c^\infty(\mathbb{R}) \), \( W \in C_c^\infty([-2,-1]\cup[1,2]) \) be a fixed non-negative even function such that \( U(x) = 1 \) for \( -2 \leq x \leq 2 \). Then we have
\[ \delta(n = 0) = \frac{1}{c} \sum_{c \equiv 1 (\text{mod } cq)} \frac{1}{c} \sum_{\alpha (\text{mod } cq)} e\left(\frac{an}{c} \right) h\left(\frac{c}{C}, \frac{n}{cCq}\right), \]
with \( c = \sum_{c \equiv 1} W_i \left(\frac{c}{C}\right) \sim C \) and
\[ h(x, y) = W(x) U(x) U(y) - W(y) U(x) U(y). \]
In particular, \( h \) is a fixed smooth function satisfying \( h(x, y) \ll \delta(|x|, |y|) \ll 1 \).

Compared with the form written in [8], Lemma 3.1 is essentially the DFI Delta method with a simpler weight function \( h \) that restricts \( |n| \ll cCq \). This particular feature enables an easier integral analysis when dual summations are applied. See [14] for such an application and [15, Ch 3] for further discussions.

### 4. Proof of Theorem 1.1

Let \( Y = X + \max(|h| : h \in \mathcal{H}) \), then \( X \leq Y \ll X^{1+\epsilon} \). Let \( \varphi \in C_c^\infty([1/2, 5/2]) \) such that \( \varphi(x) = 1 \) for \( 1 \leq x \leq 2 \). Then we have
\[ D_{a, \mathcal{H}}(X) = \frac{1}{|\mathcal{H}|} \sum_{h \in \mathcal{H}} a(h) \sum_{m=1}^{\infty} A(1, m) \lambda(r m + h) V\left(\frac{m}{X}\right) \]
\[ = \frac{1}{|\mathcal{H}|} \sum_{h \in \mathcal{H}} a(h) \sum_{m} A(1, m) V\left(\frac{m}{X}\right) \sum_{n} \lambda(n) \varphi\left(\frac{n}{Y}\right) \delta(n = r m + h). \]
4.1. Applying our Delta method. Let \( C > 0 \) be a parameter such that \( C > X^{1/2+\epsilon} \). Applying Lemma 3.1 with \( q = 1 \) and

\[
\sum_{a \pmod{c}} e \left( \frac{an}{c} \right) = \sum_{b_0, b = e a \pmod{b}} \sum_{h} A(1, m)V \left( \frac{m}{X} \right) \sum_{n} \lambda(n) \varphi \left( \frac{n}{Y} \right)
\]

we have

\[
D_{a, \mathcal{A}}(X) = \frac{1}{\mathcal{C}|\mathcal{A}|} \sum_{a \pmod{c}} \sum_{b} \frac{1}{b_0 b^2} \sum_{h \in \mathcal{A}} a(h) \sum_{m} A(1, m) V \left( \frac{m}{X} \right) \sum_{n} \lambda(n) \varphi \left( \frac{n}{Y} \right)
\]

\[
\times \sum_{a \pmod{b}} e \left( \frac{a(r m + h n)}{b} \right) h \left( \frac{b_0 b + r m + h n}{b_0 b C} \right),
\]

for some fixed smooth function \( h \) satisfying \( h(x, y) \ll \delta(|x|, |y|) \ll 1 \).

4.2. Voronoi summations. Applying Voronoi summation to the \( n \)-sum with Lemma 2.4 yields

\[
D_{a, \mathcal{A}}(X) = \frac{Y}{\mathcal{C}|\mathcal{A}|} \sum_{n \geq 1} \sum_{b} \frac{1}{b_0 b^2} \sum_{h \in \mathcal{A}} a(h) \sum_{m} A(1, m) V \left( \frac{m}{X} \right) \sum_{n} \lambda(n) \eta_2 n
\]

\[
\times \sum_{a \pmod{b}} e \left( \frac{a(r m + h n + \eta_2 \pi n)}{b} \right) \int_{0}^{\infty} V(y) h \left( \frac{ab}{C} \right) \frac{r m + h y}{b_0 b C} \text{J}_{\eta_2, \pi} \left( \frac{4\pi \sqrt{Y n y}}{b} \right) dy,
\]

with \( J_{\eta_2, \pi} \) as defined in Lemma 2.4. Repeated integration by parts gives us arbitrary saving unless

\[
1 \leq n \ll \frac{b^2 X^\epsilon}{Y} \left( 1 + \frac{Y}{b_0 b C} \right)^2.
\]

By the condition \( C \gg X^{1/2+2\epsilon} \), the above restriction becomes

\[
n \ll \frac{b^2}{Y^{1-\epsilon}} \quad \text{and} \quad b \gg Y^{1/2-\epsilon}.
\]

Write \( r_0 = (r, b), \ b = r_0 b', \ r = r_0 r' \). Applying Voronoi summation on the \( m \)-sum with Lemma 2.2 yields

\[
D_{a, \mathcal{A}}(X) = \frac{Y}{\mathcal{C}|\mathcal{A}|} \sum_{n \geq 1} \sum_{b} \frac{1}{b_0 b^2} \sum_{h \in \mathcal{A}} a(h) \sum_{m_0} \sum_{m} \sum_{m_0} \sum_{m} A(m, m_0) \sum_{n} \lambda(n) \eta_2 n
\]

\[
\times S_0(m, n; r_0 m_0, b'/m_0) F_1 \left( \frac{m_0^2 m X}{b'^3}, \frac{n Y}{b'^2}; h \right) + O(X^{-999}),
\]

where

\[
S_0(m, n; r_0 m_0, b'/m_0) = \sum_{a \pmod{b}} e \left( \frac{a h + \eta_2 \pi n}{b} \right) S(a r', \eta_1 m; b'/m_0)
\]

and

\[
F_1 \left( \frac{m_0^2 m X}{b'^3}, \frac{n Y}{b'^2}; h \right) = \int_{0}^{\infty} \int_{0}^{\infty} V(x) G_{\eta_1} \left( \frac{m_0^2 m X}{b'^3}, Y y; h \right) \text{J}_{\eta_2, \pi} \left( \frac{4\pi \sqrt{Y n y}}{b'} \right) dx dy,
\]

with \( G_{\eta_1} \left( \frac{m_0^2 m X}{b'^3}, Y y, h \right) \) as defined in Lemma 2.2 with

\[
g \left( \frac{x}{X} \right) = V \left( \frac{x}{X} \right) h \left( \frac{b_0 b + r x + h y}{b_0 b C} \right).
\]

The function \( G_{\eta_1} \) gives us arbitrary saving unless

\[
m_0^2 m \ll \frac{b'^3}{X^{1-\epsilon}}.
\]

Rewriting \( F(u, v, h; b) = u^{-2/3} F_1(u, v, h; b), (2.1) \) and \( C > X^{1/2+2\epsilon} \) gives us for \( b \gg Y^{1/2-\epsilon} \),

\[
x^j \frac{d^j}{dx^j} F(x, y, z; b) \ll (1 + x^{1/3})^j X^\epsilon
\]

for any \( j \geq 0 \).
Combining everything and rewriting \( b' \) as \( m_0 b' \), we have
\[
D_{a, \mathcal{H}}(X) = \frac{X^{2/3}}{C |\mathcal{H}|} \sum_{n_1, n_2 = \pm 1} \sum_{b_0, r_0' = r} \sum_{m_0 \gamma^{1/2 - \varepsilon} \leq b = r_0 m_0 b' \leq C} \frac{1}{b_0 m_0 b' b^2} \sum_{h \in \mathcal{H}} a(h) \sum_{m \leq m_0 b'} A(m, m_0) \frac{m_0}{m} \frac{1}{m^{1/3}} \times \sum_{n \leq \frac{m_0 b'}{b^2}} \lambda(\eta; n) S_0(m, n; h; r_0 m_0, b') F \left( \frac{m X}{m_0 b'^3}, \frac{n Y}{b^2}; h; b \right) + O \left( X^{-999} \right). \tag{4.3}
\]

4.3. Cauchy-Schwarz inequality and Poisson summation. Notice that by Corollary 2.1.1,
\[
\sum_{b_0, r_0' = r} \sum_{n \leq \frac{m_0 b'}{b^2}} \frac{1}{b_0 m_0^{1/3} b^{2/3}} \sum_{m \leq \frac{m_0 b'}{b^2}} \frac{|A(m, m_0)|^2}{m^{2/3}} \ll X^\varepsilon \sup_{M_0, C_1, C_3} \frac{1}{M_0 C_1 C_3} \sum_{m \leq \frac{m_0 b'}{b^2}} \sum_{n \leq \frac{m_0 b'}{b^2}} \frac{|A(m, m_0)|^2}{m^{2/3}} \ll X^{-1/3 + \varepsilon}.
\]

Now we apply Cauchy-Schwarz inequality twice to take out the \( r_0, m_0, m, a, b' \)-sums and then the \( n \)-sum. Together with Lemma 2.3, we have
\[
D_{a, \mathcal{H}}(X)^2 \ll \frac{X Y^{1+\varepsilon}}{C^2 |\mathcal{H}|^2} \sum_{n_1, n_2 = \pm 1} \sum_{b_0, r_0' = r} \sum_{m_0 \gamma^{1/2 - \varepsilon} \leq b = r_0 m_0 b' \leq C} \frac{1}{b_0 b^2} \sum_{n \leq \frac{m_0 b'}{b^2}} \left| \sum_{h \in \mathcal{H}} a(h) S_0(m, n; h; r_0 m_0, b') F \left( \frac{m X}{m_0 b'^3}, \frac{n Y}{b^2}; h; b \right) \right|^2. \tag{4.4}
\]

Before we proceed, we apply a lengthening trick. Let \( L \geq 1 \) be a parameter. Take \( U \in C^\infty([-2, 2]) \) be a fixed function such that \( U(x) = 1 \) for \(-1 \leq x \leq 1\), we have
\[
D_{a, \mathcal{H}}(X)^2 \ll \frac{X Y^{1-\varepsilon}}{C^2 |\mathcal{H}|^2} \sum_{n_1, n_2 = \pm 1} \sum_{b_0, r_0' = r} \sum_{m_0 \gamma^{1/2 - \varepsilon} \leq b = r_0 m_0 b' \leq C} \frac{1}{b_0 b^2} \sum_{n \leq \frac{m_0 b'}{b^2}} \left| \sum_{h \in \mathcal{H}} a(h) S_0(m, n; h; r_0 m_0, b') F \left( \frac{m X}{m_0 b'^3}, \frac{n Y}{b^2}; h; b \right) \right|^2. \tag{4.4}
\]

Remark. By adding more summation terms outside an absolute value square, this lengthening trick allow us to boost the diagonal contribution while lowering the offdiagonal contribution.

Opening the square and applying Poisson summation on the \( m \)-sum, the \( m \)-sum is equal to
\[
\sum_{m} U \left( \frac{m X^{1-\varepsilon}}{m_0 b'^3 L} \right) S_0(m_0, m, n, h_1; b') S_0(m_0, m, n, h_2; b') F \left( \frac{m X}{m_0 b'^3}, \frac{n Y}{b^2}, h_1; b \right) F \left( \frac{m X}{m_0 b'^3}, \frac{n Y}{b^2}, h_2; b \right)
- \frac{m_0 b'^3 L}{X^{1-\varepsilon}} \sum_{m} T(m, n, h_1, h_2; r_0 m_0, b') G(m, n, h_1, h_2; b),
\]
where
\[
T(m, n, h_1, h_2; r_0 m_0, b') = \frac{1}{b'} \sum_{(y \mod b')} S_0(y, n, h_1; r_0 m_0, b') S_0(y, n, h_2; r_0 m_0, b') e \left( \frac{m Y}{b'} \right)
\]
and
\[
G(m, n, h_1, h_2; b) = \int_{\mathbb{R}} U(w) F \left( L X^\varepsilon w, \frac{n Y}{b^2}, h_1; b \right) F \left( L X^\varepsilon w, \frac{n Y}{b^2}, h_2; b \right) e \left( - \frac{m_0 b'^2 L m w}{X^{1-\varepsilon}} \right) dw.
\]

With (4.2), repeated integration by parts gives us arbitrary saving unless
\[
|m| \ll \frac{X}{m_0 b'^2 L^{1/3}} X^{1+\varepsilon} \ll \frac{m_0 X^{\varepsilon}}{L^{2/3}}.
\]

Now we exploit our lengthening trick and take \( L = m_0^{1/2} X^{3\varepsilon} \). This choice gives us arbitrary saving unless \( m = 0 \).
Inserting the above analysis back into the bound for $D_{a,\mathcal{H}}(X)$, we have

$$D_{a,\mathcal{H}}(X)^2 \ll \frac{Y^{1+\varepsilon}}{C^2|\mathcal{H}|^2} \sum_{b \in \mathcal{H}} \sum_{b_0, m_0, b_0 r_0 b \equiv \ell_0 \pmod{C}} \frac{m_0 b'}{b_0 r_0} \sum_{n \ll \frac{Y^2}{b'^2}} \left( \frac{m_0 b'}{b_0 r_0} \right)^2 T(0, n, h_1, h_2; r_0 m_0, b') G(0, n, h_1, h_2; b).$$

\[
4.4. \text{ Final bound.} \text{ Applying Lemma B.1 with } \ell = 0 \text{ on } T \text{ and the bound } G \ll X^{\varepsilon} \text{ on the bound for } D_{a,\mathcal{H}}(X)^2 \text{ above, we have }
\]

$$D_{a,\mathcal{H}}(X)^2 \ll \frac{Y^{1+\varepsilon}}{C^2|\mathcal{H}|^2} \sum_{b \in \mathcal{H}} \sum_{h_1, h_2 \in \mathcal{H}} \sum_{d | h_1} \frac{m_0 b'}{b_0 r_0} \left( \frac{b^2}{Y} \right)^3 \sum_{b''} |a(h_1) a(h_2)| \delta(h_1 \equiv h_2 \pmod{b''}).$$

Applying Cauchy-Schwarz inequality, this is bounded by

$$\ll \frac{C^7}{|\mathcal{H}|^2 X^{2-\varepsilon}} \sum_{b \in \mathcal{H}} \left( \sum_{h_1, h_2 \in \mathcal{H}} |a(h_1)|^2 \delta(h_1 \equiv h_2 \pmod{b}) \right)^{1/2} \left( \sum_{h_1, h_2 \in \mathcal{H}} |a(h_2)|^2 \delta(h_1 \equiv h_2 \pmod{b}) \right)^{1/2} \ll \frac{C^7}{|\mathcal{H}|^2 X^{2-\varepsilon}} \|a\|_2^2 \sum_{b \in \mathcal{H}} \sup_{h' \in \mathcal{H}} |h' \equiv h \pmod{b} |.$$

Taking $C = X^{1/2+2\varepsilon}$ yields Theorem 1.1.

5. Proof of Theorem 1.2

With $a(h) = a_\pm(h)$ and $\mathcal{H}$ satisfying $(*)$, i.e.

$$a_\pm(h) = \sum_{d | \pm d q_1, q_2 = h-\ell} V_1 \left( \frac{d}{\ell} \right) V_2 \left( \frac{q_1}{q_2} \right) a'(q_1)$$

with $q_1 \in \mathcal{Q}_1 \subset \{1, Q_1\}$ and

$$\mathcal{H} \succ \{ h \in \mathbb{N} : a(h) \neq 0 \},$$

we can rewrite $D_{a,\mathcal{H}}(X)$ as

$$D_{a,\mathcal{H}}(X) = \frac{1}{|\mathcal{H}|} \sum_{d} V_1 \left( \frac{d}{\ell} \right) \sum_{q_2} \left( \frac{q_2}{q_2} \right) \sum_{m=1}^\infty A(1, m) \lambda(r m + \ell \pm d q_1 q_2) V \left( \frac{m}{X} \right)$$

5.1. Refinement of Lemma 1.3. We will perform a similar analysis to prove Theorem 1.2 in this section. Before that, we first use the special structure of $a(h)$ and $\mathcal{H}$ to prove the following lemma.

**Lemma 5.1.** Let $A, \varepsilon > 0$, $n$ be an integer such that $|n| \ll Y \rightarrow \infty$. Let $t = Y^{\varepsilon}$ and $C \geq Y^{1/2+5\varepsilon}$ be a parameter. Then there exists $\mathcal{C} \sim C$ and a fixed function $h$ such that

$$\mathcal{F}^\pm(n, q_1) := \sum_{d} V_1 \left( \frac{d}{\ell} \right) \sum_{q_2} \left( \frac{q_2}{q_2} \right) \delta(n \pm d q_1 q_2 = 0) = \mathcal{F}_0^\pm(n, q_1) + \mathcal{F}_1^\pm(n, q_1) + O_A(Y-A),$$

where

$$\mathcal{F}_0^\pm(n, q_1) = \frac{1}{C} \sum_{c \equiv Y^{1/2-2\varepsilon}} \sum_{q_0 \equiv \ell} \sum_{a \equiv \ell \pmod{c}} V_1 \left( \frac{d}{\ell} \right) \sum_{q_2} \left( \frac{q_2}{q_2} \right) \sum_{m=1}^\infty e \left( \frac{a(n \pm d q_1 q_2)}{c} \right) h \left( \frac{c_0 e n \pm dq_1 q_2}{c'} \right),$$

and

$$\mathcal{F}_1^\pm(n, q_1) = 0.$$
unless \( D + Q_2 \ll X^{1/2+\varepsilon} \), and in such a case, if \( 2 \not\in \{1 \text{ or Primes in } [Q_1,2Q_1]\} \) and \( Q_1 Q_2 \gg X^{1/2+\varepsilon} \), we have

\[
\mathcal{S}_1^\pm(n, q_1) = \frac{DQ_2}{c} \sum_{c_1} \sum_{\substack{0 < d + \ell \pm q_1 \leq c_1 \leq c_0 \varepsilon \leq q_2 \leq c_0 c \varepsilon}} S(n, \mp dq_2 q_1; c) \\
\times \sum_{n} A(1, m) V \left( \frac{m}{X} \right) \sum_{n} A(n) \varphi \left( \frac{n}{Y} \right) \mathcal{S}_1^\pm(r m + \ell - n, q_1')
\]

\[\times \int_0^\infty \int_0^\infty V_1(x) V_2(y) \left( \frac{c_0 c}{c} \right) \left( \frac{Y + dq_2 q_1}{c} \right) \left( \frac{r m + q_2 q_2 y}{(Dq_1 Q_2 x y)^2} \right)^{it} dx dy.
\]

For a better presentation, we delay the proof of Lemma 5.1 to the end of this section (Section 5.4).

Write \( Y = X + \ell \), then \( X \ll Y \ll X^{1+\varepsilon} \). Let \( C \gg X^{1/2+3\varepsilon} \) and write \( t = X^\varepsilon \). Applying Lemma 5.1, we have

\[\mathcal{D}_{a_+, \mathcal{A}}(X) = \mathcal{D}_0(X) + \mathcal{D}_1(X) + O_A \left( X^{-A} \right)\]

for any \( A > 0 \), where

\[\mathcal{D}_j(X) = \frac{1}{|\mathcal{A}|} \sum_{q_1 \in \mathcal{A}_j} a'(q_1) \sum_{n} A(1, m) V \left( \frac{m}{X} \right) \sum_{n} A(n) \varphi \left( \frac{n}{Y} \right) \mathcal{S}_j^\pm(r m + \ell - n, q_1')\]

for \( j = 0, 1 \), with a fixed \( \varphi \in C_c^\infty([1/2, 5/2]) \) and \( \varphi(x) = 1 \) for \( 1 \leq x \leq 2 \) as before.

### 5.2. Treatment of \( \mathcal{D}_0(X) \)

By the definition of \( \mathcal{S}_0^\pm(r m + \ell - n, q_1') \) given in Lemma 5.1, we see that the \( n \)-sum in \( \mathcal{D}_0(X) \) is given by

\[\sum_{n} \lambda(n) \varphi \left( \frac{n}{Y} \right) e \left( \frac{an}{c} \right) h \left( \frac{c_0 c}{c}, \frac{r m + \ell - n \mp dq_1 q_2}{c_0 c} \right) \left( r m + \ell - n \right)^{it}.
\]

Applying Voronoi summation (Lemma 2.4), this is equal to

\[\frac{Y}{c} \sum_{\eta \equiv \pm 1} \lambda(\eta n) e \left( - \frac{\eta n}{c} \right) \int_0^\infty \varphi(y) \left( \frac{c_0 c}{c}, \frac{r m + \ell - Y \varphi dq_1 q_2}{c_0 c} \right) \left( r m + \ell - Y n \right)^{it} dy.
\]

Now with the restriction \( c \ll Y^{1/2-2\varepsilon} \), repeated integration by parts gives us arbitrary saving unless

\[n \ll \frac{c^2}{Y^{1-\varepsilon}} \left( X^\varepsilon + \frac{Y}{c_0 c} \right)^2 \ll X^{-\varepsilon}.
\]

Hence we have

\[\mathcal{D}_0(X) \ll_A X^{-A}
\]

and

\[\mathcal{D}_{a_+, \mathcal{A}}(X) = \mathcal{D}_1(X) + O_A \left( X^{-A} \right)\] (5.1)

for any \( A > 0 \).

### 5.3. Treatment of \( \mathcal{D}_1(X) \)

At this point, we already obtain

\[\mathcal{D}_{a_+, \mathcal{A}}(X) \ll_A X^{-A}\]

when \( D + Q_2 \gg X^{1/2+\varepsilon} \) as \( \mathcal{S}_1^\pm(r m + \ell - n, q_1') = 0 \) in such a case. We are left to prove the second statement, which we now further assume \( 2 \not\in \{1 \text{ or Primes in } [Q_1,2Q_1]\} \) and \( Q_1 Q_2 \gg X^{1/2+\varepsilon} \).

Write

\[F_0(r m, n, \ell; c) = \int_0^\infty \int_0^\infty V_1(x) V_2(y) h \left( \frac{c_0 c}{c}, \frac{r m + \ell - n \pm dq_1 Q_2 x y}{c_0 c} \right) e \left( \frac{d D x + q_2 Q_2 y}{c} \right) \left( r m + \ell - n \right)^{it} dx dy,
\]

then

\[\mathcal{D}_1(X) = \frac{DQ_2}{c} \sum_{c_1} \sum_{\substack{0 < d + \ell \pm q_1 \leq c_1 \leq c_0 \varepsilon \leq q_2 \leq c_0 c \varepsilon}} \sum_{q_1 \in \mathcal{A}_1} a'(q_1) \sum_{0 < d + \ell - q_1 \leq q_2 \leq c_0 c \varepsilon} \sum_{m} A(1, m) V \left( \frac{m}{X} \right) \sum_{n} \lambda(n) \varphi \left( \frac{n}{Y} \right) S(r m + \ell - n, \mp dq_2 q_1; c) F_0(r m, n, \ell; c).
\]

By the restriction of \( c > Y^{1/2-2\varepsilon} \), \( F_0 \) is \( X^{3\varepsilon} \)-inert.
From this point onward, we proceed in the exact same way as the proof of Theorem 1.1 in Section 4. As it is repetitive, we will only write down the key steps.

5.3.1. Voronoi summations. Applying Voronoi summations in the $n$-sum and then the $m$-sum, we get

\[
\mathcal{D}_1(X) = \frac{D Q_2 X^{2/3} Y}{c |\mathcal{H}|} \sum_{\eta_1, \eta_2 = \pm 1} \sum_{a_0 \mid r' = r} \sum_{m_0 = m_{0'}} \sum_{c = r_0 m_0 c' \leq C} \sum_{q_1 \in \mathbb{Z}_1} \frac{1}{c_0 m_0 c' c_3} \sum_{q_1 \in \mathbb{Z}_1} a'(q_1) \sum_{0 < c - d < \frac{c_1}{\sqrt{2}}} \sum_{0 < c - d < \frac{c_1}{\sqrt{2}}} \sum_{\{m_0 c', r'\} = 1} A(1, m) \sum_{n \leq \frac{c}{\sqrt{2}}} \lambda(\eta_2 n) S_\ell(m, n, +d q_2 \eta_1; r_0 m_0, c') \tilde{F} \left( \frac{m X}{m_0 c' c_3}, \frac{n Y}{c^2}, d, q_1, q_2; c \right) + O_L(X^{-A}),
\]

where

\[
S_\ell(m, n, +d q_2 \eta_1; r_0 m_0, c') = \sum_{\alpha \left( \text{mod } c \right)} e \left( \frac{\alpha \ell + a(\eta_2 n + d q_2 \eta_1)}{c} \right) S(\alpha r', \eta_1 m; c')
\]

and \(\tilde{F} \left( \frac{m X}{m_0 c' c_3}, \frac{n Y}{c^2}, d, q_1, q_2; c \right)\) is some function satisfying

\[
x^j \frac{d^j}{d x^j} \tilde{F}(x, n, d, q_1, q_2; c) \ll (X^e + x^{1/3}) X^e
\]

for any \(j \geq 0\).

5.3.2. Cauchy-Schwarz inequality and Poisson summation. Applying Cauchy-Schwarz inequality to take out the $m$-sum together with a lengthening parameter \(L \geq 1\), we have

\[
\mathcal{D}_1(X)^2 \ll \frac{D Q_2 X Y^{1+\varepsilon}}{C^2 |\mathcal{H}|} \sum_{\eta_1, \eta_2 = \pm 1} \sum_{a_0 \mid r' = r} \sum_{m_0 = m_{0'}} \sum_{c = r_0 m_0 c' \leq C} \sum_{q_1 \in \mathbb{Z}_1} \frac{1}{c_0 m_0 c' c_3} \sum_{n \leq \frac{c}{\sqrt{2}}} \sum_{\{m_0 c', r'\} = 1}
\]

\[
\times \sum_{m} U \left( \frac{m X^{1-\varepsilon}}{m_0 c^2 L} \right) \sum_{q_1 \in \mathbb{Z}_1} a'(q_1) \sum_{0 < c - d < \frac{c_1}{\sqrt{2}}} \sum_{0 < c - d < \frac{c_1}{\sqrt{2}}} \sum_{\{m_0 c', r'\} = 1}
\]

\[
\times \sum_{q_1, q_1' \in \mathcal{Z}_1} \sum_{0 < c - d < \frac{c_1}{\sqrt{2}}} \sum_{0 < c - d < \frac{c_1}{\sqrt{2}}} \sum_{\{m_0 c', r'\} = 1}
\]

\[
|T_2(\ell, n, d_1 q_1 \eta_1, d_2 q_2 \eta_2; r_0 m_0, c')| = \frac{1}{c'} \sum_{\gamma \left( \text{mod } c' \right)} S_\ell(\gamma, n, +d_1 q_1 \eta_1; r_0 m_0, c') S_\ell(\gamma, n, +d_2 q_2 \eta_2; r_0 m_0, c').
\]

Opening the square, applying Poisson summation on the $m$-sum, together with \(L = m_0^{2/3} X^{5\varepsilon}\) giving arbitrary saving except the zero frequency, we get

\[
\mathcal{D}_1(X)^2 \ll \frac{D^2 Q_2 X Y^{1+\varepsilon}}{C^2 |\mathcal{H}|} \sum_{\eta_1, \eta_2 = \pm 1} \sum_{a_0 \mid r' = r} \sum_{m_0 = m_{0'}} \sum_{c = r_0 m_0 c' \leq C} \sum_{q_1 \in \mathbb{Z}_1} \frac{1}{c_0 m_0 c' c_3} \sum_{n \leq \frac{c}{\sqrt{2}}} \sum_{\{m_0 c', r'\} = 1}
\]

\[
\times \sum_{q_1, q_1' \in \mathcal{Z}_1} \sum_{0 < c - d_1 < \frac{c_1}{\sqrt{2}}} \sum_{0 < c - d_2 < \frac{c_1}{\sqrt{2}}} \sum_{\{m_0 c', r'\} = 1}
\]

\[
|T_2(\ell, n, d_1 q_1 \eta_1, d_2 q_2 \eta_2; r_0 m_0, c')|.
\]

5.3.3. Final bound. Applying Lemma B.1 on \(T_2\), we have

\[
\mathcal{D}_1(X)^2 \ll \frac{D^2 Q_2 X Y^{1+\varepsilon}}{C^2 |\mathcal{H}|} \sum_{\eta_1, \eta_2 = \pm 1} \sum_{a_0 \mid r' = r} \sum_{m_0 = m_{0'}} \sum_{c = r_0 m_0 c' \leq C} \sum_{q_1 \in \mathbb{Z}_1} \frac{1}{c_0 m_0 c' c_3} \sum_{n \leq \frac{c}{\sqrt{2}}} \sum_{\{m_0 c', r'\} = 1}
\]

\[
\times \sum_{q_1, q_1' \in \mathcal{Z}_1} \sum_{0 < c - d_1 < \frac{c_1}{\sqrt{2}}} \sum_{0 < c - d_2 < \frac{c_1}{\sqrt{2}}} \delta(d_1 q_1 q_2 \equiv d_2 q_1 q_2' \pmod{c'})
\]

\[
\ll \frac{C^2 D Q_2}{|\mathcal{H}| X^{2-\varepsilon}} \sum_{\eta_1, \eta_2 = \pm 1} \sum_{a_0 \mid r' = r} \sum_{m_0 = m_{0'}} \sum_{c = r_0 m_0 c' \leq C} \sum_{q_1 \in \mathcal{Z}_1} \sum_{0 < c - d_1 < \frac{c_1}{\sqrt{2}}} \sum_{0 < c - d_2 < \frac{c_1}{\sqrt{2}}} \delta(d_1 q_1 q_2 \equiv d_2 q_1 q_2' \pmod{c'})
\]

\[
\ll \frac{C^5 D^2 Q_2}{|\mathcal{H}|^2 X^{2-\varepsilon}} \sum_{\eta_1, \eta_2 = \pm 1} \sum_{a_0 \mid r' = r} \sum_{m_0 = m_{0'}} \sum_{c = r_0 m_0 c' \leq C} \frac{C \ C D Q_1}{C D Q_2} \left( 1 + \frac{C^2 Q_1}{c D Q_2} \right) \ll \frac{C^8}{|\mathcal{H}|^2 X^{2-\varepsilon}} \left( 1 \ + \ \frac{C Q_1}{D Q_2} \right) \|a\|_{2}^2.
\]
Here we used $|\mathcal{A}| \gg D Q_1 Q_2 X^{-\varepsilon}$.

Together with (5.1), taking $C = X^{1/2+10\varepsilon}$ yields Theorem 1.2. Now it remains to prove Lemma 5.1.

5.4. Proof of Lemma 5.1. Write $t = Y^\varepsilon$. We start by multiplying expression by

$$1 = \left( \frac{n}{d q_1 q_2} \right)^{2it} = \frac{(n)^{2it}}{(d q_1 q_2)^{2it}}.$$ 

Giving us

$$\mathcal{F}_\pm(n, q_1') = \sum_d V_1 \left( \frac{d}{D} \right) \sum_{q_2} V_2 \left( \frac{q_2}{Q_2} \right) \delta(n \pm dq_1 q_2 = 0) \frac{(n)^{2it}}{(d q_1 q_2)^{2it}}.$$

**Remark.** This is a crucial trick to eliminate some zero frequencies that come from Poisson summations from $d$ and $q_2$ sums in the upcoming steps. We would not be able to get a satisfactory bound in application to prove Theorem 1.2 if the zero frequencies were present. This crucial trick also heavily relies on the fact that our moduli is big enough.

Applying Lemma 3.1 with $C \gg Y^{1/2+5\varepsilon}$ and $q = 1$, taking out the g.c.d. $(a, c)$ in the character sum, there exists some $\mathcal{C} \sim C$ and fixed function $h$ such that

$$\mathcal{F}_\pm(n, q_1') = \frac{1}{\mathcal{C}} \sum_{d_0} \sum_{e_0} \sum_{e_0 \in \mathcal{C}} \sum_d V_1 \left( \frac{d}{D} \right) \sum_{q_2} V_2 \left( \frac{q_2}{Q_2} \right) e\left( \frac{\alpha(n \pm dq_1 q_2)}{c} \right) h\left( \frac{c_0 c}{C}, \frac{n \pm dq_1 q_2}{c_0 c} \right) \frac{(n)^{2it}}{(d q_1 q_2)^{2it}}.$$ 

Now we split the contribution depending on the size of $c$. In particular, write $\mathcal{F}_\pm(n, q_1')$ to be the contribution when $c \leq Y^{1/2-2t}$, then $\mathcal{F}_\pm(n, q_1')$ is precisely what we need in the statement of the lemma, and

$$\mathcal{F}_\pm(n, q_1') = \mathcal{F}_0^\pm(n, q_1') + \mathcal{F}_1^\pm(n, q_1'),$$

where

$$\mathcal{F}_1^\pm(n, q_1') = \frac{1}{\mathcal{C}} \sum_{d_0} \sum_{e_0} \sum_{y_{1/2-2t} < e_0 < \mathcal{C}/q_0} \sum_d V_1 \left( \frac{d}{D} \right) \sum_{q_2} V_2 \left( \frac{q_2}{Q_2} \right) e\left( \frac{\alpha(n \pm dq_1 q_2)}{c} \right) h\left( \frac{c_0 c}{C}, \frac{n \pm dq_1 q_2}{c_0 c} \right) \frac{(n)^{2it}}{(d q_1 q_2)^{2it}}.$$ 

Now we continue the analysis on $\mathcal{F}_1^\pm(n, q_1')$. Performing Poisson summation on the $q_2$-sum, we have

$$\sum_{q_2} V_2 \left( \frac{q_2}{Q_2} \right) e\left( \frac{\alpha d q_1 q_2}{c} \right) h\left( \frac{c_0 c}{C}, \frac{n \pm d q_1 q_2 y}{c_0 c} \right) q_2^{-2it} = Q_2 \sum_{q_2} \delta(ad q_1 \equiv q_2 (mod c)) \int_0^\infty V_2(y) h\left( \frac{c_0 c}{C}, \frac{n \pm d q_1 Q_2 y}{c_0 c} \right) e\left( \frac{-t}{\pi} \log(Q_2 y) + \frac{q_2 Q_2 y}{c} \right) dy.$$ 

Repeated integration by parts (Lemma 2.5) gives us arbitrary saving unless there exists $y_0 \in [1, 2]$ such that

$$\left| \frac{q Q_2}{c} - \frac{t}{\pi y_0} \right| \ll \left( 1 + \frac{D Q_1 Q_2}{c_0 c C} \right) Y^{\varepsilon/2} \ll Y^{\varepsilon/2}.$$ 

Here in the last inequality we used

$$c_0 c C \gg C Y^{1/2-2\varepsilon} \gg Y^{1+3\varepsilon}.$$ 

by the restriction of $c$. Together with $t = Y^\varepsilon$, we get arbitrary saving unless

$$0 < q_2 < C Y^{1/2-6\varepsilon}.$$ 

Similarly, we perform Poisson summation on the $d$-sum to get

$$\sum_d V_1 \left( \frac{d}{D} \right) \delta(ad q_1 \equiv q_2 (mod c)) h\left( \frac{c_0 c}{C}, \frac{n \pm d q_1 Q_2 y}{c_0 c} \right) d^{-2it} = \frac{D}{c} \sum_{y} \delta(ad q_1 \equiv q_2 (mod c)) e\left( \frac{-d y}{c} \right) \int_0^\infty V_1(x) h\left( \frac{c_0 c}{C}, \frac{n \pm D q_1 Q_2 x y}{c_0 c} \right) e\left( \frac{-t}{\pi} \log(D x) + \frac{d D x y}{c} \right) dx.$$ 

Applying the same integral analysis gives us arbitrary saving unless

$$0 < d < C Y^{1/2}.$$
Combining everything, we have
\[ S_1^\pm(n, q_1) = \frac{DQ_2}{\mathcal{C}} \sum_{q_0} \sum_{y^{1/2} < c < \mathcal{C}} \frac{1}{c_0C^2} \sum_{0 < d < \frac{c Y^{1/2}}{Q_2}} \sum_{0 < q_2 < \frac{c Y^{1/2}}{Q_2}} \mathcal{E}(n, d, q_1, q_2; c) \times \int_0^\infty \int_0^\infty V_1(x)V_2(y) h \left( \frac{c_0 c}{C}, \frac{n + Dq_1 q_2 x y}{c_0 c C} \right) e \left( \frac{d D x + q_2 y}{c} \right) \left( \frac{n^2}{(D q_1 q_2 x y)^2} \right)^{it} d x d y + O_A \left( Y^{-A} \right) \]
for any \( A > 0 \), where
\[ \mathcal{E}(n, d, q_1, q_2; c) = \sum_{\alpha} \sum_{(\bmod c)} \delta(a \gamma q_1 \equiv \pm q_2 (\bmod c)) \epsilon \left( \frac{an - d \gamma}{c} \right). \]

If \( D + q_2 > X^{1/2 + 4\epsilon} \), we immediately get
\[ S_1^\pm(n, q_1) \ll_A Y^{-A} \]
for any \( A > 0 \) by the \( d \) and \( q_2 \) sums.

Finally, we further suppose \( \mathcal{O}_1 \subseteq \{ 1 \text{ or Primes in } [Q_1, 2Q_1] \} \) and simplify the character sum. First notice if \( q_1 | c \), the congruence condition implies \( q_1 | q_2 \). However, we have \( 0 < q_2 < \frac{c Y^{1/2}}{Q_2} \ll \frac{c Y^{1/2}}{Q_2} \ll Q_1 Y^{-\epsilon} \), which is not possible. Hence we have \( (c, q_1) = 1 \). Now the congruence condition implies
\[ \gamma \equiv \pm a q_1 q_2 (\bmod c). \]
This gives us
\[ \mathcal{E}(n, d, q_1, q_2; c) = S(n, \mp d q_2 q_1; c), \]
and thus concludes the proof.

6. RECOVERING MUNSHI’S FIXED SHIFT BOUND WITHOUT JUTILA’S CIRCLE METHOD

In this section, we sketch the simple steps needed to get to Munshi’s bound for fixed shift, i.e.
\[ D_\epsilon(X) := \sum_{m=1}^\infty A(1, m) \lambda(m + \ell) V \left( \frac{m}{X} \right) \ll X^{1-1/26+\epsilon}, \]
by using his analysis in [22] together with Lemma 5.1 and Theorem 1.2, but without using the Jutila’s circle method.

Take a fixed \( U \in C^\infty_c(\mathbb{R}) \) even such that \( U(x) = 1 \) for \(-2 \leq x \leq 2\) as before. Let \( Q_1, Q_2 > 0 \) be parameters such that \( Q_1^2 \ll X^{1/2+\epsilon} \ll Q_1 Q_2, D = X^{1/2}/Q_1 Q_2 \). Let \( \mathcal{O}_1 \subseteq \{ \text{Primes in } [Q_1, 2Q_1] \} \). Applying Lemma 1.3 with \( F = U, \mathcal{O} = \mathcal{O}_1 \times [Q_2, 2Q_2] \),
\[ b(q) = \sum_{q_0 q_2 = q, q_1 \in \mathcal{O}_1} V_2 \left( \frac{q_2}{Q_2} \right) \]
with some fixed \( V_2 \neq 0 \in C^\infty_c([1, 2]) \), we obtain
\[ D_\epsilon(X) = M.T. - A.S.^+ - A.S.^-, \]
where
\[ M.T. := \frac{1}{|\mathcal{O}_1| Q_2} \sum_{q_1 \in \mathcal{O}_1} \sum_{q_2} V_2 \left( \frac{q_2}{Q_2} \right) \sum_m A(1, m) V \left( \frac{m}{X} \right) \sum_n \lambda(n) \phi \left( \frac{n}{Y} \right) \delta(m + \ell \equiv n \pmod{q_1 q_2}) U \left( \frac{m + \ell - n}{D q} \right) \]
and
\[ A.S.^+ := \frac{1}{|\mathcal{O}_1| Q_2} \sum_{q_1 \in \mathcal{O}_1} \sum_{q_2} V_2 \left( \frac{q_2}{Q_2} \right) \sum_U \left( \frac{d}{D} \right) \sum_m A(1, m) V \left( \frac{m}{X} \right) \lambda(m + \ell \pm dq) \phi \left( \frac{m + \ell \pm dq}{Y} \right). \]
as defined in Lemma 1.3.

Applying a smooth dyadic subdivision, we have
\[ A.S.^+ \ll \sup_{D' \ll D} \frac{1}{|\mathcal{O}_1| Q_2} \sum_{q_1 \in \mathcal{O}_1} \sum_{q_2} V_2 \left( \frac{q_2}{Q_2} \right) \sum_D \left( \frac{d}{D'} \right) \sum_m A(1, m) V \left( \frac{m}{X} \right) \lambda(m + \ell \pm dq) \phi \left( \frac{m + \ell \pm dq}{Y} \right) \ll \sup_{D' \ll D} D' D_{a^+, \mathcal{E}}(X), \]
with \(|a_\pm(h)|\) and \(\mathcal{H} = \{ h : a_\pm(h) \neq 0 \}\) chosen as in (*) (with \(D'\) in place of \(D\)). Then \(|\mathcal{H}| \gg D' Q_1 X^{-\varepsilon}\). Applying Theorem 1.2, we obtain the bound

\[
A.S. \ll \sup_{D' \ll D} D' \frac{X^{1+\varepsilon}}{\sqrt{D'Q_1Q_2}} \left(1 + \frac{\sqrt{X}Q_1}{D'Q_2}\right)^{1/2} \ll \frac{X^{3/2}}{Q_1Q_2}.
\]

Choosing \(Q_1Q_2 = X^{1/2+\delta}\), we obtain

\[
D_\ell(X) = M.T. + O\left(X^{-\delta}\right).
\]

As a result, we have recovered the same bound of the error term from the Jutila's circle method, and the \(M.T.\) that remains essentially matches with the main term of the Jutila's circle method. Indeed, detecting the congruence condition by additive characters

\[
\delta(m + \ell \equiv n \pmod{q_1q_2}) = \sum_{q(q_1q_2 \alpha \pmod{q})} \sum_{*} e\left(\frac{\alpha(m + \ell - n)}{q}\right),
\]

we have

\[
M.T. := \frac{1}{|\mathcal{O}_1|Q_2} \sum_{q \in \mathcal{O}_1} V_2 \left(\frac{Q_2}{Q_2}\right) \sum_{q_1q_2 \alpha \pmod{q}} \sum A(1,m)V\left(\frac{m}{X}\right) \sum_{n} A(n)\phi\left(\frac{n}{Y}\right) \sum_{*} e\left(\frac{\alpha(m + \ell - n)}{q}\right) U\left(\frac{m + \ell - n}{Dq}\right).
\]

The only difference between our \(M.T.\) here and the main term in the Jutila's circle method used by Munshi in [22] is:

- \(q_2\) sum over all integers between \(Q_2\) to \(2Q_2\) instead of primes only, and
- the modulus \(q\) is a divisor of \(q_1q_2\) instead precisely \(q_1q_2\).

However, such a difference is not essential in Munshi's treatment, and the same analysis goes through with small technicalities, giving us the same bound for \(M.T.\) as his \(\tilde{D}_h(X)\) (with \(h = \ell\) in his notation). Hence we recover Munshi's result

\[
D_\ell(X) \ll X^{1-1/26+\varepsilon}
\]

using an average shifted sum analysis instead of the Jutila's circle method.

**APPENDIX A. REFORMULATION OF THE DELTA METHOD**

**Lemma A.1** (Reformulation of the Delta method). Let \(\varepsilon > 0\), \(n, q\) be integers such that \(r > 0\) and \(|n| \ll N \rightarrow \infty\). Let \(C, D > 0\) be parameters such that \(C > N^\varepsilon\). Let \(U, W \neq 0\) be any smooth even functions such that \(U(0) = 1, W(0) = 0\) and \(U\) decays exponentially at \(\infty\). Then we have,

\[
\delta(n = 0) = S_1 - S_2,
\]

where

\[
S_1 = \frac{1}{\mathcal{C}} \sum_{c \geq 1} \delta(n \equiv 0 \pmod{cq}) W(c)U\left(\frac{n}{cDq}\right) U\left(\frac{c}{C}\right)
\]

and

\[
S_2 = \frac{1}{\mathcal{C}} \sum_{d \geq 1} \delta(n \equiv 0 \pmod{dq}) W\left(\frac{n}{dq}\right) U\left(\frac{n}{dDq}\right) U\left(\frac{d}{D}\right),
\]

with

\[
\mathcal{C} = \sum_{c \geq 1} W(c) U\left(\frac{c}{C}\right).
\]

Taking \(C = D\), \(W(x) = W'(\frac{x}{X})\) with \(W' \in C^\infty([-2,1]\cup[1,2])\) and detecting the congruence condition by additive characters yields Lemma 3.1.

**Proof.** Let

\[
\mathcal{D} = \delta(n = 0) \sum_{c \geq 1} W(c) U\left(\frac{c}{C}\right),
\]

\[
S_2 = \sum_{0 \neq d \in \mathbb{Z}} \sum_{c \geq 1} \delta(n = cdq) W(c)U\left(\frac{n}{cDq}\right) U\left(\frac{n}{Cdq}\right)
\]

and
and define

\[ S_1 = \Phi + S_2. \]

We start by rewriting \( d = \frac{n}{cq} \) in \( S_2 \), we get

\[ S_2 = \sum_{c \geq 1} \delta \left( n \equiv 0 \mod cq, n \neq m \right) W(c) U \left( \frac{n}{cDq} \right) U \left( \frac{c}{C} \right). \]

Hence adding up \( \Phi \) and \( S_2 \) with \( U(0) = 1 \), we get

\[ S_1 = \sum_{c \geq 1} \delta \left( n \equiv 0 \mod cq \right) W(c) U \left( \frac{n}{cDq} \right) U \left( \frac{c}{C} \right). \]

(A.1)

Instead, if we start by rewriting \( c = \frac{n}{dq} \) in \( S_2 \), we get

\[ S_2 = \sum_{0 \neq d \in \mathbb{Z}} \delta \left( n \equiv 0 \mod dq, \frac{n}{d} > 0 \right) W \left( \frac{n}{dq} \right) U \left( \frac{n}{Cdq} \right) U \left( \frac{d}{D} \right). \]

If \( m > n \), we get

\[ S_2 = \sum_{d \geq 1} \delta \left( n \equiv 0 \mod dq \right) W \left( \frac{n}{dq} \right) U \left( \frac{n}{Cdq} \right) U \left( \frac{d}{D} \right). \]

and if \( m < n \), we get

\[ S_2 = \sum_{d \geq 1} \delta \left( n \equiv 0 \mod dq \right) W \left( -\frac{n}{dq} \right) U \left( -\frac{n}{Cdq} \right) U \left( -\frac{d}{D} \right) \]

\[ = \sum_{d \geq 1} \delta \left( n \equiv 0 \mod dq \right) W \left( \frac{n}{dq} \right) U \left( \frac{n}{Cdq} \right) U \left( \frac{d}{D} \right) \]

as \( U \) and \( W \) are even. Combining both cases above with the case \( m = n \) giving 0 in \( S_2 \) as \( W(0) = 0 \),

\[ S_2 = \sum_{d \geq 1} \delta \left( n \equiv 0 \mod dq \right) W \left( \frac{n}{dq} \right) U \left( \frac{n}{Cdq} \right) U \left( \frac{d}{D} \right). \]

(A.2)

Defining

\[ \mathcal{C} := \sum_{c \geq 1} W(c) U \left( \frac{c}{C} \right), \]

we obtain Lemma A.1. \( \square \)

### APPENDIX B. CHARACTER SUM ANALYSIS

Let \( \ell \) be an integer. Consider the character sum

\[ \mathcal{F}(\ell, n, h_1, h_2; r_0 m_0, b') = \frac{1}{b'} \sum_{\gamma \mod \left( \frac{b'}{b'} \right)} \sum_{a_1} \sum_{a_2} \sum_{r_0 m_0 b'} e^{\left( (a_1 - a_2) \ell + a_1 h_1 - a_2 h_2 + \eta_2 (a_1 - a_2) n \right) \frac{r_0 m_0 b'}{b'}} \times S(\alpha_1 \gamma, \eta_1 \gamma; b') S(\alpha_2 \gamma, \eta_1 \gamma; b'), \]

we have

**Lemma B.1.**

\[ \mathcal{F}(\ell, n, h_1, h_2; r_0 m_0, b') \ll r_0 m_0 b' n^2 X^\epsilon \sum_{b' \mid b} b^\nu \delta(h_1 \equiv h_2 \mod b'\nu)). \]

Opening the Kloosterman sums and summing over \( \gamma \) gives us

\[ \frac{1}{b'} \sum_{\gamma \mod \left( \frac{b'}{b'} \right)} e^{\left( \eta_1 \frac{(\beta_1 - \beta_2) \gamma}{b'} \right)} = \delta(\beta_1 \equiv \beta_2 \mod \left( \frac{b'}{b'} \right)). \]
This gives us
\[
\mathcal{F}(\ell, n, h_1, h_2; r_0 m_0, b') = \sum_{\alpha_1, \alpha_2 \equiv a \ (mod \ r_0 m_0 b')} \sum_{b_0} \mu(b'_0) b_0 b_2, b_0 \sum_{a \equiv a \ (mod \ b')} e\left(\frac{(\alpha_1 - \alpha_2) \ell + \alpha_1 h_1 - \alpha_2 h_2 + \eta_2 (\alpha_1 - \alpha_2) n}{r_0 m_0 b'}\right) \sum_{\beta} e\left(\frac{\beta r (\alpha_1 - \alpha_2)}{b'}\right).
\]
Write \( b_0 = (r_0 m_0 b'_0, b'_1, b'_2) \), \( r_0 m_0 b'_1 = b_0 b_1, b'_2 = b_0 b_2 \) such that \( b_1, b_2, b_0 \) and \( (b_1 b_2 b_0) = 1 \). Then the character sum splits into
\[
\mathcal{F}(\ell, n, h_1, h_2; r_0 m_0, b') = \sum_{b_0, b_1, b_2} \mu(b'_0) b_0 b_2, b_0 \sum_{a \equiv a \ (mod \ b')} e\left(\frac{a (h_1 - h_2)}{b'}\right) \times S(h_1 + \ell, \eta_2 n b_0^2 b_1 b_2; b_1) S(h_2 + \ell, \eta_2 n b_0^2 b_1 b_2; b_1) \times \sum_{\beta_1, \beta_2 \equiv \beta \ (mod \ b_0^2 b_1 b_2)} e\left(\frac{\beta_1 (h_1 + \ell) - \beta_2 (h_2 + \ell) + \eta_2 (\beta_1 - \beta_2) n b_1 b_2}{b_0^2 b_1 b_2; b_1} \right).
\]
If \( b_1 \mid n \), then
\[
S(h_1 + \ell, \eta_2 n b_0^2 b_1 b_2; b_1) \leq \sum_{\beta \equiv \beta \ (mod \ b_0^2 b_1 b_2)} e\left(\frac{\beta n}{b_1}\right) \leq n X^\varepsilon.
\]
For \( j = 1, 2 \), if \( b_1 \mid (h_j + \ell) \), then
\[
S(h_j + \ell, \eta_2 n b_0^2 b_1 b_2; b_1) = \sum_{\beta \equiv \beta \ (mod \ b_0^2 b_1 b_2)} e\left(\frac{\beta n}{b_1}\right) \leq n X^\varepsilon.
\]
Bounding the Kloosterman sum by the Weil bound when \( b_1 \mid n(h_1 + \ell)(h_2 + \ell) \) and evaluating the Ramanujan sum, we have
\[
\mathcal{F}(\ell, n, h_1, h_2; r_0 m_0, b') \ll \sum_{b_0, b_1, b_2} b_0^2 b_1 b_2, (b_1 n^2 b_0^3 (h_1 + n^2 b_0^2 (b_1 + h_2 \equiv h_2 \ (mod \ b_0^2)))
\]
\[
\ll r_0 m_0 b' n^2 X^\varepsilon \sum_{b' \equiv b'} b' \delta(h_1 + h_2 \ (mod \ b')).
\]

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