Intermediate inflation or late time acceleration?

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Abstract

The expansion rate of ‘Intermediate inflation’ lies between the exponential and power law expansion but corresponding accelerated expansion does not start at the onset of cosmological evolution. Present study of ‘Intermediate inflation’ reveals that it admits scaling solution and has got a natural exit form it at a later epoch of cosmic evolution, leading to late time acceleration. The corresponding scalar field responsible for such feature is also found to behave as a tracker field for both gravity with canonical and some non-canonical form of kinetic term. Thus the so called Intermediate inflation should be considered as yet another dark energy model, with asymptotic de-Sitter expansion.

1 Introduction

It is now almost certain that the Universe contains 70% of dark energy, which is evolving slowly in such a manner that at present the equation of state parameter, \( w < -1/3 \), or more precisely, \( w \approx -1 \), so that the Universe is presently accelerating (see eg. [1] for a recent review). ΛCDM-model is the simplest one, that can explain the present observable features of the Universe. However, in order to comply the vacuum energy density, \( \rho_{\text{vac}} \approx 10^{74}\text{GeV}^4 \) (calculated from quantum field theory as, \( \rho_{\text{vac}} \approx \frac{m^4_{\text{pl}}}{16\pi^2} \)) with the critical density, \( \rho_{\Lambda} \approx 10^{-47}\text{GeV}^4 \) (related to the cosmological constant as - \( \rho_{\Lambda} = \frac{\Lambda m^2_{\text{pl}}}{8\pi} \)), it requires to set up yet another energy scale in particle physics. This problem is known as the ‘coincidence problem’ when stated as - ‘why \( \Lambda \) took 15 Billion years to dominate over other kinds of matter present in the Universe?’. A scalar field with dynamical equation of state \( w_{\phi} \), dubbed as quintessence field [2] appears to get rid of this problem, which during ‘slow role’ over the potential, acquires negative pressure and finally acts as effective cosmological constant (\( \Lambda_{\text{eff}} \)). Nevertheless, this quintessence field requires to be fine tuned for the energy density of the scalar field (\( \rho_{\phi} \)) or the corresponding effective cosmological constant (\( \Lambda_{\text{eff}} \)), to be comparable with the present energy density of the Universe. Tracker fields [3], [4], are introduced to overcome the fine tuning problem. Tracker fields have attractor like solutions in the sense that a wide range of initial conditions (viz., a wide range of initial values of \( \rho_{\phi} \)) rapidly converge to a common cosmic evolutionary track with \( \rho_{\Lambda} \), and finally settles down to the present observable Universe, with \( \rho_{\phi} \approx \rho_{\Lambda} \). Thus, tracker solutions avoid both the coincidence problem and the fine tuning problem without any need for defining a new energy scale.

The important parameter required to check for the existence of the tracker solutions is \( \Gamma = \frac{V'(\phi)V(\phi)}{V'(\phi)^2} \), \( V(\phi) \) being the scalar potential. For quintessence, the condition for the existence of the tracker solution with \( w_{\phi} < w_B \), (where \( w_{\phi} \) and \( w_B \) are the state parameters of the scalar field and the background field respectively) is \( \Gamma > 1 \), or equivalently, \( |\lambda| = \left| \frac{V'(\phi)}{V(\phi)} \right| \approx \left| \frac{H}{\dot{\phi}} \right| \) decreasing as \( V(\phi) \) decreases. Tacker solution further requires a nearly constant \( \Gamma \), which is satisfied if \( \left| \frac{d(\Gamma^{-1})}{H} \right| \ll |\Gamma - 1| \), or equivalently, \( |\Gamma^{-1}d(\Gamma^{-1})/H| \approx |\Gamma^{-1}d\Gamma/\dot{\phi}| \ll 1 \) [3]. The condition \( w_{\phi} < w_B \) is required for the present day acceleration of the Universe. So eventually the slope of the potential becomes sufficiently flat ensuring accelerated expansion at late times. The same condition for k-essence models, having non-canonical form of kinetic energy requires \( \Gamma > 3/2 \), and is slowly varying [5]. Such a scalar field remains subdominant until recently.

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General theory of relativity with a minimally coupled scalar field admits a solution in the form \( a = a_0 \exp (At^f) \), (where, \( a \) is the scale factor and \( a_0 > 0 \), \( A > 0 \) and \( 0 < f < 1 \) are constants) which was dubbed as intermediate inflation in the nineties \([6, 7]\). The expansion rate in the intermediate inflation is faster than power law and slower than exponential ones. Some aspects of intermediate inflation have been studied in the past years \([8, 9]\). Particularly, in a recent work \([10]\) it has been shown that such inflationary model can encounter the observational features of the three year Wilkinson Microwave Anisotropy Probe (WMAP) data \([11]\) with spectral index \( n_s = 1 \), considering non-zero tensor-to-scalar ratio \( r \). Such solutions \([6, 7]\) may also appear in other theories of gravitation. In fact, the action obtained under modification of Einstein’s theory by the introduction of higher order curvature invariant terms (which is essentially four dimensional effective action of higher dimensional string theories), has been found to be reasonably good candidate to explain the presently observed cosmological phenomena. In particular, in a recent work it has been found \([12]\) that Gauss-Bonnet interaction in four dimensions with dynamic dilatonic scalar coupling leads to a solution \( a = a_0 \exp (At^f) \) in the above form, where the Universe starts evolving with a decelerated exponential expansion. Such solutions encompasses the cosmological evolution, as the dilatonic scalar during evolution behaves as stiff fluid, radiation and pressureless dust. Solutions of these type are known as scaling solutions \([13, 14]\) in which the energy density of the scalar field \( \rho_\phi \) mimics the background matter energy density. It then comes out of the scaling regime \([13, 14]\) and eventually the Universe starts accelerating. Asymptotically, the scalar behaves as effective cosmological constant. The deceleration parameter corresponding to such solution is given by \( q = -1 + \frac{1}{4gf} \). Thus, unlike usual inflationary models with exponential or power law expansion, accelerated expansion of the scalar factor corresponding to intermediate inflation \([6, 7]\) does not start at the onset of the cosmological evolution, rather it starts after the lapse of quite some time.

Inflation should have started at the Planck epoch so that it can solve the initial conditions viz., the horizon and the flatness problems of the standard model and can lead to almost a scale invariant spectrum of density perturbation. As such the epoch at which accelerated expansion of the scale factor in intermediate inflation starts, has also been arbitrarily taken as the Planck’s era. But it is not true. Because, as observed in the context of Gauss-Bonnet gravity \([12]\), such solutions admit synchronize scaling between \( \rho_\phi \) and \( \rho_B \), which can happen long after the Planck’s era. Thus it is required to study the so called intermediate inflation in some more detail.

A comprehensive study in the present work reveals that, (1) solutions in the form \( a = a_0 \exp (At^f) \), lead to late time acceleration and therefore should be treated as dark energy model rather than inflation. The nature of such solution also reveals that the equation of state of the scalar field \( w_\phi \) follows that of the background matter \( w_B \) closely, and finally the scalar field comes out of the scaling regime \([13, 14]\), leading to accelerated expansion of the Universe. The scalar field also admits tracking condition \( \Gamma > 1 \), for standard form of kinetic energy. (2) Next, it has been observed that even for a non-canonical form of kinetic energy the same result is reproduced with the same form of potential, which also satisfies the tracker condition \( \Gamma > 3/2 \). (3) We then proceed to show that such solutions with non-canonical form of kinetic energy does not always carry a tracker field. (4) Finally, it has been shown that in the presence of background matter such solutions are admissible with a tracking field.

In the following section, we have started with a k-essence action \([15]\) in it’s simplest form, keeping only a coupling parameter \( g(\phi) \) in the kinetic energy term and write down the field equations. In section 3, instead of choosing the form of the potential, we have chosen different forms of the super-potential \( H(\phi) \) \([16]\), and presented explicit solutions in the form \( a = a_0 \exp (At^f) \), discussed above, for standard \( g = 1/2 \) and nonstandard form of kinetic energy \( g = g(\phi) \). Finally in section 4, similar solutions in the presence of background matter have been presented.

## 2 Action and the field equations

The generalized k-essence \([15]\) non-canonical Lagrangian,

\[
L = g(\phi)F(X) - V(\phi),
\]

where, \( X = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \), when coupled to gravity may be expressed in the following most simplest form

\[
S = \int d^4x \sqrt{-g} \left\{ g(\phi)\left[ \frac{R}{2\kappa^2} - g(\phi)\phi_{\mu\nu} \phi^{\mu\nu} - V(\phi) \right] \right\},
\]

(1)

where, a coupling parameter, \( g(\phi) \) is coupled with the kinetic energy term. \( g(\phi) \) has got a Brans-Dicke origin, \( g = \frac{\omega}{\phi} \) too, \( \omega(\phi) \) being the Brans-Dicke parameter. This is the simplest form of an action in which both canonical and non-canonical forms of kinetic energies can be treated. For the spatially flat Robertson-Walker space-time

\[
ds^2 = -dt^2 + a^2(t) [dr^2 + r^2 \{ d\theta^2 + \sin^2(\theta) d\phi^2 \}] 
\]
the field equations are
\[2 \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = 2\dot{H} + 3H^2 = -\kappa^2 [g\dot{\phi}^2 - V(\phi)] = -8\pi G p, \tag{2}\]

and
\[\frac{3\ddot{a}^2}{a^2} = 3H^2 = \kappa^2 [g\dot{\phi}^2 + V(\phi)] = 8\pi G \rho, \tag{3}\]

where, \(H = \dot{a}/a\), is the Hubble parameter. In addition, we have got the \(\phi\) variation equation
\[\ddot{\phi} + \frac{3}{a} \dot{a} \dot{\phi} + \frac{1}{g} g^2 \phi = 0, \tag{4}\]

which is not an independent equation, rather it is derivable from the above two equations (2) and (3). In the above, over-dot and dash (\(\cdot\)) stand for differentiations with respect to time and \(\phi\) respectively. Instead of using equations (2) and (3), it is always useful to parametrize the motion in terms of the field variable \(\phi\) \[7, 17\]. Thus, with \(\kappa^2 = 8\pi G\), the above set of equations can be expressed as,
\[\dot{\phi} = -\left(\frac{H'(\phi)}{\kappa^2 g(\phi)}\right), \tag{5}\]

and in the Hamilton-Jacobi form
\[H'(\phi)^2 - 3\kappa^2 H^2(\phi)g(\phi) + \kappa^4 V(\phi)g(\phi) = 0. \tag{6}\]

The two important parameters of the theory viz., the equation of state \(w_\phi\) and the deceleration \(q\) parameters are expressed as,
\[w_\phi = -1 - \frac{2H}{3H^2}; \quad q = -1 - \frac{\dot{H}}{H^2}. \tag{7}\]

Now, we are to solve for \(a\) (in view of \(H\)), \(\phi\), \(g(\phi)\), and \(V(\phi)\) from the above two field equations (5) and (6), and so, two additional assumptions are required. It is found that some specific forms of the super-potential \(H(\phi)\) \[10\] lead to the so called intermediate inflationary solutions.

3 Solution in the form \(a = a_0 e^{(At^f)}\) and it’s dynamics

This section is devoted in presenting the solution of the scale factor in the form \(a = a_0 e^{(At^f)}\), which was dubbed as intermediate inflation earlier. In order to study the dynamics of such solution in detail, we have tacitly assumed the presence of any form of background matter in this section.

3.1 Case-I

Let us choose the following form of the super-potential, \(H(\phi(t))\) as,
\[H = \frac{h}{\phi^n}, \tag{8}\]

where, \(h > 0\) and \(n > 0\) are constants. In view of this assumption equation (5) becomes,
\[g\dot{\phi} = \frac{nh}{\kappa^2} \phi^{-(n+1)}. \tag{9}\]

We can now solve the set of equations (6), (8) and (9), provided we choose a particular form of \(g(\phi)\) or \(V(\phi)\). In the following, we choose the standard form of kinetic energy ie., \(g = 1/2\). For the most natural choice \(g = 1/2\), the field variables are found from equations (9) and (8) as,
\[\phi = \left[\frac{2hn(n + 2)}{\kappa^2} t\right]^{\frac{1}{n+2}}; \quad H = m t^{-\left(\frac{n}{n+2}\right)}; \quad a = a_0 \exp \left[\left(\frac{m(n + 2)}{2}\right) t^{\frac{1}{n+2}}\right], \tag{10}\]
where, \( m = h^\frac{1}{(1-f)} \left( \frac{n^2}{2n(n+2)} \right)^\frac{(2n)}{m} \). The above form of the scale factor, the Hubble parameter and the scalar field \( \phi \) can be expressed respectively as,
\[
a = a_0 \exp [Af], \quad H = \frac{Af}{t(1-f)}, \quad \text{and} \quad \phi = \left[ \frac{8h(1-f)}{\kappa^2 f^2} \right]^\frac{1}{2} t^n,
\]
where, \( A = \frac{m(n+2)}{2} > 0 \) and \( 0 < f = \frac{2}{n+2} < 1 \), are related to the constants \( h \) and \( n \). In view of equation (7), the state parameter and the deceleration parameter evolve as,
\[
w_\phi = -1 + \frac{n}{3A} t^{-f}; \quad q = -1 + \frac{n}{2A} t^{-f}.
\]
The form of the potential, in view of equation (6), for such a solution is restricted to,
\[
V(\phi) = \frac{\hbar^2}{\kappa^2} \left( \frac{3\kappa^2}{\phi^{2n}} - \frac{2n^2 \kappa^2}{\phi^{2(n+1)}} \right),
\]
which has the form of double inverse power. This solution was obtained by Barrow [6] and Muslimov [7] in the nineties and was dubbed as intermediate inflation, since the expansion rate of the scale factor is greater than the power law but less than standard exponential law. For such an expansion rate,
\[
\rho + p = \frac{4n^2\hbar^2}{\kappa^4 \phi^{2(n+1)}} > 0,
\]

ie., the weak energy condition is always satisfied, and so \( w_\phi \geq -1 \). However,
\[
\rho + 3p = \frac{6}{\kappa^2} (-\dot{H} - H^2) = \frac{6Af}{\kappa^2} [(1 - f) - Aft]^t^{-(2-f)},
\]
implies that the strong energy condition is violated at
\[
t > \left( \frac{1-f}{Af} \right)^\frac{1}{t},
\]
when \( w_\phi < -1/3 \).

It is to be noted that the necessary condition for inflation, viz., \( \ddot{a} > 0 \) or more precisely, \( \frac{d}{dt}(H^{-1}) < 0 \), is satisfied under the same above condition (16). Eventually, for large \( \phi \), viz., \( \phi \gg \frac{\sqrt{\kappa^2}}{\phi} \) the potential energy starts dominating over the kinetic energy, ie., \( \dot{\phi}^2 \ll V(\phi) \), and thus the slow roll condition, \( \epsilon = -\frac{\dot{H}}{H^2} < 1 \), is also satisfied under condition (16). However, Inflation is supposed to have started at Planck’s era, so that it can solve the initial problems of the standard model, viz., the horizon and the flatness problems, the structure formation problem and lead nearly to a scale invariant spectrum. Here we observe that, accelerated expansion of the scale factor in the so called intermediate inflation does not start at the onset of cosmic evolution. So, now we can ask the question, ‘what happened prior to the onset of the accelerated expansion in the so called intermediate inflationary era?’

The solution dictates that the Universe starts evolving from an infinitely decelerated exponential expansion with \( w_\phi > 1 \). It might appear that we are considering a highly unorthodox cosmological model involving a ultra-hard equation of state and super-luminal speed of sound. This is true in some sense, since as mentioned earlier, in order to study the situations under which such solutions emerge and it’s dynamics, we have not considered the presence of any form of background matter explicitly. Nevertheless, this situation is quite similar to the phantom models where, super negative pressure gives rise to ultra negative equation of state, indicating that the effective velocity of sound in the medium, \( v = \sqrt{\frac{dp}{d\phi}} \) might become larger than the velocity of light. Likewise, here the Universe starts evolving with such a situation which actually demonstrates that corresponding era is classically forbidden and the need for invoking quantum cosmology at that era.

Now, during evolution the Universe passes through the stiff fluid era \( w_\phi = 1 \), the radiation dominated era \( w_\phi = 1/3 \), the pressureless dust era \( w_\phi = 0 \), the transition (from deceleration to acceleration) era \( w_\phi = -1/3 \), and asymptotically tends to the magic line, ie the vacuum energy dominated inflationary era \( w_\phi \approx -1 \). Thus, the equation of state \( w_\phi \) follows the matter equation of state \( w_B \) closely (which as already mentioned has been assumed in this section tacitly) and so it corresponds to the scaling solution [13, 14]. It finally comes out of the scaling regime and enters into the transition era.
Now, in the context of a realistic model in the presence of some form of background matter, if the above solution (11) is accompanied with a potential (13) then another important aspect has to be checked, and that is if the solution is tracking, which is true when the following condition is satisfied,

$$\Gamma = \frac{V''V}{V^3} > 1,$$

which is equivalent to check if

$$|\lambda| = \left| \frac{V'\phi}{\kappa V(\phi)} \right|$$

decreases with $V(\phi)$. Now, for the above form of the potential (13) the tracking condition $\Gamma > 1$ corresponds to

$$9\kappa^2\phi^4 - 6n(2n + 3)\kappa^2\phi^2 + 4n^3(n + 1) > 0,$$

ie., the solution starts tracking at the epoch when

$$\phi^2 > \frac{n(2n + 3) - n\sqrt{8n + 9}}{3\kappa^2}, \quad \text{ie., } t > \frac{2(1 - f)}{3\kappa^2 f^2}[4 - f - \sqrt{16f - 7f^2}].$$

Further,

$$|\frac{V'}{V}| = 2n \left( \frac{3\kappa^2\phi^2 - 2n(n + 1)}{3\kappa^2\phi^2 - 2n^2} \right) \left( \frac{1}{\phi} \right),$$

decreases with $V$ for large value of $\phi$, and asymptotically vanishes, ie., for $\phi \to \infty$, $|\frac{V'}{V}| = \frac{2n}{\phi} \to 0$. The other condition that $\Gamma$ should be slowly varying is satisfied provided,

$$\left| \frac{\Gamma'}{\Gamma(\frac{V'}{V})} \right| << 1.$$

This condition yields a ratio of two polynomials in $\phi$. The highest degree in the numerator is 8 while that in the denominator is 10. So the above condition is satisfied as $\phi$ evolves, since $\phi$ is a monotonically increasing function of $t$. In particular, this condition is satisfied for

$$\phi^2 > \frac{2[9n(10n^2 + 15n + 6) + 1]}{27\kappa^2(2n + 1)},$$

which corresponds to the epoch when the scalar field takes over the matter dominated era, ie., $w_\phi < w_B$.

So the corresponding scalar field is a tracker field, which at the end rolls down a sufficiently flat potential, since, $|\lambda| = \left| \frac{V'}{V} \right|$ decreases with $V$ and finally tends to zero. Thus both the coincidence and the fine tuning problems are solved. Hence, the so called intermediate inflation does not inflate the Universe at an early epoch, rather it leads to the presently observable cosmic acceleration which avoids both the coincidence and the fine tuning problem of earlier quintessence models. So the intermediate inflation should be treated as yet another dark energy model, and this proves our first claim.

### 3.2 Case-II

In this subsection we show that the same set of solutions obtained in case-I can be reproduced even for a non-canonical kinetic term. To show this, we observe that the above set of solutions does not necessarily require to fix up $g = 1/2$, rather they are found even for a functional form of $g = g(\phi)$, ie., with a non-canonical kinetic term. This can be checked easily by assuming the solution (11) of the scale factor along with the assumption (8) for the Hubble parameter. The field $\phi$, the potential $V(\phi)$, and the coupling parameter $g(\phi)$ are then found as,

$$\phi = \left( \frac{h}{Af} \right)^{\frac{1}{2}} t^{\left( \frac{1 - f}{\pi - 2f} \right)}; \quad V = \frac{h^2}{\kappa^2} \left[ \frac{3}{\phi^{2n}} - \left( \frac{h f}{\phi^{3n(1 - f)}} \right) \frac{1}{\pi - 2f} \left( 1 - f \right) \right]; \quad g = \frac{h n^2}{\kappa^2(1 - f)} \left( \frac{Af}{h} \right)^{\frac{1}{\pi - 2f}} \phi^{\frac{1}{\pi - 2f} - 2}.$$

Since $\frac{2 - f}{1 - f} > 2$, therefore the potential is in the same above form (13). To check if the potential is a tracker field, we have to consider the corresponding condition for $k$-essence models, given in [5], viz.,

$$\Gamma = \frac{V''V}{V^3} > \frac{3}{2}.$$
So let us express the potential given in (18) in the following form,

\[ V = A \phi^N - B \phi^M, \]

where, \( M = n^2 - f \) \( > N = 2n \), while the constants, \( A = \frac{3h^2}{\kappa^2} \) and \( B = \frac{1-f}{N} \left( \frac{h}{2} \right)^{\frac{1}{m}} \). Now, we observe that \( \Gamma > 3/2 \) is satisfied provided,

\[ A^2 N (2 - N) \phi^2 M + B^2 M (2 - M) \phi^2 N - 2AB (M^2 + N^2 + M + N - MN) \phi^{M+N} > 0, \]

which is true for \( N < 2 \), i.e., \( n < 1 \), since as mentioned \( M > N \) and \( \phi \) is a monotonically increasing function of time. The other condition that \( \Gamma \) is slowly varying, is also satisfied as before, since, \( |\Gamma'| \) is found as the ratio of two polynomials in \( \phi \), with a lower highest degree in the numerator than the denominator. Thus this solution has got all the features, including tracking behaviour for \( w_\phi < w_B \), of the previous solution.

Thus we find that even a non-canonical kinetic term reproduces the same set of solutions obtained with a canonical kinetic term. So, in principle it is possible to find a field theory with a non-canonical kinetic term, such that the cosmological solution is exactly the same as the solution of the said theory with canonical kinetic term. This proves our second claim which is of-course a new result.

It is to be mentioned that for \( f = \frac{2}{n^2} \), the non-canonical kinetic term turns out to be canonical and the situation arrived at in case-I is recovered.

### 3.3 Case-III

It should not be taken as granted that solution in the form \( a = a_0 e^{(At)} \) is always accompanied by a potential which is a tracker field. In this subsection we show that even a different form of the super-potential in the presence of a nonstandard form of kinetic energy, leads to similar form of the scale factor obtained in the previous subsections but it carries with it a potential which does not satisfy tracking condition. Let us choose the form of the super-potential \( H(\phi(t)) \) as,

\[ H = \kappa^2 e^{-l \phi}, \]

with \( l > 0 \). So,

\[ g \dot{\phi} = le^{-l \phi}, \quad \frac{\dot{H}}{H^2} = -\frac{l^2}{\kappa^2 g}, \quad V = -le^{-l \phi} \dot{\phi} + 3 \kappa^2 e^{-2l \phi}. \]

In view of equation (7) and (19), it is clear that for a constant \( g \), \( w_\phi \) becomes non-dynamical. So, to find the solutions explicitly, let us further make the following choice,

\[ \dot{\phi} = e^{-ml \phi}, \]

where \( m \) is a constant. Under this choice,

\[ g = le^{(m-1)l \phi}, \]

and the potential can be expressed as the algebraic sum of two inverse exponents as,

\[ V(\phi) = 3 \kappa^2 e^{-2l \phi} - le^{-(m+1)l \phi}. \]

This form of the potential \( H \), \( V \), arise as a result of compactifications in superstring models and is usually considered to exit from the scaling regime. Solutions for the scalar field and the scale factor are obtained as,

\[ \phi = \frac{1}{ml} \ln(mlt); \quad a = a_0 \exp\left[ \frac{mn^2}{(m-1)(ml)^m} \right]. \]

Thus the scale factor can be expressed in the same form (11) of the so called intermediate inflation for \( m > 1 \). Further, the equation of state and the deceleration parameters (7) are obtained as,

\[ w_\phi = -1 + \frac{2l}{3\kappa^2(mlt)} \frac{1}{m}; \quad q = -1 + \frac{l}{\kappa^2(mlt)^{\frac{1}{m}}}. \]
Now, for \( m > 1 \),
\[
\rho + p = 2l \exp[-(m - 1)l \phi] > 0,
\]
(26)
i.e., the weak energy condition is always satisfied while,
\[
\rho + 3p = 6 \left(l \exp[-(m - 1)l \phi] - \kappa^2 \exp[-2l \phi]\right),
\]
(27)
i.e., the strong energy condition is violated at \( t \geq \frac{d}{m} \left( \frac{l}{\kappa} \right)^\frac{1}{m-1} \), which corresponds to \( \exp \left((m - 1)l \phi \right) \geq \frac{l}{\kappa} \), - the epoch of transition from decelerating to the accelerating phase. Further, the necessary condition for inflation and the slow roll condition are satisfied at the same epoch. The Universe expands exponentially but decelerates from an infinitely large value. The equation of state \( w_\phi \) starts from indefinitely large value at the beginning. \( w_\phi \to \infty \), implies a greater effective velocity of sound than that of light in the corresponding medium, which also appears in phantom models having super negative pressure. So classically it has got no meaning at all. Such result only dictates the importance of invoking quantum cosmology before the equation of state reaches stiff fluid era.

As before, the field satisfies scaling solution, since during evolution it takes over the phases of stiff fluid, radiation and the matter dominated era. It then finally comes out of the scaling regime and enters into an accelerating phase before asymptotically it reaches the desired value of minus one. So this solution also has got the same feature as the previous one. However,
\[
\Gamma = \frac{V''(\phi)V(\phi)}{V'(\phi)} > 3/2,
\]
requires
\[
36\kappa^4 e^2 m l \phi + 6\kappa^2 l [((m + 1)^2 - 2)e^{(m+1)l \phi} + l^2 (m + 1)^2 e^{2l \phi} < 0,
\]
which is not satisfied for \( m > 1 \). So the potential (23) does not satisfy the tracker condition. Thus fine tuning problem cannot be avoided in this model. Note that for \( m < 1 \), the above relation may be satisfied, but then the solution does not correspond to late time acceleration, rather it is inflationary, in which the scale factor starts from zero and ends up with the value \( a_0 \). Hence, our third assertion that in non-canonical theories, the solution in the form \( a = a_0 e^{(At)^c} \) does not always carry with it a tracker field, has been established.

### 4 Presence of background matter

Observations suggest that our Universe is presently filled with 70% of dark energy, 26% of dark matter, 4% of Baryons and 0.005% of radiation \[19\]. So, to consider a realistic model, presence of background distribution of all types of Baryonic and non-Baryonic matter should be accounted for explicitly. In this section our motivation is to check if the above form of the scale factor admits viable cosmological solution in the presence of background matter. The field equations now can be arranged as,
\[
\dot{H} = -\kappa^2 [\dot{\phi}^2 + \frac{\rho_B + p_B}{2}],
\]
(28)
\[
\dot{H} + 3H^2 = \kappa^2 [V(\phi) + \frac{\rho_B - p_B}{2}],
\]
(29)
where \( \rho_B \) and \( p_B \) are the energy density and pressure of the background matter respectively. Further, since scalar field is minimally coupled to the background, so continuity equations for the background matter and the scalar field hold independently. Hence, We can write,
\[
\dot{\rho}_B + 3H(1 + w_B)\rho_B = 0 = \dot{\rho}_\phi + 3H(1 + w_\phi)\rho_\phi.
\]
(30)
Thus we have
\[
\rho_B = \rho_B^{(0)} a^{-3(1+w_B)},
\]
(31)
and
\[
\rho_\phi = \rho_\phi^{(0)} a^{-3(1+w_\phi)},
\]
(32)
where, \( \rho_B^{(0)} \) and \( \rho_\phi^{(0)} \) are the present values of the background and the scalar field energy densities. If we now plug in the solution of the scale factor in the form \( a = a_0 \exp A t \), with \( a_0 > 0, A > 0 \) and \( 0 < f < 1 \), then the potential is found in view of equation (29) as,

\[
V = \frac{1}{\kappa^2} \left[ \frac{3A^2f^2}{(\phi - \phi_0)^2(1-f)} - \frac{Af(1-f)}{(\phi - \phi_0)(2-f)} \right] - \frac{(1 - w_B)\rho_B}{2},
\]

while the kinetic term can be obtained in view of equation (28) as,

\[
g\dot{\phi}^2 = \frac{Af(1-f)}{\kappa^2(\phi - \phi_0)(2-f)} - \frac{(1 + w_B)\rho_B}{2}.
\]

In the above two equations (33) and (34), we have used the equation of state \( p_B = w_B \rho_B \), for the background matter. Now, it is not possible to find a solution of \( \phi \) in closed form for the standard form of kinetic energy \((g = 1/2)\). Thus one can make some suitable choice of the scalar field \( \phi \) to express the potential \( V(\phi) \) and the coupling parameter \( g(\phi) \) as functions of \( \phi \). It is also noticed that the first term of the potential is predominantly dominating as the Universe evolves. Thus the potential remains positive asymptotically, though it starts from an indefinitely large negative value. As an example, let us choose \( \phi \) as a monotonically increasing function of time,

\[
\phi - \phi_0 = t.
\]

So the Potential and the kinetic energy of the scalar field take the following forms respectively,

\[
V = \frac{1}{\kappa^2} \left[ \frac{3A^2f^2}{(\phi - \phi_0)^2(1-f)} - \frac{Af(1-f)}{(\phi - \phi_0)(2-f)} \right] - \frac{(1 - w_B)\rho_B^{(0)}}{2|a_0 \exp \{ A(\phi - \phi_0)f \}|^{3(1+w_B)}},
\]

and,

\[
g\dot{\phi}^2 = G(\phi) = \frac{Af(1-f)}{\kappa^2(\phi - \phi_0)(2-f)} - \frac{(1 + w_B)\rho_B^{(0)}}{2|a_0 \exp \{ A(\phi - \phi_0)f \}|^{3(1+w_B)}}.
\]

The energy density and the pressure of the scalar field are expressed as,

\[
\rho_\phi = \frac{3A^2f^2}{\kappa^2(\phi - \phi_0)^2(1-f)} - \frac{\rho_B^{(0)}}{|a_0 \exp \{ A(\phi - \phi_0)f \}|^{3(1+w_B)}},
\]

and

\[
p_\phi = \frac{1}{\kappa^2} \left[ \frac{2Af(1-f)}{(\phi - \phi_0)(2-f)} - \frac{3A^2f^2}{(\phi - \phi_0)^2(1-f)} \right] - \frac{w_B \rho_B^{(0)}}{|a_0 \exp \{ A(\phi - \phi_0)f \}|^{3(1+w_B)}}.
\]

The scale factor admits scaling solution as already noticed and the scaling of \( \rho_\phi \) becomes sloth as \( V(\phi) \) starts dominating over the kinetic energy. However, synchronized scaling of \( \rho_\phi \) and \( p_\phi \) is not enough, realistic tracking behaviour is necessary to solve the coincidence problem. Thus, it remains to be checked if the scalar field is a tracker field.

### 4.1 Presence of background Radiation

In the radiation dominated era \( w_B = w_r = 1/3 \), the state parameter of the scalar field is given by,

\[
w_\phi = \frac{3a_0^4 e^{4A(\phi - \phi_0)f} [2Af(1-f) - 3A^2f^2(\phi - \phi_0)f] - \kappa^2 \rho_r^{(0)}(\phi - \phi_0)^2(1-f)}{3A^2f^2 a_0^4 e^{4A(\phi - \phi_0)f} (\phi - \phi_0)^2(1-f) - \kappa^2 \rho_r^{(0)}(\phi - \phi_0)^2(1-f)},
\]

where, \( \rho_r \) stands for the energy density of radiation. Now, \( w_\phi \geq 1/3 \) requires,

\[
3A^4 f_0^2 [5A(\phi - \phi_0)f - 3(1-f)e^{4A(\phi - \phi_0)f} + \kappa^2 \rho_r^{(0)}(\phi - \phi_0)^2(1-f)] \geq 0.
\]

The above condition does not hold since the first term is positive and contributes dominantly for a monotonically increasing function of \( \phi \). So, \( w_\phi < w_r \). Thus we conclude that \( w_\phi < 1/3 \). The form of the potential (36) is

\[
V = \frac{1}{\kappa^2} \left[ \frac{3A^2f^2}{(\phi - \phi_0)^2(1-f)} - \frac{Af(1-f)}{(\phi - \phi_0)(2-f)} \right] - \frac{\rho_r^{(0)}}{3a_0^4 e^{4A(\phi - \phi_0)f}}.
\]
The above form of the potential is a tracker field, since $\Gamma > 3/2$, [5] requires,

$$\frac{A f^2 (1-f)}{\kappa^4} [18 A^2 f^2 (\phi - \phi_0)^2 - 3 A f (3 - f) (\phi - \phi_0)^f + \frac{(1 - f) (2 - f)}{2} e^{8 A (\phi - \phi_0)^f}$$

$$- \frac{\rho_{r(0)}}{3 \kappa^2 a_0^3} [48 A^3 f^3 (\phi - \phi_0)^{3(1+f)} - 76 A^2 f^2 (1 - f) (\phi - \phi_0)^{(2+f)} + 2 A f (1 - f) (19 - 10 f) (\phi - \phi_0)^2 -$$

$$(1 - f) (2 - f) (3 - f) (\phi - \phi_0)^{(2-f)}] e^{4 A (\phi - \phi_0)^f} + \frac{4 \rho_{r(0)}^2}{9 a_0^6} [2 A f (\phi - \phi_0)^4 - (1 - f) (\phi - \phi_0)^{(4-f)}] > 0,$$

which is obvious because, with evolution $\phi$ grows and the very first term of the above equation, which is positive definite, is most dominant. Straight forward, but lousy calculation also shows that $\left| \frac{\Gamma'}{\Gamma(\phi)} \right| \ll 1$, i.e., $\Gamma$ remains almost constant. Hence the potential (42) is indeed a tracker field.

### 4.2 Presence of Baryonic and non-Baryonic matter

During the present matter dominated era, $w_B = w_m = 0$, the state parameter of the scalar field has the following expression,

$$w = \frac{2 A f (1 - f) - 3 A^2 f^2 (\phi - \phi_0)^f}{3 A^2 f^2 (\phi - \phi_0)^f - \kappa^2 \rho_{m(0)} a_0^3 (\phi - \phi_0)^{2-f} e^{-3 A (\phi - \phi_0)^f}}.$$  \hspace{1cm} (43)

The above form of the state parameter is always negative, since, it requires, $(\phi - \phi_0)^f > \frac{2 (1-f)}{3 A f}$, which is true since $\phi$ has been chosen to increase monotonically with time. Thus, we arrive at the fact that $w < w_B$. The potential (36) in the matter dominated era takes the following form,

$$V = \frac{1}{\kappa^2} \left[ \frac{3 A^2 f^2}{(\phi - \phi_0)^{2(1-f)}} - \frac{A f (1 - f)}{(\phi - \phi_0)^{(2-f)}} \right] - \frac{\rho_{m(0)}}{2 a_0^3} e^{3 A (\phi - \phi_0)^f}.  \hspace{1cm} (44)$$

This potential is again a tracker field, since $\Gamma > 3/2$, [5] now requires

$$\frac{A f^2 (1-f)}{\kappa^4} [18 A^2 f^2 (\phi - \phi_0)^2 - 3 A f (3 - f) (\phi - \phi_0)^f + \frac{(1 - f) (2 - f)}{2} e^{8 A (\phi - \phi_0)^f}$$

$$- \frac{\rho_{m(0)}}{2 \kappa^2 a_0^3} [27 A^3 f^3 (\phi - \phi_0)^{3(1+f)} - 54 A^2 f^2 (1 - f) (\phi - \phi_0)^{(2+f)} + 3 A f (1 - f) (11 - 6 f) (\phi - \phi_0)^2 -$$

$$(1 - f) (2 - f) (3 - f) (\phi - \phi_0)^{(2-f)}] e^{4 A (\phi - \phi_0)^f} + \frac{3 \rho_{m(0)}^2}{8 a_0^6} [3 A f (\phi - \phi_0)^4 - 2 (1 - f) (\phi - \phi_0)^{(4-f)}] > 0,$$

which is obvious as before. Straight forward calculation here again shows that $\left| \frac{\Gamma'}{\Gamma(\phi)} \right| \ll 1$, i.e., $\Gamma$ remains almost constant. Hence the potential (44) is indeed a tracker field. Thus, our fourth and final assertion is proved.

### 4.3 Presence of both background radiation and the matter

In the presence of both the radiation and matter in the form of dust, we have $w < w_B$. The potential has the following form,

$$V = \frac{1}{\kappa^2} \left[ \frac{3 A^2 f^2}{(\phi - \phi_0)^{2(1-f)}} - \frac{A f (1 - f)}{(\phi - \phi_0)^{(2-f)}} \right] - \frac{\rho_{r(0)}}{3 a_0^4} e^{4 A (\phi - \phi_0)^f} - \frac{\rho_{m(0)}}{2 a_0^3} e^{3 A (\phi - \phi_0)^f}.  \hspace{1cm} (45)$$

which contains algebraic sum of two inverse powers and two inverse exponents. The condition $\Gamma > 3/2$ requires,

$$\frac{A f^2 (1-f)}{\kappa^4} [18 A^2 f^2 (\phi - \phi_0)^2 - 3 A f (3 - f) (\phi - \phi_0)^f + \frac{(1 - f) (2 - f)}{2} e^{8 A (\phi - \phi_0)^f}$$

$$+ (\text{terms which vanish as } \phi \text{ is large}) > 0.$$  

Clearly the above condition is satisfied and thus the potential is a tracker field.
5 Concluding remarks

The cosmological solution in the form \( a = a_0 e^{(At^f)} \) with \( a_0 > 0, A > 0 \) and \( 0 < f < 1 \) should be treated as dark energy model leading to late time cosmic acceleration, rather than intermediate inflation at early Universe. In the context of general theory of relativity with a minimally coupled scalar field, the accompanying potential in the form of double inverse power, is a tracker field (\( \Gamma > 1 \) and is slowly varying). Thus the coincidence problem gets solved. This has been shown in section 3.1. Even a minimally coupled theory with non-canonical kinetic energy reproduces the same set of solutions. The potential is also in the same form of double inverse power, which again satisfies the tracker condition (\( \Gamma > 3/2 \) and is slowly varying). This has been revealed in section 3.2. However, such solution of the gravitational action with non-canonical kinetic energy does not always carry a potential which is tracker. A non-tracking potential in the form of double inverse exponents has been studied in section 3.3. Such solution \( a = a_0 e^{(At^f)} \) is also admissible in the presence of background matter. The potential here is in the form of the sum of double inverse power and an inverse exponent, when the background is either radiation or dust. In the presence of both, the potential has a form of the sum of double inverse power and double inverse exponents. These completely new type of potentials are found to be tracker fields since, for \( w_\phi < w_B, \Gamma > 3/2 \) and is slowly varying. It is noticed that potential in the form of double inverse power is a tracker field while that with double inverse exponents is not. However a combination of two is again tracking. This is obvious, since as \( \phi \) increases inverse exponents become less dominant. Finally, the equation of state \( w_\phi \to -1 \) and the solution in the above form \( a = a_0 e^{(At^f)} \), asymptotically behaves as de-Sitter solution.

Thus, we conclude that a minimally coupled scalar field admitting the above form of solution of the scale factor \( a = a_0 e^{(At^f)} \), has all the nice features to account for the dark energy of the present Universe.

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