CLASSICAL AND SEMICLASSICAL PROPERTIES OF EXTREMAL BLACK HOLES WITH DILATON AND MODULUS FIELDS

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ABSTRACT

We discuss both classical and semiclassical properties of extremal black holes in theories where the dilaton and a modulus field are present. We find that the corresponding 2-dim geometry is asymptotically anti-de Sitter rather than asymptotically flat as in the purely dilatonic case. This fact has many important consequences, which we analyze at length, both for the classical behaviour and for the thermodynamical properties of the black hole. We also study the Hawking evaporation process in the semiclassical approximation. The calculations strongly indicates the emergence of a stable ground state as the end point of the process. Some comments are made about the relevance of our results for the problem of information loss in black hole physics.
1. Introduction

String inspired black hole models have been extensively studied in recent years. A first motivation for this interest is the fact that exact charged solutions of the 4-dim lowest order effective action of string theory have been found in a variety of cases [1-3].

These solutions exhibit peculiar thermodynamical properties, which are different from those of the ordinary Einstein theory black holes [4]. In particular, the extremal limit, where the values of the mass and of the charge are such that the black hole is on the edge of becoming a naked singularity, presents a behaviour which resembles that of elementary particles [5]. Even if some of these features are spoiled by going to higher order in perturbation theory [6], these models still present a relevant theoretical interest. The extremal limit of a black hole of given charge represents, in fact, the ground state for its Hawking evaporation process. The study of the scattering of low-energy particles by an extremal black hole should therefore shed some light on the problem of black hole evaporation and on the fate of the information after the evaporation process has taken place [7]. In particular, one hopes to clarify the issue whether the black hole completely disappears or a massive remnant is left. In this context, a very useful property of extremal string black hole solutions is the fact that in proximity of the horizon they split into a direct product of a 2-dim solution with a 2-sphere of constant radius [8]. It is therefore possible to study the properties of the black hole by means of a reduced 2-dimensional model, which makes the problem much easier to treat [9]. Many papers have in fact been devoted to the investigation of these 2-dim models [10,11]. If the back-reaction of the gravitational field is neglected, exact solution of the field equations including one-loop quantum corrections can be found, which describe the formation of a black hole and the early stages of its evaporation [9]. Moreover, if the back-reaction is also taken into account, the behaviour of the black hole during the final stages of the evaporation can be qualitatively described [10].

The peculiar properties of string black holes are essentially due to the non-minimal coupling of gravity and gauge fields to a massless scalar field which is contained in the low-energy spectrum of string, namely the dilaton. The non-minimality of the coupling permits to circumvent the no-hair theorems which render essentially unique the black hole solutions of Einstein theory. It is therefore interesting to consider the other non-minimally coupled massless scalars which appear in the low-energy effective action of the string [3,12]. It turns out, in fact, that modulus fields coming from the compactification of the string to 4-dimensions can acquire non minimal couplings to the gauge fields of the 4-dim action owing to string one-loop effects [13]. In a recent paper [3], we have studied a model where these effects are taken into account and found a class of exact 4-dim solutions, whose properties are slightly different from those where modulus fields are decoupled. In this paper we extend the investigation of such solutions to their extremal limit and study the black hole evaporation by means of a 2-dim effective model.

The main result of our investigation is that, contrary to the pure dilatonic case, the solutions of the 2-dim effective theory are not asymptotically flat, but have as asymptotic a space of constant negative curvature. This implies that in this approximation the 4-dim extremal black hole is screened by an infinite potential barrier, which prevents it to irradiate to the asymptotically flat region. This explains why the temperature vanishes in
the 4-dim theory. We also study the quantum evaporation of the 2-dim black hole. We find that for a wide range of the parameters which characterize the model, the Hawking radiation rate is not asymptotically mass independent as in the case of pure dilatonic 2-dim gravity, but goes to zero with the mass of the hole. This strongly suggests the emergence of a stable state at the end point of the evaporation process.

The paper is organized as follows: in section 2 we review the 4-dim solutions found in [3] and study their extremal limit. In section 3 we examine the propagation of a perturbation in the throat region of the solution and show that it is exponentially damped at infinity. In section 4 we study the 2-dimensional effective theory and discuss its solutions in different gauges. We also review 2-dim spacetimes of constant curvature. In section 5 we study the 2-dimensional theory in the conformal gauge and in section 6 we analyse the 2-dim black hole evaporation process using various methods. We state our conclusions in section 7.

2. The solution in 4 dimensions

a) The general solution

In a recent paper [3], we have studied the magnetic charged black hole solution of a 4-dim gravitational action obtained by low-energy effective string theory when moduli of the compactified manifold are taken into account.

In terms of the metric to which the string couples, which we shall use in this paper, the action had the form:

$$ S = \int \sqrt{g} \, d^4x \, e^{-2\phi} \left[ R + 4(\nabla \phi)^2 - \frac{2}{3}(\nabla \psi)^2 - F^2 - e^{2(\phi - \frac{3}{4} \psi)} F^2 \right], \quad (2.1) $$

where $\phi$ is the dilaton field, $\psi$ the compacton and $F_{\mu\nu}$ the Maxwell field strength, $q$ being a coupling constant.

Exact charged black hole solutions to this action were found in the case $\psi = \frac{3}{q} \phi + \text{const}$, which have the form:

$$ ds^2 = -\left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)^k dt^2 + \left(1 - \frac{r_+}{r}\right)^{-1} \left(1 - \frac{r_-}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, $$

$$ e^{2(\phi - \phi_0)} = \left(1 - \frac{r_-}{r}\right)^{(k-1)/2}, \quad F_{\mu\nu} = \frac{Q}{r^2} \epsilon_{\mu\nu}, \quad (2.2) $$

with

$$ k = \frac{3 - 2q^2}{3 + 2q^2}, \quad -1 \leq k \leq 1. \quad (2.3) $$

The two parameters $r_+$ and $r_-$ are related to the charge $Q$ and to the mass $M$ of the black hole by the relations:

$$ 2M = r_+ + \frac{k + 1}{2} r_-, \quad Q^2_M = \frac{1 - k}{4} r_+ r_. \quad (2.4) $$

In the limit $k = -1$ (i.e. $q \to \infty$), the solution reduces to that found in ref. [1] for the case where the compacton field is not taken into account.
The spatial part of the metric (2.2) is actually identical to that of ref. [1]: it describes an asymptotically flat region attached to a long tube (the "throat"), terminated by a horizon. In the extremal limit \( r_+ = r_- \), the tube becomes infinitely long.

In our case, however, the full spacetime has a slightly different structure with respect to ref. [1]: as we shall see, the infinitely long tube in the extremal limit is replaced by the direct product of a 2-dim negative curvature spacetime with a 2-sphere of constant radius. This fact will have important consequences on the physical properties of the model.

Also of interest are the thermodynamical properties of the black hole described by (2.2). The temperature is given by

\[
T = \frac{1}{4\pi r_+} \left( 1 - \frac{r_-}{r_+} \right)^{\frac{1+k}{2}}.
\]

which goes to zero in the extremal limit \( r_+ = r_- \), while the entropy is

\[
S = \pi r_+^2.
\]

(2.6a)

One could therefore think of the extremal limit as a non-radiating ground state. This point of view will be confirmed in the following.

It should be noticed, however, that for the "canonical" metric \( \tilde{g} = e^{-2\phi}g \), while the expression for the temperature remains the same, the entropy is given by

\[
S = \pi r_+^2 \left( 1 - \frac{r_-}{r_+} \right)^{\frac{1-k}{2}},
\]

which also vanishes for \( r_+ \to r_- \). The discrepancy is of course due to the different definition of the volume element in the two metrics.

b) The extremal limit

In the rest of this paper we shall study the extremal limit \( r_+ = r_- = Q \) of the solution (2.2), where \( Q = \frac{2}{\sqrt{1-k}}Q_M \), and the ensuing 2-dim effective theory, along the lines of ref. [8]. For this purpose, it is useful to define a new coordinate \( \sigma \), such that

\[
\sigma = \text{arcsh} \sqrt{\frac{(r-r_+)}{(r_+-r_-)}}.
\]

In terms of \( \sigma \), the metric and the dilaton field (2.2) take the form

\[
ds^2 = -4Q^2 \frac{\Delta^{k+1} \sinh^2 \sigma \cosh^{2k} \sigma}{(r_+ + \Delta \sinh^2 \sigma)^{k+1}} dt^2 + (r_+ + \Delta \sinh^2 \sigma)^2 (4d\sigma^2 + d\Omega^2),
\]

\[
e^{2(\phi-\phi_0)} = \left[ \frac{r_+ + \Delta \sinh^2 \sigma}{\Delta \cosh^2 \sigma} \right]^{\frac{1-k}{2}},
\]

where \( \Delta = r_+ - r_- \).

There are several regimes under which the extremal limit can be approached, which correspond to different solutions of the action (2.1):
1) $\sigma \gg 1$: this limit is reached by taking the asymptotically flat region and the throat fixed and allowing the horizon to move off to infinity as $\Delta \to 0$. The solution is then:

$$ds^2 = -4Q^2 \left( 1 + \frac{Q}{y} \right)^{-(k+1)} dt^2 + \left( 1 + \frac{Q}{y} \right)^2 (dy^2 + y^2 d\Omega^2),$$

$$e^{2(\phi - \phi_0)} = \left( 1 + \frac{Q}{y} \right)^{\frac{1-k}{2}}, \quad F_{\mu\nu} = \frac{\sqrt{1-k}}{2} \frac{Q}{(y+Q)^2} \epsilon_{\mu\nu},$$

(2.8)

with $y \geq 0$. The metric is everywhere regular and describes the transition between an asymptotically flat spacetime for $y \to \infty$ and one with topology $H^2 \times S^2$ for $y \to 0$, $H^2$ being 2-dim anti-de Sitter spacetime. This solution can therefore be considered a generalization of the solitonic solutions of string theory described in [14].

2) $1 \gg \sigma \gg \ln(Q/\Delta)$: this limit corresponds to the infinite throat with linear dilaton and is reached by sending to infinity both the horizon and the asymptotically flat region:

$$ds^2 = -4Q^2 e^{2(k+1)\sigma} dt^2 + Q^2 (4d\sigma^2 + d\Omega^2),$$

$$\phi = \frac{k-1}{2} \sigma.$$  

(2.9)

At variance with the GHS case, the metric in this limit does not describe an infinite cylinder, but rather the direct product of 2-dim anti-de Sitter spacetime with a 2-sphere of radius $Q$.

3) $\sigma \ll 1$: this limit is obtained by keeping the horizon and $\Delta \frac{1-k}{2} e^{-2\phi_0}$ fixed, and letting the asymptotically flat region go to infinity when $\Delta \to 0$ and describes the horizon plus the infinite throat. The solution is given by:

$$ds^2 = -4Q^2 \sinh^2 \sigma \cosh^2k \sigma dt^2 + Q^2 (4d\sigma^2 + d\Omega^2),$$

$$e^{2(\phi - \phi_0)} = \left( \frac{Q}{\cosh^2 \sigma} \right)^{\frac{1-k}{2}},$$

(2.10)

and again has the form of a direct product of a 2-dim solutions with an horizon at $\sigma = 0$ and a 2-sphere. It will therefore be useful in the discussion of black hole solutions of the 2-dim effective action. Its 2-dim sections will be discussed at length in the following.

3. Perturbations on the throat

Near the extremal limit, the two essential features of the geometry are the asymptotically flat region and the attached throat. It is therefore crucial to study the propagation of the fields along the throat. From the anti-de Sitter form of the metric can be expected that the fields are in some way confined into the throat, due to the infinitely increasing gravitational potential for $\sigma \to \infty$.

Given a perturbation $\chi$ on the throat, one can easily check that the kinetic term in the action at the linearized level is

$$S_\chi = - \int \sqrt{g} \, d^4x \, e^{-2b\phi} (\nabla \chi)^2,$$  

(3.1)
where $\tilde{g}$ is the flat metric and $b$ is a constant depending on the mode considered.

The effect of the linear dilaton background can be seen by defining the new field $\tilde{\chi} = e^{-b\phi} \chi$. The linearized action becomes therefore, modulo a total derivative,

$$S_{\tilde{\chi}} = -\int \sqrt{\tilde{g}} \, d^4 x \left[ (\nabla \tilde{\chi})^2 + \tilde{\chi}^2 (b^2 (\nabla \phi)^2 - b \nabla^2 \phi) \right] =$$

$$= -\int \sqrt{\tilde{g}} \, d^4 x \left[ (\nabla \tilde{\chi})^2 + M^2(\phi) \tilde{\chi}^2 \right], \quad (3.2)$$

where

$$M^2(\phi) = \text{const} \times \exp \left( -4 \frac{1 + k}{1 - k} \phi \right),$$

is a space dependent mass term for $\tilde{\chi}$, which becomes infinite in the region of weak coupling $e^{2\phi} \rightarrow 0$, where the excitations are thus suppressed by an infinite mass gap. They are therefore not allowed to escape to infinity, which is in our case the asymptotically flat region.

Similar results have been obtained in [4] for the "canonical" metric $\tilde{g} = e^{-2\phi} g$. In particular, these results can help to interpret the vanishing of the temperature (2.5) in the extremal limit: a potential barrier hinders the interaction between the black hole and the external fields, creating a mass gap. In our treatment, the geometrical origin of the barrier is however more evident.

### 4. The two-dimensional effective theory

**a) Dimensional reduction**

In order to investigate the quantum properties of the extremal black hole, it is useful to study the 2-dim effective model obtained by retaining only the radial modes of the 4-dim theory. This approximation is justified by the fact that the background solution, as we have seen, reduces to the direct product of two 2-dim metrics near the horizon. The 2-dim theory is renormalizable and is a generalization of the model considered in [9].

We then start with a discussion of the classical aspects of the 2-dim effective theory.\footnote{These properties will be discussed in more detail in a forthcoming paper [15]. Related 2-dim models are discussed also in [16, 11].} The action (2.1) can be dimensionally reduced by taking the angular coordinates to span a 2-sphere of constant radius $Q$: the resulting action is

$$S = \frac{1}{2\pi} \int \sqrt{\tilde{g}} \, d^2 x \, e^{-2\phi} \left[ R + 4(\nabla \phi)^2 - \frac{2}{3} (\nabla \psi)^2 + \lambda_1^2 - \lambda_2^2 \, e^{2(\phi - \frac{2}{3}\psi)} \right], \quad (4.1)$$

where

$$\lambda_1^2 = \frac{2(q^2 + 3)}{2q^2 + 3} \frac{1}{Q^2} = \frac{3 + k}{2Q^2}, \quad \lambda_2^2 = \frac{2q^2}{2q^2 + 3} \frac{1}{Q^2} = 1 - \frac{k}{2Q^2}.$$
\[ 4\nabla^2 \phi - 4(\nabla \phi)^2 - \frac{2}{3}(\nabla \psi)^2 + \lambda_i^2 + R = 0, \quad (4.2) \]
\[ \nabla^2 \psi - 2(\nabla \phi)(\nabla \psi) + \frac{q}{2} \lambda_2^2 e^{2(\phi - \frac{q}{2} \psi)} = 0. \]

With the ansatz
\[ e^{2\Phi} = \frac{q^2}{3} e^{2\phi}, \quad (4.3) \]
the 2-dim action admits the exact solution:
\[ ds^2 = -\sinh^2(\kappa \sigma) \cosh^2(\kappa \sigma)dt^2 + d\sigma^2, \]
\[ e^{2(\phi - \phi_0)} = \cosh^k(\kappa \sigma), \quad (4.4) \]
where \( \kappa = \frac{\lambda_2}{\sqrt{2(1-k)}} \), with \( k \) defined by (2.3).

These solutions are everywhere regular for any value of \( k \), and have a horizon at \( \sigma = 0 \). They of course coincide with the 2-dim section of the solutions (2.10). For \( k \neq -1 \) the solutions (4.4) behave as anti-de Sitter for \( \sigma \to \infty \). For \( k = -1 \) they are asymptotically flat and reduce to those obtained in [17].

Another class of solutions to (4.2,3) is given by the anti-de Sitter background with linear dilaton:
\[ ds^2 = -e^{2(k+1)\kappa \sigma} dt^2 + d\sigma^2, \]
\[ \phi - \phi_0 = \frac{k - 1}{2} \kappa \sigma, \quad (4.5) \]
which we shall refer as ADS linear dilaton vacuum. These solutions are asymptotic of (4.4) for \( \sigma \to \infty \) and correspond to the dimensional reduction of (2.9).

We also notice that, substituting the ansatz (4.3) directly into the action, one has:
\[ S = \frac{1}{2\pi} \int \sqrt{g} \, d^2 x \, e^{-2\phi} \left[ R + \frac{8k}{k - 1} (\nabla \phi)^2 - \lambda_2^2 \right]. \quad (4.6) \]
For \( k = 0 \) the action reduces to that of the Jackiw-Teitelboim theory [18].

b) The Schwarzchild gauge

In the following, it will be useful to write the metric in Schwarzchild coordinates. In such coordinates it is in fact possible to continue the metrics besides the horizon at \( \sigma = 0 \) and to get a more immediate insight of their physical properties. The new coordinates are defined so that the metric takes the form \( ds^2 = -\Upsilon(r) dt^2 + \Upsilon^{-1}(r) dr^2 \). The general solutions in this gauge are:
\[ ds^2 = (b^2 r^2 - a^2 r^{2k}) dt^2 + (b^2 r^2 - a^2 r^{2k})^{-1}dr^2, \]
\[ e^{2(\phi - \phi_0)} = (br)^{k - 1} \Upsilon^k, \quad (4.7) \]
where \( b = \frac{(1+k)\lambda_2}{\sqrt{2(1-k)}} \) and \( a \) is a free parameter which, from a 4-dim point of view, can be interpreted as parametrizing the departure of the solution from extremality. These coordinates can be related to the previous ones by the transformation \( br = \cosh^{1+k}(\kappa \sigma) \), with \( a = b \).

The metric asymptotes anti-de Sitter spacetime for \( r \to \infty \), and has a horizon at \( r_0 = \left(\frac{\alpha}{b}\right)^{1+k} \), which shields a singularity at \( r = 0 \), except in the special cases \( k = 0, 1 \). This is easily seen by considering the curvature:

\[
R = -2 \left( b^2 + \frac{k(1-k)}{(1+k)^2} a^2 r^{-1+k} \right).
\] (4.8)

If \( a = 0 \), the metric reduces to that of 2-dim anti-de Sitter spacetime with curvature \(-2b^2\):

\[
d s^2 = -(b r)^2 dt^2 + (br)^{-2} dr^2, \quad e^{2(\phi-\phi_0)} = (br)^{\frac{k-1}{1+k}},
\] (4.9)

which corresponds to the solution (4.5).

The mass of the solutions can easily be obtained by the ADM procedure and is given by:

\[
M = \frac{1-k}{2(1+k)} e^{-2\phi_0} a^2 b^{\frac{k-1}{1+k}},
\] (4.10a)

or, in terms of the value \( \phi_H \) of \( \phi \) at the horizon,

\[
M = \frac{\lambda_2}{2} \sqrt{\frac{1-k}{2}} \exp \left( 2 \frac{1+k}{1-k} \phi_0 - 4 \frac{1}{1-k} \phi_H \right).
\] (4.10b)

Analogously, using standard procedures, one can calculate the thermodynamical parameters:

\[
T = \frac{1}{2\pi} e^{(1+k)\phi_0} \left( \frac{2M}{1-k} \right)^{\frac{1+k}{2}} \left( \frac{b}{1+k} \right)^{\frac{1-k}{2}},
\] (4.11)

\[
S = 2\pi e^{(1+k)\phi_0} \left( \frac{1+k}{1-k} \frac{2M}{b} \right)^{\frac{1-k}{2}},
\] (4.12)

where the entropy \( S \) is computed with respect to the asymptotic anti-de Sitter solution. The temperature increases monotonically with the mass and goes to zero for \( M = 0 \), which should therefore be considered as a stable ground state for the quantum black hole. The entropy displays essentially the same behaviour as \( T \). Also is remarkable the relation \( ST = \frac{2M}{1+k} \), valid for \( \phi_0 = 0 \).

From the previous discussion is evident that for \( a \neq 0 \) all the metrics (4.7) describe black holes with asymptotically anti-de Sitter behaviour. There are however three special cases, which should be considered separately:

1) \( k = -1 \):

\[
d s^2 = -(1 - a^2 e^{-br}) dt^2 + (1 - a^2 e^{-br})^{-1} dr^2, \quad e^{2(\phi-\phi_0)} = e^{br}.
\] (4.13)
This is the asymptotically flat solution thoroughly discussed in [17]. Therefore we do not discuss it here.

2) $k = 0$:

$$ds^2 = -(b^2 r^2 - a^2) dt^2 + (b^2 r^2 - a^2)^{-1} dr^2, \quad e^{2(\phi - \phi_0)} = (br)^{-1}. \quad (4.14)$$

For $a \neq 0$ the metric, while having a constant negative curvature has different properties from anti-de Sitter spacetime. In particular, it possesses a horizon at $r = a/b$, but no singularities. We consider it in more detail below.

3) $k = 1$:

$$ds^2 = -(b^2 r^2 - a^2 r) dt^2 + (b^2 r^2 - a^2 r)^{-1} dr^2, \quad \phi = \text{const.} \quad (4.15)$$

In this case, the metric can be put in the same form as for the $k = 0$ case, by simply shifting $r$. Now, however, the dilaton is everywhere constant.

c) 2-dim spacetimes of constant negative curvature.

In order to discuss the properties of the $k = 0$ solution and for future reference, it is useful to summarize the properties of the 2-dim spacetimes of constant negative curvature. They can be easily constructed by considering parabolic hyperboloids embedded in 3-dim Minkowski spacetime, with metric $ds^2 = dz^2 - dx^2 - dy^2$. The standard anti-de Sitter spacetime is then represented by a 1-sheet hyperboloid of equation \[19]:

$$x^2 + y^2 - z^2 = 1. \quad (4.16)$$

The surface can be parameterized by the coordinates:

$$x = \cosh \sigma \sin t, \quad y = \cosh \sigma \cos t, \quad z = \sinh \sigma, \quad (4.17)$$

with $-\infty < \sigma < \infty, \ 0 \leq t \leq 2\pi$, giving rise to the metric

$$ds^2 = -\cosh^2 \sigma dt^2 + d\sigma^2, \quad (4.18a)$$

or, in the Schwarzschild gauge,

$$ds^2 = -(r^2 + 1) dt^2 + (r^2 + 1)^{-1} dr^2. \quad (4.18b)$$

Another parametrization, which however does not cover the whole hyperboloid, is given by:

$$x = e^\sigma t, \quad y = \cosh \sigma - \frac{1}{2} e^\sigma t^2, \quad z = \sinh \sigma + \frac{1}{2} e^\sigma t^2, \quad (4.19)$$

$\dagger$ For simplicity we put $\Lambda = 1$ in the following.
with $-\infty < \sigma < \infty$, $-\infty < t < \infty$. In these coordinates,

\[ ds^2 = -e^{2\sigma} dt^2 + d\sigma^2, \]

(4.20a)

or

\[ ds^2 = -r^2 dt^2 + r^{-2} dr^2. \]

(4.20b)

Contrary to the 4-dim case, in 2 dimensions there is another spacetime of constant negative curvature. This is represented by a 2-sheet hyperboloid of equation

\[ x^2 + y^2 - z^2 = -1, \]

(4.21)

embedded as before in 3-dim flat spacetime. With the choice of coordinates:

\[ x = \sinh \sigma \sin t, \quad y = \sinh \sigma \cos t, \quad z = \cosh \sigma, \]

(4.22)

$0 \leq \sigma < \infty$, $0 \leq t \leq 2\pi$, the metric becomes:\n
\[ ds^2 = -\sinh^2 \sigma dt^2 + d\sigma^2, \]

(4.23a)

or, in Schwarzschild coordinates,

\[ ds^2 = -(r^2 - 1) dt^2 + (r^2 - 1)^{-1} dr^2. \]

(4.23b)

To our knowledge, this metric has not been previously discussed in the literature, presumably because it cannot be obtained by a dimensional reduction of 4-dim anti-de Sitter spacetime (see however [20]).

From its construction, it is evident that the solution is everywhere regular. Its most striking difference from anti-de Sitter spacetime is the presence of a horizon at $r = 1$ (the vertex of the hyperboloid). The asymptotic properties instead, are identical in both cases. Thus the metric describes a non-singular black hole. Its horizon has temperature $T = \frac{1}{2\pi}$ at variance with the anti-de Sitter metric (4.18), whose temperature of course vanishes. We shall discuss in extent this feature in the following sections.

To conclude we recall some problems in the interpretation of anti-de Sitter-like metrics which emerge when a quantum theory is constructed on such background [22]. First of all, from the geometry of the spacetime and the definitions (4.17), (4.22) is evident that the time coordinate is periodic, $0 \leq t \leq 2\pi$. The consequent presence of closed timelike paths can however be avoided by going to the universal covering space, $-\infty < t < \infty$. The second problem is the lack of global hyperbolicity of anti-de Sitter spacetime, due to the fact that spacelike infinity is at finite coordinate distance in conformal coordinates. A great care is therefore needed in the choice of the boundary conditions [22]. We shall discuss at length these problems in section 6.

\[ \dagger \] In this form, the metric has been independently discussed in [20], in the context of the Jackiw-Teitelboim theory [18].
5. The solutions in the conformal gauge

a) The general solution

It is useful to write the two-dimensional black hole solutions of the previous section in the conformal gauge. In this gauge

\[ ds^2 = -e^{2\rho} dx^+ dx^-, \quad x^\pm = x^0 \pm x^1, \]

the field equations (4.2) become

\begin{align*}
\partial_+ \partial_- \rho &= 2\partial_+ \rho \partial_- \phi - 2 \partial_+ \phi \partial_- \phi - \frac{1}{3} \partial_+ \psi \partial_- \psi - \frac{\lambda_1^2}{8} e^{2\rho}, \\
\partial_+ \partial_- \phi &= 2\partial_+ \phi \partial_- \phi + \frac{\lambda_2^2}{8} e^{2\rho} - \frac{\lambda_2^2}{8} e^{2(\phi + \rho - \frac{2}{3} \psi)}, \\
\partial_+ \partial_- \psi &= \partial_+ \psi \partial_- \phi + \partial_+ \phi \partial_- \psi + q \frac{\lambda_2^2}{8} e^{2(\phi + \rho - \frac{2}{3} \psi)},
\end{align*}

(5.1)

to be solved under the constraints

\begin{align*}
\partial_+^2 \phi - 2\partial_+ \rho \partial_+ \phi - \frac{1}{3} (\partial_+ \psi)^2 &= 0, \\
\partial_-^2 \phi - 2\partial_- \rho \partial_- \phi - \frac{1}{3} (\partial_- \psi)^2 &= 0.
\end{align*}

(5.2)

The ansatz (4.3) reduces the previous equations to the form

\begin{align*}
\partial_+ \partial_- \rho &= \frac{1 + 3k}{k-1} \partial_+ \partial_- \phi + 2 \frac{1 + k}{1 - k} \partial_+ \phi \partial_- \phi, \\
\partial_+ \partial_- \phi &= 2\partial_+ \phi \partial_- \phi + \frac{\lambda_2^2}{8} e^{2\rho}, \\
\partial_+^2 \phi - 2\partial_+ \rho \partial_+ \phi - 2 \frac{1 + k}{1 - k} (\partial_+ \phi)^2 &= 0, \\
\partial_-^2 \phi - 2\partial_- \rho \partial_- \phi - 2 \frac{1 + k}{1 - k} (\partial_- \phi)^2 &= 0,
\end{align*}

(5.3)

where we have introduced the parameter \( k \) given by eq. (2.3). The constraints (5.4) may be solved for \( \rho \) in terms of \( \phi \)

\[ \rho = \frac{1}{4} \ln \left( -\frac{1}{\lambda_2^2} \partial_+ \phi \partial_- \phi \right) - \frac{1 + k}{1 - k} \phi + w_+ + w_-,
\]

\[ \partial_+ \phi = -e^{4(w_+ - w_-)} \partial_- \phi,
\]

where \( w_+(x^+), w_-(x^-) \) are two arbitrary functions of the two light-cone variables. Fixing the residual gauge freedom relative to the conformal subgroup of diffeomorphisms by
setting \( w_\pm = 0 \), one can integrate the equation of motion (5.3). The general solutions describing black holes with a regular horizon are

\[
e^{2\rho} = \frac{1}{\lambda_2} \left[ A \exp\left(4 \frac{1 + k}{1 - k} X\right) - C \exp\left(4 \frac{k}{1 - k} X\right) \right],
\]

\[
e^{2(\phi - \phi_0)} = e^{-2X},
\]

where \( X = X (x^- - x^+) \) is defined implicitly by

\[
(x^- - x^+) = - \int_{\infty}^{X} dX' \left[ A \exp\left(2 \frac{1 + k}{1 - k} X'\right) - C \exp\left(-2X'\right) \right]^{-1},
\]

\[
A = \frac{\lambda_2}{32} (1 - k),
\]

and \( C \geq 0 \) is an integration constant. Eqs. (5.5) represent the black hole solutions (4.7) written in the conformal gauge. The singularity is located at \( X \to -\infty \), the horizon at \( X = \frac{1 - k}{4} \ln \left( \frac{C}{A} \right) \). The region of weak couplings \( (e^\phi = 0) \) corresponds to \( X \to \infty \). The constant \( C \) appearing in (5.5,6) is related to the black hole mass. In fact using the formula (4.10b) we get for the ADM mass of the hole

\[
M = 8 \sqrt{\frac{2}{1 - k}} e^{-2\phi_0} C.
\]

The zero-mass solution \( (C = 0) \) corresponds to the ADS linear dilaton vacuum (4.5). In this case we can perform explicitly the integral (5.6) and write the solution (5.5) in the form

\[
e^{2\rho} = \frac{8}{\lambda_2^2} \frac{1 - k}{(1 + k)^2} (x^- - x^+)^{-2},
\]

\[
e^{2(\phi - \phi_0)} = \left[ \lambda_2^2 \frac{k + 1}{16} (x^- - x^+) \right]^{\frac{1+k}{1+k}}.
\]

The Kruskal diagram for this solution is shown in fig. 1. Note that although (5.9) is defined over the whole range of the coordinates, (5.10) exists only for \( x^- \geq x^+ \), i.e. in the left hand side of fig. 1. The line \( x^- = x^+ \) corresponds to the region of weak couplings \( e^\phi = 0 \), i.e. to the asymptotical \( \sigma \to \infty \) region of the solution (4.5). The solutions defined for \( x^+ \geq x^- \) can be obtained by changing the signs of \( x^+ \) and \( x^- \). In the following we will only consider solutions defined for \( x^- \geq x^+ \). The isometry group of the metric (5.9) is \( GL(2, R) \) realized as the fractional transformations

\[
x^+ \to \frac{ax^+ + b}{cx^+ + d}, \quad x^- \to \frac{ax^- + b}{cx^- + d},
\]

with \( ad - bc \neq 0 \). The dilaton solution (5.10) however is not invariant under this transformations.
b) The $k=0$ case

In section 4.a) we have seen how for $k = 0$ our model becomes the theory of Teitelboim and Jackiw [18]. It is also interesting to write the corresponding solutions in the conformal gauge. For $k = 0$ the equation of motion (5.3) can be written as

\[
\begin{align*}
\partial_+ \partial_- \rho &= -\frac{\lambda^2}{8} e^{2\rho}, \\
\partial_+ \partial_- \phi &= 2\partial_+ \phi \partial_- \phi + \frac{\lambda^2}{8} e^{2\rho},
\end{align*}
\]

whose general solution appears in the literature in the form [21]:

\[
\begin{align*}
e^{2\rho} &= \left(1 + \frac{\lambda^2}{32} x^+ x^- \right)^{-2}, \\
e^{-2\phi} &= \frac{\alpha_0 \left(1 - \frac{\lambda^2}{32} x^+ x^- \right) + \alpha_+ x^+ + \alpha_- x^-}{1 + \frac{\lambda^2}{32} x^+ x^-},
\end{align*}
\]

where $\alpha_0, \alpha_\pm$ are arbitrary constants. The solutions (5.13) can be written, using the residual coordinate invariance within the conformal gauge, in a form similar to the ADS linear dilaton vacuum (5.9,10)

\[
\begin{align*}
e^{2\rho} &= \frac{8}{\lambda^2} \left(x^- - x^+ \right)^{-2}, \\
e^{2(\phi-\phi_0)} &= \frac{\lambda^2}{1 - \alpha^2 x^+ x^-},
\end{align*}
\]

where $\alpha$ is a free parameter related, as we shall see in the following, to the black hole mass. The solutions (5.14,15) describe a non singular black hole. In fact the curvature tensor (4.8) is everywhere regular whereas an horizon appears where $(\nabla \phi)^2$ change sign, i.e. for

\[
x^+ = \pm \frac{1}{\alpha}, \quad x^- = \pm \frac{1}{\alpha}.
\]

The maximally extended spacetime has the structure of a chain of connected multiverse [20]. The ADM mass of the hole is

\[
M = \frac{\sqrt{2}}{4} e^{-2\phi_0} \frac{\alpha^2}{\lambda^2}.
\]

For $\alpha = 0$ (zero-mass hole) we find the ADS linear dilaton vacuum (5.9,10). Notice that the solutions describing a black hole with a given mass $M$ differ from the zero-mass solution just for the value of the dilaton, the metric part being unchanged. The Kruskal diagram of these solutions is shown in figure 2. The bold line, characterized by $x^+ = 1/\alpha^2 x^-$ does not represent a singularity of the metric but the line where the theory becomes strongly coupled
A zero-mass particle, conformally coupled to two-dimensional gravity, coming from the region of weak couplings \((x^+ = x^-)\) will hit this line and without encountering a singularity will jump in another universe. It is worth noting that the solutions (5.14,15) are invariant under the duality transformations

\[
x^+ \rightarrow \frac{1}{\alpha^2 x^+}, \quad x^- \rightarrow \frac{1}{\alpha^2 x^-},
\]

which are a subgroup of the full isometry group \(GL(2, R)\) of the metric.

6. Hawking radiation

a) Schwarzschild gauge

We have seen in the previous sections that our classical solutions represent two-dimensional black holes in anti-de Sitter (ADS) space-time. At the quantum level one expects these black holes to evaporate in the same way as black holes in flat spacetime do. However in our case the discussion of the Hawking effect is complicated by the subtleties of the quantization of fields propagating in a ADS background [22]. Anti-de Sitter spacetime is not globally hyperbolic, information can be lost or gained from spatial infinity in finite coordinate time. The effect of this loss or gain of information on the Cauchy problem is even worse in the case at hand because we do not have full control of the boundary conditions at spatial infinity. In fact our two-dimensional solutions describing an ADS throat, have to be reembedded in a four-dimensional, asymptotically flat, manifold. The details of this reembedding are crucial for setting the appropriate boundary conditions for the two-dimensional problem. We know, however, that a consistent quantization of scalar fields in anti-de Sitter space-time exists [22]. Both “transparent” or “reflective” boundary conditions may be used, the information being either “recycled” or reflected at spatial infinity. When dealing with a complete space-time such as ADS no other possibility is left. Our infinite throat, however, is embedded in a four-dimensional space-time. One way of establishing a well defined Cauchy problem is to accept that our two-dimensional manifold is incomplete and require the Cauchy data to be specified on a Cauchy surface of the surrounding spacetime. We do not know exactly how to do this because we are not able to explicitly describe the reembedding of the two-dimensional space in the four dimensional one. Nevertheless, in principle, information should be allowed to leave the two-dimensional anti-de Sitter space-time and to reach the asymptotically flat region of the four-dimensional space. In conclusion we are left with two main sets of boundary conditions for our two-dimensional problem: the information is either allowed to leave the two-dimensional spacetime or is reflected at spatial infinity (we will not consider here the rather exotic possibility that the information is ”recycled” at spatial infinity).

As a first step in the study of the Hawking effect let us discuss the Hawking temperature associated with the horizon of the solutions (4.7). As we have seen in section 4 b) the temperature of the hole, eq. (4.11), goes to zero with the mass. This indicates that, loosing mass through the Hawking radiation, the black hole will set down to a zero-mass non radiating, stable ground state which could be naturally interpreted as the ADS linear dilaton vacuum. This is strictly true only if one uses boundary conditions which allow the radiation to escape at spatial infinity. Reflecting boundary conditions will make also
possible a ground state describing a black hole in thermal equilibrium with the radiation. The local measured temperature is red-shifted by the ADS gravitational potential and decreases the further one is from the horizon.

One way for extracting more information about the Hawking evaporation process is to use, following the lines of ref. [23], the covariant conservation equations

$$\nabla_\nu T^\nu_\mu = 0, \quad (6.1)$$

for the stress tensor $T^\nu_\mu$ in the background defined by (4.7). In the two-dimensional, asymptotically flat case this gives the well known relation between the Hawking black body effect and the trace anomaly for 2D-conformal matter coupled to gravity [23]. We shall integrate the conservation equations (6.1) in the conformally flat background

$$ds^2 = \Upsilon (-dt^2 + dr_*^2), \quad (6.2)$$

which is the metric (4.7) expressed in terms of the coordinate $r_*$, defined by

$$\frac{dr}{dr_*} = \Upsilon,$$

with

$$\Upsilon = b^2 r^2 - a^2 r^{2k}.$$

The resulting form of $T^\nu_\mu$ is the following

$$T^\nu_\mu = T^{(1)\nu}_\mu + T^{(2)\nu}_\mu + T^{(3)\nu}_\mu, \quad (6.3)$$

with

$$T^{(1)\nu}_\mu = \begin{pmatrix} -\Upsilon^{-1} H_2 + T^\alpha_\alpha & 0 \\ 0 & \Upsilon^{-1} H_2 \end{pmatrix}, \quad (6.4)$$

$$T^{(2)\nu}_\mu = \frac{K}{\Upsilon} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}, \quad (6.5)$$

$$T^{(3)\nu}_\mu = \frac{J}{\Upsilon} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (6.6)$$

where

$$H_2 = \frac{1}{2} \int_{r_0}^r dr' T^\alpha_\alpha (r') \partial_r \Upsilon,$$

$r_0$ is the location of the horizon and $J, K$ are integration constants to be determined by boundary conditions. Regularity of the stress tensor on the future horizon requires $J = 0$. To be more concrete let us consider $N$ massless matter scalar fields conformally coupled to 2D-gravity. When the scalar fields are quantized in the classical background geometry (6.2) a conformal anomaly will show up. The conformal anomaly results in the non-vanishing trace for the stress tensor [24]:

$$T^\alpha_\alpha = \frac{N}{96} \mathcal{R}. \quad (6.7)$$
Using the expression (4.8) for the curvature tensor $R$ one can easily calculate the asymptotic form ($r \to \infty$) of the stress tensor (6.3)

$$\left( T_\mu^\nu \right)_{as} = -\frac{N}{192} \frac{(k + 1)^2}{1 - k} \lambda^2 \delta_\mu^\nu + T_\mu^{(2)\nu},$$  \hspace{1cm} (6.8)

with $T_\mu^{(2)\nu}$ given by eq. (6.5). The first term in eq. (6.8) represents the quantum correction to the vacuum energy of the anti-de Sitter background, whereas the second term describes the flux of Hawking radiation. The flux is red-shifted by the factor $\Upsilon^{-1}$ and is actually zero at infinity. For this reason one cannot use the boundary conditions at spatial infinity in order to determine the constant $K$ as one usually does for asymptotically flat spacetime. The expression (6.8) gives a time-independent description of the Hawking flux. It describes the Hawking radiation long after the matter has collapsed to form a black hole. In this context nothing can be said about the end point of the black hole evaporation. More details about the Hawking evaporation process can be achieved by considering the dynamics of a collapsing massless shell. This will be the subject of the following section.

b) Collapse of a massless shell

Consider $N$ scalar massless fields $f_i$ conformally coupled to the 2D-gravity model defined by the action (4.1). The classical action is

$$S = \frac{1}{2\pi} \int d^2x \sqrt{g} \left\{ R + 4(\nabla \phi)^2 - \frac{2}{3}(\nabla \psi)^2 + \lambda_1^2 - \lambda_2^2 e^{2(\phi - \frac{2}{3}\psi)} \right\} e^{-2\phi} - \frac{1}{2} \sum_{i=1}^N (\nabla f_i)^2 \right\}. \hspace{1cm} (6.9)$$

Since we are dealing with massless fields we will use the conformal gauge to study the corresponding dynamics. The classical equation of motion for the metric, the dilaton and the field $\psi$ are the same as for the matter-free case. They are given by eqs. (5.1). The field equations for the matter fields and the constraints are

$$\partial_+ \partial_- f_i = 0, \hspace{1cm} (6.10)$$

$$e^{-2\phi} \left[ \partial_+^2 \phi - 2\partial_+ \rho \partial_+ \phi - \frac{1}{3} (\partial_+ \psi)^2 \right] = \frac{1}{4} \sum_{i=1}^N \partial_+ f_i \partial_+ f_i, \hspace{1cm} (6.11)$$

$$e^{-2\phi} \left[ \partial_-^2 \phi - 2\partial_- \rho \partial_- \phi - \frac{1}{3} (\partial_- \psi)^2 \right] = \frac{1}{4} \sum_{i=1}^N \partial_- f_i \partial_- f_i.$$

The gravity system defined by the action (4.1) possesses one local degree of freedom. The question of stability becomes quite non trivial because there are several classical time-dependent matter-free solutions. However we look for solutions verifying the relation (4.3). Once the field $\psi$ as been "frozen" by means of eq. (4.3) we are left with a system devoid of local degrees of freedom. The solutions of the matter-free equations of motion (5.3) are therefore locally static and for a shell of collapsing matter we can find the solutions in a
way similar to that of ref. [9], i.e. by gluing together different solutions of the matter-free system across the history of the shell. Moreover the two-dimensional action (4.1) comes from a compactification on a four-dimensional background which does verify the relation (4.3). One is thus forced to consider also for the two-dimensional gravity system classical orbits for which eq. (4.3) holds.

Following ref. [9] we will consider solutions describing an $f$-shock wave at $x^+ = x_0^+$ travelling in the $x^-$ direction, whose only non-vanishing stress tensor component is

$$T_{++} = \frac{1}{2} \sum_{i=1}^{N} \partial_+ f_i \partial_+ f_i = 2\beta \delta(x^+ - x_0^+), \quad (6.12)$$

where $\beta$ is the magnitude of the wave. Substituting eq. (6.12) in eq. (6.11) one easily finds the solutions of the equation of motion (5.3) and of the constraints (6.11) which are continuous across the shock $x^+ = x_0^+$. They are given, for $x^+ \leq x_0^+$, by the ADS linear dilaton vacuum

$$e^{2\rho} = \frac{8}{\lambda_2^2 (1+k)^2} (x^- - x^+)^{-2},$$

$$e^{2(\phi - \phi_0)} = \left[ \frac{\lambda_2 (k+1)}{16} (x^- - x^+) \right]^{1-k}, \quad (6.13)$$

and for $x^+ \geq x_0^+$ by ($' = d/dx^-$)

$$e^{2\rho} = F'(x^-) \frac{1}{\lambda_2} \left[ A \exp(4 \frac{1+k}{1-k} X) - e^{2\phi_0} \beta \exp(4 \frac{k}{1-k} X) \right],$$

$$e^{2(\phi - \phi_0)} = e^{-2X}, \quad (6.14)$$

where $X = X(x^+, x^-)$ is defined implicitly by

$$x^+ - x_0^+ - F(x^-) = \int_\infty^X dX' \left[ A \exp(2 \frac{1+k}{1-k} X') - e^{2\phi_0} \beta \exp(-2X') \right]^{-1}, \quad (6.15)$$

$$F(x^-) = \int_{x_0^+}^{x^-} dy^- \left[ 1 - \gamma (y^- - x_0^+)^{\frac{2}{1+k}} \right]^{-1}, \quad (6.16)$$

$$\gamma = e^{2\phi_0} \beta \left[ \frac{2(1+k)}{1-k} \right]^{\frac{2}{1+k}} A^{\frac{1-k}{1+k}}. \quad (6.17)$$

The constant $A$ is given by eq. (5.7). The function $F(x^-)$ has been chosen in the form (6.16) in order to fulfil the required continuity conditions for $\rho$ and $\phi$ at $x^+ = x_0^+$. Comparing eqs. (6.14-17) with eqs. (5.5-7) one easily realizes that the redefinition $x^- \to F^{-1}(x^-)$ brings the solutions (6.14) in the form (5.5). Thus eqs. (6.14) describe a black hole of mass $M \propto \beta$. As expected the effect of the matter perturbation (6.12) on the ADS linear dilaton vacuum (6.13) is to produce a black hole of mass proportional to $\beta$. 

17
c) Hawking radiation in the conformal gauge

In section 6.b) the discussion has been purely classical. The Hawking radiation is a quantum effect which will appear when the matter fields are quantized in the classical background geometry defined by eqs (6.13,14). In the quantum theory the matter fields $f$ couple to the gravitational degrees of freedom owing to the conformal anomaly. At the one-loop level, ignoring the back-reaction of the metric, the effect of the conformal anomaly can be summarized in the non vanishing trace of the stress tensor [9]:

$$\langle T^{++} \rangle = -\frac{N}{12} \partial_+ \partial_\rho,$$

Integrating the equation of conservation for $T$ one obtains the following one loop expressions for $\langle T^{++} \rangle$ and $\langle T^{--} \rangle$ [9]:

$$\langle T^{++} \rangle = -\frac{N}{12} \left[ (\partial_+ \rho)^2 - \partial_+^2 \rho + t_+(x^+) \right],$$

$$\langle T^{--} \rangle = -\frac{N}{12} \left[ (\partial_- \rho)^2 - \partial_-^2 \rho + t_-(x^-) \right].$$

The functions $t_\pm$ appearing in eqs. (6.19) have to be fixed by means of the boundary conditions. Using eqs. (6.13-17) one obtains

$$\langle T^{++} \rangle = -\frac{N}{12} t_+(x^+),$$

$$\langle T^{--} \rangle = -\frac{N}{12} t_-(x^-),$$

for $x^+ \leq x_0^+$,

and

$$\langle T^{++} \rangle = \frac{N}{12} \left[ P + \frac{2k}{k-1} e^{2\phi_0} \beta e^{-2X} \delta(x^+ - x_0^+) - t_+ \right],$$

$$\langle T^{--} \rangle = \frac{N}{12} \left[ (F')^2 P + \frac{1}{2} \{F, x^-\} 0 - t_- \right],$$

for $x^+ \geq x_0^+$,

where

$$P = \frac{4}{k-1} e^{2\phi_0} \beta A \exp\left( \frac{4k}{1-k} X \right) - \frac{4k}{(k-1)^2} e^{4\phi_0} \beta^2 \exp(-4X),$$

and

$$\{F, x^-\} = \frac{F''}{F'} - \frac{3}{2} \left( \frac{F''}{F'} \right)^2$$

is the Schwarzian derivative of the function $F$. We will fix the boundary conditions by requiring that $T$ vanishes identically in the ADS linear dilaton region, which in turn implies $t_- = 0$, and that there should be no incoming radiation at spatial infinity for every value
of $x^-$ except for the classical $f$-wave at $x^+ = x^+_0$. Let us discuss the physical meaning of the expression (6.21) for the different values of the parameter $k$.

1. $k = 0$

One easily finds

$$< T_-- > = 0, \text{ identically.}$$

The fact that $< T_-- >$ vanishes identically over the whole space means that there is no Hawking radiation at all. This result is not surprising because, as we have seen in section 5.b), the solutions (5.14-15) describe non singular black holes. Moreover the metric part of the solutions (5.14) is the same as for the ADS linear dilaton vacuum. Thus at the classical level the effect of the shock wave on the ADS linear dilaton background is encoded in the modification of the dilaton given by eq. (5.15). The absence of Hawking radiation is the consequence of the insensitivity of the metric part of the background to the shock wave.

2. $1 < k < 0$

The flux of Hawking radiation is given by the asymptotical value of $< T_-- >$ as $X \to \infty$. Taking into account that in (6.21) $P \to 0$ as $X \to \infty$ one obtains the expression

$$< T_-- >_{as} = \frac{N}{24} \{ F, x^- \}. \quad (6.24)$$

Using the expression (6.16) for $F$ eq. (6.24) becomes

$$< T_-- >_{as} = \frac{N \gamma}{12(1+k)^2} \left[ \frac{(1-k)(x^- - x^+_0)^{2k}}{x^+ x^-} + \gamma k(x^- - x^+_0)^{\frac{2(1-k)}{x^+ x^-}} \right] \left[ 1 - \gamma(x^- - x^+_0)^{\frac{2(1+k)}{x^+ x^-}} \right]. \quad (6.25)$$

Let us discuss the expression (6.25). First we note that $< T_-- >_{as}$ blows up at the horizon, i.e. for

$$x^- = x^-_1 = x^+_0 + \gamma \frac{-(1+k)}{2}. \quad (6.26)$$

This divergence does not have an invariant physical meaning. In fact it is a consequence of the bad behaviour of our coordinate system on the horizon. The function $F$ of eq. (6.16) diverges for $x^- = x^-_1$. This divergence can be easily eliminated by going back to the original coordinate system of eq. (5.5,6), i.e by defining the new light-cone coordinate

$$\tilde{x}^- = F(x^-).$$

Expressing the tensor $< T_-- >$ in this new coordinate system we have

$$< \tilde{T}_-- > = (F')^{-2} < T_-- > \frac{N}{12} \left[ \frac{1}{2(F')^2} \{ F, x^- \} + \tilde{t}_-(x^-) \right], \quad (6.27)$$

where $< T_-- >$ is given by eq. (6.20) and (6.21) (with $t_- = 0$) for the regions $x^+ \leq x^+_0$ and $x^+ \geq x^+_0$ respectively. Requiring that $< \tilde{T}_-- >$ vanishes in the ADS linear dilaton
region and taking into account that there $< T_{--} >= 0$, one can identify the function $\tilde{t}_{--}(x^-)$

$$\tilde{t}_{--}(x^-) = -\frac{1}{2(F')^2} \{ F, x^- \}.$$  \hspace{1cm} (6.28)

With this position we can obtain, using eq. (6.24), the Hawking flux

$$< \tilde{T}_{--} >_{as} = \frac{N}{24} \frac{1}{(F')^2} \{ F, x^- \}.$$  \hspace{1cm} (6.29)

Finally substitution of $F$ gives the result

$$< \tilde{T}_{--} >_{as} = \frac{N}{12} \frac{\gamma}{(1 + k)^2} \left[ (1 - k)(x^- - x_0^+) \lambda_{2}^{2k} + \gamma k(x^- - x_0^+) \lambda_{2}^{2(1 - k)} \right].$$  \hspace{1cm} (6.30)

The expression for the Hawking flux is now well behaved for every finite value of $x^-$. The Hawking flux starts at the value

$$< \tilde{T}_{--} >_{as} = 0, \quad \text{for} \quad x^- = x_0^+,$$

and reaches at the horizon $x^- = x_1^-$ its maximum :

$$< \tilde{T}_{--} >_{as} = \frac{N}{12} \frac{\gamma^{1+k}}{(1 + k)^2},$$  \hspace{1cm} (6.31)

For $x^- \leq x_0^+$, $< \tilde{T}_{--} >_{as}$ vanishes identically because there we are in the ADS linear dilaton region. Recall that in our coordinate system spatial infinity is identified with the line $x^- = x^+$. We are therefore forced to send the shock wave adiabatically at $x^- = x_0^+$. Inserting the expression (6.17) for $\gamma$ in eq. (6.31) one easily finds the following behaviour of the Hawking flux on the horizon

$$< \tilde{T}_{--} >_{as} \propto \beta^{1+k} \lambda_2^{1-k} \propto M^{1+k} \lambda_2^{1-k},$$  \hspace{1cm} (6.32)

$M$ being the black hole mass. Thus the value of the Hawking flux on the horizon goes to zero with the mass of the hole. This behaviour has to be compared with previous results for the Hawking flux in 2-dim dilaton gravity where an asymptotically mass independent Hawking radiation rate has been found [9]. As stated in sect. 4 the 2-dim dilaton gravity theory of ref. [9] is the particular case $k = -1$ of our model. Inserting this value of $k$ in (6.32) one finds

$$< \tilde{T}_{--} >_{as} \propto \lambda_2^2,$$

i.e the asymptotically mass independent Hawking radiation rate found in ref. [9].

The fact that the Hawking flux at the horizon goes to zero with the mass of the hole strongly suggests that, loosing mass through the radiation, the black hole settles down to a stable, non radiating ground state. This conclusion is of course the only possible if, as we have seen in sect. 6.a, the temperature of the hole goes to zero with the mass. However, in the context of our semiclassical calculations, which neglect the back-reaction
of the geometry, the previous result can not be seen as a conclusive statement about the end point of the Hawking evaporation process. In fact, when the black hole has radiated away most of its initial mass, the back-reaction cannot be neglected. A check of the range of validity of our approximation can be done by calculating the value of the weak field expansion parameter \( e^\phi \) on the point where the \( f \)-wave meets the horizon. This point has evidently coordinates \((x_0^+, x_1^-)\) and a simple calculation shows that

\[
e^\phi \propto \left(\frac{\lambda_2}{\beta}\right)^{1-k} \propto \left(\frac{\lambda_2}{M}\right)^{1-k}.
\]

Thus for \( \beta \gg \lambda_2 \) or equivalently for \( M \gg \lambda_2 \), \( e^\phi \) is a small number. As expected for macroscopical holes \((M \gg \lambda_2)\) the Hawking process is seen to take place in the region of weak couplings where one is allowed to neglect the back-reaction of the metric. For \( M \sim \lambda_2 \) our calculation of the Hawking flux breaks down and we have to take into account the back-reaction in order to get meaningful results. Unfortunately we are not able to solve the system of differential equations describing the model when the back-reaction of the metric is taken into account. In conclusion, even though our calculations strongly indicate the emergence of a stable ground state at the end point of the evaporation process, a final word on the subject deserves further investigation.

3. \( 0 < k < 1 \)

For \( k > 0 \) the function \( P \) in eq. (6.21) diverges as \( X \to \infty \). Both \( < T_-^- > \) and \( < T_+^+ > \) will get therefore negative divergent asymptotical contributions. The meaning of these divergences will be clarified shortly.

In section 4.b), we found a non-vanishing value for the temperature of the hole for every value of the parameter \( k \). Moreover, the temperature was seen to go to zero with the mass. This seems in contradiction with the results of this section where we have found a zero and a negative divergent Hawking flux for \( k = 0 \) and \( k > 0 \) respectively. The solution of the puzzle lies in a careful analysis of the "quantum vacuum" for fields propagating in an ADS background. It is well known from the general theory of quantization in a curved background [25], that for a given background geometry, in general the choice of a particular coordinatization of the space will result in a particular choice for the quantum vacuum. In particular the conformal vacua belong to classes which have similar geometries but different topologies. Moreover, the classes are related by thermalization at a given temperature in the sense that an observer sitting in a vacuum corresponding to a coordinatization will regard the vacuum corresponding to the other coordinatization as a thermal bath at this temperature. Now the key point is that the vacuum defined by (4.14) is the thermalization of the vacuum defined by (5.14,15) at the temperature

\[
T = \frac{1}{4\pi} \frac{\lambda_2^2 r_0}{},
\]

(6.33),

\( r_0 \) being the location of the horizon in Schwarzschild coordinates. One can understand this by the following reasoning. The metrics (4.9) and (4.14), when written in the conformal gauge, can both be put in the form (5.9). However the coordinate transformation
required is different. The region \( r > 0 \) for these two metrics is mapped by the coordinate transformation in two different regions of the Kruskal diagram of figure 1. Thus even though the metrics (4.9) and (4.14) have the same analytical expression in terms of light-cone coordinates, they correspond to different topologies. The relationship between the two corresponding spacetimes is analog to the one between two-dimensional Rindler and Minkowski spacetime. The black hole solution (4.14) has a natural temperature associated with it given by eq. (6.33) while the ADS linear dilaton vacuum (4.9) has zero temperature. The two corresponding quantum vacua are therefore related by thermalization at temperature (6.33). Thus an observer sitting in the vacuum defined by the metric (4.14) would regard the vacuum defined by the metric (5.14) as a thermal state at this temperature. The temperature of the black hole (5.14) is therefore zero. The previous discussion explains not only why for \( k = 0 \) in the conformal gauge we have no Hawking radiation but also the meaning of the divergencies we found for \( k > 0 \). A simple consequence of the previous analysis is that in the conformal gauge the black holes with \( k = 0 \) have the same temperature (equal to zero) as the ADS linear dilaton vacuum. Looking at formula (4.11) one immediately realizes that the black hole with \( k > 0 \) have temperature lower than (6.33). They have, therefore, when described in the conformal gauge, a temperature lower than the one associated with the ADS linear dilaton vacuum i.e. a negative temperature. Thus for \( k > 0 \) the ADS linear dilaton vacuum is not the true ground state, it is essentially unstable and will tend to collapse to form a black hole.

7. Summary and outlook

In this paper we have studied the extremal limit of the four dimensional black hole solutions of the effective string theory which is obtained when in the low-energy spectrum both the dilaton and a modulus coming from the compactification of the string to 4-dim are present. These 4-dim black hole solutions evidenciate geometrical and thermodynamical properties slightly different from the purely dilatonic case. In particular, for extremal holes the temperature approaches monotonically to zero, indicating that the extremal limit has to be interpreted as a non-radiating ground state. One of the main results of this work is that this behaviour has a geometrical explanation in terms of the underlying two-dimensional effective theory. In fact we found that near the extremal point, the model is well described by an effective 2-dim gravity theory whose black hole solutions are asymptotically anti-de Sitter rather then asymptotically flat as for the purely dilatonic case. Thus the anti-de Sitter gravitational potential acts as a confining one, being responsible for the creation of a mass gap which hinders the interaction between the black hole and the external fields. Moreover the black hole solutions of the 2-dim gravity theory have a rather rich structure. They can be considered as the generalization of the two-dimensional black hole solutions found in the purely dilatonic theory. For the particular value \( k = -1 \) of the parameter characterizing the solutions we find the asymptotically flat solutions of ref [17], whereas for \( k = 0 \) our model becomes the Teitelboim-Jackiw theory whose solutions describe non singular black holes. For generic values of \(-1 < k < 1\) our solutions describe 2-dim black holes in an anti-de Sitter background. This asymptotical behaviour of the solutions not only explains the peculiar thermodynamical properties of the 4-dim black holes but has also important consequences for the thermodynamics of the 2-dim black holes. In fact we found that differently from the purely dilatonic case, the temperature of the 2-dim
hole goes to zero with the mass indicating the emergence of a stable ground state. This conclusion has been substantially confirmed by the study of the Hawking evaporation process. In fact we have found, for a wide range of values of the parameter $k$, that the Hawking radiation rate is not asymptotically mass independent as in the case of pure dilatonic 2-dim gravity but goes to zero with the mass of the hole. Even though in the calculations some subtleties are involved, mainly connected with the difficulty in setting the appropriate boundary conditions and even though, having neglected the back-reaction of the geometry, our study of the Hawking evaporation process is not complete it seems to us that our main results could hardly be spoiled by further investigations.

We conclude with some comments about the relevance of our results for the problem of information loss in black hole physics. One of the main motivations for the study of string inspired black hole models is to find a solution to the information loss puzzle in the black hole evaporation process. One possible solution is the idea that the information is retained by a stable black hole remnant. This proposal in the context of dilatonic 2-dim gravity has been discussed at length in the literature [7]. The information that resides in the black hole remnant is contained in the infinite, narrow throat attached to the spacetime which characterizes the magnetically charged black hole solutions of string theory. However if the black hole remnants have to be the solution of the puzzle, one has to solve two main problems. First, the temperature and the rate of emission of the limiting 2-dim black hole of purely dilatonic gravity does not go to zero. As pointed out by Hawking in ref. [26] this is in contradiction with the fact that the black hole settles down to a stable state. Of course one can make appeal to the back-reaction of the geometry and/or to quantum gravity effects to halt the evaporation process, so that a stable black hole remnant is left behind. The 2-dim black holes we have discussed in this paper represent an improvement in this direction. In fact they have a temperature and a rate of emission which in the limiting case goes to zero. The idea that the black hole settles down to a stable state looks more natural in this context. Second, since the remnant must be able to encode the information from an arbitrarily large initial black hole, it must have an infinite spectrum. Owing to this infinite spectrum all the ordinary physical processes would produce bursts of remnants. It has been suggested that certain remnants in string theory could prevent infinite production [27]. In the context of the 2-dim gravity theory we have studied in this paper there is a natural mechanism which suppresses the rate of production of remnant states in ordinary processes. In fact an external observer sitting in the asymptotically flat region of our 4-dim solutions finds impossible to excite the states of the infinite throat region because, as we have seen in sect. 3, this excitation are suppressed by an infinite mass gap.

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