Exponential ergodicity for infinite system of interacting diffusions

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Abstract

We develop and implement probabilistic strategy for proving exponential ergodicity for interacting diffusion processes on unbounded lattice. The technique allows us to consider cases where the generator of the particle corresponds to subelliptic operator. As a model case we present situation, where the operators arises from Heisenberg group. In the last section we list some further examples that can be handled using our methods.

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1 Introduction

The study of interacting particle systems has a long and profound history, as is well evidenced by excellent monographs [21] or [18]. Initially motivated by the problems of statistical physics the field has grown into an important area of Markov processes in itself with interesting problems and rich interplay with other subjects.

Our concern is the situation of continuous spin systems, where on each site we have a diffusion particle. In particular we are interested in such examples, where one can establish strong ergodicity properties of the Markov semigroup of entire system. Most results establishing strong ergodicity properties for interacting particle systems with unbounded state space are tied with the use of functional inequalities, see [12]. As for the diffusions, there has been two independent successful approaches to this problem in the 1990s, one by Zegarliński [31] and other by Da Prato and Zabczyk [9], each to their merit and deficiencies. The approach in [31] constructs the desired semigroup using finite dimensional approximations and ergodicity results are established via log Sobolev inequality, while more probabilistic way in [9] uses the theory of SDEs on Hilbert spaces for construction and ergodicity is tied with dissipativity properties of resulted operators.
Both these articles essentially covers only elliptic case. The question how to address some subelliptic situation has been resolved under suitable condition in \[10\] again using analytic techniques based on functional inequalities (very recently the results were extended to cover even broader class of operators in \[19\] and \[20\]). Because in such cases even in the finite dimension the strong ergodicity of the system is highly non-trivial, important part of the result lies in conquering this problem.

This article presents a new probabilistic approach to investigate these issues. The results obtained go successfully beyond Hilbert space methods in \[9\] in two directions. We can cover degenerate multiplicative noise as we present the case of Heisenberg group, but also the ergodicity results are stronger, as we establish exponential ergodicity for all bounded functions as opposed to only Lipschitz functions in \[9\].

Assume we have a space \((\mathbb{R}^n)^{\mathbb{Z}^d}\), the dynamics of the system can be then described by operator of the form

\[
\sum_{i \in \mathbb{Z}^d} A_i + q_i B_i,
\]

where \(A_i\) is second order operator acting on \(i\)-th coordinate and \(B_i\) first order operator acting on \(i\)-th coordinate. We assume that the interactions \(q_i\) affect only the drift term and are of finite range.

The desired process we construct using finite dimensional approximations by corresponding stochastic differential equations. Of course such approach is well known and nothing new in the field, see e.g. \[16\], \[11\]. The main novelty lies in the fact that we use the results of Meyn and Tweedie \[24\] to establish strong ergodicity in finite dimension. The important feature is the fact that the constants in exponential convergence doesn’t depend on the size of approximation provided we restrict ourselves to smaller class of initial configurations. In section 2 we give a proof of these finite dimensional results. Using tightness arguments we construct the process corresponding to \((1.1)\) as a solution to martingale problem. The key and hardest part is section 5, where we show under additional technical assumptions that the limit of our approximations is unique and consequently establish Markov property of our process together with strong ergodicity.

For clarity and brevity of exposition we illustrate our techniques with the specific example of the operators corresponding to Heisenberg group, but in the last section mention some other natural situation that can be dealt within our methods.
2 Outline of the proof

Let $\mathbb{H} = \mathbb{R}^3 = (x, y, z)$ be the Heisenberg group (for the detailed treatment of Heisenberg group as an example of Stratified Lie group see [8], for nice and brief account of the relation to the matrix Heisenberg group see [3]) and $X, Y$ the generators of Lie algebra on $\mathbb{H}$, i.e.

$$ X = \partial_x - \frac{1}{2} y \partial_z $$

$$ Y = \partial_y + \frac{1}{2} x \partial_z. $$

We denote $D = x \partial_x + y \partial_y + 2z \partial_z$ (so that $[X, D] = X$, $[Y, D] = Y$) the so-called dilation operator.

Consider the $d$ dimensional lattice $(\mathbb{R}^3)^{\mathbb{Z}^d}$, i.e. spin system where we have a copy of Heisenberg group at every point. We wish to study the behaviour of diffusion associated with the operator

$$ L = \sum_{i \in \mathbb{Z}^d} L_{\lambda_i} + q_{x_i} X_i + q_{y_i} Y_i, $$

(2.1)

where $X_i$ is the vector field acting on the $i$-th coordinate, $q_{s}$ is the interaction function with finite range (the more precise description will come later), $L_{\lambda_i} = X_i^2 + Y_i^2 - \lambda_i D_i$ and $\lambda_i$ are positive constants. We will see later, that whole product space $\mathbb{H}^{\mathbb{Z}^d}$ is in fact too large for us to handle and for the purposes of ergodicity results we impose our diffusion to live in a smaller space, similarly as the process in [9] lives in weighted $\ell^p$ space.

We first start with the case of diffusion on Heisenberg group. Concretely we analyse the asymptotic behaviour of the Markov process on $\mathbb{R}^3$ with generator

$$ \mathcal{L} = X^2 + Y^2 - \lambda D + q_x X + q_y Y. $$

Under suitable assumptions on $q_{s}'s$ the process can be constructed by ordinary Itô stochastic equation and using the theory of Meyn and Tweedie ([23], [24], [14]) we establish exponential convergence to the invariant measure. This result can be immediately translated to the exponential ergodicity of diffusion on $(\mathbb{R}^3)^n$ with the generator

$$ \sum_{i=1}^n \mathcal{L}_{\lambda_i} + q_{x_i} X_i + q_{y_i} Y_i. $$

The key feature of this result is in certain sense independence of the constant in the exponential convergence on the dimension $n$. More precisely if we
have a bound on the value of Lyapunov function evaluated at the initial value uniformly in \( n \), then we can draw the desired conclusion that constant is independent.

Next we consider an exhausting sequence \( \Lambda_n \subset \subset \mathbb{Z}^d \), \( \Lambda_n \nearrow \mathbb{Z}^d \) (\(|\Lambda_n| = N\)) and on every \((\mathbb{R}^3)^{\Lambda_n}\) we consider diffusion \( A^n \) that its generator extends the operator

\[
L_n = \sum_{i=1}^{N} \mathcal{L}_{\lambda_i} + q^n_{x_i} X_i + q^n_{y_i} Y_i.
\]

Unfortunately unlike in [16] we are in a situation with unbounded coefficients, so we are unable to show a limit of approximations in strong sense. Nevertheless we have tightness in appropriate weighted space \( S \), i.e. we are able to show that the distributions of the processes \( \tilde{A}^n = (A^n, 0_{j \in \mathbb{Z} \setminus \Lambda_n}) \) form a tight sequence in \( \Omega = C([0, \infty), S) \). From tightness follows the construction of family of measures \( P^a, a \in S \) such that canonical process on \( \Omega \) solves the martingale problem for (2.1). Our results are not completely satisfying since we do not prove the uniqueness of martingale problem for the operator (2.1).

Nevertheless under additional assumptions we can prove that our approximation procedure yields a unique measure. This is used to show that canonical process is a proper Markov process. Furthermore exploiting the above mentioned uniformity (in fact the reason for choosing the space \( S \) as we do so is to guarantee uniform bound in exponential convergence for every approximation) of constants in the exponential convergence we prove the existence and uniqueness of invariant measure together with the exponential convergence to the equilibrium from starting point that holds in the uniform norm.

In certain aspects therefore - such as establishing uniform exponential convergence for the limiting semigroup and necessity to assume only \( \lambda > 0 \) in relevant examples - our results compare favourably to the ones in [10], [19]. However it should be noted that our methods are only able to handle bounded interactions \( q' \)'s and we work only with much simpler generators than authors in the above mentioned articles.

### 3 Finite dimensional result

Let us now investigate the diffusion on \( \mathbb{R}^3 \) associated with the second order operator

\[
\mathcal{L} = X^2 + Y^2 - \lambda D + q_x X + q_y Y.
\] (3.1)
We will work under the following assumptions (A1):

- \( q_x, q_y \in C^\infty(\mathbb{R}^3, \mathbb{R}), \lambda > 0 \)
- \( \exists C > 0 : ||q_x|| \vee ||q_y|| \leq C \)

Under these assumptions we can construct the diffusion as a solution to the SDE

\[
dA(t) = b(A_t)dt + \sigma(A_t)dW_t.
\]

Elementary computations with vector fields and matrices reveal that the coefficients can be chosen as

\[
b = (q_x - \lambda x, q_y - \lambda y, -2\lambda z + \frac{1}{2}(q_y x - q_x y))
\]

\[
\sigma = \begin{pmatrix}
\frac{\sqrt{2}}{2} & 0 \\
0 & \frac{\sqrt{2}}{2} \\
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}
\end{pmatrix} \rightarrow \frac{1}{2}a = \frac{1}{2}\sigma^* = \begin{pmatrix}
1 & 0 & -\frac{y}{2} \\
0 & 1 & \frac{x}{2} \\
-\frac{y}{2} & \frac{x}{2} & \frac{1}{4}(x^2 + y^2)
\end{pmatrix}.
\]

The results of Meyn and Tweedie about exponential convergence of Markov processes can be stated in our diffusion context in the following way (for the precise reference see [22, Theorem 2.5] or very readable lecture notes by Rey-Bellet [7])

**Theorem 3.1** (Harris - Meyn - Tweedie). Let \( X_t \) be a Markov process on \( \mathbb{R}^n \) with transition probability \( P_t \) and generator \( L \). Suppose that following hypotheses are satisfied

H1 The Markov process is irreducible aperiodic, i. e. there exists \( t_0 \) (and then for all \( t > t_0 \)) such that

\[
P_{t_0}(x, A) > 0,
\]

for all \( x \in \mathbb{R}^n \) and open sets \( A \).

H2 For any \( t > 0 \) the Markov semigroup \( P_t \) is strong Feller, i. e. \( P_tf \in C_0 \) for any \( f \in C_0 \).

Assume there exist Lyapunov function

\[
V : \mathbb{R}^n - \rightarrow [1, \infty), \quad V(x) \xrightarrow{||x||\rightarrow\infty} +\infty
\]

and constants \( C, c > 0 \) such that

\[
LV + cV \leq C.
\]

(3.3)
Then there exists unique invariant measure \( \mu \) for the process \( X_t \) and there exist constants \( K, \alpha > 0 \) depending only on \( c \) and \( C \) such that

\[
\sup_{\{f : |f(x)| \leq V(x)\}} |E^a f(X_t) - \mu(f)| \leq KV(a)e^{-\alpha t}
\]

for any \( a \in \mathbb{R}^n \).

Every verification of the stated result is non-trivial and depends on deep results about diffusions in \( \mathbb{R}^n \). In the remainder of the section we show that the process \( A \) given by SDE with the coefficients \(^{(3.2)}\) indeed satisfies the condition of the above theorem. The existence and smoothness of transition probability density (from which strong Feller property easily follows) is the immediate consequence of the Hörmander theorem in probabilistic settings.

The version that is suitable for our purposes was first established following Hörmander work in \([17]\).

**Theorem 3.2** (Hörmander probabilistic setting, Ichihara - Kunita). Let \( X_t \) be the unique strong solution to the Stratonovich SDE

\[
dX_t = b(X_t)dt + \sum_{i=1}^{d} \sigma(X_t) \circ dW_t,
\]

where \( b, \sigma_i, 1 \leq i \leq d \in C^\infty(\mathbb{R}^n, \mathbb{R}) \). Suppose that Hörmander condition is satisfied

\[
(H) \quad \dim(Lie\{b, \sigma_1, \ldots, \sigma_d\}) = n.
\]

Then there exists probability density function \( P_t(x, dy) = p_t(x, y) \) such that \( p_t(x, y) \in C^\infty((0, \infty), \mathbb{R}^n, \mathbb{R}^n) \).

In our case \(^{(3.2)}\) the drift in the Stratonovich form is actually the same as in Itô form. In any case the Lie algebra generated by the diffusion itself is enough to satisfy the Hörmander condition as elementary computation reveals that

\[
\dim \left( \text{Lie} \left\{ \left( \sqrt{2}, 0, \frac{-y}{\sqrt{2}} \right), \left( 0, \sqrt{2}, \frac{x}{\sqrt{2}} \right) \right\} \right) = 3
\]

and thus according to the above cited theorem we have the smoothness of transition probability density for \(^{(3.2)}\).

To investigate the irreducibility of diffusion, one can use Stroock - Varadhan support theorem \(^{(29)}\), provided that we can solve the corresponding control problem. The version we will use for unbounded coefficients was proved in
Let $H$ be the subset of the absolutely continuous functions $u : [0, t] \to \mathbb{R}^d$ with $u(0) = 0$ such that $H$ contains every infinitely differentiable function form $[0, t]$ to $\mathbb{R}^d$ vanishing at zero. For the ordinary differential equation

$$\dot{x}^n(t) = b(x^n(t)) + \sum_{i=1}^d \dot{u}_i(t)\sigma_1(x^n(t))$$

$$x^n(0) = x_0 \in \mathbb{R}^n$$

we denote $O(t, x_0) = \{ y \in \mathbb{R}^n : x^n(t) = y, u \in H \}$.

**Theorem 3.3** (Stroock - Varadhan support theorem, [13]). Let $X_t$ be the solution to the Stratonovich SDE

$$dX_t = b(X_t)dt + \sum_{i=1}^d \sigma_i \circ dW, \quad X(0) = x,$$  \quad (3.5)

where the coefficients satisfy linear growth assumptions, $b$ is Lipschitz and $\sigma_i, 1 \leq i \leq d$ are smooth with bounded derivatives. Let $P_t$ be the transition probability function related to (3.5) and $O(t, x)$ be the orbit to the corresponding equation (3.4). Then $\text{supp} \, P_t(x, \cdot) = O(t, x)$.

**Lemma 3.4.** Let $P_t$ be the transition function for the equation (3.2). Then $\text{supp} \, P_t(x, \cdot) = \mathbb{R}^3$ for any $t > 0$ and $x \in \mathbb{R}^3$.

**Proof.** We make use of the classical Girsanov transform [27, pp. 166] to simplify the control problem. Concretely the support of diffusions $X_t, Y_t$

$$dX_t = b(X)dt + \sigma(X)dW$$

$$dY_t = \bar{b}(Y)dt + \sigma(Y)dW,$$  \quad (3.6)

where $\sigma$ and $b$ are as in (3.2) and

$$\bar{b} = (-\lambda x, -\lambda y, -2\lambda z)$$

is the same, because $b - \bar{b} = (q_x, q_y, \frac{1}{2}(q_yx - q_xx))$ and we have

$$\left(\begin{array}{cccc} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ -\frac{y}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{array}\right) \left(\begin{array}{c} q_x \\ q_y \\ \frac{1}{2}(q_yx - q_xx) \end{array}\right) = \left(\begin{array}{c} q_x \\ q_y \\ \frac{1}{2}(q_yx - q_xx) \end{array}\right).$$

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Hence to establish the theorem it suffices to prove the irreducibility of transition function corresponding to (3.6). Since the equation (3.6) satisfies the Theorem 3.3, we only need to prove controllability of the system

\[
\begin{align*}
\dot{x} &= \sqrt{2}\dot{u}_1 - \lambda x \\
\dot{y} &= \sqrt{2}\dot{u}_2 - \lambda y \\
\dot{z} &= -\frac{y}{\sqrt{2}}\dot{u}_1 + \frac{x}{\sqrt{2}}\dot{u}_2 - 2\lambda z
\end{align*}
\]

(3.7)

for \(u \in H\), i.e. to show that from any starting point \((x_0, y_0, z_0)\) we can choose such \(u \in H\) that \(x(t) = x_t, y(t) = y_t, z(t) = z_t\), where \((x_t, y_t, z_t) \in \mathbb{R}^3\) are prescribed ending points. If we simply choose control \(\dot{u}_1(s) = as + b\), \(\dot{u}_2(s) = cs + d\), then the problem (3.7) is reduced to solving three linear equations with four parameters, so the Lemma is proved.

The proof of existence of Lyapunov function for the operator (3.1) satisfying (3.3) is elementary, albeit bit tedious.

**Lemma 3.5.** Let \(L\) be the operator defined by (3.1) under the assumptions (A1). For the function \(V^k = ((x^2 + y^2)^2 + z^2)^k\), \(k \in \mathbb{N}\), there exist constants \(c_k, C_k > 0\) such that \(LV^k \leq c_k \leq C_k\) for any \((x, y, z) \in \mathbb{R}^3\).

**Proof.** We show the proof for \(k \geq 2\), the proof for \(k = 1\) is similar, only less laborious. Let us prepare the derivatives of \(V^k\) first (we will not write the index \(k\) as the computations are the same for any such \(k\) :

\[
\begin{align*}
V_x &= k(x^4 + 2x^2y^2 + z^2)^{k-1}(4x^3 + 4xy^2) \\
V_y &= k(x^4 + 2x^2y^2 + z^2)^{k-1}(4y^3 + 4x^2y) \\
V_z &= k(x^4 + 2x^2y^2 + z^2)^{k-1}2z \\
V_{xx} &= k(k - 1)(x^4 + 2x^2y^2 + z^2)^{k-2}2z(4x^3 + 4xy^2) \\
V_{yy} &= k(k - 1)(x^4 + 2x^2y^2 + z^2)^{k-2}2z(4y^3 + 4x^2y) \\
V_{xz} &= k(k - 1)(x^4 + 2x^2y^2 + z^2)^{k-2}2z(4x^3 + 4xy^2) \\
V_{yy} &= k(k - 1)(x^4 + 2x^2y^2 + z^2)^{k-2}2z(4y^3 + 4x^2y) \\
V_{zz} &= k(k - 1)(x^4 + 2x^2y^2 + z^2)^{k-2}4z^2 + 2k(x^4 + 2x^2y^2 + z^2)^{k-1}.
\end{align*}
\]
Then

\[ \mathcal{L}V + cV = (q_x - \lambda x)k(x^4 + 2x^2y^2 + z^2)^{k-1}(4x^3 + 4xy^2) \]
\[ + (q_y - \lambda y)k(x^4 + 2x^2y^2 + z^2)^{k-1}(4y^3 + 4x^2y) \]
\[ + (-2\lambda z + \frac{1}{2}(q_x - q_x))k(x^4 + 2x^2y^2 + z^2)^{k-1}2z \]
\[ - k(k-1)(x^4 + 2x^2y^2 + z^2)^{k-2}2yz(4x^3 + 4xy^2) \]
\[ + k(k-1)(x^4 + 2x^2y^2 + z^2)^{k-2}2xz(4y^3 + 4x^2y) \]
\[ + k(k-1)(x^4 + 2x^2y^2 + z^2)^{k-2}(4x^3 + 4xy^2)^2 + k(x^4 + 2x^2y^2 + z^2)^{k-1}(12x^2 + 4y^2) \]
\[ + k(k-1)(x^4 + 2x^2y^2 + z^2)^{k-2}(4y^3 + 4x^2y)^2 + k(x^4 + 2x^2y^2 + z^2)^{k-1}(12y^2 + 4x^2) \]
\[ + k(k-1)(x^4 + 2x^2y^2 + z^2)^{k-2}z^2(x^2 + y^2) + \frac{1}{2}(x^2 + y^2)k(x^4 + 2x^2y^2 + z^2)^{k-1} \]
\[ + c(x^4 + 2x^2y^2 + z^2)^k. \]

After factoring out the term \((x^4 + 2x^2y^2 + z^2)^{k-2}\) we make extensive use of Young inequality \((ab \leq \frac{a^p}{p} + \frac{b^q}{q})\) to handle the mixed terms that allows us to do so. \(\lesssim\) denotes for simplicity the statement smaller or equal up to a constant, i.e.

\[ A \lesssim B \iff \exists C > 0 : A \leq CB. \]

We estimate for instance

\[ |zxy^4| \lesssim |z|^{\frac{4}{5}} + |x|^{\frac{4}{5}}|y|^{\frac{2}{5}} \lesssim |z|^\frac{4}{5} + |x|^\frac{4}{5} + |y|^\frac{4}{5} \]
\[ |z^3y^2| \lesssim |z|^\frac{2}{5} + |x|^\frac{2}{5} + |y|^\frac{2}{5} \]
\[ x^5y^2 \lesssim x^7 + y^7 \]
\[ |x^3y^2z| \lesssim |z|^\frac{2}{5} + |x|^\frac{3}{5}|y|^{\frac{2}{5}} \lesssim |z|^\frac{2}{5} + |x|^\frac{7}{5} + |y|^\frac{7}{5} \]

etc. for lower order terms. Hence in the end we can write

\[ \mathcal{L}V + cV \lesssim (x^4 + 2x^2y^2 + z^2)^{k-2}\left(x^4(x^4(c - 4k\lambda + o(x^4)) \right. \]
\[ + y^4(4c - 4k\lambda + o(y^4)) \]
\[ + z^2(4c - 4k\lambda + o(z^2)) \]
\[ + x^4y^4(4c - 24k\lambda) + x^4y^6(2c - 12k\lambda) \]
\[ + x^6y^2(2c - 12k\lambda) + x^4z^2(c - 8k\lambda) \]
\[ + y^4z^2(c - 8k\lambda) + x^2y^2z^2(2c - 16k\lambda) \right). ) \] (3.8)
The estimate (3.8) shows that for any $\lambda > 0$ and $k \in \mathbb{N}, k \geq 2$ we can choose $c_k > 0$ such that $\mathcal{L}V_k + c_k V^k$ is bounded from above. Notice that it even holds

$$\mathcal{L}V^k + c_k V^k \rightarrow -\infty.$$ 

The Meyn - Tweedie theory as stated in Theorem 3.1 now ensures exponential convergence to equilibrium for diffusion corresponding to the operator (3.1). Let us now summarize the results for multidimensional case in full detail.

**Theorem 3.6 (Finite Dimensional results).** Let $(\mathbb{R}^3)^n$ be the state space and consider the operator

$$L_n = \sum_{i=1}^n \mathcal{L}_{\lambda_i} + q_x x_i + q_y y_i,$$

(3.9)

where $\lambda_i > 0, 1 \leq i \leq n$ and assumption (A1) holds

$$q_i \in C^\infty((\mathbb{R}^3)^n, \mathbb{R}), \exists C > 0 : \|q_i\| \leq C, 1 \leq i \leq n.$$

If we denote $A^n$ the diffusion corresponding to the operator (3.9), i.e. the unique solution to the Itô SDE with coefficients

$$b = (q_{x1} - \lambda_1 x_1, q_{y1} - \lambda_1 y_1, -2\lambda_1 z_1 + \frac{1}{2}(q_{y1} x_1 - q_{x1} y_1), \ldots$$

$$\ldots, q_{xn} - \lambda_n x_n, q_{yn} - \lambda_n y_n, -2\lambda_n z_n + \frac{1}{2}(q_{yn} x_n - q_{xn} y_n))$$

$$\sigma = \begin{pmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & M_n \end{pmatrix}, \quad \text{where } M_i = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}.$$

then there exists unique invariant measure $\mu_n$ for the process $A^n$. Furthermore for the function $V^k = \sum_{i=1}^n ((x_i^2 + y_i^2)^2 + z_i^2)^k + 1, k \in \mathbb{N}$ there exist constants $K_k, \alpha_k > 0$ depending only on $s = \min_{1 \leq i \leq n} \lambda_i$ and $C$ (but not on the dimension $n$!) such that the following

$$\sup_{\{f:(\mathbb{R}^3)^n \rightarrow \mathbb{R} : \|f\| \leq 1\}} |E^n f(A^n(t)) - \mu_n(f)| \leq K_k V^k(a) e^{-\alpha_k t}$$

(3.10)

holds for any $a \in (\mathbb{R}^3)^n$. 

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Proof. The proof is essentially the same as we just showed for the case of \( \mathbb{R}^3 \). The calculations and considerations needed for Lemmas 3.4, 3.5 are the same thanks to boundedness assumption \((A1)\), the smoothness of transition probability follows again immediately from Hörmander type theorem 3.2.

4 Construction of infinite dimensional measure

There are several papers dealing with infinite dimensional martingale problems (\([2, 4, 30]\)) that establishes uniqueness as well, but all are based in elliptic settings and none can be directly applied to our case. The following version of Arzelà - Ascoli theorem follows easily from the general version proved in \([25, \text{Theorem 47.1}]\).

**Theorem 4.1 (Arzelà - Ascoli).** Let \( Y \) be a complete metric space and \( f_n \in C([0, \infty), Y) \) sequence of equicontinuous functions. Endow \( C([0, \infty), Y) \) with the topology of uniform convergence on compacts. If \( \{f_n(t)\} \) is precompact in \( Y \) on a dense set of \( t \in [0, \infty) \), then \( \{f_n\} \) is precompact in \( C([0, \infty), Y) \).

To prove equicontinuity we use a variant of Kolmogorov continuity theorem (see \([5, \text{chap. 8}]\) for details).  

**Theorem 4.2.** Let \( X^n \) be continuous processes taking values in some metric space \((S, d_S)\). Suppose for any \( T > 0 \) there exists constants \( C(T), \epsilon > 0 \) and \( p > 0 \) such that

\[
\sup_n E d_S(X^n_s, X^n_t)^p \leq C(T)|t - s|^{1+\epsilon} \quad 0 \leq s \leq t \leq T.
\]

Then \( \{X^n\} \) is equicontinuous family of processes with probability 1.

The space on which we construct our measure is dictated to us by our Lyapunov function for (3.1), so that we will be able to utilize the uniform bound (3.10). However we also have to choose space such that the Theorem 4.2 will be satisfied. For the sake of completeness let us clarify, that function of \( V \) type indeed equips \( \mathbb{R}^3 \) with the metric.

**Lemma 4.3.** Endow \( \mathbb{R}^3 \) with the following operation \( d : \)

\[
d(a, b) = \sqrt[4]{((a_x - b_x)^2 + (a_y - b_y)^2 + (a_z - b_z)^2}.
\]

\((\mathbb{R}^3, d)\) is then a metric space.
Proof. The only non-trivial part is the triangle inequality. Hence we want to prove
\[
\sqrt[4]{((a_x - b_x)^2 + (a_y - b_y)^2 + (a_z - b_z)^2} \leq \sqrt[4]{((a_x - c_x)^2 + (a_y - c_y)^2 + (a_z - c_z)^2}
+ \sqrt[4]{((c_x - b_x)^2 + (c_y - b_y)^2 + (c_z - b_z)^2}).
\] (4.1)

Notice that (4.1) is clearly valid if either terms on z axis are zero, or both x and y terms are zero. Therefore it remains to prove that if for \(A, B, C, D, E, F \geq 0\)
\[
\sqrt[4]{A} \leq \sqrt[4]{B} + \sqrt[4]{C},
\sqrt[4]{D} \leq \sqrt[4]{E} + \sqrt[4]{F},
\] (4.2)

then
\[
\sqrt[4]{A + D} \leq \sqrt[4]{B + E} + \sqrt[4]{C + F}.
\] (4.3)

The left side in (4.3) is clearly maximized, if the left sides in (4.2) is maximized. This happens, if we have equality in (4.2). Hence it suffices to prove
\[
\sqrt[4]{(\sqrt[4]{B} + \sqrt[4]{C})^4 + (\sqrt[4]{E} + \sqrt[4]{F})^4} \leq \sqrt[4]{B + E} + \sqrt[4]{C + F},
\]
but this follows from ordinary Minkowski inequality for 4-norm on \(\mathbb{R}^2\).

We will denote by \(\|\cdot\|_H\) the function that assigns to \(a \in \mathbb{R}^3\) value corresponding to the metric just defined, so that \(\|a\|_H = \sqrt[4]{((a_x^2 + a_y^2)^2 + a_z^2)}\). Given \(d\) dimensional lattice \(\mathbb{Z}^d\), we introduce the weighted space
\[
S = \{a \in \mathbb{H}^{\mathbb{Z}^d} : \sum_{i \in \mathbb{Z}^d} \|a_i\|_H^8 u(i) < +\infty}\}.
\]

For now it suffices to assume about the weights (A2)
\[
\bullet \sum_{i \in \mathbb{Z}^d} u(i) < +\infty.
\]

From the Lemma above we can infer following usual considerations that \(S\) with the metric \(\|a - b\|_S = \sqrt[4]{\sum_{i \in \mathbb{Z}^d} \|a_i - b_i\|_H^8 u(i)}\), \(a, b \in S\) is complete separable metric space and so consequently \(\Omega = C([0, \infty), S)\) is Polish too.

Let us describe compact sets of \(S\).
Lemma 4.4. Let $M \subset S$. Assume that $M$ is bounded and the following condition

$$\forall \epsilon > 0 \ \forall a \in M \ \exists n_0 : \sum_{i=n_0}^{\infty} \|a_i\| u(i) < \epsilon.$$ 

Then $M$ is precompact in $S$.

Proof. We show that from any sequence $\{a^n\}$ one can extract a Cauchy sequence. By assumptions for a given $\epsilon > 0$ we find $n_0$, so we control the rest of the sequence, and on the first $n_0 - 1$ coordinates simply choose a Cauchy sequence step by step, which is possible by the boundedness assumption.

4.1 Moments estimates and tightness of approximations

Let $\Lambda_n, |\Lambda_n| = N < +\infty$ be exhausting sequence of $\mathbb{Z}^d$, i. e. $\Lambda_{n+1} \supseteq \Lambda_n$, $\bigcup_n \Lambda_n = \mathbb{Z}^d$. We wish to construct martingale solution for the operator

$$L = \sum_{i \in \mathbb{Z}^d} L_{\lambda_i} + q_{x_i} X_i + q_{y_i} Y_i.$$  \hspace{1cm} (4.4)

Suppose we have maximum norm on $\mathbb{Z}^d$ and we assume there exists constant $r > 0$ such that $q_i$ depends only on neighbours within distance $r$. More precisely we assume about interaction functions $q$’s (A1):

- $q_i \in C^\infty((\mathbb{R}^3)^{\Pi_i}, \mathbb{R})$, where $\Pi_i = \{j \in \mathbb{Z}^d : |j - i|_{\max} \leq r\}$
- $\exists C > 0 : \|q_i\| < C$.

About constants $\lambda_i$ we assume (A3):

- $s = \inf_{i \in \mathbb{Z}^d} \lambda_i > 0$.

On each space $(\mathbb{R}^3)^N$ we consider diffusion $A^n$ with generator that coincides on $C^2((\mathbb{R}^3)^{\Lambda_n})$ functions with

$$L_n = \sum_{i=1}^{N} L_{\lambda_i} + q^n_{x_i} X_i + q^n_{y_i} Y_i.$$  \hspace{1cm} (4.5)

The interaction functions $q_i$ in general depend on $n$, but in case point $i \in \mathbb{Z}^d$ has all neighbours in distance $r$, we put $q^n_i = q_i$, otherwise the functions have to be redefined, but we keep their smoothness and boundedness by $C$. Put $\bar{A}^n = (A^n, 0, i \in \mathbb{Z}^d \setminus \Lambda_n)$, then each $\bar{A}^n(t)$ has values in $S$ and therefore $\bar{A}^n$ lives in $\Omega = C([0, \infty), S)$.  

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Lemma 4.5. Let $a \in S$. Define $A^n$ as above with initial condition $A^n(0) = \pi_{\Lambda_n}(a)$ and subsequently define $\tilde{A}^n$. Assume (A1), (A2), (A3). Then there exists constants $C(T) > 0$ and for any $\delta > 0$ constants $N_0(t)$ such that following estimates hold

\[
\sup_n \forall 0 \leq s \leq t \leq T \quad E\|\tilde{A}^n(t) - \tilde{A}^n(s)\|_S^8 \leq C(T)(t - s)^2 \quad (4.6)
\]

\[
\sup_n \forall 0 \leq t \leq T \quad \sum_{i=1}^{N_0(t)+1} \|\tilde{A}^n(t)\|^8 u(i) < \delta. \quad (4.7)
\]

Proof. First notice that the assumptions lead to the existence of constant $K$ such that $(b^n, \sigma^n$ being the coefficients of SDE for $A^n)$

\[
|b^I_{i,x}(a)| \vee \|\sigma^I_{i,x}(a)\|_{\mathbb{R}^{2N}} \leq K(1 + |a_{i,x}|)
\]

\[
|b^I_{i,y}(a)| \vee \|\sigma^I_{i,y}(a)\|_{\mathbb{R}^{2N}} \leq K(1 + |a_{i,y}|)
\]

\[
|b^I_{i,z}(a)| \vee \|\sigma^I_{i,z}(a)\|_{\mathbb{R}^{2N}} \leq K(1 + \sum_{j=1}^{3} |a_{i,j}|). \quad (4.8)
\]

Suppose $0 < s, t \leq T$, we have

\[
E\|\tilde{A}^n(t) - \tilde{A}^n(s)\|_S^8 = E\sum_{i=1}^{N} \|A^n_i(t) - A^n_i(s)\|_{S^2}^8 u(i)
\]

\[
= \sum_{i=1}^{N} E((A^n_{i,x}(t) - A^n_{i,x}(s))^8 + (A^n_{i,y}(t) - A^n_{i,y}(s))^8 + (A^n_{i,z}(t) - A^n_{i,z}(s))^8)^2 u(i)
\]

\[
\lesssim \sum_{i=1}^{N} u(i) \left( E(A^n_{i,x}(t) - A^n_{i,x}(s))^8 + E(A^n_{i,y}(t) - A^n_{i,y}(s))^8 + E(A^n_{i,z}(t) - A^n_{i,z}(s))^4 \right). \quad (4.9)
\]

The $x$ term is now estimated using (L8), Burkholder - Davis - Gundy and Hölder inequalities

\[
E(A^n_{i,x}(t) - A^n_{i,x}(s))^8 = E \left( \int_s^t b^n_{i,x}(A^n(u))du + \int_s^t \sigma^n_{i,x}(A^n(u))dW(u) \right)^8
\]

\[
\lesssim |t - s|^7 E(\int_s^t |b^n_{i,x}(A^n(u))|^8 du) + |t - s|^3 E(\int_s^t \|\sigma^n_{i,x}(A^n(u))\|^8 du)
\]
Hence

\[ K \]

Invoking the Grönwall’s inequality we can deduce existence of some constant \( K \).

Altogether we derived existence of some constant \( K \).

Similarly handling the \( y \) and \( z \) we get

\[
E(A_{i,y}^n(t) - A_{i,y}^n(s))^8 \lesssim |t-s|^2 + |t-s| \int_s^t E|A_{i,x}^n(u)|^8 \, du.
\]

 individually terms we treat

\[
E|A_{i,x}^n(u)|^8 = E|a_{i,x}| + \int_0^u b_{i,x}^n(A^n(v)) \, dv + \int_0^u \sigma_{i,x}^n(A^n(v)) dW(v) \bigg| ^8
\]

\[
\lesssim |a_{i,x}|^8 + 1 + \int_0^u E|A_{i,x}^n(v)|^8 \, dv,
\]

analogously one gets

\[
E|A_{i,y}^n(u)|^8 \lesssim |a_{i,y}|^8 + 1 + \int_0^u E|A_{i,y}^n(v)|^8 \, dv
\]

\[
E|A_{i,z}^n(u)|^4 \lesssim |a_{i,z}|^4 + 1 + \int_0^u \sum_{j=1}^3 E|A_{i,j}^n(v)|^4 \, dv.
\]

Altogether we derived existence of some constant \( K(T) > 0 \) such that

\[
E|A_{i,x}^n(u)|^8 + E|A_{i,y}^n(u)|^8 + \sum_{j=1}^3 E|A_{i,j}^n(u)|^4 \leq K(T)(|a_i|^8 + 1)
\]

\[
+ K(T) \int_0^u \left( E|A_{i,x}^n(u)|^8 + E|A_{i,y}^n(u)|^8 + \sum_{j=1}^3 E|A_{i,j}^n(u)|^4 \right) \, du.
\]

Invoking the Grönwall’s inequality we can deduce existence of some constant \( K_1(T) > 0 \) such that \( \forall u \in [s,t] \)

\[
E|A_{i,x}^n(u)|^8 + E|A_{i,y}^n(u)|^8 + \sum_{j=1}^3 E|A_{i,j}^n(u)|^4 \leq K_1(T)(1 + |a_i|^8).
\] (4.10)

Hence

\[
E(A_{i,x}^n(t) - A_{i,x}^n(s))^8 + E(A_{i,y}^n(t) - A_{i,y}^n(s))^8 + E(A_{i,z}^n(t) - A_{i,z}^n(s))^4
\]
\[ |t - s|^2 + |t - s|^2 K_1(T)(1 + \|a_i\|_S^3). \]

Installing back to (4.9) we obtain thanks to (A2) and the fact that \( a \in E \) the existence of some constants \( L(T), C(T) > 0 \) such that

\[
E\|\tilde{A}^n(t) - \tilde{A}^n(s)\|_S^2 \leq \sum_{i=1}^{N} u(i)|t - s|^2 L(T)(1 + \|a_i\|_S^3)
\]

\[
\leq C(T)|t - s|^2,
\]

which we wanted to prove (4.6).

To prove (4.7) we simply utilize the key estimate (4.10) and so we have

\[
E \sum_{N_0(t)+1}^\infty \|A^n_i(t)\|_S^8 u(i) \lesssim \sum_{N_0(t)+1}^\infty u(i)K_1(t)(1 + \|a_i\|_S^8),
\]

therefore for given \( \delta > 0 \) it suffices to choose \( N_0(t) \) such that the sum \( \sum_{i=N_0(t)+1}^\infty u(i)(1 + \|a_i\|_S^8) \) is sufficiently small. \( \Box \)

**Corollary 4.6.** Let \( \tilde{A}^n \) as in Lemma 4.5. Then \( P \circ (\tilde{A}^n)^{-1}, n \geq 1 \) is tight sequence of measures in \( \Omega \).

**Proof.** The estimate (4.6) implies according to Theorem 4.2 that equicontinuity condition is satisfied. Since boundedness is immediately implied by equicontinuity and boundedness at zero, it remains to prove by Lemma 4.4 that for given \( \epsilon > 0 \)

\[
P \left( \forall t \in Q \cap (0, \infty) \forall \delta > 0 \in Q \exists N_0(t, \delta) : \sum_{i=N_0(t)+1}^\infty \|A^n_i(t)\|_S^8 u(i) < \delta \right) > 1 - \epsilon.
\]

(4.11)

But Chebyshev inequality applied in conjunction with the estimate (4.7) routinely implies that (4.11) is fulfilled. \( \Box \)

**4.2 Solution to the Martingale problem**

Now we show that weak limit of sequence \( \{P \circ (\tilde{A}^n)^{-1}\} \) can be used to construct martingale solution to the operator (4.4). We let \( A_t(w) = w(t), w \in \Omega \) be the canonical process on \( \Omega = C([0, \infty), S) \) with \( \sigma \)-algebra \( \mathcal{F} = \sigma(w(s), s \geq 0), \mathcal{F}_t = \sigma(w(s), 0 \leq s \leq t) \) denotes
the usual filtration. We further introduce spaces \( \Omega_n = C([0, \infty), (\mathbb{R}^3)^{\Lambda_n}) \),
\( B^n_t(\omega_n) = \omega_n(t) \) the canonical process on \( \Omega_n \) and the mappings
\[
i_n : (\mathbb{R}^3)^{\Lambda_n} \to S, \quad i_n(a_1, \ldots, a_N) = (a_1, \ldots, a_N, 0_{i \in \mathbb{Z}^d \setminus \Lambda_n})
\]
\[
j_n : \Omega_n \to \Omega, \quad \omega_n \to [t \to (\omega_n(t), 0_{i \in \mathbb{Z}^d \setminus \Lambda_n})].
\]
For given \( a \in S \) we denote \( A^{n,a} \) and \( \tilde{A}^{n,a} \) the processes constructed in previous section to accentuate their dependence on \( a \). In addition we denote \( P^a \) the weak limit of measures \( P \circ (\tilde{A}^{n,a})^{-1} \), to simplify the notation we denote \( \tilde{P}^a_n = P \circ (\tilde{A}^{n,a})^{-1} \) and \( P^n_a = P \circ (A^{n,a})^{-1} \), the matching expectations will then be denoted \( E^a, E^a_n \), respectively. Notice that \( \tilde{P}^a_n = P^a_n \circ j_n^{-1} \), as following calculation reveals : for \( C \in \mathcal{F} \)
\[
\tilde{P}^a_n(C) = P(\tilde{A}^{n,a}(. \in C) = P((A^{n,a}, 0)(\cdot) \in C) = P(j_n(A^{n,a}) \in C)
\]
\[
= P^a_n \circ j_n^{-1}(C).
\]
We introduce two family of functions. We say that \( f \in C^2_c, \text{Cyl}(S) \), if there exists \( \Phi_f \subset \subset \mathbb{Z}^d \) such that there is \( g \in C^2_c((\mathbb{R}^3)^{\Phi_f}, \mathbb{R}) \) (c stands for compactly supported) and \( f(a) = g(\pi_{\Phi_f}(a)) \), analogically \( f \in C^2_c, \text{Cyl}(S) \), if such \( g \in C^2((\mathbb{R}^3)^{\Phi_f}, \mathbb{R}) \). With this notation we arrive at the following theorem.

**Theorem 4.7 (Existence of solution to the martingale problem).** Let \( a \in S \). Then there exists measure probability measure \( P^a \) on \( \Omega \) such that :
\[
P(A_0 = a) = 1 \quad \text{(4.12)}
\]
\[
f(A_t) - f(A_0) - \int_s^t Lf(A_u) du \quad \text{(4.13)}
\]
is \( \mathcal{F}_t \)-martingale under \( P^a \) for any \( f \in C^2_c, \text{Cyl}(S) \) and \( \mathcal{F}_t \)-local martingale under \( P^a \) for any \( f \in C^2_c, \text{Cyl}(S) \).

**Proof.** Define \( P^a \) as above, so that we have \( \tilde{P}^a_n \xrightarrow{w} P^a \). Then with the aid of Portmanteau theorem
\[
P(A_0 = a) = 1 - \sum_k P^a(\|A_0 - a\|_S > \frac{1}{k})
\]
\[
\geq 1 - \sum_k \lim \inf \frac{n}{P(\|\tilde{A}^{n,a}(0) - a\|_S > \frac{1}{k}) = 1 - \sum_k 0,
\]

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and we see that \(4.12\) is satisfied. Let \(f \in C^2_c(S)\) be given. To prove that \(4.13\) is martingale it suffices to prove by standard technique (see [15, Lemma 3.1]) that for arbitrary \(G \in C([0, s], [0, 1]), s < t\)

\[
E^\alpha \left[ \left( f(A_t) - f(A_s) - \int_s^t Lf(A_u)du \right) G(\omega) \right] = 0. \tag{4.14}
\]

By weak convergence \(\tilde{P}_n^\alpha \Rightarrow P^\alpha\) the formula in \(4.14\) is a limit of

\[
\tilde{E}_n^\alpha \left[ \left( f(A_t) - f(A_s) - \int_s^t Lf(A_u)du \right) G(\omega) \right]. \tag{4.15}
\]

We compute

\[
\tilde{E}_n^\alpha f(A_t(\omega)) = \tilde{E}_n^\alpha f(\omega) = E_n^\alpha f([j_n(\omega_n)]_s) = E_n^\alpha(f \circ i_n)(B^n_t(\omega_n)) \tag{4.16}
\]

\[
\tilde{E}_n^\alpha G(\omega) = E_n^\alpha G((j_n(\omega_n)) = E_n^\alpha(G \circ j_n)((\omega_n)).
\]

Since \(f\) is cylindrical the operator \(L\) acting on \(f\) in fact reduces to \(L^f\), i.e. the operator

\[
L^f = \sum_{i \in \Phi_f} \mathcal{L}_{\lambda_i} + q_xX_i + q_yY_i.
\]

Consider that for \(n\) large enough every point from \(\Phi_f\) has all neighbours in \(\Lambda_n\) and hence \(L^f\) equals to \(L_n\) on \(\Phi_f\), where \(L_n\) is the operator corresponding to \(A^n\) as defined in \(4.5\). Then we adjust

\[
\tilde{E}_n^\alpha \int_s^t Lf(A_u) = E_n^\alpha \int_s^t L^f([j_n(\omega_n)]_u) = E_n^\alpha \int_s^t L^f(i_n(B^n_u(\omega_n)))
\]

\[
= E_n^\alpha \int_s^t L_n(f \circ i_n)(B^n_u(\omega_n)).
\]

Altogether we found out that \(4.15\) is equal to

\[
E_n^\alpha \left[ \left( (f \circ i_n)(B^n_t) - (f \circ i_n)(B^n_s) - \int_s^t L_n(f \circ i_n)(B^n_u)du \right) (G \circ j_n)((\omega_n)). \right]
\]

but since we know that \(P_n^\alpha\) solves the martingale problem for \(L_n\) on \(\Omega_n\), this expression equals to zero and therefore also \(4.14\) is zero. To deduce that for \(f \in C^2_c(S)\) \(4.13\) is local martingale, is the same as in finite dimension thanks to the cylindricity assumption.
5 Ergodicity results for general bounded interactions

Next we present ergodicity results that can be proven under assumptions (A1), (A2), (A3). Much more complete and satisfying results we prove in the next section, but the methods we employ to do so will force us to strengthen the assumptions.

So let \( a \in S \) be chosen and consider approximating sequence of processes \( A^{n,a} \). To each process \( A^{n,a} \) there is unique invariant measure \( \mu_n \) by Theorem 3.6. We consider measures \( \nu_n \) on \( S \) such that 

\[
\nu_n = \mu_n \circ i_n^{-1}.
\]

The first results that follows easily is the tightness of these measures.

**Lemma 5.1.** \( \{\nu_n\} \) is tight sequence of measures on \( S \).

**Proof.** Consider the Lyapunov functions \( V_2^n = 1 + \sum_{i=1}^N ((x_i^2 + y_i^2 + z_i^2)^2 u(i)) \), so that \( |V_2^n(\pi_{\Lambda_n}(a))| \leq \|a\|_S^8 \). By the proof of Lemma 3.3 we see that thanks to the assumption \( s = \inf_i \lambda_i > 0 \) the conclusions of Theorem 3.6 remain valid for this Lyapunov function as well. Hence we infer the existence of some constants \( \alpha, C(a) > 0 \) such that for \( f_n : (\mathbb{R}^3)^{\Lambda_n} \to \mathbb{R}, \|f_n\| \leq 1 \)

\[
|Ef_n(A^{n,a}(t)) - \mu_n f_n| \leq C(a)e^{-\alpha t}. \tag{5.1}
\]

For given \( \epsilon > 0 \) find \( t > 0 \) large enough, so that \( C(a)e^{-\alpha t} < \frac{\epsilon}{2} \). For this \( t \) find according to Corollary 4.6 compact set \( K \) in \( S \) such that 

\[
EI_K(\hat{A}^{n,a}(t)) \geq 1 - \frac{\epsilon}{2}.
\]

Thus following (5.1)

\[
\nu_n I_K = EI_K(\hat{A}^{n,a}(t)) - \nu_n I_K - EI_K(\hat{A}^{n,a}(t))
\]

\[
= E(I_K(\hat{A}^{n,a}(t)) - \mu_n(I_K \circ i_n) - E(I_K \circ i_n)(A^{n,a}(t))) \geq 1 - \epsilon,
\]

which proves the assertion. \( \square \)

The problem that now arises comes from the fact that in general we only know that for given \( a \in S \) there is sequence of processes \( A^{n,a} \) defined on \( (\mathbb{R}^3)^{\Lambda_n} \) such that \( P \circ (\hat{A}^{n,a})^{-1} \xrightarrow{w} P^a \), because the choice of convergent subsequence has to be done separately for every \( a \). So consequently we have to choose the limit point of \( \nu_n \) also depending on \( a \).
Corollary 5.2. For every \( a \in S \) there exists probability measure \( \nu_a \in S \) such that there are constants \( \alpha, C(a) > 0 \) and following holds \( \frac{\nu}{a} \in S \) such that there are constants \( \alpha, C(a) > 0 \)

\[
\sup_{f \in B(S), \|f\| \leq 1} |E^a f(A(t)) - \nu_a f| \leq C(a)e^{-\alpha t}. \tag{5.2}
\]

Proof. Let us fix the point \( a \in S \). From Lemma 5.1 we know there exist sequence \( \nu_n = \mu_n \circ i_n^{-1} \) such that \( \nu_n \overset{w}{\rightarrow} \nu_a, \) \( \mu_n \) is invariant measure for canonical processes \( \mathcal{B}_n \) on \( \Omega_n, \mathcal{P}_n \overset{w}{\rightarrow} \mathcal{P}_a \) and also

\[
\sup_{f:S \rightarrow \mathbb{R}, \|f\| \leq 1} |E^a_n(f \circ i_n)(B^a_n(t)) - \mu_n(f \circ i_n)| \leq C(a)e^{-\alpha t}. \tag{5.3}
\]

Let \( f \in C_B(S), \|f\| \leq 1 \) be given and we estimate

\[
|E^a f(A(t)) - \nu_a f| \leq |E^a_n f(A(t)) - \nu_n f|
+ |\tilde{E}^a_n f(A(t)) - E^a f(A(t))| + |\nu_a f - \nu_n|.
\]

Utilizing (5.3) and the facts about weak convergence we therefore obtain for arbitrary \( \epsilon > 0 \) the estimate

\[
|E^a f(A(t)) - \nu_a f| \leq C(a)e^{-\alpha t} + 2\epsilon,
\]

so that (5.2) holds for any continuous function bounded by one. (5.2) for general bounded function then follows by approximation.

6 Ergodicity results under Lipschitz assumptions

The results established in previous section can be significantly improved, but we have to adapt some additional limitations to our initial model (4.4). To make the calculation we distinguish specific approximation scheme related to the size of our interactions. Recall that \( 0 < r < \infty \) is the parameter of length of interactions for the functions \( q \)'s. We define boxes \( \Pi_n = \{i \in \mathbb{Z}^d : \max_{j \leq d} |i_j| \leq nr\}, N = |\Pi_n| = (2nr + 1)^d \).

The enhanced assumption we make about interaction function is \((A1)\):

- \( \exists C > 0 : \sup_{u \in (\mathbb{R}^3)^{2r+1}d} |q_i(u)| \leq C, \ i \in \mathbb{Z}^d \)

- \( \sup_{u \in (\mathbb{R}^3)^{2r+1}d} \sum_{j=1}^{(2r+1)} |\frac{\partial q_i}{\partial y_j}(u)|u_i| + |\frac{\partial q_i}{\partial x_j}(u)| \leq C, \ i \in \mathbb{Z}^d \)

Likewise we need to limit the growth of \( \lambda \)'s. Hence the strengthened \((A3)\) assumption:

\[
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\]
\[ \inf_{i \in \mathbb{Z}^d} \lambda_i > 0, \sup_{i \in \mathbb{Z}^d} \lambda_i < +\infty. \]

These assumptions ensure that the equation for \( A^n \) has globally Lipschitz drift. More precisely we need the following observation.

**Lemma 6.1.** Let \( \Lambda_n \supset \Pi_{k+1} \) and we denote \( b_k = (b_1, \ldots, b_K) \) (notice that this does not depend on \( n \), since we assume \( \Lambda_n \supset \Pi_{k+1} \)) the first \( K = |\Pi_k| \) coordinates of drift for the equation

\[ dA^n = b^n(A^n)dt + \sigma^n(A^n)dW_t, \]

also for an element \( c^k \in (\mathbb{R}^3)^{\Pi_k} \) we denote \( c^n_k = (c^n_{1,x}, \ldots, c^n_{K,z}) \). Then there exists constant \( L > 0 \) s. t.

\[ \|b_k(a^n) - b_k(d^n)\|^2_{(\mathbb{R}^3)^{\Pi_k}} \leq L\|a^n_{k+1} - d^n_{k+1}\|^2_{(\mathbb{R}^3)^{\Lambda_{k+1}}}, \forall a^n, d^n \in (\mathbb{R}^3)^{\Lambda_n}. \]

\( L \) is independent of \( k, n \).

**Proof.** Follows by elementary computation using assumptions (A1), (A3).

In addition we need to restrict our class of starting points \( a \in S \), so that the space includes only configurations that does not grow too fast, i. e. (A2):

- \( \sum_{i \in \mathbb{Z}^d} u(i) < +\infty, \ u(i) > 0 \)
- \( \exists \delta \in (0, 1) \exists K > 0 \) s. t.

\[ u(j) \geq \frac{K}{|i|^{1-\delta}} \quad j \in \Pi_i \setminus \Pi_{i-1}, \ i \in \mathbb{N}. \]

The key to proofs in this section are two technical Lemmas about behaviour of solutions \( A^n \) to the SDE’s related to the operator \( L_n \).

**Lemma 6.2.** Let \( a \in S \) and \( \Pi_k \) be defined as above. Suppose we have two exhausting sequences \( \{\Lambda_l\}, \{\Lambda_m\} \) and correspondingly two sequences of processes \( \{A^{m,a}_l\}, \{A^{l,a}_m\} \). We denote by \( A^{m,a}_k \) the part of \( A^{m,a} \) that lives on \( (\mathbb{R}^3)^{\Pi_k} \), i. e. \( A^{n,a}_k = (A^n_{1,x}, \ldots, A^n_{K,z}) \). For any \( \epsilon > 0 \) and \( T > 0 \) there exists \( N > 0 \) such that for any \( l, m \geq N \)

\[ E \sup_{t \in [0, T]} \|A^{l,a}_k(t) - A^{m,a}_k(t)\|^2 \leq \epsilon. \quad (6.1) \]
Proof. We release the index from norms throughout the proof as it will not lead to confusion. Also we will be little imprecise and write \( a_k = (a_{1,x}, \ldots, a_{K,z}) \) for the restriction of \( a \) to \((\mathbb{R}^3)^n_k\), in order to not overload the notation we also write \( a_j = (a_{j,x}, a_{j,y}, a_{j,z}) \) when \( j \in \mathbb{Z}^d \). Using the Lemma 6.1 one infers doing routine calculations existence of constant \( C > 0 \) so that

\[
E \sup_{t \in [0,T]} \| A^{l,a}_k(t) - A^{m,a}_k(t) \|^2 \leq CT \int_0^T E \| A^{l,a}_{k+1}(t) - A^{m,a}_{k+1}(t) \|^2 dt_1.
\]

Assuming \( l, m \) large enough so we can repeat the procedure, we obtain

\[
E \| A^{l,a}_{k+1}(t_1) - A^{m,a}_{k+1}(t_1) \|^2 \leq Ct_1 \int_0^{t_1} E \| A^{l,a}_{k+2}(t_2) - A^{m,a}_{k+2}(t_2) \|^2 dt_2 \]

\[
\cdots \leq C^{n-1} t_1 \int_0^{t_1} t_2 \int_0^{t_2} \cdots \int_0^{t_{n-1}} E \| A^{l,a}_{k+n}(t_n) - A^{m,a}_{k+n}(t_n) \|^2 dt_n \cdots dt_1.
\]

Altogether one thus obtains

\[
E \sup_{t \in [0,T]} \| A^{l,a}_k(t) - A^{m,a}_k(t) \|^2 \leq \frac{(CT^2)^n}{(2n-1)!!} K_T (1 + \| a_{n+k} \|^2),
\]

where \( K_T \) is just the constant related to the Linear growth of coefficients of our SDE (see (4.8)) and \((2n-1)!! = (2n-1) \cdot (2n-3) \cdots 3 \cdot 1 \) denotes the odd (double) factorial. Using the obvious

\[
\| a_{n+k} \|^2 \leq \sum_{j=1}^{(2(n+k)r+1)^d} 3 + \| a_j \|^8_{\mathbb{H}}
\]

we need to prove only

\[
\lim_{n \to \infty} \frac{L^n}{n!} \sum_{j=1}^{(2(n+k)r+1)^d} (1 + \| a_j \|^8_{\mathbb{H}}) = 0
\]

for arbitrary constant \( L > 0 \). Clearly it suffices to show

\[
\lim_{n} \frac{\sum_{j=1}^{(2(n+k)r+1)^d} \| a_j \|^8_{\mathbb{H}}}{n^{1-\frac{r}{2}}} = 0,
\]

where \( \delta \) is from the assumption (A2). We compute using the (A2) and \( \| a_j \|^8_{\mathbb{H}} u(j) \leq \| a \|^8_{\mathbb{H}} \)

\[
\lim_{n} \frac{\sum_{j \in \Lambda_{n+k+1} \setminus \Lambda_{n+k}} \| a_j \|^8_{\mathbb{H}}}{n^{1-\frac{r}{2}} ((n+1)^{1-\frac{r}{2}} - 1)}
\]

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The fact that (6.3) implies (6.2) is well known as Stolz - Cesàro Theorem.

**Lemma 6.3.** Let \( k \in \mathbb{N}, a \in S \text{ and } t > 0 \) be given. Let \( A^{m,a} \) be an approximating sequence defined with respect to exhausting boxes \( \Pi_m \). For any \( \varepsilon > 0 \) there exists \( \eta > 0 \) such that \( \forall m \geq k \)

\[
\|b - a\|_S < \eta \Rightarrow E\|A^{m,a}_k(t) - A^{m,b}_k(t)\|^2 < \varepsilon.
\]

(6.4)

**Proof.** Since we know that our SDE has continuous dependence on initial condition, the Lemma is nontrivial only for infinite number of \( m \) and hence we concentrate in our computations on large \( m \). Again for simplification we will not write the index to the norms through computations. Similarly to the last Lemma we get for some constants \( C > 0 \) and \( K > 0 \) (to make last sum meaningful let us formally define \((-1)!! = 1)\)

\[
E\|A^{m,a}_k(t) - A^{m,b}_k(t)\|^2 \leq C\|a_k - b_k\|^2 + C^2 t^2 \|a_{k+1} - b_{k+1}\|^2 \\
+ C t \int_0^t C t_1 \int_0^t E\|A^{m,a}_{k+2}(t_1) - A^{m,b}_{k+2}(t_2)\|^2 dt_2 dt_1 \\
\leq C\|a_k - b_k\|^2 + C^2 t^2 \|a_{k+1} - b_{k+1}\|^2 + \cdots + \frac{C n t^{2n-2}}{(2n-3)!!} \|a_{k+n-1} - b_{k+n-1}\|^2 \\
+ E \sup_{0 \leq s \leq t} \|A^{m,a}_{k+n}(s) - A^{m,b}_{k+n}(s)\|^2 \frac{(Ct^2)^n}{(2n-1)!!} \\
\leq \sum_{j=1}^n \frac{C^j t^{j-2} \|a_{k+j-1} - b_{k+j-1}\|^2}{(2j-3)!!} + K_t \left( \sum_{i \in \Lambda_{k+n}} 3 + \|a_i\|_H^8 + \|b_i\|_H^8 \right) \frac{(Ct^2)^n}{(2n-1)!!}.
\]

Same calculations like in Lemma 6.2 together with Stolz - Cesàro Theorem gives

\[
\lim_{n \to \infty} K_t \left( \sum_{i \in \Lambda_{k+n}} 3 + \|a_i\|_H^8 + \|b_i\|_H^8 \right) \frac{(Ct^2)^n}{(2n-1)!!} = 0.
\]

(6.5)

Because

\[
\lim_{n \to \infty} \frac{C n t^{2n-2} n!}{((2n-1)!!)^{\frac{3}{2}}} = 0
\]
for $l > 1$, we obtain using previously established convergence results that

$$
\sum_{j=1}^{\infty} \frac{C_j t^{2j-2} \|a_{k+j-1} - b_{k+j-1}\|^2}{(2j - 3)!!} < +\infty.
$$

(6.6)

Therefore combining (6.5) and (6.6) for given $\epsilon > 0$ we can choose $N \in \mathbb{N}$ such that

$$
\sum_{j=N}^{\infty} \frac{C_j t^{2j-2} \|a_{k+j-1} - b_{k+j-1}\|^2}{(2j - 3)!!} + \sup_{j \geq N_1} K_t \left( \sum_{i \in \Lambda_k, j} 3 + \|a_i\|_H^8 + \|b_i\|_H^8 \right) \frac{(C_j t^2)^j}{(2j - 1)!!} < \frac{\epsilon}{2}.
$$

For the first $N - 1$ terms we can choose $\eta > 0$ in (6.4) thanks to the continuous dependence on parameters for the $A^{m,a}$ in such way that

$$
\sum_{j=1}^{N-1} \frac{C_j t^{2j-2} \|a_{k+j-1} - b_{k+j-1}\|^2}{(2j - 3)!!} + \sup_{j \leq N} \sup_{0 \leq s \leq t} \|A^{m,a}_{k+j}(s) - A^{m,b}_{k+j}(s)\|^2 \frac{(C_j t^2)^j}{(2j - 1)!!} < \frac{\epsilon}{2},
$$

and the Lemma is established.

The first crucial property that follows from Lemma 6.2 is independence of the limit measure $P^a$ on the choice of convergent subsequence. Therefore by the well known properties of weak convergence this implies that the sequence $\tilde{P}_n^a$ itself weakly converges. In addition this limit doesn’t depend on the choice of approximating sequence $\Lambda_n$.

**Theorem 6.4.** Let $\tilde{A}^{m,a}, \tilde{A}^{n,a}$ be the sequences of approximating processes on $\Omega$, $a \in S$. Then there exists probability measure $P^a$ on $\Omega$ such that

$$
\lim_{m \to \infty} P \circ (\tilde{A}^{m,a})^{-1} = \lim_{l \to \infty} P \circ (\tilde{A}^{l,a})^{-1} = P^a.
$$

**Proof.** By Corollary 4.6 we know that any two such sequences has weakly convergent subsequence. So it remains to show that the limit point is the same for any two weakly convergent subsequences (to simplify notation
we call the convergent subsequences again $m$ and $l$) \{P \circ (\tilde{A}^{l,a})^{-1}\}, \{P \circ (\tilde{A}^{m,a})^{-1}\}. To prove this it clearly suffices to show that for any $f \in C_b(\Omega)$

$$\lim_l Ef(\tilde{A}^{l,a}(\cdot)) = \lim_m Ef(\tilde{A}^{m,a}(\cdot)). \quad (6.7)$$

First let $f \in C_{b,Lip}^{Cyl}(\Omega)$, i.e. there exists $k \in \mathbb{N}$ and $g \in C_{b,Lip}(\Omega_k)$ such that $f(\omega) = g((\pi_\Omega \omega)_k)$, $\Omega_k = C([0, \infty), (\mathbb{R}^3)^{H_k})$ and $g$ is Lipschitz, that is there exists constant $L > 0$ s.t.

$$|g((\omega_k)_k) - g((\tilde{\omega}_k)_k)| \leq \|\omega_k - \tilde{\omega}_k\|_{\Omega_k} \forall \omega_k, \tilde{\omega}_k \in \Omega_k.$$ 

Then we get for $m, l$ large enough

$$|Ef(\tilde{A}^{l,a}(\cdot)) - Ef(\tilde{A}^{m,a}(\cdot))|^2 = |Eg(A_k^{l,a}(\cdot)) - Eg(A_k^{m,a}(\cdot))|^2$$

$$\leq E|g(A_k^{l,a}(\cdot)) - g(A_k^{m,a}(\cdot))|^2 \leq E\|A_k^{l,a}(\cdot) - A_k^{m,a}(\cdot)\|^2,$$

hence Lemma 6.2 implies (6.7) holds for $f \in C_{b,Lip}^{Cyl}(\Omega)$.

Next let $f \in C_{b}^{Cyl}(\Omega)$, then there exists bounded sequence $f_n \in C_{b,Lip}^{Cyl}(\Omega)$ such that $f_n \to f$. Finally for $f \in C_b(\Omega)$ consider cylindrical approximation by \{f_n\}, that is $f_n(\omega) = f((\pi_\Omega \omega)_k)$ and the result follows by Lebesgue Theorem.

This results thus implies that we can in fact choose measure $\nu_a$, such that $(5.2)$ holds regardless of $a$. Later we even show, that such $\nu$ is unique invariant measure for the semigroup $P_tf(a) = E^a f(A(t))$. To translate Lemma 6.3 into desired properties, we need to recall result about strengthening of weak convergence. Its proof follows immediately from Skorokhod representation theorem (see also [28, pp. 168]).

**Lemma 6.5.** Let $P$ be a Polish space and $\mu_n, \mu$ probability measures on $P$. Suppose $\mu_n \to \mu$. Let $f_n, f \in C(P)$ such that $f_n$ are uniformly bounded and

$$x_n \to x \text{ in } P \implies f_n(x_n) \to f(x). \quad (6.8)$$

Then $\mu_n f_n \to \mu f$.

With this Lemma in hand we can now show that canonical process on $\Omega$ is true Markov process under measures $P^a$ and $\nu$ is unique invariant measure for the process.

**Theorem 6.6.** Let $A_t(w)$ be canonical process on $\Omega = C([0, \infty), S)$ and $P^a$ the unique limiting measure produced by Corollary 4.6. $(A_t, P^a)$ is then a Markov process.
Proof. Denote $\mathcal{S}$ the $\sigma$-algebra on $S$. We need to show these two properties

(I) $a \rightarrow P^a(A_t \in C)$ is measurable for any $C \in \mathcal{S}$  \hspace{1cm} (6.9)
(II) $P^a(A_{s+t} \in C | \mathcal{F}_s) = \phi(A_s), \ \phi(\cdot) = P(A_t \in B), \ \forall \ C \in \mathcal{S}, 0 \leq s \leq t.$ \hspace{1cm} (6.10)

To prove (6.9) we show that $a \rightarrow E^a f(A_t)$ is continuous function for any $f \in C_{b,Lip}^{Cyl}(S)$, the measurability for general $f \in C_b(S)$ will then follow through same procedure as in Theorem 6.3. By the uniqueness just proved, we can consider approximation $\{A^n\}$ living on the boxes $\Pi_n$. So let $f(a) = g(\pi \Pi_k(a))$ and we calculate

$$|E^a f(A_t) - E^b f(A_t)|^2 = |\lim_n E[f(\tilde{A}^{n,a}(t)) - f(\tilde{A}^{n,b}(t))]|^2$$
$$\leq \lim sup_n E|g(A_k^{n,a}(t)) - g(A_k^{n,a}(t))|^2 \leq \lim sup_n ||A_k^{n,a}(t) - A_k^{n,b}(t)||^2.$$ 

According to Lemma 6.3 this estimate implies the desired continuity.

For proving (6.10) one strives to establish $\forall f \in C_b(S)$

$$E^a[f(A_{s+t})|\mathcal{F}_s] = E^{A_s} f(A_t).$$ \hspace{1cm} (6.11)

If we denote $\varphi(\cdot) = E[f(A_t)]$ then this means - for any $C \in \mathcal{F}_s$

$$\int_C f(A_{s+t})dP^a = \int_C \varphi(A_s)dP^a.$$ 

We consider first $f \in C_{b,Lip}^{Cyl}(S)$, then we know from the first part of the proof that $\varphi(\cdot)$ is continuous. By approximation this reduces to necessity of demonstrating

$$E^a[f(A_{s+t})h(\omega.)] = E^a \varphi(A_s)h(\omega.),$$ \hspace{1cm} (6.12)

where $h$ is arbitrary, but fixed continuous bounded $\mathcal{F}_s$ - measurable function. By weak convergence $\tilde{P}^a_n \rightarrow P^a$ the left side of (6.12) is a limit of (the same calculations as we made in the proof of Theorem 4.7 are hidden there)

$$\tilde{E}^a_n[f(A_{s+t})h(\omega.)] = E^a_n[(f \circ i_n)(B^{n}_{s+t})(h \circ j_n)((\omega_n).)].$$

The finite dimensional result, i. e. the fact that $P^a_n$ solves the martingale problem on $\Omega_n$, tells us that

$$E^a_n[(f \circ i_n)(B^{n}_{s+t})(h \circ j_n)((\omega_n).)] = E^a_n[\varphi^n(i_n(B^{n}_a))(h \circ j_n)((\omega_n).)],$$
if \( \varphi^n(i_n(B^n_t)) = E_{n}^{i_n(B^n_t)}[(f \circ i_n)(B^n_t)] \). We observe that

\[
\varphi(a) = E^a f(A_t) = \lim_n E^a_n f(A_t) = \lim E^a_n[(f \circ i_n)(B^n_t)],
\]

hence (6.12) will established using Lemma 6.5 provided we can prove the implication

\[
a_n \to a \text{ in } S \implies \tilde{E}_n^a[f(A_t)] \to E^a[f(A_t)]. \tag{6.13}
\]

For given \( \epsilon > 0 \) we find \( N \) from weak convergence such that

\[
|E^a[f(A_t)] - \tilde{E}_n^a[f(A_t)]| < \frac{\epsilon}{2} \forall n \geq N.
\]

Like in the first part we also have estimate

\[
|\tilde{E}_n^a[f(A_t)] - E_n^a[f(A_t)]|^2 \leq \limsup_n \|A^{n,a}_{k}(t) - A^{n,b}_{k}(t)\|^2,
\]

so Lemma 6.3 implies we can find \( \tilde{N} \) such that

\[
|E^a[f(A_t)] - \tilde{E}_n^a[f(A_t)]| < \epsilon \forall n \geq \tilde{N}.
\]

From Lemma 6.5 we conclude that (6.12) holds for \( f \in \mathcal{C}^{C_{b,Lip}}(S) \). We infer the validity of (6.11) for general \( f \in \mathcal{C}_b(S) \) by routine approximation procedure.

This result gives us that if we set \( P^a_{t}(a,C) = P^a\{A_t \in C\} \), then it is a true transition probability function and \( P^a_{t}f(a) = E^a f(A(t)) \) is the Markov semigroup acting on all \( f \in \mathcal{B}_b(S) \) (Borel bounded functions) satisfying the Chapman - Kolmogorov equality [6, chap. I].

**Theorem 6.7.** The sequence of measures \( \{\nu_n\} \) is weakly convergent with limiting point \( \nu \). \( \nu \) is unique invariant measure for the semigroup \( P^a_{t}f(a) = E^a f(A(t)) \).

**Proof.** We fix some weakly convergent sequence of measures \( \{\nu_n\} \) and its limit point \( \nu \). First we show that any invariant measure for \( P^a_{t} \) must equal to \( \nu \) and then display that \( \nu \) is indeed invariant measure. In fact we show very strong convergence towards invariant measure if we start from different starting measure. From Corollary 5.2 we have the following estimate

\[
\sup_{f \in \mathcal{B}_b(S), \|f\|_\infty \leq 1} |E^a f(A(t)) - \nu f| \leq C(a)e^{-\alpha t}, \tag{6.14}
\]
where \( C(a) \) is constant dependent on \( a \). Let \( \vartheta \) be a probability measure on \( S \) such that \( \int_S C(a) \vartheta(a) < +\infty \) and denote \( P_t^* \) the dual semigroup acting on (signed) measures, i.e. \( P_t^* \vartheta(M) = \vartheta(P_t M) \). By (6.14)

\[
|P_t^* \vartheta(f) - \nu(f)| = \left| \int_S [P_t f(a) - \nu(f)] d\vartheta \right| \leq \vartheta(C(\cdot)) e^{-\alpha t},
\]

for \( \forall f \in B_b(S), \|f\|_\infty \leq 1 \). So if \( P_t^* \vartheta = \vartheta \), we must have \( \nu = \vartheta \). From (6.14) it obviously follows that

\[
\forall a \in S \forall f \in B_b(S), \|f\|_\infty \leq 1 \lim_{t \to \infty} P_t f(a) = \nu(f).
\]

Hence if \( \vartheta \) is arbitrary probability measure on \( S \), then Lebesgue Theorem gives

\[
\lim_{t \to \infty} \int_S [P_t f(a) - \nu(f)] d\vartheta(a),
\]

in another words

\[
\lim_{t \to \infty} P_t^* \vartheta(f) = \nu(f) \forall f \in B_b(S), \|f\|_\infty \leq 1.
\]

Provided that \( P_t^* \vartheta = \vartheta \), then obviously \( \vartheta = \nu \).

We want to prove that for any \( f \in C_b(S) \)

\[
\int_S P_t f(a) d\nu(a) = \int_S f(a) d\nu(a). \tag{6.15}
\]

We show (6.15) for \( f \in C_{b, \text{lip}}(S) \), the general case will again follow easily by approximation. Before the computation recall that \( \mu_n \) is invariant measure on \((\mathbb{R}^3)^\Lambda_n\), so that the equality

\[
\int_{(\mathbb{R}^3)^\Lambda_n} E_n^{a_n(e_n)} h(B_n^a) d\mu_n(e_n) = \int_{(\mathbb{R}^3)^\Lambda_n} h(e_n) d\mu_n(e_n) \ \forall h \in C_b((\mathbb{R}^3)^\Lambda_n)
\]

holds. Remembering the calculations (4.16) we compute

\[
\int_S f(a) d\nu(a) = \lim_n \int_S f(a) d\nu_n(a) = \lim_n \int_{(\mathbb{R}^3)^\Lambda_n} (f \circ i_n)(a_n) d\mu_n(a_n)
\]

\[
= \lim_n \int_{(\mathbb{R}^3)^\Lambda_n} E_n^{a_n i_n(a_n)} (f \circ i_n)(B_n^a) d\mu_n(a_n) = \lim_n \int_S E_n^a (f \circ i_n) (B_n^a) d\nu_n(a)
\]

\[
= \lim_n \int_S E_n^a f(A_t) d\nu_n(a) \overset{\tilde{\cdot}}{=} \int_S E_n^a f(A_t) \nu(a) = \int_S P_t f(a) d\nu(a).
\]

We can erase the question mark using the Lemma 6.5 with exactly the same line of reasoning that was required for the proof of (6.10) in previous Theorem.
For clarity we allow ourselves in the end to summarize the results obtained in a single Theorem.

**Theorem 6.8** (Infinite Dimensional results). Let $\mathbb{Z}^d$ be $d$ dimensional lattice, $r > 0$ given constant and $\Pi_n = \{i \in \mathbb{Z}^d : \|i\|_{\text{max}} \leq nr\}$. Let $q_i, i \in \mathbb{Z}^d$ be smooth functions depending on $(2r + 1)^d$ variables. Let $L$ be the operator given by

$$L = \sum_{i \in \mathbb{Z}^d} L\lambda_i + q_x X_i + q_y Y_i$$

subject to the assumptions $(A1), (A3)$:

- $\exists C > 0 : \sup_{u \in \mathbb{R}^{3(2r+1)d}} |q_i(u)| \leq C, i \in \mathbb{Z}^d$
- $\sup_{u \in \mathbb{R}^{3(2r+1)d}} \sum_{j=1}^{(2r+1)^d} \left| \frac{\partial q_i}{\partial x_j}(u) u_i \right| + \left| \frac{\partial q_i}{\partial y_j}(u) \right| \leq C, i \in \mathbb{Z}^d$
- $\inf_{i \in \mathbb{Z}^d} \lambda_i > 0, \sup_{i \in \mathbb{Z}^d} \lambda_i < \infty$.

Introduce the weighted space $S = \{a \in \mathbb{H}^{\mathbb{Z}^d} : \sum_{i \in \mathbb{Z}^d} \|a_i\|_{\mathbb{H}} u(i) < +\infty\},$ where the weights satisfy $(A2)$:

- $\sum_{i \in \mathbb{Z}^d} u(i) < +\infty, u(i) > 0$
- $\exists \delta \in (0, 1) \exists K > 0$ s. t.

$$u(j) \geq \frac{K}{i!^{1-\delta}} \quad j \in \Pi_i \setminus \Pi_{i-1}, \quad i \in \mathbb{N}.$$

Then for any $a \in S$ there exists probability measure $P^a$ on $\Omega = C([0, \infty), S)$ such that for the canonical process $A_t(\omega) = \omega_t$ we have $P^a(A_0 = a) = 1$ and the process

$$f(A_t) - f(A_0) - \int_0^t Lf(A_u)du$$

is martingale for $f \in C_c^{2,Cyl}$ under the measure $P^a$. The pair $(A_t, P^a)$ is Markov process and there exists unique invariant measure $\nu$ for the semi-group $P_t f(\cdot) = E^\nu f(A(t))$. Furthermore there exist some constants $\alpha > 0$ and $K > 0$ so that the following version of exponential ergodicity holds

$$\sup_{\{f \in C_b(S), \|f\|_{\infty} \leq 1\}} |E^\nu f(A(t)) - \nu f| \leq K(1 + \|a\|_S^8) e^{-\alpha t}.$$
7 Examples of other operators

We list some other relevant examples, that can be handled using our strategy without any additional difficulty:

- Of course the elliptic case lies naturally within our framework. Take Euclidean space $\mathbb{R}^3$ with standard Laplacian $\Delta$, $D = x\partial_x + y\partial_y + z\partial_z$, $X = \partial_x$ (etc. for $Y$, $Z$), $L_\lambda = \Delta - \lambda D$ and consider operator

$$L = \sum_{i \in \mathbb{Z}^d} L_\lambda_i + q_{i,x} X_i + q_{i,y} Y_i + q_{i,z} Z_i$$

acting on $(\mathbb{R}^3)^{\mathbb{Z}^d}$. Lyapunov function here can be chosen just $x^{2k} + y^{2k} + z^{2k}$, for $k = 2$ we get the same tightness as we had in Corollary 4.6.

- The Grushin plane [1]: Take $\mathbb{R}^2$ as the basic space and consider vector fields $X = \partial_x$, $Y = -x\partial_y$. $D$ is given by $D = x\partial_x + y\partial_y$ and operator

$$L = \sum_{i \in \mathbb{Z}^d} X_i^2 + Y_i^2 - \lambda_i D_i + q_{i,x} X_i + q_{i,y} Y_i$$

on $(\mathbb{R}^2)^{\mathbb{Z}^d}$. For the Lyapunov function works $V = x^{4k} + y^{2k}$, the tightness [1]6 works again for $k = 2$. The $\sigma$ and $u$ in Girsanov theorem to simplify the control problem can be chosen in the following way

$$\sigma = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \quad u = \begin{pmatrix} \frac{q_x}{\sqrt{2}} \\ \frac{q_y}{\sqrt{2}} \end{pmatrix}.$$

Then we have

$$\sigma u = b - \tilde{b} = (-\lambda x, -\lambda y).$$

- We cannot quite handle the example of Martinet distribution as in [10]. Take $\mathbb{R}^3$ and let $X = \partial_x - y^2\partial_z$, $Y = y\partial_y$. The problem that arises lies in the nonlinear term in $z$-axis. We can not hope for our strategy to be successful, as in the last section definitely linear growth together with strong Lipschitz condition is required. But at least the finite dimensional case is almost conquered by our methods - If one puts $D = x\partial_x + y\partial_y + z\partial_z$ and consider

$$L = X^2 + Y^2 - \lambda D + q_x X + q_y Y$$

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as operator on $\mathbb{R}^3$, then the SDE corresponding to this operator has coefficients

$$b = (q_x - \lambda x, q_y - \lambda y, -\lambda z - q_xy^2), \quad \sigma = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2}y^2 \end{pmatrix}.$$ 

Due to nonlinearities, not even global existence of process is a priori clear. However, if we set $V_k = x^{2k} + y^{6k} + z^{2k}$, we calculate that $V_k$ is the Lyapunov function giving global existence and invariant measure. The smoothness of density holds from Theorem 3.2 as well. However to our best knowledge, we are unable to investigate the irreducibility of the process.

In general we can say, that our strategy is successful whenever we can establish finite dimensional results as in (3.6) with Lyapunov function, that will enable us to construct the diffusion using tightness arguments as in chapter three. To finish the strategy with desired results, it is then essential that we can impose on the interaction such constraints leading to condition of type (6.1).

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References

[1] N. Arcozzi, A. Baldi, From Grushin to Heisenberg via an isoperimetric problem. J. Math. Anal. Appl. 340 (2008), no. 1, 165–174

[2] R. Athreya et al., Infinite dimensional stochastic differential equations of Ornstein - Uhlenbeck type, Stochastic Process. Appl. 116 (2006), no. 3, 381 - 406

[3] D. Bakry et al., On gradient bounds for the heat kernel on the Heisenberg group, Journal of Functional Analysis 255 (2008) 1905 - 1938

[4] R. Bass, E. Perkins, On uniqueness in law for parabolic SPDEs and infinite-dimensional SDEs, Electron. J. Probab. 17 (2012), no. 36, 54 pp.

[5] R. Bass, Stochastic Processes, Cambridge University Press, 2011

[6] R. Blumenthal, R. Getoor, Markov processes and potential theory, Pure and Applied Mathematics, Vol. 29 Academic Press, New York-London 1968
[7] L. Rey-Bellet, *Ergodic properties of Markov processes*, Open quantum systems. II, 1-39, Lecture Notes in Math., 1881, Springer, Berlin, 2006

[8] A. Bonfiglioli, E. Lanconelli, F. Uguzzoni, *Stratified Lie Groups and Potential Theory for Their Sub-Laplacians*, Springer-Verlag Berlin Heidelberg, 2007

[9] G. Da Prato, J. Zabczyk, *Convergence to equilibrium for classical and quantum spin systems*, Probab. Theory Related Fields 103 (1995), no. 4, 529-552

[10] F. Dragoni, V. Kontis, B. Zegarliński, *Ergodicity of Markov Semigroups with Hörmander Type Generators in Infinite Dimensions*, Potential Analysis, October 2012, Volume 37, Issue 3, 199 - 227

[11] J. Fritz, *Infinite lattice systems of interacting diffusion processes, existence and regularity properties*. Z. Wahrsch. Verw. Gebiete 59 (1982), no. 3, 291 - 309

[12] A. Guionnet, B. Zegarliński, *Lectures on logarithmic Sobolev inequalities*, Sminaire de Probabilits, XXXVI, 1-134, Lecture Notes in Math., 1801, Springer, Berlin, 2003

[13] I. Gyöngy, T. Pröhle, *On the approximation of stochastic differential equation and on Stroock-Varadhan’s support theorem*, Comput. Math. Appl. 19 (1990), no. 1, 65 - 70

[14] M. Hairer, J. C. Mattingly, *Yet another look at Harris’ ergodic theorem for Markov chains*, Seminar on Stochastic Analysis, Random Fields and Applications VI Progress in Probability Volume 63, 2011, 109 - 117

[15] M. Hofmanová, J. Seidler, *On weak solutions of stochastic differential equations*, Stoch. Anal. Appl. 30 (2012), no. 1, 100 - 121

[16] R. Holley, D. Stroock *Diffusions on an infinite dimensional torus*. J. Funct. Anal. 42 (1981), no. 1, 29 - 63

[17] K. Ichihara, H. Kunita, *A classification of the second order degenerate elliptic operators and its probabilistic characterization*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 30 (1974), 235 - 254

[18] C. Kipnis, C. Landim, *Scaling limits of interacting particle systems*, Springer - Verlag, Berlin, 1999

[19] V. Kontis, M. Ottobre, B. Zegarliński, *Markov semigroups with hypocoercive - type generator in infinite dimensions I: Ergodicity and smoothing*, preprint 2013, arXiv:1306.6452 [math-ph]

[20] V. Kontis, M. Ottobre, B. Zegarliński, *Markov semigroups with hypocoercive - type generator in infinite dimensions II: Applications*, preprint 2013, arXiv:1306.6453 [math-ph]

[21] T. Liggett, *Interacting particle systems*, Springer - Verlag, New York, 1985
[22] J. Mattingly, A. Stuart, D. Higham, *Ergodicity for SDEs and Approximations: Locally Lipschitz vector fields and degenerate noise*, Stochastic Process. Appl. 101 (2002), no. 2, 185 - 232

[23] S. Meyn, R. L. Tweedie, *Stability of Markovian processes. II. Continuous-time processes and sampled chains*, Adv. in Appl. Probab. 25 (1993), no. 3, 487 - 517

[24] S. Meyn, R. L. Tweedie, *Stability of Markovian processes. III. Foster-Lyapunov criteria for continuous-time processes*, Adv. in Appl. Probab. 25 (1993), no. 3, 518 - 548

[25] J. Munkres, *Topology*, Prentice Hall, Inc., 2nd edition, 2000

[26] D. Nualart, *The Malliavin Calculus and Related Topics*, Springer - Verlag Berlin Heidelberg, 2nd edition, 2006

[27] B. Øksendal, *Stochastic differential equations. An introduction with applications*, Sixth edition, Springer - Verlag, Berlin, 2003

[28] L. Rogers, D. Williams, *Diffusions, Markov Processes, and Martingales*, John Wiley & Sons, 1987

[29] D. Stroock, S. Varadhan, *On the support of diffusion processes with applications to the strong maximum principle*, Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability 1972, Vol. III: Probability theory, 333 - 359

[30] L. Zambotti, *An analytic approach to existence and uniqueness for martingale problems in infinite dimensions*, Probab. Theory Related Fields 118 (2000), no. 2, 147 - 168

[31] B. Zegarlinski, *The strong decay to equilibrium for the stochastic dynamics of unbounded spin systems on a lattice*, Communications in Mathematical Physics, January (III) 1996, Volume 175, Issue 2, 401 - 432