A GENERAL APPROACH TO APPROXIMATION THEORY OF OPERATOR SEMIGROUPS

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Abstract. By extending ideas from [32], we develop a general, functional calculus approach to approximation of $C_0$-semigroups on Banach spaces by bounded completely monotone functions of their generators. The approach comprises most of well-known approximation formulas, yields optimal convergence rates, and sometimes even leads to sharp constants. In an important particular case when semigroups are holomorphic, we are able to significantly improve our results for general semigroups. Moreover, we present several second order approximation formulas with rates, which in such a general form appear in the literature for the first time.

1. Introduction

The approximation formulas for $C_0$-semigroups play an important role in the study of PDEs and their numerical analysis. They are also indispensable in probability theory and in the theory of approximation of functions, and helpful in the study of semigroups themselves. Basic results on semigroup approximation can be found e.g. in [25, Chapters III.4, III.5]. While some earlier developments are thoroughly described in [11] and [41], it seems there’s no a modern and comprehensive account on approximation of $C_0$-semigroups and related matters. Some aspects of applications of semigroup approximations, although different from the setting of this paper, are presented in [1] and [2].

For quite a long time, the approximation theory of $C_0$-semigroups existed as a number of distant formulas, and each of the formulas required a separate theory. A typical illustration of that phenomena is the following famous Yosida’s formula with rates, a subject of several papers (see e.g. [32] for a relevant discussion). Recall that if $-A$ is the generator of a bounded $C_0$-semigroup $(e^{-tA})_{t\geq 0}$ on a Banach space $X$, and $\alpha \in (0, 2]$, then for all
x ∈ dom(A^α), t > 0, and n ∈ N,

\[ \|e^{-ntA(n+A)^{-1}}x - e^{-tA}x\| \leq 16M \left( \frac{t}{n} \right)^{\alpha/2} \|A^\alpha x\|, \]

where \( M := \sup_{t \geq 0} \|e^{-tA}\| \). Moreover, if \((e^{-tA})_{t \geq 0}\) is holomorphic and sectorially bounded, then for all \( \alpha \in [0, 1] \), \( x \in \text{dom}(A^\alpha) \), \( t > 0 \), and \( n \in \mathbb{N} \),

\[ \|e^{-ntA(n+A)^{-1}}x - e^{-tA}x\| \leq C(nt^{1-\alpha})^{-1} \|A^\alpha x\|, \]

where \( C > 0 \) is an absolute constant. Replacing Yosida’s approximating family \((e^{-ntA(n+A)^{-1}})_{n \geq 1}\) by another family of functions of \( A \), e.g. by Euler’s approximation \(((1 + tA/n)^{-n})_{n \geq 1}\), one can usually get a statement of a similar nature. However technical details are often annoyingly different, and it is natural to ask whether there is a way to unify all of those separate considerations. One of the aims of this paper is to show that the functional calculi theory offers a general and fruitful point of view at semigroup approximations.

A functional calculus approach to approximation of \( C_0 \)-semigroups, with an emphasis on rational approximations, has been initiated in the groundlaying papers [9] and [36], and developed further in many subsequent articles. For comparatively recent contributions to this area, based on functional calculus ideology, see e.g. [23], [24], [28], [34], [38], [40] and [46]. A fuller discussion with some other relevant references can be found in [32].

At the same time, using ideas from probability theory and probabilistic representation formulas, a unifying approach to approximation theorems for \( C_0 \)-semigroups was proposed by Butzer and Hahn [11] and Chung [17]. Somewhat closed (but avoiding an explicit probability language) studies of Dunford-Segal (or Hille) and Yosida’s formulas were done by Ditzian, [20]-[22]. The latter papers contained also a thorough discussion of convergence rates. A comprehensive probabilistic approach has been further developed in a series of papers by Pfeifer, [48]-[52], see also [26]. Pfeifer combined a number of well-known approximation formulas in terms of several probabilistic ones and derived the corresponding approximation rates (optimal according to [22] and [26]). More comments on his approach will be given below.

In this paper, we take a different route, and elaborate further the functional calculus techniques for approximation of \( C_0 \)-semigroups on Banach spaces. Being unaware of Pfeifer’s work, in [32], the first and the third author put several well-known approximation formulas for \( C_0 \)-semigroups into the framework of exponentials of (negative) Bernstein functions and the Hille-Phillips functional calculus. Recall that by Bernstein’s theorem a nonnegative function \( \varphi \) on \([0, \infty)\) is Bernstein if and only if there exists a weak star continuous convolution semigroup of Borel measures \((\mu_t)_{t \geq 0}\) on
\[ e^{-t\varphi(z)} = \int_0^\infty e^{-sz} \mu_t(ds), \quad t \geq 0. \]

for all \( z \geq 0 \). It is well-known that if \( \varphi \) is Bernstein, then \( \varphi \) extends analytically to the (open) right half-pane \( \mathbb{C}_+ \) and continuously to \( \mathbb{C}_+ \), so that (1.1) holds in fact for all \( z \in \mathbb{C}_+ \) with \( \text{Re} \, z \geq 0 \). One of the basic observations in [32] is that many approximation formulas for \( C_0 \)-semigroups follow from either of asymptotic relations

\[ e^{-tz} - e^{-n\varphi(tz/n)} \to 0, \quad n \to \infty, \]

\[ e^{-tz} - e^{-nt\varphi(z/n)} \to 0, \quad n \to \infty, \]

for all \( t > 0 \) and \( \text{Re} \, z \geq 0 \), with \( \varphi \) being a Bernstein function such that

\[ \varphi(0) = 0, \quad \varphi'(0) = 1, \quad |\varphi''(0)| < \infty. \]

In particular, Euler’s approximation corresponds to \( \varphi(z) = \log(1 + z) \) in (1.2), while Yosida’s formula arises from (1.3) with \( \varphi(z) = z/(z + 1) \). Along this way, apart from proposing a general view on semigroup approximation, we equipped the approximation formulas with optimal convergence rates depending on a “generating function” \( \varphi \), and covered in this way a number of known partial cases. For more details of these developments we refer to [32].

Note however that (1.2) and (1.3) can be put into a yet more general and apparently more natural form

\[ g_t^n(tz/n) - e^{-tz} \to 0, \quad n \to \infty, \]

where \( g_t \) is defined by either \( g_t = e^{-\varphi} \) or \( g_t = e^{-t\varphi(1/t)}, t > 0 \). Moreover, observe that each of the functions \( e^{-t\varphi}, t > 0 \), is bounded and completely monotone, that is a Laplace transform of a positive bounded Borel measure \( \mu_t \) on \( [0, \infty) \). Thus it is natural to study (1.4) for a family \( (g_t)_{t>0} \) of bounded completely monotone functions satisfying

\[ g_t(0) = 1, \quad g_t'(0) = -1, \quad g_t''(0) < \infty, \quad t > 0, \]

and not necessarily having a semigroup property with respect to \( t \). (The assumption \( t \in (0, \infty) \) is not important and can be replaced by e.g. \( t \in (0, a], a > 0 \).) The formula (1.4) is a starting point of the paper. It can also be viewed as a kind of so-called Chernoff’s product formula frequently used in the study of \( C_0 \)-semigroups, e.g. in [14], [15], [16], and [52]. Note that there is a vast literature on Chernoff type approximation formulas and their rates, see e.g. [12], [13], and [29]. Unfortunately, our methodology does not apply in this setting in view of its “noncommutative” nature.

To justify the framework of this paper, observe that the class of bounded completely monotone functions is considerably larger than the class \( \mathcal{E} = \{ e^{-\varphi} : \varphi \text{ is Bernstein} \} \), basic for [32]. For example, it is clear that if (a continuous extension of) a bounded completely monotone function \( g \) has zeros on the imaginary axis, then \( g \notin \mathcal{E} \). Another less evident condition for
$g \not\in \mathcal{E}$ is that $g$ is a Laplace transform of a bounded positive measure on $[0, \infty)$ with compact support, see e.g. [56, Lemma 7.5 and Corollary 24.4]. In general, measures $\mu$ giving rise to functions from $\mathcal{E}$ are called \textit{infinitely divisible} since according to (1.1) for any such $\mu$ there exists a weak star continuous convolution semigroup of Borel measures $(\mu_t)_{t \geq 0}$ satisfying $\mu_1 = \mu$, so that $\mu = (\mu_1/n)^n$ for every $n \in \mathbb{N}$. For more properties of this class as well as for more examples of measures that are not infinitely-divisible we refer to [57, Chapter 5] and [56, Chapters 2, 5 and 6].

Thus, relying on the functional calculi ideas from [32] and the approximation relation (1.4), we develop further a comprehensive approach to the approximation theory for bounded $C_0$-semigroups on Banach spaces. The approach allows one to unify most of known approximation formulas and equip them with optimal convergence rates, thus putting many results from [32] into their final form. Moreover, it leads to second order approximation formulas with optimal rates, and approximation results for holomorphic semigroups with sharp constants. The heart matter of our considerations is positivity of measures involved in various estimates, so the class of bounded completely monotone functions (or, equivalently, of Laplace transforms of positive bounded Borel measures) seems to provide natural limits for the techniques of this paper (and also of [32]).

Positivity aspects were also crucial in Pfeifer’s studies mentioned above. To give a flavor of his results we need to digress into the probabilistic terminology. Let $(N(\tau))_{\tau \geq 0}$ be a family of non-negative integer-valued random variables with $E(N(\tau)) = \tau \xi$ for some $\xi > 0$ such that $\sigma^2(N(\tau)) = o(\tau^2)$ for $\tau \to \infty$, where $E$ and $\sigma^2$ stand for expectation and variance, respectively. If $Y$ be a non-negative random variable with $E(Y) = \gamma$ such that $\sigma^2(Y)$ exists, then, by [50], for any bounded $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on $X$ and any $x \in X$,

\begin{equation}
(1.6) \quad e^{-tA}x = \lim_{\tau \to \infty} \psi_{N(\tau)}(E[e^{-YA/\tau}])x
\end{equation}

in the strong sense, where $\psi_{N(\tau)} = E(t^N(\tau)), t \geq 0,$ and $t = \xi \gamma$. Moreover, for $x \in \text{dom}(A)$ the rate of convergence in (1.6) can be estimated in terms of $\sigma(Y), \sigma(N(\tau))$ and $\|Ax\|$, and the same is true for $x \in \text{dom}(A^2)$ modulo a replacement of $\|Ax\|$ by $\|A^2x\|$. Similarly, if $N$ is a non-negative integer-valued random variable with $E(N) = \xi$ and $Y \geq 0$ is a real-valued random variable with $E(Y) = \gamma$, then for $t = \xi \gamma$ one has

\begin{equation}
(1.7) \quad e^{-tA}x = \lim_{n \to \infty} \{\psi_N(E[e^{-AY/n}])\}^n x,
\end{equation}

again with the convergence rate estimates. Specifying $N$ in (1.6) and (1.7) (and similar asymptotic relations in [50]), one gets a number of approximation formulas with rates that appear to be sharp by [26]. For instance, choosing $N$ to be a binomial distribution in (1.7), one gets so-called Kendall’s
approximation formula:
\[
\left\| \left( (1 - t)I + te^{-A/n} \right)^n x - e^{-tA} x \right\| \leq \frac{t(1 - t)M}{2n} \| A^2 x \|
\]
for all \( x \in \text{dom}(A^2), t \in (0, 1), \) and integers \( n > (t(1 - t))^{-1} - 6 \). For more details and explanations on probabilistic methodology we refer to Pfeifer’s papers and also to [11] and [17]. It is curious to note that in [52] Pfeifer gives an operator-probabilistic counterpart of the “product formula” (1.4). However he did not develop that result any further. All positivity aspects are somewhat hidden in his arguments, but they become transparent when the expectations and other relevant quantities are written explicitly in terms of the distribution functions.

While the probabilistic approach allowed one to encompass a number of approximation formulas similarly to our studies, it has also certain shortcomings when compared to functional calculi machinery. In particular, it does not take into account the specifics of holomorphic semigroups. Moreover, given a bounded complete function \( g \), functional calculus methods lead to explicit approximation formulas with convergence rates in terms of \( g \) and its derivatives at zero. The probabilistic arguments are somewhat less transparent since they depend on a choice of auxiliary parameters (e.g. the distribution of \( N \) as above), thus requiring certain invention. Finally, our framework includes the approximations \( (g^n_t(\cdot/n))_{n \geq 1} \) when \( g^n_t(0) \) may not be finite, and the latter falls outside the scope of limit theorems in [50] depending on finite variances and thus assuming implicitly \( g^n_t(0) < \infty \). One should note that in fact one can often read off approximations such as (1.4) from probabilistic formulas similar to (1.6) and (1.7). However that fact apparently escaped the attention of experts, and the emphasis has been put on purely probabilistic techniques. Overall, from a practical point of view, the two approaches can probably be considered as complementary rather than covering each other.

Let us now review our main results. First we present first order approximation estimates with rates, comprising a number of known approximation schemes for semigroups into a single formula with an optimal convergence rate. It is remarkable that the estimates are given in explicit a priori terms.

Theorem 1.1. Let \( -A \) be the generator of a bounded \( C_0 \)-semigroup \( (e^{-tA})_{t \geq 0} \) on \( X \), and let \( (g_t)_{t > 0} \) be a family of completely monotone functions such that for every \( t > 0 \),
\[
g_t(0) = 1, \quad g_t'(0) = -1, \quad \text{and} \quad g_t''(0) < \infty.
\]
Let \( M := \sup_{t \geq 0} \| e^{-tA} \| \). Then for all \( n \in \mathbb{N} \) and \( t > 0 \),
\[
\| g^n_t(tA/n)x - e^{-tA}x \| \leq M \frac{(g^n_t(0) - 1) t^2}{2n} \| A^2 x \|, \quad x \in \text{dom}(A^2),
\]
\[ \|g_n^t(tA/n)x - e^{-tA}x\| \leq M(g''_t(0) - 1)^{1/2} \frac{t}{\sqrt{n}} \|Ax\|, \quad x \in \text{dom}(A), \]

and, if \( \alpha \in (0, 2) \),

\[ \|g_n^t(tA/n)x - e^{-tA}x\| \leq 4M \left( (g''_t(0) - 1) \frac{t^2}{n} \right)^{\frac{\alpha}{2}} \|A^\alpha x\|, \quad x \in \text{dom}(A^\alpha). \]

Moreover, using the framework of completely monotone functions (and, implicitly, positivity of the corresponding measures), we are able to obtain second order approximation formulas, again with optimal rates. For this kind of results higher smoothness of the generating functions \( g_t \) is required.

**Theorem 1.2.** Let \(-A\) be the generator of a bounded \( C_0 \)-semigroup \((e^{-tA})_{t \geq 0}\) on \(X\), and let \((g_t)_{t > 0}\) be family of completely monotone functions such that, in addition to (1.8), one has

\[ |g'''_t(0)| < \infty \quad \text{and} \quad g''''_t(0) < \infty, \]

for every \( t > 0 \). Let \( M := \sup_{t \geq 0} \|e^{-tA}\| \). Then for all \( t > 0, n \in \mathbb{N}, \) and \( x \in \text{dom}(A^3) \),

\[ \|g_n^t(tA/n)x - e^{-tA}x - (2n)^{-1}(g''_t(0) - 1)t^2e^{-tA}A^2x\| \leq MC(g_t)t^3n^{-3/2}\|A^3 x\|, \]

where

\[ C(g_t) = \left( \frac{(g''_t(0) - 1)(g''''_t(0) - 1)}{2} \right)^{1/2}. \]

Moreover, for all \( x \in \text{dom}(A^4), t > 0 \) and \( n \in \mathbb{N} \),

\[ \|g_n^t(tA/n)x - e^{-tA}x - (2n)^{-1}(g''_t(0) - 1)t^2e^{-tA}A^2x\| \leq MC_1(g_t)n^{-2}t^3(\|A^3 x\| + t\|A^4 x\|), \]

where

\[ C_1(g_t) = g''''_t(0) - 1. \]

Theorems 1.1 and 1.2 should be compared to Theorems 4.1 and 4.4 in [50] where similar results were obtained using a different, probabilistic language.

It is of interest for applications to obtain sharper versions of the above results if the semigroup \((e^{-tA})_{t \geq 0}\) is holomorphic. In fact, it is this particular class of semigroups that attracted most of attention in the literature on semigroup approximation. By adjusting our functional calculus arguments accordingly, we obtain analogues of Theorems 1.1 and 1.2 for sectorially bounded holomorphic semigroups with improved rates (and constants) in this particular case. Here we formulate only one of these results and refer to Section 5 for other related statements.
Theorem 1.3. Let \((g_t)_{t>0}\) be as in Theorem 1.1, let \(-A\) be the generator of a sectorially bounded holomorphic \(C_0\)-semigroup \((e^{-tA})_{t\geq 0}\) on \(X\), and let for every \(\beta \geq 0\),

\[
M_\beta := \sup_{t>0} \|t^\beta A^\beta e^{-tA}\|.
\]

Then for all \(n \in \mathbb{N}\), and \(t > 0\),

\[
\|g^n_t(tA/n)x - e^{-tA}x\| \leq (2M_0 + 3M_1/2)t^n \|Ax\|, \quad x \in \text{dom}(A).
\]

Moreover, for all \(n \in \mathbb{N}\) and \(t > 0\),

\[
\|g^n_t(tA/n) - e^{-tA}\| \leq K(g''_0(0) - 1)/n,
\]

and, if \(\alpha \in (0, 1)\),

\[
\|g^n_t(tA/n)x - e^{-tA}x\| \leq 4K(g''_0(0) - 1)t^{\alpha}n^{\alpha} \|A^\alpha x\|, \quad x \in \text{dom}(A^\alpha),
\]

where \(K = 3M_0 + 3M_1 + M_2/2\).

As an illustration of our technique we revisit Euler’s approximation formula, and show in Theorem 1.4 below that the functional calculus estimates yield essentially the best constants there (matching the corresponding numerical bounds !). This is a nice illustration of the power of functional calculus.

Theorem 1.4. Let \(-A\) be the generator of a sectorially bounded holomorphic \(C_0\)-semigroup \((e^{-tA})_{t\geq 0}\) on \(X\). Then for all \(\alpha \in [0, 1]\), \(t > 0\), and \(n \in \mathbb{N}\),

\[
\left\|\left(1 + \frac{t}{n}A\right)^{-n}x - e^{-tA}x\right\| \leq M_{2-\alpha}r_{\alpha,n}t^{\alpha}n^{\alpha} \|A^\alpha x\|, \quad x \in \text{dom}(A^\alpha),
\]

where

\[
r_{\alpha,n} = \frac{1}{2n} + \frac{1 - 2\alpha}{12n^2}, \quad \alpha \in [0, 1/2), \quad r_{\alpha,n} = \frac{1}{2n}, \quad \alpha \in [1/2, 1], \quad n \in \mathbb{N}.
\]

Moreover, there exists an absolute constant \(C > 0\) such that for all \(\alpha \in [0, 1]\), \(t > 0\), and \(n \in \mathbb{N}\),

\[
\left\|\left(1 + \frac{t}{n}A\right)^{-n}x - e^{-tA}x - \frac{t^2A^2x}{2n}e^{-tA}x\right\| \leq C[M_{3-\alpha} + M_{4-\alpha}]n^{\alpha} \|A^\alpha x\|, \quad x \in \text{dom}(A^\alpha).
\]

Recall that the estimates of the form

\[
\left\|\left(1 + \frac{t}{n}A\right)^{-n}x - e^{-tA}x\right\| \leq Cn^{\alpha} \|A^\alpha x\|, \quad x \in \text{dom}(A^\alpha),
\]

where \(n \in \mathbb{N}\) and \(t > 0\), for some constant \(C = C(A)\) and integer values \(\alpha = 0\) and \(\alpha = 1\) were obtained as a partial case of more general rational approximations for sectorially bounded holomorphic \(C_0\)-semigroups in [42].
Theorems 4.2 and 4.4. See also [9, Remark, p. 693-694]. A variant of (1.11) for all $\alpha \in [0,1]$ and an explicit constant $C = C(A)$ were also proved in [32] as a consequence of functional calculus approach. In [7, Theorem 1.3], (1.11) was proved for $\alpha = 0$ and $C = 4(\max(M_0, M_1))^3$. Moreover, (1.11) has been derived in [4] in the setting of Hilbert space $m$-accretive operators of angle $\alpha \in [0, \pi/2)$ with $C = K_\alpha \cos^2 \alpha, \quad \frac{\pi \sin \pi \alpha}{2\alpha} \leq K_\alpha \leq \min(\frac{\pi}{\alpha} - 1, K_0),$ where $\frac{\pi}{2} \leq K_0 \leq 2 + \frac{2}{\sqrt{3}}$. Note also that the higher order estimate (1.10) for $\alpha = 0$ and with much worse constants was obtained in [60]. See also [32] for additional references.

Finally, it will be convenient to fix the following notation for the rest of the paper. For a closed linear operator $A$ on a complex Banach space $X$ we denote by $\text{dom}(A)$, $\text{ran}(A)$, and $\sigma(A)$ the domain, the range, and the spectrum of $A$, respectively. The norm-closure of the range is written as $\overline{\text{ran}(A)}$. The space of bounded linear operators on $X$ is denoted by $L(X)$. We let

$$C_+ = \{ z \in \mathbb{C} : \text{Re } z > 0 \}, \quad \mathbb{R}_+ = [0, \infty).$$

2. Preliminaries and further notation

Let us recall some function-theoretic facts relevant for the following. A nonnegative function $g \in C^\infty(0, +\infty)$ is called completely monotone if

$$(-1)^n \frac{d^n g(t)}{dt^n} \geq 0, \quad t > 0$$

for each $n \in \mathbb{N}$. By Bernstein’s theorem [57, Theorem 1.4] a function $g : (0, \infty) \to \mathbb{R}_+$ is completely monotone if and only if there exists a (necessarily unique) positive Laplace-transformable Borel measure $\nu$ on $\mathbb{R}_+$ such that

$$(2.1) \quad g(t) = (\mathcal{L}_\nu)(t) := \int_0^\infty e^{-ts} \nu(ds) \quad \text{for all } t > 0.$$ 

The set of completely monotone functions will be denoted by $\mathcal{CM}$, and the set of bounded completely monotone functions will be denoted by $\mathcal{BM}$. If $g \in \mathcal{CM}$ has representation (2.1), we will write $g \sim \nu$. Each completely monotone function $g$ has close relative called a Bernstein function. Recall that a nonnegative function $f \in C^\infty(0, \infty)$ is Bernstein if $f'$ is completely monotone. An exhaustive treatment of completely monotone and Bernstein functions can be found in [57].

For $g \in \mathcal{CM}$ and for $k \in \mathbb{N} \cup \{0\}$ we will denote

$$g^{(k)}(0) = \lim_{z \to 0^+} g^{(k)}(z)$$

where, in general, the limit can be infinite.

Let $M(\mathbb{R}_+)$ be the Banach algebra of bounded Borel measures on $\mathbb{R}_+$. Note that if $\mu \in M(\mathbb{R}_+)$, then the Laplace transform $\mathcal{L}_\mu$ extends to a function
holomorphic on $\mathbb{C}_+$ and continuous and bounded on $\mathbb{C}_+$. Moreover, the space

$$A^1_+(\mathbb{C}_+) := \{ \mathcal{L}\mu : \mu \in M(\mathbb{R}_+) \}$$

is a commutative Banach algebra with pointwise multiplication and with respect to the norm

$$(2.2) \quad \|\mathcal{L}\mu\|_{A^1_+} := \|\mu\|_{M(\mathbb{R}_+)} = |\mu|(\mathbb{R}_+),$$

and the mapping

$$\mathcal{L} : M(\mathbb{R}_+) \mapsto A^1_+(\mathbb{C}_+)$$

is an isometric isomorphism.

If $g \in BM$, $g = L\nu$, then by Fatou’s Lemma,

$$(2.3) \quad \infty > g(0) = \int_0^\infty \nu(ds) = \|g\|_{A^1_+},$$

due to the positivity of $\nu$.

If $-A$ is the generator of a bounded $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on a Banach space $X$, then one defines

$$(2.4) \quad g(A) := \int_0^\infty e^{-sA}\nu(ds),$$

where the integral is understood as a strong Bochner integral. The continuous algebra homomorphism

$$A^1_+(\mathbb{C}_+) \mapsto \mathcal{L}(X)$$

$$g \mapsto g(A)$$

is called the Hille-Phillips (HP-) functional calculus. Clearly

$$\|g(A)\| \leq \|g\|_{A^1_+} \sup_{t \geq 0} \|e^{-tA}\|,$$

and it is crucial that if $g \in BM$, then by (2.3) a much better estimate is available:

$$\|g(A)\| \leq g(0) \sup_{t \geq 0} \|e^{-tA}\|.$$

The basic properties of the Hille-Phillips functional calculus can be found in [37, Chapter XV]. For a general approach to functional calculi, including the HP-calculus, see [35].

By a regularization procedure, the HP-calculus admits an extension to a class of functions much larger than $A^1_+(\mathbb{C}_+)$. More precisely, if $f : \mathbb{C}_+ \mapsto \mathbb{C}$ is holomorphic such that there exists $e \in A^1_+(\mathbb{C}_+)$ with $ef \in A^1_+(\mathbb{C}_+)$ and the operator $e(A)$ is injective, then one sets

$$\text{dom}(f(A)) := \{ x \in X : (ef)(A)x \in \text{ran}(e(A)) \}$$

$$f(A) := e(A)^{-1}(ef)(A).$$

In this case $f$ is called regularizable, and $e$ is called a regularizer for $f$. Such a definition of $f(A)$ does not depend on the regularizer $e$ and $f(A)$ is a closed
operator on $X$. Moreover, the set of all regularizable functions $f$ forms an algebra $\mathcal{A}$ (depending on $A$). The procedure gives rise to the mapping

$$A \ni f \mapsto f(A)$$

from $\mathcal{A}$ into the set of all closed operators on $X$ is called the extended Hille–Phillips (HP-) functional calculus for $A$. The calculus possess a number of natural properties, and we refer to [35, Chapters 1 and 3] for their exhaustive treatment.

In this paper, the next product rule will be important: if $f$ is regularizable and $g \in A_+^1(\mathbb{C}_+)$, then

$$g(A)f(A) \subseteq f(A)g(A) = (fg)(A),$$

where products of operators are considered on their natural domains. If $f \in A_+^1(\mathbb{C}_+)$, then $f(A)$ is bounded and clearly $g(A)f(A) = (fg)(A)$. The rule will be mostly used for $f(z) = z^\alpha, \alpha > 0$, that are regularizable (e.g. by a regularizer $e(z) = (z+1)^{-\alpha}$) and thus belong to the extended HP-calculus, see e.g. [32, p. 3060] and [35, Chapter 3]. Remark that fractional powers $A^\alpha$ defined in this way coincide with the fractional powers defined by means of the widely used extended holomorphic functional calculus developed e.g. in [35, Section 3]. Thus several properties of fractional powers, well-known within the holomorphic functional calculus, will be used here without a special reference.

The following spectral mapping theorem for the HP functional calculus will also be crucial. See e.g. [37, Theorem 16.3.5] or [30, Theorem 2.2] for its proof.

**Theorem 2.1.** Let $g \in A_+^1(\mathbb{C}_+)$ and $-A$ be the generator of a bounded $C_0$-semigroup on a Banach space $X$. Then

$$\{g(\lambda) : \lambda \in \sigma(A)\} \subset \sigma(g(A)).$$

Since holomorphic semigroups will receive a special attention in our studies it would be helpful to recall some of their basic properties. If a $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on $X$ extends holomorphically to a sector $\Sigma_\theta = \{\lambda \in \mathbb{C} : |\arg \lambda| < \theta\}$ for some $\theta \in (0, \frac{\pi}{2}]$ and $e^{-tA}$ is bounded and strongly continuous in $\Sigma_\xi$ whenever $0 < \xi < \theta$, then $(e^{-tA})_{t \geq 0}$ is said to be a sectorially bounded holomorphic $C_0$-semigroup (of angle $\theta$). It is well-known that sectorially bounded holomorphic semigroups can be characterized in terms of their behavior on the real axis. Namely, $-A$ is the generator of a sectorially bounded holomorphic $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on $X$ if and only if $e^{-tA}(X) \subset \text{dom}(A), t > 0$, and

$$\sup_{t \geq 0} \|e^{-tA}\| < \infty \quad \text{and} \quad \sup_{t > 0} \|tAe^{-tA}\| < \infty,$$
see e.g. [25, Theorem 4.6]. Moreover, if \((e^{-tA})_{t \geq 0}\) is a sectorially bounded holomorphic \(C_0\)-semigroup, then for any \(\beta > 0\) one has
\[
\sup_{t > 0} \|t^{\beta} A^\beta e^{-tA}\| < \infty,
\]
see e.g. [35, Proposition 3.4.3]. The theory of \(C_0\)-semigroups, especially holomorphic ones, could essentially be put into the framework of functional calculi. However, we preferred to follow a more conventional treatment here.

3. SOME CLASSES OF BOUNDED COMPLETE MONOTONE FUNCTIONS AND THEIR ESTIMATES

To be able to “insert” an operator into relations such as (1.4) and to obtain norm bounds for the corresponding operator expressions, we will need a number of estimates for completely monotone functions. Then the HP-calculus converts them into semigroup approximation formulas with rates. Thus this section provides a function-theoretic background for subsequent functional calculi arguments.

3.1. Estimates of functions for first order approximations. To realize a functional calculus approach, we start with introducing a subclass \(\mathcal{B}_1\) of normalized bounded completely monotone functions:
\[
\mathcal{B}_1 := \{ g \in \mathcal{B}M : g(0) = 1, \quad g'(0) = -1 \}.
\]
From the definition it follows that if \(g \in \mathcal{B}_1, g \sim \nu\), then
\[
g(0) = \int_0^\infty \nu(ds) = 1, \quad -g'(0) = \int_0^\infty s \nu(ds) = 1,
\]
and for every \(z > 0\),
\[
0 \leq g(z) \leq 1, \quad -1 \leq g'(z) \leq 0.
\]
The latter inequality implies that for each \(g \in \mathcal{B}_1\),
\[
1 - g(z) \leq z, \quad z > 0.
\]
Given \(g \in \mathcal{B}_1\) define a sequence \((g_n)_{n \geq 1}\), one of the main objects of our studies, by
\[
g_n(z) := g^n(z/n), \quad z > 0,
\]
and note that \((g_n)_{n \geq 1} \subset \mathcal{B}_1\).

For sharp norm estimates of operator functions, the following subclass \(\mathcal{B}_2\) of \(\mathcal{B}_1\) will be useful:
\[
\mathcal{B}_2 := \{ g \in \mathcal{B}_1 : g''(0) < \infty \}.
\]
Note that, if \(g \in \mathcal{B}_2, g \sim \nu\), then
\[
\int_0^\infty (s - 1)^2 \nu(ds) = g''(0) - 1,
\]
hence \( g''(0) \geq 1 \) and \( g''(0) = 1 \) if and only if \( g(z) = e^{-z} \). Clearly, if \( g \in \mathcal{B}_2 \) then \((g_n)_{n \geq 1} \subset \mathcal{B}_2\). Moreover,

\[
(3.7) \quad g''_n(0) = 1 + \frac{g''(0) - 1}{n}.
\]

The next statement is similar to [32, Proposition 4.4], where we considered completely monotone functions \( g \) of the form \( g = e^{-\varphi} \), with a normalized Bernstein function \( \varphi \). It is fundamental in relating the semigroup approximation to the HP-calculus. While a result more general than (3.8) can be found in [55, Theorem C] (the reference, which was not known to the authors of [32] at the time of writing that paper), the statement below suffices for our purposes, and we provide it with an independent simple proof. For \( g \in \mathcal{C}\mathcal{M} \) and \( \alpha \in [0,2] \) define

\[
\Delta^g_\alpha(z) := \frac{g(z) - e^{-z}}{z^\alpha}, \quad z > 0.
\]

**Lemma 3.1.** Let \( g \in \mathcal{B}_1 \), \( g \sim \nu \). Then \( \Delta^g_2 \in \mathcal{C}\mathcal{M} \). Moreover

\[
(3.8) \quad \Delta^g_2(z) = \int_0^\infty e^{-zs} G(s) \, ds, \quad z > 0,
\]

where \( G \) is a function continuous on \( \mathbb{R}_+ \) and given by

\[
(3.9) \quad G(s) = \begin{cases} 
\int_0^s (s - \tau) \nu(d\tau), & s \in [0,1], \\
\int_0^\infty (\tau - s) \nu(d\tau), & s > 1.
\end{cases}
\]

If \( g \in \mathcal{B}_2 \), then

\[
(3.10) \quad \|\Delta^g_2\|_{A^1_+} = \Delta^g_2(0) = \frac{g''(0) - 1}{2}.
\]

**Proof.** First we prove the representation (3.8). Taking into account (2.11), we have for \( z > 0 \):

\[
\Delta^g_2(z) = z^{-2} \Delta^g_0(z) = \int_0^\infty e^{-z\tau} \tau \nu(d\tau) \int_0^\infty e^{-zs} \nu(ds) - \int_1^\infty e^{-zs}(s - 1) \, ds = \int_0^\infty e^{-zs} H(s) \, ds,
\]

where

\[
H(s) = \int_0^s (s - \tau) \nu(d\tau) - (s - 1)1_{[0,\infty)}(s - 1).
\]
It is clear that if \( s \leq 1 \) then \( H(s) = G(s) \). We prove that \( H(s) = G(s) \) for \( s > 1 \) as well. If \( s > 1 \), then using (3.2), we infer that

\[
H(s) = s \left( \int_0^s \nu(d\tau) - 1 \right) + 1 - \int_0^s \tau \nu(d\tau)
\]

\[
= -s \int_{s+}^{\infty} \nu(d\tau) + \int_{s+}^{\infty} \tau \nu(d\tau)
\]

\[
= \int_s^{\infty} (\tau - s) \nu(d\tau),
\]

hence \( H(s) = G(s) \). Thus (3.8) holds. Since \( G(s) > 0, s > 0 \), the function \( \Delta_2^g \) is completely monotone. The continuity of \( G \) at \( s = 1 \) follows from the identity (see (3.2))

\[
\int_{s+}^{\infty} \nu(d\tau) = \int_0^{\infty} \tau \nu(d\tau),
\]

so that

\[
G(1) = \int_0^1 (1 - \tau) \nu(d\tau) = \int_1^{\infty} (\tau - 1) \nu(d\tau) = \lim_{s \to 1^+} G(s),
\]

while the continuity of \( G \) at other points of \([0, \infty)\) is obvious.

To prove (3.10), taking into account \( \Delta_2^g \in \mathcal{CM} \) and applying Fatou’s lemma and de l’Hôpital’s rule twice, we obtain:

\[
\| \Delta_2^g \|_{\Lambda_+^1(\mathbb{C}_+)} = \lim_{z \to 0^+} \frac{g(z) - e^{-z}}{z^2} = \frac{g''(0)}{2} - 1.
\]

\[\square\]

It is useful to note several elementary properties of the density \( G \) defined above. In particular, \( G \) is non-decreasing on \([0, 1]\) and non-increasing on \( s \geq 1 \). Moreover, \( G(0) = 0 \),

(3.11) \( \max\{G(s) : s > 0\} = G(1) = \int_0^1 (1 - \tau) \nu(d\tau) \leq 1 \),

and

(3.12) \( \lim_{s \to \infty} G(s) \leq \lim_{s \to \infty} \int_s^{\infty} \tau \nu(d\tau) = 0 \).

The latter property implies

(3.13) \( \lim_{\delta \to 0} \delta \int_0^{\infty} e^{-\delta s} G(s) ds = 0 \).

by the regularity of Abel’s summation. Finally, if \( g \in \mathcal{B}_2 \), then \( \Delta_2^g \in \mathcal{L}^1(0, \infty) \), so the corresponding \( G \) satisfies

\[
\int_0^{\infty} \frac{G(s) ds}{s} < \infty
\]

by Fubini’s theorem.
Remark 3.2. Lemma 3.1 yields several simple but useful bounds for $g - e^{-z}, g \in B_1$. Since $g(z) \leq 1$ for $z \geq 0$, from Lemma 3.1 it follows that

$$0 \leq g(z) - e^{-z} \leq 1, \quad z \geq 0.$$ (3.14)

Since $g'(z) \leq 0, z \geq 0$, we also have

$$g(z) - e^{-z} = \int_0^z (g'(s) + e^{-s}) \, ds \leq z, \quad z \geq 0.$$ (3.15)

Furthermore, from (3.14) and (3.15), we conclude that

$$0 \leq g(z) - e^{-z} \leq \min \{1, z\} \leq \frac{2z}{z + 1}, \quad z \geq 0.$$ (3.16)

The next proposition shows that the sequence $(g_n)_{n \geq 1}$ defined by (3.5) may serve as an approximation of the exponent and justifies our functional calculus approach to the semigroup approximation.

**Proposition 3.3.** Let $g \in B_1$. Then for all $t > 0$ and $n \in \mathbb{N}$,

$$0 \leq g_n(t) - e^{-t} \leq t(1 + g'(t/n)),$$ (3.17)

and, in particular,

$$\lim_{n \to \infty} g_n(t) = e^{-t},$$ (3.18)

uniformly in $t$ from compact subsets of $[0, \infty)$.

**Proof.** Consider the quotients $g_n(t)/e^{-t}, n \in \mathbb{N}$. Using (3.3), (3.4) and the monotonicity of $1 + g'$ and $g$, we have

$$\log \frac{g^n(t/n)}{e^{-t}} = n \log g(t/n) + t/n = n \int_0^{t/n} \frac{g(s) + g'(s)}{g(s)} \, ds \leq n \int_0^{t/n} \frac{1 + g'(s)}{g(s)} \, ds \leq \frac{t}{g(t/n)} (1 + g'(t/n)).$$

Then, since

$$t - s \leq t \log(t/s), \quad 0 < s < t,$$

we obtain

$$0 \leq g^n(t/n) - e^{-t} \leq t(1 + g'(t/n)).$$

Remark 3.4. If $g \in B_2$, then the inequalities $1 + g'(t) \leq g''(0)t$, $t \geq 0$, and (3.17) imply that for all $n \in \mathbb{N}$ and $t \geq 0$,

$$0 \leq g_n(t) - e^{-t} \leq g''(0) \frac{t^2}{n}.$$ (3.19)

Moreover, by (3.10), we have a slightly stronger inequality:

$$0 \leq g_n(t) - e^{-t} \leq (g''(0) - 1) \frac{t^2}{2n}, \quad n \in \mathbb{N}, \quad t \geq 0.$$
We proceed with an auxiliary statement similar to Lemma 3.1. While Lemma 3.1 will be useful in the study of approximations for general $C_0$-semigroups, Lemma 3.5 below will play the same role for approximations of holomorphic $C_0$-semigroups.

**Lemma 3.5.** For every $g \in B_1$ let $s_g(z) := z^{-1}(g(z) + g'(z))$, $z > 0$. Then $s_g \in CM$. If moreover $g \in B_2$, then $s_g \in A_+^1(C_+)$ and
\[(3.20)\]
\[\|s_g\|_{A_+^1(C_+)} = g''(0) - 1.\]

**Proof.** The fact that $s_g \in CM$ for $g \in B_1$ follows from [55, Theorem A]. Again in our particular case, the argument is quite simple and we give it below. Let $g \sim \nu$. Using Fubini’s theorem, we have
\[
\frac{g(z) + g'(z)}{z} = \int_0^\infty e^{-zs} \int_0^s \nu(d\tau) ds - \int_0^\infty e^{-zs} \int_0^s \tau \nu(d\tau) ds
\]
\[= \int_0^\infty e^{-zs} G_0(s) ds, \quad z > 0,
\]
where
\[G_0(s) = \int_0^s (1 - \tau) \nu(d\tau), \quad s \in [0,1],
\]
and, by (3.2),
\[G_0(s) = \int_s^\infty (\tau - 1) \nu(d\tau) \geq 0, \quad s > 1.
\]
Thus $G_0$ is positive on $[0,\infty)$ and $s_g \in CM$ by Bernstein’s theorem. If $g \in B_2$, then (3.20) follows from Fatou’s Lemma and L’Hôpital’s rule. \hfill \qed

As a straightforward implication of Lemma 3.5 observe that for $g \in B_1$ one has
\[g(z) + g'(z) \geq 0, \quad z > 0.
\]
If moreover $g \in B_2$, then
\[g(z) + g'(z) \leq (g''(0) - 1)z, \quad z \geq 0.
\]

Let us now introduce a functional $L$ on $BM$ crucial for operator-norm estimates via the HP-calculus. For $g \in BM$, $g \sim \nu$, define
\[(3.21)\]
\[L[g] := \int_0^1 (1 - s) \nu(ds).
\]

The following statement clarifies the meaning of $L$.

**Proposition 3.6.** If $g \in B_1$, $g \sim \nu$, then $\Delta^g_1 \in A_+^1(C_+)$ and
\[(3.22)\]
\[\|\Delta^g_1\|_{A_+^1(C_+)} = 2L[g].\]

If $g \in B_2$, then
\[(3.23)\]
\[\|\Delta^g_1\|_{A_+^1(C_+)} \leq (g''(0) - 1)^{1/2},
\]
and

\begin{equation}
\| \Delta_1^g \|_{A^1_+} \leq \left( \frac{g''(0) - 1}{n} \right)^{1/2}.
\end{equation}

**Proof.** Note that

\[ \Delta_1^0(z) = z^{-1} \Delta_0^0(z) = \int_0^\infty e^{-zs} \int_0^s \nu(d\tau) \, ds - \int_1^\infty e^{-zs} \, ds. \]

Hence

\[ \| \Delta_1^0 \|_{A^1_+} = \int_1^s \int_0^s \nu(d\tau) \, ds + \int_1^\infty (1 - \int_0^s \nu(d\tau)) \, ds = \int_0^\infty |1 - \tau| \nu(d\tau) = 2 \int_0^1 (1 - \tau) \nu(d\tau), \]

i.e. (3.22) holds.

In view of (3.6) and (3.2), the estimate (3.23) follows from (3.22) by Cauchy’s inequality.

To get the convergence rates on the domains of fractional powers of (negative) semigroup generators, we will need the next interpolation result which is a partial case of [32, Lemma 4.1].

**Lemma 3.7.** Let \( F \in A^1_+(\mathbb{C}_+) \) be such that \( z^{-1} F \in A^1_+(\mathbb{C}_+) \) and

\[ \| F \|_{A^1_+} = a \quad \text{and} \quad \| z^{-1} F \|_{A^1_+} = b. \]

Then for every \( \alpha \in [0, 1] \) one has \( z^{-\alpha} F \in A^1_+(\mathbb{C}_+) \) and

\begin{equation}
\| z^{-\alpha} F \|_{A^1_+} \leq 2^{1+\alpha} a^{1-\alpha} b^\alpha.
\end{equation}

Combining now Lemma 3.7 with Proposition 3.6 we estimate \( \Delta_1^g \) with \( \alpha \) from either \([0, 1]\) or \([0, 2]\) depending on the smoothness of \( g \) at zero.

**Corollary 3.8.** If \( g \in B_1 \), then for every \( \alpha \in [0, 1] \),

\begin{equation}
\| \Delta_1^g \|_{A^1_+} \leq 8(L[g])^\alpha.
\end{equation}

If moreover \( g \in B_2 \), then for every \( \alpha \in [0, 2] \),

\begin{equation}
\| \Delta_1^g \|_{A^1_+} \leq 4 \left( g''(0) - 1 \right)^{\frac{\alpha}{2}}.
\end{equation}

**Proof.** First, let \( \alpha \in [0, 1] \) be fixed and \( g \in B_1 \). Observe that

\begin{equation}
\| \Delta_1^g \|_{A^1_+} \leq \| g \|_{A^1_+} + \| e^{-z} \|_{A^1_+} = 2,
\end{equation}

and \( \| \Delta_1^g \|_{A^1_+} = 2L[g] \) by Corollary 3.6. Then, using Lemma 3.7 with \( F = \Delta_0^g \), we obtain that

\[ \| \Delta_1^g \|_{A^1_+} \leq 4 \| \Delta_0^g \|_{A^1_+}^\alpha. \]

Thus, for \( \alpha \in [0, 1] \), (3.22) implies (3.26), and (3.27) follows from (3.23).
Next, if \( \alpha \in (1, 2] \) and \( g \in B_2 \), then (3.23), (3.10), and Lemma 3.7 with \( F = \Delta_1^g \) yield

\[
\| \Delta_1^g \|_{A_1^c(C_+)} = \| z^{-\alpha+1} \Delta_1^g \|_{A_1^c(C_+)} \\
\leq 2^\alpha \left( g''(0) - 1 \right)^{(2-\alpha)/2} \left( \frac{g''(0) - 1}{2} \right)^{\alpha-1} \\
= 2 \left( g''(0) - 1 \right)^{\frac{\alpha}{2}}.
\]

\( \square \)

The estimate in (3.26) looks rather implicit. So we proceed with giving explicit bounds for \( L[g] \) when \( g \in B_1 \).

**Lemma 3.9.** For every \( g \in B_1 \) one has

(3.29) \quad L[g] \leq \left( \frac{(1 + g'(1)) \int_0^1 g(s) \, ds}{(1 - g(1))^2} - 1 \right)^{1/2} + 2e \int_0^1 (g(s) + g'(s)) \, ds.

**Proof.** Let \( g \in B_1 \), \( g \sim \nu \). Set

\[
\beta = \int_0^1 g(s) \, ds, \quad \gamma = 1 - g(1),
\]

and define

\[
\tilde{g}(z) = \frac{1}{\beta} \int_{\beta/\gamma}^{\beta + 1} g(s) \, ds, \quad z > 0.
\]

We will estimate \( L[g] \) in terms of \( L[\tilde{g}], \beta \), and \( \gamma \). Then the bound (3.29) will follow easily.

Note that \( 0 < \gamma \leq \beta \leq 1 \), and

(3.30) \quad \tilde{g} \in B_2, \quad \tilde{g}''(0) = \frac{\beta(1 + g'(1))}{\gamma^2}.

Moreover, we have

\[
\tilde{g}(z) = \frac{1}{\beta} \int_0^\infty e^{-(\beta/\gamma)zs} \left( \frac{1 - e^{-s}}{s} \right) \nu(ds) = \int_0^\infty e^{-zs} \tilde{v}(ds), \quad z > 0,
\]

where

\[
\tilde{v}(ds) = \left( \frac{1 - e^{-((\gamma/\beta)s)}}{\gamma s} \right) \nu((\gamma/\beta) ds).
\]

Since \( \beta \leq 1 \),

\[
L[\tilde{g}] = \int_0^1 (1 - s) \left( \frac{1 - e^{-(\gamma/\beta)s}}{\gamma s} \right) \nu((\gamma/\beta) ds) \\
= \int_0^{\gamma/\beta} \left( 1 - s \frac{\beta}{\gamma} \right) \left( \frac{1 - e^{-s}}{\beta s} \right) \nu(ds) \\
\geq \int_0^{\gamma/\beta} \left( 1 - s \frac{\beta}{\gamma} \right) \left( \frac{1 - e^{-\beta s}}{\beta s} \right) \nu(ds),
\]

where
hence

\[ L[g] - L[\tilde{g}] \leq \int_0^1 (1-s) \nu(ds) - \int_0^{\gamma/\beta} \left( 1 - s \frac{\beta}{\gamma} \right) \frac{(1 - e^{-s})}{\beta s} \nu(ds) \]

\[ = \int_0^{\gamma/\beta} (1-s) \nu(ds) + \int_0^{\gamma/\beta} Q(s) \nu(ds), \]

where

\[ Q(s) = (1-s) - \left( 1 - s \frac{\beta}{\gamma} \right) \frac{(1 - e^{-s})}{\beta s} \]

\[ = \left( \frac{\beta}{\gamma} - 1 \right) \frac{(1 - e^{-s})}{\beta} + (1-s) \left\{ 1 - \frac{(1 - e^{-s})}{\beta s} \right\}. \]

Observe now that for every \( s \in [0, \gamma/\beta] \),

\[ Q(s) \leq \left( \frac{\beta}{\gamma} - 1 \right) s + (1-s) \frac{\beta s}{2} \leq \left( 1 - \frac{\gamma}{\beta} \right) + \frac{(1-s)}{2}. \]

Thus

\[ L[g] - L[\tilde{g}] \leq \left( 1 - \frac{\gamma}{\beta} \right) + \frac{1}{2} L[g], \]

and then, in view of (3.6) and (3.23),

\begin{equation}
(3.31) \quad L[g] \leq 2L[\tilde{g}] + 2 \left( 1 - \frac{\gamma}{\beta} \right)
\end{equation}

\[ \leq \left( \int_0^\infty (1-s)^2 \nu(ds) \right)^{1/2} + 2 \left( 1 - \frac{\gamma}{\beta} \right)
\]

\[ = (\tilde{g}''(0) - 1)^{1/2} + 2 \left( 1 - \frac{\gamma}{\beta} \right). \]

Since, recalling the definition of \( \gamma \) and \( \beta \),

\[ 1 - \frac{\gamma}{\beta} = \frac{\int_0^1 g(s) ds - 1 + g(1)}{\int_0^1 g(s) ds} \leq \frac{\int_0^1 g(s) ds - 1 + g(1)}{g(1)} \]

\[ \leq e \int_0^1 (g(s) + g'(s)) ds, \]

the statement follows from (3.31) and (3.30). \( \square \)

Now we are able to describe the convergence rate for \((g_n)_{n \geq 1}\) in (3.18) by means of the boundary behavior of \(g'\) at zero.

**Corollary 3.10.** Let \( g \in B_1 \). Then for all \( n \in \mathbb{N} \),

\begin{equation}
(3.32) \quad L[g_n] \leq 2e \left( 1 + \frac{1}{|g'(1/n)|} \right) \sqrt{1 + g'(1/n)}. \end{equation}

**Remark 3.11.** It is instructive to note that by (3.1) for each \( g \in B_1 \) one has \( \lim_{n \to \infty} (1 + g'(1/n)) = 0 \).
Proof. First observe that for every \( n \in \mathbb{N} \),
\[
1 - g_n(1) = \int_0^1 g_n^{-1}(s/n)|g'(s/n)| \, ds \geq e^{-1}|g'(1/n)|.
\]
Next, using the inequalities (3.17) for \( t = 1 \) and an obvious bound \( g \leq 1 \), we obtain
\[
(1 + g_n'(1/n)) \int_0^1 g_n(s) \, ds - (1 - g_n(1))^2
\]
\[
=g_n^{-1}(1/n)[g(1/n) + g'(1/n)] \int_0^1 g^n(s/n) \, ds
\]
\[
+ (1 - g^n(1/n)) \left( \int_0^1 g^n(s/n) \, ds - (1 - g^n(1/n)) \right)
\]
\[
\leq [1 + g'(1/n)] + (1 - e^{-1}) \left( \int_0^1 s(1 + g'(s/n)) \, ds + g^n(1/n) \right) - e^{-1}
\]
\[
\leq (1 + 3(1 - e^{-1})/2)(1 + g'(1/n))
\]
\[
\leq 2(1 + g'(1/n)), \quad n \in \mathbb{N}.
\]
So, by (3.29), for every \( n \in \mathbb{N} \),
\[
L[g_n] \leq e\sqrt{2\left(1 + g'(1/n)\right)^{1/2}} + 2e(1 + g'(1/n)),
\]
and (3.32) follows. \( \square \)

Thus if \( g''(0) = \infty \) then the convergence rate in Corollary 3.10 depends on the behavior of \( g \) in a neighborhood of zero. This is in contrast to Proposition 3.6 where the assumption \( g''(0) < \infty \) yields a rate estimate independent of \( g \) (up to a constant). The rates coincide when the latter condition holds, and they are optimal in this case, see Proposition 4.11.

We will use Corollary 3.10 to characterize polynomial decay rates for \( \|\Delta^m \|_{A^1_{+}(C_+)} \). To this aim the next lemma will be crucial.

Lemma 3.12. Let \( f \in CM \), \( f \sim \mu \), and
\[
\int_0^1 f(s) \, ds < \infty.
\]
If \( \gamma \in (0,1) \), the the following conditions are equivalent.

(i) There exists \( c_1 > 0 \) such that \( f(\tau) \leq c_1 \tau^{-1-\gamma} \), \( \tau \in (0,1) \).

(ii) There exists \( c_2 > 0 \) such that \( \int_0^s \mu(\tau) \, d\tau \leq c_2 s^{1-\gamma} \), \( s \geq 1 \).

(iii) There exists \( c_3 > 0 \) such that \( \int_0^{1/n} f(\tau) \, d\tau \leq c_3 n^{-\gamma} \), \( n \in \mathbb{N} \).

Proof. (i) \( \implies \) (ii): Since for every \( s \geq 1 \),
\[
\int_0^s \mu(\tau) \, d\tau \leq e \int_0^\infty e^{-\tau/s} \mu(\tau) = ef(1/s) \leq ec_1 s^{-\gamma},
\]
the implication follows.
(ii) \implies (iii) : By (3.33),
\[
\int_0^\infty \frac{\mu(d\tau)}{1 + \tau} < \infty,
\]
therefore
\[
(3.34) \quad \lim_{s \to \infty} \frac{1 - e^{-s/n}}{s} \leq \frac{2}{n + s}, \quad s > 0, \quad n \in \mathbb{N},
\]
hence, using (3.34) and integrating by parts, we obtain for every \(n \in \mathbb{N}\) and some \(c_2 > 0\):
\[
\int_0^{1/n} f(\tau) d\tau = \int_0^\infty \frac{1 - e^{-s/n}}{s} \mu(ds) \leq 2 \int_0^\infty \frac{\mu(ds)}{n + s} \leq c_2 \int_0^\infty \frac{s^{1-\gamma} ds}{(n + s)^2} = c_2 \frac{\pi(1-\gamma)}{\sin(\pi\gamma)n^{-\gamma}}.
\]

(iii) \implies (i) : If \(t \in ((n+1)^{-1}, n^{-1}), n \in \mathbb{N}\), then as \(f\) is non-increasing there is \(c_3 > 0\) such that for every \(\tau \in (0, 1]\),
\[
f(\tau) \leq f((n+1)^{-1}) \leq (n+1) \int_0^{1/n} f(s) ds \leq c_3(n+1)^{1-\gamma}2c_3\tau^{1-\gamma}.
\]

\[\square\]

Corollary 3.13. Let \(g \in B_1, g \sim \nu\). Then the following conditions are equivalent.

(i) There exists \(c_1 > 0\) such that \(g''(\tau) \leq c_1\tau^{1-\gamma}, \quad \tau \in (0, 1]\).

(ii) There exists \(c_2 > 0\) such that \(\int_0^s \tau^2 \mu(d\tau) \leq c_2 s^{1-\gamma}, \quad s \geq 1\).

(iii) There exists \(c_3 > 0\) such that \(1 + g'(1/n) \leq c_3n^{-\gamma}, \quad n \in \mathbb{N}\).

Proof. The statement follows from Lemma 3.12 applied to \(f = g''\). It suffices to note that \(g'' \in \mathcal{C}\mathcal{M}\), \(g'' \sim \tau^2 \nu(d\tau)\), \(g''\) is integrable on \([0, 1]\), and
\[
1 + g'(1/n) = \int_0^{1/n} g''(\tau) d\tau, \quad n \in \mathbb{N}.
\]

\[\square\]

Observe that if
\[
g(z) := (1 - \gamma) + \gamma(\gamma + 1) \int_0^\infty \frac{e^{-zs} ds}{(s + 1)^{2+\gamma}}, \quad \gamma \in (0, 1),
\]
then \(g \sim \nu\),
\[
\nu(ds) := (1 - \gamma)\delta_0 + \gamma(\gamma + 1) \frac{ds}{(s + 1)^{2+\gamma}}.
\]
so \( g \in B_1 \setminus B_2 \). On the other hand, by Corollary 3.13
\[
1 + g'(1/n) \leq cn^{-\gamma}, \quad \gamma \in (0, 1),
\]
for some \( c > 0 \).

### 3.2. Estimates of functions for higher order approximations and constants.

To treat higher order approximations for bounded \( C_0 \)-semigroups and obtain sharp estimates of constants in our approximation formulas, we will need several classes of completely monotone functions finer than \( B_1 \) and \( B_2 \). For \( k = 3, 4 \), let
\[
B_k = \{ g \in B_2 : |g^{(k)}(0)| < \infty \},
\]
\[
B_{k, \infty} = \left\{ g \in B_k : \int_1^\infty \frac{g(s)}{s} \, ds < \infty \right\}.
\]
Note that if \( g \in B_k \) belongs also to \( L^m(0, \infty), m \in \mathbb{N} \), then \( g \in B_{k, \infty} \). The classes \( B_k, k \in \mathbb{N} \), have been mentioned briefly in [43].

For \( g \in B_2 \) let the function \( G \) be given by Lemma 3.1. For the formulations of our higher order approximation results, it will be convenient to introduce the following functionals:
\[
a[g] := \int_0^\infty G(s) \, ds, \quad g \in B_2,
\]
\[
b[g] := \int_0^\infty (1 - s)G(s) \, ds, \quad g \in B_3.
\]
Note that for each \( g \in B_2 \),
\[
a[g] = \Delta^g_2(0) = \frac{g''(0)}{2} - 1,
\]
and, since for \( g \in B_3 \),
\[
\lim_{z \to 0^+} \frac{d}{dz} \Delta^g_2(z) = \frac{g'''(0) + 1}{6},
\]
we have
\[
b[g] = \Delta^g_2(0) + \lim_{z \to 0^+} \frac{d}{dz} \Delta^g_2(z) = \frac{3g''(0) + g'''(0) - 2}{6}.
\]

In order to obtain optimal bounds for higher order approximations of holomorphic semigroups, we will need yet another family of auxiliary functionals on \( B_{2, \infty} \).

For \( \alpha \in [0, 1] \) and \( g \in B_{2, \infty} \) define
\[
c_\alpha[g] := \frac{1}{\Gamma(2 - \alpha)} \int_0^\infty \Delta^g_{1+\alpha}(z) \, dz,
\]
where \( \Gamma \) is the Gamma-function:
\[
\Gamma(z) := \int_0^\infty t^{z-1}e^{-t} \, dt, \quad z > 0.
\]
Note that for each \( g \in \mathcal{B}_{2,\infty} \) one has \( c_\alpha[g] < \infty \), so that the mapping \( c_\alpha : \mathcal{B}_{2,\infty} \to [0, \infty) \) is well-defined. We are interested in sharp estimates of \( c_\alpha[g] \) in terms of \( g \), and several representations for \( c_\alpha \) given below will serve just that purpose.

The following proposition expresses \( c_\alpha \) in terms of the function \( G \) corresponding to \( g \) by Lemma 3.1.

**Proposition 3.14.** If \( g \in \mathcal{B}_{2,\infty} \), then for every \( \alpha \in [0, 1] \),
\[
(3.39) \quad c_\alpha[g] = \int_0^\infty \frac{G(s)}{s^{2-\alpha}} \, ds,
\]
where \( G \) is given by Lemma 3.1.

**Proof.** Observe that \( g \) admits the representation (3.8). Since \( \Delta_{1+\alpha}^0(z) = z^{1-\alpha} \Delta_2^0(z) \), taking into account the positivity of \( G(s) \) and using Fubini’s theorem, we infer from (3.8) that
\[
\int_0^\infty \Delta_{1+\alpha}^0(z) \, dz = \int_0^\infty G(s) \int_0^\infty z^{1-\alpha} e^{-zs} \, dz \, ds = \Gamma(2-\alpha) \int_0^\infty G(s) s^{\alpha-2} \, ds.
\]
\[\square\]

Similarly, in view of the boundedness of \( G \) on \((0, \infty)\), we obtain the following assertion.

**Proposition 3.15.** Let \( g \in \mathcal{B}_1 \). Then for all \( \delta > 0 \) and \( \alpha \in (0, 1) \),
\[
(3.40) \quad \frac{1}{\Gamma(1-\alpha)} \int_0^\infty \frac{\Delta_1^0(z + \delta)}{z^\alpha} \, dz = \int_0^\infty e^{-\delta s} \frac{G(s)}{s^{1-\alpha}} \, ds.
\]

Now we relate the functionals \( c_0 \) and \( c_1 \) to the mapping \( g \to s_g \) introduced in Lemma 3.5.

**Lemma 3.16.** Let \( g \in \mathcal{B}_{2,\infty} \). Then
\[
(3.41) \quad c_0[g] + c_1[g] = \int_0^\infty s_g(z) \, dz,
\]
and for every \( \alpha \in (0, 1) \),
\[
(3.42) \quad c_\alpha[g] \leq (1-\alpha)c_0[g] + \alpha c_1[g].
\]

**Proof.** Integrating by parts and using (3.39), we obtain
\[
c_0[g] = \int_0^\infty g'(z) + e^{-z} \, dz = \int_0^\infty \frac{g'(z) + g(z)}{z} \, dz - c_1[g],
\]
i.e. (3.41) holds.

From (3.39), positivity of \( G \) and Hölder’s inequality it follows that
\[
c_\alpha[g] \leq (c_0[g])^{1-\alpha} \cdot (c_1[g])^\alpha.
\]
Then a standard convexity argument implies (3.42). \[\square\]
We finish this section with the study of integrability properties for completely monotone functions. They will be required in the study of approximations for sectorially bounded holomorphic $C_0$-semigroups. In particular, the next estimate for powers of completely monotone functions will be useful. In the following we abbreviate $L^k := L^k(0, \infty), k \in \mathbb{N}$.

**Proposition 3.17.** Let $g \in \mathcal{CM}$ be such that $g \in L^k$ for some $k \in \mathbb{N}$. Then

\begin{align}
(3.43) \quad g^k(z) & \leq \|g\|_{L^k} z^{-1}, \quad z > 0, \\
(3.44) \quad g^{k+1}(z) & \leq k \|g\|_{L^k} |g'(z)|, \quad z > 0.
\end{align}

**Proof.** Let $g$ be given by (2.1) and let $k = 1$. Then by Fubini’s theorem,

\begin{equation}
(3.45) \quad \int_0^\infty s^{-1} \nu(ds) = \int_0^\infty \int_0^\infty e^{-zs} \nu(ds) dz = \int_0^\infty g(z) dz = \|g\|_{L^1}.
\end{equation}

Since $e^{-zs} \leq 1/zs$ for positive $z$ and $s$,

$$g(z) \leq z^{-1} \int_0^\infty s^{-1} \nu(ds) = \|g\|_{L^1} z^{-1}, \quad z > 0,$$

so (3.43) holds for $k = 1$. Next, using (3.45), we have

$$[g(z)]^2 \leq \left( \int_0^\infty s^{-1} e^{-zs} \nu(ds) \right) \cdot \left( \int_0^\infty s e^{-zs} \nu(ds) \right) \leq \|g\|_{L^1} |g'(z)|,$$

for every $z > 0$, and (3.44) is true for $k = 1$ as well.

Assume now that $g \in L^k$ for some $k \in \mathbb{N}$. As $g^k \in \mathcal{CM}$ and (3.43) is valid for $k = 1$, it also holds for all $k \in \mathbb{N}$. Moreover, using (3.44) with $k = 1$, we obtain

$$g^{2k}(z) \leq \|g^k\|_{L^1} |g'(z)| = k \|g^k\|_{L^1} |g^{k-1}(z)| g'(z), \quad z > 0,$$

and (3.44) holds also for each $k \in \mathbb{N}$. \(\square\)

Note that if $g \in L^p(0, \infty)$ for some $p \geq 1$ then since $g \leq 1$ we have $g \in L^q(0, \infty)$, for all $q \geq p$. Thus the choice of integers in Proposition 3.17 does not restrict generality, and it is essentially a matter of convenience.

We will employ the preceding lemma to estimate moments of sufficiently large powers of functions from $\mathcal{B.M}$. Recall that for $\alpha > 0$ and $\beta > 0$,

\begin{equation}
(3.46) \quad \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)},
\end{equation}

and

\begin{equation}
(3.47) \quad \frac{\Gamma(z)}{\Gamma(z+\beta)} = z^{-\beta} \left[ 1 + \frac{\beta(1-\beta)}{2z} + \frac{r_\beta(z)}{z^2} \right], \quad z \geq 1,
\end{equation}

where the remainder $r_\beta$ satisfies

$$r_\beta(z) \leq C_{\beta_0}, \quad z \geq 1, \quad \beta \in [0, \beta_0],$$

see e.g. [3, Chapter 1.1].
Lemma 3.18. Suppose that \( g \in \mathcal{BM} \cap L^k \) for some \( k \in \mathbb{N} \), and \( g(0) = 1 \). Then there exists \( C(g) > 0 \) such that for all \( \beta \in [0, 1] \) and \( n \geq 2k + 1, n \in \mathbb{N} \),

\[
(3.48) \quad \int_0^{\infty} g^n(z)z^\beta \, dz \leq C(g)n^{-1-\beta}.
\]

Proof. First, fix \( \beta = 0 \) and \( n \geq k + 1 \). Using (3.44) and integrating by parts, we obtain

\[
(3.49) \quad \int_0^{\infty} g^{n-k-1}(z)g^{k+1}(z) \, dz \leq k\|g^k\|_{L^1} \int_0^{\infty} g^{n-k-1}(z)|g'(z)| \, dz = \frac{k\|g^k\|_{L^1}}{n-k}.
\]

Let now \( \beta = 1 \) and \( n \geq 2k + 1 \) be fixed. Then similarly to the above,

\[
(3.50) \quad \int_0^{\infty} g^n(z) \, dz \leq \frac{k^2\|g^k\|^2_{L^1}}{n-k} \int_0^{\infty} g^{n-2k-1}(z)|g'(z)| \, dz = \frac{k^2\|g^k\|^2_{L^1}}{(n-k)(n-2k)}.
\]

Note that \( n-k \geq \frac{1}{2n} \) and \( n-2k \geq \frac{1}{3k} \). Then, by (3.49), (3.50) and Hölder’s inequality,

\[
\int_0^{\infty} g^n(z)z^\beta \, dz \leq \left( \int_0^{\infty} g^n(z) \, dz \right)^{1-\beta} \left( \int_0^{\infty} g^n(z)z \, dz \right)^{\beta} \leq C(g)n^{-1-\beta},
\]

where

\[
C(g) = 2(3k)^\beta(\|g^k\|_{L^1})^{1+\beta} \leq 6k^3(1 + \|g^k\|_{L^1})^2.
\]

This finishes the proof. \( \square \)

Lemma 3.18 yields the asymptotics of \( (c_\alpha[g_n])_{n \geq 1} \) corresponding to \( g \) with good enough integrability properties.

Theorem 3.19. Let \( g \in \mathcal{B}_2 \cap L^k \) for some \( k \in \mathbb{N} \). Then there exists \( C(g) > 0 \) such that for all \( \alpha \in [0, 1] \) and \( n \in \mathbb{N} \),

\[
(3.51) \quad c_\alpha[g_n] = \frac{g''(0) - 1}{2n} + r_\alpha(n), \quad \text{and} \quad |r_\alpha(n)| \leq \frac{C(g)}{n^2}.
\]

The proof of the above theorem is rather technical and is postponed to Appendix 2.

Remark 3.20. Note that in an important case \( g(z) = (1+z)^{-1} \) corresponding to Euler’s approximation, we have \( g \in \mathcal{B}_2 \cap L^2 \) so that \( g \) satisfies the assumptions of Theorem 3.19. Moreover, in this case we are able to improve the bounds in (3.51). Indeed, integrating by parts and using Formula
2.2.4.24], we have for every \( n \in \mathbb{N} \):

\[
\alpha c_\alpha [g_n] = \alpha \int_0^\infty \left( \frac{1}{(1 + z/n)^n} - e^{-z} \right) \frac{dz}{z^{1+\alpha}} = \Gamma(1 - \alpha) - n^{1-\alpha} \int_0^\infty \frac{dz}{(1 + z)^{n+1} z^\alpha} = \Gamma(1 - \alpha) \left( 1 - \frac{\Gamma(n + \alpha)}{n^\alpha \Gamma(n)} \right).
\]

Hence using \( \Gamma(2 - \alpha) = (1 - \alpha)\Gamma(1 - \alpha) \), it follows that

\[
(3.52) \quad c_\alpha [g_n] = \frac{1}{\alpha(1-\alpha)} \left[ 1 - \frac{\Gamma(n + \alpha)}{n^\alpha \Gamma(n)} \right].
\]

So, passing to the limit in (3.52), we infer that

\[
c_0 [g_n] = \frac{1}{\Gamma(n)} \lim_{\alpha \to 0^+} \frac{n^\alpha \Gamma(n) - \Gamma(n + \alpha)}{\alpha} = \log n - \psi(n),
\]

\[
c_1 [g_n] = \frac{1}{\Gamma(n + 1)} \lim_{\alpha \to 1^-} \frac{n^\alpha \Gamma(n) - \Gamma(n + \alpha)}{1 - \alpha} = \psi(n + 1) - \log n = \psi(n) - \log n + \frac{1}{n},
\]

where \( \psi(z) := \Gamma'(z)/\Gamma(z) \).

Using [27] Ch. 14, § 5], we conclude that

\[
\log n - \psi(n) = \frac{1}{2n} + \frac{\theta_n}{12n^2}, \quad \theta_n \in (0, 1),
\]

and then

\[
(3.53) \quad c_0 [g_n] = \frac{1}{2n} + \frac{\theta_n}{12n^2}, \quad c_1 [g_n] = \frac{1}{2n} - \frac{\theta_n}{12n^2}
\]

for every \( n \in \mathbb{N} \). Moreover, by (3.42) and (3.53),

\[
(3.54) \quad c_\alpha [g_n] \leq \frac{1}{2n} + (1 - 2\alpha) \frac{\theta_n}{12n^2}, \quad \alpha \in [0, 1].
\]

This is in agreement with a general estimate of \( c_\alpha [g_n] \) in (3.51).

4. Approximation of bounded \( C_0 \)-semigroups

4.1. First order approximations. Now we are able to formulate our first results on approximation of bounded \( C_0 \)-semigroups \( (e^{-tA})_{t \geq 0} \) on a Banach space \( X \) via completely monotone functions of \( A \). We start with an operator norm estimate for \( \Delta_g^\alpha (A) \) with \( g \in B_1 \). The estimate is obtained by a direct application of function-theoretical bounds proved in the previous section and elementary properties of the HP-calculus. The approximation formulas with rates will follow then by a simple scaling procedure.
Theorem 4.1. Let $-A$ be the generator of a bounded $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on $X$, and let $g \in B_1$, $g \sim \nu$. Let $M := \sup_{t \geq 0} \|e^{-tA}\|$. Then

\begin{equation}
\|\Delta_0^g(A)x\| \leq 2ML[g]\|Ax\|, \quad x \in \text{dom}(A),
\end{equation}

and for every $\alpha \in (0, 1)$,

\begin{equation}
\|\Delta_0^g(A)x\| \leq 8M(L[g])^\alpha \|A^\alpha x\|, \quad x \in \text{dom}(A^\alpha).
\end{equation}

Proof. Let $\alpha \in (0, 1)$ be fixed. Clearly, for every $\delta > 0$ the operator $-(A + \delta)$ generates a bounded $C_0$-semigroup $(e^{-t(A+\delta)})_{t \geq 0}$ on $X$ such that $\sup_{t \geq 0} \|e^{-t(A+\delta)}\| \leq M$. We use the HP-calculus to estimate the norm of $(A + \delta)^{-\alpha} \Delta_0^g(A + \delta)$ for each $\delta > 0$ and then pass to the limit when $\delta \to 0^+$. Using the product rule for the HP-calculus and the estimate (3.26) for functions in $A_{1+}^1(C_+)$, we infer that

\begin{equation}
\|[(\cdot + \delta)^{-\alpha} \Delta_0^g(\cdot + \delta)](A)\| = \|(A + \delta)^{-\alpha} \Delta_0^g(A + \delta)\| \leq 8M(L[g])^\alpha
\end{equation}

for every $\delta > 0$. Since $\text{dom}(A^\alpha) = \text{dom}(A + \delta)^\alpha$ (19 Proposition 3.1.9), the above inequality implies that

\begin{equation}
\|\Delta_0^g(A + \delta)x\| \leq 8M(L[g])^\alpha \|(A + \delta)^\alpha x\|, \quad x \in \text{dom}(A^\alpha), \quad \delta > 0.
\end{equation}

Moreover, by (19 Proposition 3.1.9),

\begin{equation}
\lim_{\delta \to 0^+} (A + \delta)^\alpha x = A^\alpha x, \quad x \in \text{dom}(A^\alpha).
\end{equation}

Since

\begin{equation}
\Delta_0^g(z + \delta) - \Delta_0^g(z) = \int_0^\infty e^{-zs}(e^{-\delta s} - 1) d\nu(s),
\end{equation}

by the dominated convergence theorem,

\begin{equation}
A_{1+}^1(C_+) - \lim_{\delta \to 0^+} \Delta_0^g(\cdot + \delta) = \Delta_0^g(\cdot),
\end{equation}

and then

\begin{equation}
\lim_{\delta \to 0^+} \Delta_0^g(A + \delta)x = \Delta_0^g(A)x, \quad x \in X.
\end{equation}

Now letting $\delta \to 0^+$ in (4.3) we get (4.2).

Similarly, using (3.22) instead of (3.26), we obtain a counterpart of (4.3) for $x \in \text{dom}(A)$, and then arrive at (4.1).

As a direct implication of Theorem 4.1 and Corollary 3.10 we obtain the following statement where the value of functional $L$ at $g \in B_1$ is replaced by its estimate in terms of $g'$.
**Corollary 4.2.** Let $-A$ be the generator of a bounded $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on $X$, and let $g \in B_1$. Let $M := \sup_{t \geq 0} \|e^{-tA}\|$. If $(g_n)_{n \geq 1}$ are given by (3.5), then for all $n \in \mathbb{N}$, $t > 0$ and $x \in \text{dom}(A)$,

$$
\|g^n(tA/n)x - e^{-tA}x\| \leq 4eM \left( 1 + \frac{1}{|g'(1/n)|} \right) \sqrt{1 + g'(1/n)t}\|Ax\|.
$$

Moreover, if $\alpha \in (0, 1)$, then for all $n \in \mathbb{N}$, $t > 0$ and $x \in \text{dom}(A^\alpha)$,

$$
\|g^n(tA/n)x - e^{-tA}x\| \leq 16eM \left( 1 + \frac{1}{|g'(1/n)|} \right) (1 + g'(1/n))^{\alpha/2}t^\alpha\|A^\alpha x\|.
$$

(To formulate (4.5) we used that $a^\alpha \leq a$ for $a \geq 1$ and $\alpha \in (0, 1]$.)

**Remark 4.3.** In particular, the above statement implies that if $g \in B_1$ is such that

$$
g''(\tau) \leq c_g\tau^{-1+\gamma}, \quad \tau \in (0, 1],
$$

for a fixed $c_g > 0$ and $\gamma \in (0, 1)$, and $(g_n)_{n \geq 1}$ are given by (3.5), then for all $n \in \mathbb{N}$, $t > 0$ and $x \in \text{dom}(A)$,

$$
\|\Delta^n_0(tA)x\| \leq 4(c_g/\gamma)^{1/2}eM \left( 1 + \frac{1}{|g'(1/n)|} \right) n^{-\gamma/2}t\|Ax\|.
$$

Moreover, if $\alpha \in (0, 1)$, then for all $n \in \mathbb{N}$, $t > 0$ and $x \in \text{dom}(A^\alpha)$,

$$
\|\Delta^n_0(tA)x\| \leq 16(c_g/\gamma)^{\alpha/2}eM \left( 1 + \frac{1}{|g'(1/n)|} \right) n^{-\alpha\gamma/2}t^\alpha\|A^\alpha x\|.
$$

Note that the functions from $B_1$ satisfying (4.6) are characterized in terms of their representing measures in Corollary 3.13.

**Remark 4.4.** Recall that the modulus of continuity $\omega(\epsilon, x)$ of $e^{-A}x$ over the interval $[0, \epsilon]$, $\epsilon > 0$, is defined as

$$
\omega(\epsilon, x) := \sup\{\|e^{-tA}x - x\| : t \in [0, \epsilon]\}.
$$

Write, as usual,

$$
x = x - S_\epsilon x + S_\epsilon x, \quad S_\epsilon x := \frac{1}{\epsilon} \int_0^\epsilon e^{-sA}x \, ds,
$$

where

$$
\|x - S_\epsilon\| \leq \omega(\epsilon, x), \quad \|AS_\epsilon x\| \leq \frac{\omega(\epsilon, x)}{\epsilon}.
$$
From (4.4) it follows that
\[
\| \Delta g^n(tA)x \| \leq \| \Delta g^n(tA)(x - S_\epsilon x) \| + \| \Delta g^n(tA)S_\epsilon x \|
\]
\[
\leq 2M \| x - S_\epsilon x \|
\]
\[
+ 4M \left( 1 + \frac{1}{\| g'(1/n) \|} \right) \sqrt{1 + g'(1/n)t} \| AS_\epsilon x \|
\]
\[
\leq CM \omega(\epsilon, x) \left( 1 + t \left( 1 + g'(1/n) \right)^{1/2} \right).
\] (4.7)

Then, setting
\[
\epsilon = \epsilon_n := \frac{(1 + g'(1/n))^{1/2}}{|g'(1/n)|},
\]
in (4.7), we obtain
\[
\| \Delta g^n(tA)x \| \leq 2CM(1 + t)\omega(\epsilon_n, x), \quad x \in X, \quad t > 0.
\] (4.8)

Thus, one can interpret our approximation results in terms of the modulus continuity, but we, in general, avoid using this language in the paper. However, the notion of modulus of continuity appear to be convenient in treating the optimality issues, see the end of this section.

As an illustration of Corollary 4.2 we derive a partial generalization of one of the main results of L.-K. Chung from [17] (however only for bounded $C_0$-semigroups). For every $t > 0$ consider
\[
\varphi_t(u) := \sum_{k=0}^{\infty} a_k(t)u^k,
\] (4.9)
where
\[
a_k(t) \geq 0, \quad \sum_{k=0}^{\infty} a_k(t) = 1, \quad \sum_{k=1}^{\infty} ka_k(t) = t,
\] (4.10)
and
\[
\varphi''_t(1) = \sum_{k=2}^{\infty} k(k-1)a_k(t) < \infty.
\]

By [17, Theorem 5], if $(e^{-tA})_{t \geq 0}$ is a bounded $C_0$-semigroup on $X$ then for every $x \in X$,
\[
e^{-tA}x = \lim_{n \to \infty} [\varphi_t(n(n + A)^{-1})]^n x
\] (4.11)
uniformly in $t$ from compacts in $[0, \infty)$. The analysis of the proof of [17, Theorem 5] reveals that to show that the limit in (4.11) is uniform in $t$ the author uses the additional assumption
\[
\sup_{t \in [0,b]} \varphi''_t(1) < \infty \quad \text{for all } \quad b > 0.
\] (4.12)
However, we will not require (4.12) in the following argument. Using (4.9) and (4.10), let us define the family of completely monotone functions

$$g_t(z) = \varphi_t \left( \frac{t}{t + z} \right), \quad t > 0,$$

and note that $(g_t)_{t>0} \subset B_1$. Moreover, if

$$\lim_{u \to 1} (\varphi'_t(u) - t) = 0, \quad \text{uniformly in } t \in [a, b] \text{ for all } b > a > 0,$$

so that

$$\lim_{s \to 0} (1 + g'_t(s)) = 0, \quad \text{uniformly in } t \in [0, b] \text{ for all } b > a > 0,$$

then by (4.4) we infer that

$$\|g^n_t(tA/n)x - e^{-tA}x\| \to 0, \quad n \to \infty,$$

for every $x \in X$ uniformly in $t$ from compacts in $(0, \infty)$, where

$$g_t(tA/n) = \varphi_t(n(n + A)^{-1}), \quad t > 0.$$

Thus, (4.11) holds uniformly in $t$ from compacts in $(0, \infty)$. If one assumes that in addition (4.12) holds, then the uniformity of convergence in (4.11) can be extended to compact sets from $[0, \infty)$ (with zero included), see Remark 4.7 below.

The estimates in Corollary 4.2 can be further improved if $g \in B_2$. The following statement is again a direct implication of Theorem 4.1 and Corollary 3.10, and its proof is therefore omitted.

**Theorem 4.5.** Let $-A$ be the generator of a bounded $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on $X$, and let $g \in B_2$. Let $M := \sup_{t \geq 0} \|e^{-tA}\|$. Then for all $n \in \mathbb{N}$ and $t > 0$,

$$\|\Delta_0^g(A)x\| \leq M \frac{(g''(0) - 1)}{2} \|A^2x\|, \quad x \in \text{dom}(A^2),$$

and

$$\|\Delta_0^g(A)x\| \leq M (g''(0) - 1)^{1/2} \|Ax\|, \quad x \in \text{dom}(A).$$

Moreover, if $\alpha \in (0, 2)$, then for all $n \in \mathbb{N}$, and $t > 0$,

$$\|\Delta_0^g(A)x\| \leq 4M (g''(0) - 1)^{\alpha} \|A^\alpha x\|, \quad x \in \text{dom}(A^\alpha).$$

Let now $(g_t)_{t>0}$ be a family of functions from $B_2$. The next result allows us to formulate our approximation results for families $(g_t)_{t>0}$ rather than just for a fixed function $g$. For its proof it suffices to recall the equation (3.7) and to scale Theorem 4.5 by replacing $A$ with $tA/n$.

**Corollary 4.6.** Let $-A$ be the generator of a bounded $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on $X$, and let $(g_t)_{t>0} \subset B_2$. Let $M := \sup_{t \geq 0} \|e^{-tA}\|$. Then for all $n \in \mathbb{N}$ and $t > 0$,

$$\|g^n_t(tA/n)x - e^{-tA}x\| \leq M \frac{(g''(0) - 1)}{2} \frac{t^2}{n} \|A^2x\|, \quad x \in \text{dom}(A^2),$$

and

$$\|g^n_t(tA/n)x - e^{-tA}x\| \leq M (g''(0) - 1)^{1/2} \frac{t^2}{n} \|Ax\|, \quad x \in \text{dom}(A).$$

Moreover, if $\alpha \in (0, 2)$, then for all $n \in \mathbb{N}$, and $t > 0$,

$$\|g^n_t(tA/n)x - e^{-tA}x\| \leq 4M (g''(0) - 1)^{\alpha} \frac{t^2}{n} \|A^\alpha x\|, \quad x \in \text{dom}(A^\alpha).$$
(4.17) \[ \|g^n_t(x) - e^{-tA}x\| \leq M(g''_t(0) - 1)^{1/2} \frac{t}{\sqrt{n}} \|Ax\|, \quad x \in \text{dom}(A), \]
and
(4.18) \[ \|g^n_t(x) - e^{-tA}x\| \leq 4M \left( (g''_t(0) - 1) \frac{t^2}{n} \right)^{1/4} \|A^\alpha x\|, \quad x \in \text{dom}(A^\alpha). \]

Remark 4.7. Under further assumptions on \((g_t)_{t>0}\) (e.g. \(\sup_{t \in (0,b]} g''_t(0) < \infty\) for every \(b > 0\)) one can, of course, replace \(g''_t(0) - 1\) in the corollary above by a constant depending only on \(b\). Thus we can obtain the convergence of approximation formulas with rates uniform in \(t\) from compacts in \([0,\infty)\). In particular, in this way, one can show that the convergence in (4.11) is uniform in \(t\) from compacts from \([0,\infty)\) if (4.12) is true. To simplify our formulations we omit this obvious improvement here and in the following approximation results. However, it is instructive to have in mind that the uniformity of convergence present in standard approximation results holds true in our considerations too.

Corollary 4.6 comprises a number of approximation formulas and provide them with optimal convergence rates. In particular, it covers \([32, \text{Theorem 1.3}]\), and thus the classical Dunford-Segal, Yosida and Euler formulas in \([32, \text{Corollary 1.4}]\). It also includes formulas falling outside of the scope of \([32]\), e.g. Kendall’s approximation corresponding to
\[ g_t(z) = 1 - t + t \exp(-z/t), \quad t \in (0,1), \]
and more generally any \(g_t\) with representing measures having compact supports. (Recall that the assumption \(t > 0\) in Corollary 4.6 and in the subsequent results is a matter of convenience, and it can be replaced by e.g. \(t \in (0,a]\), \(a > 0\).) An illustration of Corollary 4.6 of the same nature is provided by a spline approximating sequence, and is discussed below.

Example 4.8. Let
\[ g(z) := \int_0^1 e^{-2zs} ds = \frac{1 - e^{-2z}}{2z}, \quad z > 0. \]
Note that \(g \in B_2\), \(g''(0) = 4/3\), and that \(g\) is not an exponential of a Bernstein function (since the representing measure of \(g\) has a compact support, see a discussion in the introduction). The functions \(g^n(\cdot/n), n \in \mathbb{N}\), can be represented in terms of the classical \(B\)-splines \(B_n : \mathbb{R} \to \mathbb{R}, n \in \mathbb{N} \cup \{0\}\), defined recursively:
\[ B_0(x) = \begin{cases} 1, & x \in [0,1), \\ 0, & x \not\in [0,1), \end{cases} \]
and
\[ B_n(x) = \frac{x}{n} B_{n-1}(x) + \frac{n+1-x}{n} B_{n-1}(x-1), \quad n \in \mathbb{N}. \]
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See [8, Chapter 9] for more details. Note that since

\[(4.19)\quad B_{n-1}(x) = 0 \quad \text{if} \quad x \in (-\infty, 0) \cup (n, \infty),\]

and

\[(4.19)\quad B_n'(x) = B_{n-1}(x) - B_{n-1}(x-1),\]

one has

\[(4.20)\quad B_n(x) = \int_0^1 B_{n-1}(x-s) \, ds, \quad n \in \mathbb{N}.\]

Furthermore,

\[(4.21)\quad g^n(z) = \int_0^\infty e^{-2zs} B_{n-1}(s) \, ds, \quad n \in \mathbb{N}.\]

Indeed, arguing by induction, remark that

\begin{align*}
g(z) &= \int_0^1 e^{-2zs} \, ds = \int_0^1 e^{-2zs} B_0(s) \, ds = \int_0^\infty e^{-2zs} B_0(s) \, ds.
\end{align*}

If (4.21) holds for \(n \in \mathbb{N}\), then using (4.19) and (4.20), we obtain

\begin{align*}
g^{n+1}(z) &= \int_0^1 \int_s^\infty e^{-2z(s+t)} B_{n-1}(t) \, dt \, ds = \int_0^\infty e^{-2z\tau} \int_0^1 B_{n-1}(\tau-s) \, ds \, d\tau
\end{align*}

\begin{align*}
&= \int_0^\infty e^{-2z\tau} B_n(\tau) \, d\tau,
\end{align*}

hence (4.21) is true for \(n+1\), i.e. (4.20) holds. So, if \(g_n(z) = g^n(z/n), n \in \mathbb{N}\), and \(-A\) is the generator of a bounded \(C_0\)-semigroup, then by the HP-calculus,

\[g_n(tA/n) = n \int_0^1 e^{-2stA} B_{n-1}(ns) \, ds, \quad n \in \mathbb{N}.\]

Now Corollary 4.6 implies that for all \(n \in \mathbb{N}\), \(t > 0\), and \(\alpha \in (0, 2]\),

\[\left\| n \int_0^1 B_{n-1}(ns)e^{-2stA}x ds - e^{-tA}x \right\| \leq M \frac{t}{\sqrt{3n}} \|Ax\|, \quad x \in \text{dom}(A),\]

\[\left\| n \int_0^1 B_{n-1}(ns)e^{-2stA}x ds - e^{-tA}x \right\| \leq 4M \left(t^2 \frac{3n}{3n} \right)^{\frac{T}{2}} \|A^\alpha x\|, \quad x \in \text{dom}(A^\alpha),\]

\[\left\| n \int_0^1 B_{n-1}(ns)e^{-2stA}x ds - e^{-tA}x \right\| \leq M \frac{t^2}{6n} \|A^2 x\|, \quad x \in \text{dom}(A^2).\]

4.2. Higher order approximations. Our functional calculus approach proves to be efficient for deriving also the second order approximations formulas with rates. Given the first order formulas, the derivation becomes comparatively straightforward.
Let us define

\[ d_0[g] := \int_0^\infty (1 - s)^2 G(s) \, ds, \quad g \in \mathcal{B}_4, \quad (4.22) \]

\[ d_1[g] := \int_0^\infty \frac{(1 - s)^2(1 + s)}{s^2} G(s) \, ds, \quad g \in \mathcal{B}_{4,\infty}, \quad (4.23) \]

where \( G \) is given by (3.9). By (3.8) we have

\[ d_0[g] = \lim_{z \to 0^+} \left( \frac{d}{dz} + 1 \right)^2 \Delta_2^g(z) = \frac{-3 + 6g''(0) + 4g'''(0) + g''''(0)}{12}. \quad (4.24) \]

Similarly, using again (3.8),

\[ d_1[g] = \int_0^\infty (s - 1 - s^{-1} + s^{-2}) G(s) \, ds \]

\[ = -b[g] - c_1[g] + c_0[g], \]

where \( b[g] \) and \( c_j[g], j = 0, 1 \), are defined by (3.36) and (3.38), respectively.

We are now able to formulate our higher order approximation results for bounded \( C_0 \)-semigroups on Banach spaces. To our knowledge, apart from [50] and [60], higher order approximation formulas for \( C_0 \)-semigroups and their corresponding convergence rates have not been addressed in the literature. As in the previous section, given a bounded \( C_0 \)-semigroup \((e^{-tA})_{t \geq 0}\) on \( X \), we start with a norm estimate for \( \Delta_2^g(A) - 2^{-1}(g''(0) - 1)e^{-A}A^2, g \in \mathcal{B}_4 \), on appropriate domains. The approximation formulas will then be derived by scaling as for the first order approximations considered above.

**Theorem 4.9.** Let \(-A\) be the generator of a bounded \( C_0 \)-semigroup \((e^{-tA})_{t \geq 0}\) on \( X \), and let \( g \in \mathcal{B}_4 \). Denote \( M := \sup_{t \geq 0} \|e^{-tA}\| \), and suppose that \( b \) and \( d_0 \) are given by (3.36) and (4.24), respectively. Then for every \( x \in \text{dom}(A^3) \),

\[ \|g(A)x - e^{-tA}x - 2^{-1}(g''(0) - 1)e^{-A}A^2x\| \leq M \left( \frac{(g''(0) - 1)d_0[g]}{2} \right)^{1/2} \|A^3x\|. \quad (4.25) \]

Moreover, for every \( x \in \text{dom}(A^4) \),

\[ \|g(A)x - e^{-tA}x - 2^{-1}(g''(0) - 1)e^{-A}A^2x\| \leq M \left( |b[g]| \|A^3x\| + d_0[g] \|A^4x\| \right). \quad (4.26) \]

**Proof.** Note that by Lemma 3.1, for every \( x \in \text{dom}(A^3) \),

\[ g(A)x - e^{-A}x = \int_0^\infty G(s)e^{-sA}A^2x \, ds \]

\[ = a[g]e^{-A}A^2x + \int_0^\infty [e^{-sA} - e^{-A}]A^2xG(s) \, ds, \]

where \( a[g] \) is given by (3.36).

\[ = a[g]e^{-A}A^2x + \int_0^\infty \Delta_2^g(s)G(s) \, ds \]

\[ = a[g]e^{-A}A^2x + d_0[g]. \]

Finally, for every \( x \in \text{dom}(A^4) \),

\[ g(A)x - e^{-tA}x - 2^{-1}(g''(0) - 1)e^{-A}A^2x \]

\[ = \int_0^\infty \Delta_2^g(s)G(s) \, ds \]

\[ = \int_0^\infty \Delta_2^g(s)G(s) \, ds + b[g]e^{-A}A^2x + c_0[g]x. \]
where $a[\cdot]$ is defined by [3.35]. Since
\[ e^{-sA}x - e^{-A}x = -\int_1^s e^{-tA}Ax \, dt, \quad x \in \text{dom}(A^3), \]
we infer that
\[ (4.27) \quad g(A)x - e^{-A}x - a[g]e^{-A}A^2x = -\int_1^s \left( \int_1^s e^{-tA}A^3x \, dt \right) G(s) \, ds. \]
Hence,
\[
\| \Delta_0^g(A)x - a[g]e^{-A}A^2x \| \leq M \| A^3x \| \int_0^\infty |1 - s|G(s) \, ds
\leq M \| A^3x \| \left( \int_0^\infty (1 - s)^2 G(s) \, ds \int_0^\infty G(s) \, ds \right)^{1/2}
= M \| A^3x \| \left( a[g]d_0[g] \right)^{1/2}.
\]
It remains to recall that $a[g] = \frac{g''(0)-1}{2}$ by [3.37]. If moreover $x \in \text{dom}(A^4)$, then integrating by parts, we obtain
\[
g(A)x - e^{-A}x = \int_0^\infty e^{-sA}A^2xG(s) \, ds
= a[g]e^{-A}A^2x - e^{-A}A^3x \int_0^\infty (s - 1)G(s) \, ds
+ \int_0^\infty \left( e^{-sA} - e^{-A} + (s - 1)e^{-A}A \right) A^2xG(s) \, ds.
\]
Taking into account that
\[ e^{-sA}x - e^{-A}x + (s - 1)e^{-A}Ax = \int_1^s (s - t)e^{-tA}A^2x \, dt, \]
we have
\[ (4.28) \quad g(A)x - e^{-A}x - a[g]e^{-A}A^2x = -b[g]e^{-A}A^3x
+ \int_0^\infty \left( \int_1^s (s - t)e^{-tA}A^4x \, dt \right) G(s) \, ds. \]
Moreover,
\[ (4.29) \quad \left\| \int_0^\infty \left( \int_1^s (s - t)e^{-tA}A^4x \, dt \right) G(s) \, ds \right\| \leq M \left\| A^4x \right\| \int_0^\infty \left( \int_1^s (s - t) \, dt \right) G(s) \, ds
= M \left\| A^4x \right\| \int_0^\infty \left( (s - 1)^2 \right) G(s) \, ds
\leq Md_0[g]\left\| A^4x \right\|,
\]
and the estimate (4.26) follows. □
Replacing $b[g_n]$ and $d_0[g_n]$ by their expressions in terms of the derivatives of $g$ at zero, we formulate below Theorem 4.9 in a more explicit form.

**Corollary 4.10.** Let $-A$ be the generator of a bounded $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on $X$, and let $(g_t)_{t > 0} \subset \mathcal{B}_4$. Let $M := \sup_{t \geq 0} \|e^{-tA}\|$. Then for all $t > 0$, $n \in \mathbb{N}$, and $x \in \text{dom}(A^3)$,

$$\|g^n_t(tA/n)x - e^{-tA}x - (2n)^{-1}(g''_t(0) - 1)t^2 e^{-tA}A^2x\| \leq MC(g_t)t^3n^{-3/2}\|A^\frac{3}{2}x\|,$$

where

$$C(g_t) = \left(\frac{(g''_t(0) - 1)(g'''_t(0) - 1)}{2}\right)^{1/2}.$$  

Moreover, for all $t > 0$, $n \in \mathbb{N}$, and $x \in \text{dom}(A^4)$,

$$\|g^n_t(tA/n)x - e^{-tA}x - (2n)^{-1}(g''_t(0) - 1)t^2 e^{-tA}A^2x\| \leq MC_1(g_t)t^3n^{-2}(\|A^\frac{3}{2}x\| + t\|A^\frac{4}{2}x\|),$$

where

$$C_1(g_t) = g'''_t(0) - 1.$$  

### 4.3. Optimality for (general) $C_0$-semigroups

We finish this section with a discussion of optimality for the obtained approximation rates. A less general version of the following statement was proved in [32, Corollary 7.5] by means of the spectral mapping theorem for the HP-calculus (Theorem 2.1) and certain estimates for Bernstein functions. Here we propose a slightly different argument based on Corollary 4.10.

**Proposition 4.11.** Let $-A$ be the generator of a bounded $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on a Banach space $X$ such that $\text{ran}(A) = X$, and let $g \in \mathcal{B}_2$, $g(z) \neq e^{-z}$. If $\{|s| : s \in \mathbb{R}, is \subset \sigma(A)\} = \mathbb{R}_+$, then there exists $c > 0$ such that for all $\alpha \in (0, 2]$, $t > 0$, and $n \in \mathbb{N},$

$$\|A^{-\alpha}(g^n(tA/n) - e^{-tA})\| \geq c \left(\frac{t^2}{n}\right)^{\alpha/2}.$$  

**Proof.** Let $\alpha \in (0, 2]$ be fixed. Using Corollary 4.10 for scalar functions, let $t > 0$, $n \in \mathbb{N}$ and $s \neq 0$ be such that if

$$\epsilon = \frac{t|s|}{\sqrt{n}},$$

then $\epsilon$ is so small that $\epsilon \in (0, 1)$ and

$$|g^n(i st/n) - e^{-ist}| \geq c \left(\frac{ts}{\sqrt{n}}\right)^2 = c\epsilon^2.$$
Then, by (the spectral mapping) Theorem 2.11 we have
\[ \|A^{-\alpha}(g^n(tA/n) - e^{-tA})\| \geq \sup_{s \in \mathbb{R}} |s|^{-\alpha}(g^n(ist/n) - e^{-ist}) \]
\[ \geq c|s|^{-\alpha}e^2 = ce^{2-\alpha} \left( \frac{t^2}{n} \right)^{\alpha/2}, \]
for some constant \( c > 0 \).

It is easy to construct concrete examples of generators satisfying the assumptions of Proposition 4.11 e.g. one may consider appropriate multiplication operators on \( L^2(\mathbb{R}) \).

Let us now address the optimality of the estimate (4.30) from Corollary 4.10. In view of substantial technical difficulties the optimality will be shown only in a particular setting of \( X = C_0(\mathbb{R}) \), a shift semigroup \( (e^{-tA})_{t \geq 0} \) on \( X \), and Euler’s approximation formula.

**Example 4.12.** Let us start from abstract considerations under the assumptions of Corollary 4.10. For \( x \in \text{dom}(A^3) \), let
\[ R_{t,n}(A)x := g^n(tA/n)x - e^{-tA}x - (2n)^{-1}(g^n(0) - 1)t^2 e^{-tA}A^2x. \]
Then (4.30) asserts that
\[ \|R_{t,n}(A)x\| \leq MC(g)t^3n^{-3/2}\|A^3x\| \]
for all \( t > 0 \) and \( n \in \mathbb{N} \). If \( \omega(\epsilon, x) \) is the modulus continuity of \( e^{-tA}x \) and \( (S_\epsilon)_{\epsilon > 0} \) is the family of averaging operators defined in Remark 4.4, then for every \( x \in \text{dom}(A^2) \),
\[ \|A^2(x - S_\epsilon x)\| \leq \omega(\epsilon, A^2x), \quad \|A^3S_\epsilon x\| \leq e^{-1}\omega(\epsilon, A^2x). \]
Hence, from (4.16) and (4.32), it follows that if \( x \in \text{dom}(A^2) \) then
\[ \|R_{t,n}(A)x\| \leq \|R_{t,n}(A)(x - S_\epsilon x)\| + \|R_{t,n}(A)S_\epsilon x\| \leq Cn^{-1/2}\|A^2x - A^2S_\epsilon x\| + n^{-1/2}\|A^3S_\epsilon x\| \leq Cn^{-1/2}[\omega(\epsilon, A^2x) + n^{-1/2}e^{-1}\omega(\epsilon, A^2x)], \]
and setting \( \epsilon = t/\sqrt{n} \) we have
\[ (4.33) \quad \|R_{t,n}(A)x\| \leq Ct^2n^{-1}\omega(t/\sqrt{n}, A^2x), \quad x \in \text{dom}(A^2). \]

Now, following [22], let \( X \) be a Banach space of continuous functions on \( \mathbb{R} \), vanishing at both infinities, denoted by \( C_0(\mathbb{R}) \). Define a \( C_0 \)-semigroup \( (e^{-tA})_{t \geq 0} \) on \( C_0(\mathbb{R}) \) by
\[ (e^{-tA}f)(s) := f(s + t), \quad s \in \mathbb{R}, \]
and note that \( (Af)(s) = -f'(s) \) on a natural domain.

Consider Euler’s approximation of \( (e^{-tA})_{t \geq 0} \), i.e. choose \( g(z) = 1/(1 + z) \) in Corollary 4.10. Then
\[ g^n(z/n) = \int_0^\infty e^{-zs}v_n(s)\,ds, \quad \text{where} \quad v_n(s) := \frac{n^n}{(n - 1)!}e^{-ns}s^{n-1}, \quad n \in \mathbb{N}. \]
Making use of Lemma 3.1 and the proof of Theorem 4.9 we infer that for \( t > 0 \) and \( f \in \text{dom}(A^2) \),

\[
(R_{t,n}f)(s) = (t^2 Z_{t,n}A^2 f)(s), \quad s \in \mathbb{R},
\]

where

\[
(Z_{t,n}h)(s) := \int_{0}^{\infty} W_n(\tau) (e^{-\tau A} - e^{-tA})h(s) \, d\tau,
\]

\[
W_n(\tau) := \int_{0}^{\infty} |\tau - y| \nu_n(y) \, dy, \quad \tau > 0.
\]

Let \( u \in C_0(\mathbb{R}) \) be given by

\[
u(s) = \begin{cases} 
1 - |s|, & |s| \leq 1, \\
0, & |s| > 1,
\end{cases}
\]

and let \( f \in \text{dom}(A^2) \) be defined as

\[
f = (A^2 - I)^{-1}u,
\]

so that \( A^2 f = u + f \). Taking into that \( \omega(\epsilon, u) \leq \epsilon \) for every \( \epsilon \in (0, 1) \), we have

\[
(4.34) \quad \omega(\epsilon, A^2 f) \leq \omega(\epsilon, u) + \omega(\epsilon, f) \leq 2\epsilon, \quad \epsilon \in (0, 1).
\]

Note that

\[
(4.35) \quad (R_{t,n}f)(s) = t^2 ((Z_{t,n}u)(s) + (Z_{t,n}f)(s)), \quad s \in \mathbb{R},
\]

and (see the proof of Theorem 4.9)

\[
(4.36) \quad \|Z_{t,n}f\| \leq cMn^{-2} \left( \|Af\| + \|A^2f\| \right),
\]

for some constant \( c > 0 \). Setting \( t = 1 \) and \( s = -1 \) in (4.35), write

\[
(Z_{t,n}u)(-1) = \int_{0}^{\infty} W_n(\tau) (u(\tau - 1) - 1) \, d\tau = I_{1,n} + I_{2,n},
\]

where

\[
I_{1,n} := -\int_{0}^{2} W_n(\tau) |\tau - 1| \, d\tau, \quad I_{2,n} := -\int_{2}^{\infty} W_n(\tau) \, d\tau.
\]

Estimating the second integral from above, we obtain that for every \( n \in \mathbb{N} \),

\[
(4.37) \quad |I_{2,n}| \leq \int_{2}^{\infty} (\tau - 1)^2 W_n(\tau) \, d\tau \leq \int_{0}^{\infty} (\tau - 1)^2 W_n(\tau) \, d\tau \leq 2n^{-2}.
\]

On the other hand, using

\[
\int_{1}^{y} (y - \tau)(\tau - 1) \, d\tau = \frac{(y - 1)^3}{6},
\]
and estimating the first integral from below we have:

$$|I_{1,n}| \geq \int_{1}^{2} W_n(\tau) |\tau - 1| d\tau = \int_{1}^{2} \int_{0}^{\tau} (y - \tau) v_n(y) dy (\tau - 1) d\tau$$

$$\geq \int_{1}^{2} v_n(y) \int_{1}^{y} (\tau - y)(\tau - 1) d\tau dy$$

$$= \frac{n^n e^{-n}}{6(n - 1)!} \int_{0}^{1} e^{-ns}(1 + s)^{n-1} s^3 ds.$$  

Now Stirling’s formula and the property

$$\lim_{n \to \infty} n^2 \int_{0}^{1} e^{-ns}(1 + s)^{n-1} s^3 ds = 2,$$

see [54, p. 81], imply that

$$(4.38) \lim_{n \to \infty} n^{3/2} |I_{1,n}| \geq \frac{1}{3\sqrt{2\pi}}.$$  

So, taking into account (4.35), (4.36), (4.37) and (4.38), we infer that

$$(4.39) \lim_{n \to \infty} n^{3/2} \|R_{1,n}f\| \geq \frac{1}{3\sqrt{2\pi}}.$$  

By (4.33), we have \(\sqrt{n}\omega(1/\sqrt{n}, A^2 f) \leq 2\), so we can rewrite (4.39) in the form

$$(4.40) \lim_{n \to \infty} \frac{n}{\omega(1/\sqrt{n}, A^2 f)} \|R_{1,n}f\| \geq \frac{1}{6\sqrt{2\pi}}.$$  

The inequality (4.40) shows that (4.33) is sharp. Consequently, (4.32) is sharp too.

We believe that the optimality of (4.30) can be shown in a more general context. However we feel that the exposition will then be overloaded by unnecessary technicalities, and postpone a general argument to another occasion.

5. APPROXIMATION OF BOUNDED HOLOMORPHIC \(C_0\)-SEMIGROUPS.

As one may expect, the statements above can be improved in the framework of sectorially bounded holomorphic semigroups. Recall that for \(g = e^{-\varphi}\), where \(\varphi\) is a Bernstein function, the statements on approximation of sectorially bounded holomorphic \(C_0\)-semigroups were deduced in [32] from the results on approximation of general bounded \(C_0\)-semigroups. To this aim, given a sectorially bounded holomorphic \(C_0\)-semigroup \((e^{-tA})_{t \geq 0}\) on \(X\), we considered in [32] a bounded \(C_0\)-semigroup \((e^{-tA})_{t \geq 0}\) on \(X \oplus X\).

\[A = \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}, \quad \text{dom}(A) = \text{dom}(A) \oplus \text{dom}(A),\]

on the direct sum \(X \oplus X\), and, after certain manipulations, read off approximation results for \((e^{-tA})_{t \geq 0}\) on \(X\) from the ones for \((e^{-tA})_{t \geq 0}\) on \(X \oplus X\).
(Note that the idea to use operator matrices as above to the study of holomorphic $C_0$-semigroups goes back to [18].)

Below we propose a more direct and explicit approach. It does not depend on the approximation results for bounded, not necessarily holomorphic, $C_0$-semigroups, and it improves the results from the previous section in a particular framework of holomorphic semigroups. Moreover, it extends the results from [32] by allowing $g \in B(M)$, and sometimes offers even better estimates than those in [32] as far as the constants are concerned.

For a sectorially bounded holomorphic $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on $X$ and $\beta \geq 0$ define (see (2.6))

$$ M_\beta := \sup_{t > 0} \| t^\beta A^\beta e^{-tA} \| < \infty. $$

Thus, the constant $M_0$ has a meaning of $M$ from the previous section dealing with general $C_0$-semigroups.

Note that using moment’s inequality, for all $t > 0$ and $\gamma \in (0, 1)$, we have

$$ t^{1+\gamma} \| A^{1+\gamma} e^{-tA} x \| \leq C(\gamma) t^{1+\gamma} \| A e^{-tA} x \|^{1-\gamma} \| A^2 e^{-tA} x \|^{\gamma}, $$

$$ = C(\gamma) \| t A e^{-tA} x \|^{1-\gamma} \| t^2 A^2 e^{-tA} x \|^{\gamma} $$

$$ \leq C(\gamma) M_1^{1-\gamma} M_2^\gamma \| x \|. $$

Thus, since $M_1^{1-\gamma} M_2^\gamma \leq (1-\gamma) M_1 + \gamma M_2$ and, by [44, p. 63], $C(\gamma) \leq 3, \gamma \in [0, 1]$, we have

$$ M_1 + \gamma \leq C(\gamma) (M_1 + M_2) \leq 3(M_1 + M_2). $$

5.1. First order approximations. We start with several simple auxiliary estimates for functions of generators of holomorphic semigroups.

**Proposition 5.1.** Let $r \in B(M)$, and denote

$$ r_j(z) := z^j r(z), \quad j = 1, 2. $$

Let $-A$ be the generator of a sectorially bounded holomorphic semigroup $(e^{-tA})_{t \geq 0}$ on $X$. Then the following estimates hold:

(i) $\| r(A) \| \leq M_0 r(0)$,

(ii) $\| r_1(A) x \| \leq (M_0 + M_1) r(0) \| A x \|$, $x \in \text{dom}(A)$,

(iii) $\| r_2(A) \| \leq (2M_0 + 4M_1 + M_2) r(0)$.

**Proof.** The first assertion from (i) is evident. Next, letting $r \sim \gamma$, note that $r_1'(z) = r(z) - z \int_0^\infty s e^{-sz} \gamma(ds)$ and use (5.1) and the product rule for the (extended) HP-calculus. It follows that for all $\delta > 0$ and $x \in X$,

$$ r_1'(A + \delta) x = r(A + \delta) x - \int_0^\infty A s e^{-s} e^{-\delta s} x \gamma(ds) $$

$$ - \int_0^\infty \delta s e^{-s} e^{-\delta s} x \gamma(ds). $$
Hence
\[ \|r'_1(A + \delta)x\| \leq M_0r(0)\|x\| + M_1r(0)\|x\| + M_0\|x\|\int_0^\infty \delta se^{-\delta s} \gamma(ds). \]

Letting \( \delta \to 0^+ \) and using the dominated convergence theorem, we obtain the second assertion in (i). The other statements (ii) and (iii) can be proved similarly.

Observe that
\[ g(z) - e^{-z} = \left[ g(z) + 2g'(z) + g''(z) \right] - 2[g'(z) + g''(z)] + [g''(z) - e^{-z}], \]
and
\[ g(z) - e^{-z} = [g(z) + g'(z)] - [g'(z) + e^{-z}]. \]

Proposition 5.1 and Lemma 3.5 enable us to estimate each term in square brackets from (5.3) and (5.4), and thus to estimate \( g(A) - e^{-A} \) and \( g(A)x - e^{-A}x, x \in \text{dom}(A). \)

**Corollary 5.2.** Let \( g \in B_2 \), and let \(-A\) be the generator of a sectorially bounded holomorphic semigroup \( (e^{-tA})_{t \geq 0} \) on \( X \). Then the following estimates hold:

(i) \( \|g'(A) + g''(A)\| \leq (M_0 + M_1)(g''(0) - 1), \]
(ii) \( \|g(A) + 2g'(A) + g''(A)\| \leq M_0(g''(0) - 1), \]
(iii) \( \|e^{-A} - g''(A)\| \leq (M_0 + 2M_1 + M_2/2)(g''(0) - 1), \]
(iv) for all \( x \in \text{dom}(A), \)
\[ \|(g(A) + g'(A))x\| \leq M_0(g''(0) - 1)\|Ax\|, \]
and
\[ \|g'(A)x + e^{-A}x\| \leq (M_0 + M_1/2)(g''(0) - 1)\|Ax\|. \]

**Proof.** Lemma 3.5 implies that if \( g \in B_2 \), then \( r(z) := s g(z) \in BM \) and \( r(0) = g''(0) - 1 \). Using Proposition 5.1 (i), where \( r_1(z) := g(z) + g'(z) \), we obtain the assertion (i).

To prove (ii) let \( g \sim \nu \). Note that for
\[ q(z) := g(z) + 2g'(s) + g''(z) = \int_0^\infty (s - 1)^2 e^{-sz} \nu(ds), \quad z > 0, \]
we have \( q \in BM \) with \( q(0) = g''(0) - 1 \). Therefore,
\[ \|q(A)\| \leq M_0(g''(0) - 1), \]
that is, (ii) is true. Next, by Lemma 3.5 if \( r(z) := \Delta^2 g(z) \) then \( r \in BM \) and \( r(0) = \frac{g''(0) - 1}{2} \). Now setting \( r_2(z) = g(z) - e^{-z} \) and applying Proposition 5.1 (ii) and (iii) to \( r \), we obtain (5.6). Finally, the estimate (5.6) follows from the product rule for the (extended) HP-calculus and Lemma 3.5. \( \square \)
Remark 5.3. If \( q \) is defined by (5.7), then noting that
\[
g(z) - g''(z) = q(z) - 2(g'(z) + g''(z)),
\]
and using Proposition 5.2 (ii) and (iv), one infers that
\[
\|g(A) - g''(A)\| \leq (2M_0 + M_1)(g''(0) - 1).
\]

The following theorem is a direct implication of norm estimates for \( \Delta_0^\alpha(A) \) on appropriate domains, given by the HP-calculus.

**Theorem 5.4.** Let \( g \in B_2 \), let \( -A \) be the generator of a sectorially bounded holomorphic semigroup \( (e^{-tA})_{t \geq 0} \) on \( X \), and let \( K = 3M_0 + 3M_1 + M_2/2 \). Then
\[
\|g(A) - e^{-A}\| \leq K(g''(0) - 1),
\]
and for every \( x \in \text{dom}(A) \),
\[
\|g(A)x - e^{-A}x\| \leq (2M_0 + M_1/2)(g''(0) - 1)\|Ax\|.
\]
Moreover, for every \( \alpha \in (0,1) \) and every \( x \in \text{dom}(A^\alpha) \),
\[
\|g(A)x - e^{-A}x\| \leq 3M_0K(g''(0) - 1)\|A^\alpha x\|,
\]

**Proof.** The inequalities (5.9) and (5.10) follow from Corollary 5.2 and the representations (5.3) and (5.4).

Next, let \( \alpha \in (0,1) \), \( \beta = 1 - \alpha \), and \( \delta > 0 \). Combining moment’s inequality for the sectorial operator \( A + \delta \) with (5.9) and (5.10) and employing the (extended) HP-calculus, we obtain for any \( x \in X \):
\[
\|(A + \delta)^{-\alpha}\Delta_0^\alpha(A + \delta)x\| = \|(A + \delta)^{\beta-1}\Delta_0^\beta((A + \delta)^{-1}\Delta_0^\delta(A + \delta))x\| \\
\leq 3M_0\|\Delta_0^\beta((A + \delta)^{-1}\Delta_0^\delta(A + \delta))x\|^{\beta} \|(A + \delta)^{-1}\Delta_0^\delta(A + \delta)x\|^{1-\beta} \\
\leq 3M_0K^\beta(2M_0 + M_1/2)^{1-\beta}(g''(0) - 1)\|x\| \\
\leq 3M_0K(g''(0) - 1)\|x\|.
\]

Thus, for all \( x \in \text{dom}(A^\alpha) \),
\[
\|g(A + \delta)x - e^{-(A + \delta)x}\| \leq 3M_0K(g''(0) - 1)(A + \delta)^\alpha x\|.
\]

Letting now \( \delta \to 0^+ \), assertion (5.11) follows. \( \square \)

Now by scaling we derive our first order approximation formulas with rates for sectorially bounded holomorphic C₀-semigroups on Banach spaces.

**Corollary 5.5.** Let \( (g_t)_{t \geq 0} \subset B_2 \), and let \( -A \) be the generator of a sectorially bounded holomorphic C₀-semigroup \( (e^{-tA})_{t \geq 0} \) on \( X \). Then for all \( n \in \mathbb{N} \) and \( t > 0 \),
\[
\|g^n_t(tA/n) - e^{-tA}\| \leq K\frac{(g''(0) - 1)}{n},
\]
for \( \alpha \in (0,1) \), \( \beta = 1 - \alpha \), and \( \delta > 0 \). Combining moment’s inequality for the sectorial operator \( A + \delta \) with (5.9) and (5.10) and employing the (extended) HP-calculus, we obtain for any \( x \in X \):
\[
\|(A + \delta)^{-\alpha}\Delta_0^\alpha(A + \delta)x\| = \|(A + \delta)^{\beta-1}\Delta_0^\beta((A + \delta)^{-1}\Delta_0^\delta(A + \delta))x\| \\
\leq 3M_0\|\Delta_0^\beta((A + \delta)^{-1}\Delta_0^\delta(A + \delta))x\|^{\beta} \|(A + \delta)^{-1}\Delta_0^\delta(A + \delta)x\|^{1-\beta} \\
\leq 3M_0K^\beta(2M_0 + M_1/2)^{1-\beta}(g''(0) - 1)\|x\| \\
\leq 3M_0K(g''(0) - 1)\|x\|.
\]

Thus, for all \( x \in \text{dom}(A^\alpha) \),
\[
\|g(A + \delta)x - e^{-(A + \delta)x}\| \leq 3M_0K(g''(0) - 1)(A + \delta)^\alpha x\|.
\]

Letting now \( \delta \to 0^+ \), assertion (5.11) follows. \( \square \)
where \( K = 3M_0 + 3M_1 + M_2/2 \). Moreover for all \( n \in \mathbb{N}, t > 0, \) and \( x \in \text{dom}(A), \)

\[
(5.13) \quad \|g^n_t(tA/n)x - e^{-tA}x\| \leq (2M_0 + 3M_1/2)\left(\frac{g''_1(0) - 1}{n}\right) t\|Ax\|,
\]

and for all \( n \in \mathbb{N}, t > 0, \alpha \in (0,1), \) and \( x \in \text{dom}(A^\alpha), \)

\[
(5.14) \quad \|g^n_t(tA/n)x - e^{-tA}x\| \leq 3M_0K\left(\frac{g''_1(0) - 1}{n}\right) t\alpha\|A^\alpha x\|.
\]

The above corollary can be essentially sharpened with respect to constants in the right hand side of (5.14). The improvement requires more involved arguments and finer function classes, and is contained in the following statement.

**Theorem 5.6.** Let \(-A\) be the generator of a sectorially bounded holomorphic \(C_0\)-semigroup \((e^{-tA})_{t \geq 0}\) on \(X\), and let \(g \in B_{2,\infty}\). Then for all \(t > 0, \alpha \in [0,1]\) and \(x \in \text{dom}(A^\alpha), \)

\[
(5.15) \quad \|g^n_x(tA/n)x - e^{-tA}x\| \leq M_2 - \alpha c_\alpha\|A^\alpha x\|,
\]

where \(c_\alpha\) and \(M_2 - \alpha\) are given by (3.38) and (5.1), respectively.

**Proof.** We consider the cases \(\alpha \in [0,1)\) and \(\alpha = 1\) separately. Let \(\alpha \in [0,1)\) be fixed. By the product rule for the (extended) HP-calculus, for every \(\delta > 0, \)

\[
(5.16) \quad \Delta^\alpha_\delta(A + \delta) = (A + \delta)^2[\Delta^\alpha_\delta(z + \delta)](A),
\]

where \(\Delta^\alpha_\delta(z + \delta) \in BM, \)

\[
(5.17) \quad \Delta^\alpha_\delta(z + \delta) = \int_0^\infty e^{-zs}e^{-\delta s}G(s)\,ds, \quad \text{and} \quad \|\Delta^\alpha_\delta(z + \delta)\|_{A^1_{\text{loc}}} = \Delta^\alpha_\delta(\delta).
\]

Then, by (5.16) and (5.17), for every \(x \in \text{dom}(A^\alpha), \)

\[
(5.18) \quad \|\Delta^\alpha_0(A + \delta)x\| \leq \int_0^\infty e^{-\delta s}G(s)((A + \delta)^{2-\alpha}e^{-sA}(A + \delta)^\alpha x)\,ds.
\]

To estimate the right hand side of (5.18), recall that \(e^{-sA}(X) \subset \text{dom}(A), \)

\(s > 0, \) and by [35, Proposition 3.1.7] for all \(\beta \in [0,1]\) and \(\delta > 0, \)

\[
\|(A + \delta)^\beta x - A^\beta x\| \leq C\beta(A)\delta^\beta\|x\|.
\]
Employing (5.15) we obtain

\[
\int_0^\infty e^{-\delta s} G(s) \| (A + \delta)^{-\alpha} e^{-sA} (A + \delta)^\alpha x \| ds \\
\leq \int_0^\infty e^{-\delta s} G(s) \| (A + \delta)^{1-\alpha} e^{-sA} (A + \delta)^\alpha x \| ds \\
+ \int_0^\infty e^{-\delta s} G(s) \| [(A + \delta)^{1-\alpha} - A^{1-\alpha}] (A + \delta) e^{-sA} (A + \delta)^\alpha x \| ds \\
\leq \| (A + \delta)^\alpha x \| \int_0^\infty e^{-\delta s} G(s) \left[ \frac{M_{2-\alpha}}{s^{2-\alpha}} + \delta \frac{M_{1-\alpha}}{s^{1-\alpha}} \right] ds \\
+ \| (A + \delta)^\alpha x \| C_{1-\alpha}(A) \delta^{1-\alpha} \int_0^\infty e^{-\delta s} G(s) [M_1 s^{-1} + M_0 \delta] ds.
\]

Then, using (3.39), (3.40) and (5.19), we infer from (5.18) that

\[
\| \Delta_0^\alpha (A + \delta)x \| \leq (M_{2-\alpha} c_\alpha [g] + N_\alpha(\delta)) \| (A + \delta)^\alpha x \|, \quad x \in \text{dom}(A^\alpha),
\]

where

\[
N_\alpha(\delta) = \frac{M_{1-\alpha}}{\Gamma(1-\alpha)} \delta \int_0^\infty \frac{\Delta_2^\alpha(z + \delta)}{z^\alpha} dz \\
+ C_{1-\alpha}(A) M_1 \delta^{1-\alpha} \int_0^\infty \Delta_2^\alpha(z + \delta) dz + C_{1-\alpha}(A) M_0 \delta^{1-\alpha}.
\]

Observe now that

\[
\lim_{\delta \to 0^+} N_\alpha(\delta) = 0.
\]

Indeed, by (3.16) and the inequality \( z + 1 \geq z^\nu, \ z > 0, \) with \( \nu = (1 - \alpha)/2 \in (0, 1], \) we have

\[
\frac{\delta}{2} \int_0^\infty \frac{\Delta_2^\alpha(z + \delta)}{z^\alpha} dz \leq \delta \int_0^\infty \frac{dz}{z^\alpha(z + \delta)(z + 1)} \\
\leq \delta \int_0^\infty \frac{dz}{z^\alpha + \nu(z + \delta)} \\
= \delta^{(1-\alpha)/2} \int_0^\infty \frac{dz}{z^{(1+\alpha)/2}(z + 1)};
\]

and (5.21) follows. Thus, letting \( \delta \to 0^+ \) in (5.20), we conclude that

\[
\| \Delta_0^\alpha(A)x \| \leq M_{2-\alpha} c_\alpha(g) \| A^\alpha x \|, \quad x \in \text{dom}(A^\alpha).
\]
Finally, if $\alpha = 1$ and $x \in \text{dom}(A)$, then taking into account (3.39), we obtain:

$$\|\Delta_0^\alpha (A + \delta)x\| \leq \int_0^\infty e^{-\delta s} G(s)\|(A + \delta)e^{-sA}(A + \delta)x\| ds$$

$$\leq \|(A + \delta)x\| \int_0^\infty e^{-\delta s} G(s)[M_1s^{-1} + \delta M_0] ds$$

$$\leq \|(A + \delta)x\| \left\{ M_1c_1[g] + M_0\delta \int_0^\infty e^{-\delta s} G(s) ds \right\}.$$

Thus letting $\delta \to 0^+$ in (5.22) and using (3.13), we get (5.15) for $\alpha = 1$ as well.

Recall that, according to Corollary 5.5, (5.14) holds with a constant $3M_0(3M_0 + 3M_1 + M_2/2)$, while Theorem 5.6 in view of (5.2), Theorem 3.19 and $M_0 \geq 1$, provides a much better constant $3(M_1 + M_2)$. Note that, curiously, $M_0$ does not enter the estimate of Theorem 5.6.

Theorem 5.6 yields the following sharp approximation formula.

**Corollary 5.7.** Let $-A$ be the generator of a sectorially bounded holomorphic $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on $X$, and let $g \in B_2 \cap L^k$ for some $k \in \mathbb{N}$. Then there exists $C(g) > 0$ such that for all $\alpha \in [0, 1]$ and $x \in \text{dom}(A^\alpha)$,

$$\|g^\alpha(tA/n)x - e^{-tA}x\| \leq M_2 - \alpha \frac{g''(0)}{2n} (1 + C(g)n^{-1}) t^\alpha \|A^\alpha x\|.$$

### 5.2. Higher order approximations and sharp constants.

In this subsection we obtain a version of the second order approximation formulas from Theorem 4.9 for sectorially bounded holomorphic semigroups. While in Theorem 4.9 the constants in approximation rates were controlled by functionals $d_0$ and $b$ introduced in Section 3, in the setting of holomorphic semigroups the functional $d_1$ substitutes $d_0$.

**Theorem 5.8.** Let $-A$ be the generator of a sectorially bounded holomorphic $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on $X$. Assume that $g \in B_4 \cap L^k$ for some $k \in \mathbb{N}$. Then for all $\alpha \in [0, 1]$ and $x \in \text{dom}(A^\alpha)$,

$$\|g(A)x - e^{-A}x - \frac{g''(0)}{2} A^2 e^{-A}x\| \leq K(g, \alpha) \|A^\alpha x\|,$$

where

$$K(g, \alpha) = |b[g]|M_{3-\alpha} + 2^{-1}d_1[g]M_{4-\alpha},$$

and $d_1$ and $M_{\alpha}$ are given by (4.23) and (5.1), respectively.

**Proof.** Fix $\alpha \in [0, 1]$. By (4.28) we have

$$g(A)x - e^{-A}x - a[g]A^2 e^{-A}x = b[g]A^3 e^{-A}x$$

$$+ \int_0^\infty \left( \int_1^s (s-t)A^4 e^{-tA}x dt \right) G(s) ds.$$
Then,
\[ \| g(A)x - e^{-A}x - a[g]A^2e^{-A}x \| \]
\[ \leq |b[g]|M_{3-\alpha}\| A^\alpha x \| + M_{4-\alpha}\| A^\alpha x \| \int_0^\infty I_{\alpha}(s) G(s) \, ds, \]
where
\[ I_{\alpha}(s) := \int_1^s (s-t)t^{\alpha-4} \, dt = \frac{(2-\alpha)s^3 - (3-\alpha)s^2 + s^\alpha}{(3-\alpha)(2-\alpha)s^2}. \]
Since
\[ s^\alpha \leq \alpha s + (1-\alpha), \quad s > 0, \]
we have
\[ I_{\alpha}(s) \leq \frac{(2-\alpha)s^3 - (3-\alpha)s^2 + \alpha s + (1-\alpha)}{2(2-\alpha)s^2} \]
\[ = \left(1 - \frac{1}{2}\right)(2 - \alpha)s + (1 - \alpha) \]
\[ \leq \frac{(1-\alpha)(1+s)}{2s^2}. \]
Therefore,
\[ \| g(A)x - e^{-A}x - a[g]A^2e^{-A}x \| \]
\[ \leq |b[g]|M_{3-\alpha}\| A^\alpha x \| + \frac{d_1[g]M_{4-\alpha}}{2}\| A^\alpha x \|. \]
\[ \square \]

Now Corollary 6.2 makes it possible to replace the constants in Theorem 5.8 by their estimates in terms of \( g \) and \( M_\beta \).

**Corollary 5.9.** Let \( -A \) be the generator of a sectorially bounded holomorphic \( C_0 \)-semigroup \( (e^{-tA})_{t \geq 0} \) on \( X \). Assume that \( g \in B_k \cap L^k \) for some \( k \in \mathbb{N} \). If \( \alpha \in [0, 1] \) then there exists \( C(g) > 0 \) such that for all \( n \in \mathbb{N}, t > 0, \) and \( x \in \text{dom}(A^\alpha) \),
\[ \| g^n(tA/n)x - e^{-tA}x - a[g]n^{-1}t^2 A^2e^{-tA}x \| \]
\[ \leq C(g)[M_{3-\alpha} + M_{4-\alpha}\frac{t^\alpha}{n^2}\| A^\alpha x \|, \]
where \( a[g] = (g''(0) - 1)/2 \).

Corollary 5.9 generalizes the corresponding result from [60], where a second order Euler’s approximation (with \( \alpha = 0 \)) for holomorphic semigroups has been studied using a completely different technique.
5.3. Optimality for holomorphic \( C_0 \)-semigroups. As for general \( C_0 \)-semigroups, our approximation formulas for holomorphic semigroups are sharp, as the next proposition shows.

**Proposition 5.10.** Let \(-A\) be the generator of a sectorially bounded holomorphic \( C_0 \)-semigroup \((e^{-tA})_{t \geq 0}\) on a Banach space \(X\) such that \(\text{ran}(A) = X\) and let \(g \in \mathcal{B}_1\), \(g(z) \not\equiv e^{-z}\). If \(\mathbb{R}_+ \subset \sigma(A)\), then there exists \(c > 0\) such that for all \(\alpha \in [0,1]\), \(t > 0\), and \(n \in \mathbb{N}\),

\[
\|A^{-\alpha}(g^n(tA/n) - e^{-tA})\| \geq c\frac{t^\alpha}{n},
\]

Moreover, if in addition \(b[g] \neq 0\), then there is \(c > 0\) such that for all \(\alpha \in [0,1]\), \(t > 0\), and \(n \in \mathbb{N}\),

\[
\|A^{-\alpha}(g^n(tA/n) - e^{-tA} - a[g]n^{-1}t^2A^2e^{-tA})\| \geq c\frac{t^\alpha}{n^2}.
\]

**Proof.** Let \(\alpha \in [0,1]\) be fixed. Employing Corollary 4.10 (for scalar functions) with \(s = 1/t\) we conclude that there exists \(c > 0\) such that

\[
|g^n(1/n) - e^{-1}| \geq cn^{-1}
\]

for all \(n \in \mathbb{N}\). Then, by Theorem 2.1 we have

\[
\|A^{-\alpha}(g^n(tA/n) - e^{-tA})\| \geq \sup_{\lambda \in \mathbb{R}_+} |\lambda^{-\alpha}(g^n(\lambda t/n) - e^{-\lambda t})| \geq ct^n n^{-1},
\]

for all \(t > 0\).

Next, if \(g \in \mathcal{B}_1\) and \(b[g] \neq 0\), then by (4.20),

\[
|g^n(s/n) - e^{-s} - a[g]n^{-1}s^2e^{-s}| \geq |b[g]|s^4e^{-s} - \frac{s^4}{2}d_0[g_n] \\
\geq n^{-2}s^3(|b[g]e^{-s} - sg''(0)|), \quad s > 0.
\]

If

\[
s_0 = \min\left(1, \frac{|b[g]|}{2eg''(0)}\right),
\]

and \(s \in (0, s_0)\), then

\[
|g^n(s/n) - e^{-s} - a[g]n^{-1}s^2e^{-s}| \geq \frac{|b[g]| s^3}{2e n^2}.
\]

So, by Theorem 2.1 we infer that

\[
\|A^{-\alpha}(g^n(tA/n) - e^{-tA})\| \geq \sup_{\lambda \in \mathbb{R}_+} |\lambda^{-\alpha}(g^n(\lambda t/n) - e^{-\lambda t} - a[g]n^{-1}s^2e^{-s})| \\
\geq |t^\alpha s^{-\alpha}(g^n(s/n) - e^{-s} - a[g]n^{-1}s^2e^{-s})| \\
\geq \frac{|b[g]| s^{3-\alpha}t^\alpha}{2e n^2},
\]

where in the second line we have chosen \(\lambda = s/t\) for \(s \in (0, s_0)\). This finishes the proof. \(\square\)
Similarly to the situation of Proposition 4.11 it is straightforward to provide concrete examples of \( A \)'s satisfying the assumptions of Proposition 5.10, e.g. using multiplication operators on \( L^2(\mathbb{R}_+) \).

Remark 5.11. If \( g \in B_3 \) is of the form \( g(z) = e^{-\varphi(z)} \), \( \varphi \in \mathcal{B} \mathcal{F} \), then \( b[g] = -\varphi''(0)/6 \), and \( b[g] \neq 0 \) if \( g(z) \neq e^{-z} \). On the other hand, if

\[
g_\alpha(z) := \frac{e^{-\alpha z} + e^{-(2-\alpha)z}}{2}, \quad \alpha \in [0, 1],
\]

then \( g_\alpha \in B_3 \), \( b[g_\alpha] = 0 \), and \( g_\alpha \in L^1(\mathbb{R}_+) \) for \( \alpha \in (0, 1] \).

We finish the paper by illustrating the sharpness of our results in a particular situation of Euler’s approximation formula. Let \( g(z) = (1 + z)^{-1} \). Note that \( g \in B_2 \), \( g''(0) = 2 \) and \( g \in L^2(\mathbb{R}_+) \). Now Theorem 5.6, Corollary 5.9 and (3.54) imply the following first and second order Euler’s approximation formulas with rates for sectorially bounded holomorphic \( C_0 \)-semigroups.

**Theorem 5.12.** Let \(-A\) be the generator of a sectorially bounded holomorphic \( C_0 \)-semigroup \((e^{-tA})_{t \geq 0}\) on \( X \). Then for all \( \alpha \in [0, 1] \), \( t > 0 \), and \( n \in \mathbb{N} \),

\[
\left\| \left(1 + \frac{t}{n} A \right)^{-n} x - e^{-tA} x \right\| \leq M_{2-\alpha} r_{\alpha,n} t^\alpha \| A^\alpha x \|, \quad x \in \text{dom}(A^\alpha),
\]

where

\[
r_{\alpha,n} = \frac{1}{2n} + \frac{1 - 2\alpha}{12n^2}, \quad \alpha \in [0, 1/2), \quad r_{\alpha,n} = \frac{1}{2n}, \quad \alpha \in [1/2, 1], \quad n \in \mathbb{N}.
\]

Moreover, there exists an absolute constant \( C > 0 \) such that for all \( \alpha \in [0, 1] \), \( t > 0 \), and \( n \in \mathbb{N} \),

\[
\left\| \left(1 + \frac{t}{n} A \right)^{-n} x - e^{-tA} x - \frac{t^2 A^2 x}{2n} e^{-tA} x \right\| \leq C (M_{3-\alpha} + M_{4-\alpha}) \frac{t^\alpha}{n^2} \| A^\alpha x \|, \quad x \in \text{dom}(A^\alpha).
\]

Let us illustrate the sharpness of Theorem 5.12. Observe that by Taylor series expansion, for fixed \( t > 0 \),

\[
\frac{(1 + t/n)^{-n} - e^{-t}}{1/n} \xrightarrow{n \to \infty} e^{-t} \left[ \frac{t^3}{6n^3} + O(1/n^3) \right],
\]

Apply Theorem 5.12 to the scalar function \( e^{-t} \) when \( \alpha \) is either 0 or 1. If \( \alpha = 0 \) then uniformly in \( t \geq 0 \):

\[
|(1 + t/n)^{-n} - e^{-t}| \leq \left( \sup_{s > 0} s^2 e^{-s} \right) \left( \frac{1}{2n} + O(n^{-2}) \right) = 4e^{-2} \left( \frac{1}{2n} + O(n^{-2}) \right), \quad n \to \infty,
\]
and if $\alpha = 1$ then

\begin{equation}
(5.31) \quad |(1 + t/n)^{-n} - e^{-t}| \leq t \left( \sup_{s>0} se^{-s} \right) \left( \frac{1}{2n} + O(n^{-2}) \right)
= e^{-1}t \left( \frac{1}{2n} + O(n^{-2}) \right), \quad n \to \infty.
\end{equation}

Thus, the abstract estimate (5.30) agrees with its numerical counterpart (5.29) (where (5.30) is a uniform version of (5.31)).

6. Appendix 1: Derivatives

**Proposition 6.1.** If $g \in B_2$ and $g_n(t) = g^n(t/n), t \geq 0$, then

$$g_n(0) = 1, \quad g_n'(0) = -1, \quad g_n''(0) = 1 + \frac{g''(0) - 1}{n};$$

if $g \in B_3$, then

$$g_n'''(0) = \frac{1}{n^2}(-(n - 1)(n - 2) - 3(n - 1)g''(0) + g'''(0));$$

and if $g \in B_4$, then

$$g_n''''(0) = \frac{1}{n^3}((n - 1)(n - 2)(n - 3) + 6(n - 1)(n - 2)g''(0)
+ 3(n - 1)(g''(0))^2 - 4(n - 1)g'''(0) + g''''(0)).$$

**Proof.** The statement follows from the next explicit calculations:

$$g_n'(t) = g^{n-1}(t/n)g'(t/n),$$

$$g_n''(t) = (n - 1)(g'(t/n))^2 + g(t/n)g''(t/n)\frac{g^{n-2}(t/n)}{n},$$

$$g_n'''(t) = (n - 1)(n - 2)(g'(t/n))^3 + 3(n - 1)g(t/n)g'(t/n)g''(t/n)$$

$$+ g^2(t/n)g'''(t/n)\frac{g^{n-3}(t/n)}{n^2},$$

$$g_n''''(t) = (n - 1)(n - 2)(n - 3)(g'(t/n))^4$$

$$+ 6(n - 1)(n - 2)g(t/n)(g'(t/n))^2g''(t/n)
+ 3(n - 1)g^2(t/n)(g''(t/n))^2 + 4(n - 1)g^2(t/n)g'(t/n)g'''(t/n)$$

$$+ g^3(t/n)g''''(t/n)\frac{g^{n-4}(t/n)}{n^3}.$$

**Corollary 6.2.** Let $g \in B_3$ and the functionals $b$ and $d_0$ are given by (3.36) and (4.22), respectively. Then

\begin{equation}
(6.1) \quad b[g_n] = \frac{b[g]}{n^2}, \quad b[g] = \frac{-2 + 3g''(0) + g'''(0)}{6},
\end{equation}
and, if $g \in B_4$,

$$
(6.2) \quad d_0[g_n] = \frac{(g''(0) - 1)^2}{4n^2} + \frac{-6 + 12g''(0) - 3(g''(0))^2 + 4g'''(0) + g''''(0)}{12n^3}.
$$

Moreover,

$$
(6.3) \quad |b[g_n]| \leq \frac{g'''(0) - 1}{n^2}, \quad n \in \mathbb{N},
$$

and

$$
(6.4) \quad d_0[g_n] \leq \frac{g'''(0) - 1}{n^2}, \quad n \in \mathbb{N}.
$$

If $g \in B_4 \cap L^k$ for some $k \in \mathbb{N}$ and $d_1$ is defined by (4.28), then there exists $C(g) > 0$ such that

$$
(6.5) \quad d_1[g_n] \leq C(g)n^{-2}, \quad n \in \mathbb{N}.
$$

Proof. First note that the formulas (6.1) are (6.2) are direct consequences of Proposition 6.1. To prove (6.3) and (6.4), first recall that if $g \in \mathcal{CM}$, then

$$
|g^{(k+1)}(z)|^2 \leq g^{(k)}(z)g^{(k+2)}(z), \quad z > 0, \quad k \in \mathbb{N} \cup \{0\},
$$

see [45, Ch. XIII]. Thus, if in addition $g \in B_3$, then

$$
1 \leq g''(0) \leq (g''(0))^2 \leq |g'''(0)|.
$$

Moreover, for $g \in B_4$, one has

$$
1 \leq g''(0) \leq (g'''(0))^{1/2} \leq g''''(0), \quad |g'''(0)| \leq (g''(0)g''''(0))^{1/2} \leq g'''(0),
$$

hence for every $n \in \mathbb{N}$,

$$
|b[g_n]| \leq \frac{3(g''(0) - 1) + (g'''(0) + 1)}{6n^2} \leq \frac{4(g'''(0) - 1)}{6n^2} \leq \frac{(g'''(0) - 1)}{n^2},
$$

and

$$
\begin{align*}
d_0[g_n] & \leq \frac{(g''(0) - 1)^2}{4n^2} + \frac{-3 + 6g''(0) + 4g'''(0) + g''''(0)}{12n^3} \\
& \leq \frac{(g'''(0) - 1)}{4n^2} + \frac{6(g''(0) - 1) + (g'''(0) - 1)}{12n^3} \\
& \leq \frac{5(g'''(0) - 1)}{6n^2} \\
& \leq \frac{(g''(0) - 1)}{n^2}, \quad n \in \mathbb{N}.
\end{align*}
$$

Finally, (6.5) follows from $d_0[g] = -b[g]c_1[g] + c_2[g], (6.3)$ and Theorem 3.19. \qed
7. Appendix 2: Proof of Theorem 3.19

Since for each \( n \in \mathbb{N} \), the mapping \([0, 1] \ni \alpha \to c_\alpha[g_n] \) is continuous, it suffices to prove (3.51) only for \( \alpha \in (0, 1) \).

Fix \( \alpha \in (0, 1) \). Then for each \( n \in \mathbb{N} \),

\[
c_\alpha[g_n] = - \frac{1}{\alpha \Gamma(2 - \alpha)} \int_0^\infty (g_n(z) - e^{-z})dz^{-\alpha}
\]

\[
= - \frac{1}{\alpha \Gamma(2 - \alpha)} \int_0^\infty g_n(z)z^{-\alpha}dz + \frac{1}{\alpha \Gamma(2 - \alpha)} \int_0^\infty e^{-z}z^{-\alpha}dz.
\]

Hence

\[
(7.1) \quad c_\alpha[g_n] = n^{1-\alpha} \frac{R_\alpha(n)}{\alpha \Gamma(2 - \alpha)} + \frac{1}{\alpha(1-\alpha)},
\]

where

\[
R_\alpha(n) := \int_0^\infty g^{n-1}(z)g'(z)z^{-\alpha}dz, \quad n \in \mathbb{N}.
\]

Define

\[
f_\alpha(z) := \frac{z^\alpha(1 - \alpha a(1 - g(z))}{(1 - g(z))^\alpha}, \quad a = g''(0)/2,
\]

and write

\[
R_\alpha(n) = R_{0,\alpha}(n) + R_{1,\alpha}(n), \quad n \in \mathbb{N},
\]

where

\[
R_{0,\alpha}(n) := \int_0^\infty g^{n-1}(z)g'(z)z^{-\alpha}f_\alpha(z)dz,
\]

and

\[
R_{1,\alpha}(n) := \int_0^\infty g^{n-1}(z)g'(z)z^{-\alpha}[1 - f_\alpha(z)]dz.
\]

We will estimate \( R_{0,\alpha} \) and \( R_{1,\alpha} \) separately. To bound \( R_{0,\alpha}(n) \), letting \( s = g(z) \in (0, 1), \ z > 0 \), and using (3.46), we obtain

\[
R_{0,\alpha}(n) = - \int_0^1 g^{n-1}(1 - \alpha a(1 - s)) \frac{ds}{(1-s)^\alpha}
\]

\[
= - \frac{\Gamma(n)\Gamma(1-\alpha)}{\Gamma(n+1-\alpha)} + \frac{\alpha a}{\Gamma(n+2-\alpha)}.
\]

Thus, employing (3.47), we have

\[
Q_\alpha(n) := \frac{n^{1-\alpha}}{\alpha \Gamma(2 - \alpha)}R_{0,\alpha}(n) + \frac{1}{\alpha(1-\alpha)}
\]

\[
= \frac{n^{1-\alpha}}{\alpha \Gamma(2 - \alpha)} \left[ \frac{\Gamma(n)\Gamma(1-\alpha)}{\Gamma(n+1-\alpha)} + \alpha a \frac{\Gamma(n)\Gamma(2-\alpha)}{\Gamma(n+2-\alpha)} \right] + \frac{1}{\alpha(1-\alpha)}
\]

\[
= - \frac{n^{1-\alpha} \Gamma(n)}{\alpha(1-\alpha)\Gamma(n+1-\alpha)} + \frac{n^{1-\alpha} \Gamma(n)}{\alpha(1-\alpha)\Gamma(n+2-\alpha)} + \frac{1}{\alpha(1-\alpha)}.
\]
so that

\[(7.2) \quad Q_\alpha(n) = -\frac{1}{2n} + \frac{a}{n} + O(n^{-2}) = \frac{g''(0)}{2n} - \frac{1}{2n} + O(n^{-2}), \quad n \to \infty,\]

uniformly in \(\alpha \in (0, 1)\).

Let us now consider \(R_{1,\alpha}(n)\). Note that in view of (7.1) and (7.2), if there exists \(C(g) > 0\) such that

\[(7.3) \quad |R_{1,\alpha}(n)| \leq \frac{C(g)\alpha}{n^{\frac{3}{2} - \alpha}}, \quad n \in \mathbb{N},\]

then (3.51) follows. So let us proceed with the proof of (7.3). Let

\[m(z) := \frac{1 - g(z)}{z}, \quad z > 0.\]

Then

\[(7.4) \quad f_\alpha(z) = m^{-\alpha}(z) \left[1 - azm(z)\right],\]

and

\[(7.5) \quad f'_\alpha(z) = -\alpha m^{-1-\alpha}(z) \left[(1 - (1 - \alpha)azm(z))m'(z) + am^2(z)\right].\]

Since \(g \in B_2\), we have \(m \in BM\),

\[m(0) = 1, \quad m(1) \leq m(z) \leq 1, \quad z \in [0, 1],\]

\[(1 - g(1))/z \leq m(z) \leq 1/z, \quad z \geq 1,\]

and

\[m'(0) = -a, \quad |m'(z)| \leq a, \quad z \geq 0.\]

Therefore, \(f_\alpha \in C^2[0, \infty), f_\alpha(0) = 1,\) and \(f'_\alpha(0) = 0.\) Moreover, by (7.4) and (7.5), there exists \(C_1(g) > 0\) such that

\[(7.6) \quad |1 - f_\alpha(z)| \leq C_1(g)z^2, \quad |f'_\alpha(z)| \leq C_1(g)az, \quad z > 0.\]

Now, integrating by parts,

\[R_{1,\alpha}(n) = \frac{1}{n} \int_0^\infty z^{-\alpha}(1 - f_\alpha(z)) \, dg^a(z)\]

\[= \frac{1}{n} \int_0^\infty g^a(z) \left[\alpha z^{-1-\alpha}(1 - f_\alpha(z)) + z^{-\alpha}f'_\alpha(z)\right] \, dz.\]

Hence, using (7.6), we conclude that

\[|R_{1,\alpha}(n)| \leq C_1(g)\frac{\alpha}{n} \int_0^\infty g^a(z)z^{1-\alpha} \, dz,\]

and Lemma 3.18 implies (7.3).

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