THE ENTROPIC ERDŐS-KAC LIMIT THEOREM

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Abstract. We prove entropic and total variation versions of the Erdős-Kac limit theorem for the maximum of the partial sums of i.i.d. random variables with densities.

1. Introduction

Let \( \{X_n\}_{n \geq 1} \) be independent identically distributed (i.i.d.) random variables with mean \( \mathbb{E}X_1 = 0 \) and variance \( \mathbb{E}X_1^2 = 1 \). Put

\[
S_n := \sum_{k=1}^{n} X_k, \quad \overline{S}_n := \max_{k=1, \ldots, n} S_k, \quad n \in \mathbb{N}.
\]

Throughout we denote by \( Z \) a standard normal random variable with its density \( \varphi(x) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \) and use the symbol \( \Rightarrow \) to denote convergence in distribution.

The classical central limit theorem states that

\[
S_n / \sqrt{n} \Rightarrow Z \quad \text{as} \quad n \to \infty. \tag{1.1}
\]

In 1986 Barron \cite{Ba} established an entropic version of this result, the so-called \textit{entropic central limit theorem}. To formulate it, first let us introduce some notation.

Let \( Y \) be a random variable with density \( \psi \), and let \( X \) be a random variable whose distribution is absolutely continuous with respect to that of \( Y \). The relative entropy of \( X \) with respect to \( Y \) is defined by

\[
D(X \mid Y) := \int_{\{\psi(x) > 0\}} L \left( \frac{p(x)}{\psi(x)} \right) \psi(x) \, dx, \tag{1.2}
\]

where \( p \) is a density of \( X \), \( L(x) := x \log x \) for \( x > 0 \) and \( L(x) := 0 \) for \( x = 0 \). In case the distribution of \( X \) is not absolutely continuous with respect to that of \( Y \), put \( D(X \mid Y) := \infty \). Then the entropic central limit theorem by Barron states that

\[
D(S_n / \sqrt{n} \mid Z) \to 0 \quad \text{as} \quad n \to \infty \quad \text{if and only if} \quad D(S_{n_0} \mid Z) < \infty \quad \text{for some} \quad n_0 \in \mathbb{N}. \tag{1.3}
\]
by the distinguished property of the standard normal distribution that it maximizes (Shannon) entropy.

Barron’s result has sparked much further research on entropic limit theorems. For instance, there are several publications devoted to the rate of convergence, see Artstein, Ball, Barthe and Naor \[ABBN\], Johnson and Barron \[JB\], Johnson \[Jo\], and Bobkov, Chistyakov and Götze \[BCG3\]. Entropic limit theorems have also been derived for certain non-normal limit distributions within the class of stable laws, cf. \[Jo\], \[BCG4\], \[KHJ\].

All these limit distributions arise in connection with sums of i.i.d. random variables and are therefore infinitely divisible. Our aim is to investigate a different situation, namely for the maxima of sums of i.i.d. summands, with a limit distribution that is not infinitely divisible. Here the analogue of the classical central limit theorem is given by the Erdős-Kac limit theorem \[EK\], which states that 
\[ \frac{S_n}{\sqrt{n}} \rightarrow |Z| \quad \text{as} \quad n \rightarrow \infty. \] (1.4)

The distribution of $|Z|$, which has density $\phi_+(x) := \sqrt{\frac{2}{\pi}} e^{-x^2/2} \mathbf{1}_{(0,\infty)}(x)$, is commonly called the one-sided (or reflected) standard normal law. As explained below, this distribution plays a similar role to the normal distribution in that it maximizes entropy among all positive random variables with fixed second moment. It is therefore quite natural to ask whether the Erdős-Kac limit theorem \[EK\] also admits an entropic formulation.

To state a corresponding assertion, we introduce more notation. Given a random variable $X$ such that $\mathbb{P}(X > 0) > 0$, let $\tilde{X}$ have the same distribution as $X$ conditioned to be positive, i.e. $\mathbb{P}(\tilde{X} \in A) = \mathbb{P}(X \in A|X > 0)$ for Borel sets $A$ on the real line. Then the relative entropy of $X$ conditioned to be positive with respect to a positive random variable $Y$ with density $\psi$ is defined by
\[ D_+(X \mid Y) := D(\tilde{X} \mid Y). \] (1.5)

In the sequel, $Y$ will always be given by $|Z|$ or some scalar multiple of it. Our main result is as follows:

**Theorem 1.1.** Suppose that $X_1, X_2, \ldots$ are i.i.d. random variables with a density, mean zero and variance one. Then
\[ D_+(\frac{S_n}{\sqrt{n}} \mid |Z|) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \] (1.6)
if and only if
\[ D_+(X_1 \mid |Z|) < \infty. \] (1.7)

In fact, the assumption that the $X_j$ have a density is only for convenience and could be omitted. Note, however, that (1.7) implies that $X_1$ has a density on the positive half-line.

Let us recall that the relative entropy represents a rather strong measure of deviation of distributions. Indeed, by the Pinsker-Csiszár-Kullback inequality,
\[ D(X \mid Z) \geq \frac{1}{2} (d_{TV}(X, Z))^2, \] (1.8)
where $d_{TV}(X, Z)$ denotes the total variation distance between the distributions of $X$ and $Z$ (cf. \[Pi\], \[Cs\], \[Ku\], \[PiHT\]). Thus, (1.3) implies $d_{TV}(S_n/\sqrt{n}, Z) \rightarrow 0$
as \( n \to \infty \), and hence (1.1). Similarly, (1.6) implies \( d_{TV}(S_n/\sqrt{n}, |Z|) \to 0 \), and hence (1.4). This follows from (1.8) in combination with the well-known fact that, under our moment assumptions,

\[
P(S_n \leq 0) = O(n^{-1/2}) \tag{1.9}
\]

(cf. e.g. [Fe, pp. 414f]). In fact, for convergence in total variation distance, condition (1.7) is not needed, since we have:

**Theorem 1.2.** Suppose that \( X_1, X_2, \ldots \) are i.i.d. random variables with a density, mean zero and variance one. Then

\[
d_{TV}(S_n/\sqrt{n}, |Z|) \to 0 \quad \text{as} \quad n \to \infty.
\]

As already mentioned, both the centered and the one-sided normal distribution play a special role from the viewpoint of information theory. Let us recall that for a random variable \( X \) with density \( p \), the entropy (also called Shannon entropy or differential entropy) is defined by

\[
h(X) := - \int_{-\infty}^{\infty} L(p(x)) \, dx,
\]

where \( L \) is as in (1.2). If \( E X^2 = \sigma^2 \) is finite, then the entropy is well-defined, and

\[
h(\sigma Z) - h(X) = D(X |\sigma Z) \geq 0 \quad (Z \sim N(0, 1)),
\]

with equality if and only if \( X \) and \( \sigma Z \) have the same distribution. Thus, the centered normal distribution with second moment \( \sigma^2 \) maximizes entropy among all probability measures with the same second moment. Moreover, in

\[
D(X |\tau Z) = -h(X) + \frac{1}{2} \log(2\pi \tau^2) + \frac{1}{2} \sigma^2/\tau^2 \quad (\tau > 0)
\]

the right-hand side is minimized for \( \tau = \sigma \), so \( D(X |\sigma Z) \) may be interpreted as a measure of deviation of the distribution of \( X \) from the class of all centered normal distributions. Similarly, for a positive random variable \( X \) with finite second moment \( \mathbb{E} X^2 = \sigma^2 \),

\[
h(\sigma |Z|) - h(X) = D_+(X |\sigma |Z|) \geq 0,
\]

with equality if and only if \( X \) and \( \sigma |Z| \) have the same distribution. Hence, the one-sided normal distribution with second moment \( \sigma^2 \) maximizes entropy among all probability measures on the positive half-line with the same second moment. Also, in

\[
D_+(X |\tau |Z|) = -h(X) + \frac{1}{2} \log(\frac{1}{2} \pi \tau^2) + \frac{1}{2} \sigma^2/\tau^2 \quad (\tau > 0)
\]

the right-hand side is minimized for \( \tau = \sigma \). Therefore, as above, \( D_+(X |\sigma |Z|) \) may be interpreted as a measure of deviation of the distribution of \( X \) from the class of all one-sided normal distributions.

In this respect, note that

\[
\mathbb{E}(S_n^+ /\sqrt{n})^2 = 1 + o(1) \quad \text{as} \quad n \to \infty, \tag{1.10}
\]

see e.g. Section 6 below. Combining (1.9) and (1.10), it is easy to see that for large \( n \), \( S_n/\sqrt{n} \) conditioned to be positive has second moment approximately equal to 1, so that the comparison to \( |Z| \) in (1.6) is natural.
Finally, let us emphasize the following curious difference between the entropic central limit theorem and our Theorem 1.1. Even if $X_1$ itself has density, Barron’s characterization uses the finiteness of $D(S_{n_0} \mid Z)$ for some $n_0 \in \mathbb{N}$ (which may be any natural number); see [Ba] for an example requiring $n_0 > 1$. In contrast to that, our characterization uses $n_0 = 1$ at once. More precisely, it follows from our proof that $D_+(S_{n_0} \mid \mid Z) < \infty$ for some $n_0 \in \mathbb{N}$ if and only if this is true for $n_0 = 1$.

In the proof of (1.3) given in [Ba], entropy convolution inequalities for sums of independent random variables play a major role. In our analysis for the maxima of sums, these inequalities still play a role in the proof of Theorem 1.1, but they have less far-reaching consequences. To control the density of the maximum, we use more classical methods and results based on Fourier analysis, see Nagaev [N1, N2] and Aleshkyavichene [Al2]. This approach does not only lead to proofs of entropic limit theorems (cf. [BCG4]), but in principle, similarly as in [BCG3], it should also lead to results on the (exact) rate of convergence. Apparently, such refined results cannot be obtained by using known information-theoretic tools.

A major ingredient in our proof will be the local limit theorem for maxima of sums of i.i.d. random variables from [Al2], see also [AI1, NE, Wa] for related results. To obtain (1.6) under minimal conditions, we need to extend the result from [Al2] from bounded to unbounded densities (see Proposition 4.2).

Let us introduce some conventions for the rest of the paper. We assume that the random variables $X_j$ are i.i.d. and have a density, mean 0 and variance 1. Unless otherwise indicated, we write $p$ for their density, $F$ for their distribution function and $f$ for their characteristic function. Moreover, let $p_n, F_n, f_n$ and $\overline{p}_n, \overline{F}_n, \overline{f}_n$ denote the corresponding functions for the random variables $S_n$ and $\overline{S}_n$. We write $p^*_n$ and $\overline{p}^*_n$ for the densities of the rescaled random variables $S_n/\sqrt{n}$ and $\overline{S}_n/\sqrt{n}$.

For a real number $x$, set $x^+ := \max\{x, 0\}$ and $x^- := \max\{-x, 0\}$. Unless otherwise indicated, $O$-bounds and $o$-bounds refer to the case where $n \to \infty$ and hold uniformly in $x$ (in the region under consideration). Finally, $C_1, C_2, \ldots$ denote positive constants which may depend on the distribution of the $X_j$ and which may change from step to step.

The paper is organized as follows. Section 2 contains some preliminary remarks on relative entropy. Section 3–7 are devoted to the proof of the sufficiency part of Theorem 1.1 while the necessity part of Theorem 1.1 is proved in Section 8. Section 9 contains the proof of Theorem 1.2.

2. SOME REMARKS ON RELATIVE ENTROPY

Throughout this section, let $\psi$ be a positive probability density on the positive half-line. Given a non-negative measurable function $f$ on the real line, set

$$D(f \mid \psi) := \int_{0}^{\infty} L\left(\frac{f(x)}{\psi(x)}\right) \psi(x) \, dx,$$  \hspace{1cm} (2.1)

where $L(x)$ is the function defined in the introduction. By abuse of terminology, we will call $D(f \mid \psi)$ relative entropy even when $f$ is not a probability density on the positive half-line. Note that in this special case, we have $D(f \mid \psi) \geq 0$ by Jensen’s inequality. If $f$ is an arbitrary non-negative measurable function, this need not be true anymore, but we have at least $D(f \mid \psi) \geq \min\{L(x) : x \geq 0\} = -e^{-1}$. 

Let us collect some basic properties of relative entropy which will be used later. (Some of the proofs are straightforward, which is why we omit them.)

**Lemma 2.1.** Suppose that $\alpha$ is a positive real number and $f$ is a non-negative measurable function with $\int_0^\infty f(x) \, dx < \infty$. Then

$$D(\alpha f | \psi) = \alpha D(f | \psi) + L(\alpha) \int_0^\infty f(x) \, dx.$$ 

**Lemma 2.2.** Suppose that $\alpha_1, \ldots, \alpha_n$ are positive real numbers and $f_1, \ldots, f_n$ are non-negative measurable functions with $\int_0^\infty f_k(x) \, dx < \infty$, $k = 1, \ldots, n$. Then

$$D \left( \sum_{k=1}^n \alpha_k f_k \right) \leq \sum_{k=1}^n \alpha_k D(f_k | \psi) + \left( \log \sum_{k=1}^n \alpha_k \right) \sum_{k=1}^n \alpha_k \int_0^\infty f_k(x) \, dx.$$ 

**Lemma 2.3.** Suppose that $\psi$ is decreasing on the positive half-line and that $f$ and $g$ are probability densities on $(0, +\infty)$ and $(-\infty, 0)$, respectively. Then

$$D(f * g | \psi) \leq D(f | \psi) + e^{-1}.$$ 

**Proof of Lemma 2.3.** Since $L$ is a convex function and $g$ is a probability density on $(-\infty, 0)$, it follows from Jensen’s inequality that

$$L \left( \int_{-\infty}^0 h(y) g(y) \, dy \right) \leq \int_{-\infty}^0 L(h(y)) g(y) \, dy$$

for any non-negative measurable function $h$. We therefore obtain

$$D(f * g | \psi) = \int_0^\infty L \left( \int_{-\infty}^0 \frac{f(x-y)}{\psi(x)} g(y) \, dy \right) \psi(x) \, dx$$

$$\leq \int_0^\infty \int_{-\infty}^0 L \left( \frac{f(x-y)}{\psi(x)} \right) g(y) \, dy \psi(x) \, dx$$

$$= \int_{-\infty}^0 \int_0^\infty f(x-y) \log \left( \frac{f(x-y)}{\psi(x)} \right) \, dx \, dy.$$ 

Since $\psi(x)$ is decreasing in $x$, we have, for any $y < 0$,

$$\int_0^\infty f(x-y) \log \left( \frac{f(x-y)}{\psi(x)} \right) \, dx \leq \int_0^\infty f(x-y) \log \left( \frac{f(x-y)}{\psi(x-y)} \right) \, dx$$

$$= \int_0^\infty L \left( \frac{f(u)}{\psi(u)} \right) \psi(u) \, du - \int_0^{-y} L \left( \frac{f(u)}{\psi(u)} \right) \psi(u) \, du \leq D(f | \psi) + e^{-1}.$$ 

Combining these estimates, we get

$$D(f * g | \psi) \leq D(f | \psi) + e^{-1},$$

and the lemma is proved. $\square$

**Lemma 2.4.** Suppose that $f$ and $g$ are non-negative measurable functions with $\alpha := \int_0^\infty f(x) \, dx < \infty$ and $\beta := \int_0^\infty g(x) \, dx < \infty$. Then

$$D(f | \psi) + D(g | \psi) \leq D(f + g | \psi) \leq D(f | \psi) + D(g | \psi) + L(\alpha + \beta) - L(\alpha) - L(\beta).$$
Lemma 2.5. Suppose that measurable functions \( f, g \) with \( \int_{0}^{\infty} f(x) \, dx < \infty, \int_{0}^{\infty} g(x) \, dx < \infty \) and any \( \alpha, \beta > 0 \), we have

\[
D(f + g \mid \psi) = (\alpha + \beta) D\left( \frac{\alpha}{\alpha + \beta} f + \frac{\beta}{\alpha + \beta} g \mid \psi \right) + L(\alpha + \beta)
\]

\[
\leq (\alpha + \beta) \left[ \frac{\alpha}{\alpha + \beta} D\left( \frac{\alpha}{\alpha + \beta} f \mid \psi \right) + \frac{\beta}{\alpha + \beta} D\left( \frac{\beta}{\alpha + \beta} g \mid \psi \right) \right] + L(\alpha + \beta)
\]

\[
= \alpha D\left( \frac{\alpha}{\alpha + \beta} f \mid \psi \right) + \beta D\left( \frac{\beta}{\alpha + \beta} g \mid \psi \right) + L(\alpha + \beta)
\]

\[
= D(f \mid \psi) + D(g \mid \psi) - L(\alpha) - L(\beta) + L(\alpha + \beta).
\]

On the other hand, it is straightforward to check that \( L(x + y) \geq L(x) + L(y) \) for any \( x, y \geq 0 \), whence \( D(f + g \mid \psi) \geq D(f \mid \psi) + D(g \mid \psi) \).

In particular, it follows from Lemmas \( \ref{lem:2.1} \) and \( \ref{lem:2.2} \) that for any non-negative measurable functions \( f, g \) with \( \int_{0}^{\infty} f(x) \, dx < \infty, \int_{0}^{\infty} g(x) \, dx < \infty \) and any \( \alpha, \beta > 0 \), we have

\[
D(\alpha f + \beta g \mid \psi) < \infty \quad \text{if and only if} \quad D(f \mid \psi) < \infty \quad \text{and} \quad D(g \mid \psi) < \infty. \quad (2.2)
\]

Lemma 2.5. Suppose that \((f_n)\) and \((g_n)\) are sequences of non-negative measurable functions such that \( \int_{0}^{\infty} f_n(x) \, dx = 1 = o(1) \) and \( \int_{0}^{\infty} g_n(x) \, dx = o(1) \) as \( n \to \infty \). Then

\[
D(f_n + g_n \mid \psi) = D(f_n \mid \psi) + D(g_n \mid \psi) + o(1) \quad \text{as} \quad n \to \infty.
\]

Proof. This is an immediate consequence of Lemma \( \ref{lem:2.4} \). \( \square \)

In the following sections, \( \psi \) will always be given by the probability density \( \varphi_+(x) := \sqrt{\frac{2}{\pi}} e^{-x^2/2} (x > 0) \) or its rescaled version \( \varphi_{n, +}(x) := \sqrt{\frac{2}{\pi n}} e^{-x^2/2n} (x > 0) \), where \( n \in \{1, 2, 3, \ldots\} \). Note that \( \varphi_{n, +} \) is the density of the one-sided normal distribution with second moment \( n \). It is easy to check that for any non-negative measurable function \( f \), we have

\[
D(\sqrt{n} f(\sqrt{n} \cdot) \mid \varphi_+) = D(f \mid \varphi_{n, +}). \quad (2.3)
\]

3. Binomial Decomposition

In this section we start with the proof of sufficiency in Theorem \( \ref{thm:1.1} \). In the sequel, by a signed density we mean any measurable function \( h(x) \) defined on the real line or on the positive half-line such that \( \int_{-\infty}^{\infty} |h(x)| \, dx < \infty \). Since it is more convenient to work with bounded densities, we use a binomial decomposition of the density \( p \) to write the density \( \overline{p}_n \) (restricted to the positive half-line) as the sum of two signed densities, a bounded term \( \overline{p}_n^* \) and a remainder term \( \overline{r}_n^* \). This representation will play an important role in the proof of the sufficiency part of Theorem \( \ref{thm:1.1} \). Let us remark that binomial decompositions are a well-known tool in the investigation of the classical central limit theorem, see e.g. \( \text{[SM]} \), \( \text{[LL]} \). In connection with entropic central limit theorems, they have recently been used in \( \text{[BCG3]} \), \( \text{[BCG4]} \).

Recall that \( p \) is the density of \( X_1 \). Write

\[
p = (1 - \varrho) q_1 + \varrho q_2, \quad (3.1)
\]
where \( q_1 \) is a bounded probability density with \( \int_0^\infty q_1(x) \, dx > 0 \), \( q_2 \) is a potentially unbounded probability density, and \( 0 \leq \varrho < \frac{1}{2} \). It follows that for any \( n \geq 1 \),

\[
p_n(x) = p^{*n}(x) = \left( \sum_{k=1}^{n} \binom{n}{k} (1 - \varrho)^k \varrho^{n-k} \left( q_1^{*k} * q_2^{*(n-k)} \right)(x) \right) + \varrho^n q_2^{*n}(x) =: (1 - \varrho^n)q_{n,1}(x) + \varrho^n q_{n,2}(x), \tag{3.2}
\]

where \( q_{n,1}(x) \) and \( q_{n,2}(x) \) are again probability densities.

We now need the following formula due to Nagaev [N3, Equation (0.8)]:

For \( n \in \mathbb{N} \) and \( t \in \mathbb{R} \), we have

\[
\mathbb{E}e^{it\overline{S}_n} = \sum_{k=1}^{n} f^k(t)\overline{\varphi}_{n-k}(t), \tag{3.3}
\]

where

\[
\overline{\varphi}_0(t) := 1 \quad \text{and} \quad \overline{\varphi}_k(t) := \int_{-\infty}^{0} (1 - e^{itx}) d\overline{\varphi}_k(x) \quad (k > 0). \tag{3.4}
\]

By (3.3) and the uniqueness theorem for Fourier transforms (of signed measures), it follows that the density of \( \overline{\varphi}_n := \max\{S_1, \ldots, S_n\} \) is given by

\[
\overline{\varphi}_n(x) = \sum_{k=1}^{n} (p^{*k} * G_{n-k})(x),
\]

where

\[
G_0(dx) := \delta_0(dx), \quad G_k(dx) := \overline{\varphi}_k(0)\delta_0(dx) - \overline{\varphi}_k(x)1_{(-\infty,0)}(x) \, dx \quad \text{for} \ k > 0.
\]

and \((p^{*k} * G_{n-k})(x) := \int p^{*k}(x-y)G_{n-k}(dy)\).

Using (3.2), we may write

\[
\overline{\varphi}_n(x) = \overline{q}_n(x) + \overline{r}_n(x), \tag{3.5}
\]

where

\[
\overline{q}_n(x) := \sum_{k=1}^{n} (1 - \varrho^k) (q_{k,1} * G_{n-k})(x), \quad \overline{r}_n(x) := \sum_{k=1}^{n} \varrho^k (q_{k,2} * G_{n-k})(x). \tag{3.6}
\]

Note that each \( \overline{q}_n \) is bounded, since the \( q_{k,1} \) are bounded and the \( G_{n-k} \) are finite signed measures. The main idea is to use \( \overline{q}_n \) as a bounded approximation to \( \overline{\varphi}_n \).

Of course, \( \overline{q}_n \) and \( \overline{r}_n \) are only signed densities in general. However, they may be represented as differences of non-negative densities by writing

\[
\overline{q}_n(x) = \overline{q}_n^+(x) - \overline{q}_n^-(x) \quad \text{and} \quad \overline{r}_n(x) = \overline{r}_{n,1}(x) - \overline{r}_{n,2}(x),
\]

where \( \overline{q}_n^+ \) and \( \overline{q}_n^- \) denote the positive and negative part of \( \overline{q}_n \) and \( \overline{r}_{n,1} \) and \( \overline{r}_{n,2} \) are defined by

\[
\overline{r}_{n,j}(x) := \sum_{k=1}^{n} \varrho^k (q_{k,2} * G_{n-k}^\pm)(x)
\]

\((j = 1, 2)\), where \( \pm = + \) for \( j = 1 \), \( \pm = - \) for \( j = 2 \), and \( G_{n-k}^+ \) and \( G_{n-k}^- \) denote the positive and negative part of the signed measure \( G_{n-k} \). Note that \( \overline{r}_{n,1} \) and \( \overline{r}_{n,2} \) are not the positive and negative part of \( \overline{r}_n \) in general.
Thus, we obtain
\[ \mathcal{P}_n = (q_n^+ - q_n^-) + (\tau_{n,1} - \tau_{n,2}) \quad (3.7) \]
or (equivalently)
\[ \mathcal{P}_n + q_n^- + \tau_{n,2} = q_n^+ + \tau_{n,1}. \quad (3.8) \]
Write \( \mathcal{P}_n^*(x) := \sqrt{n} \mathcal{P}_n(\sqrt{n}x) \), \( q_n^*(x) := \sqrt{n} q_n(\sqrt{n}x) \), \( \tau_n^*(x) := \sqrt{n} \tau_n(\sqrt{n}x) \), etc. for the rescaled versions of the above densities. We then have the following result.

**Lemma 3.1.**
(a) \( \int_0^\infty |\mathcal{P}_n^*(x)| \, dx = O(n^{-1/2}) \).
(b) \( \int_0^\infty x^2 |\mathcal{P}_n^*(x)| \, dx = O(n^{-1/2}) \).
(c) If \( (1.7) \) holds then \( D(\mathcal{P}_n^* \mid \varphi_+) = D((q_n^*)^+ \mid \varphi_+) + o(1) \) as \( n \to \infty \).

**Proof.** Throughout this proof, for any measurable function \( p \), we write
\[ \|p\|_1 := \int_0^\infty |p(x)| \, dx \]
for the total variation norm (of the associated signed measure) and
\[ \|p\|_\infty := \sup_{x \in (0, \infty)} |p(x)| \]
for the supremum norm. Furthermore, if \( p \) is non-negative, we write \( D(p \mid \varphi_+) \) for the relative entropy as in \( (2.1) \). Recall the probability densities \( \varphi_{n,+} \) introduced at the end of Section 2.

**Analysis of \( \tau_{n,j}^*(x) \).** By \( (1.9) \), \( \mathcal{F}_n(0) = O(n^{-1/2}) \) as \( n \to \infty \). Thus,
\[ \|\mathcal{T}_{n,j}^*\|_1 = \|\mathcal{T}_{n,j}\|_1 \leq \sum_{k=1}^n \mathcal{F}_{n-k}(0) g_k \leq \sum_{k=1}^n \frac{C_1 g_k}{\sqrt{n-k+1}} = O(n^{-1/2}) , \quad (3.9) \]
j = 1, 2. Also, since \( G_{n-k}^\pm \) is concentrated on \((-\infty, 0]\),
\[ \int_0^\infty x^2 \mathcal{T}_{n,j}^*(x) \, dx \leq \frac{1}{n} \sum_{k=1}^n \frac{C_1 g_k}{\sqrt{n-k+1}} \int_{-\infty}^\infty x^2 q_{k,2}(x) \, dx , \]
j = 1, 2. Let \( Y_1, \ldots, Y_k \) be i.i.d. random variables with density \( q_2 \). Then
\[ \int_{-\infty}^\infty x^2 q_{k,2}(x) \, dx = \|Y_1 + \ldots + Y_k\|^2_2 \leq k^2 \|Y_1\|^2_2 , \]
and we come to the conclusion that
\[ \int_0^\infty x^2 \mathcal{T}_{n,j}^*(x) \, dx \leq \frac{1}{n} \sum_{k=1}^n \frac{C_1 g_k}{\sqrt{n-k+1}} \int_{-\infty}^\infty x^2 q_{k,2}(x) \, dx = O(n^{-3/2}) , \quad (3.10) \]
j = 1, 2. Clearly, \( (3.9) \) and \( (3.10) \) imply (a) and (b).

We will now show that if \( (1.7) \) holds then
\[ D(\mathcal{T}_{n,j}^* \mid \varphi_+) = D\left( \sum_{k=1}^n g_k q_{k,2} * G_{n-k}^\pm \mid \varphi_{n,+} \right) = o(1) , \quad (3.11) \]
j = 1, 2. We provide the details for \( \mathcal{T}_{n,2}^* \) only; the argument for \( \mathcal{T}_{n,1}^* \) being similar.
Note that $G_0^- = 0$. For $k = 1, \ldots, n - 1$, write $G_{n-k}^-(dx) = \overline{T}_{n-k}(0) s_{n-k}(x) dx$, where $s_{n-k}(x) := \overline{T}_{n-k}(x)/\overline{T}_{n-k}(0)$ $(x < 0)$ is a probability density on $(-\infty, 0)$. Also, write $q_2 = \lambda_+ q_{2,+} + \lambda_- q_{2,-}$, where $\lambda_+, \lambda_- \geq 0$, $\lambda_+ + \lambda_- = 1$, and $q_{2,+}$ and $q_{2,-}$ are probability densities on $(0, +\infty)$ and $(-\infty, 0)$, respectively. Then

$$q_{k,2} = \sum_{j=0}^{k} \binom{k}{j} \lambda_+^j \lambda_-^{k-j} q_{2,+}^* q_{2,-}^*(k-j),$$

and it follows by a two-fold application of Lemma 2.2 that

$$D \left( \sum_{k=1}^{n} \frac{q_{k,2}^* G_{n-k}^-}{\varphi_{n,+}} \right) \leq \sum_{k=1}^{n-1} q_{k,2}^* \overline{T}_{n-k}(0) D \left( q_{k,2}^* s_{n-k} \mid \varphi_{n,+} \right) + O(\log n/\sqrt{n})$$

$$\leq \sum_{k=1}^{n-1} q_{k,2}^* \overline{T}_{n-k}(0) \sum_{j=1}^{k} \binom{k}{j} \lambda_+^j \lambda_-^{k-j} D \left( q_{2,+}^* q_{2,-}^*(k-j) s_{n-k} \mid \varphi_{n,+} \right) + O(\log n/\sqrt{n}).$$

(For the last step, note that $D(q_{2,-}^k s_{n-k} \mid \varphi_{n,+}) = 0$.) Using Lemma 2.3 with $f(x) := q_{2,+}^j(x)$ and $g(x) := (q_{2,-}^*(k-j) s_{n-k})(x)$, we get

$$D \left( q_{2,+}^j q_{2,-}^*(k-j) s_{n-k} \mid \varphi_{n,+} \right) \leq D \left( q_{2,+}^j \mid \varphi_{n,+} \right) + e^{-1}.$$

Let $\mu$ and $\sigma^2$ denote the mean and variance of the probability density $q_{2,+}$, and let $\varphi_{\mu,\sigma^2}$ denote the density of the Gaussian distribution with mean $\mu$ and variance $\sigma^2$. As a consequence of the entropy power inequality (see e.g. Theorem 4 in [DCT]), we have

$$D(q_{2,+}^j \mid \varphi_{\mu,\sigma^2}) \leq D(q_{2,+} \mid \varphi_{\mu,\sigma^2}), \quad j \geq 1.$$

We therefore obtain

$$D(q_{2,+}^j \mid \varphi_{n,+}) = \int_0^\infty q_{2,+}^j \log \frac{q_{2,+}^j \varphi_{\mu,\sigma^2}}{\varphi_{n,+}} \, dx$$

$$= D(q_{2,+}^j \mid \varphi_{\mu,\sigma^2}) + \int_0^\infty q_{2,+}^j \log \frac{\varphi_{\mu,\sigma^2}}{\varphi_{n,+}} \, dx$$

$$\leq D(q_{2,+} \mid \varphi_{\mu,\sigma^2}) + O(\log n + j + 1)$$

$$= \int_0^\infty q_{2,+} \log \frac{q_{2,+} \varphi_{\mu,\sigma^2}}{\varphi_{\mu,\sigma^2}} \, dx + O(\log n + j + 1)$$

$$= D(q_{2,+} \mid \varphi_{\mu,\sigma^2}) + \int_0^\infty q_{2,+} \log \frac{\varphi_{\mu,\sigma^2}}{\varphi_{\mu,\sigma^2}} \, dx + O(\log n + j + 1)$$

$$= O(\log n + j + 1),$$

the implicit constants depending only on $q_{2,+}$. Here the last step follows from (1.7), see the remark below Lemma 2.4.
Combining the preceding estimates, it follows that
\[
D \left( \sum_{k=1}^{n} \theta^k q_{k,2} G_{n-k}^- \right) \phi_{n,+} \leq \sum_{k=1}^{n-1} \theta^k F_{n-k}(0) O(\log n + k + 1) + O(\log n/\sqrt{n}) = O(\log n/\sqrt{n}),
\]
and the proof of (3.11) is complete.

**Analysis of** \((\bar{\tau}_n^*)^\pm(x)\). To complete the proof of part (c), we will show that the relative entropy of the main terms \(\bar{\tau}_n\) and \((\bar{\tau}_n^*)^+\) in (3.8) is “stable” w.r.t. the addition of the error terms \(\bar{\tau}_{n,1}^*\), \(\bar{\tau}_{n,2}^*\) and \((\bar{\tau}_n^*)^-\). To begin with, it follows from (3.8) that
\[
(\bar{\tau}_n^*)^+ \leq \bar{\tau}_n^* + \bar{\tau}_{n,2}^* \text{ and } (\bar{\tau}_n^*)^- \leq \bar{\tau}_{n,1}^*\]
and therefore, since \(\|\bar{\tau}_n^*\|_1 = 1 - \bar{\tau}_n(0) = 1 + O(1/\sqrt{n})\) and \(\|\bar{\tau}_{n,j}^*\|_1 = O(1/\sqrt{n})\) \((j = 1, 2)\),
\[
\|(\bar{\tau}_n^*)^+\|^1_1 = 1 + O(1/\sqrt{n}) \text{ and } \|(\bar{\tau}_n^*)^-\|^1_1 = O(1/\sqrt{n}).
\]

Next we will show that
\[
D((\bar{\tau}_n^*)^- | \phi_+) = D(\bar{\tau}_n^- | \phi_{n,+}) = o(1) \tag{3.13}
\]
Since \(q_1\) is bounded by construction, \((1 - \theta^k)q_{k,1}\) is bounded uniformly in \(k \geq 1\), and we obtain
\[
\|\bar{\tau}_n\|_{\infty} = \| \sum_{k=1}^{n} (1 - \theta^k)q_{k,1} * G_{n-k} \|_{\infty} = O \left( \sum_{k=1}^{n} \frac{1}{\sqrt{n-k+1}} \right) = O(\sqrt{n}).
\]
Since \(\|\bar{\tau}_n^-\|_1 = O(1/\sqrt{n})\), it follows that
\[
D(\bar{\tau}_n^- | \phi_{n,+}) = \int_0^\infty \bar{\tau}_n^- \log(\bar{\tau}_n^-/\phi_{n,+}) dx \leq \int_0^\infty \bar{\tau}_n^- \log(C_1 \sqrt{n}/\phi_{n,+}) dx
= O\left( \frac{\log n}{\sqrt{n}} \right) + O\left( \int_0^\infty \frac{1}{n} x^2 \bar{\tau}_n(x) dx \right).
\]
Now, using (3.8) and (3.10), we have
\[
\int_0^\infty \frac{1}{n} x^2 \bar{\tau}_n^*(x) dx \leq \int_0^\infty \frac{1}{n} x^2 \bar{\tau}_{n,1}^*(x) dx = \int_0^\infty y^2 \tau_{n,1}^*(y) dy = O(n^{-3/2}).
\]
This completes the proof of (3.13).

Using (3.8), (3.11), (3.13) as well as Lemma (2.5), we now obtain
\[
D(\bar{\tau}_n^* | \phi_+) = D(\bar{\tau}_n^* + (\bar{\tau}_n^*)^- + \bar{\tau}_{n,2}^* | \phi_+) + o(1)
= D((\bar{\tau}_n^*)^+ + \bar{\tau}_{n,1}^* | \phi_+) + o(1)
= D((\bar{\tau}_n^*)^+ | \phi_+) + o(1)
\]
as \(n \to \infty\), and Lemma (3.1) is proved.
4. PROOF OF SUFFICIENCY IN THEOREM 1.1

This section contains the main part of the proof of sufficiency in Theorem 1.1. It relies on two auxiliary results which do not depend on condition (1.7) and whose proof is postponed to the following sections.

**Proposition 4.1.** For any $\varepsilon > 0$, there exists a constant $C > 0$ such that
\[
\int_{C}^{\infty} x^2 \overline{p}^*(x) \, dx \leq \varepsilon
\]
for all sufficiently large $n \in \mathbb{N}$.

**Proposition 4.2.** Under the assumptions of Theorem 1.1, there exist signed densities $r_n(x)$ such that $\|r_n\|_1 = \mathcal{O}(1/\sqrt{n})$, $\|r_n\|_\infty = \mathcal{O}(1)$ and the following holds:
\(\text{(a) Uniformly in } x \in (0, \infty),\)
\[
\overline{q}^*_n(x) = \varphi_+(x) + r_n(x) + o(1/x) \quad \text{as } n \to \infty.
\]
\(\text{(b) Uniformly in } x \in (0, e^{-1}),\)
\[
\overline{q}^*_n(x) = \varphi_+(x) + r_n(x) + \mathcal{O}\left(\log n \wedge \frac{1}{\sqrt{nx}}\right) + \mathcal{O}\left(\log x^{-1}\right) \quad \text{as } n \to \infty.
\]

Here the norms $\| \cdot \|_1$ and $\| \cdot \|_\infty$ are defined as in the proof of Lemma 3.1. Moreover, by the statement that the $\mathcal{O}$-bounds and $o$-bounds hold uniformly in $x$, we mean that for sufficiently large $n \in \mathbb{N}$, the error term is bounded by $\varepsilon_n/x$ in part (a), where $(\varepsilon_n)_{n \in \mathbb{N}}$ is a sequence of positive real numbers not depending on $x \in (0, \infty)$ such that $\lim_{n \to \infty} \varepsilon_n = 0$, and by $C_1 \left(\log n \wedge \frac{1}{\sqrt{nx}}\right) + C_2 (\log x^{-1})$ in part (b), where $C_1$ and $C_2$ are positive constants not depending on $x \in (0, e^{-1})$. Similar conventions apply to the error terms in the proof of Proposition 4.2.

Note that Proposition 4.2 may be regarded as a local version of the Erdős-Kac theorem (1.4). Moreover, part (b) is a refinement of part (a) which yields a better estimate for the error term for $x \approx 0$. Although this estimate is still unbounded, it is square-integrable near the origin. This is the crucial point for our purposes.

It should be mentioned that the proof of Proposition 4.2 closely follows that in Aleshkyavichene [A12], which is based on earlier work by Nagaev [N1, N2, N3]. Indeed, in the special case where the $X_j$ have a bounded density $p(x)$, we could take
\[
\overline{q}^*_n(x) := \overline{p}^*_n(x) \quad \text{and} \quad r_n(x) := \overline{F}_{n-1}(0) \sqrt{n} p(\sqrt{n} x) \quad (x > 0),
\]
and part (a) specializes to the following result from the literature:

**Theorem 4.3** (Aleshkyavichene [A12]). If $X_1, X_2, \ldots$ have a bounded density $p(x)$, we have $\overline{q}^*_n(x) = \varphi_+(x) + \overline{F}_{n-1}(0) \sqrt{n} p(\sqrt{n} x) + o(1/x)$, uniformly in $x \in (0, \infty)$.

**Remark.** In [A12] Theorem 4.3 is stated somewhat differently (for any $x_0 > 0$, the last term is of order $o(1)$ uniformly in $x > x_0$), but a careful analysis of the proof shows that after some minor modifications (similar to those in the proof of part (a) of Proposition 4.2 below), it also yields the result stated above.

In the general case, the definition of the signed densities $r_n(x)$ is more complicated, see Equation (7.5) below.
Proof of Sufficiency in Theorem 1.1. Suppose that (1.7) holds. Recall that $\overline{p}_n^+(x)$ is the density of $\overline{S}_n/\sqrt{n}$ (i.e. with the proper rescaling), and $\varphi_+(x) = \sqrt{2/\pi} e^{-x^2/2} (x > 0)$. Using (1.9), it is easy to see that

$$D_+ (\overline{S}_n/\sqrt{n} \mid |Z|) \to 0 \text{ if and only if } D(\overline{p}_n^+ \mid \varphi_+) \to 0. \quad (4.1)$$

Indeed, since $\overline{S}_n/\sqrt{n}$ conditioned to be positive has the density $\overline{p}_n(x)/(1 - \overline{F}_n(0))$ ($x > 0$), it follows from our definitions and Lemma 2.1 that

$$D(\overline{p}_n^+ \mid \varphi_+) = (1 - \overline{F}_n(0)) D_+ (\overline{S}_n/\sqrt{n} \mid |Z|) + L(1 - \overline{F}_n(0)),$$

so that (4.1) follows from (1.9).

Since $D_+ (\overline{S}_n/\sqrt{n} \mid |Z|) \geq 0$, it also follows from the preceding argument that

$$\liminf_{n \to \infty} D(\overline{p}_n^+ \mid \varphi_+) \geq 0.$$ \[ \text{Thus, it remains to show that} \]

$$\limsup_{n \to \infty} D(\overline{p}_n^+ \mid \varphi_+) \leq 0.$$ \[ \text{Recall that } \overline{p}_n(x) := \sqrt{n} \overline{q}_n(\sqrt{n} x), \text{ where } \overline{q}_n \text{ is defined in (3.6). By Lemma 3.1(c), it is sufficient to show that} \]

$$\limsup_{n \to \infty} D((\overline{p}_n^+)^+ \mid \varphi_+) \leq 0.$$ \[ \text{Fix } \varepsilon_0 > 0, \text{ and let } C \text{ and } c \text{ be positive real numbers with } 0 < c < 1 < C < \infty. \text{ (The precise choices will be specified below.) Then} \]

$$D((\overline{p}_n^+)^+ \mid \varphi_+) = \int_0^\infty L \left( \frac{(\overline{q}_n^+)^+(x)}{\varphi_+(x)} \right) \varphi_+(x) dx = E_1 + E_2 + E_3,$$

where $E_1, E_2, E_3$ denote the integrals over the intervals $(0, c), (c, C), (C, \infty)$, respectively. (Note that $E_1, E_2, E_3$ implicitly depend on $n$.) To complete the proof, we will show that if $C \in (1, \infty)$ is sufficiently large and $c \in (0, 1)$ is sufficiently small, then, for each $j \in \{1, 2, 3\}$, $E_j \leq \varepsilon_0$ for all sufficiently large $n \in \mathbb{N}$.

**Estimating $E_3$.** By Proposition 4.2(a), there exists a constant $M > 1$ (not depending on $n$) such that for $n \geq n_0$ and $x \geq 1$, $|\overline{p}_n(x)| \leq M$. It follows that

$$E_3 \leq \int_C^\infty \overline{p}_n(x) (|\log M + \frac{1}{2} \log \frac{x}{2} + \frac{1}{2} x^2|) dx \leq C_1 \int_C^\infty x^2 |\overline{p}_n(x)| dx,$$

where $C_1$ is a constant depending only on $M$. By Proposition 4.1, there exists a constant $C > 1$ such that

$$\int_C^\infty x^2 |\overline{p}_n(x)| dx < \varepsilon_0/C_1$$

for all sufficiently large $n \in \mathbb{N}$. By Lemma 5.1(b), this implies

$$\int_C^\infty x^2 |\overline{q}_n(x)| dx < \varepsilon_0/C_1$$

for all sufficiently large $n \in \mathbb{N}$. Thus, for $C$ sufficiently large, we have $E_3 \leq \varepsilon_0$ for all sufficiently large $n \in \mathbb{N}$. 

Thus, using Proposition 4.2 (a), it follows that

$$\varphi_+(x)$$

and using that $L(y) \leq 0$ for $y \in [0, 1]$ and $L(1 + y) \leq y + \frac{1}{2}y^2$ for $y \in (0, \infty)$, we get

$$E_1 = \int_0^c L(1 + v_n(x)) \varphi_+(x) \, dx \leq \int_0^c \left( |v_n(x)| + \frac{1}{2} |v_n(x)|^2 \right) \varphi_+(x) \, dx.$$ 

Using Proposition 4.2 (b), it follows that

$$E_1 \leq \int_0^c |\varphi_n^+(x) - \varphi_+(x)| + \frac{1}{2} |\varphi_n^+(x) - \varphi_+(x)|^2 / \varphi_+(x) \, dx$$

$$\leq C_2 \left( \int_0^c |r_n(x)| \, dx + \int_0^c \left( \log n + \frac{1}{\sqrt[n]{x}} \right) \, dx + \int_0^c (\log x^{-1}) \, dx \right)$$

$$+ C_3 \left( \int_0^c |r_n(x)|^2 \, dx + \int_0^c \left( \log n + \frac{1}{\sqrt[n]{x}} \right)^2 \, dx + \int_0^c (\log x^{-1})^2 \, dx \right).$$

By Cauchy-Schwarz inequality, it remains to control the integrals in the last line. Now, for any fixed $c \in (0, e^{-1})$, we have

$$\int_0^c |r_n(x)|^2 \, dx \leq \| r_n \|_1 \| r_n \|_\infty = o(1),$$

$$\int_0^c \left( \log n + \frac{1}{\sqrt[n]{x}} \right)^2 \, dx = \frac{\log n}{\sqrt{n}} + \frac{1}{n} (-c^{-1} + \sqrt{n} \log n) = o(1),$$

$$\int_0^c (\log x^{-1})^2 \, dx = \int_{\log(1/c)}^\infty y^2 e^{-y} \, dy < \infty.$$

Thus, for $c$ sufficiently small, we have $E_1 \leq \varepsilon_0$ for all sufficiently large $n \in \mathbb{N}$.

**Estimating $E_2$.** Let $C \in (1, \infty)$ and $c \in (0, 1)$ be the constants fixed above. The same argument as for $E_1^+$ yields

$$E_2 = \int_c^L (1 + v_n(x)) \varphi_+(x) \, dx \leq \int_c^L \left( |v_n(x)| + \frac{1}{2} |v_n(x)|^2 \right) \varphi_+(x) \, dx.$$ 

Using Proposition 4.2 (a), it follows that

$$E_2 \leq \int_c^L \left| \varphi_n^+(x) - \varphi_+(x) \right| + \frac{1}{2} \left| \varphi_n^+(x) - \varphi_+(x) \right|^2 / \varphi_+(x) \, dx$$

$$\leq C_4 \left( \int_c^L |r_n(x)| \, dx + o(1) \int_c^L x^{-1} \, dx \right)$$

$$+ C_5 \exp(C^2/2) \left( \int_c^L |r_n(x)|^2 \, dx + o(1) \int_c^L x^{-2} \, dx \right)$$

$$\leq C_4 \left( \| r_n \|_1 + o(1)(\log C - \log c) \right)$$

$$+ C_5 \exp(C^2/2) \left( \| r_n \|_1 \cdot \| r_n \|_\infty + o(1)(c^{-1} - C^{-1}) \right).$$

Thus, $E_2^+ = o(1)$ as $n \to \infty$.

This completes the proof of sufficiency in Theorem 1.1. \qed
5. SOME AUXILIARY RESULTS

Let us collect some results from the literature which will be needed for the proofs of Propositions 4.1 and 4.2.

Let $\overline{a}_k := \int_{-\infty}^0 x \, d\overline{F}_k(x)$ and $\overline{b}_k := \int_0^0 x^2 \, d\overline{F}_k(x)$, $k \geq 1$. It is known that under our standing moment assumptions the functions $\overline{\phi}_k(t)$ introduced in (3.4) satisfy the following estimates:

\begin{align*}
|\overline{\phi}_k(t)| &\leq 2F_k(0), \quad (5.1) \\
|\overline{\phi}_k(t)| &\leq |\overline{a}_k||t|, \quad (5.2) \\
|\overline{\phi}'_k(t)| &\leq |\overline{a}_k|, \quad (5.3) \\
|\overline{\phi}'_k(t) - (-it\overline{a}_k)| &\leq \frac{1}{2}|\overline{b}_k||t|^2, \quad (5.4) \\
|\overline{\phi}'_k(t) - (-it\overline{a}_k)| &\leq |\overline{b}_k||t|, \quad (5.5) \\
|\overline{\phi}''_k(t) - (-it\overline{a}_k)| &\leq |\overline{b}_k|, \quad (5.6)
\end{align*}

(see e.g. [Al2] Equations (26) and (46)), where

$$F_k(0) = \mathcal{O}(k^{-1/2}) \quad (5.7)$$

(see e.g. [Al2] Equation (39)),

$$\overline{a}_k = -(2\pi k)^{-1/2} + o(k^{-1/2}) \quad \text{and} \quad \overline{b}_k = o(1) \quad (5.8)$$

(see e.g. [Al2] Equation (1))). Let us note that the implicit constants may depend on the distribution of $X_1$.

Furthermore, we need the following classical approximations for characteristic functions of sums of i.i.d. random variables and their derivatives:

Given i.i.d. random variables $X_1, X_2, X_3, \ldots$ with mean 0, variance 1, density $p$ and characteristic function $f$, there exist positive real numbers $\gamma, \delta_1, \delta_2, \delta_3, \ldots$ (depending on the distribution of $X_1$) with $\lim_{n \to \infty} \delta_n = 0$ such that for $n \in \mathbb{N}$, $|t| \leq \gamma n^{1/2}$ and $j = 0, 1, 2$,

$$\left| \frac{d^j}{dt^j}(f^n(t/\sqrt{n}) - e^{-t^2/2}) \right| \leq \delta_n e^{-t^2/4}. \quad (5.9)$$

See e.g. [BR] Theorem 9.12. Replacing $n$ with $k$ and $t$ with $t \sqrt{k/n}$ in this estimate, we obtain, for $1 \leq k \leq n$, $|t| \leq \gamma n^{1/2}$ and $j = 0, 1, 2$,

$$\left| \frac{d^j}{dt^j}(f^k(t/\sqrt{n}) - e^{-kt^2/2n}) \right| \leq \delta_k (k/n)^{j/2} e^{-kt^2/4n}. \quad (5.9)$$

Furthermore, let $\eta \in (0, 1)$ be a constant such that

$$|t| \geq \gamma \quad \Rightarrow \quad |f(t)| \leq \eta. \quad (5.10)$$

Such a constant $\eta$ exists because $X_1$ has a density, which implies that $|f(t)| < 1$ for all $t \neq 0$ as well as $\lim_{|t| \to \infty} |f(t)| = 0$ (by the Riemann-Lebesgue lemma).
Besides that, we will repeatedly use the fact that for any $\alpha > 0$ and $n \geq k \geq 1,$
\[
\sup_{t \in \mathbb{R}} (kt^2/n)^{\alpha/2} e^{-kt^2/4n} = \mathcal{O}_\alpha(1) \tag{5.11}
\]
and
\[
\int_{-\infty}^{+\infty} (kt^2/n)^{\alpha/2} e^{-kt^2/4n} \, dt = \mathcal{O}_\alpha(\sqrt{k}), \tag{5.12}
\]
with implicit constants depending only on $\alpha.$

In addition to that, we will use the following (well-known) Gaussian tail bounds:

For any $\alpha > 0$ and $t > 0$ we have
\[
\int_{t}^{\infty} e^{-ax^2/2} \, dx \leq \sqrt{\frac{\pi}{2\alpha}} \wedge \left( \frac{1}{\alpha t} e^{-\alpha t^2/2} \right), \tag{5.13}
\]
\[
\int_{t}^{\infty} \sqrt{ax} e^{-ax^2/2} \, dx = \frac{1}{\sqrt{a}} e^{-\alpha t^2/2}, \tag{5.14}
\]
\[
\int_{t}^{\infty} ax^2 e^{-ax^2/2} \, dx \leq \sqrt{\frac{\pi}{2\alpha}} \wedge \left( \frac{1}{\alpha t} (\alpha t^2 + 1) e^{-\alpha t^2/2} \right). \tag{5.15}
\]

Moreover, we will repeatedly use the fact that
\[
\sum_{k=1}^{n-1} \frac{1}{\sqrt{k(n-k)}} = \mathcal{O} \left( \frac{1}{\sqrt{n}} \sum_{1 \leq k \leq n/2} \frac{1}{\sqrt{k}} \right) + \mathcal{O} \left( \frac{1}{\sqrt{n}} \sum_{n/2 \leq k \leq n-1} \frac{1}{\sqrt{n-k}} \right) = \mathcal{O}(1). \tag{5.16}
\]
A similar decomposition shows that if $(t_n)_{n \in \mathbb{N}}$ is a sequence of real numbers with $\lim_{n \to \infty} t_n = 0,$ we have
\[
\sum_{k=1}^{n-1} \frac{t_k}{\sqrt{k(n-k)}} = o(1). \tag{5.17}
\]

Finally, we will need the observation that the Fourier transform of the density $\varphi_+(x) := \sqrt{2}/\pi e^{-x^2/2}$ ($x > 0$) satisfies
\[
\hat{\varphi}_+(t) = e^{-t^2/2} + \frac{it}{\sqrt{2\pi n}} \int_0^n e^{-ut^2/2n} \frac{du}{\sqrt{n-u}} \tag{5.18}
\]
for all $n \in \mathbb{N}$ (see [Al2 page 452]). It follows from this that for any $x > 0,$
\[
\varphi_+(x) = \frac{1}{2\pi} \lim_{R \to \infty} \int_{-R}^R e^{-itx} \left[ e^{-t^2/2} + \frac{it}{\sqrt{2\pi n}} \int_0^n e^{-ut^2/2n} \frac{du}{\sqrt{n-u}} \right] \, dt \tag{5.19}
\]
(see [Al2 page 452]).

6. Proof of Proposition 4.1

Proposition 4.1 will be deduced from the following result:

**Proposition 6.1.** For $k = 0, 1, 2,$ we have
\[
\frac{d^k}{dt^k} \Bbb{E}(e^{it\tilde{\Sigma}_n/\sqrt{n}}) - \hat{\varphi}_+(t) = o(1)
\]
as $n \to \infty,$ uniformly in $|t| \leq \gamma n^{1/2}.$
Remarks 6.2.
(a) The Erdős-Kac theorem is equivalent to the statement that \( \mathbb{E}(e^{it\mathcal{S}_n/\sqrt{n}}) \to \hat{\varphi}_+(t) \) for any fixed \( t \in \mathbb{R} \). Thus, this theorem follows from Proposition 6.1 Let us emphasize that we do not need the existence of densities in this section.
(b) For our “application” (namely the proof of Proposition 6.1), the result for the second derivative is relevant. Indeed, for this application, it would be be sufficient to prove Proposition 6.1 for \( t = O(1) \).

Proof of Proposition 6.1. Similarly as in [Al2, Na], using (3.3) and (5.18), we have
\[
\mathbb{E}(e^{it\mathcal{S}_n/\sqrt{n}}) - \hat{\varphi}_+(t) = \left[ f^n(t/\sqrt{n}) - e^{-t^2/2} \right] + \left[ \frac{it}{\sqrt{2\pi n}} \left( \sum_{k=3}^{n-1} e^{-kt^2/2n} \frac{1}{\sqrt{n-k}} - \int_0^n e^{-ut^2/2n} \frac{du}{\sqrt{n-u}} \right) \right] + \left[ \sum_{k=3}^{n-1} f^k(t/\sqrt{n}) - e^{-kt^2/2n} \right] \varphi_{n-k}(t/\sqrt{n}) + \left[ \sum_{k=3}^{n-1} e^{-kt^2/2n} \left( -\bar{a}_{n-k} - \frac{1}{\sqrt{2\pi(n-k)}} \right) \right] \frac{it}{\sqrt{n}} + \left[ f^2(t/\sqrt{n}) \varphi_{n-2}(t/\sqrt{n}) + f(t/\sqrt{n}) \varphi_{n-1}(t/\sqrt{n}) \right].
\]

Denote the expressions in the square brackets by \( D_1(t), \ldots, D_6(t) \). (Note that all these expressions implicitly depend on \( n \)) We will show that for \( j = 1, \ldots, 6 \), uniformly in \( |t| \leq \gamma n^{1/2} \), \( D_j(t), D'_j(t), D''_j(t) \to 0 \) as \( n \to \infty \).

Convention: We always assume that \( n \geq 4 \) and \( |t| \leq \gamma n^{1/2} \). \( O \)- and \( o \)-bounds hold uniformly in this region (unless otherwise mentioned), and they may depend on the constants \( \gamma, \delta_1, \delta_2, \delta_3, \ldots \) introduced in Section 5.

On the Difference \( D_1 \). For the difference \( D_1(t) \) and its first two derivatives, the claim is immediate from (5.9) (with \( k = n \)).

On the Difference \( D_2 \). For fixed \( n \in \mathbb{N}, t \in \mathbb{R} \) and \( \beta \in \{0, 1, 2, \ldots\} \), put
\[
h_\beta(u) := (u/n)^\beta e^{-u^2/2n} \frac{1}{\sqrt{n-u}} \quad (0 < u < n).
\]

Then, for \( 1 \leq v \leq w \leq n-1 \), we have
\[
|h_\beta(w) - h_\beta(v)| = \left| \int_v^w h'_\beta(u) \, du \right| = \left| \int_v^w \left( \frac{\beta u - u^2}{2n} + \frac{1}{2(n-u)} \right) h_\beta(u) \, du \right|
\leq (w - v) \left( \frac{\beta v + v^2}{2n} + \frac{1}{2(n-w)} \right) (w/n)^\beta e^{-v^2/2n} \frac{1}{\sqrt{n-w}}.
\]
Hence, for the difference \( D_2(t) \), we get (using the above estimate with \( \beta = 0 \))
\[
\left| \frac{it}{\sqrt{2\pi n}} \left( \sum_{k=3}^{n-1} e^{-kt^2/2n} \frac{1}{\sqrt{n-k}} - \int_0^n e^{-ut^2/2n} \frac{1}{\sqrt{n-u}} \, du \right) \right|
\leq \frac{|t|}{\sqrt{2\pi n}} \sum_{k=3}^{n-2} \left| \int_k^{k+1} e^{-kt^2/2n} \frac{1}{\sqrt{n-k}} - e^{-ut^2/2n} \frac{1}{\sqrt{n-u}} \, du \right| + O(n^{-1/2})
\leq \frac{|t|}{\sqrt{2\pi n}} \sum_{k=3}^{n-2} \left( \frac{1}{(n-k-1)^{3/2}} e^{-kt^2/2n} + \frac{1}{(n-k-1)^{1/2}} e^{-k^2/2n} \right) + O(n^{-1/2})
= O\left( \sum_{k=3}^{n-2} \left( \frac{k^{3/2}(n-k-1)^{3/2}}{n} + \frac{k^{3/2}(n-k-1)^{1/2}}{n} \right) \right) + O(n^{-1/2}) = O(n^{-1/2}) .
\]

Here we have used the fact that \( (k/n)^{1/2} |t| e^{-kt^2/2n} \) and \( (k/n)^{3/2} |t|^3 e^{-kt^2/2n} \) are uniformly bounded. In particular, this fact is also used in the first step to absorb the summand for \( k = n - 1 \) and the integral over \( u \in \{ n-1, n \} \) into the \( O(n^{-1/2}) \)-term.

Furthermore, similar estimates hold for the first two derivatives of \( D_2(t) \). Indeed, these derivatives are finite linear combinations of expressions of the form
\[
\left| \frac{it^\alpha}{\sqrt{2\pi n}} \left( \sum_{k=3}^{n-1} (k/n)^\beta e^{-kt^2/2n} \frac{1}{\sqrt{n-k}} - \int_0^n (u/n)^\beta e^{-ut^2/2n} \frac{1}{\sqrt{n-u}} \, du \right) \right|
\leq \frac{|t^\alpha|}{\sqrt{2\pi n}} \sum_{k=3}^{n-2} \left| \int_k^{k+1} (k/n)^\beta e^{-kt^2/2n} \frac{1}{\sqrt{n-k}} - (u/n)^\beta e^{-ut^2/2n} \frac{1}{\sqrt{n-u}} \, du \right| + O_\beta(n^{-1/2})
\leq \frac{|t^\alpha|}{\sqrt{2\pi n}} \sum_{k=3}^{n-2} \left( \frac{(k+1)/n)^\beta}{(n-k-1)^{3/2}} e^{-kt^2/2n} \frac{1}{(n-k-1)^{1/2}} + \frac{((k+1)/n)^\beta}{(n-k-1)^{1/2}} e^{-k^2/2n} \frac{1}{(n-k-1)^{3/2}} \right)
+ O_\beta(n^{-1/2})
= O_\beta\left( \sum_{k=3}^{n-2} \left( \frac{k^{3/2}(n-k-1)^{3/2}}{n} + \frac{1}{k^{3/2}(n-k-1)^{1/2}} + \frac{1}{k^{3/2}(n-k-1)^{1/2}} \right) \right) + O_\beta(n^{-1/2})
= O_\beta(n^{-1/2}) .
\]

**On the Difference \( D_3(t) \).** For the difference \( D_3(t) \), the claim follows from (5.9) (with \( k < n \)), (5.2), (5.3) and (5.17), since
\[
\sum_{k=3}^{n-1} \left( f^k(t/\sqrt{n}) - e^{-kt^2/2n} \right) \psi_{n-k}(t/\sqrt{n})
= \mathcal{O}\left( \sum_{k=3}^{n-1} \frac{\delta_k (k/n)^{1/2} |t| e^{-kt^2/4n}}{\sqrt{k(n-k)}} \right) = \mathcal{O}\left( \sum_{k=3}^{n-1} \frac{\delta_k}{\sqrt{k(n-k)}} \right) = o(1) .
\]
Similar estimates hold for the first two derivatives. Indeed, using (5.9) (with \(k < n\), (5.2) – (5.3), (5.6), (5.8) and (5.17), we get
\[
\sum_{k=3}^{n-1} \frac{d}{dt} \left[ f^k(t/\sqrt{n}) - e^{-kt^2/2n} \right] \varphi_{n-k}(t/\sqrt{n})
\]
\[
= \sum_{k=3}^{n-1} \left[ \frac{d}{dt} \left( f^k(t/\sqrt{n}) - e^{-kt^2/2n} \right) \varphi_{n-k}(t/\sqrt{n}) \right]
\]
\[
= O \left( \sum_{k=3}^{n-1} \left[ \frac{\delta_k (k/n)^{1/2} |t| e^{-kt^2/4n}}{\sqrt{n(n-k)}} + \frac{\delta_k e^{-kt^2/4n}}{\sqrt{n(n-k)}} \right] \right)
\]
\[
= O \left( \sum_{k=3}^{n-1} \left[ \frac{\delta_k}{\sqrt{n(n-k)}} + \frac{\delta_k}{\sqrt{n(n-k)}} \right] \right) = o(1)
\]
as well as
\[
\sum_{k=3}^{n-1} \frac{d^2}{dt^2} \left[ f^k(t/\sqrt{n}) - e^{-kt^2/2n} \right] \varphi_{n-k}(t/\sqrt{n})
\]
\[
= \sum_{k=3}^{n-1} \left[ \frac{d^2}{dt^2} \left( f^k(t/\sqrt{n}) - e^{-kt^2/2n} \right) \varphi_{n-k}(t/\sqrt{n}) \right]
\]
\[
+ 2 \frac{d}{dt} \left[ f^k(t/\sqrt{n}) - e^{-kt^2/2n} \right] \varphi''_{n-k}(t/\sqrt{n})/\sqrt{n}
\]
\[
= O \left( \sum_{k=3}^{n-1} \left[ \frac{\delta_k (k/n) |t| e^{-kt^2/4n}}{\sqrt{n(n-k)}} + \frac{\delta_k (k/n)^{1/2} e^{-kt^2/4n}}{\sqrt{n(n-k)}} + \frac{\delta_k |b_{n-k}| e^{-kt^2/4n}}{n} \right] \right)
\]
\[
= O \left( \sum_{k=3}^{n-1} \left[ \frac{\delta_k \sqrt{k/n}}{\sqrt{n(n-k)}} + \frac{\delta_k \sqrt{k/n}}{\sqrt{n(n-k)}} + \frac{\delta_k |b_{n-k}|}{n} \right] \right) = o(1).
\]

**On the Difference \(D_4\).** Let \((m_n)_{n \in \mathbb{N}}\) be a sequence of natural numbers such that \(\lim_{n \to \infty} m_n = \infty\) and \(\lim_{n \to \infty} (m_n/n) \to 0\). Then, by (5.4), (5.2) and (5.8), we have
\[
\sum_{k=3}^{n-1} e^{-kt^2/2n} \varphi_{n-k}(t/\sqrt{n}) - (-\varphi_{n-k}) it/\sqrt{n}
\]
\[
\leq \sum_{k=3}^{n-m_n} (t^2/n) e^{-kt^2/2n} |\bar{b}_{n-k}| + \sum_{k=n-m_n}^{n-1} 2(|t|/\sqrt{n}) e^{-kt^2/2n} |\bar{a}_{n-k}|
\]
\[
= o \left( \sum_{k=3}^{n-m_n} (t^2/n) e^{-kt^2/2n} \right) + O \left( \sum_{k=n-m_n}^{n-1} \frac{1}{\sqrt{k(n-k)}} \right).
\]
Since $\sum_{k=3}^{\infty} xe^{-kx}$ is uniformly bounded in $x > 0$, it follows that $D_4(t) = o(1)$.

Similar estimates hold for the first two derivatives. Indeed, to this end, we have to bound, among other terms,

$$\sum_{k=3}^{n-1} e^{-kt^2/2n} \left( \frac{\varphi'_{n-k}(t/\sqrt{n})}{\sqrt{n}} - \frac{(\varphi_{n-k})'}{\sqrt{n}} \right)$$

and

$$\sum_{k=3}^{n-1} e^{-kt^2/2n} \left( \frac{\varphi''_{n-k}(t/\sqrt{n})}{\sqrt{n}} \right).$$

(For the other terms we get similar bounds as for lower-order derivatives but with extra factors $kt/n$, which are easily controlled due to the exponential factor $e^{-kt^2/2n}$.)

But, using (5.1), (5.3), (5.6), and (5.8), we get

$$\sum_{k=3}^{n-1} e^{-kt^2/2n} \left( \frac{\varphi'_{n-k}(t/\sqrt{n})}{\sqrt{n}} - \frac{(\varphi_{n-k})'}{\sqrt{n}} \right) \leq \sum_{k=3}^{n-m_n} (|t|/n)e^{-kt^2/2n}|b_{n-k}| + \sum_{k=n-m_n}^{n-1} 2e^{-kt^2/2n}|\alpha_{n-k}|/\sqrt{n}$$

$$= o \left( \sum_{k=3}^{n-m_n} \frac{t^2 + 1}{n} e^{-kt^2/2n} \right) + O \left( \sum_{k=n-m_n}^{n-1} \frac{1}{\sqrt{n(n-k)}} \right) = o(1)$$

as well as

$$\sum_{k=3}^{n-1} e^{-kt^2/2n} \left( \frac{\varphi''_{n-k}(t/\sqrt{n})}{\sqrt{n}} \right) \leq \sum_{k=3}^{n-1} (1/n)e^{-kt^2/2n}|b_{n-k}| \leq \frac{1}{n} \sum_{k=3}^{n-1} |b_{n-k}| = o(1).$$

**On the Difference $D_5$.** Similarly as above, let $(m_n)_{n\in\mathbb{N}}$ be a sequence of natural numbers such that $\lim_{n\to\infty} m_n = \infty$ and $\lim_{n\to\infty} (m_n/n) \to 0$. Then, using (5.8), we have

$$\sum_{k=3}^{n-1} e^{-kt^2/2n} \left( \frac{1}{\sqrt{2\pi(n-k)}} \right) \left| \frac{t}{\sqrt{n}} \right|$$

$$= o \left( \sum_{k=3}^{n-m_n} \frac{1}{\sqrt{k(n-k)}} \right) + O \left( \sum_{k=n-m_n}^{n-1} \frac{1}{\sqrt{k(n-k)}} \right) = o(1).$$

Again, for the derivatives, we have similar estimates involving lower powers of $t$ and/or additional factors $kt/n$.

**On the Difference $D_6$.** For fixed $k$, we have

$$(f^k)(t) = O_k(1), \quad |f_k(t)| = O_k(1), \quad |\frac{df_k}{dt}(t)| = O_k(1)$$

(as follows from our assumption $E X_1^2 < \infty$), and as $n \to \infty$,

$$\varphi_n(t) = o(1), \quad \varphi_n(t) = o(1), \quad \varphi_n(t) = o(1)$$

(as follows from (5.1), (5.3) and (5.6) - (5.8)). The claim for the difference $D_6(t)$ and its first two derivatives follows immediately from these relations.
The proof of Proposition 6.1 is complete now. \□

Proof of Proposition 4.1. To deduce Proposition 4.1 from Proposition 6.1, we use that if $X$ is a real random variable with $\mathbb{E}(X^{2k}) < \infty$, induced distribution $P_X$ and characteristic function $f_X$, then, for any $T > 0$,

$$
\int_{[-T,+T]^c} x^{2k} P_X(dx) \leq \frac{T}{2} \int_{-2/T}^{+2/T} (-1)^k (f_X^{(2k)}(0) - f_X^{(2k)}(t)) dt.
$$

For the convenience of the reader, let us recall the argument: Using that $|\sin a/a| \leq 1$ for $a \in \mathbb{R}$ and $|\sin a/a| \leq \frac{1}{2}$ for $|a| \geq 2$, we get

$$
\frac{T}{2} \int_{-2/T}^{+2/T} (-1)^k (f_X^{(2k)}(0) - f_X^{(2k)}(t)) dt

= \frac{T}{2} \int_{-2/T}^{+2/T} \int_{-\infty}^{+\infty} x^{2k} (1 - e^{itx}) dP_X(x) dt

= \int_{-\infty}^{+\infty} x^{2k} \frac{T}{2} \int_{-2/T}^{+2/T} (1 - e^{itx}) dt dP_X(x)

= \int_{-\infty}^{+\infty} x^{2k} (2 - 2 \sin(\frac{2xT}{2})/(\frac{2xT}{2})) dP_X(x)

\geq \int_{[-T,+T]^c} x^{2k} dP_X(x).
$$

Applying this inequality with $X = \frac{S_n}{\sqrt{n}}$ and $T = C$, we get

$$
\int_{C}^{\infty} x^2 \overline{p}_n(x) dx \leq \frac{C}{2} \int_{-2/C}^{+2/C} (-1) \left( f_{\frac{S_n}{\sqrt{n}}}^{(2)}(0) - f_{\frac{S_n}{\sqrt{n}}}^{(2)}(t) \right) dt

\leq 2 \sup_{|t| \leq 2/C} \left| f_{\frac{S_n}{\sqrt{n}}}^{(2)}(0) - f_{\frac{S_n}{\sqrt{n}}}^{(2)}(t) \right|.
$$

Using Proposition 6.1 it follows that for any fixed $C > 0$, we have

$$
\int_{C}^{\infty} x^2 \overline{p}_n(x) dx \leq 2 \sup_{|t| \leq 2/C} \left| \hat{\phi}_+^{(2)}(0) - \hat{\phi}_+^{(2)}(t) \right| + o(1).
$$

as $n \to \infty$. Since $\hat{\phi}_+^{(2)}(t)$ is continuous at zero, we may conclude that for $C = C(\varepsilon)$ sufficiently large, we have

$$
\int_{C}^{\infty} x^2 \overline{p}_n(x) dx \leq \varepsilon
$$

for all sufficiently large $n \in \mathbb{N}$, and the proof of Proposition 4.1 is complete. \□

Remark. Let us outline another proof of Proposition 4.1. This proof is shorter, but it is based on Spitzer’s formula and the (classical) Erdős-Kac theorem (1.4). Also, the preceding proof is more useful in that it can be modified (under higher-order moment conditions) to obtain more precise estimates on the rate of decay in Proposition 4.1.
We have to show that the sequence of random variables \(((S_n^+/\sqrt{n})^2)_{n \in \mathbb{N}}\) is uniformly integrable. It follows from the classical Erdős-Kac theorem (1.4) that \((S_n^+/\sqrt{n})^2 \Rightarrow |Z|^2\) as \(n \to \infty\). Now, it is well known that for a family of integrable random variables \(X, X_1, X_2, X_3, \ldots\) with \(X_n \Rightarrow X\),

\[
(X_n)_{n \in \mathbb{N}} \text{ is uniformly integrable if and only if } \mathbb{E}|X_n| \to \mathbb{E}|X|,
\]

see e.g. Lemma 4.11 in [Ka]. Thus, it remains to show that \(\mathbb{E}(S_n^+/\sqrt{n})^2 \to \mathbb{E}|Z|^2 = 1\) as \(n \to \infty\).

Our starting point is Spitzer’s formula (see e.g. [Fe, p. 618]), which states that for \(|s| < 1\) and \(t \in \mathbb{R}\),

\[
\sum_{n=0}^{\infty} s^n \mathbb{E}(e^{itS_n^+}) = \exp \left( \sum_{k=1}^{\infty} \frac{s^k}{k} \mathbb{E}(e^{itS_k^+}) \right).
\]

(6.2)

Differentiating twice with respect to \(t\) in (6.2), we obtain

\[
\sum_{n=0}^{\infty} s^n \mathbb{E}(S_n^+)^2 e^{itS_n^+} = \exp \left( \sum_{k=1}^{\infty} \frac{s^k}{k} \mathbb{E}(S_k^+ e^{itS_k^+}) \right) \cdot \left[ \left( \sum_{k=1}^{\infty} \frac{s^k}{k} \mathbb{E}(S_k^+ e^{itS_k^+}) \right)^2 + \sum_{k=1}^{\infty} \frac{s^k}{k} \mathbb{E}((S_k^+)^2 e^{itS_k^+}) \right]
\]

and therefore, setting \(t = 0\),

\[
\sum_{n=0}^{\infty} s^n \mathbb{E}(S_n^+)^2 = \sum_{n=0}^{\infty} s^n \cdot \left[ \left( \sum_{k=1}^{\infty} \frac{s^k}{k} \mathbb{E}(S_k^+) \right)^2 + \sum_{k=1}^{\infty} \frac{s^k}{k} \mathbb{E}((S_k^+)^2) \right].
\]

Hence, comparing coefficients, we may conclude that for any \(n \geq 1\),

\[
\mathbb{E}((S_n^+)^2) = \sum_{k \geq 1 \atop k + l \leq n} \frac{1}{k} \mathbb{E}(S_k^+) \mathbb{E}(S_l^+) + \sum_{1 \leq k \leq n} \frac{1}{k} \mathbb{E}((S_k^+)^2).
\]

Now, using the central limit theorem, the fact that \(\mathbb{E}(S_n^+ / \sqrt{n})^2 \to \mathbb{E}Z^2 = 1\) and the criterion (6.1), it is easy to see that

\[
\mathbb{E}(S_n^+ / \sqrt{n}) \to \mathbb{E}(Z^+) = \frac{1}{\sqrt{2\pi}} \quad \text{and} \quad \mathbb{E}((S_n^+ / \sqrt{n})^2) \to \mathbb{E}((Z^+)^2) = \frac{1}{2}.
\]
as $n \to \infty$. Therefore,

$$
\mathbb{E}((S_n^+/\sqrt{n})^2) = \frac{1}{n} \sum_{m=1}^{n} \sum_{k=1}^{m-1} \frac{1}{k} \mathbb{E}(S_k^+) \frac{1}{m-k} \mathbb{E}(S_{m-k}^+) + \frac{1}{n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{E}((S_k^+)^2)
$$

$$
= \frac{1}{n} \sum_{m=1}^{n} \sum_{k=1}^{m-1} \frac{1}{\sqrt{2\pi k}} \frac{1}{\sqrt{2\pi (m-k)}} + \frac{1}{n} \sum_{k=1}^{n} \frac{1}{2} + o(1)
$$

$$
= \frac{1}{n} \sum_{m=1}^{n} \frac{1}{2\pi} \int_0^1 \frac{1}{\sqrt{x(1-x)}} \, dx + \frac{1}{n} \sum_{k=1}^{n} \frac{1}{2} + o(1)
$$

$$
= 1 + o(1)
$$

as $n \to \infty$. This completes the proof of Proposition 4.1.

□

7. Proof of Proposition 4.2

Proof of Proposition 4.2. Let $p = (1 - \rho)q_1 + \rho q_2$ be as in (3.1), and let $g_1$ and $g_2$ be the Fourier transforms of $q_1$ and $q_2$, respectively. Then

$$
f^k(t) = \sum_{j=0}^{k} \binom{k}{j} (1 - \rho)^j \rho^{k-j} g_1^j(t) g_2^{k-j}(t).
$$

For $k \geq 3$, put

$$
\tilde{p}_k(x) := \sum_{j=3}^{k} \binom{k}{j} (1 - \rho)^j \rho^{k-j} (q_1^j * q_2^{(k-j)}(x))
$$

and

$$
\tilde{f}_k(t) := \sum_{j=3}^{k} \binom{k}{j} (1 - \rho)^j \rho^{k-j} g_1^j(t) g_2^{k-j}(t).
$$

Note that $\tilde{f}_n(t)$ is the Fourier transform of $\tilde{p}_n(x)$ and that $\tilde{p}_n(x)$ can be recovered from $\tilde{f}_n(t)$ by means of Fourier inversion. This follows from the fact that $g_1 \in L^2$ (being the Fourier transform of a bounded probability density) and $g_2 \in L^\infty$ (being the Fourier transform of a probability measure).

Using our moment assumptions and the fact that $\rho < \frac{1}{2}$, it is easy to see for $k \geq 3$ and $t \in \mathbb{R}$,

$$
\left| \frac{d^j}{dt^j} (f^k(t/\sqrt{n}) - \tilde{f}_k(t/\sqrt{n})) \right| = \mathcal{O}(n^{-j/2} 2^{-k}), \quad j = 0, 1, 2.
$$

It therefore follows from (5.9) that for $3 \leq k \leq n$, $|t| \leq \gamma n^{1/2}$ and $j = 0, 1, 2$,

$$
\left| \frac{d^j}{dt^j} (\tilde{f}_k(t/\sqrt{n}) - e^{-kt^2/2n}) \right| \leq \delta_k (k/n)^j/2 e^{-kt^2/4n} + \mathcal{O}(n^{-j/2} 2^{-k}). \quad (7.1)
$$
Furthermore, there exist a constant $C_0 > 0$ and a constant $\eta \in (0, 1)$ such that for $k \geq 3$ and $|t| \geq \gamma$,

$$|\tilde{f}_k(t)| \leq \sum_{j=3}^{k} \binom{k}{j} (1 - \varrho)^j \varrho^{k-j} \left| g_1^j(t) g_2^{k-j}(t) \right| \leq C_0 \eta^{k-2} |g_1(t)|^2 , \quad (7.2)$$

$$|\tilde{f}'_k(t)| \leq \sum_{j=3}^{k} \binom{k}{j} (1 - \varrho)^j \varrho^{k-j} \frac{d}{dt} \left[ g_1^j(t) g_2^{k-j}(t) \right] \leq C_0 k \eta^{k-3} |g_1(t)|^2 . \quad (7.3)$$

This follows from the fact that $g_1$ and $g_2$ also satisfy (5.10) (possibly with some modified constant $\eta$) and that $g_1'$ and $g_2'$ are bounded, $q_1$ and $q_2$ being probability measures with finite moments.

Recalling (3.2) and (3.6) and using the non-negative densities $\tilde{p}_k$ introduced above, we may write

$$\overline{q}_n(x) = \sqrt{n} \sum_{k=3}^{n} (\tilde{p}_k * G_{n-k})(\sqrt{n} x) + r_n(x) , \quad (7.4)$$

where the remainder term $r_n(x)$ is given by

$$r_n(x) := \sqrt{n} \sum_{k=1}^{n} \binom{k}{1} (1 - \varrho) \varrho^{k-1} (q_1 * q_2^{*(k-1)}) * G_{n-k} (\sqrt{n} x) + \sqrt{n} \sum_{k=2}^{n} \binom{k}{2} (1 - \varrho)^2 \varrho^{k-2} (q_1 * q_2^{*(k-2)}) * G_{n-k} (\sqrt{n} x) . \quad (7.5)$$

The functions $r_n$ are the signed densities occurring in Proposition 4.2. It is easy to see that $\|r_n\|_1 = \mathcal{O}(1/\sqrt{n})$ and $\|r_n\|_\infty = \mathcal{O}(1)$. Indeed, because $q_1$ and $q_2$ are probability densities, $q_1$ is bounded and the total variation norm of $G_n$ is of order $\mathcal{O}(1/\sqrt{n})$, we have

$$\|q_1^j * q_2^{*(k-j)} * G_{n-k}\|_1 \leq \frac{C_1}{\sqrt{n-k+1}} \quad (j = 1, 2)$$

and

$$\|q_1^j * q_2^{*(k-j)} * G_{n-k}\|_\infty \leq \frac{C_1}{\sqrt{n-k+1}} \quad (j = 1, 2) ,$$

so that the asserted properties of the densities $r_n$ follow from the estimate

$$\sum_{k=j}^{n} \binom{k}{j} (1 - \varrho)^j \varrho^{k-j} \frac{1}{\sqrt{n-k+1}} \leq \sum_{k=j}^{n} \frac{k^j \varrho^{k-j}}{\sqrt{n-k+1}} = \mathcal{O}(n^{-1/2}) \quad (j = 1, 2) .$$

Observe that all the terms in the big sum in (7.4) contain the “factor” $q_1^2(\sqrt{n} x)$ and therefore have Fourier transforms in $L^1$. Hence, similarly as in (A12), using
Fourier inversion and (5.19), we obtain the representation, for $x > 0$,

$$\mathcal{T}_n(x) - \sqrt{\frac{2}{\pi}} e^{-x^2/2} = r_n(x)$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \left( \hat{f}_n(t/\sqrt{n}) - e^{-t^2/2} \right) dt$$

$$+ \frac{1}{2\pi} \lim_{R \to \infty} \int_{-R}^{+R} e^{-itx} \frac{it}{\sqrt{2\pi n}} \left( \sum_{k=3}^{n-1} \frac{1}{\sqrt{n-k}} \int_0^n e^{-ut^2/2n} \frac{du}{\sqrt{n-u}} \right) dt$$

$$+ \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \left( \sum_{k=3}^{n-1} \left( \hat{f}_k(t/\sqrt{n}) - e^{-kt^2/2n} \right) \overline{\varphi}_{n-k}(t/\sqrt{n}) \right) dt$$

$$+ \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \left( \sum_{k=3}^{n-1} e^{-kt^2/2n} \left( \overline{\varphi}_{n-k}(t/\sqrt{n}) - (-\varphi_{n-k}) \frac{it}{\sqrt{n}} \right) \right) dt$$

$$+ \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \left( \sum_{k=3}^{n-1} e^{-kt^2/2n} \left( -\varphi_{n-k} \right) \frac{it}{\sqrt{n}} \right) dt .$$

Denote the integrals on the right-hand side by $I_1, \ldots, I_5$. Note that all the integrals implicitly depend on $n$ and $x$. We will consider each of them separately.

**Convention:** We always assume that $n \geq 4$ and $x \in (0, \infty)$ (part (a)) or $x \in (0, e^{-1})$ (part (b)). $O$- and $o$-bounds hold uniformly in these regions (unless otherwise mentioned), and they may depend on the constants $\gamma, \delta_1, \delta_2, \delta_3, \ldots$ introduced in Section 5, on the constants $C_0$ and $\eta$ in (7.2) and (7.3), and on the $L^2$-norm of the function $g_1$.

7.1. **The proof of part (a).** Throughout this subsection we assume that $n \geq 4$ and $x \in (0, \infty)$. The proof is very similar to that of Theorem 1 in [A12].

**On the Integral $I_1$.** Using integration by parts, we get

$$|I_1| = \frac{1}{x} \left| \int_{\mathbb{R}} e^{-itx} \frac{d}{dt} \left( \hat{f}_n(t/\sqrt{n}) - e^{-t^2/2} \right) dt \right|$$

$$\leq \frac{1}{x} \int_{(-\gamma \sqrt{n}, \gamma \sqrt{n})^c} \left| \frac{d}{dt} \left( \hat{f}_n(t/\sqrt{n}) - e^{-t^2/2} \right) \right| dt$$

$$+ \frac{1}{x} \int_{(-\gamma \sqrt{n}, \gamma \sqrt{n})^c} \left| \frac{d}{dt} \left( \hat{f}_n(t/\sqrt{n}) - e^{-t^2/2} \right) \right| dt .$$

By (7.4), the first integral on the right is of the order $O(\delta_n + 2^{-n}) = o(1)$. Furthermore, by (7.3), (5.14) and the fact that $g_1 \in L^2$, the second integral on the right is of the order

$$O\left( \int_{(-\gamma \sqrt{n}, \gamma \sqrt{n})^c} (n\eta^{n-3} |g_1(t/\sqrt{n})|^2 (1/\sqrt{n}) + |t| e^{-t^2/2}) dt \right)$$

$$= O(n\eta^{n-3} + e^{-n\gamma^2/2}) = o(1) .$$

Thus, $I_1 = o(1/x)$.

**On the Integral $I_2$.** By [A12 Equation (24)], we have $I_2 = O(1/(\sqrt{nx}))$. 
On the Integral $I_3$. For $k = 3, \ldots, n - 1$, let

$$I_{3,k} := \int_\mathbb{R} e^{-itx} \left( \tilde{f}_k(t/\sqrt{n}) - e^{-kt^2/2n} \right) \varphi_{n-k}(t/\sqrt{n}) \, dt.$$ 

Then, similarly as in [Al2], it follows via integration by parts that

$$|I_{3,k}| = \frac{1}{x} \int_\mathbb{R} e^{-itx} \left[ \frac{d}{dt} \left( \tilde{f}_k(t/\sqrt{n}) - e^{-kt^2/2n} \right) \right] \varphi_{n-k}(t/\sqrt{n}) \, dt \leq x^{-1} |I_{3,k,1}| + x^{-1} |I_{3,k,2}|,$$

where $I_{3,k,1}$ and $I_{3,k,2}$ denote the integrals over the sets $(-\gamma \sqrt{n}, \gamma \sqrt{n})$ and $(-\gamma \sqrt{n}, \gamma \sqrt{n})^c$, respectively. It follows from (7.1), (5.1) – (5.3), (5.7) and (5.8) that

$$|I_{3,k,1}| \leq \int_{(-\gamma \sqrt{n}, \gamma \sqrt{n})} \left| \tilde{f}_k(t/\sqrt{n}) - e^{-kt^2/2n} \right| \left| \varphi_{n-k}(t/\sqrt{n})(1/\sqrt{n}) \right| \, dt$$

$$+ \int_{(-\gamma \sqrt{n}, \gamma \sqrt{n})} \left| \frac{d}{dt} \left( \tilde{f}_k(t/\sqrt{n}) - e^{-kt^2/2n} \right) \right| \left| \varphi_{n-k}(t/\sqrt{n}) \right| \, dt$$

$$= O \left( \int_{(-\gamma \sqrt{n}, \gamma \sqrt{n})} \left[ \frac{\delta_k e^{-kt^2/4n}}{\sqrt{n(n-k)}} + \frac{2-k}{\sqrt{n(n-k)}} \right] \, dt \right)$$

$$+ O \left( \int_{(-\gamma \sqrt{n}, \gamma \sqrt{n})} \left[ \delta_k (k/n)^{1/2} |t| e^{-kt^2/4n} \right] \left( \frac{2-k}{\sqrt{n(n-k)}} \right) \, dt \right)$$

$$= O \left( \frac{\delta_k}{\sqrt{k(n-k)}} + \frac{2-k}{\sqrt{n-k}} \right).$$

Also, using (5.1), (5.3), (5.7), (5.8), (7.2) and (7.3), the Gaussian tail estimates (5.13) – (5.15) and the fact that $g_1 \in L^2$, we get

$$|I_{3,k,2}| \leq \int_{(-\gamma \sqrt{n}, \gamma \sqrt{n})^c} \left| \tilde{f}_k(t/\sqrt{n}) \right| \left| \varphi_{n-k}(t/\sqrt{n})(1/\sqrt{n}) \right| \, dt$$

$$+ \int_{(-\gamma \sqrt{n}, \gamma \sqrt{n})^c} \left| \tilde{f}'_k(t/\sqrt{n})(1/\sqrt{n}) \right| \left| \varphi_{n-k}(t/\sqrt{n}) \right| \, dt$$

$$+ \int_{(-\gamma \sqrt{n}, \gamma \sqrt{n})^c} \left| e^{-kt^2/2n} \right| \left| \varphi_{n-k}(t/\sqrt{n})(1/\sqrt{n}) \right| \, dt$$

$$+ \int_{(-\gamma \sqrt{n}, \gamma \sqrt{n})^c} \left| e^{-kt^2/2n(kt/n)} \right| \left| \varphi_{n-k}(t/\sqrt{n}) \right| \, dt$$

$$= O \left( \frac{\eta^{k-2} + k\eta^{k-3} + \frac{1}{k\gamma} e^{-k\gamma^2/2} + e^{-k\gamma^2/2}}{\sqrt{n-k}} \right).$$

Therefore,

$$I_{3,k} = O \left( \frac{\delta_k + \tilde{\eta}^k}{x \sqrt{k(n-k)}} \right), \quad (7.6)$$

where $\tilde{\eta} := \frac{1}{2}(1 + \max\{\frac{1}{2}, \eta, e^{-\gamma^2/2}\}) \in (0, 1)$. Hence, using (5.17), we get $I_3 = o(1/x)$. 

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On the Integral $I_4$. It follows from [A12, Equation (47)] that $I_4 = O(1/x)$.

For the convenience of the reader, let us briefly sketch the argument from [A12]. For $k = 3, \ldots, n - 1$, let

$$I_{4,k} := \int_{\mathbb{R}} e^{-ikx} e^{-kt^2/2n} \left( \frac{\overline{\alpha}_{n-k}(t/\sqrt{n}) - (-\alpha_{n-k}) i t/\sqrt{n}}{\sqrt{n}} \right) dt.$$  

Using integration by parts, we get

$$|I_{4,k}| \leq \frac{1}{x} \int_{\mathbb{R}} e^{-kt^2/2n} \left( \frac{k}{n} |t| \frac{\overline{\alpha}_{n-k}(t/\sqrt{n}) - (-\alpha_{n-k}) i t/\sqrt{n}}{\sqrt{n}} \right) dt.$$

We now split the integral at $\pm A$ ($A > 2$) and use the bounds [5.4] and [5.5] in the region $(-A, +A)$ and the bounds [5.2] and [5.3] in the region $(-A, +A)^c$. In combination with the Gaussian tail estimates (5.13) and (5.15), we obtain

$$\int_{\mathbb{R}} e^{-kt^2/2n} \left( \frac{k}{n} |t| \frac{\overline{\alpha}_{n-k}(t/\sqrt{n}) - (-\alpha_{n-k}) i t/\sqrt{n}}{\sqrt{n}} \right) dt \leq \int_{(-A,A)^c} \frac{\overline{\alpha}_{n-k}}{\sqrt{n}} e^{-kt^2/2n} \left( \frac{k}{n} |t|/\sqrt{n} + |t|/n \right) dt$$

$$+ \int_{(-A,A)} 2|\overline{\alpha}_{n-k}| e^{-kt^2/2n} \left( \frac{k}{n} |t|/\sqrt{n} + 1/\sqrt{n} \right) dt$$

$$= O \left( \frac{\overline{\alpha}_{n-k}}{\sqrt{n}} \right)$$

$$+ O \left( \frac{\overline{\alpha}_{n-k}}{\sqrt{n}} \sqrt{\frac{n\pi}{2k}} + \frac{n}{kA} e^{-kA^2/2n} + \frac{n}{kA} \left( \frac{k}{n} A^2 + 1 \right) e^{-kA^2/2n} \right),$$

with implicit constants not depending on $n$ or $A$. Note that the term in the square brackets is bounded by $\sqrt{2\pi n/k}$ for $k \leq n/A$ and by $(A + 2)e^{-A/2}$ for $k \geq n/A$. Thus, using [5.8], it follows that

$$|I_4| = O \left( \frac{A^{n-1}}{x} \sum_{k=3}^{n} \frac{|\overline{\alpha}_{n-k}|}{\sqrt{n}\sqrt{k}} \right)$$

$$+ O \left( \frac{1}{x} \sum_{k \leq n/A} \frac{1}{\sqrt{k(n-k)}} \right) + O \left( \frac{(A + 2)e^{-A/2}}{x} \sum_{k \geq n/A} \frac{1}{n(n-k)} \right)$$

$$= o(x^{-1} A) + O(x^{-1} A^{-1/2}) + O(x^{-1} (A + 2)e^{-A/2}).$$

Letting $A \equiv A_n \to \infty$ sufficiently slowly as $n \to \infty$, we conclude that $|I_4| = o(1/x)$.

On the Integral $I_5$. It is shown in [A12, Equation (48)] that $I_5 = o(1/x)$.

Clearly, combining the estimates for $I_1, \ldots, I_5$, we get part (a) of Proposition 4.2.
7.2. The proof of part (b). Throughout this subsection we assume that \( n \geq 4 \) and \( x \in (0, e^{-1}) \). For these values of \( x \), we can obtain somewhat better estimates by avoiding the integration-by-parts step.

**On the Integral \( I_1 \).** We have

\[
|I_1| = \left| \int_{\mathbb{R}} e^{-itx} \left[ \tilde{f}_n(t/\sqrt{n}) - e^{-t^2/2} \right] dt \right|
\]

\[
\leq \int_{(-\gamma \sqrt{n}, \gamma \sqrt{n})} |\tilde{f}_n(t/\sqrt{n}) - e^{-t^2/2}| dt
+ \int_{(-\gamma \sqrt{n}, \gamma \sqrt{n})^c} |\tilde{f}_n(t/\sqrt{n}) - e^{-t^2/2}| dt.
\]

By (7.1), the first integral on the right is of the order \( \mathcal{O}(\delta_n + \sqrt{n} 2^{-n}) = o(1) \). Furthermore, by (7.2), (5.13) and the fact that \( g_1 \in L^2 \), the second integral on the right is of the order

\[
\mathcal{O} \left( \int_{(-\gamma \sqrt{n}, \gamma \sqrt{n})} \left( \eta^{n-2} |g_1(t/\sqrt{n})|^2 + e^{-t^2/2} \right) dt \right)
= \mathcal{O}(\sqrt{n} \eta^{n-2} + \frac{1}{\sqrt{n}} e^{-n \eta^2/2}) = o(1).
\]

Thus, \( I_1 = o(1) \).

**On the Integral \( I_2 \).** We have already mentioned that \( I_2 = \mathcal{O}(1/(\sqrt{n} x)) \). Now, using (5.12) and (5.19), we also have

\[
|I_2| = \left| \lim_{R \to \infty} \int_{-R}^{+R} e^{-itx} \frac{it}{2\pi n} \left( \sum_{k=3}^{n-1} e^{-kt^2/2n} \frac{1}{\sqrt{n - k}} - \int_{0}^{n} e^{-ut^2/2n} \frac{du}{\sqrt{n - u}} \right) dt \right|
\]

\[
\leq \sum_{k=3}^{n-1} \int_{-\infty}^{+\infty} \frac{|t|}{\sqrt{2\pi n}} e^{-kt^2/2n} \frac{1}{\sqrt{n - k}} dt
+ \left| \lim_{R \to \infty} \int_{-R}^{+R} e^{-itx} \frac{it}{2\pi n} \int_{0}^{n} e^{-ut^2/2n} \frac{du}{\sqrt{n - u}} dt \right|
\]

\[
= \mathcal{O} \left( \sum_{k=3}^{n-1} \frac{1}{k \sqrt{n(n - k)}} \right) + 2\pi \varphi(x)
+ \mathcal{O} \left( \sum_{1 \leq k \leq n/2} \frac{1}{k} \right) + \mathcal{O}(1) = \mathcal{O}(\log n).
\]

Thus, \( I_2 = \mathcal{O}((\log n) \wedge (1/(\sqrt{n} x))) \).

**On the Integral \( I_3 \).** For \( k = 3, \ldots, n - 1 \), we can estimate the integral

\[
I_{3,k} := \int_{\mathbb{R}} e^{-itx} \left( \tilde{f}_k(t/\sqrt{n}) - e^{-kt^2/2n} \right) \overline{\varphi_{n-k}(t/\sqrt{n})} dt.
\]

in two different ways.
On the one hand, using integration by parts, we obtain

\[ I_{3,k} = \mathcal{O}\left(\frac{1}{x \sqrt{k(n-k)}}\right), \quad (7.7) \]

see (7.6).

On the other hand, similar estimates (without integration by parts) yield

\[ |I_{3,k}| \leq \int_{\mathbb{R}} \left| \tilde{f}_k(t/\sqrt{n}) - e^{-kt^2/2n} \varphi_{n-k}(t/\sqrt{n}) \right| dt \]

\[ = \mathcal{O}\left(\int_{(\gamma \sqrt{n}, \gamma \sqrt{n})} \frac{\delta_k |t| e^{-kt^2/4n}}{\sqrt{n(n-k)}} + \frac{2^{-k}}{\sqrt{n-k}} dt \right) + \mathcal{O}\left(\int_{(\gamma \sqrt{n}, \gamma \sqrt{n})} e^{-kt^2/2n} |\varphi_{n-k}(t/\sqrt{n})| dt \right) + \mathcal{O}\left(\int_{(\gamma \sqrt{n}, \gamma \sqrt{n})} e^{-kt^2/2n} |t/\sqrt{n}| dt \right) \]

\[ = \mathcal{O}\left(\frac{n}{k} \frac{\delta_k}{\sqrt{n(n-k)}} + 2^{-k} \frac{\sqrt{n}}{\sqrt{n-k}} \right) + \mathcal{O}\left(n^{-2} \frac{\sqrt{n}}{\sqrt{n-k}} \right) + \mathcal{O}\left(\frac{1}{k \gamma} e^{-k \gamma^2/2} \frac{\sqrt{n}}{\sqrt{n-k}} \right), \]

whence

\[ I_{3,k} = \mathcal{O}\left(\frac{n}{k} \frac{1}{\sqrt{n(n-k)}}\right). \quad (7.8) \]

Using (7.7) for \( k \leq nx^2 \) and (7.8) for \( k \geq nx^2 \) and recalling that \( x \in (0, e^{-1}) \), it follows that

\[ I_3 = \mathcal{O}\left(\frac{1}{\sqrt{nx}} \sum_{1 \leq k \leq nx^2} \frac{1}{\sqrt{k}} + \sum_{nx^2 \leq k \leq n/2} \frac{1}{k} + \sum_{n/2 \leq k \leq n-1} \frac{1}{\sqrt{n(n-k)}} \right) \]

\[ = \mathcal{O}(1) + \mathcal{O}(\log x) + \mathcal{O}(1) = \mathcal{O}(\log x). \]

Thus, \( I_3 = \mathcal{O}(\log x) \).

**On the Integral** \( I_4 \). For \( k = 3, \ldots, n-1 \), we can estimate the integral

\[ I_{4,k} := \int_{\mathbb{R}} e^{-itx} e^{-kt^2/2n} (\varphi_{n-k}(t/\sqrt{n}) - (-\varphi_{n-k}) it/\sqrt{n}) dt \]

in two different ways. On the one hand, using integration by parts and \( (5.2), (5.3) \) and \( (5.8) \), we have

\[ |I_{4,k}| \leq \frac{1}{x} \int_{\mathbb{R}} 2|\varphi_{n-k}| e^{-kt^2/2n} \left(\frac{k}{n} |t| |t|/\sqrt{n} + 1/\sqrt{n} \right) dt = \mathcal{O}\left(\frac{1}{x \sqrt{k(n-k)}}\right). \]
On the other hand, also using (5.2) and (5.8) (but without integration by parts), we have

$$|I_{4,k}| \leq \int_{\mathbb{R}} 2|\pi_{n-k}|e^{-kt^2/2n}\left(\frac{|t|}{\sqrt{n}}\right) dt = O\left(\frac{n}{k\sqrt{n(n-k)}}\right).$$

Thus, the same argument as for $I_3$ leads to the conclusion that $I_4 = O(-\log x)$.

**On the Integral $I_5$.** For $k = 3, \ldots, n-1$, we can estimate the integral

$$I_{5,k} := \int_{\mathbb{R}} e^{-itx} e^{-kt^2/2n}\left((\pi_{n-k}) - \frac{1}{\sqrt{2\pi(n-k)}} \right) \frac{it}{\sqrt{n}} dt$$

in two different ways. On the one hand, using integration by parts and (5.8), we get

$$|I_{5,k}| = O\left(\frac{1}{x} \int_{\mathbb{R}} e^{-kt^2/2n}\left(\frac{k|t|}{n(n-k)} + \frac{1}{\sqrt{n(n-k)}} \right) dt\right) = O\left(\frac{n}{k\sqrt{n(n-k)}}\right).$$

On the other hand, using (5.8) (but without integration by parts), we get

$$|I_{5,k}| = O\left(\int_{\mathbb{R}} e^{-kt^2/2n}\left(\frac{|t|}{\sqrt{n(n-k)}} \right) dt\right) = O\left(\frac{n}{k\sqrt{n(n-k)}}\right).$$

Thus, the same argument as for $I_3$ leads to the conclusion that $I_5 = O(-\log x)$.

The proof of part (b) of Proposition 4.2 is completed by combining the previous estimates. □

8. **Proof of Necessity in Theorem 1.1**

**Proof of Necessity in Theorem 1.1.** Let us quote some well-known results from the literature: Suppose that $|s| < 1$. By Spitzer’s formula (see e.g. [Fe, p. 618]), we have

$$\sum_{n=0}^{\infty} s^n \mathbb{E}(e^{itS_n}) = \frac{1}{1-s} \exp\left(\sum_{k=1}^{\infty} \frac{s^k}{k} \int_{0}^{\infty} (e^{itx} - 1) dF_k(x)\right) \quad (8.1)$$

for any $t \in \mathbb{R}$. Also (see e.g. [Fe, p. 416]), we have

$$1 + \sum_{n=1}^{\infty} s^n \mathbb{P}(S_n < 0) = \exp\left(\sum_{k=1}^{\infty} \frac{s^k}{k} \mathbb{P}(S_k < 0)\right) = \frac{1}{1-s} \exp\left(-\sum_{k=1}^{\infty} \frac{s^k}{k} \mathbb{P}(S_k \geq 0)\right).$$

Thus, Spitzer’s formula (8.1) can be rewritten as

$$\sum_{n=0}^{\infty} s^n \mathbb{E}(e^{itS_n}) = \left(1 + \sum_{n=1}^{\infty} s^n \mathbb{P}(S_n < 0)\right) \exp\left(\sum_{k=1}^{\infty} \frac{s^k}{k} \int_{[0,\infty)} e^{itx} dF_k(x)\right) \quad (8.2)$$

for any $t \in \mathbb{R}$.

Let us note that the preceding results hold without any assumptions on moments or on densities. However, if the moment assumptions stated at the beginning of the introduction are satisfied, then

$$\mathbb{P}(S_n < 0) = \Theta(n^{-1/2}) \quad (8.3)$$

for $n \geq 1$ (see e.g. [Fe, pp. 414f]). Indeed, more precise information is available.
Expanding the right-hand side of Spitzer’s formula (8.2) into a power series in $s$ and comparing coefficients, we find that for any $n \geq 1$,

$$
\mathbb{E}(e^{iS_n}) = \mathcal{F}_n(0) + \mathcal{F}_{n-1}(0) \int_0^\infty e^{itx} p_{1,+}(x) \, dx
$$

$$
+ \sum_{m=2}^n \mathcal{F}_{n-m}(0) \sum_{k_1,\ldots,k_l \geq 1 \atop k_1 + \ldots + k_l = m} \frac{1}{l!} \frac{1}{k_1 \cdots k_l} \int_0^\infty e^{itx} (p_{k_1,+} * \ldots * p_{k_l,+})(x) \, dx,
$$

where $\mathcal{F}_0(0) := 1$ and, for any $k \geq 1$, $p_{k,+}(x) := p_k(x)$ for $x > 0$ and $p_{k,+}(x) := 0$ for $x \leq 0$. Hence, by the uniqueness theorem for Fourier transforms, we have

$$
\overline{p}_n(x) = \mathcal{F}_{n-1}(0) p_1(x) + \tilde{p}_n(x)
$$

(8.4)

for almost all $x > 0$, where $\tilde{p}_n$ is a certain subprobability density on the positive half-line.

Now suppose that (1.6) holds. Then, using Lemma 2.1, we have $D(\overline{\mathcal{F}_n} \mid \varphi_+) < \infty$ for all sufficiently large $n \in \mathbb{N}$. It is easy to see that this implies $D(\overline{\mathcal{F}_n} \mid \varphi_+) < \infty$ for all sufficiently large $n \in \mathbb{N}$. Therefore, using (8.4), (8.3) and the remark (2.2) below Lemma 2.4, we may conclude that $D(p \mid \varphi_+) < \infty$, which entails (1.7) by Lemma 2.1.

9. Proof of Theorem 1.2

Proof of Theorem 1.2. Fix $\varepsilon \in (0,1)$, and let $c \in (0,1)$ and $C \in (1,\infty)$ be such that

$$
\int_c^C \varphi_+(x) \, dx > 1 - \varepsilon.\tag{9.1}
$$

Then, using Lemma 3.1(a) and Proposition 4.2(a), we have

$$
\int_c^C |\overline{\mathcal{F}_n} - \varphi_+| \, dx \leq \int_c^C |\overline{\mathcal{F}_n} - \overline{\mathcal{F}_n}| \, dx + \int_c^C |\overline{\mathcal{F}_n} - \varphi_+| \, dx = o(1)\tag{9.2}
$$

as $n \to \infty$, which implies that

$$
\int_c^C \overline{p}_n(x) \, dx > 1 - \varepsilon\tag{9.3}
$$

for all sufficiently large $n \in \mathbb{N}$. It follows from (9.1) – (9.3) that

$$
d_{TV}(\overline{\mathcal{F}_n}/\sqrt{n}, |Z|) \leq \int_R |\overline{\mathcal{F}_n} - \varphi_+| \, dx \leq \int_{(c,C)} |\overline{\mathcal{F}_n} - \varphi_+| \, dx + \int_{(c,C)^c} (\overline{\mathcal{F}_n} + \varphi_+) \, dx < 2\varepsilon
$$

for all sufficiently large $n \in \mathbb{N}$. Since $\varepsilon \in (0,1)$ is arbitrary, Theorem 1.2 is proved. □
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