Quantum Group Schrödinger Field Theory

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Abstract  

We show that a quantum deformation of quantum mechanics given in a previous work is equivalent to quantum mechanics on a nonlinear lattice with step size $\Delta x = (1 - q)x$. Then, based on this, we develop the basic formalism of quantum group Schrödinger field theory in one spatial quantum dimension, and explicitly exhibit the $SU_q(2)$ covariant algebras satisfied by the $q$-bosonic and $q$-fermionic Schrödinger fields. We generalize this result to an arbitrary number of fields.
1 Introduction

The remarkable relation between quantum groups and non-commutative geometry \[1,2\], and its realization in terms of non-commutative differential calculus \[3\] has recently acquired a noticeable place in the physics literature. The fact that non-commutative calculus can be understood as a deformation of ordinary calculus served as a paradigm to study \(q\)-deformations of standard physical systems. In particular, different aspects of quantum deformations of quantum mechanics have been studied by several authors \[4,5,6\]. In this paper, based on work in \[6\], we study Schrödinger quantum group field theory in one spatial quantum dimension. The main objective of this work is to develop a formulation of a quantum deformed second quantized field theory, which could set a basis to an analytical approach to a field theory with higher dimensional quantum space-time symmetries. In Section 2 we briefly discuss quantum mechanics for \(0 < q < 1\) and show the equivalence between the corresponding deformed Schrödinger hamiltonian with a difference equation on a nonlinear lattice. In Section 3 we develop the basic formalism of a Schrödinger quantum field theory satisfying quantum commutation relations on the nonlinear lattice, and in Section 4 we obtain explicitly the \(SU_q(2)\) covariant field algebras satisfied by \(q\)-bosonic and \(q\)-fermionic Schrödinger fields. We generalize this result to an arbitrary number of fields and conclude with some remarks.

2 Quantum Mechanics with \(0 < q < 1\)

Given \(q\) real, \(0 < q < 1\), the following deformation of the quantum Heisenberg algebra

\[ px - qx = -i, \] (2.1)

is seen to be compatible with non-commutative differential calculus on the quantum line once one defines \(p \equiv -i\hat{\partial}\), where \(\hat{\partial}\) is the quantum derivative. If we impose the hermiticity condition on both \(p\) and \(x\), then Equation (2.1) will reduce to \([p, x] = \frac{2i}{q+1}\) and therefore standard quantum mechanics. Therefore, the only two alternatives available is to require that either \(p\) or \(x\) be hermitian. As discussed in \[8\], by enforcing hermiticity on the coordinate operator, \(x = x\), we can still keep the usual meaning of position, speed, acceleration as real expectation values, allowing besides to implement the formalism with \(q\)-differentials and \(q\)-integrals. It was found that one can
define a hermitian free hamiltonian $H_0$ as

$$H_0 = g(q)p\overline{p} \quad (2.2)$$

with $\overline{p} = q^{-1}T^{-1}(x)p$, $T(x) = q^{x\partial_x}$ and $g(q = 1) = 1$, such that the time dependent deformed Schrödinger equation consistent with a time independent probability density is given by

$$\left(-x^{-1}[x\partial_x]x^{-1}[x\partial_x] + V(q, x)\right)\Phi(x, t) = i\partial_t \Phi(x, t) \quad (2.3)$$

where $[x\partial_x] = \frac{T^{1/2}(x)-T^{-1/2}(x)}{q^{1/2}-q^{-1/2}}$, and we have chosen $g(q) = q^2$. By introducing an independent variable $z$ and considering the coordinate $x$ as a function $x(z) = q^z$, Equation (2.2) can be rewritten as an equation taking values on a nonlinear lattice of variable step size

$$\Delta x(z) \equiv x(z) - x(z + 1) = (1 - q)x \quad (2.4)$$

Similarly, the difference operation on any function $y(x(z))$ is written

$$\Delta y(x(z)) \equiv y(x(z)) - y(x(z + 1)) = (1 - T(x))y(x(z)) \quad (2.5)$$

such that on the lattice the free hamiltonian $H_0$ reads

$$H_0 = -\Delta \nabla x_1 \left[\nabla y(x(z))\right] \quad (2.6)$$

where $x_1 = x(z + 1/2)$ and $\nabla x(z) = \Delta x(z - 1)$. Difference equations with operators of this type were studied in \[7\] in the context of the theory of basic hypergeometric functions.

### 3 Quantum Field Deformations

In this section we look for an action that leads to Equation (2.3). An obvious procedure is to take the usual Schrödinger action, replace the differential and integral operators by their $q$-analogs and postulate the action

$$S = \int dt \int dq x \left[ -f(q)D_x^{(+)}\Phi^* D_x^{(+)}\Phi + i\Phi^* \dot{\Phi} - V(x)\Phi^* \Phi \right] \quad (3.1)$$

where the $q$-derivative $D_x^{(+)} \equiv x^{-1} \frac{1-T(x)}{1-q}$ and the $q$-integral operator is given by the Jackson integral \[8\]

$$\int dq x f(x) \equiv (1 - q)x \sum_{n=0}^{\infty} q^n f(q^n x) \quad (3.2)$$
From the definition of the inner product introduced in [6]

\[ <\Psi, \Phi> \equiv \int \! dq \, x \, \Psi^* \Phi = (1 - q) x \sum_{n=0}^{\infty} q^n \Psi^* (q^n x) \Phi(q^n x), \tag{3.3} \]

we recall that the hermitian adjoint of the \( q \)-derivative is given by

\[ D_{x}^{(+)} = -q^{-1} D_{x}^{(-)}. \tag{3.4} \]

Integration by parts with use of the \( q \)-analog of the Leibniz rule

\[ D_{x}^{(+)} [\Phi \phi] = [D_{x}^{(+)} \Phi] T(x) \phi + \Phi [D_{x}^{(+)} \phi] \tag{3.5} \]

shows that the action \( S \) is hermitian. Variation of \( S \) with respect to \( \Phi^* \) leads to

\[
\delta S = \int dt \int dq \, x \, \delta \Phi^* \left[ f(q) D_{x}^{(+)} T^{-1}(x) D_{x}^{(+)} \Phi + i \dot{\Phi} - V \Phi \right] \\
- \int dt \left[ \delta \Phi^* T^{-1}(x) D_{x}^{(+)} \Phi \right]_x \tag{3.6}
\]

Since \( \delta \Phi^* = 0 \) at the boundary, requiring \( \delta S = 0 \) implies that

\[ - f(q) D_{x}^{(+)} T^{-1}(x) D_{x}^{(+)} \Phi = i \dot{\Phi} - V(x) \Phi, \tag{3.7} \]

which is equivalent to Equation (2.3) provided that \( f(q) = q^{1/2} \). In lattice notation, the lagrangian \( L(t) \) can be alternatively written as

\[
L(t) = \sum_{n=0}^{\infty} L_n = \sum_{n=0}^{\infty} \Delta x_n \left[ -q^{1/2} \frac{\Delta \Phi^* (x_n) \Delta \Phi (x_n)}{\Delta x_n} \right] \\
+ \sum_{n=0}^{\infty} \Delta x_n \left[ i \Phi^* (x_n) \Phi (x_n) - V(x_n) \Phi^* (x_n) \Phi (x_n) \right] \tag{3.8}
\]

where we denoted \( x_n \equiv q^n x \). The momentum density conjugate to \( \Phi \) is given by

\[ \pi(x_m) = \frac{\partial \mathcal{L}_m}{\partial \dot{\Phi}(x_m)} = i \Phi^* (x_m) \tag{3.9} \]

where \( \mathcal{L}_m \equiv \frac{L_m}{\Delta x_m} \). Similarly, the hamiltonian becomes

\[
H = \sum_{n=0}^{\infty} \Delta x_n \left[ -i q^{1/2} \frac{\Delta \pi(x_n) \Delta \Phi (x_n)}{\Delta x_n} - i V(x_n) \pi(x_n) \Phi (x_n) \right] \tag{3.10}
\]
Let us specialize to the free case, and consider $\Phi(x, t)$ as a quantum field restricted to the region $0 \leq x \leq d$. A solution to Equation (2.3) with these boundary conditions can be written as an expansion

$$\Phi(x, t) = \sum_{\kappa} \frac{a(\kappa)}{\sqrt{M_{\kappa}}} \sin(\sqrt{q}; \kappa x) e^{-i\kappa t}. \quad (3.11)$$

Since the differential operator $H_0$ is hermitian the conjugate field $\bar{\Phi}(x, t)$ satisfies the same differential equation, and it can be similarly written as

$$\bar{\Phi}(x, t) = \sum_{\kappa} \frac{\bar{\pi}(\kappa)}{\sqrt{M_{\kappa}}} \sin(\sqrt{q}; \kappa x) e^{i\kappa t}, \quad (3.12)$$

The energy eigenvalues are $\epsilon_\kappa = k\bar{k} \equiv \kappa^2$, where the allowed $\kappa$ values satisfy $\sin (\sqrt{q}; kd) = 0$. Since the operator $\hat{p}$ and the hamiltonian $H_0$ do not commute they do not share the same eigenfunctions. The variables $k$ and $\bar{k}$ denote the eigenvalues of $\hat{p}$ and $\hat{H}$ with eigenfunctions $e^{i\kappa x}$ and $e^{i\bar{k}x}$ respectively. These functions are defined in terms of the Eulerian series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \{n\} = \frac{1 - q^n}{1 - q} \quad (3.13)$$

$$\sin (\sqrt{q}; x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad (3.14)$$

The function $\sin (\sqrt{q}; \kappa x)$ satisfies the orthogonality relation

$$\frac{1}{\sqrt{M_\kappa M_{\kappa'}}} \int_0^{qd} dq x \sin \left( \sqrt{q}; \kappa x \right) \sin \left( \sqrt{q}; \kappa' x \right) = \delta_{\kappa, \kappa'} \quad (3.15)$$

with $M_\kappa = \int_0^{qd} dq x \left[ \sin \left( \sqrt{q}; \kappa x \right) \right]^2$, such that the annihilation and creation operators can be written in terms of the field $\Phi(x, t)$ as follows

$$a(\kappa) = \int_0^{qd} \frac{d_q x}{\sqrt{M_\kappa}} \Phi(x, t) \sin(\sqrt{q}; \kappa x) e^{i\kappa t}$$

$$= \sum_{n=0}^{\infty} \frac{(1 - q)d_{n+1}}{\sqrt{M_\kappa}} \Phi(d_{n+1}, t) \sin \left( \sqrt{q}; d_{n+1} \kappa \right) e^{i\kappa t} \quad (3.16)$$

and

$$\bar{a}(\kappa) = \int_0^{qd} \frac{d_q x}{\sqrt{M_{\kappa'}}} \bar{\Phi}(x, t) \sin(\sqrt{q}; \kappa x) e^{-i\kappa t}$$

$$= \sum_{n=0}^{\infty} \frac{(1 - q)d_{n+1}}{\sqrt{M_{\kappa'}}} \bar{\Phi}(d_{n+1}, t) \sin \left( \sqrt{q}; d_{n+1} \kappa \right) e^{-i\kappa t}. \quad (3.17)$$
where \( d_n \equiv q^n d \). Now, let the fields \( \Phi \) and \( \overline{\Phi} \) satisfy on the lattice the deformed commutation relations

\[
\Phi(x_n, t)\overline{\Phi}(x_m, t) - q^2 \overline{\Phi}(x_m, t)\Phi(x_n, t) = \frac{\delta_{n,m}}{\Delta x_n} \tag{3.18}
\]

which in the \( q \to 1 \) limit becomes \([\Phi(x, t), \overline{\Phi}(x, t)] \to \infty\), and the commutation relation

\[
[\Phi(x_n, t), \Phi(x_m, t)] = 0. \tag{3.19}
\]

After integration by parts the Hamiltonian operator \( H_0 \) becomes

\[
H_0 = -\frac{q}{2} \sum_{n=0}^{\infty} \Delta x_n \overline{\Phi}(x_n) \frac{\Delta}{\Delta x_n} T^{-1} \frac{\Delta \Phi(x_n)}{\Delta x_n}, \tag{3.20}
\]

such that with use of Equation (3.18) we find that the time evolution of the field \( \Phi \) can be also written as

\[
i \dot{\Phi}(x, t) = \Phi(x, t)H_0 - q^2 H_0 \Phi(x, t), \tag{3.21}
\]

where that for \( \overline{\Phi} \) is given by the Hermitian adjoint equation. The asymmetry introduced by the parameter \( q \) does not allow Equation (3.21) to apply for an arbitrary function of the fields. For example, Equation (3.21) and its Hermitian adjoint imply that the time evolution of the function \( F = \Phi \overline{\Phi} \) is written in terms of a commutator

\[
i \dot{F} = [F, H_0], \tag{3.22}
\]

and thus \( \dot{H}_0 = 0 \), as expected. From Equation (3.18) and the orthogonality relation in Equation (3.15) we find that the operators \( a(\kappa') \) and \( \overline{a}(\kappa) \) satisfy

\[
[a(\kappa'), \overline{a}(\kappa)]_{q^2} \equiv a(\kappa')\overline{a}(\kappa) - q^2 \overline{a}(\kappa)a(\kappa') = \delta_{\kappa,\kappa'} \tag{3.23}
\]

In terms of \( a(\kappa') \) and \( \overline{a}(\kappa) \) operators the Hamiltonian reads

\[
H_0 = \sum_{\kappa} \kappa^2 \overline{a}(\kappa)a(\kappa). \tag{3.24}
\]

The vacuum expectation value between two fields is given by

\[
G(x, x', t, t') \equiv <\Phi(x, t)\overline{\Phi}(x', t')>
= \sum_{\kappa} \frac{1}{M_\kappa} \sin (\sqrt{q}; \kappa x) \sin (\sqrt{q}; \kappa x') e^{i\kappa(x'-t')} \tag{3.25}
\]
From Equations (3.15) and (3.25) we obtain that \( G(x, x', t, t') \) relate the field at two lattice points \( d_n \) and \( d_m \) according to

\[
\Phi(d_m, t) = \sum_{n=0}^{\infty} (1 - q) d_n G(d_m, d_n, t, t') \Phi(d_n, t'),
\]

and therefore

\[
G(d_m, d_n, t, t') = \delta_{n,m} \Delta d_n,
\]

which itself represents a closure relation. As a simple example, we consider the case \( q \approx 0 \). A good approximation to the zeros of the basic sine function is given by

\[
\alpha_m \approx q - \frac{m - 1/4}{1 - q}
\]

with \( m = 1, 2, 3, \ldots \), and therefore Equation (3.25) can be written as an expansion over the positive integers as follows

\[
< \Phi(x, t) \Phi(x', t') >_{q \approx 0} = \frac{q}{M_1} \sum_{n=1}^{\infty} q^{-n} \sin(\sqrt{q}; \kappa_n x) \sin(\sqrt{q}; \kappa_n x') e^{i\kappa_n^2 (t' - t)}
\]

where we defined \( \kappa_n \equiv \frac{2^{-n+1/4}}{(1-q)d} \), and the coefficient

\[
M_{\kappa=\kappa_n} \equiv M_n = \int_0^{qd} dq x \sin^2(\sqrt{q}; \kappa_n x) = q^{n-1} M_1
\]

Now, once we take two lattice points \( x = d_l \) and \( x' = d_m \), we see that Equation (3.29) reduces to a finite sum. For example, if \( l \leq m \) we have

\[
< \Phi(d_l, t) \Phi(d_m, t') >_{q \approx 0} = \frac{q}{M_1} \sum_{n=1}^{l} q^{-n} \sin(\sqrt{q}; \kappa_n d_l) \sin(\sqrt{q}; \kappa_n d_m) e^{i\kappa_n^2 (t' - t)}.
\]

4 \( SU_q(2) \) covariant field algebras

In this section, based on the formalism given in Section 3, we first exhibit explicitly the covariant algebras satisfied by two \( q \)-bosonic and \( q \)-fermionic fields. Let us introduce two Schrödinger fields \( \Phi_1 \) and \( \Phi_2 \). The free full lagrangian is, of course, the sum of the two corresponding lagrangians, as given...
in Equation (3.8) with $V = 0$. It is simple to check that this Lagrangian is invariant under $SU_q(2)$ transformations

$$
\Phi'_i = T_{ij} \Phi_j
$$

(4.1)

where the $SU_q(2)$ matrix is given by

$$
T = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
$$

(4.2)

and the adjoint matrix $\overline{T}$

$$
\overline{T} = \begin{pmatrix}
d & -qb \\
-q^{-1}c & a
\end{pmatrix}
$$

(4.3)

with coefficients satisfying the well known relations

$$
ab = q^{-1}ba \quad , \quad ac = q^{-1}ca \\
b = cb \quad , \quad dc = qcd \\
db = qbd \quad , \quad da - ad = (q - q^{-1})bc \\
det_q T = ad - q^{-1}bc = 1.
$$

(4.4)

Now, a set of $q$-bosonic oscillators transforming linearly as in Equation (4.1) form a $SU_q(2)$ covariant algebra according to the following relations

$$
[a_1(\kappa'), \pi_1(\kappa)]_{q^2} = \delta_{\kappa',\kappa} + (q^2 - 1)\pi_2(\kappa)a_2(\kappa')
$$

(4.5)

$$
[a_2(\kappa'), \pi_2(\kappa)]_{q^2} = \delta_{\kappa,\kappa'}
$$

(4.6)

$$
a_2(\kappa)a_1(\kappa') = qa_1(\kappa')a_2(\kappa)
$$

(4.7)

$$
a_2(\kappa)\pi_1(\kappa') = q\pi_1(\kappa')a_2(\kappa),
$$

(4.8)

and according to Equations (3.15), (3.16) and (3.17) we obtain that the corresponding $SU_q(2)$ covariant field algebra is given by

$$
\left[\Phi_1(x_n,t), \overline{\Phi}_1(x_m,t)\right]_{q^2} = \frac{\delta_{m,n}}{\Delta x_n} + (q^2 - 1)\overline{\Phi}_2(x_m,t)\Phi_2(x_n,t)
$$

(4.9)

$$
\left[\Phi_2(x_n,t), \overline{\Phi}_2(x_m,t)\right]_{q^2} = \frac{\delta_{m,n}}{\Delta x_n}
$$

(4.10)

$$
\Phi_2(x_n,t)\Phi_1(x_m,t) = q\Phi_1(x_m,t)\Phi_2(x_n,t)
$$

(4.11)
\[ 
\Phi_2(x_n, t) \Phi_1(x_m, t) = q \Phi_1(x_m, t) \Phi_2(x_n, t) 
\]

Similarly, starting from a SU\(_q(2)\) covariant \(q\)-fermionic algebra

\[ 
b_1(\kappa)b_2(\kappa') = -qb_2(\kappa')b_1(\kappa) \quad , \quad \overline{b}_1(\kappa)b_2(\kappa') = -qb_2(\kappa')\overline{b}_1(\kappa) 
\]

\[ 
\{b_1(\kappa), b_2(\kappa')\} = \delta_{\kappa,\kappa'} - (1 - q^{-2})b_2(\kappa')b_2(\kappa) 
\]

\[ 
\{b_2(\kappa), \overline{b}_2(\kappa')\} = \delta_{\kappa,\kappa'}, 
\]

the set of \(q\)-fermionic fields \(\Psi_1\) and \(\Psi_2\) satisfying Equation (2.3), with \(V = 0\), with expansion

\[ 
\Psi_j(x, t) = \sum_{\kappa} \frac{b_j(\kappa)}{\sqrt{M_\kappa}} \sin(\sqrt{q} \kappa x) e^{-i\kappa \alpha t} , \quad j = 1, 2 
\]

satisfy the following SU\(_q(2)\) covariant field algebra

\[ 
\Psi_1(x_n, t)\Psi_2(x_m, t) = -q\Psi_2(x_m, t)\Psi_1(x_n, t) 
\]

\[ 
\overline{\Psi}_1(x_n, t)\Psi_2(x_m, t) = -q\Psi_2(x_m, t)\overline{\Psi}_1(x_n, t) 
\]

\[ 
\{\Psi_1(x_n, t), \overline{\Psi}_1(x_m, t)\} = \frac{\delta_{m,n}}{\Delta x_n} - (1 - q^{-2})\Psi_2(x_m, t)\Psi_2(x_n, t) 
\]

\[ 
\{\Psi_2(x_n, t), \overline{\Psi}_2(x_m, t)\} = \frac{\delta_{m,n}}{\Delta x_n} 
\]

\[ 
\{\Psi_1(x_n, t), \Psi_1(x_n, t)\} = 0 = \{\Psi_2(x_m, t), \Psi_2(x_n, t)\} 
\]

The operators \(\overline{a}_j\), \(a_j\) and \(\overline{b}_j\), for \(\kappa = \kappa'\) and \(j = 1, 2\) can be put in one to one correspondence with the quantum plane coordinates, differentials and derivatives in Ref. [3]. The generalization of these relations to \(n\) \(q\)-bosonic and \(q\)-fermionic fields follows. In compact form, we write

\[ 
\Omega_j(x_n)\Phi_i(x_m) = \delta_{ij} \frac{\delta_{m,n}}{\Delta x_n} \pm q^{1/2} \tilde{R}_{ikjl}(q) \Phi_i(x_m)\Phi_l(x_n) 
\]

\[ 
\Omega_i(x_n)\Omega_j(x_m) = \pm q^{1/2} \tilde{R}_{ikjl}(q) \Omega_j(x_n)\Omega_l(x_m) 
\]

\[ 
\overline{\Omega}_i(x_n)\Omega_j(x_m) = \pm q^{1/2} \tilde{R}_{ijkl}(q) \Omega_k(x_m)\Omega_l(x_n) 
\]

where \(\Omega\) denotes either \(\Phi\) or \(\Psi\), and the upper (lower) sign applies to \(\Phi\) (\(\Psi\)) field. The matrix \(\tilde{R}_{ijkl}(q)\) is given by

\[ 
\tilde{R}_{ijkl}(q) = R_{ijkl}(q^{-1}), 
\]
where $R_{ijkl}(q)$ is the $R$-matrix of $\hat{A}_{n-1}$. There is an additional covariant algebra involving $q$-bosonic and $q$-fermionic fields. In terms of the matrix $\hat{R}$ it reads

\begin{align}
\Phi_i(x_n) \Psi_j(x_m) &= q \hat{R}_{ijkl}(q) \Psi_k(x_m) \Phi_l(x_n) \quad (4.26) \\
\Psi_i(x_n) \Phi_j(x_m) &= q \hat{R}_{ijkl}(q) \Phi_l(x_m) \Psi_k(x_n) \quad (4.27) \\
\Psi_i(x_n) \Phi_j(x_m) &= q \hat{R}_{ijkl}(q) \Phi_k(x_m) \Psi_l(x_n) \quad (4.28)
\end{align}

5 Discussion

In this letter, starting from the equivalence of a quantum deformation of quantum mechanics and quantum mechanics on a nonlinear lattice, we developed the basic formalism of a $q$-bosonic ($q$-fermionic) Schrödinger quantum field theory with the fields satisfying quantum commutation (anticommutation) relations on the quantum line. We have seen that the free field case admits a field expansion in terms of orthogonal functions, in a similar way as in undeformed Schrödinger field theory. Then, based on this formalism, we considered $SU_q(2)$ as an internal quantum symmetry and obtained the covariant algebra satisfied by the fields, and generalized this result to an arbitrary number of fields. The formulation we have proposed in this work gives an additional geometrical insight into the meaning of quantum deformations and it could give a new angle to approach quantum field theory in non-commutative geometry. There are several proposals of either quantum group field theory or $q$-deformed field theory, most of them on classical space-time, which can be found in the literature [1]. It would be interesting to generalize some of the issues addressed in this paper to more general situations involving a nonzero potential and higher number of quantum dimensions.

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