SCHUR-CLASS MULTIPLIERS ON THE ARVESON SPACE: DE BRANGES-ROVNYAK REPRODUCING KERNEL SPACES AND COMMUTATIVE TRANSFER-FUNCTION REALIZATIONS

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Abstract. An interesting and recently much studied generalization of the classical Schur class is the class of contractive operator-valued multipliers $S(\lambda)$ for the reproducing kernel Hilbert space $H(k_d)$ on the unit ball $B^d \subset \mathbb{C}^d$, where $k_d$ is the positive kernel $k_d(\lambda, \zeta) = 1/(1 - \langle \lambda, \zeta \rangle)$ on $B^d$. The reproducing kernel space $H(K_S)$ associated with the positive kernel $K_S(\lambda, \zeta) = (I - S(\lambda)S(\zeta)^*) \cdot k_d(\lambda, \zeta)$ is a natural multivariable generalization of the classical de Branges-Rovnyak canonical model space. A special feature appearing in the multivariable case is that the space $H(K_S)$ in general may not be invariant under the adjoints $M^*_\lambda$ of the multiplication operators $M_\lambda : f(\lambda) \mapsto \lambda f(\lambda)$ on $H(k_d)$. We show that invariance of $H(K_S)$ under $M^*_\lambda$ for each $j = 1, \ldots, d$ is equivalent to the existence of a realization for $S(\lambda)$ of the form $S(\lambda) = D + C(I - \lambda_1 A_1 - \cdots - \lambda_d A_d)^{-1}(\lambda_1 B_1 + \cdots + \lambda_d B_d)$ such that connecting operator $U = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{bmatrix}$ has adjoint $U^*$ which is isometric on a certain natural subspace ($U$ is “weakly coisometric”) and has the additional property that the state operators $A_1, \ldots, A_d$ pairwise commute; in this case one can take the state space to be the functional-model space $H(K_S)$ and the state operators $A_1, \ldots, A_d$ to be given by $A_j = M^*_\lambda|_{H(K_S)}$ (a de Branges-Rovnyak functional-model realization). We show that this special situation always occurs for the case of inner functions $S$ (where the associated multiplication operator $M_S$ is a partial isometry), and that inner multipliers are characterized by the existence of such a realization such that the state operators $A_1, \ldots, A_d$ satisfy an additional stability property.

1. Introduction

A multivariable generalization of the Szegő kernel $k(\lambda, \zeta) = (1 - \lambda \bar{\zeta})^{-1}$ much studied of late is the positive kernel

$$k_d(\lambda, \zeta) = \frac{1}{1 - \langle \lambda, \zeta \rangle}$$

on $\mathbb{B}^d \times \mathbb{B}^d$ where $\mathbb{B}^d = \{ \lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d : \langle \lambda, \lambda \rangle < 1 \}$ is the unit ball of the $d$-dimensional Euclidean space $\mathbb{C}^d$. By $\langle \lambda, \zeta \rangle = \sum_{j=1}^d \lambda_j \bar{\zeta}_j$ we mean the standard inner product in $\mathbb{C}^d$. The reproducing kernel Hilbert space (RKHS) $H(k_d)$ associated with $k_d$ via Aronszajn’s construction is a natural multivariable analogue of the Hardy space $H^2$ of the unit disk and coincides with $H^2$ if $d = 1$. 

1991 Mathematics Subject Classification. 47A57.
Key words and phrases. Operator-valued functions, Schur-class multiplier.
For \( \mathcal{Y} \) an auxiliary Hilbert space, we consider the tensor product Hilbert space \( H_{\mathcal{Y}}(k_d) := \mathcal{H}(k_d) \otimes \mathcal{Y} \) whose elements can be viewed as \( \mathcal{Y} \)-valued functions in \( \mathcal{H}(k_d) \). Then \( H_{\mathcal{Y}}(k_d) \) can be characterized as follows:

\[
H_{\mathcal{Y}}(k_d) = \left\{ f(\lambda) = \sum_{n \in \mathbb{Z}_+^d} f_n \lambda^n : \| f \|^2 = \sum_{n \in \mathbb{Z}_+^d} \frac{n!}{|n|} \cdot \| f_n \|_{\mathcal{Y}}^2 < \infty \right\}. \tag{1.2}
\]

Here and in what follows, we use standard multivariable notations: for multi-integers \( n = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d \) and points \( \lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d \) we set

\[
|n| = n_1 + n_2 + \ldots + n_d, \quad n! = n_1!n_2! \ldots n_d!, \quad \lambda^n = \lambda_1^{n_1}\lambda_2^{n_2} \ldots \lambda_d^{n_d}. \tag{1.3}
\]

By \( \mathcal{L}(\mathcal{U}, \mathcal{Y}) \) we denote the space of all bounded linear operators between Hilbert spaces \( \mathcal{U} \) and \( \mathcal{Y} \). The space of multipliers \( \mathcal{M}_d(\mathcal{U}, \mathcal{Y}) \) is defined as the space of all \( \mathcal{L}(\mathcal{U}, \mathcal{Y}) \)-valued analytic functions \( S \) on \( \mathbb{B}^d \) such that the induced multiplication operator

\[
M_S : f(\lambda) \rightarrow S(\lambda) \cdot f(\lambda) \tag{1.4}
\]

maps \( H_d(k_d) \) into \( H_{\mathcal{Y}}(k_d) \). It follows by the closed graph theorem that for every \( S \in \mathcal{M}_d(\mathcal{U}, \mathcal{Y}) \), the operator \( M_S \) is bounded. We shall pay particular attention to the unit ball of \( \mathcal{M}_d(\mathcal{U}, \mathcal{Y}) \), denoted by

\[
S_d(\mathcal{U}, \mathcal{Y}) = \{ S \in \mathcal{M}_d(\mathcal{U}, \mathcal{Y}) : \| M_S \|_{\text{op}} \leq 1 \}.
\]

Since \( S_1(\mathcal{U}, \mathcal{Y}) \) collapses to the classical Schur class (of holomorphic, contractive \( \mathcal{L}(\mathcal{U}, \mathcal{Y}) \)-valued functions on \( \mathbb{D} \)), we refer to \( S_d(\mathcal{U}, \mathcal{Y}) \) as a generalized (\( d \)-variable) Schur class. Characterizations of \( S_d(\mathcal{U}, \mathcal{Y}) \) in terms of realizations originate to \( [11] \). We recall this result in the form presented in \( [7] \).

**Theorem 1.1.** Let \( S \) be an \( \mathcal{L}(\mathcal{U}, \mathcal{Y}) \)-valued function defined on \( \mathbb{B}^d \). The following are equivalent:

1. \( S \) belongs to \( S_d(\mathcal{U}, \mathcal{Y}) \).
2. The kernel

\[
K_S(\lambda, \zeta) = \frac{I_{\mathcal{Y}} - S(\lambda)S(\zeta)^*}{1 - \langle \lambda, \zeta \rangle} \tag{1.5}
\]

is positive on \( \mathbb{B}^d \times \mathbb{B}^d \), i.e., there exists an operator-valued function \( H : \mathbb{B}^d \rightarrow \mathcal{L}(\mathcal{H}, \mathcal{Y}) \) for some auxiliary Hilbert space \( \mathcal{H} \) so that

\[
K_S(\lambda, \zeta) = H(\lambda)H(\zeta)^*. \tag{1.6}
\]

3. There exists a Hilbert space \( \mathcal{X} \) and a unitary connecting operator (or colligation) \( U \) of the form

\[
U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \end{bmatrix} : [\mathcal{X}] \rightarrow [\mathcal{Y}^d] \tag{1.7}
\]

so that \( S(\lambda) \) can be realized in the form

\[
S(\lambda) = D + C(I_{\mathcal{X}} - \lambda_1 A_1 - \cdots - \lambda_d A_d)^{-1}(\lambda_1 B_1 + \cdots + \lambda_d B_d)
\]

\[
= D + C(I - Z(\lambda)A)^{-1}Z(\lambda)B \tag{1.8}
\]
where we set
\[
Z(\lambda) = \begin{bmatrix} \lambda_1 I_X & \ldots & \lambda_d I_X \end{bmatrix}, \quad A = \begin{bmatrix} A_1 \\ \vdots \\ A_d \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ \vdots \\ B_d \end{bmatrix}.
\]

(4) There exists a Hilbert space \( \mathcal{X} \) and a contractive connecting operator \( \mathbf{U} \) of the form (1.7) so that \( S(\lambda) \) can be realized in the form (1.8).

In analogy with the univariate case, a realization of the form (1.8) is called coisometric, isometric, unitary or contractive if the operator \( \mathbf{U} \) is respectively, coisometric, isometric, unitary or just contractive. It turns out that a more useful analogue of “coisometric realization” appearing in the classical univariate case is not that the whole connecting operator \( \mathbf{U}^* \) be isometric, but rather that \( \mathbf{U}^* \) be isometric on a certain subspace of \( \mathcal{X}^d \oplus \mathcal{Y} \).

**Definition 1.2.** A realization (1.8) of \( S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y}) \) is called weakly coisometric if the adjoint \( \mathbf{U}^* : \mathcal{X}^d \oplus \mathcal{Y} \to \mathcal{X} \oplus \mathcal{U} \) of the connecting operator is contractive and isometric on the subspace
\[
\mathcal{D} := \text{span}\{(I_X - A^* Z(\lambda)^{-1} C^*)_y : \lambda \in \mathbb{B}^d, y \in \mathcal{Y}\} \subset \mathcal{X}^d.
\]

Weakly coisometric realizations for an \( S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y}) \) can be constructed in certain canonical way as follows. Upon applying Aronszajn’s construction to the kernel \( K_S \) defined as in (1.5), (which is positive on \( \mathbb{B}^d \) by Theorem 1.1), one gets the de Branges-Rovnyak space \( \mathcal{H}(K_S) \). A weakly coisometric realization for \( S \) with the state space equal to \( \mathcal{H}(K_S) \) (and output operator \( \mathbf{C} \) equal to evaluation at zero on \( \mathcal{H}(K_S) \)) will be called a generalized functional-model realization. Here we use the term generalized functional-model realization since it may be the case that the state space \( \mathcal{H}(K_S) \) in not even invariant under the adjoints \( M_{\lambda_1}^*, \ldots, M_{\lambda_d}^* \) of the multiplication operators \( M_{\lambda_j} : f(\lambda) \mapsto \lambda_j \cdot f(\lambda) \) \( (j = 1, \ldots, d) \) on \( \mathcal{H}(K_d) \) and hence one cannot take the state operators \( A_1, \ldots, A_d \) to be given by \( A_j = M_{\lambda_j}^* \) as one would expect from the classical case. As it was shown in [7], any function \( S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y}) \) admits a generalized functional-model realization. In the univariate case, this collapses to the well known de Branges-Rovnyak functional-model realization [17, 18]. Another parallel to the univariate case is that any observable weakly coisometric realization of a Schur-class function \( S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y}) \) is unitarily equivalent to some generalized functional-model realization (observability is a minimality condition that is fulfilled automatically for every generalized functional-model realization). However, in contrast to the univariate case, this realization is not unique in general (even up to unitary equivalence); moreover, a function \( S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y}) \) may admit generalized functional-model realizations with the same state space operators \( A_1, \ldots, A_d \) and different input operators \( B_j \)'s. A curious fact is that none of the generalized functional-model realizations for \( S \) may be coisometric.

In this paper we study another issue not present in the univariate classical case, namely the distinction between commutative realizations (where the state space operators \( A_1, \ldots, A_d \) in (1.8) commute with each other) versus general realizations. Commutative realization is a natural object that appears for example in model theory for commuting row contractions [14]: the characteristic function of a commuting
row contraction \((T_1, \ldots, T_d)\) is, by definition, a Schur-class function that admits a unitary commutative realization with the state space operators \(T_1, \ldots, T_d\). It turns out that not every \(S \in \mathcal{S}_d(U, \mathcal{Y})\) can be identified as a characteristic function of a commutative row contraction; thus not every \(S \in \mathcal{S}_d(U, \mathcal{Y})\) admits a commutative unitary realization. Some more delicate arguments based on backward-shift invariance in \(H_Y(k_d)\) show that not every \(S \in \mathcal{S}_d(U, \mathcal{Y})\) admits a commutative weakly coisometric realization (see Theorem 3.5 below); more surprisingly, there are Schur-class functions that do not admit even contractive commutative realizations (see Example 3.4 below). If the Schur-class function admits a commutative weakly coisometric realization, then the associated de Branges-Rovnyak space \(H(S)\) is invariant for the backward shift operators \(M^\ast \lambda_j\) and one can arrange for a generalized functional-model realization with the additional property that the state operators \(A_1, \ldots, A_d\) are given by \(A_j = M^\ast \lambda_j|_{H(S)}\) for \(j = 1, \ldots, d\); we say that such a realization is a (non-generalized) functional-model realization. The operators \(B_1, \ldots, B_d\) are not defined uniquely by \(S(\lambda), A = (A_1, \ldots, A_d)\) and \(C\) (this is yet another distinction from the univariate case); however the nonuniqueness can be described in an explicit way. Furthermore, any observable, commutative, weakly coisometric realization for a given \(S\) is unitarily equivalent to exactly one functional-model realization (Theorem 3.6).

Inner functions, i.e., a Schur-class multiplier \(S \in \mathcal{S}_d(U, \mathcal{Y})\) for which the associated multiplication operator is a partial isometry, are special in that an inner function necessarily has a commutative weakly coisometric realization (see Theorem 3.5 below). Inner functions also play a special role as representers for (forward) shift-invariant subspaces of \(H_Y(k_d)\); for the case \(d = 1\) this is the classical Beurling-Lax-Halmos theorem \([13, 21, 22]\) while the case for general \(d\) appears more recently in the work of Arveson \([4]\) and of McCullough-Trent \([23]\) (for the general framework of a complete Nevanlinna-Pick kernel). Here we use our realization-theoretic characterization of inner multipliers to present a new proof of the \(H_Y(k_d)\)-Beurling-Lax theorem. The idea in this approach is to represent the shift-invariant subspace \(\mathcal{M}\) as the set of all \(H_Y(k_d)\)-solutions of fairly general set of homogeneous interpolation conditions, and then to construct a realization \(U = [A \ B]\) for \(S(\lambda)\) from the operators defining the homogeneous interpolation conditions. For the case \(d = 1\), this approach can be found in \([9]\) for the rational case and in \([10]\) for the non-rational case, done there in the more complicated context where the shift-invariant subspace \(\mathcal{M}\) is merely contained in the \(\mathcal{Y}\)-valued \(L^2\) space over the unit circle \(T\) and is not necessarily contained in the Hardy space \(H_Y(k_1) = H^2_Y\). We also use our analysis of the nonuniqueness of the input operator \(B\) in weakly coisometric realizations to characterize the nonuniqueness in the choice of inner-function representer \(S\) for a given shift-invariant subspace \(\mathcal{M}\) (see Theorem 4.15).

A more general version of the \(H_Y(k_d)\)-Beurling-Lax theorem, where the subspace \(\mathcal{M}\) is only contractively included in \(H_Y(k_d)\) and the representer is a not necessarily inner Schur-class multiplier, appears in the work of de Branges-Rovnyak \([17, 18]\) for the case \(d = 1\) and of the authors \([6]\) for the case of general \(d\). The realization produced by our approach here (working with \(M^\perp\) rather than directly with \(\mathcal{M}\)) is more explicit for the situation where \(\mathcal{M}\) is presented as the solution set for a homogeneous interpolation problem.
The paper is organized as follows. After the present Introduction, Section 2 recalls needed preliminaries from our earlier papers [6, 7] concerning weakly coisometric realizations (see Definition 1.2 above). Section 3 collects the results concerning such realizations where the collection of state operators $A_1, \ldots, A_d$ is commutative. Section 4 specializes the general theory to the case of inner functions. The final Section 5 discusses connections with characteristic functions and operator-model theory for commutative row contractions, a topic of recent work of Bhattacharyya, Eschmeier, Sarkar and Popescu [11, 15, 16, 27, 28], where some extensions to more general settings are also addressed.

2. Weakly coisometric realizations

Weakly coisometric realizations of Schur-class functions are closely related to range spaces of observability operators appearing in the context of Fornasini-Marchesini-type linear systems with evolution along the integer lattice $\mathbb{Z}^d$. Let $A = (A_1, \ldots, A_d)$ be a $d$-tuple of operators in $\mathcal{L}(\mathcal{X})$. If $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, then the pair $(C, A)$ is said to be an output pair. Such an output pair is said to be contractive if

$$A_1^* A_1 + \cdots + A_d^* A_d + C^* C \leq I, \quad (2.1)$$

and to be isometric if equality holds in the above relation, and to be output-stable if the associated observability operator

$$O_{C, A}: x \mapsto C(I_X - Z(\lambda) A)^{-1} x = C(I - \lambda_1 A_1 - \cdots - \lambda_d A_d)^{-1} x \quad (2.2)$$

(where $Z(\lambda)$ and $A$ are defined as in [1, 9]) maps $\mathcal{X}$ into $\mathcal{H}_Y(k_d)$. As it was shown in [6], any contractive pair $(C, A)$ is output stable and moreover, the corresponding observability operator $O_{C, A}: \mathcal{X} \to \mathcal{H}_Y(k_d)$ is a contraction. An output stable pair $(C, A)$ is called observable if the observability operator $O_{C, A}$ is injective, i.e.,

$$C(I_X - Z(\lambda) A)^{-1} x \equiv 0 \implies x = 0.$$ 

Given an output stable pair $(C, A)$, the kernel

$$K_{C, A}(\lambda, \zeta) := C(I_X - Z(\lambda) A)^{-1} (I_X - A^* Z(\zeta)^*)^{-1} C^* \quad (2.3)$$

is positive on $\mathbb{B}^d \times \mathbb{B}^d$; let $\mathcal{H}(K_{C, A})$ denote the associated RKHS. We recall (see [3]) that any positive kernel $(\lambda, \zeta) \mapsto K(\lambda, \zeta) \in \mathcal{L}(\mathcal{Y})$ on a set $\Omega \times \Omega$ (so $\lambda, \zeta \in \Omega$) gives rise to a RKHS $\mathcal{H}(K)$ consisting of $\mathcal{Y}$-valued functions on $\Omega$ with the defining property: for each $\zeta \in \Omega$ and $y \in \mathcal{Y}$, the $\mathcal{Y}$-valued function $K_y(\lambda) := K(\lambda, \zeta) y$ is in $\mathcal{H}(K)$ and has the reproducing property

$$\langle f, K_y \rangle_{\mathcal{H}(K)} = \langle f(\zeta), y \rangle_{\mathcal{Y}} \quad \text{for all } y \in \mathcal{Y}, f \in \mathcal{H}(K).$$

The following result from [6] gives the close connection between spaces of the form $\mathcal{H}(K_{C, A})$ and ranges of observability operators.

Theorem 2.1. (See [8] Theorem 3.20.) Let $(C, A)$ be a contractive pair with $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and with associated positive kernel $K_{C, A}$ given by (2.3) and the observability operator $O_{C, A}$ given by (2.1). Then:

1. The reproducing kernel Hilbert space $\mathcal{H}(K_{C, A})$ is characterized as

$$\mathcal{H}(K_{C, A}) = \text{Ran } O_{C, A}$$

with the lifted norm given by

$$\|O_{C, A} x\|_{\mathcal{H}(K_{C, A})} = \|Q x\|_{\mathcal{X}} \quad (2.4)$$

where $Q$ is the orthogonal projection onto $(\text{Ker } O_{C, A})^\perp$. 

The operator \( O_{C,A} \) is a contraction of \( X \) into \( \mathcal{H}(K_{C,A}) \). It is an isometry if and only if the pair \((C,A)\) is observable.

The space \( \mathcal{H}(K_{C,A}) \) is contractively included in the Arveson space \( \mathcal{H}_Y(k_d) \); it is isometrically included in \( \mathcal{H}_Y(k_d) \) if and only if \( O_{C,A} \) (as an operator from \( X \) into \( \mathcal{H}_Y(k_d) \)) is a partial isometry.

If \( S \) is realized as in (1.8) and \( U \) is the connecting operator given by (1.7), then the associated kernels \( K_S \) and \( K_{C,A} \) (defined in (1.5) and (2.3), respectively) are related by the following easily verified identity

\[
K_S(\lambda, \zeta) = K_{C,A}(\lambda, \zeta) + [C(I - Z(\lambda)A)^{-1}Z(\lambda) I] \frac{I - UU^*}{1 - \langle\lambda, \zeta\rangle} \left[ Z(\zeta)^*(I_X - A^*Z(\zeta)^*)^{-1}C^* \right]
\]

and then, it is easily shown (see Proposition 1.5 in [7] for details) that the second term on the right vanishes if and only if \( U^* \) is isometric on the space \( D \oplus Y \) defined as in Definition 1.2. This observation leads us to the following intrinsic kernel characterization as to when a given contractive realization is a weakly coisometric realization.

**Proposition 2.2.** A contractive realization (1.8) of \( S \in S_d(U, Y) \) is weakly coisometric if and only if the kernel \( K_S(\lambda, \zeta) \) associated to \( S \) via (1.5) can alternatively be written as

\[
K_S(\lambda, \zeta) = K_{C,A}(\lambda, \zeta)
\]

where \( K_{C,A} \) is given by (2.3).

Proposition 2.2 states that once a contractive realization \( U = [A \ B \ C \ D] \) of \( S \) is such that (2.6) holds, then this realization is weakly coisometric. The next result asserts that equality (2.6) itself guarantees the existence of weakly coisometric realizations for \( S \) with preassigned \( C \) and \( A = (A_1, \ldots, A_d) \).

**Theorem 2.3.** (See [7] Theorem 2.4.) Suppose that a Schur-class function \( S \in S_d(U, Y) \) and a contractive pair \((C,A)\) are such that (2.6) holds and let \( D := S(0) \). Then there exist operators \( B : U \rightarrow \mathcal{X}^d \) so that the operator \( U \) of the form (1.7) is weakly coisometric and \( S \) can be realized as in (1.8).

The pair \((C,A)\) for a weakly coisometric realization can be constructed in a certain canonical way.

**Theorem 2.4.** (See [7] Theorem 3.20.) Let \( S \in S_d(U, Y) \) and let \( \mathcal{H}(K_S) \) be the associated de Branges-Rovnyak space. Then:

1. There exist bounded operators \( A_j : \mathcal{H}(K_S) \rightarrow \mathcal{H}(K_S) \) such that

\[
f(\lambda) - f(0) = \sum_{j=1}^d \lambda_j (A_j f)(\lambda) \quad \text{for every } f \in \mathcal{H}(K_S) \text{ and } \lambda \in \mathbb{B}^d,
\]

and

\[
\sum_{j=1}^d \|A_j f\|_{\mathcal{H}(K_S)}^2 \leq \|f\|_{\mathcal{H}(K_S)}^2 - \|f(0)\|_{Y}^2.
\]

(2.8)
(2) There is a weakly coisometric realization \( [18] \) for \( S \) with state space \( \mathcal{X} \) equal to \( \mathcal{H}(K_S) \) with the state operators \( A_1, \ldots, A_d \) from part (1) and the operator \( C: \mathcal{H}(K_S) \to \mathcal{Y} \) defined by
\[
Cf = f(0) \quad \text{for all} \quad f \in \mathcal{H}(K_S).
\] (2.9)

Equality \( [24] \) means that the operator tuple \( \mathbf{A} = (A_1, \ldots, A_d) \) solves the Gleason problem \( [19] \) for \( \mathcal{H}(K_S) \). Let us say that \( \mathbf{A} \) is a contractive solution of the Gleason problem if in addition relation \( [28] \) holds for every \( \mathbf{A} \) is a contractive solution \( \mathbf{A} = (A_1, \ldots, A_d) \) of the Gleason problem for \( \mathcal{H}(K_S) \) gives rise to a weakly coisometric realization for \( S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y}) \) (not unique, in general). Let us call any such weakly coisometric realization a generalized functional-model realization of \( S(\lambda) \). We note that any generalized functional-model realization of \( S \) is observable and that the formula
\[
K_S(\cdot, \zeta)y = (I - A^*Z(\zeta)^*)^{-1}C^*y \quad (y \in \mathcal{Y}, \zeta \in \mathbb{B}^d)
\] (2.10)
is valid for any generalized functional-model realization. Furthermore, if
\[
\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H}(K_S) \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}(K_S)^d \\ \mathcal{Y} \end{bmatrix}
\] (2.11)
is a generalized functional model realization for an \( S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y}) \) then the space \( \mathcal{D} \) introduced in \( [1, 10] \) can be described in the following explicit functional form
\[
\mathcal{D} = \overline{\text{span}}\{Z(\zeta)^*K_S(\cdot, \zeta)y : \zeta \in \mathbb{B}^d, y \in \mathcal{Y}\}. \quad (2.12)
\]
Then a simple calculation shows that \( \mathcal{D}^\perp = \mathcal{H}(K_S)^d \ominus \mathcal{D} \) can be characterized in similar terms as
\[
\mathcal{D}^\perp = \{h \in \mathcal{H}(K_S)^d : Z(\lambda)h(\lambda) \equiv 0\}. \quad (2.13)
\]

3. Realizations with commutative state space operators

Schur-class functions that admit unitary realizations of the form \( [18] \) with commutative state space tuple \( \mathbf{A} = (A_1, \ldots, A_d) \) is a natural object appearing in the model theory for commutative row contractions (see \( [3] \)); the characteristic function of a commutative row contraction (see formula \( [6, 1] \) below) is a Schur-class function of this type (subject to an additional normalization). In the commutative context, a key role is played by the commuting \( d \)-tuple \( \mathbf{M}_\lambda := (M_{\lambda_1}, \ldots, M_{\lambda_d}) \) consisting of operators of multiplication by the coordinate functions of \( \mathbb{C}^d \) which will be called the shift (operator-tuple) of \( \mathcal{H}_Y(k_d) \), whereas the commuting \( d \)-tuple \( \mathbf{M}_\lambda := (M_{\lambda_1}^*, \ldots, M_{\lambda_d}^*) \) consisting of the adjoints of \( M_{\lambda_j} \)'s (in the metric of \( \mathcal{H}_Y(k_d) \)) will be referred to as to the backward shift. By the characterization \( [1, 2] \) and in notation \( [13] \), the monomials \( \frac{n!}{m!}\lambda^n \) form an orthonormal basis in \( \mathcal{H}(k_d) \) and then a simple calculation shows that
\[
M_{\lambda_j}^*\lambda^m = \frac{n!}{m!}\lambda^{m-e_j} \quad (m_j \geq 1) \quad \text{and} \quad M_{\lambda_j}^*\lambda^m = 0 \quad (m_j = 0)
\] (3.1)
where \( m = (m_1, \ldots, m_d) \) and \( e_j \) is the \( j \)-th standard coordinate vector of \( \mathbb{C}^d \).

Some properties of the shift tuple \( \mathbf{M}_\lambda \) needed in the sequel are listed below (for the proof, see e.g., \( [6, \text{Proposition 3.12}] \)). In the formulation and in what follows, we use multivariable power notation
\[
\mathbf{A}^n := A_1^{n_1}A_2^{n_2} \ldots A_d^{n_d}
\]
for any $d$-tuple $A = (A_1, \ldots, A_d)$ of commuting operators on a space $X$ and any $n = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d$.

**Proposition 3.1.** Let $M^*_\lambda$ be the $d$-tuple of backward shifts on $\mathcal{H}_Y(k_d)$ and let

$$G : f \mapsto f(0) \quad (f \in \mathcal{H}_Y(k_d))$$

be the operator of evaluation at the origin. Then:

1. For every $f \in \mathcal{H}_Y(k_d)$ and every $\lambda \in \mathbb{B}^d$ we have

$$f(\lambda) - f(0) = \sum_{j=1}^{d} \lambda_j (M^*_\lambda f)(\lambda).$$

2. The pair $(G, M^*_\lambda)$ is isometric and the associated observability operator is the identity operator:

$$O_{G, M^*_\lambda} = I_{\mathcal{H}_Y(k_d)}.$$  

3. The $d$-tuple $M^*_\lambda$ is strongly stable, that is,

$$\lim_{N \to \infty} \sum_{n \in \mathbb{Z}_+^d : |n| = N} \frac{N!}{n!} \| (M^*_\lambda)^n f \|_{\mathcal{H}_Y(k_d)}^2 = 0 \quad \text{for every} \quad f \in \mathcal{H}_Y(k_d).$$

We will also need the commutative analogue of Theorem 2.1 (see [6, Theorem 3.15] for the proof).

**Theorem 3.2.** Let $(C, A)$ be a contractive pair such that $C \in \mathcal{L}(X, Y)$ and the $d$-tuple $A = (A_1, \ldots, A_d) \in \mathcal{L}(X)^d$ is commutative. Let $K_{C, A}$ be the associated kernel given by (2.3). Then:

1. The reproducing kernel Hilbert space $\mathcal{H}(K_{C, A})$ is invariant under $M^*_\lambda$ for $j = 1, \ldots, d$ ($M^*_\lambda$-invariant) and the difference-quotient inequality

$$\sum_{j=1}^{d} \| (M^*_\lambda f) \|_{\mathcal{H}(K_{C, A})}^2 \leq \| f \|_{\mathcal{H}(K_{C, A})}^2 - \| f(0) \|_{Y}^2$$

holds for every $f \in \mathcal{H}(K_{C, A})$.

2. The space $\mathcal{H}(K_{C, A})$ is contractively included in $\mathcal{H}_Y(k_d)$. The inclusion is isometric exactly when the pair $(C, A)$ is isometric:

$$I_X - A_1^*A_1 - \cdots - A_d^*A_d = C^*C,$$  

and $A$ is strongly stable:

$$\lim_{N \to \infty} \sum_{n \in \mathbb{Z}_+^d : |n| = N} \frac{N!}{n!} \| A^n x \|_{X}^2 = 0 \quad \text{for all} \quad x \in X.$$  

If one drops the requirement of the connecting operator $U$ being contractive, constructing a commutative realization is not an issue not only for Schur-class functions, but even for functions from $\mathcal{H}(k_d) \otimes \mathcal{L}(U, Y)$. Indeed, for an $S \in \mathcal{H}(k_d) \otimes \mathcal{L}(U, Y)$, let

$$C = G, \quad D = S(0), \quad A_j = M^*_\lambda, \quad B_j = M^*_\lambda M_S|U \quad (j = 1, \ldots, d)$$
Proposition 3.3. Let $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{U})$ be such that the associated de Branges-Rovnyak space $\mathcal{H}(K_S)$ is finite-dimensional and is not $\mathbf{M}_\lambda^*$-invariant. Then $S$ does not have a commutative contractive realization.

Proof. Assume that $S$ admits a contractive commutative realization (1.8) with a contractive, unitary and all intermediate) realizations do exist (by Theorem 1.1), it is natural to construct commutative realizations of the same types. Note that Theorem 1.1 and more specific Theorem 2.4 give no clue as to when and how one can achieve a such a realization of a given $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$. The next proposition shows that there are Schur-class functions which do not have a commutative contractive realization.

Example 3.4. For a concrete example of a Schur-class function satisfying assumptions in Proposition 3.3, let

$$S(\lambda_1, \lambda_2) = \frac{1}{4 - \lambda_1 \lambda_2} \begin{bmatrix} 2 \sqrt{2}\lambda_1 & \sqrt{3}\lambda_2^2 & 2 - 2\lambda_1\lambda_2 & -3\lambda_2 \\ \sqrt{3}\lambda_2 & 2\sqrt{6}\lambda_2 & -3\lambda_1 & 2 - 2\lambda_1\lambda_2 \end{bmatrix}. \quad (3.8)$$
A straightforward calculation gives
\[
K_S(\lambda, \zeta) := \frac{I_2 - S(\lambda)S(\zeta)^*}{1 - \lambda_1 \zeta_1 - \lambda_2 \zeta_2}
\]
\[
= \frac{3}{(4 - \lambda_1 \lambda_2)(4 - \zeta_1 \zeta_2)} \begin{bmatrix} 2 & \lambda_2 \\ \lambda_1 & 2 \end{bmatrix} \begin{bmatrix} 2 & \zeta_1 \\ \zeta_2 & 2 \end{bmatrix}.
\]

Thus the kernel \(K_S(\lambda, \zeta)\) is positive on \(\mathbb{D}^2 \times \mathbb{D}^2\) and \(S \in \mathcal{S}_2(\mathbb{C}^4, \mathbb{C}^2)\). The associated de Branges-Rovnyak space \(\mathcal{H}(K_S)\) is spanned by rational functions
\[
f_1(\lambda) = \frac{4}{4 - \lambda_1 \lambda_2} \begin{bmatrix} 2 \\ \lambda_1 \end{bmatrix} \quad \text{and} \quad f_2(\lambda) = \frac{4}{4 - \lambda_1 \lambda_2} \begin{bmatrix} \lambda_2 \\ 2 \end{bmatrix}.
\]

Furthermore, since by (3.1) we have
\[
M^*_n(\frac{4 \lambda_1}{4 - \lambda_1 \lambda_2}) = M^*_n\left(\sum_{j=0}^{\infty} \frac{\lambda_1^{j+1} \lambda_2^j}{4^j}\right) = \sum_{j=0}^{\infty} \frac{j + 1}{2j + 1} \left(\frac{\lambda_1 \lambda_2}{4}\right)^j.
\]

The latter function is rational if and only if the single-variable function \(F(\lambda) = \sum_{j=0}^{\infty} \frac{j + 1}{2j + 1} \lambda^j\) is rational. By the well-known Kronecker theorem, \(F(\lambda)\) in turn is rational if and only if the associated infinite Hankel matrix
\[
H = [s_{i+j}]_{i,j=0}^{\infty} \quad \text{where} \quad s_k = \frac{k + 1}{2k + 1}
\]
has finite rank. However one can check that the finite Hankel matrices \(H_n = [s_{i+j}]_{i,j=0}^{n}\) have full rank for all \(n = 0, 1, 2, \ldots\) and hence \(F(\lambda)\) is not rational. We conclude that the function on the right hand side in (3.9) is not rational. Now it follows that \(M^*_n f_1\) does not belong to \(\mathcal{H}(K_S)\). Therefore \(\mathcal{H}(K_S)\) is not invariant under \(M^*_n\), and since \(\dim \mathcal{H}(K_S) = 2 < \infty\), the function \(S\) does not admit contractive commutative realizations by Proposition 3.8.

A characterization of which Schur-class functions do admit contractive commutative realizations will be given in Theorem 3.10 below. The next result gives a characterization of Schur-class functions that admit weakly coisometric commutative realizations.

**Theorem 3.5.** A Schur-class function \(S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})\) admits a commutative weakly coisometric realization if and only if the following conditions hold:

1. The associated de Branges-Rovnyak space \(\mathcal{H}(K_S)\) is \(M^*_\lambda\)-invariant, and
2. the inequality
\[
\sum_{j=1}^{d} \|M^*_n f\|^2_{\mathcal{H}(K_S)} \leq \|f\|^2_{\mathcal{H}(K_S)} - \|f(0)\|^2_{\mathcal{H}(K_S)} \quad \text{holds for all} \quad f \in \mathcal{H}(K_S).
\]

**Proof.** To prove necessity, suppose that \(S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})\) admits a weakly coisometric realization \(\mathcal{U}_S\). As noted in Proposition 2.2, it follows that \(\mathcal{H}(K_S) = \mathcal{H}(K_C, A)\). Since \(A\) is commutative, Theorem 3.2 implies that the space \(\mathcal{H}(K_S) = \mathcal{H}(K_C, A)\) is \(M^*_\lambda\)-invariant with the inequality (3.10) holding.
To prove sufficiency, suppose that \( S \in \mathcal{S}_d(U, \mathcal{Y}) \) is such that \( \mathcal{H}(K_S) \) is \( \mathbf{M}_1^\ast \)-invariant with (3.10) holding. Define operators \( A_1, \ldots, A_d : \mathcal{H}(K_S) \to \mathcal{H}(K_S) \), \( C : \mathcal{H}(K_S) \to \mathcal{Y} \) and \( D : U \to \mathcal{Y} \) by

\[
A_j = M_\lambda^\ast |_{\mathcal{H}(K_S)} \quad (j = 1, \ldots, d), \quad C : f \to f(0), \quad D = S(0).
\]  

(3.11)

Formula (3.3) tells us that the operators \( H \) for \( H \) solve the Gleason problem for \( \mathcal{H}(K_S) \). Then we apply Theorem 2.4 (part (2)) to conclude that there is a choice of \( B : U \to \mathcal{H}(K_S)^d \) with \( U \) of the form (2.7) weakly coisometric so that \( S(\lambda) = D + C(I - Z(\lambda)A)^{-1}Z(\lambda)B \). This completes the proof. \( \square \)

Note that the proof of Theorem 3.3 obtains a realization for \( S \in \mathcal{S}_d(U, \mathcal{Y}) \) of a special form under the assumption that \( \mathcal{H}(K_S) \) is \( \mathbf{M}_1^\ast \)-invariant: the state space \( \mathcal{X} \) is taken to be the de Branges-Rovnyak space \( \mathcal{H}(K_S) \) and the operators \( A = (A_1, \ldots, A_d), C, D \) are given by (3.11); only the operators \( B_j : U \to \mathcal{H}(K_S) \) remain to be determined. We shall say that any contractive realization of a given Schur-class function \( S \) of this form (i.e., with \( \mathcal{X} = \mathcal{H}(K_S) \) and \( A, C, D \) given by (3.11)) is a functional-model realization of \( S \). It is readily seen that any functional-model realization is also a generalized functional-model realization; in particular, it is weakly coisometric and observable.

Let us recall that two colligations

\[
U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{X} \oplus U \to \mathcal{X}^d \oplus \mathcal{Y} \quad \text{and} \quad \bar{U} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} : \bar{\mathcal{X}} \oplus U \to \bar{\mathcal{X}}^d \oplus \mathcal{Y}
\]

are said to be unitarily equivalent if there is a unitary operator \( U : \mathcal{X} \to \bar{\mathcal{X}} \) such that

\[
\begin{bmatrix} \oplus_{k=1}^d U & 0 \\ 0 & I_Y \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I_d \end{bmatrix}.
\]

As it was shown in [7], any observable weakly coisometric realization of a Schur-class function \( S \in \mathcal{S}_d(U, \mathcal{Y}) \) is unitarily equivalent to some generalized functional-model realization. An analogous result concerning the universality of functional-model realizations among commutative realizations is more specific.

**Theorem 3.6.** Suppose that \( S(\lambda) \in \mathcal{S}_d(U, \mathcal{Y}) \) is a Schur-class function that admits functional-model realizations. Then any commutative, observable, weakly coisometric realization of \( S \) is unitarily equivalent to exactly one functional-model realization of \( S \).

**Proof.** Let \( S(\lambda) = D + \bar{C}(I_{\bar{\mathcal{X}}} - Z(\lambda)\bar{A})^{-1}Z(\lambda)\bar{B} \) be a commutative, observable, weakly coisometric realization of \( S \). Then \( K_S(\lambda, \xi) = K_{\bar{C}, \bar{A}}(\lambda, \xi) \) by Proposition 2.2. Define operators \( A_j \)'s and \( C \) as in (3.11). Since \( S \) admits functional-model realizations (that contain \( A_j \)'s and \( C \) and are weakly coisometric), then we have also \( K_{C, A}(\lambda, \xi) = K_S(\lambda, \xi) \). Therefore \( K_{C, A} = K_{\bar{C}, \bar{A}} \). Since the the pairs \( (\bar{C}, \bar{A}) \) and \( (C, A) \) are observable, the latter equality implies (see [6, Theorem 3.17]) that there exists a unitary operator \( U : \mathcal{H}(K_S) \to \bar{\mathcal{X}} \) such that

\[
C = \bar{C}U \quad \text{and} \quad A_j = U^* \bar{A}_j U \quad \text{for} \quad j = 1, \ldots, d.
\]
Now we let \( B_j := U^* B_j : \mathcal{Y} \to \mathcal{H}(K_S) \) for \( j = 1, \ldots, d \) which is the unique choice that guarantees the realization \( U = [\hat{A} \hat{B} \hat{C} \hat{D}] \) (with \( A \) and \( B \) defined as in (3.13)) to be unitarily equivalent to the original realization \( \tilde{U} = [\hat{A} \hat{B} \hat{C} \hat{D}] \) and it is functional-model realization due to the canonical choice of \( C \) and \( A_j \)'s.

**Corollary 3.7.** Let \( U' = [A' B' C' D'] \) and \( U'' = [A'' B'' C'' D''] \) be two observable commutative weakly coisometric realizations of a Schur-class function \( S \in S_d(U, \mathcal{Y}) \). Then the pairs \((C', A')\) and \((C'', A'')\) are unitarily equivalent.

**Proof.** By Theorem 3.6, the pairs \((C', A')\) and \((C'', A'')\) are both unitarily equivalent to the canonical pair \((C, A)\) with \( C \) and \( A = (A_1, \ldots, A_d) \) defined as in (3.11). Hence \((C', A')\) and \((C'', A'')\) are unitarily equivalent to each other. \( \square \)

**Remark 3.8.** It was pointed out in [7] and justified by examples (e.g., Example 3.5) a Schur-class function may have many weakly coisometric observable realizations with associated output pairs \((C, A)\) not unitarily equivalent. Theorem 3.6 above shows that if \( S \in S_d(U, \mathcal{Y}) \) admits a commutative weakly coisometric realization, then the output pair \((C, A)\) of any commutative weakly coisometric observable realization is uniquely defined up to unitary equivalence. The example below shows that in the latter case, \( S \) may also admit many noncommutative observable weakly coisometric realizations with output pairs not unitarily equivalent. This example is of certain interest because of Theorem 4.5 below showing that in the latter case, \( S \) does not admit any unitarily equivalent observable realizations.

**Example 3.9.** Take the matrices

\[
C = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_{0,1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_{0,2} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (3.12)
\]

\[
B_{0,1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad B_{0,2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (3.13)
\]

\[
D = \begin{bmatrix} 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.14)
\]

so that the \( 7 \times 8 \) matrix

\[
U_0 = \begin{bmatrix} A_{0,1} & B_{0,1} \\ A_{0,2} & B_{0,2} \\ C & D \end{bmatrix}
\]

is coisometric. Then the characteristic function of the colligation \( U_0 \),

\[
S(\lambda) = D + C(I - \lambda_1 A_{0,1} - \lambda_2 A_{0,2})^{-1}(\lambda_1 B_{0,1} + \lambda_2 B_{0,2}) \quad (3.15)
\]

belongs to the Schur class \( S_2(\mathbb{C}^5, \mathbb{C}) \). It is readily seen that

\[
C(I - \lambda_1 A_{0,1} - \lambda_2 A_{0,2})^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix} \quad (3.16)
\]

which being substituted along with (3.13), (3.14) into (3.15) gives the explicit formula

\[
S(\lambda) = \frac{1}{2} \begin{bmatrix} \lambda_1^2 & \lambda_1 \lambda_2 & \lambda_1 \lambda_3 & \lambda_2^2 & \lambda_2 \lambda_3 & \lambda_3^2 & \sqrt{3} \end{bmatrix}. \quad (3.17)
\]
It is readily seen that the pair \((C, A_0)\) is observable (where we let \(A_0 = (A_{0,1}, A_{0,2})\)) and thus, representation (3.15) is a coisometric (and therefore, also weakly coisometric) observable realization of the function \(S \in S_2(C^5, \mathbb{C})\) given by (3.17). Then we also have
\[
K_S(\lambda, \zeta) = C(I - \lambda_1 A_{0,1} - \lambda_2 A_{0,2})^{-1}(I - \zeta_1 A_{0,1}^* - \zeta_2 A_{0,2}^*)^{-1}C^* = K_{C,A_0}(\lambda, \zeta). \tag{3.18}
\]

Now let us consider the matrices
\[
A_{\gamma,1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \gamma & 0 & 0 \end{bmatrix} \quad \text{and} \quad A_{\gamma,2} = \begin{bmatrix} 0 & 0 & 1 \\ -\gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.19}
\]
where \(\gamma \in \mathbb{C}\) is a parameter, and note that
\[
C(I - \lambda_1 A_{\gamma,1} - \lambda_2 A_{\gamma,2})^{-1} = \frac{1}{2} \cdot \begin{bmatrix} 1 & \lambda_1 & \lambda_2 \end{bmatrix}
\]
for every \(\gamma\). In particular, the pair \((C, A_\gamma)\) is observable for every \(\gamma\). The latter equality together with (3.18) gives
\[
K_S(\lambda, \zeta) = K_{C,A_\gamma}(\lambda, \zeta). \tag{3.20}
\]

Now pick any \(\gamma\) so that \(|\gamma| < \sqrt{\frac{3}{8}}\). As it is easily seen, the latter inequality is equivalent to the pair \((C, A_\gamma)\) being contractive. Thus, we have a Schur-class function \(S\) and a contractive pair \((C, A_\gamma)\) such that equality (3.20) holds. Then by Theorem 2.3 there exist operators \(B_{\gamma,1}\) and \(B_{\gamma,2}\) so that the operator
\[
U_\gamma = \begin{bmatrix} A_{\gamma,1} & B_{\gamma,1} \\ A_{\gamma,2} & B_{\gamma,2} \\ C & D \end{bmatrix}
\]
is weakly coisometric and \(S\) can be realized as
\[
S(\lambda) = D + C(I - \lambda_1 A_{\gamma,1} - \lambda_2 A_{\gamma,2})^{-1}(\lambda_1 B_{\gamma,1} + \lambda_2 B_{\gamma,2}).
\]
It remains to note that the pairs \((C, A_\gamma)\) and \((C, A_{\gamma'})\) are not unitarily equivalent (which is shown by another elementary calculation) unless \(\gamma = \gamma'\).

We conclude this section with characterizing Schur-class functions that admit contractive commutative realizations.

**Theorem 3.10.** A Schur-class function \(S \in S_d(U, \mathcal{Y})\) admits a contractive commutative realization if and only if it can be extended to a Schur-class function
\[
\tilde{S}(\lambda) = \begin{bmatrix} S(\lambda) & \tilde{S}(\lambda) \end{bmatrix} \in S_d(U \oplus \mathcal{F}, \mathcal{Y}) \tag{3.21}
\]
such that the de Branges-Rovnyak space \(\mathcal{H}(K_{\tilde{S}})\) is \(M_{\lambda}^\ast\)-invariant and the inequality
\[
\sum_{j=1}^d \|M_{\lambda_j}^\ast f\|_{\mathcal{H}(K_{\tilde{S}})}^2 \leq \|f\|_{\mathcal{H}(K_{\tilde{S}})}^2 - \|f(0)\|_{\mathcal{Y}}^2 \tag{3.22}
\]
holds for every \(f \in \mathcal{H}(K_{\tilde{S}})\).
Proof. Let $S$ admit a contractive commutative realization of the form (1.8). Extend the connecting operator $U$ of the form (1.7) to a coisometric operator

$$
\tilde{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \oplus \mathcal{F} \end{bmatrix} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{Y} \end{bmatrix}.
$$

(3.23)

The function

$$
\tilde{S}(\lambda) = \begin{bmatrix} D & \tilde{D} \end{bmatrix} + C(I - Z(\lambda)A)^{-1}Z(\lambda) \begin{bmatrix} B & \tilde{B} \end{bmatrix}
$$

is an extension of $S$ in the sense of (3.21). The latter realization is coisometric and commutative; thus $M_\lambda$-invariance of $\mathcal{H}(\tilde{S})$ and inequality (3.22) hold by Theorem 3.5.

Conversely, if $S$ can be extended to a Schur-class function $\hat{S}$ with associated de Branges-Rovnyak space $\mathcal{H}(K_{\hat{S}})$ invariant under $M_\lambda$ and satisfying property (3.22), we consider a weakly coisometric commutative realization (3.24) of $\hat{S}$ (which exists by Theorem 3.5) and restrict the input space to $\mathcal{U}$. This gives a contractive commutative realization for $S$. \hfill \square

4. REALIZATION FOR INNER MULTIPLIERS

In this section we focus on realization theory for inner multipliers. We first collect a couple of preliminary results needed for the sequel.

**Theorem 4.1.** Let $S \in S_d(\mathcal{U}, \mathcal{Y})$, let $M_S : \mathcal{H}(\mathcal{U}) \to \mathcal{H}(\mathcal{Y})$ be the multiplication operator defined in (1.4), let $K_S$ denote the positive kernel given by (1.5) and let $M_S$ denote the positive kernel

$$
M_S(\lambda, \zeta) = \frac{1}{1 - \langle \lambda, \zeta \rangle} \cdot S(\lambda)S(\zeta)^*.
$$

Then the reproducing kernel Hilbert spaces $\mathcal{H}(K_S)$ and $\mathcal{H}(M_S)$ can be characterized as

$$
\mathcal{H}(K_S) = \text{Ran} \ (I - M_S M_S^*)^{1/2}, \quad \mathcal{H}(M_S) = \text{Ran} \ M_S
$$

with respective norms

$$
\| (I - M_S M_S^*)^{1/2} f_1 \|_{\mathcal{H}(K_S)} = \| Q_1 f_1 \|_{\mathcal{H}(\mathcal{Y})} \quad \text{for all} \quad f_1 \in \mathcal{H}(\mathcal{Y}),
$$

$$
\| M_S f_2 \|_{\mathcal{H}(M_S)} = \| Q_2 f_2 \|_{\mathcal{H}(\mathcal{U})} \quad \text{for all} \quad f_2 \in \mathcal{H}(\mathcal{U})
$$

(4.1)

where $Q_1$ is the orthogonal projection onto $(\text{Ker} \ (I - M_S M_S^*)^{1/2})^\perp$ and $Q_2$ is the orthogonal projection onto $(\text{Ker} \ M_S)^\perp$.

**Proof.** The proof is based on a standard reproducing-kernel-space computation which we include for the sake of completeness.

Let $\mathcal{R}_{D_{s^*}}$ denote the space $\text{Ran} \ (I - M_S M_S^*)^{1/2}$ with norm given by the

$$
\| (I - M_S M_S^*)^{1/2} f_1 \|_{\mathcal{R}_{D_{s^*}}} = \| Q_1 f_1 \|.
$$

We also note the identity

$$
(I - M_S M_S^*)k_d(\cdot, \zeta)y = K_S(\cdot, \zeta)y \quad \text{for all} \quad \zeta \in \mathbb{B}^d \quad \text{and} \quad y \in \mathcal{Y}.
$$

It follows that the set

$$
\mathcal{D} := \text{span}\{(I - M_S M_S^*)k_d(\cdot, \zeta)y : \zeta \in \mathbb{B}^d, \ y \in \mathcal{Y}\}
$$
is dense in both $\mathcal{H}(K_S)$ and in $\mathcal{R}_{D_S}$. We then have, for all $\zeta, \zeta' \in \mathbb{B}^d$ and $y, y' \in \mathcal{Y}$,

\[
\langle (I - M_S M_S^* \zeta) d(\cdot, \zeta) y, (I - M_S M_S^* \zeta') d(\cdot, \zeta') y' \rangle_{\mathcal{R}_{D_S}} = \langle (I - M_S M_S^*) d(\cdot, \zeta) y, k_d(\cdot, \zeta') y' \rangle_{\mathcal{H}_Y(k_d)} = \langle k_d(\cdot, \zeta) y, (I - M_S M_S^*) d(\cdot, \zeta') y' \rangle_{\mathcal{H}(K_S)}.
\]

Hence the $\mathcal{R}_{D_S}$ and the $\mathcal{H}(K_S)$ inner products agree on a common dense subset. By taking closures and using completeness, it follows that $\mathcal{R}_{D_S}$ and $\mathcal{H}(K_S)$ are equal to each other isometrically.

The second statement in (4.1) follows in a similar way by using the observation

\[
M_S M_S^* k_d(\cdot, \zeta) y = M_S(\cdot, \zeta) y.
\]

□

A Schur-class function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ is said to be an inner multiplier if the multiplication operator $M_S$ (as an operator from $\mathcal{H}_d(k_d)$ into $\mathcal{H}_Y(k_d)$) is a partial isometry.

**Proposition 4.2.** Let $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$. The following are equivalent:

1. $S$ is inner.
2. $\mathcal{H}(K_S)$ is contained in $\mathcal{H}_Y(k_d)$ isometrically.
3. $\mathcal{H}(K_S) = (\text{Ran } M_S)^\perp$ isometrically.

In this case, $M_S M_S^*$ and $I_{\mathcal{H}_Y(k_d)} - M_S M_S^*$ are the orthogonal projections onto the closed subspaces Ran $M_S$ and $(\text{Ran } M_S)^\perp$ of $\mathcal{H}_Y(k_d)$, respectively.

**Proof.** The multiplier $S$ being inner is equivalent to Ran $M_S$ and Ran $(I - M_S M_S^*)^{1/2}$ being closed subspaces of $\mathcal{H}_Y(k_d)$ such that $M_S M_S^*$ and $I - M_S M_S^*$ are the orthogonal projections onto Ran $M_S$ and Ran $(I - M_S M_S^*)$ respectively. In this case the lifted-norm formulas (4.1) lead to isometric inclusions of $\mathcal{H}(K_S)$ and of $\mathcal{H}(M_S)$ in $\mathcal{H}_Y(k_d)$. The fact that $P_{\mathcal{H}(K_S)} = I - P_{\mathcal{H}(M_S)}$ tells us that $\mathcal{H}(K_S)$ and $\mathcal{H}(M_S)$ are orthogonal complements of each other. □

We are now ready for a realization characterization of inner multipliers.

**Theorem 4.3.** An $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function $S$ defined on $\mathbb{B}^d$ is an inner multiplier if and only if it admits a weakly coisometric realization (1.3) where:

1. the $d$-tuple $A = (A_1, \ldots, A_d)$ of the state space operators is commutative and is strongly stable (i.e., (3.7) holds), and
2. the output pair $(C, A)$ is isometric.

**Proof.** Suppose first that $S$ admits a realization (1.3) with $U = [A \ B \ C \ D]$ weakly coisometric with $A$ commutative and strongly stable and with (3.6) holding. By Proposition 2.2 we know that $K_S(\cdot, \zeta) = K_{C,A}(\lambda, \zeta)$. Combining this equality with Theorem 4.2 (part (2)), we conclude that the space $\mathcal{H}(K_S) = \mathcal{H}(K_{C,A})$ is included isometrically in $\mathcal{H}_Y(k_d)$. Therefore $S$ is inner by Proposition 4.2.

Conversely, suppose that $S$ is inner. Then, according to Proposition 4.2, $\mathcal{H}(K_S)$ is isometrically equal to the orthogonal complement of Ran $M_S$. As Ran $M_S$ is invariant under $M_{\lambda}$, it follows that $\mathcal{H}(K_S) = (\text{Ran } M_S)^\perp$ is $M_{\lambda}$-invariant. Hence
Theorem 4.5 applies; we let $U = \begin{bmatrix} A & B \end{bmatrix}$ be any weakly coisometric functional-model realization for $S$, that is with $A = (A_1, \ldots, A_d)$, $C$ and $D$ defined as in (3.11). Then $A$ is commutative since $M_A^*$ is commutative. As has been already observed, $\mathcal{H}(K_S) = (\text{Ran } M_S)^\perp$ is contained in $\mathcal{H}_Y(k_d)$ isometrically. Therefore $A = M_A^*|_{\mathcal{H}(K_S)}$ is strongly stable since $M_A^*$ is strongly stable on $\mathcal{H}_Y(k_d)$ by Proposition 3.11 (part (3)). By part (2) in the same proposition, the pair $(G, M_A^*)$ is isometric, i.e.,

$$I_{\mathcal{H}_Y(k_d)} - M_{\lambda_1}M_{A_1}^* - \ldots - M_{\lambda_d}M_{A_d}^* = G^*G. \quad (4.2)$$

Since $G$ and $C$ are the operators of evaluation at the origin on $\mathcal{H}_Y(k_d)$ and on $\mathcal{H}(K_S)$ respectively, we have $C = G|_{\mathcal{H}(K_S)}$. Then restricting operator equality (4.2) to $\mathcal{H}(K_S)$ we write the obtained equality in terms of $A$ and $C$ as

$$I_{\mathcal{H}(K_S)} - A_1^*A_1 - \ldots - A_d^*A_d = C^*C$$

which means that the pair $(C, A)$ is isometric. \hfill \Box

The next theorem is a variant of Theorem 3.10 for the inner case; the proof is much the same as that of Theorem 3.10 and hence will be omitted.

**Theorem 4.4.** A Schur-class function $S \in \mathcal{S}_d(U, Y)$ admits a contractive commutative realization of the form (3.11) with $A = (A_1, \ldots, A_d)$ strongly stable and $(C, A)$ isometric if and only if $S$ can be extended to an inner multiplier $\tilde{S}(\lambda) = \begin{bmatrix} S(\lambda) & \tilde{S}(\lambda) \end{bmatrix} \in \mathcal{S}_d(U \oplus \mathcal{F}, Y)$.

If $S$ is inner, then, as we have seen in the proof of Theorem 3.10 any functional-model realization for $S$ yields a commutative observable weakly coisometric realization for $S$. We now show that any observable weakly coisometric realization for $S$ necessarily is commutative.

**Theorem 4.5.** If $S \in \mathcal{S}_d(U, Y)$ is inner, then any observable weakly coisometric realization of $S$ is also commutative.

**Proof.** Let (3.11) be an observable weakly coisometric realization of $S$. Then $K_S = K_{C, A}$ (by Proposition 2.2) and therefore, since $S$ is inner, the space $\mathcal{H}(K_{C, A})$ is isometrically included into $\mathcal{H}_Y(k_d)$. By Theorem 3.10 (part (3)), the observability operator $O_{C, A} : \mathcal{X} \to \mathcal{H}_Y(k_d)$ is a partial isometry. Since the pair $(C, A)$ is observable, $O_{C, A}$ is in fact an isometry. Define the operators $T_1, \ldots, T_d$ on $\mathcal{H}(K_{C, A})$ and the operator $G : \mathcal{H}(K_{C, A}) \to Y$ by

$$T_jO_{C, A}x = O_{C, A}A_jx \quad (j = 1, \ldots, d), \quad GO_{C, A}x = Cx \quad \text{for } x \in \mathcal{X}. \quad (4.3)$$

Then for the generic element $f = O_{C, A}x$ of $\mathcal{H}(K_S) = \mathcal{H}(K_{C, A}) = \text{Ran } O_{C, A}$, we have

$$f(\lambda) = C(I - Z(\lambda)A)^{-1}x, \quad f(0) = Cx = GO_{C, A}x = Gf$$
and therefore,
\[
  f(\lambda) - f(0) = C(I - Z(\lambda)A)^{-1}x - Cx \\
  = C(I - Z(\lambda)A)^{-1}Z(\lambda)Ax \\
  = C(I - Z(\lambda)A)^{-1}\sum_{j=1}^{d}\lambda_jA_jx \\
  = \sum_{j=1}^{d}\lambda_j \cdot (\mathcal{O}_{C,A}A_jx)(\lambda) \\
  = \sum_{j=1}^{d}\lambda_j \cdot (T_j\mathcal{O}_{C,A}x)(\lambda) = \sum_{j=1}^{d}\lambda_j \cdot (T_j f)(\lambda)
\]
which means that the \(d\)-tuple \(T = (T_1, \ldots, T_d)\) solves the Gleason problem on \(\mathcal{H}(K_{C,A})\) and that \(G\) is simply the operator of evaluation at the origin. Since the pair \((C, A)\) is contractive and \(\mathcal{O}_{C,A}\) is isometric, it follows from (3.17) that the pair \((G, T)\) is also contractive. Now we recall a uniqueness result from [3] (Theorem 3.22 there): if \(\mathcal{M}\) is a backward-shift invariant subspace of \(\mathcal{H}_Y(k_d)\) isometrically included in \(\mathcal{H}_Y(k_d)\), then the \(d\)-tuple \(M_\lambda^|J = (M_\lambda^*|_{\mathcal{M}}, \ldots, M_{\lambda_d}^*|_{\mathcal{M}})\) is the only contractive solution of the Gleason problem on \(\mathcal{M}\). By this result applied to \(\mathcal{M} = \mathcal{H}(K_{C,A}) = \mathcal{H}(K_{S})\) (which is backward-shift invariant and isometrically included in \(\mathcal{H}_Y(k_d)\) since \(S\) is inner) we conclude that \(T_j = M_{\lambda_j}^|_{\mathcal{H}(K_{C,A})}\) for \(j = 1, \ldots, d\). In particular, the tuple \(T\) is commutative and therefore the original state space tuple \(A\) is necessarily commutative.

We next observe that in fact the weakly coisometric hypothesis in Theorem 4.5 can be weakened to contractive.

**Corollary 4.6.** If \(S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})\) is inner, then any observable contractive realization is commutative.

**Proof.** Assume that \(S(\lambda) = D + C(I - Z(\lambda)A)^{-1}Z(\lambda)B\) for a contractive connecting operator \(U = \{A, B\} : \mathcal{X} \oplus \mathcal{U} \to \mathcal{X}' \oplus \mathcal{Y}\). Extend \(U\) to a coisometric operator \(\hat{U}\) as in (3.24) and consider its characteristic function \(\hat{S}\) (see (3.24)) which extends \(S\) in the sense of (3.24). Since \(\hat{U}\) is a contraction, \(\hat{S}(\lambda)\) belongs to \(\mathcal{S}_d(\mathcal{U} \oplus \mathcal{F}, \mathcal{Y})\). By (3.24),
\[
  I_Y - \hat{S}(\lambda)\hat{S}(\lambda)^* = I_Y - S(\lambda)S(\lambda)^* - \hat{S}(\lambda)\hat{S}(\lambda)^* \geq 0
\]
for almost all \(\lambda \in \mathcal{S}^d\). Since \(S\) is inner, its boundary values are coisometric almost everywhere on \(\mathcal{S}^d\) (see (2.41)) and therefore, \(\hat{S}(\lambda) = 0\) almost everywhere on \(\mathcal{S}^d\). Therefore, \(\hat{S} \equiv 0\) and thus \(\hat{S}\) is inner. The formula (3.24) then gives an observable coisometric (and therefore also weakly coisometric) realization of the inner multiplier \(\hat{S}(\lambda)\). By Theorem 4.5 this realization necessarily is also commutative, i.e., \(A = (A_1, \ldots, A_d)\) is commutative. Hence the original realization \(U\) for \(S(\lambda)\) is commutative as asserted.

Theorems 3.5 and 3.6 imply that, if \(S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})\) is inner, then any contractive observable realization of \(S\) of the form (3.5) is commutative with operators \(A_1, \ldots, A_d, C\) uniquely defined (up to simultaneous unitary equivalence) and with \(D\) given by formulas (3.11). The nonuniqueness caused by possible different choices...
of $B_1, \ldots, B_d : \mathcal{U} \mapsto \mathcal{H}(K_S)$ can be described explicitly. Let $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ be inner, let $A = (A_1, \ldots, A_d)$, $C$, $D$ be given as in (3.11), let $\mathcal{D}$ be the subspace defined as in (2.12) (so that $D^\perp := \mathcal{H}(K_S)^d \ominus \mathcal{D}$ is characterized by (2.13)), and let $B : \mathcal{U} \mapsto (\mathcal{H}(K_S))^d$ be any operator so that $S$ can be realized in the form (1.8).

Then taking adjoints in (1.8) gives

$$B^* Z(\zeta)^* (I - A^* Z(\zeta)^*)^{-1} C^* = S(\zeta)^* - D^*$$

which, on account of (2.10), can be written equivalently as

$$B^* Z(\zeta)^* K_S(\cdot, \zeta)y = S(\zeta)^* y - S(0)^* y \quad (\zeta \in \mathbb{B}^d, \ y \in \mathcal{Y}).$$

Due to characterization (2.12) of $\mathcal{D}$, the latter formula completely determines the restriction of $B^*$ to $\mathcal{D}$:

$$B^*|_{\mathcal{D}} : Z(\zeta)^* K_S(\cdot, \zeta)y \to S(\zeta)^* y - S(0)^* y. \quad (4.4)$$

Write $B^* : (\mathcal{H}(K_S))^d \to \mathcal{U}$ in the form

$$B^* = \left[ X \quad B^*|_{\mathcal{D}} \right] \quad (4.5)$$

with $X = B^*|_{\mathcal{D}^\perp} : \mathcal{D}^\perp \to \mathcal{U}$. Next we note the explicit formulas for the adjoints $A_j^*$'s

$$A_j^* = \mathcal{P}_{\mathcal{H}(K_S)} M_{\lambda_j}|_{\mathcal{H}(K_S)} \quad (j = 1, \ldots, d) \quad (4.6)$$

(where $\mathcal{P}_{\mathcal{H}(K_S)}$ stands for the orthogonal projection of $\mathcal{H}_2(k_d)$ onto $\mathcal{H}(K_S)$) which are not available in the case of general (noninner) Schur-class functions. Indeed, since $\mathcal{H}(K_S)$ is isometrically included in $\mathcal{H}_2(k_d)$, we have for every $h, g \in \mathcal{H}(K_S),$

$$\langle h, A_j^* g \rangle_{\mathcal{H}(K_S)} = \langle A_j h, g \rangle_{\mathcal{H}(K_S)} = \langle M_{\lambda_j} h, g \rangle_{\mathcal{H}(K_S)} = \langle h, M_{\lambda_j} g \rangle_{\mathcal{H}_2(k_d)} = \langle h, \mathcal{P}_{\mathcal{H}(K_S)} M_{\lambda_j} g \rangle_{\mathcal{H}(K_S)}$$

and (4.6) follows. As a consequence of (4.6) we get

$$A^*|_{\mathcal{D}^\perp} = 0. \quad (4.7)$$

Indeed, if $h = \left[ \begin{array}{cccc} h_1 \\ \vdots \\ h_d \end{array} \right] \in \mathcal{D}^\perp$, it holds that $Z(\lambda)h(\lambda) \equiv 0$ (by the characterization of $\mathcal{D}^\perp$ in (2.13)) and then

$$A^* h = \sum_{j=1}^d A_j^* h_j = \mathcal{P}_{\mathcal{H}(K_S)} M_{\lambda_j} h_j = \mathcal{P}_{\mathcal{H}(K_S)}(Z h) = 0.$$

Now we define the operators $T_1 : \mathcal{D} \oplus \mathcal{Y} \to \mathcal{H}(K_S)$ and $T_2 : \mathcal{D} \oplus \mathcal{Y} \to \mathcal{U}$ by

$$T_1 = \left[ A^*|_{\mathcal{D}} \quad C^* \right] \quad \text{and} \quad T_2 = \left[ B^*|_{\mathcal{D}} \quad S(0)^* \right] \quad (4.8)$$

and combining the two latter formulas with (4.6) and (4.7), we may write the adjoint of the connecting operator $U = [A \ C \ B]$ as

$$U^* = \left[ \begin{array}{ccc} 0 & T_1 \\ X & T_2 \end{array} \right] : \left[ \begin{array}{c} \mathcal{D}^\perp \\ \mathcal{D} \oplus \mathcal{Y} \end{array} \right] \to \left[ \begin{array}{c} \mathcal{H}(K_S) \\ \mathcal{U} \end{array} \right]. \quad (4.9)$$
In the latter formula we have identified \( [\mathcal{D}^\perp : \mathcal{D} \oplus \mathcal{Y}] \) with \( \mathcal{H}(K_S)^d \). Every \( X \) such that the matrix in (4.9) is contractive leads to a contractive functional-model realization for \( S \) (due to canonical choice \( (3.11) \) of \( C \) and \( A \)) which is automatically weakly coisometric. Therefore, the restriction of \( \mathcal{U}^* \) to the space \( \mathcal{D} \oplus \mathcal{Y} \), (that is, the operator \( \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \)) is isometric:

\[
T_1^* T_1 + T_2^* T_2 = I_{\mathcal{D} \oplus \mathcal{Y}}. \tag{4.10}
\]

Since the pair \((C, A)\) is isometric, it follows from (4.9) and the formula for \( T_1 \) in (4.8) that \( T \) is coisometric:

\[
T_1 T_1^* = A^* A + C^* C = I_{\mathcal{H}(K_S)}. \tag{4.11}
\]

Then we also have

\[
T_1 T_2 T_2^* T_1^* = T_1 (I - T_1 T_1^*) T_1^* = I - I = 0,
\]

so that

\[
T_1 T_2^* = 0. \tag{4.12}
\]

Now we invoke (4.9) and make use of (4.10)–(4.12) to write the block-matrix formulas

\[
I - \mathcal{U} \mathcal{U}^* = \begin{bmatrix} I - XX^* & -X^* T_2 \\ -T_2^* X & 0 \end{bmatrix} \tag{4.13}
\]

and

\[
I - \mathcal{U}^* \mathcal{U} = \begin{bmatrix} 0 & 0 \\ 0 & I - XX^* - T_2^* T_2 \end{bmatrix}. \tag{4.14}
\]

From the formula for \( T_2 \) in (4.8), combined with the formula (4.14) for the action of \( B^*|_D \) on a generic generator of \( D \), we see that

\[
\text{Ran} \ T_2 = \text{span} \{S(\zeta)^* y : \zeta \in \mathbb{B}^d, \ y \in \mathcal{Y} \}
\]

and hence

\[
\text{Ker} \ T_2^* = (\text{Ran} \ T_2)^\perp = \{u \in \mathcal{U} : S(\lambda) u = 0\} =: \mathcal{U}_S^0. \tag{4.15}
\]

Now it follows from (4.13) that \( \mathcal{U}^* \) of the form (4.9) is contractive (isometric) if and only if \( X \) is a contraction (an isometry) from \( D^\perp \) into (onto) \( \mathcal{U}_S^0 \). Then (4.14) implies that \( \mathcal{U} \) is unitary if and only if \( X : D^\perp \to \mathcal{U}_S^0 \) is unitary. The corresponding \( B \) of the form (4.15) leads to weakly coisometric, coisometric and unitary realizations for \( S \). We are led to the following result.

**Theorem 4.7.** Let \( S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y}) \) be inner, let \( A = (A_1, \ldots, A_d) \), \( C, D \) be given as in (3.11), and let the subspaces \( \mathcal{U}_S^0 \subset \mathcal{U} \) and \( D^\perp \subset \mathcal{H}(K_S)^d \) be defined as in (4.15), (2.12) and (2.13). Then

1. \( S \) admits a coisometric functional-model realization if and only if \( \dim D^\perp \leq \dim \mathcal{U}_S^0 \).
2. \( S \) admits a unitary functional-model realization if and only if \( \dim D^\perp = \dim \mathcal{U}_S^0 \).
3. \( S \) admits a unique weakly coisometric functional-model realization if and only if \( \mathcal{U}_S^0 = \{0\} \). In this case, the operator \( B \in \mathcal{L}(\mathcal{U}, \mathcal{H}(K_S)^d) \) is defined by

\[
B^*|_D : Z(\zeta)^* K_S(\cdot, \zeta)y \to S(\zeta)^* y - S(0)^* y \quad \text{and} \quad B^*|_{D^\perp} = 0.
\]

This unique weakly coisometric functional-model realization is never coisometric.
4.1. **Beurling-Lax representation theorem for shift-invariant subspaces.**

The Beurling-Lax theorem for the context of the Arveson space $\mathcal{H}_Y(k_d)$ asserts that any closed $\mathbf{M}_X$-invariant subspace $\mathcal{M}$ of $\mathcal{H}_Y(k_d)$ can be represented in the form

$$\mathcal{M} = S \cdot \mathcal{H}_U(k_d)$$

(4.16)

for some inner multiplier $S \in \mathcal{S}_d(U, Y)$ and an appropriately chosen coefficient space $U$ (see [13, 21, 22] for the classical case $d = 1$ and [3, 23, 9] for the case of general $d$). We shall call any such $S$ a **representer** of $\mathcal{M}$. Here we present a realization-theoretic proof of the $\mathcal{H}_Y(k_d)$-Beurling-Lax theorem as an application of Theorem 4.8 (see [9, 10] for an illustration of this approach for the case $d = 1$). We first need some preliminaries.

Suppose that $A = (A_1, \ldots, A_d)$ is a commutative $d$-tuple of bounded, linear operators on the Hilbert space $\mathcal{X}$ and that $(C, A)$ is an output stable pair. We define a left-tangential functional calculus $f \to (C^*f)^{\wedge L}(A^*)$ on $\mathcal{H}_Y(k_d)$ by

$$(C^*f)^{\wedge L}(A^*) = \sum_{n \in \mathbb{Z}_+^d} A^{*n}C^*f_n \quad \text{if} \quad f = \sum_{n \in \mathbb{Z}_+^d} f_n \lambda^n \in \mathcal{H}_Y(k_d).$$

(4.17)

The computation

$$\left\langle \sum_{n \in \mathbb{Z}_+^d} A^{*n}C^*f_n, x \right\rangle_{\mathcal{X}} = \sum_{n \in \mathbb{Z}_+^d} \langle f_n, CA^n x \rangle_{\mathcal{Y}}$$

$$= \sum_{n \in \mathbb{Z}_+^d} \frac{n!}{|n|!} \langle f_n, \frac{|n|!}{n!} CA^n x \rangle_{\mathcal{Y}}$$

$$= \langle f, (\mathcal{O}_{C,A}x) \rangle_{\mathcal{H}_Y(k_d)}$$

shows that the output-stability of the pair $(C, A)$ is exactly what is needed to verify that the infinite series in the definition (4.17) of $(C^*f)^{\wedge L}(A^*)$ converges in the weak topology on $\mathcal{X}$. In fact the left-tangential evaluation with operator argument $f \to (C^*f)^{\wedge L}(A^*)$ amounts to the adjoint of the observability operator:

$$(C^*f)^{\wedge L}(A^*) = (\mathcal{O}_{C,A})^* f \quad \text{for} \quad f \in \mathcal{H}_Y(k_d).$$

(4.18)

Given an output-stable pair $(C, A)$, define a subspace $\mathcal{M}_{A^*, C^*} \subset \mathcal{H}_Y(k_d)$ by

$$\mathcal{M}_{A^*, C^*} = \{ f \in \mathcal{H}_Y(k_d) : (C^*f)^{\wedge L}(A^*) = 0 \}. \quad (4.19)$$

An easy computation (using that $A$ is commutative) shows that

$$(C^*[M_A f])^{\wedge L}(A^*) = A^*_t (C^*f)^{\wedge L}(A^*).$$

Hence any subspace $\mathcal{M} \subset \mathcal{H}_Y(k_d)$ of the form $\mathcal{M} = \mathcal{M}_{A^*, C^*}$ as in (4.19) is $\mathbf{M}_X$-invariant. We now obtain the converse.

**Theorem 4.8.** Suppose that $\mathcal{M}$ is a closed subspace of $\mathcal{H}_Y(k_d)$ which is $\mathbf{M}_X$-invariant (i.e., $\mathcal{M}$ is invariant under $\mathbf{M}_X : f(\lambda) \mapsto \lambda_j f(\lambda)$ for $j = 1, \ldots, d$). Then there is a Hilbert space $\mathcal{X}$, a commutative $d$-tuple of operators $A = (A_1, \ldots, A_d)$ on $\mathcal{X}$ and an operator $C : \mathcal{X} \to Y$ so that

1. $A$ is commutative, i.e., $A_iA_j = A_jA_i$ for $1 \leq i, j \leq d$,
2. $A$ is strongly stable, i.e., $A$ satisfies (3.7), and
3. the subspace $\mathcal{M}$ has the form $\mathcal{M}_{A^*, C^*}$ as in (4.19).
Moreover, one choice of state space $\mathcal{X}$ and operators $A_j: \mathcal{X} \to \mathcal{X}$ and $C: \mathcal{X} \to \mathcal{Y}$ is

$$\mathcal{X} = M^\perp, \quad A_j = M^*_\lambda|_{M^\perp} \quad \text{for } j = 1, \ldots, d, \quad C: f \to f(0) \quad \text{for } f \in M^\perp.$$  

(4.20)

Proof. Define $\mathcal{X}$, $A = (A_1, \ldots, A_d)$ and $C$ as in (4.20). We note that $M^\perp$ is contained in $H_2(k_d)$ isometrically and that $C = G|_{M^\perp}$, $A_j = M^*_\lambda|_{M^\perp}$ where $O_{G,M^\perp}$ is the identity on $H_2(k_d)$ (see part (2) of Proposition 3.1). Hence in particular $\text{Ran } O_{C,A} = M^\perp$. Taking orthogonal complements then gives

$$\text{Ker } (O_{C,A})^* = (M^\perp)^\perp = M$$

which in turn is equivalent to the characterization (4.19) for $\mathcal{M}$.     

The following three examples illustrate how various shift-invariant subspaces $\mathcal{M}$ are characterized by some sort of concrete homogeneous interpolation conditions can be represented in the form $\mathcal{M}_{A^\ast,C^\ast}$; more general versions of these examples are discussed in [5] in the context of nonhomogeneous interpolation problems of Nevanlinna-Pick type on domains in $\mathbb{C}^d$ of a more general form than $\mathbb{B}^d$.

Example 4.9. Let

$$\omega_1 = (\omega_{1,1}, \ldots, \omega_{1,d}), \ldots, \omega_n = (\omega_{n,1}, \ldots, \omega_{n,d}),$$

be a collection of $n$ points in the unit ball $\mathbb{B}^d$, and let $x_1, \ldots, x_n$ be a collection of linear functionals $x_j: \mathcal{Y} \to \mathbb{C}$ on $\mathcal{Y}$. Define an associated shift-invariant subspace $\mathcal{M} \subset H_2^d$ by

$$\mathcal{M} = \{ f \in H_2(k_d): x_jf(\omega_j) = 0 \text{ for } j = 1, \ldots, n \}.$$  

Then it is easy to see that $\mathcal{M} = \mathcal{M}_{A^\ast,C^\ast}$ if one takes $\mathcal{X} = \mathbb{C}^n$ and

$$A_j^* = \begin{bmatrix} \omega_{1,j} & \cdots & \omega_{n,j} \end{bmatrix} \quad \text{for } j = 1, \ldots, d, \quad C^* = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}.$$  

Example 4.10. Fix a point $\omega = (\omega_1, \ldots, \omega_d) \in \mathbb{B}^d$ and let $x_0, \ldots, x_{n-1}$ be a collection of linear functionals $x_j: \mathcal{Y} \to \mathbb{C}$ $(j = 0, \ldots, n-1)$ on $\mathcal{Y}$. Associate a shift-invariant subspace $\mathcal{M} \subset H_2(k_d)$ by

$$\mathcal{M} = \{ f \in H_2(k_d): x_0 \sum_{j:|j|=i} \frac{1}{i!} \frac{\partial^i f}{\partial \lambda^i}(\omega) + x_1 \sum_{j:|j|=i-1} \frac{1}{(i-1)!} \frac{\partial^{i-1} f}{\partial \lambda^{i-1}}(\omega) + \cdots + x_{i} f(\omega) = 0 \text{ for } i = 0, 1, \ldots, n-1 \}.$$  

Then one can check that $\mathcal{M} = \mathcal{M}_{A^\ast,C^\ast}$ if one chooses $\mathcal{X} = \mathbb{C}^n$ with

$$A_j^* = \begin{bmatrix} \omega_{j} \cr 1 & \omega_{j} & \cdots & \omega_{j} \cr 1 & \cdots & \cdots & \cdots \cr 1 & \omega_{j} \end{bmatrix} \quad \text{for } j = 1, \ldots, n, \quad C^* = \begin{bmatrix} x_0 \cr x_1 \cr \cdots \cr x_{n-1} \end{bmatrix}.$$  

Example 4.11. A more general example can be constructed as follows. Let $\omega = (\omega_1, \ldots, \omega_d)$ be a fixed point of $\mathbb{B}^d$, and let $E \subset \mathbb{Z}_d^+$ be a subset of indices which is lower inclusive, i.e.: whenever $n \in E$ and $n - e_i \in \mathbb{Z}_d^+$ (where $e_i = (0, \ldots, 1, \ldots, 0)$ is the $i$-th standard basis vector for $\mathbb{R}^d$ with $i$-th component equal to 1 and all
other components equal to zero), then it is the case that also $\mathbf{n} - \mathbf{e}_i \in E$. For each $\mathbf{n} \in E$, let $x_\mathbf{n} : \mathcal{Y} \rightarrow \mathbb{C}$ be a linear functional on $\mathcal{Y}$. Define a polynomial $x(\lambda)$ with coefficients in $\mathcal{L}(\mathcal{Y}, \mathbb{C})$ by

$$x(\lambda) = \sum_{\mathbf{n} \in E} x_\mathbf{n}(\lambda - \omega)^n.$$ 

Define a subspace $\mathcal{M} \subset H_d^2(\mathcal{Y})$ by

$$\mathcal{M} = \{ f \in \mathcal{H}_d(\mathcal{Y}) : \left( \frac{\partial^{[n]} f(\lambda) x(\lambda) f(\lambda)}{\partial \lambda^n} \right) |_{\lambda = \omega} = 0 \text{ for all } \mathbf{n} \in E \}.$$ 

Take the space $\mathcal{X}$ to be equal to $\ell^2(E)$ (complex-valued functions on the index set $E$ with norm-square-summable values) and define operators $A_j^* \in \mathcal{L}(\ell^2(E))$ via square matrices with rows and columns indexed by $E$ as

$$[A_j^*]_{\mathbf{n}, \mathbf{n}'} = \delta_{\mathbf{n}, \mathbf{n}'} \omega_j + \delta_{\mathbf{n} + \mathbf{e}_j, \mathbf{n}'} \text{ for } j = 1, \ldots, d$$

and an operator $C^* : \mathcal{Y} \rightarrow \ell^2(E)$ as the column matrix

$$C^* = \text{col}_{\mathbf{n} \in E} [x_\mathbf{n}].$$ 

Then it can be checked that $\mathcal{M} = \mathcal{M}_{A^*, C^*}$ for this choice of $(C, A)$.

We now construct an inner multiplier solving a homogeneous interpolation problem via realization theory.

**Theorem 4.12.** Suppose that $(C, A)$ is an isometric output-stable pair, with $A$ commutative and strongly stable. Let $\mathcal{M} = \mathcal{M}_{A^*, C^*} \subset \mathcal{H}_d(\mathcal{Y})$ be given by (4.19). Then there is an input space $\mathcal{U}$ and an inner Schur multiplier $S \in S_d(\mathcal{U}, \mathcal{Y})$ so that $\mathcal{M} = \text{Ran} M_S$. One such $S$ is given by

$$S(\lambda) = D + C(I - Z(\lambda)A)^{-1} Z(\lambda)B$$

where $A_1, \ldots, A_d$ and $C$ come from the given output pair $(C, A)$ and $B_1, \ldots, B_d$ are chosen so that the colligation $U = [\begin{matrix} A & B \\ C & D \end{matrix}] : \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X}^d \\ \mathcal{Y} \end{bmatrix}$ is weakly coisometric. In particular, one achieves a coisometric realization $U$ by choosing the input space $\mathcal{U}$ and $[B]$ so as to solve the Cholesky factorization problem:

$$\begin{bmatrix} B & D \\ D^* & A \end{bmatrix} = \begin{bmatrix} I_{\mathcal{X}^d} & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} - \begin{bmatrix} A & C \end{bmatrix} \begin{bmatrix} A^* & C^* \end{bmatrix}. \tag{4.21}$$

**Remark 4.13.** Note that the model output pair $(C, A)$ appearing in Theorem 4.12 is an isometric pair. In practice, however, one may give a subspace of the form $\mathcal{M}_{A^*, C}$ with $A$ commutative and strongly stable but without the pair $(C, A)$ being isometric. If however it is the case that $(C, A)$ is exactly observable in the sense that the observability gramian

$$G_{C, A} = O_{C, A}^* O_{C, A} = \sum_{\mathbf{n} \in \mathbb{Z}_d^+} \frac{|\mathbf{n}|!}{\mathbf{n}!} A^{\mathbf{n}_a} C^* C A^{\mathbf{n}}$$

is strictly positive definite, then the adjusted output pair $(\tilde{C}, \tilde{A})$ given by

$$\tilde{A}_j = H^{1/2} A_j H^{-1/2} \text{ for } j = 1, \ldots, d, \quad \tilde{C} = C H^{-1/2} \text{ where } H := G_{C, A}$$

is isometric and has all the other properties of the original output pair $(C, A)$, namely: $\tilde{A}$ is strongly stable and $\mathcal{M} = \mathcal{M}_{\tilde{A}^*, \tilde{C}^*}$. Hence in practice the requirement that the pair $(C, A)$ be isometric in Theorem 4.12 can be replaced by the condition
that \((C, A)\) is exactly observable. We note that in all the Examples \ref{ex:inner_multiplier} and \ref{ex:inner_multiplier_2} above, the associated output pair \((C, A)\), while not isometric, is exactly observable. A more complete discussion of this point can be found in \cite{Dost}. 

**Proof of Theorem \ref{thm:main_result}** Define \(S(\lambda)\) as in the statement of the theorem. By Theorem \ref{thm:main_result} \(S\) is inner. By Proposition \ref{prop:isometric} \((\text{Ran} M_S)\perp = \mathcal{H}(K_S)\) isometrically. As \(U\) is weakly coisometric, we also know that \(\mathcal{H}(K_S) = \mathcal{H}(K_{C,A})\) by Proposition \ref{prop:isometric}.

The space \(\mathcal{H}(K_{C,A})\) can in turn be identified as a set with \(\text{Ran} \mathcal{O}_{C,A}\) (see \cite[Theorem 3.14]{Dost}). By hypothesis, \(\mathcal{M} = \text{Ker} (\mathcal{O}_{C,A})^*\); hence \(\text{Ran} \mathcal{O}_{C,A} = \mathcal{M}^\perp\). Putting all this together gives \((\text{Ran} M_S)\perp = \mathcal{M}^\perp\) and therefore \(\text{Ran} M_S = \mathcal{M}\) as wanted. \(\square\)

**Remark 4.14.** By the result of \cite{Dost}, it is known that inner multipliers necessarily have nontangential boundary values on the unit sphere \(S^{d-1} = \partial \mathbb{B}^d\) which are almost everywhere (with respect to volume measure on \(S^{d-1}\)) equal to partial isometries. In the classical case \(d = 1\), inner multipliers are characterized as those Schur-class functions whose boundary values are partial isometries with a fixed initial space (see \cite[Theorem C page 97]{Dost}). There appears to be no analogous characterization in terms of boundary values of which Schur-class functions are inner multipliers for the higher dimensional case \(d > 1\); in short it is difficult to determine purely from boundary-value behavior whether a given Schur-class multiplier is an inner function or not. On the other hand, Theorem \ref{thm:main_result} enables us to write down such inner functions and Theorem \ref{thm:main_result} enables us to write down the inner representer for a given shift-invariant subspace.

It should be pointed out, however, that the analogue of a Blaschke factor on the ball as the representer for a codimension-1 \(\mathcal{M}_\lambda\)-invariant subspace of \(\mathcal{H}(k_d)\) has been known for some time (see \cite{Dost}). Indeed, for \(a = [a_1 \cdots a_d]\) a point of \(\mathbb{B}^d\) (viewed as a row matrix), with a little bit of algebra one can see that the formula for the \(1 \times d\) Blaschke factor vanishing at \(a\)

\[
    b_a(z) = (1 - aa^*)^{1/2}(1 - za^*)^{-1}(z - a)(I_d - a^*a)^{-1/2}
\]

(where the variable \(z = [z_1 \cdots z_d] \in \mathbb{B}^d\) is also viewed as a row matrix) appearing in \cite{Dost} can be written in realization form \cite{Dost} with connecting operator \(U\) given by

\[
    U = \begin{bmatrix}
        A & B \\
        C & D
    \end{bmatrix} = \begin{bmatrix}
        a^* & (1 - a^*a)^{1/2} \\
        (1 - aa^*)^{1/2} & -a
    \end{bmatrix} : \begin{bmatrix}
        C \\
        C^d
    \end{bmatrix} \to \begin{bmatrix}
        C \\
        C^d
    \end{bmatrix}.
\]

We thus see that this Blaschke factor fits the prescription of Theorem \ref{thm:main_result} with \((A^*, C^*) = (a, (1 - a^*a)^{1/2})\) and with \(B, D\) chosen to solve the Cholesky factorization problem \cite{Dost} with the resulting connecting operator \(U\) unitary. These Blaschke factors also play an important role as the characterization of automorphisms of the ball mapping the origin to a given point (see \cite[Theorem 2.2.2]{Dost} and the references there for further history).

We next show how our analysis can be used to give a description of all Beurling-Lax represenators for a given shift-invariant subspace of \(\mathcal{H}(k_d)\).

**Theorem 4.15.** Let \(\mathcal{M}\) be a closed \(\mathcal{M}_\lambda\)-invariant subspace of \(\mathcal{H}(k_d)\), let \(\mathcal{N} = \mathcal{M}^\perp = \mathcal{H}(k_d) \ominus \mathcal{M}\) and let

\[
    A_j = M_{\lambda_j}\vert_\mathcal{N} \ (j = 1, \ldots, d), \quad C: f \to f(0) \quad (f \in \mathcal{N}).
\]

Let \(\mathcal{D}\) be the subspace of \(\mathcal{N}^d\) given by \cite{Dost} and let

\[
    T := [A^*\vert_{\mathcal{D}} \quad C^*] : \mathcal{D} \oplus \mathcal{Y} \to \mathcal{N}.
\]
Then:

1. Given a Hilbert space \( \mathcal{U} \), there exists an inner multiplier \( S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y}) \) satisfying \( \text{(4.16)} \) if and only if

\[
\dim \mathcal{U} \geq \dim \text{Ran} \,(I - T^*T)^{\frac{1}{2}}.
\]  (4.22)

2. If \( \text{(4.22)} \) is satisfied, then all \( S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y}) \) for which \( \text{(4.16)} \) holds are described by the formula

\[
S(\lambda) = [C(I - Z(\lambda)A)^{-1} \, I_2] \,(I - T^*T)^{\frac{1}{2}} G^* \quad \text{(4.23)}
\]

where \( G \) is an isometry from \( (I - T^*T)^{\frac{1}{2}} \) onto \( \text{Ran} \, G \subset \mathcal{U} \).

3. If \( \dim \mathcal{U} = \dim \text{Ran} \,(I - T^*T)^{\frac{1}{2}} \), then the function \( S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y}) \) such that \( \text{(4.16)} \) holds is defined uniquely up to a constant unitary factor on the right.

Proof. If \( \mathcal{M} \) is a closed \( M_A \)-invariant subspace of \( \mathcal{H}_\lambda(k_d) \), then \( \mathcal{M}^\perp \) is isometrically included in \( \mathcal{H}_\lambda(k_d) \) and is isometrically equal to \( \mathcal{H}(K_{C,A}) \) where \( (C, A) \) are the model operators on \( \mathcal{M}^{\perp} \) as in \( \text{(4.20)} \). On the other hand, as a consequence of Proposition \( 4.15 \) we see that the Schur multiplier \( S \) is an inner-multiplier representer for \( \mathcal{M} \) if and only if \( \mathcal{H}(K_S) = \mathcal{M}^{\perp} \) isometrically. Thus the problem of describing all inner-multiplier representer \( S \) for the given \( M_A \)-invariant subspace \( \mathcal{M} \) is equivalent to the problem: describe all Schur-class multipliers \( S \) such that \( \mathcal{H}(K_S) = \mathcal{H}(K_{C,A}) \) isometrically, where \( (C, A) \) is given by \( \text{(4.20)} \). The various conclusions of Theorem \( 4.16 \) now follow as an application of Theorem 2.11 from \( 7 \) to the more special situation here (where \( A \) is strongly stable and \( \mathcal{H}(K_{C,A}) \) is contained in \( \mathcal{H}_\lambda(k_d) \) isometrically).

4.2. Examples. In this section we illustrate a number of finer points concerning realizations for inner functions with some examples.

Example 4.16. Here we give an example of an inner multiplier \( S \) with a unique, weakly coisometric, observable commutative realization \( \mathbf{U} \) which is not coisometric. Thus a fortiori \( S \) has no observable realization which is also unitary. Let

\[
S(\lambda) = \begin{bmatrix} \lambda_1^2 & \sqrt{2}\lambda_1 \lambda_2 & \lambda_2^2 \end{bmatrix} \quad \text{so that} \quad K_S(\lambda, \zeta) = 1 + \lambda_1 \zeta_1 + \lambda_2 \zeta_2. \quad (4.24)
\]

Thus \( S \in \mathcal{S}_d(\mathbb{C}^3, \mathbb{C}) \) and the functions \( \{1, \lambda_1, \lambda_2\} \) form a basis for \( \mathcal{H}(K_S) \). It is readily seen that \( \mathcal{H}(K_S) \) is invariant under the backward shifts \( M_{\lambda_1}^* \) and \( M_{\lambda_2}^* \). Define matrices

\[
A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.
\]

Straightforward calculations show that

\[
S(\lambda) = D + C(I - \lambda_1 A_1 - \lambda_2 A_2)^{-1}(\lambda_1 B_1 + \lambda_2 B_2) \quad (4.25)
\]

and that this realization is commutative, strongly stable (in fact \( A_1 \) and \( A_2 \) are the matrices of operators \( M_{\lambda_1}^* \) and \( M_{\lambda_2}^* \) restricted to \( \mathcal{H}(K_S) \) with respect to the
basis \{1, \lambda_1, \lambda_2\} of \mathcal{H}(K_S))}, observable and contractive (isometric, actually). This realization is also weakly coisometric (by Proposition 2.24), since
\[
C(I - \lambda_1 A_1 - \lambda_2 A_2)^{-1} = \begin{bmatrix} 1 & \lambda_1 & \lambda_2 \\
\end{bmatrix}
\]
and then, (4.24) can be written in the form
\[
K_S(\lambda, \zeta) = C(I - \lambda_1 A_1 - \lambda_2 A_2)^{-1}(I - \tilde{\zeta}_1 A_1^* - \tilde{\zeta}_2 A_2^*)^{-1} C^*.
\]
In fact, this example amounts to the special case of Example 4.11 where
\[
E = \{(0,0), (1,0), (0,1)\} \subset \mathbb{Z}^2_+
\]
and
\[
M = \left\{ f \in \mathcal{H}(k_2) : f(0,0) = 0, \frac{\partial f}{\partial \lambda_1}(0,0) = 0, \frac{\partial f}{\partial \lambda_2}(0,0) = 0 \right\}.
\]
Thus \(S\) is inner by Theorem 1.3. Since \(\mathcal{U}_S^2 = \{0\}\) (it is obvious), the above realization is the only realization of \(S\) (up to unitary equivalence) which is weakly coisometric, commutative and observable. It is not coisometric and therefore \(S\) does not admit coisometric (or a fortiori unitary) commutative observable realizations.

That \(S\) cannot have a unitary realization can be seen directly as follows. If \(S\) has a finite-dimensional unitary realization, then necessarily \(\dim X + \dim \mathcal{U} = 2 \cdot \dim X + \dim Y\). As \(\dim \mathcal{U} = 3\) and \(\dim Y = 1\), we then must have \(\dim X = 2\). But this is impossible since \(\dim \mathcal{H}(K_S) = 3\). On the other hand, a realization with \(\dim X = \infty\) cannot be observable.

**Example 4.17.** This example illustrates the nonuniqueness in the choice of the input operator \(B\) for a weakly coisometric, observable realization \(U\) for a given inner multiplier \(S(\lambda)\).

Let \(S(\lambda)\) be as in Example 4.16 and let \(\tilde{S}(\lambda) = [\lambda_1^2 \lambda_1 \lambda_2 \lambda_2 \lambda_1 \lambda_2]\). Then
\[
K_S(\lambda, \zeta) = 1 + \lambda_1 \tilde{\zeta}_1 + \lambda_2 \tilde{\zeta}_2 = K_S(\lambda, \zeta)
\]
and thus the de Branges-Rovnyak spaces \(\mathcal{H}(K_{\tilde{S}})\) and \(\mathcal{H}(K_S)\) coincide. We know from the previous example that \(\mathcal{H}(K_S)\) is backward-shift-invariant and isometrically included in \(\mathcal{H}(k_d)\); hence so also is \(\mathcal{H}(K_{\tilde{S}})\). Therefore \(\tilde{S}\) is inner. Hence \(\tilde{S}\) admits weakly coisometric commutative observable realizations and, up to unitary similarity, the operators \(A_1, A_2\) and \(C\) are as above while \(D = [0 \ 0 \ 0 \ 0]\). Operators \(B_1\) and \(B_2\) are not uniquely defined. Note that the space \(\mathcal{U}_{\tilde{S}}^0\) is spanned by the vector \(u = [0 \ 1 \ 0 \ -1]^T\). We already know that \(\mathcal{H}(K_{\tilde{S}})\) is 3-dimensional and \(\{1, \lambda_1, \lambda_2\}\) is an orthonormal basis for \(\mathcal{H}(K_{\tilde{S}})\). We identify \(\mathcal{H}(K_{\tilde{S}}) \oplus \mathcal{H}(K_{\tilde{S}})\) with \(\mathbb{C}^6\) upon identifying the orthonormal basis
\[
\left[ \begin{array}{c} 1 \\ 0 \\
\end{array} \right], \left[ \begin{array}{c} \lambda_1 \\ 0 \\
\end{array} \right], \left[ \begin{array}{c} \lambda_2 \\ 0 \\
\end{array} \right], \left[ \begin{array}{c} 0 \\ 1 \\
\end{array} \right], \left[ \begin{array}{c} 0 \\ \lambda_1 \\
\end{array} \right], \left[ \begin{array}{c} 0 \\ \lambda_2 \\
\end{array} \right]
\]
for \(\mathcal{H}(K_{\tilde{S}}) \oplus \mathcal{H}(K_{\tilde{S}})\) with the standard basis of \(\mathbb{C}^6\). The subspace
\[
\mathcal{D}^\perp = \text{span} \left[ \begin{array}{c} \lambda_2 \\ -\lambda_1 \\
\end{array} \right] \subset \mathcal{H}(K_{\tilde{S}})^2
\]
is identified with the one dimensional subspace of \(\mathbb{C}^6\) spanned by the vector \(x = [0 \ 0 \ 1 \ 0 \ -1 \ 0]^T\). The 5-dimensional subspace \(\mathcal{D}\) is then identified with
\[
\{x\}^\perp = \mathcal{D} = \text{span} \left\{ \left[ \begin{array}{c} \zeta_1 \\ \zeta_1^2 \\
\end{array} \right], \left[ \begin{array}{c} \bar{\zeta}_1 \\ \bar{\zeta}_1 \bar{\zeta}_2 \\
\end{array} \right], \left[ \begin{array}{c} \zeta_2 \\ \bar{\zeta}_1 \bar{\zeta}_2 \\
\end{array} \right], \left[ \begin{array}{c} \zeta_2^2 \\ \bar{\zeta}_2 \\
\end{array} \right] \right\} : \zeta_1, \zeta_2 \in \mathbb{C}\right\}.
\]
The operator $B^* = [B_1^* \ldots B_d^*]$ is defined on $\{x\}^\perp$ by
\[
B^* \begin{bmatrix}
\bar{\xi}_1 & \bar{\xi}_2 & \bar{\xi}_3 & \cdots & \bar{\xi}_d
\end{bmatrix}^T = S(\Omega)^* - S(0)^* = \begin{bmatrix}
\bar{\xi}_1 & \bar{\xi}_2 & \bar{\xi}_3 & \cdots & \bar{\xi}_d
\end{bmatrix}^T
\]
and $B^*$ must map $x$ into $U_S$ contractively. Since $\|x\| = \|u\| (= \sqrt{2})$, we set $B^*x = \alpha u$ with $|\alpha| \leq 1$ and this choice of $\alpha$ is the only freedom we have. Thus, the matrices of $B_1$ and $B_2$ with respect to the standard bases are of the form
\[
B_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & \frac{1+\alpha}{2} & 0 & \frac{1-\alpha}{2} & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & \frac{1+\alpha}{2} & 0 & \frac{1-\alpha}{2} & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]
If $|\alpha| = 1$, then the corresponding realization is unitary. Moreover, different $\alpha$’s lead to unitary realizations of $\tilde{S}$ that are not unitarily equivalent.

5. Characteristic functions of commutative row contractions

In the operator model theory for commutative row contractions (see [14, 15]), one is given a $d$-tuple of operators $T = (T_1, \ldots, T_d)$ on a Hilbert space $\mathcal{X}$ for which the associated block-row matrix is contractive:
\[
\|T\| \leq 1 \text{ where } T = [T_1 \ldots T_d] : \mathcal{X}^d \to \mathcal{X}.
\]
Under certain conditions (that $T$ be completely non-coisometric—see [15, 28]), the associated characteristic function $\theta_T(\lambda)$ is a complete unitary invariant for $T$; recently extensions of the theory to still more general settings have appeared (see [16, 27, 28]) while the fully noncommutative setting is older (see [25, 26]). All this theory can be viewed as multivariable analogues of the well-known, now classical operator model theory of Sz.-Nagy-Foias [24]. However, unlike the fully developed theory in [24] for the classical case and unlike the case for the fully noncommutative theory (see [25, 26, 12]), none of the work for the multivariable commutative setting provides a characteristicization of which Schur-class functions arise as characteristic functions.

To define $\theta_T$, we set $A = [T_1 \ldots T_d]^*$ and let $U = [A \ B] : \mathcal{X} \oplus D_T \to \mathcal{X}^d \oplus D_T$, be the Halmos unitary dilation of $A$:
\[
B = D_T|_{D_T} : D_T \to \mathcal{X}^d, \quad C = D_T^* : \mathcal{X} \to D_T^*,
\]
\[
D = -T|_{D_T} : D_T \to D_T^*,
\]
where
\[
D_T = (I_{\mathcal{X}^d} - T^*T)^{1/2}, \quad D_T = \overline{\text{Ran}} D_T \subset \mathcal{X}^d,
\]
\[
D_T^* = (I_{\mathcal{X}} - TT^*)^{1/2}, \quad D_T^* = \overline{\text{Ran}} D_T^* \subset \mathcal{X},
\]
and then $\theta_T(\lambda)$ is the transfer function associated with the colligation $U$:
\[
\theta_T(\lambda) = [-T + D_T^*(I - Z(\lambda)T^*)^{-1}Z(\lambda)D_T]|_{D_T} : D_T \to D_T^*.
\]
(5.1)
Since $U$ is unitary, it follows that $\theta_T$ is in the Schur class $S_d(D_T, D_T^*)$. More generally, a Schur-class function $S \in S_d(U, \mathcal{Y})$ is said to coincide with the characteristic function $\theta_T(\lambda)$ if there are unitary identification operators
\[
\alpha : D_T^* \to \mathcal{Y}, \quad \beta : D_T \to U
\]
such that
\[
S(\lambda) = \alpha \theta_T(\lambda) \beta^*.
\]
From our point of view, what is special about a Schur-class function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ which coincides with a characteristic function $\theta_T$ is that it is required to have a commutative unitary realization. An additional constraint follows from the fact that the unitary colligation in the construction of a characteristic function comes via the Halmos-dilation construction. The following proposition summarizes the situation. In general let us say that the Schur-class function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ is pure if
\[ \|S(0)u\| = ||u|| \text{ for some } u \in \mathcal{U} \implies u = 0. \] (5.2)

For the role of this notion in the characterization of characteristic functions for the classical case, see [24, page 188].

**Proposition 5.1.** A Schur-class function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ coincides with a characteristic function $\theta_T$ if and only if
\begin{enumerate}
  \item $S$ has a realization (1.8) with $U$ unitary and $A$ commutative, and
  \item $S$ is pure, i.e., $S$ satisfies (5.2).
\end{enumerate}

**Proof.** We first note the following general fact: if $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : X \oplus U \to X^d \oplus Y$ is unitary then the following are equivalent:
\begin{enumerate}
  \item $B$ is injective,
  \item $C^\ast$ is injective,
  \item $u \in \mathcal{U}$ with $\|Du\| = ||u||$ implies that $u = 0$.
\end{enumerate}

To see this, note that the unitary property of $U$ means that
\[ \begin{bmatrix} A^\ast A + C^\ast C & A^\ast B + C^\ast D \\ B^\ast A + D^\ast C & B^\ast B + D^\ast D \end{bmatrix} = \begin{bmatrix} I_X & 0 \\ 0 & I_Y \end{bmatrix}, \]
\[ \begin{bmatrix} AA^\ast + BB^\ast & AC^\ast + BD^\ast \\ CA^\ast + DB^\ast & CC^\ast + DD^\ast \end{bmatrix} = \begin{bmatrix} I_{X^d} & 0 \\ 0 & I_{Y^d} \end{bmatrix}. \] (5.3)

From all these relations we read off
\[ Bu = 0 \implies \|Dx\| = \|x\| \text{ and } C^\ast Dx = 0, \]
\[ \|Du\| = ||u|| \implies Bu = 0 \text{ and } C^\ast Du = 0, \]
\[ C^\ast y = 0 \implies \|D^\ast y\| = ||y|| = \|D(D^\ast y)\| \text{ and } BD^\ast y = 0. \]

Hence any one of the conditions (i), (ii) or (iii) implies the remaining ones.

Suppose now that $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ coincides with a characteristic function $\theta_T$. Then $S$ has a realization as in (1.8) and (1.7) for a connecting operator $U$ of the form
\[ U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I_{X^d} & 0 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} T^\ast & D_T \\ D_{T^\ast} & -T \end{bmatrix} \begin{bmatrix} I_X & 0 \\ 0 & \beta^\ast \end{bmatrix} \] (5.4)
for a commutative row-contraction $T = \{T_1, \ldots, T_d\}$ where $\alpha : D_{T^\ast} \to \mathcal{Y}$ and $\beta : D_T \to \mathcal{U}$ are unitary, and where $\begin{bmatrix} T^\ast \\ D_{T^\ast} \\ D_T \end{bmatrix} : X \oplus D_T \to X^d \oplus D_{T^\ast}$ is the Halmos dilation of $T^\ast$ discussed above. It is then obvious that $U$ gives a commutative, unitary realization for $S$. We also read off that any one (and hence all) of the conditions (i), (ii) and (iii) hold for $U$. As $D = S(0)$, the validity of condition (iii) implies that $S$ is pure.

Conversely suppose that $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ has a commutative, unitary realization $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : X \oplus \mathcal{U} \to X^d \oplus \mathcal{Y}$ and is pure. As $D = S(0)$ and $S$ is pure, we read off
that condition (iii) above holds, and hence also conditions (i) and (ii) hold for $U$.
From the relations we have in particular
\[ C^*C = I_X - A^*A, \quad BB^* = I_{X_d} - AA^* \]
Hence we can define unitary operators
\[ \alpha: \mathcal{D}_A \to \text{Ran} C = \mathcal{Y}, \quad \beta: \mathcal{D}_{A^*} \to \text{Ran} B^* = \mathcal{U} \]
so that
\[ \alpha D A = C \quad \text{and} \quad \beta D_{A^*} = B^* \]
Then we also have
\[ \alpha^* D \beta D_{A^*} = -\alpha^* C A^* = -D A A^* = -A^* D_{A^*} \]
from which we get
\[ D = -\alpha A^* \beta^* \]
We conclude that $U$ has the form (5.4) with $T = (A_1^*, \ldots, A_d^*)$, and hence $S$ coincides with the characteristic function $\theta_T$. □

For the inner case, we can use the results on functional-model realizations obtained above to give a more intrinsic sufficient condition for a Schur-class function to be a characteristic function.

**Theorem 5.2.** Suppose that $S \in S_d(U, \mathcal{Y})$ is inner, $\dim \mathcal{D} = \dim \mathcal{U}$ (where the subspaces $\mathcal{U}$ and $\mathcal{D}$ are defined in (1.15) and (1.10), and that $S$ is pure. Then $S$ coincides with the characteristic function of a $*$-strongly stable, commutative row-contraction.

**Proof.** Given $S$ as in the hypotheses, we see from Theorem 4.7 that $S$ has a functional-model realization $U = [A \ B] : \mathcal{H}(K_S) \oplus \mathcal{U} \to \mathcal{H}(K_S)^d \oplus \mathcal{Y}$ such that $U$ is unitary, $A$ is commutative and $A$ is strongly stable. Since by assumption $S$ is pure, we can apply Proposition 5.1 to conclude that $S$ coincides with $\theta_T$, where $T = (A_1^*, \ldots, A_d^*)$. As observed above, $A$ is strongly stable, i.e., $T$ is $*$-strongly stable, and the theorem follows. □

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