Estimating ergodization time of a chaotic many-particle system from a time reversal of equilibrium noise

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Abstract

We propose a method of estimating ergodization time of a chaotic many-particle system by monitoring equilibrium noise before and after time reversal of dynamics (Loschmidt echo). The ergodization time is defined as the characteristic time required to extract the largest Lyapunov exponent from a system’s dynamics. We validate the method by numerical simulation of an array of coupled Bose–Einstein condensates in the regime describable by the discrete Gross–Pitaevskii equation. The quantity of interest for the method is a counterpart of out-of-time-order correlators in the quantum regime.

1. Introduction

Quantitative characterization of ergodicity in many-particle systems is a long-standing challenge for the foundations of statistical physics, which dates back to the Poincaré recurrence theorem [1] and Zermelo’s paradox [2]. It was already pointed out by Boltzmann [3, 4] and since then became fairly obvious for the practitioners in the field [5, 6] that the ergodization time of many-particle systems, defined as the Poincaré recurrence time, is impractically long to be observable on experimental timescales. Instead, it is common to call many-particle systems ‘ergodic’, when they establish the Boltzmann–Gibbs equilibrium on an experimentally observable timescale. But even with such a concept in mind, it still remains a challenge to define the corresponding ergodization time and to measure this time experimentally.

In this paper, we define the ergodization time of a chaotic system as the characteristic time one needs to monitor the system in order to extract its primary chaotic parameter, namely, the largest Lyapunov exponent, which characterizes the sensitivity of a system to infinitesimal perturbations, the so-called ‘butterfly effect’. The advantage of this definition is that it is unbiased in the sense of not being coupled to any particular system’s coordinate. Our goal is to theoretically propose and numerically test a method, which can be used to experimentally determine whether the system is ergodic or not, and if it is, then to extract the system’s ergodization time. The method is based on monitoring the equilibrium noise of the system. It involves the time reversal of system’s dynamics—the so-called ‘Loschmidt echo’ [7, 8].

Various aspects of this work are relevant to the previous investigations of lattice gauge models [9–13] and spin lattice models [14–17]. We also note that our method involves the classical counterpart of out-of-time-order quantum correlators (OTOCs) [18] that have been actively investigated in recent years in the context of quantum thermalization [19–23] and many-body localization problems [24–29]. The relation between our results and OTOCs is to be discussed at the end of this paper.

2. Outline of the method

In figure 1, we outline the method. It consists of the following steps.
The exponentially-growing envelope of $t \to \infty$ is

\[
\lim_{t \to \infty} \ln |X(t)|,
\]

where $X(t)$ is represented by dashed lines. The noise is

\[
\text{Equilibrium noise of an observable } X \text{ before and after slightly imperfect time reversal.}
\]

\[
\Delta X(\Delta t) = X(\tau + \Delta t) - X(\tau - \Delta t)
\]

\[
\text{Time}
\]

\[
X
\]

\[
\Delta X
\]

\[
\Delta t
\]

\[
\tau
\]

\[
0
\]

\[
0
\]

\[
\text{Figure 1. Sketch of a slightly imperfect noise reversal. Equilibrium noise of an observable } X \text{ before and after an imperfect time reversal of a system's dynamics at time } \tau \text{ is denoted, respectively, as } X(\tau - \Delta t) \text{ (green line)} \text{ and } X(\tau + \Delta t) \text{ (red line)}, \text{ where } \Delta t = |t - \tau| \text{. In order to facilitate visual comparison, 'Time' on the horizontal axis represents } t \text{ before the time reversal and } 2 \tau - t \text{ after the time reversal. The difference between the direct and the reversed noise } \Delta X(\Delta t) = X(\tau + \Delta t) - X(\tau - \Delta t) \text{ (thick black line)} \text{ fluctuates around 0, while its amplitude grows, on average, exponentially as a function } \Delta t \text{ with a rate equal to the largest Lyapunov exponent } \lambda_{max}. \text{ The exponentially-growing envelope of } \Delta X(\Delta t) \text{ is represented by dashed lines.}
\]

\[
(i) \text{ Measuring equilibrium noise of observable } X \text{ before and after slightly imperfect time reversal. The noise is to be denoted as } X(\tau - \Delta t) \text{ and } X(\tau + \Delta t), \text{ where } \tau \text{ is the time of the dynamics' reversal, and } \Delta t = |t - \tau|.
\]

\[
(ii) \text{ Calculating the difference } \Delta X(\Delta t) = X(\tau + \Delta t) - X(\tau - \Delta t).
\]

\[
(iii) \text{ Repeating the procedure for an ensemble of randomly chosen initial conditions on an energy shell.}
\]

\[
(iv) \text{ Calculating two kinds of ensemble averages } \langle \ln |\Delta X(\Delta t)| \rangle \text{ and } \langle |\Delta X(\Delta t)| \rangle. \text{ For } \Delta t \to \infty, \text{ the former average approaches } \lambda_{max} \Delta t, \text{ while the latter one approaches } \Lambda \Delta t, \text{ where } \lambda_{max} \text{ is the largest Lyapunov exponent, and } \Lambda \text{ is a parameter to be discussed later.}
\]

\[
(v) \text{ Extracting the ergodization time } \tau_{\text{erg}}, \text{ which, as we show below, is proportional to the difference between } \Lambda \text{ and } \lambda_{max}.
\]

3. Model system

The method is generally applicable to systems where time reversal of the dynamics can be practically implemented. Here, we illustrate it for one-, two- and three-dimensional arrays of coupled Bose–Einstein condensates in the regime describable by the discrete Gross–Pitaevskii equation (DGPE):

\[
\frac{1}{i} \frac{d\psi_j}{dt} = -J \sum_k^{NN(j)} \psi_k + \beta |\psi_j|^2 \psi_j,
\]

where $\psi_j$ is the complex order-parameter, describing the condensate at site $j = 1 \ldots N$, $J$ is the hopping parameter, and $\beta$ is the nonlinear on-site interactions parameter, respectively. The summation over $k = 1 \ldots N_{\text{nn}}$ extends over the nearest-neighbors $NN(j)$ of site $j$. The DGPE generates conservative dynamics corresponding to the Hamiltonian

\[
\mathcal{H} = -J \sum_{\langle j,l \rangle} \bar{\psi}_j \psi_l + \beta \sum_j |\psi_j|^4.
\]

As a measurable quantity of interest we have chosen a set of on-site occupations $X(t) = \{ n_1, n_2, \ldots, n_N \}$, where $n_j \equiv |\psi_j|^2$.

4. Ergodization time

4.1. Definitions of Lyapunov exponents and ergodization time

The largest Lyapunov exponent is defined as $\lambda_{max} \equiv \frac{1}{\tau} \lim_{t \to \infty} D(t) \ln \frac{D(t)}{D(0)}$, where $D(t) = |\delta R(t)|$ is the distance between the two phase-space trajectories: the reference trajectory $R(t)$ and the slightly perturbed one $R(t) = R(t) + \delta R(t)$ [30].
The ratio \( \ln \left[ \frac{D(t)}{D(0)} \right] \) fluctuates in time as the reference trajectory \( R_i(t) \) explores the energy shell. We define instantaneous local stretching rates as \( \lambda(t) = \frac{d}{dt} \ln \left[ \frac{D(t)}{D(0)} \right] \). The largest Lyapunov exponent is the average of local stretching rates over a sufficiently long time: \( \lambda_{\text{max}} = \overline{\lambda(t)} \). We denote fluctuations of the local stretching rates by \( \delta \lambda(t) = \lambda(t) - \lambda_{\text{max}} \), and their autocorrelator by

\[
\varphi(t) = \langle \delta \lambda(t) \delta \lambda(0) \rangle.
\]

We propose to use the convergence of \( \overline{\lambda(t)} \) as an indicator of ergodization, and define the ergodization time as

\[
\tau_{\text{erg}} = \frac{1}{\langle \delta \lambda(t) \rangle} \int_0^\infty \varphi(t) \, dt.
\]

In numerical simulations, \( \lambda_{\text{max}} \) and \( \varphi(t) \) can be obtained from the direct calculations of \( R_i(t) \) and \( R_j(t) \). However, such an approach is impractical experimentally, because it requires tracking all phase-space coordinates of the system. An alternative, more practical approach was proposed in [16, 31]. That approach is based on monitoring the effect of Loschmidt echo on equilibrium noise of almost any observable (see appendix A).

### 4.2. Ergodization time from Loschmidt echoes

In the present setting, the Loschmidt echo is implemented by reversing the sign of Hamiltonian (2) at time \( \tau \), and simultaneously perturbing the state vector, \( \psi(t + 0) = \psi_0(t - 0) + \delta \psi_0 \), where \( \delta \psi_0 \) is a very small random perturbation. We track the equilibrium noise of the on-site occupations \( \{n_i(t)\} \) before and after the time reversal, and introduce the deviation between the reversed and direct dynamics of the on-site occupations, \( \Delta n_i(\Delta t) \equiv n_i(t + \Delta t) - n_i(t - \Delta t) \). As sketched in figure 1, the deviations \( \Delta n_i(\Delta t) \) fluctuate with an exponentially-growing envelope.

We introduce two ensemble averages of \( \Delta n_i(\Delta t) \):

\[
G(\Delta t) \equiv \left\{ \ln \left[ \sum_{i=1}^N [\Delta n_i(\Delta t)]^2 \right] \right\} \xrightarrow{\Delta t \to \infty} \lambda_{\text{max}} \Delta t
\]

and

\[
W(\Delta t) \equiv \left\{ \ln \left[ \sum_{i=1}^N [\Delta n_i(\Delta t)]^2 \right] \right\} \xrightarrow{\Delta t \to \infty} \Lambda \Delta t,
\]

where \( \Lambda \equiv \frac{1}{t} \ln \left\{ \exp \int_0^t \lambda(t') \, dt' \right\} \).

The limit (5) was verified recently in [31]. Now, we concentrate on relation (6). The reason for the difference between parameter \( \Lambda \) (sometimes referred to as the generalized maximum Lyapunov exponent [32–35]) and \( \lambda_{\text{max}} \) is the different order of operations of taking logarithm and ensemble averaging. This difference is controlled by the amplitude and the correlation time of fluctuations \( \delta \lambda(t) \). In order to demonstrate this, we first note that

\[
\Lambda - \lambda_{\text{max}} = \frac{1}{t} \ln \left\{ e^{\int_0^t \varphi(t') \, dt'} \right\}.
\]

The average on the right-hand side can be calculated analytically on the basis of the assumption that variable \( \int_0^t \delta \lambda(t') \, dt' \) is Gaussian, which gives (see appendix B):

\[
\left\{ e^{\int_0^t \varphi(t') \, dt'} \right\} \equiv e^{\int_0^\infty \varphi(t') \, dt'}.
\]

Using this relation together with equation (4), we obtain \( \Lambda - \lambda_{\text{max}} = \int_0^\infty \varphi(t') \, dt' \equiv \langle \delta \lambda^2 \rangle \tau_{\text{erg}} \). Therefore, the ergodization time can be expressed as

\[
\tau_{\text{erg}} = \frac{\Lambda - \lambda_{\text{max}}}{\langle \delta \lambda^2 \rangle}.
\]

### 4.3. Extracting the ergodization time of DGPE lattices by measuring observables only

The experimental use of equation (9) requires determining \( \lambda_{\text{max}} \) and \( \Lambda \) from equations (5) and (6) and, in addition, the knowledge of \( \langle \delta \lambda^2 \rangle \). While there might be ways of extracting \( \langle \delta \lambda^2 \rangle \) from experimental time-series, here we resort to an empirical estimate.
\[ \langle \delta X^2 \rangle \approx \frac{4\lambda_{\text{max}}^2}{N \ln N}, \]

where \( N \) is the number of nearest-neighbors for a lattice site. In section 6, we substantiate the estimate (10) on the basis of our direct numerical simulations. Why this approximation works so well for the DGPE on large lattices and whether it works for a more general class of systems needs further investigation. A possible explanation of equation (10) is that, in our simulations, the Lyapunov eigenvector corresponding to \( \lambda_{\text{max}} \) is usually localized at only a handful of sites, which is consistent with other observables of wandering localization of Lyapunov eigenvectors [36–43].

The estimate (10) leads to the following approximation for the ergodization time

\[ \tau_{\text{erg}} \approx \frac{\Lambda - \lambda_{\text{max}}}{4\lambda_{\text{max}}^2} N \ln N. \]

5. Criterion of ergodicity

When the ergodicity of a system is about to breakdown, one obvious indicator of this is an anomalously large value of the ergodization time given by equation (9). One may wonder, however, whether the Loschmidt echo response contains other signatures of broken ergodicity. In an ergodic regime, the distribution of \( \ln |\Delta X(\Delta t)| \) should be Gaussian (see appendix B), and its variance \( \sigma_G^2(\Delta t) \equiv \langle \ln^2 |\Delta X(\Delta t)| \rangle - G^2(\Delta t) \) is supposed to grow linearly in time:

\[ \sigma_G^2(\Delta t) \rightarrow 2(\Lambda - \lambda_{\text{max}}) \Delta t. \]  

In the opposite case of a non-ergodic regime, the averages in \( G(\Delta t) \) and \( W(\Delta t) \) converge poorly, which in turn leads to a non-Gaussian distribution for individual realizations of \( \ln |\Delta X(\Delta t)| \) [44], accompanied by a deviation from the linear growth of the variance \( \sigma_G^2(\Delta t) \) given by equation (12). Thus, relation (12) can be used for an experimentally feasible test of ergodization.

6. Numerical tests

For illustration, we chose three model systems: a one-dimensional chain with \( N = 100 \) sites, a two-dimensional square lattice with \( N = 10 \times 10 \) sites and a three-dimensional cubic lattice with \( N = 4 \times 4 \times 4 \). We used \( J = 1, \beta = 0.01 \). The initial conditions corresponded to the total energy \( E_{\text{total}} = 100N \) and the number of particles \( N_p = \sum_i |\psi_i|^2 = 100N \), so that the particles were distributed equally among all lattice sites \( n_i(0) = |\psi_i(0)|^2 = 100 \) with random phases. The perturbation to the state vector at the moment of time reversal was \( \psi(t + 0) = \psi(t - 0) + \delta \psi_i \) where \( \delta \psi_i \) is a random vector in the phase space subject to the constraint \( \sum_i |\delta \psi_i|^2 = 10^{-8} \).

In order to test the relation (9), we calculated the two averages of Loschmidt echoes \( G(\Delta t) \) and \( W(\Delta t) \) for one-, two- and three-dimensional DGPE lattices. The results are presented in figure 2. The values of the characteristic exponents \( \lambda_{\text{max}} \) and \( \Lambda \) extracted in each case are listed in table 1. We also collected long enough time-series of local stretching rates \( \lambda(t) \), then calculated the autocorrelation function \( \varphi(t) \) and extracted \( \langle \delta X^2 \rangle \) and \( \tau_{\text{erg}} \). In figure 3, we further demonstrate that approximation (10) for \( \langle \delta X^2 \rangle \) works well for sufficiently large lattice sizes.

Table 1 compares three values of the ergodization time: the one calculated on the basis of the definition (4), the one given by equation (9) and the one given by the approximation (11). In equation (9), we used the directly calculated value of \( \langle \delta X^2 \rangle \). For two- and three-dimensional lattices, the values of the ergodization time obtained from equations (4) and (9) agree very well. And at the same time, we observe clear discrepancy between equations (4) and (9) for the one-dimensional lattice, which indicates that the system has not ergodized on the timescale covered by the Loschmidt echo. Non-ergodized fast growing samples in \( W(\Delta t) \) from equation (6) reach a plateau significantly earlier than others: an indication of this in figure 2 is an early departure of \( W(\Delta t) \) from the linear growth regime. Overall, the ergodization time decreases with the increasing lattice dimension, being the longest in the one-dimensional case. Slow ergodization of one-dimensional chains (for Fermi–Pasta–Ulam, Klein–Gordon chains and also DGPE) has also been noticed and investigated in [45, 46].

In all three cases, we further observe that the values obtained from equation (11) give a satisfactory approximation to equation (4).

We also performed the ergodicity test associated with relation (12). The results are presented in figure 4, where the ratio \( \frac{\sigma_G^2(\Delta t)}{2\Delta t} \) is plotted as a function of the echo time \( \Delta t \). For the quickly ergodizing two- and three-
dimensional systems, the above ratio levels off rather quickly around the expected value $\Lambda = \lambda_{\text{max}}$, whereas for the slow-ergodizing one-dimensional case it never reaches the expected plateau.

7. Relation to quantum systems

Quantum-mechanical description of Loschmidt echoes involves OTOCs [16, 17, 22, 23, 28, 47–56]. We now illustrate that the parameter $\Lambda$ defined in the present work from the relation

$$\langle |\Delta X(\Delta t)| \rangle \sim \exp(\Lambda \Delta t)$$

also characterizes the growth of an OTOC in a quantum system, when the system’s constituents are describable quasi-classically. Following [16], we observe that relation (13) implies $\langle |\Delta X(\Delta t)|^2 \rangle \sim \exp(2\Lambda \Delta t)$. In other words

$$\Lambda = \frac{1}{2} \frac{d}{d(\Delta t)} \left[ \lim_{\Delta t \to 0} \ln \langle |\Delta X(\Delta t)|^2 \rangle \right],$$

where $\langle |\Delta X(\Delta t)|^2 \rangle = \langle (X(\tau + \Delta t) - X(\tau - \Delta t))^2 \rangle = 2\langle X^2 \rangle - \langle X(\tau + \Delta t)X(\tau - \Delta t) \rangle - \langle X(\tau - \Delta t)X(\tau + \Delta t) \rangle$. We focus here on the last two terms. They are equal classically, but can become different when averaged quantum-mechanically. Both of them become OTOCs in the quantum limit. Below, we show this for $\langle X(\tau + \Delta t)X(\tau - \Delta t) \rangle$.

Let us consider a quantum system in equilibrium described at $t = 0$ by Hamiltonian $\hat{H}$. The density matrix of the system is $\rho_0 \equiv \exp\left(-\frac{i}{\hbar} \hat{H} \right)$. We are interested in the fluctuations of an observable quantity represented by quantum operator $X$. Let us further assume that the Hamiltonian changes sign at $t = \tau$, and, at the same moment of time, the system components experience an infinitesimally small random perturbation describable by quantum operator $\hat{R}$ (see [16] for a concrete example). As a result, operator $\hat{X}$ evolves as

$$\hat{X}(t) = \begin{cases} e^{i\hat{H}t} \hat{X}(0) e^{-i\hat{H}t}, & t < \tau \\ e^{i\hat{H}t} \hat{R} e^{-i\hat{H}(t-\tau)} \hat{X}(0) e^{i\hat{H}(t-\tau)} e^{-i\hat{R}t}, & t > \tau \end{cases}$$

We now consider the quantum average $\langle \hat{X}(\tau + \Delta t) \hat{X}(\tau - \Delta t) \rangle \equiv \text{Tr} \{ \hat{X}(\tau + \Delta t) \hat{X}(\tau - \Delta t) \rho_0 \}$, which, with the help of equation (14) and a simple manipulation, can be transformed into

$$\langle \hat{X}(\tau + \Delta t) \hat{X}(\tau - \Delta t) \rangle = \text{Tr} \{ \hat{R}^+ \hat{X}(\tau + \Delta t) \hat{X}(\tau - \Delta t) \hat{R} \}. \tag{15}$$

Noting that $\hat{X}$, as a physical observable, must be describable by a Hermitian operator, i.e. $\hat{X}^+(t) = \hat{X}(t)$, we rewrite equation (15) as $\langle \hat{X}(\tau + \Delta t) \hat{X}(\tau - \Delta t) \rangle = \langle \hat{R}^+ \hat{X}(\tau + \Delta t) \hat{R} \hat{X}(\tau - \Delta t) \rangle$, which is the standard form of OTOC.
Table 1. Summary of numerical tests of relations (9) and (11): \( \lambda_{\text{max}} \) and \( \Lambda \) are extracted from figure 2; \( \langle \delta \lambda^2 \rangle \) is extracted either directly from a time-series of local stretching rates according to equation (4) or from empirical estimate (10); the three values of \( \tau_{\text{org}} \) are obtained on the basis of the definition (4), from the Loschmidt echo relation (9), and from the approximate relation (11).

| Lattice | \( N_{\text{int}} \) | \( \lambda_{\text{max}} \)       | \( \Lambda \)   | \( \langle \delta \lambda^2 \rangle \) | \( \tau_{\text{org}} \) |
|----------|----------------|------------------|----------------|--------------------------------|------------------|
|          |                |                  |                | Equation (4) | Equation (10) | Equation (4) | Equation (9) | Equation (11) |
| 1D, \( N = 100 \) | 2              | 0.643 ± 0.001    | 0.927 ± 0.009  | 0.362 ± 0.001 | 0.413 ± 0.001 | 0.66 ± 0.05 | 0.78 ± 0.03 | 0.69 ± 0.02 |
| 2D, \( N = 10 \times 10 \) | 4              | 0.698 ± 0.001    | 0.731 ± 0.004  | 0.104 ± 0.001 | 0.122 ± 0.001 | 0.32 ± 0.02 | 0.32 ± 0.04 | 0.27 ± 0.03 |
| 3D, \( N = 4 \times 4 \times 4 \) | 6              | 0.650 ± 0.001    | 0.670 ± 0.001  | 0.080 ± 0.001 | 0.047 ± 0.001 | 0.26 ± 0.02 | 0.25 ± 0.02 | 0.43 ± 0.03 |
Finally, we note that the quantum counterpart of the maximum classical Lyapunov exponent can be defined as

\[ \lambda_{\text{max}}^Q = \frac{1}{2} \frac{d}{d(\Delta t)} \lim_{\Delta t \to \infty, \Delta t \to 1} \text{Tr} \left[ \hat{\rho}_0 \ln \left( \hat{X}^2(\tau + \Delta t) \right) + \hat{X}^2(\tau - \Delta t) - \hat{X}(\tau + \Delta t) \hat{X}(\tau - \Delta t) - \hat{X}(\tau - \Delta t) \hat{X}(\tau + \Delta t) \right] \]

It was proposed recently in [19], that one can impose a temperature-dependent constraint on the exponential growth rate \( \Lambda \) of OTOCs (when the exponential growth regime exists, which is not always the case [16]). The constraint on \( \Lambda \), in turn, imposes a constraint on the largest Lyapunov exponent \( \lambda_{\text{max}} \) for a quantum system. As follows from the present work, as well as from [17, 22, 57], the value of \( \Lambda \) is, in general, larger than \( \lambda_{\text{max}} \). The interesting question then arises whether the difference between \( \Lambda \) and \( \lambda_{\text{max}} \) approaches zero as the number of degrees of freedom in a system increases. Our findings indicate that, for a lattice of a given dimension (1D, 2D and 3D), \( \Lambda - \lambda_{\text{max}} \) remains finite for rather large systems. Yet, this difference decreases with the increase of the lattice dimension from 1D to 2D to 3D. It is particularly small for the 3D lattice considered in this work, which is consistent with the classical spin simulations for 3D lattices done in [16], where the difference between \( \Lambda \) and \( \lambda_{\text{max}} \) was within the computational uncertainty of the simulation and, hence, was overlooked.

We further remark that the difference \( \Lambda - \lambda_{\text{max}} \) originates from the fluctuations of Loschmidt echo amplitude, which is, as shown in the present work, sensitive to ergodicity breakdown in classical systems. The counterpart of this breakdown in quantum systems is the transition from an ergodic to a many-body localized phase. It was proposed in a related study [28], that the fluctuations of a Loschmidt echo in quantum systems are sensitive to the many-body localization transition.
Finally, even though the primary agenda of the present article is to characterize ergodicity in large systems close to the thermodynamic limit, our method based on Loschmidt echoes should also be applicable to a-few-body systems. When classical a-few-body systems exhibit the breakdown of ergodicity, the ergodicity criterion proposed in section 5 should be sensitive to this. As far as a-few-body quantum systems are concerned [58–60], it is an interesting question how their energy-level spacing statistics is related to our ergodicity criterion in the classical limit. If a quantum system exhibits the Wigner–Dyson statistics of energy-level spacings in one energy range and does not exhibit it in the other one, then the respective energy shells in the classical limit likely change from ergodic to non-ergodic. In such a case, the ergodicity criterion of section 5 can be used to predict the level spacing statistics.

8. Conclusion

To summarize, we proposed a method of estimating ergodization time of a chaotic many-particle system by monitoring equilibrium noise before and after time reversal of dynamics, and validated it numerically by simulations of the DGPE. We showed that the difference between the largest Lyapunov exponent and the growth rate of the classical counterpart of OTOCs is proportional to the ergodization time of a system. We also introduced a related test for the breakdown of ergodicity.

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Appendix A. Limits (5) and (6): independence of the observable X

If an experiment can track all phase-space coordinates of a system, then it can obtain the largest Lyapunov exponent by identifying the phase-space direction $\mathbf{R}$ along which the growth of a perturbation is the quickest, i.e. the eigenvector corresponding to the largest local Lyapunov exponent. However, a realistic experiment is limited to an observable $X$. In such a case the eigenvector is unlikely to belong to the subspace of the whole phase space that contains $X$, but it is overwhelmingly likely to have a non-zero projection onto that subspace. This means that

$$ \Delta X(\Delta t) = \Delta X(0) \cos \alpha(\Delta t) e^{\int_0^t \lambda(t') dt'}, $$

(A1)

where $\alpha(\Delta t)$ is the angle between the eigenvector and the direction corresponding to $\Delta X(\Delta t)$ in the many-dimensional phase space.

Here we consider the growth of the initial difference $\Delta X(0)$ introduced by an imperfect time reversal, and justify the limits $\Delta t \to \infty$ for $G(\Delta t)$ in equation (5) (see [16])

$$ G(\Delta t) \equiv \langle |\Delta X(\Delta t)| \rangle \to \lambda_{\text{max}} \Delta t $$

(A2)

and for $W(\Delta t)$ in equation (6)

$$ W(\Delta t) \equiv \ln \langle |\Delta X(\Delta t)| \rangle \to \Lambda \Delta t. $$

(A3)

We use equation (A1) to express $G(\Delta t)$ as

$$ G(\Delta t) = \left\{ \ln |\Delta X(0)| + \ln |\cos \alpha(\Delta t)| + \ln e^{\int_0^t \lambda(t') dt'} \right\}, $$

(A4)

where the first term is constant, the second term remains limited from above after ensemble averaging over initial conditions, and the third term is the only one growing linearly with $\Delta t$. The second term $\ln |\cos \alpha(\Delta t)|$ may appear problematic for $\Delta t$ corresponding to $|\cos \alpha(\Delta t)| = 0$. However, this singularity is integrable: it vanishes after ensemble averaging. Given the definition of $\lambda_{\text{max}}$ from the main text of the article, equation (A4) implies equation (A2).

To prove the limit (A3) for $W(\Delta t)$, we assume that $|\cos \alpha(\Delta t)|$ is uncorrelated with $e^{\int_0^t \lambda(t') dt'}$ and hence factorize the average $\langle |\Delta X(0) \cos \alpha(\Delta t)| e^{\int_0^t \lambda(t') dt'} \rangle \to \langle |\Delta X(0) \cos \alpha(\Delta t)| \rangle \cdot \langle e^{\int_0^t \lambda(t') dt'} \rangle$. This assumption is, presumably, appropriate for almost any non-local observable. It is supported by the extensive numerical experience, e.g. [14–17], showing that the eigenvectors corresponding to $\lambda_{\text{max}}$ exhibit rather erratic behavior. The above factorization leads to
\[ W(\Delta t) = \ln \langle |\Delta X(0) \cos(\Delta t)| \rangle + \ln \left\langle e^{\int_0^{\Delta t} \lambda(t') dt'} \right\rangle. \] (A5)

Given the definition of \( \Lambda \), equation (A5) implies equation (A3).

**Appendix B. Derivation of equation (8)**

Here we derive equation (8)

\[ \left\langle e^{\int_0^t \delta\lambda(t') dt'} \right\rangle = e^{t \int_0^\infty \varphi(t') dt'}, \] (B1)

by a stochastic-noise method analogous to the one developed by Anderson and Weiss [61] in a different context, namely, for the calculation of exchange-narrowed magnetic resonance linewidths.

We represent the left-hand side of equation (B1) as

\[ \left\langle e^{\int_0^t \delta\lambda(t') dt'} \right\rangle = \int dYP(Y)e^Y, \] (B2)

where

\[ Y(t) = \int_0^t \delta\lambda(t') dt' = \lim_{\delta t \to 0} \delta\lambda(t_i), \] (B3)

and \( P(Y) \) is the probability distribution of \( Y(t) \). We assume that the system fluctuates near equilibrium, and, therefore, the process \( \delta\lambda(t) \) is stationary, i.e. its probability distribution \( p(\delta\lambda(t)) \) is independent of \( t \).

If \( \delta\lambda(t) \) is a Gaussian random variable, then \( Y \) is also a Gaussian random variable for all times, i.e. \( P(Y) \) is Gaussian. If \( p(\delta\lambda(t)) \) is not Gaussian, but the variable \( \delta\lambda(t) \) has a finite memory time \( \tau_{\text{erg}} \), then \( P(Y) \) still becomes Gaussian for \( t \gg \tau_{\text{erg}} \) (consequence of the central limit theorem).

Assuming Gaussianity, \( R(Y) \equiv (2\pi \langle Y^2 \rangle)^{-\frac{1}{2}} \exp \left( -\frac{Y^2}{2\langle Y^2 \rangle} \right) \). Equation (B2) now reads:

\[ \left\langle e^{\int_0^t \delta\lambda(t') dt'} \right\rangle = (2\pi \langle Y^2 \rangle)^{-\frac{1}{2}} \int dYe^{\frac{-Y^2}{2\langle Y^2 \rangle}} = e^{\langle Y^2 \rangle}. \] (B4)

We calculate the variance of \( Y \) as

\[ \langle Y^2 \rangle = \left\langle \left( \int_0^t \delta\lambda(t') dt' \right)^2 \right\rangle = \int_0^t dt' \int_0^{t'} dt'' \langle \delta\lambda(t') \delta\lambda(t'') \rangle. \] (B5)

Since \( \delta\lambda(t) \) is assumed to be stationary: \( \langle \delta\lambda(t') \delta\lambda(t'') \rangle = \langle \delta\lambda(0) \delta\lambda(t'' - t') \rangle \equiv \varphi(t'' - t') \), and equation (B5) becomes

\[ \langle Y^2(t) \rangle = \int_0^t dt' \int_{-t}^{t-t'} \varphi(t'') dt'' = \int_0^t dt' g(t'), \] (B6)

where \( g(t') = \int_{-t}^{t-t'} \varphi(t'') dt'' \). The dynamics is time-reversible, thus \( \varphi(-t') = \varphi(t') \), and \( \dot{g}(t') = -\varphi(t-t') + \varphi(-t') = -\varphi(t-t') + \varphi(t') \). Integrating equation (B6) by parts leads to

\[ \langle Y^2(t) \rangle = t \cdot g(t) - \int_0^t dt' \cdot t \dot{g}(t') \]
\[ = t \int_0^t \varphi(t') dt' - \int_0^t dt' \cdot t \varphi(t') + \int_0^t dt' \cdot t \varphi(t-t') \]
\[ = t \int_0^t \varphi(t') dt' - \int_0^t dt' \cdot t \varphi(t') + \int_0^t dt' \cdot (t-t') \varphi(t') \]
\[ = 2 \int_0^t dt' (t-t') \varphi(t'). \] (B7)

We substitute equation (B7) into equation (B4), and finally obtain

\[ \left\langle e^{\int_0^t \delta\lambda(t') dt'} \right\rangle = e^{\int_0^t dt' (t-t') \varphi(t')} \] (B8)

This integral converges if \( \varphi(t) \) decays faster than \( \frac{1}{t^2} \). In such a case, for \( t \to \infty \)

\[ \left\langle e^{\int_0^t \delta\lambda(t') dt'} \right\rangle = Ce^{\int_0^t dt' \varphi(t')}, \] (B9)

where \( C = \exp \left( -\int_0^\infty dt' \cdot t \varphi(t') \right) \).
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