D-Branes on Spaces Stratified Fibered Over Hyperbolic Orbifolds

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Abstract

We apply the methods of homology and K-theory for branes wrapping spaces stratified fibered over hyperbolic orbifolds. In addition, we discuss the algebraic K-theory of any discrete co-compact Lie group in terms of appropriate homology and Atiyah-Hirzebruch type spectral sequence with its non-trivial lift to K-homology. We emphasize the fact that the physical D-branes properties are completely transparent within the mathematical framework of K-theory. We derive criteria for D-brane stability in the case of strongly virtually negatively curved groups. We show that branes wrapping spaces stratified fibered over hyperbolic orbifolds carry charge structure and change the additive structural properties in K-homology.

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1 Introduction

In this paper we will elucidate the observation that methods of algebraic K-theory and K-homology are appropriate tools in classify branes wrapping spaces stratified fibered over hyperbolic orbifolds. Generally speaking, D-branes are complicate objects and cannot be considered as just subspace in an ambient spacetime. Their dynamics require a more abstract mathematical tools (such as of a derived category \([1]\), for example). Nevertheless in algebraic K-theory and K-homology D-branes can be realized naturally as an appropriate spacetime background with reveals various properties of their quantum dynamics that could not be otherwise detected if one classified the brane worldvolumes using only singular homology \([2, 3]\).

This paper is outlined as follows. In Sect. 2 we select the requisite mathematical material used for investigation of spaces stratified fibered over orbifolds. Our purpose there is to discuss homology with \(\text{twisted} \) spectrum coefficients. The main attention we pay for an Atiyah-Hirzebruch type spectral sequence with it relation to (ordinary) homology.

Then we continue with algebraic K-theory of spaces stratified fibered over hyperbolic orbifolds. Before discussing D-branes we begin with some background material on stratified systems of (abelian) groups over hyperbolic manifolds, and homology groups in which the obstructions lie. We follow lines and conventions of \([4]\) which apply throughout. We examine homology spectrum – \(\Omega\)-spectrum, stratified system of abelian groups and virtually negatively curved groups.

In the course of paper we follow the lines of \([2]\) and assume the following definitions: A D-brane on background \(\mathcal{B}\) is a triple \([X, F, \psi]\), where \(X\) is a \(\text{Spin}^C\) submanifold of \(\mathcal{B}\), interpreted as a D-brane in string theory, \(F\) is the Chan-Paton boundle on \(X\) and \(\psi : X \hookrightarrow \mathcal{B}\) is the embedding. A D-brane can be interpreted as the homology class of K-cycle \([X, F, \psi]\) \(\in K(\mathcal{B})\); we assume that the base space \(\mathcal{B}\) is a simply connected finite CW-complex (one can consider \(\mathcal{B}\) to be only path-connected). There are many such situations in which one is interested in the classification of D-branes in X. In fact, many of the examples considered in literature fall into this category. For instance, both the lens spaces \(L(p; q_1, \ldots, q_n)\) and the real projective spaces \(\mathbb{R}P^{2n+1}\) are circle bundles over \(\mathbb{C}P^n\) \([2]\). Furthermore, the very application of vector bundle modification identifies those D-branes such that one is a spherical fibration over the other.

In Sect. 3 we analyze the additive structure in the homology class of K-cycle and appropriate structural properties for D-branes wrapping spaces stratified fibered over hyperbolic orbifolds.

Finally in Sect. 4 we derive stability criteria for D-branes wrapping spaces stratified fibered over hyperbolic orbifolds trying to put physical notions in a mathematically rigorous setting as much as possible.
2 Branes on spaces stratified fibered over orbifolds

Let $\overline{X}$ be a closed (means compact and without boundary) connected Riemannian manifold with strictly negative sectional curvatures and let $\{X_j\}_{j=1}^\infty$, $X_1 \subset X_2 \subset X_3 \subset \ldots$ be a sequence of connected compact smooth manifolds. Let $F$ be a finite group which acting on $\overline{X}$ via isometries and on each $X_j$ via smooth maps. (The action of $F$ on $X_j$ need not to be free but on $\overline{X}$ is free.) Assume also the smooth embedding $X_j \subset X_{j+1}$ is both $F$-equivariant and $j$-connected. Let $X_\infty = \bigcup_{j=1}^\infty X_j$ and give $X$ the direct limit topology. The induced action of $F$ on $X$ is free. Assuming that $\{M^j\}_{j=1}^\infty$ is the orbit space $\overline{X} \times X_j$ under the diagonal action of $F$, and

$$M^\infty = \lim_{j} M^j.$$  

Let $X$ be the orbit space $\overline{X}/F$ and $\{p_j\}_{j=1}^\infty : M^j \to X$ be the map induced from the canonical projection of $\overline{X} \times X_j$ onto $\overline{X}$. The set $\{p_j\}$ is stratified systems of fibrations on $X$. Let as before $G(Y)$ be the stable topological pseudo-isotopy $\Omega$-spectrum associated to a topological space $Y$; then we have to define a class of maps $p_\infty$.

**Stratified system of abelian groups.** With a stratified system of fibrations $p : M \to B$ one can associate a homology $\Omega$-spectrum $H(B; G(p))$ with twisted spectrum coefficients and a map of $\Omega$-spectra: $H(B; G(p)) \to G(M)$. A functor $G$ from spaces to spectra is said to be homotopy invariant provided a homotopy equivalence of spaces $X \to Y$ induces a homotopy equivalence of spectra $G(X) \to G(Y)$ [4]. A homology $\Omega$-spectrum $H(B; G(p))$ is defined and discussed in the lectures [5]. One can construct Atiyah-Hirzebruch type spectral sequence [4] [6]:

Suppose $G$ is a homotopy invariant spectrum valued functor of spaces, and $p : M \to B$ is an stratified system of fibrations discussed above. Then the homotopy groups $\pi_q G(p^{-1}(x))$ form stratified systems of groups over $X$, denoted by $\pi_q G(p)$, and there is a (homological) spectral sequence with

$$E^2_{p,q} = H_p(B; \pi_q G(p))$$

which abuts to $\pi_{p+q} H(B; \pi_q G(p))$, where $\pi_q G(p(x))$ denotes the stratified system of abelian groups over $B$ with the group $\pi_q G(p^{-1}(x))$ corresponding to the point $x \in B$.

Let $\mathcal{A}$ be a stratified system of abelian groups. The homology groups $H_j(B; \mathcal{A})$ can be calculated as follows. A triangulation of a compact manifold $X$ consists of finite family of closed subsets $\{T_j\}_{j=1}^n$ that cover $X$, and family of homeomorphisms $\varphi_j : T'_j \to T_j$, where each $T'_j$ is a Euclidean simplex. (In the case of compact surface $B = S$ each $T'_j$ is a triangle in the plane $\mathbb{R}^2$, i.e., a compact subset of $\mathbb{R}^2$ bounded by three distinct strait lines.) Let $\mathcal{T}$ be such a triangulation that each strata $T_j$ is a subcomplex. Then a stratified system $\mathcal{A}$ restricted to each open simplex is a constant system of coefficients and

$$H(B; \mathcal{A}) = \bigoplus_j H_j(X; \mathcal{A})$$

(2.3)
is the homology of finite chain complex

\[
C := C_0 \longrightarrow C_1 \longrightarrow \ldots \longrightarrow C_n, \\
C_j = \bigoplus_{s \in T_j} A(\hat{s}).
\] (2.4)

Here \(\hat{s}\) denotes the barycenter of \(s\). In fact \(n\)-dimensional \(s\)-simplex can be decomposed on \((n + 1)!\) small \(n\)-dimensional simplexes. Vertices of new simplexes are the centers of gravity of faces of initial simplex. Set \(\{x_j\}\) of centers is a set of vertexes of some simplex of barycentric subdivision if the corresponding faces can be composed to chain of consecutively enclosed faces.

**The \(E_{0,q}^2\)-terms.** These terms in spectral sequence (2.2) can be evaluated in the following manner. Let \(\mathfrak{G}\) be the extension of \(\pi_1(X)\) determined by the action of \(F\) on \(X\). In fact the group \(\mathfrak{G}\) is isomorphic to the factor group \(\pi_1(M)/\pi_1(X)\). Let \(\mathcal{F}(\mathfrak{G})\) be the category with objects the finite subgroups of \(\mathfrak{G}\). For each element \(\gamma \in \mathfrak{G}\) we can determine a morphism \(G_1 \rightarrow G_2\) provided \(\gamma G_1 \gamma^{-1} \subseteq G_2\). Then \(E_{0,q}^2\)-terms are isomorphic to the direct limit

\[
E_{0,q}^2 \overset{\text{isomorphism}}{\longrightarrow} \lim_{G \in \mathcal{F}(\mathfrak{G})} \pi_q(G(X/p(G))),
\] (2.5)

where \(p : \mathfrak{G} \rightarrow F\) is the canonical projection and \(X/p(G)\) is the orbit space. This isomorphism can be demonstrate by using of a basic result of E. Cartan:

(i) First let us remind some conventions which will be required. Let \(A_\mu\) be the Yang-Mills connection associated with some quantum field model and let \(W\) be a (affine) space of all connections \(A_\mu\). The action of group \(G\) on \(A_\mu\) is given by: \(A_\mu \rightarrow gA_\mu g^{-1} + \partial_\mu gg^{-1}, g \in G\), where the infinite group \(G\) is the group of gauge transformations. Set of classes of equivalence of space \(W\) with respect to \(G\) action is called the orbit space. Remind also that if geodesic \(\gamma\) in \(X\) has ends belonging \(Y \subset X\) then it entirely lie in \(Y\); manifold \(Y\) satisfies this condition is called completely geodesic submanifold in \(X\).

(ii) Let us consider a map \(\varphi : G/\Gamma \rightarrow G, \varphi : g\Gamma \rightarrow \alpha(g)g^{-1}\), where \(\alpha\)-involuntary automorphism of group \(G\) satisfies: \(\alpha^2 = \text{Id}, \alpha \neq \text{Id}, (G_\alpha)_0 \subset \Gamma \subset G_\alpha\). Here \(G_\alpha\) is a set of fixed points of the automorphism \(\alpha\), and \((G_\alpha)_0\) is a component of unit of \(G_\alpha\). First note that \(\varphi\) is the diffeomorphism \(G/\Gamma\) on closed completely geodesic manifold \(X_\alpha : X_\alpha = \{g \in G | \alpha(g)g = \text{Id}\}\).

A differential \(\alpha\) in point \(\text{Id}\), \(d\alpha_{\text{Id}}\), is the involuntary automorphism of Lie algebra \(A\) of group \(G\). The operator \(d\alpha_{\text{Id}}\) decomposes the space \(A\) on two subspaces \(P\) and \(Q\) associated with eigenvalues \(\pm 1\) of \(d\alpha_{\text{Id}} : P = \{P \in A | d\alpha_{\text{Id}}(P) = P\}, Q = \{Q \in A | d\alpha_{\text{Id}}(Q) = -Q\}\).

It can be shown that the algebra \(Q\) of Lie group \(\Gamma\) is a \(P\)-orthogonal (with respect to scalar product on \(A\)) to \(Q\)-subspace in \(A\). The subspace \(P\) is invariant with respect to adjoint representation \(Ad(\Gamma)\) (see for detail [2]). The space \(X_\alpha\) is the closed submanifold in \(G\), and \(\dim X_\alpha = \dim (G/\Gamma)\). Also \(X_\alpha\) is the completely geodesic submanifold in \(G\). It is known

\[^4\] Indeed, the space \(X_\alpha\) is a kernel of the map \(\varphi\). In order to find \(\dim X_\alpha\) it is convenient to calculate the kernel of tangent map \(T_g\varphi : T_g G \rightarrow T_{\alpha(g)g} G\) in a point \(g\). \(\text{Ker} T_g \varphi = gW_g\), where \(W_g = \{P \in A | d\alpha_{\text{Id}}(P) + Ad(g)P = 0\}\). It means that \(\text{Ker} T_g \varphi = P, \text{rank} \varphi = \dim A - \dim \text{Ker} \varphi = \dim Q, \text{and it is not depend on choice of a point} g\). It follows that \(X_\alpha\) is the close submanifold in \(G\) and \(\dim X_\alpha = \dim (G/\Gamma)\).
that geodesics (with respect to invariant metric) in space $G/\Gamma$ are orbits of one-parameter subgroups of group $G$ \cite{2}. Let $\gamma$ be a geodesic in a group $G$ tangent to $X_\alpha$ in a point $g$: $\gamma_t = ge[tP]$, where $e[-] \equiv \exp(-)$, $P \in A$, and $g \cdot P \in T_gX_\alpha = \text{Ker}T_g\varphi = gWg$. We have

$$\alpha(\gamma_t)\gamma_t = \alpha(ge[tP])ge[tP] = \alpha(g)e[td\alpha(Id)(P)]ge[tP]$$

$$\implies \alpha(\gamma_t)\gamma_t = \alpha(g)ge[-tP]g^{-1}ge[tP] = \alpha(g)g = Id.$$  \hspace{1cm} (2.6)

Let us proof that $X_\alpha = \text{Im} \varphi$. We have $X_\alpha \supset \text{Im} \varphi$; it can be prooved that $X_\alpha \subset \text{Im} \varphi$. In $X_\alpha$ there is a finite number $N$ of points $\{g_j\}$ and geodesics $\{\gamma_j\}$ which joins $g_{j-1}$ with $g_j$, such that $g_N \in \text{Im} \varphi$. Let $g_{N-1} = g_N e[tP]$, $P \in W_{gN}$ for a some number of $t$. Then we have:

$$\alpha(e[-(1/2)tP]g_N)(e[-(1/2)tP]g_N)^{-1} = e[-(1/2)td\alpha(Id)(P)]\alpha(g_N)g_N^{-1}e[(1/2)tP]$$

$$= e[(1/2)t\text{Ad}(g_N)P]g_Ne[(1/2)tP] = g_NE[tP] = g_{N-1}.$$  \hspace{1cm} (2.7)

(iii) Finally repeating this procedure for $g_j, 0 \leq j \leq N - 1$ we get $g_0 \in \text{Im} \varphi$. Due to E. Cartan any orbit of a group $G$ can be imbeded in $G$ as a completely geodesic submanifold. For any compact subgroup $\Gamma \subset G$ exists involutory authomorphism $\alpha$, leaving subgroup $\Gamma$ fixed, it follows that all orbits of group $G$ can be realized as completely geodesic manifolds.

3 Strongly virtually negatively curved groups

Let $X_G(B;p)$ be a cofiber (in category of spectra) which in fact is also an $\Omega$-spectrum. In the case when $X$ is a contractible the following result holds \cite{6}:

$$Wh_n(G_M) \otimes \mathbb{Q} \cong \bigoplus_{j=0} \text{H}_j(X; Wh_{n-j}(G_y) \otimes \mathbb{Q})$$,  \hspace{1cm} (3.1)

$$K_n(\mathbb{Z}G_M) \otimes \mathbb{Q} \cong \bigoplus_{j=0} \text{H}_j(X; K_{n-j}(\mathbb{Z}G_y) \otimes \mathbb{Q})$$,  \hspace{1cm} (3.2)

where $\pi_1(M)$ has been denoted by $G_M$, $G_y = \pi_1(p^{-1}(y))$ for $y \in B$, $Wh_n$ is the Whitehead group and $Wh_{n-\ell}(G_y) \otimes \mathbb{Q}$, $K_{n-\ell}(\mathbb{Z}G_y) \otimes \mathbb{Q}$ are the suitable stratified systems of abelian groups over $B$. By definition a group $G_M$ is strongly virtually negatively curved if $G_M$ is isomorphic to $\pi_1(M)$, where $M$ is constructed as before with constraint that $X$ is contractible. Note that any discrete co-compact subgroup $\Gamma$ of a Lie group $G$, where $G$ is either $O(n,1)$, $U(n,1)$, $Sp(n,1)$, or $F_4$, is strongly virtually negatively curved.

Let $B = G/\Gamma$ be an irreducible rank one symmetric space and $\Gamma \subset G$ a maximal compact subgroup. The following statement holds \cite{6}: The algebraic K-theory of any discrete co-compact subgroup of a Lie group $G$, where $G = O(n,1)$, $U(n,1)$, $Sp(n,1)$, $F_4$ can be calculate in terms of the homology of the double coset space $G_M\setminus B \equiv G_M\setminus G/\Gamma$. As a consequence the following result follows:

$$K_n(\mathbb{Z}G_M) \otimes \mathbb{Q} \cong \bigoplus_{j=0} \text{H}_j(G_M\setminus G/\Gamma; p_{n-j})$$,  \hspace{1cm} (3.3)
where \( p_n \) is a stratified system of \( \mathbb{Q} \) vector space over \( G_M \backslash G/\Gamma \) and the vector space \( p_n(G_M \backslash g\Gamma) \) corresponding to the double coset \( G_M \backslash g\Gamma \) is isomorphic to \( K_4(\mathbb{Z}(G_M \cap g\Gamma g^{-1})) \otimes \mathbb{Q} \). In addition \( G_M \cap g\Gamma g^{-1} \) is a finite subgroup of \( G_M \). Any discrete co-compact subgroup of Lie group \( G \), where \( G \) is either \( O(n, 1), U(n, 1), Sp(n, 1) \) or \( F_4 \), is strongly virtually negatively curved. Thus we obtain the additive structure in the homology class of K-cycle and appropriate structural properties for D-branes wrapping spaces stratified fibered over hyperbolic orbifolds.

4 Conclusions

The subject of this paper the intriguing relationship between D-branes on spaces stratified fibered over negatively curved orbifolds and algebraic K-theory. The mathematical methods are based in Atiyah-Hirzebruch type spectral sequence which we used for stratified system of (abelian) groups. By the use of this technique we obtain results in the structure of the K-homology of strongly virtually negatively groups. In summary we present discussion and main results obtained in this paper:

(i) A D-brane wrapping homologically nontrivial cycle can nevertheless be unstable, if for some \( X_{j+k} \subset X \) the following condition holds [3]: \( PD(X_j \subset X_{j+k}) = \omega_k(X_{j+k}) + [H] |_{X_{j+k}} \). In this equation the left hand side denotes the Poincaré dual of \( X_j \) in \( X_{j+k} \). One can use a mathematical algorithm, the Atiyah-Hirzebruch spectral sequence (see Sect. 2), to determine which homology classes lift to K-theory classes (that is, to determine which D-branes are stable and which are not allowed) – it gives the rigorous stability criterion [2].

Suppose stratified system of fibrations on \( X \) extends through the spectral sequence as a non-trivial element of homology groups \( E^2_{p,q} \). The following result is effective for the rational calculation of many K-groups (cf. Theorem 2 of [6]): If as before \( p : M \rightarrow \mathcal{B} \) is a stratified system of fibrations and \( X \) is aspherical with \( Wh_n(\pi_1 X \times S^1) \otimes \mathbb{Q} = 0 \) for all integers \( n \), then \( XG(B;p) \otimes \mathbb{Q} = 0 \). The stratified system of fibrations could be extends through the spectral sequence as a non-trivial element of homology groups but it can have trivial lift to K-homology and it vanishes in \( E^\infty_{p,q} \).

(ii) For each of the stratified systems of fibrations: \( p : M \rightarrow \mathcal{B} \) there is homotopy equivalence of \( \Omega \)-spectra: \( G(M) \cong \mathbb{H}(B; G(p)) \times XG(B;p) \). This equation is in conformity with an Atiyah-Hirzebruch type spectral sequence (2.2). The Atiyah-Hirzebruch spectral sequence keeps track of the possible obstructions for a homology cycle of starting from \( H_p(B; \pi_q(G(p))) \) in (2.2) (the initial terms are given in (2.3)), to survive to \( E^\infty_{p,q} \). For a number of explicit examples, strongly virtually negatively curved \( G_M \), the groups \( G_y = \pi_1(p^{-1}(y)) \) are finite because they are isomorphic to subgroups of \( \Gamma \). As a consequence one can use the extensive knowledge of the algebraic K-theory of finite groups [9, 10].

\[ \text{For example we have the following result. If } G_M \text{ is a strongly virtually negatively curved then } K_n(\mathbb{Z}G_M) = 0, \forall n < -1. \text{ In the case } n = -1, K_{-1}(\mathbb{Z}G_M) \simeq \lim_{G \in \mathcal{F}(G_M)} K_{-1}(\mathbb{Z}G_M), K_{-1}(\mathbb{Z}G_M) \text{ is a finitely generated abelian group; it is generated by the images of } K_{-1}(\mathbb{Z}G_M), \text{ as } G \text{ varies over the finite subgroups of } G_M, \text{ under the map functorially induced by the inclusion of } G \text{ into } G_M. \text{ (Perhaps this statement is true for a much larger class of groups } G_M. \text{ In particular, in [6] it has been conjectured that it is true for} \]
(iii) The homotopy groups $\pi_k G(p^{-1})$ form stratified systems of groups $\pi_k G(p)$ over spaces $X$. D-branes wrapping spaces stratified fibered over hyperbolic orbifolds carry charge structure and change the additive structural properties in K-homology.

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all finitely generated subgroups $G_M$ of $GL_n(\mathbb{C})$. 7