STRONG SOLUTIONS OF STOCHASTIC MODELS FOR VISCOELASTIC FLOWS OF OLDROYD TYPE

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ABSTRACT. In this work we study stochastic Oldroyd type models for viscoelastic fluids in \( \mathbb{R}^d, d = 2,3 \). We show existence and uniqueness of strong local maximal solutions when the initial data are in \( H^s \) for \( s > d/2, d = 2,3 \). Probabilistic estimate of the random time interval for the existence of a local solution is expressed in terms of expected values of the initial data.

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1. INTRODUCTION

Over the past few years, there have been many works devoted to viscoelastic fluids in dimensions two and three. Most of these works are concerned about local existence of strong solutions, global existence of weak solutions, necessary condition for blow-up (in the spirit of well-known Beale-Kato-Majda criterion [4]) and global well-posedness for smooth solutions with small initial data.

In this work, we focus upon the classical Oldroyd type models for viscoelastic fluids (see, Oldroyd [52]) in \( \mathbb{R}^d, d = 2,3 \)

\[
\frac{\partial \bf v}{\partial t} + (\bf v \cdot \nabla)v - \nu \Delta \bf v + \nabla p = \mu_1 \nabla \cdot \tau \quad \text{in} \quad \mathbb{R}^d \times (0,T),
\]

(1.1)

\[
\frac{\partial \tau}{\partial t} + (\bf v \cdot \nabla)\tau + a\tau + Q(\tau, \nabla v) = \mu_2 D(\bf v) \quad \text{in} \quad \mathbb{R}^d \times (0,T),
\]

(1.2)

\[
\nabla \cdot \bf v = 0 \quad \text{in} \quad \mathbb{R}^d \times (0,T),
\]

(1.3)

1991 Mathematics Subject Classification. 60H15; 60H30; 76A05; 76A10; 76D03.
Key words and phrases. Oldroyd Fluid, Maximal strong solution, Lévy noise, Commutator estimates.
Here $v$ is the velocity vector field which is assumed to be divergence free, $\tau$ is the non-Newtonian part of the stress tensor (i.e., $\tau(x, t)$ is a $(d, d)$ symmetric matrix), $p$ is the pressure of the fluid, which is a scalar. The parameters $\nu$ (the viscosity of the fluid), $a$ (the reciprocal of the relaxation time), $\mu_1$ and $\mu_2$ (determined by the dynamical viscosity of the fluid, the retardation time and $a$) are assumed to be non-negative. $\mathcal{D}(v)$ is called the deformation tensor and is the symmetric part of the velocity gradient

$$\mathcal{D}(v) = \frac{1}{2} \nabla v + \nabla^t v.$$ 

$Q$ is a quadratic form in $(\tau, \nabla v)$. As remarked in Chemin and Masmoudi [13], since the equation for the stress tensor should be invariant under coordinate transformation, $Q$ cannot be most general quadratic form, and for Oldroyd fluids one usually chooses

$$Q(\tau, \nabla v) = \tau W(v) - \nu (\tau(v) \tau - b (D(v) \tau + \tau D(v))),$$

where $b \in [-1, 1]$ is a constant and $W(v) = \frac{1}{2} (\nabla v - \nabla^t v)$ is the vorticity tensor, and is the skew-symmetric part of velocity gradient.

There is growing literature devoted to these systems and it is almost impossible to provide a complete review on the topic. We shall restrict ourselves to a few significant works which are relevant to our paper. It is straightforward to observe that the formal $L^2$-energy estimate of the above system (1.1)-(1.4) is the following:

$$\frac{1}{2} \frac{d}{dt} (\mu_2 \|v(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^2}^2 + \mu_1 \|\tau(t)\|_{L^2}^2 + \nu \|\nabla v(t)\|_{L^2}^2 + a \mu_1 \|\tau(t)\|_{L^2}^2 \leq |b| \|\nabla v(t)\|_{L^\infty} \|\tau(t)\|_{L^2}^2.$$ 

Since by the Brezis-Wainger type logarithmic Sobolev inequality, $L^\infty$-norm of gradient of velocity field can be bounded by that of vorticity field for the Sobolev exponent strictly bigger than $d/2 + 1$, the difficulty here arises in getting an $L^\infty$ estimate on the vorticity. Indeed, at first glance it would seem hopeless because the vorticity equation involves a transport term as well as a nonlocal term. However, one needs to perform a losing estimate (see, Chemin-Masmoudi [13]) for the transport equation satisfied by $\tau$ that allow us to obtain a Beale-Kato-Majda ([4]) type sufficient condition of non-breakdown.

Due to the parabolic-hyperbolic coupling and the special structure of $Q$, the corresponding stationary problem is also interesting and was studied by Renardy [57]. The existence and uniqueness of local strong solutions in $H^m$ ($m$ integer) have been established by Guilloté and Saut [20]. Further, these solutions are global if the coupling between the two equations is weak as well as the initial data are small Guilloté and Saut [20]. Existence of $L^s - L^r$ solutions has been treated by Fernandez Cara, Guiñán, and Ortega [17]. Global existence of weak solutions has been shown in the corotational case (with $b = 0$) by Lions and Masmoudi [41]. Some recent works have been devoted to the proof of global well-posedness in the case of small data (e.g. see Masmoudi et al. [13, 37], Lin et al. [39], Lei et al. [35, 36]).

Let us mention the connection between the deterministic system under consideration in this work and certain other systems showing a critical coupling. A lot of works have been devoted recently to the study of two and three dimensional Boussinesq system and magnetohydrodynamics (MHD) system with partial dissipation and Ericksen-Leslie nematic liquid crystal model. In particular, a critical coupling for Boussinesq system has been studied in Hmidi et al. [22, 25], Manna and Panda [47], for MHD system in Caflisch et al. [11], for liquid crystal model in Lin and Liu [30], to name a few. The coupling in the Boussinesq system is simpler than the one in MHD system (or in liquid crystal model or viscoelastic fluid of Oldroyd type considered in this work) in the sense that the vorticity equation is forced by the gradient of the temperature but then the temperature solves an unforced convection-diffusion equation. Consequently, in the case of critical coupling, if one can find a combination of the vorticity and the temperature that has better regularity properties, it is rather easy to deduce an estimate on each individual quantity Elgindi and Rousset [15]. This is not the case for the MHD system, the liquid crystal model or for the Oldroyd model (even if $Q = 0$) since they are strongly coupled. Moreover, due to the special structure of $Q$, the Oldroyd model under
consideration possesses additional difficulty and it is evident from the lack of $L^\infty$-norm of the gradient of velocity field.

Literatures related to analysis of the above critical coupled systems perturbed by random forcing are very limited and quite recent (e.g. see Yamazaki [61] for Boussinesq system with zero dissipation, Manna et al. [46] for non-resistive MHD system, Brzeźniak et al. [8] for liquid crystal model). To the best of author’s knowledge, there is no literature available on the random perturbation of the general non-linear viscoelastic fluid of Oldroyd type (1.1)-(1.4). However, in Barbu et al. [3] and Razafimandimby [56], existence and asymptotic behaviour of a linear viscoelastic fluid equation driven by additive or multiplicative Wiener stochastic processes are studied. This equation is an integro-differential equation (of Volterra type) consisting of the Navier-Stokes equation and a hereditary (or memory) term as the integral of a linear kernel, but doesn’t possess any critical nonlinear coupling as in (1.1)-(1.2). Therefore, the standard techniques of the stochastic Navier-Stokes equation can be borrowed to establish the well-posedness and regularity of solutions.

In this context, we should make a note that the Oldroyd type viscoelastic fluid considered in this paper is purely a macroscopic model. Usually at the macroscopic level, in the case of non-Newtonian fluids such as polymeric fluids, such an equation links the stress tensor to the velocity field either through a partial differential equation (as in equations (1.1)-(1.2)) or through an integral relation (e.g. in linear Oldroyd model [3], [56]). Recently some works have been devoted for the understanding of fluid behaviour both in the macroscopic and microscopic regimes. In order to build a micro-macro model, one needs to go down to the microscopic scale and make use of kinetic theory to obtain a mathematical model for the evolution of the microstructures of the fluid (e.g. configurations of the polymer chains in the case of polymeric fluid). In mathematical terms, this micro-macro approach translates into a coupled multiscale system (simplest example of such a model is the dumbbell model) in which the polymers are modelled as dumbbells each of which consists of two beads connected by a spring (see [27] and [53] for detailed introduction on the subject). In Jourdain et al. [28], the authors analyse a stochastic finite extensible nonlinear elastic dumbbell model, and prove a local-in-time existence and uniqueness result. This work has been further extended in Jourdain et al. [29], where long-time behavior of the solution in various settings (shear flow, general bounded domain with homogeneous Dirichlet boundary conditions on the velocity, general bounded domain with non-homogeneous Dirichlet boundary conditions on the velocity) have been shown. There is also a recent trend in the community of researchers performing numerical simulations of such complex flows, where stochastic micro-macro models are considered as a numerical tool for simulating the dynamic behavior of polymeric fluids (see, [27], [53]).

In the present work, we are interested in the mathematical analysis of a stochastic version of (1.1)-(1.4). Our work is motivated by the importance of external perturbation on the dynamics of the velocity field for fluids with memory. The viscoelastic property demands that the material must return to its original shape after any deforming external force has been removed (i.e., it will show an elastic response) even though it may take time to do so. Hence the equation modelling stress tensor (i.e. the equation for $\tau$) is invariant under coordinate transformation. Therefore the question of how to incorporate a suitable perturbation modelling the stress tensor without destroying its invariance property is a delicate one.

Hence for a full understanding of the effect of fluctuating forcing field on the behaviour of the viscoelastic fluids, one needs to take into account the dynamics of $\mathbf{v}$ and $\tau$. To initiate this kind of investigation we propose a mathematical study of the following system of equations which basically describes an approximation of the system governing the viscoelastic fluids under the influence of fluctuating external forces.

\begin{align}
\frac{d\mathbf{v}(t)}{dt} + [(\mathbf{v}(t) \cdot \nabla)\mathbf{v}(t) - \nu \Delta \mathbf{v}(t) + \nabla p]dt &= \mu_1 \nabla \cdot \tau(t) + \sigma(t, \mathbf{v}(t))dW_1(t) + \int_Z G(\mathbf{v}(t-), z)\tilde{N}_1(dt, dz), \\
\frac{d\tau(t)}{dt} + [(\mathbf{v}(t) \cdot \nabla)\tau(t) + a\tau(t) + Q(\tau(t), \nabla \mathbf{v}(t))]dt &= \mu_2 D(\mathbf{v}(t))dt + (h \otimes \tau(t)) \circ dW_2(t), \\
\nabla \cdot \mathbf{v} &= 0, \\
\mathbf{v}(0, \cdot) &= \mathbf{v}_0, \quad \tau(0, \cdot) = \tau_0.
\end{align}
where \( W_1 \) is a Hilbert space valued Wiener process with nuclear operator \( Q_1, \circ dW_2(t) \) stands for the Stratonovich differential where \( W_2 \) is a real-valued Wiener processes, \( N \) is a compensated Poisson random measure and \( h \) is a bounded function. The tensor product \( h \otimes \tau(t) \) denotes usual matrix multiplication. Detailed and precise descriptions of the model are provided in the subsequent Sections.

In this paper, we study the existence and uniqueness of local (maximal) strong solutions to the incompressible, viscoelastic fluids of Oldroyd type \((1.5)-(1.8)\) in both two and three dimensions. The analysis here is significantly different from the classical one for stochastic evolution equations due to the lack of diffusion in the \( \tau \) equation \((1.0)\) and structure of \( Q \). To be a little more precise, one of the key difficulties in proving local existence with diffusion only in the \( v \) equation stems from the nonlinear terms. Since \( H^s \) is an algebra for \( s > d/2 \), so one obtains

\[
|\langle (\mathbf{v} \cdot \nabla)\tau, \varphi \rangle|_{H^s} \leq \|\mathbf{v}\|_{H^{s/2}} \|\nabla \tau\|_{H^{s/2}} \|\varphi\|_{H^s}.
\]

Thus we must estimate \( \|\nabla \tau\|_{H^s} \), and if we start with \( \tau_0 \in H^s \) we do not have any control over the \( H^s \) norm of \( \nabla \tau \) because there is no smoothing for \( \tau \). Due to the same reasons, the semigroup method to mild solutions may not work in this case and also the local \( m \)-accretivity property is not available due to the absence of a diffusive term. To the best of the authors knowledge, this work appears to be the first systematic treatment for the existence and uniqueness of the local/maximal strong solution of the stochastic viscoelastic fluid of Oldroyd type in its most general form. Global well-posedness for smooth solutions with small initial data of the viscoelastic fluids under the influence of fluctuating external forces will be addressed by the authors in near future.

The organization of the present article is as follows. In the Section 2, we introduce some notation used throughout this paper and certain known but useful results concerning fractional order Sobolev spaces, commutator estimates and stochastic analysis. After stating the hypotheses on the random noise coefficients in Section 3, we provide a full statement of the main result of this work. In the very Subsection, we also briefly outline the strategy of the proof of the main result. In Section 4, we consider an approximate system and establish an energy estimate in \( H^s, s > d/2 \). We also provide a probabilistic estimate of the stopping time. Section 5 is devoted in proving the (strong) convergence of the approximate solutions. The existence and uniqueness of local strong solution is provided in Section 6, and that of local maximal solution is proved in Section 7. In the Appendix we prove several results which are used to establish the strong convergence of the approximate solutions in Section 5.

2. Preliminaries

2.1. Notations. Throughout the paper we use the same notation \( H^s(\mathbb{R}^d) \) for both vector-valued and tensor valued functions. For notational convenience we define

\[
H^s(\mathbb{R}^d) := H^s(\mathbb{R}^d, \mathbb{R}^d) = (H^s(\mathbb{R}^d))^d \quad \text{and} \quad \mathbb{H}^s(\mathbb{R}^d) := H^s(\mathbb{R}^d; \mathbb{R}^{d \times d}).
\]

Likewise, \( L^2(\mathbb{R}^d) \) is used for both vector-valued and tensor valued functions.

2.2. Fractional Order Sobolev Spaces. For \( s \in \mathbb{R} \), let \( J^s \) denote the Bessel potential of order \( s \) which is equivalent to the operator \((I - \Delta)^{s/2}\), where \( \Delta \) is the Laplace operator, and is defined via the Fourier transform \( \mathcal{F} \) as follows

\[
\mathcal{F} [J^s f] (\xi) = (1 + |\xi|^2)^{s/2} \hat{f}(\xi).
\]

The inner product on \( H^s(\mathbb{R}^d) \) is given by

\[
(f, g)_{H^s} = \left( (1 + |\xi|^2)^{s/2} \hat{f}(\xi), (1 + |\xi|^2)^{s/2} \hat{g}(\xi) \right)_{L^2} = (\mathcal{F} [J^s f] (\xi), \mathcal{F} [J^s g] (\xi))_{L^2} = (J^s f, J^s g)_{L^2},
\]

and the norm on \( H^s(\mathbb{R}^d) \) is defined by

\[
\|f\|_{H^s} = \left( \int_{\mathbb{R}^d} \left( 1 + |\xi|^2 \right)^{s/2} |\hat{f}(\xi)|^2 \right)^{1/2} = \left( (1 + |\xi|^2)^{s/2} \hat{f}(\xi) \right)_{L^2} = \|J^s f\|_{L^2}, \quad (2.1)
\]
Remark 1. If \( s > d/2 \), then each \( f \in H^s(\mathbb{R}^d) \) is bounded and continuous and hence

\[
\|f\|_{L_\infty(\mathbb{R}^d)} \leq C \|f\|_{H^s(\mathbb{R}^d)}, \text{ for } s > d/2.
\]

Also, note that \( H^s \) is an algebra for \( s > d/2 \), i.e., if \( f, g \in H^s(\mathbb{R}^d) \), then \( fg \in H^s(\mathbb{R}^d) \), for \( s > d/2 \).

Hence, we have

\[
\|fg\|_{H^s} \leq C \|f\|_{H^s} \|g\|_{H^s}, \text{ for } s > d/2.
\]

**Remark 2.** Fix \( s > d/2 \) and let \( f, g \in H^s \) with \( \nabla \cdot f = 0 \). Then

\[
\|(f \cdot \nabla)g\|_{H^{s-1}} \leq C \|f\|_{H^s} \|g\|_{H^s}.
\]

**Proof.** Now \( f \) is divergence free, \( (f \cdot \nabla)g = \nabla \cdot (f \otimes g) \). And \( H^s \) is an algebra for \( s > d/2 \),

\[
\|(f \cdot \nabla)g\|_{H^{s-1}} = \|\nabla \cdot (f \otimes g)\|_{H^{s-1}} \leq C \|f \otimes g\|_{H^s} \leq C \|f\|_{H^s} \|g\|_{H^s}.
\]

Now we state a more generalised Lemma of Theorem 2.4.5 of Kesavan [31].

**Lemma 2.1.** (Sobolev Inequality) For \( f \in H^s(\mathbb{R}^d) \), we have

\[
\|f\|_{L^q(\mathbb{R}^d)} \leq C_{d,s,q} \|f\|_{H^s(\mathbb{R}^d)}
\]

provided that \( q \) lies in the following range

(i) if \( s < d/2 \), then \( 2 \leq q \leq \frac{2d}{d-2s} \),

(ii) if \( s = d/2 \), then \( 2 \leq q < \infty \),

(iii) if \( s > d/2 \), then \( 2 \leq q \leq \infty \).

**Remark 3.** We deduce the following result using Lemma 2.1 and this estimate will be useful in several calculations. In two dimensions, we exploit Hölder’s inequality with exponents \( 2/\epsilon \) and \( 2/(1-\epsilon) \), and Sobolev inequality for \( 0 < \epsilon < s - 1 \) to obtain

\[
\|fg\|_{L^2} \leq \|f\|_{L^{2/\epsilon}} \|g\|_{L^{2/(1-\epsilon)}} \leq C \|f\|_{H^{1-\epsilon}} \|g\|_{H^\epsilon} \leq C \|f\|_{H^1} \|g\|_{H^{1-\epsilon}}.
\]

In three dimensions, we again exploit Hölder’s inequality with exponents 6 and 3, and Sobolev inequality to obtain

\[
\|fg\|_{L^2} \leq \|f\|_{L^6} \|g\|_{L^3} \leq C \|f\|_{H^1} \|g\|_{H^{1/2}} \leq C \|f\|_{H^1} \|g\|_{H^{1/2}}.
\]

Note that, for both the two and three dimensions, we obtain the same bounds.

**Lemma 2.2.** (Interpolation in Sobolev spaces). Given \( s > 0 \), there exists a constant \( C \) depending on \( s \), so that for all \( f \in H^s(\mathbb{R}^d) \) and \( 0 < s' < s \),

\[
\|f\|_{H^{s'}} \leq C \|f\|_{L^2}^{1-s'/s} \|f\|_{H^s}^{s'/s}.
\]

For details see Theorem 9.6, Remark 9.1 of Lions and Magenes [41].

2.3. **Fourier Truncation Operator.** Let us define the Fourier truncation \( \mathcal{J}_n \) as follows:

\[
\mathcal{J}_n f(\xi) = 1_{B(0,n)}(\xi) \hat{f}(\xi),
\]

where \( B(0,n) \), a ball of radius \( n \) centered at the origin and \( 1_{B(0,n)} \) is the indicator function. We list the following properties of \( \mathcal{J}_n \) [see Chemin 12, Fefferman et. al. 14, Manna et al. 16].

1. \( \|\mathcal{J}_n f\|_{H^s(\mathbb{R}^d)} \leq \|f\|_{H^s(\mathbb{R}^d)} \). \hspace{1cm} (2.2)

2. \( \|\mathcal{J}_n f - f\|_{H^s} \leq c \left( \frac{1}{n} \right)^k \|f\|_{H^{s+k}} \). \hspace{1cm} (2.3)

3. \( \|(\mathcal{J}_n - \mathcal{J}_m) f\|_{H^s(\mathbb{R}^d)} \leq \max \left\{ \left( \frac{1}{n} \right)^k, \left( \frac{1}{m} \right)^k \right\} \|f\|_{H^{s+k}(\mathbb{R}^d)}. \hspace{1cm} (2.4)\)
2.4. Commutator Estimates. Let us now state the celebrated commutator estimate due to Kato and Ponce [30] (Lemma XI).

**Lemma 2.3.** If $s > 0$ and $1 < p < \infty$, then

$$
\|J^s (fg) - f(J^s g)\|_{L^p} \leq C_p \left( \|\nabla f\|_{L^\infty} \|J^{s-1}g\|_{L^p} + \|J^s f\|_{L^p} \|g\|_{L^\infty} \right).
$$

(2.5)

**Remark 4.** From (2.5), it can be easily seen that for $s > 0$ and $1 < p < \infty$, the nonlinear term satisfy the estimate:

$$
\|J^s [(f \cdot \nabla)g] - (f \cdot \nabla)(J^s g)\|_{L^p} \leq C_p \left( \|\nabla f\|_{L^\infty} \|J^{s-1}g\|_{L^p} + \|J^s f\|_{L^p} \|g\|_{L^\infty} \right).
$$

(2.6)

Assuming the fact that $f$ is divergence free, we have $((f \cdot \nabla)(J^s g), J^s g)_{L^2} = 0$. Hence for $p = 2$, we have the following estimate.

**Corollary 2.4.** For $s > 0$, there exists a constant $c = c(d, s)$ such that, for all $f, g \in H^s(\mathbb{R}^d)$ and $\nabla \cdot f = 0$, we have

$$
\|(J^s (f \cdot \nabla)g), J^s g\|_{L^2} \leq c (\|\nabla f\|_{L^\infty} \|g\|_{H^s} + \|f\|_{H^s} \|\nabla g\|_{L^\infty} ) \|g\|_{H^s}.
$$

(2.7)

**Remark 5.** For $s > d/2 + 1$, we have $\|\nabla f\|_{L^\infty} \leq c\|\nabla f\|_{H^{s-1}}$ and hence we get

$$
\|(J^s [(f \cdot \nabla)g], J^s g)_{L^2} \leq c (\|\nabla f\|_{H^{-1}} \|g\|_{H^s} + \|f\|_{H^s} \|\nabla g\|_{H^{-1}} ) \|g\|_{H^s} \\
\leq c\|f\|_{H^s} \|g\|_{H^s}^2.
$$

(2.8)

The next result is a partial generalization of the commutator estimates of Kato and Ponce [30] given in Theorem 1.2 of Fefferman et al. [10].

**Theorem 2.5.** Given $s > d/2$, there is a constant $c = c(d, s)$ such that, for all $f, g$ with $\nabla f, g \in H^s(\mathbb{R}^d)$ such that $\nabla \cdot f = 0$, we have

$$
\|J^s [(f \cdot \nabla)g] - (f \cdot \nabla)(J^s g)\|_{L^2} \leq c\|\nabla f\|_{H^s} \|g\|_{H^s}.
$$

(2.9)

**Corollary 2.6** (Corollary 2.1, Fefferman et al. [10]). Given $s > d/2$, there is a constant $c = c(s, d)$ such that, for all $f, g$ with $\nabla f, g \in H^s(\mathbb{R}^d)$ and $\nabla \cdot f = 0$, we have

$$
\|(J^s [(f \cdot \nabla)g], J^s g)_{L^2} \leq c\|\nabla f\|_{H^s} \|g\|_{H^s}^2.
$$

(2.10)

**Lemma 2.7.** Variants of Commutator estimates: Let $s > 0$ and $1 < p < \infty$ and $p_2, p_3 (1, \infty)$ be such that

$$
\frac{1}{p} \geq \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{p} \geq \frac{1}{p_3} + \frac{1}{p_4}.
$$

Then

$$
\|J^s (fg)\|_{L^p} \leq C (\|f\|_{L^{p_1}} \|J^s g\|_{L^{p_2}} + \|J^s f\|_{L^{p_3}} \|g\|_{L^{p_4}}).
$$

(2.11)

For details, see Lemma 2.6 of Bessaib and Ferrario [5].

**Property 2.8.** As a consequence of the commutator estimate [275], the bilinear map $Q$ satisfies the following (tame) estimate,

$$
\|Q(\tau, \nabla v)\|_{H^s} \leq C (\|\tau\|_{L^\infty} \|\nabla v\|_{H^s} + \|\nabla v\|_{L^\infty} \|\tau\|_{H^s}).
$$

2.5. Basic Stochastic Analysis. In this Subsection we are going to introduce some definitions and properties of the Hilbert space valued stochastic processes. For further details, one can refer Métrier [19], Da Prato and Zabczyk [14], Gawarécki and Mandrekar [18].

Let $U$ and $H$ be two separable Hilbert spaces. A nonnegative operator $Q \in \mathcal{L}(U, U)$ is of trace class if and only if for an orthonormal basis $\{e_j\}$ on $U$,

$$
\sum_{j=1}^{\infty} \langle Qe_j, e_j \rangle_U < \infty.
$$

A bounded linear operator $Q : U \to H$ is said to be Hilbert-Schmidt if $\sum_{k=1}^{\infty} \|Qe_k\|_H^2 < \infty$. The set $\mathcal{L}_2(U, H)$ of all Hilbert-Schmidt operators from $U$ into $H$, equipped with the norm $\|Q\|_{\mathcal{L}_2(U, H)} =$
\[
\left( \sum_{k=1}^{\infty} \|Q e_k\|_H^2 \right)^{\frac{1}{2}}
\] is a separable Hilbert space.

Further assume that \( Q \) is symmetric, positive, trace class operator on \( U \).

Let \( \mathcal{L}_Q(U, H) = \mathcal{L}_2(U_0, H) \) denote the space of all Hilbert-Schmidt operator from \( U_0 \) to \( H \) where \( U_0 = Q^{\frac{1}{2}} U \).

Let \( M \) be the totality of non-negative (possibly infinite) integral valued measures on \( (H, \mathcal{B}(H)) \) and \( \mathcal{B}_M \) be the smallest \( \sigma \)-field on \( M \) with respect to which all \( N \in M \to N(B) \in \mathbb{Z}^+ \cup \{\infty\}, B \in \mathcal{B}(H) \), are measurable.

**Definition 2.1.** An \((M, \mathcal{B}(M))\)-valued random variable \( N \) is called a Poisson random measure

1. if for each \( B \in \mathcal{B}(H) \), \( N(B) \) is Poisson distributed. i.e., \( \mathbb{P}(N(B) = n) = \frac{\eta(B)e^{-\eta(B)}}{n!}, n = 0, 1, 2, \ldots \), where \( \eta(B) = \mathbb{E}(N(B)), B \in \mathcal{B}(H) \);
2. if \( B_1, B_2, \ldots, B_n \in \mathcal{B}(H) \) are disjoint, then \( N(B_1), N(B_2), \ldots, N(B_n) \) are mutually independent.

**Definition 2.2.** A càdlàg process \((X_t)_{t \geq 0}\) is a stochastic process for which the paths \( t \mapsto X(t) \) are right continuous with left limits everywhere with probability one. \((X_t)_{t \geq 0}\), is called a Lévy process if it has stationary independent increments and is stochastically continuous.

For a \( H \)-valued Lévy process \((X_t)_{t \geq 0}\), let us define

\[
N(t, Z) = N(t, Z, \omega) = \# \{ s \in (0, \infty) : \Delta X_s(\omega) \in Z, t > 0, Z \in \mathcal{H}(H \setminus \{0\}), \omega \in \Omega \}
\]

as the Poisson random measure associated with the Lévy process where \( \Delta X_t(\omega) = X_t(\omega) - X_{t-}(\omega) \) denotes the corresponding jump for every \( \omega \in \Omega \).

The differential form of the measure \( N(t, Z, \omega) \) is written as \( N(dt, dz)(\omega) \). We call \( \tilde{N}(dt, dz) = N(dt, dz) - \lambda(dz)dt \) a compensated Poisson random measure \((\text{cPrm})\), where \( \lambda(dz)dt \) is known as compensator of the Lévy process \((X_t)_{t \geq 0}\). Here \( dt \) denotes the Lebesgue measure on \( \mathcal{B}(\mathbb{R}^+) \), and \( \lambda(dz) \) is a \( \sigma \)-finite Lévy measure on \((Z, \mathcal{B}(Z))\).

**Definition 2.3** (see Mandrekar and Rüdiger [11]). Let \( H \) and \( F \) be separable Hilbert spaces. Let \( F_t := \mathcal{B}(H) \otimes \mathcal{F}_t \) be the product \( \sigma \)-algebra generated by the semi-ring \( \mathcal{B}(H) \times \mathcal{F}_t \) of the product sets \( Z \times F, Z \in \mathcal{B}(H), F \in \mathcal{F}_t \) (where \( \mathcal{F}_t \) is the filtration of the additive process \((L_t)_{t \geq 0}\)). Let \( T > 0 \), define

\[
\mathbb{H}(Z) = \{ g : \mathbb{R}^+ \times Z \to \Omega \to F, \text{ such that } g \text{ is } F_T / \mathcal{B}(F) \text{ measurable and } g(t, z, \omega) \text{ is } \mathcal{F}_t \text{- adapted } \forall z \in Z, \forall t \in (0, T] \}.
\]

For \( p \geq 1 \), let us define,

\[
\mathbb{E}_N^p([0, T] \times Z; F) = \left\{ g \in \mathbb{H}(Z) : \int_0^T \int_Z \mathbb{E}[\|g(t, z, \omega)\|_F^p] \lambda(dz)dt < \infty \right\}.
\]

Let us denote \( D([0, T]; H) \) as the space of all càdlàg paths from \([0, T]\) into Hilbert space \( H \). The space \( D([0, T]; H) \) is endowed with the Skorokhod \( J \)-topology. For more details see Chapter 2, Metivier [12] and Chapter 3, Billingsley [13].

**Definition 2.4** (Quadratic variation process and Meyer process). Let \( M \) be a square integrable martingale with right continuous paths with values in a separable Hilbert space \( H \) on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\). Then there exists two real right continuous increasing processes \([M]\) and \( \langle M \rangle \) with \( 0 = [M]_0 = \langle M \rangle_0 \) such that

\[
\|M\|_H^2 = \|M_0\|_H^2 + 2 \int_0^t (M_{s-}, dM_s)_H + [M]_t. \tag{2.12}
\]

\( \langle M \rangle \) is the unique real right continuous increasing predictable process such that

\[
\|M\|_H^2 - \|M_0\|_H^2 - \langle M \rangle_t \text{ is a martingale.} \tag{2.13}
\]
Here $|M|$ is called the quadratic variation of $M$ and $\langle M \rangle$ the Meyer process of $M$.

**Remark 6.** If $M$ is continuous, then we have $\langle M \rangle = |M|$.

**Remark 7.** [Section 2.3, Métivier [19]] Let $H$ be Hilbert spaces and let $Q : H \to H$ be a trace class operator. Let $F_i$ for each $i$ where $W$ is a $H$-valued Wiener process, $\sigma(\cdot, \cdot) : [0, t] \times H \to L^2(H, H)$, $Z$ is a measurable subspace of $H$, $g(\cdot, \cdot) : H \times Z \to H$ and $\tilde{N}(\cdot, \cdot)$ is the compensated Poisson random measure. Then,

$$
E \left[ \int_0^t \|\sigma(s, u)\|^2_{L^2(H, H)} ds + \int_0^t \int_Z \|g(u, z)\|^2_H N(ds, dz) \right] = E \left[ \int_0^t \|\sigma(s, u)\|^2_{L^2(H, H)} ds + \int_0^t \int_Z \|g(u, z)\|^2_H \lambda(dz) ds \right].
$$

(2.14)

**Lemma 2.9.** (Burkholder-Davis-Gundy Inequality) Let $M$ be a Hilbert space valued càdlàg martingale with $M_0 = 0$ and let $p \geq 1$ be fixed. Then for any $\mathcal{F}$-stopping time $\tau$, there exists constants $c_p$ and $C_p$ such that

$$
E \left\{ |M|_{\tau}^{p/2} \right\} \leq c_p E \left\{ \sup_{0 \leq t \leq \tau} \|M_t]\|_{H}^p \right\} \leq C_p E \left\{ |M|_{\tau}^{p/2} \right\}
$$

for all $0 \leq \tau \leq \infty$, where $[M]$ is the quadratic variation of process $M$. The constants are universal (independent of $M$).

For proof see Theorem 1.1 of Marinelli and Röckner [50]. For real-valued càdlàg martingales see Theorem 3.50 of Peszat and Zabczyk [55].

### 3. The Stochastic Model and Statement of the Main Results

Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a filtered probability space, where $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ is the filtration, and this probability space satisfies the so-called usual conditions, i.e.

1. $\mathbb{P}$ is complete on $(\Omega, \mathcal{F})$,
2. for each $t \geq 0$, $\mathcal{F}_t$ contains all $(\mathcal{F}, \mathbb{P})$-null sets,
3. the filtration $\mathcal{F}$ is right-continuous.

We consider the following stochastic viscoelastic equations [131]-[134]:

\[
\begin{align*}
&dv(t) + ((v(t) \cdot \nabla)v(t) - \nu \Delta v(t) + \nabla p)dt = \mu_1 \nabla \cdot \tau(t) dt + \sigma(t, v(t))dW_1(t) + \int_Z G(v(t-), z)\tilde{N}_1(dt, dz), \\
&d\tau(t) + ((v(t) \cdot \nabla)\tau(t) + a\tau(t) + Q(\tau(t), \nabla v(t)))dt = \mu_2 D(v(t))dt + (h \otimes \tau(t)) \circ dW_2(t), \\
&\nabla \cdot v = 0, \\
&v(0, \cdot) = v_0, \quad \tau(0, \cdot) = \tau_0,
\end{align*}
\]

(3.1) \quad (3.2) \quad (3.3) \quad (3.4)

where $W_1$ is $H^s$-valued Wiener process with nuclear operator $Q_1$ and $(h \otimes \tau(t)) \circ dW_2(t)$ is understood in Stratonovich sense and $W_2 = (W_2(t))_{t \geq 0}$ is a real-valued Wiener processes on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. $h$ is an element of $L^\infty(\mathbb{R}^{d \times d})$. The tensor product $h \otimes \tau(t)$ denotes usual matrix multiplication, i.e., $(h \otimes \tau(t))_{i,j} = \sum_{k=1}^d h_{ik} \tau_{kj}(t)$ ; $i,j = 1,2,\cdots,d$. $\tilde{N}(dt, dz)$ is a compensated Poisson random measure. Further it is assumed that $W_1, W_2$ and $\tilde{N}$ mutually independent processes.

Define a linear operator $S$ from $L^2$ into itself by $S(\tau) = h \otimes \tau$. Note that $S$ is bounded and satisfies $\|S(\tau)\|_{L^2} = \|h \otimes \tau\|_{L^2} \leq \|h\|_{L^\infty} \|\tau\|_{L^2}$.

Using the relation between Stratonovich and Itô differentials (e.g. see Kuo [33], page 122 ) one has

$$
S(\tau) \circ dW_2 = \frac{1}{2} S^2(\tau) dt + S(\tau) dW_2,
$$


where \(S^2(\tau) = S \circ S(\tau) = S \circ (h \otimes \tau) = h \otimes (h \otimes \tau)\), for any \(\tau \in L^2\) and \(\circ\) is usual composition notation. Further note that
\[
\|S^2(\tau)\|_{L^2} \leq \|h\|_{L^\infty} \|h \otimes \tau\|_{L^2} \leq \|h\|^2_{L^\infty} \|\tau\|_{L^2}.
\] (3.5)

**Remark 8.** For \(s > d/2\), let \(h\) be an element of \(H^s(\mathbb{R}^{d \times d}) \subset L^\infty(\mathbb{R}^{d \times d})\), and we extend the definition of \(S\) from \(H^s\) into itself by \(S(\tau) = h \otimes \tau\) and \(\|S(\tau)\|_{H^s} \leq \|h\|_{H^s} \|\tau\|_{H^s}\) and \(\|S^2(\tau)\|_{H^s} \leq \|h\|^2_{H^s} \|\tau\|_{H^s}\).

### 3.1. Analysis of the Noise Terms

Let us assume the following properties of noise co-efficients \(\sigma\) and \(G\) namely joint continuity, linear growth and Lipschitz joint continuity.

**Assumption 3.1.** For all \(s \geq 0\), the noise co-efficient \(\sigma\) and \(G\) satisfy

- (A.1) The function \(\sigma \in C([0, T] \times H^s; L^2(\Omega; H^s))\) and \(G \in H^2([0, T] \times \mathbb{Z}; H^s(\mathbb{R}^d))\).

- (A.2) (Growth Condition) For all \(v \in H^s(\mathbb{R}^d)\) and for all \(t \in [0, T]\), there exist positive constant \(K\) such that
\[
\|\sigma(t, v)\|_{L^2(\Omega; H^s)}^2 + \int_\Omega |G(v, \omega)|^2 \lambda(d\omega) \leq K (1 + \|v\|_{H^s}^2).
\]

- (A.3) (Lipschitz Condition) For all \(t \in [0, T]\) and for all \(v_1, v_2, \in H^s(\mathbb{R}^d)\), there exist positive constant \(L\) such that
\[
\|\sigma(t, v_1) - \sigma(t, v_2)\|_{L^2(\Omega; H^s)}^2 + \int_\Omega |G(v_1, \omega) - G(v_2, \omega)|^2 \lambda(d\omega)
\leq L \|v_1 - v_2\|_{H^s}^2.
\]

**Remark 9.** For \(m > n\), by using \(\|\cdot\|_{L^2(\Omega; H^s)}\), one can note that for any \(l \geq 0\) and \(\epsilon > 0\),
\[
\|(J_n - J_m)\sigma(t, v)\|_{L^2(\Omega; H^s)}^2 = \sum_{j=1}^\infty \lambda_j \|(J_n - J_m)\sigma(t, v)\|_{H^l}^2
\leq C \sum_{j=1}^\infty \lambda_j \|\sigma(t, v)\|_{H^l}^2 = \frac{C}{n^l} \|\sigma(t, v)\|_{L^2(\Omega; H^{l+n})}^2.
\] (3.6)

We now introduce the concept of local strong solution and maximal local strong solution of \([3.1]-[3.4]\). Throughout we will assume that \(T\) is a fixed positive number.

**Definition 3.1 (Local Strong Solution).** We say that the triplet \((v, \tau, \rho_{\infty})\) is a local strong (pathwise) solution to \(\text{(3.1)-(3.4)}\) if

(i) the symbol \(\rho_{\infty}\) is a stopping time such that \(\rho_{\infty} \leq T\) a.s., and there exists a non-decreasing sequence \(\{\rho_N, N \in \mathbb{N}\}\) of stopping times with \(\rho_N \uparrow \rho_{\infty}\) a. s. as \(N \uparrow \infty\),

(ii) for \(s > d/2\) and \(t \in [0, \rho_{\infty})\), the symbols \(v\) and \(\tau\) denote progressively measurable stochastic processes such that
\[v \in L^2(\Omega; D(0, t; H^s(\mathbb{R}^d))) \cap L^2(0, t; H^{s+1}(\mathbb{R}^d)))\]
and \(\tau \in L^2(\Omega; C(0, t; H^s(\mathbb{R}^d)))\).

Moreover, for any \(t \in [0, T]\), \(N \in \mathbb{N}\), and for any \(\phi_i \in H^s(\mathbb{R}^d)\); \(i = 1, 2\) with \(\nabla \cdot \phi_1 = 0\), \(v\) and \(\tau\) satisfy the following equations with probability 1:
\[
(J^s v(t \wedge \rho_N), J^s \phi_1)_{L^2} = (J^s v_0, J^s \phi_1)_{L^2} + \int_0^{t \wedge \rho_N} (\nu \Delta J^s v - J^s (\nabla^2 v) \cdot \nabla) + \mu v \cdot J^s \tau, J^s \phi_1)_{L^2} ds
+ \int_0^{t \wedge \rho_N} (J^s \sigma(s, v(s))dW_1(s), J^s \phi_1)_{L^2}
+ \int_0^{t \wedge \rho_N} \int_\mathbb{Z} (J^s G(v(s^-), z), J^s \phi_1)_{L^2} d\tilde{N}_i(ds, dz),
\] (3.7)
\[
(J^s \tau(t \wedge \rho_N), J^s \phi_2)_{L^2} = (J^s \tau_0, J^s \phi_2)_{L^2} + \mu_2 \int_0^{t \wedge \rho_N} (J^s D(v), J^s \phi_2)_{L^2} ds
\]
Definition 3.2 (Maximal Local Strong Solution). Let \((v, \tau, \rho_\infty)\) be a local solution to (3.1)-(3.2) such that
\[
\sup_{0 \leq s \leq \rho_\infty} \|v(s)\|_{H^r}^2 + \sup_{0 \leq s \leq \rho_\infty} \|\tau(s)\|_{H^r}^2 + \int_0^{\rho_\infty} \|\nabla v(s)\|_{H^r}^2 \, ds = \infty,
\]
on the set \(\{\omega : \rho_\infty(\omega) \leq T\}\), then the local process \((v, \tau, \rho_\infty)\) is called a maximal local solution. If \(\rho_\infty < T\), the stopping time \(\rho_\infty\) is called the explosion time of the stochastic processes \((v, \tau)\).

A maximal local solution \((v^1, \tau^1, \rho^1_\infty)\) is said to be unique if for any other maximal local solution \((v^2, \tau^2, \rho^2_\infty)\), we have \(\rho^1_\infty = \rho^2_\infty\) and \(v^1(t) = v^2(t), \tau^1(t) = \tau^2(t)\) for any \(0 \leq t \leq \rho^1_\infty\) with probability 1.

3.2. Statement of the Main Result. The main results of this work are stated below and are proven in the subsections below.

Main Result 1. Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) be a given filtered probability space and let \(v_0, \tau_0 \in L^2(\Omega; H^s(\mathbb{R}^d))\) with \(\nabla \cdot v_0 = 0\), \(s > d/2\), be \(\mathcal{F}_0\)-measurable. Then there exists a unique local in time strong solution \((v, \tau, \rho_N)\) to the stochastic viscoelastic system (3.1)-(3.4), where
\[
\rho_N = \inf_{t \geq 0} \left\{ t : \mu_2 \|v(t)\|_{H^s}^2 + \mu_1 \|\tau(t)\|_{H^s}^2 + 2\mu_2 \nu \int_0^{T^{\rho_N}} \|\nabla v(r)\|_{H^s}^2 \, dr > N \right\},
\]
such that
\[
(1) \quad \mathbb{E} \left[ \sup_{0 \leq t \leq T^{\rho_N}} \mu_2 \|v(t)\|_{H^s}^2 + \sup_{0 \leq t \leq T^{\rho_N}} \mu_1 \|\tau(t)\|_{H^s}^2 + 2\mu_2 \nu \int_0^{T^{\rho_N}} \|\nabla v(t)\|_{H^s}^2 \, dt \right] < \infty \text{ for } T > 0,
\]
(2) \(\rho_N\) is a predictable strictly positive stopping time satisfying
\[
\mathbb{P} (\rho_N > \delta) \geq 1 - 2\delta e^{(\tilde{C} + C_2 \delta)} \left( 2\mathbb{E} \left( \mu_2 \|v_0\|_{H^s}^2 + \mu_1 \|\tau_0\|_{H^s}^2 \right) + 18K_2 \delta \right)
\]
for any \(\delta \in (0, 1)\), and for some positive constant \(\tilde{C}\) independent of \(\delta\).

(3) \(v \in L^2(\Omega; L^\infty(0, T \land \rho_N; H^s(\mathbb{R}^d))) \cap L^2(0, T \land \rho_N; H^{s+1}(\mathbb{R}^d)))\),

and \(\tau \in L^2(\Omega; L^\infty(0, T \land \rho_N; H^s(\mathbb{R}^d)))\),

(4) paths of the \(\mathcal{F}_t\)-adapted processes \((v, \rho_N)\) and \((\tau, \rho_N)\) are càdlàg and continuous respectively.

Moreover, there exists a unique maximal local strong solution \((v, \tau, \rho_\infty)\) to the system (3.1)-(3.4) such that
\[
\rho_\infty(\omega) := \lim_{N \to \infty} \rho_N(\omega) \text{ for almost all } \omega \in \Omega.
\]

3.3. Strategy of the Proof. We prove the main results in a few steps, which are outlined below.

Step (i) We first show that the solutions \((v_n, \tau_n)\) of smoothed version of the (Fourier) truncated stochastic Oldroyd system (3.1)-(3.4) exist and the \(H^s\)-norm of \((v_n, \tau_n)\) are uniformly bounded up to a stopping time \(\rho^n_\infty\). We also provide a probabilistic estimate of the stopping time \(\rho^n_\infty\) (see Theorem 3.3).

Step (ii) We then show that family of strong solutions \(\{(v_n, \tau_n)\}_{n \in \mathbb{N}}\) is Cauchy in \(L^2(\Omega; L^\infty(0, T \land \xi_N; L^2(\mathbb{R}^d)))\), where \(\xi_N := \lim_{n \to \infty} \rho^n_\infty\) (see Theorem 3.4.1 and Remark 3).

Step (iii) By using Sobolev interpolation, we prove \((v_n, \tau_n) \to (v, \tau)\) strongly in \(L^2(\Omega; L^\infty(0, T \land \xi_N; H^{s'}(\mathbb{R}^d)))\) for any \(0 < s' < s\). Then we show in Theorem 6.3 (a) \(\rho_N\) as the pointwise limit of \(\rho^n_\infty\) and identify \(\xi_N\) as \(\rho_N\).

(b) \((v, \tau)\) solve (3.1)-(3.4) as an equality in \(L^1(\Omega; L^2(0, T \land \rho_N; H^{s'-1}(\mathbb{R}^d)))\),
(c) $(v, \tau, \rho_N)$ is a local in time strong solution such that
\[ v \in L^2(\Omega; L^\infty(0, T \wedge \rho_N; H^s(\mathbb{R}^d))) \cap L^2(0, T \wedge \rho_N; H^{s+1}(\mathbb{R}^d)), \]
\[ \tau \in L^2(\Omega; L^\infty(0, T \wedge \rho_N; H^s(\mathbb{R}^2))). \]

(d) the $\mathcal{F}_t$-adapted paths of $(v, \rho_N)$ and $(\tau, \rho_N)$ are càdlàg and continuous respectively.

Finally, in Theorem 6.5 we show that $(v, \tau, \rho_N)$ is a unique local strong solution.

**Step (iv)** We then prove, in Theorem 7.1, the existence of a unique maximal local strong solution $(v, \tau, \rho_\infty)$ using stopping time arguments, and provide a probabilistic estimate of $\rho_\infty$.

Having proved the uniform estimates of $v_n$ and $\tau_n$ in Step (i), we could use the classical compactness theorem of Aubin and Lions to extract a subsequence $(v_{n_k}, \tau_{n_k})$ that converges strongly to $(v, \tau)$ in some sense. While this approach is natural for bounded domain, on the whole space one only obtains the requisite strong convergence on compact subsets, and one must then show that the nonlinear terms converge as required.

Our approach also deviates from the one due to Motyl [51] where martingale solution of three dimensional Navier-Stokes equations in unbounded domains has been proved using compactness method (by certain generalisation of the classical Dubinsky theorem) and Jakubowski version of the Skorokhod theorem for nonmetric spaces. Recent papers [14], [22] on stochastic Euler equation discusses the probabilistic estimate of stopping times.

We will present complete details of the proof in the remaining Sections.

### 4. Truncated Stochastic Model

We get the truncated stochastic equations of (3.1) on $\mathbb{R}^d$ as:
\[ d v_n = [\nu \Delta v_n - \nabla p_n - J_n [(v_n \cdot \nabla) v_n] + \mu_1 \nabla \cdot \tau_n] dt + J_n \sigma(t, v_n) dW(t) + \int_\mathbb{R} J_n G(n(t-)) \tilde{N}_1(dt, dz), \]
\[ d \tau_n = -\left[ J_n (v_n \cdot \nabla) \tau_n + a \tau_n + J_n Q(\tau_n, \nabla v_n) - \mu_2 D(v_n) \right] dt, \]
\[ \nabla \cdot v_n = 0, \]
\[ v_n(0, \cdot) = v_n(0), \quad \tau_n(0, \cdot) = \tau_n(0). \]

As the truncations are invariant under the flow of the equation, we ensure that $v_n, \tau_n$ lie in the space
\[ \mathcal{V}_n := \{ g \in L^2(\mathbb{R}^d) : \text{supp}(\overline{g}) \subset B(0, n) \} \]
with $\nabla \cdot v_n = 0$.

**Proposition 4.1.** Let $v_n, \tau_n \in H^s(\mathbb{R}^d)$, for $s > d/2$ with $\nabla \cdot v_n = 0$. Then the bilinear operator $J_n Q(\tau_n, \nabla v_n)$ is locally Lipschitz in $v_n$ and $\tau_n$ on the space $\mathcal{V}_n$.

**Proof.** For $s > d/2 \geq 1$, let us use integration by parts, Hölder’s inequality and Sobolev inequality for $\tau_n \in H^s(\mathbb{R}^d)$, we have
\[ |\left( J_n Q(\tau_n, \nabla v_n) - J_n Q(\tau_n, \nabla v_n^2), v_n - v_n^2 \right)_L^2 | = |\left( (Q(\tau_n, \nabla(v_n^1 - v_n^2)), J_n(v_n^1 - v_n^2) \right)_L^2 | \]
\[ \leq \| Q(\tau_n, \nabla(v_n^1 - v_n^2)) \|_L^2 \| J_n(v_n^1 - v_n^2) \|_L^2 \leq C \| \tau_n \|_{L^\infty} \| \nabla(v_n^1 - v_n^2) \|_{L^2} \| v_n^1 - v_n^2 \|_{L^2}, \]
\[ \leq C \| \tau_n \|_{H^s} \| v_n^1 - v_n^2 \|_{H^s} \| v_n^1 - v_n^2 \|_{L^2}. \]

Hence for $\tau_n \in H^s(\mathbb{R}^d)$, and for $s > d/2$,
\[ \| J_n Q(\tau_n, \nabla v_n) \|_{L^2} \leq C \| \tau_n \|_{H^s} \| v_n^1 - v_n^2 \|_{H^s}. \]

Hence, $J_n Q(\cdot, \cdot)$ is locally Lipschitz in $v_n$. Again, for $s > d/2 \geq 1$, using same arguments as before for $v_n \in H^s(\mathbb{R}^d)$, we have
\[ |\left( J_n Q(\tau_n, \nabla v_n) - J_n Q(\tau_n^2, \nabla v_n), \tau_n - \tau_n^2 \right)_L^2 | = |\left( (Q(\tau_n^2 - \tau_n^2, \nabla v_n), J_n(\tau_n^2 - \tau_n^2) \right)_L^2 | \]

\[ \leq C \| \tau_n^2 - \tau_n^2 \|_{H^s} \| \nabla v_n \|_{H^s} \| \tau_n - \tau_n^2 \|_{L^2}. \]
\[ \leq \|Q(\tau_n^1 - \tau_n^2, \nabla v_n)\|_{L^2} \|J_n(\tau_n^1 - \tau_n^2)\|_{L^2} \leq C \|\tau_n^1 - \tau_n^2\|_{L^\infty} \|\nabla v_n\|_{L^2} \|\tau_n^1 - \tau_n^2\|_{L^2} \leq C \|\tau_n^1 - \tau_n^2\|^2_{H^1} \|\nabla v_n\|_{H^1} \|\tau_n^1 - \tau_n^2\|_{L^2} \leq C \|\tau_n^1 - \tau_n^2\|^2_{H^1} \|\nabla v_n\|_{H^1} \|\tau_n^1 - \tau_n^2\|_{L^2}. \]

Hence, \(J_n Q(\cdot, \cdot)\) is locally Lipschitz in \(\tau_n\).

\[ \bigstar \]

**Remark 10.** 1. Similarly, for \(v_n, \tau_n \in H^s(\mathbb{R}^d)\), for \(s > d/2\) with \(\nabla \cdot v_n = 0\), the nonlinear operators \(J_n [(v_n \cdot \nabla) \tau_n]\) and \(J_n [(v_n \cdot \nabla) v_n]\) is locally Lipschitz in \(v_n\) and \(\tau_n\) on the space \(Y_n\).

2. By using Plancherel’s Theorem, we observe that depending on \(n\), \(\Delta v_n\) has a bounded linear growth in \(Y_n\), since

\[ \|\Delta v_n\|_{L^2(\mathbb{R}^d)} = \|\Delta v_n\|_{L^2(\mathbb{R}^d)} = \|\Delta v_n\|_{L^2(\mathbb{R}^d)} = \|\Delta v_n\|_{L^2(\mathbb{R}^d)} \leq n^2 \|v_n\|_{L^2(\mathbb{R}^d)} = n^2 \|v_n\|_{L^2(\mathbb{R}^d)}. \]

**Corollary 4.2.** By Theorem 4.9 of Mandrekar and Rüdiger [13], Ikeda and Watanabe [26] there exists a path-wise unique strong solution \((v_n, \tau_n)\) to problem (4.1)-(4.4) such that \(v_n \in L^2(\Omega; D(0, T; Y_n))\), with \(\nabla \cdot v_n = 0\) and \(\tau_n \in L^2(\Omega; C(0, T; Y_n))\), where \(T\) depends on \(n\). The solution will exist as long as \(\|v_n\|_{L^2(\Omega; H^s(\mathbb{R}^d))}, \|\tau_n\|_{L^2(\Omega; H^s(\mathbb{R}^d))}\) remain finite.

4.1. **Energy Estimates.** In this Subsection we first obtain energy estimates of approximate solutions on the time interval \([0, T \wedge \rho_N]\) for some stopping time \(\rho_N\). These estimates solely depend on the regularity of the initial data and noise terms.

**Theorem 4.3.** Let the initial data \(v_0, \tau_0 \in L^2(\Omega; H^s(\mathbb{R}^d))\), with \(\nabla \cdot v_0 = 0\), \(s > d/2\) be \(\mathcal{F}_0\)-measurable, and the Assumption 3.1 be satisfied. For each \(n \in \mathbb{N}\), let \((v_n, \tau_n)\) be the unique strong solution of (4.1)-(4.4). Define the stopping time

\[ \rho_N = \inf_{t \geq 0} \left\{ t : \mu_2 \|v_n(t)\|^2_{H^s} + \mu_1 \|\tau_n(t)\|^2_{H^s} + 2 \mu_2 \nu \int_0^t \|\nabla v_n(r)\|^2_{H^s} dr > N \right\}. \]

Then for any \(\delta\) with \(0 < \delta < 1\),

\[ \mathbb{P} (\rho_N > \delta) \geq 1 - 2 \delta e^{(\tilde{C} + C_2 \delta)} \left( 2 \mathbb{E} (\mu_2 \|v_0\|^2_{H^s} + \mu_1 \|\tau_0\|^2_{H^s}) + 18 K \mu_2 \delta \right), \]

for some positive constant \(\tilde{C}\) independent of \(\delta\).

As a consequence of the above Theorem, we have the following result.

**Remark 11.** For any \(T > 0\), the quantities

\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T \wedge \rho_N} \|v_n(t)\|^2_{H^s} \right], \mathbb{E} \left[ \sup_{0 \leq t \leq T \wedge \rho_N} \|\tau_n(t)\|^2_{H^s} \right], \mathbb{E} \left[ \int_0^{T \wedge \rho_N} \|\nabla v_n(t)\|^2_{H^s} dt \right] \]

are uniformly bounded.

**Proof.** **Step 1:**

Applying \(J^s\) to both the equations (4.1)-(4.2) we get,

\[ dJ^s v_n = [\nu \nabla J^s v_n - \nabla J^s \rho_n - J_n J^s [(v_n \cdot \nabla) \tau_n] + \mu_1 J^s \nabla \cdot \tau_n] dt + \int_{\mathbb{R}^d} J_n J^s G(v_n(t^-)) N_1(dt, dz), \]

\[ dJ^s \tau_n = -[J_n J^s (v_n \cdot \nabla) \tau_n + J_n J^s Q(\tau_n, \nabla v_n) - \mu_2 J^s \nabla D(v_n)] dt \]

\[ + \frac{1}{2} J_n J^s S(\tau_n) dt + \int_{\mathbb{R}^d} J_n J^s S(\tau_n) dW_2(t). \]

Applying Itô’s Lemma [see Brzeźniak et al. [9], Ikeda and Watanabe [26]] to the function \(\|x\|^2_{L^2}\) to the process \(J^s v_n\) and exploiting divergence free condition in (4.4) we get,

\[ d\|v_n\|^2_{L^2} = -2 \nu \|\nabla J^s v_n\|^2_{L^2} dt - 2 \langle J^s (v_n \cdot \nabla) v_n, J^s v_n \rangle_{L^2} dt \]

\[ + 2 \mu_1 \langle J^s \nabla \cdot \tau_n, J^s v_n \rangle_{L^2} dt + \|J_n J^s \sigma(t, v_n)\|_{L^2(\mathbb{R}^d)} dt + \int_{\mathbb{R}^d} \|J_n J^s G(v_n(t^-), z)\|^2_{L^2} N_1(dt, dz) \]
\[ + 2 \int_Z (J_n J^* G(v_n(t^-), z), J^* v_n(t^-))_{L^2} N_1(\text{d}t, \text{d}z) \]

Exploiting the cut off property \[ \[22 \] \] the above equality is reduced to:
\[ d\|v_n\|_{H^s}^2 + 2\nu \|\nabla J^* v_n\|_{L^2}^2 \text{d}t \leq -2(J^* (v_n \cdot \nabla) v_n, J^* v_n)_{L^2} \text{d}t + 2\mu_1 (J^* \nabla \cdot \tau_n, J^* v_n)_{L^2} \text{d}t + \|J^* \sigma(t, v_n)\|_{L^2(\Omega)}^2 \text{d}t \\
+ 2(J_n J^* \sigma(t, v_n) \text{d}W_1(t), J^* v_n)_{L^2} + \int_Z \|J^* G(v_n(t^-), z)\|_{L^2}^2 N_1(\text{d}t, \text{d}z) \\
+ 2 \int_Z (J_n J^* G(v_n(t^-), z), J^* v_n(t^-))_{L^2} N_1(\text{d}t, \text{d}z). \tag{4.9} \]

Again let us apply Itô’s Lemma to the function \( \|x\|_{L^2}^2 \) to the process \( J^* \tau_n \) in \( (4.8) \) we obtain,
\[ d\|J^* \tau_n\|_{L^2}^2 \leq -2(J^* (v_n \cdot \nabla) \tau_n, J^* \tau_n)_{L^2} \text{d}t -2(J^* Q(\tau_n, \nabla v_n), J^* \tau_n)_{L^2} \text{d}t \\
+2\mu_2 (J^* D(v_n), J^* \tau_n)_{L^2} \text{d}t + (J_n J^* S^2(\tau_n), J^* \tau_n)_{L^2} \text{d}t \\
+ \|J_n J^* S(\tau_n)\|_{L^2}^2 \text{d}t + 2(J_n J^* S(\tau_n), J^* \tau_n)_{L^2} \text{d}W_2(t) + 2a \|J^* \tau_n\|_{L^2}^2 \text{d}t. \tag{4.10} \]

Let \( \lambda_j \) be the eigenvalues of \( Q \) such that \( Qe_j = \lambda_j e_j \) for all \( j = 1, 2, \ldots \), where \( \{e_j\}_{j=1}^\infty \) are the orthonormal basis in \( L^2(\mathbb{R}^d) \). Hence we have,
\[ \|J^* \sigma(t, v_n)\|_{L^2(\Omega)}^2 \leq \sum_{j=1}^{\infty} \lambda_j \|J^* \sigma(t, v_n) e_j\|_{L^2}^2 \]
\[ = \sum_{j=1}^{\infty} \lambda_j \| \sigma(t, v_n) e_j \|_{H^s}^2 = \| \sigma(t, v_n) \|_{L^2(\Omega, H^s)}^2. \tag{4.11} \]

Multiplying \( \mu_2 \) with \( (4.9) \) and \( \mu_1 \) with \( (4.10) \) and then on adding we have
\[ d\left[ \mu_2 \|J^* v_n\|_{L^2}^2 + \mu_1 \|J^* \tau_n\|_{L^2}^2 \right] + 2\mu_2 \nu \|\nabla J^* v_n\|_{L^2}^2 \\text{d}t \]
\[ \leq -2\mu_2 (J^* (v_n \cdot \nabla) v_n, J^* v_n)_{L^2} \text{d}t -2\mu_1 (J^* \nabla \cdot \tau_n, J^* \tau_n)_{L^2} \text{d}t \\
+ \mu_2 \|J^* \sigma(t, v_n)\|_{L^2(\Omega)}^2 \text{d}t + 2\mu_2 (J_n J^* \sigma(t, v_n) \text{d}W_1(t), J^* v_n)_{L^2} \text{d}t \\
+ \mu_2 \int_Z \|J^* G(v_n(t^-), z)\|_{L^2}^2 N_1(\text{d}t, \text{d}z) + 2\mu_2 \int_Z (J_n J^* G(v_n(t^-), z), J^* v_n(t^-))_{L^2} N_1(\text{d}t, \text{d}z) \\
+ 2\mu_1 \mu_2 (J^* \nabla \cdot \tau_n, J^* v_n)_{L^2} \text{d}t + 2\mu_1 \mu_2 (J^* D(v_n), J^* \tau_n)_{L^2} \text{d}t + \mu_1 (J_n J^* S^2(\tau_n), J^* \tau_n)_{L^2} \text{d}t \\
+ \mu_1 \|h\|_{H^s}^2 \|\tau_n\|_{H^s}^2 \text{d}t + 2\mu_1 (J_n J^* S(\tau_n), J^* \tau_n)_{L^2} \text{d}W_2(t) + 2a \mu_1 \|J^* \tau_n\|_{L^2}^2 \text{d}t. \tag{4.12} \]

Using the divergence free condition of \( v_n \), we directly have \( I_0 \) is zero. We now recall the Commutator estimates (mentioned in Subsection \[ \[24 \] \] and separately estimate each \( I_i; i = 1, \ldots, 4 \). In order to estimate the term \( (I_1) \), we recall the fact that \( H^s \) is an algebra for \( s > d/2 \) to get
\[ |I_1| = |-2\mu_2 (J^* (v_n \cdot \nabla) v_n, J^* v_n)_{L^2}| \leq 2\mu_2 \| (v_n \cdot \nabla) v_n\|_{L^2} \|J^* v_n\|_{L^2} \]
\[ \leq 2\mu_2 \| (v_n \cdot \nabla) v_n\|_{H^s} \|v_n\|_{H^s} \]
\[ \leq C \mu_2 \| (v_n \cdot \nabla) v_n\|_{H^s} \|v_n\|_{H^s}^2. \tag{4.13} \]

Once again we use the fact that \( H^s \) is an algebra for \( s > d/2 \), and we recall Theorem \[ \[25 \] \]. Hence \( I_2 \) is reduced to:
\[ |I_2| = |-2\mu_1 (J^* (v_n \cdot \nabla) \tau_n, J^* \tau_n)_{L^2}| \leq 2\mu_1 \|J^* (v_n \cdot \nabla) \tau_n\|_{L^2} \|J^* \tau_n\|_{L^2} \]
\[ \leq C \mu_1 \| (v_n \cdot \nabla) \tau_n\|_{H^s} \|\tau_n\|_{H^s}^2. \tag{4.14} \]
For $I_3$, using the classical tame estimate for $Q$ (Property 2.8) we get
\[
|I_3| = | -2\mu_1 (J^*Q(\tau_n, \nabla v_n), J^*\tau_n)_{L^2} | \leq 2\mu_1 \| J^*Q(\tau_n, \nabla v_n) \|_{L^2} \| J^*\tau_n \|_{L^2} \\
\leq 2\mu_1 \| Q(\tau_n, \nabla v_n) \|_{H^s} \| \tau_n \|_{H^s} \\
\leq C\mu_1 \| \nabla v_n \|_{H^s} \| \tau_n \|_{H^s}^2. 
\]  
(4.15)

For $I_4$ we recall (4.2) and (4.5) and achieve,
\[
|I_4| = \mu_1 \left| (J_n J^* S^2(\tau_n), J^*\tau_n)_{L^2} \right| \leq \mu_1 \| J_n J^* S^2(\tau_n) \|_{L^2} \| J^*\tau_n \|_{L^2} \\
\leq \mu_1 \| J^* S^2(\tau_n) \|_{L^2} \| J^*\tau_n \|_{L^2} \\
\leq \mu_1 \| S^2(\tau_n) \|_{H^s} \| \tau_n \|_{H^s} \\
\leq \mu_1 \| h \|_{H^s} \| \tau_n \|_{H^s}^2. 
\]  
(4.16)

Using (4.13), (4.14), (4.15) and (4.16), from (4.12) we have
\[
d [\mu_2 \| v_n \|_{H^s}^2 + \mu_1 \| \tau_n \|_{H^s}^2 ] + 2\mu_2 \| \nabla v_n \|_{H^s}^2 dt \\
\leq C \left[ \mu_2 \| v_0 \|_{H^s}^2 + \mu_1 \| \tau_0 \|_{H^s}^2 \right] \| \nabla v_n \|_{H^s} + ((C+1)\| h \|_{H^s}^2 + 2\alpha) \mu_1 \| \tau_n \|_{H^s}^2 dt \\
+ \mu_2 \| \sigma(t, v_n) \|_{L^2(L^2, H^s)}^2 dt + \mu_2 \int_Z \| G(v_n(t-), \cdot) \|_{H^s}^2 N_1(dt, dz) \\
+ 2\mu_2 \int_Z \left( J_n J^* \sigma(t, v_n) dW_1(t), J^* v_n \right)_{L^2} + 2\mu_2 \int_Z \left( J_n J^* G(v_n(t-), \cdot), J^* v_n(t-) \right)_{L^2} \tilde{N}_1(dt, dz) \\
+ 2\mu_1 \int_Z \left( J_n J^* S(\tau_n), J^* \tau_n \right)_{L^2} dW_2(t). 
\]  
(4.17)

It is to be noted that
\[
\| v_n(0) \|_{H^s}^2 + \| \tau_n(0) \|_{H^s}^2 \leq \| v_0 \|_{H^s}^2 + \| \tau_0 \|_{H^s}^2. 
\]  
(4.18)

For any $T > 0$, let us integrate (4.17) from $0$ to $t$, take supremum from $0$ to $T \wedge \rho_N^t$ (where $\rho_N$ is given by (4.3)) and then take expectation in (4.17) (use Remark 3 to get
\[
E \left[ \sup_{0 \leq t \leq T \wedge \rho_N^t} \left[ \mu_2 \| v_n(t) \|_{H^s}^2 + \mu_1 \| \tau_n(t) \|_{H^s}^2 \right] \\
+ 2\mu_2 \| \nabla v_n \|_{H^s}^2 dt \right] + 2\mu_2 \int_0^{T \wedge \rho_N^t} \| \nabla v_n(r) \|_{H^s}^2 dr \\
\leq E \left[ \mu_2 \| v_0 \|_{H^s}^2 + \mu_1 \| \tau_0 \|_{H^s}^2 \right] + 2CE \left[ \sup_{0 \leq t \leq T \wedge \rho_N^t} \| \nabla v_n(t) \|_{H^s} \right] \left( \mu_2 \| v_n(t) \|_{H^s}^2 + \mu_1 \| \tau_n(t) \|_{H^s}^2 \right) dr \\
+ ((C+1)\| h \|_{H^s}^2 + 2\alpha) \mu_1 \left[ \sup_{0 \leq t \leq T \wedge \rho_N^t} \| \tau_n(t) \|_{H^s}^2 \right] dr \\
+ \mu_2 \left[ \left( \sup_{0 \leq t \leq T \wedge \rho_N^t} \| \sigma(t, v_n(r-)) \|_{L^2(L^2, H^s)}^2 \right) + \left( \int_Z \| G(v_n(r-), \cdot) \|_{H^s}^2 \lambda(dz) \right) \right] \\
+ 2\mu_2 \left[ \left( \sup_{0 \leq t \leq T \wedge \rho_N^t} \left( \int_0^t (J_n J^* \sigma(t, v_n) dW_1(t), J^* v_n(t-))_{L^2} \right) \right) \right] \\
+ 2\mu_1 \left[ \left( \sup_{0 \leq t \leq T \wedge \rho_N^t} \left( \int_0^t (J_n J^* S(\tau_n(t-)), J^* \tau_n(t-))_{L^2} dW_2(t) \right) \right) \right] \\
+ 2\mu_2 \left[ \left( \sup_{0 \leq t \leq T \wedge \rho_N^t} \left( \int_0^t \int_Z (J_n J^* G(v_n(r), \cdot), J^* v_n(r-))_{L^2} \tilde{N}_1(dr, dz) \right) \right) \right]. 
\]  
(4.19)

Now by applying Burkholder-Davis-Gundy inequality, Young’s inequality to the term (M1), we get
\[
M_1 \leq 2\sqrt{2} \mu_2 E \left( \int_0^{T \wedge \rho_N^t} \| J_n J^* \sigma(t, v_n(t)) \|_{L^2(L^2, L^2)}^2 \| J^* v_n(t) \|_{L^2}^2 dt \right)^{1/2} \\
\leq 2\sqrt{2} \mu_2 E \left( \left( \sup_{0 \leq t \leq T \wedge \rho_N^t} \| v_n(t) \|_{H^s}^2 \right)^{1/2} \left( \int_0^{T \wedge \rho_N^t} \| J^* \sigma(t, v_n(t)) \|_{L^2(L^2, L^2)}^2 dt \right)^{1/2} \right). 
\]
Again applying Burkholder-Davis-Gundy inequality, Young’s inequality in similar manner, the term $[M_3]$ is reduced to:

\[
M_2 \leq 2\sqrt{2}\mu_1 E \left( \int_0^{T\wedge\rho_N^2} \|J_nJ^*(\tau_n(t))\|_{L^2}^2 \|J^*\tau_n(t)\|_{L^2}^2 \, dt \right)^{1/2}
\]

\[
\leq \frac{\mu_1}{2} E \left( \sup_{0 \leq t \leq T\wedge\rho_N^2} \|\tau_n(t)\|_{H^s}^2 \right) + 4\mu_1 \|h\|_{H^s}^2 E \left( \int_0^{T\wedge\rho_N^2} \|\tau_n(t)\|_{H^s}^2 \, dt \right). \tag{4.21}
\]

Let us now apply Burkholder-Davis-Gundy inequality, Young’s inequality and Assumption 3.1 to the term $[M_3]$ to obtain

\[
M_3 \leq 2\sqrt{2}\mu_2 E \left[ \int_0^{T\wedge\rho_N^2} \left( \int_Z \|J_nJ^*G(\nu_n(t), z)\|_{L^2}^2 \|J^*\nu_n\|_{L^2}^2 \lambda(dz) \right) \, dt \right]^{1/2}
\]

\[
\leq \frac{\mu_2}{4} E \left( \sup_{0 \leq t \leq T\wedge\rho_N^2} \|\nu_n(t)\|_{H^s}^2 \right) + 8\mu_2 E \left( \int_0^{T\wedge\rho_N^2} \left( \int_Z \|G(\nu_n(t), z)\|_{H^s}^2 \lambda(dz) \right) \, dt \right).
\]

By using (4.20), (4.21), and (4.22) in (4.19), we get

\[
E \left( \sup_{0 \leq t \leq T\wedge\rho_N^2} \left( C_1 \nu_n(t) \right) + \mu_1 \|\tau_n(t)\|_{H^s}^2 \right) + 4\mu_2 E \left[ \int_0^{T\wedge\rho_N^2} \|\nabla \nu_n(t)\|_{H^s}^2 \, dt \right]
\]

\[
\leq 2E \left[ \mu_2 \|\nu_0\|_{H^s}^2 + \mu_1 \|\tau_0\|_{H^s}^2 \right] + 4C E \left[ \int_0^{T\wedge\rho_N^2} \|\nabla \nu_n\|_{H^s} \left( \mu_2 \|\nu_n(t)\|_{H^s}^2 + \mu_1 \|\tau_n(t)\|_{H^s}^2 \right) \, dt \right]
\]

\[
+ 2 \left( (C + 5) \|h\|_{H^s}^2 + 2a \right) \mu_1 E \left[ \int_0^{T\wedge\rho_N^2} \|\tau_n(t)\|_{H^s}^2 \, dt \right]
\]

\[
+ 18K\mu_2 E \left[ \int_0^{T\wedge\rho_N^2} \left( 1 + \|\nu_n(t)\|_{H^s}^2 + \|\tau_n(t)\|_{H^s}^2 \right) \, dt \right]. \tag{4.23}
\]

Hence, on rearranging the constants of the last two terms of (4.23), the inequality further reduces to

\[
E \left( \sup_{0 \leq t \leq T\wedge\rho_N^2} \left( C_1 \nu_n(t) \right) + \mu_1 \|\tau_n(t)\|_{H^s}^2 \right) + 4\mu_2 E \left[ \int_0^{T\wedge\rho_N^2} \|\nabla \nu_n(t)\|_{H^s}^2 \, dt \right]
\]

\[
\leq 2E \left[ \mu_2 \|\nu_0\|_{H^s}^2 + \mu_1 \|\tau_0\|_{H^s}^2 \right] + 4C E \left[ \int_0^{T\wedge\rho_N^2} \|\nabla \nu_n(t)\|_{H^s} \left( \mu_2 \|\nu_n(t)\|_{H^s}^2 + \mu_1 \|\tau_n(t)\|_{H^s}^2 \right) \, dt \right]
\]

\[
+ C_2 \left[ \int_0^{T\wedge\rho_N^2} \left( \mu_2 \|\nu_n(t)\|_{H^s}^2 + \mu_1 \|\tau_n(t)\|_{H^s}^2 \right) \, dt \right] + 18K\mu_2 T, \tag{4.24}
\]

where \(C_2 = \left( 2(C + 5) \|h\|_{H^s}^2 + 4a + \frac{18K\mu_2^2}{\mu_1} \right) \). By using Young’s inequality, we get

\[
4C \|\nabla \nu_n(t)\|_{H^s} \left( \mu_2 \|\nu_n(t)\|_{H^s}^2 + \mu_1 \|\tau_n(t)\|_{H^s}^2 \right) \leq 2\nu_2 \|\nabla \nu_n(t)\|_{H^s}^2 + \frac{C^2}{\nu_2} \left( \mu_2 \|\nu_n(t)\|_{H^s}^2 + \mu_1 \|\tau_n(t)\|_{H^s}^2 \right)^2. \tag{4.25}
\]

By using (4.23) and making use of the stopping time defined by (4.5), we obtain

\[
E \left( \sup_{0 \leq t \leq T\wedge\rho_N^2} \left( C_1 \nu_n(t) \right) + \mu_1 \|\tau_n(t)\|_{H^s}^2 \right) + 2\nu_2 E \left[ \int_0^{T\wedge\rho_N^2} \|\nabla \nu_n(t)\|_{H^s}^2 \, dt \right]
\]
\[
\leq 2E \left[ \mu_2 \| v_0 \|_{H^+}^2 + \mu_1 \| \tau_0 \|_{H^+}^2 \right] + 18K \mu_2 T + \frac{2C^2}{\nu \mu_2} E \left[ \int_0^{T^{\rho_N^0}} (\mu_2 \| v_n(t) \|_{H^+} + \mu_1 \| \tau_n(t) \|_{H^+})^2 dt \right] \\
\quad + C_2 E \left[ \int_0^{T^{\rho_N^0}} (\mu_2 \| v_n(t) \|_{H^+} + \mu_1 \| \tau_n(t) \|_{H^+}) dt \right] \\
\leq 2E \left[ \mu_2 \| v_0 \|_{H^+}^2 + \mu_1 \| \tau_0 \|_{H^+}^2 \right] + 18K \mu_2 T \\
\quad + \left( \frac{2C^2 N}{\nu \mu_2} + C_2 \right) E \left[ \int_0^{T^{\rho_N^0}} (\mu_2 \| v_n(t) \|_{H^+} + \mu_1 \| \tau_n(t) \|_{H^+}) dt \right].
\]

Finally, we have
\[
E \left[ \sup_{0 \leq t \leq T^{\rho_N^0}} \left[ \mu_2 \| v_n(t) \|_{H^+}^2 + \mu_1 \| \tau_n(t) \|_{H^+}^2 \right] \right] + 2\nu \mu_2 E \left[ \int_0^{T^{\rho_N^0}} \| \nabla v_n(t) \|_{H^+} dt \right] \\
\leq 2E \left[ \mu_2 \| v_0 \|_{H^+}^2 + \mu_1 \| \tau_0 \|_{H^+}^2 \right] + 18K \mu_2 T \\
\quad + \left( \frac{2C^2 N}{\nu \mu_2} + C_2 \right) E \left[ \int_0^{T^{\rho_N^0}} \sup_{0 \leq t \leq T^{\rho_N^0}} \left( \mu_2 \| v_n(t) \|_{H^+} + \mu_1 \| \tau_n(t) \|_{H^+} \right) dt \right]. \tag{4.26}
\]

Now for any \( T > 0 \) and \( \rho_N^0 \) defined in \( 4.39 \), an application of Gronwall’s inequality yields
\[
E \left[ \sup_{0 \leq t \leq T^{\rho_N^0}} \left[ \mu_2 \| v_n(t) \|_{H^+}^2 + \mu_1 \| \tau_n(t) \|_{H^+}^2 \right] \right] + 2\nu \mu_2 E \left[ \int_0^{T^{\rho_N^0}} \| \nabla v_n(t) \|_{H^+} dt \right] \\
\leq \left( 2E \left[ \mu_2 \| v_0 \|_{H^+}^2 + \mu_1 \| \tau_0 \|_{H^+}^2 \right] + 18K \mu_2 T \right) \exp \left\{ \left( \frac{2C^2 N}{\nu \mu_2} + C_2 \right) T \right\}. \tag{4.27}
\]

**Step II:**

We further assume that \( E \left[ \| v_0 \|_{H^+} \right] < \infty \) and \( E \left[ \| \tau_0 \|_{H^+} \right] < \infty \). By using \( 4.27 \), we get
\[
E \left[ \sup_{0 \leq t \leq \delta} \left[ \mu_2 \| v_n(t \wedge \rho_N^0) \|_{H^+}^2 + \mu_1 \| \tau_n(t \wedge \rho_N^0) \|_{H^+}^2 \right] \right] + 2\nu \mu_2 E \left[ \int_0^{\delta} \| \nabla v_n(t) \|_{H^+} dt \right] \\
\leq \left( 2E \left[ \mu_2 \| v_0 \|_{H^+}^2 + \mu_1 \| \tau_0 \|_{H^+}^2 \right] + 18K \mu_2 \delta \right) \exp \left\{ \left( \frac{2C^2 N}{\nu \mu_2} + C_2 \right) \delta \right\}, \tag{4.28}
\]

where \( C \) is a positive constant independent of \( N \) and \( \delta \). Let \( 0 < \delta < 1 \) be given. Then there exists a positive integer \( N \) such that
\[
\frac{1}{N+1} \leq \delta < \frac{1}{N}.
\]

By the definition of \( \rho_N^0 \), one can easily observe that
\[
\left\{ \sup_{0 \leq t \leq \delta} \left[ \mu_2 \| v_n(t) \|_{H^+}^2 + \mu_1 \| \tau_n(t) \|_{H^+}^2 \right] + 2\nu \mu_2 \int_0^{\delta} \| \nabla v_n(s) \|_{H^+} ds \leq N \right\} \subset \{ \rho_N^0 > \delta \}. \tag{4.29}
\]

Now as an application of Markov’s inequality for \( 0 < \delta < 1 \), and using \( 1 < \frac{N+1}{N} < 2 \), \( \frac{1}{N+1} < \delta < \frac{1}{N} \) and \( 4.28 \), we get
\[
P \left( \rho_N^0 > \delta \right) \geq P \left( \left\{ \sup_{0 \leq t \leq \delta} \left[ \mu_2 \| v_n(t) \|_{H^+}^2 + \mu_1 \| \tau_n(t) \|_{H^+}^2 \right] + 2\nu \mu_2 \int_0^{\delta} \| \nabla v_n(s) \|_{H^+} ds \leq N \right\} \right) \\
\geq 1 - \frac{1}{N} E \left( \sup_{0 \leq t \leq \delta} \left[ \mu_2 \| v_n(t) \|_{H^+}^2 + \mu_1 \| \tau_n(t) \|_{H^+}^2 \right] + 2\nu \mu_2 \int_0^{\delta} \| \nabla v_n(s) \|_{H^+} ds \right) \\
\geq 1 - \frac{1}{N} \left\{ \frac{2E \left[ \mu_2 \| v_0 \|_{H^+}^2 + \mu_1 \| \tau_0 \|_{H^+}^2 \right] + 18K \mu_2 \delta}{\left( \frac{2C^2 N}{\nu \mu_2} + C_2 \right) \delta} \right\} \\
\geq 1 - \frac{e^{C_2 \delta}}{N} \left( \frac{2E \left[ \mu_2 \| v_0 \|_{H^+}^2 + \mu_1 \| \tau_0 \|_{H^+}^2 \right] + 18K \mu_2 \delta}{\left( \frac{2C^2 N}{\nu \mu_2} + C_2 \right) \delta} \right).
where \( \tilde{C} = \frac{2C^2}{\nu p^2} \) is a constant independent of \( \delta \).

\[ (4.30) \]

5. Strong Convergence of the Truncated Solutions

We first prove that the solutions \((u_n, \tau_n)\) of (4.1)–(4.4) converge strongly in \(L^2(\Omega, L^\infty(0, T; L^2(\mathbb{R}^d)))\). In the proof we have used certain results, which have been proved in the Appendix.

**Theorem 5.1.** Let \( \rho_n^m \) be the stopping time defined in (4.5) and \( T > 0 \). Then the family of strong solutions \( \{(u_n, \tau_n)\}_{n \in \mathbb{N}} \) of (4.1)–(4.4) satisfy the following convergence results:

(i) \( \lim_{n \to \infty} \sup_{m \geq n} \mathbb{E} \left( \sup_{t \in [0, T \wedge \tau_n^m]} \|u_n - u_m\|^2_2 + \sup_{t \in [0, T \wedge \tau_n^m]} \|\tau_n - \tau_m\|^2_2 \right) = 0, \)

(ii) \( \lim_{n \to \infty} \sup_{m \geq n} \mathbb{E} \left( \int_0^{T \wedge \tau_n^m} \|\nabla (u_n - u_m)\|^2_2 dt \right) = 0, \)

where \( \tau_n^m := \tau_n \wedge \tau_m^m \).

**Proof.** We split the proof in three steps.

**Step 1:**

Let \((u_n, \tau_n)\) and \((u_m, \tau_m)\) be two strong solutions of (4.1)–(4.4) in \(V_n\) and \(V_m\) respectively. Consider the difference between the equations (4.1)–(4.4) for \(n\) and \(m\) to get

\[
d(u_n - u_m) = \nu \Delta (u_n - u_m) dt - \left( J_n [(u_n \cdot \nabla) v_n] - J_m [(u_m \cdot \nabla) v_m] \right) dt + \mu_1 \Delta (\tau_n - \tau_m) dt + \left( J_n \sigma(t, u_n) - J_m \sigma(t, u_m) \right) dW_1(t) \\
+ \int_Z \left( J_n G(u_n, z) - J_m G(u_m, z) \right) \tilde{N}_1(dt, dz),
\]

\[
d(\tau_n - \tau_m) = - (J_n [(u_n \cdot \nabla) \tau_n] - J_m [(u_m \cdot \nabla) \tau_m]) dt - (J_n Q(\tau_n, \nabla v_n) - J_m Q(\tau_m, \nabla v_m)) dt \\
+ \mu_2 D(u_n - u_m) dt + \alpha (\tau_n - \tau_m) dt + \frac{1}{2} \left[ J_n S^2(\tau_n) - J_m S^2(\tau_m) \right] dt \\
+ (J_n S(\tau_n) - J_m S(\tau_m)) dW_2(t).
\]

Applying Itô’s Lemma to the function \( \|x\|^2_2 \) and to the process \( \mu_2(u_n - u_m) \) in (5.1) and to the process \( \mu_1(\tau_n - \tau_m) \) in (5.2), and adding these two equations we get

\[
d \left( \mu_2 \|u_n - u_m\|^2_{L^2} + \mu_1 \|\tau_n - \tau_m\|^2_{L^2} \right) + 2\mu_2 \nu \|\nabla (u_n - u_m)\|^2_2 dt \\
= -2\mu_2 (J_n [(u_n \cdot \nabla) v_n] - J_m [(u_m \cdot \nabla) v_m], u_n - u_m)_{L^2} dt \\
+ \left[ 2\mu_1 \mu_2 \left( (\nabla \cdot (\tau_n - \tau_m), v_n - v_m)_{L^2} + (D(u_n - u_m), \tau_n - \tau_m)_{L^2} \right) - 2\mu_2 (\nu (p_n - p_m), v_n - v_m)_{L^2} \right] dt \\
+ \mu_2 \|J_n \sigma(t, u_n) - J_m \sigma(t, u_m)\|^2_{L^2} dt + \mu_2 ((J_n \sigma(t, u_n) - J_m \sigma(t, u_m)) dW_1(t), v_n - v_m) \\
+ \mu_2 \int_Z \left( J_n G(u_n(t-), z) - J_m G(v_m(t-), z) \right) N_1(dt, dz) \\
+ 2\mu_2 \int_Z (J_n G(u_n(t-), z) - J_m G(v_m(t-), z), v_n - v_m)_{L^2} \tilde{N}_1(dt, dz) \\
- 2\mu_1 (J_n [(u_n \cdot \nabla) \tau_n] - J_m [(u_m \cdot \nabla) \tau_m], \tau_n - \tau_m)_{L^2} dt \\
- 2\mu_1 (J_n Q(\tau_n, \nabla v_n) - J_m Q(\tau_m, \nabla v_m), \tau_n - \tau_m)_{L^2} dt - 2\alpha \mu_1 \|\tau_n - \tau_m\|^2_{L^2} dt \\
+ \mu_1 (J_n S^2(\tau_n) - J_m S^2(\tau_m), \tau_n - \tau_m)_{L^2} dt + \mu_1 \|J_n S(\tau_n) - J_m S(\tau_m)\|^2_{L^2} dt \\
+ \mu_1 (J_n S(\tau_n) - J_m S(\tau_m), \tau_n - \tau_m)_{L^2} dW_2(t).
\]

\[ (5.3) \]
Using the fact that $\nabla \cdot \mathbf{v}_n = \nabla \cdot \mathbf{v}_m = 0$ we have directly $J_0 = 0$.

\[
\begin{align*}
& d \left( \mu_2 \| \mathbf{v}_n - \mathbf{v}_m \|_{L^2}^2 + \mu_1 \| \tau_n - \tau_m \|_{L^2}^2 \right) + 2 \mu_2 \nu \| \nabla (\mathbf{v}_n - \mathbf{v}_m) \|_{L^2}^2 dt \\
& = - 2 \mu_2 \left( J_n \left[ (\mathbf{v}_n \cdot \nabla) \mathbf{v}_n \right] - J_m \left[ (\mathbf{v}_m \cdot \nabla) \mathbf{v}_m \right] , \mathbf{v}_n - \mathbf{v}_m \right)_{L^2} dt \\
& - 2 \mu_1 \left( J_n \left[ (\mathbf{v}_n \cdot \nabla) \tau_n \right] - J_m \left[ (\mathbf{v}_m \cdot \nabla) \tau_m \right] , \tau_n - \tau_m \right)_{L^2} dt \\
& - 2 \mu_1 \left( J_n \mathbf{Q}(\tau_n, \nabla \mathbf{v}_n) - J_m \mathbf{Q}(\tau_m, \nabla \mathbf{v}_m) , \tau_n - \tau_m \right)_{L^2} dt - 2 \nu \mu_1 \| \tau_n - \tau_m \|_{L^2}^2 dt \\
& + \mu_1 \left( J_n S^2(\tau_n) - J_m S^2(\tau_m) , \tau_n - \tau_m \right)_{L^2} dt + \mu_1 \| J_n S(\tau_n) - J_m S(\tau_m) \|_{L^2}^2 dt \\
& + \mu_2 \left( J_n \sigma(t, \mathbf{v}_n) - J_m \sigma(t, \mathbf{v}_m) \right)_{L^2(L^2, L^2)}^2 dt \\
& + \mu_2 \int_Z \| J_n G(\mathbf{v}_n(t) - \mathbf{v}_m(t), z) - J_m G(\mathbf{v}_m(t), z) \|_{L^2}^2 N_1(dt, dz) \\
& + 2 \mu_2 \left( (J_n \sigma(t, \mathbf{v}_n) - J_m \sigma(t, \mathbf{v}_m)) dW_1(t) , \mathbf{v}_n - \mathbf{v}_m \right)_{L^2} \\
& + 2 \mu_1 \left( J_n S(\tau_n) - J_m S(\tau_m) , \tau_n - \tau_m \right)_{L^2} dW_2(t) \\
& + 2 \mu_2 \int_Z \left( J_n G(\mathbf{v}_n(t) - \mathbf{v}_m(t), z) - J_m G(\mathbf{v}_m(t), z) , \mathbf{v}_n - \mathbf{v}_m \right)_{L^2} \tilde{N}_1(dt, dz).
\end{align*}
\]

Now using \[5.5\] and Remark \[8\], $J_4$ can be simplified further as:

\[
| \mu_1 \left( J_n S^2(\tau_n) - J_m S^2(\tau_m) , \tau_n - \tau_m \right)_{L^2} | \leq \frac{C \mu_1}{n^\epsilon} \| h \|_{H^s}^2 \| \tau_n - \tau_m \|_{L^2} + \mu_1 \| h \|_{L^\infty} \| \tau_n - \tau_m \|_{L^2}^2.
\]

Using Lemma \[A.1\] we obtain

\[
| J_1 | \leq \frac{2C \sqrt{\mu_2}}{n^\epsilon} \| \mathbf{v}_n \|_{H^s}^2 \| \mathbf{v}_n - \mathbf{v}_m \|_{L^2} + 2 \mu_2 C \| \mathbf{v}_n - \mathbf{v}_m \|_{L^2} \| \nabla \mathbf{v}_n \|_{H^s}.
\]

Using Lemma \[A.2\] $J_2$ is reduced to

\[
| J_2 | \leq \frac{2 \sqrt{\mu_2 C}}{n^\epsilon} \left( \| \mathbf{v}_n \|_{H^s} + \| \tau_n \|_{H^s} \right) \sqrt{\mu_1} \| \tau_n - \tau_m \|_{L^2} + \frac{\nu \mu_1}{2} \| \mathbf{v}_n - \mathbf{v}_m \|_{H^s}^2 + \frac{2 C \mu_1}{\nu \mu_2} \| \tau_n \|_{H^s}^2 \| \tau_n - \tau_m \|_{L^2}^2.
\]

Exploiting Lemma \[A.3\] $J_3$ becomes:

\[
| J_3 | \leq \frac{2 \sqrt{\mu_2 C}}{n^\epsilon} \left( \| \mathbf{v}_n \|_{H^s} + \| \tau_n \|_{H^s} \right) \sqrt{\mu_1} \| \tau_n - \tau_m \|_{L^2} + \frac{\nu \mu_1}{2} \| \mathbf{v}_n - \mathbf{v}_m \|_{H^s}^2 + \frac{2 C \mu_1}{\nu \mu_2} \| \tau_n \|_{H^s}^2 \| \tau_n - \tau_m \|_{L^2}^2
\]

\[
+ 2 \mu_1 \| \nabla \mathbf{v}_n \|_{H^s} \| \tau_n - \tau_m \|_{L^2}^2 + \frac{\mu_1 C}{n^\epsilon} \| \nabla \mathbf{v}_n \|_{H^s} \| \tau_n - \tau_m \|_{L^2}.
\]

Exploiting \[A.16\] in Lemma \[A.5\] $J_5$, becomes

\[
| J_5 | \leq \frac{2 \mu_1}{n^\epsilon} \| h \|_{H^s}^2 \| \tau_n \|_{H^s}^2 + 2 \mu_1 \| h \|_{L^\infty} \| \tau_n - \tau_m \|_{L^2}^2.
\]

Combining \[5.5\], \[5.6\], \[5.7\], \[5.8\] and \[5.9\] together and using the fact that $4C \| \nabla \mathbf{v}_n \|_{H^s} \leq C^2 + 2 \| \nabla \mathbf{v}_n \|_{H^s}^2$, \[5.1\] becomes:

\[
\begin{align*}
& d \left( \mu_2 \| \mathbf{v}_n - \mathbf{v}_m \|_{L^2}^2 + \mu_1 \| \tau_n - \tau_m \|_{L^2}^2 \right) + \mu_2 \nu \| \nabla (\mathbf{v}_n - \mathbf{v}_m) \|_{L^2}^2 dt \\
& \leq \frac{C}{n^\epsilon} \left( (2 \sqrt{\mu_2} + 4 \sqrt{\mu_1}) \| \mathbf{v}_n \|_{H^s}^2 + 4 \sqrt{\mu_1} \| \tau_n \|_{H^s}^2 + \sqrt{\mu_1} \| h \|_{H^s}^2 \| \tau_n \|_{H^s} + \sqrt{\mu_1} \| \nabla \mathbf{v}_n \|_{H^s}^2 \right) \\
& \times \left( \sqrt{\nu \mu_2} \| \mathbf{v}_n - \mathbf{v}_m \|_{L^2} + \sqrt{\mu_1} \| \tau_n - \tau_m \|_{L^2} \right) \\
& + \left( 2C^2 + 2a \right) + 2 \| \nabla \mathbf{v}_n \|_{H^s}^2 + \frac{2 C \mu_1}{\nu \mu_2} \left( \| \tau_n \|_{H^s}^2 + \| \tau_m \|_{H^s}^2 \right) + 2 \| h \|_{L^\infty}^2 \\
& \times \left( \mu_2 \| \mathbf{v}_n - \mathbf{v}_m \|_{L^2}^2 + \mu_1 \| \tau_n - \tau_m \|_{L^2}^2 \right) + \frac{2 \mu_1}{n^\epsilon} \| h \|_{H^s}^2 \| \tau_n \|_{H^s}^2.
\end{align*}
\]
+ \mu_2 \|J_n\sigma(t, v_n) - J_m\sigma(t, v_m)\|_{L^2(Q, L^2)}^2 dt \\
+ \mu_2 \int_z \|J_nG(v_n(t-), z) - J_mG(v_m(t-), z)\|_{L^2}^2 N_1(dt, dz) \\
+ 2\mu_2 \langle (J_n\sigma(t, v_n) - J_m\sigma(t, v_m)) \rangle dW_1(t), v_n - v_m \rangle_{L^2} \\
+ 2\mu_1 \langle J_nS(\tau_n) - J_mS(\tau_m), \tau_n - \tau_m \rangle_{L^2} dW_2(t) \\
+ 2\mu_2 \int_z \langle J_nG(v_n(t-), z) - J_mG(v_m(t-), z), v_n - v_m \rangle_{L^2} \tilde{N}_1(dt, dz). \tag{5.10}

**Step II:**

Let us define the process $\eta(t) := \exp \left( -2 \int_0^t \|\nabla v_n\|_{H^1}^2 ds \right)$, $t \in [0, T \wedge \rho^{n,m}_N)$ and apply Itô product formula (see Theorem 4.4.13, Applebaum [24]) to the process $\eta(t) \mu_2 \|v_n - v_m\|_{L^2}^2 + \mu_1 \|\tau_n - \tau_m\|_{L^2}^2$ in the interval $[0, t]$ to get

$$
\eta(t) \left( \mu_2 \|v_n - v_m\|_{L^2}^2 + \mu_1 \|\tau_n - \tau_m\|_{L^2}^2 \right) + \mu_2 \nu \int_0^t \eta(s) \|\nabla(v_n - v_m)\|_{L^2}^2 ds \\
+ \frac{C}{n^2} \int_0^t \eta(s) \left( \left( 2\sqrt{\mu_2} + 4\sqrt{\mu_1} \right) \|v_n\|_{H^2}^2 + 4\sqrt{\mu_1} \|\tau_n\|_{H^2}^2 + \sqrt{\mu_1} \|h\|_{H^2}^2 \|\tau_n\|_{H^2} + \sqrt{\mu_1} \|\nabla v_n\|_{H^2}^2 \right) \\
\times \left( \left( \sqrt{\mu_2} \|v_n - v_m\|_{L^2} + \sqrt{\mu_1} \|\tau_n - \tau_m\|_{L^2} \right) ds \\
+ \mu_1 \int_0^t \eta(s) \langle J_n\sigma(s, v_n) - J_m\sigma(s, v_m) \rangle_{L^2(Q, L^2)}^2 ds \\
+ \mu_2 \int_0^t \eta(s) \int_z \|J_nG(v_n(s-), z) - J_mG(v_m(s-), z)\|_{L^2}^2 N_1(ds, dz) \\
+ 2\mu_2 \int_0^t \eta(s) \langle (J_n\sigma(s, v_n) - J_m\sigma(s, v_m)) \rangle dW_1(s), v_n - v_m \rangle_{L^2} \\
+ 2\mu_1 \int_0^t \eta(s) \langle J_nS(\tau_n) - J_mS(\tau_m), \tau_n - \tau_m \rangle_{L^2} dW_2(s) \\
+ 2\mu_2 \int_0^t \eta(s) \int_z \langle J_nG(v_n(s-), z) - J_mG(v_m(s-), z), v_n - v_m \rangle_{L^2} \tilde{N}_1(ds, dz). \tag{5.11}
$$

It is to be noted that the quadratic variation of the product of these two adapted processes $Y_1(t) = \eta(t)$ and $Y_2(t) = \mu_2 \|v_n - v_m\|_{L^2}^2 + \mu_1 \|\tau_n - \tau_m\|_{L^2}^2$, i.e., $[Y_1, Y_2](t)$ is zero (see, Section 4.4.3, page 257 of Applebaum [24]).

Let us now take the supremum from 0 to $T \wedge \rho^{n,m}_N$, for any $T > 0$ in (5.11) and then on taking expectation and thereafter using (2.14) (in Remark 4), we get,

$$
\mathbb{E} \left[ \sup_{0 \leq t \leq T \wedge \rho^{n,m}_N} \eta(t) \left( \mu_2 \|v_n - v_m\|_{L^2}^2 + \mu_1 \|\tau_n - \tau_m\|_{L^2}^2 \right) \right] + \mu_2 \nu \mathbb{E} \left[ \int_0^{T \wedge \rho^{n,m}_N} \eta(s) \|\nabla(v_n - v_m)\|_{L^2}^2 ds \right] \\
\leq \mathbb{E} \left( \mu_2 \|v_n(0) - v_m(0)\|_{L^2}^2 + \mu_1 \|\tau_n(0) - \tau_m(0)\|_{L^2}^2 \right) \\
+ \frac{C}{n^2} \mathbb{E} \left[ \int_0^{T \wedge \rho^{n,m}_N} \eta(s) \left( \left( 2\sqrt{\mu_2} + 4\sqrt{\mu_1} \right) \|v_n\|_{H^2}^2 + 4\sqrt{\mu_1} \|\tau_n\|_{H^2}^2 + \sqrt{\mu_1} \|h\|_{H^2}^2 \|\tau_n\|_{H^2} + \sqrt{\mu_1} \|\nabla v_n\|_{H^2}^2 \right) \\
\times \left( \left( \sqrt{\mu_2} \|v_n - v_m\|_{L^2} + \sqrt{\mu_1} \|\tau_n - \tau_m\|_{L^2} \right) ds \right]
$$
Using Lemma A.4 we have
\[ |J_0| \leq \frac{\mu_2}{4} \mathbb{E} \left( \sup_{0 \leq t \leq T^{\cap \rho_N^m}} \eta(t) \| v_n(t) - v_m(t) \|_{L_2}^2 \right) + \frac{8CK \mu_2}{n^\varepsilon} \mathbb{E} \left[ \int_0^{T^{\cap \rho_N^m}} \eta(t)(1 + \| v_n \|_{L^2}^2) dt \right] + 8L \mu_2 \mathbb{E} \left[ \int_0^{T^{\cap \rho_N^m}} \eta(t) \| v_n - v_m \|_{L_2}^2 dt \right]. \] (5.13)

Exploiting Lemma A.5 we achieve
\[ |J_1| \leq \frac{\mu_1}{2} \mathbb{E} \left( \sup_{0 \leq t \leq T^{\cap \rho_N^m}} \eta(t) \| \tau_n(t) - \tau_m(t) \|_{L_2}^2 \right) + \frac{8 \mu_1}{n^\varepsilon} \mathbb{E} \left[ \int_0^{T^{\cap \rho_N^m}} \eta(t) \| h \|_{H^s}^2 \| \tau_n \|_{H^s}^2 dt \right] + 8 \mu_1 \| h \|_{L_\infty}^2 \mathbb{E} \left[ \int_0^{T^{\cap \rho_N^m}} \eta(t) \| \tau_n - \tau_m \|_{L_2}^2 dt \right]. \] (5.14)

Exploiting Lemma A.6 we have
\[ |J_2| \leq \frac{\mu_2}{4} \mathbb{E} \left( \sup_{0 \leq t \leq T^{\cap \rho_N^m}} \eta(t) \| v_n(t) - v_m(t) \|_{L_2}^2 \right) \leq \frac{8 \mu_2 CK}{n^\varepsilon} \mathbb{E} \left[ \int_0^{T^{\cap \rho_N^m}} \eta(t)(1 + \| v_n \|_{L^2}^2) dt \right] + 8 \mu_2 L \mathbb{E} \left[ \int_0^{T^{\cap \rho_N^m}} \eta(t) \| v_n - v_m \|_{L_2}^2 dt \right]. \] (5.15)

Combining (5.13), (5.14), and (5.15) and using (A.4), (A.19) and further using the fact that
\[ \sqrt{\mu_2} \| v_n - v_m \|_{L_2} + \sqrt{\mu_1} \| \tau_n - \tau_m \|_{L_2} \leq C(1 + \mu_2 \| v_n - v_m \|_{L_2}^2 + \mu_1 \| \tau_n - \tau_m \|_{L_2}^2), \]

equation (5.12) is reduced to
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T^{\cap \rho_N^m}} \eta(t) \left( \mu_2 \| v_n - v_m \|_{L_2}^2 + \mu_1 \| \tau_n - \tau_m \|_{L_2}^2 \right) \right] + 2 \mu_2 \mathbb{E} \left[ \int_0^{T^{\cap \rho_N^m}} \eta(s) \| \nabla (v_n - v_m) \|_{L_2}^2 ds \right] \\
\leq 2 \mathbb{E} \left( \mu_2 \| v_n(0) - v_m(0) \|_{L_2}^2 \right) + \mu_1 \| \tau_n(0) - \tau_m(0) \|_{L_2}^2 \right)
Note that the second term on the right hand side of (5.17) can be balanced with the first term of (5.17). Finally (5.17) becomes

Using definition of stopping time, Hölder’s inequality and rearranging, the above inequality reduces to

\[
\begin{align*}
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \eta(t) \left( \mu_2 \| \mathbf{v}_n - \mathbf{v}_m \|_{L^2}^2 + \mu_1 \| \tau_n - \tau_m \|_{L^2}^2 \right) \right] & \leq 2 \mathbb{E} \left[ \mu_2 \| \mathbf{v}_n(0) - \mathbf{v}_m(0) \|_{L^2}^2 + \mu_1 \| \tau_n(0) - \tau_m(0) \|_{L^2}^2 \right] \\
& \quad + \frac{1}{n^\nu} \left( 20 \| h \|_{H^s}^2 + 36 CK \right) \mathbb{E} \left[ \int_0^{T \wedge \rho_{n,m}} \eta(s) \left( \mu_2 + \mu_2 \| \mathbf{v}_n \|_{H^s}^2 + \mu_1 \| \tau_n \|_{H^s}^2 \right) ds \right]. \quad (5.16)
\end{align*}
\]

Note that the second term on the right hand side of (5.17) can be balanced with the first term of the left hand side of (5.17) for sufficiently large \( n \), so that \( \frac{2 \sqrt{C} \sqrt{N \sqrt{\tau_n}} \ll 1} {n} \). Therefore, using

\[
\mathbb{E} \left[ \int_0^{T \wedge \rho_{n,m}} \eta(t)dt \right] \leq \mathbb{E} \left[ \int_0^{T} \eta(t)dt \right] = \mathbb{E} \left[ \int_0^{T} \exp \left( -2C \int_0^t \| \nabla \mathbf{v}_n \|_{H^s}^2 ds \right) dt \right] \leq T,
\]

finally (5.17) becomes

\[
\begin{align*}
&\mathbb{E} \left[ \sup_{0 \leq t \leq T} \eta(t) \left( \mu_2 \| \mathbf{v}_n - \mathbf{v}_m \|_{L^2}^2 + \mu_1 \| \tau_n - \tau_m \|_{L^2}^2 \right) \right] + \frac{1}{n^\nu} \frac{\sqrt{2} C N}{\mu_2} \mathbb{E} \left[ \int_0^{T \wedge \rho_{n,m}} \eta(s) \left( \mu_2 \| \mathbf{v}_n - \mathbf{v}_m \|_{L^2}^2 + \mu_1 \| \tau_n - \tau_m \|_{L^2}^2 \right) ds \right] \\
&\leq 4 \mathbb{E} \left[ \mu_2 \| \mathbf{v}_n(0) - \mathbf{v}_m(0) \|_{L^2}^2 + \mu_1 \| \tau_n(0) - \tau_m(0) \|_{L^2}^2 \right] + \frac{1}{n^\nu} \frac{\sqrt{2} C N}{\mu_2} \left( 20 \| h \|_{H^s}^2 + 36 CK \right) \left( \mu_2 + N \right) + 2 \sqrt{2} C \left( \frac{(2 \sqrt{\tau_2} + 4 \sqrt{\tau_1}) N}{\mu_2} + \frac{4 N}{\sqrt{\mu_1}} + \| h \|_{H^s}^2 N \right) \\
&+ 2 \left( \sqrt{C^2 + a + 9 L} + \frac{2 C N}{\nu \mu_2} + 5 \| \tau_n \|_{L^2}^2 \right) + \frac{2 \sqrt{2} C}{n^\nu} \left( \frac{(2 \sqrt{\tau_2} + 4 \sqrt{\tau_1}) N}{\mu_2} + \frac{4 N}{\sqrt{\mu_1}} + \| h \|_{H^s}^2 N \right) \\
&\times \mathbb{E} \left[ \int_0^{T \wedge \rho_{n,m}} \sup_{0 \leq s \leq t} \eta(s) \left( \mu_2 \| \mathbf{v}_n - \mathbf{v}_m \|_{L^2}^2 + \mu_1 \| \tau_n - \tau_m \|_{L^2}^2 \right) dt \right]. \quad (5.18)
\end{align*}
\]
An application of standard Gronwall’s inequality yields

\[
\begin{aligned}
\E \left[ \sup_{0 \leq t \leq T-n/m} \eta(t) \left( \mu_2 \|v_n - v_m\|_{L^2}^2 + \mu_1 \|\tau_n - \tau_m\|_{L^2}^2 \right) \right] + 4\mu_2 \nu \E \left[ \int_0^{T-n/m} \eta(s) \|\nabla (v_n - v_m)\|_{L^2}^2 \, ds \right] \\
\leq \left( 4\E \left[ \mu_2 \|v_n(0) - v_m(0)\|_{L^2}^2 + \mu_1 \|\tau_n(0) - \tau_m(0)\|_{L^2}^2 \right] + C_1 \right)e^{C_2 T},
\end{aligned}
\]

where

\[
C_1 = \frac{2T}{n^2} \left( 20\|h\|_{H^s}^2 + 36CK \right) (\mu_2 + N) + 2\sqrt{2C} \left( \frac{(2\sqrt{\mu_2} + 4\sqrt{\mu_1})N}{\mu_2} + \frac{4N}{\sqrt{\mu_1}} + \|h\|_{H^s}^2 \right)
\]

and

\[
C_2 = 2 \left( 4 \left( C^2 + a + 9L \right) \frac{2CN}{\nu \mu_2} + 5\|h\|_{L^\infty}^2 \right) + \frac{2\sqrt{2C}}{n^2} \left( \frac{(2\sqrt{\mu_2} + 4\sqrt{\mu_1})N}{\mu_2} + \frac{4N}{\sqrt{\mu_1}} + \|h\|_{H^s}^2 \right).
\]

The right hand side of (5.19) tends to zero, since \(v_n(0) = J_n v_0, v_m(0) = J_m v_0, \tau_n(0) = J_n \tau_0 \) and \(\tau_m(0) = J_m \tau_0\), as \(n, m \to \infty\). Also from (5.18) and (5.19), we get

\[
\E \left[ \int_0^{T-n/m} \eta(t) \|\nabla (v_n - v_m)\|_{L^2}^2 \, dt \right] \to 0 \text{ as } n, m \to \infty.
\]

Since \(\eta(\cdot)\) is a bounded measurable \(\mathcal{F}_t\) adapted process, it directly yields the required results (i) and (ii).

\[ \square \]

**Remark 12.** As a consequence of Theorems 4.3 and 5.1 we conclude that there exists a stopping time \(\xi_N\) and processes \((v, \tau)\) such that \((v_n, \tau_n) \to (v, \tau)\) in \(L^2(\Omega; L^\infty(0, \xi_N \wedge T; L^2(\mathbb{R}^d)))\). We later (in Theorem 6.4) identify \(\xi_N\) as \(\rho_N\) (as defined in the Main Result 1), which is the pointwise limit of \(\rho_n^{n}\).

## 6. Existence and Uniqueness of Local Strong Solutions

**Proposition 6.1.** For any \(s' < s\) with \(s' > d/2\) and \(T > 0\), the following convergences hold:

(i) the family of solutions \((v_n, \tau_n) \to (v, \tau)\) strongly in the space \(L^2(\Omega; L^\infty(0, \xi_N \wedge T; H^{s'}(\mathbb{R}^d)))\) as \(n \to \infty\);

(ii) \(\nabla v_n \to \nabla v\) strongly in the space \(L^2(\Omega; L^2(0, \xi_N \wedge T; H^{s'}(\mathbb{R}^d)))\) as \(n \to \infty\);

(iii) \(\Delta v_n \to \Delta v\) strongly in the space \(L^2(\Omega; L^2(0, \xi_N \wedge T; H^{s'-1}(\mathbb{R}^d)))\) as \(n \to \infty\);

(iv) \(v \cdot \tau_n \to v \cdot \tau\) strongly in the space \(L^2(\Omega; L^\infty(0, \xi_N \wedge T; H^s(\mathbb{R}^d)))\) as \(n \to \infty\);

(v) \(D(v_n) \to D(v)\) strongly in the space \(L^2(\Omega; L^2(0, \xi_N \wedge T; H^s(\mathbb{R}^d)))\) as \(n \to \infty\).

**Proof.** We first prove (i). It follows from (5.19) that the sequence of solutions \((v_n, \tau_n) \to (v, \tau)\) strongly in \(L^2(\Omega; L^\infty(0, \xi_N \wedge T, L^2(\mathbb{R}^d)))\). Also from the estimate (5.20), we have \(\nabla v_n \to \nabla v\) strongly in \(L^2(\Omega; L^2(0, \xi_N \wedge T; L^2(\mathbb{R}^d)))\). Exploiting the interpolation inequality (Lemma 2.2 with exponents \(\frac{s}{s'-s}\) and \(\frac{s}{s'}\)) and Hölder’s inequality for \(0 < s' < s\), we obtain

\[
\E \left[ \sup_{0 \leq t \leq T-n} \|v_n - v\|_{H^s}^2 \right] \leq C \E \left[ \sup_{0 \leq t \leq T-n} \|v_n - v\|_{L^2}^2 \right]^{1-s'/s} \left( \E \left[ \sup_{0 \leq t \leq T-n} \|v_n - v\|_{H^s}^2 \right] \right)^{s'/s}
\]

\[
\leq C \left( \E \left[ \sup_{0 \leq t \leq T-n} \|v_n - v\|_{L^2}^2 \right] \right)^{1-s'/s} \left( \E \left[ \sup_{0 \leq t \leq T-n} \|v_n - v\|_{L^2}^2 \right] + \sup_{0 \leq t \leq T-n} \|v\|_{H^s}^2 \right)^{s'/s}
\]

\[
\leq (2N)^{s'/s} C \left( \E \left[ \sup_{0 \leq t \leq T-n} \|v_n - v\|_{L^2}^2 \right] \right)^{1-s'/s} \to 0, \text{ as } n \to \infty.
\]

Combining Remark 11 and Theorem 5.1 and using Sobolev interpolation for any \(s' < s\), we infer \((v_n, \tau_n) \to (v, \tau)\) strongly in \(L^2(\Omega; L^\infty(0, \xi_N \wedge T; H^{s}(\mathbb{R}^d)))\). This proves (i).
We also have $\nabla v_n \to \nabla v$ strongly in $L^2(\Omega; L^2(0, \xi \land T; L^2(\mathbb{R}^d)))$, hence for any $s' < s$, using Sobolev interpolation and similar arguments as above, we have $\nabla v_n \to \nabla v$ strongly in $L^2(\Omega; L^2(0, \xi \land T; H^{s'}(\mathbb{R}^d)))$. This directly implies (ii).

Proceeding in similar manner as above we directly have (iii), (iv) and (v).

\begin{proposition}
For $s' > d/2$ and $T > 0$,

(i) the quadratic form $J_\alpha Q(\tau_n, \nabla v_n) \to Q(\tau, \nabla v)$ strongly in $L^1(\Omega; L^\infty(0, \xi \land T; H^{s'}(\mathbb{R}^d)))$ as $n \to \infty$

(ii) the non-linear term $J_\alpha [v_n \cdot (\nabla \tau_n)] \to (v \cdot \nabla)\tau$ strongly in $L^1(\Omega; L^2(0, \xi \land T; H^{s'}(\mathbb{R}^d)))$ as $n \to \infty$.

\end{proposition}

\begin{proof}
For $s' > d/2$, by using \cite{2,2.3}, bilinear property of $Q$, Remark \cite{2} and Hölder’s inequality, for $0 < \epsilon < 1$, we have

\begin{align*}
&\mathbb{E} \left[ \sup_{0 \leq t \leq \xi \land T} \| J_\alpha Q(\tau_n, \nabla v_n) - Q(\tau, \nabla v) \|_{H^{s'}} \right] \\
&\leq \mathbb{E} \left[ \sup_{0 \leq t \leq \xi \land T} \| J_\alpha Q((\tau_n - \tau), \nabla v_n) \|_{H^{s'}} \right] + \mathbb{E} \left[ \sup_{0 \leq t \leq \xi \land T} \| J_\alpha Q(\tau, \nabla (v_n - v)) \|_{H^{s'}} \right] \\
&+ \mathbb{E} \left[ \sup_{0 \leq t \leq \xi \land T} \| J_\alpha Q(\tau, \nabla v) - Q(\tau, \nabla v) \|_{H^{s'}} \right] \\
&\leq C \mathbb{E} \left[ \sup_{0 \leq t \leq \xi \land T} \| Q((\tau_n - \tau), \nabla v_n) \|_{H^{s'}} \right] + C \mathbb{E} \left[ \sup_{0 \leq t \leq \xi \land T} \| Q(\tau, \nabla (v_n - v)) \|_{H^{s'}} \right] \\
&+ \frac{C}{n^\epsilon} \mathbb{E} \left[ \sup_{0 \leq t \leq \xi \land T} \| Q(\tau, \nabla v) \|_{H^{s'-1+\epsilon}} \right] \\
&\leq C \mathbb{E} \left[ \sup_{0 \leq t \leq \xi \land T} \| \tau_n - \tau \|_{L^\infty} \| \nabla v_n \|_{H^{s'}} \right] + \sup_{0 \leq t \leq \xi \land T} \| \tau_n - \tau \|_{H^{s'}} \| \nabla v_n \|_{L^\infty} \\
&+ C \mathbb{E} \left[ \sup_{0 \leq t \leq \xi \land T} \| \nabla (v_n - v) \|_{H^{s'}} \| \tau \|_{L^\infty} \right] + \sup_{0 \leq t \leq \xi \land T} \| \nabla (v_n - v) \|_{L^\infty} \| \tau \|_{H^{s'}} \\
&+ \frac{C}{n^\epsilon} \mathbb{E} \left[ \sup_{0 \leq t \leq \xi \land T} \| \nabla (v_n - v) \|_{H^{s'-1+\epsilon}} \right] \\
&\leq 2 C \mathbb{E} \left[ \sup_{0 \leq t \leq \xi \land T} \| \tau_n - \tau \|_{H^{s'}} \| \nabla v_n \|_{H^{s'}} \right] + 2 C \mathbb{E} \left[ \sup_{0 \leq t \leq \xi \land T} \| \nabla (v_n - v) \|_{H^{s'}} \| \tau \|_{H^{s'}} \right] \\
&+ \frac{C}{n^\epsilon} \mathbb{E} \left[ \sup_{0 \leq t \leq \xi \land T} \| \nabla (v_n - v) \|_{H^{s'-1+\epsilon}} \right] \\
&\leq 2 C \left[ \mathbb{E} \left( \sup_{0 \leq t \leq \xi \land T} \| \tau_n - \tau \|_{H^{s'}}^2 \right) \right]^{1/2} \left[ \mathbb{E} \left( \sup_{0 \leq t \leq \xi \land T} \| \nabla v_n \|_{H^{s'}}^2 \right) \right]^{1/2} \\
&+ 2 C \left[ \mathbb{E} \left( \sup_{0 \leq t \leq \xi \land T} \| \nabla (v_n - v) \|_{H^{s'}}^2 \right) \right]^{1/2} \left[ \mathbb{E} \left( \sup_{0 \leq t \leq \xi \land T} \| \tau \|_{H^{s'}}^2 \right) \right]^{1/2} \\
&+ \frac{C}{2n^\epsilon} \mathbb{E} \left[ \sup_{0 \leq t \leq \xi \land T} \| \nabla v_n \|_{H^{s'-1+\epsilon}}^2 \right] + \frac{C}{2n^\epsilon} \mathbb{E} \left[ \sup_{0 \leq t \leq \xi \land T} \| \tau \|_{H^{s'-1+\epsilon}}^2 \right] \\
&\to 0 \text{ as } n \to \infty.
\end{align*}

\end{proof}
Hence we have the convergences in 3 and 4.

Theorem 6.4. \[ \text{original problem (3.1)-(3.4)}. \]

Since for any \( t \), implementing Assumption 3.1 and Hölder’s inequality, we have

\[ \text{Hence, convergences in 1 and 2 are established.} \]

For any \( \tau \), Proposition 6.3.

we have the strong convergence in \( L^1(\Omega; L^2(0, \xi_N \cap T; H^{s'-1}(\mathbb{R}^d))) \).

This completes the proof for (i).

For \( s' > d/2 \), let \( \mathcal{F}_0 \)-measurable and \( \nabla \cdot v_0 = 0 \). Let \( v_0, \tau_0 \in L^2(\Omega; H^s(\mathbb{R}^d)) \) for \( s > d/2 \).

Then there exists a local in time strong solution of the problem (5.1)–(5.4) such that

1. \( v \in L^2(\Omega; L^\infty(0, \rho_N \wedge T; H^s(\mathbb{R}^d))) \cap L^2(0, \rho_N \wedge T; H^{s+1}(\mathbb{R}^d))), \tau \in L^2(\Omega; L^\infty(0, \rho_N \wedge T; H^s(\mathbb{R}^d))), \]

\[ \rho_N = \inf_{t \geq 0} \left\{ t : \mu_2 \| v(t) \|_{H^s} + \mu_1 \| \tau(t) \|_{H^s} + 2\mu_2 \nu \int_0^t \| \nabla v(r) \|_{H^1}^2 \, dr > N \right\}, \]

2. the \( \mathcal{F}_1 \)-adapted paths of \( (v, \rho_N) \) and \( (\tau, \rho_N) \) are càdlàg and continuous respectively.
(iii) \( \rho_N \) is a predictable strictly positive stopping time satisfying
\[
\mathbb{P}(\rho_N > \delta) \geq 1 - 2\delta e^{(C_1+C_2)\delta} \left( 2E(\mu_2\|u_0\|^2_{H^2} + \mu_1\|\theta_0\|^2_{H^2}) + 18K\mu_2\delta \right)
\]
for any \( \delta \in (0,1) \), and for some positive constant \( \mathcal{C} \) independent of \( \delta \).

Proof. Using the above Propositions, we can see from (4.1) that with probability 1
\[
(J^s\nu(t), J^s\phi_1)_{L^2} = (J^s\nu_0, J^s\phi_1)_{L^2} + \int_0^t (\nu\Delta J^s\nu - J^s[(\nu \cdot \nabla)\nu] + \mu_1 \nabla \cdot J^s\tau, J^s\phi_1)_{L^2} ds
\]
\[
+ \int_0^t (J^s\sigma(s, \nu_n(s))dW_1(s), J^s\phi_1)_{L^2} + \int_0^t \int_Z (J^sG(\nu(s-), z), J^s\phi_1)_{L^2} \tilde{N}_1(ds, dz),
\]
\[
(\nu \cdot \phi_n) = 0,
\]
are satisfied for any \( t \in [0, \rho_N^+ \wedge T) \) and \( \phi_1 \in H^s(\mathbb{R}^d) : i = 1, 2 \) with \( \nabla \cdot \phi_1 = 0 \).

Note, \( L^2(\Omega; L^\infty(0, \xi_N \wedge T; H^s(\mathbb{R}^d))) \) is the dual of \( L^2(\Omega; L^1(0, \xi_N \wedge T; H^{-s}(\mathbb{R}^d))) \) and \( L^2(\Omega; L^1(0, \xi_N \wedge T; H^{-s}(\mathbb{R}^d))) \) is separable Hilbert space (see Remark 10.1.10 and Theorem 10.1.13 of Papageorgiou and Kyritsi-Yiallourou [54]). Therefore, due to uniform boundedness of the sequences \( \nu_n \) and \( \tau_n \) from Remark 11, we can apply Banach-Alaoglu Theorem (see Theorem 4.18 of Robinson [58]), to extract subsequence \( \nu_{n_k} \) and \( \tau_{n_k} \) such that
\[
\nu_{n_k} \overset{w^*}{\rightarrow} \nu, \quad \tau_{n_k} \overset{w}{\rightarrow} \tau \quad \text{in} \quad L^2(\Omega; L^\infty(0, \xi_N \wedge T; H^s(\mathbb{R}^d)))
\]
and
\[
\nabla \nu_{n_k} \overset{w}{\rightarrow} \nabla \nu \quad \text{in} \quad L^2(\Omega; L^2(0, \xi_N \wedge T; H^s(\mathbb{R}^d))).
\]
This assures the limit satisfies
\[
\nu \in L^2(\Omega; L^\infty(0, \xi_N \wedge T; H^s(\mathbb{R}^d))) \cap L^2(\Omega; L^1(0, \xi_N \wedge T; H^{s+1}(\mathbb{R}^d))), \quad \tau \in L^2(\Omega; L^\infty(0, \xi_N \wedge T; H^s(\mathbb{R}^d))).
\]
Passing to the limit to (6.9)-(6.10) as \( n \to \infty \), it yields with probability 1
\[
(J^s\nu(t), J^s\phi_1)_{L^2} = (J^s\nu_0, J^s\phi_1)_{L^2} + \int_0^t (\nu\Delta J^s\nu - J^s[(\nu \cdot \nabla)\nu] + \mu_1 \nabla \cdot J^s\tau, J^s\phi_1)_{L^2} ds
\]
\[
+ \int_0^t (J^s\sigma(s, \nu)\nu(s))dW_1(s), J^s\phi_1)_{L^2} + \int_0^t \int_Z (J^sG(\nu(s-), z), J^s\phi_1)_{L^2} \tilde{N}_1(ds, dz),
\]
\[
(\nu \cdot \phi) = 0,
\]
for any \( t \in [0, \xi_N \wedge T) \). Hence, \( (\nu, \tau) \) solves \( (6.11)-(6.13) \) for \( s > d/2 \).

We now define
\[
\rho_N := \inf_{t \geq 0} \left\{ t : \mu_2\|\nu(t)\|_{H^2}^2 + \mu_1\|\tau(t)\|_{H^2}^2 + 2\mu_2\nu \int_0^t \|\nabla\nu(r)\|_{L^2}^2 dr > N \right\}.
\]
Claim: For fixed $N \geq 1$, $T > 0$, $\lim_{n \to \infty} \rho_N^n \land T = \rho_N \land T = \xi_N \land T$.

Proof.
Recalling the arguments as used for [6.11], we have (the subsequences still denoted by the same)

$$v_n \rightharpoonup v, \quad \tau_n \rightharpoonup \tau \quad \text{in} \quad L^2(\Omega; L^\infty(0, \xi_N \land T; H^s(\mathbb{R}^d)))$$

and $\nabla v_n \rightharpoonup \nabla v \quad \text{in} \quad L^2(\Omega; L^2(0, \xi_N \land T; H^s(\mathbb{R}^d))).$ (6.15)

Therefore by the lower semicontinuity property of weak and weak-star convergences (see Chapter 10 of Lax [5]), and using the inequality $\lim_{n \to \infty} \inf (f_n + g_n) \geq \lim_{n \to \infty} f_n + \lim_{n \to \infty} g_n$ for bounded sequence of functions $f_n$ and $g_n$, we obtain

$$\liminf_{n \to \infty} \mathbb{E} \left[ \sup_{0 \leq t \leq \xi_N \land T} \left( \mu_2 \|v_n(t)\|_{H^s}^2 + \mu_1 \|\tau_n(t)\|_{H^s}^2 \right) + 2\mu_2 \int_0^{\xi_N \land T} \|\nabla v_n(t)\|_{H^s}^2 dt \right]$$

$$\geq \mathbb{E} \left[ \sup_{0 \leq t \leq \xi_N \land T} \left( \mu_2 \|v(t)\|_{H^s}^2 + \mu_1 \|\tau(t)\|_{H^s}^2 \right) + 2\mu_2 \int_0^{\xi_N \land T} \|\nabla v(t)\|_{H^s}^2 dt \right].$$ (6.16)

Define

$$\mathcal{E}_n(t) := \mu_2 \|v(t)\|_{H^s}^2 + \mu_1 \|\tau(t)\|_{H^s}^2 + 2\mu_2 \int_0^t \|\nabla v_n(r)\|_{H^s}^2 dr,$$

and $E(t) := \mu_2 \|v(t)\|_{H^s}^2 + \mu_1 \|\tau(t)\|_{H^s}^2 + 2\mu_2 \int_0^t \|\nabla v(r)\|_{H^s}^2 dr.$ (6.17)

Therefore we may consider the case when

$$\liminf_{n \to \infty} \mathbb{E} \left[ \mathcal{E}_n(t) \right] \geq \mathbb{E} \left[ E(t) \right], \quad \text{for Lebesgue-almost all} \quad t \in [0, \xi_N \land T],$$ (6.18)

as this would imply (6.10) due to Fubini’s theorem.

By the definition of $\rho_N$ in [6.14], for each $\epsilon > 0$, there exists a $t_0 \in [0, T]$ such that $\rho_N \leq t_0 < \rho_N + \epsilon$

$$\mathcal{E}(t_0) > N.$$ (6.19)

If $[\rho_N, \rho_N + \epsilon) \subset [0, \xi_N \land T]$, by [6.18], along a subsequence of $\mathbb{E} \left[ \mathcal{E}_n(t_0) \right]$ (still denoted by the same) it converges to $\liminf_{n \to \infty} \mathbb{E} \left[ \mathcal{E}_n(t_0) \right]$, and this yields

$$\lim_{n \to \infty} \mathbb{E} \left[ \mathcal{E}_n(t_0) \right] \geq \mathbb{E} \left[ E(t_0) \right] > N.$$ (6.20)

Hence there exists $n \in \mathbb{N}$ such that $\mathbb{E} \left[ \mathcal{E}_n(t_0) \right] > N, \forall n \geq n$, from which we claim that $t_0 \geq \rho_N^n, \forall n \geq n$. If not, then there exists a natural number $n_1 > n$ such that $t_0 < \rho_N^n$. Hence $\mathcal{E}_{n_1}(t_0) \leq N$ and thus $\mathbb{E} \left[ \mathcal{E}_n(t_0) \right] \leq N$, a contradiction. Hence the claim is true.

Thus we have

$$\rho_N^n \land T \leq t_0 < (\rho_N + \epsilon) \land T, \quad \forall n \geq n.$$ (6.21)

Now, since $(v, \tau)$ is a local strong solution of [3.1]-[3.4] and the approximate equations [4.1]-[4.4] have unique strong solutions, we can identify $J_n v$ as $v_n$ and $J_n \tau$ as $\tau_n$. Since by [2.2], for every $s > 0$, $\|v_n\|_{H^s} \leq \|v\|_{H^s}$ and $\|\tau_n\|_{H^s} \leq \|\tau\|_{H^s}$, we have

$$N < \mu_2 \|v_n(t)\|_{H^s}^2 + \mu_1 \|\tau_n(t)\|_{H^s}^2 + 2\mu_2 \int_0^t \|\nabla v_n(r)\|_{H^s}^2 dr$$

$$\leq \mu_2 \|v(t)\|_{H^s}^2 + \mu_1 \|\tau(t)\|_{H^s}^2 + 2\mu_2 \int_0^t \|\nabla v(r)\|_{H^s}^2 dr, \quad \forall n \geq 1.$$ (6.22)

Hence,

$$N < \rho_N \land T \leq \rho_N^n \land T, \quad \forall n \geq n.$$ (6.23)

Therefore, for each $\epsilon > 0$,

$$(\rho_N - \epsilon) \land T \leq \rho_N^n \land T, \quad \forall n \geq 1.$$ (6.24)
Combining (6.21) and (6.24) we have for all \( n \geq \tilde{n} \) and for each \( \epsilon > 0 \),
\[
(p_n - \epsilon) \wedge T \leq \rho_n^0 \wedge T < (p_n + \epsilon) \wedge T.
\]
Taking limit as \( n \to \infty \), and using the definition of \( \xi_N \), we have
\[
(p_n - \epsilon) \wedge T \leq \xi_N \wedge T < (p_n + \epsilon) \wedge T.
\]
Since \( \epsilon > 0 \) is arbitrary, we finally infer \( \lim_{n \to \infty} \rho_n^0 \wedge T = \xi_N \wedge T = \rho_N \wedge T \). This proves the claim and (i).

From Remark 11 it is assured that \((v_n, \tau_n)\) is almost surely uniformly convergent to \((v, \tau)\) on finite interval \([0, \rho_N \wedge T]\), from which it follows that \( v \) is adapted and càdlàg (Theorem 6.2.3, Applebaum [2]) and \( \tau \) is continuous. Hence (ii) follows.

Now using the continuity argument and using (4.30) we achieve,
\[
\mathbb{P}(\rho_n > \delta) = \lim_{n \to \infty} \mathbb{P}(\rho_n^0 > \delta) \geq 1 - 2 \delta e^{(C_1 + C_2 \delta)} \left( 2 \mathbb{E}(\|v_0\|_{L^2}) + 2 \mu_1 \|\tau_0\|_{L^2} + 18 K \mu_2 \delta \right)
\]
for any \( \delta \in (0, 1) \), and for some positive constant \( C \) independent of \( \delta \).

Next we proceed to prove uniqueness of the local strong solution of (3.1)-(3.4).

**Theorem 6.5.** Let \( v_0, \tau_0 \) be \( \mathcal{F}_0 \)-measurable and \( \nabla \cdot v_0 = 0 \). Let \( v_0, \tau_0 \in L^2(\Omega; H^s(\mathbb{R}^d)) \) for \( s > d/2 \). Let \( v_i \) and \( \tau_i \) \( i = 1, 2 \) be \( \mathcal{F}_t \)-adapted càdlàg and continuous processes respectively such that \((v_i, \tau_i, \rho^N_i)\) are local strong solutions of (3.3)-(3.4) with the same initial conditions \( v_i(0) = v_0, \tau_i(0) = \tau_0 \), and \( v_i \in L^2(\Omega; L^\infty(0, \rho^N_i \wedge T; H^s(\mathbb{R}^d))) \cap L^2(0, \rho^N_i \wedge T; H^{s+1}(\mathbb{R}^d))) \), \( \tau_i \in L^2(\Omega; L^\infty(0, \rho^N_i \wedge T; H^s(\mathbb{R}^d))) \), for \( s > d/2 \). Then
\[
v_1(t) = v_2(t), \tau_1(t) = \tau_2(t) \quad \text{a.s.} \quad \forall t \in [0, \rho^N_1 \wedge T]
\]
as functions in \( L^2(\Omega; L^\infty(0, T; L^2(\mathbb{R}^d))) \). Moreover, \( \rho^N_1 = \rho^N_2 \mathbb{P} \text{-a.s.} \)

**Proof. Step I**

Let \((v_1, \tau_1, \rho^N_1)\) and \((v_2, \tau_2, \rho^N_2)\) be two local strong solutions of the system of equations (3.1)-(3.4) having common initial data \( v_1(0) = v_2(0) = v_0 \) and \( \tau_1(0) = \tau_2(0) = \tau_0 \) such that \( \mathbb{E}(\|v_0\|_{L^2}^2) < \infty \) and \( \mathbb{E}(\|\tau_0\|_{L^2}^2) < \infty \).

Considering the difference between the two equations satisfied by \((v_1, \tau_1)\) and \((v_2, \tau_2)\) we obtain
\[
\begin{align*}
\frac{d(v_1 - v_2)}{dt} &= \nu \Delta (v_1 - v_2) dt - \nabla \cdot (p_1 - p_2) dt - \|\nabla (v_1 - v_2)\| dt \\
&+ \mu_1 \nabla \cdot (\tau_1 - \tau_2) dt + (\sigma(t, v_1) - \sigma(t, v_2)) dW_1(t) \\
&+ \int_{\mathbb{R}^d} (G(v_1, z) - G(v_2, z)) \tilde{N}_1(dt, dz),
\end{align*}
\]
\[
\frac{d(\tau_1 - \tau_2)}{dt} = -d(\Delta(v_1 - v_2) + (Q(\tau_1, \nabla v_1) - Q(\tau_2, \nabla v_2)) dt \\
&+ \mu_2 D(v_1 - v_2) dt - a(\tau_1 - \tau_2) dt + \frac{1}{2} \left[ S^2(\tau_1) - S^2(\tau_2) \right] dt \\
&+ (S(\tau_1) - S(\tau_2)) dW_2(t).
\]
(6.26)
(6.27)

Let us apply Itô’s Lemma to the function \( \|x\|_{L^2}^2 \) and to the process \( \mu_2(v_1 - v_2) \) in (6.24) and to the process \( \mu_1(\tau_1 - \tau_2) \) in (6.27), and adding these two equations and further exploiting the fact that \( 2\mu_1 \mu_2 \left( \|\nabla (\tau_1 - \tau_2)\|_{L^2}^2 \right) v_1 - v_2)_{L^2} + \int (D(v_1 - v_2) - 2\mu_2(v_1 - v_2)_{L^2} v_1 - v_2)_{L^2} = 0 \) as \( \nabla \cdot v_1 = \nabla \cdot v_2 = 0 \), we achieve
\[
\begin{align*}
\frac{d}{dt} &\left( \|v_1 - v_2\|_{L^2}^2 + \mu_1 \|\tau_1 - \tau_2\|_{L^2}^2 + 2\mu_2 \nu \|\nabla (v_1 - v_2)\|_{L^2}^2 \right) dt \\
&= -2\mu_2 \left( \|\nabla (v_1 - v_2)\|_{L^2}^2 + \mu_1 \|\tau_1 - \tau_2\|_{L^2}^2 \right) dt \\
&\quad - \int_{l_5} -2\mu_1 (\|\nabla (v_1 - v_2)\|_{L^2}^2 + \mu_1 \|\tau_1 - \tau_2\|_{L^2}^2) dt
\end{align*}
\]
(6.28)
\[
-2\mu_1 \langle Q(\tau_1, \nabla v_1) - Q(\tau_2, \nabla v_2), \tau_1 - \tau_2 \rangle_{L^2} dt - 2a\mu_1 \|\tau_1 - \tau_2\|_{L^2}^2 dt \\
+ \mu_1 \langle \mathcal{S}^2(\tau_1) - \mathcal{S}^2(\tau_2), \tau_1 - \tau_2 \rangle_{L^2} dt + \mu_1 \|\mathcal{S}(\tau_1) - \mathcal{S}(\tau_2)\|_{L^2}^2 dt \\
+ \mu_2 \|\sigma(t, v_1) - \sigma(t, v_m)\|^2_{L^2(\Omega,L^2)} dt \\
+ \mu_2 \int_Z \|G(v_1(t^-), z) - G(v_2(t^-), z)\|^2_{L^2} N_1(dt, dz) \\
+ 2\mu_2 \left( (\sigma(t, v_1) - \sigma(t, v_m)) \right) dW_1(t), v_1 - v_2 \rangle_{L^2} \\
+ 2\mu_1 \langle S(\tau_1) - S(\tau_2), \tau_1 - \tau_2 \rangle_{L^2} dW_2(t) \\
+ 2\mu_2 \int_Z \left( G(v_1(t^-), z) - G(v_2(t^-), z), v_1 - v_2 \right)_{L^2} \tilde{N}_1(dt, dz). 
\] (6.28)

Using Lemmas (A.1, A.2, A.3), (A.16) in Lemma A.5 and 3.3 for \( v, \tau; i = 1, 2 \), estimating some of these integrals separately we have
\[
|I_5| \leq 2C\mu_2 \|v_1 - v_2\|_{L^2}^2 \|\nabla v_1\|_{H^s}, 
\] (6.29)
\[
|I_6| \leq \frac{\mu_2}{2} \|v_1 - v_2\|_{H^s}^2 + \frac{2\mu_1^2}{\nu\mu_2} \|\tau_1\|_{H^s}^2 \|\tau_1 - \tau_2\|_{L^2}^2, 
\] (6.30)
\[
|I_7| \leq \frac{\mu_2}{2} \|v_1 - v_2\|_{H^s}^2 + \frac{2\mu_1^2}{\nu\mu_2} \|\tau_2\|_{H^s}^2 \|\tau_2 - \tau_2\|_{L^2}^2 + 2\mu_1 C \|\nabla v_1\|_{H^s} \|\tau_1 - \tau_2\|_{L^2}^2, 
\] (6.31)
\[
|I_8| \leq \mu_1 \|h\|_{L^\infty}^2 \|\tau_1 - \tau_2\|_{L^2}^2, 
\] (6.32)
\[
|I_9| \leq 2\mu_1 \|h\|_{L^\infty}^2 \|\tau_1 - \tau_2\|_{L^2}^2. 
\] (6.33)

Combining (6.29)–(6.33) and implementing the fact that \( 4C \|\nabla v_1\|_{H^s} \leq C^2 + 2\|\nabla v_1\|_{H^s}^2 \), (6.28) reduces to:
\[
\begin{align*}
&d \left( \mu_2 \|v_1 - v_2\|_{L^2}^2 + \mu_1 \|\tau_1 - \tau_2\|_{L^2}^2 \right) + \mu_2 \nu \|\nabla(v_1 - v_2)\|_{L^2}^2 dt \\
&\leq \left( (2C^2 + 2a) + 2\|\nabla v_1\|_{H^s}^2 + \frac{2\mu_1^2}{\nu\mu_2} \|\tau_1\|_{H^s}^2 \|\tau_2\|_{H^s}^2 \right) + 2\|h\|_{L^\infty}^2 \\
&\times \left( \mu_2 \|v_1 - v_2\|_{L^2}^2 + \mu_1 \|\tau_1 - \tau_2\|_{L^2}^2 \right) \\
&+ \mu_2 \|\sigma(t, v_1) - \sigma(t, v_2)\|_{L^2(\Omega,L^2)}^2 dt \\
&+ \mu_2 \int_Z \|G(v_1(t^-), z) - G(v_2(t^-), z)\|^2_{L^2} N_1(dt, dz) \\
&+ 2\mu_2 \left( (\sigma(t, v_1) - \sigma(t, v_2)) \right) dW_1(t), v_1 - v_2 \rangle_{L^2} \\
&+ 2\mu_1 \langle S(\tau_1) - S(\tau_2), \tau_1 - \tau_2 \rangle_{L^2} dW_2(t) \\
&+ 2\mu_2 \int_Z \left( G(v_1(t^-), z) - G(v_2(t^-), z), v_1 - v_2 \right)_{L^2} \tilde{N}_1(dt, dz). 
\end{align*}
\] (6.34)

**Step II:**

For the stopping time \( \rho_N \), let us take the process \( \eta(t) = \exp \left( -2 \int_0^t \|\nabla v_1\|_{H^s}^2 ds \right), t \in [0, T \wedge \rho_N] \) and apply Itô product formula (see Theorem 4.4.13, Applebaum [2]) to the process \( \eta(t) \mu_2 \|v_1 - v_2\|_{L^2}^2 + \mu_1 \|\tau_1 - \tau_2\|_{L^2}^2 \) in the interval \([0, t]\) to get
\[
\begin{align*}
\eta(t) & \left( \mu_2 \|v_1 - v_2\|_{L^2}^2 + \mu_1 \|\tau_1 - \tau_2\|_{L^2}^2 \right) + \mu_2 \nu \int_0^t \eta(s) \left( \|\nabla v_1 - v_2\|_{L^2}^2 \right) ds \\
&\leq \int_0^t \eta(s) \left( (2C^2 + 2a) + \frac{2\mu_1^2}{\nu\mu_2} \left( \|\tau_1\|_{H^s}^2 \|\tau_2\|_{H^s}^2 \right) + 2\|h\|_{L^\infty}^2 \\
&\times \left( \mu_2 \|v_1 - v_2\|_{L^2}^2 + \mu_1 \|\tau_1 - \tau_2\|_{L^2}^2 \right) ds 
\end{align*}
\]
+ \mu_2 \int_0^t \eta(s) \frac{\|G(v_1(s), z) - G(v_2(s), z)\|^2_{L^2}}{L} \, ds
+ 2\mu_2 \int_0^t \eta(s) \left( \langle (\mathcal{J}_1(s, v_1) - \mathcal{J}_2(s, v_2)) \rangle \right) \, dW_1(s), v_1 - v_2\rangle_{L^2}
+ 2\mu_1 \int_0^t \eta(s) (S(\tau_1) - S(\tau_2), \tau_1 - \tau_2)_{L^2} \, dW_2(s)
+ 2\mu_2 \int_0^t \eta(s) \int_Z (G(v_1(s), z) - G(v_2(s), z), v_1 - v_2)_{L^2} \, d\tilde{N}_1(ds, dz). \tag{6.35}

It is to be noted that the quadratic variation of the product of these two adapted processes \(Z_1(t) = \eta(t)\) and \(Z_2(t) = \mu_2\|v_1 - v_2\|^2_{L^2} + \mu_1\|\tau_1 - \tau_2\|^2_{L^2}\), i.e., \([Z_1, Z_2](t)\) is zero (see, Section 4.4.3, page-257 of Applebaum [24]).

Let us now take the supremum from 0 to \(T \wedge \rho_N^\kappa\), for any \(T > 0\) in (6.35) and then on taking expectation and thereafter using (2.13) (in Remark 4), we get

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T \wedge \rho_N^\kappa} \eta(t) \left( \mu_2\|v_1 - v_2\|^2_{L^2} + \mu_1\|\tau_1 - \tau_2\|^2_{L^2} \right) \right] + \mu_2 \nu \mathbb{E} \left[ \int_0^{T \wedge \rho_N^\kappa} \eta(s) \|\nabla(v_1 - v_2)\|^2_{L^2} \, ds \right]
\leq \mathbb{E} \left[ \int_0^{T \wedge \rho_N^\kappa} \eta(s) \left( (2C^2 + 2a) + \frac{2C\mu_1}{\nu\mu_2} \left( \|\tau_1\|^2_{H^r} + \|\tau_2\|^2_{H^r} \right) + 2\|h\|^2_{L^\infty} \right) \right]
+ \mathbb{E} \left[ \int_0^{T \wedge \rho_N^\kappa} \eta(s) \|\sigma(s, v_1) - \sigma(s, v_2)\|^2_{ZQ(L^2, L^2)} \, ds \right]
+ \mathbb{E} \left[ \int_0^{T \wedge \rho_N^\kappa} \int_Z \eta(s) \left( \|G(v_1(s), z) - G(v_2(s), z)\|^2_{L^2} \right) \, \lambda(dz) \, ds \right]
+ 2\mu_2 \mathbb{E} \left[ \sup_{0 \leq t \leq T \wedge \rho_N^\kappa} \int_0^t \eta(s) \left( (\mathcal{J}_1(s, v_1) - \mathcal{J}_2(s, v_2)) \right) \, dW_1(s), v_1 - v_2\rangle_{L^2} \right]
+ 2\mu_2 \mathbb{E} \left[ \sup_{0 \leq t \leq T \wedge \rho_N^\kappa} \int_0^t \eta(s) (S(\tau_1) - S(\tau_2), \tau_1 - \tau_2)_{L^2} \, dW_2(s) \right]
+ 2\mathbb{E} \left[ \sup_{0 \leq t \leq T \wedge \rho_N^\kappa} \int_0^t \int_Z \eta(s) (G(v_1(s), z) - G(v_2(s), z), v_1 - v_2)_{L^2} \, d\tilde{N}_1(ds, dz) \right]. \tag{6.36}

We note that direct application of Burkholder-Davis-Gundy inequality, Young’s inequality and Assumption 3.1 produces the following estimates:

\[
|I_{10}| \leq \frac{\mu_2 L}{4} \mathbb{E} \left[ \sup_{0 \leq t \leq T \wedge \rho_N^\kappa} \eta(t)\|v_1(t) - v_2(t)\|^2_{L^2} \right] + 8\mu_2 L \mathbb{E} \left[ \int_0^{T \wedge \rho_N^\kappa} \eta(t)\|v_1 - v_2\|^2_{L^2} \, dt \right], \tag{6.37}
\]

\[
|I_{11}| \leq \frac{\mu_1}{2} \mathbb{E} \left[ \sup_{0 \leq t \leq T \wedge \rho_N^\kappa} \eta(t)\|\tau_1(t) - \tau_2(t)\|^2_{L^2} \right] + 8\mu_1 L \mathbb{E} \left( \int_0^{T \wedge \rho_N^\kappa} \eta(t)\|\tau_1 - \tau_2\|^2_{L^2} \, dt \right), \tag{6.38}
\]

and

\[
|I_{12}| \leq \frac{\mu_2}{4} \mathbb{E} \left( \sup_{0 \leq t \leq T \wedge \rho_N^\kappa} \eta(t)\|v_1(t) - v_2(t)\|^2_{L^2} \right) + 8\mu_2 L \mathbb{E} \left[ \int_0^{T \wedge \rho_N^\kappa} \eta(t)\|v_1 - v_2\|^2_{L^2} \, dt \right]. \tag{6.39}
\]
Combining (6.37) - (6.39) and Assumption 3.3, equation (6.40) is reduced to

$$\frac{1}{2} \mathbb{E} \left[ \sup_{0 \leq t \leq T \land \rho_N^1} \eta(t) \left( \mu_2 \|v_1 - v_2\|_{L^2}^2 + \mu_1 \|\tau_1 - \tau_2\|_{L^2}^2 \right) \right] + \mu_2 \nu \mathbb{E} \left[ \int_0^{T \land \rho_N^1} \eta(s) \|\nabla (v_1 - v_2)\|_{L^2}^2 ds \right]
$$

$$\leq \mathbb{E} \left[ \int_0^{T \land \rho_N^1} \eta(s) \left( 2C^2 + 2a + 18L \right) + \frac{2C_1 C_2}{\nu \mu_2} \left( \|\tau_1\|_{H^s}^2 + \|\tau_2\|_{H^s}^2 \right) + 10\|h\|_{L^\infty}^2 \right] \times \left( \mu_2 \|v_1 - v_2\|_{L^2}^2 + \mu_1 \|\tau_1 - \tau_2\|_{L^2}^2 \right) ds \right].$$

(6.40)

Using the definition of stopping time we have

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T \land \rho_N^1} \eta(t) \left( \mu_2 \|v_1 - v_2\|_{H^s}^2 + \mu_1 \|\tau(t)\|_{H^s}^2 + 2\mu_2 \nu \int_0^t \|\nabla v(r)\|_{H^s}^2 dr \leq N \right) \right].$$

(6.41)

**Step III:**

Define $A_N = \left\{ (v, \tau) : \mu_2 \|v(t)\|_{H^s}^2 + \mu_1 \|\tau(t)\|_{H^s}^2 + 2\mu_2 \nu \int_0^t \|\nabla v(r)\|_{H^s}^2 dr \leq N \right\}$. For $i = 1, 2$, let us define a process $\alpha^i = (\alpha^i(t), t \geq 0$, by

$$\alpha^i(t, \omega) := \begin{cases} 1, & \text{if } (v_1, \tau_1)(s, \omega) \in A_N, \forall s \in [0, t] \\ 0, & \text{otherwise.} \end{cases}$$

(6.42)

Note that for all $t \geq 0$, $\alpha^i(t)(v_1(0) - v_2(0)) = 0$ a.s., and $\alpha^i(t)(\tau_1(0) - \tau_2(0)) = 0$ a.s. for $i = 1, 2$. For any fixed $t \geq 0$, and for each $i = 1, 2$, $\{\alpha^i(t)\} = \{v_i, \tau_i)(s, \omega) \in A_N, \forall s \in [0, t] = \bigcap_{s \in [0, t]} \{v_i, \tau_i)(s, \omega) \in A_N \} \in \mathcal{F}$. Similarly, we can conclude $\{\alpha^i(t)\} \in \mathcal{F}$ for $i = 1, 2$. Hence for each $i = 1, 2$, $\alpha^i(t)$ is a random variable for each $t \geq 0$. Hence for each $i = 1, 2$, $\alpha^i$ is a well-defined stochastic process.

Hence from (6.41) we have

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T \land \rho_N^1} \alpha^1(t) \eta(t) \left( \mu_2 \|v_1 - v_2\|_{L^2}^2 + \mu_1 \|\tau_1 - \tau_2\|_{L^2}^2 \right) \right]
$$

$$\leq 4 \left( C^2 + a + 9L \right) + \frac{2C_1 C_2}{\nu \mu_2} \left( \|\tau_1\|_{H^s}^2 + \|\tau_2\|_{H^s}^2 \right) + 10\|h\|_{L^\infty}^2 \mathbb{E} \left[ \int_0^{T \land \rho_N^1} \sup_{0 \leq s \leq t} \alpha^1(s) \eta(s) \left( \mu_2 \|v_1 - v_2\|_{L^2}^2 + \mu_1 \|\tau_1 - \tau_2\|_{L^2}^2 \right) dt \right]
$$

$$\leq 4 \left( C^2 + a + 9L \right) + \frac{2C_1 C_2}{\nu \mu_2} \left( \|\tau_1\|_{H^s}^2 + \|\tau_2\|_{H^s}^2 \right) \mathbb{E} \left[ \int_0^{T} \sup_{0 \leq s \leq t \land \rho_N^1} \alpha^1(s) \eta(s) \left( \mu_2 \|v_1 - v_2\|_{L^2}^2 + \mu_1 \|\tau_1 - \tau_2\|_{L^2}^2 \right) dt \right].$$

(6.43)

Applying Gronwall’s inequality we obtain

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T \land \rho_N^1} \alpha^1(t) \eta(t) \left( \mu_2 \|v_1 - v_2\|_{L^2}^2 + \mu_1 \|\tau_1 - \tau_2\|_{L^2}^2 \right) \right] \leq 0.$$
and \( \mathbb{P}(\omega \in \Omega : \rho_N^1(\omega) = \rho_N^2(\omega)) = 1 \). This provides uniqueness of the local strong solution of the system \((3.1)-(3.4)\) in \( L^2(\Omega; L^\infty(0, T; L^2(\mathbb{R}^d))) \).

\[
\text{Theorem 6.6.} \quad \text{Let us construct a sequence of stopping times}
\]

\[
\tau_j \in L^2(\Omega; L^\infty(0, \rho_N^j \land T; H^s(\mathbb{R}^d))), \quad \text{for any} \quad 0 < s' < s.
\]

Then

\[
\mathbb{E}\left[ \sup_{0 \leq t \leq \rho_N^j \land \rho_N^k \land T} \| v_1 - v_2 \|_{H^{s'}}^2 \right] \leq C \left( \mathbb{E}\left[ \sup_{0 \leq t \leq \rho_N^j \land \rho_N^k \land T} \| v_1 - v_2 \|_{L^2}^2 \right] \right)^{1-s'/s} \cdot \left( \mathbb{E}\left[ \sup_{0 \leq t \leq \rho_N^j \land \rho_N^k \land T} \| v_1 - v_2 \|_{H^s}^2 \right] \right)^{s'/s} \leq (2N)^{s'/s} C \left( \mathbb{E}\left[ \sup_{0 \leq t \leq \rho_N^j \land \rho_N^k \land T} \| v_1 - v_2 \|_{L^2}^2 \right] \right)^{1-s'/s}. \quad (6.49)
\]

By Theorem 6.3,

\[
v_1(t) = v_2(t), \quad \tau_j(t) = \tau_2(t) \quad \text{a.s.} \quad \forall t \in [0, \rho_N^1 \land \rho_N^2 \land T]
\]

and \( \rho_N^1 = \rho_N^2 \), \( \mathbb{P} \)-a.s.

as functions in \( L^2(\Omega; L^\infty(0, \rho_N^1 \land T; L^2(\mathbb{R}^d))) \), we infer from (6.49) that

\[
\mathbb{E}\left[ \sup_{0 \leq t \leq \rho_N^1 \land \rho_N^2 \land T} \| v_1 - v_2 \|_{H^{s'}}^2 \right] = 0.
\]

Hence we achieve \( v_1(\cdot) = v_2(\cdot) \) in \( L^2(\Omega; L^\infty(0, \rho_N^1 \land \rho_N^2 \land T; H^{s'}(\mathbb{R}^d))) \) for any \( 0 < s' < s \).

By using \( (3.1)-(3.4) \) and Sobolev interpolation, we also get \( v_1(\cdot) = v_2(\cdot) \) in \( L^2(\Omega; L^2(0, \rho_N^1 \land \rho_N^2 \land T; H^{s'}(\mathbb{R}^d))) \). Similar calculation reveals that \( \tau_j(\cdot) = \tau_2(\cdot) \) in \( L^2(\Omega; L^\infty(0, \rho_N^1 \land \rho_N^2 \land T; H^{s'}(\mathbb{R}^d))) \) for any \( 0 < s' < s \). This implies \( \mathbb{P}(\omega \in \Omega : \rho_N^1(\omega) = \rho_N^2(\omega)) = 1 \), i.e., \( \rho_N^1 = \rho_N^2 \), \( \mathbb{P} \)-a.s. \( \Box \)

## 7. Existence and Uniqueness of Local Maximal Solutions

\[
\text{Theorem 7.1.} \quad \text{Under Assumption \( 3.1 \), Theorem 6.4 and Theorem 6.7 there exists a unique triplet \( (\mathbf{v}, \tau, \rho_S) \), which is a maximal strong solution of the system \((3.1)-(3.4)\) such that}
\]

\[
\sup_{0 \leq s \leq \rho_S} \| \mathbf{v}(s) \|_{H^s}^2 + \sup_{0 \leq s \leq \rho_S} \| \tau(s) \|_{H^s}^2 + \int_0^{\rho_S} \| \nabla \mathbf{v}(s) \|_{H^s}^2 \, ds = \infty, \text{ P - a.s. on } \{ \omega : \rho_S(\omega) < \infty \}.
\]

\[
\text{Proof.} \quad \text{Let us construct a sequence of stopping times } \{ \rho_k, k \in \mathbb{N} \} \text{ as follows:}
\]

\[
\rho_k = \inf_{t \geq 0} \left\{ t : \mu_2 \| \mathbf{v}_k(t) \|_{H^s}^2 + \mu_1 \| \tau_k(t) \|_{H^s}^2 + 2\mu_2 \nu \int_0^t \| \nabla \mathbf{v}_k(s) \|_{H^s}^2 \, ds > k \right\}. \quad (7.1)
\]
For $n > k$, let us define a sequence of stopping times $\rho^n_k$ such that
\[
\rho^n_k = \inf_{t \geq 0} \left\{ t : \mu_2\|v_n(t)\|_{H^s}^2 + \mu_1\|\tau_n(t)\|_{H^s}^2 + 2\mu_2\nu \int_0^t \|\nabla v_n(s)\|_{H^s}^2 \, ds > k \right\}, \quad k, n \in \mathbb{N}.
\] (7.3)
It is evident from the definition of $\rho_n$ that $\rho^n_k \leq \rho_n$ a.s. for $n > k$. Hence $(v_n, \tau_n, \rho^n_k)$ is a local strong solution to (6.5)-6.4 and $(v_k, \kappa_k, \rho_k)$ is also a local strong solution to (6.1)-(6.4). Hence by the uniqueness (see Theorems 6.5-6.6), we conclude that $(v_k(t), \kappa_k(t)) = (v_n(t), \tau_n(t))$ a.s. for all $t \in [0, \rho_k \wedge \rho^n_k \wedge T)$. This proves that $(v_k(t), \kappa_k(t)) = (v_n(t), \tau_n(t))$ a.s. for all $t \in [0, \rho_k \wedge T)$ and hence $\rho_k < \rho_n$ a.s. for all $k < n$. Thus \[ \{ \rho_k : k \in \mathbb{N} \} \] is an increasing sequence and has a limit $\rho_\infty := \lim_{k \to \infty} \rho_k$ a.s. By letting $k \to \infty$, let \[ \{ (v(t), \tau(t)), 0 \leq t < \rho_\infty \} \] be the stochastic processes defined by
\[
v(t) = v_k(t), \quad \tau(t) = \kappa_k(t), \quad t \in [\rho_{k-1} \wedge T, \rho_k \wedge T), \quad k \geq 1,
\] (7.4)
where $\rho_0 = 0$. Hence, $(v, \tau, \rho_\infty)$ is local strong solution to (6.1)-(6.4). We now have a triplet $(v, \tau, \rho_\infty)$ such that $(v, \tau, \rho_\infty)$ is local strong solution to (6.1)-(6.4). On the set \[ \{ \omega : \rho_\infty(\omega) < T \} \] we have
\[
\lim_{t \uparrow \rho_\infty} \left[ \sup_{0 \leq s \leq t} \mu_2\|v(s)\|_{H^s}^2 + \sup_{0 \leq s \leq t} \mu_1\|\tau(s)\|_{H^s}^2 + 2\mu_2\nu \int_0^t \|\nabla v(s)\|_{H^s}^2 \, ds \right]
\geq \lim_{k \to \infty} \left[ \sup_{0 \leq s \leq \rho_k \wedge T} \mu_2\|v(s)\|_{H^s}^2 + \sup_{0 \leq s \leq \rho_k \wedge T} \mu_1\|\tau(s)\|_{H^s}^2 + 2\mu_2\nu \int_0^t \|\nabla v(s)\|_{H^s}^2 \, ds \right]
\geq \lim_{k \to \infty} \left[ \sup_{0 \leq s \leq \rho_k \wedge T} \mu_2\|v_k(s)\|_{H^s}^2 + \sup_{0 \leq s \leq \rho_k \wedge T} \mu_1\|\kappa_k(s)\|_{H^s}^2 + 2\mu_2\nu \int_0^t \|\nabla v_k(s)\|_{H^s}^2 \, ds \right] = \infty.
\] (7.5)
Thus $(v, \tau, \rho_\infty)$ is a maximal local strong solution to (6.1)-(6.4). Now, in order to prove that this maximal strong solution is unique we let the triplet $(\tilde{v}, \tilde{\tau}, \sigma_\infty)$ be another maximal solution and \[ \{ \sigma_k, k \geq 0 \} \] is an increasing sequence of stopping times converging to $\sigma_\infty$ and is defined by
\[
\sigma_k = \inf_{t \geq 0} \left\{ t : \mu_2\|\tilde{v}(t)\|_{H^s}^2 + \mu_1\|\tilde{\tau}(t)\|_{H^s}^2 + 2\mu_2\nu \int_0^t \|\nabla \tilde{v}(s)\|_{H^s}^2 \, ds > k \right\}, \quad k \in \mathbb{N}.
\] (7.6)
Exploiting the same arguments as above and by the uniqueness Theorems 6.5 and 6.6 one can prove that $v(t) = \tilde{v}(t), \tau(t) = \tilde{\tau}(t)$ for all $t \in [0, \rho_k \wedge \sigma_k \wedge T]$ a.s. for $k \geq 0$. Hence,
\[
v(t) = \tilde{v}(t), \quad \tau(t) = \tilde{\tau}(t) \quad \text{for all } t \in [0, \rho_\infty \wedge \sigma_\infty \wedge T] \text{ a.s.}
\] (7.7)
on letting $k \uparrow \infty$. From (7.7), one can easily verify that $\rho_\infty = \sigma_\infty$ a.s. However, if not, then either $\rho_\infty > \sigma_\infty$ or $\rho_\infty < \sigma_\infty$. Now for the first case we have
\[
\lim_{t \uparrow \rho_\infty} \left[ \sup_{0 \leq s \leq t} \mu_2\|1(\sigma_\infty < \rho_\infty) v\|_{H^s}^2 + \sup_{0 \leq s \leq t} \mu_1\|1(\sigma_\infty < \rho_\infty) \tau\|_{H^s}^2 + 2\mu_2\nu \int_0^t \|\nabla 1(\sigma_\infty < \rho_\infty) v\|_{H^s}^2 \, ds \right]
= \lim_{k \to \infty} \left[ \sup_{0 \leq s \leq \rho_k \wedge T} \mu_2\|1(\sigma_\infty < \rho_\infty) v\|_{H^s}^2 + \sup_{0 \leq s \leq \rho_k \wedge T} \mu_1\|1(\sigma_\infty < \rho_\infty) \tau\|_{H^s}^2 + 2\mu_2\nu \int_0^t \|\nabla 1(\sigma_\infty < \rho_\infty) v\|_{H^s}^2 \, ds \right]
= \lim_{k \to \infty} \left[ \sup_{0 \leq s \leq \sigma_k \wedge T} \mu_2\|1(\sigma_\infty < \rho_\infty) \tilde{v}\|_{H^s}^2 + \sup_{0 \leq s \leq \sigma_k \wedge T} \mu_1\|1(\sigma_\infty < \rho_\infty) \tilde{\tau}\|_{H^s}^2 + 2\mu_2\nu \int_0^t \|\nabla 1(\sigma_\infty < \rho_\infty) \tilde{v}\|_{H^s}^2 \, ds \right]
= \infty.
\] (7.8)
and for the second case,
\[
\lim_{t \uparrow \rho_\infty} \left[ \sup_{0 \leq s \leq t} \mu_2\|1(\sigma_\infty > \rho_\infty) v\|_{H^s}^2 + \sup_{0 \leq s \leq t} \mu_1\|1(\sigma_\infty > \rho_\infty) \tau\|_{H^s}^2 + 2\mu_2\nu \int_0^t \|\nabla 1(\sigma_\infty > \rho_\infty) v\|_{H^s}^2 \, ds \right]
= \lim_{k \to \infty} \left[ \sup_{0 \leq s \leq \rho_k \wedge T} \mu_2\|1(\sigma_\infty > \rho_\infty) v\|_{H^s}^2 + \sup_{0 \leq s \leq \rho_k \wedge T} \mu_1\|1(\sigma_\infty > \rho_\infty) \tau\|_{H^s}^2 + 2\mu_2\nu \int_0^t \|\nabla 1(\sigma_\infty > \rho_\infty) v\|_{H^s}^2 \, ds \right]
= \lim_{k \to \infty} \left[ \sup_{0 \leq s \leq \rho_k \wedge T} \mu_2\|1(\sigma_\infty > \rho_\infty) \tilde{v}\|_{H^s}^2 + \sup_{0 \leq s \leq \rho_k \wedge T} \mu_1\|1(\sigma_\infty > \rho_\infty) \tilde{\tau}\|_{H^s}^2 + 2\mu_2\nu \int_0^t \|\nabla 1(\sigma_\infty > \rho_\infty) \tilde{v}\|_{H^s}^2 \, ds \right]
= \infty.
\] (7.9)
Identity \((7.8)\) contradicts the fact that \((v, \tau)\) does not explode before the time \(\rho_\infty\) and the next identity \((7.9)\) contradicts the fact that \((\tilde{v}, \tilde{\tau})\) does not explode before the time \(\sigma_\infty\). Hence, the only possibility is \(\rho_\infty = \sigma_\infty\) a. s. and this proves the uniqueness of the maximal local strong solution \((v, \tau, \rho_\infty)\) of the Stochastic equations \((3.1)-(3.3)\).

Similar ideas of proving maximal local solutions in Proposition 3.11 of Brzéniak et al. \[8\], Theorem 3.5 of Bessah et al. \[6\], and Theorem 5.4 of Manna et al. \[46\].

Finally observing that since \(\rho_k \uparrow \rho_\infty\), for any fixed \(0 < \delta < 1\) and for the choice of \(k\) with \(1 \leq \delta < \frac{1}{k}\), we infer from \((4.0)\) in Theorem 4.3

\[
P(\rho_\infty > \delta) \geq 1 - 2\delta e^{(C + C_2\delta)} \left( 2E \left( \mu_2 \|v_0\|^2_{H^s} + \mu_1 \|\tau_0\|^2_{H^s} \right) + 18K\mu_2\delta \right).
\]

APPENDIX A.

The Appendix is devoted to prove some small Lemmas which were useful in proving Theorem 5.1 and Theorem 6.5.

**Lemma A.1.** Let \(\nabla \cdot v = 0\). Then for \(m > n\) there exists \(C > 0\) (independent of \(n, m, v, \epsilon\)) such that

\[
\left| (\mathcal{J}_n[(v_n \cdot \nabla)v_n] - \mathcal{J}_m[(v_m \cdot \nabla)v_m], v_n - v_m)_{L^2} \right| \leq \frac{C}{n^\epsilon} \|v_n\|_{H^s}^2 \|v_n - v_m\|_{L^2} + C\|v_n - v_m\|_{L^2}^2 \|\nabla v_n\|_{H^s}.
\]

**Proof.** In order to estimate the term \((\mathcal{J}_n[(v_n \cdot \nabla)v_n] - \mathcal{J}_m[(v_m \cdot \nabla)v_m], v_n - v_m)_{L^2}\) we split it into three parts

\[
\left(\mathcal{J}_n - \mathcal{J}_m\right)[(v_n \cdot \nabla)v_n], v_n - v_m\right)_{L^2} + \left(\mathcal{J}_m[(v_n - v_m) \cdot \nabla)v_n], v_n - v_m\right)_{L^2}
\]

For \(m > n\), we exploit Hölder’s inequality, the cut off property [see \((2.4)\)] and \(H^s\) is an algebra for \(0 < \epsilon < s - 1\), \(s > d/2\) to the first term of \((A.2)\) to achieve,

\[
\left| (\mathcal{J}_n - \mathcal{J}_m)[(v_n \cdot \nabla)v_n], v_n - v_m\right)_{L^2} \leq \left\| (\mathcal{J}_n - \mathcal{J}_m)[(v_n \cdot \nabla)v_n]\right\|_{L^2} \|v_n - v_m\|_{L^2} \leq \frac{C}{n^\epsilon} \|v_n\|_{H^s}^2 \|v_n - v_m\|_{L^2}.
\]

Direct application of Hölder’s inequality to the second term of \((A.2)\) yields

\[
\left| (\mathcal{J}_m[(v_n - v_m) \cdot \nabla)v_n], v_n - v_m\right)_{L^2} \leq \left| (\mathcal{J}_m[(v_n - v_m) \cdot \nabla)v_n]\right| \|v_n - v_m\|_{L^2} \|\nabla v_n\|_{H^s} \leq C\|v_n - v_m\|_{H^s}^2 \|\nabla v_n\|_{H^s}.
\]

Now using the Parseval’s identity, integration by parts and \(\nabla \cdot v_n = \nabla \cdot v_m = 0\), we directly have the third term of \((A.2)\) is zero. Hence, we have \((A.1)\).

**Lemma A.2.** Let \(\nabla \cdot v = 0\). Then for \(m > n\) there exists \(C > 0\) (independent of \(n, m, v, \epsilon\)) such that

\[
\left| (\mathcal{J}_n[(v_n \cdot \nabla)\tau_n] - \mathcal{J}_m[(v_m \cdot \nabla)\tau_m], \tau_n - \tau_m)_{L^2} \right| \leq \frac{C}{n^\epsilon} \left( \|v_n\|_{H^s}^2 + \|\tau_n\|_{H^s}^2 \right) \|\tau_n - \tau_m\|_{L^2} + \frac{\nu\mu_2}{4\mu_1} \|v_n - v_m\|_{H^s}^2 + \frac{C\mu_1}{\nu\mu_2} \|\tau_n\|_{H^s}^2 \|\tau_n - \tau_m\|_{L^2}.
\]

**Proof.** Let us consider the term \((\mathcal{J}_n[(v_n \cdot \nabla)\tau_n] - \mathcal{J}_m[(v_m \cdot \nabla)\tau_m], \tau_n - \tau_m)_{L^2}\) and split it into three parts

\[
\left(\mathcal{J}_n - \mathcal{J}_m\right)[(v_n \cdot \nabla)\tau_n], \tau_n - \tau_m\right)_{L^2} + \left(\mathcal{J}_m[(v_n - v_m) \cdot \nabla)\tau_n], \tau_n - \tau_m\right)_{L^2}
\]

We use here same arguments as used in the previous Lemma \((A.1)\) Hence the first term in \((A.3)\) is reduced to

\[
\left| (\mathcal{J}_n - \mathcal{J}_m)[(v_n \cdot \nabla)\tau_n], \tau_n - \tau_m\right)_{L^2} \leq \frac{C}{n^\epsilon} \|v_n\|_{H^s} \|\tau_n\|_{H^s} \|\tau_n - \tau_m\|_{L^2}.
\]
Consider the term \((J_m)((v_n - v_m) \cdot \nabla)\tau_n, \tau_n - \tau_m)_{L^2}\) and using and use Hölder’s inequality, Remark \([3]\) to get

\[
\left|\left(\frac{\nu_2}{4\mu_1} \|v_n - v_m\|^2_{H^1} + \frac{C\mu_1}{\nu_2} \|\tau_n\|^2_{H^s} \right) \|\tau_n - \tau_m\|_{L^2} \right| \leq \frac{C}{n^\epsilon} \left( \|v_n\|^2_{H^s} + \|\tau_n\|^2_{H^s} \right) \|\tau_n - \tau_m\|_{L^2}.
\]

(A.7)

Once again on applying Parseval’s identity, integration by parts and divergence free condition on \(v_m\) we get the third term of (A.6) to be zero. Hence, we have (A.5).

Lemma A.3. For \(m > n\) there exists \(C > 0\) (independent of \(n, m, v, \epsilon\)) such that

\[
\left|\left(\frac{J_n Q(\tau_n, \nabla v_n) - J_m Q(\tau_m, \nabla v_m), \tau_n - \tau_m)_{L^2}\right)\right| \leq \frac{C}{n^\epsilon} \left( \|v_n\|^2_{H^s} + \|\tau_n\|^2_{H^s} \right) \|\tau_n - \tau_m\|_{L^2} + \frac{C\mu_1}{\nu_2} \|\tau_n\|^2_{H^s} \|\tau_n - \tau_m\|_{L^2} + C\|\nabla v_n\|_{H^s} \|\tau_n - \tau_m\|_{L^2}.
\]

(A.9)

Proof. We recall from the definition of \(Q\) that \(J_n Q(\tau_n, \nabla v_n) = J_n\left(\tau_n W(v_n)\right) - J_n\left(W(v_n)\tau_n\right) - bJ_n\left(D(v_n)\tau_n - \tau_n D(v_n)\right)\). Let us estimate the following term for \(m > n\),

\[
J_n\left(\tau_n W(v_n)\right) - J_m\left(\tau_m W(v_m)\right) = \left(\left(J_n - J_m\right)\tau_n W(v_n) + J_n\left((\tau_n - \tau_m) W(v_n)\right) + J_m\left(\tau_m (W(v_n) - W(v_m))\right)\right).
\]

Hence,

\[
\left|\left(\frac{J_n\left(\tau_n W(v_n)\right) - J_m\left(\tau_m W(v_m)\right), \tau_n - \tau_m)_{L^2}\right)\right| \leq \left|\left(\frac{\left[J_n\left(\tau_n W(v_n)\right) - J_m\left(\tau_n W(v_n)\right)\right], \tau_n - \tau_m)_{L^2}\right)\right| + \left|\left(\frac{\left[J_n\left(\tau_n W(v_n)\right)\right], \tau_n - \tau_m)_{L^2}\right)\right| + \left|\left(\frac{\left[J_m\left(\tau_m (W(v_n) - W(v_m))\right)\right], \tau_n - \tau_m)_{L^2}\right)\right|.
\]

Now, for \(0 < \epsilon < s - 1\), using Remark \([11]\) properties of \(J_n\) \((2.2), (2.3), (2.4)\) in Subsection \(2.3\) and for \(s > 0\), using the embedding \(H^s \subset H^{s-1}\) and using Young’s inequality we get

\[
|I_{13}| \leq \left(\|J_n - J_m\|_{L^2}\|\tau_n W(v_n)\|_{L^2}\|\tau_n - \tau_m\|_{L^2}\right) \leq \frac{C}{n^\epsilon} \|\tau_n W(v_n)\|_{H^s} \|\tau_n - \tau_m\|_{L^2} \leq \frac{C}{n^\epsilon} \|\nabla v_n\|_{H^s} \|\tau_n - \tau_m\|_{L^2} \leq \frac{C}{n^\epsilon} \left(\|v_n\|^2_{H^s} + \|\tau_n\|^2_{H^s}\right) \|\tau_n - \tau_m\|_{L^2} \leq \frac{C}{n^\epsilon} \left(\|v_n\|^2_{H^s} + \|\tau_n\|^2_{H^s}\right) \|\tau_n - \tau_m\|_{L^2}.
\]

Again applying similar arguments we have,

\[
|I_{14}| \leq \|\tau_n - \tau_m\|_{L^2}\|J_m(\tau_n - \tau_m)\|_{L^2} \leq C\|\nabla v_n\|_{H^s} \|\tau_n - \tau_m\|^2_{L^2}.
\]

(A.10)

Similarly the term \(I_{15}\) is reduced to

\[
|I_{15}| \leq C\|\tau_m(W(v_n) - W(v_m))\|_{L^2}\|\tau_n - \tau_m\|_{L^2} \leq C\|\tau_m\|_{L^\infty}\|W(v_n) - W(v_m)\|_{L^2}\|\tau_n - \tau_m\|_{L^2} \leq \frac{\nu_1}{16\mu_1} \|\nabla(v_n - v_m)\|^2_{L^2} + \frac{C\mu_1}{\nu_2} \|\tau_m\|^2_{H^s} \|\tau_n - \tau_m\|^2_{L^2}.
\]

(A.11)

Therefore after combining all the similar estimates for other terms of \(J_n Q(\tau_n, \nabla v_n)\), finally we have (A.9).
Lemma A.4. Let $\eta(t)$ be a stochastic process. Let $\rho^N_N$ be the stopping time given by \eqref{E}. Denote $\rho^m_N = \rho_N^m \wedge \rho^m_N$. Then

$$
E \left[ \sup_{0 \leq t \leq T_N^{\rho^m_N}} \left| \int_0^t \eta(s) \left( \left( J_n \sigma(s, v_n) - J_m \sigma(s, v_m) \right) dW_1(s), v_n(s) - v_m(s) \right) \right|_{L^2} \right] 
\leq \frac{1}{8} E \left( \sup_{0 \leq t \leq T_N^{\rho^m_N}} \eta(t) \| v_n(t) - v_m(t) \|^2_{L^2} \right) 
+ 4CK \nu^2 \int_0^{T_N^{\rho^m_N}} \eta(t)(1 + \| v_n \|^2_{H^2}) dt + 4L E \left[ \int_0^{T_N^{\rho^m_N}} \eta(t) \| v_n - v_m \|^2_{L^2} dt \right]. \tag{A.12}
$$

Proof. Let us take the term on the left hand side of the estimate \eqref{A.12} and apply Burkholder-Davis-Gundy inequality and Young's inequality to get

$$
E \left[ \sup_{0 \leq t \leq T_N^{\rho^m_N}} \left| \int_0^t \eta(s) \left( \left( J_n \sigma(s, v_n) - J_m \sigma(s, v_m) \right) dW_1(s), v_n(s) - v_m(s) \right) \right|_{L^2} \right] 
\leq 2\sqrt{2} E \left( \int_0^{T_N^{\rho^m_N}} (\eta(t))^2 \| J_n \sigma(t, v_n) - J_m \sigma(t, v_m) \|^2_{L^Q(L^2,L^2)} \| v_n(t) - v_m(t) \|^2_{L^2} dt \right)^{1/2}
\leq 2\sqrt{2} E \left( \sup_{0 \leq t \leq T_N^{\rho^m_N}} \eta(t) \| v_n(t) - v_m(t) \|^2_{L^2} \right)^{1/2} \times 
\left( \int_0^{T_N^{\rho^m_N}} \eta(t) \| J_n \sigma(t, v_n) - J_m \sigma(t, v_m) \|^2_{L^Q(L^2,L^2)} dt \right)^{1/2} 
\leq \frac{1}{8} E \left( \sup_{0 \leq t \leq T_N^{\rho^m_N}} \eta(t) \| v_n(t) - v_m(t) \|^2_{L^2} \right) 
+ 4E \left[ \int_0^{T_N^{\rho^m_N}} \eta(t) \| J_m \sigma(t, v_n) - J_n \sigma(t, v_m) \|^2_{L^Q(L^2,L^2)} dt \right]. \tag{A.13}
$$

Now, exploiting Assumption \ref{E} and cut off property (for the noise term) \ref{B.6} we have for $0 < \epsilon < s - 1$,

$$
E \left[ \int_0^{T_N^{\rho^m_N}} \eta(t) \| J_n \sigma(t, v_n) - J_m \sigma(t, v_m) \|^2_{L^Q(L^2,L^2)} dt \right] 
\leq E \left[ \int_0^{T_N^{\rho^m_N}} \eta(t) \| (J_n - J_m) \sigma(t, v_n) \|^2_{L^Q(L^2,L^2)} dt \right] 
+ E \left[ \int_0^{T_N^{\rho^m_N}} \eta(t) \| J_m \sigma(t, v_n) - \sigma(t, v_n) \|^2_{L^Q(L^2,L^2)} dt \right] 
\leq C \nu E \left[ \int_0^{T_N^{\rho^m_N}} \eta(t) \| \sigma(t, v_n) \|^2_{L^Q(L^2,H^\epsilon)} dt \right] 
+ E \left[ \int_0^{T_N^{\rho^m_N}} \eta(t) \| \sigma(t, v_n) - \sigma(t, v_m) \|^2_{L^Q(L^2,L^2)} dt \right] 
\leq CK \nu E \left[ \int_0^{T_N^{\rho^m_N}} \eta(t)(1 + \| v_n \|^2_{H^\epsilon}) dt \right] + L E \left[ \int_0^{T_N^{\rho^m_N}} \eta(t) \| v_n - v_m \|^2_{L^2} dt \right]. \tag{A.14}
$$

Lemma A.5. Let $\eta(t)$ be a stochastic process. Let $\rho^N_N$ be the stopping time given by \eqref{E}. Denote $\rho^m_N = \rho_N^m \wedge \rho^m_N$. Then

$$
E \left[ \sup_{0 \leq t \leq T_N^{\rho^m_N}} \left| \int_0^t \eta(s) \left( \left( J_n S(\tau_n) - J_m S(\tau_m), \tau_n(s) - \tau_m(s) \right) \right) dW_2(s) \right|_{L^2} \right] 
$$
Further exploiting Assumption 3.1 and (2.4), for $0 < \epsilon < s - 1$ we have

$$
\leq \frac{1}{8} E \left( \sup_{0 \leq t \leq T} \eta(t) \left\| \tau_n(t) - \tau_m(t) \right\|^2_{L^2} \right)
+ \frac{4}{n^2} \|h\|^2_{H^2} E \left( \int_0^{T \wedge \rho_{\alpha}^{n,m}} \eta(t) \| \tau_n(t) \|^2_{H^2} dt \right) + 4 \|h\|^2_{L^\infty} E \left( \int_0^{T \wedge \rho_{\alpha}^{n,m}} \eta(t) \| \tau_n(t) - \tau_m(t) \|^2_{L^2} dt \right).
$$

(A.15)

Proof. Using (2.2), (2.4), we have the following inequality

$$
\|J_n S(\tau_n) - J_m S(\tau_m)\|^2_{L^2} \leq \frac{2}{n^2} \|h\|^2_{H^2} \| \tau_n \|^2_{H^2} + 2 \|h\|^2_{L^\infty} \| \tau_n - \tau_m \|^2_{L^2}.
$$

(A.16)

Again application of Burkholder-Davis-Gundy inequality, Young’s inequality and (A.16) produces the required estimate. \qed

Lemma A.6. Let $\eta(t)$ be a stochastic process. Let $\rho_{\alpha}^n$ be the stopping time given by (L.5). Denote $\rho_{\alpha}^{n,m} = \rho_{\alpha}^n \wedge \rho_{\alpha}^m$. Then

$$
E \left[ \sup_{0 \leq t \leq T} \int_{Z} \eta(s) \left( J_n G(v_n(s-), z) - J_m G(v_m(s-), z) - v_n(s-) + v_m(s-) \right) \tilde{N}(ds, dz) \right]
\leq \frac{1}{8} E \left( \sup_{0 \leq t \leq T} \eta(t) \left\| v_n(t) - v_m(t) \right\|^2_{L^2} \right) + \frac{4CK}{n^2} E \left( \int_0^{T \wedge \rho_{\alpha}^{n,m}} \eta(t)(1 + \|v_n\|^2_{H^2}) dt \right)
+ 4 E \left( \int_0^{T \wedge \rho_{\alpha}^{n,m}} \int_Z \eta(t) \| J_n G(v_n(s-), z) - J_m G(v_m(s-), z) \|^2_{L^2} \lambda(dz) dt \right).
$$

(A.17)

Proof. An application of Burkholder-Davis-Gundy inequality, Young’s inequality produces the required estimate

$$
E \left[ \sup_{0 \leq t \leq T} \int_{Z} \eta(s) \left( J_n G(v_n(s-), z) - J_m G(v_m(s-), z) - v_n(s-) + v_m(s-) \right) \tilde{N}(ds, dz) \right]
\leq \frac{1}{8} E \left( \sup_{0 \leq t \leq T} \eta(t) \left\| v_n(t) - v_m(t) \right\|^2_{L^2} \right)
+ 4 E \left( \int_0^{T \wedge \rho_{\alpha}^{n,m}} \int_Z \eta(t) \| J_n G(v_n(s-), z) - J_m G(v_m(s-), z) \|^2_{L^2} \lambda(dz) dt \right).
$$

(A.18)

Further exploiting Assumption 3.1 and (2.4), for $0 < \epsilon < s - 1$ we have

$$
E \left[ \int_0^{T \wedge \rho_{\alpha}^{n,m}} \int_Z \eta(t) \| J_n G(v_n, z) - J_m G(v_m, z) \|^2_{L^2} \lambda(dz) dt \right]
\leq E \left[ \int_0^{T \wedge \rho_{\alpha}^{n,m}} \int_Z \eta(t) \| (J_n - J_m) G(v_n, z) \|^2_{L^2} \lambda(dz) dt \right]
+ E \left[ \int_0^{T \wedge \rho_{\alpha}^{n,m}} \int_Z \eta(t) \| J_m G(v_n, z) - G(v_m, z) \|^2_{L^2} \lambda(dz) dt \right]
\leq \frac{C}{n^2} E \left[ \int_0^{T \wedge \rho_{\alpha}^{n,m}} \int_Z \eta(t) \| G(v_n, z) \|^2_{H^2} \lambda(dz) dt \right]
+ E \left[ \int_0^{T \wedge \rho_{\alpha}^{n,m}} \int_Z \eta(t) \| G(v_m, z) \|^2_{L^2} \lambda(dz) dt \right]
\leq \frac{CK}{n^2} E \left[ \int_0^{T \wedge \rho_{\alpha}^{n,m}} \eta(t)(1 + \|v_n\|^2_{H^2}) dt \right] + L E \left[ \int_0^{T \wedge \rho_{\alpha}^{n,m}} \eta(t) \| v_n - v_m \|^2_{L^2} dt \right].
$$

(A.19)
Lemma A.7. Let $\eta(t)$ be a stochastic process. Let $\rho^{n,m}_N$ be the stopping time given by (4.15). Denote $\rho^{n,m}_N = \rho^{n}_N \wedge \rho^{m}_N$. Then

$$
E \left[ \int_0^{T \wedge \rho^{n,m}_N} \int_Z \eta(t) \| J_n G(v_n, z) - J_m G(v_m, z) \|^2_{L^2} \lambda(dz) dt \right]
\leq CK \frac{n^\epsilon}{n^t} E \left[ \int_0^{T \wedge \rho^{n,m}_N} \eta(t) (1 + \| v_n \|^2_{H^s}) dt \right] + LE \left[ \int_0^{T \wedge \rho^{n,m}_N} \eta(t) \| v_n - v_m \|^2_{L^2} dt \right]. \tag{A.20}
$$

Proof. Exploiting Assumption [3,1] and [24], for $0 < \epsilon < s - 1$, we have

$$
E \left[ \int_0^{T \wedge \rho^{n,m}_N} \int_Z \eta(t) \| J_n G(v_n, z) - J_m G(v_m, z) \|^2_{L^2} \lambda(dz) dt \right]
\leq E \left[ \int_0^{T \wedge \rho^{n,m}_N} \int_Z \eta(t) \| (J_n - J_m) G(v_n, z) \|^2_{L^2} \lambda(dz) dt \right]
+ E \left[ \int_0^{T \wedge \rho^{n,m}_N} \int_Z \eta(t) \| J_m G(v_n, z) - G(v_n, z) \|^2_{L^2} \lambda(dz) dt \right]
\leq C \frac{n^\epsilon}{n^t} E \left[ \int_0^{T \wedge \rho^{n,m}_N} \int_Z \eta(t) \| G(v_n, z) \|^2_{H^s} \lambda(dz) dt \right] + E \left[ \int_0^{T \wedge \rho^{n,m}_N} \int_Z \eta(t) \| G(v_n, z) - G(v_m, z) \|^2_{L^2} \lambda(dz) dt \right]
\leq CK \frac{n^\epsilon}{n^t} E \left[ \int_0^{T \wedge \rho^{n,m}_N} \eta(t) (1 + \| v_n \|^2_{H^s}) dt \right] + LE \left[ \int_0^{T \wedge \rho^{n,m}_N} \eta(t) \| v_n - v_m \|^2_{L^2} dt \right]. \tag{A.21}
$$

ACKNOWLEDGEMENT

The authors would like to thank Professor Zdzislaw Brzeźniak of University of York, UK for his valuable comments and pointing our attention to certain references.

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