On the Divided Power Algebra and the Symplectic Group in Characteristic 2

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Abstract

Let $V$ be an even dimensional vector space over a field $K$ of characteristic 2 equipped with a non-degenerate alternating bilinear form $f$. The divided power algebra $DV$ is considered as a complex with differential defined from $f$. We examine the cohomology modules as representations of the corresponding symplectic group.

1 Introduction

Let $V$ be a vector space of dimension $2m$ over a field $K$ equipped with a non-degenerate alternating bilinear form $f$. Let $G = Sp(V)$ be the corresponding symplectic group and $\wedge V$ the exterior algebra of $V$. There are contraction maps $\theta_k : \wedge^k V \rightarrow \wedge^{k-2} V$ defined from $f$ that are $G$-equivariant for the natural action of $G$ on $\wedge V$. By extending in the usual way, one obtains a contraction map $\theta$ defined on $\wedge V$ which is $G$-equivariant. Assume now and for the rest of the paper that the characteristic of $K$ is 2. In this case we have $\theta^2 = 0$ so that $\wedge V$ becomes a complex. It was shown by R. Gow in [2] that the complex $\wedge V$ is exact in all degrees except one and the unique non-vanishing homology module is an irreducible $G$-module. Moreover, if $K$ is algebraically closed, this module was identified as the spin module for $G$, that is, an irreducible rational representation of $G$ of highest weight $\omega_m$, where $\omega_1, \ldots, \omega_m$ are the fundamental weights of $G$ (in the numbering such that $\omega_1$ corresponds to the natural module of $G$).

The purpose of this note is to examine the case of the divided power algebra $DV$ of $V$. (We refer to [1] A2.4 for $DV$). One reason the divided power algebra arises naturally in the representation theory of $G$ is that the degree $k$ component $D_k V$ of $DV = \bigoplus_{k \geq 0} D_k V$ is a Weyl module of $G$. In order to describe our main results we need some notation.

Let $\{x_1, y_1, \ldots, x_m, y_m\}$ be a basis of $V$ such that $f(x_i, y_j) = \delta_{ij}, f(x_i, x_j) = f(y_i, y_j) = 0$ for $1 \leq i, j \leq m$. Since the characteristic of $K$ is 2, there is a well
defined $G$-equivariant map $\wedge^2 V \to D_2 V$, $u \wedge v \mapsto uv$, $u, v \in V$. Let $\omega$ be the image of the element $x_1 \wedge y_1 + \cdots + x_m \wedge y_m$ under this map and let

$$\partial_k : D_k V \to D_{k+2} V$$

be the map given by multiplication by $\omega$. We define a map $\partial : DV \to DV$ whose restriction to $D_k V$ is $\partial_k$. It easily follows that $\partial$ is $G$-equivariant for the natural action of $G$ on $DV$. Since the characteristic of $K$ is 2, we have $\partial^2 = 0$. We regard $DV$ as a complex in which $\partial$ has degree 2 and the cochain module in degree $k$ is $D_k V$.

The main result of this paper concerns the determination of the cohomology modules $H^k(DV)$. If $k < m$ or $k \not\equiv m \pmod{2}$ then $H^k(DV) = 0$ (Corollary 2.3). If $k \geq 0$ and $K$ is algebraically closed, then there is an isomorphism of $G$-modules,

$$H^{m+2k}(DV) \simeq \Delta((k\omega_1)^{(1)}) \otimes L(\omega_m)$$

where $\Delta((k\omega_1)$ is the Weyl module of $G$ of highest weight $k\omega_1$, $L(\omega_m)$ is the irreducible rational representation of $G$ of highest weight $\omega_m$ and $\Delta((k\omega_1)^{(1)})$ denotes the first Frobenius twist of $\Delta((k\omega_1)$ (Theorem 3.2).

The first and second non-vanishing cohomology modules are $H^m(DV)$ and $H^{m+2}(DV)$. We have that $H^m(DV)$ is an irreducible $G$-module of highest weight $\omega_m$. Thus $H^m(DV)$ is a spin module for $G$ and the situation here is similar to \cite{2}. We show that $H^{m+2}(DV)$ is also an irreducible $G$-module and its highest weight is $2\omega_1 + \omega_m$ (Corollary 3.4). However, it is not true in general that the nonzero $H^k(DV)$ are irreducible.

\section{The Complex $DV$}

In this section we will obtain a first description of the cohomology modules $H^k(DV) = \ker \partial_k / \text{Im} \partial_{k-2}$ of $DV$.

If $v \in V$ and $a$ is a non-negative integer, we will use the notation $v^{(a)}$ for the $a$-th divided power of $v$. If $a = 0$, we regard $v^{(0)} = 1$. If $\{v_1, \ldots, v_n\}$ is a basis of $V$, then a basis of $D_k V$ is $\{v_1^{(a_1)} \cdots v_n^{(a_n)} | a_1 + \cdots + a_n = k\}$. There are certain identities in $DV$ that we will use frequently such as

$$v^{(a)} v^{(b)} = \binom{a+b}{a} v^{(a+b)}, \quad (tu)^{(a)} = t^a u^{(a)}, \quad (u+v)^{(a)} = \sum_{i=0}^a u^{(i)} v^{(a-i)},$$

where $u, v \in V$ and $t \in K$.

We will need the following simple Lemma in the proof of Theorem 2.2 below and elsewhere.

\textbf{Lemma 2.1.} Suppose $m = 1$. Then the cohomology of the complex $DV$ is given by

$$H^k(DV) = \begin{cases} 0, & k \text{ even} \\ D_k V, & k \text{ odd.} \end{cases}$$
Proof. From the definition of the complex $DV$ it follows that

$$DV = D^0V \oplus D^1V$$

where $D^0V$ (respectively, $D^1V$) is the subcomplex of $DV$ consisting of the cochains of $DV$ that have even (respectively, odd) degrees, namely

$$D^0V : 0 \rightarrow K \xrightarrow{\partial_0} D_2V \xrightarrow{\partial_2} D_4V \xrightarrow{\partial_4} \cdots$$
$$D^1V : 0 \rightarrow V \xrightarrow{\partial_1} D_2V \xrightarrow{\partial_3} D_5V \xrightarrow{\partial_5} \cdots$$

Hence

$$H^k(DV) = \begin{cases} H^k(D^0V), & k \text{ even} \\ H^k(D^1V), & k \text{ odd} \end{cases}$$

Let $m = 1$. Suppose $x = x_1^{(a)}y_1^{(b)} \in D_kV$, $a + b = k$. Then

$$\partial_k(x) = (a + 1)(b + 1)x_1^{(a+1)}y_1^{(b+1)} \quad (1)$$

Let $k$ be odd. Then one of $a, b$ is odd and since the characteristic of $K$ is 2, we see from (1) that $\partial_k(x) = 0$ for all $x \in D_kV$. Thus $H^k(DV) = H^k(D^1V) = D_kV$.

Let $k$ be even. Then $H^k(DV) = H^k(D^0V)$. We claim that $D^0V$ is exact. Indeed, it is obvious that $\partial_0$ is injective. Let $k \geq 2$. It follows from (1) that $\ker \partial_k$ is spanned by the elements $x_1^{(a)}y_1^{(b)}$, $a + b = k$, such that both $a$ and $b$ are odd. For such an element we have $\partial_{k-2}(x_1^{(a-1)}y_1^{(b-1)}) = abx_1^{(a)}y_1^{(b)} = x_1^{(a)}y_1^{(b)}$ so that $\Im \partial_{k-2} = \ker \partial_k$. Thus $H^k(D^0V) = 0$.

For each $i = 1, \ldots, m$, let $V_i$ be the subspace of $V$ with basis $\{x_i, y_i\}$. Then the $V_i$ are non-degenerate and we consider the group $H = Sp(V_1) \times \cdots \times Sp(V_m)$ embedded in $G = Sp(V)$ in the usual manner. Each $G$-module may be considered an $H$-module upon restriction. In particular the cohomology modules $H^k(DV)$ of $DV$ are $H$-modules.

Theorem 2.2. As $H$-modules we have

$$H^k(DV) \cong \bigoplus_{a_1+\cdots+a_m = k} D_{a_1}V_1 \otimes \cdots \otimes D_{a_m}V_m$$

where the sum ranges over all $m$-tuples $(a_1, \ldots, a_m)$ of odd positive integers $a_1, \ldots, a_m$ such that $a_1 + \cdots + a_m = k$.

Proof. We use induction on $m$, the case $m = 1$ owing to Lemma 2.4.

Assume $m \geq 2$ and let $U = V_2 \oplus \cdots \oplus V_m$ so that $V = V_1 \oplus U$. We have an isomorphism

$$D_kV \cong \bigoplus_{a+b=k} D_aV_1 \otimes D_bU$$

of $GL(V_1) \times GL(U)$ modules, where $GL(V)$ denotes the general linear group of $V$. The injective maps $D_aV_1 \otimes D_bU \rightarrow D_kV$, $a + b = k$, are given by

$$D_aV_1 \otimes D_bU \ni z_1 \otimes z_2 \mapsto z_1z_2 \in D_kV.$$
Under these identifications, the differential of the complex $DV$ restricted to $D_a V_1 \otimes D_b U$ looks like

$$D_a V_1 \otimes D_b U \longrightarrow D_{a+2} V_1 \otimes D_b U \oplus D_a V_1 \otimes D_{b+2} U$$

where the horizontal map is multiplication by $x_1 y_1$ on the first factor and the identity on the second and the diagonal map is the identity on the first factor and multiplication by $x_2 y_2 + \cdots + x_m y_m$ on the second. It follows that the complex $DV$ is isomorphic to the tensor product $DV_1 \otimes DU$ of the complexes $DV_1$ and $DU$ and moreover this isomorphism is $Sp(V_1) \times Sp(U)$-equivariant. Thus

$$H^k(DV) \simeq \bigoplus_{a+b=k} H^a(DV_1) \otimes H^b(DU)$$

as $Sp(V_1) \times Sp(U)$-modules. From Lemma 2.1 $H^a(DV_1) = 0$ unless $a$ is odd in which case $H^a(DV_1) = D_a V_1$. Using this and the induction hypothesis for $H^b(DU)$, the desired result follows.

An immediate corollary is the following result.

**Corollary 2.3.** $H^k(DV) = 0$ if and only if $k < m$ or $k \not\equiv m \mod 2$.

The next result provides a basis for $H^{m+2k}(DV)$, $k \geq 0$, that will be used in Section 3.

**Corollary 2.4.** A basis for $H^{m+2k}(DV)$, $k \geq 0$, consists of the elements

$$x_1^{(b_1)} y_1^{(c_1)} \cdots x_m^{(b_m)} y_m^{(c_m)} + \text{Im} \partial_{m+2k-2}$$

where $\sum_{i=1}^m (b_i + c_i) = m + 2k$ and $b_i + c_i$ is odd for all $i$.

**Proof.** The isomorphism in the statement of Theorem 2.1 yields injective maps $D_{a_1} V_1 \otimes \cdots \otimes D_{a_m} V_m \to H^{m+2k}(DV)$, where each $a_i$ is odd and $a_1 + \cdots + a_m = m + 2k$. It follows from the proof of the Theorem, that these are induced from the injective maps $D_{a_1} V_1 \otimes \cdots \otimes D_{a_m} V_m \ni z_1 \otimes \cdots \otimes z_m \mapsto z_1 \cdots z_m \in D_{m+2k} V$, that is, they are given by $D_{a_1} V_1 \otimes \cdots \otimes D_{a_m} V_m \ni z_1 \otimes \cdots \otimes z_m \mapsto z_1 \cdots z_m + \text{Im} \partial_{m+2k-2} \in H^{m+2k}(DV)$. A basis for $D_{a_1} V_1 \otimes \cdots \otimes D_{a_m} V_m$ is

$$\{x_1^{(b_1)} y_1^{(c_1)} \otimes \cdots \otimes x_m^{(b_m)} y_m^{(c_m)} \mid b_i + c_i = a_i \text{ for all } i\}.$$ 

The union of the images of these bases under the previous injections yields the basis of $H^{m+2k}(DV)$ in the statement of the Corollary.

We determine the dimensions of the nonzero cohomology modules of $DV$.

**Corollary 2.5.** Let $k \geq 0$. Then

$$\dim H^{m+2k}(DV) = 2^m \binom{2m+k-1}{k}.$$
Proof. We shall make use of the identity

\[ \binom{n}{k} = \sum_{j=0}^{k} (j+1) \binom{n-2-j}{k-j}, \quad (3) \]

which may be proved easily by induction on \( n \).

Let \( f(m,k) = \dim H^{m+2k}(DV) \). We will show by induction on \( m \) that

\[ f(m,k) = 2^m \binom{2m+k-1}{k}. \quad (4) \]

Let \( m = 1 \). Then from Lemma 2.1 it follows that

\[ f(1,k) = \dim D_{2k+1}V = 2k+2. \]

Let \( m \geq 2 \). From the isomorphism (2) in the proof of Theorem 2.2 and from Lemma 2.1 we have

\[ H^{m+2k}(DV) \cong \bigoplus_{a \text{ odd}} D_a V_1 \otimes H^{m+2k-a}(DU). \]

By Corollary 2.3, we have \( H^{m+2k-a}(DU) = 0 \), if \( a > 2k+1 \). Hence in the above sum, \( a \) ranges over the odd integers \( 1, 3, \ldots, 2k+1 \). Thus

\[ f(m,k) = \sum_{a \text{ odd}} (a+1) f(m-1, k - \frac{a-1}{2}), \]

where \( a \) ranges as before. By induction \( f(m-1, k - \frac{a-1}{2}) = 2^{m-1} \binom{2m+k-\frac{a-1}{2}-3}{k-\frac{a-1}{2}} \)

and upon substitution we obtain

\[ f(m,k) = 2^m \sum_{a \text{ odd}} \frac{a+1}{2} \binom{2m+k-\frac{a-1}{2}-3}{k-\frac{a-1}{2}}. \quad (5) \]

Using (3), we see that (5) yields (4).

3 Main result

We will consider here the non-vanishing cohomology modules of \( DV \), that is \( H^{m+2k}(DV) \), \( k \geq 0 \), as \( G \)-modules. In this section we assume that \( K \) is algebraically closed.

First we introduce some notation. Having fixed the ordered basis \( x_1, \ldots, x_m, y_m, \ldots, y_1 \) of \( V \) we identify \( G = Sp(V) \) with \( Sp(2m,K) = \{ A \in GL(2m,K) \mid A^t JA = J \} \) where \( J \) is the matrix of the bilinear form \( f \) of the Introduction with respect to the above basis. A maximal torus in \( Sp(2m,K) \) is \( T = \{ \text{diag}(t_1, \ldots, t_m, t_m^{-1}, \ldots, t_1^{-1}) \mid t_i \neq 0 \text{ for all } i \} \). For each \( i = 1, \ldots, 2m \) let \( \varepsilon_i : T \to K - \{0\} \) be the map that sends a matrix in \( T \) to its element in position \((i,i)\). Then in the weight lattice of \( T \) we have the relations

\[ \varepsilon_i + \varepsilon_{2m+1-i} = 0, \]
where $1 \leq i \leq 2m$. The set of roots of $Sp(2m, K)$ is $\Phi = \{ \pm \varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_k \mid 1 \leq i < j \leq m, 1 \leq k \leq m \}$. Let $\Pi = \{ \varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i < m \} \cup \{ 2\varepsilon_m \}$. Then $\Pi$ is a set of simple roots in $\Phi$ and the corresponding positive roots are $\Phi^+ = \{ \varepsilon_i + \varepsilon_j, \varepsilon_i - \varepsilon_j, 2\varepsilon_k \mid 1 \leq i < j \leq m, 1 \leq k \leq m \}$. The fundamental dominant weights are $\omega_1, \ldots, \omega_m$ where $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$.

The irreducible rational representations of $Sp(2m, k)$ are parametrized by dominant weights [3, II.2]. Let $L(\lambda)$ denote the (unique up to isomorphism) irreducible module of $Sp(2m, k)$ of highest weight $\lambda$, where $\lambda$ is a dominant weight.

The Weyl module of $G$ of highest weight $\lambda$, where $\lambda$ is a dominant weight, will be denoted by $\Delta(\lambda)$. It is well known that for any integer $k \geq 0$ we have $\Delta(k\omega_1) \simeq D_k V$ [3 II 2.17].

**Proposition 3.1.** We have $H^m(DV) \simeq L(\omega_m)$ as $G$-modules.

**Proof.** From Theorem 2.2 there is an isomorphism of $H$-modules $H^m(DV) \simeq V_1 \otimes \cdots \otimes V_m$. Since $V_1 \otimes \cdots \otimes V_m$ is irreducible as an $H$-module, it follows that $H^m(DV)$ is irreducible as a $G$-module. By Corollary 2.5 a basis of $H^m(DV)$ consists of the elements $v_1 \cdots v_m + i\partial_{m-2}$, where for each $i$ we have $v_i = x_i$ or $y_i$. These are weight vectors for the action of $G$ and we see that the weights of $H^m(DV)$ are $\pm \varepsilon_1 \pm \cdots \pm \varepsilon_m$. It is easy to verify that among these, $\varepsilon_1 + \cdots + \varepsilon_m$ is the highest weight. Thus $H^m(DV) \simeq L(\omega_m)$. \hfill \qed

If $M$ is a $G$-module, we use the notation $M^{(1)}$ for the first Frobenius twist of $M$. The main result of this paper is the following.

**Theorem 3.2.** Let $k \geq 0$. There is an isomorphism of $G$-modules

$$H^{m+2k}(DV) \simeq \Delta(k\omega_1)^{(1)} \otimes L(\omega_m).$$

For the proof we will need the following Lemma.

**Lemma 3.3.** Suppose $V$ be a vector space of dimension $n$ over a field $K$ of characteristic 2 and $\{v_1, \ldots, v_n\}$ a basis of $V$. Let $k \geq 0$ and define $N$ to be the subspace of $D_{2k}V$ spanned by the monomials $v_1^{(a_1)} \cdots v_n^{(a_n)}$, $a_1 + \cdots + a_n = 2k$, where at least one of the $a_i$ is odd. Then $N$ is a $SL(n, K)$-submodule of $D_{2k}V$ and the map $\varphi : (D_k V)^{(1)} \to D_{2k}V/N$, $v_1^{(b_1)} \cdots v_n^{(b_n)} \mapsto v_1^{(2b_1)} \cdots v_n^{(2b_n)} + N$ is an isomorphism of $SL(n,K)$-modules.

**Proof.** For $t \in K$, $r, s \in \{1, \ldots, n\}$, $r \neq s$, let $g_{rs}(t) \in SL(n, K)$ be $g_{rs}(t) = I + tE_{rs}$, where $I$ is the $n \times n$ identity matrix and $E_{rs}$ in the $n \times n$ matrix with 1 in position $(r, s)$ and 0 elsewhere. Since the $g_{rs}(t)$ generate $SL(n, K)$, in order to prove the first conclusion of the lemma, it suffices to show that $g_{rs}(t)N \subseteq N$ for every $t \in K$, $r \neq s$.

We will make use of the fact that $^{a+b}_a$ is even if both $a, b$ are odd. Let $v = v_1^{(a_1)} \cdots v_n^{(a_n)} \in D_{2k}V$, $a_1 + \cdots + a_n = 2k$, such that at least one $a_i$ is odd.
We have
\[ g_{rs}(t)v = v_1^{(a_1)} \cdots (v_s + tv_r)^{(a_s)} \cdots v_n^{(a_n)} = v_1^{(a_1)} \cdots \left( \sum_{i=0}^{a_s} t^{a_s-i} v_s^{(i)} v_r^{(a_s-i)} \right) \cdots v_n^{(a_n)}. \]

Thus
\[ g_{rs}(t)v = \sum_{i=0}^{a_s} t^{a_s-i} \binom{a_r + a_s - i}{a_r} v_1^{(a_1)} \cdots v_s^{(i)} \cdots v_r^{(a_r+a_s-i)} \cdots v_n^{(a_n)}. \]

Consider a monomial \( u = v_1^{(a_1)} \cdots v_s^{(i)} \cdots v_r^{(a_r+a_s-i)} \cdots v_n^{(a_n)} \) in the right hand side of (6) where all exponents are even. By the assumption on \( v \), at least one of \( a_r, a_s \) is odd. We conclude that both of \( a_r \) and \( a_s - i \) are odd. But then the coefficient \( \binom{a_r+a_s-i}{a_r} \) of \( u \) is even and hence zero in \( K \). Thus \( g_{rs}(t)v \in N \).

We show now that the map \( \varphi : (D_k V)^{(1)} \to D_{2k} V/N \) is a map of \( SL(n, K) \)-modules. We will use the fact that \( \binom{a}{b} \equiv \binom{2a}{2b} \mod 2 \) for all integers \( a \geq b \geq 0 \).

Let \( v = v_1^{(a_1)} \cdots v_n^{(a_n)} \in D_k V \). By a similar computation as in the proof of (6) and taking into account the Frobenius twist on \( D_k V \) we have
\[ g_{rs}(t)v = \sum_{i=0}^{a_s} t^{2(a_s-i)} \binom{a_r + a_s - i}{a_r} v_1^{(a_1)} \cdots v_s^{(2i)} \cdots v_r^{(2(a_r+a_s-i))} \cdots v_n^{(2a_n)} + N. \]

On the other hand applying (6) to \( \varphi(v) \) in place of \( v \) we have
\[ g_{rs}(t)\varphi(v) = \sum_{i=0}^{2a_s} t^{2a_s-i} \binom{2a_r + 2a_s - i}{2a_r} v_1^{(2a_1)} \cdots v_s^{(i)} \cdots v_r^{(2(a_r+2a_s-i))} \cdots v_n^{(2a_n)} + N. \]

where the last equality follows from the definition of \( N \). From (7), (8) and the fact that \( \binom{a}{b} \equiv \binom{2a}{2b} \mod 2 \) we see that \( \varphi(g_{rs}(t)v) = g_{rs}(t)\varphi(v) \). It follows that \( \varphi \) is a map of \( SL(n, k) \)-modules.

Finally, it is clear that the map \( \varphi \) carries a basis of \((D_k V)^{(1)}\) to a basis of \( D_{2k} V/N \) and thus is an isomorphism.
Proof of Theorem 3.2. The $GL(V)$-map $D_{2k} V \otimes D_{m} V \rightarrow D_{m+2k} V$, $u \otimes v \mapsto uv$, induces a map $D_{2k} V \otimes \ker \partial_m \rightarrow \ker \partial_{m+2k}$ because if $\partial_m(v) = 0$, then $v\omega = 0$ so that $uv\omega = 0$ (where $\omega$ was defined in the Introduction). It is easy to check that the last map induces a map $\psi : D_{2k} V \otimes H^m(DV) \rightarrow H^{m+2k}(DV)$, $u \otimes (v + \Im \partial_{m-2}) \mapsto uv + \Im \partial_{m+2k-2}$. We claim that $\psi(N \otimes H^m(DV)) = 0$, where $N$ is defined in Lemma 3.3.

Indeed, let $u = x_1^{(a_1)} y_1^{(b_1)} \cdots x_m^{(a_m)} y_m^{(b_m)} \in N$, where $\sum_{i=1}^{m} (a_i + b_i) = 2k$. Then at least one exponent of $u$ is odd. For simplicity in notation, let us assume that this exponent is $a_1$. Let $v = x_1^{(c_1)} y_1^{(d_1)} \cdots x_m^{(c_m)} y_m^{(d_m)} + \Im \partial_{m-2} \in H^m(DV)$ be a basis element (Corollary 2.4), so that $c_i + d_i = 1$ for all $i$. Then

\[
\psi(u \otimes (v + \Im \partial_{m-2})) = qa x_1^{(a_1+c_1)} y_1^{(b_1+d_1)} \cdots x_m^{(a_m+c_m)} y_m^{(b_m+d_m)} + \Im \partial_{m+2k-2},
\]

where

\[
q = \left(\begin{array}{c}
a_1+c_1 \\
c_1 \\
\end{array}\right) \left(\begin{array}{c}
b_1+d_1 \\
d_1 \\
\end{array}\right) \cdots \left(\begin{array}{c}
a_m+c_m \\
c_m \\
\end{array}\right) \left(\begin{array}{c}
b_m+d_m \\
d_m \\
\end{array}\right).
\]

If $q$ is even, there is nothing to prove. Let $q$ be odd. Then, since we have $c_i + d_i = 1$, it follows that for each $i$ at least one of $a_i + c_i + 1, b_i + d_i + 1$ is even and in $K$ we have

\[
(a_i + c_i + 1)(b_i + d_i + 1) = 0.
\]

From the fact that $a_1$ and $q$ are odd we obtain that both $a_1 + c_1, b_1 + d_1$ are odd. Let $z \in D_{m+2k-2} V$ be

\[
z = x_1^{(a_1+c_1-1)} y_1^{(b_1+d_1-1)} x_2^{(a_2+c_2)} y_2^{(b_2+d_2)} \cdots x_m^{(a_m+c_m)} y_m^{(b_m+d_m)}.
\]

By a direct computation and using (9) we obtain

\[
\partial_{m+2k-2}(z) = (a_1+c_1)(b_1+d_1)x_1^{(a_1+c_1)} y_1^{(b_1+d_1)} \cdots x_m^{(a_m+c_m)} y_m^{(b_m+d_m)}
\]

\[
= x_1^{(a_1+c_1)} y_1^{(b_1+d_1)} \cdots x_m^{(a_m+c_m)} y_m^{(b_m+d_m)}.
\]

From (8) and (10) we have that $\psi(u \otimes (v + \Im \partial_{m-2})) = 0$ and the claim is proved.

We thus have map of $G$-modules $D_{2k}/N \otimes H^m(DV) \rightarrow H^{m+2k}(DV)$, such that $(u + N) \otimes (v + \Im \partial_{m-2}) \mapsto uv + \Im \partial_{m+2k-2}$. Using Lemma 3.3 we obtain a map $(D_k V)^{(1)} \otimes H^m(DV) \rightarrow H^{m+2k}(DV)$. Using Corollary 2.4 one easily checks that this last map is onto and hence an isomorphism because we have

\[
\dim((D_k V)^{(1)} \otimes H^m(DV)) = \binom{2m+k-1}{k} 2^m = \dim H^{m+2k}(DV)
\]

by Corollary 2.5. Finally we recall that $D_k V = \Delta(k\omega_1)$ and, by Proposition 3.1, $H^m(DV) \simeq L(\omega_m)$. \hfill \Box

**Corollary 3.4.** $H^{m+2}(DV)$ is an irreducible $G$-module of highest weight $2\omega_1 + \omega_m$. 

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Proof. By Theorem 3.2, \( H^{m+2}(DV) \simeq \Delta(\omega_1)^{(1)} \otimes L(\omega_m) \). But \( \Delta(\omega_1) \) is just the natural module of \( G \), so that \( \Delta(\omega_1)^{(1)} = L(\omega_1)^{(1)} \). By Steinberg’s tensor product theorem (see, for example, [3, II. 3.17]), \( H^{m+2}(DV) \simeq L(\omega_1)^{(1)} \otimes L(\omega_m) \simeq L(2\omega_1 + \omega_m) \).

Remark. We have seen that the first two nonzero \( H^k(DV) \) are irreducible representations of \( G \). It is not true in general that all the nonzero \( H^k(DV) \) are irreducible representations of \( G \). For example, let \( m = 1 \) so that \( G = SL(2, K) \). Then for each odd \( k \) we have \( H^k(DV) = D_kV \) from Lemma 2.1. It is well known that \( D_kV \) is irreducible if and only if \( k = 2^n - 1, n \geq 0 \).

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