Asymptotic properties of biorthogonal polynomials systems related to Hermite and Laguerre polynomials

Yan Xu

School of Mathematics and Quantitative Economics, Center for Econometric analysis and Forecasting, Dongbei University of Finance and Economics, Liaoning, 116025, PR China

Abstract

In this paper, the structures to a family of biorthogonal polynomials that approximate to the Hermite and Generalized Laguerre polynomials are discussed respectively. Therefore, the asymptotic relation between several orthogonal polynomials and combinatorial polynomials are derived from the systems, which in turn verify the Askey scheme of hypergeometric orthogonal polynomials. As the applications of these properties, the asymptotic representations of the generalized Buchholz, Laguerre, Ultraspherical (Gegenbauer), Bernoulli, Euler, Meixner and Meixner-Pilaczek are polynomials are derived from the theorems directly. The relationship between Bernoulli and Euler polynomials are shown as a special case of the characterization theorem of the Appell sequence generated by \( \alpha \) scaling functions.

Keywords: Hermite Polynomial, Laguerre Polynomial, Appell sequence, Askey Scheme, B-splines, Bernoulli Polynomial, Euler polynomials.

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Email: yan_xu@dufe.edu.cn
1. Introduction

The Hermite polynomials follow from the generating function

\[ e^{xz - \frac{z^2}{2}} = \sum_{m=0}^{\infty} \frac{H_m(x)}{m!} z^m, \quad z \in \mathbb{C}, x \in \mathbb{R} \]  

which gives the Cauchy-type integral

\[ H_m(x) = \frac{m!}{2i\pi} \oint e^{xz - \frac{z^2}{2}} z^{-(m+1)} \, dz. \]  

The derivatives of the Gaussian function, \( G(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \), produce the Hermite polynomials by the relation, \((-1)^m G^{(m)}(x) = H_m(x)G(x), m = 0, 1, \ldots\).

Therefore the orthonormal property of the Hermite polynomials,

\[ \frac{1}{m!} \int_{-\infty}^{\infty} H_m(x)H_n(x)G(x) \, dx = \delta_{m,n}, \]

can be considered as a biorthogonal relation between the derivatives of the Gaussian function, \( \{(-1)^nG^{(n)} : n = 0, 1, \ldots\} \) and the Hermite polynomials, \( \{\frac{H_m}{m!} : m = 0, 1, \ldots\} \).

The Hermite polynomials have been extensively studied since the pioneer article of C. Hermite [6] in 1864. It has many interesting properties and applications in several branches of mathematics, physical sciences and engineering.

A rich source of orthogonal polynomials, for instance, Gegenbauer [18], Laguerre [19], Charlier [13], Jacobi [14], Meixner-Pollaczek, Meixner, Krawtchouk and Hahn-type polynomials [5] have asymptotic approximations in terms of Hermite polynomials, which is known as the famous Askey scheme [21, 25, 20]. The asymptotic relations among other hypergeometric orthogonal polynomials and its q-analogue can be found in [4, 25]. The asymptotic representations of other families of polynomials, such as the generalized Bernoulli, Euler, Bessel and Buchholz polynomials are also considered [8, 15, 26].

In [27], S. L. Lee extended the biorthogonal properties between the derivatives of the Gaussian function and Hermite polynomials to a family of scaling
functions with compact support and a family of Appell sequences which approximate to the Gaussian function and Hermite polynomials respectively. The Appell polynomials are also called scaling biorthogonal polynomials which are eigenfunctions of a linear operator and the distributional derivatives of \( \phi \) are the eigenfunctions of its adjoint corresponding to the same eigenvalues \([28]\). In particular, the Appell polynomials generated by the uniform B-spline are the classical Bernoulli polynomials which asymptotic approximate to the Hermite polynomials by suitably normalized \([27]\).

The main objectives of this paper is to extend these properties to a family of non-scaling functions that approximate to the generating functions and to construct a family of biorthogonal polynomials that approximate to the Hermite polynomials and Laguerre polynomials respectively. The asymptotic properties between Hermite, Laguerre and other orthogonal polynomials which are known as Askey Scheme are derived from these theorems as simple cases. The relationship between the Appell sequence polynomials and the \( \alpha \)-scaling compact support functions are also considered.

This paper is organized as follows: In section 2, we present the framework of the biorthogonal polynomials system which related to Hermite polynomials and Gaussian function. The characterization theorem of the Appell sequence generated by the scaling functions are also shown in this section. In section 3, as the applications of section 2, the asymptotic relations among Bernoulli polynomials, Euler polynomials and B-splines are studied. The new identical relation between Bernoulli and Euler polynomials are shown as a special case. In section 4, we generalize the asymptotic relationship to a family of hypergeometric orthogonal polynomials related to Hermite polynomials. The asymptotic properties of Generalized Buchholz polynomials and Ultraspherical (Gegenbauer) polynomials are considered as the applications. In section 5, we generalize the biorthogonal systems to the Laguerre polynomials and derive several asymptotic properties in Askey scheme.
2. Biorthogonal polynomials approximate to Hermite polynomial

Let \( C^\infty(\mathbb{R}) \) denoted for the space of infinitely differentiable functions. If \( \phi : C^\infty(\mathbb{R}) \to \mathbb{R} \) is a linear functional, we shall write \( \langle \phi, \nu \rangle = \phi(\nu), \nu \in C^\infty(\mathbb{R}) \).

The linear functional \( \phi \) is continuous if and only if there is a compact subset \( K \) of \( \mathbb{R} \), a constant \( C > 0 \) and an integer \( k \geq 0 \) such that

\[
|\langle \phi, \nu \rangle| \leq C \max_{j \leq k} \sup_{x \in K} |\nu^{(j)}(x)|.
\]

We denote the space of distributions with compact support by \( \mathcal{E}'(\mathbb{R}) \). Integrable functions and measures with compact supports belong to \( \mathcal{E}'(\mathbb{R}) \). If \( f \) is a compactly supported integrable function then it is associated with the distribution, which we still denote by \( f \), defined by

\[
\langle f, \nu \rangle := \int_{\mathbb{R}} \nu(x)f(x)dx, \quad \nu \in C^\infty(\mathbb{R}).
\]

If \( m \) is a compactly supported measure on \( \mathbb{R} \), then it is associated with the distribution, which we still denote by \( m \), defined by

\[
\langle m, \nu \rangle := \int_{\mathbb{R}} \nu(x)dm(x), \quad \nu \in C^\infty(\mathbb{R}).
\]

Any \( \phi \in \mathcal{E}'(\mathbb{R}) \) has derivatives \( \phi^{(n)} \) of any order \( n \) and they are defined by

\[
\langle \phi^{(n)}, \nu \rangle = (-1)^n \langle \phi, \nu^{(n)} \rangle, \quad n = 0, 1, \ldots.
\]

Taking a compactly supported distribution \( \phi \in \mathcal{E}'(\mathbb{R}) \), then for any integer \( n \geq 0 \),

\[
\langle \phi^{(n)}, e^{(\cdot)z} \rangle = (-1)^n \langle \phi, z^n e^{(\cdot)z} \rangle = (-1)^n z^n \hat{\phi}(iz),
\]

where \( \hat{\phi}(\cdot) \) denote the Fourier transform of \( \phi(x) \).

If \( \hat{\phi}(0) \neq 0 \),

\[
\left\langle (-1)^n \phi^{(n)}, \frac{e^{(\cdot)z}}{\hat{\phi}(iz)} \right\rangle = z^n
\]

in a neighborhood of 0. Since \( \phi \) is compactly supported, \( \hat{\phi} \) is analytic. So we can define a sequence of polynomials, \( P_m \), by the generating function

\[
\frac{e^{xz}}{\hat{\phi}(iz)} = \sum_{m=0}^{\infty} \frac{P_m(x)}{m!} z^m.
\]
It follows from (2.1) and (2.2) that for any integer \( n \geq 0 \),
\[
z^n = \sum_{m=0}^{\infty} \left\langle (-1)^n \phi^{(n)}_n, \frac{P_m(x)}{m!} \right\rangle z^m,
\]
which gives the biorthogonal relation
\[
\left\langle (-1)^n \phi^{(n)}_n, \frac{P_m(x)}{m!} \right\rangle = \delta_{m,n}.
\]

**Definition 2.1.** A sequence of polynomials, \( \{P_m(x) : m \in \mathbb{N}\} \), is an Appell sequence if \( P_m(x) \) is a polynomial of degree \( m \) and
\[
P_m'(x) = mP_{m-1}(x).
\]

Differentiating (2.2) with respect to \( x \) and equating coefficients of \( z^m \) in the resulting equation gives
\[
P_m'(x) = mP_{m-1}(x), \quad m = 1, 2, \ldots ,
\]
which implies \( P_m(x) \) are Appell sequence of polynomials. Therefore the distribution \( \phi \) generates an Appell sequence of polynomials by the generating function
\[
e^{xz \hat{\phi}(iz)}.
\]

We consider a family of sequences of biorthogonal polynomials, \( \{P_{N,m} : m = 0, 1, \ldots , N = 1, 2 \ldots \} \), that are generated by a sequence of functions, \( \phi_N \), which converges to the Gaussian function
\[
e^{xz \hat{\phi}(iz)} = \sum_{m=0}^{\infty} \frac{P_{N,m}(x)}{m!} z^m.
\]

**Definition 2.2.** Let \( \tilde{\phi}_N \) be the standardized form of \( \phi_N \),
\[
\tilde{\phi}_N(x) = \sigma_N \phi_N(\sigma_N x + \mu_N),
\]
where \( \mu_N \) and \( \sigma^2_N \) are the mean and variance of \( \phi_N \). Define the standardized form of the biorthonormal polynomials, \( \tilde{P}_{N,m} \), of \( \{P_{N,m} : m = 0, 1, \ldots \} \) by
\[
\tilde{P}_{N,m}(x) = \sigma^{-m}_N P_{N,m}(\sigma_N x + \mu_N).
\]

Then the following biorthogonal relations for the standardized biorthogonal polynomials follow from (2.4):
\[
\left\langle (-1)^n \tilde{\phi}^{(n)}_N, \tilde{P}_{N,m} \right\rangle = \delta_{m,n}, \forall m, n \geq 0.
\]
Further, the generating functions of $\tilde{P}_{N,m}$ are given by
\[ \frac{e^{xz}}{\tilde{\phi}_N(iz)} = \sum_{m=0}^{\infty} \frac{\tilde{P}_{N,m}(x)}{m!} z^m. \] (2.8)

**Theorem 2.1.** Let $\tilde{\phi}_N(x)$ satisfy the following conditions:

1. There exist constant $r > 0$ such that for any $\varepsilon$, there is a sufficient large $N_0$, for any $N > N_0$, it holds

   \[ \left| \tilde{\phi}_N(iz) - e^{\frac{z^2}{2}} \right| \leq \varepsilon, \quad |z| < r. \] (2.9)

2. Let $\{\tilde{P}_{N,m}(x) : m = 0, 1, \ldots\}$ be the biorthogonal polynomials generated by the functions, $\tilde{\phi}_N$, as in (2.8).

Then for each $m = 0, 1, \ldots$, $\tilde{P}_{N,m}(x)$ converges locally uniformly to the Hermite polynomial $H_m(x)$ as $N$ goes to infinity.

**Proof.** Since $\tilde{\phi}_N(0) = 1$, we can choose a neighborhood $U$ of the origin so that $\left| \tilde{\phi}(iz) \right| \geq \frac{1}{2}$ and $\left| e^{\frac{z^2}{2}} \right| \geq \frac{1}{2}$ for all $z \in U$. Take a circle $C$ inside $U$ with center at 0 and radius $r$ so that (2.9) is satisfied.

Noting
\[ \frac{e^{xz}}{\tilde{\phi}_N(iz)} - e^{\frac{x^2}{2}} = \sum_{m=0}^{\infty} \frac{\tilde{P}_{N,m}(x) - H_m(x)}{m!} z^m. \] (2.10)

The coefficients of the Taylor series (2.10) are represented by the Cauchy’s integral formula

\[ \tilde{P}_{N,m}(x) - H_m(x) = \frac{m!}{2\pi i} \oint_C \frac{e^{xz}(e^{\frac{z^2}{2}} - \tilde{\phi}_N(iz))}{z^{m+1}\tilde{\phi}_N(iz)e^{\frac{z^2}{2}}} dz. \]

Noting $\tilde{\phi}_N(x)$ satisfy the condition (1), which means $\exists \ r > 0$, for a real number $A > 0$, there is a sufficient large $N_0$, for any $N > N_0$, it holds

\[ \left| \tilde{\phi}_N(iz) - e^{\frac{z^2}{2}} \right| \leq \frac{A}{\sigma_N}, \quad |z| < r. \] (2.11)
Therefore
\[
\left| \tilde{P}_{N,m}(x) - H_m(x) \right| \leq \frac{m!}{2\pi} \oint_{C} \frac{|e^{xz} - \phi_N(iz)|}{\phi_N(iz)|e^{xz}|} |dz|
\]
\[
\leq \frac{m!}{2\pi} \oint_{C} \frac{e^{xRe(z)} \sigma_N}{\sigma_N |\phi_N(iz)||e^{xz}|} |dz|
\]
\[
\leq \frac{4(m!)e^{rxA}}{\sigma_N r^m}
\]

Since $\sigma_N \to \infty$ as $N \to \infty$, it follows that for each $m$, $\tilde{P}_{N,m(x)} \rightarrow H_m(x)$ uniformly on compact sets. \hfill \Box

2.1. Appell sequence generated by scaling functions

The refinement equation
\[
\phi_n(x) = \int_{\mathbb{R}} \alpha \phi_n(\alpha x - y) dm_n(y), \quad x \in \mathbb{R}, \quad n = 1, 2, \ldots, \quad (2.12)
\]
where $\alpha > 1$ and $\{m_n\}$ is a sequence of probability measures with finite first and second moments. Equivalently, (2.12) can be expressed in term of Fourier transforms in the frequency domain in the form
\[
\hat{\phi}_n(\mu) = \hat{m}_n \left( \frac{\mu}{\alpha} \right) \hat{\phi}_n \left( \frac{\mu}{\alpha} \right), \quad \mu \in \mathbb{R}. \quad (2.13)
\]

**Theorem 2.2.** If two Appell sequence polynomials, $P_m(x)$ and $Q_m(x)$ are generalized by
\[
\frac{e^{xz}}{\psi(iz)} = \sum_{m=0}^{\infty} P_m(x) \frac{z^m}{m!}, \quad (2.14)
\]
and
\[
\frac{e^{xz}}{\phi(iz)} = \sum_{m=0}^{\infty} Q_m(x) \frac{z^m}{m!} \quad (2.15)
\]
respectively. Then $\phi(x)$ is a $\alpha$-scaling compact supported function with mask $\psi(x)$ if and only if
\[
\sum_{k=0}^{m} \alpha^{-m} \binom{m}{k} P_k(\alpha x) Q_{m-k}(\alpha x) = Q_m(2x). \quad (2.16)
\]
Proof. Since
\[ \frac{e^{\frac{ix}{\alpha}}}{\psi\left(\frac{iz}{\alpha}\right)} = \sum_{m=0}^{\infty} \alpha^{-m} P_m(x) \frac{z^m}{m!}, \quad \frac{e^{\frac{ix}{\alpha}}}{\phi\left(\frac{iz}{\alpha}\right)} = \sum_{m=0}^{\infty} \alpha^{-m} Q_m(x) \frac{z^m}{m!}, \]
then
\[ \frac{e^{\frac{2ix}{\alpha}}}{\psi\left(\frac{iz}{\alpha}\right)\phi\left(\frac{iz}{\alpha}\right)} = \left(\sum_{j=0}^{\infty} \alpha^{-m} P_m(x) \frac{z^m}{m!}\right) \left(\sum_{m=0}^{\infty} \alpha^{-m} Q_m(x) \frac{z^m}{m!}\right) \tag{2.17} \]
\[ = \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} \alpha^{-m} \binom{m}{k} P_k(x) Q_{m-k}(x)\right) \frac{z^m}{m!}. \tag{2.18} \]
Suppose that the Appell sequence polynomials, \( P_m(x) \) and \( Q_m(x) \), satisfy \(2.16\),
\[ \sum_{k=0}^{m} \alpha^{-m} \binom{m}{k} P_k(\alpha x) Q_{m-k}(\alpha x) = Q_m(2x). \]
Then
\[ \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} \alpha^{-m} \binom{m}{k} P_k\left(\frac{\alpha x}{2}\right) Q_{m-k}\left(\frac{\alpha x}{2}\right)\right) \frac{z^m}{m!} = \sum_{m=0}^{\infty} Q_m(x) \frac{z^m}{m!}. \tag{2.19} \]
Therefore
\[ \frac{e^{\frac{ix}{\alpha}}}{\psi\left(\frac{iz}{\alpha}\right)\phi\left(\frac{iz}{\alpha}\right)} = \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} \alpha^{-m} \binom{m}{k} P_k\left(\frac{\alpha x}{2}\right) Q_{m-k}\left(\frac{\alpha x}{2}\right)\right) \frac{z^m}{m!} \quad \tag{2.20} \]
\[ = \sum_{m=0}^{\infty} Q_m(x) \frac{z^m}{m!} = \frac{e^{\frac{ix}{\alpha}}}{\phi\left(\frac{iz}{\alpha}\right)}. \tag{2.21} \]
We see that
\[ \hat{\psi}\left(\frac{iz}{\alpha}\right) \hat{\phi}\left(\frac{iz}{\alpha}\right) = \hat{\phi}(iz), \]
which imply \( \phi(x) \) is scaling function with mask \( \psi(x) \).

Suppose that \( \phi(x) \) is a \( \alpha \)-scaling compact supported function with mask \( \psi(x) \), satisfying the scaling equation \(2.12\). Then the Fourier transform \( \hat{\phi} \) is given in \(2.13\) shows that
\[ \hat{\phi}\left(\frac{iz}{\alpha}\right) \hat{\phi}\left(\frac{iz}{\alpha}\right) = \hat{\phi}(iz). \tag{2.22} \]
It follows that

\[
\sum_{m=0}^{\infty} Q_m(2x) \frac{z^m}{m!} = e^{2xz} \frac{\phi(iz)}{\phi(iz)} \frac{\psi(iz)}{\psi(iz)} = \left( \sum_{m=0}^{\infty} \alpha^{-m} P_m(\alpha x) \frac{z^m}{m!} \right) \left( \sum_{m=0}^{\infty} \alpha^{-m} Q_m(\alpha x) \frac{z^m}{m!} \right) = \sum_{m=0}^{\infty} \left( \sum_{k=0}^{m} \alpha^{-m} \binom{m}{k} P_k(\alpha x) Q_{m-k}(\alpha x) \right) \frac{z^m}{m!}.
\]

Therefore

\[
\sum_{k=0}^{m} \alpha^{-m} \binom{m}{k} P_k(\alpha x) Q_{m-k}(\alpha x) = Q_m(2x).
\]

\[\square\]

3. Generalized Bernoulli polynomials, Euler Polynomials and B-splines

B-splines with order \(N\), which is denoted as \(B_N(\cdot)\), is defined by the induction as

\[
B_1(x) = \begin{cases} 
1 & \text{if } x \in [0, 1), \\
0 & \text{otherwise}
\end{cases}
\]

and for \(N \geq 1\)

\[
B_N = B_1 \ast B_{N-1},
\]

where \(\ast\) denotes the operation of convolution which is defined by

\[
(f \ast g)(t) := \int_{-\infty}^{+\infty} f(t-y)g(y)dy,
\]

for \(f\) and \(g\) in \(L^2(\mathbb{R})\).

The Fourier transform of \(B_N(x)\) is

\[
\hat{B}_N(\omega) = \left( \frac{1 - e^{-i\omega}}{i\omega} \right)^N, \quad \omega \in \mathbb{R}.
\]

\(B_N(x)\) also satisfies the scaling function as follow:

\[
B_N(x) = 2 \sum_{j=0}^{N} \frac{1}{2^N} \binom{N}{j} B_N(2x - j), \quad (3.1)
\]

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where the mask $\psi_N(k) := \frac{1}{\sqrt{2N}} \binom{n}{k}$. Equivalently, (3.1) can be expressed in term of Fourier transforms in the frequency domain in the form:

$$\hat{B}_N(\omega) = \hat{\psi}_N(\omega/2) \hat{B}_N(\omega/2),$$

where the Fourier transform of the mask is $\hat{\psi}_N(\omega) = \left( \frac{1 + e^{-i\omega}}{2} \right)^N$.

The asymptotic properties of B-splines have a long history going back to the physicist Arnold Sommerfeld who showed that Gaussian function can be approximated by B-splines point-wise in 1904 [1]. In 1992, Unser and his colleagues [17] proved that the sequence of normalized and scaled B-splines tends to Gaussian function in $L^p$ space as the order $N$ increases. L. H. Y. Chen, T. N. T. Goodman and S. L. Lee [16] considered the convergence orders of scaling functions which asymptotic to normality. A result due to Ralph Brinks [24] generalized Unser’s result to the derivatives of the B-splines. Yan Xu and R. H. Wang [29] gave the convergence orders of the approximation processes and showed the asymptotic relationship among B-splines, Eulerian numbers and Hermite polynomials.

**Theorem 3.1.** [29] Let be $k \in \mathbb{N}$, for $N > k + 2$, the sequence of the $k$-th derivatives, $B_N^{(k)}$, of the B-spline converges to the $k$-th derivative of the Gaussian function

$$\left( \frac{N}{12} \right)^{\frac{k+1}{2}} B_N^{(k)} \left( \sqrt{\frac{N}{12}} x + \frac{N}{2} \right) = \frac{1}{\sqrt{2\pi}} D^k \exp \left( -\frac{x^2}{2} \right) + O \left( \frac{1}{N} \right), \quad (3.2)$$

and

$$\lim_{d \to \infty} \left\{ \left( \frac{N}{12} \right)^{\frac{k+1}{2}} B_N^{(k)} \left( \sqrt{\frac{N}{12}} x + \frac{N}{2} \right) \right\} = \frac{(-1)^k}{\sqrt{2\pi}} H_k(x) G(x), \quad (3.3)$$

where the limit may be taken point-wise or in $L^p(\mathbb{R}), p \in [2, \infty)$.

Generalized Bernoulli[2, 3, 10] and Euler polynomials[22, 23] of degree $m$, order $N$ and complex argument $z$, denoted respectively by $B_m^N(z)$ and $E_m^N(z)$ can be defined by their generating functions,

$$\frac{\omega^N e^{\omega z}}{(e^\omega - 1)^N} = \sum_{m=0}^{\infty} \frac{B_m^N(z)}{m!} \omega^m, \quad |\omega| < 2\pi, \quad (3.4)$$

$$\frac{2^N e^{\omega z}}{(e^\omega + 1)^N} = \sum_{m=0}^{\infty} \frac{E_m^N(z)}{m!} \omega^m, \quad |\omega| < \pi. \quad (3.5)$$
In paper \cite{27}, S. L. Lee has proved that the Appell polynomials generated by the uniform \textit{B-spline} of order $N$ are the generalized Bernoulli polynomials of order $N$ and when suitably normalized they converge to the Hermit polynomials as $N \to \infty$. Since the B-splines approximate the Gaussian function \cite{16, 17, 24, 27, 29}, they can also be used as a filter in place of the Gaussian filter for linear scale-space \cite{30}.

Corollary 3.1. \cite{24}

$$\lim_{N \to \infty} \left( \frac{12}{N} \right)^{\frac{N}{2}} B_N^N \left( \sqrt{\frac{N}{12}} z + \frac{N}{2} \right) = H_m(z).$$

\textbf{Proof.} Recall that the uniform B-spline, $B_N(x)$, of order $N$, is the scaling function satisfying

$$B_N(x) = 2 \sum_{j=0}^{N} \frac{1}{2^N} \binom{N}{j} B_N(2x - j),$$

where the mask $\psi_N(k) := \frac{1}{2^N} \binom{N}{k}$. Equivalently, the scaling equation can be expressed in term of Fourier transforms in the frequency domain in the form:

$$\hat{B}_N(\omega) = \hat{\psi}_N(\omega/2) \hat{B}_N(\omega/2),$$

where the Fourier transform of the mask is $\hat{\psi}_N(\omega) = \left( \frac{1 + e^{-i\omega}}{2} \right)^N$. From the fourier transform of $B_N$ we have

$$\hat{B}_N(i\omega) = \left( \frac{e^{i\omega} - 1}{\omega} \right)^N. \quad (3.6)$$

By \cite{22, 24} and the generating function of $B_m^N(z)$, the generalized Bernoulli polynomials, $\{B_m^N(z) : m = 0, 1, \ldots\}$, are biorthogonal to the derivatives of the \textit{B-splines}, $B_N^{(n)}$,

$$\left( -1 \right)^n B_N^{(n)}(z), \frac{B_m^N(z)}{m!} \right) = \delta_{m,n}. \quad (3.7)$$

The standardized \textit{B-splines},

$$\hat{B}_N(x) = \sqrt{\frac{N}{12}} B_N \left( \frac{N}{12} x + \frac{N}{12} \right),$$

converges uniformly to the Gaussian function, $G(x)$, and an estimate of the rate of convergence is given in \cite{16, 27, 29}.
By Theorem 2.1 we have
\[
\lim_{N \to \infty} \left( \frac{12}{N} \right)^{\frac{m}{2}} \frac{B_m}{\sqrt{N}} \left( \sqrt{\frac{N}{12}} z + \frac{N}{2} \right) = H_m(z).
\]

It is well known that the binomial distributions converge to the normal distribution in the sense that
\[
\lim_{N \to \infty} \sum_{k=0}^{[x_N]} \frac{1}{2^N} \binom{N}{k} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt,
\]
where \( x_N = \sqrt{N}x/2 + N/2 \). Let \( \sigma_N = \sqrt{N}x/2 \) and \( \mu_N = N/2 \), then the standardized binomial distributions, \( \psi_N(x) := \sigma_N \psi_N(\sigma_N x + \mu_N) \), converges uniformly to the Gaussian function \( G(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \).

The Appell polynomials generated by the binomial distributions are the generalized Euler polynomials, \( E^N_m(z) \), that are biorthogonal to the derivatives of \( \psi_N(z) \),
\[
\langle (-1)^n \psi_N^{(n)}(z), \frac{E^N_m(z)}{m!} \rangle = \delta_{m,n}.
\]

By Theorem (2.1), the normalized generalized Euler polynomials \( \tilde{E}^N_m(z) \) converge to the Hermit polynomials, \( H_m(z) \), as \( N \to \infty \),
\[
\lim_{N \to \infty} \left( \frac{4}{N} \right)^{\frac{m}{2}} \frac{E^N_m}{\sqrt{N}} \left( \sqrt{\frac{N}{2}} z + \frac{N}{2} \right) = H_m(z).
\]
Corollary 3.3. Generalized Euler and Bernoulli polynomials satisfy
\[ B_m^N(z) = \frac{1}{2m} \sum_{k=0}^{m} \binom{m}{k} E_k^N(z) B_{m-k}^N(z). \] (3.13)

Proof. Let \( \hat{\psi}_N(\omega) = \left( \frac{1+e^{-i\omega}}{2} \right)^N \) and \( \hat{B}_N(\omega) = \left( \frac{1-e^{-i\omega}}{i\omega} \right)^N \). By \( \text{[3.4] [3.5]} \), we can see that the Euler and Bernoulli polynomials are generated by \( \hat{\psi}_N(i\omega) \) and \( \hat{B}_N(i\omega) \) respectively. Recall that the uniform B-spline, \( B_N \), of order \( N \), is the scaling function satisfying
\[ \hat{B}_N(\omega) = \hat{\psi}_N(\omega/2) \hat{B}_N(\omega/2), \]
and the Fourier transform of \( B_N \) is
\[ \hat{B}_N(\omega) = \left( \frac{1-e^{-i\omega}}{i\omega} \right)^N, \quad \omega \in \mathbb{R}. \]
Therefore by Theorem 2.2, it holds
\[ B_m^N(z) = \frac{1}{2m} \sum_{k=0}^{m} \binom{m}{k} E_k^N(z) B_{m-k}^N(z). \] (3.14)

4. Generalized Buchholz polynomials and Ultraspherical (Gegenbauer) polynomials

Theorem 4.1. Let \( \{ \tilde{P}_{N,m}(x) : m = 0, 1, \ldots \} \) be the polynomials sequence generated by \( \hat{f}_N(x,z) = \sum_{m=0}^{\infty} \frac{P_{N,m}(x) z^m}{m!} \). There are constants \( r > 0 \) and \( A > 0 \), such that for all sufficient large \( N \), it holds \( |\hat{f}_N(x,z) - e^{xz}| \leq \frac{A}{\sigma_N} \), and \( |\hat{\phi}_N(iz) - e^{iz}| \leq \frac{A}{\sigma_N} \), for \( \sigma_N \rightarrow 0 \). Then for each \( m = 0, 1, \ldots \), \( \tilde{P}_{N,m}(x) \) converges locally uniformly to the Hermite polynomial \( H_m(x) \) as \( N \) goes to infinity.

Remark 4.1. When \( \hat{f}_N(x,z) = e^{xz} \), theorem (4.1) turns to theorem (2.1).

Proof. Since \( \hat{\phi}_N(0) = 1 \), we can choose a neighborhood \( U \) of the origin so that \( |\hat{\phi}(iz)| \geq \frac{1}{2} \) and \( |e^{iz}| \geq \frac{1}{2} \) for all \( z \in U \). Take a circle inside \( U \) with center at
0 and radius \( r \), so that for all sufficient large \( N \), it holds \(| \hat{f}_N(x, z) - e^{xz} | \leq \frac{A}{\sigma_N} \) and \(| \hat{\phi}_N(ziz) - e^{z^2} | \leq \frac{A}{\sigma_N} \), for \(| z | < r \).

The coefficients of the Taylor series

\[
\frac{\hat{f}_N(x, z)}{e^{z^2}/e^{x^2}} = \sum_{m=0}^{\infty} \frac{\hat{P}_{N,m}(z)}{m!} e^{\frac{m^2}{2} z^2} = \sum_{m=0}^{\infty} \frac{\hat{P}_{N,m}(x) - H_m(x)}{m!} e^{\frac{m^2}{2} z^2}
\]

are represented by the Cauchy’s integral formula

\[
\hat{P}_{N,m}(x) - H_m(x) = \frac{m!}{2\pi i} \oint_C \frac{1}{z^{m+1}} \left( \frac{\hat{f}_N(x, z)}{\hat{\phi}(iz)} - e^{xz}e^{z^2} \right) \, dz
\]

\[
= \frac{m!}{2\pi i} \oint_C \left( e^{z^2} \frac{\hat{f}_N(x, z) - e^{xz}\hat{\phi}(iz)}{e^{z^2}} \right) \, dz.
\]

Therefore, for any given real numbers \( A > 0 \), there is a sufficient large \( N_0 \), for any \( N > N_0 \), it holds

\[
\left| \hat{P}_{N,m}(x) - H_m(x) \right| \leq \frac{m!}{2\pi} \oint_C \left( e^{z^2} \left| \frac{\hat{f}_N(x, z)}{\hat{\phi}(iz)} - e^{xz}e^{z^2} \right| \right) \, dz
\]

\[
\leq \frac{m!}{2\pi} \oint_C \left( e^{z^2} \left| \frac{\hat{f}_N(x, z) - e^{xz}e^{z^2}}{\hat{\phi}(iz)} \right| + e^{xz}e^{z^2} - e^{xz}\hat{\phi}(iz) \right) \, dz
\]

\[
\leq \frac{m!}{2\pi} \oint_C \left( e^{z^2} \left| \frac{\hat{f}_N(x, z) - e^{xz}}{\hat{\phi}(iz)} \right| + e^{xz}e^{z^2} - e^{xz}\hat{\phi}(iz) \right) \, dz
\]

\[
\leq \frac{4(m!)}{\sigma_N} \frac{A (e^{z^2} + e^{xz})}{e^{z^2}}
\]

Since \( \sigma_N \to \infty \) as \( N \to \infty \), it follows that for each \( m \), \( \hat{P}_{N,m}(x) \to H_m(x) \) uniformly on compact sets.

Generalized Buchholz and Ultraspherical (Gegenbauer) polynomials of degree \( m \), order \( N \) and complex argument \( x \), denoted respectively by \( P_m^N(x) \) and \( C_m^N(x) \), can be defined by their generating functions,

\[
e^{x(\cot z - \frac{1}{2})/2} \left( \frac{\sin z}{z} \right)^N = \sum_{m=0}^{\infty} P_m^N(x) z^m, \quad |z| < \pi
\]
and 
\[(1 - 2xz + z^2)^{-N} = \sum_{n=0}^{\infty} C_N^m(x)z^m, \quad -1 \leq x \leq 1, |\omega| < 1.\]

Buchholz polynomials are used for the representation of the Whittaker functions as convergent series expansions of Bessel functions \(\text{[9]}\). They appear also in the convergent expansions of the Whittaker functions in ascending powers of their order and in the asymptotic expansions of the Whittaker functions in descending powers of their order \(\text{[12]}\). Explicit formulas for obtaining these polynomials may be found in \(\text{[11]}\).

There are well known limits\(\text{[15]}\)
\[
\lim_{N \to \infty} \left( \frac{3}{N} \right)^{\frac{m}{2}} P_m^N(-2\sqrt{3N}x) = \frac{1}{m!} H_m(x)
\]
and
\[
\lim_{N \to \infty} (2N)^{-\frac{m}{2}} C_N^m \left( \frac{x}{\sqrt{2N}} \right) = \frac{1}{m!} H_m(x).
\]

These limits give insight in the location of the zeros for large values of the limit parameter, and the asymptotic relation with the Hermite polynomials if the parameter \(N\) become large and \(x\) is properly scaled.

Many methods are available to prove these and other limits\(\text{[15]}\). We can get these asymptotic results from theorem \(\text{[4.1]}\) as simple cases.

**Lemma 4.1.** For any \(|z| < \pi\),
\[
\lim_{N \to \infty} \text{sinc}^N \left( \frac{z}{2} \sqrt{\frac{12}{N}} \right) = \exp \left( -\frac{z^2}{2} \right).
\]

**Proof.** Set 
\[
L_N(z) := N \ln \left[ \text{sinc} \left( \frac{z}{2} \sqrt{\frac{12}{N}} \right) \right].
\]

Then with \(z_N = \frac{z}{2} \sqrt{\frac{12}{N}}\), it holds
\[
L_N(z) = N \ln \left[ \text{sinc} \left( \frac{z}{2} \sqrt{\frac{12}{N}} \right) \right] = 3z^2 \ln \left[ \text{sinc} \left( \frac{z}{2} \sqrt{\frac{12}{N}} \right) \right] = 3z^2 \ln \left[ \frac{\text{sinc}(z_N)}{z_N^2} \right].
\]
Since it holds $\text{sinc}(0) = 1$, $\text{sinc}^{(1)}(0) = 0$ and $\text{sinc}^{(2)}(0) = -\frac{1}{3}$, for the $\lim_{N \to \infty}$, and hence $z_N \to 0$, we may apply L’Hôpital’s rule twice: For any $|z| < \pi$, we have

$$
\lim_{N \to \infty} L_N(z) = 3z^2 \lim_{N \to \infty} \frac{\text{sinc}^{(1)}(z_N)}{2z_N \text{sinc}(z_N)} = 3z^2 \lim_{N \to \infty} \frac{\text{sinc}^{(2)}(z_N)}{2\text{sinc}(z_N) + 2z_N \text{sinc}^{(1)}(z_N)} = 3z^2 \frac{\text{sinc}^{(2)}(0)}{2} = -\frac{z^2}{2}.
$$

It follows: For any $|z| < \pi$,

$$
\lim_{N \to \infty} \text{sinc}^N \left( \frac{z}{2\sqrt{\frac{12}{N}}} \right) = \exp \left( -\frac{z^2}{2} \right).
$$

**Corollary 4.1.**

$$
\lim_{N \to \infty} \left( \frac{3}{N} \right)^m P_N^m(-2\sqrt{3Nx}) = \frac{1}{m!} H_m(x). \quad (4.3)
$$

**Proof.** Let $\sigma_N = \sqrt{\frac{N}{12}}$ then

$$
\sum_{m=0}^{\infty} \sigma_N^{-m} P_N^m(-12\sqrt{2x\sigma_N}) \left( \frac{\omega}{\sqrt{2}} \right)^m = e^{-6\sqrt{2x\sigma_N} \left( \frac{\omega}{\sqrt{2}} \tan \left( \frac{\omega}{\sqrt{2} \sigma_N} \right) - \frac{\sqrt{2} \sigma_N}{\omega} \right)^N}.
$$

Let $\tilde{f}_N(\omega, x) = e^{-6\sqrt{2x\sigma_N} \left( \frac{\omega}{\sqrt{2}} \tan \left( \frac{\omega}{\sqrt{2} \sigma_N} \right) - \frac{\sqrt{2} \sigma_N}{\omega} \right)}$ and $\tilde{\phi}_N(i\omega) = \left( \frac{\sin \frac{\omega}{\sqrt{2} \sigma_N}}{\frac{\omega}{\sqrt{2} \sigma_N}} \right)^{-N}$. By Taylor theorem, for any $|\omega| < \pi$ and sufficient large $N$, it holds

$$
\ln \tilde{f}_N(\omega, x) = -6\sqrt{2x\sigma_N} \left( \frac{\omega}{\sqrt{2}} \tan \left( \frac{\omega}{\sqrt{2} \sigma_N} \right) - \frac{\sqrt{2} \sigma_N}{\omega} \right) = -6\sqrt{2x\sigma_N} \left( -\frac{\omega}{3\sqrt{2} \sigma_N} + O(\sigma_N^{-3}) \right).
$$

Therefore, for $N \to +\infty$, we have

$$
\lim_{N \to \infty} \tilde{f}_N(\omega, z) = e^{2x\omega}.
$$

By lemma 4.1 we have

$$
\lim_{N \to +\infty} \tilde{\phi}_N(i\omega) = \lim_{N \to \infty} \left( \frac{\sin \frac{\omega}{\sqrt{2} \sigma_N}}{\frac{\omega}{\sqrt{2} \sigma_N}} \right)^{-N} = e^{\omega^2}.
$$
Therefore by Theorem 4.1 it holds
\[
\lim_{N \to \infty} \left( \frac{6}{N} \right)^{\frac{m}{2}} P_m^N(-2\sqrt{6N}x) = \frac{1}{m!} H_m(\sqrt{2}x) \sqrt{2^m}, \tag{4.4}
\]
equivalently,
\[
\lim_{N \to \infty} \left( \frac{3}{N} \right)^{\frac{m}{2}} P_m^N(-2\sqrt{3N}x) = \frac{1}{m!} H_m(x). \tag{4.5}
\]

Corollary 4.2.
\[
\lim_{N \to \infty} (2N)^{-\frac{m}{2}} C_m^N \left( \frac{x}{\sqrt{2N}} \right) = \frac{1}{m!} H_m(x). \tag{4.5}
\]

Proof. By the generating function of $C_m^N(x)$,
\[
(1 - 2xz + z^2)^{-N} = \sum_{n=0}^{\infty} C_m^N(x) z^m,
\]
it holds
\[
\sum_{n=0}^{\infty} (2N)^{-\frac{m}{2}} C_m^N \left( \frac{x}{\sqrt{2N}} \right) z^m = \left( 1 - \frac{xz}{\sqrt{2N}} + \frac{z^2}{2N} \right)^{-N}
\]
\[
= \left[ 1 - \left( \frac{2xz - z^2}{2N} \right) \right]^{-2N (xz - z^2/2)}.
\]

Let $g_N = \left[ 1 - \left( \frac{2xz - z^2}{2N} \right) \right]^{-2N (xz - z^2/2)}$, then $\lim_{N \to \infty} g_N = e$. Therefore
\[
\sum_{n=0}^{\infty} (2N)^{-\frac{m}{2}} C_m^N \left( \frac{x}{\sqrt{2N}} \right) z^m = g_N^{xz} g_N^{-z^2/2}.
\]
and
\[
\lim_{N \to \infty} g_N^{xz} = e^{xz} \quad \lim_{N \to \infty} g_N^{-z^2/2} = e^{-z^2/2}.
\]

By Theorem 4.1, we have
\[
\lim_{N \to \infty} (2N)^{-\frac{m}{2}} C_m^N \left( \frac{x}{\sqrt{2N}} \right) = \frac{1}{m!} H_m(x). \tag{4.5}
\]

\[\square\]
5. Biorthogonal systems relate to Generalized Laguerre polynomials

Laguerre polynomials, \( L_n(x) \), are solutions to the Laguerre differential equation

\[
xy'' + (1 - x)y' + ny = 0, \quad n \geq 0.
\]

Laguerre polynomials is a class of orthogonal polynomials with weighting function \( w(x) = e^{-x} \). The Rodrigues representation for the Laguerre polynomials is

\[
L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n}(e^{-x} x^n)
\]  

(5.1)

and the generating function for Laguerre polynomials is

\[
(1 - z)^{-\frac{1}{\alpha}} e^{\frac{xz}{1 - z}} = \sum_{m=0}^{\infty} L_m(x) z^m, \quad |z| \leq 1.
\]  

(5.2)

The generalized Laguerre polynomials, \( L_m^{(\alpha)} \), are also a class of orthogonal polynomials with weighting function \( w(x) = x^\alpha e^{-x} \) and generated by

\[
(1 - z)^{-\alpha-1} e^{\frac{xz}{1 - z}} = \sum_{m=0}^{\infty} L_m^{(\alpha)}(x) z^m, \quad |z| \leq 1.
\]  

(5.3)

The Rodrigues representation for the generalized Laguerre polynomials is

\[
L_m^{(\alpha)}(x) = \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n}(e^{-x} x^{n+\alpha}).
\]  

(5.4)

When \( \alpha = 0 \), we have \( L_m^{(0)}(x) = L_n(x) \).

The explicit formula for \( L_n^{(\alpha)}(x) \) is

\[
L_n^{(\alpha)}(x) = \frac{(\alpha + 1)_n}{n!} \sum_{j=0}^{n} \frac{(-n)_j}{(\alpha + 1)_j} x^j j!
\]  

(5.5)

where \( (\alpha)_n = \prod_{i=1}^{n} (\alpha + i - 1) \), \((\alpha)_0 = 1\), for \( n = 1, 2, 3 \ldots \). When \( \alpha = 0 \), the explicit formula for generalized Laguerre polynomials, \( L_n^{(\alpha)}(x) \), become

\[
L_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^k k!.
\]  

(5.6)

By simply computing, we have
Proposition 5.1.

\[ L_n'(x) = L_{n-1}'(x) - L_{n-1}(x). \]  

(5.7)

The orthogonality relation for the Laguerre polynomials is contained in

\[
\int_0^{\infty} L_m^{(\alpha)}(x)L_n^{(\alpha)}(x)x^\alpha e^{-x} dx = \frac{\Gamma(\alpha + n + 1)}{n!} \delta_{mn}, \quad \alpha > -1,
\]

which can be considered as a biorthogonal relation between the derivatives of \( \{ d^m / dx^m \Gamma(\alpha + n + 1) \} \) and Laguerre polynomials \( \{ L_m^{(\alpha)} : m = 0, 1, \ldots \} \),

\[
\left\langle I_m^{(\alpha)} , \frac{d^m}{dx^m} \frac{x^{\alpha+n}e^{-x}}{\Gamma(\alpha + n + 1)} \right\rangle = \delta_{m,n}.
\]

Corollary 5.1.

\[
\lim_{N \to \infty} (-1)^m (2N)^{-m/2} L_m^{(N)}(x\sqrt{2N} + N) = \frac{1}{m!} H_m(x).
\]

Proof.

\[
\sum_{m=0}^{\infty} (-1)^m (2N)^{-m/2} L_m^{(N)}(x\sqrt{2N} + N)z^m = \left(1 + \frac{z}{\sqrt{2N}}\right)^{-N-1} e^{\frac{xz}{\sqrt{2N}}} e^{\frac{x^2}{2}N}.
\]

\[
\hat{f}_N(x, z) = e^{\frac{xz}{\sqrt{2N}}} , \quad \hat{\phi}_N(iz) = \left(1 + \frac{z}{\sqrt{N}}\right)^{N+1} e^{\frac{x^2}{2}N}, \quad \text{then for } N \to \infty, \text{ we have}
\]

\[
\lim_{N \to \infty} \hat{f}_N(x, z) = e^{xz}, \quad \lim_{N \to \infty} \hat{\phi}_N(iz) = e^{\frac{x^2}{2}}.
\]

By Theorem 4.1, we have

\[
\lim_{N \to \infty} (-1)^m (2N)^{-m/2} L_m^{(N)}(x\sqrt{2N} + N) = \frac{1}{m!} H_m(x).
\]

\[
\square
\]

In this section, we consider a family of biorthogonal polynomials, \( \{ P_m(x, \alpha, \omega) : m = 0, 1, \ldots \} \), generated by a sequence of functions, \( \phi(x, \alpha, \omega) \), which converges to the generalized Laguerre polynomials, \( L_m^{(\alpha)}(x) \).
Taking a compactly supported distribution \( \phi \in \mathcal{E}'(\mathbb{R}) \), Let \( \hat{\phi} \) denote the Laplace transform of \( \phi \). Then for any integer \( n \geq 0 \),
\[
\left\langle \phi^{(n)}, (1 - z)^n e^{\frac{-z}{1-z}} \right\rangle = \left\langle \phi(x), z^n e^{\frac{-z}{1-z}} \right\rangle = z^n \hat{\phi}(\frac{z}{1-z}) \tag{5.9}
\]
If \( \hat{\phi}(0) \neq 0 \),
\[
\left\langle \phi^{(n)}, (1 - z)^n e^{\frac{-z}{1-z}} \phi(\frac{1}{1-z}) \right\rangle = z^n \tag{5.10}
\]
in a neighborhood of 0. Since \( \phi \) is compactly supported, \( \hat{\phi} \) is analytic. So we can define a sequence of polynomials, \( P_m \), by the generating function
\[
(1 - z)^n e^{\frac{-z}{1-z}} \hat{\phi}(\frac{1}{1-z}) = \sum_{m=0}^{\infty} P_m(x)z^m. \tag{5.11}
\]
It follows from (5.10) and (5.11) that for any integer \( n \geq 0 \),
\[
z^n = \sum_{m=0}^{\infty} \left\langle \phi^{(n)}, P_m(x) \right\rangle z^m, \tag{5.12}
\]
which gives the biorthogonal relation
\[
\left\langle \phi^{(n)}, P_m(x) \right\rangle = \delta_{m,n}. \tag{5.13}
\]
Differentiating (5.11) with respect to \( x \) and equating coefficients of \( z^m \) in the resulting equation gives
\[
P'_m(x) = P'_{m-1}(x) - P_{m-1}(x), \quad m = 1, 2, \ldots, \tag{5.14}
\]
which is similar to the property of Laguerre polynomials (Proposition 5.1),
\[
L'_m(x) = L'_{m-1}(x) - L_{m-1}(x).
\]
If \( \phi(x, \alpha) = \frac{x^{\alpha+n}e^{-x}}{\Gamma(\alpha+n+1)} \), the Laplace transform of \( \phi(x, \alpha) \) is \( \hat{\phi}(z, \alpha) = \frac{1}{(1-z)^{\alpha+n+1}} \), which implies \( \hat{\phi}(\frac{1}{1-z}, \alpha) = (1-z)^{\alpha+n+1} \cdot \hat{\phi}(\frac{1}{1-z}, \alpha) = \frac{1}{(1-z)^{\alpha+n+1}} \cdot \hat{\phi}(\frac{1}{1-z}, \alpha) \). We have
\[
\left\langle \phi^{(n)}(x, \alpha), \frac{e^{\frac{-z}{1-z}(1-z)^n}}{\phi(\frac{1}{1-z}, \alpha)} \right\rangle = z^n. \tag{5.15}
\]
in a neighborhood of 0. We can define a sequence of polynomials, \( L_m^{(\alpha)}(x) \), by the generating function
\[
e^{\frac{-z}{1-z}(1-z)^n} \hat{\phi}(\frac{1}{1-z}, \alpha) = (1-z)^{-\alpha-1}e^{\frac{-z}{1-z}} = \sum_{m=0}^{\infty} L_m^{(\alpha)}(x)z^m. \tag{5.16}
\]
So the biorthogonal systems generated by function \( \phi(x, \alpha) = \frac{x^{\alpha+n+1}}{\Gamma(\alpha+n+1)} \) are Laguerre polynomials.

**Theorem 5.1.** Let \( \phi(x, \alpha, \omega) \) satisfy the following conditions:

1. There are constants \( 0 < r < 1 \) and \( c \), for any \( \varepsilon > 0 \), \( \exists \delta \), such that
   \[
   |\omega - c| < \delta, \text{ it holds }
   \left| \frac{e^{-\frac{x}{1-z}}}{\phi(z, \alpha, \omega)} - (1-z)^{\alpha+n+1} \right| \leq \varepsilon, \quad |z| < r. \tag{5.16}
   \]
   Equivalently, \( \lim_{\omega \to c} \phi(x, \alpha, \omega) = \frac{e^{-x}x^{\alpha+n+1}}{\Gamma(\alpha+n+1)} \).

2. Let \( \{P_m(x, \alpha, \omega) : m = 0, 1, \ldots\} \) be the biorthogonal polynomials generated by the functions, \( \phi(z, \alpha, \omega) \) by
   \[
   \frac{e^{-\frac{x}{1-z}}(1-z)^n}{\phi(z, \alpha, \omega)} = \sum_{m=0}^{\infty} P_m(x, \alpha, \omega) z^m.
   \]
   Then for each \( m = 0, 1, \ldots \), \( \{P_m(x, \alpha, \omega) : m = 0, 1, \ldots\} \) converges locally uniformly to the generalized Laguerre polynomial, \( L_m^{(\alpha)}(x) \), as \( \omega \) goes to \( c \).

**Proof.** Since \( \phi(0, \alpha, \omega) = 1 \), we can choose a neighborhood \( U \) of the origin so that \( \left| \phi\left(\frac{z}{1-z}, \alpha, \omega\right) \right| \geq \frac{1}{2} \) and \( |(1-z)^{\alpha+1}| \geq \frac{1}{2} \) for all \( z \in U \). Take a circle \( C \) completely contained in \( U \), with centra at the origin \( 0 \) and radius \( r \), so that (5.16) is satisfied. The coefficients of the Taylor series
   \[
   \frac{(1-z)^n e^{-\frac{x}{1-z}}}{\phi\left(\frac{z}{1-z}, \alpha, \omega\right)} - \frac{e^{-\frac{x}{1-z}}}{(1-z)^{\alpha+1}} = \sum_{m=0}^{\infty} \left( P_m(x, \alpha, \omega) - L_m^{(\alpha)}(x) \right) z^m \tag{5.17}
   \]
   are represented by the Cauchy’s integral formula:
   \[
   P_m(x, \alpha, \omega) - L_m^{(\alpha)}(x) = \frac{1}{2\pi i} \oint_C \frac{e^{-\frac{x}{1-z}}(1-z)^{\alpha+n+1} - \phi\left(\frac{z}{1-z}, \alpha, \omega\right)}{z^{m+1} \phi\left(\frac{z}{1-z}, \alpha, \omega\right)(1-z)^{\alpha+1}} dz
   \]
   \[
   \left| P_m(x, \alpha, \omega) - L_m^{(\alpha)}(x) \right| \leq \frac{1}{2\pi} \oint_C e^{\frac{x}{1-z}} \frac{|(1-z)^{\alpha+n+1} - \phi\left(\frac{z}{1-z}, \alpha, \omega\right)|}{r^{m+1} |\phi\left(\frac{z}{1-z}, \alpha, \omega\right) ||(1-z)^{\alpha+1}|} dz
   \]
   \[
   \leq \frac{1}{2\pi} \oint_C \frac{e^{x Re\left(\frac{1}{1-z}\right)} \varepsilon}{r^{m+1} |\phi\left(\frac{z}{1-z}, \alpha, \omega\right) ||(1-z)^{\alpha+1}|} dz
   \]
   \[
   \leq 4e^{x Re\left(\frac{1}{1-z}\right)} \varepsilon.
   \]

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It follows that for each $m$, $P_m(x, \alpha, \omega)$ converges locally uniformly to the generalized Laguerre polynomial, $L_n^{(\alpha)}(x)$, as $\omega$ goes to $c$.

For the Meixner-Pollaczek polynomials, we have the generating function:

$$F(x, z) = (1 - e^{i\omega}z)^{-\lambda+ix} (1 - e^{-i\omega}z)^{-\lambda-ix} = \sum_{n=0}^{\infty} P^{(\lambda)}_m(x; \omega) z^n. \quad (5.18)$$

**Corollary 5.2.** The Laguerre polynomials can be obtained from Meixner-Pollaczek polynomials by the substitution $\lambda = \frac{1}{2}(\alpha + 1), x \rightarrow -\frac{1}{2}\omega^{-1}x$ and letting $\omega \rightarrow 0$.

Proof.

$$\lim_{\omega \rightarrow 0} P^{\frac{\alpha+1}{2}}_n\left(\frac{-x}{2\omega}; \omega\right) = L_n^{(\alpha)}(x).$$

Let $\tilde{\phi}(\frac{z}{1-z}, \alpha, \omega) = \left[(1 - ze^{i\omega}) (1 - ze^{-i\omega})\right]^{\frac{\alpha+1}{2}} (1 - z)^n$, it holds

$$\lim_{\omega \rightarrow 0} \tilde{\phi}(\frac{z}{1-z}, \alpha, \omega) = \lim_{\omega \rightarrow 0} \left[(1 - ze^{i\omega}) (1 - ze^{-i\omega})\right]^{\frac{\alpha+1}{2}} (1 - z)^n$$

$$= \lim_{\omega \rightarrow 0} \left[1 - 2z \cos \omega + z^2\right]^{\frac{\alpha+1}{2}} (1 - z)^n$$

$$= (1 - z)^{\alpha+n+1}.$$

Since

$$\lim_{\omega \rightarrow 0} \left(1 - ze^{i\omega}\right)^{-\frac{\alpha+1}{2}} = \lim_{\omega \rightarrow 0} \left(1 + \frac{e^{-i\omega} - e^{i\omega}}{1 - ze^{-i\omega}}\right)^{-\frac{\alpha+1}{2}}$$

$$= \lim_{\omega \rightarrow 0} \left(1 + \frac{e^{-i\omega} - e^{i\omega}}{1 - ze^{-i\omega}}\right)^{-\frac{\alpha+1}{2}}$$

$$= e^{\frac{i\pi}{2}},$$

by theorem (5.1), we have

$$\lim_{\omega \rightarrow 0} P^{\frac{\alpha+1}{2}}_n\left(\frac{-x}{2\omega}; \omega\right) = L_n^{(\alpha)}(x).$$

□
The generating function for Meixner polynomials \( M_n(x; \beta, c) \) is
\[
(1 - \frac{z}{c})^x (1 - z)^{-\beta - x} = \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} M_n(x; \beta, c) z^n.
\]

**Corollary 5.3.** The Laguerre polynomials can be obtained from Meixner polynomials by the substitution \( \beta = \alpha + 1, x \rightarrow \frac{cx}{1-c} \) and letting \( c \rightarrow 1 \).

\[
\lim_{c \rightarrow 1} M_n \left( \frac{cx}{1-c}; \alpha + 1, c \right) = \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)}.
\]

**Proof.**
\[
\sum_{n=0}^{\infty} \frac{(\alpha + 1)_n}{n!} M_n \left( \frac{cx}{1-c}; \alpha + 1, c \right) z^n
\]
\[
= \left( 1 - \frac{z}{c} \right)^{\frac{(\alpha + 1)_n}{n!}} (1 - z)^{-\alpha - 1 - \frac{cx}{1-c}}
\]
\[
= \left( \frac{1 - \frac{z}{c}}{1 - z} \right)^{\frac{(\alpha + 1)_n}{n!}} (1 - z)^{-\alpha - 1}
\]
\[
= \left( 1 + \frac{z(c - 1)}{c(1 - z)} \right)^{\frac{(\alpha + 1)_n}{n!}} (1 - z)^{-\alpha - 1}
\]
\[
= \left( 1 + \frac{z(c - 1)}{c(1 - z)} \right)^{\frac{(\alpha + 1)_n}{n!}} (1 - z)^{-\alpha - 1}
\]

Since
\[
\lim_{c \rightarrow 1} \left( 1 + \frac{z(c - 1)}{c(1 - z)} \right)^{\frac{(\alpha + 1)_n}{n!} \frac{cx}{1-c}} = e^{-x}
\]
(5.19)
Therefore, by theorem(5.1), it holds
\[
\lim_{c \rightarrow 1} (\alpha + 1)_n M_n \left( \frac{cx}{1-c}; \alpha + 1, c \right) = L_n^{(\alpha)}(x).
\]

Since
\[
L_n^{(\alpha)}(x) = \frac{(\alpha + 1)_n}{n!} \sum_{j=0}^{n} \frac{(-n)_j x^j}{(\alpha + 1)_j j!},
\]
then \( L_n^{(\alpha)}(0) = (\alpha + 1)_n \). Therefore
\[
\lim_{c \rightarrow 1} M_n \left( \frac{cx}{1-c}; \alpha + 1, c \right) = \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)}.
\]
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