The largest Erdős–Ko–Rado sets in $2 - (v, k, 1)$ designs

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Abstract An Erdős–Ko–Rado set in a block design is a set of pairwise intersecting blocks. In this article we study Erdős–Ko–Rado sets in $2 - (v, k, 1)$ designs, Steiner systems. The Steiner triple systems and other special classes are treated separately. For $k \geq 4$, we prove that the largest Erdős–Ko–Rado sets cannot be larger than a point-pencil if $r \geq k^2 - 3k + \frac{3}{4}\sqrt{k} + 2$ and that the largest Erdős–Ko–Rado sets are point-pencils if also $r \neq k^2 - k + 1$ and $(r, k) \neq (8, 4)$. For unitals we also determine an upper bound on the size of the second-largest maximal Erdős–Ko–Rado sets.

Keywords Erdős–Ko–Rado set · Block design · Steiner system · Unital

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1 Introduction

1.1 Block designs

Definition 1.1 A $t - (v, k, \lambda)$ block design, $v > k > 1$, $k \geq t \geq 1$, $\lambda > 0$, is an incidence geometry $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ with incidence relation $\mathcal{I}$, such that $|\mathcal{P}| = v$, such that any element of $\mathcal{B}$ (blocks) is incident with $k$ elements of $\mathcal{P}$ (points) and such that any $t$ points are contained in $\lambda$ common blocks. A block can be identified with the $k$-subset of $\mathcal{P}$ which it determines.

Block designs have been widely studied for many years, see for example [1,6,7,10,12] for an overview.

The following counting results are widely known.
Theorem 1.2 Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a $t - (v, k, \lambda)$ block design. Then,

- the number of blocks through an arbitrary set of $i$ points equals $\lambda_i = \lambda \binom{v-i}{t-i}/\binom{k-i}{t-i}$;
- in particular, the number of blocks through a fixed point equals $r = \lambda_1 = \lambda \binom{v-1}{t-1}/\binom{k-1}{t-1}$;
- $b = |\mathcal{B}| = \frac{vr}{k}$.

The most studied class of block designs are the $2 - (v, k, 1)$ designs, which are called Steiner systems. Among them we mention especially the $2 - (n^2 + n + 1, n + 1, 1)$ designs (the projective planes of order $n$), $n \geq 2$, the $2 - (n^2, n, 1)$ designs (the affine planes of order $n$), $n \geq 3$, and the $2 - (n^3 + 1, n + 1, 1)$ designs (the unitals of order $n$), $n \geq 2$.

By the above results, a $2 - (v, k, 1)$ design contains $b = \frac{v(v-1)}{k(k-1)}$ blocks, $r = \frac{v-1}{k-1}$ of them through a fixed point. Note that a $2 - (v, k, 1)$ design can only exist if $v \equiv 1 \pmod{k - 1}$ and $k(k - 1) \mid v(v - 1)$.

1.2 Erdős–Ko–Rado theorems

In 1961, the original Erdős–Ko–Rado theorem solved a problem in extremal combinatorics.

Theorem 1.3 ([11]). Let $\Omega$ be a set of size $n$ and $\mathcal{S}$ a family of subsets of size $k$ such that the elements of $\mathcal{S}$ are pairwise not disjoint. If $n \geq 2k$, then $|\mathcal{S}| \leq \binom{n-1}{k-1}$. If $n \geq N_0(k)$, then equality holds if and only if $\mathcal{S}$ is the set of all subsets of size $k$ containing a fixed element of $\Omega$.

In 1984, Wilson showed that the bound $n \geq 2k + 1$ is both sufficient and necessary for the above classification: the families $\mathcal{S}$ meeting the upper bound are sets of all subsets of size $k$ containing a fixed element of $\Omega$ [20].

Many generalisations of this problem have been investigated. The set $\Omega$ has often been replaced by a geometry, such as a vector space or a polar space, simultaneously replacing the subsets by subspaces. In general, an Erdős–Ko–Rado set is a set of subspaces of fixed dimension which are pairwise non-disjoint. It is called maximal if it can not be extended to a larger Erdős–Ko–Rado set. Hence, an Erdős–Ko–Rado set on a design is a set of pairwise intersecting blocks. The Erdős–Ko–Rado problem asks for the classification of the (largest) Erdős–Ko–Rado sets.

In [4, Section 2] and [9], surveys of Erdős–Ko–Rado theorems in geometrical settings can be found. Recent results on Erdős–Ko–Rado sets in projective and polar spaces can be found in e.g. [2, 3, 8, 13, 14, 16, 19].

The most important type of Erdős–Ko–Rado sets are the sets of all subsets (blocks, subspaces, ...) through a fixed point. They are called point-pencils. In a block design a point-pencil is a maximal Erdős–Ko–Rado set if $r > k$.

For general block designs, the following Erdős–Ko–Rado result was obtained by Rands.

Theorem 1.4 ([18]) Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a $t - (v, k, \lambda)$ block design and let $\mathcal{S}$ be a subset of $\mathcal{B}$ such that the blocks of $\mathcal{S}$ have pairwise at least $s$ points in common, $0 < s < t \leq k$.

- If $s < t - 1$ and $v \geq s + \binom{k}{s}(k - s + 1)(k - s)$, or
- if $s = t - 1$ and $v \geq s + \binom{k}{s}^2 (k - s)$,

then $|\mathcal{S}| \leq \lambda_s$ and equality is obtained if and only if $\mathcal{S}$ is the set of blocks through $s$ fixed points.

For an Erdős–Ko–Rado set in a $2 - (v, k, 1)$ design, this implies the following corollary.
Corollary 1.5 Let \( D \) be a \( 2 - (v, k, 1) \) block design and let \( S \) be an Erdős–Ko–Rado set of \( D \), \( k \geq 2 \). If \( v \geq 1 + k^2(k - 1) \), then \( |S| \leq r \) and \( |S| = r \) if and only if \( S \) is a point-pencil.

In the same article [18], it is claimed that the bound \( v \geq 1 + k^2(k - 1) \) can be improved to \( v > k^3 - 2k^2 + 2k \), but there is no proof of this statement. However, it is shown that the bound \( v > k^3 - 2k^2 + 2k \) is sharp. If \( v = k^3 - 2k^2 + 2k \) and \( k - 1 \) is a prime power, the \( 2 - (v, k, 1) \) design consisting of the points and lines of \( PG(3, k - 1) \) contains two different types of Erdős–Ko–Rado sets of size \( r = k^2 - k + 1 \): the set of all blocks through a fixed point and the set of blocks arising from the set of lines in a fixed plane.

In this article, we will prove the result about the bound \( v > k^3 - 2k^2 + 2k \) (see Theorem 5.1). It follows from two easy observations. The main part of this paper is devoted to the investigation of \( 2 - (v, k, 1) \) designs with \( v < k^3 - 2k^2 + 2k \) (see Theorem 5.5). It turns out that \( v = k^3 - 2k^2 + 2k \) is an isolated case. For \( 2 - (v, k, 1) \) designs with \( v \) smaller than \( k^3 - 2k^2 + 2k \) but not much, the largest Erdős–Ko–Rado sets are also point-pencils. The results are summarized in Theorem 4.4 and Corollary 5.6.

2 Some special Steiner systems

Remark 2.1 Let \( D \) be a \( 2 - (v, k, 1) \) design. For every point \( P \) in \( D \), there is a block not containing this point since \( v > k \). Each of the points on this block determines a different block through \( P \). Hence, \( r \geq k \). If \( r = k \), then \( D \) is a projective plane of order \( k - 1 \); if \( r = k + 1 \), then \( D \) is an affine plane of order \( k \). So, the projective and affine planes are the two ‘smallest’ \( 2 - (v, k, 1) \) designs.

We look at the projective and affine planes in detail.

Remark 2.2 In a projective plane, every two blocks have a point in common. Hence, in a projective plane there is only one maximal Erdős–Ko–Rado set of blocks, namely the set of all blocks. Recall that we mentioned in the introduction that a point-pencil is only maximal if \( r > k \).

Remark 2.3 In an affine plane of order \( n \), the set of blocks can be partitioned in \( n + 1 \) classes of \( n \) blocks, such that the blocks in the same class pairwise have no point in common. These are commonly called parallel classes. Two blocks of different classes always meet in a point. An Erdős–Ko–Rado set contains necessarily at most one block of each parallel class. A maximal Erdős–Ko–Rado set contains precisely one block of each parallel class. Consequently, every maximal Erdős–Ko–Rado set contains \( n + 1 \) blocks.

It should be noted that not all these maximal Erdős–Ko–Rado sets are isomorphic. Also note that the point-pencil can be described in this way.

Now we turn our attention to \( 2 - (v, k, 1) \) designs with a special property.

Definition 2.4 The \( O'Nan \) configuration in a design \( D \) is a set of four blocks, pairwise non-disjoint, such that no three contain a common point.

We will show that we can find a complete classification of the maximal Erdős–Ko–Rado sets on designs not containing an O’Nan configuration. Note that all projective planes and all affine planes of order at least 3 do contain O’Nan configurations.

We already know the point-pencil, a maximal Erdős–Ko–Rado set of size \( r \). We now give an example of a maximal Erdős–Ko–Rado set on a design without an O’Nan configuration.
Example 2.5 Let $D$ be a $2 - (v, k, 1)$ design without an O'Nan configuration. Let $P$ be a point and let $B$ be a block of $D$ such that $P \notin B$. Let $S$ be the union of $\{B\}$ and the set of all blocks through $P$ meeting $B$. It is obvious that all blocks of $S$ meet each other, hence that $S$ is an Erdős–Ko–Rado set. We call it the triangle. It contains $k + 1$ blocks. We prove that it is maximal.

Let $L$ be a block of $D$ not in $S$, meeting all blocks of $S$. The block $L$ cannot pass through $P$, hence meets all blocks of $S$ through $P$ in a different point. Since $L \notin B$, we know $k \geq 3$. Let $P'$ and $P''$ be two points on $B \setminus \{L \cap B\}$ and let $B'$ and $B''$ be the blocks of $S$ through $P$, respectively meeting $B$ in the points $P'$ and $P''$. Then the blocks $B$, $L$, $B'$ and $B''$ determine an O'Nan configuration, a contradiction.

Theorem 2.6 Let $D$ be a $2 - (v, k, 1)$ design without an O'Nan configuration and let $S$ be a maximal Erdős–Ko–Rado set on $D$. Then, $S$ is a point-pencil or a triangle.

Proof Assume that $S$ is not a point-pencil; then we can find three blocks in $S$, say $B_1$, $B_2$ and $B_3$, not through a common point. Denote $P_1 = B_2 \cap B_3$, $P_2 = B_3 \cap B_1$ and $P_3 = B_1 \cap B_2$. Any block $B \in S$ should have a non-empty intersection with as well $B_1$, $B_2$ as $B_3$. Since $D$ does not contain an O'Nan configuration, $B$ must pass through $P_1$, $P_2$ or $P_3$.

If the block $B_i' \in S$ passes through $P_1$, $B_i' \notin \{B_1, B_2, B_3\}$, and the block $B_j' \in S$ passes through $P_j$, $B_j' \notin \{B_1, B_2, B_3\}$, $1 \leq i \neq j \leq 3$, then the blocks $B_i$, $B_j$, $B_i'$ and $B_j'$ determine an O'Nan configuration, a contradiction. Hence, all blocks of $S \setminus \{B_1, B_2, B_3\}$ pass through the same point $P_i$, $1 \leq i \leq 3$. Since $S$ is maximal, it has to be a triangle based on the point $P_i$ and the block $B_i$. \hfill \square

Note that $r > k + 1$ for all $2 - (v, k, 1)$ designs without an O'Nan configuration, but the affine plane of order 2. Hence, the point-pencil is the largest Erdős–Ko–Rado set in these designs. Of course, the above result only makes sense if $2 - (v, k, 1)$ designs without an O'Nan configuration exist. We give an example.

Example 2.7 Let $\mathcal{H}(2, q^2)$ be a non-singular Hermitian variety in $\text{PG}(2, q^2)$, the Desarguesian projective plane of order $q^2$. Up to projective transformations it is defined by $X_0^{q+1} + X_1^{q+1} + X_2^{q+1} = 0$. The set of points on $\mathcal{H}(2, q^2)$ and the secant lines to $\mathcal{H}(2, q^2)$ in $\text{PG}(2, q^2)$, determine a unital. This unital is known as the classical unital or Hermitian unital.

Theorem 2.8 ([15]) A classical unital $\mathcal{U}$ does not contain an O'Nan configuration.

It is conjectured that the classical unitals are the only unitals not containing an O’Nan configuration, see [5, 17]. In [5] this conjecture is proven to be true for unitals of order 3. The unique unital of order 2 is also classical.

Corollary 2.9 On a classical unital there are only two types of maximal Erdős–Ko–Rado sets, the point-pencil and the triangle.

3 The counting arguments

In this section we will study maximal Erdős–Ko–Rado sets in $2 - (v, k, 1)$ designs that are not point-pencils.
Notation 3.1 Let $\mathcal{D}$ be a $2 - (v, k, 1)$ design and let $S$ be an Erdős–Ko–Rado set on $\mathcal{D}$. Denote the set of points of $\mathcal{D}$ covered by the blocks of $S$ by $\mathcal{P}'$.

We denote the number of points of $\mathcal{P}'$ that are contained in precisely $i$ blocks of $S$ by $k_i$. Furthermore we denote $k_S = \max\{i \mid k_i > 0\}$.

Lemma 3.2 Let $\mathcal{D}$ be a $2 - (v, k, 1)$ design and let $S$ be an Erdős–Ko–Rado set on $\mathcal{D}$. Then $|S| \leq k|S| - k + 1$. If $S$ is maximal and different from the point-pencil, then $k_S \leq k$.

Proof Fix a block $C \in S$. All blocks of $S$ have a nontrivial intersection with $C$, so

$$|S| \leq 1 + k(k_S - 1) = k|S| - k + 1.$$ 

Now we prove the second part of the lemma. For every point $P \in \mathcal{P}'$, we can find a block $B \in S$ not passing through $P$, since $S$ is maximal but not a point-pencil. Any block of $S$ through $P$ should meet $B$ and there is at most one block in $S$ through $P$ and a given point of $B$. Hence, there are at most $k$ blocks in $S$ passing through $P$. Consequently, $k_S \leq k$. □

Lemma 3.3 Choose $l \in \mathbb{N} \setminus \{0, 1\}$, and $a, b \in \mathbb{Z}$ with

$$a \geq \max \left\{-l(r - l - 1) + 1 - \frac{br}{l + 1}, -\frac{b(b - 1)}{(l + 1)l} - 2(b - 1)\right\},$$

$$a \leq \frac{rl - l^2 + l - 1}{l - 1} - \frac{b(2l^2 + 2l - r + b - 1)}{l^2 - 1}.$$

Let $n_1, \ldots, n_l \in \mathbb{N}$ be such that $\sum_{i=1}^l in_i = (a - 1)(l + 1) + br + l(l + 1)(r - l - 1)$ and $\sum_{i=2}^l i(i - 1)n_i = b(b - 1) + l(l + 1)(a + 2b - 2)$. Then $\sum_{i=2}^l (i - 1)n_i \leq \left(\frac{b}{2}\right) + (a + 2b - 2)\binom{l + 1}{2}$.

Proof Note that the inequalities $-l(r - l - 1) + 1 - \frac{br}{l + 1} \leq a$ and $-\frac{b(b - 1)}{(l + 1)l} - 2(b - 1) \leq a$ are present to ensure that both $a(l + 1) + br + l(l + 1)(r - l - 1) - l - 1$ and $b(b - 1) + l(l + 1)(a + 2b - 2)$ are nonnegative.

Using the first equality, we can express $n_1$ as a function of $l, a, b$ and $n_2, \ldots, n_l$. Note that

$$n_1 = (a - 1)(l + 1) + br + l(l + 1)(r - l - 1) - \sum_{i=2}^l in_i$$

$$\geq (a - 1)(l + 1) + br + l(l + 1)(r - l - 1) - \sum_{i=2}^l i(i - 1)n_i$$

$$= (a - 1)(l + 1) + br + l(l + 1)(r - l - 1) - b(b - 1) - l(l + 1)(a + 2b - 2)$$

$$= -a(l^2 - 1) - b(b - 1) - b(2l^2 + 2l - r) + (l + 1)(rl - l^2 + l - 1)$$

$$\geq 0,$$

by the assumption. Hence, for every choice of $n_2, \ldots, n_l$, we can find a value $n_1 \in \mathbb{N}$ such that the first equality holds. Now, we focus on the second equality. Assume that $n_j > 0$ for a value $j \geq 3$. Then define $n_j' = n_j - 1$, $n_2' = n_2 + \frac{j(j - 1)}{2}$ and $n_k' = n_k$ for $k \notin \{2, j\}$. It follows that

$$\sum_{i=2}^l i(i - 1)n_i' = \sum_{i=2}^l i(i - 1)n_i = b(b - 1) + l(l + 1)(a + 2b - 2).$$
However,  
\[
\sum_{i=2}^{l} (i-1)n_i' = \left( \sum_{i=2}^{l} (i-1)n_i \right) - (j-1) + \frac{j(j-1)}{2} > \sum_{i=2}^{l} (i-1)n_i, \\
\]

since \( j \geq 3 \). So, repeatedly applying the above construction, we find that \( \sum_{i=2}^{l} (i-1)n_i \) is maximal if \( n_i = 0 \) for all \( i \geq 3 \) and \( n_2 = \binom{l}{2} + (a + 2b - 2)(\binom{l+1}{2}) \). The lemma follows. □

**Lemma 3.4** Let \( \mathcal{D} \) be a \( 2 - (v, k, 1) \) design with replication number \( r = \frac{v-1}{k-1} \), \( k \geq 3 \), and let \( S \) be an Erdős–Ko–Rado set on \( \mathcal{D} \) such that \( |\mathcal{P}'| = k(k-1) + b \). Then  
\[
|S| \leq \max \left\{ k^2 - k + 1 - \frac{2(r-k)(k^2 - k + 1 - r)}{k(k-2)} + \frac{b(b-1)}{(k-1)(k-2)} \\
+ 2\frac{(b-1)(k^2 - k - r)}{(k-1)(k-2)}, k^2 - r - 1 + \frac{b(b-1 - r + 2k(k-1))}{k-2} \right\}.
\]

**Proof** Recall that \( \mathcal{B} \) is the set of blocks of \( \mathcal{D} \). We denote the subset of \( \mathcal{B} \) containing precisely \( i \) points of \( \mathcal{P}' \) by \( \mathcal{B}_i \) and we also denote \( m_i = |\mathcal{B}_i| \). Note that \( S \subseteq \mathcal{B}_k \). We define \( a := k^2 - k + 1 - |\mathcal{B}_k| \). Counting the tuples \((P, B)\) with \( P \in \mathcal{P}' \), \( B \in \mathcal{B} \) and \( P \) on \( B \), we find  
\[
\sum_{i=1}^{k} im_i = (k(k-1) + b)r.
\]

Now applying \( m_k = k^2 - k + 1 - a \), we find  
\[
m_1 = (k(k-1) + b)r - \sum_{i=2}^{k-1} im_i - k(k^2 - k + 1 - a) \\
= k(k-1)(r - k) + (a-1)k + br - \sum_{i=2}^{k-1} im_i.
\]

Counting the tuples \((P, P', B)\) with \( P, P' \in \mathcal{P}' \), \( B \in \mathcal{B} \), \( P \neq P' \) and both \( P \) and \( P' \) on \( B \), we find  
\[
\sum_{i=1}^{k} i(i-1)m_i = (k(k-1) + b)(k(k-1) + b - 1).
\]

Hence,  
\[
\sum_{i=2}^{k-1} i(i-1)m_i = (k(k-1) + b)(k(k-1) + b - 1) - k(k-1)(k^2 - k + 1 - a) \\
= b(b-1) + (a + 2b - 2)k(k-1).
\]

Now we consider the set \( T \) of triples \((P, P', B)\) with \( P, P' \in \mathcal{P} \setminus \mathcal{P}' \), \( B \in \mathcal{B}_1 \), \( P, P' \in B \) and \( P \neq P' \). On the one hand we know  
\[
|T| = m_1(k(k-1) + 2) = k(k-1)^2(k-2)(r - k) + (a-1)k(k-1)(k-2) \\
+ br(k(k-1) - (k-1)(k-2)\sum_{i=2}^{k-1} im_i.
\]
On the other hand, using $|\mathcal{P} \setminus \mathcal{P}'| = v - k(k - 1) - b = (r - k)(k - 1) - (b - 1)$, we can also find that

$$|T| \leq ((r - k)(k - 1) - (b - 1)) ((r - k)(k - 1) - b) - \sum_{i=2}^{k-1} (k - i)(k - i - 1)m_i.$$ 

Comparing this equality and inequality for $|T|$, we find

$$\sum_{i=2}^{k-1} (k(k - 1)(i - 1) - i(i - 1))m_i \geq k(k - 1)^2(k - 2)(r - k) + (a - 1)k(k - 1)(k - 2)$$

$$+ br(k - 1)(k - 2) - b(b - 1) - (r - k)^2(k - 1)^2 + (2b - 1)(r - k)(k - 1).$$

Using the formula for $\sum_{i=2}^{k-1} i(i - 1)m_i$, and dividing both sides by $k - 1$, it follows that

$$k \sum_{i=2}^{k-1} (i - 1)m_i \geq ak(k - 1) + bkr - k^2 + (r - k)(k^3 - 2k^2 - (r - 1)(k - 1)).$$

We distinguish between two cases. If $a > r - k + 1 + \frac{r-1}{k-2} - \frac{b(b-1-r+2k(k-1))}{k(k-2)}$, then $|S| \leq |B_k| \leq k^2 - r - \frac{r-1}{k-2} + \frac{b(b-1-r+2k(k-1))}{k(k-2)}$. If $a \leq r - k + 1 + \frac{r-1}{k-2} - \frac{b(b-1-r+2k(k-1))}{k(k-2)}$, we can apply Lemma 3.3 with $l = k - 1$. Note that the conditions $-l(r - l - 1) + \frac{br}{l+1} \leq a$ and $-\frac{b(b-1)}{l(l+1)} - 2(b - 1) \leq a$ are fulfilled since $\sum_{i=2}^{k-1} i(i - 1)m_i$ and $\sum_{i=1}^{k-1} im_i$ are nonnegative.

We find

$$k \binom{b}{2} + k(a + 2b - 2) \binom{k}{2} \geq ak(k - 1) + bkr - k^2 + (r - k)(k^3 - 2k^2 - (r - 1)(k - 1)),$$

hence

$$a \geq \frac{2(r - k)(k^2 - k + 1 - r)}{k(k - 2)} - \frac{2(b - 1)(k^2 - k - r)}{(k - 1)(k - 2)} - \frac{b(b - 1)}{(k - 1)(k - 2)}.$$

We find thus that

$$|S| \leq |B_k| \leq k^2 - k + 1 - \frac{2(r - k)(k^2 - k + 1 - r)}{k(k - 2)} + \frac{2(b - 1)(k^2 - k - r)}{(k - 1)(k - 2)}$$

$$+ \frac{b(b - 1)}{(k - 1)(k - 2)},$$

which finishes the proof. \qed

Using the substitution $R = (k - 1)^2 - r$, we can rewrite this lemma.

**Corollary 3.5** Let $\mathcal{D}$ be a $2 - (v, k, 1)$ design, $k \geq 3$, and denote $(k - 1)^2 - r = (k - 1)^2 - \frac{v - 1}{k-1}$ by $R$. Let $S$ be an Erdős–Ko–Rado set on $\mathcal{D}$ such that $|\mathcal{P}'| = k(k - 1) + b$. Then

$$|S| \leq \max \left\{ k^2 - k + 1 - \frac{2(k^2 - 3k + 1 - R)(k + R)}{k(k - 2)} + \frac{b(b - 1)}{(k - 1)(k - 2)}$$

$$+ \frac{2(b - 1)(k - 1 + R)}{(k - 1)(k - 2)}, k - 1 + R + \frac{R}{k - 2} + \frac{b(b + k^2 + R - 2)}{k(k - 2)} \right\}.$$
Lemma 3.6 Let \( \mathcal{D} \) be a \( 2 - (v, k, 1) \) design and let \( S \) be an Erdős–Ko–Rado set on \( \mathcal{D} \) with \( k_S = k \). Then \( |\mathcal{P}'| = k^2 - k + 1 \).

Proof Since \( k_S = k \), we can find a point \( P \in \mathcal{P}' \) lying on \( k \) blocks of \( S \). Denote these blocks by \( B_1, \ldots, B_k \) and let the set of points covered by these blocks by \( \mathcal{P}'' \). Any block of \( S \) not through \( P \) contains a point on each of the blocks \( B_i, i = 1, \ldots, k \). Since a block contains precisely \( k \) points, all points on such a block are contained in \( \mathcal{P}'' \). Hence, \( \mathcal{P}'' = \mathcal{P}' \) and

\[
|\mathcal{P}'| = \left| \bigcup_{i=1}^{k} B_i \right| = 1 + k(k - 1) = k^2 - k + 1.
\]

\( \square \)

Lemma 3.7 Let \( \mathcal{D} \) be a \( 2 - (v, k, 1) \) design and let \( S \) be an Erdős–Ko–Rado set on \( \mathcal{D} \) with \( k_S = k - 1 \). Write \( a' = (k - 1)^2 - |S| \). If \( a' < k - 1 \), then \( k(k - 1) \leq |\mathcal{P}'| \leq k(k - 1) + \frac{a'^2 - a'}{k-1-a'} \).

Proof First we will prove that there is a block in \( S \) containing at least two points that are on \( k - 1 \) blocks of \( S \). Assume there is no such block and choose a block \( C \). At most one point on \( C \) belongs to \( k - 1 \) blocks of \( S \). However, all blocks of \( S \) have a nontrivial intersection with \( C \), so

\[
|S| \leq 1 + (k - 2) + (k - 1)(k - 3) = (k - 1)(k - 2),
\]

hence \( a' \geq k - 1 \), which contradicts the assumption \( a' < k - 1 \).

Let \( B_1 \) be a block of \( S \) through the points \( Q_1 \) and \( Q_2 \), both on \( k - 1 \) blocks of \( S \), and let \( B_1, B_2, \ldots, B_{k-1} \) and \( B_1 = C_1, C_2, \ldots, C_{k-1} \) be the blocks of \( S \), respectively through \( Q_1 \) and \( Q_2 \). There are \( (k - 2)^2 \) points which lie on a block \( B_j \) and also on a block \( C_{j'}, 2 \leq j, j' \leq k - 1 \); there are \( k - 2 \) points which lie on a block \( B_i \) but not on a block \( C_{i'} \), and there are also \( k - 2 \) points which lie on a block \( C_i \), but not on a block \( B_{j'} \); the block \( B_1 = C_1 \) contains \( k \) points. Hence, \( |\mathcal{P}'| \geq (k - 2)^2 + 2(k - 2) + k = k(k - 1) \).

Now, recall the notation \( k_i \). By standard counting arguments we know that

\[
\sum_{i=1}^{k-1} ik_i = ((k - 1)^2 - a')k \quad \text{and} \quad \sum_{i=1}^{k-1} i(i - 1)k_i = ((k - 1)^2 - a')(k(k - 2) - a').
\]

Let \( j \in \mathbb{N} \setminus \{0\} \) be the smallest value such that \( k_j \neq 0 \) and let \( R \) be a point of \( \mathcal{P}' \) on \( j \) blocks of \( S \). Let \( B \in S \) be a block through \( R \). All blocks of \( S \) meet \( B \), hence

\[
|S| = (k - 1)^2 - a' \leq 1 + (k - 1)(k - 2) + (j - 1).
\]

It follows that \( j \geq k - 1 - a' \). Therefore, the following inequality holds:

\[
\sum_{i=1}^{k-1} (i - (k - a' - 1))(k - 1 - i)k_i \geq 0.
\]

So,

\[
0 \leq -\sum_{i=1}^{k-1} i(i - 1)k_i + (2k - a' - 3)\sum_{i=1}^{k-1} ik_i - (k - a' - 1)(k - 1)\sum_{i=1}^{k-1} k_i
\]

\[
= -((k - 1)^2 - a')(k(k - 2) - a')
\]
There is precisely one block through the points $3$ blocks.

has to be contained in $S$ the third point on this block by $R$ one of these blocks belongs to $S$.

$$\frac{(k - 1)^2 - a'}{k - a' - 1}$$

Consequently,

$$|P'| = \sum_{i=1}^{k-1} k_i \leq \frac{(k - 1)^2 - a'}{k - a' - 1} = k(k - 1) + \frac{a'^2 - a'}{k - a' - 1}$$

and the lemma follows. \qed

## 4 Classification results for $k = 3$

For $k = 2$, a $2 - (v, k, 1)$ design is a complete graph $K_v$ on $v$ vertices, the edges being the blocks. It can immediately be seen that there are precisely two different types of maximal Erdős–Ko–Rado sets on $K_v$, namely the point-pencil, which contains $v - 1$ blocks, and the triangle, a set $\{(p_1, p_2), \{p_1, p_3\}, \{p_2, p_3\}\}$ for three points $p_1, p_2, p_3 \in P$, which contains $3$ blocks.

So, the first nontrivial case is $k = 3$. A $2 - (v, 3, 1)$ design is called a Steiner triple system of size $v$. Steiner triple systems exist if and only if $v \equiv 1, 3 \pmod{6}$ and $v \geq 7$. Up to isomorphism, there is only one Steiner triple system for $v = 7$, namely the Fano plane, the projective plane of order $2$; there is only one Steiner triple system for $v = 9$, namely the affine plane of order $3$; and there are two Steiner triple systems for $v = 13$. For more details, we refer the interested reader to [7, Section II.1, Section II.2].

**Theorem 4.1** Let $D$ be a $2 - (v, 3, 1)$ design and let $S$ be a maximal Erdős–Ko–Rado set of $D$. Then $S$ belongs to one of five types. The maximal Erdős–Ko–Rado sets contain $\frac{v-1}{2}$, $4$, $5$, $6$ or $7$ blocks. Each type corresponds to a size and vice versa.

**Proof** If all blocks of $S$ pass through a common point, then $S$ is a point-pencil and it contains $\frac{v-1}{2}$ blocks. So, from now on we assume that there is no point on all blocks of $S$. Let $B_1, B_2, B_3 \in S$ be three blocks such that $B_1 \cap B_2 = \{P_3\}, B_1 \cap B_3 = \{P_2\}$ and $B_2 \cap B_3 = \{P_1\}$, with $P_1, P_2, P_3$ three different points. Let $Q_i$ be the third point on the block $B_i, i = 1, 2, 3$. There is precisely one block through the points $P_i$ and $Q_i$. We denote it by $B'_i$ and we denote the third point on this block by $R_i, i = 1, 2, 3$.

If the three points $Q_1, Q_2$ and $Q_3$ are contained in a common block $B'$, then this block has to be contained in $S$ by the maximality condition. The only other blocks that could be contained in $S$ are $B'_1, B'_2$ and $B'_3$. If all three points $R_1, R_2$ and $R_3$ are different, then only one of these blocks belongs to $S$. We find an Erdős–Ko–Rado set of size $4$ or $5$, depending on whether the block $B'$ exists. If two of the points $R_1, R_2$ and $R_3$ coincide, then we find an Erdős–Ko–Rado set of size $5$ or $6$. If $R_1 = R_2 = R_3$, then we find an Erdős–Ko–Rado set of size $6$ or $7$.

Note that the two constructions of Erdős–Ko–Rado sets of size $5$ give rise to isomorphic sets, so there is only one type of Erdős–Ko–Rado sets of size $5$. Analogously, there is also only one type of Erdős–Ko–Rado sets of size $6$.

**Remark 4.2** The five types of maximal Erdős–Ko–Rado sets in $2 - (v, 3, 1)$ designs are explicitly described in the above theorem. Apart from the point-pencil, these block sets can
be embedded in a Fano plane. However, they cannot be extended to a Fano plane by blocks of the design, due to the maximality condition. Note that the Erdős–Ko–Rado set of size 7 is a Fano plane that is embedded in the design.

Since the four types of maximal Erdős–Ko–Rado sets different from the point-pencil are determined by their size, we can denote them by $EKR_i$, $i = 4, \ldots, 7$, the index referring to their size. Note that each of the maximal Erdős–Ko–Rado sets different from the point-pencil, cover precisely 7 points of the design.

Remark 4.3 In a given $2-(v, 3, 1)$ design $D$, not necessarily all five types occur. For example, if $D$ is the Fano plane ($v = 7$), then there is only one maximal Erdős–Ko–Rado set, namely $EKR_7$, which is the set of all blocks in this case. If $D$ is not a projective plane, at least two types occur, one of which is the point-pencil.

We list the results for Erdős–Ko–Rado sets on Steiner triple systems of size $v$. For small values of $v$, the results are more detailed.

**Theorem 4.4** Let $D$ be a $2-(v, 3, 1)$ design.

- If $v = 7$, there is only one maximal Erdős–Ko–Rado set in $D$.
- If $v = 9$, there are two types of maximal Erdős–Ko–Rado sets in $D$, the point-pencil and $EKR_4$. Both contain 4 blocks.
- If $v = 13$, there are three types of maximal Erdős–Ko–Rado sets in $D$, the point-pencil, $EKR_4$ and $EKR_5$. The largest Erdős–Ko–Rado sets are the point-pencils.
- If $v = 15$, the largest Erdős–Ko–Rado sets contain 7 blocks. There are 23 nonisomorphic $2-(15, 3, 1)$ designs containing an $EKR_7$, and 57 nonisomorphic $2-(15, 3, 1)$ designs not containing an $EKR_7$. The former have two types of maximal Erdős–Ko–Rado sets of size 7; for the latter all Erdős–Ko–Rado sets of size 7 are point-pencils.
- If $v \geq 19$, the largest Erdős–Ko–Rado sets are point-pencils.

**Proof** The case $v = 7$ has been treated in Remark 4.3. If $v = 9$, then $D$ is an affine plane of order 3. One can see immediately that only two of the above types of maximal Erdős–Ko–Rado sets occur, the point-pencil and the smallest one of the others, the $EKR_4$. Both contain four blocks. Compare this result with Remark 2.3

If $v = 13$, there are two nonisomorphic $2-(v, 3, 1)$ designs. Their point sets can be denoted by $\{0, 1, \ldots, 9, a, b, c\}$. Using [7, Table II.1.27], we can write the block sets as in Table 1.

Table 1 Block sets

|     |     |     |     |     |     |     |     |     |     |     |     |     |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 1   | 1   | 1   | 1   | 1   |
| 1   | 1   | 1   | 1   | 1   | 2   | 2   | 2   | 2   | 2   | 3   | 3   | 3   |
| 4   | 4   | 4   | 4   | 4   | 4   | 4   | 4   | 4   | 4   | 4   | 4   | 4   |
| 5   | 5   | 5   | 5   | 5   | 5   | 5   | 5   | 5   | 5   | 5   | 5   | 5   |
| 6   | 6   | 6   | 6   | 6   | 6   | 6   | 6   | 6   | 6   | 6   | 6   | 6   |
| 7   | 7   | 7   | 7   | 7   | 7   | 7   | 7   | 7   | 7   | 7   | 7   | 7   |
| 8   | 8   | 8   | 8   | 8   | 8   | 8   | 8   | 8   | 8   | 8   | 8   | 8   |
| 9   | 9   | 9   | 9   | 9   | 9   | 9   | 9   | 9   | 9   | 9   | 9   | 9   |

We know that the point-pencil contains 6 blocks. By Theorem 3.4, applied for $k = 3$, $b = 1$ and $r = 6$, we know that any other maximal Erdős–Ko–Rado set contains at most 5 blocks. So, on both $2-(13, 3, 1)$ designs, at most three types of maximal Erdős–Ko–Rado sets occur.
Using the notation of Table 1, the two sets \([0, 2], [0, 3, 4], [1, 3, 5], [2, 3, 9], [2, 4, 5]\) and \([0, 1, 2], [0, 3, 4], [0, 9, a], [2, 3, 9]\) are maximal Erdős–Ko–Rado sets for both \(2 - (13, 3, 1)\) designs. Hence, there are precisely three types of maximal Erdős–Ko–Rado sets on \(2 - (13, 3, 1)\) designs.

There are 80 nonisomorphic \(2 - (15, 3, 1)\) designs, see [7, Table II.1.28] for an overview. The point-pencil contains 7 blocks in these designs. In [7, Table II.1.29] it is mentioned which of these 80 designs contains a Fano plane as subdesign; 23 of them do, and 57 do not. The statement follows.

If \(v \geq 19\), then \(r \geq 9\), hence the point-pencil contains more blocks than the Erdős–Ko–Rado sets of type \(EKR_i\), \(i = 4, \ldots, 7\).

Note that one of the 23 different \(2 - (15, 3, 1)\) designs having a Fano plane as subdesign, is the design consisting of the points and lines of PG(3, 2). Also note that the last part of Theorem 4.4 is a special case of Corollary 1.5.

5 Classification results for \(k \geq 4\)

In this section we present the main classification theorems for Erdős–Ko–Rado sets in \(2 - (v, k, 1)\) designs. In Theorem 5.1 we will provide a proof for the result claimed in [18] about \(2 - (v, k, 1)\) designs with large \(v\). Theorem 5.5 contains a classification theorem for \(2 - (v, k, 1)\) designs with \(v\) a little smaller. A survey result can be found in Corollary 5.6.

In this section we will use the parameter \(k_S\), introduced in Notation 3.1.

**Theorem 5.1** Let \(D\) be a \(2 - (v, k, 1)\) design and let \(S\) be an Erdős–Ko–Rado set on \(D\). If \(r \geq k^2 - k + 1\), then \(|S| \leq r\). If \(r = \frac{v - 1}{k - 1} > k^2 - k + 1\) and \(|S| = r\), then \(S\) is a point-pencil.

**Proof** Without loss of generality, we can assume that \(S\) is a maximal Erdős–Ko–Rado set. If \(S\) is a point-pencil, then \(|S| = r\). So, from now on, we can assume that \(S\) is not a point-pencil. By Lemma 3.2 we know that \(k_S \leq k\). However, by the same lemma we also know that \(|S| \leq k^2 - k + 1\), if \(k_S \leq k\).

Both statements in the theorem immediately follow.

As mentioned at the end of Sect. 1, there are \(2 - (v, k, 1)\) designs with \(r = k^2 - k + 1\), having a second type of Erdős–Ko–Rado sets of size \(r\).

Now, we look at Erdős–Ko–Rado sets in \(2 - (v, k, 1)\) designs with \(r \leq k^2 - k\). A classification result will be proven in Theorem 5.5. Before we prove some preparatory lemmas. In these lemmas we distinguish between the case \(4 \leq k \leq 13\) and the case \(k \geq 14\).

First, we have a look at the small cases, \(4 \leq k \leq 13\). We introduce the values \(R_k\) in Table 2.

**Lemma 5.2** Let \(D\) be a \(2 - (v, k, 1)\) design, \(4 \leq k \leq 13\), and denote \((k - 1)^2 - r = (k - 1)^2 - \frac{v - 1}{k - 1}\) by \(R\). Let \(S\) be an Erdős–Ko–Rado set on \(D\) with \(k_S = k - 1\). If \(0 \leq R \leq R_k\), then \(|S| < (k - 1)^2 - R\).

**Table 2** The values \(R_k\)

| \(k\) | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 |
|------|----|----|----|----|----|----|----|----|----|----|
| \(R_k\) | 1  | 2  | 3  | 4  | 4  | 5  | 6  | 7  | 8  | 9  |
Proof We denote \((k - 1)^2 - |S|\) by \(a'\), as in Lemma 3.7. By Lemma 3.2 we know that \(a' \geq 0\). If \(R < a'\), then \(|S| < (k - 1)^2 - R\). So, now we assume that \(a' \leq R\). Since \(R_k < k - 1\), also \(a' < k - 1\) and we know by Lemma 3.7 that \(k(k - 1) \leq |P'| \leq k(k - 1) + \frac{R(R - 1)}{k - 1 - R}\). Denoting \(|P'| - k(k - 1)\) by \(b\), it follows that \(0 \leq b \leq \frac{R(R - 1)}{k - 1 - R}\). By Lemma 3.5 we know that

\[
|S| \leq \max \left\{ k^2 - k + 1 - 2 \frac{(k^2 - 3k + 1 - R)(k + R)}{k(k - 2)} + \frac{b(b - 1)}{(k - 1)(k - 2)} + 2 \frac{(b - 1)(k - 1 + R)}{(k - 1)(k - 2)}, k - 1 + R + \frac{R}{k - 2} + \frac{b(b + k^2 + R - 2)}{k(k - 2)} \right\}.
\]

By hand or by using a computer algebra package, it can be checked that the above maximum is smaller than \((k - 1)^2 - R = r\) for all choices of \(k, R, b\) fulfilling \(4 \leq k \leq 13, 0 \leq R \leq R_k\) and \(0 \leq b \leq \frac{R(R - 1)}{k - 1 - R}\).

Extending the calculations in the above proof, we can see that the values \(R_k\) are optimal; enlarging one of these values leads to a contradiction.

Now, we look at the more general case \(k \geq 14\). We start with some inequalities which we will need in the proof of Lemma 5.4.

Lemma 5.3 Choose \(b, c, k \in \mathbb{N}\), with \(k \geq 14\), \(1 \leq c \leq \frac{4}{3}k\sqrt{k} - 2k - 2\sqrt{k}\) and \(0 \leq b \leq c\). Then

\[
k^3 - 7k^2 + 10k - 2bk - 2 - \sqrt{D(b, k)} < \frac{1 - c + \sqrt{(c - 1)^2 + 4c(k - 1)}}{2},
\]

with \(D(b, k) = (k^3 - 3k^2 - 2bk + 6k - 2)^2 - 8k(k - 1)(b - 1)(b - 2)\). Furthermore, for \(k \in \mathbb{N}\) with \(k \geq 14\),

\[
k^3 - 7k^2 + 10k - 2 - \sqrt{D(0, k)} < 0.
\]

Proof First, note that \(D(b, k) \geq 0\) for all \(0 \leq b \leq \frac{4}{3}k\sqrt{k} - 2k - 2\sqrt{k} =: C_k\), hence the above values exist.

The second part of the lemma is immediate, so we focus on the first part. Note that

\[
\frac{k^3 - 7k^2 + 10k - 2b(b + 1)k - 2 - \sqrt{D(b + 1, k)}}{4(k - 1)} - \frac{k^3 - 7k^2 + 10k - 2bk - 2 - \sqrt{D(b, k)}}{4(k - 1)} = \frac{\sqrt{D(b, k)} - \sqrt{D(b + 1, k)} - 2}{4(k - 1)}.
\]

Now,

\[
\frac{\sqrt{D(b, k)} - \sqrt{D(b + 1, k)} - 2}{4(k - 1)} \geq 0 \iff \sqrt{D(b, k)} - \sqrt{D(b + 1, k)} \geq 2k
\]

\[
\iff D(b, k) - D(b + 1, k) \geq 2k \left( \sqrt{D(b, k)} + \sqrt{D(b + 1, k)} \right)
\]

\[
\iff 2k^3 - 6k^2 + 4bk + 2k - 8b + 4 \geq \sqrt{D(b, k)} + \sqrt{D(b + 1, k)}.
\]
This final inequality is valid since \(\sqrt{D(b, k)} + \sqrt{D(b + 1, k)} \leq 2k^3 - 6k^2 - 4bk + 10k - 4\). These calculations show that
\[
\frac{k^3 - 7k^2 + 10k - 2(b + 1)k - 2 - \sqrt{D(b + 1, k)}}{4(k - 1)} \geq \frac{k^3 - 7k^2 + 10k - 2bk - 2 - \sqrt{D(b, k)}}{4(k - 1)}.
\]

Hence, it is sufficient to prove that
\[
\frac{k^3 - 7k^2 + 10k - 2ck - 2 - \sqrt{D(c, k)}}{4(k - 1)} < \frac{1 - c + \sqrt{(c - 1)^2 + 4c(k - 1)}}{2}.
\]

Since \(c \leq \frac{4}{3}k\sqrt{k} - 2k - 2\sqrt{k} < \frac{k^3 - 7k^2 + 8k}{2}\) for \(k \geq 14\), this is equivalent to
\[
\left(2(k - 1)\sqrt{(c - 1)^2 + 4c(k - 1)} + \sqrt{D(c, k)}\right)^2 > (k^3 - 7k^2 + 8k - 2c)^2
\]
\[
\iff \sqrt{(c - 1)^2 + 4c(k - 1)}\sqrt{D(c, k)} > -2k^4 + (9 + c)k^3 - (7c + 9)k^2 + (14c - 2)k + 2 - 6c.
\]

Considering the left-hand side of the inequality (1) as a function of \(c\), for a fixed value of \(k\), we can compute its second derivative. We find that this second derivative is negative on \([0, C_k]\), hence the function on the left-hand side is concave on \([0, C_k]\). Therefore, it dominates the function
\[
c \mapsto \sqrt{D(0, k)} + c\frac{\sqrt{(C_k - 1)^2 + 4C_k(k - 1)}\sqrt{D(C_k, k)} - \sqrt{D(0, k)}}{C_k}.
\]

The slope of this line is smaller than \(k^3 - 7k^2 + 14k - 6\). So, we only need to check the inequality for the largest considered value for \(c\), namely \(C_k\). It turns out that this inequality is valid if \(k \geq 14\). \(\square\)

In the final step of the argument we needed that \(k \geq 14\). This is why the cases \(4 \leq k \leq 13\) had to be treated separately. We now discuss \(2 - (v, k, 1)\) designs with \(k_S = k - 1\). These are the hardest case in the proof of Theorem 5.5.

**Lemma 5.4** Let \(D\) be a \(2 - (v, k, 1)\) design, \(k \geq 14\), and denote \((k - 1)^2 - r = (k - 1)^2 - \frac{v - 1}{k - 1}\) by \(r\). Let \(S\) be an Erdős–Ko–Rado set on \(D\) with \(k_S = k - 1\). If \(0 \leq R < \sqrt{k - 1}\) or \(\frac{1 - c + \sqrt{(c - 1)^2 + 4c(k - 1)}}{2} \leq R < \frac{-c + \sqrt{c^2 + 4(c + 1)(k - 1)}}{2}\) for a value \(c \in \mathbb{N}\), with \(1 \leq c \leq \frac{4}{3}k\sqrt{k} - 2k - 2\sqrt{k}\), then \(|S| < (k - 1)^2 - R\).

**Proof** Denote the interval \(\left[\frac{1 - c + \sqrt{(c - 1)^2 + 4c(k - 1)}}{2}, \frac{-c + \sqrt{c^2 + 4(c + 1)(k - 1)}}{2}\right]\) by \(I_c\), \(c \in \mathbb{N}\) and \(1 \leq c \leq \frac{4}{3}k\sqrt{k} - 2k - 2\sqrt{k} := C_k\), and the interval \([0, \sqrt{k - 1}]\) by \(I_0\). Recall the notation \(\mathcal{P}'\). We assume that \(R \in I_c\). From Lemma 3.7, it follows that \(|\mathcal{P}'| \leq k(k - 1) + c\). Hence, by Corollary 3.5,
\[
|S| \leq \max \left\{ \frac{k^2 - k + 1 + 2}{k - 2} \frac{(k^2 - 3k + 1 - R)(k + R)}{k(k - 2)} + \frac{b(b - 1)}{(k - 1)(k - 2)} + 2 \frac{(b - 1)(k - 1 + R)}{(k - 1)(k - 2)} k - 1 + R + \frac{R}{k - 2} k + \frac{b(b + k^2 + R - 2)}{k(k - 2)} \right\}.
\]
with \( b = k(k - 1) - \lvert P' \rvert \), hence \( 0 \leq b \leq c \). Since \( c \leq C_k \) and \( R < k - 2 \), the inequality
\[
k - 1 + R + \frac{R}{k - 2} + \frac{b(b + k^2 + R - 2)}{k(k - 2)} < (k - 1)^2 - R
\]
clearly holds in all cases. Now, we consider the inequality
\[
(k - 1)^2 - R > k^2 - k + 1 - 2\left(\frac{k^2 - 3k + 1 - R(k + R)}{k(k - 2)} + \frac{b(b - 1)}{(k - 1)(k - 2)} + 2\left(\frac{b - 1)(k - 1 + R)}{(k - 1)(k - 2)}\right)\right.
\]
\[
\left.\Leftrightarrow k + R - 2\left(\frac{k^2 - 3k + 1 - R(k + R)}{k(k - 2)} + \frac{b(b - 1)}{(k - 1)(k - 2)}\right) + 2\left(\frac{b - 1)(k - 1 + R)}{(k - 1)(k - 2)}\right)\right) .
\]
This inequality is valid if and only if
\[
\frac{k^3 - 7k^2 + 10k - 2bk - 2 - \sqrt{D(b, k)}}{4(k - 1)} < R < \frac{k^3 - 7k^2 + 10k - 2bk - 2 + \sqrt{D(b, k)}}{4(k - 1)},
\]
with \( D(b, k) = (k^3 - 3k^2 - 2bk + 6k - 2)^2 - 8k(k - 1)(b - 1)(b - 2) \). The double inequality in (2) should hold for all \( b \), with \( 0 \leq b \leq c \). Now,
\[
R < \frac{-c + \sqrt{c^2 + 4(c + 1)(k - 1)}}{2}
\]
and
\[
\frac{k^3 - 7k^2 + 10k - 2ck - 2}{4(k - 1)} \leq \frac{k^3 - 7k^2 + 10k - 2bk - 2 + \sqrt{D(b, k)}}{4(k - 1)},
\]
but the inequality
\[
\frac{-c + \sqrt{c^2 + 4(c + 1)(k - 1)}}{2} < \frac{k^3 - 7k^2 + 10k - 2ck - 2}{4(k - 1)}
\]
holds for all \( 0 \leq c \leq C_k \) since \( k \geq 14 \). Hence, the right inequality in (2) always holds. Using
\[
R \geq \frac{1 - c + \sqrt{(c - 1)^2 + 4c(k - 1)}}{2}
\]
and Lemma 5.3, also the left inequality in (2) follows. This finishes the proof. \( \square \)

**Theorem 5.5** Let \( D \) be a \( 2 - (v, k, 1) \) design, \( k \geq 4 \), and let \( S \) be an Erdős–Ko–Rado set on \( D \). If \( k^2 - k \geq r = \frac{v - 1}{k - 1} \geq k^2 - 3k + 3 \sqrt{k} + 2 \), then \( \lvert S \rvert \leq r \). If \( (r, k) \neq (8, 4) \), equality is obtained if and only if \( S \) is a point-pencil.

**Proof** Without loss of generality, we can assume that \( S \) is a maximal Erdős–Ko–Rado set. Recall the notation \( k_S \). If \( S \) is a point-pencil, then \( \lvert S \rvert = r \). So, from now on, we can assume that \( S \) is not a point-pencil. By Lemma 3.2 we know that \( k_S \leq k \). We distinguish between three cases.

- If \( k_S \equiv k - 1 \), then \( \lvert S \rvert \leq k^2 - 2k + 1 \) by Lemma 3.2. In this case, if \( k^2 - 2k + 1 < r \leq k^2 - k \), the theorem clearly holds, so we assume \( r \leq k^2 - 2k + 1 \). As before, we denote \( R = (k - 1)^2 - r \). First, assume that \( k \geq 14 \). In this case, \( 0 \leq R \leq k - \frac{3}{\sqrt{k}} - 1 \).

So, \( 0 \leq R < \sqrt{k - 1} \) or there is a value \( c \in \mathbb{N} \), with \( 1 \leq c \leq \frac{4}{3}k\sqrt{k} - 2k - 2\sqrt{k} \), such
that \( \frac{1-c+\sqrt{(c-1)^2+4(c-k)}}{2} \leq R < \frac{-c+\sqrt{c^2+4(c+1)(k-1)}}{2} \). Applying Lemma 5.4 we find that \( |S| < (k-1)^2 - R = r \).

Now assume that \( 4 \leq k \leq 13 \). In this case, \( 0 \leq R \leq R_k = \left\lfloor k - \frac{3}{4} \sqrt{k} - 1 \right\rfloor \). Applying Lemma 5.2, we find that \( |S| < (k-1)^2 - R = r \).

- If \( k_S = k \), then \( |P'| = k^2 - k + 1 \) by Lemma 3.6. So, we can apply Lemma 3.4 with \( b = 1 \). We find that

\[
|S| \leq \text{max} \left\{ k^2 - k + 1 - \frac{2(r-k)(k^2-k+1-r)}{k(k-2)}, k^2 - r - \frac{8}{k-2} + \frac{2k(k-1)-r}{k(k-2)} \right\}.
\]

The inequality \( k^2 - k + 1 - \frac{2(r-k)(k^2-k+1-r)}{k(k-2)} < r \) holds if and only if \( k^2 \geq 2r < k^2 - k + 1 \).

If \( k \geq 5 \), this condition is fulfilled since \( k^2 - k + 1 > k^2 - k \) and \( k^2 \leq k^2 - 3k + \frac{3}{4} \sqrt{k} + 2 \).

If \( k = 4 \) and \( R = 0 \), hence \( q = 9 \), then \( k^2 - k + 1 - \frac{2(r-k)(k^2-k+1-r)}{k(k-2)} = 8 < r \); if \( k = 4 \) and \( R = 1 \), hence \( q = 8 \), then \( k^2 - k + 1 - \frac{2(r-k)(k^2-k+1-r)}{k(k-2)} = 8 < r \).

Since \( k^2 - 3k + \frac{3}{4} \sqrt{k} + 2 > k^2 - k + \frac{3}{8} \) for all \( k \geq 4 \), the inequality \( k^2 - r - \frac{8}{k-2} + \frac{2k(k-1)-r}{k(k-2)} < r \) is fulfilled in all cases.

- If \( k_S \leq k - 2 \), then \( |S| \leq k^2 - 3k + 1 \) by Lemma 3.2. Clearly, \( k^2 - 3k + 1 < k^2 - 3k + \frac{3}{4} \sqrt{k} + 2 \leq r \).

Hence, for \( k \geq 5 \), in all three cases \( |S| < r \); for \( k = 4 \), in all three cases \( |S| \leq r \) and moreover \( |S| < r \) if \( r \neq 8 \). The theorem follows.

We now summarize the results of this section.

**Corollary 5.6** Let \( D \) be a \( 2 - (v, k, 1) \) design, \( k \geq 4 \), with \( r = \frac{v-1}{k-1} \geq k^2 - 3k + \frac{3}{4} \sqrt{k} + 2 \), and let \( S \) be an Erdős–Ko–Rado set on \( D \). Then \( |S| \leq r \). If \( r \neq k^2 - k + 1 \) and \( (r, k) \neq (8, 4) \), then \( |S| = r \) if and only if \( S \) is a point-pencil.

**Proof** This follows immediately from Theorems 5.1 and 5.5.

6 **Maximal Erdős–Ko–Rado sets in unitals**

The results from Lemmas 3.2, 3.4, 3.6 and 3.7 can also be used in a different way. For a fixed class of designs, with \( v \) (or equivalently \( r \)) a function of \( k \), an upper bound on the size of the largest maximal Erdős–Ko–Rado set different from a point-pencil can be computed. We show this for the unitals. Recall that a \( 2 - (q^3 + 1, q + 1, 1) \) design is a unital of order \( q \).

First we state Lemma 3.4 for a unital of order \( q \).

**Lemma 6.1** Let \( U \) be a unital of order \( q \) and let \( S \) be an Erdős–Ko–Rado set on \( U \) such that \( |P'| = q(q+1) + b \), whereby \( P' \) is the set of points covered by the elements of \( S \). Then

\[
|S| \leq \text{max} \left\{ q^2 - q + 1 + \frac{b(b-1)}{q(q-1)}, q + \frac{b(q+2)}{q^2-1} + \frac{b(b-1)}{q^2-1} \right\}.
\]

**Lemma 6.2** Let \( U \) be a unital of order \( q \) and let \( S \) be an Erdős–Ko–Rado set on \( U \) with \( k_S = q + 1 \). If \( q \geq 4 \), then \( |S| \leq q^2 - q + 1 \). If \( q = 3 \), then \( |S| \leq 8 \).

**Proof** By Lemma 3.6 we know that \( |P'| = q^2 + q + 1 \). We apply Lemma 6.1 and we find that \( |S| \leq \text{max} \left\{ q^2 - q + 1 + \frac{2}{q-1}, q + \frac{q(q+2)}{q^2-1} \right\} \). The lemma immediately follows.
Lemma 6.3 Let $U$ be a unital of order $q$ and let $S$ be an Erdős–Ko–Rado set on $U$ with $k_S = q$. If $q \geq 5$, then $|S| \leq q^2 - q + \sqrt{q^2 - \frac{2}{3} \sqrt{q}} + 1$. If $q = 3$, then $|S| \leq 7$; if $q = 4$, then $|S| \leq 13$.

Proof Denote $q^2 - |S|$ by $a'$. We can assume $a' < q$ since otherwise the lemma clearly holds. By Lemma 3.2, we know that $a' \geq 0$, and by Lemma 3.7 we know that $|P'| = q^2 + q + b$, with $0 \leq b \leq \frac{a'^2 - a'}{q-a'}$. We apply Lemma 6.1 and we find that

$$|S| \leq q^2 - q + 1 + 2 \frac{a'(a' - 1)}{(q-a')(q-1)} + \frac{a'(a' - 1)(a'^2 - q)}{q(q-1)(q-a')^2}$$

or

$$|S| \leq q + \frac{qa'(q+2)(a' - 1)}{(q^2 - 1)(q-a')} + \frac{a'(a' - 1)(a'^2 - q)}{(q-a')^2(q^2 - 1)}.$$  

Using $|S| = q^2 - a'$, the first inequality can be rewritten as

$$q(q-a' - 1)(q-a')^2(q-1) \leq a'(a' - 1)(2q^2 - 2qa' + a'^2 - q).$$

For $q = 3$, this implies $a' \geq 2$ and for $q = 4$, this implies $a' \geq 3$. For general $q$, it implies $a' \geq q - \sqrt{q^2 - \frac{2}{3} \sqrt{q}} - 1$.

Now we look at the second inequality. Using $|S| = q^2 - a'$, it can be rewritten as

$$(q^2 - q - a')(q-a')^2(q^2 - 1) \leq a'(a' - 1)(q^3 - (a' - 2)q^2 - (2a' + 1)q + a'^2).$$

Using that $0 \leq a' < q$, it follows that $a' = q - 1$.

Only one of the inequalities needs to hold, but $q - \sqrt{q^2 - \frac{2}{3} \sqrt{q}} - 1 \leq q - 1$. The lemma follows. □

Theorem 6.4 Let $U$ be a unital of order $q$ and let $S$ be a maximal Erdős–Ko–Rado set on $U$. If $q \geq 5$, then either $|S| = q^2$ and $S$ is a point-pencil, or else $|S| \leq q^2 - q + \sqrt{q^2 - \frac{2}{3} \sqrt{q}} + 1$. If $q = 4$, then either $|S| = 16 = q^2$ and $S$ is a point-pencil, or else $|S| \leq 13 = q^2 - q + 1$. If $q = 3$, then either $|S| = 9 = q^2$ and $S$ is a point-pencil, or else $|S| \leq 8$.

Proof If $S$ is a point-pencil, then it contains $q^2$ elements. From now on, we assume that $S$ is not a point-pencil. Recall the definition of $k_S$. By Lemma 3.2, $k_S \leq q + 1$. Moreover, if $k_S \leq q - 1$, then $|S| \leq q^2 - q - 1$.

First, we assume $q \geq 5$. If $k_S = q$, then $|S| \leq q^2 - q + \sqrt{q^2 - \frac{2}{3} \sqrt{q}} + 1$ by Lemma 6.3. If $k_S = q + 1$, then $|S| \leq q^2 - q + 1$ by Lemma 6.2.

The results for $q = 3, 4$ are obtained in the same way, using the results from Lemmas 6.2 and 6.3. □

Remark 6.5 Note that these results correspond with the result for classical unitals in Corollary 2.9 since the triangle contains only $q + 2$ blocks.

Note that the unitals are not covered by Corollary 1.5. However, they are covered by Theorem 5.6. So we already knew that the point-pencils are the largest Erdős–Ko–Rado sets. The above theorem thus gives a bound on the size of the second-largest maximal Erdős–Ko–Rado set.

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