MAXIMAL $L^2$-REGULARITY IN NONLINEAR GRADIENT SYSTEMS
AND PERTURBATIONS OF SUBLINEAR GROWTH

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ABSTRACT. The nonlinear semigroup generated by the subdifferential of a
convex lower semicontinuous function $\varphi$ has a smoothing effect, discovered
by H. Brézis, which implies maximal regularity for the evolution equation. We
use this and Schaefer’s fixed point theorem to solve the evolution equation
perturbed by a Nemytskii-operator of sublinear growth. For this, we need
that the sublevel sets of $\varphi$ are not only closed, but even compact. We apply
our results to the $p$-Laplacian and also to the Dirichlet-to-Neumann operator
with respect to $p$-harmonic functions.

1. INTRODUCTION

Let $H$ be a real Hilbert space, $\varphi : H \to (-\infty, +\infty]$ a proper, convex, lower
semicontinuous function and $A = \partial \varphi$ the subdifferential of $\varphi$ (see Section 2 for
more details). Then $A$ is a maximal monotone (in general, multi-valued) oper-
ator on $H$, for which the following remarkable well-posedness result holds.

Theorem 1.1 (Brézis [9]). Let $u_0 \in H$ such that $\varphi(u_0)$ is finite and $f \in L^2(0, T; H)$.
Then, there exists a unique $u \in H^1(0, T; H)$ such that

\begin{equation}
\begin{aligned}
\dot{u}(t) + Au(t) &\ni f(t) \quad \text{a.e. on } (0, T), \\
u(0) &= u_0.
\end{aligned}
\end{equation}

Our aim in this article is to study a perturbed version of (1.1). Let $\mathcal{H}$ denote
the space $L^2(0, T; H)$, $(T > 0)$, and $G : \mathcal{H} \to \mathcal{H}$ be a continuous mapping
satisfying the sublinear growth condition

\begin{equation}
\|Gv(t)\|_H \leq L \|v(t)\|_H + b(t) \quad \text{a.e. on } (0, T)
\end{equation}

and for all $v \in \mathcal{H}$.
for some constants $L, b \in L^2(0,T)$ satisfying $b(t) \geq 0$ for a.e. $t \in (0,T)$. Then we study the evolutionary problem

\[
\begin{align*}
\dot{u}(t) + Au(t) & \ni Gu(t) \quad \text{a.e. on } (0,T), \\
u(0) & = u_0.
\end{align*}
\]

(1.3)

For that, we will use a compactness argument in form of Schaefer’s fixed point theorem (see Theorem 2.1 in Section 2). Recall that lower semicontinuity of $\varphi$ is equivalent to saying that the sublevel sets $E_c := \{ u \in H \mid \varphi(u) \leq c \}$, $(c \in \mathbb{R})$, are closed. We will assume more, namely, compactness of sublevel sets $E_c$. In fact, we need this assumption only for the shifted function $\varphi_\omega$ given by $\varphi_\omega(u) = \varphi(u) + \frac{\omega}{2} \| u \|^2_H$ ($u \in H$), which is important for applications. Then our main results says the following.

**Theorem 1.2.** Let $\varphi : H \to (-\infty, +\infty]$ be a proper function such that for some $\omega \geq 0$, $\varphi_\omega$ is convex and has compact sublevel sets. Let $A = \partial \varphi$ and $G : \mathcal{H} \to \mathcal{H}$ be a continuous mapping satisfying (1.2). Then for every $u_0 \in H$ with $\varphi(u_0)$ finite, there exists $u \in H^1(0,T;H)$ solving (1.3).

We show in Example 3.6 that the solution is not unique in general. The proof of Theorem 1.2 is based on Brézis’ Theorem 1.1. However, we need it under the hypothesis that merely $\varphi_\omega$ is convex. We give a proof of this more general result (see Theorem 2.3) in the appendix of this paper. Theorem 1.2 remains also true if $u_0 \in \overline{D(\varphi)}$ where $D(\varphi) := \{ u \in H \mid \varphi(u) < +\infty \}$; however, the solution of (1.3) is merely in $H^1_{loc}((0,T);H)$ in that case.

As application, we consider $H = L^2(\Omega)$ and $G$ a Nemytskii operator. The operator $A$ may be the $p$-Laplacian ($1 \leq p < +\infty$) with possibly lower order terms and equipped with some boundary conditions (Dirichlet, Neumann, or Robin, see [13]) or a $p$-version of the Dirichlet-to-Neumann operator considered recently in [15] and via the abstract theory of $j$-elliptic functions (see [3, 4] and [12]).

**2. Preliminaries**

In this section, we define the precise setting used throughout this paper and explain our mains tools: Brezis’ result for semiconvex functions and Schaefer’s fixed point theorem.

We begin by recalling that a mapping $T$ defined on a Banach space $X$ is called *compact* if $T$ maps bounded sets in into relatively compact sets.

**Theorem 2.1 ([18], Schaefer’s fixed point theorem).** Let $X$ be a Banach space and $T : X \to X$ be continuous and compact. Assume that the “Schaefer set”

$$
S := \left\{ u \in X \mid \text{there exists } \lambda \in [0,1] \text{ s.t. } u = \lambda Tu \right\}
$$

is bounded in $X$. Then $T$ has a fixed point.

This result is a special case of Leray-Schauder’s fixed point theorem, but Schaefer gave a most elegant proof (cf [14]), which also is valid in locally compact spaces.
Given a function \( \varphi : H \to (-\infty, +\infty] \), we call the set \( D(\varphi) := \{ u \in H \mid \varphi(u) < +\infty \} \) the effective domain of \( \varphi \), and \( \varphi \) is said to be proper if \( D(\varphi) \) is non-empty. Further, we say that \( \varphi \) is lower semicontinuous if for every \( c \in \mathbb{R} \), the sublevel set
\[
E_c := \{ u \in D(\varphi) \mid \varphi(u) \leq c \}
\]
is closed in \( H \), and \( \varphi \) is semiconvex if there exists an \( \omega \in \mathbb{R} \) such that the shifted function \( \varphi_\omega : H \to (-\infty, +\infty] \) defined by
\[
\varphi_\omega(u) := \varphi(u) + \frac{\omega}{2} \| u \|_H^2, \quad (u \in H),
\]
is convex. Then, \( \varphi_\omega \) is convex for all \( \omega \geq 0 \), and \( \varphi_\omega \) is lower semicontinuous if and only if \( \varphi \) is lower semicontinuous.

Given a function \( \varphi : H \to (-\infty, +\infty] \), its subdifferential \( A = \partial \varphi \) is defined by
\[
\partial \varphi = \left\{ (u, h) \in H \times H \mid \liminf_{t \downarrow 0} \frac{\varphi(u + tv) - \varphi(u)}{t} \geq (h, v)_H \forall v \in D(\varphi) \right\},
\]
which, if \( \varphi_\omega \) is convex, reduces to
\[
\partial \varphi = \left\{ (u, h) \in H \times H \mid \varphi_\omega(u + v) - \varphi_\omega(u) \geq (h + \omega u, v)_H \forall v \in D(\varphi) \right\}.
\]
It is standard to identify a (possibly multi-valued) operator \( A \) on \( H \) with its graph and for every \( u \in H \), one sets \( Au := \{ v \in H \mid (u, v) \in A \} \) and calls \( D(A) := \{ u \in H \mid Au \neq \emptyset \} \) the domain of \( A \) and \( \text{Rg}(A) := \bigcup_{u \in D(A)} Au \) the range of \( A \).

Now, suppose \( \varphi : H \to (-\infty, +\infty] \) is proper, lower semicontinuous, and semiconvex; more precisely, let’s fix \( \omega \in \mathbb{R} \) such that \( \varphi_\omega \) is convex. Then, under those hypotheses on \( \varphi \), Brézis’ well-posedness result (Theorem 1.1) remains true.

**Remark 2.2 (Maximal \( L^2 \)-regularity).** If \( u_0 \in H \) such that \( \varphi(u_0) \) is finite, then Theorem 1.1 says that for every \( f \in L^2(0, T; H) \), the unique solution \( u \) of (1.1) has its time derivative \( \dot{u} \in L^2(0, T; H) \) and hence by the differential inclusion
\[
(2.1) \quad \dot{u}(t) + Au(t) \ni f(t) \quad \text{a.e. on } (0, T),
\]
also \( Au \in L^2(0, T; H) \). In other words, for \( f \in L^2(0, T; H) \), \( \dot{u} \) and \( Au \in L^2(0, T; H) \) admit the maximal possible regularity. For this reason, we call this property maximal \( L^2 \)-regularity, as it is customary for generators of holomorphic semigroups on Hilbert spaces (see [1] for a survey on this subject).

As before, we fix \( T > 0 \), denote by \( \mathcal{H} \) the space \( L^2(0, T; H) \), and write \( \| \cdot \|_\mathcal{H} \) for the norm \( \| \cdot \|_{L^2(0,T;H)} \).

Further, after possibly replacing \( \varphi \) by a translation, we may always assume without loss of generality that \( 0 \in D(\partial \varphi_\omega) \) and \( \varphi_\omega \) attains a minimum at \( 0 \) with \( \varphi_\omega(0) = 0 \) (for further details see [5, p159] or the appendix of this paper). By the convexity of \( \varphi_\omega \), this implies that \( (0, 0) \in \omega I_H + A \), that is,
\[
(2.2) \quad (h + \omega u, u)_H \geq 0 \quad \text{for all } (u, h) \in A.
\]

With this assumption in mind, we now state Brézis’ \( L^2 \)-maximal regularity theorem for semiconvex functions.
Theorem 2.3 (Brézis’ \(L^2\)-maximal regularity for semiconvex \(\varphi\)). Let \(u_0 \in D(\varphi)\) and \(f \in H\). Then, there exists a unique \(u \in H^1_{loc}((0,T);H) \cap C([0,T];H)\) satisfying
\[
\begin{aligned}
\dot{u}(t) + Au(t) &\ni f(t) \quad \text{a.e. on } (0,T), \\
u(0) &= u_0.
\end{aligned}
\]

Moreover, one has that \(\varphi \circ u \in W^{1,1}_{loc}((0,T]) \cap L^1(0,T)\),
\[
\|u(t)\|_H \leq \left(\|u_0\|^2_H + \int_0^T \|f(s)\|^2_H \, ds\right)^{\frac{1}{2}} e^{\frac{1+2\omega}{2} t} \text{ for every } t \in (0,T),
\]
\[
\int_0^T \varphi(u(s)) \, ds \leq \frac{1}{2} \|f\|^2_H + \frac{1+\omega}{2} \|u\|^2_H + \frac{1}{2} \|u_0\|^2_H,
\]
\[
t\varphi(u(t)) \leq \int_0^T \varphi(u(s)) \, ds + \frac{1}{2} \|\sqrt{\varphi}'\|^2_H \text{ for every } t \in (0,T),
\]
\[
\|\sqrt{\varphi}'\|_H^2 \leq 2 \int_0^T \varphi(u(t)) \, dt + \|\sqrt{\varphi}'\|^2_H.
\]

Finally, if \(u_0 \in D(\varphi)\), then \(u \in H^1(0,T;H)\).

To keep this paper self-contained, we provide a proof of this result in the appendix of this paper.

Definition 2.4. Given \(f \in H\) and \(u_0 \in H\), we call a function \(u : [0,T] \to H\) a (strong) solution of (2.3) (respectively, of (1.1)) if \(u \in H^1_{loc}((0,T);H) \cap C([0,T];H)\), \(u(0) = u_0\), and for a.e. \(t \in (0,T)\), one has that \(u(t) \in D(A)\) and \(f(t) - \dot{u}(t) \in Au(t)\).

For illustrating the theory developed in this paper, we consider the following standard example: the Dirichlet \(p\)-Laplacian perturbed by a lower order term.

Example 2.5. Let \(\Omega\) be an open subset of \(\mathbb{R}^d\), \((d \geq 1), H = L^2(\Omega)\), and for \(\frac{2}{d+2} \leq p < \infty\), let \(V = W^{1,p}_{\text{loc}}(\Omega)\) be the closure of \(C_c^1(\Omega)\) with respect to the norm \(\|u\|_V := \|\nabla u\|_{L^p(\Omega;\mathbb{R}^d)}\). Then, one has that \(V\) is continuously embedded into \(H\) (cf [11, Theorem 9.16]); we write for this \(V \hookrightarrow H\).

Further, let \(f = \beta + f_1\) be the sum of a maximal monotone graph \(\beta\) of \(\mathbb{R}\) satisfying \((0,0) \in \beta\) and a Lipschitz-Caratheodory function \(f_1 : \Omega \times \mathbb{R} \to \mathbb{R}\) satisfying \(f(x,0) = 0\); that is, for a.e. \(x \in \Omega\), \(f_1(x,\cdot)\) be Lipschitz continuous (with constant \(\omega > 0\)) uniformly for a.e. \(x \in \Omega\), and \(f_1(\cdot,u)\) is measurable on \(\Omega\) for every \(u \in \mathbb{R}\). Then, there is a proper, convex and lower semicontinuous function \(j : \mathbb{R} \to (-\infty,\infty]\) satisfying \(j(0) = 0\) and \(\partial j = \beta\) in \(\mathbb{R}\) (see [5, Example 1., p53]). We set
\[
F(u) = \phi(u) + \int_{\Omega} F_1(u(x)) \, dx \quad \text{for every } u \in H, \text{ where}
\]
\[
\phi(u) = \begin{cases}
\int_{\Omega} j(u(x)) \, dx & \text{if } j(u) \in L^1(\Omega), \\
+\infty & \text{if otherwise, and}
\end{cases}
\]
\[
F_1(u) = \int_0^{u(x)} f_1(\cdot,s) \, ds
\]
for every $u \in L^2(\Omega)$. Now, let $\varphi_1 : H \to (-\infty, +\infty]$ be given by
\[
\varphi_1(u) = \begin{cases} \frac{1}{p} \int_\Omega |u|^p \, dx + \int_\Omega F_1(u) \, dx & \text{if } u \in V, \\ +\infty & \text{if } u \in H \setminus V \end{cases}
\]
for every $u \in H$. Then the domain $D(\varphi_1)$ of $\varphi_1$ is $V$. The function $\varphi_1$ is lower semicontinuous on $H$, proper, $\varphi_{\lambda\omega}$ is convex, and for every $u \in V$, $\varphi_1$ is Gâteaux-differentiable with
\[
D_v \varphi(u) = \lim_{t \to 0^+} \frac{\varphi(u+t) - \varphi(u)}{t} = \int_\Omega |u|^{p-2} u \nabla u \nabla v + f_1(x,u) \, v \, dx
\]
for every $v \in V$. Since $V$ is dense in $H$, the operator $\partial \varphi_1$ is a single-valued operator on $H$ with domain
\[
D(\partial \varphi_1) = \left\{ u \in V \mid \exists h \in L^2(\Omega) D_v \varphi(u) = \int_\Omega h v \, dx \forall v \in V \right\},
\]
and
\[
\partial \varphi_1(u) = h = -\Delta_p u + f_1(x,u) \quad \text{in } D'(\Omega).
\]
The operator $\partial \varphi_1$ is the negative Dirichlet $p$-Laplacian $-\Delta_p^D$ on $\Omega$ with a Lipschitz continuous lower order term $f_1$. Next, we add the function $\phi$ given by (2.8) to the $\varphi_1$. For this, note that $\phi$ is proper (since for $u_0 \equiv 0$, $\phi(u_0) = 0$) with int$(D(\phi)) \neq \emptyset$, convex (since $j$ is convex), and lower semicontinuous on $H$. Thus, the function $\varphi : H \to (-\infty, +\infty]$ given by
\[
\varphi(u) = \varphi_1(u) + \phi(u) \quad \text{for every } u \in H,
\]
is convex, lower semicontinuous, and proper with domain $D(\varphi) = \{ u \in V \mid j(u) \in L^1(\Omega) \}$ and the operator $A = \partial \varphi$ is given by
\[
D(A) = \left\{ u \in D(\varphi) \mid \exists h \in L^2(\Omega) D_v \varphi(u) = \int_\Omega h v \, dx \forall v \in D(\varphi) \right\},
\]
\[
Au = h = -\Delta_p u + \beta(u) + f_1(x,u),
\]
and $A$ is single-valued provided $D(\varphi)$ is dense in $L^2(\Omega)$. Here, we note that
\[
\overline{D(A)} = D(\varphi) = \left\{ u \in H \mid \overline{j(u(x))} \in \overline{D(\beta)} \text{ for a.e. } x \in \Omega \right\}.
\]
Due to Theorem 2.3, for every $u_0 \in \overline{D(\varphi)}$ and $f \in \mathcal{H}$, there is a unique solution $u \in \dot{H}_{lo}^1((0,T];H) \cap C([0,T];H)$ of the parabolic boundary-value problem
\[
\begin{cases}
\partial_t u(t) - \Delta_p u(t) + \beta(u(t)) + f_1(\cdot, u(t)) \ni f(t) & \text{on } (0,T) \times \Omega, \\
u(t) = 0 & \text{on } (0,T) \times \partial\Omega, \\
u(0) = u_0 & \text{on } \Omega.
\end{cases}
\]
Here, we write $\partial_t u(t)$ instead of $\dot{u}(t)$ since we rewrote the abstract Cauchy problem (2.3) as an explicit parabolic partial differential equation.

3. MAIN RESULT

Throughout this section, let $\varphi : H \to (-\infty, +\infty]$ be a proper function. We assume that there is an $\omega \in \mathbb{R}$ such that $\varphi_\omega$ is convex and the sublevel set
\[
E_{\omega c} := \left\{ u \in D(\varphi) \mid \varphi_\omega(u) \leq c \right\}
\]
is compact in $H$ for every $c \in \mathbb{R}$. 
\qed
Remark 3.1. We emphasize that condition (3.1) does not imply that \( \varphi \) has compact sublevel sets. This becomes more clear if one considers as \( \varphi \) the function associated with the negative Neumann \( p \)-Laplacian \(-\Delta^p_u \) on a bounded, open subset \( \Omega \) of \( \mathbb{R}^d \) with a Lipschitz boundary \( \partial \Omega \). For \( \max\{1, \frac{2d}{d+2}\} < p < \infty \), \( (d \geq 1) \), let \( V = W^{1,p}(\Omega), H = L^2(\Omega), \mathcal{H} = L^2(0,T;L^2(\Omega)) = L^2((0,T) \times \Omega) \), and \( \varphi : H \to (-\infty, +\infty] \) be given by

\[
\varphi(u) := \begin{cases} 
\frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx & \text{if } u \in V, \\
+\infty & \text{if } u \in H \setminus V
\end{cases}
\]

for every \( u \in H \). Then, for every \( c > 0 \), the sublevel set \( E_{0,c} \) of \( \varphi \) contains the sequence \( (u_n)_{n \geq 0} \) of constant functions \( u_n \equiv n \), which does not admit any convergent subsequence in \( H \). On the other hand, for every \( \omega > 0 \) and \( c > 0 \), the sublevel set \( E_{\omega,c} \) of \( \varphi \) is a bounded set in \( V \) and by Rellich-Kondrachov’s compactness, one has that \( V \hookrightarrow H \) by a compact embedding. Thus, for every \( \omega > 0 \) and \( c > 0 \), the sublevel set \( E_{\omega,c} \) is compact in \( L^2(\Omega) \).

Let \( G : \mathcal{H} \to \mathcal{H} \) be a continuous function with \( \ell \)-\textit{sublinear growth} (sublevel sets are \( \xi \)-set for every \( \xi > 0 \)), \( G : \mathcal{H} \to \mathcal{H} \) be a continuous function satisfying

\[
\|Gv(t)\|_H \leq L\|v(t)\|_H + b(t) \quad \text{a.e. on } (0,T), \quad \text{for all } v \in \mathcal{H}.
\]

Here we let \( Gv(t) := (Gv(t)) \) to use less heavy notation. Then, our main result of this paper reads as follows.

Theorem 3.2. Let \( u_0 \in \overline{D(\varphi)} \) and \( f \in \mathcal{H} \). Then, there exists a solution \( u \in H^1_{\text{loc}}((0,T);H) \cap C([0,T];H) \) of

\[
\begin{cases} 
\dot{u}(t) + Au(t) \ni Gu(t) & \text{a.e. on } (0,T), \\
u(0) = u_0.
\end{cases}
\]

In particular, if \( u_0 \in D(\varphi) \), then problem (3.3) has a solution \( u \in H^1(0,T;H) \).

Note that \( Gu \in \mathcal{H} \). Thus, the inclusion in (3.3) means that \( Gu(t) - \dot{u}(t) \in Au(t) \) a.e. on \( (0,T) \). In particular, the following regularity estimates hold for strong solutions of (3.3).

Remark 3.3. For given \( u_0 \in \overline{D(\varphi)} \) and \( f \in \mathcal{H} \), the solution \( u \) of (3.3) satisfies

\[
\|u(t)\|_H \leq \left( \|u_0\|_{H}^2 + \|b\|_{L^2(0,T)}^2 \right)^{\frac{1}{2}} e^{\frac{\omega^2}{2} + \frac{2d}{d-2}|\xi| t} \quad \text{for all } t \in [0,T].
\]
Assume furthermore that \( g \) has sublinear growth, that is, there exist \( L \geq 0 \) and \( f \in L^2(0, T; L^2(\Omega)) \) such that

\[
|g(t, x, v)| \leq L|v| + f(t, x) \quad \text{for all } v \in \mathbb{R}, \text{ a.e. } (t, x) \in (0, T) \times \Omega.
\]

(3.5)

**Proposition 3.4.** Let \( \mathcal{H} = L^2(0, T; L^2(\Omega)) \). Then, the relation

\[
Gv \in \mathcal{H} \quad \text{for all } v \in \mathcal{H},
\]

defines a continuous operator \( G : \mathcal{H} \to \mathcal{H} \) of sublinear growth (1.2).

The proof is routine (cf [19, Proposition 26.7]) if one uses that \( f_n \to f \) in \( \mathcal{H} \) if and only if each subsequence of \( (f_n)_{n \geq 1} \) has a dominated subsequence converging to \( f \) a.e. (which is well known from the completeness proof of \( L^2 \)).

We illustrate our result by reconsidering Example 2.5 adding a perturbation of Nemytskii type.

**Example 3.5 (Example 2.5 revisited).** For \( \max\{1, \frac{2d}{d+2}\} < p < \infty \), let \( V = W^{1,p}_0(\Omega), H = L^2(\Omega), \mathcal{H} = L^2((0, T) \times \Omega) \) and let \( \varphi \) be given by (2.9). Then, there is an \( \omega > 0 \) such that \( q_\omega \) is convex and for every \( c > 0 \), the sublevel set \( E_{\omega,c} \) is compact in \( L^2(\Omega) \). Furthermore, let \( g : (0, T) \times \Omega \times \mathbb{R} \to \mathbb{R} \) be a Carathéodory function with sublinear growth and \( u_0 \in \mathcal{D}(\varphi) \). Then, there is at least one solution \( u \in H^{1, p}_{loc}((0, T]; H) \cap C([0, T]; H) \) of the parabolic boundary-value problem

\[
\begin{aligned}
\partial_t u(t, \cdot) - \Delta_p u(t, \cdot) + \beta(u(t, \cdot)) + f_1(\cdot, u(t, \cdot)) &\equiv g(t, \cdot, u(t, \cdot)) & \quad \text{on } (0, T) \times \Omega, \\
u(t, \cdot) &= 0 & \quad \text{on } (0, T) \times \partial \Omega, \\
u(0, \cdot) &= u_0 & \quad \text{on } \Omega.
\end{aligned}
\]

In general, the solutions in Example 3.5 are not unique. We give an example.

**Example 3.6 (Non-uniqueness).** Let \( \hat{g}(u) = \sqrt{|u|}, u \in \mathbb{R} \), and \( \Omega \) be an open and bounded subset of \( \mathbb{R}^d, d \geq 1 \), with a Lipschitz boundary \( \partial \Omega \). Then, there are \( L, b > 0 \) such that \( \hat{g} \) satisfies

\[
|\hat{g}(u)| \leq L|u| + b \quad \text{for every } u \in \mathbb{R}.
\]

Thus, for \( H = L^2(\Omega) \), one has that \( \mathcal{H} = L^2((0, T) \times \Omega) \) and the associated Nemytskii operator \( G : \mathcal{H} \to \mathcal{H} \) defined by (3.6) satisfies the sublinear growth condition (1.2). For \( \max\{1, \frac{2d}{d+2}\} < p < +\infty \), let \( \varphi : L^2(\Omega) \to (-\infty, +\infty) \) be the energy function (3.2) associated with the negative Neumann \( p \)-Laplacian \( -\Delta_p^N \) on \( \Omega \). Then, by Theorem 3.2, for every \( u_0 \in L^2(\Omega) \) and every \( T > 0 \), there is a solution \( u \in H^{1, p}_{loc}((0, T]; L^2(\Omega)) \cap C([0, T]; L^2(\Omega)) \) of

\[
\begin{aligned}
\partial_t u(t, \cdot) - \Delta_p^N u(t, \cdot) = \sqrt{|u(t, \cdot)|} & \quad \text{in } (0, T) \times \Omega, \\
|\nabla u(t, \cdot)|^{p-2} \nabla u(t, \cdot) &\equiv 0 & \quad \text{in } (0, T) \times \partial \Omega, \\
u(0) &= u_0 & \quad \text{on } \Omega.
\end{aligned}
\]

(3.7)

Here, \( |\nabla u|^{p-2} \nabla u \) denotes the (weak) co-normal derivative of \( u \) on \( \partial \Omega \) (cf [13]). Now, for the initial value \( u_0 \equiv 0 \) on \( \Omega \), the constant zero function \( u \equiv 0 \) is certainly a solution of (3.7). For constructing a non-trivial solution of (3.7) with
initial value \( u_0 \equiv 0 \), let \( w \in C^1[0,T] \) be a non-trivial solution of the following classical ordinary differential equation

\[
(3.8) \quad w' = \sqrt{|w|} \text{ on } (0,T), \quad w(0) = 0,
\]

For instance, one non-trivial solution is \( w(t) = t^2/4 \). Since for every constant \( c \in \mathbb{R}, -\Delta w(cI_{\Omega}) = 0 \), the function \( u(t) := w(t) \) is another non-trivial solution of (3.7) with initial value \( u_0 \equiv 0 \).

4. PROOF OF THE MAIN RESULT

For the proof of Theorem 3.2, we need some auxiliary results. The first concerns continuity and is standard (see Bénilan [8, (6.5), p87] or Barbu [5, (4.2), p128]).

**Lemma 4.1.** Let \( f_1, f_2 \in \mathcal{H}, u_1, u_2 \in H^1(0,T;\mathcal{H}) \) such that

\[
\begin{align*}
\varrho_1 + A\varrho_1 & \ni f_1 \quad \text{on } (0,T), \\
\varrho_2 + A\varrho_2 & \ni f_2 \quad \text{on } (0,T).
\end{align*}
\]

Then,

\[
(4.1) \quad \|u_1(t) - u_2(t)\|_H \leq e^{\omega t}\|u_1(0) - u_2(0)\|_H + \int_0^t e^{\omega(t-s)}\|f_1(s) - f_2(s)\|_H \, ds
\]

for every \( t \in [0,T] \).

Next, we establish the compactness of the solution operator \( P \) associated with evolution problem (2.3). Note, for convenience, we write here \( \mathcal{H} \) to denote \( \mathcal{H}, (T > 0) \), and recall that the closure \( \overline{D(q)} \) in \( H \) of the effective domain of a semiconvex function \( q \) is a convex subset of \( H \).

**Lemma 4.2.** Let \( P : \overline{D(q)} \times \mathcal{H} \to \mathcal{H} \) be the mapping defined by

\[
P(u_0, f) = "\text{solution } u \text{ of (2.3)}" \quad \text{for every } u_0 \in D(q) \text{ and } f \in \mathcal{H}.
\]

Then, \( P \) is continuous and compact.

**Proof.** (a) By Lemma 4.1, the map \( P \) is continuous from \( \overline{D(q)} \times \mathcal{H} \) to \( \mathcal{H} \).

(b) We show that \( P \) is compact. Let \((u_n^{(0)})_{n \geq 1} \subseteq \overline{D(q)} \) and \((f_n)_{n \geq 1} \subseteq \mathcal{H} \) such that \( \|u_n^{(0)}\|_H + \|f_n\|_H \leq c \) and \( u_n = P(u_n^{(0)}, f_n) \) for every \( n \geq 1 \). Then, by (2.4), (2.5) and by (2.7), for every \( \delta \in (0, T) \), there is a \( \gamma > 0 \) such that

\[
\sup_{n \geq 1} \|u_n\|_{H^1(\delta,T;\mathcal{H})} \leq \gamma.
\]

Since \( H^1(\delta,T;\mathcal{H}) \hookrightarrow C^1([\delta,T);\mathcal{H}) \) for some \( \gamma \in (0,1) \), it follows that the sequence \((u_n)_{n \geq 1}\) is equicontinuous on \([\delta,T]\) for each \( 0 < \delta < T \). Choose a countable dense subset \( D := \{t_m | m \in \mathbb{N}\} \) of \((0,T] \). Let \( m \geq 1 \). Then by (2.6),

\[
\sup_{n \geq 1} \varphi(u_n(t_m)) \quad \text{is finite}
\]

and since by (2.4), \((u_n(t_m))_{n \geq 1}\) is bounded in \( H \), there is a \( c' > 0 \) such that \((u_n(t_m))_{n \geq 1}\) is in the sublevel set \( E_{\omega,c'} \). Thus and by the assumption (3.1),
$(u_n(t_m))_{n \geq 1}$ has a convergent subsequence in $H$. By Cantor’s diagonalization argument, we find a subsequence $(u_{n_k})_{k \geq 1}$ of $(u_n)_{n \geq 1}$ such that

$$\lim_{k \to +\infty} u_{n_k}(t_m) \text{ exists in } H \text{ for all } m \in \mathbb{N}.$$  

It follows from the equicontinuity of $(u_{n_k})_{k \geq 1}$ that $u_{n_k}$ converges in $C([\delta, T]; H)$ for all $\delta \in (0, T]$. In particular, $(u_{n_k}(t))_{k \geq 1}$ converges in $H$ for every $t \in (0, T)$ and by (2.4), $(u_{n_k})_{k \geq 1}$ is uniformly bounded in $L^\infty(0, T; H)$. Thus, it follows from Lebesgue’s dominated convergence theorem that $u_{n_k} = P(u_{n_k}^{(0)}, f_{n_k})$ converges in $H$. □

**Remark 4.3.** In the previous proof, we have actually shown that $P$ is compact from $D(\varphi) \times H$ into the Fréchet space $C([0, T]; H)$.

With these preliminaries, we can now give the proof of our main result. Here, we got inspired from the linear case (cf [2]).

**Proof of Theorem 3.2.** First, let $u_0 \in \overline{D(\varphi)}$.

Let $v \in H$. Then $Gv \in H$ and so, by Brézis’ maximal $L^2$-regularity result (Theorem 2.3), there is a unique solution $u \in H^1_{\text{loc}}((0, T]; H) \cap C([0, T]; H)$ of the evolution problem

$$\begin{cases}
\dot{u}(t) + Au(t) \ni Gv(t) & \text{a.e. on } (0, T), \\
u(0) = u_0.
\end{cases}$$

Let $Tv := P(u_0, Gv)$. Then by the continuity of $G$ and since $P(u_0, \cdot) : H \to H$ is continuous and compact (Lemma 4.2), the mapping $T : H \to H$ is continuous and compact.

a) We consider the Schaefer set

$$S := \left\{ u \in H \left| \text{there exists } \lambda \in [0, 1] \text{ s.t. } u = \lambda Tu \right. \right\}.$$  

We show that $S$ is bounded in $H$. Let $u \in S$. We may assume that $\lambda \in (0, 1]$, otherwise, $u \equiv 0$. Then, one has that $u \in H^1_{\text{loc}}((0, T]; H) \cap C([0, T]; H)$ and

$$\begin{cases}
\frac{\dot{u}}{\lambda} + A \left( \frac{u}{\lambda} \right) \ni Gu & \text{on } (0, T), \\
u(0) = u_0.
\end{cases}$$

It follows from (2.2) that

$$\left( -\frac{\dot{u}}{\lambda}(t) + Gu(t) + \omega \frac{u}{\lambda}(t), \frac{u}{\lambda} \right)_H \geq 0 \quad \text{for a.e. } t \in (0, T).$$
Thus and by (1.2),
\[
\frac{d}{dt} \frac{1}{2} \|u(t)\|_H^2 = (\dot{u}(t), u(t))_H
\]
\[= (\dot{u}(t) - \lambda G u(t) - \omega \lambda u(t), u(t))_H
\]
\[+ (\lambda G u(t) + \omega \lambda u(t), u(t))_H
\]
\[\leq (\lambda G u(t) + \omega \lambda u(t), u(t))_H
\]
\[\leq \lambda \left( \|G u(t)\|_H \|u(t)\|_H + \omega \|u(t)\|^2_H \right)
\]
\[\leq \lambda \left( L \|u(t)\|^2_H + b(t) \|u(t)\|_H + \omega \|u(t)\|^2_H \right)
\]
\[\leq (2L + 1 + 2\omega) \frac{1}{2} \|u(t)\|^2_H + \frac{1}{2} b^2(t)
\]
for a.e. \( t \in (0, T) \). It follows from Gronwall’s lemma that (3.4) holds for every \( t \in [0, T] \). Thus, \( S \) is bounded in \( \mathcal{H} \). Now, Schaefer’s fixed point theorem implies that there exists \( u \in \mathcal{H} \) such that \( u = T u \); that is, \( u \in H^1_{\text{loc}}((0, T]; H) \cap C([0, T]; H) \) is a solution of the evolution problem (3.3).

b) Let \( u_0 \in D(\varphi) \). Then, by the first part of this proof, there is a solution \( u \in H^1_{\text{loc}}((0, T]; H) \cap C([0, T]; H) \) of the evolution problem (3.3). However, by Brézis’ maximal regularity result applied to \( f = Gu \in \mathcal{H} \), it follows that \( u \in H^1(0, T; H) \). This completes the proof of this theorem.

5. APPLICATION TO j-Elliptic FUNCTIONS

In the previous examples (cf Examples 2.5 and Example 3.6), \( V \) is a Banach space injected in \( H \). Recently, in [12], Chill, Hauer and Kennedy extended results of [3], [4] by Arendt and Ter Elst to a nonlinear framework of \( j \)-elliptic functions \( \varphi : V \to (-\infty, +\infty] \) generating a quasi maximal monotone operator
\[\partial_j \varphi \text{ on } H,\]
where \( j : V \to H \) is just a linear operator which is not necessarily injective. This enabled the authors of [12] to show that several coupled parabolic-elliptic systems can be realized as a gradient system in a Hilbert space \( H \) and to extend the linear variational theory of the Dirichlet-to-Neumann operator to the nonlinear \( p \)-Laplace operator (see also [6, 7, 16] for further applications and extensions of this theory).

The aim of this section is to illustrate that the main Theorem 3.2 of Section 3 can also be applied to the framework of \( j \)-elliptic functions.

Let us briefly recall some basic notions and facts about \( j \)-elliptic functions from [12]. Let \( V \) be a real locally convex topological vector space and \( j : V \to H \) be a linear operator which is merely weak-to-weak continuous (and, in general, not injective). Given a function \( \varphi : V \to (-\infty, +\infty] \), then the \( j \)-subdifferential is the operator
\[\partial_j \varphi := \left\{ (u, f) \in H \times H \middle| \exists \hat{u} \in D(\varphi) \text{ s.t. } j(\hat{u}) = u \text{ and for every } \hat{\vartheta} \in V,ight.
\[\left. \liminf_{t \searrow 0} \frac{\varphi(\hat{u} + t\hat{\vartheta}) - \varphi(\hat{u})}{t} \geq (f, j(\hat{\vartheta}))_H \right\}.
\]
The function \( \varphi \) is called \( j \)-semiconvex if there exists \( \omega \in \mathbb{R} \) such that the “shifted” function \( \varphi_{\omega} : V \to (-\infty, +\infty] \) given by
\[\varphi_{\omega}(\hat{u}) + \frac{\omega}{2} \|j(\hat{u})\|^2_H \quad \text{for every } \hat{u} \in V,
\]
is convex. If $V = H$ and $j = l_H$, then $j$-semiconvex functions $\varphi$ are the semiconvex ones (see Section 1). The function $\varphi$ is called $j$-elliptic if there exists $\omega \geq 0$ such that $\varphi_\omega$ is convex and for every $c \in \mathbb{R}$, the sublevel sets \{ \hat{u} \in V \mid \varphi_\omega(u) \leq c \} are relatively weakly compact. Finally, we say that the function $\varphi$ is lower semicontinuous if the sublevel sets \{ $\varphi \leq c$ \} are closed in the topology of $V$ for every $c \in \mathbb{R}$. It was highlighted in [12, Lemma 2.2] that (a) If $\varphi$ is $j$-semiconvex, then there is an $\omega \in \mathbb{R}$ such that
\[
\partial_j \varphi = \left\{ (u, f) \in H \times H \mid \exists \hat{u} \in D(\varphi) \text{ s.t. } j(\hat{u}) = u \text{ and for every } \hat{v} \in V \right.
qquad \left. \varphi_\omega(\hat{u} + \hat{v}) - \varphi_\omega(\hat{u}) \geq (f + \omega j(\hat{u}), j(\hat{v}))_H \right\}.
\]
(b) If $\varphi$ is Gâteaux differentiable with directional derivative $D_\varphi \varphi(\hat{v})$, then
\[
\partial_j \varphi = \left\{ (u, f) \in H \times H \mid \exists \hat{u} \in D(\varphi) \text{ s.t. } j(\hat{u}) = u \text{ and for every } \hat{v} \in V \right.
qquad \left. D_\varphi \varphi(\hat{u}) = (f, j(\hat{v}))_H \right\}.
\]

The main result in [12] is that the $j$-subdifferential $\partial_j \varphi$ of a $j$-elliptic function $\varphi$ is already a classical subdifferential. More precisely, the following holds.

**Theorem 5.1** ([12, Corollary 2.7]). Let $\varphi : V \to (-\infty, +\infty]$ be proper, lower semicontinuous, and $j$-elliptic. Then there is a proper, lower semicontinuous, semiconvex function $\varphi^H : H \to (-\infty, +\infty]$ such that $\partial_j \varphi = \partial \varphi^H$. The function $\varphi^H$ is unique up to an additive constant.

Thus the operator $A = \partial_j \varphi$ has the properties of maximal regularity we used before. The following result gives a description of $\varphi^H$ in the convex case and will be important for our intentions in this paper.

**Theorem 5.2** ([12, Theorem 2.9]). Assume that $\varphi : V \to (-\infty, +\infty]$ is convex, proper, lower semicontinuous and $j$-elliptic, and let $\varphi^H : H \to (-\infty, +\infty]$ be the function from Corollary 5.1. Then, there is a constant $c \in \mathbb{R}$ such that
\[
\varphi^H(u) = c + \inf_{\hat{u} \in j^{-1}(u)} \varphi(\hat{u}) \quad \text{for every } u \in H
\]
with effective domain $D(\varphi^H) = j(D(\varphi))$.

For our perturbation result, we need the compactness of the sublevel sets of $\varphi^H$. With the help of Theorem 5.2 we can establish a criterion in terms of the given $\varphi$ for this property.

**Lemma 5.3.** Let $\varphi : V \to (-\infty, +\infty]$ be proper, lower semicontinuous $j$-semiconvex, and $j$-elliptic. Assume that
\[
(5.1) \begin{cases}
 j : V \to H \text{ maps weakly relatively compact sets of } V \\
 \text{into relatively norm-compact sets of } H,
\end{cases}
\]
then there is an $\omega \geq 0$ such that for every $c \in \mathbb{R}$, the sublevel set
\[
E_{\omega c} = \left\{ u \in H \mid \varphi_\omega^H(u) \leq c \right\}
\]
is compact in $H$.

**Remark 5.4.** If $V$ is a normed space, then by the Eberlein-Šmulian Theorem hypothesis (5.1) is equivalent to $j$ maps weakly convergent sequences in $V$ to norm convergent sequences in $H$. This in turn is equivalent to $j$ being compact if $V$ is reflexive.
Proof of Lemma 5.3. By hypothesis, there is an \( \omega \geq 0 \) such that \( \varphi_\omega \) is convex, lower semicontinuous, and for every \( c \in \mathbb{R} \), the sublevel sets \( \{ \hat{u} \in V \mid \varphi_\omega(u) \leq c \} \) are weakly relatively compact and closed. By Corollary 5.1, there is a lower semicontinuous, proper function \( \varphi^H : H \to (-\infty, +\infty] \) such that \( \varphi^H_\omega \) is convex and \( \partial \varphi^H_\omega = \partial_j \varphi_\omega \). Applying Theorem 5.2 to \( \varphi_\omega \) and \( \varphi^H_\omega \), we have that
\[
\varphi^H_\omega(u) = d + \inf_{\hat{u} \in j^{-1}(\{u\})} \varphi_\omega(\hat{u}) \quad \text{for every } u \in H
\]
and some constant \( d \in \mathbb{R} \). For \( c \in \mathbb{R} \), let \( (u_n)_{n \geq 1} \) be an arbitrary sequence in \( E_{\text{asc}} \). By (5.2), for every \( n \in \mathbb{N} \), there is a \( \hat{u}_n \in j^{-1}(\{u_n\}) \) such that
\[
d + \varphi_\omega(\hat{u}_n) \leq c + 1.
\]
By hypothesis, all sublevel sets of \( \varphi_\omega \) are weakly relatively compact in \( V \). Thus, by our hypothesis, the image under \( j \) is relatively compact in \( H \). Consequently, there is a subsequence \( (u_{n_l})_{l \geq 1} \) of \( (u_n)_{n \geq 1} \) and a \( u \in H \) such that \( u_{n_l} = j(\hat{u}_{n_l}) \to u \) in \( H \) as \( l \to +\infty \). Since \( \varphi^H_\omega(u_{n_l}) \leq c \) and since \( \varphi^H \) is lower semicontinuous, it follows that \( \varphi^H(u) \leq c \). This shows that \( E_{\text{asc}} \) is compact. \( \square \)

Now, applying Lemma 5.3 to Theorem 3.2, we can state the following existence theorem.

**Theorem 5.5.** Let \( \varphi : V \to (-\infty, +\infty] \) be proper, lower semicontinuous \( j \)-semiconvex, and \( j \)-elliptic. Assume that the mapping \( j \) satisfies (5.1) and let \( G : \mathcal{H} \to \mathcal{H} \) be a continuous mapping of sublinear growth (1.2). Then, for \( A = \partial_j \varphi \) the nonlinear evolution problem (3.3) admits for every \( u_0 \in j(D(\varphi)) \) and \( f \in \mathcal{H} \) at least one solution \( u \in H^1_{\text{loc}}((0, T]; H) \cap C([0, T]; H) \). In particular, one has that \( \varphi \circ u \) belongs to \( W^{1,1}_{\text{loc}}((0, T]) \cap L^1(0, T) \) and inequality (3.4) holds. If \( u_0 \in D(\varphi) \), then problem (3.3) has a solution \( u \in H^1(0, T; H) \).

We complete this section by considering the following evolution problem involving the Dirichlet-to-Neumann operator associated with the \( p \)-Laplacian (cf [15, 12]).

**Example 5.6.** Let \( \Omega \) be a bounded domain with a Lipschitz continuous boundary \( \partial \Omega \). Then, for \( \frac{2d}{d+1} < p < +\infty \), the trace operator \( \text{Tr} : W^{1, p}(\Omega) \to L^2(\partial \Omega) \) is a completely continuous operator (cf [17, Théorème 6.2] for the case \( p < d \), the other cases \( p = d \) and \( p > d \) can be deduced from [17, Conséquence 6.2 & 6.3]). Now, we take
\[
V = W^{1, p}(\Omega), H = L^2(\partial \Omega), \text{ and } j = \text{Tr}.
\]
Then, \( j \) is a linear bounded mapping satisfying hypothesis (5.1). In fact, \( j \) is a prototype of a non-injective mapping. Furthermore, let \( \varphi : V \to \mathbb{R} \) be the function given by
\[
\varphi(\hat{u}) = \frac{1}{p} \int_{\Omega} |\nabla \hat{u}|^p \, dx \quad \text{for every } \hat{u} \in V.
\]
Then, \( \varphi \) is continuously differentiable on \( V \) and convex. Thus, the \( \text{Tr} \)-subdifferential operator \( \partial_{\text{Tr}} \varphi \) is given by
\[
\partial_{\text{Tr}} \varphi = \left\{(u, f) \in H \times H \mid \exists \hat{u} \in V \text{ s.t. } \text{Tr}(\hat{u}) = u \text{ and for every } \hat{\vartheta} \in V \right\}
\int_{\Omega} |\nabla \hat{u}|^{p-2} \nabla \hat{u} \nabla \hat{\vartheta} \, dx = (f, j(\hat{\vartheta}))_H \right\}.
\]
Moreover, by inequality [15, (20)], for any \( \omega > 0 \), the shifted function \( \varphi_{\omega} \) has bounded level sets in \( V \). Since \( V \) is reflexive, every level set of \( \varphi_{\omega} \) is weakly compact in \( V \). In addition, by [15, Lemma 2.1], \( \partial(D(\varphi)) \) is dense in \( H \).

Now, let \( g : (0, T) \times \Omega \times \mathbb{R} \to \mathbb{R} \) be a Carathéodory function with sublinear growth. Then by Theorem 5.5, for every \( u_0 \in L^2(\partial \Omega) \), there is at least one solution \( u \in H^1_{loc}((0, T]; L^2(\partial \Omega)) \cap C([0, T]; L^2(\partial \Omega)) \) of the elliptic-parabolic boundary-value problem

\[
\begin{aligned}
-\Delta_p \tilde{u}(t, \cdot) &= 0 & \text{on } (0, T) \times \Omega, \\
\partial_t u(t, \cdot) + |\nabla u(t, \cdot)|^{p-2} \frac{\partial}{\partial t} u(t, \cdot) &= g(t, \cdot, u(t, \cdot)) & \text{on } (0, T) \times \partial \Omega, \\
u(t, \cdot) &= u(t, \cdot) & \text{on } (0, T) \times \partial \Omega, \\
u(0, \cdot) &= u_0 & \text{on } \partial \Omega.
\end{aligned}
\]

**APPENDIX A. BRÉZIS’ MAXIMAL \( L^2 \)-REGULARITY THEOREM**

To keep this paper self-contained, we show in this appendix that Brézis’ maximal \( L^2 \)-regularity result (Theorem 2.3) remains true for proper, lower semi-continuous functions \( \varphi : H \to (-\infty, +\infty] \), which are semiconvex.

Under the above hypotheses on \( \varphi \), the subdifferential operator \( A = \partial \varphi \) is quasi maximal monotone. Note that an operator \( A \) on \( H \) is called maximal monotone if firstly, \( A \) is monotone, that is,

\[
(v_1 - v_2, u_1 - u_2)_H \geq 0 \quad \text{for all } (u_1, v_1), (u_2, v_2) \in A,
\]

and secondly, \( A \) satisfies the range condition

\[
\text{Rg}(I_H + \lambda A) = H \quad \text{for one (or, equivalently for all) } \lambda > 0.
\]

Now, an operator \( A \) is called quasi maximal monotone if there is and \( \omega \in \mathbb{R} \) such that \( \omega I_H + A \) is maximal monotone.

One important property of the class of maximal monotone operators in Hilbert spaces is that their graph is closed in \( H \times H_w \), where \( H_w \) means that \( H \) is equipped with the weak topology \( \sigma(H^*, H) \).

**Proposition A.1** ([10, Proposition 2.5]). Let \( A \) be an maximal monotone operator, \((u_n, v_n)\) \( n \geq 1 \subseteq A, u, v \in H \) such that \( u_n \rightharpoonup u \) and \( v_n \rightharpoonup v \) weakly in \( H \) as \( n \to +\infty \) and \( \limsup_{n \to +\infty} (u_n, v_n)_H \leq (u, v)_H \). Then \((u, v) \in A \) and \((u_n, v_n)_H \to (u, v)_H \) as \( n \to +\infty \).

For the class of \( \omega \)-quasi maximal monotone operators in Hilbert spaces the following existence and regularity result hold. Here, we recall [5, Theorem 4.5] in the Hilbert spaces framework and note that in Hilbert spaces monotone operators are accretive and vice versa.

**Theorem A.2** (Existence & regularity for smooth \( f \)). Let \( A \) be an \( \omega \)-quasi maximal monotone operator for some \( \omega \in \mathbb{R} \), \( f \in W^{1,1}(0, T; H) \), \( u_0 \in D(A) \). Then there is a unique solution \( u \in W^{1,\infty}(0, T; H) \) of problem (2.3).

Further, since \( \partial \varphi_{\omega} = A + \omega I_H \) has dense domain in \( D(\varphi) \) by [10, Proposition 2.11] (or [5, p.48]), the domain \( D(A) \) of the subdifferential operator \( A = \partial \varphi \) is dense in \( D(\varphi) \). For later use, we fix this observation in the next proposition.

**Proposition A.3.** Let \( \varphi : H \to (-\infty, +\infty] \) be proper, semiconvex, and lower semi-continuous. Then the domain \( D(A) \) of \( A = \partial \varphi \) is dense in \( D(\varphi) \).
We also need the following chain rule for convex functions $\varphi$.

**Lemma A.4** ([10, Lemma 3.3]). Let $\varphi : H \to (-\infty, +\infty]$ be proper, convex, and lower semicontinuous, and $u \in H^1(0,T;H)$. Assume, there is a $g \in H$ such that $(u(t), g(t)) \in \partial \varphi$ for a.e. $t \in (0,T)$. Then $\varphi \circ u$ is absolutely continuous on $[0,T]$ and

$$\frac{d}{dt} \varphi(u(t)) = (g(t), u(t))_H \quad \text{for a.e. } t \in (0,T).$$

Note, we may always assume without loss of generality that $0 \in D(\partial \varphi_u)$, $\varphi_u$ attains a minimum at 0 (that is, (2.2) holds), and $\varphi_u(0) = 0$. Otherwise, one chooses any $(u_0, v_0) \in \partial \varphi_u$ and replaces $\varphi$ by

$$\tilde{\varphi}(u) := \varphi(u + u_0) - \varphi_u(u_0) - (v_0 - \omega u_0, u)_H \quad \text{for every } u \in H.$$

Then,

$$\tilde{\varphi}_u(u) = \varphi_u(u + u_0) - \varphi_u(u_0) - (v_0, u)_H \quad \text{for every } u \in H,$$

$\varphi_u \geq 0, 0 \in D(\varphi)$, and $\tilde{\varphi}_u(0) = 0$. Moreover, for each solution $y$ of inclusion $y(t) + \partial \tilde{\varphi}(y(t)) \ni f(t) - v_0 + \omega u_0$ on $(0,T)$, the function $u(t) := y(t) + u_0$ is a solution of (2.1). This shows that there is no loss of generality by assuming that for $\varphi_u$, inequality (2.2) holds and $\varphi_u \geq 0$.

With this, we can now outline the proof of Brézis’ $L^2$-maximal regularity result.

**Proof of Theorem 2.3.** Let $f \in H$, $u_0 \in D(\varphi)$, $f_n \in H^1(0,T;H)$ such that $f_n \rightharpoonup f$ in $H$. Moreover, for every $n \geq 1$, there are $u_n(0) \in D(A)$ such that

**(A.1)**

$$\varphi_u(u_n(0)) \leq \varphi_u(u_0)$$

and $u_n(0) \to u_0$ in $H$ (see the last paragraph on [5, p.161]). By Theorem A.2, there is a unique solution $u_n \in W^{1,\infty}(0,T;H)$ of problem

$$\begin{cases} u_t + Au_n \ni f_n & \text{on } (0,T), \\ u_n(0) = u_n(0). \end{cases}$$

Then, by Lemma 4.1, $(u_n)_{n \geq 1}$ is a Cauchy sequence in $C([0,T];H)$. Hence there is a $u \in C([0,T];H)$ such that $u_n \to u$ in $C([0,T];H)$. In particular, $u(0) = u_0$.

(a) We show that $u$ satisfies (3.4). Adding $\omega u_n$ on both sides of

**(A.2)**

$$u_t + Au_n \ni f_n$$

and then multiplying the resulting inclusion by $u_n$ yields

$$\frac{d}{dt} \|u_n(t)\|_H^2 + (h + \omega u_n(t), u_n(t))_H = (f_n(t) + \omega u_n(t), u_n(t))_H$$

for every $h \in Au_n(t)$ for a.e. $t \in (0,T)$. Applying (2.2), and then integrating over $(0,t)$, for $t \in (0,T]$ leads to

$$\frac{1}{2}\|u_n(t)\|_H^2 \leq \frac{1}{2}\|u_n(0)\|_H^2 + \int_0^t \frac{1}{2}\|f_n(s)\|_H^2 ds + (1 + 2\omega) \int_0^t \frac{1}{2}\|u_n(s)\|_H^2 ds.$$

Now, the Gronwall inequality gives that $u_n$ satisfies the uniform bound (2.4) and by letting $n \to \infty$ using that $u_n \to u$ in $C([0,T];H)$, we have that $u$ satisfies (2.4).
(b) Next, we show that \( u \in H^1(0, T; H) \). First, we add \( \omega u_n \) on both sides of (A.2), and then multiply the resulting inclusion by \( \dot{u}_n \). Now, by Lemma A.4,

\[
\|\dot{u}_n(t)\|_H^2 + \frac{d}{dt} \varphi_\omega(u_n(t)) = (f_n(t) + \omega u_n(t), \dot{u}_n(t))_H
\]

for a.e. \( t \in (0, T) \). From this and by (A.1), one deduces that

\[
\frac{1}{2} \int_0^T \|\dot{u}_n(s)\|_H^2 \, ds + \varphi_\omega(u_n(t)) \leq \varphi_\omega(u_0) + \frac{1}{2} \int_0^T \|f_n(s)\|_H^2 \, ds + \frac{\omega}{2} \|u_n(t)\|_H^2 - \frac{\omega}{4} \|u_n^0\|_H^2.
\]

Note that \( \varphi_\omega \) is bounded from below by an affine function. Thus and by part (a), \( (u_n)_{n \geq 1} \) is bounded in \( H \). Since \( H \) is reflexive, \( (u_n)_{n \geq 1} \) admits a weakly convergent subsequence in \( H \). From this, by the limit \( u_n \to u \) in \( C([0, T]; H) \), we can conclude that \( u \in H^1(0, T; H) \). Moreover, by the lower semicontinuity of \( \varphi_\omega \), one see that \( u \) satisfies

\[
\frac{1}{2} \int_0^T \|\dot{u}(s)\|_H^2 \, ds + \varphi_\omega(u(t)) \leq \varphi_\omega(u_0) + \frac{1}{2} \int_0^T \|f(s)\|_H^2 \, ds + \frac{\omega}{2} \|u(t)\|_H^2 - \frac{\omega}{4} \|u^0\|_H^2
\]

for every \( t \in (0, T) \), which is equivalent to

\[
\frac{1}{2} \int_0^T \|\dot{u}(s)\|_H^2 \, ds + \varphi(u(t)) \leq \varphi(u_0) + \frac{1}{2} \int_0^T \|f(s)\|_H^2 \, ds.
\]

(c) We conclude showing that \( u \) is a solution of the evolution problem (2.1). For this, we use the lifted operator \( A \) in \( H \) given by

\[
A = \{(u, v) \in H \times H \mid v(t) \in Au(t) \text{ for a.e. } t \in (0, T)\}.
\]

Since \( \omega I_H + A = \partial \varphi_\omega \) is maximal monotone on \( H \), we have that \( A_\omega := \omega I_H + A \) is maximal monotone on \( H \) (see [10, Exemple 2.3.3]). Moreover, \( u_n \to u \) in \( H \), and after having chosen a subsequence, \( v_n := f_n + \omega u_n - \dot{u}_n \to v := f + \omega u - \dot{u} \) weakly in \( H \). Thus, by Proposition A.1, \( u \in D(A) \) and \( v \neq A_\omega u \), this is equivalent to \( u(t) \in D(A) \) and \( f(t) - \dot{u}(t) \in Au(t) \) for a.e. \( t \in (0, T) \).

(d) Next, let \( f \in H, u_0 \in D(\varphi) \), and \( u_n^0 \in D(\varphi) \) such that \( u_n^0 \to u_0 \) in \( H \). By the previous part, for every \( n \geq 1 \), there are solutions \( u_n \in H^1(0, T; H) \) of problem

\[
\begin{cases}
\dot{u}_n + Au_n \ni f & \text{on } (0, T), \\
u_n(0) = u_n^0.
\end{cases}
\]

By Lemma 4.1, \( (u_n)_{n \geq 1} \) is a Cauchy sequence in \( C([0, T]; H) \) and so, there is a \( u \in C([0, T]; H) \) such that \( u_n \to u \) in \( C([0, T]; H) \) as \( n \to +\infty \). Moreover, by the same argument as in part (a), one sees that each \( u_n \) and \( u \) satisfies (2.4).

(e) Next, we show that

\[
\int_0^T \varphi(u_n(s)) \, ds \leq \frac{1}{2} \|f\|_H^2 + \frac{1 + \omega}{4} \|u_n\|_H^2 + \frac{1}{2} \|u_n^0\|_H^2.
\]

Since \( f(t) - \dot{u}_n(t) \in \partial \varphi(u_n(t)) \), it follows from the definition of \( A = \partial \varphi \) that

\[
\varphi_\omega(v) - \varphi_\omega(u_n(t)) \geq ((f(t) - \dot{u}_n(t)) + \omega u_n(t), v - u_n(t))_H.
\]
for every $v \in H$ and a.e. $t \in (0, T)$. Thus taking $v = 0$ and using that $\varphi_\omega \geq 0$, one sees that

$$
0 \leq \varphi_\omega(u_n(t)) \leq -((f(t) - \dot{u}_n(t)) + \omega u_n(t), -u_n(t))_H
$$

$$
= (f(t), u_n(t))_H - (u_n(t), u_n(t))_H + \omega \|u_n(t)\|_H^2
$$

$$
\leq \frac{1}{2}\|f(t)\|_H^2 + (\frac{1}{2} + \omega)\|u_n(t)\|_H^2 - \frac{\delta}{n}\|u_n(t)\|_H^2
$$

for a.e. $t \in (0, T)$. Integrating over $(0, T)$, one sees that

$$
0 \leq \int_0^T \varphi_\omega(u_n(s)) \, ds \leq \frac{1}{2}\|f\|_H^2 + \frac{1+2\omega}{2}\|u\|_H^2
$$

(A.4)

From this, it follows that (A.3) holds. Then, since $u_n \to u$ in $C([0, T]; H)$ and $\varphi_\omega(u_n) \geq 0$, it follows from the lower semicontinuity of $\varphi_\omega$ and by Fatou’s lemma that (A.4) holds for $u$ and hence, $\varphi \circ u \in L^1(0, T)$ satisfying (2.5).

(f) We show that $u \in H^1_w([0, T]; H)$ with $\sqrt{\dot{u}} \in \mathcal{H}$, and there is a subsequence $(u_{n_k})_{k \geq 1}$ of $(u_n)_{n \geq 1}$ such that $\dot{u}_{n_k} \to \dot{u}$ weakly in $L^2_{loc}([0, T]; H)$. We first add $\omega u_n$ on both sides of

$$
\dot{u}_n(t) + Au_n(t) \geq f(t),
$$

and then multiply the resulting inclusion by $t \cdot \dot{u}_n(t)$. Then by Lemma A.4,

$$
\|\sqrt{t}\dot{u}_n(t)\|_H^2 + t \frac{d}{dt} \varphi_\omega(u_n(t)) = t(f(t) + \omega u_n(t), \dot{u}_n(t))_H
$$

for a.e. $t \in (0, T)$. Applying Cauchy-Schwarz’s and Young’s inequality on the right hand side of this equation, and subsequently integrating over $(0, t)$ for $t \in (0, T)$ gives

$$
\frac{1}{2} \int_0^t \|\sqrt{s}\dot{u}_n(s)\|_H^2 \, ds + t \varphi_\omega(u_n(t)) + \frac{1}{2} \int_0^t \|u_n(s)\|_H^2 \, ds
$$

$$
\leq \int_0^t \varphi_\omega(u_n(s)) \, ds + \frac{1}{2} \int_0^t s\|f(s)\|_H^2 \, ds + t \frac{\omega}{2}\|u_n(t)\|_H^2.
$$

Further, by (A.4) applied to $T = t$, one has

$$
\frac{1}{2} \int_0^t \|\sqrt{s}\dot{u}_n(s)\|_H^2 \, ds + t \varphi_\omega(u_n(t)) + \frac{1}{2} \int_0^t \|u_n(s)\|_H^2 \, ds
$$

$$
\leq \frac{1}{2}\|f\|_{L^2([0, t]; H)}^2 + \frac{1+2\omega}{2}\|u_n\|_{L^2([0, t]; H)}^2 - \frac{1}{2}\|u_n(t)\|_H^2 + \frac{1}{2}\|u_n^{(0)}\|_H^2
$$

$$
+ \frac{1}{2}\|\sqrt{f}\|_{L^2([0, t]; H)}^2 + t \frac{\omega}{2}\|u_n(t)\|_H^2.
$$

Recall that $u_n \to u$ in $C([0, T]; H)$. Thus, $(\sqrt{\dot{u}})_{n \geq 1}$ is bounded in $\mathcal{H}$ and so by the reflexivity of $\mathcal{H}$, one has that $u \in H_{loc}^{1, w}((0, T]; H)$ with $\sqrt{\dot{u}} \in \mathcal{H}$. In particular, $(u_n)_{n \geq 1}$ is bounded in $L^2(\delta, T; H)$ for every $\delta \in (0, T)$. Thus, a diagonal sequence arguments shows that there is a subsequence $(u_{n_k})_{k \geq 1}$ of $(u_n)_{n \geq 1}$ such that $\dot{u}_{n_k} \to \dot{u}$ weakly in $L^2_{loc}((0, T]; H)$.

(g) Next, we show that $u$ is a solutions of (2.3) and $\varphi \circ u \in W_{loc}^{1, w}((0, T])$. To see that $u$ is a solution of (2.3) recall that $u_0^{(0)} \to u_0$ in $H$ and the solutions $u_n$ of (A.2) converge to $u$ in $C([0, T]; H)$. Thus, $u(0) = u_0$ and since for every $\delta \in (0, T]$, $u_n \to \dot{u}$ weakly in $L^2(\delta, T; H)$, it follows by the same argument as in
part (c) from the maximal monotonicity of the operator \( \omega I_{H_t} + A_\delta \) in \( L^2(\delta, T; H) \) with

\[
A_\delta = \left\{ (u, v) \in L^2(\delta, T; H) \times L^2(\delta, T; H) \mid v(t) \in Au(t) \text{ for a.e. } t \in (\delta, T) \right\}.
\]

that \( u(t) \in D(A) \) for a.e. \( t \in (0, T) \) and \( f(t) - \dot{u}(t) \in A(u(t)) \). Moreover, since now, \( g(t) := f(t) - \dot{u}(t) + \omega u(t) \in \partial \varphi_\omega(u(t)) \) for a.e. \( t \in (0, T) \) and \( g \in L^2(\delta, T; H) \) for every \( \delta \in (0, T] \), it follows from Lemma A.4 that \( \varphi \circ u \in W^{1,1}_{\text{loc}}((0, T)) \). This completes the proof of Brézis’ \( L^2 \)-maximal regularity result for semiconvex \( \varphi \).

(h) Finally, we show that \( u \) satisfies (2.6) and (2.7). Since \( u \) is a solution of (2.3), we can add \( \omega u \) on both side of

\[\dot{u}(t) + Au(t) \ni f(t),\]

and then multiply the resulting inclusion by \( t \cdot \dot{u}(t) \). Recall, \( \sqrt{t} \dot{u} \in \mathcal{H} \). Thus by Lemma A.4,

\[\left\| \sqrt{t} \dot{u}(t) \right\|^2_{\mathcal{H}} + t \frac{d}{dt} \varphi_\omega(u(t)) = t(f(t) + \omega u(t), \dot{u}(t))_{\mathcal{H}}\]

for a.e. \( t \in (0, T) \). Next, by Cauchy-Schwarz’s and Young’s inequality, and subsequently integrating over \( (0, t) \) for \( t \in (0, T] \) gives

\[
\frac{1}{2} \int_0^t \| \sqrt{s} \dot{u}(s) \|^2_{\mathcal{H}} \, ds + t \frac{d}{dt} \varphi_\omega(u(t)) + \int_0^t s \| u(s) \|^2_{\mathcal{H}} \, ds \leq \int_0^t \varphi_\omega(u(s)) \, ds + \frac{1}{2} \int_0^t s \| f(s) \|^2_{\mathcal{H}} \, ds + t \frac{d}{dt} \| u(t) \|^2_{\mathcal{H}},
\]

from which we can conclude (2.6) and (2.7). \[\square\]

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