CUP-LENGTH ESTIMATE FOR LAGRANGIAN INTERSECTIONS

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ABSTRACT. In this paper we consider the Arnold conjecture on the Lagrangian inter-
sections of some closed Lagrangian submanifold of a closed symplectic manifold with
its image of a Hamiltonian diffeomorphism. We prove that if the Hofer’s symplectic
energy of the Hamiltonian diffeomorphism is less than a topology number defined by
the Lagrangian submanifold, then the Arnold conjecture is true in the degenerated
(non-transversal) case.

§1 Introduction and main results

Let \((M, \omega)\) be a closed symplectic manifold, \(L \subset M\) be its closed Lagrangian
submanifold. A Hamiltonian \(H : [0, 1] \times M \rightarrow \mathbb{R}\) is a \(C^\infty\) function. This function
defines a \(t\)-dependent Hamiltonian vector field \(X_{H_t}\) on \(M\) by \(\omega(\cdot, X_{H_t}) = dH_t\). The
time one map \(\varphi = \varphi^1\) of the flow generated by the Hamiltonian vector field \(X_{H_t}\)
is a symplectic automorphism of \(M\). Arnold conjecture that, for some symplectic
manifold \((M, \omega)\) and its Lagrangian submanifold \(L\), the intersection \(L \cap \varphi(L)\) con-
tains at least as many points as a topology number of \(L\). If \(L\) transversely meet
\(\varphi(L)\), then the topology number can be the rank of \(H^*(L; \mathbb{F})\) for some ring or field
\(\mathbb{F}\). In general, this topology number can be the cup-length of \(L\) which is defined by

\[
\text{cl}(L, \mathbb{F}) = \max\{k + 1 \mid \exists \alpha_i \in H^{d_i}(L, \mathbb{F}), d_i \geq 1, i = 1, \ldots, k \text{ such that } \alpha_1 \cup \cdots \cup \alpha_k \neq 0\}.
\]

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In this paper, we fixed $\mathbb{F} = \mathbb{Z}_2$ and denote the cup-length of $L$ by $\text{cl}(L)$.

It is well known that the above Arnold conjecture is not true in general. For example the “small Lagrangian torus” in a symplectic manifold can be push away by some Hamiltonian diffeomorphism. In this case the intersection $L \cap \varphi(L) = \emptyset$, but the topology number of $L$ is not zero. So we need further conditions to guarantee this version of the Arnold conjecture. The first condition was given by Floer in [F1,F2] (see also [H]). It was proved that if $\pi_2(M, L) = 0$ or $\omega(\pi_2(M, L)) = 0$, then the Arnold conjecture on the Lagrangian intersection is true. Chekanov [Ch1-Ch2] found that there is some relation between the Hofer’s bi-invariant metric of the Hamiltonian diffeomorphism and this version of Arnold conjecture.

For a Hamiltonian $H : [0, 1] \times M \to \mathbb{R}$, we can define a semi-norm of $H$ as

$$\|H\| = \int_0^1 (\max_x H(t, x) - \min_x H(t, x)) \, dt.$$  

This semi-norm is weaker than $C^0$-norm of $H$ and plays an eminent role for Hofer’s bi-invariant metric on the group of compactly supported Hamiltonian diffeomorphism. The metric is defined by

$$d(\varphi, \text{id}_M) = \inf\{\|H\| \mid \varphi \text{ is generated by } H\}.$$  

We say that $L$ is a rational Lagrangian submanifold of $M$ if there is a number $\sigma(L) > 0$ such that $\omega(\pi_2(M, L)) = \sigma(L) \cdot \mathbb{Z}$.

Chekanov in [Ch1] (see [Ch2] for a somewhat general statement) proved that if $d(\varphi, \text{id}_M) < \sigma(L)$, then $\sharp(L \cap \varphi(L)) \geq \dim H^*(L; \mathbb{Z}_2)$ provided $L$ is a rational Lagrangian submanifold of $M$ with the number $\sigma(L) > 0$ defined as above and the intersection is transverse.

The main result of this paper is the following theorem.

**Theorem.** If $L \subset M$ is a rational Lagrangian submanifold of $M$ with the number $\sigma(L)$ defined above and $d(\varphi, \text{id}_M) < \sigma(L)$, then there holds

$$\sharp(L \cap \varphi(L)) \geq \text{cl}(L).$$  

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\section*{2 J-holomorphic curves with boundary conditions}

Let $L$ be a closed embedded Lagrangian submanifold of a compact symplectic manifold $(M, \omega)$. $H : [0, 1] \times M \to \mathbb{R}$ is a smooth function, and $\varphi^t$ is the Hamiltonian flow generated by the Hamiltonian function $H$. Setting $L_1 = \varphi^1(L)$, and considering the space

\[ \Omega_1(L) = \{ \gamma \in C^\infty([0, 1], M) \mid \gamma(0) \in L, \gamma(1) \in L_1 \}, \]

restricting to this space we define a 1-form $\alpha$ by

\[ \langle \alpha(\gamma), \xi \rangle = \int_0^1 \omega(\dot{\gamma}(t), \xi(t)) \, dt. \]

This 1-form is closed. Let $\Omega^0_1(L)$ be the component of $\Omega_1(L)$ which contains the constant path. A primitive $F$ of $\alpha|_{\Omega^0_1(L)}$ is a $\mathbb{R}/\sigma\mathbb{Z}$-valued functional on $\Omega^0_1(L)$, the standard action functional of Floer's theory. It is defined up to additive constants.

For a compatible almost complex structure $J$, define a metric on $\Omega_1(L)$ as follows:

\[ \langle \xi_1, \xi_2 \rangle = \int_0^1 \omega(\xi_1(t), J\xi_2(t)) \, dt. \]

The gradient of $F$ with respect to this metric is given by

\[ \nabla F(\gamma)(t) = J(\gamma(t))\dot{\gamma}(t). \]

For a pair $(x^+, x^-)$ of critical points of $F$ which correspond to a pair of intersections of $L \cap L_1$, we consider the following moduli space which is analogue to the connect orbit space of the negative gradient flow of a Morse functional defined on a finite dimensional space

\[ \mathcal{M}(J, H, x^+, x^-) = \left\{ u : \mathbb{R} \times [0, 1] \to M \mid \begin{array}{l} \partial_s u + J\partial_t u = 0, \ u \text{ is not constant,} \\
\lim_{s \to \pm\infty} u(s, t) = x^\pm \in L \cap \varphi^1(L) \\
\end{array} \right\} \]

If $u \in \mathcal{M}(J, H, x^+, x^-)$, we define a map $\tilde{u} : \mathbb{R} \times [0, 1] \to M$ such that $u(s, t) = \varphi^t(\tilde{u}(s, t))$, then we get

\[ \partial_s \tilde{u} + \tilde{J}_t (\partial_t \tilde{u} + X_{\tilde{H}}(\tilde{u}(s, t))) = 0. \tag{2.1} \]
Here $\tilde{J}_t = (d\varphi^t)^{-1} J d\varphi^t$, $\tilde{H}(t, x) = H(t, \varphi^t(x))$ and $X_{\tilde{H}}(x) = (d\varphi^t)^{-1} X_H(\varphi^t(x))$ by definition. If $J$ is compatible with the symplectic structure $\omega$, so is for the $t$-dependent almost complex structure $\tilde{J}_t$. $\tilde{u}$ satisfies the following conditions \((2.2)\) and \((2.3)\).

\[
\begin{align*}
\tilde{u}(s, 0) & \in L, \quad \forall s \in (-\infty, +\infty) \\
\tilde{u}(s, 1) & \in L, \quad \forall s \in (-\infty, +\infty).
\end{align*}
\tag{2.2}
\]

\[
\lim_{s \to \pm\infty} \tilde{u}(s, t) = x^\pm(t),
\tag{2.3}
\]

where $x^\pm(t) = (\varphi^t)^{-1}(x^\pm)$ is a Hamiltonian flow line of the Hamiltonian function $-\tilde{H}$ and $x^\pm(0) = x^\pm \in L \cap \varphi^1(L)$. Conversely, if $\tilde{u}$ is a solution of \((2.1)\) satisfies \((2.2)\) and \((2.3)\), then $u(s, t) = \varphi^t(\tilde{u}(s, t))$ belongs to $\mathcal{M}(J, H, x^+, x^-)$. In fact, it is easy to see $u$ solves the equation

\[
\partial_s u + J \partial_t u = 0.
\tag{2.4}
\]

By definition of $u$, we have

\[
u(s, 0) = \tilde{u}(s, 0) \in L, \quad u(s, 1) = \varphi^1(\tilde{u}(s, 1)) \in \varphi^1(L),
\tag{2.5}\]

and

\[
\lim_{s \to \pm\infty} u(s, t) = \varphi^t(x^\pm(t)) = x^\pm(0) \in L \cap \varphi^1(L).
\tag{2.6}\]

Thus we can consider the following moduli space

\[
\tilde{\mathcal{M}}(J, H, x^+, x^-) = \left\{ \left. \tilde{u} \quad \begin{array}{l}
\partial_s \tilde{u} + \tilde{J}_t (\partial_t \tilde{u} + X_{\tilde{H}}(\tilde{u}(s, t))) = 0 \\
\tilde{u}(s, 0) \in L, \quad \tilde{u}(s, 1) \in L \\
\lim_{s \to \pm\infty} \tilde{u}(s, t) = x^\pm(t) \text{ is Hamiltonian flow line of } -\tilde{H} \\
x^\pm(0) = x^\pm \in L \cap \varphi^1(L)
\end{array} \right| \right\}.
\]

This moduli space $\tilde{\mathcal{M}}(J, H, x^+, x^-)$ is 1-1 correspondent with $\mathcal{M}(J, H, x^+, x^-)$.

We recall that the Hamiltonian flow line of the Hamiltonian function $-\tilde{H}$ with Lagrangian boundary condition is a solution of the following equation

\[
\begin{align*}
\dot{x}(t) &= -X_{\tilde{H}}(x(t)) \\
x(0) &\in L, \quad x(1) \in L.
\end{align*}
\tag{2.7}
\]

We can write $x(t) = (\varphi^t)^{-1}(x_0)$, then $x(0) = x_0 \in L$ and $x(1) = (\varphi^1)^{-1}(x_0) \in L$, it implies $x(0) = x_0 \in L \cap \varphi^1(L)$. The space of the solutions of \((2.7)\) is one to one correspondent with the set $L \cap \varphi^1(L)$. 

In order to find solutions of equation (2.7), we define the following spaces

\[ \tilde{\Omega}(L) = \{ x \in C^\infty([0,1], M) \mid x(0) \in L, \ x(1) \in L \}, \]
\[ \tilde{\Omega}_0(L) = \{ x \in \Omega(L) \mid [x] = 0 \in \pi_1(M, L) \}, \]

and the universal cover space of \( \tilde{\Omega}_0(L) \)

\[ \Omega_0(L) = \{ u_x : D \to M \mid u_x|_{S^+} = x, \ u_x|_{S^-} = \tilde{x} \}, \]

where \( D \) is the unit disc in \( \mathbb{C} \) with \( \partial D = S^+ \cup S^- \), and \( S^+ \) (resp. \( S^- \)) is the upper (resp. lower) half unit circle which is a part of \( \partial D \), the boundary of \( D \). \( \tilde{x} : [0,1] \to L \) is a path in \( L \) which is isotopic to \( x \) relative to the end points. On the space \( \Omega_0(L) \) we define a functional

\[ A_H(x, u_x) = \int_D u_x^* \omega + \int_0^1 \tilde{H}(t, x(t)) \, dt. \]

It is easy to see that

\[ dA_H(x)(\xi) = \int_0^1 \omega(\dot{x} + X_{\tilde{H}}(x), \xi) \]

This means that \( dA_H(x) = 0 \) implies \( \dot{x} + X_{\tilde{H}}(x) = 0 \).

The functional induces a functional \( \tilde{A} : \tilde{\Omega}_0(L) \to \mathbb{R}/\sigma\mathbb{Z} \) if \( L \) is rational with \( \omega(\pi_2(M, L)) = \sigma(L)\mathbb{Z} \) for some \( \sigma = \sigma(L) > 0 \).

§3 Morse homology and its cup product

We first recall the Morse homology theory briefly (see [MS2] for details). Let \((f, g)\) be a Morse-Smale pair on \( L \), that is, let \( f \) be a fixed Morse function and \( g \) be a generic Riemannian metric on \( L \) such that the stable and unstable manifolds \( W^s(y), W^u(x) \) for critical points \( x, y \in \text{Crit} f \) for the negative gradient flow of \((f, g)\) intersect transversely. We define the connect orbit space of \( x, y \in \text{Crit} f \) by

\[ M_{x,y}(f, g) = \{ \gamma \in C^\infty(\mathbb{R}, L) \mid \dot{\gamma} + \nabla_y f(\gamma) = 0, \ \gamma(-\infty) = x, \ \gamma(+\infty) = y \}. \]

We have \( \dim M_{x,y}(f, g) = \mu(x) - \mu(y), \) \( \mu(x) \) is the Morse index of \( x \in \text{Crit} f \), and \( M_{x,y}(f, g) \) admits a free \( \mathbb{R} \)-action by translation: \( s \cdot \gamma(\cdot) = \gamma(s + \cdot) \). We denote the quotient space by

\[ \hat{M}_{x,y}(f, g) = M_{x,y}(f, g)/\mathbb{R}. \]
Let $C^k(f)$ denote the $\mathbb{Z}_2$-free Abelian group generated by $\text{Crit}_k f = \mu^{-1}(k)$, and define the boundary operator as

$$\delta : C^k(f) \to C^{k+1}(f), \quad \delta x = \sum_{\mu(y) = \mu(x)+1} n(y, x) y$$

where $n(x, y)$ is defined by

$$n(x, y) = \sharp\mathbb{Z}_2 \hat{M}_{x, y}(f, g)$$

the modulo 2 number of $\hat{M}_{x, y}(f, g)$, it is well defined when $\mu(x) - \mu(y) = 1$. It is well known that $\delta^2 = 0$, and

$$H^*(C^*(f), \delta) \cong H^*(L; \mathbb{Z}_2). \quad (3.1)$$

Let $(f, g_i), i = 1, 2, 3$ be three generic Morse-Smale pairs on $L$ such that the following moduli spaces are $\mu(z) - \mu(x) - \mu(y)$ dimensional space for $x, y, z \in \text{Crit} f$

$$\mathcal{M}_{x, y}(f, g_1, g_2, g_3) = \{(\gamma_1, \gamma_2, \gamma_3) \in W^u(z) \times W^s(x) \times W^s(y) \mid \gamma_1(0) = \gamma_2(0) = \gamma_3(0)\}$$

and the spaces $\mathcal{M}_{x, y}(f, g_1, g_2, g_3)$ are compact in dimension 0.

Analogously to $\delta$ we define the following operation on $C^*(f, \mathbb{Z}_2)$. Given $x, y, z \in \text{Crit} f$, we set

$$n(z; x, y) = \sharp\mathcal{M}_{x, y}(f, g_1, g_2, g_3) \pmod{2} \text{ for } \mu(z) = \mu(x) + \mu(y)$$

and

$$m_2 : C^k(f, \mathbb{Z}_2) \otimes C^l(f, \mathbb{Z}_2) \to C^{k+l}(f, \mathbb{Z}_2)$$

$$m_2(x \otimes y) = \sum_{z \in \text{Crit}_{k+l} f} n(z; x, y) z. \quad (3.2)$$

$m_2$ is a chain operator and it induced a cup product of the cohomologies $H^*(L; \mathbb{Z}_2)$. These result are standard now (see for example: [MS1] section 3 for $A = 0$ thus $u$ must be a constant map, or [Fu1] for $f_1 = f_2 = f_3$ with different metrics satisfying the transversal conditions). Analogously we can define the moduli spaces for $x_0, x_1, \cdots, x_k \in \text{Crit} f$

$$\mathcal{M}_{x_0; x_1, \cdots, x_k} = \{(\gamma_0, \gamma_1, \cdots, \gamma_k) \in W^u(x_0) \times W^s(x_1) \times \cdots \times W^s(x_k) \mid \gamma_0(0) = \gamma_1(0) = \cdots = \gamma_k(0)\}$$
and

\[ m_k : C^l_1(f, \mathbb{Z}_2) \otimes \cdots \otimes C^l_k(f, \mathbb{Z}_2) \to C^{l_1+\cdots+l_k}(f, \mathbb{Z}_2) \]

\[ m_k(x_1, \ldots, x_k) = \sum_{x_0} n_k(x_0; x_1, \ldots, x_k)x_0, \]

where

\[ \mu(x_0) = \mu(x_1) + \cdots + \mu(x_k) \text{ and } n_k(x_0; x_1, \ldots, x_k) = \sharp_{\mathbb{Z}_2} M_{x_0;x_1,\ldots,x_k}. \]

$m_k$ induced $k$-fold cup-product of the cohomologies $H^*(L; \mathbb{Z}_2)$.

In this section we always assume that $(M, \omega)$ is a closed symplectic manifold. $L \subset M$ is a closed rational Lagrangian submanifold with the constant $\sigma(L) > 0$ defined as in section 2. i.e., we have $\omega(\pi_2(M, L)) = \sigma(L)\mathbb{Z}$ for some $\sigma(L) > 0$. Denote by $\mathcal{H}(M)$ the set of all Hamiltonian function $H : [0, 1] \times M \to \mathbb{R}$. Any $H \in \mathcal{H}(M)$ defines a time-dependent Hamiltonian flow $\varphi^t : M \to M$. Time one maps of such flows form a group $S(M, \omega)$ called the group of Hamiltonian symplectomorphisms of $M$. On the space $\mathcal{H}(M)$, we have a semi-normal defined by

\[ \| H \| = \int_0^1 (\max_x H(t, x) - \min_x H(t, x)) \, dt. \]

For $\varphi \in S(M, \omega)$, the energy of $\varphi$ is defined by

\[ E(\varphi) = \inf \{ \| H \| \mid \varphi \text{ is a time one flow generated by } H \in \mathcal{H}(M) \} \]

We assume that $E(\varphi) < \sigma(L)$, this condition is essential for the compactness of the moduli spaces because under this condition no bubbling-off ($J$-holomorphic sphere and disc) occurs. So we can naturally define the deformation cup product of the cohomology groups. Under the above conditions, we have the moduli space

\[ \mathcal{M}^0(J, H) = \{ \tilde{u} \in C^\infty(D, M) \mid \partial_s \tilde{u} + \tilde{J}_t (\partial_t \tilde{u} + X_{\tilde{H}}(\tilde{u}(s, t))) = 0, \ [\tilde{u}] = 0 \in \pi_2(M, L) \}. \]

Given $x_0, x_1, \ldots, x_k \in \text{Crit}_f$ we define

\[ \mathcal{M}_{x_0; x_1, \ldots, x_k}^0 = \{ (\tilde{u}, \gamma_0, \gamma_1, \ldots, \gamma_k) \in \mathcal{M}^0(J, H) \times W^u(x_0) \times W^s(x_1) \times \cdots \times W^s(x_k) \mid \]

\[ \tilde{u}(z_i) = \gamma_i(0), \ z_i \in \partial D, \ i = 0, 1, \ldots, k \}

**Theorem 3.1.** Given a Hamiltonian function $H$ with $\| H \| < \sigma(L)$, and generic pairs $(f, g_i)$, $i = 0, 1, \ldots, k$, the following operator $m^0(H)$ is well defined,

\[ m_k^0(H) : C^{l_1}(f) \otimes \cdots \otimes C^{l_k}(f) \to C^{l_1+\cdots+l_k}(f), \]
\[ m_k^0(H)(x_1 \otimes \cdots \otimes x_k) = \sum_{x_0}(\#\mathcal{M}_{x_0; x_1, \ldots, x_k \mod 2}^0) x_0 \]

Moreover, \( m^0(H) \) is a co-chain map with respect to the boundary operator \( \delta \), and the induced operation of the cohomology group is just the \( k \)-fold cup product in the sense of (3.1).

**Proof.** The essential ingredient of the proof is to prove the fact of no bubbling-off. This can be done by looking at the energy of the element \( \tilde{u} \in \mathcal{M}^0(J, H) \)

\[
E(\tilde{u}) = \int_D |\partial_s \tilde{u}|^2 dsdt = \int_D \omega(\partial_s \tilde{u}, \tilde{J}_t \partial_s \tilde{u}) \\
= -\int_D \omega(\tilde{J}_t (\partial_t \tilde{u} + X_{\tilde{H}}(\tilde{u}(s, t)), \tilde{J}_t \partial_s \tilde{u})) dsdt \\
= -\int_D \omega(X_{\tilde{H}}(\tilde{u}(s, t), \partial_s \tilde{u})) dsdt \\
= \int_D d\tilde{H}_t(\tilde{u}(s, t))(\partial_s \tilde{u}) dsdt \leq \|H\|. \tag{3.3}
\]

Here we have use the condition \([\tilde{u}] = 0 \in \pi_2(M, L)\). Since \( \|H\| < \sigma(L) \), notice that we can take \( \pi_2(M) \) as a sub-group of \( \pi_2(M, L) \), any bubbling-off must have energy at least \( \sigma(L) \), so no bubbling-off occurs. If \( H \equiv 0 \), then we have \( m^0(0) = m_k \) as defined in (3.2) which induced the \( k \)-fold cup product. Taking a suitable homotopy \( H \sim 0 \) such that the induced maps in \( H^*(L; \mathbb{Z}_2) \) satisfying \( m^0_k(H^*) = m^0_k(0^*) = m^*_k \) (see [MS1], Theorem 3.8 for similar arguments. Here we only consider \( A = 0 \)).

\[\text{§4 The proof of the main result}\]

We follow the ideas of [MS1] to prove the main result of this paper. Firstly, we modify the pair \((J, H)\) and define the “adapted solution spaces”. Given the Hamiltonian \( H \in C^\infty([0, 1] \times M, \mathbb{R}) \) and an \( \omega \)-compatible almost structure \( J \), we get a corresponding pair \((\tilde{J}, \tilde{H})\) as in section 2. Here \( \tilde{J} \) is explicitly dependent of \( t \in [0, 1] \). Pick an \( t \)-independent almost complex structure \( J_0 \) on \( TM \to M \), we extend \( J \) and \( J_0 \) to a smooth 1-parameter family \( \tilde{J} = \tilde{J}(s), s \in (-\infty, +\infty) \) as

\[
\tilde{J}(s) = \begin{cases} 
J_0, & s \leq 0, \\
\tilde{J}, & s \geq 1.
\end{cases} \tag{4.1}
\]

Let \( \beta \in C^\infty(\mathbb{R}, [0, 1]) \) be a monotone cut-off function such that

\[
\beta(s) = \begin{cases} 
0, & s \leq 0, \\
1, & s \geq 1,
\end{cases} \text{ and } \beta'(s) \geq 0.
\]
For $R \in [1, \infty)$, we defined 1-parameter pairs $(\tilde{J}_R, \tilde{H}_R)$ on $\mathbb{R} \times [0, 1] \times M$ as follows,

$$(\tilde{J}_R, \tilde{H}_R)(s, t, p) = \begin{cases} 
(J_0(p), 0), & s \leq 0, \\
(J(s, t, p), \beta(s)\tilde{H}(t, p)), & 0 < s \leq R, \\
(J(R + 1 - s, t, p), \beta(R + 1 - s)\tilde{H}(t, p)), & R < s \leq R + 1, \\
(J_0(p), 0), & s > R + 1.
\end{cases}$$

Associated to $(\tilde{J}_R, \tilde{H}_R)$ we have the Cauchy-Riemann type operator $\tilde{\partial}_R$ for $u: \mathbb{R} \times [0, 1] \to M$ satisfying the boundary conditions $u(\cdot, 0) \in L$ and $u(\cdot, 1) \in L$, and consider the following equation,

$$\tilde{\partial}_R u(s, t) := \partial_s u + \tilde{J}_R(s, t, u)(\partial_t u + X_{\tilde{H}_R}(s, t, u)) = 0. \quad (4.2)$$

We note that for $1 \leq s \leq R$, (4.2) describes the “negative gradient flow” for the action functional $A_{J,H}$, i.e., it satisfies

$$\tilde{\partial}_{J,H} u(s, t) := \partial_s u + \tilde{J}_t(\partial_t u + X_{\tilde{H}}(u(s, t))) = 0. \quad (4.3)$$

The energy of $u: \mathbb{R} \times [0, 1] \to M$ associated to $\tilde{J}_R$ is defined by

$$E_R(u) = \int_{-\infty}^{+\infty} \int_0^1 |\partial_s u|_{\tilde{J}_R}^2 \, dsdt.$$  

Since a solution $u$ of (4.2) restrict to $(-\infty, 0) \times [0, 1]$ or $(R + 1, +\infty) \times [0, 1]$ is $J_0$-holomorphic, finite energy $E_R(u) < \infty$ implies by the boundary removal of singularities (see [Oh1]) that $u$ can be extended over the disc carrying the conformal structure from $\mathbb{R} \times [0, 1]$,

$$\tilde{D} = \{-\infty\} \cup (-\infty, +\infty) \times [0, 1] \cup \{+\infty\}.$$  

Thus we can identify $\tilde{D}$ with the standard disc $(D, i)$, and for every finite energy solution $u$ of (4.2), the homotopy class $[u] \in \pi(M, L)$ is well defined. We define the adapted solution spaces associated with $R$ by

$$\mathcal{M}^0(R) = \{u \in C^\infty(\mathbb{R} \times [0, 1] \to M) | \tilde{\partial}_R(u) = 0, E_R(u) < \infty, [u] = 0 \in \pi_2(M, L)\}.$$  

For an adapted solution $u$, the following result give an estimate of the energy of $u$. 

Corollary 4.1. Every solution $u \in \mathcal{M}^0(R)$ satisfies the energy estimate

$$0 \leq E_R(u) \leq \|H\|, \quad \forall R \geq 1.$$  \hfill (4.4)

Moreover, there exists an $l \in \mathbb{R}$ such that

$$\mathcal{A}_H(u(\varrho, \cdot)) \in [l, l + \|H\|], \quad \forall \varrho \in [1, R].$$  \hfill (4.5)

Proof. These results are taken from [MS1] (Corollary 4.2) for the case of fixed points of Hamiltonian diffeomorphism. The proof is the same. We give the proof here for the readers’ convenience. For $u \in \mathcal{M}^0(R)$ and $1 \leq \sigma \leq \sigma' \leq R$, there holds

$$0 \leq E(u^-) \leq \mathcal{A}_H(u(\sigma, \cdot), [u^-]) - \int_0^1 \inf_{p \in M} H(t, p) \, dt,$$  \hfill (4.6)

$$0 \leq E(u^+) \leq -\mathcal{A}_H(u(\sigma, \cdot), [u^-]) + \int_0^1 \sup_{p \in M} H(t, p) \, dt,$$  \hfill (4.7)

$$0 \leq E(u^-) - E(u^-) = \mathcal{A}_H(u(\sigma', \cdot), [u^+]) - \mathcal{A}_H(u(\sigma, \cdot), [u^-]).$$  \hfill (4.8)

Here $u^-$ is the restriction of $u$ to $D^- := \{-\infty\} \cup (-\infty, \sigma) \times [0, 1]$ and $u^+$ is the restriction of $u$ to $D^+ := (\sigma, +\infty) \times [0, 1] \cup \{+\infty\}$. (4.6) follows by

$$E(u^-) = \int \int_{D^-} \omega(\partial_s u, \bar{J}_R \partial_s u) \, ds \, dt = \int \int_{D^-} \omega(\partial_s u, \partial_t u + \beta X_{\bar{H}}) \, ds \, dt$$

$$= \int \int_{D^-} u^* \omega + \int_{0}^{1} \tilde{H}(t, u(\sigma, t)) \, dt - \int_{-\infty}^{\sigma} \beta'(s) \, ds \int_{0}^{1} \tilde{H}(t, u(s, t)) \, dt.$$  

Thus there holds

$$\mathcal{A}_H(u(\sigma, \cdot), [u^-]) - \int_0^1 \sup_{p \in M} H(t, p) \, dt \leq E(u^-) \leq \mathcal{A}_H(u(\sigma, \cdot), [u^-]) - \int_0^1 \inf_{p \in M} H(t, p) \, dt.$$  \hfill (4.9)

Using $\mathcal{A}_H(u(\sigma, \cdot), [u^+]) = \omega([u]) - \mathcal{A}_H(u(\sigma, \cdot), [u^-])$ and $\omega([u]) = 0$, we get (4.7) analogously. (4.8) is obvious. (4.4) follows from (4.6) and (4.7). (4.5) follows from (4.9) and the fact $E(u^-) \leq E_R(u)$. 

\[\blacksquare\]
For the modified pair \((\tilde{J}_R, \tilde{H}_R)\), as in section 3, we choose an auxiliary Morse function \(f\) and 1-parameter families metrics \(g^j_s\) on \(L\), \(j = 0, 1, \cdots, k\). For any \(k + 1\)-tuple \((y_0, \cdots, y_k) \in (\text{Crit} f)^{k+1}\), we define the moduli space

\[
\mathcal{M}_{y_0; y_1, \cdots, y_k}^0 (J, H, f, (g^j_s)) = \{(u, \gamma_0, \cdots, \gamma_k) \in \mathcal{M}_{(k+1)R}^0 \times W^u_{g^0}(y_0) \times W^{s^1}_g(y_1) \times \cdots \times W^{s^k}_g(y_k) | u(-\infty) = \gamma_0(0), u(jR, 0) = \gamma_j(0), j = 1, \cdots, k \}.
\]

Here we remind that we have replace the disc \(D\) by the disc \(\tilde{D} = \{ -\infty \} \cup (-\infty, +\infty) \times [0, 1] \cup \{ +\infty \}\) with the standard complex structure \(i\), and \(z_0 = -\infty, z_j = (jR, 0)\).

An immediate consequence of Theorem 3.1 is

**Corollary 4.2.** Let \((M, \omega)\) be a closed symplectic manifold, \(L\) be its closed rational Lagrangian submanifold with the constant \(\sigma(L)\) as defined in section 2. The Hamiltonian \(H : [0, 1] \times M \to \mathbb{R}\) satisfies \(\|H\| < \sigma(L)\). Given homogeneous cohomology classes \(\alpha_1, \cdots, \alpha_k \in H^*(L)\) with nontrivial cup product \(\alpha_0 = \alpha_1 \cup \cdots \cup \alpha_k \in H^*(L)\), there exist critical points \(y_0, y_1, \cdots, y_k \in \text{Crit} f\) satisfying

\[
\mu(y_0) = \text{deg} \alpha_0, \quad \mu(y_j) = \text{deg} \alpha_j, \quad j = 1, \cdots, k
\]

such that the solution space \(\mathcal{M}_{y_0; y_1, \cdots, y_k}^0 (J, H, f, (g^j_s))\) is nonempty.

From this existence result for finite energy solutions of (4.2), we will deduce the asserted estimate for the number of critical values for the action functional \(A_H\) by considering \(R \to \infty\).

We now consider the broken flow trajectories. Let us recall the pair \((\tilde{J}, \tilde{H})\) and the Cauchy-Riemann type equation from (2.1) with \(L\) boundary conditions

\[
(\bar{\partial}_{J, H} u)(s, t) = \partial_s u + \tilde{J}(t, u)(\partial_t u + X_{\tilde{H}}(u)) = 0, \\
u(s, 0) \in L, \quad \forall s \in (-\infty, +\infty) \\
u(s, 1) \in L, \quad \forall s \in (-\infty, +\infty).
\]

and the Hamiltonian systems with the \(L\) boundary conditions from (2.7)

\[
\begin{cases}
\dot{x}(t) = -X_{\tilde{H}}(x(t)) \\
x(0) \in L, \quad x(1) \in L.
\end{cases}
\]

The set of solutions of (4.12) is 1-1 correspondent with the set of the intersection points \(L \cap \varphi^1(L)\). We denote the set of solutions of (4.12) by \(S_L(H)\).
Proposition 4.3. If the number of the above solution set $\sharp S_L(H) < \infty$, then there exists a unique limit $x \in S_L(H)$ for every solution of (4.11) restrict in the half area with the same boundary condition

$$(\partial_{J,H} u)(s,t) = \partial_s u + \bar{J}(t,u)(\partial_t u + X_{\bar{H}}(u)) = 0,$$

$u(s,0) \in L, \ \forall s \in [0, +\infty)$

$u(s,1) \in L, \ \forall s \in [0, +\infty)$

$E(u) < \infty.$ \hspace{1cm} (4.13)

that is, $u(s, \cdot) \to x$ uniformly in $C^\infty([0, 1], M)$ as $s \to \infty$.

Proof. This proposition is adapted from Proposition 4.4 of [MS1] and the proof is standard as given in [MS1]. We consider the reparametrized solution $u_n = u(\cdot + s_n, \cdot)$ for $s_n \to \infty$, we have $E(u_n|_{[-\sigma, \sigma]}) \to 0$ for all $\sigma > 0$ due to the finite energy assumption. Hence for a suitable subsequence $u_{n_k}$ converges in $C^\infty_{loc}$ and the limit is a translation invariant solution of $\partial_{J,H} u = 0$ with the mentioned boundary conditions over $\mathbb{R} \times [0, 1]$, that is constant in $s$ and therefore an $x \in S_L(H)$. Given two sequences $s_n, s'_n \to \infty$ with $u(s_n) \to x$ and $u(s'_n) \to x'$ the finiteness of $S_L(H)$ implies $x = x'$. Otherwise, one can assume that $s'_n - s_n \to \infty$ and find, after choosing suitable subsequence, a sequence $s_n < \tilde{s}_n < s'_n$ such that without loss of generality $u(\tilde{s}_n) \to \tilde{x}$ with $x \neq \tilde{x}$ and $x' \neq \tilde{x}$. Repeating this argument finitely many times leads to a contradiction. \[\blacksquare\]

Without loss of generality we can assume that $\sharp S_L(H) < \infty$. Hence for a solution of (4.11) with finite energy, there exist $x, x' \in S_L(H)$ such that

$$\lim_{s \to -\infty} u(s) = x, \ \lim_{s \to \infty} u(s) = x'.$$

We define the following connected trajectory spaces for $x, x' \in S_L(H)$

$$\mathcal{M}_{x,x'}(J, H) = \{ u : \mathbb{R} \times [0, 1] \to M \mid u \text{ solves (4.11), } \lim_{s \to -\infty} u(s) = x, \ \lim_{s \to \infty} u(s) = x' \}.$$

Similarly we define disk type solution spaces for the structure $\bar{J}$ and $\beta$ from above

$$\mathcal{M}_{x}^\pm(\bar{J}, H) = \{ u : \mathbb{R} \times [0, 1] \to M \mid \partial_x u + \bar{J}(\pm s, t, u)(\partial_t u + \beta(\pm s)X_{\bar{H}}(t,u)) = 0 \}

\begin{align*}
&u(s,0) \in L, \ u(s,1) \in L, \ \forall s \in \mathbb{R} \\
&E(u) < \infty, \ u(\pm \infty) = x
\end{align*}$$

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An element of $\mathcal{M}^-_\bar{\bar{J}}, H)$ is a map which is pseudo-holomorphic in an area containing infinity (the singularity at infinity can be removed) and is a solution of (4.13) in another area containing infinity with $x$ as its limit.

We denote the spaces of so-called broken solutions by

$$\tilde{\mathcal{M}}^0(\bar{\bar{J}}, H) = \{(u_-, u_1, \ldots, u_k, u_+)$$

$$\in \mathcal{M}_{x_0}^-(\bar{\bar{J}}, H) \times \mathcal{M}_{x_0,x_1}(J, H) \times \cdots \times \mathcal{M}_{x_{k-1},x_k}(J, H) \times \mathcal{M}_{x_k}^+(\bar{\bar{J}}, H) |$$

$$x_0, \ldots, x_k \in SL(H), \ k \geq 0, \ [u_- \# u_1 \# \cdots \# u_+] = 0 \in \pi_2(M, L)\},$$

where $\#$ is the obvious gluing operation.

Considering the solution spaces $\mathcal{M}^0(R_n)$ for $R_n \to \infty$, we say that a sequence $u_n \in \mathcal{M}^0(R_n)$ converges weakly to a broken solution $u_n \rightharpoonup (v_0, v_1, \ldots, v_k, v_{k+1}) \in \tilde{\mathcal{M}}^0(\bar{\bar{J}}, H)$ if there are sequences $\{\sigma_{i,n}\}_{n \in \mathbb{N}} \subset \mathbb{R}, i = 0, \ldots, k+1$, such that the reparametrized maps $u_n(\cdot + \sigma_{i,n}, \cdot)$ converge uniformly on compact subsets with all derivatives to $v_i$,

$$u_n(\cdot + \sigma_{i,n}, \cdot) \to v_i \text{ in } C^\infty_{\text{loc}}(\mathbb{R} \times [0, 1], M).$$

Clearly, this requires that $\sigma_{0,n} = 0$ and $\sigma_{k+1,n} = R_n + 1$ for all $n \in \mathbb{N}$. The following result is analogous to Gromov’s result about the minimal energy of $J$-holomorphic discs, [G].

**Lemma 4.4.** Given a pair $(J, H)$ with $\sharp SL(H) < \infty$, there exists a lower bound $h(J, H) > 0$ for the energy of all non-stationary finite energy trajectories, that is,

$$\partial_{J,H} u = 0, \ u(s, 0) \in L, \ u(s, 1) \in L, \ and \ \partial_s u \neq 0 \ imply \ E(u) \geq h(J, H).$$

**Proof.** We follow the ideas of [HS] to prove the result. For the case $H \equiv 0$, $u$ can be extended to a $J$-holomorphic disc. The result follows from the Gromov compactness. In fact, if there is a sequence of $J$-holomorphic discs $u_n$ with energy $E(u_n) \to 0$, then by Gromov compactness, $u_n$ weakly converges to a cusp curve with positive energy, a contradiction. If $H \neq 0$, assume that there is a sequence of solution $u_n$ with $0 \neq E(u_n) \to 0$. We prove that $\partial_s u$ converges to zero uniformly in $\mathbb{R} \times [0, 1]$ as $n$ tends to $\infty$. Otherwise there would exist a sequence $(s_n, t_n)$ such that $|\partial_s u(s_n, t_n)| \geq \delta > 0$. If $s_n$ is bounded, we can assume $s_n \to 0$ without loss of generality. Since $E(u_n)$ converges to zero no bubbling can occur and hence a subsequence of $u_n$ converges with its derivatives uniformly on compact sets to a
solution \( u : \mathbb{R} \times [0, 1] \to M \) with mentioned boundary conditions, \( \partial_s u(0, t^*) \geq \delta \) and \( E(u) = 0 \). But the latter implies that \( u(s, t) \equiv x(t) \) in contradiction to the former. If \( s_n \) is non-bounded, then we can assume \( s_n \to \infty \). We consider \( v_n(s, t) = u(s + s_n, t) \) as in the proof of Proposition 4.3, then by the finiteness condition: \( \sharp S_L(H) < \infty \), we can get \( v_n \to v \) with \( |\partial_s v(0, t^*)| \geq \delta \) and \( E(v) = 0 \), it is still a contradiction. \[\blacksquare\]

We denote the broken trajectory space by

\[
\bar{M}_{x,y}(J, H) = \{ u = (\bar{u}_1, \cdots, \bar{u}_r) \mid u_i \in \mathcal{M}_{x_{i-1}, x_i}, i = 1, \cdots, r, x_0 = x, x_r = y, r \in \mathbb{N} \}.
\]

It is the space of broken trajectories started from \( x \in S_L(H) \) and ended at \( y \in S_L(H) \). The energy of a broken trajectory \( u = (u_1, \cdots, u_r) \) satisfies

\[
E(u) = \sum_{i=1}^{r} E(u_i).
\]

If \( u \in \bar{M}_{x,x} \), then \([u] \in \pi_2(M, L)\) is well defined and \( \omega([u]) = E(u) \neq 0 \), the latter follows from the fact that the start point is just the end point, so there holds

\[
\sum_{i=1}^{r} \int_{-\infty}^{\infty} \omega(\partial_t u_i, JX_{H}(u)) \, dt = 0.
\]

Thus if \( u \neq x \), then \( E(u) \neq 0 \), it implies that \( \omega([u]) = E(u) \geq \sigma(L) \).

We define

\[
\sigma_0(\omega, H, J) = \inf \{ E(u) \mid u \in \bar{M}_{x,x}(J, H), u \neq x, x \in S_L(H) \}.
\]

**Theorem 4.5.** Let \( \sharp S_L(H) < \infty \) and \( u_n \in \mathcal{M}^0(R_n) \) be a sequence of solution with \( R_n \to \infty \) and uniformly bounded gradient \( \nabla u_n \). Then there exists a subsequence \( \{ \bar{u}_{n_k} \} \) converging weakly to a broken solution

\[
\bar{u}_{n_k} \rightharpoonup (v_-, v_1, \cdots, v_N, v_+) \in \mathcal{M}^0(\bar{J}, H).
\]

**Proof.** This result is similar to Theorem 4.5 of [MS1]. Elliptic bootstrapping implies \( C_{\text{loc}}^\infty \)-convergence for subsequences of \( \{u_n(\cdot + s_n)\} \) for any shifting sequences \( \{s_n\} \), \( s_n \to \infty \). Assume that we have already shifting sequences \( \{s_n\} \) and \( \{\bar{s}_n\} \) such that \( s_n - \bar{s}_n \to \infty \) and \( u_n(\cdot + s_n) \to v, u_n(\cdot + \bar{s}_n) \to w \) in \( C_{\text{loc}}^\infty \) with \( v \in \mathcal{M}_{x,y}(J, H) \) and \( w \in \mathcal{M}_{y', z}(J, H) \), we use the analogous argument as in the proof of Proposition 14.
4.3. We show that either \( y = y' \) or that modulo choosing a subsequence we find a sequence \( \tau_n \to \infty \) such that \( s_n < \tau_n < \bar{s}_n \) and \( u_n(\cdot + \tau_n) \to \bar{w} \in \mathcal{M}_{y,y'}(J, H) \).

This requires lifting to the covering \( \Omega_0(L) \) where the function \( \mathcal{A}_H \) is real-valued and the energy of \( u \in \mathcal{M}_{x,y}(J, H) \) is given by \( E(u) = \mathcal{A}_H(y) - \mathcal{A}_H(x) \), where \( x \) is the lifting of \( x \) in \( \Omega_0(L) \). From the total energy bound by \( \|H\| \) from Corollary 4.1 and the minimal energy \( h(J, H) > 0 \) for non-stationary trajectories from Lemma 4.4, it follows that only finite number of \( \tilde{y} \in \mathcal{S}_L(H) \), the lifting of \( \mathcal{S}_L(H) \) in \( \Omega_0(L) \), can occur between \( y \) and \( y' \). It remains to show that \( \mathcal{A}_H(y) = \mathcal{A}_H(y') \) implies \( y = y' \). This follows from the following result.

**Lemma 4.6.** Let \( \# \mathcal{S}_L(H) < \infty \), there exists a \( \gamma > 0 \) such that for every neighbourhood \( W \) of \( \mathcal{S}_L(H) \) in \( C^\infty([0, 1], M) \) there exists a number \( h = h(M) \) with the following properties:

If \( u : (r, R) \times [0, 1] \to M \) for \( -\infty \leq r < R \leq \infty \) solves

\[
\bar{\partial}_{J, H} u = 0, \quad u(\cdot, 0) \in L, \quad u(\cdot, 1) \in L, \quad [u(\frac{r+R}{2}, \cdot)] = 0 \in \pi_1(M, L),
\]

\[
E(u) \leq \gamma \quad \text{and} \quad R - r > 2h,
\]

then \( u(s) \in W \) for all \( s \in (r+h, R-h) \). Moreover, given \( k_0 \in \mathbb{N}, \epsilon > 0 \), there exists \( h = h(k_0, \epsilon) \) such that solutions of (4.14) view as a mappings into \( M \subset \mathbb{R}^N \) satisfy

\[
|D^\alpha(u(s, t) - x(t))| \leq \epsilon, \forall (s, t) \in (r+h, R-h) \times [0, 1], |\alpha| \leq k_0
\]

for a suitable \( x \in \mathcal{S}_L(H) \).

**Proof.** We prove indirectly the second assertion. Assume that given any \( \gamma > 0 \) there exist \( k(\gamma) \in \mathbb{N}, \epsilon(\gamma) > 0, h_n \to \infty, r_n < R_n \) with \( R_n - r_n \geq 2h_n \) and \( u_n : (r_n, R_n) \times [0, 1] \to M \) satisfying the boundary condition as in (4.14) and \( \bar{\partial}_{J, H} u_n = 0, \int_{r_n}^{R_n} \int_0^1 |\partial_s u_n|^2 \, ds \, dt \leq \gamma \) such that there exist \( (s_n, t_n) \in (r_n, R_n) \times [0, 1] \) and \( \alpha \leq k \) with

\[
|D^\alpha(u_n(s_n, t_n) - x(t_n))| > \epsilon
\]

for all \( n \in \mathbb{N} \) and \( x \in \mathcal{S}_L(H) \). Reparametrizing \( u_n \) so that \( v_n(s, t) = u_n(s + s_n, t) \) solves \( \bar{\partial}_{J, H} v_n = 0 \) with

\[
\int_{-h_n}^{h_n} \int_0^1 |\partial_s v_n|^2 \, ds \, dt < \gamma \quad \text{and} \quad |D^\alpha(v_n(0, t_n) - x(t_n))| > \epsilon.
\]

Without loss of generality we can replace \( t_n \) by some \( t_0 \). Choosing \( \gamma > 0 \) small enough by Gromov’s theorem about the minimal energy of pseudoholomorphic
spheres or holomorphic discs with $L$-boundary condition (see the proof of Lemma 4.4), there exists a number $c > 0$ such that

$$|\nabla v_n(s,t)| \leq c \quad \forall (s,t) \in [-\frac{3}{4} h_n, \frac{3}{4} h_n] \times [0,1], \ n \in \mathbb{N}.$$ 

Otherwise, we would obtain a pseudoholomorphic sphere or disc bubbling off with energy less than $\gamma$. Thus, choosing a suitable subsequence, without loss of generality denoted again by $n \in \mathbb{N}$, we obtain uniform convergence on compact subsets, $v_n \to (v : \mathbb{R} \times [0,1] \to M)$ in $C_{\text{loc}}^\infty$ with

$$\partial_{J,H} v = 0, \quad \int_{-\infty}^{\infty} \int_{0}^{1} |\partial_s v|^2 \leq \gamma \quad \text{and}$$

$$|D^\alpha (v(0,\cdot) - x(\cdot))|_{L^\infty([0,1])} > \epsilon, \quad v(\cdot,0) \in L, \ v(\cdot,1) \in L.$$ 

But Lemma 4.4 implies for $\gamma < \rho(J,H)$ that $\partial_s v = 0$, i.e. $v(0) \in \mathcal{S}_L(H)$ providing the contradiction. 

This also concludes the proof of Theorem 4.5 because $\mathcal{A}_H(y) = \mathcal{A}_H(y')$ implies that we can find sequences $s_n$ and $s'_n$ such that $u_n(s_n) \to y$, $u_n(s'_n) \to y'$ and $0 < s_n - s'_n$ with $E(u_n|_{(s_n,s'_n)}) \to 0$. Consequently, Lemma 4.6 yields $y = y'$. 

Remark 4.7. In our case we have $\|H\| < \sigma(L)$, and $E(u_n) \leq \|H\|$ by Corollary 4.1 for $u_n \in \mathcal{M}^0(R_n)$, bubbling-off cannot occur, thus the gradient of $u_n$ is uniformly bounded.

We denote the covering space of $\mathcal{S}_L(H)$ in the sense of section 2 by $\tilde{\mathcal{S}}_L(H)$, i.e., any element $x = (x, u_x) \in \tilde{\mathcal{S}}_L(H)$ is a critical point of $\mathcal{A}_H$ in the space $\Omega_0(L)$ and $x$ is a solution of $\dot{x} = -JH(t,x)$ with the boundary conditions $x(0) \in L$ and $x(1) \in L$. It implies $x(0) \in L \cap \varphi^1(L)$, see (2.7). The space carries a partial ordering with respect to the gradient flow of $\mathcal{A}_H$.

Definition 4.8. Given a pair $x, x' \in \tilde{\mathcal{S}}_L(H)$, we say $x \leq x'$ if there exist connecting broken flow trajectories $\mathcal{M}_{x,x'}(J,H) \neq \emptyset$. Given a Morse-Smale pair $(f, g)$, we say that $x \ll x'$ if there exist $u \in \mathcal{M}_{x,x'}(J,H)$ and $y \in \text{Crit}_f$ with $\mu(y) \geq 1$ such that $u(0,0) \in W^g_s(y)$.

If $\sharp \{L \cap \varphi^1(L)\} < \infty$, then for a generic choice of Morse function $f : L \to \mathbb{R}$ and Riemannian metric $g$ on $L$, there holds

$$\bigcup_{\mu(y) \geq 1} W^g_s(y) \cap L \cap \varphi^1(L) = \emptyset. \quad (4.15)$$

This can be prove by standard transversal analysis (see [MS1]). Thus if choose $(f, g)$ satisfying (4.15), then for $x \ll x'$ we have $x \neq x'$ thus $x < x'$ and in particular $\mathcal{A}_H(x) < \mathcal{A}_H(x')$. The latter can be seen from the proof of Theorem 4.5. By this observation we have the following result.
Corollary 4.9. Let \( k \in \mathbb{N} \) and \((f,g^i), i = 1, \cdots, k\) satisfy condition (4.15) with respect to \( H \) satisfying \( \mathcal{S}_L(H) < \infty \). Given a sequence
\[
u_n \in \mathcal{M}^0_{y_0; y_1, \cdots, y_k}((k+1)R_n), \quad R_n \to \infty
\]
with \( y_i \in \text{Crit}_f, \mu(y_i) \geq 1 \) for \( i = 0, 1, \cdots, k \), weakly converging to a broken trajectory, there exist solutions \( x_1, \cdots, x_N \in \tilde{\mathcal{S}}_L(H) \) satisfying \( x_1 \leq \cdots \leq x_N \) and \( 1 \leq n_1 \leq m_1 \leq n_2 \leq m_2 \leq \cdots \leq n_k \leq m_k \leq N \) such that \( x_{n_i} \ll x_m \), for \( i = 1, \cdots, k \). In particular, there exists an \( l \in \mathbb{R} \) such that
\[
l \leq \mathcal{A}_H(x_{n_1}) < \cdots < \mathcal{A}_H(x_{n_k}) < \mathcal{A}_H(x_{m_k}) \leq l + \|H\|.
\]
Proof. By assumption, the sequence \( u_n \in \mathcal{M}^0((k+1)R_n) \) satisfies
\[
u_n(jR_n,0) \in W^s_g(y_j), \quad j = 1, \cdots, k.
\]
Moreover, if \( u_n \) converges weakly to a broken solution
\[
(v_-,v_1,\cdots,v_N,v_+) \in \tilde{\mathcal{M}}^0(\bar{J},H)
\]
we have reparametrization sequences \( \{\sigma_{i,n}\}_{n \in \mathbb{N}} \) for \( i = 1, \cdots, N \) such that \( u_n(\cdot + \sigma_{i,n},\cdot) \to v_i \) in \( C^\infty_{\text{loc}} \) and \( u_n \to v_- \), \( u_n(\cdot - (k+1)R_n - 1,\cdot) \to v_+ \). Considering the shifted solutions \( u_{n,j} = u_n(\cdot - jR_n,\cdot) \), we thus obtain after choosing a suitable subsequence \( C^\infty_{\text{loc}} \) convergence \( u_{n,j} \to w_j \in \mathcal{M}_{x_j,x'_j}(J,H) \) for some \( x_j, x'_j \in \tilde{\mathcal{S}}_L(H), \quad j = 1, \cdots, k \). By definition, we have \( x_j \ll x'_j \) and the assumption of weak convergence implies the order
\[
x_1 \ll x'_1 \leq x_2 \ll x'_2 \leq \cdots \leq x_k \ll x'_k.
\]
We now can prove the main result of this paper. \( \quad \blacksquare \)

Theorem 4.10. Let \((M,\omega)\) be a closed symplectic manifold, and \( L \) be its closed Lagrangian submanifold satisfying the rational condition \( \omega(\pi_2(M,L)) = \sigma(L) \cdot \mathbb{Z} \), \( \sigma(L) > 0 \). \( \varphi = \varphi^J \) is a Hamiltonian automorphism of \((M,\omega)\) generated by the Hamiltonian \( H : [0,1] \times M \to \mathbb{R} \) with \( \|H\| < \sigma(L) \). Then the cup-length estimate of the Lagrangian intersection holds
\[
\sharp\{L \cap \varphi(L)\} \geq \text{cl}(L).
\]
Proof. By the assumption \( \|H\| < \sigma(L) \), for a generic almost complex structure \( J \) compatible with the symplectic structure \( \omega \), let \( k + 1 = \text{cl}(L) \), then by Corollary 4.9. \( \square \)
4.2 we find solutions \( u_n \in M^0_{y_0; y_1, \ldots, y_k}((k + 1)R_n) \) for some sequence \( R_n \to \infty \) and \( y_i \in \text{Crit} f \) where \((f, g^i)\) satisfy (4.15). By Corollary 4.9, Theorem 4.5 and Remark 4.7, there are \( k + 1 \) critical points \( x_i \in \tilde{S}_L(H) \) for \( A_H \) on \( \Omega_0(L) \) defined in section 2 such that
\[
l \leq A_H(x_1) < \cdots < A_H(x_{k+1}) \leq l + \|H\|
\]
for some \( l \in \mathbb{R} \). Due to the assumption \( \|H\| < \sigma(L) \) again, there is no broken trajectory of flow started from some solution \( x \in S_L(H) \) and ended at the same solution. In fact, the energy of this mentioned broken trajectory should be not less than the number \( \sigma(L) \), but on the other hand side, this energy should be not more than \( \|H\| \) since \( E(u_n) \leq \|H\| \). Namely, the \( k + 1 \) critical points \( x_i \) project to \( k + 1 \) different solutions \( x_i \in S_L(H) \).

Remark 4.12. As in [Ch1-Ch2], the symplectic manifold can be more generally a tame symplectic manifold, since the tameness condition allows us to deal with \( M \) as if it is compact, all the techniques are the same as in the compact case if we only consider the compactly supported Hamiltonian \( H \). We recall that \((M, \omega)\) is tame if there exists an almost complex structure \( J \) on \( M \) such that \( g(\cdot, \cdot) = \omega(\cdot, J\cdot) \) is a Riemannian metric on \( M \) satisfying the following conditions:

(i) Riemannian manifold \((M, g)\) is complete,
(ii) the sectional curvature of \( g \) is bounded,
(iii) the injectivity radius of \( g \) is bounded away from zero.

Let \( J \) be an almost complex structure on \( M \) such that \((M, \omega, J)\) is a tame almost Kähler manifold, denote by \( J \) the space of such structures. Let \( \sigma_S(M, J) \) denote the minimal area of a \( J \)-holomorphic sphere in \( M \), and \( \sigma_D(M, L, J) \) denote the minimal area of a \( J \)-holomorphic disc in \( M \) with boundary on \( L \). These numbers may equal infinity if there are no such \( J \)-holomorphic curves. Otherwise, minimals are achieves due to the Gromov compactness theorem (see [G]) and are clearly positive. Let
\[
\sigma(M, L, J) = \min(\sigma_S(M, J), \sigma_D(M, L, J))
\]
We remind the number \( \sigma_0(\omega, H, J) \) is defined just before Theorem 4.5. The following result does not require that \( L \) is rational Lagrangian submanifold of \( M \).

Theorem 4.13. If \( \|H\| < \min(\sigma_0(\omega, H, J), \sigma(M, L, J)) \), then the standard cup-length estimate is valid
\[
\#(L \cap \varphi(L)) \geq \text{cl}(M).
\]

Proof. The proof is the same as in the proof of Theorem 4.11. With the condition \( \|H\| < \min(\sigma_0(\omega, H, J), \sigma(M, L, J)) \), the bubbling-off can not occur, we can also guarantee that the different critical points \( x_i \in \tilde{S}_L(H) \) can be project to different \( x_i \in S_L(H) \) as done in the proof of Theorem 4.11.
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