MONODROMY EIGENVALUES AND ZETA FUNCTIONS WITH DIFFERENTIAL FORMS

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Abstract. For a complex polynomial or analytic function $f$, there is a strong correspondence between poles of the so-called local zeta functions or complex powers $\int |f|^{2s} \omega$, where the $\omega$ are $C^\infty$ differential forms with compact support, and eigenvalues of the local monodromy of $f$. In particular Barlet showed that each monodromy eigenvalue of $f$ is of the form $\exp(2\pi \sqrt{-1}s_0)$, where $s_0$ is such a pole. We prove an analogous result for similar $p$-adic complex powers, called Igusa (local) zeta functions, but mainly for the related algebro-geometric topological and motivic zeta functions.

Introduction

0.1. Let $f : X \to \mathbb{C}$ be a non-constant analytic function on an open part $X$ of $\mathbb{C}^n$. We consider $C^\infty$ functions $\varphi$ with compact support on $X$ and the corresponding differential forms $\omega = \varphi dx \wedge d\bar{x}$. Here and further $x = (x_1, \ldots, x_n)$ and $dx = dx_1 \wedge \cdots \wedge dx_n$. For such $\omega$ the integral

$$Z(f, \omega; s) := \int_X |f(x)|^{2s} \omega,$$

where $s \in \mathbb{C}$ with $\Re(s) > 0$, has been the object of intensive study. One verifies that $Z(f, \omega; s)$ is holomorphic in $s$. Either by resolution of singularities [At], [BG], or by the theory of Bernstein polynomials [Be], one can show that it admits a meromorphic continuation to $\mathbb{C}$, and that all its poles are among the translates by $\mathbb{Z}_{<0}$ of a finite number of rational numbers. Combining results of Barlet [Ba2], Kashiwara [Ka] and Malgrange [Ma2], the poles of (the extended) $Z(f, \omega; s)$ are strongly linked to the eigenvalues of (local) monodromy at points of $\{f = 0\}$; see §1 for the concept of monodromy.

Theorem. (1) If $s_0$ is a pole of $Z(f, \omega; s)$ for some differential form $\omega$, then $\exp(2\pi \sqrt{-1}s_0)$ is a monodromy eigenvalue of $f$ at some point of $\{f = 0\}$.

(2) If $\lambda$ is a monodromy eigenvalue of $f$ at a point of $\{f = 0\}$, then there exists a differential form $\omega$ and a pole $s_0$ of $Z(f, \omega; s)$ such that $\lambda = \exp(2\pi \sqrt{-1}s_0)$.

There are also more precise local versions in a neighbourhood of a point of $\{f = 0\}$. Similar results hold for a real analytic function $f : X(\subset \mathbb{R}^n) \to \mathbb{R}$ and integrals $\int_{X \cap \{f > 0\}} f^s \varphi dx$; we refer to e.g. [Ba3], [Ba4], [Ba6], [BM], [JM].

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0.2. Let now \( f : X \to \mathbb{Q}_p \) be a non-constant \((\mathbb{Q}_p\)-analytic function on a compact open \( X \subset \mathbb{Q}_p^n \), where \( \mathbb{Q}_p \) denotes the field of \( p\)-adic numbers. Let \( | \cdot |_p \) and \( |dx| \) denote the \( p\)-adic norm and the Haar measure on \( \mathbb{Q}_p^n \), normalized in the standard way. The \( p\)-adic integral

\[
Z_p(f; s) := \int_X |f(x)|_p^s |dx|,
\]

again defined for \( s \in \mathbb{C} \) with \( \Re(s) > 0 \), is called the \((p\)-adic\) Igusa zeta function of \( f \). Using resolution of singularities Igusa [Ig1] showed that it is a rational function of \( p^{-s} \); hence it also admits a meromorphic continuation to \( \mathbb{C} \). In this context there is an intriguing conjecture of Igusa relating poles of (the extended) \( Z_p(f; s) \) to eigenvalues of monodromy. More precisely, let \( f \) be a polynomial in \( n \) variables over \( \mathbb{Q}_p \). Then we can consider \( Z_p(f; s) \) for all prime numbers \( p \) (taking \( X = \mathbb{Z}_p^n \)).

**Monodromy conjecture** [De1]. *For all except a finite number of \( p \), we have that, if \( s_0 \) is a pole of \( Z_p(f; s) \), then \( \exp(2\pi \sqrt{-1} s_0) \) is a monodromy eigenvalue of \( f : \mathbb{C}^n \to \mathbb{C} \) at a point of \( \{ f = 0 \} \).*

This conjecture was proved for \( n = 2 \) by Loeser [Lo2]. There are by now various other partial results [ACLM1], [ACLM2], [Lo3], [RV], [Ve1], [Ve6]. (We took \( \mathbb{Q}_p \) for simplicity of notation; everything can be done over finite extensions of \( \mathbb{Q}_p \).)

0.3. There are various ‘algebro-geometric’ zeta functions, related to the \( p\)-adic Igusa zeta functions: the motivic, Hodge and topological zeta functions, for which we refer to §1. Here we just mention that the motivic zeta function specializes to the various \( p\)-adic Igusa zeta functions (for almost all \( p \)). For those zeta functions a similar monodromy conjecture can be stated; and analogous partial results are valid.

0.4. We should note that for the complex (and real) integrals in (0.1) there are more precise results of Bernstein and Barlet, involving roots of the Bernstein polynomial of \( f \) (instead of monodromy eigenvalues). Similarly there is a finer conjecture for the poles of Igusa and related zeta functions, relating them to roots of the Bernstein polynomial [De1], [DL1]. However, the results of this paper do not involve Bernstein polynomials, so we just refer the interested reader to [Be], [Ba1], [Ba2], [Ba3], [Ig2], [Lo1], [Lo2], [Lo3].

0.5. As in the complex (or real) case, one associates \( p\)-adic Igusa zeta functions, and also motivic, Hodge and topological zeta functions, to a function \( f \) and a differential form \( \omega \). In this ‘algebro-geometric’ context one considers algebraic differential forms \( \omega \); see §1.

To our knowledge a possible analogue of Theorem 0.1(2) in the context of \( p\)-adic and the related ‘algebro-geometric’ zeta functions was not studied before in the literature. For instance let \( f \) be a polynomial over \( \mathbb{Q} \) satisfying \( f(0) = 0 \), and let \( \lambda \) be a monodromy eigenvalue of \( f \) at \( 0 \). Does there exist a compact open neighbourhood \( X \) of \( 0 \) and an algebraic differential form \( \omega \) such that (the meromorphic continuation of) \( \int_X |f(x)|_p^s |\omega|_p \) has a pole \( s_0 \) satisfying \( \lambda = \exp(2\pi \sqrt{-1} s_0) \)? (If \( \omega = g(x)dx \) for some polynomial \( g \) over \( \mathbb{Q} \), the integral above is just \( \int_X |f(x)|_p^s |g(x)|_p |dx| \).)
We will concentrate in this paper on the analogous question for the topological zeta function, since a positive answer in this context automatically yields a positive answer in the context of Hodge and motivic zeta functions (see §1), and also for Igusa zeta functions (by [DL1, Théorème 2.2]). We show for instance (Theorem 3.6):

**Theorem.** Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be a nonzero polynomial function (germ). Let \( \lambda \) be a monodromy eigenvalue of \( f \) at 0. Then there exists a differential \( n \)-form \( \omega \) and a point \( P \in \{ f = 0 \} \), close to 0, such that the (local) topological zeta function at \( P \), associated to \( f \) and \( \omega \), has a pole \( s_0 \) satisfying \( \exp(2\pi \sqrt{-1} s_0) = \lambda \).

If \( f^{-1}\{ 0 \} \) has an isolated singularity at 0, then we can take 0 itself as point \( P \).

For \( n = 2 \), we construct such \( \omega \) in §2 using so-called curvettes. In arbitrary dimension we follow a similar approach, for which we first introduce a higher dimensional version of this notion (Proposition 3.2).

**0.7.** The zeta functions associated to \( f \) and the constructed \( \omega \) in the theorem above can have other poles that don’t induce monodromy eigenvalues of \( f \). So for those zeta functions the analogue of Theorem 0.1(1) is (unfortunately) not true. It would be really interesting to have a complete analogue of Theorem 0.1, roughly saying that the monodromy eigenvalues of \( f \) correspond precisely to the poles of the zeta functions associated to \( f \) and some finite list of differential forms \( \omega \) (including \( dx \)). Of course this would be a lot stronger than the (in arbitrary dimension) still wide open monodromy conjecture.

However, we indicate some examples where such a correspondence holds, for instance \( f = y^a - x^b \) with \( \gcd(a, b) = 1 \).

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1. Monodromy and zeta functions

1.1. Let \( f : \mathbb{C}^n \to \mathbb{C} \) be a non-constant polynomial function satisfying \( f(b) = 0 \). Let \( B \subset \mathbb{C}^n \) be a small enough ball with centre \( b \); the restriction \( f|_B \) is a topological fibration over a small enough pointed disc \( D \subset \mathbb{C}\setminus\{0\} \) with centre 0. The fibre \( F_b \) of this fibration is called the (local) Milnor fibre of \( f \) at \( b \); see e.g. [Mi]. The counterclockwise generator of the fundamental group of \( D \) induces an automorphism of the cohomologies \( H^q(F_b, \mathbb{C}) \), which is called the (local) monodromy of \( f \) at \( b \). By a monodromy eigenvalue of \( f \) at \( b \) we mean an eigenvalue of the monodromy action on at least one of the \( H^q(F_b, \mathbb{C}) \). It is well known that \( H^q(F_b, \mathbb{C}) = 0 \) for \( q \geq n \), and that all monodromy eigenvalues are roots of unity.

Let \( P_q(t) \) denote the characteristic polynomial of the monodromy action on \( H^q(F_b, \mathbb{C}) \).

If \( f = \prod_j f_j^{N_j} \) is the decomposition of \( f \) in irreducible components and \( d := \gcd_j N_j \), then \( P_0(t) = t^d - 1 \).

When \( b \) is an isolated singularity of \( f^{-1}\{0\} \), then \( H^q(F_b, \mathbb{C}) = 0 \) for \( q \neq 0, n - 1 \); and \( P_0(t) = t - 1 \).
1.2. Definition. The monodromy zeta function \( \zeta_{f,b}(t) \) of \( f \) at \( b \) is the alternating product of all characteristic polynomials \( P_q(t) \):

\[
\zeta_{f,b}(t) := \prod_{q=0}^{n-1} P_q(t)^{(-1)^q}.
\]

Note that there are also other conventions, see for example [A'C], [AVG].

In particular for an isolated singularity the knowledge of \( \zeta_{f,b}(t) \) and of \( P_{n-1}(t) \) are equivalent.

1.3. We recall the following interesting and useful result, which is maybe not generally known.

Lemma [De2, Lemma 4.6]. Let \( f : \mathbb{C}^n \rightarrow \mathbb{C} \) be a non-constant polynomial function. If \( \lambda \) is a monodromy eigenvalue of \( f \) at \( b \in f^{-1}\{0\} \), then there exists \( P \in f^{-1}\{0\} \) (arbitrarily close to \( b \)) such that \( \lambda \) is a zero or pole of the monodromy zeta function of \( f \) at \( P \).

It is convenient to recall also the proof in order to see how the point \( P \) is obtained. Let \( \Psi_{f,\lambda} \) be the sub-complex of the complex of nearby cycles of \( f \) corresponding to the eigenvalue \( \lambda \), where both are viewed as (shifted) perverse sheaves. Let \( \Sigma \) be the largest analytic set given by the supports of the cohomology sheaves of \( \Psi_{f,\lambda} \). Then, by perversity of \( \Psi_{f,\lambda} \), at a generic point \( P \) of \( \Sigma \) the eigenvalue \( \lambda \) appears on exactly one cohomology group of the Milnor fibre of \( f \) at \( P \).

1.4. A’Campo’s formula. Let \( f : \mathbb{C}^n \rightarrow \mathbb{C} \) be a non-constant polynomial function satisfying \( f(b) = 0 \). Take an embedded resolution \( \pi : X \rightarrow \mathbb{C}^n \) of \( f^{-1}\{0\} \) (that is an isomorphism outside the inverse image of \( f^{-1}\{0\} \)). Denote by \( E_i, i \in S \), the irreducible components of the inverse image \( \pi^{-1}(f^{-1}\{0\}) \), and by \( N_i \) the multiplicity of \( E_i \) in the divisor of \( \pi^*f \). We put \( E_i^\circ := E_i \setminus \bigcup_{j \neq i} E_j \) for \( i \in S \).

Theorem [A'C]. Denoting by \( \chi(\cdot) \) the topological Euler characteristic we have

\[
\zeta_{f,b}(t) = \prod_i (t^{N_i} - 1) \chi(E_i^\circ \cap \pi^{-1}\{b\}).
\]

1.5. Another kind of zeta functions are the topological, Hodge and motivic zeta functions, associated to a non-constant polynomial function \( f : \mathbb{C}^n \rightarrow \mathbb{C} \) and a regular differential \( n \)-form \( \omega \) on \( \mathbb{C}^n \). (More generally one can consider an arbitrary smooth quasi-projective variety \( X_0 \) instead of \( \mathbb{C}^n \) and a regular function \( f \).) We will describe these zeta functions in terms of an embedded resolution of \( f^{-1}\{0\} \cup \text{div} \omega \). Now we denote by \( E_i, i \in S \), the irreducible components of the inverse image \( \pi^{-1}(f^{-1}\{0\} \cup \text{div} \omega) \) and by \( N_i \) and \( \nu_i - 1 \) the multiplicities of \( E_i \) in the divisor of \( \pi^*f \) and \( \pi^*\omega \), respectively. We put \( E_i^\circ := \)

\((\cap_{i \in I} E_i) \setminus (\cup_{j \not\in I} E_j)\) for \(I \subset S\), in particular \(E_\emptyset^0 = X \setminus (\cup_{j \in S} E_j)\). So the \(E_i^0\) form a stratification of \(X\) in locally closed subsets.

**Definition.** The (global) topological zeta function of \((f, \omega)\) and its local version at \(b \in \mathbb{C}^n\) are

\[
Z_{\text{top}}(f, \omega; s) := \sum_{I \subset S} \chi(E_i^0) \prod_{i \in I} \frac{1}{\nu_i + sN_i},
\]

and

\[
Z_{\text{top},b}(f, \omega; s) := \sum_{I \subset S} \chi(E_i^0 \cap \pi^{-1}\{b\}) \prod_{i \in I} \frac{1}{\nu_i + sN_i},
\]

respectively, where \(s\) is a variable.

These invariants were introduced by Denef and Loeser in [DL1] for ‘trivial \(\omega\)’, i.e. for \(\omega = dx_1 \wedge \cdots \wedge dx_n\). Their original proof that these expressions do not depend on the chosen resolution is by describing them as a kind of limit of \(p\)-adic Igusa zeta functions. Later they obtained them as a specialization of the intrinsically defined motivic zeta functions [DL2]. Another technique is applying the Weak Factorization Theorem [AKMW], [Wi] to compare two different resolutions. For arbitrary \(\omega\) one can proceed analogously.

It is natural and useful to study these invariants incorporating such a more general \(\omega\), see for example [ACLM1], [ACLM2], [Ve4]. Note however that there one restricts to the situation where \(\text{supp}(\text{div} \omega) \subset f^{-1}\{0\}\). In the original context of \(p\)-adic Igusa zeta functions, see e.g. [Lo2, III 3.5].

**1.6.** There are finer variants of these zeta functions using, instead of Euler characteristics, Hodge polynomials or classes in the Grothendieck ring of varieties. We mention for instance the Hodge zeta function

\[
Z_{\text{Hod}}(f, \omega; T) = \sum_{I \subset S} H(E_i^0; u, v) \prod_{i \in I} \frac{(uv - 1)T^{N_i}}{(uv)^{\nu_i} - T^{N_i}}, \in \mathbb{Q}(u, v)(T),
\]

where \(H(\cdot; u, v) \in \mathbb{Z}[u, v]\) denotes the Hodge polynomial. Concerning Hodge and motivic zeta functions we refer to e.g. [DL2], [Ro2], [Ve5] for versions with ‘trivial \(\omega\)’; and to [ACLM1], [ACLM2], [Ve4] involving more general \(\omega\). We just mention that, in contrast with topological zeta functions, Hodge and motivic zeta functions can be defined intrinsically as formal power series (in \(T\)) with coefficients determined by the behaviour of the arcs on \(\mathbb{C}^n\) with respect to their intersection with \(f^{-1}\{0\}\) and with \(\text{div}(\omega)\). Then one shows that they are rational functions (in \(T\)) by proving explicit formulae as above in terms of an embedded resolution. However, the fact that we allow (and need) differential forms \(\omega\) with \(\text{supp}(\text{div} \omega) \not\subset f^{-1}\{0\}\), causes some technical complications. In order to avoid these, one can define Hodge and motivic zeta functions by a formula as above in terms of an embedded resolution, and use the Weak Factorization Theorem to show independency of the choice of resolution. (In fact in this paper we will only use differential forms \(\omega\) for which a given embedded resolution of \(f^{-1}\{0\}\) is also an embedded resolution of \(f^{-1}\{0\} \cup \text{div} \omega\).)
1.7. We now explain why we choose not to give details here about Hodge and motivic zeta functions. The point is that, for a given \( f \) and \( \omega \), the motivic zeta function specializes to the Hodge zeta function, which in turn specializes to the topological zeta function. (Note for instance that \( H(\cdot; 1, 1) = \chi(\cdot) \).) In particular, a pole of the topological zeta function will induce a pole of the other two. (The converse is not clear.) The problem that we want to treat here is, given a monodromy eigenvalue \( \lambda \) of \( f \), find a form \( \omega \) such that the zeta function associated to \( f \) and \( \omega \) has a pole ‘inducing \( \lambda \)’. Therefore in this paper we focus on the topological zeta function. We will succeed in proving the desired result for it, implying the analogous result for the ‘finer’ zeta functions.

2. Curves

2.1. We first prove our main result for curves. Ultimately Theorem 2.4 below will be essentially a special case of Theorems 3.5 and 3.6. It is however more precise in the case of non-isolated singularities. We also believe that it is useful to treat the case of curves first. The proof is easier and shorter, and we indicate a fact that is typical for curves.

2.2. In order to construct appropriate differential forms \( \omega \) we use curve germs (sometimes called curvettes) that intersect the exceptional components of an embedded resolution transversely; we quickly recall this notion. Let
\[
\mathbb{A}^2 \xleftarrow{\pi_1} X_1 \xleftarrow{\pi_2} X_2 \xleftarrow{\pi_3} \cdots \xleftarrow{\pi_i} X_i \xleftarrow{\pi_{i+1}} \cdots \xleftarrow{\pi_m} X_m
\]
be a composition \( \pi \) of \( m \) blowing-ups with 0 the centre of \( \pi_1 \), and the centre of all other \( \pi_i \) belonging to the exceptional locus of \( \pi_1 \circ \cdots \circ \pi_{i-1} \). In other words, all centres are points infinitely near to 0. Denote the exceptional curve of \( \pi_i \), as well as its strict transform in \( X_m \), by \( E_i \). A curvette \( C_i \) of \( E_i \) is a smooth curve (germ) on \( X_m \) satisfying \( C_i \cdot E_j = \delta_{ij} \) for all \( j = 1, \cdots, m \). So \( C_i \) intersects \( E_i \) transversely in a point not belonging to other \( E_j \). We denote \( \bar{C}_i := \pi(C_i) \) the image curve (germ) of \( C_i \) in \((\mathbb{A}^2, 0)\).

We guess that the following should be known.

2.3. Proposition. Let \( \pi^* \bar{C}_i = \sum_{j=1}^m a_{ij} E_j + C_i \) for \( i = 1, \cdots, m \). Then the determinant of the \((m \times m)\)-matrix \( (a_{ij}) \) is equal to 1. In particular \( \gcd_{1 \leq i \leq m} \{a_{ij}\} = 1 \) for all \( j \).

Proof. For all \( i, \ell \in \{1, \cdots, m\} \) we have
\[
0 = (\pi^* \bar{C}_i) \cdot E_\ell = \sum_{j=1}^m a_{ij} E_j \cdot E_\ell + \delta_{i\ell}.
\]
In other words, the matrix product \((a_{ij}) \cdot (-E_j \cdot E_\ell)\) is the identity matrix. Since minus the intersection matrix of the \( E_i \) has determinant 1, the same is true for \((a_{ij})\). □

Note. It also follows that \((a_{ij})\) is symmetric, being the inverse of a symmetric matrix.
2.4. Theorem. Let \( f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0) \) be a non-zero polynomial function (germ). Let \( \lambda \) be a monodromy eigenvalue of \( f \) at 0, i.e. \( \lambda \) is an eigenvalue for the action of the local monodromy on \( H^0(F_0, \mathbb{C}) \) or \( H^1(F_0, \mathbb{C}) \). Then there exists a differential 2-form \( \omega \) on \( (\mathbb{C}^2, 0) \) such that \( Z_{\text{top}, 0}(f, \omega; s) \) has a pole \( s_0 \) satisfying \( \exp(2\pi\sqrt{-1}s_0) = \lambda \).

Proof. Take the minimal embedded resolution \( \pi : X \to (\mathbb{C}^2, 0) \) of the curve (germ) \( f^{-1}\{0\} \). Denote the irreducible components of \( \pi^{-1}(f^{-1}\{0\}) \), i.e. the exceptional curves and the components of the strict transform, by \( E_i \) and their multiplicities in the divisors of \( \pi^*f \) and \( \pi^*(dx \wedge dy) \) by \( N_i \) and \( \nu_i - 1 \), respectively. Take for each exceptional curve \( E_i \), a (generic) curvette \( C_i \subset X \). Say \( C_i := \pi(C_i) \) is given by the (reduced) equation \( g_i = 0 \).

We first suppose that \( \lambda \) is a pole of the monodromy zeta function of \( f \) at 0. Say \( \lambda \) is a primitive \( d \)-th root of unity. By A’Campo’s formula there exists an exceptional curve \( E_{j_0} \) with \( d|N_{j_0} \) and \( \chi(E_{j_0}^\circ) < 0 \). So \( E_{j_0} \) intersects at least three times other components \( E_j \).

We associate to each curvette the following multiplicities:

\[
\text{div}(\pi^*g_\ell) = \sum_j a_{\ell j}E_j + C_\ell.
\]

Note that \( a_{\ell j} \neq 0 \) only if \( E_j \) is exceptional. We take a differential form \( \omega \) of the form \( (\prod_\ell g_\ell^{m_\ell})dx \wedge dy \). The multiplicity of \( E_i \) in the divisor of \( \pi^*\omega \) is \( \nu_i - 1 + \sum_\ell a_{\ell i}m_\ell \). So in particular the candidate pole for \( Z_{\text{top}, 0}(f, \omega; s) \) associated to \( E_{j_0} \) is

\[
s_0 = -\frac{\nu_{j_0} + \sum_\ell a_{\ell j_0}m_\ell}{N_{j_0}}.
\]

We will find suitable \( m_\ell \) such that (1) \( \exp(2\pi\sqrt{-1}s_0) = \lambda \), and (2) \( s_0 \) is really a pole.

(1) Say \( \lambda = \exp(-2\pi\sqrt{-1}\frac{b}{d}) \) with \( 1 \leq b < d \) and \( \gcd(b, d) = 1 \). Since \( \gcd_\ell\{a_{\ell j_0}\} = 1 \) by Proposition 2.3, there exist \( m_\ell \in \mathbb{Z} \) such that \( \nu_{j_0} + \sum_\ell a_{\ell j_0}m_\ell = b\frac{N_{j_0}}{d} \). Consequently there exist \( m_\ell \in \mathbb{Z}_{\geq 0} \) satisfying

\[
\nu_{j_0} + \sum_\ell a_{\ell j_0}m_\ell = b\frac{N_{j_0}}{d} \mod N_{j_0},
\]

and we can choose such \( m_\ell \) freely in their congruence class \( \mod N_{j_0} \). For all those \( m_\ell \) clearly \( \exp(2\pi\sqrt{-1}s_0) = \exp(-2\pi\sqrt{-1}\frac{b}{d}) = \lambda \).

(2) The candidate pole for \( Z_{\text{top}, 0}(f, \omega; s) \) associated to a component \( E_i \) of the strict transform is \( -\frac{1}{N_i} \), and is thus different from \( s_0 \) for ‘most’ \( m_\ell \). The candidate pole associated to another exceptional \( E_i \) is \( -\frac{\nu_i + \sum_{\ell \neq i} a_{\ell i}m_\ell}{N_i} \). Suppose that it is equal to \( s_0 \). Then

\[
\frac{\nu_i}{N_i} - \frac{\nu_{j_0}}{N_{j_0}} + \sum_\ell \left( \frac{a_{\ell i}}{N_i} - \frac{a_{\ell j_0}}{N_{j_0}} \right) m_\ell = 0.
\]

(*)
We know that $\det(a_{ij}) \neq 0$ by Proposition 2.3; in particular the vectors $(a_{\ell i})_\ell$ and $(a_{\ell j_0})_\ell$ cannot be dependent. So at least one of the coefficients of the $m_\ell$ in $(\ast)$ is non-zero, i.e. $(\ast)$ is never an empty condition. Consequently ‘most’ sets $(m_\ell)_\ell$ in our allowed lattice satisfy $s_0 \neq -\nu_i + \sum N_i a_{\ell i} m_\ell$ for all $i \neq j_0$.

The residue of $s_0$ (as candidate pole of order 1 for $Z_{\text{top},0}(f,\omega; s)$) is

$$\frac{1}{N_{j_0}} (\chi(E_{j_0}^o) - 1 + \frac{1}{1 + m_{j_0}} + \sum_i \frac{1}{\alpha_i})$$

where $\alpha_i := \nu_i - \frac{\nu_{j_0}}{N_{j_0}} N_i + \sum \ell (a_{\ell i} - a_{\ell j_0} N_i \frac{N_{j_0}}{N_{j_0}}) m_\ell$ for an $E_i$ intersecting $E_{j_0}$. (See Figure 1.)

Since $\chi(E_{j_0}^o) - 1 \neq 0$, this expression is never identically zero as function in the $m_\ell$, and hence nonzero for ‘most’ choices of $(m_\ell)_\ell$.

Secondly, if an eigenvalue $\lambda$ of $f$ at 0 is not a pole of the monodromy zeta function, it must be an eigenvalue on $H^0(F_0, \mathbb{C})$. By (1.1) the order $d$ of $\lambda$ as root of unity must divide all $N_j$ associated to components of the strict transform. But then $d$ divides all $N_i$.

Pick now any exceptional $E_{j_0}$ with $\chi(E_{j_0}^o) < 0$ and proceed as in the previous case to construct a suitable $\omega$ and a pole $s_0$ of $Z_{\text{top},0}(f,\omega; s)$ with $\exp(2\pi \sqrt{-1}s_0) = \lambda$. (There is always an exceptional $E_i$ with $\chi(E_i^o) < 0$, except in the trivial case where $f^{-1}\{0\}$ is smooth or has normal crossings at 0. But then also the theorem is quite trivial, see Example 2.6.) □

![Figure 1](image1)

**2.5. Note.** The zeta functions $Z_{\text{top},0}(f,\omega; s)$ constructed in the proof above can in general have other poles that don’t induce monodromy eigenvalues of $f$. It would be interesting to investigate the validity of the following statement, or its analogue for Hodge and motivic zeta functions.

Let $f : (\mathbb{C}^2,0) \to (\mathbb{C},0)$ be a non-zero polynomial function (germ). There exist regular differential 2-forms $\omega_1, \ldots, \omega_r$ on $(\mathbb{C}^2,0)$ such that

1. if $s_0$ is a pole of a zeta function $Z_{\text{top},0}(f,\omega_i; s)$, then $\exp(2\pi \sqrt{-1}s_0)$ is a monodromy eigenvalue of $f$ at 0, and

2. for each monodromy eigenvalue $\lambda$ of $f$ at 0, there is a differential form $\omega_i$ and a pole $s_0$ of $Z_{\text{top},0}(f,\omega_i; s)$ such that $\exp(2\pi \sqrt{-1}s_0) = \lambda$.

We present some examples of this principle.
2.6. Baby example. (1) Let \( f = x^N \) on \((\mathbb{C}^2, 0)\). The monodromy eigenvalues of \( f \) at 0 are \( \exp(2\pi \sqrt{-1} \frac{b}{N}) \) with \( 1 \leq b \leq N \). Take \( \omega_b := x^{b-1} dx \wedge dy \) for \( b = 1, \ldots, N \). We have \( Z_{\text{top},0}(f, \omega_b; s) = \frac{1}{s+b} \) with unique pole \( s_0 = -\frac{b}{N} \).

(2) Let \( f = x^{dN} y^{dN'} \) on \((\mathbb{C}^2, 0)\) with \( \gcd(N, N') = 1 \). The monodromy eigenvalues of \( f \) at 0 are \( \exp(2\pi \sqrt{-1} \frac{b}{d}) \) with \( 1 \leq b \leq d \). Take \( \omega_b := x^{bN-1} y^{bN'-1} dx \wedge dy \) for \( b = 1, \ldots, d \). We have

\[
Z_{\text{top},0}(f, \omega_b; s) = \frac{1}{(bN + sdN)(bN' + sdN')} = \frac{1}{N'N(b + sd)^2}
\]

with unique pole (of order 2) \( s_0 = -\frac{b}{d} \).

![Diagram](image_url)

**Figure 2**

2.7. Proposition. Let \( f = y^p - x^q \) on \((\mathbb{C}^2, 0)\) with \( 2 \leq p < q \) and \( \gcd(p, q) = 1 \). Take \( \omega_{ij} := x^{i-1} y^{j-1} dx \wedge dy \) for \( 1 \leq i \leq q - 1 \) and \( 1 \leq j \leq p - 1 \).

(1) If \( s_0 \) is a pole of \( Z_{\text{top},0}(f, \omega_{ij}; s) \) for some \( \omega_{ij} \), then \( \exp(2\pi \sqrt{-1} s_0) \) is a monodromy eigenvalue of \( f \) at 0.

(2) If \( \lambda \) is a monodromy eigenvalue of \( f \) at 0, then there is a form \( \omega_{ij} \) and a pole \( s_0 \) of \( Z_{\text{top},0}(f, \omega_{ij}; s) \) such that \( \exp(2\pi \sqrt{-1} s_0) = \lambda \).

Proof. Let \( \pi : X \to (\mathbb{C}^2, 0) \) be the minimal embedded resolution of \( f^{-1}\{0\} \), see Figure 2. We can consider the strict transforms \( C_1 \) and \( C_2 \) of \( C_1 := \{x = 0\} \) and \( C_2 := \{y = 0\} \) as curvettes of the exceptional curves \( E_1 \) and \( E_2 \), respectively. Moreover, the strict transform of \( f^{-1}\{0\} \) can be considered as a curvette \( C_m \) for \( E_m \).

It is well known that the multiplicities of \( E_1, E_2 \) and \( E_m \) in \( \text{div}(\pi^*f) \) are \( p, q \) and \( pq \), respectively, and that the multiplicity of \( E_m \) in \( \text{div}(\pi^*dx \wedge dy) \) is \( p + q - 1 \). Since the matrix \( (a_{ij}) \) is symmetric (where we use the notation of (2.3)) we have \( a_{1m} = a_{m1} = p \) and \( a_{2m} = a_{m2} = q \). Consequently the multiplicity of \( E_m \) in \( \text{div}(\pi^*\omega_{ij}) \) is \( p + q - 1 + (i - 1)p + (j - 1)q = ip + jq - 1 \).
The only monodromy eigenvalue on $H^0(F_0, \mathbb{C})$ is 1, and then by A’Campo’s formula the eigenvalues on $H^1(F_0, \mathbb{C})$ are

$$\exp\{-2\pi\sqrt{-1}(\frac{i}{q} + \frac{j}{p})\}$$

for $1 \leq i \leq q - 1$ and $1 \leq j \leq p - 1$.

On the other hand, for example by [Ve2], we have already that $-1$ and $-\frac{ip+jq}{pq} = -(\frac{j}{q} + \frac{i}{p})$ are the only candidate poles of $Z_{top,0}(f, \omega_{ij}; s)$. In fact they really are poles which immediately implies (1) and (2).

An elegant way to check this is the formula in [Ve3, Theorem 3.3] which yields the following compact expression (this formula remains valid in the context of arbitrary differential forms $\omega$):

$$Z_{top,0}(f, \omega_{ij}; s) = \frac{1}{ip+jq+spq}(-1 + \frac{1}{1+s} + \frac{q}{i} + \frac{p}{j})$$

$$= \frac{jq + ip + (jq + ip - ij)s}{(ip + jq + spq)(1+s)}.$$

\[\Box\]

2.8. Example. Let $f = (y^2-x^3)^2-x^6y$ on $(\mathbb{C}^2, 0)$. This is one of the simplest irreducible singularities with two Puiseux pairs. The minimal embedded resolution of $f^{-1}\{0\}$ is described in Figure 3. The numbers $(\nu_i, N_i)$ denote as usual $1+$ (the multiplicity of $E_i$ in $\text{div}(\pi^*dx \wedge dy)$) and the multiplicity of $E_i$ in $\text{div}(\pi^*f)$, respectively.

By A’Campo’s formula the monodromy eigenvalues of $f$ at 0 are 1 and all primitive roots of unity of order 6, 10, 12 and 30. Take $\omega_{ij} = x^{i-1}y^{j-1}dx \wedge dy$ for $i, j \geq 1$. We checked that the statement in (2.5) is valid for example for the sets of differential forms $\{\omega_{ij}|1 \leq i \leq 5 \text{ and } 1 \leq j \leq 3\} \cup \{\omega_{34}\}$ and $\{\omega_{ij}|1 \leq i \leq 3 \text{ and } 1 \leq j \leq 5\} \setminus \{\omega_{24}\}$.

![Figure 3](image-url)

3. Arbitrary dimension

3.1. First we construct a higher dimensional generalization of the notion of curvette such that an analogue of Proposition 2.3 is still valid.
Let $X_0$ be a smooth quasi-projective (complex) variety of dimension $n$ and let

$$X_0 \xleftarrow{\pi_1} X_1 \xleftarrow{\pi_2} X_2 \xleftarrow{\pi_3} \cdots \xleftarrow{\pi_i} X_i \xleftarrow{\pi_{i+1}} \cdots \xleftarrow{\pi_m} X_m$$

be a composition $\pi$ of $m$ blowing-ups $\pi_i$ with smooth irreducible centre $Z_{i-1}(\subset X_{i-1})$ having normal crossings with the exceptional locus of $\pi_1 \circ \cdots \circ \pi_{i-1}$. Denote the exceptional locus of $\pi_i$, as well as its consecutive strict transforms, by $E_i$.

Recall that, when created, $E_i$ has the structure of a $\mathbb{P}^k$-bundle $E_i \xrightarrow{\pi_i} Z_{i-1}$, where $k = n - 1 - \dim Z_{i-1}$. We have $\text{Pic} E_i \cong \mathbb{Z}L_i \oplus p_i^*\text{Pic} Z_{i-1}$, where $L_i$ is the divisor class corresponding to the canonical sheaf $\mathcal{O}_{E_i}(1)$ on $E_i$. The self-intersection $E_i^2$ of $E_i$ on $X_i$, considered in $\text{Pic} E_i$, is equal to $-L_i$ [Ha, Theorem II 8.24]. (When $Z_{i-1}$ is a point, $L_i$ is just the hyperplane class on $E_i \cong \mathbb{P}^{n-1}$.)

### 3.2. Proposition

One can construct consecutively for $j = 1, \cdots, m$ a smooth hypersurface $C_j$ on $X_j$ such that

1. $C_j$ has normal crossings with $E_1 \cup E_2 \cup \cdots \cup E_j$, with (the strict transform of) previously created $C_1, \cdots, C_{j-1}$, and with the next centre of blowing-up $Z_j$ (and such that $Z_j \not\subset C_j$);
2. in $\text{Pic} E_j$ we have $C_j \cap E_j = L_j + p_j^*B_j$ for some $B_j \in \text{Pic} Z_{j-1}$;
3. denoting $\tilde{C}_j := \pi_j(C_j) \subset X_{j-1}$, we have $\pi_j^*\tilde{C}_j = E_j + C_j$ in $\text{Pic} X_j$. So the multiplicity of $\tilde{C}_j$ along $Z_{j-1}$ is 1.

Note that by (1) the strict transforms in $X_m$ of all $E_j$ and $C_j$ form a normal crossings divisor.

4. Given another hypersurface $H$ on $X_m$ having normal crossings with $E_1 \cup \cdots \cup E_m$, we can choose $C_1, \cdots, C_m$ such that furthermore $H$ and all $E_j$ and $C_j$ form a normal crossings divisor on $X_m$.

**Proof.** Fix a $j \in \{1, \cdots, m\}$. Consider the sheaf $\mathcal{O}_{X_j}(1)$ on $X_j$, associated to the blowing-up map $\pi_j : X_j \to X_{j-1}$. We choose an ample invertible sheaf $\mathcal{L}$ on $X_{j-1}$. By [Ha, II Proposition 7.10] we have for some $k > 0$ that the sheaf $\mathcal{O}_{X_j}(1) \otimes \pi_j^*\mathcal{L}^k$ on $X_j$ is very ample over $X_{j-1}$. So its global sections generate a base point free linear system on $X_j$; we take $C_j$ as a general element of this linear system. By Bertini’s theorem $C_j$ satisfies (1). The inverse image on $X_m$ of this linear system is still base point free. So a general element will also satisfy the extra condition in (4).

We now verify that the intersection product $C_j \cdot E_j$, considered in $\text{Pic} E_j$, is of the form $L_j + p_j^*B_j$, which yields (2). Denote $\beta : E_j \hookrightarrow X_j$ and $\alpha : Z_{j-1} \hookrightarrow X_{j-1}$. Since the divisor class corresponding to $\mathcal{O}_{X_j}(1)$ on $X_j$ is $-E_j$ we have

$$C_j \cdot E_j = (-E_j + \pi_j^*(\cdots)) \cdot E_j = -E_j^2 + \beta^*\pi_j^*(\cdots)$$

$$= L_j + p_j^*\alpha^*(\cdots).$$

Finally we verify (3). Certainly $\pi_j^*\tilde{C}_j = \mu E_j + C_j$ where $\mu$ is the multiplicity of $\tilde{C}_j$ along $Z_{j-1}$. Intersecting with $E_j$ yields $\beta^*\pi_j^*\tilde{C}_j = \mu(-L_j) + C_j \cdot E_j$, and hence by the previous calculation $p_j^*(\cdots) = -\mu L_j + L_j + p_j^*(\cdots)$. So indeed $\mu = 1$. $\square$
3.3. The $C_j$ constructed above satisfy an analogous statement as Proposition 2.3 for curves. For the proof however we need another approach.

**Proposition.** We use the notation of (3.1) and (3.2). Denote also $ar{C}_i := \pi(C_i) \subset X_0$ and $\pi^* \bar{C}_i = \sum_{j=1}^m a_{ij} E_j + C_i$ for $i = 1, \cdots, m$. Then the determinant of the $(m \times m)$-matrix $(a_{ij})$ is equal to 1. In particular $\gcd_{1 \leq i \leq m} \{a_{ij}\} = 1$ for all $j$.

**Proof.** We proceed by induction on $m$. When $m = 1$ we have $\pi^* \bar{C}_1 = E_1 + C_1$ by Proposition 3.2(3), and so indeed $a_{11} = 1$. Take now $m > 1$. By the same proposition we have $\pi^* \bar{C}_m = E_m + C_m$, where $\bar{C}_m = \pi_m(C_m)$. Say $Z_{m-1}$ is contained in precisely $E_j, j \in J(\subset \{1, \cdots, m-1\})$. Then

$$\pi^* \bar{C}_m = \pi^*_m \left( \sum_{j=1}^{m-1} a_{mj} E_j + \bar{C}_m \right) = \left( \sum_{j \in J} a_{mj} + 1 \right) E_m + (\cdots),$$

saying that $a_{mm} = \sum_{j \in J} a_{mj} + 1$. On the other hand, since $Z_{m-1}$ is not contained in (the strict transform of) any $C_1, C_2, \cdots, C_{m-1}$, we have $a_{im} = \sum_{j \in J} a_{ij}$ for $i = 1, \cdots, m-1$. Hence

$$\det(a_{ij})_{1 \leq i \leq m} = \begin{vmatrix} a_{11} & \cdots & a_{1,m-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m-1,1} & \cdots & a_{m-1,m-1} & 0 \\ a_{m1} & \cdots & a_{m,m-1} & 1 \end{vmatrix} = \det(a_{ij})_{1 \leq i \leq m-1} = 1,$$

where the last equality is the induction hypothesis. □

**Note.** In contrast to the curve case, the matrix $(a_{ij})$ is in general not symmetric in higher dimensions, even when all $\pi_i$ are point blowing-ups. Take for example $\dim X_0 = 3$, $Z_1$ a point on $E_1$, and $Z_2$ a point on $E_1 \cap E_2$. Then

$$(a_{ij}) = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix}.$$

3.4. We now present higher dimensional versions of Theorem 2.4. We first look at zeroes or poles of monodromy zeta functions. According to Lemma 1.3, this way we treat in fact all monodromy eigenvalues. For isolated singularities we present a finer result.
3.5. Theorem. Let \( f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0) \) be a non-zero polynomial function (germ). Let \( \lambda \) be a zero or a pole of the monodromy zeta function of \( f \) at 0. Then there exists a differential \( n \)-form \( \omega \) on \( (\mathbb{C}^n, 0) \) such that \( Z_{\text{top}, 0}(f, \omega; s) \) has a pole \( s_0 \) satisfying \( \exp(2\pi \sqrt{-1}s_0) = \lambda \).

Proof. Let \( f : X_0(\subset \mathbb{C}^n) \rightarrow \mathbb{C} \) be a relevant representative of \( f \) in the sense that some embedded resolution of \( f^{-1}\{0\} \subset X_0 \) only has exceptional components that intersect the inverse image of 0. Take such an embedded resolution \( \pi : X_m \rightarrow X_0 \), which is a composition of \( m \) blowing-ups as in (3.1). Slightly abusing notation, \( E_i \) can now denote an exceptional component of \( \pi \) or an irreducible component of the strict transform. As usual \( N_i \) and \( \nu_i \) are the multiplicities of \( E_i \) in the divisor of \( \pi^*f \) and \( \pi^*(dx_1 \wedge \cdots \wedge dx_n) \), respectively.

Say \( \lambda \) is a \( d \)-th root of unity. By A’Campo’s formula there exists an exceptional component \( E_{j_0} \) with \( d | N_{j_0} \) and \( \chi(E_{j_0}^0 \cap \pi^{-1}\{0\}) \neq 0 \).

We take for \( i = 1, \ldots, m \) smooth hypersurfaces \( C_i \) as in Proposition 3.2 (considered in \( X_m \)); as extra hypersurface in 3.2(4) we take the strict transform of \( f^{-1}\{0\} \). Say the images \( \tilde{C}_i \) in \( X_0 \) of the \( C_i \) have (reduced) equation \( g_i = 0 \). As before we denote \( \pi^*\tilde{C}_i = \sum_{i=1}^m a_{ij}E_j + C_i \); the \( (a_{ij}) \) satisfy Proposition 3.3. We take for the moment a differential form \( \omega \) of the form \( (\prod_{\ell} g_i^{m_{\ell}}) dx_1 \wedge \cdots \wedge dx_n \). The multiplicity of \( E_i \) in the divisor of \( \pi^*\omega \) is \( \nu_i - 1 + \sum_{\ell} a_{i\ell}m_{\ell} \).

The candidate pole for \( Z_{\text{top}, 0}(f, \omega; s) \) associated to \( E_{j_0} \) is \( s_0 = -\frac{\nu_{j_0} + \sum_{\ell} a_{i\ell}m_{\ell}}{N_{j_0}} \). As in the proof of Theorem 2.4 we want to find suitable \( m_{\ell} \). Completely analogously as in that proof, this time using Proposition 3.3,

1. we find a lattice of nonnegative \( m_{\ell} \) ‘mod \( N_{j_0} \)’ such that \( s_0 \) satisfies \( \exp(2\pi \sqrt{-1}s_0) = \lambda \), and

2. for ‘most’ such \( m_{\ell} \) the candidate poles associated to other \( E_i \) are different from \( s_0 \).

The argument showing that \( s_0 \) is really a pole for suitable such \( m_{\ell} \) is more subtle now. We introduce some notation to describe the residue of \( s_0 \).

Let \( C_{\ell, \ell} \in J_0 \), be the hypersurfaces \( C_i \) that intersect \( E_{j_0} \cap \pi^{-1}\{0\} \) (in \( X_m \)). Denote \( C_{\emptyset}^0 := ((\cap_{j \in J} C_j) \setminus (\cup_{j \in J_0} C_j)) \cap (E_{j_0}^0 \cap \pi^{-1}\{0\}) \) for \( J \subset J_0 \); in particular \( C_{\emptyset}^0 = (E_{j_0}^0 \cap \pi^{-1}\{0\}) \setminus \cup_{i \in J_0} C_i \). These \( C_{\emptyset}^0 \) form a locally closed stratification of \( E_{j_0}^0 \cap \pi^{-1}\{0\} \). The residue of \( s_0 \) is of the form

\[
\frac{1}{N_{j_0}} \left( \chi(C_{\emptyset}^0) + \sum_{\emptyset \neq J \subset J_0} \chi(C_{\emptyset}^0) \prod_{j \in J} \frac{1}{1 + m_j} + \text{contribution of } (E_{j_0} \setminus E_{j_0}^0) \cap \pi^{-1}\{0\} \right),
\]

the last contribution also being a rational function in the \( m_{\ell} \) of negative degree. As in the proof of Theorem 2.4, if \( \chi(C_{\emptyset}^0) \neq 0 \), this expression is never identically zero as function in the \( m_{\ell} \), and so non-zero for ‘most’ choices of \( (m_{\ell})_{\ell} \).

We don’t see how to exclude the theoretical possibility that this expression is identically zero (with then necessarily \( \chi(C_{\emptyset}^0) = 0 \)). In this case we will adapt our choice of \( \omega \) to be sure to have the desired pole.

For each \( \ell \) in \( J_0 \) we construct, as in Proposition 3.2, not just one hypersurface \( C_{\ell} \), but several ones \( C_{\ell_1}, C_{\ell_2}, \ldots, C_{\ell_t} \), all general enough elements in the linear system that was
considered there. Then still all \( E_i, C_{\ell j} \), and other \( C_\ell \) will form a normal crossings divisor on \( X_m \). Say \( g_{\ell k} = 0 \) is the equation of the image of \( C_{\ell k} \) in \( X_0 \). Now we take \( \omega \) of the form \( \prod_{\ell \in J_0} (\prod_{k=1}^{m_{\ell k}} g_{\ell k}^m) \prod_{\ell \notin J_0} g_{\ell k}^m dx_1 \wedge \cdots \wedge dx_n \) such that for \( \ell \in J_0 \) the sum \( \sum_{k=1}^{m_{\ell k}} m_{\ell k} \) is an allowed \( m_{\ell} \mod N_{J_0} \), as before. The candidate pole \( s_0 \) for \( Z_{top,0}(f, \omega; s) \) associated to \( E_{j_0} \) is as above, it still satisfies \( \exp(2\pi \sqrt{-1} s_0) = \lambda \), and for ‘most’ such \( m_{\ell k} \) and \( m_{\ell} \) the candidate poles associated to other \( E_i \) are different from \( s_0 \). We now verify that for some \( t \) the expression for the residue of \( s_0 \) is not identically zero as function in the \( m_{\ell k} \) and \( m_{\ell} \).

Denote \( L_k := (\cup_{\ell \in J_0} C_{\ell k}) \cap (E_{j_0}^o \cap \pi^{-1}\{0\}) \) for \( k = 1, \cdots, t \). Since all \( C_{\ell k} \) all general elements we have that all \( \chi(L_k) \) are equal, that also the \( \chi(L_k \cap L_{k'}) \) are equal for all \( k < k' \), and more generally that the \( \chi(L_{k_1} \cap L_{k_2} \cap \cdots \cap L_{k_s}) \) are equal for all \( 1 \leq k_1 < k_2 < \cdots < k_s \leq t \). The residue of \( s_0 \) is of the form

\[
\frac{1}{N_{J_0}} \left( \chi((E_{j_0}^o \cap \pi^{-1}\{0\}) \setminus \cup_{k=1}^t L_k) + \cdots \right),
\]

where the other terms form a rational function of negative degree in the \( m_{\ell k} \) and the \( m_{\ell} \). If \( \chi((E_{j_0}^o \cap \pi^{-1}\{0\}) \setminus \cup_{k=1}^t L_k) \neq 0 \), then this residue is not identically zero and we are done. Finally we show that this must be the case for some \( t \).

Because of the normal crossings property we have \( \cap_{k=1}^T L_k = \emptyset \) for some \( T \leq n \). Suppose that \( \chi((E_{j_0}^o \cap \pi^{-1}\{0\}) \setminus \cup_{k=1}^t L_k) = 0 \) for all \( t = 1, \cdots, T \). These \( T \) conditions can be rewritten as

\[
\begin{aligned}
\chi(E_{j_0}^o \cap \pi^{-1}\{0\}) - \chi(L_1) &= 0 \\
\chi(E_{j_0}^o \cap \pi^{-1}\{0\}) - 2\chi(L_1) + \chi(L_1 \cap L_2) &= 0 \\
\chi(E_{j_0}^o \cap \pi^{-1}\{0\}) - 3\chi(L_1) + 3\chi(L_1 \cap L_2) - \chi(L_1 \cap L_2 \cap L_3) &= 0 \\
&\cdots \\
\chi(E_{j_0}^o \cap \pi^{-1}\{0\}) - (T-1)\chi(L_1) + \cdots + (-1)^{T-1}\chi(\cap_{k=1}^{T-1} L_k) &= 0 \\
\chi(E_{j_0}^o \cap \pi^{-1}\{0\}) - T\chi(L_1) + \cdots + (-1)^{T-1}T\chi(\cap_{k=1}^{T-1} L_k) + (-1)^T \cdot 0 &= 0.
\end{aligned}
\]

One easily verifies that the \((T \times T)\)-determinant of coefficients of this homogeneous linear system of equations in the \( \chi(\cdots) \) is nonzero. Hence in particular we should have \( \chi(E_{j_0}^o \cap \pi^{-1}\{0\}) = 0 \), contradicting our choice of \( E_{j_0} \). \( \square \)

3.6. Theorem. Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be a nonzero polynomial function (germ).

(a) Let \( \lambda \) be a monodromy eigenvalue of \( f \) at 0. Then there exists a differential \( n \)-form \( \omega \) and a point \( P \) in a neighbourhood of 0 such that \( Z_{top, P}(f, \omega; s) \) has a pole \( s_0 \) satisfying \( \exp(2\pi \sqrt{-1} s_0) = \lambda \). Moreover, \( P \) can be chosen as a generic point in the set \( \Sigma \) that was introduced after Lemma 1.3.

(a') If the eigenvalue \( \lambda \) appears only at 0, then there exists a differential \( n \)-form \( \omega \) such that \( Z_{top,0}(f, \omega; s) \) has a pole \( s_0 \) satisfying \( \exp(2\pi \sqrt{-1} s_0) = \lambda \).

(b) Suppose that \( f^{-1}(\{0\}) \) has an isolated singularity at 0, and let \( \lambda \) be a monodromy eigenvalue of \( f \) at 0. Then there exists a differential \( n \)-form \( \omega \) such that \( Z_{top,0}(f, \omega; s) \) has a pole \( s_0 \) satisfying \( \exp(2\pi \sqrt{-1} s_0) = \lambda \).
Proof. Parts (a) and (a’) follow immediately from Theorem 3.5 and Lemma 1.3.

Part (b) is a special case of (a’) for $\lambda \neq 1$. It could however happen in (b) that $\lambda = 1$ is not a zero or a pole of the monodromy zeta function of $f$ at 0 (when $n$ is even). In that case we pick any exceptional $E_{j_0}$ with $\chi(E_{j_0}) \neq 0$, and we proceed as in the proof of Theorem 3.5 to construct a suitable $\omega$ and a pole $s_0$ of $Z_{top,0}(f, \omega; s)$ with $\exp(2\pi \sqrt{-1} s_0) = 1$. (Note that the constructed $s_0$ is in this case indeed an integer.) By A’Campo’s formula there is always such an exceptional $E_{j_0}$, except when $n$ is even and the characteristic polynomial $P_{n-1}(t) = t - 1$. E.g. by [AGLV, page 70] this implies that $f$ has a so-called non-degenerate or Morse singularity at 0 (i.e., $f$ is in local analytic coordinates of the form $y_1^2 + y_2^2 + \cdots + y_n^2$). In this easy special case one has

$$Z_{top,0}(f, dx_1 \wedge \cdots \wedge dx_n; s) = \frac{n}{(1+s)(n+2s)} .$$

\[\Box\]

3.7. As in the curve case the zeta functions constructed in the proof of Theorem 3.5 can in general have other poles that don’t induce monodromy eigenvalues of $f$, and it would be interesting to study in arbitrary dimension the validity of the ‘principle’ in Note 2.5. We present an example below.

3.8. Example. Let $f = x^d + y^d + z^d$ on $(\mathbb{C}^3, 0)$ with $d \geq 3$. Blowing up the origin yields an embedded resolution $\pi$ of $f^{-1}\{0\}$. The exceptional surface $E \cong \mathbb{P}^2$ has multiplicities 2 and $d$ in $\text{div}(\pi^*dx \wedge dy \wedge dz)$ and $\pi^*f$, respectively. It intersects the strict transform in a smooth curve $D$ of degree $d$ and hence with Euler characteristic $3d - d^2$. By A’Campo’s formula the monodromy zeta function of $f$ at 0 is

$$\zeta_{f,0}(t) = (t^d - 1)^{d^2 - 3d + 3},$$

and the monodromy eigenvalues of $f$ at 0 are precisely all $d$-th roots of unity. Take $\omega_i := x^{i-1}dx \wedge dy \wedge dz$ for $1 \leq i \leq d$. The strict transform of $\{x = 0\}$ intersects $E$ in a line; this line intersects $D$ transversely in $d$ points. Hence

$$Z_{top,0}(f, \omega_i; s) = \frac{1}{(2+i) + sd} \left( \frac{(d-1)^2}{i} + \frac{2d-d^2}{1+s} + \frac{d}{i(1+s)} \right) = \frac{(i(d-1)^2 + 2 - d)s + (2+i)}{i((2+i) + sd)(1+s)} .$$

When $i \neq d - 2$, one easily verifies that the two candidate poles $-1$ and $-\frac{2+i}{d}$ are really poles. When $i = d - 2$ we have

$$Z_{top,0}(f, \omega_{d-2}; s) = \frac{1 + (d-2)^2s}{(d-2)(1+s)^2}$$

and $-1$ is a pole (of order 2 if $d > 3$ and of order 1 if $d = 3$).

So the set of differential forms $\{\omega_i | 1 \leq i \leq d\}$ satisfies the analogous principle as in (2.5). We can even delete $\omega_{d-2}$ from this set.
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