On explicit solutions for completely integrable classical Calogero-Moser systems in external fields

D.V.Meshcheryakov, T.D.Meshcheryakova

Physics Department, Moscow State University,
Leninskie gory, 119899, Moscow, Russia

Abstract

Explicit solutions for one completely-integrable system of Calogero-Moser type in external fields are found in case of three and four interacting particles. Relation between coupling constant, initial values of coordinates and time of falling into the singularity of potential is derived.

1. Introduction

Completely integrable finite-dimensional many-particle systems still attract considerable attention[1-6]. By now different methods of solving equations of motion have been developed for many potentials giving the property of complete integrability [1-3]. One of such methods for Calogero-
Moser systems in external fields was found in [5]. Here we apply this method in some special cases of two-parameter Calogero-Moser type potential to obtain explicit solutions of the equations of motion. We investigate three and four particle systems. We analyze explicit solutions and estimate the time of falling particles into the singularities of the potentials for different initial conditions.

Complete integrability is guaranteed by the existence of $N$ involutory constants of motion. One of these constants is Hamilton function which has the form

$$H = \sum_{i=1}^{N} \left( \frac{p_i^2}{2} + W(x_i) \right) + \sum_{i>j}^{N} V(x_i - x_j)$$ (1)

System (1) is completely integrable when and only when the external field potential $W(x)$ and pair interaction potential $V(x)$ are the following:

$$V(x) = \frac{a}{x^2}, \quad W(x) = \gamma_1 x^4 + \gamma_2 x^2 + \gamma_3 x$$ (2)

where $a, \gamma_i, i = 1, 2, 3$ are arbitrary real constants.

In [5] a connection of N-particle system with Hamiltonian (1) and potentials (2) with non-homogeneous Burgers–Hopf equation was established. This connection led to the method of solving equations of motion for systems (1)-(2). If the constants defining potentials satisfy the following constraints

$$a = -A^2, \quad \gamma_1 = -\frac{\alpha^2}{2}, \quad \gamma_2 = -\alpha \beta, \quad \gamma_3 = -\alpha A(N - 1)$$ (3)

where $A, \alpha, \beta$ are arbitrary real constants, then $N$ coordinates of the particles are $N$ real zeros of the following function
\[ \phi(x, t) = \sum_{k=0}^{N} x^k [e^{x p(t \hat{T})} \tilde{c}(0)]_k \]  

(4)

where

\[ \hat{T}_{00} = -\beta, \quad \hat{T}_{02} = -a, \quad \hat{T}_{N-1N} = -N\beta, \quad \hat{T}_{N-1N-2} = 2\alpha, \quad \hat{T}_{NN-1} = \alpha \]

\[ \hat{T}_{kk-1} = (N + 1 - k)\alpha, \quad \hat{T}_{kk+1} = -(k + 1)\beta, \quad \hat{T}_{kk+2} = \frac{(k + 1)(k + 2)}{2} a, \quad 1 \leq k \leq N - 2 \]  

(5)

and

\[ c_{N-k}(0) = (-1)^k \sum_{1 \leq \lambda_1 < \ldots < \lambda_k \leq N} x_{\lambda_1}(0) \ldots x_{\lambda_k}(0), \quad c_{N}(0) = 1. \]

Other elements of the matrix \( \hat{T} \) are zeros. In these expressions \( x_i(0), i = 1, \ldots, N \) are the initial coordinates of \( N \) particles.

2. Two-parameter class of potentials.

Further we apply the method briefly described to obtain explicit solutions for the following two-parameter class of potentials of the type (2)-(3). Consider the following constraints

\[ \beta = 0, \quad \alpha = -A^2k \]  

(6)

where \( k, A \) are arbitrary constants. The case \( k = 1 \) was considered in [6]. Therefore we have two parameters \( A \) and \( k \) in the potentials \( V(x), W(x) \). Constant \( A \) represents the intensity of pair interaction, so it can be called a coupling constants for the system in question. Parameter \( k \) defines the ratio of the pair interaction intensity to the intensity of the external field. At
$k \rightarrow 0$ particles are influenced by the pair interaction only, and at $k \rightarrow \infty$ we have no pair interaction. In this case particles are influenced by the external field only.

3. Two particles

Here we apply the method to the simplest case of two particles. For $N = 2$ from (4)-(5) one gets $\hat{T}^3 = (-2a^3k^2)E$, where $E$ is a unit matrix. In this case the row in (4) can be summed up and the coordinates are the two solutions of the following equation

$$2 \sum_{k=0}^{2} \left( \frac{x}{-a(2k^2)^{1/3}} \right)^k S_k(\theta)\tilde{T}^k\bar{c}(o)|_k = 0$$

where $a\tilde{T} = \hat{T}$, $\theta = -at(2k^2)^{1/3}$,

$$S_0(\theta) = \frac{1}{3}(e^\theta + 2e^{-\theta/2}cos(\sqrt{3}2\theta)),$$

$$S_1(\theta) = \frac{1}{3}(e^\theta - 2e^{-\theta/2}cos(\sqrt{3}2\theta + \frac{\pi}{3}))$$

$$S_2(\theta) = \frac{1}{3}(e^\theta - 2e^{-\theta/2}cos(\sqrt{3}2\theta - \frac{\pi}{3}))$$

At $k = 1$ the asymptotic form of this equation for arbitrary initial values is the following $x^2 - 2^{1/3}x + 2^{2/3} = 0$. Thus we come to the conclusion that in some finite time interval $t_0$ the particles fall into the singularity of the pair interaction. However at $k \neq 1$ for arbitrary initial values $x_1(0)$ and $x_2(0)$ there exist such values of parameter $k$ that particles do not fall into the singularity of the potential. For example, in case $x_1(0) = -c$, $x_2(0) = c$ particles do not fall into the singularity if the value of $k$ satisfies the following inequality
\[ \frac{1}{k^2} > c^2 - 1 \]

For arbitrary initial conditions the system does not collapse under the condition that both the initial coordinates and \( k \) satisfy the following inequality

\[
k^3 8x_1(0)x_2(0)(x_1(0) + x_2(0)) + k^2 4(x_1(0)^2x_2(0)^2 -
-(x_1(0)+x_2(0)) - 1) + k^4(x_1(0)+x_2(0)) + 4(x_1(0)x_2(0) - 1) - (x_1(0)+x_2(0))^2 < 0
\]

This inequality means that at some \( k \) the attraction of the particles due to the pair potential is compensated by the repelling due to the external field so that the particles do not fall into the singularity in any finite time interval.

4. Three particles

In this section we consider \( N = 3 \) system. Under constraints (6) one can get \( \hat{T}^4 = 0 \), and the algebra of T-matrix becomes nilpotent. Further we consider symmetrical initial conditions

\[ x_1(0) = -x_0, \quad x_2(0) = 0, \quad x_3(0) = x_0 \quad (8) \]

Suppose \( r = 1/(-A^2t) \). Then (8) has the following Cardano form:

\[ y^3 + uy + q = 0, \quad x = y - \frac{r^2(3 + x_0^2r)}{3Z} \quad (9) \]

where

\[ Z = r^3 - x_0^2k^2r - k^2, \quad u = \frac{-r^4(x_0^4k^2 + 9r + 3x_0^2r^2)}{3Z^2}, \]
\[ q = \frac{k \sum_{m=0}^{8} C_m r^m}{Z^3}, \]

\[ C_0 = k^4, \quad C_1 = 3k^4 x_0^2, \quad C_2 = 3k^4 x_0^4, \quad C_3 = k^2 (k^2 x_0^6 - 3), \]

\[ C_4 = -6k^2 x_0^2, \quad C_5 = -3k^2 x_0^4, \quad C_6 = 4 - \frac{2}{27} k^2 x_0^6, \]

\[ C_7 = 4x_0^2, \quad C_8 = \frac{2}{3} x_0^4 \]

Solutions of (9) are determined by the value of the discriminant \( Q = -108((u/3)^3 + (q/2)^2) \) [8]. At \( 0 \leq t < t_0 \) we thus have \( Q > 0 \) and as a consequence three real roots being the three coordinates of interacting particles:

\[ y_1 = N + M, \quad y_{2,3} = -\frac{N + M}{2} \pm i\sqrt{3} \frac{N - M}{2} \]

where

\[ N = (-\frac{q}{2} + \sqrt{\frac{-Q}{108}})^\frac{1}{3}, \quad M = (-\frac{q}{2} - \sqrt{\frac{-Q}{108}})^\frac{1}{3}, \]

provided \( MN = -u/3 \). At \( t = t_0 \) we have \( Q = 0 \), which leads to two equal coordinates of two particles. In other words at this moment two of the particles fall into the singularity of the potential of pair interaction.

For arbitrary initial conditions one can find from (3)–(5) the asymptotic form of equation (9) to be (provided \( A \neq 0 \)):

\[ (1 - kx_1(0)x_2(0)x_3(0))(x^3 - \frac{1}{k}) = 0. \]

From this relation one can easily conclude that the system in question collapses in finite time interval for any initial conditions.
Consider a limiting procedure which enables us to find exact expression for $t_0$. Suppose $A = z$, $k = 1/z^2$. Suppose $z \to 0$. This limiting procedure corresponds to infinitely weak pair interaction at finite external field. In this case it is easy to find $t_0$ explicitly. Consider initial conditions of the form $x_1(0) = x_1, x_2(0) = x_2, x_3(0) = 0$. Without pair interaction the third coordinate remains equal to zero. Two other solutions are the following

$$U(t, k, x_1, x_2) = t^2 - \frac{12^\frac{1}{3}(x_1 + x_2)}{kx_1x_2}t + \frac{12^\frac{2}{3}}{k^2x_1x_2}$$

$$x_{1,2}(t) = \frac{1}{U(t, k, x_1, x_2)} - \frac{12^\frac{1}{3}}{k}(t - \frac{12^\frac{1}{3}}{kx_{2,1}})$$

From these expressions one can get for $t_0$

$$t_0 = \left(\frac{3}{2}\right)^\frac{1}{3}\left|\frac{(x_1 + x_2)(1 - \frac{|x_1 - x_2|}{|x_1 + x_2|})}{kx_1x_2}\right|$$

Using the discriminant one can get for general initial conditions the following upper estimate for $t_0$:

$$t_0 = \frac{1}{4A^2(k + k^{-1/4})}\left|((1 - k) + 9(x_1(0)x_2(0) + x_1(0)x_3(0) + x_2(0)x_3(0)) + 3(k - 1)x_1(0)x_2(0)x_3(0) - 21(k - 1)(x_1(0) + x_2(0) + x_3(0))\frac{1}{kx_1(0)x_2(0)x_3(0)}\right|$$

This estimate is valid for $x_i(0) \neq 0$, $i = 1, 2, 3$. It is easy to see that $t_0$ increases as $A^2$ and/or $k$ decrease. For some initial conditions there exists
local maximum of $t_0$ at finite $k > 0$. For example, for $x_1(0) = -x_0$, $x_2(0) = x_1$, $x_3(0) = x_0$, if

$$x_1 > \frac{1}{129} \frac{24x_0^2(12 + \sqrt{15}) - 43}{x_0^2 + 7}$$

or

$$x_1 < \frac{1}{129} \frac{24x_0^2(12 - \sqrt{15}) - 43}{x_0^2 + 7}$$

there is a local maximum of $t_0$ at $k_0$. We do not write down the exact expression due to its complexity. A simple upper estimate reads as follows

$$k_0 = \frac{1}{8W} \left( 9(W - 8x_0^2) + \sqrt{129(W^2 - 13x_0^2W + 41x_0^4)} \right)$$

where $W = 1 + 3x_1(7 + x_0^2)$.

5. Four particles

For $N = 4$ systems equations (4)-(5) lead to $\hat{T}^5 = 18k^2a^3\hat{T}^2$. In this case it is possible to sum up the row in (4). After summation equation (7) turns into the following expression

$$\sum_{l=0}^{2} x^l S_l(\theta) \bar{T}^l \bar{c}(o)_l = 0$$

where $a\bar{T} = \hat{T}$, $\theta = -at(2)^{1/3}$,

$$S_0(\theta) = 1, \quad S_1(\theta) = (18)^{-1/3}k^{-2/3}\theta,$$

$$S_2(\theta) = \frac{9}{121^{1/3}k^{4/3}}(e^\theta - 2e^{-\theta/2}\cos(\frac{\sqrt{3}}{2}\theta - \frac{\pi}{3})) \quad S_3(\theta) = \frac{1}{54k^2}(e^\theta + 2e^{-\theta/2}\cos(\frac{\sqrt{3}}{2}\theta) - 3),$$

$$S_4(\theta) = \frac{(18)^{-1/3}}{54k^{8/3}}(e^\theta - 2e^{-\theta/2}\cos(\frac{\sqrt{3}}{2}\theta + \frac{\pi}{3}) - 3\theta)$$  \quad (13)
Therefore the four coordinates of interacting particles at any given time are defined by the real roots of

\[ \sum_{l=0}^{4} R_l x^l = 0. \]  \hspace{1cm} (14)

For initial conditions of the form \( x_1(0) = -2x_0, x_2(0) = 0, x_3(0) = x_0, x_4(0) = 2x_0 \), coefficients in (14) have the form

\[
\begin{align*}
R_0 &= 24k^3x_0^3S_3 - 8k^2x_0^2S_2 - x_0(S_1 + 30k^3S_4) + (1 + 12k^2S_3) , \\
R_1 &= 24k^3x_0^3S_2 - 8kx_0^2(S_1 + 18k^2S_4) - x_0(1 + 30k^2S_3) + 12kS_2 \\
R_2 &= 4x_0^3(3kS_1 + 54k^3S_4) - 4x_0^2(1 + 18k^2S_3) - 15kx_0S_2 + (6S_1 + 108k^2S_4) \\
R_3 &= 4x_0^3(1 + 6k^2S_3) - 8kx_0^2S_2 - 3x_0(S_1 + 10k^2S_4) + 12kS_3 \\
R_4 &= -12kx_0^3S_2 + 4x_0^2(S_1 + 18k^2S_4) + 15kx_0S_3 - 6S_2
\end{align*}
\]

At \( t \to \infty \) the asymptotic form of (14) reads as follows

\[ x^4 + k^{-1/3}(18)^{1/3}x^3 + \frac{3}{2}k^{-2/3}(12)^{1/3}x^2 + k^{-1}x - \frac{1}{2}k^{-4/3}(18)^{1/3} = 0 \]

This equation has only two real roots. Thus we come to the conclusion that for arbitrary \( x_0 \) two particles fall into the singularity of the potential in finite time interval \( t_0 \).

For arbitrary \( x_i(0) \) and \( k = 1 \) by reducing (14) to the form \( x^4 + ux^2 + qx + r = 0 \) and using the general expression for the discriminant
\[ D = 16u^4r - 4u^3q^2 - 128u^2q^2 + 144uq^2r - 27q^2 + 256r^3, \]

one can derive the following upper estimate for \( t_0 \):

\[
t_0 = \frac{(18)^{1/3}}{4A^2} \frac{\sum_{j < l} x_j^2(0)x_l^2(0)}{\sum_{j=1}^{4} x_j^2(0)} \tag{15}
\]

One can conclude from (15) that the smaller the coupling constant \( A^2 \) is, the greater \( t_0 \) is.

By now at \( N > 4 \) such estimates have not been derived yet. For example, at \( N = 5 \) one can obtain from (4)-(5)

\[
\hat{R}^6 = \hat{R}^3 + bE,
\]

where \( \hat{R} = (2^{-1}(9)^{-1/3}k^{-2/3})\hat{T} \) and \( b = (5/2)^23^{-4} \), \( E \) – identity matrix. In this case we have

\[
S_0(\theta) = 1 + \frac{b\theta^6}{6!} + \sum_{l=1}^{\infty} \frac{a_l\theta^{9+3l}}{(9+3l)!},
\]

where

\[
a_l = b \sum_{m=0}^{l-3} \frac{(l+m)!}{(l-1)!m!} b^m.
\]

Therefore summation of rows in (4) becomes more complicated.

**5.6 Summary**

We have considered Calogero-Moser systems in external fields under special constraint (6). In fact we have considered two-parameter subclass of potentials defined by (1), (2). For two-, three- and four-particle systems exact solutions have been derived and analyzed. We have derived estimates
for the collapse time of the systems in question. It is shown that $N = 2$ case is unique in the sense that there are such values of the parameters in potentials (8),(9), that the system has no finite collapse time $t_0$. For $N \geq 3$ there are no such values.

Explicit solutions obtained can be used for verification of different approximate methods and for investigation of the systems close to integrable ones. It is also possible that corresponding quantum systems will possess especially simple behavior.

REFERENCES

[1] Calogero F.// Lett.Nuovo Cim. 1975. 13. P.411; J.Moser// Adv. Math. 1975. 16. P.197.
[2] Dittrich J., Inozemtsev V.I.// J. Phys. 1993. A20. P.753.
[3] Olshanetsky M.A., Perelomov A.M.// Phys. Rep. 1983. 94. P.312.
[4] Koprak Th.T., Wagner H.J.//J.Stat.Phys. 2000. 100. P.779.
[5] Inozemtsev V.I., Meshcheryakov D.V.// Phys. Lett. 1984. A106. P.105.
[6] Meshcheryakov D.V, Tverskoy V.B.// Moscow University Phys. Bull. 2000. 1. P.66
[7] Korn G.A., Korn Th.M.// Mathematical Handbook. McGrow-Hill, New York 1965.
[8] Feferman S.// The number systems. Foundation of algebra and analysis. Addison–Wesley, Reading 1964.