Projected Euler method for stochastic delay differential equation under a global monotonicity condition

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Abstract
This paper investigates projected Euler-Maruyama method for stochastic delay differential equations under a global monotonicity condition. This condition admits some equations with highly nonlinear drift and diffusion coefficients. We appropriately generalized the idea of C-stability and B-consistency given by Beyn et al. [J. Sci. Comput. 67 (2016), no. 3, 955-987] to the case with delay. Moreover, the method is proved to be convergent with order \( \frac{1}{2} \) in a succinct way. Finally, some numerical examples are included to illustrate the obtained theoretical results.

Keywords: Stochastic delay differential equation; Projected Euler-Maruyama method; Strong convergence; C-stability; B-consistency

1. Introduction
Consider d-dimensional nonlinear stochastic delay differential equations (SDDEs)

\[ dX(t) = f(X(t), X(t - \tau))dt + g(X(t), X(t - \tau))dW(t), \quad t > 0, \]

with the initial condition given by

\[ \{X(\theta) : -\tau \leq \theta \leq 0\} = \xi \in C([-\tau, 0]; \mathbb{R}^d). \]

Here, \( X : [0, T] \times \Omega \rightarrow \mathbb{R}^d \) denotes the exact solution to (1.1), the drift term \( f : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) and diffusion term \( g : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m} \). And \( W(t) := (W_1(t), \cdots, W_m(t))^T \) is an m-dimensional Wiener process defined on given complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with a filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) under usual

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condition (i.e., it is increasing and right continuous, and \( \mathcal{F}_0 \) contains all \( \mathbb{P} \)-null sets). SDDEs can be seen as a generalization of stochastic differential equations, and they play an important role in many phenomena in physics [1–3]. In terms of well-posedness of the equation, there have been extensive study and application of SDDEs. The well known result is that the global Lipschitz condition and the linear growth condition guarantee the existence and uniqueness of analytical solution (see, [4, 5]). In 2002, Mao [6] gave the Khasminskii-type condition for SDDEs where linear growth condition was no longer necessary, and global existence and uniqueness of the solution was proved.

Most of SDDEs can not be solved analytically, so numerical calculation is particularly necessary. In the past two decades, a number of numerical methods were investigated under Lipschitz and linear growth condition (see [7–18] and references therein). Limited work has been done in SDDEs whose coefficients do not satisfy the linear growth condition, and this issue received attention only recently. The mean square stability of \( \theta \) methods for SDDEs under a coupled condition was first studied by Huang [19]. In 2018, Guo et. al [20] considered the truncated Euler-Maruyama method for nonlinear SDDEs under the generalized Khasminskii-type condition, and convergence in \( L^q \) was also derived. Zhang et al. [21] established the convergence of partially truncated Euler-Maruyama method for a class of highly nonlinear SDDEs. All their convergence analyses were under the framework given by Higham et al. [22], where complex higher moment estimation and continuous time extension of the corresponding numerical scheme should be taken into account. Recently, Beyn et al. [23, 24] proposed projected Euler-Maruyama method, projected Milstein method for SDEs by studying the C-stability and B-consistency, which can avoid those processes on the discrete time level. In this way, the convergence analysis can be simplified significantly.

Compared with implicit methods for SDEs, the explicit Euler methods process simpler algebraic structure, and can reach strong order of convergence 1/2 with cheaper computational cost. However, Hutzenthaler et. al [25] proved that strong and weak divergence in finite time of the explicit Euler method for SDEs with superlinearly growing coefficients. Subsequently, some modified Euler methods, such as tamed and truncated methods, were constructed to solve the nonlinear SDEs (see [26, 27]). The main goal of this paper is to generalize the projected Euler methods for SDDEs with superlinearly growth condition.

An outline of this paper is organized as follows. Some assumptions and projected Euler method are introduced in Section 2. Section 3 gives the main convergence theorem under the premise of stochastic C-stability and B-consistency. In Section 4, C-stability and B-consistency of projected Euler-Maruyama method are studied in detail. In Section 5, some numerical experiments are carried out to verify the theoretical results. Finally, some conclusions are drawn in the last section.

2. Preliminaries

Most of the notations in this paper come from [23]. For the sake of simplicity, we let

\[
h = \frac{\tau}{M}, \quad t_n = nh,
\]

and

\[
t_{i-M} = t_i - \tau, \quad i = 0, 1, \cdots, M,
\]
Assumption 1. There exist positive constant \( L \) and parameter \( \eta \).

Taking one step projected Euler method in [23] into account, we propose our new projected method

\[ \bar{X}_h(t_i) = X_h(t_{i-1}), \quad X_h(t_{i-M}) = \xi(t_i - \tau), \quad i = 0, \cdots, M, \]

then we say grid function \( X_h \in \mathcal{G}^2(T_h) \) is yield by the stochastic one-step method \((\Psi, h, \xi)\).

Before proceeding further, let us make the following assumptions.

**Assumption 1.** There exist positive constant \( L \) and parameter \( \eta \in (\frac{1}{2}, \infty) \) such that

\[
\langle x_1 - x_2, f(x_1, \bar{x}_1) - f(x_2, \bar{x}_2) \rangle + \eta |g(x_1, \bar{x}_1) - g(x_2, \bar{x}_2)|^2 \leq L(|x_1 - x_2|^2 + |\bar{x}_1 - \bar{x}_2|^2).
\]

The above expression is referred as to global monotonicity condition. Moreover, we assume that there is constant \( q \in (1, \infty) \) such that

\[
|f(x, \bar{x})| + |g(x, \bar{x})| \leq L(1 + |x|^q + |\bar{x}|^q),
\]

\[
|f(x_1, \bar{x}_1) - f(x_2, \bar{x}_2)| + |g(x_1, \bar{x}_1) - g(x_2, \bar{x}_2)| \leq (1 + |x_1|^{q-1} + |x_2|^{q-1})(|\bar{x}_1|^{q-1} + |\bar{x}_2|^{q-1})(|x_1 - x_2| + |\bar{x}_1 - \bar{x}_2|),
\]

for all \( x, x_1, x_2, \bar{x}_1, \bar{x}_2 \in \mathbb{R}^d \).
Assumption 2. The initial data \( \xi \) satisfies
\[
|\xi(u) - \xi(v)| \leq K_1 |u - v|^{\beta}, \quad -\tau \leq v < u \leq 0,
\]
where \( K_1 > 0 \) and \( \beta \in \left[ \frac{1}{2}, 1 \right] \) are constants.

Assumption 3. For every positive number \( R \), there exists a positive constant \( K_R \) such that
\[
|f(x, y) - f(\bar{x}, \bar{y})|^2 \vee |g(x, y) - g(\bar{x}, \bar{y})|^2 \leq K_R (|x - \bar{x}|^2 + |y - \bar{y}|^2)
\]
for those \( x, y, \bar{x}, \bar{y} \in \mathbb{R}^d \) with \( |x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq R \).

Assumption 4. There exist positive parameter \( p \in [2, \infty) \) and positive constant \( K_1 \) such that
\[
x^T f(x, y) + \frac{p - 1}{2} |g(x, y)|^2 \leq K_1 (1 + |x|^2 + |y|^2).
\]

Lemma 2.1. \((2.8)\) Assume that Assumption 2 and 4 hold. Then for any given initial data, there
is a unique global solution \( X(t) \) to (1.1) on \( t \in [-\tau, \infty) \). Moreover, the solution has the property that
\[
\sup_{-\tau \leq t \leq \tau} \mathbb{E}[X(t)]^p < \infty.
\]

Next, the concepts of C-stability and B-consistency in [23] are modified appropriately and the
corresponding definitions for SSDEs are given as follows.

Definition 2.2. A stochastic one-step method \((\Psi, h, \xi)\) for SSDEs (1.1) is said to be stochastic
C-stable if for \( \eta \in (1, \infty) \) and all random variables \( Y, Z \in L^2(\Omega, \mathcal{F}, \mathbb{P} \mid \mathbb{R}^d) \)
\[
\left\| \mathbb{E}[\Psi(Y, \bar{Y}, h) - \Psi(Z, \bar{Z}, h) \mid \mathcal{F}_t] \right\|_{L^2(\Omega; \mathbb{R}^d)}^2
+ \eta \left\| (id - \mathbb{E}[\cdot \mid \mathcal{F}]) (\Psi(Y, \bar{Y}, h) - \Psi(Z, \bar{Z}, h)) \right\|_{L^2(\Omega; \mathbb{R}^d)}^2
\leq (1 + C_{\text{stab}} h) \left\| Y - Z \right\|_{L^2(\Omega; \mathbb{R}^d)}^2 + C_{\text{stab}} h \left\| \bar{Y} - \bar{Z} \right\|_{L^2(\Omega; \mathbb{R}^d)}^2,
\]
where \( (id - \mathbb{E}[\cdot \mid \mathcal{F}] ) Y = Y - \mathbb{E}[Y \mid \mathcal{F}] \).

Definition 2.3. A stochastic one-step method \((\Psi, h, \xi)\) for SSDEs (1.1) is said to be stochastic
B-consistent of order \( \gamma \) if
\[
\left\| \mathbb{E}[X(t + h) - \Psi(X(t), X(t - \tau), h) \mid \mathcal{F}_t] \right\|_{L^2(\Omega; \mathbb{R}^d)} \leq C_{\text{cons}} h^{\gamma + 1}, \tag{2.8}
\]
and
\[
\left\| (id - \mathbb{E}[\cdot \mid \mathcal{F}]) (X(t + h) - \Psi(X(t), X(t - \tau), h)) \right\|_{L^2(\Omega; \mathbb{R}^d)} \leq C_{\text{cons}} h^{\gamma + \frac{1}{2}}, \tag{2.9}
\]
3. Convergence Theorem

The next stability lemma plays an important role in the convergence analysis.

**Lemma 3.1.** If \((\Psi, h, \xi)\) is stochastically \(C\)-stable one-step method with constants \(C_{\text{stab}}\) and \(\eta \in (1, \infty)\), then for every grid function \(Z \in \mathcal{F}^2(T_h)\),

\[
\max_{n \in \{0, \ldots, N\}} \|Z(t_n) - X_h(t_n)\|_{L^2(\Omega; \mathbb{R}^d)}^2 \\
\leq e^{2(1+C_{\text{stab}}(1+h)^\ell)} \left( \sum_{i=1}^M \|Z(t_{i-M}) - \xi(t_{i-M})\|_{L^2(\Omega; \mathbb{R}^d)}^2 + \|Z(t_0) - X_h(t_0)\|_{L^2(\Omega; \mathbb{R}^d)}^2 \\
+ \sum_{i=1}^N (1 + h^{-1}) \|\mathbb{E}[Z(t_i) - \Psi(Z(t_{i-1}), Z(t_{i-M}), h)|\mathcal{F}_{t_{i-1}}]\|_{L^2(\Omega; \mathbb{R}^d)}^2 \\
+ C \sum_{i=1}^N \| \left( id - \mathbb{E}[\cdot|\mathcal{F}_{t_{i-1}}] \right) (Z(t_i) - \Psi(Z(t_{i-1}), Z(t_{i-M}), h)) \|_{L^2(\Omega; \mathbb{R}^d)}^2 \right),
\]

where \(Z(t_{i-M}), \xi(t_{i-M}), i = 0, 1, \cdots, M, \) are defined by \(Z(t_i)\) and \(\xi(t_i)\), respectively.

**Proof.** For every \(1 \leq i \leq N\), let \(e_h(t_i) := Z(t_i) - X_h(t_i)\),

\[
\|e_h(t_i)\|_{L^2(\Omega; \mathbb{R}^d)}^2 = \|\mathbb{E}[e_h(t_i)|\mathcal{F}_{t_{i-1}}]\|_{L^2(\Omega; \mathbb{R}^d)}^2 + \|e_h(t_i) - \mathbb{E}[e_h(t_i)|\mathcal{F}_{t_{i-1}}]\|_{L^2(\Omega; \mathbb{R}^d)}^2
\]

On account of

\[
e_h(t_i) = Z(t_i) - \Psi(Z(t_{i-1}), Z(t_{i-M}), h) + \Psi(Z(t_{i-1}), Z(t_{i-M}), h) - X_h(t_i),
\]

we have

\[
\|\mathbb{E}[e_h(t_i)|\mathcal{F}_{t_{i-1}}]\|_{L^2(\Omega; \mathbb{R}^d)}^2 \leq \|\mathbb{E}[Z(t_i) - \Psi(Z(t_{i-1}), Z(t_{i-M}), h)|\mathcal{F}_{t_{i-1}}]\|_{L^2(\Omega; \mathbb{R}^d)}^2 \\
+ \|\mathbb{E}[\Psi(Z(t_{i-1}), Z(t_{i-M}), h) - X_h(t_i)|\mathcal{F}_{t_{i-1}}]\|_{L^2(\Omega; \mathbb{R}^d)}^2.
\]

By the inequality \((a + b)^2 = a^2 + 2ab + b^2 \leq (1 + h^{-1})a^2 + (1 + h)b^2\), one may derive that

\[
\|\mathbb{E}[e_h(t_i)|\mathcal{F}_{t_{i-1}}]\|_{L^2(\Omega; \mathbb{R}^d)}^2 \leq (1 + h^{-1})\|\mathbb{E}[Z(t_i) - \Psi(Z(t_{i-1}), Z(t_{i-M}), h)|\mathcal{F}_{t_{i-1}}]\|_{L^2(\Omega; \mathbb{R}^d)}^2 \\
+ (1 + h)\|\mathbb{E}[\Psi(Z(t_{i-1}), Z(t_{i-M}), h) - X_h(t_i)|\mathcal{F}_{t_{i-1}}]\|_{L^2(\Omega; \mathbb{R}^d)}^2.
\]

Repeating the same process for the item \(\|e_h(t_i) - \mathbb{E}[e_h(t_i)|\mathcal{F}_{t_{i-1}}]\|_{L^2(\Omega; \mathbb{R}^d)}^2\), and replacing \(h\) with \(h - 1\), then

\[
\|e_h(t_i) - \mathbb{E}[e_h(t_i)|\mathcal{F}_{t_{i-1}}]\|_{L^2(\Omega; \mathbb{R}^d)}^2 \leq C \| \left( id - \mathbb{E}[\cdot|\mathcal{F}_{t_{i-1}}] \right) (Z(t_i) - \Psi(Z(t_{i-1}), Z(t_{i-M}), h)) \|_{L^2(\Omega; \mathbb{R}^d)}^2 \\
+ \eta \| \left( id - \mathbb{E}[\cdot|\mathcal{F}_{t_{i-1}}] \right) (Z(t_{i-1}), Z(t_{i-M}), h) - X_h(t_i)) \|_{L^2(\Omega; \mathbb{R}^d)}^2,
\]

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where \( C_\eta = 1 + (\eta - 1)^{-1} \). Consequently, for \( 1 \leq i \leq N \),
\[
\|Z(t_i) - X_h(t_i)\|_{L^2(\Omega;\mathbb{R}^d)}^2 \\
\leq (1 + h^{-1})\|E[Z(t_i) - \Psi(Z(t_{i-1}), Z(t_{i-M}), h)]\|_{L^2(\Omega;\mathbb{R}^d)}^2 \\
+ (1 + h)\|E[\Psi(Z(t_{i-1}), Z(t_{i-M}), h) - X_h(t_i)]\|_{L^2(\Omega;\mathbb{R}^d)}^2 \\
+ C_\eta \|(id - E[\cdot|\mathcal{F}_{i-1}])(Z(t_i) - \Psi(Z(t_{i-1}), Z(t_{i-M}), h))\|_{L^2(\Omega;\mathbb{R}^d)}^2 \\
+ h\|(id - E[\cdot|\mathcal{F}_{i-1}])(\Psi(Z(t_{i-1}), Z(t_{i-M}), h) - X_h(t_i))\|_{L^2(\Omega;\mathbb{R}^d)}^2.
\]
Using the fact that \( X_h(t_i) = \Psi(X_h(t_{i-1}), X_h(t_{i-M}), h) \) and (2.7), we have
\[
\|Z(t_i) - X_h(t_i)\|_{L^2(\Omega;\mathbb{R}^d)}^2 \\
\leq (1 + h^{-1})\|E[Z(t_i) - \Psi(Z(t_{i-1}), Z(t_{i-M}), h)]\|_{L^2(\Omega;\mathbb{R}^d)}^2 \\
+ C_\eta \|(id - E[\cdot|\mathcal{F}_{i-1}])(Z(t_i) - \Psi(Z(t_{i-1}), Z(t_{i-M}), h))\|_{L^2(\Omega;\mathbb{R}^d)}^2 \\
+ (1 + (1 + C_{stab}(1 + h))h)\|Z(t_{i-1}) - X_h(t_{i-1})\|_{L^2(\Omega;\mathbb{R}^d)}^2 \\
+ C_{stab}h(1 + h)\|Z(t_{i-M}) - X_h(t_{i-M})\|_{L^2(\Omega;\mathbb{R}^d)}^2,
\]
where we have used the inequality
\[
h\|E[\Psi(Z(t_{i-1}), Z(t_{i-M}), h) - X_h(t_i)]\|_{L^2(\Omega;\mathbb{R}^d)}^2 \\
\leq h(1 + C_{stab}h)\|Z(t_{i-1}) - X_h(t_{i-1})\|_{L^2(\Omega;\mathbb{R}^d)}^2 + hC_{stab}h\|Z(t_{i-M}) - X_h(t_{i-M})\|_{L^2(\Omega;\mathbb{R}^d)}^2.
\]
Choose sufficiently small \( h \) such that \( C_{stab}h(1 + h) < 1 \). Then, summing \( i \) over 1 to \( n \) yields
\[
\|Z(t_n) - X_h(t_n)\|_{L^2(\Omega;\mathbb{R}^d)}^2 - \|Z(t_0) - X_h(t_0)\|_{L^2(\Omega;\mathbb{R}^d)}^2 \\
= \sum_{i=1}^{n} \left( \|Z(t_i) - X_h(t_i)\|_{L^2(\Omega;\mathbb{R}^d)}^2 - \|Z(t_{i-1}) - X_h(t_{i-1})\|_{L^2(\Omega;\mathbb{R}^d)}^2 \right) \\
\leq \sum_{i=1}^{n} \left( (1 + h^{-1})\|E[Z(t_i) - \Psi(Z(t_{i-1}), Z(t_{i-M}), h)]\|_{L^2(\Omega;\mathbb{R}^d)}^2 \\
+ C_\eta \|(id - E[\cdot|\mathcal{F}_{i-1}])(Z(t_i) - \Psi(Z(t_{i-1}), Z(t_{i-M}), h))\|_{L^2(\Omega;\mathbb{R}^d)}^2 \\
+ 2(1 + C_{stab}(1 + h))h\|Z(t_{i-1}) - X_h(t_{i-1})\|_{L^2(\Omega;\mathbb{R}^d)}^2 \\
+ \sum_{i=M}^{n} \|Z(t_{i-M}) - \xi(t_{i-M})\|_{L^2(\Omega;\mathbb{R}^d)}^2 \right).
\]
Finally, the desired assertion follows from (3.2) and the discrete Gronwall inequality.

The following theorem shows that convergence can be derived from stability plus consistency.
Theorem 3.1. If a stochastic one-step method \((\Psi, h, \xi)\) is stochastic C-stability and B-consistent of order \(\gamma\), then there exists a constant \(C\) such that

\[
\max_{n \in \{0, \ldots, N\}} \|X(t_n) - X_h(t_n)\|_{L^2(\Omega; \mathbb{R}^d)} \leq Ch^\gamma,
\]

where \(X\) is the exact solution of \((1.1)\) and \(X_h\) is the grid function corresponding to \((\Psi, h, \xi)\) with time step \(h\).

Proof. Due to the fact that \(X(t_{i-M}) = X_h(t_{i-M}) = \xi(t_{i-M}), i = 0, 1, \ldots, M,\) we obtain

\[
\max_{n \in \{0, \ldots, N\}} \|X(t_n) - X_h(t_n)\|_{L^2(\Omega; \mathbb{R}^d)}^2 \leq e^{2(1+C_{\text{stab}}(1+h))T} \left( \sum_{i=1}^{N} (1 + h^{-1}) \|E[Z(t_i) - \Psi(Z(t_{i-1}), Z(t_{i-M}), h)]\|_{L^2(\Omega; \mathbb{R}^d)}^2 + C_\eta \sum_{i=1}^{N} \|((id - E[\cdot, \mathcal{F}_{t_{i-1}}])(Z(t_i) - \Psi(Z(t_{i-1}), Z(t_{i-M}), h))\|_{L^2(\Omega; \mathbb{R}^d)}^2 \right).
\]

It follows from (2.8) and (2.9) that

\[
\max_{n \in \{0, \ldots, N\}} \|X(t_n) - X_h(t_n)\|_{L^2(\Omega; \mathbb{R}^d)}^2 \leq e^{2(1+C_{\text{stab}}(1+h))T} C_{\text{cons}}^2 \sum_{i=1}^{N} \left( (1 + h^{-1})h^{2(\gamma + 1)} + C_\eta h^{2\gamma + 1} \right) \leq Ch^{2\gamma}.
\]

The proof is completed now.

4. C-stability and B-consistency of the PEM Method

We follow the notation in [23]

\[
x^\circ := \min(1, h^{-\alpha}|x|^{-1})x,
\]

and denote

\[
\tilde{x}^\circ := \min(1, h^{-\alpha}|\tilde{x}|^{-1})\tilde{x},
\]

where \(x \in \mathbb{R}^d\) and step size \(h \in (0, 1]\).

Lemma 4.1. ([23]) For every \(\alpha \in (0, \infty)\) and \(h \in (0, 1]\) the mapping \(\mathbb{R}^d \ni x \mapsto x^\circ \in \mathbb{R}^d\) is globally Lipschitz continous with Lipschitz constant 1, i.e.,

\[
|x_1^\circ - x_2^\circ| \leq |x_1 - x_2|
\]

for all \(x_1, x_2 \in \mathbb{R}^d\).
Lemma 4.2. If Assumption 7 is fulfilled with $L \in (0, \infty)$, $q \in (1, \infty)$ and $\eta \in (\frac{1}{2}, \infty)$, then the functions $\bar{x}^o$, $\tilde{x}^o$ with parameter $\alpha \in (0, \frac{1}{2(q-1)})$ and $h \in (0, 1]$ satisfy
\[
|x_1^0 - x_2^0 + h(f(x_1^0, \bar{x}_1^0) - f(x_2^0, \tilde{x}_2^o))|^2 + 2\eta |h| |g(x_1^0, \bar{x}_1^0) - g(x_2^0, \tilde{x}_2^o)|^2
\leq (1 + Ch)|x_1 - x_2|^2 + Ch|\bar{x}_1 - \tilde{x}_2|^2
\]
for all $x_1, x_2 \in \mathbb{R}^d$.

Proof. By (2.2), we obtain that
\[
|x_1^0 - x_2^0 + h(f(x_1^0, \bar{x}_1^0) - f(x_2^0, \tilde{x}_2^o))|^2
\leq (1 + 2Lh)|x_1^0 - x_2^0|^2 + 2Lh|\bar{x}_1^0 - \tilde{x}_2^o|^2 - 2\eta |h| |g(x_1^0, \bar{x}_1^0) - g(x_2^0, \tilde{x}_2^o)|^2
\]
\[+ h^2(f(x_1^0, \bar{x}_1^0) - f(x_2^0, \tilde{x}_2^o))^2.
\]
Note that
\[
|f(x_1^0, \bar{x}_1^0) - f(x_2^0, \tilde{x}_2^o)|
\leq L(1 + |x_1^0|^q - 1 + |\bar{x}_1^0|^q - 1 + |\tilde{x}_2^o|^q - 1)(|x_1^0 - x_2^0| + |\bar{x}_1^0 - \tilde{x}_2^o|)
\leq L(1 + 4h^{-\alpha(q-1)})(|x_1 - x_2| + |\bar{x}_1 - \tilde{x}_2|)
\leq L(1 + 4h^{-1/2})(|x_1 - x_2| + |\bar{x}_1 - \tilde{x}_2|),
\]
where we have used (2.4), Lemma 4.1 $|x_1^0|$, $|x_2^0|$, $|\bar{x}_1^0|$, $|\tilde{x}_2^o| \leq h^{-\alpha}$ and $\alpha \in (0, \frac{1}{2(q-1)})$. Consequently,
\[
|x_1^0 - x_2^0 + h(f(x_1^0, \bar{x}_1^0) - f(x_2^0, \tilde{x}_2^o))|^2 + 2\eta |h| |g(x_1^0, \bar{x}_1^0) - g(x_2^0, \tilde{x}_2^o)|^2
\leq (1 + 2Lh)|x_1^0 - x_2^0|^2 + 2Lh|\bar{x}_1^0 - \tilde{x}_2^o|^2 + h^2 L^2 (1 + 4h^{-1/2})^2 (|x_1 - x_2|^2 + |\bar{x}_1 - \tilde{x}_2|^2)
\leq (1 + Ch)|x_1 - x_2|^2 + Ch|\bar{x}_1 - \tilde{x}_2|^2.
\]
The direct application of the above lemma can deduce that the projected Euler method is $\mathcal{B}$-consistent of order $1/2$.

Next, we show the PEM method is $\mathcal{B}$-consistent of order $1/2$.

Lemma 4.3. If Assumption 7 is fulfilled with $L \in (0, \infty)$, $q \in (1, \infty)$, and $\sup_{t \in [-\tau, T]} \|X(t)\|_{L^p(\Omega; \mathbb{R}^d)} < \infty$ holds for some positive constant $p \in [2, \infty)$, then
\[
\|X(r_1) - X(r_2)\|_{L^p(\Omega; \mathbb{R}^d)} \leq C \left(1 + 2 \sup_{t \in [-\tau, T]} \|X(t)\|_{L^p(\Omega; \mathbb{R}^d)}^q\right)|r_1 - r_2|^{1/2},
\]
for all $r_1, r_2 \in [0, T]$.

Proof. The proof can be deduced from Proposition 5.4 of [23] easily. In fact, we just need replace $\sup_{t \in [0, T]} \|X(t)\|_{L^p(\Omega; \mathbb{R}^d)}^q$ with $2 \sup_{t \in [-\tau, T]} \|X(t)\|_{L^p(\Omega; \mathbb{R}^d)}^q$. So we omit the detail of proof here. \(\square\)
Lemma 4.4. If \( f \) and \( g \) satisfy Assumption \( \mathbf{7} \) with \( L \in (0, \infty) \) and \( q \in (1, \infty) \), the exact solution of \( (1.1) \) satisfy \( \sup_{t \in [-\tau, T]} \|X(t)\|_{L^{4q-2}(\Omega; \mathbb{R}^d)} < \infty \), then for \( \forall s_1 \in [r_1, r_2] \), there exists a constant \( C \) such that

\[
\int_{r_1}^{r_2} \|f(X(s), X(s - \tau)) - f(X(s_1), X(s_1 - \tau))\|_{L^2(\Omega; \mathbb{R}^d)} ds 
\leq C \left( 1 + 4 \sup_{t \in [-\tau, T]} \|X(t)\|_{L^{4q-2}(\Omega; \mathbb{R}^d)}^{2q-1} \right) |r_1 - r_2|^{3/2},
\]

for all \( r_1, r_2 \in [0, T] \).

Proof. By (2.4) and Hölder’s inequality, we get

\[
\|f(X(s), X(s - \tau)) - f(X(s_1), X(s_1 - \tau))\|_{L^2(\Omega; \mathbb{R}^d)} 
\leq \left( 1 + |X(s)|^{q-1} + |X(s - \tau)|^{q-1} + |X(s_1)|^{q-1} + |X(s_1 - \tau)|^{q-1} \right) \|X(s) - X(s_1)\|_{L^{4q-2}(\Omega; \mathbb{R}^d)}
\]

\[
+ \left( 1 + |X(s)|^{q-1} + |X(s - \tau)|^{q-1} + |X(s_1)|^{q-1} + |X(s_1 - \tau)|^{q-1} \right) \|X(s) - X(s_1)\|_{L^{4q-2}(\Omega; \mathbb{R}^d)},
\]

where \( \rho = 2 - \frac{1}{q} \) and \( \rho' = \frac{2q-1}{q-1} \).

Without loss of generality, we discuss the last term in (4.1) with three different cases,

Case 1: \( s - \tau > 0 \) and \( s_1 - \tau < 0 \),

\[
|X(s - \tau) - X(s_1 - \tau)| \leq |X(s - \tau) - X(0)| + |X(0) - X(s_1 - \tau)|,
\]

to show

\[
\|X(s - \tau) - X(s_1 - \tau)\|_{L^{4q-2}(\Omega; \mathbb{R}^d)} 
\leq |X(s - \tau) - X(0)|_{L^{4q-2}(\Omega; \mathbb{R}^d)} + \|\xi(0) - \xi(s_1 - \tau)\|_{L^{4q-2}(\Omega; \mathbb{R}^d)}
\]

\[
\leq C \left( 1 + 2 \sup_{t \in [-\tau, T]} \|X(t)\|_{L^{4q-2}(\Omega; \mathbb{R}^d)} \right) |s - \tau|^{1/2} + K_1 |s_1 - \tau|^{\rho'},
\]

\[
\leq C \left( 1 + 2 \sup_{t \in [-\tau, T]} \|X(t)\|_{L^{4q-2}(\Omega; \mathbb{R}^d)} \right) |s - s_1|^{1/2} + K_1 |s_1 - s|^{\rho'},
\]

\[
\leq C \left( 1 + 2 \sup_{t \in [-\tau, T]} \|X(t)\|_{L^{4q-2}(\Omega; \mathbb{R}^d)} \right) |r_1 - r_2|^{1/2} + K_1 |r_1 - r_2|^{\rho'},
\]

where Assumption \( \mathbf{2} \) and Lemma \( \mathbf{4.3} \) are used.

Case 2: \( s - \tau > 0 \) and \( s_1 - \tau > 0 \), it follows from Lemma \( \mathbf{4.3} \) that

\[
\|X(s - \tau) - X(s_1 - \tau)\|_{L^{4q-2}(\Omega; \mathbb{R}^d)}
\]
If the coefficients $f$ and $g$ satisfy Assumption 2 with $L \in (0, \infty)$ and $q \in (1, \infty)$, the exact solution of \((1.1)\) satisfy $\sup_{t \in [-\tau, T]} \|X(t)\|_{L^{2q-2}(\mathbb{R}^d)} < \infty$, then there exists a constant $C$ such that

$$
\|\int_{r_1}^{r_2} g(X(s), X(s - \tau)) - g(X(r_1), X(r_1 - \tau))dW(s)\|_{L^2(\Omega; \mathbb{R}^d)} 
\leq C \left(1 + 4 \sup_{t \in [-\tau, T]} \|X(t)\|_{L^{2q-2}(\mathbb{R}^d)}^{q-1} \right) |r_1 - r_2|^{1/2}.
$$

(4.2)

for all $r_1, r_2 \in [0, T]$.

**Proof.** Itô isometry formula yields

$$
\|\int_{r_1}^{r_2} g(X(s), X(s - \tau)) - g(X(r_1), X(r_1 - \tau))dW(s)\|_{L^2(\Omega; \mathbb{R}^d)} 
\leq \left( \int_{r_1}^{r_2} \|g(X(s), X(s - \tau)) - g(X(r_1), X(r_1 - \tau))\|_{L^2(\Omega; \mathbb{R}^d)}^2 ds \right)^{1/2}.
$$

Repeating the proof in Lemma 4.4 and we get

$$
\|g(X(s), X(s - \tau)) - g(X(r_1), X(r_1 - \tau))\|_{L^2(\Omega; \mathbb{R}^d)} 
\leq L \left(1 + |X(s)|^{q-1} + |X(s - \tau)|^{q-1} + |X(r_1)|^{q-1} + |X(r_1 - \tau)|^{q-1}(|X(s) - X(r_1)|)\right) \|L^2(\Omega; \mathbb{R}^d)
+ L \left(1 + |X(s)|^{q-1} + |X(s - \tau)|^{q-1} + |X(r_1)|^{q-1} + |X(r_1 - \tau)|^{q-1}(|X(s - \tau) - X(r_1 - \tau)|)\right) \|L^2(\Omega; \mathbb{R}^d)
\leq L \left(1 + 4 \sup_{t \in [-\tau, T]} \|X(t)\|_{L^{2q-2}(\mathbb{R}^d)}^{q-1} \right) \|X(s) - X(r_1)| \|L^2(\Omega; \mathbb{R}^d)
+ L \left(1 + 4 \sup_{t \in [-\tau, T]} \|X(t)\|_{L^{2q-2}(\mathbb{R}^d)}^{q-1} \right) \|X(s - \tau) - X(r_1 - \tau)| \|L^2(\Omega; \mathbb{R}^d)
\leq C \left(1 + 4 \sup_{t \in [-\tau, T]} \|X(t)\|_{L^{2q-2}(\mathbb{R}^d)}^{q-1} \right) |r_1 - r_2|^{1/2}.
$$

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The required inequality (4.2) follows.

Before giving the consistency results, we first present the following key lemma similar to Lemma 6.5 of [23].

**Lemma 4.6.** Denote $L \in (0, \infty)$ and $\kappa \in [1, \infty)$. If for $p \in (2, \infty)$, $Y \in L^p(\Omega; \mathbb{R}^d)$, and the measurable mapping $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ has the following properties

$$|\varphi(x, \bar{x})| \leq L(1 + |x|^\kappa + |\bar{x}|^\kappa),$$

then there exists a constant $C$ which depends on $p, L$, but not on $h$ such that for all $h \in (0, 1]$

$$\|\varphi(Y, \bar{Y}) - \varphi(Y^\circ, \bar{Y}^\circ)\|_{L^2(\Omega; \mathbb{R}^d)} \leq C(1 + \|Y\|_{L^p(\Omega; \mathbb{R}^d)}^p + \||\varphi\|_{L^p(\Omega; \mathbb{R}^d)}^p)^{p/2} h^{2\alpha(p-2)\kappa},$$

where $\alpha \in (0, \infty)$, $\bar{Y}^\circ = \min(1, h^{-\alpha}|\bar{Y}|^{-1})\bar{Y}$ and $Y^\circ = \min(1, h^{-\alpha}|Y|^{-1})Y$.

**Proof.** Denote the following measurable sets by

$$A_h := \{\omega \in \Omega : |Y(\omega)| \leq h^{-\alpha}\} \in \mathcal{F},$$

$$\tilde{A}_h := \{\omega \in \Omega : |\bar{Y}(\omega)| \leq h^{-\alpha}\} \in \mathcal{F}.$$ 

Let $B_h := A_h \cap \tilde{A}_h$, and $B_h^c := \Omega \setminus B_h$, then one can see that

$$\|\varphi(Y, \bar{Y}) - \varphi(Y^\circ, \bar{Y}^\circ)\|^2_{L^2(\Omega; \mathbb{R}^d)} = \int_{\Omega} |\varphi(Y, \bar{Y}) - \varphi(Y^\circ, \bar{Y}^\circ)|^2 1_{B_h}(\omega)d\mathbb{P}(\omega).$$

By the Young inequality $ab \leq \frac{h^\alpha}{p}a^\rho + \frac{\rho}{p}h^{-\alpha}b^{\rho'}$ with $\rho, \rho' = \frac{p}{2}, \rho = \frac{p}{2(p-1)} \in (0, \infty)$, we have

$$\int_{\Omega} |\varphi(Y, \bar{Y}) - \varphi(Y^\circ, \bar{Y}^\circ)|^2 1_{B_h}(\omega)d\mathbb{P}(\omega).$$

$$\leq \frac{2h^\alpha}{p}\|\varphi(Y, \bar{Y}) - \varphi(Y^\circ, \bar{Y}^\circ)\|^p_{L^p(\Omega; \mathbb{R}^d)} + \left(1 - \frac{2}{p}\right)h^{-\alpha} \mathbb{P}(B_h^c),$$

Furthermore,

$$\|\varphi(Y, \bar{Y}) - \varphi(Y^\circ, \bar{Y}^\circ)\|_{L^p(\Omega; \mathbb{R}^d)} \leq \|\varphi(Y, \bar{Y})\|_{L^p(\Omega; \mathbb{R}^d)} + \|\varphi(Y^\circ, \bar{Y}^\circ)\|_{L^p(\Omega; \mathbb{R}^d)}$$

$$\leq 2L(1 + \|Y\|_{L^p(\Omega; \mathbb{R}^d)}^p + \||\varphi\|_{L^p(\Omega; \mathbb{R}^d)}^p)^{p/2}.$$

Besides,

$$\mathbb{P}(B_h^c) = \mathbb{E}[1_{B_h^c}]$$

$$\leq h^{\alpha/2} \mathbb{E}[1_{B_h}|Y|^{2\alpha}] + h^{\alpha/2} \mathbb{E}[1_{B_h}|\bar{Y}|^{2\alpha}]$$

$$\leq h^{\alpha/2}(\|Y\|_{L^{2\alpha}(\Omega; \mathbb{R}^d)}^{2\alpha} + \||\varphi\|_{L^\infty(\Omega; \mathbb{R}^d)}^{2\alpha}.$$
Let \( \alpha p - \frac{2r}{p-2} = v \), i.e., \( v = \alpha(p - 2)k \), we obtain

\[
\| \varphi(Y, \tilde{Y}) - \varphi(Y^\circ, \tilde{Y}^\circ) \|_{L^2(\Omega; \mathbb{R}^d)}^2 \leq \frac{2}{p} (2L)^p h^{\alpha(p - 2)k} (1 + \|Y\|^\kappa_{L^p(\Omega; \mathbb{R}^d)} + \|\tilde{Y}\|^\kappa_{L^p(\Omega; \mathbb{R}^d)})^p + \left(1 - \frac{2}{p}\right) h^{\alpha(p - 2)k} (\|\varphi(Y)\|_{L^p(\Omega; \mathbb{R}^d)} + \|\tilde{Y}\|_{L^p(\Omega; \mathbb{R}^d)})^p
\]

which completes the proof.

We conclude this section with a theorem of B-consistency of the projected Euler method.

**Theorem 4.1.** If the coefficients \( f \) and \( g \) satisfy Assumption [7] with \( L \in (0, \infty) \) and \( q \in (1, \infty) \), the exact solution of (1.1) satisfies \( \sup_{t \in [0, T]} \|X(t)\|_{L^q(t; \mathbb{R}^d)} < \infty \), then the order of B-consistent of the projected Euler method \((\Psi^{PEM}, h, \xi)\) with \( \alpha = \frac{1}{6(2q-1)} \) is \( \frac{1}{2} \).

**Proof.** By (1.1), (2.1), we have

\[
X(t + h) - \Psi^{PEM}(X(t), X(t - h), h) = \int_t^{t+h} f(X(s), X(s - \tau)) - f(X(t), X(t - \tau))ds
\]

\[
+ X(t) + h f(X(t), X(t - \tau)) - X^\circ(t) - h f(X^\circ(t), X^\circ(t - \tau))
\]

\[
+ \int_t^{t+h} g(X(s), X(s - \tau)) - g(X(t), X(t - \tau))dW(s)
\]

\[
+ (g(X(t), X(t - \tau)) - g(X^\circ(t), X^\circ(t - \tau)))d\Delta h W(t),
\]

where \( X^\circ(t) = \min(1, h^{-\alpha}|X(t)|^{-1})X(t) \), \( X^\circ(t - \tau) = \min(1, h^{-\alpha}|X(t - \tau)|^{-1})X(t - \tau) \). Moreover,

\[
\| \mathbb{E}[X(t + h) - \Psi^{PEM}(X(t), X(t - h), h) | \mathcal{F}_t] \|_{L^2(\Omega; \mathbb{R}^d)} \leq \int_t^{t+h} \| \mathbb{E}[f(X(s), X(s - \tau)) - f(X(t), X(t - \tau)) | \mathcal{F}_t] \|_{L^2(\Omega; \mathbb{R}^d)} ds
\]

\[
+ \|X(t) - X^\circ(t)\|_{L^2(\Omega; \mathbb{R}^d)} + h\|f(X(t), X(t - \tau)) - f(X^\circ(t), X^\circ(t - \tau))\|_{L^2(\Omega; \mathbb{R}^d)}
\]

From the inequality \( \| \mathbb{E}[Y | \mathcal{F}_t] \|_{L^2(\Omega; \mathbb{R}^d)} \leq \| Y \|_{L^2(\Omega; \mathbb{R}^d)} \) and Lemma 4.3, we have

\[
\int_t^{t+h} \| \mathbb{E}[f(X(s), X(s - \tau)) - f(X(t), X(t - \tau)) | \mathcal{F}_t] \|_{L^2(\Omega; \mathbb{R}^d)} ds \leq C_{con} h^{\frac{3}{2}}.
\]

Applying Lemma 4.6 to the term \( \|X(t) - X^\circ(t)\|_{L^2(\Omega; \mathbb{R}^d)} \) with \( \kappa = 1 \) and \( p = 6q - 4 \) yields

\[
\|X(t) - X^\circ(t)\|_{L^2(\Omega; \mathbb{R}^d)} \leq (1 + \|X(t)\|_{L^q(t; \mathbb{R}^d)} + \|X(t - \tau)\|_{L^q(t - \tau; \mathbb{R}^d)})^{\frac{6q - 2}{2}} h^{\frac{3}{2}}.
\]

An application of Lemma 4.6 to \( \|f(X(t), X(t - \tau)) - f(X^\circ(t), X^\circ(t - \tau))\|_{L^2(\Omega; \mathbb{R}^d)} \) with \( \kappa = q \) and \( p = 4 - \frac{2}{q} \) yields

\[
\|f(X(t), X(t - \tau)) - f(X^\circ(t), X^\circ(t - \tau))\|_{L^2(\Omega; \mathbb{R}^d)} \leq (1 + \|X(t)\|_{L^{4q - 2}(\Omega; \mathbb{R}^d)} + \|X(t - \tau)\|_{L^{4q - 2}(\Omega; \mathbb{R}^d)})^{\frac{4q - 2 - 1}{4}} h^{\frac{3}{2}}.
\]
 Altogether, and combined with (4.3), we can find (2.8) is satisfied with $\gamma = \frac{1}{2}$.

Now, we consider another estimation
\[
\| (id - \mathbb{E}[-\mathcal{F}_t])(X(t + h) - \Psi^{PEM}(X(t), X(t - h), h)) \|_{L^2(\Omega; \mathbb{R}^d)} \\
\leq \int_{t}^{t+h} \| (id - \mathbb{E}[-\mathcal{F}_t])(f(X(s), X(s - \tau)) - f(X(t), X(t - \tau))) \|_{L^2(\Omega; \mathbb{R}^d)} ds \\
+ \int_{t}^{t+h} g(X(s), X(s - \tau)) - g(X(t), X(t - \tau)) dW(s) \|_{L^2(\Omega; \mathbb{R}^d)} \\
+ \| (g(X(t), X(t - \tau)) - g(X^\circ(t), X^\circ(t - \tau))) \Delta h W(t) \|_{L^2(\Omega; \mathbb{R}^d)}.
\]
Using the inequality $\| (id - \mathbb{E}[-\mathcal{F}_t])Y \|_{L^2(\Omega; \mathbb{R}^d)} \leq \| Y \|_{L^2(\Omega; \mathbb{R}^d)}$, and Lemma 4.4, it is easy to see that
\[
\int_{t}^{t+h} \| (id - \mathbb{E}[-\mathcal{F}_t])(f(X(s), X(s - \tau)) - f(X(t), X(t - \tau))) \|_{L^2(\Omega; \mathbb{R}^d)} ds \leq C_{cons} h^2.
\]
It follows from Lemma [4.5] that
\[
\left\| \int_{t}^{t+h} g(X(s), X(s - \tau)) - g(X(t), X(t - \tau)) dW(s) \right\|_{L^2(\Omega; \mathbb{R}^d)} \leq C_{cons} h.
\]
In addition,
\[
\left\| (g(X(t), X(t - \tau)) - g(X^\circ(t), X^\circ(t - \tau))) \Delta h W(t) \right\|_{L^2(\Omega; \mathbb{R}^d)}^2 \\
= h \left\| (g(X(t), X(t - \tau)) - g(X^\circ(t), X^\circ(t - \tau))) \right\|_{L^2(\Omega; \mathbb{R}^d)}^2.
\]
And once again, we apply Lemma 4.6 to $\| g(X(t), X(t - \tau)) - g(X^\circ(t), X^\circ(t - \tau)) \|_{L^2(\Omega; \mathbb{R}^d)}$ with $\kappa = q$ and $p = 4 - \frac{2}{q}$, then
\[
\| g(X(t), X(t - \tau)) - g(X^\circ(t), X^\circ(t - \tau)) \|_{L^2(\Omega; \mathbb{R}^d)} \leq \left( 1 + \| X(t) \|_{L^{4q-2}(\Omega; \mathbb{R}^d)}^q + \| X(t - \tau) \|_{L^{4q-2}(\Omega; \mathbb{R}^d)}^q \right)^{\frac{2}{q}} h^2.
\]
In summary, (2.9) holds with $\gamma = \frac{1}{2}$.

5. Numerical Experiments

Example 1. We consider the following example [23]
\[
dy(t) = [-2y(t) + y(t - 1) - y^5(t)]dt + y^2(t) dW(t),
\]
for $t \geq 0$ with initial data $y(t) = \cos(t)$.

Example 2. Next, we consider more general SDDE
\[
dy(t) = [-2y(t) + y(t - 1) - y^5(t) - y^5(t - 1)]dt + [y^2(t) + y^2(t - 1)]dW(t),
\]
for $t \geq 0$ with initial data $y(t) = \cos(t)$.
One can see that Example 1, 2 satisfy the condition (2.2), Assumption 1 with $q = 5$ and Assumption 2 with $p = 30$. Hence, we can choose the projected parameter $\alpha = \frac{1}{2(5-1)} = \frac{1}{8}$. We use discretized Brownian paths over $[0, 2]$ with $\Delta t = 2^{-13}$. For the sake of simplicity, we regard the projected method with $h = \Delta t$ as good approximation of the exact solution. And compare it with corresponding numerical solution using $h = 128\Delta t$, $h = 64\Delta t$, $h = 32\Delta t$, and $h = 16\Delta t$ over $M = 1000$ sample paths. We measure the means of absolute errors at the endpoint $t = T = 2$, and
denote

$$e_{\Delta t}^{\text{strong}} := \frac{1}{M} \sum_{i=1}^{M} |X_N^{(i)} - X(t_N)^{(i)}|,$$

where $T - \Delta < t_N = N\Delta \leq T$.

by the endpoint error in the strong sense of the projected Euler method. In Figs 1, 2, we plot means of absolute errors $e_{\Delta t}^{\text{strong}}$ against $\Delta t$ on log-log scale. For reference, a dashed blue line is added. We can observe that the convergence rate of the projected Euler method is 1/2, which is in accordance with our theoretical results.

6. Conclusion

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