Periodic elliptic operators with asymptotically preassigned spectrum

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Abstract. We deal with operators in $\mathbb{R}^n$ of the form

$$A = -\frac{1}{b(x)} \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left( a(x) \frac{\partial}{\partial x_k} \right)$$

where $a(x), b(x)$ are positive, bounded and periodic functions. We denote by $L_{\text{per}}$ the set of such operators.

The main result of this work is as follows: for an arbitrary $L > 0$ and for arbitrary pairwise disjoint intervals $(\alpha_j, \beta_j) \subset [0, L], j = 1, \ldots, m$ ($m \in \mathbb{N}$) we construct the family of operators $\{A^\varepsilon \in L_{\text{per}}\}$ such that the spectrum of $A^\varepsilon$ has exactly $m$ gaps in $[0, L]$ when $\varepsilon$ is small enough, and these gaps tend to the intervals $(\alpha_j, \beta_j)$ as $\varepsilon \to 0$. The idea how to construct the family $\{A^\varepsilon\}$ is based on methods of the homogenization theory.

Keywords: periodic elliptic operators, spectrum, gaps, homogenization.

Introduction

Our research is inspired by the following well-known result of Y. Colin de Verdière [4]: for arbitrary numbers $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_m$ ($m \in \mathbb{N}$) and $n \in \mathbb{N} \setminus \{1\}$ there is an $n$-dimensional compact Riemannian manifold $M$ such that the first $m$ eigenvalues of the corresponding Laplace-Beltrami operator $-\Delta_M$ are exactly $\lambda_1, \ldots, \lambda_m$. In the work [16] we obtained an analogue of this fact for non-compact periodic manifolds: for an arbitrary $m$ pairwise disjoint finite intervals on the positive semi-axis ($m \in \mathbb{N}$) a periodic Riemannian manifold is constructed such that the spectrum of the corresponding Laplace-Beltrami operator has at least $m$ gaps, moreover the first $m$ gaps are close (in some natural sense) to these preassigned intervals.

The goal of the present work is to solve a similar problem for the following operators in $\mathbb{R}^n$ ($n \geq 2$):

$$A = -b^{-1} \text{div}(a \nabla) = -\frac{1}{b(x)} \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left( a(x) \frac{\partial}{\partial x_k} \right), \quad a, b \in H_{\text{per}}$$

where $H_{\text{per}}$ is a set of measurable real functions in $\mathbb{R}^n$ satisfying the conditions

$$f \in H_{\text{per}}: \begin{cases} \exists C^-, C^+ > 0: & C^- \leq f(x) \leq C^+, \forall x \in \mathbb{R}^n \quad \text{(boundedness from above and form below)} \\ \forall i \in \mathbb{Z}^n, \forall x \in \mathbb{R}^n: & f(x + i) = f(x) \quad \text{(periodicity)} \end{cases}$$

The operator $A$ acts in the space $L_{2,b}(\mathbb{R}^n) = \left\{ u \in L_2(\mathbb{R}^n), \|u\|_{L_{2,b}(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} |u(x)|^2 b(x) dx \right\}$, it is self-adjoint and positive. We denote by $L_{\text{per}}$ the set of such operators.
Operators of this type occur in various areas of physics, for example in the case $n = 3$ the operator $A$ governs the propagation of acoustic waves in a medium with periodically varying mass density $(a(x))^{-1}$ and compressibility $b(x)$.

It is well-known (see e.g. [17]) that the spectrum $\sigma(A)$ of the operator $A \in L_{\text{per}}$ has band structure, i.e. $\sigma(A)$ is the union of compact intervals $[a_k^-, a_k^+] \subset [0, \infty)$ called bands $(a_k^- = 0, a_k^+ \to \infty)$. In general the bands may overlap. The open interval $(\alpha, \beta)$ is called a gap if $(\alpha, \beta) \cap \sigma(A) = \emptyset$ and $\alpha, \beta \in \sigma(A)$.

The main result of this work is the following

**Theorem 0.1 (Main Theorem).** Let $L > 0$ be an arbitrary number and let $(\alpha_j, \beta_j)$ $(j = 1, \ldots, m, m \in \mathbb{N})$ be arbitrary intervals satisfying

$$0 < \alpha_1, \quad \alpha_j < \beta_j < \alpha_{j+1}, \quad j = 1, m-1, \quad \alpha_m < \beta_m < L \quad (0.1)$$

Let $n \in \mathbb{N} \setminus \{1\}$.

Then one can construct the family of functions $\{a^\varepsilon \in H_{\text{per}}\}$ and the function $b \in H_{\text{per}}$ such that the spectrum of the operator $A^\varepsilon = b^{-1} \text{div}(a^\varepsilon \nabla)$ has the following structure in the interval $[0, L]$ when $\varepsilon$ is small enough:

$$\sigma(A^\varepsilon) \cap [0, \ L] = [0, L] \setminus \left( \bigcup_{j=1}^{m} (\alpha_j^\varepsilon, \beta_j^\varepsilon) \right) \quad (0.2)$$

where the intervals $(\alpha_j^\varepsilon, \beta_j^\varepsilon)$ satisfy

$$\forall j = 1, \ldots, m : \quad \lim_{\varepsilon \to 0} \alpha_j^\varepsilon = \alpha_j, \quad \lim_{\varepsilon \to 0} \beta_j^\varepsilon = \beta_j \quad (0.3)$$

Moreover, $a^\varepsilon(x)$, $b(x)$ are step-functions having at most $m + 1$ values.

**Remark 0.1.** It follows from (0.1)-(0.3) that the operator $A^\varepsilon$ has exactly $m$ gaps in $[0, L]$ when $\varepsilon$ is small enough. In general, the existence of gaps in the spectra of operators from $L_{\text{per}}$ is not guaranteed, for instance in the case of constant $a(x)$, $b(x)$ the spectrum $\sigma(A)$ coincides with $[0, \infty)$. Various operators from $L_{\text{per}}$ with gaps in their spectrum were studied in the works [5–11, 22, 27] (see also the overview [12]). In these works spectral gaps are the result of high contrast either in the coefficient $a(x)$ [6, 9, 11, 27] or in the coefficient $b(x)$ [7, 8] or in both coefficients [5, 10, 22] (the last three works deal with the Laplace-Beltrami operator in $\mathbb{R}^n$ with conformally flat periodic metric; obviously, this operator belongs to $L_{\text{per}}$).

The operator $A^\varepsilon$ constructed in the present work also has high contrast in the coefficients (namely, $\lim_{\varepsilon \to 0} \left( \frac{\max_{x \in \mathbb{R}^n} a^\varepsilon(x)}{\min_{x \in \mathbb{R}^n} a^\varepsilon(x)} \right) = \infty$), but their form essentially differs from the form of the coefficients in the works mentioned above.

The idea how to construct the functions $a^\varepsilon(x)$, $b(x)$ has come from the homogenization theory. We briefly describe this construction.

Let $\varepsilon > 0$ be a small number. Let $G^\varepsilon = \bigcup_{i \in \mathbb{Z}^n} \bigcup_{j=1}^{m} G_{ij}^\varepsilon$ be a union of pairwise disjoint spherical shells $G_{ij}^\varepsilon$ lying in $\mathbb{R}^n$. It is supposed that the following conditions hold (see also Fig. 1):

- for any fixed $j \in \{1, \ldots, m\}$ the shells $G_{ij}^\varepsilon$ are centered at the nodes of $\varepsilon$-periodic lattice in $\mathbb{R}^n$,
- the shells $G_{0j}^\varepsilon$ $(j = 1, \ldots, m)$ belong to the cube $\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : 0 < x_k < \varepsilon, \forall k\}$. 

The external radius of the shells is equal to $r^e = \varepsilon r$ ($r > 0$), the thickness of their walls is equal to $d^e = \varepsilon \gamma$ ($\gamma > 3$). By $B_{ij}^e$ we denote the sphere interior to $G_{ij}^e$. We set $B^e = \bigcup_{i \in \mathbb{Z}_n} \bigcup_{j = 1}^m B_{ij}^e$.

We define the functions $a^e(x)$, $b^e(x)$ by the formulae

$$a^e(x) = \begin{cases} 1, & x \in \mathbb{R}^n \setminus G^e, \\ a_j \varepsilon^{r+1}, & x \in G_{ij}^e, \end{cases}$$

$$b^e(x) = \begin{cases} 1, & x \in \mathbb{R}^n \setminus (B^e \cup G^e), \\ b_j, & x \in B_{ij}^e \cup G_{ij}^e, \end{cases}$$

(0.4)

where $a_j, b_j$ ($j = 1, \ldots, m$) are positive constants, which will be chosen later on. We consider the operator

$$\mathcal{A}^e = -(b^e)^{-1} \text{div} (a^e \nabla) = -\frac{1}{b^e(x)} \sum_{k=1}^n \frac{\partial}{\partial x_k} \left( a^e(x) \frac{\partial}{\partial x_k} \right)$$

It will be proved (see Theorem 1.1 below) that the spectrum of $\mathcal{A}^e$ converges to the spectrum of some operator $\mathcal{A}^0$ acting in the Hilbert space $L_2(\mathbb{R}^n) \oplus \bigcup_{j=1}^m L_2(\rho_j, \sigma_j)\mathbb{R}^n_j$, where $\rho_j, \sigma_j$ ($j = 1, \ldots, m$) are positive constants. The spectrum of $\mathcal{A}^0$ coincides with the set $[0, \infty) \setminus \bigcup_{j=1}^m \left( (\sigma_j, \mu_j) \right)$, where the intervals $(\sigma_j, \mu_j)$ satisfy

$$0 < \sigma_1, \sigma_j < \mu_j < \sigma_{j+1}, \ j = 1, m-1, \ \sigma_m < \mu_m < \infty$$

and depend in a special way on $a_j$ and $b_j$.

More precisely, we will prove that for an arbitrary $L > \mu_k$ the spectrum of the operator $\mathcal{A}^e$ has the following structure in the interval $[0, L]$ when $\varepsilon$ is small enough:

$$\sigma(\mathcal{A}^e) \cap [0, L] = [0, L] \setminus \bigcup_{j=1}^m \left( \sigma_j^e, \mu_j^e \right)$$

where the intervals $(\sigma_j^e, \mu_j^e)$ satisfy

$$\forall j = 1, \ldots, m : \lim_{\varepsilon \to 0} \sigma_j^e = \sigma_j, \lim_{\varepsilon \to 0} \mu_j^e = \mu_j$$

FIG. 1.
Furthermore, we will prove (see Theorem 1.2 below) that for arbitrary intervals \((\alpha_j, \beta_j) (j = 1, \ldots, m \in \mathbb{N})\) satisfying (0.1) one can choose such \(\alpha_j, \beta_j\) in (0.4) that the following equalities hold:

\[
\forall j = 1, \ldots, m : \quad \sigma_j = \alpha_j, \mu_j = \beta_j
\]  

(0.5)

Finally we set (below \(y \in \mathbb{R}^n\))

\[
a^e(y) = \varepsilon^{-2} a^e(x), \quad b(y) = b^e(x), \quad \text{where} \quad x = y\varepsilon
\]

(0.6)

(obviously, \(b(y)\) is independent of \(\varepsilon\)). It is clear that \(a^e, b\) belong to \(H_{\text{per}}\) and are step-functions having at most \(m + 1\) values. It is easy to see that the spectra of the operator

\[
A^e = b^{-1} \text{div}(a^e \nabla)
\]

and the operator \(A^\varepsilon\) coincide (in fact, \(A^e\) is obtained from \(A^\varepsilon\) via change of variables \(x = y\varepsilon\)).

It follows from Theorem 1.1-1.2 that \(\sigma(A^e)\) satisfies (0.2)-(0.3).

We remark that the gaps open up in the spectrum of \(A^e\) because of the high contrast in the coefficient \(a^e(x)\). The coefficient \(b(x)\) is independent of \(\varepsilon\) and it is needed only in order to control the behavior of the gaps as \(\varepsilon \to 0\). In fact, the operator \(-\text{div}(a^e \nabla)\) also has at least \(m\) gaps when \(\varepsilon\) is small enough, but in general they do not converge to \((\alpha_j, \beta_j)\) as \(\varepsilon \to 0\).

**Heuristic arguments.** The classical problem of the homogenization theory (see e.g. [1–3, 18, 24–26]) is to describe the asymptotic behaviour as \(\varepsilon \to 0\) of the operator \(A^\varepsilon\) which acts in \(L^2(\Omega)\) (\(\Omega \subset \mathbb{R}^n\) is a bounded domain) and is defined by the operation

\[
A^\varepsilon_{\Omega} = -\text{div}(a^\varepsilon \nabla)
\]

and either Dirichlet or Neumann boundary conditions on \(\partial\Omega\). Here

\[
a^\varepsilon(x) = a(x\varepsilon^{-1}), \quad \text{where} \quad a \in H_{\text{per}}
\]

(0.6)

It is well-known that \(A^\varepsilon\) strongly resolvent converges to the operator (so-called “homogenized operator”)

\[
A^0_{\Omega} = -\sum_{k,l=1}^n \tilde{a}^{kl} \frac{\partial^2}{\partial x_k \partial x_l}
\]

where the constants \(\tilde{a}^{kl}\) satisfy: \(\exists C^-, C^+ > 0\) s.t. \(\forall \xi \in \mathbb{R}^n\) \(C^-|\xi|^2 \leq \tilde{a}^{kl} \xi_k \xi_l \leq C^+|\xi|^2\).

It is interesting to study the asymptotic behaviour of the operator \(A^\varepsilon\) when \(a^\varepsilon\) has more complicated form comparing with (0.6). In particular interest is the case when \(a^\varepsilon\) is bounded below but not uniformly in \(\varepsilon\). This is just our situation (see (0.4)): for fixed \(\varepsilon\) one has \(\min_{x \in \mathbb{R}^n} a^\varepsilon(x) > 0\), but \(\lim_{\varepsilon \to 0} \left(\min_{x \in \mathbb{R}^n} a^\varepsilon(x)\right) = 0\). Such type problems were widely studied in [18 Chapter 7]. In particular, the authors considered the operator \(A^{\varepsilon}_{\Omega}\) which acts in \(L^2(\Omega)\) and is defined by the operation

\[
A^{\varepsilon}_{\Omega} = -\text{div}(a^\varepsilon \nabla)
\]

and the Dirichlet boundary conditions on \(\partial\Omega\). Here \(\Omega \subset \mathbb{R}^n\) is a bounded domain, \(a^\varepsilon\) is defined by (0.4) (only the case \(m = 1\) was considered). It was proved that \(A^{\varepsilon}_{\Omega}\) converges as \(\varepsilon \to 0\) (in some sense which is close to strong resolvent convergence) to the operator \(A^{0,\text{D}}_{\Omega}\) acting in the space \(L^2(\Omega) \oplus L^2_{\rho/\sigma}(\Omega)\) and being defined by the operation

\[
A^{0,\text{D}}_{\Omega} = \begin{pmatrix}
-\tilde{a}\Delta + \rho & -\rho \\
-\sigma & \sigma
\end{pmatrix}
\]

(0.7)

and the definitional domain \(D(A^{0,\text{D}}_{\Omega}) = \{(u, v) \in H^2(\Omega) \oplus L^2_{\rho/\sigma}(\Omega) : u|_{\partial\Omega} = 0\}\). Here \(\tilde{a}, \rho, \sigma\) are positive constants that do not depend on \(\Omega\). A similar result is valid for the operator \(A^{N,\text{D}}_{\Omega}\) (the
superscripts "D" and "N" mean Dirichlet and Neumann boundary conditions): the corresponding homogenized operator $\mathcal{A}_{\Omega}^{D,0}$ is defined by operation (0.7) and the definitional domain $\mathcal{D}(\mathcal{A}_{\Omega}^{D,0}) = \{(u, v) \in H^2(\Omega) \oplus L_2\rho/\sigma(\Omega) : \, \frac{\partial u}{\partial n}|_{\partial \Omega} = 0\}$.

Although in general the strong resolvent convergence of operators does not imply the Hausdorff convergence of their spectra (see the definition at the beginning of Section 5), but suppose for a moment that this is true for the operators $\mathcal{A}_{\Omega}^{D,\varepsilon}$ and $\mathcal{A}_{\Omega}^{N,\varepsilon}$, i.e.

$\sigma(\mathcal{A}_{\Omega}^{D,\varepsilon}) \rightarrow \sigma(\mathcal{A}_{\Omega}^{D,0}), \, \sigma(\mathcal{A}_{\Omega}^{N,\varepsilon}) \rightarrow \sigma(\mathcal{A}_{\Omega}^{N,0})$ in the Hausdorff sense

We denote $\Omega_R = \{x \in \mathbb{R}^n : \, |x| < R\}$. One can prove (for example, it follows from [15 Proposition 2.3]) that

$$\forall \Omega \subset \mathbb{R}^n : \, (\sigma, \mu) \cap \sigma(\mathcal{A}_{\Omega}^{D/N,0}) = \emptyset$$

$$\forall [d^-, d^+] \subset [0, \infty) \setminus (\sigma, \mu) \, \exists R_d > 0 : \, \sigma(\mathcal{A}_{\Omega_k}^{D/N,0}) \cap [d^-, d^+] \neq \emptyset \text{ for } R > R_d$$

where $D/N$ is either $D$ or $N$, $\mu = \sigma + \rho$. These suggest that when $\varepsilon$ is small enough the operator $\mathcal{A}^\varepsilon$ has a gap in the spectrum and this gap tends to the interval $(\sigma, \mu)$ as $\varepsilon \rightarrow 0$.

The close problem was also considered in [21] where the authors studied the asymptotic behavior of the attractors for semilinear hyperbolic equation $\partial_t^2 u + \mathcal{A}_{\Omega_k}^{D,\varepsilon} u + f^\varepsilon(u) = h^\varepsilon$.

We remark that the proof of the resolvent convergence in [18] is based on the method of so-called "local energy characteristics". This method is well adapted for both periodic and non-periodic operators but it is quite cumbersome. Therefore in the present work following [16] we carry out the proof in more simple fashion via the substitution of a suitable test function into the variational formulation of the spectral problem.

In the next section we describe precisely the operator $\mathcal{A}^\varepsilon$ and formulate Theorems 1.1-1.2. Their proofs are carried out in Sections 2-7.

1. Construction of operators $\mathcal{A}^\varepsilon$ and main results

Let $n \in \mathbb{N} \setminus \{1\}, \, m \in \mathbb{N}$. Let the points $x_j \in \mathbb{R}^n \ (j = 1, \ldots, m)$ and the number $r > 0$ be such that the closed balls $B_j = \{x \in \mathbb{R}^n : \, |x - x_j| \leq r\}$ are pairwise disjoint and belong to the open cube $Y = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \, 0 < x_k < 1, \, \forall k\}$

Let $\varepsilon > 0$. We introduce the following notations (below $i \in \mathbb{Z}^n, \, j = 1, \ldots, m$):

$x^i_{ij} = \varepsilon (x_j + i)$

$$G^\varepsilon_{ij} = \{x \in \mathbb{R}^n : \, r^\varepsilon - d^\varepsilon < |x - x^i_{ij}| < r^\varepsilon\}, \quad B^\varepsilon_{ij} = \{x \in \mathbb{R}^n : \, |x - x^i_{ij}| < r^\varepsilon - d^\varepsilon\}$$

where

$$r^\varepsilon = r\varepsilon, \quad d^\varepsilon = \varepsilon^\gamma, \quad \gamma > 3$$

We also denote

$$G^\varepsilon = \bigcup_{i \in \mathbb{Z}^n} \bigcup_{j=1}^m G^\varepsilon_{ij}, \quad B^\varepsilon = \bigcup_{i \in \mathbb{Z}^n} \bigcup_{j=1}^m B^\varepsilon_{ij}, \quad F^\varepsilon = \mathbb{R}^n \setminus \left(G^\varepsilon \cup B^\varepsilon\right)$$

We will prove this statement in Section 5 (the only difference is that we will consider quasi-periodic boundary conditions, but for Dirichlet and Neumann boundary conditions the proof is similar.)
We define the piecewise constant functions \( a^\varepsilon(x) \), \( b^\varepsilon(x) \) by the formulae

\[
\begin{align*}
  a^\varepsilon(x) &= \begin{cases} 
    1, & x \in F^e \cup B^e, \\
    a^\varepsilon_j, & x \in G_{ij}^e \ (i \in \mathbb{Z}^n, \ j = 1, \ldots, m), 
  \end{cases} \\
  b^\varepsilon(x) &= \begin{cases} 
    1, & x \in F^e, \\
    b_j, & x \in B_{ij}^e \cup G_{ij}^e \ (i \in \mathbb{Z}^n, \ j = 1, \ldots, m), 
  \end{cases}
\end{align*}
\]  \tag{1.1}

where \( a_j, b_j \ (j = 1, \ldots, m) \) are positive constants.

Now we define precisely the operator \( \mathcal{A}^\varepsilon \). By \( L_{2,\varepsilon}(\mathbb{R}^n) \) we denote the Hilbert space of functions from \( L_2(\mathbb{R}^n) \) with the following scalar product:

\[
(u, v)_{L_{2,\varepsilon}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} u(x)v(x)b^\varepsilon(x)dx,
\]

Remark that

\[
C^-\|u\|_{L_{2,\varepsilon}(\mathbb{R}^n)} \leq \|u\|_{L_{2,\varepsilon}(\mathbb{R}^n)} \leq C^+\|u\|_{L_{2,\varepsilon}(\mathbb{R}^n)} \tag{1.3}
\]

where the positive constants \( C^-, C^+ \) are independent of \( \varepsilon \). By \( \eta_{2,\varepsilon}^e[u, v] \) we denote the sesquilinear form in \( L_{2,\varepsilon}(\mathbb{R}^n) \) which is defined by the formula

\[
\eta_{2,\varepsilon}^e[u, v] = \int_{\mathbb{R}^n} a^\varepsilon(x)(\nabla u, \nabla v)dx
\]

with \( \text{dom}(\eta_{2,\varepsilon}^e) = H^1(\mathbb{R}^n) \). Here \( (\nabla u, \nabla v) = \sum_{k=1}^n \frac{\partial u}{\partial x_k} \frac{\partial v}{\partial x_k} \). The form is densely defined, closed and positive. Then (see e.g. [14]) there exists the unique self-adjoint and positive operator \( \mathcal{A}^\varepsilon \) associated with the form \( \eta_{2,\varepsilon}^e[u, v] \), i.e.

\[
(\mathcal{A}^\varepsilon u, v)_{L_{2,\varepsilon}(\mathbb{R}^n)} = \eta_{2,\varepsilon}^e[u, v], \quad \forall u \in \text{dom}(\mathcal{A}^\varepsilon), \ \forall v \in \text{dom}(\eta_{2,\varepsilon}^e) \tag{1.4}
\]

Its domain \( \text{dom}(\mathcal{A}^\varepsilon) \) consists of functions \( u \) belonging to the spaces \( H^2(F^e), H^2(G_{ij}^e), H^2(B_{ij}^e) \) (for any \( i \in \mathbb{Z}^n, \ j = 1, \ldots, m \)) and satisfying the following conditions on the boundaries of the shells \( G_{ij}^e \):

\[
\begin{align*}
  (u)^+ &= (u)^-, \quad \frac{\partial u}{\partial n}^+ = a_j^\varepsilon \left( \frac{\partial u}{\partial n}^\varepsilon \right)^-, \quad x \in \partial \left( B_{ij}^e \cup G_{ij}^e \right), \\
  (u)^+ &= (u)^-, \quad a_j^\varepsilon \left( \frac{\partial u}{\partial n}^\varepsilon \right)^+ = \left( \frac{\partial u}{\partial n}^\varepsilon \right)^-, \quad x \in \partial B_{ij}^e
\end{align*}
\]  \tag{1.5}

where by + (resp. −) we denote the values of the function \( u \) and its normal derivative on the exterior (resp. interior) side of either \( \partial \left( B_{ij}^e \cup G_{ij}^e \right) \) or \( \partial B_{ij}^e \). For sufficiently smooth \( u \) the operator \( \mathcal{A}^\varepsilon \) is defined locally by the formula

\[
\mathcal{A}^\varepsilon u = -\frac{1}{b^\varepsilon(x)} \sum_{k=1}^n \frac{\partial}{\partial x_k} \left( a^\varepsilon(x) \frac{\partial u}{\partial x_k} \right)
\]  \tag{1.6}

By \( \sigma(\mathcal{A}^\varepsilon) \) we denote the spectrum of the operator \( \mathcal{A}^\varepsilon \). In order to describe the behaviour of \( \sigma(\mathcal{A}^\varepsilon) \) as \( \varepsilon \to 0 \) we introduce some additional notations.
In the domain \( F = Y \setminus \bigcup_{j=1}^{m} B_j \) we consider the following problem (below \( k = 1, \ldots, n \)):

\[
\begin{aligned}
\Delta v_k &= 0, \; x \in F \\
\frac{\partial v_k}{\partial n} &= n_k, \; x \in \partial \left( \bigcup_{j=1}^{m} B_j \right)
\end{aligned}
\]

where \( v_k, Dv_k \) are \( Y \)-periodic, i.e. \( \forall \alpha = 1, \ldots, n : \begin{cases} v_k(x) = v_k(x + e_\alpha) \\ \frac{\partial v_k}{\partial x_\alpha}(x) = \frac{\partial v_k}{\partial x_\alpha}(x + e_\alpha) \end{cases} \) for \( x = (x_1, x_2, \ldots, 0, \ldots, x_n) \)

\( \sigma \)-th place

(1.7)

where \( n = (n_1, \ldots, n_n) \) is the outward unit normal to \( \bigcup_{j=1}^{m} B_j, e_\alpha = (0, 0, \ldots, 1, \ldots, 0) \). It is known (see e.g. [3]) that the unique (up to a constant) solution \( v_k(x) \) of this problem exists. We denote

\[
\widehat{a}^k = \frac{1}{|F|} \int_{F} (\nabla(x_k - v_k), \nabla(x_l - v_l)) \, dx, \; k, l = 1, \ldots, n
\]

The matrix \( \widehat{A} = \{\widehat{a}^k\} \) is symmetric and positively defined (see e.g. [3] Chapter 1, Proposition 2.6)).

Remark 1.1. In the case when \( m = 1 \) and the center of ball \( B_1 \) coincides with the center of the cube \( Y \) the matrix \( \widehat{A} = \{\widehat{a}^k\} \) has more simple form, namely \( \widehat{A} = \overline{a}I \) where \( I \) is the identity matrix, \( \overline{a} > 0 \). This follows easily from the symmetry of the domain \( F \).

We denote

\[
\sigma_j = \frac{na_j}{rb_j}, \; \rho_j = \frac{a_j|\partial B_j|}{|F|}
\]

(1.8)

We assume that the numbers \( a_j \) and \( b_j \) in (1.1)-(1.2) are such that \( \sigma_i \neq \sigma_j \) if \( i \neq j \). For definiteness we suppose that \( \sigma_j < \sigma_{j+1}, \; j = 1, \ldots, n-1 \).

And finally let us consider the following equation (with unknown \( \lambda \in \mathbb{C} \)):

\[
\mathcal{F}(\lambda) \equiv 1 + \sum_{j=1}^{m} \frac{\rho_j}{\sigma_j - \lambda} = 0
\]

(1.9)

It is easy to prove (see Section 4) that this equation has exactly \( m \) roots \( \mu_j (j = 1, \ldots, m) \), they are real, moreover they interlace with \( \sigma_j \), i.e.

\[
\sigma_j < \mu_j < \sigma_{j+1}, \; j = 1, m-1, \; \sigma_m < \mu_m < \infty
\]

Now we are able to formulate the theorem describing the behaviour of \( \sigma(\mathcal{A}^\varepsilon) \) as \( \varepsilon \to 0 \).

**Theorem 1.1.** Let \( L \) be an arbitrary number such that \( L > \mu_m \). Then the spectrum \( \sigma(\mathcal{A}^\varepsilon) \) of the operator \( \mathcal{A}^\varepsilon \) has the following structure in \([0, L]\) when \( \varepsilon \) is small enough:

\[
\sigma(\mathcal{A}^\varepsilon) \cap [0, L] = [0, L] \setminus \bigcup_{j=1}^{m} (\sigma_j^\varepsilon, \mu_j^\varepsilon)
\]

(1.10)

where the intervals \((\sigma_j^\varepsilon, \mu_j^\varepsilon)\) satisfy

\[
\forall j = 1, \ldots, m : \lim_{\varepsilon \to 0} \sigma_j^\varepsilon = \sigma_j, \; \lim_{\varepsilon \to 0} \mu_j^\varepsilon = \mu_j
\]

(1.11)
The set $[0, \infty) \setminus \left( \bigcup_{j=1}^{m} (\sigma_j, \mu_j) \right)$ coincides with the spectrum $\sigma(\mathcal{A}^0)$ of the self-adjoint operator $\mathcal{A}^0$ which acts in the space $L_2(\mathbb{R}^n) \oplus_{j=1,m} L_{2,\rho_j/\sigma_j}(\mathbb{R}^n)$ and is defined by the formula

$$\mathcal{A}^0 U = \begin{pmatrix} -\sum_{k,l=1}^{n} a_{kl} \frac{\partial^2 u}{\partial x_k \partial x_l} + \sum_{j=1}^{m} \rho_j (u - u_j) \\ \sigma_1 (u_1 - u) \\ \sigma_2 (u_2 - u) \\ \vdots \\ \sigma_m (u_m - u) \end{pmatrix}, \quad U = \begin{pmatrix} u \\ u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} \in \text{dom}(\mathcal{A}^0) = H^2(\mathbb{R}^n) \oplus_{j=1,m} L_{2,\rho_j/\sigma_j}(\mathbb{R}^n)$$

To complete the proof of Theorem 1.1 we have to choose such $a_j$ and $b_j$ in (1.1), (1.2) that (0.5) holds.

**Theorem 1.2.** Let $(\alpha_j, \beta_j) \ (j = 1, \ldots, m)$ be arbitrary intervals satisfying (0.7). Then (0.5) holds if we choose

$$a_j = \frac{|F|}{|\partial B_j|} (\beta_j - \alpha_j) \prod_{i=1, i \neq j}^{m} \left( \frac{\beta_i - \alpha_i}{\alpha_i - \alpha_j} \right), \quad b_j = \frac{n|F|}{\ell \partial B_j} \left( \frac{\beta_j - \alpha_j}{\alpha_j} \right) \prod_{i=1, i \neq j}^{m} \left( \frac{\beta_i - \alpha_i}{\alpha_i - \alpha_j} \right)$$

(1.12)

**Remark 1.2.** Since the intervals $(\alpha_j, \beta_j)$ satisfy (0.1) then

$$\forall j : \beta_j > \alpha_j, \quad \forall i \neq j : \text{sign}(\beta_i - \alpha_j) = \text{sign}(\alpha_i - \alpha_j) \neq 0$$

Therefore $(\beta_j - \alpha_j) \prod_{i=1, i \neq j}^{m} \left( \frac{\beta_i - \alpha_i}{\alpha_i - \alpha_j} \right) > 0$ and thus the choice of $a_j$ and $b_j$ is correct.

The scheme of the proof of these theorems is as follows.

In Section 2 we introduce the functional spaces and operators that are used throughout the proof. Also we present well-known results describing the spectrum of the operator $\mathcal{A}^\varepsilon$.

In Section 3 we prove several technical lemmas.

In Section 4 we show that

$$\sigma(\mathcal{A}^0) = [0, \infty) \setminus \left( \bigcup_{j=1}^{m} (\sigma_j, \mu_j) \right)$$

(1.13)

Section 5 is a crucial part of the proof: we show that as $\varepsilon \to 0$ the set $\sigma(\mathcal{A}^\varepsilon)$ converges in the Hausdorff sense to the set $\sigma(\mathcal{A}^0)$.

In Section 6 we prove that for an arbitrary $L > 0$ the spectrum $\sigma(\mathcal{A}^\varepsilon)$ has at most $m$ gaps within the interval $[0, L]$ when $\varepsilon$ is small enough. Together with the Hausdorff convergence this fact implies the statements of Theorem 1.1.

And finally in Section 7 we prove Theorem 1.2.

**Remark 1.3.** We present the proof of Theorem 1.1 for the case $n \geq 3$ only. For the case $n = 2$ the proof is repeated word-by-word with some small modifications (for example in formula (3.10) below $r^{2-n}$ has to be replaced by $\ln r$).

2. Preliminaries: functional spaces and operators

Below $\Omega$ is a domain in $\mathbb{R}^n$ with Lipschitz boundary (if $\partial \Omega \neq \emptyset$), for simplicity we suppose that $\partial \Omega \cap \bigcup_{i,j} G_{ij} = \emptyset$. Throughout the paper we will use the following functional spaces:
• $L_{2,\varepsilon}(\Omega)$ be the Hilbert space of functions from $L_2(\Omega)$ with the scalar product
\[(u, v)_{L_{2,\varepsilon}(\Omega)} = \int_{\Omega} u(x)v(x) b^\varepsilon(x) \, dx\]

• $H^1(\Omega)$ be the subspace of $H^1(\Omega)$ consisting of functions vanishing on $\partial\Omega$,

• $C^\infty(\Omega)$ be the space of functions from $C^\infty(\Omega)$ compactly supported in $\Omega$,

• $H^{2,\varepsilon}(\Omega)$ be the space of functions from $C^\infty(\Omega)$ compactly supported in $\Omega$.

For $u, v \in H^1(\Omega)$ we denote
\[
\eta_{\Omega}^{\varepsilon}[u, v] = \int_{\Omega} a^\varepsilon(x) (\nabla u, \nabla v) \, dx \tag{2.1}
\]

By $\eta_{\Omega}^{N,\varepsilon}$ (resp. $\eta_{\Omega}^{D,\varepsilon}$) we denote the sesquilinear form defined by formula (2.1) and the definitional domain $H^1(\Omega)$ (resp. $H^1(\Omega)$).

Similarly to the operator $\mathcal{A}^\varepsilon$ (see (1.4)) we define the operator $\mathcal{A}^{N,\varepsilon}_\Omega$ (resp. $\mathcal{A}^{D,\varepsilon}_\Omega$) as the operator acting in $L_{2,\varepsilon}(\Omega)$ and associated with the form $\eta_{\Omega}^{N,\varepsilon}$ (resp. $\eta_{\Omega}^{D,\varepsilon}$). The definitional domain $\text{dom}(\mathcal{A}^{N,\varepsilon}_\Omega)$ (resp. $\text{dom}(\mathcal{A}^{D,\varepsilon}_\Omega)$) consists of functions from $H^{2,\varepsilon}(\Omega)$ satisfying the condition $\frac{\partial u}{\partial n}|_{\partial\Omega} = 0$ (resp. $u|_{\partial\Omega} = 0$) that justifies the upper index "N" (resp. "D") which indicates the Neumann (resp. Dirichlet) boundary conditions.

The spectra of the operators $\mathcal{A}^{N,\varepsilon}_\Omega$, $\mathcal{A}^{D,\varepsilon}_\Omega$ are purely discrete. We denote by $\left\{\lambda_k^{N,\varepsilon}(\Omega)\right\}_{k \in \mathbb{N}}$ (resp. $\left\{\lambda_k^{D,\varepsilon}(\Omega)\right\}_{k \in \mathbb{N}}$) the sequence of eigenvalues of $\mathcal{A}^{N,\varepsilon}_\Omega$ (resp. $\mathcal{A}^{D,\varepsilon}_\Omega$) written in the increasing order and repeated according to their multiplicity.

Now let us describe the structure of the spectrum $\sigma(\mathcal{A}^\varepsilon)$ of the operator $\mathcal{A}^\varepsilon$. The operator $\mathcal{A}^\varepsilon$ is periodic with respect to the periodic cell
\[Y_0^\varepsilon = \{ x \in \mathbb{R}^a : 0 < x_k < \varepsilon, \forall k \} \]

We denote $T^a = \{ \theta = (\theta_1, \ldots, \theta_a) \in \mathbb{C}^a : |\theta_k| = 1, \forall k \}$. For $\theta \in T^a$ we introduce the functional space $H^1_\theta(Y_0^\varepsilon)$ consisting of functions from $H^1(Y_0^\varepsilon)$ that satisfy the following condition on $\partial Y_0^\varepsilon$:
\[\forall k = 1, n : \quad u(x + \varepsilon e_k) = \theta_k u(x) \quad \text{for} \quad x = (x_1, x_2, \ldots, 0, \ldots, x_n) \tag{2.2}\]

where $e_k = (0, 0, \ldots, 1, \ldots, 0)$.

By $\eta_{Y_0^\varepsilon}^{\varepsilon}$ we denote the sesquilinear form defined by formula (2.1) (with $Y_0^\varepsilon$ instead of $\Omega$) and the definitional domain $H^1_\theta(Y_0^\varepsilon)$.

We define the operator $\mathcal{A}^{\theta,\varepsilon}_{Y_0^\varepsilon}$ as the operator acting in $L_{2,\varepsilon}(Y_0^\varepsilon)$ and associated with the form $\eta_{Y_0^\varepsilon}^{\varepsilon}$. Its definitional domain $\text{dom}(\mathcal{A}^{\theta,\varepsilon}_{Y_0^\varepsilon})$ consists of the functions from $H^{2,\varepsilon}(Y_0^\varepsilon)$ satisfying the condition (2.2) and the condition
\[\forall k = 1, n : \quad \frac{\partial u}{\partial x_k}(x + \varepsilon e_k) = \theta_k \frac{\partial u}{\partial x_k}(x) \quad \text{for} \quad x = (x_1, x_2, \ldots, 0, \ldots, x_n) \]
The operator $\mathcal{A}^\theta_{Y_0}$ has purely discrete spectrum. We denote by $\{\lambda_k^{\theta,\epsilon}(Y_0^n)\}_{k \in \mathbb{N}}$ the sequence of eigenvalues of $\mathcal{A}^\theta_{Y_0}$ written in the increasing order and repeated according to their multiplicity.

From the min-max principle (see e.g. [23]) and the enclosure $H^1(Y_0^n) \supset H^1_0(Y_0^n) \supset H^1(Y_0^n)$ one can easily obtain the inequality

$$\forall k \in \mathbb{N} : \lambda_k^{N,\epsilon}(Y_0^n) \leq \lambda_k^{D,\epsilon}(Y_0^n) \leq \lambda_k^{N,\epsilon}(Y_0^n)$$  \quad (2.3)

The following fundamental result (see e.g. [17]) establishes the relationship between the spectra of the operators $\mathcal{A}^\epsilon$ and $\mathcal{A}^\theta_{Y_0^n}$.

**Theorem.** One has

$$\sigma(\mathcal{A}^\epsilon) = \bigcup_{k=1}^{\infty} \mathcal{J}_k(\mathcal{A}^\epsilon)$$  \quad (2.4)

where $\mathcal{J}_k(\mathcal{A}^\epsilon) = \bigcup_{\theta \in \mathbb{R}^{2n}} \{\lambda_k^{\theta,\epsilon}(Y_0^n)\}$. The sets $\mathcal{J}_k(\mathcal{A}^\epsilon)$ are compact intervals.

**Remark 2.1.** It is clear that if $\epsilon^{-1} \in \mathbb{N}$ then $\mathcal{A}^\epsilon$ is also $Y$-periodic operator, i.e. $a^\epsilon(x+i) = a^\epsilon(x)$, $b^\epsilon(x+i) = b^\epsilon(x)$ for any $i \in \mathbb{Z}^n, x \in \mathbb{R}^n$. So in this case we have an analogous representation

$$\sigma(\mathcal{A}^\epsilon) = \bigcup_{k=1}^{\infty} \tilde{\mathcal{J}}_k(\mathcal{A}^\epsilon)$$  \quad (2.5)

where $\tilde{\mathcal{J}}_k(\mathcal{A}^\epsilon) = \bigcup_{\theta \in \mathbb{R}^2} \{\lambda_k^{\theta,\epsilon}(Y_0^n)\}$, $\lambda_k^{\theta,\epsilon}(Y_0^n)$ is the $k$-th eigenvalue of the operator $\mathcal{A}^{\theta,\epsilon}_{Y_0^n}$ which acts in $L_{2,\phi_{\epsilon}}(Y_0^n)$ and is defined by the operation (1.6) and the definitional domain

$$\text{dom}(\mathcal{A}^{\theta,\epsilon}_{Y_0^n}) = \left\{ u \in H^{2,\epsilon}_\partial(Y_0^n) : \forall k = 1, n \left\{ \begin{array}{ll} u(x+e_k) = \theta_k u(x) \\ \frac{\partial u}{\partial x_k}(x+e_k) = \theta_k \frac{\partial u}{\partial x_k}(x) \end{array} \right. \text{ for } x = (x_1, x_2, \ldots, 0, \ldots, x_n) \right\}$$

Studying the Hausdorff convergence of $\sigma(\mathcal{A}^\epsilon)$ as $\epsilon \to 0$ we will use the representation (2.5), while estimating the number of gaps in the interval $[0, L]$ we will use the representation (2.4).

3. **Auxiliary lemmas**

In this section we prove some technical lemmas. In order to formulate them we introduce some additional notations.

We denote

$$\kappa = \frac{1}{2} \min_{j=1,m} \text{dist} \left( B_j, \partial Y \cup \bigcup_{i \neq j} B_k \right)$$

Recall that the closed balls $B_j$ are pairwise disjoint and belong to the open cube $Y$, hence $\kappa > 0$.

We introduce the following sets (below $i \in \mathbb{Z}^n$, $j = 1, \ldots, m$):

- $Y^\epsilon_i = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : i \epsilon < x_k < (i + 1) \epsilon, \forall k\}$
- $F^\epsilon_i = Y^\epsilon_i \setminus \bigcup_{j=1}^{m} \left( B^\epsilon_{ij} \cup G^\epsilon_{ij} \right)$
- $R^\epsilon_{ij} = \{x \in \mathbb{R}^n : \epsilon < |x - x^\epsilon_{ij}| < \epsilon + \kappa \epsilon\}$
- $D^\epsilon_{ij} = \{x \in \mathbb{R}^n : |x - x^\epsilon_{ij}| < \epsilon + \kappa \epsilon\} = B^\epsilon_{ij} \cup \overline{C^\epsilon_{ij}} \cup R^\epsilon_{ij}$
- $S^\epsilon_{ij} = \{x \in \mathbb{R}^n : |x - x^\epsilon_{ij}| = \epsilon + \kappa \epsilon\} = \partial D^\epsilon_{ij}$
• $\hat{C}_{ij} = \{ x \in \mathbb{R}^n : |x - x_{ij}^e| = r^e \}$
• $\check{C}_{ij} = \{ x \in \mathbb{R}^n : |x - x_{ij}^e| = r^e - d^e \}$

We also denote
\[
I^e = \left\{ i = (i_1, \ldots, i_n) \in \mathbb{Z}^n : 0 \leq i_k \leq (\varepsilon^{-1} - 1), \forall k \right\}
\]
and set
\[
G^e = \bigcup_{i \in I^e} \bigcup_{j=1}^m G^e_{ij}, \quad B^e = \bigcup_{i \in I^e} \bigcup_{j=1}^m B^e_{ij}, \quad F^e = \bigcup_{i \in I^e} F^e_i
\]

Remark that if $\varepsilon^{-1} \in \mathbb{N}$ then $\bar{Y} = \bigcup_{i \in I^e} \bar{Y}^e_i$.

By $\langle u \rangle_B$ we denote the average value of the function $u$ over the domain $B \subset \mathbb{R}^n$ (if $|B| \neq 0$), i.e. $\langle u \rangle_B = \frac{1}{|B|} \int_B u(x) dx$. If $\Sigma \subset \mathbb{R}^n$ is a $(n-1)$-dimensional surface then the Euclidean metrics in $\mathbb{R}^n$ induces on $\Sigma$ the Riemannian metrics and measure. We denote by $ds$ the density of this measure. Again by $\langle u \rangle_{\Sigma}$ we denote the average value of the function $u$ over $\Sigma$, i.e $\langle u \rangle_{\Sigma} = \frac{1}{|\Sigma|} \int_{\Sigma} u ds$ (here $|\Sigma| = \int_{\Sigma} ds$).

If $\eta[u,v]$ is a sesquilinear form then we preserve the same notation $\eta$ for the corresponding quadratic form, i.e $\eta[u] = \eta[u,u]$.

By $\chi_\Omega$ we denote an indicator function of the domain $\Omega$, i.e. $\chi_\Omega(x) = 1$ for $x \in \Omega$ and $\chi_\Omega(x) = 0$ otherwise.

In what follows by $C, C_1, \ldots$ we denote generic constants that do not depend on $\varepsilon$.

**Lemma 3.1.** Let $D$ be a convex domain in $\mathbb{R}^n$, $d$ be the diameter of $D$, $X$ and $Y$ be arbitrary measurable subsets of $D$. Then for any $v \in H^1(D)$ the following inequality holds:
\[
|\langle v \rangle_X - \langle v \rangle_Y|^2 \leq C \| \nabla v \|^2_{L^2(D)} \frac{d^{n+2}}{|X| \cdot |Y|}
\]

**Proof.** The lemma is proved in a similar way as Lemma 4.9 from [18] p.117.

**Lemma 3.2.** Let $\varepsilon = \varepsilon_N = \frac{1}{N}$, $N = 1, 2, 3 \ldots$ Let $v^e \in H^1(Y), \| v^e \|^2_{H^1(Y)} < C$, $v^e \rightarrow v \in H^1(Y)$ strongly in $L^2(Y)$. Then $\forall j = 1, m$:
\[
\sum_{i \in I^e} \langle v^e \rangle_{S^e_{ij}^e} \chi_{S^e_{ij}^e} \rightarrow v \text{ strongly in } L^2(Y) \quad (3.1)
\]
\[
\sum_{i \in I^e} \langle v^e \rangle_{F^e_{ij}^e} \chi_{F^e_{ij}^e} \rightarrow v \text{ strongly in } L^2(Y) \quad (3.2)
\]

**Proof.** For an arbitrary $i \in I^e$ and $j \in \{1, \ldots, m\}$ one has the following inequalities:
\[
\| v^e - \langle v^e \rangle_{Y^e_i} \|^2_{L^2(Y^e_i)} \leq C \varepsilon^2 \| \nabla v^e \|^2_{L^2(Y^e_i)} \quad (3.3)
\]
\[
\varepsilon^n \| \langle v^e \rangle_{Y^e_i} - \langle v^e \rangle_{F^e_{ij}^e} \|^2 \leq C \varepsilon^2 \| \nabla v^e \|^2_{L^2(Y^e_i)} \quad (3.4)
\]
\[
\varepsilon^n \| \langle v^e \rangle_{Y^e_i} - \langle v^e \rangle_{R^e_{ij}^e} \|^2 \leq C \varepsilon^2 \| \nabla v^e \|^2_{L^2(Y^e_i)} \quad (3.5)
\]
\[
\varepsilon^n \| \langle v^e \rangle_{S^e_{ij}^e} - \langle v^e \rangle_{R^e_{ij}^e} \|^2 \leq C \varepsilon^2 \| \nabla v^e \|^2_{L^2(R^e_{ij}^e)} \quad (3.6)
\]
Inequality (3.3) is the Poincaré inequality, inequalities (3.4)-(3.5) follow directly from Lemma 3.1. Let us prove inequality (3.6). We introduce in \( R_{ij}^e \) the spherical coordinates \((r, \Theta)\), where \( r \) is a distance to \( x_{ij}^e \), \( \Theta \) are the angle coordinates. Below by \( S_{n-1} \) we denote the \((n - 1)\)-dimensional unit sphere, by \( d\Theta \) we denote the Riemannian measure on \( S_{n-1} \). One has

\[
v^e(r^e + \kappa \varepsilon, \Theta) - v^e(r, \Theta) = \int_r^{r^e + \kappa \varepsilon} \frac{\partial v^e}{\partial \rho}(\rho, \Theta)d\rho, \quad r \in (r^e, r^e + \kappa \varepsilon)
\]

We multiply this equality by \( r^{n-1} drd\Theta \), integrate from \( r^e \) to \( r^e + \kappa \varepsilon \) (with respect to \( r \)) and over \( S_{n-1} \) (with respect to \( \Theta \)), divide by \( |R_{ij}^e| \) and square. Using the Cauchy inequality we obtain

\[
\left| (v^e)_{S_{ij}^e} - (v^e)_{R_{ij}^e} \right|^2 \leq \frac{1}{|R_{ij}^e|} \int_{S_{n-1}} \int_{r^e}^{r^e + \kappa \varepsilon} \left( \int_r^{r^e + \kappa \varepsilon} \frac{\partial v^e}{\partial \rho}(\rho, \Theta)d\rho \right) r^{n-1} drd\Theta \leq C \left( \int_{S_{n-1}} \left( \int_{r^e}^{r^e + \kappa \varepsilon} \left( \int_r^{r^e + \kappa \varepsilon} \frac{\partial v^e}{\partial \rho}(\rho, \Theta)d\rho \right)^2 \rho^{n-1} d\Theta \right) \left( \int_{S_{n-1}} \frac{dp}{\rho^{n-1}} \right) \leq C_1 \left\| \nabla v^e \right\|_{L^2(R_{ij}^e)}^2 \varepsilon^{2-n}
\]

and thus (3.6) is proved.

It is clear that (3.1) follows from (3.3), (3.5), (3.6), and (3.2) follows from (3.3), (3.4).

**Lemma 3.3.** The following inequality is valid for an arbitrary \( v \in H^1(D_{ij}^e) \):

\[
\left\| v \right\|_{L^2(G_{ij}^e)}^2 \leq C \varepsilon^{\gamma - 1} \left\{ \eta_{G_{ij}^e}^e[v] + \varepsilon^2 \eta_{R_{ij}^e}^e[v] + \left\| v \right\|_{L^2(R_{ij}^e)}^2 \right\}
\]

(3.7)

**Proof.** As in the proof of Lemma 3.2 we introduce in \( G_{ij}^e \) the spherical coordinates \((r, \Theta)\). One has

\[
v(r, \Theta) = v(r^e, \Theta) + \int_{r^e}^{r} \frac{\partial v}{\partial \rho}(\rho, \Theta)d\rho, \quad r \in (r^e - d^e, r^e)
\]

Taking into account (1.1) we obtain from (3.8)

\[
\int_{S_{n-1}} \int_{r^e - d^e}^{r^e} |v(r, \Theta)|^2 r^{n-1} drd\Theta \leq 2 \left( \int_{r^e - d^e}^{r^e} r^{n-1} dr \right) \left( \int_{S_{n-1}} |v(r^e, \Theta)|^2 (r^e)^{n-1} d\Theta \right) + \int_{S_{n-1}} \left( \int_{r^e - d^e}^{r^e} \left( \int_{r^e}^{r} \frac{\partial v}{\partial \rho}(\rho, \Theta)d\rho \cdot \int_{r^e}^{r} \frac{dp}{\rho^{n-1}} \right) d\Theta \right) \leq C \left( \varepsilon^\gamma \left\| v \right\|_{L^2(G_{ij}^e)}^2 + \varepsilon^{\gamma - 1} \eta_{G_{ij}^e}^e[v] \right)
\]

Similarly we obtain

\[
\left\| v \right\|_{L^2(C_{ij})}^2 \leq C \left( \varepsilon^{-1} \left\| v \right\|_{L^2(R_{ij}^e)}^2 + \varepsilon \left\| \nabla v \right\|_{L^2(R_{ij}^e)}^2 \right)
\]

The statement of the lemma follows directly from the last two inequalities. □

**Lemma 3.4.** \( \lim_{\varepsilon \to 0} \lambda_1^e(D_{ij}^e) = \sigma_j \), where \( \sigma_j \) \((j = 1, \ldots, m)\) are defined by (1.8).
Proof. Let \( v_{ij}^\varepsilon \in \text{dom}(\mathcal{A}_{D_{ij}^\varepsilon}) \) be the eigenfunction corresponding to \( \lambda_{1,D_{ij}^\varepsilon}(D_{ij}^\varepsilon) \) and such that
\[
\int_{B_{ij}^\varepsilon} v_{ij}^\varepsilon(x)dx = |B_{ij}^\varepsilon| \tag{3.9}
\]
Instead of calculating \( v_{ij}^\varepsilon \) in the exact form we construct a convenient approximation \( \mathbf{v}_{ij}^\varepsilon \) for it.

We introduce in \( D_{ij}^\varepsilon \) the spherical coordinates \((r, \Theta, \varphi)\), \( r \in [0, r^\varepsilon + \kappa \varepsilon] \). Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be a twice-continuously differentiable function such that \( \varphi(\rho) = 1 \) as \( \rho \leq 1/2 \) and \( \varphi(\rho) = 0 \) as \( \rho \geq 1 \).

We define the function \( v_{ij}^\varepsilon \) by the formula (below we assume that \( \frac{3\varepsilon^2}{4} < r^\varepsilon - d^\varepsilon \) that is true for \( \varepsilon \) small enough)
\[
v_{ij}^\varepsilon(r, \Theta) = \begin{cases} 
1, & r \in [0, \frac{\varepsilon^2}{2}) \\
1 + \tilde{A}_j^\varepsilon \varepsilon^{2-n} \left(1 - \varphi\left(\frac{|x - x_j^\varepsilon|^2}{\varepsilon^2 r^2/4}\right)\right), & r \in \left[\frac{\varepsilon^2}{2}, r^\varepsilon - d^\varepsilon\right) \\
A_j^\varepsilon \varepsilon^{2-n} + B_j^\varepsilon, & r \in \left[r^\varepsilon - d^\varepsilon, r^\varepsilon\right) \\
(\tilde{A}_j^\varepsilon)^2 \varepsilon^{2-n} \varphi\left(\frac{|x - x_j^\varepsilon|^2}{\varepsilon^2 r^2/4}\right), & r \in \left[r^\varepsilon, r^\varepsilon + \kappa \varepsilon\right].
\end{cases} \tag{3.10}
\]

We choose the coefficients \( A_j^\varepsilon, \tilde{A}_j^\varepsilon, \hat{A}_j^\varepsilon, B_j^\varepsilon \) in such a way that \( v_{ij}^\varepsilon \) satisfies conditions (1.5):
\[
A_j^\varepsilon = 1 - a_j^\varepsilon \left[(r^\varepsilon - d^\varepsilon)^{2-n} - (r^\varepsilon)^{2-n}\right]^{-1} \sim \frac{r^\varepsilon - d^\varepsilon}{n - 2} \eta_{D_{ij}^\varepsilon}^\varepsilon \theta_{B_{ij}^\varepsilon}^\varepsilon \sim \frac{r^\varepsilon - d^\varepsilon}{n - 2} \eta_{D_{ij}^\varepsilon}^\varepsilon \theta_{B_{ij}^\varepsilon}^\varepsilon \sim \frac{r^\varepsilon - d^\varepsilon}{n - 2}
\]

It is clear that \( v_{ij}^\varepsilon \in \text{dom}(\mathcal{A}_{D_{ij}^\varepsilon}) \) and \( \mathcal{A} \mathbf{v}_{ij}^\varepsilon = 0 \) in \( D_{ij}^\varepsilon \setminus \{x : |x - x_j^\varepsilon| \in \left[\frac{3\varepsilon^2}{8}, \frac{3\varepsilon^2}{4}\right] \cup \left[r^\varepsilon + \frac{3\varepsilon^2}{4}, r^\varepsilon + \kappa \varepsilon\right]\}\).

Direct calculations lead to the following asymptotics as \( \varepsilon \to 0 \):
\[
\eta_{D_{ij}^\varepsilon}^\varepsilon \|v_{ij}^\varepsilon\|_{L_{0,\varepsilon}(B_{ij}^\varepsilon)} \sim a_j \|\partial B_j]\varepsilon, \quad \|v_{ij}^\varepsilon\|_{L_{2,\varepsilon}(B_{ij}^\varepsilon)} \sim b_j \|B_j]\varepsilon \tag{3.11}
\]
\[
\|\mathcal{A} \mathbf{v}_{ij}^\varepsilon\|_{L_{2,\varepsilon}(B_{ij}^\varepsilon)} = O(\varepsilon^\alpha), \quad \|v_{ij}^\varepsilon - 1\|_{L_{2,\varepsilon}(B_{ij}^\varepsilon)} + \|\varepsilon v_{ij}^\varepsilon\|_{L_{2,\varepsilon}(B_{ij}^\varepsilon) \cup B_{ij}^\varepsilon} = o(\varepsilon^\alpha) \tag{3.12}
\]

Using the min-max principle we get
\[
\lambda_{1,D_{ij}^\varepsilon}(D_{ij}^\varepsilon) = \frac{\eta_{D_{ij}^\varepsilon}^\varepsilon [v_{ij}^\varepsilon]^2}{\|v_{ij}^\varepsilon\|^2_{L_{2,\varepsilon}(B_{ij}^\varepsilon)}} \leq \frac{\eta_{D_{ij}^\varepsilon}^\varepsilon [v_{ij}^\varepsilon]^2}{\|v_{ij}^\varepsilon\|^2_{L_{2,\varepsilon}(B_{ij}^\varepsilon)}} \sim \frac{a_j \|\partial B_j\|^2}{b_j \|B_j\|^2} = \sigma_j \tag{3.13}
\]

One has the following estimates for the eigenfunction \( v_{ij}^\varepsilon \):
\[
\|v_{ij}^\varepsilon\|^2_{L_{2,\varepsilon}(B_{ij}^\varepsilon)} \leq C \varepsilon^2 \eta_{G_j^\varepsilon}^\varepsilon [v_{ij}^\varepsilon] \tag{3.14}
\]
\[
\|v_{ij}^\varepsilon - 1\|^2_{L_{2,\varepsilon}(B_{ij}^\varepsilon)} \leq C \varepsilon^2 \eta_{G_j^\varepsilon}^\varepsilon [v_{ij}^\varepsilon] \tag{3.15}
\]
\[
\|v_{ij}^\varepsilon\|^2_{L_{2,\varepsilon}(G_j^\varepsilon)} \leq C \varepsilon^{2-1} \left\{ \eta_{G_j^\varepsilon}^\varepsilon [v_{ij}^\varepsilon] + \varepsilon^2 \eta_{G_j^\varepsilon}^\varepsilon [v_{ij}^\varepsilon] + \|v_{ij}^\varepsilon\|^2_{L_{2,\varepsilon}(B_{ij}^\varepsilon)} \right\} \tag{3.16}
\]

The first one is the Friedrichs inequality, the second one is the Poincaré inequality and the third one follows from Lemma 3.3. Furthermore one has the equality
\[
\eta_{D_{ij}^\varepsilon}^\varepsilon [v_{ij}^\varepsilon] = \lambda_{1,D_{ij}^\varepsilon}(D_{ij}^\varepsilon) \left(\|v_{ij}^\varepsilon\|^2_{L_{2,\varepsilon}(B_{ij}^\varepsilon)} + b_j \|v_{ij}^\varepsilon\|^2_{L_{2,\varepsilon}(G_j^\varepsilon)} + b_j \left(\|v_{ij}^\varepsilon - 1\|^2_{L_{2,\varepsilon}(B_{ij}^\varepsilon)} + |B_{ij}^\varepsilon|\right)\right) \tag{3.17}
\]
It follows from (3.13)-(3.17) that
\[
\eta_{D_{ij}^\varepsilon}^\varepsilon [v_{ij}^\varepsilon] = O(\varepsilon^\alpha), \quad \|v_{ij}^\varepsilon - 1\|^2_{L_{2,\varepsilon}(B_{ij}^\varepsilon)} + \|v_{ij}^\varepsilon\|^2_{L_{2,\varepsilon}(G_j^\varepsilon) \cup B_{ij}^\varepsilon} = o(\varepsilon^\alpha) \text{ as } \varepsilon \to 0 \tag{3.18}
\]
Moreover (3.9), (3.18) imply
\[ \|v^{e}_{ij}\|^{2}_{L_{2,\mu}(\mathcal{B}^{e}_{ij})} \sim b_{j}|B_{j}|\varepsilon^{n} \] (3.19)

Now let us estimate the difference \( w^{e}_{ij} = v^{e}_{ij} - v^{e}_{ij} \). One has
\[ \|w^{e}_{ij}\|^{2}_{L_{2}(\mathcal{D}^{e}_{ij})} \leq 2 \left( \|v^{e}_{ij}\|^{2}_{L_{2}(\mathcal{G}^{e}_{ij} \cup \mathcal{R}^{e}_{ij})} + \|v^{e}_{ij}\|^{2}_{L_{2}(\mathcal{G}^{e}_{ij} \cup \mathcal{R}^{e}_{ij})} \right) + 2 \left( \|v^{e}_{ij} - 1\|^{2}_{L_{2}(\mathcal{B}^{e}_{ij})} + \|1 - v^{e}_{ij}\|^{2}_{L_{2}(\mathcal{B}^{e}_{ij})} \right) \]
and thus in view of (3.12), (3.18) we conclude that
\[ \|w^{e}_{ij}\|^{2}_{L_{2}(\mathcal{D}^{e}_{ij})} = o(\varepsilon^{n}) \] (3.20)
Furthermore using inequality (3.13) we get
\[ \eta^{e}_{ij}[w^{e}_{ij}] \leq 2(\mathcal{A}^{e}v^{e}_{ij},w^{e})_{L_{2,\mu}(\mathcal{D}^{e}_{ij})} + \left( \frac{\eta^{e}_{ij}[v^{e}_{ij}]}{\|v^{e}_{ij}\|^{2}_{L_{2,\mu}(\mathcal{D}^{e}_{ij})}}\right) \|v^{e}_{ij}\|^{2}_{L_{2,\mu}(\mathcal{D}^{e}_{ij})} - \eta^{e}_{ij}[v^{e}_{ij}] \]
and in view of (3.11), (3.12), (3.18)-(3.20) we conclude that
\[ \eta^{e}_{ij}[w^{e}_{ij}] = o(\varepsilon^{n}) \] (3.21)
The statement of the lemma follows directly from (3.11), (3.20), (3.21).

**Lemma 3.5.** \( \lim_{\varepsilon \to 0} \lambda_{2}^{D,e}(\mathcal{D}^{e}_{ij}) = \infty \)

**Proof.** We denote:
\[
\begin{align*}
\mathcal{B}^{e} &= \{ y \in \mathbb{R}^{n} : 0 \leq |y| < r - \varepsilon y \}, & \mathcal{B} &= \{ y \in \mathbb{R}^{n} : 0 \leq |y| < r \\
\mathcal{G}^{e} &= \{ y \in \mathbb{R}^{n} : r - \varepsilon y < |y| < r \} \\
\mathcal{R} &= \{ y \in \mathbb{R}^{n} : r < |y| < r + \kappa \}, & \mathcal{D} &= \{ y \in \mathbb{R}^{n} : 0 \leq |y| < r + \kappa \}
\end{align*}
\]
Also we introduce the functions \( a^{e}(y), b(y) \):
\[
\begin{align*}
a^{e}(y) &= a^{e}(y_{e} + x^{e}_{ij}), & b(y) &= b^{e}(y_{e} + x^{e}_{ij}), & y \in \mathcal{D}
\end{align*}
\]
(it is clear that \( b \) in independent of \( e \)).

By \( A^{D,e}_{D} \) we denote the operator acting in \( L_{2,\mu}(\mathcal{D}) \) and being defined by the operation
\[
A^{D,e}_{D} = -\frac{1}{b(y)} \sum_{k=1}^{n} \frac{\partial}{\partial y_{k}} \left( a^{e}(y) \frac{\partial}{\partial y_{k}} \right)
\]
and the definitional domain \( \text{dom}(A^{D,e}_{D}) \) which consists of functions \( \nu \) belonging to \( H^{2}(\mathcal{B}^{e}), H^{2}(\mathcal{G}^{e}), H^{2}(\mathcal{R}) \) and satisfying the conditions
\[
\begin{align*}
(a^{e})^{\pm}(\nu)^{\pm} &= \left( \frac{\partial \nu}{\partial n} \right)^{\pm} = a_{j}^{\pm} \left( \frac{\partial \nu}{\partial n} \right)^{\pm}, & y \in \partial \mathcal{B} \\
(a^{e})^{\pm}(\nu)^{\pm} &= \left( \frac{\partial \nu}{\partial n} \right)^{\pm} = \left( \frac{\partial \nu}{\partial n} \right)^{\pm}, & y \in \partial \mathcal{B}^{e} \\
v &= 0, & y \in \partial \mathcal{D}
\end{align*}
\]

We denote by \( \lambda_{k}^{D,e}(\mathcal{D}) \) the \( k \)-th eigenvalue of the operator \( A^{D,e}_{D} \). It is clear that
\[ \forall k \in \mathbb{N} : \lambda_{k}^{D,e}(\mathcal{D}) = \varepsilon^{2}\lambda_{k}^{D,e}(\mathcal{D}^{e}_{ij}) \] (3.22)
Below we will prove that
\[ \forall k \in \mathbb{N} : \lambda_k^{D_{\mathcal{E}}(\mathbf{D})} \rightarrow \lambda_k \]  
(3.23)
where \( \lambda_k \) is the \( k \)-th eigenvalue of the operator \( \mathbf{A} \) which acts in the space \( L_2(\mathbf{R}) \oplus L_{2,b}(\mathbf{B}) \) and is defined by the formula
\[
\mathbf{A} = -\begin{pmatrix} \Delta_{\mathbf{R}}^{D,N} & 0 \\ 0 & b_j^{-1} \Delta_{\mathbf{B}}^{N} \end{pmatrix}
\]
Here the operator \( \Delta_{\mathbf{R}}^{D,N} \) (resp. \( \Delta_{\mathbf{B}}^{N} \)) is defined by the operation \( \Delta \) and the definitional domain consisting of functions \( v \in H^2(\mathbf{R}) \) (resp. \( v \in H^2(\mathbf{B}) \)) satisfying the conditions
\[
v|_{\partial \mathcal{D}} = 0, \quad \frac{\partial v}{\partial n}|_{\partial \mathcal{R} \cup \partial \mathcal{D}} = 0 \quad \text{(resp.} \quad \frac{\partial v}{\partial n}|_{\partial \mathcal{B} \cup \partial \mathcal{D}} = 0 \text{)}
\]

It is clear that \( \lambda_1 = 0 \) (\( \lambda_1 \) coincides with the first eigenvalue of \( -b_j^{-1} \Delta_{\mathbf{B}}^{N} \)) while
\[ \lambda_2 > 0 \]  
(3.24)
(\( \lambda_2 \) coincides either with the first eigenvalue of \( -\Delta_{\mathbf{R}}^{D,N} \) or with the second eigenvalue of \( -b_j^{-1} \Delta_{\mathbf{B}}^{N} \)).

Then for any \( k \in \mathbb{N} \), the subsequence \( \varepsilon_k \) holds:
\[ \mu_k^{\varepsilon_k} \rightarrow \mu_k \]
where \( \{\mu_k^{\varepsilon_k}\}_{k=1}^{\infty} \) and \( \{\mu_k\}_{k=1}^{\infty} \) are the eigenvalues of the operators \( \mathcal{L}^\varepsilon \) and \( \mathcal{L}^0 \), which are renumbered in the increasing order with account of their multiplicity.

Let us apply this theorem. We set \( \mathcal{H}^\varepsilon = L_{2,b}(\mathbf{D}), \mathcal{H}^0 = L_2(\mathbf{R}) \oplus L_{2,b}(\mathbf{B}), \mathcal{L}^\varepsilon = (\mathbf{A}_D^{D,E} + I)^{-1}, \mathcal{L}^0 = (\mathbf{A} + I)^{-1}, \mathcal{V} = \mathcal{H}^0 \). We introduce the operator \( R^\varepsilon : \mathcal{H}^0 \rightarrow \mathcal{H}^\varepsilon \) by the formula
\[
[R^\varepsilon f](y) = \begin{cases} f^R(y), & y \in \mathbf{R}, \\ f^B(y), & y \in \mathbf{B}, \end{cases}
\]
with \( f^R, f^B \in \mathcal{V} \). Evidently conditions \( C_1 \) (with \( q = 1 \)) and \( C_2 \) hold. Let us verify condition \( C_3 \).

At first we introduce the operator \( Q^\varepsilon : H^1(\mathbf{B}^\varepsilon) \rightarrow H^1(\mathbb{R}^\varepsilon) \) by the formula
\[
[Q^\varepsilon v](y) = [Q^\varepsilon \tilde{v}](k^\varepsilon y)
\]
where \( k^\varepsilon = (r - \varepsilon^{r-1})^{-1}r \), the function \( \tilde{v}^\varepsilon \in H^1(\mathbf{B}) \) is defined by the formula \( \tilde{v}^\varepsilon(y) = v(y/k^\varepsilon) \) and \( Q : H^1(\mathbf{B}) \rightarrow H^1(\mathbb{R}^\varepsilon) \) is the operator with the following properties:
\[
\forall v \in H^1(\mathbf{B}) : \quad [Qv](y) = v(y) \quad \text{for} \ y \in \mathbf{B}, \quad \|Qv\|_{H^1(\mathbb{R}^\varepsilon)} \leq C\|v\|_{H^1(\mathbf{B})}
\]
(such an operator exists, see e.g. [19]). One has
\[\forall v \in H^1(B^ε) : \quad [Q^ε v](y) = v(y) \text{ for } y \in B^ε\]

Since \(k^ε \sim 1\) as \(ε \to 0\), then, obviously,
\[\forall v \in H^1(B^ε) : \quad \|Q^ε v\|_{H^1(\mathbb{R}^n)} \leq C_1 \|v\|_{H^1(B^ε)} \quad (3.25)\]

Let \(f = (f_R, f_B) \in H^0\). We set \(f^ε = R^ε f, \quad v^ε = L^ε f^ε\). It is clear that
\[\|v^ε\|_{L^2(D)} \leq \|f^ε\|_{L^2(D)} = \|f\|_{H^0}\]

One has the following integral equality:
\[\int_D \left[ a^ε(y)(\nabla v^ε, \nabla w^ε) + b(σ_{\varepsilon^0}^ε) (v^ε w^ε - f^ε w^ε) \right] dy = 0, \quad \forall w^ε \in H^1(D) \quad (3.27)\]

Substituting into (3.27) \(w^ε = v^ε\) and taking into account (3.26) we obtain
\[\int_D a^ε|\nabla v^ε|^2 dy \leq C \quad (3.28)\]

Let \(v^ε_R \in H^1(R)\) (resp. \(v^ε_B \in H^1(B)\)) be the restrictions of \(v^ε\) onto \(R\) (resp. the restrictions of \(Q^ε v^ε\) onto \(B\)). Since \(v^ε \in \text{dom}(A^ε_R)\) then \(v^ε_R|_D = 0\). It follows from estimates (3.25), (3.26), (3.28) that the set \(\{(v^ε_R, v^ε_B)\}_{ε}\) is bounded in \(H^1(R) \oplus H^1(B)\) uniformly in \(ε\). Therefore the set \(\{(v^ε_R, v^ε_B)\}_{ε}\) is weakly compact in \(H^1(R) \oplus H^1(B)\) and in view of the embedding theorem it is compact in \(L^2(R) \oplus L^2(B)\). Let \(ε' \subset ε\) be an arbitrary subsequence for which
\[v^ε_R \xrightarrow{ε'=ε \to 0} v_R \in H^1(R) \text{ weakly in } H^1(R) \text{ and strongly in } L^2(R), \quad v^ε_R|_D = 0\]
\[v^ε_B \xrightarrow{ε'=ε \to 0} v_B \in H^1(B) \text{ weakly in } H^1(B) \text{ and strongly in } L^2(B) \quad (3.29)\]

We will prove that
\[v = L^0 f, \quad \text{where } v = (v^ε_R, v^ε_B) \quad (3.30)\]

We define the function \(w^ε \in \tilde{H}^1(D)\) by the formula
\[w^ε(x) = (w_B(x) - w_R(x)) \varphi \left( \frac{|x - x^ε_j| - (r - ε^j - 1)}{ε^j - 1} \right) + w_R(x)\]

Here \(w_R, w_B \in C^∞(\mathbb{R}^n)\) are arbitrary functions, \(\text{supp}(w_R) \subset D, \varphi : \mathbb{R} \to \mathbb{R}\) be a smooth function such that \(\varphi(ρ) = 1\) as \(ρ \leq 1/2\) and \(\varphi(ρ) = 0\) as \(ρ \geq 1\). Substituting \(w^ε\) into (3.27) we get
\[\int_R \left[ (\nabla v^ε_R, \nabla w_R) + v^ε_R w_R - f_R w_R \right] dy + \int_B \left[ (\nabla v^ε_B, \nabla w_B) + b_j (v^ε_B w_B - f_B w_B) \right] dy + \delta(ε) = 0 \quad (3.31)\]

where
\[\delta(ε) = -\int_{G^ε} \left[ (\nabla v^ε_B, \nabla w_B) + b_j (v^ε_B w_B - f_B w_B) \right] dy + \int_{G^ε} \left[ a_j^ε(\nabla v^ε, \nabla w^ε) + b_j (v^ε w^ε - f^ε w^ε) \right] dy\]

It is clear that
\[\int_{G^ε} a_j^ε|\nabla w^ε|^2 dy + \|w^ε\|_{L^2(G^ε)}^2 \leq C(ε^2 + ε^j - 1)\]
and due to (3.25), (3.26), (3.28) we get $\delta^e \to 0$ as $\varepsilon \to 0$. Taking into account (3.29) we pass to the limit as $\varepsilon = \varepsilon \to 0$ in (3.31) and obtain

$$
\int_{\mathcal{D}} \left[ (\nabla v_R, \nabla w_R) + v_R w_R - f_R w_R \right] dy + \int_{\mathcal{B}} \left[ (\nabla v_B, \nabla w_B) + b_j (v_B w_B - f_B w_B) \right] dy = 0
$$

Hence $\Lambda_{D,N}^R v_R + v_R = f_R$ and $-b_j^j \Lambda_{B}^N v_B + v_B = f_B$. Therefore (3.30) holds. In view of (3.30) $(v_R, v_B)$ is independent of the subsequence $\varepsilon'$ and thus $(v^e_R, v^e_B)$ converges to $(v_R, v_B)$ as $\varepsilon \to 0$.

Making the substitution $x = \varepsilon x + x^e_j$ in estimate (3.7) we get

$$
\|v^e\|^2_{L^2(G^e)} \leq C \varepsilon^{2-\gamma} \int_{G^e} \alpha^e_j |\nabla v^e|^2 dy + \int_{\mathcal{R}} |\nabla v^e|^2 dy + \|v^e\|^2_{L^2(R)}
$$

and therefore in view of (3.26), (3.28) we obtain (recall that $\gamma > 3$)

$$
\|v^e\|^2_{L^2(G^e)} \to 0 \quad \text{as} \quad \varepsilon \to 0
$$

Taking into account (3.29), (3.30), (3.32) we get

$$
\|L^e R f - R^e L^0 f\|^2_{H^e} \leq \|v^e_R - v_R\|^2_{L^2(R)} + \|v^e_B - v_B\|^2_{L^2(B)} + 2 \left( \|v^e\|^2_{L^2(G^e)} + \|v_B\|^2_{L^2(G^e)} \right) \to 0 \quad \text{as} \quad \varepsilon \to 0
$$

and thus $C_3$ is proved.

Finally let us verify condition $C_4$. Let $\sup \|f^e\|_{H^e} < \infty$. We denote $v^e = L^e f^e$, it is clear that the set $\{v^e\}_e$ is bounded in $H^1(D)$ uniformly in $e$. Then the set $\{(v^e_R, v^e_B)\}_e$ is bounded in $H^1(R) \oplus H^1(B)$ uniformly in $e$ and therefore the subsequence $e' \subset e$ and $w = (w_R, w_B) \in H^1(R) \oplus H^1(B) \subset H^0$ exist such that

$$
v^e_R \to w_R \quad \text{weakly in} \quad H^1(R) \quad \text{and strongly in} \quad H^1(R)
$$

$$
v^e_B \to w_B \quad \text{weakly in} \quad L^2(B) \quad \text{and strongly in} \quad L^2(B)
$$

Moreover $v^e$ satisfies (3.32), therefore $\lim_{e \to e' \to 0} \|L^0 f^e - w\|^2 = 0$. $C_4$ is proved.

We have verified the fulfilment of conditions $C_1 - C_4$. Thus the eigenvalues $\mu^e_k$ of the operator $L^e$ converge to the eigenvalues $\mu_k$ of the operator $L^0$ as $\varepsilon \to 0$. But $\lambda^e_k(D) = (\mu^e_k)^{-1} - 1$, $\lambda_k = (\mu_k)^{-1} - 1$ that implies (3.23). The lemma is proved.

\[\square\]

4. Structure of $\sigma(\mathcal{A}^0)$

In this section we prove equality (1.13).

At first we show that

$$
\lambda \in \sigma(\mathcal{A}^0) \setminus \bigcup_{j=1}^m \{\sigma_j\} \iff \lambda \mathcal{F}(\lambda) \in \sigma(\widetilde{\mathcal{A}})
$$

where $\sigma(\widetilde{\mathcal{A}})$ is the spectrum of the operator $\widetilde{\mathcal{A}} = -\sum_{k,l=1}^n \frac{\partial^2}{\partial x_k \partial x_l}$, the function $\mathcal{F}(\lambda)$ is defined by (1.9).
Fig. 2. The graph of the function $\lambda F(\lambda)$ (for $m = 3$).

Indeed let $\lambda \in \sigma(\mathcal{A}^0) \setminus \bigcup_{j=1}^{m} \{\sigma_j\}$. Then there is nonzero $F = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \in L_2(\mathbb{R}^n) \oplus \bigoplus_{j=1,m} L_{2,\rho_j/\sigma_j}(\mathbb{R}^n)$ such that

$$F \notin \text{im}(\mathcal{A}^0 - \lambda I) \quad (4.2)$$

Let us suppose the opposite, i.e. $\lambda F(\lambda) \notin \sigma(\mathcal{A})$. Then for any $g \in L_2(\mathbb{R}^n)$ there is $u \in \text{dom}(\mathcal{A})$ such that

$$\mathcal{A}u - \lambda F(\lambda)u = g \quad (4.3)$$

We set $g = f + \sum_{j=1}^{m} \frac{\rho_j f_j}{\sigma_j - \lambda}$. It follows from (4.3) that

$$\mathcal{A}^0 U - \lambda U = F, \quad \text{where } U = \begin{pmatrix} u \\ u_1 \\ \vdots \\ u_m \end{pmatrix}, \quad u_j = \frac{\sigma_j \mu + f_j}{\sigma_j - \lambda} \quad (j = 1, \ldots, m)$$

We obtain a contradiction with (4.2), hence $\lambda F(\lambda) \in \sigma(\mathcal{A})$. Converse assertion in (4.1) is proved similarly.

It is well-known that $\sigma(\mathcal{A}) = [0, \infty)$, therefore

$$\lambda \in \sigma(\mathcal{A}^0) \setminus \bigcup_{j=1}^{m} \{\sigma_j\} \text{ iff } \lambda F(\lambda) \geq 0 \quad (4.4)$$

At first we study the function $\lambda F(\lambda)$ on $\mathbb{R}$. It is easy to get (see Fig. 2) that there are the points $\mu_j$, $j = 1, \ldots, m$ such that

$$F(\mu_j) = 0, \quad j = 1, \ldots, m - 1$$

$$\sigma_j < \mu_j < \sigma_{j+1}, \quad j = 1, \ldots, m - 1, \quad \sigma_m < \mu_m < \infty$$

$$\left\{ \lambda \in \mathbb{R} \cap \bigcup_{j=1}^{m} \{\sigma_j\} : \lambda F(\lambda) \geq 0 \right\} = [0, \infty) \cap \bigcup_{j=1}^{m} \{\sigma_j, \mu_j\}$$
Let us consider the equation $\lambda F(\lambda) = a$, where $a \in [0, \infty)$. One the one hand it is equivalent to the equation $\left(\prod_{j=1}^{m}(\sigma_j - \lambda)\right)^{-1} P_{m+1}(\lambda) = 0$, where $P_{m+1}$ is a polynomial of the degree $m + 1$, and therefore in $\mathbb{C}$ this equation at most $m + 1$ roots. On the other hand on $[0, \infty)$ the equation $\lambda F(\lambda) = a$ has exactly $m + 1$ roots (see Fig. 2). Thus the set $\{\lambda \in \mathbb{C} : \lambda F(\lambda) \geq 0\}$ belong to $[0, \infty)$.

We conclude that $\lambda \in \sigma(\mathcal{A}) \setminus \bigcup_{j=1}^{m} \{\sigma_j\}$ iff $\lambda \in [0, \infty) \setminus \bigcup_{j=1}^{m} [\sigma_j, \mu_j)$. Since $\sigma(\mathcal{A})$ is a closed set then the points $\sigma_j, j = 1, m$ also belong to $\sigma(\mathcal{A})$. This completes the proof of equality (1.13).

5. Proof of Hausdorff Convergence

This section is a main part of the proof: we show that the set $\sigma(\mathcal{A}^\varepsilon)$ converges in the Hausdorff sense to the set $\sigma(\mathcal{A})$ as $\varepsilon \to 0$, that is the following conditions \((A_H)\) and \((B_H)\) hold:

\[
\text{if } \lambda^\varepsilon \in \sigma(\mathcal{A}^\varepsilon) \text{ and } \lim_{\varepsilon \to 0} \lambda^\varepsilon = \lambda \text{ then } \lambda \in \sigma(\mathcal{A}) \quad (A_H)
\]

for any $\lambda \in \sigma(\mathcal{A})$ there exists $\lambda^\varepsilon \in \sigma(\mathcal{A}^\varepsilon)$ such that $\lim_{\varepsilon \to 0} \lambda^\varepsilon = \lambda$ \((B_H)\)

5.1. Proof of condition \((A_H)\). Let $\lambda^\varepsilon \in \sigma(\mathcal{A}^\varepsilon)$, $\lim \lambda^\varepsilon = \lambda$. We have to prove that $\lambda \in \sigma(\mathcal{A})$.

If $\lambda \in \bigcup_{\varepsilon \to 0} \{\sigma_j\}$ then \((A_H)\) holds true since $\bigcup_{j=1}^{m} \{\sigma_j\} \subset \sigma(\mathcal{A})$. Therefore we focus on the case $\lambda \notin \bigcup_{j=1}^{m} \{\sigma_j\}$.

We consider the sequence $\varepsilon_N \subset \varepsilon$, where $\varepsilon_N = \frac{\varepsilon}{N}, N = 1, 2, 3 \ldots$ For convenience we preserve the same notation $\varepsilon$ having in mind the sequence $\varepsilon_N$.

Taking into account Remark 2.1 we conclude that there exists $\theta^\varepsilon \in \mathbb{T}^n$ such that $\lambda^\varepsilon \in \sigma(\mathcal{A}^\varepsilon)$. We extract a subsequence (still denoted by $\varepsilon$) such that $\theta^\varepsilon \to \theta \in \mathbb{T}^n$.

Let $u^\varepsilon \in \text{dom}(\mathcal{A}_Y^\varepsilon)$ be the eigenfunction corresponding to $\lambda^\varepsilon$ and such that

\[
||u^\varepsilon||_{L^2,\varepsilon}(Y) = 1 \text{ (and therefore } \eta_Y^\varepsilon[u^\varepsilon] = \lambda^\varepsilon) \quad (5.1)
\]

We introduce the operator $\Pi^\varepsilon : H^1(F_Y^\varepsilon) \to H^1(Y)$ such that for each $u \in H^1(F_Y^\varepsilon)$:

\[
\Pi^\varepsilon u(x) = u(x) \text{ for } x \in F_Y^\varepsilon
\]

\[
||\Pi^\varepsilon u||_{H^1(Y)} \leq C||u||_{H^1(F_Y^\varepsilon)} \quad (5.2)
\]

It is known (see e.g. [3][18]) that such an operator exists.

Also we introduce the operators $\Pi_j^\varepsilon : L^2(\bigcup_{i \in I^\varepsilon} B_{ij}^\varepsilon) \to L^2(Y)$ ($j = 1, \ldots, m$) by the formula

\[
i \in I^\varepsilon, \ x \in Y_i^\varepsilon : \ \Pi_j^\varepsilon u(x) = \langle u \rangle_{B_{ij}^\varepsilon}
\]

(recall that $Y = \bigcup_{i \in I^\varepsilon} Y_i^\varepsilon$). Using the Cauchy inequality we obtain

\[
||\Pi_j^\varepsilon u||_{L^2(Y)} \leq C||u||_{L^2(\bigcup_{i \in I^\varepsilon} B_{ij}^\varepsilon)} \quad (5.3)
\]
Moreover due to the trace theorem and therefore

\[ \Pi^\varepsilon u^\varepsilon \to u \text{ weakly in } H^1(Y) \text{ and strongly in } L_2(Y) \]

Moreover due to the trace theorem

\[ \Pi^\varepsilon u^\varepsilon \to u \text{ strongly in } L_2(\partial Y) \] (5.4)

and therefore \( u \) belong to \( H^1_\theta(Y) \), i.e.

\[ \forall k = 1, n : u(x + e_k) = \theta_k u(x), \text{ for } x = (x_1, x_2, \ldots, 0, \ldots, x_n) \] (5.5)

We denote by \( \hat{A}_Y^\varepsilon \) the operator which is defined by the operation \( \hat{A}_Y^\varepsilon u = -\sum_{k, l=1}^n \hat{d}^{kl} \partial^2 u / \partial x_k \partial x_l \) and the definitional domain \( \text{dom}(\hat{A}_Y^\varepsilon) \) consisting of functions belonging to \( H^2(Y) \) and satisfying \( \theta \)-periodic boundary conditions, i.e.

\[ \forall k = 1, n : \begin{cases} u(x + e_k) = \theta_k u(x), \\ \sum_{l=1}^n \hat{d}^{kl} \partial u / \partial x_l (x + e_k) = \theta_k \sum_{l=1}^n \hat{d}^{kl} \partial u / \partial x_l (x), \end{cases} \text{ for } x = (x_1, x_2, \ldots, 0, \ldots, x_n) \] (k-th place)

It is clear that \( \sigma(\hat{A}_Y^\varepsilon) \subset [0, \infty) \).

**Lemma 5.1.** One has

\[ u \in \text{dom}(\hat{A}_Y^\varepsilon) \text{ and } \hat{A}_Y^\varepsilon u = \lambda F(\lambda) u \] (5.6)

**Proof.** One has the following integral equality:

\[ \int_Y \left( \hat{d}^\varepsilon(x) \nabla u^\varepsilon(x), \nabla w^\varepsilon(x) \right) dx = 0, \quad \forall w^\varepsilon \in H^1_{\partial Y}(Y) \] (5.7)

In order to prove (5.6) we plug into (5.7) a function \( w^\varepsilon \) of special type and then pass to the limit as \( \varepsilon \to 0 \).

We introduce some additional notations. Let \( v_k \in C^2(F) (k = 1, \ldots, n) \) be a function that solves the problem (1.7) in \( F \). We denote by \( \hat{v}_k \) the function that belongs to \( C^2(Y) \) and coincides with \( v_k \) in \( F \) (such a function exists, see e.g. [19]). Then we extend \( \hat{v}_k \) by periodicity to the whole \( \mathbb{R}^n \) preserving the same notation for the extended function. Using a standard regularity theory one can easily prove that \( \hat{v}_k \in C^2(\mathbb{R}^n) \). We set

\[ v_k^\varepsilon(x) = \varepsilon \hat{v}_k(x \varepsilon^{-1}) \]

Let \( \nu_{ij}^e \in C^{2,\varepsilon}(D_{ij}^\varepsilon) (i \in \mathbb{Z}^n, j = 1, \ldots, m) \) be the function which is defined in \( D_{ij}^\varepsilon \) by formula (3.10), \( \text{supp}(\nu_{ij}^e) \subset D_{ij}^\varepsilon \). We redefine it by zero in \( \mathbb{R}^n \setminus D_{ij}^\varepsilon \). Recall that \( \nu_{ij}^e \) was constructed in Lemma 3.4 as an approximation for the eigenfunction \( v_{ij}^e \) of the operator \( \mathcal{A}_{D_{ij}^\varepsilon} \) which corresponds to the first eigenvalue \( \lambda_{ij}^{D,e}(D_{ij}^\varepsilon) \) and satisfies (3.9).
Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be a twice-continuously differentiable function such that \( \varphi(\rho) = 1 \) as \( \rho \leq 1/2 \) and \( \varphi(\rho) = 0 \) as \( \rho \geq 1 \). We set
\[
\varphi^\varepsilon_i(x) = 1 - \sum_{j=1}^{m} \varphi\left(\frac{|x - x^\varepsilon_j| - (r^\varepsilon - d^\varepsilon)}{d^\varepsilon}\right), \quad x \in \mathbb{R}^n
\]
Its clear that
\[
\varphi^\varepsilon_i(x) = 0 \text{ for } x \in \bigcup_{j=1}^{m} B^\varepsilon_{ij}, \quad \varphi^\varepsilon_i(x) = 1 \text{ for } x \in \mathbb{R}^n \setminus \left(\bigcup_{j=1}^{m} (G^\varepsilon_{ij} \cup B^\varepsilon_{ij})\right)
\]
\[
|D^\alpha \varphi^\varepsilon_i| < C\varepsilon^{-\alpha r} \quad (|\alpha| = 0, 1, 2, \ldots ) \text{ in } \bigcup_{j=1}^{m} G^\varepsilon_{ij}
\]
(5.8)

We cover \( \mathbb{R}^n \) by the cubes
\[
\bar{Y}^\varepsilon_i = \left\{ x \in \mathbb{R}^n : i\varepsilon < x_k < (i+1)\varepsilon + \varepsilon^{3/2} \right\}, \quad i \in \mathbb{Z}^n
\]
Let \( \left\{ \psi^\varepsilon_i(x) \right\}_{i \in \mathbb{Z}^n} \) be a partition of unity associated with this covering, that is
\[
\psi^\varepsilon_i(x) \in C^2(\mathbb{R}^n), \quad 0 \leq \psi^\varepsilon_i(x) \leq 1, \quad \sum_{i \in \mathbb{Z}^n} \psi^\varepsilon_i(x) = 1, \quad \psi^\varepsilon_i(x) = 1 \text{ if } x \in \bar{Y}^\varepsilon_i \bigcup_{l \neq i} \bar{Y}^\varepsilon_l, \quad \psi^\varepsilon_i(x) = 0 \text{ if } x \notin \bar{Y}^\varepsilon_i
\]
Moreover, analyzing a standard procedure of the construction of the partition of unity (see e.g. [20]) we can easily construct the partition of unity satisfying the following additional conditions
\[
\forall i \in \mathbb{Z}^n, \forall x \in \mathbb{R}^n : \psi_i^\varepsilon(x) = \psi_{i0}^\varepsilon(x + i\varepsilon)
\]
(5.9)
\[
|D^\alpha \psi_0^\varepsilon(x) < C\varepsilon^{-3\alpha r/2} \quad (\alpha = 0, 1, 2) \text{ for } x \in \bar{Y}_0 \cap \bigcup_{l \neq 0} \bar{Y}_l
\]
(5.10)

We consider the function \( w^\varepsilon \) of the following form:
\[
w^\varepsilon(x) = \sum_{i \in \mathbb{Z}^n} \psi^\varepsilon_i(x) \left(g_i^\varepsilon(x) + h_i^\varepsilon(x)\right)
\]
(5.11)
where
\[
g_i^\varepsilon(x) = g(x^\varepsilon_i) + \varphi_i^\varepsilon(x) \left(\sum_{k=1}^{n} \frac{\partial g}{\partial x_k}(x^\varepsilon_i)(x_k - x_k^\varepsilon_i - v_i^\varepsilon(x)) + \frac{1}{2} \sum_{k,l=1}^{n} \frac{\partial^2 g}{\partial x_k \partial x_l}(x^\varepsilon_i)(x_k - x_k^\varepsilon_i - v_k^\varepsilon(x))(x_l - x_l^\varepsilon_i - v_l^\varepsilon(x))\right)
\]
(5.12)
\[
h_i^\varepsilon(x) = \sum_{j=1}^{m} (h_j(x^\varepsilon_j) - g(x^\varepsilon_j))v_{ij}^\varepsilon
\]
(5.13)
Here \( x^\varepsilon_i \) is the center of \( Y_i^\varepsilon, g(x), h_j(x) \) are arbitrary functions from \( C^2(\mathbb{R}^n) \) satisfying
\[
\forall x \in \mathbb{R}^n, \forall i = (i_1, \ldots , i_n) \in \mathbb{Z}^n : \begin{cases}
g(x + i) = \theta_1^{i_1} \cdots \theta_n^{i_n} g(x) \\h_j(x + i) = \theta_1^{i_1} \cdots \theta_n^{i_n} h_j(x), \quad j = 1, \ldots , m
\end{cases}
\]
(5.14)
Remark that \( \frac{\partial g_i^\varepsilon}{\partial n} = 0 \text{ on } \partial G_i^\varepsilon \). Taking this into account we conclude that \( w^\varepsilon(x) \) belongs to \( C^{2,\varepsilon}(\mathbb{R}^n) \) and in view of (5.9), (5.14) and the periodicity of \( v_i^\varepsilon \) we get
\[
\forall x \in \mathbb{R}^n, \forall i \in \mathbb{Z}^n : w^\varepsilon(x + i) = \theta_1^{i_1} \cdots \theta_n^{i_n} w^\varepsilon(x)
\]
We also introduce the notations

\[ g^\varepsilon = \sum_{i \in \mathbb{Z}^n} \psi_i^\varepsilon(x)g_i^\varepsilon(x), \quad h^\varepsilon = \sum_{i \in \mathbb{Z}^n} \psi_i^\varepsilon(x)h_i^\varepsilon(x) \]

The function \( w^\varepsilon \) belong to \( H^1_\theta(Y) \). In order to obtain the function from \( H^1_\theta(Y) \) we modify \( w^\varepsilon \) multiplying it by the function which is very close to 1 in \( Y \) as \( \varepsilon \to 0 \). Namely, we define the function \( 1^\varepsilon \in C^\infty(\mathbb{R}^n) \) by the following recurrent formulae:

\[
1^\varepsilon(x_1, \ldots, x_n) = A_\alpha(x_1, \ldots, x_{n-1})x_n + B_\alpha(x_1, \ldots, x_{n-1}),
\]

\[ \alpha = 2, \ldots, n : \begin{cases} B_\alpha(x_1, \ldots, x_{n-1}) = A_{\alpha-1}(x_1, \ldots, x_{n-2})x_n + A_{\alpha-1}(x_1, \ldots, x_{n-2}), \\ A_{\alpha}(x_1, \ldots, x_{n-1}) = (\theta_\alpha/\theta_n - 1)B_\alpha(x_1, \ldots, x_{n-1}), \\ B_1 = 1, A_1 = \theta_1/\theta_2 - 1. \end{cases} \]

It is easy to see that

\[
\max_{x \in \overline{Y}} |1^\varepsilon(x) - 1| + \max_{x \in \overline{Y}} |\nabla 1^\varepsilon(x)| \to 0 \quad \varepsilon \to 0
\]

\[ 1^\varepsilon \in H^1_\theta(Y), \text{ where } \theta_\alpha/\theta = (\theta_1/\theta_1, \ldots, \theta_n/\theta_n) \]

Finally we set

\[ w^\varepsilon(x) = w^\varepsilon(x) + (1^\varepsilon(x) - 1)w^\varepsilon(x) \]

It is clear that \( w^\varepsilon \in H^1_\theta(Y) \).

Substituting \( w^\varepsilon \) into (5.7) and integrating by parts we obtain

\[
\int_Y \left( u^\varepsilon \mathcal{A} w^\varepsilon - \lambda^\varepsilon u^\varepsilon w^\varepsilon \right) b^\varepsilon dx + \int_{\partial Y} u^\varepsilon \frac{\partial w^\varepsilon}{\partial n} ds + \int_Y \left( a^\varepsilon(\nabla u^\varepsilon, \nabla((1^\varepsilon - 1)w^\varepsilon)) - \lambda^\varepsilon b^\varepsilon u^\varepsilon (1^\varepsilon - 1)w^\varepsilon \right) dx = 0 \quad (5.16)
\]

Further we will prove that the second and the third integrals in (5.16) tend to zero as \( \varepsilon \to 0 \).

Now we focus on the first integral in (5.16). Using Lemma 3.3 and taking into account (5.1), (5.8) we obtain the estimates

\[
||u^\varepsilon||_{L^2(G^\varepsilon)} \leq Ce^{\gamma - 1} \sum_{i \in \mathcal{I}^\varepsilon} \left( \eta^\varepsilon_{G^\varepsilon} [u^\varepsilon]^2 + \varepsilon^2 \eta^\varepsilon_{G^\varepsilon} [u^\varepsilon] + ||u^\varepsilon||_{L^2}^2 \right) \leq C_1 e^{\gamma - 1} \quad (5.17)
\]

\[
||\mathcal{A} g^\varepsilon||_{L^2}^2 \leq Ce^{3\gamma} \quad (5.18)
\]

Since \( \mathcal{A} h^\varepsilon = 0 \) in \( G^\varepsilon \) then in view of (1.3), (5.17), (5.18)

\[
||u^\varepsilon, \mathcal{A} w^\varepsilon||_{L^2(G^\varepsilon)} \leq Ce \to 0 \quad (5.19)
\]

Similarly we obtain

\[
\lim_{\varepsilon \to 0} (u^\varepsilon, w^\varepsilon)_{L^2(G^\varepsilon)} = 0 \quad (5.20)
\]

We denote

\[
\overline{F}_i^\varepsilon = \{ x \in F_i^\varepsilon : i\varepsilon + \varepsilon^{3/2} < x_k < (i + 1)e \}, \quad \overline{F}_i^\varepsilon = \bigcup_{i \in \mathcal{I}^\varepsilon} \overline{F}_i^\varepsilon
\]

It is clear that \( \overline{F}_i^\varepsilon = F_i^\varepsilon \setminus \left( \bigcup_{i \neq i} Y_i^\varepsilon \right) \).
Firstly we estimate $g^\varepsilon$ in $F^\varepsilon_Y \setminus \overline{F}^\varepsilon_Y$. We represent $g^\varepsilon$ in $F^\varepsilon_Y \setminus \overline{F}^\varepsilon_Y$ in the form

$$g^\varepsilon(x) = \sum_{i \in \mathbb{Z}^n} \psi_i(x) \left[ \sum_{k=1}^n g^{i,e}_{x_k}(x_k - x^i_k) + \frac{1}{2} \sum_{k,l=1}^n g^{i,e}_{x_k x_l}(x_k - x^i_k)(x_l - x^i_l) - g(x) \right] - \sum_{k=1}^n \left( g^{i,e}_k + \sum_{l=1}^n g^{i,e}_{kl}(x_l - x^i_l) - g_k(x) \right) v^e_k(x) + \frac{1}{2} \sum_{k,l=1}^n (g^{i,e}_{kl} - g_{kl}(x)) v^e_k(x) v^e_l(x) + g(x) - \sum_{k=1}^n g_k(x) v^e_k(x) + \frac{1}{2} \sum_{k,l=1}^n g_{kl}(x) v^e_k(x) v^e_l(x) \quad (5.21)$$

Here $g^{i,e} = g(x^{i,e})$, $g_k(x) = \frac{\partial g}{\partial x_k}(x)$, $g^{i,e}_{x_k} = g^{i,e}_k(x^{i,e})$, $g_{kl}(x) = \frac{\partial^2 g}{\partial x_k \partial x_l}(x)$, $g^{i,e}_{kl} = g^{i,e}_{kl}(x^{i,e})$. It follows from (5.10), (5.21) that $|\Delta g^\varepsilon(x)| < C$ for $x \in F^\varepsilon_Y \setminus \overline{F}^\varepsilon_Y$. Since $\text{dist} \left( \bigcup_{j=1}^m D_i^{e,0}, \partial Y^e_i \right) \geq \kappa \varepsilon$ then $h^\varepsilon = 0$ in $F^\varepsilon_Y \setminus \overline{F}^\varepsilon_Y$ when $\varepsilon$ is small enough and therefore

$$\left\| (u^\varepsilon, \mathcal{A}^e w^\varepsilon)_{L^2(\overline{F}^\varepsilon_Y \setminus \overline{F}^\varepsilon_Y)} \right\| \leq C\|\Delta g^\varepsilon\|_{L^2(F^\varepsilon_Y \setminus \overline{F}^\varepsilon_Y)} \leq C_1 \sqrt{|F^\varepsilon_Y \setminus \overline{F}^\varepsilon_Y|} \leq C_2 \varepsilon^{1/4} \quad (5.22)$$

Similarly we obtain

$$\lim_{\varepsilon \to 0} (u^\varepsilon, w^\varepsilon)_{L^2(\overline{F}^\varepsilon_Y \setminus \overline{F}^\varepsilon_Y)} = 0 \quad (5.23)$$

Let us study $g^\varepsilon$ and $h^\varepsilon$ in $\overline{F}^\varepsilon_Y$. It is clear that

$$\Delta g^\varepsilon = \sum_{k,l=1}^n g^{i,e}_{kl}(\nabla(x_k - v^e_k), \nabla(x_l - v^e_l)) \text{ for } x \in \overline{F}^\varepsilon_Y \quad (5.24)$$

In view of Lemma 3.1 and the Poincaré inequality one has the following estimate:

$$\left\| u^\varepsilon - \langle u^\varepsilon \rangle_{\overline{F}^\varepsilon_Y} \right\|_{L^2(\overline{F}^\varepsilon_Y)}^2 + \varepsilon^n \left( \Pi^e u^\varepsilon \right)_{Y^e} - \langle u^\varepsilon \rangle_{\overline{F}^\varepsilon_Y} \left\| + \left\| \Pi^e u^\varepsilon - \langle \Pi^e u^\varepsilon \rangle_{Y^e} \right\|_{L^2(Y^e)}^2 \leq C \varepsilon^2 \left\| \nabla \Pi^e u^\varepsilon \right\|_{L^2(Y^e)}^2 \quad (5.25)$$

Using (5.24), (5.25) and the Poincaré inequality we get

$$(u^\varepsilon, \mathcal{A}^e w^\varepsilon)_{L^2(\overline{F}^e_Y)} = - \sum_{k,l=1}^n \left[ \left( \int_{\overline{F}^\varepsilon_Y^n} \nabla(x_k - v^e_k, \nabla(x_l - v^e_l)) \right) \varepsilon^n \sum_{i \in I^e} g^{i,e}_{kl}(\Pi^e u^\varepsilon)_{Y^e_i} \right] + o(1) =
- \sum_{k,l=1}^n a^{kl}[F] \int_{\overline{F}^\varepsilon_Y}\left( u^\varepsilon \right) \frac{\partial^2 g}{\partial x_k \partial x_l} dx + o(1) \rightarrow - \sum_{k,l=1}^n a^{kl}[F] \int_{\overline{F}^\varepsilon_Y} u^\varepsilon \frac{\partial^2 g}{\partial x_k \partial x_l} dx \quad (5.26)$$

In the same way using Lemma 3.2 (for $v^\varepsilon = \Pi^e u^\varepsilon$) we obtain

$$(u^\varepsilon, g^\varepsilon)_{L^2(\overline{F}^e_Y)} = \sum_{i \in I^e} g(x^{i,e}) u^\varepsilon_{F^\varepsilon_Y} |F| \varepsilon^n + o(1) \rightarrow |F| \int_{\overline{F}^\varepsilon_Y} u(x) g(x) dx \quad (5.27)$$

(here we also use the estimate $\varepsilon^n \|u^\varepsilon\|_{F^\varepsilon_Y} - \langle u^\varepsilon \rangle_{\overline{F}^\varepsilon_Y}^2 \leq C \varepsilon^2 \|\nabla \Pi^e u^\varepsilon\|_{L^2(Y^e)}^2$ which follows from Lemma 3.1).
Let us study $h^e$ in $\tilde{F}_Y^e$. Integrating by parts and taking into account the form of the function $v_{ij}^e$ (in particular, we have the estimate $\|A^i v_{ij}^e\|_{L^2(Y)} < Ce^n$), the Poincaré inequality and Lemma 3.2, we obtain

\[
(u^e, A^i h^e)_{L^2,\mu^e(\tilde{F}_Y^e)} = (u^e, A^i h^e)_{L^2,\mu^e(\tilde{F}_Y^e)} = \sum_{j=1}^{m} \sum_{i \in I^e} (u^e_{F^j_i}) \int_{\tilde{F}_Y^e} \frac{\partial v_{ij}^e}{\partial r} \left( h_j(x^i_j) - g(x^i_j) \right) ds + \sum_{i \in I^e} (u^e - \langle u^e \rangle_{F^j_i}, A^i h^e)_{L^2,\mu^e(\tilde{F}_Y^e)} = \sum_{j=1}^{m} a_j |\partial B_j| \sum_{i \in I^e} (u^e_{F^j_i}) \left( g(x^i_j) - h_j(x^i_j) \right) e^n + o(1) \to \sum_{j=1}^{m} a_j |\partial B_j| \int_Y u(x) \left( g(x) - h_j(x) \right) dx \quad (5.28)
\]

(here $r = |x - x^i_j|$). In the same way we get

\[
\lim_{\varepsilon \to 0} (u^e, h^e)_{L^2,\mu^e(\tilde{F}_Y^e)} = 0 \quad (5.29)
\]

Let us study $h^e$ in $B_Y^e$ ($g^e = 0$ in $B_Y^e$). Integrating by parts and using the Poincaré inequality we obtain

\[
(u^e, A^i h^e)_{L^2,\mu^e(B_Y^e)} = -\sum_{j=1}^{m} \sum_{i \in I^e} (u^e_{B^j_i}) \int_{\tilde{F}_Y^e} \frac{\partial v_{ij}^e}{\partial r} \left( h_j(x^i_j) - g(x^i_j) \right) ds + o(1) = \sum_{j=1}^{m} a_j |\partial B_j| \int_Y \Pi^{e} u^e(x)(h_j(x) - g(x))dx + o(1) \to \sum_{j=1}^{m} a_j |\partial B_j| \int_Y u_j(x) \left( h_j(x) - g(x) \right) dx \quad (5.30)
\]

In the same way we get

\[
\lim_{\varepsilon \to 0} (u^e, h^e)_{L^2,\mu^e(B_Y^e)} = \sum_{j=1}^{m} |B_j| b_j \int_{\Omega} u_j(x)h_j(x)dx \quad (5.31)
\]

Finally, let us estimate the remaining integrals in (5.16). One can easily obtain that

\[
\eta^e_Y[w^e] + ||w^e||^2_{L^2,\mu^e(Y)} < C
\]

and therefore in view of (5.15)

\[
\lim_{\varepsilon \to 0} \int_Y \left( a^e(\nabla u^e, \nabla((1^e - 1)w^e)) - A^e b^e u^e(1^e - 1)w^e \right) dx = 0 \quad (5.32)
\]

It is easy to see that the function $p^e = \frac{\partial w^e}{\partial n}_{\partial Y}$ is bounded in $L^2(\partial Y)$ uniformly in $\varepsilon$ and therefore there is a subsequence (still denoted by $\varepsilon$) and $p \in L^2(\partial Y)$ such that

\[
p^e \to p \text{ weakly in } L^2(\partial Y) \quad (5.33)
\]

Moreover it is clear that $\forall k = 1, n: \ p^e(x + e_k) = -\partial x_k p^e(x)$ for $x = (x_1, x_2, \ldots, 0, \ldots, x_n)$. Therefore

\[
\forall k = 1, n : \ p(x + e_k) = -\partial x_k p(x) \quad (5.34)
\]
Taking into account \((5.4), (5.5), (5.33), (5.34)\) we get
\[
\lim_{t \to 0} \int_{\partial Y} u^t \frac{\partial w^t}{\partial n} \, ds = \int_{\partial Y} u \, ds = 0 \tag{5.35}
\]

Then taking into account \((5.19), (5.20), (5.22), (5.23), (5.26)-(5.32), (5.35)\) we pass to the limit in \((5.16)\) and obtain the equality
\[
\int_{\Omega} \left( -u(x) |F| \sum_{k,l=1}^{n} \frac{\partial^2 g}{\partial x_k \partial x_l}(x) \right) \, dx - \lambda |F| u(x) g(x) + \sum_{j=1}^{m} \left( a_j |B_j| (g(x) - h_j(x)) u(x) + a_j |B_j| (h_j(x) - g(x)) u(x) - \lambda |B_j| h_j(x) u(x) \right) \, dx = 0 \tag{5.36}
\]

Recall that \(g, h_j \in C^2(\mathbb{R}^n)\) are arbitrary functions satisfying \((5.14)\).

Plugging \(g = 0, h_j = 0\) for \(j \neq k\) into \((5.36)\) and taking into account the equality \(|\partial B_j| = |B_j| n r^{-1}\) we get
\[
u_k = \frac{\sigma_k}{\sigma_k - \lambda} u, \quad k \in \{1, \ldots, m\} \tag{5.37}
\]

Then setting \(h_j = 0\) for all \(j = 1, \ldots, m\), integrating by parts and taking into account \((5.37)\) we get
\[
\int_{\Omega} \left( \sum_{k,l=1}^{n} \frac{\partial u}{\partial x_k} \frac{\partial g}{\partial x_l} - \lambda F(\lambda) u g \right) \, dx = 0 \tag{5.38}
\]

where the function \(F(\lambda)\) is defined by \((1.9)\).

Equality \((5.38)\) is valid for an arbitrary \(g\) belonging to \(C^\infty(\mathbb{R}^n)\) and satisfying \((5.14)\). It is clear that the set of such functions is dense in \(H^1_B(Y)\). Therefore equality \((5.38)\) implies \((5.6)\). Lemma \(5.1\) is proved.

\[\square\]

**Lemma 5.2.** \(u \neq 0\).

**Proof.** Let us introduce the spherical coordinates \((r, \Theta)\) in \(D^e_{ij}\) and the function \(u^e_{ij}\) by the formula
\[
u^e_{ij}(\rho, \Theta) = \langle u^e \rangle_{S^e_{ij}(\rho)}, \text{ where } S^e_{ij}(\rho) = \left\{ x \in \mathbb{R}^n : |x - x^e_{ij}| = \rho \right\}
\]

One has the following Poincaré inequality:
\[
\|u^e - u^e_{ij}\|^2_{L^2(S^e_{ij}(\rho))} \leq C \rho^2 \|\nabla u^e\|^2_{L^2(S^e_{ij}(\rho))} \leq C_1 e^2 \|\nabla u^e\|^2_{L^2(S^e_{ij}(\rho))}
\]

(here \(\nabla_\Theta\) is a gradient on \(S^e_{ij}(\rho)\): for example in the case \(n = 2\) one has \(\nabla_\Theta u = \frac{\partial u}{\partial \rho} \frac{\partial}{\partial \Theta}\)). Integrating it by \(\rho\) from 0 to \(r^e - d^e\) and summing by \(i\) we get
\[
\sum_{i \in I^e} \|u^e - u^e_{ij}\|^2_{L^2(B^e_{ij})} \leq C e^2 \|\nabla u^e\|^2_{L^2(\bigcup_{i \in I^e} B^e_{ij})} \leq C_1 e^2 \lambda^e \tag{5.39}
\]

We denote \(u^e_{ij} = u^e_{ij} - \langle u^e \rangle_{S^e_{ij}}\). Clearly \(u^e_{ij} \in \text{dom}(\mathcal{A}^{D^e}_{D^e_{ij}})\) and
\[
\mathcal{A}^{D^e}_{D^e_{ij}} u^e_{ij} - \lambda^e u^e_{ij} = \lambda^e \langle u^e \rangle_{S^e_{ij}}
\]
Recall that $\lambda \notin \bigcup_{j=1}^{m} \{\sigma_{j}\}$. Then in view of Lemmas 3.4, 3.5, $\lambda^{e} \notin \sigma(\mathcal{A}_{D_{ij}^{e}}^{e})$ when $\varepsilon$ is small enough. Therefore we have the following expansion:

$$u_{ij}^{e} = \sum_{k=1}^{\infty} I_{k}(\varepsilon), \text{ where } I_{k}(\varepsilon) = v_{k}^{D}(D_{ij}^{e}) \frac{\left(\lambda^{e}(u_{ij})_{S_{ij}^{e}}, v_{k}^{D}(D_{ij}^{e})\right)_{L_{2,\varepsilon}(D_{ij}^{e})}}{\|v_{k}^{D}(D_{ij}^{e})\|^{2}_{L_{2,\varepsilon}(D_{ij}^{e})}\left(\lambda_{k}^{D,\varepsilon}(D_{ij}^{e}) - \lambda^{e}\right)} \quad (5.40)$$

Here $\left\{v_{k}^{D}(D_{ij}^{e})\right\}_{k=1}^{m}$ is a system of eigenfunctions of $\mathcal{A}_{D_{ij}^{e}}^{e}$ corresponding to $\left\{\lambda_{k}^{D,\varepsilon}(D_{ij}^{e})\right\}_{k=1}^{m}$ and such that $\left(\lambda_{k}^{D}(D_{ij}^{e}), v_{k}^{D}(D_{ij}^{e})\right)_{L_{2,\varepsilon}(D_{ij}^{e})} = 0$ if $k \neq l$.

Using Lemmas 3.2, 3.3 we get (for $j \in \{1, \ldots, m\}$)

$$\sum_{i \in I_{e}} \left\|\sum_{k=1}^{\infty} I_{k}(\varepsilon)\right\|^{2}_{L_{2}(B_{ij}^{e})} \leq C \max_{k=2, \infty} \left|\lambda_{k}^{D,\varepsilon}(D_{ij}^{e}) - \lambda^{e}\right|^{-2} \sum_{i \in I_{e}} \left|\bar{u}_{ij}^{e}\right|^{2}_{S_{ij}^{e}} \varepsilon^{n} \to 0 \quad (5.41)$$

As in Lemma 3.4, we denote $v_{i}^{e} = v_{1}^{D}(D_{ij}^{e})$ assuming that $v_{i}^{e}$ is normalized by condition (3.9). Using estimates (3.16), (3.18) and Lemma 3.2 we get

$$\sum_{i \in I_{e}} \left\|I_{1}(\varepsilon)\right\|^{2}_{L_{2}(B_{ij}^{e})} \sim \sum_{i \in I_{e}} \frac{\lambda^{2}|B_{j}|}{(\sigma_{j} - \lambda)^{2}} \left|\bar{u}_{ij}^{e}\right|^{2}_{S_{ij}^{e}} \varepsilon^{n} \sim \frac{\lambda^{2}|B_{j}||u|^{2}_{L_{2}(Y)}}{(\sigma_{j} - \lambda)^{2}} \quad (5.42)$$

as $\varepsilon \to 0$. It follows from (5.40)–(5.42) that

$$\lim_{\varepsilon \to 0} \sum_{i \in I_{e}} \left\|u_{ij}^{e}\right\|^{2}_{L_{2}(B_{ij}^{e})} = \frac{\lambda^{2}|B_{j}||u|^{2}_{L_{2}(Y)}}{(\sigma_{j} - \lambda)^{2}} \quad (5.43)$$

Similarly we obtain

$$\int_{B_{ij}^{e}} u_{ij}^{e} dx \sim \frac{|\bar{u}_{ij}^{e}|_{S_{ij}^{e}} \lambda|B_{j}|}{\sigma_{j} - \lambda} \varepsilon^{n} \text{ as } \varepsilon \to 0 \quad (5.44)$$

Using (3.16), (3.18), (5.43), (5.44) and Lemma 3.2 we get

$$\sum_{i \in I_{e}} \left\|u_{ij}^{e}\right\|^{2}_{L_{2,\varepsilon}(G_{ij}^{e})} = \sum_{i \in I_{e}} \left\|u_{ij}^{e}\right\|^{2}_{L_{2}(B_{ij}^{e})} + 2\left\|u_{ij}^{e}\right\|_{L_{2}(B_{ij}^{e})} \int_{B_{ij}^{e}} u_{ij}^{e}(x) dx + \left\|\bar{u}_{ij}^{e}\right\|^{2}_{S_{ij}^{e}} \cdot |B_{j}|^{e} \varepsilon^{n} \text{ as } \varepsilon \to 0 \quad (5.45)$$

Using the Poincaré inequality and Lemma 3.2 one can easily prove that

$$\left\|\bar{u}_{ij}^{e}\right\|^{2}_{L_{2,\varepsilon}(F_{ij}^{e})} = |F| \sum_{i \in I_{e}} \left|\bar{u}_{ij}^{e}\right|_{F_{ij}^{e}} \varepsilon^{n} + o(1) \to |F| \cdot \left\|u\right\|^{2}_{L_{2}(Y)} \quad (5.46)$$

Furthermore in view of Lemma 3.3

$$\lim_{\varepsilon \to 0} \left\|u_{ij}^{e}\right\|^{2}_{L_{2,\varepsilon}(G_{ij}^{e})} = 0 \quad (5.47)$$
Finally taking into account (5.39), (5.45)-(5.47) we obtain
\[ 1 = \|u^\varepsilon\|^2_{L^2(\varepsilon)} \rightarrow \|u\|^2_{L^2(\varepsilon)} \left[ |F| + \sum_{j=1}^{m} \left( \frac{\sigma_j}{\sigma_j - \lambda} \right)^2 |B_j||b_j| \right] \]
and therefore $u \neq 0$. Lemma 5.2 is proved.

It follows from Lemmas 5.1, 5.2 that $\lambda \mathcal{F}(\lambda)$ belong to the spectrum $\sigma(\mathcal{A}_0)$ of the operator $\mathcal{A}_0$. Therefore $\lambda \mathcal{F}(\lambda) \in [0, \infty)$ and in view of (4.4), $\lambda \in \sigma(\mathcal{A}^e)$. Condition (A.1) is proved.

5.2. **Proof of condition (B.1).** Let $\lambda \in \sigma(\mathcal{A}^0)$. Let us prove that there is $\lambda^e \in \sigma(\mathcal{A}^e)$ such that $\lim_{\varepsilon \to 0} \lambda^e = \lambda$.

We assume the opposite: the subsequence (still denoted by $\varepsilon$) and $\delta > 0$ exist such that
\[ \text{dist}(\lambda, \sigma(\mathcal{A}^0)) > \delta. \quad (5.48) \]

Since $\lambda \in \sigma(\mathcal{A}^0)$ then the function $F = \left\{ f \left[ f_i \begin{array}{c} f_1 \\ \vdots \\ f_m \end{array} \right] \in L_2(\mathbb{R}^n) \bigoplus L_2(\mathbb{R}^n) \right\}$ exists such that
\[ F \notin \text{im}(\mathcal{A} - \lambda I), \text{ where I is the identical operator} \quad (5.49) \]

It follows from (5.48) that $\lambda \in \mathbb{R} \setminus \sigma(\mathcal{A}^e)$. Then $\text{im}(\mathcal{A}^e - \lambda I) = L_{2,\rho}^e(\mathbb{R}^n)$ and hence for an arbitrary $f^e \in L_{2,\rho}^e(\mathbb{R}^n)$ there is the unique $u^e \in \text{dom}(\mathcal{A}^e)$ such that
\[ \mathcal{A}^e u^e - \lambda u^e = f^e \quad (5.50) \]

We substitute the following $f^e(x) \in L_{2,\rho}^e(\mathbb{R}^n)$ into (5.50):
\[ f^e(x) = \begin{cases} (\langle f \rangle)^e, & x \in F_i^e, \\ (\langle f_j \rangle)^e, & x \in B_{ij}^e, \\ 0, & x \in \bigcup_{i,j} G_{ij}^e. \end{cases} \]

It is clear that the norms $\|f^e\|^2_{L_{2,\rho}^e(\mathbb{R}^n)}$ are bounded uniformly in $\varepsilon$. Then in view of (5.48) $u^e$ satisfies the inequality
\[ \|u^e\|^2_{L_{2,\rho}^e(\mathbb{R}^n)} \leq \delta^{-1} \|f^e\|^2_{L_{2,\rho}^e(\mathbb{R}^n)} \leq C \]

Furthermore
\[ \|\nabla u^e\|^2_{L^2(\mathbb{R}^n)} \leq \|f^e\|^2_{L_{2,\rho}^e(\mathbb{R}^n)} + |\lambda| \cdot \|u^e\|^2_{L_{2,\rho}^e(\mathbb{R}^n)} \leq C \]

Hence a subsequence (still denoted by $\varepsilon$) and $u \in H^1(\mathbb{R}^n), u_j \in L_2(\mathbb{R}^n)$ such that
\[ \Pi^e u^e \rightarrow u \text{ weakly in } H^1(\mathbb{R}^n) \text{ and strongly in } L_2(G) \text{ for any compact set } G \subset \mathbb{R}^n \]
\[ \Pi_j^e u^e \rightarrow u_j \text{ weakly in } L_2(\mathbb{R}^n) \text{, } (j = 1, \ldots, m) \]

where $\Pi^e, \Pi_j^e \quad (j = 1, \ldots, m)$ are the operators introduced above in the proof of condition (A.1).

For an arbitrary $w^e \in C^\infty(\mathbb{R}^n)$ one has the following integral equality:
\[ \int_{\mathbb{R}^n} \left( \phi^e(x)(\nabla u^e(x), \nabla w^e(x)) - \lambda^e b^e(x)u^e(x)w^e(x) - b^e(x)f^e(x)w^e(x) \right) dx = 0 \quad (5.51) \]
We substitute into (5.51) the function \( u^e \) of the form (5.11)-(5.13), but with \( g, h_j \in \mathcal{C}^\infty(\mathbb{R}^n) \). Making the same calculations as in the proof of condition (A) we obtain

\[
\int_{\mathbb{R}^n} \left[-u(x) \sum_{k,l=1}^n \partial^2 g \frac{\partial^2 g}{\partial x_k \partial x_l}(x) - \lambda|F|u(x)g(x) - |F|f(x)g(x) + \sum_{j=1}^m (a_j|\partial B_j|(g(x) - h_j(x)))u(x) + a_j|\partial B_j|(h_j(x) - g(x))u_j(x) - \lambda|B_j|b_j u_j(x)h_j(x) - |B_j|b_j f_j(x)h_j(x)\right]dx = 0 \tag{5.52}
\]

for an arbitrary \( g, h_j \in \mathcal{C}^\infty(\mathbb{R}^n) \). It follows from (5.52) that

\[
U = \begin{pmatrix} u \\ u_1 \\ \cdots \\ u_m \end{pmatrix} \in \text{dom}(\mathcal{A}^0) \quad \text{and} \quad \mathcal{A}^0 U - \lambda U = F
\]

We obtain a contradiction with (5.49). Condition (B) is proved.

6. End of proof of Theorem 1.1

In general the Hausdorff convergence of \( \sigma(\mathcal{A}^e) \) to \( \sigma(\mathcal{A}^0) \) does not imply (1.10)-(1.11). However if we prove that \( \sigma(\mathcal{A}^e) \) has at most \( m \) gaps in \([0, L]\) when \( \varepsilon \) is less some \( \varepsilon_L \) then this implication holds true. More precisely the following simple proposition is valid.

**Proposition 6.1.** Let \( \mathcal{B}^e = [0, L) \setminus \left( \bigcup_{j=1}^m (\alpha_j^e, \beta_j^e) \right) \), \( \mathcal{B} = [0, L) \setminus \left( \bigcup_{j=1}^m (\alpha_j, \beta_j) \right) \), where \( L < \infty \) and

\[
\begin{align*}
0 &\leq \alpha_1^e, \quad \alpha_j^e < \beta_j^e \leq \alpha_{j+1}^e, \quad j = 1, m^e - 1, \quad \alpha_{m^e}^e \leq L \\
0 &< \alpha_1, \quad \alpha_j < \beta_j < \alpha_{j+1}, \quad j = 1, m - 1, \quad \alpha_m < L \\
m^e &\leq m
\end{align*}
\]

\( \mathcal{B}^e \) converges to \( \mathcal{B} \) in the Hausdorff sense as \( \varepsilon \to 0 \)

Then \( m^e = m \) when \( \varepsilon \) is small enough and

\[
\forall j = 1, \ldots, m : \quad \lim_{\varepsilon \to 0} \alpha_j^e = \alpha_j, \quad \lim_{\varepsilon \to 0} \beta_j^e = \beta_j
\]

We introduce the notation \([a_k^-(\varepsilon), a_k^+(\varepsilon)] := \bigcup_{\theta \in \mathbb{R}^n} \left\{ \lambda_k^\theta(\varepsilon Y_0^e) \right\} \).

**Lemma 6.1.** \( \lim_{\varepsilon \to 0} a_{m+1}^+(\varepsilon) = \infty \)

**Proof.** In the same way as in the proof Lemma 5.5 we obtain the following equality

\[
\lim_{\varepsilon \to 0} \varepsilon^2 \lambda_k^{N,\varepsilon}(\varepsilon Y_0^e) = \lambda_k, \quad k = 1, 2, 3, \ldots \tag{6.1}
\]

\footnote{For example, the set \( \sigma^e := \sigma(\mathcal{A}^0) \cap \left( \bigcup_{k \in \mathbb{N}} [\varepsilon k, \varepsilon (k + \frac{1}{2})] \right) \) also converges to \( \sigma(\mathcal{A}^0) \) in the Hausdorff sense, but the number of gaps in \( \sigma^e \cap [0, L] \) tends to infinity as \( \varepsilon \to 0 \).}
where \( \{ \lambda_k \}_{k \in \mathbb{N}} \) are the eigenvalues of the operator \( A \) which acts in the space \( L_2(F) \oplus \bigoplus_{j=1}^{m} L_{2,\beta_j}(B_j) \) and is defined by the operation

\[
A = \begin{pmatrix}
\Delta_F^N & 0 & \cdots & 0 \\
0 & b_1^{-1} \Delta_{B_1}^N & & \\
& \cdots & \ddots & \\
0 & 0 & \cdots & b_m^{-1} \Delta_{B_m}^N
\end{pmatrix}
\]

(here \( \Delta_F^N \) and \( \Delta_{B_j}^N \) are the Neumann Laplacians in \( F \) and \( B_j \)). It is clear that \( \lambda_j = 0 \) for \( j = 1, \ldots, m + 1 \) while \( \lambda_{m+2} > 0 \). Then using (6.1) and taking into account (2.3) we get

\[
\lim_{\epsilon \to 0} a_{m+2}^-(\epsilon) \geq \lim_{\epsilon \to 0} A_{m+2}(Y_0^\epsilon) = \lambda_{m+2} \lim_{\epsilon \to 0} \epsilon^{-2} = \infty
\]

Suppose that there is a subsequence (still denoted by \( \epsilon \)) such that the numbers \( a_{m+1}^+(\epsilon) \) are bounded uniformly in \( \epsilon \). Let the numbers \( L, L_1 \) be such that \( \mu_m < L < L_1 \) and \( a_{m+1}^+(\epsilon) < L \). Since \( \lim_{\epsilon \to 0} a_{m+2}^-(\epsilon) = \infty \) then \( a_{m+2}^-(\epsilon) > L_1 \) when \( \epsilon \) is small enough. Hence \( \sigma(\mathcal{A}^\epsilon) \cap [L, L_1] = \emptyset \) when \( \epsilon \) is small enough. We obtain a contradiction with condition (B_1) of the Hausdorff convergence. Thus \( \lim_{\epsilon \to 0} a_{m+1}^+(\epsilon) = \infty \). □

Lemma 6.1 implies that for an arbitrary \( L > 0 \) the spectrum \( \sigma(\mathcal{A}^\epsilon) \) has at most \( m \) gaps in the interval \([0, L]\) when \( \epsilon \) is small enough:

\[
\sigma(\mathcal{A}^\epsilon) \cap [0, L] = [0, L] \setminus \bigcup_{j=1}^{m^\epsilon} (\sigma_j^\epsilon, \mu_j^\epsilon)
\]

where \( (\sigma_j^\epsilon, \mu_j^\epsilon) \subset [0, L] \) are some pairwise disjoint intervals, \( m^\epsilon \leq m \). Here the intervals are renumbered in the increasing order.

We have proved that \( \sigma(\mathcal{A}^\epsilon) \) converges to \([0, \infty) \setminus \bigcup_{j=1}^{m} (\sigma_j, \mu_j)\) in the Hausdorff sense as \( \epsilon \to 0 \). Let \( L \) be an arbitrary number such that \( L > \mu_m \). Then, evidently, \( \sigma(\mathcal{A}^\epsilon) \cap [0, L] \) converges to \([0, L] \setminus \bigcup_{j=1}^{m} (\sigma_j, \mu_j)\) in the Hausdorff sense. By Proposition 6.1 \( m^\epsilon = m \) when \( \epsilon \) is small enough and

\[
\forall j = 1, \ldots, m : \lim_{\epsilon \to 0} \sigma_j^\epsilon = \sigma_j, \quad \lim_{\epsilon \to 0} \mu_j^\epsilon = \mu_j
\]

Theorem 1.1 is proved.

7. Proof of Theorem 1.2

Substituting \( a_j, b_j \) into (1.8) we get

\[
\sigma_j = \alpha_j
\]

(i.e. the first equality in (6.5) holds) and

\[
\rho_j = (\beta_j - \alpha_j) \prod_{i=1, i \neq j}^{m} \frac{\beta_i - \alpha_j}{\alpha_i - \alpha_j}
\] (7.1)
Recall that \( \mu_j (j = 1, m) \) are the roots of equation (1.9), therefore in order to prove the equalities \( \mu_j = \beta_j (j = 1, m) \) we have to show that
\[
\forall k = 1, \ldots, m : \sum_{j=1}^{m} \frac{\rho_j}{\beta_k - \alpha_j} = 1 \tag{7.2}
\]

Let us consider (7.2) as a system of \( m \) linear algebraic equations (\( \rho_j, j = 1, \ldots, m \) are unknowns). It is clear that (7.2) follows from the following

**Lemma 7.1.** The system (7.2) has the unique solution \( \rho_1, \ldots, \rho_m \) which is defined by (7.1).

**Proof.** We prove the lemma by induction. For \( m = 1 \) its validity is obvious. Suppose that we have proved it for \( m = N - 1 \). Let us prove it for \( m = N \).

Multiplying the \( k \)-th equation in (7.2) (\( k = 1, \ldots, N \)) by \( \beta_k - \alpha_N \) and then subtracting the \( N \)-th equation from the first \( N - 1 \) equations we obtain a new system
\[
\forall k = 1, \ldots, N - 1 : \sum_{j=1}^{N-1} \hat{\rho}_j = 1
\]
where the new variables \( \hat{\rho}_j, j = 1, \ldots, N - 1 \) are expressed in terms of \( \rho_j \) by the formula
\[
\hat{\rho}_j := \rho_j \frac{\alpha_N - \alpha_j}{\beta_N - \alpha_j}, \quad j = 1, \ldots, N - 1 \tag{7.3}
\]
Hence \( \hat{\rho}_j, j = 1, N - 1 \) satisfy the system (7.2) with \( m = N - 1 \). By the induction
\[
\hat{\rho}_j = (\beta_j - \alpha_j) \prod_{i=1,N-1\neq j} \left( \frac{\beta_i - \alpha_j}{\alpha_i - \alpha_j} \right) \tag{7.4}
\]
It follows from (7.3), (7.4) that \( \rho_j (j = 1, \ldots, N - 1) \) satisfy (7.1) (with \( m = N \)). The validity of this formula for \( \rho_N \) follows easily from the symmetry of system (7.2). Lemma 7.1 is proved. \( \square \)

**Theorem 1.2** is proved.

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