The Consistency of Adversarial Training for Binary Classification

Natalie S. Frank
Courant Institute
New York University
New York, NY 10012
nf1066@nyu.edu

Jonathan Niles-Weed
Courant Institute
New York University
New York, NY 10012
jnw@cims.nyu.edu

Abstract

Robustness to adversarial perturbations is of paramount concern in modern machine learning. One of the state-of-the-art methods for training robust classifiers is adversarial training, which involves minimizing a supremum-based surrogate risk. The statistical consistency of surrogate risks is well understood in the context of standard machine learning, but not in the adversarial setting. In this paper, we characterize which supremum-based surrogates are consistent for distributions absolutely continuous with respect to Lebesgue measure in binary classification. Furthermore, we obtain quantitative bounds relating adversarial surrogate risks to the adversarial classification risk. Lastly, we discuss implications for the $H$-consistency of adversarial training.

1 Introduction

A central issue in the study of neural nets is their susceptibility to adversarial perturbations—perturbations imperceptible to the human eye can completely derail neural networks [Szegedy et al., 2013]. As these models are used in applications such as self-driving cars and medical imaging [Paschali et al., 2018, Li et al., 2021], a central question is training classifiers robust to adversarial perturbations. Many studies attempt to tackle this problem.

In standard classification, one seeks to minimize the classification risk, which penalizes incorrectly classified data with a modified 0-1 loss. Since minimizing the classification risk is typically computationally intractable, a common approach is to minimize a surrogate risk, obtained by replacing the modified 0-1 loss by a better-behaved alternative. If a minimizing sequence of the surrogate risk also minimizes the classification risk, the surrogate risk is called consistent. Many classic papers study the consistency of surrogate risks in the standard classification setting [Bartlett et al., 2006, Lin, 2004, Steinwart, 2007, Philip M. Long, 2013, Mingyuan Zhang, 2020]. On the other hand, in the adversarial scenario, obtaining a robust classifier requires minimizing the adversarial classification risk, which incurs a penalty at a point if it is close to lying in the opposite class. Minimizing this risk selects classifiers that are robust to adversaries who can add a small perturbation to each example. As in the standard case, minimizing this risk directly is computationally intractable, so one typically minimizes a surrogate risk. Unlike the standard case, however, little is known about the consistency of surrogate risks in the adversarial context. Though this question has been partially studied in the literature [Bhattacharjee and Chaudhuri, 2020, 2021, Awasthi et al., 2021a, c, Meunier et al., 2022], a general theory is lacking.

We leverage recent results on the theory of adversarial learning to give a surprisingly simple and general characterization: in the context of binary classification and data distributions absolutely continuous with respect to Lebesgue measure, we prove that a surrogate loss is adversarially consistent when learning over the class of all measurable functions if and only if is non-adversarially
consistent. To the best of the authors’ knowledge, this paper is the first to prove that a loss-based learning procedure is consistent for a wide range of distributions in the adversarial setting. Subsequently, we prove an approximate form of $H$-consistency for the $\rho$-margin loss and the ramp loss. Interestingly, Awasthi et al. [2021a] show that for a well-motivated class of linear functions, no supremum-based surrogate is adversarially consistent. This result suggests that finding surrogates which satisfy the $H$-consistency criterion may not be feasible in the adversarial setting. Lastly, we use the $\Psi$-transform of [Bartlett et al. 2006] to bound the excess adversarial classification risk in terms of a duality gap. This inequality could be used to bound the difference between the adversarial classification risk of a classifier and the minimum possible adversarial classification risk.

2 Related Works

Many previous works have studied the consistency of surrogate risks [Bartlett et al., 2006, Lin, 2004, Steinwart, 2007, Philip M. Long, 2013, Mingyuan Zhang, 2020]. The classic papers by [Bartlett et al., 2006, Lin, 2004, Zhang, 2004] explore the consistency of surrogate risks over all measurable functions. The works [Philip M. Long, 2013, Mingyuan Zhang, 2020] study $H$-consistency, which is consistency restricted to a smaller set of functions. Steinwart [2007] generalizes some of these results into a framework referred to as calibration. Awasthi et al. [2021a], Bao et al. [2021], Awasthi et al. [2021c], Meunier et al. [2022] then use this framework to analyze the calibration of adversarial surrogate losses. However, calibration analysis does not imply adversarial consistency without distributional assumptions. Furthermore, Awasthi et al. [2021a] show that no supremum-based risk is $H$-consistent for a well-motivated function class. Lastly, Bhattacharjee and Chaudhuri [2020, 2021] use a different set of techniques to study the consistency of non-parametric methods in adversarial scenarios.

Our results rely on recent works establishing the properties of minimizers to surrogate adversarial risks. [Awasthi et al., 2021a, Pydi and Jog, 2021, Bungert et al., 2021] all proved the existence of minimizers to the adversarial risk and Pydi and Jog [2021] proved a minimax theorem for the zero-one loss. Building on their work, Frank [2022a] later proved similar the existence and minimax statements to arbitrary surrogate losses. Lastly, Trillos and Murray [2020, Frank, 2022b] study further properties of the minimizers to the adversarial classification loss.

While this paper focuses on consistency, there have been several other directions studying adversarial learning from a theoretical perspective. The papers [Bubeck et al., 2018b, Nakkarin, 2019, Degwekar et al., 2019, Tsipras et al., 2018] explore statistical and computational bottlenecks in adversarial learning. There have been several attempts to guarantee the performance of adversarial learning algorithms. Raghunathan et al. [2018], Weng et al. [2018], Zhang et al. [2018], Wong and Kolter [2018], Sinha et al. [2020] approach this problem by certifying certificates to robustness. Furthermore, several works study the sample complexity in the adversarial setting; [Kim and Loh, 2018, Yin et al., 2019, Awasthi et al., 2020, Montasser et al., 2019] study the Rademacher complexity and VC-dimension of adversarial learning while Xing et al. [2021] uses algorithmic stability to give an upper bound on the generalization of adversarial training.

3 Problem Setup

3.1 Surrogate Risks

This paper studies binary classification on $\mathbb{R}^d$. Explicitly, labels are $\{-1, +1\}$ and the data is from a distribution $D$ on the set $\mathbb{R}^d \times \{-1, +1\}$. The measures $P_1, P_0$ define the probabilities of finding points with a given label in a region of $\mathbb{R}^d$. Formally, let

$$P_1(A) = D(A \times \{+1\}), P_0(A) = D(A \times \{-1\}).$$

Throughout this paper, we assume that

**Assumption 1.** $P_1, P_0$ are absolutely continuous with respect to Lebesgue measure.

Typical classification algorithms involve learning a real-valued function $f$ and then classifying according to the sign of $f$, which we define as

$$\text{sign } \alpha = \begin{cases} +1 & \text{if } \alpha > 0 \\ -1 & \text{if } \alpha \leq 0 \end{cases}.$$
The choice \( \text{sign} 0 = -1 \) is arbitrary. The classification risk \( R(f) \) is then the probability of misclassifying a point under \( D \): \( R(f) = D(\text{sign} \ f(x) \neq y) \). The classification risk in terms of the marginal distributions \( P_0, P_1 \) is

\[
R(f) = \int 1_{f(x) \leq 0} dP_1 + \int 1_{f(x) > 0} dP_0. \tag{1}
\]

In the adversarial setting, every \( x \)-value is perturbed by a malicious adversary before undergoing classification by \( f \). We assume that these perturbations are bounded by \( \epsilon \) in some norm \( \| \cdot \| \) and furthermore, we assume that the adversary knows both our classifier \( f \) and the true label of the point \( x \). In the machine learning literature, this attack model is called the evasion attack. In other words, \( f \) misclassifies \( x \) when there is a point \( x' \in B_\epsilon(x) \) for which \( f(x') \neq y \). Hence, a point \( x \) with label \( +1 \) is misclassified when there is a point \( \| x' - x \| \leq \epsilon \) for which \( f(x') \leq 0 \) and a point \( x \) with label \( -1 \) is misclassified when there is a point \( \| x' - x \| \leq \epsilon \) for which \( f(x') > 0 \). The expected fraction of errors made by the classifier \( f \) under this model is therefore

\[
R^\epsilon(f) = \int \sup_{\|h\| \leq \epsilon} 1_{f(x+h) \leq 0} dP_1 + \int \sup_{\|h\| \leq \epsilon} 1_{f(x+h) > 0} dP_0, \tag{2}
\]

which is called the adversarial classification risk.

Next, we discuss surrogate risks for the standard classification problem. In general, optimizing the empirical version of the classification risk is an intractable problem [Ben-David et al. 2003]. Instead, we consider minimizing the risk

\[
R_\phi(f) = \int \phi(f) dP_1 + \int \phi(-f) dP_0 \tag{3}
\]

over all measurable functions. A point \( x \) is then classified according to \( \text{sign} f \). There are many possible choices for \( \phi \)—typically one chooses a loss that is easy to optimize. In this paper, we assume that

**Assumption 2.** \( \phi \) is decreasing, non-negative, lower semi-continuous, and \( \lim_{\alpha \to \infty} \phi(\alpha) = 0 \).

Most surrogate losses in machine learning satisfy this assumption. If \( f \) is always a minimizer of \( R \) when \( f \) is a minimizer of \( R_\phi(f) \), we call \( R_\phi \) a consistent risk and \( \phi \) a consistent loss.\(^4\) Consistent losses have been studied in many prior works; most prominently [Bartlett et al. 2006] showed that a convex decreasing loss \( \phi \) is consistent iff \( \phi \) is differentiable at zero and \( \phi'(0) \neq 0 \).

Similarly, because optimizing (2) is computationally intractable, one defines a surrogate for the adversarial classification risk (2) by

\[
R^\epsilon_\phi(f) = \int \sup_{\|x' - x\| \leq \epsilon} \phi(f(x')) dP_1 + \int \sup_{\|x' - x\| \leq \epsilon} \phi(-f(x')) dP_0 \tag{4}
\]

Due to the supremum in this expression, we refer to this risk as a supremum-based surrogate. We define adversarial consistency analogously.

**Definition 1.** Let \( P_0, P_1 \) define a data distribution. The loss \( \phi \) is adversarially \( \mathcal{H} \)-consistent for \( P_0, P_1 \) if every sequence \( h_n \) which minimizes \( R^\epsilon_\phi \) over \( \mathcal{H} \) also minimizes \( R^\epsilon \) over \( \mathcal{H} \). We say that \( \phi \) is adversarially consistent for \( P_0, P_1 \) if it is \( \mathcal{H} \)-consistent for the class of all measurable functions.

When referring to a loss as adversarially consistent, we will typically not specify \( P_0, P_1 \) as the data distribution should be clear from the context.

We conclude by recalling a characterization of minimizers for \( R_\phi \) and \( R \) in the non-adversarial setting. Finding minimizers of \( R \) is straightforward – by defining \( P = P_0 + P_1 \) and \( \eta = dP_1/dP \), one can write

\[
R(f) = \int \eta(x) 1_{f(x) \leq 0} + (1 - \eta(x)) 1_{f(x) > 0} dP \]

\(^4\) In the context of standard (non-adversarial) learning, the concept we defined as consistency is often referred to as calibration, see for instance [Bartlett et al. 2006], [Steinwart 2007]. We opt for the term ‘consistency’ as the prior works [Awasthi et al. 2021a,c], [Meunier et al. 2022] use calibration to refer to a different but related concept in the adversarial setting.
Thus, $f$ minimizes $R$ so long as $f(x) \leq 0$ when $\eta(x) \leq 1 - \eta(x)$ and $f(x) > 0$ when $\eta(x) > 1 - \eta(x)$. The same procedure finds minimizers of $R_\phi$: in terms of $\eta$ and $P$, the loss $R_\phi$ is
\[ R_\phi(f) = \int \eta(x)\phi(f(x)) + (1 - \eta(x))\phi(-f(x))dP = \int C(\eta)(x), f(x)dP \tag{5} \]
where
\[ C(\eta, \alpha) := \eta\phi(\alpha) + (1 - \eta)\phi(-\alpha) \]
Thus $f$ is a minimizer of $R_\phi$ iff $f$ minimizes $C(\eta)(x), \alpha)$ a.e., or in other words, $f(x) \in \arg\min_{x} C(\eta)(x), \alpha)$ almost everywhere. However, over the set $\mathbb{R}$, the function $C(\eta)(x), \alpha)$ may not always have a minimizer. For instance, consider the exponential loss $\psi(\alpha) = e^{-\alpha}$ for $\eta = 1$. On the other hand, if $\phi$ is lower semi-continuous, $C(\eta, \alpha)$ always has a minimizer in the extended real numbers $\mathbb{R}^\infty := \mathbb{R} \cup -\infty, +\infty$. Therefore, in the remainder of this paper, we assume that functions $f$ assume values in $\mathbb{R}$. However, if the values $R_\phi(f), R_\phi^*(f)$ are finite, $\phi \circ f, \phi \circ -f$ must be finite outside a set measure 0, so at minimizers of $R_\phi$, $R_\phi^*$ integrands still assume finite values.

Note that integrals of functions assuming values in $\mathbb{R} \cup \{\infty\}$ can still be defined using standard measure theory, see for instance [Folland [1999]]. Because we consider only lower semi-continuous $\phi$, $\arg\min_{x} C(\eta, \alpha)$ is always non-empty.

This alternative view of the risks $R$ and $R_\phi$ provides a pointwise criterion for consistency:

**Lemma 1.** The loss $\phi$ is consistent iff for each $\eta$, every minimizing sequence $\alpha_n$ for $C(\eta, \alpha)$ is also a minimizing sequence of $C(\eta, \alpha)$.

We prove this lemma in Appendix [R]

### 3.2 Measurability

We briefly discuss the issue of measurability. In order to define the integral in (4), the function $\sup_{\|\xi - x\| \leq \epsilon} \phi(f(\xi))$ must always be measurable. This is not always the case. For any $\epsilon > 0$, [Pydi and Jog, 2021] find a Borel set $A$ for which $\sup_{\|x - x\| \leq \epsilon} 1_A(x')$ is not Borel measurable. In order to define the integrals in (2) and (4), we work in a different sigma algebra named the *universal sigma algebra*. We adopt the treatment of [Nishiura, 2010] for our definitions. Let $B(\mathbb{R}^d)$ denote the Borel sigma algebra and let $\nu$ be a Borel measure. We denote the completion of $B(\mathbb{R}^d)$ under $\nu$ by $\mathcal{L}_\nu(\mathbb{R}^d)$, which is the $\sigma$-algebra of sets Lebesgue measurable by $\nu$. Let $\mathcal{M}(\mathbb{R}^d)$ be the set of Borel measures on $\mathbb{R}^d$. Then we define the *universal sigma algebra* $\mathcal{W}(\mathbb{R}^d)$ as $\mathcal{W}(\mathbb{R}^d) = \bigcap_{\nu\in\mathcal{M}(\mathbb{R}^d)} \mathcal{L}_\nu(\mathbb{R}^d)$. In other words, $\mathcal{W}(\mathbb{R}^d)$ is the $\sigma$-algebra of sets which are measurable with respect to the completion of every Borel measure. Frank [2022a] prove the following theorem:

**Theorem 1.** Let $f$ be a universally measurable function. Then the function $S_\epsilon(f)(x)$ defined by $S_\epsilon(f)(x) = \sup_{\|h\| \leq \epsilon} f(x + h)$ is universally measurable as well.

Hence, if $f$ is a universally measurable function, then the integrals of (2) and (4) are well-defined. Therefore, unless otherwise noted, functions which are composed with losses are assumed to be universally measurable.

### 3.3 Existence and Minimax for Adversarial Risks

We will prove that $\phi$ is adversarially consistent by comparing the minimizers of $R_\phi^*$ with those of $R^*$. In the next section, in order to compare these minimizers, we will attempt to re-write the adversarial loss in a "pointwise" manner similar to (5). In order to achieve this representation of the adversarial loss, we apply minimax theorems from [Pydi and Jog, 2021, Frank, 2022a]. Before presenting these theorem, we introduce the $\infty$-Wasserstein metric from optimal transport. For two measures $\mathbb{Q}_0, \mathbb{Q}_1$ satisfying $\mathbb{Q}_0(\mathbb{R}^d) = \mathbb{Q}_1(\mathbb{R}^d)$, let $\Pi(\mathbb{Q}_0, \mathbb{Q}_1)$ be the set of couplings of $\mathbb{Q}_0, \mathbb{Q}_1$:
\[ \Pi(\mathbb{Q}_0, \mathbb{Q}_1) := \{\nu: \nu(A \times \mathbb{R}^d) = \mathbb{Q}_0(A), \nu(\mathbb{R}^d \times A) = \mathbb{Q}_1(A)\} \]

The $W_\infty$ distance between $\mathbb{Q}_0$ and $\mathbb{Q}_1$ is defined as
\[ W_\infty(\mathbb{Q}_0, \mathbb{Q}_1) = \inf_{\nu \in \Pi(\mathbb{Q}_0, \mathbb{Q}_1)} \text{ess sup} ||x - y|| \]
We denote the \( \infty \)-Wasserstein ball around a measure \( \mathbb{Q} \) by \( B_\infty^\infty(\mathbb{Q}) \). Theorem 8 of [Pydi and Jog [2021]] states the following existence and minimax theorem for the adversarial classification risk:

**Theorem 2.** Assume that Assumption[7] holds. Let

\[
\hat{R}(f, \mathbb{P}_0, \mathbb{P}_1) = \int 1_{f(x) \leq 0} d\mathbb{P}_1 + \int 1_{f(x) > 0} d\mathbb{P}_0
\]

Then

\[
R^*_{\phi} = \inf_{f \in \mathcal{F}} \sup_{\mathbb{P} \in B_\infty^\infty(\mathbb{P}_0)} \hat{R}(f, \mathbb{P}_0, \mathbb{P}_1) = \sup_{\mathbb{P} \in B_\infty^\infty(\mathbb{P}_0)} \inf_{f \in \mathcal{F}} \hat{R}(f, \mathbb{P}_0, \mathbb{P}_1)
\]

and furthermore equality is attained for some Borel measurable \( \tilde{f} \) and \( \hat{\mathbb{P}}_1, \hat{\mathbb{P}}_0 \) with \( W_\infty(\hat{\mathbb{P}}_0, \mathbb{P}_0) \leq \epsilon \) and \( W_\infty(\hat{\mathbb{P}}_1, \mathbb{P}_1) \leq \epsilon \).

[Frankl 2022a] prove a similar statement for the surrogate risk \( R^*_\phi \). They start by showing the existence of a minimizer which assumes its extrema over \( \epsilon \)-balls.

**Theorem 3.** Assume that Assumptions[7] and[2] hold. Let \( \mathcal{F} \) be the set of all universally measurable functions on \( \mathbb{R}^d \). Then there is a Borel measurable function \( f^* \) that minimizes \( R^*_{\phi} \):

\[
\inf_{f \in \mathcal{F}} R^*_{\phi}(f) = R^*_0(f^*).
\]

Furthermore, this \( f^* \) assumes its minimum on every \( B_\epsilon(x) \) \( \mathbb{P}_1 \)-a.e. and its maximum on every \( B_\epsilon(x) \) \( \mathbb{P}_0 \)-a.e.

Using the well-behaved minimizer from Theorem[3] [Frankl 2022a] then proved a minimax result analogous to Theorem[5]. Notice that if \( T : \mathbb{R}^d \to \mathbb{R}^d \) is any transformation with \( ||T(x) - x|| \leq \epsilon \) and \( \mathbb{Q} \) is any measure, then \( W_\infty(\mathbb{Q}, \mathbb{Q} \circ T^{-1}) \leq \epsilon \).

**Theorem 4.** Assume that Assumptions[7] and[2] hold, and let \( f^* \) be the minimizer of Theorem[3].

Define the risk \( \hat{R}_\phi \) as

\[
\hat{R}_\phi(f, \mathbb{P}_0, \mathbb{P}_1) = \int \phi(f(x)) d\mathbb{P}_1 + \int \phi(-f(x)) d\mathbb{P}_0.
\]

Then

\[
R^*_{\phi} = \inf_{f \in \mathcal{F}} \sup_{\mathbb{P} \in B_\infty^\infty(\mathbb{P}_0)} \hat{R}_\phi(f, \mathbb{P}_0, \mathbb{P}_1) = \sup_{\mathbb{P} \in B_\infty^\infty(\mathbb{P}_0)} \inf_{f \in \mathcal{F}} \hat{R}_\phi(f, \mathbb{P}_0, \mathbb{P}_1)
\]

and furthermore equality is attained for some Borel measurable \( f^* \) and \( \mathbb{P}_0, \mathbb{P}_1 \) with \( W_\infty(\mathbb{P}_0, \mathbb{P}_0) \leq \epsilon \) and \( W_\infty(\mathbb{P}_1, \mathbb{P}_1) \leq \epsilon \). Additionally, \( \mathbb{P}^*_i = \mathbb{P}_i \circ T_i^{-1} \) for which \( T_i : \mathbb{R}^d \to \mathbb{R}^d \) are universally measurable functions with \( ||T_i(x) - x|| \leq \epsilon \) and \( \inf_{||y-x|| \leq \epsilon} f^*(y) = f^*(T_1(x)) \) \( \mathbb{P}_1 \)-a.e. and \( \sup_{||y-x|| \leq \epsilon} f^*(y) = f^*(T_0(x)) \) \( \mathbb{P}_0 \)-a.e.

Notice that in this theorem, the fact that \( f^* \) assumes its local extrema over \( \epsilon \)-balls is essential to defining the measures \( \mathbb{P}^*_0, \mathbb{P}^*_1 \).

We will now discuss an interpretation of Theorems[2] and[4]. These theorems suggest that the adversarial training problem can be interpreted as a game between the adversary and the learner in which the adversary selects adversarial perturbations and the learner picks a function \( f \). Theorems[2] and[4] imply that in this game, the player which chooses first does not have an advantage.

We assume that \( \mathbb{P}_0, \mathbb{P}_1 \) are absolutely continuous with respect to Lebesgue measure due to the assumptions in Theorems[2] and[4]. If one could generalize Theorems[2] and[4] to other distributions, then all the results of this paper would generalize to those distributions as well. The absolute continuity assumption is one of the main limitations of our results.
4 The Consistency of $R_{\phi}^*$

In this section, we prove our main result: if $\phi$ is consistent, it is adversarially consistent as well. We begin by stating two duality results which are direct consequences of Theorems 5 and 6. Define

$$C_0^*(\eta) = \inf \limits_{\alpha} C_\phi(\eta, \alpha), \quad C^*(\eta) = \min(\eta, 1 - \eta)$$

and

$$\hat{R}_{\phi}(\mathbb{P}_0', \mathbb{P}_1') = \int C_\phi \left( \frac{d\mathbb{P}_1'}{d(\mathbb{P}_1' + \mathbb{P}_0')} \right) d(\mathbb{P}_0' + \mathbb{P}_1'), \quad \hat{R}^c(\mathbb{P}_0', \mathbb{P}_1') = \int C^* \left( \frac{d\mathbb{P}_1'}{d(\mathbb{P}_1' + \mathbb{P}_0')} \right) d(\mathbb{P}_0' + \mathbb{P}_1').$$

**Theorem 5.** Let $\mathbb{P}_0, \mathbb{P}_1$ be absolutely continuous with respect to Lebesgue measure. Then

$$\hat{R}_{\phi}^* = \inf \limits_f R^*(f) = \sup \limits_{\mathbb{P}_1' \in \mathcal{B}_c^\infty(\mathbb{P}_1)} \hat{R}_{\phi}^*(\mathbb{P}_0', \mathbb{P}_1')$$

Equality is assumed at the $\hat{f}, \hat{\mathbb{P}}_0, \hat{\mathbb{P}}_1$ of Theorem 2. Furthermore, if $g$ is any minimizer of $R^*$, then

$$g(x) \in \arg \min \limits_\alpha C_\phi \left( \frac{d\mathbb{P}_1'}{d(\mathbb{P}_1' + \mathbb{P}_0')}, \alpha \right)$$

$\mathbb{P}_0 + \mathbb{P}_1$-a.e. and furthermore, $g_n(x)$ is a minimizing sequence of $R_{\phi}^*$, $g_n(x)$ iff it minimizes

$$C(d\mathbb{P}_1/d(\mathbb{P}_1 + \mathbb{P}_0), \cdot) \mathbb{P}_0 + \mathbb{P}_1$$.a.e.

This result follows from Theorem 2. An analogous duality result holds for $R_{\phi}^*$:

**Theorem 6.** Assume that Assumptions 1 and 2 are satisfied. Then

$$R_{\phi}^* = \inf \limits_f R_{\phi}^*(f) = \sup \limits_{\mathbb{P}_1' \in \mathcal{B}_c^\infty(\mathbb{P}_1)} \hat{R}_{\phi}^*(\mathbb{P}_0', \mathbb{P}_1')$$

Equality is assumed at $f^*, \mathbb{P}_0^*, \mathbb{P}_1^*$ for which $\mathbb{P}_1^* = \mathbb{P}_1 \circ (T_1^*)^{-1}$ with $\|T_1^*(x) - x\| \leq \epsilon$ and

$$\sup \limits_{\|y-x\| \leq \epsilon} \phi(f^*(y)) = \phi(f^*(T_1^*(x))), \sup \limits_{\|y-x\| \leq \epsilon} \phi(-f^*(y)) = \phi(-f^*(T_0^*(x)))$$

If $g$ is any minimizer of $R_{\phi}^*$, then

$$g(x) \in \arg \min \limits_\alpha C_\phi \left( \frac{d\mathbb{P}_1^*}{d(\mathbb{P}_1^* + \mathbb{P}_0^*)}(x), \alpha \right)$$

$\mathbb{P}_0^* + \mathbb{P}_1^*$-a.e. and furthermore, if $g_n(x)$ is any minimizing sequence of $R_{\phi}^*$, $g_n(x)$ must minimize

$$C_\phi(d\mathbb{P}_1^*/d(\mathbb{P}_1^* + \mathbb{P}_0^*), \cdot) \mathbb{P}_0^* + \mathbb{P}_1^*$$.a.e.

This theorem follows from Theorems 3 and 4. We formally prove these results in Appendix C.

In the non-adversarial setting, minimizers to $R_0$ depend heavily on the form of $\phi$. One would expect the same to hold for the adversarial loss $R_{\phi}^*$. However, maximizers of $\hat{R}_{\phi}$ only depend on $C_0^*(\eta)$. Notice that (12) of Theorem 6 implies that the maximizers of $R_{\phi}^*$ determine the minimizers of $R_{\phi}^*$. Therefore, if two different loss functions have the same $C_0^*$, then either both are consistent or both are not consistent. Notice that for the ramp loss $\max(1 - \alpha, 0)$, the $\rho$-margin loss $\min(1, \max(1 - \alpha/\rho, 0))$, and one-half the sigmoid loss $1/2 \cdot 1/(1 + e^\alpha)$, $C_0^*(\eta) = C^*(\eta) = \min(\eta, 1 - \eta)$. This calculation together with Lemma 1 and (12) prove that the ramp loss, the margin loss, and the sigmoid loss are all adversarially consistent.

This observation suggest that there are $\mathbb{P}_0^*, \mathbb{P}_1^*$ which always maximize $\hat{R}_{\phi}$ independently of the choice of $\phi$. We will prove this statement in the next two lemmas.

**Lemma 2.** Assume that Assumptions 3 and 2 hold. Let $\mathbb{P}_0^* \in \mathcal{B}_c^\infty(\mathbb{P}_0), \mathbb{P}_1^* \in \mathcal{B}_c^\infty(\mathbb{P}_1)$ be given by $\mathbb{P}_1^* = \mathbb{P}_1 \circ (T_1^*)^{-1}$ with $\|T_1^*(x) - x\| \leq \epsilon$. If $\mathbb{P}_1^*, T_1^*$ satisfy

$$\inf \limits_{\|y-x\| \leq \epsilon} \frac{d\mathbb{P}_1^*}{d(\mathbb{P}_0^* + \mathbb{P}_1^*)}(y) = \frac{d\mathbb{P}_1^*}{d(\mathbb{P}_0^* + \mathbb{P}_1^*)}(T_1^*(x)) \mathbb{P}_1$$.a.e.

$$\sup \limits_{\|y-x\| \leq \epsilon} \frac{d\mathbb{P}_1^*}{d(\mathbb{P}_0^* + \mathbb{P}_1^*)}(y) = \frac{d\mathbb{P}_1^*}{d(\mathbb{P}_0^* + \mathbb{P}_1^*)}(T_0^*(x)) \mathbb{P}_0$$.a.e.,

then $\mathbb{P}_0^*, \mathbb{P}_1^*$ maximize both $\hat{R}_{\phi}^*$ and $\hat{R}^c$.\n
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To prove this result, we construct a primal certificate—we find an \( f^* \) for which \( R^*(f^*) = \tilde{R}^*(P_0^*, P_1^*) \). Because \( R_0^*(f) \geq R_{0,*}^* \geq R_0(f, P_0^*, P_1^*) \) for all \( f, P_0^*, P_1^* \in \mathcal{B}_c^\infty(P_0^*) \), and \( P_1^* \in \mathcal{B}_c^\infty(P_1^*) \), the equality \( R^*(f^*) = \tilde{R}^*(P_0^*, P_1^*) \) implies that \( f^*, P_0^*, P_1^* \) are optimal. To construct this \( f^* \), we use the following lemma proved in Appendix [A].

**Lemma 3.** Let \( \phi \) satisfy Assumption 1 Then there is an increasing function \( \alpha : [0, 1] \rightarrow \mathbb{R} \) for which \( \alpha(\eta) \in \text{argmin}_\eta C(\eta, \alpha) \).

**Proof of Lemma 2** We will show that \( P_0^*, P_1^* \) maximize \( \tilde{R}_\phi^* \). To show that they also maximize \( \tilde{R}^* \), notice that \( \tilde{R}^* = R_\phi^* \) when \( \phi \) is the ramp loss.

Define \( P^* = P_0^* + P_1^* \) and \( \eta^* = dP_0^*/dP^* \). Let \( f^*(x) = \alpha(\eta^*(x)) \). We will show that \( R_\phi^*(f^*) = R_\phi^*(P_0^*, P_1^*) \). We follow the steps of the proof of Theorem 6 in reverse. To start,

\[
\tilde{R}^*(P_0^*, P_1^*) = \int C^*_\phi \left( \frac{dP_0^*}{dP_1^* + P_0^*} \right) d(P_0^* + P_1^*)
\]

\[
= \int \eta^*(x) \phi(\alpha(\eta^*(x))) + (1 - \eta^*(x)) \phi(-\alpha(\eta^*(x))) d(P_0^* + P_1^*) \quad \text{(definition of } C^*_\phi )
\]

\[
= \int \eta^*(x) \phi(f^*(x)) + (1 - \eta^*(x)) \phi(-f^*(x)) d(P_0^* + P_1^*) \quad \text{(definition of } f^* )
\]

\[
= \int \phi(f^*(x)) dP_1^* + \int \phi(-f^*(x)) dP_0^* \quad \text{(Definition of } \eta^*, P^* )
\]

Next we can apply a change of variables to conclude that

\[
\tilde{R}^*(P_0^*, P_1^*) = \int \phi(f^*(T^*_1(x))) dP_1^* + \int \phi(-f^*(T^*_0(x))) dP_0^* \quad (14)
\]

However, the conditions (13) imply that \( P_1 \)-a.e.,

\[
f^*(T^*_1(x)) = \alpha(\eta^*(T^*_1(x))) = \alpha \left( \inf_{\|y-x\| \leq \epsilon} \eta^*(y) \right) = \inf_{\|y-x\| \leq \epsilon} \alpha(\eta^*(y)) = \inf_{\|y-x\| \leq \epsilon} f^*(y)
\]

and similarly \( f^*(T^*_0(x)) = \sup_{\|y-x\| \leq \epsilon} f^*(y) \) \( P_0 \)-a.e. Thus by substituting these two expressions into (14), we conclude that

\[
\tilde{R}^*(P_0^*, P_1^*) = \int \sup_{\|y-x\| \leq \epsilon} \phi(f^*(y)) dP_1^* + \int \sup_{\|y-x\| \leq \epsilon} \phi(-f^*(y)) dP_0^* = R_{\phi,*}^*(f^*) \quad (15)
\]

The duality result of Equation 10 implies that for any \( f, P_0^*, P_1^* \in \mathcal{B}_c^\infty(P_0), P_1^* \in \mathcal{B}_c^\infty(P_1) \), \( R_{\phi,*}^*(f) \leq R_{\phi,*}^*(P_0^*, P_1^*) \). Therefore, (15) implies that \( \tilde{R}^*(P_0^*, P_1^*) = R_{\phi,*}^*(f^*) = R_{\phi,*}^* \).

\[\square\]

We have shown that if \( P_1^*, T_1 \) satisfy the conditions (13), then \( P_0^*, P_1^* \) are optimal. Next we show that there exist \( P_1^*, T_1 \) which satisfy these criteria.

**Lemma 4.** Assume that \( P_0, P_1 \) are absolutely continuous with respect to Lebesgue measure. Then there exist \( P_0^* \in \mathcal{B}_c^\infty(P_0) \) and \( P_1^* \in \mathcal{B}_c^\infty(P_1) \) which are given by \( P_1^* = P_1 \circ (T^*_1)^{-1} \) with \( \|T_1(x) - x\| \leq \epsilon \) and these \( T_1^* \) satisfy (13).

**Proof.** Let \( \psi \) be the exponential loss \( \psi(\alpha) = e^{-\alpha} \). Then for each \( \eta \), one can show that the unique minimizer of \( C_\psi(\eta, \alpha) = \frac{1}{2} \log \eta/(1-\eta) \). The function \( \alpha \) is strictly increasing. Equation 12 of Theorem 6 implies that \( f^* = \alpha(dP_1^*/dP_0^* + P_0^*) \) minimizes \( R_{\phi,*}^* \), where \( P_1 = P_1 \circ (T^*_1)^{-1} \), and the \( T_1^* \) satisfy (11). Because \( \psi \) is strictly decreasing and \( \alpha \) is strictly increasing, the result follows. \[\square\]

Finally, we use Lemmas 1, 2, and 3 to prove consistency.

**Theorem 7.** Assume that Assumptions 1 and 2 hold. If \( \phi \) is consistent then it is adversarially consistent for \( P_0, P_1 \).
Proof. Let \( P_0, P_1 \) be the measures given by Lemma\[2\] Define \( \mathbb{P}^*_0 = P_0 + P_1 \) and \( \eta^* = d\mathbb{P}^*_1 / d(P_1 + P_0) \). By Lemma\[2\] the measures \( P_0, P_1 \) maximize the risks \( R^*, R_0^* \). The last statement of Theorem\[6\] then implies that if \( g_n \) is any minimizing sequence of \( R^*_\phi \), then \( g_n(x) \) must minimize \( C_\phi(\eta^*(x), \cdot) \) \( P_0 + P_1 \) a.e.

Thus if \( \phi \) is consistent, Lemma\[1\] implies that \( g_n(x) \) will also minimize \( C(\eta^*(x), \alpha) = \eta^*(x)1_{\alpha \leq 0} + (1 - \eta^*(x))1_{\alpha > 0} P_0, P_1 \) a.e. Theorem\[5\] then implies that \( g_n \) is a minimizing sequence for \( R^* \).

\[\square\]

Lastly, by constructing a counterexample, one can show that if \( \phi \) is not consistent it is not adversarially consistent. We prove the following theorem in Appendix\[3\].

Theorem 8. If \( \phi \) is not consistent, then it is not adversarially consistent.

Lemmas\[2\] and Theorem\[6\] are also related to the theory of \( F \)-divergences. For a convex function \( F \) with \( F(1) = 0 \), the \( F \)-divergence between two measures \( \mu, \nu \) with \( \mu \ll \nu \) is \( D_F(\mu \| \nu) = \int F(d\mu / d\nu) d\nu \). Bartlett et al.\[2006\] show that the function \( C^*(\eta) \) is always a concave function which assumes its supremum at 1/2 and furthermore \( C^*(\eta) = C^*(\eta - 1/2) \). Thus \( C^*(1/2) - R_0^*(P_0, P_1) = C^*(1/2) - D_{\phi^*}(P_1^* \| P_0 + P_1^*) \) where \( F_{\phi^*}(\eta) = 1 - C^*_\phi(\eta - 1/2) / C^*_\phi(1/2) \). Lemma\[3\] then implies that there always exist \( \mathbb{P}^*_0, \mathbb{P}^*_1 \) which minimize \( D_{\phi^*} \) over \( \mathcal{B}_\infty^0(P_0), \mathcal{B}_\infty^\infty(P_1) \) independently of \( \phi \). Theorem\[6\] finds the dual to this minimization problem, and Lemma\[2\] describes conditions which characterize the optimal \( \mathbb{P}^*_0, \mathbb{P}^*_1 \).

5 Quantitative Bounds

In this section, we study how bounding the excess adversarial \( \phi \)-risk bounds the adversarial classification risk. Such bounds describe the efficiency of minimizing the surrogate \( R^*_\phi \) as a procedure for finding a minimizer of \( R^* \). Furthermore, quantitative bounds allow us to study consistency with respect to function classes \( \mathcal{H} \) smaller than the set of all measurable functions. We start with a simple observation involving the ramp and \( \rho \)-margin losses. As mentioned in Section\[4\] if \( \phi \) is either the ramp loss or the \( \rho \)-margin loss, then \( C^*_\phi = C^* \). Therefore, \( R^*_\phi = R^* \). However, since \( \phi(\alpha) \geq 1_{\alpha \leq 0} \) and \( \phi(-\alpha) \geq 1_{\alpha > 0} \) for any function \( f, R^*(f) \leq R^*_\phi(f) \). Combining these observations results in the following theorem:

Theorem 9. Assume that Assumption\[7\] is satisfied. Let \( \phi \) be the ramp loss or the \( \rho \)-margin loss. Then \( R^*(f) - R^*_\phi(f) \leq R^*_\phi(f) - R^*_\phi \).

Let \( \mathcal{H} \) be a function class with a small approximation error. Formally, assume that \( \inf_{h \in \mathcal{H}} R^*_\phi(h) - R^*_\phi \leq \delta \). Then Theorem\[9\] implies that for the \( \rho \)-margin loss or the ramp loss,

\[ R^*(f) - \inf_{h \in \mathcal{H}} R^*(h) \leq R^*_\phi(f) - \inf_{h \in \mathcal{H}} R^*_\phi(h) + \delta \]

In other words, if the approximation error of the class \( \mathcal{H} \) under the adversarial risk \( R^*_\phi \) is small, then consistency is a good proxy for \( \mathcal{H} \) consistency. Generalizing Theorem\[9\] to other risks would generalize this statement about approximate \( \mathcal{H} \)-consistency. A result of Awasthi et al.\[2013\] suggests that improving upon this approximate form of \( \mathcal{H} \)-consistency would be difficult. They prove that for the class \( \mathcal{H} = \{ x \mapsto w \cdot x : w \in \mathbb{R}^2, \|w\|_2 \leq 1 \} \), \( R^*_\phi \) is not \( \mathcal{H} \)-consistent for any continuous \( \phi \).

Next, for more general \( \phi \), we bound \( R^*(f) - R^*_\phi \) in terms of a duality gap of the surrogate risk. We use the \( \Psi \)-transform of Bartlett et al.\[2006\].

Theorem 10. Let \( H^*_\phi(\eta) = \inf_{\alpha : \alpha(2\eta - 1) \leq 0} C_\phi(\eta, \alpha) \). Define \( \Psi(\theta) = \text{conv}(H^*_\phi((1+\theta)/2) - C^*_\phi((1+\theta)/2)) \), where \( \text{conv} G \) is the largest convex function \( F \) with \( F \leq G \). Then

\[ \Psi(R(f) - R^*_\phi) \leq R^*_\phi(f) - R^*_\phi \]

Furthermore, \( \Psi \) is increasing and if \( \phi \) is consistent, then \( \theta > 0 \) implies that \( \Psi(\theta) > 0 \).

Using the \( \Psi \)-transform, we prove the following theorem:
Theorem 11. Assume that Assumptions 1 and 2 are satisfied and furthermore assume that \( \phi \) is non-constant on \([0, \alpha]\) for any \( \alpha > 0 \).

Let \( f \) be a function that assumes its minimum and maximum on every closed \( \epsilon \)-ball. Then there exist universally measurable \( T_1, T_0 \) for which \( f(T_1(x)) = \inf_{\|y-x\|} f(y) \) \( \mathbb{P}_1 \) a.e. and \( f(T_0(x)) = \sup_{\|y-x\|} f(y) \) \( \mathbb{P}_0 \) a.e. Define \( \mathbb{P}_0' = \mathbb{P}_0 \circ T_0^{-1}, \mathbb{P}_1' = \mathbb{P}_1 \circ T_1^{-1} \). Then

\[
\Psi(R^*_e(f) - R^*_e) \leq R^*_e(f) - R^*_e(\mathbb{P}_0', \mathbb{P}_1')
\]

where \( \Psi \) is the \( \Psi \)-transform of Theorem 10.

One can estimate the quantity \( \tilde{R}_e^*(\mathbb{P}_0', \mathbb{P}_1') \) by estimating the densities of \( \mathbb{P}_0' \) and \( \mathbb{P}_1' \). Thus on a trained model, (16) can be used to estimate the gap between the error of a classifier \( f \) and the minimum adversarial risk.

\[\text{Proof.}\] The existence of the universally measurable \( T_0, T_1 \) follows from Lemma 6 in Appendix C. Next, we will argue that

\[
\int_{f(T_1(x)) \leq 0} 1_f(y) + \int_{f(T_1(x)) > 0} 1_f(y) \leq \int_{f(T_0(x)) > 0} 1_f(y)
\]

hold \( \mathbb{P}_1 \)-a.e. and \( \mathbb{P}_0 \)-a.e. respectively. Assume that \( \int_{f(T_1(x)) \leq 0} 1_f(y) < \int_{f(T_0(x)) > 0} 1_f(y) \). In other words, \( f(T_1(x)) > 0 \) while there is a point \( y \in \mathcal{B}_\epsilon(x) \) for which \( f(y) \leq 0 \). If \( \phi(f(T_1(x))) = \phi(f(y)) \), then \( \phi \) must constant on \([0, \alpha]\) for some \( \alpha \). This statement contradicts the hypotheses of this theorem, so (17) must hold. An analogous argument implies (18).

Next, because \( \Psi \) is increasing, the duality relation (8) implies that

\[
\Psi(R^*_e(f) - R^*_e) \leq \Psi(R^*_e(f) - \tilde{R}_e^*(\mathbb{P}_0', \mathbb{P}_1')).
\]

Therefore,

\[
\Psi(R^*_e(f) - \tilde{R}_e^*(\mathbb{P}_0', \mathbb{P}_1')) = \Psi\left(\int \sup_{\|y-x\| \leq \epsilon} 1_f(y) - \int \sup_{\|y-x\| > \epsilon} 1_f(y) - \int C^*(\eta)d\mathbb{P}'\right)
\]

\[\text{(Equations 17,18)}\]

\[
= \Psi\left(\int 1_f(T_1(x))\leq 0d\mathbb{P}_1 - \int 1_f(T_0(x)) > 0d\mathbb{P}_0 - \int C^*(\eta)d\mathbb{P}'\right)\]

\[\text{(Change-of-variable)}\]

\[
= \Psi\left(\int 1_f(x)\leq 0d\mathbb{P}_1 - \int 1_f(x) > 0d\mathbb{P}_0 - \int C^*(\eta)d\mathbb{P}'\right)
\]

Notice that in Theorem 10 \( R_e = \int C^*(\eta)d(\mathbb{P}_0 + \mathbb{P}_1) \). Thus the argument of \( \Psi \) in (19) is of the form \( R(f) - R_e \) for the distribution given by \( \mathbb{P}_0', \mathbb{P}_1' \). Therefore, Theorem 10 implies that

\[
\Psi(R^*_e(f) - \tilde{R}_e^*(\mathbb{P}_0', \mathbb{P}_1')) \leq \int \phi(f(x))d\mathbb{P}' + \int \phi(-f(x))d\mathbb{P}' - \int C^*_\phi(\eta)d\mathbb{P}'
\]

Thus we proved that for distributions absolutely continuous with respect to Lebesgue measure, the adversarial training procedure is consistent. Furthermore, we found quantitative bounds in terms of surrogate risks which allows us to study \( \mathcal{H} \)-consistency in the adversarial setting. Additionally, our theorems are also related to the theory of \( F \)-divergences. We hope that insights to consistency and the structure of adversarial learning will lead to the design of better adversarial learning algorithms.

6 Conclusion

In conclusion, we proved that for distributions absolutely continuous with respect to Lebesgue measure, the adversarial training procedure is consistent. Furthermore, we found quantitative bounds in terms of surrogate risks which allows us to study \( \mathcal{H} \)-consistency in the adversarial setting. Additionally, our theorems are also related to the theory of \( F \)-divergences. We hope that insights to consistency and the structure of adversarial learning will lead to the design of better adversarial learning algorithms.

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Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper’s contributions and scope? [Yes] We prove all of our claims
   (b) Did you describe the limitations of your work? [Yes] In Section 3.3, we explain why we need to assume that $P_0, P_1$ are absolutely continuous with respect to Lebesgue measure. In section 5, we discuss the necessity of generalizing Theorem 9.
   (c) Did you discuss any potential negative societal impacts of your work? [N/A]
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]

2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes] The central assumptions of this paper are highlighted in Section 5, namely Assumptions [1] and [2].
(b) Did you include complete proofs of all theoretical results? [Yes]

3. If you ran experiments...

(a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [N/A]

(b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [N/A]

(c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [N/A]

(d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [N/A]

4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...

(a) If your work uses existing assets, did you cite the creators? [N/A]

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5. If you used crowdsourcing or conducted research with human subjects...

(a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]

(b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]

(c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]
A Proof of Lemma 3

We will show that there is an increasing function \( \alpha : [0, 1] \rightarrow \mathbb{R} \) for which \( C_{\phi}(\eta, \alpha(\eta)) = C^*_\phi(\eta) \) and
\[
\eta \leq \frac{1}{2}, \ y < \alpha(\eta) \Rightarrow C_{\phi}(\eta, y) > C_{\phi}(\eta, \alpha(\eta)) \quad \text{and} \quad \eta > \frac{1}{2}, \ y > \alpha(\eta) \Rightarrow C_{\phi}(\eta, y) > C_{\phi}(\eta, \alpha(\eta)).
\]
In other words, on \([0, 1/2]\), the function \( \alpha(\eta) \) is the smallest minimizer of \( C_{\phi}(\eta, \cdot) \) and on \((1/2, 1]\) is the largest minimizer of \( C_{\phi}(\eta, \cdot) \).

**Proof of Lemma 3.** We will consider \( \eta \leq 1/2 \), one can show the statement for \( \eta > 1/2 \) by swapping the the roles of \( \eta \) and \( 1-\eta \). First, notice that because \( \phi \) is lower semi-continuous, \( C_{\phi}(\eta, \cdot) \) always has a minimizer. Next, we show that if \( \eta_2 < \eta_1 \leq 1/2 \), then \( C^*_\phi(\eta_2) \leq C^*_\phi(\eta_1) \). If \( \alpha_1 \) is a minimizer of \( C_{\phi}(\eta_1, \alpha), \) then \( \eta_1 \leq 1/2 \) and \( C_{\phi}(\eta_1, \alpha_1) \leq C_{\phi}(\eta_1, -\alpha_1) \), we have that \( \phi(-\alpha_1) \leq \phi(\alpha_1) \). Therefore
\[
C^*_\phi(\eta_1) = C_{\phi}(\eta_1, \alpha_1) = \eta_1 \phi(\alpha_1) + (1-\eta_1)\phi(-\alpha_1) \geq \eta_2 \phi(\alpha_1) + (1-\eta_2)\phi(-\alpha_1) = C^*_\phi(\eta_2)
\]
We will show that if \( \eta_2 \leq \eta_1 \), then the smallest minimizer of \( C_{\phi}(\eta_2, \cdot) \) is less than any minimizer of \( C_{\phi}(\eta_1, \cdot) \). Therefore, if we define \( \alpha(\eta) \) on \([0, 1/2]\) as
\[
\alpha(\eta) = \inf \left( \arg\min_{\alpha} C_{\phi}(\eta, \alpha) \right),
\]
then \( \alpha(\eta) \) is increasing and \( C_{\phi}(\eta, \alpha(\eta)) = C^*_\phi(\eta) \).

Let \( \alpha_2 \) be the smallest minimizer of \( C_{\phi}(\eta_2, \cdot) \)— specifically, set
\[
\alpha_2 = \inf \left( \arg\min_{\alpha} C_{\phi}(\eta_2, \alpha) \right)
\]
Now for any \( \alpha, \)
\[
C_{\phi}(\eta_2, \alpha) - C_{\phi}(\eta_2, \alpha_2) = \eta_2(\phi(\alpha) - \phi(\alpha_2)) + (1-\eta_2)(\phi(-\alpha) - \phi(-\alpha_2)) \geq 0 \tag{20}
\]
If in fact \( \alpha_2 = -\infty \), then every number in \( \mathbb{R} \) is larger than \( \alpha_2 \), and thus every minimizer of \( C_{\phi}(\eta_1, \alpha) \) is larger than \( \alpha_2 \).

Now assume that \( \alpha_2 > -\infty \). Consider an \( \alpha \) for which \( \alpha < \alpha_2 \). If \( \alpha \) is not a minimizer of \( C_{\phi}(\eta_2, \cdot) \), then \( C_{\phi}(\eta_2, \alpha) \neq C^*_\phi(\eta_2) \). Therefore, \( C_{\phi}(\eta_2, \alpha) - C_{\phi}(\eta_2, \alpha_2) > 0 \). Furthermore, \( \phi(\alpha) - \phi(\alpha_2) \geq 0 \) and \( \phi(-\alpha) - \phi(-\alpha_2) \leq 0 \). Specifically, this implies that if \( \eta_1 \geq \eta_2 \),
\[
C_{\phi}(\eta_1, \alpha) - C_{\phi}(\eta_1, \alpha_2) \geq C_{\phi}(\eta_2, \alpha) - C_{\phi}(\eta_2, \alpha_2) > 0
\]
Therefore, if \( \alpha < \alpha_2 \), then \( \alpha \) cannot be a minimizer of \( C_{\phi}(\eta_1, \cdot) \). Thus any minimizer of \( C_{\phi}(\eta_1, \cdot) \) is larger than or equal to \( \alpha_2 \).

\[\square\]

B Proof of Lemma 1

We start by expressing the optimal values of the standard risks \( R_{\phi}, R \) in terms of \( C^*_\phi \) and \( C^* \). The proof is essentially the “pointwise optimality” argument referenced in the text.

**Lemma 5.** The optimal value of \( R_{\phi} \) is \( \inf_{f} R_{\phi}(f) = \int C^*_\phi(f) \, d\mathbb{P} \). Furthermore, if \( g_n \) is a minimizing sequence of \( R_{\phi} \), then \( \lim_{n \to \infty} C_{\phi}(\eta(x), g_n(x)) = C^*_\phi(\eta(x)) \, \mathbb{P}\text{-a.e.} \)

Similarly, the optimal value of \( R \) is \( \inf_{f} R(f) = \int C^*(\eta(x)) \, d\mathbb{P} \) and if \( g_n \) is a minimizing sequence of \( R \), then \( \lim_{n \to \infty} C(\eta(x), g_n(x)) = C^*(\eta(x)) \, \mathbb{P}\text{-a.e.} \)

**Proof.** First notice that \( R_{\phi}^* = \int C^*_\phi(\eta(x)) \, d\mathbb{P} \). Clearly, for any function \( f \), \( C_{\phi}(\eta(f(x)), \cdot) \) so \( R_{\phi}(f) = \int C_{\phi}(\eta(f(x)), f(x(x))) \, d\mathbb{P} \geq \int C^*_\phi(\eta(x)) \, d\mathbb{P} \). However, if one chooses \( f(x) = \alpha(\eta(x)) \), where \( \alpha \) is the function of Lemma 3, we see that \( R_{\phi}(f) = \int C^*_\phi(\eta(x)) \, d\mathbb{P} \) and thus \( R_{\phi}^* = \int C^*(\eta(x)) \, d\mathbb{P} \).
Now let \( g_n(x) \) be a sequence of functions that minimizes \( R_\phi \). Then \( C_\phi(\eta(x), g_n(x)) \) must approach \( C_\phi^*(\eta(x)) \) \( \mathbb{P} \)-a.e., as otherwise one could argue that \( \lim_{n \to \infty} R_\phi(g_n) > \int C_\phi^*(\eta(x))d\mathbb{P} \).

An analogous argument implies the result for the classification risk \( R \).

We use this result to prove Lemma 7.

**Proof of Lemma 7** Assume that \( \phi \) is consistent. In other words, every minimizing sequence of \( R_\phi \) must also minimize \( R \). By Lemma 5, \( g_n(x) \) must minimize \( C(\eta(x), \cdot) \) \( \mathbb{P} \)-a.e. Similarly, \( C(\eta(x), g_n(x)) \) must approach \( C_\phi^*(\eta(x)) \) \( \mathbb{P} \)-a.e. By choosing a distribution for which \( \text{supp}\ \mathbb{P} \) is a single point \( \{x\} \), we can conclude that for each \( \eta \), every minimizing sequence of \( C_\phi(\eta, \cdot) \) must also be a minimizing sequence of \( C(\eta, \cdot) \).

Now assume that every sequence \( \alpha_n \) that minimizes \( C_\phi(\eta, \cdot) \) also minimizes \( C(\eta, \cdot) \). Let \( g_n \) be a minimizing sequence of \( R_\phi \). We will argue that the same sequence minimizes \( R \). Lemma 5 implies that \( g_n(x) \) minimizes \( C(\eta(x), \alpha) \) \( \mathbb{P} \)-a.e. By our assumption on \( C_\phi \) and \( C \), this same sequence must also minimize \( C(\eta(x), \cdot) \). Recall that \( C(\eta(x), f(x)) \leq 1 \) for all \( f \). Thus the bounded convergence theorem then implies that

\[
\lim_{n \to \infty} R(g_n) = \lim_{n \to \infty} \int C(\eta(x), g_n(x))d\mathbb{P} = \int \lim_{n \to \infty} C(\eta(x), g_n(x))d\mathbb{P} = \int C_\phi^*(\eta(x), g_n(x))d\mathbb{P} = R^*
\]

Therefore \( g_n \) is a minimizing sequence for \( R \). 

**C  Proof of Theorems 5 and 6**

We start with the proof of Theorem 5.

**Proof of Theorem 5** To start, we show the second equality of (9). Theorem 2 states that

\[
R_\phi^* = \sup_{\mathbb{P}_1' \in B_\phi^*(\mathbb{P}_1)} \inf_{\mathbb{P}_0' \in B_\phi^*(\mathbb{P}_0)} \int \mathbf{1}_{f(x) \leq 0} d\mathbb{P}_1' + \int \mathbf{1}_{f(x) > 0} d\mathbb{P}_0'
\]

By setting \( \mathbb{P}_1' = \mathbb{P}_1 + \mathbb{P}_0' \), \( \eta \hat{=} d\mathbb{P}_1'/d\mathbb{P}_1 \), we can re-write this expression as

\[
R_\phi^* = \sup_{\mathbb{P}_1' \in B_\phi^*(\mathbb{P}_1)} \inf_{\mathbb{P}_0' \in B_\phi^*(\mathbb{P}_0)} \int \eta'(x) \mathbf{1}_{f(x) \leq 0} + (1 - \eta'(x)) \mathbf{1}_{f(x) > 0} d\mathbb{P}_1' = \sup_{\mathbb{P}_1' \in B_\phi^*(\mathbb{P}_1)} \inf_{\mathbb{P}_0' \in B_\phi^*(\mathbb{P}_0)} \int C_\phi^*(\eta'(x)) d\mathbb{P}_1' = \sup_{\mathbb{P}_1' \in B_\phi^*(\mathbb{P}_1)} \inf_{\mathbb{P}_0' \in B_\phi^*(\mathbb{P}_0)} \hat{R}_\phi^*(\mathbb{P}_1', \mathbb{P}_0')
\]

Next, the equality is assumed at the specified \( f, \mathbb{P}_1' \), and \( \mathbb{P}_0' \) by Theorem 2.

We will now show (9). Let \( \hat{\mathbb{P}} = \hat{\mathbb{P}}_0 + \hat{\mathbb{P}}_1, \hat{\eta} = d\mathbb{P}_1'/d(\hat{\mathbb{P}}_1 + \hat{\mathbb{P}}_0) \). Then because \( R_\phi^* = \hat{R}(\hat{f}, \hat{\mathbb{P}}_0, \hat{\mathbb{P}}_1) \),

\[
R_\phi^* = \sup_{\hat{f}} \inf_{\mathbb{P}_1' \in B_\phi^*(\mathbb{P}_1)} \int \mathbf{1}_{f \leq 0} d\mathbb{P}_1' + \int \mathbf{1}_{f > 0} d\mathbb{P}_0'
\]

\[
= \inf_{\hat{f}} \int \mathbf{1}_{\hat{f} \leq 0} d\hat{\mathbb{P}}_1 + \int \mathbf{1}_{\hat{f} > 0} d\hat{\mathbb{P}}_0 = \inf_{\hat{f}} \int \hat{\eta}(\hat{f} \leq 0) + (1 - \hat{\eta}(\hat{f} > 0)) d\hat{\mathbb{P}}
\]

\[
= \int C_\phi^*(\hat{\eta})d\hat{\mathbb{P}}
\]

(21)

The integrand can be minimized in a pointwise manner, so (9) holds. Similarly, if \( g_n(x) \) is a minimizing sequence of \( R_\phi \), Lemma 5 implies that it must minimize \( C(d\hat{\mathbb{P}}_1'/d(\hat{\mathbb{P}}_0 + \hat{\mathbb{P}}_1), \alpha) \hat{\mathbb{P}}_0 + \hat{\mathbb{P}}_1 \)-a.e.

To prove the converse, we use the bounded convergence theorem. Let \( g_n(x) \) be a sequence of functions that minimizes \( C(\hat{\eta}(x), \cdot) \) \( \hat{\mathbb{P}} \)-a.e. Then because \( \hat{\eta}(x) \mathbf{1}_{g_n(x) \leq 0} + (1 - \hat{\eta}(x)) \mathbf{1}_{g_n(x) > 0} \leq 1 \), the bounded convergence theorem implies that

\[
\lim_{n \to \infty} R_\phi^*(g_n) = \lim_{n \to \infty} \int \hat{\eta}(x) \mathbf{1}_{g_n(x) \leq 0} + (1 - \hat{\eta}(x)) \mathbf{1}_{g_n(x) > 0} d\hat{\mathbb{P}} = \int C_\phi^*(\hat{\eta})d\hat{\mathbb{P}}
\]

Therefore, (21) implies that \( g_n \) is a minimizing sequence of \( R_\phi^* \).
The proof of Theorem 6 is identical.

Proof of Theorem 6 To start, we show the second equality of (10). Theorem 4 states that

\[ R_{\phi,*} = \sup_{\mathcal{P}'} \inf_{\mathcal{P}} \int \phi(f) d\mathcal{P}' + \int \phi(-f) d\mathcal{P} \]

By setting \( \mathcal{P}' = \mathcal{P}_1 + \mathcal{P}_0, \) \( \eta' = d\mathcal{P}_1/d\mathcal{P}' \), we can re-write this expression as

\[ R_{\phi,*} = \sup_{\mathcal{P}'_1, \mathcal{P}'_0} \inf_{\mathcal{P}} \int \eta'(x) \phi(f(x)) + (1 - \eta'(x)) \phi(-f(x)) d\mathcal{P}' = \sup_{\mathcal{P}'_1, \mathcal{P}'_0} \int \mathcal{C}_{\phi}^*(\eta') d\mathcal{P}' = \sup_{\mathcal{P}'_1, \mathcal{P}'_0} \mathcal{R}_{\phi}(\mathcal{P}'_0, \mathcal{P}'_1) \]

Next, the equality is assumed at the specified \( f, \mathcal{P}_0, \mathcal{P}_1 \) by Theorem 4.

We will now show (12). Let \( \mathcal{P}^* = \mathcal{P}_0 + \mathcal{P}_1, \eta^* = d\mathcal{P}_1^*/d(\mathcal{P}_0^* + \mathcal{P}_1^*) \). Then because \( R_{\phi,*} = \mathcal{R}_{\phi}(f^*, \mathcal{P}_0^*, \mathcal{P}_1^*) \),

\[ R_{\phi,*} = \inf_{\mathcal{P}} \sup_{\mathcal{P}'_1, \mathcal{P}'_0} \int \phi(f) d\mathcal{P}' + \int \phi(-f) d\mathcal{P} \]

\[ = \inf_{\mathcal{P}} \int \phi(f) d\mathcal{P}' + \int \phi(-f) d\mathcal{P} = \inf_{\mathcal{P}} \int \phi(f) + (1 - \eta^*) \phi(-f) d\mathcal{P} \]

The integrand can be minimized in a pointwise manner, so (12) holds. Similarly, if \( g_n(x) \) is a minimizing sequence of \( R_{\phi} \), Lemma 5 implies that it must minimize \( C_{\phi}(d\mathcal{P}^*_1/d(\mathcal{P}^*_0 + \mathcal{P}^*_1, \alpha))\mathcal{P}^*_0 + \mathcal{P}^*_1 \)-a.e.

D Proof of Theorem 8

Proof of Theorem 8 Notice that \( C^*(\eta) = C^*(1 - \eta) \), which implies that \( C^* \) is symmetric with respect to 1/2. If \( \phi \) is not consistent, then there is an \( k > 1/2 \) for which \( C(k, \alpha) \) is minimized at an \( \alpha^* \) with \( \alpha^* \leq 0 \). Define a distribution \( P_0, P_1 \) for which \( P = P_0 + P_1 \) is the uniform distribution and \( \eta(x) \equiv k \). Then \( R_{\phi} \) is minimized by \( f(x) \equiv \alpha^* \), which does not minimize the classification risk \( R \). Because \( f \) is constant, \( R_{\phi}(f) = R_{\phi}(f) \). However, for all functions \( g \), \( R_{\phi}(g) \geq R_{\phi}(f) \). Thus \( f \) must minimize \( R_{\phi} \) as well. The same argument implies that \( f \equiv \alpha^* \) does not minimize \( R^* \). Therefore, \( \phi \) is not adversarially consistent.

E Lemma 6 used in the proof of Theorem 11

In the proof of Theorem 11, we use the following technical result from [Frank 2022a]:

Lemma 6. Let \( Q \) be an arbitrary Borel measure and let \( f \) be a non-negative Borel measurable function which assumes its infimum on every closed \( \epsilon \)-ball \( Q \)-a.e. Then

\[ \int \sup_{|x' - x| \leq \epsilon} \phi(f(x')) dQ = \sup_{Q' \in \mathcal{B}_{\infty}^n(Q)} \int \phi(f(x)) dQ' \quad (22) \]

The supremum on the right hand side is assumed by a measure given by \( Q \circ T^{-1} \) for a universally measurable map \( T \). Furthermore, this \( T \) satisfies

\[ \sup_{|x' - x| \leq \epsilon} \phi(f(x')) = \phi(f(T(x))) \]

\( Q \)-a.e.