SPECIAL CUBIC BIRATIONAL TRANSFORMATIONS OF PROJECTIVE SPACES

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ABSTRACT. We extend our classification of special Cremona transformations whose base locus has dimension at most three to the case when the target space is replaced by a (locally) factorial complete intersection.

INTRODUCTION

A birational transformation of a complex projective space into a not too singular irreducible projective variety is called special if its base locus scheme is smooth and connected. The study of special birational transformations started by Semple and Tyrrell in [ST68, ST69, ST70], who in particular constructed the main examples of special Cremona transformations having two-dimensional base locus. Later, a more systematic treatment has been provided by Crauder, Katz, Ein, and Shepherd-Barron in [Kat87, CK89, ESB89, CK91], and further contributions has been given in [HKS92]. More recently, other authors and ourselves obtained more results on the theory, focusing mainly on classification problems; see [Rus09, Sta12, AS13, Sta13, Sta15, Li16, AS16, FH18, Sta16].

In this paper, we consider the problem of classifying special birational transformations whose base locus has dimension at most three. When the base locus has dimension three, a preliminary analysis shows that there are only three important classes of transformations (see Corollary 1.3):

- quintic transformations of \( \mathbb{P}^5 \);
- quadratic transformations of \( \mathbb{P}^8 \);
- cubic transformations of \( \mathbb{P}^6 \).

This situation was basically already obtained in [CK91], and the classification of the special Cremona transformations as in the first case has been achieved in [ESB89, Theorem 3.2]. Recently, Alzati and Sierra in [AS16] extended this result to the case in which the image \( Z \) of the map is a prime Fano manifold.

The transformations as in the second case have been classified by ourselves under the hypothesis that the image \( Z \) is a (locally) factorial variety; see [Sta12, Sta13, Sta15]. In the particular case when \( Z \) is a factorial complete intersection, the classification is summarized in Table 4.

In [Sta16] we initiated the study of the transformations as in the third and last case, by classifying the special cubic Cremona transformations of \( \mathbb{P}^6 \), and hence completing the classification of the special Cremona transformations with three-dimensional base locus. This paper is a continuation to [Sta16]. Here we classify special cubic birational transformations of \( \mathbb{P}^6 \) into a factorial complete intersection \( Z \), and hence we complete the classification of all special birational transformations into a factorial complete intersection and whose base locus has dimension at most three. The main result of this paper is the following:

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Theorem 0.1. There are exactly $28 = 9 + 15 + 4$ types (listed, respectively, in Tables 2, 4, and 5) of special birational transformations of a projective space into a factorial complete intersection such that the base locus has dimension at most three and the inverse map is not linear.

The choice of our hypothesis on $Z$ is motivated not only by the desire of simplifying the problem, but also because this condition is satisfied in most of the known examples; see also Remarks 2.6, 5.2, and 6.7.

We give also results on special birational transformations whose inverse map is defined by linear forms, even if these transformations from our point of view are less interesting, and never occur in the case of Cremona transformations. We have the following:

Theorem 0.2. There are exactly 9 types (listed in Table 6) of special birational transformations of a projective space into a factorial complete intersection such that the base locus has dimension at most three and the inverse map is linear.

The paper is organized as follows. In Section 1, by following arguments from [CK89, ESB89], we provide some general results on special birational transformations, giving an account of what is known and what is unknown about the problem of classification when the base locus has dimension at most three. In Section 2 we focus on the main contribution of this paper, by classifying special cubic birational transformations into a factorial complete intersection and whose base locus has dimension at most three, see Corollary 2.5. In Section 3, in order to make our classification effective, we construct new examples of special cubic birational transformations, see Examples 3.1, 3.2, 3.3; we also review some constructions in [Sta16] such as that of a special cubic Cremona transformation having a conic bundle as base locus, see Example 3.6. We provide explicit equations defined over $\mathbb{Q}$ for all our examples. In Section 4 we illustrate briefly a connection between the study of special cubic birational transformations of $\mathbb{P}^6$ and the celebrated problem about the rationality of cubic fourfolds. In Section 5 we recall the analogous classification results for special quadratic transformations, and in Section 6 we analyze the special transformations of $\mathbb{P}^5$ into a complete intersection, which was classified in [AS16] under the hypothesis that the image $Z$ of the map is a prime Fano manifold. Since we allow singularities on $Z$, we get a new type of transformation with respect to their classification, which is the restriction to a general hyperplane of the inverse of a special cubic Pfaffian Cremona transformation of $\mathbb{P}^6$. Finally, in Section 7 we give some results on transformations having linear inverse.

1. General results on special birational transformations

Let $\varphi : \mathbb{P}^n \rightarrow Z \subseteq \mathbb{P}^N$ be a birational transformation into a factorial projective variety $Z \subseteq \mathbb{P}^N$. Without loss of generality we can always take $Z \subseteq \mathbb{P}^N$ to be non-degenerate (i.e., not contained in any hyperplane). We say that $\varphi$ is special if its base locus $\mathcal{B} \subset \mathbb{P}^n$ is smooth and connected.

In this case, the singular locus of $Z$ is set-theoretically contained in the base locus of $\varphi^{-1}$, and we also always require that this inclusion is strict, i.e. that $Z$ is “not too singular”.

Remark 1.1. In this paper we are mainly interested in the case when $Z \subseteq \mathbb{P}^{n+a}$ is an $n$-dimensional non-degenerate complete intersection, hence it is an intersection of hypersurfaces of degrees $e_1, \ldots, e_a \geq 2$. We recall here some well-known facts.

If the singular locus of $Z$ has dimension less than $n - 3$, then $Z$ is factorial (cf. Grothendieck’s parafactoriality theorem (Samuel’s conjecture) [Gro68, XI, 3.14]). In this case, it is normal and hence projectively normal (cf. [Har77, Exercise II 8.4]). Moreover, we can assume $n \geq 3$, so that
we have that $\text{Cl}(Z) \simeq \text{Pic}(Z) \simeq \mathbb{Z}\langle \mathcal{O}_Z(1) \rangle$ (cf. [Har70, Corollary 3.2]), and hence we also have that the homogeneous coordinate ring of $Z$, $S(Z)$, is a unique factorization domain (cf. [Har77, Exercise II 6.3]).

This implies that if $\varphi : \mathbb{P}^n \dashrightarrow Z \subseteq \mathbb{P}^{n+a}$ is a (special) birational map into a factorial complete intersection $Z$, then the inverse map of $\varphi$ is uniquely represented up to proportionality by a vector $(F_0, \ldots, F_n) \in S(Z)^{n+1}$ of forms of the same degree $d = d_2$ without common factors; in other words the degree sequence of $\varphi^{-1}$ has length one and consists of $d$ only, see [Sim04].

Finally, let us recall that if $U$ is the smooth locus of $Z$, we have $\omega_U = \mathcal{O}_U(c - n - 1)$, where $c = \sum_{j=1}^n e_j - a$ is the coinindex of $Z$. We have $c \geq 0$; moreover, $c = 0$ if and only if $Z = \mathbb{P}^n$, and $c = 1$ if and only if $Z$ is a quadric hypersurface (see also [KO73]).

We say that a special birational transformation $\varphi : \mathbb{P}^n \dashrightarrow Z$ is of type $(d_1, d_2)$ if its multidegree has the form $(1, d_1, \ldots, d_2 \Delta, \Delta)$, where $\Delta = \deg(Z)$ and $d_2 \in \mathbb{Z}$. By Remark 1.1, when $Z$ is a factorial complete intersection then the type of $\varphi$ is always well-defined, i.e. $d_2 \in \mathbb{Z}$. We shall also say that $\varphi$ is quadro-quadric, quadro-cubic, cubo-quadric, and so on to indicate that its type is $(2, 2), (2, 3), (3, 2), \text{ and so on.}$

The following proposition is a simple generalization of results from [ESB89, CK89].

**Proposition 1.2.** Let $\varphi : \mathbb{P}^n \dashrightarrow Z$ be a special birational transformation of type $(d_1, d_2)$ into a factorial variety $Z$. Denoting by $r$ (respectively by $r'$) the dimension of the base locus of $\varphi$ (respectively of $\varphi^{-1}$), we have:

\[ r = nd_1d_2 - nd_2 - d_1d_2 - d_2 - c + 2 \]

or equivalently

\[ d_1 = \frac{r' + 2}{n - r - 1}, \quad d_2 = \frac{r - c + 2}{n - r' - 1}, \]

where $c$ is a non-negative integer defined in the proof below.

**Proof.** Let $\mathcal{B} \subseteq \mathbb{P}^n$ be the base locus of $\varphi$. Denote by $\pi : \tilde{\mathbb{P}}^n = \text{Bl}_{\mathcal{B}}(\mathbb{P}^n) \to \mathbb{P}^n$ the natural projection of the blow-up of $\mathbb{P}^n$ along $\mathcal{B}$, $E = \pi^{-1}(\mathcal{B})$ the exceptional divisor, and $H$ a divisor in $|H^0(\pi^*(\mathcal{O}_{\mathbb{P}^n}(1)))|$. Let also $\pi' : \tilde{\mathbb{P}}^n \to Z$ be the composition of $\pi$ and $\varphi$, $\mathcal{B}' \subseteq Z$ the base locus scheme of $\varphi^{-1}$, $E' = \pi'^{-1}(\mathcal{B}')$, and $H'$ a divisor in $|H^0(\pi'^*(\mathcal{O}_Z(1)))|$. Since $\text{Sing}(Z)_{\text{red}} \subseteq \mathcal{B}'_{\text{red}}$, from the same argument used in the proof of [ESB89, Proposition 2.1] (see also [Sta12, Proposition 4.1]), we deduce that

\[ \text{Pic}(\tilde{\mathbb{P}}^n) = \mathbb{Z}\langle H' \rangle \oplus \mathbb{Z}\langle E \rangle = \mathbb{Z}\langle H' \rangle \oplus \mathbb{Z}\langle E' \rangle, \]

and we have the change of basis formulas:

\[ \begin{pmatrix} H' \\ E' \end{pmatrix} = \begin{pmatrix} d_1 & -1 \\ d_1d_2 - 1 & -d_2 \end{pmatrix} \begin{pmatrix} H \\ E \end{pmatrix}. \]

Let us notice that the restriction of $\pi'$ induces an isomorphism of $\tilde{\mathbb{P}}^n \setminus E'$ onto $Z \setminus \mathcal{B}'$ and that $\text{codim}(\mathcal{B}') \geq 2$. Thus we have $\text{Pic}(Z) \simeq \text{Pic}(Z \setminus \mathcal{B}') \simeq \text{Pic}(\tilde{\mathbb{P}}^n \setminus E') \simeq \mathbb{Z}\langle \mathcal{O}_{\tilde{\mathbb{P}}^n}(H') \rangle$, and we can define the integer $c = c(Z)$ such that $\omega_{Z, \mathcal{B}'} \simeq \mathcal{O}_{Z, \mathcal{B}'}(c - n - 1)$. Using this invariant we can write the canonical divisor of $\tilde{\mathbb{P}}^n$ in two different ways (see again the argument in the proof of [ESB89, Proposition 2.1]):

\[ K_{\tilde{\mathbb{P}}^n} = (-n - 1)H + (n - r - 1)E = (c - n - 1)H' + (n - r' - 1)E'. \]
The formulas (1.1) and (1.2) follow directly from (1.3), (1.4), and (1.5).

The following result extends that of [CK91, Corollary 1], and in the case when $Z$ is a prime Fano manifold it is a consequence of [AS16, Propositions 5, 8, 11]. Its proof follows immediately from Proposition 1.2 by taking into account that the dimensions $r$ and $r'$ are at most $n - 2$. See also Table 1.

| Case | $n$ | $r$ | $r'$ | $d_1$ | $d_2$ | $c$ |
|------|----|----|-----|-----|-----|----|
| (1a) | 3  | 1  | 1   | 3   | 2   | 1  |
| (1b) | 3  | 1  | 1   | 3   | 3   | 0  |
| (1c) | 4  | 1  | 2   | 2   | 2   | 1  |
| (1d) | 4  | 1  | 2   | 2   | 3   | 0  |
| (2a) | 4  | 2  | 1   | 3   | 2   | 0  |
| (2b) | 4  | 2  | 2   | 4   | 2   | 2  |
| (2c) | 4  | 2  | 2   | 4   | 3   | 1  |
| (2d) | 5  | 2  | 2   | 2   | 2   | 0  |
| (2e) | 6  | 2  | 4   | 2   | 2   | 1  |
| (2f) | 6  | 2  | 2   | 4   | 3   | 0  |
| (3a) | 5  | 3  | 3   | 2   | 2   | 1  |
| (3b) | 5  | 3  | 3   | 5   | 2   | 3  |
| (3b) | 5  | 3  | 3   | 5   | 3   | 2  |
| (3c) | 5  | 3  | 3   | 5   | 5   | 1  |
| (3d) | 6  | 3  | 4   | 3   | 2   | 3  |
| (3d) | 6  | 3  | 4   | 3   | 3   | 2  |
| (3d) | 6  | 3  | 4   | 3   | 4   | 1  |
| (3d) | 6  | 3  | 4   | 3   | 5   | 0  |
| (3f) | 7  | 3  | 4   | 2   | 2   | 1  |
| (3g) | 8  | 3  | 6   | 2   | 2   | 3  |
| (3g) | 8  | 3  | 6   | 2   | 3   | 2  |
| (3g) | 8  | 3  | 6   | 2   | 4   | 1  |
| (3h) | 8  | 3  | 6   | 2   | 5   | 0  |

Table 1. Preliminary classification of special birational transformations of type $(d_1, d_2)$ into a factorial variety, with $d_2 > 1$ and $r \leq 3$ (the notation is as in Proposition 1.2).

**Corollary 1.3.** Let $\varphi : \mathbb{P}^n \dashrightarrow Z$ be a special birational transformation of type $(d_1, d_2)$, with $d_2 \geq 1$, into a factorial variety $Z$.

1. If the base locus of $\varphi$ has dimension one, then one of the following holds:
   
   (a) $n = 3$ and $\varphi$ is a cubo-quadric transformation into a quadric hypersurface;
   (b) $n = 3$ and $\varphi$ is a cubo-cubic Cremona transformation;
   (c) $n = 4$ and $\varphi$ is a quadro-quadric transformation into a quadric hypersurface;
   (d) $n = 4$ and $\varphi$ is a quadro-cubic Cremona transformation.
(2) If the base locus of \( \phi \) has dimension two, then one of the following holds:

(a) \( n = 4 \) and \( \phi \) is a cubo-quadric Cremona transformation;
(b) \( n = 4 \) and \( \phi \) is a quartic transformation with \( (d_2, c) \in \{(2, 2), (3, 1)\} \);
(c) \( n = 4 \) and \( \phi \) is a quarto-quartic Cremona transformation;
(d) \( n = 5 \) and \( \phi \) is a quadro-quadric Cremona transformation;
(e) \( n = 6 \) and \( \phi \) is a quadratic transformation with \( (d_2, c) \in \{(2, 2), (3, 1)\} \);
(f) \( n = 6 \) and \( \phi \) is a quadro-quartic Cremona transformation.

(3) If the base locus of \( \phi \) has dimension three, then one of the following holds:

(a) \( n = 5 \) and \( \phi \) is a quarto-quadric transformation into a quadric hypersurface;
(b) \( n = 5 \) and \( \phi \) is a quintic transformation with \( (d_2, c) \in \{(2, 3), (3, 2), (4, 1)\} \);
(c) \( n = 5 \) and \( \phi \) is a quinto-quintic Cremona transformation;
(d) \( n = 6 \) and \( \phi \) is a cubic transformation with \( (d_2, c) \in \{(2, 3), (3, 2), (4, 1)\} \);
(e) \( n = 6 \) and \( \phi \) is a cubo-quadric Cremona transformation;
(f) \( n = 7 \) and \( \phi \) is a quadro-quadric transformation into a quadric hypersurface;
(g) \( n = 8 \) and \( \phi \) is a quadratic transformation with \( (d_2, c) \in \{(2, 3), (3, 2), (4, 1)\} \);
(h) \( n = 8 \) and \( \phi \) is a cubo-quintic Cremona transformation.

**Remark 1.4.** The special Cremona transformations having base locus of dimension at most two have been classified in [CK89] (see also [Kat87, CK91, HKS92]). So we have a classification of the transformations as in the cases (1b), (1d), (2a), (2c), (2d), and (2f) of Corollary 1.3. Moreover, the classification of the transformations as in case (3c) follows from [ESB89, Theorem 3.2].

The special quadratic birational transformations having base locus of dimension at most three have been classified in [Sta12, Sta13, Sta15] (the final classification is summarized in [Sta15, Table 1]; see also Table 4 below). So, we also have a classification of the transformations as in the cases (1c), (2e), (3f), (3g), and (3h).

The cases (1a), (2b), and (3a) are simpler to handle (see the proofs of Corollary 2.5 and Propositions 6.1 and 6.2), and furthermore, under the assumption that the image \( Z \) is a smooth prime Fano variety, the transformations as in case (3b) have been classified in [AS16]. It remains to consider the cases (3d) and (3e), which, as it also follows a posteriori from the examples, are the most intricate ones.

Recently, in [Sta16, Theorem 0.5] we classified the transformations as in case (3e), and in the proof of [Sta16, Theorem 0.6] we studied the transformations as in a particular type of case (3d) with \( (d_2, c) = (3, 2) \), that is cubo-cubic transformations into a cubic hypersurface. In Section 2 we follow the approach used in [Sta16] to study more transformations as in case (3d), namely when \( Z \) is a factorial complete intersection. This will allow us to obtain a complete effective classification of all special birational transformations into a factorial complete intersection whose base locus has dimension at most three.

**Remark 1.5.** Using the arguments of [ESB89, Proposition 2.3], it is not difficult to show that if \( \mathcal{B} \subset \mathbb{P}^n \) is the base locus of a special birational transformation \( \phi : \mathbb{P}^n \dashrightarrow Z \) of type \( (d_1, d_2) \) into a factorial variety \( Z \), then the \( d_1 \)-th secant variety of \( \mathcal{B} \), \( \text{Sec}_{d_1}(\mathcal{B}) \), i.e. the closure of the union of all the \( d_1 \)-secant lines of \( \mathcal{B} \), is a hypersurface in \( \mathbb{P}^n \) of degree \( d_1 d_2 - 1 \); moreover, if \( p \in \text{Sec}_{d_1}(\mathcal{B}) \) is a general point, then the union of all the \( d_1 \)-secants of \( \mathcal{B} \) through \( p \) is an \((n - r' - 1)\)-dimensional linear space \( L_p \) and \( L_p \cap \mathcal{B} \subset L_p \) in a hypersurface of degree \( d_1 \), where \( r' \) denotes the dimension of the base locus of \( \phi^{-1} \). However, we will not need this geometric property in the sequel, except in the case when \( d_2 = 1 \). Indeed notice that if \( d_2 = 1 \) then the
above property implies that Sec$_{d_1}(\mathcal{B})$ is the unique hypersurface of degree $d_1 - 1$ containing $\mathcal{B}$; in particular $h^0(\mathbb{P}^n, \mathcal{I}_{\mathcal{B}}(d_1 - 1)) = 1$.

**Remark 1.6.** We recall here a special case of the well-known relationship between the multi-degree of a rational map between projective varieties and the push-forward to projective space of the Segre class of its base locus; see [Ful84, Proposition 4.4], [CK89, p. 291], and [Do111, Subsection 2.3].

Let $\varphi : \mathbb{P}^n \dasharrow Z$ be a special birational transformation of type $(d_1, d_2)$. Let $\mathcal{B}$ denote its base locus, $r = \dim(\mathcal{B})$, and let $s_j(\mathcal{N}_{\mathcal{B}, \mathbb{P}^n}^{r-j})$ be the degree of the $i$-th Segre class of the normal bundle of $\mathcal{B}$. If $(\delta_0, \delta_1, \ldots, \delta_{n-1}, \delta_n) = (1, d_1, \ldots, d_2, \Delta, \Delta)$ is the multi-degree of $\varphi$, then the following formula holds for $k = 0, \ldots, n$:

\begin{equation}
\delta_{n-k} = d_1^{n-k} \left( \frac{n-k}{r-k} \right) d_1^{r-k} \deg(\mathcal{B}) - \sum_{i=k}^{r-1} \left( \frac{n-k}{i-k} \right) d_1^{i-k} s_{r-i}(\mathcal{N}_{\mathcal{B}, \mathbb{P}^n}^{r-i}) H_{\mathcal{B}}^r.
\end{equation}

The following general result puts restrictions on the Hilbert polynomial of the base locus of a special birational transformation (see also [Stal12, Proposition 4.4] and [Ver01, Corollary 2.12]).

**Lemma 1.7.** Let $\varphi : \mathbb{P}^n \dasharrow Z \subseteq \mathbb{P}^N$ be a special birational transformation of type $(d_1, d_2)$ into a non-degenerate and linearly normal factorial variety $Z$. Then we have

\begin{equation}
\chi(\mathcal{O}_\mathcal{B}(d_1)) = \left( \frac{n+d_1}{d_1} \right) - N - 1 \quad \text{and} \quad \chi(\mathcal{O}_\mathcal{B}(d_1 - 1)) = \left( \frac{n+d_1 - 1}{d_1 - 1} \right) + \left\lceil \frac{d_2 - 1}{d_2} \right\rceil - 1.
\end{equation}

Moreover, if the dimension $r'$ of the base locus of $\varphi^{-1}$ is at most $n - 3$, we also have:

\begin{equation}
\chi(\mathcal{O}_\mathcal{B}(d_1 - j)) = \left( \frac{n+d_1 - j}{d_1 - j} \right), \quad \text{whenever } \frac{2}{j} \leq j \leq r' - 1.
\end{equation}

**Proof.** Let the notation be as in the beginning of the proof of Proposition 1.2. If $V$ denotes the $(N+1)$-dimensional vector space associated to the linear system defining $\varphi$, then we have:

\begin{equation}
V \subseteq H^0(\mathbb{P}^n, \mathcal{O}_{\mathcal{B}}(d_1)) = H^0(\mathbb{P}^n, d_1 H - E) = H^0(\mathbb{P}^n, H') = H^0(Z, \pi'_* (\mathcal{O}_{\mathbb{P}^n} \otimes \mathcal{O}_Z(1))) = H^0(Z, \mathcal{O}_Z(1)),
\end{equation}

where the last two equalities follow respectively from the projection formula [Har77, Exercise II 5.1] and from the Zariski’s Main Theorem [Har77, Corollary III 11.4]. Now, from the non-degeneracy and linear normality of $Z$, we get $h^0(Z, \mathcal{O}_Z(1)) = N + 1$, and therefore we have

\begin{equation}
h^0(\mathbb{P}^n, \mathcal{O}_{\mathcal{B}}(d_1)) = N + 1.
\end{equation}

Moreover, if $d_2 > 1$ then we clearly have $h^0(\mathbb{P}^n, \mathcal{O}_{\mathcal{B}}(d_1 - 1)) = 0$, while if $d_2 = 1$ then it follows from Remark 1.5 that $h^0(\mathbb{P}^n, \mathcal{O}_{\mathcal{B}}(d_1 - 1)) = 1$; thus, in formulas, we have

\begin{equation}
h^0(\mathbb{P}^n, \mathcal{O}_{\mathcal{B}}(d_1 - 1)) = 1 - \left\lceil \frac{d_2 - 1}{d_2} \right\rceil.
\end{equation}

Let now $r = \dim \mathcal{B}$ and $K = K_{\mathbb{P}^n}$. Using (1.4) and (1.5), we can write for each $t \in \mathbb{Z}$:

\begin{equation}
tH - E = K + (tH - E - K) = K + (t - ((n-r)d_1 - n - 1))H + (n-r)H'.
\end{equation}

For $t \geq (n-r)d_1 - n - 1$, we have that $(t - ((n-r)d_1 - n - 1))H$ is nef, $(n-r)H'$ is nef and big, and therefore their sum is nef and big (see e.g. [Deb01, p. 20]). Thus from the Kawamata-Viehweg vanishing theorem (see e.g. [Fuj90, Corollary 4.13]) we deduce the following: (Let
us notice that this follows from [BEL91, Proposition 1] when \( t \geq (n - r)d_1 - n \); see also the beginning of the proof of [BEL91, Theorem 1.2].)

\[
(1.11) \quad h^i(\mathbb{P}^n, \mathcal{I}_{\mathcal{B}}(t)) = h^i(\mathbb{P}^n, tH - E) = 0, \text{ whenever } i > 0 \text{ and } t \geq (n - r)d_1 - n - 1.
\]

Now, by (1.1) and (1.11), using the exact sequence \( 0 \rightarrow \mathcal{I}_{\mathcal{B}} \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_{\mathcal{B}} \rightarrow 0 \), we obtain:

\[
(1.12) \quad \chi(\mathcal{O}_{\mathcal{B}}(t)) = h^0(\mathbb{P}^n, \mathcal{O}_{\mathcal{B}}(t)) - h^0(\mathbb{P}^n, \mathcal{I}_{\mathcal{B}}(t)), \text{ whenever } n - r' \geq d_1 + 1 - t,
\]

and the assertion follows by (1.9) and (1.10). \( \square \)

2. Special cubic birational transformations

In this section we prove our main result, that is Theorem 2.3. We first need to show that there are a finite number of possible cases to be analyzed.

**Lemma 2.1.** Let \( \varphi : \mathbb{P}^6 \rightarrow Z \subseteq \mathbb{P}^{6+a} \) be a special birational transformation of type \((3,d)\) into a non-degenerate, linearly normal, factorial variety \( Z \subseteq \mathbb{P}^{6+a} \) of degree \( \Delta \) and codimension \( a \).

Let \( \mathcal{B} \subseteq \mathbb{P}^6 \) be the base locus of \( \varphi \) and assume that it is three-dimensional (which is automatic when \( d \neq 1 \)). Let us define \( \varepsilon(d) = 1 \) if \( d = 1 \) and \( \varepsilon(d) = 0 \) otherwise, and denote by \( \lambda \) and \( g \), respectively, the degree and sectional genus of \( \mathcal{B} \). Then the following hold:

1. The multidegree of \( \varphi \) is given by \((1,3,9,-\lambda + 27,-7\lambda + 2g + 79,\Delta d, \Delta)\).
2. If \( K_{\mathcal{B}} \) and \( H_{\mathcal{B}} \) denote, respectively, a canonical divisor and a hyperplane section divisor of \( \mathcal{B} \), then we have:

\[
K_{\mathcal{B}}H_{\mathcal{B}}^2 = -2\lambda + 2g - 2, \quad K_{\mathcal{B}}^3H_{\mathcal{B}} = -39\lambda + 14g + 3\Delta d - 12a + 12\varepsilon(d) + 331,
\]

\[
K_{\mathcal{B}}^3 = \lambda^2 - 77\lambda + 14g - 3\Delta d - 12a + 60\varepsilon(d) + 688.
\]

3. If \( \mathcal{T}_{\mathcal{B}} \) denotes the tangent bundle of \( \mathcal{B} \) then we have:

\[
c_1(\mathcal{T}_{\mathcal{B}})H_{\mathcal{B}} = 2\lambda - 2g + 2, \quad c_2(\mathcal{T}_{\mathcal{B}})H_{\mathcal{B}} = -29\lambda + 16g - \Delta d + 227,
\]

\[
c_3(\mathcal{T}_{\mathcal{B}}) = 230\lambda - 102g + 11\Delta d - \Delta - 1842.
\]

4. If \( S \subset \mathbb{P}^5 \) denotes a smooth hyperplane section of \( \mathcal{B} \), then we have:

\[
K_{\mathcal{B}}H_{S} = -\lambda + 2g - 2, \quad K_{S} = -42\lambda + 18g + \Delta d - 12a + 12\varepsilon(d) + 327,
\]

\[
\chi(\mathcal{O}_{\mathcal{S}}) = -6\lambda + 3g - a + \varepsilon(d) + 46, \quad c_2(\mathcal{T}_{\mathcal{S}}) = -30\lambda + 18g - \Delta d + 225.
\]

**Proof.** The proof is similar to that of [Sta16, Lemma 3.2], but we include it here for the sake of completeness. From the Hirzebruch-Riemann-Roch formula [Har77, Exercise A 6.7], we have:

\[
(2.1) \quad K_{\mathcal{B}}H_{\mathcal{B}}^2 = -c_1(\mathcal{T}_{\mathcal{B}})H_{\mathcal{B}}^2 = 2(g - 1 - H_{\mathcal{B}}),
\]

\[
(2.2) \quad K_{\mathcal{B}}^3H_{\mathcal{B}} = 12(\chi(\mathcal{O}_{\mathcal{B}}(H_{\mathcal{B}}))) - \chi(\mathcal{O}_{\mathcal{B}})) - 2H_{\mathcal{B}}^3 + 3K_{\mathcal{B}}H_{\mathcal{B}}^2 - c_2(\mathcal{T}_{\mathcal{B}})H_{\mathcal{B}}.
\]

Since \( \mathcal{B} \) is embedded in \( \mathbb{P}^n \) with \( n \leq 6 \), we also have (see [LS86, p. 543]):

\[
(2.3) \quad K_{\mathcal{B}}^3 = c_1(\mathcal{T}_{\mathcal{B}}) + 7c_2(\mathcal{T}_{\mathcal{B}})H_{\mathcal{B}} - 48\chi(\mathcal{O}_{\mathcal{B}}) + (H_{\mathcal{B}}^3)^2 - 35H_{\mathcal{B}}^3 - 21K_{\mathcal{B}}H_{\mathcal{B}}^2 - 7K_{\mathcal{B}}^2H_{\mathcal{B}}.
\]
Moreover, from the exact sequence $0 \to \mathcal{F}_B \to \mathcal{F}_B|_B \to \mathcal{N}_{B|B} \to 0$ and since $s(\mathcal{N}_{B|B}) = c(\mathcal{N}_{B|B})^{-1}$, we get:

(2.4) \hspace{1cm} c_1(\mathcal{F}_B)H_B^3 = 7H_B^3 + s_1(\mathcal{N}_{B|B})H_B^2,

(2.5) \hspace{1cm} c_2(\mathcal{F}_B)H_B^3 = 21H_B^3 + 7s_1(\mathcal{N}_{B|B})H_B^2 + s_2(\mathcal{N}_{B|B})H_B,

(2.6) \hspace{1cm} c_3(\mathcal{F}_B) = 35H_B^3 + 21s_1(\mathcal{N}_{B|B})H_B^2 + 7s_2(\mathcal{N}_{B|B})H_B + s_3(\mathcal{N}_{B|B}).

Since $\mathcal{B}$ is the base locus of a special birational transformation of type $(3, d)$, by (1.6) we have:

(2.7) \hspace{1cm} \Delta = -540H_B^3 - 135s_1(\mathcal{N}_{B|B})H_B^2 - 18s_2(\mathcal{N}_{B|B})H_B - s_3(\mathcal{N}_{B|B}) + 729,

(2.8) \hspace{1cm} \Delta d = -90H_B^3 - 15s_1(\mathcal{N}_{B|B})H_B^2 - 2s_2(\mathcal{N}_{B|B})H_B + 243.

Finally, from the exact sequence $0 \to \mathcal{F}_S \to \mathcal{F}_B|_S \to \mathcal{O}_S(H_S) \to 0$, we get:

(2.9) \hspace{1cm} c_2(\mathcal{F}_S) = c_2(\mathcal{F}_B)H_B + K_SH_S = c_2(\mathcal{F}_B)H_B + K_BH_B + H_B^3,

and by [Har77, Example A 4.1.2] we also have:

(2.10) \hspace{1cm} K_S^2 = 12\chi(\mathcal{O}_S) - c_2(\mathcal{F}_S) = 12(\chi(\mathcal{O}_B) - \chi(\mathcal{O}_B(-H_B))) - c_2(\mathcal{F}_S).

Now, using the formulas (1.7), the proof is reduced to solving a system of linear equations. □

**Lemma 2.2.** In the hypothesis and notation of Lemma 2.1, if $d \neq 1$ then the possible 5-tuples $(\lambda, g, \Delta, d, a)$ belong to a subset $\Gamma_{2480}^3 \subset \mathbb{Z}^5$ of cardinality 2480; if $d = 1$ then the possible 4-tuples $(\lambda, g, \Delta, a)$ belong to a subset $\Gamma_{1139}^4 \subset \mathbb{Z}^4$ of cardinality 1139.

**Proof.** We have $3 \leq \lambda \leq 27$ as $\mathcal{B}$ is a non-degenerate variety cut out by cubics and of codimension 3. Moreover, $g$ is non-negative and limited from above by the Castelnuovo bound [Cas89]:

(2.11) \hspace{1cm} g \leq \left\lfloor \frac{\lambda - 2}{3} \right\rfloor \left(\lambda - 4 - \left\lfloor \frac{\lambda - 2}{3} \right\rfloor - 1\right) \frac{3}{2}.

From Proposition 1.2 we obtain that $d \leq 5$; moreover, $d = 5$ if and only if $\Delta = 1$, and $d = 4$ if and only if $\Delta = 2$. Of course we also have that if $\Delta = 1$ then $a = 0$, if $\Delta = 2$ then $a = 1$, and if $\Delta \geq 3$ then $a \geq 1$. Furthermore, the multidegree $(\delta_0, \ldots, \delta_6)$ of $\phi$ must satisfy a series of inequalities (see [Laz04, Corollary 1.6.3] and [Dol11, Subsection 1.4]), some of which are the following: $\delta_1 \delta_3 \geq \delta_1$, $\delta_1 \delta_4 \geq \delta_5$, $\delta_2^2 \geq \delta_2 \delta_4$, $\delta_5^2 \geq \delta_5 \delta_6$, $\delta_2^2 \geq \delta_1 \delta_6$. Using Lemma 2.1, these five inequalities become:

(2.12) \hspace{1cm} 2\lambda - g + 1 \geq 0,

(2.13) \hspace{1cm} -21\lambda + 6g - \Delta d + 237 \geq 0,

(2.14) \hspace{1cm} \lambda^2 + 9\lambda - 18g + 18 \geq 0,

(2.15) \hspace{1cm} 49\lambda^2 - 28\lambda g + 4g^2 + (\Delta d - 1106)\lambda + 316g - 27\Delta d + 6241 \geq 0,

(2.16) \hspace{1cm} 7\Delta \lambda - 2\Delta g + \Delta^2 d^2 - 79\Delta \geq 0.
Finally from [LS86, Theorem 0.7.3] and Lemma 2.1, we deduce the following four inequalities involving functions of $\lambda, g, \Delta, d, a$:

\begin{align*}
(2.17) & \quad \lambda^2 + 7\lambda - 10g - 2\Delta d + 12a - 12\varepsilon(d) - 92 \geq 0, \\
(2.18) & \quad -10\lambda + 8g + 2\Delta d - 12a + 12\varepsilon(d) + 94 \geq 0, \\
(2.19) & \quad -290\lambda + 140g - 13\Delta d + \Delta + 2290 \geq 0, \\
(2.20) & \quad -147\lambda + 74g + 12\Delta d - \Delta - 84a + 108\varepsilon(d) + 1183 \geq 0.
\end{align*}

All these conditions together define a subset of $\mathbb{Z}^5$ consisting of 3619 elements, of which 2480 have $d$-coordinate different from 1.

**Theorem 2.3.** Let $\phi : \mathbb{P}^n \dasharrow \mathbb{Z} \subseteq \mathbb{P}^{n+a}$ be a special birational transformation of type $(3, d)$, $d > 1$, into a non-degenerate factorial complete intersection. If the base locus $\mathcal{B}$ of $\phi$ has dimension 3, then $n = 6$ and one of the following cases holds:

- $a = 0$, $\phi$ is a cubo-quintic Cremona transformation, and we have two cases according to [Sta16, Theorem 0.5]:
  - $\mathcal{B}$ is a threefold of degree 14, sectional genus 15, with trivial canonical bundle, and given by the Pfaffians of a skew-symmetric matrix;
  - $\mathcal{B}$ is a threefold of degree 13 and sectional genus 12, which admits the structure of a conic bundle over $\mathbb{P}^2$.

- $a = 1$, $\mathcal{B}$ is a threefold of degree 12 and sectional genus 10, and two cases occur:
  - $\phi$ is a cubo-cubic transformation into a cubic hypersurface and $\mathcal{B}$ admits the structure of a fibration over $\mathbb{P}^1$ whose generic fiber is a sextic del Pezzo surface;
  - $\phi$ is a cubo-quartic transformation into a quadric hypersurface and $\mathcal{B}$ is the blow-up at one point of a threefold of degree 13 and sectional genus 10 which admits the structure of a fibration over $\mathbb{P}^1$ whose generic fiber is a quintic del Pezzo surface.

- $a = 2$, $\phi$ is a cubo-cubic transformation into a complete intersection of two quadrics in $\mathbb{P}^8$ and $\mathcal{B}$ is a threefold of degree 11 and sectional genus 8 obtained by blowing-up 3 points on a 3-dimensional linear section of $\mathbb{G}(1, 5) \subset \mathbb{P}^{18}$.

- $a = 3$, $\phi$ is a cubo-quadratic transformation into a complete intersection of three quadrics in $\mathbb{P}^9$ and $\mathcal{B}$ is a threefold of degree 10 and sectional genus 6, which admits the structure of a scroll over $\mathbb{P}^2$ with four double points blown up.

**Proof.** The fact that $n = 6$ follows by (1.1). Thus we are in the situation of Lemmas 2.1 and 2.2.

If $\mathcal{Z}$ is a complete intersection of type $(e_1, \ldots, e_n)$, then $\Delta = \prod_{i=1}^n e_i$ and $c = \sum_{i=1}^n e_i - a$. From Proposition 1.2 we see that $a = \sum_{i=1}^n e_i + d - 5$ and in particular this forces $a + d \leq 5$. By imposing these conditions on the set $\Gamma_3^{2480}$ given in Lemma 2.2, we get a set $\Gamma_3^{174}$ consisting of 174 not excluded 5-tuples $(\lambda, g, \Delta, d, a)$, and of which the not excluded triples $(\Delta, d, a)$ are the following:

- $(1, 5, 0)$, $(4, 2, 1)$, $(3, 3, 1)$, $(2, 4, 1)$, $(6, 2, 2)$, $(4, 3, 2)$, $(8, 2, 3)$; moreover, this forces $7 \leq \lambda \leq 18$ and $0 \leq g \leq 28$.

In the following we apply general results from adjunction theory for which we refer mainly to [Fuj90, BS95, Ion84, Ion86b, Som86, SV87, BBS89].

Consider the complete linear system $|K_{\mathcal{B}} + 2H_{\mathcal{B}}|$ on $\mathcal{B}$. By [Som79, SV87], it is base-point free unless $(\mathcal{B}, H_{\mathcal{B}})$ is one of the following:

1. $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$;
2. a quadric $(\mathcal{Q}^3, \mathcal{O}_{\mathcal{Q}^3}(1))$;
2.21 \[ (K_{\mathcal{B}} + 3H_{\mathcal{B}})^3 = (K_{\mathcal{B}} + 3H_{\mathcal{B}})^2 H_{\mathcal{B}} = 0, \]

which, using Lemma 2.1, translate into two polynomial equations in the variables \((\lambda, g, \Delta, d, a)\) without common solutions in the set \(\Gamma^5_{174}\). So we deduce that \(|K_{\mathcal{B}} + 2H_{\mathcal{B}}|\) is base-point free.

Now we show that \(K_{\mathcal{B}} + 2H_{\mathcal{B}}\) is nef and big with the exception of one case. By [Som86, SV87], we have that \(K_{\mathcal{B}} + 2H_{\mathcal{B}}\) is nef and big unless \((\mathcal{B}, H_{\mathcal{B}})\) has the structure of one of the following:

1. a del Pezzo variety;
2. a quadric fibration over a smooth curve;
3. a scroll over a smooth surface.

In case (1) we get three equations:
\[ K_{\mathcal{B}} H_{\mathcal{B}}^2 + 2H_{\mathcal{B}}^3 = K_{\mathcal{B}}^2 H_{\mathcal{B}} + 2K_{\mathcal{B}} H_{\mathcal{B}}^2 = K_{\mathcal{B}}^3 + 2K_{\mathcal{B}}^2 H_{\mathcal{B}} = 0. \]

In case (2) we get two equations:
\[ (K_{\mathcal{B}} + 2H_{\mathcal{B}})^3 = (K_{\mathcal{B}} + 2H_{\mathcal{B}})^2 H_{\mathcal{B}} = 0. \]

In case (3) we get one equation:
\[ (K_{\mathcal{B}} + 2H_{\mathcal{B}})^3 = 0. \]

Using Lemma 2.1 all these equations translate into polynomial equations in \((\lambda, g, \Delta, d, a)\). Then one can see that the systems (2.22) and (2.23) have no solutions in \(\Gamma^5_{174}\), while (2.24) has exactly the two solutions: \((10, 6, 8, 2, 3)\) and \((12, 11, 8, 2, 3)\). Let us recall that in case (3) we have \(\mathcal{B} = \mathbb{P}_Y(\mathcal{E})\) and the numerical invariants of the polarized surface \((Y, \det \mathcal{E})\) can be determined in terms of those of \((\mathcal{B}, H_{\mathcal{B}})\), see [BB05b, Lemma 5.2]. Thus we get two cases:

- \((\lambda, g, \Delta, d, a) = (10, 6, 8, 2, 3), \chi(\mathcal{E}|_Y) = 1, K_Y^2 = 5, c_1(\mathcal{E})^2 = 20, \) and \(c_2(\mathcal{E}) = 10;\)
- \((\lambda, g, \Delta, d, a) = (12, 11, 8, 2, 3), \chi(\mathcal{E}|_Y) = 4, K_Y^2 = 18, c_1(\mathcal{E})^2 = 29, \) and \(c_2(\mathcal{E}) = 17.\)

The first case is possible and corresponds to a cubo-quadric transformation into a complete intersection of three quadrics, see Example 3.4. Instead, the second case does not occur since \(Y\) must be non-ruled and hence we would have \(c_1(\mathcal{E})^2 \leq 2g - 2\) (see also [BT15, Proposition 5.3]).

Thus, from now on we can assume that \(K_{\mathcal{B}} + 2H_{\mathcal{B}}\) is nef and big. It follows that \((\mathcal{B}, H_{\mathcal{B}})\) admits a unique (minimal) reduction, that is there exists a pair \((\mathcal{R}, H_{\mathcal{B}})\), where \(\mathcal{R}\) is a smooth irreducible threefold and \(H_{\mathcal{B}}\) an ample divisor on \(\mathcal{R}\), such that there is a morphism \(\mathcal{B} \to \mathcal{R}\) expressing \(\mathcal{B}\) as the blowing-up of \(\mathcal{R}\) at a finite number \(\nu\) of distinct points \(p_1, \ldots, p_{\nu}\) and \(H_{\mathcal{B}} \simeq \pi^* H_{\mathcal{R}} - \sum_{i=1}^{\nu} \pi^{-1}(p_i)\). Moreover, \(K_{\mathcal{R}} + 2H_{\mathcal{R}}\) is very ample, where \(K_{\mathcal{R}}\) denotes the canonical divisor of \(\mathcal{R}\).

We have the following upper bound for the number \(\nu\) of exceptional divisors on \((\mathcal{B}, H_{\mathcal{B}})\):\(^1\)
\[ \nu \leq (1/8) \lambda^4 - (1/2) \lambda^2 \Delta d + (1/2) \Delta^2 d^2 + (3/4) \lambda^3 - 3 \lambda^2 g + (1/2) \lambda \Delta d + 5 g \Delta d + 3 \lambda^2 a - 6 \Delta d a - (433/8) \lambda^2 + 20 \lambda g + 13 g^2 + (25/2) \Delta d - 7 \lambda a - 30 g a + 18 a^2 + (2825/4) \lambda - 98 g - 21 a - 2969. \]

\(^1\)Notice that the difference between the right and left side of the inequality (2.25) coincides with the sum \(\Sigma_i ((5j^3)/4)\), where \(l\) runs over all lines contained in the surface \(S \subset \mathbb{P}^3\) having self-intersection \(\leq -2\). Thus (2.25) is a strict inequality if and only if \(S\) contains a line with self-intersection \(\leq -6\).
Indeed the inequality (2.25) is obtained by applying to the general hyperplane section $S$ of $\mathfrak{B}$ the formula, due to P. Le Barz (see [LB80, LB81]), calculating the number of 4-secant lines of a surface in $\mathbb{P}^3$, and taking into account that, since $S$ is cut out by cubics, it cannot have 4-secant lines; see also [Sta16, Subsection 2.4]. In particular, it follows that there are a finite number of not excluded 6-tuples $(\lambda, g, \nu, \Delta, d, a)$, which actually are 4237. Let $\Gamma^6_{4237}$ denote the set of all these 6-tuples. In this set we have $11 \leq \lambda \leq 18$, $7 \leq g \leq 28$, and $0 \leq \nu \leq 181$.

Now we show that $K_{\mathfrak{B}}+H_{\mathfrak{B}}$ is nef and big in exactly one case. For $j = 0, \ldots, 3$ we define the $j$-th pluridegree of $(\mathfrak{B}, H_{\mathfrak{B}})$ as $d_j(\mathfrak{B}) = (K_{\mathfrak{B}}+H_{\mathfrak{B}})^j H_{\mathfrak{B}}^{3-j}$. Notice that we have $K_{\mathfrak{B}} H_{\mathfrak{B}}^2 = K_{\mathfrak{B}} H_{\mathfrak{B}}^2 - 2\nu$, $K_{\mathfrak{B}}^2 H_{\mathfrak{B}} = K_{\mathfrak{B}}^2 H_{\mathfrak{B}} + 4\nu$, and $K_{\mathfrak{B}}^3 = K_{\mathfrak{B}}^3 - 8\nu$, so that from Lemma 2.1 we can express the pluridegrees of $(\mathfrak{B}, H_{\mathfrak{B}})$ as polynomial functions of $(\lambda, g, \nu, \Delta, d, a)$. Now, if $K_{\mathfrak{B}}+H_{\mathfrak{B}}$ is nef and big, we have the following inequalities (see [BBS89]):

\begin{align}
(2.26) & \quad d_1(\mathfrak{B}) - 1 \geq 0, \quad d_2(\mathfrak{B}) - 1 \geq 0, \quad d_3(\mathfrak{B}) - 1 \geq 0, \quad d_4(\mathfrak{B})^2 - d_2(\mathfrak{B}) d_0(\mathfrak{B}) \geq 0,
(2.27) & \quad -\lambda + 2g - \nu - 3 \geq 0,
(2.28) & \quad \Delta d - 42\lambda + 18g + \nu - 12a + 326 \geq 0,
(2.29) & \quad \lambda^2 - 192\lambda + 62g - \Delta - 48a + 1674 \geq 0,
(2.30) & \quad -\lambda \Delta d - \nu \Delta d + 43\lambda^2 - 22\lambda g + 4g^2 + 43\lambda \nu - 22g \nu + 12\lambda a + 12\nu a - 323\lambda - 8g - 323\nu + 4 \geq 0.
\end{align}

The 6-tuples $(\lambda, g, \nu, \Delta, d, a)$, belonging to the set $\Gamma^6_{4237}$ that satisfy (2.27), (2.28), (2.29), and (2.30) are the two following: $(14, 15, 0, 1, 5, 0)$ and $(18, 28, 0, 3, 3, 1)$. The last one does not occur: indeed we have $d_1(\mathfrak{B})^2 - d_2(\mathfrak{B}) d_0(\mathfrak{B}) = 0$ and $d_2(\mathfrak{B})^2 - d_3(\mathfrak{B}) d_1(\mathfrak{B}) = 1521 \neq 0$, and this is a contradiction by [BBS89, Lemma 1.1, (1.1.2)] (see also the proof of [Sta16, Theorem 0.6]).

The tuple $(14, 15, 0, 1, 5, 0)$ corresponds to a Cremona transformation and we can apply [Sta16, Theorem 0.5], see also Example 3.5.

We now can assume that $K_{\mathfrak{B}}+H_{\mathfrak{B}}$ is nef and big. Then by [Som86] we have that $(\mathfrak{B}, H_{\mathfrak{B}})$ has the structure of one of the following:

1. $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$;
2. $(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$;
3. a Veronese fibration over a curve;
4. a Mukai variety;
5. a del Pezzo fibration over a curve;
6. a conic bundle over a surface.

One see easily that cases (1) and (2) do not occur (by applying for instance [BB05b, Lemma 3.3]). In case (3) we have two equations:

\begin{align}
(2.31) & \quad (2K_{\mathfrak{B}}+3H_{\mathfrak{B}})^3 = (2K_{\mathfrak{B}}+3H_{\mathfrak{B}})^2 H_{\mathfrak{B}} = 0,
\end{align}

which admit no solutions in $\Gamma^6_{4237}$. In case (4) we have three equations:

\begin{align}
(2.32) & \quad K_{\mathfrak{B}}^3 + H_{\mathfrak{B}}^3 = K_{\mathfrak{B}}^2 H_{\mathfrak{B}} - H_{\mathfrak{B}}^3 = K_{\mathfrak{B}} H_{\mathfrak{B}}^2 + H_{\mathfrak{B}}^3 = 0,
\end{align}

which admit in $\Gamma^6_{4237}$ the unique solution $(11, 8, 3, 4, 3, 2)$, corresponding to a cubo-cubic transformation into a complete intersection of two quadrics, see Example 3.3. In case (5) we have
two equations and an inequality:

\[(2.33) \quad d_3(\mathcal{R}) = 0, \quad d_2(\mathcal{R}) = 0, \quad \text{and} \quad d_1(\mathcal{R}) - 1 \geq 0.\]

The tuples in the set \(\Gamma^6_{4237}\) that satisfy (2.33) are the two following: \((12, 10, 0, 3, 3, 1)\) and \((12, 10, 1, 2, 4, 1)\). Moreover one has that in these two cases \((\mathcal{R}, H_{\mathcal{R}})\) is a fibration over \(\mathbb{P}^1\) whose generic fibre is a del Pezzo surface of degree 6 and 5, respectively (see [BB05b, p. 13]; see also [BT15, Proposition 5.16]). Both these cases are possible, see Examples 3.1 and 3.2. Finally, in case (6) we have one equation and two inequalities:

\[(2.34) \quad d_3(\mathcal{R}) = 0, \quad d_2(\mathcal{R}) - 1 \geq 0, \quad \text{and} \quad d_1(\mathcal{R})^2 - d_2(\mathcal{R}) d_0(\mathcal{R}) \geq 0.\]

There is only one tuple in \(\Gamma^6_{4237}\) that satisfy (2.34), which is \((13, 12, 0, 1, 5, 0)\). This corresponds to a Cremona transformation and we can apply [Sta16, Theorem 0.5], see also Example 3.6. This conclude the proof.

**Remark 2.4.** The proof of Theorem 2.3 can be easily translated into a computer program. Our code written for MACAULAY2 [GS19] is available as an ancillary file included in the arXiv submission. Although our code is not optimal, it executes all of the proof in a few seconds.

**Corollary 2.5.** Table 2 classifies all the special birational transformations of type \((3, d), d > 1\), into a factorial complete intersection and whose base locus has dimension at most three.

**Proof.** If the dimension of the base locus \(\mathfrak{B}\) is three, this is the content of Theorem 2.3. In the case when the dimension of \(\mathfrak{B}\) is two, from Corollary 1.3 we see that the transformation must be a cubo-quadric Cremona transformation, and then we apply [CK89]. If the dimension of \(\mathfrak{B}\) is one, from Corollary 1.3 we have either a cubo-cubic Cremona transformation or a cubo-quadric transformation into a quadric. The first case is classical (cf. [Kat87]). In the second case we apply Lemma 1.7; and to get an example we can take the restriction to a hyperplane of a special cubo-quadric Cremona transformation of \(\mathbb{P}^4\). \(\square\)

| \(r\) | \(n\) | \(a\) | \(\lambda\) | \(g\) | Abstract structure of \(\mathfrak{B}\) | \(d\) | \(\Delta\) | Existence |
|---|---|---|---|---|---|---|---|---|
| I | 1 | 3 | 6 | 3 | Determinantal curve | 3 | 1 | [Kat87] |
| II | 1 | 3 | 1 | 5 | Elliptic curve | 2 | 2 | [CK89] |
| III | 2 | 4 | 0 | 5 | Elliptic scroll | 2 | 1 | [CK89] |
| IV | 3 | 6 | 0 | 14 | 15 | Pfaffian threefold | 5 | 1 | Example 3.5 |
| V | 3 | 6 | 0 | 13 | 12 | Conic bundle over \(\mathbb{P}^2\) | 5 | 1 | Example 3.6 |
| VI | 3 | 6 | 1 | 12 | 10 | Sextic del Pezzo fibration over \(\mathbb{P}^1\) | 3 | 3 | Example 3.1 |
| VII | 3 | 6 | 1 | 12 | 10 | Blow up at one point of a quintic del Pezzo fibration over \(\mathbb{P}^1\) | 4 | 2 | Example 3.2 |
| VIII | 3 | 6 | 2 | 11 | 8 | Blow up at three points of \(G(1, 5) \cap \mathbb{P}^9 \subset \mathbb{P}^9\) | 3 | 4 | Example 3.3 |
| IX | 3 | 6 | 3 | 10 | 6 | Scroll over \(\mathbb{P}^2\) with four double points blown up | 2 | 8 | Example 3.4 |

**Table 2.** All the types of special birational transformations \(\mathbb{P}^r \dasharrow \mathfrak{Z} \subseteq \mathbb{P}^{r+a}\) of type \((3, d)\), with \(d > 1\) and base locus \(\mathfrak{B}\) of dimension \(r \leq 3\), degree \(\lambda\), sectional genus \(g\), into a factorial complete intersection \(\mathfrak{Z} \subseteq \mathbb{P}^{r+a}\) of degree \(\Delta\).
Remark 2.6. Let \( \varphi : \mathbb{P}^n \to Z \subseteq \mathbb{P}^n + a \) be a special birational transformation of type \((3, d)\), with \( d > 1 \) and base locus of dimension three, into a prime Fano manifold \( Z \). If \( Z \) is not a complete intersection, then \( \varphi \) is a cubo-cubic transformation into \( \mathbb{G}(1, 4) \subseteq \mathbb{P}^9 \) and its base locus is a threefold of degree 10 and sectional genus 6 with the structure of a scroll over \( \mathbb{P}^2 \).

Proof. Indeed, by Corollary 1.3 we see that \( n = 6 \) and the coindex \( c \) of \( Z \) satisfies \( c = 5 - d \leq 3 \). Thus such manifolds \( Z \) are completely classified [Fuj90, Muk89] and we get conditions on the possible \( \Delta \) and \( a \). By imposing these conditions on the set \( \Gamma_{2480}^a \setminus \Gamma_{174}^a \) (see Lemma 2.2 and the proof of Theorem 2.3), we get a set \( \Gamma_{88}^a \), consisting of 88 not excluded 5-tuples \((\lambda, g, \Delta, d, a)\). Then, by proceeding as in the proof of Theorem 2.3, one gets that \( \varphi \) is a cubo-cubic transformation into \( \mathbb{G}(1, 4) \subseteq \mathbb{P}^9 \) and its base locus is as in one of the two following cases:

- a threefold of degree 10 and sectional genus 6 with the structure of a scroll over \( \mathbb{P}^2 \);
- a minimal threefold of log-general type of degree 16 and sectional genus 23.

The first case actually occurs: the map defined by the maximal minors of a generic \( 3 \times 5 \) matrix of linear forms on \( \mathbb{P}^n \) gives us an example (see also [Sta16, Subsection 4.4.1]). The second case does not occur. Indeed such a threefold must be linked inside the complete intersection of three cubics to a smooth threefold \( X \subseteq \mathbb{P}^6 \) of degree 11 and sectional genus 13 (see e.g. [Oko94, p. 423]). By Lemma 1.7 and [Har94, Proposition 4.7], we can determine the Hilbert polynomial of \( X \), so that in particular we deduce that \( \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-1)) = 8 \). But this is a contradiction by the maximal list of invariants of threefolds of degree 11 obtained in [BSS92, BB05a]. □

Remark 2.7. Let \( \varphi : \mathbb{P}^n \to Z \) be a special birational transformation of type \((3, d)\), with \( d > 1 \) and base locus of dimension four, into a factorial variety \( Z \). From Proposition 1.2, it follows that \( n = 7 \), the base locus of \( \varphi^{-1} \) has dimension four, and one of the following holds:

- \( \varphi \) is a cubo-cubic Cremona transformation;
- \( \varphi \) is a cubo-quadratic transformation into a variety \( Z \) of coindex 2.

To the best of the author’s knowledge, no examples are known of such transformations. Notice that they look like extensions to \( \mathbb{P}^n \) of special cubic transformations of \( \mathbb{P}^6 \). In [Sta16, Theorem 0.6], we showed that the base locus of a special Cremona transformation of \( \mathbb{P}^7 \) must necessarily be a fourfold of degree 12 and sectional genus 10 with a structure of sextic del Pezzo fibration over \( \mathbb{P}^1 \).

2.1. Cubo-linear birational transformations. In this subsection, we describe the special cubic birational transformations into a factorial complete intersection and whose base locus has dimension at most three that are not considered in Corollary 2.5 and Table 2. The following result tells us that there are no relevant transformations.

Theorem 2.8. Let \( \varphi : \mathbb{P}^n \to Z \subseteq \mathbb{P}^n + a \) be a special birational transformation of type \((3, 1)\) into a factorial complete intersection and whose base locus \( \mathcal{B} \) has dimension \( r \leq 3 \). Then \( n \in \{3, 4, 5\} \), \( \mathcal{B} \) has codimension 2, degree 5, and sectional genus 2, and \( Z \subseteq \mathbb{P}^n + a \) is a complete intersection of two quadrics. Moreover, if \( n = 4 \) then \( \mathcal{B} \) is the image of \( \mathbb{P}^2 \) via the linear system of quartic curves with 7 simple base points and one double point; if \( n = 5 \) then \( \mathcal{B} \) admits the structure of quadric fibration over \( \mathbb{P}^1 \).

Proof. From Proposition 1.2 we deduce that there are two cases:

1. \((r, n) \in \{(1, 3), (2, 4), (3, 5)\}, r' = 1, \) and \( c = 2 \), so that \((\Delta, a) \in \{(4, 2), (3, 1)\}\);
2. \(r = 3, n = 6, r' = 4, \) and \( c = 4 \).
Case (1). Since \( r' = 1 \), by Lemma 1.7 we have \( n - 1 = r + 1 \) conditions on the Hilbert polynomial of \( \mathcal{B} \), and hence we can write it as a function of \( a \). Thus we see that \( \lambda = 7 - a \) and \( g = 6 - 2a \), and this leaves two cases: either \( (\lambda, g, \Delta, a) = (5, 2, 4, 2) \) or \( (\lambda, g, \Delta, a) = (6, 4, 3, 1) \). In the former case, we can apply [Ion84]. In the latter case, \( \mathcal{B} \) must be a complete intersection of a quadric and a cubic, so that \( Z \) is a cubic hypersurface with a double point \( p \) and \( \varphi^{-1} \) is the projection from \( p \). This case is excluded because we have that the singular locus of \( Z \) coincides with the base locus of \( \varphi^{-1} \).

Case (2). Let the notation be as in Lemmas 2.1 and 2.2. If \( Z \) is a complete intersection of type \( (e_1, \ldots, e_n) \), then \( \Pi_{i=1}^n e_i = \Delta \) and \( \sum_{i=1}^n e_i = a + c = a + 4 \). Thus we have that \((\Delta, a) \in \{(5, 1), (8, 2), (9, 2), (12, 3), (16, 4)\} \), and the only 4-tuples \((\lambda, g, \Delta, a)\) in the set \( \Gamma^4_{1139} \) that satisfy this are: \((15, 21, 16, 4), (15, 20, 16, 4), (14, 17, 16, 4), (13, 14, 16, 4), (15, 19, 12, 3), (18, 28, 9, 2) \). Now, we can repeat step by step the proof of Theorem 2.3 in order to conclude that the first three cases do not occur, while in the other three ones we have that \( \mathcal{B} \) is a minimal threefold of log-general type. In these three remaining cases, the multidegree of \( \varphi \) is respectively: \((1, 3, 9, 14, 16, 16, 16), (1, 3, 9, 12, 12, 12, 12), (1, 3, 9, 9, 9, 9, 9) \), from which follows that the dimension \( r' \) of the base locus of \( \varphi^{-1} \) is respectively \( 3, 2, \) and \( 1 \). This is a contradiction since we must have \( r' = 4 \) (see also Remark 2.10).

The following example shows that all the cases of Theorem 2.8 actually occur.

**Example 2.9.** Let \( P \subset \mathbb{P}^n, n \in \{3, 4, 5\} \), be a two-codimensional subspace, and let \( W \subset \mathbb{P}^n \) be a generic complete intersection of a quadric and a cubic containing \( P \). Then the residual intersection \( X = \overline{W \setminus P} \) is a two-codimensional smooth irreducible variety of degree 5 and sectional genus 2, cut out by one quadric and two cubics. The linear system of cubics through \( X \) defines a birational map \( \mathbb{P}^n \dashrightarrow Z \subset \mathbb{P}^{n+2} \), where \( Z \) is a smooth complete intersection of two quadrics.

**Remark 2.10.** Here we give three examples of birational maps of type \((3, 1)\) with smooth and irreducible base locus from \( \mathbb{P}^6 \) into a complete intersection, having multidegrees as in the three cases considered in the final part of the proof of Theorem 2.8. These maps are not special since the singular locus of the image coincides set-theoretically with the base locus of the inverse map.

1. Let \( X \subset \mathbb{P}^6 \) be a generic complete intersection of one quadric and two cubics. Then \( X \) is a smooth irreducible threefold of degree 18 and sectional genus 28, and the linear system of cubics through \( X \) defines a birational map into a complete intersection \( Z \) of two cubics with \( \text{Sing}(Z) = 1 \).

2. Let \( Y \subset \mathbb{P}^4 \subset \mathbb{P}^6 \) be a generic cubic hypersurface. Let \( W \subset \mathbb{P}^6 \) be a generic complete intersection of one quadric and two cubics containing \( Y \). Then the residual intersection \( X = \overline{W \setminus Y} \) is a smooth irreducible threefold of degree 15 and sectional genus 19, cut out by one quadric and three cubics. The linear system of cubics through \( X \) defines a birational map into a complete intersection \( Z \) of two quadrics and one cubic with \( \text{Sing}(Z) = 2 \).

3. Let \( Y \subset \mathbb{P}^5 \subset \mathbb{P}^6 \) be a generic complete intersection of two quadrics. Let \( W \subset \mathbb{P}^6 \) be a generic complete intersection of one quadric and two cubics containing \( Y \). Then the residual intersection \( Y' = \overline{W \setminus Y} \) is a smooth irreducible threefold of degree 14 and sectional genus 16, cut out by one quadric and three cubics. The linear system of cubics through \( Y' \) defines a birational map into a 6-fold \( Z \subset \mathbb{P}^9 \) of degree 13 and sectional genus 14, cut out by one quadric and four cubics. Let \( X \subset \mathbb{P}^6 \) be a generic three-dimensional linear section of \( Z \). Then the linear system of cubics through \( X \) defines a birational map \( \mathbb{P}^6 \dashrightarrow Z \subset \mathbb{P}^{10} \) into a complete intersection \( Z \) of four quadrics with \( \text{Sing}(Z) = 3 \).
3. NEW AND REVISED EXAMPLES OF SPECIAL CUBIC TRANSFORMATIONS

In this section we give explicit constructions for all the examples of special cubic birational transformations of $\mathbb{P}^6$ as in Table 2.

Most of the calculations in the examples below are done using MACAULAY2 [GS19] with the package Cremona [Sta18a]; some others, as singular locus computations, are done using SINGULAR [DGPS18]. We point out that the version 4.2.3 or later of the MACAULAY2 package Cremona provides the method specialCubicTransformation, which takes an integer between 1 and 9 and returns an explicit example of special cubic birational transformation over $\mathbb{Q}$ in accordance to Table 2.

**Example 3.1.** Let $Y = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$ be the Segre embedding of three $\mathbb{P}^1$. It is classically known (see [Edg32, CMR04]) that $Y$ is a threefold with one apparent double point, that is for the general point $p \in \mathbb{P}^7$ there is a unique secant line $L \subset \mathbb{P}^7$ to $Y$ passing through $p$. Using this property, we can define a Cremona involution $T : \mathbb{P}^7 \dashrightarrow \mathbb{P}^7$, by sending $p$ to the point $T(p) \in L$ such that $\{p, T(p)\}$ is harmonically conjugate to $L \cap Y$ (see e.g. [Dol11, Lecture 4]). The dual variety to $Y$ is a hypersurface defined by a quartic form $F$, known as Cayley’s hyperdeterminant, and the involution $T$ coincides with the map defined by the partial derivatives of $F$.\(^2\) The base locus of $T$ is a reducible fourfold $X \subset \mathbb{P}^7$ of degree 12 and sectional arithmetic genus 10 consisting of the union of three $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$ intersecting pairwise in $Y$. Thus, although $T$ is far from being special, it has the same invariants as a hypothetical special Cremona transformation of $\mathbb{P}^7$, by [Sta16, Theorem 0.6]; see also Remark 2.7.

Now we deform $X$ by applying generic linkages. Take $W$ to be a generic complete intersection of three cubics containing $X$. Then one has that $W = X \cup X'$, where $X' \subset \mathbb{P}^7$ is an irreducible fourfold of degree 15 and sectional genus 16 cut out by 5 cubics. Take now $W'$ to be a generic complete intersection of three cubics containing $X'$. Then one has that $W' = X' \cup \tilde{X}$, where $\tilde{X} \subset \mathbb{P}^7$ is an irreducible fourfold of degree 12 and sectional genus 10, cut out by 9 cubics that define a Cremona transformation $\tilde{T} : \mathbb{P}^7 \dashrightarrow \mathbb{P}^7$. Unfortunately, $\tilde{X}$ is still singular and hence $\tilde{T}$ is not special. However, the singular locus of $\tilde{X}$ consists of only a finite number of points, so that the restriction of $\tilde{T}$ to a general hyperplane $\mathbb{P}^6 \subset \mathbb{P}^7$ gives us an example of special cubo-cubic birational transformation $\mathbb{P}^6 \dashrightarrow Z \subset \mathbb{P}^7$ into a cubic hypersurface $Z$, as in line VI of Table 2. One verifies that the singular locus of $Z$ has dimension one.

**Example 3.2.** Let $S^{10} \subset \mathbb{P}^{15}$ be the 10-dimensional Spinor variety parametrizing one of the families of $\mathbb{P}^4$’s contained in a smooth quadric in $\mathbb{P}^3$. Let $Y = S^{10} \cap \mathbb{P}^8 \subset \mathbb{P}^8$ be the intersection of $S^{10} \subset \mathbb{P}^{15}$ with a generic $\mathbb{P}^8 \subset \mathbb{P}^{15}$. So that $Y \subset \mathbb{P}^8$ is a smooth threefold of degree 12 and sectional genus 7, cut out by 10 quadrics (these 10 quadrics define a special quadro-quartic birational transformation into a smooth quadric as in line XIII of Table 4). Let $Y' \subset \mathbb{P}^7$ be the projection of $Y$ in $\mathbb{P}^7$ from a general point on $Y$. Then $Y'$ is a smooth threefold of degree 11 and sectional genus 7, cut out by 5 quadrics. Take $W$ to be a generic complete intersection of type $(2,2,2,3)$ containing $Y'$. We have that $W = Y' \cup X'$, where $X' \subset \mathbb{P}^7$ is a smooth threefold of degree 13 and sectional genus 10, cut out by four quadrics and two cubics. Let now $X \subset \mathbb{P}^6$ be the projection of $X'$ in $\mathbb{P}^6$ from a general point on $X'$. Then $X \subset \mathbb{P}^6$ is a smooth threefold.

\(^2\)For further computational details, see the online documentation of the methods abstractRationalMap from Cremona [Sta18a], and dualVariety from Resultants [Sta18b].
of degree 12 and sectional genus 10, cut out by 8 cubics. These 8 cubics define a special cubo-quartic birational transformation \( \mathbb{P}^6 \dashrightarrow Z \subset \mathbb{P}^7 \) into a quadric hypersurface \( Z \) singular along a line. Thus we have an example of transformation as in line VII of Table 2.

**Example 3.3.** Let \( G(1,5) \subset \mathbb{P}^{14} \) be the Grassmannian of lines in \( \mathbb{P}^5 \). Let \( Y = G(1,5) \cap \mathbb{P}^9 \subset \mathbb{P}^9 \) be the intersection of \( G(1,5) \subset \mathbb{P}^{14} \) with a generic \( \mathbb{P}^9 \subset \mathbb{P}^{14} \). So that \( Y \subset \mathbb{P}^9 \) is a smooth threefold of degree 14 and sectional genus 8, cut out by 15 quadrics. Let \( X \subset \mathbb{P}^6 \) be the projection of \( Y \) in \( \mathbb{P}^6 \) from a plane generated by three general points on \( Y \). Then \( X \subset \mathbb{P}^6 \) is a smooth threefold of degree 11 and sectional genus 8, cut out by 9 cubics. These 9 cubics define a special cubo-cubic birational transformation \( \mathbb{P}^6 \dashrightarrow Z \subset \mathbb{P}^8 \) into a complete intersection of two quadrics which has singular locus of dimension one. Thus we have an example of transformation as in line VIII of Table 2. (Let us recall that the projection of \( Y \) in \( \mathbb{P}^8 \) from a general point on \( Y \) is a threefold cut out by 9 quadrics that define a special quadro-quintic Cremona transformation of \( \mathbb{P}^8 \) as in line X of Table 4.)

**Example 3.4.** Here we give an example of a special cubo-quadratic birational transformation into a complete intersection of three quadrics in \( \mathbb{P}^9 \), as in line IX of Table 2. This example has already been constructed in [Sta16, Subsection 4.4.2] but here we provide a slightly different construction. The base locus of this transformation is a threefold scroll in lines of degree 10 and sectional genus 6 whose existence had been left undecided in [FL97].

The main idea in [Sta15] provides an algorithm to construct special birational transformations whose base locus is a threefold scroll \( \chi = \mathbb{P}(\delta) \) over a surface \( \chi \). In our specific case, the steps to follow are the following:

- Take the map \( f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^{15} \) defined by the linear system of sextic curves with 4 general double points. Then we have \( Y = f(\mathbb{P}^2) \).
- Take 10 general points \( p_1, \ldots, p_{10} \) on \( \mathbb{P}^2 \) and let \( h : \mathbb{P}^{15} \dashrightarrow \mathbb{P}^5 \) be the projection from the linear span of the 10 points \( f(p_1), \ldots, f(p_{10}) \). Then \( S = (h \circ f)(\mathbb{P}^2) \) is a general hyperplane section of \( X \).
- Let \( \psi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^{10} \) be the map defined by the cubics through \( S \). This is a cubo-quadratic birational map into a complete intersection of 4 quadrics. Then the linear system generated by these 4 quadrics together with the 6 quadrics defining the inverse of \( \psi \) defines a quadro-cubic Cremona transformation of \( \mathbb{P}^9 \).
- Compute the inverse \( \Psi : \mathbb{P}^9 \dashrightarrow \mathbb{P}^9 \) of the Cremona transformation in the previous step. Then \( \Psi \) is an extension map of \( \psi \), and its base locus is an irreducible 6-fold of degree 10, sectional genus 6, and with singular locus of dimension 2. By restricting \( \Psi \) to a general \( \mathbb{P}^6 \subset \mathbb{P}^9 \) we get a special birational transformation \( \mathbb{P}^6 \dashrightarrow Z \subset \mathbb{P}^9 \) into a complete intersection \( Z \) of three quadrics; the singular locus of \( Z \) has dimension one.

**Example 3.5** (Subsection 4.2 of [ESB89]; see also Subsection 4.1 of [Sta16]). Let \( G(1,6) \subset \mathbb{P}^{20} \) be the Grassmannian of lines in \( \mathbb{P}^6 \), and let Sec\((G(1,6)) \subset \mathbb{P}^{20} \) denote the variety of secant lines to \( G(1,6) \). We have that Sec\((G(1,6)) \subset \mathbb{P}^{20} \) is an irreducible variety of codimension three, degree 14, sectional genus 15, cut out by 8 cubics, and with singular locus equal to \( G(1,6) \). The intersection Sec\((G(1,6)) \cap \mathbb{P}^6 \subset \mathbb{P}^6 \) of Sec\((G(1,6)) \) with a generic \( \mathbb{P}^6 \subset \mathbb{P}^{20} \) is a smooth threefold of degree 14 and sectional genus 15 which is the base locus of a special cubo-quintic Cremona transformation of \( \mathbb{P}^6 \), hence as in line IV of Table 2. Notice that the equations of Sec\((G(1,6)) \subset \mathbb{P}^{20} \) are given by the Pfaffians of the principal \( 6 \times 6 \) minors of a generic \( 7 \times 7 \) skew-symmetric matrix of variables.
Example 3.6 (Subsection 4.2 of [Sta16]). Here, we recall and simplify the construction of a special Cremona transformation of $\mathbb{P}^6$ whose base locus is a threefold of degree 13 and sectional genus 12, hence as in line V of Table 2.

We first construct an irreducible singular threefold in $\mathbb{P}^6$ of degree 13 and sectional genus 12. Let $C \subset \mathbb{P}^3$ be the isomorphic projection of the quintic rational normal curve together with an embedded point $p_0$, and let $p_1, \ldots, p_5$ be 5 general points on a general plane passing through $p_0$. Then the homogeneous ideal of the non-reduced scheme $C \cup \{p_1, \ldots, p_5\} \subset \mathbb{P}^3$ is generated by 7 quartics, which define a map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^6$. The image of this map is an irreducible threefold $X \subset \mathbb{P}^6$ of degree 13 and sectional genus 12, cut out by 7 cubics, and with singular locus of dimension two.

Now we proceed as in Example 3.1 in order to deform the threefold $X$ into another threefold $\tilde{X}$. Let $W \subset \mathbb{P}^6$ be a generic complete intersection of three cubics containing $X$. Then we have $W = X \cup X'$, where $X' \subset \mathbb{P}^6$ is an irreducible threefold of degree 14 and sectional genus 14, cut out by 6 cubics. Now let $W' \subset \mathbb{P}^6$ be a generic complete intersection of three cubics containing $X'$. Then we have $W' = X' \cup \tilde{X}$, where $\tilde{X} \subset \mathbb{P}^6$ is an irreducible threefold of degree 13 and sectional genus 12, cut out by 7 cubics, and with only one singular point. Even if we repeat the procedure starting from $\tilde{X}$ we still get one singular point.

We remove the singularity of $\tilde{X}$ in this way. Take $S \subset \mathbb{P}^5$ to be a generic hyperplane section of $\tilde{X}$. The cubics through $S$ define a cubo-quintic birational map into a quintic hypersurface. Taking together the quintic defining the image and the 6 quintics defining the inverse, we get a linear system of quintics on $\mathbb{P}^5$ which defines a quinto-cubic Cremona transformation. The inverse of this map has as base locus a smooth irreducible threefold of degree 13 and sectional genus 12.

4. Connection with cubic fourfolds

In all our examples of special cubic birational transformations of $\mathbb{P}^6$ (see Table 2), a generic cubic in the linear system defining the map turns out to be singular at a finite number of points. Thus its generic hyperplane section $X \subset \mathbb{P}^5$ is a smooth cubic hypersurface containing a very special surface $S \subset \mathbb{P}^3$, the general hyperplane section of the base locus of the map.

Let $\mathcal{H}$ be an irreducible component of the Hilbert scheme $\text{Hilb}_{\mathbb{P}^5}^2(\mathcal{O}_{\mathbb{P}^5}(1))$ containing $[S]$. In lucky cases, as it turns out to be in all our cases, via some explicit calculations on a particular pair $(S, X)$, and via a semicontinuity argument illustrated in [Nue15], one can demonstrate that the cubic hypersurfaces in $\mathbb{P}^5$ containing a surface belonging to $\mathcal{H}$ form an irreducible locus in $\mathbb{P}^{35} = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^5}(3)))$ of codimension one. The closures in $\mathbb{P}^{35}$ of these loci belong to a countable family of divisors $\{C_\delta\}_\delta$ indexed by the integers $\delta > 6$ with $\delta \equiv 0, 2 \pmod{6}$. These divisors have been introduced and studied by Hassett in [Has99, Has00] (see also [Has16]). With our examples, we cover the first four members of this family, that is $C_8, C_{12}, C_{14},$ and $C_{18}$, thus providing alternative geometric descriptions of them. We include in Table 3 some more details.

The value of $\delta$ can be calculated in terms of the invariants of the surface $S$ (see [Has00, Section 4.1]). Since we are assuming that $S$ is the general hyperplane section of the base locus of a special cubic birational transformation, we can apply Lemma 2.1 (4) to get a formula depending only by the simplest invariants of the transformation. In the notation of Lemma 2.1, we have:

\begin{equation}
\delta = -\lambda^2 + 6\lambda d - 27\lambda + 18g - 36a + 36e(d) + 288.
\end{equation}
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\[ \delta = h_0 \left( \mathcal{I}_{S/P^5} (3) \right) \]

### Remark 2.6

Example 3.5 (Table 2, line IV) 14 7 77 29
Example 3.6 (Table 2, line V) 14 7 68 20
Example 3.1 (Table 2, line VI) 18 8 63 16
Example 3.2 (Table 2, line VII) 12 8 65 18
Example 3.3 (Table 2, line VIII) 14 9 60 14
Example 3.4 (Table 2, line IX) 14 10 55 10
Example 7.3 18 19 39 3
Example 7.4 18 13 51 9
Example 7.5 14 12 58 15
Example 7.6 14 11 63 19

**Table 3.** Surfaces \( S \subset P^5 \) contained in a cubic fourfold \( [X] \in C_\delta \) and obtained as general hyperplane section of the base locus of a cubic transformation of \( P^6 \).

Notice that (4.1) forces \( \delta \equiv 0, 2 \) (mod 6), but not \( \delta > 6 \). By imposing this last condition on the sets \( \Gamma_{1480} \) and \( \Gamma_{1139} \) given in Lemma 2.2, and proceeding as in the proof of Theorem 2.3, one can see that no other values of \( \delta \), besides 8, 12, 14, 18, can be achieved by a special birational transformation of \( P^6 \) of type \((3, d)\) into a factorial variety.

One of the most challenging open problems in classical and modern algebraic geometry is the rationality of cubic fourfolds. The works by Hassett, Kuznetsov, Addington, and Thomas (see [Kuz10, AT14, Kuz16, Has16]) lead to the following problem (usually called Kuznetsov Conjecture): *The generic cubic fourfold \([X] \in C_\delta \) is rational if and only if \( \delta \) is an admissible value.* The admissible values are the even integers \( \delta > 6 \) not divisible by 4, by 9 and nor by any odd prime of the form \( 2 + 3m \), so that the first admissible values are 14, 26, 38. The rationality for \( C_{14} \) was shown in the classical works by Morin and Fano (see [Mor40, Fan43]; see also [BRS15]), and in the recent paper [RS17], Russo and ourselves showed the rationality for \( C_{26} \) and \( C_{38} \). The decisive step of our discovery was to find a description for \( C_\delta \) in terms of some surface \( S \) contained in the generic \([X] \in C_\delta \) and which admits (for some \( e \geq 1 \)) a congruence of \((3e - 1)\)-secant rational curves of degree \( e \), that is, through a general point \( p \in P^5 \) there passes a unique rational curve \( C_p \) of degree \( e \) which is \((3e - 1)\)-secant to \( S \); see [RS17] for precise and more general definitions.

This property is a very rare phenomenon for the surfaces in \( P^5 \), but not so rare for our surfaces. Indeed, the surfaces corresponding to the lines IV and V of Table 2 admit a congruence of 14-secant rational normal quintic curves (see [RS18, Subsection 4.3]); and the surface corresponding to the line VIII admits a congruence of 8-secant twisted cubics (see [RS18, Subsection 4.2]). Furthermore, the isomorphic projection in \( P^5 \) of the surface corresponding to the line V of Table 4 admits a congruence of 5-secant conics (see [RS17, Theorem 2]); and the surface of \( P^5 \), base locus of the inverse map corresponding to the line II of Table 5, admits a congruence of 2-secant lines [Fan43].

### 5. Special Quadratic Birational Transformations

For the convenience of the reader, in this section we collect some of the results concerning special quadratic birational transformations. For proofs and details, see [Sta12, Sta13, Sta15].
Theorem 5.1 ([Sta12, Sta13, Sta15]). Table 4 classifies all the special birational transformations of type \((2, d)\), \(d > 1\), into a factorial complete intersection and whose base locus has dimension at most three.

|   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|
| I | 1 | 4 | 0 | 5 | 1 | Elliptic curve | 3 | 1 |
| II | 2 | 4 | 1 | 4 | 0 | Rational normal curve | 2 | 2 |
| III | 2 | 5 | 0 | 4 | 0 | Veronese surface | 2 | 1 |
| IV | 2 | 6 | 0 | 7 | 1 | Elliptic scroll | 4 | 1 |
| V | 2 | 6 | 0 | 8 | 3 | Blow up of \(\mathbb{P}^2\) at 8 simple points | 4 | 1 |
| VI | 1 | 6 | 7 | 7 | Blow up of \(\mathbb{P}^2\) at 5 simple points and one double point | 3 | 2 |
| VII | 2 | 6 | 2 | 6 | 1 | Blow up of \(\mathbb{P}^2\) at 3 simple points | 2 | 4 |
| VIII | 3 | 7 | 1 | 6 | 1 | Hyperplane section of \(\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8\) | 2 | 2 |
| IX | 2 | 8 | 0 | 12 | 6 | Scroll over rational surf. with \(K^2 = 5\) | 5 | 1 |
| X | 3 | 8 | 0 | 13 | 8 | Blow up at one point of \(G(1, 5) \cap \mathbb{P}^9 \subset \mathbb{P}^9\) | 5 | 1 |
| XI | 3 | 8 | 1 | 11 | 5 | Blow-up of \(Q^2\) at 5 points | 3 | 3 |
| XII | 3 | 8 | 1 | 11 | 5 | Scroll over \(\mathbb{F}_1\) | 4 | 2 |
| XIII | 3 | 8 | 1 | 12 | 7 | Linear section of the spin, \(\mathbb{S}_{10} \subset \mathbb{P}_{15}\) | 4 | 2 |
| XIV | 3 | 8 | 2 | 10 | 4 | Scroll over \(Q^2\) | 3 | 4 |
| XV | 3 | 8 | 3 | 9 | 3 | Scroll over \(\mathbb{P}^2\) | 2 | 8 |

Table 4. All the types of special birational transformations \(\mathbb{P}^n \rightarrow Z \subseteq \mathbb{P}^{n+a}\) of type \((2, d)\), with \(d > 1\) and base locus \(\mathcal{B}\) of dimension \(r \leq 3\), degree \(\lambda\), sectional genus \(g\), into a factorial complete intersection \(Z \subseteq \mathbb{P}^{n+a}\) of degree \(\Delta\).

Remark 5.2. Let \(\varphi : \mathbb{P}^n \rightarrow Z \subseteq \mathbb{P}^{n+a}\) be a special birational transformation of type \((2, d)\), with \(d > 1\) and base locus of dimension three, into a prime Fano manifold \(Z\). If \(Z\) is not a complete intersection, then we immediately deduce from the classification in [Sta15, Table 1] that \(\varphi\) is a quadro-quadric transformation of \(\mathbb{P}^8\) into \(G(1, 5) \subset \mathbb{P}^{14}\) and its base locus is a rational normal scroll of degree 6. This kind of transformations has been classically studied in [Sem31].

By Remark 1.5, the following proposition follows easily from results on quadratic entry locus varieties, which are proved in [Rus09, CMR04, IR10].

Proposition 5.3 ([Sta13]). Let \(\varphi : \mathbb{P}^n \rightarrow Z \subseteq \mathbb{P}^{n+a}\) be a special birational transformation of type \((2, 1)\) into a factorial complete intersection and whose base locus \(\mathcal{B}\) has dimension \(r \leq 3\). Then \(n \in \{3, 4, 5\}\), \(\mathcal{B} \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n\) is a quadric of codimension 2, and \(Z \subset \mathbb{P}^{n+1}\) is a quadric hypersurface.

We also point out that in the quadratic case, a number of results on special birational transformations have been obtained, without imposing any condition on the dimension of the base locus. Among these we have classifications for quadro-quadric Cremona transformations [ESB89] (those whose base locus is a Severi variety [Zak93, Chapter IV]), quadro-cubic Cremona transformations [Rus09], quadro-quadric transformations into a smooth quadric [Sta12, AS13],
quadro-quintic Cremona transformations [Sta13, Sta15, Rus09], quadro-cubic transformations into a factorial del Pezzo variety [Sta13, Sta15], quadro-linear transformations into a prime Fano manifold [FH18].

6. SPECIAL QUARTIC AND QUINTIC BIRATIONAL TRANSFORMATIONS

Taking into account Corollary 1.3, Remark 1.4, Corollary 2.5, and Theorem 5.1, in order to complete the classification of the special birational transformations \( \varphi : \mathbb{P}^n \rightarrow Z \subseteq \mathbb{P}^{n+a} \) whose base locus \( \mathcal{B} \) has dimension at most three and where \( Z \) is a factorial complete intersection, we only need to classify the transformations as in the cases (2b), (3a) and (3b) of Corollary 1.3. We treat the case (2b) in Proposition 6.1, the case (3a) in Proposition 6.2, and the case (3b) in Theorem 6.4. We recall that these kinds of transformations have been classified in [AS16] under the hypothesis that \( Z \) is a smooth prime Fano variety.

**Proposition 6.1.** Let \( \varphi : \mathbb{P}^4 \rightarrow Z \subseteq \mathbb{P}^{4+a} \) be a special birational transformation of type \((4,d)\), with \( d > 1 \) and base locus \( \mathcal{B} \) (necessarily of dimension two), into a non-degenerate factorial complete intersection \( Z \subseteq \mathbb{P}^{4+a} \). Then one of the two following cases holds:

- \( \varphi \) is a quarto-quartic Cremona transformation and \( \mathcal{B} \) is a determinantal surface of degree \( \lambda = 10 \) and sectional genus \( g = 11 \) given by the vanishing of the \( 4 \times 4 \) minors of a \( 4 \times 5 \) matrix of linear forms;
- \( \varphi \) is a quarto-quadratic transformation into a cubic hypersurface and \( \mathcal{B} \) is a \( K_3 \) surface with five \((-1)\)-lines of degree \( \lambda = 9 \) and sectional genus \( g = 8 \).

**Proof.** If \( a = 0 \) we apply [CK89, Theorem 3.3]. So we can assume \( a > 1 \). Denoting by \( \Delta \) the degree of \( Z \), by Corollary 1.3, case (2b), we have to consider the following cases:

- \( d = 3, a = 1, \Delta = 2 \);
- \( d = 2, a = 1, \Delta = 3 \);
- \( d = 2, a = 2, \Delta = 4 \).

By (1.7) we can express \( \chi(\mathcal{O}_\mathcal{B}) \) and the sectional genus \( g \) as functions of \( \lambda \) and \( a \) as follows:

\[
(6.1) \quad \chi(\mathcal{O}_\mathcal{B}) = 6\lambda + 3a - 55, \quad g = 4\lambda + a - 29.
\]

Since \( \mathcal{B} \) is a surface embedded in \( \mathbb{P}^4 \), we can compute \( K_\mathcal{B}^2 \) and \( c_2(\mathcal{T}_\mathcal{B}) \) (see [Har77, p. 434]):

\[
(6.2) \quad K_\mathcal{B}^2 = (1/2)\lambda^2 + (27/2)\lambda + 13a - 180, \quad c_2(\mathcal{T}_\mathcal{B}) = -(1/2)\lambda^2 + (117/2)\lambda + 23a - 480.
\]

Using (6.2) one can easily compute that

\[
(6.3) \quad s_1(\mathcal{N}_{\mathcal{B},\mathbb{P}^4}) H_\mathcal{B} = -12\lambda - 2a + 60, \quad s_2(\mathcal{N}_{\mathcal{B},\mathbb{P}^4}) = -(1/2)\lambda^2 + (217/2)\lambda + 33a - 780.
\]

From the formula (1.6) (with \( (n,k) = (4,0) \)) using (6.3) we get

\[
(6.4) \quad \Delta = (1/2)\lambda^2 - (25/2)\lambda - a + 76.
\]

In the cases when \( (a,\Delta) \in \{(1,2), (2,4)\} \), the equation (6.4) has no integral solutions, while if \( (a,\Delta) = (1,3) \) we get \( \lambda = 9 \) or \( \lambda = 16 \). Of course, \( \lambda = 16 \) is impossible since \( \mathcal{B} \) is cut out by quartics and is not a complete intersection, therefore we have:

\[
d = 2, a = 1, \Delta = 3, \lambda = 9, g = 8, K_\mathcal{B}^2 = -5, \chi(\mathcal{O}_\mathcal{B}) = 2. \]

Now the assertion follows from [AR]. \( \square \)
Proposition 6.2. There are no special birational transformations $\mathbb{P}^5 \dashrightarrow Z \subseteq \mathbb{P}^{5+a}$ of type $(4, d)$, with $d > 1$, into a linearly normal factorial variety $Z \subseteq \mathbb{P}^{5+a}$.

Moreover, if $\mathbb{P}^5 \dashrightarrow Z \subseteq \mathbb{P}^{5+a}$ is a special birational transformation of type $(4, 1)$ into a factorial complete intersection $Z$, then its base locus is a threefold of degree 9 and sectional genus 9, which is linked to a cubic scroll in the complete intersection of a cubic and a quartic, and $Z \subseteq \mathbb{P}^8$ is a complete intersection of three quadrics.

Proof. Let $\varphi : \mathbb{P}^5 \dashrightarrow Z \subseteq \mathbb{P}^{5+a}$ be a special birational transformation of type $(4, d)$ into a non-degenerate linearly normal factorial variety $Z$ of degree $\Delta$. From Proposition 1.2, we obtain that either $d = 2$ and $c(Z) = 1$ (so that $Z$ is a quadric hypersurface), or $d = 1$ and $c(Z) = 3$; moreover, in both cases the base locus $\mathcal{B}$ of $\varphi$ is a threefold of a certain degree $\lambda$, and the base locus of $\varphi^{-1}$ has dimension two. It follows that the multidegree of $\varphi$ is given by

\[ (6.5) \quad (1, 4, 16 - \lambda, \Delta d^2, \Delta d, \Delta). \]

Now, by Lemma 1.7 we can express the Hilbert polynomial of $\mathcal{B}$ as a function of $\lambda$ and $a$. In particular, the sectional genus of $\mathcal{B}$ is given by $g = 4\lambda - 2\epsilon(d) + a - 28$, where $\epsilon(1) = 1$ and $\epsilon(2) = 0$. Then, just as in the proof of Lemma 2.1, we can compute that

\[
\begin{align*}
\ell_1(N_{\mathcal{B}, \mathbb{P}^5})H^2_{\mathcal{B}} & = -12\lambda - 2a + 4\epsilon(d) + 58, \\
\ell_2(N_{\mathcal{B}, \mathbb{P}^5})H^2_{\mathcal{B}} & = -\Delta d + 96\lambda + 32a - 64\epsilon(d) - 672, \\
\ell_3(N_{\mathcal{B}, \mathbb{P}^5}) & = 20\Delta d - 640\lambda - \Delta - 320a + 640\epsilon(d) + 5184,
\end{align*}
\]

and therefore by (1.6) we deduce that the multidegree of $\varphi$ must be

\[ (6.6) \quad (1, 4, 16 - \lambda, 2a - 4\epsilon(d) + 6, \Delta d, \Delta). \]

By comparing (6.5) and (6.6) we get:

\[ (6.7) \quad a = (1/2)\Delta d^2 + 2\epsilon(d) - 3. \]

On the other hand, from the fact that the general hyperplane section of $\mathcal{B}$ is a surface embedded in $\mathbb{P}^4$, we get the following relation (see [Har77, p. 434]):

\[ (6.8) \quad \lambda^2 - 2\Delta d - 25\lambda - 2a + 16\epsilon(d) + 150 = 0. \]

Thus, we conclude that $\Delta = d = 2$ is impossible. Therefore it must be that

\[ (6.9) \quad d = 1, \quad \Delta = (1/3)\lambda^2 - (25/3)\lambda + 56, \quad a = (1/6)\lambda^2 - (25/6)\lambda + 27, \]

and the first assertion follows.

Now, since we have $\lambda < 16$ and $g \geq 0$, we deduce that the invariants of $\varphi$ are as in one of the following cases ($m$ stands for multidegree):

(i) $(\lambda, g, \Delta) = (6, 2, 18), m = (1, 4, 10, 18, 18, 18)$;
(ii) $(\lambda, g, \Delta) = (7, 4, 14, 6), m = (1, 4, 9, 14, 14, 14)$;
(iii) $(\lambda, g, \Delta) = (9, 9, 8, 3), m = (1, 4, 7, 8, 8, 8)$;
(iv) $(\lambda, g, \Delta) = (10, 12, 6, 2), m = (1, 4, 6, 6, 6, 6)$;
(v) $(\lambda, g, \Delta) = (12, 19, 4, 1), m = (1, 4, 4, 4, 4, 4)$;
(vi) $(\lambda, g, \Delta) = (13, 23, 4, 1), m = (1, 4, 3, 4, 4, 4)$;
(vii) $(\lambda, g, \Delta) = (15, 32, 6, 2), m = (1, 4, 1, 6, 6, 6)$. 

In case (iii) we can apply [FL94], and one easily verifies that this case occurs with $Z$ smooth. In cases (i) and (ii), $Z$ is not a complete intersection. Cases (iv) and (v) are in contradiction with the fact that the base locus of the inverse map has dimension (at least) two. Finally, cases (vi) and (vii) are excluded since the multidegree does not satisfy the Hodge inequalities. 

The following proposition is established in a similar way as we showed Propositions 6.1 and 6.2; see also [AS16, Proposition 10].

**Proposition 6.3.** If $\mathbb{P}^4 \dashrightarrow Z \subseteq \mathbb{P}^{4+a}$ is a special birational transformation of type $(4, 1)$ into a factorial complete intersection $Z$, then its base locus is a surface of degree 9 and sectional genus 9, which is linked to a cubic scroll in the complete intersection of a cubic and a quartic, and $Z \subseteq \mathbb{P}^7$ is a complete intersection of three quadrics.

We also omit the proof of the following result because the methods we used in Section 2 can be applied here more efficiently, since the numerical invariants of a threefold in $\mathbb{P}^5$ satisfy more conditions of those of a threefold in $\mathbb{P}^6$. Alternatively, one can obtain a proof from the results of Alzati and Sierra in [AS16]. Indeed, even if they require that $Z$ is a smooth prime Fano variety, some of their arguments also apply to our situation with obvious changes.

**Theorem 6.4** ([ESB89, AS16]). Let $\phi : \mathbb{P}^5 \dashrightarrow Z \subseteq \mathbb{P}^{5+a}$ be a special birational transformation of type $(5, d)$ into a non-degenerate factorial complete intersection $Z \subseteq \mathbb{P}^{5+a}$. Then its base locus $\mathcal{B}$ has dimension three, and one of the three following cases holds:

- $\phi$ is a quinto-quintic Cremona transformation, $\mathcal{B}$ has degree 15 and sectional genus 26 and is given by the vanishing of the $5 \times 5$ minors of a $5 \times 6$ matrix of linear forms;
- $\phi$ is a quinto-cubic transformation into a cubic hypersurface and $\mathcal{B}$ has degree 14 and sectional genus 22;
- $\phi$ is a quinto-linear transformation into a complete intersection of four quadrics and $\mathcal{B}$ has degree 14 and sectional genus 23.

**Remark 6.5.** In the second case of Theorem 6.4, $\mathcal{B}$ is linked inside the complete intersection of two quintics to a threefold of degree 11 and sectional genus 13, from which it follows that the ideal sheaf $\mathcal{I}_{\mathcal{B}}$ is given by a resolution $0 \rightarrow \mathcal{I}_{\mathbb{P}^5}(-7) \oplus \mathcal{O}_{\mathbb{P}^5}(-6) \rightarrow \mathcal{O}_{\mathbb{P}^5}(-5)^{\oplus 7} \rightarrow \mathcal{I}_{\mathcal{B}} \rightarrow 0$; see [BSS92] and [AS16, Lemma 7]. Moreover the cubic hypersurface $Z \subseteq \mathbb{P}^6$ turns out to be singular. In Example 6.6 we shall give an explicit example where $Z$ is factorial, being singular only at a finite number of points.

In the third case of Theorem 6.4, an explicit example can be constructed by using that $\mathcal{B}$ is linked inside the complete intersection of two quintics to a threefold of degree 11 and sectional genus 14, which is linked inside the complete intersection of two quartics to a threefold of degree 5 and sectional genus 2 as in Example 2.9. Even in this case one sees that $\mathcal{Z}$ is singular only at a finite number of points. Moreover, the ideal sheaf $\mathcal{I}_{\mathcal{B}}$ is given by a resolution $0 \rightarrow \mathcal{O}_{\mathbb{P}^5}(-6)^{\oplus 4} \rightarrow \mathcal{O}_{\mathbb{P}^5}(-4) \oplus \mathcal{O}_{\mathbb{P}^5}(-5)^{\oplus 4} \rightarrow \mathcal{I}_{\mathcal{B}} \rightarrow 0$; see [BSS92] and [AS16, Lemma 8].

**Example 6.6.** Let $\phi : \mathbb{P}^6 \dashrightarrow \mathbb{P}^6$ be a special cubo-quintic Cremona transformation as in Example 3.5. Let $Z \subset \mathbb{P}^6$ be a generic cubic hypersurface containing the base locus of $\phi$. One sees that the singular locus of $Z$ is zero-dimensional, and in particular $Z$ is factorial. Now, the base locus of the inverse map is an irreducible fourfold of degree 14, sectional genus 22, and with zero-dimensional singular locus. Hence the restriction of $\phi$ to $Z$ gives a cubo-quintic birational transformation $\phi|_Z : Z \dashrightarrow \mathbb{P}^5 \subset \mathbb{P}^6$ whose inverse map is special and with three-dimensional base locus. Thus we have examples of transformations as in line IV of Table 5.
Some of the relevant results of this section are summarized in Table 5.

|   | r | n | a | \(\lambda\) | g | Abstract structure of \(\mathcal{B}\) | \(d_1\) | \(d\) | \(\Delta\) | Existence |
|---|---|---|---|---|---|---|---|---|---|---|
| I | 2 | 4 | 0 | 10 | 11 | Determinantal surface | 4 | 4 | 1 | [CK89] |
| II | 2 | 4 | 1 | 9 | 8 | K3 surface with 5 \((-1)\)-lines | 4 | 2 | 3 | [Fan43] |
| III | 3 | 5 | 0 | 15 | 26 | Determinantal threefold | 5 | 5 | 1 | [ESB89] |
| IV | 3 | 5 | 1 | 14 | 22 | See Remark 6.5 | 5 | 3 | 3 | Example 6.6 |

**Table 5.** All the types of special birational transformations \(\mathbb{P}^n \dashrightarrow Z \subseteq \mathbb{P}^{n+a}\) of type \((d_1,d)\), with \(d_1 \in \{4,5\}\), \(d > 1\), and base locus \(\mathcal{B}\) of dimension \(r \leq 3\), degree \(\lambda\), sectional genus \(g\), into a factorial complete intersection \(Z \subseteq \mathbb{P}^{n+a}\) of degree \(\Delta\).

**Remark 6.7.** Let \(\varphi : \mathbb{P}^n \dashrightarrow Z \subseteq \mathbb{P}^{n+a}\) be a special birational transformation of type \((d_1,d)\), with \(d_1 \in \{4,5\}\), \(d > 1\), and base locus of dimension three, into a prime Fano manifold \(Z\). If \(Z\) is not a complete intersection, then from [AS16, Theorem 8] we deduce that \(\varphi\) is a quintic-quadratic transformation into a linear section \(Z \subset \mathbb{P}^{11}\) of \(G(1,5) \subset \mathbb{P}^{14}\) and such that the base locus \(\mathcal{B}\) has degree 12 and sectional genus 16; moreover, the ideal sheaf \(\mathcal{I}_\mathcal{B}\) is given by a resolution 

\[0 \rightarrow \mathcal{O}_{\mathbb{P}^5}(-5)^3 \oplus \mathcal{O}_{\mathbb{P}^5}(-6) \rightarrow \Omega_{\mathbb{P}^5}(-3) \rightarrow \mathcal{I}_\mathcal{B} \rightarrow 0.\]

### 7. Special birational transformations whose inverse is linear

We summarize some of our results in the following:

**Theorem 7.1.** Table 6 classifies all the special birational transformations of type \((d,1)\) into a factorial complete intersection and whose base locus has dimension at most three.

**Proof.** The proof follows immediately by applying Proposition 1.2 and then Theorem 2.8, Proposition 5.3, Proposition 6.2, Proposition 6.3, and Theorem 6.4. \(\square\)

|   | r | n | d | \(\lambda\) | g | a | \(\Delta\) | Abstract structure of \(\mathcal{B}\) |
|---|---|---|---|---|---|---|---|---|
| I | 1 | 3 | 2 | 2 | 0 | 1 | 2 | Conic |
| II | 1 | 3 | 3 | 5 | 2 | 2 | 4 | Curve of genus 2 |
| III | 2 | 4 | 2 | 2 | 0 | 1 | 2 | Two-dimensional quadric |
| IV | 2 | 4 | 3 | 5 | 2 | 2 | 4 | Blow up of \(\mathbb{P}^2\) at 7 simple points and one double point |
| V | 2 | 4 | 4 | 9 | 9 | 3 | 8 | Surface linked to a cubic scroll in the c. i. of type \((3,4)\) |
| VI | 3 | 5 | 2 | 2 | 0 | 1 | 2 | Three-dimensional quadric |
| VII | 3 | 5 | 3 | 5 | 2 | 2 | 4 | Quadric fibration over \(\mathbb{P}^1\) |
| VIII | 3 | 5 | 4 | 9 | 9 | 3 | 8 | Threefold linked to a cubic scroll in the c. i. of type \((3,4)\) |
| IX | 3 | 5 | 5 | 14 | 23 | 4 | 16 | See Remark 6.5 |

**Table 6.** All the types of special birational transformations \(\mathbb{P}^n \dashrightarrow Z \subseteq \mathbb{P}^{n+a}\) of type \((d,1)\) and base locus \(\mathcal{B}\) of dimension \(r \leq 3\), degree \(\lambda\), sectional genus \(g\), into a factorial complete intersection \(Z \subseteq \mathbb{P}^{n+a}\) of degree \(\Delta\).
Remark 7.2. Let $\varphi : \mathbb{P}^n \dasharrow Z \subset \mathbb{P}^{n+a}$ be a special birational transformation of type $(d, 1)$ into a prime Fano manifold $Z$ and whose base locus $\mathcal{B}$ has dimension three. Assume further that $Z$ is not a complete intersection. Then we have $d \in \{2, 3, 4, 5\}$, by Proposition 1.2.

If $d = 2$, by [FH18] we deduce that there are two types of transformations: either

- $\mathcal{B} = \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5 \subset \mathbb{P}^6$ and $Z = G(1, 4) \subset \mathbb{P}^9$; or
- $\mathcal{B} \subset \mathbb{P}^6$ is a linear section of $G(1, 4) \subset \mathbb{P}^9 \subset \mathbb{P}^{10}$ and $Z \subset \mathbb{P}^{12}$ is a linear section of the Spinor variety $S^{10} \subset \mathbb{P}^{15}$.

If $d = 3$, then one sees that $n = 6$ and $Z \subset \mathbb{P}^{6+a}$ has coindex 4. The program used in Section 2 gives here a long list of not excluded cases. Some of these cases occur really but $Z$ is not necessarily smooth. Other cases can be easily excluded using the classification of threefolds in $\mathbb{P}^6$ of degree at most twelve, see [Ion84, Ion86a, Ion90, FL94, FL97, BB05a, BT15]. We give a list of examples in Subsection 7.1, which provides probably the maximal list of types of special cubo-linear birational transformations of $\mathbb{P}^6$ into a (factorial) variety $Z \subset \mathbb{P}^{6+a}$ and having base locus of dimension three.

If $d = 4$ or $d = 5$, then we have $n = 5$. By [AS16, Theorem 8] we obtain that $d = 5$, and that there are two types of transformations: either

- $\mathcal{B}$ has degree 13 and sectional genus 19, and $Z \subset \mathbb{P}^{10}$ has degree 21 and coindex 4; or
- $\mathcal{B}$ has degree 11 and sectional genus 13, and $Z \subset \mathbb{P}^{15}$ is a linear section of $G(1, 6) \subset \mathbb{P}^{20}$.

An explicit example of such a threefold $\mathcal{B} \subset \mathbb{P}^5$ of degree 13 and sectional genus 19 can be constructed by taking the linked inside the complete intersection of a quartic and a quintic to a septic Palatini scroll $\mathcal{F} \subset \mathbb{P}^5$ over a cubic surface in $\mathbb{P}^3$ (see [Ion84, Ort92]). By taking instead the linked $Y \subset \mathbb{P}^5$ to $\mathcal{B} \subset \mathbb{P}^5$ inside the complete intersection of two quartics, and then the linked to $Y \subset \mathbb{P}^5$ inside the complete intersection of a quartic and a quintic, we get an explicit example of threefold of degree 11 and sectional genus 13 as above.

7.1. Further examples of special cubic transformations. We conclude by giving four examples of special cubo-linear birational transformations $\varphi : \mathbb{P}^6 \dasharrow Z \subset \mathbb{P}^{6+a}$ of $\mathbb{P}^6$ into an irreducible variety $Z \subset \mathbb{P}^{6+a}$. In all the examples, the base locus $\mathcal{B} \subset \mathbb{P}^6$ of $\varphi$ is a smooth irreducible threefold and the base locus $\mathcal{B}' \subset \mathcal{Z}$ of $\varphi^{-1}$ is an irreducible fourfold, equal to the image under $\varphi$ of the unique quadric hypersurface containing $\mathcal{B}$. It turns out that the singular locus of $Z$ is strictly contained in $\mathcal{B}'$, as one can see by computing the tangent space of $Z$ at a general point of $\mathcal{B}'$. We are not able to check whether $Z$ is factorial.

Example 7.3. Let $Y \subset \mathbb{P}^7$ be a smooth hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$. Let $X \subset \mathbb{P}^6$ be the isomorphic projection of $Y$ from a point not belonging to the secant variety of $Y$. Then $X$ is a smooth threefold of degree 6, sectional genus 1, and cut out by 13 cubics and one quadric. The linear system of cubics through $X$ defines a birational map into a variety $Z \subset \mathbb{P}^{19}$ of degree 57 and cut out by 62 quadrics. Notice that from [Fuj82] it follows that there are no other special cubic birational transformations of $\mathbb{P}^6$ whose base locus is a non-linearly normal threefold.

Example 7.4. Let $Y \subset \mathbb{P}^{13} \subset \mathbb{P}^{14}$ be the image of a smooth quadric hypersurface in $\mathbb{P}^4$ under the quadratic Veronese embedding of $\mathbb{P}^4$. Let $X \subset \mathbb{P}^6$ be the projection of $Y$ in $\mathbb{P}^6$ from a linear subspace $\mathbb{P}^6 \subset \mathbb{P}^{13}$ spanned by 7 general points on $Y$. Then one has that $X$ is a smooth threefold of degree 9, sectional genus 5, and cut out by 7 cubics and one quadric. The linear system of cubics through $X$ defines a birational map into a variety $Z \subset \mathbb{P}^{13}$ of degree 30 and cut out by 14 quadrics.
Example 7.5. Let $Y \subset \mathbb{P}^8$ be a general 3-dimensional linear section of the 10-dimensional Spinor variety $\mathbb{S}_{10}^2 \subset \mathbb{P}^{15}$. Let $X \subset \mathbb{P}^6$ be the projection of $Y$ in $\mathbb{P}^6$ from a line spanned by two general points on $Y$. Then $X \subset \mathbb{P}^6$ is a smooth threefold of degree 10, sectional genus 7, and cut out by 6 cubics and one quadric. The linear system of cubics through $X$ defines a birational map into a variety $Z \subset \mathbb{P}^{12}$ of degree 25, cut out by 10 quadrics, and whose singular locus has dimension 1.

Example 7.6. Let $Y \subset \mathbb{P}^6$ be a general 3-dimensional linear section of the Grassmannian $\mathbb{G}(1, 4) \subset \mathbb{P}^6$. Take $W$ to be a generic complete intersection of two quadrics and one cubic containing $Y$, and let $Y' = W \setminus Y$ be the residual intersection. Take now $W'$ to be a generic complete intersection of one quadric and two cubics containing $Y'$, and let $X = W' \setminus Y'$ be the residual intersection. Then $X \subset \mathbb{P}^6$ is a smooth threefold of degree 11, sectional genus 9, and cut out by 5 cubics and one quadric. The linear system of cubics through $X$ defines a birational map into a variety $Z \subset \mathbb{P}^{11}$ of degree 21, cut out by 6 quadrics and one cubic, and whose singular locus has dimension 1.

Remark 7.7. The existence of the threefold of degree 9 and sectional genus 5 given in Example 7.4 had been left undecided in [FL94].

The existence of the threefold of degree 10 and sectional genus 7 given in Example 7.5 reveals a missing case to the classification obtained in [FL97] (it seems that the claim in [FL97, Proposition 6.3] is wrong).

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