Information and Estimation over Binomial and Negative Binomial Models

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Abstract

In recent years, a number of results have been developed which connect information measures and estimation measures under various models, including, predominantly, Gaussian and Poisson models. More recent results due to Taborda and Pérez-Cruz relate the relative entropy to certain mismatched estimation errors in the context of binomial and negative binomial models, where, unlike in the case of Gaussian and Poisson models, the conditional mean estimates concern models of different parameters than those of the original model. In this note, a different set of results in simple forms are developed for binomial and negative binomial models, where the conditional mean estimates are produced through the original models. The new results are more consistent with existing results for Gaussian and Poisson models.

I. INTRODUCTION

Since a simple differential relationship between the mutual information and the minimum mean square error over a scalar Gaussian model was discovered [1], a number of similar results have been developed for several other models, e.g., Poisson models [2], [3]. In the context of Gaussian and Poisson models, it has also been found that the relative entropy can be expressed as the integral of the increase of the estimation error due to mismatched prior distribution [3]–[5].

More recently, Taborda and Pérez-Cruz [6] developed results in the context of binomial and negative binomial models. The key result expresses the derivative of the relative entropy of two output distributions of the same binomial (or negative binomial) model induced by two different inputs in terms of certain mismatched estimation errors. As in the case of Gaussian and Poisson models, the errors concern conditional mean estimates of the input given the output. However, the conditional mean is produced using binomial (or negative binomial) models with modified parameters. In particular, in case of the binomial model, the modified model is of one fewer trial than the original model; in case of the negative model, the modified model is of one more failure than the original model.

In this note, we develop a different set of results concerning essentially the same binomial and negative binomial models as in [6]. The results put the derivative of the relative entropy in a simple form concerning some average difference of conditional mean estimates due to mismatched prior distribution. In contrast to results of [6], the conditional mean estimates here are based on the original binomial and negative binomial models. The results are thus more consistent with existing results for Gaussian and Poisson models.
II. THE BINOMIAL MODEL

The binomial model is based on the binomial distribution, Binomial\((n, q)\), which describes the probability of having \(k\) successful trials in \(n\) independent Bernoulli trials, each with probability \(q\) to succeed:

\[
P(Y = k) = \binom{n}{k} q^k (1 - q)^{n-k}, \quad k = 0, \ldots, n. \tag{1}
\]

With some hindsight, we let the binomial model be a random transformation from a random variable \(X\) which takes its value on \((a, \infty)\) to another random variable \(Y\), where, conditioned on \(X = x\), \(Y\) follows the distribution of Binomial\((n, a/x)\). The conditional probability mass function is given by

\[
p_{Y|X}(y|x) = \binom{n}{y} \left( \frac{a}{x} \right)^y \left( 1 - \frac{a}{x} \right)^{n-y}, \quad y = 0, \ldots, n. \tag{2}
\]

The variables \(X\) and \(Y\) are viewed as the input and output of the binomial model. Here the input \(X\) controls the probability of success of an individual Bernoulli trial, namely, for fixed \(X\), the success-to-failure ratio is \(a : X\). The larger \(X\) is, the fewer trials succeed on average. The parameter \(a\) is viewed as a scaling of the input.

If the prior distribution of the input \(X\) is \(P_X\), the corresponding output distribution is denoted by \(P_Y\); if the prior distribution of \(X\) is \(Q_X\), the corresponding output distribution is denoted by \(Q_Y\). Throughout this note, we use \(E\{\cdot\}\) and \(E\{\cdot | \cdot\}\) to denote expectation and conditional expectation under distribution \(P\), whereas we use \(E_Q\{\cdot\}\) and \(E_Q\{\cdot | \cdot\}\) to denote expectation and conditional expectation under distribution \(Q\). Thus the conditional mean of \(X\) given \(Y\) is denoted by \(E\{X | Y\}\) under distribution \(P\) and by \(E_Q\{X | Y\}\) under distribution \(Q\).

We also define the following function

\[
g(t) = t - 1 - \log t. \tag{3}
\]

This function is convex on \((0, \infty)\), and achieves its unique minimum, 0, at \(t = 1\). For two positive numbers, \(x\) and \(\hat{x}\), the function \(g(x/\hat{x})\) can be viewed as a measure of their difference, in the sense that it is always nonnegative, and that it is equal to 0 if and only if \(x = \hat{x}\). Moreover, \(g(x/\hat{x})\) increases monotonically as \(x/\hat{x}\) departs from 1 in either direction of the axis.

**Theorem 1.** Let \(P_Y\) and \(Q_Y\) be the output distribution of the binomial model (2) induced by input distributions \(P_X\) and \(Q_X\), respectively, where \(P_X\) and \(Q_X\) put no probability mass on \((-\infty, a]\). Then

\[
\frac{d}{da} D(P_Y||Q_Y) = E \left\{ \frac{Y}{a} \cdot g \left( \frac{E_Q\{X | Y\}}{a} - \frac{E\{X | Y\}}{a} \right) \right\}. \tag{4}
\]

**Lemma 1.** Let \(P_Y\) be the probability mass function of the output of the binomial model described by (2), where the input follows distribution \(P_X\) with zero mass on \((-\infty, a]\). For every \(y = 0, \ldots, n\),

\[
\frac{d}{da} P_Y(y) = \frac{1}{a} \left( yP_Y(y) - (y + 1)P_Y(y + 1) \right) \tag{5}
\]

where we use the convention that \(p_Y(n + 1) = 0\). The result remains true if \(P_Y\) is replaced by \(Q_Y\) in (5).

**Proof:** We start with

\[
P_Y(y) = E \left\{ \binom{n}{y} \left( \frac{a}{X} \right)^y \left( 1 - \frac{a}{X} \right)^{n-y} \right\}. \tag{6}
\]
Using (6) again, we arrive at (5) from (10).

Evidently, 
\[
\frac{d}{da} P_Y(y) = E \left\{ \binom{n}{y} \frac{d}{da} \left( \frac{a}{X} \right)^y \left( 1 - \frac{a}{X} \right)^{n-y} \right\} = E \left\{ \binom{n}{y} \left( \frac{a}{X} \right)^y \frac{d}{da} \left( 1 - \frac{a}{X} \right)^{n-y} \right\} + E \left\{ \binom{n}{y} \left( \frac{a}{X} \right)^{y+1} \left( 1 - \frac{a}{X} \right)^{n-y-1} \right\} - \frac{n-y}{a} E \left\{ \binom{n}{y} \left( \frac{a}{X} \right)^{y+1} \left( 1 - \frac{a}{X} \right)^{n-y-1} \right\}
\]

We next prove Theorem 1.

**Proof of Theorem 1**

From
\[
D(P_Y || Q_Y) = \sum_{y=0}^{n} P_Y(y) \log \frac{P_Y(y)}{Q_Y(y)},
\]

it is not difficult to show that
\[
\frac{d}{da} D(P_Y || Q_Y) = \sum_{y=0}^{n} \left( \log \frac{P_Y(y)}{Q_Y(y)} \right) \frac{dP_Y(y)}{da} - \frac{P_Y(y)}{Q_Y(y)} \frac{dQ_Y(y)}{da}
\]

where
\[
A = a \sum_{y=0}^{n} \left( \log \frac{P_Y(y)}{Q_Y(y)} \right) \frac{dP_Y(y)}{da}
\]

Using (6) again, we arrive at (5) from (10).

Since (5) holds for any input distribution \( P_X \), it remains true if \( P_X \) is replaced by another distribution \( Q_X \), as long as the input is always greater than \( a \).

Lemma 1 resembles a result for Gaussian models in [1], where the derivative with respect to the scaling parameter translates to the derivative with respect to the output variable. For the binomial model, the output is discrete and the result consists of the difference of the output distribution (modulated by the variable \( y \)) in lieu of derivative.

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and

\[ B = a \sum_{y=0}^{n} P_Y(y) \frac{dQ_Y(y)}{Q_Y(y)} \]

\[ = \sum_{y=0}^{n} P_Y(y) \left( yQ_Y(y) - (y + 1)Q_Y(y + 1) \right) \]

\[ = \sum_{y=1}^{n} yP_Y(y) - \sum_{y=0}^{n-1} P_Y(y) \left( yQ_Y(y) - (y + 1)Q_Y(y + 1) \right) \]

\[ = \sum_{y=1}^{n} yP_Y(y) \left( 1 - \frac{P_Y(y-1)Q_Y(y)}{P_Y(y)Q_Y(y-1)} \right). \]

Moreover,

\[ P_Y(y-1) = E \left\{ \left( \frac{y}{n-y+1} \right) \left( \frac{a}{X} \right)^{y-1} \left( 1 - \frac{a}{X} \right)^{n-y+1} \right\} \]

\[ = E \left\{ \left( \frac{y}{n-y+1} \right) \left( \frac{a}{X} \right)^{-1} \left( 1 - \frac{a}{X} \right) \left( \frac{n}{y} \right) \left( \frac{a}{X} \right)^{y} \left( 1 - \frac{a}{X} \right)^{n-y} \right\} \]

\[ = \frac{y}{n-y+1} E \left\{ X | Y = y \right\} P_Y(y). \]

Similarly,

\[ Q_Y(y-1) = \frac{y}{n-y+1} E_Q \left\{ X | Y = y \right\} Q_Y(y). \]

Therefore,

\[ \frac{P_Y(y-1)Q_Y(y)}{P_Y(y)Q_Y(y-1)} = \frac{E \{ X | Y = y \} - a}{E_Q \{ X | Y = y \} - a}. \]

Plugging (28) into (18) and (23) and subsequently (13), we have

\[ \frac{d}{da} \mathcal{D}(P_Y||Q_Y) = a^{-1} \sum_{y=1}^{n} yP_Y(y)(T(y) - 1 - \log T(y)) \]

where \( T(y) \) is a shorthand for the function defined as the RHS of (28). By definition (3), we have established (4) in Theorem 1.
III. The Negative Binomial Model

The negative binomial distribution is defined by the following probability mass function

\[ P(Y = y) = \binom{y + r - 1}{y} (1 - q)^r q^y, \quad y = 0, 1, \ldots \]  

which is the probability that \( y \) successful trials are seen before the \( r \)-th failure is observed, where the trials are independent Bernoulli trials each with probability \( q \) to succeed. We denote this distribution as \( \text{-Binomial}(r, q) \).

With some hindsight, we define a negative binomial model based on random transformation from random variable \( X \) to random variable \( Y \), where, conditioned on \( X = x \), \( Y \) has distribution \( \text{-Binomial}(r, b/(b+x)) \). That is, the random transformation is given by conditional probability mass function

\[ P_{Y|X}(y|x) = \binom{y + r - 1}{y} \left( \frac{x}{b+x} \right)^r \left( \frac{b}{b+x} \right)^y, \quad y = 0, 1, \ldots . \]  

Theorem 2. Let \( P_Y \) and \( Q_Y \) be the output distribution of the binomial model \( (31) \) induced by input distributions \( P_X \) and \( Q_X \), respectively, where \( P_X \) and \( Q_X \) put no probability mass on \( (-\infty, 0] \). Then

\[ \frac{d}{db} D(P_Y || Q_Y) = \mathbb{E} \left\{ \frac{Y}{b} \cdot g \left( \frac{\mathbb{E} \{ X | Y \} + b}{\mathbb{E} Q \{ X | Y \} + b} \right) \right\}. \]  

Lemma 2. Let \( P_Y \) be the probability mass function of the output of the negative binomial model described by \( (31) \), where the input is always positive and follows distribution \( P_X \). For every \( y = 0, 1, \ldots \),

\[ \frac{d}{db} P_Y(y) = \frac{1}{b} (yP_Y(y) - (y + 1)P_Y(y + 1)). \]  

The result remains true if \( P_Y \) is replaced by \( Q_Y \) in \( (33) \).

**Proof:** We start with

\[ P_Y(y) = \mathbb{E} \left\{ \binom{y + r - 1}{y} \left( \frac{X}{b+X} \right)^r \left( \frac{b}{b+X} \right)^y \right\}. \]  

Evidently,

\[ \frac{d}{db} P_Y(y) = \mathbb{E} \left\{ \binom{y + r - 1}{y} \frac{d}{db} \left( \binom{X}{b+X} \right)^r \left( \frac{b}{b+X} \right)^y \right\} \]  

\[ = \mathbb{E} \left\{ \binom{y + r - 1}{y} \left( \frac{X}{b+X} \right)^r \left( \frac{b}{b+X} \right)^y \left( \frac{-r}{b+X} + \frac{yX}{b(b+X)} \right) \right\} \]  

\[ = \mathbb{E} \left\{ \binom{y + r - 1}{y} \left( \frac{X}{b+X} \right)^r \left( \frac{b}{b+X} \right)^y \left( \frac{y - y + r}{b} \right) \right\} \]  

\[ = \frac{y}{b} P_Y(y) - \frac{y+1}{b} \mathbb{E} \left\{ \binom{y + r}{y+1} \left( \frac{X}{b+X} \right)^r \left( \frac{b}{b+X} \right)^{y+1} \right\} \]  

\[ = \frac{1}{b} (yP_Y(y) - (y + 1)P_Y(y + 1)). \]
Since (33) holds for any input distribution \( P_X \), it remains true if \( P_X \) is replaced by another distribution \( Q_X \), as long as the input is always nonnegative.

It is interesting to see that (33) is literally identical to (5) if the two parameters \( a \) and \( b \) are identical.

The proof of Theorem 2 based on Lemma 2 resembles that of Theorem 1.

Proof of Theorem 2: Using similar techniques as in the proof of Theorem 2, we arrive at

\[
\frac{d}{db} D(P_Y \| Q_Y) = \frac{1}{b} \sum_{y=1}^{\infty} y P_Y(y) \left( T(y) - 1 - \log T(y) \right)
\]

where

\[
T(y) = \frac{P_Y(y - 1) Q_Y(y)}{P_Y(y) Q_Y(y - 1)}.
\]

Moreover,

\[
P_Y(y - 1) = E \left\{ \left( \frac{y + r - 2}{y - 1} \right) \left( \frac{X}{b + X} \right)^r \left( \frac{b}{b + X} \right)^{y-1} \right\}
\]

\[
= E \left\{ \frac{y}{y + r - 1} \frac{b + X}{b} \left( y + r - 1 \right) \left( \frac{X}{b + X} \right)^r \left( \frac{b}{b + X} \right)^y \right\}
\]

\[
= \frac{y}{y + r - 1} E \left\{ 1 + \frac{X}{b} \mid Y = y \right\} P_Y(y).
\]

Similarly,

\[
Q_Y(y - 1) = \frac{y}{y + r - 1} E_Q \left\{ 1 + \frac{X}{b} \mid Y = y \right\} Q_Y(y).
\]

Therefore,

\[
T(y) = \frac{E \left\{ X \mid Y = y \right\} + b}{E_Q \left\{ X \mid Y = y \right\} + b}.
\]

Theorem 2 is thus established using (33), (41) and (47).

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