A Simple Lattice Version of the Nonlinear Schrodinger Equation and its Deformation with Exact Quantum Solution

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Abstract A lattice version of quantum nonlinear Schrodinger (NLS) equation is considered, which has significantly simple form and fulfills most of the criteria desirable for such lattice variants of field models. Unlike most of the known lattice NLS, the present model belongs to a class which does not exhibit the usual symmetry properties. However this lack of symmetry itself seems to be responsible for the remarkable simplification of the relevant objects in the theory, such as the Lax operator, the Hamiltonian and other commuting conserved quantities as well as their spectrum. The model allows exact quantum solution through algebraic Bethe ansatz and also a straightforward and natural generalisation to the vector case, giving thus a new exact lattice version of the vector NLS model. A deformation representing a new quantum integrable system involving Tamm-Dancoff like q-boson operators is constructed.
1. Introduction

Among integrable systems discrete models represent a special class, interest to which has been revived in recent years [1-3]. In the context of quantum integrable systems, apart from their own right as solvable quantum lattice models, they also play an important role by providing lattice regularised versions of the corresponding continuum models. Thus the lattice nonlinear Schrödinger (NLS) [4-9], lattice sine-Gordon [7] models etc. are useful for finding out the exact quantum solutions of the related field models through the quantum inverse scattering method (QISM) [10]. Moreover, these lattice versions are often able to unveil the hidden algebraic structures of the original field models [1,11].

Ideally a candidate for such quantum integrable discrete models represented by the Lax operator \( L(n \mid \lambda) \) should fulfill the following basic criteria:

i) It should satisfy exactly the quantum Yang-Baxter equation (QYBE)

\[
R(\lambda - \mu) \ L(\lambda) \otimes L(\mu) = L(\mu) \otimes L(\lambda) \ R(\lambda - \mu),
\]

(1.1)

ii) It should have the same quantum \( R \)-matrix as the corresponding field model.

iii) The discrete Lax operator should yield the continuum one at \( \Delta \to 0 \):

\( L(n \mid \lambda) \to 1 + \Delta L(x, \lambda) \), \( \Delta \) being the lattice constant.

iv) The Hamiltonian of the discrete model should be ‘local’ and give back the local field model at the continuum limit.

Moreover, as a desirable physical requirement the Lax operator as well as the Hamiltonian and the energy spectrum should be as simple as possible.

A discrete version of NLS model was first suggested by Ablowitz and Ladik [5]. However the corresponding \( R \)-matrix, a key object in QISM, is expressed through trigonometric functions [12] and does not coincide with the well known rational \( R \)-matrix of the NLS field model, which reads:

\[
R(\lambda) = \lambda P - i \kappa I = \begin{pmatrix}
\lambda - i \kappa & -i \kappa \\
\lambda & -i \kappa \\
\end{pmatrix},
\]

(1.2)

That is, criterion ii) laid down above is not satisfied. Subsequently, a different discrete NLS was proposed [6], constructed through the Holstein-Primakoff transformation (HPT) applied to an infinite dimensional irreducible representation of \( su(2) \) with the classical Lax operator given by

\[
L(n \mid \lambda) = \begin{pmatrix}
c - i \frac{\kappa}{2} \Delta + \frac{\kappa}{2} \Delta^2 \psi(n)^* \psi(n) & -i \psi(n)^* \Delta \sqrt{\kappa(c + \frac{\kappa}{4} \Delta^2 \psi(n)^* \psi(n))} \\
i \Delta \sqrt{\kappa(c + \frac{\kappa}{4} \Delta^2 \psi(n)^* \psi(n))} \psi(n) & c + i \frac{\kappa}{2} \Delta + \frac{\kappa}{2} \Delta^2 \psi(n)^* \psi(n)
\end{pmatrix},
\]

(1.3)
related to spin parameter \( s = -\frac{2}{\kappa \Delta} \) with \( c = 1 \) and canonical Poisson brackets
\[
\{\psi(n), \psi(m)^\ast\} = \delta_{nm}.
\]

This model is free from the earlier drawback, namely it satisfies ii), though it now fails to fulfill the locality criterion iv) at the quantum level. As a remedy another version of lattice NLS was introduced \([7]\), represented by a Lax operator which explicitly depends on lattice points and may be expressed as a product \( l(n \mid \lambda) = L(2n \mid \lambda)L(2n-1 \mid \lambda) \), where \( L(n \mid \lambda) \) is taken as (1.3) with \( c = 1 + \frac{1}{4}(-1)^n \kappa \Delta \), that is:
\[
L(n \mid \lambda) = 
\begin{pmatrix}
1 + \frac{1}{4}(-1)^n \kappa \Delta - \frac{i}{2} \lambda \Delta + \frac{\kappa}{2} \Delta^2 \psi(n)^* \psi(n) & \psi(n)^* \Delta \sqrt{\kappa(1 + \frac{1}{4}(-1)^n \kappa \Delta + \frac{\kappa}{2} \Delta^2 \psi(n)^* \psi(n))}

\frac{i \Delta}{\kappa(1 + \frac{1}{4}(-1)^n \kappa \Delta + \frac{\kappa}{2} \Delta^2 \psi(n)^* \psi(n))} \psi(n) & 1 + \frac{1}{4}(-1)^n \kappa \Delta + \frac{i}{2} \lambda \Delta + \frac{\kappa}{2} \Delta^2 \psi(n)^* \psi(n)
\end{pmatrix}
\]

This model satisfies the required criteria, but looks involved to comply with the physical requirement of simplicity. Finally, in a further investigation \([8]\), a relatively simpler model was proposed, where the Lax operator was given directly by (1.4). However the important criterion iii) is now not fulfilled and the simplification achieved is also not fully satisfactory, as it is evident from the form of the Lax operator and structure of the following Hamiltonian:
\[
H = -\frac{4}{3 \kappa \Delta^3} \sum_n (t_n + \bar{t}_n^\dagger + \frac{8 - \kappa \Delta}{8 - 2 \kappa \Delta}) + \left( \frac{4}{3 \Delta^2} + \frac{\kappa^2}{12} \right) \sum_n \Delta \psi_n^\dagger \psi_n,
\]

where the local density \( t_n \) again has different expressions, depending whether it corresponds to even or odd sites. For odd \( n \) it takes the form
\[
t_n = (\alpha^\dagger(n+2)\alpha(n+1))^{-1}(\alpha^\dagger(n)\alpha(n-1))^{-1}(\alpha^\dagger(n+1)\alpha(n))^{-1}
\]

\[
(\alpha^\dagger(n+1)\sigma_3 \alpha(n-1))(\alpha^\dagger(n+2)\alpha(n+1))^{-1},
\]

with
\[
\bar{\alpha}(n) = (-i \sqrt{\frac{\kappa}{2}} \Delta \psi_n^\dagger, \sqrt{2}(1 - \frac{\kappa \Delta}{2} - \frac{\kappa \Delta^2}{4} \psi_n^\dagger \psi_n)^{\frac{1}{2}})
\]

and at even \( n \) sites different, though similar expressions hold for both \( t_n \) and \( \bar{\alpha}(n) \) \([8]\). The energy spectrum of this model obtained through the Bethe ansatz is also rather complicated, and gets simplified only at the continuum limit. As far as we know, until now not many other proposals were invoked to improve this situation \([9]\), particularly to achieve simpler forms of local conserved quantities at the lattice level. However, a completely different approach was formulated in ref.\([4]\) through equivalence between NLS and spin models using intertwiners of quantum spaces.

Our primary aim here is to consider a quantum integrable lattice model, which at the continuum limit yields the more general AKNS system \([13]\) and as an allowed reduction the
NLS field model. The system considered fulfills all the desirable requirements of a discrete quantum system listed above and most importantly, exhibits considerably simple expressions for the related conserved quantities at the lattice level. Indeed it satisfies the QYBE exactly with the same rational $R$-matrix (1.2) allowing also solution via QISM, yields local Hamiltonian and the Lax operator of the NLS field model at the continuum limit. Moreover, it has an extremely simple structure, which induces an almost trivial form for the projector required for the construction of local Hamiltonians. Remarkably, this projector turns out to be field-independent and symmetric. As a further relevant feature, our model also allows a natural vector generalisation at the lattice level. Finally it admits an integrable deformation involving Tamm-Dancoff type $q$-bosons. On the other hand, for achieving all these nice properties, one has to pay some price which is reflected in the non-hermitian nature of the physical observables at the lattice level. At the same time the associated Lax operator lacks the usual $SU(2)(SU(1, 1))$ symmetry.

We should stress here that such Lax operators with lesser symmetries were found also to be significant in generating a large class of quantum integrable models [11].

2. The classical model

The model under scrutiny may be given through the Lax operator of the form

$$L(n \mid \zeta) = \begin{pmatrix}
\zeta + \Delta \kappa \phi(n) \psi(n) & -i\phi(n)(\kappa \Delta)^{\frac{1}{2}} \\
-i\psi(n)(\kappa \Delta)^{\frac{1}{2}} & 1
\end{pmatrix}.$$ \hspace{1cm} (2.1)

Its simplified structure compared to (1.3) is explicit, though due to no-conjugacy relation between between $\phi$ and $\psi$, it is obviously not hermitian. Note that similar forms of $L$ operators appear also in analysing discrete self-trapping systems [14] as well as integrable systems close to Toda lattice [15].

Recently, the bi-hamiltonian structure of of the classical system corresponding to (2.1) has been determined and its complete integrability has been established [16] rigorously through the explicit construction of action-angle variables using the $r$-matrix approach. Recall that at the classical limit the QYBE (1.1) reduces to the classical Yang-Baxter equation

$$\{L(n \mid \zeta) \otimes L(m \mid \eta)\} = \{r(\zeta, \eta), L(n \mid \zeta) \otimes L(m \mid \eta)\}\delta_{nm}.$$ \hspace{1cm} (2.2)

For the present model, the quantum $R$-matrix is given by (1.2) and is related to its classical counterpart $r$ by:

$$\frac{1}{\zeta} PR(\zeta) = I - i\kappa r(\zeta), \quad r(\zeta) = \frac{\Delta}{\zeta} P.$$ \hspace{1cm} (2.3)

Now to show the transition of the Lax operator to that of the continuum model one should put $\zeta = 1 + i\Delta \lambda$ and $\psi(n) \rightarrow i\sqrt{\Delta} \psi(n), \phi(n) \rightarrow -i\sqrt{\Delta} \phi(n)$, which introducing...
ψ(n) = \frac{1}{\Delta} \int_{x_n}^{x_{n+\Delta}} \psi(x) \, dx$, and a similar expression for φ, would yield from (2.1): $L(n \mid \zeta) = 1 + \Delta L(x, \lambda) + O(\Delta^2)$.

$L(x, \lambda)$ is the Lax operator of the corresponding field model, given by

$$L(x, \lambda) = \begin{pmatrix} i\lambda & \kappa \frac{\Delta}{2} \phi \\ \kappa \frac{\Delta}{2} \psi & 0 \end{pmatrix}. \quad (2.2)$$

It may be easily checked that the conserved quantities associated with this system are the same as those of the AKNS system [13]; moreover, since their Poisson structures coincide, one may conclude that the two systems are equivalent. In fact through a simple gauge transformation

$$\mathcal{L} \rightarrow h \mathcal{L} h^{-1} + h_x h^{-1}, \quad h = e^{-i \frac{\Delta}{2} x} \quad (2.3),$$

this $\mathcal{L}$-operator can be changed into the standard Lax operator of continuum NLS:

$$L(x, \lambda) = \begin{pmatrix} i\frac{\lambda}{2} & \kappa \frac{\Delta}{2} \phi \\ \kappa \frac{\Delta}{2} \psi & -i\frac{\lambda}{2} \end{pmatrix} = i\frac{\lambda}{2} \sigma^3 + \kappa \frac{\Delta}{2} \phi \sigma^+ + \kappa \frac{\Delta}{2} \psi \sigma^-, \quad (2.4)$$

restoring the unitary symmetry, since as is well known, the AKNS system allows the reduction $\phi = \psi^*$. 

3. Quantum model

Recently, more general forms of $L$-operator of discrete quantum integrable models corresponding to standard $R$-matrices have been proposed [11]. Such class of $L$ operators associated with the rational $R$-matrix (1.2) and satisfying the QYBE may be given by the following expression which clearly lacks the unitary symmetry:

$$L = \begin{pmatrix} K_1 + i\frac{\Delta}{2} K_2 & K_- \\ K_+ & K_3 + i\frac{\Delta}{2} K_4 \end{pmatrix}, \quad (3.1)$$

where $K$ operators satisfy the algebra

$$[K_+, K_-] = (K_1 K_4 - K_2 K_3), \quad [K_1, K_3] = 0$$

$$[K_1, K_\pm] = \pm K_\pm K_2, \quad [K_3, K_\pm] = \mp K_\pm K_4, \quad (3.2)$$

with $K_2, K_4$ as central elements. It may be seen that when $K_1 = -K_3, \ K_2 = K_4 = 1, \ K_+ = (K_-)^\dagger$, (3.2) reduces to the standard $su(2)$ algebra and one can get back the known lattice NLS (1.3) through HPT. However the $L$ operator (3.1) in general gives the possibility of generating other quantum integrable models which do not exhibit such symmetry. Quantum Toda chain is one of the main examples [11]. It is interesting to observe that the quantum version of the NLS model (2.1) considered here, also falls into this class and can be obtained from (3.1) through the following realisation:

$$K_1 = \Delta^2 \kappa \phi \psi + 1, \quad K_2 = -\Delta \kappa, \quad K_3 = 1, \quad K_4 = 0, \quad K_+ = i\Delta \sqrt{\kappa} \psi, \quad K_- = -i\Delta \sqrt{\kappa} \phi, \quad (3.3)$$

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where the operators $\psi, \phi$ obey the canonical commutation relation $[\psi(n), \phi(m)] = \frac{1}{\Delta} \delta_{nm}$.

This quantum model, represented by the Lax operator

$$L(n | \lambda)_{\text{NLS}} = \begin{pmatrix}
1 - i\lambda\Delta + \Delta^2\kappa\phi(n)\psi(n) & -i\Delta\kappa^{1/2}\phi(n) \\
i\Delta\kappa^{1/2}\phi(n) & 1
\end{pmatrix},$$

(3.4),
as descendant of the integrable ‘ancestor’ model (3.1), is naturally quantum integrable and satisfies the QYBE with the same $R$ matrix (1.2) as the NLS field model.

In exact analogy with the classical case, (3.4) allows transition to the Lax operator of the AKNS system and through allowed reduction to that of the continuum quantum NLS model. Indeed, the gauge transformation (2.3), being independent of the field operators, is clearly applicable to the quantum case as well.

It is known [7] that the conserved quantities $C_l$ may be obtained from the transfer matrix $\tau(\lambda) = tr\left(\prod^1_N L(n | \lambda)\right)$, through an expansion at a special point $\nu$, in the form:

$$C_l = \frac{1}{\kappa l!} \frac{\partial^l}{\partial \lambda^l} \log \tau(\lambda) |_{(\lambda=\nu)}.$$ 

In what follows we use the method developed in [7,8]. The locality of the Hamiltonian and other conserved quantities can be achieved provided that at this special point $\nu$ the operator $L(\lambda)$ be expressible both as a “direct” and an “inverse” one-dimensional projector [7,9]. This in turn implies the vanishing of its quantum determinant [6] $det_q L$ at this point, where

$$det_q L = tr(P_- (L(\lambda) \otimes L(\lambda + i\kappa)))$$

$$= \frac{1}{2} [(L_{11}\tilde{L}_{22} + L_{22}\tilde{L}_{11}) - (L_{21}\tilde{L}_{12} + L_{12}\tilde{L}_{21})]$$

with $P_- = \frac{1}{4}(1 - \sum_a \sigma_a \otimes \sigma_a)$ being the antisymmetriser and $L \equiv L(\lambda), \tilde{L} \equiv L(\lambda + i\kappa)$.

We observe that for Lax operator (3.4) one gets $det_q L = 1 - i\lambda\Delta$, giving a single degeneracy point $\nu_1 = -\frac{i}{\Delta}$. The resulting projector depends on the field operators and one cannot avoid the implementation of the involved procedure discovered and applied in [6-8] and elaborated in [9]. Fortunately however, under an irrelevant scaling of the Lax operator $L \to \tilde{L} = (\frac{i}{\Delta})L$, which evidently does not affect the QYBE and thus can only give equivalent lattice models, the quantum determinant becomes

$$det_q \tilde{L} = -\frac{1}{\Delta^2} \left(1 - i\lambda\Delta\right) \frac{\xi(\xi + \Delta)}{\Delta^2(1 + \kappa\xi)},$$

where $\xi = \frac{i\lambda}{\Delta}$. That is, another degeneracy point $\xi = \nu_2 = 0$ naturally appears.
The rescaled operator  $\hat{L}$ takes the form

$$\hat{L}(n \mid \xi) = \left( \begin{array}{cc} 1 + \frac{N(n)\xi}{\Delta} & -i\kappa \frac{\phi(n)\xi}{\Delta} \\ i\kappa \frac{\psi(n)\xi}{\Delta} & \frac{1}{\Delta}\xi \end{array} \right),$$

with $N(k) = 1 + \kappa \Delta^2 \phi(k) \psi(k)$. At the new degeneracy point $\xi = \nu_2 = 0$, it becomes remarkably simple as it turns into a field independent projector:

$$\hat{L}(0) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) = \mathcal{P}$$

(3.5)

The above procedure amounts essentially to choose the expansion point at $\lambda = \infty$. We emphasize that the existence of such an exceptional expansion point where the projector becomes field-independent is possible only due to the asymmetry of the present model. We note incidentally that an analogous property also holds at the classical level [16]. As a consequence, due to the almost trivial form of $\hat{L}(0)$ (3.5), as we will see now, not only the required locality condition is satisfied, but also the derivation as well as the expression of the Hamiltonian and the other conserved quantities become extremely simple.

For explicit calculations we use now $\hat{L}$ and expand around $\xi = 0$ assuming periodic boundary conditions, dropping however the hat sign from all subsequent expressions. This gives

$$\tau(0) = tr\left( \prod L(k \mid \xi) \mid_{(\xi=0)} \right) = 1, \quad (3.5a)$$

$$\frac{\partial}{\partial \xi} \tau(\xi)_{\xi=0} \equiv \tau'(0) = tr \sum_k (L(N \mid \xi) \cdots L'(k \mid \xi) \cdots L(1 \mid \xi))_{(\xi=0)}, \quad (3.5b)$$

$$= \frac{1}{\Delta} tr \sum_k (\mathcal{P} \cdots N(k) \mathcal{P} \cdots \mathcal{P})$$

$$= \frac{1}{\Delta} \sum_k N(k).$$

In a similar way one gets

$$\tau''(0) = 2 \left( \frac{1}{\Delta^2} \sum_{i > k} N(i)N(k) + \kappa \sum_k \phi(k+1) \psi(k) \right), \quad (3.5c)$$

where a factor 2 appears due to the identity

$$(\cdots L(i) \cdots L'(k) \cdots)'_{\xi=0} = (\cdots L'(i) \cdots L(k) \cdots)'_{\xi=0}$$

valid since $L''(\xi) = 0$. Continuing further we get

$$\tau'''(0) = \frac{6}{\Delta} \left( \frac{1}{\Delta^2} \sum_{i > j > k} N(i)N(j)N(k) + \kappa \left( \sum_{i,k(i \neq k, \neq k+1)} N(i) \phi(k+1) \psi(k) + \sum_k \phi(k+1) \psi(k-1) \right) \right), \quad (3.5d)$$

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etc. Notice that the conserved quantities $C_k$ may be given through the above expressions (3.5) in the following form

\[
C_1 = \frac{1}{\kappa}(\log\tau(\xi))' \mid (\xi = 0) = \frac{1}{\kappa} \tau(0)^{-1}\tau'(0)
\]

\[
C_2 = \frac{1}{2\kappa}(\log\tau(\xi))'' \mid (\xi = 0) = \frac{1}{2\kappa} [\tau(0)^{-1}\tau''(0) - (\tau(0)^{-1}\tau'(0))^2]
\]

\[
C_3 = \frac{1}{3\kappa}(\log\tau(\xi))''' \mid (\xi = 0)
\]

\[
= \frac{1}{6\kappa}[2(\tau^{-1}\tau'(0))^3 + \tau^{-1}\tau'''(0) - 2(\tau^{-1}\tau'(0))\tau^{-1}\tau''(0)) - (\tau^{-1}\tau''(0))(\tau^{-1}\tau'(0))],
\]

where $\tau^{-1} = \tau^{-1}(0)$ Inserting now the expressions (3.5) one finally gets the required observables

\[
N = C_1 = \frac{1}{\Delta\kappa} \sum_k N(k)
\]

as the ‘Number’ operator,

\[
P = C_2 \equiv \sum_k p_k = \sum_k (\phi(k+1)\psi(k) - \frac{1}{2\kappa\Delta^2}N(k)^2),
\]

as the ‘Momentum’ operator and

\[
H = C_3 \equiv \sum_k h_k = \frac{1}{\Delta} \sum_k (\phi(k+1)\psi(k-1) - (N(k) + N(k+1))\phi(k+1)\psi(k) + (3\kappa\Delta^2)^{-1}N(k)^3),
\]

as the Hamiltonian of the system.

It may be noted that the above conserved quantities are not symmetric in $\phi$ and $\psi$, which is a consequence of the asymmetry of the Lax operator. On the other hand their locality is explicit and it is interesting to observe that even though expressions (3.5) given through expansion of $\tau(\xi)$ were all nonlocal, in the corresponding conserved quantities (3.6) all such nonlocal terms get cancelled among themselves leaving only the local ones, as it occurs also in the classical case. We stress again that the evident simplicity of expressions (3.6 a,b,c) for the conserved quantities, is the most prominent feature of the present model.

The transition of these conserved quantities to those of the NLS field model is easily achieved at the continuum limit by taking

\[
N = \left. \left( \frac{1}{\Delta\kappa} \sum_k (N(k) - 1) \right) \right|_{(\Delta \to 0)} = \int dx \phi(x)\psi(x),
\]
\[ P = 2 \left( \sum_k (p_k + \frac{1}{2\kappa\Delta^2}) \right) |_{\Delta \to 0} = \int dx (\phi_x \psi - \phi \psi_x), \quad (3.7b) \]

\[ H = - \left( \sum_k (h_k - \frac{1}{3\kappa\Delta^2}) \right) |_{\Delta \to 0} \]

\[ = \int dx (\phi_x \psi_x + \kappa(\phi \psi)^2), \quad (3.7c) \]

with the standard assumption of vanishing boundary condition. It is worth remarking that the continuous conserved quantities (3.7) of the AKNS type system are now symmetric in \( \phi \) and \( \psi \), which allows therefore the reduction \( \phi = \psi^\dagger \) yielding the known expressions for the NLS field model.

The evident closeness between the conserved quantities (3.6) of the lattice version with those of (3.7) related to the continuum model is a noticable feature of the present model.

For solving the eigenvalue problem for the Hamiltonian of the discrete model exactly, we go along the well established steps [8] of algebraic Bethe ansatz forming the basic tool of QISM [10]. Defining the monodromy matrix as

\[ T(\lambda) = \left( \begin{array}{cc} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{array} \right) = \prod_k^N L(k \mid \lambda) \]

we get the expression for transfer matrix as

\[ \tau(\lambda) = \text{tr} (T(\lambda)) = A(\lambda) + D(\lambda), \]

which generates the conserved quantities, while \( B(\lambda), C(\lambda) \) acts as ‘creation’ and ‘annihilation’ operators, respectively. The \( n \)-particle eigenstates may be defined as

\[ | n \rangle = \prod_i^N B(\lambda_i) | 0 \rangle \]

with the property of the ‘vacuum’:

\[ C(\lambda) | 0 \rangle = 0, \quad A(\lambda) | 0 \rangle = a(\lambda)^N | 0 \rangle, \quad D(\lambda) | 0 \rangle = d(\lambda)^N | 0 \rangle. \]

The QYBE for the monodromy matrix is given again by equation (1.1), with \( L \) operators replaced by the corresponding \( T \) operators. In elementwise form this equation yields the ‘commutation’ relations

\[ [A(\lambda), A(\mu)] = [D(\lambda), D(\mu)] = [B(\lambda), B(\mu)] = [C(\lambda), C(\lambda)] = 0 \]

\[ A(\lambda)B(\mu) = \frac{1}{c(\mu, \lambda)} B(\mu)A(\lambda) - \frac{b(\mu, \lambda)}{c(\mu, \lambda)} B(\lambda)A(\mu) \]

\[ D(\lambda)B(\mu) = \frac{1}{c(\lambda, \mu)} B(\mu)D(\lambda) - \frac{b(\lambda, \mu)}{c(\lambda, \mu)} B(\lambda)D(\mu) \]

with

\[ b(\lambda, \mu) = \frac{i\kappa}{\lambda - \mu - i\kappa}, \quad c(\lambda, \mu) = \frac{\lambda - \mu}{\lambda - \mu - i\kappa}. \]

The eigenvalues of \( \tau(\lambda) \) giving the physical observables may be obtained by using the commutation relations (3.8) between \( A, B \) and \( D, B \) and the properties of the ‘vacuum’ stated above. Skipping out the details we present only the main results as follows.

\[ \tau(\lambda) | n \rangle = \mathcal{E}(\lambda, \{\lambda_j\}) | n \rangle, \quad (3.9) \]
where
\[ \mathcal{E}(\lambda, \{\lambda_j\}) = a(\lambda)^N \prod_{l} \frac{1}{c(\lambda_l, \lambda)} + d(\lambda)^N \prod_{l} \frac{1}{c(\lambda, \lambda_l)} \] (3.10)

Note that the form of eigenvalues (3.10) is obtained provided the parameters \( \lambda_i \) satisfy the condition [10]
\[ \left( \frac{a(\lambda_j)}{d(\lambda_j)} \right)^N = \prod_{l \neq j} \frac{c(\lambda_l, \lambda_j)}{c(\lambda_j, \lambda_l)}, \quad i, j = 1, \ldots, n. \] (3.11)

For the present case one obtains \( a(\xi) = (1 + \frac{\xi}{\Delta}) \), \( d(\xi) = \frac{\xi}{\Delta} \). We define the Hamiltonians by
\[ C_k = \frac{1}{n! \kappa} \frac{\partial}{\partial \xi_k} \log(\tau(\xi)a^{-N}(\xi)) \big|_{(\xi=0)}, \]
where \( a^{-N} \) is included to remove irrelevant constant terms and to avoid linear combinations of conserved quantities. Thus we get from (3.10) \[ C_k = \frac{1}{n! \kappa} \frac{\partial}{\partial \xi_k} \log \tilde{\tau}(\xi) \big|_{(\xi=0)}, \] where
\[ \tilde{\tau}(\xi) = \prod_{k} \frac{1 + i(\lambda_k - i\kappa)\xi}{1 + i\lambda_k\xi}, \] (3.12)

which finally yields
\[ \mathcal{N} = -C_1 = n, \]
\[ \mathcal{P} = -i(C_2 + \frac{\kappa}{2} C_1) = \sum_{k=1}^{n} \lambda_k \] (3.13)
\[ \mathcal{E} = C_3 - \kappa C_2 + \frac{\kappa^2}{6} C_1 = \sum_{k=1}^{n} \lambda_k^2. \]

We observe that the energy is proportional to \( \lambda_k^2 \), the momentum is proportional to \( \lambda_k \) and the number of particles is equal to the quasiparticle excitation number, as required. Note again that this result concerning the discrete model under consideration is much similar to that of the NLS field model [10] including the combinations of different conserved quantities to determine the momentum and the energy spectrum. However, contrary to the continuum case, here the values of \( \lambda_k \)'s are not arbitrary and should be determined from the equations (3.11), which for the present model reads:
\[ (1 - i\lambda_j \Delta)^N = \prod_{l \neq j} \frac{\lambda_j - \lambda_k - i\kappa}{\lambda_j - \lambda_k + i\kappa}, \quad i, j = 1, \ldots, n \] (3.14)

We should emphasise that the energy spectrum of this model obtained above and the related constraints on \( \lambda_k \) are indeed extremely simple.

4. Vector generalisation of the model
It is interesting to observe that the models violating the $SU(2)$ type symmetry, proposed in [11] can be easily generalised for the $gl(N)$ case. Out of such generalised system one might then construct quantum integrable models, like multi-component Toda chain, vector NLS model etc., as realisations through a set of independent bosonic operators. Such generalised systems are given by the Lax operator

$$L = \sum_l (K_l^+ + \frac{i\lambda}{\kappa}K_l^-)e_{ll} + \sum_{j\neq l} K_{lj}e_{jl}, \quad (4.1)$$

where $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$ are the generators of $gl(N)$. It can be shown that the above $L$ operator, associated with the rational $(N^2 \times N^2)$ $R$-matrix: $R(\lambda) = 1 + \frac{i\lambda}{\kappa}\Pi$, where $\Pi = \sum_{lk} e_{kl} \otimes e_{lk}$, satisfies the QYBE if the generators $K$ yield the following algebra:

$$\begin{align*}
[K_{mk}, K_{kl}] &= K_k^- K_{ml}, & [K_{kl}, K_{lk}] &= K_k^+ K_l^- - K_k^- K_l^+,
[K_k^+, K_{kl}] &= K_{kl} K_k^-, & [K_k^+, K_{lk}] &= -K_{lk}K_k^-,
[K_k^+, K_{lm}] &= [K_{kl}, K_{km}] = K_{kl}, K_{ml} = [K_{kl}, K_{mn}] = 0,
\end{align*} \quad (4.2)$$

where $K_k^-$ commute with all other generators; $[K_k^-, K_{ij}] = 0$ and thus are central elements, while $K_k^\pm$ form an abelian subalgebra. We may notice again that in general this is not a $su(N)$ algebra, which however is recovered at some particular symmetric reduction.

Different realisations of this algebra would generate through (4.1) different quantum integrable models, which would share the same rational $R$-matrix but generically would not exhibit unitary symmetry. Consider now a realisation of (4.2) through a set of independent operators with the commutation relations $[\psi_l, \phi_k] = \delta_{lk}$, $[\psi_l, \psi_k] = [\phi_l, \phi_k] = 0$, in the form

$$\begin{align*}
K_1^- &= -1, & K_1^+ &= \sum_j \phi_j \psi_j, & K_i^+ &= 1_{ii}, & K_i^- &= 0, \quad (i = 2, \ldots, N) \\
K_{1j} &= \psi_j, & K_{j1} &= \phi_j, & K_{ij} &= 0, & 1 < (i,j) \leq N
\end{align*} \quad (4.3)$$

The corresponding Lax operator (4.1) will then read

$$L(\lambda) = \begin{pmatrix}
-\frac{i\lambda}{\kappa} + (\tilde{\phi}\tilde{\psi}) \\
\tilde{\phi}
\end{pmatrix}.$$ 

This Lax operator, which yields a quantum integrable lattice model, gives the vector NLS model [17,18] at the continuum limit and is a natural generalisation of (3.4) to the vector case. The associated $R$-matrix also coincides with that of the field model [18]. Thus (4.4) is related to the Lax operator of a new exactly integrable lattice version of the vector NLS model. The corresponding classical system has been considered in [16].

5. A novel quantum integrable Tamm-Dancoff $q$-bosonic model
A number of lattice models involving $q$-oscillators, which are integrable at the quantum level have already been discovered [11,19]. Most of these models are related to the quantum group structures associated with the trigonometric $R$ matrix, which forms a separate class entirely different from the NLS model with rational $R$ matrix (1.2). We present here an integrable deformation of the discrete NLS model (3.4), which involves Tamm-Dancoff (TD) [20] type $q$-boson operator, but at the same time is related to a rational $R$-matrix.

It has been shown in [11] that for a 'Symmetry breaking' transformation [21]:

$$R_{kl}^{ij}(\lambda) \rightarrow e^{i\theta(j-k)} R_{kl}^{ij}(\lambda),$$

of the original $R$-matrix (1.2), where $\theta$ is some constant parameter, the algebra (3.1) of $K$ operators is also deformed in an interesting way. We find a realisation of this deformed algebra through TD type $q$-bosonic operators $b,c,N$:

$$[b,N] = b, \quad [c,N] = -c, \quad bc - qcb = q^N, \quad (5.1)$$

where $b$ and $c$ are not in general hermitian conjugate of each other. Here we have introduced the parameter $q = e^{i2\theta}$. One may compare the above TD type $q$-deformed bosons with the standard $q$-oscillator algebra [22]:

$$[a,N] = a, \quad [a^\dagger,N] = -a^\dagger, \quad aa^\dagger - qa^\dagger a = q^{-N}.$$  

The algebraic relations (5.1) yield the Lax operator of the corresponding model as

$$L^q(\lambda) = \left( \begin{array}{ccc} (1 + \kappa N - i\lambda)f(N) & -i\kappa^{\frac{1}{2}} c \\ i\kappa^{\frac{1}{2}} b & f(N) \end{array} \right), \quad (5.2)$$

where $f(N) = q^{\frac{1}{2}(N-\frac{1}{2})}$.

Note that it represents a quantum integrable system, which satisfies the QYBE with the deformed rational $R$ matrix

$$R^q(\lambda) = \left( \begin{array}{ccc} \lambda - i\kappa & -i\kappa & \lambda q^{-\frac{1}{2}} \\ -i\kappa & \lambda q^{\frac{1}{2}} & -i\kappa \\ \lambda^{-i\kappa} & -i\kappa & \lambda - i\kappa \end{array} \right). \quad (5.3)$$

Evidently at $q \rightarrow 1$ one recovers from (5.3) the Lax operator (3.4) of our discrete NLS and also gets $R^q \rightarrow R$ as in (1.2).

There is a simple mapping from such TD-deformed operators to the operators of the original lattice NLS model as $b = f(N)\psi, c = f(N)\phi$ recovering the canonical relation $[\psi,\phi] = 1$; accordingly the Lax operator (5.2) is mapped into (3.4) by:

$$L^q(n) = q^{\frac{1}{2}(N(n)-\frac{1}{2})} L(n)_{NLS}.$$ 

Hence, this Tamm-Dancoff type deformed bosonic system represents a new quantum integrable lattice model which can be solved through the algebraic Bethe ansatz using the results reported in sect.3.
6. Concluding remarks

It might be worthwhile to summarize here the main results obtained in this paper, and to stress once again its underlying ideology. We have presented a quantum integrable lattice model, which in general corresponds to the AKNS type system. It may also be considered as a lattice NLS model, that shares with the continuum field model the same quantum R-matrix, yields local and quite simple expressions for the conserved quantities including the Hamiltonian and allows to determine their spectrum by the standard quantum inverse scattering method. Remarkably, the present lattice NLS admits a natural vector generalisation giving a new exact lattice version of the vector NLS model, which is much simpler than all other vector generalisations available in the literature. Finally, the model can be easily deformed, giving rise to a Tamm-Dancoff type q-boson model, exactly solvable at the quantum level. The price we had to pay to achieve all the previous results was the breaking of unitary symmetry, which is however restored at the continuum limit. We note that the same advantages and the same drawback characterise the classical version investigated in [16]. For those who think, as we do, that the price we paid was an acceptable one, the ideology underlying the construction of the present model might be a fruitful one.

After the completion of this work ref. [23,24] were brought to our notice. In [23], as a significant contribution, a most general form of $L$ operator for the lattice NLS was found, which provides the basis for classification of all $L$ operators related to the $R$-matrix (1.2). The physical and mathematical properties of lattice NLS along with many other models have been discussed in great detail in [24]. We note that the model presented here is consistent with the general form of the $L$ operator of [23], which thus gives another basis for its validity.

Acknowledgements: One of the authors (AK) acknowledges with thanks the support of Alexander von Humboldt Foundation. The research reported in the present paper has been carried out in the framework of the national research program “Problemi matematici della Fisica”, supported by the italian Ministry of University and Scientific and Technological Research (MURST). The authors are also thankful to Prof. V. Korepin for essential constructive remarks and for pointing out the ref. [23,24].
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