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On the period mod $m$ of polynomially-recursive sequences: a case study

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Abstract

Many polynomially-recursive sequences have a periodic behavior mod $m$. In this paper, we analyze the period mod $m$ of a class of second-order polynomially-recursive sequences. Starting with a problem originally coming from an enumeration of avoiding pattern permutations, we give a generalization which appears to be linked with nice number theory notions (the Carmichael function, algebraic integers, Wieferich primes).

1 Introduction

In his analysis of sorting algorithms, Knuth introduced the notion of forbidden pattern in permutations, which later became a field of research per se [11]. By studying the basis of such forbidden patterns for permutations reachable with $k$ right-jumps from the identity permutation, the authors of [1] discovered that the permutations of size $n$ in this basis were enumerated by the sequence of integers $(b_n)_{n \geq 0}$ given by $b_0 = 1$, $b_1 = 0$,

$$b_{n+2} = 2nb_{n+1} + (1 + n - n^2)b_n \quad \text{for all} \quad n \geq 0. \quad (1)$$

This is sequence A265165 in the OEIS$^1$; it starts like 0, 1, 2, 7, 32, 179, 1182, 8993, 77440, 744425, 7901410, 91774375, . . .

Such a sequence defined by a recurrence with polynomial coefficients in $n$ is called $P$-recursive (for polynomially recursive). Some authors also call such sequences holonomic, or D-finite (see, e.g., [5, 7, 13, 16]). The D-finite (for differentially finite) terminology comes from the fact that a sequence $(f_n)_{n \geq 0}$ satisfies a linear recurrence with polynomial coefficients in $n$ if and only if its generating function $F(z) = \sum_{n \geq 0} f_n z^n$ satisfies a linear differential equation with polynomial coefficients in $z$. Accordingly, $P$-recursive sequences and D-finite functions satisfy many closure properties: this contributes to make them ubiquitous in combinatorics, number theory, analysis of algorithms, computer algebra, mathematical physics, etc. It is not always the case that such sequences have a closed form. In our case, the generating function of $(b_n)_{n \geq 0}$ has in fact a nice closed form involving the golden ratio. Indeed, putting

$$\alpha := \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \beta := \frac{1 - \sqrt{5}}{2}$$

for the two roots of the quadratic equation $x^2 - x - 1 = 0$, it was shown in [1] that the exponential generating function of $(b_n)_{n \geq 0}$, namely

$$B(x) = \sum_{n \geq 0} b_n \frac{x^n}{n!}, \quad \text{satisfies} \quad B(x) = \frac{\beta}{\beta - \alpha} (1 - x)^\alpha + \frac{\alpha}{\alpha - \beta} (1 - x)^\beta - 1. \quad (2)$$

It should be stressed here that our sequence $(b_n)_{n \geq 0}$ is an instance of a noteworthy phenomenon: it is one of the rare combinatorial sequences exhibiting an irrational exponent

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$^1$OEIS stands for the On-Line Encyclopedia of Integer Sequences, see https://oeis.org.
in its asymptotics:

\[
\frac{b_n}{n!} \sim \frac{\alpha}{\sqrt{5\Gamma(\alpha - 1)}} n^{\alpha - 2} (1 + o(1)) \quad \text{as} \quad n \to \infty,
\]

where \( \Gamma(z) = \int_0^{+\infty} t^{z-1} \exp(-t) dt \) is the Euler gamma function. We refer to the wonderful book of Flajolet and Sedgewick [5] for a few other examples of such a phenomenon in analytic combinatorics, and to [1, Section 4] for further comments on the links between G-functions and (ir)rational exponents in the asymptotics of the coefficients.

P-recursive sequences are also of interest in number theory, where there is a vast literature analyzing the modular congruences of famous sequences, e.g., for the binomial coefficients, or the Fibonacci, Catalan, Motzkin, Apéry numbers, see [3, 6, 9, 14, 19]. For example, the Apéry numbers satisfy

\[
A(p^e q) = A(p^{e-1} q) \mod p^{3e},
\]

in which the exponent \( 3e \) in the modulus grows faster than the exponent \( e \) in the function argument. This phenomenon is sometimes called “supercongruence”, and find roots in seminal works by Kummer and Ramanujan (see [8, 12, 17] for more recent advances on this topic). Accordingly, many articles consider sequences modulo \( m = 2^r \), or \( m = 3^r \), or variants of power of a prime number.

We now restate an important result which holds for any \( m \) (not necessarily the power of a prime number).

**Theorem 1** (Congruences and periods for P-recursive sequences [1, Theorem 7]).

Consider any P-recurrence of order \( r \):

\[
P_0(n) u_n = \sum_{i=1}^{r} P_i(n) u_{n-i},
\]

where the polynomials \( P_0(n), \ldots, P_r(n) \) belong to \( \mathbb{Z}[n] \), and where the polynomial \( P_0(n) \) is invertible mod \( m \). Then the sequence \( (u_n \mod m)_{n \geq 0} \) is eventually periodic\(^2\). In particular, sequences such that \( P_0(n) = 1 \) are periodic mod \( m \). Additionally, the preperiod and the period \( p \) are bounded by \( m^{2r+1} \), therefore one can efficiently compute them via the Knuth–Floyd cycle-finding algorithm (the tortoise and the hare algorithm).

N.B.: It is not always the case that P-recursive sequences are periodic mod \( p \). E.g., it was proven in [10] that Motzkin numbers are not periodic mod \( m \), and it seems that

\[(n + 3)(n + 2)u_n = 8(n - 1)(n - 2)u(n - 2) + (7n^2 + 7n - 2)u(n - 1),\quad u_0 = 0, u_1 = 1,
\]

is also not periodic mod \( m \), for any \( m > 2 \) (this P-recursive sequence counts a famous class of permutations, namely, the Baxter permutations). This is coherent with Theorem 1, as the leading term in the recurrence (the factor \((n + 3)(n + 2)\)) is not invertible mod \( m \), for infinitely many \( n \).

---

\(^2\)An eventually periodic sequence of period \( p \) is a sequence for which \( u_{n+p} = u_n \) for all \( n \geq n^* \) (\( n^* \) is called the preperiod). Some authors use the terminology “ultimately periodic” instead. In the sequel, as the context is clear, we will often omit the word “eventually”.

\[3\]
For our sequence \((b_n)_{n \geq 1}\) (defined by recurrence (1)), this theorem explains the periodic behavior of \(b_n \mod m\). Thanks to the bounds mentioned in Theorem 1, we can get \(b_n \mod m\), by brute-force computation, for any given \(m\). For example \(b_n \mod 15\) is periodic of period 12 (after a preperiod \(n^* = 9\)):

\[(b_n \mod 15)_{n \geq 9} = (10, 5, 10, 0, 10, 5, 0, 5, 0, 5)\infty.

The period can be quite large, for example \(b_n \mod 3617\) has period 26158144. More generally, for every positive integer \(m\), the sequence \((b_n \mod m)_{n \geq 1}\) is eventually periodic, for some period \(p\) depending on \(m\), as defined in the footnote on the previous page. For each \(m\), let \(T_m\) be the smallest possible period \(p\). In this paper, we study some properties of \((T_m)_{m \geq 1}\).

This is sequence A306699 in the OEIS; here are its first few values \(T_2, \ldots, T_{100}\):

\[2, 12, 8, 1, 12, 84, 8, 36, 2, 1, 24, 104, 84, 12, 16, 544, 36, 1, 8, 84, 2, 1012, 24, 1, 104, 108, 168, 1, 12, 1, 32, 12, 544, 72, 2664, 2, 312, 8, 1, 36, 12, 8, 36, 12, 1012, 4324, 48, 588, 2, 1632, 104, 5512, 108, 1, 168, 12, 2, 1, 24, 1, 2, 252, 64, 104, 12, 2948, 544, 3036, 84, 1, 72, 10512, 2664, 12, 8, 84, 312, 1, 16, 324, 1, 96, 18624, 588, 36, 8.

Do you detect some hidden patterns in this sequence? This is what we tackle in the next section.

## 2 Periodicity mod \(m\) and links with number theory

Our main result is the following.

**Theorem 2.** Let \((b_n)_{n \geq 0}\) be the sequence defined by the recurrence of Formula 1. The period \(T_m\) of this sequence \(b_n \mod m\) satisfies:

a) If \(m = p_1^{e_1} \cdots p_k^{e_k}\) (where \(p_1, \ldots, p_k\) are distinct primes), then\(^3\)

\[T_m = \text{lcm}(T_{p_1^{e_1}}, \ldots, T_{p_k^{e_k}}).
\]

b) We have \(T_m = 1\) if and only if \(m\) is the product of primes \(p \equiv 0, 1, 4 \pmod{5}\).

c) For every prime \(p\), we have \(T_p \mid 2p \text{ord}_5(p)\).

d) If \(T_m > 1\) then \(2 \mid T_m\) if \(m\) is even, and \(4 \mid T_m\) if \(m\) is odd.

e) For \(m \geq 3\), we have \(T_m = 2\) if and only if \(m\) is even and \(\frac{m}{2}\) is the product of primes \(p \equiv 0, 1, 4 \pmod{5}\).

f) For every prime \(p\), we have \(T_{p^k} \mid 2p^k(p - 1)\).

\(^3\)As usual, lcm stands for the least common multiple.
The function $T_m$ thus shares some similarities with the Carmichael function introduced in [2, p. 39], and it is expected that its asymptotic behavior is also similar (following, e.g., the lines of [4]). In this article, we focus on the rich arithmetic properties of this function. Note that Theorem 2 allows computing $T_m$ in a much faster way than the brute-force algorithm mentioned in Section 1: the complexity goes from $m^{2r+1}$ via brute-force to $\ln(m)^3$ via Shor’s factorization algorithm [15] (or to sub-exponential complexity in $\ln(m)$ with other efficient algorithms, if one does not want to rely on the use of quantum computers!).

**Proof of Part a).** The proof will use a little preliminary result. We call $T_m$ the “eventual period of the sequence mod $m$”, or, for short, the “period”, even if the sequence starts with some terms which does not satisfy the periodic pattern. The following lemma holds for all eventually periodic sequences of integers.

**Lemma 3.** $T_m$ divides all other periods of $(u_n)_{n \geq 0}$ modulo $m$.

**Proof.** Let $T_m = a$ and assume there is $b$ (not a multiple of $a$) which is also a period modulo $m$. Thus, there are $n_a$, $n_b$ such that $u_{n+a} \equiv u_n \pmod{m}$ for all $n > n_a$ and $u_{n+b} \equiv u_n \pmod{m}$ for all $n > n_b$. Let $d = \gcd(a, b)$. By Bézout’s identity, one has then $d = Aa + Bb$ for some integers $A$, $B$. Let $n_{a,b} = \max\{n_a, n_b\} + |A|a + |B|b$ and assume that $n > n_{a,b}$. Then $u_{a+d} = u_{n+Aa+Bb} \equiv u_{n+|A+a+|B+1|b} \pmod{m} \equiv u_{n+a} \pmod{m}$ so $d < a$ is a period of $(u_n)_{n \geq 0}$ modulo $m$, contradicting the minimality of $a$. \qed

An immediate consequence is the following:

**Corollary 4.** We have $T_{\lcm(m_1, \ldots, m_r)} = \lcm(T_{m_1}, \ldots, T_{m_r})$.

**Proof.** First consider $r = 2$, and let $a := m_1$, $b := m_2$. Since $\lcm(T_a, T_b)$ is a multiple of both $T_a$ and $T_b$, it follows that it is a period of $(u_n)_{n \geq 0}$ modulo both $a$ and $b$, so modulo $\lcm(a, b)$. It remains to prove that it is the minimal one. To this aim, suppose that $T_{\lcm(a,b)} < \lcm(T_a, T_b)$. Then either $T_a \nmid T_{\lcm(a,b)}$ or $T_b \nmid T_{\lcm(a,b)}$. Since the two cases are similar, we only deal with the first one. In this case we would have that both $T_a$ and $T_{\lcm(a,b)}$ would be periods modulo $a$. By the previous lemma, this would force $\gcd(T_a, T_{\lcm(a,b)}) < T_a$, which would obviously be a contradiction. Now, a trivial induction on the number $r \geq 2$ gives that $T_{\lcm(m_1, \ldots, m_r)} = \lcm(T_{m_1}, \ldots, T_{m_r})$

holds for all positive integers $m_1, \ldots, m_r$. \qed

In particular Part a) of Theorem 2 holds: $T_m = \lcm(T_{p_1^{e_1}}, \ldots, T_{p_k^{e_k}})$. Let us now tackle the proofs of Parts b)–f).

**Proof of Part b).** We use the generating function (2), which tells us that

$$[x^n]B(x) = \frac{b_n}{n!} = \frac{(-1)^n}{\sqrt{5}} \left( \binom{\beta}{n} - \binom{\alpha}{n} \right),$$

(3)
Thus,
\[ b_n = \frac{(-1)^{n-1}}{\sqrt{5}} (\beta \alpha (\alpha - 1) \cdots (\alpha - (n - 1)) - \alpha \beta (\beta - 1) \cdots (\beta - (n - 1))). \quad (4) \]

By Fermat’s little theorem,
\[ \prod_{k=0}^{p-1} (X - k) = X^p - X \pmod{p}. \quad (5) \]

Now, assume that \( p \equiv 1, 4 \pmod{5} \). Then
\[ \prod_{k=0}^{p-1} (\alpha - k) \equiv \alpha^p - \alpha \pmod{p} \equiv 0 \pmod{p}, \]
where for the last congruence we used the law of quadratic reciprocity: since \( p \equiv 1, 4 \pmod{5} \), we have
\[ \left( \frac{5}{p} \right) = \left( \frac{p}{5} \right) = 1, \]
where \( \left( \frac{\bullet}{p} \right) \) is the Legendre symbol. Thus,
\[ \alpha^p = \left( \frac{1 + \sqrt{5}}{2} \right)^p \equiv \frac{1 + \sqrt{5} \cdot 5^{(p-1)/2}}{2^p} \pmod{p} \equiv \alpha \pmod{p}, \quad (6) \]
because \( 5^{(p-1)/2} \equiv \left( \frac{5}{p} \right) \equiv 1 \pmod{p} \) by Euler’s criterion.

In the above and in what follows, for two algebraic integers \( \delta, \gamma \) and an integer \( m \) we write \( \delta \equiv \gamma \pmod{m} \) if the number \( (\delta - \gamma)/m \) is an algebraic integer. This shows that
\[ \frac{1}{p} \prod_{k=0}^{p-1} (\alpha - k) \]
is an algebraic integer. The same is true with \( \alpha \) replaced by \( \beta \). Now take \( r \geq 1 \) be any integer and take \( n \geq pr \). Then, for each \( \ell = 0, 1, \ldots, r - 1 \), we have that both
\[ \frac{1}{p} \prod_{k=0}^{p-1} (\alpha - (p\ell + k)) \quad \text{and} \quad \frac{1}{p} \prod_{k=0}^{p-1} (\beta - (p\ell + k)) \]
are algebraic integers. Thus, if \( n \geq pr \), then
\[ \frac{\sqrt{5}b_n}{p^n} = (-1)^{n-1} \left( \beta \prod_{\ell=0}^{r-1} \prod_{k=0}^{p-1} (\alpha - (p\ell + k)) \prod_{k=pr}^{n-1} (\alpha - k) - \alpha \prod_{\ell=0}^{r-1} \prod_{k=0}^{p-1} (\beta - (p\ell + k)) \prod_{k=pr}^{n-1} (\beta - k) \right) \]

\[ 6 \]
is an algebraic integer. Thus, $5b_n^2/p^{2r}$ is an algebraic integer and a rational number, so an integer. Since $p \neq 5$, it follows that $p^{2r} \mid b_n^2$, so $p^r \mid b_n$ for $n \geq pr$. This shows that $T_{p^r} = 1$ for all such primes $p$ and positive integers $r$. The same is true for $p = 5$. There we use that $\alpha - 3 = \sqrt{5}\beta$, so $\sqrt{5} \mid \alpha - 3$. Thus, if $n \geq 10r$, we have that

$$\prod_{k=1}^{n}(\alpha - k) \text{ is a multiple of } \prod_{\ell=0}^{2r-1}(\alpha - (3 + 5\ell)) \text{ in } \mathbb{Z}[(1 + \sqrt{5})/2],$$

which in turn is a multiple of $5^r = \sqrt{5}2r$ in $\mathbb{Z}[(1 + \sqrt{5})/2]$. Thus, if $n \geq 10r$, then $5^r \mid b_n$.

This shows that also $T_{5^r} = 1$ and in fact, $m \mid b_n$ for all $n > n_m$ if $m$ is made up only of primes $0, 1, 4 \pmod{5}$. This finishes the proof of b).

**Proof of Part c.** The claim is satisfied for $p = 2$, as $(b_n \mod{2})_{n \geq 0} = (1, 0)^{\infty}$, thus $T_2 = 2 \mid 4$. Consider now $p > 2$. By Part b), it suffices to consider odd primes $p \equiv 2, 3 \pmod{5}$. Evaluating Formula (5) at $\alpha = 1 + \sqrt{5}/2$, one has

$$\frac{p-1}{k=0} \prod (\alpha - k) \equiv \alpha^p - \alpha \pmod{p},$$

Since $5^{(p-1)/2} \equiv -1 \pmod{p}$, the argument from (6) shows that $\alpha^p \equiv \beta \pmod{p}$. Thus

$$\prod_{k=1}^{2p}(\alpha - k) = \prod_{k=1}^{p}(\alpha - k) \prod_{k=p+1}^{2p}(\alpha - k) \equiv (\beta - \alpha)^2 \pmod{p} \equiv 5 \pmod{p}.$$

The same is true for $\alpha$ replaced by $\beta$. Thus, it follows that for $n > 2p$, we have

$$b_{n+2p} = \frac{(-1)^{n+2p-1}}{\sqrt{5}} \left( \beta \prod_{k=0}^{n+2p-1}(\alpha - k) - \alpha \prod_{k=0}^{n+2p-1}(\beta - k) \right)$$

$$\equiv \frac{(-1)^{n-1}}{\sqrt{5}} 5 \left( \beta \prod_{k=0}^{n-1}(\alpha - k) - \alpha \prod_{k=0}^{n-1}(\beta - k) \right) \pmod{p}$$

$$\equiv 5b_n \pmod{p}.$$

Applying this $k$ times, we get

$$b_{n+2pk} \equiv 5^kb_n \pmod{p}.$$

Taking $k = p - 1$ and applying Fermat’s little theorem $5^{p-1} \equiv 1 \pmod{p}$, we get $T_p \mid 2p(p-1)$. We can optimize this idea by taking $k = \text{ord}_p(5)$, where $\text{ord}_p(5)$ is the order of 5 modulo $p$ (the smallest $k > 0$ such that $5^k \equiv 1 \pmod{p}$), this gives the stronger wanted claim: $T_p \mid 2p\text{ord}_p(5)$.  

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Proof of Part d). By a), we know that $T_p \mid T_{pn}$. Taking $p = 2$, one gets $2 \mid T_m$. Now, if $T_m > 1$, by b), there is at least a prime $p = 2, 3 \pmod{5}$ such that $p \mid m$. We then have $T_p \mid T_m$ by a). We now prove by contradiction that $T_p$ is a multiple of 4.

Take a prime $p \geq 3$ and assume $\nu_2(T_p) < 2$, where $\nu_q(a)$ is the exponent of $q$ in the factorization of $a$. That is, $T_p$ would either be odd or 2 times an odd number. Since $T_p \mid 2(p-1)$, it would follow that if we write $p-1 = 2^a k$, where $k$ is odd, then $T_p \mid 2pk$. Thus, one would have

$$b_n \equiv b_{n+2pk} \equiv 5^kb_n \pmod{p} \quad (7)$$

for all $n > n_p$. Since $p = 2, 3 \pmod{5}$, 5 is not a quadratic residue, and thus $5^k \not\equiv 1 \pmod{p}$ (since $-1 \equiv 5^{(p-1)/2} \equiv (5^k)^{2^{a-1}} \pmod{p}$). So, the above congruence (7) would imply that $p \mid (5^k-1)b_n$ but $p \nmid 5^k-1$, so $b_n \equiv 0 \pmod{p}$ for all large $n$. Take $n$ and $n+1$ and rewrite what we got, i.e., $b_n \equiv b_{n+1} \equiv 0 \pmod{p}$ in $\mathbb{Z}[\alpha]/p\mathbb{Z}[\alpha]$ as

$$b_n = \beta \prod_{k=0}^{n-1} (\alpha - k) - \alpha \prod_{k=0}^{n-1} (\beta - k) \equiv 0 \pmod{p},$$

$$b_{n+1} = \beta \left( \prod_{k=0}^{n-1} (\alpha - k) \right) (\alpha - n) - \beta \left( \prod_{k=0}^{n-1} (\beta - k) \right) (\beta - n) \equiv 0 \pmod{p}.$$

We treat this as a linear system in the two unknowns

$$(X, Y) = \left( \beta \prod_{k=0}^{n-1} (\alpha - k), \alpha \prod_{k=0}^{n-1} (\beta - k) \right)$$

in the field with $p^2$ elements $\mathbb{Z}[\alpha]/p\mathbb{Z}[\alpha]$. This is homogeneous. None of $X$ or $Y$ is 0 since $p$ cannot divide $\beta \prod_{k=0}^{n-1} (\alpha - k)$. Thus, it must be that the determinant of the above matrix is 0 modulo $p$, but this is

$$\begin{vmatrix} 1 & -1 \\ \alpha - n & -(\beta - n) \end{vmatrix} = \sqrt{5},$$

which is invertible modulo $p$. Thus, indeed, it is not possible that $b_n$ and $b_{n+1}$ is a multiple of $p$ for all large $n$, getting a contradiction. This shows that $T_p$ is a multiple of 4.

Proof of Part e). Let $m$ be of shape different from the one required in Part b), i.e., $m$ has now at least one prime $p \equiv 2, 3 \pmod{5}$ such that $p \mid m$. Then $4 \mid T_p$ by what we have done above, and so $4 \mid T_m$ by a). Thus, such $m$ cannot participate in the situations described either at d) or e). Further, one has $T_4 = 8$ as $(b_n \pmod{4}_{n \geq 0} = (1, 0, 1, 2, 3, 0, 3, 2)^\infty$. Thus, if $4 \mid m$, then $8 \mid T_m$. Hence, if $T_m = 2$, then the only possibility is that $2 \mid m$ and $m/2$ is a product of primes congruent to 0, 1, 4 modulo 5. Conversely, if $m$ has such structure then $T_m = 2$ by a) and the fact that $T_2 = 2$ and $T_{p^r} = 1$ for all odd prime power factors $p^r$ of $m$. This ends the proof of e).
Proof of Part f). Finally, f) is based on a preliminary result: a slight generalization of (5), namely
\[ p^{r-1} \prod_{k=0}^{p^r-1} (X - k) \equiv (X^p - X)^{p^{r-1}} \quad (\text{mod } p^r) \] (8)
valid for all odd primes \( p \) and \( r \geq 1 \). Let us prove (8) by induction on \( r \). We first prove it for \( r = 2 \). We return to (5) and write
\[ p^{r-1} \prod_{k=0}^{p^r-1} (X - k) = X^p - X + pH_1(X), \]
where \( H_1(X) \in \mathbb{Z}[X] \). Changing \( X \) to \( X - p\ell \) for \( \ell = 0, 1, \ldots, p - 1 \), we get that
\[ p^{r-1} \prod_{k=0}^{p^r-1} (X - (p\ell + k)) = (X - p\ell)^p - (X - p\ell) + pH(X - p\ell) \equiv (X^p - X - pH(X)) - p\ell \quad (\text{mod } p^2). \]
In the above, we used the fact that \( H(X - p\ell) \equiv H(X) \quad (\text{mod } p) \). Thus,
\[ p^{2-1} \prod_{k=0}^{p^2-1} (X - k) = \prod_{\ell=0}^{p-1} \prod_{k=0}^{p-1} (X - (p\ell + k)) \]
\[ \equiv \prod_{k=0}^{p-1} ((X^p - X - pH(X)) - p\ell) \quad (\text{mod } p^2) \]
\[ \equiv (X^p - X - pH(X))^p - (X^p - X - pH(X))^{p-1} p \left( \sum_{\ell=0}^{p-1} \ell \right) \quad (\text{mod } p^2) \]
\[ \equiv (X^p - X)^p - (X^p - X - pH(X))^{p-1} p \left( \frac{p(p-1)}{2} \right) \quad (\text{mod } p^2) \]
\[ \equiv (X^p - X)^p \quad (\text{mod } p^2). \]
In the above, we used the fact that \( p \) is odd so \( p(p-1)/2 \) is a multiple of \( p \). This proves (8) for \( r = 2 \). Now, assuming that (8) holds for \( p^r \), for some \( r \geq 2 \), we get that for all \( \ell \geq 0 \), we have
\[ p^{r-1} \prod_{k=0}^{p^r-1} (X - (p^r\ell + k)) \equiv ((X - p^r\ell)^p - (X - p^r\ell))^{p^{r-1}} + p^r H_r(X - p^r\ell) \quad (\text{mod } p^{r+1}) \]
\[ \equiv (X^p - X)^{p^{r-1}} + p^r H_r(X) \quad (\text{mod } p^{r+1}), \]
where \( H_r(X) \in \mathbb{Z}[X] \). This allows concluding the induction step, and thus the generaliza-
tion (8) that we wanted:
\[
\prod_{k=0}^{p^{r+1}-1} (X - k) = \prod_{\ell=0}^{p} \prod_{k=0}^{p^{r}-1} (X - (p^r \ell + k))
\equiv ((X^p - X)^{p^{r-1}} + p^r H_r(X))^p \pmod{p^{r+1}}
\equiv (X^p - X)^{p^r} \pmod{p^{r+1}}.
\]

Equipped with this preliminary result, letting \( p > 2 \) be congruent to \( 2, 3 \pmod{5} \), evaluating the above identity in \( \alpha \), and using that \( \alpha^p \equiv \beta \pmod{p} \), we get that
\[
\prod_{k=0}^{p^{r-1}} (\alpha - k) \equiv (X^p - X)^{p^{r-1}} \pmod{p^r} \equiv (\alpha^p - \alpha)^{p^{r-1}} \pmod{p^r} \equiv (\beta - \alpha)^{p^{r-1}} \pmod{p^r}.
\]

This shows that
\[
\prod_{k=0}^{2p^r - 1} (\alpha - k) \equiv (\beta - \alpha)^{2p^r - 1} \pmod{p^r} \equiv 5^{p^{r-1}} \pmod{p^r}.
\]

The same is true for \( \beta \); this leads to
\[
b_{n+2p^r} \equiv \frac{(-1)^n + 2p^r - 1}{\sqrt{5}} 5^{p^{r-1}} \left(\beta \prod_{k=0}^{n-1} (\alpha - k) - \alpha \prod_{k=0}^{n-1} (\beta - k)\right) \pmod{p^r} \equiv 5^{p^{r-1}} b_n \pmod{p^r}.
\]

Thus, applying this \( k \) times, we get
\[
b_{n+2p^r k} \equiv 5^{p^{r-1} k} b_n \pmod{p^r}.
\]

By Euler’s theorem \( a^{\phi(n)} \equiv 1 \pmod{n} \), one has \( 5^{p^{r-1} (p-1)} \equiv 1 \pmod{p^r} \). Thus, taking \( k = p - 1 \) in (9), we get \( b_{n+2p^r (p-1)} \equiv b_n \pmod{p^r} \). Therefore, \( T_{p^r} | 2p^r (p-1) \).

N.B.: As in the proof of c), we can optimize this idea; indeed \( \text{ord}_5(p^r) = p^{r-1} \text{ord}_5(p) \) and thus taking \( k = \text{ord}_5(p) \), one gets \( T_{p^r} | 2p^r \text{ord}_5(p) \).

Finally, it remains to prove f) for \( p = 2 \). Here, by inspection, we have
\[
\prod_{k=0}^{7} (X - k) \equiv (X^2 - X)^4 \pmod{4}.
\]

By induction on \( r \geq 2 \), one shows that
\[
\prod_{k=0}^{2^{r+1}-1} (X - k) \equiv (X^2 - X)^{2^r} \pmod{2^r}.
\]
Evaluating this in \( \alpha \), we get
\[
2^r + 1 - \prod_{k=0}^{2^r-1} (\alpha - k) \equiv (\alpha^2 - \beta)^{2^r - 1} \pmod{2^r}.
\]
The same holds for \( \beta \), so
\[
b_{n+2^r+1} = \frac{(-1)^{n+2^r+1-1}}{\sqrt{5}} 5^{2^r-1} \left( \beta \prod_{k=0}^{n-1} (\alpha - k) - \alpha \prod_{k=0}^{n-1} (\beta - k) \right) \pmod{2^r}
\]
\[
\equiv 5^{2^r-1} b_n \pmod{2^r} \equiv b_n \pmod{2^r}
\]
showing that \( T_{2^r} \mid 2^r+1 \) for all \( r \geq 2 \).

3 Comments and generalizations

Along the proof of our main result we showed that if \( p \equiv 2 \) or \( 3 \) (mod 5), then
\[
b_{n+2p} \equiv 5b_n \pmod{p}.
\]
From here we deduced that \( T_p \mid 2p(p-1) \) via the fact that \( 5^{p-1} \equiv 1 \pmod{p} \). One may ask whether it can be the case that
\[
T_{p^2} \mid 2p(p-1), \text{ for some prime } p?
\]
Well, first of all, it implies that \( 5^{p-1} \equiv 1 \pmod{p^2} \). This makes \( p \) a base-5 Wieferich prime\(^4\). Despite the fact that it is conjectured that there are infinitely many such primes, only 7 base-5 Wieferich primes are currently known! (They are listed as A123692). Amongst them, only \( p = 2, 40487, 1645333507 \), and \( 6692367337 \) are additionally congruent to \( 2 \) (mod 5), and none is known to be congruent to \( 3 \) (mod 5). Note that the condition of \( p \equiv 2 \) or \( 3 \) (mod 5) being base-5 Wieferich is not sufficient to have the divisibility property (10). So, how many other primes could lead to \( T_{p^2} \mid 2p(p-1) \)? A close analysis of our arguments show that in addition to be a base-5 Wieferich prime, it should also hold that
\[
\prod_{k=0}^{2p-1} (\alpha - k) - 5 \equiv 0 \pmod{p^2},
\]
and if this is the case then indeed \( T_{p^2} \mid 2p(p-1) \). Since the integer
\[
\frac{1}{p} \left( \prod_{k=0}^{2p-1} (\alpha - k) - 5 \right) \in \mathbb{Z}[\alpha]
\]
\(^4\)A prime \( p \) is a Wieferich prime in base \( b \) if \( b^{p-1} \equiv 1 \pmod{p^2} \). This notion was introduced (with \( b = 2 \)) by Arthur Wieferich in 1909 in his work on Fermat’s last theorem [18].
should be the zero element in the finite field $\mathbb{Z}[\alpha]/p\mathbb{Z}[\alpha]$, with $p^2$ elements, it could be that the “probability” that this condition happens is $1/p^2$. By the same logic, the “probability” that $p$ is base-5 Wieferich should be $1/p$. Assuming these events to be independent, we could infer that the probability that both these conditions hold is $1/p^3$. Then, as the series

$$\sum_{p \equiv 2, 3 \pmod{5}} \frac{1}{p^3}$$

is convergent, this heuristically suggests that there should be only finitely many primes $p \equiv 2$ or $3 \pmod{5}$ such that $T_{p^2} | 2p(p - 1)$.

Finally, our results apply to other sequences as well. More precisely, let $a, b$ be integers and let $\alpha, \beta$ be the roots of $x^2 - ax - b$. Let

$$B(x) = \frac{\beta}{\beta - \alpha}(1 - x) + \frac{\alpha}{\alpha - \beta}(1 - x)^\beta - 1 = \sum_{n \geq 0} b_n \frac{x^n}{n!}.$$ 

Accordingly, the sequence $(b_n)_{n \geq 0}$ satisfies $b_0 = 1$, $b_1 = 0$, and, for $n \geq 0$

$$b_{n+2} = (2n - a + 1)b_{n+1} + (b + an - n^2)b_n.$$

What are the periods mod $m$ of such sequences?

- In case $\alpha$ and $\beta$ are rational (hence, integers), $B(x)$ is a rational function, so $b_n = n!u_n$, where $(u_n)_{n \geq 0}$ is binary recurrent with constant coefficients. It then follows that $b_n \equiv 0 \pmod{m}$ for all $m$ provided $n > n_m$ is sufficiently large. Thus, $T_m = 1$.

- In case $\alpha, \beta$ are irrational, then we get a result similar to Theorem 2 (where we had $(a, b) = (1, 1)$). Namely, $b_n \equiv 0 \pmod{m}$ for all $n$ sufficiently large whenever $m$ is the product of odd primes $p$ for which the Legendre symbol $\left(\frac{\Delta}{p}\right) = 0, 1$, where $\Delta = a^2 + 4b$ is the discriminant of the quadratic $x^2 - ax - b$. In case $p$ is odd and $\left(\frac{\Delta}{p}\right) = -1$, we have that $T_p | 2p(p - 1)$ and $T_p$ is a multiple of 4. Also, $T_{p^r} | 2p^r(p - 1)$ for all $r \geq 1$ in this case. The proofs are similar. In the case of the prime 2, one needs to distinguish cases according to the parities of $a, b$. For example, if $a$ and $b$ are odd, then $\Delta \equiv 5 \pmod{8}$, so 2 is not a quadratic residue modulo $\Delta$, so $T_{2^r} | 2^{r+1}$ for all $r \geq 1$, whereas if $a$ is odd and $b$ is even then $T_2 = 1$.

This concludes our analysis of the periodicity of such P-recursive sequences mod $m$. 

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