Quantum black holes and holography

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Abstract. It is technically difficult (if not impossible) to write down and solve self-consistently the semiclassical Einstein equations in the case of evaporating black holes. These difficulties can in principle be overcome in an apparently very different context, the Randall-Sundrum braneworld models in Anti-de Sitter space. Use of Maldacena’s AdS/CFT correspondence led us to formulate a holographic conjecture for black holes localised on a brane, for which 4D quantum corrected black holes are dual to classical 5D black holes. This duality is applied to the computation of the correction to the newtonian potential on the brane, with new results on the semiclassical side, and a prediction about the existence of static large mass braneworld black holes is made.

1. Introduction
One of the most urgent problems a self-consistent quantum gravity theory will have to solve is a proper understanding of the quantum properties of black holes. In the standard approach one uses the semiclassical theory, in which matter fields are quantized in a classical curved background (see for instance [1]). In this context the Einstein equations are replaced by the semiclassical ones

\[ G_{\mu\nu} = 8\pi \langle T_{\mu\nu} \rangle, \]

where the source term is provided by the expectation values of the stress energy tensor operator for the matter fields in a suitably defined quantum state. The problem with these equations is that the exact expression of \( \langle T_{\mu\nu} \rangle \) for an arbitrary geometry, needed to solve self-consistently Eqs. (1), is not known in four dimensions. What one usually does is to evaluate \( \langle T_{\mu\nu} \rangle \) in a fixed classical background, such as for instance the Schwarzschild spacetime (but even in this case one has to resort to approximations), and then use it as source in (1) to compute the \( O(\hbar) \) corrections to the classical geometry. This provides a good approximation to the exact semiclassical solution when the quantum terms are small compared to the classical ones.

Unfortunately, in the physically most interesting situation, the black holes evaporation process [2], the fixed background approximation cannot always be trusted. In particular, it looses validity at the late stages of the evolution, and this prevents us to make quantitative predictions about the fate of quantum black holes. One might object that when the black hole reaches the Planck scale a full quantum gravitational treatment is needed to properly address this issue. To find a way out to this problem one usually argues that quantum gravitational effects should always be negligible compared to those due to a large number \( N \) of matter fields.

In this (almost) hopeless situation, three and a half years ago a new framework was proposed [3] which allows in principle to address this difficult issue in an apparently very different context,
namely braneworld models in Anti-de Sitter space (in particular, the Randall-Sundrum model RS2 [4]). The tool used is the so called holographic interpretation in AdS braneworlds, which results from the application of Maldacena’s AdS/CFT correspondence [5] to RS2. This is briefly reviewed in Section 2. Application to the search of black holes localised on the brane (Section 3) is quite straightforward, and leads to the exciting possibility that the quantum properties of 4D black holes can be understood by solving 5D classical Einstein equations. An important verification of the holographic interpretation in AdS braneworlds is the computation of the correction to the Newtonian potential \( \phi \) on the brane (Section 4), which from the AdS/CFT perspective can be calculated in two very different, but equivalent ways, i.e. classical in 5D [6] and quantum mechanical in 4D [7], both in the weak field limit. Using the semiclassical framework Eqs. (1) (Sections 5 and 6) we shall see how, via a numerical computation of \( \langle T_{\mu\nu} \rangle \) in the Schwarzschild spacetime for matter fields in the (zero temperature) Boulware state, to rederive the result for \( \phi \) (a nontrivial computation indeed) from the asymptotic quantum corrected Schwarzschild geometry. Moreover, since our approach allows to go beyond the weak field limit we shall make a prediction about the existence and the spacetime structure of large mass black holes in AdS braneworlds. Finally, in Section 7 we end by highlighting that this new framework gives us the way to overcome the technical difficulties in the semiclassical theory to deal with evaporating black holes.

2. Holographic interpretation in AdS braneworlds

The AdS/CFT duality [5] predicts a one-to-one correspondence between a quantum gravity theory defined in Anti-de Sitter space and a Conformal Field Theory living in its boundary at infinity. The relevant duality we shall be interested in is between Type IIB String Theory in \( \text{AdS}_5 \times S^5 \) (which arises as the near-horizon limit of \( N \) coincident D3-branes) and \( \mathcal{N} = 4 \) SU(\( N \)) super Yang-Mills theory. At leading order (large \( N \) or, equivalently, large \( \text{AdS} \) length \( L \)) the correspondence is between a classical theory in the bulk and a strongly coupled CFT in the ‘t Hooft planar limit \( \lambda \equiv g_{\text{YM}}^2 N \gg 1 \). To apply this formalism to AdS braneworlds we shall need a further ingredient, which is provided by the so called UV-IR connection [8]. By expressing AdS metric in Poincaré coordinates

\[
ds^2 = \frac{L^2}{z^2} \left[ \eta_{\mu\nu} dx^\mu dx^\nu + dz^2 \right],
\]

where (timelike) infinity corresponds to \( z = 0 \), an infrared cutoff in (2) at \( z = \delta \) corresponds to a UV cutoff in the dual CFT at distances \( \delta \).

On the other hand, in the Randall-Sundrum model RS2 [4] our universe (the brane) is identified with a hypersurface in five dimensional AdS space (the bulk). In the language of modified AdS/CFT (called cutoff AdS/CFT [9]) the brane becomes the boundary of AdS space and its position fixes the cutoff scale \( \delta = L \). In addition, by studying the linear gravitational perturbations \( h_{\mu\nu} \) around the Randall-Sundrum vacuum (2) one finds that the dependence on the extra-dimensional coordinate \( z \) is expressed via the following Schrödinger type equation (the brane is now placed at \( z = 0 \))

\[
\left[ -\partial_z^2 + V(z) \right] h_{\mu\nu} = m^2 h_{\mu\nu},
\]

where \( V(z) = \frac{15}{8L^2(z^2/L^2 + 1)} - \frac{3}{2L} \delta(z) \) is called the Minkowski volcano potential. The zero mode \( m = 0 \) gets trapped on the brane, reproducing 4D gravity there, and moreover there is a continuum of Kaluza-Klein modes with no gap.

To sum up the results presented in this section, we have seen that the RS2 model provides a realization of the cutoff AdS/CFT correspondence with the brane, our universe, playing the role of the boundary of AdS space. In the brane we have both the dual cutoff CFT and Einstein
gravity. In the large $N$ (or $L$) limit this leads to the formulation of the holographic interpretation in RS braneworlds, which states that the dual of the classical bulk theory is a (strongly coupled) quantum CFT in the planar limit coupled to classical gravity. In this view the bulk KK modes are dual to the brane CFT modes. We will see a concrete realization of this last duality in the computation of the correction to the Newtonian potential in Section 4.

3. The holographic conjecture for black holes on the brane

It is now quite straightforward to apply the formalism just presented to the search of black hole solutions in AdS braneworlds [3]: 4D black holes (with large mass) localised on the brane found by solving the classical bulk equations in $AdS_5$ are quantum corrected black holes and not classical ones. By quantum corrected black holes we mean, of course, solutions to the semiclassical Einstein equations (1). This conjecture has great appeal, as it allows to use known semiclassical results to predict the behavior of classical 5D solutions (see Section 6) and the other way around, i.e. use of the bulk classical Einstein equations to overcome the technical difficulties due to the incomplete knowledge of Eqs. (1).

We must warn, however, that the semiclassical results are mostly derived from the assumption that the matter fields are weakly coupled (free), whereas in this context the CFT is strongly coupled. Despite this fact, there is an excellent agreement between the exact solutions for black holes localised on a 2 brane [10] and the 2+1 semiclassical solutions describing quantum corrected conical singularities and quantum corrected BTZ black holes (see for instance [11]). In the most interesting case of $AdS_5$ with a 3 brane, in [12] the Oppenheimer-Snyder model (collapsing dust) on the brane was considered. Unlike in standard 4D gravity, where due to Birkhoff’s theorem the interior solution for the star can be smoothly matched to a unique exterior represented by Schwarzschild, the effective equations on the brane do not allow the matching with a static exterior. Moreover, despite the many efforts [13, 14] it hasn’t been possible to find static black hole solutions to the bulk equations. One possible explanation is of course that the problem is technically difficult. However, these results can be naturally explained in terms of the holographic conjecture for black holes on the brane: due to the inclusion of quantum effects black holes cannot be static, as they evaporate via the Hawking effect. However, it is important to stress that no actual proof of it has been provided so far in $AdS_5$.

4. Correction to the Newtonian potential $\phi$: 5D classical vs. 4D quantum computation

An important verification of the holographic interpretation in AdS braneworlds concerns the computation of the correction to the Newtonian potential on the brane. This was computed classically in $AdS_5$ in [6] with the result

$$\phi = \frac{M}{r} \left( 1 + \frac{2}{3} \frac{L^2}{r^2} \right),$$

(4)

where the Newtonian term comes from the zero mode in the decomposition (3) and the additional contribution is derived by integrating over all the continuum KK modes. Since these latter are supposed to correspond to the modes of the dual CFT, this same result must be derivable in a 4D quantum context by computing the one graviton exchange diagram with the insertion of matter loops [15]. In the case of conformal fields the calculation was performed long ago by Duff [16]

$$\phi = \frac{M}{r} \left( 1 + \frac{\alpha}{r^2} \right),$$

(5)

where the coefficient $\alpha$ depends on the spin of the field

$$\alpha = \frac{1}{45 \pi} \left( 12N_1 + 3N_{1/2} + N_0 \right).$$

(6)
We observe that the correction term \( \frac{1}{r^3} \) decays in the same way as in (4), but to show complete equivalence between the two results one needs to check the numerical coefficients. This can be done by specifying the matter content of the CFT, i.e. \( N_1 = N^2, N_{1/2} = 4N^2 \) and \( N_0 = 6N^2 \) for \( \mathcal{N} = 4 \) SU(N) SYM, plus the relation \( N^2 = \pi L^2 \) (derived from AdS/CFT and the RS relation between 4D and 5D Newton’s constants). Summing the contribution of all the fields involved (note that this is a weak coupling calculation) the result (4) is exactly reproduced [7].

5. Computation of \( \phi \) via the 4D backreaction equations

We shall now see how nontrivial is to reproduce the result (5) in the semiclassical context. The idea is to read off \( \phi \) directly from the quantum corrected Schwarzschild solution. To solve Eqs. (1) we need an expression for \( \langle T_{\mu \nu} \rangle \) in the Schwarzschild spacetime for, say, a conformal scalar field (only the leading order term is important).

The first problem we face is: what is the correct choice for the quantum state of the matter field? We know we have three physically distinct possibilities. The first is the Boulware state [17], constructed by requiring that at infinity it reduces to Minkowski ground state. In this state \( \langle T_{\mu \nu} \rangle \) vanishes at large \( r \), but is strongly divergent at the horizon. It is possible to construct a state, the Hartle-Hawking state [18], regular at the horizon, but again we pay a price: \( \langle T_{\mu \nu} \rangle \) is nonvanishing asymptotically, where it describes thermal radiation at the Hawking temperature \( T_H = 1/8\pi M \). Physically, this is a black hole in thermal equilibrium with its own radiation. Finally, the most interesting case is the Unruh state [19], which reproduces the late-time properties of the radiation emitted in the process of black hole formation (Hawking radiation). Clearly, only the Boulware vacuum gives a semiclassical configuration which is static and asymptotically flat (Hartle-Hawking gives a static non asymptotically flat solution and Unruh an asymptotically flat time dependent one). This is indeed what we need for our purposes.

Let us now look at the analytic approximations proposed in the literature for \( \langle T_{\mu \nu} \rangle \) in Boulware state to check the leading order behavior. Brown and Ottewill [20] and Frolov and Zelnikov [21] derived two distinct expressions valid only for the conformal case, the first using the transformation properties of the stress tensor in two conformally related spacetimes and the second, the Killing approximation, writing down the most general tensor with the correct trace anomaly via the Killing vector, the curvature and their derivatives up to a certain order. Anderson, Hiscock and Samuel [22] developed an approximation valid for all couplings \( \xi \) with the scalar curvature based on the WKB approximation. They all agree that the leading term is

\[
\langle T_{\mu \nu} \rangle \sim O(M^2/r^6),
\]

which was considered to be a reasonable result since it is of the same order as the trace anomaly. However, it is not difficult to show that this would give a quantum correction to \( \phi \) of the order \( M^2/r^4 \) and not \( M/r^3 \) as in (5)!

A numerical computation was then performed in [23] till large values of \( r \) based on the method developed in [22]. The results of this investigation indicate that the correct leading order term is not (7) but rather \( M/r^5 \). Moreover, fitting the numerical results up to two significant digits we find that

\[
\begin{align*}
\langle T_t^t \rangle &= \frac{M}{60\pi^2 r^5} \left( \xi - \frac{1}{6} \right) \frac{3M}{4\pi^2 r^5}, \\
\langle T_r^r \rangle &= \frac{M}{120\pi^2 r^5} \left( \xi - \frac{1}{6} \right) \frac{3M}{4\pi^2 r^5}, \\
\langle T_{\phi \phi} \rangle &= -\frac{M}{80\pi^2 r^5} + \left( \xi - \frac{1}{6} \right) \frac{3M}{8\pi^2 r^5}.
\end{align*}
\]
It is now straightforward to insert the above expressions into Eqs. (1) to find the asymptotic quantum corrected Schwarzschild metric

\[ ds^2 \simeq - \left[ 1 - \frac{2M}{r} \left( 1 + \frac{\alpha}{r^2} \right) \right] dt^2 + \left[ 1 - \frac{2M}{r} \left( 1 + \frac{\beta}{r^2} \right) \right]^{-1} dr^2 + r^2 d\Omega^2 , \]  

(9)

where

\[ \alpha = \frac{1}{45\pi} - \left( \xi - \frac{1}{6} \right) \frac{1}{6\pi} , \quad \beta = \frac{1}{30\pi} + \left( \xi - \frac{1}{6} \right) \frac{1}{2\pi} , \]  

(10)

and read from the \( g_{tt} \) component the result for \( \phi \)

\[ \phi = \frac{M}{r} \left( 1 + \frac{\alpha}{r^2} \right) . \]  

(11)

Remarkably, our result (11) agrees with (5) in the conformal case \( \xi = 1/6 \) and also matches the known result [24] in the minimally coupled case \( \xi = 0 \). Of course, to completely reproduce the Randall Sundrum result (4) one would need to perform the numerical analysis for spin 1/2 and spin 1 fields as well.

6. Prediction about the existence of large mass static braneworld black holes

The successful check discussed in the previous section gives us the confidence to try to go beyond the weak field limit, which is the major limitation of the previous methods encountered, and make a prediction about the existence of static braneworld black holes with large mass. It is well known that at the horizon \( r_H = 2M \) the Boulware state stress tensor diverges as [25]

\[ \langle T_{\mu\nu} \rangle \sim \frac{2N_1 + \frac{4}{3}N_{1/2} + N_0}{30 \frac{212\pi^2 M^4 f^2}{f^2}} (1, -1/3, -1/3, -1/3) , \quad f = (1 - 2M/r) . \]  

(12)

It would be interesting to understand whether and how strong coupling effects modify this result. Naive insertion of (12) in Eqs. (1) indicates that the quantum terms destroy the horizon and replace it with a curvature singularity. This is confirmed by a careful numerical analysis of the backreaction equations in the s-wave approximation [26]. The quantum corrected solution is very similar to Schwarzschild till very close to \( r = 2M \), but then big differences emerge: in particular there appears a bouncing point for the radial function \( r \) (the radius of the two-spheres), which prevents the formation of an event horizon, and, beyond it, a curvature singularity.

But do we expect, on physical grounds, quantum effects to be large at the horizon for a large mass black hole? Since all curvature invariants are small the natural answer is no, at least in the semiclassical approximation. What the peculiar properties of the static solution just discussed indicate is that the natural thing for a black hole is to be time dependent and evaporate. Indeed, the usual physical interpretation of the Boulware state is that is describes vacuum polarization due to matter fields around a static star (whose radius is bigger than \( 2M \)) and not a black hole. One possible exception to this conclusion could be if the system has charges (say, gauge charges) allowing the presence of zero-temperature configurations. Indeed, it has been shown in [27] that the stress energy tensor is regular at the horizon of an extremal Reissner-Nordström black hole.

Coming back, finally, to the dual problem in 5D we are thus led to the following prediction: either macroscopic static braneworld black holes do not exist or, if they do, they have zero temperature.

7. Open questions

The main qualitative improvement holography is giving us in our effort to understand the quantum properties of black holes is to provide a well defined set of equations (five dimensional Einstein equations with a negative cosmological constant), as opposed to the incomplete
knowledge of the exact form of Eqs. (1). Of course its resolution in the physically interesting cases is not an easy task, and the main results will probably come via a numerical analysis. Kudoh, Tanaka and Nakamura [14] gave numerical evidence for the existence of small static 5D black holes \( r_H \sim L \), but not of large ones. According to the holographic conjecture the deviations from staticity should be explained in terms of Hawking radiation in the dual theory. A check of this claim will give strong quantitative support to the conjecture.

The most important application is to solve the full 5D equations for the time dependent problem corresponding to gravitational collapse on the brane. This is a hard problem, but, possibly, with a high reward: to understand the details of black holes evaporation and, maybe, to have some clues concerning the information loss paradox.

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