Mixing rates of particle systems with energy exchange

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Abstract

A fundamental problem of non-equilibrium statistical mechanics is the derivation of macroscopic transport equations in the hydrodynamic limit. The rigorous study of such limits requires detailed information about rates of convergence to equilibrium for finite sized systems. In this paper, we consider the finite lattice \{1, 2, \ldots, N\}, with an energy $x_i \in (0, \infty)$ associated with each site. The energies evolve according to a Markov jump process with nearest neighbour interaction such that the total energy is preserved.

We prove that for an entire class of such models the spectral gap of the generator of the Markov process scales as $O(N^{-2})$. Furthermore, we provide a complete classification of reversible stationary distributions of product type. We demonstrate that our results apply to models similar to the billiard lattice model considered in Gaspard and Gilbert (2009 J. Stat. Mech.: Theory Exp. 2009 24), and hence provide a first step in the derivation of a macroscopic heat equation for a microscopic stochastic evolution of mechanical origin.

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1. Introduction

1.1. Motivation and related works

A fundamental problem of non-equilibrium statistical mechanics is the derivation of effective equations in the hydrodynamic limit. Often these are hydrodynamic equations (Euler, Navier–Stokes), or related transport equations (Burgers equation, heat equation). There are very few models for which rigorous results exist. They include particle models like simple exclusion, zero range processes, see [25] and references therein, and continuous systems like the Ginzburg–Landau equation [15, 16, 21] and the model of [26].
The rigorous study of hydrodynamic limits requires detailed information about rates of convergence to equilibrium for finite sized systems, especially if the system is of non-gradient type. In particular, the scaling of the spectral gap of the generator with the system size \( N \) is of crucial importance. Obtaining good estimates (in terms of the system size) on the spectral gap of the generator is highly non-trivial. For example, to obtain the corresponding results for the Kac model [24] it took almost half a century [23] (using Yau’s martingale method [34, 35]) and [6–9, 20].

Recently there has been a growing interest in and hope for establishing hydrodynamic limits for systems that are either purely deterministic or originate (somehow) from deterministic, in particular mechanical, models. A program to obtain information about the stationary distributions under the influence of stochastic boundary conditions was proposed in [14]. Another approach was suggested in the recent series of papers [17, 19] for a billiard lattice model, whose ergodicity was proved in [3]. We note that history shows that the advantage of billiard models is that, while being mathematically rather technical, they seem to represent a family of mechanical systems for which obtaining rigorous proofs is the most realistic. Some examples are the works of [31] on ergodicity of semi-dispersing billiards; of [30] on that of hard ball systems; of [36] and [11] on exponential correlation decay for 2D dispersing billiards; of [4] and of [32] on diffusion and, respectively, superdiffusion of 2D Lorentz processes; and of [2] on derivation of the linear Boltzmann equation for the 2D Lorentz process; see also the collection [33].

The approach proposed in [17, 19] is based on a two-step procedure. In the first step of the proposed two-step procedure the deterministic dynamics is rescaled, without assuming additional stochastic boundary conditions, to obtain a mesoscopic stochastic model (also referred to as master equation). In a second step the hydrodynamic behaviour of the mesoscopic stochastic model should be derived.

For neither of the two steps proposed in [17, 19] rigorous results are available. Deriving master equations from interacting mechanical models is a very difficult problem. Only recently some rigorous results in this direction were obtained in [13], where the weak interaction limit is considered opposed to the rare interaction limit of [17, 19]. The second step, i.e. deriving the hydrodynamic limit from the master equation, seems a much more tractable mathematical problem. This paper is a first attempt in this direction by providing information about the spectral gap of the generator of an entire class of models, which are of similar type as the master equation of the billiard lattice model considered in [17, 19]. Moreover, the order of dependence of this bound on the system size is optimal. In particular, the energy exchange mechanism of the model [19] belongs to the class of models we are considering, and the obtained spectral bound is exactly the necessary one for which the derivation of the hydrodynamic limit is feasible.

1.2. Outline of the paper

In section 1.3 we give a detailed description of the class of models we consider in this paper. Informally these models can be described as follows. To each site of the lattice \( \{1, \ldots, N\} \) carries a positive real value, called energy, and to each pair of neighbouring sites we associate independent Poissonian clock with rate function \( \Lambda \). Whenever a clock rings the corresponding pair of energies is updated according to a transition kernel \( P \), which preserves the total energy. Then the clocks are reset, so that evolution of the energies defines a Markov process whose infinitesimal generator is precisely defined in (1).

The purpose of this paper is to present a dynamical and geometric approach to establish the scaling of the spectral gap of the generator (1) with respect to the lattice size \( N \) under rather
general assumptions on the rate function $\Lambda$ and transition kernel $P$. The strategy we adopt is as follows. In section 3 we show that for a large class of rates $\Lambda$ and transition operators $P$ the scaling of the spectral gap of the corresponding generator (1) can be obtained by considering only the special case of a constant rate $\Lambda$ and a state independent transition kernel $P$. The precise statement is formulated in theorem 3.1, which we prove under the two key assumptions: the reversibility of the process $X(t)$, and the existence of a lower bound on the rate function $\Lambda$. The requirement of a lower bound on the rate function seems to be a technical condition, but it cannot be removed at present.

In section 5 we show that (a slight modification of) the three-dimensional stochastic billiard lattice model of [19] is a special case of the general model considered in this paper, provided that one introduces a lower cut-off for the rate function originally considered in [19]. In particular, we show that it then follows that the spectral gap scales as $O(N^{-2})$.

Since we assume reversibility of the stationary distribution to derive the spectral properties, we provide in section 4 a classification of reversible stationary distributions of product type. Such measures are of particular interest in the hydrodynamic limit, and appear naturally in mechanical models and statistical mechanics in form of Gibbs measures. We show in theorem 4.3 that if a model of the class (1) considered in this paper admits a reversible product distributions, then this measure must necessarily be a product Gamma distributions (or a single atom). This is precisely the type of product measures considered in statistical mechanics for mechanical models.

The main part of the paper deals with establishing the scaling of the spectral gap of the generator for the process with constant rates $\Lambda$ and state independent transition kernel $P$. This case is studied in section 2. The key difference of our analysis when compared with the above-mentioned related works is that instead of focusing directly on $L^2$ convergence, for example by analysing the associated Dirichlet form, we first establish weak convergence towards a stationary distribution. For the later part it is crucial that this weak convergence is made quantitative in a sufficiently strong metric for the weak topology. For this purpose we use the Vaserstein distance and prove in theorem 2.9 that there is an exponential rate of convergence of $X(t)$ to equilibrium, which scales as $O(N^{-2})$ in the system size $N$. The key step in the proof is to construct an adapted metric on the state space of $X(t)$, for which the contraction property can be established. This requires special coordinates and a coupling argument, which is presented in propositions 2.6 and 2.8.

The advantage of first establishing exponential convergence in the weak sense is that it allows us to include very general transition kernels $P$ (for example, non-absolutely continuous kernels), and does not make reference to the invariant measure. Instead it relies on a very natural geometric property of the interaction mechanism of $X(t)$.

In a second step we assume reversibility of the constructed unique invariant measure, and show that the $L^2$ convergence occurs at an exponential rate, which is explicitly related to the rate of convergence in Vaserstein metric. In particular, this shows that the spectral gap scales as $O(N^{-2})$ in the lattice size $N$. The precise statement is given in theorem 2.12, whose proof relies on the Kantorovich–Rubinstein duality property of the Vaserstein metric, see lemma 2.11. This is another manifestation of the usefulness of the weak convergence in Vaserstein distance in the study of the spectral gap for interacting particle systems.

Section 6 contains final comments and conclusions.

1.3. Description of the model

The model we consider in this paper is as follows. Let $N \geq 2$ be an integer, and consider the lattice $\{1, 2, \ldots, N\}$. To every site $i$ of this lattice we associate an energy $x_i$, which is a positive
real number. The collection of all the energies will be denoted by \( x = (x_1, \ldots, x_N) \in \mathbb{R}_+^N \). To each nearest neighbour pair of the lattice we associate an independent exponential clock with a rate \( \Lambda \) that depends on the total energy of this pair. As soon as one of the \( N-1 \) clocks rings, say for the pair \((i, i+1)\), then a number \( 0 \leq \alpha \leq 1 \) is drawn, independently of everything else, according to a distribution \( P \), that only depends on the two energies \( x_i, x_{i+1} \). The update of the energies is then such that the new energy at site \( i \) is \( \alpha (x_i + x_{i+1}) \), the new energy at site \( i+1 \) is \( (1-\alpha) (x_i + x_{i+1}) \), and all other energies remain unchanged.

This procedure defines a continuous-time Markov jump process \( X(t) \) on \( \mathbb{R}_+^N \). More formally, we define the process \( X(t) \) by its infinitesimal generator \( \mathcal{L} \), acting on bounded functions \( A: \mathbb{R}_+^N \to \mathbb{R} \) as

\[
\mathcal{L}A(x) = \sum_{i=1}^{N-1} \Lambda(x_i, x_{i+1}) \int P(x_i, x_{i+1}, d\alpha) \left[ A(T_{i,\alpha}x) - A(x) \right],
\]

where \( \Lambda: \mathbb{R}_+^2 \to \mathbb{R}_+ \) is continuous, and \( P(x_i, x_{i+1}, d\alpha) \) is a probability measure on \([0, 1]\), which depends continuously on \((x_i, x_{i+1}) \in \mathbb{R}_+^2 \). The maps \( T_{i,\alpha} \) model the energy exchange between the neighbouring sites \( i \) and \( i+1 \), and are defined by

\[
T_{i,\alpha}(x) = x + [\alpha x_{i+1} - (1-\alpha) x_i][e_i - e_{i+1}],
\]

where \( e_i \) denotes the \( i \)th unit vector of \( \mathbb{R}^N \).

In particular, the process \( X(t) \) preserves the total energy, i.e. for any two times \( t_1 \) and \( t_2 \) the identity \( \sum_{i=1}^{N} X_i(t_1) = \sum_{i=1}^{N} X_i(t_2) \) holds. Therefore, we introduce for any \( \epsilon > 0 \) the sets

\[
S_{\epsilon,N} = \left\{ x \in \mathbb{R}_+^N : \frac{1}{N} \sum_{i=1}^{N} x_i = \epsilon \right\},
\]

which are invariant for the process \( X(t) \). The value of \( \epsilon \) represents the mean energy per site.

Since \( S_{\epsilon,N} \) is compact and invariant the assumed continuity of \( \Lambda \) and \( P \) guarantees the existence of at least one stationary distribution \( \pi_{\epsilon,N} \) for \( X(t) \) on each \( S_{\epsilon,N} \). As we pointed out, the scaling of the rate of convergence towards the stationary distribution in terms of the lattice size \( N \) is of crucial importance in studying the hydrodynamic limit of this model rigorously.

2. Analysis of a special case

In this section, we consider a special case of the class of processes defined by generators of the form (1). Namely we consider the case where the rate function \( \Lambda \) is constant, and the transition kernel \( P \) is state independent. In other words we consider a process \( X(t) \) with infinitesimal generator

\[
\mathcal{L}A(x) = \Lambda \sum_{i=1}^{N-1} \int P(d\alpha) \left[ A(T_{i,\alpha}x) - A(x) \right]
\]

acting on the space of bounded observables \( A: \mathbb{R}_+^N \to \mathbb{R} \).

As was already mentioned the process \( X(t) \) preserves the total energy. This implies that the process cannot have a unique stationary state on all of \( \mathbb{R}_+^N \). However, we will show below that the restriction of the process to any of the invariant sets \( S_{\epsilon,N} \) has a unique stationary distribution.

The first step in this direction is to introduce more convenient coordinates on \( S_{\epsilon,N} \), which is the purpose of the next result.

\(^3\) Throughout this paper we will always assume that the various functions are Borel measurable without stating this assumption explicitly. This will not lead to confusion, since higher regularity assumptions (like continuity or integrability) are stated explicitly.
Lemma 2.1 (x in terms of u). Let $N$ and $\epsilon$ be fixed. Then any $x \in S_{\epsilon,N}$ can be uniquely written as

$$x = \epsilon 1 + \sum_{i=1}^{N-1} u_i [e_i - e_{i+1}]$$

for some $u \in \mathbb{R}^{N-1}$, where $1$ denote the vector $(1, \ldots, 1)$. Furthermore, via this change of coordinates the set $S_{\epsilon,N} \subset \mathbb{R}^N$ is in one-to-one correspondence with the set

$$\hat{S}_{\epsilon,N} = \{u \in \mathbb{R}^{N-1} : -\epsilon \leq u_1, u_{i-1} \leq \epsilon + u_i, u_{N-1} \leq \epsilon\}.$$

Note that the vectors $e_i - e_{i+1}$ for $i = 1, \ldots, N-1$ span the simplex $S_{\epsilon,N}$, but they are not mutually orthogonal. However, they almost are in the sense that any two of them are perpendicular as soon as they correspond to two values of $i$, which differ by at least 2.

In the following we will also need the inverse coordinate transformation, which expresses $u$ in terms of $x$.

Lemma 2.2 (u in terms of x). Let $x \in \mathbb{R}^N$ be given. Then the corresponding $\epsilon$ is given by

$$\epsilon = \sum_{i=1}^N \frac{1}{N} x_i,$$

and the corresponding $u$ is the solution to the discrete Poisson equation with Dirichlet boundary conditions

$$u_{i-1} - 2u_i + u_{i+1} = x_{i+1} - x_i \quad \text{for} \quad i = 1, \ldots, N-1,$$

where we formally set $u_0 \equiv u_N \equiv 0$. More explicitly

$$u_i = \sum_{k=1}^{i} (x_k - \epsilon) = \left[1 - i \frac{1}{N} \right] \sum_{k=1}^{i} x_k - \frac{i}{N} \sum_{k=1}^{i+1} x_k \quad \text{for all} \quad 1 \leq i \leq N-1$$

is the expression for the corresponding $u \in \mathbb{R}^{N-1}$.

Proof. Clearly, $x \in S_{\epsilon,N}$ if and only if $\epsilon$ is given by the claimed formula. Furthermore, it follows immediately from the definition of the coordinates $u$, that $x_i = \epsilon + u_i - u_{i-1}$ for all $i$, where we use the convention $u_0 \equiv u_N \equiv 0$. This implies that $u$ must solve the discrete Poisson equation with zero Dirichlet boundary conditions.

On the other hand, we can sum up the expression for $x_i$ in terms of $u$ and obtain a telescoping sum, which yields

$$u_i = \sum_{k=1}^{i} (u_k - u_{k-1}) = \sum_{k=1}^{i} (x_k - \epsilon)$$

for all $i = 1, \ldots, N-1$.

And since $\epsilon N = \sum_{i=1}^{N} x_i$ we can replace $\epsilon$ in terms of this sum, and thus obtain the second expression for $u_i$. \hfill \Box

The point of the change of coordinates from $x$ to $\epsilon$ and $u$ is to separate out the conserved quantity $\epsilon$, and consider only the evolution of the non-trivial part $U(t)$ of the process $X(t)$

$$X(t) = \epsilon 1 + \sum_{i=1}^{N-1} U_i(t) [e_i - e_{i+1}],$$

namely the $u$-coordinate vector corresponding to $X(t)$. Since $\epsilon$ is conserved it follows that $U(t)$ is also a homogeneous Markov process (for each $\epsilon$ separately). Using the results of lemmas 2.1 and 2.2 we can now derive the infinitesimal generator of $U(t)$. 
Lemma 2.3 (The generator of $U(t)$). Let $N$ and $\epsilon$ be fixed. Then the process $U(t)$ is a homogeneous Markov process on $\hat{S}_{\epsilon,N}$, whose infinitesimal generator $\hat{\mathcal{L}}_{\epsilon,N}$ is given by

$$\hat{\mathcal{L}}_{\epsilon,N}A(u) = \Lambda \sum_{i=1}^{N-1} \int P(d\alpha) [A(\hat{T}^\epsilon_{i,\alpha}u) - A(u)],$$

where

$$\hat{T}^\epsilon_{i,\alpha}u - u = [(1 - \alpha) u_{i-1} + \alpha u_{i+1} + (2\alpha - 1) \epsilon - u_i] e_i \in \mathbb{R}^{N-1}$$

with the convention $u_0 \equiv u_N \equiv 0$.

Proof. From its definition (2) we have $T_{i,\alpha}(x) = x + [\alpha x_{i+1} - (1 - \alpha) x_i] [e_i - e_{i+1}]$. Note that $[T_{i,\alpha}x]_k$ agrees with $x_k$ for all $k$ different from $i$ and $i+1$, and $[T_{i,\alpha}x]_i + [T_{i,\alpha}x]_{i+1}$ equals $x_i + x_{i+1}$ (local energy conservation). Therefore, $[\hat{T}^\epsilon_{i,\alpha}u]_k$ equals $u_k$ for all $k \neq i$, because by lemma 2.2 we have $u_i = \sum_{k=1}^{i} (x_k - \epsilon)$.

So it remains to consider $[\hat{T}^\epsilon_{i,\alpha}u]_i$. Using the above two expressions for $u$ and $T_{i,\alpha}(x)$ we obtain

$$[\hat{T}^\epsilon_{i,\alpha}u]_i - u_i = \sum_{k=1}^{i} ([T_{i,\alpha}(x)]_k - \epsilon) - \sum_{k=1}^{i} (x_k - \epsilon) = [T_{i,\alpha}(x)]_i - x_i$$

$$= \alpha x_{i+1} - (1 - \alpha) x_i.$$

Using lemma 2.1 we can express $x$ in terms of $u$ as $x_i = \epsilon + u_i - u_{i-1}$, where we used the convention $u_0 \equiv u_N \equiv 0$. Substituting this expression in the previous formula yields the claimed expression for $\hat{T}^\epsilon_{i,\alpha}u - u$. Furthermore, this (trivially) also shows the claimed expression for the infinitesimal generator of $U(t)$. \hfill \Box

2.1. Weak convergence

Fix again the values of $\epsilon$ and $N$. To study the existence of and rate of convergence to a stationary distribution we consider a bivariate Markov process $(U(t), U'(t))$ on $\hat{S}_{\epsilon,N} \times \hat{S}_{\epsilon,N}$, whose infinitesimal generator $\hat{\mathcal{L}}$ reads

$$\hat{\mathcal{L}}A(u, u') = \Lambda \sum_{i=1}^{N-1} \int P(d\alpha) [A(\hat{T}^\epsilon_{i,\alpha}u, \hat{T}^\epsilon_{i,\alpha}u') - A(u, u')]$$

for any (bounded) observable $A$ on $\hat{S}_{\epsilon,N} \times \hat{S}_{\epsilon,N}$. Note that this is a special Markov coupling of two copies of the Markov chains generated by $\hat{\mathcal{L}}$.

In order to analyse the weak convergence of the process $X(t)$ towards a stationary distribution we consider the Vaserstein metric on the probability measures on $S_{\epsilon,N}$. This requires, however, a metric $d(\cdot, \cdot)$ on $S_{\epsilon,N}$. We equip $S_{\epsilon,N}$ with the Euclidean metric

$$\hat{d}(u, u') := \left[ \sum_{i=1}^{N-1} (u_i - u'_i)^2 \right]^{1/2},$$

which corresponds to the metric

$$d(x, x') = \left[ \sum_{i=1}^{N-1} \left( \sum_{k=1}^{i} x_k - x'_k \right)^2 \right]^{1/2} \equiv \hat{d}(u, u')$$

on $S_{\epsilon,N}$. In particular, we then have following estimate on the diameter of $S_{\epsilon,N}$. 

Lemma 2.4 (Diameter of $S_{\epsilon,N}$). Let $\epsilon$ and $N$ be fixed. Then

$$\max_{x,x' \in S_{\epsilon,N}} d(x, x') = \max_{u,u' \in \hat{S}_{\epsilon,N}} \hat{d}(u, u') \leq \epsilon N \sqrt{N-1}$$

holds.

Proof. By lemma 2.1 it follows that for any $u \in \hat{S}_{\epsilon,N}$ the inequality $-\epsilon \leq u_i \leq \epsilon (N - i)$ holds for all $i = 1, \ldots, N - 1$. Therefore,

$$\hat{d}(u, u')^2 = \sum_{i=1}^{N-1} (u_i - u'_i)^2 \leq \sum_{i=1}^{N-1} (\epsilon)^2 (N - i)^2 = \epsilon^2 N^2 (N - 1)$$

for any two $u$ and $u'$, which implies the claim. \qed

The following proposition 2.6 provides the first step to estimate $\hat{d}(U(t), U'(t))$. A particular role will be played by the matrix

$$C(N) = \begin{pmatrix}
2 & 0 & -1 \\
0 & 2 & 0 & -1 \\
-1 & 0 & 2 & 0 & -1 \\
& \ddots & \ddots & \ddots & \ddots \\
-1 & 0 & 2 & 0 & -1 \\
-1 & 0 & 2 & 0 & 2
\end{pmatrix} \in \mathbb{R}^{N \times N}. \quad (7)
$$

The spectral properties of $C^{(N-1)}$ are provided by the following lemma 2.5.

Lemma 2.5 (Spectrum of $C^{(N-1)}$). If $N$ is odd, then the eigenvalues of $C^{(N-1)}$ are given by

$$4 \sin^2 \left[ \frac{\pi k}{N+1} \right] \quad \text{for} \quad k = 1, \ldots, \frac{N-1}{2},$$

where each has multiplicity two. If $N$ is even, then the eigenvalues of $C^{(N-1)}$ are given by

$$4 \sin^2 \left[ \frac{\pi k}{N} \right] \quad \text{for} \quad k = 1, \ldots, \frac{N}{2} - 1,$$

$$4 \sin^2 \left[ \frac{\pi k}{N+2} \right] \quad \text{for} \quad k = 1, \ldots, \frac{N}{2},$$

each of multiplicity one.

Proof. By the definition of $C^{(N)}$ we see that the even and odd indices separate. In fact, it is readily seen that the action of $C^{(N)}$ on the odd indexed $(u_1, u_3, \ldots)$ and the even indexed $(u_2, u_4, \ldots)$ entries of $u$ is given by the action of the tridiagonal matrix $A$ with diagonal entries equal to 2 and the off diagonal entries equal to $-1$

$$A = \begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
& \ddots & \ddots & \ddots \\
& & -1 & 2 & -1 \\
& & & -1 & 2
\end{pmatrix}.$$
It is readily verified that if $A \in \mathbb{R}^{m \times m}$, then for $k = 1, \ldots, m$ the vectors $(\sin[k \pi k m^{-1}], \ldots, \sin[k \pi k m^{-1}])$ are eigenvectors of $A$ corresponding to the eigenvalues
\[
4 \sin^2 \left( \frac{\pi k}{2 (m + 1)} \right) \quad \text{for} \quad k = 1, \ldots, m.
\]

If $N$ is odd, say $N = 2m + 1$ for some $m \geq 1$, then there are $m$ odd and $m$ even indexed entries in $u \in \mathbb{R}^{N-1}$. Therefore, the eigenvalues of $C^{(2m)}$ are given by
\[
4 \sin^2 \left( \frac{\pi k}{2 (m + 1)} \right) \quad \text{for} \quad k = 1, \ldots, m
\]
where each has multiplicity two.

If $N$ is even, say $N = 2m + 2$ for some $m \geq 1$, then there are $m + 1$ odd and $m$ even indexed entries in $u \in \mathbb{R}^{N-1}$. Therefore, the eigenvalues of $C^{(2m+1)}$ are given by
\[
4 \sin^2 \left( \frac{\pi k}{2 (m + 1)} \right) \quad \text{for} \quad k = 1, \ldots, m,
\]
\[
4 \sin^2 \left( \frac{\pi k}{2 (m + 2)} \right) \quad \text{for} \quad k = 1, \ldots, m + 1,
\]
where each has multiplicity one.

\[\square\]

**Proposition 2.6 (Average contraction rate).** Assume that the transition kernel $P$ satisfies
\[\int P(d\alpha) \alpha = \frac{1}{2}.\]
Then
\[\mathcal{L} \hat{d}(u, u')^2 \leq -\Lambda [1 - 4 \sigma_P^2] \sin^2 \left( \frac{\pi}{N + 2} \right) \hat{d}(u, u')^2\]
holds for any two states $u$ and $u'$, where $\sigma_P^2$ denotes the variance of $P$.

**Remark 2.7.** Since $P$ is supported on $[0, 1]$ and is assumed to have mean $\int P(d\alpha) \alpha = \frac{1}{2}$ it follows that the variance of $P$ satisfies $0 \leq 1 - 4 \sigma_P^2 \leq 1$.

**Proof of proposition 2.6.** From the definition of the generator $\mathcal{L}$ and the distance $\hat{d}(\cdot, \cdot)$ it follows
\[\mathcal{L} \hat{d}(u, u')^2 = \Lambda \sum_{i=1}^{N-1} \int P(d\alpha) [\hat{d}(\hat{T}_{i,a}^\varepsilon u, \hat{T}_{i,a}^\varepsilon u')^2 - \hat{d}(u, u')^2]\]
and
\[\hat{d}(\hat{T}_{i,a}^\varepsilon u, \hat{T}_{i,a}^\varepsilon u')^2 - \hat{d}(u, u')^2 = \sum_{k=1}^{N-1} \left[ (\hat{T}_{i,a}^\varepsilon u)_k - (\hat{T}_{i,a}^\varepsilon u')_k \right]^2 - (u_k - u'_k)^2\]
\[= \sum_{k=1}^{N-1} \left[ (\hat{T}_{i,a}^\varepsilon u - u)_k - (\hat{T}_{i,a}^\varepsilon u' - u')_k \right] \cdot \left[ (\hat{T}_{i,a}^\varepsilon u - u)_k - (\hat{T}_{i,a}^\varepsilon u' - u')_k + 2 (u_k - u'_k) \right].\]

Making use of the explicit expression for $\hat{T}_{i,a}^\varepsilon u - u$ provided by lemma 2.3
\[\hat{T}_{i,a}^\varepsilon u - u = [(1 - \alpha) u_{i-1} + \alpha u_{i+1} + (2 \alpha - 1) \epsilon - u_i] e_i\]
the above sum simplifies to
\[
\hat{d}(\hat{T}_{i_2,\alpha} u, \hat{T}_{i_1,\alpha}' u')^2 - \hat{d}(u, u')^2 = \left[ (1 - \alpha) [u_{i_2} - u_{i_1}' - u_{i_1} - u_{i_2}'] \right] \\
\cdot \left[ \hat{T}_{i_2,\alpha} u - u_{i_1} - [\hat{T}_{i_1,\alpha}' u' - u_{i_1}'] + 2 (u_i - u_i') \right] \\
= \left[ (1 - \alpha) [u_{i_2} - u_{i_1}' - u_{i_1} - u_{i_2}'] \right] \\
\cdot \left[ (1 - \alpha) [u_{i_1} - u_{i_1}' - u_{i_1} - u_{i_2}'] \right] + \alpha [u_{i_1} - u_{i_1}' + u_{i_2} - u_{i_2}'] \\
= \left[ (1 - \alpha) [u_{i_2} - u_{i_1}' - u_{i_1} - u_{i_2}'] \right] \left[ (1 - \alpha) [u_{i_1} - u_{i_1}' - u_{i_1} - u_{i_2}'] \right] \\
+ 2 \alpha (1 - \alpha) [u_{i_1} - u_{i_1}' - u_{i_1} - u_{i_2}'] [u_{i_1} - u_{i_1}'] \quad [u_{i_1} - u_{i_2}'],
\]
which in particular shows that the above expression depends only on the difference vector \( u - u' \).

Performing now the sum over \( i \) yields
\[
\sum_{i=1}^{N-1} \left[ \hat{d}(\hat{T}_{i_2,\alpha} u, \hat{T}_{i_1,\alpha}' u')^2 - \hat{d}(u, u')^2 \right] = (1 - \alpha)^2 \sum_{i=1}^{N-2} \left[ u_i - u_i' \right]^2 + \alpha^2 \sum_{i=2}^{N-1} \left[ u_i - u_i' \right]^2 \\
+ \alpha (1 - \alpha) \sum_{i=2}^{N-2} 2 \left[ u_{i-1} - u_{i-1}' \right] \left[ u_{i+1} - u_{i+1}' \right] - \sum_{i=1}^{N-1} \left[ u_i - u_i' \right]^2,
\]
where we made use of the convention \( u_0 \equiv u_N \equiv u_0' \equiv u_N' \equiv 0 \).

Note that the assumption \( \int P(\text{d}\alpha) \alpha = \frac{1}{2} \) implies
\[
\int P(\text{d}\alpha) \alpha^2 = \int P(\text{d}\alpha) (1 - \alpha)^2 = \sigma^2_P + \frac{1}{2}, \quad \int P(\text{d}\alpha) \alpha (1 - \alpha) = \frac{1}{2} - \sigma^2_P
\]
and hence
\[
\frac{1}{\Lambda} \mathcal{E} \left[ \hat{d}(u, u')^2 \right] = \int P(\text{d}\alpha) (1 - \alpha)^2 \sum_{i=1}^{N-2} \left[ u_i - u_i' \right]^2 + \int P(\text{d}\alpha) \alpha^2 \sum_{i=2}^{N-1} \left[ u_i - u_i' \right]^2 \\
+ \int P(\text{d}\alpha) \alpha (1 - \alpha) \sum_{i=2}^{N-2} 2 \left[ u_{i-1} - u_{i-1}' \right] \left[ u_{i+1} - u_{i+1}' \right] - \sum_{i=1}^{N-1} \left[ u_i - u_i' \right]^2 \\
= \frac{1 - 4 \sigma^2_P}{4} \left[ \sum_{i=1}^{N-1} \left[ u_i - u_i' \right]^2 - \sum_{i=2}^{N-2} 2 \left[ u_{i-1} - u_{i-1}' \right] \left[ u_{i+1} - u_{i+1}' \right] \right] \\
- \frac{1 + 4 \sigma^2_P}{4} \left[ \left[ u_1 - u_1' \right]^2 + \left[ u_{N-1} - u_{N-1}' \right]^2 \right].
\]
It is now straightforward to verify that
\[
\mathcal{E} \left[ \hat{d}(u, u')^2 \right] = \frac{1 - 4 \sigma^2_P}{4} \left[ u - u' \right]^T C^{(N-1)} \left[ u - u' \right] \\
- \frac{1 + 4 \sigma^2_P}{4} \left[ \left[ u_1 - u_1' \right]^2 + \left[ u_{N-1} - u_{N-1}' \right]^2 \right],
\]
where the matrix \( C^{(N-1)} \) was defined in (7).
Observe that by lemma 2.5 the smallest eigenvalue of $C^{(N-1)}$ equals $4 \sin^2 \left[ \frac{\pi}{N+2} \right]$ if $N$ is odd, and $4 \sin^2 \left[ \frac{\pi}{N+2} \right]$ if $N$ is even. Therefore,

$$
\tilde{L}[\hat{d}(u, u')] \leq -\Lambda \frac{1 - 4 \sigma_p^2}{4} [u - u']^T C^{(N-1)} [u - u']
$$

follows from the fact that $C^{(N-1)}$ is a symmetric matrix, and $0 \leq 1 - 4 \sigma_p^2$.

Proof. Let $U$ and $U'$ be any two random variables on $\mathcal{S}_{\epsilon,N}$ with distribution denoted by $\mu$ and $\mu'$, respectively. Recall that for $p \geq 1$ the Vaserstein-$p$ distance is defined by

$$
\rho_p(U, U') \equiv \rho_p(\mu, \mu') = \inf \Gamma \left[ \int_{\mathcal{S}_{\epsilon,N} \times \mathcal{S}_{\epsilon,N}} \Gamma(du, du') \hat{d}(u, u')^p \right]^{\frac{1}{p}},
$$

where the infimum is taken over all couplings $\Gamma$ of $\mu$ and $\mu'$. To shorten the notation we set $\rho(\mu, \mu') \equiv \rho_1(\mu, \mu')$ in the special case $p = 1$.

Proposition 2.8 (Rate of convergence in Vaserstein-$2$ distance). Assume that the transition kernel $P$ satisfies $\int P(du)\alpha = \frac{1}{2}$. Let $U(t)$ and $U'(t)$ be two Markov chains generated by $\hat{L}$ on $\mathcal{S}_{\epsilon,N}$. Then for all $t \geq 0$

$$
\rho_2(U(t), U'(t)) \leq \rho_2(U(0), U'(0)) \exp \left( -\frac{1}{2} \Lambda [1 - 4 \sigma_p^2] \sin^2 \left[ \frac{\pi}{N+2} \right] t \right)
$$

Proof. Denote the distribution of the bivariate Markov process $(U(t), U'(t))$ with generator $\hat{L}$ by $\Gamma_t(du, du')$, and denote by $\mu_t(du)$ and $\mu'_t(du')$ the two marginals.

Observe that the generator $\hat{L}$ of this bivariate process $(U(t), U'(t))$ is constructed in such a way that $U(t)$ and $U'(t)$ are Markov chains with generator $\hat{L}$ whose distributions are given by $\mu_t(du)$ and $\mu'_t(du')$, respectively.

Therefore, $\Gamma_t(du, du')$ is a coupling of the two distributions $\mu_t(du)$ and $\mu'_t(du')$ for all $t \geq 0$. In particular,

$$
\rho_2(U(t), U'(t))^2 \leq \int_{\mathcal{S}_{\epsilon,N} \times \mathcal{S}_{\epsilon,N}} \Gamma_t(du, du') \hat{d}(u, u')^2
$$

follows from the very definition of the Vaserstein distance.

By the Markov property of the bivariate chain

$$
\hat{d}(U(t), U'(t))^2 - \hat{d}(U(0), U'(0))^2 - \int_0^t \hat{L} \hat{d}(U(s), U'(s))^2 ds
$$

is a centred martingale. Hence for all $t \geq 0$

$$
\mathbb{E} \hat{d}(U(t), U'(t))^2 = \mathbb{E} \hat{d}(U(0), U'(0))^2 + \int_0^t \mathbb{E} \hat{L} \hat{d}(U(s), U'(s))^2 ds.
$$

Differentiating with respect to $t$ and applying the estimate of proposition 2.6 yields

$$
\frac{d}{dt} \mathbb{E} \hat{d}(U(t), U'(t))^2 \leq -\Lambda [1 - 4 \sigma_p^2] \sin^2 \left[ \frac{\pi}{N+2} \right] \mathbb{E} \hat{d}(U(t), U'(t))^2.
$$

Gronwall’s inequality shows that

$$
\rho_2(U(t), U'(t))^2 \leq \mathbb{E} \hat{d}(U(0), U'(0))^2 \exp \left( -\Lambda [1 - 4 \sigma_p^2] \sin^2 \left[ \frac{\pi}{N+2} \right] t \right) \mathbb{E} \hat{d}(U(0), U'(0))^2
$$

for any initial distribution $\Gamma_0$ of the bivariate chain.
Taking in the infimum over all couplings $\Gamma_0$ of $\mu_0$ and $\mu'_0$ yields

$$\rho_2(U(t), U'(t))^2 \leq \exp\left(-\Lambda [1 - 4 \sigma^2_\mathcal{F} \sin^2 \left(\frac{\pi}{N + 2}\right)t\right) \rho_2(U(0), U'(0))^2$$

$$\leq \epsilon^2 N^2 (N - 1) \exp\left(-\Lambda [1 - 4 \sigma^2_\mathcal{F} \sin^2 \left(\frac{\pi}{N + 2}\right)t\right),$$

where the second inequality is due to the estimate on the diameter of $\mathcal{S}_{\epsilon,N}$ provided in lemma 2.4.

By definition of the metric $d(., .)$ on $\mathcal{S}_{\epsilon,N}$ in terms of $\hat{d}(., .)$ it follows immediately from proposition 2.8 that there is at most one stationary distribution for $X(t)$ on each $\mathcal{S}_{\epsilon,N}$, and that the rate of convergence in the associated Vaserstein distance is the same as the rate of convergence for $U(t)$.

Furthermore, by assumption the process $X(t)$ on $\mathcal{S}_{\epsilon,N}$ generated by $\mathcal{L}$ is Feller. Hence the compactness of $\mathcal{S}_{\epsilon,N}$ guarantees that there is at least one stationary distribution. This proves the following theorem 2.9.

**Theorem 2.9 (Ergodicity and mixing rate of $X(t)$ on each $\mathcal{S}_{\epsilon,N}$).** If the transition kernel $P$ satisfies $\int P(dx) \alpha = \frac{1}{2}$, and $\sigma^2_\mathcal{F} < \frac{1}{2}$, then there exists a unique stationary distribution $\pi_{\epsilon,N}$ on $\mathcal{S}_{\epsilon,N}$. Furthermore,

$$\rho_2(X(t), \pi_{\epsilon,N}) \leq \rho_2(X(0), \pi_{\epsilon,N}) \exp\left(-\frac{1}{2} \Lambda [1 - 4 \sigma^2_\mathcal{F} \sin^2 \left(\frac{\pi}{N + 2}\right)t\right)$$

$$\leq \epsilon N \sqrt{N - 1} \exp\left(-\frac{1}{2} \Lambda [1 - 4 \sigma^2_\mathcal{F} \sin^2 \left(\frac{\pi}{N + 2}\right)t\right)$$

holds for all $t$, and any initial distribution of $X(0)$ on $\mathcal{S}_{\epsilon,N}$.

2.2. $L^2_{\pi_{\epsilon,N}}$-spectral gap

In order to analyse the spectrum of $\mathcal{L}$ in $L^2_{\pi_{\epsilon,N}}$, we will make an extra assumption on the invariant measure $\pi_{\epsilon,N}$. Recall that a measure $\mu$ is called reversible under $\mathcal{L}$ if for all bounded $f: \mathcal{S}_{\epsilon,N} \times \mathcal{S}_{\epsilon,N} \rightarrow \mathbb{R}$

$$\int \mu(dx) [\mathcal{L} f (., x)](x) = \int \mu(dx) [\mathcal{L} f (x, .)](x)$$

(8)

holds. In particular, considering functions $f$ of the form $f(x, x') = F(x)$ for some bounded $F: \mathcal{S}_{\epsilon,N} \rightarrow \mathbb{R}$ shows that $\mu$ must be invariant under $\mathcal{L}$.

Furthermore, $\mathcal{L}$ acts on $L^2_{\pi_{\epsilon,N}}$ as a bounded, self-adjoint negative semi-definite operator. An estimate on the size of its spectral gap is provided in theorem 2.12. Because the result of the following lemma 2.10 will play a central role in the proof of theorem 2.12 we include the details of this well-known result for completeness.

**Lemma 2.10 (Auxiliary estimate on the spectrum of a self-adjoint operator).** Let $H$ be a real (or complex) Hilbert space and $T: H \rightarrow H$ a bounded, self-adjoint linear operator. Suppose there exists a constant $0 \leq \gamma$ and a dense subspace $G \subset H$ on which for all $g \in G$ and $f \in H$ there exists a constant $C_{f,g} > 0$ such that $|\langle f, T^n g \rangle | \leq C_{f,g} \gamma^n$ for all $n \geq 1$. Then the spectrum of $T$ is contained in $[-\gamma, \gamma]$.

**Proof.** The classical spectral theory of bounded self-adjoint linear operators [12] states that the spectrum $\sigma(T)$ of $T$ is a compact interval in $[-\|T\|, \|T\|]$, and there exists a unique spectral measure $E(d\lambda)$ such that for any $f, g \in H$

$$1 = \int_{\mathbb{R}} E(d\lambda), \quad T^n = \int_{\mathbb{R}} \lambda^n E(d\lambda), \quad \langle T^n f, g \rangle = \int_{\mathbb{R}} \lambda^n \langle E(d\lambda)f, g \rangle.$$
where $E(d\lambda)$ is supported on $\sigma(T)$, and $m_{f,g}(d\lambda) \equiv \langle E(d\lambda), f, g \rangle$ is a finite signed measure on $\sigma(T)$, whose total variation norm satisfies $|m_{f,g}|_{TV} \leq \|f\| \cdot \|g\|$.

Suppose that the spectrum $\sigma(T)$ of $T$ is not contained in $[-\gamma, \gamma]$. Then there exists $s > \gamma$ such that for $S_s = (-\infty, -s) \cup (s, \infty)$ the projection $E(S_s)$ is non-zero. Hence there exists a non-zero $f_s \in H$ with $E(S_s) f_s = f_s$. In particular,

$$\|f_s\|^2 = \int_{\sigma(T)} m_{f_s,f_s}(d\lambda) = \int_{S_s} m_{f_s,f_s}(d\lambda) > 0,$$

because the support of the measure $m_{f_s,f_s}(d\lambda)$ is contained in $S_s$ by choice of $f_s$. In particular, $m_{f_s,f_s} \neq 0$.

For any $g \in G$, and all $n \geq 0$ we have

$$\frac{1}{\gamma^{2n}} \langle f_s, T^{2n} g \rangle = \frac{1}{\gamma^{2n}} \langle T^{2n} f_s, g \rangle = \int_{S_s} \frac{\lambda}{\gamma} |\lambda|^{2n} m_{f_s,g}(d\lambda).$$

Due to the assumption on $G$ we also have that

$$\left| \frac{1}{\gamma^{2n}} \langle f_s, T^{2n} g \rangle \right| \leq C_{f,g}$$

Since $m_{f,g}$ is a finite measure, and $|\frac{\lambda}{\gamma}| \geq \frac{\gamma}{\gamma} > 1$ on its support, the boundedness of the above expression for all $n$ can only be satisfied if in fact $m_{f,g} = 0$.

Thus we have shown that $m_{f_s,f_s} \neq 0$, but $m_{f,g} = 0$ for all $g \in G$. Since $m_{f,g}$ is continuous in $g$ (in fact linear and bounded) the denseness of $G$ implies that there exists a sequence $(g_n)_{n \geq 1} \subseteq G$ such that $g_n \rightarrow f_s$ in $H$, and hence $0 = m_{f_s,g_n} \rightarrow m_{f_s,f_s} \neq 0$. This is a contradiction to continuity. Therefore the assumption on $s$ must have been wrong, so that for all $s > \gamma$ the projection $E(S_s)$ must be zero. And since $\lambda \in \mathbb{R}$ is in the resolvent set of $T$ if and only if there exists an open neighbourhood $S$ of $\lambda$ such that $E(S) = 0$ it follows that $\sigma(T) \subseteq [-\gamma, \gamma]$.

\begin{lemma}[Lipschitz contraction] Let $A : S_{\epsilon,N} \rightarrow \mathbb{R}$ be a Lipschitz continuous function with respect to the distance $d(\cdot, \cdot)$, and set $A_t(x) = E[A(X(t)) | X(t) = x]$ for all $t \geq 0$ and $x \in S_{\epsilon,N}$. Then $A_t$ is Lipschitz continuous with Lipschitz constant

$$\text{Lip}(A_t) \leq \text{Lip}(A) \exp \left( -\frac{1}{2} \Lambda [1 - 4\sigma^2] \sin^2 \left[ \frac{\pi}{N + 2} t \right] \right)$$

for all $t \geq 0$.

\end{lemma}

\begin{proof}

By Jensen’s inequality it follows immediately from the very definition of the Vaserstein distance that $\rho_1(X(t), X'(t)) \leq \rho_2(X(t), X'(t))$ for all $1 \leq \rho_1 \leq \rho_2$. Therefore it follows from proposition 2.8 that

$$\rho_1(X(t), X'(t)) \leq \rho_2(X(0), X'(0)) \exp \left( -\frac{1}{2} \Lambda [1 - 4\sigma^2] \sin^2 \left[ \frac{\pi}{N + 2} t \right] \right)$$

for any joint distribution of $(X(0), X'(0))$ on $S_{\epsilon,N} \times S_{\epsilon,N}$.

Note that $S_{\epsilon,N}$ is compact, and hence

$$\sup_{\text{Lip}(A) \leq 1} |E(A(X(t)) - E(A(X'(t)))| = \rho_1(X(t), X'(t))$$

which is the well-known Kantorovich–Rubinstein duality theorem for the Vaserstein-1 metric.

Using the specific initial distribution $(X(0), X'(0)) = (x, x')$ on $S_{\epsilon,N} \times S_{\epsilon,N}$ we obtain

$$|A_t(x) - A_t(x')| \leq \text{Lip}(A) \rho_1(X(t), X'(t))$$

$$\leq \text{Lip}(A) d(x, x') \exp \left( -\frac{1}{2} \Lambda [1 - 4\sigma^2] \sin^2 \left[ \frac{\pi}{N + 2} t \right] \right)$$
because in this case $\rho_2(X(0), X'(0)) = d(x, x')$. And since $x, x' \in S_{\epsilon, N}$ are arbitrary we see that $A_t$ is Lipschitz continuous with the claimed estimate on its Lipschitz constant.

Combining now the result of lemma 2.11 with that of lemma 2.10 we are in a position to estimate the spectral gap of $L$ acting on $L^2_{\pi_{\epsilon, N}}$, provided we assume that the stationary distribution $\pi_{\epsilon, N}$ is reversible. In this case $L$ is a self-adjoint, bounded, negative semi-definite operator on $L^2_{\pi_{\epsilon, N}}$.

**Theorem 2.12 ($L^2_{\pi_{\epsilon, N}}$-spectral gap for reversible $\pi_{\epsilon, N}$).** Suppose that $P$ satisfies $\int P(\mathrm{d}x) \alpha = \frac{1}{2}$ and $\sigma_\rho^2 < \frac{1}{2}$. If the stationary distribution $\pi_{\epsilon, N}$ of $X(t)$ on $S_{\epsilon, N}$ is reversible, then

$$\sigma(L) \subset \left(-\infty, -\frac{1}{2} \Lambda \left[1 - 4 \sigma_\rho^2\right] \sin^2\left[\frac{\pi}{N + 2}\right]\right] \cup \{0\},$$

where 0 is a simple eigenvalue corresponding to the constant eigenfunction.

**Proof.** By assumption $L$ generates a self-adjoint, positive semi-definite contraction semigroup $e^{tL}$ on $L^2_{\pi_{\epsilon, N}}$, which satisfies $e^{tL}1 = 1$. Therefore, the subspace $H$ of $L^2_{\pi_{\epsilon, N}}$ consisting of functions perpendicular to the constant functions is invariant. Hence, the decomposition $L^2_{\pi_{\epsilon, N}} = H \oplus \text{span}[1]$ is invariant under $e^{tL}$, and $e^{tL}$ may be restricted to $H$.

Furthermore, it is a consequence of Lusin’s theorem [29] that the set of Lipschitz continuous functions on $S_{\epsilon, N}$ is dense in $L^2_{\pi_{\epsilon, N}}$. Hence the set $G$ of Lipschitz continuous functions $A$ on $S_{\epsilon, N}$ with $\int \pi_{\epsilon, N}(\mathrm{d}x) A(x) = 0$ is dense in $H$.

By lemma 2.4 and the mean value theorem, for any $f \in H$ and $g \in G$ we have

$$|\langle f, g \rangle| \leq \|f\| \|g\| \leq \|f\| \text{diam} S_{\epsilon, N} \text{Lip}(g) \leq \|f\| \epsilon N \sqrt{N - 1} \text{Lip}(g)$$

and hence

$$\|f, e^{tL}g\| \leq \|f\| \epsilon N \sqrt{N - 1} \text{Lip}(g) \exp\left(\frac{-\frac{1}{2} \Lambda \left[1 - 4 \sigma_\rho^2\right] \sin^2\left[\frac{\pi}{N + 2}\right] t}\right)$$

follows from lemma 2.11 for all $n \geq 0$.

Since $e^{tL}$ is a positive operator the result of lemma 2.10 yields

$$\sigma(e^{tL}|_H) \subset \left(0, \exp\left(\frac{-\frac{1}{2} \Lambda \left[1 - 4 \sigma_\rho^2\right] \sin^2\left[\frac{\pi}{N + 2}\right] t}\right)\right].$$

This implies

$$\sigma(L|_H) = \frac{1}{t} \log \sigma(e^{tL}) \subset \left(-\infty, -\frac{1}{2} \Lambda \left[1 - 4 \sigma_\rho^2\right] \sin^2\left[\frac{\pi}{N + 2}\right]\right],$$

which completes the proof.

From the proof of theorem 2.12 it is clear that the abstract result lemma 2.10 shows that an estimate on the exponential rate of weak convergence of $X(t)$ in Vaserstein-1 distance automatically yields an estimate on the spectral gap of $L$ on $L^2_{\pi_{\epsilon, N}}$, provided that the stationary distribution $\pi$ is reversible. This is strategy is well known to be very efficient to estimate the spectral gap, e.g. [10, 27] and references therein.

**Remark 2.13.** All results of this section are essentially consequences of proposition 2.6 and lemma 2.10. And since the statement of proposition 2.6 is readily rephrased for the embedded discrete time Markov chain with transition operator

$$\mathcal{P}A(x) = A(x) + \frac{1}{N - 1} \frac{1}{\Lambda} L A(x) = \sum_{i=1}^{N-1} \frac{1}{N - 1} \int P(\mathrm{d}x) A(T_{i, \alpha}x)$$

the results of this section all carry over (essentially verbatim) to the discrete time setting. One only has to multiply the rate of convergence (and hence the spectral gap) by $\frac{1}{N - 1}$ in the results for continuous time to obtain the corresponding results for the discrete time setting.
3. Spectral gap in $L^2_{\pi_{\epsilon,N}}$ for the general case

Now we consider the general situation where the continuous-time Markov process $X(t)$ is generated by the infinitesimal generator $L$ given in (1). Suppose that $\pi_{\epsilon,N}$ is a reversible measure for $L$. Then the associated Dirichlet form

$$D_{\epsilon,N}(A) = \int \pi_{\epsilon,N}(dx) A(x) [L A](x)$$  \hspace{1cm} (9a)

is defined for all $A \in L^2_{\pi_{\epsilon,N}}$, and has the representation

$$D_{\epsilon,N}(A) = \frac{1}{2} \sum_{i=1}^{N-1} \int \pi_{\epsilon,N}(dx) \Lambda(x_i, x_{i+1}) \int P(x_i, x_{i+1}, d\alpha) [A(T_i, \alpha x) - A(x)]^2.$$  \hspace{1cm} (9b)

Clearly, the constant functions are in $L^2_{\pi_{\epsilon,N}}$, and are eigenfunctions of $L$ to the eigenvalue 0. Since $L$ is self-adjoint on $L^2_{\pi_{\epsilon,N}}$, the rate of convergence of $X(t)$ to its equilibrium distribution is determined by the remaining part of the spectrum of $L$. Therefore we define

$$\gamma_{\epsilon,N} = \inf \left\{ \frac{D_{\epsilon,N}(A)}{\text{Var}_{\epsilon,N}(A)} : A \in L^2_{\pi_{\epsilon,N}}, \text{Var}_{\epsilon,N}(A) \neq 0 \right\},$$  \hspace{1cm} (10)

where $\text{Var}_{\epsilon,N}(A)$ denotes the variance of $A$ with respect to $\pi_{\epsilon,N}$. The quantity $\gamma_{\epsilon,N}$ is equal to the spectral gap of $L$ on $L^2_{\pi_{\epsilon,N}}$. For the spectrum of $L$ on $L^2_{\pi_{\epsilon,N}}$, denoted by $\sigma(L)$, it then follows that $-\gamma_{\epsilon,N} \in \sigma(L)$, and $\sigma(L) \subset (-\infty, -\gamma_{\epsilon,N}] \cup \{0\}$.

The basic idea to prove convergence rates for $X(t)$ is to compare the spectral gap of its generator $L$ to a suitably chosen reference process of type (3) considered in section 2. In order to distinguish these two generators we use a superscript $\star$

$$L^\star A(x) = \Lambda^\star \sum_{i=1}^{N-1} \int P^\star(\alpha x) [A(T_i, \alpha x) - A(x)],$$

$$D^\star_{\epsilon,N}(A) = \frac{1}{2} \int \pi^\star_{\epsilon,N}(dx) \sum_{i=1}^{N-1} \Lambda^\star [A(T_i, \alpha x) - A(x)]^2,$$

to denote the invariant measure, the generator and the corresponding Dirichlet form of the reference process.

**Theorem 3.1 (Spectral gap for $L$).** Fix $\epsilon > 0$ and $N$, and let $\pi_{\epsilon,N}$ be a reversible stationary distribution of $L$ on $S_{\epsilon,N}$. Suppose that there exist a constant $\Lambda^\star > 0$ and a probability measure $P^\star$ on $[0, 1]$ with mean $\int P^\star(\alpha x) \alpha = \frac{1}{2}$ and variance $\sigma_{P^\star}^2 < \frac{1}{4}$ such that the following are satisfied:

(i) The rate function $\Lambda$ satisfies $\Lambda(x_i, x_{i+1}) \geq \Lambda^\star$ for $\pi_{\epsilon,N}$-almost all $x \in S_{\epsilon,N}$, and all $1 \leq i \leq N - 1$.

(ii) There exists a constant $c > 0$ such that $P$ satisfies the minorization condition $P(x_i, x_{i+1}) \geq c P^\star(.)$ for $\pi_{\epsilon,N}$-almost all $x \in S_{\epsilon,N}$, and all $1 \leq i \leq N - 1$.

(iii) The unique (recall theorem 2.9) stationary distribution $\pi^\star_{\epsilon,N}$ of $L^\star$ on $S_{\epsilon,N}$ (corresponding to $\Lambda^\star$ and $P^\star$) is reversible.

(iv) The measures $\pi_{\epsilon,N}$ and $\pi^\star_{\epsilon,N}$ are uniformly equivalent, i.e. there exist two constants $0 < C^\star_{\epsilon} \leq C^\star_{\epsilon} < \infty$ such that their Radon–Nikodym derivative satisfies $C^-_{\epsilon} = \frac{\pi_{\epsilon,N}(dx)}{\pi^\star_{\epsilon,N}(dx)} \leq C^+_{\epsilon}$ for all $N$. 

Then spectral gap of \( \mathcal{L} \) on \( L^2_{\pi_{\epsilon,N}} \) satisfies
\[
\gamma_{\epsilon,N} \geq c \frac{C^-_\epsilon}{C^+_\epsilon} \Lambda \frac{1}{2} \left[ 1 - 4 \sigma^2_{P_{\epsilon,N}} \right] \sin^2 \left( \frac{\pi}{N+2} \right),
\]
and hence 0 is a simple eigenvalue of \( \mathcal{L} \).

**Remark 3.2.** Later we will see that—apart from condition (i)—the conditions of theorem 3.1 are fulfilled in a wide range of models of mechanical origin, interesting to us. Indeed, in theorem 4.3 we will prove a characterization of reversible measures of a particular type. Among others, it will provide the existence of reversible stationary measures for a large class rate functions \( \Lambda \) and transition kernels \( P \). This result, in particular, addresses conditions iii) and (iv) in the above theorem 3.1 in quite satisfactory generality. Also, in section 5, we show that (ii) is satisfied, for instance, in the Gaspard–Gilbert model with three-dimensional balls. Finally, (i) is the consequence of our method. Nevertheless establishing hydrodynamical limit transition is a great challenge even under our conditions and for doing it our theorems serve as an excellent background. Finally we note that the applicability of the statement in its present form seems to be restricted to models where \( \pi_{\epsilon,N} = \pi^*_{\epsilon,N} \) therefore the weakening of condition (iv) would also be desirable.

**Proof.** By assumption we can compare the measures \( \pi_{\epsilon,N} \), \( \pi^*_{\epsilon,N} \), and \( P \), \( P^* \), so that for the Dirichlet form, recall (9), we obtain the estimate
\[
D_{\epsilon,N}(A) = \frac{1}{2} \sum_{i=1}^{N-1} \pi_{\epsilon,N}(dx) \Lambda(x_i, x_{i+1}) \int P(x_i, x_{i+1}, d\alpha) \left[ A(T_i, \alpha x) - A(x) \right]^2 \geq c \frac{C^-_\epsilon}{C^+_\epsilon} \sum_{i=1}^{N-1} \pi^*_\epsilon(dx) \Lambda^* \int P^*(d\alpha) \left[ A(T_i, \alpha x) - A(x) \right]^2,
\]
which is nothing else but \( D_{\epsilon,N}(A) \geq c \frac{C^-_\epsilon}{C^+_\epsilon} D^*_{\epsilon,N}(A) \) for all \( A \in L^2_{\pi_{\epsilon,N}} \).

Furthermore, the variational characterization of the variance yields the estimate
\[
\text{Var}_{\epsilon,N}(A) = \inf_{q \in \mathbb{R}} \int \pi_{\epsilon,N}(dx) \left[ A(x) - q \right]^2 = \inf_{q \in \mathbb{R}} \int \pi^*_{\epsilon,N}(dx) \frac{\pi_{\epsilon,N}(dx)}{\pi^*_{\epsilon,N}(dx)} \left[ A(x) - q \right]^2 \leq C^*_\epsilon \text{Var}^*_{\epsilon,N}(A)
\]
for all \( A \in L^2_{\pi_{\epsilon,N}} \).

Combining both of the above estimates shows
\[
\frac{D_{\epsilon,N}(A)}{\text{Var}_{\epsilon,N}(A)} \geq c \frac{C^-_\epsilon}{C^+_\epsilon} \frac{D^*_{\epsilon,N}(A)}{\text{Var}^*_{\epsilon,N}(A)}
\]
for any \( A \in L^2_{\pi_{\epsilon,N}} \) with \( \text{Var}_{\epsilon,N}(A) \neq 0 \). In other words, the spectral gap of \( \mathcal{L} \) admits the estimate
\[
\gamma_{\epsilon,N} \geq c \frac{C^-_\epsilon}{C^+_\epsilon} \inf \left\{ \frac{D^*_{\epsilon,N}(A)}{\text{Var}^*_{\epsilon,N}(A)} : A \in L^2_{\pi_{\epsilon,N}}, \ \text{Var}_{\epsilon,N}(A) \neq 0 \right\}.
\]

Finally, note that the assumed bounds \( C^-_\epsilon \leq \frac{\pi_{\epsilon,N}(dx)}{\pi^*_{\epsilon,N}(dx)} \leq C^*_\epsilon \) imply that \( L^2_{\pi_{\epsilon,N}} = L^2_{\pi^*_{\epsilon,N}} \) so that the above estimate for \( \gamma_{\epsilon,N} \) can be rewritten as
\[
\gamma_{\epsilon,N} \geq c \frac{C^-_\epsilon}{C^*_\epsilon} \gamma^*_{\epsilon,N}
\]
where \( \gamma^*_{\epsilon,N} \) denotes the spectral gap of \( \mathcal{L}^* \) in \( L^2_{\pi^*_{\epsilon,N}} \).
Now recall that by theorem 2.12
\[ \gamma^*_{\epsilon,N} \geq \frac{1}{2} \Lambda^* \left[ 1 - 4 \sigma^2 \right] \sin^2 \left[ \frac{\pi}{N+2} \right] \]
which in turn shows for the spectral gap of \( L \)
\[ \gamma_{\epsilon,N} \geq c \frac{C'}{C} \Lambda^* \frac{1}{2} \left[ 1 - 4 \sigma^2 P^* \right] \sin^2 \left[ \frac{\pi}{N+2} \right], \]
which completes the proof. □

4. Classification of reversible product measures

In this section we will characterize reversible product measures of \( X(t) \). It is worth recalling at this point that for any \( N \) fixed the sets \( S_{\epsilon,N} \subset \mathbb{R}^N_+ \) are invariant for the process for any choice of \( \epsilon > 0 \). And since these are simplexes there are no (non-trivial) product measures \( \mu(dx_1) \cdots dx_N \) supported by a single \( S_{\epsilon,N} \). However, conditioning an invariant product measure on all of \( \mathbb{R}^N_+ \) to any \( S_{\epsilon,N} \) yields an invariant measure on \( S_{\epsilon,N} \).

Therefore, we will consider product measures on all of \( \mathbb{R}^N_+ \) (canonical measures) instead on the ergodic components \( S_{\epsilon,N} \) (micro-canonical measures). And since our main convergence result theorem 3.1 is for reversible invariant measures, we consider here only reversible product measures.

The first step in classifying all of them is provided by lemma 4.1, which says that it suffices to consider \( N = 2 \).

**Lemma 4.1 (Reversible product measures and system size).** Let \( \nu \) be a probability measure on \( \mathbb{R}_+ \). Then the product (probability) measure \( \mu(dx) = \nu(dx_1) \cdots \nu(dx_N) \) on \( \mathbb{R}_+^N \) is reversible for \( X(t) \) (with generator (1)) for some \( N \) if and only if it is reversible for \( N = 2 \).

**Proof.** Let \( A: \mathbb{R}_+^N \times \mathbb{R}_+^N \to \mathbb{R} \) be bounded. To shorten the notation we use \( \mathcal{L}A(x, .) \) to denote the function obtained by the action of the generator \( L \) on second variable of the function \( A(x, x') \), while treating the first variable as a parameter. Further we use \( \mathcal{L}A(x, .)(x') \) to denote the evaluation of the function \( \mathcal{L}A(x, .) \) at the point \( x' \). Correspondingly, in \( \mathcal{L}A(., x') \) the second variable is treated as a parameter.

By definition (1) of the generator \( L \) we have
\[
\int_{\mathbb{R}_+^N} \mu(dx) [\mathcal{L}A(., x)](x) = \sum_{i=1}^{N-1} \int_{\mathbb{R}_+^N} \nu(dx_1) \cdots \nu(dx_N) \Lambda(x_i, x_{i+1})
\]
\[
\cdot \int P(x_i, x_{i+1}, d\alpha) [A(T, \alpha x, x) - A(x, x)]
\]
\[
= \sum_{i=1}^{N-1} \int_{\mathbb{R}_+^2} \nu(dx_i) \nu(dx_{i+1}) \Lambda(x_i, x_{i+1}) \int P(x_i, x_{i+1}, d\alpha)
\]
\[
\cdot \left[ A_{i,i+1}(\alpha [x_i + x_{i+1}], (1 - \alpha) [x_i + x_{i+1}], x_i, x_{i+1})
\right.
\]
\[
- A_{i,i+1}(x_i, x_{i+1}, x_i, x_{i+1}) \right],
\]
where we used the short hand notation
\[
A_{i,i+1}(x_i, x_{i+1}, x_i', x_{i+1}') = \int_{\mathbb{R}_+^{N-2}} \nu(dx_1) \cdots \nu(dx_{i-1}) \nu(dx_{i+2}) \cdots \nu(dx_N) A(x, z_i),
\]
\[
z_i \equiv (x_1, \ldots, x_{i-1}, x_i', x_{i+1}', x_{i+2}, \ldots, x_N).
Recall that reversibility means \( \int_{\mathbb{R}^N} \mu(dx) [\mathcal{L}A(\cdot, x)](x) = \int_{\mathbb{R}^N} \mu(dx) [\mathcal{L}A(x, \cdot)](x) \), so that reversibility holds if and only if

\[
\sum_{i=1}^{N-1} \int_{\mathbb{R}^2} v(dx_i) v(dx_{i+1}) A(x_i, x_{i+1}) \int P(x_i, x_{i+1}, d\alpha) \cdot A_{i,i+1} (\alpha [x_i + x_{i+1}], (1 - \alpha) [x_i + x_{i+1}], x_i, x_{i+1})
\]

\[
= \sum_{i=1}^{N-1} \int_{\mathbb{R}^2} v(dx_i) v(dx_{i+1}) \Lambda(x_i, x_{i+1}) \int P(x_i, x_{i+1}, d\alpha) \cdot A_{i,i+1} (x_i, x_{i+1}, \alpha [x_i + x_{i+1}], (1 - \alpha) [x_i + x_{i+1}])
\]

for any bounded \( A: \mathbb{R}_+^N \times \mathbb{R}_+^N \rightarrow \mathbb{R} \).

In the particular case where \( A(x, x') = \phi(x_1, x'_1) \) for some bounded \( \phi: \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R} \)

\[
A_{1,2}(x_1, x_2, x'_1, x'_2) = \phi(x_1, x'_1)
\]

\[
A_{i,i+1}(x_i, x_{i+1}, x'_i, x'_{i+1}) = \int_{\mathbb{R}_+} v(dx_i) \phi(x_1, x_i) \equiv \text{const}
\]

for all \( i = 2, \ldots, N - 1 \). Hence reversibility requires

\[
\int_{\mathbb{R}^2} v(dx_1) v(dx_2) A(x_1, x_2) \int P(x_1, x_2, d\alpha) \phi(\alpha [x_1 + x_2], x_1) = \int_{\mathbb{R}^2} v(dx_1) v(dx_2) \Lambda(x_1, x_2) \int P(x_1, x_2, d\alpha) \phi(x_1, \alpha [x_1 + x_2]).
\]

Consider now \( A(x, x') = \psi(x_1, x_2, x'_1, x'_2) \) for some bounded \( \psi: \mathbb{R}_+^2 \times \mathbb{R}_+^2 \rightarrow \mathbb{R} \). Then

\[
A_{1,2}(x_1, x_2, x'_1, x'_2) = \psi(x_1, x_2, x'_1, x'_2),
\]

\[
A_{2,3}(x_2, x_3, x'_2, x'_3) = \int_{\mathbb{R}_+} v(dx_1) \psi(x_1, x_2, x_1, x'_2) \equiv \hat{\psi}(x_2, x'_2),
\]

\[
A_{i,i+1}(x_i, x_{i+1}, x'_i, x'_{i+1}) = \int_{\mathbb{R}^2} v(dx_1) v(dx_2) \psi(x_1, x_2, x_i, x_{i+1}) \equiv \text{const}
\]

for all \( i = 3, \ldots, N - 1 \). Combining this with the previous special case (applied to \( \phi = \hat{\psi} \)) shows that reversibility requires

\[
\int_{\mathbb{R}^2} v(dx_1) v(dx_2) A(x_1, x_2) \int P(x_1, x_2, d\alpha) \cdot \psi(\alpha [x_1 + x_2], (1 - \alpha) [x_1 + x_2], x_1, x_2)
\]

\[
= \int_{\mathbb{R}^2} v(dx_1) v(dx_2) \Lambda(x_1, x_2) \int P(x_1, x_2, d\alpha) \cdot \psi(x_1, x_2, \alpha [x_1 + x_2], (1 - \alpha) [x_1 + x_2])
\]

for any bounded test function \( \psi: \mathbb{R}_+^2 \times \mathbb{R}_+^2 \rightarrow \mathbb{R} \). And since this is also sufficient for reversibility, it follows that reversibility of the product measure holds if and only if the above equality holds for all \( \psi \).

Finally, observe that this last expression is precisely the reversibility condition for \( N = 2 \), which completes the proof. \( \square \)

The final expression in the above proof actually shows that reversibility of the product measure is equivalent to a slightly stronger statement than the one stated in lemma 4.1. Namely,
because both integrands agree at \((0, 0)\) the reversibility of the product measure is equivalent to
\[
\int_{\mathbb{R}^2 \setminus \{(0, 0)\}} \nu(dx_1) \nu(dx_2) \Lambda(x_1, x_2) \int P(x_1, x_2, d\alpha) \cdot \psi(\alpha [x_1 + x_2], (1 - \alpha) [x_1 + x_2], x_1, x_2)
\]
\[
= \int_{\mathbb{R}^2 \setminus \{(0, 0)\}} \nu(dx_1) \nu(dx_2) \Lambda(x_1, x_2) \int P(x_1, x_2, d\alpha)
\]
\[
\cdot \psi(x_1, x_2, \alpha [x_1 + x_2], (1 - \alpha) [x_1 + x_2]).
\]  
(11)

This simplification is relevant, because so far we have not ruled out yet the possibility of \(\nu\) having an atom at 0.

For the further analysis we will need to assume that the rate function \(\Lambda\) and the transition kernel \(P\) are of the form
\[
\Lambda(x_i, x_{i+1}) = \Lambda_x(x_i + x_{i+1}) \Lambda_r \left( \frac{x_i}{x_i + x_{i+1}} \right),
\]
\[
P(x_i, x_{i+1}, d\alpha) = P \left( \frac{x_i}{x_i + x_{i+1}}, d\alpha \right).
\]  
(12)

Here the subscripts \(s\) and \(r\) stand for ‘sum’ and ‘ratio’, respectively. Note that \(\frac{x_i}{x_i + x_{i+1}}\) makes sense everywhere on \(\mathbb{R}^2 \setminus \{(0, 0)\}\), and by the above this set is all that we need to consider. In section 5 we will see that the representation (12) naturally occurs in models originating from mechanical systems.

We have already shown that in order to classify reversible product measures for arbitrary \(N\) it is enough to study the case \(N = 2\). This, however, is still not a completely straightforward problem, since the answer might depend on the rate functions \(\Lambda_x\) and \(\Lambda_r\). The next corollary 4.2 simplifies this issue.

**Corollary 4.2 (Reversible product measures and rate functions).** If \(\Lambda_x(\eta) > 0\) for all \(0 < \eta < \infty\), then the process has a reversible stationary product measure \(\mu\) (as in lemma 4.1) if and only if
\[
\int_{\mathbb{R}^2 \setminus \{(0, 0)\}} \nu(dx_1) \nu(dx_2) \Lambda_x(x_1 + x_2) \Lambda_r \left( \frac{x_1}{x_1 + x_2} \right) \int P \left( \frac{x_1}{x_1 + x_2}, \eta (x_1 + x_2, \alpha, \frac{x_1}{x_1 + x_2}) \right)
\]
\[
= \int_{\mathbb{R}^2 \setminus \{(0, 0)\}} \nu(dx_1) \nu(dx_2) \Lambda_x(x_1 + x_2) \Lambda_r \left( \frac{x_1}{x_1 + x_2} \right) \int P \left( \frac{x_1}{x_1 + x_2}, \eta (x_1 + x_2, \alpha) \right)
\]
holds for all bounded \(\eta: \mathbb{R}^+ \setminus [0] \times [0, 1]^2 \rightarrow \mathbb{R}\).

**Proof.** By (11) reversibility of the product measure is equivalent to
\[
\int_{\mathbb{R}^2 \setminus \{(0, 0)\}} \nu(dx_1) \nu(dx_2) \Lambda_x(x_1 + x_2) \Lambda_r \left( \frac{x_1}{x_1 + x_2} \right) \int P \left( \frac{x_1}{x_1 + x_2}, \alpha [x_1 + x_2], (1 - \alpha) [x_1 + x_2], x_1, x_2 \right)
\]
\[
\cdot \psi(\alpha [x_1 + x_2], (1 - \alpha) [x_1 + x_2], x_1, x_2)
\]
\[
= \int_{\mathbb{R}^2 \setminus \{(0, 0)\}} \nu(dx_1) \nu(dx_2) \Lambda_x(x_1 + x_2) \Lambda_r \left( \frac{x_1}{x_1 + x_2} \right) \int P \left( \frac{x_1}{x_1 + x_2}, \alpha [x_1 + x_2], (1 - \alpha) [x_1 + x_2] \right)
\]
for any (non-negative) test function \(\psi: \mathbb{R}^2 \setminus \{(0, 0)\} \times \mathbb{R}^2 \rightarrow \mathbb{R}\). On \(\mathbb{R}^2 \setminus \{(0, 0)\}\) the change of coordinates \((x_1, x_2) \mapsto (x_1 + x_2, \frac{x_1}{x_1 + x_2})\) is one-to-one, hence any such function \(\psi\) may be recast as
\[
\psi(x_1, x_2, x'_1, x'_2) \equiv \eta \left( x_1 + x_2, \frac{x_1}{x_1 + x_2}, x'_1 + x'_2, \frac{x'_1}{x'_1 + x'_2} \right).
\]
for some function $\eta: (\mathbb{R}_+ \times [0, 1])^2 \to \mathbb{R}$. Therefore reversibility holds if and only if
\[
\int_{\mathbb{R}_+^2 \setminus (0, 0)} \nu(dx_1) \nu(dx_2) \Lambda_r(x_1 + x_2) \frac{x_1}{x_1 + x_2} \int P\left(\frac{x_1}{x_1 + x_2}, \alpha\right) \cdot \eta\left(x_1 + x_2, \alpha, \frac{x_1}{x_1 + x_2}\right)
\]
\[= \int_{\mathbb{R}_+^2 \setminus (0, 0)} \nu(dx_1) \nu(dx_2) \Lambda_r(x_1 + x_2) \frac{x_1}{x_1 + x_2} \int P\left(\frac{x_1}{x_1 + x_2}, \alpha\right) \cdot \eta\left(x_1 + x_2, \alpha, \frac{x_1}{x_1 + x_2}\right)
\]
holds for all $\eta: \mathbb{R}_+ \setminus (0, 0) \times [0, 1]^2 \to \mathbb{R}$.

And since $x_1 + x_2 > 0$ our assumption on $\Lambda_r$ implies that $\Lambda_r$ is strictly positive, and hence may as well be combined with $\eta$, because $\eta$ is arbitrary. This completes the proof. $\square$

With lemma 4.1, and corollary 4.2 we are now in a position to classify all reversible product measures, which is the content of the following theorem 4.3. This classification relies on a well-known fact [28] about Gamma distributions. Namely, suppose that $X_1$ and $X_2$ are two non-constant, independent, positive random variables. Then $X_1 + X_2$ and $\frac{x_1}{x_1 + x_2}$ are independent if and only if $X_1$ and $X_2$ are independent, identically Gamma-distributed random variables.

In the theorem below we use the following notation: For $\epsilon > 0$ we denote by $\delta(\epsilon, d\alpha)$ the Dirac measure concentrated at $\epsilon$.

**Theorem 4.3 (Reversible product measures).** Suppose that the Markov chain on $[0, 1]$ with transition kernel $P(\beta, d\alpha)$ has a unique invariant distribution, say $p(.)$. Let $N$ be arbitrary, and suppose further that $\Lambda_r$ is such that $\Lambda_r(\sigma) > 0$ for all $\sigma > 0$, and $\Lambda_r(\beta) > 0$ for all $0 < \beta < 1$. Then the product measure $\mu(dx) = \nu(dx_1) \cdots \nu(dx_N)$ is reversible for $X(t)$ if and only if $\nu$ is a reversible measure for the Markov chain generated by $P$, and either of the following two holds:

(i) There exists $T > 0$ and $d > 0$ such that
\[
\nu(dx_1) = \frac{d_1}{T} \left[\frac{x_1}{T}\right]^{d-1} e^{-\frac{x_1}{T}},
\]
\[
p(d\beta) = d\beta [(1 - \beta)]^{d-1} \frac{\Gamma(d)}{\Gamma\left(\frac{d}{2}\right)^2} \Lambda_r(\beta) \frac{1}{Z},
\]
where $Z$ is the normalizing constant.

(ii) There exists $\epsilon > 0$ such that $\nu(dx_1) = \delta(\epsilon, dx_1)$, $p(d\alpha) = \delta(\frac{1}{2}, d\alpha)$, and $P(\frac{1}{2}, d\alpha) = \delta(\frac{1}{2}, d\alpha)$.

**Proof.** From lemma 4.1 we know that it suffices to consider $N = 2$, and corollary 4.2 shows—as it is also clear intuitively—that the choice of $\Lambda_r$ is irrelevant, and that we only need to consider the process on $\mathbb{R}_+^2 \setminus \{(0, 0)\}$.

Using the change of variables $\sigma = x_1 + x_2$, $\beta = \frac{x_1}{x_1 + x_2}$ on $\mathbb{R}_+^2 \setminus \{(0, 0)\}$ we can disintegrate the product measure $\nu(dx_1) \nu(dx_2)$ such that for any (bounded) $\eta: \mathbb{R}_+ \to \mathbb{R}$ we have
\[
\int_{\mathbb{R}_+^2 \setminus (0, 0)} \nu(dx_1) \nu(dx_2) \eta(x_1, x_2) = \int_{\mathbb{R}_+ \setminus (0, 0)} \nu_\eta(d\sigma) \int_{(0, 1]} \nu_r(\sigma, \beta) \eta(\beta \sigma, (1 - \beta) \sigma),
\]
where $\nu_r(.)$ is the distribution of the sum $x_1 + x_2$ and $\nu_r(\sigma, .)$ is the conditional distribution of the ratio $\frac{x_1}{x_1 + x_2}$ given that $x_1 + x_2 = \sigma$. 


Using this notation the condition for the reversibility of the product measure of corollary 4.2 takes on the form
\[
\int v_s(d\sigma) \int v_r(\sigma, d\beta) \Lambda_r(\beta) \int P(\beta, d\alpha) \eta(\sigma, \alpha, \beta) = \int v_s(d\sigma) \int v_r(\sigma, d\beta) \Lambda_r(\beta) \int P(\beta, d\alpha) \eta(\sigma, \beta, \alpha).
\]
This holds if and only if for \(v_s\)-almost every \(\sigma\)
\[
\int v_r(\sigma, d\beta) \Lambda_r(\beta) \int P(\beta, d\alpha) \tilde{\eta}(\alpha, \beta) = \int v_r(\sigma, d\beta) \Lambda_r(\beta) \int P(\beta, d\alpha) \tilde{\eta}(\beta, \alpha)
\]
for all bounded \(\tilde{\eta} : [0, 1] \times [0, 1] \rightarrow \mathbb{R}\).

Suppose now that the product measure is reversible. The special choice \(\eta(\alpha, \beta) = \psi(\alpha)\) for some \(\psi : [0, 1] \rightarrow \mathbb{R}\) thus shows that
\[
\int v_r(\sigma, d\beta) \Lambda_r(\beta) \int P(\beta, d\alpha) \psi(\alpha) = \int v_r(\sigma, d\beta) \Lambda_r(\beta) \psi(\beta)
\]
for all \(\psi\). In other words, the (not normalized) non-negative measure \(v_r(\sigma, d\beta)\Lambda_r(\beta)\) must be invariant under \(P\). And since by assumption \(P\) has a unique invariant distribution, denote it by \(p\), it thus follows that
\[
\frac{1}{Z} v_r(d\sigma) \Lambda_r(\beta) = p(d\beta), \quad Z = \int v_r(d\sigma) \Lambda_r(\beta)
\]
for \(v_s\)-almost every \(\sigma\), where \(Z > 0\) by assumption on \(\Lambda_r\).

In particular, this means that the conditional distribution \(v_r(\sigma, \cdot)\) of the ratio \(\frac{x_1}{x_1 + x_2}\) given that \(\sigma = x_1 + x_2\) actually is the same for all values of \(\sigma\). In other words the sum \(x_1 + x_2\) and the ratio \(\frac{x_1}{x_1 + x_2}\) are independent. And since also \(x_1\) and \(x_2\) are independent (by assumption) we conclude [28] that either \(\nu\) is a point mass, i.e. \(\nu(dx_1) = \delta(\epsilon, dx_1)\) for some \(\epsilon > 0\), or \(\nu\) is a Gamma distribution, i.e.
\[
\nu(dx_1) = \frac{dx_1}{T} \left[\frac{x_1}{T}\right]^{d-1} e^{-\frac{x}{T}} \frac{\Gamma(d)}{\Gamma(d/2)^2} (0 < x_1 < \infty)
\]
for some \(T > 0\) and \(d > 0\).

In the former case it follows
\[
v_s(d\sigma) = \delta(2 \epsilon, d\sigma), \quad p(d\beta) = v_r(d\beta) = \delta(\frac{x_1}{x_1 + x_2}, d\beta)
\]
for \(v_s, v_r\). Hence the reversibility condition (13) becomes \(\int P(\frac{x_1}{x_1 + x_2}, d\alpha) \eta(\alpha, \frac{x_1}{x_1 + x_2}) = \int P(\frac{x_1}{x_1 + x_2}, d\alpha) \eta(\frac{x_1}{x_1 + x_2}, \alpha)\) for all \(\eta\), which is equivalent to
\[
P(\frac{x_1}{x_1 + x_2}, d\alpha) = \delta(\frac{x_1}{x_1 + x_2}, d\alpha).
\]
Similarly, in the latter case
\[
v_s(d\sigma) = \frac{d\sigma}{T} \left[\frac{\sigma}{T}\right]^{d-1} \frac{\Gamma(d)}{\Gamma(d/2)^2}, \quad v_r(d\beta) = d\beta (1 - \beta) \left[\frac{\beta(1 - \beta)}{d}\right]^{d-1} \frac{\Gamma(d)}{\Gamma(d/2)^2}
\]
follows for \(v_s, v_r\), where we used the well-known properties of Gamma and Beta distributions.
The reversibility condition (13) becomes
\[
\int_0^1 d\beta (1 - \beta)^{d-1} \frac{\Gamma(d)}{\Gamma(d/2)^2} \Lambda_r(\beta) \int P(\beta, d\alpha) \eta(\alpha, \beta) = \int_0^1 d\beta (1 - \beta)^{d-1} \frac{\Gamma(d)}{\Gamma(d/2)^2} \Lambda_r(\beta) \int P(\beta, d\alpha) \eta(\beta, \alpha)
\]
for all $\eta$, and
\[
p(d\beta) = d\beta \frac{\Gamma(d)\Lambda_r(\beta)}{\Gamma\left(\frac{d}{2}\right)^2} \frac{1}{Z}
\]

must be the expression for the unique stationary distribution of $P$.

This proves that if the product measure is reversible, then $\nu$ is either constant, or a Gamma distribution, and the transition kernel must have the claimed stationary distribution.

To complete the proof it remains to consider the converse. Assume either of the two possible distributions for $\nu$ and also the corresponding assumption on $P$. For these special distributions it is well known (and easily verified) that the sum and the ratio are independent with the distributions as considered above. Hence we see that the reversibility condition (13) is indeed satisfied.

\[\square\]

**Remark 4.4.** If the product measure $\mu(dx)$ on $\mathbb{R}_+^N$ of type (i) in the above theorem 4.3 is invariant for $X(t)$ for some $d > 0$ and $T > 0$, then its restriction $\pi_{\epsilon,N}$ to $S_{\epsilon,N}$ is independent of the value of $T$ for all $\epsilon$. In fact $\pi_{\epsilon,N}$ is, apart from the normalization of $S_{\epsilon,N}$, the Dirichlet distribution parameters $(d_2, \ldots, d_N)$ [22]. Therefore, $\mu(dx)$ on $\mathbb{R}_+^N$ is invariant for $X(t)$ for any other value of $T > 0$ as long as $d$ remains unchanged.

Note that in the statement of theorem 4.3 there is an assumption on the kernel $P$ that appears in the generator of the process $X(t)$. By lemma 4.1 and corollary 4.2 it suffices to consider the reversibility of the product measure for $N = 2$ and constant rates. Upon restricting this process to any of the invariant sets $S_{\epsilon,2}$, the embedded discrete time Markov chain is precisely the Markov chain on $[0, 1]$ with transition kernel $P(\beta, d\alpha)$. Therefore, the assumption in theorem 4.3 on the kernel $P$ is equivalent to saying that for $N = 2$ and constant rates the process $X(t)$ has a unique stationary distribution on any of the $S_{\epsilon,2}$. A sufficient condition for this uniqueness is to assume that $P$ satisfies a uniform minorization condition, i.e. there exists a constant $c > 0$ and a probability measure $P^*\alpha = \frac{1}{2}$ and $\sigma \alpha < \frac{1}{4}$ such that $P(\beta, \alpha) \geq cP^*(\alpha)$ for all $\beta \in [0, 1]$. This is same condition $P$ assumed in theorem 3.1.

We complete this section by showing that if the process $X(t)$ has a stationary measure of product type it is possible to establish an upper bound on the spectral gap of $L$, which is of the same order as the lower bound provided by theorem 3.1.

**Proposition 4.5 (Upper bound on the spectral gap).** Suppose that $X(t)$ has a non-degenerate reversible stationary distribution of product type, and let the corresponding $d > 0$ be as in case (i) of theorem 4.3. Fix $\epsilon > 0$. If there exists a constant $0 < c < 1$ such that
\[
I_{\epsilon,c}(\epsilon) = \frac{4d^d}{\Gamma(d)} \int_0^\infty dw \int_0^\infty e^{-w} \Lambda_r(2\epsilon w) < \infty
\]
and if also
\[
I_\epsilon = \int_0^1 du \frac{\Gamma(d)}{\Gamma\left(\frac{d}{2}\right)^2} u^{\frac{d}{2}-1} (1-u)^{\frac{d}{2}-1} \Lambda_r(u) < \infty,
\]
then the spectral gap of $L$ on $L^2_{\pi_{\epsilon,N}}$ satisfies
\[
\gamma_{\epsilon,N} \leq (d+1) \sin^2 \left(\frac{\pi}{2N}\right) I_{\epsilon,c}(\epsilon)
\]
for all $N \geq \frac{3}{2} \frac{\Gamma(d+1)}{\Gamma\left(\frac{d}{2}\right)}$. This upper bound is of the same order (with respect to $N$) as the lower bound derived in theorem 3.1.
Proof. Fix $\epsilon > 0$ and $N$, and let $\pi_{\epsilon,N}$ be a reversible stationary distribution of $L$ on $S_{\epsilon,N}$. Since we assume reversibility the generator is self-adjoint, so that the spectral gap $\gamma_{\epsilon,N}$ of $L$ acting on $L_{\pi_{\epsilon,N}}^2$ has a variational characterization, recall (10). Thus, evaluating $D_{\epsilon,N}(A)$ for any specific $A$ provides an upper bound for $\gamma_{\epsilon,N}$.

For any real numbers $a_1, \ldots, a_N$ consider the observable $A(x) = \sum_{j=1}^{N} a_{j} x_{j}$. It follows from (2) that

$$A(T_{\epsilon,N}x) - A(x) = \sum_{j=1}^{N} a_{j} [T_{\epsilon,N}x - x]_{j} = [\alpha x_{i+1} - (1 - \alpha) x_{i}] [a_{i} - a_{i+1}]$$

holds for all $i$ and $\alpha$. In particular, the Dirichlet form evaluated at $A$, recall (9), can thus be rewritten as

$$D_{\epsilon,N}(A) = \frac{1}{2} \sum_{i=1}^{N-1} [a_{i} - a_{i+1}]^{2} \int \pi_{\epsilon,N}(dx) \Lambda(x_{i}, x_{i+1}) [x_{i} + x_{i+1}]^{2} \cdot$$

$$\cdot \int P(x_{i}, x_{i+1}, dx_{i+1}) \left[ \alpha - \frac{x_{i}}{x_{i} + x_{i+1}} \right]^{2}.$$ 

Since for all $i = 1, \ldots, N - 1$ the equality

$$\int \pi_{\epsilon,N}(dx) \Lambda(x_{i}, x_{i+1}) [x_{i} + x_{i+1}]^{2} = \int \pi_{\epsilon,N}(dx) \Lambda(x_{1}, x_{2}) [x_{1} + x_{2}]^{2}$$

holds, any choice of real numbers $a_{1}, \ldots, a_{N}$ yields the estimate

$$D_{\epsilon,N}(A) \leq \frac{1}{2} \int \pi_{\epsilon,N}(dx) \Lambda(x_{1}, x_{2}) [x_{1} + x_{2}]^{2} \sum_{i=1}^{N-1} [a_{i} - a_{i+1}]^{2}.$$ 

Note that

$$\int \pi_{\epsilon,N}(dx) \Lambda(x_{1}, x_{2}) [x_{1} + x_{2}]^{2} = \int \pi_{\epsilon,N}(dx) \Lambda_{\epsilon}(x_{1}, x_{2}) \Lambda_{\epsilon}(\frac{x_{1}}{x_{1} + x_{2}}) [x_{1} + x_{2}]^{2}$$

$$= \int \pi_{\epsilon,N}(dx) \Lambda_{\epsilon}(\epsilon N \frac{x_{1} + x_{2}}{\epsilon N}) \Lambda_{\epsilon}(\frac{x_{1}}{x_{1} + x_{2}}) [\epsilon N \frac{x_{1} + x_{2}}{\epsilon N}]^{2}$$

$$= \int \mu(dx) \Lambda_{\epsilon}(\epsilon N \frac{x_{1} + x_{2}}{\epsilon N + \ldots + x_{N}}) \Lambda_{\epsilon}(\frac{x_{1}}{x_{1} + x_{2}}) [\epsilon N \frac{x_{1} + x_{2}}{\epsilon N + \ldots + x_{N}}]^{2}$$

and that by theorem 4.3 there exists $T > 0$ and $d > 0$ such that $\mu(dx) = \nu(dx_{1}) \cdots \nu(dx_{N})$ where

$$\nu(dx_{1}) = \frac{dx_{1}}{T} \left[ \frac{x_{1}}{T} \right]^{\nu-1} e^{-x_{1}/T} \frac{1}{\Gamma(\frac{\nu}{2})}.$$ 

By elementary properties of the Gamma distribution, e.g. [22], it follows that $u = \frac{x_{1}}{\epsilon N + \ldots + x_{N}}$, $v = \frac{x_{1} + x_{2}}{\epsilon N + \ldots + x_{N}}$ are independent random variable with beta distribution. Their respective densities read

$$\frac{\Gamma(d)}{\Gamma(\frac{d}{2})^{2}} u^{d-1} (1 - u)^{d-1}, \quad \frac{\Gamma(Nd)}{\Gamma(d) \Gamma(Nd - d)} v^{d-1} (1 - v)^{\frac{Nd - d}{2} - 1}$$

for all $0 < u, v < 1$ and $N > 2$. Therefore

$$\int \mu(dx) \Lambda_{\epsilon}(\epsilon N \frac{x_{1} + x_{2}}{x_{1} + \ldots + x_{N}}) \Lambda_{\epsilon}(\frac{x_{1}}{x_{1} + x_{2}}) [\epsilon N \frac{x_{1} + x_{2}}{x_{1} + \ldots + x_{N}}]^{2} = \epsilon^{2} J_{\nu}, J_{\epsilon,N},$$

where $J_{\nu}$ is the $\alpha$th joint moment of independent beta random variables with parameters $\epsilon N - \frac{x_{1} + x_{2}}{x_{1} + \ldots + x_{N}}$ and $\epsilon N + \frac{x_{1} + x_{2}}{x_{1} + \ldots + x_{N}}$.
With respect to the stationary distribution. Due to the structure of the product measure \( \pi_{\epsilon,N} \) for all \( \epsilon \), respectively, yields

\[
\int_0^1 du \frac{\Gamma(d)}{\Gamma(\frac{d}{2})} u^{\frac{d}{2}-1} (1-u)^{\frac{d}{2}-1} \Lambda_{\epsilon}(\epsilon N v) \left[ N v \right]^2,
\]

The change of variables \( w = \frac{N}{2} v \) and \( 1 - v = e^{-\frac{1}{2} x} \) in the numerator and denominator of \( J_{\epsilon,N} \), respectively, yields

\[
J_{\epsilon,N} = \int_0^1 du \frac{\Gamma(d)}{\Gamma(\frac{d}{2})} u^{\frac{d}{2}-1} (1-u)^{\frac{d}{2}-1} \Lambda_{\epsilon} (2 \epsilon w) 4 w^2
\]

Using the inequality \( 1 + x \leq e^x \) for all \( x \in \mathbb{R} \) it thus follows that

\[
J_{\epsilon,N} \leq \int_0^1 \int_0^\infty dw \frac{\Gamma(d)}{\Gamma(\frac{d}{2})} w^{d-1} e^{-w d (1 - \frac{1}{2} \frac{d}{d + 1})} \Lambda_{\epsilon} (2 \epsilon w) 4 w^2
\]

for all \( N > 2 \frac{d + 1}{d} \). Therefore, for all \( N > 2 \frac{d + 1}{d} \) the Dirichlet form \( D_{\epsilon,N}(A) \) satisfies

\[
D_{\epsilon,N}(A) \leq \frac{\epsilon^2}{2} \sum_{i,j=1}^N [a_{ij} - a_{i+1}]^2 \int_0^1 du \frac{\Gamma(d)}{\Gamma(\frac{d}{2})} u^{\frac{d}{2}-1} (1-u)^{\frac{d}{2}-1} \Lambda_{\epsilon}(u)
\]

\[
\int_0^\infty dw \frac{\Gamma(d)}{\Gamma(\frac{d}{2})} w^{d-1} e^{-w d (1 - \frac{1}{2} \frac{d}{d + 1})} \Lambda_{\epsilon} (2 \epsilon w) 4 w^2.
\]

In order to estimate the spectral gap from above it remains to derive a lower bound of the variance \( \text{Var}_{\epsilon,N}(A) \), which reads

\[
\text{Var}_{\epsilon,N}(A) = \sum_{i,j=1}^N a_{ij} \text{Cov}_{\epsilon,N}(x_i, x_j)
\]

where \( \text{Cov}_{\epsilon,N}(x_i, x_j) = \pi_{\epsilon,N}(x_i, x_j) - \pi_{\epsilon,N}(x_i) \pi_{\epsilon,N}(x_j) \) denotes the covariance of \( x_i \) and \( x_j \) with respect to the stationary distribution. Due to the structure of the product measure \( \mu \) it is clear that under \( \pi_{\epsilon,N} \) the scaled vector \( \frac{x}{N} \) has a Dirichlet distribution of type \( (\frac{d}{2}, \ldots, \frac{d}{2}) \). Hence

\[
\text{Cov}_{\epsilon,N}(x_i, x_j) = \epsilon^2 N^2 \text{Cov}_{\epsilon,N}(x_i \epsilon N, x_j \epsilon N) = \epsilon^2 N \delta_{ij} - \frac{1}{N^2} \sum_{i,j=1}^N \left[ a_{ij} - \frac{1}{N} \sum_{k=1}^N a_{ik} \right]^2,
\]

which shows

\[
\frac{1}{\epsilon^2 \left( \frac{d}{2} + \frac{1}{N} \right)} \text{Var}_{\epsilon,N}(A) = \sum_{i,j=1}^N a_{ij} \frac{N \delta_{ij} - \frac{1}{N}}{N} = \sum_{i=1}^N [a_i - \frac{1}{N} \sum_{k=1}^N a_k]^2.
\]
and thus
\[ \frac{D_{c,N}(A)}{\text{Var}_{c,N}(A)} \leq \frac{1}{2} \left( d + \frac{1}{N} \right) \sum_{i=1}^{N-1} \int_0^1 \frac{\Gamma(d)}{\Gamma(\frac{d}{2})} u^{\frac{d}{2} - 1} (1 - u)^{\frac{d}{2} - 1} \Lambda_r(u) \]
\[ \cdot \int_0^\infty dw \frac{4 d^d}{\Gamma(d)} u^{d+1} e^{-w d(1 - \frac{d}{2})} \Lambda_s(2 \epsilon w) \]

for any choice of real numbers \( a_1, \ldots, a_N \), which are not all equal, and all \( N > 2 \frac{d+1}{d} \).

Finally, it remains to choose the numbers \( a_1, \ldots, a_N \) such that the ratio \( \frac{\sum_{i=1}^{N-1} \frac{1}{N} [a_i - a_{i+1}]^2}{\sum_{i=1}^{N} \frac{1}{N} [a_i - a_{i+1}]^2} \) is \( O(N^{-2}) \). This is clearly the case if we set \( a_i = \phi(i \frac{1}{N}) \) for any smooth function \( \phi : [0, 1] \rightarrow \mathbb{R} \). However, we can even find the optimal choice for \( a_1, \ldots, a_N \). Namely, it is readily verified that the symmetric matrix corresponding to the quadratic form \( \sum_{i=1}^{N-1} [a_i - a_{i+1}]^2 \) has eigenvalues \( c_k \) and corresponding eigenvectors \( a^{(k)} \) given by
\[ c_k = 4 \sin^2 \frac{\pi k}{2N}, \quad a^{(k)}_i = \cos \left( \pi k \frac{i}{N} \right), \quad i = 1, \ldots, N, \quad k = 0, \ldots, N - 1. \]

Therefore, the smallest possible value of \( \frac{\sum_{i=1}^{N-1} \frac{1}{N} [a_i - a_{i+1}]^2}{\sum_{i=1}^{N} \frac{1}{N} [a_i - a_{i+1}]^2} \) for any vector \( a \in \mathbb{R}^N \) corresponds to \( a = a^{(1)} \) and equals to \( c_1 = 4 \sin^2 \left( \frac{\pi}{2N} \right) \).

The conclusion then is that for every \( N > 2 \frac{d+1}{d} \) we found an observable \( A \) such that
\[ \frac{D_{c,N}(A)}{\text{Var}_{c,N}(A)} \leq \frac{1}{2} \left( d + \frac{1}{N} \right) 4 \sin^2 \left( \frac{\pi}{2N} \right) \int_0^1 \frac{\Gamma(d)}{\Gamma(\frac{d}{2})} u^{\frac{d}{2} - 1} (1 - u)^{\frac{d}{2} - 1} \Lambda_r(u) \]
\[ \cdot \int_0^\infty dw \frac{4 d^d}{\Gamma(d)} u^{d+1} e^{-w d(1 - \frac{d}{2})} \Lambda_s(2 \epsilon w). \]

The variational form of the spectral gap then shows that this bound is also an upper bound on the spectral gap of \( L^2_{\mathcal{E},N} \).

To finish the proof, observe that the assumed constant \( 0 < c < 1 \) allows for the further estimate \( e^{-w d(1 - \frac{d}{2})} \leq e^{-w d \epsilon} \) in the second integral, provided that \( N \geq \frac{2}{\epsilon c} \). And since this condition on \( N \) automatically implies that \( N > 2 \frac{d+1}{d} \) and thus \( \frac{d}{2} + \frac{1}{N} \leq \frac{d(2-\epsilon)}{2(d+1)} \leq \frac{d(2+\epsilon)}{2(d+1)} \), we obtain the claimed estimate on the spectral gap.

5. Example: the rarely interacting billiard lattice

Here we illustrate the use of theorems 3.1 and 4.3 with the billiard lattice model studied in [19], which was one of the main motivations for our work presented in this paper. It was argued in [19] that in the limit of rare collisions the dynamics of a billiard lattice becomes a Markov jump process. The notation used in [19] differs from ours in that we separate the rate of interaction \( \Lambda, \Lambda_r \) from the transition probability kernel \( P \), whereas in [19] the product \( \Lambda, \Lambda_r \) is denoted by \( W \), and the rate function \( \Lambda, \Lambda_r \) is denoted by \( v \). Changing equations (61) and (62) of [19] to our notation yields
\[ P(\beta, d\alpha) = \frac{3}{2} \left( 1 + \sqrt{\frac{q(1-w)}{p(1-p)}} \right) \Lambda_r(\beta) = \frac{\sqrt{2\pi}}{6} \frac{1}{\sqrt{\beta}} \vee (1 - \beta), \quad \Lambda_r(\beta) = \frac{\sqrt{2\pi}}{6} \frac{1}{\sqrt{\beta}} \vee (1 - \beta), \quad \Lambda_r(s) = \sqrt{s} \]

for the transition kernel \( P \) and the rate functions \( \Lambda, \Lambda_r \), respectively. In the above we use \( \vee \) and \( \wedge \) to denote the maximum and the minimum of two numbers, respectively.
Since the underlying mechanical model has a three-dimensional configuration space for each of the constituent particles it follows that

\[ d = 3, \quad v(dx_1) = \frac{dx_1}{T} \sqrt{\frac{x_1}{T}} e^{-\frac{x_1}{\sqrt{\pi}}}, \]

\[ p(da) = da \sqrt{a (1 - a)} \frac{8}{\pi} \Lambda_r(a) \frac{1}{Z} = da \frac{5}{2\sqrt{2}} \sqrt{a \wedge (1 - a)} \left[ \frac{1}{2} + a \vee (1 - a) \right], \]

as in theorem 4.3 provides an invariant product measure (which is the canonical Gibbs measure corresponding to the temperature \( T \)) for the mechanical model, and thus must also be invariant for the limiting jump process.

Another general property that the jump process inherits from the underlying mechanical model is that the rate function \( \Lambda \) is proportional to the square root of the total energy of the two sites that interact, i.e. \( \Lambda_r(\sigma) = \sqrt{\sigma} \) as mentioned above. This cannot be avoided when taking scaling limits of interacting mechanical models, because it corresponds to the kinematic scaling relation between the energy and the velocity (and hence the time scale). However, a rate function without a uniform lower bound leads to serious technical complications at various levels. See, for example, [13] for how this issue seriously complicates the rigorous derivation of the weak interaction limit of a related deterministic model.

Furthermore, such a rate function also complicates the rigorous analysis of the rate of convergence to equilibrium. In fact, in order to apply the results established in this paper we need to have \( \Lambda_r \) bounded from below. Recall that we showed in lemma 4.1 that the above reversible product measure is also a reversible stationary distribution for the process generated by the infinitesimal generator corresponding to any other function \( \Lambda_r \) (while keeping \( \Lambda_s \) and \( P \) unchanged). And since \( \Lambda_r \) represents the kinematic scaling, and not the nature of the energy exchange during an interaction, we will change the model of [19] in that we change \( \Lambda_r \) for all \( \alpha, \beta \) for all \( \alpha \in [0, 1] \). Hence \( p(d\beta) P(\beta, da) = p(da) P(\alpha, d\beta) \), i.e.

\[ \int p(d\beta) \int P(\beta, da) \psi(\alpha, \beta) = \int p(d\beta) \int P(\beta, da) \psi(\beta, \alpha) \]

holds for all \( \psi : [0, 1]^2 \to \mathbb{R} \).
Consider now the reference process with generator defined by
\[
\mathcal{L}^* A(x) = \Lambda^* \sum_{i=1}^{N-1} \int P^*(d\alpha) \left[ A(T_i \alpha x) - A(x) \right]
\]
with
\[
\Lambda^* = \inf_{\eta > 0} \Lambda_\alpha(\eta) \inf_{0 < \beta < 1} \Lambda_\beta(\beta) = \frac{\sqrt{\pi}}{3} \inf_{\eta > 0} \Lambda_\alpha(\eta) \quad \text{and} \quad P^*(d\alpha) = \frac{8}{\pi} \sqrt{\alpha (1 - \alpha)}.
\]
By theorem 4.3 it is clear that \(\mathcal{L}^*\) has the same stationary distribution of product type as \(\mathcal{L}\), and hence \(C^-_\epsilon = C^+_\epsilon = 1\) in the notation of theorem 3.1.

Furthermore, the estimate
\[
\frac{P(\beta, d\alpha)}{P^*(d\alpha)} = \frac{3\pi}{16} \frac{1}{\frac{1}{2} + \beta \vee (1 - \beta) \sqrt{\alpha (1 - \alpha)}} \frac{1}{\frac{1}{2} + \beta \wedge (1 - \beta) \sqrt{\beta \wedge (1 - \beta)}}
\]
provides the minorization condition \(P(\beta, d\alpha) \geq \frac{\pi}{4} P^*(d\alpha)\). In particular, this implies that the Markov chain on \([0, 1]\) with transition kernel \(P\) has a unique invariant measure.

Therefore, it follows then from theorem 4.3 that \(\mu(dx)\) is a reversible product measure, and must be unique up to the choice of \(T > 0\). Combining this with the above minorization condition for \(P\) and
\[
\int P^*(d\alpha) \alpha = \frac{1}{2}, \quad \sigma_{P^*}^2 = \int_0^1 d\alpha \frac{8}{\pi} \sqrt{\alpha (1 - \alpha)} \left[ \alpha - \frac{1}{2} \right]^2 = \frac{1}{16}
\]
we see that all assumptions of theorem 3.1 are satisfied, which provides the lower bound on \(\gamma_{\epsilon, N}\)
\[
\gamma_{\epsilon, N} \geq c \frac{C^-_\epsilon}{C^+_\epsilon} \Lambda^* \frac{1}{2} \left(1 - 4 \sigma_{P^*}^2\right) \sin^2 \left[ \frac{\pi}{N+2} \right] = \frac{\sqrt{\pi}}{32} \inf_{\eta > 0} \Lambda_\alpha(\eta) \sin^2 \left[ \frac{\pi}{N+2} \right]
\]
Using the notation as in proposition 4.5 our assumption on \(\Lambda_\alpha\) means that \(I_{s,\alpha}(\epsilon) < \infty\). Furthermore, using the explicit expression for \(\Lambda_\alpha\) we see that
\[
I_{\alpha} = \int_0^1 du \frac{\Gamma(d)}{\Gamma(\frac{d}{2})^2} u^{\frac{d}{2}-1} (1 - u)^{\frac{d}{2}-1} \Lambda_\alpha(u) = \frac{16}{15} \frac{\sqrt{\pi}}{\sqrt{\alpha}}
\]
which proves the upper bound on \(\gamma_{\epsilon, N}\).

The significance of the result of lemma 5.1 is that it provides an interesting model that fits the conditions of theorem 3.1. We would like to point out that previous to [19] the analogous two-dimensional billiard network was studied in [18]. However, in this case the uniform mixing condition (ii) of theorem 3.1 fails to hold, which is why we restricted our attention in the above to the three-dimensional setting.

6. Conclusion

The authors of [19] suggested a two-step strategy for deriving the heat equation from a mechanical model. Motivated by that we have introduced in this work a class of stochastic
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models with the aim to implement the second step of their strategy: the derivation of the heat equation from a mesoscopic stochastic model.

At present it is widely understood that a necessary ingredient to rigorously establish the hydrodynamical limit is a sharp bound on the dependence of the spectral gap of the generator on the system size. Such a bound is one of the main results of this paper.

In addition to the importance of this bound for the hydrodynamic limit, an additional value of our result is that for systems with continuous state space such bounds are hard and scarce, e.g. [9, 23]. As in those works, our method requires to assume that the rates are bounded away from zero.

In more detail: according to our main result the spectral gap of the infinitesimal generator of the process scales as \( O(N^{-2}) \) in terms of the systems size \( N \). This is precisely the kind of scaling which allows for a diffusive scaling limit, and hence the study of the hydrodynamic limit. However, we do not study the hydrodynamic limit in this paper, because it requires different ideas and techniques, and results on the spectral gap are of interest in their own right.

To keep the model as close to the mechanical ones as possible (see section 5 and [19]) it is desirable to remove the assumption on existence of a uniform lower bound of the rate function. Such results are know for certain models with degenerate rates, e.g. [1, 5]. Numerical simulations suggest that the \( O(N^{-2}) \) scaling of the spectral gap remains true also for rate functions that can approach zero. In particular for the square root of the total energy of the interacting pair, which is the rate function that appears in mechanical models due to the kinematic scaling of the velocity with the energy. However, for the model considered in this paper we do not have a rigorous proof of such a statement available at present.

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