Regular Methods of Summability and the Banach-Saks Property for Double Sequences

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Abstract. A Banach space $B$ is said to satisfy the Banach-Saks property with respect to a regular summability method if every bounded subsequence has a summable subsequence. We show that if a Banach space satisfies the Banach-Saks property with respect to a Robison-Hamilton regular summability method, for every bounded double sequence there exists a $\beta$-subsequence whose subsequences are all summable to the same limit.

1. Introduction

A Banach space $B$ is said to have the Banach-Saks property with respect to a regular summability method \((a_{i,j})_{i,j}\) if for every bounded sequence, there exists a summable subsequence. Erdős and Magidor showed that if the Banach space $B$ has the Banach-Saks property with respect to a summability method \((a_{i,j})\) then every bounded sequence has a summable subsequence such that every subsequence of the subsequence is also \((a_{i,j})\)-summable [2]. In this short note, we take advantage of a new type of subsequence of a double sequence recently introduced by Dumitru and Franco [1] to generalize the result of Erdős and Magidor to double sequences and Robison-Hamilton regular summability methods.

1.1. Definitions and Notation

In [1], a new type of double subsequence of a double sequence was introduced. Let $\psi: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be defined recursively in the following way

$$
\psi(1,n) = (n-1)^2 + 1,
$$
$$
\psi(m,1) = m^2,
$$
$$
\psi(m,n) = \begin{cases} 
\psi(m-1,n) + 1 & \text{if } 1 < m \leq n, \\
\psi(m,n-1) - 1 & \text{if } 1 < n < m.
\end{cases}
$$

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In matrix form, this looks like the following,

\[
\begin{pmatrix}
\psi(1, 1) & \psi(1, 2) & \psi(1, 3) & \psi(1, 4) & \cdots \\
\psi(2, 1) & \psi(2, 2) & \psi(2, 3) & \psi(2, 4) & \cdots \\
\psi(3, 1) & \psi(3, 2) & \psi(3, 3) & \psi(3, 4) & \cdots \\
\psi(4, 1) & \psi(4, 2) & \psi(4, 3) & \psi(4, 4) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}
= 
\begin{pmatrix}
1 & 2 & 5 & 10 & \cdots \\
4 & 3 & 6 & 11 & \cdots \\
9 & 8 & 7 & 12 & \cdots \\
16 & 15 & 14 & 13 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}
\]

Then, define a $\beta$-section $S_\beta \subseteq \mathbb{N} \times \mathbb{N}$ by

\[
S_\beta := \left\{ (m, n) \in \mathbb{N} \times \mathbb{N} \mid \frac{1}{\beta} \leq \frac{m}{n} \leq \beta \right\}.
\]

**Definition 1.1** ($\beta$-subsequence [1]). Let $x = [x_{k,l}]$ be a double sequence and let $\beta > 1$ be an extended real. The double sequence $y^{(\pi, \beta)}$ is called a $\beta$-subsequence of the double sequence $x$ if and only if there exists a strictly increasing function $\pi : \psi(S_\beta) \to \psi(S_\beta)$ such that

\[
y^{(\pi, \beta)}_{\pi(p), \pi(q)} =
\begin{cases}
\psi(p, q), & \text{if } 1 - \frac{1}{\beta} < \frac{p}{q} \text{ or } \frac{p}{q} > \beta \\
\psi(x(p, q)), & \text{if } 1 - \frac{1}{\beta} \leq \frac{p}{q} \leq \beta
\end{cases}
\]

where $z_i = x^{\psi^{-1}(1)}$. If $\beta = +\infty$, the inequalities are understood in the limit sense.

**Definition 1.2** (Summability Method [6]). Let $A$ be a four dimensional summability method that maps the complex double sequences $x$ into the double sequence $Ax$ where the $m, n$-th term of $Ax$ is given by

\[
(Ax)_{m,n} = \sum_{k,l=1}^{\infty} a_{m,n,k,l} x_{k,l}.
\]

**Definition 1.3** (P-convergence [5]). A double sequence $x = [x_{k,l}]$ has a Pringsheim limit $L$ if and only if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

\[
|x_{k,l} - L| < \epsilon,
\]

whenever $k, l > N$. In this case, we say $x$ is P-convergent and we denote it by

\[
L = \lim_{k,l \to \infty} x_{k,l}.
\]

Unless otherwise specified, the notation $\lim$ is reserved in this article to limits in the Pringsheim sense.

**Definition 1.4** (RH-regular [6]). Let $A$ be a four dimensional matrix. $A$ is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

Hamilton and Robison provide a characterization of RH-regularity that will be useful for the rest of the article.

**Theorem 1.5** (Hamilton [4], Robison [6]). A 4-dimensional matrix $A$ is RH-regular if and only if

(RH1) $\lim_{m,n \to \infty} a_{m,n,k,l} = 0$ for each $(k, l) \in \mathbb{N}^2$;

(RH2) $\lim_{m,n \to \infty} \sum_{k,l=0}^{\infty} a_{m,n,k,l} = 1$;

(RH3) $\lim_{m,n \to \infty} \sum_{k=0}^{\infty} |a_{m,n,k,l}| = 0$, for each $l \in \mathbb{N}$. 

(RH4) \[ \lim_{m,n \to \infty} \sum_{l=0}^{\infty} |a_{m,n,k,l}| = 0, \text{ for each } k \in \mathbb{N}; \]

(RH5) \[ \lim_{m,n \to \infty} \sum_{k,l=0}^{\infty} |a_{m,n,k,l}| \text{ is } P\text{-convergent;} \]

(RH6) there exist finite positive integers \( A \) and \( B \) such that
\[ \sum_{k>B \atop l>B} |a_{m,n,k,l}| < A \]
for each \((m,n) \in \mathbb{N}^2\).

In order to keep our notation consistent to [3] and [2], we introduce the following definitions.

**Definition 1.6.** Let \( S \) be a set and \( \kappa \) a cardinal. Then,

1. \( 2^S := \{ X | X \subseteq S \} \) and
2. \([S]^{\kappa} = \{ X \subseteq S | |X| = \kappa \} \).

Let \( \omega \) denote the set of natural numbers and let \( P(\omega) \) denote the set of all infinite subsets of \( \omega \).

**Definition 1.7.** A subset \( S \) of \( 2^\omega \) is Ramsey if and only if there exists \( M \in [\omega]^{<\omega} \) such that either \([M]^{<\omega} \subseteq S \) or \([M]^{<\omega} \subseteq 2^\omega \setminus S \).

In other words, an infinite subset \( S \) of \( 2^\omega \) is Ramsey if and only if there exists an infinite subset of the natural numbers \( M \) such that every infinite subset of \( M \) belongs to \( S \) or every infinite subset of \( M \) does not belong to \( S \). Lastly, in the proof of the following theorem we use the concept of a Borel set. Therefore, we remind the reader of this definition.

**Definition 1.8 (Borel Sets).** Let \( X \) be a topological space. The Borel \( \sigma \)-algebra of \( X \) is the smallest \( \sigma \)-algebra that contains all open sets of \( X \). Elements of the Borel \( \sigma \)-algebra are called Borel sets.

We remark that all Borel sets in \( P(\omega) \) are Ramsey sets [3].

2. Main Theorem

**Theorem 2.1.** Let \( \langle e_{i,j} \rangle_{i,j \in \mathbb{N}} \) be a bounded double sequence of elements in a Banach space \( B \) and \( \langle a_{i,j,k,l} \rangle_{i,j,k,l \in \mathbb{N}} \) a RH-regular summability method. Then, there exists a \( \beta \)-subsequence \( \langle e_{i,j} \rangle_{\gamma,\delta \in \mathbb{N}} \) such that:

1. every \( \beta \)-subsequence of \( \langle e_{i,j} \rangle_{\gamma,\delta \in \mathbb{N}} \) is summable with respect to \( \langle a_{i,j,k,l} \rangle_{i,j,k,l \in \mathbb{N}} \), where they all are summed to the same limit; or
2. no \( \beta \)-subsequence of \( \langle e_{i,j} \rangle_{\gamma,\delta \in \mathbb{N}} \) is summable with respect to \( \langle a_{i,j,k,l} \rangle_{i,j,k,l \in \mathbb{N}} \).

**Proof.** The proof is adapted from [2]. As in [2], we consider the topology on \( P(\omega) \) generated by the subbasis \( \{ A_n \}_{n \in \omega} \cup \{ B_n \}_{n \in \omega} \), where
\[ A_n = \{ X \in P(\omega) | n \notin X \}, \quad B_n = \{ X \in P(\omega) | n \in X \}. \]

There exists a unique bijective and increasing map \( \tau : \psi(S_\beta) \to \mathbb{N} \) (see Figure 1). We impose the topology on \( P(\psi(S_\beta)) \) induced by this map and the topology on \( P(\omega) \).

Consider a set \( X \in P(\psi(S_\beta)) \). It is clear that there exists a unique bijective and monotonically increasing function from \( \psi(S_\beta) \) to \( X \). Denote this function by \( \tau_X : \psi(S_\beta) \to X \). Now, we consider \( \beta \)-subsequence of \( \langle e_{i,j} \rangle_{i,j \in \mathbb{N}} \) corresponding to \( X \) to be the \( \beta \)-subsequence \( \langle e_{i,j}^{(\tau_X,\beta)} \rangle_{i,j \in \mathbb{N}} \) as defined in Definition 1.1.
Partition $P(\omega)$ into two sets,

\[ A = \{ X \in P(\omega) \mid \langle e^{(\pi-1)(\omega, \theta)} \rangle_{i,j \in \mathbb{N}} \text{ is } \langle a_{i,j,k,l} \rangle_{i,j,k,l \in \mathbb{N}} \text{ summable}, \right. \]

\[ B = P(\omega) \setminus A. \]

We will show next that $A$ is a Ramsey set. If this is the case, then there exists an $M \in P(\omega)$ such that either all infinite subsets of $M$ are in $A$, or else they all are not in $A$. Since each of those $M$’s corresponds to a $\beta$-subsequence of $\langle e^{(\pi-1)(\omega, \theta)} \rangle_{i,j \in \mathbb{N}}$, then they would all be either $\langle a_{i,j,k,l} \rangle_{i,j,k,l \in \mathbb{N}}$-summable, or else they would all be not $\langle a_{i,j,k,l} \rangle_{i,j,k,l \in \mathbb{N}}$-summable.

It suffices to show that $A$ is a Borel set in $P(\omega)$.

To simplify the notation, define

\[ \langle d^X_{i,j} \rangle_{i,j \in \mathbb{N}} := \langle e^{(\pi-1)(\omega, \theta)} \rangle_{i,j \in \mathbb{N}} \]

and consider

\[ B_{c,m,n,p,q} = \left\{ X \in P(\omega) \mid \left\| \sum_{i,j=1}^{\infty, \infty} a_{m,n,k,l} d^X_{i,j} - \sum_{i,j=1}^{\infty, \infty} a_{p,q,k,l} d^X_{i,j} \right\| < \epsilon \right\}. \]

With respect to this definition,

\[ A = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcup_{m,n,p,q} B_{1,k,m,n,p,q}. \]

As a result, to show that $A$ is a Borel set, it suffices to show that $B_{c,m,n,p,q}$ is open. Let $\epsilon' > 0$ be such that

\[ \left\| \sum_{i,j=1}^{\infty, \infty} a_{m,n,k,l} d^X_{i,j} - \sum_{i,j=1}^{\infty, \infty} a_{p,q,k,l} d^X_{i,j} \right\| < \epsilon' < \epsilon. \]

Let $T > 0$ be an upper bound of $\langle e_{i,j} \rangle_{i,j \in \mathbb{N}}$ and by (RH2) pick $J > 0$ large enough so that following inequalities
It can be verified that if
\[ \left| a_{m,n,k,l} \right| + \sum_{i,j=1}^{\infty} \left| a_{p,q,k,l} \right| < \frac{e - e'}{4}, \]
by (RH3),
\[ \left| \sum_{i,j=1}^{\infty} a_{m,n,k,l} + \sum_{i,j=1}^{\infty} a_{p,q,k,l} \right| < \frac{e - e'}{4}, \]
by (RH4),
\[ \left| \sum_{i,j=1}^{\infty} a_{m,n,k,l} + \sum_{i,j=1}^{\infty} a_{p,q,k,l} \right| < \frac{e - e'}{4}, \]
by (RH5).

Let \( X \in B_{e,m,n,p,q} \). We construct next an open neighborhood \( C \) of \( X \) such that \( C \subseteq B_{e,m,n,p,q} \). We start by defining the set
\[ S_K = \{ c \in \omega | \pi_{\tau^{-1}(X)} \circ \tau^{-1}(c) < \psi(K, K) \} , \]
where \( K = \max(p \in \mathbb{N} | 1/\beta \leq p/\beta \leq \beta) \).

Finally, we define
\[ C = \{ Y \in P(\omega) | \ Y \cap S_K = X \cap S_K \} . \]

It can be verified that if \( Y \in C \), then \( d_{k,l}^X = d_{k,l}^Y \). In particular,
\[ \left| \sum_{i,j=1}^{\infty} a_{m,n,k,l}d_{k,l}^Y - \sum_{i,j=1}^{\infty} a_{p,q,k,l}d_{k,l}^Y \right| = \left| \sum_{i,j=1}^{\infty} a_{m,n,k,l}d_{k,l}^X - \sum_{i,j=1}^{\infty} a_{p,q,k,l}d_{k,l}^X \right| . \]

The set \( C \) is open in the topology on \( P(\omega) \) and clearly \( X \in C \). We now show that \( C \subseteq B_{e,m,n,p,q} \).
\[ \left| \sum_{i,j=1}^{\infty} a_{m,n,k,l}d_{k,l}^Y - \sum_{i,j=1}^{\infty} a_{p,q,k,l}d_{k,l}^Y \right| \leq \left| \sum_{i,j=1}^{\infty} a_{m,n,k,l}d_{k,l}^X - \sum_{i,j=1}^{\infty} a_{p,q,k,l}d_{k,l}^X \right| + T \left| \sum_{i,j=1}^{\infty} a_{m,n,k,l} + \sum_{i,j=1}^{\infty} a_{p,q,k,l} \right| . \]
and thus
\[
\left\| \sum_{i,j=1}^{\infty} a_{m,n,k,j}d_{ij}^X - \sum_{i,j=1}^{\infty} a_{p,q,k,j}d_{ij}^X \right\| < \left\| \sum_{i,j=1}^{\infty} a_{m,n,k,j}d_{ij}^X - \sum_{i,j=1}^{\infty} a_{p,q,k,j}d_{ij}^X \right\|
+ \left\| \sum_{i,j=1}^{\infty} a_{m,n,k,j}d_{ij}^X - \sum_{i,j=1}^{\infty} a_{p,q,k,j}d_{ij}^X \right\|
+ \frac{3(\epsilon - \epsilon')}{4} < \epsilon' + \epsilon - \epsilon' = \epsilon.
\]

Therefore, \( C \subseteq B_{c,m,p,q} \). Hence every element of \( B_{c,m,p,q} \) has an open neighborhood \( C \) included in \( B_{c,m,p,q} \), therefore \( B_{c,m,p,q} \) is open.

As noted above, this implies that \( A \) is a Ramsey set. Hence there exists an infinite subset of the natural numbers \( M \) such that every infinite subset of \( M \) belongs to \( A \) or every infinite subset of \( M \) does not belong to \( A \). If \( M \not\subseteq A \), then for any infinite \( X \subset M \) the subsequence \( \langle d_{ij}^X \rangle_{r \in \mathbb{N}} \) is not \( \langle a_{i,j,k} \rangle_{i,j,k \in \mathbb{N}} \)-summable. In this case, conclusion (2) is obtained.

Otherwise, if \( M \in A \) it is clear that for all infinite \( X \subset M \) the subsequence \( \langle d_{ij}^X \rangle_{r \in \mathbb{N}} \) is \( \langle a_{i,j,k} \rangle_{i,j,k \in \mathbb{N}} \)-summable. Moreover, one can argue in the same way as in [2] to show that for all infinite \( X \subset M \) the subsequences \( \langle d_{ij}^X \rangle_{r \in \mathbb{N}} \) sum to the same limit. \( \square \)

**Corollary 2.2.** Assume that \( \langle e_{ij} \rangle_{i,j \in \mathbb{N}} \) and \( \langle a_{i,j,k} \rangle_{i,j,k \in \mathbb{N}} \) are as in Theorem 2.1. Assume further, that \( B \) satisfies the Banach-Saks property with respect to the summability method \( \langle a_{i,j,k} \rangle_{i,j,k \in \mathbb{N}} \). Then, there exists a \( \beta \)-subsequence \( \langle e_{ij} \rangle_{r \in \mathbb{N}} \) such that every \( \beta \)-subsequence of \( \langle e_{ij} \rangle_{r \in \mathbb{N}} \) is summable with respect to \( \langle a_{i,j,k} \rangle_{i,j,k \in \mathbb{N}} \), where they all are summed to the same limit.

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