abstract. It is usually assumed that any consistent interaction either deforms or retains the gauge symmetries of the corresponding free theory. We propose a simple model where an obvious irreducible gauge symmetry does not survive an interaction, while the interaction is consistent as it preserves the number of physical degrees of freedom. The model turns out admitting a less obvious reducible set of gauge generators which is compatible with the interaction and smooth in coupling constant. Possible application to gravity models is discussed.

1. Introduction

The concept of a consistent interaction first and foremost implies that the free field equations and the nonlinear ones describe the same number of physical degrees of freedom. It also assumes that the Lagrangian and its gauge transformations are smooth in coupling constants. Proceeding from these assumptions, one can seek for consistent interactions by adding vertices to a quadratic Lagrangian and deforming simultaneously the gauge symmetry transformations. The deformation technique is known as the Noether procedure or cohomological perturbation theory, see [1] for review. If no vertices are compatible with any deformation of gauge symmetry, this is considered as a no-go theorem for a consistent interaction. Various no-go results are known for gravitational interactions in various models, see [2], [3], [4], [5] and references therein.

Thus, according to the Noether procedure the interaction is considered inadmissible unless a deformation exists for the free gauge symmetry such that leaves the full Lagrangian invariant. In the next section, we suggest a simple model that does not correspond to this wide-spread opinion. In this example, the irreducible (and most obvious) parametrization of gauge symmetry obstructs interactions, while the reducible (and less obvious) form of gauge symmetry is compatible with the interaction. At the free level, both gauge symmetries are equivalent in the sense that they gauge out the same number of degrees of freedom, while they are inequivalent
with respect to inclusion of interaction. In Section 3, we discuss a more complex model of
topological gravity where a similar phenomenon can be expected to appear.

2. AN EXAMPLE OF THE MODEL WITH MULTIPLE CHOICE OF GAUGE GENERATORS

Consider the following action for the scalar and vector fields in 2d Minkowski space:

\[ S[\phi, A] = \int d^2 x \phi \left( \partial_\mu A^\mu + \frac{g}{2} A_\mu A^\mu \right). \tag{1} \]

The field equations read

\[ \partial_\mu A^\mu + \frac{g}{2} A_\mu A^\mu = 0, \quad D_\mu \phi = 0, \tag{2} \]

where \( D_\mu^\pm = \partial_\mu \pm g A_\mu \). The parameter \( g \) plays the role of the coupling constant. The commu-
tator of the “covariant derivatives” \( D^-_\mu \) gives the “curvature” of the vector field \( F = \epsilon^{\mu\nu} \partial_\mu A_\nu \),

where \( \epsilon^{\mu\nu} = -\epsilon^{\nu\mu} \) is the 2d Levi-Civita symbol.

In the free limit \( g \to 0 \), the field equations (2) have the obvious general solution \(1\):

\[ A_\mu = \epsilon^{\mu\nu} \partial_\nu \varrho, \quad \phi = C, \tag{3} \]

where \( \varrho(x) \) is an arbitrary scalar field, and \( C \) is an arbitrary constant. So, we see that the
scalar \( \phi \) carries no local degrees of freedom. The topological mode described by the constant \( C \) is fixed by the boundary conditions. Below we will always impose zero boundary conditions,
so that the unique solution will be \( \phi = C = 0 \).

Unlike \( \phi \), which is just an auxiliary field vanishing on-shell, the vector field \( A \) may assume
arbitrary values at each given instant of time. This is a direct consequence of the fact that
the two components of \( A \) are bound by a single equation. So, if one solves the field equation
for \( \partial_0 A^0 \), the right hand side will essentially involve the arbitrary function \( A^1 \), that makes the
solution arbitrary for \( A^0 \), unless any further equation is imposed. For example, if we were
restricted to the special solutions with \( F = 0 \), \( \varrho \) would not be arbitrary functional parameter
in the general solution (3), rather it would be subject to the D’Alambert equation \( \square \varrho = 0 \).
Below, we suppose that no special conditions are imposed on \( A \) like that. In particular, we
consider the general solutions with \( F(x) \neq 0 \) at almost all space-time points. (This is quite

\(1\)In this limit (1) becomes the action of the 2d abelian BF-model, if one identifies the scalar \( B \) with the field
\( \phi \) and expresses the vector field \( A^\mu \) in terms of its Hodge-dual \( \tilde{A}^\mu = \epsilon^{\mu\nu} A_\nu \). Then, \( F = \partial_\mu \tilde{A}^\mu \).
similar to the non-degeneracy assumption for the metric tensor in general relativity.) Clearly, in the space of all solutions the vector fields with \( F = 0 \) form a subspace of measure zero.

When \( g = 0 \), the action (1) enjoys the irreducible gauge symmetry

\[
\delta_\phi \phi = 0, \quad \delta_\phi A^\mu = \epsilon^{\mu\nu} \partial_\nu \phi,
\]

with \( \phi \) being the scalar gauge parameter. A simple count shows that the gauge transformation (4) leaves no room for the local physical degrees of freedom. So, the theory is topological as might be expected from the analysis of the general solution (3).

With the interaction switched on (\( g \neq 0 \)) the field \( \phi \) is still fixed on the general solution for \( A \). Indeed, the second equation in (2) has the differential consequence

\[
\epsilon^{\mu\nu} D^-_\mu D^-_\nu \phi = -gF \phi = 0.
\]

As the above equation states that the product of two factors \( F \cdot \phi \) vanishes, the dynamics bifurcates into two branches: either \( F = 0 \) or \( \phi = 0 \). If \( F \neq 0 \), then \( \phi = 0 \). This branch has a smooth limit to the case \( g = 0 \), where \( \phi \) vanishes, while \( F \neq 0 \) for general solutions of this branch. The alternative option \( F = 0 \) is not smoothly connected with the free case, as it corresponds to the solution \( A_\mu = \partial_\mu \rho \) with \( \rho \) subject to the D’Alambert equation \( \Box \rho = 0 \). As we are going to have the general solution in the free limit, hereafter we opt for the branch with \( \phi = 0 \). Then, we still have the first equation in (2). It is a single equation imposed on two components of \( A_\mu \):

\[
\partial_0 A_0 + g/2(A_0)^2 = \partial_1 A_1 + g/2(A_1)^2.
\]

The system is under-determined, so the general solution obviously involves arbitrary function. With arbitrary \( A_1 \) in the right hand side, the solution exists for a single unknown function \( A_0 \). The general solution for \( A_0 \) essentially depends on the arbitrary function \( A_1 \). From this mere fact one can expect that the vector field \( A \) is pure gauge as it has been in the free theory for the same reason. Also notice that the field equation (6) has no differential consequences, in particular, it by no means implies \( F = 0 \).

Equation (3) is smooth in \( g \) and describes a system with no local degrees of freedom for any value of \( g \), including \( g = 0 \). Proceeding from that, one could expect that the gauge symmetry of the model with \( g \neq 0 \) is a deformation of the transformation (4) for \( g = 0 \). These expectations,
however, do not come true. The point is that the quadratic vertex \( \frac{g}{2} A^2 \) in the equation (6) is \textit{not} invariant under the gauge transformation (4) even modulo the free equation,

\[
\delta_{\varrho} \left( \frac{g}{2} A^2 \right) = g A_\mu \epsilon^{\mu \nu} \partial_\nu \varrho \neq 0.
\]  

This means that the gauge symmetry (4) can’t be deformed to make it consistent with the quadratic vertex in Eq. (6). If the paradigm of cohomological perturbation theory was naively applied to this case, it could be interpreted as a no-go theorem for the interaction.

In our recent paper [7], the existence of a local gauge symmetry has been proven for any under-determined regular system of 2\(d\) field equations. In the case under consideration the corresponding gauge transformations read

\[
\delta_\varepsilon \phi = 0, \quad \delta_\varepsilon A^\mu = g \varepsilon^\mu - \epsilon^{\mu \nu} D^+ \nu (F^{-1} D^+ \lambda \varepsilon^{\lambda}),
\]

\(\varepsilon^\mu\) being an arbitrary vector parameter. Unlike (4), these gauge transformations are reducible.

The corresponding gauge-for-gauge transformations read

\[
\delta_\kappa \varepsilon^\mu = \epsilon^{\mu \nu} D^+ \nu \kappa,
\]

where \(\kappa\) is an arbitrary scalar parameter. The infinitesimal gauge transformations (8) form a closed gauge algebra with the following commutation relations:

\[
[\delta_\varepsilon_2, \delta_\varepsilon_3] = \delta_\varepsilon_4, \quad \varepsilon_3^\mu = \left( \frac{D^+ \varepsilon^{\lambda}_2}{F} \right) D^+ \mu \left( \frac{D^+ \varepsilon^{\lambda}_1}{F} \right) + g \epsilon^{\mu \nu} \varepsilon_{1 \nu} \left( \frac{D^+ \varepsilon^{\lambda}_2}{F} \right) - (\varepsilon_1 \leftrightarrow \varepsilon_2).
\]

Notice that the gauge parameter \(\varepsilon_3\) is defined here only modulo the reducibility relation (9).

Again, a covariant count of physical degrees of freedom (using, for example, the general formulae from [6]) shows that the transformations (8), (9) gauge out all the degrees of freedom. So, the model (1) is indeed topological for any \(g\).

It is instructive to consider the limit \(g \to 0\) for the transformation (8, 9):

\[
\delta_\varepsilon \phi = 0, \quad \delta_\varepsilon A^\mu = -\epsilon^{\mu \nu} \partial_\nu (F^{-1} \partial_\lambda \varepsilon^{\lambda}), \quad \delta_\kappa \varepsilon^{\lambda} = \epsilon^{\lambda \mu} \partial_\mu \kappa.
\]

As is seen, this reproduces the transformation of the free theory (1) with \(\varrho = -F^{-1} \partial_\lambda \varepsilon^{\lambda}\).

Since the gauge parameters \(\varepsilon^\mu\) enter these transformations through a single function \(g\), the

\[\text{The transformations can be made regular in a vicinity of } F = 0 \text{ by rescaling the gauge parameter: } \varepsilon^{\lambda} \to F^2 \varepsilon^{\lambda}. \text{ Then the special field configurations (} F = 0 \text{) are precisely those that are unaffected by the infinitesimal gauge transformations.}\]
gauge symmetry appears to be reducible. Altogether, the transformations (11) gauge out as many degrees of freedom as the single gauge transformation (4). So, one may regard (11) as a weird form of the “simplest”, i.e., irreducible gauge transformation (4). We see that the free limit of the model (1) admits a multiple choice for the gauge generators, including reducible and irreducible options. Both the options equally well gauge out the degrees of freedom at the free level, while they are inequivalent from the viewpoint of interaction\textsuperscript{3}. The simplest irreducible choice (4) does not survive the interaction, while the less obvious reducible choice of the gauge transformations (11) turns out compatible with the cubic vertex. Some other examples of multiple-choice of gauge symmetry has been recently noticed in [8] for free models of various spin fields. As we see here, the distinctions between the different forms of free gauge transformations can become crucial at the level of interaction.

This example demonstrates a potential way of bypassing the “no-go” theorems for the existence of consistent interactions in various field-theoretical models. Most of these theorems are deduced from obstructions to deformation of a particular set of gauge generators. Similar to the example above, the simplest set of gauge generators may happen to obstruct any nontrivial deformation, while a less obvious alternative set can be compatible with reasonable interactions.

3. A POSSIBLE MODEL FOR TOPOLOGICAL GRAVITY

One can regard the action (1) as a pattern for constructing more realistic physical models demonstrating the multiple-choice gauge symmetry phenomenon. Below, we briefly discuss a theory involving the metric tensor $g$ and the scalar field $\phi$. The action reads

$$S[\phi, g] = \int \phi R \sqrt{-g} d^4x,$$

where $R$ is the scalar curvature. Rescaling the metric $g \rightarrow \phi g$, it is even possible to induce a kinetic term for the scalar field, so that the theory may resemble the Brans-Dicke gravity [9].

\textsuperscript{3}Strictly speaking relations (11) do not define a complete set of gauge transformations for the free equations, since the gauge transformation (4) can’t be obtained by specifying the gauge parameter $\varepsilon^\lambda$ in (11). Nonetheless, the set of gauge generators (11) is big enough to gauge out all the degrees of freedom and the generators form a closed gauge algebra, whose commutation relations follow from (10) by setting $g = 0$.\n
The equations of motion resulting from (12) are equivalent to

\[ R = 0 , \quad (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box - R_{\mu\nu})\phi = 0 . \]  

(13)

Consider the linearization of these equations over the background of \( \phi = 0 \) and and flat metric. The linearized system includes a single scalar equation for the metric perturbation and the overdetermined system \((\partial_\mu \partial_\nu - \eta_{\mu\nu} \Box)\phi = 0\) for the perturbation of the scalar field. The general solution for the latter system is the linear function \( \phi = C_\mu x^\mu + C \), with \( C_\mu \) and \( C \) being arbitrary constants. Imposing zero boundary conditions, we get \( \phi = 0 \). So, there is no propagating degrees of freedom associated with the scalar field. This fully corresponds to the model of previous section, with the only difference that the overdetermined system for the scalar field is now of the second order. One can verify that \( \phi \) remains an auxiliary field on the curved background, though the analysis is more cumbersome comparing to the previous example. As \( \phi \) does not propagate, the metrics tensor satisfies the only equation \( R = 0 \), which is analogous to the equation \( \partial_\mu A^\mu + (g/2)A^2 = 0 \) from the previous section. A regular single equation essentially involving more than one unknown field should describe a pure gauge system with no local degrees of freedom. As a result the model (12) must have a rich gauge symmetry, which is by no means exhausted by the general coordinate transformations. (The four parameters of diffeomorphisms are clearly insufficient for gauging out ten components of the metric tensor.)

Furthermore, the gauge transformations in question can be reconstructed by the gauge symmetries of the equation \( R = 0 \) alone. Indeed, if \( \delta_\varepsilon g \) is such a symmetry, then \( \delta_\varepsilon R = \hat{A}R \) for some differential operator \( \hat{A} \) depending on \( g, \varepsilon \), and their derivatives. Denoting by \( \hat{A}^* \) the formal adjoint of the differential operator \( \hat{A} \) with respect to the integration measure \( \sqrt{-g} d^4 x \), we can extend the transformation \( \delta_\varepsilon g \) to the gauge invariance of the action (12) by setting \( \delta_\varepsilon \phi = -\hat{A}^* \phi \).

Finding a complete set of gauge generators for the action (12) and their reducibility relations (if any) appears to be a rather nontrivial problem, which yet to be solved. The free gauge transformations, being taken in the most simple form, resist any deformation to the nonlinear ones, much as it happens in the example of the previous section. So, there should exist another set of gauge generators that does not reduce in the flat limit to the simplest generators of the free theory. We are going to address this issue elsewhere.
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REFERENCES

[1] M. Henneaux, *Consistent Interactions Between Gauge Fields: The Cohomological Approach*, Contemp. Math. 219 (1998) 93.

[2] M. Henneaux, G. L. Gomez and R. Rahman, *Gravitational Interactions of Higher-Spin Fermions*, JHEP 1401 (2014) 087.

[3] E. Joung and M. Taronna, *Cubic-interaction-induced deformations of higher-spin symmetries*, JHEP 03 (2014) 103.

[4] X. Bekaert, N. Boulanger and M. Henneaux, *Consistent deformations of dual formulations of linearized gravity: A no-go result*, Phys.Rev. D67 (2003) 044010.

[5] N. Boulanger, T. Damour, L. Gualtieri and M. Henneaux, *Inconsistency of interacting, multi-graviton theories*, Nucl.Phys. B597 (2001) 127-171.

[6] D. S. Kaparulin, S. L. Lyakhovich and A. A. Sharapov, *Consistent interactions and involution*, JHEP 1301 (2013) 097.

[7] S. L. Lyakhovich and A. A. Sharapov, *Gauge symmetries in 2D field theory*, [arXiv:1312.2671[math-ph]].

[8] D. Francia, S. L. Lyakhovich and A. A. Sharapov, *On the gauge symmetries of Maxwell-like higher-spin Lagrangians*, Nucl.Phys. B881 (2014) 248.

[9] C. H. Brans and R. H. Dicke, *Mach’s Principle and a Relativistic Theory of Gravitation*, Phys. Rev. 124 (1961) 925-935.

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