The Radon Transformation and Its Application in Tomography

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Abstract. In a mathematical setting, the radon transformation is in the form of integration, which was proposed by Johann Radon in 1917. Radon transform has been applied in multiple fields of study, especially in medical research. The Radon transform can represent the data obtained from tomographic scans, so the inverse of Radon transform can be used to reconstruct the original projection properties, which is useful in computed axial tomography, electron microscopy, reflection seismology, and in the solution of hyperbolic partial differential equations. This paper summarizes the indispensable role of the radon transformation and gives a simple proof of the back-projection formula.

1. Introduction

X-rays, similar to visible light, are a form of electromagnetic radiation. The wavelengths of the X-rays range from $3 \times 10^{10}$Hz to $3 \times 10^{19}$Hz [1]. Tomography is a modern technology scanning a three or two dimensional object by imagining multiple one-dimensional slices of the object through X-rays. In a computerized tomography (CT) scan, these one-dimensional slices are parallel X-ray beams. Two measurement data observed by a CT scanner are initial intensity $I_0$ of each X-ray beam around the radiation source and the final intensity $I_f$ obtained by the radiation detector [2]. Intuitively, the denser solid object X-ray beams pass through, the greater decrease in the intensity will happen. Thus, by measuring the change of the intensity of a single beam, knowledge about the density of the medium can be gained. As for more complex object, taking a human body as an example, X-ray beams can pass through many different objects, such as blood, bone, and tissue. This causes the failure of gaining enough information by just measuring the changes of intensities of a single beam in a single direction. By changing the relative position of the source and the detector, more information on the object can be gleamed.

This survey will give proof of the backprojection formula, which is one of the indispensable tools used in the reconstruction process of the medical imaging. One can refer to [3-12] and the references therein for more details.

First, we give several useful definitions.

Definition 1. The intensity of the beam, $I(x)$, at a distance $x$ from the source is defined as:

$$I(x) = N(x) \cdot E$$

(1)

where $E$ is the energy level of an X-ray and $N(x)$ is the rate of photon propagation.

Definition 2. The amount of energy absorbed per unit thickness of material at a distance $x$ from the source is called Attenuation coefficient $A(x)$.

The Beer-Lambert Law reveals the relationship between intensity and attenuation coefficient.
Definition 3. Beer-Lambert Law the intensity $I(x)$ of a non-refractive, monochromatic X-ray beam that crosses through a homogeneous medium is:

$$I(x) = e^{-A(x)x}$$  \hspace{1cm} (2)

Rewriting beer-Lambert Law in a differential form:

$$\frac{dl}{dx} = -A(x)I(x)$$

We denote the initial intensity as $I(x_0) = I_0$ and the final intensity at $x_1$ as $I(x_1)$

Thus, we further get:

$$\int_{x_0}^{x_1} \frac{dl}{I(x)} = - \int_{x_0}^{x_1} A(x)dx$$

Therefore, CT scan algorithm is used for obtaining the attenuation coefficient.

2. Parameterization for a line

In $\mathbb{R}^2$, a general line can be expressed as $ax+by=c$, where $a, b, c \in \mathbb{R}$ and $a^2 + b^2 \neq 0$. Since $a^2 + b^2 
eq 0$, we can write the equation of a line as:

$$\frac{a}{\sqrt{a^2 + b^2}}x + \frac{b}{\sqrt{a^2 + b^2}}y = \frac{c}{\sqrt{a^2 + b^2}}$$  \hspace{1cm} (3)

Notice that:

$$\left(\frac{a}{\sqrt{a^2 + b^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2 + b^2}}\right)^2 = 1$$

We define $\cos(\theta) = \frac{a}{\sqrt{a^2 + b^2}}$ and $\sin(\theta) = \frac{b}{\sqrt{a^2 + b^2}}$ where $\theta \in [0, 2\pi)$

Denote $\vec{t} = (\cos(\theta), \sin(\theta))$

Notice that $\frac{c}{\sqrt{a^2 + b^2}}$ is the distance between the line and the origin. And we let $t = \frac{c}{\sqrt{a^2 + b^2}}$

We can write the line as $l_{t, \theta} \rightarrow x \cdot \cos(\theta) + y \cdot \sin(\theta) = t$  \hspace{1cm} (4)

Further, we can describe $l_{t, \theta}$ in terms of real number s:

$$(x(s), y(s)) = (t \cdot \cos(\theta) - s \cdot \sin(\theta), t \cdot \sin(\theta) + s \cdot \cos(\theta)), s \in \mathbb{R}$$

3. The radon transform

Definition 4. For a function $f(t, \theta) \in C_c(\mathbb{R}^2)$, we define Radon Transform of $f$, $Rf$, where $t \in \mathbb{R}, \theta \in (0, 2\pi]$ as a line integration along $l_{t, \theta}$

$$Rf(t, \theta) = \int_{-\theta}^{\theta} f(ds) = \int_{-\theta}^{\theta} f(x(s), y(s))ds$$

If $f(t, \theta) \in C_c(\mathbb{R}^2)$ means $f$ has a compact support, i.e., $f$ vanishes outside a compact set in $\mathbb{R}^2$. The reason we add such a requirement on $f$ is that volume of an object in the medical imaging context is finite. Since the Radon transform is a line integration, it is a powerful tool to calculate the total density of a function along a given line $l$. Also, if we take the Radon transform along different lines with different tilt angles, we are able to get several density functions for the object. In mathematical language, $\ln \frac{(I_0)}{(I)}$ is known information measured by CT scanner, but we need to compute $A(x)$.

Before we investigate inversion formulas for the Radon transform, it is necessary to provide some useful properties of the Radon transform.

Theorem 5. $\alpha, \beta \in \mathbb{R}, f, g \in C_c(\mathbb{R}^2)$, then we have
\[ R(\alpha f + \beta g) = \alpha R(f) + \beta R(g) \]

\[ Rf(t, \theta) = Rf(-t, -\theta) \]

\[ Rf(t, \theta) = \int_{-\infty}^{\infty} f(t \cdot \cos(\theta) - s \cdot \sin(\theta), t \cdot \sin(\theta) + s \cdot \cos(\theta)) \, ds \]

From this theorem, we know the radon transform is linear and even.

**Proof.** The first one can be proved by linearity of line integration. Suppose \( I_L(f) \) means integration of \( f \) along a line \( L \). Then by definition of the line integration, it follows that \( I_L(\alpha f + \beta g) = \alpha I_L(f) + \beta I_L(g) \). The second statement holds since the line \( \theta \) and \( \overrightarrow{\theta} \) are the same line. As for the third one, it can be proved directly from the definition of the line integration and parameterization of \( \overrightarrow{\theta} \).

4. The Fourier transform

**Definition 6.** If \( f \) is an absolutely integrable function on \( R \), the Fourier Transform of \( f \), denoted as \( Ff \), is defined as:

\[ Ff(y) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i xy} \, dx \]

**Theorem 7.** \( \alpha, \beta \in R, f, g \in L^1(R) \), then we have

\[ F(\alpha f + \beta g) = \alpha F(f) + \beta F(g) \]

\[ Ff(\omega) < \infty \]

**Definition 8.** If \( f \) is an absolutely integrable function on \( R \), the inverse Fourier Transform of \( f \), denoted as \( F^{-1}f \), is defined as:

\[ F^{-1}f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(f)(x) e^{i\omega x} \, d\omega \]

**Definition 9 (Schwartz Space \( S \)).** The Schwartz Space consists of all infinitely differentiable functions on \( R^{-1} \) satisfying: \( \forall k, L \in N^+ \)

\[ \sup_{x \in R} \left| x^k \frac{\partial^L}{\partial x^L} f(x) \right| < \infty \]

**Theorem 10. (The Fourier Inversion Theorem).** If \( f \in S \), then \( F^{-1}(Ff)(x) = f(x) \)

**Definition 11.** For an absolutely integrable function \( f \) defined on \( R^2 \), the 2-dimensional Fourier transform of \( f \), denoted as \( F_2f \), is defined as:

for \( (x, y) \in R^2 \)

\[ F_2f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-i(s \cdot x + t \cdot y)} \, ds \, dt \]

**The central slice theorem

**Theorem 12.** If \( f \) is absolutely integrable on \( R^2 \), then \( \forall \theta \in [0,2\pi) \), we have:

\[ F_2f(s \cdot \cos(\theta), s \cdot \sin(\theta)) = F(Rf)(s, \theta) \]

**Proof.** Notice that:

\[ F_2f(s \cdot \cos(\theta), s \cdot \sin(\theta)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-is(x \cdot \cos(\theta) + y \cdot \sin(\theta))} \, dx \, dy \]

We parameterized the line \( \overrightarrow{\theta} \) as:

\[ x(s) = t \cdot \cos(\theta) - s \cdot \sin(\theta), \]

\[ y(s) = t \cdot \sin(\theta) + s \cdot \cos(\theta), \]

\[ t = x \cdot \cos(\theta) + y \cdot \sin(\theta) \]
Computing the determinant of the Jacobian for \( x(s) \) and \( y(s) \), we have:

\[
\det \begin{vmatrix}
\frac{\partial x}{\partial t} & \frac{\partial x}{\partial s} \\
\frac{\partial y}{\partial t} & \frac{\partial y}{\partial s}
\end{vmatrix} = 1
\]

Thus, \( ds/dt = dx/dy \). Then we have:

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-is(x\cos(\theta) + y\sin(\theta))} dxdy
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t \cdot \cos(\theta) - s \cdot \sin(\theta), t \cdot \sin(\theta) + s \cdot \cos(\theta)) e^{-ist} dsdt
\]

\[
= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(t \cdot \cos(\theta) - s \cdot \sin(\theta), t \cdot \sin(\theta) + s \cdot \cos(\theta)) ds \right) e^{-ist} dt
\]

\[
= \int_{-\infty}^{\infty} \left( Rf(t, \theta) \right) e^{-ist} dt
\]

\[
= F(Rf)(s, \theta)
\]

5. Recovering attenuation coefficient function

In this section, we’ll discuss how to recover the attenuation coefficient. In the physical setting, the Radon transform \( Rf(t, \theta) \) represents the information of total intensity along a line \( L_{t, \theta} \). With the help of a CT scanner, we can measure the changes of intensity of an X-ray beam that passes through the object along \( L_{t, \theta} \). By changing direction \( \theta \), multiple slices are created, which can help us better gain knowledge about the object the X-ray that shot through.

**Definition 13.** Let \( f = f(t, \theta) \). The backprojection, \( Bf \), at the position \((x, y)\), is defined as:

\[
Bf(x, y) = \frac{1}{\pi} \int_{0}^{\pi} f(x \cdot \cos(\theta) + y \cdot \sin(\theta), \theta) d\theta
\]

We define backprojection on the radon transform, we get:

\[
BRf(x, y) = \frac{1}{\pi} \int_{0}^{\pi} Rf(x \cdot \cos(\theta) + y \cdot \sin(\theta), \theta) d\theta
\]

**Theorem 14.** Suppose \( f \) is an absolutely integrable function defined on \( R^2 \), then:

\[
f(x, y) = \frac{1}{2} BF^{-1}[|s|]F(Rf)(s, \theta)(x, y)
\]

Proof. Notice that:

\[
f(x, y) = F_2^{-1} F_2 f(x, y)
\]

\[
= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_2 f(X, Y) e^{i(xX + yY)} dXdY
\]

Notice that the formula of changing variable from Cartesian to polar coordinated is:

\[
X = S \cdot \cos(\theta), Y = S \cdot \sin(\theta)
\]

Thus, Jacobin determinant is:

\[
\det \begin{vmatrix}
\frac{\partial X}{\partial S} & \frac{\partial X}{\partial \theta} \\
\frac{\partial Y}{\partial S} & \frac{\partial Y}{\partial \theta}
\end{vmatrix} = |S|
\]

Which means \( dXdY = |S| dSd\theta \). We further get:

\[
\frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_2 f(X, Y) e^{i(xX + yY)} dXdY
\]

\[
= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_2 f(S \cdot \cos(\theta), S \cdot \sin(\theta)) e^{is(x\cos(\theta) + y\sin(\theta))} |S| dSd\theta
\]

By using the central slice theorem, we have:
\[
\frac{1}{4\pi^2} \int_0^\pi \int_{-\infty}^{\infty} F_2(S \cdot \cos(\theta), S \cdot \sin(\theta)) e^{iS(x\cos(\theta) + y\sin(\theta))} |S| dS d\theta
\]

\[
= \frac{1}{4\pi^2} \int_0^\pi \int_{-\infty}^{\infty} F(Rf(S, \theta)) e^{iS(x\cos(\theta) + y\sin(\theta))} |S| dS d\theta
\]

Notice that:

\[
\int_{-\infty}^{\infty} F(Rf(S, \theta)) e^{iS(x\cos(\theta) + y\sin(\theta))} |S| dS
\]

\[
= 2\pi \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} F(Rf(S, \theta)) e^{iS(x\cos(\theta) + y\sin(\theta))} |S| dS \right)
\]

\[
= 2\pi F^{-1} [ |S| F(Rf)(S, \theta) ] (x \cdot \cos(\theta) + y \cdot \sin(\theta), \theta)
\]

Thus,

\[
\frac{1}{4\pi^2} \int_0^\pi \int_{-\infty}^{\infty} F(Rf(S, \theta)) e^{iS(x\cos(\theta) + y\sin(\theta))} |S| dS d\theta
\]

\[
= \frac{1}{2\pi} \int_0^\pi F^{-1} [ |S| F(Rf)(S, \theta) ] (x, y) |S| dS d\theta
\]

6. Discussion

We discuss the Radon transform and its simple properties at first. We also review the definition and Fourier transform and its properties analogous to the properties of the Fourier transform. We prove the central slice theorem and backprojection formula at last.

7. Conclusion

By Radon transform, we can go inverse and deduce the properties of the projection of a X-ray through the material. And we can also figure out the density of the material. Such a method can be used to identify the material. What’s more, this method can be applied in spatially resolved spectroscopy, radio-astronomy, diagnostic radiology, and computer vision for object recognition. With this variety of applications, we look forward to seeing more examples of its usage. Since Radon transform is usually accompanied by Fourier transform, we would further explore the relationship between these two inspiring methods and their usage.

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