Renormalization Group Improving the Effective Action

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Abstract: The existence of fluctuations together with interactions leads to scale-dependence in the couplings of quantum field theories for the case of quantum fluctuations, and in the couplings of stochastic systems when the fluctuations are of thermal or statistical nature. In both cases the effects of these fluctuations can be accounted for by solutions of the corresponding renormalization group equations. We show how the renormalization group equations are intimately connected with the effective action: given the effective action we can extract the renormalization group equations; given the renormalization group equations the effects of these fluctuations can be included in the classical action by using what is known as improved perturbation theory (wherein the bare parameters appearing in tree-level expressions are replaced by their scale-dependent running forms). The improved action can then be used to reconstruct the effective action, up to finite renormalizations, and gradient terms.

I. INTRODUCTION:

The important role of the effective action, and its specialization to constant fields, the effective potential, as fundamental constructs in quantum field theory has now been appreciated for quite some time\textsuperscript{[1–4]}. The main purpose of the present paper is to show how radiative corrections and (quantum) fluctuations may be taken into account in the effective action and effective potential by means of renormalization group improved perturbation theory, a concept originally introduced long ago within the context of QED in the landmark work by Gell-Mann and Low\textsuperscript{[5]}. (For discussions see\textsuperscript{[6–10]}.)

An expression is said to be renormalization group improved if the bare parameters in the corresponding (typically tree-level) expression are replaced by their scale-dependent running forms (calculated to some given order in perturbation theory). Thus, an improved quantity (whether it be the effective action, effective potential, a Green function, \textit{etc.}) is one which combines perturbation theory plus the renormalization group. Such improved quantities are extremely useful since they allow us to go beyond the strict limitations of ordinary perturbation theory. In practice, an improved quantity is calculated to a certain number in \textit{loops} but to \textit{all} orders in the couplings and includes much more physics than the same quantity calculated to a given finite order in the coupling. In the loop-expansion, we are in effect, summing up infinite subclasses of diagrams. Although the concept of improved perturbation theory has been around since the work of Gell-Mann and Low\textsuperscript{[5]}, and has achieved the status of “folklore”, it is rather surprising that explicit treatments in the literature are somewhat sparse\textsuperscript{[11]}. The purpose of this paper is to amend this omission and at the same time to show how RG improvement may be carried out in a straightforward manner by computing the effective action and the renormalization group equations directly from the path integral \textit{without the need to handle Feynman diagrams}.

We shall show that solving the renormalization group equations and constructing the improved action is enough to imply a “reconstruction theorem” whereby the most important pieces of the effective action (the leading logarithms) can be deduced even if the effective action is itself unknown. (The importance of the leading logarithms in perturbation theory has been appreciated since the work of the Russian School in the early days of quantum field theory. See\textsuperscript{[8]} for an accessible discussion.) We shall demonstrate how the renormalization group equations may be obtained by inspection of the effective action and how renormalization group \textit{improving} the bare action yields the leading logarithms in the renormalized effective action. (More precisely, the \textit{n}-loop effective \textit{potential} is sufficient to determine almost all the renormalization group equations to \textit{n}-loops [all but that for the wavefunction renormalization]. And conversely the renormalization group equations are enough to enable us to first improve the classical potential and then to reconstruct the effective potential up to finite renormalizations.) This reconstruction theorem is very important: Often it is much easier to calculate the divergent parts of the bare effective action than it is to calculate the renormalized effective action. The logarithmically divergent pieces are however enough to yield the renormalization group equations, which then can be used to improve the classical action, and finally via the reconstruction theorem we can deduce the leading logarithms in the effective action itself. This specifies the effective action up to finite renormalizations and gradient terms.

Since we want this article to be widely accessible, we shall be as clear and explicit as possible. We shall first carry out the demonstration explicitly for $\lambda(\phi^3)_4$ ($\lambda\phi^4$ in $n = 4$ dimensions) at one loop, and then move on to calculate the improved action for $\lambda(\phi^3)_6$ ($\lambda\phi^6$ in $n = 6$ Euclidean spacetime) where the nontrivial wavefunction renormalization must be taken into account. For completeness we add some comments concerning the $\lambda(\phi^3)_3$ and $\lambda(\phi^n)_2$ theories.
The plan of this paper is as follows. In the next section we briefly develop and exhibit the formal unrenormalized expression for the effective action valid for any scalar field theory and show how the associated renormalization group equations (RGE) may be obtained from it by inspection. The recipe is carried out explicitly for $\lambda \phi^4$ in $n=4$ Euclidean spacetime employing a simple momentum cutoff scheme that is well suited for treating homogeneous background fields and computing effective potentials (which are the homogeneous field limits of the effective action). We extract the RGE’s by inspection, solve them and use their solutions to improve the classical action. When we expand out the improved classical action (to one loop) we can use it to reconstruct the leading logarithm terms in the effective action (to one loop). We next turn to the Schwinger proper-time regularization scheme (useful for calculating the effective action when the background field is not necessarily homogeneous) and re-derive the RGE’s for $\lambda \phi^3$ as a check. We then use this formalism to calculate the leading logarithms in the effective action for $\lambda \phi^3$ in $n=6$, where non-trivial wavefunction renormalization effects are known to show up. Once again, we deduce the associated RGE’s by inspection and may use their solutions to improve the classical action. The formalism is then applied to $\lambda (\phi^n)_2$, and write down the renormalization group equations, effective potential, and leading logarithm portion of the effective action for arbitrary polynomial interactions.

We conclude with a summary and brief comments on the use of improved perturbation theory in the context of stochastic field theories.

II. PRELIMINARIES

We consider a general renormalizable scalar field theory. At the quantum level its generating functional $Z[J]$ can be written as

$$Z[J] = N \int [D\phi] \exp \left\{ -\frac{S[\phi]}{\hbar} + \int d^n x J(x) \phi(x) \right\} .$$

where $N$ is a normalization factor, and $\hbar$ is (for the purposes of this article) a dimensionless loop-counting parameter. (If we wish to convert the final answers back to physical units, the $\hbar$ will of course convert back into the physical Planck constant.) By following the standard procedures, we can define

$$W[J] = +\hbar \ln Z[J] ,$$

the generating functional for the connected correlation functions, and its Legendre transform $\Gamma[\bar{\phi}]$, with $J$ and $\bar{\phi}$ being conjugate variables to each other (in the Legendre transform sense of the word). The loop expansion for $\Gamma[\bar{\phi}]$ is based on the following background field decomposition of the field $\phi$

$$\phi = \bar{\phi} + \varphi ,$$

in which $\bar{\phi}[J]$ is considered a mean field which satisfies the classical equation of motion with source $J$,

$$\left( \frac{\delta S}{\delta \bar{\phi}} \right)_{\bar{\phi}[J]} = J ,$$

and $\varphi$ is the quantum fluctuation of the field about the given background. It is usually assumed that this equation has unique solutions $\bar{\phi}[J]$, at least for small $J$, and further that for vanishing source $J = 0$, the unique solution is the constant zero mean field $\bar{\phi} = 0$. This certainly is valid for a symmetric vacuum in scalar field theories, but it is not an essential aspect of the formalism. For instance in gravity, zero source corresponds to flat Minkowski space, $g_{\mu\nu} = \eta_{\mu\nu} \neq 0$, while in sigma-models zero source corresponds to constant position on the field manifold.

As is well known, the calculation of the first quantum correction to the effective action reduces to evaluating a Gaussian functional integral. The normalization factor turns out to be important in that it will cancel out the divergent vacuum energy density (i.e., the cosmological constant term) from the effective action. From the normalization condition

$$Z[0] = 1 \rightarrow N^{-1} = \int [D\phi] \exp \left\{ -\frac{S[\phi]}{\hbar} \right\} ,$$

we derive the following equation for the generating functional of the connected Green functions $W[J]$
\[ W[J] = W_0[J] + \hbar W_1[J] + O(\hbar^2), \]  
\[ W_0[J] = - \{ S[\phi[J]] - S[\phi[0]] \} + \int d^n x \ J(x) \phi(x), \]  
and
\[ W_1[J] = \sqrt{\frac{\det S_2(\phi[0])}{\det S_2(\phi[J])}} = - \frac{1}{2} \left\{ \text{tr} \ln \frac{S_2(\phi[J])}{S_2(\phi[0])} - \text{tr} \ln \frac{\mu^2_{\infty}}{\mu^2_{\infty}} \right\}. \]

Here \( \phi[J] \) is the solution to the classical equations of motion with source \( J \), and \( \phi[0] \) the classical solution with no source, that is
\[ \left( \frac{\delta S}{\delta \phi} \right)_{\phi[J]} = J, \quad \text{and} \quad \left( \frac{\delta S}{\delta \phi} \right)_{\phi[0]} = 0. \]

Furthermore
\[ S_2(x_1, x_2; \phi) = \left( \frac{\delta^2 S[\phi = \bar{\phi}]}{\delta \phi(x_1) \delta \phi(x_2)} \right), \]
and we have introduced an arbitrary dimensional scale factor \( \mu_{\infty} \) to keep the argument of the logarithm dimensionless. This \( \mu_{\infty} \) does not necessarily have anything to do with the running scale that will be introduced later; it has dimensions of mass.

In a symmetric vacuum we typically have \( S[\phi[0]] = 0 \), unless there is an explicit cosmological constant. To this order we also have for the effective action
\[ \Gamma[\bar{\phi}] \equiv -W[J] + \int d^n x \ J(x) \bar{\phi}(x) = -W_0[J] + \int d^n x \ J(x) \bar{\phi}(x) - \hbar W_1[J] + O(\hbar^2), \]
so that we can write
\[ \Gamma[\bar{\phi}] = \Gamma_0[\bar{\phi}] + \hbar \Gamma_1[\bar{\phi}] + O(\hbar^2), \]
with
\[ \Gamma_0[\bar{\phi}] = S[\bar{\phi}] - S[\phi[0]], \]
and
\[ \Gamma_1[\bar{\phi}] = -W_1[J[\bar{\phi}]], \]
so that
\[ \Gamma_1[\bar{\phi}] = \frac{1}{2} \left\{ \text{tr} \ln \frac{S_2(\phi)}{S_2(\phi[0])} \right\}. \]

Even if an explicit cosmological constant is present in the tree level action, it automatically drops out of the one-loop effective action. The “tr” in the expression for \( \Gamma_1[\bar{\phi}] \) involves an integration over both \( x_1 \) and \( x_2 \) as well as an identification of these two points by means of a delta function \( \delta^2(x_1, x_2) \).

If we were working to higher than one-loop order we would have to be careful to define \( \phi[J] \) via
\[ \phi[J, x] = \frac{\delta W[J]}{\delta J(x)}, \]
which implies
\[ \left( \frac{\delta \Gamma}{\delta \phi} \right)_{\phi[J]} = J. \]
Since the explicit calculations in this article are all to one-loop order, equation (18) is sufficient for our purposes.

We know that as we shall be carrying out improved perturbation theory later, this Gaussian (or one-loop) approximation is sufficient. By this we mean that the renormalization group equations will give us the (one-loop) improved and renormalized theory to all orders in the coupling $\lambda$ in the leading log approximation. (In more general situations there could be many coupling constants.) The first important thing to notice is the fact that the quantity $\Gamma_{1}[\bar{\phi}]$ is divergent, and we will need to analyze its divergences. This will be done in the next section. The second important remark is the fact that the bare theory does not depend on the arbitrary scale introduced by the renormalization scheme; therefore we shall be able to derive the renormalization group equations from the identity

$$\mu \frac{d \Gamma_{1}[\phi]}{d \mu} = 0 = \mu \frac{d S[\phi]}{d \mu} + h \mu \frac{d \Gamma_{1}[\phi]}{d \mu} + O(h^2).$$

This equation yields a polynomial in the background field whose coefficients are precisely the RGE’s. Although we shall be using quantum scalar field theory to illustrate how the mechanism of improved perturbation theory works, it is important to keep in mind that this latter identity (18) holds for all renormalizable field theories, of both the quantum and stochastic varieties, and is independent of the nature of the fluctuations that drive the scale-dependence of the couplings appearing in the theory. This is because the effective action (or effective potential) is always a renormalization group invariant. Of course, when applied to a general stochastic field theory, the loop counting parameter will no longer be $\hbar$, but will be instead the amplitude of the noise two-point function. We shall come back to this important point in our final section.

### III. HOMOGENEOUS BACKGROUND FIELDS: REGULARIZATION

This section closely follows the two basic early references that treat the study of the effective potential, namely [3] and [4], in which the extraction of the RGE’s directly from the effective action (or potential) was pointed out and used for the very first time. The limit of the effective action for constant fields $\bar{\phi}(x) = \phi_{0}$ is the effective potential and generates the connected one-particle irreducible Feynman graphs for zero external momentum. Since $\lambda \phi^{4}$ in $n = 4$ dimensions has no wavefunction renormalization at one loop, we shall consider the homogeneous field limit and calculate the effective potential in a simple way using a momentum cutoff procedure. Notice the fact that as the background field is homogeneous, the only divergences showing up in the one-loop-contribution to the effective action come from the effective potential term, and not the kinetic piece.

The specific scalar Lagrangian we are considering is a $\lambda \phi^{4}$ theory in Euclidean $n$-dimensional spacetime is $s$

$$\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \frac{1}{2} m^{2} \phi^{2} + \frac{\lambda}{4!} \phi^{4},$$

(19)

and the corresponding classical action is given by

$$S[\phi] = \int d^{n}x \left[ \frac{1}{2} \partial_{\mu} \phi(x) \partial^{\mu} \phi(x) + \frac{1}{2} m^{2} \phi^{2}(x) + \frac{\lambda}{4!} \phi^{4}(x) \right].$$

(20)

We first calculate $S_{2}(x_{1}, x_{2}; \bar{\phi})$ for the action (20) to obtain

$$S_{2}(x_{1}, x_{2}; \bar{\phi}) = \left[ -\partial_{\mu} \partial^{\mu} + m^{2} + \frac{\lambda}{2} \bar{\phi}^{2} \right] \delta^{n}(x_{1}, x_{2}),$$

(21)

and

$$S_{2}(x_{1}, x_{2}; \bar{\phi} = 0) = \left[ -\partial_{\mu} \partial^{\mu} + m^{2} \right] \delta^{n}(x_{1}, x_{2}).$$

(22)

The one-loop contribution to the effective action is therefore

$$\Gamma_{1}[\bar{\phi}] = \frac{1}{2} \text{tr} \ln \left\{ \frac{[-\partial_{\mu} \partial^{\mu} + m^{2} + \frac{\lambda}{2} \bar{\phi}^{2}] \delta^{n}(x_{1}, x_{2})}{\mu_{\infty}^{2}} \right\} - \frac{1}{2} \text{tr} \ln \left\{ \frac{[-\partial_{\mu} \partial^{\mu} + m^{2}] \delta^{n}(x_{1}, x_{2})}{\mu_{\infty}^{2}} \right\}.$$  

(23)

A simple integral representation for the effective action may be had by going to a momentum representation for the operators and using the homogeneity of the background field:
\[ \Gamma_1[\tilde{\phi}] = \frac{1}{2} \left\{ \text{tr} \ln \frac{S_2(\tilde{\phi})}{\mu_2^2} - \text{tr} \ln \frac{S_2(\tilde{\phi} = 0)}{\mu_2^2} \right\} \]

\[ = \frac{1}{2} \int d^n x \langle x | \ln \frac{S_2(\tilde{\phi})}{\mu_2^2} - \ln \frac{S_2(\tilde{\phi} = 0)}{\mu_2^2} | x \rangle \]

\[ = \frac{1}{2} \int d^n x \int d^n q_1 \int d^n q_2 \langle x | q_1 \rangle \left\{ \ln \left[ \frac{q_1^2 + m^2 + \frac{1}{2} \tilde{\phi}^2}{\mu_2^2} \right] - \ln \left[ \frac{q_1^2 + m^2}{\mu_2^2} \right] \right\} \delta^n(q_1, q_2, | x \rangle \]

By making use of the definition of \( S_2(\tilde{\phi}) \) in momentum representation, we can write

\[ \Gamma_1[\tilde{\phi}] = \frac{1}{2} \int d^n x \int d^n q_1 \int d^n q_2 \langle x | q_1 \rangle \left\{ \ln \left[ \frac{q_1^2 + m^2 + \frac{1}{2} \tilde{\phi}^2}{\mu_2^2} \right] - \ln \left[ \frac{q_1^2 + m^2}{\mu_2^2} \right] \right\} \delta^n(q_1, q_2) \langle q_2 | x \rangle. \]

We have to be careful here. The operator \( S_2(\tilde{\phi}) \) is a lucky accident for fewer than six dimensions: blindly replacing \( \bar{\lambda} \) from the effective action. We will have more to say about this point later on in the discussion, when we analyze the leads to the correct renormalization group equations, and is in error only insofar as it drops all (finite) gradient terms.

By carrying out an expansion in powers of \( 1/\Lambda \), we are able now to calculate \( \mathcal{I}(\mathcal{M}^2) \) and \( \mathcal{I}(m^2) \) explicitly, where we have also introduced a momentum cutoff \( \Lambda \) to regulate the expressions and render the theory finite.

\[ \mathcal{I}(\mathcal{M}^2) = \frac{1}{16 \pi^2} \left\{ \frac{1}{2} \mathcal{M}^2 \Lambda^2 - \frac{1}{4} \Lambda^4 - \frac{1}{2} \mathcal{M}^4 \ln \left( \frac{\Lambda^2 + \mathcal{M}^2}{\mu_2^2} \right) + \frac{1}{2} \Lambda^4 \ln \left( \frac{\Lambda^2 + \mathcal{M}^2}{\mu_2^2} \right) + \frac{1}{2} \mathcal{M}^4 \ln \left( \frac{\mathcal{M}^2}{\mu_2^2} \right) \right\} \]

\[ \mathcal{I}(m^2) = \frac{1}{16 \pi^2} \left\{ \frac{1}{2} m^2 \Lambda^2 - \frac{1}{4} \Lambda^4 - \frac{1}{2} m^4 \ln \left( \frac{\Lambda^2 + m^2}{\mu_2^2} \right) + \frac{1}{2} \Lambda^4 \ln \left( \frac{\Lambda^2 + m^2}{\mu_2^2} \right) + \frac{1}{2} m^4 \ln \left( \frac{m^2}{\mu_2^2} \right) \right\}, \]

where we have defined

\[ \mathcal{M}^2 \equiv m^2 + \frac{\Lambda}{2} \tilde{\phi}^2. \]

We are interested in separating the finite pieces from the divergent ones in \( \Gamma_1[\tilde{\phi}] \) (the divergent terms will be taken care of in the renormalization procedure chosen; see the following section), therefore we must consider limits such as

\[ \lim_{\Lambda \to +\infty} \mathcal{I}(\mathcal{M}^2). \]

By carrying out an expansion in powers of \( 1/\Lambda \) we obtain
\[ (16\pi^2) \lim_{\Lambda \to +\infty} I(M^2) = \frac{1}{2} M^2 \Lambda^2 - \frac{1}{4} \Lambda^4 - \frac{1}{2} M^4 \ln \left[ \frac{\Lambda^2}{\mu^2} \left(1 + \frac{M^2}{\Lambda^2}\right) \right] + \frac{1}{2} \Lambda^4 \ln \left[ \frac{\Lambda^2}{\mu^2} \left(1 + \frac{M^2}{\Lambda^2}\right) \right] + \frac{1}{2} M^4 \ln \frac{M^2}{\mu^2} \]

\[ = \frac{1}{2} M^2 \Lambda^2 - \frac{1}{4} \Lambda^4 + \frac{1}{2} (\Lambda^4 - M^4) \ln \frac{\Lambda^2}{\mu^2} + \frac{1}{2} (\Lambda^4 - M^4) \left( \frac{M^2}{\Lambda^2} - \frac{M^4}{2\Lambda^4} + \ldots \right) + \frac{1}{2} M^4 \ln \frac{M^2}{\mu^2} \]

\[ = \frac{1}{2} M^2 \Lambda^2 - \frac{1}{4} \Lambda^4 + \frac{1}{2} (\Lambda^4 - M^4) \ln \frac{\Lambda^2}{\mu^2} + \frac{M^2 \Lambda^2}{2} - \frac{M^4}{4} + \frac{1}{2} M^4 \ln \frac{M^2}{\mu^2} + O(\Lambda^{-2}), \quad (31) \]

Therefore

\[ \lim_{\Lambda \to +\infty} I(M^2) = \frac{\Lambda^4}{16\pi^2} \left( \ln \frac{\Lambda}{\mu_\infty} - \frac{1}{4} \right) + \frac{M^2 \Lambda^2}{16\pi^2} + \frac{M^4}{32\pi^2} \left( \ln \frac{M^2}{\Lambda^2} - \frac{1}{2} \right) + O(\Lambda^{-2}), \quad (32) \]

and

\[ \lim_{\Lambda \to +\infty} I(m^2) = \frac{\Lambda^4}{16\pi^2} \left( \ln \frac{\Lambda}{\mu_\infty} - \frac{1}{4} \right) + \frac{m^2 \Lambda^2}{16\pi^2} + \frac{m^4}{32\pi^2} \left( \ln \frac{m^2}{\Lambda^2} - \frac{1}{2} \right) + O(\Lambda^{-2}). \quad (33) \]

This means that the regulated one-loop contribution to the effective action for \( \lambda \phi^4 \) in \( n = 4 \) is given by

\[ \Gamma_1[\bar{\phi}] = \frac{1}{32\pi^2} \int d^4x \left\{ (M^2 - m^2)\Lambda^2 + \frac{M^4}{2} \left[ \ln \frac{M^2}{\Lambda^2} - \frac{1}{2} \right] - \frac{m^4}{4} \left[ \ln \frac{m^2}{\Lambda^2} - \frac{1}{2} \right] + O(\Lambda^{-2}) \right\}. \quad (34) \]

Note that \( \mu_\infty \) has disappeared, as of course it should, since it was only introduced in the first place to make the argument of the logarithm dimensionless.

A particularly useful separation between finite and divergent pieces is obtained by introducing a new arbitrary scale \( \mu \) and writing

\[ \frac{M^2}{\Lambda^2} = \frac{\mu^2}{\mu^2} \Lambda^2, \quad \text{and} \quad \frac{m^2}{\Lambda^2} = \frac{m^2}{\mu^2} \Lambda^2, \quad (35) \]

so that

\[ \Gamma_1[\bar{\phi}] = \frac{1}{32\pi^2} \int d^4x \left\{ (M^2 - m^2)\Lambda^2 + \frac{(M^4 - m^4)}{2} \left[ \ln \frac{\mu^2}{\Lambda^2} - \frac{1}{2} \right] + \frac{M^4}{2} \ln \frac{M^2}{\mu^2} - \frac{m^4}{4} \ln \frac{m^2}{\mu^2} + O(\Lambda^{-2}) \right\}. \quad (36) \]

This \( \mu \) is logically independent from the previous \( \mu_\infty \), and is being used for a different purpose; \( \mu \) is being used in order to collect all the divergent contributions in one place to separate them from the interesting finite pieces of the effective action. We have now developed all the basic tools that we need to proceed to the next section, where we carry out the renormalization.

**IV. EFFECTIVE POTENTIAL AND RENORMALIZATION**

We recall here that the one-loop effective action is given by

\[ \Gamma[\bar{\phi}] = S[\bar{\phi}] + \hbar \Gamma_1[\bar{\phi}] + O(\hbar^2), \quad (37) \]

and the one-loop effective potential is obtained by calculating the one-loop effective action for a homogeneous field \( \bar{\phi} = \phi_0 \), and dividing by the volume of spacetime \( \Omega = \int d^4x \). That is

\[ V[\phi_0] \equiv \frac{\Gamma[\bar{\phi} = \phi_0]}{\Omega} = V[\bar{\phi} = \phi_0] + \hbar \frac{\Gamma_1[\bar{\phi} = \phi_0]}{\Omega} + O(\hbar^2), \quad (38) \]

with

\[ V[\phi] = \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4, \quad (39) \]
the classical potential.

From (36) we can write

$$V[\phi_0] = V[\phi_0] + \frac{\hbar}{32\pi^2} \left[ \lambda^2 \frac{\lambda}{2} \phi_0^2 + \frac{(M_0^4 - m^4)}{2} \left( \ln \frac{\mu^2}{\Lambda^2} - \frac{1}{2} \right) + \frac{M_0^4}{2} \ln \frac{M_0^4}{\mu^2} - m^4 \ln \frac{m^2}{\mu^2} + O(\Lambda^{-2}) \right] + O(\hbar^2)$$

$$= \frac{m^2}{2} \phi_0^2 + \frac{\lambda}{4!} \phi_0^4 + \frac{\hbar}{32\pi^2} \left[ \lambda^2 \frac{\lambda}{2} \phi_0^2 + \frac{(M_0^4 - m^4)}{2} \left( \ln \frac{\mu^2}{\Lambda^2} - \frac{1}{2} \right) + \frac{M_0^4}{2} \ln \frac{M_0^4}{\mu^2} - m^4 \ln \frac{m^2}{\mu^2} + O(\Lambda^{-2}) \right] + O(\hbar^2),$$

with $M_0^2 = m^2 + \frac{\lambda}{2} \phi_0^2$. At this stage the expressions are all given in terms of bare (unrenormalized) parameters and the dependence on the cutoff is explicit. We have introduced the arbitrary scale $\mu$ to get the cutoff dependent pieces concentrated in one place. The effective potential is seen by inspection to be independent of $\mu$.

Since we are dealing with a renormalizable theory, we know that we can absorb the cutoff dependence into the parameters $m$ and $\lambda$. Because there is no wavefunction renormalization, $Z = 1 + O(\hbar^2)$ to one loop in $\lambda(\phi^4)_4$ field theory, we do not need to worry about this particular complication. Otherwise we would have directly gone to investigating the effective action $\Gamma[\phi_0]$. Specifically, let us write

$$m^2 = m^2(\mu) + \hbar[\delta m^2](\mu) + O(\hbar^2),$$

$$\lambda = \lambda(\mu) + \hbar[\delta \lambda](\mu) + O(\hbar^2).$$

For a specific and useful choice of separation into renormalized parameters and counterterms the classical potential $V[\phi_0]$ may then be written as

$$V[\phi_0] = V[\phi_0, m(\mu), \lambda(\mu)] - \frac{\hbar}{64\pi^2} \left[ \lambda^2 \frac{\lambda}{2} \phi_0^2 + \frac{(M_0^4 - m^4)}{2} \left( \ln \frac{\mu^2}{\Lambda^2} - \frac{1}{2} \right) \right] + O(\hbar^2),$$

where the counter terms have been so chosen so that they will render the one-loop effective potential finite and cutoff independent. Notice also the very important fact that in the $O(\hbar)$ term above, the parameters can with equal facility be taken to be either the bare ones or the renormalized ones, since the difference will contribute to the full expression only at $O(\hbar^2)$.

We conclude that the effective potential, up to one loop, is given by

$$V[\phi_0] = V[\phi_0, m(\mu), \lambda(\mu)] + \frac{\hbar}{64\pi^2} \left( \frac{M_0^4}{\mu^2} \ln \frac{M_0^4}{\mu^2} - m^4 \ln \frac{m^2}{\mu^2} \right) + O(\hbar^2)$$

$$= \frac{m^2(\mu)}{2} \phi_0^2 + \frac{\lambda(\mu)}{4!} \phi_0^4 + \frac{\hbar}{64\pi^2} \left( \frac{M_0^4(\mu)}{\mu^2} \ln \frac{M_0^4(\mu)}{\mu^2} - m^4(\mu) \ln \frac{m^2(\mu)}{\mu^2} \right) + O(\hbar^2).$$

The parameters $m(\mu)$ and $\lambda(\mu)$ are the renormalized ones, whereas the parameter $M_0^2 = m^2 + \frac{\lambda}{2} \phi_0^2$ is still written in terms of the bare parameters $m$ and $\lambda$. (This is actually convenient for some purposes!) Of course, to this order in the loop expansion, it is equally valid (and considerably more elegant) to write

$$V[\phi_0] = \frac{m^2(\mu)}{2} \phi_0^2 + \frac{\lambda(\mu)}{4!} \phi_0^4 + \frac{\hbar}{64\pi^2} \left\{ \frac{\lambda(\mu)^2 \phi_0^2}{4} \ln \frac{\lambda(\mu) \phi_0^2}{2\mu^2} - m^4(\mu) \ln \frac{m^2(\mu)}{\mu^2} \right\} + O(\hbar^2).$$

Perhaps the key “miracle” of renormalization is this: although there are many occurrences of the parameter $\mu$ on the right hand side of this equation, the left hand side is completely independent of $\mu$. After all, the above expression is a re-writing of equation (40). The “miracle” is of course no miracle: it has been enforced by explicit construction, and will be the underpinning of our derivation of the renormalization group equations.

Note that the above represents a particular prescription for renormalization. We can always change the division between running parameters and counterterms in equation (44) by arbitrary finite quantities without disturbing the elimination of the cutoff dependencies. The prescription chosen here has the advantages of both providing a clean analytic form for the one-loop effective potential and simultaneously being well adapted to the massless limit. Indeed if we let the mass go to zero we get

$$V[\phi_0] = \frac{\lambda(\mu)}{4!} \phi_0^4 + \frac{\hbar}{64\pi^2} \left\{ \frac{\lambda(\mu) \phi_0^2}{4} \ln \frac{\lambda(\mu) \phi_0^2}{2\mu^2} \right\} + O(\hbar^2).$$
Equivalently

\[ V[\phi_0] = \frac{\lambda(\mu)}{4!} \phi_0^4 \left( 1 + \frac{3 \lambda(\mu)}{32 \pi^2} \ln \frac{\lambda(\mu)}{2\mu^2} \right) + O(h^2). \] (49)

We shall use this massless effective potential when we show how to reconstruct the one-loop effective potential from the one-loop renormalization group improved bare potential.

To make the ambiguity under finite renormalizations explicit, suppose we replace (44) above by

\[ V[\phi_0] = V[\phi_0, m(\mu), \lambda(\mu)] - \frac{h}{32 \pi^2} \left[ \frac{\lambda^2}{2} \phi_0^2 + \left( \frac{M_0^4 - m^4}{2} \right) \left( \ln \frac{\mu^2}{\Lambda^2} - \frac{1}{2} \right) \right] + h \epsilon_1 m^2(\mu) \phi_0^2 + h \epsilon_2 \phi_0^4 + O(h^2), \] (50)

where \( m(\mu) \) and \( \lambda(\mu) \) are new and slightly different renormalized couplings. (They will, however, satisfy the same renormalization group equations, at least to one loop.) We have also used dimensional analysis to constrain the ambiguities and place all the arbitrariness into two dimensionless parameters \( \epsilon_1 \) and \( \epsilon_2 \). Then the previous analysis continues to hold except that the effective potential is replaced by

\[ V[\phi_0] = \frac{m^2(\mu)}{2} \phi_0^2 + \frac{\lambda(\mu)}{4!} \phi_0^4 + \frac{h}{64 \pi^2} \left( M_0^4(\mu) \ln \frac{M_0^2(\mu)}{\mu^2} - m^4(\mu) \ln \frac{m^2(\mu)}{\mu^2} \right) \]

\[ + h \epsilon_1 m^2(\mu) \phi_0^2 + h \epsilon_2 \phi_0^4 + O(h^2). \] (51)

This two-parameter ambiguity in the effective potential is an intrinsic and unavoidable side-effect of renormalization: this ambiguity may be used to force the effective potential to have certain simplifying properties, and the choice made previously [in (44)] was exactly one such choice. Another common choice, if \( m(\mu) \neq 0 \), is to fix the derivatives of the effective potential at zero field to be

\[ \frac{d^{(4)}V}{d\phi_0^4} \bigg|_{\phi_0=0} = \lambda(\mu) \] (52)

\[ \frac{d^{(2)}V}{d\phi_0^2} \bigg|_{\phi_0=0} = m^2(\mu). \] (53)

These renormalization conditions are equivalent to particular choices of \( \epsilon_1 \) and \( \epsilon_2 \). See, for example, pages 453–454 of Itzykson and Zuber [19]. If \( m(\mu) = 0 \), a popular choice is

\[ \frac{d^{(4)}V}{d\phi_0^4} \bigg|_{\phi_0=\mu} = \lambda(\mu). \] (54)

See, for example, page 454 of Itzykson and Zuber [19]. We shall eschew such specific choices and stay with the simple form (47) above.

V. RENORMALIZATION BY DIFFERENTIATION AND SUBSEQUENT INTEGRATION

For comparison, we include here some remarks that we believe to be helpful, regarding an alternative approach to the divergent structure of the theory. The main reference for this section is Weinberg [14], (see chapter 16) though related discussions can be found in many field theory textbooks. In Feynman diagram language the idea is to differentiate with respect to some external parameter a sufficient number of times and so render the relevant integrals finite. These differentiations may be with respect to external momenta (where they underly the BPHZ renormalization program), with respect to some particle mass, or (for the case we are interested in) the derivative may be with respect to the external field. After differentiation has rendered the integrals finite, the result is re-integrated an equal number of times. Each integration introduces a new constant of integration; these arbitrary constants of integration are the counterterms.

In our case, \( V_1 \) contains ultraviolet divergences. Since this theory is perturbatively renormalizable, these divergences can be absorbed into a renormalization of the parameters of the theory. \( V_1 \) is divergent, and by power counting, it can be seen that it is made convergent by differentiating three times with respect to \( M^2 \). We start by writing
\[ V_1[\phi_0] = W_1(M_0^2) - W_1(m^2), \]  
with
\[ W_1[a] = \int \frac{d^4 q}{(2\pi)^4} \ln \left( \frac{q^2 + a}{\mu^2} \right). \]

Differentiating thrice
\[ \frac{d^3 W_1[a]}{da^3} = \frac{1}{8\pi^2} \int \frac{dy}{(y + a)^2} = \frac{1}{32\pi^2 a}. \]

Integrating the previous equation three times, and adding the corresponding constants of integration, we obtain
\[ W_1[a] = \frac{a^2}{64\pi^2} \ln \frac{a}{\mu^2} + \kappa_1(\mu)a^2 + \kappa_2a + \kappa_3, \]

with the \( \kappa \)'s being divergent coefficients. The dimensional parameter \( \mu \) must be introduced to keep the argument of the logarithm dimensionless, and implies a \( \mu \) dependence in \( \kappa_1 \) in such a manner that the left hand side above is independent of \( \mu \). Since the \( \mu \) independence of the left hand side holds for all values of \( a \), the same logic implies that \( \kappa_2 \) and \( \kappa_3 \) cannot be functions of \( \mu \). (In this formalism we do not need to keep \( \mu \) and \( \mu_\infty \) distinct, and we may choose to conflate them if we wish.) Therefore
\[ V = \frac{m^2}{2} \phi_0^2 + \frac{\lambda}{4!} \phi_0^4 + \hbar \left[ W_1[M_0^2] - W_1(m^2) \right] + O(\hbar^2) \]
\[ = \frac{m^2}{2} \phi_0^2 + \frac{\lambda}{4!} \phi_0^4 + \frac{\hbar}{64\pi^2} \left\{ \mathcal{M}_0^2 \ln \left( \frac{\mathcal{M}_0^2}{\mu^2} \right) - m^4 \ln \frac{m^2}{\mu^2} \right\} + \hbar \kappa_1(\mu) \left( \lambda m^2 \phi_0^2 + \frac{\lambda^2 \phi_0^4}{4} \right) + \hbar \kappa_2 \frac{\lambda}{2} \phi_0^2 + O(\hbar^2) \]
\[ = m^2 + 2h\kappa_1(\mu) m^2 + h\kappa_2 \lambda \]
\[ \frac{2}{2} \phi_0^2 + \frac{\lambda + 6h\kappa_1(\mu) \lambda^2}{4!} \phi_0^4 + \frac{\hbar}{64\pi^2} \left\{ \mathcal{M}_0^2 \ln \left( \frac{\mathcal{M}_0^2}{\mu^2} \right) - m^4 \ln \frac{m^2}{\mu^2} \right\} + O(\hbar^2). \]

Everything is here still expressed in terms of unrenormalized quantities and cutoff dependent parameters \( \kappa_i \). The theory is renormalizable, so that we expect to be able to write \( V \) as a function of the renormalized parameters, with no explicit divergences, though we will just have to live with the presence of the scale \( \mu \). Define renormalized parameters by
\[ m^2(\mu) = m^2 + \hbar [2\lambda m^2 \kappa_1(\mu) + \lambda \kappa_2] + O(\hbar^2), \]
\[ \lambda(\mu) = \lambda + \hbar [6\lambda^2 \kappa_1(\mu)] + O(\hbar^2). \]

Remember that the divergences are all buried in the \( \kappa_i \)’s. This choice of renormalized parameters is again to some extent arbitrary, but is a particularly simple one based on keeping the form of the effective potential invariant. Any other choice of renormalization prescription will differ from this one by only finite quantities and will at worst lead to finite renormalization ambiguities in the effective action. In terms of these particular renormalized parameters we can write the partially renormalized expression
\[ V = \frac{m^2(\mu)}{2} \phi_0^2 + \frac{\lambda(\mu)}{4!} \phi_0^4 + \frac{\hbar}{64\pi^2} \left\{ \mathcal{M}_0^2(\mu) \ln \frac{\mathcal{M}_0^2(\mu)}{\mu^2} - m^4(\mu) \ln \frac{m^2(\mu)}{\mu^2} \right\} + O(\hbar^2), \]

where in the \( O(\hbar) \) terms we have kept the bare parameters. Since bare and renormalized parameters differ at \( O(\hbar) \) this is completely equivalent to the expression written completely in terms of renormalized parameters
\[ V = \frac{m^2(\mu)}{2} \phi_0^2 + \frac{\lambda(\mu)}{4!} \phi_0^4 + \frac{\hbar}{64\pi^2} \left\{ \mathcal{M}_0^2(\mu) \ln \frac{\mathcal{M}_0^2(\mu) \lambda^2}{\mu^2} - m^4(\mu) \ln \frac{m^2(\mu)}{\mu^2} \right\} + O(\hbar^2). \]

All the \( \mu \) dependencies on the right hand side of the above cancel exactly: the left hand side is known by construction to be independent of \( \mu \). Finally notice that all these results agree with the arguments in the previous section.
VI. RENORMALIZATION GROUP EQUATIONS FOR THE $\lambda\phi^4$-THEORY

As we remarked earlier, there is no wavefunction renormalization at this order in the loop expansion (for the $\lambda\phi^4$ theory in 4 dimensions), and the condition

$$\mu \frac{d \Gamma[\bar{\phi}]}{d\mu} = 0 \quad \Rightarrow \quad \mu \frac{d V[\phi_0]}{d\mu} = 0. \quad (66)$$

This can be seen more directly, by noting (at this stage without proof) that the general form of the renormalized effective action at one-loop order is

$$\Gamma[\bar{\phi}] = \int d^n x \left[ \frac{1}{2} Z(\mu) \partial^{\mu} \bar{\phi} \partial_{\mu} \bar{\phi} + V(\bar{\phi}) + \mathcal{O}(h^2) \partial^{\mu} \bar{\phi} \partial_{\mu} \bar{\phi} \right]. \quad (67)$$

At one-loop order for a $\lambda\phi^4$ theory there is no wavefunction renormalization. That is $Z(\mu) = 1 + O(h^2)$, and $\phi$ is itself the renormalized and improved field (it does not depend on the scale $\mu$).

$$\tilde{\phi} = Z^{1/2}(\mu) \phi_0 = \phi_0. \quad (68)$$

It is now easy to see that the scale independence of the one-loop effective action implies (in this particular case) the scale independence of the one-loop effective potential, where

$$V[\phi_0] = V[\phi_0, \mu] + \frac{\hbar}{64\pi^2} \left\{ \mathcal{M}_4(\mu) \ln \frac{\mathcal{M}_2^2(\mu)}{\mu^2} - m^4(\mu) \ln \frac{m^2(\mu)}{\mu^2} \right\} + O(h^2), \quad (69)$$

and

$$V[\phi_0, \mu] = \frac{m^2(\mu)}{2} \phi_0^2 + \frac{\lambda(\mu)}{4!} \phi_0^4. \quad (70)$$

Notice that in all the expressions after this one, we mean for $m$ and $\lambda$ the renormalized, scale dependent parameters $m(\mu)$, and $\lambda(\mu)$. Differentiating, we get

$$0 = \mu \frac{d}{d\mu} \left( \frac{m^2}{2} \phi_0^2 + \frac{\lambda}{4!} \phi_0^4 \right) + \frac{\hbar}{64\pi^2} \mu \frac{d}{d\mu} \left\{ \mathcal{M}_4(\mu) \ln \frac{\mathcal{M}_2^2(\mu)}{\mu^2} - m^4(\mu) \ln \frac{m^2(\mu)}{\mu^2} \right\} + O(h^2), \quad (71)$$

which implies

$$\mu \frac{d}{d\mu} \left( \frac{m^2}{2} \phi_0^2 + \frac{\lambda}{4!} \phi_0^4 \right) = \frac{\hbar}{32\pi^2} (\mathcal{M}_4^2 - m^4) + O(h^2) \quad (72)$$

$$= \frac{\hbar}{32\pi^2} \left( \frac{\lambda^2}{4} \phi_0^4 + \lambda m^2 \phi_0^2 \right) + O(h^2). \quad (73)$$

Note that derivatives of $\mathcal{M}$ and $m$ with respect to $\mu$ on the right hand side of the two equations above lead to contributions only at $O(h^2)$ and so can be neglected at the order we are interested in. Equivalently, we could apply the same differentiation to the semi-renormalized effective potential, equations (67) or (64). This derivative technique was first used in the work of Fujimoto, O’Raifeartaigh, and Parravicini [3], and was extensively developed in the work of Gato, León, Perez–Mercader, and Quirós [4]. It is a very simple and powerful technique that deserves wider use.

Finally, we conclude

$$\frac{\phi_0^2}{2} \mu \frac{d^2 m^2}{d\mu^2} + \frac{\phi_0^4}{4!} \mu \frac{d\lambda}{d\mu} = \frac{\hbar}{128\pi^2} \phi_0^2 \lambda^2 + \frac{\hbar}{32\pi^2} \phi_0^2 m^2 \lambda + O(h^2). \quad (74)$$

We can now easily obtain the renormalization group equations for $m(\mu)$ and $\lambda(\mu)$, and solve them. We need only identify the terms with the same powers of $\phi_0$. This is because the two sides of (74) are self-consistent, as a result of the renormalizability of the potential.

(For a non-renormalizable theory [e.g., $\lambda(\phi^4)_4$] we would find different functional forms on the two sides of the equation, since $\mathcal{M}_4$ now contains terms such as $\phi_0^4$) thus indicating that we had not included enough terms in the classical potential $V(\phi)$. If we attempt to fix this by adding additional terms to $V(\phi)$ even higher order terms show up in $\mathcal{M}_4$ and we are forced to bootstrap ourselves into a situation where $V(\phi)$ contains all powers of $\phi$. Viewed
as a fundamental theory the resulting model is generally condemned as being "non-predictive" though viewed as an effective theory it is more interesting.)

Had we not taken into account the normalization factor for the generating functional $Z[J]$, the term involving $m^4$ would not have canceled out and we would have had to carry along an additional RGE for the vacuum energy density or cosmological constant, while adding a tree-level cosmological constant to our bare action. The careful inclusion of the normalization factor $N$ in the partition function takes care of subtracting the divergences corresponding to the vacuum.

We have then

$$\frac{dm^2}{d\mu} = \frac{\hbar}{16\pi^2} m^2 \lambda + O(h^2),$$

$$\frac{d\lambda}{d\mu} = \frac{3\hbar}{16\pi^2} \lambda^2 + O(h^2).$$

These are the one-loop renormalization group equations for the theory. It is standard to define

$$\beta \left( \frac{\lambda}{\mu} \right) \equiv \frac{\partial \lambda}{\partial \mu},$$

$$\gamma_m \left( \frac{\lambda}{\mu} \right) \equiv \frac{1}{2} \frac{\partial \ln m^2}{\partial \mu},$$

$$\gamma_d \left( \frac{\lambda}{\mu} \right) \equiv \frac{1}{2} \frac{\partial \ln Z}{\partial \mu}.$$ (77, 78, 79)

Unfortunately, we must warn the reader that various authorities use a different sign convention for $\gamma_m$. With these conventions we obtain

$$\beta \left( \frac{\lambda}{\mu} \right) = \frac{3\hbar \lambda^2}{16\pi^2} + O(h^2),$$

$$\gamma_m \left( \frac{\lambda}{\mu} \right) = \frac{\hbar \lambda}{32\pi^2} + O(h^2),$$

$$\gamma_d \left( \frac{\lambda}{\mu} \right) = 0 + O(h^2).$$ (80, 81, 82)

These results agree with Ramond [20] Chapter 4, with Zinn–Justin [39] Chapter 11, and with Le Bellac [35] Chapters 2, 5, 6, and 7. (We have not actually derived $\gamma_d$ at this stage but merely asserted the result. A proof will be forthcoming shortly.)

The solution of these renormalization group equations is straightforward: they are given by

$$\lambda = \lambda_0 \left( 1 - \frac{3\hbar}{16\pi^2} \lambda_0 \ln \frac{\mu}{\mu_0} + O(h^2) \right)^{-1},$$

$$m^2 = m_0^2 \left( \mu/\mu_0 \right) \frac{\lambda_0}{32\pi^2} \exp(O(h^2)).$$ (83, 84)

where $\lambda_0 = \lambda(\mu_0), m_0 = m(\mu_0)$ and $\mu_0$ denotes an arbitrary initial renormalization scale.

**VII. CONSISTENCY CHECK ON THE EFFECTIVE POTENTIAL**

We now present a simple consistency check to verify that the effective potential is independent of the renormalization scale $\mu$. Start with the formula

$$V[\phi_0, \mu] = \frac{m^2(\mu)}{2} \phi_0^2 + \frac{\lambda(\mu)}{4!} \phi_0^4,$$

and insert the running couplings we have just derived. Then

$$V[\phi_0, \mu] = \frac{1}{2} m_0^2 \left( \mu/\mu_0 \right) \frac{\lambda_0}{32\pi^2} \exp(O(h^2)) \phi_0^2 + \lambda_0 \left( 1 - \frac{3\hbar}{16\pi^2} \lambda_0 \ln \frac{\mu}{\mu_0} + O(h^2) \right)^{-1} \phi_0^4.$$(85, 86)
Expanding this to $O(h^2)$

$$V[\phi_0, \mu] = \frac{1}{2} m_0^2 \left( 1 + \frac{\hbar}{16\pi^2} \lambda_0 \ln \frac{\mu}{\mu_0} \right) \phi_0^2 + \frac{\lambda_0}{4!} \left( 1 + \frac{3\hbar}{16\pi^2} \lambda_0 \ln \frac{\mu}{\mu_0} \right) \phi_0^4 + O(h^2). \quad (87)$$

This can be re-expressed as

$$V[\phi_0, \mu] = V[\phi_0, \mu_0] + \frac{\hbar}{32\pi^2} \left\{ \frac{1}{2} m_0^2 \lambda_0 \phi_0^2 + \frac{1}{8} \lambda_0^2 \phi_0^4 \right\} \ln \frac{\mu^2}{\mu_0^2} + O(h^2), \quad (88)$$

or better yet

$$V[\phi_0, \mu] = V[\phi_0, \mu_0] + \frac{\hbar}{64\pi^2} \left\{ \mathcal{M}_0^4(\mu_0) - m_0^4 \right\} \ln \frac{\mu^2}{\mu_0^2} + O(h^2). \quad (89)$$

The point is that after inserting the running couplings, as deduced from the renormalization group, the $\mu$ dependence in the renormalized classical potential cancels exactly (to $O(h^2)$) the $\mu$ dependence in $V_1$, so that

$$V[\phi_0] = V[\phi_0, \mu] + \frac{\hbar}{64\pi^2} \left\{ \mathcal{M}_0^4(\mu) \ln \frac{\mathcal{M}_0^2(\mu)}{\mu^2} - m^4(\mu) \ln \frac{m^2(\mu)}{\mu^2} \right\} + O(h^2),$$

$$= V[\phi_0, \mu_0] + \frac{\hbar}{64\pi^2} \left\{ \mathcal{M}_0^4(\mu_0) \ln \frac{\mathcal{M}_0^2(\mu_0)}{\mu_0^2} - m^4(\mu_0) \ln \frac{m^2(\mu_0)}{\mu_0^2} \right\} + O(h^2). \quad (91)$$

This may be a little tedious, but it has the virtue of being explicit, and verifying the consistency of the whole approach.

**VIII. THE IMPROVED POTENTIAL: RECONSTRUCTION**

We have just seen how knowledge of the one-loop effective potential gives the one-loop RGE’s; and have verified that the running of the couplings implied by the RGE’s is consistent with the renormalization scale independence of the effective potential. We shall now point out that this is a two-way street: Suppose that (by hook or by crook) we have been provided with the RGE’s but have somehow forgotten how to calculate the effective potential. Then the RGE’s can be used to renormalization group improve the classical potential, and this improved potential can then be used to reconstruct the one-loop effective potential up to finite renormalizations.

It is important to realize that there are calculational techniques, we shall present one such later in this article, that provide the (one-loop) RGE’s without calculating the (one-loop) effective potential. The reconstruction technique we are about to present is then the fastest way of deriving the (one-loop) effective action.

We start with the original classical potential

$$V[\phi_0] = \frac{1}{2} m^2 \phi_0^2 + \frac{\lambda}{4!} \phi_0^4, \quad (92)$$

The improvement of this classical potential consists in substituting all the bare parameters (mass, coupling constant, wavefunction normalization) by their renormalized running forms, that is

$$m \rightarrow m_{\text{imp}} = m(\mu), \quad (93)$$

$$\lambda \rightarrow \lambda_{\text{imp}} = \lambda(\mu), \quad (94)$$

$$\phi \rightarrow \phi_{\text{imp}} = \phi(\mu) = Z^{1/2}(\mu) \phi, \quad (95)$$

to obtain

$$V_{\text{imp}}[\phi_0] = \frac{1}{2} m^2(\mu)\phi_{\text{imp}}^2 + \frac{\lambda(\mu)}{4!} \phi_{\text{imp}}^4$$

$$= \frac{1}{2} m_0^2 \frac{\lambda_0}{4!} \exp(O(h^2)) \phi_0^2 + \frac{\lambda_0}{4!} \left( 1 - \frac{3\hbar}{16\pi^2} \lambda_0 \ln \frac{\mu}{\mu_0} + 0(h^2) \right)^{-1} \phi_0^4. \quad (96)$$

This procedure yields an improved potential with running, scale-dependent parameters. To one-loop we can carry out a leading-log expansion:
We wish to see what we can deduce about the function \( X[\phi_0, \mu(m), \lambda(\mu)] \). From the improved potential discussed above we know we can write
\[
V[\phi_0] = V[\phi_0, m(\mu), \lambda(\mu)] + \hbar X[\phi_0, m(\mu), \lambda(\mu)] + O(\hbar^2). \tag{99}
\]

Since the above is (despite naive appearances), independent of \( \mu \), we deduce that there must exist functions \( X_1[\phi_0, \mu(m), \lambda(\mu)] \) and \( X_2[\phi_0, \mu(m), \lambda(\mu)] \), which can no longer explicitly depend on \( \mu \), such that
\[
X[\phi_0, m(\mu), \lambda(\mu)] = \frac{\hbar}{64\pi^2} \{ \mathcal{M}_0^4(\mu) - m^4 \} \ln \frac{X_1[\phi_0, m(\mu), \lambda(\mu)]}{\mu^2} + \frac{\hbar}{64\pi^2} \{ \mathcal{M}_0^4(\mu) - m^4 \} \ln \frac{X_2[\phi_0, m(\mu), \lambda(\mu)]}{\mu^2} + \hbar X[\phi_0, m(\mu), \lambda(\mu)] + O(\hbar^2). \tag{100}
\]

We now see the beginnings of similarity with the explicit one-loop effective potentials previously calculated. We shall complete the job by using dimensional analysis plus the existence of appropriate limits to constrain the functions \( X_1 \) and \( X_2 \).

For clarity of exposition it is easiest to first deal with the massless case when the above simplifies to
\[
V[\phi_0] = V[\phi_0, \mu(\lambda)] + \frac{\hbar}{64\pi^2} \{ \mathcal{M}_0^4(\mu) - m^4 \} \ln \frac{X_1[\phi_0, \mu(\lambda)]}{\mu^2} + \frac{\hbar}{64\pi^2} \{ \mathcal{M}_0^4(\mu) - m^4 \} \ln \frac{X_2[\phi_0, \mu(\lambda)]}{\mu^2} + \hbar X[\phi_0, m(\mu), \lambda(\mu)] + O(\hbar^2). \tag{103}
\]

But then by dimensional analysis
\[
X_1[\phi_0, \lambda(\mu)] = \alpha_1(\lambda(\mu)) \phi_0^2, \tag{104}
\]
\[
X_2[\phi_0, \lambda(\mu)] = \alpha_2(\lambda(\mu)) \phi_0^4, \tag{105}
\]
with the \( \alpha \)'s being dimensionless functions of the dimensionless variable \( \lambda \). This tells us that there exists an \( \alpha_3 \) such that
\[
V[\phi_0] = V[\phi_0, \mu(\lambda)] + \frac{\hbar}{64\pi^2} \left\{ \frac{\lambda(\mu)^2 \phi_0^4(\mu)}{4} \right\} \ln \frac{\phi_0^2}{\mu^2} + \hbar \alpha_3(\lambda(\mu)) \phi_0^4 + O(\hbar^2). \tag{106}
\]

Comparing this with equation (48) we see that we have recovered the one-loop massless effective potential up to an unknown term proportional to \( \phi_0^4 \). (That is, up to a finite renormalization). This is of course exactly what we should expect: the RGE's (and therefore the improved potential) are sensitive only to the divergent terms in the renormalization, and so working backwards from the RGE's we cannot possibly recover the terms depending on finite renormalizations.

Repeating this procedure for the massive theory is trickier and more tedious, but the basic ideas and results remain the same. Step back to the general result
\[
V[\phi_0] = V[\phi_0, m(\mu), \lambda(\mu)] + \frac{\hbar}{64\pi^2} \{ \mathcal{M}_0^4(\mu) - m^4 \} \ln \frac{X_1[\phi_0, m(\mu), \lambda(\mu)]}{\mu^2} + \frac{\hbar}{64\pi^2} \{ \mathcal{M}_0^4(\mu) - m^4 \} \ln \frac{X_2[\phi_0, m(\mu), \lambda(\mu)]}{\mu^2} + \hbar X[\phi_0, m(\mu), \lambda(\mu)] + O(\hbar^2). \tag{107}
\]

Then there exists a function \( X_3[\phi_0, m(\mu), \lambda(\mu)] \) such that
\[ V[\phi_0] = V[\phi_0, m(\mu), \lambda(\mu)] + \frac{\hbar}{64\pi^2} \left\{ \mathcal{M}_0^4(\mu) \ln \frac{\mathcal{M}_0^2(\mu)}{\mu^2} - m^4(\mu) \ln \frac{m^2(\mu)}{\mu^2} \right\} + \hbar X_3[\phi_0, m(\mu), \lambda(\mu)] + O(\hbar^2). \] (108)

From the massless case we have just investigated we know that there exists a dimensionless function \( \alpha_4 \) such that

\[ \lim_{m \to 0} X_3[\phi_0, m(\mu), \lambda(\mu)] = \alpha_4(\lambda(\mu)) \phi_0^4. \] (109)

But we also know (because we constructed the effective action to be zero at zero field) that

\[ \lim_{\phi_0 \to 0} X_3[\phi_0, m(\mu), \lambda(\mu)] = 0. \] (110)

This is enough to tell us that there exists a dimensionless function \( \alpha_5(\lambda(\mu)) \) such that

\[ X_3[\phi_0, m(\mu), \lambda(\mu)] = \alpha_4(\lambda(\mu)) \phi_0^4 + \alpha_5(\lambda(\mu)) m^2(\mu) \phi_0^2. \] (111)

The last step is to assemble everything to see that

\[ V[\phi_0] = V[\phi_0, m(\mu), \lambda(\mu)] + \frac{\hbar}{64\pi^2} \left\{ \mathcal{M}_0^4(\mu) \ln \frac{\mathcal{M}_0^2(\mu)}{\mu^2} - m^4(\mu) \ln \frac{m^2(\mu)}{\mu^2} \right\} \]
\[ + \hbar \left\{ \alpha_4(\lambda(\mu)) \phi_0^4 + \alpha_5(\lambda(\mu)) m^2(\mu) \phi_0^2 \right\} + O(\hbar^2). \] (112)

This completes the task we set out to perform: we have used the one-loop improved classical potential to reconstruct the one-loop effective potential up to finite renormalizations.

IX. INHOMOGENEOUS FIELDS: REGULARIZATION

The simple momentum cutoff technique used above works well if we are only interested in computing the effective potential, for which the background field is taken to be homogeneous (and/or whenever wavefunction renormalization is not an issue). For the more general case of the full effective action where wavefunction renormalization is important, a different method for regulating and computing the effective action is required (or at least, preferable). Improvement of the complete effective action and the easy extraction of the associated RGE’s is as simple as in the case of the effective potential. The following is an example of one such method based on Schwinger’s proper time representation of the effective action. Complete background details may be found in the book by Zinn–Justin [39], (Appendix to Chapter 8 in second edition, Appendix to Chapter 9 in third edition), in which the divergences of the effective action (up to one-loop) are calculated explicitly.

For the most general scalar field action we can write

\[ S[\phi] = \int d^n x \left\{ \frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) + V[\phi] \right\}, \] (113)

with

\[ V[\phi] = \sum_{j=0}^{N} \frac{\lambda_j}{j!} \phi^j. \] (114)

The operator \( S_2 \) takes the form

\[ S_2(x_1, x_2; \phi) = [-\partial_\mu \partial^\mu + V''[\phi(x_1)]] \delta^n(x_1, x_2), \] (115)

and

\[ S_2(x_1, x_2; \phi = 0) = [-\partial_\mu \partial^\mu + m^2] \delta^n(x_1, x_2), \] (116)

with the special notation that \( m^2 = \lambda_2 = V''[\phi = 0] \).

We now make use of the result derived by Zinn–Justin [39] for the divergent term in the one-loop correction to the effective action (using Schwinger’s proper time regularization):
\[ \Gamma_1[\phi] = \frac{1}{2} \left( \frac{1}{(4\pi)^{n/2}} \right) \left\{ -\frac{\epsilon^{1-n/2}}{1 - n/2} \int d^n x \left[ V''(x) - m^2 \right] + \frac{1}{2} \frac{\epsilon^{2-n/2}}{2 - n/2} \int d^n x \left[ (V'')^2(x) - m^4 \right] \right\} - \frac{1}{64 \pi^2} \left\{ \int d^n x \left[ (V'')^3(x) - m^6 + \frac{1}{2} \partial_{\mu} V''(x) \partial^{\mu} V''(x) \right] \right\} + \ldots \] (117)

See (A8.15) in the Appendix to Chapter 8 of the second edition, or (A9.15) in the Appendix to Chapter 9 of the third edition. Here a comparison with the usual cutoff regulator can be made by replacing \( \epsilon \to \Lambda^{-2} \). Additionally, whenever \( n \) is an even integer \( \epsilon^0/0 \) is to be understood as \( \ln(1/\epsilon) \), and for comparison with the cutoff regulator should be replaced by \( \ln(\Lambda^2/\mu^2) \). This expression gives all the divergences for \( n \leq 6 \).

We are now ready to apply this result to a particular model. We shall first repeat the calculation of the RGE’s for \( \lambda \phi^4 \) theory in four dimensions, in order to compare with the results obtained using the momentum cutoff method in the previous section. Notice that the relevance of the expression (114) is the fact that we can carry out the renormalization at the level of the effective action, and not merely the effective potential, therefore we can deal with the kinetic term and with wavefunction renormalizations, as can be appreciated by the appearance of the gradient terms.

### X. Effective Action for \( \lambda \phi^4 \)

The divergent terms of the effective action for the \( \lambda \phi^4 \) theory in four dimensions are

\[ \Gamma_1[\phi] = \frac{1}{64 \pi^2} \left\{ \Lambda^2 \int d^4 x \lambda \phi^2(x) - \ln \frac{\Lambda^2}{m^2} \int d^4 x \left[ \frac{\lambda^2}{4} \phi^4(x) + \lambda m^2 \phi^2(x) \right] \right\} \] (118)

\[ = \frac{1}{64 \pi^2} \left\{ \Lambda^2 \int d^4 x \lambda \phi^2(x) - \ln \frac{\Lambda^2}{m^2} \int d^4 x \left[ \mathcal{M}^4(\phi(x)) - m^4 \right] \right\}. \] (119)

Here \( \mathcal{M}^4(\phi(x)) = m^2 + (\lambda/2)\phi(x)^2 \) is the position dependent generalization of the \( \mathcal{M}(\phi_0) \) defined in the previous analysis. For a constant field of course, this reproduces the divergent pieces exhibited in equation (40). The fact that there is no divergence containing a gradient term is the justification for our statement that the one-loop wavefunction renormalization for \( \lambda \phi^4 \) vanishes. By this derivation the result is clearly much more general: for any scalar field theory in dimensions five or less (renormalizable or not) the one-loop wavefunction renormalization vanishes. (In particular for \( \lambda \phi^6 \) and \( \lambda \phi^2 \) at one loop all the divergences can be collected in the effective potential.) Putting all this aside for now, we focus attention on \( \lambda \phi^4 \) and check that the RGE’s are completely independent of the regularization scheme.

In the one-loop approximation

\[ \Gamma[\phi] = S[\phi] - S[0] + \hbar \Gamma_1[\phi] + O(\hbar^2), \] (120)

with

\[ \Gamma_1[\phi] = \Gamma_1^0[\phi] + \Gamma_1^{\text{finite}}[\phi]. \] (121)

Since this is a renormalizable theory, we can hope to absorb the divergences into redefinitions of the parameters. For instance we can write

\[ S[\phi] = S[\phi, \mu] - \frac{\hbar}{64 \pi^2} \left\{ \Lambda^2 \int d^4 x \lambda \phi^2(x) - \ln \frac{\Lambda^2}{\mu^2} \int d^4 x \left[ \mathcal{M}^4(\phi(x)) - m^4 \right] + O(\hbar^2) \right\}. \] (122)

so that the effective action may be rewritten as

\[ \Gamma[\phi] = S[\phi, \mu] + \frac{1}{32 \pi^2} \ln \frac{m}{\mu} \int d^4 x \left[ \frac{\lambda^2}{4} \phi^4(x) + \lambda m^2 \phi^2(x) \right] + \Gamma_1^{\text{finite}}[\phi]. \] (123)

This is not quite the same decomposition into renormalized parameters and counterterms as made previously [equation (41)], differing from that prescription by a finite renormalization. This does not matter and does not change any physics. Also note that \( \Gamma_1^{\text{finite}}[\phi] \) contains both non-gradient terms (equivalent to the previously derived effective potential up to finite renormalizations) and gradient terms that were absent in the effective potential calculation. (These gradient terms start out at \( O(\hbar \phi^2 (\partial \phi)^2) \), and contain arbitrarily high powers of both the field and gradients of the field, but the coefficients of these terms are both finite and [in principle] calculable, which is what makes renormalizable theories predictively useful.)
We are now ready to derive the renormalization group equations, but before doing so, and for the sake of clarity, we give the expression for the scale-dependent classical action, that is the classical action with scale-dependent parameters.

\[
S[\phi, \mu] = \int d^4x \frac{1}{2} Z(\mu) \partial_\mu \phi(x) \partial^\mu \phi(x) + \frac{1}{2} Z(\mu) m^2(\mu) \phi^2(x) + \frac{\lambda(\mu)}{4!} Z^2(\mu) \phi^4(x). \tag{124}
\]

As we have already seen, the renormalization group equations follow from

\[
\mu \frac{d\Gamma[\phi]}{d\mu} = 0 = \mu \frac{dS[\phi, \mu]}{d\mu} + \hbar \mu \frac{d\Gamma[\phi]}{d\mu} + O(\hbar^2), \tag{125}
\]

which yields

\[
\frac{\mu}{2} \frac{dZ(\mu)}{d\mu} (\partial_\mu \phi \partial^\mu \phi) + \frac{\mu}{2} Z(\mu) \frac{dm^2(\mu)}{d\mu} \phi^2 + \frac{\mu}{2} m^2(\mu) \frac{dZ(\mu)}{d\mu} \phi^2 + \frac{\mu}{4!} Z^2(\mu) \frac{d\lambda(\mu)}{d\mu} \phi^4 + 2\mu \frac{\lambda(\mu)}{4!} Z(\mu) \frac{dZ(\mu)}{d\mu} \phi^4 = \frac{\hbar}{32\pi^2} \left( \frac{\lambda^2}{4} \phi^3 + \lambda m^2 \phi^2 \right) + O(\hbar^2). \tag{126}
\]

We now proceed to identify terms in the left and right hand sides of the previous equation:

\[
\frac{\mu}{2} \frac{dZ(\mu)}{d\mu} (\partial_\mu \phi \partial^\mu \phi) = 0 + O(\hbar^2) \tag{127}
\]

\[
\frac{\mu}{2} Z(\mu) \frac{dm^2(\mu)}{d\mu} \phi^2 + \frac{\mu}{2} m^2(\mu) \frac{dZ(\mu)}{d\mu} \phi^2 = \frac{\hbar}{32\pi^2} \lambda m^2 \phi^2 + O(\hbar^2) \tag{128}
\]

\[
\frac{\mu}{4!} Z^2(\mu) \frac{d\lambda(\mu)}{d\mu} \phi^4 + 2\mu \frac{\lambda(\mu)}{4!} Z(\mu) \frac{dZ(\mu)}{d\mu} \phi^4 = \frac{\hbar}{32\pi^2} \frac{\lambda^2}{4} \phi^4 + O(\hbar^2). \tag{129}
\]

We can conclude then

\[
\frac{dZ}{d\mu} = 0 + O(\hbar^2) \tag{130}
\]

\[
\frac{\mu}{2} Z(\mu) \frac{dm^2(\mu)}{d\mu} \phi^2 = \frac{\hbar}{32\pi^2} \lambda m^2 \phi^2 + O(\hbar^2) \tag{131}
\]

\[
\frac{\mu}{4!} Z^2(\mu) \frac{d\lambda(\mu)}{d\mu} \phi^4 = \frac{\hbar}{32\pi^2} \frac{\lambda^2}{4} \phi^4 + O(\hbar^2). \tag{132}
\]

From the first equation, and not to change the normalization from that of the bare fields, we can choose \(Z(\mu) = 1 + O(\hbar^2)\), which yields the following two renormalization group equations for \(m^2(\mu)\) and \(\lambda(\mu)\)

\[
\mu \frac{dm^2}{d\mu} = \frac{\hbar}{16\pi^2} m^2 \lambda + O(\hbar^2) \tag{133}
\]

\[
\mu \frac{d\lambda}{d\mu} = \frac{3\hbar}{16\pi^2} \lambda^2 + O(\hbar^2), \tag{134}
\]

which can be seen to be the same as the ones \([72, 73]\) derived using the effective potential approach with a momentum cutoff regularization prescription. [This verifies (at one loop) the scheme independence of the RGE’s.] Notice that here we have by no means made use of the assumption that the field was homogeneous. Also, although the wavefunction renormalization is \(Z(\mu) = 1 + O(\hbar^2)\) to this order, the proper-time formalism is well-suited to handling divergences in the kinetic energy term of the effective action. To see how this works, we will subsequently apply this technique to the \(\lambda \phi^4\) theory in 6 dimensions, where there is a one-loop wavefunction renormalization.

Before leaving the \(\lambda(\phi^4)\) theory, we will explicitly point out that with the current technique we have just computed the one-loop RGE’s without calculating the one-loop effective action (or even the one-loop effective potential). This
situation is made to order for our reconstruction technique, now using the renormalization group improved classical action. In the same way that we improved the classical potential we now consider

\[ S[\phi(x)] = \int d^4x \left\{ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right\}. \]  

We improve this classical action by substituting all the parameters by the improved running parameters, that is

\[ m \to m_{\text{imp}}(\mu) \]  
\[ \lambda \to \lambda_{\text{imp}}(\mu) \]  
\[ \phi \to \phi_{\text{imp}}(\mu) = Z^{3/2}(\mu) \phi, \]

to obtain

\[ S_{\text{imp}} = \int d^4x \left\{ \frac{1}{2} \partial_\mu \phi_{\text{imp}} \partial^\mu \phi_{\text{imp}} + \frac{1}{2} m^2(\mu)\phi_{\text{imp}}^2 + \frac{\lambda(\mu)}{4!} \phi_{\text{imp}}^4 \right\}. \]  

We now follow exactly the same procedure as for the effective potential, expanding the above in leading logarithms to deduce the \( \mu \) dependence of the improved action, which then gives information about the \( \mu \) dependence of the one-loop contribution to the effective action. Simply repeating the previous steps now yields

\[ \Gamma[\phi(x)] = S[\phi(x), m(\mu), \lambda(\mu)] + \int d^4x \left\{ \frac{\hbar}{64\pi^2} \left( \mathcal{M}^2(\phi(x), \mu) \ln \frac{\mathcal{M}^2(\phi(x), \mu)}{\mu^2} - m^2(\mu) \ln \frac{m^2(\mu)}{\mu^2} \right) \right\} \]
\[ + \hbar \left\{ \alpha_4(\lambda(\mu)) \phi(x)^4 + \alpha_5(\lambda(\mu)) m^2(\mu) \phi(x)^2 \right\} + O(\hbar \phi^2(\partial \phi)^2) + O(\hbar^2 \phi^2(\partial \phi)^2) \]
\[ + O(\hbar^2). \]

The novelty here is that the field \( \phi(x) \) is allowed to be position dependent. We have the same finite renormalization ambiguities as in the constant field (effective potential) case, but there are now additional unknowns, even at one loop, coming from higher orders in the gradient expansion. The coefficients of the \( O(\hbar \phi^2(\partial \phi)^2) \) and \( O(\hbar^2 \phi^2(\partial \phi)^2) \) terms are finite and in principle calculable, and so are not accessible via RGE techniques—we will have to resort to a Feynman diagram or related type of calculation to extract the actual coefficient. For completeness, we mention that Itzykson and Zuber report the \( O(\hbar \phi^2(\partial \phi)^2) \) term to be

\[ \frac{\hbar \lambda^2}{192\pi^2} \phi^2(\partial \phi)^2 \]

(see p. 455, eq. (9-130) of [13].)

The significance of the results presented so far lies not in that we have complete information regarding the effective potential (which we do not), but rather in that after this long (because we have been very explicit) build-up, we have a method for very rapidly extracting the one-loop effective action with a minimum of actual calculation.

We shall now apply this technique to \( \lambda(\phi^3) \) (where we have to worry about wavefunction renormalization even at one loop), to \( \lambda(\phi^5) \) (where everything is particularly simple at one loop), and to \( \lambda(\phi^n) \) (where in 2 dimensions almost anything is renormalizable).

**XI. RENORMALIZATION GROUP EQUATIONS FOR \( \lambda(\phi^3) \) THEORY**

Let us consider now \( \lambda(\phi^3) \). The bare action is

\[ S[\phi] = \int d^6x \left\{ \frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) + \frac{\lambda_1}{1!} \phi(x) + \frac{1}{2!} m^2 \phi^2(x) + \frac{\lambda_3}{3!} \phi^3(x) \right\}. \]

Therefore

\[ V'' = \lambda_2 + \lambda_3 \phi \overset{\text{def}}{=} m^2 + \lambda_3 \phi. \]  

17
The divergent terms of the effective action for the $\lambda \phi^3$ theory in 6 dimensions are

$$
\Gamma_1^A[\phi] = \frac{1}{128\pi^3} \left\{ \frac{\Lambda^4}{2} \int d^6x \lambda_3 \phi(x) - \frac{\Lambda^2}{2} \int d^6x [\lambda_3^2 \phi^2(x) + 2m^2 \lambda_3 \phi(x)] 
+ \frac{1}{3} \ln \frac{\Lambda}{m} \int d^6x \left[ \lambda_3^3 \phi(x) + 3 \lambda_3^2 m^2 \phi^2(x) + 3m^4 \lambda_3 \phi(x) + \frac{\lambda_3^4}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) \right] \right\}. 
$$

(144)

At the one-loop approximation

$$
\Gamma[\phi] = S[\phi] - S[0] + h \Gamma_1[\phi] + O(h^2),
$$

(145)

with

$$
\Gamma_1[\phi] = \Gamma_1^A[\phi] + \Gamma_1^\text{finite}[\phi].
$$

(146)

A minor caution: $S[0]$ means $S[\phi[J = 0]]$, which is not zero for an asymmetric theory like this. Fortunately, $S[0]$ is by definition a field-independent offset to the effective action, which when written in terms of bare parameters is manifestly independent of the renormalization scale $\mu$. It therefore does not contribute to the RGE’s (at any number of loops), and can be quietly ignored.

This is a renormalizable theory, and therefore we can split the bare parameters into renormalized parameters and counterterms and write

$$
S[\phi] = S[\phi, \mu] - \frac{h}{128\pi^3} \left\{ \frac{\Lambda^4}{2} \int d^6x \lambda_3 \phi(x) - \frac{\Lambda^2}{2} \int d^6x [\lambda_3^2 \phi^2(x) + 2m^2 \lambda_3 \phi(x)] 
+ \frac{1}{3} \ln \frac{\Lambda}{m} \int d^6x \left[ \lambda_3^3 \phi(x) + 3 \lambda_3^2 m^2 \phi^2(x) + 3m^4 \lambda_3 \phi(x) + \frac{\lambda_3^4}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) \right] \right\} + O(h^2),
$$

(147)

so that the effective action can be rewritten as

$$
\Gamma[\phi] = S[\phi, \mu] - S[0] + \frac{h}{384\pi^3} \ln \frac{\mu}{m} \int d^6x \left[ \lambda_3^3 \phi^3(x) + 3 \lambda_3^2 m^2 \phi^2(x) + 3m^4 \lambda_3 \phi(x) + \frac{\lambda_3^4}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) \right] 
+ h \Gamma_1^\text{finite}[\phi] + O(h^2).
$$

(148)

We are now ready to derive the corresponding renormalization group equations, but before doing so, and for the sake of clarity, we give the expression for the scale-dependent classical action, that is, the classical action written in terms of scale-dependent parameters. (This expression is exact, though as a practical matter calculations are typically carried out at some fixed order in the loop expansion.)

$$
S[\phi, \mu] = \int d^6 x \frac{1}{2} Z(\mu) \partial_\nu \phi(x) \partial^\nu \phi(x) + \frac{\lambda_1(\mu)}{1!} Z^{1/2}(\mu) \phi^4(x) + \frac{m^2(\mu)}{2!} Z(\mu) \phi^2(x) + \frac{\lambda_3(\mu)}{3!} Z^{3/2}(\mu) \phi^3(x).
$$

(149)

Again, the renormalization group equations follow immediately from

$$
\frac{d}{d \mu} \frac{\mu}{d \Gamma[\phi]} = 0 = \mu \frac{d}{d \mu} S[\phi, \mu] + h \mu \frac{d}{d \mu} \Gamma_1[\phi] + O(h^2),
$$

(150)

which yields

$$
\frac{\mu}{d \mu} \frac{d Z(\mu)}{d \mu} (\partial_\nu \phi \partial^\nu \phi) 
+ \mu \frac{d Z^{1/2}(\mu)}{d \mu} \frac{d \lambda_1(\mu)}{d \mu} \phi + \frac{\mu}{2} \lambda_1(\mu) Z^{-1/2}(\mu) \frac{d Z(\mu)}{d \mu} \phi 
+ \mu \frac{d m^2(\mu)}{d \mu} \phi^2 + \frac{\mu}{2} m^2(\mu) \frac{d Z(\mu)}{d \mu} \phi^2 
+ \frac{\mu}{3!} Z^{3/2}(\mu) \frac{d \lambda_3(\mu)}{d \mu} \phi^3 + \frac{3 \mu \lambda_3(\mu)}{2} Z^{1/2}(\mu) \frac{d Z(\mu)}{d \mu} \phi^3 
= -\frac{h}{384\pi^3} \left[ \lambda_3^3 \phi^3 + 3 \lambda_3^2 m^2 \phi^2 + 3m^4 \lambda_3 \phi + \frac{\lambda_3^4}{2} \partial_\mu \phi \partial^\mu \phi \right] + O(h^2).
$$

(151)
Matching like powers of $\phi$ on both sides of this equation, as we did previously, we deduce the following set of (1-loop) renormalization group equations:

\[
\frac{dZ}{d\mu} = -\frac{h}{384\pi^3}\lambda_1^2 + O(h^2) \tag{152}
\]

\[
\mu Z \frac{d\lambda_1}{d\mu} + \frac{\mu}{2} \lambda_1 \frac{dZ}{d\mu} = -\frac{h}{128\pi^3} \lambda_3 m^2 Z^{1/2} + O(h^2) \tag{153}
\]

\[
\mu Z \frac{dm^2}{d\mu} + \mu m^2 \frac{dZ}{d\mu} = -\frac{h}{64\pi^3} \lambda_3^2 m^2 + O(h^2) \tag{154}
\]

\[
\mu Z \frac{d\lambda_3}{d\mu} + \frac{3\mu}{2} \lambda_3 \frac{dZ}{d\mu} = -\frac{h}{64\pi^3} \lambda_3^3 Z^{-1/2} + O(h^2). \tag{155}
\]

We now define the following functions

\[
\beta_1 \left( \lambda_1, \lambda_3, \frac{m}{\mu} \right) \overset{\text{def}}{=} \mu \frac{\partial \lambda_1}{\partial \mu} \tag{156}
\]

\[
\beta_3 \left( \lambda_1, \lambda_3, \frac{m}{\mu} \right) \overset{\text{def}}{=} \mu \frac{\partial \lambda_3}{\partial \mu} \tag{157}
\]

\[
\gamma_m \left( \lambda_1, \lambda_3, \frac{m}{\mu} \right) \overset{\text{def}}{=} \mu m^2 \frac{\partial m^2}{\partial \mu} \tag{158}
\]

\[
\gamma_d \left( \lambda_1, \lambda_3, \frac{m}{\mu} \right) \overset{\text{def}}{=} \mu \frac{\partial \ln Z}{\partial \mu}, \tag{159}
\]

to obtain

\[
\beta_1 \left( \lambda_1, \lambda_3, \frac{m}{\mu} \right) = +\frac{h}{768\pi^3} \left( \frac{\lambda_1 \lambda_3^3}{128\pi^3} - \lambda_3 m^2 \right) + O(h^2) \tag{160}
\]

\[
\beta_3 \left( \lambda_1, \lambda_3, \frac{m}{\mu} \right) = -\frac{3h\lambda_3}{256\pi^3} + O(h^2) \tag{161}
\]

\[
\gamma_m \left( \lambda_1, \lambda_3, \frac{m}{\mu} \right) = -\frac{5h\lambda_3}{384\pi^3} + O(h^2) \tag{162}
\]

\[
\gamma_d \left( \lambda_1, \lambda_3, \frac{m}{\mu} \right) = -\frac{h\lambda_3^2}{384\pi^3} + O(h^2). \tag{163}
\]

These results can be checked (for example) against chapter 7, section 3, of Collins [23]. (We again warn the reader that there are conflicting semi-standard definitions for the anomalous dimensions $\gamma$, our choice is opposite to that for Collins [23], but the same as that of Ramond [20] and Zinn–Justin [39].) We do not need to check at the level of the primitive parameters $m^2$, $Z$ and $\lambda$, because it is more efficient to do so at the level of the parameters $\beta_3$, $\gamma_m$, and $\gamma_d$. In most treatments the equation for $\beta_1$ is not taken into consideration. It is traditionally assumed that there is no term linear in $\phi$ in the original action, and furthermore, that when renormalizing, we should impose the condition that all the tadpole diagrams vanish: that is, that the renormalized and scale dependent $\lambda_1(\mu)$ must vanish. This assumption is tantamount to fine-tuning the bare parameter $\lambda_1$, order by order in perturbation theory, to force $\lambda_1(\mu) = 0$.

A simplifying approach that does not involve fine-tuning is to go to the massless and zero external bias limit. That is, to simultaneously set $m = 0$ and $\lambda_1 = 0$, this being a fixed-plane of the one-loop RGE’s. Then we can safely restrict attention to the $\beta_3-\gamma_d$ plane. (We can also easily check that if we simultaneously set both bare parameters $\lambda_1$ and $m$ equal to zero, then renormalization effects will not generate such parameters, at least to one loop order in six dimensions.)

**XII. IMPROVED ACTION FOR THE MASSLESS ZERO-BIAS $\lambda(\phi^3)_6$ THEORY**

We start with the bare massless Lagrangian density for $\lambda(\phi^3)_6$:

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{\lambda}{3!} \phi^3. \tag{164}
\]
The *improvement* of this Lagrangian density consists in substituting all the bare parameters by the renormalization group improved ones, that is

\[
\lambda \rightarrow \lambda_{\text{imp}} = \lambda(\mu) \\
\phi \rightarrow \phi_{\text{imp}} = \phi(\mu) = Z^{1/2}(\mu) \phi,
\]

(165)

(166)
to obtain

\[
\mathcal{L}_{\text{imp}} = \frac{1}{2} \partial_\mu \phi_{\text{imp}} \partial^\mu \phi_{\text{imp}} + \frac{\lambda(\mu)}{3!} \phi_{\text{imp}}^3 \\
= \frac{1}{2} Z(\mu) \partial_\mu \phi \partial^\mu \phi + \frac{\lambda(\mu)}{3!} Z^{3/2}(\mu) \phi^3.
\]

(167)

In this massless zero-bias case the RGE’s are straightforward to solve. We have

\[
\lambda(\mu) = \lambda(\mu_0) \left(1 + \frac{3\hbar\lambda^2(\mu_0)}{128\pi^3} \ln \frac{\mu}{\mu_0} + O(\hbar^2)\right)^{-1/2},
\]

(168)

and

\[
Z(\mu) = Z(\mu_0) \left(\mu/\mu_0\right)^{-\frac{\hbar\lambda^2(\mu_0)}{32\pi^3}} \exp(O(\hbar^2)).
\]

(169)

Inserting this into the improved action and expanding to \(O(\hbar^2)\) we see

\[
\mathcal{L}_{\text{imp}} = \frac{1}{2} Z(\mu_0) \partial_\mu \phi \partial^\mu \phi + \frac{\lambda(\mu_0)}{3!} Z(\mu_0)^{3/2} \phi^3 \\
+ \hbar \left\{ -\frac{\lambda^2(\mu_0)}{768\pi^3} Z(\mu_0) \partial_\mu \phi \partial^\mu \phi - \frac{\lambda^2(\mu_0)}{384\pi^3} Z(\mu_0)^{3/2} \phi^3 \right\} \ln \frac{\mu}{\mu_0} + O(\hbar^2)
\]

(170)

\[
= \frac{1}{2} \partial_\mu \phi(\mu_0) \partial^\mu \phi(\mu_0) + \frac{\lambda(\mu_0)}{3!} \phi(\mu_0)^3 \\
+ \hbar \left\{ -\frac{\lambda^2(\mu_0)}{768\pi^3} \partial_\mu \phi(\mu_0) \partial^\mu \phi(\mu_0) - \frac{\lambda^2(\mu_0)}{384\pi^3} \phi(\mu_0)^3 \right\} \ln \frac{\mu}{\mu_0} + O(\hbar^2).
\]

(171)

As we have seen before in other examples, the explicit \(\mu\) dependence in the improved action must cancel against the explicit \(\mu\) dependence in the one-loop contribution. Without repeating the details (already presented in the \(\lambda(\phi^4)_4\) calculation), this constrains the form of the one-loop effective potential and implies that up to finite renormalizations

\[
\mathcal{L}_{\text{effective}} = \frac{1}{2} \partial_\mu \phi(\mu) \partial^\mu \phi(\mu) + \frac{\lambda(\mu)}{3!} \phi(\mu)^3 + \hbar \left\{ -\frac{\lambda(\mu)^2}{1536\pi^3} \partial_\mu \phi(\mu) \partial^\mu \phi(\mu) - \frac{\lambda(\mu)^3}{768\pi^3} \phi(\mu)^3 \right\} \ln \frac{\phi(\mu)}{\mu^2}
\]

(172)

+ \(O(\hbar^2) + O(\hbar^2)).

Part of this expression can be checked against the effective potential. (The logarithms in the effective action have to match up with the logarithmically divergent terms in the regulated one-loop effective potential.) But the term involving gradients that arises from the one-loop wavefunction renormalization is completely inaccessible via effective potential techniques.

**XIII. RGE’S AND EFFECTIVE ACTION FOR \(\lambda(\phi^6)_3\)**

The key result can be very succinctly stated: at one loop nothing runs in 3 dimensions. Once we have seen this result once, it becomes obvious from equation \(^{[117]}\) that this is generic to any odd-dimensional spacetime at one loop. In any odd number of dimensions there will be no logarithmic divergences, and hence no terms logarithmic in the renormalization scale in the one-loop effective action.

So let us turn to considering our simple example: symmetric \(\lambda(\phi^6)_3\). The bare action is

\[
S[\phi] = \int d^3x \left\{ \frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) + \frac{1}{2} \mu^2 \phi^2(x) + \frac{\lambda_4}{4!} \phi^4(x) + \frac{\lambda_6}{6!} \phi^6(x) \right\},
\]

(173)
Therefore
\[ V'' = m^2 + \frac{1}{2!} \lambda_4 \phi^2 + \frac{1}{4!} \lambda_6 \phi^4. \] (174)

The divergent terms of the effective action for the $\lambda \phi^6$ theory in 3 dimensions are
\[ \Gamma^\Lambda_1[\phi] = \frac{1}{(4\pi)^{3/2}} \left\{ \Lambda \int d^3x \left[ \frac{1}{2!} \lambda_4 \phi^2 + \frac{1}{4!} \lambda_6 \phi^4 \right] + O(\Lambda^{-1}) \right\}. \] (175)

Note that there are no logarithmic divergences (this will play a very important role!). At the one-loop approximation, we again know that
\[ \Gamma[\phi] = S[\phi] - S[0] + h \Gamma_1[\phi] + O(h^2), \] (176)
with
\[ \Gamma_1[\phi] = \Gamma^\Lambda_1[\phi] + \Gamma^\text{finite}_1[\phi]. \] (177)

This is a renormalizable theory, and therefore we can split the bare parameters into renormalized parameters and counterterms to write
\[ S[\phi] = S[\phi, \mu] - \frac{h}{(4\pi)^{3/2}} \left\{ \Lambda \int d^3x \left[ \frac{1}{2!} \lambda_4 \phi^2 + \frac{1}{4!} \lambda_6 \phi^4 \right] + O(\Lambda^{-1}) \right\} + O(h^2), \] (178)
so that the effective action can be rewritten as
\[ \Gamma[\phi] = S[\phi, \mu] - S[0] + h \Gamma^\text{finite}_1[\phi] + O(h^2). \] (179)

We are now ready to derive the corresponding renormalization group equations. Since the renormalization scale does not appear in $\Gamma^\text{finite}_1$ above, the RGE’s will be trivial to one loop. Differentiating
\[ \mu \frac{d \Gamma[\phi]}{d \mu} = 0 = \mu \frac{d S[\phi, \mu]}{d \mu} + h \mu \frac{d \Gamma_1[\phi]}{d \mu} + O(h^2), \] (180)
yields
\[ \mu \frac{dZ}{d\mu} = 0 + O(h^2) \] (181)
\[ \mu \frac{dm^2}{d\mu} = 0 + O(h^2) \] (182)
\[ \mu \frac{d\lambda_4}{d\mu} = 0 + O(h^2) \] (183)
\[ \mu \frac{d\lambda_6}{d\mu} = 0 + O(h^2). \] (184)

Thus all the $\beta$ functions vanish at one loop. This does not mean that this theory is one-loop finite. It does mean that the theory does not “run” at one loop, and that the renormalized coupling constants are renormalization scale independent at this order. The effective potential is simply equal to the classical potential, up to finite renormalization ambiguities. The effective action, explicitly exhibiting finite renormalization ambiguities, is simply
\[ \mathcal{L}_{\text{effective}} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2!} m^2 \phi^2(x) + \frac{\lambda_4}{4!} \phi^4(x) + \frac{\lambda_6}{6!} \phi^6(x) \]
\[ + \h \left\{ \epsilon_1 \frac{1}{2!} m^2 \phi^2(x) + \epsilon_2 \frac{\lambda_4}{4!} \phi^4(x) + \epsilon_3 \frac{\lambda_6}{6!} \phi^6(x) \right\} \]
\[ + O(h \phi(\partial \phi)^2) + O(h^2). \] (185)
In two dimensions any scalar field theory with polynomial interactions is renormalizable. The field \( \phi \) is canonically dimensionless, and for the bare action we have

\[
S[\phi] = \int d^2x \left\{ \frac{1}{2} \partial_{\mu} \phi(x) \partial^{\mu} \phi(x) + V(\phi) \right\}.
\]

Here \( V(\phi) \) is an arbitrary polynomial in the field \( \phi \). All the coefficients in this polynomial have the same canonical dimension, that of \( m^2 \). The divergent terms of the effective action in 2 dimensions are

\[
\Gamma_1^A[\phi] = -\frac{1}{4\pi} \ln \frac{\Lambda}{m} \int d^2x \left\{ V''(\phi(x)) - m^2 \right\} + O(\Lambda^{-2}).
\]

(187)

Where we have defined \( m^2 = V''(\phi = 0) \). Note that there is now only a logarithmic divergence. The wavefunction renormalization again vanishes to one loop, and we have

\[
\Gamma[\phi] = S[\phi] - S[0] + \hbar \Gamma_1[\phi] + O(\hbar^2),
\]

(188)

with

\[
\Gamma_1[\phi] = \Gamma_1^A[\phi] + \Gamma_1^{\text{finite}}[\phi].
\]

(189)

This is a renormalizable theory, and therefore we can split the bare parameters into renormalized parameters and counterterms and write

\[
S[\phi] = S[\phi, \mu] + \frac{\hbar}{4\pi} \ln \frac{\Lambda}{\mu} \int d^2x \left\{ V''(\phi(x)) - m^2 \right\} + O(\hbar \Lambda^{-2}) + O(\hbar^2),
\]

(190)

so that the effective action can be rewritten as

\[
\Gamma[\phi] = S[\phi, \mu] - S[0] - \frac{\hbar}{4\pi} \ln \frac{\mu}{m} \int d^2x \left\{ V''(\phi(x)) - m^2 \right\} + \hbar \Gamma_1^{\text{finite}}[\phi] + O(\hbar \Lambda^{-2}) + O(\hbar^2).
\]

(191)

The corresponding renormalization group equations are most easily collected together and presented as

\[
\mu \frac{dV(\phi(x), \mu)}{d\mu} = \frac{\hbar}{4\pi} \left\{ V''(\phi(x), \mu) - m^2(\mu) \right\} + O(\hbar^2).
\]

(192)

Integrating yields

\[
V(\phi(x), \mu) = V(\phi(x), \mu_0) + \frac{\hbar}{4\pi} \left\{ V''(\phi_0, \mu) - m^2(\mu_0) \right\} \ln \frac{\mu}{\mu_0} + O(\hbar^2).
\]

(193)

We can use this to renormalization group improve the classical potential, but the reconstruction technique we have applied previously does not enable us to recover the full one-loop effective potential. The best we can do with the reconstruction technique is to obtain

\[
\mathcal{V}(\phi_0, \mu) = V(\phi_0, \mu_0) + \frac{\hbar}{8\pi} \left\{ V''(\phi_0, \mu) - m^2(\mu) \right\} \ln \frac{\mu}{m(\mu)} + \hbar m^2(\mu) X[V(\phi_0, \mu)/m^2(\mu), \mu] + O(\hbar^2).
\]

(194)

Here \( X[V(\phi_0, \mu)/m^2(\mu), \mu] \) is an arbitrary dimensionless function of the indicated variables. Unfortunately, since \( \phi_0 \) is now dimensionless we cannot use dimensional analysis to further pin down the behaviour of \( X \). To reduce the remaining freedom, it is best to backtrack to our earlier result (274) which for two dimensions yields

\[
\mathcal{V}(\phi_0, \mu) = V(\phi_0, \mu_0) + \frac{\hbar}{8\pi} \left\{ V''(\phi_0, \mu) \ln \frac{V''(\phi_0, \mu)}{m^2(\mu)} - m^2(\mu) \ln \frac{m^2(\mu)}{m^2(\mu)} \right\} + O(\hbar^2),
\]

(195)

up to finite renormalizations. This latter result can now be inserted into what we know about the effective action to determine (up to finite renormalizations)

\[
\Gamma[\phi(x)] = S[\phi(x), m(\mu), \lambda(\mu)] + \int d^2x \left[ \frac{\hbar}{8\pi} \left\{ V''(\phi(x), \mu) \ln \frac{V''(\phi(x), \mu)}{m^2(\mu)} - m^2(\mu) \ln \frac{m^2(\mu)}{m^2(\mu)} \right\} + O(\hbar \phi^2(\partial \phi)^2) + O(\hbar (\partial \phi)^4) \right] + O(\hbar^2).
\]

(196)

This finally gives us the leading term in the gradient expansion for the one-loop effective action in \( \lambda(\phi^n)_2 \).
Improvement consists of substituting the bare parameters appearing in a (renormalizable) field theory by their running, scale-dependent forms, calculated to some order in the coupling(s). The very scale dependence of these couplings is of course a consequence of the fluctuations and interactions present in the theory. The scale dependence is handled by the renormalization group whose aim is to describe how the dynamics of a system evolves as we change the scale at which the phenomena is being observed. Improved perturbation theory then results from combining the tools of renormalization group with perturbation theory and allows us to go beyond the strict limitations imposed by conventional perturbation theory alone.

When cross-fertilized with zeta function technology, the ideas presented in this article provide an exceptionally clean and compact formalism for extracting all of one-loop physics from an appropriate Seeley-DeWitt coefficient. Details will be presented elsewhere [46].

Applications of these powerful techniques are by no means limited to “just” quantum field theory. A vast range of non-linear physical phenomena subject to fluctuations, not necessarily of a quantum mechanical nature, may also be investigated using much of the same technology already developed for quantum field theory. We have in mind processes subject to thermal or statistical noise which abound in phenomenology ranging from, but not limited to, the problem of pattern formation, convection and hydrodynamic turbulence, chaos, chemical instabilities, and morphogenesis [47,48]. All such phenomena can be modelled by non-linear reaction-diffusion equations of one rubric or another, with non-potential and/or derivative interactions as well as with conventional polynomial potentials. The inevitable stochasticity inherent in these systems can be incorporated by means of a noise source, and we have to consider in general stochastic nonlinear parabolic equations. The dynamics encoded in these equations can be equivalently and profitably re-cast in terms of generating functional integrals, thus converting stochastic dynamics per se into a field-theory language that can be calculationally exploited in a maximally efficient manner [49–51].

A most interesting formal analogy that arises between quantum and stochastic field theories is in the identification of a loop-counting parameter. It turns out that at least for processes subject to white Gaussian noise, the modulus of the noise two-point function is the loop-counting parameter in a field theory formulation of statistical fluctuations, and this is the analog of Planck’s constant $\hbar$ which is the loop counting parameter when the fluctuations are of a quantum nature. But there is much more. As the system is subject to fluctuations, we can expect the parameters appearing in the stochastic equations to run with distance or momentum scale as dictated by corresponding renormalization group equations. These equations can be just as simply calculated for stochastic field theory as they are for quantum field theory by way of the effective action in the manner shown here. And, just as in the case of quantum field theory, their solutions can be used to improve expressions based on tree-level stochastic equations. We shall report on these developments elsewhere [44] [48].
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