$\mathbb{F}_q$-Linear Calculus over Function Fields

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Abstract

We define analogues of higher derivatives for $\mathbb{F}_q$-linear functions over the field of formal Laurent series with coefficients in $\mathbb{F}_q$. This results in a formula for Taylor coefficients of a $\mathbb{F}_q$-linear holomorphic function, a definition of classes of $\mathbb{F}_q$-linear smooth functions which are characterized in terms of coefficients of their Fourier-Carlitz expansions. A Volkenborn-type integration theory for $\mathbb{F}_q$-linear functions is developed; in particular, an integral representation of the Carlitz logarithm is obtained.

Key words: $\mathbb{F}_q$-linear function; Carlitz basis; Carlitz logarithm; Volkenborn integral; difference operator; Bargmann-Fock representation.
1 INTRODUCTION

It is well known that any non-discrete, locally compact field of characteristic \( p \) is isomorphic to the field \( K \) of formal Laurent series with coefficients from the Galois field \( \mathbb{F}_q \), \( q = p^\gamma \), \( \gamma \in \mathbb{Z}_+ \).

The foundations of analysis over \( K \) were laid in a series of papers by Carlitz (see, in particular, [3-5]), Wagner [20, 21], Goss [6-8], and Thakur [17, 18].

An interesting property of many functions introduced in the above works as analogues of classical elementary and special functions, is their \( \mathbb{F}_q \)-linearity. Recall that a function \( f : K \to K \) (where \( K \) is a completion of an algebraic closure of \( K \)) is called \( \mathbb{F}_q \)-linear if \( f(t_1 + t_2) = f(t_1) + f(t_2) \) and \( f(\alpha t) = \alpha f(t) \) for any \( t, t_1, t_2 \in K, \alpha \in \mathbb{F}_q \). In a similar way one can define a notion of a \( \mathbb{F}_q \)-linear function on any \( \mathbb{F}_q \)-subspace of \( K \), and also a notion of a \( \mathbb{F}_q \)-linear operator on a vector space over \( K \).

Tools of classical calculus are not sufficient to study behavior of \( \mathbb{F}_q \)-linear functions. For example, if such a function \( f \) is differentiable then \( f'(t) \equiv \text{const} \), and all the higher derivatives vanish irrespective of possible properties of \( f \). In particular, one cannot reconstruct the Taylor coefficients of a holomorphic function – the classical formula contains the expression \( \frac{f^{(n)}(t)}{n!} \) where both the numerator and denominator vanish.

Thanks to Carlitz, we know the correct analogue of the factorial. A counterpart of a higher derivative is given in this paper. In fact, its main ingredient, the difference operator \( \Delta^{(n)} \),

\[
(\Delta^{(n)} f)(t) = \Delta^{(n-1)} f(xt) - x^{q^n-1} \Delta^{(n-1)} f(t), \quad n \geq 1; \quad \Delta^{(0)} = \text{id},
\]

where \( x \) is a prime element in \( K \), was introduced by Carlitz and often used subsequently.

Our approach is based on identities for the difference operator \( \Delta = \Delta^{(1)} \), which emerge as a function field analogue of the Bargmann-Fock representation of the canonical commutation relations of quantum mechanics [9, 14]. Note that an analogue of the Schrödinger representation was obtained in [12].

We will also use \( \Delta^{(n)} \) in order to characterize “smoothness” of \( \mathbb{F}_q \)-linear functions and obtain an exact correspondence between the degree of smoothness (or analyticity) and the decay rate for coefficients of Fourier-Carlitz expansions. Note that in the \( p \)-adic case (where smoothness is understood in a conventional sense) such a correspondence was found in [1, 2, 15].

Having a new “derivative”, we can introduce a kind of an antiderivative, and a Volkenborn type integral (see [15, 19] for the case of zero characteristic), which leads, in particular, to an integral representation of the Carlitz logarithm, establishing a direct connection between the latter and the Carlitz module operation.

2 PRELIMINARIES

2.1 Carlitz Basis [3, 4, 6, 12, 20]

Denote by \( | \cdot | \) the non-Archimedean absolute value on \( K \); if \( z \in K \),

\[
z = \sum_{i=n}^{\infty} \zeta_i x^i, \quad n \in \mathbb{Z}, \quad \zeta_i \in \mathbb{F}_q, \quad \zeta_n \neq 0,
\]
then $|z| = q^{-n}$. It is well known that this valuation can be extended onto $\overline{K}_c$. Let $O = \{z \in K : |z| \leq 1\}$ be the ring of integers in $K$. The ring $\mathbb{F}_q[x]$ of polynomials (in the indeterminate $x$) with coefficients from $\mathbb{F}_q$ is dense in $O$.

Let $C_0(O, \overline{K}_c)$ be the Banach space of all $\mathbb{F}_q$-linear continuous functions $O \rightarrow \overline{K}_c$, with the supremum norm $\| \cdot \|$. Any function $\varphi \in C_0(O, \overline{K}_c)$ admits a unique representation as a uniformly convergent series

$$\varphi = \sum_{i=0}^{\infty} c_i f_i, \quad c_i \in \overline{K}_c, \ c_i \to 0,$$

satisfying the orthonormality condition

$$\|\varphi\| = \sup_{i \geq 0} |c_i|.$$

Here $\{f_i\}$ is the sequence of the normalized Carlitz $\mathbb{F}_q$-linear polynomials defined as follows.

Let $e_0(t) = t$,

$$e_i(t) = \prod_{m \in \mathbb{F}_q[x]} (t - m), \quad i \geq 1. \quad (1)$$

It is known [3, 6] that

$$e_i(t) = \sum_{j=0}^{i} (-1)^{i-j} \left[ \begin{array}{c} i \\ j \end{array} \right] t^{q^j}$$

where

$$\left[ \begin{array}{c} i \\ j \end{array} \right] = \frac{D_i}{D_j L_{i-j}^{q^j}},$$

the elements $D_i, L_i \in K$ are defined as

$$D_i = [i][i-1]^q \ldots [1]^{q^{i-1}}; \ L_i = [i][i-1] \ldots [1] (i \geq 1); \ D_0 = L_0 = 1,$$

and $[i] = x^q - x \in O$. Finally,

$$f_i(t) = D_i^{-1} e_i(t), \quad i = 0, 1, 2, \ldots.$$

The basis $\{f_i\}$ can be complemented up to an orthonormal basis $\{h_j\}$ of the space $C(O, \overline{K}_c)$ of all continuous $\overline{K}_c$-valued functions on $O$, as follows. Let us write any natural number $j$ as

$$j = \sum_{i=0}^{\nu} \alpha_i q^i, \quad 0 \leq \alpha_i < q, \quad (2)$$

and set

$$h_j(t) = \frac{G_j(t)}{\Gamma_j}, \quad G_j(t) = \prod_{i=0}^{\nu} (c_i(t))^{\alpha_i}, \quad \Gamma_j = \prod_{i=0}^{\nu} D_i^{\alpha_i}.$$

It is clear that $h_j$ is a polynomial of degree $j$, $h_{q^i} = f_i$. 

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Lemma 1. The elements $D_m$, $L_m$, and $\Gamma_{q^{m-1}}$ are connected by the identity

$$\Gamma_{q^{m-1}}L_m = D_m.$$ 

Proof. By the definition,

$$\Gamma_{q^{m-1}} = (D_0 \ldots D_{m-1})^{q-1}.$$

We have $D_m = \prod_{i=1}^{m} [i]^{q^{m-i}}$, $L_m = \prod_{i=1}^{m} [i]$, $D_0 \ldots D_{m-1} = \prod_{j=1}^{m-1} \prod_{i=1}^{j} [i]^{q^{j-i}} = \prod_{i=1}^{m-1} \prod_{j=i}^{m} [i]^{q^{j-i}} = \prod_{i=1}^{m} \prod_{j=1}^{m-1} [i]^{\frac{m-i}{q-1}}$, so that

$$\Gamma_{q^{m-1}}L_m = [m] \prod_{i=1}^{m-1} [i]^{q^{m-i}} = D_m. \quad \square$$

Another system of polynomials introduced by Carlitz is given by

$$g_j(t) = \prod_{i=0}^{\nu} g_{\alpha_i q^i}(t), \quad g_0(t) \equiv 1,$$

where $j$ is expanded as in (2),

$$g_{\alpha_i q^i}(t) = \begin{cases} e_{\alpha_i}(t), & \text{if } \alpha_i < q - 1, \\ e_{q-1}^i(t) - D_{q-1}^i, & \text{if } \alpha_i = q - 1. \end{cases}$$

Lemma 2. The system of polynomials

$$\tau_m(t) = \frac{g_{q^{m-1}}(t)}{\Gamma_{q^{m-1}}}, \quad m = 0, 1, 2, \ldots, t \in O,$$

is orthonormal.

Proof. By the definition,

$$g_{q^{m-1}}(t) = \prod_{i=0}^{m-1} (e_{q-1}^i(t) - D_{q-1}^i),$$

whence

$$\tau_m(t) = \prod_{i=0}^{m-1} (f_{q-1}^i(t) - 1),$$

so that $\|\tau_m\| \leq 1$. Since $\tau_m(0) = (-1)^{m-1}$, we have $|\tau_m(0)| = 1$ and $\|\tau_m\| = 1$.

In order to prove orthogonality, it suffices to show that for any natural $m$, and any $\lambda_1, \ldots, \lambda_m \in K_c$

$$\left\| \sum_{k=0}^{m} \lambda_k \tau_k \right\| \geq |\lambda_m| \quad (3)$$

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(see Proposition 50.4 in [15]).

It is known [6] that
\[ g_{q^m-1}(t) = \sum_{j=0}^{q^m-1} G_j(t) g_{q^m-1-j}(0). \]

It follows from the definition of \( \Gamma_j \) that
\[ \Gamma_{q^m-1} = \Gamma_{q^m-1-j} \Gamma_j, \quad 0 \leq j \leq q^m - 1, \]

so that
\[ \tau_m(t) = \sum_{j=0}^{q^m-1} \sigma_{m,j} h_j(t), \quad \sigma_{m,j} = \frac{g_{q^m-1-j}(0)}{\Gamma_{q^m-1-j}}. \]

Now
\[ \sum_{k=0}^{m} \lambda_k \tau_k(t) = \sum_{j=0}^{q^m-1} h_j(t) \sum_{\log_q(j+1) \leq k \leq m} \lambda_k \sigma_{k,j}, \]

\[ \left| \sum_{k=0}^{m} \lambda_k \tau_k \right| \geq |\lambda_m \sigma_{m,q^m-1}|, \]
due to the orthonormality of \( \{ h_j \} \). Since \( \sigma_{m,q^m-1} = \frac{g_0(0)}{\Gamma_0} = 1 \), we come to (3). □

2.2 Canonical Commutation Relations [12]

In the quantum mechanics of harmonic oscillator (see e.g. [16]) a creation operator transforms a stationary state into a stationary state of the next (higher) energy level, an annihilation operator acts in the opposite way. In quantum field theory these properties are used to obtain operators which change the number of particles.

The analogues of the creation and annihilation operators (in the Schrödinger representation) for our situation are as follows. Let \( R_q \) be a \( \mathbb{F}_q \)-linear operator on \( C_0(O, \overline{K}_c) \) of the form \( R_q u = u^q \). Consider the operators
\[ a^+ = R_q - I, \quad a^- = \sqrt{\Delta} \circ \Delta \]
where \( I \) is the identity operator, \( \Delta = \Delta^{(1)} \) (see Introduction), while \( \sqrt{\Delta} \) is the inverse to \( R_q \).

Both operators \( a^+ \) and \( a^- \) are \( \mathbb{F}_q \)-linear and continuous on \( C_0(O, \overline{K}_c) \) and
\[ a^- a^+ - a^+ a^- = [1]^{1/q} I. \]

The operator \( a^+ a^- \) possesses the orthonormal eigenbasis \( \{ f_i \} \),
\[ (a^+ a^-) f_i = [i] f_i, \quad i = 0, 1, 2, \ldots ; \]

\( a^+ \) and \( a^- \) act upon the basis as follows:
\[ a^+ f_{i-1} = [i] f_i, \quad a^- f_i = f_{i-1}, \quad i \geq 1; \quad a^- f_0 = 0. \]
We look at (4)-(6) as an analogue of the Schrödinger representation (for an exposition see [14, 16] or other standard texts on mathematical foundations of quantum mechanics) since in non-Archimedean analysis spaces of continuous functions (or sequence spaces isomorphic to them) are often used instead of the Hilbert spaces of square integrable functions where the operators of conventional quantum mechanics are defined. Note also that a $p$-adic analogue of the Schrödinger representation was found in [11].

It is interesting that the above operators possess remarkable algebraic properties. The operator $\Delta$ is a derivation on the ring of $\mathbb{F}_q$-linear functions from $O$ to itself, with composition being the multiplication in the ring. The operator $R_q - I$ is an $R_q$-derivation in the sense of [10].

The Bargmann-Fock representation is a realization of a structure like (4)-(6) by operators on a space of holomorphic functions. A simple construction for the case of a function field is given below.

## 3 HOLOMORPHIC FUNCTIONS AND THE BARGMANN-FOCK REPRESENTATION

Consider a Banach space $H$ over the field $\mathbb{K}_c$ consisting of $\mathbb{F}_q$-linear power series

$$u(t) = \sum_{n=0}^{\infty} a_n \frac{t^{q^n}}{D_n}, \quad a_n \in \mathbb{K}_c, \quad |a_n| \to 0.$$  \hfill (7)

The norm in $H$ is given by

$$\|u\|_A = \sup_n |a_n|.$$  

Since

$$|D_n| = q^{-1} (q^{-1})^q \ldots (q^{-1})^{q^{n-1}} = q^{-\frac{q^n - 1}{q - 1}} = q^{-1/(q-1)} q^{n-1},$$

a series (7) defines a holomorphic function for $|t| \leq q^{-1/(q-1)}$.

It is obvious that the sequence of the functions $\tilde{f}_n(t) = \frac{t^{q^n}}{D_n}$, $n = 0, 1, 2, \ldots$, is an orthonormal basis of $H$.

The desired representation is given by the following operators on the space $H$:

$$\tilde{a}^+ = R_q, \quad \tilde{a}^- = \sqrt[\varphi]{\Delta}$$

(note that the form of the operators $\tilde{a}^\pm$ is only slightly different from the one of $a^\pm$, but they act on a different Banach space!).

By a straightforward computation based on the identities

$$\Delta \left( t^{q^n} \right) = \lfloor n \rfloor t^{q^n}, \quad D_{n+1} = \lceil n + 1 \rceil D_n^q,$$

we show that the relations (4)-(6) hold for the operators $\tilde{a}^\pm$, with $\tilde{f}_n$ substituted for $f_n$.

Now we can find a reconstruction formula for coefficients of the series (7).

**Theorem 1.** If $u \in H$ then

$$a_n = \lim_{t \to 0} \frac{\Delta^{(n)}u(t)}{t^{q^n}}, \quad n = 0, 1, 2, \ldots.$$  \hfill (8)
Proof. We proceed from the identities
\[ \tilde{a} - (\lambda \tilde{f}_n) = \lambda^{1/q} \tilde{f}_{n-1}, \quad n \geq 1, \quad \lambda \in \overline{K}_c; \quad \tilde{a} - \tilde{f}_0 = 0. \]

It follows that
\[ (\tilde{a}^-)^n u(t) = \sum_{k=n}^{\infty} a_k^{1/q^n} \frac{t^{q^{k-n}}}{D_{k-n}} \]
whence
\[ a_n^{1/q^n} = \lim_{t \to 0} \frac{(\tilde{a}^-)^n u(t)}{t}. \]

Now (8) is a consequence of the identity \((\tilde{a}^-)^n = q^{n-\Delta n} \) (see [6]). □

4 SMOOTH FUNCTIONS

Let \( u \in C^0_0(O, \overline{K}_c), \)
\[ D^k u(t) = t^{-q^k} \Delta^{(k)} u(t), \quad t \in O \setminus \{0\}. \]

We will say that \( u \in C^{k+1}_0(O, \overline{K}_c) \) if \( D^k u \) can be extended to a continuous function on \( O. \)

\( C^{k+1}_0(O, \overline{K}_c) \) can be considered as a Banach space over \( \overline{K}_c, \) with the norm
\[ \sup_{t \in O} \left( |u(t)| + |D^k u(t)| \right). \]

Note that \( C^{k+1}_0(O, \overline{K}_c) \) coincides with the set of all differentiable \( F_q \)-linear functions \( O \to \overline{K}_c. \)

In this section we will obtain a characterization of functions from \( C^{k+1}_0(O, \overline{K}_c) \) in terms of coefficients of the expansion \( u = \sum_{n=0}^{\infty} c_n \tilde{f}_n. \) First we prove two auxiliary results.

**Lemma 3.** If a function \( v : O \setminus \{0\} \to \overline{K}_c \) is continuous and bounded, and \( v \) admits a pointwise convergent expansion
\[ v(t) = \sum_{n=0}^{\infty} v_n \tau_n(t), \quad t \in O \setminus \{0\}, \quad (9) \]

\( v_n \in \overline{K}_c, \) then
\[ \sup_{t \in O \setminus \{0\}} |v(t)| = \sup_{0 \leq n < \infty} |v_n|. \]

**Proof.** It is clear that
\[ |v(t)| \leq \sup_n |v_n|, \quad t \neq 0. \]

In order to prove the inverse inequality we will perform the summation of both sides in (9) through all monic polynomials \( t \) with \( \deg t = m, \) and use the identity
\[ \sum_{\deg t = m, \ t \ monic} g_l(t) G_k(t) = \begin{cases} 0, & \text{if } k + l \neq q^m - 1, \\ (-1)^m \frac{P_m}{P_{2m}}, & \text{if } k + l = q^m - 1, \end{cases} \]
valid for $k < q^m$, $l < q^m$ (see [4, 6]). In particular, we will need the case when $k = 0$, $l = q^n - 1$, for which we get

$$\sum_{\deg t = m} g_{q^n-1}(t) = \begin{cases} 0, & \text{if } n < m, \\ (-1)^m \frac{D_m}{L_m}, & \text{if } n = m. \end{cases}$$

If $n > m$ then [4, 6] $g_{q^n-1}(t) = t^{-1}e_n(t) = 0$ for $\deg t = m$ by the definition of $e_n(t)$.

Now the summation in (9) yields

$$\sum_{\deg t = m} v(t) = (-1)^m \frac{D_m}{L_m} \cdot \frac{v_m}{\Gamma_{q^m-1}}.$$  

Using Lemma 1, we obtain that $|v_m| \leq \sup_{t \in O \setminus \{0\}} |v(t)|$.  \(\square\)

**Lemma 4.** Let a function $w \in C(O, \overline{K}_c)$ be such that the function $\gamma(t) = tw(t)$ is $\mathbb{F}_q$-linear. Then

$$w(t) = \sum_{n=0}^{\infty} w_n \gamma_n(t), \quad w_n \in \overline{K}_c,$$

and the series (10) is uniformly convergent on $O$.

**Proof.** Consider an expansion

$$\gamma(t) = \sum \gamma_n f_n(t), \quad t \in O,$$

where $\gamma_n \in \overline{K}_c$, $\gamma_n \to 0$. The fact that the function $t^{-1}\gamma(t)$ is continuous at $t = 0$ means that $\gamma(t)$ is differentiable at $t = 0$. By a result of Wagner [21], $\gamma_n L_n^{-1} \to 0$ or, equivalently, $|\gamma_n| q^n \to 0$ for $n \to \infty$.

Dividing both sides of (11) by $t$, we obtain the expansion (10) with $w_n = D_n^{-1} \gamma_n \Gamma_{q^n-1}$, so that $w_n \to 0$.  \(\square\)

Now we are in a position to prove the characterization result.

**Theorem 2.** A function $u = \sum_{n=0}^{\infty} c_n f_n \in C_0(O, \overline{K}_c)$ belongs to $C_0^{k+1}(O, \overline{K}_c)$ if and only if

$$q_{nq^k}|c_n| \to 0 \quad \text{for } n \to \infty.$$  

In this case

$$\sup_{t \in O} |\mathcal{D}^k u(t)| = \sup_{n \geq k} q^{(n-k)q^k} |c_n|. \quad (13)$$

**Proof.** In view of (6)

$$(-a)^k f_n = \begin{cases} f_{n-k}, & \text{if } n \geq k, \\ 0, & \text{if } n < k. \end{cases}$$
Since \((a^{-})^k = \sqrt[k]{\Delta} \circ \Delta^{(k)}\), we find that
\[
\mathcal{D}^k u(t) = t^{-q^k} \sum_{n=k}^{\infty} c_n f_{n-k}^k(t), \quad t \neq 0,
\]
which implies the identity
\[
(\mathcal{D}^k u(t))^{q^{-k}} = \sum_{n=k}^{\infty} c_n^{q^{-k}} \Gamma_{q^{n-k}-1} D_{n-k}^{-1} \tau_{n-k}(t), \quad t \neq 0,
\]
(15)

As before, an easy computation shows that \(|\Gamma_{q^{n-1}} D_n^{-1}| = q^n\). Now, if (12) is satisfied, then by Lemma 2 the right-hand side of (15) is a continuous function on \(O\), which means that \(u \in C_0^{k+1}(O, \mathcal{K})\). The equality (13) follows from Lemma 3.

Conversely, suppose that \(\mathcal{D}^k u\) is continuous on \(O\). Let \(w(t) = (\mathcal{D}^k u(t))^{q^{-k}}\). By (14), the function \(\gamma(t) = tw(t)\) has the form
\[
\gamma(t) = \sum_{n=k}^{\infty} c_n^{q^{-k}} f_{n-k}(t), \quad t \neq 0.
\]
(16)
Since \(w\) is continuous on \(O\), we see that \(\gamma(0) = 0\). On the other hand, \(f_n(0) = 0\) for all \(n\), so that the equality (16) holds for all \(t \in O\), and the function \(\gamma\) is \(F_q\)-linear (however, we cannot claim the uniform convergence of the series in (16)).

Now Lemma 4 implies the representation (10) with \(w_n \to 0\). It follows from (10) and (16) that
\[
\sum_{n=0}^{\infty} \left( w_n - c_n^{q^{-k}} L_{n-k}^{-1} \right) \tau_n(t) = 0
\]
for all \(t \neq 0\). By Lemma 3, this means that
\[
w_n = c_n^{q^{-k}} L_{n-k}^{-1}
\]
for \(n \geq 0\), which implies (12). \(\square\)

Note that for the case \(k = 0\) our characterization is equivalent to Wagner’s theorem which was used above in the course of proving our general result.

5 ANALYTIC FUNCTIONS

Denote by \(A(O, \mathcal{K})\) the space of all functions \(O \to \mathcal{K}\) of the form
\[
u(t) = \sum_{n=0}^{\infty} a_n t^{q^n}, \quad \mathcal{K} \ni a_n \to 0.
\]
(17)
It is clear that \(A(O, \mathcal{K}) \subset C_0(O, \mathcal{K})\).

Theorem 3. A function \(u = \sum_{n=0}^{\infty} c_n f_n \in C_0(O, \mathcal{K})\) belongs to \(A(O, \mathcal{K})\) if and only if
\[
q^{\frac{a^n}{n!}} |c_n| \to 0 \quad \text{for} \ n \to \infty.
\]
(18)
Proof. Let \( u \in A(O, \mathcal{K}_c) \) be represented by the series (17). Let us find an expression for its Fourier-Carlitz coefficients \( c_n \).

It is known [6] that

\[
t^{q^n} = \sum_{j=0}^{n} b_{nj} e_j(t)
\]

where

\[
b_{nj} = \frac{\left( \Delta(j) t^{q^n} \right)_{|t=1}}{D_j}.
\]

We have seen (Sect. 3) that

\[
(\sqrt{q} \circ \Delta(j)) \left( \frac{t^{q^n}}{D_n} \right) = (\tilde{a})^j \left( \frac{t^{q^n}}{D_n} \right) = \frac{t^{q^{n-j}}}{D_{n-j}}.
\]

whence

\[
(\Delta(j) t^{q^n})_{|t=1} = \frac{D_n}{D_{n-j}^{q^n}}.
\]

so that

\[
t^{q^n} = \sum_{j=0}^{n} \frac{D_n}{D_{n-j}^{q^n}} f_j(t).
\]

Substituting this into (17) and changing the order of summation we find that

\[
c_n = \sum_{k=n}^{\infty} \frac{D_k}{D_{k-n}^{q^n}} a_k.
\]

Since \( |D_k| = q^{\frac{k}{q-1}} \), we obtain that

\[
|c_n| \leq q^{\frac{n}{q-1}} \sup_{k \geq n} |a_k|
\]

which implies (18).

Conversely, (18) means that \( c_n D_n^{-1} \to 0 \) for \( n \to \infty \). Using (1') we can write

\[
u(t) = \sum_{n=0}^{\infty} c_n \frac{D_n}{D_{n-j}} \sum_{j=0}^{n} (-1)^{n-j} \frac{D_n}{D_{j} L_{n-j}^{q^n}} t^{q^j}.
\]

It is easy to get that

\[
\left| \frac{D_n}{D_{j} L_{n-j}^{q^n}} \right| = q^{s_{nj}}
\]

where

\[
s_{nj} = \begin{cases} (n-j)q^j - q^{j-1} - \cdots - q^1, & \text{if } n \geq j + 1, \\ 0, & \text{if } n = j. \end{cases}
\]
Since $s_{nj} \leq q^j [(n - j - 1) - q^{n-j-1}]$ if $n \geq j + 1$, and the function $z \mapsto q^z - z$ increases for $z \geq 1$, we have $s_{nj} < 0$. Now we may rewrite (19) in the form (17) with

$$|a_n| \leq \max_{k \geq n} \left| \frac{c_k}{D_k} \right| \to 0, \quad n \to \infty. \quad \Box$$

A similar theorem can be obtained if $K$ is a completion of $\mathbf{F}_q(x)$ corresponding to an arbitrary irreducible polynomial $\pi \in \mathbf{F}_q[x]$, $\deg \pi = d$. In this case one gets the condition $q^{dn-1} |c_n| \to 0$ for $n \to \infty$ instead of (18). The proof is based on the fact that the absolute value $|\alpha_i|_{\pi}$ equals $q^{-1}$ if $d$ divides $i$, or 1 in the opposite case (which can be deduced from Lemma 2.13 in [13]).

6 INDEFINITE SUM

Viewing the operator $a^-$ as a kind of a derivative, it is natural to introduce an appropriate antiderivative. Following the terminology used in the analysis over $\mathbb{Z}_p$ (see [15]) we call it the indefinite sum.

Consider in $C_0(O, \mathring{K}_c)$ the equation

$$a^- u = f, \quad f \in C_0(O, \mathring{K}_c). \quad (20)$$

Suppose that $f = \sum_{k=0}^{\infty} \varphi_k f_k$, $\varphi_k \in \mathring{K}_c$. Looking for $u = \sum_{k=0}^{\infty} c_k f_k$ and using the fact that

$$a^- u = \sum_{k=1}^{\infty} c_k^{1/q} f_{k-1} = \sum_{l=0}^{\infty} c_{l+1}^{1/q} f_l$$

we find that $c_{l+1} = \varphi_l^q$, $l = 0, 1, 2, \ldots$. Therefore $u$ is determined by (20) uniquely up to the term $c_0 f_0(t) = c_0 t$, $c_0 = u(1)$.

Fixing $u(1) = 0$ we obtain a $\mathbf{F}_q$-linear bounded operator $S$ on $C_0(O, \mathring{K}_c)$, the operator of indefinite sum: $Sf = u$. It follows from Theorem 2 that $S$ is also a bounded operator on each space $C^k_0(O, \mathring{K}_c)$, $k = 1, 2, \ldots$. Note that $S f_k = f_{k+1}$, $k = 0, 1, \ldots$, so that $S$ is not compact.

Another possible procedure to find $Sf$ is an interpolation. If $u = Sf$ then

$$u(xt) - xu(t) = f^q(t), \quad u(1) = 0.$$

Setting successively $t = 1, x, x^2, \ldots$, we get

$$u(x) = f^q(1),$$
$$u(x^2) = f^q(x) + xf^q(1),$$
$$u(x^3) = f^q(x^2) + xf^q(x) + x^2 f^q(1),$$
$$
.............
$$
$$u(x^n) = f^q(x^{n-1}) + xf^q(x^{n-2}) + \ldots + x^{n-1} f^q(1).$$
Since \( u \) is assumed \( \mathbb{F}_q \)-linear, this determines \( u(t) \) for all \( t \in \mathbb{F}_q[x] \). Extending by continuity, we find \( u(t) \) for any \( t \in O \): if
\[
t = \sum_{n=0}^{\infty} \zeta_n x^n, \quad \zeta_n \in \mathbb{F}_q,
\]
then
\[
u(t) = \sum_{n=0}^{\infty} \zeta_n u(x^n) = \sum_{n=1}^{\infty} \zeta_n \sum_{j=1}^{n} x^{j-1} f^q(x^{n-j}) = \sum_{j=1}^{\infty} \sum_{n=j}^{\infty} \zeta_n x^{j-1} f^q(x^{n-j}),
\]
so that
\[
u(t) = \sum_{n=0}^{\infty} x^n f^q\left(\sum_{m=0}^{\infty} \zeta_{m+n+1} x^m\right).
\]

7 VOLKENBORN-TYPE INTEGRAL

The Volkenborn integral of a function on \( \mathbb{Z}_p \) was introduced in [19] (see also [15]) in order to obtain relations between some objects of \( p \)-adic analysis resembling classical integration formulas of real analysis. Here we extend this approach to the function field situation. Our definition is based essentially on the Carlitz difference operator \( \Delta \), which shows again its close connection with basic structures of analysis over the field \( K \).

The integral of a function \( f \in C^1_0(O, \overline{K}_c) \) is defined as
\[
\int_O f(t) \, dt \overset{\text{def}}{=} \lim_{n \to \infty} Sf(x^n)/x^n = (Sf)'(0).
\]

It is clear that the integral is a \( \mathbb{F}_q \)-linear continuous functional on \( C^1_0(O, \overline{K}_c) \),
\[
\int_O c f(t) \, dt = c^q \int_O f(t) \, dt, \quad c \in \overline{K}_c.
\]

Since a power series \( \sum_{n=0}^{\infty} a_n t^{q^n} \) with \( a_n \to 0 \) converges in \( C^1_0(O, \overline{K}_c) \), it can be integrated termwise:
\[
\int_O \sum_{n=0}^{\infty} a_n t^{q^n} \, dt = \sum_{n=0}^{\infty} a_n^q \int_O t^{q^n} \, dt.
\]

The integral possesses the following “invariance” property (related, in contrast to the case of \( \mathbb{Z}_p \), to the multiplicative structure):
\[
\int_O f(xt) \, dt = x \int_O f(t) \, dt - f^q(1), \quad (21)
\]

Indeed, let \( g(t) = f(xt) \). Then
\[
Sg(x^n) = g^q(x^{n-1}) + xg^q(x^{n-2}) + \cdots + x^{n-1}g^q(1)
= f^q(x^n) + xf^q(x^{n-1}) + \cdots + x^{n-1} f^q(x) = (Sf)(x^{n+1}) - x^n f^q(1)
\]
whence
\[
\frac{Sg(x^n)}{x^n} = x \cdot \frac{(Sf)(x^{n+1})}{x^{n+1}} - f(q)(1),
\]
and (21) is obtained by passing to the limit for \( n \to \infty \).

Using (20) we obtain by induction that
\[
\int_0 f(x^n t) dt = x^n \int_0 f(t) dt - x^{n-1} f(q)(1) - x^{n-2} f(q)(x) - \cdots - f(q)(x^{n-1}).
\]

This equality implies the following invariance property. Suppose that a function \( f \) vanishes on all elements \( z \in F_q[x] \) with \( \deg z < n \). Then, if \( g \in F_q[x] \), \( \deg g \leq n \), we have
\[
\int_0 f(gt) dt = g \int_0 f(t) dt.
\]

Our last result will contain the calculation of integrals for some important functions on \( O \). Let us recall the definitions of some special functions introduced by Carlitz (see [8]).

The function \( C_s(z) \), \( s \in F_q[x] \), \( z \in K \), defining the Carlitz module, is given by the formula
\[
C_s(z) = \sum_{i=0}^{\deg s} f_i(s) z^{q^i}.
\]

If \( |z| < 1 \), \( C_s(z) \) can be extended with respect to \( s \):
\[
C_s(z) = \sum_{i=0}^{\infty} f_i(s) z^{q^i}, \quad s \in O,
\]
(note that \( f_i(s) = 0 \) for \( i > \deg s \) if \( s \in F_q[x] \)). Since \( s^{-1} f_i(s) = L_i^{-1} \tau_i(s) \) and \( |L_i^{-1}| = q^i \), the function \( C_s(z) \) is differentiable with respect to \( s \).

The Carlitz logarithm \( \log_C(z) \) is defined as
\[
\log_C(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{q^n}}{L_n}, \quad |z| < 1.
\]

The Carlitz exponential is defined by the power series
\[
e_C(z) = \sum_{n=0}^{\infty} \frac{z^{q^n}}{D_n}, \quad |z| < 1.
\]

**Theorem 4.** (i) For any \( n = 0, 1, 2, \ldots \)
\[
\int_0 t^{q^n} dt = -\frac{1}{[n + 1]}.
\]
(ii) For any \( n = 0, 1, 2, \ldots \)

\[
\int_0 f_n(t) \, dt = \frac{(-1)^{n+1}}{L_{n+1}}.
\]  
(23)

(iii) If \( z \in K, \, |z| < 1 \), then

\[
\int_0 C_s(z) \, ds = \log_C(z) - z.
\]  
(24)

\textbf{Proof.} (i) We have seen that

\[
a_q^-(\frac{t^q^n}{D_n}) = \frac{t^{q^n}-1}{D_{n-1}}, \quad n \geq 1.
\]

It follows from the definition of the operator \( S \) that

\[
S\left(\frac{t^q^n}{D_n}\right) = \frac{t^{q^n+1} - t}{D_{n+1}}
\]

whence

\[
S\left(\frac{t^q^n}{D_n}\right) (x^k) = \frac{x^{k(q^n+1)} - 1}{[n+1]} \quad \rightarrow \quad -\frac{1}{[n+1]}, \quad k \rightarrow \infty.
\]

(ii) We have

\[
\int_0 f_n(t) \, dt = (Sf_n)'(0) = f_{n+1}'(0).
\]

According to (1'), the linear term in the expression for \( f_i \) is

\[
(-1)^i \frac{[1]}{D_i} t = \frac{(-1)^i}{L_i} t.
\]

(25)

The differentiation yields (23).

(iii) Applying the operator \( a_q^- \) (with respect to the variable \( s \)) to the function \( C_s(z) \) we find that

\[
a_q^- C_s(z) = \sum_{i=1}^\infty f_{i-1}(s) z^{q^{i-1}} = C_s(z)
\]

whence

\[
S_s C_s(z) = C_s(z) - C_1(z) s = \sum_{i=0}^\infty f_i(s) z^{q_i} - z s.
\]

Fixing \( z \) and denoting \( \varphi(s) = S_s C_s(z) \) we obtain that

\[
\frac{\varphi(x^n)}{x^n} = \sum_{i=0}^\infty \frac{f_i(x^n)}{x^n} z^{q_i} - z.
\]

(26)
It is seen from (1′) and (25) that
\[ \frac{f_i(x^n)}{x^n} \to \frac{(-1)^i}{L_i} \quad \text{for } n \to \infty. \]

On the other hand,
\[ t^{-1}f_i(t) = D_i^{-1}g_{q^{-1}}(t) = D_i^{-1}\Gamma_{q^{-1}}\tau_i(t). \]
As we know, \(|\tau_i(t)| \leq 1, |D_i^{-1}\Gamma_{q^{-1}}| = q^i\), so that
\[ \left| \frac{f_i(x^n)}{x^n} \right| \leq q^i \]
for all \(n\). If \(z \in O, |z| < 1\), then \(|z| \leq q^{-1}\), and the series in (26) converges uniformly with respect to \(n\). Passing to the limit \(n \to \infty\) in (26) we come to (24). \(\Box\)

Setting in (24) \(z = e_C(t), |t| < 1\), we get an identity for the Carlitz exponential:
\[ \int_0 e_C(st) \, ds = t - e_C(t). \]

On the other hand, the formula (24) implies a more general formula (conjectured by D.Goss).

**Corollary.** If \(a \in O, z \in K, |z| < 1\), then
\[ \int_O C_{sa}(z) \, ds = a \log C(z) - C_a(z). \] \hspace{1cm} (27)

**Proof.** Since it is shown easily that (for each fixed \(z\)) the mapping \(O \to C_0^1(O, K)\) of the form \(a \mapsto C_{sa}(z)\) is continuous, it is sufficient to prove (27) for \(a = x^n, n = 1, 2, \ldots\). Using (24), we find that
\[ \int_O C_{sa}(z) \, ds = x^n(\log C(z) - z) - \sum_{k=1}^{n} x^{n-k}C_{x^{k-1}}^q(z). \] \hspace{1cm} (28)

Let \(t = \log C(z)\). Then \(z = e_C(t)\) (see [8, 12]). It follows from properties of \(e_C\) [8] that
\[ \sum_{k=1}^{n} x^{n-k}C_{x^{k-1}}^q(z) = \sum_{k=1}^{n} x^{n-k}(e_C(x^k t) - xe_C(x^{k-1} t)) = e_C(x^n t) - x^n e_C(t) = C_a(x^n(z) - x^n z). \]
Substituting this into (28) we come to (27). \(\Box\)

As \(C_{sa}(z) = C_s(C_a(z))\), equation (24) also implies that
\[ \int_O C_{sa}(z) \, ds = \log C(C_a(z)) - C_a(z). \]
Comparing this with (27) implies
\[ a \log C(z) = \log C(C_a(z)) \]
which is precisely the functional equation of \(\log C(z)\).
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