ON ENERGY-MOMENTUM AND SPIN/HELCITY 
OF QUARK AND GLUON FIELDS

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Abstract

In special relativity, quantum matter can be classified according to mass-energy and spin. The corresponding field-theoretical notions are the energy-momentum-stress tensor $\mathcal{T}$ and the spin angular momentum tensor $\mathcal{S}$. Since each object in physics carries energy and, if fermionic, also spin, the notions of $\mathcal{T}$ and $\mathcal{S}$ can be spotted in all domains of physics. We discuss the $\mathcal{T}$ and $\mathcal{S}$ currents in Special Relativity (SR), in General Relativity (GR), and in the Einstein-Cartan theory of gravity (EC). We collect our results in 4 theses: (i) The quark energy-momentum and the quark spin are described correctly by the canonical (Noether) currents $\mathcal{T}$ and $\mathcal{S}$, respectively. (ii) The gluon energy-momentum current is described correctly by the (symmetric and gauge invariant) Minkowski type current. Its (Lorentz) spin current vanishes, $\mathcal{S} = 0$. However, it carries helicity of plus or minus one. (iii) GR contradicts thesis (i), but is compatible with thesis (ii). (iv) Within the viable EC-theory, our theses (i) and (ii) are fulfilled and, thus, we favor this gravitational theory.

1 Introduction

The nucleon spin and how it is built up in terms of spin and orbital angular momentum contributions of the quark and gluon fields is still under discussion. Recently, in this context, the problem has been addressed of the appropriate energy-momentum and spin tensors of quark and gluon fields, see the review paper of Leader and Lorcé [1]. They emphasize the importance of the splitting of the angular momentum of the gluon field into orbital and spin parts. However, since the energy-momentum and angular momentum distributions of a field are interrelated via the orbital angular momentum, the angular momentum question can only be answered if the energy-momentum distribution is treated at the same time. This is an expression of the semi-direct product structure of the Poincaré group $P(1, 3) := T(4) \rtimes SO(1, 3)$; here $T(4)$ denotes the translation group and $SO(1, 3)$ the Lorentz group.

These facts are, of course, recognized by Leader and Lorcé [1] perfectly well, as can be seen by their discussion of the so-called Belinfante and the canonical energy-momentum tensors of both, the gluon and the quark fields. Even though they mention general relativity (GR) in this context, their main arguments are taken from special-relativistic quantum field theory. On the other side it is known—we only remind of Weyl’s verdict [2] that only “the process of variation to be applied to the metrical structure of the world, leads to a true definition of the energy” of matter—that an appropriate gravitational theory is obligatory in order to get a clear insight into the energy-momentum distribution of matter.

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[1] Invited talk delivered at the XV Workshop on High Energy Spin Physics ‘DSPIN-13’ in Dubna, Russia, 08–12 October 2013 (file DubnaSpin2013_6.tex, 02 Feb. 2014).
Why is this true? In Newton’s gravitational theory the mass density of matter is the source of gravity; in GR, by an appropriate generalization, it is the (symmetric) Hilbert energy-momentum tensor $H^t_{ij}$, which is computed by variation of the matter Lagrangian $\mathcal{L}$ with respect to the metric tensor $g_{ij}$, namely $H^t_{ij} := \frac{\delta \mathcal{L}}{\delta g^{ij}}$. Consequently, we have to assume that the energy-momentum distribution is, in the classical limit, a measurable quantity and that this localized energy-momentum distribution, with its 10 components, is the source of the gravitational field. As long as we subscribe to GR, the Hilbert energy-momentum tensor is the only viable energy-momentum tensor of matter, a fact that is put into doubt by Leader and Lorcè.

Teryaev (3) already pointed out that the energy-momentum tensor of matter will play a decisive role at the interface between quantum chromodynamics (QCD) and gravity, see also (4). He discussed the gravitational moments of Dirac particles, as done earlier by Kobzarev and Okun (5) and by Hehl et al. (6).

Let us recall the eminent importance of the Poincaré group. Wigner’s mass-spin classification of elementary (or fundamental) particles (7) is at the basis of the standard model of particle physics, the quark and the gluon are particular examples of it. The mass-spin classification, by means of a scalar and a vector quantity, underlines the particle aspect of matter. The corresponding notions for elementary fields in classical field theory, are the energy-momentum current and the spin current. Thus, the mass-spin classification of matter is mirrored on the field-theoretical side by the canonical (Noether) energy-momentum current $T^t_{ik}$, with $4 \times 4$ components, and the canonical (Noether) spin current $S_{ijk} = -S_{jik}$ with its $6 \times 4$ components. The relation of the Hilbert and the Noether energy-momentum currents will be discussed further down.

On the gravitational side, there took some developments place that are not without implications for the understanding of the energy-momentum and the spin distribution of matter fields, see also the thermodynamic considerations of Becattini & Tinti (8). GR got a competitor in the Einstein-Cartan(-Sciama-Kibble) theory of gravitation (EC) or, more generally, in the Poincaré gauge theory of gravitation (PG). A short outline and the classical papers of the subject can be found in Blagojević and Hehl (9), see also the review paper (10).

The EC is a viable gravitational theory that can be distinguished from GR at very high densities or at very small distances occurring in early cosmology. The critical distance is $\ell_{EC} \approx (\frac{\lambda_{Co}^2 \ell_{Pl}^2}{\lambda_{Co}^2 + \lambda_{Pl}^2})^{1/3}$, with the Compton wave length $\lambda_{Co}$ of the particle involved, about $10^{-26}$ cm for the nucleon, and the Planck length $\ell_{Pl} \approx 10^{-33}$ cm. Mukhanov (11) has argued the the data of the Planck satellite support GR up to distances of the order of $10^{-27}$ cm, that is, the same order of magnitude where the deviations of EC are supposed to set in.

The EC-theory is a simple case of a PG-theory. The PG-theory is formulated in a Riemann-Cartan (RC) spacetime with torsion $C_{ij}^k (=-C_{ji}^k)$ and curvature $R_{ij}^{kl} (=-R_{ji}^{kl} = -R_{ij}^{lk})$. The gravitational Lagrangian of PG-theory is, in general, quadratic in the field strengths torsion and curvature. EC-theory is the simplest case, when the Lagrangian, apart from the cosmological term, consists only of a linear curvature piece $\sim R_{ij}^{ji}$ (summation!), the Riemann-Cartan generalization of the Hilbert-Einstein Lagrangian. Then, additionally to the gravitational effects of GR, we find a very weak spin-spin-contact interaction that is governed by Einstein’s gravitational constant. But what is more relevant in the present context is that in PG-theory—hence also in EC-theory—the source of the Newton-Einstein type gravity is the canonical energy-momentum and the source of a Yang-Mills type strong gravity the canonical spin.

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2"...we feel that the fundamental versions are the canonical and the Belinfante ones, since they involve at least local fields..."; see (1), page 92.
3We use, for energy-momentum and spin, the notions ‘tensor’ and ‘current’ synonymously.
However, one has to be careful in the details: Gauge field, like the electromagnetic or the
The quark current, as spin 1/2 current, should be of a similar type as the superfluid $^3$He in the A-phase. That is, the (physically correct) energy-momentum current of the quark field should be asymmetric and most probably the canonical (Noether) current $\mathcal{T}_{ij}$ of Eq.(1).
In electromagnetism, only $\mathcal{T}_{ij}$ survives (9 components), since it is massless, that is, $\mathcal{T}_{k}^{k} = 0$, and carries helicity, but no (Lorentz) spin, i.e., $\mathcal{T}_{[ij]} = 0$. The analogous should be true for the gluon field, since, like the Maxwell (photon) field, it is a gauge field, see below for some more details.

Where took Einstein the symmetry of the energy-momentum tensor from? Einstein, in [17] on the pages 48 and 49, discussed the symmetry of the energy-momentum tensor of Maxwell’s theory. Subsequently, on page 50, he argued: “We can hardly avoid making the assumption that in all other cases, also, the space distribution of energy is given by a symmetrical tensor, $T_{\mu\nu}$,...”. This is hardly a convincing argument if one recalls that the Maxwell field is massless. As we saw, the A-phase of $^3$He contradicts Einstein’s assumption. Asymmetric energy-momentum tensors are legitimate quantities in physics and, the symmetry of an energy-momentum tensor has to retire as a generally valid rule.

3 Lorentz invariance

Invariance under 3+3 infinitesimal Lorentz transformations, $x'^{i} = x^{i} + \omega^{ij}x_{j}$, with $\omega^{ij} = 0$, yields, via the Noether theorem and $\delta L / \delta \Psi = 0$, angular-momentum conservation,

$$\partial_{k}\left( \mathcal{S}_{ij}^{k} + x_{i}^{[j} \mathcal{T}_{j]}^{k} \right) = 0, \quad \mathcal{S}_{ij}^{k} := -\frac{\partial L}{\partial \partial_{k}\Psi} f_{ij} \Psi = -\mathcal{S}_{ji}^{k}. \quad (5)$$

The canonical (Noether) spin $\mathcal{S}_{ij}^{k}$, the spin current density, is a tensor of type $(\frac{1}{2} \otimes 4 \oplus 4 \oplus 4)$, plays a role in the interpretation of the Einstein-de Haas effect (1915). If we differentiate in (5) the second term and apply $\partial_{k}\mathcal{T}_{ij}^{k} = 0$, then we find a form of angular momentum conservation that can be generalized to curved and contorted spacetimes ($x^{i}$ is not a vector in general):

$$\partial_{k}\left( \mathcal{S}^{ijk} + x^{[i} \mathcal{T}^{j]k} \right) = 0 \implies \partial_{k}\mathcal{S}^{ijk} - \mathcal{T}^{[ij]} = 0. \quad (6)$$

If $\mathcal{S}^{ijk} = 0$, then $\mathcal{T}^{[ij]} = 0$, that is, the energy-momentum tensor is symmetric, but not necessarily vice versa.

The irreducible decomposition, with the axial vector piece $\text{AX} \mathcal{S}_{ijk} := \mathcal{S}_{[ijk]}$ and the vector piece $\text{VEC} \mathcal{S}_{ij}^{k} := \frac{2}{3} \mathcal{S}_{[i|\ell} \delta_{j]}^{k}$, reads:

$$\mathcal{S}_{ij}^{k} = \text{TEN} \mathcal{S}_{ij}^{k} + \text{VEC} \mathcal{S}_{ij}^{k} + \text{AX} \mathcal{S}_{ij}^{k}, \quad (7)$$

$$24 = 16 \oplus 4 \oplus 4,$$

The Weyssenhoff ansatz for a classical spin fluid is again of the convective type. We take

$$\mathcal{S}_{ij}^{j} = p_{i}^{j} \quad \text{mom. curr. d.} \quad \text{and} \quad \mathcal{S}_{ij}^{k} = \mathcal{S}_{ij}^{k} \quad \text{spin d.}$$

The momentum density $p_{i}$ is no longer proportional to the velocity, as it was in (5). Usually, the constraint $s_{ij}u^{j} = 0$ is assumed.

For the Dirac field, the spin current is totally antisymmetric, $\mathcal{D} \mathcal{S}_{ijk} = \mathcal{D} [\mathcal{S}_{ijk}]$. Thus, only the axial vector spin current survives, $\text{AX} \mathcal{D} \mathcal{S}_{ijk} \neq 0$. The Dirac field is highly symmetric. Accordingly, we can introduce the spin flux vector

$$\mathcal{S}^{i} := \frac{1}{3!} \epsilon^{ijkl} \mathcal{S}_{jkl} \sim (\text{spin flux density 1 comp., spin density 3 comps.}). \quad (9)$$
The spin density distribution is spatially isotropic.

4 Poincaré invariance

We collect our results: The Poincaré invariance of the action yields the $4 + 6$ conservation laws,

\begin{align}
\partial_k \mathcal{T}_i^k &= 0 \quad \text{(energy-momentum conservation),} \quad (10) \\
\partial_k \mathcal{G}_{ij}^k - \mathcal{T}_{[ij]} &= 0 \quad \text{(angular momentum conservation).} \quad (11)
\end{align}

These field theoretical notions $\mathcal{T}_i^k$ and $\mathcal{G}_{ij}^k$ have their analogs in a the particle description of matter. The Lie algebra of the Poincaré group reads (see [18] for details, $\hbar = 1$):

\begin{align}
[P_i, P_j] &= 0, \\
[J_{ij}, P_k] &= 2i g_{k[i} P_{j]} \quad \text{(transl. and Lorentz transf. mix, as in } \mathcal{G}_{ijk} + x_{[i}\mathcal{T}_{j]k}), \\
[J_{ij}, J_{kl}] &= 2i (g_{k[i} J_{j]l} - g_{l[i} J_{j]k}).
\end{align}

We recognize its semidirect product structure, as it is manifest in the existence of orbital angular momentum. The “square roots” of the Casimir operators $P^2$ (mass square) and $W^2$ (spin square), with the Pauli-Lubanski vector $W^i := \frac{1}{2} \epsilon^{ijkl} J_{jk} P_l$, correspond to $\mathcal{T}_i^k$ and $\mathcal{G}_{ij}^k$.

5 Exterior calculus in a Riemann-Cartan (RC) space, the electromagnetic/gluon energy-momentum, and the Dirac field

We introduce the generally covariant calculus of exterior differential forms that is valid not only in Minkowski space, but also in the RC-spacetime of the Poincaré gauge theory of gravity, see [19]. We work with an orthonormal coframe (tetrad) $\vartheta^\alpha = e^i_\alpha dx^i$ and a Lorentz connection $\Gamma^\alpha_{\beta\gamma} = \Gamma^{\alpha^{\beta\gamma}} dx^i = -\Gamma^{\beta\alpha}$; the fields are exterior forms (0-forms, 1-forms,..., 4-forms) with values in the algebra of some Lie group; the frame (or anholonomic) indices are in Greek, $\alpha, \beta, \cdots = 0, 1, 2, 3$. The electromagnetic potential is a 1-form $A = A_i dx^i$, the field strength a 2-form $F := dA = \frac{1}{2} F_{ij} dx^i \wedge dx^j$, the exterior derivative is denoted by $d$, the gauge covariant exterior derivative is by $D$, for details see [20].

The matter currents translate from tensor to exterior calculus as follows: Energy-momentum 3-form $\mathcal{T}_\alpha = \mathcal{T}_\alpha^* \star \partial_\gamma = \delta L_{\text{mat}}/\delta \partial^\alpha$, spin 3-form $\mathcal{S}_{\alpha\beta} = \mathcal{S}_{\alpha\beta}^* \star \partial_\gamma = \delta L_{\text{mat}}/\delta \Gamma^{\alpha\beta}$, with the Hodge star $\star$. Here we displayed already the variational expression, which will be explained below.

Maxwell’s vacuum field $A(x)$ is a 1-form, a geometrical object independent of coordinates and frames. As such, it has vanishing Lorentz-spin, $\mathcal{S}_{\alpha\beta} = 0$, but helicity $\pm 1$. The analogous is true for the gluon field. As a consequence, in exterior calculus, its canonical (i.e. Noether) energy-momentum 3-form is symmetric and gauge invariant directly, see [19], footnote 53. Conventionally, see [14], the coordinate dependent components $A_i$ of $A$ are used in the Lagrangian formalism, see also the clarifying considerations of Benn et al. [21].

**Thesis 1:** The energy-momentum current 3-form of the free gluon field $F = DA$ is given by the Minkowski type expression [22]

\begin{align}
\mathcal{T}_\alpha &= \frac{1}{2} [F \wedge (e_\alpha^* F) - *F \wedge (e_\alpha^* F)] \quad \text{or} \quad \mathcal{T}_i^j = \frac{1}{4} \epsilon^{ijkl} F_{kl} F_{ij} - F_{ik} F_{jk}. \quad (13)
\end{align}
The (Lorentz) spin current of the gluon field vanishes, $\mathcal{S}_{\alpha\beta\gamma} = 0$, the gluon orbital angular momentum current is given by $x_i^j \tilde{\mathfrak{s}}_{\alpha j i}^\gamma$ and represents the total angular momentum. As a gauge potential, the gluon is described by a 1-form and has helicity $\pm 1$.

The second example, Dirac field in exterior calculus for illustration. Its Lagrangian reads, 

$$L_D = \frac{i}{2} (\bar{\Psi} \gamma^a D_a \Psi + \bar{\Psi} D \Psi \gamma^5 + \bar{\Psi} \gamma^a \gamma^5 \Psi),$$

with $\gamma := \gamma_\alpha \gamma^\alpha$ and $\gamma_{(\alpha\beta)} = \epsilon_{\alpha\beta\gamma} \gamma^\gamma$. The 3-forms of the canonical momentum and spin current densities are $(D_\alpha := e_\alpha \gamma D$, here $\gamma$ denotes the interior product sign): 

$$\xi_\alpha := \frac{i}{2} (\bar{\Psi} \gamma^a \gamma^5 D_a \Psi + \bar{\Psi} D \Psi \gamma^5 \gamma^a), \quad \mathcal{S}_{\alpha\beta} := \frac{1}{4} \epsilon_{\alpha\beta\gamma} \bar{\Psi} \gamma^5 \gamma^\gamma \Psi.$$ 

In Ricci calculus $\mathcal{S}_{\alpha\beta\gamma} = \mathcal{S}_{[\alpha\beta\gamma]} = \frac{1}{4} \epsilon_{\alpha\beta\gamma\delta} \bar{\Psi} \gamma^5 \gamma^\delta \Psi$. Because of the equivalence principle, the inertial currents $\xi_\alpha$ and $\mathcal{S}_{\alpha\beta}$ are, at the same time, the gravitational currents of the classical Dirac field. A decomposition of $(\xi_\alpha, \mathcal{S}_{\alpha\beta})$ à la Gordon, yields the gravitational moment densities of the Dirac field [6]; it is a special case of relocalization, see below.

**Thesis 2:** The canonical (Noether) energy-momentum and the canonical (Noether) spin current 3-forms of a Dirac/quark field are given by the expressions in Eq. (15).

## 6 Relocalization of energy-momentum and spin distribution

We redefine the canonical currents $\xi_\alpha^i$ and $\mathcal{S}_{ij}^k$ by adding curls, see [6, 23],

$$\tilde{\xi}_i^j := \xi_i^j + \partial_k Y^j_{ik}, \quad \tilde{\mathcal{S}}_{ij}^k := \mathcal{S}_{ij}^k + Y_{[ij]}^k + \partial_k Z^k_{ij},$$

with the arbitrary antisymmetric super-potentials $Y^j_{ik} = -Y^k_{ij}$ and $Z^k_{ij} = -Z^k_{ji} = -Z^k_{ij}$. We substitute (16), and the partial derivative of (16)2 into (10) and (11), Then we recognize that these relocalized currents fulfill the original conservation laws:

$$\partial_j \tilde{\xi}_i^j = 0, \quad \partial_k \tilde{\mathcal{S}}_{ij}^k - \tilde{\xi}_{[ij]} = 0.$$

The integrated total energy-momentum and the total angular momentum of an insular material system are invariant under relocalization [23]. However, “relocalization invariance” under the transformations specified in (16) is not a generally valid physical principle. It should rather be understood as a formal trick to compute the total energy-momentum and angular momentum in a most convenient way.

It is convenient to introduce a new superpotential $U$ that is equivalent to $Y$ by

$$U^k_{ij} := -Y^k_{ij} = -U^k_{ji} \implies Y^j_{ik} = -U^j_{ik} + U^j_{ki} - U^j_{k}\cdot.$$

The Belinfante relocalization (1939) is a special case: Belinfante [24] effectively required $\tilde{\mathcal{S}}_{kl}^j = 0$. Then, by (16)2 and (18), $\tilde{\mathcal{S}}_{ij}^k = U^k_{ij} - \partial_l Z_{ij}^k$ and the relocalized energy-momentum, $\tilde{\xi}_i^j := \tilde{\xi}_i^j$, with $\tilde{\mathcal{S}}_{ij}^k = 0$, reads

$$\tilde{\xi}_i^j = \xi_i^j - \partial_k \left( \mathcal{S}_{ij}^k - \mathcal{S}_{ij}^k + \mathcal{S}_{ij}^k \right) \quad \text{with} \quad \partial_j \tilde{\xi}_i^j = 0, \quad \tilde{\mathcal{S}}_{kl}^j = 0.$$

For the Dirac field, because of the total antisymmetry of $\mathcal{S}_{ijk}$, we find simply $\tilde{\xi}_{ij} = \tilde{\mathcal{S}}_{(ij)}$, see [25]. Incidentally, the Gordon relocalization, mentioned above, differs from the Belinfante relocalization.
Dynamic Hilbert energy-momentum in general relativity

How can we choose amongst the multitude of the relocalized energy-momentum tensors and spin tensors? After all, as physicists we are convinced that the energy and the spin distribution of matter (but not of gravity!) are observable quantities, at least in the classical domain. There must exist physically correct and unique energy-momentum and spin tensors in nature. The Belinfante recipe was to kill $\Sigma_{[kl]}$ in order to tailor the energy-momentum for the application in Einstein’s field equation.

Already in 1915, Hilbert defined the dynamic energy-momentum as the response of the matter Lagrangian to the variation of the metric [26]:

$$H_i t_{ij} := \frac{2}{\delta} \frac{\delta \mathcal{L}_{\text{mat}}(g, \Psi, \nabla \Psi)}{\delta g_{ij}};$$  (20)

$g^{ij}$ (or its reciprocal $g_{kl}$) is the gravitational potential in GR. The matter Lagrangian is supposed to be minimally coupled to $g^{ij}$, in accordance with the equivalence principle. Only in gravitational theory, in which spacetime can be deformed, we find a real local definition of the material energy-momentum tensor. The Hilbert definition is analogous to the relation from elasticity theory “stress $\sim \delta$(elastic energy)/$\delta$(strain)”. Recall that strain is defined as $\varepsilon_{ab} := \frac{1}{2} (\text{deformed} g^{ab} - \text{unfluctuated} g^{ab})$, see [27]. Even the factor 2 is reflected in the Hilbert formula.

Rosenfeld (1940) has shown [28], via Noether type theorems, that the Belinfante tensor $\text{Bel}_{t_{ij}}$, derived within SR, coincides with the Hilbert tensor $H_{ij} t_{ij}$ of GR. Thus, the Belinfante-Rosenfeld recipe yields...

**Thesis 3:** In the framework of GR, the Hilbert energy-momentum tensor

$$H_i t_{ij}^j = \text{Bel}_{t_{ij}} = \Sigma^j_i - \nabla_k \left( \mathcal{G}^{ij}_k - \mathcal{G}^{jk}_i + \mathcal{G}^{kj}_i \right) = H_{ij} t_{ij},$$  (21)

localizes the energy-momentum distribution correctly; here $(\Sigma^j_i, \mathcal{G}^{ij}_k)$ are the canonical Noether currents. The spin tensor attached to $H_{ij} t_{ij}$ vanishes.

The Rosenfeld formula (21) identifies the Belinfante with the Hilbert tensor. In other words, the Belinfante tensor provides the correct source for Einstein’s field equation. As long as we accept GR as the correct theory of gravity, the localization of energy-momentum and spin of matter is solved. This state of mind is conventionally kept till today by most theoretical physicists. In passing, one should note that the spin of matter has a rather auxiliary function in this approach. After all, the spin of the Hilbert-Belinfante-Rosenfeld tensor simply vanishes.

However, the Poincaré gauge theory of gravity (PG; Sciama, Kibble 1961, see [9] for a review), in particular the viable Einstein-Cartan theory (EC) with the curvature scalar as gravitational Lagrangian, has turned the Rosenfeld formula (21) upside down.

Dynamic Sciama-Kibble spin in Poincaré gauge theory

The gauging of the Poincaré group identifies as gauge potentials the orthonormal coframe $\vartheta^\alpha = e_i^\alpha dx_i$ and the Lorentz connection $\Gamma^\alpha{}_{\beta\gamma} = \Gamma_i^\alpha{}_{\beta\gamma} dx_i = -\Gamma^\beta{}_{\alpha}$. The spacetime arena of the emerging Poincaré gauge theory of gravity (PG) is a Riemann-Cartan space with Cartan’s torsion and with Riemann-Cartan curvature as gauge field strength, respectively [10]:

$$C_{ij}^\alpha := \nabla_i e_j^\alpha, \quad R_{ij}^{\alpha\beta} := \nabla^\gamma \Gamma_{ij}^{\alpha\beta} \quad (or \ C^\alpha = D\vartheta^\alpha, \ R^{\alpha\beta} = "D\Gamma^\alpha{}_{\beta}).$$  (22)
The energy-momentum and angular momentum laws generalize to
\[^*\nabla_k \mathcal{T}^k_i = C_{ik\ell} \mathcal{T}_\ell^k + R_{iklm} \mathcal{G}_{lm}^k, \quad \nabla_k \mathcal{G}_{ij}^k - \mathcal{T}_{[ij]} = 0; \quad (23)\]

here \[^*\nabla_k := \nabla_k + C_{k\ell}\ell\]. GR is the subcase for \(\mathcal{G}_{ij}^k = 0\), see also the reviews \([29, 30]\). The material currents are defined by variations with respect to the potentials:
\[\mathcal{T}_{\alpha}^i = \delta \mathcal{L}_{\text{mat}}(e, \Gamma, \Psi, D \Psi) / \delta e^\alpha_i, \quad \mathcal{G}_{\alpha\beta}^i = \delta \mathcal{L}_{\text{mat}}(e, \Gamma, \Psi, D \Psi) / \delta \Gamma^\alpha_i \beta. \quad (24)\]

This Sciama-Kibble definition of the spin (1961) is only possible in the Riemann-Cartan space-time of PG. It is analogous to the relation “moment stress \(\sim \delta(\text{elastic energy}) / \delta(\text{contortion})\)” in a Cosserat type medium, the contortion being a “rotational strain”, see \([31]\).

The application of the Lagrange-Noether machinery to the minimally coupled action function yields, after a lot of algebra, the final result, see \([19]\):
\[\mathcal{T}_{\alpha}^i = \mathcal{T}_{\alpha}^i = \mathcal{H}_i^\alpha, \quad \mathcal{G}_{\alpha\beta}^i = \mathcal{G}_{\alpha\beta}^i. \quad (25)\]

The dynamically defined energy-momentum and spin currents à la Sciama-Kibble coincide with the canonical Noether currents of classical field theory.

**Thesis 4:** Within PG, the quark energy-momentum and the quark spin are distributed in accordance with the canonical Noether currents \(\mathcal{T}_{\alpha}^i\) and \(\mathcal{G}_{\alpha\beta}^i\), respectively.

This is in marked contrast to the doctrine in the context of GR.

We express the canonical energy-momentum tensor in terms of the Hilbert one (see \([32]\)):
\[\mathcal{T}_{\alpha}^i = \mathcal{T}_{\alpha}^i = \mathcal{H}_i^\alpha + \nabla_k (\mathcal{G}_{\alpha\beta}^i - \mathcal{G}_{ik}^\alpha + \mathcal{G}_{k\alpha}^i), \quad \mathcal{G}_{\alpha\beta}^i = \mathcal{G}_{\alpha\beta}^i. \quad (26)\]

The new Rosenfeld formula \((26)_1\) reverses its original meaning in \((21)\). Within PG, the canonical tensor \(\mathcal{T}_{\alpha}^i\) represents the correct energy-momentum distribution of matter and the (sym)metric Hilbert tensor now plays an auxiliary role. In GR, it is the other way round. Moreover, we are now provided with a dynamic definition of the canonical spin tensor. In GR, the spin was only a *kinematic* quantity floating around freely.

These results on the correct distribution of material energy-momentum and spin in the framework of PG are are independent of a specific choice of the gravitational Lagrangian. However, if we choose the RC curvature scalar as a gravitational Lagrangian, we arrive at the Einstein-Cartan(-Sciama-Kibble) theory of gravitation, which is a viable theory of gravity competing with GR.

### 9 An algebra of the momentum and the spin currents?

We discussed exclusively classical field theory. Can we learn something for a corresponding quantization of gravity? Our classical analysis has led us to the gravitational currents \(\mathcal{T}_{\alpha}\) and \(\mathcal{G}_{\alpha\beta}\). They represent the sources of gravity.

In strong and in electroweak interaction, before the standard model had been worked out, one started with the current algebra of the phenomenologically known strong and the electroweak currents (see Sakurai \([33]\), Fritzsch et al. \([34]\), and also Cao \([35]\)).
Schwinger (1963) studied, for example, the equal time commutators of the components of the Hilbert energy-momentum tensor [36]. Should one try to include also the spin tensor components and turn to the canonical tensors?

In the Sugawara model (1968), “A field theory of currents” was proposed [37] with 8 vector and 8 axial vector currents for strong interaction and a symmetric energy-momentum current for gravity that was expressed bilinearly in terms of the axial and the vector currents. Now, when we have good arguments that the gravitational currents are $T_\alpha$ and $S_{\alpha\beta}$, one may want to develop a corresponding current algebra by determining the equal time commutator of these currents....

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