A CONSTRUCTIVE APPROACH TO ONE-DIMENSIONAL GORENSTEIN k-ALGEBRAS

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Abstract. Let $R$ be the power series ring or the polynomial ring over a field $k$ and let $I$ be an ideal of $R$. Macaulay proved that the Artinian Gorenstein $k$-algebras $R/I$ are in one-to-one correspondence with the cyclic $R$-submodules of the divided power series ring $\Gamma$. The result is effective in the sense that any polynomial of degree $s$ produces an Artinian Gorenstein $k$-algebra of socle degree $s$. In a recent paper, the authors extended Macaulay’s correspondence characterizing the $R$-submodules of $\Gamma$ in one-to-one correspondence with Gorenstein $d$-dimensional $k$-algebras. However, these submodules in positive dimension are not finitely generated. Our goal is to give constructive and finite procedures for the construction of Gorenstein $k$-algebras of dimension one and any codimension. This has been achieved through a deep analysis of the $G$-admissible submodules of $\Gamma$. Applications to the Gorenstein linkage of zero-dimensional schemes and to Gorenstein affine semigroup rings are discussed.

1. Introduction

Gorenstein rings were introduced by A. Grothendieck and they are a generalization of complete intersections, indeed the two notions coincide in codimension two by a well known result by Serre. Codimension three Gorenstein rings are completely described by Buchsbaum and Eisenbud’s structure theorem, [2], but despite many attempts the general construction of Gorenstein rings remains an open problem in higher codimension. A. Kustin and M. Miller, in a series of papers, studied the structure of Gorenstein ideals of codimension 4, see [24]. More recently, M. Reid studied their projective resolution aiming to extend the result of D. Buchsbaum and D. Eisenbud, see [30]. Gorenstein rings are of great interest in many areas of mathematics and they have appeared as an important component in a significant number of problems with applications to commutative algebra, singularity theory, number theory and more recently to combinatorics, among other areas. The lack of a general structure of Gorenstein rings is the main obstacle in several problems, among them the Gorenstein linkage. For a complete and interesting presentation of Gorenstein rings, see H. Bass’s and C. Huneke’s papers, [1] and [19].

Let $R$ denote the power series ring $k[z_1, \ldots, z_n]$ or the polynomial ring $k[z_1, \ldots, z_n]$ over a field $k$ and let $I \subset R$ be an ideal (homogeneous if $R$ is a polynomial ring). We denote by $\mathcal{M}$ the maximal ideal of $R$ generated by $z_1, \ldots, z_n$.

As an effective consequence of Matlis duality, it is known that an Artinian ring $R/I$ is a Gorenstein $k$-algebra if and only if $I$ is the set of solutions of a system of polynomial

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differential operators with constant coefficients. Macaulay, at the beginning of the 20th century, proved that the Artinian Gorenstein $k$-algebras are in correspondence with the cyclic $R$-submodules of the divided powers ring $\Gamma = k_{DP}[Z_1, \ldots, Z_n]$ where the elements of $R$ act as derivatives (or contraction) on $\Gamma$, see [13], [21].

Thanks to the effective construction of the Artinian Gorenstein algebras, in the last twenty years several authors have applied this device to several problems, among others: Waring’s problem [15], the n-factorial conjecture in combinatorics and geometry [17], the cactus rank [29], the geometry of the punctual Hilbert scheme of Gorenstein schemes [21], the classification up to analytic isomorphism of Artinian Gorenstein rings [12], and the Koszulness of $k$-algebras [11].

Recently in [10] the authors extended Macaulay’s correspondence characterizing the submodules of $\Gamma$, called $G$-admissible submodules, in one-to-one correspondence with Gorenstein d-dimensional $k$-algebras (Theorem 2.3). These submodules in positive dimension are not finitely generated. Clearly this is an obstacle to the effective construction of Gorenstein algebras of positive dimension.

In this paper we give a constructive and finite procedure for producing Gorenstein one-dimensional $k$-algebras of any codimension. Thanks to a deep analysis of the structure of the $G$-admissible $R$-submodules of $\Gamma$, it is possible to write an algorithmic procedure for constructing step-by-step a finite subset of $\Gamma$, called a $G$-admissible set, as a good candidate for being extended to a $G$-admissible submodule. In the graded case, similarly to the Artinian case, a suitable DP-polynomial $H \in \Gamma$ uniquely determines a one-dimensional Gorenstein ring and hence a $G$-admissible $R$-submodule, see Proposition 2.11, Remark 3.2. This is no longer true in the local case where more sophisticated techniques will be necessary. By using the theory of the standard bases (or Hironaka bases), Theorem 3.3 gives necessary and sufficient effective conditions to build up the Gorenstein one-dimensional $k$-algebras from a finite subset of $\Gamma$. In particular, starting from a suitable finite $G$-admissible set $H$, we can construct all the ideals $I$ of $R$ such that $R/I$ is a Gorenstein one-dimensional $k$-algebra and the dual contains $H$. Example 3.7 shows that two Gorenstein rings sharing the same $H$ in general are not analytically isomorphic, even if they share the same associated graded ring. In the last two sections we apply the results to two classes of examples: the construction of the inverse system of $d$ distinct Gorenstein points of $\mathbb{P}^3_k$ and the inverse system of a class of Gorenstein semigroup rings in $A^3_k$. The computations are performed by using the computer algebra system Singular [5] and the Singular library INVERSE-SYST.lib [9].

2. THE STRUCTURE OF THE INVERSE SYSTEM

Let $V$ be a vector space of dimension $n$ over a field $k$ where, unless specified stated otherwise, $k$ is an infinite field of any characteristic. If $V$ denotes the $k$-vector space $\langle z_1, \ldots, z_n \rangle$, then we denote by $V^* = \langle Z_1, \ldots, Z_n \rangle$ the dual basis. Let $P = \text{Sym}^kV = \bigoplus_{i \geq 0} \text{Sym}^i_k V$ be the standard graded polynomial ring in $n$ variables over $k$ and $\Gamma = D_k(V^*) = \bigoplus_{i \geq 0} D^k_i(V^*) = \bigoplus_{i \geq 0} \text{Hom}_k(P_i, k)$ be the graded $P$-module of graded $k$-linear homomorphisms from $P$ to $k$, hence $\Gamma \simeq k_{DP}[Z_1, \ldots, Z_n]$ the divided power ring. In particular $\Gamma_j = \langle \{ Z^L | |L| = j \} \rangle$ is the span of the dual generators to $z^L = z_1^{l_1} \cdots z_n^{l_n}$, where $L$ denotes the multi-index $L = (l_1, \ldots, l_n) \in \mathbb{N}^n$ of length $|L| = \sum_i l_i$. If $L \in \mathbb{Z}^n$ then we set $Z^L = 0$ if any component of $L$ is negative. The monomials $Z^L$ are called
divided power polynomials (DP-polynomials) and the elements $F = \sum_L b_L Z^{[L]}$ of $\Gamma$ the divided power polynomials (DP-polynomials).

We recall that $\Gamma$ is an $R$-module with $R$ acting on $\Gamma$ by contraction as follows. This action is also called apolarity.

**Definition 2.1.** If $h = \sum_M a_M z^M \in R$ and $F = \sum_L b_L Z^{[L]} \in \Gamma$, then the contraction of $F$ by $h$ is defined as

$$h \circ F = \sum_{M,L} a_M b_L Z^{[L-M]}$$

For short, from now on we write $Z^k$ instead of $Z^{[L]}$.

Recall that the injective hull $E_R(k)$ of $k$ as an $R$-module is isomorphic as an $R$-module to the divided power ring $\Gamma$ (see [14], [28]). For detailed information see [6], [13], [21], Appendix A. If the characteristic of the field $k$ is zero, then there is a natural isomorphism of $R$-algebras between $(\Gamma, \circ)$, where $\circ$ is the contraction already defined, and the usual polynomial ring $P$ replacing contraction with taking partial derivatives. In this paper we always consider $\Gamma$ as an $R$-module by contraction.

If $I \subset R$ is an ideal of $R$ then $(R/I)^\vee = \text{Hom}_R(R/I, E_R(k))$ is the $R$-submodule of $\Gamma$

$$I^\perp = \{g \in \Gamma \mid I \circ g = 0\}.$$  

This submodule of $\Gamma$ is called *Macaulay’s inverse system of $I$*. Given an $R$-submodule $W$ of $\Gamma$, then the dual $W^\vee = \text{Hom}_R(W, E_R(k))$ is the ring $R/\text{Ann}_R(W)$ where

$$\text{Ann}_R(W) = \{f \in R \mid f \circ g = 0 \text{ for all } g \in W\}$$

is an ideal of $R$. If $I$ is a homogeneous ideal of $R$ (resp. $W$ is generated by homogeneous polynomials) then $I^\perp$ is generated by homogeneous polynomials of $\Gamma$ (resp. $\text{Ann}_R(W)$ is an homogeneous ideal of $R$) and $I^\perp = \oplus I_j^\perp$ where $I_j^\perp = \{F \in \Gamma_j \mid h \circ F = 0 \text{ for all } h \in I_j\}$. Notice that the Hilbert function of $R/I$ can be computed by $I^\perp$, see for instance [10] Section 2. Macaulay in [25] IV proved a particular case of Matlis duality, called Macaulay’s correspondence, between the ideals $I \subseteq R$ such that $R/I$ is an Artinian local ring and $R$-submodules $W = I^\perp$ of $\Gamma$ which are finitely generated. Macaulay’s correspondence is an effective method for computing Gorenstein Artinian rings, see [3], Section 1, [20], [15] and [21]. We summarize with the statement *Artinian Gorenstein $k$-algebras $A = R/I$ of socle degree $s$ correspond to cyclic $R$-submodules of $\Gamma$ generated by a DP-polynomial $F \neq 0$ of degree $s$*.

The authors extended Macaulay’s correspondence characterizing the $d$-dimensional local Gorenstein $k$-algebras in terms of suitable submodules of $\Gamma$, see [10]. This result in the one-dimensional case will be our starting point and for completeness we include here the statement.

**Definition 2.2.** An $R$-submodule $M$ of $\Gamma$ is called $G$-admissible if it admits a countable system of generators $\{H_l\}_{l \in \mathbb{N}_+}$ satisfying the following conditions

1. There exists a linear form $z \in R$ such that for all $l \in \mathbb{N}_+$

   $$z \circ H_l = \begin{cases} H_{l-1} & \text{if } l > 1 \\ 0 & \text{otherwise.} \end{cases}$$

2. $\text{Ann}_R(\langle H_l \rangle) \circ H_{l+1} = \langle H_1 \rangle$ for all $l \in \mathbb{N}_+$. 

If this is the case, we say that \( M = \langle H_l, l \in \mathbb{N}_+ \rangle \) is a \( G \)-admissible \( R \)-submodule of \( \Gamma \) with respect to the linear form \( z \in R \).

Notice that if \( R/I \) has positive depth, then there always exists a linear form \( z \in R \) which is regular modulo \( I \) because \( k \) is infinite.

**Theorem 2.3.** There is a one-to-one correspondence \( C \) between the following sets:

(i) one-dimensional Gorenstein \( k \)-algebras \( A = R/I \),

(ii) non-zero \( G \)-admissible \( R \)-submodules \( M = \langle H_l, l \in \mathbb{N}_+ \rangle \) of \( \Gamma \).

In particular, given an ideal \( I \subset R \) with \( A = R/I \) satisfying (i) and \( z \) a linear regular element modulo \( I \), then

\[
C(A) = I^\perp = \langle H_l, l \in \mathbb{N}_+ \rangle \subset S \quad \text{with} \quad \langle H_l \rangle = (I + (z^l))^\perp
\]
is \( G \)-admissible. Conversely, given an \( R \)-submodule \( M \) of \( \Gamma \) satisfying (ii), then

\[
C^{-1}(M) = R/I \quad \text{with} \quad I = \operatorname{Ann}_R(M) = \bigcap_{l \in \mathbb{N}_+} \operatorname{Ann}_R(\langle H_l \rangle).
\]

The main goal of this paper is to construct \( G \)-admissible \( R \)-submodules of \( \Gamma \).

Notice that if we fix any DP-polynomial \( H_1 \in \Gamma \) such that \( z \circ H_1 = 0 \), then \( M = \langle Z^l H_1, l \in \mathbb{N} \rangle \) is \( G \)-admissible with respect to \( z \), but the corresponding one-dimensional Gorenstein ring is not of great interest because it is a cone over the Gorenstein Artinian ring corresponding to the DP-polynomial \( H_1 \), see [10], Proposition 4.1. In the next remark we observe that the choice of \( H_1 \) is very important in the construction of a \( G \)-admissible module, because it encodes much information on the corresponding Gorenstein ring.

**Remark 2.4.** Let \( A = R/I \) be a 1-dimensional Gorenstein ring and let \( z \) be a linear regular element modulo \( I \). Let \( I^\perp = \langle H_l, l \in \mathbb{N}_+ \rangle \) be the corresponding \( G \)-admissible dual module with respect to \( z \). We recall that by Theorem 2.3, we have \( \langle H_1 \rangle = (I + (z))^\perp \). Hence by Macaulay’s correspondence, \( \deg(H_1) \) coincides with the socle degree of the Artinian reduction \( A/zA \). On the other hand we have the following inequality on the multiplicity \( e(A) \) of \( A \):

\[
e(A) \leq \operatorname{Length}_R(A/zA) = \dim_k(\langle H_1 \rangle).
\]

In particular the equality holds if \( z \) is a superficial element of \( A \). If \( A \) is a standard graded \( k \)-algebra which is Gorenstein, then the above equality holds for every regular element \( z \) modulo \( I \). Hence \( e(A) = \dim_k(\langle H_1 \rangle) \) (the dimension as a \( k \)-vector space of the \( R \)-module generated by \( H_1 \)) and \( \text{reg}(A) = \deg H_1 \) where \( \text{reg}(A) \) is the Castelnuovo-Mumford regularity of \( A \).

As a consequence, in the graded case, important geometric information such as the multiplicity, the arithmetic genus, or more generally, the Hilbert polynomial of the Gorenstein \( k \)-algebra can be controlled by the choice of \( H_1 \), the first step in the construction of a \( G \)-admissible dual module.

Recall that if \( I \) is an homogeneous ideal, then the dual \( R \)-submodule \( I^\perp \) of \( \Gamma \) can be generated by homogeneous (in the usual meaning) DP-polynomials and conversely if the \( R \)-submodule of \( \Gamma \) is homogeneous, then the ideal \( I \) is homogeneous. Hence the correspondence will be between one-dimensional Gorenstein standard graded \( k \)-algebras and \( G \)-admissible homogeneous \( R \)-submodules of \( \Gamma \). The following result refines Theorem 2.3 in the case of graded \( k \)-algebras.
Theorem 2.5. There is a one-to-one correspondence \( \mathcal{C} \) between the following sets:

(i) one-dimensional Gorenstein standard graded \( k \)-algebras \( A = R/I \) of multiplicity \( e = e(A) \) (resp. Castelnuovo-Mumford regularity \( r = \text{reg}(A) \))

(ii) non-zero \( G \)-admissible homogeneous \( R \)-submodules \( M = \langle H_l, l \in \mathbb{N}_+ \rangle \) of \( \Gamma \) such that \( \dim_k \langle H_1 \rangle = e \) (resp. \( \deg H_1 = r \))

In the construction of \( G \)-admissible \( R \)-modules, the following remark suggests to us to proceed step by step starting from \( H_1 \).

Remark 2.6. Given two DP-polynomials \( H, G \in \Gamma \), we say that \( G \) is a primitive of \( H \) with respect to \( z \in R \) if \( z \circ G = H \). From the definition of the contraction \( \circ \), we will get

\[
G = ZH + C
\]

for some \( C \in \Gamma \) such that \( z \circ C = 0 \). We remark that \( ZH \) denotes the usual multiplication in a polynomial ring and we do not use the internal multiplication in \( \Gamma \) as DP-polynomials. Hence, according to Definition 2.2, part (1), given a \( G \)-admissible module generated by \( \{ H_l \}_{l \in \mathbb{N}_+} \), we have that \( H_{l+1} \) is a primitive of \( H_l \) with respect to \( z \) for every positive integer \( l \). In particular \( H_2 \) is a primitive of \( H_1 \) and so on. Hence there exist \( C_1, \ldots, C_l \) in \( \Gamma \) (depending on condition (2) of Definition 2.2) such that \( z \circ C_i = 0 \) and for all positive \( l \)

\[
H_{l+1} = z^l H_1 + z^{l-1} C_1 + \cdots + z C_{l-1} + C_l
\]

In the effective construction of a Gorenstein ring \( R/I \), note that we may assume \( \Gamma = k_{DP}[Z_1, \ldots, Z_n] \) and take \( z = z_1 \). This presentation has the advantage that we may assume \( H_1, C_1, \ldots, C_l \) in \( k_{DP}[Z_2, \ldots, Z_n] \).

According to Remark 2.6, starting from a DP-polynomial \( H_1 \), in each step we have to choose the “constants” \( C_i \) imposing condition (2) of Definition 2.2. We have implemented a Singular routine to determine the possible DP-polynomials \( H_{l+1} \) from \( H_l \), if they exist.

Guided by the construction of one-dimensional Gorenstein \( k \)-algebras \( R/I \) in a finite and effective number of steps, we state the following questions:

Question 2.7. When can a finite set \( \{ H_1, \ldots, H_{l+1} \} \) of DP-polynomials of \( \Gamma \) verifying conditions (1) and (2) of Definition 2.2 be completed to a system of generators of a \( G \)-admissible \( R \)-module \( M = \langle H_l, l \in \mathbb{N}_+ \rangle \)?

Question 2.8. Does there exist an integer \( t \) such that the finite subset \( \{ H_1, \ldots, H_{l+1} \} \) of a \( G \)-admissible \( R \)-module \( M = \langle H_l, l \in \mathbb{N}_+ \rangle \) determines (uniquely?) the Gorenstein ideal \( I = \text{Ann}_R(M) \)?

According to the definition of a \( G \)-admissible \( R \)-submodule of \( \Gamma \), see Definition 2.2, we say:

Definition 2.9. A finite set of DP-polynomials \( \mathcal{H} = \{ H_l \mid 0 < l \leq t_0 \} \) is a \( G \)-admissible set with respect to a linear element \( z \in R \) if it satisfies conditions (1) and (2) of Definition 2.2 with respect to \( z \in R \).

Note that an algorithm in Singular has been implemented for computing \( G \)-admissible sets with respect to \( z \in R \) starting from a DP-polynomial \( H_1 \) such that \( z \circ H_1 = 0 \).
Remark 2.10. Notice that in $\mathcal{H} = \{H_l; 0 < l \leq t_0\}$ the last polynomial $H_{t_0}$ determines the full sequence $H_1, \cdots, H_{t_0-1}$ by contraction with respect to $z$. Thus $\mathcal{H}$ is identified by the DP-polynomial $H_{t_0}$. But $H_{t_0}$ is not an arbitrary polynomial simply of the shape described in Remark 2.6. In fact, in Definition 2.2, condition (1) does not imply condition (2) as the following example shows. Let us consider the finite set: $\mathcal{H}$ described in Remark 2.6. In fact, in Definition 2.2, condition (1) does not imply condition (2) with respect to $y$. and it satisfies condition (2) except for the last polynomial since $\text{Ann}_R(\langle H_4 \rangle) \cap H_5 \nsubseteq \langle H_1 \rangle$. However, if we replace $H_5$ with $H_5 + X^6$ we get a G-admissible set with respect to $y$.

From now on if $J$ is an ideal of $R$ (not necessarily homogeneous), we denote by $J_{\leq s}R$ the ideal of $R$ generated by all the elements of $J$ in $\mathcal{M}^s \setminus \mathcal{M}^{s+1}$ if any, otherwise (0). In the homogeneous case, it means generated by forms of $J$ of degree at most $s$. The next result gives a first positive answer to Question 2.8 in the homogeneous case.

Proposition 2.11. Let $t_0$ be a positive integer and let $\mathcal{H} = \{H_l; 0 < l \leq t_0\}$ be a homogeneous $G$-admissible set with respect to $z \in R$. Let $\deg H_1 = r$ and assume $t_0 \geq r + 2$. If $\mathcal{H}$ can be extended to a $G$-admissible $R$-module, then the corresponding graded Gorenstein $k$-algebra $A = R/I$ is uniquely determined and

$$I = \text{Ann}_R(H_{t_0})_{\leq r+1}R.$$ 

The previous result was proved in [10], Proposition 4.2. It depends on the fact that the maximum degree of a minimal system of generators of $I$ is at most $\text{reg}(A) + 1$ and it coincides with the socle degree $\text{socdeg}(A/zA) + 1 = r + 1$. Clearly it is also true that

$$I = \text{Ann}_R(H_{t_0})_{\leq r+1}R.$$ 

The previous result can be improved if we know the maximum degree $t$ of the generators of $I$ in which case one can replace $r = \deg H_1$ by $t - 1(\leq r)$, hence

$$I = \text{Ann}_R(H_{t+1})_{\leq t}R.$$ 

Remark 2.12. As in the Artinian case, the previous result tells us that there is a one-to-one correspondence between Gorenstein graded $k$-algebras of dimension one and socle degree $r$ and suitable homogeneous DP-polynomials $H_{r+2}$ of degree $2r + 1$.

We present here an example.

Example 2.13. We consider the following $G$-admissible set of DP-polynomials in $\Gamma = k_{PD}[X,Y]$ with respect to $y$ of $R = k[x,y]$:

$$\mathcal{H} = \{H_1 = X^2, H_2 = X^2Y, H_3 = X^4 + X^2Y^2, H_4 = X^4Y + X^2Y^3\}$$

It is a subset of the $G$-admissible $R$-submodule $M$ of $\Gamma$ generated by

$$\{H_t = \sum_{i=0}^{2i \leq t-1} X^{2+2i}Y^{t-1-2i} \mid t \geq 1\}$$

In this case $r = \deg H_1 = 2$, hence according to Proposition 2.11

$$I = \text{Ann}_R(H_4)_{\leq 3}R.$$
In fact, \( I = \text{Ann}_R(H_4) \leq 3R = (x^3 - xy^2) \) is a Gorenstein ideal, \( y \) is a non-zero divisor in \( A = R/I \) and \( I^\perp \) is \( M \). We remark that in particular \( \mathcal{H} \) (actually \( H_4 \)) determines the Gorenstein ideal \( I \) and hence the \( G \)-admissible \( R \)-module \( I^\perp \).

We end this section with a very partial answer (for the moment) to Question 2.7 in the homogeneous case. We will need the following proposition.

**Proposition 2.14.** Let \( H = \{ H_l; 0 < l \leq t_0 \} \) be a finite \( G \)-admissible set with respect to \( z \in R \). Then

\[
\text{Ann}_R(H_{t+1}) + (z^i) = \text{Ann}_R(H_t)
\]

for all \( t = 1, \cdots, t_0 - 1 \). In particular \( \text{Ann}_R(H_{t+1}) + (z) = \text{Ann}_R(H_1) \).

**Proof.** Since \( z \circ H_{t+1} = H_t \) we have \( \langle H_l \rangle \subset \langle H_{t+1} \rangle \), and then \( \text{Ann}_R(H_{t+1}) \subset \text{Ann}_R(H_t) \).

From condition (1) we deduce that \( z^i \circ H_t \) is 0 so \( (z^i) \subset \text{Ann}_R(H_t) \).

Let now \( a \in \text{Ann}_R(H_t) \), from conditions (2) and (1) there exists \( b \in R \) such that

\[
a \circ H_{t+1} = b \circ H_1 = b \circ (z^i \circ H_{t+1}).
\]

Hence \( a - bz^i \in \text{Ann}_R(H_{t+1}) \), so \( a \in \text{Ann}_R(H_{t+1}) + (z^i) \). \( \square \)

**Corollary 2.15.** Assume \( \dim R = 2 \). Let \( t_0 \) be a positive integer and let \( \mathcal{H} = \{ H_l; 0 < l \leq t_0 \} \) be a \( G \)-admissible homogeneous set with respect to \( z \in R \). Assume \( t_0 \geq r + 2 \) where \( r = \deg H_1 \). Then \( \mathcal{H} \) can be extended to a \( G \)-admissible module and the corresponding graded Gorenstein \( k \)-algebra \( A = R/I \) is uniquely determined by

\[
I = \text{Ann}_R(H_{r+2})_{\leq r+1}R.
\]

**Proof.** Denote by \( (x, z) \) a minimal system of generators of the maximal ideal of \( R \). Because \( z \circ H_1 = 0 \) and \( r = \deg H_1 \), we may write \( H_1 = x^r \) and hence \( \text{Ann}(H_1) = (x^{r+1}, z) \). By Proposition 2.14 we have \( \text{Ann}(H_{t+1}) + (z) = \text{Ann}(H_1) \) for every \( t \geq 1 \), hence \( \text{Ann}(H_{t+1}) \subset (x^{r+1}, z) \). We also know that \( \text{Ann}(H_1) \) is generated by a regular sequence, say \( (F_t, G_t) \).

Hence we may assume \( F_t = x^{r+1} + zM_t \) for some form \( M_t \) of degree \( r \) and \( G_t = zN_t \) for some form \( N_t \in R \). We prove \( \deg G_t \geq t+1 \) for every \( t = 0, \cdots, t_0 \). We proceed inductively on \( t \). If \( t = 0 \), then \( M_0 = 0 \) and \( G_0 = z \). Assume \( \text{Ann}(H_{j+1}) = (x^{r+1} + zM_{j+1}, zN_{j+1}) \) with \( \deg N_j \geq j \) and prove that \( \text{Ann}(H_{j+2}) = (x^{r+1} + zM_{j+1}, zN_{j+1}) \) with \( \deg N_{j+1} \geq j + 1 \). Now \( zN_{j+1} \circ H_{j+2} = 0 \) implies \( N_{j+1} \circ H_{j+1} = 0 \). Hence \( N_{j+1} \in (x^{r+1} + zM_j, zN_j) \) and we conclude. Hence \( \text{Ann}_R(H_{r+2})_{\leq r+1}R = (x^{r+1} + zM_{r+1}) = I \). \( \square \)

Unfortunately, Proposition 2.11 and Proposition 2.15 cannot be extended to the non-homogeneous case, even in the algebraic case. In fact the information on the multiplicity is not enough to determine the singularity. The following example is a counterexample in the non-graded case to both the above results.

**Example 2.16.** For all \( n \geq 2 \) we consider the one-dimensional local ring \( A_n = k[[x, y]]/(f_n) \) with \( f_n = y^2 - x^n \). Notice that, for all \( n \geq 1 \), \( A_n \) is an algebraic Gorenstein ring of multiplicity \( e(A_n) = 2 \). On the other hand \( A_n/xA_n = k[y]/(y^2) \) is an Artinian reduction of \( A_n \), so \( H_1 = Y \) and hence \( \deg H_1 = 1 \). If \( n \geq 4 \), it is clear that we cannot obtain the ideal \( (f_n) \) after \( \deg H_1 + 2 = 3 \) steps as Proposition 2.11 and Corollary 2.15 suggest.
3. One dimensional Gorenstein local or graded rings

The aim of this section is to give constructive answers to Questions 2.8 and 2.7 in the
local and graded case. It will be useful to recall some well known facts concerning the
standard bases in the local case. Let

\[ P = gr_M(R) \]

is the associated graded ring of \( R \). For every element \( f \in R \setminus \{0\} \) we can
write \( f = f_1 + f_{v+1} + \cdots \), where \( f_v \) is not zero and \( f_j \) is a homogeneous polynomial of
degree \( j \) in \( P \) for every \( j \geq v \). We say that \( v(f) \) is the order of \( f \) (or \( M \)-valuation),
denote \( f_v \) by \( f^* \) and call it the initial form of \( f \). If \( f = 0 \) we agree that its order is \( \infty \). We
denote by \( I^* \) the ideal of \( P \) generated by all the initial forms of the elements in \( I \). A set of
generators (not necessarily minimal) of \( I \), say \( \{f_1, \ldots, f_r\} \), is called a standard basis
(or Hironaka’s basis) of \( I \) if their initial forms generate \( I^* \). Notice that the associated graded
ring of \( A = R/I \) with respect to the maximal ideal is isomorphic to \( P/I^* \). It is known that
the Krull dimension of \( A \) and of \( gr_M(R/I) = P/I^* \) coincide, as well, by definition, their
Hilbert function. Unfortunately, even if \( A \) is Gorenstein, then in general \( gr_M(R/I) \) is no
longer Gorenstein nor even Cohen-Macaulay, see for instance the examples in Section 5.
Notice that if \( I \) is a homogeneous ideal in \( P \), then any homogeneous system of generators
of \( I \) is a standard basis. A set of generators of \( I \) is called a minimal standard basis
of \( I \) if the initial forms minimally generate \( I^* \). Notice that the orders of the elements
in a minimal system of generators of \( I \) are not uniquely determined, instead the orders
of the elements of a minimal standard basis of \( I \) (and the number of generators) are
uniquely determined by the first graded Betti numbers of \( I^* \). If \( \{f_1, \ldots, f_r\} \) is a minimal
standard basis of \( I \), then \( v(f_i) = \deg(f_i^*) \) and \( r \) is the minimal number of generators of the
homogeneous ideal \( I^* \). We recall that an element \( z \in M \setminus M^2 \) is superficial for \( A \) if and
only if \( z^* \) is a homogeneous filter-regular element of degree one in \( G = gr_M(R/I) = P/I^* \)
or equivalently \( (0 :_G z^*)_j = 0 \) for all large degree \( j \). If depth \( A > 0 \), then a superficial
element is also a regular element modulo \( I \), see for instance [31], Section 1.2.

The next Theorem has a central role in this paper. In particular it extends and it
refines Proposition 2.11 in the local case.

**Theorem 3.1.** Let \( A = R/I \) be a one-dimensional Gorenstein ring and let \( I^+ = \langle H_i \mid i \in \mathbb{N}_+ \rangle \) be the correspondent \( G \)-admissible module with respect to \( z \) (a regular linear element modulo \( I \)). Given \( t \geq e = \dim_k(H_1) \), let \( \{h_1, \ldots, h_r\} \) be the elements of a minimal standard basis of \( \text{Ann}_R(H_{t+1}) \) such that \( v(h_i) \leq t \). Then

(i) \( (\text{Ann}_R(H_{t+1})^*)_{\leq t} = (I^*)_{\leq t} \).

(ii) There exist \( \alpha_1, \ldots, \alpha_r \in R \) such that \( \{h_1 + \alpha_1 z^{t+1}, \ldots, h_r + \alpha_r z^{t+1}\} \) is a minimal standard basis of \( I \).

(iii) \( \text{Ann}_R(H_{t+1}) = I^* + (z^{t+1}) = \langle h_1, \ldots, h_r \rangle + (z^{t+1}) \).

Assume that \( z \) is a superficial element of \( A \) and denote by \( J \) the ideal generated by
\( \{h_1, \ldots, h_r\} \) and \( B = R/J \).

(iv) \( B \) is a one-dimensional Gorenstein ring and \( z \) is a regular element modulo \( J \). In
particular \( J^* = I^* \) and hence the Hilbert functions of \( A \) and \( B \) agree.

(v) Assume that \( \{h_1, \ldots, h_s\} \) is a minimal system of generators of \( J \), then \( \{h_1 + \alpha_1 z^{t+1}, \ldots, h_s + \alpha_s z^{t+1}\} \) is a minimal system of generators of \( I \).
Proof. (i) Since $I + (z^{t+1}) = \text{Ann}_R(H_{t+1})$ we get that $I^* \subset \text{Ann}_R(H_{t+1})^*$. Conversely, let $F \in \text{Ann}_R(H_{t+1})^*\leq t$ be a homogeneous form. Then there are $f \in I$ and $\beta \in R$ such that $(f + \beta z^{t+1})^* = F$. Since the degree of $F$ is less than or equal to $t$, we deduce that $F = f^* \in I_{\leq t}^*$.

(ii) Let $g_1, \ldots, g_p$ be a minimal standard basis of $I$. Since $t \geq e = \dim_k(H_1) \geq e(R/I)$ and $I^*$ is minimally generated by forms of degree less than or equal to $e(R/I)$, \[8\], we have that $\deg g_i^* \leq t$, $i = 1, \ldots, p$.

Let $\{h_1, \ldots, h_r\}$ be the elements of a minimal standard basis of $\text{Ann}_R(H_{t+1})$ such that $\deg h_i^* \leq t$. From (i) we get $$(g_1^*, \ldots, g_p^*)\leq t = I_{\leq t}^* = \text{Ann}_R(H_{t+1})_{\leq t}^* = (h_1^*, \ldots, h_r^*)_{\leq t}$$ and then

$$(2) \quad I^* = (g_1^*, \ldots, g_p^*) = (h_1^*, \ldots, h_r^*).$$

Thus $h_1^*, \ldots, h_r^*$ is a minimal system of generators of $I^*$.

For all $h_i \in I + (z^{t+1})$ we can write, $i = 1, \ldots, r$, $h_i = \gamma_i - \alpha_i z^{t+1}$ with $\gamma_i \in I$ and $\alpha_i \in R$. From this identity we have $h_i + \alpha_i z^{t+1} = \gamma_i \in I$, so

$$h_i^* = (h_i + \alpha_i z^{t+1})^* \in I^*$$

for $i = 1, \ldots, r$. Hence $\{h_i + \alpha_i z^{t+1}\}_{i=1,\ldots,r}$ is a minimal standard basis of $I$.

(iii) It is a consequence of (ii).

(iv) We consider the morphism $\pi$

$$\pi : X = \text{Spec} \left( \frac{R[w]}{(h_1 + w^{t+1} \alpha_1 z^{t+1}, \ldots, h_r + w^{t+1} \alpha_r z^{t+1})} \right) \rightarrow \text{Spec}(k[w])$$

Notice that $\pi^{-1}(0)$ is $B$ and that for all $a \neq 0 \in k$ we have $\pi^{-1}(a) \cong A$.

First we prove that $B$ is one-dimensional.

We assume that $z$ is a superficial element of degree one of $A$, so $e$ is the multiplicity of $A$. We denote by $m$ (resp. $n$) the maximal ideal of $A$ (resp. $B$).

From the identity $I + (z^{t+1}) = J + (z^{t+1})$, $t \geq e$, we deduce $I + M^{e+1} = J + M^{e+1}$, so $m^j/m^{j+1} = n^j/n^{j+1}$ for $j = e - 1, e$. Since the multiplication by $z$ induces an isomorphism

$$\frac{m^{e-1}}{m^e} \xrightarrow{z} \frac{m^e}{m^{e+1}}$$

we get an isomorphism

$$\frac{n^{e-1}}{n^e} \xrightarrow{z} \frac{n^e}{n^{e+1}}$$

By Nakayama’s lemma we deduce $n^e = z n^{e-1}$, so $n^j = z n^{j-1}$ for all $j \geq e$.

From this we get that $\dim(B) \leq 1$. Assume that $\dim(B) = 0$. From the upper semi-continuity of the dimension over the ground field $k$ of the fibers of the morphism $\pi$ we get $\dim(A) = 0$ which is a contradiction with the hypothesis $\dim(A) = 1$.

From the identity $I + (z^{t+1}) = J + (z^{t+1})$ we get $I + (z) = J + (z)$, so the multiplicity $e'$ of $B$ satisfies

$$e' \leq \dim_k(R/J + (z)) = \dim_k(R/I + (z)) = e.$$
We denote by $e'T - e'_1$ (resp. $eT - e_1$) the Hilbert Polynomial of $B$ (resp. $A$). Assume $e' < e$ and let $l$ be an integer such that

$$e'l - e'_1 < el - e_1.$$ 

Consider now the morphism

$$\pi_l : X_l = \text{Spec} \left( \frac{R[w]}{(h_1 + w^{t+1} \alpha_1 z^{t+1}, \ldots, h_r + w^{t+1} \alpha_r z^{t+1}) + M^{t+1} R[w]} \right) \longrightarrow \text{Spec}(k[w])$$

We have $\dim_k(\pi_l^{-1}(0)) = e'l - e'_1$ and that $\dim_k(\pi_l^{-1}(a)) = el - e_1$ for all $a \neq 0 \in k$. By upper semi-continuity of the dimension over the ground field $k$ of the fibers of the morphism $\pi_l$ we get a contradiction, so $e' = e$.

We know that $n^j = zn^{j-1}$ for all $j \geq e = e'$ so

$$\frac{n^{j-1}}{n^j} \to \frac{n^j}{n^{j+1}}$$

is an isomorphism for all $j \geq e' = e$. Hence $z$ is a superficial degree one element of $B$. Since $\dim_k(B/(z)) = e'$ we get that $z$ is a nonzero divisor of $B$ by [31], Proposition 1.2(3).

Since $A$ and $B$ are one-dimensional Cohen-Macaulay local rings of multiplicity $e$, $I^*$ and $J^*$ are minimally generated by forms of degree less than or equal to $e$. On the other hand

$$J^*_{\leq e} = (J + (z^{t+1}))_{\leq e} = (I + (z^{t+1}))_{\leq e} = (h_1^*, \ldots, h_r^*)_{\leq e} = I_{\leq e}^*$$

Hence $I^* = J^*$ and the Hilbert functions of $A$ and $B$ agree.

(v) Assume that $\{h_1, \ldots, h_s\}$, $s \leq r$, is a minimal system of generators of the ideal $J$. Assume now that $s < r$ and let $s < j \leq r$. Then there exist $a_j^i \in R, i = 1, \ldots, s$, such that

$$h_j = \sum_{i=1}^{s} a_j^i h_i$$

Thus

$$h_j + \alpha_j z^{t+1} - \sum_{i=1}^{s} a_j^i (h_i + \alpha_i z^{t+1}) = z^{t+1}(\alpha_j - \sum_{i=1}^{s} a_j^i \alpha_i) \in I$$

Since $z$ is a non-zero divisor modulo $I$ we deduce

$$\alpha_j - \sum_{i=1}^{s} a_j^i \alpha_i = \beta_j \in I$$

and then

$$h_j + \alpha_j z^{t+1} = \sum_{i=1}^{s} a_j^i (h_i + \alpha_i z^{t+1}) + \beta_j z^{t+1}$$

for $j = s + 1, \ldots, r$. Since $\beta_j z^{t+1} \in MI$, by Nakayama's Lemma $I$ is generated by $\{h_i + \alpha_i z^{t+1}\}_{i=1, \ldots, s}$.

Assume that $\{h_i + \alpha_i z^{t+1}\}_{i=1, \ldots, s}$ is not a minimal system of generators, for instance if

$$h_s + \alpha_s z^{t+1} = \sum_{i=1}^{s-1} a_j^i (h_i + \alpha_i z^{t+1})$$
for some \( a_j^i \in R, i = 1, \ldots, s \), then
\[
h_s - \sum_{i=1}^{s-1} a_j^i h_i = z^{t+1}(-\alpha_s + \sum_{i=1}^{s-1} a_j^i \alpha_i)
\]
Since \( z \) is a non-zero divisor modulo the ideal \( J \) we get
\[
\alpha_s - \sum_{i=1}^{s-1} a_j^i \alpha_i = \beta_j \in J
\]
and then
\[
h_s - \sum_{i=1}^{s-1} a_j^i h_i \in mJ
\]
which is a contradiction with the assumption that \( \{h_1, \ldots, h_s\} \) is a minimal system of generators of \( J \).

\[\square\]

Notice that in Theorem \ref{thm:annihilators} the ideal \( \text{Ann}_R(H_{t+1}) = I + (z^{t+1}) \) has generators of valuation \( v \leq t \) because \( t \geq e(R/I) \). Hence the assumptions of this result are consistent.

**Remark 3.2.** If the ideal \( I \) is homogeneous, then \( \text{Ann}_R(H_{t+1}) = \text{Ann}_R(H_{t+1})^* \) and \( I = I^* \). In this case, in the previous result, we have necessarily \( \alpha_i = 0 \) for every \( i \). Hence \( H_{e+1} \) determines the ideal \( I \) where \( e = e(R/I) \). This confirms Proposition \ref{prop: annihilators} because \( e \geq r + 1 \) where \( r = \deg H_1 \). We remark that in the homogeneous case, \( H_{r+2} \) determines the Gorenstein ideal \( I \), while instead in the non homogeneous case, the DP-polynomial \( H_{e+1} \) determines a Gorenstein ideal \( I \) (but not uniquely). We will see that \( H_1, \ldots, H_{e+1} \) can be completed in different ways to a \( G \)-admissible module, see Example \ref{ex:admissible}.

The next result gives a sufficient condition for obtaining a Gorenstein 1-dimensional ring from a finite sequence \( \mathcal{H} \) only requiring that the DP-polynomials are primitive of each other (in the sense of Remark \ref{rem:primitive}), i.e. we only require that they satisfy condition (1) of Definition \ref{def:primitive}.

**Proposition 3.3.** Let \( \mathcal{H} = \{H_1, \ldots, H_{t+1}\} \) be a finite set of DP-polynomials satisfying condition (1) of Definition \ref{def:primitive} with respect to a linear form \( z \) of \( R \). Then \( z^i \in \text{Ann}_R(H_i) \) for all \( i = 1, \ldots, t + 1 \).

Let \( I \) be an ideal of \( R \) such that \( \text{Ann}_R(H_{t+1}) = I + (z^{t+1}) \). Assume \( z \) regular modulo \( I \), then
\[
\text{Ann}_R(H_i) = I + (z^i)
\]
for all \( i = 1, \ldots, t + 1 \). In particular \( R/I \) is a one-dimensional Gorenstein \( k \)-algebra and \( I^\perp \) contains \( \{H_1, \ldots, H_{t+1}\} \).

**Proof.** First of all, \( z^i \in \text{Ann}_R H_i \) for all \( i = 1, \ldots, t + 1 \) since \( z^i \circ H_i = 0 \) by assumption. We proceed now by descending recurrence. Given \( a \in I + (z^i), i \leq t \), we have \( az \in I + (z^{i+1}) = \text{Ann}_R(H_{i+1}) \). Hence
\[
(az) \circ H_{i+1} = a \circ (z \circ H_{i+1}) = a \circ H_i = 0
\]
so \( a \in \text{Ann}_R(H_i) \). For all \( a \in \text{Ann}_R(H_i) \) we have \( 0 = a \circ H_i = (az) \circ H_{i+1} \). Hence \( az \in \text{Ann}_R(H_{i+1}) = I + (z^{i+1}) \). Since \( z \) is a non-zero divisor modulo \( I \) we get \( a \in I + (z^i) \).
It follows that $R/I$ is one-dimensional Gorenstein $k$-algebra because $I^e(z) = \text{Ann}_R H_1$ is Artinian Gorenstein and $z$ is $I$-regular. We conclude by Theorem 2.3. □

We give an answer to Questions 2.7 and 2.8 and we describe all the Gorenstein ideals $I$ of $R$ of multiplicity $e = e(R/I)$ such that $I^e$ contains a $G$-admissible set $\mathcal{H} = \{H_1, \ldots, H_{t+1}\}$ with $t \geq e = \dim_k(H_1)$. In particular this allows us to find a system of generators of the Gorenstein ideal $I$ by a finite effective procedure. We note that a procedure has been implemented in Singular using the next result.

**Theorem 3.4.** Let $\mathcal{H} = \{H_1, \ldots, H_{t+1}\}$ be a $G$-admissible set with respect to a linear form $z$ with $t \geq e = \dim_k(H_1)$. Then the following conditions are equivalent:

(i) There exists a $G$-admissible $R$-submodule of $\Gamma$ with respect to $z \in R$, say $M_{\mathcal{H}}$, obtained by completion of $\mathcal{H}$. In particular if $I = \text{Ann}_R(M_{\mathcal{H}})$, then $A = R/I$ is one-dimensional and Gorenstein, $z$ is regular modulo $I$, and $\text{Ann}(H_{t+1}) = I + (z^{t+1})$.

(ii) $\text{Ann}_R(H_{t+1}) = (h_1, \ldots, h_r) + (z^{t+1})$ where $h_1, \ldots, h_r$ are the elements of a minimal standard basis of $\text{Ann}_R(H_{t+1})$ with $v(h_i) \leq t$ and there exist $\alpha_1, \ldots, \alpha_r \in R$ such that $z$ is regular modulo $(h_1 + \alpha_1 z^{t+1}, \ldots, h_r + \alpha_r z^{t+1})$.

If this is the case, then $I = (h_1 + \alpha_1 z^{t+1}, \ldots, h_r + \alpha_r z^{t+1})$ and $I^e = M_{\mathcal{H}}$.

**Proof.** Assume (i), then there exists a $G$-admissible $R$-submodule of $\Gamma$ with respect to $z \in R$, say $M_{\mathcal{H}}$, obtained by completion of $\mathcal{H}$. Let $I = \text{Ann}_R(M_{\mathcal{H}})$ be the corresponding ideal. By Theorem 2.3, $R/I$ is one-dimensional and Gorenstein, $z$ is regular modulo $I$, and $\text{Ann}_R(H_{t+1}) = I + (z^{t+1})$. By Theorem 3.1, given $h_1, \ldots, h_r$ the elements of a minimal standard basis of $\text{Ann}_R(H_{t+1})$ with $v(h_i) \leq t$, we know that there is a family of elements $\alpha_1, \ldots, \alpha_r \in R$ such that $(h_1 + \alpha_1 z^{t+1}, \ldots, h_r + \alpha_r z^{t+1})$ is a minimal standard basis of $I$ and hence $z$ is a non-zero divisor modulo $(h_1 + \alpha_1 z^{t+1}, \ldots, h_r + \alpha_r z^{t+1}) = I$. Conversely, if we assume (ii) and consider the ideal $I = (h_1 + \alpha_1 z^{t+1}, \ldots, h_r + \alpha_r z^{t+1})$, then $\text{Ann}_R(H_{t+1}) = (h_1, \ldots, h_r) + (z^{t+1}) = I + (z^{t+1})$. Because $z$ is regular modulo $I$, then by Lemma 3.3, $R/I$ is Gorenstein one-dimensional, hence $I^e$ is $G$-admissible with respect to $z$ and it contains $\mathcal{H}$. Hence $I^e = M_{\mathcal{H}}$. □

**Remark 3.5.** We remark that the previous result is effective. Given a finite set $\mathcal{H} = \{H_1, \ldots, H_{t+1}\}$ with $t \geq e = \dim_k(H_1)$, by using [9] we can check if $\mathcal{H}$ is $G$-admissible with respect to a linear form $z$. We can compute $\text{Ann}_R(H_{t+1})$ and determine the elements $h_1, \ldots, h_r$ of a minimal standard basis of $\text{Ann}_R(H_{t+1})$ with $v(h_i) \leq t$. Let $\{h_1, \ldots, h_s\}$ be a minimal system of generators of $J = (h_1, \ldots, h_r)$. Let $\alpha_1, \ldots, \alpha_s$ be elements of $R$ such that $z$ is a regular superficial element modulo $I_{\alpha} = (h_1 + \alpha_1 z^{t+1}, \ldots, h_s + \alpha_s z^{t+1})$, then $R/I_{\alpha}$ and $R/J$ are Gorenstein one-dimensional rings with $I_{\alpha}^e = J^e$ sharing the same Hilbert function. In fact $R/I_{\alpha}$ is Gorenstein one dimensional by Proposition 3.3 and we conclude concerning $R/J$ by Theorem 3.1.

**Example 3.6.** Consider $R = k[x, y, z]$. According to Remark 3.5, we wish to construct the ideals $I$ in $R$ such that $R/I$ are Gorenstein local rings of dimension 1 and multiplicity 5 sharing the same $G$-admissible set with respect to $x \in R$. Consider a polynomial $H_1 \in \Gamma = k_{PD}[Y, Z]$ such that $\dim_k(H_1) = 5$. Let $H_1 = Z^2 + Y^3$. One can verify that

$$\mathcal{H} = \{H_1, H_2 = XH_1, H_3 = X^2H_1, H_4 = X^3H_1 + Y^4Z + YZ^3, H_5 = XH_4, H_6 = XH_5\}$$
is a finite $G$-admissible set with respect to $x$.

The elements of a minimal standard basis of $\text{Ann}_R(H_6)$ of valuation $\leq e = 5$ are $h_1 = yz - x^3, h_2 = z^2 - y^3, h_3 = y^4 - x^3z$. We verify that $x$ is regular modulo the ideal $J = (h_1, h_2, h_3) = (h_1, h_2)$, hence by Theorem 3.3 $R/J$ is one-dimensional Gorenstein. Notice that the ideal $J = (h_1, h_2, h_3) = (h_1, h_2)$ is the defining ideal in $R$ of the semigroup ring $k[t^5, t^6, t^9]$. Consider now the ideal $I_\alpha = (h_1 + \alpha_1 x^6, h_2 + \alpha_2 x^6)$ with $\alpha_1, \alpha_2 \in R$. Since $x$ is regular modulo $I_\alpha$ for every $\alpha_1, \alpha_2 \in R$, then $I_\alpha$ describes all the ideals of $R$ such that $R/I_\alpha$ is a one-dimensional Gorenstein ring of multiplicity 5 and the dual is a completion of $H$. In particular the family of Gorenstein ideals $I_\alpha$ share the same associated graded ring, hence the same Hilbert function, because they have the same tangent cone.

In general, it is hard to prove that two ideals are non analytically isomorphic. In the next example we give two deformations $I_1, I_2$ of $J$, see Remark 3.5 such that $I_1, I_2, J$ are pairwise non-analytically isomorphic and they come from the same $G$-admissible set.

**Example 3.7.** Consider $R = k[x, y, z]$ and the $G$-admissible set with respect to $x \in R$

$$H = \{H_1 = Y^3 + Z^2, H_i = X^{i-1}H_1, i = 2, \ldots, 6\}.$$  

Hence $H_6 = X^5Y^3 + X^5Z^2$ and $\text{Ann}_R(H_6) = (yz - x^7, z^2 - y^3, x^6)$. Following the notations of Remark 3.5, $J = (h_1, h_2)$ with $h_1 = yz - x^7, h_2 = z^2 - y^3$. Notice that $J$ is the defining ideal of the Gorenstein monomial curve $k[t^5, t^{14}, t^{21}]$. Consider the following two deformations of $J$: $I_1 = (h_1 + x^6, h_2)$ and $I_2 = (h_1, h_2 + x^6)$, that is respectively $(\alpha_1, \alpha_2) = (x^6, 0)$ and $(\alpha_1, \alpha_2) = (0, x^6)$. Both ideals define reduced one-dimensional Gorenstein rings. By using Singular we compute the multiplicity sequences of the above ideals: $\{5, 5, 4, 1, \ldots\}$ for $J$, $\{5, 5, 2, 2, 1, \ldots\}$ for $I_1$, and $I_2$ defines an ordinary singularity with 5 non-singular branches. Hence $J, I_1, I_2$ are pairwise non-analytically isomorphic.

4. 0-dimensional Gorenstein schemes

As an application of the previous results, in this section we will present some examples and discussions that we hope will be useful in the theory of Gorenstein linkage (G-linkage). Note that the lack of a general structure of homogeneous Gorenstein ideals of higher codimension is the main obstacle to extending the Gorenstein liaison theory in codimension at least three; the codimension two Gorenstein liaison case is well understood, see [23]. See, for instance, [21], [24] and [22] for some constructions of particular families of Gorenstein algebras.

We say that two schemes $X$ and $Y$ in $\mathbb{P}_k^n$ that are reduced and without common components are directly Gorenstein-linked if their union is arithmetically Gorenstein. More generally, if we call $I(G)$ the Gorenstein ideal in $R = k[x_0, \ldots, x_n]$, then

$$I(G) : I(X) = I(Y) \quad \text{and} \quad I(G) : I(Y) = I(X).$$

Equivalence classes in Gorenstein linkage are determined by the equivalence relation generated by direct Gorenstein linkage. In terms of the inverse system, it is easy to verify that $I(G) : I(X) = I(Y)$ is equivalent to

$$I(X)^\perp = I(Y) \circ I(G)^\perp.$$
We say $X$ is glicci if it is in the Gorenstein linkage equivalence class of a complete intersection. In analogy to the codimension 2 theory (where Gorenstein is complete intersection), one hopes that every arithmetically Cohen-Macaulay scheme is glicci and, in particular, every finite set of points in $\mathbb{P}^3$ should be glicci, see for instance [18], [23]. This was verified by R. Hartshorne in [18] and by D. Eisenbud, R. Hartshorne and F.O. Schreyer in [7] if $d = |X| \leq 33$, $d = 37, 38$.

Usually it is very difficult to construct a set of distinct points $G$ whose defining ideal is Gorenstein. General points are very far in general from being Gorenstein. The problem becomes even more difficult if the set $G$ must contain a given set $X$ of points as the linkage theory requires. We take advantage of the results of this paper for giving explicit examples of reduced one dimensional Gorenstein $k$-algebras $A = k[x_0, \ldots, x_n]/I(G)$ with given Hilbert function. Our approach aims to be constructive, but for the moment we cannot deduce a theoretic approach.

Our goal is the following. Given the integers $d; e$ and an appropriate Hilbert function of degree $d+e$ admissible for a Gorenstein graded $k$-algebra of dimension one, we will construct some examples of reduced Gorenstein schemes $G$ in $\mathbb{P}^3_k$ with the given Hilbert function. We verify that $G$ contains a degree $d$ subscheme $X$ with generic Hilbert function whose complement $Y$ of degree $e$ also has generic Hilbert function. It turns out that there is only a finite number of possibilities for the Hilbert function of such Gorenstein schemes. We recall that in our construction the Hilbert function is determined by the suitable choice of $H_1$. In fact the entries of the $h$-vector are determined by $\dim_k(H_1)_j$. We say that a 0-dimensional scheme in $\mathbb{P}^3_k$ has generic Hilbert function if it is maximal, that is $HF_X(j) = \text{Max}\{\deg X, \binom{n+j}{j}\}$. In general $I(X)$ is level, hence also the structure of $I(X)^\perp$ is known by [20]. Different methods for constructing points which are glicci in special cases are known. Our hope is that our computations could suggest some theoretical methods to use more abstractly.

In this section let $R = k[x, y, z, w]$ and hence $\Gamma = k_{DP}[X, Y, Z, W]$. We assume the ground field $k$ is of characteristic zero. We may assume without loss of generality that the points we want to construct do not lie on $w = 0$. Hence we want to find a $G$-admissible set $\mathcal{H} = \{H_1, \ldots, H_{t+1}\}$ in $\Gamma$ with respect to $w$ such that $\dim_k(H_1) = d + e$. Moreover if $t = \max$ degree of the generators of $I(G)$ or $t = \deg H_1$, then by the previous results:

$$I(G) = \text{Ann}_R(H_{t+1}) \leq_t R.$$  

If $X$ is glicci, then there exists a $G$-admissible sequence $\mathcal{H} = \{H_1, \ldots, H_{t+1}\}$ such that

$$(I(X) + (w^i))^\perp \subseteq \langle H_i \rangle$$

for all $i = 1, \ldots, t + 1$.

**Example 4.1.** (5 Gorenstein points) We know that a set of 4 points in $\mathbb{P}^3_k$ is glicci, in fact they are linked to a complete intersection by a Gorenstein scheme of length 5. This is a very special case, in fact if we consider a set $X$ of 4 points in linear general position and we take a sufficiently general point $Y$, then $G = X \cap Y$ consists of 5 points which are Gorenstein (but not a complete intersection). If we consider $H_1 = X^2 + Y^2 + Z^2$, then $\dim_k(H_1) = \dim_k(1, X, Y, Z, H_1) = 5$ and we can take the $G$-admissible set $\mathcal{H} = \{H_1, H_2 = WH_1 + X^3 + Y^3 + Z^3, H_3 = WH_2 + X^4 + Y^4 + Z^4\}$ with respect to $w \in R$. 


In this case, then $I = \text{Ann}_R(H_3)_{\leq 2}R$ is radical and it is the defining ideal of a Gorenstein set of 5 points containing 4 points in linear general position.

**Example 4.2.** (Gorenstein set of 14 points) It is known that a set of 10 points in $\mathbb{P}^3_\mathbb{K}$ is glicci, in fact they are linked to a set of 4 points by a Gorenstein scheme of length 14. In this example we use the inverse system to find a Gorenstein zero-scheme consisting of $d + e = 14$ points which are the union of two reduced subschemes $X$ and $Y$ respectively of $d = 10$ and $e = 4$ points with maximal Hilbert function.

Consider $H_1 = X^3Y + Y^3Z + XZ^3 \in \mathbb{K}_{DP}[X, Y, Z]$ and we observe that $\dim_k(H_1) = 14$ with Hilbert function $h_i = \dim_k(H_1)_i: \{1, 3, 6, 3, 1\}$. Recall that $\text{Ann}_R(H_1) = I(G) + (w)$. We want to construct a 1-dimensional Gorenstein graded algebra $R/I(G)$ from a $G$-admissible set $\mathcal{H} = \{H_1, H_2, H_3, H_4\}$. Notice that in Theorem [3.3] we can choose $t = 3$ because by the Hilbert function we know that the ideal is generated in degree 3. By using Singular’s library [9], given $H_j$ with $j = 1, \ldots, t$, it is possible to find the admissible elements $C_j \in \mathbb{K}_{DP}[X, Y, Z]$ such that $H_{j+1} = WH_j + C_j$. In the first step we prove that we can choose $C_1$ any homogeneous form of degree 5 in the variables $X, Y, Z$. Take $C_1 = XY^2Z^2$.

Consider $H_2 = WH_1 + XY^2Z^2$. We realize computationally that this choice of $C_1$ fixes the successive elements $C_2 = -2X^4Y^2 - XY^4Z + 2Z^6$ and $C_3 = -XY^6 - 2X^5YZ - 2X^2Y^3Z^2 - 2X^3Z^4$, then $H_4 = W^3H_1 + W^2C_1 + WC_2 + C_3$. Hence

$$I = \text{Ann}_R(H_4)_{\leq 3}R = (x^3 - y^2z + z^2w, x^2y - z^3 + 2xw^2, \ xy^2 - xzw + 2yw^2, y^3 - xz^2 + yzw, x^2z + 2w^3, xyz - y^2w + zw^2, yz^2 - x^2w)$$

is Gorenstein and $\dim R/I = 1$. Using Singular, we can verify that the ideal is radical, hence it is the defining ideal of a Gorenstein set of 14 distinct points. We can also prove that it is the union of two subsets of points comprised of 10 and 4 points with maximal Hilbert function.

It would be more interesting to have a theoretic argument proving that starting from a given subscheme $X$ of $d = 10$ distinct points with maximal Hilbert function, then it is always possible to find $e = 4$ distinct points, say $Y$, such that the union defines a 0-dimensional Gorenstein scheme of length $d + e = 14$.

In our case $I(Y)$ is generated by 6 quadrics $q_1 = x^2 + wL_1, q_2 = xy + wL_2, \ldots, q_6 = z^2 + wL_6$ with $L_i$ suitable linear forms in $R$ and $I(X) = \text{Ann}_R(q_i \circ H_4, \ldots, q_6 \circ H_4)_{\leq 3}R$ where $\mathcal{H} = \{H_1, H_2, H_3, H_4\}$ is a $G$-admissible set. Our approach suggests to find $J = (q_1, \ldots, q_6) = I(4)$ imposing the following conditions: $(I(X) + (w^j))^{\perp} = J \circ H_i, i = 1, \ldots, 4$, with $\mathcal{H} = \{H_1, H_2, H_3, H_4\} G$-admissible.

In our particular case the system has a solution and we have: $L_1 = 3x + 12y + 3z + 7w, L_2 = 6x + 11y + 9z + 11w, L_3 = 12x + 10y + 11z + 7w, L_4 = 9x + 7y + 5z + 11w, L_5 = x + 10y + 4z + 3w, L_6 = 10x + 2y + 10z + 2w$ and $I(X) = (-x^3 + 3x^2y + 3y^3 - y^2z + 3xz^2 + 3z^3 + xyz, -2x^3 + 6x^2y - 4y^3 - 2y^2z - 4xz^2 + yz^2 + 6z^3 + x^2w, -3x^3 - x^2y - 2y^3 - 2y^2z - 2x^2z - z^3 + z^2w, -6x^3 - 4x^2y + 5y^3 - 6y^2z + 6xz^2 - 4z^3 + yzw, -3x^3 + x^2y + 4y^3 + xyz - 3y^2z + 4xz^2 + 3z + y^2w + 2x^3 - 3x^2y + xy^2 - 3y^3 + 2y^2z - 3xz^2 - 3z^3 + xzw)$.

**Example 4.3.** (Gorenstein sets of 30 and 55 points) This case was considered by Hartshorne in [18] to be a possible counterexample to the “conjecture”. But Eisenbud, Hartshorne and Schreyer in [7] proved that a set of 20 points in $\mathbb{P}^3_\mathbb{K}$ is glicci, in fact they are linked to a set of 10 points by a Gorenstein scheme of length 30. In this example we use the
inverse system to find a set $G$ of $d + e = 30$ Gorenstein points which are the union of two reduced subschemes $X$ and $Y$ respectively of $d = 20$ and $e = 10$ points with maximal Hilbert function. We may proceed as in the previous example.

In this case $\deg H_1 = 6$ because the socle degree of $R/I(G)$ is 6. We consider $H_1 = X^6 + Y^6 + Z^6 + X^5 Y + Y^5 Z + X Z^5 \in \mathbb{k}[X, Y, Z]$ because $\dim_k(H_1) = 30$ and the Hilbert function is: $\{1, 3, 6, 10, 6, 3, 1\}$. Recall that $\Ann_R(H_1) = I(G) + (w)$. We want to construct a 1-dimensional Gorenstein graded algebra $R/I(G)$ from a $G$-admissible set $\mathcal{H} = \{H_1, \ldots, H_5\}$. Notice that in Theorem 3.4 we can choose $t = 4$ because by the Hilbert function we know that the ideal is generated in degree 4. By using Singular’s library [9] we can find $C_1, \ldots, C_4$ and we get $H_5 = W^4 H_1 + W^3 C_1 + \cdots + WC_3 + C_4$. At the first step we choose $C_1 = X^2 Y^2 Z^2$, then as before we realized that all the successive forms are uniquely determined.

In the same paper, Eisenbud, Hartshorne and Schreyer stated that the degree of the smallest collection of general points in $\mathbb{P}^3_k$ not yet known to be glicci is 34. As before, what we are able to construct, with our methods, is a set $G$ of 55 Gorenstein points containing a subscheme of 34 distinct points with generic Hilbert function, whose complementary 21 points has also generic Hilbert function.

In this case $\deg H_1 = 8$ because the socle degree of $R/I(G)$ is 8. We consider $H_1 = X^8 + Y^8 + Z^8 + X^3 Y^3 Z^2 + X^2 Y^3 Z^3 + X^3 Y^2 Z^3 \in \mathbb{k}[X, Y, Z]$ because $\dim_k(H_1) = 55$ and its Hilbert function is: $\{1, 3, 6, 10, 15, 10, 6, 3, 1\}$. Recall that $\Ann_R(H_1) = I(G) + (w)$. We want to construct a 1-dimensional Gorenstein graded algebra $R/I(G)$ from a $G$-admissible set $\mathcal{H} = \{H_1, \ldots, H_6\}$. Notice that in Theorem 3.4 we can choose $t = 5$ because by the Hilbert function we know that the ideal is generated in degree $\leq 5$. By using Singular’s library [9] we can find $C_1, \ldots, C_5$ and we get $H_6 = W^5 H_1 + W^4 C_1 + \cdots + WC_4 + C_5$. We choose $C_1 = XY^4 Z^4$, then by the algorithm we realize that, as in the previous examples, all the successive “constants” $C_i$ are uniquely determined. For the moment we cannot understand the reason for this constraint. We think that this could be the key of a constructive general argument.

5. GORENSTEIN SEMIGROUP RINGS

Semigroup rings are a broad class of one-dimensional domains and they are a test case for many open problems. Hence we end this paper with the construction of the inverse system of the defining ideal of a family of semigroup rings. In particular we study a corresponding $G$-admissible set in $\Gamma$ according to Theorem 3.4. Let $b \geq 2$ be an integer. Consider

$$A = \mathbb{k}[[t^{3b}, t^{3b+1}, t^{6b+3}]].$$

It is easy to see that $A = \mathbb{k}[[x, y, z]]/I$ where $I = (x z - y^3, z^b - x^{2b+1})$. Thus $A$ is a one-dimensional Gorenstein local domain of type $(2, b)$ and multiplicity $e = 3b$, see [10].

Example 5.5. It is an interesting class of ideals because the associated graded ring is not Cohen-Macaulay even if the ideal is a complete intersection. The Hilbert function of $A$
We compute the inverse system $I^\perp = \langle H_t, t \in \mathbb{N}_+ \rangle$ of $I$ with respect to $x$. Notice that the ideal $I$ is homogeneous with respect to $\deg_b$, the DP-polynomial $H_i$ is homogeneous as well. Notice that $e = 3b = \dim_k(H_1)$. We determine that $\mathcal{H} = \{H_1, \cdots, H_{3b+1}\}$ is a $G$-admissible set in $\Gamma = k_{DP}[X, Y, Z]$ with respect to $x$ which determines $I$.

Claim: $H_1 = Y^2Z^{b-1}$ and $\deg_b(H_1) = 6b^2 + 3bt - 1$, $t \geq 1$.

In fact we know that $H_1$ is a generator of $(I + (x))^\perp$. Since $I + (x) = (x, y^3, z^b)$, an easy computation give us $H_1 = y^2z^{b-1}$. Recall that $H_t = x \circ H_{t+1}$. From this we get the second part of the claim.

We write $\delta(b, t) = 6b^2 + 3bt - 1$, for $b \geq 2$ and $t \geq 1$; we denote by $\mathcal{M}_{b,t}$ the set of homogeneous monomials of degree $\delta(b, t)$ in $\Gamma = k_{DP}[X, Y, Z]$. For all $t \geq 1$ consider the DP-polynomial of degree $\delta(b, t)$

$$H_t = \sum_{m \in \mathcal{M}_{b,t}} m.$$

Claim: $\text{Ann}_R(H_{3b+1}) = I + (x^{3b+1})$.

Notice that we only have to prove that

$$\begin{cases} 
(1) & x^{3b+1} \circ H_{3b+1} = 0 \\
(2) & (xz - y^3) \circ H_{3b+1} = 0 \\
(3) & (z^b - x^{2b+1}) \circ H_{3b+1} = 0
\end{cases}$$

We set $\delta = \delta(b, 3b + 1) = 15b^2 + 3b - 1$. Recall that $\deg(H_{3b+1}) = \delta$.

(1) For all monomials $m = X^iY^jZ^k$ of degree $\delta$ we have to prove that $i < 3b + 1$. Assume that $i \geq 3b + 1$. The Diophantine equation

$$(3b + 1)j + (6b + 3)k = \delta - 3bi$$

has the following solutions:

$$(j, k) = (-2, 1)(\delta - 3bi) + \lambda(6b + 3, -(3b + 1))$$

for all $\lambda \in \mathbb{Z}$. Since $j, k \geq 0$ we deduce $\delta - 3bi \geq 0$ and

$$A = \frac{2(\delta - 3bi)}{6b + 3} \leq \lambda \leq B = \frac{\delta - 3bi}{3b + 1}.$$
In particular \( i < 5b + 1 \) and then \( 3b + 1 \leq i < 5b + 1 \). We can perform the corresponding Euclidean divisions

\[
A = 5b - i + r_1, B = 5b - i + r_2
\]

with \( 0 < r_1 = \frac{3i - 9b - 2}{6b + 3} < r_2 = \frac{i - 2b + 1}{3b + 1} < 1 \). Hence there are no integers \( \lambda \) such that \( A \leq \lambda \leq B \). Hence \( i < 3b + 1 \).

(2) We prove that \( xz \circ M_{b,3b+1} = y^3 \circ M_{b,3b+1} \). Given \( m = X^i Y^j Z^k \in \mathcal{M}_{b,3b+1} \) we have that \( xz \circ m = 0 \) if \( i = 0 \) or \( k = 0 \). Hence \( xz \circ M_{b,3b+1} \) is the set of monomials \( m' = X^{i-1} Y^j Z^{k-1} \) such that \( m = X^i Y^j Z^k \in \mathcal{M}_{b,3b+1} \) and \( i, k \geq 1 \). Since \( m' = y^3 \circ (Y^3m') \) and \( y^3m' \in \mathcal{M}_{b,3b+1} \) we get \( xz \circ M_{b,3b+1} \subset y^3 \circ M_{b,3b+1} \). Given \( m = X^i Y^j Z^k \in \mathcal{M}_{b,3b+1} \) we have that \( y^3 \circ m = 0 \) if \( j \leq 2 \). Hence \( y^3 \circ M_{b,3b+1} \) is the set of monomials \( m' = X^i Y^j Z^k \) such that \( m = X^i Y^j Z^k \in \mathcal{M}_{b,3b+1} \) and \( j \geq 3 \). Since \( m' = xz \circ (XZm') \) and \( XZm' \in \mathcal{M}_{b,3b+1} \) we get \( y^3 \circ M_{b,3b+1} \subset xz \circ M_{b,3b+1} \).

(3) We prove that \( z^b \circ M_{b,3b+1} = x^{2b+1} \circ M_{b,3b+1} \). We proceed as for (2).

Similarly one can prove that \( \text{Ann}_R(H_t) = I + (x^t) \) for every \( t \geq 1 \).

For instance if \( b = 3 \) we get that

\[
H_{10} = x^2 y^2 z^5 + xy^5 z^4 + y^8 z^3 + x^9 y^2 z^2 + x^8 y^5 z + x^7 y^8.
\]

In this case

\[
\text{Ann}_R(H_{10}) = (xz - y^3, z^3 - x^7) + (x^{10}) = I + (x^{10}).
\]

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