A 4D Duality Web

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Abstract

We construct a web of non-supersymmetric dualities in four spacetime dimensions for theories with theta and Maxwell terms. Our construction mirrors the recipe used in (2+1)-dimensions to demonstrate a similar duality for theories with Chern-Simons terms. As in the 3-dimensional case, the web consists of boson-boson, fermion-fermion and boson-fermion dualities, any one of which can be taken as a seed for the remaining two. To be concrete, we begin by constructing the basic bosonization duality which we then use to generate the web, using an analog of the 3-dimensional massive construction given in [1]. The resulting 4-dimensional duality web is best understood as an extension of the, by now well-known, 3-dimensional case due to its somewhat singular nature. We conjecture that this is the final thread in a multi-dimensional duality web that connects low energy theories in 4-, 3- and 2-spacetime dimensions and speculate on its application to 3-dimensional topological insulators.
1 Introduction

Duality, the idea of two entirely different descriptions of the same physical degrees of freedom, is one of the most far-reaching and consequential concepts in physics. Among the many examples that come to mind, we count:

- **Electric-magnetic duality in 4-dimensional spacetime** - perhaps the first and most easily recognisable example, the mapping \((E, B) \mapsto (B, -E)\) leaves the vacuum Maxwell equations,

\[
\begin{align*}
\nabla \cdot E &= 0, \\
\nabla \cdot B &= 0, \\
\n\nabla \times E &= -\frac{\partial B}{\partial t}, \\
\n\nabla \times B &= \frac{\partial E}{\partial t},
\end{align*}
\]

invariant. This seemingly innocuous observation, the precursor of S-duality, lies at the heart of Kapustin and Witten’s physical interpretation of the Geometric Langlands conjecture [2].

- **Gauge/Gravity duality** - arguably, the most celebrated of the duality relations, the gauge theory/gravity duality manifests itself most clearly in the AdS/CFT correspondence [3] between a theory of quantum gravity (such as superstring theory) in \(AdS_d \times \mathcal{M}^{10-d}\) and a gauge theory on the boundary of the AdS component (such as supersymmetric Yang-Mills on \(\partial AdS_d\)). Directly or indirectly, it has been the primary catalyst for most of the advances both in gravity research as well as strongly-coupled quantum field theory over the past two decades.

- **T-duality** - originally discovered in the context of the string worldsheet sigma model in string theory, T-duality is a map that exchanges the geometric data associated with one 2-dimensional sigma model, say \((\mathcal{X}, g, B)\) with another \((\tilde{\mathcal{X}}, \tilde{g}, \tilde{B})\) such that the quantum field theories induced from such data are equivalent. For example, the conformal field theory of a free boson on a circle of radius \(R\) is equivalent to that of a free boson on the circle of radius \(1/R\) under T-duality.

- **Bosonization** - is the remarkable observation, made more-or-less simultaneously in the condensed matter and particle physics communities [4–7], that a theory of relativistic Dirac fermions with the usual anti-commutation relations, may be replaced by a bosonic field theory that captures the same physics. This is particularly clean in one spatial dimension where, because of the linear one-particle dispersion near the Fermi surface, particle-hole pairs exhibit a

\[^1\text{In this language } \mathcal{X} \text{ is a smooth manifold, equipped with a Riemannian metric } g \text{ and Kalb-Ramond field } B.\]
quasi-particle-like dispersion near zero momentum and propagate as a coherent bosonic particle.

Common to all of these examples is the fact that the duality maps one theory at strong (weak) coupling to another at weak (strong) coupling. Moreover, all but the gauge/gravity duality in the list above can be expressed formally in terms of a path integral. Roughly, one starts with a path integral,

\[ Z_{\text{master}} = \int \mathcal{D} \Phi \mathcal{D} \Lambda \exp \left( S_{\text{master}}[\Phi, \Lambda] \right), \]

over a space of auxiliary fields and shows that integrating out one or other set of fields produces the path integral corresponding to the action function \( S[\Phi] \) or its dual \( \tilde{S}[\Lambda] \). While this path integral formulation was well known for T-duality, it was not until the seminal work of Burgess and Quevedo [8] that it was understood that 1+1-dimensional bosonization could also be understood as a duality in this sense. With a bose-fermi duality (bosonization) and a bose-bose duality (T-duality) in place, it seems like it was only a matter of time before the dots were connected to reveal a duality web in 2-dimensional spacetime. Surprisingly, this was only carried out in the pedagogical work of Karch et.al. in 2019 [9], with the use of some sophisticated tools from the theory of quadratic forms and lessons learnt from condensed matter theory, in the intervening years.

One such lesson was the realization that matter coupled to Chern-Simons gauge theory in 2+1-dimensions exhibits a remarkably rich structure. This is beautifully captured by Aharony’s conjectured Chern-Simons-matter dualities [10],

\[
\begin{align*}
N_f \text{ fermions} + U(k)_{-N+N_f, -N+k+N_f} & \longleftrightarrow N_f \text{ scalars} + SU(N)_k, \\
N_f \text{ fermions} + SU(k)_{-N+N_f} & \longleftrightarrow N_f \text{ scalars} + U(N)_{k,k}, \\
N_f \text{ fermions} + U(k)_{-N+N_f, -N-k+N_f} & \longleftrightarrow N_f \text{ scalars} + U(N)_{k,k+N},
\end{align*}
\]

where the ‘+’ denotes a coupling between the matter field and a Chern-Simons field with a particular gauge group, and \( U(N)_{k,l} \equiv (SU(N)_k \times U(1)_l) / \mathbb{Z}_N \). In particular, the choice \( N_f = N = k = 1 \) identifies two interesting cases; a scalar at the Wilson-Fisher fixed point is dual to a fermion coupled to an abelian Chern-Simons gauge field, and a free fermion is dual to a gauged Wilson-Fisher scalar.

Another lesson came from the condensed matter physics of a (3+1)-dimensional topological insulator whose (2+1)-dimensional boundary exhibits dual descriptions in terms of (i) a massless Dirac fermion coupled to an external gauge field, \( \mathcal{L} = i \bar{\Psi} \partial_\mu \Psi \), and (ii) another Dirac fermion coupled to a gauge field, say \( a \), which is itself coupled
to the external gauge field\(^2\), \(\mathcal{L}_{\text{dual}} = \i\chi\partial_a \chi + \frac{1}{4\pi} Ada\). This was recognised in [12, 13] as a fermionic version of the more well-known bosonic particle-vortex duality [14,15] between the \(O(2)\) Wilson-Fisher scalar \(\mathcal{L}_{\text{WF}} = |D_B \Phi|^2 - |\Phi|^4\) and its gauged counterpart, \(\mathcal{L}_{\text{AH}} = |D_b \phi|^2 - |\phi|^4 + \frac{1}{2\pi} b dB\), the 3-dimensional abelian Higgs model.

These three, seemingly disparate, threads were recognised to be part of an larger web of non-supersymmetric dualities in the summer of 2016 [11, 16, 17], precipitating a slew of results from both high energy and condensed matter communities that have led to a far deeper understanding of the phases of quantum matter. For an excellent and pedagogical account of the arguments leading to the 3 dimensional duality web, and many of the subsequent developments, we refer the interested reader to the excellent reviews [18, 19] and references therein. As so eloquently put in [9]; dualities beget dualities. Much of this begetting is realised through path integral manipulations [20] that relate the partition functions in the dual pair and, on the face of it, and up to the existence of certain topological invariants, none of these manipulations appear to depend in a crucial way on the number of dimensions in the problem, which begs the question:

*If duality webs exist in 2- and 3-dimensional spacetimes, could one also exist in 4D?*

Given the utility of the 3 dimensional duality web in driving developments in ultra quantum matter - mostly planar electronic matter residing on the boundary of bulk (3+1)-dimensional topological insulators, but also other novel quantum matter such as graphene - over the past few years, it is clear that a similar set of relations describing the quantum phases of bulk physics would be equally important. There are, however, some subtle issues that need unpacking first.

The most pressing among these is that, both in 2- and 3-dimensions, the seed of the duality web, from which the rest of the duality relations could be derived, is bosonization. In \(D = d + 1\) spacetime dimensions, the duality approach to bosonization of [20] maps a theory with a massive Dirac fermion to one for a Kalb-Ramond \((d-1)\)-form \(B\). This in turn can be formulated as a scalar field theory with derivative interactions. However, while the bosonic theory retains several desirable properties such as gauge invariance under \(B \rightarrow B + d\omega\) with \((d-2)\)-form \(\omega\); others, such as locality, are lost for \(D \geq 4\). For example, in the large \(m\) limit, and in the notation\(^2\) as pointed out in [11], this statement is not precisely true since the gauge-invariance (mod 2\(\pi\)) of the \(Ada/4\pi\) term is incompatible with standard Dirac quantization. For a detailed discussion, and resolution, of this subtle issue, see [11].
above,

\[-\bar{\psi}(\partial + m + i\phi)\psi \leftrightarrow -\frac{1}{k_D} \Omega \square \Omega + \Omega a,\]

where $a$ is an external gauge field to which the fermion couples, $k_D$ is a $D$-dependent constant and the bosonic field $\Omega$ is the Hodge dual of the field strength for the Lagrange multiplier field $\Lambda$. Still, there are reasons to be optimistic. The most compelling was already pointed out in [11] where the authors provide solid, yet circumstantial, arguments for the embedding of the 3-dimensional duality web into a 4-dimensional spacetime in which the various dualities (particle-vortex, bose-fermi and fermi-fermi) reside on a 2+1 dimensional boundary, coupled through a complex coupling $\tau \equiv \theta + \frac{2\pi i}{e^2}$ to a half-space bulk action for the gauge fields. Inspired by this argument, the recent construction of a 4-dimensional fermion-fermion duality in [21], and following the 3-dimensional construction in given in [1], we attempt here to provide a direct construction of a 4-dimensional duality web.

While the path integral manipulations we employ are of course not new, the central premise of this article namely the existence of a fully 4 dimensional duality web is, to the best of our knowledge, a new and novel addition to the existing 2- and 3-dimensional duality webs. For its pedagogical value, we show how our construction fits in with the arguments given by Seiberg et.al in [11] and perform the dimensional reduction to the 3-dimensional web. A key feature of our construction, which it inherits from that in [1], is that the duality is performed away from criticality; we gap the theory with a non-zero mass, carry out the duality at the level of the path integral and then tune the mass (which we will view as a regulator, of sorts) to zero.

The paper is organised as follows; section 2 reviews the 3-dimensional construction of the basic bosonization map given in [1], paying attention to the details of how this is used to thread together a web of boson-fermion, boson-boson and fermion-fermion dualities. In section 3, we lift this to (3+1)-dimensions where (modulo the locality issue above) the 4-dimensional analog of bosonization, at least in its realization as a duality, is known. More tricky to define is the higher dimensional analog of the bosonic particle-vortex duality. Here also, we review the recent fermion-fermion duality of [21], pointing out some subtleties that make it not quite what we want for the duality web. In section 4, we bring these ingredients together to construct, directly at the path integral level, the 4-dimensional duality web. We conclude with a discussion of some implications for the condensed matter physics of bulk topological insulators.
2 A review of the 3-dimensional web construction

Before proceeding to the 4-dimensional case of interest to us, it will be useful to establish some conventions and familiarise ourselves with the path integral manipulations required by reviewing the construction of the 3-dimensional duality web. While our primary interest will be with 4-dimensional spacetime, much of our notation will carry over from 3 dimensions. To begin, bosons will generally be denoted $\phi$, fermions by $\psi$, external (non-dynamical) gauge fields by capitalised letters like $A = A_\mu \, dx^\mu$ and emergent gauge fields (usually arising from gauging some internal symmetries) by lower case letters such as $a = a_\mu \, dx^\mu$. A massive scalar has action

$$S_{\text{scalar}}[\phi; A] = \int d^3 x \left( |D \phi|^2 - m^2 |\phi|^2 \right),$$

where the operator $D \equiv d - iA$ minimally couples the scalar to an external $U(1)$ gauge field. We will also have need to add a quartic operator $-\alpha |\phi|^4$ term to this, in which case we’ll denote the action $S_{\text{boson}}[\phi; A; \alpha]$. Similarly, the action for a gapped Dirac fermion, coupled to an external $U(1)$ gauge field, will be given by

$$S_{\text{fermion}}[\psi; A] = \int d^3 x \, i \overline{\psi} (\not{D} + m) \psi,$$

with $\not{D} = \gamma^\mu D_\mu$ as usual. Central to the transmutation of statistics in 2+1-dimensions is the idea of flux attachment; adding a single flux quantum to a (fermion) boson changes it into a (boson) fermion. In a relativistic theory such as the ones considered in this section, this is accomplished by coupling to a Chern-Simons term\(^3\)

$$S_{\text{CS}}[A] = \frac{k}{4\pi} \int A \, dA,$$

with Chern-Simons level $k \in \mathbb{Z}$, to ensure that this term is gauge invariant if $F = dA$ is canonically normalized. More relevant for our 4-dimensional construction is the related BF-coupling

$$S_{\text{BF}}[A; B] = \frac{1}{2\pi} \int A \, dB = S_{\text{BF}}[B; A] + \text{boundary term},$$

with the coefficient chosen so that the flux $\frac{1}{2\pi} \int dB$ has unit charge under $A$. Having set up our notation, let us be clear about what we mean by ‘duality’. We will call two theories dual to each other in the infrared, if their partition functions are equal. Establishing this equality is non-trivial but, fortunately, at least algorithmic. Following [20], to dualise a given theory, say theory $A$, with action $S_A[\phi]$, we:

\(^3\)In this, and what follows, expressions like $A \, dA$ are shorthand for $A \wedge dA$ which, in turn, is the differential form notation for the indexed expression $\epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$. 

5
• Gauge an internal symmetry of the theory, which introduces a dynamical gauge field \( a \) over which a functional integral is to be performed.

• Constrain \( a \) to be pure gauge by imposing the flatness condition \( f = da = 0 \) through a Lagrange multiplier \( \Lambda \) in the path integral. This ensures that we are not adding any additional degrees of freedom. Taken together, this produces a master action \( S_{\text{master}}[\phi; a; \Lambda] \).

• Integrating out \( \Lambda \), followed by \( a \) results in the original theory \( A \). On the other hand, reversing the order and integrating out \( a \), followed by \( \phi \) produces a new theory, \( B \) say, with action \( S_B[\Lambda] \), for the dual field \( \Lambda \).

This entire process, including subtleties introduced into the path integral measure through gauge-fixing and a Fadeev-Popov determinant, is then summarised, rather tersely, as \( S_A[\phi] \leftrightarrow S_B[\Lambda] \). To illustrate this construction let us now review the 3-dimensional duality web, following closely the form derived in [1,16,22]. As stated above, the basic web consists of three interconnected dualities - bose-bose, fermi-fermi and bose-fermi - and while it is true that any of these can serve as a seed for the full web, we will usually begin with the last and work our way back.

### 2.1 Bose-Fermi

With this in mind, let us define the basic 3-dimensional bosonization duality. In its mass deformed version\(^4\), as defined in [22], it is a map between a massive Dirac fermion coupled to an external gauge field \( A = A_\mu \, dx^\mu \) whose Chern-Simons flux we denote \( \text{CS}[A] \), and a complex scalar coupled to the same gauge field. At the level of the partition function, this is a statement of equality between the fermion partition function,

\[
Z_{\text{fermion}}[A; m] \equiv \int D\psi \exp \left( iS_{\text{fermion}}[\psi; A] \right); \quad (2.5)
\]

and that of the scalar with flux attached,

\[
Z_{\text{scalar+flux}}[A] = \int DaD\phi \exp \left( iS_{\text{scalar}}[\phi; a] + i\text{CS}[a] + iS_{\text{BF}}[a, A] \right) \quad (2.6)
\]

via the relation

\[
Z_{\text{fermion}}[A; m]e^{-\frac{i}{2}\text{CS}[A]} = Z_{\text{scalar+flux}}[A]. \quad (2.7)
\]

\(^4\)Here the mass term for the fermion is to be thought of as a regulator taking one away from the quantum critical point at \( m = 0 \). We shall have more to say about this in the 4-dimensional case later.
At this juncture, there are a few points that warrant mention. The first is that (2.7) is, as alluded to earlier, really the result of implementing the full duality algorithm above. The second is that this equality of partition functions holds in the infra-red limit in which we send the quartic scalar coupling $\alpha \to \infty$ while simultaneously tuning the scalar mass to zero in order to hit the Wilson-Fisher fixed point. Writing the complex scalar as $\phi = \phi_0 e^{i\theta}$, the relevant relation is then really

$$Z_{\text{scalar+flux}}[A] = \lim_{\alpha \to \infty, E \ll \alpha} \int \mathcal{D}a \mathcal{D}\phi_0 \mathcal{D}\theta \mathcal{D}\sigma \times \exp \left\{ iS_{\text{scalar}}[\theta, a; \phi_0] + iS_{\text{CS}}[a] + iS_{\text{BF}}[a, A] ight\} = \int \mathcal{D}a \mathcal{D}\theta \exp \left( iS_{\text{scalar}}[\theta, a; \phi_0] + iS_{\text{CS}}[a] + iS_{\text{BF}}[a, A] \right),$$

(2.8)

with $S_{\text{scalar}}[\theta, a; \phi_0] = -\frac{1}{2} \int d^3x \, (\partial_\mu \phi_0)^2 + \sigma (\phi_0^2 - m) + \frac{\sigma^2}{2\alpha}$. Incidentally, this same limit suppresses the Maxwell term $(da)^2/4e^2$ for the dynamical gauge field relative to its Chern-Simons term since $e \to \infty$. Finally, to construct the duality web, we will need the time-reversed version of (2.7). Since, under the time-reversal operator, each of the BF and CS terms pick up a minus sign only, we find that

$$Z_{\text{fermion}}[A] e^{+\frac{i}{2}S_{\text{CS}}[A]} = \tilde{Z}_{\text{scalar+flux}}[A] \equiv \int \mathcal{D}\phi \mathcal{D}a \exp \left( iS_{\text{scalar}}[\phi, a] - iS_{\text{CS}}[a] - iS_{\text{BF}}[a, A] \right).$$

(2.9)

Equations (2.7) and (2.9) can then be taken as seeds for the remaining fermi-fermi and bose-bose dualities as follows. At this point, they are postulated, but in the final subsection of this section we will show how they were proven, in a certain limit, in [1].

### 2.2 Fermi-Fermi

The fermi-fermi thread of the duality web builds on the seminal conjecture of Son that the composite fermion that so successfully explains a number of phenomena associated to the fractional quantum Hall effect is a Dirac fermion. In a nutshell, this duality posits that a massless Dirac fermion $\psi$, coupled to an external field $A$ is dual to a composite Dirac fermion $\chi$ coupled to a dynamical gauge field $a$ which is itself coupled to $A$ through a BF coupling. The latter theory is simply QED$_3$ with a single fermion flavour and an additional BF coupling between the external gauge
field and a fluctuating one\textsuperscript{5}. It is described by the partition function

\[
Z_{\text{QED}_3}[A; m] = \int \mathcal{D} \chi \mathcal{D} a \exp \left( i S_{\text{fermion}}[\chi, a] + \frac{i}{2} S_{\text{BF}}[a, A] \right),
\]

so that Son’s duality reads

\[
Z_{\text{QED}_3}[A] = Z_{\text{fermion}}[A].
\]

This relation follows from the basic bosonization duality as follows:

- Starting from the basic bosonization relation (2.7), we promote external gauge field $A$ to a dynamical one $a$ with a BF-coupling $\frac{i}{2} S_{\text{BF}}[\bar{\alpha}; A]$ to a new external field, also denoted by $A$ and then integrate over $a$ in the functional integral.

- With this, the left-hand side of (2.7) becomes $Z_{\text{QED}_3}[A]$ as above. The right hand side on the other hand becomes

\[
Z_{\text{scalar+fluxes}}[A] = \int \mathcal{D} \phi \mathcal{D} \bar{\alpha} \exp \left( i S_{\text{scalar}}[\phi; \bar{\alpha}] + i S_{\text{CS}}[\bar{\alpha}] ight.
\]

\[
+ \left. i S_{\text{BF}}[\bar{\alpha}, a] + \frac{i}{2} S_{\text{BF}}[a, A] + \frac{i}{2} S_{\text{CS}}[\bar{\alpha}] \right).
\]

- Now, integrating over the statistical field $a$, using its equation of motion, $da = -(dA + 2d\bar{\alpha})$, substituting back into the action, and collecting terms gives

\[
e^{-\frac{i}{2} S_{\text{CS}}[A]} \int \mathcal{D} \phi \mathcal{D} \bar{\alpha} \exp \left( i S_{\text{scalar}}[\phi; \bar{\alpha}] - i S_{\text{CS}}[\bar{\alpha}] - i S_{\text{BF}}[\bar{\alpha}, A] \right),
\]

in which we recognise the functional integral as the scalar side of the time-reversed bosonization relation (4.1). Together with the contact interaction $e^{+\frac{i}{2} S_{\text{CS}}[A]}$ then, this equates finally to $Z_{\text{fermion}}[A]$, establishing the fermionic particle-vortex duality.

### 2.3 Bose-Bose

The final thread in the 3D duality web is the duality between two bosonic theories. The derivation of this duality hinges on an intermediate bosonization step that amounts to attaching flux to a fermion in the process transmuting it into a boson so it will be worth our while to clarify this first before moving on. To attach a background flux to a fermion, we start with the fermion partition function $Z_{\text{fermion}}[A]$, promote the external gauge field to a dynamical one, $b$ say, then couple the latter to attachments $\frac{i}{2} S_{\text{BF}}[\bar{\alpha}; A]$ to the functional integral.

\textsuperscript{5}Strictly speaking, one should call this “BF-QED$_3$”, but we will drop the “BF” in what follows, for simplicity.
another external gauge field $A$ through a BF coupling and, finally, integrate over $b$ in the functional integral. The result is that

$$Z_{\text{fermion+flux}}[A] = \int \mathcal{D} b \ Z_{\text{fermion}}[b] e^{-\frac{i}{2} S_{\text{CS}}[b] - i S_{\text{BF}}[b,A]} . \quad (2.14)$$

On the other hand, the basic bosonization duality (2.7), replaces $Z_{\text{fermion}}[b]$ with $Z_{\text{scalar+flux}}[b] e^{\frac{i}{2} S_{\text{CS}}[b]}$ in the functional integral, and since the 1-form $b$ appears linearly, the integration over $b$ can be carried out using its equation of motion $db = dA$ or, in the absence of any holonomies\footnote{We are glossing over some important, and technical points here since they are not relevant for our present discussion. [16] is an absolute treasure-trove of insight on this and other subtleties.}, substituting $b = A$ into the partition function to obtain $Z_{\text{scalar}}[A] e^{i S_{\text{CS}}[A]}$. In summary then, we have the new bosonization relation

$$Z_{\text{fermion+flux}}[A] = Z_{\text{scalar}}[A] e^{i S_{\text{CS}}[A]} , \quad (2.15)$$

which also has a handy time-reversed version where, as usual, we change the sign of the CS and BF couplings to give

$$Z_{\text{scalar}}[A] e^{-i S_{\text{CS}}[A]} = \tilde{Z}_{\text{fermion+flux}}[A] \equiv \int \mathcal{D} a \ Z_{\text{fermion}}[a] \exp \left( i \frac{1}{2} S_{\text{CS}}[a] + i S_{\text{BF}}[a,A] \right) . \quad (2.16)$$

Now we play exactly the same game with (2.15) in the form

$$Z_{\text{fermion+flux}}[A] e^{-i S_{\text{CS}}[A]} = Z_{\text{scalar}}[A] . \quad (2.17)$$

On the left hand side, promoting the background $A$ to a dynamical $b$, coupling in a new background field, $\tilde{A}$ and functionally integrating results in

$$\int \mathcal{D} a \mathcal{D} b \ Z_{\text{fermion}}[b] \exp \left( -i \frac{1}{2} S_{\text{CS}}[a] - i S_{\text{BF}}[a;b] - i S_{\text{CS}}[b] - i S_{\text{BF}}[b,A] \right) . \quad (2.18)$$

Again, since $b$ enters this expression linearly, we can integrate it out via its equation of motion, $db = dA - da$. With the no-holonomies caveat in place we can replace $b$ with $A - a$ in the partition function to find (on the left-hand side)

$$\int \mathcal{D} a Z_{\text{fermion}}[A] e^{\frac{i}{2} S_{\text{CS}}[a] + i S_{\text{BF}}[a,A] + i S_{\text{CS}}[A]} = \tilde{Z}_{\text{fermion+flux}}[A] e^{\frac{i}{2} S_{\text{CS}}[A]} = Z_{\text{scalar}}[A] . \quad (2.19)$$

On the right hand side of (2.15), the process of promoting the external gauge field to a dynamical one (and subsequent coupling to another external field) is another way of saying that we have *gauged* the global background $U(1)$. Consequently, the theory on the right hand side is simply scalar QED$_3$. Finally then, the boson-boson map reads

$$Z_{\text{scalar-\text{QED}_3}}[S] = Z_{\text{scalar}}[S] . \quad (2.20)$$

If the scalar is massless with a quartic self-interaction, this is the statement that a Wilson-Fisher scalar is dual to a gauged Wilson-Fisher scalar, otherwise known as particle-vortex duality.
2.4 A constructive ‘proof’ of the bosonization step

To prove [1] the bosonization step from the first subsection (and, by extension, the whole duality web, at least in the low energy limit to be defined shortly), one starts with the particle-vortex duality in the formulation of [15], which is the duality between the particle partition function

\[ Z_{\text{particle}}[\phi_0, A] = \int \mathcal{D} \theta e^{i S[\phi_0, \theta, A]} \]

\[ = \int \mathcal{D} \theta \exp \left[ -i \int d^3 x \frac{1}{2} \left( (\partial_\mu \phi_0)^2 + \phi_0^2 (\partial_\mu \theta_{\text{smooth}} + \partial_\mu \theta_{\text{vortex}} + A_\mu)^2 \right) \right] \quad (2.21) \]

and the vortex partition function

\[ Z_{\text{vortex}}[\phi_0, A] = \int \mathcal{D} \lambda \mu e^{i S_{\text{dual}}[\phi_0, \lambda, A]} \]

\[ = \int \mathcal{D} \lambda \mu \exp \left[ -i \int d^3 x \left( \frac{1}{2} (\partial_\mu \phi_0)^2 + \frac{1}{4(2\pi \phi_0)^2} \Lambda_{\mu\nu} \Lambda^{\mu\nu} \right. \right. \]

\[ + \left. \left. \frac{1}{2\pi} \epsilon^{\mu\nu\rho} \lambda_\mu \partial_\nu A_\rho + j_{\text{vortex}}^{\mu} \lambda_\mu \right) \right] \quad , \quad (2.22) \]

with \( \Lambda_{\mu\nu} \equiv \partial_\mu \lambda_\nu \). If, in addition we consider \( \phi_0 \) constant and at low energies where \( E \ll \phi_0^2 \), we can drop the first two terms in the above.

Then we add \( S_{CS}[A] + S_{BF}[A; C] \) to the actions in both particle and vortex path integrals (which adds “flux” to both sides), and then integrate over \( A_\mu \) (formerly the electromagnetic field, now integrated over, so it is a “statistical” gauge field) to obtain a function of \( C_\mu \), which now plays the role of the electromagnetic field. On the “particle” side of the duality, we obtain the scalar+flux side of the basic bosonization step that we wanted to prove,

\[ Z'_{\text{particle+flux}}[C] = \int \mathcal{D} A_\mu \mathcal{D} \theta \exp \left( i S_{\text{scalar}}[\theta, A; \phi_0] + i S_{CS}[A] + i S_{BF}[A; C] \right) \]

\[ = Z_{\text{scalar+flux}}[C] \quad , \quad (2.23) \]

while on the “vortex” side we obtain

\[ Z'_{\text{vortex+flux}}[C] = \int \mathcal{D} A_\mu \mathcal{D} \lambda_\mu \exp \left( i S_{BF}[, \lambda; A] + i S_{BF}[A; C] + i S_{CS}[A] + i \int d^3 x j^{\mu}_{\text{vortex}} \lambda_\mu \right) \]

\[ = Z_{\text{vortex+flux}}[C] \quad . \quad (2.24) \]
Since the action is quadratic in $A_\mu$, to carry out the integral over $A_\mu$ we can use its equation of motion, $dA = -(dC + d\lambda)$, leading to

$$Z'_{\text{vortex+flux}}[C] = \int D\lambda_\mu \exp \left( -iS_{\text{CS}}[\lambda] - iS_{\text{BF}}[\lambda, C] + i\int d^3x j^\mu_{\text{vortex}}\lambda_\mu \right).$$

(2.25)

After a redefinition of $\lambda_\mu \to \lambda_\mu = \sqrt{2}\tilde{\lambda}_\mu + C_\mu (-1 + 1/\sqrt{2})$, we finally obtain the gauge side of the BQ map, with a CS term for $C_\mu$ and a vortex term (which was not included in the original duality web),

$$Z'_{\text{vortex+flux}}[C] = Z'_{\text{gauge+flux}}[C] \exp \left[ -\frac{i}{2}S_{\text{CS}}[\lambda] - i\int d^3x j^\mu_{\text{vortex}}C_\mu \left( -1 + \frac{1}{\sqrt{2}} \right) \right].$$

(2.26)

The BQ map defined in [8, 20], is between the fermion partition function coupled to the field $C_\mu$ and a gauge action coupled to the same,

$$Z_{\text{fermion}}[C; m] = Z_{\text{gauge}}[C] = \int D\lambda_\mu \exp \left( -\frac{i}{2k_3}\epsilon^{\mu\nu\rho}\lambda_\mu \partial_\nu \lambda_\rho - i\epsilon^{\mu\nu\rho}\lambda_\mu \partial_\nu C_\rho \right)
=\int D\tilde{\lambda}_\mu \exp \left( -2iS_{\text{CS}}[\tilde{\lambda}] - iS_{\text{BF}}[\tilde{\lambda}, C] \right).$$

(2.27)

In order to find the gauge+vortex partition function above, we must replace $C_\mu \to C_\mu + \partial_\mu \theta_{\text{vortex}}$, giving

$$Z_{\text{fermion+flux}}[C; m] = \int D\psi D\bar{\psi} \exp \left[ i\int \bar{\psi} (\partial + m + C + \partial \theta_{\text{vortex}}) \psi \right],$$

$$Z_{\text{gauge+flux}}[C] = \int D\tilde{\lambda}_\mu \exp \left[ -2iS_{\text{CS}}[\tilde{\lambda}] - iC_{\text{BF}}[\tilde{\lambda}, C] - i\int d^3x j^\mu_{\text{vortex}}\tilde{\lambda}_\mu \right],$$

(2.28)

$$Z_{\text{fermion+flux}}[C; m] = Z_{\text{gauge+flux}}[C].$$

In summary then, we get the basic bosonization duality step we wanted plus a vortex component,

$$Z_{\text{scalar+flux}}[C] = Z'_{\text{particle+flux}}[C] = Z'_{\text{vortex+flux}}[C],$$

$$= Z'_{\text{gauge+vortex}}[C] \exp \left[ -\frac{i}{2}S_{\text{CS}}[C] - i\int d^3x j^\mu_{\text{vortex}}C_\mu \left( -1 + \frac{1}{\sqrt{2}} \right) \right],$$

(2.29)

$$= Z'_{\text{fermion+vortex}}[C] \exp \left[ -\frac{i}{2}S_{\text{CS}}[C] - i\int d^3x j^\mu_{\text{vortex}}C_\mu \left( -1 + \frac{1}{\sqrt{2}} \right) \right].$$

We see that the basic bosonization step was proven constructively, at least in the low-energy limit $E \ll m = \phi^2_0$, $E \ll \alpha$, the same parameter as in (2.8). Of course, this does not imply a proof outside this limit (for $m \to 0$, in particular), which is what one really wants for the duality web.
In four dimensions, we will follow this same construction of the basic bosonization step, in order to establish a duality web. We will find that it only holds in certain limits so there again, we do not claim to prove the 4-dimensional duality web, but rather to define it, constructively in the appropriate limit.

3 Four-dimensional dualities

So, the 3-dimensional duality web furnishes an interconnected set of relations between bosonic and fermionic theories. As we have reviewed in the previous section, there are three such classes of relations; bose-bose, fermi-fermi and bose-fermi, together with their time-reversed counterparts. From this construction, we have learnt a few interesting lessons; the first is that the Chern-Simons coupling, peculiar to odd-dimensional spacetimes, is a key player in this construction and the second is that, while any of the dualities can seed the whole web, it really is easiest to begin with a bosonization. Each of these is important to keep in mind for extending the web away from three dimensions. In this section we will follow the procedure laid out in the previous subsection in order to directly construct a basic 4-dimensional bosonization step analogous to the one in three dimensions, which will then be used to seed a 4 dimensional web.

The ingredients for this construction are the 4-dimensional analogs of particle-vortex duality and the bose-fermi duality that we call the BQ map, described in [20]. Fortunately for us, the latter was already constructed in [20]. The analog of particle-vortex duality is, however, a little more subtle. Particle-vortex duality (see, for example, [15]) in three dimensions is, of course, based on the Poincaré duality between a scalar and a 1-form gauge field. One dimension higher, a scalar is Poincaré dual to a 2-form “gauge field” corresponding to the antisymmetric tensor $B_{\mu\nu}$. Physically, this would point to a particle/vortex-string duality. An attempt at such a duality was considered in [23]) in the context of disordered superfluids in higher dimensions where it was shown that a particle-vortex duality persists in any $D \geq 3$ with the role of the “vortex” played by codimension-2 branes. While certainly intriguing, for our purposes of constructing a 4-dimensional duality web, we construct a new version of this 4 dimensional particle-vortex-type duality inspired by the related construction in [21]. It will be instructive to review the latter before proceeding further.
3.1 A 4 dimensional fermi-fermi duality

In an interesting recent article [21], the author argues for the equality between the actions for a Dirac fermion and a composite fermion in 4-dimensional spacetime. The argument starts from the action for a massive Dirac fermion coupled to a gauge field

\[ S_1[\psi, A_\mu] = \int d^4x \left[ \bar{\psi} \Gamma^\mu (\partial_\mu + iA_\mu) \psi + m \bar{\psi} \psi \right]. \]  

(3.1)

Integration of the massive fermion and defining, as usual, \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) results in a low-energy effective action

\[ S_{\text{eff}}^1[A_\mu] = \frac{1}{32\pi} \int d^4x \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}, \]  

(3.2)

of the \( \theta F \tilde{F} \) form. Here, the mass \( m < 0 \) of the fermion is to be thought of as a regulator to be removed at the end of the computation. Similarly, the action

\[ S_{\text{eff}}^2[A_\mu, a_\mu, B_{\mu\nu}] = \int d^4x \epsilon^{\mu\nu\rho\sigma} \left[ \frac{1}{32\pi} F_{\mu\nu} f_{\rho\sigma} - \frac{1}{8\pi} B_{\mu\nu} (F_{\rho\sigma} + f_{\rho\sigma}) + \frac{1}{32\pi^2} H_{\mu\nu\rho} H^{\mu\nu\rho} \right], \]  

(3.3)

with \( f_{\mu\nu} \equiv \partial_\mu a_\nu - \partial_\nu a_\mu \), \( H_{\mu\nu\rho} \equiv \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}, \) and where \( \chi \) is a real parameter, can be thought of as the low-energy effective action, obtained by integrating out the massive fermions in the action for a composite neutral Dirac fermion, \( \Psi \) coupled to an emergent gauge field \( a_\mu \) and a new antisymmetric tensor gauge field \( B_{\mu\nu} \),

\[ S_2[\Psi, A_\mu, a_\mu, B_{\mu\nu}] = \int d^4x \left[ \bar{\Psi} \Gamma^\mu (\partial_\mu + ia_\mu) \Psi + m \bar{\Psi} \Psi - \frac{1}{8\pi} \epsilon^{\mu\nu\rho\sigma} B_{\mu\nu} (F_{\rho\sigma} + f_{\rho\sigma}) + \frac{1}{32\pi^2} \chi H_{\mu\nu\rho} H^{\mu\nu\rho} \right]. \]  

(3.4)

Here again the fermion mass, \( m < 0 \) is interpreted as a regulator to be removed at the end of the calculation. We note that, from the first two terms in \( S_{\text{eff}}^2 \), \( B_{\mu\nu} \) has dimension 2, which means that the last term is a dimension 6 operator, with \( \chi \) a dimension 2 parameter. The upshot of this is that at low energies we can ignore the \( H^2 \) term in the effective action. Consequently, \( B_{\mu\nu} \) acts as a Lagrange multiplier that enforces the constraint \( F_{\mu\nu} = -f_{\mu\nu}, \) or since the gauge fields are Abelian and there are no nontrivial topological issues lurking, \( A_\mu = -a_\mu. \) The implication then is that \( S_{\text{eff}}^2 \) and \( S_{\text{eff}}^1 \) are identical. It is then concluded in [21], correctly, that the Dirac fermion and composite fermion actions (3.1) and (3.4) are equal. Palumbo

\[ ^7 \text{In this section we retain the indices in various expressions to disambiguate between contractions with the metric and those involving the completely anti-symmetric tensor } \epsilon_{\mu\nu\lambda\sigma}. \]
goes further and shows that, at the 3-dimensional boundary of the 4-dimensional spacetime, the effective action

\[ S_1^{\text{eff}}[A_\mu] = \int_{M_4} d\mathcal{L}_{CS} = S_{CS}[A] = \frac{1}{8\pi} \int_{\partial M_4} A dA, \]

(3.5)

while the effective action

\[ S_2^{\text{eff}}[A_\mu, a_\mu, B_{\mu\nu}] = \int_{M_4} d\mathcal{L}_{CS-BF} = S_{CS-BF}[A, a, b] = \frac{1}{8\pi} \int_{\partial M_4} \left( a da - b d(a + A) \right), \]

(3.6)

which is of the mixed CS-BF form. He then notes that the 3-dimensional Chern-Simons action \( S_{CS}[A] \) is the effective action for a Dirac fermion \( \psi \) coupled to \( A \), the trivial 3 dimensional version of \( S_1 \), while \( S_{CS-BF}[A, a, b] \) is the effective action for a composite neutral Dirac fermion coupled to a (dynamical) emergent gauge field \( a \), plus BF terms,

\[ S_2[\Psi, A, a, b] = \int \left[ \bar{\Psi} D_a \Psi + m \bar{\Psi} \Psi - \frac{1}{8\pi} b d(a + A) \right]. \]

(3.7)

All of this analysis is certainly correct but, while the similarities are certainly there, it is important to note that this is not the same as the duality in Son’s conjecture. There, in the composite fermion action, the BF term couples the dynamical gauge field \( a \) to the external field \( A \) via the term \( +\frac{1}{4\pi} A da \), and not \( b \) to \( a \) and \( A \) as in the above. Nevertheless, the idea proposed in [21] is quite neat and we take inspiration from it to propose an alternative. As we will show below, a similar 4-dimensional duality can be realised by forgetting about the fermions, concentrating instead on the low energy bosonic duality between (3.2) and (3.3), and embedding it into another, 4-dimensional analog of the 3-dimensional particle-vortex duality combined with a 4-dimensional version of the BQ map [20].

### 3.2 Bosonization in 4 dimensions

In order to understand how to lift 3-dimensional particle-vortex duality to four dimensions, note that the former is a map between the Lagrangian

\[ -\frac{(\partial_\mu \phi_0)^2}{2} - \frac{\phi_0^2}{2} (\partial_\mu \theta_{\text{vortex}} + \partial_\mu \theta_{\text{smooth}} + A_\mu)^2, \]

(3.8)

To put the action in this form, we can either first peel off the derivative acting on \( a + A \), then in the 3-dimensional action define the 2-form field strength \( B \equiv db \), and partially integrate to put \( S_{CS-BF} \) in the above form, or equivalently, first write \( B \) in terms of \( b \) and peel off the derivative acting on \( b \) to directly obtain the action.
\[-\frac{(\partial_\mu \phi_0)^2}{2} - \frac{H_{\mu\nu}}{4\phi_0^2} + \epsilon^{\mu\nu\rho\sigma} b_\mu \partial_\nu (a_\rho + A_\rho).\] (3.9)

Here we have written the complex scalar \( \phi(x) = \phi_0(x)e^{i\theta(x)} \) and split the phase \( \theta \) into a “smooth” part \( \theta_{\text{smooth}} \) and a part, \( \theta_{\text{vortex}} \), that encodes the non-trivial monodromy of the vortex. \( b_\mu \) is a Lagrange multiplier that comes from gauging the global \( U(1) \) symmetry of the first Lagrangian, and \( H_{\mu\nu} \) is its associated field strength. Finally, the (fluctuating) statistical gauge field \( a_\mu \equiv \partial_\mu \theta_{\text{vortex}} \). We have also dropped the Maxwell term for the external gauge field \( A_\mu \) in the low energy limit where the duality is valid. This form of the duality extends in a natural way to a 4-dimensional low energy duality between

\[-\frac{(\partial_\mu \phi_0)^2}{2} - \frac{H_{\mu\nu}}{4\phi_0^2} - \frac{1}{8\pi} \epsilon^{\mu\nu\rho\sigma} B_{\mu\nu} \partial_\rho (a_\sigma + A_\sigma) + \frac{1}{32\pi} \epsilon^{\mu\nu\rho\sigma} f_{\mu\nu} f_{\rho\sigma}.\] (3.10)

and

\[-\frac{(\partial_\mu \phi_0)^2}{2} - \frac{\phi_0^2}{2} (\partial_\mu \theta + A_\mu)^2 + \frac{1}{32\pi} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}.\] (3.11)

Here again, we use the low energy condition to ignore the kinetic term for \( B_{\mu\nu} \), so this is consistent with the 3-dimensional case. The only extra ingredient is the existence of the \( \theta \)-term. As it stands, this is still a boson-boson duality. Fortunately, the 4-dimensional Burgess-Quevedo (BQ) map in [20] provides a relation between a fermion coupled to an external \( U(1) \) gauge field and a 2-form field \( B_{\mu\nu} \) (which is dual to a pseudoscalar in four dimensions) coupled to the same gauge field so, by composing these two duality relations, \textit{like we saw that we did when constructively proving the bosonization step in 3 dimensions}, we should get the fundamental bosonization step for the 4-dimensional duality web.

Before proceeding further, let’s pause to make a comment. After the dust has settled, we will see that the combined BQ + PV map takes a Dirac fermion to a complex scalar (modulo some topological fluxes), essentially because the BQ duality maps a Dirac fermion to a scalar, dual to the gauge field. In three dimensions, as far as degrees of freedom are concerned, this makes sense since a complex scalar has the same number of on-shell degrees of freedom (1 complex, or 2 real) as a Dirac fermion (2 complex components, reduced on-shell to 1). In four dimensions, however, the duality still maps a Dirac fermion to a complex scalar, only now, with the Dirac fermion carrying 2 complex degrees of freedom and the scalar carrying 1 complex degree of freedom, there appears to be a mismatch of on-shell degrees of freedom. How then do we make sense of the duality? It refers to the response to the outside coupling (electromagnetism, with source \( A_\mu \)), as understood by Burgess and Quevedo.
is to say, we understand it as the equality of the partition functions for the systems coupled to an electromagnetic source $A_\mu$. This was precisely the case in the 3-dimensional web, only there the number of degrees of freedom of the systems coupled to external electromagnetic source were the same. The latter is nice to have, but not necessary.

In more detail, the particle/vortex-string duality in 4-dimensional spacetime is an equality between the partition functions for “particle” variables,

$$Z_{\text{particle}}[A] = \int \mathcal{D}\theta \exp \left\{ i \int d^4x \left[ -\frac{(\partial_\mu \phi_0)^2}{2} - \frac{\phi_0^2}{2} (\partial_\mu \theta + A_\mu)^2 + \frac{1}{32\pi} \epsilon^{\mu \nu \rho \sigma} F_{\mu \nu} F_{\rho \sigma} \right] \right\}, \quad (3.12)$$

and that for “vortex” variables,

$$Z_{\text{vortex}}[A] = \int \mathcal{D}a_\mu \mathcal{D}B_{\mu \nu} \exp \left\{ i \int d^4x \left[ -\frac{(\partial_\mu \phi_0)^2}{2} - H_{\mu \nu \rho}^2 4\phi_0^2 \right. \right. \left. - \frac{1}{8\pi} \epsilon^{\mu \nu \rho \sigma} B_{\mu \nu} \partial_\rho (a_\sigma + A_\sigma) + \frac{1}{32\pi} \epsilon^{\mu \nu \rho \sigma} f_{\mu \nu} f_{\rho \sigma} \right\}. \quad (3.13)$$

When $\phi_0$ is constant, and at energies $E \ll \phi_0$, we can drop the two first terms, just as in three dimensions. Then, imitating what we did in three dimensions, we add some terms like $S_{CS}[A]$ and $S_{BF}[A,C]$ and integrate over $A_\mu$ to get a function of $C_\mu$ only. On the vortex side, modulo some subtlties, this should furnish the gauge side of the BQ map, while the particle side would define the scalar part of the basic bosonization step of the duality web. The $A_\mu$ integration produces, among others, an $\epsilon^{\mu \nu \rho \sigma} B_{\mu \nu} G_{\rho \sigma}$ term, with $G_{\mu \nu} = \partial_\mu C_\nu$. More generally though, in 4-dimensional spacetime, consider the situation when the Maxwell terms will dominate over the topological terms $\epsilon FF$ and $\epsilon FG$, so the latter can be ignored in most situations, i.e.,

$$\frac{1}{g_4^3} \left( F_{\mu \nu} F^{\mu \nu} + F_{\mu \nu} G^{\mu \nu} \right) + \frac{1}{32\pi} \epsilon^{\mu \nu \rho \sigma} \left( F_{\mu \nu} F_{\rho \sigma} + F_{\mu \nu} G_{\rho \sigma} \right), \quad (3.14)$$

with $1/g_4^2 \gg \theta$. Absorbing the $g_4$ into $A_\mu$ and $C_\mu$, and the $1/(4\pi)$ into $B_{\mu \nu}$ and carrying out the path integral over $A_\mu$ in the partition function, we find

$$F_{\mu \nu} = \frac{\epsilon_{\mu \nu \rho \sigma} B^{\rho \sigma} - G_{\mu \nu}}{2}, \quad (3.15)$$

which, when replaced in the action gives

$$-B_{\mu \nu} B^{\mu \nu} + \frac{1}{64\pi} \epsilon^{\mu \nu \rho \sigma} B_{\mu \nu} G_{\rho \sigma} - \frac{1}{4} G_{\mu \nu} G^{\mu \nu}. \quad (3.16)$$
However, it is a well-known fact that, unlike the local CS and BF terms in three dimensions, in $D \geq 4$, the BQ map [20] produces a non-local $\Box^{-1}$ interaction. Specifically,

$$Z_{\text{fermion}}[C, m] = Z_{\text{gauge}}[C]$$

$$= \int \mathcal{D} B_{\mu\nu} \exp \left( -i \int d^4 x \left[ \frac{1}{2k_4} \epsilon^{\mu\nu\rho\sigma} \partial_{\nu} B_{\rho\sigma} + \epsilon^{\mu\nu\rho\sigma} C_{\mu} \partial_{\nu} B_{\rho\sigma} \right] \right)$$

$$= \int \mathcal{D} B_{\mu\nu} \exp \left( -i \int d^4 x \left[ \frac{1}{3k_4} H_{\mu\nu\rho} \Box H_{\mu\nu\rho} + \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} B_{\mu\nu} G_{\rho\sigma} \right] \right) .$$

(3.17)

While this expression doesn’t look much like what we had above, it is in fact equivalent. To see this, let $\Omega_{\mu} = \epsilon^{\mu\nu\rho\sigma} \partial_{\nu} B_{\rho\sigma}$. Then, the path integration over $B_{\mu\nu}$ produces the $\Omega$-equation of motion,

$$\Omega_{\mu} = -k_4 \Box C_{\mu} .$$

(3.18)

When replaced back into the path integral, this results in a $+ \frac{1}{2} k_4 C_{\mu} \Box C_{\mu}$ term. On the other hand, the same variation with respect to $B_{\mu\nu}$ in (3.16) gives $B_{\mu\nu} = + \frac{1}{4} \epsilon_{\mu\nu\rho\sigma} G_{\rho\sigma}$ and a corresponding contribution of $+ \frac{1}{4} C_{\mu\nu} \Box G_{\mu\nu}$ in the path integral. In Lorenz gauge $\partial^{\mu} C_{\mu} = 0$, this then reduces to the same action as above. To summarise, we have obtained an action equivalent to that in the gauge side of the BQ map in four dimensions. Note that the action in $Z_{\text{gauge}}[S]$ in the BQ map is 4-dimensional, but it is easy to imagine adding a 3-dimensional boundary term of the form $\epsilon^{\mu\nu\rho} b_{\mu} \partial_{\nu} b_{\rho}$, which can be neglected in the bulk, but not on the boundary, and where $b_{\mu}$ is path integrated in three dimensions. This would then facilitate a dimensional reduction to three dimensions.

Now, connecting the pieces of this argument; the scalar part of this basic bosonization step is given by the particle plus flux side of the particle/vortex-string duality,

$$Z_{\text{scalar+flux}} \equiv Z_{\text{particle+flux}}[C] = \int \mathcal{D} A_{\mu} \mathcal{D} \theta \exp \left\{ i \int d^4 x \left[ -\frac{1}{2} \left( \partial_{\mu} \phi_0 \right)^2 - \frac{\phi_0^2}{2} \left( \partial_{\mu} \theta + A_{\mu} \right)^2 \right] + \frac{1}{16\pi} \epsilon^{\mu\nu\rho\sigma} [F_{\mu\nu} F_{\rho\sigma} + F_{\mu\nu} G_{\rho\sigma}] + \frac{1}{g_4^2} [F_{\mu\nu} F_{\mu\nu} + F_{\mu\nu} G_{\mu\nu}] \right\} ,$$

(3.19)

where we have added an $\epsilon FG$ term that matches the $\epsilon FF$ already present (and which, as we point out above, is subleading to the Maxwell term at small $g_4$ in four dimensions). Note that the first two terms in the action correspond to a complex scalar coupled to $A_{\mu}$ at the fixed point, as in the 3-dimensional story. The fermion
side of the duality starts with the vortex + flux side,

\[
Z'_{\text{vortex+flux}}[C] = \int \mathcal{D}A_\mu \mathcal{D}B_{\mu\nu} \mathcal{D}a_\mu \exp \left\{ i \int d^4x \left[ -\frac{(\partial_\mu \phi_0)^2}{2} - \frac{H^2_{\mu\nu}(B)}{4\phi_0^2} - \frac{1}{8\pi} \epsilon^{\mu\nu\rho\sigma} B_{\mu\nu} \partial_\rho (a_\sigma + A_\sigma) \right. \right.
\]
\[
+ \frac{1}{32\pi} \epsilon^{\mu\nu\rho\sigma} (f_{\mu\nu} f_{\rho\sigma} + f_{\mu\rho} G_{\rho\sigma}) + \frac{1}{g_4^2} (F_{\mu\nu} F^{\mu\nu} + F_{\mu\nu} G^{\mu\nu}) \left. \right\} ,
\]
(3.20)

where the first two terms are neglected in matching with the gauge side of the BQ map, and where we have converted $\epsilon FG$ from the particle side into a $\epsilon fG$ term on the vortex side, since at low energies $a_\mu = -A_\mu$. We note that the 3-dimensional vortex current, $j^{\mu}_{\text{vortex}} C_\mu$ which came from $j^{\mu}_{\text{vortex}} \lambda_\mu$ corresponds to the term $\epsilon^{\mu\nu\rho\sigma} B_{\mu\nu} \partial_\rho \partial_\sigma \theta_{\text{vortex}}$ in four dimensions, with $\partial_\sigma \theta_{\text{vortex}}$ standing in for the statistical gauge field $a_\sigma$. Also note that for a vortex (or, more precisely in four dimensions, a vortex string) $\partial_\mu \theta_{\text{vortex}} + A_\mu = 0$. Finally, carrying out the $A_\mu$ and $B_{\mu\nu}$ path integrals (and neglecting the $\epsilon B \partial a$ term as small relative to the Maxwell terms) leads to the fermion partition function,

\[
Z_{\text{fermion}} = \int \mathcal{D}a_\mu \left\{ Z_{\text{fermion}}[C; m] \times \exp \left[ + \frac{1}{32\pi} \epsilon^{\mu\nu\rho\sigma} (f_{\mu\nu} f_{\rho\sigma} - f_{\mu\rho} G_{\rho\sigma}) \right] \right\} ,
\]
(3.21)

precisely the fermion side of the basic bosonization step, as anticipated.

### 3.3 Dimensional reduction to 3 dimensions

Let us check now how these statements translate into three dimensions. First, we would like to understand how the $\epsilon B \partial A$ term in four dimensions (in the vortex side of the particle-vortex duality) reduces to the correct $\epsilon b A$ term in three dimensions. To state the question more precisely, suppose that the boundary is in the $x^4$ direction in the 4-dimensional spacetime and take (with $i, j = 1, 2, 3$)

\[
B_{4i} = \partial_i b_i , \quad B_{ij} = 0 , \quad a_4 = A_4 = 0.
\]
(3.22)

With this ansatz, $\epsilon^{\mu\nu\rho} \partial_\mu b_\nu \partial_\rho (a_\sigma + A_\sigma) = \epsilon^{iijk} \partial_4 [b_i \partial_j (A_k + a_k)]$, while the field strength for $B_{\mu\nu}$ satisfies

\[
H_{4ij} = \partial_i B_{4j} + \partial_j B_{4i} + \partial_j B_{4i} = \partial_4 (\partial_i b_j - \partial_j b_i) \equiv \partial_4 k_{ij} ,
\]
(3.23)

with all remaining $H_{ijk} = 0$. Moreover, since

\[
\frac{H_{\mu\nu}\chi^H_{\mu\nu}}{\chi} = \partial_4 k_{ij} \partial_4 k_{ij} ,
\]
(3.24)
this term as still be safely neglected. This leaves just the nonlocal $H_{\mu\nu\rho} \frac{1}{\Box} H_{\mu\nu\rho}$ term. At low energies in three dimensions (and with an obvious definition of $\Box_{(D)}$),

$$\Box_{(4)} H_{ij} = (\partial_i^2 + \Box_{(3)} H_{i}) = (\partial_i^2 - l^2) H_{ij} \simeq \partial_i^2 H_{ij},$$

(3.25)

where small energies means $k^2 \ll \partial_i^2$. From this last expression then we deduce that in this limit, $1/\Box_{(4)} \simeq 1/\partial_i^2$, so

$$H_{\mu\nu\rho} \frac{1}{\Box_{(4)}} H_{\mu\nu\rho} = H_{i} \frac{1}{\Box_{(4)}} H_{i} \simeq \partial_i^2 k_{ij} \frac{1}{\partial_i^2} \partial_i^2 k_{ij} = -k_{ij} k^{ij},$$

(3.26)

after an integration by parts. This term is clearly negligible in three dimensions. Indeed, integrating over the (finite) 4-dimensional bulk gives $\int dx \equiv \frac{1}{\phi_0^2}$, one of the terms that was neglected in the 3-dimensional calculation. Of course, the correct term to be added in the gauge side of the BQ map in three dimensions is $CS[b]$, which has to be added by hand as a boundary term in the 4-dimensional action. We have checked that the remaining terms in the action dimensionally reduces correctly, so that the basic 4-dimensional bosonization map reduces to the appropriate map in three dimensions.

4 Constructing the duality web

With all the pieces in place, we can now construct a 4-dimensional web of dualities, by a combination of the basic bosonization duality and its time-reversed version. In this section, we will demonstrate this by constructing the fermi-fermi duality and bose-bose duality through repeated application of the basic bosonization duality, imitating closely the 3-dimensional version in [16].

4.1 A 4 dimensional fermi-fermi duality

In order to imitate the 3-dimensional procedure in 4 dimensions, we need one more ingredient; the time-reversed version of the bosonization step. Reversing the time direction, gives the $\epsilon FF$ and $\epsilon FG$ terms a minus sign while leaving all the rest unchanged, so that the time-reversed version of bosonization is

$$\int \mathcal{D}a_{\mu} \left\{ \sum_{\text{fermion}} [C; m] \times \exp \left[ \frac{-1}{32\pi} \epsilon^{\mu\nu\rho\sigma} \left( f_{\mu\nu} f_{\rho\sigma} - f_{\mu\nu} g_{\rho\sigma} \right) \right] \right\} = Z_{\text{scalar+flux}} [C] = \sum_{A_{\mu}} \mathcal{D} \left[ i \int d^4 x \left[ \frac{1}{2} \left( \partial_\mu \phi_0 \right) \partial_\mu \phi_0 - \frac{\phi_0^2}{2} \left( \partial_\mu \theta + A_\mu \right)^2 \frac{-1}{32\pi} \epsilon^{\mu\nu\rho\sigma} \left( F_{\mu\nu} F_{\rho\sigma} + F_{\mu\nu} G_{\rho\sigma} \right) + \frac{1}{g_4^2} \left( F_{\mu\nu} F^{\mu\nu} + F_{\mu\nu} G^{\mu\nu} \right) \right] \right\}.$$

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To construct the 4-dimensional equivalent of Son’s fermi-fermi duality, we take the original bosonization duality, promote the external gauge field $C_\mu$ to a fluctuating one $\bar{\mathcal{A}}^\mu$, (with field strength $\bar{\mathcal{F}} = d\bar{\mathcal{A}}$), add the terms

$$\frac{1}{2} \bar{\mathcal{F}} \mathcal{F} - \beta \bar{\mathcal{F}} \mathcal{G} + \alpha \epsilon \bar{\mathcal{F}} \mathcal{G} + \frac{\gamma}{2} \epsilon \bar{\mathcal{F}} \mathcal{F},$$

(4.2)

with constants $\alpha, \beta, \gamma$ that will be fixed in due course, and then integrate out the $\bar{\mathcal{A}}^\mu$. We find that

$$\int \mathcal{D} \bar{\mathcal{A}}^\mu \mathcal{D} a^\mu \left\{ Z_{\text{fermion}}[\bar{A}; m] \times \exp \left[ \frac{1}{32\pi} \epsilon^{\mu\nu\rho\sigma} \left( f_{\mu\nu} f_{\rho\sigma} - f_{\mu\nu} \bar{F}^\rho \bar{F}^\sigma \right) + \alpha \bar{F}_{\mu\nu} \mathcal{G}^\rho \bar{G}^\sigma + \frac{\gamma}{2} \bar{F}_{\mu\nu} \mathcal{F}^\rho \mathcal{F}^\sigma \right] \right\}$$

$$= \int \mathcal{D} \bar{\mathcal{A}}^\mu \mathcal{D} A^\mu \mathcal{D} \theta \exp \left\{ i \int d^4 x \left[ -\frac{(\partial_\mu \phi_0)^2}{2} - \frac{\phi_0^2}{2} (\partial_\mu \theta + A^\mu)^2 + \frac{1}{32\pi} \epsilon^{\mu\nu\rho\sigma} \left( F^\mu_{\nu\rho}(A) F^\rho_{\nu\sigma}(A) + F^\mu_{\nu\rho}(A) F^\rho_{\nu\sigma}(\bar{A}) + \alpha \bar{F}_{\mu\nu} \bar{F}_{\rho\sigma} + \frac{\gamma}{2} \bar{F}_{\mu\nu} \bar{F}_{\rho\sigma} \right) + \frac{1}{2 \mathcal{g}_4^2} \left[ F_{\mu\nu} F^{\mu\nu} + 2 \beta F_{\mu\nu} \mathcal{G}^\mu \mathcal{G}^\nu - \beta^2 \mathcal{G}^\mu \mathcal{G}^\nu \right] \right] \right\}.$$ 

(4.3)
\[
\frac{3}{2} \bar{F}_{\mu \nu} G_{\rho \sigma} - \bar{F}_{\mu \nu} \bar{F}_{\rho \sigma} + \frac{1}{g_4^2} \left( \bar{F}_{\mu \nu} \bar{F}^{\mu \nu} - \bar{F}_{\mu \nu} G^{\mu \nu} \right) \}
\]
\[
= \int \mathcal{D} a_\mu \left\{ \right. \\
Z_{\text{fermion}}[C; m] \times \exp \left[ - \frac{1}{32\pi} \epsilon^{\mu \nu \rho \sigma} (f_{\mu \nu} f_{\rho \sigma} - f_{\mu \nu} G_{\rho \sigma} \\
+ \frac{1}{2} G_{\mu \nu} G_{\rho \sigma} - \frac{1}{4g_4^2} G_{\mu \nu} G^{\mu \nu} \right] \right. \\
+ \left. \frac{1}{2} G_{\mu \nu} G_{\rho \sigma} \right) \}.
\] (4.6)

It is this then that is the 4-dimensional equivalent of the 3-dimensional fermi-fermi duality of Son, and that will fulfill the analogous role in the 4 dimensional duality web.

### 4.2 Bose-bose duality

Following the strategy in [16], we need to find another version of the bosonization duality that can be used twice to obtain a bose-bose duality. To do so, we need to modify the basic bosonization step in order to find just the scalar partition function on one side. Continuing as above, we rename the external field \( C_\mu \rightarrow \bar{A}_\mu \), then add

\[
\frac{1}{32\pi} \epsilon^{\mu \nu \rho \sigma} G_{\mu \nu} \bar{F}_{\rho \sigma} + \frac{1}{g_4^2} G_{\mu \nu} \bar{F}^{\mu \nu}
\] (4.7)

to the action in the bosonization step, then path integrate over \( \bar{A} \). On the scalar side

\[
\int \mathcal{D} \bar{A}_\mu \mathcal{D} A_\mu \mathcal{D} \theta \exp \left\{ i \int d^4 x \left[ \frac{(\partial_\mu \Phi_0)^2}{2} - \frac{\Phi_0^2}{2} (\partial_\mu \theta - S_\mu)^2 \\
+ \frac{1}{32\pi} \epsilon^{\mu \nu \rho \sigma} (F_{\mu \nu} F_{\rho \sigma} + F_{\mu \nu} G_{\rho \sigma} + G_{\mu \nu} \bar{F}_{\rho \sigma}) + \frac{1}{g_4^2} \left( \bar{F}_{\mu \nu} F^{\mu \nu} + F_{\mu \nu} G^{\mu \nu} + G_{\mu \nu} \bar{F}^{\mu \nu} \right) \right] \right\}
\] (4.8)

As before, \( \bar{A}_\mu \) can be integrated out by solving for its equation of motion, which requires that \( F_{\mu \nu} = -G_{\mu \nu} \). On substituting, and doing the integral over \( A_\mu \) (against the delta function given by the integration of the Lagrange multiplier), we get

\[
\int \mathcal{D} \theta \exp \left\{ i \int d^4 x \left[ \frac{(\partial_\mu \Phi_0)^2}{2} - \frac{\Phi_0^2}{2} (\partial_\mu \theta - S_\mu)^2 + \frac{1}{32\pi} \epsilon^{\mu \nu \rho \sigma} G_{\mu \nu} G_{\rho \sigma} + \frac{1}{g_4^2} G_{\mu \nu} G^{\mu \nu} \right] \right\}
\] = \( Z_{\text{scalar}}(-C) \exp \left\{ i \int d^4 x \left[ + \frac{1}{32\pi} \epsilon^{\mu \nu \rho \sigma} G_{\mu \nu} G_{\rho \sigma} + \frac{1}{g_4^2} G_{\mu \nu} G^{\mu \nu} \right] \} \). (4.9)

Finally, adding in the requisite extra terms on the fermion side, we obtain

\[
Z_{\text{scalar}}(-C) = \int \mathcal{D} \bar{A}_\mu \mathcal{D} a_\mu Z_{\text{fermion}}(\bar{A}, m) \exp \left\{ i \int d^4 x \left[ + \frac{1}{32\pi} \epsilon^{\mu \nu \rho \sigma} \times \\
\times (f_{\mu \nu} f_{\rho \sigma} - f_{\mu \nu} G_{\rho \sigma} + \bar{F}_{\mu \nu} G_{\rho \sigma} - G_{\mu \nu} G_{\rho \sigma}) + \frac{1}{g_4^2} \left( \bar{F}_{\mu \nu} G^{\mu \nu} - G_{\mu \nu} G^{\mu \nu} \right) \right] \right\}.
\]
At this point we would proceed by using the above bosonization step: promoting the external field $C$ to a dynamical field, say, $b$; adding in a new external field $A$, as well as the appropriate terms in the exponent and integrating over $b$. We would then expect to find the same fermion side of the bosonization step, except in terms of the external field $A$ and a time-reversed version, equal to another scalar partition function. Carrying out these steps results, however, not in the time-reversed version, but the original partition function! This is so because we cannot change the sign of the $\epsilon_{\text{ff}}$ term, as would be needed for a time-reversed relation. To circumvent this problem, we will try to use the same bosonization step, only now in reverse instead of the time-reversed version. Adding only the $1/g_4^2$ terms (with $k_{\mu\nu} \equiv \partial_{[\mu} b_{\nu]}$)

$$\frac{1}{g_4^2} \left( \beta k_{\mu\nu} F^{\mu\nu} + \gamma k_{\mu\nu} k^{\mu\nu} - \mu F_{\mu\nu} F^{\mu\nu} \right)$$

(4.11)

to the action yields a fermion side of the equality that takes the form

$$\int \mathcal{D} b_\mu \mathcal{D} \bar{A}_\mu \mathcal{D} a_\mu \mathcal{Z}_{\text{fermion}}[\bar{A}, m] \exp \left\{ i \int d^4 x \left[ \frac{1}{2(1-\gamma)} \bar{F}_{\mu\nu} F^{\mu\nu} + \frac{1}{4(1-\gamma)} \bar{F}_{\mu\nu} \bar{F}^{\mu\nu} + \left( \frac{\beta^2}{4(1-\gamma)} - \mu \right) F_{\mu\nu} F^{\mu\nu} \right] \right\}.$$  

(4.12)

The equation of motion for $b_\mu$ on this side is

$$2(1-\gamma) k_{\mu\nu} = \bar{F}_{\mu\nu} + \beta F_{\mu\nu} + \frac{g_4^2}{32\pi} \epsilon_{\mu\nu\rho\sigma} \left( \bar{F}_{\rho\sigma} - f_{\rho\sigma} \right).$$

(4.13)

Substituting this into the fermion side of the equality gives an exponent

$$\frac{1}{g_4^2} \left[ \frac{\beta}{2(1-\gamma)} \bar{F}_{\mu\nu} F^{\mu\nu} + \frac{1}{4(1-\gamma)} \bar{F}_{\mu\nu} \bar{F}^{\mu\nu} + \left( \frac{\beta^2}{4(1-\gamma)} - \mu \right) F_{\mu\nu} F^{\mu\nu} \right]$$

$$+ \frac{1}{32\pi} \epsilon_{\mu\nu\rho\sigma} \left[ \frac{1}{2(1-\gamma)} \bar{F}_{\mu\nu} \bar{F}_{\rho\sigma} + \frac{\beta}{2(1-\gamma)} \bar{F}_{\mu\nu} F_{\rho\sigma} + f_{\mu\nu} f_{\rho\sigma} \right]$$

$$+ \frac{1}{2(1-\gamma)} f_{\mu\nu} [\bar{F}_{\rho\sigma} + \beta F_{\rho\sigma}] .$$

(4.14)

Now we note that in order to obtain the correct fermionic side of the bosonization step, we require that $-2\gamma = 2\mu = \beta \to \infty$. Indeed, in this case the exponent in the fermion side of the equality is

$$\frac{1}{g_4^2} \left( F_{\mu\nu} F^{\mu\nu} \right) + \frac{1}{32\pi} \epsilon_{\mu\nu\rho\sigma} \left( F_{\mu\nu} F_{\rho\sigma} + f_{\mu\nu} f_{\rho\sigma} - f_{\mu\nu} F_{\rho\sigma} \right),$$

(4.15)

exactly what we have above, with $C$ replaced by $A$, and with the extra terms on the scalar side. Finally then, we can put this together as the bose-bose duality,

$$\int \mathcal{D} C_\mu \mathcal{Z}_{\text{scalar}}[C] \exp \left\{ i \int d^4 x \left[ \frac{\beta}{g_4^2} \left( G_{\mu\nu} F^{\mu\nu} - \frac{1}{2} G_{\mu\nu} G^{\mu\nu} - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \right) \right] \right\} = \mathcal{Z}_{\text{scalar}}[-A] \exp \left\{ i \int d^4 x \left[ \frac{1}{32\pi} \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} + \frac{1}{g_4^2} F_{\mu\nu} F^{\mu\nu} \right] \right\}.$$ 

(4.16)
At this point, we need to point out, however, that this relation holds only in the admittedly, rather singular $\beta \to \infty$ (in addition to the $g_4 \to 0$) limit.

## 5 Conclusions

Even though the sparks were there for many years prior, it was the discovery of the 3-dimensional duality web in the summer of 2016 [11, 16, 17] that really re-ignited the area of low-energy dualities, both in the high energy community studying the structure of the gauge/gravity correspondence, as well a condensed matter community looking to probe the properties of novel new materials like topological insulators and superconductors. The resulting surge of activity has precipitated a plethora of new results at both low and high energies as well as an acceleration of the blurring of boundaries between the two disciplines\(^9\). Among these results was a wonderful article by Karch, Tong and Turner [9] that realised a 2-dimensional web by collecting several well-known dualities and connecting the dots with a non-trivial topological invariant; the Arf invariant that plays a similar role in two dimensions as the Chern-Simons invariant in three. Inspired by these 2- and 3-dimensional developments, as well as the more recent 4-dimensional fermi-fermi duality of Palumbo [21], in this article we demonstrate the existence of a similar 4-dimensional web of low energy, non-supersymmetric abelian dualities. As with the former dualities, the 4-dimensional web consists of bose-bose and fermi-fermi dualities as well as a 4-dimensional version of bosonization. Any of these can be used to seed the full web. In addition, we have shown that, through dimensional reduction, the 4 dimensional web can be related to the known 3 dimensional one. Since the latter was shown to descend to the identified 2 dimensional web in [9], our construction adds the final piece to what is really an inter-dimensional web that relates low-dimensional theories in four, three and two spacetime dimensions. As in the previous duality web constructions, there remain some outstanding issues. Among these, we list (in no particular order):

- As we pointed out above, the 4 dimensional duality web holds in the infrared. This much was to be expected from the 3 dimensional case. However, in addition to the low-energy limit, in order to realise the 4-dimensional bose-bose duality, and consequently the full web also, we needed to take the (singular) $\beta \to \infty$ limit and while the physics of the $g_4 \to 0$ limit is clear, it is unclear to us how to interpret the former. Clarifying this point would be important in

\(^9\)As an example, we point to the SYK model, currently one of the hottest topics in high energy physics because of its leading role in our current understanding of the black hole information problem and the fact that each of S, Y and K are condensed matter theorists.
understanding the limits of validity of the 4 dimensional web, as well as how it relates to the existing lower-dimensional duality webs.

- Soon after its discovery, a series of articles by Kachru et.al. [24,25] argued that 3-dimensional bosonization, understood as a seed of the duality web, could be derived from a single starting point: the 3 dimensional mirror symmetry that equates a free chiral superfield and $N = 2$ supersymmetric QED$_3$ with a single charged superfield. There, it was shown that when the supersymmetry was broken in a controlled way through a D-term deformation, the chiral duality flows precisely to the 3 dimensional bosonization duality

$$i \Psi \mathcal{D}_A \Psi - \frac{1}{8\pi} AdA \quad \leftrightarrow \quad |D_a \phi|^2 + \frac{1}{4\pi} ada + \frac{1}{2\pi} Ada,$$

in the infrared. From here, additional 3D dualities can be generated through the action of the modular group of transformations, $(S, T)$, which act on the Lagrangian of a general CFT as

$$S : \mathcal{L}(\Phi, A) \mapsto \mathcal{L}(\Phi, a) - \frac{1}{2\pi} B da$$

$$T : \mathcal{L}(\Phi, A) \mapsto \mathcal{L}(\Phi, A) + \frac{1}{4\pi} AdA.$$

While the methods utilised in [24, 25] are essentially 3-dimensional, it is interesting to note that the (2+1)-dimensional superspace can be obtained from an $N = 1$ superspace in (3+1) dimensions, by dimensional reduction. The resulting $N = 2$ superspace has the two basic superfields: A chiral superfield $(\phi, \psi, F)$, where $\phi$ is a complex scalar, $\psi$ a 2-component Dirac fermion and $F$ is an auxiliary complex field, and a vector superfield made up of a gauge field $A_\mu$, a real scalar $\sigma$, a 2-component Dirac fermion $\lambda$, and an auxiliary real scalar $D$. Given this, and our extensive knowledge of dualities in 4-dimensional supersymmetric theories [26], it begs the question as to whether a similar unified description of the 4-dimensional duality web exists?

- In addition to fuelling some impressive advances in the understanding of Chern-Simons matter [27–29], the 3-dimensional duality web furnishes a powerful non-perturbative framework within which to study a spectrum of strongly correlated quantum critical points. The obvious example here is Son’s explanation [30] of the particle-hole symmetry exhibited by fractional quantum Hall (FQH) states at $\nu = \frac{1}{2}$. More generally, since dual quantum critical points necessarily share a phase diagram, dualities can be used to diagnose the presence of exotic gapped phases of quantum matter. Two more examples that come to mind here are the construction of non-abelian Read-Rezayi FQH states given in [31]
and the study of so-called deconfined quantum critical points [32], exhibited by (2+1)-dimensional quantum magnets such as the non-compact $\mathbb{CP}^1$ sigma model coupled to a dynamical non-compact $U(1)$ gauge field $a$ through the covariant derivative $D_a = d - ia$,

$$L_{\mathbb{CP}^1} = (D_a z) (D_a z) - z^\dagger z.$$  \hspace{1cm} (5.4)

Key insights into such field theories were gained by realizing them on the boundary of (3+1)-dimensional bulk symmetry protected topological phases, with their 3 dimensional dualities descending from the 4 dimensional bulk theory. In this case - as in the examples presented in section 6 of [11] - the bulk theory plays an auxiliary role, regulating the boundary theory in a way that preserves the full internal symmetries of the IR fixed point. On the other hand, the IR dualities that are the subject of this article are genuinely 4-dimensional. Can they (or their non-abelian extensions) then be used to predict novel exotic gapped phases in (3+1) dimensions? Relatedly, boundaries remain an interesting issue for bosonization [33, 34] and the duality web more generally [35]. In this case, it would be of interest to know if the class of 4-dimensional theories we consider here can be used to construct phenomenologically relevant, anomaly-free theories in (2+1)-dimensions?

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