Three-Slit Interference: A Duality Relation

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Abstract

The issue of interference and which-way information is addressed in the context of 3-slit interference experiments. A new path distinguishability \( D_Q \) is introduced, based on Unambiguous Quantum State Discrimination (UQSD). An inequality connecting the interference visibility and which-path distinguishability, \( V + \frac{2D_Q}{3-D_Q} \leq 1 \), is derived which puts a bound on how much fringe visibility and which-way information can be simultaneously obtained. It is argued that this bound is tight.

Keywords: Complementarity, Wave-particle duality, Three-slit interference

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1. Introduction

The two-slit interference experiment with particles has become a cornerstone for studying wave-particle duality. So fundamental is the way in which the two-slit experiment captures the essence of quantum theory, that Feynman ventured to state that it is a phenomenon “which has in it the heart of quantum mechanics; in reality it contains the only mystery” of the theory [1]. That radiation and massive particles can exhibit both wave nature and particle nature in different experiment, had become quite clear in the early days of quantum mechanics. However, Niels Bohr emphasized that wave-nature, characterized by two-slit interference, and the particle-nature, characterized by the knowledge of which slit the particle passed through, are mutually exclusive [2]. In doing this he raised this concept to the level of a new fundamental principle.

Much later, this principle was made quantitatively precise by deriving a bound on to what extent the two natures could be observered simultaneously, independently by Greenberger and Yasin [3] and Englert [4]. Here one characterizes the extent to which one can distinguish which of the two slits the particle passed through by a quantity \( D \), and the visibility of the interference by \( V \). Then the relation putting a bound on the two is given by the so-called Englert-Greenberger-Yasin (EGY) duality relation [3, 4]

\[
V^2 + D^2 \leq 1.
\]  

Thus one can see that \( D \) and \( V \), which can take values between 0 and 1, are dependent on each other. A full which-way information (\( D = 1 \)) would surely wash out the interference (\( V = 0 \)). Eqn. [1] can be thought to be a quantitative statement of Bohr’s complementarity principle. It smoothly interpolates between the two extreme scenarios discussed by Bohr, namely, full which-way information and no which-way information.

A dramatic manifestation of Bohr’s complementarity principle has been demonstrated in the so-called quantum eraser. Here, “erasing” the which-way information after the particle has passed through the slits, allows one to recover the lost interference fringes [5].

Various aspects of complementarity were also explored by Jaeger, Shimony and Vaidman [6] where they also explored the case where the particle can follow multiple path, and not just two, before interfering. Bohr’s prin-
principle of complementarity should surely apply to mul-

ti-slit experiments too. However, one might wonder if one can find a quantitatively precise statement of it for multi-slit experiments. Various attempts have been made to formulate a quantitatively precise statement of complementarity, some kind of a duality relation, for the case of multibeam interferometers [7, 8, 9]. However, the issue is still not satisfactorily resolved. Beyond the well studied two-slit experiment, the simplest multi-beam case is the 3-slit interference experiment. The EOE relation was derived only in the context of 2-slit experiments, and one would like an analogous relation for the case of 3-slit experiments. That is the focus of this paper. Of late there has been a newly generated focus on the three-slit interference experiments [10, 11, 12], albeit for a different reason.

2. Three-slit interference

Three slit interference is somewhat more involved than its 2-slit counterpart simply due to the fact that while the two-slit interference is the result of interference between two parts coming from the two slits, in the 3-slit interference there are three parts which interfere in different ways. Assuming the slit separation to be $d$, there are two interferences from slit 1 and 2 and from slit 2 and slit 3. Both these interferences involve slit separation $d$. In addition there is an interference between parts from slit 1 and 3, which involves a slit separation of $2d$.

We expect additional complicacy in interpreting Bohr’s complementarity because if we know that the particle did not go through (say) slit 3, it may not imply complete loss of interference as there is still ambiguity regarding which of the other two slits, 1 or 2, the particle went through.

2.1. Which-way information

First we would like to have a way of knowing which of the three slits the particle passed through. Any which-way detector should have three states which should correlate with the particle passing through each slit. Let these states be $|d_1\rangle, |d_2\rangle, |d_3\rangle$, which correspond to particle passing through slits 1, 2 and 3, respectively. Without loss of generality we assume that the states $|d_1\rangle, |d_2\rangle, |d_3\rangle$ are normalized, although they may not necessarily be mutually orthogonal. The combined state of the particle and the which-way detector can be written as

$$|\Psi\rangle = |\psi_1\rangle|d_1\rangle + |\psi_2\rangle|d_2\rangle + |\psi_3\rangle|d_3\rangle,$$  \hspace{1cm} (2)

where $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle$ are the amplitudes of the particle passing through the slit 1, 2 and 3, respectively. If $|d_1\rangle, |d_2\rangle, |d_3\rangle$ are mutually orthogonal, we can find a Hermitian operator (and thus, a measurable quantity) which will give us different eigenvalues corresponding to $|d_1\rangle, |d_2\rangle, |d_3\rangle$, and thus to the particle passing through each of the three slit. In this case, which slit the particle went through, can be known without ambiguity.

If $|d_1\rangle, |d_2\rangle, |d_3\rangle$ are not mutually orthogonal, the three ways of particle going through the slits will not be fully distinguishable. One needs to define a distinguishability of the three different paths the particle. Defining distinguishability in multi-slit experiments has been a thorny issue [6, 7, 8, 9].

2.2. Unambiguous Quantum State Discrimination

As one can see from (2), the problem of distinguishing the three paths of the particle boils down to distinguishing between the three states $|d_1\rangle, |d_2\rangle, |d_3\rangle$. In the following we describe a well established method of unambiguously discriminating between two non-orthogonal quantum states, which goes by the name of Unambiguous Quantum State Discrimination (UQSD) [13, 14, 15, 16, 17]. If two states $|p\rangle, |q\rangle$ are not orthogonal, it is impossible to distinguish between the two with certainty. What UQSD does is to separate the measurement results into two categories. First category is the one in which the discrimination fails. The second one distinguishes between the two states without any error. Let the first system, whose states are $|p\rangle, |q\rangle$ interact with a second system.
which is initially in a state $|s_0\rangle$. The interaction and time evolution leads to the following entangled states

$$
U|p\rangle|s_0\rangle = a|p_1\rangle|s_1\rangle + \beta|p_2\rangle|s_2\rangle,
$$

$$
U|q\rangle|s_0\rangle = \gamma|q_1\rangle|s_1\rangle + \delta|q_2\rangle|s_2\rangle,
$$

where $U$ is a unitary operator such that $\langle s_1|s_2\rangle = 0$ and $\langle p_1|q_1\rangle = 0$. If one measures an observable of the second system which has two eigenstates $|s_1\rangle, |s_2\rangle$ with different eigenvalues, the result $|s_1\rangle$ lands us into a situation where the non-orthogonal states $|p\rangle, |q\rangle$ have been replaced by orthogonal $|p_1\rangle, |q_1\rangle$. The orthogonal states $|p_1\rangle, |q_1\rangle$ can be distinguished with hundred percent accuracy, thus distinguishing the original $|p\rangle, |q\rangle$ without error. However, the other result for system 2, $|s_2\rangle$, leads us to states $|p_2\rangle, |q_2\rangle$ which are not orthogonal, and the discrimination of $|p\rangle, |q\rangle$ fails. So, either the process fails, or it distinguishes between $|p\rangle, |q\rangle$ without error. The probability of successfully distinguishing between $|p\rangle$ and $|q\rangle$ depends on the constants $\alpha, \beta, \gamma, \delta$. It can be easily shown that the maximum probability of successfully distinguishing between $|p\rangle$ and $|q\rangle$ is given by [14]

$$
P = 1 - |\langle p|q\rangle|.
$$

It should be mentioned that if one tries to distinguish between $|p\rangle$ and $|q\rangle$ by doing projective measurements on them, the probability of distinguishing is less than or equal to that given by the above relation [14]. In other words, UQSD is the best bet for discriminating between two non-orthogonal states. This fact has also been experimentally demonstrated recently [18] [19] [20].

### 2.3. Distinguishability

The preceding analysis suggests a natural definition of distinguishability. In a two-slit experiment, if the combined state of the particle and the which way detector can be written as $|\Psi\rangle = |\psi_1\rangle|d_1\rangle + |\psi_2\rangle|d_2\rangle$, the two paths can be distinguished if the two states $|d_1\rangle, |d_2\rangle$ can be distinguished. The probability of successfully telling which slit the particle went through is just the probability with which $|d_1\rangle$ can be unambiguously distinguished. We thus define distinguishability of the two paths in a double-slit experiment as the maximum probability of unambiguously distinguishing between $|d_1\rangle$ and $|d_2\rangle$. Thus we define a new path distinguishability for a two-slit experiment as

$$
D_Q \equiv 1 - |\langle d_1|d_2\rangle|.
$$

We denote it by $D_Q$ to distinguish it from the $D$ used in [1].

UQSD has also been generalized to the case of N non-orthogonal states. The probability of unambiguously distinguishing between N non-orthogonal quantum states is bounded by [20]

$$
P_N \leq 1 - \frac{1}{N-1} \sum_{i\neq j} \sqrt{p_i p_j} |\langle \psi_i|\psi_j\rangle|,
$$

where $|\langle \psi_i|\rangle$ are the N non-orthogonal states, and $p_i$ are their respective a-priori probabilities. For three non-orthogonal states, the above reduces to

$$
P_3 \leq 1 - \frac{1}{3} (|\langle \psi_1|\psi_2\rangle| + |\langle \psi_2|\psi_3\rangle| + |\langle \psi_1|\psi_3\rangle|),
$$

where we have assumed $p_1 = p_2 = p_3 = 1/3$.

The entangled state [2] implies that the probability with which one can tell which of the three slits the particle went through, is the probability with which one can distinguish between $|d_1\rangle, |d_2\rangle, |d_3\rangle$. We define the path-distinguishability for the 3-slit experiment as the maximum probability with which one can distinguish between the three states $|d_1\rangle, |d_2\rangle, |d_3\rangle$, which is now given by

$$
D_Q \equiv 1 - \frac{1}{3} (|\langle d_1|d_2\rangle| + |\langle d_2|d_3\rangle| + |\langle d_1|d_3\rangle|),
$$

whose value lies in the range $0 \leq D_Q \leq 1$. We consider probabilities of all the three paths to be equal, simply because that is the case which gives the sharpest interference. Even in the two-slit interference, unequal beams reduce the visibility of interference.

### 2.4. Interference and which-way information

In order to obtain a tight bound on visibility of interference, given a particular amount of which-way information, we do a rigorous wave-packet analysis. Assuming a particle traveling along the z-direction and passing through the three-slit, with a slit separation $d$, and then interacting with a which-path detector through a unitary evolution. The evolution of the state is given by the time-dependent Schrödinger equation

$$
\text{i} \hbar \partial_t |\Psi(t)\rangle = \hat{H} |\Psi(t)\rangle,
$$

where $\hat{H}$ is the Hamiltonian of the system (the energy operator).
Our strategy is the following. The motion of the particle along the z-axis is redundant, as it only translates the position of the particle from the triple-slit to the screen. What is relevant is the motion and dispersion of the particle along the x-direction. Without going into the details of the which-way detector, we assume that it is a device having 3 states $|d_1\rangle$, $|d_2\rangle$, $|d_3\rangle$, which get entangled with the three amplitudes of the particle passing through slits 1, 2, and 3, respectively. This entanglement is a necessary condition for the which-way detector to get any information about which slit the particle went through.

Using this strategy, the combined state of the particle and the which-path detector, when the particle emerges from the triple-slit (time $t = 0$), will be of form

$$\psi(x, 0) = A \left( |d_1\rangle e^{-\frac{(x-d_1)^2}{4\sigma^2}} + |d_2\rangle e^{-\frac{(x-d_2)^2}{4\sigma^2}} + |d_3\rangle e^{-\frac{(x-d_3)^2}{4\sigma^2}} \right).$$

(10)

where $A = \frac{1}{\sqrt{3}}(2\pi e^2)^{-1/4}$.

After a time $t$, the state of the particle and the detector evolves to

$$\psi(x, t) = A \left( |d_1\rangle e^{-\frac{(x-d_1)^2}{4\sigma^2+2\alpha t/m}} + |d_2\rangle e^{-\frac{(x-d_2)^2}{4\sigma^2+2\alpha t/m}} + |d_3\rangle e^{-\frac{(x-d_3)^2}{4\sigma^2+2\alpha t/m}} \right),$$

(11)

where $A = \frac{1}{\sqrt{3}} \sqrt{\pi}(e^{iht/2m})^{-1/2}$.

The probability of finding the particle at position $x$ on the screen is given by

$$|\psi(x, t)|^2 = |A|^2 \left( e^{-\frac{(x-d_1)^2}{4\sigma^2+2\alpha t/m}} + e^{-\frac{(x-d_2)^2}{4\sigma^2+2\alpha t/m}} + e^{-\frac{(x-d_3)^2}{4\sigma^2+2\alpha t/m}} + 2|\langle d_1|d_2 \rangle| e^{-\frac{(x-d_1)^2}{4\sigma^2+2\alpha t/m}} + 2|\langle d_1|d_3 \rangle| e^{-\frac{(x-d_1)^2}{4\sigma^2+2\alpha t/m}} + 2|\langle d_2|d_3 \rangle| e^{-\frac{(x-d_2)^2}{4\sigma^2+2\alpha t/m}} \right),$$

(12)

where $\sigma^2 = \sigma^2 + (ht/2me)^2$ and $\Omega^2 = e^2 + (ht/2m)^2$. For simplicity, we assume that $|\langle d_i|d_j \rangle|$ are all real, where $i, j \in (1, 3)$. The probability density then reduces to

$$|\psi(x, t)|^2 = |A|^2 \left( e^{-\frac{(x-d_1)^2}{4\sigma^2+2\alpha t/m}} + e^{-\frac{(x-d_2)^2}{4\sigma^2+2\alpha t/m}} \right) + 2|\langle d_1|d_2 \rangle| e^{-\frac{(x-d_1)^2}{4\sigma^2+2\alpha t/m}} \cos \left( \frac{xt}{4m\Omega^2} - \beta \right) + 2|\langle d_1|d_3 \rangle| e^{-\frac{(x-d_1)^2}{4\sigma^2+2\alpha t/m}} \cos \left( \frac{xt}{4m\Omega^2} + \beta \right),$$

(13)

where $\beta = \frac{ht/2m}{\Omega^2}$.

Visibility of the interference fringes is conventionally defined as

$$V = \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}},$$

(14)

where $I_{\text{max}}$ and $I_{\text{min}}$ represent the maximum and minimum intensity in neighboring fringes, respectively. Maxima and minima of $|\langle d_1|d_2 \rangle|$ will occur at points where the value of each cosine is 1 and -1/2, respectively, provided we ignore $\beta$. The ideal visibility can then be written down as

$$V_I = \frac{3}{\alpha} \left( |\langle d_1|d_2 \rangle| e^{\frac{t^2}{\sigma^2}} + |\langle d_1|d_3 \rangle| e^{-\frac{t^2}{\sigma^2}} + |\langle d_2|d_3 \rangle| e^{-\frac{t^2}{\sigma^2}} \right),$$

(15)

where $\alpha = 2 \left( e^{\frac{t^2}{\sigma^2}} + 2 \cosh(xd/\sigma^2)^2 e^{-\frac{t^2}{\sigma^2}} \right)$. In reality, fringe visibility will depend on many things, including the width of the slits. For example, if the width of the slits is very large, the fringes may not be visible at all. In an ideal situation, the maximum visibility one can theoretically get will be in the case when $d \ll \sigma$. Actual fringe visibility will be less than or equal to that, and can be written as

$$V \leq \frac{3}{2} \left( |\langle d_1|d_2 \rangle| + |\langle d_1|d_3 \rangle| + |\langle d_2|d_3 \rangle| \right).$$

(16)

Using (8) the above equation gives

$$V + \frac{2DQ}{3 - DQ} \leq 1.$$  

(17)

Eqn. (17) is a new duality relation which puts a bound on how much which-way information we can obtain and how
much fringe visibility we consequently get. It is straightforward to check that $V = 1$ is possible only for $D_Q = 0$, and $D_Q = 1$ implies $V = 0$.

2.5. Specific cases

Let us look at some special cases arising from the fact that there are not two, but three slits. Suppose we have a which-way detector which can detect with certainty if the particle has passed through slit 1 or not. If the particle has not passed through slit 1, the detector is unable to say which of the other two slits, 2 or 3, has the particle taken. Such a scenario can occur, for example, if we have a tiny camera in front of slit 1, which, for each particle, can say if the particle has gone through slit 1 or not. If the particle has gone through slit 1 or not. In this case $d_1$ is orthogonal to both $d_2$ and $d_3$, and $d_2$ and $d_3$ are parallel. Thus, in this case, $|d_2d_3⟩ = 1$ and $|d_1d_2⟩ = |d_1d_3⟩ = 0$. The distinguishability $D_Q$, in this case is

$$D_Q = 1 - \frac{1}{3}(|d_1d_2⟩ + |d_2d_3⟩ + |d_1d_3⟩) = \frac{2}{3}, \quad (18)$$

Consequently, the fringe visibility is limited by

$$V \leq \frac{3}{7}. \quad (19)$$

Physically what is happening is the following. Particle going through slits 2 and 3 gives rise to a sharp interference pattern, however, particle going through slit 1 gives rise to a uniform background particle count, thus reducing the overall visibility of the fringes arising from slits 2 and 3.

Let us consider another case where $|d_1⟩$ and $|d_2⟩$ are orthogonal to each other, but both have equal overlap with $|d_3⟩$. Such a case can be exemplified by $|d_1⟩ = \frac{1}{\sqrt{2}}(|↑⟩ + |↓⟩), |d_2⟩ = \frac{1}{\sqrt{2}}(|↑⟩ - |↓⟩)$ and $|d_3⟩ = |↑⟩$, where $|↑⟩, |↓⟩$ form an orthonormal set. In this case

$$D_Q = 1 - \frac{1}{3}(|d_1d_2⟩ + |d_2d_3⟩ + |d_1d_3⟩) = 1 - \frac{\sqrt{2}}{3}, \quad (20)$$

Consequently, the fringe visibility is limited by

$$V \leq \frac{3\sqrt{2}}{6 + \sqrt{2}}. \quad (21)$$

2.6. The two-slit experiment

Just for completeness, here we wish to derive a duality relation for a two-slit experiment in the case where one defines which-way distinguishability based on UQSD, which is given by (5), and reads $D_Q = 1 - |⟨d_1|d_2⟩|$.

By carrying out an analysis similar to the one earlier in this section, one can show that the distinguishability and fringe visibility, in the two-slit experiment, are bounded by

$$V + D_Q \leq 1. \quad (22)$$

This relation looks different from the EGY relation (1) only because distinguishability has been defined differently. In Englert’s analysis, distinguishability is given by $D = \sqrt{1 - |⟨d_1|d_2⟩|^2}$ (4). It can be related to $D_Q$ in (5) by the relation

$$D_Q = 1 - |1 - D|^2. \quad (23)$$

Substituting the above in (22), the latter can be shown to reduce to the EGY relation (1). This new relation (22) appears to be more versatile for two-slit experiments, because it also applies to certain modified two-slit experiments in which the which-way detector is replaced by a “quantum device” (23).

3. Conclusion

In the analysis carried out in this letter, we have introduced a new which-way distinguishability $D_Q$, based on UQSD, which is just the maximum probability with which one can unambiguously distinguish between the the quantum states of the which-way detector correlated with the paths of the particle. Consequently, it is the maximum probability with which one can unambiguously tell which slit the particle went through. We carried out a wave-packet evolution of a particle through a triple-slit. Calculating fringe-visibility through an exact analysis, we relate it to the which-way distinguishability and derive a new duality relation $V + \frac{2D_Q}{1 - D_Q} \leq 1$. Because of the way in which the analysis is carried out, this should be the tightest possible bound on distinguishability and fringe visibility for the 3-slit experiment. Lastly, we feel that (7) suggests a straightforward definition of distinguishability
for multi-slit interference experiments:

\[ D_Q = 1 - \frac{1}{N(N-1)} \sum_{i \neq j} |\langle \psi_i | \psi_j \rangle|, \]

(24)

where we assume equal probability for all \( N \) paths.

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