WAVE BREAKING AND PERSISTENT DECAY OF SOLUTION
TO A SHALLOW WATER WAVE EQUATION

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ABSTRACT. As we all know, wave breaking of the water wave is important and interesting to physicist and mathematician. In the article, we devote to the study of blow-up phenomena, the decay of solution and traveling wave solution to a shallow water wave equation. First, based on the blow-up scenario, some new blow-up phenomena is derived. By virtue of a weighted function, the persistent decay of solution is established. Finally, we explore the analytic solutions and traveling wave solutions.

1. Introduction. In this paper, we consider the Cauchy problem of the following nonlinear partial differential equation

\[
\begin{cases}
    u_t - \mu \frac{u_{txx}}{16} + u_x + \frac{3\epsilon}{2} uu_x + \frac{\mu}{16} u_{xxx} = \frac{3}{16} \epsilon^2 (2u^2 u_x - \epsilon u^3 u_x)
    \\
    - \frac{7\mu}{2} (2u_x u_{xx} + uu_{xxx}),
    \quad t > 0, \quad x \in \mathbb{R},
    \\
    u(0, x) = u_0(x), \quad x \in \mathbb{R},
\end{cases}
\]

which describes the propagation of surface waves of moderate amplitude in shallow water [9]. It was discovered very recently by Constantin and Lannes explored the model equation for the evolution of the surface elevation [9]

\[
\varsigma_t + \varsigma_x + \frac{3}{2} \epsilon \varsigma \varsigma_x - \frac{3}{8} \epsilon^2 \varsigma_x^2 + \frac{3}{16} \epsilon^3 \varsigma_x + \mu (\alpha \varsigma_{xxx} + \beta \varsigma_{txx}) = \epsilon \mu (\gamma \varsigma_{xxx} + \delta \varsigma_x) \varsigma_x.
\]

In order to construct an approximate solution consistent with the Green-Naghdi equations

\[
\begin{cases}
    \varsigma_t + ((1 + \epsilon) \nu) \nu_x = 0, \quad x \in \mathbb{R}
    \\
    \nu_x + \nu_x + \epsilon \nu \nu_x = \frac{\mu}{2(1 + \epsilon)} [(1 + \epsilon) \nu_t + \epsilon \nu \nu_{xx} - \epsilon \nu_x^2],
\end{cases}
\]

the constants \(\alpha, \beta, \gamma\) and \(\delta\) satisfy

\[
\alpha = p, \quad \beta = p - \frac{1}{6}, \quad \gamma = -\frac{3}{2} p - \frac{1}{6}, \quad \delta = -\frac{9}{2} p - \frac{5}{24}, \quad p \in \mathbb{R}.
\]

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Let \( p = \frac{1}{12} \) and \( \zeta = u \), Eq. (12) is equivalent to Eq. (1.1). One of the most prominent relatives is the Camassa-Holm (CH) equation \[2\] which derived by Camassa and Holm

\[
y_t + u y_x + 2u_x y = 0, \quad y = u - u_{xx}
\]

from an asymptotic approximation to the Hamiltonian for the Green–Naghdi equations in shallow water theory, which approximates to the incompressible Euler equation at the next order beyond the KdV equation \[9\]. It has a bi-Hamiltonian structure \[17\], a Lax pair based on a linear spectral problem of second order, and is completely integrable \[5\]. Moreover, Eq. (1.4) not only has wave breaking and global existence of strong solution \[5, 6, 7, 10\], but also exists peakon solutions \[3\], which are orbital stable \[11\].

The CH equation is not the only integrable PDE of its kind, being a shallow water equation whose dispersionless version has weak solitons. Degasperis and Procesi \[14\] used an asymptotic integrability approach to isolate integrable third order equations, and discovered the Degasperis–Procesi (DP) equation

\[
y_t + u y_x + 3u_x y = 0, \quad y = u - u_{xx}.
\]

(1.5)

The DP equation can be regarded as a model for nonlinear shallow water dynamics \[20\]. Degasperis, Holm and Hone \[13\] prove the formal integrability of Eq. (1.5) by constructing a Lax pair. They also show \[13\] that it has a bi-Hamiltonian structure and an infinite sequence of conserved quantities, and admits exact peakon solutions.

Despite the form of DP is similar to the CH equation, it should be emphasized that these two equations are truly different, such as the conservation laws of Eq. (1.5) are weaker than them of Eq. (1.4) \[15\]. One of the important features of Eq. (1.5) is that it has not only peakon solitons \[13\], i.e. solutions at the form \( u(t, x) = ce^{-|x-ct|} \) and periodic peakon solutions \[35\], but also has shock peakons \[4, 24\] which are given by

\[
u(t, x) = \begin{cases} 
-\frac{1}{t+k}\text{sgn}(x)e^{-|x|}, & k > 0, \\
0, & x \in \mathbb{Z}.
\end{cases}
\]

Compared with the CH and DP equations, by the Hamiltonian structure, Eq. (1.1) has the following infinite sequence of conserved quantities

\[
H(u) = \int_{\mathbb{R}} (u_x^2 + \frac{\mu}{12} u_{xx}^2) dx = H_0, \quad (1.6)
\]

and admits solitary travelling wave solutions decaying at infinity \[18\]. The orbital stability of it has been recently obtained by Mutluabas and Geyer in \[25\] using the Hamiltonian structure of Eq. (1.1) as \( \epsilon = 4, \mu = 12 \). The well-posedness of Eq. (1.1) in space \( H^s, s > \frac{3}{2} \) or \( B_{p,r}^s, s > \max\{\frac{3}{2}, 2 + \frac{1}{p}\} \) is the same as it of the CH and DP equations \[9, 12, 26, 34\]. Moreover, Eq. (1.1) itself is not symmetrical, i.e. \((u, x) \rightarrow (-u, -x)\), some results of the equation are truly different with the CH and DP equations \[9\].

In this paper, based on the conservation law (1.6) and blow-up criterion,

\[
\limsup_{t \uparrow T} \left\{ \sup_{x \in \mathbb{R}} u_x(t, x) \right\} = \infty, \quad \text{if} \quad \epsilon > 0,
\]
or

$$\liminf_{t \uparrow T} \left\{ \inf_{x \in \mathbb{R}} u_x(t, x) \right\} = -\infty,$$

if the slope of $u_0$ satisfies

$$\left\{ \left( 8K^2M \frac{2}{\pi^2} \right)/(7\epsilon(2n - 1)) \right\} \frac{2n-1}{2n} < \int_{\mathbb{R}} u_{0x}^{2n+1} dx, \epsilon > 0,$$

or

$$\int_{\mathbb{R}} u_{0x}^{2n+1} dx < - \left\{ \left( 8K^2M \frac{2}{\pi^2} \right)/(7\epsilon(2n - 1)) \right\} \frac{2n-1}{2n}, \epsilon < 0,$$

then the solution $u$ of Eq.(1.1) with the initial datum $u_0$ blows up in finite time $T$. Moreover, if there exists a point $x_0 \in \mathbb{R}$ such that

$$u_0'(x_0) > \sqrt{\frac{4K_1^2}{7\epsilon}}, \text{ if } \epsilon > 0$$

or

$$u_0'(x_0) < -\sqrt{\frac{4K_1^2}{7\epsilon}}, \text{ if } \epsilon < 0,$$

then the solution $u$ blows up in finite time, and the lifespan $T$ of solution satisfies

$$\frac{2\sqrt{7} \pi}{\sqrt{7}\epsilon K_1} \left( \frac{\pi}{2} - \arctan \sqrt{\frac{7\pi}{2K_1} M(0)} \right) \leq T \leq \frac{1}{K_1 \sqrt{7}\epsilon} \log \left[ \frac{\sqrt{7}\epsilon}{2} M(0) + K_1 \right], \text{ if } \epsilon > 0,$$

or if $\epsilon < 0$, then

$$\frac{2\sqrt{7} |\epsilon|}{\sqrt{7}|\epsilon| K_1} \left( \frac{\pi}{2} + \arctan \sqrt{\frac{7|\epsilon|}{2K_1} M(0)} \right) \leq T \leq \frac{1}{K_1 \sqrt{7} |\epsilon|} \log \left[ \frac{\sqrt{7} |\epsilon|}{2} M(0) - K_1 \right].$$

In particular, the blow-up rate of blow-up solution is $\frac{1}{\sqrt{\epsilon}}$. Next, in view of a weighted function which is defined in [1], the persistence decay of solution is established. The analyticity of solutions for water wave equation has been studied extensively [8, 19, 21, 28, 30]. We finally explore the existence and uniqueness of analytic solutions and traveling wave solutions.

The remainder of the paper is organized as follows. In Section 2, note that the conservation law (1.6) is equivalent to the $H^1$-norm of solution, we derive the blow-up scenario of solution by Theorem 2.1 and Theorem 2.2. Then some new results of wave breaking phenomena are investigated. In Section 3, by the weight function, we explore the persistent decay of solution in a weight $L^p$-spaces. As the argument which was used in [23, 30], in Section 4, the existence and uniqueness of analytic solutions to Eq.(1.1) with the analytic initial data is obtained, we also prove that Eq.(1.1) has a family traveling wave solutions.

2. Global existence and wave breaking. In this subsection, based on a conservation law of strong solution, we will derive the blow-up scenario and establish the singularity of strong solution to Eq.(1.1).

At first, if $\mu > 0$, then the operator $(1 - \frac{\mu}{12} \partial_x^2)$ is reversible, one can rewrite Eq.(1.1) as follows

$$\begin{cases}
    u_t - (u_x + \frac{7}{2} \epsilon uu_x) = -\partial_x (1 - \frac{\mu}{12} \partial_x^2)^{-1} f(u), & t > 0, x \in \mathbb{R}, \\
    u(0, x) = u_0(x), & x \in \mathbb{R},
\end{cases}$$

(2.1)
where the function $f(u) = 2u + \frac{5}{2}u^2 - \frac{1}{5}c^2u^3 + \frac{3}{4}ε^3u^4 - \frac{7}{48}εmu_x^2$.

Next, we present two important and useful lemmas.

**Lemma 2.1.** [22] Assume that $s > 0$. Then we have
\[ ||[Λ^s g]||_L^2 \leq c(||∂xg||_L^s ||Λ^{s-1}f||_L^2 + ||Λ^s g||_L^2 ||f||_L^∞), \]
where $c$ is constant depending only on $s$, and $f, g \in L^∞ \cap H^{s-1}$.

**Lemma 2.2.** [6] Let $T > 0$ and $u \in C^1([0, T); H^2)$. Then for every $t \in [0, T)$, there exist at least one pair points $ξ(t), ζ(t) \in ℝ$, such that
\[ m(t) = \inf_{ξ(t) \in ℝ} u(t, ξ(t)), \quad M(t) = \sup_{ξ(t) \in ℝ} u(t, ξ(t)), \]
and $m(t), M(t)$ are absolutely continuous in $[0, T)$. Moreover,
\[ \frac{dm(t)}{dt} = u_t(t, ξ(t)), \quad \frac{dM(t)}{dt} = u_t(t, ξ(t)), \quad a.e. \quad [0, T). \]

The well-posedness of Eq.(1.1) in $H^s(ℝ)$, $s > \frac{3}{2}$ can be established by the Kato semigroup theory (or transport equation theory), similar result can be found in [32, 33, 34]. We prove here the blow-up scenario and global existence of solution to Eq.(1.1).

**Theorem 2.1.** Assume $u_0 \in H^s(ℝ)$, $s > \frac{3}{2}$. Let $T$ be the lifespan of corresponding solution $u(t, x)$ to Eq.(1.1) of the initial datum, if $\int_0^T ||u_x(s)||_{L^∞}ds < ∞$, then the solution $u$ globally exists. Moreover, the solution $u$ blows up in finite time $T$ if and only if
\[ \lim_{t \to T} ||u_x(t)||_{L^∞} = ∞. \]

**Proof.** If $u(t, x)$ is the solution of Eq.(1.1) with the initial datum $u_0 \in H^s(ℝ)$, $s > \frac{3}{2}$, then applying the operator $Λ^s = (1 - ∂_x^2)^{s}$ on both side of Eq.(2.1), after multiplying by $Λ^s u$, and integration by parts on $ℝ$ yields that
\[ \frac{d}{dt} \langle u, u \rangle_{H^s} = \langle u_x, u \rangle_{H^s} + \frac{7}{2}ε \langle uu_x, u \rangle_{H^s} - \langle Λ^s(1 - \frac{μ}{12}∂_x^2)^{-1}f(u), u \rangle_{H^s} \]
\[ = \frac{7}{2}ε \langle uu_x, u \rangle_{H^s} - \langle Λ^s(1 - \frac{μ}{12}∂_x^2)^{-1}f(u), u \rangle_{H^s}. \]

On the one hand, the first term of RHS in (2.2) can be dealt with as follows
\[ \frac{7}{2}ε \langle uu_x, u \rangle_{H^s} = \frac{7}{2}ε \langle [Λ^s (uu_x)], Λ^s u \rangle_{L^2} \]
\[ \leq c(||Λ^s u||_{L^2} ||Λ^s u||_{L^2} + ||uΛ^s u_x||_{L^2}) \]
\[ = c(||u_x||_{L^∞} ||Λ^s u||_{L^2} + ||u_x||_{L^∞} ||Λ^{s-1}u_x||_{L^2}) ||Λ^s u||_{L^2} \]
\[ \leq c \left( ||u_x||_{L^∞} ||Λ^s u||_{L^2} + \frac{1}{2} ||u_x||_{L^∞} ||Λ^{s-1}u_x||_{L^2} \right), \]
\[ \text{where we have used Lemma 2.1 in the second inequality}. \]

On the other hand, note that
\[ ||u||_{L^2}^2 + \frac{μ}{12} ||u_x||_{L^2}^2 = ||u_0||_{L^2}^2 + \frac{μ}{12} ||u_0x||_{L^2}^2 := E_0^2 \]
is a conservation law of Eq. (1.1), then it follows that
\[ \|u\|_{L^\infty}^2 \leq \max \left\{ \frac{1}{\mu}, \frac{1}{2}E_0^2 \right\}. \]  
(2.4)

Therefore, we estimate the second term of RHS in (2.2) to yield
\[
\langle \partial_x (1 - \frac{\mu}{12} \partial_x^2)^{-1} f(u), u \rangle_{H^s}
\]
\[
\leq c \left( 2u + \frac{5}{2} \epsilon u^2 - \frac{1}{8} \epsilon^2 u^3 + \frac{3}{64} \epsilon^3 u^4 - \frac{7}{48} \epsilon \mu u_x^2, u \right)_{H^{s-1}}
\]
\[
\leq c \|u\|_{L^\infty} + \|u_x\|_{L^\infty}^2 \|u\|_{H^{s-1}}
\]
\[
\leq c(1 + \|u_x\|_{L^\infty})\|u\|_{H^s}^2,
\]
where the last inequality follows by (2.4) and \( s > \frac{3}{2} \), which guarantees that \( H^{s-1}(\mathbb{R}) \) is a Banach algebra.

Inserting (2.3) and (2.5) into (2.2), one has that
\[
\frac{d}{dt}\|u\|_{H^s} \leq c\|u_x\|_{L^\infty}\|u\|_{H^s}.
\]  
(2.6)

By virtue of Gronwall’s inequality to (2.6) is given by
\[
\|u\|_{H^s} \leq \|u_0\|_{H^s} e^{cE_0^2} c\|u_x(t)\|_{L^\infty} ds.
\]

If there exist a \( M > 0 \) such that \( \lim_{t \uparrow T}\|u_x(t)\|_{L^\infty} \leq M \), then by the Gronwall inequality to (2.6) obtains that
\[
\|u\|_{H^s} \leq \|u_0\|_{H^s} e^{cM T}.
\]

Assume \( \lim_{t \uparrow T}\|u_x(t)\|_{L^\infty} = \infty \), by Sobolev’s embedding theorem, we deduce that the solution \( u(t, x) \) will blow up in finite time \( T \). This completes the proof of Theorem 2.1.

**Theorem 2.2.** Given \( u_0 \in H^s(\mathbb{R}) \), \( s > \frac{3}{2} \) and \( T \) be the maximal existence time of solution \( u(t, x) \) to Eq. (2.1) with the initial datum \( u_0 \). Then the solution \( u \) blows up in finite time \( T \) if and only if
\[
\limsup_{t \uparrow T} \left\{ \sup_{x \in \mathbb{R}} u_x(t, x) \right\} = \infty, \quad \text{if} \quad \epsilon > 0,
\]  
(2.7)

or
\[
\liminf_{t \uparrow T} \left\{ \inf_{x \in \mathbb{R}} u_x(t, x) \right\} = -\infty, \quad \text{if} \quad \epsilon < 0.
\]  
(2.8)

**Proof.** If the slope of the solution \( u \) satisfies (2.7) or (2.8) in finite time, then by Theorem 2.1 and Sobolev’s embedding theorem, the solution \( u \) will blow up in space \( H^s(\mathbb{R}) \) in finite time \( T \).

Differentiating Eq. (2.1) with respect to \( x \) variable. Let the Green function \( p = (3/\mu)^\frac{1}{2} e^{-2(3/\mu)^{\frac{1}{2}}|x|} \), in view of the identity \((1 - \frac{\mu}{12} \partial_x^2)p * g \in L^2(\mathbb{R})\), we have
\[
u_{tt} = u_{xx} + \frac{7}{4} \epsilon u_x^2 + \frac{7}{4} \epsilon uu_{xx} + \frac{\mu}{12} \left( 2u + \frac{5}{2} \epsilon u^2 - \frac{1}{8} \epsilon^2 u^3 + \frac{3}{64} \epsilon^3 u^4 - p * f \right). \tag{2.9}
\]
By Young’s inequality and Hölder’s inequality, one has that
\[
\|2u + \frac{5}{2}u^2 - \frac{1}{8}e^2u^3 + \frac{3}{64}e^3u^4 - p * f\|_{L^\infty} \\
\leq c(\epsilon, \mu, E_0) + \|p * f\|_{L^\infty} \\
\leq c\|p\|_{L^2}\|u\|_{L^2} + c\|p\|_{L^\infty}(\|u^2\|_{L^1} + \|u_x^2\|_{L^1}) \\
\leq C_0.
\] (2.10)

As \(\epsilon > 0\), define \(m(t) = u_x(t, \xi(t)) = \inf_{x \in \mathbb{R}} u_x(t, x)\). Since \(u_{xx}(t, \xi(t)) = 0\), it follows for a.e. \(t \in [0, T]\) that
\[
m'(t) \geq \frac{7}{4} m^2(t) - C_0.
\] (2.11)

If \(m(t) < -\sqrt{\frac{4C_0}{7 \epsilon}}\), then \(m'(t) > 0\) and \(m(t)\) is an increasing function. Otherwise, \(-\sqrt{\frac{4C_0}{7 \epsilon}} \leq m(t) \leq 0\). Consequently,
\[
m(t) \geq \min \left\{ m(0), -\sqrt{\frac{4C_0}{7 \epsilon}} \right\},
\]
i.e. the slope of solution is bounded from below. Thanks to Theorem 2.1, the solution \(u\) blows up in finite time \(T\) if and only if
\[
\limsup_{t \uparrow T} \left\{ \sup_{x \in \mathbb{R}} u_x(t, x) \right\} = \infty, \quad \text{if} \quad \epsilon > 0,
\]
On the other hand, if \(\epsilon < 0\), let \(M(t) = u_x(t, \zeta(t)) = \sup_{x \in \mathbb{R}} u_x(t, x)\). Since \(u_{xx}(t, \zeta(t)) = 0\), in view of (2.9) and (2.10), for a.e. \(t \in [0, T]\), we deduce that
\[
M'(t) \leq \frac{7}{4} \epsilon M^2(t) + C_0.
\] (2.12)

If \(M(t) > \sqrt{\frac{4C_0}{7 \epsilon}}\), then \(M'(t) < 0\) and \(M(t)\) is a decreasing function. Otherwise, \(0 \leq M(t) \leq \sqrt{\frac{4C_0}{7 \epsilon}}\). Immediately,
\[
M(t) \leq \max \left\{ M(0), \sqrt{\frac{4C_0}{7 \epsilon}} \right\},
\]
i.e. the slope of solution is bounded from above. Thanks to Theorem 2.1, the solution \(u\) blows up in finite time \(T\) if and only if
\[
\liminf_{t \uparrow T} \left\{ \inf_{x \in \mathbb{R}} u_x(t, x) \right\} = -\infty, \quad \text{if} \quad \epsilon < 0,
\]
which derives the result of Theorem 2.2.

Based on the method which comes from [29], we now present some results of wave breaking of Eq.(2.1).

**Theorem 2.3.** Let \(u_0 \in H^s(\mathbb{R})\), \(s > \frac{3}{2}\) and \(\epsilon > 0\). Assume \(u(t, x)\) be corresponding solution of Eq.(2.1) with the initial datum \(u_0\). If for \(n \in \mathbb{N}^+\), the slope of \(u_0\) satisfies
\[
\left\{ (8K^2 M \frac{2}{2n+1})/(7\epsilon(2n-1)) \right\}^{2n-1} < \int_{\mathbb{R}} u_{0x}^{2n+1} dx,
\]
Thus there exists the lifespan $T < \infty$ such that the solution $u$ blows up in finite time $T$. Moreover, the above bound of lifespan $T$ is estimated by

$$T \leq \int_{h(0)}^{\infty} \frac{dy}{\frac{7\epsilon(2n-1)}{8M} y^{\frac{2n}{2n-1}} - K^2},$$

where $h(0) := \int_R u_x^{2n+1} dx$ and the constant $K$ satisfies (2.16).

**Proof.** In view of (2.9), for all $n \in \mathbb{N}/\{0\}$, it follows that

$$\frac{d}{dt} \int_R u_x^{2n+1} dx = (2n + 1) \int_R u_x^2 u_{tx} dx = \frac{7\epsilon}{4}(2n - 1) \int_R u_x^{2n+2} dx + (2n + 1) \frac{\mu}{12} \int_R u_x^2 (2u + \frac{5}{2} \epsilon u^2 - \frac{1}{8} \epsilon^2 u^3 + \frac{3}{64} \epsilon^3 u^4 - p * f) dx.$$

(2.13)

Thanks to the conservation law, there exists $M$ such that

$$\|u\|_{L^\infty} \leq \|u\|_{H^1} \leq M.$$

Thus

$$\|u_x^{2n}(2u + \frac{5}{2} \epsilon u^2 - \frac{1}{8} \epsilon^2 u^3 + \frac{3}{64} \epsilon^3 u^4 - p * f)\|_{L^1}$$

$$\leq \|u_x^{2n}\|_{L^1} \left( (\|p\|_{L^1} + 1)(2M + \frac{5}{2} \epsilon M^2 + \frac{1}{8} \epsilon^2 M^3 + \frac{3}{64} \epsilon^3 M^4) + \frac{7}{48} \epsilon \mu \|p\|_{L^\infty} \|u_x^{2n}\|_{L^1} \right)$$

$$\leq \|u_x\|_{L^2} \|u_x^{2n+2}\|_{L^{2n+2}} \left( 4M + 5\epsilon M^2 + \frac{1}{4} \epsilon^2 M^3 + \frac{3}{32} \epsilon^3 M^4 + \frac{7}{48} \epsilon \mu (\frac{3}{\mu})^\frac{1}{2} M^2 \right)$$

$$\leq \frac{n - 1}{n} \epsilon \frac{4}{\mu} \|u_x\|_{L^{2n+2}} \left( 4M + 5\epsilon M^2 + \frac{1}{4} \epsilon^2 M^3 + \frac{3}{32} \epsilon^3 M^4 + \frac{7}{48} \epsilon \mu (\frac{3}{\mu})^\frac{1}{2} M^2 \right),$$

(2.14)

where we have used $\|p\|_{L^1} = 1$, $\|p\|_{L^\infty} = (3/\mu)\frac{1}{2}$, the second inequality comes from Gagliardo–Nirenberg’s inequality, the last inequality is guaranteed by the Young’s inequality with $\epsilon$.

Let $\epsilon = (\frac{21\epsilon n(2n-1)}{2n(n-1)(2n+1)})^{\frac{n-1}{n}}$, combining (2.13) with (2.14) to deduce

$$\frac{d}{dt} \int_R u_x^{2n+1} dx \geq \frac{7\epsilon}{8}(2n - 1) \int_R u_x^{2n+2} dx - K^2,$$

(2.15)

where the constant $K$ satisfies

$$K = \left( \frac{\mu(2n+1)M^2}{12n} \right)^{1/2} \left( \frac{21\epsilon n(2n-1)}{2\mu(n-1)(2n+1)} \right)^{n-1} \left( 4M + 5\epsilon M^2 + \frac{1}{4} \epsilon^2 M^3 + \frac{3}{32} \epsilon^3 M^4 + \frac{7}{48} \epsilon \mu (\frac{3}{\mu})^\frac{1}{2} M^2 \right)^{1/2}.$$

(2.16)

By virtue of the following interpolation inequality

$$\|u_x\|_{L^{2n+1}} \leq C \|u_x\|_{L^2}^{\frac{1}{2n+1}} \|u_x\|_{L^{2n+2}}^{\frac{n}{2n+1}};$$
i.e.

\[
\left( \int_{\mathbb{R}} u_{x}^{2n+1} dx \right)^{\frac{2n}{2n-1}} \leq c \left( \int_{\mathbb{R}} u_{x}^{2} dx \right)^{\frac{1}{2n-1}} \int_{\mathbb{R}} u_{x}^{2n+2} dx \\
\leq c(M) \left( \int_{\mathbb{R}} u_{x}^{2} dx \right)^{\frac{2n}{2n-1}} \int_{\mathbb{R}} u_{x}^{2n+2} dx.
\] (2.17)

Plugging (2.17) into (2.16), let \( h(t) := \int_{\mathbb{R}} u_{x}^{2n+1} dx \), it can be transformed into

\[
\partial_{t} h(t) \geq \frac{7\epsilon(2n-1)}{8M^{\frac{2n}{2n-1}}}(h(t))^{\frac{2n}{2n-1}} - K^{2}.
\] (2.18)

By virtue of the assumption of Theorem 2.3

\[
h(0) > \left\{ \left( 8K^{2}M^{\frac{2n}{2n-1}}} \right)/(7\epsilon(2n-1) \right\}^{\frac{2n-1}{2n}},
\]

then \( h'(t) > 0 \) and \( h(t) \) is a increasing function. If the solution \( u \) globally exists, then there is \( t_{1} > 0 \) such that

\[
h'(t) \geq \frac{7\epsilon(2n-1)}{16M^{\frac{2n}{2n-1}}}(h(t))^{\frac{2n}{2n-1}}, \quad t \geq t_{1}.
\] (2.19)

Consequently

\[
0 < \frac{1}{(h(t))^{\frac{1}{2n-1}}} \leq \frac{1}{(h(t_{1}))^{\frac{1}{2n-1}}} - \frac{7\epsilon}{16M^{\frac{2n}{2n-1}}}(t - t_{1}), \quad t \geq t_{1}.
\] (2.20)

Since \( h(t_{1}) > 0 \), let \( t \geq t_{1} \) large enough, the above inequality will lead a contradiction. There exists lifespan \( T < \infty \) such that \( \lim_{t \uparrow T} h(t) = \infty \). Due to

\[
\left| \int_{\mathbb{R}} u_{x}^{2n+1} dx \right| \leq \|u_{x}\|_{L^{\infty}}^{2n-1} \int_{\mathbb{R}} u_{x}^{2} dx \leq M^{2}\|u_{x}\|_{L^{\infty}}^{2n-1}.
\]

Thanks to Theorem 2.2, it follows that

\[
\limsup_{t \uparrow T} \left\{ \sup_{x \in \mathbb{R}} u_{x}(t, x) \right\} = \infty.
\]

By solving (2.18), one can easily check that the lifespan \( T \) satisfies

\[
T \leq -\int_{0}^{T} \frac{h'(t)dt}{\frac{7\epsilon(2n-1)}{8M^{\frac{2n}{2n-1}}}(h(t))^{\frac{2n}{2n-1}}} - K^{2}.
\]

Since \( \lim_{t \uparrow T} h(t) = \infty \), the above inequality can be changed into

\[
T \leq \int_{h(0)}^{\infty} \frac{dy}{\frac{7\epsilon(2n-1)}{8M^{\frac{2n}{2n-1}}}y^{\frac{2n}{2n-1}}} - K^{2}.
\]

\[\square\]

**Remark 2.1.** Note that \( \int_{\mathbb{R}} u_{x}dx = \int_{\mathbb{R}} u_{0x}dx \) is a conservation law of Eq.(1.1), thus the number \( n \in \mathbb{N}/\{0\} \). Moreover, if \( n = 1 \), the lifespan \( T \) of solution is estimated above by

\[
T \leq \sqrt{\frac{14\epsilon}{7\epsilon K}} M \log \frac{\frac{\sqrt{14\epsilon}}{7\epsilon K} h(0) + K}{\frac{\sqrt{14\epsilon}}{7\epsilon K} h(0) - K}.
\]
Theorem 2.4. Suppose \( u_0 \in H^s(\mathbb{R}) \), \( s > \frac{3}{2} \) and \( \epsilon < 0 \). Let \( u(t, x) \) be corresponding solution of Eq.\((2.1)\) with the initial datum \( u_0 \). If the slope of \( u_0 \) satisfies
\[
\int_{\mathbb{R}} u_0^{2n+1} dx < -\left\{ \frac{8K^2M^\frac{2}{n-1}}{7|\epsilon|(2n-1)} \right\} \frac{2n-1}{2n},
\]
then there exists the lifespan \( T < \infty \) such that the solution \( u \) blows up in finite time \( T \). In particular, the above bound of lifespan \( T \) is estimated by
\[
T \leq \int_{h(0)}^{\infty} \frac{dy}{\frac{7|\epsilon|(2n-1)}{8M^\frac{2}{n-1}} y^{\frac{2n}{2n-1}} - K^2},
\]
where \( h(0) := \int_{\mathbb{R}} u_0^{2n+1} dx \) and the constant \( K \) satisfies (2.16).

Proof. Proceeding exactly as the estimate of (2.15), one has that
\[
\frac{d}{dt} \int_{\mathbb{R}} u_x^{2n+1} dx \leq \frac{7\epsilon}{8} (2n-1) \int_{\mathbb{R}} u_x^{2n+2} dx + K^2,
\]
using inequality (2.17), due to \( \epsilon < 0 \), we immediately deduce that
\[
\partial_t h(t) \leq \frac{7\epsilon(2n-1)}{8M^\frac{2}{n-1}} (h(t)) \frac{2n}{2n-1} + K^2.
\]
In view of the assumption of Theorem 2.4
\[
h(0) < -\left\{ \frac{8K^2M^\frac{2}{n-1}}{7|\epsilon|(2n-1)} \right\} \frac{2n-1}{2n},
\]
then \( h'(t) < 0 \) and \( h(t) \) is a decreasing function. Similar to process of dealing with (2.19), there exists lifespan \( T < \infty \) such that \( \lim_{t \uparrow T} h(t) = -\infty \) and
\[
\lim \inf_{t \uparrow T} \left\{ \inf_{x \in \mathbb{R}} u_x(t, x) \right\} = -\infty.
\]
In particular, solving inequality (2.22) derives that the above bound of lifespan of solution
\[
T \leq \int_{h(0)}^{\infty} \frac{dy}{\frac{7|\epsilon|(2n-1)}{8M^\frac{2}{n-1}} y^{\frac{2n}{2n-1}} - K^2}.
\]
This completes the proof of Theorem 2.4. \( \square \)

Remark 2.2. As the slope of initial datum satisfies some certain condition, if we choose different \( n \), Theorem 2.3 and 2.4 give a class of blow-up solutions to Eq.\((2.1)\) and give the formula of the above bound of lifespan \( T \).

Next, we will derive another blow-up phenomena to Eq.\((2.1)\).

Theorem 2.5. Given \( u_0 \in H^s(\mathbb{R}) \), \( s > \frac{3}{2} \). Assume \( u(t, x) \) be corresponding solution of Eq.\((2.1)\) with the initial datum \( u_0 \). If there exists a point \( x_0 \in \mathbb{R} \) such that
\[
u'_0(x_0) > \sqrt{\frac{4K^2_1}{7\epsilon}}, \quad \text{if } \epsilon > 0
\]
or
\[
u'_0(x_0) < -\sqrt{\frac{4K^2_1}{7|\epsilon|}}, \quad \text{if } \epsilon < 0,
\]
then the solution $u$ blows up in finite time. Moreover the lifespan $T$ of solution satisfies

$$
\frac{2\sqrt{\tau \epsilon}}{7\epsilon K_1} \left( \frac{\pi}{2} - \arctan \frac{\sqrt{\tau \epsilon}}{2K_1} M(0) \right) \leq T \leq \frac{1}{K_1 \sqrt{\tau \epsilon}} \log \frac{\frac{\sqrt{\tau \epsilon}}{2} M(0) + K_1}{\frac{\sqrt{\tau \epsilon}}{2} M(0) - K_1}, \text{ if } \epsilon > 0,
$$

or if $\epsilon < 0$, then

$$
\frac{2\sqrt{\tau |\epsilon|}}{7|\epsilon| K_1} \left( \frac{\pi}{2} + \arctan \frac{\sqrt{\tau |\epsilon|}}{2K_1} m(0) \right) \leq T \leq \frac{1}{K_1 \sqrt{\tau |\epsilon|}} \log \frac{\frac{\sqrt{\tau |\epsilon|}}{2} m(0) - K_1}{\frac{\sqrt{\tau |\epsilon|}}{2} m(0) + K_1},
$$

where $M(0) = \sup_{\epsilon \in \mathbb{R}} u_0'(x)$, $m(0) = \inf_{x \in \mathbb{R}} u_0(x)$ and

$$
K_1 = \sqrt{\frac{\mu}{12} \left( 4M + 5\epsilon M^2 + \frac{1}{4} \epsilon^2 M^3 + \frac{3}{32} \epsilon^3 M^4 + \frac{7}{48} \epsilon \mu \left( \frac{3}{2} \frac{1}{M^2} \right) \right)}.
$$

Proof. Let $M(t) = u_x(t, \zeta(t)) = \sup_{x \in \mathbb{R}} u_x(t, x)$. Since $u_{xx}(t, \zeta(t)) = 0$, by virtue of (2.9) is given by

$$
M'(t) = \frac{\mu}{12} (2u + 5 \epsilon u_2 - \frac{1}{8} \epsilon^2 u_3 + \frac{3}{64} \epsilon^3 u_4 - p \ast f)(t, \zeta(t)).
$$

By Young’s inequality and Hölder’s inequality

$$
\begin{align*}
\frac{\mu}{12} \|2u + 5 \epsilon u_2 - \frac{1}{8} \epsilon^2 u_3 + \frac{3}{64} \epsilon^3 u_4 - f\|_{L^\infty} &\leq \frac{\mu}{12} (4M + 5\epsilon M^2 + \frac{1}{4} \epsilon^2 M^3 + \frac{3}{32} \epsilon^3 M^4 + \frac{7}{48} \epsilon \mu \left( \frac{3}{2} \frac{1}{M^2} \right) \frac{1}{M^2}) \tag{2.25} \\
&:= K_1'.
\end{align*}
$$

Plugging (2.25) into (2.24), we have

$$
\frac{7}{4} \epsilon M^2(t) - K_1' \leq M'(t) \leq \frac{7}{4} \epsilon M^2(t) + K_1'. \tag{2.26}
$$

If $\epsilon > 0$, in view of the assumption of Theorem 2.5 to yield $M(0) > \sqrt{\frac{4K_1^2}{\tau \epsilon}}$. By solving the LHS of inequality (2.26), it follows that

$$
\frac{\sqrt{\tau \epsilon} M(0) - K_1}{\frac{\sqrt{\tau \epsilon}}{2} M(0) + K_1} e^{K_1 \sqrt{\tau \epsilon} t} - 1 \leq \frac{-2K_1}{\sqrt{\tau \epsilon} M(t) + K_1} \leq 0, \tag{2.27}
$$

and $\lim_{t \to T} M(t) = \infty$, i.e. the solution blows up in finite time $T$ and

$$
T \leq \frac{1}{K_1 \sqrt{\tau \epsilon}} \log \frac{\frac{\sqrt{\tau \epsilon}}{2} M(0) + K_1}{\frac{\sqrt{\tau \epsilon}}{2} M(0) - K_1}.
$$

Similarly, by the RHS of inequality (2.26), it easily check that the lower bound of lifespan $T$ satisfies

$$
T \geq \frac{2\sqrt{\tau \epsilon}}{7 \epsilon K_1} \left( \frac{\pi}{2} - \arctan \frac{\sqrt{\tau \epsilon}}{2K_1} M(0) \right).
$$

If $\epsilon < 0$, define $m(t) = u_x(t, \xi(t)) = \inf_{x \in \mathbb{R}} u_x(t, x)$. Since $u_{xx}(t, \xi(t)) = 0$, we also can get

$$
\frac{7}{4} \epsilon M^2(t) - K_1^2 \leq m'(t) \leq \frac{7}{4} \epsilon m^2(t) + K_1^2. \tag{2.28}
$$
By virtue of the assumption of Theorem 2.5, we have $m(0) < -\sqrt{\frac{4K_1^2}{7|\epsilon|}}$, proceeding as estimate of (2.26), one can easily check from (2.28) that $\lim_{t \to T} m(t) = -\infty$, i.e. the solution blows up in finite time $T$. Moreover, the lifespan of solution satisfies

$$2\sqrt{\frac{7|\epsilon|}{7|\epsilon|K_1}} \left( \frac{\pi}{2} + \arctan \frac{\sqrt{7|\epsilon|}}{2K_1} m(0) \right) \leq T \leq \frac{1}{K_1\sqrt{7|\epsilon|}} \log \frac{\sqrt{7|\epsilon|} m(0) - K_1}{\sqrt{7|\epsilon|} m(0) + K_1},$$

which concludes the proof of Theorem 2.5. \hfill \Box

**Corollary 2.1.** Let $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. If the solution $u$ to Eq. (2.1) with the initial datum $u_0$ blows up in finite time $T$, which is guaranteed by Theorem 2.3-2.5, then the blow-up rate of solution satisfies

$$\lim_{t \to T} \left\{ (T-t) \sup_{x \in \mathbb{R}} u_x(t, x) \right\} = \frac{4}{\epsilon}, \quad \text{if } \epsilon > 0, \quad (2.29)$$

$$\lim_{t \to T} \left\{ (T-t) \inf_{x \in \mathbb{R}} u_x(t, x) \right\} = \frac{4}{\epsilon}, \quad \text{if } \epsilon < 0, \quad (2.30)$$

while the $L^\infty$-norm of solution remains bounded.

**Proof.** By the assumption $T < \infty$ and Theorem 2.2, it follows that

$$\limsup_{t \to T} \left\{ \sup_{x \in \mathbb{R}} u_x(t, x) \right\} = \infty, \quad \text{if } \epsilon > 0, \quad (2.31)$$

or

$$\liminf_{t \to T} \left\{ \inf_{x \in \mathbb{R}} u_x(t, x) \right\} = -\infty, \quad \text{if } \epsilon < 0. \quad (2.32)$$

As $\epsilon > 0$, in view of (2.31), choosing $\epsilon \in (0, 1)$, it easily find a $t_0 < T$ such that

$$M(t_0) \geq \sqrt{\frac{4K_1^2}{7|\epsilon|} + \frac{K_1^2}{\epsilon}}. \quad (2.33)$$

By the LHS of (2.26), it yields that $M(t)$ is increasing on the domain $[t_0, T)$. Multiplying (2.26) by $\frac{1}{M(t)}$, thanks to (2.33), we have

$$-\frac{7}{4} \epsilon - \epsilon \leq \frac{d}{dt} \left( \frac{1}{M(t)} \right) \leq -\frac{7}{4} \epsilon + \epsilon.$$

Integrating the above inequality on $[t, T]$ with time variable $s \in [t_0, T)$ is given by

$$(T-t) \left( -\frac{7}{4} \epsilon - \epsilon \right) \leq -\frac{1}{M(t)} \leq (T-t) \left( -\frac{7}{4} \epsilon + \epsilon \right),$$

i.e.

$$\frac{1}{\frac{7}{4} \epsilon + \epsilon} \leq (T-t)M(t) \leq \frac{1}{\frac{7}{4} \epsilon - \epsilon}. \quad (2.34)$$

Since $\epsilon \in (0, 1)$ is arbitrary, the inequality (2.34) implies the result (2.29).

On the other hand, if $\epsilon < 0$, we deal with (2.28) exactly as the estimate of (2.26) to end up with (2.30). \hfill \Box
3. The persistent decay of solution. By a weight function, our aim is to investigate the persistent decay of solution in a weight $L^p$-spaces. At first, we recall some definitions which comes from [1].

**Definition 3.1.** A weight function $v : \mathbb{R}^n \to \mathbb{R}$ is sub-multiplicative, if and only if for all $x, y \in \mathbb{R}^n$, it satisfies $v(x + y) \leq v(x)v(y)$. A positive function $\varphi$ is called a $v$-moderate if there exists $c_0 > 0$ such that

$$\varphi(x + y) \leq c_0 v(x)\varphi(y), \quad \forall x, y \in \mathbb{R}.$$  

**Remark 3.1.** If choosing a standard weight functions

$$\varphi(x) = \varphi_{abcd}(x) = e^{a|x|^b}(1 + |x|^c \log(e + |x|)^d), \quad (3.1)$$

then we easily check that:

(a). As $b \in [0, 1]$ and $a, c, d \in [0, \infty)$, then the weight $\varphi_{abcd}(x)$ is sub-multiplicative.
(b). As $b \in [0, 1]$ and $a, c, d \in \mathbb{R}$, then the weight $\varphi_{abcd}(x)$ is $\varphi_{a, b, c, d}$-moderate for $|a| \leq \alpha$, $|b| \leq \beta$, $|c| \leq \gamma$, $|d| \leq \delta$.

**Definition 3.2.** Assume $\varphi : \mathbb{R} \to \mathbb{R}$ be locally absolutely continuous, such that $|\varphi(x)|^p \leq A\varphi(x)$ for some $A > 0$. In addition, if $\varphi(x)$ is $v$-moderate, for sub-multiplicative weight function $v$ satisfying $\inf_{\mathbb{R}} v > 0$ and

$$v(x)e^{-|x|} \in L^1(\mathbb{R}),$$

then $\varphi(x)$ is admissible weight function of Eq.(1.1).

We now establish the main result in this subsection.

**Theorem 3.1.** Assume the initial datum $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. Then Eq.(1.1) has a solution $u(t, x)$ which belongs to $C([0, T); H^s) \cap C^1([0, T); H^{s-1})$ with $u_0$. In addition, for any admissible weight function $\varphi$ of Eq.(1.1) and $p \in [2, \infty]$, if the initial datum $u_0$ satisfies

$$\|u_0\varphi\|_{L^p} + \|u_0x\varphi\|_{L^p} \leq C_0,$$

then, for all $t \in [0, T]$, the following estimate of solution holds

$$\|u\varphi\|_{L^p} + \|ux\varphi\|_{L^p} \leq C_0(\|u_0\varphi\|_{L^p} + \|u_0x\varphi\|_{L^p}),$$

where the constant depends on $T, v, \varphi$ and $\|u_x\|_{L^\infty}$.

**Proof.** Let $\phi(x) = \min\{N, \varphi(x)\}$ for any $N \in \mathbb{Z}^+$. Then the function $\phi$ is uniformly $v$-moderate with respect to $N$. Multiplying Eq.(2.1) by the admissible weight function $\phi$, after applying $p|u\phi|^{p-2}u\phi$, integration by parts to variable $x$,

$$\partial_t \int_{\mathbb{R}} |u\phi|^pdx = p \int_{\mathbb{R}} (u_x\phi)|u\phi|^{p-2}u\phi dx + \frac{7}{2} ep \int_{\mathbb{R}} u_x|u\phi|^p dx - p \int_{\mathbb{R}} \partial_x(1 - \frac{\mu}{12}\partial_x^2)^{-1}f\phi|u\phi|^{p-2}u\phi dx. \quad (3.2)$$

Note that

$$\partial_t \int_{\mathbb{R}} |u\phi|^pdx = p\|u\phi\|_{L^p}^{-1}\partial_t\|u\phi\|_{L^p},$$
in view of Hörder’s inequality to inequality (3.2). Let $G = (3/\mu)^{\frac{1}{2}} e^{-2(3/\mu)^{\frac{1}{2}}} |\cdot|$, since $G * f = (1 - \frac{\mu}{12} \partial_x^2)^{-1} f$, it yields that

\[
\partial_t \|u\phi\|_{L^p} \leq \|u_x \phi\|_{L^p} + \frac{7}{2} \epsilon \|u_x\|_{L^\infty} \|u\phi\|_{L^p} + \|\partial_x (1 - \frac{\mu}{12} \partial_x^2)^{-1} f \phi\|_{L^p} \\
\leq \|u_x \phi\|_{L^p} + \frac{7}{2} \epsilon \|u_x\|_{L^\infty} \|u\phi\|_{L^p} + \|\partial_x (G * f) \phi\|_{L^p} \\
\leq c \left(2 + \frac{5}{2} \epsilon \|u\|_{L^\infty} + \frac{1}{8} \epsilon^2 \|u\|_{L^\infty}^2 + \frac{3}{64} \epsilon^2 \|u\|_{L^\infty}^3 + \frac{7}{2} \epsilon \|u\|_{L^\infty} \right) \\
\times \|u\phi\|_{L^p} + \left(1 + \frac{7}{48} \epsilon \|u_x\|_{L^\infty} \|u\phi\|_{L^p} \right),
\]

where we have used $\|u\|_{L^\infty}, \|u_x\|_{L^\infty} \leq M = \sup_{x \in [0, T]} \|u(t)\|_{H^s}$ and

\[
\|\partial_x (G * f) \phi\|_{L^p} \leq \|\partial_x G \phi\|_{L^p} \|f \phi\|_{L^p} \\
\leq c \left(2 + \frac{5}{2} \epsilon \|u\|_{L^\infty} + \frac{1}{8} \epsilon^2 \|u\|_{L^\infty}^2 + \frac{3}{64} \epsilon^2 \|u\|_{L^\infty}^3 \right) \\
\times \|u\phi\|_{L^p} + \frac{7}{48} \epsilon \|u_x\|_{L^\infty} \|u\phi\|_{L^p}.
\]

On the other hand, we deal with (2.9) as the above process

\[
\partial_t \int_{\mathbb{R}} |u_x \phi|^p \text{d}x = p \int_{\mathbb{R}} (u_x \phi)|u_x \phi|^{p-2} u_x \phi \text{d}x + \frac{7}{2} \epsilon p \int_{\mathbb{R}} u u_x \phi|u_x \phi|^{p-2} u_x \phi \text{d}x \\
- \frac{7}{4} \epsilon p \int_{\mathbb{R}} u_x^2 \phi|u_x \phi|^{p-2} u_x \phi \text{d}x \\
+ \frac{12}{\mu} p \int_{\mathbb{R}} [f_1 \phi - (G * f) \phi]|u_x \phi|^{p-2} u_x \phi \text{d}x,
\]

where $f_1(u) = f(u) + \frac{7}{4} \epsilon \mu u_x^2 = 2u + \frac{5}{2} \epsilon u^2 - \frac{1}{8} \epsilon^2 u^3 + \frac{3}{64} \epsilon^2 u^4$.

Since

\[
\int_{\mathbb{R}} (u_x \phi)|u_x \phi|^{p-2} u_x \phi \text{d}x = - \int_{\mathbb{R}} (u_x \phi_x)|u_x \phi|^{p-2} u_x \phi \text{d}x \\
\leq A \int_{\mathbb{R}} |u_x \phi|^p \text{d}x,
\]

\[
\int_{\mathbb{R}} u u_x \phi|u_x \phi|^{p-2} u_x \phi \text{d}x = \int_{\mathbb{R}} u|u_x \phi|^{p-2} u_x \phi (u_x \phi_x - u_x \phi_x) \text{d}x \\
= \frac{1}{p} \int_{\mathbb{R}} u |u_x \phi|^p \text{d}x - \int_{\mathbb{R}} u|u_x \phi|^{p-2} u_x \phi u_x \phi \text{d}x \\
\leq \left( \frac{1}{p} \|u_x\|_{L^\infty} + A \|u\|_{L^\infty} \right) \|u_x \phi\|_{L^p}^p.
\]

Thus, one can easily check that

\[
\partial_t \|u_x \phi\|_{L^p} \leq \frac{12}{\mu} (1 + c) \left(2 + \frac{5}{2} \epsilon \|u\|_{L^\infty} + \frac{1}{8} \epsilon^2 \|u\|_{L^\infty}^2 + \frac{3}{64} \epsilon^2 \|u\|_{L^\infty}^3 \right) \|u\phi\|_{L^p} \\
+ \left( A + \frac{7}{4} (1 + c) \|u_x\|_{L^\infty} + \frac{7}{2} \epsilon \left( \frac{1}{p} \|u_x\|_{L^\infty} + A \|u\|_{L^\infty} \right) \right) \|u_x \phi\|_{L^p}.
\]
Combining (3.3) with (3.4), by Gronwall’s inequality, let \( N \uparrow \infty \), there exists \( C_1 \) and \( C_2 \) depending only on \( \varphi, v \) and \( M \) such that
\[
\|u_\varphi\|_{L^p} + \|u_x\varphi\|_{L^p} \leq C_1(\|u_0\varphi\|_{L^p} + \|u_0x\varphi\|_{L^p})e^{C_2t}.
\]

\[\square\]

**Remark 3.2.** If we choose \( \varphi = \varphi_{abcd} \) as in (3.1) and satisfies
\[
a \geq 0, \quad 0 \leq b \leq 1, \quad c, d \in \mathbb{R}, \quad ab < 1,
\]
then \( \varphi_{abcd} \) is the admissible weight function of Eq.(1.1). Let \( \varphi = \varphi_{0000} \) with \( c > 0 \) and \( p = \infty \). Then Theorem 3.1 deduces that the uniform algebra decay of solution. If we take \( \varphi = \varphi_{a100} \) with \( 0 < a < 1 \) and \( p = \infty \), then the solution decays exponentially as \( |x| \to \infty \).

4. Analytic solutions and traveling wave solutions.

**Definition 4.1.** \( H^{0, \rho} \) is the Banach space of all the function \( f(x) \) such that
(i) \( f \) is analytic in \( D(\rho) = \mathbb{R} \times (-\rho, \rho) = \{ x \in \mathbb{C} : \tilde{x} \in (-\rho, \rho) \} \);
(ii) \( f \in L^2 (\Gamma(\tilde{x})) \) for \( \tilde{x} \in (-\rho, \rho) \); i.e. if \( \tilde{x} \in (-\rho, \rho) \), then \( f(\tilde{x}x + i\tilde{x}) \) is a square integrable function of \( \tilde{x} \), where \( \Gamma(b) = \{ x \in \mathbb{C} : \tilde{x} = b \} \);
(iii) \( |f|_\rho = \sup_{x \in (-\rho, \rho)} ||f(\cdot + i\tilde{x})||_{L^2(\Gamma(\tilde{x}))} < \infty \).

**Definition 4.2.** \( H^{k, \rho} \) is the Banach space of all the function \( f(x) \) such that
(i) \( \partial_x^j f \in H^{0, \rho} \) for all \( 0 \leq j \leq k \);
(ii) \( ||f||_{k, \rho} = \sum_{0 \leq j \leq k} ||\partial_x^j f||_\rho < \infty \).

In view of the Fourier transform, the norm of \( H^{k, \rho} \) is equivalent to
\[
||f||_{k, \rho} = \left( \int_{\mathbb{R}} e^{2\rho|\xi|} (1 + |\xi|^2)^k |\hat{f}(\xi)|^2 d\xi \right)^{1/2}.
\]

It is not difficult to check the following properties.

\begin{itemize}
  \item For any \( 0 < \rho_1 \leq \rho_2 \), \( H^{k, \rho_2} \subseteq H^{k, \rho_1} \) and \( \| \cdot \|_{k, \rho_1} \leq \| \cdot \|_{k, \rho_2} \);
  \item \( H^{1, \rho} \subset H^k \), \forall \rho > 0 \) and \( \| \partial_x(u - v) \|_{1, \rho} \leq \epsilon \|u - v\|_{\rho - \rho_0} \), \( 0 < \rho < \rho_1 \);
  \item If \( u, v \in H^{1, \rho} \), \( \rho < \rho_1 \), then \( \|u\partial_x u - v\partial_x v\|_{1, \rho} \leq \epsilon \|u - v\|_{\rho - \rho_0} \).
\end{itemize}

**Theorem 4.1.** Assume \( u_0(x) \in H^{1, \rho_0} \). Then there exists \( \beta > 0 \), such that for any \( 0 < \rho < \rho_0 \), and a unique continuously differential (w.r.t. time) solution with the initial data \( u_0(x) \) to Eq.(1.1). Moreover,
\[
\begin{align*}
  u(t, \cdot) \in H^{1, \rho} \quad \text{and} \quad \partial_t u(t, \cdot) \in H^{1, \rho} , \quad \text{if} \ t \in \left[ 0, \frac{\rho_0 - \rho}{\beta} \right].
\end{align*}
\]

Let
\[
F(t, u) = u_0 + \int_0^t \left( \frac{7}{2} \varepsilon uu_x - \partial_x (1 - \frac{\mu}{12} \partial_x^2)^{-1} f(u) \right) ds,
\]
with \( f(u) = 2u + \frac{5}{2} \varepsilon u^2 - \frac{1}{3} \varepsilon^2 u^3 + \frac{3}{8} \varepsilon^3 u^4 - \frac{7}{32} \varepsilon \mu u_x^2 \). We can rewrite Eq.(2.1) as follows
\[
\begin{align*}
  \begin{cases}
    u = F(t, u), & t > 0, \\
    u(0, x) = u_0(x).
  \end{cases}
\end{align*}
\]
We now state ACK Theorem in another form given by Safonov in \cite{27}. Then Theorem 4.1 is a straightforward consequence of the following result.
Proposition 4.1. [23, 27] Consider the problem: \( u = F(t, u) \). Let \( \exists R > 0, \rho_0 > 0 \) and \( \beta_0 > 0 \) such that if \( 0 < t \leq \frac{\rho_0}{\beta_0} \), \( F(t, u) \) satisfies the following conditions:

(i) For any \( 0 < \rho_1 < \rho \leq \rho_0 \) and \( u \in \{ u \in X_\rho : \sup_{0 \leq t \leq T} |u(t)|_\rho \leq R \} \), the function \( F(t, u) : [0, T] \mapsto X_\rho \) is continuous.

(ii) For any \( 0 < \rho < \rho_0 \) the function \( F(t, u) : [0, \rho_0/\beta_0] \mapsto \{ u \in X_\rho : \sup_{0 \leq t \leq T} |u(t)|_\rho \leq R \} \) is a continuous and satisfies

\[
|F(t, 0)|_\rho \leq R_0 < R.
\]

(iii) For any \( 0 < \rho_1 < \rho(s) < \rho_0 \) and \( u, v \in \{ u \in X_\rho : \sup_{0 \leq t \leq T} |u(t)|_{\rho-\beta_0 t} \leq R \} \),

\[
|F(t, u) - F(t, v)|_{\rho_1} \leq C \int_0^t \frac{|u - v|_{\rho_1(s)} ds}{\rho(s) - \rho_1}.
\]

Then \( \exists \beta > \beta_0 \) such that for any \( 0 < \rho < \rho_0 \), \( u = F(t, u) \) has a unique solution \( u(t) \in X_\rho \) with \( 0 \leq t \leq (\rho_0 - \rho)/\beta \). Moreover

\[
\sup_{\rho < \rho_0 - \beta t} |u(t)|_\rho \leq R.
\]

It is easy to check the conditions (i) and (ii) is true to Eq.(4.2). In order to obtain the result of Theorem 4.1, it suffice to derive the following result.

Lemma 4.1. Let \( R > 0 \), for any \( 0 < \rho_1 < \rho(s) < \rho_0 \) and \( u, v \in \{ u \in H^{1, \rho} : \sup_{0 \leq t \leq T} ||u(t)||_{1, \rho-\beta_0 t} \leq R \} \). There exists a constant \( C > 0 \) such that

\[
||F(t, u) - F(t, v)||_{1, \rho_1} \leq C \int_0^t \frac{||u - v||_{1, \rho_1(s)} ds}{\rho(s) - \rho_1},
\]

where the constant \( C \) depends on \( ||u||_{1, \rho}, ||v||_{1, \rho} \).

Proof. Thanks to the equivalent norm of \( H^{1, \rho} \) (4.1), for any \( 0 < \rho_1 < \rho \), in view of simple calculation, one has that

\[
||u^2 - v^2||_{1, \rho_1} \leq c||u - v||_{1, \rho_1} \leq \frac{c||u - v||_{1, \rho}}{\rho - \rho_1},
\]

\[
||u^3 - v^3||_{1, \rho_1} \leq c||u - v||_{1, \rho_1} \leq \frac{c||u - v||_{1, \rho}}{\rho - \rho_1},
\]

\[
||u^4 - v^4||_{1, \rho_1} \leq c||u - v||_{1, \rho_1} \leq \frac{c||u - v||_{1, \rho}}{\rho - \rho_1},
\]

\[
||\partial_x(u^2 - v^2)||_{1, \rho_1} \leq c(||u - v||_{1, \rho_1} + ||u_x - v_x||_{1, \rho_1}) \leq \frac{c||u - v||_{1, \rho}}{\rho - \rho_1}.
\]

In view of above relations and Proposition 4.1, it follows that

\[
||F(t, u) - F(t, v)||_{1, \rho_1} \leq \int_0^t \left(||u_x - v_x||_{1, \rho_1} + \frac{7}{4}c||\partial_x(u^2 - v^2)||_{1, \rho_1}\right) ds
\]

\[
+ \int_0^t \frac{||\partial_x(1 - \frac{1}{12} \rho_1^2) f(u - f(v))||_{1, \rho_1}}{\rho(s) - \rho_1} ds,
\]

which completes the proof of Lemma 4.1. \( \square \)

Next, we will prove that Eq.(1.1) have a family traveling wave solutions. First, it is necessary to present two definitions.
Definition 4.3. A solution $u(t,x)$ to Eq. (1.1) is $x$-symmetric if there exists a function $b(t) \in C^{1}(\mathbb{R}^+)$ such that

$$u(t,x) = u(t, 2b(t) - x), \forall t \in [0, \infty),$$

for almost every $x \in \mathbb{R}$. We say that $b(t)$ is the symmetric axis of $u(t,x)$.

Definition 4.4. Assume that $u(t,x) \in X(\mathbb{R})$ and satisfy

$$
\int_{\mathbb{R}^+} \int_{\mathbb{R}} \left[ u(1 - \partial_x^2) \varphi_t + g(u) \varphi_x + \left( u^4 + 7\epsilon u^2 \right) \partial_x^3 \varphi \right] dt dx = 0,
$$

for all $\varphi \in C^\infty_0(\mathbb{R}^+ \times \mathbb{R})$. Then $u(t,x)$ is a weak solution to Eq.(1.1), where $X(\mathbb{R}) = \{ u : u \in C(\mathbb{R}^+, H^1(\mathbb{R})) \}$ and

$$g(u) = u + \frac{3}{4} \epsilon u^2 + \frac{1}{8} \epsilon^2 u^3 + \frac{3}{64} \epsilon^4 - \frac{7}{48} \epsilon \mu u^2.$$

Remark 4.1. Since $C^\infty_0(\mathbb{R}^+ \times \mathbb{R})$ is dense in $C^1_0(\mathbb{R}^+, H^1(\mathbb{R}))$. By the density argument, we only need to consider the test functions belong to $C^1_0(\mathbb{R}^+, H^1(\mathbb{R}))$.

As the method proving Theorem 4.1 in [31], we can derive

**Theorem 4.2.** Let $u(t,x)$ be $x$-symmetric. If $u(t,x)$ be a unique weak solution of Eq.(1.1), then $u(t,x)$ is a traveling wave.

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