Non-adiabatic collapse of a quasi-spherical radiating star

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A model is proposed of a collapsing quasi-spherical radiating star with matter content as shear-free isotropic fluid undergoing radial heat-flow with outgoing radiation. To describe the radiation of the system, we have considered both plane symmetric and spherical Vaidya solutions. Physical conditions and thermodynamical relations are studied using local conservation of momentum and surface red-shift. We have found that for existence of radiation on the boundary, pressure on the boundary is not necessary.

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I. INTRODUCTION

In Einstein gravity, gravitational collapse with realistic astronomical matter distribution is an important problem of astrophysics. Usually, the formation of compact stellar objects such as white dwarf and neutron star are preceded by a period of radiative collapse. Hence for astrophysical collapse, it is necessary to describe the appropriate geometry of interior and exterior regions and to determine proper junction conditions which allow the matching of these regions.

The study of gravitational collapse was started long ago in 1939 by Oppenheimer and Snyder [1]. They studied dust collapse with a static Schwarzchild exterior while interior space-time is described by Friedman like solution. Since then several authors have extended the above study of collapse of which important and realistic generalizations are the following: (i) the static exterior was studied by Misner and Sharp [2] for a perfect fluid in the interior, (ii) using the idea of outgoing radiation of the collapsing body by Vaidya [3], Santos and collaborations [4-9] included the dissipation in the source by allowing radial heat flow (while the body undergoes radiating collapse). Recently, Ghosh and Deskar [10] have considered collapse of a radiating star with a plane symmetric boundary (which has a close resemblance with spherical symmetry [11]) and have concluded with some general remarks.

So far most of the studies have considered radiating star with interior geometry as spherical. But in the real astrophysical situation the geometry of the interior of a star may not be exactly spherical, rather quasi-spherical in form. So it will be interesting to study quasi-spherical interior geometry of a radiating star. In this paper, we have considered the interior space-time $V^-$ by Szekeres’ model [12, 13] while for exterior geometry $V^+$ we have considered plane symmetric Vaidya space-time. The plan of the paper is as follows: Exact heat flux solution in Szekeres’ model has been presented in section II. The junction conditions, physical properties and thermodynamical relations are shown in section III. The paper ends with a discussion in section IV.

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II. SOLUTION IN SZEKERES MODEL WITH HEAT FLUX

The space-time metric for Szekeres’ model in \((n + 2)\) dimension is in the form [13, 14]

\[
ds_+^2 = -dt^2 + e^{2\alpha}dr^2 + e^{2\beta} \sum_{i=1}^{n} dx_i^2 \tag{1}
\]

where \(\alpha\) and \(\beta\) are functions of all the \((n + 2)\) space-time variables. The stress-energy tensor of a non-viscous heat conducting fluid has the expression

\[
T_{\mu\nu} = (\rho + p)u_{\mu}u_{\nu} + pg_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu \tag{2}
\]

where \(\rho, p, q_\mu\) are the fluid density, isotropic pressure and heat flow vector. We take the heat flow vector \(q_\mu\) to be orthogonal to the velocity vector i.e., \(q_\mu u^\mu = 0\). For comoving co-ordinate system we choose \(u^\mu = (1, 0, 0, 0, \ldots, 0)\) and \(q^\mu = (0, q_1, q_2, \ldots, q_n)\) where \(q = q(t, r, x_1, \ldots, x_n)\) and \(q_i = q_i(t, r, x_1, \ldots x_n), i = 1, 2, \ldots, n\). Now the non-vanishing components of the Einstein field equation

\[
G_{\mu\nu} = T_{\mu\nu} + \Lambda g_{\mu\nu}
\]

for the above space-time model (1) with matter field in the form of (2) are [13, 14]

\[
n\dot{\alpha}'\dot{\beta} - \frac{1}{2}n(n-1)\dot{\beta}'^2 - e^{-2\beta} \sum_{i=1}^{n} \left\{ \alpha_i^2 + \frac{1}{2} \left( n(n-1)(n-2)\beta_i^2 + (n-2)\alpha_i\beta_i + \alpha_i x_i + \alpha_i x_i, \right) + (n-1)\beta_{x_i} \right\} + e^{-2\alpha} \left\{ n\alpha'\beta' - \frac{1}{2}n(n+1)\beta'^2 - n\beta'' \right\} = \Lambda + \rho \tag{3}
\]

\[
\frac{1}{2}n(n+1)\dot{\beta}^2 + n\ddot{\beta} - \frac{1}{2}n(n-1)e^{-2\alpha}\beta'^2 - e^{-2\beta} \sum_{i=1}^{n} \left\{ \frac{1}{2} \left( n(n-1)(n-2)\beta_i^2 + (n-1)\beta_{x_i} \right) \right\} = \Lambda - p \tag{4}
\]

\[
\alpha_i^2 + \alpha + (n-1)\alpha\dot{\beta} + \frac{1}{2}n(n-1)\beta'^2 + (n-1)\dot{\beta} + e^{-2\alpha} \left\{ (n-1)\alpha'\beta' - \frac{1}{2}n(n-1)\beta'^2 - (n-1)\beta'' \right\} - e^{-2\beta} \sum_{i \neq j}^{n} \left\{ \alpha_{x_i} + \frac{1}{2} \left( n-2)(n-3)\beta_{x_j}^2 + \alpha_{x_j x_j} + (n-2)\beta_{x_j x_j} + (n-3)\alpha_{x_j} \beta_{x_j} \right) \right\} - e^{-2\beta} \left\{ (n-1)\alpha_{x_i} \beta_{x_i} + \frac{1}{2} (n-1)(n-2)\beta_{x_i}^2 \right\} = \Lambda - p \tag{5}
\]

\[
\alpha_{x_i} \left( -\alpha_{x_i} + \beta_{x_i} \right) + \beta_{x_i} \left( \alpha_{x_i} + (n-2)\beta_{x_i} \right) - \alpha_{x_i x_j} - (n-2)\beta_{x_i x_j} = 0, \quad (i \neq j) \tag{6}
\]

\[
\left( \dot{\beta} - \dot{\alpha} \right) \beta' + \beta' = \frac{1}{n} q e^{2\alpha} \tag{7}
\]

\[
\left( \dot{\alpha} - \dot{\beta} \right) \alpha_i + \alpha_i + (n-1)\beta_{x_i} = q_i e^{2\beta} \tag{8}
\]

\[
\alpha_{x_i} \beta' - \beta'_{x_i} = 0 \tag{9}
\]
where dot, dash and subscript stands for partial derivatives with respect to $t$, $r$ and the corresponding variables respectively (e.g. $\beta_{x_i} = \frac{\partial \beta}{\partial x_i}$) and $i, j = 1, 2, 3, ..., n$.

From equations (7) and (9) after differentiating with respect to $x_i$ and $t$ respectively, we have the integrability condition

$$\frac{\partial q}{\partial x_i} + q \alpha_{x_i} = n \beta \dot{\beta}_{x_i} e^{-2\alpha}, \quad (i = 1, 2, ..., n)$$

(10)

This equation can not be solved in general. So we have assumed $\beta' \neq 0, \ddot{\beta}_{x_i} = 0$. Then the form of $\beta$ is

$$e^\beta = R(t, r)e^{\nu(r,x_1,x_2,...,x_n)}$$

(11)

and from equation (9) we have the solution for $\alpha$ as

$$e^\alpha = \frac{R' + R\nu'}{D(t, r)}$$

(12)

where $R$ and $D$ are functions of $t$, $r$ only. From equations (4) and (5) using equations (11) and (12) we have the differential equations of $R$ and $D:

$$2R\ddot{R} + (n - 1)(R^2 - D^2) - \frac{2}{n}(\Lambda - p)R^2 = (n - 1)f(r)$$

(13)

and

$$R\dot{D} = f(r) e^{-2\alpha}$$

(14)

where $f(r)$ is the arbitrary function of $r$.

The function $\nu$ satisfies the equation

$$e^{-\nu} = A(r) \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} B_i(r)x_i + C(r)$$

(15)

where $A$, $B_i$, $C$ are arbitrary functions of $r$ alone with the restriction

$$\sum_{i=1}^{n} B_i^2 - 4AC = f(r)$$

(16)

Now from equations (7), (8) and (10) we have the components of heat flux vector as

$$q = \frac{n}{R} \dot{D} e^{-\alpha}$$

(17)

and

$$q_i = \frac{\dot{D}}{D} \alpha_{x_i} e^{-\beta}$$

(18)

From the above solution, we can see that the field equation (6) is automatically satisfied.

Using (3), (4) and (5) we have the expression for density as

$$\rho = -\frac{n}{(n-1)}(\dot{\alpha} + n\dot{\beta} + \ddot{\alpha}^2 + n\dot{\beta}^2) + \frac{(n+1)}{(n-1)} p + \frac{2\Lambda}{(n-1)}$$

(19)
However, from equation (14) we note that as $R$ and $D$ are functions of $t$ and $r$ only, so $\alpha$ is independent of the space co-ordinates $x_i$'s (i.e., $\alpha_{x_i} = 0$, $\forall$ $i = 1, 2, ..., n$). Hence from equation (18) we have $q_i = 0$ and from equation (17) we have seen that $q = q(t, r)$ i.e., $q$ is a function of $t$ and $r$ only. Thus only radial heat flow is possible for the choice of the metric as we consider.

III. JUNCTION CONDITIONS AND CONSEQUENCE FOR SZEKERES MODEL WITH PLANE SYMMETRIC V AI DAYA METRIC

Let us consider a time-like $(n+1)D$ hypersurface $\Sigma$, which divides $(n+2)D$ space-time into two distinct $(n+2)D$ manifolds $V^-$ and $V^+$. For junction conditions we follow the modified version of Israel [15] by Santos [4, 5]. Now the geometry of the space-time $V^-$ is given by equation (1) while $V^+$ and the boundary $\Sigma$ are characterised by the metric ansatzs as

\[ ds^2_+ = \frac{2m(v)}{(n-1)z^{n-1}}dv^2 - 2dvdz + z^n \sum_{i=1}^{n} dx_i^2 \]  

and

\[ ds^2_\Sigma = -d\tau^2 + A^2(\tau) \sum_{i=1}^{n} dx_i^2 \]  

where the arbitrary function $m(v)$ in the Vaidya metric represents the mass at retarded time $v$ inside the boundary surface $\Sigma$. Now Israel's junction conditions (as described by Santos) are

(i) The continuity of the line element i.e.,

\[ (ds^2_\Sigma) = (ds^2_+) = ds^2_\Sigma \]  

where ($\_\Sigma$) means the value of ($\_\Sigma$) on $\Sigma$.

(ii) The continuity of extrinsic curvature over $\Sigma$ gives

\[ [K_{ij}] = K_{ij}^+ - K_{ij}^- = 0 , \]  

where due to Eisenhart the extrinsic curvature has the expresion

\[ K_{ij}^\pm = -n_\mu^\pm \frac{\partial^2 \chi_\sigma^\pm}{\partial \xi^i \partial \xi^j} - n_\mu^\pm \Gamma^\mu_{\lambda\nu} \frac{\partial \chi_\lambda^\pm}{\partial \xi^i} \frac{\partial \chi_\nu^\pm}{\partial \xi^j} \]  

Here $\xi^i = (\tau, x_1, x_2, ..., x_n)$ are the intrinsic co-ordinates to $\Sigma$, $\chi_\pm^\sigma$, $\sigma = 0, 1, 2, ..., n + 1$ are the co-ordinates in $V^{\pm}$ and $n_\mu^\pm$ are the components of the normal vector to $\Sigma$ in the co-ordinates $\chi_\pm^\sigma$. It is to be noted that the above continuity conditions are equivalent to junction conditions due to Lichnerowicz and O’ Brien and synge.

Now for the interior space-time described by the metric (1) the boundary of the interior matter distribution (i.e., the surface $\Sigma$) will be characterized by

\[ f(r, t) = r - r_\Sigma = 0 \]  

where $r_\Sigma$ is a constant. As the vector with components $\frac{\partial f}{\partial \chi^-}$ is orthogonal to $\Sigma$ so we take

\[ n^-_\mu = (0, e^\sigma, 0, ..., 0). \]
So comparing the metric ansatzs given by equations (1) and (21) for \( dr = 0 \) we have from the continuity relation (22)

\[
\frac{dt}{d\tau} = 1, \quad A(\tau) = e^\beta \quad \text{on} \quad r = r_\Sigma
\]  

(26)

Also the components of the extrinsic curvature for the interior space-time are

\[
K^\tau_\tau = 0 \quad \text{and} \quad K^\tau_i = [\beta' e^{2\beta - \alpha}]_{\Sigma}, \quad i = 1, 2, \ldots, n.
\]  

(27)

On the other hand for the exterior Vaidya metric described by the equation (20) with its exterior boundary, given by

\[
f(z, v) = z - z_\Sigma(v) = 0
\]  

(28)

the unit normal vector to \( \Sigma \) is given by

\[
n^+ = \left( -\frac{2m(v)}{(n-1)z^{n-1}} + \frac{dz}{dv} \right)^{-1/2} \left( -\frac{dz}{dv}, 1, 0, \ldots, 0 \right)
\]  

(29)

and the components of the extrinsic curvature are

\[
K^+\tau = \left[ \frac{d^2v}{d\tau^2} \left( \frac{dv}{d\tau} \right)^{-1} - \frac{m}{z^n d\tau} \right]_{\Sigma}
\]  

(30)

and

\[
K^+_i = \left[ z \frac{dz}{d\tau} - \frac{2m(v)}{(n-1)z^{n-2} d\tau} \right]_{\Sigma}, \quad i = 1, 2, \ldots, n
\]  

(31)

Hence the continuity of the extrinsic curvature due to junction condition (see eq. (23)) gives

\[
\left[ \frac{2m}{(n-1)} e^{-(n-2)\beta} \frac{dv}{d\tau} \right]_{\Sigma} = \left[ e^{2\beta} \left( \dot{\beta} - \beta' e^{-\alpha} \right) \right]_{\Sigma}
\]  

(32)

and

\[
\left[ \frac{1}{d\tau} \right]_{\Sigma} = \left[ e^\beta \left( \ddot{\beta} + \beta' e^{-\alpha} \right) \right]_{\Sigma}
\]  

(33)

Now using the junction condition (32) with the help of equation (33), we have

\[
m(v) = \frac{1}{2} \left[ (n-1) e^{(n+1)\beta} \left( \dot{\beta}^2 - \beta'^2 e^{-2\alpha} \right) \right]_{\Sigma}
\]  

(34)

We can interprete this as the total energy bounded within the surface \( \Sigma \) and is equivalent to the well known mass function in spherical symmetry due to Cahill and McVittie [16].

Further using (7), (11), (15), (16), (32), (33) and (34) we obtain (after simplification)

\[
p_\Sigma = (qc^\alpha)_{\Sigma} + \frac{1}{2} n(n-1) \left[ f(r)R^{-2} \right]_{\Sigma}
\]  

(35)

where the second term on the r.h.s. does not vanish on \( \Sigma \). So on the boundary vanishing of the isotropic pressure does not imply the vanishing of the heat flux. Thus
for a quasi-spherical shearing distribution of a collapsing fluid, undergoing dissipation in the form of heat flow, the isotropic pressure on the surface of discontinuity $\Sigma$ does not balance the radiation. Hence in the absence of isotropic pressure there may still be radiation on the boundary and the exterior space-time $V^+$ will still be Vaidya space-time.

Moreover, the total luminosity for an observer at rest at infinity is

$$L_\infty = \lim_{r \to 0} \left( \frac{n-1}{n} \right) z^n \epsilon = - \left( \frac{dm}{dv} \right)_\Sigma$$

$$= \left[ \frac{(n-1)}{n} e^{(n+2)\beta} \frac{(R+D)^2}{R^2} \left\{ p - \frac{1}{2} n(n-1)f(r)R^{-2} \right\} \right]_\Sigma$$

(36)

If we now consider an observer on the boundary $\Sigma$ then the luminosity for that observer is

$$L_\Sigma = \left( \frac{n-1}{n} \right) z^n \epsilon_\Sigma = - \left[ \left( \frac{dv}{d\tau} \right)^2 \frac{dm}{dv} \right]_\Sigma$$

(37)

$$= \left[ \frac{(n-1)}{n} e^{n\beta} \left\{ p - \frac{1}{2} n(n-1)f(r)R^{-2} \right\} \right]_\Sigma$$

(38)

Thus the boundary red-shift ($Z_\Sigma$) of the radiation emitted by a star can be written as

$$Z_\Sigma = \sqrt{\frac{L_\Sigma}{L_\infty}} - 1 = \left[ \frac{e^{-\beta R}}{(R+D)} \right]_\Sigma - 1$$

(39)

Hence the luminosity measured by an observer at rest at infinity is reduced by the red-shift in comparison to the luminosity observed on the surface of the collapsing body. Also when $R + D = 0$ then the boundary red-shift attains unlimited value (i.e., $Z_\Sigma \to \infty$) and the luminosity vanishes at infinity (i.e., $L_\infty \to 0$).

We now discuss the thermodynamical relations for a collapsing star. We have seen above that when the star produces unpolarized radiation while its non-adiabatic fluid collapses then the junction condition (35) between pressure $p$ and heat flux $q$ has to be justified. Also for physically reasonable fluid we should have (i) $0 < p < \rho < \infty$ for $0 \leq r \leq r_\Sigma$, (ii) both $dm/d\rho$ and $d\rho/d\varepsilon$ are negative for $r > 0$, while $0 < \frac{d\rho}{d\varepsilon} < 1$ for $0 \leq r \leq r_\Sigma$. Further from thermodynamical point of view, we should have the following relations [17]:

(a) $(\rho, u^\mu; \rho) = 0$ (equation of conservation of matter) where the effective rest mass density (measured in the rest frame of $u^\mu$), $\rho_E$ is related to the internal energy density $U$ by the relation

$$\rho = \rho_E (1 + U)$$

(b) Gibbs equation:

$$TdS = dU + p d(1/\rho_E)$$

where as usual $S$ is the entropy and $T$ is the temperature [18] of the collapsing star.

(c) Second law of thermodynamics:

$$S_{\mu}^{\mu} \geq 0$$
where the entropy flux $S^\mu$ is defined by

$$S^\mu = \rho E u^\mu + \frac{1}{T} q^\mu$$

$(d)$ Temperature gradient law:

$$q^\mu = -\kappa(g^{\mu\nu} + u^\mu u^\nu)(T_{,\nu} + T u_{,\nu} u^\nu)$$

with positive thermal conductivity $\kappa$. But we note that the second law of thermodynamics can be derived from the temperature gradient law, so there is no need to satisfy it.

Now from the conservation of mass density we have for the present model

$$\rho_E = \rho_0(r, x_1, ..., x_n) e^{-(\alpha + \beta^2)}$$

with $\rho_0$ as effective rest mass density in the infinite past.

Also from the equation for temperature gradient law the radial heat flow has the form

$$q = -\kappa \frac{\partial T}{\partial r} e^{-2\alpha}$$

Further, if we assume the thermal conductivity $\kappa$ as a polynomial in temperature i.e.,

$$\kappa = \gamma T^\Omega \geq 0$$

then from the comparative study of the expressions for $q$ in equations (17) and (39) we have the expression for temperature

$$T^{\Omega + 1} = -\frac{n(\Omega + 1)}{\gamma} \int \left( \frac{R'}{R} + \nu' \right) \frac{\partial}{\partial t} (\log D)dr + T_0(t)$$

with $T_0(t)$ an arbitrary function of $t$.

### IV. DISCUSSION

In this work we have found a general solution for Szekeres’ $(n + 2)$-D space-time model with perfect fluid and heat flux. Here we may mention that this solution is not only a higher dimensional generalization of Goode [19] but also it is general in the sense that here heat flux is directed along all spatial directions. However, for the study of junction conditions with exterior Vaidya model we have considered only radial heat flow. We have studied both the physical conditions and the thermodynamical relations for the collapse of a radiating star model.

Moreover, in the Szekeres’ model if we consider the co-ordinate transformation [13, 14] $(x_1, x_2, ..., x_n) \rightarrow (\theta_1, \theta_2, ..., \theta_n)$ by

$$x_1 = \sin \theta_n \sin \theta_{n-1} ... \sin \theta_2 \cot \frac{1}{2} \theta_1$$
$$x_2 = \cos \theta_n \sin \theta_{n-1} ... \sin \theta_2 \cot \frac{1}{2} \theta_1$$
$$x_3 = \cos \theta_{n-1} \sin \theta_{n-2} ... \sin \theta_2 \cot \frac{1}{2} \theta_1$$
$$... ... ... ... ... ...$$
$$x_{n-1} = \cos \theta_3 \sin \theta_2 \cot \frac{1}{2} \theta_1$$
$$x_n = \cos \theta_2 \cot \frac{1}{2} \theta_1$$
then the form of the \((n+2)\)-D metric equation (1) becomes

\[
ds^2_- = -dt^2 + e^{2\alpha}dr^2 + \frac{1}{4} e^{2\beta} \csc^4(\theta_1/2)(d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \ldots + \sin^2\theta_1 \sin^2\theta_{n-1} d\theta_n^2)
\]

(42)

Now if we match this interior metric with the spherical \((n+2)\)-D Vaidya metric i.e.,

\[
ds^2_+ = -\left(1 - \frac{2m(v)}{(n-1)z^{n-1}}\right)dv^2 - 2vdz + z^2(d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \ldots + \sin^2\theta_1 \sin^2\theta_{n-1} d\theta_n^2)
\]

(43)

then on the boundary the relation between pressure and heat flux is

\[
p_k = (qe^\alpha)_{\Sigma},
\]

which is identical in form to the earlier works of Santos et al [4, 5] and Ghosh et al [11]. Also the other conclusions are very much similar to their results. So we have not mentioned them here.

Therefore, for quasi-spherical radiating star with plane symmetric boundary the shear-free distribution of collapsing fluid undergoing dissipation in the form of heat flux, it is not necessary to have non-vanishing isotropic pressure on the boundary for existence of heat flow on it and the result strongly depends on the value of \(f(r)\).

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