A SIMPLE PROOF OF THE STRONG INTEGRALITY FOR FULL COLORED HOMFLY INVARIANTS

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ABSTRACT. By using the HOMFLY skein theory, we prove a strong integrality theorem for the reduced colored HOMFLYPT invariants defined by a basis in the full HOMFLY skein of the annulus.

1. INTRODUCTION

Let \((\mathcal{L}_+, \mathcal{L}_-, \mathcal{L}_0)\) be the standard notation for the Conway triple of link diagrams. The reduced Jones polynomial \(J(\mathcal{L}; q)\) of a link (diagram) \(\mathcal{L}\) can be determined by the following skein relation

\[
q^2J(\mathcal{L}_+; q) - q^{-2}J(\mathcal{L}_-; q) = (q - q^{-1})J(\mathcal{L}_0; q),
\]

and the normalization condition \(J(U; q) = 1\), where \(U\) denotes the trivial knot, i.e. the unknot throughout this paper. By induction on the number of the components of the link \(\mathcal{L}\). It is easy to obtain, for an orient link \(\mathcal{L}\) with \(L\) components, \(J(\mathcal{L}; q) \in \mathbb{Z}[q^{\pm 2}]\) for \(L\) odd, and \(qJ(\mathcal{L}; q) \in \mathbb{Z}[q^{\pm 2}]\) if \(L\) even. In particular, for a knot \(K\) (i.e. a link with only one component), we have \(J(K; q) \in \mathbb{Z}[q^{\pm 2}]\). The Jones polynomial \(J(\mathcal{L}; q)\) can be viewed as the \(U_q(sl_2)\) quantum group invariant with the fundamental representations \([18]\). More generally, for a simple complex Lie algebra \(\mathfrak{g}\), T. Le [8] showed that, with a suitable normalization, the quantum group \(U_q(\mathfrak{g})\) invariants with any irreducible representations lie in the ring \(\mathbb{Z}[q^{\pm 2}]\) which is called the strong integrality (see Theorem 2.2 in [8]).

In this paper, we formulate a similar strong integrality theorem for the reduced colored HOMFLYPT invariant defined by a basis in the full HOMFLY skein of the annulus. We use the results from the HOMFLY skein theory due to Morton, Aiston, Lukac etc [1, 2, 16, 9] instead of using the quantum group theory by [11, 3]. So our proof is completely different from Le’s [8].

Let \(\mathcal{H}(\mathcal{L}; q, a)\) be the framed HOMFLYPT invariant of \(\mathcal{L}\) determined by the local relations as showed in Figure 1. Let \(\mathcal{L}\) be the full HOMFLY skein of the annulus, we will recall the definition in Section 2. In [7], Hadji and H. R. Morton constructed the basis elements \(Q_{\lambda, \mu}\) in the full skein \(\mathcal{C}\). For a link \(\mathcal{L}\) with \(L\) components, we choose \(L\) basis elements \(Q_{\lambda^\alpha, \mu^\alpha}\), \(\alpha = 1, \ldots, L\) in the skein \(\mathcal{C}\). We construct the satellite link \(\mathcal{L} \ast \otimes_{\alpha=1}^{L} Q_{\lambda^\alpha, \mu^\alpha}\). Then the full colored HOMFLYPT invariant for a link \(\mathcal{L}\) is given by

\[
W_{[\lambda^1, \mu^1], [\lambda^2, \mu^2], \ldots, [\lambda^L, \mu^L]}(\mathcal{L}; q, a) = q^{-\sum_{\alpha=1}^{L}(\kappa_{\lambda^\alpha} + \kappa_{\mu^\alpha})w(k_{\alpha})} - \sum_{\alpha=1}^{L}w(k_{\alpha})\mathcal{H}(\mathcal{L} \ast \otimes_{\alpha=1}^{L} Q_{\lambda^\alpha, \mu^\alpha}; q, a).
\]

In particular, when all \(\mu^\alpha = \emptyset\) for \(\alpha = 1, \ldots, L\), \(W_{[\lambda^1, \emptyset], [\lambda^2, \emptyset], \ldots, [\lambda^L, \emptyset]}(\mathcal{L}; q, a)\) is the ordinary colored HOMFLYPT invariant \(W_{[\lambda^1, \lambda^2, \ldots, \lambda^L]}(\mathcal{L}; q, a)\) studied in [21, 3]. We refer to Section 2 and 3 for a review of the HOMFLY skein theory and the definition of the full colored
HOMFLYPT invariant for an oriented link. By the definition, \( W_{[\lambda^1,\mu^1],...[\lambda^L,\mu^L]}(\mathcal{L}; q, a) \) is not a polynomial of \( q^{\pm 1}, a^{\pm 1} \) in general, because of the factors \( (q^k - q^{-k}) \) in the denominator. In order to kill these factors, we introduce the notion of reduced full colored HOMFLYPT invariants.

For the case of a knot \( \mathcal{K} \), it is natural to define the reduced full colored HOMFLYPT invariant of \( \mathcal{K} \) by

\[
P_{[\lambda,\mu]}(\mathcal{K}; q, a) = \frac{W_{[\lambda,\mu]}(\mathcal{K}; q, a)}{W_{[\lambda,\mu]}(U; q, a)}.
\]

Then, we have

**Theorem 1.1.** For any knot \( \mathcal{K} \), \( P_{[\lambda,\mu]}(\mathcal{K}; q, a) \in \mathbb{Z}[q^{\pm 2}, a^{\pm 2}] \).

In fact, Theorem 1.1 is a special case of the following Theorem 1.2. Theorem 1.1 can also be proved directly by using the integrality theorem \([16]\) (i.e Theorem 1 in \([16]\)) and the Theorem 1.3 in the following for the case of knot.

It seems natural to generalize the definition of the reduced full colored HOMFLYPT invariants for a general link \( \mathcal{L} \) with \( L \) components as follow:

\[
P_{[\lambda^1,\mu^1],...[\lambda^L,\mu^L]}(\mathcal{L}; q, a) = \frac{W_{[\lambda^1,\mu^1],...[\lambda^L,\mu^L]}(\mathcal{L}; q, a)}{\prod_{\alpha=1}^L W_{[\lambda^\alpha,\mu^\alpha]}(U; q, a)}.
\]

Unfortunately, \( P_{[\lambda^1,\mu^1],...[\lambda^L,\mu^L]}(\mathcal{L}; q, a) \) does not lie in the ring \( \mathbb{Z}[q^{\pm 1}, a^{\pm 1}] \) in general. The recent works by S. Nawata et al. \([17, 15]\) motivate us to define the following \([\lambda^\alpha, \mu^\alpha]\)-reduced HOMFLYPT invariants for a link \( \mathcal{L} \):

\[
Q_{[\lambda^1,\mu^1],...[\lambda^L,\mu^L]}^{[\lambda^\alpha, \mu^\alpha]}(\mathcal{L}; q, a) = \frac{W_{[\lambda^1,\mu^1],...[\lambda^L,\mu^L]}(\mathcal{L}; q, a)}{W_{[\lambda^\alpha,\mu^\alpha]}(U; q, a)} \quad \text{for} \quad \alpha = 1, ..., L.
\]

However, \( Q_{[\lambda^1,\mu^1],...[\lambda^L,\mu^L]}^{[\lambda^\alpha, \mu^\alpha]}(\mathcal{L}; q, a) \) still contains the factors of the forms \( (q^k - q^{-k}) \) in denominator. In order to kill these factors, we add a factor to \( Q_{[\lambda^1,\mu^1],...[\lambda^L,\mu^L]}^{[\lambda^\alpha, \mu^\alpha]}(\mathcal{L}; q, a) \), and introduce the notation of the normalized \([\lambda^\alpha, \mu^\alpha]\)-reduced invariant as

\[
P_{[\lambda^1,\mu^1],...[\lambda^L,\mu^L]}^{[\lambda^\alpha, \mu^\alpha]}(\mathcal{L}; q, a) = \prod_{\beta=1, \beta \neq \alpha}^L \prod_{\rho^\beta, \nu^\beta} \prod_{x \in \rho^\beta, \nu^\beta} (a \cdot q^{cn(x)})(q^{hl(x)} - q^{-hl(x)}) \cdot Q_{[\lambda^1,\mu^1],...[\lambda^L,\mu^L]}^{[\lambda^\alpha, \mu^\alpha]}(\mathcal{L}; q, a).
\]

See Section 3.1 for the definitions of \( cn(x), hl(x) \) in formula (1.6).

The main goal of this paper is to prove the following:

**Theorem 1.2.** For any \( \alpha = 1, ..., L \), we have

\[
P_{[\lambda^1,\mu^1],...[\lambda^L,\mu^L]}^{[\lambda^\alpha, \mu^\alpha]}(\mathcal{L}; q, a) \in \mathbb{Z}[q^{\pm 2}, a^{\pm 2}].
\]
Moreover, if we take the last paragraph in page 42 in [5]. In fact, the definition of $P$ Theorem 1.2 implies 

$$P_{\chi^1,...,\chi^L}(\mathcal{L}; q, a) = \prod_{\beta=1}^{L} \prod_{\alpha \neq x \in \chi^\beta} (a \cdot q^{cn(x)}) (q^{hl(x)} - q^{-hl(x)}) \frac{W_{\chi^1,...,\chi^L}(\mathcal{L}; q, a)}{W_{\chi^\alpha}(U; q, a)}.$$ 

Moreover, if we take $\chi^1 = \chi^2 = \cdots = \chi^L = (r^\alpha)$, by some straight calculations, we have

$$\prod_{x \in (r^\alpha)} (q^{hl(x)} - q^{-hl(x)}) = \prod_{i=0}^{\rho-1} \{r + i\}, \quad \prod_{x \in (r^\alpha)} (a \cdot q^{cn(x)}) = a^{r\rho} q^{\frac{1}{2} r \rho (r - \rho)},$$

where $\{n\}! = \prod_{i=1}^{n} \{i\}$, and $\{i\} = (q^i - q^{-i})$.

Hence,

$$P_{(r^\alpha),...,(r^\alpha)}(\mathcal{L}; q, a) = \left( a^{r\rho} q^{\frac{1}{2} r \rho (r - \rho)} \prod_{i=0}^{\rho-1} \{r + i\} \right)^{L-1} \frac{W_{(r^\alpha),...,(r^\alpha)}(\mathcal{L}; q, a)}{W_{(r^\alpha)}(U; q, a)}.$$ 

Theorem 1.2 implies $P_{(r^\alpha),...,(r^\alpha)}(\mathcal{L}; q, a) \in \mathbb{Z}[q^{\pm 2}, a^{\pm 1}]$. This is just a statement in [5] (see the last paragraph in page 42 in [5]). In fact, the definition of $P_{(r^\alpha)}(\mathcal{L}; q, a)$ given by formula (3.2) in [3] is equal to $(-1)^{r\rho (L-1)} P_{(r^\alpha),...,(r^\alpha)}(\mathcal{L}; q, a)$. According to the work [5], $P_{(r^\alpha),...,(r^\alpha)}(\mathcal{L}; q, a) \in \mathbb{Z}[q^{\pm 2}, a^{\pm 1}]$ suggests that $P_{(r^\alpha),...,(r^\alpha)}(\mathcal{L}; q, a)$ may have a categorification whose homology is finite dimensional. Similarly, Theorem 1.2 provides us a fact that the categorification of the general invariant $P_{[\chi^1,\mu^1],...,[\chi^L,\mu^L]}(\mathcal{L}; q, a)$ has a finite dimensional homology.

In order to prove Theorem 1.2, firstly, we give a simple proof of the following symmetries for the full colored HOMFLYPT invariants.

**Theorem 1.3.** For any link $\mathcal{L}$ with $L$-components, we have

$$W_{[\chi^1,\mu^1],...,[\chi^L,\mu^L]}(\mathcal{L}; q^{-1}, a) = (-1)^{\sum_{n=1}^{L} (|\chi^1| + |\mu^1|)} W_{[\chi^1,\mu^1],...,[\chi^L,\mu^L]}(\mathcal{L}; q, a),$$

$$W_{[\chi^1,\mu^1],...,[\chi^L,\mu^L]}(\mathcal{L}; q, a) = (-1)^{\sum_{n=1}^{L} (|\chi^1| + |\mu^1|)} W_{[\chi^1,\mu^1],...,[\chi^L,\mu^L]}(\mathcal{L}; q, a),$$

$$W_{[\chi^1,\mu^1],...,[\chi^L,\mu^L]}(\mathcal{L}; q, a) = (-1)^{\sum_{n=1}^{L} (|\chi^1| + |\mu^1|)} W_{[\chi^1,\mu^1],...,[\chi^L,\mu^L]}(\mathcal{L}; q, a).$$

**Remark 1.4.** When all the $\mu^i = \emptyset$, it is well-known that the above symmetries hold for the ordinary colored HOMFLYPT invariants $W_{\chi^1,...,\chi^L}(\mathcal{L}; q, a)$ of a link $\mathcal{L}$:

$$W_{\chi}(\mathcal{L}; q^{-1}, t) = (-1)^{||\chi||} W_{\chi}(\mathcal{L}; q, t),$$

$$W_{\chi}(\mathcal{L}; q, -t) = W_{\chi}(\mathcal{L}; q, t),$$

$$W_{\chi}(\mathcal{L}; q^{-1}, -t) = (-1)^{||\chi||} W_{\chi}(\mathcal{L}; q, t).$$

The first proof of these symmetries was given in the paper [12] [13], a different proof for (1.4) and (1.5) was given in [21]. Then, in the joint paper with Q. Chen, K. Liu and P. Peng [4], we provide a new simple proof for (1.14), (1.15), (1.16). Recent, D. Tubbenhauer, P. Vaz and P. Wedrich [20] also give a different proof for (1.14). In this paper, we follow the method used in [4] to prove Theorem 1.3 in Section 4.
Then, by using the integrality theorem in [16] (i.e. Theorem 1.1 in [16]) with a slight modification, we finish the proof of Theorem 1.2 in Section 5.

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2. **The Skein models**

Given a planar surface \( F \), the framed HOMFLY skein \( S(F) \) of \( F \) is the \( \Lambda \)-linear combination of orientated tangles in \( F \), modulo the two local relations as showed in figure 1, where \( z = q - q^{-1} \), the coefficient ring \( \Lambda = \mathbb{Z}[q^\pm 1, a^\pm 1] \) with the elements \( q^k - q^{-k} \) admitted as denominators for \( k \geq 1 \). The local relation showed in figure 2 is a consequence of

\[
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\text{Figure 1. Two local relations}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{Figure 2. Removal of a null-homotopic closed curve}
\end{array}
\end{array}
\]

the above relations. It follows that the removal of a null-homotopic closed curve without crossings is equivalent to time a scalar \( s = \frac{a-a^{-1}}{q-q^{-1}} \).

2.1. **The plane.** When \( F = \mathbb{R}^2 \), it is easy to follow that every element in \( S(F) \) can be represented as a scalar in \( \Lambda \). For a link \( \mathcal{L} \) with a diagram \( D_{\mathcal{L}} \), the resulting scalar \( \langle D_{\mathcal{L}} \rangle \in \Lambda \) is the framed HOMFLYPT polynomial of the link \( \mathcal{L} \). In the following, we will also use the notation \( \mathcal{H}(\mathcal{L}; q, a) \) to denote the \( \langle D_{\mathcal{L}} \rangle \) for simplicity. In particular, as to the unknot \( U \), we have \( \mathcal{H}(U; q, a) = \frac{a-a^{-1}}{q-q^{-1}} \).
2.2. The annulus. Let $\mathcal{C}$ be the HOMFLY skein of the annulus, i.e. $\mathcal{C} = S(S^1 \otimes I)$. $\mathcal{C}$ is a commutative algebra with the product induced by placing one annulus outside another. $\mathcal{C}$ is freely generated by the set $\{A_m : m \in \mathbb{Z}\}$, $A_m$ for $m \neq 0$ is the closure of the braid $\sigma_{|m|-1} \cdots \sigma_2 \sigma_1$, and $A_0$ is the empty diagram $\emptyset$. It follows that $\mathcal{C}$ contains two subalgebras $\mathcal{C}_+$ and $\mathcal{C}_-$ which are generated by $\{A_m : m \in \mathbb{Z}, m \geq 0\}$ and $\{A_m : m \in \mathbb{Z}, m \leq 0\}$. The algebra $\mathcal{C}_+$ is spanned by the subspace $\mathcal{C}_{n,0}$. There is a good basis $\{Q_\lambda\}$ of $\mathcal{C}_+$ consisting of the closures of certain idempotents of Hecke algebra $H_{n,0}$.

In [7], R. Hadji and H. Morton constructed the basis elements $\{Q_{\lambda,\mu}\}$ explicitly for $\mathcal{C}$. We will review this construction in next section.

3. Full colored HOMFLYPT invariants

3.1. Partitions and symmetric functions. A partition $\lambda$ is a finite sequence of positive integers $(\lambda_1, \lambda_2, \ldots)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots$. The length of $\lambda$ is the total number of parts in $\lambda$ and denoted by $l(\lambda)$. The degree of $\lambda$ is defined by $|\lambda| = \sum_{i=1}^{l(\lambda)} \lambda_i$. If $|\lambda| = d$, we say $\lambda$ is a partition of $d$ and denoted as $\lambda \vdash d$. The automorphism group of $\lambda$, denoted by $\text{Aut}(\lambda)$, contains all the permutations that permute parts of $\lambda$ by keeping it as a partition. Obviously, $\text{Aut}(\lambda)$ has the order $|\text{Aut}(\lambda)| = \prod_{i=1}^{l(\lambda)} m_i(\lambda)!$ where $m_i(\lambda)$ denotes the number of times that $i$ occurs in $\lambda$. We can also write a partition $\lambda$ as $\lambda = (1^{m_1(\lambda)})(2^{m_2(\lambda)})(\cdots)$. We denote by $\mathcal{P}$ the set of all the partitions.

Every partition can be identified as a Young diagram. The Young diagram of $\lambda$ is a graph with $\lambda_i$ boxes on the $i$-th row for $j = 1, 2, \ldots, l(\lambda)$, where we have enumerate the rows from top to bottom and the columns from left to right. The $j$-th box in the $i$-th row has the coordinates $(i, j)$. The content $c_n(x)$ of the box $x = (i, j)$ is defined to be $j - i$.

Given a partition $\lambda$, we define the conjugate partition $\lambda'$ whose Young diagram is the transposed Young diagram of $\lambda$ which is derived from the Young diagram of $\lambda$ by reflection in the main diagonal. For the box $x = (i, j) \in \lambda$, the hook length is defined to be $h(x) = \lambda_i + \lambda'_j - i - j + 1$.

The following numbers associated with a given partition $\lambda$ are used frequently in this paper: $z_\lambda = \prod_{j=1}^{l(\lambda)} j^{m_j(\lambda)} m_j(\lambda)!$, $\kappa_\lambda = \sum_{j=1}^{l(\lambda)} \lambda_j(\lambda_j - 2j + 1)$. Obviously, $\kappa_\lambda$ is an even number and $\kappa_\lambda = -\kappa_\lambda'$.

The $m$-th complete symmetric function $h_m$ and elementary symmetric function are defined by the following two generating functions

$$H(t) = \sum_{m \geq 0} h_m t^m = \prod_{i \geq 1} \frac{1}{(1 - x_i t)}, \quad E(t) = \sum_{m \geq 0} e_m t^m = \prod_{i \geq 1} (1 + x_i t),$$

respectively.

The power sum symmetric function of infinite variables $x = (x_1, \ldots, x_N, \ldots)$ is defined by $p_n(x) = \sum_i x_i^n$. Given a partition $\lambda$, we define $p_\lambda(x) = \prod_{j=1}^{l(\lambda)} p_{\lambda_j}(x)$. The Schur function $s_\lambda(x)$ is determined by the Frobenius formula

$$s_\lambda(x) = \sum_{|\mu|=|\lambda|} \frac{\chi_\lambda(C_\mu)}{\chi_\mu} p_\mu(x),$$

where $\chi_\lambda$ is the character of the irreducible representation of the symmetric group $S_{|\lambda|}$ corresponding to $\lambda$. $C_\mu$ denotes the conjugate class of symmetric group $S_{|\mu|}$ corresponding...
to partition $\mu$. The orthogonality of character formula gives
\[(3.3) \quad \sum_A \chi_A(C_\mu)\chi_A(C_\nu) = \delta_{\mu\nu}.
\]

For $\lambda, \mu, \nu \in \mathcal{P}$, we define the littlewood-Richardson coefficient $c^\nu_{\lambda,\mu}$ as
\[(3.4) \quad s_\lambda(x)s_\mu(x) = \sum_\nu c^\nu_{\lambda,\mu}s_\nu(x).
\]

It is easy to see that $c^\nu_{\lambda,\mu}$ can be expressed by the characters of symmetric group by using the Frobenius formula
\[(3.5) \quad c^\nu_{\lambda,\mu} = \sum_{\rho,\tau} \frac{\chi_\lambda(C_\rho)}{z_\rho} \frac{\chi_\mu(C_\tau)}{z_\tau} \chi_\nu(C_{\rho,\tau}).
\]

3.2. Basic elements in $\mathcal{C}$. Given a permutation $\pi \in S_m$ with the length $l(\pi)$, let $\omega_\pi$ be the positive permutation braid associated to $\pi$. We have $l(\pi) = w(\omega_\pi)$, the writhe number of the braid $\omega_\pi$.

We define the quasi-idempotent element in Hecke algebra $H_m$,
\[(3.6) \quad a_m = \sum_{\pi \in S_m} q^{l(\pi)}\omega_\pi.
\]

Let element $h_m \in \mathcal{C}_{m,0}$ be the closure of the elements $\frac{1}{\alpha_m}a_m \in H_m$, i.e $h_m = \frac{1}{\alpha_m} \hat{a}_m$. Where $\alpha_m$ is determined by the equation $a_m a_m = \alpha_m a_m$, it gives $\alpha_m = q^{m(m-1)/2} \prod_{i=1}^m \frac{q^i - q^{-i}}{q-q^{-1}}$.

The skein $\mathcal{C}_+$ ($\mathcal{C}_-$) is spanned by the monomials in $\{h_m\}_{m \geq 0}$ ($\{h_k^*\}_{k \geq 0}$). The whole skein $\mathcal{C}$ is spanned by the monomials in $\{h_m, h_k^*\}_{m,k \geq 0}$. $\mathcal{C}_+$ can be regarded as the ring of symmetric functions in variables $x_1, \ldots, x_N$,.. with the coefficient ring $\Lambda$. In this situation, $\mathcal{C}_{m,0}$ consists of the homogeneous functions of degree $m$. The power sum $P_m = \sum x_i^m$ are symmetric functions which can be represented in terms of the complete symmetric functions, hence $P_m \in \mathcal{C}_{m,0}$. Moreover, the following identity was first obtained in $[1]$, and see $[15]$ for a different proof:
\[(3.7) \quad [m]P_m = X_m = \sum_{j=0}^{m-1} A_{m-1-j,j},
\]

where $[m] = \frac{q^m - q^{-m}}{q-q^{-1}}$ and $A_{i,j}$ is the closure of the braid $\sigma_{i+j}\sigma_{i+j-1} \cdots \sigma_j^{-1} \cdots \sigma_i^{-1}$. Given a partition $\mu$, we define
\[(3.8) \quad P_\mu = \prod_{i=1}^{l(\mu)} P_{\mu_i}.
\]

Then, in $[7]$, Hadji and Morton constructed the basis $Q_{\lambda,\mu}$ on the whole skein $\mathcal{C}$ as follow. Given two partitions $\lambda, \mu$ with $l$ and $r$ parts. We first construct a $(l+r) \times (l+r)$-matrix $M_{\lambda,\mu}$ with entries in $\{h_m, h_k^*\}_{m,k \in \mathbb{Z}}$ as follows, where we have let $h_m = 0$, if $m < 0$ and $h_k^* = 0$ if $k < 0$. 
It is easy to note that the subscripts of the diagonal entries in the $h$-rows are the parts $\lambda_1, \lambda_2, ..., \lambda_l$ of $\lambda$ in order, while the subscripts of the diagonal entries in the $h^*$-rows are the parts $\mu_1, \mu_2, ..., \mu_r$ of $\mu$ in reverse order.

Then, $Q_{\lambda, \mu}$ is defined as the determinant of the matrix $M_{\lambda, \mu}$, i.e.

\begin{equation}
Q_{\lambda, \mu} = \det M_{\lambda, \mu}.
\end{equation}

Usually, we write $Q_{\lambda} = Q_{\lambda, \emptyset}$ and $Q_{\mu}^* = Q_{\emptyset, \mu}$, we have

\begin{equation}
Q_{\lambda, \mu} = \sum_{\sigma, \rho, \nu} (-1)^{\sigma} c_{\lambda, \rho} c_{\rho, \nu}^* Q_{\rho} Q_{\nu}^*.
\end{equation}

The Frobenius formula (3.2) implies

\begin{equation}
Q_{\lambda} = \sum_{\mu} \frac{\chi_{\lambda}(\mu)}{z_{\mu}} P_{\mu}.
\end{equation}

3.3. Full colored HOMFLYPT invariants. Let $\mathcal{L}$ be a framed link with $L$ components with a fixed numbering. For diagrams $Q_1, ..., Q_L$ in the skein model of annulus with the positive oriented core $\mathcal{C}$, we define the decoration of $\mathcal{L}$ with $Q_1, ..., Q_L$ as the link

\begin{equation}
\mathcal{L} \otimes_{\alpha=1}^{L} Q_{\alpha}
\end{equation}

which derived from $\mathcal{L}$ by replacing every annulus $\mathcal{L}$ by the annulus with the diagram $Q_{\alpha}$ such that the orientations of the cores match. Each $Q_{\alpha}$ has a small backboard neighborhood in the annulus which makes the decorated link $\mathcal{L} \otimes_{\alpha=1}^{L} Q_{\alpha}$ into a framed link.

In particular, when $Q_{\lambda^\alpha, \mu^\alpha} \in C_{d_{\alpha}, t_{\alpha}}$, where $\lambda^\alpha, \mu^\alpha$ are the partitions of positive integers $d_{\alpha}$ and $t_{\alpha}$ respectively, for $\alpha = 1, ..., L$.

**Definition 3.1.** The full colored HOMFLYPT invariant of $\mathcal{L}$ is defined as follow:

\begin{equation}
W_{[\lambda^1, \mu^1], [\lambda^2, \mu^2], ..., [\lambda^L, \mu^L]}(\mathcal{L}; q, a) = q^{-\sum_{\alpha=1}^{L}(\kappa_{\lambda^\alpha} + \kappa_{\mu^\alpha})w(\kappa_{\alpha}) - \sum_{\alpha=1}^{L}(|\lambda^\alpha| + |\mu^\alpha|)w(\kappa_{\alpha})} \mathcal{H}(\mathcal{L} \otimes_{\alpha=1}^{L} Q_{\lambda^\alpha, \mu^\alpha}; q, a).
\end{equation}

$W_{[\lambda^1, \mu^1], [\lambda^2, \mu^2], ..., [\lambda^L, \mu^L]}(\mathcal{L}; q, a)$ is a framing independent invariant of link $\mathcal{L}$. In fact, by the result in [6], the framing factor for $Q_{\lambda, \mu}$ is $q^{\kappa_{\lambda} + \kappa_{\mu} |\lambda| + |\mu|}$.

4. The symmetries

Before giving the proof of Theorem 1.3, we need an observation which is an easy consequence of the formula (3.7). We introduce the notation $\{m\} = q^m - q^{-m}$, and for a partition $\lambda \in \mathcal{P}$, we let $\{\lambda\} = \prod_{i=1}^{l(\lambda)} \{\lambda_i\}$.
Lemma 4.1. Given any two partitions $\lambda$ and $\mu$, for a knot $K$, we have

$$ (q - q^{-1}) \{\lambda\} \{\mu\} H(K \ast P_\lambda P_\mu^*; q, a) \in \mathbb{Z}[(q - q^{-1})^2, a^{\pm 1}]. $$

Proof. By using the skein relations in Figure 1 for framed HOMFLYPT invariant $H(L; q, a)$, we have, for a link $L$ with $L$ components,

$$ (q^{-1} - q)^L H(L; q, a) \in \mathbb{Z}[(q - q^{-1})^2, a^{\pm 1}]. $$

The formula (3.7) gives:

$$ (q - q^{-1})^L H(L; q, a) = (q - q^{-1}) \sum_{m=0}^{m-1} A_{m-1-j,j}. $$

Therefore, by using formula (4.2), we get

$$ \{\lambda\} \{\mu\} H(K \ast P_\mu^*; q, a) = (q - q^{-1}) \sum_{m=0}^{m-1} H(K \ast A_{m-1-j,j}; q, a) \in \mathbb{Z}[(q - q^{-1})^2, a^{\pm 1}], $$

since $K \ast A_{m-1-j,j}$ is an one-component link for $j = 0, \ldots, m - 1$. Similarly, it is straightforward to obtain the formula (4.1) in the same way.

By using the above integrality result, we prove the following symmetries for the full colored HOMFLYPT invariants.

Theorem 4.2. For any link $L$ with $L$-components,

$$ (4.5) \quad W[\lambda_1, \mu_1] \ast \cdots \ast [\lambda_L, \mu_L](L; q^{-1}, a) = (-1)^{\sum_{i=1}^{L}(|\lambda^i| + |\mu^i|)} W[|\lambda^1\rangle, |\mu^1\rangle, \ldots, |\lambda^L\rangle, |\mu^L\rangle](L; q, a), $$

$$ (4.6) \quad W[\lambda_1, \mu_1] \ast \cdots \ast [\lambda_L, \mu_L](L; -q, a) = (-1)^{\sum_{i=1}^{L}(|\lambda^i| + |\mu^i|)} W[|\lambda^1\rangle, |\mu^1\rangle, \ldots, |\lambda^L\rangle, |\mu^L\rangle](L; q, a), $$

$$ (4.7) \quad W[\lambda_1, \mu_1] \ast \cdots \ast [\lambda_L, \mu_L](L; q, -a) = (-1)^{\sum_{i=1}^{L}(|\lambda^i| + |\mu^i|)} W[|\lambda^1\rangle, |\mu^1\rangle, \ldots, |\lambda^L\rangle, |\mu^L\rangle](L; q, a). $$

Proof. For convenience, we only provide the proof for the case of a knot $K$. The method can be easily generalized to the case for a general link. By Lemma 4.1, for any two partitions $\tau$ and $\delta$, we have

$$ (4.8) \quad \{\tau\} \{\delta\} H(K \ast P_\tau P_\delta^*; q, a) \in \mathbb{Z}[(q - q^{-1})^2, a^{\pm 1}]. $$

Combing with the identity $\{\tau\} \{\delta\} \rightarrow -\delta = (-1)^{|\tau| + |\delta|} \{\tau\} \{\delta\},$ we obtain

$$ (4.9) \quad H(K \ast P_\tau P_\delta^*; -q, a) = (-1)^{|\tau| + |\delta|} H(K \ast P_\tau P_\delta^*; q, a). $$

by the formula (3.12),

$$ (4.10) \quad Q_\rho = \sum_{|\tau| = |\rho|} \frac{\chi_{\rho}(\tau)}{|\tau|} P_\tau, \quad Q_\nu = \sum_{|\delta| = |\nu|} \frac{\chi_{\nu}(\delta)}{|\delta|} P_\delta^*. $$

So we have

$$ (4.11) \quad H(K \ast Q_\rho Q_\nu^*; -q, a) = (-1)^{|\rho| + |\nu|} H(K \ast Q_\rho Q_\nu^*; q, a). $$
By the formula (3.11) for $Q_{\lambda,\mu}$,
\begin{equation}
\mathcal{H}(K \ast Q_{\lambda,\mu}; -q, a) = \sum_{\sigma, \rho, \nu} (-1)^{\sigma} c^\lambda_{\sigma, \rho} c^\mu_{\sigma, \nu} \mathcal{H}(K \ast Q_{\rho} Q^*_\nu; -q, a)
\end{equation}
\begin{equation}
= \sum_{\sigma, \rho, \nu} (-1)^{\sigma} c^\lambda_{\sigma, \rho} c^\mu_{\sigma, \nu} (-1)^{|\sigma| + |\rho|} \mathcal{H}(K \ast Q_{\rho} Q^*_\nu; q, a)
\end{equation}
\begin{equation}
= (-1)^{|\lambda| + |\mu|} \sum_{\sigma, \rho, \nu} (-1)^{\sigma} c^\lambda_{\sigma, \rho} c^\mu_{\sigma, \nu} \mathcal{H}(K \ast Q_{\rho} Q^*_\nu; q, a)
\end{equation}
\begin{equation}
= (-1)^{|\lambda| + |\mu|} \mathcal{H}(K \ast Q_{\lambda,\mu}; q, a),
\end{equation}
where the third “=” is from the observation that $c^\lambda_{\sigma, \rho} = 0$ if $|\lambda| \neq |\sigma| + |\rho|$. Therefore, by the definition of the full colored HOMFLYPT invariants $W_{[\lambda,\mu]}(\mathcal{L}; q, a)$ in formula (3.14), we obtain
\begin{equation}
W_{[\lambda,\mu]}(K, -q, a) = (-1)^{|\lambda| + |\mu|} W_{[\lambda,\mu]}(K, q, a),
\end{equation}
since $\kappa_{\lambda}$ is even for any partition $\lambda$.

Similarly, by formula (4.8), we have
\begin{equation}
\mathcal{H}(K \ast P_\tau P^*_\delta; q^{-1}, a) = (-1)^{l(\tau) + l(\delta)} \mathcal{H}(K \ast P_\tau P^*_\delta; q, a).
\end{equation}
By using the identities,
\begin{equation}
\chi_{\rho^t}(\tau) = (-1)^{|\rho|-l(\tau)} \chi_{\rho}(\tau), \quad \chi_{\nu^t}(\delta) = (-1)^{|\nu|-l(\delta)} \chi_{\rho}(\delta),
\end{equation}
we have
\begin{equation}
\mathcal{H}(K \ast Q_{\rho} Q^*_\nu; q^{-1}, a) = (-1)^{|\rho| + |\nu|} \mathcal{H}(K \ast Q_{\rho} Q^*_\nu; q, a).
\end{equation}
By the formula (3.5) for $c^\lambda_{\sigma, \rho}$, it is easy to get
\begin{equation}
c^\lambda_{\sigma, \rho} = c^{\lambda^t}_{\sigma^t, \rho^t}, \quad c^\mu_{\sigma, \nu} = c^\mu_{\sigma, \nu^t}.
\end{equation}
Therefore,
\begin{equation}
\mathcal{H}(K \ast Q_{\lambda,\mu}; q^{-1}, a) = (-1)^{|\lambda| + |\mu|} \mathcal{H}(K \ast Q_{\lambda,\mu}; q, a),
\end{equation}
i.e.
\begin{equation}
W_{[\lambda,\mu]}(K, q^{-1}, a) = (-1)^{|\lambda| + |\mu|} W_{[\lambda,\mu]}(K, q, a),
\end{equation}
since $\kappa_{\lambda} = -\kappa_{\lambda^t}$ and $\kappa_{\mu} = -\kappa_{\mu^t}$.

Finally, in order to show the last identity (4.7), by using the Lemma 4.3 proved in [4], we have (see the formula (4.22) in [4]):
\begin{equation}
\mathcal{H}(K \ast P_\tau P^*_\delta; q, -a) = (-1)^{(|\tau| + |\delta|)(w(K)-1)} \mathcal{H}(K \ast P_\tau P^*_\delta; q, a).
\end{equation}
Then it is direct to show
\begin{equation}
\mathcal{H}(K \ast Q_{[\lambda,\mu]}; q, -a) = (-1)^{|\lambda| + |\mu|)(w(K)-1)} \mathcal{H}(K \ast Q_{[\lambda,\mu]}; q, a),
\end{equation}
i.e.
\begin{equation}
W_{[\lambda,\mu]}(K, q, -a) = (-1)^{|\lambda| + |\mu|} W_{[\lambda,\mu]}(K, q, a).
\end{equation}
5. Proof of the strong integrality

Let us first consider the case of a knot $K$. Recall the definition of the reduced full colored HOMFLYPT invariant

\[ P[\lambda,\mu](K; q, a) = W[\lambda,\mu](K; q, a) \]

The main result in [16] (see Theorem 1 in [16]) can be written in the following form:

**Proposition 5.1** (Morton). For any knot $K$, $P[\lambda,\mu](K; q, a)$ is a 2-variable integer Laurent polynomial, i.e. $P[\lambda,\mu](K; q, a) \in \mathbb{Z}[q^\pm 1, a^\pm 1]$.

By the formulas (4.6) and (4.7) in Theorem 4.2, we immediately obtain:

\[ P[\lambda,\mu](K; -q, a) = P[\lambda,\mu](K; q, a), \quad P[\lambda,\mu](K; q, -a) = P[\lambda,\mu](K; q, a). \]

Therefore, $P[\lambda,\mu](K; q, a)$ satisfies the following strong integrality:

**Theorem 5.2.** For any knot $K$, $P[\lambda,\mu](K; q, a) \in \mathbb{Z}[q^\pm 2, a^\pm 2]$.

In fact, Proposition 5.1 can be generalized slightly to the case of any link as follow. For a link $L$ with $L$ components $K_1, \ldots, K_L$, if we only decorate the $\alpha$-th component $K_\alpha$ with $Q_{\lambda^\alpha,\mu^\alpha}$, then its full colored HOMFLYPT invariant is denoted by $W[\lambda^\alpha,\mu^\alpha](L; q, a)$. Let

\[ P[\lambda^\alpha,\mu^\alpha](L; q, a) = \frac{W[\lambda^\alpha,\mu^\alpha](L; q, a)}{W[\lambda^\alpha,\mu^\alpha](U; q, a)}, \]

we also have

**Proposition 5.3.** $P[\lambda^\alpha,\mu^\alpha](L; q, a) \in \mathbb{Z}[q^{\pm 1}, a^{\pm 1}]$.

**Proof.** A slight modification for the proof of Proposition 5.1 presented in [16] (see page 333-334) will give the proof of Proposition 5.3. In fact, given a link $L$ with $L$ components $K_1, \ldots, K_L$, we denote it by $L = K_1 \vee \cdots \vee K_L$. We cut the component $K_\alpha$ open and get a 1-tangle, so we can draw $L$ in the annulus as the closure of this 1-tangle. Decorating $K_\alpha$ with a diagram $Q_\alpha$ gives a diagram $K_1 \vee \cdots \vee K_\alpha \star Q_\alpha \vee \cdots \vee K_L$ in $C$, it induces a linear map $T_{K_\alpha}^C : C \to C$. Refer to the page 332 in [16] for this construction in the case of $K$. If $Q_\alpha$ is an eigenvector of $T_{K_\alpha}^C$ with eigenvalue $a_{K_\alpha}^C$, then $K_1 \vee \cdots \vee K_\alpha \star Q_\alpha \vee \cdots \vee K_L = T_{K_\alpha}^C(Q_\alpha) = a_{K_\alpha}^C Q_\alpha = a_{K_\alpha}^C U \star Q_\alpha$ implies that

\[ a_{K_\alpha}^C = \frac{\mathcal{H}(K_1 \vee \cdots \vee K_\alpha \star Q_\alpha \vee \cdots \vee K_L; q, a)}{\mathcal{H}(U \star Q_\alpha; q, a)}. \]

The eigenvectors of $T_{K_\alpha}^C$ are given by $Q_{\lambda,\mu}$ since the map $T_{K_\alpha}^C$ commutes with the meridian map $\phi$, see [7]. We denote the eigenvalue of $T_{K_\alpha}^C$ corresponding to eigenvector by $a_{K_\alpha}^C(\lambda, \mu)$, the proof of Theorem 1 in [16] (see page 334) shows that

\[ a_{K_\alpha}^C(\lambda, \mu) \in \mathbb{Z}[q^{\pm 1}, a^{\pm 1}]. \]

Moreover, Proposition 5.3 immediately implies that

**Proposition 5.4.** For any link $L$, the full colored HOMFLYPT invariants $W[\lambda^1,\mu^1,\ldots,\lambda^L,\mu^L](L; q, a)$ contains a factor $W[\lambda^\alpha,\mu^\alpha](U; q, a)$ where $\alpha$ can be chosen to be $1, \ldots, L$. 
This phenomenon has been showed in [17] after doing many detailed calculations for colored HOMFLYPT invariants with symmetric representations by Chern-Simons theory.

Now, we recall the definition of normalized $[\lambda^\alpha, \mu^\alpha]$-reduced full colored HOMFLYPT invariant in Section 1.

\begin{equation}
\text{Proof.} \quad P_{[\lambda^1, \mu^1], \ldots, [\lambda^L, \mu^L]}^{[\lambda^\alpha, \mu^\alpha]}(L; q, a) = \prod_{\beta=1, \beta \neq \alpha}^L \prod_{\rho^\beta, \mu^\beta} \prod_{x \in \rho^\beta, \mu^\beta} \left(a \cdot q^{c_{n}(x)}(q^{h_{l}(x)} - q^{-h_{l}(x)}) \right) \cdot Q_{[\lambda^\alpha, \mu^\alpha]}^{[\lambda^1, \mu^1], \ldots, [\lambda^L, \mu^L]}(L; q, a),
\end{equation}

where

\begin{equation}
Q_{[\lambda^\alpha, \mu^\alpha]}^{[\lambda^1, \mu^1], \ldots, [\lambda^L, \mu^L]}(L; q, a) = \frac{W_{[\lambda^1, \mu^1], \ldots, [\lambda^L, \mu^L]}(L; q, a)}{W_{[\lambda^\alpha, \mu^\alpha]}(U; q, a)} \text{ for } \alpha = 1, \ldots, L.
\end{equation}

We have the following strong integrality:

**Theorem 5.5.** For $\alpha = 1, \ldots, L$, we have

\begin{equation}
P_{[\lambda^1, \mu^1], \ldots, [\lambda^L, \mu^L]}^{[\lambda^\alpha, \mu^\alpha]}(L; q, a) \in \mathbb{Z}[q^{\pm 2}, a^{\pm 1}].
\end{equation}

**Proof.** By formula (3.11) and (3.12),

\begin{equation}
Q_{\lambda, \mu} = \sum_{\sigma, \rho, \nu} (-1)^{\sigma} c_{\sigma, \rho} c_{\sigma, \nu} Q_{\rho} Q_{\nu}^*,
\end{equation}

\begin{equation}
= \sum_{\sigma, \rho, \nu, \tau, \delta} (-1)^{\sigma} c_{\sigma, \rho} c_{\sigma, \nu} \frac{\chi_{\rho}(\nu) \chi_{\nu}(\delta)}{z_{\tau} z_{\delta}} \frac{1}{\{\tau\} \{\delta\}} X_{\tau} X_{\delta}^*,
\end{equation}

Therefore,

\begin{equation}
L \ast \otimes_{\beta=1}^L Q_{\lambda^\alpha, \mu^\alpha} = K_1 \ast Q_{\lambda^1, \mu^1} \vee \cdots \vee K_\alpha \ast Q_{\lambda^\alpha, \mu^\alpha} \vee \cdots \vee K_L \ast Q_{\lambda^L, \mu^L}
= \sum_{\tau^\beta, \delta^\beta, \beta \neq \alpha} C_{\tau^1, \delta^1, \ldots, \tau^\alpha, \delta^\alpha, \ldots, \tau^L, \delta^L} \prod_{\beta=1, \beta \neq \alpha} \frac{1}{\{\tau^\beta\} \{\delta^\beta\}}
\cdot K_1 \ast X_{\tau^1} X_{\delta^1}^* \vee \cdots \vee K_\alpha \ast Q_{[\lambda^\alpha, \mu^\alpha]} \vee \cdots \vee K_L \ast X_{\tau^L} X_{\delta^L}^*,
\end{equation}

where

\begin{equation}
C_{\tau^1, \delta^1, \ldots, \tau^\alpha, \delta^\alpha, \ldots, \tau^L, \delta^L} = \sum_{\sigma^\beta, \rho^\beta, \nu^\beta, \tau^\gamma, \delta^\gamma, \beta \neq \alpha, \beta \neq \alpha} \prod_{\sigma^\beta, \rho^\beta, \nu^\beta, \tau^\gamma, \delta^\gamma} (-1)^{\sigma^\beta} c_{\sigma^\beta, \rho^\beta} c_{\sigma^\beta, \nu^\beta} \frac{\chi_{\rho^\beta}(\nu^\beta) \chi_{\nu^\beta}(\delta^\beta)}{z_{\tau^\beta} z_{\delta^\beta}},
\end{equation}

and $\tau^\alpha, \delta^\alpha$ denote the indexes $\tau^\alpha, \delta^\alpha$ do not appear in the summation.

We denote the link $L_{\tau^1, \delta^1, \ldots, \tau^\alpha, \delta^\alpha, \ldots, \tau^L, \delta^L} = K_1 \ast X_{\tau^1} X_{\delta^1}^* \vee \cdots \vee K_\alpha \vee \cdots \vee K_L \ast X_{\tau^L} X_{\delta^L}^*$.

By formula (5.5),

\begin{equation}
a_{K_\alpha}^{L_{\tau^1, \delta^1, \ldots, \tau^\alpha, \delta^\alpha, \ldots, \tau^L, \delta^L}}(\lambda, \mu) \in \mathbb{Z}[q^{\pm 1}, a^{\pm 1}],
\end{equation}
and

\[ \mathcal{H}(L \star \otimes_{\beta=1}^{L} Q_{[\lambda^\alpha, \mu^\alpha]}; q, a) = \sum_{\tau^\beta, \delta^\beta, \beta \neq \alpha} C_{\tau^1, \delta^1, \ldots, \tau^L, \delta^L} \prod_{\beta=1, \beta \neq \alpha} \frac{1}{\tau^\beta} \{\delta^\beta\} \cdot Q_{[\lambda^\alpha, \mu^\alpha]}(\lambda, \mu) \mathcal{H}(U \star Q_{[\lambda^\alpha, \mu^\alpha]}; q, a). \]

Recall that we have two expressions for the colored HOMFLYPT invariant for unknot \( U \), the first one is given by (see [12]),

\[ \mathcal{H}(U \star Q_{\lambda}; q, a) = \sum_{\mu} \frac{\chi_{\lambda}(\mu)}{z_{\mu}} \prod_{i=1}^{l(\mu)} a^{\mu_i} - a^{-\mu_i} q^{\mu_i} - q^{-\mu_i}. \]

Another one is given by (see Lemma 3.6.1 in page 51 in [10]),

\[ \mathcal{H}(U \star Q_{\lambda}; q, a) = \prod_{x \in \lambda} \frac{a^{-1} q^{cn(x)} - aq^{-cn(x)}}{q^{h(x)} - q^{-h(x)}}. \]

Therefore, we have

\[ \prod_{x \in \lambda} (q^{h(x)} - q^{-h(x)}) \sum_{\mu} \frac{\chi_{\lambda}(\mu)}{z_{\mu}} \prod_{i=1}^{l(\mu)} \frac{1}{q^{\mu_i} - q^{-\mu_i}} \in \mathbb{Z}[q^{\pm 1}, a^{\pm 1}]. \]

So for any \( \beta \neq \alpha \),

\[ \prod_{x \in \rho^\beta, \nu^\beta} \{hl(x)\} \sum_{\rho^\beta, \nu^\beta} \frac{\chi_{\rho^\beta}(\tau^\beta)}{z_{\tau^\beta}} \frac{\chi_{\nu^\beta}(\delta^\beta)}{z_{\delta^\beta}} \frac{1}{\tau^\beta} \{\delta^\beta\} \in \mathbb{Z}[q^{\pm 1}, a^{\pm 1}]. \]

Therefore, we have

\[ \prod_{\beta=1, \beta \neq \alpha} \prod_{|\lambda^\beta| - |\rho^\beta| = |\mu^\beta| - |\nu^\beta|} \{hl(x)\} \sum_{\tau^\beta, \delta^\beta, \beta \neq \alpha} C_{\tau^1, \delta^1, \ldots, \tau^L, \delta^L} \prod_{\beta=1, \beta \neq \alpha} \frac{1}{\tau^\beta} \{\delta^\beta\} \in \mathbb{Z}[q^{\pm 1}, a^{\pm 1}]. \]

Hence,

\[ P_{[\lambda^1, \mu^1], \ldots, [\lambda^L, \mu^L]}(L; q, a) \in \mathbb{Z}[q^{\pm 1}, a^{\pm 1}]. \]

On the other hand, by formulas (4.6) and (4.7), we have

\[ Q_{[\lambda^1, \mu^1], \ldots, [\lambda^L, \mu^L]}(L; \pm q, \mp a) = (-1)^{\sum_{\beta \neq \alpha} (|\lambda^\beta| + |\mu^\beta|)} Q_{[\lambda^1, \mu^1], \ldots, [\lambda^L, \mu^L]}(L; q, a). \]

Combing the two expressions (5.14) and (5.15) for colored HOMFLYPT invariant of the unknot \( U \), for any partition \( \lambda \), we obtain the identity:

\[ (-1)^{|\lambda|} = (-1)^{\sum_{x \in \lambda} (hl(x) + cn(x))}. \]
Thus
\begin{equation}
P_{[\lambda^1, \mu^1] \ldots [\lambda^L, \mu^L]}(\mathcal{L}; \pm q, \mp a) = P_{[\lambda^1, \mu^1] \ldots [\lambda^L, \mu^L]}(\mathcal{L}; q, a).
\end{equation}

Combing formulas (5.20) and (5.23), we obtain
\begin{equation}
P_{[\lambda^1, \mu^1] \ldots [\lambda^L, \mu^L]}(\mathcal{L}; q, a) \in \mathbb{Z}[q^{\pm 2}, a^{\pm 2}].
\end{equation}

□

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