$L^p$-Fourier asymptotics, Hardy-type inequality and fractal measures

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Received: 31 May 2016 / Accepted: 22 June 2017 / Published online: 29 June 2017
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Abstract Suppose $\mu$ is an $\alpha$-dimensional fractal measure for some $0 < \alpha < n$. Inspired by the results proved by Strichartz (J. Funct. Anal. 89:154–187, 1990), we discuss the $L^p$-asymptotics of the Fourier transform of $f \, d\mu$ by estimating bounds of

$$\liminf_{L \to \infty} \frac{1}{L^k} \int_{|\xi| \leq L} |\hat{f} \, d\mu(\xi)|^p \, d\xi,$$

for $f \in L^p(d\mu)$ and $2 < p < 2n/\alpha$. In a different direction, we prove a Hardy type inequality, that is,

$$\int \frac{|f(x)|^p}{(\mu(E_x))^{p-2}} \, d\mu(x) \leq C \liminf_{L \to \infty} \frac{1}{L^{n-\alpha}} \int_{B_L(0)} |\hat{f} \, d\mu(\xi)|^p \, d\xi$$

where $1 \leq p \leq 2$ and $E_x = E \cap (-\infty, x_1) \times (-\infty, x_2) \times \cdots \times (-\infty, x_n)$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ generalizing the one dimensional results by Hudson and Leckband (J. Funct. Anal. 108:133–160, 1992).

Keywords Supports of Fourier transform · Hausdorff dimension · Minkowski content · Salem sets · Ahlfors–David regular sets · Hardy type inequality

Mathematics Subject Classification Primary 42B10; Secondary 37F35 · 40E05 · 28A78

Communicated by A. Constantin.

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1 Introduction

One of the basic questions in harmonic analysis is to study the decay properties of the Fourier transform of measures or distributions supported on thin sets (sets of Lebesgue measure zero) in $\mathbb{R}^n$. Let $f \in C^\infty_c(\mathbb{R}^n)$ and $d\sigma$ be the surface measure on the sphere $S^{n-1} \subset \mathbb{R}^n$. Then using the properties of Bessel functions,

$$|\hat{f}d\sigma(\xi)| \leq C (1 + |\xi|)^{-\frac{n-1}{2}}.$$  

It follows that $\hat{f}d\sigma \in L^p(\mathbb{R}^n)$ for all $p > \frac{2n}{n-1}$. This result can be extended to compactly supported measures on $(n-1)$-dimensional manifolds with appropriate assumptions on the curvature. On the other hand, the results in [2] show that $\hat{f}d\sigma \notin L^p(\mathbb{R}^n)$ for $1 \leq p \leq \frac{2n}{n-1}$. Similar results are known for measures supported in lower dimensional manifolds in $\mathbb{R}^n$ under appropriate curvature conditions (see pages 347–351 in [13]). However, the picture for fractal measures is far from complete.

In [12], we related the integrability of the function and the fractal dimension of the support of its Fourier transform by proving the following theorem and its sharpness.

**Theorem 1.1** [12] Let $f \in L^1 \cap L^p(\mathbb{R}^n)$ be such that $\hat{f}$ is supported in a set $E \subset \mathbb{R}^n$. Suppose $E$ is a set of finite $\alpha$-packing measure, $0 < \alpha < n$. Then $f$ is identically zero, provided $p \leq 2n/\alpha$.

Inspired by results in [14], we look for quantitative estimates for Fourier transform of fractal measures. Let $E$ be a compact set of finite upper Minkowski’s $\alpha$-content. In this paper, we obtain certain quantitative versions of Theorem 1.1 by estimating the $L^p$ norm of the Fourier transform of appropriate fractal measures $\mu$ supported in $E$ over a ball centered at origin with large radius for $1 \leq p < 2n/\alpha$, that is, by obtaining bounds for the following:

$$\limsup_{L \to \infty} \frac{1}{L^k} \int_{||\xi|| \leq L} |\hat{f}d\mu(\xi)|^p d\xi,$$

where $k$ depends on $\alpha$, $p$ and $n$.

If $\mu$ is a compactly supported locally uniformly $\alpha$-dimensional measure, that is, $\mu(B_r(x)) \leq ar^\alpha$ for all $0 < r \leq 1$ and some non-zero finite constants $a$, then in [14], Strichartz proved that there exists constant $C$ independent of $f$ such that

$$\limsup_{L \to \infty} \frac{1}{L^{n-\alpha}} \int_{||\xi|| \leq L} |\hat{f}d\mu(\xi)|^2 d\xi \leq C \|f\|_{L^2(d\mu)}^2. \tag{1.1}$$

The authors in [9] and [10] have generalized (1.1) for a general class of measures. If a locally uniformly $\alpha$-dimensional measure $\mu$ is supported in a quasi $\alpha$-regular set, then in [14] the author proved that there exists a non-zero constant independent of $f$ such that

$$\|f\|_{L^2(d\mu)}^2 \leq C \liminf_{L \to \infty} \frac{1}{L^{n-\alpha}} \int_{||\xi|| \leq L} |\hat{f}d\mu(\xi)|^2 d\xi. \tag{1.2}$$
Using Hölder’s inequality, we note from (1.2) that if \( f \in L^2(d\mu) \), where \( \mu \) is a locally uniformly \( \alpha \)-dimensional measure, then for \( 1 \leq p \leq 2 \),

\[
\| f \|_{L^2(d\mu)}^2 \leq C \limsup_{L \to \infty} \frac{1}{L^{n-\alpha p/2}} \int_{|\xi| \leq L} |\hat{f}d\mu(\xi)|^p d\xi. \tag{1.3}
\]

Suppose \( \mu \) is a finite measure supported on a set \( E \) such that \( \alpha \)-dimensional upper Minkowski content of non-zero \( \mu \)-measure bounded subsets \( S \) of \( E \) is non-zero and bounded above by \( \mu(S) \) for some \( 0 < \alpha < n \), that is,

\[
\limsup_{\epsilon \to 0} |S(\epsilon)|^{\alpha-n} \leq C \mu(S).
\]

Note that \( \mu \) need not be locally uniformly \( \alpha \)-dimensional measure. We prove (1.3) for \( \mu \):

**Theorem 1.2** Let \( f \in L^2(d\mu) \) be a positive function. Then for \( 2 \leq p < 2n/\alpha \),

\[
\int_{\mathbb{R}^n} |f(x)|^2 d\mu(x) \leq C \liminf_{L \to \infty} \left( \frac{1}{L^{n-\alpha p/2}} \int_{|\xi| \leq L} |\hat{f}d\mu(\xi)|^p d\xi \right)^{2/p}. \tag{1.4}
\]

Next we prove that the above estimate (1.4) is optimal.

**Theorem 1.3** Let \( u \) be a tempered distribution supported in a set \( E \) such that \( \alpha \)-dimensional upper Minkowski content of all non-zero \( \mu \)-measure bounded subsets \( S \) of \( E \) is non-zero and bounded above by \( \mu(S) \) where \( \mu \) is a finite measure supported on \( E \) and for \( 2 \leq p < 2n/\alpha \),

\[
\limsup_{L \to \infty} \frac{1}{L^{n-\alpha p/2}} \int_{|\xi| \leq L} |\hat{u}(\xi)|^p d\xi < \infty.
\]

Then \( u \) is an \( L^2 \) density \( u_0 d\mu \) on \( E \) and

\[
\left( \int_{E} |u_0|^2 d\mu \right)^{p/2} \leq C \limsup_{L \to \infty} \frac{1}{L^{n-\alpha p/2}} \int_{|\xi| \leq L} |\hat{u}(\xi)|^p d\xi < \infty.
\]

Related type of results for tempered distributions supported in a smooth manifold of dimension \( d < n \) can be found in [1], where the authors proved that if \( u \) is a tempered distribution such that \( \hat{u} \in L^2_{\text{loc}}(\mathbb{R}^n) \),

\[
\limsup_{L \to \infty} \frac{1}{L^{n-d}} \int_{|\xi| \leq L} |\hat{u}(\xi)|^2 d\xi < \infty
\]

and the restriction of \( u \) to an open subset \( X \) of \( \mathbb{R}^n \) is supported by a \( C^1 \) submanifold \( M \) of codimension \( k = n - d \), then it is an \( L^2 \)-density \( u_0 dS \) on \( M \) and
\[
\int_M |u_0|^2 dS \leq C \limsup_{L \to \infty} \frac{1}{L^k} \int_{|\xi| \leq L} |\hat{u}(\xi)|^2 d\xi,
\]
where \(C\) only depends on \(n\).

In a different direction, we consider the results of Hudson and Leckband in [7] that gives a Hardy type inequality for discrete measures. Let \(\|u\|_{B^p,a,p} = \lim_{L \to \infty} L^{-1} \int_{-L}^L |u(x)|^p dx\), \(f \, d\mu_0\) be the zero-dimensional measure \(f(x) = \sum_1^\infty c_k \delta(x - a_k)\) and let \(1 < p \leq 2\) where \(c_k\) is a sequence of complex numbers and \(d_k\) is a sequence of real numbers not necessarily increasing. Assume that \(u(x) = \hat{f} \, d\mu_0(x)\). The authors in [7] proved that if \(c_k^*\) denote the nonincreasing rearrangement of the sequence \(|c_k|\), then

\[
\sum_{k} \frac{|c_k|^p}{k^{2-p}} \leq \sum_{k} \frac{|c_k|^p}{k^{2-p}} \leq C \|u\|_{B^p,a,p}^p. \tag{1.5}
\]

To extend the above result from zero-dimensional sets to a fractional dimensional set, the authors in [7] defined \(\alpha\)-coherent set in \(\mathbb{R}^n\): for \(0 < \alpha < 1\), a set \(E \subset \mathbb{R}^n\) of finite \(\alpha\)-dimensional Hausdorff measure is called \(\alpha\)-coherent if for all \(x\),

\[
\limsup_{\epsilon \to 0} |E_{x}^0(\epsilon)| \epsilon^{\alpha-n} \leq C_E \mathcal{H}_\alpha(E_{x}^0),
\]

where \(E_{x}^0 = \{y \in E : y \leq x\} \text{ and } 2^{-\alpha} \leq \limsup_{\delta \to 0} \frac{\mathcal{H}_\alpha(E \cap (y-\delta,y+\delta))}{\delta^{\alpha}} \leq 1\) and \(E_{x}^0(\epsilon)\) denotes the \(\epsilon\)-distance set of \(E_{x}^0\).

Let \(E \subset \mathbb{R}\) be either an \(\alpha\)-coherent set or a quasi \(\alpha\)-regular set of finite \(\alpha\)-dimensional Hausdorff measure, for \(0 < \alpha < 1\), and \(f \in L^1(d\mu)\), where \(\mu = \mathcal{H}_\alpha|_E\). Then the authors in [7] proved that

\[
\int_E \frac{|f(x)|}{\mu(E_x)} \, d\mu(x) \leq C \liminf_{L \to \infty} \frac{1}{L^{1-\alpha}} \int_{-L}^L |\hat{f} \, d\mu(\xi)| \, d\xi, \tag{1.6}
\]

where \(C\) is a constant independent of \(f\).

Using the upper Minkowski content and finding a continuous analogue of the arguments in [7], we extend (1.5) and (1.6) to \(\mathbb{R}^n\) and also generalize (1.5) to any \(\alpha\)-dimensional measure \((0 < \alpha < n)\) and \(n \geq 1\) with a slight modification in the hypothesis. For \(E \subset \mathbb{R}^n\), let

\[
E_x = E \cap (-\infty, x_1] \times \cdots \times (-\infty, x_n]
\]

for \(x = (x_1, \ldots, x_n) \in \mathbb{R}^n\):

**Theorem 1.4** Let \(0 < \alpha < n\). Let \(\mu\) be a finite measure supported in the set \(E\) such that \(\alpha\)-dimensional upper Minkowski content of all non zero \(\mu\)-measure bounded subsets \(S\) of \(E\) is non-zero and bounded by \(\mu(S)\). Let \(f \in L^p(d\mu)\) \((1 \leq p \leq 2)\) be a
positive function. Then there exists a constant $C$ independent of $f$ such that

$$\int \frac{|f(x)|^p}{(\mu(E_x))^{2-p}} d\mu(x) \leq C \liminf_{L \to \infty} \frac{1}{L^{n-\alpha}} \int_{B_L(0)} |\hat{f}(\xi)|^p d\xi.$$

We recollect some definitions and notations in the rest of this section from [4,5] and [11]. We study the $L^p$-asymptotics of the Fourier transform of the fractal measures for $p \geq 2$ in the Sect. 2 and for $1 \leq p \leq 2$, we prove generalized Hardy inequality in the Sect. 3.

Let $X$ be a metric space, $F$ a family of subsets of $X$ such that for every $\delta > 0$ there are $E_1, E_2, \ldots \in F$ such that diameter of $E_k$ is less than or equal to $\delta$ for all $k$ and $X = \bigcup_k E_k$. For $0 < \delta \leq \infty$ and $A \subset X$, we define $s$-dimensional Hausdorff measure as

$$\mathcal{H}_s(A) = \lim_{\delta \to 0} \mathcal{H}_s^\delta(A),$$

where

$$\mathcal{H}_s^\delta(A) = \inf \left\{ \sum_{i=1}^{\infty} d(E_i)^s : A \subset \bigcup_i E_i, d(E_i) \leq \delta, E_i \in F \right\}$$

and $d(E)$ denotes the diameter of the set $E$.

The Hausdorff dimension of a set $A$ is given by

$$\dim_H(A) = \sup \{ s : \mathcal{H}_s(A) > 0 \} = \sup \{ s : \mathcal{H}_s(A) = \infty \} = \inf \{ t : \mathcal{H}_t(A) < \infty \} = \inf \{ t : \mathcal{H}_t(A) = 0 \}.$$

For a non-empty subset $A$ of $\mathbb{R}^n$, let $A(\epsilon) = \{ x \in \mathbb{R}^n : \inf_{y \in A} |x - y| < \epsilon \}$ denote the closed $\epsilon$-neighborhood of $A$. Some authors call $A(\epsilon)$, the $\epsilon$-parallel set of $A$ or $\epsilon$-distance set of $A$. Let $E$ be a non-empty bounded subset of $\mathbb{R}^n$. The $\epsilon$-covering number of $E$ denoted by $N(E, \epsilon)$, is the smallest number of open balls of radius $\epsilon$ needed to cover $E$. The $\epsilon$-packing number of $E$ denoted by $P(E, \epsilon)$ is the largest number of disjoint open balls of radius $\epsilon$ with centres in $E$. The $\epsilon$-packing of $E$ is any collection of disjoint balls $\{B_{r_k}(x_k)\}_k$ with centres $x_k \in E$ and radii satisfying $0 < r_k \leq \epsilon/2$. Then we have the following lemma:

**Lemma 1.5** [11] Fix $\epsilon > 0$. Let $A$ be a non-empty bounded subset of $\mathbb{R}^n$ and $|A(\epsilon)|$ denote the Lebesgue measure of $A(\epsilon)$, where $A$ is a non-empty bounded subset of $\mathbb{R}^n$. Then,

1. $N(A, 2\epsilon) \leq P(A, \epsilon) \leq N(A, \epsilon/2)$.
2. $\Omega_n P(A, \epsilon) \epsilon^n \leq |A(\epsilon)| \leq \Omega_n N(A, \epsilon)(2\epsilon)^n$,
   where $\Omega_n$ denotes the volume of the unit ball in $\mathbb{R}^n$.
3. For $0 \leq s < \infty$, $P(A, \epsilon/2) \epsilon^s \leq P^s(A)$.

(see pages 78–79 in [11].)
The $s$-dimensional **upper** and **lower Minkowski contents** of $A$ are defined by

$$\mathcal{M}^{ss}(A) = \limsup_{\epsilon \to 0} (2\epsilon)^{s-n}|A(\epsilon)|, \quad (1.7)$$

$$\mathcal{M}^{*s}(A) = \liminf_{\epsilon \to 0} (2\epsilon)^{s-n}|A(\epsilon)|. \quad (1.8)$$

**Definition 1.6** [8] A similitude $S$ is a map $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(x) = sR(x) + b, \ x \in \mathbb{R}^n$$

for some isometry $R, b \in \mathbb{R}^n$ and $0 < s < 1$. The number $s$ is called contraction ratio or dilation factor of $S$. Let $S = \{S_1, \ldots, S_m\}, \ m \geq 2$ be a collection of finite set of similitudes with dilation factors $s_1, \ldots, s_m$ (so that $S_j = s_j R_j + b_j$ where $R_j$ denotes an isometry and $b_j \in \mathbb{R}^n$). We say that a non-empty compact set $K$ is **invariant** under $S$ if

$$K = \bigcup_{j=1}^m S_j K.$$

$S$ satisfies the **open set condition** if there is a non-empty open set $O$ such that $\bigcup_{j=1}^m S_j(O) \subset O$ and $S_j(O) \cap S_k(O) = \emptyset$ for $j \neq k$. We call the invariant set $K$ under $S$ to be **self-similar** if with $\alpha = \dim_H(K)$,

$$\mathcal{H}_\alpha(S_{j_1}(K) \cap S_{j_2}(K)) = 0 \quad \text{for} \ j_1 \neq j_2.$$

**2 $L^p$-asymptotic properties of fractal measures for $2 \leq p < 2n/\alpha$**

Let $\mu$ denote a fractal measure supported in an $\alpha$-dimensional set $E \subset \mathbb{R}^n$ and $f \in L^q(d\mu) \ (1 \leq q \leq \infty)$. Suppose $2 < p \leq 2n/\alpha$. In this section, we obtain the bounds for

$$\frac{1}{L^k} \int_{|\xi| \leq L} |\hat{f}d\mu(\xi)|^p d\xi \quad (2.1)$$

for some positive $k$.

We start recalling a few results proved in [14]:

**Theorem 2.1** [14] Let $\mu$ be a locally uniformly $\alpha$-dimensional measure and $f \in L^2(d\mu)$. Then

$$\sup_{0 < t \leq 1} t^{(n-\alpha)/2} \int_{\mathbb{R}^n} e^{-t|\xi|^2/2} |\hat{f}d\mu(\xi)|^2 d\xi \leq C \|f\|^2_{L^2(d\mu)}.$$

Also, with the use of mean quadratic variation, Lau in [9] investigated the fractal measures by defining a class of complex valued $\sigma$-finite Borel measures $\mu$ on $\mathbb{R}^n$, $\mathcal{M}^p_\sigma$, for $1 \leq p < \infty$ with \[ Springer \]
\[ \|\mu\|_{M_0^p} = \sup_{0 < \delta \leq 1} \left( \frac{1}{\delta^{n+\alpha(p-1)}} \int_{\mathbb{R}^n} |\mu(Q_\delta(x))|^p \right)^{1/p} < \infty \]

and

\[ \|\mu\|_{M_0^\infty} = \sup_u \sup_{0 < \delta < 1} \frac{1}{(2\delta)\alpha} |\mu|(Q_\delta(u)) < \infty, \]

where \( Q_\delta(u) \) denotes the half open cube \( \prod_{j=1}^n (x_j - \delta, x_j + \delta) \). For \( 1 \leq p < \infty, \) \( \alpha < n, \) \( B^p_\alpha \) denotes the set of all locally \( p \)-th integrable function \( f \) in \( \mathbb{R}^n \) such that

\[ \|f\|_{B^p_\alpha} = \sup_{L \geq 1} \left( \frac{1}{L^{n-\alpha}} \int_{B_L} |f|^p \right)^{1/p} < \infty. \]

For \( 0 \leq \alpha \leq \beta < n, \) we have from [10], \( B^p_\beta \subseteq B^p_\alpha \subseteq L^p(dx/(1 + |x|^{n+1})). \)

For \( \delta > 0, \) we define the transformation \( W_\delta \) as

\[ (W_\delta f)(x) = \int_{\mathbb{R}^n} f(y) E_\delta(y) e^{2\pi i x \cdot y} dy, \]

where \( E_\delta(y) = \int_{|\xi| \leq \delta} e^{2\pi i y \cdot \xi} d\xi = 2\pi (\delta |y|^{-1})^{n/2} J_{n/2}(2\pi \delta |y|) \) and \( J_{n/2} \) is the Bessel function of order \( n/2. \) If \( \mu \) is a bounded Borel measure on \( \mathbb{R}^n \) and \( f = \hat{\mu}, \) then for \( \delta > 0 \) and for any ball \( B_\delta(x), \) \( \mu(B_\delta(x)) = (W_\delta f)(x) \) for almost all \( x \in \mathbb{R}^n \) with respect to \( n \)-dimensional Lebesgue measure. Lau studied the asymptotic properties of measures in \( M_\alpha^0 \) in [9]:

**Theorem 2.2** [9]

1. Let \( 1 \leq p \leq 2, \) \( 1/p + 1/p' = 1 \) and \( 0 \leq \alpha < n. \) Suppose \( \mu \in M_\alpha^p \) then \( \hat{\mu} \in B^{p'}_\alpha \) with

\[ \sup_{L \geq 1} \left( \frac{1}{L^{n-\alpha}} \int_{B_L} |\hat{\mu}|^{p'} \right)^{1/p'} = \|\hat{\mu}\|_{B^{p'}_n} \leq C \|\mu\|_{M_\alpha^p}, \]

for some constant \( C \) depending on \( \mu. \)

2. Let \( \mu \) be a positive \( \sigma \)-finite Borel measure on \( \mathbb{R}^n \) and \( f \) be any Borel \( \mu \)-measurable function on \( \mathbb{R}^n. \) Let \( d\mu_f = f d\mu. \) \( \mu \) is locally uniformly \( \alpha \)-dimensional if and only if \( \|\mu_f\|_{M_\alpha^p} \leq C \|f\|_{L^p(d\mu)} \) for all \( f \in L^p(d\mu), \) \( p > 1 \) and \( C \) is a non-zero constant dependent on \( p. \)

In this paper, we concentrate on obtaining lower bounds for (2.1) as \( L \) grows to infinity. Strichartz proved in [14] an analogue of Radon–Nikodym theorem for positive measure with no infinite atoms:

**Theorem 2.3** [14] Let \( \mu \) be a measure with no infinite atoms, and let \( \nu \) be \( \sigma \)-finite and absolutely continuous with respect to \( \mu. \) Then there exists a unique decomposition \( \nu = \nu_1 + \nu_2 \) such that \( d\nu_1 = \phi d\mu \) for a non-negative measurable function \( \phi \) and \( \nu_2 \) is null with respect to \( \mu, \) that is, \( \nu_2(A) = 0 \) whenever \( \mu(A) < \infty. \)
Remark 2.4 As observed in [14], any locally uniformly $\alpha$-dimensional measure $\mu$ can be written as $d\mu = \phi d\mathcal{H}_\alpha + d\nu$ where $\nu$ is null with respect to $\mathcal{H}_\alpha$ and $\phi$ is a non negative measurable function belonging to $L^1(\mathbb{R}^n)$.

In [14], the author studied the asymptotic properties of locally uniformly $\alpha$-dimensional measures and proved a Plancherel type theorem:

Theorem 2.5 [14] Let $\mu' = \mu + \nu$ be a locally uniformly $\alpha$-dimensional measure on $\mathbb{R}^n$ where $\mu = \mathcal{H}_\alpha|_E$ and $\nu$ is null with respect to $\mathcal{H}_\alpha$. If $E$ is quasi regular, then for fixed $y$ and constant $c$ independent of $y$,

$$
c \int_E |f|^2 d\mathcal{H}_\alpha \leq \liminf_{L \to \infty} \frac{1}{L^{n-\alpha}} \int_{B_L(y)} |\hat{f} d\mu'|^2.
$$

These results are analogous to the results proved by Agmon and Hormander in [1] when $\alpha$ is an integer. Applying Holder’s inequality to the inequality in the Theorem 2.5, we obtain:

Corollary 2.6 Let $f \in L^2(\mu)$ be supported in a quasi $\alpha$-regular set $E$ of non-zero finite $\alpha$-dimensional Hausdorff measure ($0 < \alpha < n$), where $\mu = \mathcal{H}_\alpha|_E$ is a locally uniformly $\alpha$-dimensional measure. Then for $p \geq 2$,

$$
\| f \|_{L^2(\mu)}^p \leq c \limsup_{L \to \infty} \frac{1}{L^{n-\frac{\alpha p}{2}}} \int_{|\xi| \leq L} |\hat{f} d\mu(\xi)|^p d\xi
$$

where $c$ is a non zero finite constant depending on $n$, $\alpha$ and $p$.

The above results are proved for locally uniformly $\alpha$-dimensional measure. Suppose $\mu$ is a finite measure supported on a sparse set $E$. If $\mu$ is such that the $\alpha$-dimensional upper Minkowski content of $S$ is non-zero and bounded above by $\mu(S)$, for any $S \subset E$ such that $\mu(S) \neq 0$, then $\mu$ might not be a locally uniformly $\alpha$-dimensional measure. We prove an analogue result with this measure $\mu$ to the above corollary for the range $2 \leq p < 2n/\alpha$.

Theorem 2.7 Let $\mu$ be a finite measure supported in a sparse set $E$ such that the $\alpha$-dimensional upper Minkowski content of $S$ is non-zero and bounded above by $\mu(S)$, for any $S \subset E$ such that $\mu(S) \neq 0$ and let $f \in L^2(d\mu)$ be a positive function. Then,

$$
\int_{\mathbb{R}^n} |f(x)|^2 d\mu(x) \leq C \liminf_{L \to \infty} \frac{1}{L^{n-\alpha}} \int_{B_L(0)} |\hat{f} d\mu(\xi)|^2 d\xi
$$

and

$$
\int_{\mathbb{R}^n} |f(x)|^2 d\mu(x) \leq C' \liminf_{L \to \infty} \frac{1}{L^{n-\alpha}} \int_{\mathbb{R}^n} e^{-\frac{\|x\|^2}{2L^2}} |\hat{f} d\mu(\xi)|^2 d\xi,
$$

where the constants $C$ and $C'$ are independent of $f$. 

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Proof Since $E$ is compact, without loss of generality we assume that $E$ is contained in a large cube in the positive quadrant, that is, there exists smallest positive integer $m$ such that for all $x = (x_1, \ldots, x_n) \in E, 0 < x_j < m$. Let $M = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_j \leq m, \forall j\}$.

Fix $0 < \epsilon < 1$. For $k = (k_1, \ldots, k_n), (0 < k_j \in \mathbb{Z})$,

\[ Q_k = \{x = (x_1, \ldots, x_n) \in M : (k_j - 1)\epsilon < x_j \leq k_j\epsilon, j = 1, \ldots, n\}. \]

Let $Q_0$ be the collection of all such $Q_k$’s whose intersection with $E$ has non-zero $\mu$-measure, that is, $\mu(Q_k) \neq 0$. Since $E$ is compact, there exists finite number of $Q_k$’s in $Q_0$. Let $\delta_0 = \min_{Q_k \in Q_0} \{\mu(Q_k)\}$. Then $E = \cup(Q_k \cap E) \cup E'$ where the union is finite and $\mu(E') = 0$.

\[
\int_E |f(x)|^2 d\mu(x) = \sum_{Q_k \in Q_0} \int_{Q_k} |f(x)|^2 d\mu(x)
\leq 2 \sum_{Q_k \in Q_0} \int_{Q_k} \left|f(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} f(y) d\mu(y)\right|^2 d\mu(x)
+ 2 \sum_{Q_k \in Q_0} \frac{1}{\mu(Q_k)} \left(\int_{Q_k} f(y) d\mu(y)\right)^2. \tag{2.2}
\]

Now by the hypothesis on $\mu$, for each $k$, there exists $\delta_k$ such that

\[
|(Q_k \cap E)(\delta)| \delta^{n-\alpha} \leq C_n \mu(Q_k \cap E) + C_n \delta_0 \epsilon
\leq 2C_n \mu(Q_k \cap E) = 2C_n \mu(Q_k), \tag{2.3}
\]

for all $\delta \leq \delta_k$. Fix $\delta_0 = \min\{\epsilon, \delta_0, \delta_1, \delta_2, \ldots\}$. Since there are finite $Q_k$’s, $\delta_0 > 0$. Let $\phi$ be a positive Schwartz function such that $\hat{\phi}(0) = 1$, support of $\hat{\phi}$ is supported in the unit ball and there exists $r_1 > 0$ such that

\[
\int_{A_{r_1}(0)} \phi(x) dx = \frac{1}{2n+1}, \tag{2.4}
\]

where $A_{r_1}(0) = \{x = (x_1, \ldots, x_n) : -r_1 < x_j \leq 0, \forall j\}$. Denote $\phi_L(x) = \phi(Lx)$ for all $L > 0$. Let $r = n^{\frac{1}{2}} r_1$. Fix $L$ large such that $r/L \leq \delta_0$. Then we have,

\[
\left(\int_{Q_k} f(y) d\mu(y)\right)^2 = 2^{2(n+1)} \left(\int_{Q_k} \int_{A_{r_1}(0)} \phi(x) dx f(y) d\mu(y)\right)^2
= 2^{2(n+1)} L^{2n} \left(\int_{Q_k} \int_{A_{r_1/L}(y)} \phi_L(x - y) dx f(y) d\mu(y)\right)^2
= 2^{2(n+1)} L^{2n} \left(\int_{Q_k} \int_{Q_k} \phi_L(x - y) f(y) d\mu(y) dx\right)^2, \tag{2.5}
\]

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where

\[ Q_k^L = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \exists y = (y_1, \ldots, y_n) \in E \text{ such that } y_j = x_j \leq y_j, \forall j \}. \]

Then \(|Q_k^L| \leq |(Q_k \cap E)(r/L)|\), where \((Q_k \cap E)(r/L)\) denotes the \(r/L\)-distance set of \(Q_k \cap E\) (since \(r = \sqrt{n}r_1\)). Also since \(\phi\) and \(f\) are positive,

\[
\int_{Q_k^L} \int_{Q_k} \phi_L(x - y)f(y)d\mu(y)dx \leq \int_{Q_k^L} \phi_L * f d\mu(x)dx.
\]

Thus from (2.5),

\[
\frac{1}{2^{2(n+1)}} \left| \int_{Q_k} f(y)d\mu(y) \right|^2 \leq L^{2\alpha} \left| \int_{Q_k^L} \phi_L * f d\mu(x)dx \right|^2.
\]

\[
\leq L^{2\alpha}|Q_k^L| \int_{Q_k^L} |\phi_L * f d\mu(x)|^2dx
\]

\[
\leq L^{2\alpha}|(Q_k \cap E)(r/L)| \int_{Q_k^L} |\phi_L * f d\mu(x)|^2dx
\]

\[
\leq 2Cn r^{n-\alpha} L^{n+\alpha} \mu(Q_k) \int_{Q_k^L} |\phi_L * f d\mu(x)|^2dx \text{ by (2.3).}
\]

Thus there exists a constant \(C_1\) independent of \(\epsilon, L\) and \(f\) such that

\[
\frac{1}{\mu(Q_k)} \left| \int_{Q_k} f(y)d\mu(y) \right|^2 \leq C_1 L^{n+\alpha} \int_{Q_k^L} |\phi_L * f d\mu(x)|^2dx.
\]

Hence, from (2.2)

\[
\int_{E} |f(x)|^2 d\mu(x) \leq 2 \sum_k \int_{Q_k} \left| f(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} f(y)d\mu(y) \right|^2 d\mu(x)
\]

\[
+ 2C_1 L^{n+\alpha} \sum_{Q_k \in Q} \int_{Q_k^L} |\phi_L * f d\mu(x)|^2dx.
\]

By the choice of \(r/L < \delta_0 < \epsilon\), any \(x \in Q_k^L\) belongs to at most \(2^n\) number of other \(Q_k^L\)’s in \(Q_0\). Hence there exists a constant \(C = 2C_1 2^n\) independent of \(f, \epsilon\) and \(L\) such that for all \(r/L \leq \delta_0\)

\[
\int_{E} |f(x)|^2 d\mu(x) \leq 2e\epsilon + CL^{n+\alpha} \int_{E(r/L)} |\phi_L * f d\mu(x)|^2dx, \quad (2.6)
\]
where
\[
e_\epsilon = \sum_k \int_{Q_k} |f(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} f(y) d\mu(y)|^2 d\mu(x).
\]

For given \(\epsilon\), let \(g \in C_\infty_c(d\mu)\) be such that \(\|f - g\|_{L^2(d\mu)}^2 < \epsilon\). Then,
\[
e_\epsilon = \sum_k \int_{Q_k} \left| f(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} f(y) d\mu(y) \right|^2 d\mu(x)
\leq 2 \sum_k \int_{Q_k} \left| f(x) - g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} f(y) d\mu(y) \right|^2 d\mu(x)
+ 2 \sum_k \int_{Q_k} \left| g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y) d\mu(y) \right|^2 d\mu(x)
\leq 4 \sum_k \int_{Q_k} |f(x) - g(x)|^2 + \left| \frac{1}{\mu(Q_k)} \int_{Q_k} f(y) - g(y) d\mu(y) \right|^2 d\mu(x)
+ 2 \sum_k \int_{Q_k} \left| g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y) d\mu(y) \right|^2 d\mu(x)
\leq 8 \sum_k \int_{Q_k} |f(x) - g(x)|^2 d\mu(x)
+ 2 \sum_k \int_{Q_k} \left| g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y) d\mu(y) \right|^2 d\mu(x).
\]

Since \(E = \cup_k (Q_k \cap E) \cup E'\),
\[
e_\epsilon \leq 8\|f - g\|_{L^2(d\mu)}^2 + 2 \sum_k \int_{Q_k} \left| g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y) d\mu(y) \right|^2 d\mu(x)
\leq \epsilon + 2 \sum_k \int_{Q_k} \left| g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y) d\mu(y) \right|^2 d\mu(x). \tag{2.7}
\]

Since \(g\) is compactly supported continuous function, \(g\) is uniformly continuous and
\[
|g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y) d\mu(y)| \to 0
\]
uniformly in $x$ and $Q_k$ as $\mu(Q_k) \to 0$. As $\epsilon \to 0$, we have $\mu(Q_k) \to 0$. Hence

\[
\sum_k \int_{Q_k} \left| g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y) d\mu(y) \right|^2 d\mu(x) \leq \sum_k \mu(Q_k) \sup_{Q_k \in Q_0} \sup_{x \in Q_k} \left| g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y) d\mu(y) \right|^2
\]

which goes to zero as $\epsilon$ goes to zero. Therefore, from (2.7), $e_\epsilon$ goes to zero as $\epsilon$ goes to zero. Letting $\epsilon$ to 0, we have $r_1/L \leq \delta_0 \to 0$. Thus (2.6) becomes

\[
\int_E |f(x)|^2 d\mu(x) \leq C \liminf_{L \to \infty} \frac{1}{L^{n+\alpha}} \int_{E(r/L)} |\phi_L \ast f d\mu(x)|^2 dx,
\]

Since the support of $\hat{\phi}$ is in the unit ball, we have

\[
\int_E |f(x)|^2 d\mu(x) \leq C \|\phi\|^2_{L^1(\mathbb{R}^n)} \liminf_{L \to \infty} \frac{1}{L^{n-\alpha}} \int_{B_L(0)} \left| \hat{f} \right|^2 d\mu(\xi).
\]

The assumption on the support of $\hat{\phi}$ to be in the unit ball is used only in the last step. Consider $\phi(x) = e^{-|x|^2/2}$. Proceeding in a similar way, we have

\[
\int_E |f(x)|^2 d\mu(x) \leq C \liminf_{L \to \infty} \frac{1}{L^{n-\alpha}} \int_{\mathbb{R}^n} e^{-|\xi|^2/2L^2} |\hat{f} d\mu(\xi)|^2 d\xi.
\]

Hence the proof. \hfill \Box

By an application of Holder’s inequality, we obtain the following Corollary.

**Corollary 2.8** Let $f \in L^2(d\mu)$ be a positive function where $\mu$ is a finite measure supported in a set $E$ such that the $\alpha$-dimensional upper Minkowski content of all non-zero $\mu$-measure subsets $S$ of $E$ is non-zero and bounded above by $\mu(S)$. Then for $2 \leq p < 2n/\alpha$,

\[
\int_{\mathbb{R}^n} |f(x)|^2 d\mu(x) \leq C \liminf_{L \to \infty} \left( \frac{1}{L^{n-\alpha p/2}} \int_{B_L(0)} |\hat{f} d\mu(\xi)|^p d\xi \right)^{2/p}
\]
and

\[ \int_{\mathbb{R}^n} |f(x)|^2 d\mu(x) \leq C' \liminf_{L \to \infty} \left( \frac{1}{L^{n-\alpha p/2}} \int_{\mathbb{R}^n} e^{-\frac{|\xi|^2}{L^2}} |\hat{f}(\xi)|^p d\xi \right)^{2/p}, \]

where the constants \(C\) and \(C'\) are independent of \(f\).

Now, to prove that Theorem 2.7 is optimal we prove the following lemma by closely following the arguments in [1] (also see page 174 of [6]).

**Lemma 2.9** Let \(u\) be a tempered distribution supported in a compact set \(E\). Let \(\chi\) be a radial \(C^\infty_c\) function supported in the unit ball and \(\int_{\mathbb{R}^n} \chi(x) dx = 1\). Denote \(\chi_\varepsilon(x) = \varepsilon^{-n} \chi(x/\varepsilon)\) and \(u_\varepsilon = u * \chi_\varepsilon\). Let \(\sigma_u(r) = \int_{S^{n-1}} |\hat{u}(r\omega)|^2 d\omega\). Then,

\[ \|u_\varepsilon\|^2 \leq C \varepsilon^{(\alpha-n)(1-\frac{1}{q})} \left( \sup_{\varepsilon L > 1} \frac{1}{L^k} \int_0^L (\sigma_u(r))^p r^{n-1} dr \right)^{\frac{2}{p}}, \]

for some non-zero finite constants \(C\) independent of \(\varepsilon\) and \(k = n - \frac{\alpha p}{2} - (n - \alpha) \frac{P}{2q} \) with \(1 < q \leq \infty\) and \(2 \leq p < 2n/\alpha\).

**Proof** By the Plancherel theorem,

\[
\|u_\varepsilon\|^2 = \int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 |\hat{\chi}(\varepsilon \xi)|^2 d\xi \\
= \int_0^{\varepsilon^{-1}} (\sigma_u(r)) |\hat{\chi}(\varepsilon r)|^2 r^{n-1} dr + \sum_{j=1}^\infty \int_{2^{-j-1} \varepsilon}^{2^{-j-1}} (\sigma_u(r)) |\hat{\chi}(\varepsilon r)|^2 r^{n-1} dr.
\]

\[
\leq \left( \int_0^{\varepsilon^{-1}} (\sigma_u(r))^{p/2} r^{n-1} dr \right)^{\frac{2}{p}} \left( \int_0^1 |\hat{\chi}(r)|^{1-\frac{2}{p}} r^{n-1} dr \right)^{1-\frac{2}{p}}
\]

\[
+ \sum_{j=1}^\infty \left( \int_{2^{-j-1} \varepsilon}^{2^{-j-1}} \sigma_u(r)^{p/2} r^{n-1} dr \right)^{\frac{2}{p}} \left( \int_{2^{-j-1}}^1 |\hat{\chi}(r)|^{1-\frac{2}{p}} r^{n-1} dr \right)^{1-\frac{2}{p}}
\]

\[
\leq \varepsilon^{(\alpha-n)(1-\frac{1}{q})} \left( \sum_{j=0}^\infty a_j \left( \sup_{\varepsilon L > 1} \frac{1}{L^k} \int_0^L (\sigma_u(r))^p r^{n-1} dr \right)^{\frac{2}{p}} \right),
\]

where, for all \(j > 0\)

\[
a_j = \left( 2^{\frac{2kj}{p-2}} \int_{2^{j-1}}^{2^j} |\hat{\chi}(r)|^{\frac{2p}{p-2}} r^{n-1} dr \right)^{1-\frac{2}{p}}
\]
and \( a_0 = \left( \int_0^1 |\hat{\chi}(r)| \frac{2p}{n - \frac{ap}{2}} r^{n-1} dr \right)^{1 - \frac{2}{p}} \). We have \( \sum_j a_j \) is finite. Thus

\[
\|u_\varepsilon\|^2 \leq \varepsilon^{(\alpha - n)(1 - \frac{1}{q})} C \left( \sup_{\|L\| > 1} \frac{1}{L} \int_0^{2L} \sigma_u(r) \frac{p}{r} r^{n-1} dr \right)^{\frac{2}{p}}
\]

since \( k = n - \frac{ap}{2} - (n - \alpha) \frac{p}{2q} \).

**Theorem 2.10** Fix \( 0 < \alpha < n \). Let \( \mu \) be a finite measure supported in a compact set \( M \) such that the \( \alpha \)-dimensional upper Minkowski content of \( S \) is non-zero and bounded above by \( \mu(S) \) for any \( S \subset M \) with \( \mu(S) \neq 0 \). Let \( u \) be a tempered distribution such that \( \supp u \) is contained in \( M \) and \( \sigma_u(r) = \int_{|x| < r} |\hat{u}(r)|^2 \, dr \). Let \( 2 \leq p < \frac{2n}{\alpha} \). Then

\[
\|u\|_1^p \leq C \limsup_{L \to \infty} \frac{1}{L^{n - \frac{ap}{2} - (n - \alpha) \frac{p}{2q}}} \int_0^{L} (\sigma_u(r))^{\frac{p}{2}} r^{n-1} \, dr \leq C' \limsup_{L \to \infty} \frac{1}{L^{n - \frac{ap}{2} - (n - \alpha) \frac{p}{2q}}} \int_{|\xi| < L} |\hat{u}(\xi)|^p \, d\xi,
\]

where \( \|u\|_1 = \sup \{ <u, \psi> : \psi \in C_c^\infty(\mathbb{R}^n), \|\psi\|_{L^\infty(\mathbb{R}^n)} \leq 1 \}, C \) and \( C' \) are non zero finite constants depending only on \( n, \alpha \) and \( p \).

In general, for \( 2 \leq p < \frac{2n}{\alpha + \frac{ap}{2q}} \), where \( 1 < q \leq \infty \)

\[
\|u\|_r^p \leq C \limsup_{L \to \infty} \frac{1}{L^{n - \frac{ap}{2} - (n - \alpha) \frac{p}{2q}}} \int_0^{L} (\sigma_u(r))^{\frac{p}{2}} r^{n-1} \, dr
\]

\[
\leq C' \limsup_{L \to \infty} \frac{1}{L^{n - \frac{ap}{2} - (n - \alpha) \frac{p}{2q}}} \int_{|\xi| < L} |\hat{u}(\xi)|^p \, d\xi,
\]

where \( \frac{1}{r} + \frac{1}{sq} = 1 \), \( \|u\|_r = \sup \{ <u, \psi> : \|\psi\|_{L^q(\mathbb{R}^n)} \leq 1 \}, C \) and \( C' \) are non zero finite constants depending on \( n, \alpha, p \) and \( q \).

**Proof** Choose an even function \( \chi \in C_c^\infty(\mathbb{R}^n) \) with support in unit ball and \( \int_{\mathbb{R}^n} \chi(x) \, dx = 1 \). Let \( \chi_\varepsilon(x) = \varepsilon^{-n} \chi(x/\varepsilon) \) and \( u_\varepsilon = u \ast \chi_\varepsilon \). Then by Lemma 2.9,

\[
\|u_\varepsilon\|^2 \leq C \varepsilon^{(\alpha - n)(1 - \frac{1}{q})} \left( \sup_{\|L\| > 1} \frac{1}{L} \int_0^{2L} (\sigma_u(r))^{\frac{p}{2}} r^{n-1} \, dr \right)^{\frac{2}{p}}.
\]

Let \( \psi \in C_c^\infty(\mathbb{R}^n) \). Let \( S = \text{supp} u \cap \text{supp} \psi \) where \( \text{supp} \psi \) is contained in a ball \( B_{R_\psi}(0) \) of radius \( R_\psi \). Since \( S \) is a bounded subset of \( M \), by hypothesis, we have

\[
\limsup_{\varepsilon \to 0} |S_\varepsilon| \varepsilon^{\alpha - n} \leq c \mu(S) < \infty.
\]
For given $0 < \delta < 1$, there exists $\epsilon_0$ such that for all $\epsilon < \epsilon_0$, $|S(\epsilon)|\epsilon^{\alpha - n} \leq C(\mu(S) + \delta) \leq C_M$. So, for $k = n - \frac{\alpha p}{2} - (n - \alpha) \frac{p}{2q}$,

$$|\langle u_\epsilon, \psi \rangle|^2 \leq \|u_\epsilon\|^2_2 \int_{S_\epsilon} |\psi|^2$$

$$\leq \|u_\epsilon\|^2_2 \left( \int_{\mathbb{R}^n} |\psi|^{2q} \right)^{\frac{1}{q}} |S_\epsilon|^{1 - \frac{1}{q}}$$

$$\leq C_M \|\psi\|^2_{2q} \epsilon^{(n-\alpha)(1-\frac{1}{q})} \|u_\epsilon\|^2_2$$

$$\leq C \|\psi\|^2_{2q} \left( \sup_{\epsilon \leq \epsilon_0} \frac{1}{L^k} \int_0^L (\sigma_u(r))^\frac{p}{n} r^{n-1} dr \right)^\frac{2}{p}. $$

Thus

$$\|u\|^p \leq C \limsup_{L \to \infty} \frac{1}{L^{n - \frac{ap}{2} - (n - \alpha) \frac{p}{2q}}} \int_0^L (\sigma_u(r))^p 2r^{n-1} dr$$

$$\leq C' \limsup_{L \to \infty} \frac{1}{L^{n - \frac{ap}{2} - (n - \alpha) \frac{p}{2q}}} \int_{|\xi| < L} |\hat{u}(\xi)|^p d\xi.$$ 

Note that $1 \leq r < 2$ in the above result. We now prove that the lower bound $\|u\|^r$ in the above theorem can be improved and that Theorem 2.7 is optimal.

**Theorem 2.11** Let $\nu$ be a finite Radon measure supported in a compact set $E$ such that the $\alpha$-dimensional upper Minkowski content of $S$ is non-zero and bounded above by $\nu(S)$ for any $S \subset E$ with $\nu(S) \neq 0$. Let $u$ be a tempered distribution supported in $E$ such that for some $2 \leq p < 2n/\alpha$,

$$\limsup_{L \to \infty} \frac{1}{L^{n - \frac{ap}{2} - (n - \alpha) \frac{p}{2q}}} \int_{|\xi| \leq L} |\hat{u}(\xi)|^p d\xi < \infty.$$ 

Then $u$ is an $L^2$ density $u_0 \, d\nu$ on $E$ and

$$\left( \int_E |u_0|^2 \, d\nu \right)^{p/2} \leq C \limsup_{L \to \infty} \frac{1}{L^{n - \frac{ap}{2}}} \int_{|\xi| \leq L} |\hat{u}(\xi)|^p d\xi < \infty.$$ 

**Proof** Let $\psi \in C_c^\infty(\mathbb{R}^n)$. Let $S = supp u \cap supp \psi$. Then $S$ is bounded and let $M$ be the smallest closed cube that contains $S$. As in Theorem 2.7, for $0 < \delta < 1$, let $Q_0$ be the collection of all half open cubes $Q_k = \{x = (x_1, \ldots, x_n) \in M : (k_j - 1)\delta < x_j \leq k_j \delta\}$, $(k = (k_1, \ldots, k_n), \, k_j \in \mathbb{Z})$ and $Q_0$ be the collection of all $Q_k \in Q_0$ such that $\nu(Q_k \cap E) \neq 0$. Denote $\mu = \nu|_S$. Since $S$ is bounded, there are finite $Q_k$’s in $Q_0$. Let $\delta_0 = \min_{Q_k \in Q_0} \{\mu(Q_k)\}$. By hypothesis, for each $k$, there exists $\delta_k$ such that
for all $\epsilon \leq \delta_k$. Fix $\epsilon_0 = \min\{\delta, \delta_0, \delta_1, \delta_2, \ldots\}$. For every $\epsilon < \epsilon_0$, let $Q_0^c$ denote the collection of all $Q_k$ in $\tilde{Q}_0$ such that $|Q_k \cap S(\epsilon)| \neq 0$.

$$
\epsilon^{a-n} \int_{S(\epsilon)} |\psi(x)|^2 \, dx = \epsilon^{a-n} \sum_{Q_k \in Q_0^c \setminus Q_0} \int_{Q_k \cap S(\epsilon)} |\psi(x)|^2 \, dx
$$

$$
\leq \epsilon^{a-n} \sum_{Q_k \in Q_0^c \setminus Q_0} \int_{Q_k \cap S(\epsilon)} |\psi(x)|^2 \, dx
$$

$$
+ 2\epsilon^{a-n} \sum_{Q_k \in Q_0} \int_{Q_k \cap S(\epsilon)} \left| \psi(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} \psi(y) \, d\mu(y) \right|^2 \, dx
$$

$$
+ 2\epsilon^{a-n} \sum_{Q_k \in Q_0} \int_{Q_k \cap S(\epsilon)} \left| \frac{1}{\mu(Q_k)} \int_{Q_k} \psi(y) \, d\mu(y) \right|^2 \, dx.
$$

Since, for $Q_k \in Q_0^c \setminus Q_0$, $\mu(Q_k) = 0$, from (2.9),

$$
\epsilon^{a-n} \sum_{Q_k \in Q_0^c \setminus Q_0} \int_{Q_k \cap S(\epsilon)} |\psi(x)|^2 \, dx
$$

$$
\leq 2C_n \|\psi\|_{\infty}^2 \sum_{Q_k \in Q_0^c \setminus Q_0} \mu(Q_k) = 0.
$$

Hence,

$$
\epsilon^{a-n} \int_{S(\epsilon)} |\psi(x)|^2 \, dx \leq 2\epsilon^{a-n} \sum_{Q_k \in Q_0} \int_{Q_k \cap S(\epsilon)} \left| \psi(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} \psi(y) \, d\mu(y) \right|^2 \, dx
$$

$$
+ 2\epsilon^{a-n} \sum_{Q_k \in Q_0} \int_{Q_k \cap S(\epsilon)} \left| \frac{1}{\mu(Q_k)} \int_{Q_k} \psi(y) \, d\mu(y) \right|^2 \, dx
$$

$$
\leq \epsilon_\delta + 2 \sum_{Q_k \in Q_0} \epsilon^{a-n} |Q_k \cap S(\epsilon)| \frac{1}{\mu(Q_k)} \int_{Q_k} |\psi(y)|^2 \, d\mu(y),
$$

where

$$
e_\delta = 2 \sum_{Q_k \in Q_0} \epsilon^{a-n} \int_{Q_k \cap S(\epsilon)} \left| \psi(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} \psi(y) \, d\mu(y) \right|^2 \, dx.
$$

By (2.9),

$$
\epsilon^{a-n} |Q_k \cap S(\epsilon)| \leq \epsilon^{a-n} |(Q_k \cap S)(\epsilon)| \leq 2C_n \mu(Q_k).
$$
Hence

\[ e^{\alpha-n} \int_{S(\epsilon)} |\psi(x)|^2 dx \leq e_\delta + 4C_n \sum_{Q_k \in Q_0} \int_{Q_k} |\psi(y)|^2 d\mu(y) \]

\[ = e_\delta + 4C_n \int_E |\psi(y)|^2 d\mu(y). \quad (2.11) \]

Now, since \( \psi \) is compactly supported continuous function, 

\[ \int_{Q_k} \psi(y) d\mu(y) \to 0 \text{ uniformly as } \delta \text{ goes to zero and hence } \sup_{x \in S(\epsilon)} |\psi(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} \psi(y) d\mu(y)| \to 0 \text{ as } \delta \text{ goes to zero.} \]

\[ e_\delta = e^{\alpha-n} \sum_{Q_k \in Q_0} \int_{Q_k \cap S(\epsilon)} |\psi(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} \psi(y) d\mu(y)|^2 dx \]

\[ \leq e^{\alpha-n} \sum_{Q_k \in Q_0} |Q_k \cap S(\epsilon)| \sup_{x \in S(\epsilon)} \left| \psi(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} \psi(y) d\mu(y) \right|^2 \]

\[ \leq e^{\alpha-n} |S(\epsilon)| \sup_{x \in S(\epsilon)} \left| \psi(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} \psi(y) d\mu(y) \right|^2. \]

Then together with (2.10), \( e_\delta \) goes to zero as \( \delta \) goes to zero. Thus from (2.11), for given \( 0 < \delta < 1 \), there exists small \( \epsilon_0 \) such that for all \( \epsilon < \epsilon_0 \),

\[ e^{\alpha-n} \int_{S(\epsilon)} |\psi(x)|^2 dx \leq e_\delta + 4C_n \int_E |\psi(y)|^2 d\nu(y) \]

\[ = e_\delta + 4C_n \|\psi\|_{L^2(d\nu)}^2. \quad (2.12) \]

where \( e_\delta \) tends to zero as \( \delta \) tends to zero.

Now we proceed as in the Theorem 2.10. Choose an even function \( \chi \in C_c^\infty(R^n) \) with support in unit ball and \( \int_{R^n} \chi(x) dx = 1 \). Let \( \chi_\epsilon(x) = \epsilon^{-n} \chi(x/\epsilon) \) and \( u_\epsilon = u \ast \chi_\epsilon \). Then by Lemma 2.9,

\[ \|u_\epsilon\|^2 \leq C \epsilon^{\alpha-n} \left( \sup_{\epsilon L > 1} \frac{1}{L^{n-\alpha p/2}} \int_0^L (\sigma_u(r))^{\frac{p}{2}} r^{n-1} dr \right)^{\frac{2}{p}} \]

\[ \leq C \epsilon^{\alpha-n} \left( \sup_{\epsilon L > 1} \frac{1}{L^{n-\alpha p/2}} \int_{|\xi| < L} |\hat{u}(\xi)|^p d\xi \right)^{\frac{2}{p}}. \quad (2.13) \]
We have $\epsilon \to 0$ as $\delta \to 0$. Thus

$$| < u, \psi > |^2 = \lim_{\epsilon \to 0} | < u_\epsilon, \psi > |^2 \leq \lim_{\epsilon \to 0} \| u_\epsilon \|_2^2 \int_{S_\epsilon} |\psi|^2 \leq \lim_{\epsilon \to 0} \| u_\epsilon \|_2^2 e^{(n-\alpha)} (e^{\delta} + C \| \psi \|_{L^2(d\nu)})$$

from (2.12).

Thus letting $\delta$ go to zero, together with (2.13),

$$| < u, \psi > |^2 \leq C \| \psi \|_{L^2(d\mu)}^2 \left( \limsup_{L \to \infty} \frac{1}{L^{n-ap/2}} \int_{|\xi| \leq L} |\hat{u}(\xi)|^2 d\xi \right)^{2/p}$$

Thus $u$ is an $L^2$ density $u_0 \, d\nu$ on $E$ and

$$\left( \int_E |u_0|^2 \, d\nu \right)^{p/2} \leq C \limsup_{L \to \infty} \frac{1}{L^{n-ap/2}} \int_{|\xi| \leq R} |\hat{u}(\xi)|^p d\xi < \infty.$$ 

\[ \square \]

In [1], the authors studied the Fourier asymptotics of measures supported in a smooth manifold and proved similar results:

Let $u$ be a tempered distribution such that $\hat{u} \in L^2_{loc}$ and

$$\limsup_{L \to \infty} \frac{1}{L^k} \int_{|\xi| \leq L} |\hat{u}(\xi)|^2 d\xi < \infty.$$ 

If the restriction of $u$ to an open subset $X$ of $\mathbb{R}^n$ is supported by a $C^1$-submanifold $M$ of codimension $k$, then it is an $L^2$-density $u_0 \, dS$ on $M$ and

$$\int_M |u_0|^2 \, dS \leq C \limsup_{L \to \infty} \frac{1}{L^k} \int_{|\xi| \leq R} |\hat{u}(\xi)|^2 d\xi,$$

where $C$ only depends on $n$.

3 $L^p$-asymptotic properties of fractal measures for $1 \leq p \leq 2$

Let $\mu$ denote a fractal measure supported in an $\alpha$-dimensional set $E \subset \mathbb{R}^n$ and $f \in L^q(d\mu)$ ($1 \leq q \leq \infty$). Suppose $1 \leq p \leq 2$ dependent on $q$. In this section, we obtain lower bounds for

$$\liminf_{L \to \infty} L^{-k} \int_{|\xi| \leq L} |\hat{f \, d\mu}(\xi)|^p d\xi.$$
for positive \( k = n - \alpha \) and prove generalized Hardy inequality for fractal measures on \( \mathbb{R}^n \) of dimension \( 0 < \alpha < n \).

Consider the generalized Hardy inequality for discrete measures proved by the authors in [7]:

**Theorem 3.1** [7] Let \( c_k \) be a sequence of complex numbers, \( a_k \) be a sequence of real numbers and \( f d\mu_0 \) denote the zero dimensional measure \( f(x) = \sum_{i=1}^{\infty} c_k \delta(x - a_k) \) where \( \delta \) is the usual Dirac measure at zero.

1. Let \( a_1 < a_2 < \cdots \) and assume \( \hat{f} d\mu_0 = \sum_{i=1}^{\infty} c_k e^{ia_k x} \) belongs to the class of almost periodic functions. Then,

\[
\sum_{i=1}^{\infty} \frac{|c_k|}{k} \leq C \lim_{L \to \infty} L^{-1} \int_{-L}^{L} |\hat{f} d\mu_0(x)| dx.
\]

2. Let \( a_k \) be a sequence of real numbers, not necessarily increasing and \( 1 < p \leq 2 \). Assume that \( u(x) = \hat{f} d\mu_0(x) \) converges to \( \sum_{i=1}^{\infty} c_k e^{ia_k x} \) in the class of almost periodic functions. Then

\[
\sum_{i=1}^{\infty} \frac{|c_k|^p}{k^{2-p}} \leq \sum_{i=1}^{\infty} \frac{|c'_k|^p}{k^{2-p}} \leq C \lim_{L \to \infty} L^{-1} \int_{-L}^{L} |u(x)|^p dx,
\]

where \( c'_k \) is the nonincreasing rearrangement of the sequence \( |c_k| \).

The authors also proved generalized Hardy inequality for fractal measures \( f d\mu \) on \( \mathbb{R}^1 \) of dimension \( \alpha (0 < \alpha < 1) \) in [7] by generalizing part (1) of the above theorem with additional hypothesis on \( \mu \). To prove the same, they introduced \( \alpha \)-coherent sets in \( \mathbb{R} (0 < \alpha < 1) \). Given \( x \in \mathbb{R} \) and a set \( E \subset \mathbb{R} \), let \( E_x = E \cap (-\infty, x] \). Let \( s = \sup\{x : \mathcal{H}_\alpha(E_x) < \infty\} \), \( E^0 = (E_x)^* \) where, for a set \( E \),

\[
E^* = \{x \in E : 2^{-\alpha} \leq \overline{d}(\mathcal{H}_\alpha|E, x) \leq 1\}.
\]

The set \( E \subset \mathbb{R} \) is \( \alpha \)-coherent \( (0 < \alpha < 1) \), if there is a constant \( C \) such that for all \( x \leq s \),

\[
\limsup_{\delta \to 0} |E^0_x(\delta)| \delta^{\alpha - 1} \leq C \mathcal{H}_\alpha(E^0_x),
\]

where \( |E^0_x(\delta)| \) denotes the one dimensional Lebesgue measure of the \( \delta \)-distance set \( E^0_x(\delta) \) of \( E^0_x \). The following was proved in [7].

**Theorem 3.2** [7] Suppose \( 0 < \alpha < 1 \), \( f \in L^1(d\mathcal{H}_\alpha) \) and \( \mu = \mathcal{H}_\alpha|E \) where \( E \) is either \( \alpha \)-coherent or quasi \( \alpha \)-regular. Then, there exists a non-zero finite constant independent of \( f \) such that

\[
\int_E \frac{|f(x)| d\mu(x)}{\mathcal{H}_\alpha(E^0_x)} \leq C \liminf_{L \to \infty} L^{\alpha - 1} \int_{-L}^{L} |\hat{f} d\mu(x)| dx.
\]
Remark 3.3 Examples in [7] show that there are quasi regular sets in \( \mathbb{R} \) which are not \( \alpha \)-coherent and there are \( \alpha \)-coherent sets which are not quasi regular, for given \( 0 < \alpha < 1 \).

In this section, using the upper Minkowski content and finding a continuous analogue of the arguments used in the proof of the Theorem 3.2, we prove an analogous version of part(2) of the Theorem 3.1 for \( 0 < \alpha < n, n \geq 1 \) and \( 1 \leq p \leq 2 \) with a slight modification in the hypothesis:

**Theorem 3.4** Let \( E \subset \mathbb{R}^n \) be a compact set such that the \( \alpha \)-dimensional upper Minkowski content of \( S \) is non-zero and bounded above by \( \mathcal{H}_\alpha(S) \) for any \( S \subset E \) with \( \mathcal{H}_\alpha(S) \neq 0 \) for some \( 0 < \alpha < n \) and let \( \mu = \mathcal{H}_\alpha|E \). Let \( f \in L^p(d\mu) \) be a positive function, for \( 1 \leq p \leq 2 \). Then there exists a constant \( C \) independent of \( f \) such that

\[
\int_E |f(x)|^p \frac{d\mu(x)}{[\mu(E_x)]^{2-p}} \leq C \liminf_{L \to \infty} \frac{1}{L^{n-\alpha}} \int_{|\xi| \leq L} |\hat{f}d\mu(\xi)|^p d\xi, \tag{3.1}
\]

where \( E_x = E \cap [(-\infty, x_1] \times \cdots \times (-\infty, x_n)] \) for \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \).

First we prove the following lemma:

**Lemma 3.5** Suppose \( L > 1 \) and \( 0 < \delta = r/L < 1 \) are given constants. Let \( g_L \in L^1(\mathbb{R}^n) \) and \( S_\delta = \bigcup_{i=1}^s \Delta_i^\delta \) be the union of disjoint cubes such that \( 0 < |\Delta_i^\delta| < \delta^n \). Then, there exists a non-zero finite constant \( C_2 \) independent of \( g_L, s, \delta \) and \( L \) such that

\[
\frac{\delta^{-n}}{P_\delta} \int_{S_\delta} |g_L(x)| dx \leq C_2 \int_{\mathbb{R}^n} |\hat{g_L}(\xi)| d\xi, \tag{3.2}
\]

where \( P_\delta > 1 \) is a constant dependent on \( \delta \).

**Proof** For all \( i = 1, \ldots, s \), construct \( f_i \in L^2(\mathbb{R}^n) \) such that

\[
|\hat{f_i}(x)| = \frac{\delta^{-n}}{P_\delta} \quad \text{for} \quad x \in \Delta_i^\delta \\
= 0 \quad \text{for} \quad x \notin \Delta_i^\delta \\
\hat{f_i}(x)g_L(x) \geq 0.
\]

Since \( |\Delta_i^\delta| \leq \delta^n \) and \( P_\delta > 1 \), \( \|\hat{f_i}\|_1 \leq 1 \) and hence for all \( \xi \), \( |f_i(\xi)| \leq 1 \). Denote \( F_0 \equiv 0 \). For all \( i = 1, \ldots, s \), let

\[
F_i(\xi) = \frac{4}{5}F_{i-1}(\xi) \exp\left(\frac{-1}{4\delta^2} |f_i(\xi)|\right) + \frac{f_i(\xi)}{20}
\]

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and denote $F \equiv F_s$. Since $|f_i(\xi)| \leq 1$ for all $i$, we have $|F_1(\xi)| \leq 1/4$. Note that for all $0 \leq t \leq 1$ and $s \geq 1$,

$$\frac{4}{5} \exp \left( -\frac{t}{4s^2} \right) \leq 1 - \frac{t}{5}$$

$$\frac{1}{5} \exp \left( -\frac{t}{4s^2} \right) + \frac{t}{20} \leq \frac{1}{4}.$$

Since for all $\xi$, $|f_2(\xi)| \leq 1$, we have

$$|F_2(\xi)| \leq \frac{1}{5} \exp \left( -\frac{|f_2(\xi)|}{4s^2} \right) + \frac{|f_2(\xi)|}{20} \leq \frac{1}{4}.$$

Then by induction $\|F\|_{\infty} \leq 1/4$. By construction, we have

$$\hat{F}(x) = \sum_{k=1}^{s-1} \left[ \frac{4^{s-k} f_k(\xi)}{5^{s-k} 20} \exp \left( -\frac{1}{4s^2} \sum_{l=k+1}^{s} |f_l(\xi)| \right) \right] \hat{g}_L(x) + \frac{f_s(\xi)}{20}.$$  (3.3)

Now consider $\hat{F}$,

$$\hat{\hat{F}}(x) = \sum_{k=1}^{s-1} \left[ \frac{4^{s-k} f_k(\xi)}{5^{s-k} 20} \exp \left( -\frac{1}{4s^2} \sum_{l=k+1}^{s} |f_l(\xi)| \right) \right] \hat{\hat{g}}_L(x) + \frac{\hat{f}_s(x)}{20}$$

$$= \sum_{k=1}^{s-1} \left[ \frac{4^{s-k} f_k(\xi)}{5^{s-k} 20} \left( \exp \left( -\frac{1}{4s^2} \sum_{l=k+1}^{s} |f_l(\xi)| \right) - 1 \right) \right] \hat{\hat{g}}_L(x)$$

$$+ \sum_{k=1}^{s-1} \frac{4^{s-k} \hat{f}_k(x)}{5^{s-k} 20} \hat{\hat{g}}_L(x).$$

By the construction of $f'_i s$, for all $x \in \Delta_{i_0}^s$, $|\hat{f}_i(x)| = 0$ for all $i \neq i_0$ and $\hat{f}_i(x) g_L(x) \geq 0$. Hence

$$\text{Re}(\hat{F}(x) g_L(x))$$

$$\leq \sum_{k=1}^{s-1} \left[ \frac{4^{s-k} f_k(\xi)}{5^{s-k} 20} \left( \exp \left( -\frac{1}{4s^2} \sum_{l=k+1}^{s} |f_l(\xi)| \right) - 1 \right) \right] \hat{\hat{g}}_L(x) |g_L(x)|$$

$$+ \frac{4^{s-i_0}}{5^{s-i_0} 20} \hat{f}_{i_0}(x) g_L(x)$$

$$\leq \sum_{k=1}^{s-1} \frac{\|f_k\|_2}{20} \left( \exp \left( -\frac{1}{4s^2} \sum_{l=k+1}^{s} |f_l(\xi)| \right) - 1 \right) \left\|g_L(x)\right\|_2$$

$$+ \frac{1}{20} \hat{f}_{i_0}(x) g_L(x).$$
That is, for \( x \in \Delta^\delta_{t_0} \),

\[
Re(20\hat{F}(x)g_L(x)) - \hat{f}_0(x)g_L(x) \\
\leq \sum_{k=1}^{s-1} \|f_k\|_2 \left\| \left( \exp \left( \frac{-1}{4s^2} \sum_{l=k+1}^{s} |f_l(\xi)| \right) - 1 \right) \right\|_2 |g_L(x)|. \tag{3.4}
\]

Since for all \( a > 0, \left| \exp(\frac{-a}{2}) + \frac{1}{a} \right| \leq 1 \) and for all \( i, \|f_i\|_2 \leq \delta^{-1/n} \) we have

\[
\sum_{k=1}^{s-1} \|f_k\|_2 \left\| \left( \exp \left( \frac{-1}{4s^2} \sum_{l=k+1}^{s} |f_l(\xi)| \right) - 1 \right) \right\|_2 \leq \sum_{k=1}^{s-1} \|f_k\|_2 \left( \sum_{l=k+1}^{s} \frac{\|f_l\|_2}{4s^2} \right) \leq \frac{\delta^{-n}}{8}.
\]

Thus from (3.4), for \( x \in \Delta^\delta_{t_0} \)

\[
\delta^{-n}|g_L(x)| = |\hat{f}_0(x)g_L(x)| \\
\leq |\hat{f}_0(x)g_L(x) - Re(20\hat{F}(x)g_L(x))| + Re(20\hat{F}(x)g_L(x)) \\
\leq \frac{\delta^{-n}}{8}|g_L(x)| + Re(20\hat{F}(x)g_L(x)).
\]

Thus for all \( i \) and \( x \in \Delta^\delta_{t_0}, 0 \leq \delta^{-n}|g_L(x)| \leq 40Re(\hat{F}(x)g_L(x)). \) Hence, for all \( x, 0 \leq \delta^{-n}|g_L(x)| \leq 40Re(\hat{F}(x)g_L(x)) \) and

\[
\int_{S_\delta} \delta^{-n}|g_L(x)|dx \leq 40Re \left( \int_{\mathbb{R}^n} \hat{F}(x)g_L(x)dx \right) \leq 40 \int_{\mathbb{R}^n} |F(\xi)||\hat{g}_L(\xi)|d\xi.
\]

Also we have \( \|F\|_{\infty} \leq 1/4. \) Then,

\[
\int_{S_\delta} \delta^{-n}|g_L(x)|dx \leq C_2 \int_{\mathbb{R}^n} |\hat{g}_L(\xi)|d\xi.
\]

Hence the proof. \( \Box \)

**Proof of Theorem 3.4:**

Since \( E \) is a bounded set, without loss of generality we assume that \( \tilde{m} > 1 \) is the smallest integer such that for all \( x = (x_1, \ldots, x_n) \in E, 1 \leq x_j \leq \tilde{m}, j = 1, \ldots, n. \) Fix \( 0 < \epsilon < 1 \) and \( m = \tilde{m} + 1. \) Then \( E(\epsilon), \) the \( \epsilon \)-distance set of \( E \) is contained in \( M = (0, m) \times \cdots \times (0, m). \)

As in Theorem 2.7, we approximate \( f d\mu \) with a Schwartz function on a fine decomposition of \( E(r/L), r/L \)-distance set of \( E \) for very small \( r/L \) depending on \( \epsilon. \) First,
we consider a self-similar Cantor type fractal \( \tilde{C}_\varepsilon \) as in the proof of Theorem 3.2 in [7] such that \( C_\varepsilon \) has small \( \alpha \)-Hausdorff measure and set \( C_\varepsilon \) the \( n \)-times cartesian product of \( \tilde{C}_\varepsilon \).

Construct a self-similar Cantor-type set \( C \) in \([-2/\varepsilon, -1/\varepsilon] \times \ldots \times [-2/\varepsilon, -1/\varepsilon] \subset \mathbb{R}^n \) satisfying open set condition with dilation factor \( 0 < \eta < 1 \) such that \( N\eta^\alpha = 1 \) and \( \mathcal{H}_\alpha(C) = 1 \). (See Definition 1.6.) Let \( C_\varepsilon \) denote the \( \varepsilon \)-dilated \( C \) such that \( C_\varepsilon \subset [-2, -1] \times \ldots \times [-2, -1] = M_1 \) and \( \mathcal{H}_\alpha(C_\varepsilon) = \varepsilon^\alpha \mathcal{H}_\alpha(C) = \varepsilon^\alpha \).

Denote \( E' = E \cup C_\varepsilon \). Thus for all \( x \in E, \mu(E'_x) = \mu(E_x) + \mathcal{H}_\alpha(C_\varepsilon) \) and also \( E' \) is such that the upper Minkowski content of \( E'_x \) is nonzero and bounded above by their Hausdorff measure. Hence

\[
\int_E \frac{|f(x)|^p}{\mu(E_x)^{2-p}}d\mu(x) = \lim_{\varepsilon \to 0} \int_E \frac{|f(x)|^p}{(\mu(E_x) + (\eta^{-1}\varepsilon)^\alpha + \varepsilon)^{2-p}}d\mu(x) \\
\leq \lim_{\varepsilon \to 0} \int_E \frac{|f(x)|^p}{(\mu(E'_x) + \varepsilon)^{2-p}}d\mu(x) \quad (3.5)
\]

Now to approximate \( fd\mu \) with a Schwartz function, we proceed as in the Theorem 2.7.

Fix \( \varepsilon_1 < \varepsilon/2 \). For each \( k = (k_1, \ldots, k_n), (0 < k_j \in \mathbb{Z}) \) denote \( Q_k = \{x = (x_1, \ldots, x_n) \in M : (k_j - 1)\varepsilon_1 < x_j \leq k_j\varepsilon_1\} \). Let \( Q_0 \) denote the finite collection of all such cubes whose intersection with \( E \) has non zero measure, that is, \( \mu(Q_k) \neq 0 \). For every \( k = (k_1, \ldots, k_n), \) denote \( x_k = ((k_1 - 1)\varepsilon_1, \ldots, (k_n - 1)\varepsilon_1), E_k = E_{x_k} = E \cap \prod_{j=1}^n(\varepsilon_1, k_j\varepsilon_1 - \varepsilon_1) \) and \( E'_k = E'_{x_k} = E' \cap \prod_{j=1}^n(-\varepsilon_1, (k_j - 1)\varepsilon_1]. \) Then for all \( Q_k \in Q_0 \) and \( x \in Q_k, \mu(E'_k) \leq \mu(E'_x) \). Also for all \( k, \mu(E'_k) = \mu(E_k) + \mathcal{H}_\alpha(C_\varepsilon) \geq 0. \) Since \( E \) is compact, \( Q_0 \) has finite disjoint collection of half open cubes. Hence

\[
\int_E \frac{|f(x)|^p}{\mu(E'_x)^{2-p}}d\mu(x) = \sum_{Q_k \in Q_0} \int_{Q_k} \frac{|f(x)|^p}{(\mu(E'_x) + \varepsilon)^{2-p}}d\mu(x) \\
\leq \sum_{Q_k \in Q_0} \int_{Q_k} \frac{|f(x)|^p}{(\mu(E'_k) + \varepsilon)^{2-p}}d\mu(x) \\
\leq \frac{C_p}{(\varepsilon_1)^{2-p}} \sum_{Q_k \in Q_0} \int_{Q_k} |f(x)|^p \\
- \frac{1}{(\mu(Q_k))^{p}} \int_{Q_k} f(y)d\mu(y) \bigg|_{Q_k}^{p} \bigg| d\mu(x) \\
+ \sum_{Q_k \in Q_0} \frac{\mu(Q_k)^{1-p}}{(\mu(E'_k) + \varepsilon)^{2-p}} \int_{Q_k} f(y)d\mu(y) \bigg|_{Q_k}^{p} \bigg| .
\]

(3.6)

Let \( i_{\varepsilon_1} = \inf_{Q \in Q_0} \mu(Q). \) Since infimum is taken over cubes in \( Q_0 \), which is a finite collection and \( \mu(Q) \neq 0 \), we have \( i_{\varepsilon_1} > 0. \) Now, for each \( k \), there exists \( \delta_k \) such that
\[(Q_k \cap E)(\delta) \leq C_n \mu(Q_k \cap E) + C_n i_{\epsilon_1} \epsilon, \quad (3.7)\]
\[|E_k(\delta)| \delta^{\alpha-n} \leq C_n \mu(E_k') + C_n \epsilon, \quad (3.8)\]

for all \(\delta \leq \delta_k\). Let \(\tilde{\delta}_1 \leq \min \{\delta_k\}\).

Let \(\phi\) be a positive Schwartz function such that \(\hat{\phi}(0) = 1\), support of \(\hat{\phi}\) is supported in the unit ball and there exists \(r_1 > 0\) such that

\[
\int_{A_{r_1}(0)} \phi(x) dx = 1/2^{n+1}, \quad (3.9)
\]

where \(A_{r_1}(0) = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : -r_1 < x_j \leq 0, \forall j\}\). Denote \(\phi_L(x) = \phi(Lx)\) for all \(L > 0\). Fix \(\delta_0 \leq \min(\epsilon, \delta_1), r = n^{1/2}r_1\) and \(L\) large such that \(r/L \leq \delta_0\). Then we have,

\[
\left| \int_{Q_k} f(y) d\mu(y) \right|^p = 2^{p(n+1)} \left| \int_{Q_k} \int_{A_{r_1}(0)} \phi(x) dx f(y) d\mu(y) \right|^p
\]
\[
= 2^{p(n+1)} L^{np} \left| \int_{Q_k} \int_{A_{r_1/L}(y)} \phi_L(x-y) dx f(y) d\mu(y) \right|^p
\]
\[
= 2^{p(n+1)} L^{np} \left| \int_{Q_k E_L} \int_{Q_k} \phi_L(x-y) f(y) d\mu(y) dx \right|^p.
\]

where \(Q_k E_L = \{x = (x_1, \ldots, x_n) \in M : \exists y = (y_1, \ldots, y_n) \in E, \text{ such that } y_j - r_1/L < x_j \leq y_j \forall j\}\). Note that \(|Q_k E_L| \leq |(Q_k \cap E)(r/L)|\), where \((Q_k \cap E)(r/L)\) denotes the \(r/L\)-distance set of \(Q_k \cap E\) (since \(r = n^{1/2}r_1\)). Since \(\phi\) and \(f\) are positive,

\[
\int_{Q_k E_L} \int_{Q_k} \phi_L(x-y) f(y) d\mu(y) dx \leq \int_{Q_k E_L} \phi_L * f d\mu(x) dx.
\]

Thus

\[
\left| \int_{Q_k} f(y) d\mu(y) \right|^p \leq 2^{p(n+1)} L^{np} \int_{Q_k E_L} \phi_L * f d\mu(x) dx
\]
\[
\leq 2^{p(n+1)} L^{np} \left| (Q_k E_L) \right|^{p-1} \int_{Q_k E_L} |\phi_L * f d\mu(x)|^p dx
\]
\[
\leq 2^{p(n+1)} r^{(n-a)(p-1)} L^{n+a(p-1)} \left| (Q_k \cap E)(r/L) \right|^{(\alpha-n)(p-1)}
\]
\[
\times \int_{Q_k E_L} |\phi_L * f d\mu(x)|^p dx.
\]

By (3.7), there exists a constant \(\tilde{C}\) independent of \(f\), \(\epsilon\), and \(L\) such that

\[
\frac{1}{\mu(Q_k)^{p-1}} \left| \int_{Q_k} f(y) d\mu(y) \right|^p \leq \tilde{C} L^{n+a(p-1)} \int_{Q_k E_L} |\phi_L * f d\mu(x)|^p dx.
\]

(3.10)
Let

\[ e_{\epsilon_1} = \sum_{Q_k \in Q_0} e_k = \sum_{Q_k \in Q_0} \int_{Q_k} \left| f(x) - \frac{1}{(\mu(Q_k))^p} \int_{Q_k} f(y) d\mu(y) \right|^p d\mu(x). \]

(3.11)

Then from (3.6), (3.8) and (3.10), there exists a constant \( \tilde{C}_1 \) independent of \( f, \epsilon \) and \( L \) such that

\[
\int_E \frac{|f(x)|^p}{(\mu(E') + \epsilon)^{2-p}} d\mu(x) \\
\leq C_p \epsilon^{p-2} e_{\epsilon_1} + C_p \tilde{C}_1 L^{n+(p-1)} \sum_{Q_k \in Q_0} \int_{Q_k E_L} \frac{\phi_L * f d\mu(x)}{(\mu(E_k') + \epsilon)^{2-p}} dx \\
\leq C_p \epsilon^{p-2} e_{\epsilon_1} + \tilde{C}_1 L^{n+(p-1)+\alpha} \sum_{Q_k \in Q_0} \int_{Q_k E_L} \frac{\phi_L * f d\mu(x)}{(\mu(E_k') + \epsilon)^{2-p}} dx, \quad (3.12)
\]

For given \( \epsilon_1 \), let \( g \in C_c^\infty (d\mu) \) be such that \( \| f - g \|_{L^p(d\mu)} < \epsilon_1 \). Then, as in the proof of Theorem 2.7, we have

\[
e_{\epsilon_1} \leq 2C_p^2 \sum_k \int_{Q_k} |f(x) - g(x)|^p d\mu(x) \\
+ C_p \sum_k \int_{Q_k} \left| g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y) d\mu(y) \right|^p d\mu(x).
\]

Since \( E = \cup_k (Q_k \cap E) \),

\[
e_{\epsilon_1} \leq 2C_p^2 \| f - g \|_{L^p(d\mu)}^p + C_p \sum_k \int_{Q_k} \left| g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y) d\mu(y) \right|^p d\mu(x) \\
\leq 2C_p \epsilon_1 + 2 \sum_k \int_{Q_k} \left| g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y) d\mu(y) \right|^2 d\mu(x). \quad (3.13)
\]

Since \( g \) is compactly supported continuous function, \( g \) is uniformly continuous and

\[
|g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y) d\mu(y)| \to 0
\]

uniformly in \( x \) and \( Q_k \) as \( \mu(Q_k) \to 0 \). As \( \epsilon_1 \to 0 \), we have \( \mu(Q_k) \to 0 \) for all \( k \). Hence
\[
\sum_k \int_{Q_k} \left| g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y) d\mu(y) \right|^p d\mu(x) \\
\leq \sum_k \mu(Q_k) \sup_{Q_k \in \mathcal{Q}_0} \sup_{x \in Q_k} \left| g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y) d\mu(y) \right|^p \\
= \mu(E) \sup_{Q_k \in \mathcal{Q}_0} \sup_{x \in Q_k} \left| g(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} g(y) d\mu(y) \right|^p ,
\]
which goes to zero as \( \epsilon_1 \) goes to zero. Therefore, from (3.13), \( e_{\epsilon_1} \) goes to zero as \( \epsilon_1 \) goes to zero.

Since \( r/L < \epsilon \), for each \( k = (k_1, \ldots, k_n) \), \( Q_k E_L \) intersects with at most \( 2^n - 1 \) other cubes \( Q_m \cap E(r/L) \), where \( m = (m_1, \ldots, m_n) \), \( k_j - 1 < m_j < k_j \). Hence for each \( k \), \( Q_k E_L \) is the union of \( Q_m \cap E(r/L) \) and at most \( 2^n - 1 \) other sets \( Q_m \cap E(r/L) \). Then for all such \( m \), \( |E_m(r/L)| \leq |E_k(r/L)| \). Thus for each \( k \),
\[
\int_{Q_k \cap E(r/L)} \frac{\left| \phi_L * f d\mu(x) \right|^p}{|E'_k(r/L)|^{2-p}} dx
\]
repeats at most \( 2^n \) times. Let \( \tilde{\mathcal{Q}}_0 \) denote the collection of all \( Q_k = \{x = (x_1, \ldots, x_n) \in M : (k_j - 1)\epsilon_1 < x_j < k_j\epsilon_1\} \) where \( k = (k_1, \ldots, k_n) \), \( 0 < k_j \in \mathbb{Z} \) such that \( |Q_k \cap E(r/L)| \neq 0 \). Thus from (3.12) and (3.6), there exists a constant \( C_0 \) independent of \( f, \epsilon \) and \( L \) such that for all \( r/L \leq \delta_0 \),
\[
\int_E \frac{|f(x)|^p}{(\mu(E'_k) + \epsilon)^{2-p}} d\mu(x) \\
\leq C_p \epsilon^{p-2} e_{\epsilon_1} + C_0 L^n (p-1) + \alpha \sum_{Q_k \in \tilde{\mathcal{Q}}_0} \int_{Q_k \cap E(r/L)} \frac{\left| \phi_L * f d\mu(x) \right|^p}{|E'_k(r/L)|^{2-p}} dx , \tag{3.14}
\]
where \( e_{\epsilon_1} \) goes to zero as \( \epsilon_1 \) goes to zero.

Denote \( \delta = r/L \). By the construction of \( C_\epsilon \), for all \( k \), \( |C_\epsilon(\delta)| < |E'_k(\delta)| \). Also, by Lemma 1.5, \( |C_\epsilon(\delta)| \geq C_n P(C_\epsilon, \delta) \delta^n \). Denote \( P_\delta = P(C_\epsilon, \delta) > 1 \), the \( \delta \)-packing number of \( C_\epsilon \). For \( j = 0, 1, \ldots, J \), let \( S_j \) be the sub-collection of all \( Q_k \in \tilde{\mathcal{Q}}_0 \) such that \( 2^j P_\delta \delta^n \leq |E'_k(\delta)| < 2^{j+1} P_\delta \delta^n \). We consider only nonempty collections. Denote \( g_L(x) = \phi_L * f d\mu(x) \). Then
\[
\sum_{Q_k \in \tilde{\mathcal{Q}}_0} \int_{Q_k \cap E(\delta)} \frac{|g_L(x)|^p}{|E'_k(\delta)|^{2-p}} = \sum_j \sum_{Q_k \in S_j} \int_{Q_k \cap E(\delta)} \frac{|g_L(x)|^p}{|E'_k(\delta)|^{2-p}} \\
\leq \sum_j (2^j P_\delta \delta^n)^{p-2} \sum_{Q_k \in S_j} \int_{Q_k \cap E(\delta)} |g_L(x)|^p dx .
\]
For each \( j \), we can write \( \cup_{Q_k \in S_j} Q_k \cap E(\delta) = S_j = \cup_{i=1}^{j+1} \Delta_i^\delta \) as the finite disjoint union of non-empty sets intersected with cubes of volume \( \delta^n \), that is, \( 0 < |\Delta_i^\delta| \leq \delta^n \).
Then
\[ \sum_{Q_k \in CQ_0} \int_{Q_k \cap E(\delta)} \frac{|g_L(x)|^p}{|E_k'(\delta)|^{2-p}} \leq \sum_j (2^j P^{\delta n})^{p-2} \int_{S_j} |g_L(x)|^p dx \quad (3.15) \]

For every \( j \), applying Lemma 3.5, we have
\[ \frac{\delta^{-n}}{P^{\delta}} \int_{S_j} |g_L(x)| dx \leq C \int_{\mathbb{R}^n} |\hat{g}_L(\xi)| d\xi. \quad (3.16) \]

We recall the following interpolation theorem due to Stein (See page 213 in [3] for the proof):

**Theorem 3.6** Let \((\mathcal{R}, \mu)\) and \((\mathcal{S}, \nu)\) be totally \(\sigma\)-finite measure spaces and let \( T \) be a linear operator defined on the \(\mu\)-simple functions on \(\mathcal{R}\) taking values in the \(\nu\)-measurable functions on \(\mathcal{S}\). Suppose that \( u_i, v_i \) are positive weights on \(\mathcal{R}\) and \(\mathcal{S}\) respectively, and that \( 1 \leq p_i, q_i \leq \infty \), \((i = 0, 1)\). Suppose
\[ \| (Tf) v_i \|_{q_i} \leq M_i \| fu_i \|_{p_i}, \quad (i = 0, 1) \]
for all \(\mu\)-simple functions \( f \). Let \( 0 \leq \theta \leq 1 \) and define
\[ \frac{1}{p} = \frac{1}{p_0} - \theta \frac{1}{p_1}, \quad \frac{1}{q} = \frac{1}{q_0} - \theta \frac{1}{q_1} \]
and
\[ u = u_0^{1-\theta} u_1^\theta, \quad v = v_0^{1-\theta} v_1^\theta. \]

Then, if \( p < \infty \), the operator \( T \) has a unique extension to a bound linear operator from \( L_u^p \) into \( L_v^q \) which satisfies
\[ \| (Tf) v \|_q \leq M_0^{1-\theta} M_1^\theta \| fu \|_p, \]
for all \( f \in L_u^p \)

Let \( v_0 = \frac{\delta^{-n}}{P^{\delta}} \chi_{S_0} \) and \( v_1 = u_0 = u_1 = 1 \), where \( \chi_{S_j} \) denotes the characteristic function on \( S_j \). Let \( T \) be defined as \( T(\psi) = \hat{\psi} \), the inverse Fourier transform of \( \psi \). By (3.16), we have for each \( j \) and \( L \),
\[ \| (Tg_L)v_0 \|_1 \leq C \| \hat{g}_L \|_1 \]

By Plancherel theorem, we have
\[ \| (Tg_L)v_1 \|_2 \leq \| \hat{g}_L \|_2 \]
Then applying the Theorem 3.6, for $1 < p < 2$, we have

$$(\delta^n P_\delta)^{p-2} \int_{S_j} |\phi L * f d\mu(x)|^p dx \leq C' \int_{\mathbb{R}^n} |\phi L * f d\mu(\xi)|^p d\xi. \quad (3.17)$$

where $C'$ is a non-zero finite constant independent of $f$. Using (3.15), (3.14) and (3.17), there exists a constant $C$ independent of $f$, $\epsilon$ and $L$ such that for very large $L$

$$\int_E \frac{|f(x)|^p}{(\mu(E'_x) + 2\epsilon)^{2-p}} d\mu(x) \leq e\epsilon_1\epsilon^{p-2} + CL^{n(\alpha+1)+\alpha} \int_{|\xi|\leq L} |\widehat{f d\mu}(\xi)|^p d\xi,$$

for all $r/L \leq \delta_0$, where $\delta_0$ goes to zero as $\epsilon_1 < \epsilon/2 \to 0$. Hence letting $\epsilon_1$ to zero, we have

$$\int_E \frac{|f(x)|^p}{(\mu(E'_x) + 2\epsilon)^{2-p}} d\mu(x) \leq C \liminf_{L \to \infty} \frac{1}{L^{n-\alpha}} \int_{|\xi|\leq L} |\widehat{f d\mu}(\xi)|^p d\xi.$$

Letting $\epsilon$ go to zero, using (3.5), we have

$$\int_E \frac{|f(x)|^p}{(\mu(E'_x))^2} d\mu(x) \leq C \liminf_{L \to \infty} \frac{1}{L^{n-\alpha}} \int_{|\xi|\leq L} |\widehat{f d\mu}(\xi)|^p d\xi.$$ 

Hence the proof.

In [7], the authors proved that the above result (3.1) fails for $n = 1$ without any restriction on the set $E$. It is not known for what optimal value of $k$ and optimal range of $p$, $\limsup_{L \to \infty} \frac{1}{L^k} \int_{|\xi|\leq L} |\widehat{f d\mu}(\xi)|^p d\xi$ is non-zero and finite.

Acknowledgements The author would like to express her sincere gratitude to her research supervisor, Prof. E. K. Narayanan for his guidance and immense support. The author also wishes to thank Prof. Malabika Pramanik and Prof. Robert Strichartz for their valuable remarks. The author is grateful to Prof. Kaushal Verma for the financial assistance provided and UGC-CSIR for financial support. This work is a part of PhD dissertation and is supported in part by UGC Centre for Advanced Studies.

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