Quasi-Nambu-Goldstone modes in nonrelativistic systems

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When a continuous symmetry is spontaneously broken in nonrelativistic systems, there appear either type-I or type-II Nambu-Goldstone modes (NGMs) with linear or quadratic dispersion relation, respectively. When equation of motion or the potential term has an enhanced symmetry larger than that of Lagrangian or Hamiltonian, there can appear quasi-NGMs if it is spontaneously broken. We construct a theory to count the numbers of type-I and type-II quasi-NGMs and NGMs, when the potential term has a symmetry of a non-compact group. We show that the counting rule based on the Watanabe-Brauner matrix is valid only in the absence of quasi-NGMs because of non-hermitian generators, while that based on the Gram matrix \cite{DT & MN, arXiv:1404.7696} is still valid in the presence of quasi-NGMs. We show that there exist two types of type-II gapless modes, a genuine NGM generated by two conventional zero modes (ZMs) originated from the Lagrangian symmetry, and quasi-NGM generated by a coupling of one conventional ZM and one quasi-ZM, which is originated from the enhanced symmetry, or two quasi-ZMs. We find that, depending on the moduli, some NGMs can change to quasi-NGMs and vice versa with preserving the total number of gapless modes. The dispersion relations are systematically calculated by a perturbation theory. The general result is illustrated by the complex linear $O(N)$ model, containing the two types of type-II gapless modes and exhibiting the change between NGMs and quasi-NGMs.

I. INTRODUCTION

Symmetry principle is one of the most important concepts for modern physics. When a continuous symmetry of Hamiltonian or Lagrangian is not preserved in the ground state, spontaneous symmetry breaking (SSB) occurs\cite{1,2}. SSB is ubiquitous in nature from magnetism, superfluidity and superconductivity to quantum field theories, in which it is the most important basis to achieve unification of fundamental forces. When such a SSB occurs, there must appear gapless modes known as Nambu-Goldstone modes (NGMs)\cite{1–3}. NGMs are the most important degrees of freedom at low-energy\cite{4–6}. In relativistic systems, dispersion relations of NGMs are always linear. On the other hand, the dispersion relation can be either linear ($\propto |k|$) or quadratic ($\propto k^2$) in nonrelativistic systems. They are called type-I and type-II NGMs, respectively\cite{7}. Prime examples are given by the Heisenberg ferromagnets and antiferromagnets, which give one type-II and two type-I NGMs, respectively.\cite{35,36} Spinor Bose-Einstein condensates (BECs) of ultracold atoms\cite{33} provide a variety of examples of type-II NGMs\cite{3}. In high energy physics, type-II NGMs appear in dense quark matter\cite{11–13}.

The number of NGMs coincides with the number of generators of broken symmetries in relativistic theories. On the other hand, the number of NGMs in nonrelativistic systems has been unclear until recently. Nielsen and Chadha gave the inequality among the numbers of type-I and II NGMs and broken generators\cite{7}. With the idea of Nambu\cite{14}, Watanabe and Brauner gave a conjecture in Ref.\cite{15} stating that the number of type-II NGMs is a half the rank of the Watanabe-Brauner (WB) matrix, whose components are commutators of generators corresponding to broken symme-

tries, sandwiched by the ground state. Then, the equality of the Nielsen-Chadha inequality and the Watanabe-Brauner conjecture have been proved recently by using the effective Lagrangian approach based on a coset space\cite{16}, by Mori's projection operator method\cite{17}, and later by the Bogoliubov conjecture have been proved recently by using the effective Lagrangian based on coset spaces is very powerful because everything can be described in terms of only symmetry\cite{4–6,38}. However, it does not work in the presence of additional zero modes other than NGMs such as quasi-NGMs\cite{39,40}. This is the case that we discuss in this paper.

Experiments contain vortices in scalar BECs, helium superfluids\cite{37} and dense quark matter\cite{31}, a domain wall in ferromagnets\cite{32} and two-component BECs\cite{33,34}, and a skyrmion line in ferromagnets\cite{35,36}. Among these cases, when zero modes are non-normalizable, there appear non-integer power dispersion relation, such as $\propto k^{3/2}$ for a domain wall in two-component BECs\cite{33} and $\propto -k \log k$ for a vortex in scalar BECs or helium superfluids\cite{37}. However, these dispersion relation become quadratic so they are type-II NGMs, when transverse sizes are small enough as shown in Ref.\cite{10,30} for a vortex and in Ref.\cite{10} for a domain wall. It has been also shown in Ref.\cite{10} that non-integer dispersion does not occur in the uniform ground states.

Among various approaches, the effective Lagrangian based on coset spaces is very powerful because everything can be described in terms of only symmetry\cite{4–6,38}. However, it does not work in the presence of additional zero modes other than NGMs such as quasi-NGMs\cite{39,40}. This is the case that we discuss in this paper.

Quasi-NGMs appear when the symmetry of potential term or equation of motion is larger than the symmetry of Lagrangian or Hamiltonian and it is spontaneously broken in the
ground state. In relativistic theories, they appear in technicolor models [41] and supersymmetric field theories [43,48]. When Lagrangian in supersymmetric theories has a symmetry $G$, the superpotential always has an enlarged symmetry $G^2$, a complexification of $G$. As a consequence, as proved in Refs. [43,44], there must appear at least one quasi-NGM when a global symmetry is spontaneously broken in supersymmetric theories (in the absence of gauge interaction [47]).

In nonrelativistic systems, quasi-NGMs appear in condensed matter systems such as $A$-phase of $^1$He superfluids [49] and $F = 2$ spinor BECs [50], and color superconductivity of dense quark matter [51].

In our previous paper [10], we presented the Bogoliubov theory approach to formulate general treatment of NGMs in nonrelativistic systems. The advantages of this approach are that one can deal with additional zero modes such as quasi-NGMs in the same ground with NGMs on one hand, and that one can also deal with NGMs for space-time symmetry breaking in the same manner on the other hand.

In this paper, we discuss quasi-NGMs in the Bogoliubov theory. In the presence of quasi-NGMs, there are two interesting physics that the effective field theory approach cannot deal with:

1. There can exist type-II modes consisting of one genuine zero mode and one quasi zero mode.

2. Some genuine zero modes can turn to quasi zero modes with keeping the total number of zero modes.

Apparently, the effective Lagrangian based on coset space cannot deal with the first point even if one ignores quasi-NGMs, because of type-II mode which contains only one symmetry generator. It is the same for the second point.

We focus on the cases that the potential term has noncompact symmetry whose Lie algebra inevitably contains non-hermitian generators, which is motivated by quasi-NGMs in supersymmetric theories [52], and/or that the symmetry of the gradient term is reduced by multiple components with different particle masses. We show that the WB matrix does not work to count type-II modes in this case. On the other hand, we use the Gram matrix in the Bogoliubov theory. This reduces to the WB matrix only when all generators are hermitian. In general case, we still can count the number of type-II mode by using the Gram matrix. We present the perturbation theory to calculate dispersion relations of (quasi-)NGMs. We find in general that there exist type-II mode made of one NGM and one quasi-NGM in addition to usual case of two NGMs. We call the former quasi-NGMs and the latter conventional NGMs. We demonstrate this theory by an explicit example exhibiting the above two features, that is, the complex linear $O(N)$ model [53] consisting of $N$ complex scalar fields with $O(N)$ symmetry.

This paper is organized as follows. In Sec. II we give models and the Gross-Pitaevskii-(like) and Bogoliubov equations. In Sec. III we give our general framework to obtain (quasi-)NGMs and their dispersion relations. In Sec. IV we give an example of the complex linear $O(N)$ model consisting of $N$ complex scalar fields with $O(N)$ symmetry, to demonstrate our theory. Sec. V is devoted to a summary and discussion. In Appendix A we give detailed calculations for perturbation theory to obtain dispersion relations of (quasi-)NGMs.

II. THE MODEL AND BOGOLIUBOV EQUATIONS

Here we construct a generalized theory of (quasi-)NGMs when the masses of kinetic terms are not necessarily equal to each other and/or the symmetry of the potential term is represented by a noncompact group. In such a situation, the counting by the WB matrix [15] is no longer applicable due to the non-hermitian properties of generators of a noncompact group, while the counting based on the Gram matrix [10] is still valid.

A. Model

For definiteness, we consider the following Hamiltonian describing the $N$-component scalar fields:

$$\mathcal{H} = \mathcal{T} + \mathcal{V},$$

$$\mathcal{T} = \frac{1}{2} \int d^dx \left( 2M_{ij} \nabla \psi_i \nabla \psi_j + L_{ij} \nabla \psi_i \nabla \psi_j + L_{ij} \nabla \psi_i \nabla \psi_j \right),$$

$$\mathcal{V} = \int d^Dx F(\psi^* \psi).$$

Here, $M_{ij} = M_{ji}^*$ and $L_{ij} = L_{ji}$. The repeated indices imply a summation over those indices. Here and hereafter, we use the vectorial notation $\psi = (\psi_1, \ldots, \psi_N)^T$, and $F(\psi^* \psi)$ is an abbreviation of $F(\psi_1^*, \ldots, \psi_N^*, \psi_1, \ldots, \psi_N)$. The function $F(\psi^* \psi)$ is assumed to have the following symmetry

$$F(\psi^* \psi) = F(g^* \psi^* \psi, g \psi),$$

for $g \in G_N$, where the group $G_N$ is a subgroup of $GL(N, \mathbb{C})$, which is not necessarily to be a compact group, and hence $g$ need not be unitary. In order to guarantee the stability of the system, we require that the kinetic term $\mathcal{T}$ is always non-negative. This imposes the condition that the coefficient matrix

$$\tilde{M} = \begin{pmatrix} M & L \\ L^T & M^* \end{pmatrix}, \quad M = M^T, L = L^T,$$

must be positive-definite, where $M$ and $L$ are $N \times N$ matrices whose $(i, j)$-components are given by $M_{ij}$ and $L_{ij}$. Since $\tilde{M}$ is positive-definite, from the theorem of Ref. [54], there exist a symplectic transformation

$$\left( \begin{array}{c} \psi \\ \psi^* \end{array} \right) = U \left( \begin{array}{c} \phi \\ \phi^* \end{array} \right),$$

$$U^{-1} = \sigma^* U^* \sigma, \quad U = \tau U^* \tau,$$

$$\sigma = \begin{pmatrix} I_N & 0 \\ 0 & -I_N \end{pmatrix}, \quad \tau = \begin{pmatrix} I_N & 0 \\ 0 & I_N \end{pmatrix}$$

in this case.
such that $\tilde{M}$ is transformed into a diagonal matrix:
\begin{equation}
U^T \tilde{M} U = \text{diag}\left(\frac{1}{2m_1}, \ldots, \frac{1}{2m_N}, \frac{1}{2m_1}, \ldots, \frac{1}{2m_N}\right).
\end{equation}

Here, $m_i$'s can be interpreted as particle masses of $N$-species. By positive-definiteness, the particle masses $m_i$'s are all positive.

Thus, in order to avoid confusions, we give a few remarks on terminologies and conventions. The matrix $U$ satisfying Eq. (2.7) is called “paraunitary” in Refs. [54–56], while it is called “Bogoliubov-unitary (B-unitary)” in our work [10] since it represents a Bogoliubov transformation of bosonic field operators. The well-known symplectic transformation can be obtained by
\begin{equation}
S = U^{-1} U_0, \quad U_0 = \frac{1}{\sqrt{2}} \left( I_N, iI_N \right).
\end{equation}

Then, $S$ is a real-valued matrix satisfying $S^T JS = J$ with $J = \sigma_T$. See also Appendix B of Ref. [10].

In the diagonal form in Eq. (2.10), if all masses $m_i$'s are different from each other, $\mathcal{T}$ is invariant only under the phase multiplication of each component $\phi_i \to e^{i\theta_i} \phi_i$, and hence the symmetry group of $\mathcal{T}$, henceforth we write as $G_\mathcal{T}$, is given by $G_\mathcal{T} = U(1)^N$. When some $m_i$ are degenerate, the symmetry group of $\mathcal{T}$ is enhanced. For instance, if $m_1 = m_2$ but all remaining $m_3, \ldots, m_N$ are different, $G_\mathcal{T} = U(2) \times U(1)^{N-2}$. If all masses are the same, $m_1 = \cdots = m_N$, the symmetry group is given by $G_\mathcal{T} = U(N)$, which was treated in our previous work [10]. Most generally, if there are $p_i$ tuples consisting of $N_i$ components with having the same mass, the symmetry group is given by
\begin{equation}
G_\mathcal{T} = \prod_i U(N_i)^{p_i}, \quad \sum_i p_i N_i = N. \tag{2.12}
\end{equation}

Although we can always transform $\mathcal{T}$ to the diagonal form in Eq. (2.10), the choice of the field $\phi_1, \ldots, \phi_N$ which diagonalizes the kinetic term $\mathcal{T}$ is not always convenient for consideration of the potential term $\mathcal{V}$. Thus, henceforth, we construct a general theory with $\mathcal{T}$ in the form of Eq. (2.3).

For the potential term $\mathcal{V}$, we allow it to have a symmetry of a noncompact group $G_\mathcal{V}$. We emphasize that the total Hamiltonian $\mathcal{H} = \mathcal{T} + \mathcal{V}$ only has a symmetry of a compact group $G_\mathcal{H} = G_\mathcal{T} \cap G_\mathcal{V}$, since $G_\mathcal{T}$ is a subgroup of the unitary group $U(N)$.

The symmetry groups $G_\mathcal{T}$ and $G_\mathcal{V}$ of the kinetic term $\mathcal{T}$ and the potential term $\mathcal{V}$ generally have no inclusion relation, i.e., $G_\mathcal{T} \nsubseteq G_\mathcal{V}$ and $G_\mathcal{V} \nsubseteq G_\mathcal{T}$ may hold simultaneously. In this case, the Hamiltonian may have no continuous symmetry except for spacetime ones, i.e. $G_\mathcal{H} = \{e\}$, where $\{e\}$ is a trivial group consisting only of an identity. It has no Noether conservation law except for energy and momentum. Even in this extreme case, there can exist gapless modes, i.e., quasi-NGMs, as we see below. This fact implies that the concepts of Noether charges/currents are not indispensable in the formulation and proof of counting rule of NGMs and quasi-NGMs. Indeed, in our previous work [10], the concept of symmetry was necessary only when we derive SSB-originated zero-modes and the conservation law was not used directly.

### B. Gross-Pitaevskii and Bogoliubov equations

Let us derive the fundamental equations and clarify the problem. The Hamilton equation describing the $N$-component order parameter $\psi = (\psi_1, \ldots, \psi_N)^T$ is given by
\begin{equation}
i\partial_t \psi_i = -iM_i \nabla^2 \psi_i - L_i \nabla^2 \psi_i^* + \frac{\partial F}{\partial \psi_i^*}, \tag{2.13}
\end{equation}
\begin{equation}
-i\partial_t \psi_i^* = -iM_i^* \nabla^2 \psi_i^* - L_i^* \nabla^2 \psi_i + \frac{\partial F}{\partial \psi_i}. \tag{2.14}
\end{equation}

Borrowing the terms from condensed matter physics, we call the above equation as the Gross-Pitaevskii (GP) equation, and writing the linearized fields as $\delta \phi_i = \delta u_i$, $\delta \phi_i^* = \delta v_i$, we obtain
\begin{equation}
i\partial_t \delta u_i = -M_i \nabla^2 \delta u_i - L_i \nabla^2 \delta v_i + F_i \delta u_i + G_{ij} \delta v_j, \tag{2.15}
\end{equation}
\begin{equation}
i\partial_t \delta v_i = -M_i^* \nabla^2 \delta v_i - L_i^* \nabla^2 \delta u_i + F_i^* \delta v_i + G_{ij} \delta u_j \tag{2.16}
\end{equation}

with
\begin{equation}
F_{ij} = \frac{\partial^2 F}{\partial \psi_i \partial \psi_j}, \quad G_{ij} = \frac{\partial^2 F}{\partial \psi_i^* \partial \psi_j^*}. \tag{2.17}
\end{equation}

We also call Eqs. (2.15) and (2.16) the Bogoliubov equation in accordance with condensed matter physics. Assuming the spacetime-independent $\psi$, and the plane-wave solution of the form $(u, v) \propto e^{i(kx - \omega t)}$, we obtain the eigenvalue problem of the $2N \times 2N$ matrix:
\begin{equation}
\epsilon \left( \begin{array}{c} u \\ v \end{array} \right) = (H_0 + M_0 k^2) \left( \begin{array}{c} u \\ v \end{array} \right), \tag{2.18}
\end{equation}
\begin{equation}
H_0 = \begin{pmatrix} F & G \\ -G^* & F^* \end{pmatrix}, \quad M_0 = \sigma \tilde{M} = \begin{pmatrix} M & L \\ -L^* & -M^* \end{pmatrix}. \tag{2.19}
\end{equation}

where $F$ and $G$ are the matrices whose $(i, j)$-components are given by $F_{ij}$ and $G_{ij}$, satisfying $F = F^T$ and $G = G^T$. What we want to know is the dispersion relation $\epsilon(k)$. We solve this problem by perturbation theory by regarding $H_0$ as an unperturbed part and $M_0$ as a perturbation term. If $M_0 = 0$, the problem reduces to the one which was solved in Ref. [10].

### III. GENERAL THEORY OF (QUASI)-NAMBU-GOLDSTONE MODES

#### A. Conventional and quasi zero-mode solutions

The SSB-originated zero-mode solutions are the most important key concept in classification and perturbative calculations of dispersion relations of NGMs in the formulation by
the Bogoliubov theory \[10\]. Here we generalize them for the case of quasi-NGMs.

First, let us consider the conventional SSB-originated zero-mode solutions derived from the symmetry of the total Hamiltonian \( G_H \). Let \( \psi \) be a solution of the GP equation \( 2.13 \) and \( 2.14 \), and let \( \hat{Q}_j (j = 1, \ldots, n) \) be a generator of \( G_H \) with \( n = \text{dim} \, G_H \). Since \( G_H \) is a subgroup of the unitary group \( U(N) \), \( \hat{Q}_j \) must be hermitian. We can immediately find the following property:

\[ \psi \text{ is a solution of the GP equation.} \quad \Leftrightarrow \quad \phi = e^{i\hat{Q}_j} \psi \text{ is also a solution.} \quad (3.1) \]

Here \( \alpha \) is a real parameter. Then, by differentiating the GP equation with substituted \( \phi \) by \( \alpha \), and by setting \( \alpha = 0 \) after differentiation, we obtain the following particular solution for the Bogoliubov equation \( 2.15 \) and \( 2.16 \):

\[ \begin{pmatrix} u \\ v \end{pmatrix} = \hat{Q}_j \begin{pmatrix} \psi \\ \psi^* \end{pmatrix}, \quad j = 1, \ldots, n. \quad (3.2) \]

In particular, if we consider a time-independent \( \psi \), we obtain the zero-energy solution of the Bogoliubov equation. In order to distinguish them from that originated from the symmetry of \( G_V \), henceforth we call them conventional zero-mode (conventional ZM) solutions. (Here, in order to make the name short, we omit “SSB-originated”.) We note that if \( \psi \) does not break the symmetry with respect to \( \hat{Q}_j \), i.e., if \( e^{i\hat{Q}_j} \psi = \psi \), Eq. (3.2) only gives a zero vector. Therefore, if we write a number of broken symmetry as \( m (\leq n) \), we obtain \( m \) linearly independent conventional ZMs. We also note that the conventional ZM solution exists even when \( \psi \) has a spatial dependence, i.e., when it is written as \( \psi = \psi(x) \).

Next, let us derive the zero-mode solutions originated from the symmetry of the potential term \( G_V \). We henceforth call such solutions quasi-zero-mode (quasi-ZM) solutions. Let \( \psi = (\psi_1, \ldots, \psi_N)^T \) be a spacetime-independent solution of the GP equation \( 2.13 \). Let \( \hat{Q}_j (j = 1, \ldots, n') \) be a generator of \( G_V \) but not that of \( G_H \), where \( n' = \text{dim} \, G_V - \text{dim} \, G_H \). As already mentioned, \( \hat{Q}_j \) need not be hermitian. Then, following the same argument with \( G_H \), we can show

\[ \psi \text{ is a solution of the GP equation.} \quad \Leftrightarrow \quad \phi = e^{i\hat{Q}_j} \psi \text{ is also a solution.} \quad (3.3) \]

Also, by the same argument with conventional ZMs, we obtain the particular solution of the Bogoliubov equation

\[ \begin{pmatrix} u \\ v \end{pmatrix} = \hat{Q}_j \begin{pmatrix} \psi \\ \psi^* \end{pmatrix}, \quad j = 1, \ldots, n', \quad (3.4) \]

which we call a quasi-ZM.

We note that the property in Eq. (3.3) holds \textit{only when} \( \psi \) \textit{does not have a spatial dependence}, because the kinetic term \( T \) is not invariant under the symmetry operation of \( G_V \). If the order parameter has a spatial dependence as \( \psi(r) \), then \( \phi(r) = e^{i\hat{Q}_j} \psi(r) \) is no longer a solution of the GP equation. This fact implies that the quasi-NGMs are expected to be fragile and are not robust against a perturbation inducing a spatial nonuniformity such as potential walls, vortices, and solitons.

At least in the systematic derivation of dispersion relations by perturbation theory, the distinction of the concept between conventional ZMs and quasi-ZMs is unimportant, as will be seen in the next subsection.

B. Gram matrix and dispersion relations

Let the linearly-independent conventional ZMs and quasi-ZMs derived in the previous subsection be \( q_1, \ldots, q_m \) and \( \tilde{q}_1, \ldots, q_{m'} \). For simplicity, we define \( q_{m+l} = \tilde{q}_l \) for \( l = 1, \ldots, m' \). Then, we introduce a Gram matrix \( P \) of size \( m + m' \), whose \((i, j)\)-component is given by

\[ P_{ij} = (q_i, q_j), \quad (3.5) \]

where the \( \sigma \)-inner product is defined by \[10\]

\[ (x, y)_\sigma = x^\dagger \sigma y, \quad \sigma = \begin{pmatrix} I_N & 0 \\ 0 & -I_N \end{pmatrix}. \quad (3.6) \]

If \((x, y)_\sigma = 0\), \( x \) and \( y \) are said to be \( \sigma \)-orthogonal. If \((x, x)_\sigma \neq 0\), \( x \) is said to have finite norm. If not, it is said to have zero norm.

Let us block-diagonalize this Gram matrix. Since \( P \) is a pure-imaginary hermitian matrix, there exists a real orthogonal matrix \( O \) of size \( m + m' \) giving the following block-diagonal form:

\[ O^{-1} P O = (-\nu_1 \sigma_1) \otimes \cdots \otimes (-\nu_s \sigma_s) \otimes O_r, \quad \sigma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (3.7) \]

where \( r + 2s = m + m' \) and \( \nu_1, \ldots, \nu_s > 0 \). Then the rank of \( P \) becomes

\[ \text{rank} \, P = 2s. \quad (3.8) \]

As shown below, \( s \) gives the number of type-II gapless excitations. In the new basis giving this block-diagonal form in Eq. (3.7), we write the first \( 2s \) vectors as \( x_1^{(1)}, x_1^{(2)}, \ldots, x_s^{(1)}, x_s^{(2)} \) and the rest \( r \) vectors as \( y_1, \ldots, y_r \). Generally, they may be a linear combination of conventional ZMs and quasi-ZMs, i.e., \( q_i \)'s and \( \tilde{q}_j \)'s, and the mixing between conventional ZMs and quasi-ZMs can occur.

We can construct a finite-norm vector \( x_i = \sum_{l=1}^{m'+1} (x_i^{(1)} - i x_i^{(2)}) \). These zero-mode solutions, \( y_1, \ldots, y_r \) and \( x_1, \ldots, x_s \), become a seed of gapless excitations, i.e., a solution of the Bogoliubov equation Eq. (2.13) with finite momentum \( k \) and the dispersion relation \( \epsilon(k) \) can be obtained by perturbation theory \[10\]. Since the calculation is a little long and complicated, we show this in Appendix A. Here we only show the main result.

The zero-mode solutions introduced above satisfy

\[ (x_i, x_j)_\sigma = \delta_{ij}, \quad (3.9) \]

\[ (y_i, y_j)_\sigma = (y_i, x_j)_\sigma = 0. \quad (3.10) \]
While \( x_i \)'s have finite norm, \( y_i \)'s have zero norm. All of them are \( \sigma \)-orthogonal to each other. Whether a given zero mode have finite or zero norm is crucial for classification of NGMs \cite{10}. Let us assume that \( \sigma H_0 \) is positive-semidefinite and \( \sigma M_0 \) is positive-definite, where \( H_0 \) and \( M_0 \) are given in Eqs. 2.18 and 2.19. This assumption ensures that the ground state has a linear stability \cite{10}. As we show in Appendix A, we can always find the following basis without changing the \( \sigma \)-orthogonal relations Eqs. (3.9) and (3.10):

\[
(x_i, M_0 x_i)_\sigma = \frac{1}{\mu_i} \delta_{ij}, \quad \mu_1, \ldots, \mu_s > 0, \tag{3.11}
\]

\[
(y_i, M_0 y_i)_\sigma = 2 \kappa_i \delta_{ij}, \quad \kappa_1, \ldots, \kappa_t > 0, \tag{3.12}
\]

\[
(x_i, M_0 y_i)_\sigma = 0. \tag{3.13}
\]

Using this basis, we can perturbatively solve the Bogoliubov relation

\[
(\epsilon + \frac{1}{\mu_i} k^2 + O(k^3)) \delta_{ij} = \mu_1 \mu_2 \ldots \mu_s \sigma_{ij} + \kappa_1 \kappa_2 \ldots \kappa_t \sigma_{ij} \equiv \rho_{ij}^{\text{WB}}, \tag{3.14}
\]

and the gapless mode arising from \( y_i \) has a type-I dispersion relation

\[
\epsilon = \sqrt{2 \kappa_i k + O(k^2)}. \tag{3.15}
\]

Thus we have \( r \) type-I and \( s \) type-II gapless excitations, and the rank of \( P \) describes the number of type-II modes. See Appendix A for more detailed and complete description.

Now let us give a more precise definition for conventional and quasi-NGMs. As stated above, \( x_i \)'s and \( y_i \)'s are generally written as a linear combination of conventional ZMs \( q_1, \ldots, q_m \) and quasi-ZMs \( \tilde{q}_1, \ldots, \tilde{q}_m \). If the zero mode solution \( y_i \) is written by only using \( q_j \)'s, then a type-I gapless mode arising from \( y_i \) is called a type-I NGM. If \( y_i \) contains \( \tilde{q}_j \)'s, then the type-I gapless mode arising from \( y_i \) is called a type-I quasi-NGM. In the same way we define type-II NGMs and type-II quasi-NGMs depending on whether \( x_i \) includes \( \tilde{q}_j \)'s or not. The classification explained here is summarized in Table I.

| type-I NGM: \( y_i = \sum \alpha_i q_i \) | \( (y_i, y_i)_\sigma = 0 \) |
|----------|------------------|
| type-I quasi-NGM: \( y_i = \sum \alpha_i q_i + \sum \sigma_i \tilde{q}_i \) | \( (y_i, y_i)_\sigma = 0 \) |
| type-II NGM: \( x_i = \sum \alpha_i q_i \) | \( (x_i, x_i)_\sigma = 1 \) |
| type-II quasi-NGM: \( x_i = \sum \alpha_i q_i + \sum \sigma_i \tilde{q}_i \) | \( (x_i, x_i)_\sigma = 1 \) |

Thus, it cannot be expressed as “an expectation value of commutators”. In this case, the WB matrix is no longer equivalent to the Gram matrix and does not work anymore to count type-II modes. Even in such the case, as demonstrated above, we can derive zero-mode solutions by differentiation with respect to parameters in the noncompact group, and can count the numbers of type-I and II modes by the Gram matrix in the same way with Ref. \cite{10}.

We note that if NGMs are classified based on not dispersion relations but whether zero modes are paired (type-B) or unpaired (type-A) \cite{16}, the criterion based on the WB matrix is still intact, though the dispersion relations cannot be predicted correctly.

### IV. EXAMPLE: COMPLEX LINEAR \( O(N) \) MODEL

In this section, we demonstrate the general theory given above by an explicit example, the complex linear \( O(N) \) model. This model is also interesting in the point that it exhibits NGM-quasi-NGM changes, \( i.e. \), some of NGMs change to quasi-NGMs in particular points in the target space, with preserving the total number of NGMs and quasi-NGMs.

#### A. Complex linear \( O(N) \) model

Let us start with the complex linear \( O(N) \) model with the Lagrangian

\[
\mathcal{L}(\psi_i(x), \dot{\psi}_i(x)) = \int \left( \frac{i \bar{\psi}_i \dot{\psi}_i - i \dot{\psi}_i^* \psi_i}{2} \right) - \mathcal{T} - \mathcal{V}, \tag{4.1}
\]

\[
\mathcal{T} = \int dx \nabla \psi_i^* \nabla \psi_i, \tag{4.2}
\]

\[
\mathcal{V} = \int dx F(\psi_i^*, \dot{\psi}_i) \tag{4.3}
\]

Here, the spatial dimension is arbitrary and the repeated indices imply the summation over \( 1 \leq i \leq N \). The potential

---

**TABLE I. Classification of genuine and quasi-NGMs based on the properties of seed zero-mode solutions.**

- \( q_i \)'s are conventional ZMs obtained from the symmetry of the Hamiltonian \( G_N \), and \( \tilde{q}_i \)'s are quasi-ZMs from the symmetry of the potential \( V \). A given gapless mode is a NGM (quasi-NGM) if the seed zero-mode solution does not include (includes) quasi-ZMs in its linear combination. The dispersion relations are determined by the norm of zero-mode. The coefficients of type-II (quasi-)NGMs may be complex to make the norm finite.
function $F(s, s^*)$ is assumed to be real $F(s, s^*) = F(s^*, s)$ and written only by the $O(N, \mathbb{C})$ singlet

$$s := \sum_{i=1}^{N} \psi_i \overline{\psi}_i. \quad (4.4)$$

By this assumption, while the symmetry group of the total Lagrangian is $G_L = O(N, \mathbb{R})$, the symmetry group of the potential term $V$ is $G_V = O(N, \mathbb{C})$. The enhancement of the symmetry in the potential term is crucial for emergence of quasi-NGMs. The symmetry groups for each term and the total Lagrangian are summarized as

$$G_T = U(N), \quad (4.5)$$
$$G_V = O(N, \mathbb{C}), \quad (4.6)$$
$$G_L = G_T \cap G_V = O(N, \mathbb{R}). \quad (4.7)$$

Although we do not have to specify the form of the potential term, here we give two examples. The simplest example is given by

$$F(s, s^*) = \lambda|s - r^2 e^{2i\theta}|^2, \quad (4.8)$$

where $r$ and $\lambda$ are positive and real, and $\theta$ is real. A simple example of $F$ with an additional $U(1)$ symmetry, $G_V = U(1) \times O(N, \mathbb{C})$, is given by

$$F(s, s^*) = |s|^4 - 2r^2|s|^2 \quad (4.9)$$

with a real constant $r$.

In order to apply the general results obtained in the previous section, let us move on to the Hamiltonian formalism. The canonical momentum fields for $\psi_i(x)$’s are given by

$$\pi_i(x) = \frac{\delta L}{\delta \dot{\psi}_i(x)} = \frac{\lambda \psi_i(x)\overline{\psi}_i(x) + \lambda \psi_i(x)\overline{\psi}_i(x)}{2}, \quad \pi_i^*(x) = \frac{\delta L}{\delta \dot{\psi}_i^*(x)} = \frac{-\beta\overline{\psi}_i(x)}{2}. \quad (4.10)$$

Then, the Hamiltonian is introduced by the Legendre transformation, which coincides with $T + V$:

$$\mathcal{H} = \int dx \left( \pi_i \dot{\psi}_i + \pi_i^* \dot{\psi}_i^* \right) - L = T + V. \quad (4.11)$$

The symmetry of the Hamiltonian is the same with that of the Lagrangian: $G_H = G_L$. The Hamilton equation for this system is

$$i\dot{\psi}_i = \frac{\delta H}{\delta \overline{\psi}_i} = -\nabla^2 \psi_i + 2\psi_i \frac{\partial F(s, s^*)}{\partial s} \bigg|_{s=\overline{\psi}_i, s^*=\overline{\psi}_i}, \quad (4.12)$$
$$-i\dot{\psi}_i^* = \frac{\delta H}{\delta \psi_i} = -\nabla^2 \psi_i^* + 2\psi_i \frac{\partial F(s, s^*)}{\partial s} \bigg|_{s=\overline{\psi}_i, s^*=\overline{\psi}_i}. \quad (4.13)$$

This is an analog of the GP equation describing Bose condensates, though the current system does not necessarily conserve a “particle density” $\rho = \sum_i \psi_i^* \psi_i$, because of the absence of the $U(1)$ symmetry. The potential term in Eq. (4.12) is a case without $U(1)$-symmetry. The particle density is conserved in the case with the $U(1)$ symmetry, for instance for the potential term in Eq. (4.9).

Next, we determine the ground state. Let us assume that the ground state of $\psi_i$ is spatially uniform. Then, the ground state solely determined by the minimization of the potential $V$. From Eqs. (4.12) and (4.13), $\frac{\partial F}{\partial s} = \frac{\partial F}{\partial s^*} = 0$ hold in the stationary state.

We can generally show that any $N$-component complex vector $\psi = (\psi_1, \cdots, \psi_N)^T$ can be transformed into the following form by $O(N, \mathbb{R})$ transformation:

$$\psi = r e^{i\theta} \begin{pmatrix} \cosh \varphi \\ i \sinh \varphi \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (4.14)$$

where $r, \theta, \varphi \in \mathbb{R}$ and $r > 0$, $\varphi > 0$. Thus, without loss of generality, we assume that the solution of Eqs. (4.12) and (4.13) is given with Eq. (4.14). Note that the scalar $s$ is given by

$$s = \psi_i \overline{\psi}_i = r^2 e^{2i\theta}, \quad (4.15)$$

which does not depend on $\varphi$. Therefore, the order parameter space consisting of ground states has a residual degree of freedom represented by $\varphi$, in addition to the NGM degree of freedom due to $O(N, \mathbb{R})$-rotation symmetry. This degree of freedom is directly related to the emergence of quasi-NGMs. We can further understand it by an enhanced group symmetry $G_V$ as follows.

When we use $G_V = O(N, \mathbb{C})$, $\psi$ can be transformed to

$$\psi = re^{i\phi} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (4.16)$$

that is, $\varphi$ can be taken to be zero. The unbroken symmetry is then $H_V = O(N - 1, \mathbb{C})$, and the order parameter manifold is

$$\frac{G_V}{H_V} = \frac{O(N, \mathbb{C})}{O(N-1, \mathbb{C})} \simeq T^* \frac{O(N, \mathbb{R})}{O(N-1, \mathbb{R})} \simeq T^* S^{N-1}. \quad (4.17)$$

Since the gradient term is invariant only under $O(N, \mathbb{R})$, this space does not have an $O(N, \mathbb{C})$ isometry but only an $O(N, \mathbb{R})$ isometry. The unbroken symmetry $H_L$ of Lagrangian is not unique, depending on $\varphi$. It is

$$H_L = \begin{cases} O(N-1, \mathbb{R}) & \text{for } \varphi = 0, \\ O(N-2, \mathbb{R}) & \text{for } \varphi \neq 0. \end{cases} \quad (4.18)$$

Therefore, the number of NGMs varies depending on $\varphi$. This can be understood by noting that the unbroken symmetry $H_{V\varphi}$ depends on $\varphi$ as $H_{V\varphi} = g H_{V\varphi} g^{-1}$ with $g \in G_V$ and the unbroken symmetry of the potential, $H_{V\varphi}$, at each $\varphi$ is isomorphic to each other, while the unbroken symmetry of Lagrangian,

$$H_L = H_{V\varphi} \cap U(N), \quad (4.19)$$
does not have to be isomorphic to each other for every \( \varphi \).

When the manifold in Eq. (4.17) is endowed with a Ricci-flat Kähler metric, it is the Eguchi-Hanson space \([57]\) for \( N = 3 \), the deformed conifold \([58]\) for \( N = 4 \), and the Stenzel metric \([53,59]\) for general \( N \).

### B. The Bogoliubov equation

The linearization of the GP equation yields the Bogoliubov equation. That is, substituting \((\psi_i, \psi_i^*) = (\psi_i + \delta \psi_i, \psi_i^* + \delta \psi_i^*)\) to Eqs. (4.12) and (4.13) and ignoring the higher-order terms w.r.t. \( \delta \psi_i \)'s and \( \delta \psi_i^\ast \)'s and rewriting \((\delta \psi_i, \delta \psi_i^\ast) = (u_i, v_i)\), we get

\[
\partial_t u_i = -\nabla^2 u_i + 4 \frac{\partial^2 F}{\partial s^2} \psi_i^* \psi_j u_j + \left( 2 \frac{\partial F}{\partial s} \delta_{ij} + 4 \frac{\partial^2 F}{\partial s^2} \psi_i \psi_j \right) v_j,
\]

(4.20)

\[
-\partial_t v_i = -\nabla^2 v_i + 4 \frac{\partial^2 F}{\partial s^2} \psi_i \psi_j v_j + \left( 2 \frac{\partial F}{\partial s} \delta_{ij} + 4 \frac{\partial^2 F}{\partial s^2} \psi_i \psi_j \right) u_j,
\]

(4.21)

where the notations of substitution \( |x = \psi_i, s = \psi_i^* \) for derivatives of \( F \) are omitted.

Then the stationary Bogoliubov equation with an eigenenergy \( \epsilon \) can be obtained by substitution \((u_i, v_i) \propto e^{i(kx - \epsilon t)}\), yielding

\[
\epsilon \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} F + k^2 & G \\ -G^* & -F - k^2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},
\]

(4.22)

where \( u = (u_1, \ldots, u_N)^T \) and \( v = (v_1, \ldots, v_N)^T \) and \( F \) and \( G \) are \( N \times N \) matrices whose components are given by

\[
F_{ij} = 4 \frac{\partial^2 F}{\partial s^2} \psi_i \psi_j, \quad G_{ij} = 4 \frac{\partial^2 F}{\partial s^2} \psi_i \psi_j^*.
\]

(4.23)

Here, we concentrate on the simplest model, the \( O(3) \) model. When \( \psi_i \) is given by Eq. (4.14), the matrices in Eq. (4.22) reduce to

\[
F = 4r^2 \frac{\partial^2 F}{\partial s^2} \begin{pmatrix} \cosh^2 \varphi & i \cosh \varphi \sinh \varphi & 0 \\ -i \cosh \varphi \sinh \varphi & \sinh^2 \varphi & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

(4.24)

\[
G = 4r^2 e^{-2s} \frac{\partial^2 F}{\partial s^2} \begin{pmatrix} \cosh^2 \varphi & -i \cosh \varphi \sinh \varphi & 0 \\ i \cosh \varphi \sinh \varphi & -\sinh^2 \varphi & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

(4.25)

Solving the Bogoliubov equation (4.22), we soon have the following dispersion relations:

\[
\epsilon = k^2 \quad (\text{doubly degenerate}),
\]

(4.26)

\[
\epsilon = \sqrt{16(F_{sss}^2 - F_{ss} F_{s' s'}) r^4 \cosh^2(2\varphi) + 8 F_{ss} F_{s' s'} r^2 \cosh(2\varphi) k^2 + k^4}
\]

(4.27)

Here, \( F_{ss} = \frac{\partial F}{\partial s}, \ F_{ss} = \frac{\partial^2 F}{\partial s^2}, \) and \( F_{s' s'} = \frac{\partial^2 F}{\partial s^2} \) and we have only shown the positive dispersion relations. Thus, we have two type-II and one gaupl excitations.

The gapful mode given in Eq. (4.27) becomes a type-I mode, when the relation

\[
F_{ss}^2 - F_{ss} F_{s' s'} = 0
\]

(4.28)

holds. This corresponds to the emergence of the \( U(1) \)-symmetry as follows: If \( F(s, s') \) is a function depending only on \( |s|^2 \), i.e., if \( F \) can be written as \( F(s, s') = \tilde{F}(|s|^2) \), the potential is also invariant under the \( U(1) \) transformation \( \psi \to e^{i\theta} \psi \) and \( G_{\psi \psi} \) becomes \( G_{\psi \psi} = U(1) \times O(3, \mathbb{C}) \). In this case, the following holds:

\[
s \frac{\partial F}{\partial s} = s' \frac{\partial F}{\partial s'}, \quad \epsilon = |s|^2 \tilde{F}(|s|^2).
\]

(4.29)

Differentiating Eq. (4.29) by \( s \) and \( s' \) and using the stationary condition \( \frac{\partial F}{\partial s} = \frac{\partial F}{\partial s'} = 0 \), we have

\[
\frac{\partial^2 F}{\partial s^2} = s \frac{\partial^2 F}{\partial s^2}, \quad \frac{\partial^2 F}{\partial s'^2} = s' \frac{\partial^2 F}{\partial s'^2},
\]

(4.30)

which leads Eq. (4.28). Thus, the emergence of the type-I mode can be explained by the emergence of the \( U(1) \) symmetry.

The above result for general potential \( F(s, s') \) can be checked by the specific examples of the potential terms given in Eqs. (4.3) and (4.9). In the next subsection, we investigate conventional ZMs and quasi-ZMs and identify the origin of the type-II modes, given in Eq. (4.26).

### C. Zero-mode solutions

Let us apply the result of Subsec. 3.B to the current model. The symmetry of the total Lagrangian or Hamiltonian is given by Eq. (4.3). \( G_L = G_H = O(3, \mathbb{R}) \) has generators \( T_1, T_2, \) and \( T_3 \), where \( T_i \) is a generator of rotation with respect to \( i \)-axis, and its components are given by \( (T_i)_{jk} = -i \epsilon_{ijk} \) with \( \epsilon_{ijk} \) being the Levi-Civita tensor. The symmetry of the potential is given by Eq. (4.4), \( G_{\psi \psi} = O(3, \mathbb{C}) \) is six-dimensional and the generators are given by \( i T_1, i T_2, \) and \( i T_3 \) in addition to those of \( G_L \). Thus, we have at most six zero-mode solutions:

\[
\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} Q \psi \\ -Q^* \psi^\ast \end{pmatrix}, \quad Q = T_1, T_2, T_3, iT_1, iT_2, \text{ and } iT_3.
\]

(4.31)

These are the solutions of the Bogoliubov equation Eq. (4.22) with \( \epsilon = 0 \) for \( k = 0 \). If \( Q \) is a linear combination of \( T_1, T_2, T_3 \), then the zero mode solution becomes a conventional ZM. If \( iT_1, iT_2, iT_3 \) are included, it becomes a quasi-ZM. Any state \( \psi \) represented by Eq. (4.13) preserves \( H_{\psi \psi} = O(2, \mathbb{C}) \) unbroken symmetry, because

\[
\alpha (\cosh \varphi L_1 + i \sinh \varphi L_2) \psi = 0, \quad \alpha \in \mathbb{C}.
\]

(4.32)

So, the number of broken continuous symmetry in \( G_{\psi \psi} \) is four and there are only four linearly-independent solutions in Eq.
Whether Eq. (4.31) includes the symmetry within $G_L$ or not depends on the value of $\phi$. If $\phi \neq 0$, two elements in Eq. (4.31) are non-hermitian and it has no symmetry operation in $G_L$, and hence $H_L = [e]$. On the other hand, if $\phi = 0$, it has a hermitian element $T_1$ and $H_L = O(2, \mathbb{R})$. Thus, the numbers of conventional ZMs and quasi-ZMs change depending on whether $\phi = 0$ or not, with keeping the total number of zero modes.

If $\phi \neq 0$, we have three conventional ZMs

$$q_i = \begin{pmatrix} T_i \psi \\ -T_i^\dagger \psi^* \end{pmatrix}, \quad i = 1, 2, 3,$$  (4.33)

and one quasi-ZM

$$q_3 = \begin{pmatrix} i T_3 \psi \\ i T_3^\dagger \psi^* \end{pmatrix}.$$  (4.34)

The other modes written by $i T_1$ and $i T_2$ are not independent of those of $T_1$ and $T_2$. We remark that the quasi-ZM $q_3$ can be also obtained by differentiation by a parameter $\phi$, i.e., $q_3 \propto \partial_\phi (\psi, \psi^*)^T$. From them, we can construct finite-norm vectors as

$$x_1 = \frac{1}{2r \sinh \varphi} q_1 - \frac{i}{2r \cosh \varphi} q_2 = (0, 0, e^{i\theta}, 0, 0, 0)^T,$$  (4.35)

$$x_2 = \frac{q_1 - iq_3}{2r} = (\sinh \varphi e^{i\theta}, i \cosh \varphi e^{i\theta}, 0, 0, 0, 0)^T.$$  (4.36)

These zero-mode solutions give rise to type-II modes, if we solve the equation Eq. (4.22) with $k \neq 0$ perturbatively, as shown in Subsec. III B and Appendix A. Since $x_1$ can be written by a linear combination of conventional ZMs, the type-II mode arising from $x_1$ is a conventional NGM. On the other hand, $x_2$ is a linear combination of a conventional ZM and quasi-ZM, and hence the type-II mode arising from $x_2$ is a quasi-NGM. We thus obtain the two type-II modes in Eq. (4.26) from zero-mode analysis, and identified one to be a genuine type-II NGM and the other to be a quasi-NGM made of one conventional ZM and one quasi-ZM.

Next, let us consider the case $\phi = 0$. In this case, since $T_1 \psi = 0$, the number of conventional ZMs is two:

$$q_i = \begin{pmatrix} T_i \psi \\ -T_i^\dagger \psi^* \end{pmatrix}, \quad i = 2, 3.$$  (4.37)

Instead, we have two quasi-ZMs:

$$q_i = \begin{pmatrix} i T_i \psi \\ i T_i^\dagger \psi^* \end{pmatrix}, \quad i = 2, 3.$$  (4.38)

The finite-norm eigenvectors are given by

$$x_1 = \frac{q_2 - iq_3}{2r} = (0, 0, -ie^{i\theta}, 0, 0, 0),$$  (4.39)

$$x_2 = \frac{q_3 - iq_1}{2r} = (0, ie^{i\theta}, 0, 0, 0, 0)^T.$$  (4.40)

Both the modes are written as a linear combination of a conventional ZM and quasi-ZM, thus the two type-II modes in Eq. (4.26) are both quasi-NGMs.

While we have concentrated on the complex $O(3)$ model, the analysis can be easily extended to the complex $O(N)$ model. At $\phi = 0$, there are $N - 1$ type-II quasi-NGMs consisting of $N - 1$ conventional ZMs and $N - 1$ quasi-ZMs, and at $\phi \neq 0$, there are $2N - 3$ conventional ZMs and one quasi-ZM, yielding $N - 2$ type-II NGM and one type-II quasi-NGM. With the $U(1)$ symmetric potential such as Eq. (4.9), there is also one type-I NGM. These are summarized in Table II.

V. SUMMARY AND DISCUSSION

We have presented a framework in the Bogoliubov theory to study NGMs and quasi-NGMs in the same ground. We have found two phenomena of quasi-NGMs that the effective Lagrangian approach based on coset spaces cannot deal with. There exist two kinds of type-II gapless modes with quadratic dispersion relations, a genuine NGM consisting of two conventional ZMs and a quasi-NGM consisting of one conventional ZM and one quasi-ZM or two quasi-ZMs. Depending on the moduli, genuine NGMs can change into quasi-NGMs with preserving the total number of gapless modes. We have discussed the cases that the potential term has non-compact symmetry, whose Lie algebra inevitably contains non-hermitian generators, and/or that the symmetry of the gradient term is reduced. We have shown that the WB matrix can count only NGMs, while the Gram matrix in our framework can count both NGMs and quasi-NGMs. We have presented perturbation theory to obtain dispersion relations. We have demonstrated the theory by the complex linear $O(N)$ model consisting of $N$ complex scalar fields with $O(N)$ symmetry.

Some comments on quasi-NGMs are addressed here. Quasi-NGMs can be also localized in the vicinity a topological soliton. An example can be found in a baby Skyrmion line [36]. In this case, dilatation and $U(1)$ phase rotation are symmetries of equations of motion and of Lagrangian, respectively. They are spontaneously broken in the presence of the baby Skyrmion, and a type-II NGM, dilaton-magnon, consisting of quasi ZM (the dilatation) and conventional ZM (the $U(1)$ phase) is localized around it.

Quasi-NGMs are fragile against quantum corrections and will be gapped because the gradient (kinetic) term is not invariant under the enlarged symmetry of the potential, while type-II NGMs remain gapless in quantum corrections even in lower dimensions [60]. It will be important to study the fate of type-II modes consisting of one conventional ZM and one quasi-ZM under quantum corrections. When the quasi-ZM is gapped by quantum corrections, such a type-II mode may change to a type-I NGM.

Quasi-NGMs are also fragile against spatial (or temporal) gradients because of the same reason. Quasi-NGMs in the bulk may be gapped for instance in the vicinity of a topological soliton. Detailed discussion on this direction remains as a future problem.
TABLE II. The numbers of conventional ZMs, quasi-ZMs, type-II NGMs and quasi-NGMs in the complex linear $O(N)$ model for the cases $\varphi = 0$ and $\varphi \neq 0$ in Eq. (A.14). Here we assume that $G_\tau$ does not have a $U(1)$-symmetry.

| $H_e$ | $H_\nu$ | # of conventional ZMs | # of quasi-ZMs | # of type-II NGMs | # of type-II quasi-NGMs |
|-------|---------|------------------------|---------------|------------------|------------------------|
| $\varphi = 0$ | $O(N - 1, \mathbb{R})$ | $O(N - 1, \mathbb{C})$ | $N = 1$ | $N = 1$ | $0$ | $N = 1$ |
| $\varphi \neq 0$ | $O(N - 2, \mathbb{R})$ | $O(N - 1, \mathbb{C})$ | $2N = 3$ | $1$ | $N = 2$ | $1$ |

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Appendix A: Perturbation theory

In this appendix, we present a perturbation theory for the matrix of the Bogoliubov equation $H_0 + M_0 k^2$ [Eq. (2.17)]. We solve the eigenvalue problem of this matrix by regarding $H_0$ as an unperturbed part and $M_0 k^2$ as a perturbation term, with knowing the zero-energy eigenvectors of $H_0$, i.e., conventional ZMs and quasi-ZMs derived in Subsec. III A.

First we derive a Colpa’s standard form [55] for $H_0$. Let us assume that $H_0$ is a B-hermitian matrix such that $\sigma H_0$ is positive-semidefinite, and the eigenvectors with zero eigenvalue $H_0$ is exhausted by $y_1, \ldots, y_k, \bar{x}_1, \ldots, \bar{x}_r$. Following the result by Colpa [55] (See also Sec. 3 of Ref. [10]), for each $y_i$, there exists a unique generalized eigenvector $z_i$, satisfying the relations $H_0 z_i = 2 y_i \langle y_i, z_i \rangle = 2 \delta_{ij}$ [55]. We also write the eigenvector with the positive eigenvalue $\lambda_i$ as $w_i$. We introduce the following B-unitary matrix using the vectors defined so far:

$$U = \begin{pmatrix}
\frac{y_1 + z_1}{2} & \ldots & \frac{y_k + z_k}{2} & x_1 & \ldots & x_m & w_1 & \ldots & w_m
\end{pmatrix},$$

(A.2)

Since the column vectors in this $U$ form a $\sigma$-orthonormal basis, the following $\sigma$-orthogonal relations hold:

$$(x_i, x_j)_{\sigma} = -(\tau x_i^\ast, \tau x_j^\ast)_{\sigma} = \delta_{ij}, \quad (y_i, z_j)_{\sigma} = 2 \delta_{ij},$$

$$(y_i, y_j)_{\sigma} = (z_i, z_j)_{\sigma} = (y_i, x_j)_{\sigma} = (y_i, \tau x_i^\ast)_{\sigma} = 0, \quad (z_i, x_j)_{\sigma} = (z_i, \tau x_i^\ast)_{\sigma} = (x_i, \tau x_i^\ast)_{\sigma} = 0,$$

(A.3)

where the relations for $w_i$’s are omitted. Using this $U$, Colpa’s standard form [55] for $H_0$ is given by

$$U^{-1} H_0 U = \begin{pmatrix}
I_r & I_r \\
-I_r & \Lambda \\
-I_r & \Lambda
\end{pmatrix},$$

(A.4)

and the spectral decomposition of $H_0$ is given by

$$H_0 = \sum_{i=1}^{m} \lambda_i w_i w_i^\ast + \sum_{i=1}^{m} \lambda_i \tau w_i w_i^\ast \tau \sigma + \sum_{i=1}^{r} y_i y_i^\ast \sigma.$$  

(A.5)

Note that this standard form is slightly different from our previous work [10]. In Ref. [10], if we use $\bar{y}_i = \sqrt{\kappa_i} y_i$ and $\bar{z}_i = z_i / \sqrt{\kappa_i}$ instead of $y_i$ and $z_i$, and if we omit tildes, then we obtain the expression in Eq. (A.5) [61]. The standard form in Ref. [10] is unique under a different constraint, $(y_i, y_j)_{\sigma} = 2 \delta_{ij}$, and this choice is convenient if the kinetic term is given by $M_0 = \sigma$. If the kinetic term is given by a more general matrix, however, this convention is not so convenient.

Next, let us calculate eigenvectors and eigenvalues of the matrix $H_0 + M_0 k^2$ for finite momentum $k \neq 0$ by perturbation theory. Let us expand eigenvectors and eigenvalues as $\xi = \xi_0 + k \xi_1 + k^2 \xi_2 + \cdots$ and $\epsilon = \epsilon_0 + k \epsilon_1 + k^2 \epsilon_2 + \cdots$. Henceforth we are only interested in the cases where $\xi_0$ is an eigenvector of $H_0$ with zero eigenvalue. Thus we set $\epsilon_0 = 0$, and the perturbation equations up to $O(k^2)$ is given by

$$H_0 \xi_1 = \epsilon_1 \xi_0,$$

$$M_0 \xi_0 + H_0 \xi_2 = \epsilon_2 \xi_0 + \epsilon_1 \xi_1.$$  

(A.6)

(A.7)
Since $\xi_0$ is given by an eigenvector of $H_0$ with zero eigenvalue, and since the components of zeroth-order solutions in the higher-order terms $\xi_i$ with $i \geq 1$ can be always eliminated, we can set

$$\xi_0 = \sum_{j=1}^{s} a_j x_j + \sum_{j=1}^{s} b_j \tau x_j + \sum_{j=1}^{r} c_j y_j, \quad (A.8)$$

$$\xi_i = \sum_{j=1}^{r} d_{ij} z_j + \sum_{j=1}^{N-r-s} a_{ij} w_j + \sum_{j=1}^{N-r-s} b_{ij} \tau w_j, \quad i \geq 1. \quad (A.9)$$

Form the first order equation $(A.6)$, we immediately have

$$2d^{(1)}_{i} - \epsilon_1 c_i = 0, \quad \epsilon_1 a_i = \epsilon_1 b_i = 0, \quad \alpha^{(1)}_i = \beta^{(1)}_i = 0. \quad (A.10)$$

The next discussion differs depending on whether $\epsilon_1$ is zero or not.

We first consider the case $\epsilon_1 \neq 0$. Then we obtain $a_i = b_i = 0$ and $d^{(1)}_{i} = \frac{1}{2} \epsilon_1 c_i$. Thus, the eigenvector up to $O(k^2)$ can be written as

$$\xi_0 = \sum_{j=1}^{r} c_j y_j, \quad \xi_1 = \sum_{j=1}^{r} c_j z_j \quad (A.11)$$

$$\xi = \sum_{j=1}^{r} c_j y_j + k \xi_0 y_j + O(k^2). \quad (A.12)$$

Taking the $\sigma$-inner product between $y_i$ and the second-order equation $(A.7)$, we obtain

$$\sum_{j=1}^{r} (y_i, M_0 y_j) \sigma c_j = \epsilon_1^2 c_i. \quad (A.13)$$

If we define $r \times r$ matrix $Y$ whose $(i, j)$-component is given by $Y_{ij} = (y_i, M_0 y_j)\sigma$, the above is the eigenvalue problem of $Y$. Since $\sigma M_0$ is assumed to be positive-definite, the matrix $Y$ is positive-definite, real, and symmetric matrix. The fact that $Y$ is real can be checked as follows. If we write $y_j = (\phi_j, -\phi_j)^T$, then

$$(y_i, M_0 y_j)\sigma = 2 \Re \left( \phi^\dagger_i M \phi_j - \phi^\dagger_j M \phi_i \right), \quad (A.14)$$

which is obviously real. Therefore, there exist a real orthogonal matrix $R$ such that $R^T Y R$ becomes diagonal, and the eigenvalues are all real and positive. If we introduce a new basis by $\tilde{y}_i = \sum y_j R_{ji}$ and $\tilde{z}_i = \sum z_j R_{ji}$, and write the eigenvalues as $2k_1, \ldots, 2k_r(> 0)$,

$$(\tilde{y}_i, M_0 \tilde{y}_j)\sigma = 2\epsilon_1 \delta_{ij}, \quad 2k_1, \ldots, 2k_r > 0. \quad (A.15)$$

Thus, the first order eigenvalue is given by $\epsilon_1 = \pm \sqrt{2k_1}$, giving the linear dispersion $\epsilon = \pm \sqrt{2k_1} + O(k^2)$, and the eigenvector is given by $\tilde{y}_i \pm k_1 \sqrt{2} \tilde{z}_i + O(k^2)$. Here we note that the tilde-added vectors, $\tilde{y}_i$’s and $\tilde{z}_i$’s also satisfy the same $\sigma$-orthogonal relations in Eq. $(A.3)$.

Next, let us consider the case $\epsilon_1 = 0$. From Eq. $(A.10)$, we have $d^{(1)}_{i} = a^{(1)}_i = \beta^{(1)}_i = 0$ and hence $\xi_1 = 0$. Thus the perturbation equation begins from the second-order, given by

$$M_0 \xi_0 + H_0 \xi_2 = \epsilon_2 \xi_0. \quad (A.16)$$

We first introduce the following vectors $\tilde{x}_i$’s by the Gram-Schmidt-like process:

$$\tilde{x}_i = x_i - \sum_{j=1}^{r} \frac{(\tilde{y}_j, M_0 x_i)\sigma}{2k_j} \tilde{y}_j. \quad (A.17)$$

The corresponding $\tau \tilde{x}_i$ can be written in the same form:

$$\tau \tilde{x}_i = \tau x_i - \sum_{j=1}^{r} \frac{(\tilde{y}_j, M_0 \tau x_i)\sigma}{2k_j} \tilde{y}_j. \quad (A.18)$$

This can be shown as follows. Since $M_0$ and $\sigma$ are $B$-hermitian, $\tau M_0^\ast \tau = -M_0$ and $\tau \sigma \tau = -\sigma$ hold. Noting them and the relation $\tilde{y}_j = -\tau \tilde{y}_j$, we have

$$\tilde{y}_j^\ast M_0 \tilde{y}_j = \tilde{y}_j^\ast \sigma M_0 \tilde{y}_j = (\tilde{y}_j^\ast r) (\tilde{y}_j^\ast \sigma M_0 \tilde{y}_j) \tau \tilde{x}_i = -(\tilde{y}_j, M_0 \tau x_i)\sigma. \quad (A.19)$$

The new basis $\tilde{x}_i$, $\tau \tilde{x}_i$ do not change the $\sigma$-orthogonal relations in Eq. $(A.3)$, and further satisfy the following:

$$(\tilde{x}_i, M_0 \tilde{y}_j)\sigma = (\tau \tilde{x}_i, M_0 \tilde{y}_j)_{\sigma} = 0. \quad (A.20)$$

Since $M_0$ is $B$-hermitian, the relation $(M_0 \tilde{x}_i, \tilde{y}_j)_{\sigma} = (M_0 \tau \tilde{x}_i, \tilde{y}_j)_{\sigma} = 0$ also holds. Then, let us redefine the starting zeroth order eigenvector $\xi_0$ as

$$\xi_0 = \sum_{j=1}^{r} a_j \tilde{x}_j + \sum_{j=1}^{r} b_j \tau \tilde{x}_j + \sum_{j=1}^{r} c_j \tilde{y}_j. \quad (A.21)$$

This redefinition does not change the result of the first-order perturbation calculations in Eq. $(A.10)$. Then, taking the $\sigma$-inner product between the second-order equation $(A.16)$ and $\tilde{y}_j$, and using Eq. $(A.15)$, we obtain

$$c_j = 0, \quad j = 1, \ldots, r. \quad (A.22)$$

Next, taking the $\sigma$-inner products between Eq. $(A.16)$ and $\tilde{x}_i$ or $\tau \tilde{x}_i$, we obtain

$$\sum_{j=1}^{r} (\tilde{x}_i, M_0 \tilde{x}_j)_{\sigma} a_j + \sum_{j=1}^{r} (\tilde{x}_i, M_0 \tau \tilde{x}_j)_{\sigma} b_j = \epsilon_2 a_i, \quad (A.23)$$

$$-\sum_{j=1}^{r} (\tau \tilde{x}_i, M_0 \tilde{x}_j)_{\sigma} a_j - \sum_{j=1}^{r} (\tau \tilde{x}_i, M_0 \tau \tilde{x}_j)_{\sigma} b_j = \epsilon_2 b_i. \quad (A.24)$$

Now, let $X$ and $\Xi$ be $s \times s$ matrices whose $(i, j)$-component is given by $X_{ij} = (\tilde{x}_i, M_0 \tilde{x}_j)_{\sigma}$ and $\Xi_{ij} = (\tilde{x}_i, M_0 \tau \tilde{x}_j)_{\sigma}$, respectively. Then, the above equations are interpreted as the eigenvalues problem of the following $B$-hermitian matrix $Z$:

$$Z = \begin{pmatrix} X & \Xi \\ -\Xi^\ast & -X^\ast \end{pmatrix}. \quad (A.25)$$

Due to the assumption that $\sigma M_0$ is positive-definite, $\sigma Z$ is also positive-definite. Thus, from the theorem of Ref. [54] (or from
Theorem 3.4 of Ref. [10], there exists a B-unitary matrix $U$ such that

$$U^{-1}ZU = \text{diag}(\mu_1^{-1}, \ldots, \mu_s^{-1}, -\mu_1^{-1}, \ldots, -\mu_s^{-1}),$$

$$\mu_1, \ldots, \mu_s > 0.$$  \hfill (A.26)

If we write new basis vectors diagonalizing $Z$ as $\tilde{x}_i$, $\tau \tilde{x}_i$, the dispersion relation of type-II quasi-NGM arising from $\tilde{x}_i$ is given by $\epsilon = \mu_i^{-1}k^2 + O(k^4)$, and that from $\tau \tilde{x}_i$ is given by $\epsilon = -\mu_i^{-1}k^2 + O(k^4)$. We thus obtain type-II dispersion relations.

Finally we add a remark. If we rewrite the tilde-added vectors $\tilde{y}_j$, $\tilde{x}_i$ with tildeless notations as $y_j$, $x_i$, then they satisfy the following $\sigma$-orthogonal relations:

$$(x_i, M_0 x_j)_\sigma = (\tau x_i, M_0 \tau x_j)_\sigma = \frac{1}{\mu_i} \delta_{ij}, \quad (x_i, M_0 x_j)_\sigma = 0,$$  \hfill (A.27)

$$(y_i, M_0 y_j)_\sigma = 2 \delta_{ij}, \quad (y_i, M_0 x_j)_\sigma = (y_i, M_0 \tau x_j)_\sigma = 0.$$  \hfill (A.28)

If we set $M_0 = \sigma$ in these relations, it becomes a revisit of the $\sigma$-orthogonal relations given in Subsec. 4.1 of Ref. [10]. The derivation shown here is also applicable to the case $M_0 = \sigma$. The derivation here means that the perturbative calculations and derivations of type-I and type-II dispersion relations do not need the block-diagonalization of the WB matrix, if we appropriately solve the perturbative equation for degenerate zero eigenvalues. However, in the special case $M_0 = \sigma$, as was shown in Subsec. 2.3 of Ref. [10], the choice of the basis such that the WB matrix becomes block-diagonal makes perturbative calculations a little easier.

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