SPECIAL RELATIVISTIC HYDRODYNAMICS WITH GRAVITATION

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ABSTRACT

Special relativistic hydrodynamics with weak gravity has hitherto been unknown in the literature. Whether such an asymmetric combination is possible has been unclear. Here, the hydrodynamic equations with Poisson-type gravity, considering fully relativistic velocity and pressure under the weak gravity and the action-at-a-distance limit, are consistently derived from Einstein’s theory of general relativity. An analysis is made in the maximal slicing, where the Poisson’s equation becomes much simpler than our previous study in the zero-shear gauge. Also presented is the hydrodynamic equations in the first post-Newtonian approximation, now under the general hypersurface condition. Our formulation includes the anisotropic stress.

Key words: gravitation – hydrodynamics – relativistic processes

1. INTRODUCTION

Dealing with the full machinery of Einstein’s gravity in astrophysical situations is often a formidable task both conceptually and practically, especially in the numerical study. It is desirable to have approximation methods if the situation allows. One such well-known approximation is the post-Newtonian (PN) method (Chandrasekhar 1965; Chandrasekhar & Nutku 1969; Chandrasekhar & Esposito 1970; Weinberg 1972; Blanchet et al. 1990; Shibata & Asada 1995; Asada et al. 1996; Asada & Futamase 1997; Takada & Futamase 1999; Hwang et al. 2008, HNP hereafter). The PN approximation takes into account the effect of Einstein’s gravity as a correction term in Newtonian equations; thus, it is weakly relativistic but fully nonlinear. In this work, we will present the first PN (1PN) hydrodynamic equations in the general temporal-gauge (hypersurface or slicing) condition, Section 4.

Besides the 1PN approximation, our main objective in this work is to present another approximation, that is, the special relativistic hydrodynamics in the presence of weak self-gravity with the action-at-a-distance limit. Because we take into account the fully relativistic pressure, anisotropic stress, and velocity, this approximation is complementary to the 1PN method and is fully nonlinear as well; compared to the 1PN approximation, our gravity and the action-at-a-distance conditions are weaker. Our equations will reveal how the relativistic pressure, anisotropic stress, and velocity gravitate (Section 2). The equations will be derived consistently from Einstein’s equation (Section 3).

Einstein’s gravity causes the results to depend on the choice of temporal gauge. Previously, we studied it in the zero-shear gauge (ZSG; Hwang & Noh 2013c, HNc hereafter; Hwang et al. 2016). Here, we take the maximal slicing (MS), which sets the trace of extrinsic curvature equal to zero (Smarr & York 1978). The MS causes a difference only in the Poisson’s equation, which becomes much simpler than the previous one in the ZSG and suitable for numerical study. Here we include the anisotropic stress.

We use the terms temporal gauge, hypersurface, and slicing condition interchangeably. In cosmology the MS corresponds to the uniform-expansion gauge taking the perturbed part of the expansion scalar in the normal frame (which is the same as the perturbed part of trace of extrinsic curvature with a minus sign) to be zero. The gauge degrees of freedom are completely fixed in this slicing (as well as the ZSG) together with our (spatial) gauge conditions imposed in the spatial part of the metric, and each variable in this gauge has a unique corresponding gauge-invariant combination to the fully nonlinear order (Bard- een 1988; Noh & Hwang 2004; Hwang & Noh 2013a, HNa hereafter).

Previously, we presented a fully nonlinear and exact cosmological perturbation theory in the absence of spatially transverse-trace-free metric perturbation (HNa). Although the formulation was made in the cosmological context, it is applicable to the nonexpanding background by setting the scale factor equal to one, and allowing the background Robertson-Walker metric to be that of Minkowsky. We have shown that, in the c-goes-to-infinity limit, our perturbation formulation of Einstein’s gravity properly reproduces the Newtonian hydrodynamics in two gauge conditions (Hwang & Noh 2013b, HNb hereafter): the ZSG and the uniform-expansion gauge (the MS in Minkowsky background). The 1PN equations are also properly recovered without fixing the temporal-gauge condition (Noh & Hwang 2013).

Because we have a curious difference between the presence/absence of the \(3p/c^2\) term in the Poisson’s equation depending on the gauge (compare Equations (4) and (11)), we pay attention to it in the corresponding PN formulation (Section 4) as well as in the gauge transformation to the linear order (Section 5).

In Appendix A, we present the special relativistic hydrodynamics in the Minkowsky spacetime now including the anisotropic stress; as far as we are aware, the case with the stress is not known in the literature. In Appendix B, we present the fully nonlinear and exact perturbation equations in a gauge-ready (i.e., without taking the temporal gauge) form, ignoring the transverse-trace-free-type spatial metric in the nonexpanding background; this is also new in the literature.

Section 2 is a summary of the newly proposed special relativistic hydrodynamic formulation with gravity. Section 3 derives the formulation from full Einstein’s gravity in the two gauge conditions. Section 4 presents the complementary 1PN equations without taking the slicing condition.

For readers interested in numerical implementation, here we summarize the fundamental sets of equations derived in
this work. Equations (1)–(4) are the complete set of equations for special relativistic hydrodynamics with gravity in the MS; for a more generally valid form, see Equations (31)–(33). An alternative expression is Equations (89)–(91) and (86). In the post-Newtonian approximation Equations (58)–(63) are the complete set with the general gauge condition in Equation (68).

2. SPECIAL RELATIVISTIC HYDRODYNAMICS WITH GRAVITATION: SUMMARY

The special relativistic extension of the mass conservation, the energy conservation, the momentum conservation, and the Poisson’s equations, respectively, in the MS is

$$\frac{d\rho}{dt} + \nabla \cdot \mathbf{v} + \frac{d}{dt} \ln \gamma = 0, \quad (1)$$

The index of $\chi_j$ is raised and lowered by $\delta_{ij}$ as the metric. Keeping $\chi^i$ in the metric is essentially important in the MS analysis; see the below Equation (52). We ignored the transverse-trace-free perturbation in the spatial metric, and imposed spatial gauge conditions so that the spatial part of the metrics becomes as simple as in Equation (7) to the fully nonlinear orders (Bardeen 1988; Noh & Hwang 2004, HNa); in the spatial part of the metric tensor

$$g_{ij} = \left(1 + \frac{2\Psi}{c^2}\right)\delta_{ij} + 2\eta_{ij} + C^{(v)}_{ij} + C^{(\nu)}_{ij} + 2C^{(e)}_{ij}, \quad (8)$$

with $C^{(v)}_{ij} \equiv 0$ and $C^{(\nu)}_{ij} \equiv 0 \equiv C^{(e)}_{ij}$, we imposed the spatial gauge conditions as

$$\gamma \equiv 0 \equiv C^{(\nu)}, \quad (9)$$

and set $C^{(e)}_{ij} = 0$ because we ignore the transverse-trace-free part of the metric. We take the same spatial gauge in the 1PN approximation, see Equation (64) (HNP). Although we ignore the transverse-trace-free part of the metric, we keep the corresponding part of the anisotropic stress. Thus we are considering situations where the tensor part (gravitational waves) of the metric is not significantly excited; for example, in the PN approximation, the $C^{(e)}_{ij}$ part appears from the 2.5PN order (Chandrasekhar & Esposito 1970).

The $\Psi$ does not affect the fluid motions and is determined by

$$\Delta \Phi + 4\pi G \rho = -\frac{4\pi G}{c^2} \left[\left(\rho + \frac{p}{c^2}\right)\gamma \nabla v^2 + \Pi^i_i\right], \quad (10)$$

Thus, with relativistic pressure, anisotropic stress or velocity, $\Psi$ in general differs from $\Phi$. Notice how the relativistic pressure, anisotropic stress, and velocity gravitate. We may call Equations (4) and (10) the extended Newtonian and the PN Poisson’s equations, respectively, where $\Phi$ and $\Psi$ may be termed as the Newtonian and the PN gravitational potentials, respectively; in the Newtonian limit, $\Psi$ does not play a role in Newtonian dynamics, but we have $\Psi = \Phi$.

In the case of the ZSG, which sets the longitudinal part of $\chi_j$ equal to zero, the only difference appears in the Newtonian Poisson’s equation. Instead of Equation (4), we have

$$\Delta \Phi + 4\pi G \rho = -\frac{8\pi G}{c^2} \left[\left(\rho + \frac{p}{c^2}\right)\gamma \nabla v^2 + \Pi^i_i\right] + \frac{12\pi G}{c^2} \Delta^{-1}\nabla_i \nabla^i \Gamma \left(\rho + \frac{p}{c^2}\right)\gamma \nabla v^i + \Pi^i_i, \quad (11)$$

and Equations (1)–(3) and (10) remain the same (Hwang et al. 2016). Notice the complication involving the inverse Laplacian operator $\Delta^{-1}$ and the absence of the $\mathcal{E}_2/c^2$ term in Equation (11) compared to Equation (4) in the MS.

In order to derive the above equations we assume the weak gravity and the action-at-a-distance conditions

$$\frac{\Phi}{c^2} \ll 1, \quad \frac{\Psi}{c^2} \ll 1, \quad \frac{\gamma^2 \ell^2}{k} \ll 1, \quad (12)$$
where \( t_\kappa \sim 1/\sqrt{G\varrho} \) and \( t_\ell \sim \ell/c \sim 2\pi/(kc) \) are gravitational timescale and the light propagating timescale of a characteristic length scale \( \ell \), respectively; \( k \) is the wave number with \( \Delta = -k^2 \). The action-at-a-distance condition implies that we keep the action-at-a-distance nature of Newtonian theory. In our approximation presented above we ignore these dimensionless quantities compared to the order of unity.

We have

\[
\gamma^2 \frac{t_\kappa^2}{c^4} \sim \gamma^2 G\varrho c^2 \sim \frac{\phi}{c^2} \sim \frac{\Psi}{c^2}.
\]

(13)

where, in the last two steps, we used our special relativistic Poisson’s equations in Equations (4), (10), and (11); \( M \) is the characteristic mass scale. This estimate shows that all of the conditions in Equation (12) are of the same order. Thus, even for ultra-relativistic velocity \( (\gamma \gg 1) \) the action-at-a-distance condition is comparable to the weak gravity conditions due to the relativistic boost corrections in the Poisson’s equations. Still, in such an ultra-relativistic situation, we can make an argument that, unless we have a steep gradient in the gravitational potential, the gravity term is negligible in the hydrodynamic equations, see the below Equation (50); without the steep gradient in \( \Phi \), the \( \nabla^2 \Phi \) term in Equation (3) is negligible compared to the convective term in the left-hand side due to the weak gravity condition. This is true despite the fact that in Equations (4), (10), and (11) the Lorentz boosted density and pressure in the right-side sides of these equations dominate the gravity. In the presence of steep gradients of \( \Phi \) and \( \Psi \), we have a more general extended hydrodynamic conservation equation derived in Equations (31)–(33), while the Poisson’s equation in Equations (4), (10), and (11) remains the same. The gravity might have non-negligible roles in situations with mildly relativistic velocity (caused by a non-gravitational means) with \( \gamma \sim 1 \).

### 2.1. Newtonian Limit

By taking the \( c \)-goes-to-infinity limit, we recover the Newtonian hydrodynamic equations in both gauges (HNb). We have

\[
\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho v) = 0,
\]

(14)

\[
\frac{\partial \varrho}{\partial \tau} + v \cdot \nabla \varrho = -\Delta \Phi = -\frac{1}{\varrho} (\nabla \varrho + \nabla \Psi j/),
\]

(15)

\[
\Delta \Psi + 4\pi G\varrho = 0.
\]

(16)

Even in the Newtonian context, the internal energy Equation (6) in the \( c \)-infinity limit becomes

\[
\varrho \frac{d \Pi}{d t} + p \nabla \cdot \varrho = -\frac{1}{\varrho} (\nabla \varrho + \nabla \Psi j/).
\]

(17)

Including the internal energy, the energy conservation in Equation (2) becomes

\[
\frac{d \varrho}{dt} + \left( \varrho + \frac{p}{c^2} \right) \nabla \cdot \varrho = -\frac{1}{\varrho} \nabla \varrho + \nabla \Psi j/.
\]

(18)

### 2.2. Slow-motion and Newtonian Stress Limit

Now, for non-relativistic velocity \( (v^2/c^2 \rightarrow 0) \) and Newtonian anisotropic stress, but still with fully relativistic pressure, Equations (1)–(4) become

\[
\frac{d \varrho}{dt} + \varrho \nabla \cdot v = \frac{\varrho}{\varrho + p/c^2} \frac{1}{c^2} v \cdot \nabla p,
\]

(19)

\[
\frac{d \varrho}{dt} + \left( \varrho + \frac{p}{c^2} \right) \nabla \cdot \varrho = \frac{1}{\varrho} \nabla \varrho + \nabla \Psi j/.
\]

(20)

\[
\frac{d \varrho}{dt} = \nabla \varrho - \frac{1}{\varrho + p/c^2} \left( \nabla \varrho + \frac{1}{c^2} \nabla \Psi j/ \right).
\]

(21)

\[
\Delta \varrho + 4\pi G \left( \varrho + 3 \frac{p}{c^2} \right) = 0,
\]

(22)

and Equation (10) becomes

\[
\Delta \Psi + 4\pi G\varrho = 0.
\]

(23)

In the ZSG, instead of Equation (22), we have (HNe)

\[
\Delta \varrho + 4\pi G\varrho = 0.
\]

(24)

The only difference between the two gauges is the presence or absence of \( 3p/c^2 \)-term in the Poisson’s equation. Equations (19) and (20) give

\[
\varrho \frac{d \Pi}{d t} + p \nabla \cdot \varrho = \frac{p/c^2}{\varrho + p/c^2} v \cdot \nabla p - \Pi j/\nabla j/.
\]

(25)

### 2.3. Static Limit

In a static equilibrium situation with all time derivatives vanishing, Equation (3) leads to

\[
\nabla \varrho = \frac{1}{\varrho + p/c^2} (\nabla \varrho + \nabla \Psi j/).
\]

(26)

Equations (4) and (10) become

\[
\Delta \varrho = -4\pi G \left( \varrho + 3 \frac{p}{c^2} \right),
\]

(27)

\[
\Delta \Psi = -4\pi G\varrho,
\]

(28)

where we used Equation (37).

Further assuming a spherical symmetry, we have

\[
\frac{dp}{dr} = \frac{4\pi G}{r^2} \left( \varrho + \frac{p}{c^2} \right) \int_0^r \left( \varrho + \frac{3p}{c^2} \right) r^2 dr,
\]

(29)

which is the well-known Tolman–Oppenheimer–Volkoff relation in the weak gravity limit (Oppenheimer & Volkoff 1939; Tolman 1939). The presence of \( 3p/c^2 \)-terms in the slow-motion limit of Equations (4) and (22) are consistent with the historically known results (Whittaker 1935; McCrea 1951; Harrison 1965).

It is curious to notice, however, that this \( 3p/c^2 \)-term is absent in the ZSG: see Equations (11) and (24). In the ZSG, instead of Equation (27), we have

\[
\Delta \varrho = -4\pi G \left( \varrho + \frac{3}{c^2} \Delta \nabla \Psi j/ \nabla \Psi j/ \right).
\]

(30)

This makes the ZSG result inconsistent with the Tolman–Oppenheimer–Volkoff relation by missing the \( 3p/c^2 \)-term in Equation (29), which can be regarded as a serious
disqualification in the spherically symmetric situation. At the moment, the reason why we have such an inconsistent result in the ZSG is unclear to us; derivations in the ZSG in HNc and Hwang et al. (2016) are consistent with full Einstein’s equations just as in the case for the MS performed in this work; the ZSG case is also derived in parallel below. This has motivated us to pay special attention to this difference (presence or absence of the 3p/c^2 term in the Poisson’s equation depending on hypersurface) by analyzing the 1PN approximation and gauge transformation in perturbation theory: the difference is real, see the paragraphs below Equation (68) and also Section 5.

3. DERIVATION

Here, we derive Equations (1)–(4), (10), and (11) from full Einstein’s equations assuming Equation (12). There is no a priori reason for a formulation based on such assumptions to be possible. The proof based on full Einstein’s equation will guarantee the consistency as a result. We will use the fully nonlinear and exact perturbation formulation of the perturbation equations summarized in Appendix B.

In our derivation of the conservation equations, we apply the weak gravity condition, but kept the \( \nabla \cdot \nu / c^2 \) term compared to \( \nabla \cdot \nu \) and the \( \nu \cdot \nabla / c^2 \) term compared to \( \nu \cdot \nabla \), etc. (however, not in the combination with the anisotropic stress), although these terms are practically negligible in our weak gravity condition, see the below Equation (50). We ignore the \( \Psi / c^2 \) term compared to unity. Thus, instead of Equations (1)–(3), we will derive the following, more general, forms

\[
\frac{d\bar{\gamma}}{dt} + \bar{\gamma} \left[ \nabla \cdot \nu + \frac{d}{dt} \ln \gamma + \frac{1}{c^2} \nabla (\Psi - \Phi) \right] = 0, \tag{31}
\]

\[
\frac{d\bar{\rho}}{dt} + \left( \rho + \frac{p}{c^2} \right) \left[ \nabla \cdot \nu + \frac{d}{dt} \ln \gamma + \frac{1}{c^2} \nabla (\Psi - \Phi) \right] = -\frac{1}{c^2} \Pi^i_j \nu^{j,i} - \frac{1}{c^4} \Pi_\theta \nu^\theta, \tag{32}
\]

\[
\frac{dv}{dt} = \nabla \Phi - \frac{1}{c^2} \nu v^i \nabla \Phi + \frac{v^2}{c^2} \nabla \Psi - \frac{\rho}{\rho + p/c^2} \left( \nabla \rho + \frac{1}{c^2} \nu \rho \right) + \frac{1}{\rho + p/c^2} \left( -\Pi^j_i + \frac{1}{c^2} \nu (\Pi^j_i \nu) \right) - \frac{1}{\gamma^2} (\Pi^i_j \nu^j) + \frac{1}{c^2} \nu (\Pi^i_j \nu^j \nu^k). \tag{33}
\]

Equations (31) and (32) give

\[
\frac{d\Pi}{\bar{\nu}} + \frac{p}{\bar{\nu}} \left[ \nabla \cdot \nu + \frac{d}{dt} \ln \gamma + \frac{1}{c^2} \nu \cdot \nabla (\Psi - \Phi) \right] = -\Pi^i_j \nu^{j,i} - \frac{1}{c^4} \Pi_\theta \nu^\theta. \tag{34}
\]

In this more general form, the relativistic pressure, anisotropic stress, and velocity, in general, demand the presence of two gravitational potentials, \( \Phi \) and \( \Psi \). Although these will be derived consistently from Einstein’s gravity, we will argue that these additional terms are practically negligible, unless we have steep gradients in \( \Psi \) and \( \Psi \), see the below Equation (50). By ignoring these terms, thus strictly applying the weak gravity conditions in the \( \rho \), we have Equations (1)–(3).

We consider a general fluid energy-momentum tensor

\[
T_{ab} = (\rho^e + p) u_a u_b + p g_{ab} + \tau_{ab}, \tag{35}
\]

with \( \rho^e \equiv \rho / c^2 \). The anisotropic stress is introduced as

\[
\tau_{ij} \equiv \Pi_i \nu_j, \tag{36}
\]

where the indices of \( \Pi_i \) are raised and lowered by \( \delta_{ij} \) as the metric. From \( \Pi^i_i \equiv 0 \), we have

\[
\Pi^i_i = \Pi_i \nu^i \nu_j/c^2. \tag{37}
\]

The fluid four-vector is introduced as

\[
u_i \equiv \nabla \nu_i/c, \quad \gamma \equiv -n_i u^i = \frac{1}{\sqrt{1 - \frac{v_i^2}{c^2}}}, \tag{38}
\]

where \( \nu_i \) is the fluid three-velocity measured by the Eulerian observer (HNa); the index of \( \nu_i \) is raised and lowered by \( \delta_{ij} \) as the metric; \( n_i \) is the normal-frame four-vector with \( n_i \equiv 0 \). In the weak gravity approximation, the Lorentz factor \( \gamma \) becomes negligible in the one in Equation (5).

The ADM (Arnowitt–Deser–Misner) momentum constraint in Equation (108) gives

\[
\frac{2}{3} \kappa_{ij} + c \left( \frac{1}{2} \Delta \chi_i + \frac{1}{6} \chi^j_{,ij} \right) = -\frac{8\pi G}{c^2} \left[ \left( \rho + \frac{p}{c^2} \right) \nabla^2 \nu_i + \frac{1}{c^2} \Pi^i_j \nu^j \right]. \tag{39}
\]

We decompose \( \chi_i \) into the longitudinal and transverse parts as \( \chi_i = \chi^l_i + \chi^t_i \) with \( \chi^{t(i)}_{;ij} = 0 \). The MS takes \( \kappa \equiv 0 \), and the ZSG takes \( \kappa \equiv 0 \) as the slicing condition; \( \kappa \) is the trace of extrinsic curvature, and the trace-free part of extrinsic curvature vanishes for \( \chi_i = 0 \), see Equation (B7) in HNa. In the MS, thus \( \kappa \equiv 0 \), this equation can be decomposed as

\[
\Delta \chi^l_i = -\frac{12\pi G}{c^3} \left[ \left( \rho + \frac{p}{c^2} \right) \nabla^2 \nu_i + \frac{1}{c^2} \Pi^i_j \nu^j \right], \tag{40}
\]

\[
\Delta \chi^t_i = \frac{16\pi G}{c^3} \left[ \left( \rho + \frac{p}{c^2} \right) \nabla^2 \nu_i + \frac{1}{c^2} \Pi^i_j \nu^j \right]. \tag{41}
\]

Similarly, in the ZSG, thus \( \chi \equiv 0 \), we have

\[
\kappa = \frac{12\pi G}{c^2} \Delta^{-1} \nu^i \left[ \left( \rho + \frac{p}{c^2} \right) \nabla^2 \nu_i + \frac{1}{c^2} \Pi^i_j \nu^j \right], \tag{42}
\]

and \( \chi^{t(i)} \) is the same as in Equation (41). From these, we can estimate

\[
\frac{1}{c} \Delta^{-1} \kappa_{ij} \sim \chi^l_j \sim \chi^t_j \left[ \frac{\nabla^2 \nu_i \tau_{ij}}{c^2} \right]. \tag{43}
\]
where it is understood that the MS and the ZSG consider \( \kappa \equiv 0 \) and \( \chi \equiv 0 \), respectively. Thus, under the action-at-a-distance condition \( \Delta^{-1} \kappa_i \partial / \partial c \), \( \chi_i \) and \( \chi_i^{(v)} \) are negligible compared with \( v_i / c \).

3.1. Conservation Equations

Using Equations (12) and (43), the covariant momentum conservation in Equation (114) gives

\[
\frac{1}{\gamma} \frac{d}{dt}(\gamma v) = \nabla \Phi + \frac{v^2}{c^2} \nabla \Psi - \frac{1}{\gamma^2} \nabla p + \frac{v}{c^2} \frac{dp}{dt} + \frac{1}{\gamma^2} \sum_{i,j} \frac{\nabla^2 \Pi_{ij}}{c^2} + \frac{1}{\gamma^2} [\nabla \cdot (v \Pi_{ij} v^j)] - \frac{1}{\gamma^2} [\nabla \cdot (\Pi_{ij} v^j)] + \frac{1}{c^2} \nabla \Pi_{ik} v^k,
\]

with \( v = v_i \). This can be arranged to give Equation (33) and

\[
\frac{d}{dt} \ln \gamma = \frac{1}{c^2} v \cdot \nabla \Phi - \frac{1}{\gamma^2} \frac{dp}{dt} + \frac{1}{\gamma^2} \frac{v}{c^2} \nabla \Psi + \frac{1}{\gamma^2} \sum_{i,j} \frac{\nabla^2 \Pi_{ij}}{c^2} + \frac{1}{\gamma^2} [\nabla \cdot (v \Pi_{ij} v^j)] - \frac{1}{\gamma^2} [\nabla \cdot (\Pi_{ij} v^j)] + \frac{1}{c^2} \nabla \Pi_{ik} v^k.
\]

The continuity equation \( \overline{\gamma} \overline{u^i} = 0 \) can be written as \( \overline{\gamma} + \overline{\gamma} \overline{u} = 0 \), and leads to Equation (31). We used

\[
\overline{\gamma} = c \partial / \partial c, \overline{\gamma} = \frac{d}{dt} \overline{\theta} = 0, \quad \overline{\theta} \equiv cu c_x,
\]

\[
\overline{\nabla} \cdot v + \frac{d}{dt} \ln \gamma = \frac{1}{c^2} v \cdot \nabla (\Psi - \Phi),
\]

which follows from Equations (C2), (C3), (D10), and (D11) in HNa. Equation (118) also gives Equation (31) directly. Using Equation (45), we have

\[
\frac{d\overline{\gamma}}{dt} + \overline{\nabla} \cdot v = -\overline{\gamma} \frac{1}{c^2} v \cdot \nabla \Psi + \frac{\overline{\gamma}}{c^2} \frac{1}{\overline{\gamma}^2} \frac{dp}{dt} + \frac{1}{\gamma^2} \sum_{i,j} \frac{\nabla^2 \Pi_{ij}}{c^2} - \frac{1}{\gamma^2} \sum_{i,j} \frac{\nabla \cdot (v \Pi_{ij} v^j)}{c^2} + \frac{1}{\gamma^2} [\nabla \cdot (\Pi_{ij} v^j)] - \frac{1}{\gamma^2} [\nabla \cdot (\Pi_{ij} v^j)] + \frac{1}{c^2} \nabla \Pi_{ik} v^k,
\]

The covariant energy conservation in Equation (113) gives Equation (32). Using Equation (45), we have

\[
\frac{d\overline{\gamma}}{dt} + \left[ (\overline{\gamma} + \frac{p}{c^2} \right] \overline{\nabla} \cdot v = -\overline{\gamma} + \frac{p}{c^2} \frac{1}{c^2} \frac{v \cdot \nabla \Psi}{\gamma^2} + \frac{1}{\gamma^2} \frac{dp}{dt} - \frac{1}{\gamma^2} \frac{v \cdot \nabla \Psi}{\gamma^2} + \frac{1}{\gamma^2} \sum_{i,j} \frac{\nabla^2 \Pi_{ij}}{c^2} - \frac{1}{\gamma^2} \sum_{i,j} \frac{\nabla \cdot (v \Pi_{ij} v^j)}{c^2} + \frac{1}{\gamma^2} [\nabla \cdot (\Pi_{ij} v^j)] - \frac{1}{\gamma^2} [\nabla \cdot (\Pi_{ij} v^j)] + \frac{1}{c^2} \nabla \Pi_{ik} v^k.
\]

For later use, we present alternative expressions of the conservation equations

\[
\frac{\partial}{\partial t} \left[ (\gamma + \frac{p}{c^2}) \gamma^2 \right] + \overline{\nabla} \cdot \left[ (\gamma + \frac{p}{c^2}) \gamma^2 \right] = \left( \gamma + \frac{p}{c^2} \right) \gamma^2 \overline{\nabla} \cdot (\overline{\nabla} \Psi) + \gamma^2 \overline{\nabla} \cdot (\overline{\nabla} \Psi) - \frac{1}{c^2} \overline{\nabla} \cdot (\Pi_{ij} v^j) - \frac{1}{c^2} (\Pi_{ij} v^j) - \frac{1}{c^2} (\Pi_{ij} v^j)\overline{\nabla} \cdot (\overline{\nabla} \Psi)
\]

\[
\frac{\partial}{\partial t} \left[ (\gamma + \frac{p}{c^2}) \gamma^2 \right] + \overline{\nabla} \cdot \left[ (\gamma + \frac{p}{c^2}) \gamma^2 \right] = \left( \gamma + \frac{p}{c^2} \right) \gamma^2 \overline{\nabla} \cdot (\overline{\nabla} \Psi) + \gamma^2 \overline{\nabla} \cdot (\overline{\nabla} \Psi) - \frac{1}{c^2} \overline{\nabla} \cdot (\Pi_{ij} v^j) - \frac{1}{c^2} (\Pi_{ij} v^j) - \frac{1}{c^2} (\Pi_{ij} v^j)\overline{\nabla} \cdot (\overline{\nabla} \Psi)
\]

which also follow from Equations (111) and (112), respectively. These conservation equations are valid in both gauges. The presence of the post-Newtonian potential \( \Psi \) in Equations (31)–(33) demands two Poisson’s equations. In Hwang et al. (2016), however, we have argued that although the \( (v^2/c^2) \overline{\nabla} \Psi \) term in Equation (44) is comparable to the \( \overline{\nabla} \Phi \) term in our approximation, it is negligible compared to the convective term \( v \cdot \overline{\nabla} \Psi \) in the left-hand side due to the weak gravity condition. (In the ultra-relativistic case \( v \approx c \), the \( (v^2/c^2) \overline{\nabla} \Psi \) term is negligible compared to the convective term by the weak gravity condition, unless we have steep gradients in the gravitational potentials \( \Phi \) and \( \Psi \); in this case, the \( \Psi \) term can be negligible as well. In the relativistic case (we keep \( v^2/c^2 \) order, but \( v^2 \ll c^2 \) we have the \( (v^2/c^2) \overline{\nabla} \Psi \) term much smaller than the \( \overline{\nabla} \Phi \)-term, and the latter is again much smaller than the convective term: thus, although we may have to keep the \( \overline{\nabla} \Psi \) term, the \( (v^2/c^2) \overline{\nabla} \Psi \) term can be ignored.) This applies similarly to \( \Psi \) and \( c^{-2} v \cdot \overline{\nabla} \Phi \) terms in Equations (31)–(34), (45), and (48)–(50), and \( \overline{\nabla} (\Phi - \Psi) \) terms in Equations (49) and (50), see Equation (52).

Thus, the basic set of conservation equations effectively becomes Equations (1)–(3). We have kept these in practice negligible terms because, first it is consistent to keep these terms in our derivation from Einstein’s field equations, second these general equations might be relevant in the presence of
steep gradients of gravitational potentials, and third the results are also consistent with the 1PN expansion to be presented in the next section (see the last paragraph in Section 4).

Now we examine the remaining Einstein’s field equations in the two gauge conditions separately and show the full consistency of the derived equations.

3.2. Einstein’s Equations in the MS

In the MS, the ADM energy constraint and the trace of ADM propagation in Equations (109) and (107) give Equations (4) and (10), respectively. In order to show the consistency of these two equations, we need to examine the trace-free ADM propagation in Equation (110). This gives

\[
\left(\nabla^i \nabla_j - \frac{1}{3} \delta^i_j \Delta \right) (\Phi - \Psi) = - \frac{8 \pi G}{c^2} \left[ \left( \rho + \frac{p}{c^2} \right) \gamma^2 \left( \nu^i \nu_j \right) - \frac{1}{3} \delta^i_j \Pi^k_k - \frac{1}{3} \delta^i_j \Pi^k_k \right] - \frac{1}{3} \delta^i_j \nu_j
\]

\[
= - \frac{4 \pi G}{c^2} \Delta^{-1} \frac{\partial}{\partial t} \left\{ 2 \nabla^i \left[ \left( \rho + \frac{p}{c^2} \right) \gamma^2 \nu_j + \frac{1}{c^2} \Pi^k_{ij} \right] 
+ 2 \nabla^i \left[ \left( \rho + \frac{p}{c^2} \right) \gamma^2 \nu_j + \frac{1}{c^2} \Pi^k_{ij} \right] 
- \left( \nabla^i \nabla_j - \frac{1}{3} \delta^i_j \Delta \right) (\Phi - \Psi) \right\},
\]

(51)

where we used Equations (40), (41), and (50). Thus, we have

\[
\Delta (\Phi - \Psi) = - \frac{4 \pi G}{c^2} \left( 3 \rho + \left( \rho + \frac{p}{c^2} \right) \gamma^2 \nu_j + \frac{1}{3} \delta^i_j \Pi^k_k \right).
\]

(52)

Using Equation (52), Equations (4) and (10) become consistent. We note that \( \chi_i \) has a nontrivial role only in Equations (51) and (52); Equation (51) still has a contribution from the \( \chi_i \) part, but the term has no role in deriving Equation (52). In this sense, we have to keep \( \chi_i \) in our metric, though it has no role in deriving the final result.

One remaining equation to be checked for consistency is the trace of extrinsic curvature in Equation (106). This gives

\[
c \Delta \chi = - 3 \frac{\Psi}{c^2}.
\]

(53)

We can show that this is consistent with Equations (10), (40), and (49). This completes the proof of full consistency of Equations (31)–(33), (4), and (10) in Einstein’s gravity, under the assumptions of Equation (12).

3.3. Einstein’s Equations in the ZSG

The case of ZSG proceeds similarly. Equation (107) gives Equation (10). Equation (107) gives

\[
\Delta \Phi + 4 \pi G \left( \rho + \frac{p}{c^2} \right) = \kappa - \frac{8 \pi G}{c^2} \left[ \left( \rho + \frac{p}{c^2} \right) \gamma^2 \nu_j^2 + \Pi^k_k \right].
\]

(54)

From Equation (42), using Equation (50), we can derive Equation (11). In order to show the consistency between Equation (10) and Equation (11), we need to examine Equation (110). This gives

\[
\left(\nabla^i \nabla_j - \frac{1}{3} \delta^i_j \Delta \right) (\Phi - \Psi) = \left( \rho + \frac{p}{c^2} \right) \gamma^2 \nu_j^2 + \Pi^k_k - \frac{1}{3} \delta^i_j \Pi^k_k
\]

\[
= - \frac{4 \pi G}{c^2} \Delta^{-1} \frac{\partial}{\partial t} \left\{ 2 \nabla^i \left[ \left( \rho + \frac{p}{c^2} \right) \gamma^2 \nu_j + \frac{1}{c^2} \Pi^k_{ij} \right] 
+ 2 \nabla^i \left[ \left( \rho + \frac{p}{c^2} \right) \gamma^2 \nu_j + \frac{1}{c^2} \Pi^k_{ij} \right] 
- \left( \nabla^i \nabla_j - \frac{1}{3} \delta^i_j \Delta \right) (\Phi - \Psi) \right\},
\]

(55)

where we used Equations (40), (41), and (50). Thus we have

\[
\Delta (\Phi - \Psi) = - \frac{12 \pi G}{c^2} \Delta^{-1} \frac{\partial}{\partial t} \left\{ \left( \rho + \frac{p}{c^2} \right) \gamma^2 \nu_j^2 + \Pi^k_k - \frac{1}{3} \delta^i_j \Pi^k_k \right\}.
\]

(56)

Using Equation (56), we can show that Equations (10) and (11) are consistent. We note that \( \chi_i \) has a nontrivial role only in Equations (55) and (56); Equation (55) still has contribution from the \( \chi_i \) part, but the term has no role in deriving Equation (52). In this sense, we have to keep \( \chi_i \) in our metric even though it has no role in deriving the final result.

One remaining equation to be checked for consistency is the trace of extrinsic curvature in Equation (106). This gives

\[
\kappa = - \frac{3 \Psi}{c^2}.
\]

(57)

We can show that this is consistent with Equations (10), (40), and (49).
4. 1PN APPROXIMATION: COMPLETE EQUATIONS WITHOUT FIXING SLICING

Now, we switch our subject to the hydrodynamic 1PN approximation. The main purpose of the presentation is to show the consistency of our special-relativity-with-gravity approximation presented in the previous sections. However, the general hydrodynamic set of 1PN equations without fixing the hypersurface condition presented in this section is also new in the literature; previously, it was derived in the cosmological context (HNP). The complete set of 1PN equations without taking the hypersurface condition is

\[ \ddot{\Omega} + \nabla \cdot (\nabla \Omega) = - \frac{1}{c^2} \left( \frac{\partial}{\partial t} + \Omega \cdot \nabla \right) \left( \frac{1}{2} \dot{\Omega}^2 + 3 U \right), \] (58)

\[ \ddot{\Omega} + \nabla \cdot (\nabla \Omega) = - \frac{1}{c^2} \left[ \left( \frac{\partial}{\partial t} + \Omega \cdot \nabla \right) \left( \frac{1}{2} \dot{\Omega}^2 + 3 U \right) + \frac{3}{2} \nabla^2 U \right], \] (59)

\[ \ddot{\Omega} = \nabla \cdot \nabla \Omega - \nabla U - \frac{1}{c^2} \left( \nabla p + \nabla \Pi_i^i \right) \]

\[ = \frac{1}{c^2} \left\{ -2 \nabla (U^2 - \Phi) + \dot{P} + \sigma (P_{ij} - P_{ji}) \right\} \]

\[ - \sigma \left( \frac{\partial}{\partial t} + \Omega \cdot \nabla \right) \left( \frac{1}{2} \dot{\Omega}^2 + 3 U \right) + \sigma^2 \nabla U \]

\[ + \frac{1}{c^2} \left[ \left( \sigma^2 + 4U + \Pi + \frac{P}{c^2} \right) \nabla^2 p + \nabla \Pi_i^i \right] \]

\[ - \sigma \left( \frac{\partial}{\partial t} + \Omega \cdot \nabla \right) \sigma^2 \nabla U \]

\[ - \nabla \left( \Pi_i^i \sigma^2 \right) - \nabla \sigma \nabla p + 2 \nabla \Pi_i^i \left( \frac{1}{2} \dot{\Omega}^2 + 3 U \right) \right\}, \]

\[ \Delta U + 4\pi G\sigma = - \frac{1}{c^2} \left[ 3 U - 2 U \Delta U + 2 \Delta \phi \right] \]

\[ + \dot{P} + 8 \pi G \left( \sigma^2 + \frac{1}{2} \sigma^2 + \frac{3}{2} r \right) \] (61)

\[ 0 = \frac{1}{4} (P_{ij} - \Delta P_{ij}) + \nabla \dot{U} - 4\pi G \nabla \sigma, \] (62)

\[ 0 = U - V. \] (63)

These follow from the continuity, the energy conservation, the momentum conservation, the trace of ADM propagation, the ADM momentum constraint, and the trace-free ADM propagation equations, respectively: see Equations (62), (114), (115), (119), (120), and (91), respectively, in HNP, which were presented in the cosmological background. In our non-cosmological background, we set \( a = 1 \) and \( g_{00} = 0 \equiv p_{00} \), etc., for the background quantities. As mentioned, these 1PN equations also follow from the fully nonlinear perturbation formulation (Noh & Hwang 2013).

The left-hand and right-hand sides of Equations (58)–(63) are the 0PN (Newtonian) and 1PN orders, respectively. From Equations (58) and (59), we have \( \Delta U + 4\pi G \sigma = 0 \). Our 1PN metric convention follows Chandrasekhar (1965)

\[ g_{00} = -\left[ 1 - \frac{1}{c^2} 2U + \frac{1}{c^4} (2U^2 - 4\Phi) \right], \]

\[ g_{ij} = \left[ 1 + \frac{1}{c^2} 2V \right] \delta_{ij}, \]

(64)

where the index of \( P_i \) is raised and lowered by \( \delta_{ij} \) as the metric. Compared to Equation (7), we have

\[ \Phi = U - \frac{1}{c^2} (U^2 - 2\Phi), \quad \chi_i = \frac{1}{c^2} P_i, \quad \Psi = V. \] (65)

The velocity in the 1PN equations is defined as

\[ u^i \equiv u^\rho \frac{p_i}{c}, \]

(66)

where \( \rho^i \) is the fluid coordinate three-velocity (HNP); the index of \( \sigma \) is raised and lowered by \( \delta_{ij} \) as the metric. Compared with \( \nu_i \), defined in Equation (38), we have

\[ \nu_i = \sigma_i + \frac{1}{c^2} [(U + 2V)_{\sigma_i} - P_i]. \] (67)

The above set of equations is still in a gauge-ready form, i.e., without taking the temporal gauge condition yet. The general 1PN hypersurface condition is (HNP)

\[ P_{ij} + n U = 0, \]

(68)

The choice of \( n \) corresponds to taking different temporal gauges: notable ones are the harmonic gauge (\( n = 4 \)) in Weinberg (1972), the MS (\( n = 3 \)) in Chandrasekhar (1965) and the ZSG (\( n = 0 \)) (HNP).

Now, we show the gauge dependence of \( 3p/c^2 \) term in the Poisson’s Equation (61). To 0PN order, Equation (61) gives \( \Delta U = -4\pi G \sigma \). From this, using Equation (59), we have \( \Delta U = 4\pi G \nabla \cdot (\nabla \sigma) \), which also follows from Equation (62). From this, using Equations (59) and (60), we can show

\[ \dot{U} = -4\pi G p + 4\pi G \Delta^{-1} \left[ \nabla \cdot (\nabla \sigma \nabla U) \right. \]

\[ - \nabla \nabla (\nabla \sigma \nabla \dot{U}) \right]. \] (69)

Imposing the general gauge condition in Equation (68), thus still in a gauge-ready form, Equation (61) gives

\[ \Delta U + 4\pi G \sigma + \frac{1}{c^2} \left[ - (n - 3) \dot{U} + 12 \pi G p + \ldots \right] = 0. \] (70)

Using Equation (69), Equation (70) can be arranged as

\[ \Delta U + 4\pi G \sigma + \frac{1}{c^2} \left[ 4\pi G p + \ldots \right] = 0. \] (71)

From Equations (70) and (71), we notice that both the propagation speed of the gravitational potential and the presence of a pressure term in the 1PN Poisson’s equation depend on the gauge condition, and these are related. Under the general gauge condition the propagation speed can be naïvely read from Equation (70) as \( c / \sqrt{n - \frac{3}{2}} \), which becomes the speed of light only in the harmonic gauge with \( n = 4 \), and the pressure term can be read from Equation (71) as \( np/c^2 \).
The gauge dependence of the propagation speed of gravitational potential is not surprising because the potential is gauge dependent, whereas the propagation speed of the (physical) Weyl tensor naturally does not depend on the gauge choice and is always $c$: see Section 7 of HNP. We have an exact analogy in the classical electromagnetism, where the propagation speed of the field potential depends on the gauge choice, whereas the propagation speed of the (physical) field strength is always $c$ (Jackson 2002).

Likewise, the form of the 1PN pressure term also depends on the gauge choice. It vanishes in the ZSG, and becomes $3p/c^2$ and $4p/c^2$ in the MS and the harmonic gauge, respectively. In the MS with $P^i,_{i} = -3U$ the $3U$ in Equation (61) cancels by the $P^j,_{j}$, thus leaving $12\pi G p$. Similarly, in the ZSG with $P^i,_{i} = 0$, the $12\pi G p$ in Equation (61) cancels by the $3U$ using Equation (69).

We can show that Equations (31)–(33), (4), (10), and (11) in the two gauge conditions are consistent with the 1PN Equations (58)–(63), in the overlapping regime (i.e., for the weak gravity and the action-at-a-distance, and 1PN for the pressure, anisotropic stress, and velocity) of the two approximations. In both gauges, according to the comment above Equation (12), we have $v = \bar{v}$, but $\bar{v} \cdot \nabla \bar{v} = v \cdot \nabla v - 3v \cdot \nabla \Phi/c^2$ and $\nabla \cdot (\bar{v} \nabla) = \nabla \cdot (\dot{\bar{v}} v) - 3\bar{v} \cdot \nabla \Phi/c^2$. In the MS the Equations (58)–(63) become

$$\tilde{\bar{v}} + \nabla \cdot (\bar{v} \nabla) = \frac{1}{c^2} (-\bar{v} \cdot \nabla \Phi + v \cdot \nabla p + v \nabla_j \Pi^j_i), \quad (72)$$

$$\tilde{\bar{v}} + \nabla \cdot (\bar{v} \nabla) = \frac{1}{c^2} \left( -\bar{v} \cdot \nabla \Phi - \frac{d\Pi}{dt} + v \cdot \nabla p - p \nabla \cdot v + 3v \cdot (\nabla \Pi^j_i - \Pi^j_i \nabla_j \nu^i) \right), \quad (73)$$

$$\frac{d}{dt} \bar{v} - \nabla \Phi + \frac{1}{\tilde{\bar{v}}} (\nabla p + \nabla_j \Pi^j_i) = \frac{1}{c^2} \left( v^2 \nabla \Phi - v \nabla \Phi \right) + \frac{1}{\tilde{\bar{v}}} \left[ -\dot{\bar{v}} + \left( v^2 + \Pi^j_i \frac{p}{\tilde{\bar{v}}} \right) (\nabla \bar{p} + \nabla_j \Pi^j_i) + v \nabla_j (\Pi^j_i \nu^i) - (\Pi^j_i \nu^i) \right], \quad (74)$$

$$\Delta \Phi + 4\pi G \left( \frac{\rho + 3}{c^2} \right) = \frac{-8\pi G}{c^2} \bar{v}^2, \quad (75)$$

$$0 = \Delta \Psi + 4\pi G \bar{v}, \quad (76)$$

$$0 = \nabla \Phi - 4\pi G \bar{v} \nu_i - \frac{1}{4c^2} \Delta \chi^{(\nu)}_i. \quad (77)$$

Equation (77) is consistent with Equations (53), (57), (40), and (42) valid to the 0PN order. In the ZSG, instead of Equation (75), we have

$$\Delta \Phi + 4\pi G \rho = \frac{-8\pi G}{c^2} \bar{v}^2 + \frac{12\pi G}{c^2} \Delta^{-1} \nabla \nabla_j (\bar{v} \nu^j + \Pi^j_i), \quad (78)$$

and the other equations remain the same.

5. GAUGE TRANSFORMATION

Now, we can show that, to the linear order in perturbation, the presence or absence of the $3p/c^2$ term in the extended Newtonian Poisson’s equation in Equations (4) or (11) depends on the temporal gauge, while Equations (1)–(3) and (10) remain the same. Under the gauge transformation $\tilde{x} = x + \xi$, in a non-expanding background, the gauge transformation properties in Equation (252) of Noh & Hwang (2004) give

$$\tilde{\Phi} = \Phi + c\xi^j, \quad \tilde{\Psi} = \Psi, \quad \tilde{\chi} = \chi - \xi^j,$$

$$= \kappa + c\Delta \xi^j, \quad \tilde{\rho} = \rho, \quad \tilde{p} = p, \quad \tilde{v} = v - c\xi^j, \quad (79)$$

where the spatial gauge transformation is already fixed by our spatial gauge condition (Bardeen 1988); similarly, we can add $\tilde{\gamma} = \gamma$ for the rest-mass density; we introduced $\nu_i \equiv \nu_i + \nu_i^{(0)}$ with $\nu_i^{(0)} \equiv 0$. Thus, variables $\Psi, \gamma, \rho, p$ and $v$ are invariant under the gauge transformation. Let's consider $\tilde{x}^c$ and $x^c$ coordinates as the MS and the ZSG, respectively. Then, using $\chi_{ZSG} \equiv 0$ and $\kappa_{MS} \equiv 0$, we have

$$\Phi_{MS} = \Phi_{ZSG} + c\xi^j, \quad \chi_{MS} = \chi_{ZSG} + c\xi^j,$$

$$\kappa_{MS} = -c\Delta \xi^j, \quad \nu_{MS} = \nu_{ZSG} - c\xi^j, \quad (80)$$

and $\Psi_{MS} = \Psi_{ZSG}$ etc. Using these gauge transformation properties, we can show the following. To the linear order, Equation (10) is apparently gauge invariant. The forms of Equations (1)–(3) to the linear order remain the same in both gauges in our approximation in Equation (12). From Equation (80), we have

$$\Delta \Phi_{MS} - \Delta \Phi_{ZSG} = c\Delta \xi^j = -c\Delta \chi_{MS} = -\kappa_{ZSG}$$

$$= \frac{3}{c^2} - 12\pi G \frac{p}{c^2}, \quad (81)$$

where we used Equations (53), (57), (2)–(4), and (11), and the approximation in Equation (12). Thus, we exactly recover the pressure difference.

6. OTHER EXPRESSIONS

In a conventional notation of special relativistic hydrodynamics suitable for numerical study (Wilson & Mathews 2003; Ryu et al. 2006), Equations (31)–(33) can be arranged using Equations (49) and (50), as

$$\frac{D}{c^2} + \nabla (Dv^i) = \frac{1}{c^2} \nabla \Phi - \nabla \Psi, \quad (82)$$

$$\frac{E}{c^2} + \nabla \cdot [(E + p)v] = \frac{1}{c^2} (E + p)v \cdot \nabla \Phi + \nabla \cdot (F - \Psi), \quad (83)$$

$$\frac{M}{c^2} + \nabla \cdot (Mv^i + p\delta^i_j) = \frac{1}{c^2} (E + p) \left[ \nabla \Phi + \nabla \cdot (\Psi - F) + \frac{v^2}{c^2} \nabla \Psi \right] - \nabla_j (\Pi^j_i \nu^i) - \frac{1}{c^2} (\Pi^j_i v^j \nu^i), \quad (84)$$

and the other equations remain the same.
where
\[ D \equiv \gamma \mathcal{E}, \quad E \equiv \gamma^2 c^2 (\mathcal{E} - p), \quad M_i \equiv \gamma^2 \mathcal{E} v_i, \]  
(85)
are the coordinate (Eulerian) mass density, the ADM energy density and the coordinate momentum density, respectively, with \( h \equiv 1 + \frac{1}{2} \) the specific enthalpy; we have \( E + p = \gamma^2 (\mathcal{E} + p) \). The gravitational potential \( \Phi \) is determined by Equation (4) in the MS and Equation (11) in the ZSG, and \( \Psi \) by Equation (10). In terms of the above notation, we have
\[ \Delta \Phi + 4\pi G \left[ \frac{1}{c^2} (E + 3p) + \frac{M^i M_i}{E + p} + \frac{2}{c^2} \Pi^i_i \right] = 0, \quad (86) \]
\[ \Delta \Psi + 4\pi G \left[ \frac{1}{c^2} (E + \Pi) \right] = 0, \quad (87) \]
in the MS; in the case of ZSG, instead of Equation (86), we have
\[ \Delta \Phi + 4\pi G \left[ \frac{1}{c^2} E + \frac{M^i M_i}{E + p} + \frac{2}{c^2} \Pi_i^i \right] - 3 \Delta^{-1} \nabla_i \nabla_j \left( \frac{M^i M_j}{E + p} + \Pi_i^j \right) = 0. \quad (88) \]

If we strictly apply the weak gravity limit in Equation (12), we can ignore \( \nabla(\Phi - \Psi) \) and \( \nabla \Psi \) terms in Equations (82)–(84). Although these terms are consistently derived in our special-relativity-with-gravity approximation, as we argue below Equation (50), the strengths of these terms are, in practice, negligible compared to other terms in the equations. Thus, in practice, we have
\[ \dot{D} + \nabla_i (D v^i) = 0, \quad (89) \]
\[ \dot{E} + \nabla_i [(E + p) v^i] = \frac{1}{c^2} (E + p) v \cdot \nabla \Phi - \nabla_j (\Pi_i^j v^i) - \frac{1}{c^2} (\Pi_{ij}^j v^i \cdot \dot{y}), \quad (90) \]
\[ M_i + \nabla_j (M_j v^j + p \delta_i^j) = \frac{1}{c^2} (E + p) \nabla \Phi - \nabla_j \Pi_i^j - \frac{1}{c^2} (\Pi_{ij}^j v^i), \quad (91) \]
which correspond to Equations (1)–(3); \( \Phi \) is determined by Equation (86) in the MS, and by Equation (88) in the ZSG. The gravity term in Equation (90) is present even in Newtonian hydrodynamics, see Equation (2.32) in Shu (1992). This is because \( E \) contains both the \( \gamma c^2 \) order term together with the \( \gamma \Pi \) and \( p \) terms,\(^4\) thus, in order to derive this term, we have to use Equation (31) instead of Equation (1).

7. DISCUSSION

We presented two hydrodynamic approximations of Einstein’s gravity. The first method, which is new, is the special relativistic hydrodynamics with fully relativistic pressure and velocity in the presence of self-gravity. In the MS, Equations (1)–(4) are the complete set, and in the ZSG Equation (11) replaces Equation (4).

The second method is the 1PN approximation: our new contribution here is the gauge-ready expression in a non-expanding background. Previously, the hydrodynamic 1PN approximation was studied by Chandrasekhar (1965) in the MS and by Weinberg (1972) in the harmonic gauge; we made the gauge-ready extension in the cosmological context (HNP). Equations (58)–(63) are the complete set of 1PN equations with the general temporal-gauge condition in Equation (68).

The fully general relativistic hydrodynamic equations in terms of \( D, E, \) and \( M_i \) are known in the literature (Wilson & Mathews 2003; Alcubierre 2008; Bonu et al. 2009; Baumgarte & Shapiro 2010; Gourgoulhon 2012; Rezzolla & Zanotti 2013). The numerical relativity generally demands heavy computational resources with some conceptual difficulties compared to the Newtonian hydrodynamics. Our special relativistic hydrodynamics with self-gravity is formally quite similar to the Newtonian one. The special relativistic hydrodynamics with weak gravity was not known in the literature.

We can imagine many astrophysical situations where the special relativistic effect is important, while the gravity is weak. Except near neutron stars or black holes, in most celestial bodies (planets, stars, star cluster, galaxies, galaxy clusters, interstellar, and intergalactic media, and even including astrophysical regions some distances away from pulsar, quasars, jets, etc.) the gravity is weak. The strength of the gravity is measured by \( \Phi / c^2 \sim GM / (\ell c^2) \), where \( M \) and \( \ell \) are the characteristic mass and size dimensions, respectively. In the celestial bodies mentioned above this, dimensionless gravitational strength is of the order of \( \Phi / c^2 \sim 10^{-5} \sim 10^{-10} \) and even smaller. In such situations, the gravity is indeed weak. We anticipate that our two approximations may find wide applications in such astrophysical situations, where the special relativistic activity is important.

In most realistic astrophysical situations, where special relativity is important, however, the magnetic activity is often also involved. Extending our special-relativity-with-gravity approximation, including special relativistic MHD (magneto-hydrodynamics), is currently in progress.

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APPENDIX A
SPECIAL RELATIVISTIC HYDRODYNAMICS

The special relativistic hydrodynamic equations without the anisotropic stress are known in the literature (Weinberg 1972; Battaner 1996). Here we present the full equations in the presence of the anisotropic stress. We consider Minkowsky spacetime, thus \( g_{ab} = \eta_{ab} \). The general energy-momentum tensor is in Equation (35). The normalized \( (u^a u_a \equiv -1) \) four-
vector is
\[ u_i = \gamma v_i, \quad u^i = \gamma^{-1} v^i, \quad u_0 = -u^0 = -\gamma; \]
\[ \gamma \equiv -n_i u^i = \frac{1}{\sqrt{1 - v^2}}, \quad (92) \]

with the index of \( v^i \) raised and lowered by \( \delta_{ij} \) and its inverse; we have \( v^2 \equiv |v|^2 \equiv v_i v^i \) with \( v \equiv v^i \); the normal four-vector is
\[ n_i \equiv 0 = n^i, \quad n_0 = -n^0 = -1. \]
(93)
The anisotropic stress \( (\tau_{ab} u^b = 0) \) gives
\[ \pi_{ij} = \Pi_{ij}, \quad \pi_{00} = -\Pi_{ij} v_i v_j, \quad \pi^0_0 = \Pi^0_j v_j; \quad \pi^i_j = \Pi_{ij}, \]
\[ \pi^0_i = \Pi_{ij} v_j, \quad \pi^0_0 = -\Pi^0_j v_j; \quad \pi^i_0 = -\Pi^i_j v_j. \]
(94)

with the index of \( \Pi_{ij} \) raised and lowered by \( \delta_{ij} \) and its inverse. The condition \( \pi^i_c \equiv 0 \) gives
\[ \Pi^i_j = \Pi_{ij} \frac{v_i v^j}{c^2}. \]
(95)
The continuity equation \( (\square u^a)_{,a} = 0 \) gives
\[ (\square \gamma) + \nabla \cdot (\square \gamma v) = 0. \]
(96)

The energy and momentum conservation equation \( T_{ab}^{,b} = 0 \), for \( a = 0 \) and \( i \), respectively, give
\[ \left[ \left( \rho + \frac{p}{c^2} \right) \gamma^2 \right] + \nabla \cdot \left[ \left( \rho + \frac{p}{c^2} \right) \gamma^2 v \right] = \frac{p}{c^2} - \frac{1}{c^2} \nabla \cdot (\Pi_{j}^i v^i) - \frac{1}{c^4} (\Pi_{j}^i v^i v^j), \]
\[ \left[ \left( \rho + \frac{p}{c^2} \right) \gamma^2 v_i \right] + \nabla \cdot \left[ \left( \rho + \frac{p}{c^2} \right) \gamma^2 v v_i \right] = -\nabla_p \rho - \Pi^j_i - \frac{1}{c^2} (\Pi_{j}^i v^j). \]
(97)
\( (98) \)

From these, we can derive alternative forms
\[ \left( \rho + \frac{p}{c^2} \right) \gamma^2 \frac{dv_i}{dt} = -\nabla_p - v_i \frac{\rho}{c^2} - \Pi^j_i \]
\[ + \frac{1}{c^2} v_j \nabla_j (\Pi_{k}^i v^k) - \frac{1}{c^2} (\Pi_{j}^i v^j) \]
\[ + \frac{1}{c^2} v_i \nabla_j (\Pi_{k}^j v^k), \]
\[ \left( \rho + \frac{p}{c^2} \right) \frac{d}{dt} \ln \gamma = -\frac{1}{c^2} \left( \frac{dp}{dt} - \frac{1}{\gamma^2 \rho} \right) \]
\[ - \frac{1}{c^2} \left[ \Pi^j_i v^j - \frac{v^2}{c^2} (\Pi_{j}^i v^j) \right] \]
\[ - \frac{1}{c^2} \left[ \Pi_{j}^i v^j - \frac{v^2}{c^2} (\Pi_{j}^i v^j) \right]. \]
(99)
(100)
\[ \left( \rho + \frac{p}{c^2} \right) \frac{d}{dt} \ln \gamma = -\frac{1}{c^2} \left( \frac{dp}{dt} - \frac{1}{\gamma^2 \rho} \right) \]
\[ - \frac{1}{c^2} \left[ \Pi^j_i v^j - \frac{v^2}{c^2} (\Pi_{j}^i v^j) \right] \]
\[ - \frac{1}{c^2} \left[ \Pi_{j}^i v^j - \frac{v^2}{c^2} (\Pi_{j}^i v^j) \right]. \]
(101)

The covariant energy and momentum conservation equations in Equations (113) and (114) give Equations (102) and (101), respectively. The ADM energy and momentum conservation equations in Equations (111) and (112) give Equations (97) and (98), respectively.

**APPENDIX B**

**FULLY NONLINEAR PERTURBATIONS IN THE NONEXPANDING BACKGROUND**

Here, we present the fully nonlinear and exact perturbation equations in the Minkowsky background, ignoring the transverse-trace-free tensor-type perturbation. As the metric convention, we take
\[ ds^2 = -(1 + 2\alpha) c^2 dt^2 - 2\chi_i c dt dx^i + (1 + 2\varphi) \delta_{ij} dx^i dx^j, \]
(102)

where \( \alpha, \varphi, \) and \( \chi_i \) are functions of spacetime with arbitrary amplitude; the index of \( \chi_i \) is raised and lowered by \( \delta_{ij} \) as the metric. Without losing any generality, we have taken the spatial gauge condition as in Equation (9). We can derive the exact inverse metric and the rest of the algebra is straightforward (HN2014). The lapse function and the Lorentz factor are
\[ N \equiv \frac{1}{\sqrt{-g^{00}}} = \sqrt{1 + 2\alpha + \chi^2 \chi_k / (1 + 2\varphi)}, \]
\[ \gamma \equiv -u^k n_k = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}(1 + 2\varphi)}}, \]
(105)

where \( v^i \) is introduced as in Equation (38). The equations follow from the cosmological ones presented in Equations (6)–(11) and (14)–(19) of Hwang et al. (2016); compared to our previous notations in the cosmological context, we now set \( a \equiv 1 \) and \( \varrho = \bar{\varrho}, \rho = \bar{\rho}, \vartheta = \bar{\vartheta}, \) etc. We have not taken the temporal gauge condition yet; as the temporal gauge condition, we may impose the synchronous gauge (\( \alpha \equiv 0 \)), the zero-shear gauge (\( \chi_j \equiv 0 \)), the comoving gauge (\( \vartheta_j \equiv 0 \)), and the uniform-expansion gauge or the MS (\( \kappa \equiv 0 \)), see Equation (79); the synchronous gauge leaves remnant gauge mode even after imposing the gauge condition, see Equation (79). Definition of
ADM energy conservation:
\[
\frac{\mathcal{N}}{\mathcal{N}^2(1 + 2\varphi)^2} \left\{ \frac{1}{2} \mathcal{N}^j \mathcal{N}^{i j} \varphi_{i j} - \frac{1}{6} \mathcal{N}^i \mathcal{N}^j \varphi_{i j} - \mathcal{N}^i \varphi_{i j} \right\}
\]
+ \frac{1}{1 + 2\varphi} \left\{ \frac{2}{(1 + 2\varphi)^2} \left[ \varphi_{i j} \mathcal{N}^i - \frac{1}{3} \delta_{i j} \mathcal{N}^i \varphi_{i j} \right] \right\} = -\Pi^{(\text{ADM})i}. \tag{111}
\]

ADM momentum conservation:
\[
\left\{ \frac{1}{\mathcal{N}} \frac{\partial}{\partial t} \mathcal{N} - \kappa \right\} \left[ (\mu + p) \gamma^2 v_i \right] + \frac{\mathcal{N}}{\mathcal{N}^2(1 + 2\varphi)^2} \left[ \frac{1}{2} \mathcal{N}^i \mathcal{N}^{i j} \varphi_{i j} - \frac{1}{6} \mathcal{N}^i \mathcal{N}^j \varphi_{i j} - \mathcal{N}^i \varphi_{i j} \right]
\]
+ \left( \frac{3\mathcal{N}^i}{1 + 2\varphi} \right) \left( \mu + p \right) \gamma^2 v_i
\]
+ \frac{\mathcal{N}^i}{\mathcal{N}^2(1 + 2\varphi)^2} \left[ \varphi_{i j} \mathcal{N}^i - \frac{1}{3} \delta_{i j} \mathcal{N}^i \varphi_{i j} \right] = -\Pi^{(\text{ADM})i}. \tag{112}
\]

Covariant energy conservation:
\[
\left\{ \frac{\partial}{\partial t} + \frac{1}{1 + 2\varphi} (\mathcal{N} \dot{\varphi} + c \dot{\varphi} \mathcal{N}) \right\} \mu + (\mu + p) \left\{ -\mathcal{N} \kappa \right\}
\]
Here we add the continuity equation, \((\overline{\gamma} u^a)_{,c} = 0\),

\[
\begin{align*}
\frac{\partial}{\partial t} + \frac{1}{1 + 2\phi} (N_{ij} + c\chi^i) \nabla_j \overline{\gamma} &= 0, \\
+ \frac{\overline{\gamma}}{c(1 + 2\phi)} \left\{ -N_k + \frac{N_{ij}v^j}{1 + 2\phi} + \frac{N_{ij}v^j}{(1 + 2\phi)^2} + \frac{1}{1 + 2\phi} \right\} \frac{\partial}{\partial t} \overline{\gamma} &= 0.
\end{align*}
\]

This can be derived using Equations (C2), (C3), (D10), and (D11) in HNa.

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