DYNAMICAL BEHAVIORS AND OBLIQUE RESONANT
NONLINEAR WAVES WITH DUAL-POWER LAW
NONLINEARITY AND CONFORMABLE
TEMPORAL EVOLUTION

Md. Golam Hafez
Department of Mathematics, Chittagong University of Engineering and Technology
Chittagong-4349, Bangladesh

Sayed Allamah Iqbal
Department of Electrical and Electronic Engineering
International Islamic University Chittagong, Chattogram-4318, Bangladesh

Asaduzzaman
Department of Computer Science and Engineering
Chittagong University of Engineering and Technology
Chittagong-4349, Bangladesh

Zakia Hammouch*
Division of Applied Mathematics, Thu Dau Mot University
Binh Duong Province, Vietnam
Department of Medical Research
China Medical University Hospital Taichung 40402, Taiwan
Department of Sciences, École Normale Supérieure
Moulay Ismail University of Meknes, Morocco

Abstract. In this article, the oblique resonant traveling waves and dynamical behaviors of (2+1)-dimensional Nonlinear Schrödinger equation along with dual-power law nonlinearity, and fractal conformable temporal evolution are reported. The considered equation is converted to an ordinary differential equation by taking the traveling variable wave transform and properties of Khalil’s conformable derivative into account. The modified Kudryashov method is implemented to divulge the oblique resonant traveling wave of such an equation. It is found that the obliqueness is only affected on width, but not on amplitude and phase portraits of resonant nonlinear propagating wave dynamics. The research outcomes are very helpful for analyzing the obliquely propagating nonlinear resonant wave phenomena and their dynamical behaviors in several nonlinear systems having Madelung fluids and optical bullets.

1. Introduction. It is well known that nonlinear Schrödinger equations (NLSEs) having dual-power law of nonlinearity are occurred for describing the physical issues not only in optical theories but also in many branches of mathematical physics. Such equations are mainly implemented to analyze the nonlinear refractive index

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* Corresponding author: Zakia Hammouch, email: hammouch_zakia@tdmu.edu.vn.
(NRI) saturation. For instance, Biswas [3] have reported the propagation of ultra-short optical pulses by considering the NRI as
\[ n = n_0 + n_2 |E|^2 + n_4 |E|^4 \]
in a nonlinear medium. He has been easily derived from the dimensionless NLSE by taking NRI into account. However, the response of a quasi-continual oscillation having complex amplitude is appeared to describe low dimensional gravity in a physical system. It is followed that resonant NLSE (RNSE) is occurred for interpreting the propagation with Bohm’s quantum potential in various nonlinear optical dynamical systems [29, 8]. According to the hidden variable theory, such potential may be produced for developing the quantum behavior not only in quantum mechanics but also in dynamical systems [5]. Besides, RNSEs are also occurred for investigating the uniaxial nonlinear wave phenomena in the plasmas. Lee et al. [25] and Lee and Pashaev [24] have reported the basic characteristics of nonlinear magneto-acoustic wave propagation in a cold collisionless plasma system base on the traveling solution of RNSE. Due to the potentiality of resonant dynamics, references [22]-[6] have already focused to study such phenomena by considering the RNSEs. But, the majority of researchers [19]-[6] have reported resonant solitons by displaying only the analytical solutions of (1+1)-dimensional RNLSE without considering the dynamical behaviors of such resonant wave dynamics.

However, an optical pulse maintains its spatio-temporal coherent structures even as propagating in localized space and time. It is mainly produced by balancing the diffraction and dispersion with optical nonlinearity, where the dispersion and diffraction are attempted to spread the burst respectively in the lengthwise and transversal specification. The controlling evolution equations are an arena at this stage for analyzing the physical scenarios not only in an optical bullet but also in various nonlinear systems. In such situations, one may be considered either two- or three-dimension based on whether diffraction is inadequate to one or two transverse dimensions. Hence, one can be assumed the following RNSE with fractal conformable temporal evolution for investigating the optical resonant traveling waves and their dynamical behaviors in many dynamical systems as mentioned earlier [29, 8]:

\[
iD_\lambda^t \Psi + \eta \nabla^2 \Psi + \sigma f(s) |\Psi|^2 \Psi + s\delta \left\{ \frac{\nabla^2 |\Psi|}{|\Psi|} \right\} \Psi = 0, \tag{1}
\]

where the nonlinear physical systems become nonlocal or non-conservative and \( \nabla^2 = \partial_{xx} + \partial_{yy} + \partial_{zz} \). It is noted here that \( \Psi(x,y,z,t) \) measures the intricate wave unknown, \( i=\sqrt{-1} \) and \( D_\lambda^t \) measures the derivative of fractal conformable order, where \( 0 < \lambda \leq 1 \), respectively. The other real-valued parameters \( \eta, \sigma \) and \( \delta \) are indicated respectively the coefficient of group speed, nonlinearity of non-Kerr and nonlinearity due to the resonant. The above equation is integrable for any constant values of \( s \). Also, \( s<1 \) indicates that Eq.(1) is producible to NLSE, whereas \( s>1 \) indicates that Eq.(1) can not producible to conventional NLSE.

On the other hand, the appearance of angles and thicknesses is mainly produced comparatively immense conversion productivity in the interior of the nonlinear material. Accordingly, Li et al. [26] have investigated the productivity of second-harmonic production based on the structural power pump, second-order susceptibility, and field augmentation in photonic bandgap shaped. For a prearrange incidence viewpoint and sample depth, the upper bounds on pump intensity may occur as shown in [26]. Very recently, Ferdous et al. [8] have reported the oblique
resonant solitons with fractional temporal evolution by considering Kerr, and parabolic low nonlinearity into account. Additionally, the oblique optical resonant enhancements of electromagnetic waves in the presence of static ambient magnetic fields are produced not only in photonic metamaterials, optical bullets, etc. but also a wide frequency ranges through micro-waves to plasmas. At these stages, nonlinear resonant dynamics may be essential in two-dimensional and only two directions of wave propagations are of relevance, that is, parallel and perpendicular to the magnetic field viz \( \nabla = (\partial_x, 0, \partial_z) \). In such situations, one may be supposed that perturbations are considered to evolve and propagate in xz-plane, without loss of overview. Hence, Eq.(1) is reduced, taking the dual-power low nonlinearity as \( f(s) = s^N + \rho s^{2N} \) into account, to

\[
 i D^\alpha \Psi + \eta (\partial_{xx} + \partial_{zz}) \Psi + \left( \sigma |\Psi|^{2N} + \sigma \rho |\Psi|^{4N} \right) \Psi + \delta \left\{ \frac{(\partial_{xx} + \partial_{zz}) |\Psi|}{|\Psi|} \right\} \Psi = 0, \quad (2)
\]

where \( \rho \) is the dual-power law nonlinearity parameter. The traveling wave solution of this model may become unstable and decay in the unstable region \( 1 < N < 2 \), whereas collapse in a finite time for \( N \leq 2 \). Eq.(2) are occurred by considering NRI as a model equation for describing wave propagation not only in the context of the optical fiber but also in photovoltaic-photorefractive materials, e.g., lithium niobate, in many organic and polymer materials, in collisionless plasmas and water wave dynamics.

But, no work has been reported the oblique resonant traveling wave with fractal conformable time evolution and dynamical behaviors of such wave by considering dual-power law nonlinearity into account. Being motivated from the above description and significance, this research work is reported the oblique resonant traveling wave propagation and their dynamical behaviors in the vicinity of the considered equation for sophisticated forgiving intricate physical issues arising in many branches of science and engineering as mentioned previously. The influence of physical parameters, fractal conformable parameter and obliqueness on the resonant wave propagation and phase patriots are visualized graphically with physical interpretation. Thus, the presented manuscript is structured as: In section 2, the RNLSE with time fractal conformable evolution and dual-power law nonlinearity is converted to integer-order ordinary differential equations. In section 3, the analytical solutions of the considered equation are presented. In Section 4, the formation of a dynamical system is provided. In section 5, the possible outcomes are displayed graphically with the physical description. Finally, the concluding remarks are drown in Section 6.

2. Converted ODEs from time fractal conformable RNLSE. In the past, mathematician researchers observed the concept of fractional calculus (derivative and integer) as a purely theoretic field. In modern times, the subject has attracted many researchers because it has been established that there are more applications as it was not in the past. Many physical problems have been recently modeled using the concept of fractional calculus [20]-[15]. Several definitions of fractional calculus have evolved including Weyl, Riesz, Feller, Riemann, Liouville, Caputo, Gruwald-Letnikov, in this work, we use the fractal conformable derivative introduced by The Jordanian group in [23]. To obtain ODEs from the RNLSE with Conformable Fractal Derivatives (CFDs), Khalil et al.[23] have yielded the useful definition of
CFD based on the conformable function as

\[ D^\lambda_\chi (f_c) (\chi) = \lim_{\Delta \chi \to 0} \frac{f_c (\chi + \Delta \chi \chi^{1-\lambda}) - f_c (\chi)}{\Delta \chi}, \text{ for all } \chi > 0, \]

with the properties

(i) \( D^\lambda_\chi (a_1 f_c (\chi) + a_2 g_c (\chi)) = a_1 D^\lambda_\chi f_c (\chi) + a_2 D^\lambda_\chi g_c (\chi) \), for all \( a_1, a_2 \in \mathbb{R} \),

(ii) \( D^\lambda_\chi (\chi^\gamma) = \gamma \chi^{\gamma-\lambda} \), for all \( \gamma \in \mathbb{R} \),

(iii) \( D^\lambda_\chi (f_c (\chi) g_c (\chi)) = f_c (\chi) D^\lambda_\chi g_c (\chi) + g_c (\chi) D^\lambda_\chi f_c (\chi) \)

and

(iv) \( D^\lambda_\chi (f_c (\chi) / g_c (\chi)) = (g_c (\chi) D^\lambda_\chi f_c (\chi) - f_c (\chi) D^\lambda_\chi g_c (\chi)) / g_c^2 (\chi) \).

It is noted here that CFD is very effective for describing the non-locality as well as non-conservative dynamical systems in many physical problems. Now, one can assume the traveling wave variable transform as

\[ \Psi (x, z, t) = e^{i \Omega (\xi)} , \]

where

\[ \xi = x \cos \theta + z \sin \theta + c \left( \frac{t^\lambda}{\lambda} \right) , \]

\[ \Omega = k (x \cos \theta + z \sin \theta) + \omega \left( \frac{t^\lambda}{\lambda} \right) , \]

\[ \cos^2 \theta + \sin^2 \theta = 1. \]

Here, \( c, k, \omega \) and \( \theta \) are the speed of reference frame, wave number and angular frequency, and \( \theta \) is used for measuring the obliqueness. Hence, Eq.(2) is converted to the following ODE of integer order by taking the properties (ii) of CFD, and Eq.(3) and Eq.(4) into account:

\[ (\eta + \delta) F''(\xi) - (\omega + \eta k^2) F(\xi) + \sigma F^{2N+1} (\xi) + \sigma \rho G^{4N+1} (\xi) = 0 \]

where \( c = -2\eta k \) and prime indicate the differentiation with regards to \( \xi \). For simplicity, the highest order nonlinearity can be minimized from Eq.(5) by considering the transformation as \( F(\xi) = G^{1/2N} (\xi) \) and yields

\[ (\eta + \delta) \left\{ \frac{1}{2N} G G'' + \frac{1-2N}{4N^2} (G')^2 \right\} - (\omega + \eta k^2) G^2 + \sigma G^3 + \sigma \rho G^4 = 0 \]

3. Oblique resonant traveling waves via mKM. It is found that nonlinear coherent structures are examined by evaluating analytical solutions of Nonlinear Model equations (NLMEs) via several types of mathematical techniques [12]-[10]. Besides, many scholars [17, 18] have recently shown that the mKM is very effective and easily applicable for contracting traveling wave solutions of NLMEs. Hence, the Modified Kudryashov Method (MKM) is implemented to reveal the analytical solutions of RNLSE having dual-power law nonlinearity and time fractality.

According to MKM, the analytical solution of Eq.(6) can be controlled by the polynomial expansion as

\[ G(\xi) = \sum_{j=0}^{n} e_j V^j (\xi), e_n \neq 0, \]

where \( e_j \) indicates the real constraints to be evaluated later. Additionally, \( V (\xi) = 1 / (1 + d \xi) \) is satisfied the following auxiliary ODE:

\[ V' (\xi) = (V^2 (\xi) - V (\xi)) (lnr ) , \ r \neq 0, \ r \neq 1, \]
where \( r \) and \( d \) are respectively the real numbers. One can be determined the value of \( n \) by applying the balancing principle between the nonlinearity of the highest order with the derivative of higher-order from Eq.(6) and gives \( n = 1 \). Hence, Eq.(7) becomes in the following form:

\[
G (\xi) = e_0 + e_1 \mathcal{V} (\xi) , \tag{9}
\]

Now, substituting Eq.(9) together with Eq.(8) into Eq.(6), the following nonlinear algebraic equations are evaluated by taking the different power of \( \mathcal{V} (\xi) \) with the aid of Maple software:

\[
(V (\xi))^0 : \quad \rho \sigma e_0^4 - e_0^2 \eta k^2 + \sigma e_0^2 - e_0^4 \omega = 0 , \\
(V (\xi))^1 : \quad - 2 e_0 e_1 + 3 \sigma e_0^2 e_1 - 2 e_0 e_1 \eta k^2 + 4 \rho e_0 e_1 + \frac{\eta e_1 (\ln r)^2 e_0}{2N} + \frac{\delta e_1 (\ln r)^2 e_0}{2N} = 0 , \\
(V (\xi))^2 : \quad - e_1^2 \eta k^2 + 3 \sigma e_0 e_1^2 - e_1^2 \omega + \frac{\eta e_1^2 (\ln r)^2}{4N^2} + \frac{\delta e_1^2 (\ln r)^2}{4N^2} + 6 \rho \sigma e_0 e_1^2 \\
\quad - 3 \eta e_1 (\ln r)^2 e_0 - 3 \delta e_1 (\ln r)^2 e_0 = 0 , \\
(V (\xi))^3 : \quad 6 e_1^3 + 4 \rho \sigma e_0 e_1^2 - \frac{\eta e_1^2 (\ln r)^2}{2N} - \frac{\eta e_1^2 (\ln r)^2}{2N} - \frac{\delta e_1^2 (\ln r)^2}{2N} - \frac{\delta e_1^2 (\ln r)^2}{2N} = 0 , \\
(V (\xi))^4 : \quad \rho \sigma e_0^4 + \frac{\eta e_1^2 (\ln r)^2}{N} + \frac{\eta e_1^2 (\ln r)^2}{4N^2} + \frac{\delta e_1^2 (\ln r)^2}{2N} + \frac{\delta e_1^2 (\ln r)^2}{4N^2} = 0 .
\]

Simplifying the above evaluated nonlinear algebraic equations by using the computational software, like Maple, one can determine the following values for \( e_0, e_1, r \) and \( \omega \):

**Type 1:**

\[
\omega = - \left[ \eta k^2 + \frac{(2N + 1) \sigma^2}{4 \rho (N + 1)^2} \right] , \quad e_0 = 0 , \quad e_1 = - \frac{(2N + 1) \sigma}{2 \rho (N + 1)} , \quad r = \exp \left\{ \pm \frac{\sigma N}{N + 1} \sqrt{\frac{(2N + 1)}{\rho (\eta + \delta)}} \right\} .
\]

**Type 2:**

\[
\omega = - \left[ \eta k^2 + \frac{(2N + 1) \sigma^2}{4 \rho (N + 1)^2} \right] , \quad e_0 = - \frac{(2N + 1) \sigma}{2 \rho (N + 1)} , \quad e_1 = \frac{(2N + 1) \sigma}{2 \rho (N + 1)} , \quad r = \exp \left\{ \pm \frac{\sigma N}{N + 1} \sqrt{\frac{(2N + 1)}{\rho (\eta + \delta)}} \right\}.
\]

Hence, the following resonance oblique analytical solutions are archived for \((2+1)\)-dimensional RNLS having fractal temporal evolution by combining Eq.(3), Eq.(4), Eq.(9) and the solution of Eq.(8) along with Type 1 and Type 2, respectively:

\[
\Psi_1 (x, z, t) = e^{i \left\{ k(x \cos \theta + z \sin \theta) + \omega \left( \frac{\omega}{\sigma} \right) \right\}}
\times \left[ \frac{(2N + 1) \sigma}{2 \rho (N + 1)} \times \frac{1}{1 + de \left\{ \pm \frac{\sigma N}{N + 1} \sqrt{\frac{(2N + 1)}{\rho (\eta + \delta)}} (x \cos \theta + z \sin \theta + c \left( \frac{\omega}{\sigma} \right)) \right\}} \right]^{\pm N} ,
\]

\[
(10)
\]
and
\[ \Psi_2(x, z, t) = e^{i \left( k(x \cos \theta + z \sin \theta) + \omega \left( \frac{i}{\lambda} \right) \right)} \left\{ \frac{(2N + 1) \sigma}{2 \rho (N + 1)} \right\} \left[ -1 + \frac{1}{1 + de} \left\{ \pm \frac{\sigma N}{\lambda (N + 1)} \sqrt{-\frac{(2N + 1) \rho}{\sigma N \lambda (N + 1)}} \right\} \right\} \frac{1}{\lambda} \] .

4. Formation of dynamical system from RNLSE. To report the dynamical system and existence condition of oblique traveling wave solutions, one can be assumed that \( X = F(\xi) \) and \( Y = F'(\xi) \). Eq.(5) can then be converted to the following dynamical system:
\[ \frac{dX}{d\xi} = Y, \quad \frac{dY}{d\xi} = \left( \omega + \eta k^2 \right) X - \frac{\sigma}{(\eta + \delta)} X^{2N+1} - \frac{\sigma \rho}{(\eta + \delta)} X^{4N+1} = R, \] (12)
which can be converted to the planar Hamiltonian system with the following first integral:
\[ H(X, Y) = \frac{1}{2} Y^2 + \left( \frac{\omega + \eta k^2}{2(\eta + \delta)} \right) X - \frac{\sigma}{(\eta + \delta)} X^{2N+1} - \frac{\sigma \rho}{(\eta + \delta)} X^{4N+1} = R, \] (13)
where \( R \) is the integrating constant. It is noted that Eq. (12) is independent of the conformable parameter \( \lambda \), which is indicated that the bifurcation properties remain unchanged with the changes of \( \lambda \).

To obtain the equilibrium point for studying the phase portraits of the dynamical system as mentioned in Eq.(12), one can be considered as
\[ f(X) = \left( \frac{\omega + \eta k^2}{(\eta + \delta)} \right) X - \frac{\sigma}{(\eta + \delta)} X^{2N+1} - \frac{\sigma \rho}{(\eta + \delta)} X^{4N+1}, \] (14)
It is interesting to be noted that the number of equilibrium points and enveloped separative layers can be remarkably generated by the phase portraits of a dynamical system [12]. A dynamical system may be formed one wave solution for the corresponding NLMEs by depending on any orbit in the phase portrait of such systems. It is, therefore, finding out the singular points of the formed dynamical system, as mentioned in Eq.(12) and yields the following three singular points:
\[ X (0, 0), \]
\[ X_+ \left( \left\{ \frac{\sigma - \sqrt{\sigma^2 + 4 \sigma \rho (\omega + \eta k^2)}}{2 \sigma \rho} \right\} \frac{i}{\lambda}, 0 \right), \]
\[ X_- \left( \left\{ \frac{\sigma + \sqrt{\sigma^2 + 4 \sigma \rho (\omega + \eta k^2)}}{2 \sigma \rho} \right\} \frac{i}{\lambda}, 0 \right), \] (15)
Based on the qualitative theory of ODEs [28], various types of orbits, e.g. (i) O is a center and \( X_+ \), and \( X_- \) are saddle point. (ii) \( X_+ \) and \( X_- \) are center and O is a saddle point, etc. can be obtained for the dynamical system by depending on the considered values of \( N, \eta, \sigma, \omega, k \) and \( \rho \). The phase portraits of Eq.(12) will be discussed later.
Moreover, multiplying Eq. (5) by \( 2F' (\xi) \), which yields the integral form after simplification as
\[
\frac{1}{2} \left( F' (\xi) \right)^2 + V (F (\xi)) = 0,
\]
with
\[
V (F (\xi)) = -\frac{(\omega + \eta k^2)}{2(\eta + \delta)} F^2 (\xi) + \frac{\sigma}{(2N + 2)(\eta + \delta)} F^{2N+2} (\xi)
\]
\[
+ \frac{\sigma \rho}{(4N + 2)(\eta + \delta)} F^{4N+2} (\xi),
\]
From a physical point of view, one can be considered Eq. (16) as the so-called, energy integral, of oscillating objects together with velocity \( F' (\xi) \) and position \( F (\xi) \) in the barrier with potential \( V (F (\xi)) \) for describing the physical phenomena not only in optical fiber, optical bullet, etc. but also in plasmas. It is seen that the first and second terms of Eq. (16) are indicated, respectively, the kinetic and potential energies.

Now, one can be obtained the differentiation of \( V (F (\xi)) \) with regards to \( F (\xi) \) as
\[
V' (F (\xi)) = F (\xi) \left\{ -\frac{(\omega + \eta k^2)}{\eta + \delta} + \frac{\sigma}{(\eta + \delta)} F^{2N} (\xi) - \frac{\sigma \rho}{(\eta + \delta)} F^{4N} (\xi) \right\}.
\]
It is observed that \( \left( F' (\xi) \right)^2 /2 \) involved in Eq. (16) is always a positive quantity and provides \( V (F (\xi)) = 0 \) in \( F_{\min} < F < 0 \) or \( 0 < F < F_{\max} \) for the resonant traveling wave propagation in its entire motion. Hence, the exact solutions of Eq. (2) is represented oblique propagating resonant traveling wave whenever the condition as \( [V'' (F (\xi))]_{F=0} = -\frac{(\omega + \eta k^2)}{(\eta + \delta)} < 0 \) is existed. However, the exact solutions of Eq. (2) will be provided oscillatory, that is, periodic waves.

5. Result and discussion. In this section, the oblique resonant traveling wave and their dynamical behavior are described with graphical representation. The appearance of obliqueness in many physical situations, e.g., photonic metamaterials, optical bullet, collisionless plasmas, water wave dynamics, etc. are already provided by many researchers in their studies. It is inspired to study the traveling waves and phase portraits of resonant dynamics by considering the RNLSE with time fractal evolution and dual-power law nonlinearity into account. The obtained analytical solution of the Eq. (2) via the mKM are discussed and represented graphically for demonstrating the significance of obliqueness.

Fig. 1(a) and Fig. 1(b) explores the resonant traveling wave structures of \( |\Psi| \) by changing of \( \lambda \) and \( t \) together with the fixed values of the other physical parameters. While Fig. 1(c) and Fig. 1(d) explores the resonant traveling wave structures of \( |\Psi| \) with respect \( x \) and \( \theta \) keeping \( z \)-axis constant and with respect \( z \) and \( \theta \) keeping \( x \)-axis constant by assuming the remaining parameters constant. On the other hand, the resonant traveling wave structures of \( |\Psi| \) for different values of \( \theta \) and the real and imaginary part of \( |\Psi| \) with \( \theta = 30^0 \) are explored in Fig. 2 by assuming the remaining parameters constant. It is observed from Fig. 1 and Fig. 2 that the dark solitons that are, kink-shaped structures are generated for the considered equations in which the widths of kink-shaped structures are significantly changed. Still, the amplitudes of kink-shaped structures are unchanged with the changes of obliqueness. The kink-shaped coherent structures are also changing with the changes of time and fractal
It is shown that the discontinuity redis arisen and produced shock waves in any varied dynamical system by the considered equation. Fig.3 produces the resonant periodic wave traveling waves of \(|\Psi|\) for changing of \(\theta\) by assuming the typical parameters constants and \(t = 1\). It is observed from Fig.3 that the oblique resonant periodic waves are also significantly changed with the changes of obliqueness. It is also found that the changes of resonant kink-shaped and periodic waves in the \(x\) \((z)\) direction are increasing (decreasing) with the increase of obliqueness. The kink-shaped and periodic traveling wave structures are only obtained whenever the condition \((\omega + \eta k^2) / (\eta + \delta) > 0\) or \(\rho<0\) and \((\omega + \eta k^2) / (\eta + \delta) < 0\) or \(\rho>0\) are existed, which is in good agreement with the theoretical investigations.

In order to examine the phase portraits, Eq.(12) can be rewritten as

\[
G(X, Y) = \frac{(\omega + \eta k^2)}{(\eta + \delta)} X - \frac{\sigma}{(\eta + \delta)} X^{2N+1} - \frac{\sigma \rho}{(\eta + \delta)} X^{4N+1},
\]

Using the following Jacobian matrix,

\[
J(X, Y) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix},
\]

and from equation Eq. (18), one obtains

\[
J(X, Y) = \begin{pmatrix} \frac{(\omega + \eta k^2)}{(\eta + \delta)} & 0 \\ -\frac{\sigma (2N+1)}{(\eta + \delta)} X^{2N} & -\frac{\sigma \rho (4N+1)}{(\eta + \delta)} X^{4N} & 1 \\ 0 & 0 \end{pmatrix}.
\]

It is noted that one can get the eigenvalues from Eq.(18) to analysis the phase portrait of the dynamical system as mentioned in Eq.(12), by using critical points (15) and corresponding values of \(N, \omega, \eta, k, \delta, \sigma\) and \(\rho\) in the Jacobian matrix as mentioned in Eq.(20).

When \(N = 0.5, \omega = 2, \eta = 0.1, k = 0.4, \delta = 1, \sigma = 0.5\) and \(\rho = 0.09\), one obtains three points \(x_+(3.14, 0), x_-(-14.2541, 0)\) and \((0, 0)\) with the corresponding eigenvalues are \(r_{1,2}^+ = \pm 1.496i, r_{1,2}^- = \pm 3.19i\), and \(r_{1,2}^- = \pm 1.35\), respectively. It is found that the eigenvectors \(x_+(3.14, 0)\) and \(x_-(-14.2541, 0)\) is obtained corresponding to purely complex eigenvalues, which yields that these critical points of the system are center and stable. On the other hand, the other critical point \((0, 0)\) is an unstable saddle-node because corresponding eigenvalues are real and opposite. The phase portrait and its vector field, in this case, are shown in Fig.4(a) and Fig.4(b), respectively, which is in good agreement with the above theoretical finding. When \(N = 0.5, \omega = 2, \eta = 0.1, k = 0.4, \delta = 1, \sigma = 0.5\), and \(\rho = 0.01\), the three eigenvectors are obtained as \(x_+(3.88, 0), x_-(-103.88, 0)\) and \((0, 0)\) and their corresponding eigenvalues are obtained as \(r_{1,2}^+ = \pm 1.38i, r_{1,2}^- = \pm 7.13i\), and \(r_{1,2}^- = \pm 1.35\), respectively. It is observed that the first two points are given purely complex eigenvalues. So, the critical points \(x_+(3.88, 0)\) and \(x_-(-103.88, 0)\) of the system are presented in center and stable, whereas another point \((0, 0)\) is unstable and saddle due to the corresponding eigenvalues are real and opposite. Fig.4(c) and Fig.4(d) are clearly indicated such centers and saddle node in the phase portrait and its vector field, respectively, when \(N = 0.5, \omega = 2, \eta = 0.1, k = 0.4, \delta = 1, \sigma = 0.5\) and \(\rho = -0.09\), one can also be obtained the three critical points the corresponding to the eigenvalues as \(r_{1,2}^+ = \pm 0.67 \mp 1.26i, r_{1,2}^- = \pm 0.67 \pm 1.26i\) and \(r_{1,2}^- = \pm 1.35\). It is investigated that the first two critical are unstable points because of their corresponding eigenvalues is provided opposite sign real parts, while another point
(0, 0) is also saddle node unstable because of the corresponding eigenvalues are real and opposite sign. Fig.4(e), and Fig.4(f) are clearly indicated such unstable nodes in the phase portrait and its vector field, respectively. Besides, the phase portrait and its vector field, respectively are shown in Fig.5(a) and Fig.5(b) based on the oscillatory condition with $N = 0.5$, $\omega = -2$, $\eta = 0.1$, $k = 0.4$, $\delta = 1$, $\sigma = 0.5$ and $\rho = 0.09$. It is found Fig. 5 that the two critical points are unstable due to the corresponding eigenvalues $r_{1,2}^+ = \pm 1.24 \mp 0.67i$, and $r_{1,2}^- = \pm 1.24 \mp 0.67i$, whereas the point (0, 0) is saddle node unstable point due to the real and opposite sign of the corresponding eigenvalues $r_{1,2} = \pm 1.34$. It is noted here that the above phase portraits are described for nonlinear refractive index saturation $N = 0.5$. 

Now, when $N = 1.5$, $\omega = 1$, $\eta = 0.1$, $k = 0.4$, $\delta = 1$, $\sigma = 0.5$ and $\rho = 0.09$, the eigenvalues are found as $r_{1,2}^+ = \mp 1.77i$, $r_{1,2}^- = \mp 4.80i$ and $r_{1,2} = \pm 0.96$. In this case, two critical points $x_+$ and $x_-$ becomes stable based on the pure complex corresponding eigenvalues and the other point (0, 0) becomes an unstable saddle based on the real and opposite sign of corresponding eigenvalue, which is obviously predicted in the phase portrait and its vector field as displayed in Fig.6(a) and Fig.6(b), respectively. Finally, Fig.7(a) and Fig.7(b) respectively display the phase portrait and its vector field by taking $N = 3$, $\omega = 1$, $\eta = 0.1$, $k = 0.4$, $\delta = 1$, $\sigma = 0.5$ and $\rho = 0.09$, and determining the eigenvalues as $r_{1,2}^+ = \mp 2.51i$, $r_{1,2}^- = \mp 6.80i$ and $r_{1,2} = \pm 0.96$. Fig. 8 is clearly shown that two critical points $x_+$ and $x_-$ becomes stable and the other point (0, 0) becomes an unstable saddle node. The systems equilibrium points is limited to the boundary $-1$ to $+1$ on the $x$ axis when $N$ is approaching to large number ($N > 2$) and other than the system equilibrium points expands to the out of the boundary $-1$ to $+1$. It is provided that the system is produced the stable or unstable nodes with regarding the values of $N$.

It is also predicted form all possible phase portraits of the dynamical system that the resonant traveling waves of NLME may become unstable and decay when $N$ lies between 0 to 1, but collapse in a finite time for $N$ greater than equal 2. It can be concluded that the theoretical results obtained in this article would be very helpful for analyzing the nature of resonant wave phenomena and their various orbits in several nonlinear systems, e. g. photonic metamaterials, nonlinear optics, telecommunications industry, collisionless plasmas, optical bullet, etc. [3, 29, 25, 24, 33].

6. Conclusions. The (2+1)-dimensional RNSE with dual-power law nonlinearity has considered for reporting the nonlinear wave dynamics and their dynamical properties with time fractal evolution and obliqueness. The MKM has been implemented for constructing the resonant traveling waves of these equations. It has been found that the width of traveling waves is significantly modified, but the amplitude remains unchanged with the influences of obliqueness. The kink-shaped and periodic traveling waves have only occurred by deadening on the provided existence condition. On the other hand, the nonlinear dynamical system has been derived from the considered equation using the traveling wave transformation. All possible phase portraits of the obtained dynamical system have displayed graphically with the physical description by depending on the equilibrium points’ eigenvalues. It is found that the phase portraits of the dynamical system are strongly dependent on the parameters involved in the RNLSE, but not obliqueness. In addition, the resonant traveling waves have produced for the considered equation by depending on the orbits of phase patriots of the obtained nonlinear dynamical system. Hence,
the theoretical investigations presented in this manuscript have been used for better examining the nature of oblique resonant waves and their dynamical behaviors by considering RNLSE in any visible nonlinear systems.
Figure 2. Resonant kink-shaped structures of (a) $|\Psi|$ for $\theta = 5^0$ (red) and $\theta = 30^0$ (orange), (b) $|\Psi|$ for $\theta = 55^0$ (orange) and $\theta = 85^0$ (red), (c) real part of $|\Psi|$ with $\theta = 30^0$ and (d) imaginary part of $|\Psi|$ with $\theta = 30^0$. The remaining parameters are selected as $\lambda = 0.5$, $\eta = -1$, $\sigma = 0.5$, $N = 1$, $\delta = 2$, $\rho = -0.5$, $k = 0.5$, $t = 10$ and $d = 1$.

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Figure 3. Resonant periodic wave structures of (a) $|\Psi|$ for $\theta = 45^0$, (b) $|\Psi|$ for $\theta = 80^0$ and (c) $|\Psi|$ for $\theta = 45^0$ (red) and $\theta = 80^0$ (orange). The remaining parameters are selected as $\eta = -0.1$, $\sigma = 0.5$, $N = 1.5$, $\delta = 1$, $\rho = 0.09$, $k = 1$, $t = 1$ and $d = 1$.

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Figure 4. The phase portraits and its vector fields of nonlinear dynamical system as mentioned in Eq. (12) by assuming different values of $\rho = 0.09$ (Figs. 4(a) and (b)), $\rho = 0.01$ (Figs. 4(c) and (d)) and $\rho = -0.09$ (Figs. 4(e) and (f)) with $N = 0.5$, $\omega = 2$, $\eta = 0.1$, $k = 0.4$, $\delta = 1$ and $\sigma = 0.5$. 
Figure 5. The (a) phase portrait and its (b) vector fields of the nonlinear dynamical system as mentioned in Eq.(12) by assuming for the values of parameters $N = 0.5$, $\omega = -2$, $\eta = 0.1$, $k = 0.4$, $\delta = 1$, $\sigma = 0.5$, and $\rho = 0.09$.

Figure 6. The (a) phase portrait and its (b) vector fields of the nonlinear dynamical system as mentioned in Eq.(12) by assuming for the values of parameters $N = 1.5$, $\omega = 1$, $\eta = 0.1$, $k = 0.4$, $\delta = 1$, $\sigma = 0.5$, and $\rho = 0.09$.

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Figure 7. The (a) phase portrait and its (b) vector fields of the nonlinear dynamical system as mentioned in Eq.(12) by assuming for the values of parameters $N = 3$, $\omega = 1$, $\eta = 0.1$, $k = 0.4$, $\delta = 1$, $\sigma = 0.5$, and $\rho = 0.09$. 

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E-mail address: hafez@cuet.ac.bd, golam_hafez@yahoo.com
E-mail address: sayed.allamah.iqbal@gmail.com
E-mail address: asad@cuet.ac.bd
E-mail address: hammouch_zakia@tdmu.edu.vn