WAVE BREAKING OF PERIODIC SOLUTIONS TO THE FORNBERG-WHITHAM EQUATION

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ABSTRACT. Based on recent well-posedness results in Sobolev (or Besov spaces) for periodic solutions to the Fornberg-Whitham equations we investigate here the questions of wave breaking and blow-up for these solutions. We show first that finite maximal life time of a solution necessarily leads to wave breaking. Second, we prove that for a certain class of initial wave profiles the corresponding solutions do indeed blow-up in finite time.

1. Introduction. We investigate the qualitative properties regarding blow-up and wave breaking of (spatially) periodic solutions to the so-called Fornberg-Whitham equation, which was introduced as a shallow water wave model that is comparably simple and yet showed indications of wave breaking (cf. [10, 11, 5, 9]). For non-periodic solutions to the Fornberg-Whitham equation on the real line, rigorous blow-up results and wave breaking have been proved in [3]. Here, we show that similar results hold also in the periodic case.

Let \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \) be the one dimensional torus group. Functions on \( \mathbb{T} \) may be identified with 1-periodic functions on \( \mathbb{R} \). We will consider the wave height described basically by a function of space and time \( u: \mathbb{T} \times \mathbb{R} \to \mathbb{R}, (x,t) \mapsto u(x,t) \), though in the relevant cases of a finite time of existence of a wave solution \( u \), the time domain will be confined to a bounded closed interval \( [0,T_0] \), \( T_0 > 0 \), or to a half-open interval \( [0,T[, T > 0 \). We will often write \( u(t) \) to denote the function \( x \mapsto u(x,t) \).

The Cauchy problem for the Fornberg-Whitham equation reads

\[
\begin{align*}
  u_{txx} - u_t + \frac{9}{2} u_x u_{xx} + \frac{3}{2} uu_{xxx} - \frac{3}{2} uu_x + u_x &= 0, \\
  u(x,0) &= u_0(x).
\end{align*}
\]

If we suppose that \( u \) is at least continuous with respect to the time variable and of some Sobolev regularity with respect to the spatial variable, then we may employ the linear continuous operator \( Q := (\text{id} - \partial_x^2)^{-1}: H^r(\mathbb{T}) \to H^{r+2}(\mathbb{T}) \), for any \( r \in \mathbb{R} \) (acting on the spatial part \( u(.,t) \) at any fixed time \( t \)), to rewrite the above Cauchy

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problem in the following non-local form
\[ u_t + \frac{3}{2} uu_x = Qu_x, \quad (3) \]
\[ u(x, 0) = u_0(x), \quad (4) \]
which also requires less spatial regularity of a prospective solution \( u \).

**Remark 1.** Concerning normalization factors and signs we followed here in Equations (3) and (1) the detailed form as used in the recent publications [7, 8], which agrees with Equation (29) in [5] (in case \( \nu = 1 \)). Note that there was a sign error in the latter equation with the linear term involving \( u_x \) (compare with Equation (4) in [5] or with earlier papers cited there). However, replacing \( u(x, t) \) by \(-u(x, -t)\) transforms solutions of either sign variant of the equation into solutions for the other. Moreover, if \( u \) solves (3), then \( v := 3u/2 \) is a solution to \( v_t + vv_x = Qu_x \).

Well-posedness results for (3-4) with spatial regularity according to Sobolev or Besov scales have been obtained in [7, 8]. Here we will make use of the following simpler consequence:

*If \( s > 3/2 \) and \( u_0 \in H^s(\mathbb{T}) \), then there exists \( T_0 > 0 \) such that (3-4) possesses a unique solution \( u \in C([0, T_0], H^s(\mathbb{T})) \cap C^1([0, T_0], H^{s-1}(\mathbb{T})) \). (5)*

The map \( u_0 \mapsto u \) is continuous \( H^s(\mathbb{T}) \to C([0, T_0], H^s(\mathbb{T})) \) and
\[ \sup_{t \in [0, T_0]} \| u(t) \|_{H^s(\mathbb{T})} < \infty. \quad (6) \]

The life span \( T_0 \) can be guaranteed to be above some a priori lower bound depending only on \( s \) and the \( H^s \)-norm of \( u_0 \). For given and fixed Sobolev index \( s > 3/2 \) and \( u_0 \in H^s(\mathbb{T}) \) we will consider the maximal life span \( T \) for a unique solution \( u \), defined as the supremum of all possible \( T_0 \) in the above well-posedness result. Thus, a unique solution that is global in time corresponds to the case \( T = \infty \).

The observations made in [8, Theorem 1.5 and its proof] include the following result, which can be considered preparatory of a blow-up situation: *Let \( s > 3/2 \) and \( T \) be the maximal life span \((0 < T \leq \infty)\) for the solution \( u \in C([0, T], H^s(\mathbb{T})) \) to (3-4) corresponding to the initial data \( u_0 \in H^s(\mathbb{T}) \). If \( T < \infty \), then*
\[ \limsup_{t \uparrow T} \| u(t) \|_{H^s(\mathbb{T})} = \infty \quad \text{and} \quad \int_0^T \| u_x(t) \|_{L^\infty(\mathbb{T})} \, dt = \infty. \quad (7) \]

Section 2 will be concerned with the detailed proof that a finite maximal life span \( T < \infty \) for a solution \( u \) necessarily implies wave breaking for this solution at time \( T \). Recall (cf. [2, Definition 6.1]) that wave breaking is said to occur for \( u \) at time \( T > 0 \), if the wave itself remains bounded while its slope becomes unbounded, i.e.,
\[ \sup_{t \in [0, T]} \| u(t) \|_{L^\infty(\mathbb{T})} < \infty \quad \text{and} \quad \limsup_{t \uparrow T} \| u_x(t) \|_{L^\infty(\mathbb{T})} = \infty. \quad (8) \]

In Section 3 we show that for a certain class of initial configurations \( u_0 \), the maximal time of existence is indeed finite, hence wave breaking does occur for these initial values.

In the sequel, we will occasionally simplify notation by dropping \( \mathbb{T} \) in referring to the spaces \( H^s(\mathbb{T}) \) or \( L^p(\mathbb{T}) \).
2. Wave breaking in case of finite maximal life span. In this section, we will show that the assumption of a finite maximal life span always implies wave breaking. Until stated otherwise, we will use the following convention throughout:

Let $s > 3/2$, $u_0 \in H^s$, $u$ be the corresponding unique solution to (3-4), and denote by $T$ its maximal life span.

We begin by drawing a simple immediate consequence from (7).

**Proposition 1.** If $T < \infty$, then

$$\limsup_{t \uparrow T} \|u_x(t)\|_{L^\infty} = \infty. \quad (9)$$

**Proof.** For every $T_0 \in \mathbb{R}$ with $0 < T_0 < T$, we have $u_x \in C([0, T_0], H^{s-1}) \subset C([0, T_0], L^\infty)$, since $s - 1 > 1/2$ implies $H^{s-1}(\mathbb{T}) \subset L^\infty(\mathbb{T})$. We conclude that

$$\int_{T_0}^{T} \|u_x(t)\|_{L^\infty} dt < \infty$$

and the second part of (7) then yields

$$\forall T_0 \in \mathbb{R}, 0 < T_0 < T : \int_{T_0}^{T} \|u_x(t)\|_{L^\infty} dt = \infty.$$

Therefore, the continuous function $t \mapsto \|u_x(t)\|_{L^\infty}$, $[0, T] \mapsto [0, \infty]$ is unbounded on every interval $[T_0, T]$ with $0 < T_0 < T$. In other words, for every $n \in \mathbb{N}$, $n > 1/T$, we can find $t_n \in [T - \frac{1}{n}, T]$ such that $\|u_x(t_n)\|_{L^\infty} > n$, which proves (9). \qed

To obtain a wave breaking result, we need to show that $\|u(t)\|_{L^\infty}$ remains bounded as $t$ approaches $T$ from the left. As a preparation we first prove boundedness for the $L^2$ norms on finite time intervals.

**Lemma 2.1.** For every $t \in [0, T]$ we have

$$\|u(t)\|_{L^2} \leq e^t \|u_0\|_{L^2}. \quad (10)$$

**Proof.** We know that

$$u_t(t) + \frac{3}{2} u(t) u_x(t) = Q(u_x(t))$$

holds with equality in $L^2(\mathbb{T})$ for every $t$. We may multiply the above equation by $u(t)$ (note that $u(t)^2 u_x(t) \in H^1 \cdot H^1 \cdot L^2 \subset H^1 \cdot L^2 \subset L^2 \cdot L^2 \subset L^2$, since $H^1(\mathbb{T})$ is an algebra) and obtain, with equality in $L^1$,

$$\frac{1}{2} \frac{d}{dt} (u(t)^2) + \frac{3}{2} \partial_x (u(t)^3) = u(t) Q(\partial_x u(t)),$$

which upon spatial integration over $\mathbb{T}$ and noting that $(Q \circ \partial_x) u(t) \in H^{s+1} \subset H^1$ gives

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}^2 = \langle u(t) | (Q \circ \partial_x) u(t) \rangle$$

$$\leq \|u(t)\|_{L^2} \|Q \circ \partial_x) u(t)\|_{L^2} \leq \|u(t)\|_{L^2} \|Q \circ \partial_x) u(t)\|_{H^1}.$$ The operator $Q \circ \partial_x$ is bounded $L^2(\mathbb{T}) \rightarrow H^1(\mathbb{T})$, in fact, $\|Q \circ \partial_x) v\|_{H^1} \leq \|v\|_{L^2}$ holds for any $v \in L^2(\mathbb{T})$ (as can be seen from the definition of $H^1(\mathbb{T})$ via Fourier series representation and from Parseval’s formula), hence we deduce further that

$$\frac{d}{dt} \|u(t)\|_{L^2}^2 \leq \|Q \circ \partial_x) u(t\|_{L^2} \|u(t)\|_{L^2} = \|u(t)\|_{L^2}^2,$$

which implies $\|u(t)\|_{L^2}^2 \leq e^{2t} \|u_0\|_{L^2}^2$ as claimed. \qed

Now we are in a position to show boundedness of the $L^\infty$ norm of the solution in case of a finite maximal life span. An interesting aspect of the following proof is that it does make use of the method of characteristics.
Proposition 2. If $T < \infty$, then
\[ \sup_{t \in [0,T]} \| u(t) \|_{L^\infty} < \infty. \] (11)

Proof. Let $y \in \mathbb{T}$ and $\tau \in [0,T]$ be arbitrary. We consider the characteristic ordinary differential equation with initial condition for a curve $\gamma: [0,T] \to \mathbb{T}$ corresponding to the solution $u$, that is,
\[ \dot{\gamma}(s) = u(\gamma(s), s), \quad \gamma(\tau) = y. \] (12)

Note that $u \in C([0,T], H^s(\mathbb{T})) \subseteq C([0,T], C^1(\mathbb{T}))$, since $H^s(\mathbb{T}) \subset C^1(\mathbb{T})$, hence $(x,t) \mapsto u(x,t)$ is continuous on $[0,T] \times \mathbb{T}$ and globally Lipschitz with respect to $x$. By compactness of $\mathbb{T}$ we therefore have a unique global solution $\gamma \in C^1([0,T], \mathbb{T})$ to the characteristic initial value problem above.

By standard reasoning we obtain
\[ \frac{d}{ds}(u(\gamma(s), s)) = u_x(\gamma(s), s) \dot{\gamma}(s) + u_t(\gamma(s), s) = u_x(\gamma(s), s) u(\gamma(s), s) + u_t(\gamma(s), s) = (Qu_x)(\gamma(s), s) \]
and therefore,
\[ u(y, \tau) = u(\gamma(\tau), \tau) = u(\gamma(0), 0) + \int_0^\tau (Qu_x)(\gamma(s), s) \, ds \]
\[ = u_0(\gamma(0)) + \int_0^\tau (Qu_x)(\gamma(s), s) \, ds, \]
which implies
\[ |u(y, \tau)| \leq \| u_0 \|_{L^\infty} + \int_0^\tau |(Qu_x)(\gamma(s), s)| \, ds. \] (13)

Observe that for every $r \in [0,T]$ we have $Qu_x(r) \in Q(H^{s-1}) = H^{s+1} \subset H^1 \subset L^\infty$ and we may again employ boundedness of $Q \circ \partial_x$ as operator $L^2 \to H^1$ to obtain
\[ |(Qu_x)(\gamma(s), s)| \leq \| Qu_x(s) \|_{L^\infty} \leq \| Qu_x(s) \|_{H^1} = \| (Q \circ \partial_x) u(s) \|_{H^1} \leq \| u(s) \|_{L^2}. \]

Inserting this into (13) yields
\[ |u(y, \tau)| \leq \| u_0 \|_{L^\infty} + \int_0^\tau \| u(s) \|_{L^2} \, ds \]
and applying (10) then gives
\[ |u(y, \tau)| \leq \| u_0 \|_{L^\infty} + \| u_0 \|_{L^2} \int_0^\tau e^s \, ds = \| u_0 \|_{L^\infty} + \| u_0 \|_{L^2} (e^\tau - 1) \]
\[ \leq \| u_0 \|_{L^\infty} + \| u_0 \|_{L^2} (e^T - 1). \]

Since $y$ and $\tau$ were arbitrary and the upper bound obtained is independent of these ingredients the proof is complete. \qed

In combination of the previous two propositions we directly conclude as follows.

Corollary 1. If $T < \infty$ then wave breaking occurs for the solution $u$ at time $T$.

We will now prove a more precise result in case of a slightly more regular initial value, namely $u_0 \in H^2(\mathbb{T})$ in place of $u_0 \in H^s(\mathbb{T})$ with merely $s > 3/2$. 


Theorem 2.2. If $s = 2$ and $T < \infty$ then the wave solution $u$ breaks with negative
infinite slope at time $T$. More precisely, we have
\[
\sup_{t \in [0, T]} \| u(t) \|_{L^\infty(T)} < \infty
\]
while
\[
\liminf_{t \uparrow T} \left( \inf_{x \in \mathbb{T}} u_x(x, t) \right) = -\infty.
\]

Proof. The boundedness of $\| u(t) \|_{L^\infty(T)}$ follows from (11). The strategy of proof is
to show that the negation of the last assertion, i.e.,
\[
\exists M \geq 0: \quad u_x(x, t) \geq -M \quad \forall x \in \mathbb{T} \forall t \in [0, T],
\]
leads to the boundedness of $\| u(t) \|_{H^2}$ as $t$ approaches $T$, which then causes a
contradiction due to the first part of (7) and the fact that $T$ is supposed to be the
maximal life span.

To reach the contradictory conclusion about the $H^2$ norm, we employ the following
line of arguments: Since $T < \infty$ we deduce from (10) that $\sup_{t \in [0, T]} \| u(t) \|_{L^2} < \infty$;
moreover, an application of the Poincaré-Wirtinger inequality ([1, p. 312]) to
the 1-periodic function $u_x(t) \in H^1(\mathbb{T})$ (and noting that $\int_{\mathbb{T}} u_x(x, t) \, dx = 0$) gives
$\| u_x(t) \|_{L^2} \leq C \| u_{xx}(t) \|_{L^2}$ for some constant $C > 0$. Therefore, in the current
situation we may note the validity of the implication
\[
\sup_{t \in [0, T]} \| u_{xx}(t) \|_{L^2} < \infty \implies \sup_{t \in [0, T]} \| u(t) \|_{H^2} < \infty.
\]
Thus, it suffices to show that (14) implies $\sup_{t \in [0, T]} \| u_{xx}(t) \|_{L^2} < \infty$ and the proof
will be complete. More precisely, we will show the following
Claim. (14) \implies \sup_{t \in [0, T]} \| u_{xx}(t) \|_{L^2} \leq e^{(1+\frac{15M}{2})T} \| u_0'' \|_{L^2}.

As a final technical reduction, we note that, due to well-posedness in the solution
space $C([0, T], H^2)$, the inequality asserted above is stable under $H^2$-limits $u_{0, \varepsilon} \to
u_0$ (as $\varepsilon \to 0$) of regularizations $u_{0, \varepsilon} \in H^3(\mathbb{T})$ of the initial value. Therefore,
it suffices to establish the claim for $u_0 \in H^3(\mathbb{T})$, in which case we have for the
solution $u \in C([0, T], H^3) \cap C^1([0, T], H^2)$.

We may apply $\partial_x^2$ to Equation (3) and obtain
\[
Q u_{xxx}(t) = \partial_x^2 \left( u_t(t) + \frac{3}{2} u(t) u_x(t) \right) = u_{xxx}(t) + \frac{9}{2} u_x(t) u_{xx}(t) + \frac{3}{2} u(t) u_{xxx}(t),
\]
which holds as an equation in $L^2(\mathbb{T})$ for every $t \in [0, T]$, since $Q$ maps $L^2$ into $H^2 \subset
H^2 \cdot H^1 \subset L^2$, and $u(t) u_{xx}(t) \in H^3 \cdot L^2 \subset L^\infty \cdot L^2 \subset L^2$.

Multiplication of the above equation by $u_{xx}(t)$ and integrating over $\mathbb{T}$ gives
\[
\langle u_{xx}(t) | (Q \circ \partial_x)(u_{xx}(t)) \rangle = \frac{1}{2} \frac{d}{dt} \| u_{xx}(t) \|_{L^2}^2 + \frac{9}{2} \int_{\mathbb{T}} u_x(x, t) u_{xx}(x, t)^2 \, dx + \frac{3}{2} \int_{\mathbb{T}} u(x, t) u_{xx}(x, t) u_{xxx}(x, t) \, dx,
\]
where the last integrand is of the form $u \ u_{xx} u_{xxx} = u \partial_x(u_{xx}^2)/2$ and an integration
by parts yields
\[
\langle u_{xx}(t) | (Q \circ \partial_x)(u_{xx}(t)) \rangle = \frac{1}{2} \frac{d}{dt} \| u_{xx}(t) \|_{L^2}^2 + \frac{15}{4} \int_{\mathbb{T}} u(x, t) u_{xx}(x, t)^2 \, dx.
\]
We rewrite this in the form

\[
\frac{d}{dt} \|u_{xx}(t)\|^2_{L^2} = 2\|u_{xx}(t)((Q \circ \partial_x)(u_{xx}(t))) + \frac{15}{2} \int_T (-u_x(x,t)) u_{xx}(x,t)^2 \, dx
\]

and by (14) deduce (again using that \(Q \circ \partial_x\) is bounded \(L^2 \to H^1\))

\[
\frac{d}{dt} \|u_{xx}(t)\|^2_{L^2} \leq 2\|u_{xx}(t)\|^2_{L^2}((Q \circ \partial_x)(u_{xx}(t))\|_{L^2} + \frac{15M}{2} \|u_{xx}(t)\|^2_{L^2} \leq
\]

\[
\leq 2\|u_{xx}(t)\|^2_{L^2} + \frac{15M}{2} \|u_{xx}(t)\|^2_{L^2} = \left(2 + \frac{15M}{2}\right) \|u_{xx}(t)\|^2_{L^2}.
\]

Integration with respect to time and Gronwall’s lemma now imply

\[
\forall t \in [0, T]: \quad \|u_{xx}(t)\|^2_{L^2} \leq \|u_{xx}(0)\|^2_{L^2} e^{(2+\frac{15M}{2})t} \leq \|u_0''\|^2_{L^2} e^{2(1+\frac{15M}{2})T},
\]

which proves the claim. \(\square\)

**Remark 2.** In retrospect, we have shown in Corollary 1 that any solution according to (5) with a finite maximal life span and with initial value \(u_0 \in H^s(T)\), \(s > 3/2\), suffers wave breaking. Under the condition \(s \geq 2 > 3/2\) we add in Theorem 2.2 the qualitative information that this breaking wave develops a singularity with **negative infinite slope**. The reason for requiring the slightly higher regularity on the initial data is technical, because the proof aims at establishing the crucial inequality \(\sup_{t \in [0, T]} \|u_{xx}(t)\|^2_{L^2} \leq C \|u_0''\|^2_{L^2}\) upon operating with a second order spatial derivative on the basic equation (3) and using classical energy estimates with some care. The result might still hold with the relaxed condition \(s > 3/2\), but we expect that a proof would require a more heavy machinery from function space theory and would be less direct. If we focus on qualitative aspects of the solutions and in view of the equally important blow-up scenario established in the following section, which uses \(s \geq 3\), the setting of Theorem 2.2 suffices.

3. **Blow-up for a class of initial data.** We will show below that for a considerable class of initial wave profiles the maximal life span of the solution is finite and blow-up occurs in the form of wave breaking. The result as well as a large part of the reasoning leading to it is similar to corresponding statements and proofs in [3] for the case of the real line and initial data in \(H^\infty(\mathbb{R})\) (matching with the well-posedness result from [9]). The main differences now lie in alternative estimates required for the convolution kernel of the operator \(Q\) on the torus and in the fact that here we may work with initial data of lower regularity due to the more recent and improved well-posedness results available ([7, 8]).

As a preparation we collect a few details about the convolution kernel \(K\) implementing the translation invariant operator \(Q = (id - \partial_x^2)^{-1}\) on the one-dimensional torus \(T\). The operator \(Q\) corresponds to the Fourier multiplier \(\hat{K}(k) = 1/(1+4\pi^2k^2)\) \((k \in \mathbb{Z})\), which satisfies \(\hat{K} \in l^1(\mathbb{Z}) \subset l^2(\mathbb{Z})\). Therefore \(K\) possesses the Fourier series representation

\[
\sum_{k \in \mathbb{Z}} \frac{e^{2\pi ikx}}{1+4\pi^2k^2}
\]

in the sense of \(L^2(T)\) and the Fourier series itself converges uniformly due to a classic theorem by Weierstraß, since the sum of the \(L^\infty\)-norms is convergent. Let the uniform limit be denoted by \(F \in C(T)\). Since \(K\) and \(F\) both belong to \(L^2(T)\) and have the same Fourier series, they agree as classes in \(L^2(T)\), in particular \(K(x) = F(x)\).
Lemma 3.1. If extremum is attained is obvious in this case, and the statement on differentiability

Embodiment Subsection 6.3.2 or Escher's lecture in [4]). In fact, due to compactness of

In the inital version proved for functions on the real line in [3, Theorem 2.1] (see also [2, Subsection 6.3.2] or Escher's lecture in [4]). In fact, due to compactness of \( \mathbb{T} \) and the embedding \( H^2(\mathbb{T}) \subset C^1(\mathbb{T}) \), the assertion about existence of a location where the extremum is attained is obvious in this case, and the statement on differentiability almost everywhere is proven in exactly the same way.

The formula confirms again that \( F \) is continuous (since \( F(0) = \lim_{t \to 0} F(t) \)), but also reveals that \( F \) is \( C^1 \) on \( \mathbb{T} \setminus \{0\} \), piecewise \( C^1 \) on \( \mathbb{T} \) in the sense that the derivative is continuous off \( x = 0 \) and possesses one-sided limits at \( x = 0 \), and has a non-differentiable peak at \( x = 0 \). Thus \( K \) is absolutely continuous with piecewise continuous derivative \( K' \) (possessing one-sided limits at the only point of discontinuity \( x = 0 \)). We have

\[
K'(x) = \frac{e^x - e^{1-x}}{2(e - 1)} \quad \forall x \neq 0. \tag{15}
\]

Another substantial ingredient in the proof of the blow-up result below is an accurate description of the evolution of spatial extrema of functions \( v \in C^1([0, T], H^2(\mathbb{T})) \), which we may transfer to periodic functions without essential changes from the original version proved for functions on the real line in [3, Theorem 2.1] (see also [2, Subsection 6.3.2] or Escher's lecture in [4]). In fact, due to compactness of \( \mathbb{T} \) and the embedding \( H^2(\mathbb{T}) \subset C^1(\mathbb{T}) \), the assertion about existence of a location where the extremum is attained is obvious in this case, and the statement on differentiability almost everywhere is proven in exactly the same way.

Lemma 3.1. If \( T > 0 \) and \( v \in C^1([0, T], H^2(\mathbb{T})) \), then for every \( t \in [0, T] \) there is \( \xi(t) \in \mathbb{T} \) such that

\[
m(t) := \min_{x \in \mathbb{T}} v_x(x, t) = v_x(\xi(t), t).
\]

The function \( m : [0, T] \to \mathbb{R} \) is Lipschitz continuous, in particular, differentiable almost everywhere on \([0, T] \), and satisfies

\[
m'(t) = v_{xx}(\xi(t), t) \quad \text{for almost every } t \in [0, T].
\]
The same statement clearly holds for the maximum in place of the minimum.

After all these preparations, we formulate and prove the main result.

**Theorem 3.2.** Let \( u_0 \in H^3(\mathbb{T}) \) and \( u \) be the unique solution to (3-4) with maximal life span \( T \). If

\[
\min_{x \in \mathbb{T}} u_0'(x) + \max_{x \in \mathbb{T}} u_0'(x) < -\frac{2}{3},
\]

then \( T < \infty \) and we observe wave breaking for \( u \) at time \( T \).

**Proof.** Since \( u_0 \in H^3(\mathbb{T}) \), we have \( u \in C^1([0,T],H^2) \cap C([0,T],H^3) \) and therefore Lemma 3.1 is applicable with appropriate \( \xi_1(t) \) and \( \xi_2(t) \) in \( \mathbb{T} \) \((t \in [0,T])\) to the functions

\[
m_1(t) := \min_{x \in \mathbb{T}} u_x(x,t) = u_x(\xi_1(t),t),
\]

\[
m_2(t) := \max_{x \in \mathbb{T}} u_x(x,t) = u_x(\xi_2(t),t).
\]

Note that \( u_x(t) \in H^2(\mathbb{T}) \subset C^1(\mathbb{T}) \) and \( u_x(\xi_j(t),t) = 0 \) holds due to the choice of \( \xi_j(t) \) \((j = 1,2)\) as locations of extrema. By periodicity of the \( C^1 \) function \( u(.,t) \), we necessarily have \( m_1(t) \leq 0 \leq m_2(t) \) for every \( t \in [0,T] \). (For example, \( m_1(t) > 0 \) is absurd, because the function \( u(.,t) \) would then have to be strictly increasing and periodic, which contradicts continuity.)

Differentiating Equation (3) with respect to \( x \) leads to

\[
u_{xt}(t) + \frac{3}{2} u_x(t)^2 + \frac{3}{2} u(t) u_{xx}(t) = K * (u_{xx}(t))
\]

and inserting \( x = \xi_j(t) \) then gives (recall that \( u_x(\xi_j(t),t) = 0 \))

\[
m_j'(t) + \frac{3}{2} m_j(t)^2 = \int_{\mathbb{T}} K(y) u_{xx}(\xi_j(t) - y) dy \quad \text{for almost every } t \in ]0,T[., \ j = 1,2.
\]

The properties of \( K \) allow for an integration by parts, hence we obtain \( t \)-a.e.

\[
m_j'(t) + \frac{3}{2} m_j(t)^2 = -\int_{\mathbb{T}} K'(y) u_x(\xi_j(t) - y) dy =
\]

\[
= -\frac{1}{2(e-1)} \int_0^1 e^y u_x(\xi_j(t) - y) dy + \frac{e}{2(e-1)} \int_0^1 e^{-y} u_x(\xi_j(t) - y) dy \leq
\]

\[
- \frac{m_1(t)}{2(e-1)} \int_0^1 e^y dy + \frac{e m_2(t)}{2(e-1)} \int_0^1 e^{-y} dy = \frac{1}{2} (m_2(t) - m_1(t)),
\]

which in turn yields

\[
m_1'(t) \leq -\frac{3}{2} m_1(t)^2 + \frac{1}{2} (m_2(t) - m_1(t)), \quad (17)
\]

\[
m_2'(t) \leq -\frac{3}{2} m_2(t)^2 + \frac{1}{2} (m_2(t) - m_1(t)). \quad (18)
\]

Thus, we are now in a situation perfectly analogous with [3, Theorem 3.2, p. 237], but for convenience of the reader we repeat the remaining steps of the conclusion.

The sum of the inequalities in (17) and (18) gives (almost everywhere on \([0,T]\))

\[
(m_1 + m_2)' \leq -\frac{3}{2} (m_1^2 + m_2^2) + (m_2 - m_1) = (m_2 - m_1)(1 + \frac{3}{2} (m_1 + m_2)) - 3 m_2^2.
\]

The function \( m_1 + m_2 \) is absolutely continuous on \([0,T], m_2 - m_1 \geq 0 \) and the hypothesis of the theorem implies at time \( t = 0 \) the condition \( 1 + \frac{3}{2} (m_1(0) + m_2(0)) < \)
0. By the above inequality, the corresponding condition must hold for all time, i.e.,

$$\forall t \in [0, T[: 1 + \frac{3}{2}(m_1(t) + m_2(t)) < 0,$$

which we put to use in (17) to deduce (a.e. on $[0, T[$)

$$m'_1 \leq -\frac{3}{2}m_1 + \frac{1}{2}m_1 + \frac{1}{2}m_2 < -\frac{3}{2}m_1 - \frac{1}{2}m_1 + \frac{1}{2}\left(-m_1 - \frac{2}{3}\right) =$$

$$= -\frac{3}{2}\left(m_1^2 + \frac{2}{3}m_1 + \frac{2}{9}\right) = -\frac{3}{2}\left(m_1 + \frac{1}{3}\right)^2 + \frac{1}{9} \leq -\frac{3}{2}\left(m_1 + \frac{1}{3}\right)^2.$$

Putting $M(t) := m_1(t) + \frac{1}{3}$ we have $M(0) = m_1(0) + \frac{1}{3} < -\frac{2}{3} - m_2(0) + \frac{1}{3} = -\frac{1}{3} - m_2(0) < 0$ and

$$M'(t) = m'_1(t) \leq -\frac{3}{2}M(t)^2$$

for almost every $t \in [0, T[$, which implies that $M(t) < 0$ throughout. Finally, we obtain a.e. with respect to $t$,

$$\frac{d}{dt}\left(\frac{1}{M(t)}\right) = -\frac{M'(t)}{M(t)^2} \geq \frac{3}{2}$$

and therefore upon integration for every $t \in [0, T[$,

$$\frac{1}{M(t)} \geq \frac{1}{M(0)} + \frac{3}{2}t.$$ 

Observing $M(0) < 0$, we conclude that $M(t) \to -\infty$ as $t \to 2/(3|M(0)|)$. Thus, $T < \infty$, $\lim_{t \to T} m_1(t) = -\infty$, and from Proposition 2 we know that the $L^\infty$-norm of $u$ stays bounded as $t$ approaches $T$, which proves wave breaking with negative slope unbounded from below. \qed 

REFERENCES

[1] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer-Verlag, New York, 2011.

[2] A. Constantin, *Nonlinear Water Waves with Applications to Wave-Current Interactions and Tsunamis*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 2011.

[3] A. Constantin and J. Escher, *Wave breaking for nonlinear nonlocal shallow water equations*, *Acta Math.*, 181 (1998), 229–243.

[4] A. Constantin, J. Escher, R. S. Johnson and G. Villari, *Nonlinear Water Waves*, Springer-Verlag, Florence, 2016.

[5] B. Fornberg and G. B. Whitham, *A numerical and theoretical study of certain nonlinear wave phenomena*, *Philos. Trans. Roy. Soc. London Ser. A.*, 289 (1978), 373–404.

[6] L. Grafakos, *Classical and Modern Fourier Analysis*, Pearson Education, Upper Saddle River, 2004.

[7] J. Holmes, *Well-posedness of the Fornberg-Whitham equation on the circle*, *J. Differential Equations*, 260 (2016), 8530–8549.

[8] J. Holmes and R. C. Thompson, *Well-posedness and continuity properties of the Fornberg–Whitham equation in Besov spaces*, *J. Differential Equations*, 263 (2017), 4355–4381.

[9] P. I. Naumkin and I. A. Shirshov, *Nonlinear Nonlocal Equations in the Theory of Waves*, American Mathematical Society, Providence, 1994.

[10] R. L. Seliger, *A note on the breaking of waves*, *Proc. Roy. Soc. A.*, 303 (1968), 493–496.

[11] G. B. Whitham, *Linear and Nonlinear Waves*, Wiley-Interscience, New York-London-Sydney, 1974.

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