Invariant subspaces of algebras of analytic elements associated with periodic flows on W*-algebras

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ABSTRACT. We consider an action of the circle group, T on a W*- algebra, M with . Similarly to the case when \( M = L^\infty(T) \) is acted upon by translations, we define the generalized Hardy space \( H_+ \subset H \) where \( H \) is the Hilbert space of a standard representation of \( M \) and the subalgebra \( M_+ \) of analytic elements of \( M \) with respect to the action. We prove that \( M_+ \subset B(H_+) \) is a reflexive algebra of operators if the Arveson spectrum is finite or, if the spectrum is infinite, the spectral subspace corresponding to the least positive element contains an unitary operator. We also prove that \( M_+ \) is reflexive if \( M \) is an abelian W*- algebra. Examples include the algebra of analytic Toeplitz operators, \( w^* \)-crossed products, reduced \( w^* \)-semicrossed products and some reflexive nest subalgebras of von Neumann algebras.

1 Introduction

Let \( A \subset B(X) \) denote a weakly closed algebra of operators on the Hilbert space \( X \). Denote by \( Lat(A) \) the lattice of closed subspaces of \( X \) that are invariant for all operators \( a \in A \). Let \( algLat(A) = \{ b \in B(X) | bK \subset K \text{ for all } K \in Lat(A) \} \). The algebra is called reflexive if \( A = algLat(A) \) (for more information on reflexive operator algebras, we recommend the book by H. Radjavi and P. Rosenthal [7]). Thus, a reflexive operator algebra is completely determined by the lattice of its invariant subspaces. The problem of whether a weakly closed algebra of operators is reflexive started to be studied in the 1960s. In particular, Sarason [8], proved that that the algebra of analytic Toeplitz operators on the Hardy space \( H^2(T) \) where \( T \) is the the unit circle \( T = \{ z \in \mathbb{C} | |z| = 1 \} \), is reflexive and so is any of its weakly closed subalgebras. In [6] we extended this result to the case of \( H^p(T), 1 < p \leq \infty \). The same paper contains a study of reflexivity of subalgebras of analytic elements of crossed products of finite von

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Neumann algebras on non commutative Hardy spaces $H^p$ associated with the $W^*$-dynamical system. Further, in [3], E. Kakariadis has considered the more general case of semi crossed products and, among other results, he has extended the particular case of our main result in [6] for $p = 2$ to the semicrossed product setting. In this paper we study a related problem in a more general setting than the crossed product or the reduced $w^*$-semicrossed product considered in [6] and [3]. We consider a $W^*$-dynamical system $(M, T, \alpha)$ where $M$ is a $\sigma$-finite $W^*$-algebra and a standard covariant representation of the system on a certain Hilbert space, $H$, as constructed below. We then define the non commutative Hardy space, $H_+$ associated with this representation. We show that if the Arveson spectrum of the action is finite, then the algebra of analytic elements, $M_+$, is a reflexive operator algebra on $H_+$ (Theorem 5). We consider next the case of infinite Arveson spectrum and show that if the spectral subspace corresponding to the smallest positive element of the spectrum contains a unitary element, then, again, the algebra $M_+$ is reflexive (Theorem 8). Finally, we consider the case of abelian von Neumann algebras and we prove that in this case $M_+$ is always reflexive (Theorem 13). Examples and particular cases of our results include the classical result of Sarason, the crossed product [6], the reduced $w^*$-semicrossed product [3], the algebra of $n \times n$ upper triangular matrices and other nest subalgebras of von Neumann algebras.

2 Notations and preliminary results

Let $(M, T, \alpha)$ be a $W^*$-dynamical system, where $M$ is a $W^*$-algebra with separable predual, $T$ is the circle group, $T = \{ z \in \mathbb{C} | |z| = 1 \}$ and $\alpha$ a $w^*$-continuous action of $T$ on $M$. For each $n \in \mathbb{Z}$, denote by

$$M_n = \left\{ \int T^* \alpha_z(m)dz | m \in M \right\}$$

where the integral is taken in the $w^*$-topology. Therefore,

$$M_{-n} = \{ m^* | m \in M_n \}$$

It can immediately be checked that

$$M_n = \{ m \in M | \alpha_z(m) = z^n m \}$$

and

$$M_{-n} = (M_n)^*,$$  where $(M_n)^* = \{ m^* | m \in M_n \}$

In particular, $M_0$ is the algebra of the fixed points of $M$ under the action $\alpha$. It is clear that the mapping $P_n^M : M \to M_0$ defined by $P_n^M(m) = \int T^* \alpha_z(m)dz$ is a projection of $M$ onto the closed subspace $M_0 \subset M$. Thus $P_0^M : M \to M_0$ is a faithful normal conditional expectation of $M$ onto the subalgebra $M^\alpha$. 

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It is well known that $M$ is the $w^*$-closed linear span of $\{M_n|n \in \mathbb{Z}\}$. The set $\{n \in \mathbb{Z}|M_n \neq \{0\}\}$ is called the Arveson spectrum of the action $\alpha$ and will be denoted by $sp(\alpha)$ (for more information about the Arveson spectrum of an action, see [1], [5]). Since $M$ is $\sigma$-finite, it follows, in particular, that $M_0$ has a faithful normal state, $\varphi_0$. This is the case, in particular, when $M_0$ has separable predual. Then, $\varphi = \varphi_0 \circ P_0^M$ is a faithful normal $\alpha$-invariant state of $M$. Let $(H_\varphi, \pi_\varphi, \xi_\varphi)$ be the GNS representation of $M$ corresponding to the state $\varphi$. Clearly, $\pi_\varphi$ is a faithful representation of $M$ on $H_\varphi$ and $\xi_\varphi$ is a cyclic and separating vector for $M$. In the rest of the paper we will identify $\pi_\varphi(M)$ with $M$ and will write $m$ instead of $\pi_\varphi(m)$, $m \in M$. Also, we will denote $H_\varphi$ by $H$ and $\xi_\varphi$ by $\xi_0$. In other words, we will consider a von Neumann algebra, $M \subset B(H)$, that has a cyclic and separating unit vector $\xi_0$ and the vector state $\varphi(m) = \langle m\xi_0, \xi_0 \rangle$ is $\alpha$-invariant. If we define $U_z(m\xi_0) = \alpha_z(m)\xi_0$, for all $z \in T$, then it is easy to see that $\{U_z\}_{z \in T}$ is a group of unitary operators on $H$ that implements the action $\alpha$.

If $S$ and $F$ are the closures of the Tomita-Takesaki conjugate linear densely defined operators:

$$S_0(m\xi_0) = m^*\xi_0, m \in M$$
$$F_0(m'\xi_0) = (m')^*\xi_0, m' \in M'$$

then, it is known (see for instance [2], Section 9.2) that $S^* = F$, the positive self adjoint operator $\Delta = S^*S$ has an inverse $\Delta^{-1} = SS^*$ and there is a conjugate linear isometry $J$ from $H$ onto $H$ such that

$$S = J\Delta^{\frac{1}{2}}$$

From the definition of the unitaries $U_z, z \in T$, it immediately follows that all $U_z, z \in T$, commute with $S$ and therefore with $S^* = F$. Indeed

$$U_zS(m\xi_0) = U_z(m^*\xi_0) = \alpha_z(m^*)\xi_0 = SU_z(m\xi_0)$$

and therefore, $U_zS = SU_z, z \in T$. Thus, all $U_z, z \in T$ commute with $J$ as well.

We can now state the following

**Proposition 1** With the notations above, we have the following:

i) $\alpha_z'(m') = U_z'm'U_z^*$ is an action of $T$ on $M'$, where $M'$ is the commutant of $M$ in $B(H)$

ii) If $z \in T$, then $U_z$ commutes with $J$, and $J\alpha_z(m)J = \alpha_z'(JmJ)$, $m \in M, z \in T$

iii) $(M')_n = JM_{-n}J, n \in \mathbb{Z}$

iv) $U_z(m'\xi_0) = \alpha_z'(m')\xi_0$

v) $sp(\alpha) = sp(\alpha')$

**Proof.** i) Indeed, if $m \in M$, $m' \in M'$ and $a \in M$, we have:

$$U_zm'U_z^*m(a\xi_0) = U_zm'U_z^*SU_zU_z(a\xi_0) = U_zm'(\alpha_z(m)(\alpha_z(a)\xi_0)) =$$

$$U_z\alpha_z'(m)(m')\alpha_z(a)(\xi_0) = v)sp(\alpha) = sp(\alpha') =$$
= mU_zm'U_z^∗z(αz(a)ξ_0) = mU_zm'U_z^∗z(aξ_0)

Hence \( α'_z(m') = U_zm'U_z^∗z \in M' \).

ii) The fact that the unitaries \( U_z \) commute with \( J \) was proved above. For the second part of ii), notice that

\[ Jα_z(m)J = JU_zmU_z^∗zJ = U_zJmJU_z^∗z \]

iii) follows immediately from i) and ii).

iv) Let \( m' \in M' \). Then, \( m' = JmJ \) for some \( m \in M \). Then we have

\[ U_z(mξ_0) = U_z(JmJξ_0) = JU_z(mξ_0) = J(α_z(m)ξ_0) = α'_z(m')ξ_0 \]

v) Let \( n \in sp(α) \). Then, since \( M_{−n} = \{ m^* | m \in M_n \} \) we have \( n \in sp(α') \). From iii) it then follows that \( n \in sp(α') \). The proof of the other inclusion is similar  

Let \( H_n = \left\{ \int z^nU_z(ξ)dz | ξ ∈ H \right\} = \{ ξ ∈ H | U_zξ = z^nξ \} \). Then, the map \( P_n^H \) from \( H \) to \( H_n \) defined as follows

\[ P_n^H(ξ) = \int z^nU_z(ξ)dz | n ∈ Z, ξ ∈ H \]

is an orthogonal projection of \( H \) onto the closed subspace \( H_n \). Also the closed subspaces \( (M')_n ⊂ M' \) and the projections \( P_n^{M'} \) from \( M' \) onto \( (M')_n \) can be defined similarly with \( M_n \) and \( P_n^M \).

**Proposition 2** i) If \( n ≠ k \), then \( H_n \) and \( H_k \) are orthogonal

ii) For every \( ξ ∈ Z \) we have \( M_nξ_0 = H_n \), where \( M_nξ_0 = \{ mξ_0 | m ∈ M_n \} \)

iii) For every \( ξ ∈ Z \) we have \( (M')_nξ_0 = H_n \), where \( (M')_nξ_0 = \{ m'ξ_0 | m' ∈ (M')_n \} \)

iv) The direct sum of Hilbert spaces \( \sum H_n \) equals \( H \)

**Proof.** i) Let \( ξ ∈ H_n, η ∈ H_k \). Then \( U_z(ξ) = z^nξ \) and \( U_z(η) = z^kη \), for all \( z ∈ T \). Since \( U_z \) are unitary, we have

\[ \langle ξ, η \rangle = \langle U_zξ, U_zη \rangle = z^{n−k} \langle ξ, η \rangle \], \( z ∈ T \)

Since \( n ≠ k \) it follows that \( \langle ξ, η \rangle = 0 \)

ii) Since \( ξ_0 \) is cyclic for \( M \), the subspace \( Mξ_0 = \{ mξ_0 | m ∈ M \} \) is dense in \( H \). Then, if \( P_n^M \) and \( P_n^M \) are the above projections, we have

\[ M_nξ_0 = P_n^M(M)ξ_0 = \left\{ \int z^nα_z(m)(ξ_0)dz | m ∈ M \right\} = \left\{ \int z^nU_z(ξ)dz | ξ = mξ_0, m ∈ M \right\} = \{ P_n^H(ξ) | ξ = mξ_0, m ∈ M \} \]
Since the subspace $M\xi_0$ is dense in $H$ and $P_n^H$ is an orthogonal projection, the result stated in ii) follows

iii) Since $\xi_0$ is cyclic for $M'$, the subspace $M'\xi_0 = \{m'\xi_0 | m' \in M'\}$ is dense in $H$. Then, if $P_n^H$ and $P_n^M$ are the above projections, we have

$$(M')_n\xi_0 = P_n^{M'} (M') \xi_0 = \left\{ \int \overline{\alpha}(m')((\xi_0)dz | m' \in M') \right\} =
= \left\{ \int \overline{\alpha}U_z(\xi)dz | m'\xi_0, m' \in M \right\} = \{P_n^H(\xi)| \xi = m'\xi_0, m' \in M'\}
$$

Since $\xi_0$ is cyclic for $M'$, the result follows as in ii).

iv) Let $\eta \in H$ be such that $\eta \perp H_n$ for all $n \in \mathbb{Z}$. Since $M$ is the $w^*$-closed linear span of $\{M_n | n \in \mathbb{Z}\}$, it follows from ii) that $\eta \perp M\xi_0$, so, since $\xi_0$ is cyclic for $M$, $\eta = 0$.

3 Invariant subspaces of subalgebras of analytic elements on generalized Hardy spaces

Let $(M, T, \alpha), M \subset B(H)$ and $\xi_0$ be as in the previous section. If $sp(\alpha) = \{0\}$, then $\alpha$ is trivial in the sense that $\alpha_z = id$ for every $z \in T$, where $id$ is the identity automorphism, so $M = M_0$ and $H = H_0$. Since every von Neumann algebra is reflexive, this case has no interest from the point of view of this paper. Suppose $sp(\alpha) \neq \{0\}$. Then, by Proposition 1 v), we have $sp(\alpha') = sp(\alpha) \neq \{0\}$. We now define the generalized Hardy space, $H_+$ and the subalgebras of analytic elements of $M$, $M_+ \subset B(H_+)$ and $(M')_+ \subset B(H_+)$ which are generalizations of the algebra of analytic Toeplitz operators to the framework of periodic $W^*$-dynamical systems:

$$H_+ = \sum_{n \geq 0} H_n$$

If $p_+$ denotes the projection of $H$ onto $H_+$, $\vee_{n \geq 0} M_n$ the $wo$-closed algebra generated by $\{M_n | n \geq 0\}$ and $\vee_{n \geq 0} (M')_n$ the $wo$-closed algebra generated by $\{(M')_n | n \geq 0\}$, we define

$$M_+ = p_+(\vee_{n \geq 0} M_n)|_{H_+}$$

$$(M')_+ = p_+(\vee_{n \geq 0} (M')_n)|_{H_+}$$

In this section we will find conditions for the reflexivity of $M_+$ [Theorems 5, 8 and 13]. We consider first the case when $sp(\alpha)$ is finite.

**Lemma 3** If $sp(\alpha)$ is finite, then $(M')_+ = M_+$

**Proof.** Clearly, $M_+ \subset ((M')_+)'$. Let now $x \in (M')_+ \subset B(H_+)$. Then $x$ commutes, in particular, with $(M')_k, k \geq 0$. Consider the following dense linear subspace of $H$

$H' = \{ \xi \in H | P_k^H(\xi) \in (M')_k \xi_0 \text{ and } P_k^H(\xi) = 0 \text{ for all but finitely many } k \in \mathbb{Z} \}$

Then $x$ commutes with $H'$. Since $H'$ is dense in $H$, we have $x = 0$. Therefore, $M_+ = (M')_+$. The lemma follows.
Define the operator $\hat{x}$ on $H'$ as follows

$$\hat{x}(m'_k\xi_0) = m'_k x \xi_0, k \in \mathbb{Z}$$

Since $\xi_0$ is a separating vector for $M'$ and $x$ commutes with $(M')^+$, $\hat{x}$ is well defined. Due to the obvious facts that, for every operator $a \in B(H)$ and $\eta \in H$, $||a\eta|| = ||a||\eta ||$ where $|a|$ is the absolute value of $a$, and $|m'_k| \in (M')_0 \subset (M')^+$, it follows that, if $m'_k \in (M')_k, k \in \mathbb{Z}$, then

$$||\hat{x} m'_k \xi_0|| = ||m'_k x \xi_0|| = ||m'_k| x \xi_0|| = ||x|| ||m'_k| \xi_0|| = ||x|| ||m'_k \xi_0||$$

Therefore $\hat{x}$ is bounded on $H_k, k \in \text{sp}(\alpha)$ and $||\hat{x}|| \leq ||x||$, so, since $\text{sp}(\alpha)$ is finite, $\hat{x}$ is bounded on $H$. On the other hand, if $n, k \in \mathbb{Z}, m'_n, m'_k \in (M')_n, m'_k \in (M')_k$ we have $m'_n m'_k = m'_{n+k} \in (M')_{n+k}$, so

$$m'_n \hat{x}(m'_k \xi_0) = m'_n m'_k x \xi_0 = m'_{n+k} x \xi_0 = \hat{x}(m'_{n+k} \xi_0) = \hat{x}(m'_n m'_k \xi_0)$$

Therefore, $\hat{x} \in (M')' = M$. Thus, $p_+ \hat{x} p_+ = x \in M_+$ and we are done. 

Suppose that $\text{sp}(\alpha)$ is finite. In this case, since, as noticed above, for every $n, k \in \text{sp}(\alpha), m_n, m_k \in M_k$ we have $m_n m_k \in M_{n+k}$, it follows that every element of $M_n, n \in \text{sp}(\alpha) \setminus \{0\}$ is nilpotent. Notice also that, by Proposition 1 v), we have $\text{sp}(\alpha) = \text{sp}(\alpha')$.

**Lemma 4** Let $n \in \text{sp}(\alpha) \setminus \{0\}, n > 0$. and $w \in (M')_n$ a partial isometry. If $w$ is nilpotent, then $w \in (\text{algLat}(M_+))'$

**Proof.** We will use a method similar with the one used in the proof of [4, Proposition 19 i)]. Suppose $w^k = 0$ for some $k \in \mathbb{N}, k \geq 2$. We will prove the Lemma by induction. Let first $k = 2$, so $w^2 = 0$. Let $p = 1 - w^* w \in (M')_0$, be the orthogonal projection of $H_+$ onto the kernel of $w$. Then, $p, 1 - p \in \text{Lat}(M_+)$ and therefore, with respect to the decomposition $I_{H_+} = p + (1 - p)$, we have

$$w = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}$$

$$m = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \forall m \in M_+$$

$$x = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}, \forall x \in \text{algLat}(M_+)$$

Since $mw = wm$, we have $m_1 c = cm_2$. Therefore the subspace $K = \{c \xi \oplus \xi | \xi \in H_+\}$ is invariant for $M_+$. Since $x \in \text{algLat}(M_+)$, it follows that $K$ is invariant for $x$. Thus $x_1 c = cx_2$, so $wx = xw$ and we are done with the case $k = 2$. Suppose next that for every partial isometry $w \in (M')_n$ with $w^k = 0$ it follows that $w \in (\text{algLat}(M_+))'$ and let $w \in (M')_n$ with $w^{k+1} = 0$. Let $p$ denotes the orthogonal projection on $\ker w, p = 1 - w^* w$, so that $p \in (M')_0$. Since $wp = 0$
it follows that

\[(1 - p)w = (1 - p)w(1 - p)\]

and therefore

\[(1 - p)w^k = ((1 - p)w(1 - p))^k, k \in \mathbb{N}.\]

Since \(w^{k+1} = 0\) we have \(w^k(H_+) \subset \ker w\) and therefore

\[0 = (1 - p)w^k = ((1 - p)w(1 - p))^k.\]

By hypothesis, \((1 - p)w = (1 - p)w(1 - p) \in (\text{algLat}(M_+))'\). On the other hand, since \(w \in (M')_1\) and \(p_0 \in (M')_0\) we have \(pw \in (M')_1\). Since obviously \((pw)^2 = 0\), by the previous arguments for \(k = 2\), it follows that \(pw \in (\text{algLat}(M_+))'\).

Therefore

\[w = pw + (1 - p)w \in (\text{algLat}(M_+))'\]

and the proof is completed.

We can now state and prove the following

**Theorem 5** Let \((M, T, \alpha)\) be as above. If \(\text{sp}(\alpha)\) is finite, then \(M_+\) is reflexive.

**Proof.** Let \(x \in \text{algLat}(M_+)\). Since \(pH_+ \in \text{Lat}(M_+)\) for every projection \(p \in (M')_0\), it follows that \(x \in (M')_0\). If \(a \in (M')_n\) for some \(n > 0\), then \(a\) has a polar decomposition \(a = w|a|\), where obviously \(w \in (M')_n\) and \(|a| \in (M')_0\).

Since \(\text{sp}(\alpha)\) is finite, \(w\) is nilpotent and therefore by the previous lemma \(xw = wx\), so \(x \in (M')_+\). Applying Lemma 3 it follows that \(x \in M_+\) and we are done.  

**Example 6** Let \(M\) be a von Neumann algebra and \(\{p_1, p_2, ..., p_k\}\) a family of mutually orthogonal projections of \(M\). Let \(U_z = \sum z^n p_i, z \in T\), where \(\{n_1, n_2, ..., n_k\} \subset \mathbb{Z}\) is a decreasing finite set. Consider the periodic flow \(\alpha_z(m) = U_z m U_z^*, z \in T\) on \(M\). It is easy to show that

\[M_+ = \left\{ m \in M | m(\sum_{j=1}^i p_j H) \subset \sum_{j=1}^i p_j H, i = 1, ..., k \right\}\]

is the reflexive nest subalgebra of \(M\) corresponding to the nest \(p_1 \leq p_1 + p_2 \leq ... \leq \sum_{j=1}^k p_j\) and \(\text{sp}(\alpha)\) is finite.

Let \(n_0 = \min \{ n \in \text{sp}(\alpha) | n > 0 \}\). Suppose that \(M_{n_0}\) contains a unitary operator, \(v_0\).
**Proposition 7** In the above conditions, we have

i) \( M_{k_{0}} = \{ v_{0}^{k}a | a \in M_{0} \}, k \in \mathbb{Z} \)

ii) \( H_{k_{0}} = \{ v_{0}^{k}a_{0} | a \in M_{0} \}, k \in \mathbb{Z} \)

iii) \((M')_{k_{0}}\) contains an unitary operator, \( w_{0} \)

iv) \((M')_{k_{0}} = \{ w_{0}^{k}b | b \in (M')_{0} \}, k \in \mathbb{Z} \)

v) \( sp(\alpha) \) is a subgroup of \( \mathbb{Z} \)

**Proof.** Let \( m \in M_{k_{0}} \). Then, \((v_{0}^{*})^{k}m = a \in M_{0}\)

ii) follows from i) and Proposition 2, ii)

iii) By Proposition 1, iii) the unitary operator \( w_{0} = Jv_{0}^{*}J \) is in \((M')_{k_{0}}\)

iv) Similar with i)

v) From i) it follows that \( \{ kn_{0} | k \in \mathbb{Z} \} \subseteq sp(\alpha) \). On the other hand, if \( l \in \mathbb{Z}, l \in sp(\alpha) \), then there exists \( k_{0} \in \mathbb{Z} \) such that \( k_{0}n_{0} \leq l \leq (k_{0} + 1)n_{0} \). It can be easily verified that, \((v_{0}^{*})^{k} \in M_{k_{0}}^{*} = M_{-k_{0}} \) and, since \( v_{0} \) is unitary, we have \((v_{0}^{*})^{k_{0}}M_{l} \subseteq M_{l-k_{0}n_{0}} \), so \( l-k_{0}n_{0} \in sp(\alpha) \). Since \( l-k_{0}n_{0} \leq n_{0} \) and \( n_{0} = \min \{ n | n \in sp(\alpha), n > 0 \} \), it follows that either \( l = k_{0}n_{0} \) or \( l = (k_{0}+1)n_{0} \). Therefore, \( sp(\alpha) = \{ kn_{0} | k \in \mathbb{Z} \} \) and we are done.

**Theorem 8** Let \((M, T, \alpha), H_{+}, M_{+}, v_{0} \) be as above. Then, the algebra \( M_{+} \subseteq B(H_{+}) \) is reflexive.

The proof of this result will be given after a series of lemmas. The next lemma is a substitute for Lemma 3 for the case when \( M_{k_{0}}, \) (and therefore \((M')_{k_{0}}\), contains an unitary operator \( v_{0} \) (respectively \( w_{0} \)).

**Lemma 9** i) \((M')_{+} = (M')_{+}\)

ii) \((M')_{+} \subseteq (M')_{+}\)

**Proof.** i) Let \( x \in (M')_{n} \) and \( m \in M_{k}, n, k \geq 0 \). Then \( p_{+}xp_{+}mp_{+} = p_{+}xmp_{+} = p_{+}mp_{+}xp_{+} \), so \((M')_{+} \subseteq (M')_{+}\). Let now \( x \in (M')_{+} \subseteq (M')_{+}\). Consider the following linear subspace of \( H \)

\[ H' = \{ \xi \in H | P_{n}^{H}P_{n}^{H}(\xi) \in M_{n}^{H}, n \in \mathbb{Z} \text{ and } P_{n}^{H}(\xi) = 0 \text{ for all but finitely many } n \} \]

Using Proposition 2 iii) and Proposition 7 iv), it follows that \( H' \) is a dense subspace of \( H \). Let \( \xi \in H' \cap H_{+} \). Define the following operator, \( \tilde{x} \) on \( H' \)

\[ \tilde{x}(w_{0}^{n}\xi) = (w_{0}^{k})^{n}x\xi, n > 0 \]

Then, \( \tilde{x} \) is well defined. Indeed if \( \xi \in H_{+} \) and \( \xi' \in H_{+} \) are such that \((w_{0}^{n})^{n}\xi = (w_{0}^{n})^{k}\xi' \) for \( n, k \in \mathbb{Z}, \) say \( n \leq k \), then \( (w_{0}^{n})^{n-k}\xi = (w_{0}^{n})^{k-n}\xi = \xi' \). Since \((w_{0}^{n})^{k-n} \in M_{+}, \) and \( x \) commutes with \( M_{+} \) we have

\[ (w_{0}^{n})^{n-k}x\xi = (w_{0}^{n})^{k-n}x(\xi) = x(0)^{k-n}\xi = x\xi' \]

Therefore,

\[ (w_{0}^{n})^{n}\xi = (w_{0}^{n})^{k}\xi' \]
so

$$\hat{x}((w_0^*)^n \xi) = \hat{x}((w_0^*)^k \xi')$$

Next we prove that the operator $\hat{x}$ is bounded. Indeed

$$\|\hat{x}((w_0^*)^n \xi)\| = \|((w_0^*)^n x)\xi\| = \|x\xi\| \leq \|x\| \|\xi\| = \|x\| \|((w_0^*)^n \xi)\|$$

Since $w_0 \in M'$, it is straightforward to prove that $\hat{x} \in M'$ and $p_+ \hat{x} p_+ = x$. Thus $x \in (M')_+$ and we are done

ii) By replacing $M$ with $M'$ in i), and using Proposition 2, iii), the result follows

For every $\lambda \in \mathbb{C}, |\lambda| < 1$, let $K_\lambda = \left\{x(\lambda,b) = \sum_{n \geq 0} \lambda^nw_0^nb | b \in (M')_0 \right\}$ and $K = \overline{lin\{K_\lambda | \lambda \in \mathbb{C}, |\lambda| < 1\}}$, where the closure is in the norm topology of $M'$.

**Lemma 10** With the above notations, $(M')_0 \subset K$ and $w_0 \in K$.

**Proof.** Clearly, for $\lambda = 0$, we get $K_0 = (M')_0 \subset K$. We will prove now that $w_0^n \in K$ for every $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$ and let $0 < \epsilon < 1$. Denote by $\lambda_0, \lambda_1, ..., \lambda_n$ the complex roots of the equation $\lambda^{n+1} = \epsilon^{n+1}$. Let $\{\mu_0, \mu_1, ..., \mu_n\}$ be the solution of the following system of linear equations:

$$\sum_{i=0}^{i=n} \mu_i \lambda_i^j = \delta_{jn}, j = 0, 1, ..., n$$

where $\delta_{jn}$ is the Kronecker symbol. Using Cramer’s rule, we get

$$\mu_i = \frac{\pm 1}{\prod_{k \neq i} (\lambda_k - \lambda_i) \prod_{k \geq i} (\lambda_i - \lambda_k)}$$

The choices of $\{\lambda_i\}$ and $\{\mu_i\}$ imply that

$$\sum_{i=0}^{n} \mu_ix(\lambda_i,1) = w_0^n + \sum_{j=n+1}^{\infty} \left(\sum_{i=0}^{n} \mu_i \lambda_i^j\right)w_0^j.$$

Furthermore, for $i \neq k, k \geq 1$ we have

$$|\lambda_i - \lambda_k| \geq |\lambda_k - \lambda_k| = |\lambda_k - \lambda_k| = 2\epsilon \sin \frac{\pi}{n+1}$$

Therefore

$$|\mu_i| = \frac{1}{2^n \epsilon^n (\sin \frac{\pi}{n+1})^n}, i = 0, 1, ..., n$$

Thus

$$\left|\sum_{i=0}^{n} \mu_i \lambda_i^j\right| \leq \frac{\epsilon^j}{2^n \epsilon^n (\sin \frac{\pi}{n+1})^n}, j \geq n + 1$$
It follows that
\[ \left\| \sum_{i=0}^{n} \mu_i x(\lambda_i, 1) - w_0^n \right\| \leq \sum_{j=n+1}^{\infty} (\sum_{i=0}^{n} \mu_i \lambda_i^j) w_0^j \leq \sum_{j=n+1}^{\infty} 2^n \epsilon^n (\sin \frac{\pi}{n+1})^n = 2^n (\sin \frac{\pi}{n+1})^n (1 - \epsilon) \]

Hence \( w_0 \in K \) and we are done. \( \blacksquare \)

For \( \lambda \in \mathbb{C}, |\lambda| < 1 \), denote
\[ \tilde{H}_+^\lambda = \left\{ \sum_{n=0}^{\infty} \lambda^j w_0^j \xi \in H_0 \right\} \]

**Corollary 11** The linear subspace, \( \text{lin} \left\{ \tilde{H}_+^\lambda | \lambda \in \mathbb{C}, |\lambda| < 1 \right\} \), spanned by \( \left\{ \tilde{H}_+^\lambda | \lambda \in \mathbb{C}, |\lambda| < 1 \right\} \)
is dense in \( H_+ \).

**Proof.** By Proposition 2 ii), \( H_0 = (M')_0 \xi_0 \) and therefore, by Proposition 7 iv), \( H_{k \xi_0} = w_0^k H_0 \). The result follows from the above lemma. \( \blacksquare \)

We can give now the

**Proof.** of Theorem 8. By Proposition 2 iii), \( (M')_0 p_+ \subset B(H_+) \). Since \( v_0 \in M_{n_0} \subset M \), it follows that \( v_0 e = ev_0 \) for every projection \( e \in (M')_0 \). Therefore, for every projection \( e \in (M')_0, eH_+ \in \text{Lat}(M_+) \). Let \( x \in \text{algLat}(M_+) \). Then, in particular, \( xe = ex \) for every projection \( e \in (M')_0 \), hence, \( x^* e = e x^* \) for every projection \( e \in (M')_0 \), where \( x^* \) is the adjoint of \( x \) in \( B(H_+) \). It is clear that \( x^* \in \text{algLat}(M^*_+) \), where \( M^*_+ \subset B(H_+) \) is the algebra of the adjoint elements of \( M_+ \subset B(H_+) \). On the other hand, since \( v_0 e = ev_0 \) for all projections \( e \in (M')_0 \), we have that \( v_0^* e = e v_0^* \) for \( e \in (M')_0 \) and \( v_0^* \in M_{-n_0} \subset B(H) \). If we denote by \( S_{v_0} \) the operator \( v_0 \) on \( H_+ \), we have
\[ S_{v_0}^* (\xi) = 0 \text{ if } \xi \in H_0 \text{ and} \]
\[ S_{v_0}^* (\xi) = v_0^* \xi \text{ if } \xi \in H_n, n > 0 \]

It follows that \( S_{v_0}^* e = e S_{v_0}^* \) for all \( e \in (M')_0 \). Hence \( eH_+ \in \text{Lat}(M^*_+) \) for all \( e \in (M')_0 \). On the other hand, by the definition of \( w_0 \), we have \( v_0 w_0 = w_0 v_0 \). Thus
\[ S_{v_0}^* S_{w_0}^* = S_{w_0}^* S_{v_0}^* \text{ and } a S_{w_0}^* = S_{w_0}^* a, a \in M_0 \]

If we denote
\[ H^\lambda_+ = \{ \zeta \in H_+ | S_{w_0}^* \zeta = \lambda \zeta \} \]

it is clear that
\[ \tilde{H}_+^\lambda \subset H^\lambda_+ \text{ for all } \lambda \in \mathbb{C}, |\lambda| < 1 \]

Since for every \( \lambda \in \mathbb{C}, |\lambda| < 1 \), we have \( H^\lambda_+ \in \text{Lat}(M^*_+) \), where \( H^\lambda_+ \) denotes, as above, the closed subspace of all eigenvectors of \( S_{w_0}^* \) corresponding to the
eigenvalue $\lambda$, and $x^* \in \text{algLat}(M_+^1)$, it follows that $x^*H_+^\lambda \subset H_+^\lambda$, for all $\lambda \in \mathbb{C}, |\lambda| < 1$. We will prove next that

$$x^*S_{w_0}^* = S_{w_0}^* x^*$$

and then apply Lemma 9 to conclude that $x \in M_+$. Since $x^*H_+^\lambda \subset H_+^\lambda$, it immediately follows that $x^*S_{w_0}^* = S_{w_0}^* x^*$ on $H_+^\lambda$. By Corollary 12, the linear span of $\{\tilde{H}_+^\lambda|\lambda \in \mathbb{C}, |\lambda| < 1\}$ is dense in $H_+$. Since $\tilde{H}_+^\lambda \subset H_+^\lambda$, we have that the linear span of $\{H_+^\lambda|\lambda \in \mathbb{C}, |\lambda| < 1\}$ is dense in $H_+$. Therefore $x^*S_{w_0}^* = S_{w_0}^* x^*$ on $H_+$. Hence $x \in (M'_+)'$. From Lemma 9 ii) it follows that $x \in M_+$ and we are done. \[\square\]

**Remark 12** Let $A$ be a weakly closed algebra of operators on the Hilbert space $X$. Suppose that there exists a family of mutually orthogonal projections $\{q_i\} \subset A$ such that:

i) $q_i$ commutes with $A$ for every $i$

ii) $q_iAq_i \subset B(q_iX)$ is reflexive for every $i$ and

iii) $\sum q_i = I_X$ where $I_X$ is the identity of $B(X)$

Then $A \subset B(X)$ is reflexive.

**Proof.** Let $x \in \text{algLat}(A)$. Since $q_i, 1 - q_i \in \text{Lat}(A)$, we have $xq_i = q_i xq_i \in \text{algLat}(q_iAq_i)$. The reflexivity of $q_iAq_i$ implies that $xq_i, q_iAq_i = Aq_i$ for all $i$.

Therefore $x = \sum xq_i \in A$. \[\square\]

The above techniques allow us to prove the following

**Theorem 13** Let $(M, T, \alpha)$ be a $W^*$-dynamical system with $M$ abelian in the standard form considered above. Then $M_+$ is reflexive.

**Proof.** Let $n_0 = \min\{n \in sp(\alpha)|n > 0\}$. Let $v \in M_{n_0}$ be a partial isometry and $e = v^*v = vv^* \in M_0$. Then, $ve$ is an unitary element of $eM_{n_0}e \subset B(eH_+)$. By Theorem 8, we have that $eM_+e$ is a reflexive subalgebra of $B(eH_+)$. If $(1 - e)M_{n_0}(1 - e) \neq \{0\}$, then there is a partial isometry $u \in M_{n_0}$ such that $u^*u = uu^* \leq 1 - e$. Let $\{e_{\gamma}\}_{\gamma \in \Gamma} \subset M_0$ be a maximal family of orthogonal projections such that for every $\gamma \in \Gamma_1, e_{\gamma} = v_{\gamma} v_{\gamma}^* = v_{\gamma}^* v_{\gamma}$ for some partial isometry $v_{\gamma} \in M_{n_0}$. Set $f_1 = \sum_{\gamma \in \Gamma} e_{\gamma}$. Then, by Remark 12 $f_1M_+f_1$ is a reflexive subalgebra of $B(f_1H_+)$ and $f_1M_{n_0}f_1 = \{0\}$. Suppose, by induction, that for every $k \in \mathbb{N}, k \geq n_0$ there is a projection $f_k \in M_0, f_k \leq 1 - f_{k-1}$, such that

$$f_kM_+f_k \text{ is a reflexive subalgebra of } B(f_kH_+) \text{ and } f_kM_i f_k = \{0\} \text{ for all } i, n_0 \leq i \leq k$$

Then, as in the step $k = n_0$, we can find $f_{k+1} \leq 1 - \sum_{i=n_0+1}^{k} f_i$ such that

$$f_{k+1}M_+f_{k+1} \text{ is a reflexive subalgebra of } B(f_{k+1}H_+) \text{ and } f_{k+1}M_i f_{k+1} = \{0\} \text{ for all } i, n_0 \leq i \leq k + 1$$

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Let $f = \sum_{i \geq n_0} f_i$. Then, applying again Remark 12 it follows that $fM_+ f$ is a reflexive subalgebra of $B(fH_+)$ and $fM_i f = \{0\}$ for all $i \in \mathbb{N}$. Therefore,

$$(1 - f)M_+ (1 - f) = (1 - f)M_0 (1 - f)$$

which is a von Neumann subalgebra of $B((1 - f)H_+)$ and is therefore reflexive. Thus, by Remark 12 we have that

$$M_+ = fM_+ \oplus (1 - f)M_+$$

is reflexive.

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