On Equivalence of Critical Collapse of Non-Abelian Fields

Piotr Bizoń\textsuperscript{1}, Tadeusz Chmaj\textsuperscript{2}, and Zbiślaw Tabor\textsuperscript{1}

\textsuperscript{1}Institute of Physics, Jagiellonian University, Cracow, Poland
\textsuperscript{2}Institute of Nuclear Physics, Cracow, Poland

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Abstract

We continue our study of the gravitational collapse of spherically symmetric skyrmions. For certain families of initial data, we find the discretely self-similar Type II critical transition characterized by the mass scaling exponent $\gamma \approx 0.20$ and the echoing period $\Delta \approx 0.74$. We argue that the coincidence of these critical exponents with those found previously in the Einstein-Yang-Mills model is not accidental but, in fact, the two models belong to the same universality class.

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This is a second paper in the series devoted to the study of critical gravitational collapse in the Einstein-Skyrme (ES) model. The main motivation for using this model is an attempt to understand the role of different length scales and stationary solutions in the dynamics of Einstein’s equations at the threshold of black hole formation. In the first paper [1] we showed that the presence of sphaleron solutions gives rise to Type I critical behavior for certain families of initial data. In particular, we found a new kind of first order phase transition in which subcritical data relax to the static gravitating skyrmion. Here, we focus our attention on Type II critical behavior. After providing the numerical evidence for the existence of such transition, we discuss the universality of critical behavior with respect to intrinsic scales present in the model. We show that the two length scales become gradually irrelevant on small spacetime scales near criticality and consequently the near-critical solutions can be asymptotically self-similar. Interestingly enough, the critical exponents characterizing the Type II transition, the mass-scaling exponent $\gamma \approx 0.20$ and the echoing period $\Delta \approx 0.74$, agree (within the numerical error limits) with those previously found in the Einstein-Yang-Mills (EYM) model [2]. We argue that this fact is a reflection of equivalence of Type II critical transitions between the EYM and the ES models, that is the critical solutions in both models are asymptotically identical. To our knowledge, this is the first example of universality of critical collapse across two physically fundamentally different systems (the universality classes observed previously in collapse simulations comprised models differing only by potential terms in a lagrangian).

For this paper to be self-contained let us briefly recall the setup for the spherically symmetric ES model we used in [1]. Adopting the polar time slicing and the areal radial coordinate, the spacetime metric can be written as

$$ds^2 = -N(r,t)e^{-2\delta(r,t)}dt^2 + N^{-1}(r,t)dr^2 + r^2d\Omega^2.$$  \hfill (1)

As a matter source we take the $SU(2)$-valued scalar field $U(x)$ (called the chiral field) and assume the hedgehog ansatz $U = exp(i\vec{\sigma} \cdot \hat{r} F(r,t))$, where $\vec{\sigma}$ is the vector of Pauli matrices. The components of stress-energy tensor $T_{ab}$ expressed in the orthonormal frame determined
by the metric (1) are

$$T_{00} = \frac{u}{2r^2} (NF'^2 + N^{-1}e^{2\delta}F'^2) + \frac{\sin^2 F}{r^2} \left( f^2 + \frac{\sin^2 F}{2e^2r^2} \right),$$

(2)

$$T_{11} = \frac{u}{2r^2} (NF'^2 + N^{-1}e^{2\delta}F'^2) - \frac{\sin^2 F}{r^2} \left( f^2 + \frac{\sin^2 F}{2e^2r^2} \right),$$

(3)

$$T_{01} = \frac{u}{r^2} e^\delta \dot{F} F',$$

(4)

where overdots and primes denote $\partial/\partial t$ and $\partial/\partial r$ respectively, and $u = f^2r^2 + \frac{2}{e^2}\sin^2 F$. The two coupling constants $f^2$ and $e^2$ have dimensions: $[f^2] = ML^{-1}$ and $[e^2] = M^{-1}L^{-1}$ (we use units in which $c = 1$). In order to write the evolution equations in the first order form, we define an auxiliary variable $P = ue^{\delta}N^{-1}\dot{F}$. Then, the full set of ES equations is

$$\dot{F} = e^{-\delta}N\frac{P}{u},$$

(5)

$$\dot{P} = (e^{-\delta}NuF')' - \sin(2F)e^{-\delta} \left[ f^2 + \frac{1}{e^2} \left( NF'^2 - N\frac{P^2}{u^2} + \frac{\sin^2 F}{r^2} \right) \right],$$

(6)

$$N' = \frac{1 - N}{r} - 8\pi Gr T_{00},$$

(7)

$$\dot{N} = -8\pi Gre^{-\delta}N T_{01},$$

(8)

$$\delta' = -4\pi Gr N^{-1}(T_{00} + T_{11}).$$

(9)

In order to make an identification of certain terms in the equations easier, all the coupling constants are displayed, however we remind that, apart from the overall scale, solutions depend on these constants only through the dimensionless parameter $\alpha = 4\pi Gf^2$. We solve Eqs. (5-9) for regular asymptotically flat initial data. The condition of regularity at the center $N(r, t) = 1 + O(r^2)$ is ensured by the boundary condition $F(r, t) = O(r)$ for $r \to 0$. The asymptotic flatness of initial data $N(r, 0) = 1 + O(1/r)$ for $r \to \infty$ is guaranteed by the initial condition $F(r, 0) = B\pi + O(1/r^2)$, where the integer $B$, usually referred to as the baryon number, is the topological degree of the chiral field. Since the baryon number is dynamically preserved, the Cauchy problem falls into infinitely many superselection sectors labelled by $B$ - here we concentrate on the $B = 0$ sector. A typical one-parameter family of
initial data in this class, interpolating between black-hole and no-black-hole spacetimes, is
an initially incoming “Gaussian”
\[
F(r, 0) = A r^3 \exp \left[ - \left( \frac{r - r_0}{s} \right)^4 \right],
\]
where one of the parameters \(A, s,\) or \(r_0\) is varied (hereafter this parameter is denoted by \(p\)) while the others are fixed. As usually, the critical value \(p^*\) is located by performing a bisecting search in \(p\). In order to get into close proximity of \(p^*\) we have implemented an adaptive mesh refinement algorithm. This code was essential in probing the critical region with sufficient resolution. Our numerical results demonstrate the existence of Type II critical transition with its two main characteristic features, first observed by Choptuik in the massless scalar field collapse [3], namely:

- **Mass scaling:** For supercritical data, the final black hole mass scales as \(m_{BH} \sim C|p - p^*|^\gamma\) with the exponent \(\gamma \approx 0.20\). As shown in Fig. 1 this scaling law holds over two orders of magnitude of mass.

- **Echoing:** For near-critical data, the solutions approach (for sufficiently small \(r\)) a certain universal intermediate attractor which is discretely self-similar with the echoing period \(\Delta \approx 0.74\). This is illustrated in Fig. 2.

We find that the critical exponents, \(\gamma\) and \(\Delta\), are universal not only with respect to initial data, but also with respect to the parameter \(\alpha\). The universality of Type II critical collapse with respect to a dimensionless parameter is a rare phenomenon which can occur only if the parameter enters evolution equations through terms which become “irrelevant” near criticality (see the discussion of this issue in Gundlach’s review [5]). Now, we would like to show that this is exactly what happens for \(\alpha\) in the ES equations. Along the way, we will see that discrete self-similarity is compatible with the presence of two length scales in the model. Our basic assumption is that \(F(r, t)/\sqrt{r}\) is an echoing quantity (scaling variable). This assumption is not only justified empirically (see Fig. 2) but, as follows from a simple dimensional analysis of Einstein’s equations (7-9), it seems to be the only possibility.
compatible with the discrete self-similarity of metric functions \( N \) and \( \delta \). We introduce a unit of length \( L = \sqrt{4\pi G/e} \) and dimensionless coordinates

\[
\tau = \ln \left( \frac{t^* - t}{L} \right) \quad \text{and} \quad \xi = \ln \left( \frac{r}{t^* - t} \right) ,
\]

where \( t^* \) is the accumulation time of the infinite number of echos. We also define new dimensionless variables

\[
\Phi(\tau, \xi) = \frac{F(r, t)}{\sqrt{r/L}}, \quad Z(\tau, \xi) = \sqrt{rL} F'(r, t), \quad \Pi(\tau, \xi) = \sqrt{rL} e^\delta N^{-1} \dot{F}(r, t).
\]

By assumption \( \Phi, Z, \) and \( \Pi \) are the scaling variables, that is they are asymptotically periodic in \( \tau \): \( \Phi(\tau, \xi) \approx \Phi(\tau + \Delta, \xi) \) etc. for large negative \( \tau \) and fixed \( \xi \). Rewriting Eqs.(5-9) in these new variables, we find that \( \alpha \) is always multiplied by \( e^\tau \), and the only other terms depending explicitly on \( \tau \) appear through the combination

\[
X = e^{-\frac{1}{2}(\tau + \xi)} \sin \left( e^{\frac{1}{2}(\tau + \xi)} \Phi \right) .
\]

For example, the hamiltonian constraint (7) takes the form

\[
1 - N - \frac{\partial N}{\partial \xi} = N(\alpha e^{\tau + \xi} + 2X^2)(Z^2 + \Pi^2) + X^2(2\alpha e^{\tau + \xi} + X^2).
\]

In the limit \( \tau \to -\infty \), the terms containing \( \alpha \) become negligible (“irrelevant” in the language of renormalization group theory), and therefore the critical behavior does not depend on \( \alpha \).

In the same limit, \( X \to \Phi \), so the equations become asymptotically autonomous in \( \tau \), and thereby scale invariant. The equivalence of Type II critical behavior between ES and EYM models is an immediate consequence of the universality with respect to \( \alpha \), because, as we pointed out in [4], for \( \alpha = 0 \) Eqs.(5-9) reduce (after the substitution \( w = \cos F \)) to the EYM equations. This means that the EYM critical solution constructed by Gundlach [6] is valid (to the leading order) in the ES case as well, and hence the critical exponents in both models are the same. As mentioned above, this theoretical retrodiction is confirmed numerically.

We conclude with two remarks concerning possible extensions of the research presented here and in [1]. First, we would like to point out that by setting all terms in Eqs.(5-9)
containing the coupling constant \( e^2 \) (the Skyrme terms) to zero, one obtains the \( \sigma \)-model coupled to gravity. This model is scale invariant, so it does not admit a Type I transition but a Type II transition is expected to exist. If so, the natural scaling variables will be

\[
\tilde{\Phi}(\tau, \xi) = F(r, t), \quad \tilde{Z}(\tau, \xi) = rF'(r, t), \quad \tilde{\Pi}(\tau, \xi) = re^\delta N^{-1}\dot{F}(r, t). \tag{15}
\]

It is easy to check that in this case the terms proportional to \( \alpha \) are not irrelevant. For example the analogue of Eq. (14) is

\[
1 - N - \frac{\partial N}{\partial \xi} = \alpha N(\tilde{Z}^2 + \tilde{\Pi}^2) + 2\alpha \sin^2 \tilde{\Phi}. \tag{16}
\]

Therefore, the critical solution, and \textit{eo ipso} the critical exponents, are anticipated to depend strongly on \( \alpha \). We have been informed that the group of researchers led by Peter Aichelburg is in the process of investigating this and related problems \footnote{\textsuperscript{[7]}}. If their results confirm our expectation, the universality with respect to \( \alpha \) in the ES model could be interpreted as another nonperturbative effect of the Skyrme correction to the \( \sigma \)-model.

Second, we recall that, in contrast to Type II, the Type I critical transition in the ES model is manifestly nonuniversal with respect to \( \alpha \) because the critical solution (the sphaleron) changes with \( \alpha \), in particular, it exists only for sufficiently small \( \alpha \). Thus, for large \( \alpha \) only Type II behavior is possible, while for small \( \alpha \) the two types of critical behavior can coexist. In the latter case, one can anticipate the existence of crossover effects at the border of basins of attractions of Type I and Type II critical solutions. We leave an investigation of these fascinating effects to other researchers who, as we have recently accidentally found out \footnote{\textsuperscript{[8]}}\footnote{\textsuperscript{[9]}}, are also interested in the ES model. We would like to emphasize that a full description of an extremely rich phenomenology of the ES model is not a \textit{per se} goal our studies – for us this model is only a testing ground for addressing certain issues of the dynamics of Einstein’s equations in the presence of intrinsic scales and stationary (stable and unstable) solutions.

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FIG. 1. The black hole mass scaling. The logarithm of black hole mass $M_{BH}$ is plotted versus the logarithmic distance from criticality $\ln |p-p^*|$ for supercritical solutions generated from initial data (10) for $\alpha = 0.02$. The power-law fit is indicated by the solid line with the slope $\gamma \approx 0.20$. 

\[
\ln(M_{BH}) = A + \gamma \ln|p-p^*|
\]

$\gamma \approx 0.20$
FIG. 2. Numerical evidence for echoing. For a marginally critical solution generated from initial data (10) for $\alpha = 0.02$ we plot the profile of $\Phi = \frac{F}{\sqrt{r}}$ versus $\rho = \ln(r)$ for some arbitrary late time $t_1$ and superimpose the profile of the first echo at time $t_2$ shifted by $\rho \rightarrow \rho + \Delta_\rho$. The time $t_2$ and the radial echoing period $\Delta_\rho$ are chosen to minimize the mean square difference between the two profiles. By repeating this calculation for a sequence of pairs $(t_1, t_2)$, we estimated the temporal echoing period $\Delta_\tau$ from the slope of the line $t_2 = t^*(1 - e^{-\Delta_\tau}) + e^{-\Delta_\tau}t_1$, confirming the expectation that $\Delta_\rho = \Delta_\tau$. 

$\Delta_\rho \approx 0.74$