Complete Solution of $SU(2)$ Chern-Simons Theory

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Abstract

Explicit and complete topological solution of $SU(2)$ Chern-Simons theory on $S^3$ is presented.

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Chern-Simons theories have been of immense interest in recent times. These find application in many areas of physics. Pure Chern-Simons theories are also closely related to knots and links. Knot theory is relevant, for example, in physics of polymers as well as to the study of some properties of biological molecules.

Chern-Simons action is given by

$$kS = \frac{k}{4\pi} \int_{S^3} tr(AdA + \frac{2}{3} A^3)$$

where A is the matrix valued connection one-form of gauge group $SU(2)$. The topological operators of this theory are the Wilson link operators. For a link L made of oriented knots $(C_1, C_2, ..., C_s)$ carrying $SU(2)$ spins $j_1, j_2, ..., j_s$ on them respectively, the Wilson link operator is

$$W_{j_1 j_2 ... j_s}[L] = \prod_{i=1}^{s} tr_{j_i} P \exp \oint_{C_i} A$$

Here the one-form A carries matrix representations corresponding to respective spin $j_i$ on the knot $C_i$. We are interested in the expectation value of these operators:

$$V_{j_1 j_2 ... j_s}[L] = Z^{-1} \int [dA] W_{j_1 j_2 ... j_s}[L] e^{ikS}, \quad Z = \int [dA] e^{ikS}$$

The partition function is given by $^4$:

$$Z = \left[\frac{2}{k+2}\right]^{1/2} \sin(\pi/(k+2))$$

Placing doublet representations on all components, Witten has also shown that these link invariants satisfy the same skein relations as those by Jones polynomials. These skein relations can be recursively solved to obtain Jones polynomials for an arbitrary link. Generalization of skein relations to the cases other than spin half have also been obtained. But unfortunately these cannot be solved recursively to obtain the expectation values (3) for an arbitrary link. In refs. we developed a direct method to obtain such invariants for links which
are related to 4-strand braids. But links which cannot be so constructed stayed elusive. In the following we present a method which is applicable to arbitrary links in $S^3$. To do so we shall make use of theory of coloured-oriented braids and the duality properties of correlators of $SU(2)_k$ Wess-Zumino conformal field theory on an $S^2$.

An $n$-braid is a collection of $n$ non-intersecting strands connecting two parallel rigid rods. When the strands are oriented and coloured, we have coloured-oriented braids. We shall colour the strands by putting $SU(2)$ spins on them. A general braid can thus be specified by giving $n$ assignments, $\hat{j}_i = (j_i, \epsilon_i)$ representing the spin $j_i$ and orientation $\epsilon_i (= \pm 1$ for the strand going into or away from the rod $)$ on the $n$ points where the strands meet the upper rod and also $n$ spin-orientation assignments $\hat{l}_i = (l_i, \eta_i)$ on the $n$ points on the lower rod. This braid will be represented symbolically as $B_n(\hat{j}_1 \hat{j}_2 ... \hat{j}_n \hat{j}_1^* \hat{j}_2^* ... \hat{j}_n^*)$. For a given spin-orientation assignment $\hat{j}_i = (j_i, \epsilon_i)$, we define the conjugate assignment as $\hat{j}_i^* = (j_i, -\epsilon_i)$. Then assignments $(\hat{l}_i)$ are just a permutation of $(\hat{j}_i^*)$.

An arbitrary $n$-braid can be generated by applying braiding generators $B_i, i = 1, 2, ... n - 1$ on the “identity” braids $I_n(\hat{j}_1 \hat{j}_1^* \hat{j}_2 \hat{j}_2^* ... \hat{j}_n \hat{j}_n^*)$. These are represented in fig. 1. There are more than one identity braid. Further, composition between two braids is defined only if there is colour-orientation matching along the rods that are merged. These braids form a groupoid instead of a group. The generators, however, still satisfy the same generating relations as for the ordinary braids. Also platting construction of ordinary braids\textsuperscript{10} can also be extended to coloured-oriented braids. Consider a $2m$-braid with special spin-orientation assignments as $B_{2m}(\hat{j}_1 \hat{j}_1^* \hat{j}_2 \hat{j}_2^* ... \hat{j}_m \hat{j}_m^*)$. Platting then constitutes of pairwise joining the the successive strands $(2i - 1, 2i), i = 1, 2, ... m$ from above and below the two
rods. There is a theorem due to Birman for ordinary braids which relates links with plats.\textsuperscript{10}. This theorem can obviously also be stated for our coloured-oriented braids:

**Proposition 1**: A coloured-oriented knot or link can be represented (though not uniquely) by a platting of an oriented-coloured braid $B_{2m}\left(\hat{j}_1 \hat{l}_1 \hat{j}_1^* \hat{l}_1^* \ldots \hat{j}_m \hat{l}_m \hat{j}_m^* \hat{l}_m^*\right)$.

In addition to this property, we need one more ingredient for our discussion. This is the duality properties of correlators of $SU(2)_k$ Wess-Zumino conformal field theory on an $S^2$. For example, the four-point correlators for primary fields with spins $j_1, j_2, j_3, j_4$ (such that these four spins form a singlet) can be represented in three different equivalent ways. Two such ways are shown in figs. 2(a, b). The trivalent points in these diagrams all satisfy fusion rules which for sufficiently large $k$ are same as the triangle conditions for $SU(2)$ spins. These two linearly independent sets of conformal blocks, $\phi_j(j_1j_2j_3j_4)$ and $\phi'_l(j_1j_2j_3j_4)$ are related by duality\textsuperscript{11,7}:

$$\phi_j(j_1j_2j_3j_4) = \sum_l a_{jl} \left[\begin{array}{ccc} j_1 & j_2 & j_3 \\ j_3 & j_4 & \end{array}\right] \phi'_l(j_1j_2j_3j_4)$$ (4)

where duality matrices are explicitly given in terms of $SU(2)$ quantum Racah coefficients\textsuperscript{12}.

$$a_{jl} \left[\begin{array}{ccc} j_1 & j_2 & j_3 \\ j_3 & j_4 & \end{array}\right] = (-)^{j+l+min(j_1+j_3,j_2+j_4)} \sqrt{[2j+1][2l+1]} \left[\begin{array}{ccc} j_1 & j_2 & j_3 \\ j_3 & j_4 & \end{array}\right]$$ (5a)

$$\begin{array}{c}
\pmatrix{j_1 & j_2 & j_3 \\ j_3 & j_4 & j_23} = \triangle(j_1j_2j_12) \triangle(j_3j_4j_12) \triangle(j_1j_4j_23) \triangle(j_3j_2j_23)
\times \sum_{m \geq 0} (-)^m [m+1]!![m-j_1-j_2-j_12]!![m-j_3-j_4-j_12]!!
\times [m-j_1-j_4-j_23]!![m-j_3-j_2-j_23]!![j_1+j_2+j_3+j_4-m]!!
\times [j_1+j_3+j_12+j_23-m]!![j_2+j_4+j_12+j_23-m]!!\end{array}^{-1}$$ (5b)
\[
\triangle (abc) = \sqrt{\frac{[-a + b + c]![a - b + c]![a + b - c]!}{[a + b + c + 1]!}}
\]  

(5c)

Here the square brackets represent the q-numbers, 

\[
[x] = (q^{x/2} - q^{-x/2})/(q^{1/2} - q^{-1/2})
\]  

with \( q = \exp(2\pi i/(k + 2)) \).

This duality relation can be extended to arbitrary 2m-point correlators. In particular we shall be interested in the correlators \( \phi_{(p; r)}(j_1...j_{2m}) \) and \( \phi'_{(q; s)}(j_1...j_{2m}) \) represented in figs. 3(a, b) for primary fields carrying spins \( j_1j_2...j_{2m} \). We have used a compact notation for spins on the internal lines in these diagrams as \( p = (p_0p_1...p_{m-1}) \), \( r = (r_1r_2...r_{m-3}) \) and similarly \( q = (q_0q_1...q_{m-1}) \), \( s = (s_1s_2...s_{m-3}) \). These two figures represent two equivalent ways of combining 2m spins \( j_1j_2...j_{2m} \) into singlets and are related by duality:

\[
\phi'_{(q; s)}(j_1j_2...j_{2m}) = \sum_{(p; r)} a_{(p; r)(q; s)} \begin{bmatrix}
    j_1 & j_2 \\
    j_3 & j_4 \\
    \vdots & \vdots \\
    j_{2m-1} & j_{2m}
\end{bmatrix} \phi_{(p; r)}(j_1j_2...j_{2m})
\]  

(6a)

The duality matrices here are given in terms of those given in eqns.5 for four-point correlators:

\[
a_{(p; r)(q; s)} \begin{bmatrix}
    j_1 & j_2 \\
    \vdots & \vdots \\
    j_{2m-1} & j_{2m}
\end{bmatrix} = \sum_{t_1, t_2, t_{m-2}} \prod_{i=1}^{m-2} (a_{t_i, p_i} \begin{bmatrix}
    r_{i-1} & j_{2i+1} \\
    j_{2i+2} & r_i
\end{bmatrix} a_{t_i, s_{i-1}} \begin{bmatrix}
    t_{i-1} & q_i \\
    s_i & j_{2m}
\end{bmatrix})
\]  

\times \prod_{l=0}^{m-2} a_{r_lq_{l+1}} \begin{bmatrix}
    t_l & j_{2l+2} \\
    j_{2l+3} & t_{l+1}
\end{bmatrix}
\]  

(6b)

where \( r_0 \equiv p_0, r_{m-2} \equiv p_{m-1}, t_0 \equiv j_1, t_{m-1} \equiv j_{2m}, s_0 \equiv q_0, s_{m-2} \equiv q_{m-1} \) and \( \tilde{j}_1 + \tilde{j}_1 + ... + \tilde{j}_{2m-1} = \tilde{j}_{2m} \) and spins meeting at the trivalent points in figs.3 satisfy the triangle relations. The proof of this statement can be developed by repeated applications of the duality transformation involving four points given by eq.(4).
Now we are in a position to develop the solution of the SU(2) Chern-Simons theory. Let us consider an $S^3$ from which two 3-balls have been removed. This is a manifold with two boundaries, each an $S^2$. Let $2m (m = 1, 2, 3, \ldots)$ Wilson lines carrying spin-orientations $\hat{j}_1 \hat{j}_2 \ldots \hat{j}_{2m}$ without any entanglements connect them as shown in fig.4(a). Thus we have placed the identity braids $I_{2m}$ inside this manifold. The Chern-Simons functional integral over this manifold, following Witten$^4$, can be thought of as a state in the tensor product of the vector spaces, $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$, associated with the two boundaries. These vector spaces are related to the space of conformal blocks for $2m$-point correlators with spin assignments $(j_1 j_2 \ldots j_{2m})$ of the SU(2)$_k$ Wess-Zumino theory on these boundaries. Corresponding to the conformal blocks shown in figs. 3(a and b), we have two possible sets of basis vectors for each of these vector spaces, $| \phi_{(p;r)}(\hat{j}_1 \hat{j}_2 \ldots \hat{j}_{2m}) >$ and $| \phi'_{(q;s)}(\hat{j}_1 \hat{j}_2 \ldots \hat{j}_{2m}) >$ respectively. The corresponding bases for the dual vector spaces associated with oppositely oriented boundaries will be represented by $< \phi_{(p;r)}(\hat{j}_1 \hat{j}_2 \ldots \hat{j}_{2m}) |$ and $< \phi'_{(q;s)}(\hat{j}_1 \hat{j}_2 \ldots \hat{j}_{2m}) |$ respectively. Gluing two manifolds along two such oppositely oriented boundaries represents a natural product of the vectors. The bases vectors under this product are normalized so that

$$< \phi_{(p;r)}(\hat{j}_1 \hat{j}_2 \ldots \hat{j}_{2m}) | \phi_{(p';r')} (\hat{j}_1 \hat{j}_2 \ldots \hat{j}_{2m}) > = \delta_{(p,p')} \delta_{(r,r')}$$

$$< \phi'_{(q;s)} (\hat{j}_1 \hat{j}_2 \ldots \hat{j}_{2m}) | \phi'_{(q';s')} (\hat{j}_1 \hat{j}_2 \ldots \hat{j}_{2m}) > = \delta_{(q,q')} \delta_{(s,s')} \quad (7)$$

Further, the primed and unprimed bases for each of the vector spaces are related by the duality matrices of eqns.6:

$$| \phi'_{(q;s)} (\hat{j}_1 \hat{j}_2 \ldots \hat{j}_{2m}) > = \sum_{(p,r)} a_{(p;r)(q;s)} \begin{bmatrix} \hat{j}_1 \\ \hat{j}_2 \\ \vdots \\ \hat{j}_{2m} \end{bmatrix} | \phi_{(p;r)} (\hat{j}_1 \hat{j}_2 \ldots \hat{j}_{2m}) > \quad (8)$$
Now the Chern-Simons functional integral over the three-manifold of fig. 4(a) can thus be expanded in terms of one of these bases for each boundary as

\[ \nu_I = \sum_{(p,r)} \left| \phi^{(1)}_{(p,r)} (\hat{j}_1^* \hat{j}_2^* \ldots \hat{j}_{2m}) > \right| \left| \phi^{(2)}_{(p,r)} (\hat{j}_1 \hat{j}_2 \ldots \hat{j}_{2m}) > \right| \]  

(9)

Here we have put superscripts (1) and (2) explicitly on the basis vectors to indicate that these belong to the vector spaces \( \mathcal{H}^{(1)} \) and \( \mathcal{H}^{(2)} \) associated with the two boundaries respectively.

The conformal blocks shown in fig. 3 and therefore the corresponding bases \( |\phi_{(p,r)} > \) and \( |\phi'_{(q,s)} > \) are eigen functions of the odd-indexed braid generators \( B_{2l+1} \), \( l = 0, 1, \ldots, m - 1 \) and even-indexed generators, \( B_{2l}, l = 1, 2, \ldots, m - 1 \), respectively:

\[ B_{2l+1} |\phi_{(p,r)} (\hat{j}_1 \hat{j}_2 \ldots \hat{j}_{2l+1} \hat{j}_{2l+2} \ldots \hat{j}_{2m}) > = \lambda_{ql}(\hat{j}_{2l+1}, \hat{j}_{2l+2}) |\phi_{(p,r)} (\hat{j}_1 \hat{j}_2 \ldots \hat{j}_{2l+2} \hat{j}_{2l} \ldots \hat{j}_{2m}) > \]  

\[ B_{2l} |\phi'_{(q,s)} (\hat{j}_1 \hat{j}_2 \ldots \hat{j}_{2l} \hat{j}_{2l+1} \ldots \hat{j}_{2m}) > = \lambda_{ql}(\hat{j}_{2l}, \hat{j}_{2l+1}) |\phi'_{(q,s)} (\hat{j}_1 \hat{j}_2 \ldots \hat{j}_{2l+1} \hat{j}_{2l} \ldots \hat{j}_{2m}) > \]  

(10)

The eigenvalues of these half-twist matrices depend on the relative orientation of the strands involved \(^7,11\)

\[ \lambda_l(\hat{j}, \hat{j}') = \lambda_l^{(+)}(j, j') \equiv (-)^{j+j'-t} q^{(c_j+c_{j'})/2+c_{\min(j,j')}-c_t/2} \quad if \quad \epsilon \epsilon' = +1 \]

\[ = (\lambda_l^{(-)}(j, j'))^{-1} \equiv (-)^{j-j'}-t q^{c_j-c_{j'}|/2-c_t/2} \quad if \quad \epsilon \epsilon' = -1 \]  

(11)

where \( c_j = j(j+1) \).

These braiding generators can be applied to the identity braids of fig.(4a) to obtain a general braid inside the manifold. Thus the braid represented by the shaded box in fig.(4b)
can be written as a word $\mathcal{B}$ in terms of these generators. Using eqn.9 we can represent the functional integral over this three-manifold as:

$$\nu_{\mathcal{B}} = \sum_{(p;r)} |\phi_{(p;r)}^{(1)}>\mathcal{B}|\phi_{(p;r)}^{(2)}>$$  \hspace{1cm} (12)$$

To plat this braid consider the Chern-Simons functional integral over the ball shown in fig. 4c. This functional integral can again be represented by a vector in the vector space associated with the boundary. It is proportional to the basis vector $|\phi_{(0,0)}(\hat{j}_1\hat{j}_1^*...\hat{j}_m\hat{j}_m^*)>$ which is the eigen function of the odd indexed braiding generators with eigenvalue 1. Further gluing two copies of this manifold onto each other along oppositely oriented boundaries yields $m$ untangled unknots. The invariant for these is simply the product of invariants for individual unknots$^7$, $[2j_i+1]$, $i = 1, 2...m$. Thus the functional integral (normalised by multiplying by $Z^{-1/2}$) for the 3-ball of fig. 4c is

$$\nu = (\prod_{i=1}^{m}[2j_i+1]^{1/2}) |\phi_{(0,0)}(\hat{j}_1\hat{j}_1^*...\hat{j}_m\hat{j}_m^*)>$$  \hspace{1cm} (13)$$

Now we are ready to plat the braid shown in fig. 4(b) by gluing it from above and below by two copies of 3-ball of fig. 4c along oppositely oriented boundaries with spin-orientation assignments matching properly. This leads us to our main result which we now state:

**Proposition 2**: The expectation value of a Wilson operator for a link $L$ obtained by platting an oriented-coloured $2m$-braid $\mathcal{B}_{2m}$ $\left(\hat{j}_1 \hat{j}_1^* \hat{j}_2 \hat{j}_2^* ... \hat{j}_m \hat{j}_m^*\right)$ represented by a word in terms of the braid generators is given by

$$V[L] = (\prod_{i=1}^{m}[2j_i+1]) <\phi_{(0,0)}^{(0)}(\hat{l}_1^*\hat{l}_1...\hat{l}_m^*\hat{l}_m)> |\mathcal{B}_{2m}\left(\hat{j}_1 \hat{j}_1^* \hat{j}_2 \hat{j}_2^* ... \hat{j}_m \hat{j}_m^*\right)|\phi_{(0,0)}^{(0)}(\hat{j}_1\hat{j}_1^*...\hat{j}_m\hat{j}_m^*)>$$  \hspace{1cm} (14)$$
Propositions 1 and 2 along with the fact that two bases $|\phi_{(p,r)}\rangle$ and $|\phi'_{(q,s)}\rangle$ are related by the duality matrices as given by eqns. (8) and (6b) above allow us to calculate explicitly the functional average (3) for any arbitrary link. This provides the complete topological solution of the $SU(2)$ Chern-Simons theory on $S^3$. The method has obvious generalization to other gauge groups as well as other three-manifolds.

Placing spin 1/2 representation on all the component knots of a link, gives us the Jones polynomial. Placing other representations on the knots yields a whole variety of new invariants. These invariants are more powerful than Jones invariant as these do distinguish knots which are represented by the same Jones polynomial. The discussion of these aspects as well as more elaborate details of the proofs above will be presented elsewhere.

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Figure Captions

Fig. 1. Identity braids and braid generators

Fig. 2. Duality transformation of 4-point correlators

Fig. 3. Two equivalent sets of conformal blocks for 2m-point correlators

Fig. 4. Functional integrals over manifolds with boundaries