Twisting functors for quantum group modules

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Abstract
We construct twisting functors for quantum group modules. First over the field $\mathbb{Q}(v)$ but later over any $\mathbb{Z}[v, v^{-1}]$-algebra. The main results in this paper are a rigorous definition of these functors, a proof that they satisfy braid relations and applications to Verma modules.

1 Introduction

Twisting functors was first introduced by S. Arkhipov (as a preprint in 2001 and published in [Ark04]). H. Andersen quantized the construction of twisting functors in [And03]. Each twisting functor $T_w$ is defined via a so called semi-regular bimodule $S_w^v$. By the definition in [And03] its right module structure is not clear. Our first goal is to demonstrate that $S_w^v$ is in fact a bimodule. We verify this by constructing an explicit isomorphism to an inductively defined right module. The calculations are in fact rather complicated and involve several manipulations with root vectors, see Section 2 below. At the same time these calculations will be essential in [Ped15a] and [Ped15b].

Once we have established the definition of the twisting functors we prove that they satisfy braid relations, see Proposition 3.11. In the ordinary (i.e. non-quantum) case the corresponding result was obtained by O. Khomenko and V. Mazorchuck in [KM05]. Our approach is similar but again the quantum case involves new difficulties, see Section 3. This section also contains an explicit proof of the fact that for the longest word $w_0 \in W$ the twisting functor $T_{w_0}$ takes a Verma module to its dual, see Theorem 3.9.

The above results have several applications in the representation theory of quantum group: They enable us to construct so called twisted Verma modules and Jantzen filtrations of (twisted) Verma modules with arbitrary (non-integral) weights and to derive the sum formula for these. In turn this simplifies the linkage principle in quantum category $O_q$, $q$ being a non-root of unity in an arbitrary field.

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1.2 Notation

In this paper we work with a quantum group over a semisimple Lie algebra $g$ defined as in [Jan96]. Let $\Phi$ (resp. $\Phi^+$ and $\Phi^-$) denote the roots (resp. positive and negative roots) and let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ denote the simple roots. The quantum group has generators $\{E_{\alpha}, F_{\alpha}, K_{\alpha} | \alpha \in \Pi\}$ with relations as found in [Jan96]. Let $\Lambda = \mathbb{Z}\Phi$ denote the root lattice. Let $(\alpha_{ij})$ be the cartan matrix for $g$ and let $(-\cdot\cdot)$ be the standard invariant bilinear form. At first we work with the quantum group $U_q(g)$ defined over $\mathbb{Q}(v)$ but later we will specialize to an arbitrary field and any nonzero $q$ in the field. This is done by considering Lusztig’s $A$-form $U_A$ where $A = \mathbb{Z}[v, v^{-1}]$, see Section 8. For any $A$-algebra $R$: $U_R = U_A \otimes_A R$. We will later need the automorphism $\omega$ of $U_a$ and the antipode $S$ defined as in [Jan96] along with the definition of quantum numbers $[\nu]_q$ and quantum binomial coefficients. We use the notation $E^{(r)} = \frac{E^r}{[r]!}$ and similarly for $F$. The Weyl group $W$ is generated by the simple reflections $s_i = s_{\alpha_i}$. As usual we define for a weight $\mu \in \Lambda$ the weight space $(U_v)_\mu := \{u \in U_v | K_{\alpha}u = v^{(\alpha)}(\mu)u \text{ for all } \alpha \in \Pi\}$. For a weight $\mu$, $K_\mu$ is defined as follows: $K_\mu = \prod_{i=1}^n K_\alpha_i$ if $\mu = \sum_{i=1}^n a_i \alpha_i$. There is a braid group action on the quantum group $U_v$ usually denoted by $T_s$, where $s_i$ is the reflection with respect to the simple root $\alpha_i$. In this paper we will reserve the $T$ for twisting functors so we will call this braid group action $R$ instead. That is we have automorphisms $R_{s_i}$ such that

\[
R_{s_i}E_{\alpha_i} = -F_{\alpha_i}K_{\alpha_i},
\]

\[
R_{s_i}E_{\alpha_j} = \begin{cases} 
(\alpha_{ij}) \neq 0 \text{ if } i \neq j & (\alpha_{ij}) \neq 0 \text{ if } i \neq j \end{cases}, \]

\[
R_{s_i}F_{\alpha_i} = -K_{\alpha_i}^{-1} E_{\alpha_i},
\]

\[
R_{s_i}F_{\alpha_j} = \begin{cases} 
(\alpha_{ij}) \neq 0 \text{ if } i \neq j & (\alpha_{ij}) \neq 0 \text{ if } i \neq j \end{cases}, \]

\[
R_{s_i}K_{\mu} = K_{s_\mu(\mu)},
\]

Our definition of braid operators follows the definition in [Jan96]. Note that this definition differs slightly from the definition in [Lus90] (cf. [Jan96] Warning 8.14).

The inverse to $R_{s_i}$ is given by

\[
R_{s_i}^{-1}E_{\alpha_i} = - K_{\alpha_i}^{-1} F_{\alpha_i},
\]

\[
R_{s_i}^{-1}E_{\alpha_j} = \begin{cases} 
(\alpha_{ij}) \neq 0 \text{ if } i \neq j & (\alpha_{ij}) \neq 0 \text{ if } i \neq j \end{cases}, \]

\[
R_{s_i}^{-1}F_{\alpha_i} = - E_{\alpha_i}K_{\alpha_i},
\]

\[
R_{s_i}^{-1}F_{\alpha_j} = \begin{cases} 
(\alpha_{ij}) \neq 0 \text{ if } i \neq j & (\alpha_{ij}) \neq 0 \text{ if } i \neq j \end{cases}, \]

\[
R_{s_i}^{-1}K_{\mu} = K_{s_\mu(\mu)}.
\]

For $w \in W$ with a reduced expression $s_{i_1} \cdots s_{i_r}$, $R_w$ is defined as $R_{s_{i_1}} \cdots R_{s_{i_r}}$. This is independent of the reduced expression of $w$. An important property of the braid operators is that if $\alpha_{i_1}, \alpha_{i_2} \in \Pi$ and $w(\alpha_{i_1}) = \alpha_{i_2}$ then $R_w(F_{\alpha_{i_1}}) = F_{\alpha_{i_2}}$. These properties are proved in Chapter 8 in [Jan96].
We define $\beta_j := s_{i_1} \cdots s_{i_j-1}(\alpha_i)$, $j = 1, \ldots, N$

In this way we get $\{\beta_1, \ldots, \beta_N\} = \Phi^+$. We could just as well have used the opposite reduced expression $v_0 = s_{i_r} \cdots s_{i_1}$. In the following we will sometimes use the numbering $s_{i_1} \cdots s_{i_N}$, and sometimes the numbering $s_{i_N} \cdots s_{i_1}$. Note that if $w = s_{i_1} \cdots s_{i_r}$ and we expand this to a reduced expression $s_{i_1} \cdots s_{i_r} s_{i_{r+1}} \cdots s_{i_N}$, we get $\{\beta_1, \ldots, \beta_N\} = \Phi^+ \cap w(\Phi^-)$. We can define ‘root vectors’ $F_{\beta_j}, j = 1, \ldots, N$

Note that this definition depends on the chosen reduced expression. For a different reduced expression we might get different root vectors. As mentioned above if $\beta \in \Pi$ then the root vector $F_{\beta}$ defined above is the same as the generator with the same notation (cf. e.g [Jan96, Proposition 8.20]) so the notation is not ambiguous in this case. Let $w \in W$ and let $s_{i_r} \cdots s_{i_1}$ be a reduced expression of $w$. Define $F_{\beta_j}$ by choosing a reduced expression $s_{i_1} \cdots s_{i_r} s_{i_{r+1}} \cdots s_{i_N}$ of $w_0$ starting with the reduced expression $s_{i_1} \cdots s_{i_r}$ of $w^{-1}$. We define a subspace $U_v^-(w)$ of $U_v^-$ as follows:

$$U_v^-(w) := \text{span}(v) \left\{ F_{\beta_1}^{\alpha_1} \cdots F_{\beta_r}^{\alpha_r} | a_j \in \mathbb{N} \right\}$$

where $F_{\beta_j} = R_{s_{i_1}} \cdots R_{s_{i_{j-1}}}(F_{\alpha_i})$ as before. The definition of $U_v^-(w)$ seems to depend on the reduced expression of $w$. But the subspace is independent of the chosen reduced expression. This is shown in [Jan96, Proposition 8.22]. We will show below that $U_v^-(w)$ is a subalgebra of $U_v^-$ and that

$$U_v^-(w) = \text{span}(v) \left\{ F_{\beta_r}^{\alpha_r} \cdots F_{\beta_1}^{\alpha_1} | a_j \in \mathbb{N} \right\}$$

For a subalgebra $N \subset U_v$ we define $N^* = \bigoplus_{\mu} N^*_\mu$ (i.e. the graded dual) with the action given by $(u f)(x) = f(u x)$ for $u \in U_v, f \in N^*, x \in N$. We define ‘the semiregular bimodule’ $S^w_v := U_v \otimes_{U_v^-} U_v^-(w)^*$. Proving that this is a $U_v$-bimodule will be the first main result of this paper. We will show that there exists a right module structure on $S^w_v$ such that as a right module $S^w_v$ is isomorphic to $U_v^-(w)^* \otimes_{U_v^-} U_v$.

2 Calculations with root vectors

For use in the theorem below define:

**Definition 2.1** Let $A = \mathbb{Z}[v, v^{-1}]$ and let $A'$ be the localization of $A$ in $[2]$ (and/or $[3]$) if the Lie algebra contains any $B_n, C_n$ or $F_4$ part (resp. any $G_2$ part). Let $w \in W$ have a reduced expression $s_{i_r} \cdots s_{i_1}$. Define $\beta_j$ and $F_{\beta_j}$, $j = 1, \cdots, r$ as above: $\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_i)$ and $F_{\beta_j} = R_{s_{i_1}} \cdots R_{s_{i_{j-1}}}(F_{\alpha_i})$.

We define

$$U_{A'}(w) = \text{span}_{A'} \left\{ F_{\beta_1}^{\alpha_1} \cdots F_{\beta_r}^{\alpha_r} | a_1, \ldots, a_r \in \mathbb{N} \right\}$$
We can assume that we can get from one of the reduced expression to the other. This subspace is independent of the reduced expression for \( w \). This can be proved in the same way as Lemma 2.2 using the rank 2 calculations done in [Lus90].

Let \( A = \mathbb{Z}[v, v^{-1}] \). Lusztigs \( A \)-form is defined to be the \( A \) subalgebra of \( U_v \) generated by the divided powers \( E^{(n)}_\alpha \) and \( F^{(n)}_\alpha \) for \( n \in \mathbb{N} \) and \( K_{i \pm 1}^\pm \).

We want to define \( U_A(w) = \text{span}_A \left\{ F^{(a_1)}_{\beta_1} \cdots F^{(a_r)}_{\beta_r} | a_i \in \mathbb{N} \right\} \) where the \( F_{\beta_i} \) are defined from a reduced expression of \( w \) like earlier. We have \( U_A'(w_0) = U_A' \) so we want a similar property over \( A' \): \( U_A'(w_0) = U_A' \) where \( U_A' \) is the \( A' \)-subalgebra generated by \( \left\{ F^{(n)}_{\alpha} \mid n \in \mathbb{N}, \alpha = 1, \ldots, n \right\} \). This is shown very similar to the way it is shown for \( U_v \) in [Jan96].

**Lemma 2.2** Assume \( g \) does not contain any \( G_2 \) components:

1. The subspace \( \text{span}_A \left\{ F^{(a_1)}_{\beta_1} \cdots F^{(a_r)}_{\beta_r} | a_i \in \mathbb{N} \right\} \) depends only on \( w \), not on the reduced expression chosen for \( w \).

2. Let \( \alpha \) and \( \beta \) be two distinct simple roots. If \( w \) is the longest element in the subgroup of \( W \) generated by \( s_\alpha \) and \( s_\beta \), then the span defined as before is the subalgebra of \( U_A \) generated by \( F^{(a)}_{\alpha} \) and \( F^{(b)}_{\beta} \), \( a, b \in \mathbb{N} \).

**Proof.** Claim 2. is shown on a case by case basis. We will show first that the second claim implies the first.

We show this by induction on \( l(w) \). If \( l(w) \leq 1 \) then there is only one reduced expression of \( w \) and there is nothing to show. Assume \( l(w) > 1 \) and that \( w \) has two reduced expressions \( w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_r} \) and \( w = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_r} \). We can assume that we can get from one of the reduced expression to the other by an elementary braid move \( (s_{\alpha_1} s_{\alpha_2} \cdots s_{\beta_2} s_{\alpha_2} \cdots s_{\beta_2} s_{\beta_1}) \). Set \( \alpha = \alpha_1 \) and \( \gamma = \gamma_1 \).

If \( \alpha = \gamma \), set \( w' = s_{\alpha} w \). Then the subspace spanned by the elements as in the lemma is for both expressions equal to:

\[
\left( \sum_{a \geq 0} F^{(a)}_{\alpha} \right) \cdot R_{s_{\alpha}}(U_A'(w'))
\]

(1)

If \( \alpha \neq \gamma \) then the elementary move must take place at the beginning of the reduced expression for both reduced expressions. Let \( w'' \) be the longest element generated by \( s_{\alpha} \) and \( s_{\beta} \) then we must have \( w = w'' w' \) for some \( w' \) with \( l(w'') + l(w') = l(w) \) and the reduced expression for \( w' \) in both reduced expressions are equal whereas the reduced expressions for \( w'' \) are the two possible combinations for the two different reduced expressions. So the span of the products is given by \( U_A'(w') R_{w''}(U_A'(w'')) \) which is independent of the reduced expression by the second claim.

We turn to the proof of the second claim: First assume we are in the simply laced case. Then \( w = s_{\alpha_1} s_{\beta} s_{\alpha_2} = s_{\beta}s_{\alpha_1} s_{\beta} \). Lets work with the reduced expression \( s_{\alpha_1} s_{\beta} s_{\alpha_2} \). The other situation is symmetric by changing the role of \( \alpha \) and \( \beta \). We want to show that

\[
B := \left\langle F^{(n_1)}_{\alpha}, F^{(n_2)}_{\beta} \mid n_1, n_2 \in \mathbb{N} \right\rangle_A = \text{span}_A \left\{ F^{(a_1)}_{\alpha} F^{(a_2)}_{\alpha_2} F^{(a_3)}_{\beta} | a_i \in \mathbb{N} \right\} =: V
\]

(2)
where \( F^{(a)}_{\alpha+\beta} = R_\alpha(F^{(a)}_{\beta}) \). By [Lus90] section 5 we have that \( F^{(a)}_{\alpha+\beta} \in U^-_A \) for all \( a \in \mathbb{N} \) and we see that

\[
F^{(k)}_\beta F^{(k')}_{\alpha} = \sum_{t,s \geq 0} (-1)^s v^{-tr-s} F^{(r)}_{\alpha} F^{(s)}_{\alpha+\beta} F^{(t)}_{\beta}
\]

where the restrictions on the sum is \( s+t = k' \) and \( s+t = k \). Lusztig calculates for the \( E_i \)'s but just use the anti-automorphism \( \Omega \) (defined in Section 1 of [Lus90]) on the results to get the corresponding formulas for the \( F_i \)'s. Also we get the \((-1)^s\) from the fact that (using the notation of [Lus90]) \( E_{12} = -R_{-2}(E_{01}) \) because of the difference in the definition of the braid operators. Since \( F^{(a)}_{\alpha+\beta} \in U^-_A \) we have that \( V \subset B \). If we show that \( V \) is invariant by multiplication from the left with \( F^{(a)}_{\alpha} \) and \( F^{(a)}_{\beta} \) for all \( a \in \mathbb{N} \) then we must have \( B \subset V \). For \( F^{(a)}_{\alpha} \) this is clear. For \( F^{(k)}_{\beta}, k \in \mathbb{N} \) we use the formula above:

\[
F^{(k)}_\beta F^{(a_1)}_{\alpha_1} F^{(a_2)}_{\alpha_2} F^{(a_3)}_{\alpha_3} = \sum_{t,s \geq 0} (-1)^s v^{-d(tr+s)} F^{(r)}_{\alpha_1} F^{(s)}_{\alpha_2} F^{(t)}_{\beta}
\]

We see that \( F^{(k)}_\beta V \subset V \) so \( V = B \).

In the not simply laced case we have to use the formulas in [Lus90] section 5.3 (d)-(i) but the idea of the proof is the same. If there were similar formulas for the \( G_2 \) case it would be possible to show the same here. I do not know if similar formulas can be found in this case. The important part is just that if you ‘v-commute’ two of the ‘root vectors’ \( F^{(k)}_{\beta_i} \) and \( F^{(k')}_{\beta_j} \) you get something that is still in \( U^-_A \).

\[\]

Lemma 2.3

\[ U^-_A (w_0) = U^-_A \]

Proof. It is clear that \( U^-_A (w_0) \subset U^-_A \). We want to show that \( F^{(k)}_\alpha U^-_A (w_0) \subset U^-_A (w_0) \) for all simple \( \alpha \).

\( U^-_A (w_0) \) is independent of the chosen reduced expression so we can choose a reduced expression for \( w_0 \) such that \( s_\alpha \) is the last factor. Then the first ‘root vector’ \( F_{\beta_1} \) is equal to \( F_\alpha \). Then it is clear that \( F^{(k)}_\alpha U^-_A (w_0) \subset U^-_A (w_0) \). Since this was for an arbitrary simple root \( \alpha \) the proof is finished. (This argument is sketched in the appendix of [Lus90].)

\[\]

Corollary 2.4 We get a basis of \( U^-_A \) by the products of the form \( F^{(a_1)}_{\beta_1} \cdots F^{(a_N)}_{\beta_N} \) where \( a_1, \ldots, a_N \in \mathbb{N} \).

Corollary 2.5 \( U^-_A (w) = U^-_w (w) \cap U^-_A \).

Proof. Assume the length of \( w \) is \( r \) and define for \( k \in \mathbb{N}^r \)

\[
F^{(k)} = F^{(k_1)}_{\beta_1} \cdots F^{(k_r)}_{\beta_r}
\]

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It is clear that $U_A^{-}(w) \subseteq U_A^{-}(w) \cap U_A$. Assume $x \in U_A^{-}(w) \cap U_A$. Since $x \in U_A^{-}(w)$ we have constants $c_k \in \mathbb{Q}(v)$, $k \in \mathbb{N}^r$ such that

$$x = \sum_{k \in \mathbb{N}^r} c_k F^{(k)}$$

Assume the length of $w_0$ is $N$ and denote for $n \in \mathbb{N}^N$, $F^{(n)}$ like above for $w$. $U_A^{-}(w) \cap U_A \subseteq U_A^{-}(w_0) \cap U_A = U_A^{-}(w_0)$ ($U_A^{-}(w_0) \subset U_A^{-}(w_0) \cap U_A$ clearly and $U_A^{-}(w_0)$ is invariant under multiplication by $U_A$) so there exists $b_n \in A$, $n \in \mathbb{N}^N$ such that

$$x = \sum_{k \in \mathbb{N}^N} b_k F^{(k)}$$

But then we have two expressions of $x$ in $U_A^{-}(w)$ expressed as a linear combination of basis elements. So we must have that the multidindices $b_k$ are zero on coordinates $\geq r$ and that all the $c_k$ are actually in $A$. This proves the corollary. □

**Definition 2.6** Let $x \in (U_v)_\mu$ and $y \in (U_v)_\gamma$ then

$$[x, y]_v := xy - v^{-|\mu|\gamma}yx$$

**Proposition 2.7** For $x_1 \in (U_v)_{\mu_1}$, $x_2 \in (U_v)_{\mu_2}$ and $y \in (U_v)_\gamma$ we have

$$[x_1 x_2, y]_v = x_1 [x_2, y]_v + v^{-(\gamma|\mu_2)} [x_1, y]_v x_2$$

and

$$[y, x_1 x_2]_v = v^{-(\gamma|\mu_1)} x_1 [y, x_2]_v + [y, x_1]_v x_2$$

**Proof.** Direct calculation. □

We have the following which corresponds to the Jacobi identity. Note that putting $v = 1$ recovers the usual Jacobi identity for the commutator.

**Proposition 2.8** for $x \in (U_v)_\mu$, $y \in (U_v)_\nu$ and $z \in (U_v)_\gamma$ we have

$$[[x, y]_v, z]_v = [x, [y, z]_v]_v - v^{-(\mu|\nu)} [y, [x, z]_v]_v + v^{-(\nu|\mu)} \left( v^{-(\gamma|\nu)} - v^{-(\gamma|\mu)} \right) [x, z]_v y$$

**Proof.** Direct calculation. □

The main tool that will be used in this project is the following theorem from [DP93] thm 9.3 originally from [LS91] Proposition 5.5.2:

**Theorem 2.9** Let $F_{\beta_j}$ and $F_{\beta_i}$ be defined as above. Let $i < j$. Let $A = \mathbb{Z}[v, v^{-1}]$ and let $A'$ be the localization of $A$ in $[2]$ (and/or $[3]$) if the Lie algebra contains any $B_n, C_n$ or $F_4$ part (resp. any $G_2$ part). Then

$$[F_{\beta_j}, F_{\beta_i}]_v = F_{\beta_j} F_{\beta_i} - v^{-(\beta_i|\beta_j)} F_{\beta_i} F_{\beta_j} \in \text{span}_{A'} \left\{ F_{\beta_{i+1}}^{a_i+1}, \ldots, F_{\beta_j-1}^{a_j-1} \right\}$$

**Proof.** We shall provide the details of the proof sketched in [DP93]. The rank 2 case is handled in [Lus90]. Note that in [Lus90] we see that when $\mu = 2$ (in his
notation) we get second divided powers and when \( \mu = 3 \) we get third divided powers. This is one reason why we need to be able to divide by [2] and [3].

So we assume the rank 2 case is proven. In particular we can assume there is no \( G_2 \) component. Let \( k \in \mathbb{N}, k < j \). Then \([F_{\beta_j}, F_{\beta_k}] = R_{s_i} \cdots R_{s_{i-k}} [R_{s_{i-k}} \cdots R_{s_{i-1}} (F_{\alpha_i}), F_{\alpha_i}]_v\) so we can assume in the above that \( i = 1 \). We can then assume that \( j > 2 \) because otherwise we would be in the rank 2 case. We will show by induction over \( l \in \mathbb{N} \) that

\[
[F_{\beta_j}, F_{\beta_l}]_v = F_{\beta_l} F_{\beta_j} - v^{-(\beta_j|\beta_l)} F_{\beta_l} F_{\beta_j} \in \text{span}_A \left\{ F_{\beta^2} \cdots F_{\beta^{l-1}} \right\}.
\]

for all \( 1 < t \leq l \). The induction hypothesis that

\[
[F_{\beta_j}, F_{\beta_l}]_v = F_{\beta_l} F_{\beta_j} - v^{-(\beta_j|\beta_l)} F_{\beta_l} F_{\beta_j} \in \text{span}_A \left\{ F_{\beta^2} \cdots F_{\beta^{l-1}} \right\}
\]

for \( t \leq l \). We need to prove the result for \( l + 1 \). We have \( \beta_{l+1} = s_{i_l} \cdots s_{i_1} (\alpha_{i_1}) \). Now define \( i = i_1 \) and \( j = i_1 + 1 \). Set \( w = s_{i_1} \cdots s_{i_{l-1}} \). So \( \beta_{l+1} = ws_i (\alpha_j) \) and \( F_{\beta_{l+1}} = R_w R_s (F_{\alpha_j}) \). Define \( \alpha = \alpha_{i_1} \). So we need to show that

\[
[R_w R_s (F_{\alpha_j}), F_{\alpha_i}]_v \in \text{span}_A \left\{ F_{\beta^2} \cdots F_{\beta_{l-1}} \right\}
\]

We divide into cases:

Case 1) \((\alpha_i|\alpha_j) = 0\): In this case \( R_w R_s (F_{\alpha_j}) = R_w (F_{\alpha_j}) \). Since \( s_{i_l} s_j \) there is a reduced expression for \( w_0 \) starting with \( s_{i_1} \cdots s_{i_{l-1}} s_j s_i \). So the induction hypothesis gives us that \([R_w (F_{\alpha_j}), F_{\alpha_i}]_v\) can be expressed by linear combinations of ordered monomials involving only \( F_{\beta_2} \cdots F_{\beta_{l-1}} \).

Case 2) \((\alpha_i|\alpha_j) = -1 \) and \( l(w s_j) > l(w) \): In this case \( w s_j (\alpha_j) = w(\alpha_j) > 0 \) so there is a reduced expression for \( w_0 \) starting with \( s_{i_1} \cdots s_{i_{l-1}} s_j s_i s_j \). So we have by induction that \([R_w (F_{\alpha_j}), F_{\alpha_i}]_v\) is a linear combination of ordered monomials only involving \( F_{\beta_2} \cdots F_{\beta_{l-1}} \).

Observe that we have

\[
F_{\beta_{l+1}} = R_w R_s (F_{\alpha_j}) = R_w (F_{\alpha_j}) F_{\alpha_i} - v F_{\alpha_i} F_{\alpha_j} = R_w (F_{\alpha_j}) F_{\beta_j} - v F_{\beta_j} R_w (F_{\alpha_j}) = [R_w (F_{\alpha_j}), F_{\beta_j}]_v
\]

so by Proposition 2.8 we get

\[
[F_{\beta_{l+1}}, F_{\alpha_i}]_v = [[R_w (F_{\alpha_j}), F_{\beta_j}]_v, F_{\alpha_i}]_v = [R_w (F_{\alpha_j}), [F_{\beta_j}, F_{\alpha_i}]_v]_v - v^{-(w(\alpha_j))} [F_{\beta_j}, [R_w (F_{\alpha_j}), F_{\alpha_i}]_v]_v + v^{-(\beta_j|\alpha + w(\alpha_j))} (v^{-1} - v) [R_w (F_{\alpha_j}), F_{\alpha_i}]_v F_{\beta_j}.
\]

By induction (and Proposition 2.7) \([R_w (F_{\alpha_j}), F_{\beta_j}]_v\) \( [F_{\beta_j}, [R_w (F_{\alpha_j}), F_{\alpha_i}]_v]_v \) and \([F_{\beta_j}, [R_w (F_{\alpha_j}), F_{\alpha_i}]_v]_v \) are linear combinations of ordered monomials containing only \( F_{\beta_2} \cdots F_{\beta_{l-1}} \) so we have proved this case.

Case 3) \((\alpha_i|\alpha_j) = -1 \) and \( l(w s_j) < l(w) \): In this case write \( u = w s_j \). We claim \( l(us_j) > l(u) \). Assume \( l(us_j) < l(u) \) then

\[
l(w) + 2 = l(w s_j s_i) = l(us_j s_i s_j) < l(u) + 2 = l(w) + 1
\]
A contradiction. So there is a reduced expression of $w_0$ starting with $ws_i$. We have $F_{\beta_{l+1}} = R_w R_{s_i} (F_{\alpha_j}) = R_w (F_{\alpha_i})$ so we get

$$[F_{\beta_{l+1}}, F_{\alpha}]_v = [R_w (F_{\alpha_i}), F_{\alpha}]_v$$

Now we claim that either $u^{-1}(\alpha) = \alpha_j$ or $u^{-1}(\alpha) = 0$: Indeed $w^{-1}(\alpha) = 0$ so $u^{-1}(\alpha)$ is $< 0$ unless $w^{-1}(\alpha) = s_j w^{-1}(\alpha) = s_j (-\alpha_j) = \alpha_j$. If $\alpha = u(\alpha_j)$ we get

$$[R_w (F_{\alpha_i}), F_{\alpha}]_v = R_w ([F_{\alpha_i}, F_{\alpha}]_v) = R_w (R_{s_j} (F_{\alpha_i})) = R_w (F_{\alpha_i}) = F_{\beta_l}$$

In the other case we know from induction that

$$[R_w (F_{\alpha_i}), F_{\alpha}]_v \in U^-_w (u^{-1})$$

Now $U^-_w (u^{-1}) \subset U^-_w (s_j u^{-1}) = U^-_w (w^{-1})$ so we get that $[R_w (F_{\alpha_i}), F_{\alpha}]_v$ can be expressed as a linear combination of monomials involving $F_{\alpha} = F_{\beta_1}$ and the terms $F_{\beta_2} \cdots F_{\beta_{l-1}}$. Assume that a monomial of the form $F_{\alpha} F_{\beta_2} \cdots F_{\beta_{l-1}}$ appears with nonzero coefficient. The weights of the left and right hand side must agree so we have $ws_i (\alpha_j) + \alpha = \sum_{k=2}^{l-1} a_k \beta_k + m \alpha$ or

$$ws_i (\alpha_j) = \sum_{k=2}^{l-1} a_k \beta_k + (m-1) \alpha$$

Since $w^{-1}(\beta_k) < 0$ for $k = 1, 2, \ldots, l-1$ (and $\alpha = \beta_1$) we get

$$\alpha_i + \alpha_j = w^{-1} ws_i (\alpha_j) = \sum_{k=2}^{l-1} a_k w^{-1}(\beta_k) + (m-1) w^{-1}(\alpha) < 0.$$

Which is a contradiction.

Case 4) $(\alpha_i, \alpha_j^y) = -1$, $(\alpha_i | \alpha_j) = -2$ and $l(ws_j) > l(w)$: Here we get

$$F_{\beta_{l+1}} = R_w R_{s_i} (F_{\alpha_j}) = R_w (F_{\alpha_j} F_{\alpha_i} - v^2 F_{\alpha_i} F_{\alpha_j}) = R_w (F_{\alpha_j} F_{\beta_1} - v^2 F_{\beta_1} R_w (\alpha_j)) = [R_w (F_{\alpha_j}), F_{\beta_1}]_v$$

From here the proof goes exactly as in case 2.

Case 5) $(\alpha_i, \alpha_j^y) = -2$, and $l(ws_j) > l(w)$: First of all since $l(ws_j) > l(w)$ we can deduce that $l(ws_i s_j s_j) = l(w) + 4$: We have $-\beta_{l+1} + 2 ws_i s_j (\alpha_i) = ws_i s_j s_j (\alpha_i) = w(\alpha_j) > 0$ showing that we must have $ws_i s_j (\alpha_i) > 0$.

We have

$$F_{\beta_{l+1}} = R_w R_{s_i} (F_{\alpha_j}) = R_w (F_{\alpha_j} F_{\alpha_i}^{(2)} - v F_{\alpha_j} F_{\alpha_i} + v^2 F_{\alpha_i}^{(2)} F_{\alpha_j})$$

We claim that we have

$$R_{s_i} (F_{\alpha_i}) = \frac{1}{[2]} (R_{s_i} R_{s_j} (F_{\alpha_i}) F_{\alpha_i} - F_{\alpha_i} R_{s_i} R_{s_j} (F_{\alpha_i}))$$

This is shown by a direct calculation. First note that

$$R_{s_i} R_{s_j} (F_{\alpha_i}) = R_{s_j}^{-1} R_{s_j} R_{s_i} R_{s_j} (F_{\alpha_i}) = R_{s_j}^{-1} (F_{\alpha_i}) = F_{\alpha_j} F_{\alpha_i} - v^2 F_{\alpha_i} F_{\alpha_j}$$
Then which by induction and the above is a linear combination of ordered monomials involving only
\[ F_{\beta_i+1} = \frac{1}{2} \left( R_u R_s R_j (F_{\alpha_i}) - F_{\beta_i} R_u R_s R_j (F_{\alpha_i}) \right) \]

Therefore
\begin{align*}
F_{\beta_i+1} &= \frac{1}{2} \left( R_u R_s R_j (F_{\alpha_i}) - F_{\beta_i} R_u R_s R_j (F_{\alpha_i}) \right) \\
&= \frac{1}{2} \left( R_u R_s R_j (F_{\alpha_i}) - F_{\beta_i} R_u R_s R_j (F_{\alpha_i}) \right) \\
&= \frac{1}{2} \left( R_u R_s R_j (F_{\alpha_i}) - F_{\beta_i} R_u R_s R_j (F_{\alpha_i}) \right)
\end{align*}

By Proposition 2.8 and the above we get
\[ [R_u R_s R_j (F_{\alpha_i}), F_{\alpha}] = [R_u R_s R_j (F_{\alpha_i}), F_{\beta_i}] v \\
&= [R_u R_s R_j (F_{\alpha_i}), F_{\alpha}] v + v^2 [F_{\beta_i}, [R_u R_s R_j (F_{\alpha_i}), F_{\alpha}] v] v \\
&+ v^{2-\langle \alpha | \beta_i \rangle} \left( v^{-2} - v^2 \right) [R_u R_s R_j (F_{\alpha_i}), F_{\alpha}] v F_{\beta_i}
\]

which by induction is a linear combination of ordered monomials involving only
\[ F_{\beta_2}, \ldots, F_{\beta_i} \]

Case 6) \( \langle \alpha_i | \alpha_j \rangle = -2 \), \( l(ws_j) < l(w) \) and \( l(ws_j s_i) < l(ws_j) \): Set \( u = ws_j s_i \). We claim \( l(ws_j s_i) > l(u) \). Indeed suppose the contrary then \( l(u) + 2 = l(ws_j s_i) = l(ws_j s_i s_j) < l(u) + 4 = l(w) + 2 \). We reason like in case 3: We have \( F_{\beta_i+1} = R_u R_s R_j (F_{\alpha_i}) = R_u R_s R_j (F_{\alpha_i}) = R_u R_s R_j (F_{\alpha_i}) \). Now either \( u^{-1} (\alpha) = \alpha_i \), \( u^{-1} (\alpha) = s_i (\alpha_j) \) or \( u^{-1} (\alpha) < 0 \). If \( u^{-1} (\alpha) < 0 \) we get by induction that \( [F_{\alpha_i}, R_u (F_{\alpha_i})], v \) is in \( U_{\mathcal{A}} (u^{-1}) \subset U_{\mathcal{A}} (w^{-1}) \) and by essentially the same weight argument as in case 3 we are done.

If \( \alpha = u (\alpha_i) \) then
\[ [R_u (F_{\alpha_i}), F_{\alpha}] v = [R_u (F_{\alpha_i}), R_u (F_{\alpha_i})] v \\
&= R_u (F_{\alpha_i}, F_{\alpha_i} - u^2 F_{\alpha_i} F_{\alpha_i}) \\
&= \left\{ \begin{array}{ll}
R_u (F_{\alpha_i}, F_{\alpha_i}) & \text{if } \langle \alpha_j, \alpha_j \rangle = -1 \\
R_u (F_{\alpha_i}, F_{\alpha_i}) & \text{if } \langle \alpha_j, \alpha_j \rangle = -2
\end{array} \right.
\]

So \( [F_{\alpha_i}, R_u (F_{\alpha_i})] v \in U_{\mathcal{A}} (s_i s_j s_i u^{-1}) = U_{\mathcal{A}} (s_i w^{-1}) \). Assume we have a monomial of the form \( F_{\alpha_i}^{m_2} F_{\beta_2}^{m_2} \cdots F_{\beta_l}^{m_l} \) with \( m \) nonzero in the expression of \([R_u (F_{\alpha_i}), F_{\alpha}] v \).

Then
\[ ws_j (\alpha_j) = \sum_{k=2}^{l} a_k \beta_k + (m - 1) \alpha \]
and we get
\[ \alpha_j = \sum_{k=2}^{l} a_k s_i w^{-1}(\beta_k) + (m - 1) s_i w^{-1}(\alpha) < 0. \]

A contradiction.

If \( \alpha = u s_i(\alpha_j) \) then
\[ [R_u(F_{\alpha_j}), F_{\alpha}]_v = R_u[F_{\alpha_j}, R_{s_i}(F_{\alpha})]_v = R_u(F_{\alpha_j} R_{s_i}(F_{\alpha}) - v^{-2} R_{s_i}(F_{\alpha}) F_{\alpha_j}) = R_u(R_s R_{s_i}(F_{\alpha_j}) R_{s_i}(F_{\alpha}) - v^{-2} R_{s_i}(F_{\alpha_j}) R_s R_{s_i}(F_{\alpha})) \]

Which is in \( U_A^-(s_i s_j s_i w^{-1}) = U_A^-(s_i w^{-1}) \) by the rank 2 case. By the same weight argument as above we are done.

Case 7) \( (\alpha_j)(\alpha_j) = -2, l(ws_j) < l(w) \) and \( l(ws_j s_i) > l(ws_j) \): Set \( u = ws_j \).

Like in case 3) we get that either \( u^{-1}(\alpha) = \alpha_j \) or \( u^{-1}(\alpha) < 0 \). If \( \alpha = u(\alpha_j) \):
\[ [F_{\beta_{i+1}}, F_{\alpha}]_v = R_u[R_s R_{s_i}(F_{\alpha_j}), F_{\alpha}]_v \in U_A^-(s_i s_j u^{-1}) = U_A^-(s_i w^{-1}) \]

And by a weight argument as above we are done.

If \( u^{-1}(\alpha) < 0 \) then \( \alpha = \beta_i^l \) for some \( i \in \{1,\ldots,l-2\} \) where the \( \beta_i^s \)s are defined as above but using a reduced expression of \( u \). Set \( \beta_{i-1}^u = u(\alpha_j) \), \( \beta_i^l = us_j(\alpha) \) and \( \beta_{i+1}^u = us_j s_i(\alpha_j) = ws_i(\alpha_j) = \beta_{i+1} \). Then
\[ [F_{\beta_{i+1}}, F_{\alpha}]_v = [F_{\beta_{i+1}^u}, F_{\beta_i^l}]_v \in U_A^-(s_i s_j u^{-1}) = U_A^-(s_i w^{-1}) \]

by induction and by a weight argument as above we are done. \( \square \)

First some considerations about the elements \( F_{\beta_j}, j = 1,\ldots,r \):

**Lemma 2.10** Let \( w_l = s_{i_1} \cdots s_{i_N} \) and let \( F_{\beta_j} = R_{s_{i_1}} \cdots R_{s_{i_j}}(F_{\alpha_j}) \) let \( l, r \in \{1,\ldots,N\} \) with \( l \leq r \). Then
\[ \text{span}_{\mathbb{Q}(v)} \left\{ F_{\beta_j^{e_1}} \cdots F_{\beta_j^{e_l}} | a_j \in \mathbb{N} \right\} = \text{span}_{\mathbb{Q}(v)} \left\{ F_{\beta_j^{e_1}} \cdots F_{\beta_j^{e_r}} | a_j \in \mathbb{N} \right\} \]

and the subspace is invariant under multiplication from the left by \( F_{\beta_i}, i = l,\ldots,r \).

**Proof.** If \( r - l = 0 \) the lemma obviously holds. Assume \( r - l > 0 \). For \( k \in \mathbb{N}^{r-l}, k = (k_1,\ldots,k_r) \) let \( F^k = F_{\beta_{j_1}^{e_1}} \cdots F_{\beta_{j_r}^{e_r}} \). We will prove the statement that \( F^k \in \text{span}_{\mathbb{Q}(v)} \left\{ F_{\beta_j^{e_1}} \cdots F_{\beta_j^{e_r}} | a_j \in \mathbb{N} \right\} \) by induction over \( k_1 + \cdots + k_r \). If \( k = 0 \) the statement holds. We have
\[ F^k = F_{\beta_{j_1}^{e_1}} F_{\beta_{j_1}^{e_1}}^{-1} F_{\beta_{j_1}^{e_1}+1} \cdots F_{\beta_r} \]

by induction \( F_{\beta_{j_1}^{e_1}}^{-1} F_{\beta_{j_1}^{e_1}+1} \cdots F_{\beta_r} \in \text{span}_{\mathbb{Q}(v)} \left\{ F_{\beta_j^{e_1}} \cdots F_{\beta_j^{e_r}} | a_j \in \mathbb{N} \right\} \) so if we show that \( F_{\beta_{j_1}} F_{\beta_{j_1}}^{e_1} \cdots F_{\beta_{j_r}}^{e_r} \in \text{span}_{\mathbb{Q}(v)} \left\{ F_{\beta_j^{e_1}} \cdots F_{\beta_j^{e_r}} | a_j \in \mathbb{N} \right\} \) for all \( b_{j_i}, i = l,\ldots,r \) then we have shown the first inclusion.

We use downwards induction on \( j \) and induction on \( b_1 + \cdots + b_r \). If \( j = r \) then this is obviously true. If \( j < r \) we use theorem 2.9 to conclude that \( F_{\beta_{j_1}} F_{\beta_{j_1}} - v^{-1}\beta_j) F_{\beta_j} \in \text{span}_{\mathbb{Q}(v)} \left\{ F_{\beta_{j_1}^{e_1}} \cdots F_{\beta_{j_1}^{e_r}} | a_j \in \mathbb{N} \right\} \)
If \( b_r = 0 \) the induction over \( j \) finishes the claim. We get now if \( b_r \neq 0 \)

\[
F_{\beta j}^b F_{\beta j}^{b} \cdots F_{\beta j}^{b} = v(\beta, 1_{\beta j}) \left( F_{\beta j} F_{\beta j} F_{\beta j}^{b-1} \cdots F_{\beta j}^{b} + \Sigma F_{\beta j}^{b-1} \cdots F_{\beta j}^{b} \right)
\]

where \( \Sigma \in \text{span}_{Q(v)} \left\{ F_{\beta j}^{a_1} \cdots F_{\beta j}^{a_{j+1}} | a_i \in \mathbb{N} \right\} \). By the induction on \( b_r + \cdots + b_t \)

\[
F_{\beta j} F_{\beta j}^{b-1} \cdots F_{\beta j}^{b} \in \text{span}_{Q(v)} \left\{ F_{\beta j}^{a} \cdots F_{\beta j}^{a_{j+1}} a_i \in \mathbb{N} \right\}
\]

and the induction on \( j \) ensures that

\[
\Sigma F_{\beta j}^{b-1} \cdots F_{\beta j}^{b} \in \text{span}_{Q(v)} \left\{ F_{\beta j}^{a} \cdots F_{\beta j}^{a_{j+1}} a_i \in \mathbb{N} \right\}
\]

since \( \Sigma \) contains only elements generated by \( F_{\beta j}^{b-1} \cdots F_{\beta j}^{b} \).

We have now shown that

\[
\text{span}_{Q(v)} \left\{ F_{\beta j}^{a_1} \cdots F_{\beta j}^{a_{j+1}} a_j \in \mathbb{N} \right\} \subset \text{span}_{Q(v)} \left\{ F_{\beta j}^{a} \cdots F_{\beta j}^{a_{j+1}} a_j \in \mathbb{N} \right\}
\]

The other inclusion is shown symmetrically. In the process we also proved that the subspace is invariant under left multiplication by \( F_{\beta j} \).

\[ \square \]

**Remark** The above lemma shows that \( U^v_c \) is an algebra.

**Definition 2.11** Define \( \text{ad}(F_{\beta j})(F_\alpha) := \left[ \cdots [F_{\alpha}, F_{\beta}], \cdots, \alpha, F_{\beta} \right]_v \) and \( \widetilde{\text{ad}}(F_{\beta j})(F_\alpha) := [F_{\beta j}, \cdots [F_{\beta j}, F_{\alpha}, \cdots]_v \) where the ‘\( v \)-commutator’ is taken \( t \) times from the left and right respectively.

**Proposition 2.12** Let \( u \in (U_A)_{\mu, \beta} \in \Phi^+ \) and \( F_{\beta j} \) a corresponding root vector. Set \( r = (\mu, \beta)^{\nu} \). Then in \( U_A \) we have the identity

\[
\text{ad}(F_{\beta j}^i)(u) = [i]! \sum_{n=0}^{i} (-1)^n \nu_{\beta j}^{n(1-i-r) F_{\beta}^{(n)} u F_{\beta}^{(i-n)}}
\]

and

\[
\widetilde{\text{ad}}(F_{\beta j})(u) = [i]! \sum_{n=0}^{i} (-1)^n \nu_{\beta j}^{n(1-i-r) F_{\beta}^{(i-n)} u F_{\beta}^{(i-n)}}
\]

**Proof.** This is proved by induction. For \( i = 0 \) this is clear. The induction step for the first claim:

\[
[i]! \sum_{n=0}^{i} (-1)^n \nu_{\beta j}^{n(1-i-r) F_{\beta}^{(n)} u F_{\beta}^{(i-n)}} F_{\beta}^{(n)} u F_{\beta}^{(i-n)}
\]

\[- \nu_{\beta j}^{r-2i} F_{\beta}^{(i)} [i]! \sum_{n=0}^{i} (-1)^n \nu_{\beta j}^{n(1-i-r) F_{\beta}^{(n)} u F_{\beta}^{(i-n)}} F_{\beta}^{(i-n)}
\]

\[= [i]! \sum_{n=0}^{i} (-1)^n \nu_{\beta j}^{n(1-i-r)} [i+1 - n] F_{\beta}^{(n)} u F_{\beta}^{(i+1-n)}
\]

\[- [i]! \sum_{n=0}^{i} (-1)^n \nu_{\beta j}^{n(1-i-r)-r-2i} [n+1] F_{\beta}^{(n+1)} u F_{\beta}^{(i-n)}
\]

\[= [i]! \sum_{n=0}^{i+1} (-1)^n \nu_{\beta j}^{n(1-i-r)} \left( \nu_{\beta j}^{[i+1 - n]} + \nu_{\beta j}^{n-i-1} [n] \right) F_{\beta}^{(n)} u F_{\beta}^{(i+1-n)}
\]

\[= [i+1]! \sum_{n=0}^{i+1} (-1)^n \nu_{\beta j}^{n(1-i-r)} F_{\beta}^{(n)} u F_{\beta}^{(i+1-n)}.
\]

The other claim is shown similarly by induction. \[ \square \]
So we can define \( \text{ad}(F^{(i)}_{\beta})(u) := ([i])^{-1} \text{ad}(F^{(i)}_{\beta})(u) \in U_A \) and \( \tilde{\text{ad}}(F^{(i)}_{\beta})(u) := ([i])^{-1}\tilde{\text{ad}}(F^{(i)}_{\beta})(u) \in U_A \).

**Proposition 2.13** Let \( a \in \mathbb{N}, u \in (U_A)_\mu \) and \( r = \langle \mu, \beta^v \rangle \). In \( U_A \) we have the identities

\[
u F_{\beta}^{(a)} = \sum_{i=0}^{a} v_{\beta}^{(i-a)(r+i)} F_{\beta}^{(a-i)} \text{ad}(F^{(i)}_{\beta})(u)
\]

and

\[
u F_{\beta}^{(a)} u = \sum_{i=0}^{a} v_{\beta}^{(i-a)(r+i)} \tilde{\text{ad}}(F^{(i)}_{\beta})(u) F_{\beta}^{(a-i)}
\]

**Proof.** This is proved by induction. For \( a = 0 \) this is obvious. The induction step for the first claim:

\[
[a + 1]_{\beta} u F_{\beta}^{(a+1)} = u F_{\beta}^{(a)} F_{\beta}
\]

\[
= \sum_{i=0}^{a} v_{\beta}^{(i-a)(r+i)} F_{\beta}^{(a-i)} \text{ad}(F^{(i)}_{\beta})(u) F_{\beta}
\]

\[
= \sum_{i=0}^{a} v_{\beta}^{(i-a)(r+i)-r-2}[a + 1 - i]_{\beta} F_{\beta}^{(a+1-i)} \text{ad}(F^{(i)}_{\beta})(u) + \sum_{i=0}^{a} v_{\beta}^{(i-a)(r+i)} [i + 1]_{\beta} F_{\beta}^{(a-i)} \text{ad}(F^{(i+1)}_{\beta})(u)
\]

\[
= \sum_{i=0}^{a} v_{\beta}^{(i-a-1)(r+i)-i} [a + 1 - i]_{\beta} F_{\beta}^{(a+1-i)} \text{ad}(F^{(i)}_{\beta})(u) + \sum_{i=1}^{a+1} v_{\beta}^{(i-a-1)(r+i-1)} [i]_{\beta} F_{\beta}^{(a+1-i)} \text{ad}(F^{(i)}_{\beta})(u)
\]

\[
= \sum_{i=0}^{a+1} v_{\beta}^{(i-a-1)(r+i)} \left( v_{\beta}^{r}[a + 1 - i]_{\beta} + v_{\beta}^{a+1-i}[i]_{\beta} \right) F_{\beta}^{(a+1-i)} \text{ad}(F^{(i)}_{\beta})(u)
\]

\[
= [a + 1]_{\beta} \sum_{i=0}^{a+1} v_{\beta}^{(i-a-1)(r+i)} F_{\beta}^{(a+1-i)} \text{ad}(F^{(i)}_{\beta})(u).
\]

So the induction step for the first identity is done. The other three identities are shown similarly by induction. \( \square \)

Let \( s_1, \ldots, s_N \) be a reduced expression of \( w_0 \) and construct root vectors \( F_{\beta_i}, \) \( i = 1, \ldots, N. \) In the rest of the section \( F_{\beta_i} \) refers to the root vectors constructed as such. In particular we have an ordering of the root vectors.
Proposition 2.14 Let $1 \leq i < j \leq N$.

$$[F^{a}_{\beta_{i}}, F^{b}_{\beta_{j}}]_{v} \in \text{span}_{Q(v)} \left\{ F^{a_{1}}_{\beta_{i}} \cdots F^{a_{j}}_{\beta_{j}} | a_{i} \in \mathbb{N}, a_{i} < a, a_{j} < b \right\}$$

Proof. From Theorem 2.13 we get the $a = 1$, $b = 1$ case. We will prove the general case by 2 inductions.

If $j - i = 1$ then $[F^{a}_{\beta_{i}}, F^{b}_{\beta_{j}}]_{v} = 0$ for all $a$. We will use induction over $j - i$.

We have by Proposition 2.17 that

$$[F^{a}_{\beta_{i}}, F^{b}_{\beta_{j}}]_{v} = v^{((a-1)\beta_{i}, \beta_{j})} F^{a-1}_{\beta_{i}} [F^{a}_{\beta_{i}}, F^{b}_{\beta_{j}}]_{v} + [F^{a}_{\beta_{i}}, F^{b}_{\beta_{j}}]_{v} F^{a-1}_{\beta_{i}}$$

The first term is in the correct subspace by Theorem 2.13. On the second we use the fact that $[F^{a}_{\beta_{i}}, F^{b}_{\beta_{j}}]_{v}$ only contains factors $F^{a_{k+1}}_{\beta_{i}} \cdots F^{a_{j-1}}_{\beta_{j-1}}$ and the induction over $j - i$ as well as induction over $a$ to conclude that we can commute the $F^{a-1}_{\beta_{i}}$ to the correct place and be in the correct subspace.

Now just make a similar kind of induction on $i - j$ and $b$ to get the result that

$$[F^{a}_{\beta_{i}}, F^{b}_{\beta_{j}}]_{v} \in \text{span}_{Q(v)} \left\{ F^{a_{1}}_{\beta_{i}} \cdots F^{a_{j}}_{\beta_{j}} | a_{i} \in \mathbb{N}, a_{i} < a, a_{j} < b \right\} \quad \square$$

Corollary 2.15 Let $1 \leq i < j \leq N$.

$$[F^{a}_{\beta_{i}}, F^{b}_{\beta_{j}}]_{v} \in \text{span}_{A} \left\{ F^{a_{1}}_{\beta_{i}} \cdots F^{a_{j}}_{\beta_{j}} | a_{i} \in \mathbb{N}, a_{i} < a, a_{j} < b \right\}$$

Proof. The Proposition 2.14 tells us that there exists $c_{k} \in \mathbb{Q}(v)$ such that

$$[F^{a}_{\beta_{i}}, F^{b}_{\beta_{j}}]_{v} = \sum_{k} c_{k} F^{a_{1}}_{\beta_{i}} \cdots F^{a_{j}}_{\beta_{j}}$$

with $a_{k} < a$ and $b_{k} < b$ for all $k$. But since $[F^{a}_{\beta_{i}}, F^{b}_{\beta_{j}}]_{v} \in U_{A}^{\perp}$ there exists $b_{i} \in A$ such that

$$[F^{a}_{\beta_{i}}, F^{b}_{\beta_{j}}]_{v} = \sum_{k} b_{k} F^{a_{1}}_{\beta_{i}} \cdots F^{a_{j}}_{\beta_{j}}$$

Now we have two expressions of $[F^{a}_{\beta_{i}}, F^{b}_{\beta_{j}}]_{v}$ in terms of a basis of $U_{\mathbb{Q}(v)}$. So we must have that the $c_{k}$'s are equal to the $b_{k}$'s. Hence $c_{k} \in A$ for all $k$ \quad \square

Lemma 2.16 Let $n \in \mathbb{N}$. Let $1 \leq j < k \leq N$.

$$\text{ad}(F^{(i)}_{\beta_{j}})(F^{(n)}_{\beta_{k}}) = 0 \text{ and } \overrightarrow{\text{ad}}(F^{(i)}_{\beta_{j}})(F^{(n)}_{\beta_{k}}) = 0 \text{ for } i \gg 0.$$ 

Proof. We will prove the first assertion. The second is proved completely similar.

We can assume $\beta_{j} = 1$ because

$$\text{ad}(F^{(i)}_{\beta_{j}})(F^{(n)}_{\beta_{k}}) = T_{s_{1}} \cdots T_{s_{j-1}} \left( \text{ad}(F^{(i)}_{\beta_{j}})(T_{s_{j}} \cdots T_{s_{k-1}} (F^{(n)}_{\beta_{k}})) \right).$$

So we assume $\beta_{j} = \beta_{1} =: \beta \in \Pi$ and $\alpha := \beta_{k} = s_{i_{1}} \cdots s_{i_{k-1}} (\alpha_{i_{j}}) \in \Phi^{+}$. We have

$$\text{ad}(F_{\beta})(F^{(n)}_{\alpha}) \in \text{span}_{A} \left\{ F^{a_{1}}_{\beta_{2}} \cdots F^{a_{j}}_{\beta_{j}} | a_{i} \in \mathbb{N}, a_{k} < n \right\},$$
hence the same must be true for \( \text{ad}(F^{(i)}_{\beta})(F^{(n)}_{\alpha}) \). By homogeneity if the monomial \( F^{(a_2)}_{\beta_2} \cdots F^{(a_k)}_{\beta_k} \) appears with nonzero coefficient then we must have

\[
i\beta + n\alpha = \sum_{s=2}^{k} a_s \beta_s
\]

or equivalently

\[
(n - a_k)\alpha = \sum_{s=2}^{k-1} a_s \beta_s - i\beta
\]

use \( s_{\beta} \) on this to get

\[
(n - a_k)s_{\beta}(\alpha) = \sum_{s=2}^{k-1} a_s s_{\beta}(\beta_s) + i\beta.
\]

By the way the \( \beta_s \)'s are chosen \( s_{\beta}(\beta_s) > 0 \) for \( 1 < s < k \). So this implies that a positive multiple \( (n - a_k) \) of a positive root must have \( i\beta \) as coefficient. If we choose \( i \) greater than \( nd \) where \( d \) is the maximal possible coefficient of a simple root in any positive root then this is not possible. Hence we must have for \( i > nd \) that \( \text{ad}(F^{(i)}_{\beta})(F^{(n)}_{\alpha}) = 0 \). □

In the next lemma we will need to work with inverse powers of some of the \( F_{\beta} \)'s. We know from e.g. [And03] that \( \{F^{\alpha}_{\beta} | \alpha \in \Pi\} \) is a multiplicative set so we can take the Ore localization in this set. Since \( R_\omega \) is an algebra isomorphism of \( U_\omega \) we can also take the Ore localization in one of the 'root vectors' \( F_{\beta_j} \). We will denote the Ore localization in \( F_{\beta} \) by \( U_{v_{\beta}}(F_{\beta}) \).

**Lemma 2.17** Let \( \beta \in \Phi^+ \) and \( F_{\beta} \) a root vector. Let \( u \in (U_{\nu_{\beta}})_{\mu} \) be such that \( \tilde{\text{ad}}(F^i_{\beta})(u) = 0 \) for \( i > 0 \). Let \( a \in \mathbb{N} \) and set \( r = \langle \mu, \beta^\vee \rangle \). Then in the algebra \( U_{v_{\beta}}(F_{\beta}) \) we get

\[
u F_{\beta}^{-a} = \sum_{i \geq 0} v_{\beta}^{-ar-(a+1)i} \left[ a + i - 1 \atop i \right] F_{\beta}^{-i-a} \tilde{\text{ad}}(F^i_{\beta})(u)
\]

and if \( u' \in (U_{\nu_{\beta}})_{\mu} \) is such that \( \text{ad}(F^i_{\beta})(u') = 0 \) for \( i > 0 \)

\[
u F_{\beta}^{-a} u' = \sum_{i \geq 0} v_{\beta}^{-ar-(a+1)i} \left[ a + i - 1 \atop i \right] \text{ad}(F^i_{\beta})(u') F_{\beta}^{-i-a}
\]

**Proof.** First we want to show that

\[
\tilde{\text{ad}}(F^i_{\beta})(u) F_{\beta}^{-1} = \sum_{k=i}^{\infty} v_{\beta}^{-r-2k} F_{\beta}^{-k+i+1} \tilde{\text{ad}}(F^k_{\beta})(u)
\]

(3)

Remember that \( \tilde{\text{ad}}(F^k_{\beta})(u) = 0 \) for \( k \) big enough so this is a finite sum. This is shown by downwards induction on \( i \). If \( i \) is big enough this is 0 = 0. We have

\[
u F_{\beta} \tilde{\text{ad}}(F^i_{\beta})(u) = \tilde{\text{ad}}(F^{i+1}_{\beta})(u) + v_{\beta}^{-r-2i} \tilde{\text{ad}}(F^i_{\beta})(u) F_{\beta}
\]
\[ \tilde{\text{ad}}(F_\beta^i)(u)F_{\beta}^{-1} = F_{\beta}^{-1}\tilde{\text{ad}}(F_{\beta}^{i+1})(u)F_{\beta}^{-1} + v_{\beta}^{-r-2i}F_{\beta}^{-1}\tilde{\text{ad}}(F_{\beta}^i)(u) \]
\[ = \sum_{k=1}^{\infty} v_{\beta}^{-r-2k}F_{\beta}^{-k+i-1}\tilde{\text{ad}}(F_{\beta}^k)(u) + v_{\beta}^{-r-2i}F_{\beta}^{-1}\tilde{\text{ad}}(F_{\beta}^i)(u) \]
\[ = \sum_{k=1}^{\infty} v_{\beta}^{-r-2k}F_{\beta}^{-k+i-1}\tilde{\text{ad}}(F_{\beta}^k)(u) \]

Setting \( i = 0 \) in the above we get the induction start:
\[ uF_{\beta}^{-1} = \sum_{k>0} v_{\beta}^{-r-2k}F_{\beta}^{-k-1}\tilde{\text{ad}}(F_{\beta}^k)(u) \]

For the induction step assume
\[ uF_{\beta}^{-a} = \sum_{i\geq 0} v_{\beta}^{-ar-(a+1)i} \left[ \frac{a+i-1}{i} \right] F_{\beta}^{-a+i}\tilde{\text{ad}}(F_{\beta}^i)(u) \]

Then
\[ uF_{\beta}^{-a-1} = \sum_{i\geq 0} v_{\beta}^{-ar-(a+1)i} \left[ \frac{a+i-1}{i} \right] F_{\beta}^{-a+i}\tilde{\text{ad}}(F_{\beta}^i)(u)F_{\beta}^{-1} \]
\[ = \sum_{i\geq 0} v_{\beta}^{-ar-(a+1)i} \left[ \frac{a+i-1}{i} \right] F_{\beta}^{-a+i}\sum_{k\geq i} v_{\beta}^{-r-2k}F_{\beta}^{-k+i-1}\tilde{\text{ad}}(F_{\beta}^k)(u) \]
\[ = \sum_{k\geq 0} \sum_{i=0}^k v_{\beta}^{-(a+1)r-(a+1)i-2k} \left[ \frac{a+i-1}{i} \right] F_{\beta}^{-a-1-k}\tilde{\text{ad}}(F_{\beta}^k)(u) \]
\[ = \sum_{k\geq 0} v_{\beta}^{-(a+1)r-(a+2)k} \left( \sum_{i=0}^k v_{\beta}^{-(a+1)i+ak} \left[ \frac{a+i-1}{i} \right] \right) F_{\beta}^{-a-1-k}\tilde{\text{ad}}(F_{\beta}^k)(u) \]

The induction is finished by observing that
\[ \sum_{i=0}^k v_{\beta}^{-(a+1)i+ak} \left[ \frac{a+i-1}{i} \right] = v_{\beta}^k + \sum_{i=1}^k v_{\beta}^{-(a+1)i+ak} \left( v_{\beta}^{a+i} \left[ \frac{a+i-1}{i} \right] - v_{\beta}^{a+i-1} \left[ \frac{a+i-1}{i} \right] \right) \]
\[ = v_{\beta}^k + \sum_{i=1}^k v_{\beta}^{ai+ak} \left[ \frac{a+i}{i} \right] - \sum_{i=1}^k v_{\beta}^{a(i+1)+ak} \left[ \frac{a+i-1}{i-1} \right] \]
\[ = v_{\beta}^k + \sum_{i=1}^k v_{\beta}^{ai+ak} \left[ \frac{a+i}{i} \right] - \sum_{i=0}^{k-1} v_{\beta}^{ai+ak} \left[ \frac{a+i}{i} \right] \]
\[ = \left[ a + k \right] \]

The other identity is shown similarly by induction. \( \square \)

**Definition 2.18** Let \( \beta \in \Phi^+ \) and let \( \beta \) be \( F_\beta \) a root vector. We define for \( n \in \mathbb{N} \) in \( U_v(F_\beta) \)
\[ F_{\beta}^{(n)} = [n]! F_{\beta}^{-n} \]

i.e. \( F_{\beta}^{(n)} = \left( F_{\beta}^{(n)}^{-1} \right) \).
Corollary 2.19 Let $\beta \in \Phi^+$ and $F_\beta$ a root vector. Let $u \in \{U_v\}_\mu$ be such that $\text{ad}(F^{(i)}_\beta)(u) = 0$ for $i \gg 0$. Let $a \in \mathbb{N}$ and set $r = \langle \mu, \beta^\vee \rangle$. Then in the algebra $U_{v(F_\beta)}$ we get

$$uF^{-a}_\beta F^{-1}_\beta = \sum_{i \geq 0} v^{-a-(a+1)r-(a+2)i}_\beta F^{(i-a)}_\beta F^{-1}_\beta \text{ad}(F^{(i)}_\beta)(u)$$

and if $w' \in \{U_v\}_\mu$ is such that $\text{ad}(F^{(i)}_\beta)(w') = 0$ for $i \gg 0$

$$F^{-a}_\beta F^{-1}_\beta w' = \sum_{i \geq 0} v^{-a-(a+1)r-(a+2)i}_\beta \text{ad}(F^{(i)}_\beta)(w') F^{(i-a)}_\beta F^{-1}_\beta$$

3 Twisting functors

In this paper we are following the paper [And03] closely. The definition of twisting functors for quantum group modules given later and the ideas in this section are mostly coming from this paper.

We will start by showing that the semiregular bimodule $S^w_v$ is a bimodule isomorphic to $U^*_w(w) \otimes_{U_w^*(w)} U_v$ as a right module.

Recall how $U_v(w)$, $S^w_v$ and $S_v(F)$ are defined: Let $s_{i_1} \cdots s_{i_k}$ be a reduced expression for $w$ and $F_{\beta_{i_j}} = R_{s_{i_1}} \cdots R_{s_{i_{j-1}}} (F_{\alpha_{i_j}})$ as usual then

$$U^*_w(w) = \text{span}_{\mathbb{Q}(v)} \{ F^{\alpha_1}_{\beta_{i_1}} \cdots F^{\alpha_r}_{\beta_{i_r}} | a_i \in \mathbb{N} \}$$

$$S^w_v = U_v \otimes_{U^*_w(w)} U^*_w(w)^*$$

and for $F \in U^*_w$ such that $\{ F^a | a \in \mathbb{N} \}$ is a multiplicative set

$$S_v(F) = U_{v(F)}/U_v$$

where $U_{v(F)}$ denotes the Ore localization in the multiplicative set $\{ F^a | a \in \mathbb{N} \}$.

In the following proposition we will define a left $U_v$ isomorphism between $S^w_v$ and $S_v(F_{\beta_k}) \otimes_{U_v} S^w_v$ where $w' = s_{i_k} w$. We will need some notation. Let $m \in \mathbb{N}$. We denote by $f^{(r)}_m \in (Q(v)[F_{\beta_k}])^*$ the linear function defined by $f^{(r)}_m(F^{\alpha}_{\beta_{i_k}}) = \delta_m, a$. We will drop the $(r)$ from the notation in most of the following. For $g \in U^*_w(w)^*$ we define $f_m g$ to be the linear function defined by: For $x \in U^*_w(w)$, $(f_m g)(xF^{\alpha}_{\beta_{i_k}}) = f_m(F^{\alpha}_{\beta_{i_k}}) g(x)$. From the definition of $U^*_w(w)$ and because we are taking graded dual every $f \in U^*_w(w)^*$ is a linear combination of functions on the form $f_m \cdot g$ for some $m \in \mathbb{N}$ and $g \in U^*_w(w)$ (by induction this implies that every function in $U^*_w(w)$ is a linear combination of functions of the form $f^{(m)}_{m_1} \cdots f^{(2)}_{m_2} f^{(1)}_{m_1}$ for some $m_1, \ldots, m_r \in \mathbb{N}$). Note that the definition of $f_m$ makes sense for $m < 0$ but then $f_m = 0$.

Proposition 3.1 Assume $w = s_{i_k} \cdots s_{i_1} = s_{i_k} w'$, where $k$ is the length of $w$, then as a left $U_v$ module

$$S^w_v \cong S_v(F_{\beta_k}) \otimes_{U_v} S^w_v$$
by the following left $U_v$ isomorphism

$$\varphi_k : S^w_v \rightarrow S_v(F_{\beta_k}) \otimes_{U_v} S^w_v$$

defined by:

$$\varphi_k(u \otimes f_m \cdot g) = uF^{-m-1}_{\beta_k}K_{\beta_k} \otimes (1 \otimes g), \quad u \in U_v, m \in \mathbb{N}, g \in U_v^{-}(u')^*$$

The inverse to $\varphi_k$ is the left $U_v$-homomorphism $\psi_k : S_v(F_{\beta_k}) \otimes_{U_v} S^w_v \rightarrow S^w_v$ given by:

$$\psi_k(uF^{-m}_{\beta_k} \otimes (1 \otimes g)) = v^{(m\beta_k|\beta_k)}uF^{-1}_{\beta_k} \otimes f_{m-1} \cdot g, \quad u \in U_v, m \in \mathbb{N}, g \in U_v^{-}(u')^*$$

**Proof.** The question is if $\varphi_k$ is welldefined. Let $f = f_m \cdot g$. We need to show that the recipe for $uF_{\beta_j} \otimes f$ is the same as the recipe for $u \otimes F_{\beta_j} f$ for $j = 1, \ldots, k$. For $j = k$ this is easy to see. Assume from now on that $j < k$. We need to figure out what $F_{\beta_j} f$ is. We have by Proposition 2.13 (setting $r = (\beta_j, \beta_k')$)

$$(F_{\beta_j} f)(xF_{\beta_k}^a) = f(xF_{\beta_k}^a F_{\beta_j})$$

so

$$F_{\beta_j} f = \sum_{i \geq 0} v^{\beta}_{\beta} -m(r+i) \left[ m + i \right] \left[ \beta \right] f_{m+i} \cdot \left( \tilde{\text{ad}}(F_{\beta_k}^a)(F_{\beta_j}) \right) g (xF_{\beta_k}^a)$$

Note that the sum is finite because of Lemma 2.16.

On the other hand we have that $uF_{\beta_j} \otimes f$ is sent to (using Lemma 2.17)

$$uF_{\beta_j} F^{-m-1}_{\beta_k}K_{\beta_k} \otimes (1 \otimes g)$$

$$= u \sum_{i \geq 0} v^{\beta}_{\beta} -(m+1)r-(m+2)i \left[ m + i \right] \left[ \beta \right] F^{-i-m-1}_{\beta_k} \tilde{\text{ad}}(F_{\beta_k}^a)(F_{\beta_j}) K_{\beta_k} \otimes (1 \otimes g)$$

$$= u \sum_{i \geq 0} v^{\beta}_{\beta} -mr-mi \left[ m + i \right] \left[ \beta \right] F^{-i-m-1}_{\beta_k} \tilde{\text{ad}}(F_{\beta_k}^a)(F_{\beta_j}) \otimes (1 \otimes g)$$

Using the fact that $\tilde{\text{ad}}(F_{\beta_k}^a)(F_{\beta_j})$ can be moved over the first and the second tensor we see that the two expressions $uF_{\beta_j} \otimes f$ and $u \otimes F_{\beta_j} f$ are sent to the same.

So $\varphi_k$ is a welldefined homomorphism. It is clear from the construction that $\varphi_k$ is a $U_v$ homomorphism.

We also need to prove that $\psi_k$ is welldefined. We prove that $uF^{-m}_{\beta_k} F_{\beta_j} \otimes (1 \otimes g)$ is sent to the same as $uF^{-m}_{\beta_k} \otimes (1 \otimes g)$ by induction over $k-j$. If $j = k-1$ we
see from Lemma 2.17 and Theorem 2.9 that \( F_{\beta_{k-1}}^{-a}F_{\beta_k}^{-a} = v^{-(a\beta_k|\beta_{k-1})}F_{\beta_{k-1}}^{-a}F_{\beta_k}^{-a} \) and therefore \( uF_{\beta_k}^{-m}F_{\beta_{k-1}} \odot (1 \odot g) \) is sent to

\[
u(m\beta_k - \beta_{k-1})_uK_{\beta_k}^{-1}F_{\beta_{k-1}} \odot f_{m-1} \cdot g
\]

\[
= \nu(m\beta_k + (m-1)\beta_k - \beta_{k-1})_uK_{\beta_k}^{-1} \odot f_{m-1} \cdot g
\]

Note that because we have \( \overline{\text{ad}}(F_{\beta_k}) = 0 \) for all \( i \geq 1 \) we get \( F_{\beta_{k-1}}(f_{m-1} \cdot g) = \nu^{-(\beta_{k-1}(1)\beta_k)}f_{m-1} \cdot (F_{\beta_{k-1}} \cdot g) \). Using this we see that \( uF_{\beta_k}^{-m}F_{\beta_{k-1}} \odot (1 \odot g) \) is sent to the same as \( uF_{\beta_k}^{-m} \odot (1 \odot F_{\beta_{k-1}} g) \).

Now assume \( j - k > 1 \). To calculate what \( uF_{\beta_k}^{-m}F_{\beta_j} \odot (1 \odot g) \) is sent to we need to calculate \( F_{\beta_k}^{-m}F_{\beta_j} \). By lemma 2.17

\[
F_{\beta_k}^{-m}F_{\beta_j} = u^{mr}F_{\beta_j}F_{\beta_k}^{-m} - \sum_{i \geq 1} v_{\beta}^{(m+1)i} \left[ \frac{m+i-1}{i} \right] F_{\beta_k}^{-m-i} \beta \circ \alpha
\]

so

\[
u(m\beta_k - \beta_{k-1})_uK_{\beta_k}^{-1} \odot f_{m-1} \cdot g
\]

Note that, because we have \( \overline{\text{ad}}(F_{\beta_k}) = 0 \) for all \( i \geq 1 \) we get \( F_{\beta_{k-1}}(f_{m-1} \cdot g) = \nu^{-(\beta_{k-1}(1)\beta_k)}f_{m-1} \cdot (F_{\beta_{k-1}} \cdot g) \).

By the induction over \( k - j \) (remember that \( \overline{\text{ad}}(F_{\beta_k})(u) \) is a linear combination of ordered monomials involving only the elements \( F_{\beta_{j+1}} \cdots F_{\beta_{k-1}} \)) this is sent to the same as

\[
u(m\beta_k - \beta_j)_{u}K_{\beta_k}^{-1} \odot f_{m-1} \cdot g
\]

which is sent to

\[
u(m\beta_k - \beta_j)_{u}K_{\beta_k}^{-1} \odot f_{m-1} \cdot g
\]

But this is what \( uF_{\beta_k}^{-m} \odot (1 \odot F_{\beta_j} g) \) is sent to. We have shown by induction that \( \psi_k \) is well defined. It is easy to check that \( \psi_k \) is the inverse to \( \varphi_k \). \( \square \)

**Proposition 3.2** Let \( w = s_{i_1} \cdots s_{i_1} \) etc. There exists an isomorphism of left \( U_v \)-modules

\[
S^w_v \cong S_v(F_{\beta_k}) \otimes U_v \cdots \otimes U_v \cdot S_v(F_{\beta_1})
\]

**Proof.** The proof is by induction of the length of \( w \). Note that \( S^w_v \cong U_v \otimes k^* \cong U_v \) so proposition 3.1 with \( w' = e \) gives the induction start.

Assume the length of \( w \) is \( r > 1 \). By proposition 3.1 we have \( S^w_v \cong S_v(F_{\beta_r}) \otimes U_v \cdots \otimes U_v \cdot S_v(F_{\beta_1}) \). By induction \( S^w_v \cong S_v(F_{\beta_{r-1}}) \otimes U_v \cdots \otimes U_v \cdot S_v(F_{\beta_1}) \). This finishes the proof. \( \square \)
We can now define a right action on $S^w_v$ by the isomorphism in proposition [3.2]. By first glance this might depend on the chosen reduced expression for $w$. But the next proposition proves that this right action does not depend on the reduced expression chosen.

**Proposition 3.3** As a right $U_v$ module $S^w_v \cong U_v^-(w)^* \otimes_{U_v} U_v$.

**Proof.** All isomorphisms written in this proof are considered to be right $U_v$ isomorphisms. This is proved in a very similar way to proposition [3.1]. We will sketch the proof here.

Define $S^1_v = (U^1_v)^* \otimes_{U_v} U_v$, where $U^1_v = \text{span}_{Q(v)} \left\{ F_{\alpha_1} \cdots F_{\alpha_n} | a_i \in \mathbb{N} \right\}$. Note that $S^1_v = U^-_v(w)^* \otimes_{U_v} U_v$. We want to show that $(U^1_v)^* \otimes_{U_v} U_v \cong S^{1+1}_v \otimes_{U_v} S_v(F_{\beta_1})$. If we prove this we will have $S^1_v \cong S^2_v \otimes_{U_v} S_v(F_{\beta_1}) \cong \cdots \cong S_v(F_{\beta_n}) \otimes_{U_v} \cdots \otimes_{U_v} S_v(F_{\beta_1}) \cong S^w_v$ as a right module and we are done.

Let $r = (\beta_j, \beta'_j)$. From Proposition [2.13] we have

\[ F_{\beta_j} F_{\beta'_j} = \sum_{i=0}^{a} v_{\beta_j}^{(i-a)(r+1)} \left( \frac{a}{i} \right)_{\beta_j} F_{\beta_j}^{-i} \text{ad}(F_{\beta_j})(F_{\beta'_j}) \]

and by Lemma [2.17] we have

\[ F_{\beta'_j}^{-a} F_{\beta_j} = \sum_{i \geq 0} v_{\beta'_j}^{(a-r-1)+(i+1)} \left( \frac{a-i+1}{i} \right)_{\beta'_j} \text{ad}(F_{\beta_j})(F_{\beta'_j}) F_{\beta'_j}^{-1-a} \]

We define the right homomorphism $\varphi_l$ from $(U^1_v)^* \otimes_{U_v} U_v \to S^{1+1}_v \otimes_{U_v} S_v(F_{\beta_1})$ by

\[ \varphi_l(g \cdot f_{m_1} \otimes u) = (g \otimes 1) \otimes K_{\beta_l} F_{\beta_1}^{-m_1-1} u \]

Like in the previous proposition we can use the above formulas to show that this is well defined and we can define an inverse like in the previous proposition only reversed. The inverse is:

\[ \psi_l((g \otimes 1) \otimes F_{\beta_1}^{-m_1-1} u) = v^{-(m_1+1)\beta_1} \beta_l g \cdot f_m \otimes K_{\beta_1}^{-1} u \]

So we have now that $S^w_v$ is a bimodule isomorphic to $U_v \otimes_{U^-_v(w)} U^-_v(w)^* \otimes_{U^-_v(w)} U_v$ as a left module and isomorphic to $U^-_v(w)^* \otimes_{U^-_v(w)} U_v$ as a right module. We want to examine the isomorphism between these two modules. For example what is the left action of $K_{\alpha_0}$ on $f \otimes 1 \in (U^-_v(w))^* \otimes_{U^-_v(w)} U_v$.

Assume $f = f^{(r)}_m \cdots f^{(1)}_{m_1}$ i.e. that $f(F_{\beta_1} \cdots F_{\beta'_1}) = \delta_{m_1,a_1} \cdots \delta_{m_0,a_0}$. Then we get via the isomorphism $(U^-_v(w))^* \otimes_{U^-_v(w)} U_v \cong S_v(F_{\beta_1}) \otimes_{U_v} \cdots \otimes_{U_v} S_v(F_{\beta_1})$ that $f \otimes u$ is sent to

\[ K_{\beta_r} F_{\beta_r}^{-m_r-1} \otimes \cdots \otimes K_{\beta_1} F_{\beta_1}^{-m_1-1} u \]

We want to investigate what this is sent to under the isomorphism $S_v(F_{\beta_r}) \otimes_{U_v} \cdots \otimes_{U_v} S_v(F_{\beta_1}) \cong U_v \otimes_{U^-_v(w)} (U^-_v(w))^*$. To do this we need to commute $u$ with $F_{\beta_1}^{-m_1-1}$, then $F_{\beta_2}^{-m_2-1}$ and so on. So we need to find $\tilde{u}$ and $m'_1, \ldots, m'_r$ such that

\[ K_{\beta_r} F_{\beta_r}^{-m_r-1} \cdots K_{\beta_1} F_{\beta_1}^{-m_1-1} u = \tilde{u} K_{\beta_r} F_{\beta_r}^{-m'_r-1} \cdots K_{\beta_1} F_{\beta_1}^{-m'_1-1} \]

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or equivalently

\[ uF_{\beta_1}^{m_1+1}K_{\beta_1}^{-1} \cdots F_{\beta_r}^{m_r+1}K_{\beta_r}^{-1} = f_{\beta_1}^{m_1+1}K_{\beta_1}^{-1} \cdots F_{\beta_r}^{m_r+1}K_{\beta_r}^{-1} \]

Assume we have found such \( \tilde{u} \) and \( m'_1, \ldots, m'_r \) then the above tensor is sent to

\[ v \Sigma_{i=1}^r (m'_i + 1) \beta_i \otimes \tilde{f} \]

where \( \tilde{f} = f(m'_1) \cdots f(m'_r) \). So in conclusion we have that \( f \otimes u \in (U_v^-(w))^* \otimes U_v^-(w) \) \( U_v \) maps to \( v \Sigma_{i=1}^r (m'_i + 1) \beta_i \otimes \tilde{f} \in U_v \otimes U_v^-(w) \). So if we want to figure out the left action of \( u \) on a tensor \( f \otimes 1 \) we need to first use the isomorphism \( (U_v^-(w))^* \otimes U_v^-(w) \) \( U_v \to U_v \otimes U_v^-(w) \). Then use \( f \) on \( U_v \) and then use the isomorphism \( U_v \otimes U_v^-(w) \) \( (U_v^-(w))^* \otimes U_v^-(w) \) \( U_v \) back again.

In particular if \( u = K_\alpha \) we have \( \overline{f} = f \) and \( \overline{f} = v \Sigma_{i=1}^r (m'_i + 1) \beta_i K_\alpha \). Note that if \( f = f(m) \cdots f(1) \) then the grading of \( f \) is \( \sum_{i=1}^r m_i \beta_i \) so \( K_\alpha (f \otimes 1) = v(\gamma + \sum_{i=1}^r \beta_i \alpha) f \otimes K_\alpha \) for \( f \in (U_v^-(w))^* \).

**Definition 3.4** Let \( w \in W \). For a \( U_v \)-module \( M \) define a ‘twisted’ version of \( M \) called \( {}^wM \). The underlying space is \( M \) but the action on \( {}^wM \) is given by:

\[ u \cdot m = R_{w^{-1}}(u)m \]

Note that if \( w, s \in W \) and \( l(sw) > l(w) \) then \( {}^s( {}^wM ) = {}^{sw}M \) since for \( u \in U_v \) and \( m \in {}^{sw}(M) \):

\[ u \cdot m = R_s u \cdot m = R_{w^{-1}}(R_s(u))m = R_{w^{-1}}(u)m \]

**Definition 3.5** The twisting functor \( T_w \) associated to an element \( w \in W \) is the following:

\[ T_w : U_v - \text{Mod} \to U_v - \text{Mod} \] is an endofunctor on \( U_v - \text{Mod} \). For a \( U_v \)-module \( M \):

\[ T_w M = {}^w(S_v^w \otimes U_v \ M) \]

**Definition 3.6** Let \( M \) be a \( U_v \)-module and \( \lambda : U_v^0 \to \mathbb{Q}(v) \) a character (i.e. an algebra homomorphism into \( \mathbb{Q}(v) \)). Then

\[ M_\lambda = \{ m \in M | \forall u \in U_v^0, um = \lambda(u)m \} \]

Let \( X \) denote the set of characters. Let \( \text{wt} M \) denote all the weights of \( M \), i.e. \( \text{wt} M = \{ \lambda \in X | M_\lambda \neq 0 \} \). We define for \( \mu \in \Lambda \) the character \( v^\mu \) by \( v^\mu(K_\alpha) = v^{(\mu, \alpha)} \). We also define \( v^\mu = v^{(\mu, \alpha)} \). We say that \( M \) only has integral weights if all its weights are of the form \( v^\mu \) for some \( \mu \in \Lambda \).

\( W \) acts on \( X \) by the following: For \( \lambda \in X \) define \( w \lambda \) by

\[ (w \lambda)(u) = \lambda(R_{w^{-1}}(u)) \]
Note that $w^{v\mu} = v^{w(\mu)}$.

We will also need the dot action. It is defined as such: For a weight $\mu \in X$ and $w \in W$, $w, \mu = v^{\rho} w(v^{\mu} \mu)$ where $\rho = \frac{1}{2} \sum \beta \in \Phi^{\vee} \beta$ as usual. The Verma module $M(\lambda)$ for $\lambda \in \Lambda$ is defined as $M(\lambda) = U_v \otimes U^\geq K_{\lambda}$ where $K_{\lambda}$ is the onedimensional module with trivial $U_v^+$ action and $U_v^0$ action by $\lambda$ (i.e. $K_{\mu \cdot 1} = \lambda(K_{\mu}))$. $M(\lambda)$ is a highest weight module generated by $v_{\lambda} = 1 \otimes 1$.

Note that $R_{w^{-1}}$ sends a weight space of weight $\mu$ to the weight space of weight $w(\mu)$ since if we have a vector $v$ with weight $\mu$ in a module $M$ we get in $w(M)$ that

$$K_{\alpha} \cdot v = R_{w^{-1}}(K_{\alpha}) v = K_{w^{-1}(\alpha)} v = v^{(w^{-1}(\alpha))|\mu} = v^{(\alpha|w(\mu))} v$$

We define the character of a $U_v$-module $M$ as usual: The character is a map $\text{ch} M : X \to \mathbb{N}$ given by $\text{ch} M(\mu) = \dim M_{\mu}$. Let $\epsilon^\mu$ be the delta function $\epsilon^{\mu}(\gamma) = \delta_{\mu, \gamma}$. We will write $\text{ch} M$ as the formal infinite sum

$$\text{ch} M = \sum_{\mu \in X} \dim M_{\mu} \epsilon^\mu$$

For more details see e.g. [Hum08]. Note that if we define $w(\sum_\mu a_\mu \epsilon^\mu) = \sum_\mu a_\mu v^{w(\mu)}$ then $\text{ch} M = w(\text{ch} M)$ by the above considerations.

**Proposition 3.7**

$$\text{ch} T_w M(\lambda) = \text{ch} M(w, \lambda)$$

**Proof.** To determine the character of $T_w M(\lambda)$ we would like to find a basis. We will do this by looking at some vectorspace isomorphisms to a space where we can easily find a basis. Then use the isomorphisms back again to determine what the basis looks like in $T_w M(\lambda)$. So assume $w = s_{i_N} \cdots s_{i_1}$ is a reduced expression for $w$. Expand to a reduced expression $s_{i_N} \cdots s_{i_{r+1}} s_{i_r} \cdots s_{i_1}$ for $w_0$.

Let $U_v = \text{span}_{Q(v)} \{ F^{a_{r+1}}_\beta \cdots F^{a_N}_\beta | a_i \in \mathbb{N} \}$. We have the canonical vector space isomorphisms

$$U_v^w(w)^* \otimes U_v(w) \cong U_v(w^* \otimes U_v(w)) \otimes \mathbb{C} \cong U_v(w^* \otimes k)$$

The map from the last vectorspace to the first is easily seen to be $f \otimes u \otimes 1 \mapsto f \otimes u \otimes v_\lambda$. $f \in U_v^w(w)^*$, $u \in U_v^w$ and $v_\lambda = 1 \otimes 1 \in U_v \otimes k$ is a highest weight vector in $M(\lambda)$.

So we see that a basis of $T_w M(\lambda) = \{ U_v^w(w)^* \otimes U_v(w) \otimes U_v M \}$ is given by the following: Choose a basis $\{ f_i \}_{i \in I}$ for $U_v(w)^*$ and a basis $\{ u_j \}_{j \in J}$ for $U_v^w$. Then a basis for $T_w M(\lambda)$ is given by

$$\{ f_i \otimes u_j \otimes v_\lambda \}_{i \in I, j \in J}$$

So we can find the weights of $T_w M(\lambda)$ by examining the weights of $f \otimes u \otimes v_\lambda$ for $f \in U_v(w)^*$ and $u \in U_v^w$. By the remarks before this proposition we have that $K_{\alpha}(f \otimes 1) = v^{(\gamma + \sum_{i=1}^r \beta_i)|\alpha} f \otimes K_{\alpha}$ for $f \in U_v^w(w)^*$. So for such $f$ and for $u \in (U_v^w)^w$ the weight of $f \otimes u \otimes v_\lambda$ is $v^{(\gamma + \sum_{i=1}^r \beta_i) \cdot \lambda}$. After the twist with $w$ the
Then we claim that $w^w(\gamma + \mu)w.\lambda$. The weights $\gamma$ and $\mu$ are exactly such that $w(\gamma) < 0$ and $w(\mu) < 0$ so we see that the weights of $T_u M(\lambda)$ are \{ $w^w.\lambda | \mu < 0$ \} each with multiplicity $\mathcal{P}(\mu)$ where $\mathcal{P}$ is Kostant’s partition function. This proves that the character is the same as the character for the Verma module $M(w.\lambda)$. 

**Definition 3.8** Let $\lambda \in X$ and $M(\lambda)$ the Verma module with highest weight $\lambda$. Let $w \in W$. We define

$$M^w(\lambda) = T_u M(w^{-1}.\lambda)$$

Recall the duality functor $D: U_v - \text{Mod} \to U_v - \text{Mod}$. For a $U_v$ module $M$ $DM = \text{Hom}(M, k)$ is the graded dual module with action given by $(xf)(m) = f(S(\omega(m)))$ for $x \in U_v$, $f \in DM$ and $m \in M$. By this definition we have $\text{ch} DM = \text{ch} M$ and $D(\text{ch} M) = M$.

**Theorem 3.9** Let $w_0$ be the longest element in the Weyl group. Let $\lambda \in X$. Then

$$T_{w_0} M(\lambda) \cong DM(w_0.\lambda)$$

**Proof.** We will show that $DT_{w_0} M(w_0.\lambda) \cong M(\lambda)$ by showing that $DT_{w_0} M(w_0.\lambda)$ is a highest weight module with highest weight $\lambda$. We already know that the characters are equal by proposition 3.7 so all we need to show is that $DT_{w_0} M(w_0.\lambda)$ has a highest weight vector of weight $\lambda$ that generates the whole module over $U_v$. Consider the function $g_{\lambda} \in DM^{w_0}(\lambda)$ given by:

$$g_{\lambda}(F_{\beta_N}^{a_N - 1} \otimes \cdots \otimes F_{\beta_1}^{a_1 - 1} \otimes v_{w_0.\lambda}) = \begin{cases} 1 & \text{if } a_N = \cdots = a_1 = 0 \\ 0 & \text{otherwise} \end{cases}$$

We claim that $F_{\beta_N}^{a_N - 1} \otimes \cdots \otimes F_{\beta_1}^{a_1 - 1} \otimes v_{w_0.\lambda}$ with $a_i \in \mathbb{N}$ defines a basis for $M^{w_0}(\lambda)$ so this defines a function on $M^{w_0}(\lambda)$. In the proof of proposition 3.7 we see that a basis is given by $f \otimes 1 \otimes v_\lambda \in U_v (w_0) \otimes U_v \otimes M(\lambda) = T_{w_0} M(\lambda)$. We know that elements of the form $f^{(N)} \cdot f^{(1)}$ defines a basis of $(U_v)$. So we see that the weights of $\mu < 0$ so we see that the weights of $\mathcal{P}(\mu)$ where $\mathcal{P}$ is Kostant’s partition function. This proves that the character is the same as the character for the Verma module $M(w.\lambda)$.

We have shown that $(F_{\beta_N}^{a_N - 1} \otimes \cdots \otimes F_{\beta_1}^{a_1 - 1} \otimes v_{w_0.\lambda})$ is a basis of $M^{w_0}(\lambda)$.

The action on a dual module $DM$ is given by $uf(u') = f(S(\omega(u)u'))$. Remember that the action on $M^{w_0}(\lambda)$ is twisted by $R_{w_0}$ so we get that

$$u g_{\lambda}(F_{\beta_N}^{a_N - 1} \otimes \cdots \otimes F_{\beta_1}^{a_1 - 1} \otimes v_{w_0.\lambda}) = g_{\lambda}(R_{w_0}(S(\omega(u)))F_{\beta_N}^{a_N - 1} \otimes \cdots \otimes F_{\beta_1}^{a_1 - 1} \otimes v_{w_0.\lambda}).$$

In particular for $u = K_\mu$ we get

$$K_\mu g_{\lambda}(F_{\beta_N}^{a_N - 1} \otimes \cdots \otimes F_{\beta_1}^{a_1 - 1} \otimes v_{w_0.\lambda}) = g_{\lambda}(K_{w_0(\mu)}F_{\beta_N}^{a_N - 1} \otimes \cdots \otimes F_{\beta_1}^{a_1 - 1} \otimes v_{w_0.\lambda}) = \nu^e(u.\lambda)(K_{w_0(\mu)})g_{\lambda}(F_{\beta_N}^{a_N - 1} \otimes \cdots \otimes F_{\beta_1}^{a_1 - 1} \otimes v_{w_0.\lambda}).$$
where

\[ c = (w_0(\mu)) \sum_{i=1}^{N} a_i \beta_i + \sum_{i=1}^{N} \beta_i \]

we have

\[
v^\mu(w_0, \lambda)(K_{w_0(\mu)}) = v^\mu(w_0(\mu)) \sum_{i=1}^{N} a_i \beta_i + \sum_{i=1}^{N} \beta_i \left( v^{-\rho(\mu)}(v^\mu \lambda) \right) (K_{w_0(\mu)})
\]

\[
= v^\mu(w_0(\mu)) \sum_{i=1}^{N} a_i \beta_i + \sum_{i=1}^{N} \beta_i \left( v^{-\rho(\mu)}(v^\mu \lambda) \right) (K_{\mu})
\]

\[
= v^\mu(w_0(\mu)) \sum_{i=1}^{N} a_i \beta_i + \sum_{i=1}^{N} \beta_i \left( v^{-\rho(\mu)}(v^\mu \lambda) \right) (K_{\mu})
\]

\[
= v^\mu(w_0(\mu)) \sum_{i=1}^{N} a_i \beta_i + \sum_{i=1}^{N} \beta_i \lambda(K_{\mu})
\]

Setting the \(a_i\)'s equal to zero we get \(\lambda(K_{\mu})\). So \(g_\lambda\) has weight \(\lambda\). We want to show that \(g_\lambda\) generates \(DM^{w_0}(\lambda)\) over \(U_v\).

Let \(M \in \mathbb{N}^N\), \(M = (m_1, \ldots, m_N)\). An element in \(DM^{w_0}(\lambda)\) is a linear combination of elements of the form \(g_M\) defined by:

\[ g_M(F_{\beta_N}^{-a_N-1} \otimes \cdots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0, \lambda}) = \delta_{a_1,m_1} \cdots \delta_{a_N,m_N} \]

This is because of the way the dual module is defined (as the graded dual). We want to show that \(g_M \in U_v g_\lambda\) by using induction over \(m_1 + \cdots + m_N\). Note that \(g_{(0, \ldots, 0)} = g_\lambda\) so this gives the induction start. Assume \(M = (m_1, \ldots, m_N) \in \mathbb{N}^N\). Let \(j\) be such that \(m_j = \cdots = m_{j+1} = 0\) and \(m_j > 0\). By induction we get for \(M' = (0, \ldots, 0, m_j - 1, m_{j-1}, \ldots, m_1)\) that \(g_{M'} \in U_v g_\lambda\). Now let \(u_j = \omega(S^{-1}(R_{w_0}^{-1}(F_{\beta_j})))\). Then

\[ u_j g_\lambda(F_{\beta_N}^{-a_N-1} \otimes \cdots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0, \lambda}) = g_\lambda(F_{\beta_j} F_{\beta_N}^{-a_N-1} \otimes \cdots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0, \lambda}) \]

From lemma \[\text{2.17}\] we get for \(r > j\) (setting \(k = (\beta_j, \beta_r^\nu)\))

\[ F_{\beta_j} F_{\beta_r} = v^{a} F_{\beta_r} F_{\beta_j} + \sum_{i \geq 1} u_{i \beta_r}^{a - (a + 1) i} \left[ \frac{a + i - 1}{i} \right] F_{\beta_r}^{-i} a \text{ad}(F_{\beta_j})(u) \]

But \(g_{M'}\) is zero on every \(F_{\beta_N}^{-a_N-1} \otimes \cdots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0, \lambda}\) where one of the \(a_i\)'s with \(i > j\) is strictly greater than zero. This coupled with the observation above gives us that

\[ u_j g_M'(F_{\beta_N}^{-a_N-1} \otimes \cdots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0, \lambda}) \]

\[ = g_M(v^\nu F_{\beta_N}^{-a_N-1} \otimes \cdots \otimes F_{\beta_1}^{-a_1-1-1} \otimes \cdots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0, \lambda}) \]

\[ = v^\nu g_M(F_{\beta_N}^{-a_N-1} \otimes \cdots \otimes F_{\beta_1}^{-a_1-1} \otimes \cdots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0, \lambda}) \]

where \(c\) is some constant coming from the commutations. We see that \(g_M = v^{-c} u_j g_{M'}\) which finishes the induction step.

So in conclusion we have that \(DM^{w_0}(\lambda)\) is a highest weight module with highest weight \(\lambda\). So we have a surjection from \(M(\lambda)\) to \(DM^{w_0}(\lambda)\). But since the two modules have the same character and the weight spaces are finite dimensional the surjection must be an isomorphism.
We want to define twisting functors so they make sense to apply to $U_A$ modules. Note first that the maps $R_s$ send $U_A$ to $U_A$.

Recall that for $n \in \mathbb{N}$ with $n > 0$ and $F_\beta$ a root vector we have defined in $U_v(F_\beta)$

$$F_\beta^{(-n)} = [n]_\beta ! F_\beta^{-n}$$

i.e. $F_\beta^{(-n)} = \left(F_\beta^{(n)}\right)^{-1}$ (4)

Proposition 3.10 Let $M$ be a $U_v$-module, $\beta \in \Phi^+$ and let $w \in W$. Assume $s_r \cdots s_1$ is a reduced expression of $w$ and $F_\beta = R_{s_1} \cdots R_{s_r} (F_\alpha)$ for some $\alpha \in \Pi$ such that $l(s_\alpha w) > l(w)$ (so we have $w(\beta) = \alpha$). Then

$$w(S_v(F_\beta) \otimes U_v M) \cong S_v(F_\alpha) \otimes U_v w M$$

Proof. Define the map $\varphi : S_v(F_\alpha) \otimes w M \rightarrow w(S_v(F_\beta) \otimes M)$ by

$$\varphi(u F_\alpha^{(-m)} \otimes m) = R_{w^{-1}}(u) F_\beta^{(-m)} \otimes m$$

This is obviously a $U_v$-homomorphism if it is well-defined and it is a bijection because $R_{w^{-1}}$ is a $U_v$-isomorphism. We have to check that if $u F_\alpha^{(-m)} = u' F_\alpha^{(-m')}$ then $R_{w^{-1}}(u) F_\beta^{(-m)} = R_{w^{-1}}(u') F_\beta^{(-m')}$ and that $\varphi(u F_\alpha^{(-m)} \otimes m) = \varphi(u F_\alpha^{(-m)} \otimes R_{w^{-1}}(u') m)$ but $u F_\alpha^{(-m)} = u' F_\alpha^{(-m')}$ if and only if $F_\alpha^{(m')} u = F_\alpha^{(m')} u'$. Using the isomorphism $R_{w^{-1}}$ on this we get $F_\beta^{(-m')} R_{w^{-1}}(u) = F_\beta^{(-m')} R_{w^{-1}}(u')$ which implies $R_{w^{-1}}(u) F_\beta^{(-m)} = R_{w^{-1}}(u') F_\beta^{(-m')}$. For the other equation: Since we only have the definition of $\varphi$ on elements on the form $u F_\alpha^{(-m)} \otimes m$ assume $F_\alpha^{(-m)}' u = \tilde{u} F_\beta^{(-m)}$. This is equivalent to $u' F_\alpha^{(-m)} = F_\alpha^{(-m)} \tilde{u}$. Use $R_{w^{-1}}$ on this to get $R_{w^{-1}}(u') F_\beta^{(-m)} = F_\beta^{(-m)} \tilde{u}$ or equivalently $F_\beta^{(-m)} R_{w^{-1}}(u) = R_{w^{-1}}(\tilde{u}) F_\beta^{(-m)}$. Now we can calculate:

$$\varphi(u F_\alpha^{(-m)} u' \otimes m) = \varphi(u \tilde{u} F_\alpha^{(-m)} \otimes m)$$

$$= R_{w^{-1}}(u \tilde{u}) F_\beta^{(-m)} \otimes m$$

$$= R_{w^{-1}}(u) R_{w^{-1}}(\tilde{u}) F_\beta^{(-m)} \otimes m$$

$$= R_{w^{-1}}(u) F_\beta^{(-m)} \otimes R_{w^{-1}}(u') m = \varphi(u F_\alpha^{(-m)} \otimes R_{w^{-1}}(u)m) \quad \Box$$

Proposition 3.11 $w \in W$. If $s$ is a simple reflection such that $sw > w$ then

$$T_{sw} = T_s \circ T_w$$

Proof. Let $\alpha$ be the simple root corresponding to the simple reflection $s$. By proposition 3.12 we get for $M$ a $U_v$-module:

$$T_{sw} M = s w (S_v^{sw} \otimes U_v M) \cong s w (S_v(R_{w^{-1}}(F_\alpha)) \otimes U_v S_v^{sw} \otimes U_v M)$$

$$\cong s w (S_v(R_{w^{-1}}(F_\alpha)) \otimes U_v S_v^{sw} \otimes U_v M)$$

$$\cong s (S_v(F_\alpha) \otimes U_v w (S_v^{sw} \otimes U_v M))$$

Where the last isomorphism is the one from proposition 3.10 \quad \Box

4 Twisting functors over Lusztig's A-form

We want to define twisting functors so they make sense to apply to $U_A$ modules.
Definition 4.1 Let $s$ be a simple reflection corresponding to a simple root $\alpha$. Let $S^s_A$ be the $U_A$-sub-bimodule of $S_v = S_v(F_\alpha)$ generated by the elements $\{F_\alpha^{(-n)}F_\alpha^{-1}|n \in \mathbb{N}\}$.

Note that $S^s_A \otimes_A \mathbb{Q}(v) = S^s_v$

Proposition 4.2 In $U_v(sl_2)$ let $E, K, F$ be the usual generators and define as in [Lus90] the elements

$$\left[ K; c \right] = \prod_{n=1}^{t} \frac{Kv^c - K^{-1}u^{-c}}{u^s - u^{-s}}$$

then

$$F^{(-s)}F^{-1}E^{(r)} = \sum_{t=0}^{r} E^{(r-t)} \left[ K; r - s - t - 2 \right] F^{(-s-t)}F^{-1}$$

Proof. This is proved by induction over $r$. We define as in [Jan96]

$$[K; c] = \left[ K; c \right] = \frac{Kv^c - K^{-1}u^{-c}}{u^s - u^{-s}}$$

From [Jan96] we get $EF^{s+1} = F^{s+1}E + [s + 1]E^s[K; -s]$ so

$$F^{(-s)}E = EF^{-s+1} + [s+1]F^{-1}E[F^{(-s)}]F^{-s-1} = EF^{(-s)} + [s+1][K; -2-s]F^{-s-2}$$

And multiplying through with $[s]!$ we get

$$F^{(-s)}F^{-1}E = EF^{(-s)}F^{-1} + [K; -2-s]F^{(-s-1)}F^{-1}.$$  

This is the induction start. The rest is the induction step. In the process you have to use that

$$\frac{1}{[r]} \left( [r - t; \begin{array}{c} K; r - s - t \\ t \end{array}] + [K; r - 1 - s - t; \begin{array}{c} t \\ t - 1 \end{array}] [K; -s - t] \right) = [K; r - s - t - 1]$$

or equivalently that

$$[r - t][K; r - s - t] + [t][K; -s - t] = [r][K; r - s - 2t]$$

This can be shown by a direct calculation.  \Box

We could have proved this in the other way around instead too to get

Proposition 4.3

$$E^{(r)}F^{(-s)}F^{-1} = \sum_{t=0}^{r} F^{(-s-t)}F^{-1} \left[ K; s + t - r + 2 \right] E^{(r-t)}$$

The above and Corollary 2.19 shows that $S_A(F)$ is a bimodule if there is no $G_2$ part. It may very well be possible to show the same for $G_2$ but the formulas will be even longer in this case. I have not done these calculations so that is why we exclude the $G_2$ case.

We can now define the twisting functor $T^s_A$ corresponding to $s$:
Definition 4.4 Let $s$ be a simple reflection corresponding to a simple root $\alpha$. The twisting functor $T_A^s : U_A \text{-Mod} \to U_A \text{-Mod}$ is defined by: Let $M$ be a $U_A$ module, then:

$$T_A^s(M) = \mathbb{Q}(F_\alpha \otimes U_A M)$$

Note that $T_A^s(M) \otimes_A \mathbb{Q}(v) = T_s(M \otimes_A \mathbb{Q}(v))$ so that if $M$ is a $\mathbb{Q}(v)$ module then $T_A^s = T_s$ on $M$.

We want to define the twisting functor for every $w \in W$ such that if $w$ has a reduced expression $w = s_i \cdots s_{i_1}$ then $T_w^A = T_{s_{i_1}} \circ \cdots \circ T_{s_{i_1}}$. As before we define a 'semiregular bimodule' $S_A^w = U_A \otimes_{U_A^-} U_A^-(w)^*$ and show this is a bimodule isomorphic to $S_A(F_{\beta_i}) \otimes_{U_A} \cdots \otimes_{U_A} S_A(F_{\beta_1})$.

Theorem 4.5 $S_A^w := U_A \otimes_{U_A^-} U_A^-(w)^*$ is a bimodule isomorphic to $S_A(F_{\beta_i}) \otimes_{U_A} \cdots \otimes_{U_A} S_A(F_{\beta_1})$ and the functors $T_A^w$, $s \in \Pi$ satisfy braid relations.

Proof. Note that $U_A^-(w)$ can be seen as an $A$-submodule of $U^-_w(w)$ and similarly $U_A^-(w)^*$ can be seen as a submodule of $U^-_w(w)^*$. So we have an injective $A$ homomorphism $S_A^w \to S_A^v$.

Assume the length of $w$ is $r$ and $w = s_i w'$, $l(w') = r - 1$. We want to show that the isomorphism $\varphi_r$ from proposition 3.2 restricts to an isomorphism $S_A^w \to S_A(F_{\beta_r}) \otimes_{U_A} S_A^v$.

Assume $f \in U_A^-(w)$ is such that $f = g \cdot f_n'$ meaning that $f(xF_{\beta_r}^n) = g(x)\delta_{m,n}$, $x \in U_A^-(w')$, $n \in \mathbb{N}$ where $g \in U_A^-(w')^*$. Then $f_n' = [m]_{\beta_r} f_m$ where $f_m$ is defined like in proposition 3.2 and for $u \in U_A$ we have therefore

$$\varphi_r(u \otimes f) = uF_{\beta_r}^{-m}F_{\beta_r}^{-1} \otimes (1 \otimes g)$$

which can be seen to lie in $S_A(F_{\beta_r}) \otimes_{U_A} S_A^v$. The inverse also restricts to a map to the right space:

$$\psi_r(uF_{\beta_r}^{-(m)}F_{\beta_r}^{-1} \otimes (1 \otimes g)) = \psi_r(u[m]_{\beta_r}F_{\beta_r}^{-(m-1)} \otimes (1 \otimes g)) = [m]_{\beta_r} u \otimes f_m \cdot g = u \otimes f_m \cdot g$$

The maps are well defined because they are restrictions of well defined maps and it is easy to see that they are inverse to each other.

As in the generic case we get a right module action on $S_A^w$ in this way. This is the right action coming from $S_A^w$ restricted to $S_A^w$. So now we have $S_A^w = S_A(F_{\beta_r}) \otimes_{U_A} \cdots \otimes_{U_A} S_A(F_{\beta_1})$. Showing that the twisting functors then satisfy braid relations is done in the same way as in proposition 3.11.

Now we can define $T_A^w = T_{s_{i_1}} \circ \cdots \circ T_{s_{i_1}}$ if $w = s_{i_1} \cdots s_{i_1}$ is a reduced expression of $w$. By the previous theorem there is no ambiguity in this definition since the $T_A^s$'s satisfy braid relations.

It is now possible for any $A$ algebra $R$ to define twisting functors $U_R \text{-Mod} \to U_R \text{-Mod}$. Just tensor over $A$ with $R$. 26
F.x. let $R = \mathbb{C}$ with $v \mapsto 1$. $S_A(F_\beta) \otimes_A \mathbb{C}$ is just the normal $S^n = U(g_\lambda)/U$ via the isomorphism $uF_{\beta}^{(-n)}F_{\beta}^{-1} \otimes 1 \mapsto \varpi y_{\beta}^{-n-1}$ where $\varpi$ is given by the isomorphism between $U^{1}_{\lambda} \otimes_A \mathbb{C}$ and $U$.

**Theorem 4.6** Let $R$ be an $A$-module with $v \in A$ being sent to $q \in R \setminus \{0\}$. Let $\lambda : U^{1}_{R} \to R$ be an $R$-algebra homomorphism and let $M_{R}(\lambda) = U_{R} \otimes_{U^{1}_{R}} R_{\lambda}$ be the $U_{R}$ Verma module with highest weight $\lambda$ where $R_{\lambda}$ is the rank 1 free $U^{1}_{R}$-module with $U^{1}_{R}$ acting trivially and $U^{0}_{R}$ acting as $\lambda$. Let $D : U_{R} \to U_{R}$ be the duality functor on $U_{R} \to \text{Mod}$ induced from the duality functor on $U_{A} \to U_{A}$. Then

$$T_{w_0}^{R}M_{R}(\lambda) \cong DM_{R}(w_0, \lambda)$$

**Proof.** The proof is the almost the same as the proof of theorem 3.9. We have by Corollary 2.19 (setting $k = \langle \beta, \beta' \rangle$)

$$F_{\beta_1}F_{\beta}^{(-a)}F_{\beta}^{-1} = q^{-(a+1)(\beta, \beta')}F_{\beta}^{(-a)}F_{\beta}^{-1}F_{\beta} + \sum_{j \geq 1} q_{\beta_j}^{-(a+1)k-(a+2)j}F_{\beta}^{(-a-1)}F_{\beta}^{-1}d(F_{\beta}^{(j)}(u))$$

Define for $M = (m_1, \ldots, m_N) \in \mathbb{N}$ the function

$$g_{M}(F_{\beta_1}^{(-a_N)}F_{\beta_1}^{-1} \otimes \cdots \otimes F_{\beta_1}^{(-a_1)}F_{\beta_1}^{-1} \otimes w_{0, \lambda}) = \begin{cases} 1 & \text{if } a_1 = m_1 \ldots a_N = m_N \\ 0 & \text{otherwise} \end{cases}$$

Note that $g(0, \ldots, 0) = g_{\lambda}$ from theorem 3.9. In particular it has weight $\lambda$. We want to show that $DM_{R}(\lambda) = U_{R}g(0, \ldots, 0)$. We use induction on the number of nonzero entries in $M$. Assume $j$ is such that $m_N = \cdots = m_{j+1} = 0$ and $m_j = n > 0$. Let $M' = (0, \ldots, 0, m_j-1, \ldots, m_1)$. By induction $g_{M'} \in U_{R}g(0, \ldots, 0)$.

Set $u = \omega(S^{-1}(R_{w_0}^{1}(F_{\beta_j}^{(n)}))).$ Then

$$ug_{M'}(F_{\beta_1}^{(-a_N)}F_{\beta_1}^{-1} \otimes \cdots \otimes F_{\beta_1}^{(-a_1)}F_{\beta_1}^{-1} \otimes w_{0, \lambda})$$

$$= g_{M'}(F_{\beta_1}^{(n)}F_{\beta_1}^{(-a_N)}F_{\beta_1}^{-1} \otimes \cdots \otimes F_{\beta_1}^{(-a_1)}F_{\beta_1}^{-1} \otimes w_{0, \lambda})$$

$$= g_{M'}(1_{[n]}_{\beta_j}F_{\beta_1}^{(-a_N)}F_{\beta_1}^{-1} \otimes \cdots \otimes F_{\beta_1}^{(-a_1)}F_{\beta_1}^{-1} \otimes w_{0, \lambda})$$

$$= g_{M'}(q_{a_j}^{1_{[n]}_{\beta_j}}F_{\beta_1}^{(-a_N)}F_{\beta_1}^{-1} \otimes \cdots \otimes F_{\beta_j}^{(-a_N)}F_{\beta_j}^{-1} \otimes \cdots \otimes F_{\beta_1}^{(-a_1)}F_{\beta_1}^{-1} \otimes w_{0, \lambda})$$

$$\vdots$$

$$\vdots$$

$$= g_{M'}(q_{a_j}^{1_{[n]}_{\beta_j}}F_{\beta_1}^{(-a_N)}F_{\beta_1}^{-1} \otimes \cdots \otimes F_{\beta_j}^{(-a_N)}F_{\beta_j}^{-1} \otimes \cdots \otimes F_{\beta_1}^{(-a_1)}F_{\beta_1}^{-1} \otimes w_{0, \lambda})$$

$$= \begin{cases} g_{M'}(q_{a_j}^{1_{[n]}_{\beta_j}}F_{\beta_1}^{(-a_N)}F_{\beta_1}^{-1} \otimes \cdots \otimes F_{\beta_j}^{(-a_N)}F_{\beta_j}^{-1} \otimes \cdots \otimes F_{\beta_1}^{(-a_1)}F_{\beta_1}^{-1} \otimes w_{0, \lambda}) & \text{if } n \leq a_j \\ 0 & \text{otherwise} \end{cases}$$

for some appropriate integers $c_1, \ldots, c_n \in \mathbb{Z}$. $g_{M'}$ is nonzero on this only when $n = a_j$. So we get in conclusion that $ug_{M'} = v^{-c_\alpha}g_{M}$. This finishes the induction step. $\square$
5 \mathfrak{sl}_2 calculations

Assume \( g = \mathfrak{sl}_2 \). Let \( r \in \mathbb{N} \). Let \( M_A(v^r) \) be the \( U_A(\mathfrak{sl}_2) \) Verma module with highest weight \( v^r \in \mathbb{Z} \) i.e. \( M_A(v^r) = U_A \otimes_{U_A^r} A_v \) where \( A_v \) is the free module of rank 1, \( U_A^r \) module with \( U_A^r \) acting trivially and \( K \cdot 1 = q^r \). Inspired by [And03] we see that in \( \mathfrak{sl}_2 \) acting trivially and \( K \cdot 1 = q^r \). In the following section we will try to say something about the composition factors of a Verma module so it is natural to consider first \( \mathfrak{sl}_2 \) Verma modules.

Definition 5.1 Let \( g = \mathfrak{sl}_2 \). Let \( r \in \mathbb{N} \). Then \( H_A(v^r) \) is defined to be the free \( U_A(\mathfrak{sl}_2) \)-module of rank \( r + 1 \) with basis \( e_0, \ldots, e_r \) defined as follows:

\[
K e_i = v^{−2i} e_i, \quad F^{(e)} e_i = \begin{bmatrix} K; c \end{bmatrix} e_i = \begin{bmatrix} r − 2i + c \end{bmatrix} e_i
\]

\[F^{(n)} e_i = \left[ \begin{array}{c} i \end{array} \right] e_{i−n}, \quad n \in \mathbb{N}
\]

\[F^{(n)} e_i = \left[ \begin{array}{c} r − i \end{array} \right] e_{i+n}, \quad n \in \mathbb{N}
\]

for \( i = 0, \ldots, r \). Where \( e_{<0} = 0 = e_{>r} \).

Lemma 5.2 Let \( g = \mathfrak{sl}_2 \). Let \( r \in \mathbb{N} \). Then we have a short exact sequence:

\[0 \to DM_A(v^{−r−2}) \to M_A(v^r) \to H_A(v^r) \to 0
\]

Proof. We use the fact that \( DM_A(v^{−r−2}) = T^A_r M_A(v^r) \) by theorem 4.6. Let \( e_i = F^{(i)} w_0 \) where \( w_0 \) is a highest weight vector in \( M_A(v^r) \). We will construct a \( U_A \)-homomorphism \( span_A \{ e_i | i > r \} \to DM_A(−r − 2) \). Let \( \tau \) be as defined in [Jan96] Chapter 4. Note that in \( U_A(F) S(\tau(F)) \) is invertible so we can consider \( S \) and \( \tau \) as automorphisms of \( U_A(F) \). We define a map by

\[e_{r+i} \mapsto (−1)^{r+i} S(\tau(F(−i−1)))w_0
\]

Note that for \( \mathfrak{sl}_2 \) \( R_\omega = S \circ \tau \circ \omega \). Using this and the formula in proposition 4.2 it is straightforward to check that this is a \( U_A \)-homomorphism. \( \square \)

If we specialize to an \( A \)-algebra \( R \) with \( R \) being a field where \( v \) is sent to a non-root of unity \( q \in R \) we get that \( M_R(q^r) = U_R \otimes_{U_A} M_A(v^r) \) is simple for \( k < 0 \). So in the above with \( r \in \mathbb{N} \), \( DM_R(q^{−r−2}) = M_R(q^{−r−2}) = L_R(q^{−r−2}) \) and actually we see also that \( H_R(q^r) = L_R(q^r) \). So there is an exact sequence

\[0 \to L_R(q^{−r−2}) \to M_R(q^r) \to L_R(q^r) \to 0
\]
where \( U \) is an isomorphism if the exact sequence
\[
M \rightarrow \cdots \rightarrow 0.
\]
which is similar to earlier.

\[\lambda X : U^q \rightarrow B \text{ to be the weight defined by } \lambda X = \lambda(K_\mu)X \text{ and we define } M_B(\lambda X) = U_B \otimes_{U_B \otimes \mathbb{Z}_+} B_{AX} \text{ to be the Verma module with highest weight } \lambda X.\]

Note that \( M_B(\lambda X) \otimes_B \mathbb{C} \cong M(\lambda) \) when we consider \( \mathbb{C} \) as a \( B \)-algebra via the specialization \( X \mapsto 1 \).

For a simple root \( \alpha = \alpha_i \in \Pi \) we define \( M_{B,i}(\lambda X) := U_B(i) \otimes_{U_B^0} B_{AX} \), where \( U_B(i) \) is the subalgebra generated by \( U_B^0 \) and \( F_{\alpha_i} \). We define \( M_{B,i}^n(\lambda) := s_i((U_B(i) \otimes_{U_B(s_i)} U_B(s_i)^*) \otimes_{U_B(i)} M_{B,i}(s,\lambda)) \) where the module \( (U_B(i) \otimes_{U_B(s_i)} U_B(s_i)^*) \) is a \( U_B(i) \)-bimodule isomorphic to \( S_{B,i}(s,\lambda) = (U_B(i)(F_{\alpha_i})/U_B(i) \otimes_{U_B(i)} s,\lambda) \) by similar arguments as earlier.

**Proposition 6.1** There exists a nonzero homomorphism \( \varphi : M_B(\lambda X) \rightarrow M_B^n(\lambda X) \) which is an isomorphism if \( q^\alpha \lambda(K_\alpha) \notin \pm q^\mathbb{Z}_+ \) and otherwise we have a short exact sequence
\[
0 \rightarrow M_B(\lambda X) \xrightarrow{\varphi} M_B^n(\lambda X) \rightarrow M(s_\alpha,\lambda) \rightarrow 0.
\]
where we have identified the cokernel \( M_B^n(s_\alpha,\lambda X)/(X - 1)M_B(s_\alpha,\lambda X) \) with \( M(s_\alpha,\lambda) \).

Furthermore there exists a nonzero homomorphism \( \psi : M_B^n(\lambda X) \rightarrow M_B(\lambda X) \) which is an isomorphism if \( q^\alpha \lambda(K_\alpha) \notin \pm q^\mathbb{Z}_+ \) and otherwise we have a short exact sequence
\[
0 \rightarrow M_B^n(\lambda X) \xrightarrow{\psi} M_B(\lambda X) \rightarrow M(\lambda)/M(s_\alpha,\lambda) \rightarrow 0.
\]

**Proof.** We will first define a map from \( M_B,i(\lambda X) \) to
\[
M_B^n,i(\lambda X) = s_i((U_B(i)(F_{\alpha_i})/U_B(i) \otimes_{U_B} M_B,i(s_\alpha,\lambda X)) .
\]
Setting \( \lambda' = \lambda X \) define
\[
\varphi(F_{\alpha}^{(n)} v_{\lambda'}) = a_n F_{\alpha}^{(-n)} F_{\alpha}^{-1} \otimes v_{s_\alpha,\lambda'}
\]
where
\[
a_n = (-1)^n q_{\alpha}^{-n(n+1)} \frac{\pi_t^1 \lambda'(K_\alpha)^{n} \prod_{t=1}^n q_{\alpha}^{-t} \lambda'(K_\alpha) - q_{\alpha}^{-1} \lambda'(K_\alpha)^{-1}}{q_{\alpha}^n - q_{\alpha}^{-t}}.
\]
So we need to check that this is a homomorphism: First of all for $K$

$$K_\mu \cdot a_n F_\alpha^{(-n)} F_\alpha^{(-n)-1} \otimes v_{s_n, \nu} = a_n K_{s_n(\mu)} F_\alpha^{(-n)} F_\alpha^{(-n)-1} \otimes v_{s_n, \nu}$$

$$= q^{(n+1)(s_\mu)\alpha} (s_\alpha, \nu)(K_{s_n(\mu)} F_\alpha^{(-n)} F_\alpha^{(-n)-1}) \otimes v_{s_n, \nu}$$

$$= q^{-(n+1)(\mu)\alpha} q^{-\rho(\rho, s_n(\mu))} q^{(\rho, \mu)\nu}(K_\mu) F_\alpha^{(-n)} F_\alpha^{(-n)-1} \otimes v_{s_n, \nu}$$

$$= q^{-n(\mu)\alpha} X(K_\mu) F_\alpha^{(-n)} F_\alpha^{(-n)-1} \otimes v_{s_n, \nu}$$

$$= \varphi(K_\mu F_\alpha^{(n)} v_\lambda)$$

We have

$$E_\alpha \cdot a_n F_\alpha^{(-n)} F_\alpha^{(-n)-1} \otimes v_{s_n, \nu} = a_n R_n(E_\alpha) F_\alpha^{(-n)} F_\alpha^{(-n)-1} \otimes v_{s_n, \nu}$$

$$= - a_n F_\alpha^{(-n)+1} \otimes v_{s_n, \nu}$$

$$= q^{2n} \alpha(K_\mu)[n] a_n F_\alpha^{(-n+1)} F_\alpha^{(-n)-1} \otimes v_{s_n, \nu}$$

and

$$\varphi(E_\alpha F_\alpha^{(n)} v_\lambda) = \varphi\left( F_\alpha^{(n+1)} \frac{q_\alpha^{1-n} - q_\alpha^{n-1} K^{-1}(K_\alpha)}{q_\alpha - q_\alpha} \right) \otimes v_\lambda$$

so we see that $\varphi(E_\alpha F_\alpha^{(n)} v_\lambda) = E_\alpha \cdot \varphi(F_\alpha^{(n)} v_\lambda)$. Clearly $\varphi(E_\alpha F_\alpha^{(n)} v_\lambda) = 0 = E_\alpha \cdot a_n F_\alpha^{(-n)} F_\alpha^{(-n)-1} \otimes v_{s_n, \nu}$ for any simple $\alpha' \neq \alpha$ so what we have left is $F_\alpha$: By Proposition 4.3

$$F_\alpha \cdot a_n F_\alpha^{(-n)} F_\alpha^{(-n)-1} \otimes v_{s_n, \nu}$$

$$= a_n R_n(F_\alpha) F_\alpha^{(-n)} F_\alpha^{(-n)-1} \otimes v_{s_n, \nu}$$

$$= - a_n K^{-1} F_\alpha^{(-n)} F_\alpha^{(-n)-1} \otimes v_{s_n, \nu}$$

$$= - a_n K^{-1} F_\alpha^{(-n)+1} \otimes v_{s_n, \nu}$$

$$= - a_n q_\alpha^{-2(n+1)} s_\alpha(K_\mu)^{-1} \frac{n_\alpha^{n+2} - q_\alpha^{n+2} s_\alpha(K_\mu)^{-1}}{q_\alpha - q_\alpha} \otimes v_{s_n, \nu}$$

and

$$\varphi(F_\alpha F_\alpha^{(n)} v_\lambda) = [n + 1]_\alpha \varphi(F_\alpha^{(n+1)} v_\lambda)$$

$$= [n + 1]_\alpha a_n F_\alpha^{(-n)+1} \otimes v_\lambda$$

so we see that $\varphi(F_\alpha F_\alpha^{(n)} v_\lambda) = F_\alpha \cdot \varphi(F_\alpha^{(n)} v_\lambda)$. Now note that if $\lambda(K_\mu) \notin \pm q_\alpha^n$ then $X - 1$ does not divide $a_n$ for any $n \in \mathbb{N}$ implying that $a_n$ is a unit. So when $\lambda(K_\mu) \notin \pm q_\alpha^n$, $\varphi$ is an isomorphism. If
\( \lambda(K_{n}) = \varepsilon q_{\varepsilon}^{n} \) for some \( \varepsilon \in \{ \pm 1 \} \) and \( r \in \mathbb{N} \) we see that \( X - 1 \) divides \( a_{n} \) for any \( n > r \) so the image of \( \varphi \) is

\[
\text{span}_B \left\{ F_{\alpha}^{(-n)} F_{\alpha} \otimes v_{s_{n}, \lambda} | n \leq r \right\} + (X - 1) \text{span}_B \left\{ F_{\alpha}^{(-n)} F_{\alpha}^{-1} \otimes v_{s_{n}, \lambda} | n > r \right\}.
\]

Thus the cokernel \( M_{B,3}^{nd}(\lambda)/\text{Im} \varphi \) is equal to

\[
\text{span}_B \left\{ F_{\alpha}^{(-n)} F_{\alpha}^{-1} \otimes v_{s_{n}, \lambda} | n > r \right\} / (X - 1) \text{span}_B \left\{ F_{\alpha}^{(-n)} F_{\alpha}^{-1} \otimes v_{s_{n}, \lambda} | n > r \right\}
\]

which is seen to be isomorphic to \( M_{B,3}^{nd}(s_{r}, \lambda')/(X - 1)M_{B,3}^{nd}(s_{r}, \lambda') \).

If \( \lambda(K_{n}) \notin \pm q_{\varepsilon}^{n} \) then obviously we can define an inverse to \( \varphi, \psi : M_{B,3}^{nd}(\lambda') \to M_{B,3}(\lambda') \). If \( \lambda(K_{n}) = \varepsilon q_{r}^{n} \) for some \( \varepsilon \in \{ \pm 1 \} \) and some \( r \in \mathbb{N} \) we define \( \psi : M_{B,3}^{nd}(\lambda') \to M_{B,3}(\lambda') \) by

\[
\psi \left( F_{\alpha}^{(-n)} F_{\alpha}^{-1} \otimes v_{s_{n}, \lambda'} \right) = \left( \frac{X - 1}{a_{n}} \right) F_{\alpha}^{(n)} v_{\lambda'}. \]

(Note that for all \( \lambda \) and all \( n \in \mathbb{N}, (X - 1)^{2} / a_{n} \) so \( (X - 1)^{2} / a_{n} \in B \). This implies \( \varphi \circ \psi = (X - 1) \text{id} \) and \( \psi \circ \varphi = (X - 1) \text{id} \). Using that \( \varphi \) is a \( U_{q} \)-homomorphism we show that \( \psi \) is: For \( u \in U_{q} \) and \( v \in M_{B,3}^{nd}(\lambda') \):

\[
(X - 1) \psi(uv) = \psi(u \varphi(v))) = \psi(\varphi(u \psi(v))) = (X - 1) u \psi(v).
\]

Since \( B \) is a domain this implies \( \psi(uv) = u \psi(v) \).

We see that \( X - 1 \) divides \( \frac{1}{a_{n}} \) for any \( n \leq r \) so the image of \( \psi \) is

\[
(X - 1) \text{span}_B \left\{ F_{\alpha}^{(n)} v_{\lambda} | n \leq r \right\} + \text{span}_B \left\{ F_{\alpha}^{(n)} v_{\lambda} | n > r \right\}.
\]

Thus the cokernel \( M_{B,3}(\lambda)/\text{Im} \psi \) is equal to

\[
\text{span}_B \left\{ F_{\alpha}^{(n)} v_{\lambda} | n \leq r \right\} / (X - 1) \text{span}_B \left\{ F_{\alpha}^{(n)} v_{\lambda} | n \leq r \right\}
\]

which is seen to be isomorphic to

\[
M_{B,3}(\lambda)/M_{B,3}(s_{r}, \lambda)
\]

Now we induce to the whole quantum group: We have that

\[
M_{B}(\lambda') = U_{B} \otimes_{U_{B}(i)} M_{B,3}(\lambda')
\]

and

\[
M_{B}(\lambda') = s_{1} \left( (U_{B} \otimes_{U_{B}(i)} U_{B}(s_{r})) \otimes_{U_{B}(i)} U_{B} \otimes_{U_{B}(i)} B_{\lambda'} \right)
\]

\[
\cong s_{1} \left( (U_{B} \otimes_{U_{B}(i)} U_{B}(i)) \otimes_{U_{B}(i)} U_{B}(s_{r}) \otimes_{U_{B}(i)} B_{\lambda'} \right)
\]

\[
\cong U_{B} \otimes_{U_{B}(i)} s_{1} \left( (U_{B}(i) \otimes_{U_{B}(i)} U_{B}(s_{r})) \otimes_{U_{B}(i)} B_{\lambda'} \right)
\]

\[
\cong U_{B} \otimes_{U_{B}(i)} M_{B,3}(\lambda')
\]

so by inducing to \( U_{B} \)-modules using the functor \( U_{B} \otimes_{U_{B}(i)} - \) we get a map \( \varphi : M_{B}(\lambda') \to M_{B}(\lambda') \) and a map \( \psi : M_{B}(\lambda') \to M_{B}(\lambda') \). This functor is exact on \( M_{B,3}(\lambda') \) and \( M_{B,3}(\lambda') \) so the proposition follows from the above calculations. \( \square \)
Proposition 6.2 Let $\lambda : U^0_q \to \mathbb{C}$ be a weight. Set $\lambda' = \lambda X$. Let $w \in W$ and $\alpha \in \Pi$ such that $w(\alpha) > 0$. There exists a nonzero homomorphism $\varphi : M^u_B(\lambda') \to M^w_B(\lambda')$ that is an isomorphism if $q^\rho \lambda(K_{w(\alpha)}) \not\in \pm q^Z$ and otherwise we have the short exact sequence

$$0 \to M^u_B(\lambda') \xrightarrow{\varphi} M^w_B(\lambda') \to M^w(s_{w(\alpha)} : \lambda) \to 0$$

where the cokernel $M^u_B(\lambda')/(X - 1)M^w_B(\lambda')$ is identified with $M^w(s_{w(\alpha)} : \lambda)$.

Furthermore there exists a nonzero homomorphism $\psi : M^w_B(\lambda X) \to M^w_B(\lambda X)$ which is an isomorphism if $q^\rho \lambda(K_{w(\alpha)}) \not\in \pm q^Z$ and otherwise we have a short exact sequence

$$0 \to M^w_B(\lambda X) \xrightarrow{\psi} M^w_B(\lambda X) \to M^w(s_{w(\alpha)} : \lambda) \to 0.$$

Proof. Let $\mu = w^{-1} \lambda$ and $\mu' = \mu X$ then from Proposition 6.1 we get a homomorphism $M_B(\mu') \to M^u_B(\mu')$ and a homomorphism $M^u_B(\mu') \to M_B(\mu')$. Observe that

$$q^\mu \lambda(K_{\alpha}) = w^{-1} \lambda(K_{\alpha})$$

$$= w^{-1}(q^\rho \lambda)(K_{\alpha})$$

$$= q^{(\rho | w(\alpha))} \lambda(K_{w(\alpha)})$$

$$= (q^\rho \lambda)(K_{w(\alpha)})$$

so $M_B(\mu') \to M^u_B(\mu')$ and $M^u_B(\mu') \to M_B(\mu')$ are isomorphisms if $(q^\rho \lambda)(K_{w(\alpha)}) \not\in \pm q^Z$ and otherwise we have the short exact sequences

$$0 \to M_B(\mu') \to M^u_B(\mu') \to M(\mu') \to 0,$$

and

$$0 \to M^u_B(\mu') \to M_B(\mu') \to M(\mu')/M(s_{\alpha} : \mu') \to 0.$$

Now we use the twisting functor $T_w$ on the homomorphisms $M_B(\mu') \to M^u_B(\mu')$ and $M^u_B(\mu') \to M_B(\mu')$ to get homomorphisms $\varphi : M^u_B(\lambda) \to M^u_B(\lambda)$ and $\psi : M^u_B(\lambda X) \to M^u_B(\lambda X)$ (using the fact that $T_w \circ T_w = T_{w w}$). We are done if we show that $T_w$ is exact on Verma modules. But

$$T_w M_B(\mu') = (U^-_B(\mu') \otimes_{U^+_B(\mu')} U_B) \otimes_{U^+_B(\mu')} U_B \otimes_{U^-_B(\mu')} B_{\mu'}$$

$$\cong w \left( (U^-_B(\mu') \otimes_{U^+_B(\mu')} U_B) \otimes_{U^-_B(\mu')} B_{\mu'} \right)$$

$$\cong w \left( U^-_B(\mu') \otimes_{U^+_B(\mu')} U_B \otimes_{U^-_B(\mu')} B_{\mu'} \right)$$

as vectorspaces and $U^0_B$ modules. Observing that $U^-_B$ is free over $U^-_B(w)$ we get the exactness. \[\square\]

Fix a weight $\lambda : U^0_q \to \mathbb{C}$ and a $w \in W$. Define $\Phi^+(w) := \Phi^+ \cap w(\Phi^-) = \{\beta \in \Phi^+ | w^{-1} \beta < 0\}$ and $\Phi^+(\lambda) := \{\beta \in \Phi^+ | q^\rho \lambda(K_\beta) \in \pm q^Z\}$. Choose a
reduced expression of $w_0 = s_{i_1} \cdots s_{i_N}$ such that $w = s_{i_n} \cdots s_{i_1}$. Set

$$
\beta_j = \begin{cases} 
-w s_{i_1} \cdots s_{i_j-1}(\alpha_{i_j}), & \text{if } j \leq n \\
s_{i_1} \cdots s_{i_j-1}(\alpha_{i_j}), & \text{if } j > n
\end{cases}
$$

Then $\Phi^+ = \{\beta_1, \ldots, \beta_N\}$ and $\Phi^+(w) = [\beta_1, \ldots, \beta_n]$. We denote by $\Psi_B^w(\lambda)$ the composite

$$
M_B^w(\lambda X) \xrightarrow{\varphi^w_2(\lambda)} M_B^{ws_1}(\lambda X) \xrightarrow{\varphi^w_2(\lambda)} \cdots \xrightarrow{\varphi^w_N(\lambda)} M_B^{wus_n}(\lambda X)
$$

where the homomorphisms are the ones from Proposition 6.2 i.e. the first $n$ homomorphisms are the $\psi$’s and the last $N - n$ homomorphisms are the $\varphi$’s from Proposition 6.2. We denote by $\Psi^w(\lambda)$ the $U_q$-homomorphism $M^w(\lambda X) \rightarrow M^{wus}(\lambda X)$ induced by tensoring the above $U_B$-homomorphism with $\mathbb{C}$ considered as a $B$ module by $X \mapsto 1$.

In analogy with Theorem 7.1 in [AL03] and Proposition 4.1 in [And03] we have

**Theorem 6.3** Let $\lambda : U^0_q \rightarrow \mathbb{C}$ be a weight. Let $w \in W$. Then there exists a filtration of $M^w(\lambda)$, $M^w(\lambda) \supset M^w(\lambda)^1 \supset \cdots \supset M^w(\lambda)^r$ such that $M^w(\lambda)/M^w(\lambda)^1 \cong \text{Im} \Psi^w(\lambda) \subset M^{wus}(\lambda)$ and

$$
\sum_{i=1}^r \text{ch} M^w(\lambda)^i = \sum_{\beta \in \Phi^+(\lambda) \setminus \Phi^+(w)} \text{ch}(M(\lambda) - \text{ch}(s_\beta \lambda)) + \sum_{\beta \in \Phi^+(\lambda) \setminus \Phi^+(w)} \text{ch}(s_\beta \lambda)
$$

**Proof.** Set $\lambda' = \lambda X$. Define for $i \in \mathbb{N}$

$$
M_B^w(\lambda')^i = \{m \in M_B^w(\lambda') : \Psi_B^w(\lambda)(m) \in (X - 1)^i M_B^{wus}(\lambda')\}.
$$

Set $M^w(\lambda)^i = \pi(M_B^w(\lambda')^i)$ where $\pi : M_B^w(\lambda) \rightarrow M^w(\lambda)$ is the canonical homomorphism from $M_B^w(\lambda)$ to $M_B^w(\lambda)/(X - 1)M_B^w(\lambda) \cong M^w(\lambda)$. This defines a filtration of $M^w(\lambda)$. We have $M^w(\lambda)^N + 1 = 0$ so the filtration is finite.

Let $\mu : U^0_q \rightarrow \mathbb{C}$ be a weight. Set $\mu' = \mu X$. The maps $\varphi^w_\mu(\lambda)$ restrict to weight spaces. Denote the restriction $\varphi^w_\mu(\lambda)_{\mu'}$. Let $\Psi_B^w(\lambda)_{\mu'} : M_B^w(\lambda)_{\mu'} \rightarrow M_B^{wus}(\lambda)_{\mu'}$ be the restriction of $\Psi_B^w(\lambda)$ to the $\mu'$ weight space. We have a nondegenerate bilinear form $(\cdot, \cdot)$ on $M(\lambda)_{\mu'}$ given by $(x, y) = (\Psi_B^w(\lambda)_{\mu'}(x))(y)$. It is nondegenerate since $\Psi_B^w(\lambda)$ is injective. Let $\nu : B \rightarrow \mathbb{C}$ be the $(X - 1)$-adic valuation i.e. $\nu(b) = m$ if $b = (X - 1)^m b'$, $(X - 1) \nmid b'$. We have by [Hum08] Lemma 5.6 (originally Lemma 5.1 in [Jan9])

$$
\sum_{j \geq 1} \text{dim}(M_j)_{\mu} = \nu(\det \Psi_B^w(\lambda)_{\mu'}).
$$

Clearly $\nu(\det \Psi_B^w(\lambda)_{\mu'}) = \sum_{j=1}^N \nu(\det \varphi^w_j(\lambda)_{\mu'})$ and the result follows when we show:

$$
\nu(\det \varphi^w_j(\lambda)_{\mu'}) = \dim_{\mathbb{C}} \text{coker} \varphi^w_j(\lambda)_{\mu'}.
$$

Fix $\varphi := \varphi^w_j(\lambda)_{\mu'}$ and let $M$ and $N$ be the domain and codomain respectively. $M$ and $N$ are free $B$ modules of finite rank. Let $d$ be the rank. We can choose
bases \( m_1, \ldots, m_d \) and \( n_1, \ldots, n_d \) such that \( \varphi(m_i) = a_i n_i, \ i = 1, \ldots, d \) for some \( a_i \in B \). Set \( C = \ker \varphi \cong \bigoplus_{i=1}^d B/(a_i) \) and set \( C_B = C \otimes_B (B/(X - 1)B) = C \otimes_B \mathbb{C} \) where \( C \) is considered a \( B \)-module by \( X \mapsto 1 \). Note that

\[
B/(a_i) \otimes_B \mathbb{C} = \begin{cases} \mathbb{C}, & \text{if } (X - 1)a_i \\ 0, & \text{otherwise} \end{cases}
\]

so \( \dim \mathbb{C} C_B = \# \{ i | \nu(a_i) > 0 \} \). Since there exists a \( \psi : N \to M \) such that \( \varphi \circ \psi = (X - 1) \) id we get \( \nu(a_i) \leq 1 \) for all \( i \). So then \( \dim \mathbb{C} C_B = \nu(\det \varphi) \) and the claim has been shown. \( \square \)

7 Linkage principle

Let \( R \) be a field that is an \( A \)-module and \( q \in R \) the nonzero element that \( v \) is sent to. As usual we can define the Verma modules: Assume \( \lambda : U_R^0 \to R \) is a homomorphism. Then we define \( M_R(\lambda) = U_R \otimes_{U_R^0} R_\lambda \) where \( R_\lambda \) is the one-dimensional \( R \)-module with trivial action and \( U_R^+ \) and \( U_R^0 \) acting as \( \lambda \). There is a unique simple quotient \( L_R(\lambda) \) of \( M_R(\lambda) \).

Let \( \alpha = \alpha_i \in \Pi \). Consider the parabolic Verma module \( M_{R,i}(\lambda) := U_R(i) \otimes_{U_R^0} R_\lambda \), where \( U_R(i) \) is the submodule generated by \( U_R^+ \otimes_{U_R^0} F_\alpha \). We get a map \( M_{R,i}(\lambda) \to M_{R,i}(\lambda) := \psi((U_R(i) \otimes_{U_R^0} U_R(s_i)) \otimes_{U_R(i)} M_{R,i}(s,\lambda)) \) where the module \( (U_R(i) \otimes_{U_R^0} U_R(s_i)) \otimes_{U_R(i)} M_{R,i}(s,\lambda) \) is an \( \mathbf{U}_R(i) \)-bimodule by the similar arguments as earlier. Inducing to the whole quantum group and using \( T_w \) we get a homomorphism

\[
M_{R}(\lambda) \to M_{R}(\lambda)
\]

So we can construct a sequence of homomorphisms \( \varphi_1, \ldots, \varphi_N \)

\[
M_R(\lambda) \xrightarrow{\varphi_1} M_{R}(\lambda) \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_N} M_{R}(\lambda) = DM_R(\lambda)
\]

We denote the composition by \( \Psi \). Note that the image of \( \Psi \) must be the unique simple quotient \( L_R(\lambda) \) of \( M_R(\lambda) \) since every map \( M_R(\lambda) \to DM_R(\lambda) \) maps to the unique simple quotient of \( M(\lambda) \) (by the usual arguments f.x. like in [Hum08, thm 3.3]).

First we want to consider some facts about the map \( \varphi : M_R(\lambda) \to M_{R}(\lambda) \). Let \( M_{\alpha}(\lambda) \) denote the \( U_R(\mathfrak{sl}_2) \) Verma module with highest weight \( \lambda(K_{\alpha}) \). We will use the notation \( M_{\alpha}(\lambda) \) for the parabolic \( U_R(i) \) Verma module \( U_R(i) \otimes_{U_R^0} R_\lambda \). The map \( \varphi \) was constructed by first inducing the map of parabolic modules and then using the twisting functor \( T_w \).

Assume the sequence of \( U_R(\mathfrak{sl}_2) \) modules \( M_{\alpha}(\lambda) \to M_{\alpha}(\lambda) \to Q_{\alpha}(\lambda) \to 0 \) is exact (i.e. \( Q_{\alpha}(\lambda) \) is the cokernel of the map \( M_{\alpha}(\lambda) \to M_{\alpha}(\lambda) \)). Inflating to the parabolic situation we get an exact sequence \( M_{\alpha}(\lambda) \to M_{\alpha}(\lambda) \to Q_{\alpha}(\lambda) \to 0 \) where \( Q_{\alpha}(\lambda) \) is just the inflation of \( Q_{\alpha}(\lambda) \) to the corresponding parabolic module.

Inducing from a parabolic module to the whole module is done by applying the functor \( M \mapsto U_R \otimes_{U(\mathfrak{sl}_2)} M \). This is right exact so we get the exact sequence \( M_R(\lambda) \to M_R(\lambda) \to Q_R(\lambda) \to 0 \) where \( Q_R(\lambda) = U_R \otimes_{U(\mathfrak{sl}_2)} Q_{\alpha}(\lambda) \).
Assume we have a finite filtration of $Q_\alpha(\lambda)$:

$$0 = Q_0 \subset Q_1 \subset \cdots \subset Q_r = Q_\alpha(\lambda)$$

such that $Q_{i+1}/Q_i \cong L_\alpha(\mu_i)$. So we have after inflating:

$$0 = Q_{p_0,0} \subset Q_{p_0,1} \subset \cdots \subset Q_{p_0,r} = Q_{p_0}(\lambda)$$

such that $Q_{p_0,i+1}/Q_{p_0,i} \cong L_{p_0}(\mu_i)$.

That is we have short exact sequences of the form

$$0 \to Q_{p_0,i} \to Q_{p_0,i+1} \to L_{p_0}(\mu_i) \to 0.$$ 

Since induction is right exact we get the exact sequence

$$Q_{R,i+1} \to Q_{R,i} \to L_{p_0}(\mu_i) \to 0.$$ 

Where $Q_{R,i}$ is the induced module of $Q_{p_0,i}$ and $L_{p_0}(\mu_i)$ is the induced module of $L_{p_0}(\mu_i)$.

Starting from the top we have

$$Q_{R,r-1} \to Q_{R,r} \to L_{p_0}(\mu_{r-1}) \to 0.$$ 

So we see that the composition factors of $Q_{R}^{\text{wr}}(\lambda)$ are contained in the set of composition factors of $L_{p_0}(\mu_{r-1})$ and the composition factors of $Q_{R,r-1}$. By induction we get then that the composition factors of $Q_{R,r-1}$ are composition factors of $L_{p_0}(\mu_i)$, $i = 0, \ldots, r-2$. The conclusion is that we can get a restriction on the composition factors of $Q_{R}(\lambda)$ by examining the composition factors of induced simple modules.

Let $L = L_{p_0}(\mu)$ be a simple parabolic module and let $\mathcal{L}$ be the induction of $L$. Then because induction is right exact we have

$$M_{R}(\mu) \to \mathcal{L} \to 0.$$ 

So the composition factors of $\mathcal{L}$ are composition factors of $M_{R}(\mu)$. This gives us a restriction on the composition factors of $M_{R}(\lambda)$:

Use the above with $w^{-1}\lambda$ in place of $\lambda$ and use the twisting functor $T_{w}^{R}$ on the exact sequence $M_{R}(w^{-1}\lambda) \to M_{R}^{R}(w^{-1}\lambda) \to Q_{R}(w^{-1}\lambda) \to 0$ to get

$$M_{R}^{w}(\lambda) \to M_{R}^{w}(\lambda) \to Q_{R}^{w}(\lambda) \to 0.$$ 

where $Q_{R}^{w}(\lambda) = T_{w}^{R}(Q_{R}(w^{-1}\lambda))$. Add the kernel to get the 4-term exact sequence

$$0 \to K_{R}^{w}(\lambda) \to M_{R}^{w}(\lambda) \to M_{R}^{w}(\lambda) \to Q_{R}^{w}(\lambda) \to 0.$$ 

Since $\text{ch} M_{R}^{w}(\lambda) = \text{ch} M_{R}^{w}(\lambda)$ we must have $\text{ch} K_{R}^{w}(\lambda) = \text{ch} Q_{R}^{w}(\lambda)$.

So we have a sequence of homomorphisms $\varphi_i$

$$M_{R}(\lambda) \xrightarrow{\varphi_i} M_{R}^{w}(\lambda) \xrightarrow{\varphi_i} \cdots \xrightarrow{\varphi_i} M_{R}^{\text{we}}(\lambda) = DM_{R}(\lambda)$$ 

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and these maps each fit into a 4-term exact sequence

$$0 \to K^w_R(\lambda) \to M^w_R(\lambda) \to M^w_R(\lambda) \to Q^w_R(\lambda) \to 0$$

where $\text{ch}K^w_R(\lambda) = \text{ch}Q^w_R(\lambda)$. In particular $M^w_R(\lambda) \to M^w_R(\lambda)$ is an isomorphism if the corresponding $\mathfrak{sl}_2$ map $M_\alpha(w^{-1},\lambda) \to DM_\alpha(w^{-1},\lambda)(= M^*_\alpha(w^{-1},\lambda))$ is an isomorphism. If the $\mathfrak{sl}_2$ map is not an isomorphism then we have a restriction on the composition factors that can get killed by the map $M_\beta(w^{-1},\lambda) \to M^*_\beta(w^{-1},\lambda)$ by the above. To get to the map $M^w_R(\lambda) \to M^w_R(\lambda)$ we use $T_\nu$ which is right exact so we get a restriction on the composition factors killed by $M^w_R(\lambda) \to M^w_R(\lambda)$ too.

Fix $\alpha$. From the above we know that a composition factor of $Q^R_R(\lambda)$ is a composition factor of $L_{\beta_{0}}(\mu)$ for some $\mu$ where $L_{\alpha}(\mu)$ is a composition factor of $M_{\alpha}(\lambda)$. Use this for $w^{-1},\lambda$ and use $T_w$. So we get that a composition factor of $Q^w_R(\lambda)$ is a composition factor of $L_w L_{\beta_{0}}(\mu)$ with $\mu$ as before. Since $T_w$ is right exact we have that

$$T_w M_\mu \to T_w L_{\beta_{0}}(\mu) \to 0$$

is exact. Since $\text{ch}T_w M_\mu = \text{ch} M(w,\mu)$ we see that a composition factor of $Q^w_R(\lambda)$ must be a composition factor of $M_{\alpha}(w,\mu)$ where $\mu$ is such that $L_{\alpha}(\mu)$ is a composition factor of $M_{\alpha}(w^{-1},\lambda)$.

**Definition 7.1** We define a partial order on weights. We say $\mu \leq \lambda$ if $\mu^{-1} \lambda = q^{\sum_{i=1}^{n} a_i, \alpha_i}$ for some $a_i \in \mathbb{N}$.

For a weight $\nu$ of the form $\nu = q^{\sum_{i=1}^{n} a_i, \alpha_i}$ with $a_i \in \mathbb{N}$ we call $\sum_{i=1}^{n} a_i$ the height of $\nu$.

Note that for a Verma module $M(\lambda)$ we have $\mu \leq \lambda$ for all $\mu \in \text{wt} M(\lambda)$ where $\text{wt} M(\lambda)$ denotes the weights of $M(\lambda)$.

**Definition 7.2** Let $\mu, \lambda \in \Lambda$. Define $\mu \uparrow \lambda$ to be the partial order induced by the following: $\mu$ is less than $\lambda$ if there exists a $w \in W$, $\alpha \in \Pi$ and $\nu \in \Lambda$ such that $\mu = w.\nu < \lambda$ and $L_{\alpha}(\nu)$ is a composition factor of $M_{\alpha}(w^{-1},\lambda)$.

i.e. $\mu \uparrow \lambda$ if there exists a sequence of weights $\mu = \mu_1, \ldots, \mu_r = \lambda$ such that $\mu_i$ is related to $\mu_{i+1}$ as above.

We have established the following

**Proposition 7.3** If $L_\mu(\mu)$ is a composition factor of $M_\mu(\lambda)$ then $\mu \uparrow \lambda$.

**Proof.** Choose a reduced expression of $w_0$ and construct the maps $\varphi_i$ as above. If $L_\mu(\mu)$ is a composition factor of $M_\mu(\lambda)$ it must be killed by one of the maps $\varphi_i$ since the image of $\Psi$ is $L_\lambda(\lambda)$. So $L_\mu(\mu)$ must be a composition factor of one of the modules $Q^w_R(\lambda)$. We make an induction on the height of $\mu^{-1} \lambda$. If $\mu^{-1} \lambda = 1$ then $\lambda = \mu$ and we are done. Otherwise we see that $L_\mu(\mu)$ is a composition factor of one of the $Q^w_R(\lambda)$’s. But every composition factor of $Q^w_R(\lambda)$ is a composition factor of $M(\nu)$ where $\nu \uparrow \lambda$ and $\nu < \lambda$. Since $\nu < \lambda$ the height of $\mu^{-1} \nu$ is less then the height of $\mu^{-1} \lambda$ so we are done by induction. □

In the non-root of unity case $\uparrow$ is equivalent to the usual strong linkage: $\mu$ is strongly linked to $\lambda$ if there exists a sequence $\mu_i$ with $\mu = \mu_1 < \mu_2 < \cdots < \mu_r = \lambda$ and $\mu_i = s_{\beta_i} \mu_{i+1}$ for some positive roots $\beta_i$ (remember that if $\beta = w(\alpha)$ then $s_\beta = ws_\alpha w^{-1}$).
In the nonroot of unity case we see that $M_\alpha(w^{-1},\lambda)$ is simple if
\[
q^\alpha w^{-1}.\lambda(K_\alpha) \notin \pm q^{-Z_{\alpha}^{>0}}.
\]
Otherwise there is one composition factor in $M_\alpha(w^{-1},\lambda)$ apart from $L_\alpha(w^{-1},\lambda)$, namely $L_\alpha(s_\alpha w^{-1},\lambda)$. So the composition factors of $Q^w_R$ are composition factors of $M_\alpha(ws_\alpha w^{-1},\lambda) = M_R(s_\alpha(\alpha),\lambda)$. Actually $Q^w_R = M_{ws}(s_\alpha(\alpha),\lambda)$ in this case:

Let's consider the construction of the maps $\varphi_i$ in the previous section. We start with the map $M_\alpha(\lambda) \to M^s_\alpha(\lambda)$ and then inflate to $M_{p_\alpha}(\lambda) \to M^s_{p_\alpha}(\lambda)$. In the case where $q$ is not a root of unity it is easy to see that if $q^\alpha(\lambda)(K_\alpha) \notin \pm q^{-Z_{\alpha}^{>0}}$ then this is an isomorphism and otherwise the kernel (and the cokernel) is isomorphic to $M_{p_\alpha}(s\lambda)$ which is a simple module. So after inducing we get the 4-term exact sequence

\[
0 \to M^w_R(s,\lambda) \to M^w_R(\lambda) \to M^w_R(s,\lambda) \to 0
\]
since induction is exact on Verma modules. Use these observations on $w^{-1},\lambda$ and the fact that $T_w$ is exact on Verma modules and we get the map $M^w_R(\lambda) \to M^w_R(\lambda)$ which is an isomorphism if $q^\alpha(\lambda)(K_\alpha) \notin \pm q^{-Z_{\alpha}^{>0}}$ and otherwise we have the 4-term exact sequence

\[
0 \to M^w_R(s,\lambda) \to M^w_R(\lambda) \to M^w_R(s,\lambda) \to 0
\]

**Theorem 7.4** Let $R$ be a field (any characteristic) and let $q \in R$ be a non-root of unity. $R$ is an $A$-algebra by sending $v$ to $q$. Let $\lambda : U_q^0 \to R$ be an algebra homomorphism.

$M_R(\lambda)$ has finite Jordan-Holder length and if $L_R(\mu)$ is a composition factor of $M_R(\lambda)$ then $\mu \uparrow \lambda$ where $\uparrow$ is the usual strong linkage.

**Proof.** This will be proved by induction over $\uparrow$. If $\lambda$ is anti-dominant (i.e. $q^\alpha(\lambda)(K_\alpha) \notin \pm q^{-Z_{\alpha}^{>0}}$ for all $\alpha \in \Pi$) then we get that all the maps $\varphi_i$ are isomorphisms and so $M_R(\lambda)$ is simple. Now assume $\lambda$ is not anti-dominant. A composition factor $L_R(\mu)$ must be killed by one of the $\varphi_i$’s so must be a composition factor of $Q^w_R$ for some $w$. By the above calculations we see that if $q^\alpha(\lambda)(K_\alpha) \notin \pm q^{-Z_{\alpha}^{>0}}$ then $M^w_R(\lambda) \to M^w_R(\lambda)$ is an isomorphism and otherwise $Q^w_R = M^w_R(s_\alpha,\lambda)$. By induction all the Verma modules with highest weight $\mu$ strongly linked to $\lambda$ has finite length and the composition factors are strongly linked to $\mu$. This finishes the induction. $\square$

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