Comparison of compression vs pure shearing for a simple model of athermal frictionless disks in suspension

Anton Peshkov\textsuperscript{1} and S. Teitel\textsuperscript{1}

\textsuperscript{1}Department of Physics and Astronomy, University of Rochester, Rochester, NY 14627

(Dated: May 26, 2022)

Using a simplified model for a non-Brownian suspension, we numerically study the response of athermal, overdamped, frictionless disks in two dimensions to isotropic and uniaxial compression, as well as to pure shearing, all at finite constant strain rates $\dot{\epsilon}$. We show that isotropic and uniaxial compression result in the same jamming packing fraction $\phi_J$, while pure shear induced jamming occurs at a slightly higher $\phi_J^*$, consistent with that found previously for simple shearing. A critical scaling analysis of pure shearing gives critical exponents consistent with those previously found for both isotropic compression and simple shearing. Using orientational order parameters for contact bond directions, we compare the anisotropy of the force and contact networks. We find that the main difference between uniaxial compression and pure shearing is given by the relative magnitude of the isotropic to nematic terms. Higher orientational moments are similar in the two cases.

I. INTRODUCTION

In a recent work \cite{Peshkov2022} we considered isotropic vs uniaxial compression, within a simple granular model of bidisperse non-Brownian spheres in suspension, as a means for numerically studying the effect of stress anisotropy on the jamming transition of frictionless particles. Isotropic compression at a finite rate results in configurations with an isotropic stress; there is a finite pressure but no shear stress. Uniaxial compression at a finite rate results in configurations with an anisotropic stress; there is both a finite pressure and a finite shear stress, similar to the case of sheared systems. Our analysis found that, in three dimensions, jamming via isotropic and uniaxial compression display the same universal critical behavior, despite the difference in stress symmetry.

In this work we consider more generally the differences between isotropically compressed, uniaxially compressed, and pure sheared configurations, when driven at a finite strain rate $\dot{\epsilon}$, as one approaches and goes above the jamming transition $\phi = \phi_J$. For simplicity we consider the case of circular disks in two dimensions, using the same simple idealized model of a non-Brownian suspension as we used previously \cite{Peshkov2022}.

We compare the pressure $p$ and shear stress $\sigma$ arising from such deformations. Just below the jamming $\phi_J$, we find the pressure from isotropic and uniaxial compression to be equal within some range of $\phi$ depending on the initial sample preparation, while the pressure from pure shearing is roughly an order of magnitude smaller. The pressures in all three cases converge as $\phi$ increases above $\phi_J$. Below $\phi_J$, the shear stress for uniaxial compression becomes greater than for pure shear as $\phi$ decreases, while above $\phi_J$ it is reversed. From a comparison of the stress for these three deformations, we infer that the jamming $\phi_J$, and the critical exponents at jamming, are the same for isotropic and uniaxial compression. However we argue that the jamming $\phi_J^*$ for pure shear is slightly larger than the $\phi_J$ for compression. A critical scaling analysis for the case of pure shear gives a value for $\phi_J^*$ consistent with that previously found for simple shearing, while the critical exponents are consistent with those found for both simple shearing and for isotropic compression.

We also consider geometrical measures of the configurational contact network, particularly that average number of contacts per particle $Z$, and the fraction of contacts between the different types of particles in our bidisperse system. We find that, comparing the three deformations, these geometrical measures show small differences when one is below the jamming $\phi_J$, but that they become equal above $\phi_J$.

Finally, we compare the system anisotropy that results from uniaxial compression with that from pure shearing. We show that the stress tensor anisotropy, measured by the macroscopic friction $\mu = \sigma/p$, behaves quite differently for these two cases. For pure shearing $\mu$ is monotonically decreasing as $\phi$ increases, while for uniaxial compression $\mu$ has a sharp minimum at $\phi_J$. The anisotropy of the contact network, as measured by the fabric tensor, shows similar behavior. We generalize these anisotropy measures to higher order orientational order parameters of both the force and contact network. We find that the main difference between uniaxial compression and pure shearing is the relative magnitude of the isotropic to nematic terms, with higher orientational moments behaving similarly.

The remainder of our paper is organized as follows. In Sec. \ref{sec:system} we present our model and numerical methods. In Sec. \ref{sec:results} we present our results for the system stress, for uniaxial compression, isotropic compression, and pure shear. In Sec. \ref{sec:discussion} we present a discussion of the anisotropy of the configurational contact and force networks in these cases. In Sec. \ref{sec:conclusions} we summarize our results. In Appendix A we provide a more complete discussion of the compression ensembles we use, discussing the dependence of the stress on the initial packing fraction $\phi_{\text{init}}$ from which compression begins, and considering the $\phi_{\text{init}} \to 0$ limit. In Appendix B we provide a more detailed discussion of pure shearing, including a critical scaling analysis. In Appendix C we include fur-
ther discussion of the anisotropy of the contact and force networks.

II. MODEL AND METHODS

Our model has been described in detail elsewhere [1,5]. We simulate athermal ($T = 0$), bidisperse, frictionless soft-core disks in two dimensions, with equal numbers of big and small disks with diameter ratios $d_b/d_s = 1.4$ [3]. Particles, with centers of mass at positions $r_i$, interact with a one-sided harmonic contact repulsion,

$$U(r_{ij}) = \begin{cases} \frac{1}{2}k_c \left(1 - \frac{r_{ij}}{d_{ij}}\right)^2, & r_{ij} < d_{ij} \\ 0, & r_{ij} > d_{ij} \end{cases}$$

where $k_c$ is a stiffness constant, $r_{ij} = |r_i - r_j|$ and $d_{ij} = (d_i + d_j)/2$. The elastic force acting on particle $i$ due to its contact with $j$ is then,

$$f^e_{ij} = -\frac{dU(r_{ij})}{dr_i}.$$  

As a simplified model for particles in solution, we add a dissipative force due to the viscous drag on the particle with respect to the local velocity of the suspending host medium [4, 6, 7].

$$f^d_i = -k_d V_i \left[ \frac{dr_i}{dt} - v_{host}(r_i) \right],$$

where $k_d$ is a dissipative constant, $V_i$ is the area of particle $i$, and $v_{host}(r)$ is the velocity of the host medium at position $r$. Particle motion is then determined from these forces using Newton’s equation. We take particle masses to be proportional to their area, $m_i \propto V_i$. Because our particles are circular and frictionless, we ignore particle rotations.

For the linear deformations we consider in this work, the background host velocity can be expressed in terms of the strain rate tensor $\dot{\epsilon}$,

$$v_{host}(r) = \dot{\epsilon} \cdot r.$$  

We will consider the three cases of uniaxial compression, isotropic compression, and pure shear, with respective strain rate tensors,

$$\dot{\epsilon}_{uni} = \begin{bmatrix} \dot{\epsilon} & 0 \\ 0 & 0 \end{bmatrix}, \quad \dot{\epsilon}_{iso} = -\frac{1}{2} \begin{bmatrix} \dot{\epsilon} & 0 \\ 0 & \dot{\epsilon} \end{bmatrix},$$

$$\dot{\epsilon}_{ps} = -\frac{1}{2} \begin{bmatrix} \dot{\epsilon} & 0 \\ 0 & -\dot{\epsilon} \end{bmatrix}.$$  

For uniaxial compression, we compress along the $\hat{x}$ direction, while for pure shearing we compress along $\hat{x}$ while expanding along $\hat{y}$. Note, the factor of $1/2$ in $\dot{\epsilon}_{iso}$ is so that the rate of area change is the same for $\dot{\epsilon}_{iso}$ as for $\dot{\epsilon}_{uni}$. The factor of $1/2$ in $\dot{\epsilon}_{ps}$ is so that we can then view uniaxial compression as a superposition of an isotropic compression plus a pure shear,

$$\dot{\epsilon}_{uni} = \dot{\epsilon}_{iso} + \dot{\epsilon}_{ps}.$$  

Our particles are placed in a rectangular box with side lengths $L = (L_x, L_y)$, centered at $r = 0$. As we make our elastic deformations, the box lengths vary according to

$$\frac{dL}{dt} = \dot{\epsilon} \cdot L.$$  

At each integration step, particles that would fall outside the system box are returned to the box using periodic boundary conditions [1].

To carry out our numerical simulations, we recast our model in terms of three dimensionless parameters [5]. The first is the packing fraction $\phi$,

$$\phi = \frac{1}{L_x L_y} \sum_i V_i.$$  

For both isotropic and uniaxial compression, $\phi$ increases with time as the system gets compressed. For the area-preserving pure shear deformation, $\phi$ stays constant.

The second is the quality factor $Q$, which measures the strength of the dissipative force relative to the elastic force. If $\tau_d = m_s k_d V_s$ and $\tau_e = (m_s d_s^2)/k_e$ are the time scales associated with the dissipative and elastic forces [5], we have,

$$Q = \frac{\tau_d}{\tau_e} = \frac{\sqrt{m_s k_e}}{k_d V_s d_s}.$$  

As $Q$ decreases, inertial effects decrease. For $Q$ sufficiently small, behavior becomes independent of the particular value of $Q$ and one enters the overdamped limit corresponding to massless particles, $m_s \rightarrow 0$ [8, 9]. For our simulations we will use $Q = 1$, which is sufficiently small to put us in this overdamped limit [8].

In the $m_s \rightarrow 0$ overdamped limit both $\tau_e$ and $\tau_d \rightarrow 0$, however we can define a time scale that remains finite [8],

$$\tau_0 = \frac{\tau_e^2}{\tau_d} = \frac{\tau_e}{Q} = \frac{k_d V_s d_s^2}{k_e}.$$  

Our third dimensionless parameter is then the dimensionless strain rate,

$$\dot{\epsilon} \tau_0.$$  

Henceforth we will take our unit of length to be $d_s = 1$, and our unit of time to be $\tau_0 = 1$. Quoted values of $\dot{\epsilon}$ are therefore the same as $\dot{\epsilon} \tau_0$. We consider strain rates spanning the range $\dot{\epsilon} = 10^{-8.5}$ to $10^{-4}$.

We use LAMMPS [10] to integrate the equations of motion, using a time step of $\Delta t/\tau_0 = 0.1$. Unless otherwise noted, we use $N = 32768$ total particles. Our
simulations for isotropic and uniaxial compression start with an initial configuration at low packing $\phi_{\text{init}} = 0.4$, constructed as follows. We place particles down one by one at random, but making sure that there are no particle overlaps; if an overlap occurs, we discard that particle and try again until all $N$ particles are placed in the box. For isotropic compression, we use a square box with $L_x = L_y$. For uniaxial compression we start at $\phi_{\text{init}}$ with a rectangular box with $L_y < L_x$, such that the box becomes roughly square by the time we have compressed to the jamming $\phi_J$. For uniaxial and isotropic compression we average our results over 20 independent initial configurations. For our pure shear simulations we start at each $\phi$ with a configuration generated by our uniaxial compression protocol, using the same strain rate $\dot{\epsilon}$. For each $\dot{\epsilon}$ we average our results over 10 independent initial configurations, except for our slowest rates $\dot{\epsilon} \leq 10^{-7}$, where we use only a single initial configuration.

### III. RESULTS: STRESS

In this section we consider the stress generated in the system by the elastic deformations. We consider only the stress arising from the elastic forces, since this is the dominant term at low strain rates. The stress tensor can be expressed in terms of the force moments as \[ P = \frac{1}{L_x L_y} \left( \sum_{i<j} \mathbf{f}_{ij} \otimes (\mathbf{r}_i - \mathbf{r}_j) \right), \] (13)
where $(\cdots)$ denotes an average over our independent runs. A dimensionless stress tensor can be defined as \[ p = \frac{\tau^2 d_s}{m_s} P. \] (14)
For our geometry, and the three deformations of Eqs. [5] and [6], the stress tensor $P$ will have the simple form, \[ P = \begin{bmatrix} p + \sigma & 0 \\ 0 & p - \sigma \end{bmatrix}, \] (15)
where the pressure $p$ is the isotropic part of the stress, and the shear stress $\sigma$ is the anisotropic part of the stress. For isotropic compression, symmetry gives $\sigma = 0$. For both uniaxial compression and pure shear we have $\sigma > 0$.

The area-preserving process of shearing at a finite rate defines a steady-state ensemble of configurations that becomes independent of the initial starting configuration, provided one shears sufficiently long. This has previously been observed for the case of simple shearing \[12, 14\], and in Appendix B we confirm that it is also the case for pure shearing. Our results below for $p$ and $\sigma$ from pure shearing represent a time average over configurations, once this steady-state limit has been reached. The resulting values of $p$ and $\sigma$ are determined solely by the parameters $\phi$, $Q$ and $\dot{\epsilon}$.

For isotropic and uniaxial compression, however, the situation is not as simple. As one compresses, $\phi$ increases, and the ensemble of configurations one passes through can depend on the ensemble of initial configurations one starts the compression from. In our case, where we start from configurations of non-overlapping particles at an initial packing $\phi_{\text{init}}$, our values of $p$ and $\sigma$ can depend not only on the parameters $\phi$, $Q$, and $\dot{\epsilon}$, but also on the additional parameter $\phi_{\text{init}}$. In Appendix A we consider this dependence of the stress on $\phi_{\text{init}}$. We find that as $\phi_{\text{init}}$ decreases, the stress for both isotropic and uniaxial compression approaches a well defined $\phi_{\text{init}} \to 0$ limit. For $\phi_{\text{init}}$ not too small, the resulting $p(\phi)$ and $\sigma(\phi)$ approach this limiting curve as $\phi$ reaches the dense limit, just below jamming. Since using very small $\phi_{\text{init}}$ can be computationally expensive for the large $N = 32768$ system size that we wish to use, to avoid finite size effects near jamming, here we use $\phi_{\text{init}} = 0.4$. We find that this $\phi_{\text{init}}$ is sufficiently small that our results are roughly independent of $\phi_{\text{init}}$ once $\phi > 0.8$. Further details are presented in Appendix A.

In Fig. 1(a) we plot $p$ vs $\phi$, for several different strain rates $\dot{\epsilon}$, for our three types of deformation: uniaxial compression, isotropic compression, and pure shear. The vertical dashed line in this figure shows the jamming transition for isotropic compression, $\phi_J = 0.8415$, as we have determined previously \[5\]. As found before \[1, 5\], we find here (not shown) that the three types of deformation all have a linear rheology, $p \propto \dot{\epsilon}$, provided one is not too close to jamming, while $\phi$ approaches a constant as $\dot{\epsilon} \to 0$. Comparing the three cases, for $\phi < \phi_J$, we see that $p_{\text{uni}}$ is equal to $p_{\text{uni}}$, but $p_{\text{ps}}$ is about a factor 10 smaller than the other two. For $\phi > \phi_J$, however, we see that the $p$ for all three cases are becoming equal as $\dot{\epsilon} \to 0$.

In Fig. 1(b) we similarly plot $\sigma$ vs $\phi$ at different $\dot{\epsilon}$, for uniaxial compression and pure shear ($\sigma_{\text{iso}} = 0$ by symmetry). As with the pressure $p$, the shear stress $\sigma \propto \dot{\epsilon}$ if one is not too close to jamming, while $\sigma$ approaches a constant as $\dot{\epsilon} \to 0$ above jamming. Here we see that $\sigma_{\text{uni}}$ and $\sigma_{\text{ps}}$ are generally of the same order of magnitude, but $\sigma_{\text{ps}} < \sigma_{\text{uni}}$ for $\phi < \phi_J$, while $\sigma_{\text{ps}} > \sigma_{\text{uni}}$ for $\phi > \phi_J$. As $\dot{\epsilon} \to 0$, $\sigma_{\text{ps}}/\sigma_{\text{uni}} \to 1$ if $\phi \approx \phi_J$.

Next we consider some geometrical properties of our configurations. In Fig. 2(a) we plot the average number of contacts per particle $Z$ vs $\phi$ for our three types of deformation, at two small strain rates $\dot{\epsilon}$ $= 10^{-7}$ and $10^{-8}$. As was observed before for isotropic compression \[11\] and simple shearing \[12, 13\], we find that, as $\dot{\epsilon} \to 0$, $Z$ stays finite and varies roughly linearly with $\phi$ for $\phi < \phi_J$, while at $\phi_J$ and above we see the square root singularity, $(Z - Z_J) \sim (\phi - \phi_J)^{1/2}$ associated with jamming \[11, 13\]. Here our value of $Z_J$ at jamming is slightly below the isostatic value of $Z_{\text{isostatic}} = 2d = 4$ since, for simplicity, we have not excluded rattler particles when computing $Z$ \[12, 13\].

Our observation that $Z$ approaches a constant as $\dot{\epsilon} \to 0$ indicates that, at low strain rates, the system forms a...
pure shear ($\varepsilon$) the isotropic compression-driving jamming to $p$ $Z \propto$ the contacts varies $\sigma$ $\phi$ these are between two small particles, two big particles that are exactly the same critical packing fraction $\phi$ different types of deformation. To investigate this we compute the stress ratios between the different types of deformation.

In Fig. 2(b) we plot the fraction of the particle contacts that are between two small particles, two big particles, and between one small and one big particle vs $\phi$ for the strain rate $\dot{\varepsilon} = 10^{-7}$. Similar to $Z$ and $p$, we see that above $\phi_J$ these fractions become equal for all three cases. Below $\phi_J$ we see that pure shearing produces more small-small contacts and fewer big-big contacts than does isotropic or uniaxial compression. Our above results thus show that, in the jammed state above $\phi_J$, it is only the shear stress $\sigma$ that clearly distinguishes between the three different types of deformation.

Next we consider whether all three cases jam at exactly the same critical packing fraction $\phi_J$. To investigate this we compute the stress ratios between the different cases. In Fig. 2(a) we plot $p_{\text{uni}}/p_{\text{iso}}$ vs $\phi$, for several different strain rates $\dot{\varepsilon}$. We see no particular features as $\phi$ passes through $\phi_J$. Since, as $\dot{\varepsilon} \rightarrow 0$, the bulk viscosity $\zeta \equiv p/\dot{\varepsilon} \sim (\phi_J - \phi)^{-\beta}$ diverges at $\phi_J$ with the critical exponent $\beta$, the absence of any features in $p_{\text{uni}}/p_{\text{iso}}$ near $\phi_J$ strongly suggests that $p_{\text{uni}}$ and $p_{\text{iso}}$ jam at exactly the same $\phi_J$ and their $\zeta$ diverge with the same exponent $\beta$. This conclusion is in agreement with what we explicitly demonstrated for three dimensions in an earlier work [1].

In Fig. 2(b) we similarly plot $p_{\text{iso}}/p_{\text{ps}}$ vs $\phi$. Here we see a very different behavior. We find that $p_{\text{iso}}/p_{\text{ps}}$ develops a peak just below $\phi_J$: as $\dot{\varepsilon}$ decreases, the height of this peak increases and the location of the peak moves closer to $\phi_J$. The same behavior is found for $p_{\text{uni}}/p_{\text{ps}}$, shown in Fig. 2(c), and for $\sigma_{\text{uni}}/\sigma_{\text{ps}}$ in 2(d).

Two possible explanations for such behavior are: (i) Pure shearing jams at a slightly higher $\phi_J$ than the $\phi_J$ for compression. In this case we would expect, in the limit $\dot{\varepsilon} \rightarrow 0$, that $p_{\text{iso}}/p_{\text{ps}}$ diverges as $\phi \rightarrow \phi_J$ from below, stays infinite for $\phi_J < \phi < \phi_J^*$, and approaches a finite constant for all $\phi > \phi_J^*$. (ii) Pure shearing jams at the same $\phi_J$ as does compression, but with a smaller exponent $\beta^* < \beta$. In this case we would expect, in the limit $\dot{\varepsilon} \rightarrow 0$, that $p_{\text{iso}}/p_{\text{ps}}$ diverges as $\phi \rightarrow \phi_J$ from below, but approaches a finite constant for all $\phi > \phi_J$. Our data is more consistent with the possibility (i), since we see that there remains a small interval above the compressive $\phi_J$ where the stress ratio continues to increase as $\dot{\varepsilon}$ decreases. Prior work [7-10] has demonstrated that, for simple shearing with $\varepsilon = \dot{\varepsilon} \times \phi$, our model jams at the packing $\phi_J^* = 0.8435 > \phi_J = 0.8415$, and that the exponent $\beta$ is the same as found for compression [5]. In Appendix B we present a detailed critical scaling analysis of our pure shearing data that confirms that pure shearing indeed behaves the same as simple shearing, with a $\phi_J^* > \phi_J$, but the same exponent $\beta$.

We have previously noted in Eq. (7) that, with regard to the strain rate tensor $\varepsilon$, uniaxial compression can be regarded as a superposition of isotropic compression plus pure shearing. It is therefore natural to wonder whether a similar superposition holds for the resulting stresses in these flowing states, if one is in the region where the rheology is linear. Our results in Fig. 3 however, show that it does not. Were a superposition of stress to hold, we would expect $p_{\text{uni}} = p_{\text{iso}} + p_{\text{ps}}$ and $\sigma_{\text{uni}} = \sigma_{\text{iso}}$ (since $\sigma_{\text{iso}} = 0$). Our results in Fig. 3 as well as our earlier results in Ref. [5] (see Fig. 1(b) of that work), show that for $\dot{\varepsilon} \leq 10^{-7}$ we remain in the linear rheology region for $\phi$ up to at least $\sim 0.82$. However below $\phi = 0.82$ we see from Figs. 3(a) and 3(d) that $p_{\text{uni}} = p_{\text{iso}}$ and $\sigma_{\text{uni}} \neq \sigma_{\text{iso}}$.
FIG. 3. Stress ratios comparing uniaxial compression, isotropic compression, and pure shear. Pressure ratios (a) $p_{\text{uni}}/p_{\text{iso}}$, (b) $p_{\text{ps}}/p_{\text{iso}}$, (c) $p_{\text{uni}}/p_{\text{ps}}$, and (d) shear stress ratio $\sigma_{\text{uni}}/\sigma_{\text{ps}}$ vs $\phi$ for different strain rates $\dot{\epsilon}$. The vertical dashed line indicates the isotropic compression-driving jamming $\phi_J = 0.8415$. The system has $N = 32768$ particles.

$\sigma_{\text{uni}} \approx 2.5\sigma_{\text{ps}}$. In general, we see from Fig. 3(d) that $\sigma_{\text{uni}} = \sigma_{\text{ps}}$ only at an isolated point close to $\phi_J$. Thus there is no principle of superposition for stress in the flowing states below $\phi_J$.

IV. RESULTS: ANISTROPY

In this section we compare the anisotropy of configurations undergoing uniaxial compression with those being pure sheared. First we consider the anisotropy of the stress tensor, parameterized by the macroscopic friction $\mu \equiv \sigma/p$. In Fig. 4(a) we plot $\mu$ vs $\phi$ for different strain rates $\dot{\epsilon}$; open symbols indicate uniaxial compression while closed symbols are for pure shear. In both cases $\mu$ approaches a limiting, finite value, as $\dot{\epsilon} \to 0$. However, as we noted previously for three dimensions [1], we find a distinct difference between the two cases. For pure shearing the $\dot{\epsilon} \to 0$ limiting curve is monotonically decreasing as $\phi$ increases, just as was seen previously for simple shearing [17]. In contrast, for uniaxial compression, this curve develops a cusp-like minimum at $\phi_J$. For pure shear we find the value of $\mu$ at $\phi_J$ to be, $\mu_{\text{ps}}^{\text{uni}} \approx 0.1$, the same as found for simple shearing [17]. For uniaxial compression we find $\mu_{\text{ps}}^{\text{uni}} \approx 0.02$. Thus the ratio $\mu_{\text{ps}}^{\text{uni}}/\mu_{\text{ps}}^{\text{uni}} \approx 5$. However, a key point is that in both cases $\mu_J$ stays finite; the system remains anisotropic at jamming and above.

For $\phi < \phi_J$, the smaller value of $\mu$ for uniaxial compression is primarily due to the much larger pressure. As seen in Fig. 3(c), close below $\phi_J$ we have $p_{\text{uni}}/p_{\text{ps}} \approx 10$. In contrast, as seen in Fig. 3(d), the shear stress is $\sigma_{\text{uni}}/\sigma_{\text{ps}} \approx 2.7$. So $\mu_{\text{ps}}^{\text{uni}}/\mu_{\text{ps}}^{\text{uni}} \approx 3.7$; upon approaching $\phi_J$, this ratio increases. Above $\phi_J$, we see from Fig. 3(c) that $p_{\text{uni}}/p_{\text{ps}} \approx 1$, while $\sigma_{\text{uni}}/\sigma_{\text{ps}} \approx 0.5$. Thus, above $\phi_J$ we have $\mu_{\text{ps}}^{\text{uni}}/\mu_{\text{ps}}^{\text{uni}} \approx 2$, and this difference is now due entirely to the difference in the shear stress.

It is interesting to ask how much of this difference in anisotropy, comparing uniaxial compression to pure shearing, is due to anisotropy in the force network, as measured by the stress tensor, vs how much is due to the geometrical anisotropy of the contact network. We therefore consider the behavior of the fabric tensor [18]. If $\mathbf{r}_{ij} = (\mathbf{r}_i - \mathbf{r}_j)/r_{ij}$ is the unit vector pointing along a bond connecting two particles in contact, and $M_b = NZ/2$ is the total number of contact bonds in the configuration, the fabric tensor can be defined as,

$$F = \left\langle \frac{1}{M_b} \sum_{(i,j)} \mathbf{f}_{ij} \otimes \hat{\mathbf{r}}_{ij} \right\rangle,$$

where the sum is over all bonds $(i,j)$ in the contact network. Since, for circular particles, the elastic contact force $f_{ij}^{\text{el}}$ is always parallel to $\mathbf{f}_{ij}$, if we define the force moment as $h_{ij} = \mathbf{f}_{ij} \times \mathbf{r}_{ij}$, we can rewrite the stress tensor of Eq. (15) as,

$$P = \left\langle \frac{1}{M_b} \sum_{(i,j)} h_{ij} \mathbf{f}_{ij} \otimes \hat{\mathbf{r}}_{ij} \right\rangle.$$

We thus see that the fabric tensor is similar to the stress tensor, but without weighting each bond by its force moment. The fabric tensor is thus a purely geometric measure of the contact network.

For our uniaxial compression we compress along the $\hat{x}$ direction; for pure shear we also compress along $\hat{x}$ while
expanding along $\mathbf{\hat{y}}$. If we define $\theta_{ij}$ as the angle $\mathbf{r}_{ij}$ makes with respect to $\mathbf{\hat{x}}$, then we can write,

$$
\mathcal{F} = \begin{bmatrix}
\langle \cos^2 \theta_{ij} \rangle & \langle \cos \theta_{ij} \sin \theta_{ij} \rangle \\
\langle \cos \theta_{ij} \sin \theta_{ij} \rangle & \langle \sin^2 \theta_{ij} \rangle
\end{bmatrix}
$$

(18)

$$
= \frac{1}{2} \mathbf{I} + \frac{1}{2} \begin{bmatrix}
\langle \cos 2\theta_{ij} \rangle & \langle \sin 2\theta_{ij} \rangle \\
\langle \sin 2\theta_{ij} \rangle & -\langle \cos 2\theta_{ij} \rangle
\end{bmatrix}
$$

(19)

where $\mathbf{I}$ is the identity tensor, and now $\langle \cdots \rangle$ represents a combined average over both bonds within a given configuration and over different independent configurations. The first piece $\mathbf{I}/2$ is the isotropic part of $\mathcal{F}$, while the second piece gives the anisotropic part.

The eigenvalues of $\mathcal{F}$ are then,

$$
\lambda_{\pm} = \frac{1}{2} (1 \pm \Delta \lambda),
$$

(20)

where

$$
\Delta \lambda = \frac{\lambda_+ - \lambda_-}{\lambda_+ + \lambda_-} = \sqrt{\langle \cos 2\theta_{ij} \rangle^2 + \langle \sin 2\theta_{ij} \rangle^2}.
$$

(21)

Comparing to Eq. (15) we see that $\Delta \lambda$ for the fabric tensor is analogous to $\mu = \sigma/p$ for the stress tensor. For our geometry, the reflection symmetry of our deformations $y \leftrightarrow -y$, which implies a symmetry $\theta_{ij} \leftrightarrow -\theta_{ij}$, leads to the conclusion that $\langle \sin 2\theta_{ij} \rangle = 0$, and thus $\Delta \lambda = |\langle \cos 2\theta_{ij} \rangle|$.

In Fig. 3b we plot $\Delta \lambda$ vs $\phi$ for different strain rates $\dot{\varepsilon}$, for both uniaxial compression (open symbols) and pure shear (closed symbols). We see qualitatively the same behavior as found for $\mu$. $\Delta \lambda^{ps}$ is monotonically decreasing as $\phi$ increases, while $\Delta \lambda^{uni}$ has a cusp-like minimum at $\phi_{ij}$, and $\Delta \lambda^{ps}_{ij}/\Delta \lambda^{uni}_{ij} \approx 5$. The close correspondence of the behavior of $\Delta \lambda$ with that of $\mu$ suggests that it is the geometry of the contact network that is the primary mechanism for the anisotropy in the systems.

From Eq. (21) we see that $\Delta \lambda$ can also be viewed as the magnitude of the nematic order parameter for contact bond directions [19]. We can then generalize the fabric tensor to consider higher order moments of the bond orientational order. If we define $\mathcal{P}(\theta)$ as the probability density that a bond angle $\theta_{ij} = \theta$, then since bonds have no head nor tail, i.e. we can equally view the bond as $\mathbf{r}_{ij}$ or $\mathbf{r}_{ji} = -\mathbf{r}_{ij}$, then $\mathcal{P}$ must have the symmetry $\mathcal{P}(\theta) = \mathcal{P}(\theta + \pi)$. Therefore we define $\mathcal{P}(\theta)$ as a function on the range $\theta \in [0, \pi]$ only, and normalize it so that $\int_0^\pi d\theta \mathcal{P}(\theta) = 1$. We can then expand $\mathcal{P}(\theta)$ in terms of a Fourier series [20]. We have,

$$
\mathcal{P}(\theta) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{n=1} A_n \cos 2n\theta + B_n \sin 2n\theta
$$

(22)

where the Fourier coefficients are given by,

$$
A_n = \int_0^\pi d\theta \mathcal{P}(\theta) \cos 2n\theta = \langle \cos 2n\theta \rangle
$$

(23)

$$
B_n = \int_0^\pi d\theta \mathcal{P}(\theta) \cos 2n\theta = \langle \sin 2n\theta \rangle.
$$

(24)

We can then rewrite Eq. (22) as,

$$
\mathcal{P}(\theta) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{n=1} S_{2n} \cos(2n[\theta - \theta_{2n}])
$$

(25)

where the amplitude $S_{2n}$ is given by

$$
S_{2n} = \sqrt{A_n^2 + B_n^2} = \sqrt{\langle \cos 2n\theta \rangle^2 + \langle \sin 2n\theta \rangle^2}
$$

(26)

and the orientation $\theta_{2n}$ is given by,

$$
\tan(2n\theta_{2n}) = \frac{B_n}{A_n} = \frac{\langle \sin 2n\theta \rangle}{\langle \cos 2n\theta \rangle}.
$$

(27)

We thus see that $S_{2n}$ is just the magnitude of the $2n$-fold orientational order parameter for the bond directions of the geometrical contact network, and $\theta_{2n}$ is the angle of orientation of the order parameter [19]. Odd order orientational order parameters (i.e., $S_{2n+1}$) all vanish due to the symmetry $\mathcal{P}(\theta) = \mathcal{P}(\theta + \pi)$. $S_{2} = \Delta \lambda$ gives nematic order, $S_{4}$ gives tetratic order, and $S_{6}$ gives hexatic order.

For our geometry, the symmetry in $\theta_{ij} \leftrightarrow -\theta_{ij}$ results in $B_n = \langle \sin 2n\theta \rangle = 0$. So $S_{2n} = |A_n| = |\langle \cos 2n\theta \rangle|$ and $\tan(2n\theta_{2n}) = 0$. Therefore, $\theta_{2n} = 0$ when $\langle \cos 2n\theta \rangle > 0$, and $\theta_{2n} = \pi/2n$ when $\langle \cos 2n\theta \rangle < 0$. Because $\theta_{2n}$ is restricted to only these two possible values, we will drop the absolute value sign in the definition of $S_{2}$ and henceforth adopt the notation,

$$
S_{2n} = \langle \cos 2n\theta \rangle, \quad \begin{cases} 
\theta_{2n} = 0 & \text{when } S_{2n} > 0 \\
\theta_{2n} = \pi/2n & \text{when } S_{2n} < 0.
\end{cases}
$$

(28)

In Fig. 5a we plot $S_{2n}$ vs $\phi$, for $n = 1, 2, 3,$ and 4, at the fixed strain rate $\dot{\varepsilon} = 10^{-7}$. Closed symbols represent pure shear, while open symbols give uniaxial compression. The $n = 1$ nematic order parameter $S_{2}$ is the same as the $\Delta \lambda$ previously shown in Fig. 3b. We see that the $n = 2$ tetratic order parameter $S_{4}$ is comparable in size to the $n = 1$ nematic order, $|S_{4}| \approx |S_{2}|$, while the $n = 3$ hexatic ordering $S_{6}$ is noticeable but smaller. $S_{8}$ and higher order terms are generally quite small. That $S_{2}, S_{6} > 0$ indicates that the nematic and hexatic orderings are oriented at $\theta_{2}, \theta_{6} = 0$, while $S_{4} < 0$ means that the tetratic ordering is at $\theta_{4} = \pi/4$, along the diagonal. These results indicate the expected conclusion that bonds prefer to orient along the compressive direction $\mathbf{\hat{x}}$, and are least likely to orient along the transverse direction $\mathbf{\hat{y}}$. As noted earlier for the nematic ordering, we see that for all moments the orientational ordering of pure shearing is greater than for uniaxial compression, $|S_{2n}^{ps}| > |S_{2n}^{uni}|$. 
We can similarly generalize the stress tensor. If $\mathcal{P}(\vartheta, h)$ is the joint probability distribution for a bond to be in direction $\vartheta$ and have a force moment $f_{ij}^{\text{rel}} r_{ij} = h$, then we define,

$$\tilde{h}(\vartheta) = \int_0^\infty dh \mathcal{P}(\vartheta, h).$$

(29)

$\tilde{h}(\vartheta)$ is the average force moment per radian at angle $\vartheta$. We can then expand $\tilde{h}(\vartheta)$ in a Fourier series to get,

$$\tilde{h}(\vartheta) = \frac{C_0}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left[ C_n \cos 2n\vartheta + D_n \sin 2n\vartheta \right]$$

(30)

$$= C_0 \left[ \frac{1}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} S_{2n} \cos(2n(\vartheta - \vartheta_{2n})) \right],$$

(31)

where the Fourier coefficients are given by,

$$C_n = \int_0^{\pi} d\vartheta \tilde{h}(\vartheta) \cos 2n\vartheta = \langle h \cos 2n\vartheta \rangle$$

(32)

$$D_n = \int_0^{\pi} d\vartheta \tilde{h}(\vartheta) \sin 2n\vartheta = \langle h \sin 2n\vartheta \rangle$$

(33)

the magnitude $S_{2n}$ is given by,

$$S_{2n} = \sqrt{C_n^2 + D_n^2} \frac{C_0}{C_n} = \sqrt{\langle h \cos 2n\vartheta \rangle^2 + \langle h \sin 2n\vartheta \rangle^2}$$

(34)

and the orientation $\vartheta_{2n}$ is given by,

$$\tan(2n\vartheta_{2n}) = \frac{D_n}{C_n} = \frac{\langle h \sin 2n\vartheta \rangle}{\langle h \cos 2n\vartheta \rangle}$$

(35)

The magnitude and orientation, $(S_{2n}, \vartheta_{2n})$, therefore give the $2n$-fold orientational order parameters for the force network; they are a generalization of orientational order to the case where the individual directors (in our case the $h_{ij} f_{ij}$) have varying magnitudes (in our case $h_{ij} = f_{ij}^{\text{rel}} r_{ij}$), rather than all being fixed at unity.

From the definition of $\tilde{h}$ in Eq. (29), the symmetry of $\mathcal{P}$ in $\vartheta \leftrightarrow -\vartheta$ results in $D_n = 0$, and so we have, $S_{2n} = |\langle h \cos 2n\vartheta \rangle|/\langle h \rangle$, and $\vartheta_{2n} = 0$ when $\langle h \cos 2n\vartheta \rangle > 0$ and $\vartheta_{2n} = \pi/2n$ when $\langle h \cos 2n\vartheta \rangle < 0$. As we did in Eq. (28), we will adopt the notation that,

$$S_{2n} = \langle h \cos 2n\vartheta \rangle$$

(36)

We can now relate the coefficients $C_n$ to the components of the stress tensor. Using the definition of $\mathbf{p}$ in Eq. (17), and making the corresponding steps that led to Eq. (19), we can write,

$$\mathbf{p} = \frac{M_b}{2L_x L_y} \begin{bmatrix} \langle h \rangle + \langle h \cos 2\vartheta \rangle & 0 \\ 0 & \langle h \rangle - \langle h \cos 2\vartheta \rangle \end{bmatrix}.$$  

(37)

Comparing with Eq. (15) we then have,

$$\mu = \frac{\sigma}{p} = \frac{\langle h \cos 2\vartheta \rangle}{\langle h \rangle} = \frac{C_1}{C_0} = S_2.$$  

(38)

Thus $\mu = S_2$ is like the nematic order parameter of the force network. The higher moments $S_{2n}$ give higher order force-orientational information.

In Fig. 5(b) we plot $S_{2n}$ vs $\varphi$, for $n = 1, 2, 3, 4$, at the fixed strain rate $\dot{\varepsilon} = 10^{-7}$. Closed symbols represent pure shear, while open symbols denote uniaxial compression. $S_2$ is the same as $\mu$ previously shown in Fig. 4(a). Comparing to the $S_{2n}$ from the fabric, shown in Fig. 5(a), we see that $S_{2n}$ is generally smaller than $S_2$ for $n > 1$, and thus the dominant mode of anisotropy in the force network is from the $S_2 = \mu$ nematic term. This indicates that weighting the contact bonds by their force moment $h_{ij}$ serves to reduce the non-nematic components of the anisotropy present in the contact network geometry. As with $S_{2n}$, we see that $S_4$ is generally negative while $S_6$ is positive.

From our above results, shown in Figs. 4 and 5, it is clear that the difference in anisotropy, comparing the two cases of uniaxial compression and pure shearing, is to a great extent due to the difference between the relative magnitudes of the isotropic part to the anisotropic part of the stress and fabric tensors. It is therefore interesting to subtract off the isotropic part, and to see how only the anisotropic parts for these two cases compare with each other. To do this we consider the ratio of the $2n$-fold orientational order parameter to the lowest nematic order, i.e. $S_{2n}/S_2 = A_n/A_1$ for the contact network, and $S_{2n}/S_2 = C_n/C_1$ for the force network. In Figs. 6(a) and 6(b) we plot these ratios vs $\varphi$, for the fixed strain rate.
FIG. 6. (a) Ratio of orientational order parameters $S_{2n}/S_2$ for the bond directions in the geometrical contact network, and (b) ratio of generalized orientational order parameters $S_{2n}/S_2$ for the force-weighted bond directions in the force network, vs $\phi$ for strain rate $\dot{\epsilon} = 10^{-7}$. Results are shown for tettratic ($n = 2$), hexatic ($n = 3$), and 8-fold ($n = 4$) orientational order. Closed symbols denote pure shearing while open symbols denote uniaxial compression. The vertical dashed line indicates the isotropic compression-driving jamming $\phi_J = 0.8415$. The system has $N = 32768$ particles.

$\dot{\epsilon} = 10^{-7}$, for tettratic ($n = 2$), hexatic ($n = 3$), and 8-fold ($n = 4$) order. We see from these plots that there is now relatively little difference between uniaxial compression and pure shearing, and that the magnitudes of these higher order orientational terms are relatively small for the force network at all $\phi$ (see Fig. 6(b)), though not for the contact network (see Fig. 6(a)). We thus conclude that the main difference in anisotropy, comparing uniaxial compression with pure shearing, is due to differences in the nematic ordering. We give some additional discussion of the system anisotropy in Appendix B.

V. SUMMARY

We have carried out numerical simulations of athermal, frictionless, overdamped, bidisperse circular disks in two dimensions, within a simple model for a non-Brownian suspension, as the packing fraction is increased though the jamming transition. We compare the stresses that result when the system is deformed by isotropic compression, uniaxial compression, and pure shearing, all applied at a fixed strain rate $\dot{\epsilon}$.

Below jamming, the pressure $p$ arising from uniaxial compression is found to be roughly equal that of isotropic compression, while the pressure from pure shearing is roughly an order of magnitude smaller. Above jamming, all three cases approach roughly the same pressure. The shear stress $\sigma$ for isotropic compression is, by symmetry, equal to zero. The shear stress for uniaxial compression is greater than that of pure shearing below jamming, but above jamming it is pure shearing that gives the larger shear stress. The two shear stresses are of the same order of magnitude.

By comparing the stress ratios of the three types of deformation in Fig. 6 we have argued that isotropic compression and uniaxial compression have the same jamming packing $\phi_J$, with bulk viscosities $\zeta = p/\gamma$ that diverge with the same critical exponent $\beta$. However the jamming packing $\phi^*_J$ for pure shearing is slightly larger than the $\phi_J$ for compression. In Appendix B we provide a critical scaling analysis of our pure shearing results that finds that the pure shearing $\phi^*_J$ is indeed greater than $\phi_J$, and that $\phi^*_J$ is equal to the jamming packing previously found for simple shearing. We further find that pure shearing has the same critical exponents, for example $\beta$, as found for compression. Thus stress-isotropic jamming is in the same critical universality class as stress-anisotropic jamming in two dimensions.

The strain rate tensor for uniaxial compression can be viewed as a superposition of the strain rate tensors for isotropic compression plus pure shearing, $\epsilon_{uni} = \epsilon_{iso} + \epsilon_{ps}$. We have therefore asked if there is any similar superposition for the resulting stresses in the linear rheology region below jamming. Our conclusion is that there is no such superposition for stresses.

Finally, we have compared the anisotropy of configurations produced by uniaxial compression with that of configurations produced by pure shear. We have considered both the stress anisotropy $\mu = \sigma/p$ and the anisotropy of the fabric tensor $\Delta \lambda$ of the contact network. Both parameters approach a finite limiting curve as $\dot{\epsilon} \to 0$, demonstrating that the systems remain anisotropic both at jamming and above. We find that $\mu$ and $\Delta \lambda$ behave qualitatively the same as a function of packing $\phi$ and strain rate $\dot{\epsilon}$, indicating that anisotropy is driven primarily by the geometry of the contact network. However we found that there is a big difference comparing pure shearing with uniaxial compression. The anisotropy parameters $\mu$ and $\Delta \lambda$ are smaller for uniaxial compression than for pure shearing, by a factor of order 3 – 5. For pure shearing, $\mu$ and $\Delta \lambda$ are monotonically decreasing as $\phi$ increases, while for uniaxial compression there is a kink with a sharp minimum at $\phi_J$.

We have shown that $\Delta \lambda$ can be viewed as the nematic order parameter for bond directions in the contact network, while $\mu$ can be viewed as a generalized nematic order parameter for bond directions when the bonds are weighted by the force moment on each bond; we call this the force network. We have then generalized to higher order, $2n$-fold orientational order parameters (tetratic, hexatic, etc.) for a more complete parameterization of the anisotropy of the configurations. We find that these $2n$-fold orientational order parameters tend to be larger for the contact network as compared to the force network. However, when subtracting off the isotropic term and measuring these $2n$-fold orientational order parameters relative to the nematic order parameter, then the anisotropic parts of the stress (i.e., the force network) and fabric (i.e., the contact network) become comparable. We thus conclude that the main difference in network anisotropy, comparing uniaxial compression to pure shearing, is due to the nematic ordering.
ACKNOWLEDGMENTS

We thank Brendan Barrow for contributions at early stages of this work. This work was supported by National Science Foundation Grant Nos. DMR-1809318 and PHY-1757062. Computations were carried out at the Center for Integrated Research Computing at the University of Rochester.

APPENDIX A: COMPRESSION ENSEMBLES

In this appendix we describe in greater detail our compression ensemble and its limiting behaviors. Our compressions start from random configurations of non-overlapping disks, constructed as described at the end of Sec. II, at a given initial packing \( \phi \) over a range of \( \epsilon \) of Sec. II, at a given initial packing \( \phi \) over a range of \( \epsilon \) and its limiting behaviors. Our

...rheology region (p, \( \sigma \) \( \propto \) \( \dot{\epsilon} \)) for \( \phi \) \( \lesssim \) 0.82, which covers the region of our primary interest.

In Fig. 7(a) we plot the resulting \( p \) vs \( \phi \) for isotropic compression, for values of \( \phi_{\text{init}} \) = 0.01 to 0.50. In Fig. 7(b) we replotted these results as \( p \) vs \( \phi_{\text{init}} \) at several different values of \( \phi \). Fitting the data of \( p \) vs \( \phi_{\text{init}} \) to a cubic polynomial (shown as the solid curves in Fig. 7(b)), we then extrapolate to determine the \( \phi_{\text{init}} \) \( \rightarrow \) 0 limiting value of \( p(\phi) \); these are shown as the black dots and dashed line in Fig. 7(a). The corresponding plots for uniaxial compression are shown in Figs. 7(c) and 7(d) for the pressure \( p \), and in Figs. 7(e) and 7(f) for the shear stress \( \sigma \).

We see that in all cases the stress (whether \( p \) or \( \sigma \)) approaches a well defined limiting curve as \( \phi_{\text{init}} \) \( \rightarrow \) 0. There is a clear dependence of the stress on the particular value of \( \phi_{\text{init}} \) at small \( \phi \), however this dependence goes away as \( \phi \) increases, and the curves for all \( \phi_{\text{init}} \) approach the limiting curve as one enters the dense region just below jamming. As \( \phi \) decreases from this dense region, the curves for different \( \phi_{\text{init}} \) start to peel away from this limiting curve, vanishing as \( \phi \rightarrow \phi_{\text{init}} \); the smaller \( \phi \) is \( \phi_{\text{init}} \), the wider \( \phi \) is \( \phi_{\text{init}} \), the higher \( \phi \) of \( \phi_{\text{init}} \) and \( \phi \) of \( \phi_{\text{init}} \) curve is a good approximation for the \( \phi_{\text{init}} \) \( \rightarrow \) 0 limiting curve. Such behavior suggests that \( \phi_{\text{init}} \) is an irrelevant variable in the sense of critical scaling, and that using a finite value for \( \phi_{\text{init}} \) will not effect the critical behavior at jamming, provided one restricts data to be sufficiently close to \( \phi_{\text{init}} \).

In Fig. 8 we show similar plots of the average contact number per particle \( Z \) for different \( \phi_{\text{init}} \). We include rattler particles in our computation of \( Z \) so that it remains well defined even at low \( \phi \). We see the same qualitative behavior as we found for the stress. \( Z \) approaches a well defined limit as \( \phi_{\text{init}} \) \( \rightarrow \) 0, and the curves for finite \( \phi_{\text{init}} \) all approach this limiting curve as \( \phi \) increases towards jamming. From Figs. 7 and 8 we see that, for the \( \phi_{\text{init}} \) = 0.40 that we use in the main body of this work, effects due to the finite value of \( \phi_{\text{init}} \) should be rather small once \( \phi \gtrsim 0.80 \).

The results shown in Figs. 7 and 8 show qualitatively similar behavior, with respect to the dependence on \( \phi_{\text{init}} \), for both isotropic and uniaxial compression. However we find an interesting result if we directly compare the pressure of the two cases. In Fig. 9 we plot the uniaxial to isotropic pressure ratio \( p_{\text{uni}}/p_{\text{iso}} \) vs \( \phi \), for different values of \( \phi_{\text{init}} \). Just below \( \phi_{\text{J}} \) and above, we find \( p_{\text{uni}}/p_{\text{iso}} \) = 1, within the estimated errors, as we reported in Sec. III. However, as \( \phi \) decreases, we see that \( p_{\text{uni}}/p_{\text{iso}} \) eventually increases above unity. This increase from unity shifts down to lower packings the smaller is the value of \( \phi_{\text{init}} \). We conjecture that \( p_{\text{uni}} = p_{\text{iso}} \) for all \( \phi \), as \( \phi_{\text{init}} \) \( \rightarrow \) 0.
FIG. 8. (a) Average contact number $Z$ (including rattlers) vs packing $\phi$ at a strain rate $\dot{\epsilon} = 10^{-7}$, for isotropic compression of $N = 8192$ particles starting from different initial packings $\phi_{\text{init}}$. The vertical dashed line indicates the compression-driving jamming $\phi_J = 0.8415$. The width of each curve indicates the estimate error. (b) Data of panel (a) replotted as $Z$ vs $\phi_{\text{init}}$, at several different packings $\phi$. The solid lines are cubic polynomial fits. The black dots and dashed line in (a) are the extrapolated values of $Z$ as $\phi_{\text{init}} \to 0$, obtained from such fits. (c) and (d) are analogous plots of $Z$ for the case of uniaxial compression.

FIG. 9. Pressure ratio $p_{\text{uni}}/p_{\text{iso}}$ vs packing $\phi$, comparing uniaxial to isotropic compression, at a strain rate $\dot{\epsilon} = 10^{-7}$ for $N = 8192$ particles starting from different initial packings $\phi_{\text{init}}$. (b) An expanded view of (a), looking closer in the vicinity of the jamming transition $\phi_J$. The vertical dashed lines indicate the compression-driving jamming $\phi_J = 0.8415$. For clarity, data points are shown only at intervals of $\Delta\phi = 0.01$ and representative error bars are shown only on a subset of those points.

APPENDIX B: CRITICAL SCALING FOR PURE SHEARING

In this appendix we provide more details of our pure shearing simulations, defined by the strain rate tensor of Eq. $[6]$. Most prior work studying the effect of shearing on the jamming transition has considered simple shearing $[7, 8, 11, 14, 16, 17, 21, 24]$, where the strain rate tensor for flow in the $\hat{x}$ direction is given by $\dot{\epsilon} = \hat{c} \hat{x} \otimes \hat{y}$. Simple shearing can be viewed as a superposition of pure shearing plus a system rotation with angular velocity $\dot{\epsilon}/2$.

Both simple and pure shearing preserve the system area. To pure shear, we compress the system in the $\hat{x}$ direction, while expanding it in the $\hat{y}$ direction, both at the same rate $\dot{\epsilon}/2$. Unlike simple shear, where the system can be sheared indefinitely via the use of Lees-Edwards boundary conditions $[27]$, we can only pure shear to a certain total strain $\epsilon = \dot{\epsilon} t$ before the system becomes too narrow in the $\hat{x}$ direction and finite size effects become important. For our system size of $N = 32768$ particles, however, we find that we can always shear to at least $\epsilon = 2$ with no apparent finite size effects, and that this is sufficient to reach steady-state behavior.

For the results reported in the main text, we pure sheared from an initial configuration obtained from uniaxial compression at the same rate $\dot{\epsilon}$. For $\dot{\epsilon} > 10^{-7}$ we averaged results over 10 independent initial configurations, while for $\dot{\epsilon} \leq 10^{-7}$ we used only a single initial configuration. In all cases, the reported steady-state values were obtained by averaging results over some strain interval ($\epsilon_1, \epsilon_2$) within the steady-state region. In contrast, to illustrate the evolution of the stress under pure shearing, in Fig. 10 we show instantaneous results vs $\epsilon$ for configurations sheared at $\dot{\epsilon} = 10^{-7}$, averaged over 10 independent initial configurations. We compare the case where the initial configurations were obtained from uniaxial compression, and so have some finite initial shear stress $\sigma > 0$, to the case where the initial configurations were obtained from isotropic compression, and so have $\sigma = 0$.

In Figs. 10(a), 10(b), and 10(c) respectively, we plot $p$, $\sigma$ and the contact number $Z$ (rattlers included) vs strain $\epsilon$ for $\dot{\epsilon} = 10^{-7}$, at $\phi = 0.80$ below jamming. In Figs. 10(d), 10(e), and 10(f) we plot the same quantities at $\phi = 0.86$ above jamming. For the case of $\phi < \phi_J$, where the stress is due entirely to the finite strain rate, i.e., $p, \sigma \propto \dot{\epsilon}$, we find that the initial discontinuous change in the deformation (from uniaxial or isotropic compression to pure shear) results in an essentially instantaneous change in $p, \sigma$, and $Z$. Following this initial instantaneous change, these parameters show a non-monotonic behavior as $\epsilon$ increases and the system relaxes to its steady state. We find this non-monotonic behavior to be limited to a fairly narrow window of $\phi$ below $\phi_J$.

For the case $\phi > \phi_J$, where there remains a finite stress even as $\dot{\epsilon} \to 0$, the initial change in $p, \sigma$ and $Z$ is still relatively rapid, though it is now smooth and continuous. For $\phi$ both above and below jamming, we see that, as $\epsilon$ increases, the system reaches a steady state, where these quantities plateau to roughly constant values. The time needed to reach the steady state increases, and in principle diverges, as one approaches the jamming critical point, $\phi = \phi_J$ and $\dot{\epsilon} \to 0$. We also see in Fig. 10 that the values in the steady-state are independent of
the starting initial configuration, as has been previously noted for simple shearing [11].

In Fig. 10 of Sec. III we argued that the jamming packing fraction $\phi^*_j$ for pure shearing is slightly larger than the $\phi^*_j = 0.8415$ for uniaxial or isotropic compression. We now give further evidence for this. A main characteristic of the jamming transition is that as $\dot{\epsilon} \to 0$, then below $\phi^*_j$ the stress $p, \sigma \to 0$ vanish, while above $\phi^*_j$ the stress $p, \sigma \to p_0, \sigma_0$ stays finite. Thus, at small $\dot{\epsilon}$, curves of $p$ and $\sigma$ vs $\dot{\epsilon}$ will be concave for $\phi < \phi^*_j$, but convex for $\phi > \phi^*_j$. In Fig. 11 we plot the steady-state values of $p$ and $\sigma$ from pure shearing vs $\dot{\epsilon}$ for several different values of $\phi$ near jamming. Applying the above criterion to the pressure $p$ in Fig. 11(a), we clearly see that the jamming point for pure shearing satisfies $0.842 < \phi^*_j < 0.844$, and is thus larger than the jamming $\phi^*_j = 0.8415$ found by us previously [3] for isotropic compression. The curves of shear stress $\sigma$ in Fig. 11(b) similarly argue for $0.842 < \phi^*_j < 0.844$, even though drawing conclusions from $\sigma$ can be complicated by larger corrections to scaling than exist for $p$ [17, 9].

To determine the specific value of $\phi^*_j$ we can fit our data to the assumed critical scaling equation. Since corrections-to-scaling have been found to be smaller for $p$ than for $\sigma$, we fit our data for pressure to the leading scaling form [1, 7, 9],

$$p(\phi, \dot{\epsilon}) = \dot{\epsilon}^\beta \left( \frac{\phi - \phi^*_j}{\dot{\epsilon}^{1/z\nu}} \right)^\gamma,$$

(39)

FIG. 11. (a) Pressure $p$ and (b) shear stress $\sigma$ vs strain rate $\dot{\epsilon}$, in steady-state pure shearing, for several different values of packing $\phi$ near jamming. The system has $N = 32768$ particles.
shear, for different strain rates $\dot{\varepsilon}$ of Fig. 12(a) to the scaling form of Eq. (39), for different is obtained using $\phi = 0.84319$. The scaling collapse is obtained using $\phi = 0.84319$, $q = 0.505$, and $1/z\nu = 0.268$. The system has $N = 32768$ particles.

![Graph](image)

**APPENDIX C: ORIENTATIONAL ORDERING DISTRIBUTIONS**

In this appendix we present some additional results concerning the anisotropy of the contact and force networks in uniaxial compression and pure shearing. In Sec. [IV](#) we studied the geometrical contact network, described by the function $P(\theta)$, which gives the probability density for a bond in the contact network to be oriented at an angle $\theta$ with respect to the compressive direction $\hat{x}$. We also considered the force network, described by the function $h(\theta)$ defined in Eq. (29), which gives the average force moment $h_{n}(f_{ij} \hat{x}_{ij})$ per unit radian on a bond oriented at angle $\theta$. For these two functions, we analyzed their Fourier coefficients, which defined respectively the $2n$-fold orientational order parameters $S_{2n}$ for bond directions and the generalized (force-weighted) $2n$-fold orientational order parameters $\tilde{S}_{2n}$.

Here we consider directly the functions $P(\theta)$ and $h(\theta)$, and related quantities. In Figs. [14a](#) and [14b](#) we plot $P(\theta)$ vs $\theta$ for several different packing fractions $\phi$ at the strain rate $\dot{\varepsilon} = 10^{-7}$, for uniaxial compression and pure shearing respectively. In Figs. [15a](#) and [15b](#) we similarly plot the corresponding $h(\theta)/(h)$; we normalize $h(\theta)$ by $h$ so that all curves have a common average of $1/\pi$. We use a common scale for the vertical axes of both uniaxial and pure shear cases, so as to allow an easy visual comparison between the two. From Figs. [14](#) and [15](#) one can easily visualize many of the conclusions of Sec. [IV](#). The anisotropy decreases as one approaches $\phi \approx 0.84$. Pure shearing results in greater anisotropy than uniaxial compression. The anisotropy of the contact network, given by $P(\theta)$, involves larger, higher order, Fourier components than does the force network, given by $h(\theta)$, par-
The system has a bond directed at angle $\theta$ with respect to the compressive direction $\hat{x}$. Results are plotted vs $\theta$ for several different packing fractions $\phi$ at strain rate $\dot{\epsilon} = 10^{-7}$. (a) is for uniaxial compression, while (b) is for pure shearing. The solid horizontal black line at $h(h)/\pi = 1/\pi$ represents the average value. The system has $N = 32768$ particles.

The function $\tilde{h}(\theta) = \int dh P(\theta, h) h$ was defined as the average force-moment per radian at angle $\theta$; this expression incorporates in its definition the probability that there will indeed be contact bonds at angle $\theta$. Alternatively we can ask, what is the average force-moment on bonds at angle $\theta$ independent of the likelihood that there are bonds at that orientation. If $P(\theta, h)$ is the joint probability that there is a bond at angle $\theta$ with force-moment $h$, then we can write, $P(\theta, h) = P(h|\theta) P(\theta)$, where $P(h|\theta)$ is the conditional probability to find a force-moment $h$, given that there is a bond at $\theta$. We can then define,

$$h(\theta) = \int_0^\infty dh P(h|\theta) h = \frac{\tilde{h}(\theta)}{P(\theta)}. \quad (40)$$

To illustrate the difference between $h(\theta)$ and $\tilde{h}(\theta)$, imagine that all bonds had the same force-moment $h$; then we would have $\tilde{h}(\theta) = hP(\theta)$ while $h(\theta) = h$ would be constant.

In Figs. 16(a) and 16(b) we plot $h(\theta)/\pi(\theta)$ vs $\theta$ for the same parameters as in Figs. 14 and 15. We normalize $h(\theta)$ by $1/\pi(\theta)$ so that all curves have the same average $1/\pi$ as the $\tilde{h}(\theta)$ curves in Fig. 15 Comparing to $\tilde{h}(\theta)$ we see that $h(\theta)$ has a somewhat smaller anisotropy, yet the anisotropy in the force-moments remains sizeable. As $\phi \rightarrow \phi_J$, we see, as might be expected, that the forces are greater than average for $0 \leq \theta \lesssim \pi/4$, and less than average for $\pi/4 \lesssim \theta \leq \pi$.

Next we consider the purely anisotropic parts of $P(\theta)$ and $\tilde{h}(\theta)$. From Eqs. (22) and (30) we define the anisotropic parts by, $\Delta P(\theta) = P(\theta) - 1/\pi$ and $\Delta h(\theta) = \tilde{h}(\theta) - C_0/\pi$. By construction, $\Delta P(\theta)$ and $\Delta \tilde{h}(\theta)$ both average to zero when averaging over $\theta$. We then normalize these functions by their leading (nematic) Fourier coefficient, so as to compare how their shape varies as the packing $\phi$ varies. In Figs. 17(a) and 17(b) we plot $\Delta P(\theta)/A_1$ vs $\theta$ for different $\phi$ at strain rate $\dot{\epsilon} = 10^{-7}$, for uniaxial compression and pure shearing, respectively. In Figs. 18(a) and 18(b) we make similar plots of $\Delta h(\theta)/C_1$. In both Figs. 17 and 18 the solid black line is the functional form, $(2/\pi) \cos 2\theta$, that one would have if only the nematic ($n = 1$) term was present. In Fig. 17 we see that $\Delta P(\theta)$ involves significant higher order terms beyond the nematic, however, qualitatively, there does not appear to be much difference between the two cases of uniaxial compression and pure shearing. This is in agreement with what was seen for the order parameters $S_{2n}/S_2$ in Fig. 6(a). In contrast, Fig. 18 shows that the nematic term does give a reasonable approximation, and so higher order terms are relatively small. Again there is little qualitative differences between uniaxial compression and pure shear, in agreement with what was seen for the order parameters $S_{2n}/S_2$ in Fig. 18(b). We thus conclude that there is little difference in the anisotropic parts of either the contact network or the force network, when comparing the two cases of uniaxial compression and pure shearing, and the main difference between these two cases
FIG. 17. Anisotropic part of the contact network bond orientation probability \( \Delta P(\theta) = P(\theta) - \frac{1}{\pi} \), normalized by the magnitude of the nematic order Fourier coefficient \( A_1 = \langle \cos 2\theta \rangle \), vs \( \theta \) for several different packing fractions \( \phi \) at strain rate \( \dot{\epsilon} = 10^{-7} \). (a) is for uniaxial compression, while (b) is for pure shearing. The solid black line gives the functional form for the purely nematic term, \( \frac{2}{\pi} \cos 2\theta \). The system has \( N = 32768 \) particles.

FIG. 18. Anisotropic part of the average force-moment orientation \( \Delta \tilde{h}(\theta) = \tilde{h}(\theta) - \langle \tilde{h} \rangle / \pi \), normalized by the magnitude of the nematic order Fourier coefficient \( C_1 = \langle \tilde{h} \cos 2\theta \rangle \), vs \( \theta \) for several different packing fractions \( \phi \) at strain rate \( \dot{\epsilon} = 10^{-7} \). (a) is for uniaxial compression, while (b) is for pure shearing. The solid black line gives the functional form for the purely nematic term, \( \frac{2}{\pi} \cos 2\theta \). The system has \( N = 32768 \) particles.

lies in the relative magnitude of the anisotropic term to the isotropic term, i.e., \( \Delta \lambda = S_2 \) and \( \mu = \sigma / p = S_2 \).
[1] A. Peshkov and S. Teitel, Universality of stress-anisotropic and stress-isotropic jamming of frictionless spheres in three dimensions: Uniaxial versus isotropic compression, Phys. Rev. E 105, 024902 (2022).
[2] A. J. Liu and S. R. Nagel, The jamming transition and the marginally jammed solid, Annu. Rev. Condens. Matter Phys. 1, 347 (2010).
[3] C. S. O’Hern, L. E. Silbert, A. J. Liu, and S. R. Nagel, Jamming at zero temperature and zero applied stress: The epitome of disorder, Phys. Rev. E 68, 011306 (2003).
[4] P. Olsson and S. Teitel, Critical scaling of shear viscosity at the jamming transition, Phys. Rev. Lett. 99, 178001 (2007).
[5] A. Peshkov and S. Teitel, Critical scaling of compression-driven jamming of athermal frictionless spheres in suspension, Phys. Rev. E 103, L040901 (2021).
[6] D. J. Durian, Foam mechanics at the bubble scale, Phys. Rev. Lett. 75, 4780 (1995) and Bubble-scale model of foam mechanics: Melting, nonlinear behavior, and avalanches, Phys. Rev. E 55, 1739 (1997).
[7] P. Olsson and S. Teitel, Critical scaling of shearing rheology at the jamming transition of soft-core frictionless disks, Phys. Rev. E 83, 030302(R) (2011).
[8] D. Vägberg, P. Olsson, and S. Teitel, Dissipation and rheology of sheared soft-core frictionless disks below jamming, Phys. Rev. Lett. 112, 208303 (2014).
[9] D. Vägberg, P. Olsson, and S. Teitel, Critical scaling of Bagnold rheology at the jamming transition of frictionless two-dimensional disks, Phys. Rev. E 93, 052902 (2016).
[10] See: https://lammps.sandia.gov/
[11] D. Vägberg, P. Olsson, and S. Teitel, Glassiness, rigidity, and jamming of frictionless soft core disks, Phys. Rev. E 83, 031307 (2011).
[12] C. Heussinger and J.-L. Barrat, Jamming transition as probed by quasistatic shear flow, Phys. Rev. Lett. 102, 218303 (2009).
[13] P. Olsson, Relaxation times and rheology in dense athermal suspensions, Phys. Rev. E 91, 062209 (2015).
[14] P. Olsson, Dimensionality and viscosity exponent in shear-driven jamming, Phys. Rev. Lett. 122, 108003 (2019).
[15] M. Wyart, L. E. Silbert, S. R. Nagel, and T. A. Witten, Effects of compression on the vibrational modes of marginally jammed solids, Phys. Rev. E 72, 051306 (2005).
[16] P. Olsson and S. Teitel, Herschel-Bulkley shearing rheology near the athermal jamming transition, Phys. Rev. Lett. 109, 108001 (2012).
[17] D. Vägberg, P. Olsson, and S. Teitel, Universality of jamming criticality in overdamped shear-driven frictionless disks, Phys. Rev. Lett. 113, 148002 (2014).
[18] J. Zhang, T. Majmudar, A. Tordesillas, and R. Behringer, Statistical properties of a 2D granular material subjected to cyclic shear, Granul. Matter 12, 159 (2010).
[19] A. Donev, J. Burton, F. H. Stillinger, and S. Torquato, Tetradic order in the phase behavior of a hard-rectangle system, Phys. Rev. B 73, 034109 (2006).
[20] E. Azema and F. Radjai, Stress-strain behavior and geometrical properties of packings of elongated particles, Phys. Rev. E 81, 051304 (2010); Force chains and contact network topology in sheared packings of elongated particles, Phys. Rev. E 85, 031303 (2012); and Internal Structure of Inertial Granular Flows, Phys. Rev. Lett. 112, 078001 (2014).
[21] T. Hatano, Growing length and time scales in a suspension of athermal particles, Phys. Rev. E 79, 050301(R) (2009).
[22] M. Otsuki and H. Hayakawa, Critical behaviors of sheared frictionless granular materials near the jamming transition, Phys. Rev. E 80, 011308 (2009).
[23] E. Lerner, G. Düring, and M. Wyart, A Unified framework for non-Brownian suspension flows and soft amorphous solids, Proc. Natl. Acad. Sci. U.S.A. 109, 4798 (2012).
[24] E. DeGiuli, G. Düring, E. Lerner, and M. Wyart, Unified theory of inertial granular flows and non-Brownian suspensions, Phys. Rev. E 91, 062206 (2015).
[25] T. Kawasaki, D. Coslovich, A. Ikeda, and L. Berthier, Diverging viscosity and soft granular rheology in non-Brownian suspensions, Phys. Rev. E 91, 012203 (2015).
[26] B. Andretotti, J.-L. Barrat, and C. Heussinger, Shear flow of non-brownian suspensions close to jamming, Phys. Rev. Lett. 109, 105001 (2012).
[27] D. J. Evans and G. P. Morriss, Statistical Mechanics of Nonequilibrium Liquids (Academic Press, London, 1990).