Abstract

Let \((M_k, Q_k)_{k \in \mathbb{N}}\) be independent copies of a random vector \(\mathbb{R}^2\)-valued random vector. It is known that if \(Y_n := Q_1 + M_1Q_2 + \ldots + M_{n-1}Q_n\) converges a.s. to a random variable \(Y\), then the law of \(Y\) satisfies the stochastic fixed-point equation \(Y \overset{d}{=} Q_1 + M_1Y\), where \((Q_1, M_1)\) is independent of \(Y\). In the present paper we consider the situation when \(|Y_n|\) diverges to \(\infty\) in probability because \(|Q_1|\) takes large values with high probability, whereas the multiplicative random walk with steps \(M_k\)'s tends to zero a.s. Under a regular variation assumption we show that \(\log |Y_n|\), properly scaled and normalized, converge weakly in the Skorokhod space equipped with the \(J_1\)-topology to an extremal process. A similar result also holds for the corresponding Markov chains. Proofs rely upon a deterministic result which establishes the \(J_1\)-convergence of certain sums to a maximal function and subsequent use of the Skorokhod representation theorem.

Keywords: extremal process; functional limit theorem; perpetuity; random difference equation

2000 Mathematics Subject Classification: Primary: 60F17
Secondary: 60G50

1 Introduction

Let \((M_k, Q_k)_{k \in \mathbb{N}}\) be independent copies of a random vector \((M, Q)\) with arbitrary dependence of the components, and let \(X_0\) be a random variable which is independent of \((M_k, Q_k)_{k \in \mathbb{N}}\). Then the sequence \((X_n)_{n \in \mathbb{N}_0}\) defined by

\[ X_n = M_nX_{n-1} + Q_n, \quad n \in \mathbb{N}, \]  

is a homogeneous Markov chain. In view of the representation

\[
\begin{align*}
X_n &= \Psi_n(X_{n-1}) = \Psi_n \circ \ldots \circ \Psi_1(X_0) \\
&= Q_n + M_nQ_{n-1} + \ldots + M_nM_{n-1} \ldots M_2Q_1 + M_nM_{n-1} \ldots M_1X_0
\end{align*}
\]

for \(n \in \mathbb{N}\), where \(\Psi_n(t) := Q_n + M_nt\) for \(n \in \mathbb{N}\), \((X_n)_{n \in \mathbb{N}}\) is nothing else but the forward iterated function system. Closely related is the backward iterated function system

\[ Y_n := \Psi_1 \circ \ldots \circ \Psi_n(0) = Q_1 + M_1Q_2 + \ldots + M_1M_2 \ldots M_{n-1}Q_n, \quad n \in \mathbb{N}. \]
In the case that \( X_0 = 0 \) a.s. it is easily seen that \( X_n \) has the same law as \( Y_n \) for each fixed \( n \).

Put

\[ \Pi_0 := 1, \quad \Pi_n := M_1 M_2 \cdots M_n, \quad n \in \mathbb{N} \]

and assume that

\[ \mathbb{P}\{M = 0\} = 0 \quad \text{and} \quad \mathbb{P}\{Q = 0\} < 1 \quad (1.2) \]

and

\[ \mathbb{P}\{Q + Mr = r\} < 1 \quad \text{for all} \quad r \in \mathbb{R}. \quad (1.3) \]

Then according to Theorem 2.1 in [8] the series \( \sum_{k \geq 1} \Pi_{k-1} Q_k \) is absolutely a.s. convergent provided that

\[ \lim_{n \to \infty} \Pi_n = 0 \quad \text{a.s. and} \quad I := \int_{(1, \infty)} \frac{\log x}{A(\log x)} \mathbb{P}\{|Q| \in dx\} < \infty, \quad (1.4) \]

where \( A(x) := \mathbb{E}(\log^{-} |M| \wedge x), \ x > 0. \) The sum, say, of the series is then called perpetuity.

It is also well-known what happens in the 'trivial cases' when at least one of conditions (1.2) and (1.3) does not hold.

(a) If \( \mathbb{P}\{M = 0\} > 0 \), then \( \tau := \inf\{k \in \mathbb{N} : M_k = 0\} < \infty \text{ a.s.}, \) and the perpetuity trivially converges, the limit being an a.s. finite random variable \( \sum_{k=1}^\tau \Pi_{k-1} Q_k. \) Plainly, its law is a unique invariant measure for \( (X_n) \).

(b) If \( \mathbb{P}\{Q = 0\} = 1 \), then \( \sum_{k \geq 1} \Pi_{k-1} Q_k = 0 \) a.s.

(c) If \( \mathbb{P}\{Q + Mr = r\} = 1 \) for some \( r \in \mathbb{R} \), then either \( \delta_r \) is a unique invariant probability measure for \( (X_n) \) or every probability law is an invariant measure, or every symmetric around \( r \) probability law is an invariant measure (see Theorem 3.1 in [8] for the details).

Under assumptions (1.2), (1.3) and (1.4) the Markov chain \( (X_n) \) has a unique invariant probability measure which is the law of the perpetuity. Equivalently, the law of \( Y \) is a unique solution to the stochastic fixed-point equation

\[ Y \overset{d}{=} Q + MY, \quad (1.5) \]

where the vector \( (M, Q) \) is assumed independent of \( Y \), sometimes called the random difference equation. Equations (1.5) appear in diverse areas of both applied and pure mathematics and various properties of \( Y \) have attracted considerable attention. Papers [1, 8, 18] give pointers to relevant literature.

For \( (X_n) \) defined by (1.1), we write \( X^v_n \) to indicate that \( X_0 = v \) for \( v \in \mathbb{R} \). If the first part of (1.4) is in force we infer \( |X^v_n - X^w_n| = \Pi_n |v-w| \to 0 \text{ a.s. as } n \to \infty, \) for any \( v, w \in \mathbb{R}. \) Therefore, the case when \( \lim_{n \to \infty} \Pi_n = 0 \text{ a.s. will be called contractive.} \)

In the present paper we are interested in the case when conditions (1.2), (1.3) and

\[ \lim_{n \to \infty} \Pi_n = 0 \quad \text{a.s. and} \quad I = \infty \quad (1.6) \]

hold, i.e., the model is still contracting, yet the second condition in (1.4) is violated. By Theorem 2.1 in [8] \( (Y_n) \) is then a divergent perpetuity in the sense that \( |Y_n| = |\sum_{k=1}^n \Pi_{k-1} Q_k| \overset{P}{\to} \infty \text{ as } n \to \infty. \) The purpose of the present paper is to prove functional limit theorems for the Markov chains \( (X_n) \) and for the divergent perpetuities \( (Y_n) \) under the aforementioned assumptions.

As far as we know Grincevičius [2] was the first to prove a limit theorem for \( Y_n \) in the case \( \mathbb{E}\log |M| = 0 \) under the assumption that \( M > 0 \text{ a.s.} \) Also, weak convergence
of one-dimensional distributions of divergent perpetuities has been investigated in \[3, 10, 13, 15\] under various assumptions on \( M \) and \( Q \). To the best of our knowledge, (a) functional limit theorems for divergent perpetuities have not been obtained so far; (b) \[13\] is the only contribution to case \((1.6)\) which deals with one-dimensional convergence. We would like to stress that outside the area of limit theorems we are only aware of two papers \[12\] and \[19\] which investigate case \((1.6)\). Unlike \((1.6)\) the critical non-contractive case \( \mathbb{E} \log |M| = 0 \) has received more attention in the literature, see \[2, 4, 5, 6, 9, 10, 15\].

Assuming that the tail of \( \log -|M| \) is lighter than that of \( \log +|Q| \) we state two functional limit theorems thereby covering a variety of situations. In particular, we do not require finiteness of \( \mathbb{E} \log |M| \). Under \((1.6)\) the complementary case is also possible where the tail of \( \log -|M| \) is not lighter than that of \( \log +|Q| \).

Take, for instance, \( \mathbb{P}\{\log -|M| > x\} \sim x^{-\alpha} \log x, \ x \to \infty \), and \( \mathbb{P}\{\log |Q| \in dx\} = \alpha x^{-\alpha-1} 1_{(1,\infty)}(x) \) dx for some \( \alpha \in (0, 1) \). Even though this situation is beyond the scope of the present work we note without going into details that it is unlikely that there is functional convergence in the Skorokhod space equipped with one of the standard topologies like \( J_1 \) or \( M_1 \). Also, it is worth to stress that unlike some previous papers on limit theorems for perpetuities we allow \( M \) and \( Q \) to take values of both signs.

For \( c > 0 \) and \( \alpha > 0 \), let \( N(c, \alpha) := \sum_k \varepsilon_{\{ \tau_{c, \alpha}(k), j_{c, \alpha}(k) \}} \) be a Poisson random measure on \([0, \infty) \times (0, \infty]\) with mean measure \( LEB \times \mu_{c, \alpha} \), where \( \varepsilon_{(t, x)} \) is the probability measure concentrated at \((t, x) \in [0, \infty) \times (0, \infty]\), \( LEB \) is the Lebesgue measure on \([0, \infty)\), and \( \mu_{c, \alpha} \) is a measure on \((0, \infty] \) defined by \( \mu_{c, \alpha}(x, \infty] = cx^{-\alpha}, \ x > 0 \). Let \( D := D[0, \infty) \) denote the Skorokhod space of right-continuous functions defined on \([0, \infty)\) with finite limits from the left at positive points. Throughout the paper we use \( \Rightarrow \) to denote weak convergence in the Skorokhod space \( D \) equipped with the \( J_1 \)-topology. We write \( \Rightarrow \) in \( S \) to denote weak convergence in a space \( S \) other than \( D \). Also, we stipulate hereafter that the supremum over the empty set is equal to zero.

Theorem 1.1 treats the situation in which both \( M_k \)’s and \( Q_k \)’s affect the limit behavior of the processes in question, whereas in the situation of Theorem 1.5 only the contribution of \( Q_k \)’s persists in the limit.

**Theorem 1.1.** Assume that

\[
\mathbb{E} \log |M| = -a \in (-\infty, 0), \tag{1.7}
\]

that

\[
\lim_{x \to \infty} x \mathbb{P}\{\log |Q| > x\} = c \tag{1.8}
\]

for some \( c > 0 \). If

\[
\mathbb{P}\{Y_k = 0\} = 0 \tag{1.9}
\]

for each \( k \in \mathbb{N} \), then

\[
\frac{\log |Y_{[n] + 1}|}{an} \Rightarrow \sup_{t_{k(c, a, 1)}^{(c, a, 1)} \leq \cdot} ( -t_{k(c, a, 1)}^{(c, a, 1)} + j_{k(c, a, 1)}^{(c, a, 1)} ), \ n \to \infty, \tag{1.10}
\]

and if

\[
\mathbb{P}\{X_k = 0\} = 0 \tag{1.11}
\]
for each \( k \in \mathbb{N} \), then

\[
\frac{\log |X_{[n]+1}|}{an} \Rightarrow g(t) + \sup_{t_k(c/a,1) \leq t} \left( t_k^{(c/a,1)} + j_k^{(c/a,1)} \right), \quad n \to \infty, \tag{1.12}
\]

where \( g(t) := -t \cdot e^{-t} \), \( t \geq 0 \).

**Remark 1.2.** Conditions (1.9) and (1.11) ensure that the paths of \( \log |Y_{[n]+1}| \) and \( \log |X_{[n]+1}| \) belong to \( D \). While a simple sufficient condition for (1.9) to hold is continuity of the law of \( Q \), (1.11) holds if either \( X_0 = 0 \) a.s. and the law of \( Q \) is continuous or the law of \( X_0 \) is continuous. Condition (1.9) ((1.11)) is not needed if (a) we replace \( \log \) with \( \log^+ \) in (1.10) ((1.12)); (b) consider weak convergence in \( D(0, \infty) \) rather than \( D \). The same remark also concerns Theorem 1.5 given below.

**Remark 1.3.** Since \( X_n = Y_n \) for each \( n \in \mathbb{N} \) provided that \( X_0 = 0 \) a.s., the one-dimensional distributions of the limit processes in (1.10) and (1.12) must coincide. Moreover, they can be explicitly computed and are given by

\[
P\left\{ \sup_{t_k(c/a,1) \leq u} \left( -t_k^{(c/a,1)} + j_k^{(c/a,1)} \right) \leq x \right\} = P\left\{ -u + \sup_{t_k(c/a,1) \leq u} \left( t_k^{(c/a,1)} + j_k^{(c/a,1)} \right) \leq x \right\} = \left( \frac{x}{x + u} \right)^{c/a} \tag{1.13}
\]

for \( x \geq 0 \) and \( u > 0 \).

Indeed, for \( x \geq 0 \), the probability on the left-hand side equals

\[
P\{N(c/a,1)((t, y) : t \leq u, -t + y > x) = 0\} = \exp \left( -\mathbb{E}N(c/a,1)((t, y) : t \leq u, -t + y > x) \right)
\]

because \( N(c/a,1)((t, y) : t \leq u, -t + y > x) \) is a Poisson random variable. It remains to note that

\[
\mathbb{E}N(c/a,1)((t, y) : t \leq u, -t + y > x) = \int_0^u \int_{[0, \infty)} 1_{\{-t+y>x\}} \mu_{c/a,1}(dy)dt
\]

\[
= (c/a) \int_0^u (x + t)^{-1}dt = (c/a)(\log(x + u) - \log x).
\]

**Remark 1.4.** Theorem 5(ii) in [13] states that, for fixed \( a > 0 \),

\[
\lim_{n \to \infty} P\left\{ \log \left( \sum_{k=0}^n e^{-ak |Q_{k+1}|} \right) \leq anx \right\} = \left( \frac{x}{x + 1} \right)^{c/a}, \quad x \geq 0 \tag{1.14}
\]

provided that

\[
\lim_{x \to \infty} x(1 - \mathbb{E}\exp(-e^{-x |Q|})) = c \in (0, \infty).
\]

By an Abelian-Tauberian argument the last relation is equivalent to (1.8). This implies that convergence (1.13) follows from (1.10) and (1.12).

**Theorem 1.5.** Suppose that \( P\{M = 0\} = 0 \), \( \lim_{n \to \infty} \Pi_n = 0 \) a.s., and that

\[
P\{\log |Q| > x\} \sim x^{-\alpha} \ell(x), \quad x \to \infty \tag{1.15}
\]
for some $\alpha \in (0, 1]$ and some $\ell$ slowly varying at $\infty$. Let $(b_n)$ be a positive sequence which satisfy $\lim_{n \to \infty} nP\{\log |Q| > b_n\} = 1$. In the case $\alpha = 1$ assume additionally\footnote{Among other things this implies $E \log^+ |Q| = \infty$.} that $\lim_{x \to \infty} \ell(x) = +\infty$. In the case $E \log^+ |M| = \infty$ assume that

$$\lim_{x \to \infty} \frac{E(\log^+ |M| \wedge x)}{xP\{\log |Q| > x\}} = 0.$$  \hfill (1.16)

If condition (1.9) holds, then

$$\frac{\log |Y_{[n] + 1}|}{b_n} \Rightarrow \sup_{i_k^{(1, \alpha)} \leq} j_k^{(1, \alpha)}, \quad n \to \infty,$$  \hfill (1.17)

and if condition (1.11) holds, then

$$\frac{\log |X_{[n] + 1}|}{b_n} \Rightarrow \sup_{i_k^{(1, \alpha)} \leq} j_k^{(1, \alpha)}, \quad n \to \infty.$$  \hfill (1.18)

Remark 1.6. Theorem 5(iii) in \cite{13} states that, for fixed $a > 0$,

$$\lim_{n \to \infty} P\left\{ \log \left( \sum_{k=0}^{n} e^{-ak} |Q_{k+1}| \right) \leq b_n x \right\} = \exp(-x^{-\alpha}), \quad x \geq 0$$  \hfill (1.19)

provided that the function $x \mapsto 1 - E \exp(-e^{-x}|Q|)$ is regularly varying at $\infty$ with index $-\alpha$, $\alpha \in (0, 1)$, and $(b_n)$ satisfies $n(1 - E \exp(-e^{-b_n}|Q|)) = 1$. By an Abelian theorem,

$$1 - E \exp(-e^{-x}|Q|) \sim P\{\log |Q| > x\}, \quad x \to \infty.$$  

Therefore, (1.19) follows from (1.17) after noting that

$$P\left\{ \sup_{i_k^{(1, \alpha)} \leq} j_k^{(1, \alpha)} \leq x \right\} = P\{N^{(1, \alpha)}((t, y) : t \leq u, y > x) = 0\} = \exp(-ux^{-\alpha}), \quad x \geq 0$$  \hfill (1.20)

for each $u > 0$.

The rest of the paper is structured as follows. In Section 2 we state and prove Theorem 2.1, a deterministic result which is our key tool for dealing with the functional limit theorems. With this at hand, Theorem 1.1 and Theorem 1.5 are then proved in Section 3 and Section 4, respectively.

## 2 Main technical tool

Denote by $M_p$ the set of Radon point measures $\nu$ on $[0, \infty) \times (0, \infty]$ which satisfy

$$\nu([0, T] \times [\delta, \infty]) < \infty$$  \hfill (2.1)

for all $\delta > 0$ and all $T > 0$. The $M_p$ is endowed with the vague topology. Denote by $M_p^*$ the set of $\nu \in M_p$ which satisfy

$$\nu([0, T] \times (0, \infty]) < \infty$$
for all $T > 0$. Define the mapping $G$ from $D \times M_p$ to $D$ by\footnote{Assumption (2.1) ensures that $G(f, \nu) \in D$. If (2.1) does not hold, $G(f, \nu)$ may lost right-continuity.}

$$G(f, \nu)(t) := \begin{cases} 
\sup_{k: \tau_k \leq t} (f(\tau_k) + y_k), & \text{if } \tau_k \leq t \text{ for some } k, \\
\exp(c_n f(0)), & \text{otherwise},
\end{cases}$$

where $\nu = \sum_k \epsilon(\tau_k, y_k)$. Also, for each $n \in \mathbb{N}$, we define the mapping $F_n$ from $D \times M_p^*$ to $D$ by

$$F_n(f, \nu)(t) := \begin{cases} 
\exp(c_n^{-1} \log^+ |\sum_k: \tau_k \leq t \pm \exp(c_n(f(\tau_k) + y_k))|), & \text{if } \tau_k \leq t \text{ for some } k, \\
\exp(c_n f(0)), & \text{otherwise},
\end{cases}$$

where the signs $+$ and $-$ are arbitrarily arranged, and $(c_n)$ is some sequence of positive numbers. The definition of $F_n$ in the case of empty sum stems from the fact that we define $|\sum_k: \tau_k \leq t \pm \exp(c_n(f(\tau_k) + y_k))| := \exp(c_n f(0))$ if there is no $k$ such that $\tau_k \leq t$.

**Theorem 2.1.** For $n \in \mathbb{N}$, let $f_n \in D$ and $\nu_n \in M_p$. Let $(\tau_k^{(n)}, y_k^{(n)})$ be the points of $\nu_n$, i.e., $\nu_n = \sum_k \epsilon(\tau_k^{(n)}, y_k^{(n)})$. Assume that $f_0$ is continuous with $f_0(0) = 0$ and

(A1) $\nu_0(\{0\} \times (0, \infty]) = 0$ and $\nu_0((r_1, r_2) \times (0, \infty]) \geq 1$ for all positive $r_1$ and $r_2$ such that $r_1 < r_2$;

(A2) $\nu_0 = \sum_k \epsilon(\tau_k^{(0)}, y_k^{(0)})$ does not have clustered jumps, i.e., $\tau_k^{(0)} \neq \tau_j^{(0)}$ for $k \neq j$;

(A3) if not all the signs under the sum defining $F_n$ are the same, then

$$f_0(\tau_i^{(0)}) + y_i^{(0)} \neq f_0(\tau_j^{(0)}) + y_j^{(0)} \text{ for } i \neq j \quad (2.2)$$

and

$$\sup_{\tau_k^{(0)} \leq T, y_k^{(0)} \leq \gamma} (f_0(\tau_k^{(0)}) + y_k^{(0)}) > 0 \quad (2.3)$$

for each $T > 0$ such that $\nu_0(\{T\}, (0, \infty]) = 0$ and small enough $\gamma > 0$;

(A4) $\lim_{n \to \infty} c_n = \infty$ and

$$\lim_{n \to \infty} c_n^{-1} \log \#\{k: \tau_k^{(n)} \leq T\} = 0 \quad (2.4)$$

for each $T > 0$ such that $\nu_0(\{T\}, (0, \infty]) = 0$;

(A5) $\lim_{n \to \infty} f_n = f_0$ in $D$ in the $J_1$-topology.

(A6) $\lim_{n \to \infty} \nu_n = \nu_0$ in $M_p$.

Then

$$\lim_{n \to \infty} F_n(f_n, \nu_n) = G(f_0, \nu_0) \quad (2.5)$$

in $D$ in the $J_1$-topology.
Proof. It suffices to prove convergence (2.5) in \(D[0,T]\) for any \(T > 0\) such that \(\nu_0(T \times (0,\infty)) = 0\) because the last condition ensures that \(F(f_0,\nu_0)\) is continuous at \(T\).

If all the signs under the sum defining \(F_n\) are the same, then

\[
G(f_0,\nu_n)(t) \leq F_n(f_0,\nu_n)(t) \leq c_n^{-1} \log^+ \# \{k : \tau_k^{(n)} \leq t\} + G(f_0,\nu_n)(t)
\]

for all \(t \in [0,T]\). In this case, (2.5) is a trivial consequence of Theorem 1.3 in [11], which treats the convergence \(\lim_{n \to \infty} G(f_n,\nu_n) = G(f_0,\nu_0)\) in \(D\).

In what follows we thus assume that not all the signs are the same. Let \(\rho = \{0 = s_0 < s_1 < \cdots < s_m = T\}\) be a partition of \([0,T]\) such that

\[
\nu_0(\{s_k\} \times (0,\infty)) = 0, \quad k = 1, \ldots, m.
\]

Pick now \(\gamma > 0\) so small that

\[
\nu_0((s_k,s_{k+1}) \times (\gamma,\infty)) \geq 1, \quad k = 0, \ldots, m - 1
\]

and that \(\sup_{\tau_k \in \tau, y_k \in \gamma}(f_0(\tau_k^{(0)}) + y_k^{(0)}) > 0\). The latter is possible because \(\sup_{\tau_k \in \tau, y_k \in \gamma}(f_0(\tau_k^{(0)}) + y_k^{(0)}) > 0\) as a consequence of (2.5).

Condition (A6) implies that \(\nu_0([0,T] \times (\gamma,\infty)) = \nu_0([0,T] \times (\gamma,\infty]) = p\) for large enough \(n\) and some \(p \geq 1\). Denote by \((\bar{\tau}_i,\bar{y}_i)_{1 \leq i \leq p}\) an enumeration of the points of \(\nu_0\) in \([0,T] \times (\gamma,\infty]\) with \(\bar{\tau}_1 < \bar{\tau}_2 < \cdots < \bar{\tau}_p\) and by \((\tau_i^{(n)},\bar{y}_i^{(n)})_{1 \leq i \leq p}\) the analogous enumeration of the points of \(\nu_n\) in \([0,T] \times (\gamma,\infty]\). Then

\[
\lim_{n \to \infty} \sum_{i=1}^{p} (|\bar{\tau}_i^{(n)} - \bar{\tau}_i| + |\bar{y}_i^{(n)} - \bar{y}_i|) = 0
\]

and more importantly

\[
\lim_{n \to \infty} \sum_{i=1}^{p} (|f_n(\tau_i^{(n)}) - f_0(\bar{\tau}_i)| + |\bar{y}_i^{(n)} - \bar{y}_i|) = 0
\]

because (A5) and the continuity of \(f_0\) imply that \(\lim_{n \to \infty} f_n = f_0\) uniformly on \([0,T]\).

Define \(\lambda_n\) to be continuous and strictly increasing functions on \([0,T]\) with \(\lambda_n(0) = 0, \lambda_n(T) = T, \lambda_n(\tau_i^{(n)}) = \bar{\tau}_i\) for \(i = 1, \ldots, p\), and let \(\lambda_n\) be linearly interpolated elsewhere on \([0,T]\). For \(n \in \mathbb{N}\) and \(t \in [0,T]\), set

\[
V_n(t) := \sum_{\tau_i = \lambda_n(\tau_i^{(n)}) \leq t} \pm \exp \left( c_n(f_n(\tau_i^{(n)}) + \bar{y}_i^{(n)}) \right)
\]

and

\[
W_n(t) := \sum_{\lambda_n(\tau_k^{(n)}) \leq t} \pm \exp \left( c_n(f_n(\tau_k^{(n)}) + y_k^{(n)}) \right) - V_n(t).
\]

With this at hand we have

\[
d_T(F_n(f_n,\nu_n),G(f_0,\nu_0)) \leq \sup_{t \in [0,T]} |\lambda_n(t) - t| + c_n^{-1} \sup_{t \in [0,T]} \left| \log^+ |V_n(t) + W_n(t) - \log^+ |V_n(t)| \right|
\]

\[
+ \sup_{t \in [0,T]} \left| c_n^{-1} \log^+ |V_n(t)| - \log(f_0(\bar{\tau}_i) + \bar{y}_i) \right|
\]

\[
+ \sup_{t \in [0,T]} \left| \sup_{\tau_i \leq t} (f_0(\bar{\tau}_i) + \bar{y}_i) - \sup_{\tau_k^{(0)} \leq t} (f_0(\tau_k^{(0)}) + y_k^{(0)}) \right|
\]

for any \(T > 0\).
where $d_T$ is the standard Skorokhod metric on $D[0, T]$.

We treat the terms on the right-hand side of (2.9) separately.

1ST TERM. The relation $\limsup_{n \to \infty} (\lambda_n(t) - t) = 0$ is easily checked.

2ND TERM. We denote the second term by $I_n(\gamma)$ and use inequality

$$|\log^+ |x| - \log^+ |y|| \leq \log(1 + |x - y|), \quad x, y \in \mathbb{R}$$

which yields

$$I_n(\gamma) \leq c_n^{-1} \sup_{t \in [0, T]} \log \left(1 + |W_n(t)|\right)$$

$$\leq c_n^{-1} \log \left(1 + \sum_{\lambda_n(\tau_k^{(n)}) \leq t, \tau_k^{(n)} \neq \bar{\tau}_i^{(n)}} \exp \left(c_n(f_n(\tau_k^{(n)}) + y_k^{(n)})\right)\right)$$

$$\leq c_n^{-1} \log \left(1 + \#\{k : \tau_k^{(n)} \leq T, \tau_k^{(n)} \neq \bar{\tau}_i^{(n)}\}\right)$$

$$\times \sup_{\tau_k^{(n)} \leq T, \tau_k^{(n)} \neq \bar{\tau}_i^{(n)}} \exp \left(c_n(f_n(\tau_k^{(n)}) + y_k^{(n)})\right)$$

$$\leq c_n^{-1} \log \#\{k : \tau_k^{(n)} \leq T\} + \sup_{\tau_k^{(n)} \leq T, \tau_k^{(n)} \neq \bar{\tau}_i^{(n)}} \left(f_n(\tau_k^{(n)}) + y_k^{(n)}\right)$$

$$+ \left(c_n \#\{k : \tau_k^{(n)} \leq T, \tau_k^{(n)} \neq \bar{\tau}_i^{(n)}\}\right)^{-1}$$

$$\times \exp \left(-c_n \sup_{\tau_k^{(n)} \leq T, \tau_k^{(n)} \neq \bar{\tau}_i^{(n)}} \left(f_n(\tau_k^{(n)}) + y_k^{(n)}\right)\right)$$

having utilized $\log(1 + x) \leq \log x + 1/x$, $x > 0$ and that $\lambda_n(\tau_k^{(n)}) \leq T$ iff $\tau_k^{(n)} \leq T$. The first term on the right-hand side of (2.9) converges to zero in view of (2.4). As to the second, we apply Theorem 1.3 in [1] to infer

$$\sup_{\tau_k^{(n)} \leq T, \tau_k^{(n)} \neq \bar{\tau}_i^{(n)}} \left(f_n(\tau_k^{(n)}) + y_k^{(n)}\right) = \sup_{\tau_k^{(n)} \leq T, y_k^{(n)} \leq \gamma} \left(f_n(\tau_k^{(n)}) + y_k^{(n)}\right)$$

$$\rightarrow \sup_{\tau_k^{(0)} \leq T, y_k^{(0)} \leq \gamma} \left(f_0(\tau_k^{(0)}) + y_k^{(0)}\right),$$

(2.10)

as $n \to \infty$. The latter goes to zero as $\gamma \to 0$ because $f_0 = 0$ by assumption. Finally, the last term on the right-hand side of (2.9) tends to zero as $n \to \infty$ for the principal factor of exponential growth does so as a consequence of (2.10) and the assumption $\sup_{\tau_k^{(0)} \leq T, y_k^{(0)} \leq \gamma} (f_0(\tau_k^{(0)}) + y_k^{(0)}) > 0$. Summarizing we have proved that $\lim_{\gamma \to 0} \limsup_{n \to \infty} I_n(\gamma) = 0$.

3RD TERM. Denote the third term of (2.8) by $J_n$. We shall use the inequality

$$J_n \leq \sup_{t \in [0, T]} A_n(t) + c_n^{-1} \sup_{t \in [0, T]} \log^+ |V_n(t)|,$$

where $A_n(t) := \left|c_n^{-1} \log |V_n(t)| - \sup_{\tau_i \leq t} (f_0(\tau_i) + \bar{y}_i)\right|$, $t \in [0, T]$.

If $t \in [0, \bar{\tau}_1]$, then $A_n(t) = \left|f_n(0) - f_0(0)\right| \rightarrow 0$ as $n \to \infty$ by the definition of the functionals. Let now $t \in [\bar{\tau}_k, \bar{\tau}_{k+1}]$, $k = 1, \ldots, p - 1$ or $t \in [\bar{\tau}_p, T]$. Since all $\exp(f_0(\bar{\tau}_1) + \bar{y}_1), \ldots, \exp(f_0(\bar{\tau}_k) + \bar{y}_k)$ are distinct by (2.2) and

$$\lim_{n \to \infty} \exp(f_n(\bar{\tau}_j^{(n)}) + \bar{y}_j^{(n)}) = \exp(f_0(\bar{\tau}_j) + \bar{y}_j), \quad j = 1, \ldots, k,$$


by (2.7), we conclude that \( \exp(f_n(\bar{\tau}_1^{(n)}) + \bar{y}_1^{(n)}), \ldots, \exp(f_n(\bar{\tau}_k^{(n)}) + \bar{y}_k^{(n)}) \) are all distinct, for large enough \( n \). Denote by \( a_{k,n} < \ldots < a_{1,n} \) their increasing rearrangement\(^3\) and put

\[
B_n(t) := c_n^{-1} \log \left| 1 \pm \left( \frac{a_{2,n}}{a_{1,n}} \right)^c_n \pm \ldots \pm \left( \frac{a_{k,n}}{a_{1,n}} \right)^c_n \right|.
\]

Since \( \lim_{n \to \infty} \left( \pm \left( \frac{a_{2,n}}{a_{1,n}} \right)^c_n \pm \ldots \pm \left( \frac{a_{k,n}}{a_{1,n}} \right)^c_n \right) = 0 \), there is an \( N_k \) such that

\[
|B_n(t)| \leq c_n^{-1} \quad \text{for } n \geq N_k.
\]

Summarizing we have

\[
\sup_{t \in [0,T]} |B_n(t)| \leq c_n^{-1} \quad \text{for all } n \geq \max(N_1, \ldots, N_p).
\] (2.11)

With these at hand we can proceed as follows

\[
A_n(t) = \left| \sup_{\bar{\tau}_i \leq t} \left( f_n(\bar{\tau}_i^{(n)}) + \bar{y}_i^{(n)} \right) + B_n(t) - \sup_{\bar{\tau}_i \leq t} \left( f_0(\bar{\tau}_i) + \bar{y}_i \right) \right|
\]

\[
\leq \left| \sup_{\bar{\tau}_i \leq t} \left( f_n(\bar{\tau}_i^{(n)}) + \bar{y}_i^{(n)} \right) - \sup_{\bar{\tau}_i \leq t} \left( f_0(\bar{\tau}_i) + \bar{y}_i \right) \right| + |B_n(t)|
\]

\[
\leq \sum_{i=1}^p \left( \left| f_n(\bar{\tau}_i^{(n)}) - f_0(\bar{\tau}_i) \right| + \left| \bar{y}_i^{(n)} - \bar{y}_i \right| \right) + |B_n(t)|.
\]

In view of (2.7) and (2.11) the right-hand side tends to zero uniformly in \( t \in [0,T] \) as \( n \to \infty \).

We already know that

\[
\lim_{n \to \infty} \sup_{t \in [0,T]} c_n^{-1} \log |V_n(t)| = \sup_{\bar{\tau}_i \leq T} (f_0(\bar{\tau}_i) + \bar{y}_i).
\]

Recalling that

\[
\sup_{\bar{\tau}_i \leq T} (f_0(\bar{\tau}_i) + \bar{y}_i) = \sup_{\tau_k^{(0)} \leq T, y_k^{(0)} > \gamma} (f_0(\tau_k^{(0)}) + y_k^{(0)}) > 0
\]

we infer \( \lim_{n \to \infty} \sup_{t \in [0,T]} |V_n(t)| = +\infty \) and thereupon \( \sup_{t \in [0,T]} \log^+ |V_n(t)| = 0 \) for large enough \( n \). Hence \( \lim_{n \to \infty} J_n = 0 \).

4TH TERM. In the proof of Theorem 1.3 in [11] it is shown that\(^4\)

\[
\sup_{t \in [0,T]} \left| \sup_{\bar{\tau}_i \leq t} (f_0(\bar{\tau}_i) + \bar{y}_i) - \sup_{\tau_k^{(0)} \leq t} (f_0(\tau_k^{(0)}) + y_k^{(0)}) \right| \leq \omega f_0(2|\rho|) + \gamma,
\]

where \( |\rho| := \max_i (s_{i+1} - s_i) \) and \( \omega f_0(\varepsilon) := \sup_{|u - v| \leq \varepsilon, u, v \geq 0} |f_0(u) - f_0(v)| \) is the modulus of continuity of \( f_0 \). Of course, the right-hand side of the last inequality tends to zero on sending \( |\rho| \) and \( \gamma \) to zero.

Collecting pieces together and letting in (2.8) \( n \to \infty \) and then \( |\rho| \) and \( \gamma \) tend to zero we arrive at the desired conclusion

\[
\lim_{n \to \infty} d_T(F_n(f_n, \nu_n), G(f_0, \nu_0)) = 0.
\]

\(^3\) Although \( a_{j,n} \)'s depend on \( t \) we suppress this dependence for the sake of clarity.

\(^4\) Condition (2.6) is only used in this part of the proof.
3 Proof of Theorem 1.1

Proof of (1.10). We first show that
\[
\frac{\log^{-} |Y_{[n]}|}{an} \Rightarrow h(\cdot), \tag{3.1}
\]
where \(h(t) = 0, t \geq 0\). To this end, we intend to check that conditions (1.2), (1.3) and (1.6) hold. If they do, then, as \(n \to \infty\), \(|Y_n| \xrightarrow{P} \infty\) by Theorem 2.1 in [7].

Condition (3.1) entails \(\sup_{t \leq k \leq nT+1} |Y_k| = \sup_{t \leq k \leq nT+1} |Y_k| \xrightarrow{P} \infty\) for each \(T > 0\). This entails \(\sup_{t \in [0,T]} \log^{-} |Y_{[n]}| = 0\) for each \(T > 0\) and large enough \(n\) which proves (3.1). Assumption (1.7) entails \(\lim_{n \to \infty} \Pi_n = 0\) a.s. and \(P\{M = 0\} = 0\). Condition \(P\{Q = 0\} = 0\) is a part of (1.9). Suppose \(Q + Mr = r\) a.s. for some \(r \in \mathbb{R}\). In view of \(P\{Q = 0\} = 0\) we have \(r \neq 0\) and then \(|Q|/|r| = |1 - M| \leq 1 + |M|\) a.s. Since \(E \log(1 + |M|) < \infty\) by (1.7) we must have \(E \log^{+} |Q| < \infty\). This contradiction completes the proof of (3.1).

For \(k \in \mathbb{N}_0\), set \(S_k := \log |\Pi_k|\) and \(\eta_{k+1} := \log |Q_{k+1}|\). As a consequence of the strong law of large numbers,
\[
\frac{S_{[n]}}{an} \Rightarrow g(\cdot), \quad n \to \infty, \tag{3.2}
\]
where \(g(t) := -t, t \geq 0\) (actually, in (3.2) the a.s. convergence holds, see Theorem 4 in [7]). According to Corollary 4.19 (ii) in [10] condition (1.8) entails
\[
\sum_{k \geq 0} \mathbb{1}_{\{\eta_{k+1} > 0\}} \varepsilon_{n^{-1}k, (an)^{-1}\eta_{k+1}} \Rightarrow N(c/a, 1), \quad n \to \infty \tag{3.3}
\]
in \(M_p\), see Section 2 for the definition of \(M_p\). Now relations (3.2) and (3.3) can be combined into the joint convergence
\[
\left(\left(\frac{an}{n}ight)^{-1}S_{[n]}, \sum_{k \geq 0} \mathbb{1}_{\{\eta_{k+1} > 0\}} \varepsilon_{n^{-1}k, (an)^{-1}\eta_{k+1}} \right) \Rightarrow (g(\cdot), N(c/a, 1)) \quad \text{as} \quad n \to \infty
\]
in \(D[0, \infty) \times M_p\). By the Skorokhod representation theorem there are versions which converge a.s. Retaining the original notation for these versions we want to apply Theorem 2.1 with \(f_n(\cdot) = (an)^{-1}S_{[n]}, f_0 = g; \nu_n = \sum_{k \geq 0} \mathbb{1}_{\{\eta_{k+1} > 0\}} \varepsilon_{(n^{-1}k, (an)^{-1}\eta_{k+1}}\), \(\nu_0 = N(c/a, 1), c_n = an\) and the signs \pm defined by \(\text{sgn}(\Pi_k Q_{k+1})\) to conclude
\[
\frac{\log^{+} |Y_{[n]}|}{an} \Rightarrow \sup_{i_k(c/a, 1)} (-i_k(c/a, 1) + j_k(c/a, 1)),
\]

Of course, this together with (3.1) proves (1.10).

Thus it remains to check that all the assumptions of Theorem 2.1 hold. We already know that conditions (A5) and (A6) are fulfilled. Condition (2.4) holds trivially. Further \(N(c/a, 1)(([0, T] \times [\delta, \infty]) < \infty\) a.s. for all \(\delta > 0\) and all \(T > 0\) because \(\mu_{c/a, 1}(\delta, \infty) < \infty\). Plainly, \(N(c/a, 1)(\{0\} \times (0, +\infty]) = 0\) a.s., and \(N(c/a, 1)((r_1, r_2) \times (0, \infty]) \geq 1\) a.s. whenever \(0 < r_1 < r_2\) because \(\mu_{c/a, 1}(0, \infty]) = \infty\). This gives (A1).

Next we check (2.2). Our argument is similar to that given on p. 223 in [7]. We fix any \(T > 0, \delta > 0\) and use the representation
\[
N(c/a, 1)([0, T] \times (\delta, \infty]) = \sum_{k \geq 1} \varepsilon_{(U_k, V_k)}(\cdot),
\]

10
where \((U_i)\) are i.i.d. with the uniform \([0, T]\) distribution, \((V_j)\) are iid with \(\mathbb{P}\{V_1 \leq x\} = (1 - \delta/x) \mathbb{1}_{(\delta, \infty)}(x)\), and \(N\) has the Poisson distribution with parameter \(Tc/(a\delta)\), all the random variables being independent. It suffices to prove that

\[
I := \mathbb{P}\{N \geq 2, -U_k + V_k = -U_i + V_i \text{ for some } 1 \leq k < j \leq N\} = 0.
\]

This is a consequence of the fact that \(-U_1 + V_1\) has a continuous distribution which implies \(\mathbb{P}\{-U_1 + V_1 = -U_2 + V_2\} = 0\). Indeed,

\[
I = \sum_{n \geq 2} \mathbb{P}\{-U_k + V_k = -U_i + V_i \text{ for some } 1 \leq k < j \leq n\} \mathbb{P}\{N = n\}
= \sum_{n \geq 2} \binom{n}{2} \mathbb{P}\{-U_1 + V_1 = -U_2 + V_2\} \mathbb{P}\{N = n\} = 0
\]

An analogous working leads to the conclusion that \(N^{(c/a, 1)}\) does not have clustered jumps a.s., i.e., (A2) holds. The last thing that needs to be checked is condition \(\text{(2.3)}\). Arguing as in Remark 1.3 we infer

\[
\text{for every } T > 0 \text{ and any } \gamma \in (0, T).
\]

**Proof of \((1.2)\).** Without loss of generality we assume that \(X_0 = 0\) a.s. and use the representation

\[
X_{[n] + 1} = \Pi_{[n] + 1} \sum_{k = 0}^{[n]} \Pi_k^* Q_k^*, \tag{3.4}
\]

where \(\Pi_k^* := \Pi_k^{-1}, k \in \mathbb{N}_0\) and \(Q_k^* := Q_k/M_k\) (with generic copy \(Q^*\), \(k \in \mathbb{N}\). Observe that

\[
\sup_{0 \leq t \leq T} \frac{|S_{[nt] + 1} - S_{[nt]}|}{n} = \max_{1 \leq k \leq [nT] + 1} \frac{\log |M_k|}{n} \overset{P}{\to} 0, \quad n \to \infty
\]

for every \(T > 0\), because \(\lim_{x \to \infty} x \mathbb{P}\{\log |M| > x\} = 0\) as a consequence of \(\mathbb{E} \log |M| < \infty\). This together with \((3.2)\) proves

\[
\frac{\log |\Pi_{[n] + 1}|}{an} \Rightarrow g(\cdot), \quad n \to \infty, \tag{3.5}
\]

where \(g(t) = -t, t \geq 0\). Further, write, for \(\varepsilon \in (0, 1)\) and \(x > 0\),

\[
\mathbb{P}\{\log |Q| > (1 + \varepsilon)x\} - \mathbb{P}\{\log |M| > \varepsilon x\} \leq \mathbb{P}\{\log |Q| - \log |M| > x\} \leq \mathbb{P}\{\log |Q| > (1 - \varepsilon)x\} + \mathbb{P}\{\log^{-} |M| > \varepsilon x\}. \tag{3.6}
\]

Multiplying the inequality by \(x\), sending \(x \to \infty\) and then \(\varepsilon \to 0\) yields

\[
\mathbb{P}\{|Q^*| > x\} = \mathbb{P}\{\log |Q| - \log |M| > x\} \sim \mathbb{P}\{\log |Q| > x\} \sim cx^{-1}, \quad x \to \infty.
\]
Set $M^* := 1/M$. Conditions (1.2) and (1.3) with $(M, Q)$ replaced by $(M^*, Q^*)$ are easily checked. Also, we have $\lim_{n \to \infty} \Pi_n^* = \infty$ a.s. Hence $\prod_{k=1}^{n} \Pi_{k-1}^* \to \infty$ as $n \to \infty$ by Theorem 2.1 in [8]. Arguing in the same way as in the proof of (1.10) we see that

$$\frac{\log^- | \sum_{k=0}^{[n]} \Pi_k^* Q_{k+1}^* |}{b_n} \Rightarrow h(\cdot), \quad n \to \infty.$$ 

An application of Theorem 2.1 gives

$$\frac{\log^+ | \sum_{k=0}^{[n]} \Pi_k^* Q_{k+1}^* |}{b_n} \Rightarrow \sup_{\ell(\cdot)} (\ell^{(e/1)}(c/a, 1) + \ell^{(c/a, 1)}(c/a, 1)), \quad n \to \infty.$$ 

Now (1.12) follows by a combination of the last two relations and (3.5).

### 4 Proof of Theorem 1.5

The proof proceeds along the lines of that of Theorem 1.1 but is simpler for the contribution of $M_k$’s is negligible. Therefore we only provide details for fragments which differ principally from the corresponding ones in the proof of Theorem 1.1.

Observe that

$$\lim_{n \to \infty} \frac{b_n}{n} = +\infty. \quad (4.1)$$

Indeed, since $(b_n)$ is a regularly varying sequence of index $1/\alpha$, this is trivial when $\alpha \in (0, 1)$. If $\alpha = 1$, this follows from the relation $b_n/n \sim \ell(b_n)$ as $n \to \infty$ and our assumption that $\lim_{x \to \infty} \ell(x) = \infty$.

**Proof of (1.17).** As far as

$$\frac{\log^- | Y_{[n] + 1} |}{b_n} \Rightarrow h(\cdot), \quad n \to \infty \quad (4.2)$$

is concerned which is the counterpart of (3.1) we have to check two things that are not obvious in the case when $\mathbb{E} \log^- | M | = \infty$: condition (1.3) and $I = \int_{(1, \infty)} \frac{\log x}{A(\log x)} \mathbb{P}\{|Q| \leq x\} dx = \infty$.

Assume first that $\mathbb{P}\{|Q + M r = r\} = 1$ for some $r \not= 0$. In view of $|Q - r| = |M||r|$, the tails of $\log^+ |Q|$ and $\log^+ |M|$ must exhibit the same asymptotics. However, this is not a case, for the tail of $\log^+ |Q|$ is heavier than that of $\log^+ |M|$.

Next, according to (1.16), for any $B > 0$ there exists $x_0 > 0$ such that

$$\frac{\log x}{A(\log x)} \geq \frac{B}{\mathbb{P}\{|Q| > x\}}$$

whenever $x \geq x_0$. Hence,

$$I \geq B \int_{[x_0, \infty)} \frac{\mathbb{P}\{|Q| \in \text{d}x\}}{\mathbb{P}\{|Q| > x\}} = \infty.$$ 

Thus, (4.2) holds.

---

*We omit details which are very similar to but simpler than those appearing in the proof of 1.10.*
To proceed we recall the already used notation $S_k := \log |\Pi_k|$ and $\eta_{k+1} := \log |Q_{k+1}|$, $k \in \mathbb{N}_0$. According to Corollary 4.19 (ii) in [10] condition (1.15) entails
\[
\sum_{k \geq 0} \mathbb{1}_{\{\eta_{k+1} > 0\}} \varepsilon_{(n^{-1}k, b_n^{-1} \eta_{k+1})} \Rightarrow N(1, \alpha), \ n \to \infty
\] (4.3)
in $M_p$. If we can prove that
\[
\frac{S_{[nT]} - 0}{b_n} \Rightarrow h(\cdot), \ n \to \infty,
\] (4.4)
where $h(t) = 0$, $t \geq 0$, then relations (4.3) and (4.4) can be combined into the joint convergence
\[
\left( b_n^{-1} S_{[nT]}, \sum_{k \geq 0} \mathbb{1}_{\{\eta_{k+1} > 0\}} \varepsilon_{(n^{-1}k, b_n^{-1} \eta_{k+1})} \right) \Rightarrow \left( h(\cdot), N(1, \alpha) \right), \ n \to \infty
\]
in $D \times M_p$. By the Skorokhod representation theorem there are versions which converge a.s. Retaining the original notation for these versions we apply Proposition 2.4 with $f_n(\cdot) = b_n^{-1} S_{[nT]}$, $f_n = h$, $\nu_n = \sum_{k \geq 0} \mathbb{1}_{\{\eta_{k+1} > 0\}} \varepsilon_{(n^{-1}k, b_n^{-1} \eta_{k+1})}$, $\nu_0 = N(1, \alpha)$, $c_n = b_n$ and the signs $\pm$ defined by $\text{sgn}(\Pi_k Q_{k+1})$ which gives (1.17) with log replaced with $\log^+$. The latter in combination with (4.2) proves (1.17).

It only remains to check (4.4). To this end, it suffices to prove that
\[
\sup_{0 \leq t \leq T} |S_{[nt]}| \leq \max_{0 \leq k \leq [nT]} |S_k| \quad \rightarrow \quad 0, \quad n \to \infty
\] (4.5)
for every $T > 0$. Set
\[
S^+_0 = S^-_0 := 0, \ S^+_n := \log^+ |M_1| + \ldots + \log^+ |M_n|, \ S^-_n := \log^- |M_1| + \ldots + \log^- |M_n|
\]
for $n \in \mathbb{N}$. Since $(b_n)$ is a regularly varying sequence and
\[
\max_{0 \leq k \leq [nT]} |S_k| \leq \max_{0 \leq k \leq [nT]} S^+_k + \max_{0 \leq k \leq [nT]} S^-_k = S^+_{[nT]} + S^-_{[nT]},
\]
(4.5) follows if we prove that $\lim_{n \to \infty} (S^+_n / b_n) = 0$ in probability. While doing so, we treat two cases separately.

CASE WHEN $E \log^- |M| < \infty$. Then necessarily $E \log^+ |M| < \infty$ for otherwise $\lim_{n \to \infty} \Pi_n = \infty$ a.s. Therefore we have $\lim_{n \to \infty} n^{-1} S^+_n = E \log^+ |M|$ by the strong law of large numbers. Invoking (4.1) proves (4.3).

CASE WHEN $E \log^- |M| = \infty$. Condition (1.16) entails $\lim_{n \to \infty} \frac{n}{b_n} E \left( (\log^- |M|) \wedge b_n \right) = 0$. Since
\[
\frac{n}{b_n} E \left( (\log^- |M|) \wedge b_n \right) = n P \{ \log^- |M| > b_n \} + \frac{n}{b_n} E \log^- |M| \mathbb{1}_{\{\log^- |M| \leq b_n \}},
\]
we infer
\[
\lim_{n \to \infty} n P \{ \log^- |M| > b_n \} = 0
\] (4.6)
and
\[
\lim_{n \to \infty} n \frac{b_n}{b_n} E \left( \log^- |M| \mathbb{1}_{\{\log^- |M| \leq b_n \}} \right) = 0.
\] (4.7)
Using (4.7) together with Markov’s inequality proves
\[ \lim_{n \to \infty} \frac{\sum_{k=1}^{n} \log^{-} |M_k| \mathbb{1}_{\{\log^{-} |M_k| \leq b_n\}}}{b_n} = 0 \quad \text{in probability.} \]

Since
\[ \mathbb{P}\left\{ b_n^{-1} \sum_{k=1}^{n} \log^{-} |M_k| \neq b_n^{-1} \sum_{k=1}^{n} \log^{-} |M_k| \mathbb{1}_{\{\log^{-} |M_k| \leq b_n\}} \right\} \leq \sum_{k=1}^{n} \mathbb{P}\{\log^{-} |M_k| > b_n\} = n \mathbb{P}\{\log^{-} |M| > b_n\}, \]

(4.6) implies that the left-hand side tends to zero as \( n \to \infty \). Therefore \( \lim_{n \to \infty} (S_n^{-}/b_n) = 0 \) in probability.

Left with proving that \( \lim_{n \to \infty} (S_n^{+}/b_n) = 0 \) in probability we suppose immediately that \( \mathbb{E} \log^{+} |M| = \infty \) for the complementary case can be treated in exactly the same way as above (use the strong law of large numbers). Since \( \lim_{n \to \infty} S_n^{-} = -\infty \) a.s. by the assumption, Lemma 8.1 in [14] tells us that \( \lim_{n \to \infty} S_n^{+}/S_n^{-} = 0 \) a.s. which together with \( \lim_{n \to \infty} (S_n^{-}/b_n) = 0 \) in probability implies \( \lim_{n \to \infty} (S_n^{+}/b_n) = 0 \) in probability. The proof of (4.4) is complete. Hence so is that of (1.17).

Proof of (1.18) follows the pattern of that of (1.12) but is simpler. Referring to (1.12) the only things that need to be checked are that
\[ \frac{\log |\Pi_{[n-1]}+1|}{b_n} \Rightarrow h(\cdot), \quad n \to \infty, \]

where \( h(t) = 0, \ t \geq 0, \) and that
\[ \mathbb{P}\{\log |Q| - \log |M| > x\} \sim \mathbb{P}\{\log |Q| > x\} \sim x^{-\alpha} \ell(x), \ x \to \infty. \]

To prove the first of these, write
\[ \sup_{0 \leq t \leq T} \frac{|S_{[nt]+1} - S_{[nt]}|}{b_n} \leq \sup_{0 \leq t \leq T} \frac{|S_{[nt]+1}|}{b_n} + \sup_{0 \leq t \leq T} \frac{|S_{[nt]}|}{b_n} \]

and use (4.5) to infer
\[ \sup_{0 \leq t \leq T} \frac{|S_{[nt]+1} - S_{[nt]}|}{b_n} \overset{p}{\to} 0, \quad n \to \infty \]

for every \( T > 0 \). To check the second we shall use (3.6).

CASE \( \mathbb{E} \log^{-} |M| < \infty \). We have \( \lim_{x \to \infty} x \mathbb{P}\{\log^{-} |M| > \varepsilon x\} = 0 \) whereas \( \lim_{x \to \infty} x \mathbb{P}\{\log |Q| > x\} = \infty \) (recall that in the case \( \alpha = 1 \) we assume that \( \lim \ell(x) = \infty \)). Therefore,
\[ \lim_{x \to \infty} \frac{\mathbb{P}\{\log^{-} |M| > \varepsilon x\}}{\mathbb{P}\{\log |Q| > x\}} = 0. \quad (4.8) \]

Since \( \mathbb{E} \log^{-} |M| < \infty \) entails \( \mathbb{E} \log^{+} |M| < \infty \), the same argument proves (4.8) for the tail of \( \log^{+} |M| \).

CASE \( \mathbb{E} \log^{-} |M| = \infty \) AND \( \mathbb{E} \log^{+} |M| < \infty \). It suffices to check (4.8) which is a consequence (1.16).
Case $\mathbb{E} \log^- |M| = \mathbb{E} \log^+ |M| = \infty$. We only have to prove that
\[
\lim_{x \to \infty} \frac{\mathbb{P}\{\log^+ |M| > \varepsilon x\}}{\mathbb{P}\{\log |Q| > x\}} = 0.
\]
Since $\lim_{n \to \infty} S_n = -\infty$ a.s. by the assumption, we have
\[
\mathbb{E} \frac{\log^+ |M|}{A(\log^+ |M|)} < \infty
\]
(see Proposition 2.6 in [8]). Therefore
\[
\lim_{x \to \infty} \frac{x \mathbb{P}\{\log^+ |M| > \varepsilon x\}}{\mathbb{E}(\log^- |M| \wedge x)} = 0
\]
and the desired relation follows by an application of (1.16).

Acknowledgements A.I. thanks Alexander Marynych and Andrey Pilipenko for useful discussions. D.B. was partially supported by the NCN grant DEC-2012/05/B/ST1/00692.

References

[1] Alsmeyer, G., Iksanov, A. and Rößler, U. (2009). On distributional properties of perpetuities. J. Theor. Probab. 22, 666–682.
[2] Babillot, M., Bougerol, P. and Elie, L. (1997). The random difference equation $X_n = A_n X_{n-1} + B_n$ in the critical case. Ann. Probab. 25, 478–493.
[3] Basu, R. and Rotterstein, A. (2013). Divergent perpetuities modulated by regime switches. Stoch. Models. 29, 129–148.
[4] Brofferio, S. (2003). How a centered random walk on the affine group goes to infinity. Ann. I. H. Poincaré. 39, 371-384.
[5] Brofferio, S. and Buraczewski D. (2014+). On unbounded invariant measures of stochastic dynamical systems. Ann. Probab., to appear.
[6] Buraczewski, D. (2007). On invariant measures of stochastic recursions in a critical case. Ann. Appl. Probab. 17, 1245–1272.
[7] Glynn, P. W. and Whitt, W. (1988). Ordinary CLT and WLLN versions of $L = \lambda W$. Math. Oper. Res. 13, 674–692.
[8] Goldie, C. M. and Maller, R. A. (2000). Stability of perpetuities. Ann. Probab. 28, 1195–1218.
[9] Grincevičius, A. K. (1975). Limit theorems for products of random linear transformations on the line. Lithuanian Math. J. 15, 568–579.
[10] Hiczenko, P. and Wesolowski, J. (2011). Renorming divergent perpetuities. Bernoulli. 17, 880–894.
[11] Iksanov, A. and Pilipenko, A. (2014). On the maximum of a perturbed random walk. Stat. Probab. Letters. 92, 168–172.
[12] Kellerer, H. G. (1992). Ergodic behaviour of affine recursions I: criteria for recurrence and transience. Technical report, University of Munich, Germany. Available at http://www.mathematik.uni-muenchen.de/~kellerer/
[13] Pakes, A. G. (1983). Some properties of a random linear difference equation. Austral. J. Statist. 25, 345–357.
[14] Pruitt, W. E. (1981). General one-sided laws of the iterated logarithm. Ann. Probab. 9, 1–48.
[15] Rachev, S. T. and Samorodnitsky, G. (1995). Limit laws for a stochastic process and random recursion arising in probabilistic modelling. Adv. Appl. Probab. 27, 185–202.
[16] Resnick, S. (1987). Extreme values, regular variation, and point processes. New York: Springer-Verlag.
[17] Resnick, S. I. (2007). *Heavy-tail phenomena: Probabilistic and statistical modeling*. New York: Springer.

[18] Vervaat, W. (1979). On a stochastic difference equation and a representation of non-negative infinitely divisible random variables. *Adv. Appl. Probab.* 11, 750–783.

[19] Zeevi, A. and Glynn, P. W. (2004). Recurrence properties of autoregressive processes with super-heavy-tailed innovations. *J. Appl. Probab.* 41, 639–653.