Continuous-time trading and
emergence of volatility

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Abstract
This note continues investigation of randomness-type properties emerging in idealized financial markets with continuous price processes. It is shown, without making any probabilistic assumptions, that the strong variation exponent of non-constant price processes has to be 2, as in the case of continuous martingales.

1 Introduction
This note is part of the recent revival of interest in game-theoretic probability (see, e.g., [7] [8] [4] [2] [3]). It concentrates on the study of the “$\sqrt{dt}$ effect”, the fact that a typical change in the value of a non-degenerate diffusion process over short time period $dt$ has order of magnitude $\sqrt{dt}$. Within the “standard” (not using non-standard analysis) framework of game-theoretic probability, this study was initiated in [9]. In our definitions, however, we will be following [10], which also establishes some other randomness-type properties of continuous price processes. The words such as “positive”, “negative”, “before”, and “after” will be understood in the wide sense of $\geq$ or $\leq$, respectively; when necessary, we will add the qualifier “strictly”.

The latest version of this working paper can be downloaded from the web site http://probabilityandfinance.com (Working Paper 25).

2 Null and almost sure events
We consider a perfect-information game between two players, Reality (a financial market) and Sceptic (a speculator), acting over the time interval $[0, T]$, where $T$ is a positive constant fixed throughout. First Sceptic chooses his trading strategy and then Reality chooses a continuous function $\omega : [0, T] \rightarrow \mathbb{R}$ (the price process of a security).
Let $\Omega$ be the set of all continuous functions $\omega : [0, T] \to \mathbb{R}$. For each $t \in [0, T]$, $\mathcal{F}_t$ is defined to be the smallest $\sigma$-algebra that makes all functions $\omega \mapsto \omega(s)$, $s \in [0, t]$, measurable. A process $S$ is a family of functions $S_t : \Omega \to [-\infty, \infty]$, $t \in [0, T]$, each $S_t$ being $\mathcal{F}_t$-measurable (we drop the adjective “adapted”). An event is an element of the $\sigma$-algebra $\mathcal{F}_T$. Stopping times $\tau : \Omega \to [0, T] \cup \{\infty\}$ w.r.t. to the filtration $(\mathcal{F}_t)$ and the corresponding $\sigma$-algebras $\mathcal{F}_\tau$ are defined as usual; $\omega(\tau(\omega))$ and $S_\tau(\omega)$ will be simplified to $\omega(\tau)$ and $S_\tau$, respectively (occasionally, the argument $\omega$ will be omitted in other cases as well).

The class of allowed strategies for Sceptic is defined in two steps. An elementary trading strategy $G$ consists of an increasing sequence of stopping times $\tau_1 \leq \tau_2 \leq \cdots$ and, for each $n = 1, 2, \ldots$, a bounded $\mathcal{F}_{\tau_n}$-measurable function $h_n$. It is required that, for any $\omega \in \Omega$, only finitely many of $\tau_n(\omega)$ should be finite. To such $G$ and an initial capital $c \in \mathbb{R}$ corresponds the elementary capital process

$$K_{t}^{G,c} := c + \sum_{n=1}^{\infty} h_n(\omega)(\omega(\tau_{n+1} \wedge t) - \omega(\tau_n \wedge t)), \quad t \in [0, T]$$

(with the zero terms in the sum ignored); the value $h_n(\omega)$ will be called the portfolio chosen at time $\tau_n$, and $K_t^{G,c}(\omega)$ will sometimes be referred to as Sceptic’s capital at time $t$.

A positive capital process is any process $S$ that can be represented in the form

$$S_t(\omega) := \sum_{n=1}^{\infty} K_{t}^{G_n,c_n}(\omega), \quad (1)$$

where the elementary capital processes $K_{t}^{G_n,c_n}(\omega)$ are required to be positive, for all $t$ and $\omega$, and the positive series $\sum_{n=1}^{\infty} c_n$ is required to converge. The sum $\sum_{n=1}^{\infty} c_n$ is always positive but allowed to take value $\infty$. Since $K_{0}^{G_n,c_n}(\omega) = c_n$ does not depend on $\omega$, $S_0(\omega)$ also does not depend on $\omega$ and will sometimes be abbreviated to $S_0$.

The upper probability of a set $E \subseteq \Omega$ is defined as

$$\mathbb{P}(E) := \inf \{ S_0 \mid \forall \omega \in \Omega : S_T(\omega) \geq I_E(\omega) \},$$

where $S$ ranges over the positive capital processes and $\mathbb{I}_E$ stands for the indicator of $E$.

We say that $E \subseteq \Omega$ is null if $\mathbb{P}(E) = 0$. A property of $\omega \in \Omega$ will be said to hold almost surely (a.s.), or for almost all $\omega$, if the set of $\omega$ where it fails is null.

Upper probability is countably (and finitely) subadditive:

**Lemma 1.** For any sequence of subsets $E_1, E_2, \ldots$ of $\Omega$,

$$\mathbb{P}\left( \bigcup_{n=1}^{\infty} E_n \right) \leq \sum_{n=1}^{\infty} \mathbb{P}(E_n).$$

In particular, a countable union of null sets is null.
3 Main result

For each \( p \in (0, \infty) \), the strong \( p \)-variation of \( \omega \in \Omega \) is

\[
\var_p(\omega) := \sup_{\kappa} \sum_{i=1}^{n} |\omega(t_i) - \omega(t_{i-1})|^p,
\]

where \( n \) ranges over all positive integers and \( \kappa \) over all subdivisions \( 0 = t_0 < t_1 < \cdots < t_n = T \) of the interval \([0, T]\). It is obvious that there exists a unique number \( \text{vex}(\omega) \in [0, \infty] \), called the strong variation exponent of \( \omega \), such that \( \var_p(\omega) \) is finite when \( p > \text{vex}(\omega) \) and infinite when \( p < \text{vex}(\omega) \); notice that \( \text{vex}(\omega) \notin (0, 1) \).

The following is a game-theoretic counterpart of the well-known property of continuous semimartingales (Lepingle [5], Theorem 1 and Proposition 3; Lévy [6] in the case of Brownian motion).

Theorem 1. For almost all \( \omega \in \Omega \),

\[
\text{vex}(\omega) = 2 \text{ or } \omega \text{ is constant.} \tag{2}
\]

(Alternatively, (2) can be expressed as \( \text{vex}(\omega) \in \{0, 2\} \).)

4 Proof

The more difficult part of this proof (\( \text{vex}(\omega) \leq 2 \) a.s.) will be modelled on the proof in [11], which is surprisingly game-theoretic in character. The proof of the easier part is modelled on [11]. (Notice, however, that our framework is very different from those of [1] and [11], which creates additional difficulties.) Without loss of generality we impose the restriction \( \omega(0) = 0 \).

Proof that \( \text{vex}(\omega) \geq 2 \) for non-constant \( \omega \) a.s.

We need to show that the event \( \text{vex}(\omega) < 2 \) \& \( \text{nc}(\omega) \) is null, where \( \text{nc}(\omega) \) stands for “\( \omega \) is not constant”. By Lemma [11] it suffices to show that \( \text{vex}(\omega) < p \) \& \( \text{nc}(\omega) \) is null for each \( p \in (0, 2) \). Fix such a \( p \). It suffices to show that \( \var_p(\omega) < \infty \) \& \( \text{nc}(\omega) \) is null and, therefore, it suffices to show that the event \( \var_p(\omega) < C \) \& \( \text{nc}(\omega) \) is null for each \( C \in (0, \infty) \). Fix such a \( C \). Finally, it suffices to show that the event

\[
E_{p,C,A} := \left\{ \omega \in \Omega \left| \var_p(\omega) < C \& \sup_{t \in [0,T]} |\omega(t)| > A \right. \right\}
\]

is null for each \( A > 0 \). Fix such an \( A \).

Choose a small number \( \delta > 0 \) such that \( A/\delta \in \mathbb{N} \), and let \( \Gamma := \{k\delta \mid k \in \mathbb{Z}\} \) be the corresponding grid. Define a sequence of stopping times \( \tau_n \) inductively by

\[
\tau_{n+1} := \inf \{ t > \tau_n \mid \omega(t) \in \Gamma \setminus \{\omega(\tau_n)\} \}, \quad n = 0, 1, \ldots,
\]
with \( \tau_0 := 0 \) and \( \inf \emptyset \) understood to be \( \infty \). Set \( T_A := \inf\{t \mid |\omega(t)| = A\} \), again with \( \inf \emptyset := \infty \), and

\[
  h_n(\omega) := \begin{cases} 
    2\omega(\tau_n) & \text{if } \tau_n(\omega) < T \land T_A(\omega) \text{ and } n + 1 < C/\delta^p \\
    0 & \text{otherwise.}
  \end{cases}
\]

The elementary capital process corresponding to the elementary gambling strategy \( G := (\tau_n, h_n)_{n=1}^\infty \) and initial capital \( c := \delta^{2-p}C \) will satisfy

\[
  \omega^2(\tau_{n+1}) - \omega^2(\tau_n) = 2\omega(\tau_n) (\omega(\tau_{n+1}) - \omega(\tau_n)) + (\omega(\tau_{n+1}) - \omega(\tau_n))^2
  = K_{\tau_{n+1}}^{G,c}(\omega) - K_{\tau_n}^{G,c}(\omega) + \delta^2
\]

provided \( \tau_{n+1}(\omega) \leq T \land T_A(\omega) \) and \( n + 1 < C/\delta^p \), and so satisfy

\[
  \omega^2(\tau_N) = K_{\tau_N}^{G,c}(\omega) - K_0^{G,c} + N\delta^2 = K_{\tau_N}^{G,c}(\omega) - \delta^{2-p}C + \delta^{2-p}N\delta^p \leq K_{\tau_N}^{G,c}(\omega) \quad (3)
\]

provided \( \tau_N(\omega) \leq T \land T_A(\omega) \) and \( N < C/\delta^p \). On the event \( E_{p,C,A} \) we have \( T_A(\omega) < T \) and \( N < C/\delta^p \) for the \( N \) defined by \( \tau_N = T_A \). Therefore, on this event

\[
  A^2 = \omega^2(T_A) \leq K_{T_A}^{G,c}(\omega) = K_T^{G,c}(\omega).
\]

We can see that \( K_T^{G,c}(\omega) \) increases with \( \delta^{2-p}C \), which can be made arbitrarily small by making \( \delta \) small, to \( A^2 \) over \([0,T]\); this shows that the event \( E_{p,C,A} \) is null.

The only remaining gap in our argument is that \( K_t^{G,c} \) may become strictly negative strictly between some \( \tau_n < T \land T_A \) and \( \tau_{n+1} \) with \( n + 1 < C/\delta^p \) (it will be positive at all \( \tau_N \in [0, T \land T_A] \) with \( N < C/\delta^p \), as can be seen from (3)). We can, however, bound \( K_t^{G,c} \) for \( \tau_n < t < \tau_{n+1} \) as follows:

\[
  K_t^{G,c}(\omega) = K_{\tau_n}^{G,c}(\omega) + 2\omega(\tau_n) (\omega(t) - \omega(\tau_n)) \geq 2|\omega(\tau_n)| (-\delta) \geq -2A\delta,
\]

and so we can make the elementary capital process positive by adding the negligible amount \( 2A\delta \) to Sceptic’s initial capital.

**Proof that** \( \text{vex}(\omega) \leq 2 \) **a.s.**

We need to show that the event \( \text{vex}(\omega) > 2 \) is null, i.e., that \( \text{vex}(\omega) > p \) is null for each \( p > 2 \). Fix such a \( p \). It suffices to show that \( \text{var}_p(\omega) = \infty \) is null, and therefore, it suffices to show that event

\[
  E_{p,A} := \left\{ \omega \in \Omega \mid \text{var}_p(\omega) = \infty \& \sup_{t \in [0,T]} |\omega(t)| < A \right\}
\]

is null for each \( A > 0 \). Fix such an \( A \).

The rest of the proof follows [1] closely. Let \( M_t(f, (a,b)) \) be the number of upcrossings of the open interval \((a,b)\) by a continuous function \( f \in \Omega \) during the time interval \([0,t]\), \( t \in [0,T] \). For each \( \delta > 0 \) we also set

\[
  M_t(f, \delta) := \sum_{k \in \mathbb{Z}} M_t(f, (k\delta, (k+1)\delta)).
\]
The strong \( p \)-variation \( \text{var}_p(f, [0, t]) \) of \( f \in \Omega \) over an interval \( [0, t], t \leq T \), is defined as
\[
\text{var}_p(f, [0, t]) := \sup_{\kappa} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^p,
\]
where \( n \) ranges over all positive integers and \( \kappa \) over all subdivisions \( 0 = t_0 < t_1 < \cdots < t_n = t \) of the interval \( [0, t] \) (so that \( \text{var}_p(f) = \text{var}_p(f, [0, T]) \)). The following key lemma is proved in \[1\] (Lemma 1; in fact, this lemma only requires \( p > 1 \)).

**Lemma 2.** For all \( f \in \Omega \), \( t > 0 \), and \( q \in [1, p) \),
\[
\text{var}_p(f, [0, t]) \leq \frac{2^{p+q+1}}{1 - 2^{-p}} (2c_{q, \lambda, t}(f) + 1) \lambda^p,
\]
where
\[
\lambda \geq \sup_{s \in [0, t]} |f(s) - f(0)|
\]
and
\[
c_{q, \lambda, t}(f) := \sup_{k \in \mathbb{N}} 2^{-kq} M_t(f, \lambda 2^{-k}).
\]

Another key ingredient of the proof is the following game-theoretic version of Doob’s upcrossings inequality:

**Lemma 3.** Let \( c < a < b \) be real numbers. For each elementary capital process \( S \geq c \) there exists a positive elementary capital process \( S^* \) that starts from \( S_0^* = a - c \) and satisfies, for all \( t \in [0, T] \) and \( \omega \in \Omega \),
\[
S_t^*(\omega) \geq (b - a) M_t(S(\omega), (a, b)),
\]
where \( S(\omega) \) stands for the sample path \( t \mapsto S_t(\omega) \).

**Proof.** The following standard argument is easy to formalize. Let \( G \) be an elementary gambling strategy leading to \( S \) (when started with initial capital \( S_0 \)). An elementary gambling strategy \( G^* \) leading to \( S^* \) (with initial capital \( a - c \)) can be defined as follows. When \( S \) first hits \( a \), \( G^* \) starts mimicking \( G \) until \( S \) hits \( b \), at which point \( G^* \) chooses portfolio 0; after \( S \) hits \( a \), \( G^* \) mimics \( G \) until \( S \) hits \( b \), at which point \( G^* \) chooses portfolio 0; etc. Since \( S \geq c \), \( S^* \) will be positive. \( \square \)

Now we are ready to finish the proof of the theorem. Let \( T_A := \inf\{t \mid \omega(t) = A\} \) be the hitting time for \( A \) (with \( T_A := T \) if \( A \) is not hit). By Lemma 3 for each \( k \in \mathbb{N} \) and each \( i \in \{-2^k + 1, \ldots, 2^k\} \) there exists a positive elementary capital process \( S^{k,i}_{T_A} \) that starts from \( A + (i - 1)A2^{-k} \) and satisfies
\[
S^{k,i}_{T_A} \geq A2^{-k} M_{T_A}(\omega, ((i - 1)A2^{-k}, iA2^{-k})).
\]
Summing $2^{-kq}S^{k,i}/A2^{-k}$ over $i \in \{-2^{k} + 1, \ldots, 2^{k}\}$, we obtain a positive elementary capital process $S^{k}$ such that

$$S^{k}_{0} = 2^{-kq} \sum_{i=-2^{k}+1}^{2^{k}} \frac{A + (i-1)A2^{-k}}{A2^{-k}} \leq 2^{-kq}2^{2k+1}$$

and

$$S^{k}_{T_{A}} \geq 2^{-kq}M_{T_{A}}(\omega, A2^{-k}).$$

Next, assuming $q \in (2, p)$ and summing over $k \in \mathbb{N}$, we obtain a positive capital process $S$ such that

$$S_{0} = \sum_{k=1}^{\infty} 2^{-kq}2^{2k+1} = \frac{2^{3-q}}{1-2^{2-q}}$$

and

$$S_{T_{A}} \geq c_{q,A,T_{A}}(\omega).$$

On the event $E_{p,A}$ we have $T_{A} = T$ and so, by Lemma 2, $c_{q,A,T_{A}}(\omega) = \infty$. This shows that $S_{T} = \infty$ on $E_{p,A}$ and completes the proof.

5 Conclusion

Theorem 1 says that, almost surely,

$$\text{var}_{p}(\omega) \begin{cases} < \infty & \text{if } p > 2 \\ = \infty & \text{if } p < 2 \text{ and } \omega \text{ is not constant.} \end{cases}$$

The situation for $p = 2$ remains unclear. It would be very interesting to find the upper probability of the event \{var$_{2}(\omega) < \infty$ and $\omega$ is not constant\}. (Lévy’s \[6\] result shows that this event is null when $\omega$ is the sample path of Brownian motion, while Lepingle \[5\] shows this for continuous, and some other, semi-martingales.)

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