ON \((h - m)\)-CONVEXITY AND HADAMARD-TYPE INEQUALITIES

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Abstract. In this paper, a new class of convex functions as a generalization of convexity which is called \((h - m)\)-convex functions and some properties of this class is given. We also prove some Hadamard’s type inequalities.

1. INTRODUCTION

The concept of \(m\)-convexity has been introduced by Toader in [12], intermediate between the ordinary convexity and starshaped property, as following:

Definition 1. The function \(f : [0, b] \rightarrow \mathbb{R}, b > 0\), is said to be \(m\)-convex, where \(m \in [0, 1]\), if we have

\[ f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y) \]

for all \(x, y \in [0, b]\) and \(t \in [0, 1]\). We say that \(f\) is \(m\)-concave if \(-f\) is \(m\)-convex.

Several papers have been written on \(m\)-convex functions and we refer the papers [11], [12], [13], [14], [15] and [17]. In [13], Dragomir and Toader proved following inequality for \(m\)-convex functions.

Theorem 1. Let \(f : [0, \infty) \rightarrow \mathbb{R}\) be a \(m\)-convex function with \(m \in (0, 1]\). If \(0 \leq a < b < \infty\) and \(f \in L_1[a, b]\), then one has the inequality:

\[
\frac{1}{b - a} \int_a^b f(x) \, dx \leq \min \left\{ \frac{f(a) + mf\left(\frac{b}{m}\right)}{2}, \frac{f(b) + mf\left(\frac{a}{m}\right)}{2} \right\}.
\]

In [17], Dragomir established following inequalities of Hadamard-type similar to above.

Theorem 2. Let \(f : [0, \infty) \rightarrow \mathbb{R}\) be a \(m\)-convex function with \(m \in (0, 1]\). If \(0 \leq a < b < \infty\) and \(f \in L_1[a, b]\), then one has the inequality:

\[
f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x) + mf\left(\frac{x}{m}\right) \, dx \leq \frac{m + 1}{4} \left[ \frac{f(a) + f(b)}{2} + m f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right) \right].
\]
Theorem 3. Let \( f : [0, \infty) \to \mathbb{R} \) be a \( m \)-convex function with \( m \in (0, 1] \). If \( f \in L_1[a, b] \) where \( 0 \leq a < b < \infty \), then one has the inequality:

\[
\frac{1}{m+1} \left[ \int_a^b f(x) \, dx + \frac{mb-a}{b-m} \int_m^b f(x) \, dx \right] \leq (mb-a) \frac{f(a) + f(b)}{2}.
\]

In [16], Breckner introduced a new class of convex functions, a generalization of the ordinary convexity, is called \( s \)-convexity, as following:

**Definition 2.** Let \( s \) be a real number, \( s \in (0, 1] \). A function \( f : [0, \infty) \to [0, \infty) \) is said to be \( s \)-convex (in the second sense), or that \( f \) belongs to the class \( K_2^s \), if

\[
f(\alpha x + (1-\alpha)y) \leq \alpha^s f(x) + (1-\alpha)^s f(y)
\]

for all \( x, y \in [0, \infty) \) and \( \alpha \in [0, 1] \).

Some properties of \( s \)-convexity have been given in [4] and Kırmacı et al. proved some inequalities for \( s \)-convex functions in [10]. In [9], Dragomir and Fitzpatrick established the following Hadamard’s type inequalities:

**Theorem 4.** Suppose that \( f : [0, \infty) \to [0, \infty) \) is an \( s \)-convex function in the second sense, where \( s \in (0, 1] \) and let \( a, b \in [0, \infty) \), \( a < b \). If \( f \in L_1[0, 1] \), then the following inequalities hold:

\[
2^{s-1} f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{s+1}.
\]

The constant \( k = \frac{1}{s+1} \) is the best possible in the second inequality in (1.4). The above inequalities are sharp.

In [3], Godunova and Levin introduced the following class of functions.

**Definition 3.** A function \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) is said to belong to the class of \( Q(I) \) if it is nonnegative and, for all \( x, y \in I \) and \( \lambda \in (0, 1) \) satisfies the inequality:

\[
f(\lambda x + (1-\lambda)y) \leq \frac{f(x)}{\lambda} + \frac{f(y)}{1-\lambda}
\]

In [2], Dragomir et al. defined following new class of functions.

**Definition 4.** A function \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) is \( P \) function or that \( f \) belongs to the class of \( P(I) \), if it is nonnegative and for all \( x, y \in I \) and \( \lambda \in [0, 1] \), satisfies the following inequality:

\[
f(\lambda x + (1-\lambda)y) \leq f(x) + f(y)
\]

In [2], Dragomir et al. proved two inequalities of Hadamard’s type for class of Godunova-Levin functions and \( P \)-functions.

**Theorem 5.** Let \( f \in Q(I) \), \( a, b \in I \), with \( a < b \) and \( f \in L_1[a, b] \). Then the following inequality holds:

\[
f \left( \frac{a+b}{2} \right) \leq 4 \int_a^b f(x) \, dx
\]
Theorem 6. Let \( f \in P(I), a, b \in I, \) with \( a < b \) and \( f \in L_1[a, b] \). Then the following inequality holds.

\[
(1.6) \quad f \left( \frac{a + b}{2} \right) \leq \frac{2}{b - a} \int_a^b f(x)dx \leq 2[f(a) + f(b)]
\]

On all of these, in \([5]\), Varošanec defined \( h \)-convex functions and gave some properties of this class of functions.

Definition 5. Let \( h : J \subseteq \mathbb{R} \to \mathbb{R} \) be a positive function. We say that \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) is \( h \)-convex function, or that \( f \) belongs to the class \( SX(h, I) \), if \( f \) is nonnegative and for all \( x, y \in I \) and \( t \in (0, 1) \) we have

\[
(1.7) \quad f (tx + (1 - t)y) \leq h(t) f (x) + h(1 - t) f (y).
\]

If inequality (1.7) is reversed, then \( f \) is said to be \( h \)-concave, i.e., \( f \in SV(h, I) \). Obviously, if \( h(t) = t \), then all nonnegative convex functions belong to \( SX(h, I) \) and all nonnegative concave functions belong to \( SV(h, I) \); if \( h(t) = \frac{1}{t} \), then \( SX(h, I) \subseteq Q(I) \); if \( h(t) = 1 \), \( SX(h, I) \supseteq P(I) \); and if \( h(t) = t^s \), where \( s \in (0, 1) \), then \( SX(h, I) \supseteq K_s^2 \).

Theorem 7. (See [1], Theorem 6) Let \( f \in SX(h, I), a, b \in I, \) with \( a < b \) and \( f \in L_1([a, b]). \) Then

\[
(1.8) \quad \frac{1}{2h \left( \frac{1}{t} \right)} f \left( \frac{a + b}{2} \right) \leq \frac{1}{(b - a)} \int_a^b f(x)dx \leq [f(a) + f(b)] \int_0^1 h(\alpha) d\alpha
\]

For some recent results for \( h \)-convex functions we refer the interest of reader to the papers [1], [6], [7] and [8].

The main aim of this paper is to give a new class of convex functions and to give some properties of this functions, by using a similar way to proof of properties of \( h \)-convexity (see [5]). Therefore, some inequalities of Hadamard-type related to this new class of convex functions are given.

2. MAIN RESULTS

We will introduce a new class of convex functions in the following definition.

Definition 6. Let \( h : J \subseteq \mathbb{R} \to \mathbb{R} \) be a non-negative function. We say that \( f : [0, b] \to \mathbb{R} \) is a \((h - m)\)-convex function, if \( f \) is non-negative and for all \( x, y \in [0, b] \), \( m \in [0, 1] \) and \( \alpha \in (0, 1) \), we have

\[
(2.1) \quad f (\alpha x + m (1 - \alpha) y) \leq h(\alpha) f (x) + mh(1 - \alpha) f (y).
\]

If the inequality (2.1) is reversed, then \( f \) is said to be \((h - m)\)-concave function on \([0, b]\).

Obviously, if we choose \( m = 1 \), then we have \( h \)-convex functions. If we choose \( h(\alpha) = \alpha \), then we obtain non-negative \( m \)-convex functions. If we choose \( m = 1 \) and \( h(\alpha) = \{\alpha, 1, \frac{1}{\alpha}, \alpha^s\} \), then we obtain the following classes of functions, non-negative convex functions, \( P \)-functions, Godunova-Levin functions and \( s \)-convex functions (in the second sense), respectively.

Remark 1. Let \( h \) be a non-negative function such that

\[ h(\alpha) \geq \alpha \]
for all $\alpha \in (0, 1)$. If $f$ is a non-negative $m$-convex function on $[0, b]$, then for all $x, y \in [0, b]$, $m \in [0, 1]$ and $\alpha \in (0, 1)$, we have

$$f (\alpha x + m (1 - \alpha) y) \leq h (\alpha) f (x) + m h (1 - \alpha) f (y).$$

This shows that $f$ is a $(h - m)$-convex function. By a similar way, one can see that, if

$$h (\alpha) \leq \lambda f$$

for all $\alpha \in (0, 1)$. Then, all non-negative $m$-concave functions are $(h - m)$-concave function on $[0, b]$.

**Proposition 1.** Let $h_1, h_2$ be non-negative functions defined on $J \subseteq \mathbb{R}$ such that

$$h_2 (t) \leq h_1 (t)$$

for $t \in (0, 1)$. If $f$ is $(h_2 - m)$-convex, then $f$ is $(h_1 - m)$-convex.

**Proof.** If $f$ is $(h_2 - m)$-convex, then for all $x, y \in [0, b]$ and $\alpha \in (0, 1)$, we can write

$$f (\alpha x + m (1 - \alpha) y) \leq h_2 (\alpha) f (x) + m h_2 (1 - \alpha) f (y) \leq h_1 (\alpha) f (x) + mh_1 (1 - \alpha) f (y).$$

Which completes the proof of $(h_1 - m)$-convexity of $f$. □

**Proposition 2.** If $f, g$ are $(h - m)$-convex functions and $\lambda > 0$, then $f + g$ and $\lambda f$ are $(h - m)$-convex functions.

**Proof.** By using definition of $(h - m)$-convex functions, we can write

(2.2) \quad $f (\alpha x + m (1 - \alpha) y) \leq h (\alpha) f (x) + mh (1 - \alpha) f (y)$

and

(2.3) \quad $g (\alpha x + m (1 - \alpha) y) \leq h (\alpha) g (x) + mh (1 - \alpha) g (y)$

for all $x, y \in [0, b]$, $m \in [0, 1]$ and $\alpha \in (0, 1)$. If we add (2.2) and (2.3), we get

$$(f + g) (\alpha x + m (1 - \alpha) y) \leq h (\alpha) (f + g) (x) + mh (1 - \alpha) (f + g) (y).$$

This shows that $f + g$ is $(h - m)$-convex function. Therefore, to prove $(h - m)$ -convexity of $\lambda f$, from the definition, we have

$$\lambda f (\alpha x + m (1 - \alpha) y) \leq h (\alpha) \lambda f (x) + mh (1 - \alpha) \lambda f (y).$$

This completes the proof. □

The following inequality of Hadamard-type for $(h - m)$-convex functions holds.

**Theorem 8.** Let $f : [0, \infty) \to \mathbb{R}$ be a $(h - m)$-convex function with $m \in (0, 1]$, $t \in [0, 1]$. If $0 \leq a < b < \infty$ and $f \in L_1 [a, b]$, then the following inequality holds:

(2.4) \quad \frac{1}{b-a} \int_a^b f(x)dx \leq \min \left\{ f (a) \int_0^1 h (t) dt + mf \left( \frac{b}{m} \right) \int_0^1 h (1 - t) dt, \right.

\quad \left. f (b) \int_0^1 h (t) dt + mf \left( \frac{a}{m} \right) \int_0^1 h (1 - t) dt \right\}.
Proof. From the definition of \((h - m)\)–convex functions, we can write
\[
f(tx + m(1 - t)y) \leq h(t)f(x) + mh(1 - t)f(y)
\]
for all \(x, y \geq 0\). It follows that; for all \(t \in [0, 1]\),
\[
f(ta + (1 - t)b) \leq h(t)f(a) + mh(1 - t)f\left(\frac{b}{m}\right)
\]
and
\[
f(tb + (1 - t)a) \leq h(t)f(b) + mh(1 - t)f\left(\frac{a}{m}\right).
\]
Integrating these inequalities on \([0, 1]\), with respect to \(t\), we obtain
\[
\int_0^1 f(ta + (1 - t)b)\,dt \leq f(a) \int_0^1 h(t)\,dt + mf\left(\frac{b}{m}\right) \int_0^1 h(1 - t)\,dt
\]
and
\[
\int_0^1 f(tb + (1 - t)a)\,dt \leq f(b) \int_0^1 h(t)\,dt + mf\left(\frac{a}{m}\right) \int_0^1 h(1 - t)\,dt.
\]
It is easy to see that;
\[
\int_0^1 f(ta + (1 - t)b)\,dt = \int_0^1 f(tb + (1 - t)a)\,dt = \frac{1}{b - a} \int_a^b f(x)\,dx.
\]
Using this equality, we obtain the required result. \(\square\)

Corollary 1. If we choose \(h(t) = 1\) in (2.4), we obtain the following inequality;
\[
\frac{1}{b - a} \int_a^b f(x)\,dx \leq \min\left\{ f(a) + mf\left(\frac{b}{m}\right), f(b) + mf\left(\frac{a}{m}\right) \right\}.
\]

Corollary 2. If we choose \(m = 1\) in (2.4), we obtain the following inequality;
\[
\frac{1}{b - a} \int_a^b f(x)\,dx \leq \min\left\{ f(a) \int_0^1 h(t)\,dt + f(b) \int_0^1 h(1 - t)\,dt, \right. \]
\[
\left. f(b) \int_0^1 h(t)\,dt + f(a) \int_0^1 h(1 - t)\,dt \right\}.
\]

Remark 2. If we choose \(h(t) = t\) in (2.4), we obtain the inequality (1.1).

Remark 3. If we choose \(m = 1\) and \(h(t) = t\) in (2.4), we obtain the right hand side of the Hadamard’s inequality. If we choose \(m = 1\) and \(h(t) = 1\) in (2.4), we obtain the right hand side of the inequality (1.0). If we choose \(m = 1\) and \(h(t) = t^s\) in (2.4), we obtain the right hand side of the inequality (1.4).

Another result of Hadamard-type for \((h - m)\)–convex functions is embodied in the following theorem.
Theorem 9. Let \( f : [0, \infty) \to \mathbb{R} \) be a \((h - m)\)–convex function with \( m \in (0, 1] \), \( t \in [0, 1] \). If \( 0 \leq a < b < \infty \) and \( f \in L_1[a, b] \), then the following inequality holds:

\[
(2.5) \quad f \left( \frac{a + b}{2} \right) \leq h \left( \frac{\alpha}{b - a} \right) \int_a^b \left[ f(x) + mf \left( \frac{x}{m} \right) \right] dx
\]

\[
\leq h \left( \frac{1}{2} \right) \left[ f(a) + mf \left( \frac{b}{m} \right) + mf \left( \frac{a}{m} \right) + m^2 f \left( \frac{b^2}{m^2} \right) \right]
\]

Proof. For \( x, y \in [0, \infty) \) and \( \alpha = \frac{1}{2} \), we can write definition of \((h - m)\)–convex function as following:

\[
f \left( \frac{x + y}{2} \right) \leq h \left( \frac{1}{2} \right) f(x) + mh \left( \frac{1}{2} \right) f \left( \frac{y}{m} \right)
\]

If we choose \( x = ta + (1 - t)b \) and \( y = tb + (1 - t)a \), we get

\[
f \left( \frac{a + b}{2} \right) \leq h \left( \frac{1}{2} \right) f \left( ta + (1 - t)b \right) + mh \left( \frac{1}{2} \right) f \left( (1 - t) \frac{a}{m} + t \frac{b}{m} \right)
\]

for all \( t \in [0, 1] \). By integrating the result on \([0, 1]\) with respect to \( t \), we have

\[
(2.6) \quad f \left( \frac{a + b}{2} \right) \leq h \left( \frac{1}{2} \right) \int_0^1 f \left( ta + (1 - t)b \right) dt + mh \left( \frac{1}{2} \right) \int_0^1 f \left( (1 - t) \frac{a}{m} + t \frac{b}{m} \right) dt.
\]

By the facts that

\[
\int_0^1 f \left( ta + (1 - t)b \right) dt = \frac{1}{b - a} \int_a^b f(x) dx
\]

and

\[
\int_0^1 f \left( (1 - t) \frac{a}{m} + t \frac{b}{m} \right) dt = \frac{m}{b - a} \int_a^b f \left( \frac{x}{m} \right) dx = \frac{1}{b - a} \int_a^b f \left( \frac{x}{m} \right) dx.
\]

Using these equalities in (2.6), we obtain the first inequality of (2.5). By the \((h - m)\)–convexity of \( f \), we can write

\[
(2.7) \quad h \left( \frac{1}{2} \right) \left[ f \left( ta + (1 - t)b \right) + mf \left( (1 - t) \frac{a}{m} + t \frac{b}{m} \right) \right]
\]

\[
\leq h \left( \frac{1}{2} \right) \left[ tf(a) + m(1 - t) f \left( \frac{b}{m} \right) + m(1 - t) f \left( \frac{a}{m} \right) + m^2 tf \left( \frac{b^2}{m^2} \right) \right].
\]

Integrating the inequality (2.7) on \([0, 1]\) with respect to \( t \), we have

\[
\frac{h \left( \frac{1}{2} \right)}{b - a} \left[ \int_a^b f(x) dx + \int_a^b f \left( \frac{x}{m} \right) dx \right]
\]

\[
\leq h \left( \frac{1}{2} \right) \left[ f(a) + mf \left( \frac{b}{m} \right) + mf \left( \frac{a}{m} \right) + m^2 f \left( \frac{b^2}{m^2} \right) \right]
\]

which completes the proof. \( \square \)
Corollary 3. If we choose \( h(t) = 1 \) in (2.3), we obtain the following inequality:

\[
 f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) + m f \left( \frac{x}{m} \right) \, dx \\
\leq \left[ f(a) + m f \left( \frac{a}{m} \right) + m f \left( \frac{b}{m} \right) + m^2 f \left( \frac{b}{m^2} \right) \right].
\]

Corollary 4. If we choose \( m = 1 \) and \( h(t) = t^s \) in (2.5), we obtain the following inequality:

\[
2^{s-1} f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}
\]

which is similar to (1.4).

Remark 4. If we choose \( m = 1 \) in (2.4), we obtain the right hand side of the inequality (1.8).

Remark 5. If we choose \( m = 1 \) and \( h(t) = t \) in (2.5), we obtain the Hadamard’s inequality.

Remark 6. If we choose \( h(t) = t \) in (2.5), we obtain the inequality (1.2).

The following inequality also holds for \((h - m)\)–convex functions.

**Theorem 10.** Let \( f : [0, \infty) \to \mathbb{R} \) be a \((h - m)\)–convex function with \( m \in (0, 1], t \in [0, 1] \). If \( 0 \leq a < b < \infty \) and \( f \in L_1 [ma, b] \), then the following inequality holds:

\[
\frac{1}{m+1} \left[ \frac{1}{mb - a} \int_a^{mb} f(x) \, dx + \frac{1}{b - ma} \int_{ma}^{b} f(x) \, dx \right] \leq \frac{f(a) + f(b)}{2} \left[ \int_0^1 h(t) \, dt + \int_0^1 h(1-t) \, dt \right].
\]

**Proof.** From definition of \((h - m)\)–convex functions, we can write

\[
f \left( a + (1-t)b \right) \leq h(t) f(a) + mh(1-t) f(b)
\]

\[
f \left( (1-t)a + m(1-t)b \right) \leq h(1-t) f(a) + mh(t) f(b)
\]

\[
f \left( t(1-t)a + m(1-t)b \right) \leq h(t) f(b) + mh(1-t) f(a)
\]

and

\[
f \left( (1-t)b + mta \right) \leq h(1-t) f(b) + mh(t) f(a)
\]
for all \( t \in [0, 1] \). By summing these inequalities and integrating on \([0, 1]\) with respect to \( t \), we obtain

\[
\int_0^1 f(ta + m(1-t)b) \, dt + \int_0^1 f((1-t)a + mtb) \, dt
\]

\[
+ \int_0^1 f(tb + m(1-t)a) \, dt + \int_0^1 f((1-t)b + mta) \, dt
\]

\[
\leq (f(a) + f(b))(m+1) \left[ \int_0^1 h(t) \, dt + \int_0^1 h(1-t) \, dt \right].
\]

(2.9)

It is easy to show that

\[
\int_0^1 f(ta + m(1-t)b) \, dt = \int_0^1 f((1-t)a + mtb) \, dt = \frac{1}{mb-a} \int_a^{mb} f(x) \, dx
\]

and

\[
\int_0^1 f(tb + m(1-t)a) \, dt = \int_0^1 f((1-t)b + mta) \, dt = \frac{1}{b-ma} \int_{ma}^{b} f(x) \, dx.
\]

By using these equalities in (2.9), we get the desired result. \( \square \)

**Corollary 5.** If we choose \( h(t) = 1 \) in (2.8), we obtain the following inequality;

\[
\frac{1}{m+1} \left[ \frac{1}{mb-a} \int_a^{mb} f(x) \, dx + \frac{1}{b-ma} \int_{ma}^{b} f(x) \, dx \right] \leq f(a) + f(b).
\]

**Remark 7.** If we choose \( m = 1 \) and \( h(t) = t \) in (2.8), we obtain the right hand side of the Hadamard’s inequality. If we choose \( m = 1 \) and \( h(t) = 1 \) in (2.8), we obtain the right hand side of the inequality (1.6). If we choose \( m = 1 \) and \( h(t) = t^s \) in (2.8), we obtain the right hand side of the inequality (1.4).

**Remark 8.** If we choose \( h(t) = t \) in (2.8), we obtain the inequality (1.3).

**References**

[1] M.Z. Sarikaya, A. Saglam and H. Yldrm, On some Hadamard-type inequalities for \( h \)-convex functions, Journal of Mathematical Inequalities, 2, 3 (2008), 335-341.

[2] S.S. Dragomir, J. Pecaric, L.E. Persson, Some inequalities of Hadamard type, Soochow J.Math. 21 (1995) 335–341.

[3] E.K. Godunova, V.I. Levin, Neravenstva dlja funkciĭ širokogo klassa, soderžačego vypuklye, monotomnye i nekotorye drugie vidy funkciĭ, in: Vycislitel. Mat. i. Mat. Fiz. Mežvuzov. Sb. Nauc. Trudov, MGPI, Moskva, 1985, pp. 138–142.

[4] H. Hudzik and L. Maligranda, Some remarks on \( s \)-convex functions, Aequationes Math. 48 (1994) 100–111.

[5] S. Varosanec, On \( h \)-convexity, J. Math. Anal. Appl., 326 (2007), 303–311.

[6] M. Bombardelli and S. Varosanec, Properties of \( h \)-convex functions related to the Hermite-Hadamard-Fejér inequalities, Computers and Mathematics with Applications, 58 (2009), 1869–1877.

[7] M.Z. Sarikaya, E. Set and M.E. Özdemir, On some new inequalities of Hadamard type involving \( h \)-convex functions, Acta Math. Univ. Comenianae, Vol. LXXIX, 2 (2010), pp. 265-272.
[8] P. Burai and A. Hazy, On approximately $h$–convex functions, Journal of Convex Analysis, 18, 2 (2011).

[9] S.S. Dragomir and S. Fitzpatrick, The Hadamard’s inequality for $s$–convex functions in the second sense, Demonstratio Math., 32 (4) (1999), 687-696.

[10] U.S. Kirmaci, M.K. Bakula, M.E. Özdemir and J. Pecaric, Hadamard-type inequalities for $s$–convex functions, Applied Mathematics and Computation, 193 (2007), 26-35.

[11] M.E. Özdemir, M. Avci and E. Set, On some inequalities of Hermite-Hadamard type via $m$–convexity, Applied Mathematics Letters, 23 (2010), 1065-1070.

[12] G. Toader, Some generalization of the convexity, Proc. Colloq. Approx. Opt., Cluj-Napoca, (1985), 329-338.

[13] S.S. Dragomir and G. Toader, Some inequalities for $m$–convex functions, Studia Univ. Babeş-Bolyai Math., 38 (1) (1993), 21-28.

[14] G. Toader, On a generalization of the convexity, Mathematica, 30 (53) (1988), 83-87.

[15] M.K. Bakula, J. Pecaric and M. Ribicic, Companion inequalities to Jensen’s inequality for $m$–convex and $(\alpha, m)$–convex functions, J. Ineq. Pure and Appl. Math., 7 (5) (2006), Art. 194.

[16] W.W. Breckner, Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Raumen, Pupl. Inst. Math., 23 (1978), 13–20.

[17] S.S. Dragomir, On some new inequalities of Hermite-Hadamard type for $m$–convex functions, Tamkang Journal of Mathematics, 33 (1) (2002).

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