Towards the Small Quasi-Kernel Conjecture

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Abstract

Let \( D = (V, A) \) be a digraph. A vertex set \( K \subseteq V \) is a quasi-kernel of \( D \) if \( K \) is an independent set in \( D \) and for every vertex \( v \in V \setminus K \), \( v \) is at most distance 2 from \( K \). In 1974, Chvátal and Lovász proved that every digraph has a quasi-kernel. P. L. Erdős and L. A. Székely in 1976 conjectured that if every vertex of \( D \) has a positive indegree, then \( D \) has a quasi-kernel of size at most \( |V|/2 \). This conjecture is only confirmed for narrow classes of digraphs, such as semicomplete multipartite, quasi-transitive, or locally semicomplete digraphs. In this note, we state a similar conjecture for all digraphs, show that the two conjectures are equivalent, and prove that both conjectures hold for a class of digraphs containing all orientations of 4-colorable graphs (in particular, of all planar graphs).

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1 Introduction and notation

The digraphs in this note may have antiparallel arcs, but do not have loops. Let \( D \) be a digraph. We denote by \( V(D) \) and \( A(D) \) the vertex set and the arc set of \( D \), respectively.

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We say \( D \) is weakly connected if the underlying graph of \( D \) is connected. Let \( x \in V(D) \). The open (closed) outneighborhood and inneighborhood of \( x \) in \( D \), denoted \( N^+_D(x) \) (\( N^-_D(x) \)) and \( N^-_D(x) \) (\( N^+_D[x] \)) are defined as follows.

\[
N^+_D(x) = \{ y \in V(D) \mid xy \in A(D) \}, \quad N^+_D[x] = N^+_D(x) \cup \{x\}, \\
N^-_D(x) = \{ y \in V(D) \mid yx \in A(D) \}, \quad N^-_D[x] = N^-_D(x) \cup \{x\}.
\]

The outdegree of \( x \) in \( D \) is \( d^+_D(x) = |N^+_D(x)| \), and the indegree of \( x \) in \( D \) is \( d^-_D(x) = |N^-_D(x)| \). Vertices of indegree zero in \( D \) are called sources of \( D \) and vertices of outdegree zero in \( D \) are called sinks of \( D \). By \( \delta^+(D) \) (respectively, \( \delta^-(D) \)) we denote the minimum outdegree (respectively, indegree) in \( D \) among all vertices of \( D \). For each \( X \subseteq V(D) \), we let

\[
N^+_D(X) = \bigcup_{x \in X} N^+_D(x) \setminus X, \quad N^+_D[X] = N^+_D(X) \cup X, \\
N^-_D(X) = \bigcup_{x \in X} N^-_D(x) \setminus X, \quad N^-_D[X] = N^-_D(X) \cup X.
\]

Let \( u, v \in V(D) \) and \( K \subseteq V(D) \). The distance from \( u \) to \( v \) in \( D \), denoted \( \text{dist}_D(u, v) \), is the length of a shortest directed path from \( u \) to \( v \). The distance from \( K \) to \( v \) in \( D \), is \( \text{dist}_D(K, v) = \min \{ \text{dist}_D(x, v) \mid x \in K \} \). We say \( K \) is a kernel of \( D \) if \( K \) is independent in \( D \) and for every \( v \in V(D) \setminus K \), \( \text{dist}_D(K, v) = 1 \). We say \( K \) is a quasi-kernel of \( D \) if \( K \) is independent in \( D \) and for every \( v \in V(D) \setminus K \), \( \text{dist}_D(K, v) \leq 2 \).\(^1\)

A digraph \( D \) is kernel-perfect if every induced subdigraph of it has a kernel. Richardson proved the following result.

**Theorem 1** (Richardson [10]). *Every digraph without directed odd cycles is kernel-perfect.*

The proof gives rise to an algorithm to find one. On the other hand, Chvátal [4] showed that in general it is NP-complete to decide whether a digraph has a kernel, and by a result of Fraenkel [6] it is NP-complete even in the class of planar digraphs of degree at most 3. While not every digraph has a kernel, Chvátal and Lovász [5] proved that every digraph has a quasi-kernel. In 1976, P.L. Erdős and S. A. Székely made the following conjecture on the size of a quasi-kernel in a digraph.

**Conjecture 2** (Erdős–Székely [1]). *Every \( n \)-vertex digraph \( D \) with \( \delta^+(D) \geq 1 \) has a quasi-kernel of size at most \( \frac{n}{2} \).*

If \( D \) is an \( n \)-vertex digraph consisting of the disjoint union of directed 2- and 4-cycles, then every kernel or quasi-kernel of \( D \) has size exactly \( \frac{n}{2} \). Thus, Conjecture 2 is sharp.

In 1996, Jacob and Meyniel [9] showed that a digraph without a kernel contains at least three distinct quasi-kernels. Gutin et al. [7] characterized digraphs with exactly one and two quasi-kernels, thus provided necessary and sufficient conditions for a digraph to have

\(^1\)Our definition of a kernel is the digraph dual of what was originally defined in [6], and it is “consistent” with the definition of a quasi-kernel.
at least three quasi-kernels. However, these results do not discuss the sizes of the quasi-kernels. Heard and Huang [8] in 2008 showed that each digraph \( D \) with \( \delta^+(D) \geq 1 \) has two disjoint quasi-kernels if \( D \) is semicomplete multipartite (including tournaments), quasi-transitive (including transitive digraphs), or locally semicomplete. As a consequence, Conjecture 2 is true for these three classes of digraphs.

We propose a conjecture which formally implies Conjecture 2. It suggests a bound for digraphs that may have sources. Note that each quasi-kernel of a digraph contains all of its source vertices and hence contains no outneighbors of the source vertices.

**Conjecture 3.** Let \( D \) be an \( n \)-vertex digraph, and let \( S \) be the set of sources of \( D \). Then \( D \) has a quasi-kernel \( K \) such that

\[
|K| \leq \frac{n + |S| - |N_D^+(S)|}{2}.
\]

To show that the upper bound above is best possible, consider the following examples.

- Let \( S \) be a nonempty set of isolated vertices, and let \( D \) be a digraph obtained from a directed triangle by adding an arc from every vertex in \( S \) to the same vertex in the triangle. Then every quasi-kernel of \( D \) has size \( |S| + 1 = \frac{(|S|+3)+|S|-1}{2} \).

- Let \( D \) be an orientation of a connected bipartite graph with parts \( S \) and \( T \) where each arc goes from \( S \) to \( T \). Then \( S \) forms a quasi-kernel of \( D \) of size \( |S| = \frac{(|S|+|T|)+|S|-|T|}{2} \).

In this paper, we support Conjectures 2 and 3 by showing the following results.

**Theorem 4.** Let \( D \) be an \( n \)-vertex digraph and \( S \) be the set of sources of \( D \). Suppose that \( V(D) \setminus N_D^+(S) \) has a partition \( V_1 \cup V_2 \) such that \( D[V_i] \) is kernel-perfect for each \( i = 1, 2 \). Then \( D \) has a quasi-kernel of size at most \( \frac{n + |S| - |N_D^+(S)|}{2} \).

Since by Theorem 1, every digraph without directed odd cycles is kernel-perfect, Theorem 4 immediately yields:

**Corollary 5.** Conjectures 2 and 3 hold for every orientation of each graph with chromatic number at most 4.

By the Four Color Theorem [2, 3], Corollary 5 yields that Conjectures 2 and 3 hold for every digraph whose underlying graph is planar.

**Theorem 6.** If Conjecture 3 fails and \( D \) is a counterexample to it with the minimum number of vertices, then \( D \) has no source.

Since Conjecture 3 implies Conjecture 2, Theorem 6 implies that the two conjectures are equivalent.

In the next section we prove Theorem 4 and in Section 3 prove Theorem 6.
2 Proof of Theorem 4

Let $D' = D - N_D^+[S]$ be the digraph obtained by removing the source vertices and their outneighbors, and $V_1 \cup V_2 = V(D')$ be a partition of $V(D')$ such that $D[V_1]$ is kernel-perfect for each $i = 1, 2$. In addition, we choose such a partition so that $|V_2|$ is as small as possible. Observe that adding a source vertex $v$ to a kernel-perfect digraph $H$ results in a new kernel-perfect digraph: let $H'$ be the resulting digraph, and let $F$ be a subdigraph of $H'$ that contains $v$. Then $K \cup \{v\}$ is a kernel of $F$ where $K$ is any kernel of $F - N_H^+[v]$ in $H$.

If there exists some $v \in V_2$ with no inneighbors in $V_1$, then we may move $v$ from $V_2$ to $V_1$, and obtain a new partition of $V(D')$ into kernel-perfect subgraphs with a smaller $V_2$ by Theorem 1. Thus, by the choice of $V_2$,

$$N_{D'}(v) \cap V_1 \neq \emptyset \quad \text{for every } v \in V_2. \quad (1)$$

For a digraph $F$ and an independent set $R \subseteq V(F)$, we say $R_0 \subseteq R$ is a concise set of $R$ in $F$ if $N_F^+(R_0) = N_F^+(R)$ and $|R_0| \leq |N_F^+(R)|$. Indeed, every independent set has a concise set—iteratively add vertices $v$ from $R$ to $R_0$ if and only if $|N_F^+(R_0 \cup \{v\})| > |N_F^+(R_0)|$.

Since $D[V_1]$ is kernel-perfect, it has a kernel $K$. Let $R_0$ be a concise set of $R$ in $D'$. Let $D'' = D' - (R_0 \cup N_{D'}^+(R_0)) = D' - N_{D'}^+[R_0]$. We partition $R \setminus R_0$ into sets $S''$ and $T$ of sources and non-sources in $D''$ respectively. Note that since each $v \in S''$ was not a source in the original digraph $D$, $v$ must have an inneighbor in $V(D) - V(D'')$.

Set $K = S \cup R_0 \cup T$. We will show that $K$ is a quasi-kernel of $D$. We first show that it is independent. Indeed, $K \cap R$ is independent, since $R$ was a kernel of $D[V_1]$. There are no arcs from $K \cap R$ to $K \setminus R = S$ because each vertex in $S$ is a source in $D$. Similarly, there are no arcs from $S$ to $K \setminus S$. Finally, there are no arcs from $S$ to $K \setminus S$ because $K \setminus S \subseteq V(D') = V(D) - N_D^+[S]$.

Now we check that each vertex is at distance at most 2 from $K$. For any $v \in N_D^+[K]$, we have $\text{dist}_D(K, v) \leq 1$. Consider $v \in V_1 \setminus N_D^+[K]$. Recall that $R$ is a kernel of $D[V_1]$, so $V_1 \subseteq N_D^+(R)$. It follows that since $R_0$ is a concise set of $R$, the vertex $v$ must be contained in $K \cap R = S''$. Therefore $v$ has an inneighbor in $N_D^+[S] \cup N_{D'}^+[R_0] \subseteq N_D^+[K]$, hence $\text{dist}_D(K, v) \leq 2$.

Now suppose $v \in V_2 \setminus N_D^+[K]$. By (1), $v$ has an inneighbor $u \in V_1$. If $u \in N_D^+[K]$, then $\text{dist}_D(K, v) \leq 2$. So we may assume $u \in V_1 \setminus N_D^+[K] = S''$. Since $S'' \subseteq R$, $v \in N_{D'}^+[R]$. But $R_0$ is a concise set of $R$, so $v \in N_{D'}^+[R_0] \subseteq N_D^+[K]$. We get $\text{dist}_D(K, v) \leq 1$.

Therefore, $K$ is a quasi-kernel of $D$. If $|T| \leq |V(D'') \setminus T|$ (so $2|T| \leq |V(D'') \cup T| = |V(D'')|$), then using the fact that $R_0$ is a concise set,

$$|K| = |S| + |R_0| + |T| \leq |S| + \frac{1}{2}|R_0 \cup N_{D'}^+(R)| + \frac{1}{2}|V(D'')| \leq |S| + \frac{1}{2}|V(D) \setminus N_D^+[S]| \leq \frac{1}{2}(n + |S| - |N_D^+(S)|),$$

and the theorem holds. Thus, assume that $|T| > |V(D'') \setminus T|$ (so $|V(D'') \setminus T| < |V(D'')|/2$). Note that $V(D'') \setminus T = (V_2 \setminus N_{D'}^+(R)) \cup S''$. Since $D[V_2]$ is kernel-perfect and adding source
vertices preserves kernel-perfectness, the digraph $D'' - T$ is also kernel-perfect. Let $W$ be a kernel of $D'' - T$ and set $K' = (S \cup R_0 \cup W) \setminus N_D^+(W)$.

Similarly to $K$, the set $K'$ is independent in $D$. Since $|T| > |V(D'') \setminus T|$, \[ |K'| \leq |S| + |R_0| + |W| \leq |S| + \frac{1}{2} |R_0 \cup N_D^+(R)| + \frac{1}{2} |V(D'')| \leq \frac{n + |S| - |N_D^+(S)|}{2}. \]

We now show that dist$_D(K', v) \leq 2$ for every $v \in V(D) \setminus K'$.

Observe that $S'' \subseteq W$ since the vertices in $S''$ are sources in $D'' - T$. Clearly, we have that each vertex $v \in V(D'' - T)$ has dist$_D(K', v) \leq 1$. Now suppose $v \in T$. Since $v$ is not a source in $D''$, it has an inneighbor in $V(D'')$, and this neighbor cannot be in $T$ because $T \subseteq R$ is independent. Hence dist$_D(K', v) \leq 2$.

We have dist$_D(K', v) \leq 1$ for all $v \in N_D^+(S)$. It remains to consider $v \in V(D') \setminus V(D'') = N_D^+(R_0)$. If $v \in R_0$, then either $v \in K'$ or $v \in N_D^+(W)$. Hence dist$_D(K', v) \leq 1$. It follows that dist$_D(K', v) \leq 2$ for all $v \in N_D^+(R_0)$. Therefore $K'$ is a quasi-kernel of $D$.

\hfill $\square$

3 Proof of Theorem 6

Assume Conjecture 3 fails and $D$ is a counterexample to it with the fewest vertices. Let $n = |V(D)|$. We assume $n \geq 4$ as the cases $n \leq 3$ are verifiable by hand. By the minimality of $n$, $D$ is weakly connected. Let $S$ be the set of sources of $D$. We show that $S = \emptyset$. Assume instead that $S \neq \emptyset$.

**Case 1:** $|N_D^+[S]| \geq 3$. Let $D_1$ be obtained from $D$ by deleting all vertices in $N_D^+[S]$, adding two new vertices $x$ and $y$, adding an arc from $y$ to every vertex of $D - N_D^+[S]$ that is an outneighbor of some vertex of $N_D^+(S)$ in $D$, and adding an arc from $x$ to $y$. Then $x$ is the only source vertex of $D_1$, and $N_{D_1}^+(x) = \{y\}$. Since $|V(D_1)| = |V(D)| - |N_D^+[S]| + 2 \leq |V(D)| - 1$, the minimality of $n$ implies that $D_1$ has a quasi-kernel $K_1$ of size at most $\frac{n - |N_D^+[S]| + 2 + 1 - 1}{2}$. Then $K = (K_1 \setminus \{x\}) \cup S$ is a quasi-kernel of $G$ that has size at most \[ \frac{n - |N_D^+[S]| + 2 + 1 - 1}{2} - 1 + |S| = \frac{n + |S| - |N_D^+(S)|}{2}, \]

as desired.

**Case 2:** $|N_D^+[S]| \leq 2$. Since $D$ is weakly connected, and $|S| \geq 1$, we get $|S| = 1$ and $|N_D^+(S)| = 1$. Let $D_1 = D - N_D^+[S]$. If $D_1$ has no sources, then by the minimality of $D$, digraph $D_1$ has a quasi-kernel $K_1$ with $|K_1| \leq \frac{n - 2}{2}$. Then $K = K_1 \cup S$ is a desired quasi-kernel of $D$. Therefore, we assume that $D_1$ has a source. Let \[ S_1 = \{v \in V(D_1) \mid d_{D_1}(v) = 0\}. \]

If $|N_D^+(S_1)| \leq |S_1|$, we let $D_2 = D_1 - S_1$. By the minimality of $D$, $D_2$ has a quasi-kernel $K_1$ of size at most $\frac{n - 2 - |S| + |N_D^+(S_1)|}{2} \leq \frac{n - 2}{2}$. Then $K = K_1 \cup S$ is a desired quasi-kernel of $D$. Thus, we assume that $|N_D^+(S_1)| > |S_1|$. Let $D_2$ be obtained from $D_1$ by deleting all
vertices in $N^+_{D_1}[S_1]$, adding two new vertices $x$ and $y$, adding an arc from $y$ to every vertex of $D_1 - N^+_{D_1}[S_1]$ that is an outneighbor of some vertex of $N^+_{D_1}(S_1)$ in $D_1$, and adding an arc from $x$ to $y$. Note that $x$ is the only source of $D_2$, and $N^+_{D_2}(x) = \{y\}$. Again, by the minimality of $D$, $D_2$ has a quasi-kernel $K_1$ of size at most $\frac{n - 2 - |N^+_{D_1}[S_1]| + 2 + 1 - 1}{2}$. Then $K = (K_1 \setminus \{x\}) \cup S \cup S_1$ is a quasi-kernel of $D$ that has size at most $\frac{n - 1}{2}$, as desired.

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