W*-SUPERRIGIDITY FOR COINDUCED ACTIONS

DANIEL DRIMBE

Abstract. We prove W*-superrigidity for a large class of coinduced actions. We prove that if Σ is an amenable almost-malnormal subgroup of an infinite conjugacy class (icc) property (T) countable group Γ, the coinduced action Γ ↷ X from an arbitrary probability measure preserving action Σ ↷ X₀ is W*-superrigid. We also prove a similar statement if Γ is an icc non-amenable group which is measure equivalent to a product of two infinite groups. In particular, we obtain that any Bernoulli action of such a group Γ is W*-superrigid.

1. Introduction and statement of the main results

1.1. Introduction. To every measure preserving action Γ ↷ (X, µ) of a countable group Γ on a standard probability space (X, µ), one associates the group measure space von Neumann algebra $L^∞(X)⋊Γ$ [MvN36]. If the action Γ ↷ X is free, ergodic and probability preserving (pmp), then $L^∞(X)⋊Γ$ is a II₁ factor which contains $L^∞(X)$ as a Cartan subalgebra, i.e. a maximal abelian von Neumann algebra whose normalizer generates $L^∞(X)⋊Γ$. The classification of group measure space II₁ factors $L^∞(X)⋊Γ$ is a central problem in the theory of von Neumann algebras. Two free ergodic pmp actions Γ ↷ (X, µ) and Λ ↷ (Y, ν) on standard probability spaces (X, µ) and (Y, ν) are said to be W*-equivalent if $L^∞(X)⋊Γ$ is isomorphic to $L^∞(Y)⋊Λ$.

If the groups are amenable, the classification up to W*-equivalency has been completed in the 1970s. More precisely, the celebrated theorem of Connes [Co76] asserts that all II₁ factors arising from free ergodic pmp actions of countable amenable groups are isomorphic to the hyperfinite II₁ factor. In contrast, the non-amenable case is much more challenging and it has led to a beautiful rigidity theory in the sense that one can deduce conjugacy from W*-equivalence. A major breakthrough in the classification of II₁ factors was made by Popa between 2001-2004 through the invention of deformation/rigidity theory (see [Po07, Va10a, Io12a] for surveys). In particular, he obtained the following W*-rigidity result: let Γ ↷ X be a free ergodic pmp action of an infinite conjugacy class (icc) countable group Γ which has an infinite normal subgroup with the relative property (T) and let Λ ↷ Y := Y₀Λ be a Bernoulli action of a countable group Λ. Popa proved that if the two actions are W*-equivalent, then the actions are conjugate [Po03, Po04], i.e. there exist a group isomorphism $d : Γ → Λ$ and a measure space isomorphism $θ : X → Y$ such that $θ(gx) = d(g)θ(x)$ for all $g ∈ Γ$ and almost everywhere (a.e.) $x ∈ X$.

The most extreme form of rigidity for an action Γ ↷ (X, µ) is W*-superrigidity, i.e. whenever Λ ↷ (Y, ν) is a free ergodic pmp action W*-equivalent to Γ ↷ (X, µ), then the two actions are conjugate. A few years ago, Peterson was able to show the existence of virtually W*-superrigid actions [Pe09]. Soon after, Popa and Vaes discovered the first concrete families of W*-superrigid actions [PV09]. Ioana then proved in [Io10] a general W*-superrigidity result for Bernoulli actions.

Theorem (Ioana, [Io10]). If Γ is an icc property (T) group and (X₀, µ₀) is a non-trivial standard probability space, then the Bernoulli action Γ ↷ (X₀, µ₀)Γ is W*-superrigid.

#Acknowledgments

The author was partially supported by NSF Career Grant DMS #1253402.
The main ingredient of his proof was the discovery of a beautiful dichotomy result for abelian subalgebras of II\textsubscript{1} factors coming from Bernoulli actions.

Using a similar method, Ioana, Popa and Vaes were able to prove later that any Bernoulli action of an icc non-amenable group which is a product of two infinite groups is also W\textsuperscript{*}-superrigid [PV10]. A few years ago Boutonnet extended these results to Gaussian actions in [Bo12]. Several other classes of W\textsuperscript{*}-superrigid actions have been found in [PV10, CP10, HPV10, Va10b, CS11, CSU11, PV11, PV12, CIK13, CIK15, Dr15, GITD16].

1.2. Statement of the main results. Our first theorem is a generalization of Ioana’s W\textsuperscript{*}-superrigidity result [Io10] Theorem A] to coinduced actions. Before stating the theorem, we explain first the terminology that we use starting with the notion of coinduced actions (see e.g. [Io06b]).

Definition 1.1. Let \( \Gamma \) be a countable group and let \( \Sigma \) be a subgroup. Let \( \phi : \Gamma / \Sigma \to \Gamma \) be a section. Define the cocycle \( c : \Gamma \times \Gamma / \Sigma \to \Sigma \) by the formula

\[
c(g, i) = \phi^{-1}(gi)g\phi(i),
\]

for all \( g \in \Gamma \) and \( i \in \Gamma / \Sigma \).

Let \( \Sigma \overset{\sigma_0}{\longrightarrow} (X_0, \mu_0) \) be a pmp action, where \( (X_0, \mu_0) \) is a non-trivial standard probability space. We define an action \( \Gamma \overset{\sigma}{\longrightarrow} X_0^\Gamma / \Sigma \), called the coinduced action of \( \sigma_0 \), as follows:

\[
\sigma_g((x_i)_{i \in \Gamma / \Sigma}) = (x_i')_{i \in \Gamma / \Sigma}, \quad \text{where } x_i' = c(g^{-1}, i)^{-1}x_{g^{-1}i}.
\]

Note the following remarks:

- \( \sigma \) is a pmp action of \( \Gamma \) on the standard probability space \( X_0^{\Gamma / \Sigma} \).
- if we consider the trivial action of \( \Lambda = \{e\} \) on \( X_0 \), then the coinduced action of \( \Gamma \) on \( X_0^{\Gamma / \{e\}} = X_0^\Gamma \) is the Bernoulli action.

Recall that an inclusion \( \Gamma_0 \subset \Gamma \) of countable groups has the relative property (T) if for every \( \epsilon > 0 \), there exist \( \delta > 0 \) and a finite subset \( F \subset \Gamma \) such that if \( \pi : \Gamma \to U(K) \) is a unitary representation and \( \xi \in K \) is a unit vector satisfying \( \|\pi(g)\xi - \xi\| < \delta \), for all \( g \in F \), then there exists \( \xi_0 \in K \) such that \( \|\xi - \xi_0\| < \epsilon \) and \( \pi(h)\xi_0 = \xi_0 \), for all \( h \in \Gamma_0 \). The group \( \Gamma \) has the property (T) if the inclusion \( \Gamma \subset \Gamma_0 \) has the relative property (T). To give some examples, note that \( \mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes SL_2(\mathbb{Z}) \) has the relative property (T) and \( SL_n(\mathbb{Z}) \), \( n \geq 3 \), has the property (T) [Ka67, Ma82].

Finally, we say that a subgroup \( \Sigma \) of a countable group \( \Gamma \) is called \( n \)-almost malnormal if for any \( g_1, g_2, ..., g_n \in \Gamma \) such that \( g_i^{-1}g_j \notin \Sigma \) for all \( i \neq j \), the group \( \cap_{i=1}^n g_i\Sigma g_i^{-1} \) if finite. The subgroup \( \Sigma \) is called almost malnormal if it is \( n \)-almost malnormal for some \( n \geq 1 \).

Theorem A. Let \( \Gamma \) be an icc group which admits an infinite normal subgroup \( \Gamma_0 \) with relative property (T) and \( \Sigma \) be an amenable almost malnormal subgroup of \( \Gamma \). Let \( \sigma_0 \) be a pmp action of \( \Sigma \) on a non-trivial standard probability space \( (X_0, \mu_0) \) and denote by \( \sigma \) the coinduced action of \( \Gamma \) on \( X := X_0^{\Gamma / \Sigma} \). Then \( \Gamma \overset{\sigma}{\longrightarrow} X \) is W\textsuperscript{*}-superrigid.

Example 1.2. In particular, Theorem A can be applied for \( \Gamma = SL_3(\mathbb{Z}) \) and \( \Sigma = \langle A \rangle \), where

\[
A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}
\]

[PV06, Section 7]. See [PV06] for more concrete examples of amenable almost malnormal subgroups of \( PSL_n(\mathbb{Z}) \), \( n \geq 3 \). See also [RS10, Theorem 1.1], a result which proves the existence of amenable almost malnormal subgroups of torsion-free uniform lattices in connected semisimple real algebraic groups with no compact factors.
We now generalize Ioana-Popa-Vaes’ result [IPV10, Theorem 10.1] to coinduced actions. First, recall that two countable groups $\Gamma$ and $\Lambda$ are called measure equivalent in sense of Gromov if there exist two commuting free measure preserving actions of $\Gamma$ and $\Lambda$ on a standard measure space $(\Omega, m)$, such that the actions of $\Gamma$ and $\Lambda$ each admit a finite measure fundamental domain [Gr91]. Natural examples of measure equivalent groups are provided by pairs of lattices $\Gamma, \Lambda$ in an unimodular locally compact second countable group.

**Theorem B.** Let $\Gamma$ be an icc non-amenable group which is measure equivalent to a product of two infinite groups. Let $\Sigma$ be an amenable almost malnormal subgroup and let $\sigma_0$ be a pmp action of $\Sigma$ on a non-trivial standard probability space $(X_0, \mu_0)$ and denote by $\sigma$ the coinduced action of $\Gamma$ on $X := X_0^{\Gamma/\Sigma}$.

Then $\Gamma \actson X$ is $W^*$-superrigid.

See Theorem 6.3 for a more general statement in which it is assumed instead that $\Gamma$ is measure equivalent to a group $\Lambda_0$ whose group von Neumann algebra $L(\Lambda_0)$ is not prime. Note that Theorems A and B provide a complementary class of $W^*$-superrigid coinduced actions from the one found in [Dr15, Corollary 1.4].

**Example 1.3.** A more general statement of Theorem B can be applied for $\Sigma \subset \Gamma = \Delta \wr \Sigma$ with $\Delta$ non-amenable and $\Sigma$ amenable (see Remark 6.4).

The following remark shows that if $\Sigma$ is not almost malnormal, the action $\Gamma \actson X$ is not necessary $W^*$-superrigid. To put this in context, we recall first the notion of OE-superrigidity and Singer’s result [Si55]. Two actions $\Gamma \actson X$ and $\Lambda \actson Y$ are orbit equivalent (OE) if there exists a measure space isomorphism $\theta : X \to Y$ such that $\theta(\Gamma x) = \Lambda \theta(x)$, for a.e. $x \in X$. A pmp action $\Gamma \actson X$ is OE-superrigid if whenever $\Lambda \actson Y$ is a free ergodic pmp action which is OE to $\Gamma \actson X$, then the two actions are conjugate.

Singer proved in [Si55] that two free ergodic pmp actions $\Gamma \actson X$ and $\Lambda \actson Y$ are OE if and only if there exists an isomorphism of the group measure space algebras $L^\infty(X) \rtimes \Gamma$ and $L^\infty(Y) \rtimes \Lambda$ which preserves the Cartan algebras $\mathcal{L}^\infty(X)$ and $\mathcal{L}^\infty(Y)$. In particular, $W^*$-superrigidity implies OE-superrigidity.

**Remark 1.4.** If $\Sigma$ is not almost malnormal, the action $\Gamma \actson X$ may fail to be $W^*$-superrigid. Indeed, suppose $\Gamma$ is an icc group which splits as a direct product $\Gamma = \Sigma \times \Delta$, with $\Sigma$ amenable and $\Delta$ a non-amenable group. Connes and Jones have found in [CJ82] a class of groups $\Sigma$ and a class of free ergodic pmp actions $\Sigma \actson X_0$ for which the coinduced action $\Gamma \actson X$ of $\sigma_0$ is not $W^*$-superrigid. Precisely, they have proven that $\mathcal{M} := L^\infty(X) \rtimes \Gamma$ is McDuff, i.e. $\mathcal{M} \simeq M\bar{\otimes} R$, where $R$ is the hyperfinite $\Pi_1$ factor. However, [Dr15, Corollary 1.3] implies that $\Gamma \actson X$ is OE-superrigid.

Note that Theorem B extends the class of groups whose Bernoulli actions are $W^*$-superrigid. Therefore we record the following result.

**Corollary C.** Let $\Gamma$ be an icc non-amenable group which is measure equivalent to a product of two infinite groups. Let $(X_0, \mu_0)$ be a non-trivial standard probability space. Then the Bernoulli action $\Gamma \actson X_0^{\Gamma}$ is $W^*$-superrigid.

We recall the well known theorem due to Borel which asserts that every connected non-compact semisimple Lie group contains a lattice (see [Bo63] and [Ra72, Theorem 14.1]). Using this, we obtain an immediate consequence of Corollary C.

**Corollary D.** Let $\Gamma$ be an icc lattice in a product $G = G_1 \times \cdots \times G_n$ of $n \geq 2$ connected non-compact semisimple Lie groups and let $(X_0, \mu_0)$ be a non-trivial standard probability space. Then the Bernoulli action $\Gamma \actson X_0^{\Gamma}$ is $W^*$-superrigid.
Note that a combination of Popa’s cocycle superrigidity theorem for product groups \cite{Po06} and the results on uniqueness of Cartan subalgebras from \cite{PV12} already proves Corollary D, but only in the case when each factor $G_1, \ldots, G_n$ is of rank one.

1.3. Comments on the proof of Theorem B. For obtaining the proofs of Theorem A and Theorem B, we adapt the proofs used by Ioana \cite{Io10} and Ioana-Popa-Vaes \cite{IPV10} to the context of coinduced actions. We outline briefly and informally the proof of Theorem B since it has as a consequence Corollary C.

To this end, let $\Gamma$ be an icc group and let $\Sigma$ be an almost malnormal subgroup. Assume $\Gamma$ is measure equivalent to a product $\Lambda_0 = \Lambda_1 \times \Lambda_2$ of two countable groups. By \cite{Fu99}, $\Gamma$ and $\Lambda_0$ must have stably orbit equivalent actions. To simplify notation, assume there exist free ergodic pmp actions of $\Gamma$ and $\Lambda_0$ on a probability space $(Y_0, \mu)$ whose orbits are equal, almost everywhere. Thus, $L^\infty(Y_0) \rtimes \Gamma = L^\infty(Y_0) \rtimes \Lambda_0$.

Suppose $\Sigma \htimes X_0$ is a pmp action on a non-trivial standard probability space and let $\Gamma \htimes X := X_0^{\Gamma/\Sigma}$ be the corresponding coinduced action. Our goal is to show that $\Gamma \htimes X$ is $W^\ast$-superrigid. Assume that $\Lambda \htimes Y$ is an arbitrary free ergodic pmp action such that $M := L^\infty(X) \rtimes \Gamma = L^\infty(Y) \rtimes \Lambda$.

First, we reduce the problem to showing that the Cartan subalgebras $L^\infty(X)$ and $L^\infty(Y)$ are unitarily conjugated. We do this by proving in Section 4 a cocycle superrigidity theorem for $\Gamma \htimes X$. Combined with \cite[Theorem 5.6]{Po05}, we obtain that $\Gamma \htimes X$ is OE-superrigid. Therefore, by a result of Singer \cite{Si55} it is enough to show that $L^\infty(Y)$ is unitarily conjugate to $L^\infty(X)$ in $M$. We note that this is actually equivalent to $L^\infty(Y) \prec_M L^\infty(X)$, by \cite[Theorem A.1]{Po06b}. See Section 2.2 for the definition of Popa’s intertwining symbol “\prec”.

As is \cite{Io10}, we make use of the decomposition $M = L^\infty(Y) \rtimes \Lambda$ via the comultiplication $\Delta : M \to M \bar{\otimes} M$ defined by $\Delta(b\lambda) = b\lambda \otimes \lambda$, for all $b \in L^\infty(Y)$ and $\lambda \in \Lambda$, introduced in \cite{PV09}. Here we denote by $\{v_\lambda\}_{\lambda \in \Lambda}$ the canonical unitaries implementing the action of $\Lambda$ on $L^\infty(Y)$. The next step is to prove that there exists a unitary $u \in M \bar{\otimes} M$ such that

$$u\Delta(L(\Gamma))u^* \subset L(\Gamma) \bar{\otimes} L(\Gamma).$$

(1.1)

This is obtained in two steps. A main technical contribution of our paper is to use the rigidity of $\Gamma$ inherited from the product structure of $\Lambda_0$ through measure equivalence. We do this in Section 4 by introducing an ”amplified” version of the comultiplication map $\Delta$ which is defined on the larger von Neumann algebra $(L^\infty(Y_0) \bar{\otimes} L^\infty(X)) \rtimes \Gamma$. Combined with the spectral gap rigidity theorem for coinduced actions (Theorem 3.1) proved in Section 3 we obtain the conclusion (1.1).

In Section 5, following Ioana’s idea \cite{Io10}, we obtain a dichotomy theorem for certain abelian algebras. The result is a straightforward adaptation of \cite[Theorem 5.1]{IPV10} to coinduced actions and has two consequences. First, we obtain

$$\Delta(L^\infty(X))' \cap (M \bar{\otimes} M) \prec L^\infty(X) \bar{\otimes} L^\infty(X).$$

Second, it implies a weaker version of Popa’s conjugacy criterion adapted to coinduced actions. This will altogether prove Theorem B.

1.4. Acknowledgements. I am very grateful to my advisor Adrian Ioana for all the help given through valuable discussions. I would also like to thank Rémi Boutonnet for helpful comments about the paper.
2. Preliminaries

2.1. **Terminology.** A von Neumann algebra $M$ is called **tracial** if it is equipped with a faithful normal tracial state $\tau$. We denote by $L^2(M)$ the completion of $M$ with respect to the norm $\|x\|_2 = \sqrt{\tau(x^*x)}$. For $Q \subset M$, a unital von Neumann subalgebra of $M$, we denote by $e_Q : L^2(M) \to L^2(Q)$ the orthogonal projection onto $L^2(Q)$. We denote by $E_Q : M \to Q$, the conditional expectation onto $Q$. Jones’ basic construction of the inclusion $Q \subset M$ is defined as the von Neumann subalgebra of $\mathcal{B}(L^2(M))$ generated by $M$ and $e_Q$.

Denote by $\mathcal{U}(M)$ the group of unitary elements of $M$ and by $\mathcal{N}_M(Q) = \{u \in \mathcal{U}(M) | uu^* = Q\}$ the normalizer of $Q$ inside $M$. Denote also by $Q' \cap M = \{x \in M | xq = qx, \text{ for all } q \in Q\}$ the relative commutant of $Q$ in $M$ and by $\mathcal{Z}(M) = M \cap M'$ the center of $M$.

2.2. **Popa’s intertwining by bimodules.** We recall from [Po03] Theorem 2.1 and Corollary 2.3] Popa’s intertwining by bimodules tehnique which is fundamental to deformation/rigidity theory.

**Theorem 2.1.** [Po03] Let $P$ and $Q$ be (not necessarily unital) subalgebras of a tracial von Neumann algebra $M$. The following are equivalent:

- There exist non-zero projections $p \in P, q \in Q$, a $*$-homomorphism $\varphi : pMp \to qQq$ and a non-zero partial isometry $v \in pMp$ such that $xv = v\varphi(x)$, for all $x \in pMp$.
- For any group $\mathcal{U} \subset \mathcal{U}(P)$ such that $\mathcal{U}'' = P$ there is no sequence $\{u_n\} \subset \mathcal{U}$ such that $\|E_Q(xu_ny)\|_2 \to 0$, for all $x, y \in M$.

If one of these conditions holds true, then we write $P \preceq_M Q$ and we say that a corner of $P$ embeds into $Q$. If $Pp' \preceq_M Q$ for any non-zero projection $p' \in P' \cap pMp$, then we write $P \preceq^\times_M Q$.

2.3. **Bimodules and weak containment.** Let $M, N$ be tracial von Neumann algebras. An $M$-$N$-bimodule $M\mathcal{H}_N$ is a Hilbert space $\mathcal{H}$ equipped with two commuting normal unital $*$-homomorphisms $M \to B(\mathcal{H})$ and $N^\text{op} \to B(\mathcal{H})$. An $M$-$N$-bimodule $M\mathcal{H}_N$ is weakly contained in a $M$-$N$-bimodule $M\mathcal{K}_N$ and we write $M\mathcal{H}_N \subset_{\text{weak}} M\mathcal{K}_N$ if for any $\epsilon > 0$, finite subsets $F \subset M, G \subset N$ and $\xi \in \mathcal{H}$, there exist $\eta_1, \ldots, \eta_n \in \mathcal{K}$ such that

$$\|\langle x\eta_y, \xi \rangle - \sum_{i=1}^n \langle x\eta_i, \eta_i \rangle\| \leq \epsilon, \text{ for all } x \in F, y \in G.$$

Given two bimodules $M\mathcal{H}_N$ and $N\mathcal{K}_P$, one can define the Connes tensor product $\mathcal{H} \otimes_N \mathcal{K}$ which is an $M$-$P$ bimodule (see [Co94] V.Appendix B]). If $M\mathcal{H}_N \subset_{\text{weak}} M\mathcal{K}_N$, then $M\mathcal{H} \otimes_N \mathcal{L}_P \subset_{\text{weak}} M\mathcal{K} \otimes_N \mathcal{L}_P$, for any $N$-$P$ bimodule $\mathcal{L}$.

2.4. **Relative amenability.** Let $(M, \tau)$ be a tracial von Neumann algebra. Let $p \in M$ be a projection and $P \subset pMp, Q \subset M$ be von Neumann subalgebras. Following [OP07] Definition 2.2], we say that $P \subset pMp$ is amenable relative to $Q$ inside $M$ if there exists a positive linear functional $\varphi : p\langle M, e_Q \rangle \to \mathbb{C}$ such that $\varphi|_{pMp} = \tau$ and $\varphi$ is $P$-central. We say that $M$ is amenable if $M$ is amenable relative to $\mathcal{C}_1$ inside $M$.

By [OP07] Section 2.2, $P$ is amenable relative to $Q$ inside $M$ if and only if $M L^2(Mp)p$ is weakly contained in $M L^2(M\langle e_Q \rangle)p$.

A von Neumann subalgebra $P \subset pMp$ is strongly non-amenable relative to $Q$ if for all non-zero projections $p_1 \in P' \cap pMp$, the von Neumann algebra $p_1 P$ is non-amenable relative to $Q$.

For $B \subset M$ a von Neumann subalgebra, we have $L^2(M) \otimes_B L^2(M) \cong L^2(M\langle e_B \rangle)$ as $M$-$M$-bimodules. Note that $B$ is amenable if and only if $M L^2(M) \otimes_B L^2(M)_{\text{weak}} \subset_{\text{weak}} M L^2(M) \otimes L^2(M)_{\text{weak}}$. 


Recall that a countable group $\Gamma$ is amenable if and only if every unitary representation of $\Gamma$ is weakly contained in the left regular representation (\cite[Theorem G.3.2]{HV08}). The next lemma is the analogous statement for amenable von Neumann algebras. The result is likely well-known, but for a lack of reference, we include a proof.

**Lemma 2.2.** Let $A$ be a tracial von Neumann algebra. Then $A$ is amenable if and only if every $A$-$A$-bimodule $K$ is weakly contained in the coarse bimodule $L^2(A) \otimes L^2(A)$.

**Proof.** Suppose $A$ is amenable and let $K$ be an $A$-$A$-bimodule. Then the trivial bimodule $A L^2(A)_A$ is weakly contained in the coarse bimodule $A L^2(A) \otimes L^2(A)_A$. Since $L^2(A) \otimes_A K$ identifies with $K$ as $A$-$A$ bimodules, we obtain that

\begin{equation}
A K_A \subset_{\text{weak}} A L^2(A) \otimes K_A.
\end{equation}

Now, since any right module of $A$ is contained in $\bigoplus_N L^2(A)$ as a right $A$-submodule, we have that

\begin{equation}
C K_A \subset_{\text{weak}} C L^2(A)_A.
\end{equation}

Thus, (2.1) and (2.2) implies that $A K_A$ is weakly contained in the coarse $A$-$A$-bimodule. The converse is clear by taking $K = L^2(A)$, the trivial $A$-$A$-bimodule.

We end this subsection by recording an immediate corollary of \cite[Lemma 2.6]{DHI16}. We provide a proof for the reader’s convenience.

**Lemma 2.3.** \cite[Lemma 2.6]{DHI16} Let $P$ and $Q$ be two von Neumann subalgebras of a tracial von Neumann algebra $(M, \tau)$. If $P$ is non-amenable relative to $Q$, then there exists a non-zero projection $z \in \mathcal{N}_M(P)' \cap M$ such that $Pz$ is strongly non-amenable relative to $Q$.

**Proof.** Using Zorn’s lemma and a maximality argument, we can find a projection $z \in P' \cap M$ such that $Pz$ is strongly non-amenable relative to $Q$ and $P(1-z)$ is amenable relative to $Q$. Using \cite[Lemma 2.6]{DHI16} there exists $z_1 \in \mathcal{N}_M(P)' \cap M$ such that $1-z \leq z_1$ and $Pz_1$ is amenable relative to $Q$. Therefore, $P(z_1 - (1-z))$ is amenable relative to $Q$, which implies that $1-z = z_1 \in \mathcal{N}_M(P)' \cap M$.

\section{Intertwining of rigid algebras}

\subsection{The free product deformation for coinduced actions.} In \cite{Io06a} Ioana introduced a malleable deformation for general Bernoulli actions, where the base is any tracial von Neumann algebra. This is a variant of the malleable deformation discovered by Popa \cite{Po03} in the case of Bernoulli actions with abelian or hyperfinite base and it was used in the context of coinduced actions in \cite{Dr15}.

Coinduced actions for tracial von Neumann algebras are defined as in Section \cite{Io06a}. More precisely, let $\Gamma$ be a countable group and let $\Sigma$ be a subgroup. Let $\phi : \Gamma / \Sigma \to \Gamma$ be a section. Define the cocycle $c : \Gamma \times \Gamma / \Sigma \to \Sigma$ by the formula

\[ c(g, i) = \phi^{-1}(gi)g\phi(i), \]

for all $g \in \Gamma$ and $i \in \Gamma / \Sigma$.

Let $\Sigma \overset{\sigma_0}{\curvearrowright} (A_0, \tau_0)$ be a trace preserving action, where $(A_0, \tau_0)$ is a tracial von Neumann algebra. We define an action $\Gamma \overset{\sigma}{\curvearrowright} A_0^{\Gamma / \Sigma}$, called the coinduced action of $\sigma_0$, as follows:

\[ \sigma_\gamma(a_i)_{i \in \Gamma / \Sigma} = (a'_i)_{i \in \Gamma / \Sigma}, \text{ where } a'_i = (\sigma_0)c(\gamma^{-1}, i)\gamma^{-1}(a_{\gamma^{-1}i}). \]
Note that $\sigma$ is a trace preserving action of $\Gamma$ on the tracial von Neumann algebra $A_0^{\Gamma/\Sigma}$.

Consider the free product $A_0 \ast L(\mathbb{Z})$ with respect to the natural traces. Extend canonically $\sigma_0$ to an action on $A_0 \ast L(\mathbb{Z})$. Denote by $\tilde{M} = (A_0 \ast L(\mathbb{Z}))^{\Gamma/\Sigma} \rtimes_{\sigma_0} \Gamma$ the corresponding crossed product of the coinduced action $\Gamma \rtimes (A_0 \ast L(\mathbb{Z}))^{\Gamma/\Sigma}$ of $\sigma_0$.

Take $u \in L(\mathbb{Z})$ the canonical generating Haar unitary. Let $h = h^* \in L(\mathbb{Z})$ be such that $u = \exp(ih)$ and set $u_t = \exp(it h)$ for all $t \in \mathbb{R}$. Define the deformation $(\alpha_t)_{t \in \mathbb{R}}$ by automorphisms of $\tilde{M}$ by

$$\alpha_t(u_g) = u_g \quad \text{and} \quad \alpha_t(\otimes h \in \Gamma/\Sigma a_h) = \otimes h \in \Gamma/\Sigma \text{Ad}(u_t)(a_h),$$

for all $g \in \Gamma, t \in \mathbb{R}$ and $\otimes h \in \Gamma/\Sigma a_h \in (A_0 \ast L(\mathbb{Z}))^{\Gamma/\Sigma}$ an elementary tensor.

### 3.2. Spectral gap rigidity for coinduced actions.

**Theorem 3.1.** Let $\Gamma$ be an icc countable group and let $\Sigma$ be an almost malnormal subgroup. Let $\sigma_0$ be a pmp action of $\Sigma$ on a non-trivial standard probability space $(X_0, \mu_0)$. Denote by $M = L^\infty(X) \rtimes \Gamma$ the crossed-product von Neumann algebra of the coinduced action $\Gamma \rtimes (X_0, \mu)^{\Gamma/\Sigma}$ associated to $\Sigma \rtimes_{\sigma_0} (X_0, \mu)$. Let $N$ be an arbitrary tracial von Neumann algebra and suppose $Q \subset p(M \otimes N)p$ is a von Neumann subalgebra such that $Q' \cap p(M \otimes N)p$ is strongly non-amenable relative to $1 \otimes N$.

Then,

$$\sup_{b \in \mathcal{M}(Q)} \| (\alpha_t \otimes id)(b) - b \|_2 \text{ converges to } 0 \text{ as } t \to 0.$$

Theorem 3.1 and its proof are similar with other results from the literature [Po06, Lemma 5.1], [IPV10, Corollary 4.3] and especially with [BV12, Theorem 3.1] (where the generalized Bernoulli action might have non amenable stabilizers) and with [KV15, Theorem 2.6] (which is another version of this result for coinduced actions).

**Proof of Theorem 3.1.**

Put $\mathcal{M} := M \otimes N$ and $\tilde{M} := \tilde{M} \otimes N$. The proof of this theorem goes along the same lines as the proof of [BV12, Theorem 3.1]. Therefore, instead of working with the bimodule $\mathcal{M}L^2(\tilde{M} \otimes \mathcal{M})_\mathcal{M}$, we use the following $\mathcal{M}$-$\mathcal{M}$-submodule

$$\mathcal{K} := \mathcal{M} \{ (\otimes_{i \in \mathcal{F}} a_i)u_n \otimes n \mid \mathcal{F} \subset \Gamma/\Sigma \text{ with } k \leq |\mathcal{F}| < \infty, n \in \mathbb{N} \text{ and } g \in \Gamma \text{ for all } i \in \mathcal{F}, \quad a_i \in A_0 \ast L(\mathbb{Z}) \text{ for at least } k \text{ elements } i \in \mathcal{F} \}.$$ 

**Claim 1.** The $\mathcal{M}$-$\mathcal{M}$-bimodule $\mathcal{K}$ is weakly contained in the bimodule $L^2(\mathcal{M}) \otimes_{1 \otimes N} L^2(\mathcal{M})$.

**Proof of Claim 1.** Let $\mathcal{A} \subset A_0 \otimes \mathbb{C}$ be an orthonormal basis of $L^2(A_0) \otimes \mathbb{C}$ and denote by $u$ the canonical Haar unitary of $L(\mathbb{Z})$. Define the orthonormal set $\tilde{\mathcal{A}} \subset L^2(A_0 \ast L(\mathbb{Z})) \otimes L^2(A_0)$ by

$$\tilde{\mathcal{A}} := \{ u^{n_1}a_1u^{n_2}a_2 \ldots u^{n_{k-1}}a_{k-1}u^{n_k}a_k | k \geq 1, n_j \in \mathbb{Z} \setminus \{0\}, a_j \in \mathcal{A} \text{ for all } j \}$$

This gives us the following orthogonally decomposition of $L^2(A_0 \ast L(\mathbb{Z}))$ into $A_0$-$A_0$ submodules:

$$L^2(A_0 \ast L(\mathbb{Z})) = L^2(A_0) \oplus \bigoplus_{a \in \tilde{\mathcal{A}}} A_0aA_0. \quad (3.1)$$

If we denote

$$\mathcal{C} := \{ (\otimes_{i \in \mathcal{F}} c_i) \otimes 1 | \mathcal{F} \text{ finite }, k \leq |\mathcal{F}| < \infty, c_i \in \tilde{\mathcal{A}}, \text{ for all } i \in \mathcal{F} \},$$

then the decomposition (3.1) implies that the bimodule $\mathcal{K}$ can be written as the linear span $\mathcal{K} = \mathcal{M} \mathcal{C} \mathcal{M}$. To finish the proof of this claim, note that it is enough to consider an element
DANIEL DRIMBE

\[ c \in C \text{ and prove that the } M-M\text{-bimodule } \mathfrak{sp}McM \text{ is weakly contained in the coarse bimodule } L^2(M) \otimes L^2(M). \]

Let \( c = (\otimes_{i \in F} e_i) \otimes 1 \in C. \) We denote by \( \Gamma_0 := \{ g \in \Gamma | gf = f, \text{ for all } f \in F\}, \) the stabilizer of \( F \) for the action \( \Gamma \rhd \Gamma \otimes \Sigma \) and by \( \Gamma_1 := \{ g \in \Gamma | g \cdot F = F\}, \) the normalizer of \( F \) for the same action. Since \( \Sigma \) is \( k \)-almost malnormal and \( \Gamma_0 \) is a finite index subgroup of \( \Gamma_1, \) we obtain that \( \Gamma_1 \) is a finite group.

Denote \( P = A \rtimes \Gamma_1. \) Since \( P \) is amenable, Lemma \[2.2\] implies that the \( P-P\)-bimodule \( \mathfrak{sp}McM \) is weakly contained in the coarse bimodule \( L^2(P) \otimes L^2(P). \) Thus, for each \( \epsilon > 0, F \subset \Gamma_1 \text{ and } E \subset A \) finite subsets, there exist \( \eta_1, \eta_2, \ldots, \eta_n \in L^2(P) \otimes L^2(P) \) such that

\[
(3.2) \quad \| (au_g c (bu_h)^*)^* - \sum_{i=1}^n (au_g \eta_i (bu_h)^*)^* \| \leq \epsilon,
\]

for all \( g, h \in F \) and \( a, b \in E. \)

Using the canonical inclusion \( L^2(P) \subset L^2(M), \) we obtain that \( \| au_g \eta_i (bu_h)^* \| = 0, \) for all \( (g, h) \in (\Gamma \times \Gamma) \setminus (\Gamma_1 \times \Gamma_1) \) and \( a, b \in A. \) Note that also \( \| au_g c (bu_h)^* \| = 0, \) for all \( (g, h) \in (\Gamma \times \Gamma) \setminus (\Gamma_1 \times \Gamma_1) \) and \( a, b \in A. \) Using these observations together with \[3.2\], we obtain that the \( M-M\)-bimodule \( \mathfrak{sp}McM \) is weakly contained in the coarse bimodule \( L^2(M) \otimes L^2(M). \) This finishes the proof of the claim.

Denote by \( P_K \) the orthogonal projection of \( L^2(\hat{M}) \) onto the closed subspace \( K. \)

**Claim 2.** \( \sup_{b \in U(Q)} \| P_K(\alpha_t \otimes \text{id})(b) \|_2 \) converges to 0 as \( t \to 0. \)

**Proof of Claim 2.** Suppose the claim is false. Then there exist \( \delta > 0, \) a sequence of positive numbers \( t_n \to 0, \) as \( n \to \infty, \) and a sequence of unitaries \( b_n \in U(Q) \) such that \( \sup_{b \in U(Q)} \| P_K((\alpha_t \otimes \text{id})(b_n)) \|_2 \geq \delta, \) for all \( n \geq 1. \)

Define \( \xi_n = P_K((\alpha_t \otimes \text{id})(b_n)). \) For all \( x \in \mathcal{Q}' \cap pMp, \) we have \( \| \xi_n x - x \xi_n \| \to 0, \) as \( n \to \infty. \) Note also that \( \lim \inf_{n \to \infty} \| \xi_n \|_2 \geq \delta \) and \( \| x \xi_n \|_2 \leq \| x \|_2, \) for all \( x \in \mathcal{M}. \) Then, \[Ho15\] Lemma 2.3 implies that there exists a projection \( q \in \mathcal{Z}(\mathcal{Q}' \cap pMp) \) such that the \( \mathcal{M}-(\mathcal{Q}' \cap pMp)q \) bimodule \( L^2(Mq) \) is weakly contained in \( K. \) Claim 1 implies now that the \( \mathcal{M}-(\mathcal{Q}' \cap pMp)q \) bimodule \( L^2(Mq) \) is weakly contained in the bimodule \( L^2(\mathcal{M}) \otimes_{1 \otimes N} L^2(M). \) This implies that \( (\mathcal{Q}' \cap pMp)q \) is amenable relative to \( 1 \otimes N \) inside \( \mathcal{M}, \) which contradicts the hypothesis. This proves the claim.

In order to finish the proof of the theorem we need a variant of Popa’s transversality property. In the proof of \[BV12\] Theorem 3.1 it is proven the following fact for generalized Bernoulli actions: if \( \sup_{b \in U(Q)} \| P_K((\alpha_t \otimes \text{id})(b)) \|_2 \) converges to 0 as \( t \to 0, \) then \( \sup_{b \in U(Q)} \| (\alpha_t \otimes \text{id})(b) - b \|_2 \) converges to 0 as \( t \to 0. \) With the same proof we obtain the same result for coinduced actions. Claim 2 completes now the proof of the theorem.

For \( Q \subset M \) a von Neumann subalgebra, we define \( \mathcal{Q}N_M(Q) \subset M \) to be the set of all elements \( x \in M \) for which there exist \( x_1, \ldots, x_n, y_1, \ldots, y_n \) satisfying \( xQ \subset \sum_{i=1}^n Qx_i \) and \( Qx \subset \sum_{i=1}^n y_iQ. \) The weak closure of \( \mathcal{Q}N_M(Q) \) is called the **quasi-normalizer of \( Q \) inside \( M \)** and note that it is a von Neumann subalgebra of \( M \) which contains both \( Q \) and \( Q' \cap M. \)

The proof of \[IPV10\] Theorem 4.2 carries over verbatim and gives us the following result.

**Theorem 3.2.** Let \( \Gamma \) be an icc countable group and let \( \Sigma \) be an almost malnormal subgroup. Let \( \sigma_0 \) be a pmp action of \( \Sigma \) on a non-trivial standard probability space \( (X_0, \mu_0). \) Denote by \( M = L^\infty(X) \rtimes \Gamma \) the cross-product von Neumann algebra of the coinduced action \( \Gamma \rhd (X_0, \mu) \otimes \Sigma \) associated to \( \Sigma \otimes (X_0, \mu). \) Let \( N \) be a II_1 factor and suppose \( Q \subset p(M \otimes N)p \) is a von Neumann subalgebra. Denote by \( P \) the quasi-normalizer of \( Q \) in \( p(M \otimes N)p. \)
If there exist $0 < t < 1$ and $\delta > 0$ such that
\[
\tau(b^*(\alpha_t \otimes \text{id})(b)) \geq \delta, \text{ for all } b \in \mathcal{U}(Q),
\]
then one of the following statements is true:

- $Q < \mathcal{1} \subset N$,
- $P < (A \rtimes \Sigma)\subset N$,
- there exists a unitary $u \in M\bar{\otimes}N$ such that $uPu^* \subset L(\Gamma)\otimes N$.

### 3.3. Controlling intertwiners and relative commutants

In the Appendix of his PhD thesis [Bo14], Boutonnet has presented a unified approach to the notion of mixing for von Neumann algebras. As a consequence, we obtain results which give us good control over intertwiners between certain subalgebras of von Neumann algebras arising from coinduced actions.

#### Definition 3.3.
Let $A \subset N \subset M$ be an inclusion of finite von Neumann algebras. We say that the inclusion $N \subset M$ is mixing relative to $A$ if for any sequence of unitaries $\{x_n\} \subset \mathcal{U}(N)$ with $\|E_A(yx_nz)\|_2 \to 0$ for all $y, z \in N$, we have
\[
\|E_N(m_1x_nm_2)\|_2 \to 0 \text{ for all } m_1, m_2 \in M \otimes N.
\]

#### Proposition 3.4. [Bo14] Appendix A
Let $A \subset N \subset M$ be an inclusion of finite von Neumann algebras such that $N \subset M$ is mixing relative to $A$. Let $Q \subset pMp$ be a subalgebra such that $Q \not\subset_M A$. Denote by $P$ the quasi-normalizer of $Q$ in $pMp$.

1. If $Q \subset N$, then $P \subset N$.
2. If $Q \prec N$, then there exists a non-zero partial isometry $v \in pM$ such that $vv^* \subset P$ and $v^*Pv \subset N$.
3. If $N$ is a factor and if $Q \prec_M N$, then there exists a unitary $u \in \mathcal{U}(M)$ such that $uPu^* \subset N$.

#### Lemma 3.5.
Let $\Sigma$ be a subgroup of a countable group $\Gamma$. Let $\Sigma \lhd \prec A_0$ be a tracial action on a non-trivial von Neumann algebra $A_0$ and let $\Gamma \lhd \prec A := A_0^{\Gamma/\Sigma}$ be the coinduced action of $\sigma_0$. Let $\Gamma \lhd \prec C$ be another tracial action. Then $C \times \Gamma \subset (C\bar{\otimes}A) \times \Gamma$ is mixing relative to $C \times \Sigma$.

**Proof.** Denote $M := (C\bar{\otimes}A) \times \Gamma$ and $I := \Gamma/\Sigma$. Let $\{x_n\} \subset \mathcal{U}(C \times \Gamma)$ be a sequence of unitaries such that $\|E_{C\times\Gamma}(yx_nz)\|_2 \to 0$, for all $y, z \in C \times \Gamma$. Let $a, b \in M \subset (C \times \Gamma)$. We have to show that $\|E_{C\times\Gamma}(ax_nb)\|_2 \to 0$. Since $E_{C\times\Gamma}$ is $C \times \Gamma$-bimodular, we can assume $a, b \in A$. Moreover, we can suppose that there exist a finite subset $J \subset I$ and $\zeta \in \mathcal{J}$ such that $a = \bigotimes_{j \in J} bj_j \in A_0$ with $b_j \in A_0 \otimes \mathcal{C}$. If $j_0 = g_0\Sigma$ and $J = \{g_1\Sigma, \ldots, g_n\Sigma\}$ note that $\Sigma_0 := \{g \in \Gamma | g_0 \in J\} = \cup_{i=1}^n g_i\Sigma g_0^{-1}$. Now, since $ax_nb = \sum_{g \in \Sigma_0} aE_C(x_nu_g^*\sigma_g(b)u_g$, we have
\[
\|E_{C\times\Gamma}(ax_nb)\|_2^2 = \sum_{g \in \Sigma_0} |\tau(\sigma_g(b))|^2\|E_C(x_nu_g^*)\|_2^2 \leq \|a\|_2^2\|b\|_2^2 \sum_{i=1}^n\|E_{C\times\Sigma}(u_g^*x_nu_g)\|_2^2,
\]
which goes to zero because of the assumption. This proves the lemma. \[\qed\]

Proposition 3.3 together with Lemma 3.5 give the following result.

#### Corollary 3.6.
Let $\Sigma$ be a subgroup of a countable group $\Gamma$. Let $\Sigma \lhd \prec A_0$ be a tracial action on a non-trivial von Neumann algebra $A_0$ and let $\Gamma \lhd \prec A := A_0^{\Gamma/\Sigma}$ be the coinduced action of $\sigma_0$. Let $\Gamma \lhd \prec C$ be another tracial action and let $N$ be an arbitrary factor. Define $M := (C\bar{\otimes}A) \times \Gamma$. Suppose $Q \subset p(M\bar{\otimes}N)p$ is a von Neumann subalgebra such that $Q \not\subset (C \times \Sigma)\bar{\otimes}N$. Denote by $P$ the quasi-normalizer of $Q$ inside $p(M\bar{\otimes}N)p$.

1. If $Q \subset p((C \times \Gamma)\bar{\otimes}N)p$, then $P \subset p((C \times \Gamma)\bar{\otimes}N)p$. 

Throughout this section we will use many times the following easy observation:

\[\forall v \in P \text{ and } v^*Pv \subset (C \times \Gamma) \bar{\otimes} N.\]

The proof of the following proposition is similar to [Bo12a, Corollary 3.7] and we leave it to the reader.

**Proposition 3.7.** Let \(\Gamma \lhd C\) be a tracial action and denote \(M_0 = C \times \Gamma\). Let \(\Sigma\) be an almost malnormal subgroup of \(\Gamma\). Suppose \(Q \subset pM_0p\) is a von Neumann subalgebra such that \(Q \lhd C \times \Sigma\) and \(Q \neq C\). Denote by \(P\) the quasi-normalizer of \(Q\) inside \(pM_0p\).

Then \(P \lhd C \times \Sigma\).

### 4. Rigidity coming from measure equivalence

In this section we establish some results needed in the proof of Theorem B. Throughout the section, we will work with coinduced actions satisfying the following:

**Assumption 4.1.** Let \(\Sigma\) be a subgroup of a countable icc group \(\Gamma\). Let \(\sigma_0\) be a pmp action of \(\Sigma\) on a non-trivial standard probability space \((X_0, \mu_0)\) and denote by \(\sigma\) the coinduced action of \(\Gamma\) on \(X := X_0^{\Gamma/\Sigma}\). Suppose:

- \(\Gamma\) is a non-amenable icc group which is measure equivalent to a group \(\Lambda_0\) for which the group von Neumann algebra \(L(\Lambda_0)\) is not prime.
- \(\Sigma\) is almost malnormal.

Note that since \(\Sigma\) is almost malnormal in \(\Gamma\), we have that \([\Gamma : \Sigma] = \infty\). Before stating the results of this section, we need to introduce some notation.

**Notation 4.2.** The group von Neumann algebra \(L(\Lambda_0)\) is not prime, therefore there exist von Neumann algebras \(R_1\) and \(R_2\), both not of type I, such that \(L(\Lambda_0) = R_1 \bar{\otimes} R_2\). Since \(L(\Lambda_0)\) is diffuse and non-amenable, there exists \(z_0 \in \bar{Z}(L(\Lambda_0))\) such that \(R_1z_0\) and \(R_2z_0\) are diffuse and \(L(\Lambda_0)z_0\) is non-amenable.

The group \(\Gamma\) is measure equivalent to \(\Lambda_0\). By [Fu99, Lemma 3.2], \(\Gamma\) and \(\Lambda_0\) admit stably orbit equivalent free ergodic pmp actions. Thus, we may find a free ergodic pmp action \(\Gamma \lhd (Z_0, \nu)\) and \(\ell \geq 1\), such that the following holds: consider the product action \(\Gamma \times \mathbb{Z}/\ell\mathbb{Z} \lhd (Z_0 \times \mathbb{Z}/\ell\mathbb{Z}, \nu \times c)\), where \(\mathbb{Z}/\ell\mathbb{Z}\) acts on itself by addition and \(c\) denotes the counting measure on \(\mathbb{Z}/\ell\mathbb{Z}\). Then there exist a non-negligible measurable set \(Y_0 \subset Z_0 \times \mathbb{Z}/\ell\mathbb{Z}\) and a free ergodic measure preserving action \(\Lambda_0 \lhd Y_0\) such that

\[R(\Lambda_0 \lhd Y_0) = R(\Gamma \times \mathbb{Z}/\ell\mathbb{Z} \lhd Z_0 \times \mathbb{Z}/\ell\mathbb{Z}) \cap (Y_0 \times Y_0).\]

We put \(C_0 = L^\infty(Y_0), M_0 = L^\infty(Z_0 \times \mathbb{Z}/\ell\mathbb{Z}) \times (\Gamma \times \mathbb{Z}/\ell\mathbb{Z}), p = 1_{Y_0}\), and note that \(C_0 \times \Lambda_0 = pM_0p\).

We identify \(L^\infty(Z_0 \times \mathbb{Z}/\ell\mathbb{Z}) \times \mathbb{Z}/\ell\mathbb{Z} = M_\ell(C), \) and use this identification to write \(M_0 = C \times \Gamma\), where \(C = L^\infty(Z_0) \otimes M_\ell(C)\) and \(\Gamma\) acts trivially on \(M_\ell(C)\).

Denote \(A = L^\infty(X)\) and let \(\{u_g\}_{g \in \Gamma} \subset (C \bar{\otimes} A) \rtimes \Gamma\) denote the canonical unitaries implementing the diagonal action of \(\Gamma\) on \(C \bar{\otimes} A\).

**Remark 4.3.** Throughout this section we will use many times the following easy observation (see [Va08, Lemma 3.4]). Let \(P \subset pM_0p\) and \(Q \subset qMq\) be von Neumann subalgebras of a tracial von Neumann algebra \((M, \tau)\). Then:

- if \(p_0Pp_0 \lhd Q\) for a non-zero projection \(p_0 \in P\), then \(P \lhd Q\).
Lemma 4.4. Let \( w : \Gamma \to \mathcal{U}(A \bar{\otimes} N) \) be a cocycle for the action \( \sigma \otimes \text{id} \), where \( N \) is a II_1 factor. Define the \(*\)-homomorphism \( d : C \times \Gamma \to (A \times \Gamma) \bar{\otimes} N \bar{\otimes} (C \times \Gamma) \) by \( d(cu_g) = w_g u_g \otimes c u_g, g \in \Gamma, c \in C \). Let \( Q \subset pM_p \) be a subalgebra and let \( \Sigma_0 \subset \Gamma \) be a subgroup. The following hold:

1. If \( Q \not\subset C \), then \( d(Q) \neq 1 \otimes N \bar{\otimes} (C \times \Gamma) \).
2. If \( |\Gamma : \Sigma_0| = \infty \), then \( d(L(\Lambda_0)) \neq (A \times \Sigma_0) \bar{\otimes} N \bar{\otimes} (C \times \Gamma) \).
3. If \( Q \) is non-amenable, then \( d(Q) \) is non-amenable relative to \( 1 \otimes N \bar{\otimes} (C \times \Gamma) \).

Proof. Denote \( \mathcal{M} = (A \times \Gamma) \bar{\otimes} N \bar{\otimes} M_0 \) and \( \mathcal{N} = 1 \otimes N \bar{\otimes} M_0 \).

1. Let \( \{u_n\}_{n \geq 1} \subset \mathcal{U}(Q) \) be a sequence of unitaries such that \( \|E_C(u_n u_g)\|_2 \to 0 \), for all \( g \in \Gamma \). We claim that

\[
\|E_{1 \otimes N \bar{\otimes} M_0}(xd(u_n)y)\|_2 \to 0,
\]

for all \( x, y \in \mathcal{M} \).

Since \( E_N \) is \( \mathcal{N} \)-bimodular, by Kaplansky’s density theorem we may assume \( x = au_g \otimes 1 \otimes 1 \), \( y = bu_h \otimes 1 \otimes 1 \) for some \( a, b \in A \) and \( g, h \in \Gamma \). Then for all \( n \geq 1 \), we have

\[
xd(u_n)y = \sum_{k \in \Gamma} a_{\sigma_g(w_k)}\sigma_{\sigma h}(b) u_{ghk} \otimes E_{C}(u_n u^*_k) u_k.
\]

Therefore, \( \|E_N(xd(u_n)y)\|_2 \leq \|a\|\|b\|\|E_C(u_n u^*_h)\|_2 \to 0 \).

2. Assume \( d(L(\Lambda_0)) \subset (A \times \Sigma_0) \bar{\otimes} N \bar{\otimes} M_0 \). Since \( d(C_0) \subset 1 \otimes 1 \otimes C_0 \), we obtain \( d(C_0 \times \Lambda_0) \subset (A \times \Sigma_0) \bar{\otimes} N \bar{\otimes} M_0 \). Therefore \( d(L(\Gamma)) \subset (A \times \Sigma_0) \bar{\otimes} N \bar{\otimes} M_0 \), which implies \( L(\Gamma) \subset L(\Sigma_0) \). Indeed, suppose by contrary that \( L(\Gamma) \not\subset L(\Sigma_0) \).

Then there exists a sequence \( u_n \in \mathcal{U}(L(\Gamma)) \) such that \( \|E_{L(\Sigma_0)}(uv_n)\|_2 \to 0 \), for all \( x, y \in L(\Gamma) \). We would like to prove that

\[
\|E_{L(\Sigma_0) \bar{\otimes} N M_0}(xd(u_n)y)\|_2 \to 0,
\]

for all \( x, y \in (A \times \Gamma) \bar{\otimes} N \bar{\otimes} M_0 \). For proving (4.1), it is enough to consider \( x = u_g \otimes 1 \otimes 1 \) and \( y = u_h \otimes 1 \otimes 1 \), with \( g, h \in \Gamma \). In this case one can check that

\[
\|E_{(A \times \Sigma_0) \bar{\otimes} N M_0}(xd(u_n)y)\|_2 = \|E_{L(\Sigma_0)}(u_g u_n u_h)\|_2,
\]

which goes to 0. Therefore (4.1) is proven and we obtain that \( d(L(\Gamma)) \not\subset (A \times \Sigma_0) \bar{\otimes} N \bar{\otimes} M_0 \), contradiction.

Thus \( L(\Gamma) \subset L(\Sigma_0) \), which implies that \( \Sigma_0 \) has finite index in \( \Gamma \) by [DHI06 Lemma 2.5].

3. Suppose by contrary that \( d(Q) \) is amenable relative to \( \mathcal{N} \). Then there exists a positive linear functional \( \phi : (p)(\mathcal{M}, e_N) d(p) \to \mathbb{C} \) such that \( \phi_d(p, M_d(p)) = \tau \) and \( \phi \) is \( d(Q) \)-central. Define now \( \varphi : p(M_0, e_C)p \to \mathbb{C} \) by

\[
\varphi(\sum_{i=1}^{N} m_i e_{C n_i}) = \phi(\sum_{i=1}^{N} d(m_i) e_{N d(n_i))},
\]

where \( N \geq 1, m_i, n_i \in M_0, i \in \{1, \ldots, N\} \). Note that \( \varphi \) is a well defined positive linear functional. Indeed, suppose \( \sum_{i=1}^{N} m_i e_{C n_i} = 0 \), with \( m_i, n_i \in M_0 \), for all \( 1 \leq i \leq N \). This implies \( \sum_{i=1}^{N} d(m_i) \tau(n_i) = 0 \). Since \( E_N(d(m)) = \tau(m) \), for all \( m \in M_0 \), we obtain \( \sum_{i=1}^{N} d(m_i) E_N(d(n_i)) = 0 \), which implies \( \sum_{i=1}^{N} m_i e_{N d(n_i)} = 0 \). Therefore, \( \varphi \) is a positive linear functional which is \( Q \)-central and \( \varphi |_{pM_0p} = \tau \). We obtain \( Q \) is amenable, contradiction. 

Denote by \( \mathcal{U}_{\text{fin}} \) the class of Polish groups which arise as closed subgroups of the unitary groups of II_1 factors [FP05]. In particular, all countable discrete groups and all compact Polish groups belong to \( \mathcal{U}_{\text{fin}} \).
Theorem 4.5. (Cocycle superrigidity.) Let $\Gamma \curvearrowright X$ be as in Assumption 4.1. Then any cocycle $w : \Gamma \times X \to \Lambda$ valued in a group $\Lambda \in \mathcal{U}_{\text{fin}}$ untwists, i.e., there exists a measurable map $\varphi : X \to \Lambda$ and a group homomorphism $d : \Gamma \to \Lambda$ such that $w(g, x) = \varphi(gx)d(g)\varphi(x)^{-1}$ for all $g \in \Gamma$ and $x \in X$.

This result was proven in [PS09] for Bernoulli actions using deformations obtained from closable derivations. In our case, we will provide a direct proof for Theorem 4.5 which uses only the free product deformation $\alpha_t$ defined in Section 3.1.

Proof. Define $A := L^\infty(X)$ and let $N$ be a II$_1$ factor such that $\Lambda \subset \mathcal{U}(N)$. We associate to $w : \Gamma \times X \to \mathcal{U}(N)$ the cocycle $w : \Gamma \to \mathcal{U}(A \bar{\otimes} N)$, given by $w_g(x) = w(g, g^{-1}x)$. Define $Q = \{(w_g u_g)''\}_{g \in \Gamma}$.

Claim. We have $\sup_{b \in \mathcal{U}(Q)} \|(\alpha_t \otimes \text{id})(b) - b\|_2$ converges to 0 as $t \to 0$.

Proof of the Claim. As in Lemma 4.4 we define the $*$-homomorphism $d : C \otimes \Gamma \to (A \otimes \Gamma) \bar{\otimes} N \bar{\otimes} (C \otimes \Gamma)$ by $d(c u_g) = w_g u_g \otimes c u_g, g \in \Gamma, c \in C$. Denote $M = (A \otimes \Gamma) \bar{\otimes} N \bar{\otimes} M_0$. Without loss of generality assume that $R_1 z_0$ is non-amenable. Lemma 4.4 implies that $d(R_1 z_0)$ is non-amenable relative to $1 \otimes N \bar{\otimes} (C \otimes \Gamma)$. By Lemma 2.3 there exists a non-zero projection $z \in N_{d(z_0) M d(z_0)} d(R_1 z_0)' \cap d(z_0) M d(z_0)$ such that $d(R_1) z$ is strongly non-amenable relative to $1 \otimes N \bar{\otimes} (C \otimes \Gamma)$. Using Theorem 3.1 we obtain that $\sup_{b \in \mathcal{U}(d(R_2) z)} \|(\alpha_t \otimes \text{id} \otimes \text{id})(b) - b\|_2$ converges to 0 as $t \to 0$.

and therefore by Theorem 3.2 we obtain that one of the following hold:

1. $d(R_2) z < 1 \otimes N \bar{\otimes} (C \otimes \Gamma)$,
2. $d(L(\Lambda_0)) z < (A \otimes \Sigma) \bar{\otimes} N \bar{\otimes} (C \otimes \Gamma)$,
3. $d(L(\Lambda_0)) z < L(\Gamma) \bar{\otimes} N \bar{\otimes} (C \otimes \Gamma)$.

Note that (1) and (2) are not possible by Lemma 4.4 since $R_2 z_0$ is diffuse and $[\Gamma : \Sigma] = \infty$. Therefore (3) is true.

Now, together with the remark that $d(C) \subset 1 \otimes 1 \otimes C$ we obtain that $d(C \otimes \Gamma) < L(\Gamma) \bar{\otimes} N \bar{\otimes} (C \otimes \Gamma)$. One can check directly this fact or use [BV12] Lemma 2.3. Proceeding in the same way, we obtain actually $d(C \otimes \Gamma) \not< M L(\Gamma) \bar{\otimes} N \bar{\otimes} (C \otimes \Gamma)$. Lemma 4.4 implies that $d(C \otimes \Gamma) \not< L(\Sigma) \bar{\otimes} N \bar{\otimes} (C \otimes \Gamma)$, so by Corollary 3.6 we obtain $\sup_{b \in \mathcal{U}(Q)} \|(\alpha_t \otimes \text{id})(b) - b\|_2$ converges to 0 as $t \to 0$.

Proof.

Using a result which goes back to Popa [Po05], the claim implies that the cocycle $w$ untwists (see [Dr15] Theorem 2.15, the proof of [Dr15] Proposition 3.2 and [Dr15] Remark 3.3]).

Theorem 4.6. Let $\Gamma \curvearrowright X$ be as in Assumption 4.1 and suppose that $\Sigma$ is amenable. Let $\Lambda \curvearrowright B$ be a tracial action on a non-trivial abelian von Neumann algebra $B$ such that $A \otimes \Gamma = B \otimes \Lambda$. Denote by $\Delta : B \times \Lambda \to (B \otimes \Lambda) \bar{\otimes} L(\Lambda)$ the comultiplication $\Delta(b \psi) = b \psi \otimes \psi$ for all $b \in B$ and $\lambda \in \Lambda$ (we let $\{\psi\}_\lambda \subset B \times \Lambda$ denote the canonical unitaries implementing the action of $\Lambda$ on $B$).

Then there exists a unitary $u \in \mathcal{U}((A \otimes \Gamma) \bar{\otimes} (A \otimes \Gamma))$ such that $u \Delta(L(\Gamma)) u^* \subset L(\Gamma \otimes \Gamma)$.
Define $M := (C \otimes A) \rtimes \Gamma$ and $\theta : M \to M \otimes M \otimes M$ by $\theta(c a u_g) = c a u_g \otimes \Delta(a u_g)$, for all $c \in C, a \in A$ and $g \in \Gamma$. In the following lemma we record some properties of the unital $*$-homomorphism $\theta$ which are similar to the ones of \cite{Io10} Lemma 10.2).

**Lemma 4.7.** Let $Q \subset q M q$. The following hold:

1. If $Q$ is diffuse, then $\theta(Q) \not\prec M \otimes 1 \otimes M$.
2. If $Q \not\prec B$, then $\theta(Q) \not\prec M \otimes M \otimes 1$.
3. If $Q$ has no amenable direct summand, then $\theta(Q)$ is strongly non-amenable relative to $M \otimes M \otimes 1$ and $M \otimes 1 \otimes M$.

We continue now with the proof of Theorem 4.6 and we will give the proof of Lemma 4.7 at the end of this section.

**Proof of Theorem 4.6.** Without loss of generality we can assume that $R_1 z_0$ is non-amenable. Take $z \in \mathcal{Z}(R_1 z_0)$ such that $R_1 z$ has no amenable direct summand.

**Claim 1.** We have $\sup_{\theta \in \mathcal{U}(\Delta(L(\Gamma)))} \| (id \otimes \alpha_t)(b) - b\|_2$ converges to 0 as $t \to 0$.

**Proof of Claim 1.** Note that $\theta(R_1 z)$ is strongly non-amenable relative to $M \otimes M \otimes 1$ by Lemma 4.7. Therefore by Theorem 3.1 we obtain

$$\sup_{b \in \mathcal{U}(\theta(R_1 z))} \| (id \otimes id \otimes \alpha_t)(b) - b\|_2$$ converges to 0 as $t \to 0$.

Using Theorem 3.2 we obtain that one of the following three conditions holds:

1. $\theta(R_2 z) \prec M \otimes M \otimes 1$,
2. $\theta(L(A_0) z) \prec M \otimes M \otimes (A \rtimes \Sigma)$,
3. there exists a unitary $u \in M$ such that $u \theta(L(A_0) z) u^* \subset M \otimes M \otimes L(\Gamma)$.

If (1) holds,Lemma 4.7 implies $R_2 z \prec_M B$. By applying \cite{Va08} Lemma 3.5, we obtain $B \prec_M z M z \cap (R_2 z)'$. Note that if $R_2 z \prec_M C \rtimes \Sigma$, Proposition 3.6 implies that $L(A_0) \prec C \rtimes \Sigma$. Using \cite{BV12} Lemma 2.3 we deduce that $C \rtimes \Sigma \prec_M C \rtimes \Sigma$. This is a contradiction since $[\Gamma : \Sigma] = \infty$.

Therefore $R_2 z \not\prec_M C \rtimes \Sigma$ and Corollary 3.6 implies that $z M z \cap (R_2 z)' \subset C \rtimes \Gamma$, so $B \not\prec_M C \rtimes \Gamma$. On the other hand, since $B \subset A \rtimes \Gamma$, we obtain $B \not\prec_{A \rtimes \Gamma} L(\Gamma)$. Proposition 3.7 implies that $B \not\prec_{A \rtimes \Gamma} L(\Sigma)$. Finally, using Corollary 3.6 we obtain that $A \rtimes \Gamma \not\prec_{A \rtimes \Gamma} L(\Gamma)$, which is a contradiction.

Now, if (2) holds, we obtain $\theta(L(A_0)) \prec_M M \otimes M \otimes (A \rtimes \Sigma)$. Together with $\theta(C) \subset C \otimes 1 \otimes 1$, we obtain $\theta(M_0) \prec_M M \otimes M \otimes (A \rtimes \Sigma)$. Since $\Sigma$ is amenable, it implies that $\theta(M_0)$ is not strongly non-amenable relative to $M \otimes M \otimes 1$. Now, $M_0$ is a factor, so Lemma 4.7 gives that $M_0$ is amenable, which is a contradiction.

Thus, (3) holds. Since $\theta(C_0) \subset C_0 \otimes 1 \otimes 1$, we obtain

$$\theta(M_0) \prec_M M \otimes M \otimes L(\Gamma)$$

With the same computation, we obtain $\theta(M_0) \prec_M M \otimes M \otimes L(\Sigma)$.

Lemma 4.7 implies that $\theta(M_0) \not\prec_M M \otimes M \otimes L(\Sigma)$ since $\Sigma$ is amenable and $M_0$ is a factor. By Corollary 3.6 we obtain that $\sup_{\theta \in \mathcal{U}(\Delta(L(\Gamma)))} \| (id \otimes \alpha_t)(b) - b\|_2$ converges to 0 as $t \to 0$. \qed

**Claim 2.** We have $\sup_{\theta \in \mathcal{U}(\Delta(L(\Gamma)))} \| (\alpha_t \otimes id)(b) - b\|_2$ converges to 0 as $t \to 0$.

**Proof of Claim 2.** As in Claim 1, by applying Lemma 4.7 Theorem 3.1 and Theorem 3.2 we obtain that one of the following conditions hold:

1. $\theta(R_2 z) \prec_M 1 \otimes M$,
2. $\theta(L(A_0) z) \prec M \otimes (A \rtimes \Sigma) \otimes M$,
(3) there exists a unitary $u \in \mathcal{M}$ such that $u\theta(L(\Lambda_0))z u^* \subset M \bar{\otimes} L(\Gamma) \bar{\otimes} M$.

Note that by Lemma 4.7 (1) is not possible since $R_2z$ is diffuse. As before, (2) is not possible, which implies (3) holds true and by reasoning as before we obtain the claim. \hfill \Box

Notice that $\Delta(L(\Gamma))$ is a factor since $\Gamma$ is icc. Using Claim 1 and 2 and by applying twice Theorem 3.2 and [PV10, Lemma 10.2] we obtain the conclusion. \hfill ■

**Proof of Lemma 4.4.** The proofs of (1) and (2) are similar to the proof of Lemma 4.1 (see also the proof of [PV10, Lemma 10.2]). For proving (3), denote $\mathcal{M} := M \bar{\otimes} M \bar{\otimes} (A \times \Gamma)$ and $\psi : M \rightarrow M \bar{\otimes} M$, by $\psi(cau_g) = cu_g \otimes au_g$ for all $c \in C, a \in A$ and $g \in \Gamma$.

**Claim 1.** We have $\mathcal{M}L^2(\mathcal{M}) \otimes_{M \bar{\otimes} M \bar{\otimes} 1} L^2(\mathcal{M})_{\theta(M)} \subseteq \mathcal{M}L^2(\mathcal{M}) \otimes L^2(A \times \Gamma)_{\psi(M)_{1,2,4}}$, (here we consider that $\psi(M) \subset M \bar{\otimes} M$ acts to the right on $L^2(\mathcal{M}) \otimes L^2(M) \otimes L^2(\mathcal{M}) \otimes L^2(M)$ on the first and fourth positions.)

**Proof of the Claim 1.** Note that we have the identification

$$\mathcal{M}L^2(\mathcal{M}) \otimes_{M \bar{\otimes} M \bar{\otimes} 1} L^2(\mathcal{M})_{\theta(M)} \simeq_{M \bar{\otimes} M \bar{\otimes} 1} L^2(\mathcal{M}) \otimes L^2(A \times \Gamma)_{\theta(M)_{1,2,4}}$$

as $\mathcal{M}$-$\mathcal{M}$-bimodules. Therefore, it is enough to show that

$$L^2(\mathcal{M}) \otimes L^2(A \times \Gamma)_{\theta(M)} \subseteq \mathcal{M}L^2(\mathcal{M}) \otimes L^2(M) \otimes L^2(\mathcal{M}) \otimes L^2(M)_{\psi(M)_{1,3}}$$

Let $B$ be an orthonormal basis for $L^2(B)$ and note that we have the following orthogonal decomposition into $(\mathcal{M})$-$\mathcal{M}$-bimodules:

$$L^2(\mathcal{M}) \otimes L^2(A \times \Gamma) = \bigoplus_{b \in B} \mathcal{M}L^2(\mathcal{M}) \otimes L^2(A \times \Gamma)_{\theta(M)}$$

First, notice that for a fixed $b \in B$ we have

$$\mathcal{M}L^2(\mathcal{M}) \otimes L^2(M) \otimes L^2(A \times \Gamma)_{\psi(M)_{1,3}} \subseteq \mathcal{M}L^2(\mathcal{M}) \otimes L^2(\mathcal{M}) \otimes B L^2(A \times \Gamma)_{\psi(M)_{1,3}}$$

as $(\mathcal{M})$-$\mathcal{M}$-bimodules. Indeed, let $m_1, m_2, m_3 \in M$ and let us prove that

$$\langle (m_1 \otimes m_2 \otimes 1)(1 \otimes 1 \otimes b)\theta(m_3), 1 \otimes 1 \otimes b \rangle = \langle (m_1 \otimes m_2 \otimes 1)(1 \otimes 1 \otimes B 1)\psi(m_3), 1 \otimes 1 \otimes B 1 \rangle$$

We may assume $m_3 = cau_g$ for some $c \in C, a \in A$ and $g \in \Gamma$. Write $au_g = \sum_{l \in \Lambda} b_lv_l \in B \times \Lambda$, with $b_l \in B$ for all $l \in \Lambda$. Therefore, the LHS of (4.3) equals to

$$\tau((m_1 \otimes m_2 \otimes b^* b)\theta(m_3)) = \tau(m_1 cu_g \otimes ((m_2 \otimes b^* b)\Delta(au_g))) = \tau(m_1 cu_g)\tau(m_2 b_e).$$

On the other hand, the RHS of (4.3) equals to

$$\tau((m_1 \otimes E_B(m_2))\psi(m_3)) = \tau(m_1 cu_g \otimes E_B(m_2)au_g) = \tau(m_1 cu_g)\tau(m_2 b_e),$$

which proves (4.3).

Now since $B$ is amenable, we obtain that

$$L^2(\mathcal{M}) \otimes L^2(M) \otimes B L^2(A \times \Gamma)_{\psi(M)_{1,3}} \subseteq \mathcal{M}L^2(\mathcal{M}) \otimes L^2(\mathcal{M}) \otimes L^2(A \times \Gamma)_{\psi(M)_{1,3}}.$$

This finishes the proof of the claim. \hfill \Box

**Claim 2.** We have $\mathcal{M}L^2(\mathcal{M}) \otimes L^2(M)_{\psi(M)_{1,4}} \subseteq \mathcal{M}L^2(\mathcal{M}) \otimes L^2(\mathcal{M})_{\psi(M)_{1,4}}$.

**Proof of the Claim 2.** First, note that it is enough to prove

$$\mathcal{M}L^2(\mathcal{M}) \otimes L^2(M)_{\psi(M)_{1,4}} \subseteq \mathcal{M}L^2(\mathcal{M}) \otimes L^2(M)_{\psi(M)_{1,4}}.$$
Let \( C \) be an orthonormal basis for \( L^2(C) \) and note that we have the following orthogonal decomposition into \( M-M \)-bimodules:

\[
L^2(M) \otimes L^2(M) = \bigoplus_{c \in C} \mathcal{M} (1 \otimes c) d(M).
\]

Note that \( \mathcal{M} (1 \otimes c) d(M) \cong L^2(M) \otimes_C L^2(M) \) as \( M-M \)-bimodules. Indeed, let us take \( m_1 = c_1 a_1 u_{g_1}, m_2 = c_2 a_2 u_{g_2} \), and note that

\[
\langle m_1 (1 \otimes c) \psi(m_2), 1 \otimes c \rangle = \langle c_1 a_1 u_{g_1} c_2 a_2 u_{g_2} 1 \otimes c \rangle = \delta_{g_1,e} \delta_{g_2,e} \tau(c_1 c_2) \tau(a_1) \tau(a_2)
\]

and

\[
\langle m_1 e_C m_2, e_C \rangle = \tau(E_C(c_1 a_1 u_{g_1}) c_2 a_2 u_{g_2}) = \delta_{g_1,e} \delta_{g_2,e} \tau(c_1 c_2) \tau(a_1) \tau(a_2).
\]

This implies that \( \mathcal{M} (1 \otimes c) \psi(M) \cong L^2(M) \otimes_C L^2(M) \) as \( M-M \)-bimodules. Since \( C \) is amenable, the claim is proven.

Now, assume that \( \theta(Q) \) is not strongly non-amenable relative to \( M \bar{\otimes} M \otimes 1 \). Then there exists a non-zero projection \( p \in \theta(Q) \cap \theta(q) \mathcal{M} \theta(q) \) such that

\[
\mathcal{M} L^2(M_p)_{\theta(q)} \subset \mathcal{M} L^2(M) \otimes_{M \bar{\otimes} M \otimes 1} L^2(M)_{\theta(q)}.
\]

Using Claim 1 and 2, we obtain now that \( \mathcal{M} L^2(M_p)_{\theta(q)} \subset \mathcal{M} L^2(M) \otimes L^2(Q)_{Q} \).

Take \( z \in Q \) such that \( \theta(z) \) is the support projection of \( E_{\theta(q)}(p) \). Note that \( z \) is a non-zero central projection in \( Q \) and that \( \theta \) embeds the trivial \( Q_z-Q_z \)-bimodule into \( \theta(q) L^2(\theta(Q_z))_{\theta(q)} \). Therefore, \( Q_z L^2(Q_z)_{Q_z} \subset \mathcal{M} L^2(M) \otimes L^2(Q_z)_{Q_z} \). Finally, we obtain \( Q_z L^2(Q_z)_{Q_z} \subset \mathcal{M} L^2(M) \otimes L^2(Q)_{Q} \), which means that \( Q_z \) is amenable, contradiction.

In a similar way, one can prove that \( \theta(Q) \) is strongly non-amenable relative to \( M \bar{\otimes} 1 \bar{\otimes} M \). This ends the proof.

5. INTERTWINING OF ABELIAN SUBALGEBRAS

Throughout this section we will use the following notation. Let \( \Gamma \) be a countable group. Let \( \Sigma \) be an almost malnormal subgroup and let \( \sigma_0 \) be a tracial action of \( \Sigma \) on a non-trivial abelian von Neumann algebra \( A_0 \). Denote by \( \sigma \) the coinduced action of \( \Gamma \) on \( A := A_0^{\Gamma/\Sigma} \). Finally, denote \( M = A \rtimes \Gamma \).

The next result is a localization theorem for coinduced actions which goes back to [Io10, Theorem 6.1]. The form presented in this paper is very similar to [IPV10, Theorem 5.1], but written with coinduced actions instead of generalized Bernoulli ones.

**Theorem 5.1.** Assume that \( D \subset M \bar{\otimes} M \) is an abelian von Neumann subalgebra which is normalized by a group of unitaries \( (\gamma(s))_{s \in A} \) that belong to \( L(\Gamma) \bar{\otimes} L(\Gamma) \). Denote by \( P \) the quasi-normalizer of \( D \) inside \( M \bar{\otimes} M \). We make the following assumptions:

1. \( D \not\subset M \otimes 1 \) and \( D \not\subset 1 \otimes M \),
2. \( P \not\subset M \bar{\otimes} (A \rtimes \Sigma) \) and \( P \not\subset (A \rtimes \Sigma) \bar{\otimes} M \),
3. \( P \not\subset M \bar{\otimes} L(\Gamma) \) and \( P \not\subset L(\Gamma) \bar{\otimes} M \),
4. \( \gamma(\Lambda)^{''} \not\subset L(\Gamma) \bar{\otimes} L(\Sigma) \) and \( \gamma(\Lambda)^{''} \not\subset L(\Sigma) \bar{\otimes} L(\Gamma) \).

Define \( C := D' \cap (M \bar{\otimes} M) \). Then for every non-zero projection \( q \in Z(C) \) we have \( Cq \prec A \bar{\otimes} A \).
The proof is identically with the one of [IPV10, Theorem 5.1], since essentially the same computations still hold once we replace generalized Bernoulli actions by coinduced ones.

Next, we obtain a similar statement if one considers an abelian von Neumann algebra in $M$ and not in $M\overline{\otimes}M$.

**Theorem 5.2.** Assume that $D \subset M$ is an abelian von Neumann subalgebra which is normalized by a group of unitaries $(\gamma(s))_{s \in \Lambda}$ that belong to $L(\Gamma)$. Denote by $P$ the quasi-normalizer of $D$ inside $M$. We make the following assumptions:

1. $D$ is diffuse,
2. $P \not\subset A \rtimes \Sigma$,
3. $P \not\subset L(\Gamma)$,
4. $\gamma(\Lambda)'' \not\subset L(\Sigma)$.

Define $C := D' \cap M$. Then for every non-zero projection $q \in \mathcal{Z}(C)$ we have $Cq \prec A$.

As noticed in [Io10], we obtain as a corollary a weaker version of Popa's conjugacy criterion adapted in this case to coinduced actions.

**Theorem 5.3.** Suppose $\Gamma$ is icc and $\Sigma$ is amenable. Let $\Lambda \rhd B$ be another tracial action of a countable group $\Lambda$ on a non-trivial abelian von Neumann algebra $B$ such that $M = A \rtimes \Gamma = B \rtimes \Lambda$ and $L(\Lambda) \subset L(\Gamma)$.

Then $B \prec A$.

**Proof.** The proof is a direct application of Theorem 5.2. Note that the quasi-normalizer of the abelian algebra $B$ is $M$. Now, notice that if $M \prec A \rtimes \Sigma$, by [DHI16, Lemma 2.5.1] we obtain that $[\Gamma : \Sigma] < \infty$. This is not possible since $\Sigma$ is almost malnormal in $\Gamma$. Also $L(\Lambda) \not\subset L(\Sigma)$ since $\Sigma$ is amenable and therefore we obtain $B \prec A$.

$\blacksquare$

6. Proof of the main results

In [Io10], Ioana has proven that any Bernoulli action of an arbitrary icc property (T) group is $W^*$-superrigid. The strategy of his proof was successfully applied also in [IPV10] and [Bo12].

6.1. A general method for obtaining $W^*$-superrigidity. Using Ioana’s proof, we identify a couple of steps for proving that a certain free ergodic pmp action $\Gamma \rhd X$ is $W^*$-superrigid (see also the introduction of [Bo12]). Consider an arbitrary free ergodic pmp action $\Lambda \rhd Y$ such that $M := A \rtimes \Gamma = B \rtimes \Lambda$, where $A = L^\infty(X)$ and $B = L^\infty(Y)$. Define the comultiplication $\Delta : M \to M \hat{\otimes} L(\Lambda)$ by $\Delta(bv_\lambda) = bv_\lambda \otimes v_\lambda$, for all $b \in B, \lambda \in \Lambda$, where we denote by $v_\lambda, \lambda \in \Lambda$, the canonical unitaries corresponding to the action of $\Lambda$.

**Step 1.** One has to show that $\Gamma \rhd X$ is OE superrigid. From now on, using Singer’s result [Si55], it is enough to assume that $B$ is not unitarily conjugated to $A$ in $M$, which is equivalent to $B \not\subset M A$ [Po06b, Theorem A.1].

**Step 2.** One can also assume that there exists a non-zero projection $s_0 \in L(\Lambda)' \cap M$ such that $L(\Lambda)s_0 \not\subset L(\Gamma)$.

**Step 3.** One shows that there exists a unitary $u \in \mathcal{U}(M \hat{\otimes} M)$ such that $u\Delta(L(\Gamma))u^* \subset L(\Gamma \times \Gamma)$.
Step 4. Next, one proves that the algebra $C := \Delta(A)' \cap (M \bar{\otimes} M)$ satisfies

$$Cq \preceq_{M \bar{\otimes} M} A \bar{\otimes} A \quad \text{for all } q \in Z(C).$$

Step 5. Using the previous steps together with a generalization of [Po04 Theorem 5.2], one essentially obtains that there exist a unitary $v \in \mathcal{U}(M \bar{\otimes} M)$, a group homomorphism $\delta : \Gamma \to \Gamma \times \Gamma$ and a character $\omega : \Gamma \to \mathbb{C}$ such that $vCv^* = A \bar{\otimes} A$ and $v\Delta(u_g)v^* = \omega(g)u_{\delta(g)}$, for all $g \in \Gamma$ (the precise statement is the Step 3 of the proof [IPV10 Theorem 10.1]).

Step 6. Using Step 5, one proves that for every sequence $(x_n)_n$ in $M$ for which the Fourier coefficient (w.r.t. the decomposition $M = A \times \Gamma$) converges to 0 pointwise in $\| \cdot \|_2$, then the Fourier coefficient of $\Delta(x_n)$ (w.r.t. the decomposition $M \bar{\otimes} M = (M \bar{\otimes} A) \times \Gamma$) also converges to 0 pointwise in $\| \cdot \|_2$. This shows $B \prec A$ and Step 1 implies that $\Gamma \curvearrowright X$ is $W^*$-superrigid.

6.2. Proof of Theorem [A]. We record first the following observation.

Remark 6.1. Since $\Sigma$ is almost normal in $\Gamma$, using [Dr15 Lemma 5.3], the action $\Gamma \curvearrowright X$ is free (see also [Io06b Lemma 2.1]).

Proof of Theorem [A] Assume that $\Lambda \curvearrowright (Y, \nu)$ is an arbitrary free ergodic pmp action such that

$$M := L^\infty(X) \rtimes \Gamma = L^\infty(Y) \rtimes \Lambda.$$  

We put $A = L^\infty(X)$, $B = L^\infty(Y)$. Define $\Delta : M \to M \bar{\otimes} M$ by $\Delta(bv_s) = bv_s \otimes v_s$, for all $b \in B$ and $s \in \Lambda$, where we denote by $v_s, s \in \Lambda$, the canonical unitaries corresponding to the action of $\Lambda$.

Since the action $\Gamma \curvearrowright X$ is OE superrigid (using [Dr15 Theorem A] and [Po05 Theorem 5.6]), Step 1 is completed. To prove Step (2), suppose $L(\Lambda)q \prec L(\Gamma)$ for all $q \in L(\Lambda)' \cap M$. Since $\Sigma$ is amenable, $L(\Lambda) \not\preceq L(\Sigma)$, so by Corollary 3.6 there exists a unitary $u \in \mathcal{U}(M)$ such that $uL(\Lambda)u^* \subset L(\Gamma)$.

Based on Step 1, Theorem 5.3 proves that $\Gamma \curvearrowright X$ is $W^*$-superrigid. This completes Step 2. Therefore, we take a non-zero projection $q_0 \in L(\Lambda)' \cap M$ such that $L(\Lambda)q_0 \not\prec L(\Gamma)$. Step (3) is obtained by combining Theorem 3.2 and [IPV10 Lemma 10.2.5].

Proof of Step (4). Note that Theorem 5.1 proves this step by considering the abelian subalgebra $D_0 := \Delta(A)(1 \otimes q_0)$. For showing this, denote $C_0 = D_0' \cap (M \bar{\otimes} q_0 M q_0)$, $C = \Delta(A)' \cap (M \bar{\otimes} M)$ and note that $C_0 = C(1 \otimes q_0)$. Since $L(\Lambda)q_0 \not\prec L(\Gamma)$, [Jo08 Lemma 9.2.4] implies that $\Delta(M)(1 \otimes q_0) \not\subset M \bar{\otimes} L(\Gamma)$. Using [IPV10 Lemma 10.2], we see that all the conditions of Theorem 5.1 are satisfied. Therefore, we obtain that $C_0q \prec A \bar{\otimes} A$, for all $q \in Z(C)$.

Proof of Step (5). For proving Step (3) of the proof of [IPV10 Theorem 10.1], one only needs to show:

- If $H$ is a subgroup of $\Gamma \times \Gamma$ such that $H$ acts non-ergodically on $A \bar{\otimes} A$, then $\Delta(L(\Gamma)) \not\subset (A \bar{\otimes} A) \times H$.

Suppose by contrary that $\Delta(L(\Gamma)) \not\subset (A \bar{\otimes} A) \times H$. It is easy to prove that there exists a finite set $T \subset \Gamma$ such that $H \subset (\cup_{t \in T} \Sigma) \times \Gamma$ or $H \subset \Gamma \times (\cup_{t \in T} \Sigma)$. This implies that $\Delta(L(\Gamma)) \not\prec (A \times \Sigma) \bar{\otimes} M$ or $\Delta(L(\Gamma)) \not\prec M \bar{\otimes}(A \times \Sigma)$. By applying [IPV10 Lemma 10.2.5], we obtain a contradiction.

Step (6) works in general once the other steps are proven. This finishes the proof of the theorem.
6.3. Proof of Theorem B

In this subsection we will prove a more general statement of Theorem B.

Assumption 6.2. Let $\Sigma$ be a subgroup of a countable icc group $\Gamma$. Let $\sigma_0$ be a pmp action of $\Sigma$ on a non-trivial standard probability space $(X_0, \mu_0)$ and denote by $\sigma$ the coinduced action of $\Gamma$ on $X := X_0^{\Gamma/\Sigma}$. Suppose:

- $\Gamma$ is a non-amenable icc group which is measure equivalent to a group $\Lambda_0$ for which the group von Neumann algebra $L(\Lambda_0)$ is not prime.
- $\Sigma$ is amenable and almost malnormal.

Theorem 6.3. Let $\Gamma \curvearrowright X$ be as in Assumption 6.2. Then $\Gamma \curvearrowright X$ is $W^*$-superrigid.

Proof. The proof of this theorem goes along the same lines as the proof of Theorem A. We point out only the differences. The action $\Gamma \curvearrowright X$ is OE superrigid using Theorem 4.5 and [Po05, Theorem 5.6]. Step (3) follows by Theorem 4.6 All the other steps follow as in the proof of Theorem A, which finishes the proof.

Remark 6.4. A careful handling of Theorem 4.6 shows that Assumption 6.2 can be improved by supposing the weaker assumption that $L(\Lambda_0)$ contains a commuting pair of diffuse subalgebras $P_1$ and $P_2$ such that $P_2$ is non-amenable and $\mathcal{N}_{L(\Lambda_0)}(P_1 \lor P_2)' = L(\Lambda_0)$ (see also Step 1 of the proof of [IPV10, Theorem 8.2]).

Corollary 6.5. Let $\Gamma$ be an icc non-amenable group which is measure equivalent to a group $\Lambda_0$ for which $L(\Lambda_0)$ is not prime. Then the Bernoulli action $\Gamma \curvearrowright (X, \mu)$ is $W^*$-superrigid, where $(X, \mu)$ is a non-trivial standard probability space.

References

[BHV08] B.Bekka, P. de la Harpe, A.Valette: Kazhdan’s Property (T), (New Mathematical Monographs, 11), Cambridge University Press, Cambridge, 2008.

[Bo63] A. Borel: Compact Clifford-Klein forms of symmetric spaces, Topology 2 (1963), 111-122.

[Bo12a] R. Boutonnet: On solid ergodicity for Gaussian actions, J. Funct. Anal., 263 (2012) 1040-1063.

[Bo12] R. Boutonnet: $W^*$-superrigidity of mixing Gaussian actions of rigid groups, Adv. Math. 244 (2013)

[Bo14] R. Boutonnet: Plusieurs aspects de rigidité des algèbres de von Neumann, PhD thesis (2014).

[BV12] M. Berbec, S. Vaes: $W^*$-superrigidity for group von Neumann algebras or left-right wreath products, Proceedings of the London Mathematical Society 108 (2014), 1116-1152.

[CI08] I. Chifan, A. Ioana: Ergodic subequivalence relations induced by a Bernoulli action, Geometric and Functional Analysis, 20(1): 53-67, 2010.

[CI17] I. Chifan, A. Ioana: Amalgamated free product rigidity for group von Neumann algebras, Preprint, arxiv:1705.07350.

[CIK13] I. Chifan, A. Ioana, Y. Kida: $W^*$-superrigidity for arbitrary actions of central quotients of braid groups, Math. Ann. 361 (2015), 925-959.

[CJ82] A. Connes, V.F.R. Jones: A $II_1$ factor with two non-conjugate Cartan subalgebras, Bull. Amer. Math. Soc. 6 (1982), 211-212.

[CK15] I. Chifan, Y. Kida: OE and $W^*$ superrigidity results for actions by surface braid groups, Proceedings of the London Mathematical Society, in press.

[Co76] A. Connes: Classification of Injective Factors Cases $II_1, II_\infty, III_\lambda, \lambda \neq 1$, Ann. of Math., 104 (1976), 73-115.

[Co94] A. Connes: Noncommutative Geometry, Academic Press, 1994.

[CP10] I. Chifan, J. Peterson: Some unique group-measure space decomposition results, Duke Math. J. 162 (2013), no. 11, 1923-1966.

[CS11] I. Chifan, T. Sinclair: On the structural theory of $II_1$ factors of negatively curved groups, Ann. Sci. c. Norm. Supr. (4) 46 (2013), 1-33.
[CSU11] I. Chifan, T. Sinclair, B. Udrea: On the structural theory of II_1 factors of negatively curved groups, II. Actions by product groups, Adv. Math. 245 (2013), 208-236.

[DHI16] D. Drimbe, D. Hoff, A. Ioana: Prime II_1 factors arising from irreducible lattices in products of rank one simple Lie groups to appear in J. Reine. Angew. Math.

[dHW14] P. de la Harpe, C. Weber: Malnormal subgroups and Frobenius groups: basics and examples, with an appendix by Denis Osin, Confluentes Math. 6 (2014), no. 1, 6576

[Dr15] D. Drimbe: Cocycle and orbit equivalence superrigidity for coinduced actions, to appear in Ergodic Theory Dynam. Systems.

[Fu99] A. Furman: Orbit equivalence rigidity, Ann. of Math. (2) 150 (1999), no. 3, 1083-1108.

[FV10] P. Fima, S. Vaes: HNN extensions and unique group measure space decomposition of II_1 factors Trans. Amer. Math. Soc. 354 (2012) 2601-2617.

[Ho15] D. Hoff: Von Neumann algebras of equivalence relations with nontrivial one-cohomology, J. Funct. Anal. 270 (2016), no. 4, 1501-1536.

[GITD16] D. Gaboriau, A. Ioana, R. Tucker-Drob: Cocycle superrigidity for translation actions of product groups, submitted, arXiv 1603.07616 (2016).

[Gr91] M. Gromov: Asymptotic invariants of infinite groups, Geometric group theory, Vol. 2 (Sussex, 1991), London Math. Soc. Lecture Note Ser., vol. 182, Cambridge Univ. Press, Cambridge, 1993, pp. 1-295.

[HPV10] C. Houdayer, S. Popa, S. Vaes: A class of groups for which every action is W*-superrigid, Groups Geom. Dyn. 7 (2013), 577-590.

[Io06a] A. Ioana: Rigidity results for wreath product of II_1 factors, J. Funct. Anal. 252 (2007), 763-791.

[Io06b] A. Ioana: Orbit inequivalent actions for groups containing a copy of II_1 factors with at most one Cartan subalgebra.

[Io10] A. Ioana: W*-superrigidity for Bernoulli actions of property (T) groups J. Amer. Math. Soc. 24 (2011), 1175-1226.

[Io12a] A. Ioana: Classification and rigidity for von Neumann algebras, European Congress of Mathematics, EMS (2013), 601-625.

[IPV10] A. Ioana, S. Popa, S. Vaes: A Class of superrigid group von Neumann algebras, Ann. of Math. (2) 178 (2013), 231-286

[Ka67] D. Kazhdan: On the connection of the dual space of a group with the structure of its closed subgroups, Funct. Anal. and its Appl. 1 (1967), 63-65.

[KV15] A. Krogager, S. Vaes: A class of II_1 factors with exactly two crossed product decompositions, preprint arXiv:1512.06677.

[Ma82] G. Margulis: Finitely-additive invariant measures on Euclidian spaces, Ergodic Theory Dynam. Systems 2(1982), 383-396.

[MvN36] F. J. Murray, J. von Neumann, On rings of operators. Ann. of Math. 37 (1936). 116-229.

[OP07] N. Ozawa, S. Popa: On a class of II_1 factors with at most one Cartan subalgebra, Ann. of Math. (2) 172 (2010), no. 1, 713-749.

[Pe09] J. Peterson: Examples of group actions which are virtually W*-superrigid, Preprint. arXiv:1002.1745.

[Po01] S. Popa: On a class of type II_1 factors with Betti numbers invariants, Ann. of Math. 163 (2006), 809-899.

[Po03] S. Popa: Strong rigidity of II_1 factors arising from malleable actions of w-rigid groups. I., Invent. Math. 165 (2006), 369-408.

[Po04] S. Popa: Strong rigidity of II_1 factors arising from malleable actions of w-rigid groups. II., Invent. Math. 165(2) (2006), 409-451.

[Po05] S. Popa: Cocycle and orbit equivalence superrigidity for malleable actions of w-rigid groups, Invent. Math. 170 (2007), 243-295.

[Po06b] S. Popa: On a class of type II_1 factors with Betti numbers invariants, Ann. of Math. 163 (2006), 809-889.

[Po06] S. Popa: On the superrigidity of malleable actions with spectral gap, J. Amer. Math. Soc. 21 (2008), 981-1000.

[Po07] S. Popa: Deformation and rigidity for group actions and von Neumann algebras, In Proceedings of the ICM (Madrid, 2006), Vol. I, European Mathematical Society Publishing House, 2007, 445-477.

[PS09] J. Peterson, T. Sinclair: On cocycle superrigidity for Gaussian actions Erg. Th. and Dyn. Sys. 32 (2012), no. 1, 249-272.

[PV06] S. Popa, S. Vaes: Strong rigidity of generalized Bernoulli actions and computations of their symmetry groups, Adv. Math. 217 (2008), 833-872.

[PV09] S.Popa, S. Vaes: Group measure space decomposition of II_1 factors and W*-superrigidity, Invent. Math. 182 (2010), no. 2, 371-417.

[PV11] S. Popa, S. Vaes: Unique Cartan decomposition for II_1 factors arising from arbitrary actions of free groups, Acta Mathematica 212 (2014), 141-198.
[PV12] S. Popa, S. Vaes: *Unique Cartan decomposition for II$_1$ factors arising from arbitrary actions of hyperbolic groups*, Journal fur die reine und angewandte Mathematik, 690 2014, 433-458.

[Ra72] M. S. Raghunathan: *Discrete subgroups of Lie groups*, Springer-Verlag, New York, 1972, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68.

[RS10] G. Robertson, Tim Steger: *Malnormal subgroups of lattices and the Pukanszky invariant in group factors*, J. Funct. Anal. 258-8 (2010), 2708-2713.

[Si55] I. M. Singer: *Automorphism of finite factors*, Amer. J. Math. 77 (1955), 117-133.

[Va08] S. Vaes: *Explicit computations of all finite index bimodules for a family of II$_1$ factors*, Ann. Sci. Éc. Norm. Supér. (4) 41 (2008), no. 5, 743-788.

[Va10a] S. Vaes: *Rigidity for von Neumann algebras and their invariants*, Proceedings of the ICM (Hyderabad, India, 2010), Vol. III, Hindustan Book Agency (2010), 1624-1650.

[Va10b] S Vaes: *One-cohomology and the uniqueness of the group measure space decomposition of a II$_1$ factor*, Math. Ann. 355 (2013), 661-696.

Mathematics Department; University of California, San Diego, CA 90095-1555 (United States).

E-mail address: ddrimbe@ucsd.edu