Lie algebras constructed with Lie modules and their positively and negatively graded modules

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Abstract

In this paper, we shall give a way to construct a graded Lie algebra \( L(g, \rho, V, B_0) \) from a standard pentad \((g, \rho, V, B_0)\) consisting of a Lie algebra \(g\) which has a non-degenerate invariant bilinear form \(B_0\) and \(g\)-modules \((\rho, V)\) and \(V \subset \text{Hom}(V, \mathfrak{k})\) all defined over a field \(\mathfrak{k}\). In general, we do not assume that these objects are finite-dimensional. We can embed the objects \(g, \rho, V, B_0\) into \(L(g, \rho, V, B_0)\). Moreover, we construct specific positively and negatively graded modules of \(L(g, \rho, V, B_0)\). Finally, we give a chain rule on the embedding rules of standard pentads.

1 Introduction

A standard quadruplet is a quadruplet of the form \((g, \rho, V, B_0)\), where \(g\) is a finite-dimensional reductive Lie algebra, \((\rho, V)\) a finite-dimensional representation of \(g\) and \(B_0\) a non-degenerate symmetric invariant bilinear form on \(g\) all defined over the complex number field \(\mathbb{C}\), which satisfies the conditions that \(\rho\) is faithful and completely reducible and that \(V\) does not have a non-zero invariant element. In [9], the author proved that any standard quadruplet \((g, \rho, V, B_0)\) has a graded Lie algebra, denoted by \(L(g, \rho, V, B_0) = \bigoplus_{n \in \mathbb{Z}} V_n\), such that \(V_0 \simeq g, V_1 \simeq V\) and \(V_{-1} \simeq \text{Hom}(V, \mathbb{C})\) (see [9, Theorem 2.11]). That is, any finite-dimensional reductive Lie algebra and its finite-dimensional faithful and completely reducible representation can be embedded into some (finite or infinite-dimensional) graded Lie algebra. We call a graded Lie algebra of the form \(L(g, \rho, V, B_0)\) the Lie algebra associated with a standard quadruplet. Some well-known Lie algebras correspond to some standard quadruplet, for example, finite-dimensional semisimple Lie algebras and loop algebras. Moreover, the bilinear form \(B_0\) can be also embedded into \(L(g, \rho, V, B_0)\), i.e. there exists a non-degenerate symmetric invariant bilinear form on \(L(g, \rho, V, B_0)\) whose restriction to \(V_0 \times V_0\) coincides with \(B_0\) (see [9, Proposition 3.2]). By the way, H. Rubenthaler obtained some similar results in [8] using the Kac theory in [3].

The first purpose of this paper is to extend the theory of standard quadruplets to the cases where the objects are infinite-dimensional. For this, we need to consider pentads \((g, \rho, V, V, B_0)\)

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instead of quadruplets, where \( g \) is a finite or infinite-dimensional Lie algebra, \( \rho : g \otimes V \rightarrow V \) a representation of \( g \) on a finite or infinite-dimensional vector space \( V \), \( V \) a \( g \)-submodule of \( \text{Hom}(V, \mathfrak{t}) \), \( B_0 \) a non-degenerate invariant bilinear form on \( g \) all defined over a field \( \mathfrak{t} \). In general, we do not assume that \( B_0 \) is symmetric. We define the notion of standard pentads by the existence of a linear map \( \Phi_\rho : V \otimes V \rightarrow g \) satisfying \( B_0(a, \Phi_\rho(v \otimes \phi)) = \langle \rho(a \otimes v), \phi \rangle \) for any \( a \in g \), \( v \in V \) and \( \phi \in V \). A standard quadruplet \((g, \rho, V, B_0)\) can be naturally regarded as a standard pentad \((g, \rho, V, \text{Hom}(V, \mathbb{C}), B_0)\), and, thus, we can say that the notion of standard pentads is an extension of the notion of standard quadruplets. Then, by a similar argument to the argument in [9], we can construct a graded Lie algebra from an arbitrary standard pentad \((g, \rho, V, B_0)\) denoted by \( L(g, \rho, V, V, B_0) = \bigoplus_{n \in \mathbb{Z}} V_n \) such that the objects \( g, \rho, V, V \) can be embedded into it. We call such a graded Lie algebra a Lie algebra associated with a standard pentad. This is the first main result of this paper. Of course, the graded Lie algebra associated with a standard quadruplet \((g, \rho, V, B_0)\) is isomorphic to the graded Lie algebra associated with a standard pentad \((g, \rho, V, \text{Hom}(V, \mathbb{C}), B_0)\).

Moreover, if the bilinear form \( B_0 \) of \((g, \rho, V, \mathfrak{t}, B_0)\) is symmetric, then \( B_0 \) can be also embedded into \( L(g, \rho, V, V, B_0) \), i.e. there exists a non-degenerate symmetric invariant bilinear form \( B_L \) on \( L(g, \rho, V, V, B_0) \) whose restriction to \( V_0 \times V_0 \) coincides with \( B_0 \).

When \( B_0 \) is symmetric, we can expect that a Lie algebra of the form \( L(g, \rho, V, V, B_0) \) (not necessary finite-dimensional) and its representation can be embedded into some graded Lie algebra using \( B_L \). The second purpose is to construct positively graded modules and negatively graded modules of \( L(g, \rho, V, V, B_0) \) which can be embedded into some graded Lie algebra under some assumptions. In general, it is known that for any graded Lie algebra \( l = \bigoplus_{n \in \mathbb{Z}} l_n \) and \( l_0 \)-module \( U \), there exists a positively (respectively negatively) graded \( l \)-module such that the base space (respectively top space) is the given \( l_0 \)-module \( U \) (see [2] Theorem 1.2)). In this paper, we shall try to construct such \( L(g, \rho, V, V, B_0) \)-modules from a \( g \)-module \((\pi, U)\) using a similar way to the construction of a Lie algebra associated with a standard pentad. Precisely, we inductively construct a positively (respectively negatively) graded \( L(g, \rho, V, V, B_0) \)-module \((\tilde{\pi}^+, \tilde{U}^+) \), \( \tilde{U}^+ = \bigoplus_{m \geq 0} U^+_m \) (respectively \((\tilde{\pi}^-, \tilde{U}^-) \), \( \tilde{U}^- = \bigoplus_{m \leq 0} U^-_m \)) such that the “base space” \( U_0^+ \) (respectively the “top space” \( U_0^- \)) is the given \( g \)-module \( U \). In general, the modules \( \tilde{U}^+ \) and \( \tilde{U}^- \) are infinite-dimensional. We shall try to embed \( L(g, \rho, V, V, B_0) \) and its module of the form \( \tilde{U}^+ \) into some graded Lie algebra. If we assume that \( B_0 \) is symmetric and that \( U \) has a \( g \)-submodule \( \mathcal{U} \) of \( \text{Hom}(U, \mathfrak{t}) \) such that \((g, \pi, U, U, B_0)\) is a standard pentad, then we can embed the objects \( L(g, \rho, V, V, B_0) \) and \( \tilde{U}^+ \) into some graded Lie algebra. Precisely, under these assumptions, we have that a pentad \((L(g, \rho, V, V, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{U}^-, B_L)\) is also standard, and, thus, we can embed the objects \( L(g, \rho, V, V, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{U}^- \) into the graded Lie algebra \( L(g, \rho, V, V, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{U}^-, B_L \). In this situation, we have a “chain rule” of the Lie algebras constructed with a standard pentad. This is the second main result of this paper.

This paper consists of three sections.

In section 2 we shall study the Lie algebras associated with a standard pentad. First, in section 2.1 we define the notion of standard pentads (see Definition 2.2) and construct a graded Lie al-
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gebra from a standard pentad \((g, \rho, V, V, B_0)\), which is denoted by \(L(g, \rho, V, V, B_0) = \bigoplus_{n \in \mathbb{Z}} V_n\) (see Theorem 2.15). In section 2.2 we consider some properties of Lie algebras of the form \(L(g, \rho, V, V, B_0)\) such that \(B_0\) is symmetric. In these cases, we can also embed the bilinear form \(B_0\) into \(L(g, \rho, V, V, B_0)\), i.e., we can obtain a non-degenerate symmetric invariant bilinear form on \(L(g, \rho, V, V, B_0)\) whose restriction to \(V_0 \times V_0\) coincides with \(B_0\) (see Proposition 2.13). Moreover, the Lie algebra \(L(g, \rho, V, V, B_0)\) can be characterized by the transitivity and the existence of such a bilinear form (see Theorem 2.20). Finally, we give two lemmas on derivations on \(L(g, \rho, V, V, B_0)\) (see Lemmas 2.37 and 2.38).

In section 3, we shall study positively and negatively graded modules of a Lie algebra of the form \(L(g, \rho, V, V, B_0)\). First, in sections 3.1 and 3.2 we shall construct positively graded \(L(g, \rho, V, V, B_0)\)-module and negatively graded \(L(g, \rho, V, V, B_0)\)-module from a \(g\)-module \((\pi, U)\), i.e., we shall give another proof of [2] Theorem 1.2] in the special cases where the graded Lie algebra is of the form \(L(g, \rho, V, V, B_0)\). In section 3.1 we construct a family of \(g\)-modules \(\{U_m^+\}_{m \geq 0}\) (respectively \(\{U_m^-\}_{m \leq 0}\)) from the pentad \((g, \rho, V, V, B_0)\) and the \(g\)-module \((\pi, U)\) by induction. In section 3.2 we define a structure of positively (respectively negatively) graded \(L(g, \rho, V, V, B_0)\)-module on \(\hat{U}^+ := \bigoplus_{m \geq 0} U_m^+\) (respectively \(\hat{U}^- := \bigoplus_{m \leq 0} U_m^-\)). We call this positively (respectively negatively) graded module of \(L(g, \rho, V, V, B_0)\) the positive extension (respectively negative extension) of \(U\) with respect to \((g, \rho, V, V, B_0)\) (see Theorems 3.12 and 3.14). These modules are transitive and characterized by their transitivity (see Theorem 3.17). In sections 3.3 and 3.4 we try to construct a standard pentad which contains a Lie algebra of the form \(L(g, \rho, V, V, B_0)\) and its module of the form \(\hat{U}^+\). For this, we need to assume that \(B_0\) is symmetric and \(U\) is embedded into some standard pentad \((g, \pi, U, U, B_0)\). In section 3.3 for the \(g\)-submodule \(U\) of \(\text{Hom}(U, U)\), we shall extend the canonical pairing \(U \times U\) to \(\hat{U}^+ \times \hat{U}^-\). Moreover, in section 3.4 we shall construct the \(\Phi\)-map of \((L(g, \rho, V, V, B_0), \tilde{\pi}^+, \tilde{\hat{U}}^+, \tilde{\hat{U}}^-, B_L)\) from the \(\Phi\)-map of the pentad \((g, \pi, U, U, B_0)\) inductively. Consequently, under the assumptions that \((g, \rho, V, V, B_0)\) and \((g, \pi, U, U, B_0)\) are standard pentads and that their bilinear form \(B_0\) is symmetric, we can embed the Lie algebra \(L(g, \rho, V, V, B_0)\) and its module \(\hat{U}^+\) into a standard pentad \((L(g, \rho, V, V, B_0), \tilde{\pi}^+, \tilde{\hat{U}}^+, \tilde{\hat{U}}^-, B_L)\). Finally, in section 3.5 we consider the graded Lie algebra \(L(L(g, \rho, V, V, B_0), \tilde{\pi}^+, \tilde{\hat{U}}^+, \tilde{\hat{U}}^-, B_L)\) under the situation of sections 3.3 and 3.4. From the constructions of \(L(g, \rho, V, V, B_0), \tilde{\hat{U}}^+\) and \(\tilde{\hat{U}}^-\), we can write this graded Lie algebra using the data \(g, \rho, V, V, B_0\) and \(U, U\). Precisely, we have the following result on the structures of Lie algebras:

\[
L(L(g, \rho, V, V, B_0), \tilde{\pi}^+, \tilde{\hat{U}}^+, \tilde{\hat{U}}^-, B_L) \simeq L(g, \rho \otimes \pi, V \otimes U, V \otimes U, B_0).
\]

This is a chain rule in the theory of standard pentads (see Theorem 3.20).

**Notation 1.1.** In this paper, we regard a representation \(\rho\) of a Lie algebra \(L\) on \(V\) as a linear map \(\rho : L \otimes V \to V\) which satisfies that

\[
\rho([a, b] \otimes v) = \rho(a \otimes \rho(b \otimes v)) - \rho(b \otimes \rho(a \otimes v))
\]
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for any $a, b \in l$ and $v \in V$.

**Definition 1.2.** In this paper, we say that a Lie algebra $l$ is a $\mathbb{Z}$-graded Lie algebra or simply a graded Lie algebra if and only if there exist vector subspaces $l_n$ of $l$ for all $n \in \mathbb{Z}$ such that:

- $l = \bigoplus_{n \in \mathbb{Z}} l_n$ and $[l_n, l_m] \subset l_{n+m}$ for any $n, m \in \mathbb{Z}$,
- $l$ is generated by $l_{−1} \oplus l_0 \oplus l_1$.

In general, we do not assume that each $l_n$ is finite-dimensional (cf. [3, Definition 1]).

Moreover, if $l$ satisfies the following two conditions, we say that $l$ is transitive (see [3, Definition 2]):

- for $x \in l_i$, $i \geq 0$, $[x, l_{−1}] = \{0\}$ implies $x = 0$,
- for $x \in l_i$, $i \leq 0$, $[x, l_1] = \{0\}$ implies $x = 0$.

**Definition 1.3.** In this paper, we say that a module $(\varpi^+, W)$, $W = \bigoplus_{m \geq 0} W_m$ (respectively $(\varpi^-, W)$, $W = \bigoplus_{m \leq 0} W_m$) of a graded Lie algebra $\bigoplus_{n \in \mathbb{Z}} l_n$ is positively graded (respectively negatively graded) when $\varpi^+(l_n \otimes W_m) \subset W_{n+m}$ (respectively $\varpi^-(l_n \otimes W_m) \subset W_{n+m}$) for any $n, m$ (cf. [2, Definition 0.1]), and, moreover, we say that a positively graded module $(\varpi^+, W)$ (respectively a negatively graded module $(\varpi^-, W)$) is transitive when the following condition holds (cf. [2, Definition 1.1]):

$$\text{for } w \in W_m, m \geq 1, \varpi^+(V_{−1} \otimes w) = \{0\} \text{ implies } w = 0$$

(respectively $\text{for } w \in W_m, m \leq −1, \varpi^-(V_{1} \otimes w) = \{0\} \text{ implies } w = 0$).

**Notation 1.4.** In this paper, we denote the set of all natural numbers, integers and complex numbers by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{C}$ respectively. We denote the set of matrices of size $n \times m$ ($n, m \in \mathbb{N}$) whose entries are belong to a ring $R$ by $M(n, m; R)$, the unit matrix and the zero matrix of size $n$ by $I_n$ and $O_n$ respectively. Moreover, $\delta_{kl}$ stands for the Kronecker delta, $\text{Tr}(A)$ stands for the trace of a square matrix $A$.

## 2 Standard pentads and corresponding Lie algebras

### 2.1 Standard pentads

Let us start with the definitions of $\Phi$-map and standard pentads.

**Definition 2.1 ($\Phi$-map).** Let $\mathfrak{g}$ be a non-zero Lie algebra with non-degenerate invariant bilinear form $B_0$, $\rho : \mathfrak{g} \otimes V \to V$ a representation of $\mathfrak{g}$ on a vector space $V$ and $\mathcal{V}$ a $\mathfrak{g}$-submodule of $\text{Hom}(V, \mathfrak{g})$ all defined over a field $\mathfrak{k}$. We denote the canonical pairing between $V$ and $\text{Hom}(V, \mathfrak{g})$ by $\langle \cdot , \cdot \rangle$ and...
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the canonical representation of $\mathfrak{g}$ on $V$ by $\varrho$. Then, if a pentad $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ has a linear map $\Phi_\rho : V \otimes V \to \mathfrak{g}$ which satisfies an equation

$$B_0(a, \Phi_\rho(v \otimes \phi)) = \langle \rho(a \otimes v), \phi \rangle = -\langle v, \varrho(a \otimes \phi) \rangle$$

for any $a \in \mathfrak{g}$, $v \in V$ and $\phi \in \mathcal{V}$ (cf. [9, Definition 1.9]), we call it a $\Phi$-map of the pentad $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$. Moreover, when a pentad $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ has a $\Phi$-map, we define a linear map $\Psi_\rho : V \otimes V \to \mathfrak{g}$ by:

$$B_0(a, \Psi_\rho(\phi \otimes v)) = \langle v, \varrho(a \otimes \phi) \rangle = -\langle \rho(a \otimes v), \phi \rangle.$$ 

We call this map $\Psi_\rho$ a $\Psi$-map of $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$.

In general, a pentad might not have a $\Phi$-map. If a pentad $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ has a $\Phi$-map, then the equation (2.1) determines the linear map $\Phi_\rho$ uniquely. Moreover, we have an equation

$$\Phi_\rho(v \otimes \phi) + \Psi_\rho(\phi \otimes v) = 0$$

for any $v \in V$ and $\phi \in \mathcal{V}$.

**Definition 2.2** (Standard pentads). We retain to use the notation of Definition 2.1. If a pentad $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ satisfies the following conditions, we call it a standard pentad:

- the restriction of $\langle \cdot, \cdot \rangle$ to $V \times V$ is non-degenerate, (2.3)
- there exists a $\Phi$-map from $V \otimes V$ to $\mathfrak{g}$. (2.4)

**Lemma 2.3.** Under the notation of Definitions 2.1 and 2.2, we have the following claims:

- if $V$ is finite-dimensional, then a vector space $\mathcal{V}$ satisfying (2.3) coincides with $\text{Hom}(V, k)$, (2.5)
- if $\mathfrak{g}$ is finite-dimensional, then any pentad $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ satisfies the condition (2.4). (2.6)

In particular, if both $\mathfrak{g}$ and $V$ are finite-dimensional, then any quadruplet $(\mathfrak{g}, \rho, V, B_0)$ can be naturally regarded as a standard pentad $(\mathfrak{g}, \rho, V, \text{Hom}(V, k), B_0)$.

**Proof.** The claim (2.5) is clear. Let us show the claim (2.6). If $\mathfrak{g}$ is finite-dimensional, then the dual space of $\mathfrak{g}$ can be identified with $\mathfrak{g}$. Precisely, if $\mathfrak{g}$ is finite-dimensional, then any linear map $f : \mathfrak{g} \to \mathfrak{k}$ corresponds to some element $A \in \mathfrak{g}$ such that

$$f(a) = B_0(a, A)$$

for any $a \in \mathfrak{g}$. Thus, for any $v \in V$ and $\phi \in \mathcal{V}$, there exists an element of $\mathfrak{g}$ which corresponds to a linear map $\mathfrak{g} \to \mathfrak{k}$ defined by

$$a \mapsto \langle \rho(a \otimes v), \phi \rangle.$$ 

It means that the pentad $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ has the $\Phi$-map. 

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Remark 2.4. If \( V \) is infinite-dimensional, then a submodule \( \mathcal{V} \) of \( \text{Hom}(V, \mathfrak{g}) \) satisfying the condition \( (2.3) \) does not necessary coincide with \( \text{Hom}(V, \mathfrak{g}) \).

Remark 2.5. In general, a Lie algebra \( \mathfrak{g} \) and its module \((\rho, V)\) might not have a \( \mathfrak{g} \)-submodule \( V \subset \text{Hom}(V, \mathbb{C}) \) and a bilinear form \( B_0 \) such that a pentad \((\mathfrak{g}, \rho, V, V, B_0)\) is standard.

Example 2.6. Let \( \mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \), \( K \) be the Killing form on \( \mathfrak{g} \) and \( \mathcal{L}(\mathfrak{g}) = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \) be the loop algebra (see [4, Ch.7]). Let \( \mathcal{K} \) be a bilinear form on \( \mathcal{L}(\mathfrak{g}) \) defined by:

\[
K_{\mathcal{K}}(t^n \otimes X, t^m \otimes Y) := \delta_{n+m,0}K(X, Y).
\]

Clearly, the bilinear form \( K_{\mathcal{K}} \) is non-degenerate and invariant. Thus, we can regard \( \mathcal{L}(\mathfrak{g}) \) itself as a \( \mathcal{L}(\mathfrak{g}) \)-submodule of \( \text{Hom}(\mathcal{L}(\mathfrak{g}), \mathbb{C}) \) via the non-degenerate invariant bilinear form \( K_{\mathcal{K}} \). Then, a pentad \((\mathcal{L}(\mathfrak{g}), \text{ad}, \mathcal{L}(\mathfrak{g}), \mathcal{K})\), where \( \text{ad} \) stands for the adjoint representation, is standard. In fact, we have the condition \( (2.3) \) clearly, and, we can identify the bracket product \( \mathcal{L}(\mathfrak{g}) \times \mathcal{L}(\mathfrak{g}) \to \mathcal{L}(\mathfrak{g}) \) with the \( \Phi \)-map of \((\mathcal{L}(\mathfrak{g}), \text{ad}, \mathcal{L}(\mathfrak{g}), \mathcal{K})\), denoted by \( \Phi^1_{\text{ad}} \).

However, a pentad \((\mathcal{L}(\mathfrak{g}), \text{ad}, \mathcal{L}(\mathfrak{g}), \text{Hom}(\mathcal{L}(\mathfrak{g}), \mathbb{C}), \mathcal{K})\) is not standard since it does not have the \( \Phi \)-map. In fact, if we assume that this pentad might have the \( \Phi \)-map, denoted by \( \Phi^2_{\text{ad}} \), and put

\[
H_0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_0 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y_0 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{g},
\]

\[
\phi_{Y_0} \in \text{Hom}(\mathcal{L}(\mathfrak{g}), \mathbb{C}), \quad \langle t^n \otimes X, \phi_{Y_0} \rangle := K(Y_0, X),
\]

then an element \( \Phi^2_{\text{ad}}((1 \otimes X_0) \otimes \phi_{Y_0}) \in \mathcal{L}(\mathfrak{g}) \) satisfies the equation

\[
K_{\mathcal{K}}(t^n \otimes H_0, \Phi^2_{\text{ad}}((1 \otimes X_0) \otimes \phi_{Y_0})) = \langle [t^n \otimes H_0, 1 \otimes X_0], \phi_{Y_0} \rangle = \langle t^n \otimes 2X_0, \phi_{Y_0} \rangle = K(Y_0, 2X_0) = 8 \quad (2.7)
\]

for any \( n \in \mathbb{Z} \). The Lie algebra \( \mathcal{L}(\mathfrak{g}) \) does not have an element satisfying \( (2.7) \) for any \( n \in \mathbb{Z} \), and, thus, the pentad \((\mathcal{L}(\mathfrak{g}), \text{ad}, \mathcal{L}(\mathfrak{g}), \text{Hom}(\mathcal{L}(\mathfrak{g}), \mathbb{C}), \mathcal{K})\) does not have the \( \Phi \)-map.

On the \( \Phi \)-map and \( \Psi \)-map of a standard pentad, we have similar properties to ones of the \( \Phi \)-map and \( \Psi \)-map of a standard quadruplet (see [9]).

Proposition 2.7. The \( \Phi \)-map and the \( \Psi \)-map of a standard quadruplet \((\mathfrak{g}, \rho, V, V, B_0)\) are homomorphisms of Lie modules. (cf. [9, Proposition 1.3]).

Proof. We can prove it by the same way to [9 Proposition 1.3].
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**Definition 2.8.** Let \((g, \rho, V, \mathcal{V}, B_0)\) be a standard pentad. For each element \(v \in V\) and \(\phi \in \mathcal{V}\), we define linear maps \(\Phi_{\rho,v} \in \text{Hom}(\mathcal{V}, g)\) and \(\Psi_{\rho,\phi} \in \text{Hom}(V, g)\) by:

\[
\Phi_{\rho,v}(\psi) := \Phi_{\rho}(v \otimes \psi), \quad \Psi_{\rho,\phi}(u) := \Psi_{\rho}(\phi \otimes u)
\]

for any \(u \in V\) and \(\psi \in \mathcal{V}\). Moreover, we define the following linear maps:

\[
\Phi_{\rho}^g : V \to \text{Hom}(\mathcal{V}, g) \\
\phi \mapsto \Phi_{\rho,\phi}.
\]

\[
\Psi_{\rho}^g : \mathcal{V} \to \text{Hom}(V, g) \\
v \mapsto \Phi_{\rho,v}.
\]

To simplify, we denote \(\Phi_{\rho,v}(\psi)\) and \(\Psi_{\rho,\phi}(u)\) by \(v(\psi)\) and \(\phi(u)\) respectively.

**Definition 2.9.** Let \((g, \rho, V, \mathcal{V}, B_0)\) be a standard pentad. Put \(V_0 := g\), \(V_1 := V\) and \(V_{-1} := \mathcal{V}\) and denote the canonical representations of \(g\) on \(V_0\) and \(V_{\pm 1}\) by \(\rho_0\) and \(\rho_{\pm 1}\). We define homomorphisms of \(g\)-modules \(p_0\) and \(q_0\) by:

\[
p_0 : V_1 \otimes V_0 \to V_1 \\
v_1 \otimes a \mapsto -\rho_1(a \otimes v_1),
\]

\[
q_0 : V_{-1} \otimes V_0 \to V_{-1} \\
\phi_{-1} \otimes b \mapsto -\rho_{-1}(b \otimes \phi_{-1}).
\]

Moreover, we define homomorphisms of \(g\)-modules \(p_1\) and \(q_{-1}\) by:

\[
p_1 : V_1 \otimes V_1 \to \text{Hom}(V_{-1}, V_1) \\
v_1 \otimes u_1 \mapsto (\eta_{-1} \mapsto \rho_1(v_1(\eta_{-1}) \otimes u_1) - \rho_1(u_1(\eta_{-1}) \otimes v_1)),
\]

\[
q_{-1} : V_{-1} \otimes V_{-1} \to \text{Hom}(V_1, V_{-1}) \\
\phi_{-1} \otimes \psi_{-1} \mapsto (\xi_{i} \mapsto \rho_{-1}(\phi_{-1}(\xi_{i}) \otimes \psi_{-1}) - \rho_{-1}(\psi_{-1}(\xi_{i}) \otimes \phi_{-1})),
\]

where \(v_1(\eta_{-1}) \in V_0\) and \(\phi_{-1}(\xi_{i}) \in V_0\) stand for \(\Phi_{\rho,v_1}(\eta_{-1})\) and \(\Psi_{\rho,\phi_{-1}}(\xi_{i})\) respectively.

Moreover, suppose that \(i \geq 2\) and there exist \(g\)-modules \((\rho_{-i+1}, V_{-i+1})\) and \((\rho_{-i}, V_{-i})\) and homomorphisms of \(g\)-modules \(p_{i-1} : V_1 \otimes V_{i-1} \to \text{Hom}(V_{-1}, V_{i-1})\) and \(q_{-i+1} : V_{-1} \otimes V_{-i+1} \to \text{Hom}(V_1, V_{-i+1})\). Then, we put \(V_i := \text{Im} p_{i-1}, V_{-i} := \text{Im} q_{-i+1}\) and define linear maps \(p_i, q_{-i}\) by:

\[
p_i : V_1 \otimes V_i \to \text{Hom}(V_{-1}, V_i) \\
v_1 \otimes u_i \mapsto (\eta_{-1} \mapsto \rho_i(v_1(\eta_{-1}) \otimes u_i) + p_{i-1}(v_1 \otimes u_i(\eta_{-1}))),
\]

\[
q_{-i} : V_{-1} \otimes V_{-i} \to \text{Hom}(V_i, V_{-1}) \\
\phi_{-1} \otimes \psi_{-i} \mapsto (\xi_{1} \mapsto \rho_{-i}(\phi_{-1}(\xi_{1}) \otimes \psi_{-i}) + q_{-i+1}(\phi_{-1} \otimes \psi_{-i}(\xi_{1}))),
\]

where \(u_i(\eta_{-1}) \in V_{-i+1}\) and \(\psi_{-i}(\xi_{1}) \in V_{-i+1}\) are the images of \(\eta_{-1}\) and \(\xi_{i}\) via \(u_i\) and \(\psi_{-i}\) respectively. Then, the linear maps \(p_i\) and \(q_{-i}\) are homomorphisms of \(g\)-modules (cf. [9 Proposition 1.10]). We
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denote the images of \( p_i \) and \( q_{-1} \) by \( V_{i+1} \) and \( V_{-i-1} \) and the canonical representations of \( \mathfrak{g} \) on \( V_{i+1} \) and \( V_{-i-1} \) by \( \rho_{i+1} \) and \( \rho_{-i-1} \) respectively. Thus, inductively, we obtain \( \mathfrak{g} \)-modules \( V_n \) and representations \( \rho_n \) of \( \mathfrak{g} \) on \( V_n \) for all \( n \in \mathbb{Z} \). We call \( V_n \) the \( n \)-\textit{graduation} of \((\mathfrak{g}, \rho, V, \mathcal{V}, B_0)\).

**Remark 2.10.** For any \( v_1 \in V_1 \) and \( \phi_{-1} \in V_{-1} \), we have

\[
\begin{align*}
p_1(v_1 \otimes v_1)(\eta_{-1}) &= \rho_1(v_1(\eta_{-1}) \otimes v_1) - \rho_1(v_1(\eta_{-1}) \otimes v_1) = 0, \\
q_{-1}(\phi_{-1} \otimes \phi_{-1})(\xi_{1}) &= \rho_{-1}(\phi_{-1}(\xi_{1}) \otimes \phi_{-1}) - \rho_{-1}(\phi_{-1}(\xi_{1}) \otimes \phi_{-1}) = 0.
\end{align*}
\]

In general, we do not assume that \( \rho \) and \( q \) are surjective, i.e. we do not assume that \( V_1 = \text{Im} \rho_0 \) and \( V_{-1} = \text{Im} q_0 \). In particular cases where these linear maps are surjective, we have the following proposition.

**Proposition 2.11.** If \( \rho : \mathfrak{g} \otimes V \to V \) and \( q : \mathfrak{g} \otimes V \to V \) are surjective, then \( \Phi_{\rho}^q \) and \( \Psi_{\rho}^q \) are injective, and, thus, \( V \) and \( V \) can be regarded as \( \mathfrak{g} \)-submodules of \( \text{Hom}(V_{-1}, V_0) \) and \( \text{Hom}(V_{-1}, V_0) \) respectively.

**Proof.** To show this proposition, we use the condition \( \text{(2.3)} \). Let us show that the linear map \( \Phi_{\rho}^q \) is injective. We take an arbitrary element \( v \in V \) which satisfies that \( \Phi_{\rho,v} = 0 \). Then we have

\[
0 = B_0(a, \Phi_{\rho,v}(\phi)) = \langle \rho(a \otimes v), \phi \rangle = -\langle v, \phi(a \otimes \phi) \rangle
\]  
for all \( a \in \mathfrak{g} \) and \( \phi \in \mathcal{V} \). By the condition \( \text{(2.3)} \) and the assumption that \( \rho \) is surjective, we have that \( v = 0 \). Therefore, we obtain that \( \Phi_{\rho}^q \) is injective. Similarly, we can show that \( \Psi_{\rho}^q \) is injective. \( \blacksquare \)

**Definition 2.12.** We define the following bilinear maps

\[
\begin{align*}
[\cdot, \cdot]_0^n : & V_0 \times V_n \to V_n, \\
[\cdot, \cdot]_1^n : & V_1 \times V_n \to V_{n+1}, \\
[\cdot, \cdot]_{-1}^{-1} : & V_{-1} \times V_n \to V_{n-1}
\end{align*}
\]
by:

\[
\begin{align*}
[a_0, z_n]_0^n &:= \rho_n(a_0 \otimes z_n), \\
[x_1, z_n]_1^n &:= \begin{cases} 
p_n(x_1 \otimes z_n) & (n \geq 0) \\
-z_n(x_1) & (n \leq -1) \end{cases}, \\
[y_{-1}, z_n]_{-1}^{-1} &:= \begin{cases} 
-z_n(y_{-1}) & (n \geq 1) \\
q_n(y_{-1} \otimes z_n) & (n \leq 0) \end{cases}
\end{align*}
\]
where \( a_0 \in V_0, x_1 \in V_1, y_{-1} \in V_{-1} \) and \( z_n \in V_n \). Note that \( z_n(x_1) \) stands for \( \Psi_{\rho, z_{-1}}(x_1) \) when \( n = -1 \) and the image of \( x_1 \) via \( z_n \in \text{Hom}(V_1, V_{n+1}) \) when \( n \leq -2 \). Moreover, for \( i \geq 1 \), we define the following bilinear maps

\[
[\cdot, \cdot]_{i+1}^i : V_{i+1} \times V_n \to V_{i+n+1}, \quad [\cdot, \cdot]_{i-1}^{-i} : V_{i-1} \times V_n \to V_{i+n-1}
\]
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by:

\[ p_i(x_1 \otimes z_i), w_n]_{n+1}^i := [x_1, [z_i, w_n]_{n+1}^1] - [z_i, [x_1, w_n]_{n+1}^1] \quad (2.9) \]

\( (x_1 \in V_1, z_i \in V_i, w_n \in V_n) \)

and

\[ q_{-i}(y_{-1} \otimes \omega_{-i}), w_n]_{n-1}^{-i} := [y_{-1}, [\omega_{-i}, w_n]^{-i}]_{n+1} - [\omega_{-i}, [y_{-1}, w_n]^{-i}]_{n+1} \quad (2.10) \]

\( (y_{-1} \in V_{-1}, \omega_{-i} \in V_{-i}, w_n \in V_n) \)

inductively. Then the bilinear maps (2.9) and (2.10) are well-defined. It can be shown by the same argument to the argument of [9, Propositions 2.5 and 2.6]. Consequently, we can define a bilinear map \([\cdot, \cdot]_m : V_n \times V_m \to V_{n+m}\) for any \(n, m \in \mathbb{Z}\).

**Definition 2.13.** For a standard pentad \((g, \rho, V, V, B_0)\), we denote a direct sum of its \(n\)-graduations by \(L(g, \rho, V, V, B_0)\), i.e.

\[ L(g, \rho, V, V, B_0) := \bigoplus_{n \in \mathbb{Z}} V_n. \]

Moreover, we define a bilinear map \([\cdot, \cdot]_n : L(g, \rho, V, V, B_0) \times L(g, \rho, V, V, B_0) \to L(g, \rho, V, V, B_0)\) by

\[ [x_n, y_m] := [x_n, y_m]_n \quad (2.11) \]

for any \(n, m \in \mathbb{Z}, x_n \in V_n\) and \(y_m \in V_m\).

**Proposition 2.14.** This bilinear map \([\cdot, \cdot]_n\) satisfies the following equations

\[ [x, y] + [y, x] = 0, \quad (2.12) \]

\[ [x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \quad (2.13) \]

for any \(x, y, z \in L(g, \rho, V, V, B_0)\).

**Proof.** We can prove it by the same argument to the argument of [9, Propositions 2.9 and 2.10].

As a corollary of Proposition 2.14, we have the following theorem.

**Theorem 2.15** (Lie algebra associated with a standard pentad). Let \((g, \rho, V, V, B_0)\) be a standard pentad over a field \(k\). Then the vector space \(L(g, \rho, V, V, B_0) = \bigoplus_{n \in \mathbb{Z}} V_n\) is a graded Lie algebra with a bracket product \([\cdot, \cdot]_n\) defined in Definition 2.13. We call this graded Lie algebra the Lie algebra associated with \((g, \rho, V, V, B_0)\) (cf. [9, Theorem 2.11]).

**Proof.** If the characteristic of \(k\) is not 2, our claim follows from Proposition 2.14 immediately.
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Assume that the characteristic of \( \mathfrak{k} \) is 2. To prove our claim, it is sufficient to show that the bilinear form \([\cdot, \cdot]\) satisfies the equation

\[
[x,x] = 0 \tag{2.14}
\]

for any \( x \in L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0) \). Let us show the equation (2.14) for the cases where \( x = x_n \in V_n \) for some \( n \in \mathbb{Z} \). For \( n = 0 \), the restriction of \([\cdot, \cdot] : L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0) \times L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0) \to L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)\)
to \( V_0 \times V_0 \) coincides with the bracket product of \( \mathfrak{g} \). Thus we have \([x_0, x_0] = 0\) for any \( x_0 \in V_0 \). For \( n \geq 1 \), by Proposition 2.11 we have

\[
[[x_n, x_n], \phi_{-1}] = [[x_n, \phi_{-1}], x_n] + [x_n, [x_n, \phi_{-1}]] = 0 \tag{2.15}
\]

for any \( \phi_{-1} \in V_{-1} \) and \( x_n \in V_n \). Thus, we have the equation \([x_n, x_n] = 0\) in \( V_{2n} = \text{Hom}(V_{-1}, V_{2n-1}) \) for any \( x_n \in V_n \). Similarly, we have the same equation \([x_n, x_n] = 0\) for any \( n \leq -1 \) and \( x_n \in V_n \).

Therefore, we have the equation (2.14) for the cases where \( x = x_n \in V_n \) for some \( n \in \mathbb{Z} \).

Now, we take an arbitrary element \( x \in L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0) \). There exist integers \( n_1, \ldots, n_k \) and elements \( x_{n_1} \in V_{n_1}, \ldots, x_{n_k} \in V_{n_k} \) such that \( x = x_{n_1} + \cdots + x_{n_k} \). Then we have

\[
[x, x] = [x_{n_1} + \cdots + x_{n_k}, x_{n_1} + \cdots + x_{n_k}] = \sum_{i=1}^{k} [x_{n_i}, x_{n_i}] + \sum_{i \neq j} \left( [x_{n_i}, x_{n_j}] + [x_{n_j}, x_{n_i}] \right) = 0.
\]

Thus we have the equation (2.14) for any \( x \in L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0) \). This completes the proof. \( \blacksquare \)

**Remark 2.16.** Note that we can prove Theorem 2.15 without the assumption that the bilinear form \( B_0 \) is symmetric.

Note that \( V_0 = \mathfrak{g} \) and that the \( V_0 \)-modules \( V_0, V_1, V_{-1} \) are isomorphic to \( \mathfrak{g}, V, \mathcal{V} \) respectively. In this sense, we can say that the objects \( \mathfrak{g}, (\rho, V), (\phi, \mathcal{V}) \) can be embedded into \( L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0) \).

In particular, when \( \rho \) and \( \phi \) are faithful and surjective, we have a similar result on the structure of a graded Lie algebra of the form \( L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0) \) to the result which is obtained by H. Rubenthaler in [3 Proposition 3.4.2]. We can show the following proposition by Proposition 2.11 immediately.

**Proposition 2.17.** Let \((\mathfrak{g}, \rho, V, \mathcal{V}, B_0)\) be a standard pentad. If both \( \rho : \mathfrak{g} \otimes V \to V \) and \( \phi : \mathfrak{g} \otimes \mathcal{V} \to \mathcal{V} \) are faithful and surjective, then the graded Lie algebra \( L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0) \) is transitive.

### 2.2 Standard pentads with a symmetric bilinear form

In the previous section, we proved that for any standard pentad \((\mathfrak{g}, \rho, V, \mathcal{V}, B_0)\), there exists a graded Lie algebra such that \( \mathfrak{g}, \rho, V \) and \( \mathcal{V} \) can be embedded into it. In this section, we discuss cases where \( B_0 \) is symmetric. In these cases, we can also embed \( B_0 \) into \( L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0) \) and we can obtain some useful properties.
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**Proposition 2.18.** Let \((g, \rho, V, \mathcal{V}, B_0)\) be a standard pentad such that \(B_0\) is symmetric. We define a symmetric bilinear form \(B_L\) on \(L(g, \rho, V, \mathcal{V}, B_0)\) inductively as follows:

\[
\begin{align*}
B_L(a, b) &= B_0(a, b), \\
B_L(v, \phi) &= \langle v, \phi \rangle, \\
B_L(p_i(v_1 \otimes u_i), q_{-1}(\phi_{-1} \otimes \psi_{-1})) &= B_L(u_i, [q_{-1}(\phi_{-1} \otimes \psi_{-1}), v_1]), \\
B_L(x_n, y_m) &= 0
\end{align*}
\]

for any \(a, b \in V_0, v \in V, \phi \in \mathcal{V}, i \geq 1, v_1 \in V_1, \phi_{-1} \in V_{-1}, u_i \in V_i, \psi_{-i} \in V_{-i}, n, m \in \mathbb{Z}, n + m \neq 0, x_n \in V_n\) and \(y_m \in V_m\). Then \(B_L\) is a non-degenerate symmetric invariant bilinear form on \(L(g, \rho, V, \mathcal{V}, B_0)\) (cf. [9, Proposition 3.2]).

**Proof.** Note that it is clear that the restriction of \(B_L\) to \(V_0 \times V_0\) and \(V_1 \times V_{-1}\) is well-defined. Let us show the well-definedness of \(B_L\) on \(V_2 \times V_{-2}\). For any \(v_1, u_1 \in V_1\) and \(\phi_{-1}, \psi_{-1} \in V_{-1}\), we have

\[
B_L(u_1, [q_{-1}(\phi_{-1} \otimes \psi_{-1}), v_1]) = B_L(u_1, [[\phi_{-1}, v_1], \psi_{-1}] + [\phi_{-1}, [\psi_{-1}, v_1]])
\]

\[
= \langle u_1, [[\phi_{-1}, v_1], \psi_{-1}] + [\phi_{-1}, [\psi_{-1}, v_1]] \rangle
\]

\[
= B_0([\phi_{-1}, v_1], \psi_{-1}(u_1)) - B_0([\psi_{-1}, v_1], \phi_{-1}(u_1))
\]

\[
= B_0([\phi_{-1}, v_1], \psi_{-1}(u_1)) - B_0([\psi_{-1}, v_1], \phi_{-1}(u_1)) \quad \text{(by the assumption that \(B_0\) is symmetric)}
\]

\[
= B_0([v_1, \phi_{-1}], u_1(\psi_{-1})) - B_0(u_1(\phi_{-1}), [v_1, \psi_{-1}])
\]

\[
= \langle [v_1, \phi_{-1}], u_1 \rangle + [v_1, [u_1, \phi_{-1}]], \psi_{-1})
\]

\[
= B_L([p_1(v_1 \otimes u_1), \phi_{-1}], \psi_{-1}). \quad (2.16)
\]

Thus, if \(v_1^1, \ldots, v_1^l, u_1^1, \ldots, u_1^k \in V_1\) and \(\phi_{-1}^1, \ldots, \phi_{-1}^l, \psi_{-1}^1, \ldots, \psi_{-1}^k \in V_{-1}\) satisfy the equation

\[
\sum_{s=1}^l p_1(v_1^s \otimes u_1^s) = 0, \quad \sum_{t=1}^k q_{-1}(\phi_{-1}^t \otimes \psi_{-1}^t) = 0,
\]

then

\[
\sum_{s=1}^l B_L(u_1^s, [q_{-1}(\phi_{-1} \otimes \psi_{-1}), v_1]) = \sum_{s=1}^l B_L([p_1(v_1^s \otimes u_1^s), \phi_{-1}], \psi_{-1}) = 0,
\]

\[
\sum_{t=1}^k B_L(u_1, [q_{-1}(\phi_{-1}^t \otimes \psi_{-1}^t), v_1]) = 0
\]

for any \(v_1, u_1 \in V_1\) and \(\phi_{-1}, \psi_{-1} \in V_{-1}\), that is, we have the well-definedness of \(B_L\) on \(V_2 \times V_{-2}\). This \(B_L \mid_{V_2 \times V_{-2}}\) is \(g\)-invariant. Moreover, by a similar argument, we have the well-definedness of \(B_L\) on \(V_i \times V_{-i}\) for each \(i \geq 3\) by induction (see [9 section 1.2]). Consequently, we can show the well-definedness of \(B_L\) on the whole \(L(g, \rho, V, \mathcal{V}, B_0)\) and that \(B_L\) is non-degenerate symmetric invariant by the same argument as the argument in [9 section 1.2 and Proposition 3.2].
Remark 2.19. We need the assumption that $B_0$ is symmetric to show that the bilinear form $B_L$ is $L(g, r, V, V, B_0)$-invariant. Precisely, we need this assumption to show an equation

$$B_L(v_1, [\phi_{-1}, a]) = B_L([v_1, \phi_{-1}], a)$$

for any $a \in V_0, v_1 \in V_1, \phi_{-1} \in V_{-1}$.

Under the assumption that $B_0$ is symmetric, the graded Lie algebra is characterized by the existence of such a bilinear form. The following is a proposition concerning the “universality” and “uniqueness” of Lie algebras associated with a standard pentad with a symmetric bilinear form.

Theorem 2.20. Let $\mathfrak{L} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{L}_n$ be a graded Lie algebra which has a non-degenerate symmetric invariant bilinear form $B_\Sigma$. If $\mathfrak{L}$ and $B_\Sigma$ satisfy the following conditions, then a pentad $(\mathfrak{L}_0, \text{ad}, \mathfrak{L}_1, \mathfrak{L}_{-1}, B_\Sigma |_{\mathfrak{L}_0 \times \mathfrak{L}_0})$ is standard and $\mathfrak{L}$ is isomorphic to $L(\mathfrak{L}_0, \text{ad}, \mathfrak{L}_1, \mathfrak{L}_{-1}, B_\Sigma |_{\mathfrak{L}_0 \times \mathfrak{L}_0})$:

$$\mathfrak{L}_{i+1} = [\mathfrak{L}_1, \mathfrak{L}_i], \mathfrak{L}_{i-1} = [\mathfrak{L}_{-1}, \mathfrak{L}_i] \text{ for all } i \geq 1,$$

the restriction of $B_\Sigma$ to $\mathfrak{L}_i \times \mathfrak{L}_{-i}$ is non-degenerate for any $i \geq 0,$

where $\text{ad}$ stands for the adjoint representation of $\mathfrak{L}$ on itself (cf. [9, Proposition 3.3]).

Proof. First of all, let us check that the pentad $(\mathfrak{L}_0, \text{ad}, \mathfrak{L}_1, \mathfrak{L}_{-1}, B_\Sigma |_{\mathfrak{L}_0 \times \mathfrak{L}_0})$ is standard. By (2.18), we can obtain that $B_\Sigma |_{\mathfrak{L}_0 \times \mathfrak{L}_0}$ is non-degenerate and that $\mathfrak{L}_1$ and $\mathfrak{L}_{-1}$ satisfy the condition (2.19). It is easy to show that we can identify the restriction of the bracket product $[\cdot, \cdot]$ of $\mathfrak{L}$ to $\mathfrak{L}_1 \times \mathfrak{L}_{-1} \to \mathfrak{L}_0$ with the $\Phi$-map of the pentad $(\mathfrak{L}_0, \text{ad}, \mathfrak{L}_1, \mathfrak{L}_{-1}, B_\Sigma |_{\mathfrak{L}_0 \times \mathfrak{L}_0})$. Thus, the condition (2.20) holds.

We denote the $n$-graduation of $(\mathfrak{L}_0, \text{ad}, \mathfrak{L}_1, \mathfrak{L}_{-1}, B_\Sigma |_{\mathfrak{L}_0 \times \mathfrak{L}_0})$ by $(\mathfrak{L})_n$ for any $n \in \mathbb{Z}$ and a bilinear form on $L(\mathfrak{L}_0, \text{ad}, \mathfrak{L}_1, \mathfrak{L}_{-1}, B_\Sigma |_{\mathfrak{L}_0 \times \mathfrak{L}_0})$ obtained in Proposition 2.18 by $(B)_\Sigma$. Let $\sigma_0 : (\mathfrak{L})_0 \to \mathfrak{L}_0$ and $\sigma_{\pm 1} : (\mathfrak{L})_{\pm 1} \to \mathfrak{L}_{\pm 1}$ be the identity maps respectively. Then the linear maps $\sigma_0$ and $\sigma_{\pm 1}$ satisfy the following equations:

$$\sigma_0(a), \sigma_{\pm 1}(x_{\pm 1})] = \sigma_{\pm 1}([a, x_{\pm 1}]),$$

$$\sigma_1(x_1), \sigma_{-1}(x_{-1})] = \sigma_0([x_1, x_{-1}])$$

for any $a \in (\mathfrak{L})_0$ and $x_{\pm 1} \in (\mathfrak{L})_{\pm 1}$. Indeed, the equation (2.19) is clear, and, we have

$$B_\Sigma(\sigma_0(b), [\sigma_1(x_1), \sigma_{-1}(x_{-1})]) = B_\Sigma([\sigma_0(b), \sigma_1(x_1)], \sigma_{-1}(x_{-1})) = B_\Sigma(\sigma_1([b, x_1]), \sigma_{-1}(x_{-1})) = (B_\Sigma([b, x_1], x_{-1}) = (B_\Sigma(b, [x_1, x_{-1}]) = B_\Sigma(\sigma_0(b), \sigma_0([x_1, x_{-1}])$$

for any $b \in (\mathfrak{L})_0$. Thus, we can obtain the equation (2.20).

For each $i \geq 1$, we define linear maps $\sigma_{i+1} : (\mathfrak{L})_{i+1} \to \mathfrak{L}_{i+1}$ and $\sigma_{i-1} : (\mathfrak{L})_{i-1} \to \mathfrak{L}_{i-1}$ by:

$$\sigma_{i+1} : p_i(x_1 \otimes z_i) \mapsto [\sigma_1(x_1), \sigma_i(z_i)],$$

$$\sigma_{i-1} : q_i(x_1 \otimes z_i) \mapsto [\sigma_{-1}(x_{-1}), \sigma_{-i}(z_i)]$$

(2.23)
for any $x_{\pm 1} \in (\mathfrak{L})_{\pm 1}$ and $z_{\pm 1} \in (\mathfrak{L})_{\pm 1}$ inductively. Note that it follows from (2.19) that the linear maps $\sigma_1$ and $\sigma_{-1}$ on $\rho(\mathfrak{g} \otimes V)$ and $\rho(\mathfrak{g} \otimes V)$ defined by the same equations as (2.22) and (2.23) where $i = 0$ coincide with the identity maps respectively. We can prove that the linear maps $\sigma_n$ $(n \in \mathbb{Z})$ are well-defined and satisfy
\[
[\sigma_0(a), \sigma_n(z_n)] = \sigma_n([a, z_n]), \quad (2.24)
\]
\[
[\sigma_{\pm 1}(x_{\pm 1}), \sigma_n(z_n)] = \sigma_{n \pm 1}([x_{\pm 1}, z_n]) \quad (2.25)
\]
for any $n \in \mathbb{Z}$, $a \in (\mathfrak{L})_0$, $x_{\pm 1} \in (\mathfrak{L})_{\pm 1}$ and $z_n \in (\mathfrak{L})_n$ by a similar argument to the argument of [9, Proposition 3.3]. Then a linear map $\sigma : L(\mathfrak{L}_0, \text{ad}, \mathfrak{L}_1, \mathfrak{L}_2, \mathcal{B}_\mathfrak{L}|_{\mathfrak{L}_0 \times \mathfrak{L}_0}) \to \mathfrak{L}$ defined by
\[
\sigma(z_n) := \sigma_n(z_n), \quad (2.26)
\]
where $n \in \mathbb{Z}$ and $z_n \in (\mathfrak{L})_n \subset L(\mathfrak{L}_0, \text{ad}, \mathfrak{L}_1, \mathfrak{L}_2, \mathcal{B}_\mathfrak{L}|_{\mathfrak{L}_0 \times \mathfrak{L}_0})$, is an isomorphism of Lie algebras. We can also prove this by a similar argument to the argument of [9, Proposition 3.3].

As a corollary of Theorem 2.20 we can say that the theory of standard pentads is an extension of the theory of standard quadruplets.

**Proposition 2.21.** Let $(\mathfrak{g}, \rho, V, B_0)$ be a standard quadruplet (see [9, Definition 1.9]). Then the Lie algebra $L(\mathfrak{g}, \rho, V, B_0)$ associated with $(\mathfrak{g}, \rho, V, B_0)$ (see [9, Theorem 2.11]) is isomorphic to the Lie algebra $L(\mathfrak{g}, \rho, V, \text{Hom}(V, \mathbb{C}), B_0)$.

**Definition 2.22.** Let $(\mathfrak{g}^1, \rho^1, V^1, B^1_0)$ and $(\mathfrak{g}^2, \rho^2, V^2, B^2_0)$ be standard pentads. We say that these pentads are *equivalent* if and only if there exists an isomorphism of Lie algebras $\tau : \mathfrak{g}^1 \to \mathfrak{g}^2$, linear isomorphisms $\sigma : V^1 \to V^2$, $\zeta : V^1 \to V^2$ and a non-zero element $c \in \mathfrak{t}$ such that
\[
\sigma(\rho^1(a^1 \otimes x^1)) = \rho^2(\tau(a^1) \otimes \sigma(x^1)), \quad (2.27)
\]
\[
\zeta(\rho^1(a^1 \otimes y^1)) = \rho^2(\tau(a^1) \otimes \zeta(y^1)), \quad (2.28)
\]
\[
\langle x^1, y^1 \rangle^1 = \langle \sigma(x^1), \zeta(y^1) \rangle^2, \quad (2.29)
\]
\[
B^1_0(a^1, b^1) = cB^2_0(\tau(a^1), \tau(b^1)) \quad (2.30)
\]
where $a^1, b^1 \in \mathfrak{g}^1$, $x^1 \in V^1$, $y^1 \in V^1$ and $\langle \cdot, \cdot \rangle^i$ stands for the pairing between $V^i$ and $V^i$ ($i = 1, 2$). We denote this equivalence relation by
\[
(\mathfrak{g}^1, \rho^1, V^1, B^1_0) \simeq (\mathfrak{g}^2, \rho^2, V^2, B^2_0). \quad (2.31)
\]

**Remark 2.23.** Note that if $V$ is finite-dimensional, then linear isomorphisms $\tau$, $\sigma$ satisfying (2.27) induce a linear isomorphism from $V^1 = \text{Hom}(V^1, \mathfrak{t})$ to $V^2 = \text{Hom}(V^2, \mathfrak{t})$ satisfying (2.28) and (2.29).
Proposition 2.24. If standard pentads \((g^1, \rho^1, V^1, B_0^1)\) and \((g^2, \rho^2, V^2, B_0^2)\) are equivalent, then the Lie algebras associated with them are isomorphic, i.e. we have

\[
L(g^1, \rho^1, V^1, B_0^1) \simeq L(g^2, \rho^2, V^2, B_0^2)
\]

(cf. [9, Proposition 3.6]).

Proof. We denote the \(n\)-gradation of \((g^1, \rho^1, V^1, V^1, B_0^1)\) by \(V^1_n\) \((i = 1, 2)\) for all \(n \in \mathbb{Z}\) and the bilinear forms on \(L(g^1, \rho^1, V^1, V^1, B_0^1)\) defined in Proposition 2.18 by \(B^1_i\) \((i = 1, 2)\). Under the notation of Definition 2.22, we define linear maps \(\sigma_0 := \tau : V^1_0 \rightarrow V^2_0\), \(\sigma_1 := \frac{1}{c}\sigma : V^1_1 \rightarrow V^2_1\) and \(\sigma_{-1} := \varsigma : V^1_{-1} \rightarrow V^2_{-1}\). Then, these linear maps \(\sigma_0\) and \(\sigma_{\pm 1}\) satisfy the same equations as (2.19) and (2.20). In fact, the equation (2.19) is clear, and, we have

\[
B^1_0(\sigma_0(a_0), [a_1(x)], \sigma_{-1}(y_{-1})) = B^1_0(\sigma_{1}(a_0, x)), \sigma_{-1}(y_{-1})
\]

\[
= \frac{1}{c}B^1_0([a_0, x], y_{-1}) = \frac{1}{c}B^1_0(a_0, [x, y_{-1}]) = B^1_0(\sigma_0(a_0), \sigma_0([x, y_{-1}]])
\]

for any \(a_0 \in V^1_0, x \in V^1_1\) and \(y_{-1} \in V^1_{-1}\). Thus, we have the equation (2.20). Then, by the same argument as the argument in proof of Theorem 2.20, we can construct an isomorphism \(L(g^1, \rho^1, V^1, B_0^1)\) to \(L(g^2, \rho^2, V^2, B_0^2)\).

\[
\text{Remark 2.25. The converse of Proposition 2.22 is not true. In fact, we have an example of two non-equivalent pentads such that the corresponding Lie algebras are isomorphic (see [9] pp. 398–399).}
\]

Definition 2.26. Let \((g^1, \rho^1, V^1, B_0^1)\) and \((g^2, \rho^2, V^2, B_0^2)\) be standard pentads. Let \(\rho^1 \boxplus \rho^2\) and \(\phi^1 \boxplus \phi^2\) be representations of \(g^1 \oplus g^2\) on \(V^1 \oplus V^2\) and \(V^1 \oplus V^2\) defined by:

\[
(\rho^1 \boxplus \rho^2)((a^1, a^2) \otimes (v^1, v^2)) := (\rho^1(a^1 \otimes v^1), \rho^2(a^2 \otimes v^2)),
\]

\[
(\phi^1 \boxplus \phi^2)((b^1, b^2) \otimes (\phi^1, \phi^2)) := (\phi^1(b^1 \otimes \phi^1), \phi^2(b^2 \otimes \phi^2))
\]

where \(a^i, b^i \in g^i, v^i \in V^i, i = 1, 2\). Let \(B_0^1 \boxplus B_0^2\) be a bilinear form on \(g^1 \oplus g^2\) defined by:

\[
(B_0^1 \boxplus B_0^2)((a^1, a^2), (b^1, b^2)) := B_0^1(a^1, b^1) + B_0^2(a^2, b^2)
\]

where \(a^i, b^i \in g^i, i = 1, 2\). Then, clearly, a pentad \((g^1 \oplus g^2, \rho^1 \boxplus \rho^2, V^1 \oplus V^2, V^1 \oplus V^2, B_0^1 \boxplus B_0^2)\) is also a standard pentad. We call it a \textit{direct sum} of \((g^1, \rho^1, V^1, B_0^1)\) and \((g^2, \rho^2, V^2, B_0^2)\) and denote it by \((g^1, \rho^1, V^1, V^1, B_0^1) \boxplus (g^2, \rho^2, V^2, V^2, B_0^2)\).

Proposition 2.27. Let \((g^1, \rho^1, V^1, V^1, B_0^1)\) and \((g^2, \rho^2, V^2, V^2, B_0^2)\) be standard pentads. Then the Lie algebra \(L((g^1, \rho^1, V^1, V^1, B_0^1) \boxplus (g^2, \rho^2, V^2, V^2, B_0^2))\) is isomorphic to \(L(g^1, \rho^1, V^1, V^1, B_0^1) \oplus L(g^2, \rho^2, V^2, V^2, B_0^2)\) (cf. [9] Proposition 3.9)).
Proof. We retain to use the notation of Proposition 2.24. Then, we have the following $\mathbb{Z}$-grading of $L(g^1, \rho^1, V^1, B^1_0) \oplus L(g^2, \rho^2, V^2, B^2_0)$:

$$L(g^1, \rho^1, V^1, B^1_0) \oplus L(g^2, \rho^2, V^2, B^2_0) = \bigoplus_{n \in \mathbb{Z}} (V^1_n \oplus V^2_n). \quad (2.34)$$

By Theorem 2.20, we have our claim.

**Definition 2.28.** Let $(g, \rho, V, B_0)$ be a standard pentad. We say that $(g, \rho, V, B_0)$ is decomposable if and only if there exist standard pentads $(a, \rho_a, V_a, B_{a,0})$ and $(b, \rho_b, V_b, B_{b,0})$ such that

$$(\dim a + \dim V_a)(\dim b + \dim V_b) \neq 0, \quad (2.35)$$

$$(g, \rho, V, B_0) \simeq (a, \rho_a, V_a, B_{a,0}) \oplus (b, \rho_b, V_b, B_{b,0}). \quad (2.36)$$

If $(g, \rho, V, B_0)$ is not decomposable, we say that $(g, \rho, V, B_0)$ is indecomposable.

**Definition 2.29.** Let $(g, \rho, V, B_0)$ be a standard pentad. We say that $(g, \rho, V, B_0)$ is reducible if and only if there exist an ideal $a$ of $g$ and $g$-submodules $V_a$ and $V_a$ of $V$ and $V$ satisfying that:

$$\{0\} \neq V_a \oplus a \oplus V_a \subseteq V \oplus a \oplus V, \quad (2.37)$$

$$\rho(a \otimes V), \rho(g \otimes V) \subset V_a \text{ and } \rho(a \otimes V), \rho(g \otimes V) \subset V_a, \quad (2.38)$$

$$\Phi_\rho(V_a \otimes V), \Phi_\rho(V \otimes V_a) \subset a. \quad (2.39)$$

And, we say that $(g, \rho, V, B_0)$ is irreducible if and only if it is not reducible.

**Remark 2.30.** If a standard pentad is irreducible, then it is indecomposable.

**Proposition 2.31.** Let $(g, \rho, V, B_0)$ be an irreducible standard pentad. Then the representations $\rho: g \otimes V \rightarrow V$, $\varrho: g \otimes g \rightarrow V$ and the $\Phi$-map $\Phi_\rho : V \otimes V \rightarrow g$ are surjective.

**Proof.** If $\rho(g \otimes V) \oplus \Phi_\rho(V \otimes V) \oplus \rho(g \otimes V) = \{0\}$, it follows that $\dim V = \dim g = \dim V = 0$ from the assumption that $(g, \rho, V, B_0)$ is irreducible. In particular, we have $\rho(g \otimes V) = V = \{0\}$ and $\rho(g \otimes V) = V = \{0\}$. If $\rho(g \otimes V) \oplus \Phi_\rho(V \otimes V) \oplus \rho(g \otimes V) \neq \{0\}$, since it satisfies the conditions (2.38) and (2.39), we have $\rho(g \otimes V) \oplus \Phi_\rho(V \otimes V) \oplus \rho(g \otimes V) = V \oplus g \otimes V$.

**Proposition 2.32.** Let $(g, \rho, V, B_0)$ be an irreducible standard pentad whose representation $\rho$ is faithful and denote the Lie algebra associated with it by $L(g, \rho, V, B_0) = \bigoplus_{n \in \mathbb{Z}} V_n$. Let $N$ (respectively $M$) be an integer such that $V_{N+1}$ is not $\{0\}$ (respectively $V_{-M-1}$ is not $\{0\}$). Then for any non-zero element $z_N \in V_N$ (respectively $z_{-M} \in V_{-M}$), there exists an element $x_1 \in V_1$ such that $[x_1, z_N] \neq 0$ (respectively $y_{-1} \in V_{-1}$ such that $[y_{-1}, z_{-M}] \neq 0$) (cf. [9, Proposition 3.11]).
Proof. When $N \leq -1$, we have our claim by Propositions 2.11, 2.17 and 2.31. When $N = 0$, we have our claim by the assumption that $\rho$ is faithful. Assume that $N \geq 1$ and $V_{N+1} \neq \{0\}$ and put $a_N := \{a_N \in V_N \mid [x_1, a_N] = 0 \text{ for any } x_1 \in V_1\}$ and $a_n := \{a_n \in V_n \mid [x_1, a_n] \in a_{n+1} \text{ for any } x_1 \in V_1\}$ for $n \leq N - 1$ inductively. Then $a_n$ is a $V_0$-submodule of $V_n$ for each $n$, i.e. $[V_0, a_n] \subset a_n$, and we have that $[V_{n+1}, a_n] \subset a_{n+1}$ for any $n \in \mathbb{Z}$ (see [9, the proof of Proposition 3.11]). In particular, $a_{-1} \oplus a_0 \oplus a_1$ satisfies the conditions (2.38) and (2.39). If $a_{-1} \oplus a_0 \oplus a_1 = \mathcal{V} \oplus \mathfrak{g} \oplus V$, then we have $a_N = V_N$ and a contradiction to the assumption that $V_{N+1} \neq \{0\}$. Thus we have $a_1 = \{0\}$, and, thus, $a_2 = \{0\}, \ldots, a_N = \{0\}$ by the transitivity of $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$. Similarly, we have our result for $M$ such that $V_{-M-1} \neq \{0\}$.

Proposition 2.33. Let $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ be an irreducible standard pentad whose representation $\rho$ is faithful. If the Lie algebra $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ is finite-dimensional, then $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ is simple (cf. [7, Proposition 3.12]). Moreover, if $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ is defined over $\mathbb{C}$ and $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ is a finite-dimensional simple Lie algebra, then a triplet $(\mathfrak{g}, \rho, V)$ corresponds to some prehomogeneous vector space of parabolic type (see [7, Theorem 3.13]).

Proof. We can show this by Proposition 2.32 and the same argument to the argument of [7, Proposition 3.12 and Theorem 3.13].

A prehomogeneous vector space of parabolic type (abbrev. a PV of parabolic type) is a PV which can be obtained from a $\mathbb{Z}$-graded finite-dimensional semisimple Lie algebra. PVs of parabolic type are classified by H. Rubenthaler (see [5, 6, 7]).

Example 2.34. Let $m \geq 2$ and $\mathfrak{g} = \mathfrak{gl}_1(\mathbb{C}) \oplus \mathfrak{sl}_m(\mathbb{C})$, $\rho = \Lambda_1$ a representation of $\mathfrak{g}$ on $\mathbb{C}^m$ defined by

$$A_1((a, A) \otimes v) := av + Av \quad (a \in \mathfrak{gl}_1, A \in \mathfrak{sl}_m, v \in V),$$

$B_0 = \kappa_m$ a bilinear form on $\mathfrak{g}$ defined by

$$\kappa_m((a, A), (a', A')) := \frac{m}{m+1}aa' + \text{Tr}(AA') \quad (a, a' \in \mathfrak{gl}_1, A, A' \in \mathfrak{sl}_m).$$

Then, a pentad $(\mathfrak{g}, \rho, V, \text{Hom}(V, \mathbb{C}), B_0) = (\mathfrak{gl}_1 \oplus \mathfrak{sl}_m, \Lambda_1, \mathbb{C}^m, \mathbb{C}^m, \kappa_m)$ is a standard pentad which has a $(m^2 + 2m)$-dimensional graded simple Lie algebra $L(\mathfrak{gl}_1 \oplus \mathfrak{sl}_m, \Lambda_1, \mathbb{C}^m, \mathbb{C}^m, \kappa_m) = V_{-1} \oplus V_0 \oplus V_1$ (see [9, Example 1.14]). This Lie algebra $L(\mathfrak{gl}_1 \oplus \mathfrak{sl}_m, \Lambda_1, \mathbb{C}^m, \mathbb{C}^m, \kappa_m)$ is isomorphic to $\mathfrak{sl}_{m+1}$. Indeed, from the classification of PVs of parabolic type (see [5, 6, 7]) and the dimension of $L(\mathfrak{g}, \rho, V, \text{Hom}(V, \mathbb{C}), B_0)$, it is isomorphic to $\mathfrak{sl}_{m+1}$.

Example 2.35. Put $\mathfrak{g} := \mathfrak{gl}_1(\mathbb{C}) \oplus \mathfrak{gl}_1(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$, $V := \mathbb{C}^2 = M(2, 1; \mathbb{C})$, $\mathcal{V} := \mathbb{C}^2$ and define representations $\rho : \mathfrak{g} \otimes V \rightarrow V$, $\varrho : \mathfrak{g} \otimes \mathcal{V} \rightarrow \mathcal{V}$ by

$$\rho((a, b, A) \otimes v) := bv + Av, \quad \varrho((a, b, A) \otimes \phi) := -b\phi - \phi A \phi$$

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Lie algebras constructed with Lie modules and their modules

for any \((a, b, A) \in \mathfrak{g}, \, v \in V, \, \phi \in \mathcal{V}\). We can identify \(\mathcal{V}\) with \(\text{Hom}(V, \mathbb{C})\) via the following bilinear map \(\langle \cdot, \cdot \rangle_{\mathcal{V}} : V \times \mathcal{V} \to \mathbb{C}\) defined by:

\[
\langle v, \phi \rangle_{\mathcal{V}} := t v \phi.
\]

Let \(B_0\) be a bilinear form on \(\mathfrak{g}\) defined by:

\[
B_0((a, b, A), (a', b', A')) := \frac{3}{4} aa' + bb' + \frac{1}{2}(ab' + a'b) + \text{Tr}(AA').
\]

Then, a pentad \((\mathfrak{g}, \rho, V, B_0)\) is a standard pentad whose \(\Phi\)-map is defined by:

\[
\Phi_{\rho}(v \circ \phi) = (-tv \phi, \frac{3}{2}v \phi, v' \phi - \frac{1}{2}v \phi I_2).
\]

The Lie algebra \(L(\mathfrak{g}, \rho, V, B_0)\) is isomorphic to \(\mathfrak{gl}_1 \oplus \mathfrak{sl}_3\). Indeed, if we put \(\mathfrak{g}_V^1 := \mathbb{C} \cdot (1, 0, O_2)\), \(\mathfrak{g}_V^2 := \mathbb{C} \cdot (-\frac{3}{4}, 1, O_2) \oplus \mathfrak{sl}_2\), then we have

\[
L(\mathfrak{g}, \rho, V, B_0) \cong L(\mathfrak{g}_V^1, \rho |_{\mathfrak{g}_V^1}, \{0\}, \{0\}, B_0 \{\mathfrak{g}_V^1 \times \mathfrak{g}_V^1\}) \oplus (\mathfrak{g}_V^2, \rho |_{\mathfrak{g}_V^2}, V, V, B_0 \{\mathfrak{g}_V^2 \times \mathfrak{g}_V^2\})
\]

\[
\cong \mathfrak{g}_V^1 \oplus L(\mathfrak{g}_V^2, \rho |_{\mathfrak{g}_V^2}, V, V, B_0 \{\mathfrak{g}_V^2 \times \mathfrak{g}_V^2\}) \cong \mathfrak{g}_V^1 \oplus \mathfrak{V} \oplus \mathfrak{g}_V^2 \oplus \mathfrak{V}
\]

from Example 2.34. Moreover, under this identification, the bilinear form \(B_L\) on \(L(\mathfrak{g}_V^2, \rho |_{\mathfrak{g}_V^2}, V, V, B_0 \{\mathfrak{g}_V^2 \times \mathfrak{g}_V^2\})\) is given by \(B_L(\hat{A}, \hat{A}') = \text{Tr}(\hat{A} \hat{A}')\) \((\hat{A}, \hat{A}' \in \mathfrak{sl}_3)\). In fact, if we put

\[
h := (0, 0, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) \in \mathfrak{g}_V^2,
\]

then \(B_0(h, h) = 2\). On the other hand, we can obtain \(\text{Tr}(\text{ad} h \circ \text{ad} h) = 12\), where \(\text{ad}\) stands for the adjoint representation of \(L(\mathfrak{g}_V^2, \rho |_{\mathfrak{g}_V^2}, V, V, B_0 \{\mathfrak{g}_V^2 \times \mathfrak{g}_V^2\})\), by a direct calculation. Since any non-degenerate invariant bilinear form on \(\mathfrak{sl}_3\) is a scalar multiple of the Killing form, we can obtain that \(B_L\) is 1/6 times the Killing form of \(\mathfrak{sl}_3\), i.e. \(B_L(\hat{A}, \hat{A}') = \text{Tr}(\hat{A} \hat{A}')\).

**Proposition 2.36.** Let \((\mathfrak{g}, \rho, V, B_0)\) be a standard pentad whose representation \(\rho\) is faithful. Under this assumption, the pentad \((\mathfrak{g}, \rho, V, B_0)\) is irreducible if and only if the Lie algebra \(L(\mathfrak{g}, \rho, V, B_0)\) does not have a non-zero proper graded ideal.

**Proof.** Assume that \((\mathfrak{g}, \rho, V, B_0)\) is reducible. Under the notation of Definition 2.29, we put \(a_{-1} := \mathcal{V}_a, \, a_0 := \mathfrak{a}, \, a_1 := \mathfrak{V}_a\). Moreover, we put \(a_n := [V_{1}, a_{n-1}]\) for all \(n \geq 2\) and \(a_m := [V_{-1}, a_{m+1}]\) for all \(m \leq -2\) inductively. Then a direct sum \(\mathfrak{A} := \bigoplus_{n \in \mathbb{Z}} a_n\) is a non-zero proper graded ideal of \(L(\mathfrak{g}, \rho, V, V, B_0)\). In fact, by the assumption that \([V_{1}, a_{i}] \subseteq a_{i+j}\) for any \(-1 \leq i, j, i+j \leq 1\), we can easily show that \([V_0, \mathfrak{A}], [V_{\pm 1}, \mathfrak{A}] \subseteq \mathfrak{A}\) by induction. Since \(L(\mathfrak{g}, \rho, V, B_0)\) is generated by \(V_0\) and \(V_{\pm 1}\), we have \([L(\mathfrak{g}, \rho, V, V, B_0), \mathfrak{A}] \subseteq \mathfrak{A}\). Thus, \(\mathfrak{A}\) is a graded ideal. Since \(\{0\} \neq a_{-1} \oplus a_0 \oplus a_1 \subseteq \mathfrak{V} \oplus \mathfrak{g} \oplus V\), we have \(\{0\} \neq \mathfrak{A} \subseteq L(\mathfrak{g}, \rho, V, V, B_0)\).
Conversely, assume that \((g, \rho, V, V, B_0)\) is irreducible. Let \(b = \sum_{n \in \mathbb{Z}} (b \cap V_n)\) be a non-zero graded ideal of \(L(g, \rho, V, V, B_0)\) and put \(b_n := b \cap V_n\). Then, by Proposition 2.32, we can obtain that \(b_0 \neq \{0\}\). In fact, since \(b \neq \{0\}\), there exists an integer \(n \in \mathbb{Z}\) and a non-zero element \(z_n \in b_n\). For example, if \(n \geq 1\), then there exist elements \(y^1_{n-1}, \ldots, y^n_{n-1} \in V_{-1}\) such that \(y^n_{n-1}, \ldots, [y^1_{n-1}, z_n, \ldots] \in b_0 \setminus \{0\}\). Since \(b_{-1} \oplus b_0 \oplus b_1\) satisfies the conditions (2.38) and (2.39), it coincides with \(V_{-1} \oplus V_0 \oplus V_1\), and, thus, \(b = L(g, \rho, V, V, B_0)\).

The following lemmas are to construct a derivation on \(L(g, \rho, V, V, B_0)\). They are used in Theorem 3.26.

**Lemma 2.37.** Let \((g, \rho, V, V, B_0)\) be a standard pentad, \(L(g, \rho, V, V, B_0) = \bigoplus_{n \in \mathbb{Z}} V_n\) be the Lie algebra associated with it. Let \(\alpha_i : V_i \to V_i\) \((i = 0, \pm 1)\) be linear maps which satisfy

\[
\alpha_{i+j}([a_i, b_j]) = [\alpha_i(a_i), b_j] + [a_i, \alpha_j(b_j)]
\]  

(2.40)

for any \(-1 \leq i, j, i + j \leq 1\) and elements \(a_i \in V_i, b_j \in V_j\). Then, there exists a linear map \(\alpha : L(g, \rho, V, V, B_0) \to L(g, \rho, V, V, B_0)\) such that \(\alpha\) is a derivation on \(L(g, \rho, V, V, B_0)\) and its restriction to \(V_i\) \((i = 0, \pm 1)\) coincides with \(\alpha_i\).

**Proof.** First, let us construct linear maps \(\alpha_i : V_i \to V_i\) for all \(i \in \mathbb{Z}\) by induction. Let \(i \geq 1\) and assume that the integer \(i\) satisfies the condition that we have linear maps \(\alpha_j : V_j \to V_j\) for all \(0 \leq j \leq i\) which satisfy the following equations:

\[
\alpha_j([a_0, b_j]) = [\alpha_0(a_0), b_j] + [a_0, \alpha_j(b_j)],
\]

\[
\alpha_j([x_1, b_{j-1}]) = [\alpha_1(x_1), b_{j-1}] + [x_1, \alpha_{j-1}(b_{j-1})],
\]

\[
\alpha_{j-1}([y_{j-1}, b_j]) = [\alpha_{j-1}(y_{j-1}), b_j] + [y_{j-1}, \alpha_j(b_j)]
\]

for any \(0 \leq j \leq i\), \(a_0 \in V_0, x_1 \in V_1, y_{j-1} \in V_{-1}, b_j \in V_j\) and \(b_{j-1} \in V_{j-1}\). By the assumption (2.40), when \(i = 1\) the given linear maps \(\alpha_0, \alpha_{\pm 1}\) satisfy these equations. Then we define a linear map \(\alpha_{i+1} : V_{i+1} \to V_{i+1}\) by:

\[
\alpha_{i+1}([x_1, b_i]) := [\alpha_i(x_1), b_i] + [x_1, \alpha_i(b_i)]
\]

(2.41)

for any \(x_1 \in V_1\) and \(b_i \in V_i\). Let us check the well-definedness of \(\alpha_{i+1}\). In fact, for any \(y_{-1} \in V_{-1}, x_1 \in V_1\) and \(b_i \in V_i\), we have

\[
[y_{-1}, [\alpha_i(x_1), b_i] + [x_1, \alpha_i(b_i)]] = [y_{-1}, [\alpha_i(x_1), b_i]] + [y_{-1}, [x_1, \alpha_i(b_i)]]
\]

\[
= [[y_{-1}, \alpha_i(x_1)], b_i] + [\alpha_i(x_1), [y_{-1}, b_i]] + [[y_{-1}, x_1], \alpha_i(b_i)] + [x_1, [y_{-1}, \alpha_i(b_i)]]
\]

\[
= [\alpha_0([y_{-1}, x_1]), b_i] - [[\alpha_{-1}(y_{-1}), x_1], b_i] + [\alpha_1(x_1), [y_{-1}, b_i]]
\]

\[
+ [[y_{-1}, x_1], \alpha_i(b_i)] + [x_1, \alpha_{i-1}([y_{-1}, b_i])] - [x_1, [\alpha_{-1}(y_{-1}), b_i]]
\]

\[
= \alpha_i([y_{-1}, x_1], b_i) + \alpha_i([x_1, [y_{-1}, b_i]]) - [\alpha_{-1}(y_{-1}), [x_1, b_i]]
\]

(2.42)
Thus, if $x_1^1, \ldots, x_1^l \in V_1$ and $b_1^1, \ldots, b_1^l \in V_1$ satisfy $\sum_{s=1}^l [x_1^s, b_1^s] = 0$, then we have

$$\sum_{s=1}^l [y_{-1}, [\alpha_1(x_1^s), b_1^s] + [x_1^s, \alpha_1(b_1^s)]] = 0$$

for any $y_{-1} \in V_{-1}$. Therefore, we have $\sum_{s=1}^l ([\alpha_1(x_1^s), b_1^s] + [x_1^s, \alpha_1(b_1^s)]) = 0$ and the well-definedness of $\alpha_{i+1}$. Moreover, $\alpha_{i+1}$ satisfies the following equations:

$$\begin{align*}
\alpha_{i+1}([a_0, b_{i+1}]) &= [\alpha_0(a_0), b_{i+1}] + [a_0, \alpha_{i+1}(b_{i+1})], \quad (2.43) \\
\alpha_{i+1}(x_1, b_i) &= [\alpha_1(x_1), b_i] + [x_1, \alpha_i(b_i)], \quad (2.44) \\
\alpha_i([y_{-1}, b_{i+1}]) &= [\alpha_{-1}(y_{-1}), b_{i+1}] + [y_{-1}, \alpha_{i+1}(b_{i+1})] \quad (2.45)
\end{align*}$$

for any $a_0 \in V_0$, $x_1 \in V_1$, $y_{-1} \in V_{-1}$, $b_i \in V_i$ and $b_{i+1} \in V_{i+1}$. In fact, for any $a_0 \in V_0$, $x_1 \in V_1$, and $b_i \in V_i$, we have

$$\begin{align*}
\alpha_{i+1}([a_0, x_1, b_i]) &= \alpha_{i+1}([x_1[a_0, b_i]]) + \alpha_{i+1}([x_1[a_0, b_i]]) \\
&= [\alpha_1([a_0, x_1], b_i) + [a_0, \alpha_i(b_i)] + [\alpha_1(x_1), b_i] + [x_1, \alpha_i([a_0, b_i])] \\
&= [a_0(a_0), x_1, b_i] + [a_0, \alpha_1(x_1), b_i] + [a_0, x_1, \alpha_i(b_i)] \\
&\quad + [\alpha_1(x_1), a_0, b_i] + [x_1, [a_0(a_0), b_i]] + [y_{-1}, [a_0, \alpha_i(b_i)] \\
&= [a_0(a_0), x_1, b_i] + [a_0, \alpha_1(x_1), b_i] + [a_0, x_1, \alpha_i(b_i)] \\
&\quad = [a_0(a_0), x_1, b_i] + [a_0, \alpha_{i+1}([x_1, b_i])]. \quad (2.46)
\end{align*}$$

Thus, we can obtain the equation (2.43). The equation (2.44) is clear. The equation (2.45) follows from (2.42). Thus, inductively, we can obtain linear maps $\alpha_i$ for all $i \geq 0$, and, similarly, we can construct linear maps $\alpha_{-i} : V_{-i} \rightarrow V_{-i}$ for all $i \geq 0$. Consequently, we have linear maps $\alpha_n : V_n \rightarrow V_n$ for all $n \in \mathbb{Z}$ which satisfy

$$\begin{align*}
\alpha_n([a_0, b_n]) &= [a_0(a_0), b_n] + [a_0, \alpha_n(b_n)], \quad (2.47) \\
\alpha_{n+1}(x_1, b_n) &= [\alpha_1(x_1), b_n] + [x_1, \alpha_n(b_n)], \quad (2.48) \\
\alpha_{n-1}([y_{-1}, b_n]) &= [\alpha_{-1}(y_{-1}), b_n] + [y_{-1}, \alpha_n(b_n)] \quad (2.49)
\end{align*}$$

for any $a_0 \in V_0$, $x_1 \in V_1$, $y_{-1} \in V_{-1}$ and $b_n \in V_n$.

We define a linear map $\alpha : L(g, \rho, V, V, B_0) \rightarrow L(g, \rho, V, V, B_0)$ by:

$$\alpha(a_n) := \alpha_n(a_n) \quad (2.50)$$

for any $n \in \mathbb{Z}$ and $a_n \in V_n$. Then $\alpha$ is a derivation of Lie algebras. In fact, we can show the following equation

$$\alpha([a_n, b_m]) = [\alpha(a_n), b_m] + [a_n, \alpha(b_m)] \quad (2.51)$$

for any $n, m \in \mathbb{Z}$, $a_n \in V_n$ and $b_m \in V_m$ by the equations (2.47), (2.48), (2.49) inductively.  

Lemma 2.38. Let \((g, \rho, V, \mathcal{V}, B_0)\) be a standard pentad and \(\alpha\) be a derivation on \(L(g, \rho, V, \mathcal{V}, B_0)\). If \(\alpha\) satisfies the equation

\[
B_L(\alpha(z), \omega) = -B_L(z, \alpha(\omega))
\]

for any \(z = z_n \in V_n\) \((n = 0, \pm 1)\) and \(\omega \in L(g, \rho, V, \mathcal{V}, B_0)\), then we have the same equation for any \(z, \omega \in L(g, \rho, V, \mathcal{V}, B_0)\).

Proof. We argue our claim in the cases where \(z = z_n \in V_n\) for some \(n\) and prove it by induction on \(n\). Suppose that \(n \geq 0\). If \(n = 0, 1\), then our claim follows from the assumption. Suppose that \(n \geq 2\). Then, by the induction hypothesis, we have

\[
\begin{align*}
B_L(\alpha([x_1, z_{n-1}]), \omega) & = B_L([\alpha(x_1), z_{n-1}], \omega) + B_L([x_1, \alpha(z_{n-1})], \omega) \\
& = -B_L(z_{n-1}, [\alpha(x_1), \omega]) - B_L(0, [x_1, \omega]) \\
& = -B_L(z_{n-1}, [\alpha(x_1), \omega]) + B_L(z_{n-1}, [x_1, \alpha(\omega)]) \\
& = B_L(z_{n-1}, [x_1, \alpha(\omega)]) \\
& = B_L([x_1, z_{n-1}], \alpha(\omega))
\end{align*}
\]

for any \(x_1 \in V_1, z_{n-1} \in V_{n-1}\). Since \(V_n = [V_1, V_{n-1}]\), we have our claim for \(n\). Thus, by induction, we have our claim for all \(n \geq 0\). Similarly, we can show our claim for \(n \leq -1\). \(\blacksquare\)

3 Graded modules of \(L(g, \rho, V, \mathcal{V}, B_0)\)

3.1 A construction of vector spaces \(\bar{U}^+\) and \(\bar{U}^-\)

As mentioned in section \(1\) the purpose of this and the next section is to construct a positively graded module and a negatively graded module of \(L(g, \rho, V, \mathcal{V}, B_0)\) from a given \(g\)-module \(U\), which will be denoted by \(\bar{U}^+\) and \(\bar{U}^-\). First, we construct \(\bar{U}^+\) and \(\bar{U}^-\) as vector spaces by induction.

Definition 3.1. Let \((g, \rho, V, \mathcal{V}, B_0)\) be a standard pentad and \(L(g, \rho, V, \mathcal{V}, B_0) = \bigoplus_{n \in \mathbb{Z}} V_n\) be the Lie algebra associated with it. Let \(\pi : g \otimes U \to U\) be a representation of \(g = V_0\) on a vector space \(U\) over \(\mathfrak{t}\). We put \(U_0^+ = U_0^+ := U, \pi_0^+ = \pi_0 := \pi\) and define linear maps \(r_0^+ : V_1 \otimes U_0^+ \to \text{Hom}(V_{-1}, U_0^+)\) and \(r_0^- : V_1 \otimes U_0^- \to \text{Hom}(V_{-1}, U_0^-)\) by:

\[
\begin{align*}
r_0^+ : V_1 \otimes U_0^+ & \to \text{Hom}(V_{-1}, U_0^+) \\
x_1 \otimes u_0 & \mapsto (\eta_{-1} \mapsto \pi_0^+([\eta_{-1}, x_1] \otimes u_0)), \quad (3.1) \\
r_0^- : V_1 \otimes U_0^- & \to \text{Hom}(V_{-1}, U_0^-) \\
y_1 \otimes u_0 & \mapsto (\xi_{-1} \mapsto \pi_0^-([\xi_{-1}, y_1] \otimes u_0)). \quad (3.2)
\end{align*}
\]

Proposition 3.2. The maps \(r_0^+\) and \(r_0^-\) are homomorphisms of \(g\)-modules.
Proof. We prove for \( r_0^+ \). For any elements \( a \in \mathfrak{g}, x_1 \in V_1, \eta_{-1} \in V_{-1} \) and \( u_0 \in U_0^+ \), we have
\[
r_0^+([a, x_1] \otimes u_0 + x_1 \otimes \pi^+_0(a \otimes u_0)) = \pi^+_0([\eta_{-1}, [a, x_1]] \otimes u_0) + \pi^+_0([\eta_{-1}, x_1] \otimes \pi^+_0(a \otimes u_0))
\]
Thus \( r_0^+ \) is a homomorphism of \( \mathfrak{g} \)-modules. Similarly, we can prove that \( r_0^- \) is a homomorphism of \( \mathfrak{g} \)-modules.

It follows from Proposition 3.2 that the linear spaces \( U_1^+ := \text{Im} \ r_0^+ \) and \( U_{-1}^- := \text{Im} \ r_0^- \) have the canonical \( \mathfrak{g} \)-module structures. We denote these canonical representations by \( \pi_1^+ \) and \( \pi_{-1}^- \) respectively. Moreover, we inductively construct \( \mathfrak{g} \)-modules \( U_2^+, U_3^+, \ldots \) by using the following proposition.

Proposition 3.3. Assume that there exist \( \mathfrak{g} \)-modules \( (\varpi^+, W^+), (\varpi^-, W^-) \) and \( \mathfrak{g} \)-module homomorphisms \( \lambda^+: V_1 \otimes W^+ \rightarrow \text{Hom}(V_{-1}, W^+) \) and \( \lambda^-: V_{-1} \otimes W^- \rightarrow \text{Hom}(V_1, W^-) \). We put \( \hat{W}^+: = \text{Im} \lambda^+, \hat{W}^-: = \text{Im} \lambda^- \) and denote the canonical representations of \( \mathfrak{g} \) on them by \( \varpi^+ \) and \( \varpi^- \) respectively. Then the following linear maps are \( \mathfrak{g} \)-module homomorphisms:
\[
\hat{\lambda}^+: V_1 \otimes \hat{W}^+ \rightarrow \text{Hom}(V_{-1}, \hat{W}^+)
\]
\[
x_1 \otimes \hat{\varpi}^+ \mapsto (\eta_{-1} \mapsto \hat{\varpi}^+([\eta_{-1}, x_1] \otimes \hat{\varpi}^+(\eta_{-1}))),
\]
\[
\hat{\lambda}^-: V_{-1} \otimes \hat{W}^- \rightarrow \text{Hom}(V_1, \hat{W}^-)
\]
\[
y_{-1} \otimes \hat{\varpi}^- \mapsto (\xi_1 \mapsto \hat{\varpi}^-([\xi_1, y_{-1}] \otimes \hat{\varpi}^-(\xi_1))).
\]

Proof. We can prove it by a similar argument to the argument of [3] Proposition 1.10. Take any elements \( a \in \mathfrak{g}, x_1 \in V_1, \eta_{-1} \in V_{-1} \) and \( \hat{\varpi}^+ \in \hat{W}^+ \). Then we have
\[
(\hat{\lambda}^+([a, x_1] \otimes \hat{\varpi}^+) + \hat{\lambda}^+(x_1 \otimes \hat{\varpi}^+(a \otimes \hat{\varpi}^+)))(\eta_{-1})
\]
\[
= \hat{\varpi}^+([\eta_{-1}, [a, x_1]] \otimes \hat{\varpi}^+(\eta_{-1})) + \hat{\varpi}^+([\eta_{-1}, x_1] \otimes \hat{\varpi}^+(a \otimes \hat{\varpi}^+(\eta_{-1}))) + \hat{\varpi}^+([a, x_1] \otimes \hat{\varpi}^+(\eta_{-1}))
\]
Thus, \( \hat{\lambda}^+ \) is a homomorphism of \( \mathfrak{g} \)-modules. By the same way, we can prove that \( \hat{\lambda}^- \) is also a \( \mathfrak{g} \)-module homomorphism.
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**Definition 3.4.** Suppose that \( j \geq 1 \) and there exist \( \mathfrak{g} \)-modules \((\pi^{+}_{j-1}, U^{+}_{j-1}), (\pi^{-}_{j+1}, U^{-}_{j+1})\) and homomorphisms of \( \mathfrak{g} \)-modules \( r^{+}_{j-1} : V_{1} \otimes U^{+}_{j-1} \to \text{Hom}(V_{1}, U^{+}_{j}) \) and \( r^{-}_{j+1} : V_{-1} \otimes U^{-}_{j+1} \to \text{Hom}(V_{1}, U^{-}_{j+1}) \). Put \( U^{+}_{j} := \text{Im} r^{+}_{j-1} \) and \( U^{-}_{j} := \text{Im} r^{-}_{j+1} \). Then we define linear maps \( r^{+}_{j} \) and \( r^{-}_{j} \) by:

\[
r^{+}_{j} : V_{1} \otimes U^{+}_{j} \to \text{Hom}(V_{1}, U^{+}_{j}),
\]

\[
x_{1} \otimes u^{+}_{j} \mapsto (\eta \mapsto \pi^{+}_{j}([\eta^{-1}, x_{1}] \otimes u^{+}_{j}) + r^{+}_{j-1}(x_{1} \otimes u^{+}_{j}(\eta^{-1}))), \tag{3.5}
\]

\[
r^{-}_{j} : V_{-1} \otimes U^{-}_{j} \to \text{Hom}(V_{1}, U^{-}_{j}),
\]

\[
y_{-1} \otimes u^{-}_{j} \mapsto (\xi \mapsto \pi^{-}_{j}([\xi_{1}, y_{-1}] \otimes u^{-}_{j}) + r^{-}_{j+1}(y_{-1} \otimes u^{-}_{j}(\xi_{1}))). \tag{3.6}
\]

Then, by Proposition 3.3, \( r^{+}_{j} \) and \( r^{-}_{j} \) are homomorphisms of \( \mathfrak{g} \)-modules. We denote by \( U^{+}_{j+1} \) and \( U^{-}_{j-1} \) the images of \( r^{+}_{j} \) and \( r^{-}_{j} \), and the canonical representations of \( \mathfrak{g} \) on \( U^{+}_{j+1} \) and \( U^{-}_{j-1} \) by \( \pi^{+}_{j+1} \) and \( \pi^{-}_{j-1} \) respectively. Moreover, we put

\[
U^{+}_{j} := \{0\}, \quad U^{-}_{j} := \{0\}
\]

for \( j \geq 1 \). We denote the zero representations of \( \mathfrak{g} \) on \( U^{+}_{j} \) and \( U^{-}_{j} \) by \( \pi^{+}_{j} \) and \( \pi^{-}_{j} \) for all \( j \geq 1 \). Thus, inductively, we obtain \( \mathfrak{g} \)-modules \((\pi^{+}_{m}, U^{+}_{m}), (\pi^{-}_{m}, U^{-}_{m})\) for all \( m \in \mathbb{Z} \). Under these notation, we define linear spaces \( \tilde{U}^{+} \) and \( \tilde{U}^{-} \) by:

\[
\tilde{U}^{+} := \bigoplus_{m \in \mathbb{Z}} U^{+}_{m}, \quad \tilde{U}^{-} := \bigoplus_{m \in \mathbb{Z}} U^{-}_{m}. \tag{3.7}
\]

Throughout this paper, we use these notation.

### 3.2 A construction of representations of \( L(\mathfrak{g}, \rho, V, \mathcal{V}, B_{0}) \) on \( \tilde{U}^{+} \) and \( \tilde{U}^{-} \)

In this section, we define \( L(\mathfrak{g}, \rho, V, \mathcal{V}, B_{0}) \)-module structures on vector spaces \( \tilde{U}^{+} \) and \( \tilde{U}^{-} \) constructed in (3.7). For this, we start with the following definition.

**Definition 3.5.** We define the following linear maps:

\[
\pi^{+}_{0,m} : V_{0} \otimes U^{+}_{m} \to U^{+}_{m}, \quad \pi^{+}_{1,m} : V_{1} \otimes U^{+}_{m} \to U^{+}_{m+1}, \quad \pi^{-}_{1,m} : V_{-1} \otimes U^{+}_{m} \to U^{+}_{m-1}
\]

by:

\[
\pi^{+}_{0,m}(a \otimes u^{+}_{m}) := \pi^{+}_{m}(a \otimes u^{+}_{m}) \quad (m \in \mathbb{Z}), \tag{3.8}
\]

\[
\pi^{+}_{1,m}(x_{1} \otimes u^{+}_{m}) := \begin{cases} r^{+}_{m}(x_{1} \otimes u^{+}_{m}) & (m \geq 0) \\ 0 & (m \leq -1) \end{cases}, \tag{3.9}
\]

\[
\pi^{-}_{1,m}(y_{-1} \otimes u^{+}_{m}) := \begin{cases} u^{+}_{m}(y_{-1}) & (m \geq 1) \\ 0 & (m \leq 0) \end{cases} \tag{3.10}
\]

where \( a \in V_{0}, x_{1} \in V_{1}, y_{-1} \in V_{-1} \) and \( u^{+}_{m} \in U^{+}_{m} \).
By the above definition, we can obtain the following proposition.

**Proposition 3.6.** Under the above notation, we have the following equations:

\[
\begin{align*}
\pi_{1,m}^{+}([x_{1},a] \otimes u_{m}^{+}) &= \pi_{1,m}^{+}(x_{1} \otimes \pi_{0,m}^{+}(a \otimes u_{m}^{+}))-\pi_{0,m+1}^{+}(a \otimes \pi_{1,m}^{+}(x_{1} \otimes u_{m}^{+})), \\
\pi_{-1,m}^{+}([y_{-1},a] \otimes u_{m}^{+}) &= \pi_{1,m}^{+}(y_{-1} \otimes \pi_{0,m}^{+}(a \otimes u_{m}^{+}))-\pi_{0,m-1}^{+}(a \otimes \pi_{1,m}^{+}(y_{-1} \otimes u_{m}^{+})), \\
\pi_{1,m-1}^{+}(x_{1} \otimes \pi_{-1,m}^{+}(y_{-1} \otimes u_{m}^{+})) &= \pi_{0,m}^{+}([x_{1},y_{-1}] \otimes u_{m})+\pi_{-1,m+1}^{+}(y_{-1} \otimes \pi_{1,m}^{+}(x_{1} \otimes u_{m})).
\end{align*}
\]  
\(3.11\)  
\(3.12\)  
\(3.13\)

**Proof.** Let us show (3.13). The equations (3.11) and (3.12) can be shown similarly. If \(m \leq -1\), then (3.13) is clear. If \(m = 0\), then the left hand side equals to 0. For the right hand side, we have

\[
\begin{align*}
\pi_{0,0}^{+}([x_{1},y_{-1}] \otimes u_{0}) + \pi_{1,1}^{+}(y_{-1} \otimes \pi_{0,0}^{+}(x_{1} \otimes u_{0})) &\end{align*}
\]

Thus we have (3.13) when \(m = 0\). For \(m \geq 1\), the equation (3.13) follows from definition.

**Definition 3.7.** We define the following linear maps for \(i \geq 1\) inductively:

\[
\begin{align*}
\pi_{i+1,m}^{+} : V_{i+1} \otimes U_{m}^{+} &\rightarrow U_{i+1,m}^{+} \\
p_{i}(x_{i} \otimes z_{i}) \otimes u_{m}^{+} &\rightarrow \pi_{1,i+m}^{+}(x_{i} \otimes \pi_{i,m}^{+}(z_{i} \otimes u_{m}^{+}))-\pi_{i+1,m}^{+}(z_{i} \otimes \pi_{1,m}^{+}(x_{1} \otimes u_{m}^{+})), \\
\pi_{i-1,m}^{+} : V_{i-1} \otimes U_{m}^{+} &\rightarrow U_{i-1,m}^{+} \\
q_{i}(y_{-i} \otimes \omega_{-i}) \otimes u_{m}^{+} &\rightarrow \pi_{1,-i+m}^{+}(y_{-i} \otimes \pi_{-i,m}^{+}(\omega_{-i} \otimes u_{m}^{+}))-\pi_{i-1,m}^{+}(\omega_{-i} \otimes \pi_{1,m}^{+}(y_{1} \otimes u_{m}^{+})).
\end{align*}
\]  
\(3.14\)  
\(3.15\)

Note that the linear maps \(\pi_{0,m}^{+}, \pi_{1,m}^{+}\) defined in Definition 3.6 satisfy the same equations as (3.14) and (3.15) in the cases where \(i = 0\) by Proposition 3.6. For \(i \geq 1\), we must show the well-definedness of Definition 3.7. To prove it, let us show the following two propositions.

**Proposition 3.8.** (The well-definedness of \(\pi_{i+1,m}^{+}\) given in (3.13)) Suppose that \(i \geq 0\). Suppose that the linear map \(\pi_{i,m}^{+}\) defined in (3.12) is well-defined for any \(m \in \mathbb{Z}\) and satisfies the following equations:

\[
\begin{align*}
\pi_{0,i+m}^{+}(a \otimes \pi_{i,m}^{+}(z_{i} \otimes u_{m}^{+})) &= \pi_{i,m}^{+}([a,z_{i}] \otimes u_{m}^{+})+\pi_{i,m}^{+}(z_{i} \otimes \pi_{0,m}^{+}(a \otimes u_{m}^{+})), \\
\pi_{i,m-1}^{+}(z_{i} \otimes \pi_{-1,m}^{+}(y_{-1} \otimes u_{m}^{+})) &= \pi_{i-1,m}^{+}([z_{i},y_{-1}] \otimes u_{m}^{+})+\pi_{i-1,m+1}^{+}(y_{-1} \otimes \pi_{i,m}^{+}(z_{1} \otimes u_{m}^{+})).
\end{align*}
\]  
\(3.16\)  
\(3.17\)

If \(x_{i}^{1}, \ldots, x_{i}^{l} \in V_{i}\) and \(z_{i}^{1}, \ldots, z_{i}^{l} \in V_{i}\) satisfy \(\sum_{s=1}^{l} p_{i}(x_{i}^{s} \otimes z_{i}^{s}) = 0\), then we have

\[
\sum_{s=1}^{l} \left(\pi_{i+1,m}^{+}(x_{i}^{s} \otimes \pi_{i,m}^{+}(z_{i}^{s} \otimes u_{m}^{+}))-\pi_{i+1,m+1}^{+}(z_{i}^{s} \otimes \pi_{i,m}^{+}(x_{1} \otimes u_{m}^{+}))\right) = 0
\]  
\(3.18\)
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for all \( m \in \mathbb{Z} \) and \( u^+_m \in U^+_m \). In particular, we can obtain the well-definedness of the linear map \( \pi^+_n \) defined in (3.13) for any \( m \in \mathbb{Z} \). Moreover, the linear maps \( \pi^+_{i,m} \) (\( m \in \mathbb{Z} \)) satisfy the following equations:

\[
\begin{align*}
\pi^+_{0,i,m+1}(a \otimes \pi^+_{i+1,m}(z_{i+1} \otimes u^+_m)) &= \pi^+_{i+1,m}(a \otimes \pi^+_{i+1,m}(z_{i+1} \otimes u^+_m)) + \pi^+_{i+1,m}(z_{i+1} \otimes \pi^+_{0,m}(a \otimes u^+_m)) \\
\pi^+_{i+1,m-1}(z_{i+1} \otimes \pi^+_{0,m}(y_1 \otimes u^+_m)) &= \pi^+_{i+1,m-1}(z_{i+1} \otimes \pi^+_{0,m}(y_1 \otimes u^+_m)) + \pi^+_{i+1,m-1}(y_1 \otimes \pi^+_{i,m}(z_{i+1} \otimes u^+_m)).
\end{align*}
\]

(3.19)

(3.20)

**Proof.** We argue by induction on \( i \). For \( i = 0 \), our claim follows from Proposition 3.6. Suppose that \( i \geq 1 \). We fix \( i \) and argue by induction on \( m \). First, if \( m \leq -1 \), then the equation (3.18) is clear. If \( m \geq 0 \), then we have

\[
\begin{align*}
\pi^+_{i+1,m+1}(y_1 \otimes \pi^+_{i,m}(z_i \otimes u^+_m)) &= \pi^+_{i+1,m+1}(y_1 \otimes \pi^+_{i,m}(z_i \otimes u^+_m)) + \pi^+_{i+1,m+1}(y_1 \otimes \pi^+_{i,m}(z_i \otimes u^+_m)) \\
&= \pi^+_{i+1,m+1}(y_1 \otimes \pi^+_{i,m}(z_i \otimes u^+_m)) + \pi^+_{i+1,m+1}(y_1 \otimes \pi^+_{i,m}(z_i \otimes u^+_m)).
\end{align*}
\]

(3.21)

for any \( x_1 \in V_1 \), \( z_i \in V_i \), \( y_1 \in V_{-1} \) and \( u^+_m \in U^+_m \). By the induction hypotheses on \( i \) and \( m \), if we take elements \( x_1, \ldots, x_i \in V_{1} \) and \( z_1, \ldots, z_i \in V_i \) satisfying \( \sum_{s=1}^{i} p_s(x^+_s \otimes z^+_s) = 0 \), then we have

\[
\begin{align*}
\sum_{s=1}^{i} \pi^+_{s,m}([x^+_s, z^+_s], y_1) &= 0 \quad \text{(by the induction hypothesis on } i), \quad (3.22) \\
\sum_{s=1}^{i} (\pi^+_{s+1,m-1}(x^+_s \otimes \pi^+_{s,m-1}(z^+_s \otimes u^+_m))) &= 0 \quad \text{(by the induction hypothesis on } m). \quad (3.23)
\end{align*}
\]

Thus, we have

\[
\begin{align*}
\sum_{s=1}^{i} \pi^+_{i+1,m+1}(y_1 \otimes \pi^+_{i,m}(z_i \otimes u^+_m)) &= \pi^+_{i+1,m+1}(y_1 \otimes \pi^+_{i,m}(z_i \otimes u^+_m)) + \pi^+_{i+1,m+1}(y_1 \otimes \pi^+_{i,m}(z_i \otimes u^+_m)) \\
&= \pi^+_{i+1,m+1}(y_1 \otimes \pi^+_{i,m}(z_i \otimes u^+_m)) + \pi^+_{i+1,m+1}(y_1 \otimes \pi^+_{i,m}(z_i \otimes u^+_m)).
\end{align*}
\]

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from ([3.21]). Since \( i + m + 1 \geq 1 \), we can obtain that

\[
\sum_{s=1}^{l} (\pi_{i+1,m}(x_{s}^* \otimes r_{m+1}(y_{s}^* \otimes u_{m+1})) - \pi_{i,m+1}(z_{s}^* \otimes \pi_{1,m}(x_{1} \otimes r_{m+1}(y_{1} \otimes u_{m+1})))))
= 0 \in U_{i+1,m+1}^{+} \subset \text{Hom}(V_{-1}, U_{i+1,m}^{+}).
\] (3.25)

Therefore we can obtain the well-definedness of the linear map \( \pi_{i+1,m}^{\pm}: V_{i+1} \otimes U_{m}^{+} \to U_{i+1,m+1}^{+} \) given in ([3.14]) for any \( m \).

In order to complete the proof, we must show the equations ([3.19]) and ([3.20]). Let us show ([3.19]). Under the above notation, for any \( m \in \mathbb{Z} \), we have

\[
\pi_{0,i+1,m+1}(a \otimes \pi_{i+1,m}^{+}(p_{i}(x_{1} \otimes z_{1}) \otimes u_{m}))
= \pi_{0,i+1,m+1}(a \otimes \pi_{i+1,m}^{+}(x_{1} \otimes \pi_{i,m}^{+}(z_{1} \otimes u_{m}))) - \pi_{0,i+1,m+1}(a \otimes \pi_{i,m+1}^{+}(z_{1} \otimes \pi_{1,m}^{+}(x_{1} \otimes u_{m})))
= \pi_{i,i+1,m}(a, x_{1}) \otimes \pi_{i,m}^{+}(z_{1} \otimes u_{m}) + \pi_{i,i+1,m}(x_{1} \otimes \pi_{i,m}^{+}(z_{1} \otimes u_{m}))
- \pi_{i,m+1}^{+}(z_{1} \otimes \pi_{0,m}^{+}(a \otimes u_{m}))
= \pi_{i+1,m}^{+}([a, x_{1}] \otimes \pi_{i,m}^{+}(z_{1} \otimes u_{m})) + \pi_{i,i+1,m}(x_{1} \otimes \pi_{i,m}^{+}([a, z_{1}] \otimes u_{m}))
- \pi_{i,m+1}^{+}(z_{1} \otimes \pi_{0,m}^{+}(a \otimes u_{m}))
= \pi_{i+1,m}^{+}([a, x_{1}] \otimes \pi_{i,m}^{+}(z_{1} \otimes u_{m})) + \pi_{i+1,m}^{+}([x_{1}, a, z_{1}] \otimes u_{m}) + \pi_{i+1,m}^{+}([x_{1}, z_{1}] \otimes \pi_{0,m}^{+}(a \otimes u_{m}))
= \pi_{i+1,m}^{+}([a, x_{1}] \otimes \pi_{i,m}^{+}(z_{1} \otimes u_{m})) + \pi_{i+1,m}^{+}([x_{1}, a, z_{1}] \otimes u_{m}) + \pi_{i+1,m}^{+}([x_{1}, z_{1}] \otimes \pi_{0,m}^{+}(a \otimes u_{m}))

Thus we have ([3.19]). The equation ([3.20]) follows from ([3.21]). This completes the proof.

Proposition 3.9. (The well-definedness of \( \pi_{i+1,m}^{\pm} \) given in ([3.17])) Suppose that \( i \geq 0 \). Suppose that the linear map \( \pi_{i,m}^{\pm} \) defined in ([3.13]) is well-defined for any \( m \in \mathbb{Z} \) and satisfies the following equations:

\[
\pi_{0,-i,m}^{+}(a \otimes \pi_{i,m}^{+}(\omega_{-i} \otimes u_{m}^{+})) = \pi_{0,-i,m}^{+}([a, \omega_{-i}] \otimes u_{m}^{+}) + \pi_{0,-i,m}^{+}(\omega_{-i} \otimes \pi_{0,m}^{+}(a \otimes u_{m}^{+})),
\]

(3.26)

\[
\pi_{i+1,m+1}^{\pm}(\omega_{-i} \otimes \pi_{-i,m}^{\pm}(x_{1} \otimes u_{m}^{+})) = \pi_{i+1,m+1}^{\pm}([\omega_{-i}, x_{1}] \otimes u_{m}^{+}) + \pi_{1,-i-m}^{\pm}(x_{1} \otimes \pi_{-i,m}^{\pm}(\omega_{-i} \otimes u_{m}^{+})).
\]

(3.27)

If \( y_{1}^{'}, \ldots, y_{l}^{'} \in V_{-i} \) and \( \omega_{1}^{'}, \ldots, \omega_{l}^{'} \in V_{-i} \) satisfy \( \sum_{s=1}^{l} q_{-i}(y_{s}^{'} \otimes \omega_{-s}^{'}) = 0 \), then we have

\[
\sum_{s=1}^{l} (\pi_{-i-1,m}^{+}(y_{s}^{'} \otimes \pi_{-i,m}^{+}(\omega_{-s}^{'}, u_{m}^{+})) - \pi_{-i-1,m}^{+}(\omega_{-s}^{'}, \pi_{1,m}^{+}(y_{s}^{'}, u_{m}^{+}))) = 0
\] (3.28)

for all \( m \in \mathbb{Z} \) and \( u_{m}^{+} \in U_{m}^{+} \). In particular, we can obtain the well-definedness of the linear map \( \pi_{-i-1,m}^{+} \) defined in ([3.17]) for any \( m \in \mathbb{Z} \). Moreover, the maps \( \pi_{-i-1,m}^{+} \) (\( m \in \mathbb{Z} \)) satisfy the
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following equations:

\[\pi^+_{0,-i+m-1}(a \otimes \pi^+_{i-1,m}((x_{i-1} \otimes u^+_m)))
= \pi^+_{i-1,m-1}(a \otimes \pi^+_{i-1,m+1}(x_{i-1} \otimes u^+_m))
\]

\[\pi^+_{i-1,m+1}(\omega_{i-1} \otimes \pi^+_{1,m}(x_{i} \otimes u^+_m))
= \pi^+_{i,m}(\omega_{i-1} \otimes \pi^+_{i-1,m-1}(x_{i} \otimes u^+_m)). \tag{3.29}\]

**Proof.** If \(i = 0\), then our claim immediately follows from the definition. Suppose that \(i \geq 1\). We fix \(i\) and discuss by induction on \(m\). If \(m \leq 0\), the equation (3.29) is clear. Suppose that \(m \geq 1\). Then, for any \(x_1 \in V_1\), \(y_1 \in V_{-1}\), \(\omega_1 \in V_{-i}\) and \(u^+_{m-1} \in U_{m-1}\), we have

\[\pi^+_{i-1,m-1}(y_{-1} \otimes \pi^+_{i,m-1}(x_{i} \otimes u^+_{m-1})))
= \pi^+_{i-1,m-1}(y_{-1} \otimes \pi^+_{i-1,m}(x_{i} \otimes u^+_{m-1})
\]

\[\pi^+_{i-1,m-1}(y_{-1} \otimes \pi^+_{i-1,m}(x_{i} \otimes u^+_{m-1}))
= \pi^+_{i-1,m-1}(y_{-1} \otimes \pi^+_{i-1,m}(x_{i} \otimes u^+_{m-1})))
\]

By the induction hypotheses on \(i\) and \(m\), if we take elements \(y^1_{-1}, \ldots, y^l_{-1} \in V_{-1}\) and \(\omega^1_{-i}, \ldots, \omega^l_{-i} \in V_{-i}\), we have

\[\pi^+_{i-1,m-1}(y_{-1} \otimes \pi^+_{i,m-1}(x_{i} \otimes u^+_{m-1})))
= \pi^+_{i-1,m-1}(y_{-1} \otimes \pi^+_{i-1,m}(x_{i} \otimes u^+_{m-1})))
\]

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$V_{-i}$ satisfying $\sum_{s=1}^{l} q_{-i}(y_{-s,1} \otimes \omega_{-s,i}) = 0$, then we have

$$\sum_{s=1}^{l} \pi_{-i,m-1}^{+}([y_{-s,1}, \omega_{-s,i}], x_{1} \otimes u_{m-1}^{+}) = 0 \quad \text{(by the induction hypothesis on } i),$$

$$\sum_{s=1}^{l} (\pi_{-i,m-2}^{+}(x_{1} \otimes \pi_{-i,m-1}^{+}(y_{-1} \otimes \pi_{-i,m-1}^{+}(\omega_{-i} \otimes u_{m-1}^{+}))))$$

$$\quad - \pi_{-i,m-2}^{+}(x_{1} \otimes \pi_{-i,m-1}^{+}(\omega_{-i} \otimes \pi_{-i,m-1}^{+}(y_{-1} \otimes u_{m-1}^{+})))) = 0 \quad \text{(by the induction hypothesis on } m).$$

Thus, we have

$$\sum_{s=1}^{l} (\pi_{-i,m}^{+}(y_{-s,1} \otimes \pi_{-i,m}^{+}(\omega_{-s,i} \otimes \pi_{1,m-1}^{+}(x_{1} \otimes u_{m-1}^{+}))))$$

$$\quad - \pi_{-i,m}^{+}(\omega_{-s,i} \otimes \pi_{1,m-1}^{+}(y_{-1} \otimes \pi_{1,m-1}^{+}(x_{1} \otimes u_{m-1}^{+})))) = 0 \quad (3.34)$$

from (3.31). Since $\pi_{1,m-1}^{+} : V_{1} \otimes U_{m-1}^{+} \rightarrow U_{m}^{+}$ is surjective, we can obtain the equation (3.32). Moreover, we have the well-definedness of the linear map $\pi_{i-1,m}^{+} : V_{-i-1} \otimes U_{m}^{+} \rightarrow U_{i+m-1}^{+}$ given in (3.15) for any $m$.

The equation (3.32) can be shown by a similar way to the proof of Proposition 3.8. Moreover, the equation (3.30) follows from (3.31).

**Definition 3.10.** By the above propositions, Propositions 3.8 and 3.9, we define a linear map $\pi^{+} : L(\mathfrak{g}, \rho, V, \mathcal{V}, B_{0}) \otimes \hat{U}^{+} \rightarrow \hat{U}^{+}$ by:

$$\pi^{+}(z_{n} \otimes u_{m}^{+}) := \pi_{n,m}^{+}(z_{n} \otimes u_{m}^{+})$$

where $n, m \in \mathbb{Z}$, $z_{n} \in V_{n}$ and $u_{m}^{+} \in U_{m}^{+}$.

This linear map $\pi^{+}$ satisfies the following equations:

$$\pi^{+}([a, z_{n}] \otimes u_{m}^{+}) = \pi^{+}(a \otimes \pi^{+}(z_{n} \otimes u_{m}^{+})) - \pi^{+}(z_{n} \otimes \pi^{+}(a \otimes u_{m}^{+})),$$

$$\pi^{+}([x_{1}, z_{n}] \otimes u_{m}^{+}) = \pi^{+}(x_{1} \otimes \pi^{+}(z_{n} \otimes u_{m}^{+})) - \pi^{+}(z_{n} \otimes \pi^{+}(x_{1} \otimes u_{m}^{+})),$$

$$\pi^{+}([y_{-1}, z_{n}] \otimes u_{m}^{+}) = \pi^{+}(y_{-1} \otimes \pi^{+}(z_{n} \otimes u_{m}^{+})) - \pi^{+}(z_{n} \otimes \pi^{+}(y_{-1} \otimes u_{m}^{+}))$$

for any $n, m \in \mathbb{Z}$, $a \in V_{0}$, $x_{1} \in V_{1}$, $y_{-1} \in V_{-1}$, $z_{n} \in V_{n}$ and $u_{m}^{+} \in U_{m}^{+}$. Moreover, we have the following proposition on $\pi^{+}$.

**Proposition 3.11.** The map $\pi^{+}$ satisfies the following equation:

$$\pi^{+}([x, y] \otimes u) = \pi^{+}(x \otimes \pi^{+}(y \otimes u)) - \pi^{+}(y \otimes \pi^{+}(x \otimes u)) \quad (3.35)$$

for any $x, y \in L(\mathfrak{g}, \rho, V, \mathcal{V}, B_{0})$ and $u^{+} \in \hat{U}^{+}$. 27
Thus, we have our result for any $n = p_{n-1}(x_1 \otimes z_{n-1})$ for some $x_1 \in V_1$ and $z_{n-1} \in V_{n-1}$ without loss of generality. Then, by the induction hypothesis, we have

$$
\hat{\pi}^+(p_{n-1}(x_1 \otimes z_{n-1}), y \otimes u^+) = \hat{\pi}^+([x_1, [z_{n-1}, y]] \otimes u^+) - \hat{\pi}^+([z_{n-1}, [x_1, y]] \otimes u^+)
$$

$$
= \hat{\pi}^+([x_1, [z_{n-1}, y]] \otimes u^+)) - \hat{\pi}^+([z_{n-1}, [x_1, y]] \otimes u^+))
$$

$$
- \hat{\pi}^+(z_{n-1} \otimes \hat{\pi}^+([x_1, y] \otimes u^+)) + \hat{\pi}^+([x_1, y] \otimes \hat{\pi}^+(z_{n-1} \otimes u^+))
$$

$$
= \hat{\pi}^+([x_1, [z_{n-1}, y]] \otimes u^+)) - \hat{\pi}^+([z_{n-1}, [x_1, y]] \otimes u^+))
$$

$$
- \hat{\pi}^+(z_{n-1} \otimes \hat{\pi}^+([x_1, y] \otimes u^+)) + \hat{\pi}^+([x_1, y] \otimes \hat{\pi}^+(z_{n-1} \otimes u^+))
$$

$$
= \hat{\pi}^+([x_1, [z_{n-1}, y]] \otimes u^+)) - \hat{\pi}^+([z_{n-1}, [x_1, y]] \otimes u^+))
$$

$$
- \hat{\pi}^+(z_{n-1} \otimes \hat{\pi}^+([x_1, y] \otimes u^+)) + \hat{\pi}^+([x_1, y] \otimes \hat{\pi}^+(z_{n-1} \otimes u^+))
$$

$$
= \hat{\pi}^+([x_1, [z_{n-1}, y]] \otimes u^+)) - \hat{\pi}^+([z_{n-1}, [x_1, y]] \otimes u^+))
$$

$$
- \hat{\pi}^+(z_{n-1} \otimes \hat{\pi}^+([x_1, y] \otimes u^+)) + \hat{\pi}^+([x_1, y] \otimes \hat{\pi}^+(z_{n-1} \otimes u^+))
$$

$$
= \hat{\pi}^+([x_1, [z_{n-1}, y]] \otimes u^+)) - \hat{\pi}^+([z_{n-1}, [x_1, y]] \otimes u^+))
$$

$$
- \hat{\pi}^+(z_{n-1} \otimes \hat{\pi}^+([x_1, y] \otimes u^+)) + \hat{\pi}^+([x_1, y] \otimes \hat{\pi}^+(z_{n-1} \otimes u^+))
$$

Thus, we have our result for any $n \geq 0$.

Similarly, we can obtain our result for any $n \leq -1$. This completes the proof. }

From Proposition 3.11, we have the following theorem.

**Theorem 3.12.** The vector space $\hat{U}^+ = \bigoplus_{m \in \mathbb{Z}} U^+_m = \bigoplus_{m \geq 0} U^+_m$ has a structure of a positively graded $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$-module whose representation is $\hat{\pi}^+$. We call the module $(\hat{\pi}^+, \hat{U}^+)$ the positive extension of $U$ with respect to a standard pentad $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$. (This is a special case of [28, Theorem 1.2].)

By the same argument, we can obtain a negatively graded Lie module of $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$.

**Definition 3.13.** We define the following linear maps:

$$
\pi_{0,m} : V_0 \otimes U^+_m \rightarrow U^+_m, \quad \pi_{1,m} : V_1 \otimes U^+_m \rightarrow U^+_m, \quad \pi_{-1,m} : V_{-1} \otimes U^+_m \rightarrow U^+_m
$$
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by:

\[ \pi_{0,m}(a \otimes u_m^-) := \pi_m^-(a \otimes u_m), \quad (3.36) \]
\[ \pi_{1,m}(x_1 \otimes u_m^-) := \begin{cases} 0 & (m \geq 0) \\ u_m^-(x_1) & (m \leq -1) \end{cases}, \quad (3.37) \]
\[ \pi_{-1,m}(y_{-1} \otimes u_m^-) := \begin{cases} 0 & (m \geq 1) \\ r_m^-(y_{-1} \otimes u_m^-) & (m \leq 0) \end{cases} \quad (3.38) \]

where \( m \in \mathbb{Z}, a \in V_0, x_1 \in V_1, y_{-1} \in V_{-1} \) and \( u_m^- \in U_m^- \).

**Theorem 3.14.** The vector space \( \hat{U}^- = \bigoplus_{m \in \mathbb{Z}} U_m^- = \bigoplus_{m \leq 0} U_m^- \) has a structure of a negatively graded \( L(\mathfrak{g}, \rho, V, V, B_0) \)-module whose representation is \( \hat{\pi}^- \). We call the module \((\hat{\pi}^-, \hat{U}^-)\) the negative extension of \( U \) with respect to a standard pentad \((\mathfrak{g}, \rho, V, V, B_0)\). (This is a special case of \([2, \text{Theorem 1.2}])\.)

Note that an arbitrary module of \( L(\mathfrak{g}, \rho, V, V, B_0) \) is not necessary written in the form of \( \hat{U}^+ \) or \( \hat{U}^- \). For example, the adjoint representation of a loop algebra \( \mathcal{L}(\mathfrak{sl}_2, \text{ad}, \mathfrak{sl}_2, \mathfrak{sl}_2, K_{\mathfrak{sl}_2}) = \mathcal{L}(\mathfrak{sl}_2(\mathbb{C})) = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{sl}_2(\mathbb{C}) = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} t^n \otimes \mathfrak{sl}_2 \), where \( K_{\mathfrak{sl}_2} \) is the Killing form of \( \mathfrak{sl}_2 \), cannot be written in the form of positively or negatively graded module. Indeed, \( \mathcal{L}(\mathfrak{sl}_2(\mathbb{C})) \) does not have a non-zero element which commutes with any element of the form \( t \otimes X \) or \( t^{-1} \otimes X \) \((X \in \mathfrak{sl}_2)\).

**Proposition 3.15.** Under the notation of Theorems 3.12 and 3.14, \( L(\mathfrak{g}, \rho, V, V, B_0) \)-modules \( \hat{U}^+ \) and \( \hat{U}^- \) have the following properties:

\[ \hat{U}^+ \text{ and } \hat{U}^- \text{ are transitive}, \quad (3.39) \]
\[ \hat{U}^+ \text{ and } \hat{U}^- \text{ are } L(\mathfrak{g}, \rho, V, V, B_0)\text{-irreducible if and only if } U = U_0^+ = U_0^- \text{ is } \mathfrak{g}\text{-irreducible}. \quad (3.40) \]

(This is a special case of \([2, \text{Theorem 1.1}]\).)

**Proof.** By the definition, we can show \((3.39)\) immediately.

Let us show \((3.40)\). Assume that \( U \) is an irreducible non-zero \( L(\mathfrak{g}, \rho, V, V, B_0) \)-submodule of \( \hat{U}^+ \). Then we have that \( W \cap U_0^+ \neq \{0\} \) (cf. \([2, \text{Corollary 1.2}]\)). In fact, take a non-zero element \( w \in W \). Then there exist integers \( 0 \leq m_1 < \cdots < m_k \) and \( w_{m_1} \in W \cap U_{m_1}^+, \ldots, w_{m_k} \in W \cap U_{m_k}^+ \) such that \( w = w_{m_1} + \cdots + w_{m_k} \). Since \( \hat{U}^+ \) is transitive, we can take \( y_{m_1} \in V_{-1} \) such that \( 0 \neq \hat{\pi}^+(y_{m_1} \otimes \cdots \otimes \hat{\pi}^+(y_{m_k} \otimes w) \cdots) \in W \cap U_0^+ \). By the assumption that \( U \) is irreducible, we have \( W \cap U_0^+ = U \). Since \( \hat{U}^+ \) is generated by \( U = U_0^+ \) and \( V_0, V_1 \), we have that \( W \) coincides with \( \hat{U}^+ \).

Conversely, assume that \( \hat{U}^+ \) is an irreducible \( L(\mathfrak{g}, \rho, V, V, B_0) \)-module. Take a non-zero \( \mathfrak{g} \)-submodule \( W \) of \( U \). Then a submodule \( \overline{W} \) of \( \hat{U}^+ \) which is generated by \( V_0, V_1, W \) is a non-zero \( L(\mathfrak{g}, \rho, V, V, B_0) \)-submodule of \( \hat{U}^+ \). Thus, \( \overline{W} = \hat{U}^+ \), and, in particular, \( W = W \cap U_0^+ = U \). Similarly, we can show \((3.40)\) for the negative extension \( \hat{U}^- \).
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**Example 3.16.** We retain to use the notation of Example 2.35. Put $U := \mathbb{C}$ and define a representation $\pi : g \otimes U \to U$ by:

$$\pi((a, b, A) \otimes u) := au$$

for any $u \in U$. Then, the positive extension $\tilde{U}^+$ of $U$ with respect to $(g, \rho, V, V, B_0)$ is 3-dimensional irreducible representation of $L(g, \rho, V, V, B_0) = V_{-1} \oplus V_0 \oplus V_1 \simeq gl_3 \oplus sl_3$. In fact, for any $v \in V_1 = V$, $\phi \in V_{-1} = V$ and $u \in U$, we have

$$\tilde{\pi}^+(\phi \otimes \tilde{\pi}^+(v \otimes u)) = -\tilde{\pi}^+((\Phi, \rho, \phi) \otimes u) = -\pi((-^t v \phi, \frac{3}{2}v^t \phi, v^t \phi - \frac{1}{2}v \phi I_2) \otimes u) = \tilde{t} v \phi u.$$

Thus, the element $\tilde{\pi}^+(v \otimes u)$ can be identified with $uv \in V_1 = V$ via $\langle \cdot, \cdot \rangle$, in particular, $U_1^+$ is 2-dimensional. Moreover, we have

$$\tilde{\pi}^+(\phi \otimes \tilde{\pi}^+(v' \otimes \tilde{\pi}^+(v \otimes u)))$$

$$= -\tilde{\pi}^+((-^t v' \phi, \frac{3}{2}v'' \phi, v'' \phi - \frac{1}{2}v' \phi I_2) \otimes \tilde{\pi}^+(v \otimes u) + \tilde{\pi}^+(v' \otimes \tilde{\pi}^+(\phi \otimes \tilde{\pi}^+(v \otimes u)))$$

$$= -\tilde{\pi}^+((v' \phi \cdot v \otimes u \otimes u) + \tilde{\pi}^+(v' \otimes v' \phi u) + \tilde{\pi}^+(v' \otimes v' \phi u))$$

$$= 0$$

for any $v, v' \in V_1$, $\phi \in V_{-1}$ and $u \in U$. Therefore, the positive extension $\tilde{U}^+ = U_0^+ \oplus U_1^+$ is a 3-dimensional irreducible representation (see Proposition 3.15).

The positive and negative extensions of $U$ are characterized by the transitivity.

**Theorem 3.17.** Let $(g, \rho, V, V, B_0)$ be a standard pentad. Let $(g, U) = (g, \bigoplus_{m \geq 0} U_m)$ (respectively $(g, U) = (g, \bigoplus_{m \leq 0} U_m)$) be a positively graded Lie module (respectively a negatively graded Lie module) of $L(g, \rho, V, V, B_0)$. If the $L(g, \rho, V, V, B_0)$-module $(g, U)$ (respectively $(g, U)$) is transitive and generated by $V_0$, $V_1$ and $U_0$ (respectively generated by $V_0$, $V_{-1}$ and $U_0$), then $U_0$ is isomorphic to the positive extension of $U_0$ with respect to $(g, \rho, V, V, B_0)$ (respectively $U_0$ is isomorphic to the negative extension of $U_0$ with respect to $(g, \rho, V, V, B_0)$). (This is a special case of 2. Theorem 1.2.)

**Proof.** We denote the positive extension of $U_0$ with respect to $(g, \rho, V, V, B_0)$ by

$$\tilde{U}_0^+ = \bigoplus_{m \geq 0} (U)^+_m$$

and the canonical representation of $L(g, \rho, V, V, B_0)$ on $\tilde{U}_0^+$ by $\tilde{\pi}^+$. Note that $(U)^+_0 = U_0$. We let $\tau_0 : (U)^+_0 \to U_0$ be the identity map on $(U)^+_0 = U_0$ and define linear maps $\tau_i : (U)^+_i \to U_i$ by

$$\tau_i(x \otimes U_i) := \tilde{\pi}(x \otimes \tau_{i-1}(U_{i-1}))$$

for $i \geq 0$.
for \( i \geq 1 \) and any \( x_i \in V_1 \) and \( u_{i-1}^+ \in (U)_i^+ \) inductively. These \( \tau_i \)'s are well-defined and satisfy the following equation:

\[
\tau_{i+j}(\tilde{x}^+(a_j \otimes u_i^+)) = \tilde{x}(a_j \otimes \tau_i(u_i^+))
\]  

(3.41)

for \( j = 0, \pm 1 \) and any \( a_j \in V_j, u_i^+ \in (U)_i^+ \). Let us show it by induction on \( i \). It is clear that the equation (3.41) holds when \( i = 0 \) and \( j = 0, -1 \). In order to show the equation (3.41) for \( i = 0 \) and \( j = 1 \), let us show that \( \tau_1 \) is well-defined. Take an arbitrary element \( y_{-1} \in V_{-1} \), then we have

\[
\begin{align*}
\tilde{x}(y_{-1} \otimes \tilde{x}(x_1 \otimes \tau_0(u_i^+))) &= \tilde{x}([y_{-1}, x_1] \otimes \tau_0(u_i^+) + \tilde{x}(x_1 \otimes \tilde{x}(y_{-1} \otimes \tau_0(u_i^+)))) \\
&= \tilde{x}([y_{-1}, x_1] \otimes \tau_0(u_i^+)) = \tau_0(\tilde{x}^+([y_{-1}, x_1] \otimes u_i^+)) = \tau_0(\tilde{x}^+ (y_{-1} \otimes r_0^+ (x_1 \otimes u_i^+))).
\end{align*}
\]  

(3.42)

Thus, if \( x_1^1, \ldots, x_1^s \in V_1 \) and \( u_{i-1}^1, \ldots, u_{i-1}^s \in (U)_i^+ \) satisfy \( \sum_{s=1}^i x_1^s \otimes u_{i-1}^s = 0 \), then we have

\[
\sum_{s=1}^i \tilde{x}(y_{-1} \otimes \tilde{x}(x_1^s \otimes \tau_0(u_{i-1}^s))) = 0
\]

for any \( y_{-1} \in V_{-1} \). Since \( (\tilde{x}, U) \) is transitive, it follows that \( \sum_{s=1}^i \tilde{x}(x_1^s \otimes \tau_0(u_{i-1}^s)) = 0 \), and thus, we have the well-definedness of \( \tau_1 \). By the equation (3.42), we can obtain the equation (3.41) where \( i = 0 \) and \( j = 1 \).

Let \( i \geq 1 \) and assume that \( \tau_0, \ldots, \tau_i \) are well-defined and that \( \tau_i \) satisfies the equation (3.41) for \( j = 0, -1 \). Then for any \( y_{-1} \in V_{-1} \), we have

\[
\begin{align*}
\tilde{x}(y_{-1} \otimes \tilde{x}(x_1 \otimes \tau_i(u_i^+))) &= \tilde{x}([y_{-1}, x_1] \otimes \tau_i(u_i^+) + \tilde{x}(x_1 \otimes \tilde{x}(y_{-1} \otimes \tau_i(u_i^+)))) \\
&= \tau_i(\tilde{x}^+([y_{-1}, x_1] \otimes u_i^+) + \tilde{x}(x_1 \otimes \tilde{x}(y_{-1} \otimes u_i^+))) \\
&= \tau_i(\tilde{x}^+(y_{-1} \otimes x_1 \otimes u_i^+) + \tau_i(\tilde{x}^+(y_{-1} \otimes r_0^+ (x_1 \otimes u_i^+)))) \\
&= \tau_i(\tilde{x}^+(y_{-1} \otimes \tilde{x}^+ (x_1 \otimes u_i^+))) = \tau_i(y_{-1} \otimes r_0^+ (x_1 \otimes u_i^+)).
\end{align*}
\]  

(3.43)

Thus, by the same argument to the argument of the case where \( i = 0 \) and \( j = 1 \), we have the well-definedness of \( \tau_{i+1} \), i.e. \( \tau_i \) satisfies the equation (3.41) for \( j = 1 \), and that \( \tau_{i+1} \) satisfies the equation (3.41) for \( j = -1 \). Moreover, by a similar argument to the argument of (3.43), we have that \( \tau_{i+1} \) satisfies the equation (3.41) for \( j = 0 \). Therefore, by induction on \( i \), we can obtain the well-definedness of \( \tau_i \) and the equation (3.41) for all \( i \geq 0 \) and \( j = 0, \pm 1 \).

We define a linear map \( \tau : U_i^+ \to U_j^+ \) by

\[
\tau(u_i^+) := \tau_i(u_i^+)
\]

(3.44)

for any \( i \geq 0 \) and \( u_i^+ \in (U)_i^+ \). This \( \tau \) is an isomorphism of \( L(\mathfrak{g}, \rho, V, V_0) \)-modules. In fact, by the assumption that \( U_i \) is generated by \( V_1 \) and \( U_{i-1}^+ \), we have the surjectivity of \( \tau \). Moreover, by the equation (3.41) in the cases where \( i \geq 1 \) and \( j = -1 \) and the definition of \( \tau_0 \) and the transitivity of the positive extension of \( U_{i-1}^+ \), we have the injectivity of \( \tau \). Thus, \( \tau \) is bijective. Moreover, since \( L(\mathfrak{g}, \rho, V, V_0) \) is generated by \( V_0, V_{\pm 1} \), it follows that \( \tau \) is a homomorphism of \( L(\mathfrak{g}, \rho, V, V_0) \)-modules from the equation (3.41). Therefore \( U_i \) is isomorphic to \( U_i^+ \) as \( L(\mathfrak{g}, \rho, V, V_0) \)-modules.

By the same argument, we can prove our claim for \( (\tilde{x}, U) \).
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As an application of Theorem 3.17, we have the following proposition.

**Proposition 3.18.** Let \((\mathfrak{g}, \rho, V, \mathcal{V}, B_0)\) be a standard pentad and \(U, W\) (respectively \(\mathcal{U}, \mathcal{W}\)) be \(\mathfrak{g}\)-modules. Then the positive extension of \(U \oplus W\) (respectively the negative extension of \(U \oplus W\)) with respect to \((\mathfrak{g}, \rho, V, \mathcal{V}, B_0)\) is isomorphic to a direct sum of positive extensions of \(U\) and \(W\) (respectively negative extensions of \(U\) and \(W\)) with respect to \((\mathfrak{g}, \rho, V, B_0)\), i.e.

\[
(U \oplus W)^+ \simeq \hat{U}^+ \oplus \hat{W}^+ \quad \text{(respectively } (U \oplus W)^- \simeq \hat{U}^- \oplus \hat{W}^-).\]

**3.3 A pairing between \((\tilde{\pi}^+, \hat{U}^+)\) and \((\tilde{\omega}^-, \hat{U}^-)\)**

In the previous section, we constructed positively and negatively graded \(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)\)-modules. Next, let us try to embed these modules into some graded Lie algebra. For this, we need to embed \(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)\) and \((\tilde{\pi}^+, \hat{U}^+)\) into some standard pentad. However, as mentioned in Remark 2.3, the objects \(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)\) and \(\hat{U}^+\) might not have a submodule of \(\text{Hom}(\hat{U}^+, \mathfrak{k})\) and a bilinear form on \(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)\) satisfying the conditions \((2.3)\) and \((2.4)\). In the present and next sections, we only consider the cases where \(B_0\) is symmetric and \(U\) has a submodule \(\mathcal{U} \subset \text{Hom}(U, \mathfrak{k})\) such that \((\mathfrak{g}, \pi, U, \mathcal{U}, B_0)\) is standard. Then, we can show that a pentad \((L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \hat{U}^+, \hat{U}^-, B_L)\) is standard. First, in this section, we consider the negative extension \(\hat{U}^-\) of \(U\) and construct a non-degenerate invariant bilinear form \(\hat{U}^+ \times \hat{U}^- \rightarrow \mathfrak{k}\) under the assumption \((2.3)\) inductively (cf. \([2]\) Remark 1.4]). In the next section, we shall construct the \(\Phi\)-map of the pentad \((L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \hat{U}^+, \hat{U}^-, B_L)\).

**Definition 3.19.** Let \((\tilde{\pi}^+, \hat{U}^+)\) and \((\tilde{\omega}^-, \hat{U}^-)\), \(\mathcal{U} \subset \text{Hom}(U, \mathfrak{k})\) be \(\mathfrak{g}\)-modules such that the restriction of the canonical pairing \(\langle \cdot, \cdot \rangle_0 : U \times U \rightarrow \mathfrak{k}\) is non-degenerate, and, let \(\hat{U}^+\) and \(\hat{U}^-\) be the positive and negative extensions of \(U\) and \(\mathcal{U}\) respectively. We define a bilinear map \(\langle \cdot, \cdot \rangle_0^i\) by:

\[
\langle \cdot, \cdot \rangle_0^0 : U_0^+ \times U_0^- \rightarrow \mathfrak{k}
\]

\[
(u_0^+, w_0^-) \mapsto \langle u_0^+, w_0^- \rangle_0.
\]

(3.45)

Moreover, for \(i \geq 1\), we define a bilinear map \(\langle \cdot, \cdot \rangle_i^\downarrow \) by:

\[
\langle \cdot, \cdot \rangle_i^\downarrow : U_i^+ \times U_i^- \rightarrow \mathfrak{k}
\]

\[
(r_{i-1}^+(x_1 \otimes u_{i-1}^+), r_{i-1}^-(y_{i-1} \otimes w_{i-1}^-)) \mapsto -(\hat{\pi}^+ (y_{i-1} \otimes r_{i-1}^+(x_1 \otimes u_{i-1}^+)), w_{i-1}^-)_{i-1}^i
\]

(3.46)

inductively.

The well-definedness of Definition 3.19 can be obtained by the following proposition.
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**Proposition 3.20.** Let \( j \geq 0 \). Assume that the bilinear map \( \langle \cdot, \cdot \rangle_{j}^{\pm} \) defined in (3.46) is well-defined and satisfies the following equations:

\[
\langle \tilde{\pi}^{+}(a \otimes u_{j}^{+}), w_{-j}^{+} \rangle_{j}^{\pm} + \langle u_{j}^{+}, \tilde{\pi}^{-}(a \otimes w_{-j}^{+}) \rangle_{j}^{\pm} = 0,
\]

\[
\langle \tilde{\pi}^{+}(y_{-1} \otimes \tilde{\pi}^{+}(x_{1} \otimes u_{j}^{+})), w_{-j}^{+} \rangle_{j}^{\pm} = \langle u_{j}^{+}, \tilde{\pi}^{-}(x_{1} \otimes \tilde{\pi}^{-}(y_{-1} \otimes w_{-j}^{+})) \rangle_{j}^{\pm}
\]

for any \( a \in g = V_{0} \subset L(g, \rho, V, \mathcal{V}, B_{0}) \), \( x_{1} \in V_{1}, y_{-1} \in V_{-1}, u_{j}^{+} \in U_{j+1}^{+}, u_{j}^{+} \in U_{j}^{+} \) and \( w_{-j}^{+} \in \mathcal{U}_{-j}^{+} \).

Then the bilinear map \( \langle \cdot, \cdot \rangle_{j+1}^{\pm} \) defined in (3.46) is also well-defined and satisfies the following equations:

\[
\langle \tilde{\pi}^{+}(a \otimes u_{j+1}^{+}), w_{-j-1}^{+} \rangle_{j+1}^{\pm} + \langle u_{j+1}^{+}, \tilde{\pi}^{-}(a \otimes w_{-j-1}^{+}) \rangle_{j+1}^{\pm} = 0,
\]

\[
\langle \tilde{\pi}^{+}(y_{-1} \otimes \tilde{\pi}^{+}(x_{1} \otimes u_{j}^{+})), w_{-j-1}^{+} \rangle_{j+1}^{\pm} = \langle u_{j+1}^{+}, \tilde{\pi}^{-}(x_{1} \otimes \tilde{\pi}^{-}(y_{-1} \otimes w_{-j-1}^{+})) \rangle_{j+1}^{\pm}
\]

for any \( a \in g = V_{0} \subset L(g, \rho, V, \mathcal{V}, B_{0}) \), \( x_{1} \in V_{1}, y_{-1} \in V_{-1}, u_{j}^{+} \in U_{j}^{+}, u_{j+1}^{+} \in U_{j+1}^{+} \) and \( w_{-j-1}^{+} \in \mathcal{U}_{-j-1}^{+} \).

**Proof.** First, we let \( j = 0 \). It is clear that \( \langle \cdot, \cdot \rangle_{0}^{\pm} \) satisfies (3.47) and (3.49). Let us show that \( \langle \cdot, \cdot \rangle_{0}^{\pm} \) satisfies (3.48). Indeed, under the above notation, we have

\[
\langle \tilde{\pi}^{+}(y_{-1} \otimes \tilde{\pi}^{+}(x_{1} \otimes u_{0}^{+})), w_{0} \rangle_{0}^{\pm} = \langle \tilde{\pi}^{+}([y_{-1}, x_{1}] \otimes u_{0}^{+}), w_{0} \rangle_{0}^{\pm} = \langle u_{0}^{+}, \tilde{\pi}^{-}([x_{1}, y_{-1}] \otimes w_{0}) \rangle_{0}^{\pm}
\]

Thus, the bilinear map \( \langle \cdot, \cdot \rangle_{0}^{\pm} \) satisfies the assumptions of Proposition 3.20.

Next, let us show that the bilinear map \( \langle \cdot, \cdot \rangle_{-1}^{-} \) is well-defined. Take arbitrary natural numbers \( \nu, \mu \in \mathbb{N} \) and elements \( x_{1}^{\nu}, \ldots, x_{1}^{\nu} \in V_{1}, u_{0}^{+, \nu}, \ldots, u_{0}^{+, \nu} \in U_{0}^{\nu}, y_{-1}^{\nu, \mu}, \ldots, y_{-1}^{\nu, \mu} \in V_{-1}^{\nu, \mu}, w_{0}^{+, \nu, \mu}, \ldots, w_{0}^{+, \nu, \mu} \in U_{0}^{\nu, \mu} \) satisfying

\[
\sum_{s=1}^{\nu} r_{s}^{+}(x_{1}^{s} \otimes w_{0}^{+, s}) = 0, \quad \sum_{l=1}^{\mu} r_{l}^{-}(y_{-1}^{l} \otimes w_{0}^{-, l}) = 0.
\]

Then, for any \( y_{-1} \in V_{-1}, w_{0}^{-} \in U_{0}^{-}, x_{1} \in V_{1} \) and \( u_{0}^{+} \in U_{0}^{+} \), we have

\[
\langle \sum_{s=1}^{\nu} r_{s}^{+}(x_{1}^{s} \otimes u_{0}^{+, s}),(y_{-1}), w_{0}^{-} \rangle_{0}^{0} = 0,
\]

and, by the equation (3.53), we have

\[
\sum_{l=1}^{\mu} \langle r_{l}^{-}(x_{1} \otimes u_{0}^{+}), (y_{-1}^{-}, w_{0}^{-}) \rangle_{0}^{0} = \sum_{l=1}^{\mu} \langle u_{0}^{+}, r_{l}^{-}(y_{-1}^{-} \otimes w_{0}^{-}) \rangle_{0}^{0} = 0.
\]
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By (3.54) and (3.55), we can obtain that \( \langle \cdot, \cdot \rangle^1_{-1} \) is well-defined.

Let us consider properties of \( \langle \cdot, \cdot \rangle^1_{-1} \). By (3.53), we have that \( \langle \cdot, \cdot \rangle^1_{-1} \) satisfies

\[
\langle \tilde{\pi}^+(x_1 \otimes u^+_1), w^-_{-1} \rangle^1_{-1} = -\langle u^+_0, \tilde{\pi}^-(x_1 \otimes w^-_{-1}) \rangle^1_{-1} \tag{3.56}
\]

for any \( x_1 \in V_1, u^+_0 \in U^+_0 \) and \( w^-_{-1} \in U^-_1 \), i.e., \( \langle \cdot, \cdot \rangle^1_{-1} \) satisfies the equation (3.52). Moreover, \( \langle \cdot, \cdot \rangle^1_{-1} \) satisfies the equations (3.50) and (3.51). In fact, for all \( a \in V_0, x_1 \in V_1, y_{-1} \in V_{-1}, u^+_0 \in U^+_0 \) and \( w^-_{0} \in U^-_1 \), we have

\[
\langle \tilde{\pi}^+(a \otimes r^+_0 (x_1 \otimes u^+_0)), r^-_0 (y_{-1} \otimes w^-_{0}) \rangle^1_{-1} = -\langle \tilde{\pi}^+((y_{-1} \otimes \tilde{\pi}^+(a \otimes \tilde{\pi}^+(x_1 \otimes u^+_0))), r^-_0 y_{-1} \rangle^0_0 \\
= -\langle \tilde{\pi}^+(a \otimes \tilde{\pi}^+(y_{-1} \otimes \tilde{\pi}^+(x_1 \otimes u^+_0))), r^-_0 y_{-1} \rangle^0_0 + \langle \tilde{\pi}^+([a, y_{-1}] \otimes \tilde{\pi}^+(x_1 \otimes u^+_0)), r^-_0 y_{-1} \rangle^0_0 \\
= \langle \tilde{\pi}^+(y_{-1} \otimes \tilde{\pi}^+(x_1 \otimes u^+_0)), r^-_0 y_{-1} \rangle^0_0 - \langle \tilde{\pi}^-((x_1 \otimes u^+_0), r^-_0 y_{-1} \rangle^1_{-1} \\
= -\langle \tilde{\pi}^+((x_1 \otimes u^+_0), r^-_0 y_{-1} \rangle^1_{-1} - \langle \tilde{\pi}^+((x_1 \otimes u^+_0), r^-_0 y_{-1} \rangle^1_{-1} \\
= -\langle r^-_0 (x_1 \otimes u^+_0), r^-_0 y_{-1} \rangle^1_{-1} - \langle \tilde{\pi}^-((x_1 \otimes u^+_0), r^-_0 y_{-1} \rangle^1_{-1} \tag{3.57}
\]

Thus \( \langle \cdot, \cdot \rangle^1_{-1} \) satisfies (3.50). And, from (3.56) and (3.57), we have

\[
\langle \tilde{\pi}^+(y_{-1} \otimes \tilde{\pi}^+(x_1 \otimes u^+_0)), w^-_{-1} \rangle^1_{-1} \\
= \langle \tilde{\pi}^+((y_{-1} \otimes x_1) \otimes u^+_0), w^-_{-1} \rangle^1_{-1} + \langle \tilde{\pi}^+(x_1 \otimes \tilde{\pi}^+(y_{-1} \otimes u^+_0)), w^-_{-1} \rangle^1_{-1} \\
= -\langle u^+_1, \tilde{\pi}^-((y_{-1} \otimes x_1) \otimes w^-_{-1}) \rangle^1_{-1} - \langle \tilde{\pi}^+((y_{-1} \otimes x_1) \otimes u^+_0), \tilde{\pi}^-((y_{-1} \otimes x_1) \otimes w^-_{-1}) \rangle^1_{-1} \\
= \langle u^+_1, \tilde{\pi}^-((y_{-1} \otimes x_1) \otimes w^-_{-1}) \rangle^1_{-1} \\
= \langle u^+_1, \tilde{\pi}^-((y_{-1} \otimes x_1) \otimes w^-_{-1}) \rangle^1_{-1} \tag{3.58}
\]

for any \( x_1 \in V_1, y_{-1} \in V_{-1}, u^+_0 \in U^+_1 \) and \( w^-_{-1} \in U^-_1 \). Thus \( \langle \cdot, \cdot \rangle^1_{-1} \) satisfies (3.51).

We let \( j \geq 1 \). Suppose that the bilinear map \( \langle \cdot, \cdot \rangle^{j+1}_{-j} \) is well-defined and satisfies the equations (3.47), (3.48) and (3.49). Let us show the well-definedness of \( \langle \cdot, \cdot \rangle^{j+1}_{-j-1} \). Take arbitrary natural numbers \( \nu, \mu \in \mathbb{N} \) and elements \( x_1^{\nu}, \ldots, x_1^{\nu} \in V_1, u_j^{\mu+1}, \ldots, u_j^{\mu+1} \in U^+_1, y_{-j}^{\mu}, \ldots, y_{-j}^{\mu} \in V_{-1}, w_{-j}^{\nu}, \ldots, w_{-j}^{\mu} \in U^-_1 \) satisfying

\[
\sum_{s=1}^{\nu} r_j^+(x_1^s \otimes u_j^{s+}) = 0, \quad \sum_{t=1}^{\mu} r_{-j}^-(y_{-j}^t \otimes w_{-j}^{-t}) = 0. \tag{3.59}
\]

Then, by the equation (3.48) and the same argument to the argument of (3.54) and (3.55), we have the following equations:

\[
\sum_{s=1}^{\nu} r_j^+(x_1^s \otimes u_j^{s+})(y_{-1}^t)_{-j-1}^{t} = 0, \quad \sum_{t=1}^{\mu} (r_j^+(x_1^t \otimes u_j^{t+}))_{-j-1}^{t} = 0. \tag{3.60}
\]

Thus, we have that the bilinear map \( \langle \cdot, \cdot \rangle^{j+1}_{-j-1} \) is well-defined.

From the equation (3.48), we have

\[
\langle \tilde{\pi}^+(x_1 \otimes u_j^t), w_{-j-1}^{-t} \rangle^{j+1}_{-j-1} = -\langle u_j^t, \tilde{\pi}^-(x_1 \otimes w_{-j-1}^{-t}) \rangle^{j}_{-j-1} \tag{3.61}
\]
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for any \( x_1 \in V_1, u_j^+ \in U_j^+ \) and \( w_{-j-1} \in U_{-j-1} \). We can show that the bilinear map \( \langle \cdot, \cdot \rangle^{j+1}_{j-1} \) satisfies the equation (3.52) from the equation (3.61) and that it also satisfies the equations (3.50) and (3.51) by the same argument to the argument of the case where \( j = 0 \).

By Proposition 3.20 we can obtain pairings \( \langle \cdot, \cdot \rangle_j^j \) for all \( j \geq 0 \) inductively. Then, we can define a pairing between \((\tilde{\pi}^+, \tilde{U}^+)\) and \((\tilde{\pi}^-, \tilde{U}^-)\).

Definition 3.21. We define a bilinear map \( \langle \cdot, \cdot \rangle : \tilde{U}^+ \times \tilde{U}^- \rightarrow \mathfrak{k} \) by:

\[
\langle u_n^+, w_m^- \rangle := \begin{cases} 
\langle u_n^+, w_m^- \rangle_n & (n = m) \\
0 & (n \neq m)
\end{cases}
\]

(3.62)

for any \( n, m \geq 0 \), \( u_n^+ \in U_n^+ \subset \tilde{U}^+ \) and \( w_m^- \in U_m^- \subset \tilde{U}^- \).

By Definition 3.19 and Proposition 3.20, we have that \( \langle \cdot, \cdot \rangle \) satisfies

\[
\langle \tilde{\pi}^+(z_j \otimes u^+), \tilde{\pi}^-(w^-) \rangle = -\langle \tilde{\pi}^+(z_j \otimes w^-), \tilde{\pi}^-(w^-) \rangle
\]

(3.63)

for \( j = 0, \pm 1 \) and any \( z_j \in V_j, \tilde{u}^+ \in \tilde{U}^+, \tilde{w}^- \in \tilde{U}^- \).

Proposition 3.22. The bilinear form \( \langle \cdot, \cdot \rangle : \tilde{U}^+ \times \tilde{U}^- \rightarrow \mathfrak{k} \) is non-degenerate and \( L(g, \rho, V, \mathcal{V}, B_0) \)-invariant (cf. [2 Definition 1.4 and Remark 1.4]).

Proof. First, let us show that the bilinear form \( \langle \cdot, \cdot \rangle \) is non-degenerate. For this, it is sufficient to show that the bilinear map \( \langle \cdot, \cdot \rangle_j^j : U_j^+ \times U_j^- \rightarrow \mathfrak{k} \) is non-degenerate for each \( j \geq 0 \). We show it by induction on \( j \). For \( j = 0 \), it follows that \( \langle \cdot, \cdot \rangle_0^0 \) is non-degenerate from the assumption. For \( j + 1 \), we take an element \( u_{j+1}^+ \in U_{j+1}^+ \) which satisfies \( \langle u_{j+1}^+, r_{-j}(y_{-1} \otimes w_{-j}) \rangle_{j+1}^{j+1} = 0 \) for any \( y_{-1} \in V_{-1} \) and \( w_{-j} \in U_{-j} \). Then, we have

\[
0 = \langle u_{j+1}^+, r_{-j}(y_{-1} \otimes w_{-j}) \rangle_{j+1}^{j+1} = -\langle \tilde{\pi}^+(y_{-1} \otimes u_{j+1}^+), w_{-j} \rangle_{j+1}^j = -\langle u_{j+1}^+(y_{-1}), w_{-j} \rangle_{j+1}^j.
\]

By the induction hypothesis that \( \langle \cdot, \cdot \rangle_j^j \) is non-degenerate, we can obtain that \( u_{j+1}^+(y_{-1}) = 0 \) for any \( y_{-1} \in V_{-1} \), and, thus, we have \( u_{j+1}^+ = 0 \in U_{j+1}^+ \subset \text{Hom}(V_{-1}, U_{j+1}^+) \). Similarly, we can show that an element \( w_{-j-1}^- \in U_{-j-1}^- \) which satisfies \( \langle r_j^- (x_1 \otimes u_j^+), w_{-j-1}^- \rangle_{j-1}^j = 0 \) for any \( x_1 \in V_1 \) and \( u_j^+ \in U_j^+ \) is 0 by (3.63). Summarizing the above argument, we can obtain that the map \( \langle \cdot, \cdot \rangle_{j+1}^1 \) is non-degenerate. Therefore, by induction, we can obtain that the bilinear map \( \langle \cdot, \cdot \rangle : \tilde{U}^+ \times \tilde{U}^- \rightarrow \mathfrak{k} \) is non-degenerate.

Next, let us show that the bilinear map \( \langle \cdot, \cdot \rangle \) is \( L(g, \rho, V, \mathcal{V}, B_0) \)-invariant. For this, it is sufficient to show that the following equation holds:

\[
\langle \tilde{\pi}^+(x_j \otimes u_n^+), w_{-n-j}^- \rangle_n^{n+j} + \langle u_n^+, \tilde{\pi}^-(x_j \otimes w_{-n-j}^-) \rangle_n^{n+j} = 0
\]

(3.64)
for any \(j, n \in \mathbb{Z}, x_j \in V_j, u_n \in U_n^+\) and \(w_{-n-j} \in \mathcal{U}_{-n-j}\). We shall show it by induction on \(j\). Assume that \(j \geq 0\). For \(j = 0, 1\), the equation (3.64) follows from (3.63) immediately. For \(j + 1\), by induction hypothesis, we have

\[
\langle \hat{\pi}^+(v_1 \otimes x_j) \otimes u_n^+, w_{-n-j-1}^{-}\hat{\pi}^{-1}(v_1 \otimes x_j) \rangle_{n-j-1}^{n+j+1} = \langle \hat{\pi}^+(v_1 \otimes x_j) \otimes u_n^+), w_{-n-j-1}^{-}\hat{\pi}^{-1}(v_1 \otimes x_j) \rangle_{n-j-1}^{n+j+1} = \langle \hat{\pi}^+(v_1 \otimes x_j) \otimes u_n^+, \hat{\pi}^{-1}(v_1 \otimes x_j) \rangle_{n-j}^{n+1} + \langle \hat{\pi}^{-1}(v_1 \otimes x_j) \rangle_n^{-n-j} \rangle_{n}^{n+1} - \langle \hat{\pi}^+(v_1 \otimes x_j) \otimes u_n^+, \hat{\pi}^{-1}(v_1 \otimes x_j) \rangle_{n-j}^{n+1} \rangle_n
\]

for any \(n \in \mathbb{Z}, x_j \in V_j, v_1 \in V_1, u_n^+ \in U_n^+\) and \(w_{-n-j-1}^{-} \in \mathcal{U}_{-n-j-1}^{-}\). Thus, by induction, we can show the equation (3.64) for all \(j \geq 0\). Similarly, we can obtain the equation (3.64) for all \(j \leq 0\). Thus, we have the equation (3.64) for all \(j \in \mathbb{Z}\). Therefore the bilinear map \(\langle \cdot, \cdot \rangle : \mathcal{U}^+ \times \mathcal{U}^- \to \mathfrak{k}\) is \(L(g, \rho, V, \mathcal{V}, B_0)\)-invariant.

By Proposition 3.22, we can regard \(\mathcal{U}^-\) as an \(L(g, \rho, V, \mathcal{V}, B_0)\)-module of \(\text{Hom}(\mathcal{U}^+, \mathfrak{k})\).

### 3.4 The \(\Phi\)-map between \((\hat{\pi}^+, \mathcal{U}^+)\) and \((\hat{\pi}^-, \mathcal{U}^-)\)

We retain to assume that a pentad \((g, \pi, U, \mathcal{U}, B_0)\) is standard and the bilinear form \(B_0\) is symmetric. As I proved in section 3.3, a pentad \((L(g, \rho, V, \mathcal{V}, B_0), \hat{\pi}^+, \mathcal{U}^+, B_L)\) satisfies the condition (2.3).

Let us construct the \(\Phi\)-map of the pentad \((L(g, \rho, V, \mathcal{V}, B_0), \hat{\pi}^+, \mathcal{U}^+, \mathcal{U}^-, B_L)\) and show that it is standard.

**Definition 3.23.** Assume that pentads \((g, \rho, V, \mathcal{V}, B_0)\) and \((g, \pi, U, \mathcal{U}, B_0)\) are standard and that \(B_0\) is symmetric. We define a linear map \(\Phi_0 : U_0^+ \otimes U_0^- \to V_0\) as:

\[
\Phi_0(u_0^+ \otimes w_0^-) := \Phi_\pi(u_0^+ \otimes w_0^-)
\]

where \(x_1 \in V_1, y_{-1} \in V_{-1}, u_0^+ \in U_0^+, w_0^- \in \mathcal{U}_{-1}^-\) and \(\Phi_\pi\) is the \(\Phi\)-map of \((g, \pi, U, \mathcal{U}, B_0)\).

Moreover, for each \(i \geq 0\), we inductively define a linear map \(\Phi_{i+1} : U_{i+1}^+ \otimes \mathcal{U}_{i+1}^- \to V_{i+1}\) by:

\[
\Phi_{i+1}(x_i \otimes u_i^+ \otimes w_i^-) := [x_i, \Phi_\pi(u_i^+ \otimes w_i^-)],
\]

where \(x_i \in V_i, y_{-1} \in V_{-1}, u_i^+ \in U_i^+\) and \(w_i^- \in \mathcal{U}_i^-\).

Assume that an integer \(j \geq 0\) satisfies a condition that we have linear maps \(\Phi^k_{j} : U_{k}^+ \otimes \mathcal{U}_{-j}^- \to V_{k-j}\) for all \(k \geq 0\). Then, for any \(k \geq 0\), we define a linear map \(\Phi^k_{j-1} : U_{k}^+ \otimes \mathcal{U}_{-j-1}^- \to V_{k-j-1}\) by:

\[
\Phi^k_{j-1}(u_k^+ \otimes w_{-j}) := \begin{cases} [y_{-1}, \Phi^k_{j}(u_k^+ \otimes w_{-j})] & (k = 0) \\ [y_{-1}, \Phi^k_{j}(u_k^+ \otimes w_{-j})] - \Phi^{k-1}_{j-1}(\hat{\pi}^+(y_{-1} \otimes u_k^+ \otimes w_{-j})) & (k \geq 1) \end{cases}
\]

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where \( y_{-1} \in V_{-1}, u^+_k \in U^+_k \) and \( w^-_j \in U^-_j \).

Consequently, we can define linear maps \( \tilde{\Phi}^i_{-j} : U^+_i \otimes U^-_j \rightarrow V_{i-j} \) for all \( i,j \geq 0 \).

**Proposition 3.24.** The linear map \( \tilde{\Phi}^i_{-j} \) is well-defined and satisfies the following equation:

\[
B_L(a_{-i+j}, \tilde{\Phi}^i_{-j}(u^+_i \otimes w^-_j)) = \langle \tilde{\pi}^+(a_{-i+j} \otimes u^+_i), w^-_j \rangle
\]

for any \( i,j \geq 0, a_{-i+j} \in V_{-i+j}, u^+_i \in U^+_i \) and \( w^-_j \in U^-_j \).

**Proof.** Let us show that the linear maps defined by the equations (3.66), (3.67) and (3.68) satisfy our claim by induction. First, let us show that the linear map \( \tilde{\Phi}^{i+1}_0 \) defined in (3.67) is well-defined by induction on \( i \). For \( i = 0 \), under the above notation, we have

\[
B_L(a_{-1}, \tilde{\Phi}^0_0(u^+_0 \otimes w^-_0)) = B_L([a_{-1}, x_1], \tilde{\Phi}^0_0(u^+_0 \otimes w^-_0)) = \langle \tilde{\pi}^+(a_{-1} \otimes u^+_0), w^-_0 \rangle
= \langle r^+_0(x_1 \otimes u^+_0)(a_{-1}), w^-_0 \rangle = \langle \tilde{\pi}^+(a_{-1} \otimes r^+_0(x_1 \otimes u^+_0)), w^-_0 \rangle
\]

for any \( a_{-1} \in V_{-1} \). Thus, if \( x_1, \ldots, x_l \in V_1 \) and \( u^+_{-l}, \ldots, u^+_{0} \in U^+_0 \) satisfy \( \sum_{s=1}^l r^+_0(x_s^1 \otimes u^+_0) = 0 \), then we have

\[
\sum_{s=1}^l B_L(a_{-1}, [x_s^1, \tilde{\Phi}^0_0(u^+_0 \otimes w^-_0)]) = 0
\]

for any \( a_{-1} \in V_{-1} \). Since the restriction of \( B_L \) to \( V_{-1} \times V_1 \) is non-degenerate, we have

\[
\sum_{s=1}^l [x_s^1, \tilde{\Phi}^0_0(u^+_0 \otimes w^-_0)] = 0,
\]

and, thus, the map \( \tilde{\Phi}^0_0 \) is well-defined. The equation (3.69) follows from (3.70).

For \( i \geq 1 \), under the notation of (3.67), we have

\[
B_L(a_{-i-1}, [x_1, \tilde{\Phi}^0_0(u^+_0 \otimes w^-_0)]) = B_L([a_{-i-1}, x_1], \tilde{\Phi}^0_0(u^+_0 \otimes w^-_0)) = \langle \tilde{\pi}^+(a_{-i-1} \otimes u^+_i), w^-_0 \rangle
= \langle \tilde{\pi}^+(a_{-i-1} \otimes \tilde{\pi}^+(x_1 \otimes u^+_i)), w^-_0 \rangle - \langle \tilde{\pi}^+(x_1 \otimes \tilde{\pi}^+(a_{-i-1} \otimes u^+_i)), w^-_0 \rangle
= \langle \tilde{\pi}^+(a_{-i-1} \otimes r^+_i(x_1 \otimes u^+_i)), w^-_0 \rangle
\]

by the induction hypothesis for any \( a_{-i-1} \in V_{-i-1} \). Thus, by the same argument to the argument of the case where \( i = 0 \), we have the well-definedness of \( \tilde{\Phi}^{i+1}_0 \) and that \( \tilde{\Phi}^{i+1}_0 \) satisfies the equation (3.69). Therefore, by induction, we can obtain our claim on \( \tilde{\Phi}^{i+1}_0 \) for all \( i \geq 0 \).

Let us show that the linear maps defined in (3.68) are well-defined. We assume that an integer \( i \geq 0 \) satisfies the condition that we have linear maps \( \tilde{\Phi}^k_{-i} : U^+_k \otimes U^-_i \rightarrow V_{k-i} \) for all \( k \geq 0 \) which satisfy the equation (3.69). When \( i = 0 \), it has been shown that this assumption holds. Then, we can show the well-definedness of the linear maps \( \tilde{\Phi}^{k+1}_{-i} (k \geq 0) \) by induction on \( k \). When \( k = 0 \), we
can show that $\tilde{\Phi}_{k-1}^0$ is well-defined and satisfies (3.69) by a similar argument to the argument of (3.67). When $k \geq 1$, we have
\[
B_L(a_{-k+1}, [y_{-1}, \tilde{\Phi}_k^0(u_k^+ \otimes w_0^-)] - \tilde{\Phi}_k^{k-1}(\tilde{\pi}^+(y_{-1} \otimes u_k^+) \otimes w_0^-))
= B_L((a_{-k+1}, y_{-1}], \tilde{\Phi}_k^0(u_k^+ \otimes w_0^-)) - B_L(a_{-k+1}, \tilde{\Phi}_k^{k-1}(\tilde{\pi}^+(y_{-1} \otimes u_k^+) \otimes w_0^-))
= \langle \tilde{\pi}^((a_{-k+1}, y_{-1} \otimes u_k^+), w_0^0 \rangle - \langle \tilde{\pi}^+(a_{-k+1} \otimes \tilde{\pi}^+(y_{-1} \otimes u_k^+)), w_0^- \rangle
= -\langle \tilde{\pi}^+(y_{-1} \otimes \tilde{\pi}^+(a_{-k+1} \otimes u_k^+)), w_0^- \rangle
= \langle \tilde{\pi}^+(a_{-k+1} \otimes u_k^+), \tilde{\pi}^-(y_{-1} \otimes w_0^-) \rangle = \langle \tilde{\pi}^+(a_{-k+1} \otimes u_k^+), r_0^0 (y_{-1} \otimes w_0^-) \rangle
\] (3.74)
for any $k \geq 1$ and $a_{-k+1} \in V_{-k+1}$ under the notation of (3.68). Thus, by a similar argument to the argument of the equation (3.69), we have the well-definedness of $\tilde{\Phi}_k^0$ for all $k \geq 1$ and that $\tilde{\Phi}_k^0$ satisfies the equation (3.69). For $i \geq 1$, by the same argument to the argument of the case where $i = 0$, we have the well-definedness of $\tilde{\Phi}_k^{k-1}$ for all $k \geq 0$ and that $\tilde{\Phi}_k^{k-1}$ satisfies the equation (3.69). Thus, by induction, we have linear maps $\tilde{\Phi}_{i,j}$ for all $i, j \geq 0$ which satisfies the equation (3.69). This completes the proof.

As a corollary of Propositions 3.22 and 3.24, we have the following theorem.

**Theorem 3.25.** Let $(g, \rho, V, \mathcal{V}, B_0)$ and $(g, \pi, U, \mathcal{U}, B_0)$ be standard pentads and assume that $B_0$ is symmetric. Then a pentad $(L(g, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{U}^-, B_L)$ is also a standard pentad whose $\Phi$-map, denoted by $\Phi_{\pi}^+$, is defined by:
\[
\tilde{\Phi}_{\pi}^+(u_i^+ \otimes w_j^-) := \tilde{\Phi}_{i,j}^+(u_i^+ \otimes w_j^-)
\] (3.75)
for any $i, j \geq 0$, $u_i^+ \in U_i^+$ and $w_j^- \in U_j^-$, where $\tilde{\Phi}_{i,j}^+$ is the linear map defined in Definition 3.23.

### 3.5 Chain rule

Under the assumptions of sections 3.3 and 3.4, let us construct the Lie algebra associated with a standard pentad of the form $(L(g, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{U}^-, B_L)$. To find the structure of the Lie algebra $L(L(g, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{U}^-, B_L)$, we give the following theorem.

**Theorem 3.26** (chain rule). Let $(g, \rho, V, B_0)$ and $(g, \pi, U, \mathcal{U}, B_0)$ be standard pentads. Assume that $B_0$ is symmetric. Then a pentad $(L(g, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{U}^-, B_L)$ is also a standard pentad and the Lie algebra associated with it is isomorphic to $L(g, \rho \oplus \pi, V \oplus U, \mathcal{V} \oplus \mathcal{U}, B_0)$, i.e. we have
\[
L(L(g, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{U}^-, B_L) \simeq L(g, \rho \oplus \pi, V \oplus U, \mathcal{V} \oplus \mathcal{U}, B_0)
\] (3.76)
as Lie algebras.

**Proof.** Note that the pentad $(g, \rho \oplus \pi, V \oplus U, \mathcal{V} \oplus \mathcal{U}, B_0)$ is a standard pentad whose $\Phi$-map $\Phi_{\rho \oplus \pi}$ is given by:
\[
\Phi_{\rho \oplus \pi}((v, u) \otimes (\phi, \psi)) = \Phi_{\rho}(v \otimes \phi) + \Phi_{\pi}(u \otimes \psi)
\]
Denote a bilinear form on $(U, L)$, which satisfy Lemmas 2.37 and 2.38, we can define derivations $\alpha$ and $\beta$ on $L(g, \rho, V, B_0)$ which satisfy

$$\alpha(v_n) = nv_n, \quad \alpha(u_+^i) = iu_+^i, \quad \alpha(w^-_j) = -jw^-_j, \quad \beta(\hat{u}_m^+) = m\hat{u}_m^+$$

Moreover, we denote $(\hat{U}^+)_1$ and $(\hat{U}^+)_-1$ by:

$$(\hat{U}^+)_1 = \bigoplus_{i \geq 0} U^+_i, \quad (\hat{U}^+)_-1 = \bigoplus_{j \geq 0} U^-_j.$$  

Denote a bilinear form on $L(g, \rho, V, B_0)$, we can define derivations $\alpha$ and $\beta$ on $L(g, \rho, V, B_0)$ which satisfy

$$\alpha(v_n) = nv_n, \quad \alpha(u_+^i) = iu_+^i, \quad \alpha(w^-_j) = -jw^-_j, \quad \beta(\hat{u}_m^+) = m\hat{u}_m^+$$

and

$$B_L(\alpha(\varphi), \varphi) + B_L(\beta(\varphi), \varphi) = B_L(\beta(\varphi), \alpha(\varphi)) = 0$$

for any $n, m \in \mathbb{Z}$, $i, j \geq 0$, $v_n \in V_n$, $u_+^i \in U^+_i \subset (\hat{U}^+)_1$, $w^-_j \in U^-_j \subset (\hat{U}^+)_-1$, $\hat{u}_m^+ \in (\hat{U}^+)_m$ and $\varphi, \psi \in L(g, \rho, V, B_0)$. Since $L(g, \rho, V, B_0)$ is generated by $L(g, \rho, V, B_0)$ and $(\hat{U}^+)_\pm 1$ and since $L(g, \rho, V, B_0)$ and $(\hat{U}^+)_\pm 1$ are generated by $V_0$, $V_{\pm 1}$, $U = U^+_0$ and $U = U^-_0$, we have that $L(g, \rho, V, B_0) = \hat{U}^+_0$, $\hat{U}^-_0$. Put

$$W_{n,m} := \{ \varphi \in L(g, \rho, V, B_0) | \alpha(\varphi) = n\varphi, \beta(\varphi) = m\varphi \}$$

for any $n, m \in \mathbb{Z}$. Then we can easily show that all eigenvalues of $\alpha$ and $\beta$ are integers by induction and that $[W_{n,m}, W_{k,l}] \subset W_{n+k,m+l}$. Thus, we can obtain the following $\mathbb{Z}$-grading of $L(g, \rho, V, B_0)$ induced by the eigenspace decomposition of $\gamma := \alpha + \beta$:

$$L(g, \rho, V, B_0) = \bigoplus_k \left( \bigoplus_{n+m=k} W_{n,m} \right).$$

If we put $W_k^\gamma := \{ \varphi | \gamma(\varphi) = k\varphi \}$, then we have $W_k^\gamma = \bigoplus_{n+m=k} W_{n,m}$ and, thus, we can obtain the following $\mathbb{Z}$-grading of $L(g, \rho, V, B_0)$:

$$L(g, \rho, V, B_0) = \bigoplus_k W_k^\gamma.$$
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In particular,

\[ W_0^\gamma = V_0, \quad W_1^\gamma = V_1 \oplus U_0^+, \quad W_{-1}^\gamma = V_{-1} \oplus U_0^-. \]  

We can easily show that \( W_{k+1}^\gamma = [W_1^\gamma, W_k^\gamma], \) \( W_{-k-1}^\gamma = [W_{-1}^\gamma, W_k^\gamma] \) for all \( k \geq 1 \) and that the restriction of \( \mathfrak{B}_L \) to \( W_k^\gamma \times W_{-k}^\gamma \) is non-degenerate for any \( k \in \mathbb{Z} \) from (3.80). Therefore, by Theorem 2.20, we have the equation (3.76).

**Example 3.27.** We retain to use the notations of Examples 2.35 and 3.16. Put \( \mathfrak{g} \to \mathfrak{u} \to \mathcal{U} \) and a bilinear map \( \langle \cdot, \cdot \rangle_U : \mathcal{U} \times \mathcal{U} \to \mathbb{C} \) by:

\[ \varphi((a,b,A) \otimes w) := -aw, \quad \langle u, w \rangle_U := uw. \]

We can identify \( \mathcal{U} \) with \( \text{Hom}(U, \mathbb{C}) \) via \( \langle \cdot, \cdot \rangle_U \). Then pentads \((L(g, \rho, V, V, B_0), \bar{\pi}^+, \bar{\nu}^+, \bar{\nu}^-, B_L)\) and \((g, \rho \oplus \pi, V \oplus U, V \oplus U, B_0)\) are standard. Let us show that the Lie algebra \( L(g, \rho \oplus \pi, V \oplus U, V \oplus U, B_0) \) is isomorphic to \( \mathfrak{sl}_4 \). Put elements

\[ H_0 := \begin{pmatrix} \frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & -\frac{1}{4} & 0 \\ -\frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix}, \quad H_1 := \begin{pmatrix} \frac{3}{4} & -1 & 0 \\ -1 & \frac{3}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}, \quad H_2 := \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \in \mathfrak{sl}_4. \]

Then we can obtain a \( \mathbb{Z} \)-grading of \( \mathfrak{sl}_4 \) by the eigenspace decomposition of \( \text{ad} H_0 \):

\[ \mathfrak{sl}_4 = \bigoplus_{i=-2}^{2} \mathfrak{l}_i \quad (\mathfrak{l}_i := \{ X \in \mathfrak{sl}_4 \mid [H_0, X] = iX \}). \]  

In particular,

\[ \mathfrak{l}_0 = \{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & A \end{pmatrix} \mid a, b \in \mathbb{C}, A \in \mathfrak{g}k_2, a + b + \text{Tr}(A) = 0 \} \cong \mathfrak{gl}_1 \oplus \mathfrak{gl}_1 \oplus \mathfrak{sl}_2, \]

\[ \mathfrak{l}_1 = \{ \begin{pmatrix} 0 & u & 0 & 0 \\ 0 & v_1 & v_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid u, v_1, v_2 \in \mathbb{C} \}, \quad \mathfrak{l}_{-1} = \{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ \psi & 0 & 0 & 0 \\ 0 & \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 & 0 \end{pmatrix} \mid \psi, \phi_1, \phi_2 \in \mathbb{C} \}. \]

Then, we have that \( \mathfrak{l}_0 \cong \mathbb{C}H_1 \oplus \mathbb{C}H_2 \oplus \mathfrak{sl}_2 \) and that the restriction of a bilinear form \( T \), defined by \( T(X, X') := \text{Tr}(XX') \) \( (X, X' \in \mathfrak{sl}_4) \), to \( \mathfrak{l}_0 \times \mathfrak{l}_0 \) satisfies:

\[ T \mid_{\mathfrak{l}_0 \times \mathfrak{l}_0} ((a, b, A), (a', b', A')) = \frac{3}{4}ab' + bb' + \frac{1}{2}(ab' + a'b) + \text{Tr}(AA'), \]
where \( a, a' \in \mathbb{C}H_1, b, b' \in \mathbb{C}H_2, A, A' \in \mathfrak{sl}_2 \). Thus, we can easily show that the grading \((3.82)\) and the Killing form of \(\mathfrak{sl}_4\), denoted by \(K_{\mathfrak{sl}_4}\), satisfy the assumptions of Theorem 2.20 and that a pentad \((l_0, \text{ad}, l_1, l_{-1}, K_{\mathfrak{sl}_4} |_{l_0 \times l_0})\) is equivalent to \((\mathfrak{g}, \rho \oplus \pi, V \oplus U, V \oplus U, B_0)\) (cf. \([5, 6, 7, \text{the theory of prehomogeneous vector spaces of parabolic type}]\)). Thus, by Theorems 2.20 and 3.26, we have

\[
L(L(\mathfrak{g}, \rho, V, V, B_0), \pi^+, \pi^+, \pi^+, B_L) \simeq L(\mathfrak{g}, \rho \oplus \pi, V \oplus U, V \oplus U, B_0) \simeq \mathfrak{sl}_4.
\]

In this case, we can directly check that the Lie algebra \(L(L(\mathfrak{g}, \rho, V, V, B_0), \pi^+, \pi^+, \pi^+, B_L)\) is isomorphic to \(\mathfrak{sl}_4\) using Examples 2.34, 2.35 and 3.16. In fact, by the results of Examples 2.34 and 3.16 we have that the pentad \((L(\mathfrak{g}, \rho, V, V, B_0), \pi^+, \pi^+, \pi^+, B_L)\) is equivalent to the pentad \((\mathfrak{gl}_1 \oplus \mathfrak{sl}_3, \Lambda_1, \mathbb{C}^3, \mathbb{C}^3, \kappa_3)\), which is defined in Example 2.34. Thus, we have that the Lie algebra \(L(L(\mathfrak{g}, \rho, V, V, B_0), \pi^+, \pi^+, \pi^+, B_L)\) is isomorphic to \(\mathfrak{sl}_4\).

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