Sparse Symmetric Linear Arrays with Contiguous Sum and Difference Co-array

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Abstract—Sparse arrays can resolve significantly more scatterers or sources than sensor by utilizing the co-array — a virtual array structure consisting of pairwise differences or sums of sensor positions. Although several sparse array configurations have been developed for passive sensing applications, far fewer active array designs exist. In active sensing, the sum co-array is typically more relevant than the difference co-array, especially when the scatterers are fully coherent. This paper proposes a general symmetric array configuration suitable for both active and passive sensing. We first derive necessary and sufficient conditions for the sum and difference co-array of this array to be contiguous. We then study two specific instances based on the Nested array and the Klove-Mosigge basis, respectively. In particular, we establish the relationship between the minimum-redundancy solutions of the two resulting symmetric array configurations, and the previously proposed Concatenated Nested Array (CNA) and Klove Array (KA). Both the CNA and KA have closed-form expressions for the sensor positions, which means that they can be easily generated for any desired array size. The two arrays structures also achieve low redundancy, and a contiguous sum and difference co-array, which allows resolving vastly more coherent scatterers or incoherent sources than sensors.

Keywords—Active sensing, sparse array configuration, symmetry, sum co-array, difference co-array, minimum redundancy.

I. INTRODUCTION

Sensor arrays are a key technology in, for example, radar, wireless communication, medical imaging, radio astronomy, sonar, and seismology [2]. The key advantages of arrays include spatial selectivity and the capability to mitigate interference. However, conventional uniform array configurations may become prohibitively expensive, when a high spatial resolution facilitated by a large electrical aperture, and consequently a large number of sensors is required.

Sparse arrays allow for significantly reducing the number of sensors and costly RF-IF chains, whilst resolving vastly more signal sources or scatterers than sensors. This is facilitated by a virtual array model called the co-array [3, 4], which is commonly defined in terms of the pairwise differences or sums of the physical sensor positions [4]. Uniform arrays have a redundant co-array, which allows the number of physical sensors to be reduced without affecting the number of unique co-array elements. This allows sparse arrays to identify up to $O(N^2)$ signal sources using only $N$ sensors [5–7]. Typically, a contiguous, i.e., uniform co-array is desired, since it maximizes the number of virtual sensors for a given array aperture, and allows for statistically efficient array processing algorithms developed for conventional arrays to be employed [8, 9].

Typical sparse array designs, such as the Minimum-Redundancy Array (MRA) [10, 11], seek to maximize the array aperture, subject to a contiguous co-array and a given number of sensors $N$. Although optimal, the MRA lacks a closed-form expression for its sensor positions, and quickly becomes impractical to compute, as the search space of the combinatorial optimization problem that needs to be solved grows exponentially with $N$. Consequently, large sparse arrays have to be constructed by sub-optimal means. For instance, smaller (but computable) MRAs can be extended to larger apertures by repeating regular substructures in the array [12], or by recursively nesting them in a fractal manner [12]. Recent research into such recursive or fractal arrays [13–16] has also revived interest in symmetric array configurations. Symmetry or translational invariance in either the physical or co-array domain can further simplify array design [17], calibration [18], and processing [19–22, 2, p. 721]. Parametric array designs are also of great interest because their sensor positions have closed-form expressions, which facilitates simple optimization of the array geometry. Notable parametric arrays include, e.g., the Wichmann [23–25], Co-prime [26], Nested [8], and Super nested array [27, 28].

Sparse array configurations have been developed mainly for passive sensing, where the difference co-array can be exploited if the source signals are incoherent or weakly correlated. Fewer works consider the sum co-array, which is more relevant in active sensing applications, such as imaging or radar, where scatterers may be fully coherent [41, 29, 40]. Some of our recent work has aimed at filling this gap by proposing symmetric extensions to existing parametric array configurations [51–53]. Yet a unifying analysis of such symmetric arrays is still lacking from the literature.

A. Contributions and organization

This paper focuses on the design of sparse linear active arrays with a contiguous sum co-array. The main contributions of the paper are twofold. Firstly, we propose a general symmetric sparse linear array design. We establish necessary and sufficient conditions under which the sum and difference co-array of this array are contiguous. We also determine sufficient conditions that greatly simplify array design by allowing for array configurations with a contiguous difference co-array to be leveraged. This connects our work to the abundant literature on mostly asymmetric sparse arrays with a contiguous...
difference co-array [8], [25], [49], [53]. Moreover, it provides a unifying framework for symmetric configurations relevant to both active and passive sensing [31], [33].

The second main contribution is a detailed study of two specific instances of this symmetric array — one based on the Nested array [8], and the other on the Kløve-Mossige basis from number theory [5]. In particular, we clarify the connection between these symmetric arrays and the recently studied Concatenated Nested Array (CNA) [31], [37] and Kløve Array (KA) [11]. We also derive the minimum redundancy parameters for both the CNA and KA. Additionally, we show that the minimum-redundancy symmetric nested array reduces to the CNA. Both the CNA and KA can be generated for practically any number of elements, as their properties and sensor positions have closed-form expressions.

The paper is organized as follows. Section II introduces the signal model and the considered array figures of merit. In Section III, we briefly review the MRA and some of its characteristics and extensions. Section IV then presents the general definition of the proposed symmetric array, and outlines both necessary and sufficient conditions for its sum co-array to be contiguous. In Section V, we study two interesting special cases of this array, and derive their minimum-redundancy parameters. Finally, we compare the discussed array configurations numerically in Section VI, before concluding the paper and discussing future work in Section VII.

B. Notation

We denote matrices by bold uppercase, vectors by bold lowercase, and scalars by unbolded letters. Sets are denoted by calligraphic letters. The set of integers from $a \in \mathbb{Z}$ to $c \in \mathbb{Z}$ in steps of $b \in \mathbb{N}$ is denoted \{a : b : c\} = \{a, a+b, a+2b, \ldots, c\}. Shorthand \{a : c\} denotes \{a : 1 : c\}.

II. Preliminaries

In this section, we briefly review the active sensing and sum co-array models. We then define the two considered array figure of merit, the primary one being redundancy $R$, and the secondary one being the $d$-spacing multiplicity $S(d)$.

A. Signal model and sum co-array

Consider a linear array of $N$ transmitting and receiving sensors, whose normalized positions are given by the set of positive integers $D = \{d_n\}_{n=1}^N \subseteq \mathbb{N}$. The first element of the array is located at $d_1 = 0$, and the last element at $d_N = L \in \mathbb{N}$. The (normalized) array aperture is thus $\max D = L$. This array is used to actively sense $K$ far field scatterers with reflectivities $\\{\gamma_k\}_{k=1}^K \subseteq \mathbb{C}$ in azimuth directions $\\{\varphi_k\}_{k=1}^K \subseteq [-\pi/2, \pi/2]$. Each transmitter illuminates the scattering scene using narrowband radiation in a sequential or simultaneous (orthogonal MIMO) manner [29], [38]. The reflectivities are assumed fixed during the coherence time of the scene, which may consist of one or several snapshots (pulses). The single snapshot received signal after matched filtering is then [29]

$$x = (A \odot A)\gamma + n,$$  \hspace{1cm} (1)

where $\odot$ denotes the Khatri-Rao (columnwise Kronecker) product. $A \in \mathbb{C}^{N \times K}$ is the array steering matrix, $\gamma = [\gamma_1, \ldots, \gamma_K]^T \in \mathbb{C}^K$ is the scattering coefficient vector, and $n \in \mathbb{C}^{N \times 1}$ is a receiver noise vector following a zero-mean white complex circularly symmetric normal distribution. A typical array processing task is to estimate parameters $\{\varphi_k, \gamma_k\}_{k=1}^K$, or some functions thereof, from the measurements $x$.

The effective steering matrix in (1) is given by $A \odot A$. Assuming ideal omnidirectional sensors, we have

$$[A \odot A]_{(n-1)N+m,k} = e^{j2\pi(d_n+d_m)\delta \sin \varphi_k},$$

where $\delta$ is the unit inter sensor spacing in carrier wavelengths (typically $\delta = 1/2$). Consequently, the entries of $A \odot A$ are supported on a virtual array, known as the sum co-array, which consists of the pairwise sums of the physical element locations.

Definition 1 (Sum co-array). The virtual element positions of the sum co-array of physical array $A$ are

$$D_S = D + D = \{d_n + d_m \mid d_n, d_m \in D\}.$$  \hspace{1cm}

The relevance of $D_S$ is that it may have up to $N(N+1)/2$ unique elements, which is vastly more than the number of physical sensors $N$. This implies that $O(N^2)$ coherent scatterers can be identified from (1), provided the set of physical sensor positions $D$ is judiciously designed.

A closely related concept to the sum co-array is that of the difference co-array [3], [4]. Defined as the set of pairwise element position differences, the difference co-array mainly arises in passive sensing applications, such as direction finding, where the incoherence of the source signals is leveraged. Also, more exotic co-arrays, such as the difference of the sum co-array [40], [41], and the union of the sum and difference co-array [42], [43] have been shown to emerge in other circumstances.

The sum co-array is uniform or contiguous, if it equals a virtual Uniform linear array (ULA) of aperture $2L = 2\max D$.

Definition 2 (Contiguous sum co-array). The sum co-array of $D$ is contiguous if $D_S = \{0 : 2\max D\}$.

A contiguous (sum or difference) co-array is generally desirable for two reasons. Firstly, it maximizes the number of unique co-array elements for a given physical aperture. Second, many efficient and widely used array processing algorithms are designed for uniform arrays. For example, co-array MUSIC [8], [9] only utilizes the uniform section of the co-array.

B. Array figures of merit

1) Redundancy, $R$, quantifies the multiplicity of the co-array elements. A non-redundant array achieves $R = 1$, whereas $R > 1$ holds for a redundant array.

Definition 3 (Redundancy [11]). The redundancy of an $N$ sensor array with aperture $L$ and a contiguous sum co-array is

$$R(N) = \frac{N(N+1)/2}{2L(N)+1}.$$  \hspace{1cm}

\textsuperscript{1}Up to $O(N^4)$ incoherent scatterers can be resolved by utilizing the second-order statistics of [4] and the difference of the sum co-array [29].
The numerator of $R$ is the maximum number of unique pairwise sums generated by $N$ numbers. The denominator is the number of elements in the sum co-array, which is $|D_C| = 2L + 1$. Since the redundancy $R$ is a function of $N$, it is often convenient to compute the asymptotic redundancy

$$R_\infty = \lim_{N \to \infty} \frac{N^2}{4L(N)}.$$ 

If the sum co-array is not contiguous, then the expression for $R$, respectively $R_\infty$, needs to be modified. In particular, the denominator in Definition 3 should be replaced by the number of contiguous sum co-array elements, in a similar fashion to the definition of redundancy of the difference co-array in [10].

2) The d-spacing multiplicity, $S(d)$, enumerates the number of inter-element spacings of a given length $d$ in the array [44]. For linear arrays, $S(d)$ simplifies to the weight or multiplicity function [8], [45] of the difference co-array (when $d \geq 1$).

**Definition 4 (d-spacing multiplicity [44]).** The multiplicity of separation $d \geq 1$ in a linear array $D$ is

$$S(d) = \frac{1}{2} \sum_{d_n \in D} \sum_{d_m \in D} \mathbb{1}(|d_n - d_m| = d).$$

It is straightforward to see that $0 \leq S(d) \leq \min(N - 1, [L/d]), d \in \mathbb{N}_+$. Additionally, if the difference co-array is contiguous, then $S(d) \geq 1$ for $1 \leq d \leq L$.

Typically, a low value for $S(d)$ is desired for small $d$, as elements that are closer to each other interact more strongly and exhibit coupling [18]. Consequently, the severity of mutual coupling effects deteriorating the array performance may be controlled by decreasing $S(d)$ [27], [35], [46], [47]. This simplifies array design, but has its limitations, since it neglects important practical factors impacting coupling, such as the individual element gain patterns and the mounting platform, as well as the scan angle and uniformity of the array [48].

Since treating such non-linear effects in a mathematically tractable way is challenging, proxies like the number of unit spacings $S(1)$ are often considered instead to simplify array design.

### III. Minimum-Redundancy Array

In this section, we present the sparse array design problem solved by the Minimum-redundancy array (MRA). We then review some properties of the MRA, and discuss a low complexity extension called the Reduced-redundancy array.

The MRA maximizes the array aperture, subject to a given number of elements and a contiguous sum co-array. This is equivalent to minimizing the redundancy of the array.

**Definition 5 (Minimum-redundancy array (MRA)).** The element positions of the MRA are given by the solution to

maximize $\max D$

subject to $|D| = N$ and $D + D = \{0 : 2 \max D\}$. \hspace{1cm} (P1)

Problem (P1) is equivalent to finding an extremal restricted additive 2-basis with $N$ elements. Such additive bases, corresponding to “sum MRAs” [11], have been extensively studied in number theory [11], [56], [57], [49], [50], as have restricted difference bases [23], [51] and the corresponding “difference MRAs” in array processing [10]. Solving (P1) is nevertheless challenging, since the size of the search space grows exponentially with the number of elements $N$. Consequently, solutions are currently only known for $N \leq 48$ [50], [52]. Note that the MRA can alternatively be defined for a fixed aperture. In this case, the MRA minimizes $N$ subject to a contiguous co-array. The difference MRA following this definition is also referred to as the sparse ruler [53]. For a given $N$, several such rulers of different length may exist. For simplicity, however, we only consider optimization problems similar to (P1), where the number of elements, rather than the aperture, is fixed.

#### A. Key properties

The lack of a closed-form expression for the sensor positions of the MRA make its properties difficult to analyze precisely. Nevertheless, it is straightforward to see that a linear array with a contiguous sum co-array necessarily contains two elements in each end of the array, as shown by the following lemma.

**Lemma 1 (Necessary sensors).** Let $N \geq 2$. If $D$ has a contiguous sum co-array, then $D \supseteq \{0, 1, L - 1, L\}$.

**Proof:** Clearly, $D + D \supseteq \{0, 1, 2L - 1, 2L\}$ $\iff$ $D \supseteq \{0, 1, L - 1, L\}$. Similarly, $D' - D \supseteq \{0, 1, L - 1, L\}$ $\iff$ $D' \supseteq \{0, 1, L - 1, L\}$. We may write $D \supseteq \{0, 1, L\}$ without loss of generality, since any $D \supseteq \{0, L - 1, L\}$ can be mirrored to satisfy $L - D \supseteq \{0, 1, L\}$. \hfill $\blacksquare$

**Lemma 1** directly implies that any array with a contiguous sum co-array and $N \geq 4$ sensors has at least two sensor pairs separated by a unit inter-element spacing, i.e., $S(1) \geq 2$.

It is also easy to derive bounds on the minimum number of elements $N$, or maximum aperture $L$ of the MRA by considering the perfect array (PA). The PA is an idealized array with a contiguous sum co-array achieving redundancy $R = 1$. Since any MRA satisfies $R \geq 1$, Definition 5 yields

$$L \leq (N^2 + N - 2)/4$$

$$N \geq 2\sqrt{L} + 16/1 - 2.$$ 

Unlike the MRA, the PA is not realizable in practice apart from two trivial exceptions.

**Corollary 1 (Perfect arrays).** The only two perfect arrays are $\{0\}$ and $\{0, 1\}$. All other MRAs are redundant.

**Proof:** By Lemma 1 the element at position $L$ of the sum co-array can be represented in at least two ways, namely $L = L + 0 = (L - 1) + 1$. Consequently, if $N \geq 4$, then $R > 1$ must hold. The MRAs for $N \leq 3$ are $\{0\}$, $\{0, 1\}$, and $\{0, 1, 2\}$. Only the first two of these satisfy $R = 1$. \hfill $\blacksquare$

Most importantly, however, the asymptotic redundancy of the MRA can be bounded from below and above as follows:

**Theorem 1 (Asymptotic redundancy of MRA [1], [54]).** The asymptotic redundancy of the MRA satisfies the inequality

$$1.190 < R_\infty < 1.917.$$
TABLE I. TWO MRAS WITH N = 8 ELEMENTS [50]. THE FIRST MRA HAS FEWER CLOSELY SPACED ELEMENTS AND HENCE A LOWER \( \varsigma \).

| Configuration | \( S(1) \) | \( S(2) \) | \( S(3) \) | \( \varsigma \) |
|---------------|-----------|-----------|-----------|--------|
| \( (0, 1, 2, 5, 8, 11, 12, 13) \) | 4         | 2         | 3         | 0.040203 \ldots |
| \( (0, 1, 3, 4, 9, 10, 12, 13) \) | 4         | 2         | 4         | 0.040204 \ldots |

Proof: The lower bound \( R_{\infty} \geq \frac{11}{7 + \sqrt{5}} > 1.190 \) follows directly from [54] Theorem 1.2, and the upper bound \( R_{\infty} \leq \frac{24}{13} < 1.917 \) from [1] Theorem, p. 177. 

An array with a non-contiguous sum co-array can actually have lower redundancy than the MRA [50]. For example, the asymptotic redundancy of (unrestricted) extremal additive 2-bases satisfies \( R_{\infty} \leq 1.73 \) [56]. Naturally, also difference bases/MRAs can be restricted or unrestricted [10, 51]. The restricted MRA has nevertheless become more widely adopted, for the reasons listed in Section II-A. We note that sum MRAs are generally more redundant than difference MRAs due to the commutativity of the sum \((a+b = b+a, \text{but } a-b \neq b-a)\).

B. Uniqueness

Problem [P1] may have several solutions for a given \( N \), which means that the MRA of Definition 5 is not necessarily unique [50]. In order to guarantee uniqueness, we introduce a secondary optimization criterion. In particular, we consider the MRA with the fewest closely spaced elements. This MRA is found by minimizing a weighted sum of \( d \)-spacing multiplicities (see Section II-B) among the solutions to [P1], which is equivalent to subtracting a regularizing term from the objective function of [P1]. This regularizer \( \varsigma \geq 0 \) is defined as

\[ \varsigma = \sum_{d=1}^{L} S(d)10^{-d\lceil \log L \rceil}, \tag{2} \]

where \( L \) is the (integer-valued) array aperture. Consequently, any two solutions to (the unregularized) problem [P1], say \( a \) and \( b \), satisfy \( \varsigma_a > \varsigma_b \), if and only if \( S_a(n) > S_b(n) \) and \( S_a(d) = S_b(d) \) for all \( 1 \leq d < n \). In words: (2) promotes large sensor displacements by prioritizing the value of \( S(1) \), then \( S(2) \), then \( S(3) \), etc. For example, Table I shows two MRAs with equal \( S(1) \) and \( S(2) \), but different \( S(3) \). The MRA with the smaller \( S(3) \), and therefore lower value of \( \varsigma \), is preferred.

C. Symmetry and the Reduced Redundancy Array

The majority of currently known sum MRAs are actually symmetric [50, 52]. In fact, there exist at least one symmetric solution to [P1] for each \( N \leq 48 \). Moreover, the solution with the fewest closely spaced elements (lowest \( \varsigma \)) turns out to always be symmetric for \( N \leq 48 \). Indeed, symmetry seems to arise naturally from the additive problem structure, whereas difference MRAs are generally asymmetric [10, 12].

Imposing symmetry on the array design problem has the main advantage of reducing the size of the search space [17, 36]. In case of the MRA, this can be achieved by adding constraint \( D = \max \mathcal{D} - D \to [P1] \). Unfortunately, the search space of this symmetric MRA still scales exponentially with \( N \). Fortunately, another characteristic of the MRA may readily be exploited. Namely, MRAs tend to have a sparse mid section consisting of uniformly spaced elements [50]. The reduced redundancy array (RRA) extends this uniform mid section, resulting in an array with a larger aperture than the original MRA [11, 12].

Definition 6 (Reduced redundancy array (RRA) [11]). The element positions of the RRA for a given MRA are given by

\[ D_{\text{RRA}} = \mathcal{P} \cup (\mathcal{M} + \max \mathcal{P}) \cup (\mathcal{S} + \max \mathcal{P} + \max \mathcal{M}), \]

where \( \mathcal{P} \) is the prefix and \( \mathcal{S} \) the suffix of the MRA, and

\[ M = \{0 : M : (N - |\mathcal{P}| - |\mathcal{S}| + 1)M\} \]

is the mid subarray, with inter-element spacing \( M \) in \( \mathbb{N}_+ \).

The prefix \( \mathcal{P} \), suffix \( \mathcal{S} \), and the inter-element spacing \( M \) of \( M \) are determined by the generator MRA, i.e., the MRA that is extended. For example, the MRA with \( N = 7 \) elements is

\[ \{0, 1, 2, 5, 8, 9, 10\} = \{0, 1, 2\} \cup \{2, 3 : 8\} \cup \{8, 9, 10\}. \]

Note that \( \mathcal{S} = \max \mathcal{P} - \mathcal{P} \) holds, if the MRAs is symmetric as above. Also, the RRA has a contiguous sum co-array.

The RRA has a low redundancy when \( |D_{\text{RRA}}| \approx |D_{\text{MRA}}| \). However, the redundancy of the RRA goes to infinity as \( |D_{\text{RRA}}| \) grows, since the aperture of the RRA only increases linearly with the number of sensors. Consequently, we will next consider a general class of symmetric arrays which scale better and admit a solution in polynomial time, provided the design space is constrained judiciously. In particular, we will show that this class of symmetric arrays naturally extends many of the established array configurations designed for passive sensing to the active sensing setting.

IV. SYMMETRIC ARRAY — GENERAL DESIGN AND CONDITIONS FOR CONTIGUOUS CO-ARRAY

In this section, we establish a general framework for symmetric arrays with a contiguous sum (and difference) co-array. The proposed symmetric array with generator \( \mathcal{G} \) (S-\( \mathcal{G} \)) is constructed by taking the union of generator array \( \mathcal{G} \) and its mirror image shifted by some non-negative integer \( \lambda \).

Definition 7 (Symmetric array with generator \( \mathcal{G} \) (S-\( \mathcal{G} \))). The element positions of the S-\( \mathcal{G} \) are given by

\[ D_{\mathcal{G}-\mathcal{G}} = \mathcal{G} \cup (\max \mathcal{G} - \mathcal{G} + \lambda), \tag{3} \]

where \( \mathcal{G} \) is the generator array and \( \lambda \in \mathbb{N} \) a shift parameter.

Fig. 1(a) shows a schematic of the S-\( \mathcal{G} \). Note that the number of elements satisfies \( |\mathcal{G}| \leq N \leq 2|\mathcal{G}| \), depending on the overlap between \( \mathcal{G} \) and \( \mathcal{L} - \mathcal{G} \), whereas the aperture \( L \) is

\[ L = \max \mathcal{G} + \lambda. \tag{4} \]

\[ \text{This terminology is adopted from the literature on fractal arrays [13, 57].} \]

\[ \text{Note that } \lambda \leq 0 \text{ is equivalent to considering the mirrored generator } \max \mathcal{G} - \mathcal{G} \text{ for } \lambda \geq 0. \text{ Therefore, we may set } \lambda \geq 0 \text{ without loss of generality.} \]

\[ ^{2} \text{This is slightly tighter than the bound } R_{\infty} < 1.75 \text{ presented in [11] (and repeated in [31]), which is based on an additive basis by Moses [55].} \]
The exact properties of the $S$-$G$ are determined by the particular choice of $G$ and $\lambda$, which is a subject that we will examine in more detail in Section V. Next, however, we establish necessary and sufficient conditions that any $G$ and $\lambda$ must fulfill for $S$-$G$ to have a contiguous sum co-array.

A. Necessary and sufficient conditions for contiguous co-array

Fig. 1(b) illustrates the difference co-array of the $S$-$G$, which is composed of the difference co-array and shifted sum co-arrays of the generator $G$. By symmetry, the sum and difference co-array of the $S$-$G$ are equivalent up to a shift. This fact, along with (3), allows us to express the necessary and sufficient condition for the contiguity of both co-arrays in terms of the generator $G$ and shift parameter $\lambda$. Moreover, we may conveniently decompose the condition into two simpler subconditions, as shown by the following theorem.

**Theorem 2** (Conditions for contiguous co-array). The sum (and difference) co-arrays of the $S$-$G$ is contiguous if and only if

1. $(G - G) \cup (G + G - L) \cup (L - (G + G)) \supseteq \{0 : \text{max } G\}$
2. $G + G \supseteq \{0 : \lambda - 1\}$

where $L$ is the array aperture given by (4).

**Proof:** By symmetry of the physical array, the sum co-array is contiguous if and only if the difference co-array is contiguous (e.g., see [44, Lemma 1]), that is,

$$D_{S,G} + D_{S,G} = \{0 : 2L\} \iff D_{S,G} - D_{S,G} = \{-L : L\}.$$  

By symmetry of the difference co-array, this is equivalent to requiring that $D_{S,G} - D_{S,G} \supseteq \{0 : L\}$, or using (3) that

$$D_{S,G} - D_{S,G} = G \cup (L - G) - G \cup (L - G) = (G - G) \cup (G + G - L) \cup (L - (G + G)) \supseteq \{0 : L\}.$$  

Conditions (C2) and (C1) then directly follow from (4).

Note that (C2) may also be reformulated as $\lambda \leq H$, where

$$H = \arg \max_{h \in \mathbb{N}} \{ h \mid \{0 : h - 1\} \subseteq G + G\}.$$  

Here, $H$ is the position of the first hole in $G + G$, i.e., the number of elements in the largest contiguous subarray contained in the sum co-array of $G$. Since $H \leq 2 \cdot \text{max } G + 1$, it is straightforward to verify that for the sum co-array of $D_{S,G}$ to be contiguous, $\lambda$ necessarily satisfies

$$\lambda \leq 2 \cdot \text{max } G + 1.$$  

B. Sufficient conditions for contiguous co-array

It is also instructive to consider some simple sufficient conditions for satisfying (C1) and (C2) as outlined in the following two corollaries to Theorem 2.

**Corollary 2** (Sufficient conditions for (C1)). Condition (C1) is satisfied if either of the following holds:

1. $G$ has a contiguous difference co-array.
2. $G$ has a contiguous sum co-array and $\lambda \leq \text{max } G$.

**Corollary 3** (Sufficient conditions for (C2)). Condition (C2) is satisfied if any of the following hold:

1. $\lambda \leq 1$
2. $\lambda \leq 3$ and $|G| \geq 2$
3. $\lambda \leq 2 \cdot \text{max } G + 1$ and $G$ has a contiguous sum co-array.

**Proof:** Firstly, if the difference co-array of $G$ is contiguous, then $G - G \supseteq \{0 : \text{max } G\}$ holds by definition, implying (C1). Secondly, if the sum co-array is contiguous, then $G + G \supseteq \{0 : 2 \cdot \text{max } G\}$ holds, which implies that

$$G + G - L = \{-L : \text{max } G - \lambda\}$$

$$L - (G + G) = \{\text{max } G + \lambda : L\}.$$  

If $\lambda \leq \text{max } G$, then the union of these two sets cover $\{-L : L\} \supseteq \{0 : \text{max } G\}$, implying (C1).

**Corollary 3** (Sufficient conditions for (C2)). Condition (C2) is satisfied if any of the following hold:

1. $\lambda \leq 1$
2. $\lambda \leq 3$ and $|G| \geq 2$
3. $\lambda \leq 2 \cdot \text{max } G + 1$ and $G$ has a contiguous sum co-array.

**Proof:** Firstly, $\lambda \leq 1$ implies (C2) since $G \supseteq \{0\}$. Secondly, if $|G| \geq 2$, then $G \supseteq \{0,1\}$ holds by Lemma 1. Consequently, $|G| \geq 2$ and $\lambda \leq 3$ implies (C2). Thirdly, if $G$ has a contiguous sum co-array and $\lambda \leq 2 \cdot \text{max } G + 1$, then $G + G = \{0 : 2 \cdot \text{max } G\} \supseteq \{0 : \lambda - 1\}$ holds by definition.

We note that if $N \geq 2$, then $\lambda \leq \text{max } G + 2$ is actually sufficient in Item [iii] of Corollary 2 (cf. Lemma 1). This is also sufficient for satisfying (C2) by Item [iii] of Corollary 3. Item [i] of Corollary 2 is of special interest, since it states that any $G$ with a contiguous difference co-array satisfies (C1). This greatly simplifies array design, as shown in the next section, where we develop two arrays leveraging this property.

V. LOW-REDUNDANCY SYMMETRIC ARRAY DESIGNS USING PARAMETRIC GENERATORS

Similarly to the MRA in (P1), we wish to find the $S$-$G$ with maximal aperture that satisfies the conditions in Theorem 2. Given a number of elements $N$, and a class of generator arrays $G$, such that $G \in G$, this minimum-redundancy $S$-$G$ design is found by solving the following optimization problem:

$$\begin{align*}
\text{maximize} & \quad \text{max } G + \lambda \\
\text{subject to} & \quad |G \cup (\text{max } G + \lambda)| = N \\
& \quad \text{(C1)} \text{ and (C2)}
\end{align*}$$

(P2)

In general, (P2) is a non-convex problem, whose difficulty depends on the choice of $G$. Solving (P2) may therefore require a grid search over $\lambda$ and the elements of $G$, which can have exponential complexity at worst. At best, however, a solution can be found in polynomial time, or even in closed form.

We will now focus on a family of choices for $G$, such that each $G \in G$ has the following two convenient properties:

1. $G$ has a contiguous difference co-array
2. $G$ has a closed-form expression for its aperture.

Item [a] greatly simplifies (P2) by directly satisfying condition (C1) (see Item [i] of Corollary 2), whereas Item [b] enables the straightforward optimization of the array parameters. Condition (C2) is typically easy to satisfy for an array with these two properties. This is the case with many sparse array configurations in the open literature, such as the Nested [8] and Wichmann array [23]–[25]. Symmetrizing array geometries with a contiguous difference co-array is thus a practical way to synthesize configurations with a contiguous sum co-array.
Definition 8 (Symmetric Nested Array (S-NA)). The element positions of the S-NA are given by
\[ \mathcal{S} = \mathcal{D}_1 \cup (\mathcal{D}_2 + N_1), \]
with \( \mathcal{D}_1 = \{0 : N_1 - 1\}; \mathcal{D}_2 = \{0 : N_1 + 1 : (N_2 - 1)(N_1 + 1)\}; and array parameters \( N_1, N_2 \in \mathbb{N} \).

Special cases of the S-NA have also been previously proposed. For example, [58] Definition 2] considered the case \( \lambda = 0 \) for improving the robustness of the NA to sensor failures. Next briefly discuss a special case that is more relevant from the minimum-redundancy point of view.

1) Concatenated Nested Array (CNA): The S-NA coincides with a restricted additive 2-basis studied by Rohrbach in the 1930’s [37, Satz 2], when the shift parameter \( \lambda \) satisfies
\[ \lambda = (N_1 + 1)k + N_1, \]  
where \( k \in \{0 : N_2\} \). Unaware of Rohrbach’s early result, we later derived the same configuration independently (based on the NA), calling it the Concatenated Nested Array (CNA) [31].

Definition 9 (Concatenated Nested Array (CNA) [31]). The element positions of the CNA are given by
\[ \mathcal{C} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_1 \cup (\mathcal{D}_2 + N_1), \]
where \( N_1, N_2 \in \mathbb{N} \), and \( \mathcal{D}_1, \mathcal{D}_2 \) follow Definition 8

The CNA is illustrated in Fig. 2(a). When \( N_1 = 0 \) or \( N_2 \in \{0, 1\} \), the array degenerates to the ULA. In the interesting case when \( N_1 + N_2 \geq 1 \), the aperture of the CNA is [31]
\[ L = (N_1 + 1)(N_2 + 1) - 2. \]  
Furthermore, the number of elements is [31]
\[ N = \begin{cases} N_1, & \text{if } N_2 = 0 \\ 2N_1 + N_2, & \text{otherwise}, \end{cases} \]  
and the number of unit spacings evaluates to
\[ S(1) = \begin{cases} N_1 - 1, & \text{if } N_2 = 0 \\ N_2 - 1, & \text{if } N_1 = 0 \\ 2N_1, & \text{otherwise} \end{cases} \]  
(9)

2) Minimum-redundancy solution: The S-NA solving (P2) is actually a CNA. This follows from the fact that any S-NA that has a contiguous sum co-array, but is not a CNA, has redundant elements, due to the reduced number of overlapping elements between \( \mathcal{S} \) and its shifted mirror image.

Proposition 1. The S-NA solving (P2) is a CNA.

Proof: We write [C1] and [C2] as a constraint on \( \lambda \), which we then show implies the proposition under reparametrization. Firstly, [C1] holds by Item [17] of Corollary 2 since the NA has a contiguous difference co-array [38]. Secondly, the location of the first hole in the sum co-array of the NA yields that [C2] is satisfied if and only if
\[ \lambda \leq \begin{cases} 2(N_1 + N_2) - 1, & \text{if } N_1 = 0 \text{ or } N_2 = 0 \\ N_2(N_1 + 1) + N_1, & \text{otherwise}. \end{cases} \]  
(10)

If \( 0 \leq \lambda \leq N_1 - 1 \), then a CNA with the same number of element \( N = 2|\mathcal{S}| - 2 \), but a larger aperture \( L = \max \mathcal{G} + \lambda \) is achieved by instead setting \( \lambda = \max \mathcal{G} - N_1 \). Otherwise,
it is straightforward to verify that the S-NA either reduces to the CNA, or a CNA can be constructed with the same \( N \) but a larger \( L \), by satisfying (10) with equality.

By Proposition 1 problem (P2) simplifies to maximizing the aperture of the CNA for a given number of elements:

\[
\text{maximize } N_1N_2 + N_1 + N_2 \quad \text{s.t. } 2N_1 + N_2 = N. \quad \text{(P3)}
\]

Problem (P3) actually admits a closed-form solution (31). In fact, we may state the following more general result for any two-variable integer program similar to (P3).

**Lemma 2.** Let \( f \) be a concave function. The solution to

\[
\text{maximize } f(x, y) \text{ subject to } g(x) = y \quad \text{(P4)}
\]

is given by \( x = [z] + k \) and \( y = g(x) \), where \( z \) solves

\[
\text{maximize } f(z, g(z)),
\]

and \( |k| \) is the smallest non-negative integer satisfying \( g([z] + k) \in \mathbb{N} \), where \( k \in \{-[z], -[z] + 1, \ldots, 1\} \). By concavity of \( f \), the smallest \( |k| \) satisfying \( g([z] + k) \in \mathbb{N} \) then yields the global optimum of (P4).

**Proof:** Since \( f(z, g(z)) \) is concave, \( x = [z] + k \) maximizes \( f([z], g([z])) \) among all \( x \in \mathbb{N} \). This is a global optimum of (P4), if \( g([z]) \in \mathbb{N} \). Generally, the optimal solution can be expressed as \( x = [z] + k \), where \( k \in \{-[z], -[z] + 1, \ldots, 1\} \). By concavity of \( f \), the smallest \( |k| \) satisfying \( g([z] + k) \in \mathbb{N} \) then yields the global optimum of (P4).

Lemma 2 is useful for solving two-variable integer programs similar to (P3) in closed-form. Such optimization problems often arise in, e.g., sparse array design (32), (33). In our case, Lemma 2 allows expressing the minimum-redundancy parameters of the CNA directly in terms of \( N \) as follows.

**Theorem 3 (Minimum-redundancy parameters of CNA).** The parameters of the CNA solving (P3) are

\[
\begin{align*}
N_1 &= (N - \alpha)/4 \\
N_2 &= (N + \alpha)/2,
\end{align*}
\]

where \( N = 4m + k, m \in \mathbb{N}, k \in \{0 : 3\} \), and

\[
\alpha = (k + 1) \mod 4 - 1.
\]

**Proof:** By Lemma 2 the optimal solution to (P3) is given by \( N_1 = ([N - 1]/4) \) and \( N_2 = N - 2N_1 = 31 \). Since any \( N \in \mathbb{N} \) may be expressed as \( N = 4m + k \), where \( m \in \mathbb{N} \) and \( k \in \{0 : 3\} \), we have

\[
N_1 = \left\lceil \frac{m + k - 1}{4} \right\rceil = \left\{ \begin{array}{ll} m, & k \in \{0, 1, 2\} \\ m + 1, & k = 3. \end{array} \right.
\]

Since \( m = (N - k)/4 \), we have \( N_1 = m = (N - k)/4 \), when \( k \in \{0, 1, 2\} \). Similarly, when \( k = 3 \), we have \( N_1 = m + 1 = (N - k + 4)/4 = (N + 1)/4 \), which yields (11) and (13).

Substituting (11) into (2) then yields (12).

By Theorem 3 the properties of the minimum-redundancy CNA can also be written compactly as follows.

**Corollary 4 (Properties of minimum-redundancy CNA).** The aperture \( L \), number of elements \( N \), and number of unit spacings \( S(1) \) of the CNA solving (P3) are

\[
\begin{align*}
L &= (N^2 + 6N - 7)/8 - \beta \\
N &= 2\sqrt{2L + 2 + \beta} - 3 \\
S(1) &= (N - \alpha)/2,
\end{align*}
\]

where \( \beta = (\alpha - 1)^2/8 \) and \( \alpha \) is given by (13).

**Proof:** This follows from Theorem 3 and (7) - (9).

B. Symmetric Kløve-Mossige array (S-KMA)

In the 1980’s, Kløve and Mossige proposed an additive 2-basis with some interesting properties (36). In particular, the basis has a contiguous difference co-array (see Appendix A) and a low asymptotic redundancy \( R_\infty = 1.75 \), despite having a non-contiguous sum co-array (see Appendix B). We call this construction the Kløve-Mossige Array (KMA). As shown in Fig. 3(b), the KMA contains a CNA, and can therefore be interpreted as another extension of the NA. However, selecting the KMA as the generator in (3) yields the novel Symmetric Kløve-Mossige Array (S-KMA), shown in Fig. 3(b).

**Definition 10 (Symmetric Kløve-Mossige Array (S-KMA)).** The element positions of the S-KMA are given by (3), with

\[
\begin{align*}
\mathcal{G} &= \mathcal{D}_{\text{CNA}} \cup (D_3 + 2\max D_{\text{CNA}} + 1), \\
D_3 &= \{0 : N_1 : N_1^2\} + \bigcup_{i=1}^{N_3}\{(i - 1)(N_1^2 + \max D_{\text{CNA}} + 1)\},
\end{align*}
\]

\( \mathcal{D}_{\text{CNA}} \) following Definition 9 and parameter \( N_1, N_2, N_3 \in \mathbb{N} \).

1) Kløve array: The structure of the S-KMA simplifies substantially when the shift parameter \( \lambda \) is of the form

\[
\lambda = 2\max D_{\text{CNA}} + 1 + \max (\max D_{\text{CNA}} + N_1^2)k,
\]

where \( k \in \{0 : N_3\} \). In fact, this S-KMA coincides with the Kløve array (KA), which is based on a class of restricted additive 2-bases proposed by Kløve in the context of additive combinatorics (11) (see also (33)).

**Definition 11 (Kløve array (KA)).** The element positions of the KA with parameters \( N_1, N_2, N_3 \in \mathbb{N} \) are given by

\[
\begin{align*}
\mathcal{D}_{\text{KA}} &= \mathcal{D}_{\text{CNA}} \cup (D_3 + 2\max D_{\text{CNA}} + 1) \\
&\quad \cup (D_{\text{CNA}} + (N_3 + 2)\max D_{\text{CNA}} + N_3(N_1^2 + 1) + 1),
\end{align*}
\]

where \( D_{\text{CNA}} \) follows Definition 9 and \( D_3 \) Definition 10.

Fig. 3(c) illustrates the KA, which consists of two CNAs connected by a sparse mid-section consisting of \( N_3 \) widely separated and sub-sampled ULAs with \( N_1 + 1 \) elements each. The KA reduces to the CNA when \( N_1 = 0 \) or \( N_2 = 0 \). When

\[\text{(13)}\]

An equivalent result is given in (34–36, Eq. (7) and (80) and (37, Eq. (13)), but in a more inconvenient format due to the use of the rounding operator.

*The S-KMA is undefined for \( N_1 = N_2 = 0 \), as \( D_{\text{CNA}} = \emptyset \). Consequently, we will not consider this case further. However, we note that Definition 10 is easily modified so that the S-KMA degenerates to a ULA even in this case.*
Problem (P5) is an integer program with three unknowns that is challenging to solve in closed form for any $N$, in contrast to (P3), which only has two integer unknowns (cf. Theorem 3). Nevertheless, some $N$ readily yield a closed-form solution to (P5), as shown by the following theorem.

**Theorem 4** (Minimum-redundancy parameters of KA). The parameters of the KA solving (P5) are

\[
N_1 = (N + 3)/23
\]
\[
N_2 = 5(N + 3)/23
\]
\[
N_3 = (9N - 42)/(N + 26),
\]

when $N \in \{20, 43, 66, 112, 250\}$.

**Proof**: Under certain conditions, we may obtain a solution to (P5) by considering a relaxed problem. Specifically, solving (16) for $N_3$, substituting the result into (15), and relaxing $N_1, N_2 \in \mathbb{R}$ leads to the following concave quadratic program:

\[
\text{maximize } (N_1 + N_2)(N + 3) - 3N_1N_2 - 4N_1^2 - 2N_2^2.
\]

At the critical point of the objective function we have

\[
\partial L/\partial N_1 = N - 8N_1 - 3N_2 + 3 = 0
\]
\[
\partial L/\partial N_2 = N - 3N_1 - 4N_2 + 3 = 0.
\]

Solving these equations for $N_1$ and $N_2$ yields (18) and (19), which when substituted into (16) yields (20). These are also solutions to (P5) if:

(i) $N_1 \in \mathbb{N}$, which holds when $N = 23k - 3, k \in \mathbb{N}_+$, i.e., $N = 20, 43, 66, 89, 112, 135, 158, 181, 204, 227, 250, \ldots$

(ii) $N_2 \in \mathbb{N}$, which holds when $N_1 \in \mathbb{N}$, as $N_2 = 5N_1 \in \mathbb{N}$

(iii) $N_3 \in \mathbb{N}$, which holds when $N = (26l + 42)/(9 - l) \in \mathbb{N}$ and $l \in \{0 : 8\}$, i.e., $N = 20, 43, 66, 112, 250$.

The only integer-valued $N$ satisfying all these conditions are $N = 20, 43, 66, 112, 250$, as stated. The KA in Theorem 4 also has the following properties:

**Corollary 5** (Properties of minimum-redundancy KA). The aperture $L$, number of elements $N$, and number of unit spacings $S(1)$ of the KA solving (P5) are

\[
L = (3N^2 + 18N - 19)/23
\]
\[
N = \sqrt{23/3\sqrt{L} + 2} - 3
\]
\[
S(1) = \begin{cases} (N + 22)/6 & N_1 = 1 \\ 4(N + 3)/23 & N_1 \geq 2, \end{cases}
\]

when $N \in \{20, 43, 66, 112, 250\}$.

**Proof**: This follows from Theorem 4 and (15)–(17).
Furthermore, the minimum-redundancy KA achieves the asymptotic redundancy $R_\infty = 23/12$, as shown in the following proposition, which is a reformulation of [1] Theorem, p. 177.

**Proposition 2** (Asymptotic redundancy of KA [1]). The asymptotic redundancy of the solution to (P5) is

$$R_\infty = 23/12 < 1.9167.$$  

**Proof:** Let $N = 23k+9$, where $k \in \mathbb{N}$. A feasible KA that is equivalent to the minimum-redundancy KA when $N \to \infty$ is then given by the choice of parameters $N_1 = (N-9)/23$, $N_2 = 5N_1$, and $N_3 = 9$. Substitution of these parameters into (15) yields $L = 3N^2/23 + O(N)$, i.e., $R_\infty = 23/12$.  

3) **Polynomial time grid search:** Although solving (P5) in closed form for any $N$ is challenging, we can nevertheless obtain the solution in at most $O(N \log N)$ objective function evaluations. This follows from the fact that the feasible set of (P5) only has $O(N \log N)$ or fewer elements.

**Proposition 3** (Cardinality of feasible set in (P5)). The cardinality of the feasible set in (P5) is at most $O(N \log N)$.

**Proof:** We may verify from (16) that $0 \leq N_1 \leq (N-2)/4$ and $0 \leq N_3 \leq (N-4N_1)/(N_1+1)$. Consequently, the number of grid points that need to be checked is

$$V = \sum_{N_1=0}^{N/2} \left[ \frac{N-4N_1}{N_1+1} \right] + 1 \leq \int_0^{N/2} \frac{N-3x+5/2}{x+1/2} dx.$$  

The upper bound follows from ignoring the floor operations, and substituting $N_1 = x-1/2$, where $x \in \mathbb{R}$, to account for the rectangular integration implied by the sum. Finally,

$$V \leq (N+4) \log(N/2+2)-3(N+2)/4 = O(N \log N)$$

follows from integration by parts.

Algorithm 1 summarizes a simple grid search that finds the solution to (P5) in $O(N \log N)$ steps, as implied by Proposition 3. We iterate over $N_1$ and $N_3$, because this choice yields the least upper bound on the number of grid points that need to be checked. Since the solution to (P5) is not necessarily unique, we select the KA$_R$ with the fewest closely spaced elements, similarly to the MRA in Section II-B. Note that computing the regularizer $Z$ requires $O(N^2)$ floating point operations (flops), whereas evaluating the aperture $L$ only requires $O(1)$ flops. Consequently, the time complexity of finding the KA$_R$ with the fewest closely spaced elements is $O(N^2)$, whereas finding any KA$_R$, that is, solving (P5) in general, has a worst case complexity of $O(N \log N)$.

Finally, we point out that in a related work, we also developed a KA with a constraint on the number of unit spacings [31]. This constant unit spacing Kløve Array (KA$_S$) achieves $S(1) = 8$ for any $N$ at the expense of a slight increase in redundancy (asymptotic redundancy $R_\infty = 2$). The minimum-redundancy parameters of the KA$_S$ can be found in closed form by application of Lemma 2, similarly to the CNA.

---

7Selecting $N_1$ and $N_2$, or $N_2$ and $N_3$, yields $O(N^2)$ points.

8Actually, $\zeta$ only needs to be computed when $L=L^*$ in Algorithm 1.

---

C. Other generator choices

In addition to the NA and KMA, there are naturally many other choices for generator $G$ that may lead to a low-redundancy symmetric array configuration. For example, the Wichmann generator (satisfying (C1) by Item (i) of Corollary 2) with shift $\lambda = 0$ (satisfying (C2) by Item (i) of Corollary 3) yields the Interleaved Wichmann array (IWA) with fewer unit spacings than the CNA, but the same asymptotic redundancy [32]. Although finding the minimum-redundancy parameters of the general symmetric Wichmann array (S-WA) in closed-form is cumbersome, numerical optimization of $\lambda$ and the other array parameters can slightly improve the non-asymptotic redundancy of the S-WA compared to the IWA. However, the KA still achieves both lower $R$ and $S(1)$.

Numerical experiments also suggest that some array configurations, such as the Super nested array [27] or Co-prime array [26], are not as well suited as generators $G$, at least from the redundancy point of view (mirroring $G$ does not help either). Nevertheless, several other generators, both with and without contiguous difference co-arrays, remain to be explored in future work. For example, the difference MRA [10] and minimum-hole array [59, 60] are interesting candidates.

VI. COMPARISON OF ARRAY FIGURES OF MERIT

In this section, we compare the sparse array configurations presented in Sections III and VI in terms of the array figures of merit introduced in Section II-B. For numerical examples demonstrating the ability of these arrays to resolve more scatterers than sensors, see the companion paper [33].

Table II summarizes the key properties of the different sparse array configurations, including the asymptotic element redundancy, defined as

$$\eta_\infty = \lim_{L \to \infty} \frac{N(L)}{N_{MRA}(L)} = \sqrt{\frac{R_\infty}{R_{MRA,\infty}}}.$$  

The ratio $\eta_\infty$ is the limit of the relative number of elements of a given array with respect to the MRA of equivalent aperture $L$. Note that $\eta_\infty \geq 1$, since the MRA provides a lower bound on the number of elements for a given aperture, subject to a contiguous sum co-array (see Section III). By Theorem 1, the KA$_R$ has between 0 and 27/6 more elements than then MRA (as $L \to \infty$). This is the tightest bound on the asymptotic

---

**Algorithm 1** Grid search of size $O(N \log N)$ solving (P5)

1: procedure $\text{KA}_R(N)$  
2: \{$L^*, \zeta^*$\} ← \{0, $\infty$\}  
3: for $N_1 \in \{\lfloor (N-2)/4 \rfloor \}$ do  
4: \hspace{0.5cm} for $N_3 \in \{\lfloor (N-4N_1)/(N_1+1) \rfloor \}$ do  
5: \hspace{1cm} $N_2 \leftarrow (N - (N_1+1)N_3)/2 - 2N_1$  
6: \hspace{1cm} if $N_2 \mod 1 = 0$ then  
7: \hspace{1.5cm} Compute $L$ and $\zeta$ using (15) and (2)  
8: \hspace{1cm} if $L - \zeta < L^* - \zeta^*$ then  
9: \hspace{2cm} \{$L^*, \zeta^*$\} ← \{$L, \zeta$\}  
10: \hspace{2cm} \{$N_1^*, N_2^*, N_3^*$\} ← \{$N_1, N_2, N_3$\}  
11: return $N_1^*, N_2^*, N_3^*$

---
TABLE II. KEY PROPERTIES OF SPARSE ARRAYS CONFIGURATIONS DISCUSSED IN SECTIONS III AND VI. THE ARRAYS ARE SYMMETRIC AND HAVE A CONTIGUOUS SUM AND DIFFERENCE CO-ARRAY (A SYMMETRIC MRA EXISTS AT LEAST FOR ALL N ≤ 48). THE KAR HAS AT MOST 27% MORE SENSORS THAN THE MRA (AS L → ∞), WHICH IS LESS THAN OTHER KNOWN ARRAYS WITH CLOSED-FORM SENSOR POSITIONS.

| Array configuration                  | Max. aperture, L | Min. no. of elements, N | No. of unit spacings, S(1) | Asymptotic redundancy, R∞ | Element redundancy, r∞ |
|--------------------------------------|------------------|-------------------------|-----------------------------|--------------------------|------------------------|
| Reduced-Redundancy Array (RRA)       | 15N − 353        | (L + 353)/15            | 10                          | ∞                        | 1.02 − 1.30            |
| Concatenated Nested Array (CNA)      | (N^2 + 6N − 7)/8 | 2√2/√L + 3/2            | 3/2                         | N/4 + 1                  | 1.02 − 1.30            |
| Interweaved Wichmann Array (IWA)     | (N^2 + 3N − 4)/8 | 2√2/√L + 27/32          | 3/2                         | N/4 + 1                  | 1.02 − 1.30            |
| Constant unit spacing Kløve Array (KAA) | (N^2 + 10N − 87)/8 | 2√2/√L + 14 − 5         | 8                           | 2                        | 1.02 − 1.30            |
| Minimum-Redundancy Kløve Array (KAR) | (3N^2 + 18N − 19)/23 | √23/3√N + 4 − 3       | 4N/23 + 12/24               | 1.92                     | 1 − 1.27               |

*Given the minimum-redundancy parameters assumed in the two previous columns.

Fig. 4. The KAR achieves the lowest redundancy, when the number of array elements is N ≥ 72. When 40 ≤ N ≤ 71, the RRA is less redundant. MRAs are only known for N ≤ 48.

Element redundancy of a parametric sparse array configuration with a contiguous sum co-array that is currently known.

Fig. 4 shows the redundancy R as a function of the number of array elements N. By definition, the MRA is the least redundant array configuration. However, it is also prohibitively expensive to compute for large N, and therefore only known for N ≤ 48. For N ≥ 49, one currently has to resort to alternative configurations that are cheaper to generate, such as the KAR. The KAR achieves the lowest redundancy when N ≥ 72. Asymptotically, the KAR has 1 − √23/24 ≈ 2.1% fewer elements than the KAS, CNA, and IWA, when 49 ≤ N ≤ 71, the KAR is the least redundant configuration. However, the redundancy of the RRA diverges to infinity with increasing N. The discussed array configurations (excluding the MRA) with N = 96 sensors are shown in Fig. 5 for illustration.

Fig. 5 shows the number of unit spacings S(1) as a function of N. In general, S(1) increases linearly with N, and the KAR has the smallest rate of growth. The two exceptions are the KAS and RRA, which have a constant number of unit spacings regardless of N, although only the KAS has a bounded redundancy with increasing N (see Fig. 4). As discussed in Section II-B2, S(1) may be used as a simplistic indicator of the robustness of the array to mutual coupling effects. Assessing the actual degree of coupling ultimately requires measurements, or simulations using an electromagnetic model including the mounting platform and the antenna type.

A detailed study of mutual coupling is beyond the scope of this paper.

Obviously, many other important figures of merit are omitted here for brevity of presentation. For example, fragility and the achievable beampattern are natural criteria for array design or performance evaluation. Fragility quantifies the sensitivity of the co-array to sensor failures.

The array configurations studied in this paper only have essential sensors, and therefore high fragility, since the difference (and sum) co-array ceases to be contiguous if a sensor is removed. This is the cost of low redundancy. The beampattern is of interest in applications employing linear processing. For example, in adaptive beamforming, the one-way (transmit or receive) beampattern is critical, whereas in active imaging, the two-way (combined transmit and receive) beampattern is more relevant. Although the one-way beampattern of a sparse array generally exhibits high sidelobes, a wide range of two-way beampatterns may be achieved using one or several transmissions and receptions. The arrays discussed in this paper can achieve the same effective beampattern as the ULA of equivalent aperture, by employing multiple transmissions and receptions.

VII. CONCLUSIONS AND FUTURE WORK

This paper proposed a general symmetric sparse linear array design suitable for both active and passive sensing. We established a necessary and sufficient condition for the sum and difference co-array to be contiguous, and identified sufficient conditions that substantially simplify the array design. We studied two special cases in detail, the CNA and KA, both of which achieve a low redundancy and can be generated for any number of sensors N. The KA achieves the lowest asymptotic redundancy among the considered array configurations. This also yields an upper bound on the redundancy of the MRA, whose exact value remains an open question. The upper bound may be tightened further by novel sparse array designs, possibly suggested by the proposed symmetric array framework.

In future work, it would be of interest to characterize the redundancy of other asymmetric generators with a contiguous difference co-array, as well as symmetric recursive/fractal arrays with a contiguous sum co-array. Another related direction is investigating the advantages of symmetric array configurations over co-array equivalent asymmetric arrays in more detail. This could further increase the relevance of the symmetric sparse array configurations studied in this paper.

APPENDIX A

CONTIGUOUS DIFFERENCE CO-ARRAY OF THE KMA

We now show that the Kløve-Mossige array G in Definition 10 has a contiguous difference co-array. By symmetry of...
By Definitions 9 and 10, we have
\[
\mathcal{C} = (\mathcal{D}_{\text{CNA}} - \mathcal{D}_{\text{CNA}}) \cup (D_3 - \mathcal{D}_{\text{CNA}} + 2 \max \mathcal{D}_{\text{CNA}} + 1)
\supseteq \{0 : \max \mathcal{D}_{\text{CNA}}\} \cup (D_3 + \mathcal{D}_{\text{CNA}} + \max \mathcal{D}_{\text{CNA}} + 1),
\]
where the second line follows from the fact that the CNA is symmetric and has a contiguous difference co-array. Consequently, \(\mathcal{C} \supseteq \{0 : \max \mathcal{G}\}\) holds if and only if
\[
D_3 + \mathcal{D}_{\text{CNA}} = \{0 : \max \mathcal{G} - \max \mathcal{D}_{\text{CNA}} - 1\}.
\]
Due to the periodicity of \(D_3\), this condition simplifies to
\[
D_3 + \mathcal{D}_{\text{CNA}} = \{0 : \max \mathcal{D}_{\text{CNA}} + N_1^2\},
\]
where \(D_4 = \{0 : N_1 : N_1^2\}\), and by Definition 9 we have
\[
D_4 + \mathcal{D}_{\text{CNA}} = \{D_4 + D_1\} \cup \{D_4 + D_2 + N_1\} \cup \{D_4 + D_1 + N_2(N_1 + 1)\}.
\]
As \(D_1 + D_4 = \{0 : N_1(N_1 + 1) - 1\}\), it suffices to show that
\[
D_4 + D_2 \supseteq \{N_1^2 : (N_2 - 1)(N_1 + 1)\}.
\]
By Definitions 9 and 10 we have
\[
D_4 + D_2 = \{kN_1 + l(N_1 + 1) \mid k \in \{0 : N_1\} ; l \in \{0 : N_2 - 1\}\} = \{i(N_1 + 1) - k \mid k \in \{0 : N_1\} ; i - k \in \{0 : N_2 - 1\}\} \supseteq \{i(N_1 + 1) - k \mid k \in \{0 : N_1\} ; i \in \{N_1 : N_2 - 1\}\} \supseteq \{N_1^2 : (N_2 - 1)(N_1 + 1)\},
\]
which implies that the difference co-array of \(\mathcal{G}\) is contiguous.

**APPENDIX B**

**FIRST HOLE IN THE SUM CO-ARRAY OF THE KMA**

Let \(\mathcal{G}\) denote the Kløve-Mossige array (KMA) as in Definition 10. Furthermore, let \(H \in \mathbb{N}\), as defined in (5), be the first hole in \(\mathcal{G} + \mathcal{G}\). In the following, we show that
\[
H = \begin{cases} 
2 \max \mathcal{G} + 1, & \text{if } N_1 + N_2 = 1 \\
h + 1, & \text{if } N_1 \geq 1 \text{ and } N_2 = 1 \\
h, & \text{otherwise,}
\end{cases}
\]
where the non-negative integer \(h\) is defined as
\[
h = \max \mathcal{G} + \max \mathcal{D}_{\text{CNA}} + 1 = N_3(\max \mathcal{D}_{\text{CNA}} + 1 + N_1^2) + 2 \max \mathcal{D}_{\text{CNA}} + 1.
\]

The first case, which we only briefly mention here, follows trivially from the fact that \(\mathcal{G}\) degenerates to the ULA when either \(N_1 = 0\) and \(N_2 = 1\), or \(N_1 = 1\) and \(N_2 = 0\). We prove the latter two cases by contradiction, i.e., by showing that \(h + 1 \in \mathcal{G} + \mathcal{G}\) or \(h \in \mathcal{G} + \mathcal{G}\) leads to an impossibility.

We start by explicitly writing the sum co-array of \(\mathcal{G}\) as
\[
\mathcal{G} + \mathcal{G} = (\mathcal{D}_{\text{CNA}} + \mathcal{D}_{\text{CNA}}) \cup (\mathcal{D}_{\text{CNA}} + D_3 + 2 \max \mathcal{D}_{\text{CNA}} + 1)
\cup (D_3 + D_3 + 4 \max \mathcal{D}_{\text{CNA}} + 2).
\]
Note that the CNA has a contiguous sum co-array, that is,
\[
\mathcal{D}_{\text{CNA}} + \mathcal{D}_{\text{CNA}} = \{0 : 2 \max \mathcal{D}_{\text{CNA}}\}.
\]
Furthermore, it was shown in Appendix A that
\[
\mathcal{D}_{\text{CNA}} + D_3 + 2 \max \mathcal{D}_{\text{CNA}} + 1 = \{2 \max \mathcal{D}_{\text{CNA}} + 1 : h - 1\}.
\]
Consequently, \(h \in \mathcal{G} + \mathcal{G}\) holds if and only if
\[
h \in D_3 + D_3 + 4 \max \mathcal{D}_{\text{CNA}} + 2.
\]
By Definition 10 there must therefore exist non-negative integers \(k \in \{0 : 2N_1\}\) and \(l \in \{0 : 2(N_3 - 1)\}\) such that
\[
h = kN_1 + l(\max \mathcal{D}_{\text{CNA}} + 1 + N_1^2) + 4 \max \mathcal{D}_{\text{CNA}} + 2.
\]
Substituting (21) into (22) and rearranging the terms yields
\[
(N_3 - l)(\max \mathcal{D}_{\text{CNA}} + 1 + N_1^2) = 2 \max \mathcal{D}_{\text{CNA}} + 1 + kN_1.
\]
Since \(k \in \{0 : 2N_1\}\), the following inequality must hold:
\[
2 \max \mathcal{D}_{\text{CNA}} + 1 \leq N_3 - l \leq 2N_1^2 + 2 \max \mathcal{D}_{\text{CNA}} + 1.
\]
This reduces to \(0 < N_3 - l < 2\), or more conveniently, \(N_3 - l = 1\), since \(N_3 - l\) is an integer. Consequently, we have
\[
N_1(N_1 - k) = \max \mathcal{D}_{\text{CNA}},
\]
where \( \max D_{\text{CNA}} \geq 0 \) leads to \( N_1 - k \in \{ 0 : N_1 \} \). Substituting \( \max D_{\text{CNA}} = L \) in (7) into (23) yields

\[
N_1 - k = N_2 + 1 + \frac{N_2 - 1}{N_1}.
\]

(24)

Combined with \( N_1 - k \leq N_1 \), this implies that

\[
N_1 \geq \frac{N_2 + 1 + \sqrt{(N_2 + 1)^2 + 4(N_2 - 1)}}{2} \geq N_2 + 1,
\]

since \( N_1, N_2 \geq 1 \). We identify the following two cases:

- If \( N_2 = 1 \), then (24) yields that \( N_1 - k = 2 \), implying that \( H > h \). However, it is straightforward to verify that \( H = h + 1 \) from the fact that when \( h \) is replaced by \( h + 1 \) in (22), no integer-valued \( N_1 \geq 2 \) satisfies the equation.

- If \( N_2 \geq 2 \), then \( N_1 \leq N_2 - 1 \) follows from (24), since \( (N_2 - 1) / N_1 \) must be an integer. This leads to a contradiction, since both \( N_1 \geq N_2 + 1 \) and \( N_1 \leq N_2 - 1 \) cannot hold simultaneously. Consequently, \( H = h \) holds.

Finally, \( H = h \) also holds when \( N_1 = 0 \) and \( N_2 \geq 2 \), or \( N_1 \geq 2 \) and \( N_2 = 0 \), since \( G \) degenerates into the NA in this case. This covers all of the possible values of \( H \).

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