New Extremal bounds for Reachability and Strong-Connectivity Preservers under failures

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What are Fault-tolerant Preservers?

A sparse subgraph “preserving” a given property of input graph even after bounded number of failures. 

at most $k$
Example: Connectivity

Under NO Failures

Spanning tree $T$ is sparse subgraph preserving connectivity

Under $k = 1$ Failures

“$G - (x, y)$” is connected

$T - (x, y)$ is NOT connected
Can we make structures fault-tolerant?
This Talk..

Sparsifying graphs “to preserve” strong-connectivity and pairwise-reachability after $k$-failures.
Fault-tolerant Strong-connectivity Preservers

- **Given**: digraph $G$, parameter $k$.

- **Aim**: to compute a *sparse* subgraph $H$ of $G$ such that for any set $F$ of $k$ failures:

  $\text{SCCs of } (G - F) = \text{SCCs of } (H - F)$

\[
(G - F) \quad \xrightarrow{\text{Diagram}} \quad (H - F)
\]
### Fault-tolerant Strong-connectivity Preservers

#### Related Work: Connectivity in undirected graphs

| Undirected graphs (k failures) | $O(kn)$ edges |

Can we get $O(n^{2-\varepsilon})$ space for digraphs?
Fault-tolerant Strong-connectivity Preservers

| Directed graphs (\(k\) failures) | \(\tilde{O}(k \ 2^k \ n^{2-1/k})\) edges |
|-----------------------------------|---------------------------------------|

Our Results: Strong-Connectivity

\[ O(n) \text{ for } k = 1 \]
Our Techniques: FT SCC Preserver

- **SCC in** $G - F$
  - **Small SCC in** $G - F$
  - **Large SCC in** $G - F$

Main Ideas:
- Color-coding technique (also used in FT-spanner construction by Dinitiz & Krauthgamer)
- Hitting set property of random sets.
Sample a random set $J$ of size $\gamma_1$ and add to $H$. Take union of SCC certificates in $(G - J)$ and add to $H$. Repeat.
Large SCC in $G - F$

Sample a random set $R$ of size $\gamma_2$

Take union of $k$-FT IN- and OUT-Reachability preserver of vertices in $R$, and add to $H$

$k$-FT OUT-Reachability Preserver of $r$

A sparse subgraph $H^r$ s.t. for any set $F$ of $k$ failures:

Vertices reachable from $r$ in $(G - F) = \text{Vertices reachable from } r \text{ in } (H^r - F)$

$k$-FT IN-Reachability Preserver of $r$

A sparse subgraph $H^r$ s.t. for any set $F$ of $k$ failures:

Vertices having path to $r$ in $(G - F) = \text{Vertices having path to } r \text{ in } (H^r - F)$

Number of edges in $H^r = O(2^k n)$
Fault-tolerant Pair-wise Reachability Preservers

- **Given**: digraph $G = (V, E)$, parameter $k$, pair-set $\mathcal{P} \subseteq (V \times V)$.

- **Aim**: to compute a sparse subgraph $H$ of $G$ such that for any set $F$ of $k$ failures:

  for each $(x, y) \in \mathcal{P}$:

  $$(G - F) \quad \quad (H - F)$$
### What is known?

| Under No Failures | $O(n + \min\{p\sqrt{n}, n^2p^2\})$ edges |
|-------------------|------------------------------------------|
| Pair-set $P$ of size $p$ |                                           |

For $\sqrt{n}$ pairs, size bound is $O(n)$.

| $k$-FT Single Source Reachability ($\{s\} \times V$) | $O(2^k n)$ edges |
|---------------------------------------------------|------------------|
| $k$ failures                                       |                  |

Size increases by 2, on increasing $k$ by 1
For single failure, for **upto how many pairs** can we get $O(n)$ sized preserver?

How does size of preserver change when moving from single to dual failure?

Can we get $o(n^2)$ size preserver for sub-quadratic number of pairs?
### OUR RESULTS

#### Upper Bound (k=1)

- **Pair-set $\mathcal{P}$ of size $p$**
  - $O(n + \min\{p\sqrt{n}, n\sqrt{p}\})$ edges

#### Lower Bound (k=2)

- $\Omega(n^\epsilon)$ pairs
  - $\Omega(n^{1+\epsilon/8})$ edges

For $\sqrt{n}$ pairs, size bound is $O(n)$.

For $o(n^2)$ pairs, size bound is $o(n^2)$.

Polynomial increment on increasing $k$ by 1.

For polynomial-number of pairs, preserver has super-linear size.
Main Idea: $O(n + |\mathcal{P}| \sqrt{n})$ bound

How does paths in $\text{Preserver}(s, t)$ and $\text{Preserver}(s', t')$ interact?
Main Idea: \( O(n + |\mathcal{P}| \sqrt{n}) \) bound

\( H_{\text{opt}} = \text{optimal subgraph satisfying } (H_{\text{opt}} + H_{\text{scc}}) \text{ is Preserver for } \mathcal{P}. \)

**Proposition 1:** Only one edge out of \( h_1 \) and \( h_2 \) lies in \( H_{\text{opt}} \).
Main Idea: \( O(n + |\mathcal{P}| \sqrt{n}) \) bound

\( H_{opt} = \) optimal subgraph satisfying \( (H_{opt} + H_{scc}) \) is Preserver for \( \mathcal{P} \).

**Proposition 2:** Only one edge out of \( h_1 \) and \( h_2 \) lies in \( H_{opt} \).
Main Idea: $O(n + |\mathcal{P}|\sqrt{n})$ bound

$H_{opt} = \text{optimal subgraph satisfying } (H_{opt} + H_{scc}) \text{ is Preserver for } \mathcal{P}.$

Key Property: Among all $s_i - t_i$ paths intersecting $v$ with incident-edge in $H_{opt}$, at most two intersects $s - t$ path more than once.

Size of $H_{scc}$ is $O(n)$.

Size of $H_{opt}$ is $O(|\mathcal{P}|\sqrt{n})$. 
Can we extend this to $k > 1$ failures?

NO

Preserver($s,t$) $\neq$ Union of $o(n)$ number of $s - t$ paths!
Future Work

FT-Reachability-Preserver: Sparseness under multiple failures?

FT-Reachability-Preserver: \( O(n + \min\{p\sqrt{n}, \ n^{2/3}p^{2/3}\}) \) size for \( k = 1 \)？

FT Strong-connectivity Preserver: Linear size for constant \( k( > 1) \)？