On a class of distributions generated by stochastic mixture of the extreme order statistics of a sample of size two

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Abstract

This paper considers a family of distributions constructed by a stochastic mixture of the order statistics of a sample of size two. Various properties of the proposed model are studied. We apply the model to extend the exponential and symmetric Laplace distributions. An extension to the bivariate case is considered.

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1 Introduction

Different methods may be used to introduce a new parameter to a family of distributions to increase flexibility for modeling purposes. Marshall and Olkin \cite{10} introduced a method for adding a parameter to a family of distributions and applied it to the exponential and Weibull models. Jones \cite{5} used the distribution of order statistics to provide new families of distributions with extra parameters. The well-known Farlie–Gumbel-Morgenstern (FGM, for short) family of bivariate distributions with the given univariate marginal distributions $F_1$ and $F_2$, is defined by

$$H(x, y) = F_1(x)F_2(y)\{1 + \lambda \bar{F}_1(x)\bar{F}_2(y)\},$$

where $\lambda \in [-1, 1]$; see, Drouet-Mari and Kotz (\cite{6}, Chapter 5) for a good review. For a given univariate cumulative distribution function $F$, the univariate version of (1) may be considered as

$$G_\lambda[F](x) = F(x)\{1 + \lambda \bar{F}(x)\},$$

for all $x$ and $-1 \leq \lambda \leq 1$. The family of distributions defined by (2) is comparable with the Marshall-Olkin \cite{10} family of distributions, which also called the proportional odds model \cite{7,10}, given by

$$H(x) = \frac{F(x)}{1 - (1 - \alpha)\bar{F}(x)}, \quad -\infty < x < \infty, \quad \alpha > 0.$$
Note that $H$ with $0 < \alpha < 1$ could be written as
\[ H(x) = F(x) \sum_{k=0}^{\infty} (1 - \alpha) \bar{F}(x)^k, \]
and hence for $0 < \lambda < 1$, (2) is a first-order approximation to the proportional odds model.

The aim of the present paper is to investigate different properties of (2). We first provide a physical interpretation for this model in Section 3. Some preservation results of stochastic orderings and aging properties are given in Section 4. A generalization of the ordinary exponential distribution which exhibits both increasing and decreasing hazard rate functions and a skew extension of the symmetric Laplace distribution are given in Section 5. Bivariate case is discussed in Section 5. In Section 1 we recall some notions that will be used in the sequel.

## 2 Preliminaries

Let us recall some notions of stochastic orderings and aging concepts that will be useful in this paper. Let $X$ be a continuous random variable with the cdf $F$, the survival function $\bar{F} = 1 - F$, the probability density function (pdf) $f$, the residual life survival function $\bar{F}_t(x) = P(X > x + t | X > t)$ and the hazard rate function $h_F(x) = f(x)/\bar{F}(x)$. Then $F$ is said to have: (i) increasing (decreasing) hazard rate IHR (DHR) if $h_F(x)$ is increasing (decreasing) in $x$; (ii) increasing (decreasing) hazard rate average IHRA (DHRA) if $\int_0^t h_F(x) dx/t$ is increasing (decreasing) in $t$; (iii) new better (worse) than used NBU (NWU) property if $\bar{F}_1(x) \leq (\geq) \bar{F}(x)$, for all $x \geq 0$ and $t \geq 0$. The implications
\[ \text{IHR} \implies \text{IHRA} \implies \text{NBU} \quad \text{and} \quad \text{DHR} \implies \text{DHRA} \implies \text{NWU}, \]
are well known. See [2] for more detail. The following definitions will be used for various stochastic comparisons. Let $F_1$ and $F_2$ be two cdfs with the corresponding pdfs $f_1$ and $f_2$, the hazard rate functions $h_{F_1}$, $h_{F_2}$, and the quantile functions $F_1^{-1}$ and $F_2^{-1}$, respectively, where $F_i^{-1} = \sup\{x | F_i(x) \leq u\}$, for $0 \leq u \leq 1$. The cdf $F_1$ is said to be smaller than $F_2$ in (i) stochastic order ($F_1 \prec_{st} F_2$) if $F_1(x) \geq F_2(x)$ for all $x$; (ii) hazard rate order ($F_1 \prec_{hr} F_2$) if $h_{F_1}(x) \geq h_{F_2}(x)$ for all $x$; (iii) likelihood ratio order ($F_1 \prec_{lr} F_2$) if $f_2(x)/f_1(x)$ is non-decreasing in $x$; (iv) convex transform order ($F_1 \prec_c F_2$) if $F_2^{-1}F_1(x)$ is convex in $x$ on the support of $F_1$; (v) star order ($F_1 \prec_* F_2$) if $F_2^{-1}F_1(x)/x$ is increasing in $x \geq 0$; (vi) superadditive order ($F_1 \prec_{su} F_2$) if $F_2^{-1}F_1(x+y) \geq F_2^{-1}F_1(x) + F_2^{-1}F_1(y)$; (vii) dispersive order ($F_1 \prec_{disp} F_2$) if $F_2^{-1}F_1(x) - x$ increases in $x$. The implications
\[ F_1 \prec_{lr} F_2 \implies F_1 \prec_{hr} F_2 \implies F_1 \prec_{st} F_2 \]
are well known. See [12] for an extensive study of these notions.
3 Genesis of family (2)

Let $X_1$ and $X_2$ be two independent and identically distributed random variables having the survival function $\bar{F} = 1 - F$. For $-1 \leq \lambda \leq 1$, let $Z$ be a Bernoulli random variable, independent of $X_i$s, with $P(Z = 1) = \frac{1+\lambda}{2}$ and $P(Z = 0) = \frac{1-\lambda}{2}$. Consider the stochastic mixture

$$U = ZX_{(1)} + (1 - Z)X_{(2)},$$

(5)

where $X_{(1)} = \min(X_1, X_2)$ and $X_{(2)} = \max(X_1, X_2)$ are the corresponding order statistics of $X_1$ and $X_2$. Since the distribution functions of $X_{(2)}$ and $X_{(1)}$ are given by $F_2(x) = F^2(x)$ and $F_1(x) = 2F(x) - F^2(x)$, respectively, then the cdf of $U$, denoted by $G_\lambda[F]$, is given by

$$G_\lambda[F](x) = \frac{1+\lambda}{2}F_1(x) + \frac{1-\lambda}{2}F_2(x)$$

$$= F(x)\{1 + \lambda \bar{F}(x)\},$$

(6)

for all $x$ and $-1 \leq \lambda \leq 1$. Clearly $G_0[F] = F$, $G_{-1}[F] = F_2$, and $G_1[F] = F_1$. Since $G_\lambda[F](.)$ is increasing in $\lambda$, we have the inequality

$$F_2(x) \leq G_\lambda[F](x) \leq F_1(x),$$

for all $x$ and $-1 \leq \lambda \leq 1$.

In the following result we show that the transformation (2) is “unique”, in the sense that given a distribution $F$, this generates a unique distribution or a family of distributions.

**Proposition 1.** Let $F_1$ and $F_2$ be two distribution functions such that $G_\lambda[F_1] = G_\lambda[F_2]$ for every $\lambda \in [-1, 1]$. Then $F_1 = F_2$.

**Proof.** Suppose that $\lambda > 0$ (the case $\lambda = 0$ is trivial and the case $\lambda < 0$ the result could be proved similar). Then, $G_\lambda[F_1] = G_\lambda[F_2]$, is equivalent to

$$[F_1(x) - F_2(x)][1 - \lambda(F_1(x) + F_2(x) - 1)] = 0,$$

(7)

for each $x$. Suppose there exist a point $x_0 \in R$ such that —without loss of generality— $F_1(x_0) < F_2(x_0)$. Then the equality (7) is equivalent to $F_1(x_0) + F_2(x_0) = \frac{1}{\lambda} + 1$. Since $1 \leq \frac{1}{\lambda}$ and $F_1(x_0) < F_2(x_0) < 1$, we must have $F_2(x_0) > 1$. This absurd, so that we conclude that $F_1 = F_2$.

4 Properties

The survival function, the probability density function and the hazard rate function corresponding to (2) are given by

$$\bar{G}_\lambda[F](x) = \bar{F}(x)\{1 - \lambda F(x)\},$$

(8)
\[ g_\lambda[F](x) = f(x)\{1 + \lambda(1 - 2F(x))\} \] (9)

and

\[ h_G(x; \lambda) = \frac{g_\lambda[F](x)}{G_\lambda[F](x)} = h_F(x) \left( 1 + \frac{\lambda F(x)}{1 - \lambda F(x)} \right), \] (10)

respectively, where, \( h_F(x) \) is the hazard rate function of \( F \).

It follows from (10) that

\[ \lim_{x \to -\infty} h_G(x; \lambda) = (1 + \lambda) \lim_{x \to -\infty} h_F(x), \quad \lim_{x \to \infty} h_G(x; \lambda) = \lim_{x \to \infty} h_F(x), \]

\[ h_F(x) \leq h_G(x; \lambda) \leq (1 + \lambda)h_F(x), \quad (-\infty < x < \infty, 0 \leq \lambda \leq 1), \]

\[ (1 + \lambda)h_F(x) \leq h_G(x; \lambda) \leq h_F(x), \quad (-\infty < x < \infty, -1 \leq \lambda \leq 0). \]

Let \( \bar{F}_t(x) = \frac{F(x+t)}{F(t)} \) be the residual life survival function corresponding to cdf \( F \). Then from (8), the residual life survival function of the generated distribution \( G_\lambda[F] \), denoted by \( \bar{G}_{\lambda,t}[F](x) \), is given by

\[ \bar{G}_{\lambda,t}[F](x) = \frac{G_\lambda[F](x+t)}{G_\lambda[F](t)} \]

\[ = \bar{F}_t(x) \left( \frac{1 - \lambda F(x+t)}{1 - \lambda F(t)} \right) \]

\[ = \bar{F}_t(x) \left( 1 - \frac{\lambda F(t)}{1 - \lambda F(t)} F(x + t) - F(t) \right) \]

\[ = \bar{F}_t(x) \{1 - \beta F_t(x)\} \]

\[ = \bar{G}_\beta[F_t](x), \] (11)

where \( \beta = \beta(t) = \frac{\lambda F(t)}{1 - \lambda F(t)} \) and \( F_t(x) = 1 - \bar{F}_t(x) \). Thus the residual life survival function of \( G_\lambda[F] \) is the transformed version of the residual life survival function of \( F \) under (2), with a new parameter.

By solving the equation \( F(x)\{1 + \lambda(1 - F(x))\} = G_\lambda[F](x) \), with respect to \( F \), one gets

\[ F(x) = \frac{1 + \lambda - \sqrt{(1 + \lambda)^2 - 4\lambda G_\lambda[F](x)}}{2\lambda}, \]

which gives the the quantile function of \( G_\lambda[F] \) as

\[ G_\lambda^{-1}[F](q) = F^{-1} \left( \frac{1 + \lambda - \sqrt{(1 + \lambda)^2 - 4\lambda q}}{2\lambda} \right), \quad 0 \leq q \leq 1. \] (12)

Note that \( \lim_{\lambda \to 0} G_\lambda^{-1}[F](q) = F^{-1}(q) \). In particular, the median of \( G_\lambda[F] \) is given by

\[ G_\lambda^{-1}[F](0.5) = F^{-1} \left( \frac{1 + \lambda - \sqrt{1 + \lambda^2}}{2\lambda} \right). \]
4.1 Stochastic comparisons

In this section we provide some results for stochastic orderings and aging properties of a given cdf under the transformation \( \lambda \).

**Proposition 2.** For a given cdf \( F \), we have

\( a) \) If \( F \) is IHR (IHRA, NBU) and \(-1 \leq \lambda \leq 0\), then \( G_\lambda[F] \) is IHR (IHRA, NBU).

\( b) \) If \( F \) is DHR (DHRA, NWU) and \( 0 \leq \lambda \leq 1 \), then \( G_\lambda[F] \) is DHR (DHRA, NWU).

\( c) \) The parametric family \( \{G_\lambda[F]\} \) of distributions is decreasing in \( \lambda \) in the likelihood ratio order. Consequently, \( G_\lambda[F] \) is decreasing in the hazard rate and stochastic orders.

**Proposition 3.** Suppose that \( F_1 \) and \( F_2 \) be two given CDFs such that \( F_1 \prec_{st} F_2 \). Then \( G_\lambda[F_1] \prec_{st} G_\lambda[F_2] \) for every \( \lambda \in [-1, 1] \).

**Proof.** Since \( F_1 \prec_{st} F_2 \) implies that \( F_1(x) \geq F_2(x) \) and \( \bar{F}_1(x) \leq \bar{F}_2(x) \), for all \( x \); we have \( G_\lambda[F_1](x) = F_1(x) \frac{1}{1+\lambda} \geq \bar{F}_2(x) \frac{1}{1+\lambda} = G_\lambda[F_2](x) \) for \( \lambda < 0 \) and \( G_\lambda[F_1](x) = \bar{F}_1(x) \frac{1}{1-\lambda} \leq \bar{F}_2(x) \frac{1}{1-\lambda} = G_\lambda[F_2](x) \) for \( \lambda > 0 \), which completes the proof.

**Proposition 4.** Let \( F_1 \) and \( F_2 \) be two given cdfs and let \( G_\lambda[F_1] \) and \( G_\lambda[F_2] \) be their transformed versions using (2). Then

\[ F_1 \prec_{\text{order}} F_2 \Rightarrow G_\lambda[F_1] \prec_{\text{order}} G_\lambda[F_2], \]

where \( \prec_{\text{order}} \) is any one of the orders \( \prec_c, \prec_s, \prec_{su} \) and \( \prec_{disp} \).

**Proof.** From (2) it is easy to see that

\[ G_\lambda^{-1}[F_2] (G_\lambda[F_1](x)) = F_2^{-1} (F_1^{-1}(x)), \]

for all \( x \), which gives the required result.

4.2 A symmetry property

The transformation map (2) can be applied to any symmetric or asymmetric distribution. The following result shows the effect of this transformation on the symmetry property of the parent distribution.

**Proposition 5.** Let \( X \) with the cdf \( F \), be a symmetric random variable about zero (i.e., \( X \) and \(-X \) have the same distribution) and let \( Y_\lambda \) be a random variable distributed according to \( G_\lambda[F] \), \(-1 \leq \lambda \leq 1 \). Then \( Y_{-\lambda} \) and \(-Y_\lambda \) have the same distribution.
Proof. Since \( X \) is symmetric about zero, then \( F(x) = 1 - F(-x) = \tilde{F}(-x) \) for all \( x \). From (2) and (8) we have

\[
P(-Y_\lambda \leq y) = \tilde{G}_\lambda[F](y) \\
= \tilde{F}(-y)\{1 - \lambda F(-y)\} \\
= F(y)\{1 - \lambda \tilde{F}(y)\} \\
= G_{-\lambda}[F](y) \\
= P(Y_{-\lambda} \leq y),
\]

which completes the proof.

5 Examples

5.1 The transformed exponential distribution

In particular case that \( F \) is an exponential distribution with the parameter \( \theta \), the two–parameter distribution generated using (2) has the cdf

\[
G(x; \lambda, \theta) = (1 - e^{-\theta x})(1 + \lambda e^{-\theta x}), \quad x, \theta > 0, -1 \leq \lambda \leq 1,
\]

and the corresponding density function

\[
g(x; \lambda, \theta) = \theta e^{-\theta x}\{1 + \lambda(2e^{-\theta x} - 1)\}.
\]

For the density function \( g \), we have that \( \log g(x; \lambda, \theta) \), is concave for \(-1 \leq \lambda \leq 0\) and convex for \( 0 \leq \lambda \leq 1 \). As a result for \( 0 \leq \lambda \leq 1 \), \( g(x; \lambda, \theta) \) is decreasing, and for \(-1 \leq \lambda < 0 \), \( g(x; \lambda, \theta) \) is unimodal. By solving the equation \( d \log g(x; \lambda, \theta)/dx = 0 \), it is readily verified that the density function \( g(x; \lambda, \theta) \) has the mode equal to zero for \( \lambda > -\frac{1}{3} \) and \(-\frac{1}{\theta} \ln(\frac{\lambda+1}{4\lambda}) \) for \( \lambda < -\frac{1}{3} \).

From (10), the hazard rate function of this distribution is given by

\[
h(x; \lambda, \theta) = \frac{\theta\{1 + \lambda(2e^{-\theta x} - 1)\}}{1 + \lambda(e^{-\theta x} - 1)}.
\]

It may be noticed that while the exponential distribution has a constant hazard rate function, the generated cdf \( G \), has increasing hazard rate for \(-1 \leq \lambda < 0 \), and decreasing hazard rate for \( 0 < \lambda \leq 1 \), which follows using the log-convexity and the log-concavity of the density function.

From (11) the residual life survival function corresponding to (13), is given by

\[
\tilde{G}_t(x; \lambda, \theta) = e^{-\theta x}\{1 + \beta(e^{-\theta x} - 1)\},
\]

where \( \beta = \beta(t) = \lambda e^{-\theta t}\{1 + \lambda(e^{-\theta t} - 1)\}^{-1} \). The limit distribution as \( t \to \infty \) is an ordinary exponential distribution because the limit of \( \beta(t) \) is 0.
From (15) the mean residual life function of a random variable $X$ having cdf (13), could be obtained as

$$m(t; \lambda, \theta) = E(X - t|X > t) = \int_{0}^{\infty} G_t(x; \lambda, \theta) dx = \frac{1 + \lambda(\frac{1}{2}e^{-\theta t} - 1)}{\theta[1 + \lambda(e^{-\theta t} - 1)]},$$

which is increasing in $t$ for $0 \leq \lambda \leq 1$ and decreasing for $-1 \leq \lambda \leq 0$, with $\lim_{t \to \infty} m(t; \lambda, \theta) = 1/\theta = E(X; 0, \theta)$ and $\lim_{t \to 0} m(t; \lambda, \theta) = (2 - \lambda)/2\theta = E(X; \lambda, \theta)$; and hence

$$\frac{1}{\theta} \leq m(t; \lambda, \theta) \leq \frac{2 - \lambda}{2\theta} \quad (-1 \leq \lambda \leq 0),$$

and

$$\frac{2 - \lambda}{2\theta} \leq m(t; \lambda, \theta) \leq \frac{1}{\theta} \quad (0 \leq \lambda \leq 1).$$

The moment generating function of this distribution is given by

$$M(t) = E(e^{tx}) = \frac{\theta(2\theta - (1 + \lambda)t)}{(\theta - t)(2\theta - t)}.$$

By straightforward integration the raw moments are found to be

$$E(X^r) = \frac{(1 + \lambda(2^r - 1))r!}{\theta^r},$$

for $r \in \mathbb{N}$.

Since for the exponential distribution with the parameter $\theta$ we have $F^{-1}(q) = -\frac{1}{\theta} \ln(1 - q)$, $0 < q < 1$, then from (12) the quantile function of the generated distribution is given by

$$G^{-1}(q) = -\frac{1}{\theta} \ln \left(\frac{\lambda - 1 + \sqrt{(1 + \lambda)^2 - 4\lambda q}}{2\lambda}\right).$$

Note that if $\lambda \to 0$, then $G^{-1}(q) \to -\frac{1}{\theta} \ln(1 - q)$.

It may be noticed that for the generated distribution, median($X$), mode($X$) and $E(X)$ are all decreasing in $\lambda, \theta$ and mod($X$) $\leq$ median($X$) $\leq$ $E(X)$.

5.2 A class of skew–Laplace distributions

The classical symmetric Laplace distribution has the pdf

$$f(x; \theta) = \frac{1}{2\theta} e^{-|x|/\theta},$$

$$0 < \theta < \infty.$$
and cdf
\[ F(x; \theta) = \begin{cases} \frac{1}{2}e^{\frac{x}{\theta}}, & x \leq 0, \\ 1 - \frac{1}{2}e^{-\frac{x}{\theta}}, & x \geq 0, \end{cases} \]
where \(-\infty < x < \infty\) and \(\theta > 0\). The symmetric Laplace distribution has been used as an alternative to the normal distribution for modeling heavy tails data. Different forms of the skewed Laplace distributions have been introduced and studied by various authors. Recently, Kozubowski and Nadarajah \[9\] identified over sixteen variations of the Laplace distribution. In the following we propose a new version of the skewed Laplace distribution using (2). The cdf and pdf of the generated model are given by
\[ G_\lambda(x; \theta) = \begin{cases} \frac{1}{2}e^{\frac{x}{\theta}}\{1 + \lambda(1 - \frac{1}{2}e^{\frac{x}{\theta}})\}, & x \leq 0, \\ 1 - \frac{1}{2}e^{-\frac{x}{\theta}}\{1 - \lambda(1 - \frac{1}{2}e^{-\frac{x}{\theta}})\}, & x \geq 0, \end{cases} \]
and
\[ g_\lambda(x; \theta) = \begin{cases} \frac{1}{2\theta}e^{\frac{x}{\theta}}\{1 + \lambda(1 - e^{\frac{x}{\theta}})\}, & x \leq 0, \\ \frac{1}{2\theta}e^{-\frac{x}{\theta}}\{1 - \lambda(1 - e^{-\frac{x}{\theta}})\}, & x \geq 0, \end{cases} \]
respectively. The moment generating function of \(G_\lambda\), is given by
\[ M(t) = \frac{1 - \lambda \theta t}{1 - (\theta t)^2} + \frac{\lambda \theta t}{4 - (\theta t)^2}, \]
and the raw moments are found to be
\[ E(X^r) = \begin{cases} \frac{\lambda \theta r\theta((1-2^{r+1})}{2^{r+1}}, & \text{if } r \text{ is odd}, \\ r!\theta^r, & \text{if } r \text{ is even}. \end{cases} \]
The expectation, variance, skewness and the kurtosis are given by
\[ E(X) = -\frac{3}{4}\lambda \theta, \]
\[ \text{Var}(X) = \theta^2(1 - \frac{9}{16}\lambda^2), \]
\[ \text{Skewness}(X) = \frac{18\lambda(4 + 3\lambda^2)}{(9\lambda^2 - 32)\sqrt{32 - 9\lambda^2}}, \]
\[ \text{Kurtosis}(X) = \frac{6144 - 243\lambda^4 - 2592\lambda^2}{(32 - 9\lambda^2)^2}. \]
It may be noticed that the skewness of \(G_\lambda\) is decreasing in \(\lambda\), and then \(-1.1423 \leq \text{Skewness}(X) \leq 1.1423\). It is positive for \(-1 \leq \lambda \leq 0\), and negative for \(0 \leq \lambda \leq 1\).

### 6 Bivariate case

#### 6.1 Construction

A large number of bivariate distributions have been proposed in literature. A very wide survey on bivariate distributions are given in [1] and [8]. The method used to construct the family of
distributions given by (2) also lends itself well to the construction of bivariate distributions whose univariate marginal cdf are of the form (2).

**Proposition 6.** Let $F$ be a bivariate cdf with the univariate marginal cdfs $F_1$, $F_2$ and the associated survival function $\overline{F}(x, y) = 1 - F_1(x) - F_2(y) + F(x, y)$. Then, for every $-1 \leq \lambda \leq 1$, the function $G_\lambda : \mathbb{R}^2 \to [0, 1]$, defined by

$$G_\lambda(x, y) = (1 + \lambda) \left( F_1(x)F_2(y) + F(x, y)\overline{F}(x, y) \right) - \lambda F^2(x, y),$$

is a bivariate cdf with the univariate marginal distributions

$$G_1(x) = F_1(x)\{1 + \lambda \overline{F}_1(x)\} \quad \text{and} \quad G_2(y) = F_2(y)\{1 + \lambda \overline{F}_2(y)\}.$$  

**Proof.** To prove this, let $(X_1, Y_1)$ and $(X_2, Y_2)$ be two independent random vector having common bivariate cdf $F$ and the univariate marginal cdfs $F_1$ (of $X_i$) and $F_2$ (of $Y_i$), $i = 1, 2$. Let $X_1, X_2$ and $Y_1, Y_2$ be their corresponding order statistics. For $-1 \leq \lambda \leq 1$, consider the random pair $(V_1, V_2) = (X_1, Y_1)$ with probability $\frac{1+\lambda}{2}$ and $(V_1, V_2) = (X_2, Y_2)$ with probability $\frac{1-\lambda}{2}$. Then, it is straightforward to verify that $(V_1, V_2)$ have the joint cdf \((16)\) with $G_\lambda(x, \infty) = G_1(x)$ and $G_\lambda(\infty, y) = G_1(y)$.

Note that the special case $F(x, y) = F_1(x)F_2(y)$, the cdf \((16)\) reduces to

$$G_\lambda(x, y) = F_1(x)F_2(y) \left\{ F_1(x)F_2(y) + (1 + \lambda)(\overline{F}_1(x) + \overline{F}_2(y)) \right\},$$

which may serve as a competitor to the FGM family of distributions with the univariate margins of the form \((17)\).

### 6.2 Underlying copula

A bivariate distribution $F$ can be written in the form $F(x, y) = C\{F_1(x), F_2(y)\}$, where $C$ is the copula associated with $F$; see Nelsen [11] for more detail. The function $\hat{C}$ defined by $\hat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v) = \overline{C}(1 - u, 1 - v)$, is the survival copula associated with $C$ and, moreover $\overline{F}(x, y) = \hat{C}\{\overline{F}_1(x), \overline{F}_2(y)\} = \overline{C}\{F_1(x), F_2(y)\}$.

The following result shows the relationship between the copula associated with the baseline cdf $F$ and the copula of generated cdf $G_\lambda$.

**Proposition 7.** Let $C_\lambda$ be the copula of the cdf $G_\lambda$ defined by \((16)\) and let $D$ be the copula of the baseline cdf $F$. Then

$$C_\lambda(\psi(u), \psi(v)) = (1 + \lambda) \left\{ uv + D(u, v)\overline{D}(u, v) \right\} - \lambda D^2(u, v),$$

for all $0 < u, v < 1$, and $-1 \leq \lambda \leq 1$, where

$$\psi(t) = t + \lambda t(1 - t), \quad 0 < t < 1.$$
Proof. Notice that using the definition of a copula, the bivariate cdf (16) can be rewritten as

\[ C_\lambda \{ G_1(x), G_2(y) \} = (1 + \lambda) \left( F_1(x)F_2(y) + D\{ F_1(x), F_2(y) \} \right) - \lambda D^2 \{ F_1(x), F_2(y) \}, \]

where \( G_1 \) and \( G_2 \) are given by (17). By applying the transformations \( u = F_1(x) \) and \( v = F_2(y) \) on both sides of (19) we readily obtain the required result.

Remark 1. Note that if the baseline copula \( D \) is symmetric, i.e., \( D(u, v) = D(v, u) \), for all \( u, v \in (0, 1) \), then the generated copula \( C_\lambda \) defined in (18) is symmetric.

Proposition 8. The family of copulas \( \{ C_\lambda \} \) defined in (18) is positively ordered for all \(-1 < \lambda \leq 1\) and any baseline copula \( D \); i.e., \( C_{\lambda_1}(u, v) \geq C_{\lambda_2}(u, v) \) for all \( u, v \in (0, 1) \) whenever \( \lambda_1 \geq \lambda_2 \).

Proof. For any two copulas \( C_{\lambda_1} \) and \( C_{\lambda_2} \) of the form (18), one has
\[
C_{\lambda_1} \{ \psi_{\lambda_1}(u), \psi_{\lambda_1}(v) \} - C_{\lambda_2} \{ \psi_{\lambda_2}(u), \psi_{\lambda_2}(v) \} = (\lambda_1 - \lambda_2) \{ D(u, v)(1-u)(1-v) + uv(1-D(u,v)) \},
\]
where \( \psi_{\lambda}(t) = t + \lambda t(1-t), 0 < t < 1 \). Since \( D(u, v)(1-u)(1-v) + uv(1-D(u,v)) \geq 0 \) for all \( u, v \in (0, 1) \) and the function \( \psi_{\lambda}(t) = t + \lambda t(1-t) \) is increasing in \( \lambda \) for all \( t \in (0, 1) \), it is easy to see that
\[
C_{\lambda_1}(u, v) - C_{\lambda_2}(u, v) \geq 0,
\]
for all \( u, v \in (0, 1) \) and \( \lambda_1 \geq \lambda_2 \), which completes the proof.

We now consider some special cases.

Example 1. For the special case that \( D(u, v) = uv \), i.e., \( F(x, y) = F_1(x)F_2(y) \), we have
\[
C_\lambda \{ \psi(u), \psi(v) \} = uv\{ uv + (1 + \lambda)(2 - u - v) \}.
\]

Example 2. Suppose that \( D = M \), where \( M(u, v) = \min(u, v) \), is the Fréchet–Hoeffding upper bound copula (see (17)); which means that the baseline cdf \( F \) is the cdf of two perfect positive dependent random variable \( X \) and \( Y \). Since \( M(u, v) = M(1-u, 1-v) \) for every \( u, v \in (0, 1) \), it is easy to verify that \( M(u, v)M(u, v) = M(u, v) - uv \). By applying (18) to \( M \), from the fact that for non-decreasing function \( \psi \), \( \min \{ \psi(u), \psi(v) \} = \psi(\min(u, v)) \) we obtain
\[
C_\lambda(\psi(u), \psi(v)) = M(u, v) \{ 1 + \lambda(1 - M(u, v)) \} = \psi \{ M(u, v) \} = M(\psi(u), \psi(v)),
\]
that is for all \( \lambda \in (-1, 1) \),
\[
C_\lambda(u, v) = M(u, v).
\]
Thus the functional transformation (10) preserves the perfect dependence of the parent distribution.
Example 3. Suppose that $D = W$, where $W(u,v) = \max(u+v-1,0)$, is the Fréchet–Hoeffding lower bound copula (see [11]); which means that the baseline cdf $F$ is the cdf of two perfect negative dependent random variable $X$ and $Y$. It is easy to verify that $W(u,v) = 0$, for every $u,v \in (0,1)$. Thus (18) gives

$$C_\lambda(\psi(u),\psi(v)) = uv + \lambda\{uv - W^2(u,v)\}.$$ 

7 Discussion

We have introduced a method for constructing a new family of distributions from any given one. We deliberately restricted our attention to the study of some general properties of the proposed model in univariate as well as the bivariate case. The attentive reader will agree that the construction presented here leaves room for more studies beyond what accomplished in this work. In our next investigation we aim to make deeper contributions to the distribution theory connected to the bivariate case.

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