On diagonal pluriharmonic metrics of $G$-Higgs bundles

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Abstract

Let $(E, \Phi) \to (X, \omega_X)$ be a Higgs bundle over a compact Kähler manifold. We suppose that the holomorphic vector bundle $E$ decomposes into a direct sum of holomorphic line bundles. In this paper, we give the necessary and sufficient condition for the existence of a diagonal metric which is a solution to the Hermitian-Einstein equation. Our theorem can easily be generalized to $G$-Higgs bundles. We also describe the relationship between the stability condition and our condition using the torus action on the space of Higgs fields.

1 Main theorem and proof

Let $X$ be a compact connected Riemann surface of genus $g(X) \geq 2$ with the canonical bundle $K_X \to X$. We define a holomorphic vector bundle $E$ as the direct sum of holomorphic line bundles $L_1, \ldots, L_r$ over $X$ with $\deg(L_1) + \cdots + \deg(L_r) = 0$. Let $\Phi_j$ be a holomorphic section of $(L_j^{-1} L_{j+1}) \otimes K_X$ for each $j = 1, \ldots, r-1$, and $\Phi_r$ a holomorphic section of $(L_r^{-1} L_1) \otimes K_X$. We define $\Phi$ as the sum of all the $\Phi_j$, $j = 1, \ldots, r$, and it belongs to the space $H^0(\text{End} E \otimes K_X)$. The pair $(E, \Phi)$ is called a cyclic Higgs bundle [3, 6]. Suppose that $\Phi_j \neq 0$ for all $j = 1, \ldots, r$. Then $(E, \Phi)$ is stable, and thus, there uniquely exists a harmonic metric $h$ on $(E, \Phi)$. The harmonic metric $h$ splits as $h = (h_1, \ldots, h_r)$ concerning to the decomposition since the Higgs field $\Phi$ is an eigensection of the gauge transformation $g = \text{diag}(1, \omega, \ldots, \omega^{r-1})$ ($\omega = e^{2\pi \sqrt{-1}/r}$) [8]. This property of the harmonic metric makes it possible to use techniques of maximum principles when we investigate it (see [4]). The present paper aims to establish the following two results regarding Higgs bundles over compact Kähler manifolds: First, we provide a necessary and sufficient condition for the existence of a diagonal metric that solves the Hermitian-Einstein equation (see Theorem 1). Second, we establish the relationship between the condition which we give in Theorem 1 and the usual stability condition of Higgs bundles (see Proposition 6 and Proposition 7). The precise statement of Theorem 1 is as follows: Let $(E, \Phi) \to (X, \omega_X)$ be a Higgs bundle over a compact Kähler manifold. Suppose that the holomorphic vector bundle $E$ decomposes as $E = L_1 \oplus \cdots \oplus L_r$ with holomorphic line bundles $L_1, \ldots, L_r \to X$. Then the Higgs field $\Phi$ decomposes as $\Phi = \Phi_0 + \sum_{i,j=1,\ldots,r} \Phi_{i,j}$, where $\Phi_{i,j}$ is a holomorphic $(1,0)$-form with values
in $L_j^{-1}L_i$ and $\Phi_0$ is the diagonal part. For each $j = 1, \ldots, r$, let $\gamma_j$ be the degree of the holomorphic line bundle $L_j$ with respect to the Kähler form $\omega_X$. We assume that $\deg_{\omega_X}(E) = \gamma_1 + \cdots + \gamma_r = 0$ for simplicity. Let $V$ be a real vector space defined as $V := \{(x_1, \ldots, x_r) \in \mathbb{R}^r \mid x_1 + \cdots + x_r = 0\}$. For each $i, j = 1, \ldots, r$, let $v_{i,j} \in V$ be a vector defined as $v_{i,j} := u_i - u_j$, where $u_1, \ldots, u_r$ is the canonical basis of $\mathbb{R}^r$. We define a vector $\gamma \in V$ as $\gamma := (\gamma_1, \ldots, \gamma_r)$. Then the following holds:

**Theorem 1.** The Higgs bundle $(E, \Phi)$ admits a solution to the Hermitian-Einstein equation which is diagonal concerning the decomposition $E = L_1 \oplus \cdots \oplus L_r$ if and only if the following (i) and (ii) hold:

(i) The off-diagonal part of $\Lambda[\Phi \wedge \Phi^h]$ vanishes for a diagonal metric $h = (h_1, \ldots, h_r)$, where $\Lambda$ denotes the adjoint of $\omega_X \wedge$.

(ii) The following holds:

$$-\gamma \in \sum_{i,j=1,\ldots,r, i \neq j} \mathbb{R}_{>0}v_{i,j}. \quad (1)$$

In particular, a Higgs bundle satisfying (i) and (ii) is polystable.

**Remark 2.** From [10, Proposition 3.4], if the above condition (i) and (ii) hold and in addition, $c_2(E) = 0$, then the solution of the Hermitian-Einstein equation is a diagonal pluriharmonic metric. The corresponding pluriharmonic map from the universal covering space $\tilde{X}$ to $\text{SL}((r, \mathbb{C})/\text{SU}(r)$ naturally lifts to the map to $\text{SL}((r, \mathbb{C})/T$, where $T$ is the maximal torus of $\text{SU}(r)$ which consists of all diagonal matrices of $\text{SU}(r)$ (see [3, 4]).

**Remark 3.** Theorem 1 can easily be generalized to $G$-Higgs bundles. In particular, the following holds: let $G$ be a complex simple Lie group with Lie algebra $\mathfrak{g}$. Let $H \subseteq G$ be a maximal torus with Lie algebra $\mathfrak{h}$. Let $\alpha_1, \ldots, \alpha_l$ be a base of the root space $\Delta \subseteq \mathfrak{h}^*$ and $\delta$ the highest root. Let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ be the root space decomposition. Consider a $G$-Higgs bundle $(P_G, \Phi) \to (X, \omega_X)$ with holomorphic $G$-bundle $P_G$ which admits a reduction $P_H \subseteq P_G$ to a holomorphic $H$-subbundle $P_H$. Suppose that the Higgs field $\Phi$ is of the following form:

$$\Phi = \sum_{\alpha \in \{-\alpha_1, \ldots, -\alpha_l, \delta\}} \Phi_\alpha.$$  

We call $(P_G, \Phi)$ a $G$-cyclic Higgs bundle over $(X, \omega_X)$. For a $G$-cyclic Higgs bundle, if $\Phi_\alpha \neq 0$ for all $\alpha \in \{-\alpha_1, \ldots, -\alpha_l, \delta\}$, then there uniquely exists a diagonal solution to the Hermitian-Einstein equation of the $G$-cyclic Higgs bundle $(P_G, \Phi)$. This can be verified by the following observation: For each $\alpha \in \{-\alpha_1, \ldots, -\alpha_l, \delta\}$, let $h_\alpha$ be the coroot of $\alpha$. Then we can observe that

$$\sum_{\alpha \in \{-\alpha_1, \ldots, -\alpha_l, \delta\}} \mathbb{R}_{>0}h_\alpha = \sqrt{-1}t,$$  

where
where $t$ is the Lie algebra of the maximal compact torus $T \subseteq H$. Therefore, no matter which direction the vector $\gamma$ points in, we see that condition $\mathbf{11}$ holds for the $G$-cyclic Higgs bundle. This implies the claim (see also [3] Section 2).

**Remark 4.** We prove Theorem $\mathbf{11}$ by applying [3] Theorem 1. However, Theorem $\mathbf{11}$ can also be derived by showing the stability of quiver bundles $\mathbf{11}$ coincides with condition $\mathbf{11}$ in the setting we are considering. We also note that the same condition as $\mathbf{11}$ is obtained in [2] for a slightly different PDE.

We prove Theorem $\mathbf{11}$. We can easily check that condition $\mathbf{11}$ is a necessary condition for the existence of a diagonal metric which solves the Hermitian-Einstein equation. Suppose that $\mathbf{11}$ holds. We fix a diagonal metric $h = (h_1, \ldots, h_r)$ such that $\det(h) = 1$. Let $\xi := (f_1, \ldots, f_r)$ $(f_1 + \cdots + f_r = 0)$ be a pair of $\mathbb{R}$-valued functions. Then the Hermitian-Einstein equation for a metric $(e^{h_1}h_1, \ldots, e^{h_r}h_r)$ is the following:

$$
\Delta_{\omega_X} \xi + \sum_{i,j=1,\ldots,r} 4|\Phi_{i,j}|^2_{h,\omega_X} e^{(\nu_{ij} - \xi)} \nu_{ij} = -2\sqrt{-1}AF_h,
$$

where we denote by $F_h$ the curvature of the Chern connection of $h$ and by $\Delta_{\omega_X}$ the geometric Laplacian. We apply [3] Theorem 1 to equation (2). In order to apply [3] Theorem 1, we must check that for each $i, j = 1, \ldots, r$, if $\Phi_{i,j}$ is not a zero section, then $\log |\Phi_{i,j}|^2_{h,\omega_X}$ is integrable. This follows from the following lemma, which is an immediate consequence of the fact that plurisubharmonic functions are locally integrable:

**Lemma 5.** Let $U \subseteq \mathbb{C}^n$ be a domain and $f : U \to V$ a holomorphic section of a trivial bundle $V = U \times \mathbb{C}^r \to U$. Then for any smooth Hermitian metric $h_V$ on $V$, $\log |f|_{h_V} \in L^1_{\text{loc}}(U)$.

**Proof.** Let $\nu_1, \ldots, \nu_r : U \to V$ be a holomorphic frame of $V$. Then $f$ is denoted as $f = f_1\nu_1 + \cdots + f_r\nu_r$. We denote by $h_V : U \to \text{Herm}(r)$ the Hermitian matrix valued smooth function whose $(i, j)$ component is $h_V(\nu_i, \nu_j)$. Then $h_V$ is diagonalized as $\hat{B}h_V = \text{diag}(\lambda_1, \ldots, \lambda_r)$ for a unitary matrix valued function $B$ and positive functions $\lambda_1, \ldots, \lambda_r$ over $U$, where we denote by $\text{diag}(\lambda_1, \ldots, \lambda_r)$ the diagonal matrix whose diagonal entries are $\lambda_1, \ldots, \lambda_r$. We set $(f_1, \ldots, f_r) := (f_1, \ldots, f_r)B$. Let $F \subseteq U$ be a compact subset of $U$. We define a positive constant $C$ as $C := \min_{1 \leq i \leq r} \{\min_{x \in F} \lambda_1(x), \ldots, \min_{x \in F} \lambda_r(x)\}$. Then we have

$$
\log |f|^2_{h_V} = \log \{C|f_1|^2 + \cdots + C|f_r|^2\}
\geq \log \{C|f_1|^2 + \cdots + C|f_r|^2\}
= C + \log \{C|f_1|^2 + \cdots + C|f_r|^2\}
= C + \log \{|f_1|^2 + \cdots + |f_r|^2\}
$$

for each point of $F$. Since $f_1, \ldots, f_r$ are holomorphic functions, $\log \{|f_1|^2 + \cdots + |f_r|^2\}$ is a plurisubharmonic function (see [3]). In particular, it is in $L^1_{\text{loc}}(U)$. This implies the claim.


Then we prove Theorem 1.

Proof of Theorem 1. As already remarked, under the assumption of (i), the Hermitian-Einstein equation for a diagonal metric \( h = (e^{f_1}h_1, \ldots, e^{f_r}h_r) \) is equation (2). Then from [8, Theorem 1], equation (2) has a smooth solution if and only if (ii) holds. This implies the claim. \( \square \)

2 Relationship between condition (I) and the stability condition

We describe the relationship between condition (I) and the usual stability condition of Higgs bundles by using the torus action on the space of Higgs fields. Suppose that \( \gamma_1, \ldots, \gamma_r \) are all rational numbers. For each subset \( I \subseteq \{1, \ldots, r\} \), we define a subbundle \( E_I \subseteq E \) as \( E_I := \bigoplus_{i \in I} L_i \). Then the following holds:

**Proposition 6.** The following are equivalent:

(i) The following holds:

\[
-\gamma \in \sum_{i,j=1,\ldots,r, \Phi_{i,j} \neq 0} Q_{\geq 0} v_{i,j}.
\] (3)

(ii) For any subset \( I \subseteq \{1, \ldots, r\} \), if \( E_I \) is preserved by \( \Phi \), then \( \deg_{\omega_X}(E_I) \leq 0 \).

In particular, if \( (E, \Phi) \) is semistable, then \( (E, \Phi) \) satisfies (3) for any holomorphic splitting.

**Proposition 7.** The following (i) and (ii) are equivalent:

(i) The following holds:

\[
-\gamma \in \sum_{i,j=1,\ldots,r, \Phi_{i,j} \neq 0} Q_{> 0} v_{i,j}.
\] (4)

(ii) There exist subsets \( I_1, \ldots, I_k \subseteq \{1, \ldots, r\} \) such that

- The holomorphic vector bundle \( E \) is a direct sum of \( E_{I_1}, \ldots, E_{I_k} \): \( E = E_{I_1} \oplus \cdots \oplus E_{I_k} \) and for each \( j = 1, \ldots, k \), \( E_{I_j} \) is preserved by \( \Phi \).
- For each \( j = 1, \ldots, k \), \( \deg_{\omega_X}(E_{I_j}) = 0 \).
- For each \( j = 1, \ldots, k \), if there exists a subset \( I \subsetneq I_j \) and if \( E_I \) is preserved by \( \Phi \), then \( \deg_{\omega_X}(E_I) < 0 \).

In particular, if \( (E, \Phi) \) is stable, then \( (E, \Phi) \) satisfies (3) for any holomorphic splitting.
Remark 8. The choice of the holomorphic splitting of $E$ is of course not unique. For example, suppose that the rank of $E$ is 2, $E$ decomposes as $E = L_1 \oplus L_2$, and there exists a non-trivial holomorphic bundle map $f : L_1 \to L_2$. Then $E = L'_1 \oplus L_2$ is another decomposition, where $L'_1$ denotes a holomorphic line bundle defined as $L'_1 := \{(v_1, v_2) \in L_1 \oplus L_2 \mid v_2 = f(v_1)\}$.

Before starting the proof of Proposition 6 and Proposition 7, we make some preparations. Let $T$ be the maximal torus of $SU(r)$ which consists of all diagonal matrices of $SU(r)$. We denote by $t$ the Lie algebra of $T$. Let $H \subseteq SL(r, \mathbb{C})$ be the complexification of $T$ with Lie algebra $\mathfrak{h} = t \oplus \sqrt{-1}t$. We define a lattice $\mathfrak{h}_\mathbb{Z}$ of $\sqrt{-1}t$ as $\mathfrak{h}_\mathbb{Z} := \{\text{diag}(n_1, \ldots, n_r) \mid n_1, \ldots, n_r \in \mathbb{Z}, n_1 + \cdots + n_r = 0\}$. Note that $\mathfrak{h}_\mathbb{Z}$ coincides with the kernel of the exponential map $\text{Exp} : \mathfrak{h} \to H$. We regard the vector $\gamma$ and the vectors $v_{i,j}$ ($i = 1, \ldots, r$) as elements of $\mathfrak{h}_\mathbb{Q} := \mathfrak{h}_\mathbb{Z} \otimes \mathbb{Q}$ by identifying the canonical basis $u_1, \ldots, u_r$ with $\text{diag}(1, \ldots, 0, \ldots, 0)$. For each $i,j = 1, \ldots, r$, let $\alpha_{i,j}$ be the element of $\mathfrak{h}^*$ defined as $\alpha_{i,j} := \lambda_i - \lambda_j$, where $\lambda_1, \ldots, \lambda_r \in \mathfrak{h}^*$ is the dual basis of $u_1, \ldots, u_r$. For each $s \in \sqrt{-1}t$, we define a subset $A_s := \{(i,j) \mid i,j \in \{1, \ldots, r\}, \alpha_{i,j}(s) \geq 0\}$. Then we start the proof of Proposition 6 and Proposition 7.

Proof of Proposition 6. Condition (3) holds if and only if for each $z \neq 0$, the closure of $H \cdot (\Phi, z)$ does not intersects $\mathbb{C} \times \{0\}$ (see [8, Proposition 3]). Furthermore, this is equivalent to the following:

- It holds that $\gamma^\vee(s) \geq 0$ for any $s \in \sqrt{-1}t$ such that

$$\Phi \in H^0(\bigoplus_{(i,j) \in A_s} L_j^{-1}L_i \otimes \Lambda^{1,0}).$$

(5)

Let $s = \text{diag}(s_1, \ldots, s_r)$ be an element of $\sqrt{-1}t$ such that (3) holds. We may assume that $s_1 \geq \cdots \geq s_r$. Let $I_j \subseteq \{1, \ldots, r\}$ be a subset defined as $I_j := \{1, \ldots, j\}$. From (7), the Higgs field $\Phi$ preserves the sequence of subbundles $0 \subseteq E_{I_{j_1}} \subseteq E_{I_{j_2}} \subseteq \cdots \subseteq E_{I_{j_k}} \subseteq E$, where $1 \leq j_1 < \cdots < j_k < r$ are all the elements of $J_s := \{j \in \{1, \ldots, r\} \mid s_j > s_{j+1}\}$. Then $\gamma^\vee(s)$ can be
calculated as
\[
\gamma^\vee(s) = -n \text{Tr}(\gamma s) \\
= -n(\gamma_1 s_1 + \cdots + \gamma_r s_r) \\
= -n(\gamma_1(s_1 - s_2) + (\gamma_1 + \gamma_2)(s_2 - s_3) + \cdots \\
+ (\gamma_1 + \cdots + \gamma_{r-1})(s_{r-1} - s_r)) \\
= -n \sum_{j \in J_s} (s_j - s_{j+1}) \text{deg}_{\omega,X}(E_{t_j}).
\]

Therefore, it holds that $\gamma^\vee(s) \geq 0$ if and only if $\text{deg}_{\omega,X}(E_{t_j}) \leq 0$ for all $j \in J_s$. This implies the claim.

**Proof of Proposition 7.** Condition (4) holds if and only if $H \cdot (\Phi, z)$ is closed for each $z \neq 0$ (see [8, Proposition 4]). Furthermore, this is equivalent to that the following holds for all $s \in \sqrt{-1}t$ such that (3) holds:

- The element $s$ of $\sqrt{-1}t$ satisfies $\gamma^\vee(s) \geq 0$ and if $\gamma^\vee(s) = 0$, then $\Phi$ lies in $H^0(\bigoplus_{(i,j) \in A^0_s} L_i^{-1} L_i \otimes A^{1,0})$, where $A^0_s$ is defined as $A^0_s := \{(i,j) \mid i, j \in \{1, \ldots, r\}, \alpha_{i,j}(s) = 0\}$.

Then from (6), we see that (i) and (ii) are equivalent.

**Remark 9.** For the proof of Proposition 6 and Proposition 7, the author referred the definition of the stability of Higgs bundles by using parabolic subalgebras (see, for example, [7]). We refer the reader to [9] for the relationship between the condition (4) and Donaldson’s functional restricted to diagonal metrics on Higgs bundles.

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