ON GENUS ONE CURVES OF DEGREE 5 WITH SQUARE-FREE DISCRIMINANT

TOM FISHER AND MOHAMMAD SADEK

Abstract. We study genus one curves of degree 5 defined by Pfaffians. We give new formulae for the invariants, and prove the equivalence of two different definitions of minimality. As an application we show that transformations between models with square-free discriminant are necessarily integral. This result is used by Bhargava and Shankar in their work on the average ranks of elliptic curves.

1. Introduction

Let $E$ be an elliptic curve over a number field $\mathbb{K}$. An $n$-covering of $E$ is a pair $(C, \pi)$ where $C$ is a smooth curve of genus one and $\pi : C \to E$ is a morphism such that $\pi = [n] \circ \psi$ for some isomorphism $\psi : C \to E$ defined over $\mathbb{K}$. If $C$ is everywhere locally soluble then by [6, Theorem 1.3] there exists a $\mathbb{K}$-rational divisor $D$ on $C$ such that $D$ is linearly equivalent to $\psi^*([n]_E)$. The linear system $|D|$ defines a morphism $C \to \mathbb{P}^{n-1}$. If $n \geq 3$ then this morphism is an embedding, and the image is called a genus one normal curve of degree $n$. The word “normal” refers to the fact the curve is projectively normal, i.e. the homogeneous co-ordinate ring is integrally closed. This should not be confused with the fact $C$ is normal, which is automatic since $C$ is smooth.

When $n = 2, 3, 4$ the curve $C$ is represented by a binary quartic, ternary cubic, or pair of quadrics in 4 variables. In this paper we take $n = 5$, in which case $C$ is represented by data of the following form.

A Pfaffian model $\Phi$ over a ring $R$ is a $5 \times 5$ alternating matrix of linear forms in $R[x_1, \ldots, x_5]$. We write $X_5(R)$ for the space of all Pfaffian models over $R$. Two models $\Phi$ and $\Phi'$ are $R$-equivalent if $\Phi' = [A, B]\Phi$ for some $A, B \in \text{GL}_5(R)$. The action of $A$ is given by $\Phi \mapsto A\Phi A^T$, and the action of $B$ is given by

$$(\Phi_{ij}(x_1, \ldots, x_5)) \mapsto (\Phi_{ij}(x'_1, \ldots, x'_5))$$

where $x'_j = \sum_{i=1}^5 B_{ij}x_i$. We define $\det[A, B] = (\det A)^2 \det B$. The models $\Phi$ and $\Phi'$ are properly $R$-equivalent if $\det[A, B] = 1$. The invariants $c_4, c_6, \Delta \in \mathbb{Z}[X_5]$ are

\begin{align*}
c_4 &= \frac{1}{20} \sum_{i,j,k,l,m=1}^5 x_ix_jx_kx_lx_m^2 \\
c_6 &= \frac{1}{120} \sum_{i,j,k,l,m,n=1}^5 x_ix_jx_kx_lx_mx_n^3 \\
\Delta &= \text{det}([A, B])
\end{align*}
certain integer coefficient polynomials in the 50 coefficients of a Pfaffian model. We give formulae for these in Section 2.

We work over a discrete valuation field \( K \) with valuation ring \( \mathcal{O}_K \), normalised valuation \( v : K^\times \to \mathbb{Z} \), uniformiser \( \pi \), and residue field \( k = \mathcal{O}_K / \pi \mathcal{O}_K \).

Our main result is the following. It answers a question of Bhargava, and is used in the work of Bhargava and Shankar \([3, \text{Proposition 11}]\) on the average size of the 5-Selmer group of an elliptic curve.

**Theorem 1.1.** Let \( \Phi, \Phi' \in X_5(\mathcal{O}_K) \) be Pfaffian models with \( v(\Delta(\Phi)) \leq 1 \) and \( v(\Delta(\Phi')) \leq 1 \). If \( \Phi' = [A, B]\Phi \) for some \( A, B \in \text{GL}_5(K) \) then \( A, B \in K^\times \text{GL}_5(\mathcal{O}_K) \).

In particular

(i) If \( \Phi \) and \( \Phi' \) are \( K \)-equivalent then they are \( \mathcal{O}_K \)-equivalent.

(ii) The stabiliser of \( \Phi \) in \( \text{GL}_5(K) \times \text{GL}_5(K) \) is contained in the subgroup generated by \( \text{GL}_5(\mathcal{O}_K) \times \text{GL}_5(\mathcal{O}_K) \) and \( [\pi^{-1}I_5, \pi^2I_5] \).

To indicate how Theorem 1.1 is useful, we give the following global application. We take \( K = \mathbb{Q} \), but note that the result generalises immediately to any number field with class number 1. We say that a Pfaffian model \( \Phi \) has the same invariants as an elliptic curve \( E \) if the invariants \( c_4(\Phi), c_6(\Phi), \Delta(\Phi) \) are the same as the invariants \( c_4, c_6, \Delta \) of a minimal Weierstrass equation for \( E \).

**Theorem 1.2.** Let \( E/\mathbb{Q} \) be an elliptic curve with square-free minimal discriminant. Then the 5-Selmer group \( S^{(5)}(E/\mathbb{Q}) \) is in bijection with the set of Pfaffian models over \( \mathbb{Z} \) with the same invariants as \( E \), up to proper \( \mathbb{Z} \)-equivalence.

In Sections 3 and 4 we introduce two different definitions of minimality, and show that if they agree then Theorem 1.1 is a natural consequence. The agreement of the two definitions is proved in Sections 5, 6 and 7. This extends \([17, \text{Theorem 4.1}]\) from genus one curves of degrees 2, 3 and 4, to degree 5. In Section 8 we give a short alternative proof of Theorem 1.1 that is motivated by the ideas in the rest of this paper, but avoids nearly all the scheme-theoretic machinery.

2. Pfaffians and invariants

In this section we briefly describe how the equations for a genus one normal curve of degree 5 can be written in terms of Pfaffians. We then give some new formulae for the invariants of a Pfaffian model, that are simpler than the evaluation algorithms in \([9, \text{Section 8}]\).

The Pfaffian of an alternating matrix is an integer coefficient polynomial in the entries of the matrix, whose square is the determinant. We only need to consider
Pfaffians of $4 \times 4$ matrices, in which case
\[
\text{pf} \begin{pmatrix}
0 & a_{12} & a_{13} & a_{14} \\
0 & a_{23} & a_{24} & \\
0 & & a_{34} & \\
0 & & & 
\end{pmatrix} = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.
\]

If $\Phi$ is an $5 \times 5$ alternating matrix then the row vector of submaximal Pfaffians of $\Phi$ is $\text{Pf}(\Phi) = (p_1, \ldots, p_5)$ where $p_i = (-1)^i \text{pf}(\Phi^{(i)})$ and $\Phi^{(i)}$ is the matrix obtained by deleting the $i$th row and column of $\Phi$. It can be shown, for example by direct calculation, that $\text{Pf}(\Phi)\Phi = 0$, $\text{adj}(\Phi) = \text{Pf}(\Phi)^T$ and $\text{Pf}(A\Phi A^T) = \text{Pf}(\Phi)\text{adj}(A)$ for all $5 \times 5$ matrices $A$.

In this section we work over any field $K$. Let $C \subset \mathbb{P}^4_K$ be a genus one normal curve, i.e. a smooth curve of genus one embedded by a complete linear system of degree 5. Let $R = K[x_1, \ldots, x_5] = \oplus_{d \geq 0} R_d$ be the polynomial ring with its usual grading by degree. Let $R(d)$ be the graded free $R$-module of rank 1 with $R(d) = R_{d+c}$. By the Buchsbaum-Eisenbud structure theorem [4], [5], or the treatment specific to this case in [10], the coordinate ring of $C$ has minimal free resolution
\[
0 \longrightarrow R(-5) \xrightarrow{\text{Pf}(\Phi)^T} R(-3)^5 \xrightarrow{\Phi} R(-2)^5 \xrightarrow{\text{Pf}(\Phi)} R
\]
for some $\Phi \in X_5(K)$. In particular the homogeneous ideal of $C$ is generated by the $4 \times 4$ Pfaffians of $\Phi$. More generally, for any $\Phi \in X_5(K)$, we let $C_\Phi \subset \mathbb{P}^4_K$ be the subscheme defined by its $4 \times 4$ Pfaffians. We say that $\Phi$ is non-singular if $C_\Phi$ is a smooth curve of genus one. We write $K[X_5]$ for the polynomial ring in the 50 coefficients of a Pfaffian model. A polynomial $F \in K[X_5]$ is an invariant of weight $k$ if $F \circ g = (\det g)^k F$ for all $g \in \text{GL}_5 \times \text{GL}_5$.

**Theorem 2.1.** There are invariants $c_4, c_6, \Delta \in \mathbb{Z}[X_5]$ of degrees 20, 30, 60 and weights 4, 6, 12, satisfying $c_4^2 - c_6^2 = 1728\Delta$, with the following properties.

(i) If $\text{char}(K) \neq 2, 3$ then the ring of invariants in $K[X_5]$ is generated by (the images of) $c_4$ and $c_6$.

(ii) A model $\Phi \in X_5(K)$ is non-singular if and only if $\Delta(\Phi) \neq 0$.

(iii) There exist $a_1, a_2, a_3, a_4, a_6, b_2, b_4, b_6 \in \mathbb{Z}[X_5]$ satisfying

\[
\begin{align*}
b_2 &= a_1^2 + 4a_2, & b_4 &= a_1a_3 + 2a_4, & b_6 &= a_3^2 + 4a_6, \\
c_4 &= b_2^2 - 24b_4, & c_6 &= -b_2^3 + 36b_2b_4 - 216b_6,
\end{align*}
\]

such that if $\Phi \in X_5(K)$ is non-singular then $C_\Phi$ has Jacobian
\[
y^2 + a_1(\Phi)xy + a_3(\Phi)y = x^3 + a_2(\Phi)x^2 + a_4(\Phi)x + a_6(\Phi).
\]
**Proof:** This is [9, Theorem 4.4] together with [11, Theorem 1.1].

It is shown in [9, Section 5.4] that if char \( K \neq 2 \) and \( \Phi \in X_5(K) \) is non-singular then there is an invariant differential \( \omega_\Phi \) on \( C_\Phi \) given by

\[
(3) \quad \omega_\Phi = \frac{x_i^2 d(x_j/x_i)}{Q(x_1, \ldots, x_5)}, \quad \text{where } Q = \frac{\partial P}{\partial x_k} \frac{\partial \Phi}{\partial x_l} \frac{\partial P^T}{\partial x_m}.
\]

\( P = \text{Pf}(\Phi) \), and \((i, j, k, l, m)\) is any even permutation of \((1, 2, 3, 4, 5)\). In the definition of \( Q \), it is understood that by the partial derivative of a matrix we mean the matrix of partial derivatives. As we show in Remark 7.6 the restriction char \( K \neq 2 \) is not needed.

In [12, Section 7] an alternative description of the invariant differential is given in terms of a certain covariant. We now give an explicit construction of this covariant, based in part on ideas in [2, Section 4]. For \((i, j, k, l, m)\) an even permutation of \((1, 2, 3, 4, 5)\) we define

\[
\Omega_{ij} = \frac{\partial P}{\partial x_k} \frac{\partial \Phi}{\partial x_l} \frac{\partial P^T}{\partial x_m} + \frac{\partial P}{\partial x_m} \frac{\partial \Phi}{\partial x_k} \frac{\partial P^T}{\partial x_l} + \frac{\partial P}{\partial x_l} \frac{\partial \Phi}{\partial x_m} \frac{\partial P^T}{\partial x_k}.
\]

Now \( \Omega = (\Omega_{ij}) \) is an alternating matrix of quadratic forms. We define an action of \( \text{GL}_5 \times \text{GL}_5 \) on the space of such matrices via

\[
[A, B] : \Omega \mapsto B^{-T}(\Omega_{ij}(x_1', \ldots, x_5'))B^{-1}
\]

where \( x_j' = \sum_{i=1}^{5} B_{ij}x_i \). In particular the first copy of \( \text{GL}_5 \) acts trivially. Recall that for \( g = [A, B] \) we defined \( \det g = (\det A)^2 \det B \).

**Lemma 2.2.** The map \( \Phi \mapsto \Omega \) is a covariant of weight 1, in the sense that

\[
g\Phi \mapsto (\det g)g\Omega
\]

for all \( g \in \text{GL}_5 \times \text{GL}_5 \).

**Proof:** If we replace \( \Phi \) by \( A\Phi A^T \) then \( P \) is replaced by \( P \text{adj} A \) and \( \Omega \) is multiplied by \( (\det A)^2 \). So it suffices to consider \( g = [I_5, B] \) for \( B \) running over a set of generators for \( \text{GL}_5 \). Since the cases where \( B \) is a diagonal matrix or a permutation matrix are easy, this reduces us to considering \( B = I_5 + \lambda E_{12} \), where \( \lambda \in K \) and \( E_{12} \) is the elementary matrix with a 1 in position \((1, 2)\) and all other entries 0. This corresponds to the substitution \( x_2 \leftarrow x_2 + \lambda x_1 \). In the definition of \( \Omega_{ij} \) we replace \( \frac{\partial P}{\partial x_1} \) by \( \frac{\partial P}{\partial x_1} + \lambda \frac{\partial P}{\partial x_2} \) and \( \frac{\partial \Phi}{\partial x_1} \) by \( \frac{\partial \Phi}{\partial x_1} + \lambda \frac{\partial \Phi}{\partial x_2} \). This has the effect of replacing \( \Omega_{r2} \) by \( \Omega_{r2} - \lambda \Omega_{r1} \) and \( \Omega_{2r} \) by \( \Omega_{2r} - \lambda \Omega_{1r} \) for \( r = 3, 4, 5 \). A calculation, using the fact \( \Phi \) is alternating, shows that the other entries of \( \Omega \) do not change. Thus \( \Omega \) changes to \( g\Omega \) as required. \( \square \)
We put
\[ M_{ij} = \sum_{r,s=1}^{5} \partial \Omega_{ir} \partial x_{r} \partial \Omega_{js} \partial x_{s} \quad \text{and} \quad N_{ijk} = \sum_{r=1}^{5} \partial M_{ij} \partial x_{r} \partial \Omega_{rk}. \]

**Theorem 2.3.** The invariants \( c_4, c_6 \in \mathbb{Z}[X_5] \) are given by
\[
    c_4(\Phi) = \frac{1}{27340} \sum_{i,j,r,s=1}^{5} \frac{\partial^2 M_{ij}}{\partial x_{r} \partial x_{s}} \frac{\partial^2 M_{rs}}{\partial x_{i} \partial x_{j}}
\]
and
\[
    c_6(\Phi) = \frac{-1}{1036800} \sum_{i,j,r,s,t=1}^{5} \frac{\partial^3 N_{ijk}}{\partial x_{r} \partial x_{s} \partial x_{t}} \frac{\partial^3 N_{rst}}{\partial x_{i} \partial x_{j} \partial x_{k}}.
\]

**Proof:** It may be checked using Lemma 2.2 that these polynomials are invariants of degrees 20 and 30. By Theorem 2.1 it only remains to show they are scaled as specified in [9]. We can do this by computing a single numerical example. \( \square \)

We may compute the discriminant \( \Delta \) either as \((c_4^3 - c_6^2)/1728\), or directly using the method at the end of [9, Section 8].

### 3. Minimal Pfaffian models

In this section we make some remarks about minimal Pfaffian models, and more specifically those with square-free discriminant. We also explain how Theorem 1.2 follows from Theorem 1.1.

From now on \( K \) will be a discrete valuation field, with ring of integers \( \mathcal{O}_K \), and normalised valuation \( v : K^\times \to \mathbb{Z} \). We fix a uniformiser \( \pi \) and write \( k \) for the residue field. Let \( S = \text{Spec} \mathcal{O}_K \). For the proof of Theorem 1.1 we are free to replace \( K \) by any unramified extension. We may therefore assume when convenient that \( K \) is complete, and \( k \) is algebraically closed.

A Pfaffian model \( \Phi \in X_5(K) \) is integral if \( \Phi \in X_5(\mathcal{O}_K) \), i.e. it has coefficients in \( \mathcal{O}_K \). It follows from Theorem 2.1 that if \( \Phi \) is non-singular and integral then \( v(\Delta(\Phi)) = v(\Delta_E) + 12\ell(\Phi) \) where \( \Delta_E \) is the minimal discriminant of the Jacobian \( E \), and \( \ell(\Phi) \geq 0 \) is an integer called the level. We say that \( \Phi \) is minimal if \( v(\Delta(\Phi)) \) is minimal among all integral models \( K \)-equivalent to \( \Phi \). If \( \Phi' = g\Phi \) for \( g = [A,B] \) with \( A, B \in \text{GL}_5(K) \) then \( \ell(\Phi') = \ell(\Phi) + v(\det g) \).

**Theorem 3.1.** (Minimisation theorem) Let \( \Phi \in X_5(K) \) be non-singular. If \( C_\Phi(K) \neq \emptyset \) then \( \Phi \) is \( K \)-equivalent to an integral model of level 0.

**Proof:** This is [11, Theorem 2.1(i)]. \( \square \)
The proof of Theorem 3.1 is rather short. In \[11\] the first author also investigated to what extent the hypothesis \(C_\Phi(K) \neq \emptyset\) can be weakened, and gave an algorithm for minimising.

**Lemma 3.2.** Let \(\Phi \in X_5(\mathcal{O}_K)\) with \(v(\Delta(\Phi)) \leq 1\).

(i) The Jacobian \(E\) of \(C_\Phi\) has Kodaira symbol \(I_0\) or \(I_1\).

(ii) If \(K\) is a \(p\)-adic field then \(C_\Phi(K) \neq \emptyset\).

**Proof:** (i) By Theorem 2.1 we have \(v(\Delta(E)) \leq 1\). It follows by Tate’s algorithm that the Kodaira symbol is either \(I_0\) or \(I_1\).

(ii) Since \(v(\Delta(\Phi)) < 1\) we have \(\ell(\Phi) = 0\). Then by \[11\], Theorem 7.1 we have \(C_\Phi(\mathbb{Q}_p) \neq \emptyset\) where \(\mathbb{Q}_p\) is the maximal unramified extension. By (i) we know that \(E/K\) has Tamagawa number 1. Therefore, as explained in \[13\], Lemma 2.1, solubility over \(\mathbb{Q}_p\) is equivalent to solubility over \(K\).

**Remark 3.3.** To prove Theorem 1.1 it suffices to show that \(B \in K^\times \text{GL}_5(\mathcal{O}_K)\). The reason for this is as follows. By Lemma 5.2 we know that if \(\Phi\) is minimal then its \(4 \times 4\) Pfaffians are linearly independent mod \(\pi\). So if \(\Phi\) and \(\Phi' = [A, \lambda I_5] \Phi\), then from \(\text{Pf}(\Phi') = \lambda^2 \text{Pf}(\Phi) \text{adj}(A)\) we deduce that \(A \in K^\times \text{GL}_5(\mathcal{O}_K)\). The final statements (i) and (ii) of Theorem 1.1 are immediate, since \(v(\det[A,B]) = 0\) and the transformations \([\lambda I_5, \lambda^{-2} I_5]\) for \(\lambda \in K^\times\) act trivially on the space of Pfaffian models.

We now explain how Theorem 1.2 follows from Theorem 1.1.

**Theorem 3.4.** Let \(E/\mathbb{Q}\) be an elliptic curve. The 5-Selmer group \(S^{(5)}(E/\mathbb{Q})\) is in bijection with the proper \(\mathbb{Q}\)-equivalence classes of Pfaffian models \(\Phi \in X_5(\mathbb{Q})\) with the same invariants as \(E\) and \(C_\Phi(\mathbb{Q}_p) \neq \emptyset\) for all primes \(p\).

**Proof:** This is a special case of \[12\], Theorem 6.1. \(\square\)

**Proof of Theorem 1.2:** By Theorem 3.1 and strong approximation, each of the classes in Theorem 3.4 contains a model with coefficients in \(\mathbb{Z}\). Since \(\Delta_E\) is square-free, Theorem 1.1 shows that the map from proper \(\mathbb{Z}\)-equivalence classes to proper \(\mathbb{Q}\)-equivalence classes is injective. Moreover the condition \(C_\Phi(\mathbb{Q}_p) \neq \emptyset\) is automatically satisfied by Lemma 3.2. \(\square\)

Let \(\Phi \in X_5(\mathcal{O}_K)\) have reduction \(\phi \in X_5(k)\). We write \(C_\phi \subset \mathbb{P}^4\) for the \(S\)-scheme defined by the \(4 \times 4\) Pfaffians. It has generic fibre \(C_\Phi\) and special fibre \(C_\phi\).

Suppose the entries of \(\phi\) span \((x_1, \ldots, x_5)\). If \(P\) is \(k\)-point on \(C_\phi\) then by an \(\mathcal{O}_K\)-equivalence we may assume \(P = (1 : 0 : \ldots : 0)\). We may further assume \(\phi_{12} = x_1\) and all other \(\phi_{ij}\) (for \(i < j\)) are linear forms in \(x_2, \ldots, x_5\). The tangent space to \(C_\phi\) at \(P\) is \(\{\phi_{34} = \phi_{35} = \phi_{45} = 0\} \subset \mathbb{P}_k^4\).
Lemma 3.5. Let \( P \in C_\Phi \) as above. The following are equivalent.

(i) The tangent space to \( C_\Phi \) at \( P \) has dimension at most 2.

(ii) Every linear combination \( r\Phi_{34} + s\Phi_{35} + t\Phi_{45} \) (where \( r, s, t \in \mathcal{O}_K \), not all in \( \pi\mathcal{O}_K \)) that vanishes mod \( \pi \) has coefficient of \( x_1 \) not divisible by \( \pi^2 \).

Proof: By (i) we mean \( \dim(m_P/m_P^2) \leq 2 \) where \( m_P \) is the maximal ideal of the local ring at \( P \). The lemma is proved by a straightforward calculation.

The following lemma will be used both to show that \( C_\Phi \) is regular, and in the elementary proof of Theorem 1.1 in Section 8.

Lemma 3.6. If \( \Phi \in X_5(\mathcal{O}_K) \) with \( v(\Delta(\Phi)) \leq 1 \) then every \( k \)-point \( P \) on \( C_\Phi \) satisfies the conditions in Lemma 3.5.

Proof: If the entries of \( \Phi \) fail to span \( \langle x_1, \ldots, x_5 \rangle \) then \( \Phi \) is clearly not minimal and \( v(\Delta(\Phi)) \geq 12 \). Therefore an \( \mathcal{O}_K \)-equivalence brings us to the situation considered in Lemma 3.5. Let \( d \) be the dimension of the tangent space to \( C_\Phi \) at \( P \). If \( d = 1 \) we are done. If \( d \geq 3 \) we may assume \( \phi_{34} \in \langle x_5 \rangle \) and \( \phi_{35} = \phi_{45} = 0 \). Then

\[
[\text{Diag}(\pi^{1/2}, \pi^{1/2}, 1, 1, 1), \pi^{-1/2}\text{Diag}(\pi^{-1/2}, 1, 1, 1, \pi^{1/2})] \Phi
\]

has coefficients in \( \mathcal{O}_K[\pi^{1/2}] \). So in this case \( v(\Delta(\Phi)) \geq 6 \).

Now suppose \( d = 2 \). We may assume \( \phi_{34} = x_4 \), \( \phi_{35} = x_5 \) and \( \phi_{45} = 0 \). To complete the proof we show that if \( \Phi_{45} \) has coefficient of \( x_1 \) divisible by \( \pi^2 \) then \( v(\Delta(\Phi)) \geq 2 \). Checking this directly, using the formulae for the invariants in Section 2 is unfortunately not practical. Instead we argue as follows. By making substitutions of the form \( x_4 \leftarrow x_4 + \lambda x_1 \) and \( x_5 \leftarrow x_5 + \mu x_1 \) for suitable \( \lambda, \mu \in \pi\mathcal{O}_K \) we may arrange that \( \Phi_{34} \) and \( \Phi_{35} \) also have their coefficients of \( x_1 \) divisible by \( \pi^2 \). Then substituting for \( x_1 \) we have

\[
\Phi = \begin{pmatrix}
0 & x_1 & \alpha_1 & \alpha_2 & \alpha_3 \\
0 & \beta_1 & \beta_2 & \beta_3 \\
0 & \ell_3 & -\ell_2 \\
- & 0 & \ell_1 \\
0 & & & & \\
\end{pmatrix}
\]

where \( \ell_1 \equiv 0 \pmod{\pi} \), the coefficient of \( x_1 \) in each of the \( \alpha_i \) and \( \beta_i \) vanishes mod \( \pi \), and the coefficient of \( x_1 \) in each of the \( \ell_i \) vanishes mod \( \pi^2 \). By subtracting suitable multiples of the first two rows/columns from the last three rows/columns we may further assume that the coefficient of \( x_1 \) in each of the \( \alpha_i \) and \( \beta_i \) vanishes mod \( \pi^2 \). Since it only matters what the coefficients are mod \( \pi^2 \), we may now
assume that none of the \( \alpha_i, \beta_i \) and \( \ell_i \) involve \( x_1 \). By \cite{11} Lemma 2.4, \( \Phi \) has the same discriminant as the quadric intersection

\[
\ell_1 \alpha_1 + \ell_2 \alpha_2 + \ell_3 \alpha_3 = 0 \\
\ell_1 \beta_1 + \ell_2 \beta_2 + \ell_3 \beta_3 = 0.
\]

Since \( \ell_1 \equiv 0 \pmod{\pi} \), the reduction of this quadric intersection mod \( \pi \) contains a line. It can then be checked (for example by a brute force calculation) that the discriminant vanishes mod \( \pi^2 \). This completes the proof. \( \square \)

4. Geometric minimality and an application

In this section we define the notion of \emph{geometric minimality} and explain the role it has to play in the proof of Theorem \cite{11}. We assume from now on that the residue field \( k \) is algebraically closed. Following \cite{15} Definition 8.3.1 we have

\textbf{Definition 4.1.} A fibred surface \( C/S \) is an integral projective flat \( S \)-scheme of dimension 2.

\textbf{Lemma 4.2.} Let \( C \subset \mathbb{P}^{n-1}_K \) be a smooth projective curve and \( C \) its closure in \( \mathbb{P}^{n-1}_S \). Then \( C \) is a fibred surface. Moreover \( C \) is normal if and only if

(i) \( C \) is Cohen-Macaulay, and
(ii) there are only finitely many non-regular points on the special fibre.

\textbf{Proof:} The coordinate ring of \( C \) is a subring of that of \( C \). Since \( C \) is integral it follows that \( C \) is integral. Then \( C \to S \) is flat and \( \dim C = 2 \) by \cite{15} Corollaries 4.3.10 and 4.3.14. By definition \( C \) is projective. Since \( \dim C = 2 \) and the generic fibre is smooth, (i) and (ii) are equivalent to the conditions \((S_2)\) and \((R_1)\) in Serre’s criterion \cite{15} Theorem 8.2.23). \( \square \)

Let \( C/S \) be a fibred surface. Lipman \cite{11} showed that if \( K \) is complete then \( C \) admits a desingularisation (i.e. resolution of singularities). If \( C \) has smooth generic fibre then the hypothesis that \( K \) is complete may be removed, as described in \cite{15} Corollary 8.3.51. If in addition \( C \) is normal then by \cite{15} Proposition 9.3.32 it admits a minimal desingularisation.

\textbf{Definition 4.3.} Let \( C \subset \mathbb{P}^{n-1}_K \) be a genus one normal curve of degree \( n \), with Jacobian \( E \). Let \( C \) be the closure of \( C \) in \( \mathbb{P}^{n-1}_S \). We say that \( C \) is \emph{geometrically minimal} if \( C \) is normal, and the minimal desingularisation of \( C \) is isomorphic (as an \( S \)-scheme) to the minimal proper regular model of \( E \).
This definition is not invariant under changes of co-ordinates defined over $K$. We remark that if $C$ is geometrically minimal then $C(K) \neq \emptyset$, and $C$ is obtained from the minimal proper regular model of $E$ by contracting some of the irreducible components of the special fibre.

Before explaining how geometric minimality is used in the proof of Theorem 4.1, we quote the following lemma.

**Lemma 4.4.** Let $C$ be a projective $S$-scheme, and $L$ an invertible sheaf on $C$.

(i) The natural map $H^0(C, L) \otimes_{\mathcal{O}_K} K \rightarrow H^0(C_K, L_K)$ is an isomorphism.

(ii) We have the inequality $\dim_k H^0(C_k, L_k) \geq \dim_K H^0(C, L_K)$.

(iii) If equality holds in (ii) then $H^0(C, L)$ is a free $\mathcal{O}_K$-module and the natural map $H^0(C, L) \otimes_{\mathcal{O}_K} k \rightarrow H^0(C_k, L_k)$ is an isomorphism.

**Proof:** Part (i) is [15, Corollary 5.2.27] with $A = \mathcal{O}_K$ and $B = K$. The rest is [15, Lemma 5.2.31 and Theorem 5.3.20].

**Theorem 4.5.** Let $C_1 \subset \mathbb{P}^{n-1}_K$ and $C_2 \subset \mathbb{P}^{n-1}_K$ be genus one normal curves of degree $n$. Suppose that $C_1$ and $C_2$ are isomorphic via a change of coordinates given by $B \in \text{GL}_n(K)$. If $C_1$ and $C_2$ are geometrically minimal, and their Jacobian $E$ has Kodaira symbol $I_0$ or $I_1$, then $B \in K^\times \text{GL}_n(\mathcal{O}_K)$.

**Proof:** Since the Jacobian $E$ has Kodaira symbol $I_0$ or $I_1$ the special fibre of $E$ (the minimal proper regular model of $E$) is either a smooth curve of genus one, or a rational curve with a node. Let $C_i$ be the closure of $C_i$ in $\mathbb{P}^{n-1}_S$. Then $C_i$ is obtained from $E$ by contracting some of the irreducible components of the special fibre. Since $E_k$ is irreducible and $C_{i,k}$ is a curve it follows that $C_i \cong E$. We now write $f_i : E \rightarrow \mathbb{P}^{n-1}_S$ for the embedding with image $C_i$ and let $L_i = f_i^*\mathcal{O}(1)$.

Since $C_i = C_{i,K}$ is a genus one curve of degree $n$ we have $\dim_K H^0(E, L_{i,K}) = n$. Since $C_{i,k}$ is either a genus one curve or a rational curve with a node, and it has degree $n$ by [14, Chapter III, Corollary 9.10], we have $\dim_k H^0(E_k, L_{i,k}) = n$. Then Lemma 4.4 shows that $H^0(E, L_i) \cong \mathcal{O}_K^\times$. Our choice of co-ordinates on $\mathbb{P}^{n-1}_S$ corresponds to a choice of bases for $H^0(E, L_1)$ and $H^0(E, L_2)$. By hypothesis $L_{1,K} \cong L_{2,K}$, and the isomorphism $H^0(E, L_{1,K}) \cong H^0(E, L_{2,K})$ is given, relative to our chosen bases, by some scalar multiple of $B$.

Let $L = L_1 \otimes L_2^{-1}$. By Lemma 4.4(ii) both $L_k$ and its dual $L_k^{-1}$ have non-zero global sections. Since $E_k$ is irreducible it follows that $L_k$ is trivial. Then by Lemma 4.4(iii) both $L$ and $L^{-1}$ have global sections that are nowhere vanishing on the special fibre. Therefore $L$ is trivial and so $L_1 \cong L_2$. Taking global sections gives an isomorphism of $\mathcal{O}_K$-modules $H^0(E, L_1) \cong H^0(E, L_2)$. This isomorphism
is again given, relative to our chosen bases, by a scalar multiple of \( B \). It follows that \( B \in K^{\times} \text{GL}_n(\mathcal{O}_K) \). 

**Proof of Theorem 1.1:** We saw in Lemma 3.2(i) that for \( \Phi \in X_5(\mathcal{O}_K) \) with \( v(\Delta(\Phi)) \leq 1 \), the Jacobian of \( C_\Phi \) has Kodaira symbol \( I_0 \) or \( I_1 \). We are free to replace \( K \) by an unramified extension. So by [11, Theorem 7.1] we may assume that \( C_\Phi(K) \neq \emptyset \) and likewise for \( \Phi' \). In Sections 5, 6 and 7 we show that, since \( \Phi \) and \( \Phi' \) are minimal, \( C_\Phi \) and \( C_{\Phi'} \) are geometrically minimal. Theorem 4.5 then shows that \( B \in K^{\times} \text{GL}_5(\mathcal{O}_K) \) and we are done by Remark 3.3. 

\[ \blacksquare \]

5. **Minimal Pfaffian models are flat**

Let \( \Phi \in X_5(\mathcal{O}_K) \) with reduction \( \phi \in X_5(k) \). In this section we show that if \( \Phi \) is minimal then \( C_\phi \) is a fibred surface.

**Lemma 5.1.** If \( \Phi \in X_5(\mathcal{O}_K) \) is non-singular then the following are equivalent.

(i) \( C_\Phi \) is the closure of \( C_\Phi \) in \( \mathbb{P}^4_k \).

(ii) \( C_\Phi \) is a fibred surface.

(iii) \( C_\phi \) is a curve.

**Proof:** (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii). See Lemma [12] and [15] Corollary 4.3.14.

(iii) \( \Rightarrow \) (i). Let \( R = k[x_1, \ldots, x_5] \). With notation as in Section 2 there is a complex of graded free \( R \)-modules

\[ 0 \longrightarrow R(-5) \xrightarrow{\text{Pf}(\phi^T)} R(-3)^5 \xrightarrow{\phi} R(-2)^5 \xrightarrow{\text{Pf}(\phi)} R. \]

Since \( C_\phi \) is a curve, this complex is exact by the Buchsbaum-Eisenbud acyclicity criterion [8, Theorem 20.9].

Let \( \text{Pf}(\Phi) = (p_1, \ldots, p_5) \). Let \( \mathcal{I} \) be the ideal in \( \mathcal{R} = \mathcal{O}_K[x_1, \ldots, x_5] \) generated by \( p_1, \ldots, p_5 \). We must show that if \( f \in \mathcal{R} \) and \( \pi f \in \mathcal{I} \) then \( f \in \mathcal{I} \). We write \( \pi f = \sum_{i=1}^5 f_ip_i \) for some \( f_i, \ldots, f_5 \in \mathcal{R} \). Then \( \sum_{i=1}^5 f_ip_i \equiv 0 \) (mod \( \pi \)). Since (4) is exact it follows that \( f_i = \pi g_i + \sum_{j=1}^5 \Phi_{ij}h_j \) for some \( g_1, \ldots, g_5, h_1, \ldots, h_5 \in \mathcal{R} \). Then \( \pi f = \sum_{i=1}^5 f_ip_i = \pi \sum_{i=1}^5 g_ip_i \) and so \( f \in \mathcal{I} \) as required. 

\[ \blacksquare \]

**Lemma 5.2.** If \( \Phi \in X_5(\mathcal{O}_K) \) is minimal then

(i) The \( 4 \times 4 \) Pfaffians of \( \phi \) are linearly independent.

(ii) The subscheme \( C_\phi \subset \mathbb{P}^4_k \) does not contain a plane.

(iii) The entries of \( \phi \) span \( \langle x_1, \ldots, x_5 \rangle \).

**Proof:** This is [11] Lemma 7.8. 

\[ \blacksquare \]
Lemma 5.3. If $\phi \in X_5(k)$ satisfies conditions (i) and (ii) in Lemma 5.2 then $C_\phi$ is a curve.

Proof: By [9, Lemma 5.8] every irreducible component of $C_\phi$ has dimension at least 1. We must show there are no components of dimension 2 or more. Let $\text{Sing} C_\phi$ be the set of points of $C_\phi$ with tangent space of dimension at least 2. This contains all components of $C_\phi$ of dimension 2 or more. If $\text{Sing} C_\phi$ is contained in a line then we are done. So suppose $P_1, P_2, P_3 \in \text{Sing} C_\phi$ span a plane $\Pi$. If $C_\phi$ contains each of the lines $P_iP_j$ then it must contain $\Pi$, since $C_\phi$ is defined by quadrics. But this is impossible by (ii). We may therefore suppose $P_1P_2 \not\subset C_\phi$.

A change of co-ordinates gives $P_1 = (1 : 0 : \ldots : 0)$ and $P_2 = (0 : 1 : \ldots : 0)$. If we write $\phi = \sum x_i M_i$ then $M_1$ and $M_2$ have rank 2, but their sum has rank 4. Therefore $\phi$ is equivalent to a model with $\phi_{12} = x_1$, $\phi_{34} = x_2$ and all other $\phi_{ij}$ (for $i < j$) linear forms in $x_3, x_4, x_5$. Since $P_1, P_2 \in \text{Sing} C_\phi$ it follows that $\phi_{35}$ and $\phi_{45}$ are linearly dependent, and $\phi_{15}$ and $\phi_{25}$ are linearly dependent. Therefore the space of linear forms spanned by the entries of the last row/column of $\phi$ has dimension at most 2. Replacing $\phi$ by a $k$-equivalent model brings us to the case

$$\phi = \begin{pmatrix} 0 & \xi & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & \beta_1 & \beta_2 & \beta_3 \\ 0 & \eta & 0 \\ - & 0 & 0 \\ 0 \end{pmatrix}$$

where $\xi, \eta, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ are linear forms in $x_1, \ldots, x_5$. By (i) the linear forms $\alpha_3$ and $\beta_3$ are linearly independent, and $\eta \neq 0$. Therefore $C_\phi$ is the union of

$$\Gamma_2 = \{ \alpha_3 = \beta_3 = \xi \eta - \alpha_1 \beta_2 + \alpha_2 \beta_1 = 0 \}$$

and

$$\Gamma_3 = \left\{ \text{rank } \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{pmatrix} \leq 1 \right\} \cap \{ \eta = 0 \}.$$ 

We may think of $\Gamma_2$ as a degenerate conic, and $\Gamma_3$ as a degenerate twisted cubic. It remains to show that these degenerations are still curves. In the case of $\Gamma_2$ this is clear by (ii). In the case of $\Gamma_3$ we use the following lemma. The conditions of the lemma are satisfied by (i) and (ii).

Lemma 5.4. Let $\psi$ be a $2 \times 3$ matrix of linear forms in $x_1, \ldots, x_4$. Let $\Gamma_3 \subset \mathbb{P}^3$ be defined by $\text{rank } \psi \leq 1$. Suppose that
(i) The $2 \times 2$ minors of $\psi$ span a vector space of dimension at least 2.
(ii) The subscheme $\Gamma_3 \subset \mathbb{P}^3$ does not contain a plane.

Then $\Gamma_3$ is a curve.

**Proof:** Since $\Gamma_3$ is defined by quadrics, any irreducible component of dimension 2 would have degree 1 or 2. These possibilities are ruled out by (ii) and (i). \qed

**Theorem 5.5.** If $\Phi \in X_5(O_K)$ is minimal then $C_\Phi$ is a fibred surface.

**Proof:** This is immediate from Lemmas 5.1, 5.2 and 5.3. \qed

### 6. Minimal Pfaffian models are normal

We have seen that if $\Phi \in X_5(O_K)$ is minimal then $C_\Phi$ is a fibred surface. In this section we show that $C_\Phi$ is regular, and hence normal. To treat the general case we check the conditions in Lemma 4.2.

**Lemma 6.1.** If $\Phi \in X_5(O_K)$ is minimal then

(i) $C_\Phi$ is a local complete intersection,

(ii) $C_\Phi$ is Cohen-Macaulay.

**Proof:** (i) Since $C_\Phi \subset \mathbb{P}^4_S$ has codimension 3 we must show it is locally defined by 3 equations. Let $\text{Pf}(\Phi) = (p_1, \ldots, p_5)$. Since $\Phi$ is alternating, the relations $\sum_{i=1}^5 p_i \Phi_{ij} = 0$ for $j = 4, 5$, show that the intersection $C_\Phi \cap \{\Phi_{45} \neq 0\}$ is defined by $p_1 = p_2 = p_3 = 0$. By Lemma 5.2(iii) the affine pieces $\{\Phi_{ij} \neq 0\}$ cover $\mathbb{P}^4_S$.

(ii) This follows from (i) and [15, Corollary 8.2.18]. \qed

We prepare to check the second condition in Lemma 4.2. Recall that we assume $k$ is algebraically closed.

**Lemma 6.2.** Let $\phi \in X_5(k)$ satisfy the conditions of Lemma 5.2. Suppose $C_\phi$ has a multiple component $\Gamma$. Then after replacing $\phi$ by a $k$-equivalent model, we are in one of the following cases

\[
(\text{i}) \begin{pmatrix} 0 & x_1 & x_2 & * & * \\
0 & * & * & 0 \\
0 & * & 0 \\
\vdots & 0 & x_5 \\
0 & & & & 0 \end{pmatrix} \quad (\text{ii}) \begin{pmatrix} 0 & x_1 & x_2 & x_3 & 0 \\
0 & x_3 & x_4 & 0 \\
0 & 0 & x_4 \\
\vdots & 0 & x_5 \\
0 & & & & 0 \end{pmatrix}
\]
where the entries * are linear forms in $x_3, x_4, x_5$. Moreover $\Gamma = \{x_3 = x_4 = x_5 = 0\}$ in cases (i), (ii), (iii), and $\Gamma = \{x_1 x_3 - x_2^2 = x_4 = x_5 = 0\}$ in case (iv).

**Proof:** Lemma 5.3 shows that $C_\phi$ is a curve and so the complex (H) is exact. From this minimal free resolution we compute that $C_\phi$ has Hilbert polynomial

$$h(t) = \binom{t+4}{4} - 5\binom{t+2}{4} + 5\binom{t+1}{4} - \binom{t-1}{4} = 5t.$$  

In particular $C_\phi \subset \mathbb{P}^4$ has degree 5. The multiple component $\Gamma$ must therefore be a line or a conic.

Case $\Gamma$ is a line. We may assume $\Gamma = \{x_3 = x_4 = x_5 = 0\}$. Then $\phi = \sum x_i M_i$ where all linear combinations of $M_1$ and $M_2$ have rank at most 2. By hypothesis, $M_1, \ldots, M_5$ are linearly independent. So we are either in case (iii), or $\phi$ takes the form

\[
\begin{pmatrix}
0 & x_1 & x_2 & \ast & \ast \\
0 & \ast & \ast & \ast & \ast \\
0 & \ast & \ast & \ast & \ast \\
- & 0 & 0 & 0 & 0
\end{pmatrix}
\]

where the entries $\alpha, \beta, \gamma, \delta$ and * are linear forms in $x_3, x_4, x_5$. By row and column operations (and substitutions for $x_1$ and $x_2$) we may suppose $\alpha, \beta, \gamma, \delta$ do not involve $x_5$. We write $\alpha = \alpha_3 x_3 + \alpha_4 x_4$ and likewise for $\beta, \gamma, \delta$. As shown in (\text{[II]} Section 4], $\Gamma$ is a multiple component if and only if the determinant of

\[
\begin{pmatrix}
\gamma_3 s - \alpha_3 t & \gamma_4 s - \alpha_4 t \\
\delta_3 s - \beta_3 t & \delta_4 s - \beta_4 t
\end{pmatrix}
\]

vanishes as a polynomial in $s$ and $t$. If the rows of this matrix are linearly dependent (over $k$) then we may reduce to case (i). Otherwise the columns are linearly dependent, and we may reduce to the case $\alpha_3 = \beta_3 = \gamma_3 = \delta_3 = 0$ yet $\alpha_4 \delta_4 - \beta_4 \gamma_4 \neq 0$. Since $C_\phi$ does not contain the plane $\{x_4 = x_5 = 0\}$ it follows
that \( \phi_{23} \), and at least one of \( \phi_{14} \) and \( \phi_{15} \), involves \( x_3 \). By a substitution for \( x_3 \) we may assume \( \phi_{23} = x_3 \). By row and columns operations (and substitutions for \( x_1 \) and \( x_2 \)) we may assume \( \phi_{14} \) and \( \phi_{15} \) are multiples of \( x_3 \). Replacing the 4th and 5th rows/columns by suitable linear combinations, and likewise for the 2nd and 3rd rows/columns, brings us to case (ii).

Case \( \Gamma \) is a conic. We may assume \( \Gamma = \{ x_1x_3 - x_2^2 = x_4 = x_5 = 0 \} \). Let \( \text{Pf}(\phi) = (p_1, \ldots, p_5) \). Replacing \( \phi \) by an equivalent model we have \( p_i(x_1, x_2, x_3, 0, 0) = 0 \) for \( i = 1, 2, 3, 4 \) and \( p_5(x_1, x_2, x_3, 0, 0) = x_1x_3 - x_2^2 \). Since \( \text{Pf}(\phi) \phi = 0 \) and \( C_\phi \) is a curve, we may suppose the last row/column of \( \phi \) has entries \( x_4, x_5, 0, 0, 0 \). As shown in [11, Section 4], \( \Gamma \) is a multiple component if and only if \( \phi_{34}(x_1, x_2, x_3, 0, 0) = 0 \). In this case \( \phi \) is equivalent to a model of the form

\[
\begin{pmatrix}
0 & \xi & x_1 + \langle x_4, x_5 \rangle & x_2 + \langle x_4, x_5 \rangle & x_4 \\
0 & x_2 + \langle x_4, x_5 \rangle & x_3 + \langle x_4, x_5 \rangle & \langle x_4, x_5 \rangle & x_5 \\
0 & \langle x_4, x_5 \rangle & 0 \\
- & 0 & 0 & 0
\end{pmatrix}
\]

where each \( \langle x_4, x_5 \rangle \) denotes some linear combination of \( x_4 \) and \( x_5 \). Subtracting multiples of the last three rows/columns from the first two rows/columns we may suppose \( \xi = 0 \). Since the \( 4 \times 4 \) Pfaffians of \( \phi \) are linearly independent we cannot have \( \phi_{34} = 0 \). So making substitutions for \( x_4 \) and \( x_5 \) brings us to the case

\[
\begin{pmatrix}
0 & 0 & x_1 + \langle x_4, x_5 \rangle & x_2 + \langle x_4, x_5 \rangle & \langle x_4, x_5 \rangle \\
0 & x_2 + \langle x_4, x_5 \rangle & x_3 + \langle x_4, x_5 \rangle & \langle x_4, x_5 \rangle & x_5 \\
0 & \langle x_4, x_5 \rangle & 0 & x_5 & 0 \\
- & 0 & 0 & 0 & 0
\end{pmatrix}
\]

If \( P \in \text{GL}_2(k) \) then

\[
P \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} P^T = \begin{pmatrix} x'_1 & x'_2 \\ x'_2 & x'_3 \end{pmatrix}
\]

where \( x'_1, x'_2, x'_3 \) are linear combinations of \( x_1, x_2, x_3 \). Acting on \( \phi \) by a matrix of the form \( \text{Diag}(P, P, 1) \) we may therefore reduce to the case \( \phi_{15} = x_4 \) and \( \phi_{25} = x_5 \). Subtracting multiples of the 5th row/column from the 3rd and 4th rows/columns...
we may assume \( \phi_{14} = \phi_{23} \). Then making substitutions for \( x_1, x_2, x_3 \) brings us to case (iv).

The following lemma and its proof could also be used to extend the algorithms for testing local solubility in [13] to genus one curves of degree 5.

**Lemma 6.3.** If \( \Phi \in X_5(\mathcal{O}_K) \) is minimal then each multiple component \( \Gamma \) of the special fibre \( C_\phi \) has at most three non-regular points.

**Proof:** We split into the four cases in Lemma 6.2.

(i) We put

\[
\Phi_{25} \equiv \pi(\alpha_1 x_1 + \alpha_2 x_2 + \ldots) \pmod{\pi^2}
\]

\[
\Phi_{35} \equiv \pi(\beta_1 x_1 + \beta_2 x_2 + \ldots) \pmod{\pi^2}
\]

where \( \alpha_1, \alpha_2, \beta_1, \beta_2 \in k \). We find that \((s : t : 0 : 0 : 0) \in \Gamma \) is a non-regular point if and only if the linear form

\[
\beta_1 s^2 + (\alpha_1 - \beta_2)st - \alpha_2 t^2 = 0.
\]

If \( \phi_{24} = \phi_{34} = 0 \) then \( C_\phi \) is not a curve. If the quadratic form in \( s \) and \( t \) vanishes identically then, after subtracting a multiple of the 1st row/column from the 5th row/column, we may assume \( \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0 \). Since \( \phi_{45} = x_5 \) we may assume by a substitution for \( x_5 \) that \( \Phi_{45} = x_5 \). Then the transformation

\[
[\text{Diag}(\pi, 1, 1, 1, \pi^{-1}), \pi^{-1} \text{Diag}(1, 1, \pi, \pi, \pi^2)]
\]

shows that \( \Phi \) is not minimal.

(ii) We put

\[
\Phi_{24} \equiv x_4 + \pi(\alpha_1 x_1 + \alpha_2 x_2 + \ldots) \pmod{\pi^2}
\]

\[
\Phi_{25} \equiv \pi(\beta_1 x_1 + \beta_2 x_2 + \ldots) \pmod{\pi^2}
\]

\[
\Phi_{34} \equiv \pi(\gamma_1 x_1 + \gamma_2 x_2 + \ldots) \pmod{\pi^2}
\]

\[
\Phi_{35} \equiv x_4 + \pi(\delta_1 x_1 + \delta_2 x_2 + \ldots) \pmod{\pi^2}
\]

We find that \((s : t : 0 : 0 : 0) \in \Gamma \) is a non-regular point if and only if

\[
\gamma_1 s^3 + (\alpha_1 - \gamma_2 - \delta_1)s^2 t - (\alpha_2 + \beta_1 - \delta_2)st^2 + \beta_2 t^3 = 0.
\]

Making a substitution for \( x_4 \) we may assume \( \alpha_2 = \delta_1 = 0 \). Subtracting a multiple of the 1st row/column from the 5th row/column we may assume \( \alpha_1 = \delta_2 = 0 \). If the cubic form in \( s \) and \( t \) vanishes identically then \( \beta_1 = \beta_2 = \gamma_1 = \gamma_2 = 0 \). Since \( \phi_{45} = x_5 \) we may assume by a substitution for \( x_5 \) that \( \Phi_{45} = x_5 \). Then the transformation

\[
[\text{Diag}(\pi, 1, 1, \pi^{-1}, \pi^{-1}), \pi^{-1} \text{Diag}(1, 1, \pi, \pi^2, \pi^3)]
\]
shows that $\Phi$ is not minimal.

(iii) We put

$$\Phi_{45} \equiv \pi(\alpha_1 x_1 + \alpha_2 x_2 + \ldots) \pmod{\pi^2}.$$  

We find that $(s : t : 0 : 0 : 0) \in \Gamma$ is a non-regular point if and only if $s\phi_{34} - t\phi_{24}$ and $s\phi_{35} - t\phi_{25}$ are linearly dependent, or $\alpha_1 s + \alpha_2 t = 0$. If the first of these possibilities is true for all $s$ and $t$, then $C_\phi$ is not a curve. If $\alpha_1 = \alpha_2 = 0$ then the transformation

$$[\text{Diag}(1, 1, 1, \pi^{-1}, \pi^{-1}), \text{Diag}(1, 1, \pi, \pi, \pi)]$$

shows that $\Phi$ is not minimal.

(iv) We put

$$\Phi_{35} \equiv \pi(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \ldots) \pmod{\pi^2},$$

$$\Phi_{45} \equiv \pi(\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \ldots) \pmod{\pi^2}.$$

We find that $(s^2 : st : t^2 : 0 : 0) \in \Gamma$ is a non-regular point if and only if

$$\beta_1 s^3 + (\beta_2 - \alpha_1)s^2 t + (\beta_3 - \alpha_2)st^2 - \alpha_3 t^3 = 0.$$  

Subtracting a multiple of the first two rows/columns from the last row/column we may assume $\alpha_1 = \alpha_2 = 0$. If the cubic form in $s$ and $t$ vanishes identically then $\alpha_i = \beta_i = 0$ for $i = 1, 2, 3$. Then the transformation

$$[\text{Diag}(\pi, \pi, 1, \pi^{-1}, \pi^{-1}), \pi^{-1} \text{Diag}(1, 1, 1, \pi, \pi)]$$

shows that $\Phi$ is not minimal.

\[\square\]

**Theorem 6.4.** If $\Phi \in X_5(\mathcal{O}_K)$ is minimal then $C_\Phi$ is a normal fibred surface.

**Proof:** In Section 5 we showed that $C_\Phi \subset \mathbb{P}^4_S$ is the closure of $C_\Phi$ and hence a fibred surface. The conditions for normality in Lemma 4.2 were checked in Lemmas 6.1 and 6.3. \[\square\]

7. **Minimal Pfaffian models are geometrically minimal**

In this section we show that if $\Phi \in X_5(\mathcal{O}_K)$ is minimal and $C_\Phi(K) \neq \emptyset$ then $C_\Phi \subset \mathbb{P}^4_K$ is geometrically minimal. This extends [17, Theorem 4.1] from genus one curves of degrees 2, 3 and 4, to degree 5, and could also be used to prove results analogous to those in [16].

**Definition 7.1.** Let $E/K$ be an elliptic curve with minimal Weierstrass equation $y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$. Then

$$\omega_E = \frac{dx}{2y + a_1 x + a_3} \in H^0(K, \Omega^1_E)$$
is called a \textit{Néron differential} on $E$. It is uniquely determined up to multiplication by elements of $\mathcal{O}_K^\times$.

Let $\mathcal{C}$ be a fibred surface over $S = \text{Spec} \, \mathcal{O}_K$. If $\mathcal{C} / S$ is a local complete intersection then we can define the canonical sheaf $\omega_{\mathcal{C} / S}$ as in \cite[Definition 6.4.7]{15}. This is an invertible sheaf on $\mathcal{C}$. If $\mathcal{C}$ has generic fibre $E$ then $H^0(\mathcal{C}, \omega_{\mathcal{C} / S})$ is a sub-$\mathcal{O}_K$-module of the 1-dimensional $K$-vector space $H^0(E, \Omega^1_{E/K})$.

The following theorem and its proof is closely related to \cite[Theorem 9.4.35]{15}. See also \cite{7}.

\textbf{Theorem 7.2.} Let $E/K$ be an elliptic curve, with minimal proper regular model $\mathcal{E} / S$. Let $\mathcal{C} / S$ be a normal fibred surface with generic fibre isomorphic to $E$ and minimal desingularisation $\tilde{\mathcal{C}}$. Suppose $\mathcal{C}$ is a local complete intersection and $\omega_{\mathcal{C} / S} = \omega_{\mathcal{O}_C}$ for some $\omega \in H^0(\mathcal{E}, \Omega^1_{E/K})$. The following are equivalent.

(i) We have $\omega_{\mathcal{C} / S} = \omega_E \mathcal{O}_C$ where $\omega_E$ is a Néron differential on $E$.

(ii) The morphism $\tilde{\mathcal{C}} \to \mathcal{E}$ (which exists by definition of $\mathcal{E}$) is an isomorphism.

\textbf{Proof:} (i) $\Rightarrow$ (ii). Let $f : \tilde{\mathcal{C}} \to \mathcal{E}$ be the morphism in (ii) and $g : \tilde{\mathcal{C}} \to \mathcal{C}$ the minimal desingularisation. We are assuming that $\omega_{\mathcal{C} / S} = \omega_E \mathcal{O}_C$, whereas \cite[Theorem 9.4.35]{15} gives that $\omega_{\mathcal{E} / S} = \omega_E \mathcal{O}_E$. Therefore
\begin{equation}
(5) \quad f^* \omega_{\mathcal{E} / S} \cong \omega_E \mathcal{O}_E \cong g^* \omega_{\mathcal{C} / S}.
\end{equation}

Let $\Gamma$ be an exceptional divisor (or $(-1)$-curve) on $\tilde{\mathcal{C}}$. Since the desingularisation $g : \tilde{\mathcal{C}} \to \mathcal{C}$ is minimal, it does not contract $\Gamma$. Therefore
\begin{equation}
\omega_{\tilde{\mathcal{C}} / S} |_{\Gamma} \cong g^* \omega_{\mathcal{C} / S} |_{\Gamma}
\end{equation}

By \cite[Corollary 9.3.27]{15} we know that $\omega_{\mathcal{E} / S}$ is globally free. Therefore each of the sheaves in (5) is globally free. Writing $K_{\tilde{\mathcal{C}} / S}$ for a canonical divisor on $\tilde{\mathcal{C}} / S$ we have
\begin{equation}
K_{\tilde{\mathcal{C}} / S} \cdot \Gamma = \deg(\omega_{\tilde{\mathcal{C}} / S}) = \deg(g^* \omega_{\mathcal{C} / S} |_{\Gamma}) = 0.
\end{equation}

On the other hand \cite[Proposition 9.3.10]{15} shows that $K_{\tilde{\mathcal{C}} / S} \cdot \Gamma < 0$. This is a contradiction. We deduce that $\tilde{\mathcal{C}}$ has no exceptional divisors. It follows by the factorisation theorem \cite[Theorem 9.2.2]{15} that $f : \tilde{\mathcal{C}} \to \mathcal{E}$ is an isomorphism.

(ii) $\Rightarrow$ (i). Let $F$ be the exceptional locus of the minimal desingularisation $g : \tilde{\mathcal{C}} \to \mathcal{C}$. Then
\begin{equation}
(6) \quad H^0(\tilde{\mathcal{C}}, \omega_{\tilde{\mathcal{C}} / S}) \subset H^0(\tilde{\mathcal{C}} \setminus F, \omega_{\tilde{\mathcal{C}} / S}) = H^0(\mathcal{C} \setminus g(F), \omega_{\mathcal{C} / S}) = H^0(\mathcal{C}, \omega_{\mathcal{C} / S})
\end{equation}

where the last equality uses that $\mathcal{C}$ is normal: see \cite[Lemma 9.2.17]{15}. We are assuming that $\tilde{\mathcal{C}} \cong \mathcal{E}$ and $\omega_{\mathcal{E} / S} = \omega_{\mathcal{O}_C}$. Therefore $H^0(\tilde{\mathcal{C}}, \omega_{\tilde{\mathcal{C}} / S}) = \omega_E \mathcal{O}_K$ and $H^0(\mathcal{C}, \omega_{\mathcal{C} / S}) = \omega_k \mathcal{K}$. The inclusion (6) shows that $\omega_E = h \omega$ for some $h \in \mathcal{O}_K$. 

Since the sheaves $\omega_{\tilde{C}/S}$ and $g^*\omega_C/S$ are identical on $\tilde{C} \setminus F$, the divisor $\text{div}(h)$ on $\tilde{C} \cong E$ is a sum of irreducible components of $F$. On the other hand, $\text{div}(h)$ is a multiple of the special fibre. Since not all of the irreducible components of the special fibre are contracted, it follows that $h \in \mathcal{O}_K^\times$ as required. \hfill $\Box$

**Remark 7.3.** Following the proof of [15, Theorem 9.4.35(a)], we have the following alternative proof of “(ii) $\Rightarrow$ (i)”. Let $\Gamma_1, \ldots, \Gamma_r$ be the irreducible components of the special fibre that are contracted by $g : \tilde{C} \rightarrow C$. By [15, Theorem 9.1.27] the intersection matrix $(\Gamma_i \cdot \Gamma_j)$ is negative definite. Since $\tilde{C} \cong E$ is minimal, an argument using Castelnuovo’s criterion and the adjunction formula (see [15, Example 9.4.19] or [18, Chapter IV, Theorem 8.1(b)]) shows that $K_{\tilde{C}} \cdot \Gamma_i = 0$ for all $i$. Therefore the contraction morphism $g : \tilde{C} \rightarrow C$ satisfies the hypotheses of [15, Corollary 9.4.18]. As a consequence $g_*\omega_{\tilde{C}}/S = \omega_C/S$ and $g^*\omega_C/S = \omega_{\tilde{C}}/S$. Therefore $H^0(C, \omega_C/S) = H^0(\tilde{C}, \omega_{\tilde{C}}/S) = H^0(E, \omega_E/S) = \omega_E\mathcal{O}_K$.

**Theorem 7.4.** Let $\Phi \in X_5(\mathcal{O}_K)$ be non-singular with reduction $\phi \in X_5(k)$. Suppose $C = C_\Phi$ is a fibred surface, and the entries of $\phi$ span $(x_1, \ldots, x_5)$. Then $C$ is a local complete intersection with $\omega_{C}/S = \omega_\Phi \mathcal{O}_C$ where $\omega_\Phi$ is defined by (3).

**Proof:** Exactly as in the proof of Lemma 6.1, the affine piece $C \cap \{\Phi_{ij} \neq 0\}$ is defined by $p_1 = p_2 = p_3 = 0$. The restriction of the canonical sheaf to this affine piece is as claimed by [15, Corollary 6.4.14] and the next lemma. Since the definition (3) of $\omega_\Phi$ is invariant under all even permutations of the subscripts, and the affine pieces $\{\Phi_{ij} \neq 0\}$ cover $\mathbb{P}_5^4$, the theorem follows. \hfill $\Box$

**Lemma 7.5.** Let $R$ be any ring. Let $\phi \in X_5(R)$ with $\text{Pf}(\phi) = (p_1, \ldots, p_5)$. Let $I$ be the ideal in $R[x_1, \ldots, x_5]$ generated by $p_1, \ldots, p_5$. Then

$$\frac{\partial(p_1, p_2, p_3)}{\partial(x_1, x_2, x_3)} = \phi_{45} \sum_{i,j=1}^5 \frac{\partial p_i \partial \phi_{ij} \partial p_j}{\partial x_1 \partial x_2 \partial x_3} \quad (\text{mod } I).$$

**Proof:** We have $\sum_{i=1}^5 p_i \phi_{ij} = 0$ for all $1 \leq j \leq 5$. Differentiating with respect to $x_k$, and working mod $I$, this gives

$$\sum_{i=1}^5 \frac{\partial p_i}{\partial x_k} \phi_{ij} = 0.$$
Using first that \( \phi \) is alternating, and then (8), we compute
\[
\phi_{45} \sum_{i=1}^{3} \sum_{j=4}^{5} \frac{\partial p_i}{\partial x_1} \frac{\partial \phi_{ij}}{\partial x_2} \frac{\partial p_j}{\partial x_3} = \sum_{i=1}^{3} \sum_{j=4}^{5} \frac{\partial p_i}{\partial x_1} \left( \frac{\partial \phi_{i4}}{\partial x_2} \phi_{j5} - \frac{\partial \phi_{i5}}{\partial x_2} \phi_{j4} \right) \frac{\partial p_j}{\partial x_3}
\]
\[
= -\sum_{i,j=1}^{3} \frac{\partial p_i}{\partial x_1} \left( \frac{\partial \phi_{i4}}{\partial x_2} \phi_{j5} - \frac{\partial \phi_{i5}}{\partial x_2} \phi_{j4} \right) \frac{\partial p_j}{\partial x_3}.
\]
Subtracting the same identity with \( \frac{\partial}{\partial x_1} \) and \( \frac{\partial}{\partial x_3} \) switched gives
\[
\phi_{45} \sum_{i=1}^{3} \sum_{j=4}^{5} \frac{\partial \phi_{ij}}{\partial x_2} \frac{\partial (p_i, p_j)}{\partial (x_1, x_3)} = \sum_{i,j} \left( -\phi_{i4} \phi_{j5} + \phi_{i5} \phi_{j4} \right) \frac{\partial (p_i, p_j)}{\partial (x_1, x_3)}
\]
where we write \( \sum_{i<j} \) for the sum over all \( 1 \leq i < j \leq 3 \). Again using (8),
\[
\sum_{i,j=4}^{5} \frac{\partial p_i}{\partial x_1} \frac{\partial \phi_{ij}}{\partial x_3} = -\sum_{i=4}^{5} \sum_{j=1}^{3} \frac{\partial p_i}{\partial x_1} \phi_{ij} \frac{\partial p_j}{\partial x_3} = \sum_{i,j=1}^{3} \frac{\partial p_i}{\partial x_1} \phi_{ij} \frac{\partial p_j}{\partial x_3}.
\]
Therefore
\[
\phi_{45} \frac{\partial (p_4, p_5)}{\partial (x_1, x_3)} = \sum_{i<j} \phi_{ij} \frac{\partial (p_i, p_j)}{\partial (x_1, x_3)}.
\]
We break up the sum on the right of (7) as
\[
\sum_{i,j} \frac{\partial \phi_{ij}}{\partial x_2} \frac{\partial (p_i, p_j)}{\partial (x_1, x_3)} + \sum_{i=1}^{3} \sum_{j=4}^{5} \frac{\partial \phi_{ij}}{\partial x_2} \frac{\partial (p_i, p_j)}{\partial (x_1, x_3)} + \phi_{45} \frac{\partial (p_4, p_5)}{\partial (x_1, x_3)}
\]
Then by (9) and (10), the right hand side of (7) is
\[
\sum_{i<j} \frac{\partial}{\partial x_2} \left( \phi_{ij} \phi_{45} - \phi_{i4} \phi_{j5} + \phi_{i5} \phi_{j4} \right) \frac{\partial (p_i, p_j)}{\partial (x_1, x_3)}.
\]
Since for \( i, j, k \) an even permutation of \( 1, 2, 3 \) we have \(-p_k = \phi_{ij} \phi_{45} - \phi_{i4} \phi_{j5} + \phi_{i5} \phi_{j4} \) the result follows.
\[
\square
\]
**Remark 7.6.** We keep the notation of the lemma. Differentiating the relation \( \sum_{j=1}^{5} \phi_{ij} p_j = 0 \) with respect to \( x_4 \) and \( x_5 \) we have
\[
\sum_{j=1}^{5} \frac{\partial \phi_{ij}}{\partial x_4} \frac{\partial p_j}{\partial x_5} + \sum_{j=1}^{5} \frac{\partial \phi_{ij}}{\partial x_5} \frac{\partial p_j}{\partial x_4} + \sum_{j=1}^{5} \phi_{ij} \frac{\partial^2 p_j}{\partial x_4 \partial x_5} = 0.
\]
We multiply by $\frac{\partial}{\partial x_4}$ and sum over $i$. By [9] and the fact $\phi$ is alternating, the second two terms vanish mod $I$. Therefore

$$\sum_{i,j=1}^{5} \frac{\partial p_i \partial \phi_{ij} \partial p_j}{\partial x_4 \partial x_i \partial x_5} \equiv 0 \pmod I.$$

This shows that the restriction $\text{char } K \neq 2$ in [9 Section 5.4] is not needed.

**Lemma 7.7.** Let $\Phi \in \mathcal{X}_5(K)$ be non-singular with $C_\Phi(K) \neq \emptyset$. Then $\Phi$ has level 0 if and only if $\omega_\Phi$ is a Néron differential on $C_\Phi \cong E$.

**Proof:** Let $E/K$ have minimal Weierstrass equation

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6.$$

The complete linear system $|4.0_E|$ defines a morphism $\alpha : E \to \mathbb{P}^3$. It is given by $(x, y) \mapsto (1 : x : y : x^2)$. The image is $C_4 = \{Q_1 = Q_2 = 0\} \subset \mathbb{P}^3$ where

$$Q_1 = x_1 x_4 - x_2^2,$$

$$Q_2 = x_3^2 + a_1 x_2 x_3 + a_3 x_1 x_3 - x_2 x_4 - a_2 x_2^2 - a_4 x_1 x_2 - a_6 x_1^2,$$

and an invariant differential $\omega_4$ on $C_4$ is given by

$$\omega_4 = x_1^2 d(x_2/x_1).$$

We claim that (i) $\Delta(Q_1, Q_2) = \Delta_E$ and (ii) $\omega_4$ is a Néron differential on $C_4 \cong E$. Indeed the invariants were scaled in [9] so that (i) is true, whereas for (ii) it is easy to see that $\alpha^* \omega_4 = dx/(2y + a_1 x + a_3)$.

Since $C_\Phi(K) \neq \emptyset$ we may identify $C_\Phi \cong E$. The hyperplane section is linearly equivalent to $4.0_E + P$ for some $P \in E(K)$. Let $\Psi \in \mathcal{X}_5(K)$ be the Pfaffian model constructed from the quadric intersection $(Q_1, Q_2)$ by “unprojection centred at $P$” as described in [11 Lemma 2.3]. By [11 Lemma 2.4] and its proof, we have (i) $\Delta(\Psi) = \Delta_E$ and (ii) $\omega_\Psi$ is a Néron differential on $C_\Psi \cong E$.

The curves $C_\Phi$ and $C_\Psi$ differ by a change of co-ordinates defined over $K$. So by [10 Theorem 4.1(ii)], the Pfaffian models $\Phi$ and $\Psi$ are $K$-equivalent, say $\Phi = g \Psi$ for some $g \in \text{GL}_5(K) \times \text{GL}_5(K)$. Since $\Delta$ is an invariant of weight 12 we have $\Delta(\Phi) = (\det g)^{12} \Delta(\Psi)$. Let $\gamma : C_\Phi \to C_\Psi$ be the isomorphism described by $g$. By [9 Proposition 5.19] we have $\gamma^* \omega_\Psi = (\det g) \omega_\Phi$. Therefore both the conditions in the statement of the lemma are equivalent to $v(\det g) = 0$. \hfill $\Box$

**Remark 7.8.** If $\text{char } K \neq 2, 3$ then [9 Proposition 5.23] shows that $(C_\Phi, \omega_\Phi)$ and $(E, \omega)$ are isomorphic over $\overline{K}$, where $E$ is the elliptic curve $[2]$ and $\omega = dx/(2y + a_1(\Phi)x + a_3(\Phi))$. This gives an alternative proof of Lemma 7.7. The isomorphism
$C_\Phi \cong E$ might not be defined over $K$, but differs from an isomorphism that is defined over $K$ by an automorphism of the curve $E$. The latter might rescale $\omega$ by a root of unity, but won’t change whether it is a Néron differential.

**Theorem 7.9.** Let $\Phi \in X_5(O_K)$ be non-singular with $C_\Phi(K) \neq \emptyset$. Suppose $C_\Phi$ is a fibred surface, and the entries of $\phi$ span $\langle x_1, \ldots, x_5 \rangle$. Then $\Phi$ is minimal if and only if $C_\Phi$ is geometrically minimal.

**Proof:** Lemma 5.1 shows that $C = C_\Phi$ is the closure of $C_\Phi$ in $\mathbb{P}^4$. By either Definition 4.3 or Theorem 6.4 we may suppose $C$ is normal. Let $E$ be the Jacobian of $C_\Phi$. Since $C_\Phi(K) \neq \emptyset$ we have $C_\Phi \cong E$. Theorem 3.1 and Lemma 7.7 show that $\Phi$ is minimal if and only if $\omega_\Phi$ is a Néron differential on $C_\Phi \cong E$. The theorem now follows from Theorems 7.2 and 7.4.

By Lemma 5.2, Theorem 5.5 and Theorem 7.9 we have

**Corollary 7.10.** If $\Phi \in X_5(O_K)$ is minimal and $C_\Phi(K) \neq \emptyset$ then $C_\Phi$ is geometrically minimal.

8. **An alternative proof of Theorem 1.1**

We give a short alternative proof of Theorem 1.1 that avoids using schemes, except for the definition of a regular point. It would however be rather hard to motivate this proof without the work in earlier sections.

By putting the matrices $A, B \in \text{GL}_5(K)$ in Smith normal form (and making use of Remark 3.3), Theorem 1.1 is equivalent to the following.

**Theorem 8.1.** Let $\Phi, \Phi' \in X_5(O_K)$ with $v(\Delta(\Phi)) \leq 1$ and $v(\Delta(\Phi')) \leq 1$. If

$$\Phi' = [\text{Diag}(\pi^{-r_1}, \ldots, \pi^{-r_5}), \text{Diag}(\pi^{s_1}, \ldots, \pi^{s_5})] \Phi$$

for some $r_1, \ldots, r_5, s_1, \ldots, s_5 \in \mathbb{Z}$ then $s_1 = s_2 = \ldots = s_5$.

For the proof we may assume the residue field $k$ is algebraically closed. As before we write $\phi \in X_5(k)$ for the reduction of $\Phi$ mod $\pi$. For the purposes of this section, a $k$-point $P$ on $C_\phi$ is regular if it satisfies the conditions in Lemma 3.5, and otherwise non-regular. Since $\dim C_\Phi = 2$ this agrees with the standard terminology, but we don’t need to know this.

**Lemma 8.2.** If $v(\Delta(\Phi)) \leq 1$ then $C_\Phi$ contains no lines or conics.
PROOF: If $C_\phi$ contains a line or conic then, arguing as in the proof of Lemma 6.2 we may assume

$$\phi = \begin{pmatrix}
0 & x_1 & x_2 & * & * \\
0 & * & * & * & * \\
0 & * & * & * & * \\
- & 0 & * & * & * \\
0 & & & & \\
\end{pmatrix} \text{ or } \begin{pmatrix}
0 & * & * & * & * \\
0 & * & * & * & * \\
0 & * & * & * & * \\
- & 0 & 0 & & \\
0 & & & & \\
\end{pmatrix}$$

where the entries * on the left are linear forms in $x_3, x_4, x_5$, and on the right are linear forms in $x_1, \ldots, x_5$. In the first case we apply the transformation

$$[\text{Diag}(\pi, 1, 1, 1, 1) \pi^{-1} \text{Diag}(1, 1, \pi, \pi)].$$

Then $\phi_{14} = \phi_{15} = 0$ and an $O_K$-equivalence brings us to the second case. In the second case we may assume $\phi_{34} \in \langle x_1 \rangle$. Applying the transformation

$$[\text{Diag}(\pi, \pi, 1, 1, 1) \pi^{-1} \text{Diag}(1, 1, 1, 1)]$$

gives a model with a non-regular point at $(1 : 0 : \ldots : 0)$. Since all transformations we have used preserve (the valuation of) the discriminant, we are done by Lemma 3.6.

Lemma 8.3. Let $\Phi, \Phi' \in X_5(O_K)$ be Pfaffian models satisfying

$$\Phi' = [\text{Diag}(\pi^{-r_1}, \ldots, \pi^{-r_5}), \text{Diag}(\pi^{s_1}, \ldots, \pi^{s_5})]\Phi$$

for some $r_1 \leq \ldots \leq r_5$ and $s_1 \leq \ldots \leq s_5$.

(i) If $C_\phi$ contains no lines then $r_1 + r_4 \leq s_2$, $r_2 + r_3 \leq s_2$ and $r_2 + r_4 \leq s_3$.

(ii) If $C_\phi$ contains no lines or conics then $r_1 + r_5 \leq s_3$, $r_2 + r_5 \leq s_4$, $r_3 + r_4 \leq s_4$ and $r_3 + r_5 \leq s_5$.

PROOF: (i) If $r_1 + r_4 > s_2$ then all entries of $\phi$ outside the top left $3 \times 3$ submatrix are linear forms in $x_3, x_4, x_5$. So $C_\phi$ contains the line $\{x_3 = x_4 = x_5 = 0\}$. If $r_2 + r_3 > s_2$ then all entries of $\phi$ outside the first row/column are linear forms in $x_3, x_4, x_5$. So $C_\phi$ contains the line $\{x_3 = x_4 = x_5 = 0\}$. If $r_2 + r_4 > s_3$ then $C_\phi$ contains the line $\{\phi_{23} = x_4 = x_5 = 0\}$.

(ii) If $r_1 + r_5 > s_3$, $r_2 + r_5 > s_4$ or $r_3 + r_5 > s_5$ then the entries of the last row/column of $\phi$ are in $\langle x_4, x_5 \rangle$, $\langle \phi_{15}, x_5 \rangle$ or $\langle \phi_{15}, \phi_{25} \rangle$. If $r_3 + r_4 > s_4$ then the bottom right $3 \times 3$ submatrix of $\phi$ has entries in $\langle x_5 \rangle$. So in all these cases $\phi$ is $k$-equivalent to a model with $\phi_{35} = \phi_{45} = 0$. Let $p_5$ be the Pfaffian of the top left
Lemma 8.4. Let $\Phi$ and $\Phi'$ be as in Theorem 8.1 and suppose $0 = r_1 \leq \ldots \leq r_5$ and $s_1 \leq \ldots \leq s_5$. Then the $r_i$ and $s_i$ are given by

$$
\begin{array}{cccccc}
 r_1 & r_2 & r_3 & r_4 & r_5 & s_1 \\
 0 & \alpha & 2\alpha & 3\alpha & 4\alpha & s_2 \\
 & & & & & s_3 \\
 & & & & & s_4 \\
 & & & & & s_5
\end{array}
\leq
\begin{array}{cccccc}
 2\alpha & 3\alpha & 4\alpha & 5\alpha & \geq & 6\alpha
\end{array}
$$

for some $\alpha \geq 0$.

Proof: The inequalities in Lemma 8.3 together with the inequalities obtained when we replace $(r_1, \ldots, r_5; s_1, \ldots, s_5)$ by $(-r_5, \ldots, -r_1; -s_5, \ldots, -s_1)$ give

$$s_2 = r_1 + r_4 = r_2 + r_3 \implies r_2 - r_1 = r_4 - r_3$$
$$s_3 = r_1 + r_5 = r_2 + r_4 \implies r_2 - r_1 = r_5 - r_4$$
$$s_4 = r_2 + r_5 = r_3 + r_4 \implies r_3 - r_2 = r_5 - r_4$$

Therefore $r_1, \ldots, r_5$ are in arithmetic progression. The other statements follow. 

Lemma 8.5. Let $\Phi, \Phi' \in X_5(\mathcal{O}_K)$ be Pfaffian models satisfying

$$\Phi' = [\text{Diag}(\pi^{-r_1}, \ldots, \pi^{-r_5}), \text{Diag}(\pi^{s_1}, \ldots, \pi^{s_5})] \Phi$$

for some $r_1 \leq \ldots \leq r_5$ and $s_1 \leq \ldots \leq s_5$.

(i) If $r_1 + r_4 > s_1$ and $r_4 + r_5 > s_5 > s_1$ then $C_\Phi$ has a non-regular point.

(ii) If $r_1 + r_3 > s_1$ and $r_3 + r_4 > s_3 > s_1$ then $C_\Phi$ has a non-regular point.

(iii) If $r_2 + r_5 < s_5$ and $r_1 + r_2 < s_1 < s_5$ then $C_{\Phi'}$ has a non-regular point.

(iv) If $r_3 + r_5 < s_5$ and $r_2 + r_3 < s_3 < s_5$ then $C_{\Phi'}$ has a non-regular point.

Proof: (i) Since $r_1 + r_4 > s_1$ the only entries of $\phi$ involving $x_1$ are in the top left $3 \times 3$ submatrix. So $P = (1 : 0 : \ldots : 0)$ is a point on $C_\Phi$. Since $r_4 + r_5 > s_5$ we have $\phi_{45} = 0$ and so $P$ is a singular point. Since $r_4 + r_5 > s_1 + 1$ the coefficient of $x_1$ in $\Phi_{45}$ vanishes mod $\pi^2$. Therefore $P$ is a non-regular point.

(ii) Since $r_1 + r_3 > s_1$ the only entries of $\phi$ involving $x_1$ are in the top left $2 \times 2$ submatrix. So $P = (1 : 0 : \ldots : 0)$ is a point on $C_\Phi$. Since $r_3 + r_4 > s_3$ we have $\phi_{34}, \phi_{35}, \phi_{45} \in \langle x_4, x_5 \rangle$ and so $P$ is a singular point. Since $r_3 + r_4 > s_1 + 1$ the coefficient of $x_1$ in each of $\Phi_{34}, \Phi_{35}$ and $\Phi_{45}$ vanishes mod $\pi^2$. Therefore $P$ is a non-regular point.

(iii), (iv) These follow from (i) and (ii) by switching the roles of $\Phi$ and $\Phi'$. 

Proof of Theorem 8.1. We may assume \( r_1 \leq \ldots \leq r_5 \) and \( s_1 \leq \ldots \leq s_5 \). Replacing \( r_i \) by \( r_i + \lambda \) and \( s_i \) by \( s_i + 2\lambda \) still gives the same transformation. So we may assume \( r_1 = 0 \). Then the \( r_i \) and \( s_i \) are as given in Lemma 8.4.

If \( \alpha = 0 \) then \( r_1 = \ldots = r_5 \) and the conclusion \( s_1 = \ldots = s_5 \) follows from the fact \( \Phi \) and \( \Phi' \) are minimal. We assume for a contradiction that \( \alpha \geq 1 \). Since \( r_1 + r_4 = 3\alpha > s_1 \) it follows by Lemmas 3.6 and 8.5(i) that \( r_4 + r_5 \leq s_5 \). Since \( r_2 + r_3 = 3\alpha < s_3 \) it follows by Lemmas 3.6 and 8.5(iv) that \( r_3 + r_5 \geq s_5 \). Putting these together we have

\[
r_4 + r_5 \leq s_5 \leq r_3 + r_5.
\]

Therefore \( r_3 = r_4 \) and this contradicts our assumption that \( \alpha \geq 1 \). \( \Box \)

References

[1] M. Artin, Lipman’s proof of resolution of singularities for surfaces, in Arithmetic geometry, G. Cornell and J.H. Silverman (eds), Springer-Verlag, New York, 1986.
[2] M. Bhargava, Higher composition laws, IV. The parametrization of quintic rings. Ann. of Math. (2) 167 (2008), no. 1, 53–94.
[3] M. Bhargava and A. Shankar, The average size of the 5-Selmer group of elliptic curves is 6, and the average rank is less than 1, preprint 2013, arXiv:1312.7859v1
[4] D.A. Buchsbaum and D. Eisenbud, Gorenstein ideals of height 3, Seminar D. Eisenbud/B. Singh/W. Vogel, Vol. 2, pp. 30–48, Teubner-Texte zur Math., 48, Teubner, Leipzig, 1982.
[5] D.A. Buchsbaum and D. Eisenbud, Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3, Amer. J. Math. 99 (1977) 447-485.
[6] J.W.S. Cassels, Arithmetic on curves of genus 1, IV. Proof of the Hauptvermutung, J. reine angew. Math. 211 (1962) 95–112.
[7] B. Conrad, Minimal models for elliptic curves, 2005, math.stanford.edu/~conrad/papers/minimalmodel.pdf
[8] D. Eisenbud, Commutative algebra with a view toward algebraic geometry, GTM 150, Springer-Verlag, New York, 1995.
[9] T.A. Fisher, The invariants of a genus one curve, Proc. Lond. Math. Soc. (3) 97 (2008) 753–782.
[10] T.A. Fisher, Explicit 5-descent on elliptic curves, in ANTS X: Proceedings of the Tenth Algorithmic Number Theory Symposium (San Diego), E.W. Howe and K.S. Kedlaya (eds), Mathematical Sciences Publishers, 2013, http://msp.org/obs/2013/1-1/
[11] T.A. Fisher, Minimisation and reduction of 5-coverings of elliptic curves, Algebra & Number Theory 7 (2013), no. 5, 1179–1205.
[12] T.A. Fisher, Invariant theory for the elliptic normal quintic, I. Twists of X(5), Math. Ann. 356 (2013), no. 2, 589–616.
[13] T.A. Fisher and G.F. Sills, Local solubility and height bounds for coverings of elliptic curves, Math. Comp. 81 (2012), no. 279, 1635–1662.
[14] R. Hartshorne, Algebraic geometry, GTM 52, Springer-Verlag, New York-Heidelberg, 1977.
[15] Q. Liu, Algebraic geometry and arithmetic curves, Oxford University Press, 2002.
[16] M. Sadek, Counting models of genus one curves, *Math. Proc. Cambridge Philos. Soc.* 150 (2011), no. 3, 399–417.

[17] M. Sadek, Minimal genus one curves, *Funct. Approx. Comment. Math.* 46 (2012), part 1, 117–131.

[18] J.H. Silverman, *Advanced topics in the arithmetic of elliptic curves*, GTM 151, Springer-Verlag, New York, 1994.

University of Cambridge, DPMMS, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WB, UK

E-mail address: T.A.Fisher@dpmms.cam.ac.uk

American University in Cairo, Mathematics and Actuarial Science Department, AUC Avenue, New Cairo, Egypt

E-mail address: mmsadek@aucegypt.edu