Biharmonic Obstacle Problem: Guaranteed and Computable Error Bounds for Approximate Solutions

D. E. Apushkinskaya\textsuperscript{a,b,*} and S. I. Repin\textsuperscript{c,d,**}

\textsuperscript{a}Saarland University, P.O. Box 151150, Saarbrücken, 66041 Germany
\textsuperscript{b}Peoples' Friendship University of Russia (RUDN University), Moscow, 117198 Russia
\textsuperscript{c}Steklov Institute of Mathematics at St. Petersburg, St. Petersburg, 191023 Russia
\textsuperscript{d}University of Jyväskylä, P.O. Box 35 (Agora), Jyväskylä, 40014 Finland

\textsuperscript{*}e-mail: darya@math.uni-sb.de
\textsuperscript{**}e-mail: repin@pdmi.ras.ru

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Abstract—The paper is concerned with an elliptic variational inequality associated with a free boundary obstacle problem for the biharmonic operator. We study the bounds of the difference between the exact solution (minimizer) of the corresponding variational problem and any function (approximation) from the energy class satisfying the prescribed boundary conditions and the restrictions stipulated by the obstacle. Using the general theory developed for a wide class of convex variational problems we deduce the error identity. One part of this identity characterizes the deviation of the function (approximation) from the exact solution, whereas the other is a fully computed value (it depends only on the data of the problem and known functions). In real life computations, this identity can be used to control the accuracy of approximate solutions. The measure of deviation from the exact solution used in the error identity contains terms of different nature. Two of them are the norms of the difference between the exact solutions (of the direct and dual variational problems) and corresponding approximations. Two others are not representable as norms. These are nonlinear measures vanishing if the coincidence set defined by means of an approximate solution satisfies certain conditions (for example, coincides with the exact coincidence set). The error identity is true for any admissible (conforming) approximations of the direct variable, but it imposes some restrictions on the dual variable. We show that these restrictions can be removed, but in this case the identity is replaced by an inequality. For any approximations of the direct and dual variational problems, the latter gives an explicitly computable majorant of the deviation from the exact solution. Several examples illustrating the established identities and inequalities are presented.

Keywords: variational inequalities, estimates of the distance to the exact solution, a posteriori estimates

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1. INTRODUCTION

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^d \) with Lipschitz continuous boundary \( \partial \Omega \). By \( \nu \) we denote an outward unit normal to \( \partial \Omega \), and \( \varphi \) denotes a given function (obstacle) in \( C^2(\overline{\Omega}) \) such that \( \varphi \leq 0 \) on \( \partial \Omega \). Throughout the paper, we use the standard notation for the Lebesgue and Sobolev spaces of functions. By \( g_{\overline{\Omega}} \) we denote \( \max\{g, 0\} \).

Consider the variational problem (\( \mathcal{P} \)): minimize the functional

\[
J(\nu) = \int_{\Omega} \left( \frac{1}{2} |\Delta \nu|^2 - f \nu \right) dx
\]

over the closed convex set

\[
\mathcal{K} = \left\{ \nu \in H^2(\Omega) \left| \nu|_{\partial \Omega} = \frac{\partial \nu}{\partial \nu_{\partial \Omega}} = 0, \nu \geq \varphi \text{ a.e. in } \Omega \right. \right\}.
\]

Here \( f \in L^1(\Omega) \), and \( \varphi \) is a given function (obstacle) such that \( \varphi \in C^2(\overline{\Omega}) \) and \( \varphi \leq 0 \) on \( \partial \Omega \).
Problem (P) is called the **biharmonic obstacle problem**. It has many applications in elasticity theory (frictionless contact problems of elastic plates or beams over a rigid obstacle) and in fluid mechanics (incompressible fluid flow at low Reynolds number on a plane). By standard results (see, e.g., [1, 2]), the problem (P) has a unique solution \( u \), which satisfies a.e. in \( \Omega \) the following relations:

\[
\Delta^2 u \geq f, \quad u \geq \varphi, \quad (\Delta^2 u - f)(u - \varphi) = 0. \tag{1.2}
\]

In particular, due to the well-known works [3] and [4] we have the following a priori regularity

\[
u \in H^3_{\text{loc}}(\Omega) \cap W^{2,\infty}_{\text{loc}}(\Omega) \quad \text{and} \quad \Delta u \in W^{2,\infty}_{\text{loc}}(\Omega). \tag{1.3}
\]

In general, the domain \( \Omega \) is divided in two subdomains \( \Omega_0 \) and \( \Omega_\varphi \), in which \( u \) has different properties. In \( \Omega_0 \) the equation \( \Delta^2 u = f \) holds, while in \( \Omega_\varphi \) the solution \( u \) coincides with the obstacle (this set is called the coincidence set). The interface between these sets is a priori unknown, so that the problem \( P \) belongs to the class of free boundary problems.

The biharmonic obstacle problems have been actively studied by many authors, starting with the pioneering works of Landau and Lifshitz [5], Frehse [4, 6], Cimatti [7], Stampacchia [8], Brézis and Stampacchia [8]. We mention also the well-known monographs by Duvaut and Lions [10], and by Rodrigues [11], where some examples of the problem of bending a plate over an obstacle are considered. Notice that most of the studies of the fourth order obstacle problems were mainly focused either on regularity of minimizers or on the properties of the respective free boundaries (see [3, 4, 6, 7, 12–14]).

Approximation methods for the biharmonic obstacle problem have been developed within the framework of computational methods for variational inequalities (see, e.g., [15–19]) and related optimal control problems [20, 21]. Hence, in principle, it is known how to construct a sequence of approximations converging to the exact minimizer of this nonlinear variational problem.

In this paper, we are concerned with a different question. Our goal is to deduce a guaranteed and fully computable bound for the distance (measured in terms of natural energy norm) between the exact solution \( u \in \mathbb{K} \) and an approximate solution \( v \in \mathbb{K} \). To obtain such a bound, we apply the same method as was used in [22] for the derivation of guaranteed error bounds of the difference between exact solution of the linear biharmonic problem and any function in the energy admissible class of functions. In [23–26] and some other publications this method was applied to obstacle problems associated with elliptic operators of the second order. Below we show that it is also quite efficient for higher order operators. It generates a natural error measure (which characterizes the deviation of any function from the corresponding energy space from the exact solution) and provides the fully computable estimates of this measure.

### 2. ESTIMATES OF THE DISTANCE TO THE EXACT SOLUTIONS

#### 2.1. General Form of the Error Identity

Consider the functional spaces

\[
V := \left\{ w \in H^2(\Omega) \bigg| w|_{\partial \Omega} = \frac{\partial w}{\partial n}|_{\partial \Omega} = 0 \right\}
\]

and \( N := L^2(\Omega, \mathcal{M}^{d\times d}_{\text{Sym}}) \), where \( \mathcal{M}^{d\times d}_{\text{Sym}} \) denotes the space of \( d \times d \) symmetric matrices. The corresponding conjugate (dual) spaces are \( V^* = H^{-2}(\Omega) \) and \( N^* = N \), respectively.

It is easy to see that the functional \( J \) in the problem (1.1) can be represented in the form

\[
J(v) = G(\Lambda v) + F(v), \tag{2.1}
\]

where the operator \( \Lambda \) and functionals \( G \) and \( F \) are defined as follows:

\[
\Lambda : V_0 \to N, \quad \Lambda := \nabla \nabla; \quad G : N \to \mathbb{R}, \quad G(n) := \frac{1}{2} \int_{\Omega} |n|^2 \, dx; \nonumber
\]

\[
F : V_0 \to \mathbb{R}, \quad F(v) := -\int_{\Omega} f v \, dx + \chi_{K}(v), \quad \chi_{K}(v) = \begin{cases} 0, & v \in \mathbb{K}, \\ +\infty, & v \notin \mathbb{K}. \end{cases}
\]
Hence, we can use the general theory presented in [27] and [28]. Further, we denote by \( p^* \) the exact solution of the dual variational problem, which is to maximize the dual functional (cf. [29])

\[
I^*(n^*) := -G^*(n^*) - F^*(-\Lambda^* n^*)
\]  

over the set \( N^* \). Here the operator \( \Lambda^* \) is defined as follows:

\[
\Lambda^* : N \to V_0^*, \quad \Lambda^* := \text{div} \, \text{Div},
\]

while the functionals \( G^* : N^* \to \mathbb{R} \) and \( F^* : V^* \to H^{-2}(\Omega) \to \mathbb{R} \) are the Young–Fenchel transforms of \( G \) and \( F \), respectively.

In view of the duality relation \( J(u) = I^*(p^*) \) (which is fulfilled for the given variational problem) and the identities (7.2.13)–(7.2.14) from [28], we have for an arbitrary \( v \in \mathbb{K} \) and \( n^* \in N^* \) the following relations:

\[
J(v) - J(u) = J(v) - I^*(p^*) = D_G(\Lambda v, p^*) + D_F(v, -\Lambda^* p^*),
\]

\[
I^*(p^*) - I^*(n^*) = J(u) - I^*(n^*) = D_G(\Lambda u, n^*) + D_F(u, -\Lambda^* n^*).
\]

Here \( D_G \) and \( D_F \) denote the so-called compound functionals which are determined by the relations

\[
D_G(\Lambda v, n^*) = G(\Lambda v) + G^*(n^*) - (\Lambda v, n^*),
\]

\[
D_F(v, -\Lambda^* n^*) = F(v) + F^*(n^*) + \langle \Lambda n^*, v \rangle,
\]

where \( \langle v^*, v \rangle \) stands for a linear functional coupling the elements \( v \in V \) and \( v^* \in V^* \). It follows from the definition of a conjugate functional that the compound functional is always non-negative.

Relations (2.3) and (2.4) directly imply the error identity (see [27] for more details)

\[
D_G(\Lambda v, p^*) + D_F(v, -\Lambda^* p^*) + D_G(\Lambda u, n^*) + D_F(u, -\Lambda^* n^*) = D_G(\Lambda v, n^*) + D_F(v, -\Lambda^* n^*),
\]

which holds for any \( v \in \mathbb{K} \) and \( n^* \in N^* \). The left-hand side of (2.7) contains four terms that can be considered as deviation measures of the functions \( v \) and \( n^* \) from \( u \) and \( p^* \), respectively. The first two terms can be treated as a measure \( \mu(v) \) characterizing the error of approximation \( v \), while another two terms can be regarded as a measure \( \mu^*(n^*) \) indicating the error of dual approximation \( n^* \). The right-hand side consists of two terms that do not contain unknown exact \( u \) and \( p^* \), and, therefore, it can be calculated explicitly. Moreover, from (2.3) and (2.4) it follows that the r.h.s. of (2.7) coincides with the so-called duality gap \( J(v) - I^*(p^*) \). Notice that the sequences of approximations \( \{v_k\}, \{n^*_k\} \), constructed with the help of variational methods, should minimize this gap. Therefore, the identity (2.7) shows that the measures on the l.h.s. of (2.7) must tend to zero, if the sequences \( \{v_k\}, \{n^*_k\} \) are constructed correctly and converge to exact solutions. Hence, the measures on the l.h.s. of the error identity are an adequate characteristic of the quality of approximations.

Identity (2.7) holds for any variational problem with the functional of the form (2.1). We establish it’s form in terms of the studied problem. It is easy to see that \( G^* : N^* \to \mathbb{R} \) is defined by the equality

\[
G^*(n^*) := \frac{1}{2} \|p^*\|^2 \quad \text{(hereinafter, \( \| \cdot \| \) denotes the norm in the spaces \( L_2(\Omega) \) for scalar, vector, and matrix functions). Therefore, the first terms on the right hand sides of (2.3) and (2.4) are computed easily:
\]

\[
D_G(\Lambda v, p^*) = \frac{1}{2} \|\nabla v - p^*\|^2 = \frac{1}{2} \|\nabla (v - u)\|^2,
\]

\[
D_G(\Lambda u, n^*) = \frac{1}{2} \|\nabla u - n^*\|^2 = \frac{1}{2} \|p^* - n^*\|^2.
\]

A computation of the last summands on the right hand sides of (2.3) and (2.4) requires more work.
To compute $\langle v^*, v \rangle$, we need the intermediate Hilbert space $\mathcal{V} := L^2(\Omega)$. It is clear that $\mathcal{V}$ satisfies the inclusions $V \subset \mathcal{V} \subset V^*$. If $v^* \in \mathcal{V}$ then the scalar product $\langle v^*, v \rangle$ is identified with scalar product in the space $\mathcal{V}$, i.e.,

$$\langle v^*, v \rangle = \int_{\Omega} v^* v \, dx.$$ 

Notice that this integral is well-defined for any $v^* \in \mathcal{V}$ and $v \in V$.

In accordance with the definition of the conjugate functional, for $n^* \in N$ we have

$$F^*(-\Lambda^* n^*) = \sup_{v \in \mathbb{C}} \left\{ -\langle v^*, v \rangle + (f, v) \right\} = \sup_{v \in \mathbb{C}} \left\{ -\langle n^*, \Lambda v \rangle + (f, v) \right\} = \sup_{v \in \mathbb{C}} \left\{ \int_{\Omega} (f v - n^* : \nabla \nabla v) \, dx \right\}.$$

Observe that a function $n^*$ satisfies the additional restriction

$$n^* \in H(\Omega, \text{div Div}) := \left\{ m^* \in N^* : \text{div Div} m^* \in L^2(\Omega) \right\}. \tag{2.10}$$

Taking into account the condition $\frac{\partial v}{\partial n} \bigg|_{\partial \Omega} = 0$ and using integration by parts we conclude that

$$0 = \int_{\Omega} v(n^* \nabla v) \, ds = \int_{\Omega} \text{div}(n^* \nabla v) \, dx = \int_{\Omega} (\nabla v \text{div} n^* + n^* : \nabla \nabla v) \, dx.$$

Hence

$$\int_{\Omega} (f v - n^* : \nabla \nabla v) \, dx = \int_{\Omega} (f v + \nabla v \cdot \text{div} n^*) \, dx. \tag{2.11}$$

Combining (2.11) with the formula

$$0 = \int_{\partial \Omega} (\text{Div} n^* \cdot v) \, ds = \int_{\Omega} \text{div}(v \text{div} n^*) \, dx = \int_{\Omega} (\text{Div} n^* \cdot \nabla v + v \text{div Div} n^*) \, dx, \tag{2.12}$$

we arrive at

$$\int_{\Omega} (f v - n^* : \nabla \nabla v) \, dx = \int_{\Omega} (f - \text{div Div} n^*) v \, dx.$$

Thus, we have

$$F^*(-\Lambda^* n^*) = \sup_{v \in \mathbb{C}} \left\{ \int_{\Omega} (f - \text{div Div} n^*) v \, dx \right\} \quad \forall n^* \in H(\Omega, \text{div Div}). \tag{2.13}$$

Let $\hat{v} \in \mathbb{K}$ be a given function. Then the function $\hat{v} + w$ with

$$w \in V^+(\Omega) := \{ w \in V(\Omega) : w \geq 0 \text{ a.e. in } \Omega \},$$

also belongs to $\mathbb{K}$. It is easy to see that

$$F^*(-\Lambda^* n^*) \geq \int_{\Omega} (f - \text{div Div} n^*) \hat{v} \, dx + \sup_{w \in V^+(\Omega)} \left\{ \int_{\Omega} (f - \text{div Div} n^*) w \, dx \right\}.$$

Therefore, this expression is finite if and only if $n^* \in Q^*$, where

$$Q^* := \left\{ m^* \in H(\Omega, \text{div Div}) \left| \int_{\Omega} (f - \text{div Div} m^*) w \, dx \leq 0 \quad \forall w \in V^+(\Omega) \right. \right\}. \tag{2.14}$$

The integral condition in the definition of $Q^*$ means that $f - \text{div Div} n^* \leq 0$ almost everywhere in $\Omega$. Indeed, suppose that $n^* \in Q^*$ and $f - \text{div Div} n^* > 0$ on some nonzero measure set $\omega$. Then there exists a ball $B(x_0, \rho) \subset \omega$ where this inequality holds. Consider a compact function $w \in V^+(\Omega)$ having support in this ball. The function $w$ is positive inside the ball, and, consequently,

$$\int_{\Omega} (f - \text{div Div} m^*) w \, dx > 0.$$
We get a contradiction that proves the validity of the statement. Let \( n^* \in Q^*_\Omega \). It is clear that

\[
\sup_{v \in K_\Omega} \int_\Omega (f - \text{div} \text{Div} n^*)v dx \leq \int_\Omega (f - \text{div} \text{Div} n^*)\varphi dx.
\]

Moreover, there exists a sequence of functions \( v_k \in K_\Omega \) such that \( v_k \to \varphi \) in \( L^2(\Omega) \). We conclude that

\[
F^*(-\Lambda^*n^*) = \begin{cases} \int_\Omega \varphi (f - \text{div} \text{Div} n^*) dx & \text{if } n^* \in Q^*_\Omega, \\ +\infty, & \text{otherwise}. \end{cases}
\]

Thus, the compound functional \( \mathcal{D}_F(v,-\Lambda^*n^*) \) is finite if and only if the condition

\[
f - \text{div} \text{Div} n^* \leq 0 \tag{2.15}
\]

is satisfied almost everywhere in \( \Omega \).

Therefore, for \( n^* \in H(\Omega,\text{div} \text{Div}) \) satisfying (2.15), the compound functional \( \mathcal{D}_F(v,-\Lambda^*n^*) \) has the form

\[
\mathcal{D}_F(v,-\Lambda^*n^*) = \int_\Omega (f - \text{div} \text{Div} n^*)(\varphi - v) dx. \tag{2.16}
\]

Our next goal is to compute \( \mathcal{D}_F(v,-\Lambda^*p^*) \) for any function \( v \in K \). We cannot use the previous formula since \( p^* \), in general, does not satisfy the condition (2.10). Indeed, due to (1.3) we only know that \( \text{div} \text{Div} p^* \) is a square integrable function on the set \( \Omega_u \) (where \( u > \varphi \) and the equality \( \text{div} \text{Div} p^* = f \) holds) and on the coincidence set \( \Omega_c = \{ u = \varphi \} = \Omega \setminus \Omega_u \). However, the square integrability of \( \text{div} \text{Div} p^* \) does not hold in the whole domain. By this reason, we use a different argument. Setting \( v = u \) in (2.3) we have

\[
\mathcal{D}_F(u,-\Lambda^*p^*) = 0,
\]

which implies

\[
F^*(-\Lambda^*p^*) = -F(u) - \langle \Lambda^*p^*,u \rangle = \int_\Omega (fu - p^* : \nabla \nabla u) dx.
\]

Now, using the last equality and arguing in the same way as in deriving (2.11), we conclude that

\[
\mathcal{D}_F(v,-\Lambda^*p^*) = F(v) + F^*(-\Lambda^*p^*) + \langle \Lambda^*p^*,v \rangle = \int_\Omega f(u - v) - p^* : \nabla \nabla (u - v) dx = \int_\Omega f(u - v) + \nabla (u - v) \cdot \text{Div} p^* dx.
\]

We have

\[
\int_\Omega (f(u - v) + \text{Div} p^* \cdot \nabla (u - v)) dx = \int_{\{ u > \varphi \}} (\ldots) dx + \int_{\{ u = \varphi \}} (\ldots) dx.
\]

Let \( v_{r_u} \) denote the exterior unit normal to \( \partial \Omega_\varphi \) and \( e := u - v \). Since

\[
\int_{\{ u > \varphi \}} (fe + \text{Div} p^* \cdot \nabla e) dx = \int_{\{ u > \varphi \}} (f - \text{div} \text{Div} p^*) edx - \int_{\Gamma_u} (\text{Div} p^* \cdot v_{r_u}) e ds = 0
\]

and

\[
\int_{\{ u = \varphi \}} (fe + \text{Div} p^* \cdot \nabla e) dx = \int_{\{ u = \varphi \}} (f - \text{div} \text{Div} p^*) edx + \int_{\Gamma_u} (\text{Div} p^* \cdot v_{r_u}) e ds,
\]

we obtain

\[
\mathcal{D}_F(v,-\Lambda^*p^*) = \int_{\Gamma_u} [\text{Div} p^* \cdot v_{r_u}](u - v) ds + \int_{\{ u = \varphi \}} (f - \text{div} \text{Div} p^*)(u - v) dx, \tag{2.17}
\]

where \( [\gamma] := \gamma_{\Gamma_u}(\Omega_\varphi) - \gamma_{\Gamma_u}(\Omega_u) \) denotes the jump of \( \gamma \) across the free boundary \( \Gamma_u := \partial \Omega_\varphi \) (in our case \( \gamma = \text{Div} p^* \cdot v_{r_u} \)). Thus, we have to take into consideration an additional integral term arising on the free boundary \( \Gamma_u \).
Combining (2.3)–(2.9) with (2.16), (2.17), we get explicit expressions for the measures in the left-hand side of (2.7). For any function \( \nu \in \mathcal{K} \) we have
\[
\mu(\nu) := \mathcal{D}_G(\Lambda \nu, p^*) + \mathcal{D}_F(\nu, -\Lambda^* p^*) = \frac{1}{2}\|\nabla\nabla(u - \nu)\|^2 + \mu_\varphi(\nu),
\] (2.18)
where
\[
\mu_\varphi(\nu) := \int_{[u = \varphi]} (\nabla \cdot (\nabla \nabla u - f))(\nu - u)dx - \int_{\Gamma_u} [\nabla \nabla u \cdot \nu_{\Gamma_u}] (\nu - u)ds.
\] (2.19)
The first term in (2.18) controls the deviation from \( u \) in the \( H^2 \)-norm. The second term \( \mu_\varphi(\nu) \) defined by (2.19) can be viewed as an additional (nonlinear) measure of the difference \( \nu - u \). This term is nonnegative and vanishes if \( \nu = u \). Indeed, it is well-known that the problem (\( \mathcal{P} \)) is equivalent to the biharmonic variational inequality: find \( u \in \mathcal{K} \) such that
\[
\int_{\Omega} \{\nabla \nabla u : \nabla \nabla (\nu - u) - f(\nu - u)\} dx \geq 0 \quad \forall \nu \in \mathcal{K}.
\] (2.20)
Applying integration by parts two times and arguing in the same manner as in deriving (2.17), we transform the inequality (2.20) to the form
\[
\mu_\varphi(\nu) \geq 0 \quad \forall \nu \in \mathcal{K}.
\]
Notice that the first term in \( \mu_\varphi(\nu) \) is quite analogous to that in the classical obstacle problem (see [25]). The expression \( \mu_\varphi(\nu) \) plays the role of a "weight" function which is negative, so that the whole integral is zero or positive. The second term in \( \mu_\varphi(\nu) \) is of a new type, which vanishes if \( \nu = u \) on \( \Gamma_u \).

It is easy to see that \( \mu_\varphi(\nu) = 0 \) if \( \Omega^*_\varphi \subset \{ \xi \in \Omega | \nu(\xi) = \varphi(\xi) \} \), i.e., if the exact coincidence set contains in the coincidence set generated by the \( \nu \). In other cases, this measure will be positive. Thus, the measure \( \mu_\varphi(\nu) \) can be considered as a characteristic of how accurately the set \( \{ \nu = \varphi \} \) approximate the exact coincidence set \( \Omega^*_\varphi \). Obviously, this component is very weak and does not provide the desired information about the free boundary \( \Gamma_u \). However, it is impossible to get more information about the free boundary in the framework of the standard variational approach. Indeed, in view of the equality \( \mu(\nu) = J(\nu) - J(u) \), the measure \( \mu(\nu) \) tends to zero at all minimizing sequences. Moreover, this measure is the strongest among all measures that possess such a property.

Similarly, we see that if \( p^* \) is the maximizer of the dual variational problem (2.2) and \( n^* \in H(\Omega, \nabla \nu) \) is its approximation satisfying the condition (2.15), then the corresponding deviation measure has the form
\[
\mu^*(n^*) := \mathcal{D}_G(\Lambda u, n^*) + \mathcal{D}_F(u, -\Lambda^* n^*) = \frac{1}{2}\|p^* - n^*\|^2 + \mu_\varphi^*(n^*),
\] (2.21)
where
\[
\mu_\varphi^*(n^*) := \int_{\Omega} (f - \nabla \cdot (\nabla n^*)) (\varphi - u)dx.
\] (2.22)
The integral in (2.22) is non-negative. It can be considered as a measure penalizing (in weak integral sense) an incorrect behavior of the dual variable on the set \( \Omega^*_\varphi \) (where \( n^* \) must satisfy the differential equation). Finally, we note that \( \mu^*(n^*) = I^*(p^*) - I^*(n^*) \). Therefore, a sequence \( n^*_k \) is a maximizing sequence in the dual problem if and only if this error measure tends to zero. Regrettably, both measures \( \mu_\varphi \) and \( \mu^* \) are too weak to estimate how accurately the free boundary \( \Gamma_u \) is reproduced by the approximate solution. This fact shows limitations of direct variational methods in reconstruction of free boundaries (see also [25]).

The equalities (2.7), (2.13), (2.16), (2.18), and (2.21) imply the following result:

**Theorem 2.1.** For a function \( n^* \in H(\Omega, \nabla \nu) \) satisfying the condition (2.15) and a function \( \nu \in \mathcal{K} \) the identity
\[
\mu(\nu) + \mu^*(n^*) = \frac{1}{2}\|\nabla\nabla \nu - n^*\|^2 + \int_{\Omega} (f - \nabla \cdot (\nabla n^*)) (\varphi - u)dx,
\] (2.23)
holds. The left-hand side of (2.23) is a measure of the deviation of \( \nu \) from \( u \) and of \( n^* \) from \( p^* \), while the right-hand side of the above identity is a fully computable expression.
2.2. Extension of the Admissible set for $n^*$

Equality (2.23) provides a simple and transparent form of the error identity, but it operates with the functions $n^* \in H(\Omega, \text{div Div})$ satisfying the condition (2.15). This functional set is rather narrow and inconvenient if we wish to use in practice. In this subsection, we overcome this drawback and extend the admissible set for $n^*$.

**Lemma 2.1.** For any function $\tilde{n}^* \in H(\Omega, \text{div Div})$ the projection inequality

$$
\inf_{n^* \in Q^*} \left\| n^* - \tilde{n}^* \right\|^2 \leq C_{F_0} \left\| f - \text{div Div } \tilde{n}^* \right\|_\circ
$$

(2.24)

holds. Here $C_{F_0}$ is the constant defined by (2.27), and the set $Q^*$ is determined in (2.14).

**Proof.** For any function $m^* \in H(\Omega, \text{div Div})$ the equality

$$
\sup_{w \in V'(\Omega)} \int_\Omega \left( \frac{1}{2} |m^* - \tilde{n}^*|^2 + fw - m^* : \nabla \nabla w \right) dx = \frac{1}{2} \|n^* - \tilde{n}^*\|^2 + \sup_{w \in V'(\Omega)} \int_\Omega (f - \text{div Div } m^*) w dx
$$

holds. Therefore

$$
\inf_{n^* \in Q^*} \frac{1}{2} \left\| n^* - \tilde{n}^* \right\|^2 = \inf_{n^* \in N^*} \sup_{w \in V'(\Omega)} \int_\Omega \left( \frac{1}{2} |n^* - \tilde{n}^*|^2 + fw - n^* : \nabla \nabla w \right) dx.
$$

(2.25)

The Lagrangian defining the minimax formulation (2.25) is linear w.r.t. $w$ and convex w.r.t. $n^*$. For $w = 0$ it is coercive w.r.t. the dual variable. The space $N^*$ is a Hilbert one, and $V'(\Omega)$ is a convex closed subset of the reflexive space $V$. Using the well-known sufficient conditions providing the possibility of permutation inf and sup (see, e.g., [29, § 2, Ch. IV]), we can rewrite (2.25) in the form

$$
\inf_{n^* \in Q^*} \frac{1}{2} \left\| n^* - \tilde{n}^* \right\|^2 = \inf_{n^* \in N^*} \sup_{w \in V'(\Omega)} \int_\Omega \left( \frac{1}{2} |n^* - \tilde{n}^*|^2 + fw - n^* : \nabla \nabla w \right) dx.
$$

Examination of the infimum w.r.t. $n^* \in N^*$ is reduced to an algebraic problem whose solution satisfies the equation $n^* = \tilde{n}^* + \nabla \nabla w$ almost everywhere in $\Omega$. Using this equation and integrating by parts, we obtain

$$
\inf_{n^* \in Q^*} \frac{1}{2} \left\| n^* - \tilde{n}^* \right\|^2 = \sup_{w \in V'(\Omega)} \int_\Omega \left( -\frac{1}{2} |\nabla \nabla w|^2 + fw - \tilde{n}^* : \nabla \nabla w \right) dx
$$

= \sup_{w \in V'(\Omega)} \int_\Omega \left( -\frac{1}{2} |\nabla \nabla w|^2 + (f - \text{div Div } \tilde{n}^*) w \right) dx \leq \sup_{w \in V'(\Omega)} \int_\Omega \left( -\frac{1}{2} |\nabla \nabla w|^2 + (f - \text{div Div } \tilde{n}^*)_\circ w \right) dx.
$$

(2.26)

Successive application of the Friedrich’s type inequality

$$
\|w\| \leq C_{F_0} \|\nabla \nabla w\| \leq C_{F_0}^2 \|\nabla \nabla w\|
$$

(2.27)

allows us to estimate the last integral as follows:

$$
\int_\Omega (\text{div Div } \tilde{n}^* + f)_\circ w dx \leq C_{F_0}^2 \|f - \text{div Div } \tilde{n}^*\|_\circ \|\nabla \nabla w\|.
$$

(2.28)

Denoting $t := \|\nabla \nabla w\|$ and combining (2.26) with (2.28), we see that the supremum in (2.26) can be estimated from above by the quantity

$$
\sup_{t \geq 0} \left( -\frac{1}{2} t^2 + C_{F_0}^2 \|f - \text{div Div } \tilde{n}^*\|_\circ \|t\| \right) = \frac{1}{2} C_{F_0}^2 \|f - \text{div Div } \tilde{n}^*\|_\circ^2,
$$

which implies (2.24).
2.3. Majorant of the Measure of Deviation from the Exact Solution

Now we use Lemma 2.1 and identity (2.23) to obtain an estimate that holds for \( \tilde{v}^* \in H(\Omega, \text{div Div}) \). First of all, we have to transform the expressions for measure \( \mu^*(n^*) \) defined by the formula (2.21). Using the Young inequality (with the parameter \( \beta \)), we get the following lower bound for \( \mu^*(n^*) \):

\[
\mu^*(n^*) = \frac{1}{2} \left\| p^* - n^* + \tilde{n}^* - \tilde{v}^* \right\|^2 + \int_{\Omega} (f - \text{div Div}(n^* - \tilde{n}^*)(\varphi - u))dx \\
\geq \mu_\beta^*(\tilde{n}^*) + \frac{1}{2} \left( 1 - \frac{1}{\beta} \right) \left\| n^* - \tilde{n}^* \right\|^2 - \int_{\Omega} (\varphi - u) \text{div Div}(n^* - \tilde{n}^*)dx,
\]

(2.29)

where

\[
\mu_\beta^*(\tilde{n}^*) := \frac{1}{2} \left( 1 - \frac{1}{\beta} \right) \left\| p^* - \tilde{n}^* \right\|^2 + \mu_\varphi(\tilde{n}^*). 
\]

Now, we recall the identity (2.23). For the right-hand side, we obtain the upper bound

\[
\mu(v) + \mu^*(n^*) = \frac{1}{2} \left\| \nabla \nabla v - n^* \right\|^2 + \int_{\Omega} (f - \text{div Div} \tilde{n}^*)(\varphi - v)dx \\
+ \int_{\Omega} (f - \text{div Div}(n^* - \tilde{n}^*))(\varphi - v)dx \leq \frac{1}{2} (1 + \beta) \left\| \nabla \nabla v - \tilde{n}^* \right\|^2 + \frac{1}{2} \left( 1 + \frac{1}{\beta} \right) \left\| n^* - \tilde{n}^* \right\|^2 \\
+ \int_{\Omega} (f - \text{div Div} \tilde{n}^*)(\varphi - v)dx - \int_{\Omega} (\varphi - v) \text{div Div}(n^* - \tilde{n}^*)dx.
\]

(2.30)

The bounds (2.29) and (2.30) are valid for any function \( \tilde{n}^* \in H(\Omega, \text{div Div}) \).

Putting together (2.29) and (2.30), shifting the terms to the opposite side, and combining the similar terms, we get

\[
\mu(v) + \mu_\beta^*(\tilde{n}^*) \leq \frac{1}{2} (1 + \beta) \left\| \nabla \nabla v - \tilde{n}^* \right\|^2 + \int_{\Omega} (f - \text{div Div} \tilde{n}^*)(\varphi - v)dx \\
+ \frac{1}{\beta} \left\| n^* - \tilde{n}^* \right\|^2 + \int_{\Omega} (v - u) \text{div Div}(n^* - \tilde{n}^*)dx.
\]

(2.31)

Successive application of Hölder’s inequality and Young’s inequality (with parameter \( \beta \)) to the last term on the right hand side of (2.31) yields the inequality

\[
\int_{\Omega} (v - u) \text{div Div}(n^* - \tilde{n}^*)dx \leq \frac{1}{2} \left\| \nabla \nabla (u - v) \right\|^2 + \frac{1}{2\beta} \left\| n^* - \tilde{n}^* \right\|^2.
\]

(2.32)

Relations (2.31), (2.32), and (2.24) provide the desired estimate (2.33) where the right-hand side contains only known functions and can be computed explicitly.

**Theorem 2.2.** For any \( v \in K \) and \( \tilde{n}^* \in H(\Omega, \text{div Div}) \), the full measure of deviation of these functions from the exact solutions of direct and dual problems \( (u \text{ and } p^*) \), respectively is subject to the estimate

\[
\frac{1}{2} \left( 1 - \frac{1}{\beta} \right) \left\| \nabla \nabla (u - v) \right\|^2 + \left\| p^* - \tilde{n}^* \right\|^2 + \mu_\varphi(v) + \mu_\beta^*(\tilde{n}^*) \leq \mathcal{M}(v, \tilde{n}^*, f, \varphi, \beta),
\]

(2.33)

where

\[
\mathcal{M}(v, \tilde{n}^*, f, \varphi, \beta) := \frac{1}{2} (1 + \beta) \left\| \nabla \nabla v - \tilde{n}^* \right\|^2 + \frac{3}{2\beta} C_{\tilde{v}_0} \left\| (f - \text{div Div} \tilde{n}^*) \right\|^2 + \int_{\Omega} (f - \text{div Div} \tilde{n}^*)(\varphi - v)dx,
\]

a parameter \( \beta \in (0, 1] \), and \( C_{\tilde{v}_0} \) is the same constant as in Lemma 2.1.
Remark 2.1. In (2.29)–(2.32), we used Young’s inequality with the same constant $\beta$. In general, the constants can be taken different. Then, after an optimization (with respect to the constants) we get a more accurate (but also more cumbersome) expression for the majorant $M$, which we do not include here.

3. NUMERICAL EXAMPLES

In this section, we consider two examples that demonstrate how the identity (2.23) and the estimate (2.33) work in practice.

First, we consider a model 1D problem, where the exact solution is known and, therefore, we can explicitly compute approximation errors associated with the primal and dual variables. In this example, the approximate solution has essentially smaller coincidence set than the exact one. Nevertheless the error identity holds and error estimates computed for a regularized dual approximation are quite sharp.

Another example is motivated by an obstacle problem with a plane obstacle for radially symmetric plate which is fixed on the boundary. Here, the results are similar to those obtained in the 1D model problem and illustrate the validity of the error identity (2.23).

Certainly, it will be interesting to apply these estimates for those cases, where approximations are constructed by some standard (e.g., FEM) approximations of the biharmonic obstacle problem. However, this question is beyond the framework of the present paper.

3.1. Model 1D Problem

Let $\Omega = (-1,1)$, let $\varphi \equiv -1$, and let $f \equiv c$. For $c = -1152$, the minimizer of the problem (1.1) has the form

$$u(x) = \begin{cases} -8(x+1)^2(6x^2 + 4x + 1) & \text{if } -1 \leq x < -0.5, \\ 1 & \text{if } x \in \Omega_\varphi := [-0.5,0.5], \\ -8(x-1)^2(6x^2 - 4x + 1) & \text{if } 0.5 < x \leq 1. \end{cases}$$

This function satisfies the boundary conditions $u(\pm1) = u'(\pm1) = 0$ and the equation $u'''' + c = 0$ in $\Omega_0 = (-1,-0.5) \cup (0.5,1)$. Notice also that the flux

$$p^* = u^{\prime\prime}(x) = \begin{cases} -48(2x+1)(6x + 5) & \text{if } -1 < x < -0.5, \\ 0 & \text{if } x \in \Omega_\varphi, \\ -48(2x-1)(6x - 5) & \text{if } 0.5 < x < 1, \end{cases}$$

does not satisfy the condition (2.10) which in this case reduces to $(n^*)'' \in L^2(-1,1)$. Function $p^*$ and its derivative are shown in Fig. 1. It is easy to see that $(p^*)'' \not\in L^2(-1,1)$.
Consider the function

\[ v_1(x) = \begin{cases} 
\frac{-16}{27}(x+1)^2(1-8x) & \text{if } -1 < x < -0.25, \\
-1 & \text{if } -0.25 \leq x \leq 0.25, \\
\frac{-16}{27}(x-1)^2(1+8x) & \text{if } 0.25 < x < 1.
\end{cases} \]

Obviously, \( v_1 \in \mathbb{K} \) and \( \{x \in \Omega \mid v_1(x) = -1\} \subset \Omega_\varphi \) (see Fig. 2).

As an approximation of the flux, we first consider the function

\[ n^*(x) = \begin{cases} 
20(2x+1)^2(5+6x) & \text{if } -1 < x < -0.5, \\
0 & \text{if } -0.5 \leq x \leq 0.5, \\
20(2x-1)^2(5-6x) & \text{if } 0.5 < x < 1.
\end{cases} \]

Notice that \( n^* \) satisfies the conditions (2.10) and (2.15) (see Fig. 3). According to (2.18) and (2.19) the measure \( \mu(v_1) \) consists of two terms. In this example they can be calculated:

\[
\frac{1}{2} \left\| \nabla(u - v_1) \right\|^2 = \int_0^{0.5} (u'' - v_1'')^2 \, dx = \int_{0.25}^{0.5} (v_1'')^2 \, dx + \int_{0.5}^1 (u'' - v_1'')^2 \, dx \approx 125.15
\]

(3.1)

and

\[
\mu_\varphi(v_1) = \int_{-0.5}^{0.5} \left( \text{div} \, \nabla \varphi(f - v_1 - \varphi) - [u'''(-0.5)](v_1 - u) \big|_{x=-0.5} - [u'''(0.5)](v_1 - u) \right)_{x=0.5} = 152.89.
\]

(3.2)

Here, \([u'''(a)] := [u'''(a-0) - u'''(a+0)]\) denotes the jump at the point \( a \).

The measure \( \mu^*(n^*) \) is calculated in accordance with (2.21) and (2.22):

\[
\frac{1}{2} \left\| p - n^* \right\|^2 = \int_0^1 (p - n^*)^2 \, dx = 74.74,
\]

(3.3)

\[
\mu_\varphi^*(n^*) = 2 \int_{0.5}^1 (f' - n'')(\varphi - u) \, dx = 156.8.
\]

(3.4)

Combination of (3.1), (3.2), (3.3), and (3.4) implies the following full error measure for deviations of the functions \( v_1 \) and \( n^* \) from the exact solutions of direct and dual problems, respectively. We obtain

\[
\mu(v_1) + \mu^*(n^*) = 278.04 + 231.54 = 509.58.
\]

(3.5)
Consider the right-hand side of the error identity (2.23) (since the chosen function $n^*$ satisfies the condition (2.15) this identity holds). Direct calculation
\[
\frac{1}{2} \left\| \nabla \nu_1 - n^* \right\|^2 = \int_0^1 (\nu_1'' - n^*)^2 \, dx = \frac{1}{25} \int_0^{0.25} (\nu_1'' - n^*)^2 \, dx + \frac{1}{0.5} \int_0^{0.5} (\nu_1'' - n^*)^2 \, dx \approx 23.063
\]
and
\[
\int_{-1}^1 (f - \text{div} \nabla n^*)(\varphi - \nu_1) \, dx = 2 \int_{0.25}^{0.5} f(\varphi - \nu_1) \, dx + 2 \int_{0.5}^{1} (f - \nu_1'')(\varphi - \nu_1) \, dx \approx 486.515.
\]
Thus, the sum of these terms gives the same value 509.58 as the sum of measures (3.5).

Next, we take $\tilde{n}^*$ such that this function satisfies the condition (2.10), but $\tilde{n}^*$ does not satisfy the condition (2.15). Set $\tilde{n}^*$ by the formula
\[
\tilde{n}^*(x) = \begin{cases} 
8(3x + 1)^2(6x + 5) & \text{if } -1 \leq x < -1/3, \\
0 & \text{if } -1/3 \leq x \leq 1/3, \\
8(3x - 1)^2(5 - 6x) & \text{if } 1/3 < x \leq 1.
\end{cases}
\]
On the set $x \in (1, 17/18) \cup [17/18, 1)$ the condition (2.15) does not hold. Therefore, we cannot use the error identity (2.23) but can use the estimate (2.33). Let us verify how accurately it holds.

By direct calculations we obtain
\[
\frac{1}{2} \left\| p^* - \tilde{n}^* \right\|^2 = \frac{1}{25} \int_{0.25}^{0.5} (\tilde{n}^*)^2 \, dx + \frac{1}{0.5} \int_{0.5}^{1} (p^* - \tilde{n}^*)^2 \, dx \approx 24.9137,
\]
\[
\mu_\varphi^p(\tilde{n}^*) = 2 \int_{0.25}^{0.5} (f - \text{div} \nabla \tilde{n}^*)(\varphi - u) \, dx = 2 \int_{0.25}^{0.5} (f - \tilde{n}^*)'\varphi' - u) \, dx \approx 72.0,
\]
\[
\left\| \nabla \nu_1 - \tilde{n}^* \right\|^2 = 2 \int_{0.25}^{0.5} (\nu_1'')^2 \, dx + 2 \int_{0.5}^{1} (\nu_1'' - \tilde{n}^*)^2 \, dx \approx 66.16,
\]
\[
\frac{1}{2} \left\| (f - \text{div} \nabla \tilde{n}^*)_\varphi \right\|^2 = \int_{17/18}^{1} (f - \tilde{n}^*)'\varphi' \, dx \approx 384.0,
\]
\[
\int_{-1}^{1} (f - \text{div} \nabla \tilde{n}^*)(\varphi - \nu_1) \, dx = 2 \int_{0.25}^{0.5} f(\varphi - \nu_1) \, dx + 2 \int_{0.5}^{1} (f - \tilde{n}^*)'\varphi' \, dx \approx 51.0947 + 268.267 = 319.36.
\]
Recall that for $\Omega = (-1,1)$ we have $C_{f_0} = 4/\pi^2$. Thus, according to (2.33) for any $\beta \in (0,1]$ the majorant $\mathcal{M}(v_1, \bar{n}^*, f, \varphi, \beta)$ has the form

$$\mathcal{M}(v_1, \bar{n}^*, -1152, -1, \beta) = 352.44 + 33.08\beta + 189.22\frac{1}{\beta}.$$ 

Taking into account (3.1) and (3.2), we get the expression $524.95 - 150.06\beta$ for the left-hand side of the inequality for (2.33). Thus, for any $\beta \in (0,1]$, this inequality takes the form

$$524.95 - 150.06\beta \leq 352.44 + 33.08\beta + 189.22\frac{1}{\beta}. \quad (3.6)$$

In particular, for $\beta = 0.5$ and $\beta = 1$ the ratio of the majorant (the r.h.s. of (3.6)) to the deviation measure (the l.h.s. of (3.6)) is characterized by the values 1.66 and 1.53, respectively.

Further, we consider the approximations (see Fig. 4)

$$v_\varepsilon(x) = \begin{cases} -\frac{4}{(2\varepsilon + 1)^2}(1 + x)^2(-4x + 6\varepsilon - 1) & \text{if } -1 \leq x < \varepsilon - \frac{1}{2}, \\ -1 & \text{if } \varepsilon - \frac{1}{2} \leq x \leq \frac{1}{2} - \varepsilon, \\ -\frac{4}{(2\varepsilon + 1)^2}(1 - x)^2(4x + 6\varepsilon - 1) & \text{if } \frac{1}{2} - \varepsilon < x \leq 1, \end{cases}$$

where $\varepsilon$ is the parameter satisfying $0 \leq \varepsilon \leq 1/2$. For these approximations we have $\{x \in \Omega | v_\varepsilon = -1\} \subset \Omega_\varphi$. Notice that we get a better approach of the coincidence set $\{x \in \Omega | u = -1\}$ as $\varepsilon \to 0$. Moreover, the function $v_0$ coincides with $\varphi$ on the exact coincidence set (but this function does not coincide with $u$).

Approximations $n_\varepsilon^*$ of the exact flux $p^*$ are constructed by smoothing the second derivative of $v_\varepsilon$ (which replace $\nabla\nabla v_\varepsilon$) such that $(n_\varepsilon^*)'' \in L^2(-1,1)$. The latter corresponds to the condition $n_\varepsilon^* \in H(\text{div Div}, \Omega)$. In particular, if we take

$$n_\varepsilon^*(x) = \begin{cases} \frac{48}{(2\varepsilon + 1)^3}(-2x + 2\varepsilon - 1)(4x - 2\varepsilon + 3) & \text{if } -1 \leq x < \varepsilon - \frac{1}{2}, \\ 0 & \text{if } \varepsilon - \frac{1}{2} \leq x \leq \frac{1}{2} - \varepsilon, \\ \frac{48}{(2\varepsilon + 1)^3}(2x + 2\varepsilon - 1)(-4x - 2\varepsilon + 3) & \text{if } \frac{1}{2} - \varepsilon < x \leq 1, \end{cases}$$

then (2.15) is satisfied, and we can use the error identity (2.23).
Table 1. Components of the measure $\mu(v_\varepsilon)$

| $\varepsilon$ | $\frac{1}{2}\|
abla\nabla(u - v_\varepsilon)\|^2$ | $\mu(\varphi_\varepsilon)$ | $J(\varphi_\varepsilon) - J(u)$ | $k(\varphi_\varepsilon)$ [%] |
|-----|---------------------------------|---------------------------|-------------------|------------------|
| 0.35 | 134.060                         | 250.280                   | 384.340           | 65.12            |
| 0.25 | 125.156                         | 152.889                   | 278.044           | 54.99            |
| 0.15 | 109.904                         | 68.192                    | 178.096           | 38.29            |
| 0.05 | 81.474                          | 9.852                     | 91.326            | 1.06             |
| 0.00 | 57.600                          | 0                         | 57.60             | 0                |

Table 2. Components of the measure $\mu^*(n_\varepsilon^*)$

| $\varepsilon$ | $\frac{1}{2}\|p^* - n_\varepsilon\|^2$ | $\mu^*(n_\varepsilon^*)$ | $I^*(p^*) - I^*(n_\varepsilon^*)$ | $k(n_\varepsilon^*)$ [%] |
|-----|-----------------------------------|---------------------------|------------------|------------------|
| 0.35 | 119.444                           | 422.937                   | 542.381          | 77.98            |
| 0.25 | 109.443                           | 400.119                   | 509.562          | 78.52            |
| 0.15 | 95.972                            | 357.378                   | 453.349          | 78.83            |
| 0.05 | 78.510                            | 270.053                   | 348.563          | 77.48            |
| 0.00 | 68.571                            | 192                       | 260.571          | 73.68            |

Table 1 contains results related to the components of $\mu(v_\varepsilon)$ computed for $\varepsilon = 0.05j$, $j = 7, 5, 3, 1$, and 0. Both terms (quadratic and nonlinear) decrease as $\varepsilon \to 0$. However, the first term remains positive since the sequence of approximate solutions does not tend to the exact solution $u$, while the second term tends to zero, because the corresponding sequence of the approximated coincidence sets tends to the exact set $\Omega_\varphi$. It is easy to see that the sum of these two terms (constituting a measure $\mu(v_\varepsilon)$) is equal to the deviation of $J(\varphi_\varepsilon)$ from the exact minimum of the direct variational problem. In the last column of Table 1, we present the relative contribution of the nonlinear measure $\mu(\varphi_\varepsilon)$ expressed by the quantity

$$k(\varphi_\varepsilon) := 100 \frac{\mu(\varphi_\varepsilon)}{\mu(v_\varepsilon)} \text{ [%]}.$$

Table 2 contains similar data related to the measure $\mu^*(n_\varepsilon^*)$. As in the case of the measure $\mu(\varphi_\varepsilon)$, both quadratic and nonlinear terms decrease as $\varepsilon \to 0$. Nevertheless, this nonlinear measure does not tend to zero. The measure controls the violation of the equation $\text{div} \, \text{Div} \, n_\varepsilon^* = f$ on the set $\Omega_\varphi$. The function $n_\varepsilon^*$ does not satisfy the latter equation for any $\varepsilon$, and, consequently, $\mu^*(n_\varepsilon^*) > 0$. Moreover, in this case the measure $\mu^*(n_\varepsilon^*)$ evidently dominates the first term. This fact is confirmed by the quantity

$$k(n_\varepsilon^*) := 100 \frac{\mu^*(n_\varepsilon^*)}{\mu^*(n_\varepsilon^*)} \text{ [%]}$$

presented in the last column.

Table 3 reports on the error identity (2.23). First two columns correspond to the two terms forming the right-hand side of (2.23). Here, $N(v_\varepsilon, n_\varepsilon^*)$ denotes the term $\int_{\Omega} (f - (n_\varepsilon^*)) (\varphi - v_\varepsilon) d\Omega$ corresponding to the summand $\int_{\Omega} (f - \text{div} \, \text{Div} \, n_\varepsilon^*) (\varphi - v_\varepsilon) d\Omega$ in the identity. The sum of these terms is given in the third column. It coincides exactly with the sum of measures given in Tables 1 and 2. Observe that the values in the first two columns of Table 3 are computed directly by functions $v_\varepsilon$ and $n_\varepsilon^*$. These functions can be considered as approximate solutions constructed with the help of some computational procedure. The table shows that the sum

$$\frac{1}{2}\|\nabla\nabla v_\varepsilon - n_\varepsilon^*\|^2 + N(v_\varepsilon, n_\varepsilon^*)$$

is a good and easily computed characteristic of the quality of approximate solutions.
3.2. Bending of a Circular Plate

Let \( \Omega = B_3 \subset \mathbb{R}^2 \), where \( B_3 \) denotes the open ball centered at the origin with radius 3. In this case, the problem \( \mathcal{P} \) can be considered as a simplified version of the bending problem for a clamped elastic circular plate above the plane obstacle \( \varphi \equiv -1 \) under the action of an external force \( f \). Simplification is that we replace the tensor of the elastic constants by the unit tensor. In the context of the considered issues, such a simplification does not play a significant role. We set

\[
\varphi \equiv -1, \quad f \equiv \frac{c_1}{c_2}, \quad \text{where} \quad c_1 = 9 \ln 3 - 4, \quad c_2 = 208 - 216 \ln 3 + 9 \ln^2 3.
\]

Notice that for the given data we can explicitly define the radial solution of the problem. In the polar coordinates \((r, \theta)\), the minimizer has the following form:

\[
u(r, \theta) = \begin{cases} 
-1 & \text{if} \quad 0 < r < 1, \\
\frac{(r^2 - 1)(128 + c_1(r^2 - 3)) + 4(c_1 - 32(1 + r^2)) \ln r}{4c_2} - 1 & \text{if} \quad 1 \leq r \leq 3.
\end{cases}
\]

It is clear that \( \Delta^2 u = f_0 \) in \( B_3 \setminus B_1 \), \( u \geq -1 \) in \( B_1 \), and \( u(3, \theta) = \frac{\partial u}{\partial r}(3, \theta) = 0 \) for all \( \theta \in [0, 2\pi) \). An elementary calculation shows that

\[
p^* = \left( \begin{array}{c} p_{11}^* \\ p_{12}^* \\ p_{21}^* \\ p_{22}^*
\end{array} \right),
\]

where \( p_{11}^* = p_{12}^* = p_{21}^* = p_{22}^* = 0 \) in the ball \( B_1 \), while for \((r, \theta) \in B_3 \setminus B_1\), the components of \( p^* \) are defined by the formulas

\[
p_{11}^* = \frac{(r^2 - 1) \cos(2\theta)(c_1(r^2 + 1) - 32) + 2r^2(c_1(r^2 - 1) - 32 \ln r)}{c_2 r^2},
\]

\[
p_{12}^* = p_{21}^* = \frac{(r^2 - 1) \sin(2\theta)(c_1(r^2 + 1) - 32)}{c_2 r^2},
\]

\[
p_{22}^* = \frac{(1 - r^2) \cos(2\theta)(c_1(r^2 + 1) - 32) + 2r^2(c_1(r^2 - 1) - 32 \ln r)}{c_2 r^2}.
\]

It is easy to check that \( \text{div} \text{Div} p^* \notin L^2(B_3) \) and \( \text{div} \text{Div} p^* = f_0 \) on the set \( B_3 \setminus B_1 = \Omega \setminus \{u = \varphi\} \).

Further, we define the function

\[
v_2(r, \theta) = u(r, \theta) + \begin{cases} 
0.5[1 - \cos(\pi(3 - r))] & \text{if} \quad 1 \leq r \leq 3, \\
0 & \text{if} \quad 0 < r < 1.
\end{cases}
\]

It is clear that \( v_2 \in L^2 \) and \( v_2 \geq u \) in \( \Omega \) and \( v_2 = u \) in \( B_1 \). Thus, \( v_2 \) has the same coincidence set \( B_1 \) as the minimizer \( u \) (see Fig. 5).
By direct computations of the measure (2.18), we get
\[ \mu(v_2) = \frac{1}{2} \| \nabla \nabla (u - v_2) \|^2 + \mu_\phi(v_2) = \frac{1}{2} \| \nabla \nabla (u - v_2) \|^2 = 157.19. \] (3.7)

We point out that \( \mu_\phi(v_2) = 0 \), since \( \{ x \in \Omega | v_2 = -1 \} = \Omega_\phi \).

Let us set now
\[ \hat{\mu} = \begin{pmatrix} \hat{\mu}_{11}^* \\ \hat{\mu}_{12}^* \\ \hat{\mu}_{21}^* \\ \hat{\mu}_{22}^* \end{pmatrix}, \]

where
\[ \hat{\mu}_{11}^* = \hat{\mu}_{22}^* = \begin{cases} 0 & \text{if } (r, \theta) \in B_1, \\ (r - 1)^3 \cos(2\theta) & \text{if } (r, \theta) \in B_2 \setminus B_1, \end{cases} \]
\[ \hat{\mu}_{12}^* = \hat{\mu}_{21}^* = \begin{cases} 0 & \text{if } (r, \theta) \in B_1, \\ 9(r - 1)^3 \sin(2\theta) & \text{if } (r, \theta) \in B_2 \setminus B_1. \end{cases} \]

The function \( \hat{\mu}^* \) satisfies the condition (2.15). Therefore, we can verify the validity of the identity (2.23).

Computing the error measure \( \mu^* (\hat{\mu}^*) \) defined by (2.21) and (2.22), we get
\[ \mu^* (\hat{\mu}^*) = \frac{1}{2} \| \rho^* - \hat{\mu}^* \|^2 + \int_{B_1 \setminus B_1} (f - \text{div} \, \hat{\mu}^*)(-1 - u)rdrd\theta = 14.84 + 63.46 = 78.30. \] (3.8)

Combination (3.7) and (3.8) implies the following value for the full error measure (the result is rounded to two decimal places)
\[ \mu(v_2) + \mu^* (\hat{\mu}^*) = 157.19 + 78.30 = 235.49. \] (3.9)

To compute the terms on the right-hand side of the identity (2.23), we use only the functions \( v_2 \) and \( \hat{\mu}^* \) (known approximations of the exact solutions). The sum of these terms gives the same value as (3.9):
\[ \frac{1}{2} \| \nabla^2 v_2 - \hat{\mu}^* \|_2^2 - \int_{B_1} (f - \text{div} \, \hat{\mu}^*)(1 + v_2)rdrd\theta = 111.15 + 124.34 = 235.49. \]

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\text{Fig. 5.} \text{ The exact solution } u \text{ (a) and the function } v_2 \text{ (b).}
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