On some free boundary problems of the prey-predator model

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Abstract. In this paper we investigate some free boundary problems for the Lotka-Volterra type prey-predator model in one space dimension. The main objective is to understand the asymptotic behavior of the two species (prey and predator) spreading via a free boundary. We prove a spreading-vanishing dichotomy, namely the two species either successfully spread to the entire space as time $t$ goes to infinity and survive in the new environment, or they fail to establish and die out in the long run. The long time behavior of solution and criteria for spreading and vanishing are also obtained. Finally, when spreading successfully, we provide an estimate to show that the spreading speed (if exists) cannot be faster than the minimal speed of traveling wavefront solutions for the prey-predator model on the whole real line without a free boundary.

Keywords: Prey-predator model; Free boundary problems; Spreading-vanishing dichotomy; Long time behavior; Criteria.

AMS subject classifications (2000): 35K51, 35R35, 92B05, 35B40.

1 Introduction

The expanding of the new or invasive species is one of the most important topics in mathematical ecology. A lot of mathematicians have made efforts to develop various invasion models and investigated them from a viewpoint of mathematical ecology. In this paper we consider three free boundary problems for the Lotka-Volterra type prey-predator model.

In the real world, the following phenomena will happen constantly:

(i) At the initial state, one kind of pest species (prey) occupied some bounded area (initial habitat). In order to control the pest species, the biological method is to put some natural enemies (predator) in this area;

(ii) There is some kind of species (prey) in a bounded area (initial habitat), and at some time (initial time) another kind of species (the new or invasive species, predator) enters this area.

In general, both prey and predator have a tendency to emigrate from the boundary to obtain their new habitat, i.e., they will move outward along the unknown curve (free boundary) as time increases. We argue that such prey in this model is the most favored food of the predator, and its spreading behavior has such a dominant influence of spreading of the predator that they roughly share the same spreading front. It is reasonable to assume that the free boundary invades at a rate that is proportional to the magnitudes of the prey and predator populations gradients there. We want to realize the dynamics/variations of prey, predator and free boundary. For simplicity, we assume that the interaction between prey and predator obeys the Lokta-Volterra law, and restrict

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our problem to the one dimensional case. Under the suitable rescaling, the model we are concerned here becomes one of the following three free boundary problems:

(i) Left boundary is fixed with zero Dirichlet boundary conditions and the right boundary is free

\[
\begin{align*}
\begin{cases}
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= u(1-u-av), & t > 0, & 0 < x < h(t), \\
\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} &= v(c-v+bu), & t > 0, & 0 < x < h(t), \\
u &= v = 0, & t > 0, & x = 0, \\
u &= v = 0, & h'(t) = -\mu(u_x + \rho v_x), & t > 0, & x = h(t), \\
u(0,x) &= u_0(x), & v(0,x) &= v_0(x), & x \in [0,h_0], \\
h(0) &= h_0;
\end{cases}
\end{align*}
\]

(DFB)

(ii) Left boundary is fixed with zero Neumann boundary conditions and the right boundary is free

\[
\begin{align*}
\begin{cases}
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= u(1-u-av), & t > 0, & 0 < x < h(t), \\
\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} &= v(c-v+bu), & t > 0, & 0 < x < h(t), \\
u_x &= v_x = 0, & t > 0, & x = 0, \\
u &= v = 0, & h'(t) = -\mu(u_x + \rho v_x), & t > 0, & x = h(t), \\
u(0,x) &= u_0(x), & v(0,x) &= v_0(x), & x \in [0,h_0], \\
h(0) &= h_0;
\end{cases}
\end{align*}
\]

(NFB)

(iii) With double free boundaries

\[
\begin{align*}
\begin{cases}
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= u(1-u-av), & t > 0, & g(t) < x < h(t), \\
\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} &= v(c-v+bu), & t > 0, & g(t) < x < h(t), \\
u &= v = 0, & g'(t) = -\mu_l(u_x + \rho_l v_x), & t > 0, & x = g(t), \\
u &= v = 0, & h'_r(t) = -\mu_r(u_x + \rho_r v_x), & t > 0, & x = h(t), \\
u(0,x) &= u_0(x), & v(0,x) &= v_0(x), & x \in [-h_0,h_0], \\
g(0) &= -h_0, & h(0) &= h_0.
\end{cases}
\end{align*}
\]

(TFB)

In the above three problems, \(x = g(t)\) and \(x = h(t)\) represent the left and right moving boundaries, respectively, which are to be determined, \(a,b,c,d,h_0,\mu,\rho,\mu_l,\mu_r,\rho_l\) and \(\rho_r\) are given positive constants. The initial functions \(u_0(x),v_0(x)\) satisfy

(DFB1) \(u_0, \quad v_0 \in C^2([0,h_0]), \quad u_0(0) = v_0(0) = u_0(h_0) = v_0(h_0) = 0, \quad u_0(x), v_0(x) > 0 \) in \((0,h_0)\) for the problem (DFB);

(NFB1) \(u_0, \quad v_0 \in C^2([0,h_0]), \quad u_0'(0) = v_0'(0) = u_0(h_0) = v_0(h_0) = 0, \quad u_0(x), v_0(x) > 0 \) in \((0,h_0)\) for the problem (NFB);

(TFB1) \(u_0, \quad v_0 \in C^2([-h_0,h_0]), \quad u_0(\pm h_0) = v_0(\pm h_0) = 0, \quad u_0(x), v_0(x) > 0 \) in \((-h_0,h_0)\) for the problem (TFB).

In both problems (DFB) and (NFB), it is assumed that the species can only invade further into the environment from the right end of the initial region. While in the problem (TFB), it is assumed that the species can invade further into the environment from two ends of the initial region.
The ecological backgrounds of free boundary conditions in the above problems can also refer to [1]. Such kind of free boundary conditions have been used in [14, 16, 21–24] and [29].

Recently, Wang and Zhao [29, 30] studied the similar free boundary problems to (TFB) with double free boundaries in which the prey lives in the whole space but the predator lives in the region enclosed by the free boundary. Especially, in [30], the authors dealt with the higher dimension and heterogeneous environment case. They have established the spreading-vanishing dichotomy, long time behavior of solution and criteria for spreading and vanishing.

Since the solution of (TFB) has the same properties as that of (NFB), we only discuss problems (DFB) and (NFB) in the following. For the global existence and uniqueness of solution, similar to the proof of Theorem 2.1, Lemma 2.1 and Theorem 2.2 in [29], we can prove the following theorem.

**Theorem 1.1** Any one of (DFB) and (NFB) has a unique global solution, and for any \(\alpha \in (0, 1)\) and \(T > 0\),

\[
(u, v, h) \in [C^{1+\alpha,1+\alpha}(\mathbb{R}^2 \cap D_T)]^2 \times C^{1+\alpha,1+\alpha}([0, T]),
\]

where

\[
D_T = \left\{ (t, x) \in \mathbb{R}^2 : t \in (0, T], x \in (0, h(t)) \right\}.
\]

Furthermore, there exists a positive constant \(M\) such that

\[
0 < u(t, x), v(x, t) \leq M \quad \text{for} \quad 0 < t < \infty, \quad 0 < x < h(t),
\]

\[
0 < h'(t) \leq M \quad \text{for} \quad 0 < t < \infty.
\]

In the absence of \(v\), problems (DFB) and (NFB) are reduced to the one phase Stefan problems which were studied by Kaneko and Yamada [17] and Du and Lin [11]. The well-known Stefan condition has been used in the modeling of a number of applied problems. For example, it was used to describe the melting of ice in contact with water [28], the modeling of oxygen in the muscle [5], the wound healing [3], the tumor growth [4], and the spreading of species [7, 8, 10, 11, 13, 14, 16, 17, 21, 29, 30]. There is a vast literature on the Stefan problems, and some important theoretical advances can be found in [2, 5] and the references therein.

Some similar free boundary problems have been used in two-species models in several earlier papers; please refer to, for example, [15, 16, 21, 22, 23, 24] over a bounded spatial interval, and [14] over the half spatial line for the competition model. For the study of free boundary problems of other type ecological models, we refer to, for instance, [4, 9, 12, 14, 26] and references cited therein.

The organization of this paper is as follows. To study the long time behavior of solution to the problem (DFB), in Section 2 we discuss its stationary solutions. Section 3 is devoted to the long time behavior of \((u, v)\) and get a spreading-vanishing dichotomy. To establish the criteria for spreading and vanishing, in Section 4 we provide some comparison principles. The criteria for spreading and vanishing will be given in Section 5. In Section 6, we study the estimation of asymptotic spreading speed. The last section is a brief discussion.
2 Positive solutions of the corresponding elliptic problems in half line

To discuss the long time behaviour of solution to the problem (DFB), we first discuss its stationary solutions. The stationary problem of (DFB) is the following elliptic problem in the half line

\[
\begin{cases}
-u'' = u(1 - u - av), & 0 < x < \infty, \\
-dv'' = v(c - v + bu), & 0 < x < \infty, \\
u(0) = v(0) = 0.
\end{cases}
\] (2.1)

The main purpose of this section is to study the existence of positive solutions of (2.1). To this aim, we first study the existence and uniqueness of positive solution to the following problem of single equation:

\[
\begin{cases}
-du'' = u(f(x) - \lambda u), & 0 < x < \infty, \\
u(0) = 0,
\end{cases}
\] (2.2)

where \(d\) and \(\lambda\) are positive constants. When \(f\) is a positive constant, by Proposition 2.1 of [1] or Proposition 4.1 of [11], the problem (2.2) has a unique positive solution \(u(x)\). Moreover, \(u'(x) > 0\) and \(\lim_{x \to \infty} u(x) = f/\lambda\).

2.1 The existence, uniqueness and stability of positive solution to (2.2)

**Theorem 2.1** Assume that \(f\) satisfies

\[f \in C_\alpha([0, \infty)) \text{ with } 0 < \alpha < 1, \quad \inf_{x \geq 0} f(x) := f_0 > 0, \quad \|f\|_{\infty} < \infty.\] (2.3)

Then the problem (2.2) has a unique positive solution \(u(x)\). Furthermore,

(i) if \(f(x)\) is increasing in \(x\), so is \(u(x)\) and \(\lim_{x \to \infty} u(x) = \frac{1}{\lambda} \lim_{x \to \infty} f(x)\);

(ii) if \(f(x)\) is decreasing in \(x\), then either \(u(x)\) is increasing in \(x\), or there exists \(x_0 > 0\) such that \(u(x)\) is increasing in \((0, x_0)\) and \(u(x)\) is decreasing in \((x_0, \infty)\). Therefore, \(\lim_{x \to \infty} u(x) = \frac{1}{\lambda} \lim_{x \to \infty} f(x)\).

**Proof.** We first analyse the properties of positive solution \(u\) to the problem (2.2). It is obvious that \(u'(0) > 0\). Moreover,

(a) if \(u(x_1) > \frac{1}{\lambda} \|f\|_{\infty}\) and \(u'(x_1) \geq 0\) for some \(x_1 \in (0, \infty)\), then \(u(x)\) must approach infinity as \(x\) approaches some finite \(x_2\). This is impossible since \(u(x)\) is defined in \([0, \infty)\);

(b) if \(u(x_1) < f_0/\lambda\) and \(u'(x_1) \leq 0\) for some \(x_1 \in (0, \infty)\), then \(u\) must vanish at some finite \(x_2\). This is also impossible as \(u(x)\) is positive in \((0, \infty)\).

So we have

\[u'(0) > 0, \quad \sup_{x \geq 0} u(x) \leq \frac{1}{\lambda} \|f\|_{\infty},\] (2.4)

and there exists some positive constant \(\tau\), depending on \(d, \lambda\) and \(f\), such that

\[u(x) \geq \tau, \quad \forall \ x \geq 1.\] (2.5)
Next, we prove the existence. It is well known that, when \( l > \pi \sqrt{d/f_0} \), the problem
\[
\begin{aligned}
-du'' &= u(f(x) - \lambda u), \quad 0 < x < l, \\
u(0) &= 0 = u(l)
\end{aligned}
\] (2.6)
has a uniqueness positive solution, denoted by \( u_l \), and \( u_l \) satisfies
\[
\sup_{0 \leq x \leq l} u_l(x) \leq \frac{1}{\lambda} \|f\|_\infty.
\]
Moreover, by the comparison principle ([6, Lemma 5.2], or [18, Proposition 2.1]) we have that \( u_l(x) \) is increasing in \( l \). In view of the regularity theory and compactness argument, it follows that there exists a positive function \( u \), such that \( u_l \to u \) in \( C^2_{\text{loc}}([0, \infty)) \) as \( l \to \infty \), and \( u \) solves (2.2).

The uniqueness is followed from the following comparison principle (Proposition 2.1).

At last, notice that for any \( x \in (0, \infty) \), \( u(x) \leq \frac{1}{\lambda} f(x) \) if \( u''(x) \leq 0 \), while \( u(x) \geq \frac{1}{\lambda} f(x) \) if \( u''(x) \geq 0 \), it is easily seen that conclusions (i) and (ii) hold. \( \square \)

Now we give the comparison principle. Let \( d \) and \( \lambda \) be positive constants. Assume that \( f_i \in C^\alpha_{\text{loc}}([0, \infty)) \) and satisfies \( 0 < \inf_{x \geq 0} f_i(x) \leq \|f_i\|_\infty < \infty \) for \( i = 1, 2 \). By Theorem 2.1, the problem
\[
\begin{aligned}
-du'' &= u(f_i(x) - \lambda u), \quad 0 < x < \infty, \\
u(0) &= 0
\end{aligned}
\]
has a positive solution, denoted by \( u_i \).

**Proposition 2.1 (Comparison principle)** Under the above conditions, if \( f_1(x) \leq f_2(x) \) for all \( x \geq 0 \), then we have that
\[
u_1(x) \leq u_2(x), \quad \forall \ x \geq 0.
\]

**Proof.** First, by (2.4) and (2.5), we have that \( u_i'(0) > 0 \) and \( 0 < u_i(x) \leq \frac{1}{\lambda} \sup_{x \geq 0} f_i(x) < \infty \) for all \( x > 0 \), and \( u_i(x) \geq \tau \) for all \( x \geq 1 \) and some positive constant \( \tau \). Hence, there exists a constant \( k \geq 1 \) such \( u_1(x) \leq ku_2(x) \) for all \( x \geq 0 \). Let
\[
k_0 = \inf \{k > 0 : u_1(x) \leq ku_2(x), \ \forall \ x \geq 0\}, \quad \text{i.e.,} \quad k_0 = \sup_{x \geq 0} \frac{u_1(x)}{u_2(x)}.
\]
Then \( u_1(x) \leq k_0 u_2(x) \) for all \( x \geq 0 \). If \( k_0 \leq 1 \), then our conclusion is true.

We assume that \( k_0 > 1 \) and shall derive a contradiction. Let \( \varphi(x) = k_0 u_2(x) - u_1(x) \). Then \( \varphi(x) \geq 0 \) for all \( x \geq 0 \), and \( \varphi \) satisfies
\[
\begin{aligned}
-d\varphi'' &= k_0 u_2(f_2(x) - \lambda u_2) - u_1(x)(f_1(x) - \lambda u_1) \\
&\geq f_1(x)\varphi - \lambda (k_0 u_2^2 - u_2^2) \\
&= [f_1(x) - \lambda (k_0 u_2 + u_1)]\varphi + \lambda k_0 u_2^2(k_0 - 1), \quad 0 < x < \infty
\end{aligned}
\] (2.7)
\[
\begin{aligned}
&\geq [f_1(x) - \lambda (k_0 u_2 + u_1)]\varphi \\
&\geq \frac{1}{2} f_1(0)\varphi, \quad 0 < x \ll 1
\end{aligned}
\] (2.8)
since \( f_1(0) > 0 \) and \( u_1(x) \to 0 \) as \( x \to 0 \). Thanks to \( \varphi(0) = 0 \), we have \( \varphi'(0) \geq 0 \). If \( \varphi'(0) = 0 \), it follows from (2.8) that \( \varphi'(x) < 0 \) as \( 0 < x \ll 1 \). Consequently, \( \varphi(x) < 0 \) as \( 0 < x \ll 1 \). This is
contradiction. So, \( \varphi'(0) > 0 \), i.e., \( k_0 u_2'(0) > u_1'(0) \). Remember this fact and the definition of \( k_0 \), it is easily seen that at least one of the following happens:

(i) There exists some \( x_0 \in (0, \infty) \) such that \( k_0 u_2(x_0) = u_1(x_0) \),

(ii) \( k_0 u_2(x) - u_1(x) > 0 \) for all \( x > 0 \) and \( \liminf_{x \to \infty} [k_0 u_2(x) - u_1(x)] = 0 \).

When the case (i) occurs, then \( \varphi(x_0) = 0 \), \( \varphi'(x_0) = 0 \) and \( \varphi''(x_0) \geq 0 \). It is derived from (2.7) that \( k_0 \leq 1 \). This is a contradiction.

When the case (ii) happens, then \( \liminf_{x \to \infty} \varphi(x) = 0 \). Remember that \( \varphi(x) > 0 \) for all \( x > 0 \), it is not hard to prove that there exists a sequence \( \{x_n\} \) with \( x_n \to \infty \) such that

\[
\varphi''(x_n) \geq 0, \quad \lim_{n \to \infty} \varphi(x_n) = 0.
\]

By passing to a subsequence, we may assume that \( u_2(x_n) \to \sigma \) for some positive constant \( \sigma \). It follows from (2.7) that

\[
0 \geq [f_1(x_n) - \lambda(k_0 u_2(x_n) + u_1(x_n))]\varphi(x_n) + \lambda k_0 u_2^2(x_n)(k_0 - 1).
\]

Letting \( n \to \infty \) we get

\[
0 \geq \lambda k_0 \sigma^2 (k_0 - 1) > 0.
\]

This is also a contradiction. The proof is finished. \( \square \)

In the following, we denote the unique positive solution of (2.2) by \( \hat{u}(x) \). Now we discuss its global stability.

**Theorem 2.2** Assume that \( d \) and \( \lambda \) are positive constants, and \( f(x) \) satisfies (2.3). Suppose that \( \phi \neq 0 \) is a bounded, continuous and nonnegative function. Let \( u(t, x) \) be the unique solution of the following parabolic problem

\[
\begin{aligned}
&u_t - du_{xx} = u(f(x) - \lambda u), & t > 0, & 0 < x < \infty, \\
u(t, 0) = 0, & t > 0, \\
u(0, x) = \phi(x), & 0 \leq x < \infty.
\end{aligned}
\]

Then

\[
\lim_{t \to \infty} u(t, x) = \hat{u}(x) \text{ uniformly in any compact subset of } [0, \infty).
\]

**Proof** By the positivity of parabolic equations we have that \( u(t, x) > 0 \) for all \( t > 0 \) and \( x > 0 \), and \( u_x(t, 0) > 0 \) for all \( t > 0 \). We may assume that \( \phi(x) > 0 \) for all \( x > 0 \), and \( \phi(0) = 0 \) and \( \phi'(0) > 0 \). It is well known that, when \( l > \pi \sqrt{d/f_0} \), where \( f_0 = \inf_{x \geq 0} f(x) > 0 \), the problem (2.6) has a unique positive solution, denoted by \( \hat{u}_l(x) \). For \( l > \pi \sqrt{d/f_0} \), consider the following initial-boundary value problem

\[
\begin{aligned}
&u_t - du_{xx} = u(f(x) - \lambda u), & t > 0, & 0 < x < l, \\
u(t, 0) = 0 = u(t, l), & t > 0, \\
u(0, x) = \phi(x), & 0 \leq x \leq l.
\end{aligned}
\]
Let \( u_l(t, x) \) be the unique solution of (2.10). Then
\[
    u(t, x) \geq u_l(t, x), \quad \forall x \in [0, l], \ t \geq 0.
\] (2.11)

It is obvious that a large positive constant \( M \) is an upper solution of (2.6). Note that \( l > \pi \sqrt{d/f_0} \), it can be proved that \( \delta_1 \sin \frac{\pi x}{l} \) is a lower solution of (2.6) provided that \( 0 < \delta_1 < 1 \). Since \( \phi(x) > 0 \) for \( x > 0 \) and \( \phi'(0) > 0 \), there is a small positive constant \( \delta_2 \) such that \( \delta_2 \sin \frac{\pi x}{l} \leq \phi(x) \) for all \( x \in [0, l] \). Set \( \delta = \min \{ \delta_1, \delta_2 \} \), then \( \delta \sin \frac{\pi x}{l} \leq \phi(x) \) for all \( x \in [0, l] \) and \( \delta \sin \frac{\pi x}{l} \) is a lower solution of (2.6). Let \( \bar{u}_l(t, x) \) and \( \underline{u}_l(t, x) \) be the unique solution of (2.10) with \( \phi(x) = M \) and \( \phi(x) = \delta \sin \frac{\pi x}{l} \), respectively. Then
\[
    \underline{u}_l(t, x) \leq u_l(t, x) \leq \bar{u}_l(t, x), \quad \forall x \in [0, l], \ t \geq 0,
\] (2.12)
and \( \bar{u}_l(t, x) \) is decreasing and \( \underline{u}_l(t, x) \) is increasing in \( t \). Moreover, both limits \( \lim_{t \to \infty} \bar{u}_l(t, x) = \bar{u}_l(x) \) and \( \lim_{t \to \infty} \underline{u}_l(t, x) = \underline{u}_l(x) \) are positive solutions of (2.6). By the uniqueness we have \( \bar{u}_l(x) \equiv u_l(x) \equiv \bar{u}_l(x) \). It follows from (2.12) that
\[
    \lim_{t \to \infty} u_l(t, x) = \hat{u}_l(x) \quad \text{uniformly in } [0, l].
\]

Hence, by (2.11),
\[
    \liminf_{t \to \infty} u(t, x) \geq \hat{u}_l(x) \quad \text{uniformly in } [0, l].
\] (2.13)

From the proof of Theorem 2.1 we know that \( \hat{u}_l \to \hat{u} \) in \( C^{2+\alpha}_{\text{loc}}([0, \infty)) \) as \( l \to \infty \). Consequently, by (2.13), we arrive at
\[
    \liminf_{t \to \infty} u(t, x) \geq \hat{u}(x) \quad \text{uniformly in any compact subset of } [0, \infty). \] (2.14)

Let \( M = \max \{ \| \phi \|_{\infty}, \| f \|_{\infty}/\lambda \} \) and \( \bar{u}(t, x) \) be the unique solution of
\[
    \begin{cases}
        u_t - du_{xx} = u(f(x) - \lambda u), & t > 0, \ 0 < x < \infty, \\
        u(t, 0) = 0, & t > 0, \\
        u(0, x) = M, & 0 \leq x < \infty.
    \end{cases}
\]

Then we have the following conclusions:

(a) \( \bar{u}(t, x) \) is monotone decreasing in \( t \);
(b) \( u(t, x) \leq \bar{u}(t, x) \) for \( x \geq 0 \) and \( t \geq 0 \);
(c) \( \bar{u}(t, x) \geq \bar{u}(x) \) for \( x \geq 0 \) and \( t \geq 0 \) since \( \hat{u}(x) < M \);
(d) \( \lim_{t \to \infty} \bar{u}(t, x) = u^*(x) \) uniformly in any compact subset of \([0, \infty)\) for some positive solution \( u^*(x) \) of (2.2).

Because \( \hat{u}(x) \) is the unique positive solution of (2.2), one has \( u^*(x) = \hat{u}(x) \). Therefore,
\[
    \limsup_{t \to \infty} u(t, x) \leq \hat{u}(x) \quad \text{uniformly in any compact subset of } [0, \infty).
\]

This combines with (2.14) to derive (2.9). \( \square \)

Before ending this subsection, we provide two propositions that will be used in the study of the long time behaviour of solution to the problem (DFB).
Proposition 2.2  Assume that $d$ and $\lambda$ are positive constants, $f(x)$ satisfies (2.3). For any given constant $K > \frac{1}{3}\|f\|_{\infty}$ and any $l \gg 1$, let $\bar{u}_l(x)$ be the unique positive solution of
\[
\begin{cases}
-du'' = u(f(x) - \lambda u), & 0 < x < l, \\
 u(0) = 0, & u(l) = K.
\end{cases}
\]
Then we have
\[
\lim_{l \to \infty} \bar{u}_l(x) = \hat{u}(x) \quad \text{uniformly in any compact subset of } [0, \infty).
\]

Proof. First, the existence and uniqueness of $\bar{u}_l(x)$ can be obtained by the upper and lower solutions method and the comparison principle ([6, Lemma 5.2], [18, Proposition 2.1]), respectively.

Since $\bar{u}_l(x) \leq K$ in $[0, l]$, by the comparison principle we see that $\bar{u}_l(x) \geq \hat{u}(x)$ in $[0, l]$, and $\bar{u}_l(x)$ is decreasing in $l$. Similar to the proof of Theorem 2.1, $\bar{u}_l \rightarrow \hat{u}$ in $C^2_{\text{loc}}([0, \infty))$ as $l \rightarrow \infty$ because $\hat{u}$ is the unique positive solution of (2.2). \qed

Using Proposition 2.1 and the regularity theory and compactness argument, we can prove the following proposition.

Proposition 2.3  Assume that $d$ and $\lambda$ are positive constants, $f(x)$ satisfies (2.3). Let $0 < \varepsilon \ll 1$ and $u_{\varepsilon}^{\pm}(x)$ be the unique positive solution of
\[
\begin{cases}
-du'' = u(f(x) \pm \varepsilon - \lambda u), & 0 < x < \infty, \\
u(0) = 0.
\end{cases}
\]
Then
\[
\lim_{\varepsilon \to 0} \tilde{u}_{\varepsilon}^{\pm}(x) = \hat{u}(x) \quad \text{uniformly in any compact subset of } [0, \infty).
\]

2.2  The existence of positive solution to (2.1)

Theorem 2.3  Assume that $a(b+c) < 1$. Then the problem (2.1) has a positive solution. Moreover, any positive solution $(u, v)$ of (2.1) satisfies
\[
\bar{u}(x) \leq u(x) \leq \bar{u}(x), \quad \bar{v}(x) \leq v(x) \leq \bar{v}(x), \quad \forall x \geq 0,
\]
where $\bar{u}, \bar{v}, \underline{u}$ and $\underline{v}$ will be given in the following proof.

Proof. Step 1: The construction of $\underline{u}, \underline{v}, \bar{u}$ and $\bar{v}$.
Let $\bar{u}$ be the unique positive solution of
\[
\begin{cases}
-u'' = u(1 - u), & 0 < x < \infty, \\
u(0) = 0.
\end{cases}
\]
Then
\[
\bar{u}'(x) > 0, \quad \text{and } \lim_{x \to \infty} \bar{u}(x) = 1.
\]
In view of Theorem 2.1, the problem
\[
\begin{cases}
-\frac{d^2v}{dx^2} = v[c - v + b\bar{u}(x)], & 0 < x < \infty, \\
v(0) = 0
\end{cases}
(2.17)
\]
has a unique positive solution, denoted by \(\bar{v}(x)\). Then \(\bar{v}'(x) > 0\) and \(\lim_{x \to \infty} \bar{v}(x) = b + c\). Since \(a(b + c) < 1\), by Theorem 2.1 again, the problem
\[
\begin{cases}
-u'' = u(1 - a\bar{v}(x) - u), & 0 < x < \infty, \\
u(0) = 0
\end{cases}
\]
has a unique positive solution, denoted by \(u(x)\). Then \(u(x) < 1\) and \(\lim_{x \to \infty} u(x) = 1 - a(b + c)\). By virtue of Theorem 2.1 once again, the problem
\[
\begin{cases}
-\frac{dv}{dx} = v[c - v + bu(x)], & 0 < x < \infty, \\
v(0) = 0
\end{cases}
\]
has a unique positive solution, denoted by \(v(x)\). Then \(v(x) < b + c\).

Applying Proposition 2.1 we have that \(u(x) \leq \bar{u}(x)\) for all \(x \geq 0\). Consequently, \(v(x) \leq \bar{v}(x)\) for all \(x \geq 0\) by use of Proposition 2.1 once again.

**Step 2:** Existence of positive solution.

The conclusion of Step 1 shows that \(u, v, \bar{u}, \bar{v}\) are the coupled ordered lower and upper solutions of (2.1). For any given \(l > 0\), it is obvious that \(u, v, \bar{u}, \bar{v}\) and \(\bar{v}\) are also the coupled ordered lower and upper solutions of the following problem
\[
\begin{cases}
-u'' = u(1 - u - av), & 0 < x < l, \\
-\frac{dv}{dx} = v(c - v + bu), & 0 < x < l, \\
u(0) = u(0), & v(0) = v(0), \\
u(l) = \bar{u}(l), & v(l) = \bar{v}(l).
\end{cases}
(2.18)
\]
By the standard upper and lower solutions method we know that the problem (2.18) has at least one positive solution, denoted by \((u_l, v_l)\), and
\[
\bar{u}(x) \leq u_l(x) \leq \bar{u}(x), \quad \bar{v}(x) \leq v_l(x) \leq \bar{v}(x), \quad \forall 0 \leq x \leq l.
\]
Applying the local estimation and compactness argument, it can be concluded that there exists a pair \((u, v)\), such that \((u_l, v_l) \to (u, v)\) in \(C^{2}_{loc}([0, \infty))\), and \((u, v)\) solves (2.1). It is obvious that \((u, v)\) satisfies (2.15).

By use of Proposition 2.1 successively, we also see that if \(a(b + c) < 1\) then any positive solution \((u, v)\) of (2.1) satisfies (2.15). \(\square\)

**3 Long time behavior of \((u, v)\) and spreading-vanishing dichotomy**

It follows from Theorem 1.1 that \(x = h(t)\) is monotonic increasing. Therefore, \(\lim_{t \to \infty} h(t) = h_\infty \in (0, \infty]\). To discuss the long time behavior of \((u, v)\), we first derive an estimate.
Theorem 3.1  Let \((u,v,h)\) be the solution of (DFB) or (NFB). If \(h_\infty < \infty\), then there exists a constant \(M > 0\), such that
\[
\|u(t, \cdot), v(t, \cdot)\|_{C^1[0, h(t)]} \leq M, \quad \forall \ t > 1.
\]
Moreover,
\[
\lim_{t \to \infty} h'(t) = 0. \tag{3.1}
\]

**Proof.** The proof is similar to that of Theorem 4.1 of [29]. We omit the details.

3.1  Vanishing case \((h_\infty < \infty)\)

To discuss the asymptotic behaviors of \(u\) and \(v\), we first prove a general result. Let \(d\), \(\beta\) and \(g_0\) be positive constants and \(C \in \mathbb{R}\). Assume that \(w_0 \in C^2([0, g_0])\) and satisfies \(w_0'(0) = 0\) (or \(w_0(0) = 0\)), \(w_0(g_0) = 0\) and \(w_0(x) > 0\) in \((0, h_0)\). Let
\[
g \in C^{1+\alpha}([0, \infty)), \quad w \in C^{1+\alpha, 1+\alpha}([0, \infty) \times [0, g(t)])
\]
for some \(\alpha > 0\), and satisfy \(g(t) > 0, w(t, x) > 0\) for all \(0 \leq t < \infty\) and \(0 < x < g(t)\).

**Proposition 3.1** Under the above conditions, we further suppose that
\[
\lim_{t \to \infty} g(t) = g_\infty < \infty, \quad \lim_{t \to \infty} g'(t) = 0, \tag{3.2}
\]
and
\[
\|w(t, \cdot)\|_{C^1[0, g(t)]} \leq M, \quad \forall \ t > 1 \tag{3.3}
\]
for some constant \(M > 0\). If \((w, g)\) satisfies
\[
\begin{cases}
  w_t - dw_{xx} \geq w(C - w), & t > 0, \quad 0 < x < g(t), \\
  w_x = 0 \ (or \ w = 0), & t > 0, \quad x = 0, \\
  w = 0, \quad g'(t) \geq -\beta w_x, & t > 0, \quad x = g(t), \\
  w(0, x) = w_0(x), & x \in [0, g_0], \\
  g(0) = g_0,
\end{cases}
\]
then
\[
\lim_{t \to \infty} \max_{0 \leq x \leq g(t)} w(t, x) = 0. \tag{3.4}
\]

**Proof.** The proof is similar to that of [29, Theorem 4.2]. For the convenience to reader, we shall give the details because of this is a very important conclusion in the study of free boundary problems for systems.

On the contrary we assume that there exist \(\sigma > 0\) and \(\{(t_j, x_j)\}_{j=1}^\infty\), with \(0 \leq x_j < g(t_j)\) and \(t_j \to \infty\) as \(j \to \infty\), such that
\[
w(t_j, x_j) \geq 4\sigma, \quad j = 1, 2, \ldots \tag{3.5}
\]
Since \( 0 \leq x_j < g_\infty \), there exist a subsequence of \( \{x_j\} \), denoted by itself, and \( x_0 \in [0, g_\infty] \), such that \( x_j \to x_0 \) as \( j \to \infty \). We claim that \( x_0 \in [0, g_\infty] \). If \( x_0 = g_\infty \), then \( x_j - g(t_j) \to 0 \) as \( j \to \infty \). By use of the inequality (3.5) firstly and the inequality (3.3) secondly, we have that

\[
\left| \frac{4\sigma}{x_j - g(t_j)} \right| \leq \left| \frac{w(t_j, x_j) - w(t_j, g(t_j))}{x_j - g(t_j)} \right| = |w_x(t_j, \bar{x}_j)| \leq M,
\]

where \( \bar{x}_j \in (x_j, g(t_j)) \). It is a contradiction since \( x_j - g(t_j) \to 0 \). Similarly, it is easily deduced that \( x_0 \neq 0 \) if the boundary condition is \( w(t, 0) = 0 \).

By use of (3.3) and (3.5), there exists \( \delta > 0 \) such that \( [x_0, x_0 + \delta] \subset [0, g_\infty) \) and

\[
w(t_j, x) \geq 2\sigma, \quad \forall x \in [x_0, x_0 + \delta]
\]

for all large \( j \). Since \( g(t_j) \to g_\infty \) as \( j \to \infty \), without loss of generality we may think that \( g(t_j) > x_0 + \delta \) for all \( j \).

Let \( r_j(t) = x_0 + \delta + t - t_j \). Then \( r_j(t_j) < g(t_j) \). Set

\[
\tau_j = \inf \{ t > t_j : g(t) = r_j(t) \}.
\]

Take advantage of \( g_\infty < \infty \) and \( r_j(t) \to \infty \) as \( t \to \infty \), we have \( \tau_j < \infty \). It is easy to obtain that \( \tau_j < t_j - \delta + g_\infty \). This implies

\[
x_0 + \delta \leq r_j(t) \leq g(t) \quad \text{in} \quad [t_j, \tau_j]. \tag{3.6}
\]

Define \( y_j(t, x) = (\pi - \varepsilon) \frac{x - x_0}{\delta + t - t_j} \) and

\[
w_j(t, x) = \sigma e^{-kt_j} [\cos y_j(t, x) + \cos \varepsilon], \quad (t, x) \in \overline{\Omega}_{t_j},
\]

where \( \varepsilon \) (\( \varepsilon < \pi/8 \)) and \( k \) are positive constants to be chosen later, and

\[
\Omega_{t_j} = \{(t, x) : t_j < t < \tau_j, \ x_0 < x < r_j(t)\}.
\]

It is obvious that \( w_j(t, x_0) < 2\sigma \), \( w_j(t_j, r_j(t_j)) = 0 \), and \( 0 \leq y_j(t, x) \leq \pi - \varepsilon \) for \( (t, x) \in \overline{\Omega}_{t_j} \), the latter implies \( w_j(t, x) \geq 0 \) in \( \Omega_{t_j} \).

We want to compare \( w(t, x) \) and \( w_j(t, x) \) in \( \overline{\Omega}_{t_j} \). According to (3.6), it follows that

\[
w(t, x_0) \geq 2\sigma > w_j(t, x_0), \quad w(t, r_j(t)) \geq 0 = w_j(t, r_j(t)), \quad \forall t \in [t_j, \tau_j].
\]

On the other hand, it is obvious that

\[
w(t_j, x) \geq 2\sigma \geq w_j(t_j, x), \quad \forall x \in [x_0, x_0 + \delta].
\]

Thus, if the positive constants \( \varepsilon \) and \( k \) can be chosen independent of \( j \) such that

\[
w_j(t) - dw_jxxx \leq w_j(C - w_j) \quad \text{in} \quad \Omega_{t_j}, \tag{3.7}
\]

it can be deduced that \( w_j(t, x) \leq w(t, x) \) for \( (t, x) \in \Omega_{t_j} \) by applying the comparison principle to \( w \) and \( w_j \) over \( \Omega_{t_j} \). Since \( w(\tau_j, g(\tau_j)) = 0 = w_j(\tau_j, r_j(\tau_j)) \) and \( g(\tau_j) = r_j(\tau_j) \), it follows that \( w_x(\tau_j, g(\tau_j)) \leq w_{jx}(\tau_j, r_j(\tau_j)) \). Thanks to \( \varepsilon < \pi/8 \) and \( \delta + \tau_j - t_j < g_\infty \), we have

\[
w_{jx}(\tau_j, r_j(\tau_j)) = -\frac{\sigma (\pi - \varepsilon)}{\delta + \tau_j - t_j} e^{-k(\tau_j - t_j)} \sin(\pi - \varepsilon) \leq -\frac{7\sigma \pi}{8g_\infty} e^{-k g_\infty} \sin \varepsilon.
\]
Note the boundary condition \(-\beta w_{x}(\tau_{j}, g(\tau_{j})) \leq g'(\tau_{j})\), we have

\[ g'(\tau_{j}) \geq \frac{7\beta\sigma\pi}{8g_{\infty}} e^{-k\delta} \sin \varepsilon, \]

which implies \(\limsup_{t \to \infty} g'(t) > 0\) since \(\lim_{j \to \infty} \tau_{j} \to \infty\). This contradicts to (3.2), and (3.4) is obtained.

Now we prove that if \(\varepsilon \) and \(k\) satisfy

\[ \varepsilon < \frac{\pi}{8}, \quad \sin \varepsilon \leq \frac{3\pi d\delta^{2}}{8g_{\infty}^{2}}, \quad (3.8) \]

\[ k \geq \frac{\pi g_{\infty}}{\delta^{2}(\cos \varepsilon - \cos 2\varepsilon)} + 2\sigma + |C| + d\left(\frac{\pi}{\delta}\right)^{2}, \quad (3.9) \]

then (3.7) holds for all large \(j\). Hence, (3.3) holds. Direct computations yield

\[
\begin{align*}
\frac{\partial w_{j} \partial t - d \frac{\partial^{2} w_{j}}{\partial x^{2}} - w_{j}(C - w_{j})}{
\begin{align*}
&= -kw_{j} - \sigma e^{-k(t-t_{j})} y_{j} \sin y_{j} + d\sigma e^{-k(t-t_{j})} y_{j}^{2} \cos y_{j} - w_{j}(C - w_{j}) \\
&\leq (2\sigma + |C| + dy_{j}^{2} - k)w_{j} - d\sigma e^{-k(t-t_{j})} y_{j}^{2} \cos \varepsilon - \sigma e^{-k(t-t_{j})} y_{j} \sin y_{j} \\
&\leq \left(2\sigma + |C| + d\left(\frac{\pi}{\delta}\right)^{2} - k\right)w_{j} + \sigma e^{-k(t-t_{j})} \left[\frac{\pi(x - x_{0})}{\delta^{2}} \sin y_{j} - d\left(\frac{\pi - \varepsilon}{g_{\infty}}\right)^{2} \cos \varepsilon\right] \\
&:= I.
\end{align*}\]
\end{align*}
\]

By (3.9), we have \(2\sigma + |C| + d(\pi/\delta)^{2} - k < 0\).

Since \(0 \leq y_{j} \leq \pi - \varepsilon\) when \((t, x) \in \Omega_{tj}\), we can decompose \(\Omega_{tj} = \Omega_{tj}^{1} \cup \Omega_{tj}^{2}\), where

\[
\begin{align*}
\Omega_{tj}^{1} &= \{(t, x) \in \Omega_{tj} : t_{j} < t < \tau_{j}, \; \pi - 2\varepsilon < y_{j}(t, x) < \pi - \varepsilon\}, \\
\Omega_{tj}^{2} &= \{(t, x) \in \Omega_{tj} : t_{j} < t < \tau_{j}, \; 0 \leq y_{j}(t, x) \leq \pi - 2\varepsilon\}.
\end{align*}
\]

It is obvious that \(\sin y_{j} \leq \sin 2\varepsilon\) when \((t, x) \in \Omega_{tj}^{1}\), and \(\cos y_{j} \geq -\cos 2\varepsilon\) when \((t, x) \in \Omega_{tj}^{2}\). Because of \(\varepsilon < \pi/8\, w_{j}(t, x) \geq 0\) and \(x - x_{0} \leq g_{\infty}\) in \(\Omega_{tj}\), in view of (3.8) and (3.9), we conclude

\[ I \leq \sigma e^{-k(t-t_{j})} \left(\frac{\pi g_{\infty}}{\delta^{2}} \sin 2\varepsilon - d\frac{3\pi^{2}}{4g_{\infty}^{2}} \cos \varepsilon\right) < 0 \]

when \((t, x) \in \Omega_{tj}^{1}\), and

\[ I \leq \sigma e^{-k(t-t_{j})} \left(\frac{2\sigma + |C| + d\left(\frac{\pi}{\delta}\right)^{2} - k}{\cos \varepsilon - \cos 2\varepsilon} + \frac{\pi g_{\infty}}{\delta^{2}}\right) < 0 \]

when \((t, x) \in \Omega_{tj}^{2}\). The proof is completed.

\[\square\]

**Theorem 3.2** Let \((u, v, h)\) be any solution of (DFB) or (NFB). If \(h_{\infty} < \infty\), then

\[ \lim_{t \to \infty} \|u(t, \cdot), v(t, \cdot)\|_{C([0, h(t)])} = 0. \quad (3.10) \]

This result shows that if both prey and predator can not spread into the infinity, then they will die out eventually.
We should remark that this theorem plays key roles in the following two aspects: (i) affirming that the two species disappear eventually; (ii) determining the criteria for spreading and vanishing (see the following Section 5).

**Proof of Theorem 3.2** Since \(u(t, x) > 0\) and \(u_x(t, h(t)) < 0\), we see that \(v\) satisfies

\[
\begin{cases}
  v_t - dv_{xx} \geq v(c - v), & t > 0, \ 0 < x < h(t), \\
  v = 0 \ (\text{or} \ v_x = 0), & t > 0, \ x = 0, \\
  v = 0, \ h'(t) \geq -\mu rv_x, & t > 0, \ x = h(t), \\
  v(0, x) = v_0(x), & x \in [0, h_0], \\
  h(0) = h_0.
\end{cases}
\]

In view of Theorem 3.1 and Proposition 3.1 we have that \(\lim_{t \to \infty} \|v(t, \cdot)\|_{C([0, h(t)])} = 0\). Hence, there exists \(T \gg 1\), such that

\[v(t, x) < 1/(2a) \ \forall \ t \geq T, \ 0 \leq x \leq h(t).\]

Remember that \(v_x(t, h(t)) < 0\) for all \(t \geq T\), we see that \(u(t, x)\) satisfies

\[
\begin{cases}
  u_t - u_{xx} > u(1/2 - u), & t \geq T, \ 0 < x < h(t), \\
  u = 0 \ \text{or} \ u_x = 0, & t \geq T, \ x = 0, \\
  u = 0, \ h'(t) > -\mu u_x, & t \geq T, \ x = h(t), \\
  u(t, x) = u(T, x), & t = T, \ x \in [0, h(t)].
\end{cases}
\]

Applying Theorem 3.1 and Proposition 3.1 again, we conclude that \(\lim_{t \to \infty} \|u(t, \cdot)\|_{C([0, h(t)])} = 0\). The proof is finished. \(\blacksquare\)

### 3.2 Spreading case \((h_\infty = \infty)\)

We first consider the problem (NFB). In the same way as the proofs of Theorems 4.3 and 4.4 in [29], we can prove the following two theorems.

**Theorem 3.3** Let \((u, v, h)\) be the solution of (NFB). If \(h_\infty = \infty\), then for the weakly hunting case \(ac < 1, \ ab < 1\), we have

\[
\lim_{t \to \infty} u(t, x) = \frac{1 - ac}{1 + ab}, \quad \lim_{t \to \infty} v(t, x) = \frac{b + c}{1 + ab}
\]

uniformly in any compact subset of \([0, \infty)\).

We remark that the conditions \(ac < 1\) and \(ab < 1\) are similar to the weak competition conditions, see [14].

**Theorem 3.4** Let \((u, v, h)\) be the solution of (NFB). If \(h_\infty = \infty\), then for the strongly hunting case \(ac \geq 1\), we have

\[
\lim_{t \to \infty} u(t, x) = 0, \quad \lim_{t \to \infty} v(t, x) = c
\]

uniformly in any compact subset of \([0, \infty)\).
In the rest of this section, we consider the problem (DFB). This is the main part of this section.

**Theorem 3.5** Assume that \( h(\infty) = \infty \). If \( a(b + c) < 1 \), then the solution \((u(t, x), v(t, x))\) of (DFB) satisfies

\[
\liminf_{t \to \infty} u(t, x) \geq \underline{u}(x), \quad \limsup_{t \to \infty} u(t, x) \leq \bar{u}(x) \quad \text{uniformly in any compact subset of } [0, \infty), \quad (3.11)
\]

\[
\liminf_{t \to \infty} v(t, x) \geq \underline{v}(x), \quad \limsup_{t \to \infty} v(t, x) \leq \bar{v}(x) \quad \text{uniformly in any compact subset of } [0, \infty), \quad (3.12)
\]

where \( \bar{u}, \bar{v}, \underline{u} \) and \( \underline{v} \) are given in the proof of Theorem 2.3.

**Proof.** *Step 1* Define

\[
\phi(x) = \begin{cases} 
  u_0(x), & 0 \leq x \leq h_0, \\
  0, & x \geq h_0,
\end{cases}
\]

and let \( w(t, x) \) be the unique positive solution of

\[
\begin{cases}
  w_t - w_{xx} = w(1 - w), & t > 0, \quad 0 < x < \infty, \\
  w(t, 0) = 0, & t > 0, \\
  w(0, x) = \phi(x), & x \geq 0.
\end{cases}
\]

By the comparison principle, \( u(t, x) \leq w(t, x) \) for all \( t > 0 \) and \( 0 \leq x \leq h(t) \). In view of Theorem 2.2, \( \lim_{t \to \infty} w(t, x) = \bar{u}(x) \) uniformly in any compact subset of \([0, \infty)\), where \( \bar{u}(x) \) is the unique positive solution of (2.10). Thanks to \( h(\infty) = \infty \), we get the second limit of (3.11).

*Step 2* For any given \( 0 < \varepsilon \ll 1 \) and \( l \gg 1 \), there exists a large \( T \), such that

\[
h(t) > l, \quad u(t, x) < \bar{u}(x) + \varepsilon, \quad \forall \ 0 \leq x \leq l, \ t \geq T.
\]

Let \( v^i(t, x) \) be the unique positive solution of

\[
\begin{cases}
  v_t - dv_{xx} = v[c - v + b(\bar{u}(x) + \varepsilon)], & t > T, \quad 0 < x < l, \\
  v(t, 0) = 0, \quad v(t, l) = K, & t > T, \\
  v(T, x) = K, & x \in [0, l],
\end{cases}
\]

where \( K > \max \{M, c + b(1 + \varepsilon)\} \), and \( M \) is given by Theorem 1.1. Since \( v(t, x) \leq M \), by the comparison principle we have

\[
v(t, x) \leq v^i(t, x), \quad \forall \ 0 \leq x \leq l, \ t \geq T. \tag{3.13}
\]

Let \( v_\varepsilon(x) \) be the unique positive solution of (2.2) with \( \lambda = 1 \) and \( f(x) = c + b(\bar{u}(x) + \varepsilon) \), and let \( v_1(x) \) be the unique positive solution of

\[
\begin{cases}
  -dv'' = v[c - v + b(\bar{u}(x) + \varepsilon)], & 0 < x < l, \\
  v(0) = 0, \quad v(l) = K.
\end{cases} \tag{3.14}
\]

Thanks to \( v_\varepsilon(x) < K \), the comparison principle asserts

\[
v_\varepsilon(x) \leq v_1(x), \quad v_\varepsilon(x) \leq v^i(t, x), \quad \forall \ t \geq T, \ 0 \leq x \leq l.
\]
Moreover, when \( l \) solution of (2.17). These facts and (3.15) imply the second limit of (3.12). We can prove that 2.2 and 2.3 successively. We can prove that 

\[ \lim_{t \to \infty} v(t, x) \leq v_l(x) \quad \text{uniformly on} \quad [0, l]. \]

(3.15)

By Proposition 2.2, \( \lim_{t \to \infty} v_l(x) = v_\varepsilon(x) \) uniformly in any compact subset of \([0, \infty)\). By Proposition 2.3, \( \lim_{\varepsilon \to 0} v_\varepsilon(x) = \bar{v}(x) \) uniformly in any compact subset of \([0, \infty)\), where \( \bar{v}(x) \) is the unique positive solution of (2.17). These facts and (3.15) imply the second limit of (3.12).

**Step 3** Since \( a(b + c) < 1 \), choose \( \varepsilon_0 > 0 \) such that \( a(b + c + \varepsilon_0) < 1 \). For any given \( 0 < \varepsilon < \varepsilon_0 \) and \( l \gg 1 \), there exists a large \( T \) such that

\[ h(t) > l, \quad v(t, x) < \bar{v}(x) + \varepsilon, \quad \forall \ 0 \leq x \leq l, \ t \geq T. \]

Moreover, when \( l \gg 1 \), the problem

\[
\begin{cases}
-u'' = u[1 - u - a(b + c + \varepsilon)], & 0 < x < l, \\
u(0) = 0 = u(l)
\end{cases}
\]

has a unique positive solution, denoted by \( u_\varepsilon^*(x) \). Thanks to \( u_\varepsilon(T, 0) > 0 \) and \( u(T, l) > 0 \), there is a positive constant \( \sigma < 1 \) such that \( u(T, x) \geq \sigma u_\varepsilon^*(x) \) for all \( 0 \leq x \leq l \). As \( \bar{v}(x) \leq b + c \), it is easy to see that \( \sigma u_\varepsilon^*(x) \) is a lower solution of the problem

\[
\begin{cases}
-u'' = u[1 - u - a(\bar{v}(x) + \varepsilon)], & 0 < x < l, \\
u(0) = 0 = u(l). \tag{3.16}
\end{cases}
\]

Let \( u_l(t, x) \) be the unique solution of

\[
\begin{cases}
u_t - u_{xx} = u[1 - u - a(\bar{v}(x) + \varepsilon)], & t > T, \ 0 < x < l, \\
u(t, 0) = 0 = u(t, l), & t > T, \\
u(T, x) = \sigma u_\varepsilon^*(x), & x \in [0, l].
\end{cases}
\]

Then \( u(t, x) \geq u_l(t, x) \) for \( 0 \leq x \leq l \) and \( t \geq T \), and \( u_l(t, x) \) is increasing in \( t \). Similar to the above, it can be deduced that the limit \( \lim_{t \to \infty} u_l(t, x) := u_l(x) \) exists and is the unique positive solution of (3.16). Moreover, such limit holds uniformly on \([0, l]\). Hence,

\[ \lim_{t \to \infty} \inf_{l} u(t, x) \geq u_l(x) \quad \text{uniformly on} \quad [0, l]. \]

(3.17)

Similarly to the argument of Step 2, let \( l \to \infty \) firstly and \( \varepsilon \to 0 \) secondly, and apply Propositions 2.2 and 2.3 successively. We can prove that

\[ \lim_{l \to \infty} u_l(x) = u(x) \quad \text{uniformly in any compact subset of} \quad [0, \infty). \]

This fact combined with (3.17) allows us to derive the first limit of (3.11).

Similarly, we can prove the first limit of (3.12). The proof is finished.

\[ \square \]
4 Comparison principles

In this section we shall provide some comparison principles which will be used to estimate the solution \((u,v,h)\) and determine the criteria governing spreading and vanishing.

**Lemma 4.1 (Comparison principle)** Let \(\bar{h} \in C^1([0,\infty))\) and \(\bar{h}(t) > 0\) in \([0,\infty)\). Let \(\bar{u}, \bar{v} \in C(\overline{D}) \cap C^{1,2}(O)\), with \(O = \{(t,x) : t > 0, 0 < x < \bar{h}(t)\}\). Assume that \((\bar{u}, \bar{v}, \bar{h})\) satisfies

\[
\begin{cases}
\bar{u}_t - \bar{u}_{xx} \geq \bar{u}(1 - \bar{u}), & t > 0, \ 0 < x < \bar{h}(t), \\
\bar{v}_t - d\bar{v}_{xx} \geq \bar{v}(c - \bar{v} + b\bar{u}), & t > 0, \ 0 < x < \bar{h}(t), \\
\bar{u}(t,0) \geq 0, \ \bar{v}(t,0) \geq 0, & t > 0, \\
\bar{u}(t,\bar{h}(t)) = \bar{v}(t,\bar{h}(t)) = 0, & t > 0, \\
\bar{h}'(t) \geq -\mu[\bar{u}_x(t,\bar{h}(t)) + \rho \bar{v}_x(t,\bar{h}(t))], & t > 0.
\end{cases}
\]

(4.1)

If \(\bar{h}(0) \geq h_0\), \(\bar{u}(0,x), \bar{v}(0,x) \geq 0\) on \([0,\bar{h}(0)]\), and \(\bar{u}(0,x) \geq u_0(x), \bar{v}(0,x) \geq v_0(x)\) on \([0,h_0]\). Then the solution \((u,v,h)\) of (DFB) satisfies \(h(t) \leq \bar{h}(t)\) on \([0,\infty)\), and \(u \leq \bar{u}, v \leq \bar{v}\) on \(D\), where \(D = \{(t,x) : t \geq 0, 0 \leq x \leq h(t)\}\).

If, in (4.1), the conditions \(\bar{u}(t,0) \geq 0\) and \(\bar{v}(t,0) \geq 0\) are replaced by \(\bar{u}_x(t,0) \leq 0\) and \(\bar{v}_x(t,0) \leq 0\), then the conclusion still holds for the solution of (NFB).

**Proof.** The proof can proceed as the argument of [14, Lemma 5.1] with minor modification.

We first consider that \(\bar{h}(0) > h_0\). Then \(\bar{h}(t) > h(t)\) for small \(t > 0\). We can derive that \(\bar{h}(t) > h(t)\) for all \(t \geq 0\). If this is not true, there exists \(t_0 > 0\) such that \(\bar{h}(t_0) = h(t_0)\) and \(\bar{h}(t) > h(t)\) for all \(t \in (0,t_0)\). Thus, \(\bar{h}'(t_0) \leq h'(t_0)\). Set

\[D_{t_0} = \{(t,x) : t \in (0,t_0), \ 0 < x < h(t)\} \]

Recall that \(\bar{u}(0,x) \geq u_0(x)\) on \([0,h_0]\) and \(\bar{u}(0,h(t_0)) = 0 = u(t_0,h(t_0))\), the strong maximal principle yields that \(\bar{u} > u\) in \(D_{t_0}\), and \(\bar{u}_x(t_0,h(t_0)) < u_x(t_0,h(t_0))\). Similarly, \(\bar{v} > v\) in \(D_{t_0}\), and \(\bar{v}_x(t_0,h(t_0)) < v_x(t_0,h(t_0))\). However,

\[\bar{h}'(t_0) \geq -\mu[\bar{u}_x(t_0,h(t_0)) + \rho \bar{v}_x(t_0,h(t_0))] = -\mu[u_x(t_0,h(t_0)) + \rho v_x(t_0,h(t_0))] = h'(t_0).\]

We get a contradiction. Hence, \(\bar{h}(t) > h(t)\) for all \(t \geq 0\), and \(u \leq \bar{u}, v \leq \bar{v}\) on \(\overline{D}\).

When \(\bar{h}(0) = h_0\), the process is the same as that of [14, Lemma 5.1].

In the same way as the proof of Lemma 4.1 we can prove the following lemma.

**Lemma 4.2 (Comparison principle)** Let \(h \in C^1([0,\infty))\) with \(h(t) > 0\) for all \(t \in [0,\infty)\), and \(v \in C(\overline{O_1}) \cap C^{1,2}(O_1)\) with \(O_1 = \{(t,x) : t > 0, 0 < x < h(t)\}\). Assume that \((v,h)\) satisfies

\[
\begin{cases}
v_t - d v_{xx} \leq v(c - v), & t > 0, \ 0 < x < h(t), \\
v(t,0) = v(t,h(t)) = 0, & t > 0, \\
h'(t) \leq -\mu \rho v_x(t,h(t)), & t > 0.
\end{cases}
\]

(4.2)
and $0 < h(0) \leq h_0$, $0 \leq v(0, x) \leq v_0(x)$ on $[0, h(0)]$. Then the solution $(u, v, h)$ of (DFB) satisfies $h(t) \geq h(t)$ on $[0, \infty)$, and $v(t, x) \geq v(t, x)$ on $\Omega_1$.

If, in (4.2), the condition $v(t, 0) = 0$ is replaced by $v(t, 0) \geq 0$, then the conclusion still holds for the solution of (NFB).

5 The criteria governing spreading and vanishing

In this section we study the criteria governing spreading and vanishing for the problems (DFB) and (NFB), respectively. We first give a necessary condition of vanishing.

**Theorem 5.1** If $h_\infty < \infty$, then $h_\infty \leq \pi \min \{\sqrt{d/c}, 1\}$ for the problem (DFB), and $h_\infty \leq \frac{\pi}{2} \min \{\sqrt{d/c}, 1\}$ for the problem (NFB).

Define

$$
\Lambda = \begin{cases} 
\pi \min \{\sqrt{d/c}, 1\} & \text{for the problem (DFB)}, \\
\frac{\pi}{2} \min \{\sqrt{d/c}, 1\} & \text{for the problem (NFB)}. 
\end{cases}
$$

Then $h_0 \geq \Lambda$ implies $h_\infty = \infty$ due to $h'(t) > 0$ for $t > 0$.

**Proof.** We only deal with the problem (DFB), since the problem (NFB) can be treated by the similar way. By Theorem 3.2, $h_\infty < \infty$ implies $\lim_{t \to \infty} \|u(t, \cdot), v(t, \cdot)\|_{C[0, h(t)]} = 0$. We assume $h_\infty > \Lambda$ to get a contradiction.

If $h_\infty > \pi$, then there exists $\varepsilon > 0$ such that $h_\infty > \pi \sqrt{1/(1 - a\varepsilon)}$. For such $\varepsilon$, there exists $T \gg 1$ such that $h(T) > \pi \sqrt{1/(1 - a\varepsilon)}$ and

$$
v(t, x) \leq \varepsilon, \quad \forall \ t \geq T, \ x \in [0, h(T)].
$$

Set $l = h(T)$ and let $w = w(t, x)$ be the unique positive solution of the following initial boundary value problem with fixed boundary:

$$
\begin{cases} 
w_t = w_{xx} + w (1 - w - a\varepsilon), & t > T, \ 0 < x < l, \\
w(t, 0) = w(t, l) = 0, & t > T, \\
w(T, x) = u(T, x), & 0 \leq x \leq l.
\end{cases}
$$

By the comparison principle,

$$
w(t, x) \leq u(t, x), \quad \forall \ t \geq T, \ 0 \leq x \leq l.
$$

Since $l > \pi \sqrt{1/(1 - a\varepsilon)}$, it is well known that $w(t, x) \to W(x)$ as $t \to \infty$ uniformly in any compact subset of $(0, l)$, where $W$ is the unique positive solution of

$$
\begin{cases} 
W_{xx} + W (1 - a\varepsilon - W) = 0, & 0 < x < l, \\
W(0) = W(l) = 0.
\end{cases}
$$

Hence, $\lim \inf_{t \to \infty} u(t, x) \geq \lim_{t \to \infty} w(t, x) = W(x) > 0$ in $(0, l)$. This is a contradiction to (3.10).
If $h_\infty > \pi \sqrt{d/c}$, then there exists $T \gg 1$ such that $h(T) > \pi \sqrt{d/c}$. Set $l = h(T)$ and let $z = z(t, x)$ be the unique positive solution of the following initial boundary value problem with fixed boundary:

$$\begin{cases}
z_t = d z_{xx} + z(c - z), & t > T, \ 0 < x < l, \\
z(t, 0) = z(t, l) = 0, & t > T, \\
z(T, x) = v(T, x), & 0 \leq x \leq l.
\end{cases}$$

By the comparison principle,

$$z(t, x) \leq v(t, x), \ \forall \ t \geq T, \ 0 \leq x \leq l.$$

Since $l > \pi \sqrt{d/c}$, similarly to the above, we can get a contradiction to (3.10). □

Now we discuss the case $h_0 < \Lambda$.

**Lemma 5.1** Suppose that $h_0 < \Lambda$. For the problem (DFB), if

$$\mu \geq \mu^0 := \max\left\{1, \frac{1}{c} \|v_0\|_\infty \right\} \frac{d}{\rho} \left( \frac{\pi^2 d}{c} - h_0^2 \right) \left( \int_0^{h_0} x v_0(x) dx \right)^{-1},$$

then $h_\infty = \infty$. For the problem (NFB), if

$$\mu \geq \mu^0 := \max\left\{1, \frac{1}{c} \|v_0\|_\infty \right\} \frac{d}{\rho} \left( \frac{\pi}{2} \sqrt{\frac{c}{d}} - h_0 \right) \left( \int_0^{h_0} v_0(x) dx \right)^{-1},$$

then $h_\infty = \infty$.

**Proof.** For the problem (DFB), we consider the following auxiliary problem

$$\begin{cases}
v_t - d v_{xx} = v(c - v), & t > 0, \ 0 < x < h(t), \\
v(t, 0) = 0, \ v(t, h(t)) = 0, & t > 0, \\
h'(t) = -\mu \rho v_x(t, h(t)), & t > 0, \\
v(0, x) = v_0(x), & 0 \leq x \leq h_0, \\
h(0) = h_0.
\end{cases}$$

It follows from Lemma 4.2 that

$$h(t) \leq h(t), \ v(t, x) \leq v(t, x), \ \forall \ t > 0, \ 0 < x < h(t).$$

Recall that $h_0 < \Lambda \leq \pi \sqrt{d/c}$, and $\mu \geq \mu^0$, in view of the Proposition 4.8 in [17], it yields $h(\infty) = \infty$. Therefore, $h_\infty = \infty$.

For the problem (NFB), we consider the following auxiliary problem

$$\begin{cases}
v_t - d v_{xx} = v(c - v), & t > 0, \ 0 < x < h(t), \\
v_x(t, 0) = 0, \ v(t, h(t)) = 0, & t > 0, \\
h'(t) = -\mu \rho v_x(t, h(t)), & t > 0, \\
v(0, x) = v_0(x), & 0 \leq x \leq h_0, \\
h(0) = h_0.
\end{cases}$$
By Lemma 3.2
\[ h(t) \leq h(t), \quad v(t, x) \leq v(t, x), \quad \forall \ t > 0, \ 0 < x < h(t). \]

Note that \( h_0 < \Lambda \leq \frac{\pi}{2} \sqrt{d/c} \), and \( \mu \geq \mu^0 \), by use of the Lemma 3.7 of [11], we have \( h(\infty) = \infty \). Therefore, \( h_\infty = \infty \). \( \square \)

**Lemma 5.2** Assume that \( h_0 < \Lambda \). Then there exists \( \mu_0 > 0 \), depending also on \( u_0(x) \) and \( v_0(x) \), such that \( h_\infty < \infty \) when \( \mu \leq \mu_0 \) for both problems (DFB) and (NFB).

**Proof.** We shall use the argument from Ricci and Tarzia [27] to construct suitable upper solutions and use Lemma 4.1 to derive the desired conclusion.

**Step 1** We first consider the problem (NFB) and adopt the following functions constructed by Du and Lin [11] (see also Guo and Wu [14]):

\[
\sigma(t) = h_0 \left( 1 + \delta - \frac{\delta}{2} e^{-\beta t} \right), \quad t \geq 0; \quad V(y) = \cos \frac{\pi y}{2}, \quad 0 \leq y \leq 1,
\]

\[
\bar{u}(t, x) = M e^{-\beta t} V \left( \frac{x}{\sigma(t)} \right), \quad \bar{v}(t, x) = b M e^{-\beta t} V \left( \frac{x}{\sigma(t)} \right), \quad 0 \leq x \leq \sigma(t).
\]

It is obvious that

\[
\bar{u}_x(t, 0) = \bar{v}_x(t, 0) = 0, \quad \bar{u}(t, \sigma(t)) = \bar{v}(t, \sigma(t)) = 0, \quad \forall \ t \geq 0.
\]

Recall that \( h_0 < \frac{\pi}{2} \min \{ \sqrt{d/c}, 1 \} \) in the present case and \( \bar{v}(t, x) = b \bar{u}(t, x) \). Similarly to the proof of Corollary 1(iii) in [14] (pp.892), we can verify that, for suitable small positive constants \( \delta \) and \( \beta \), and large positive constant \( M \), the pair \((\bar{u}, \bar{v})\) satisfies

\[
\begin{cases}
\bar{u}_t - \bar{u}_{xx} - \bar{u}(1 - \bar{u}) \geq 0, & t > 0, \ 0 < x \leq \sigma(t), \\
\bar{v}_t - d \bar{v}_{xx} - \bar{v}(c - \bar{v} + b \bar{u}) \geq 0, & t > 0, \ 0 < x \leq \sigma(t), \\
\bar{u}(0, x) \geq u_0(x), \ \bar{v}(0, x) \geq v_0(x), & 0 \leq x \leq h_0(1 + \delta/2).
\end{cases}
\]

Moreover, for such fixed constants \( \delta \), \( \beta \), and \( M \), there exists \( \mu_0 > 0 \) such that

\[
\sigma'(t) + \mu (\bar{u}_x + \rho \bar{v}_x) \big|_{x=\sigma(t)} \geq 0
\]

for all \( \mu \leq \mu_0 \).

By Lemma 4.1 \( \sigma(t) \geq h(t) \). Taking \( t \rightarrow \infty \) we have \( h_\infty \leq \sigma(\infty) = h_0(1 + \delta) < \infty \).

**Step 2** Now we discuss the problem (DFB). Recall that \( h_0 < \pi \min \{ \sqrt{d/c}, 1 \} \) for our present case. We can verify that there exist two positive constants \( \delta, \beta \ll 1 \) such that

\[
\begin{align*}
\frac{1}{h_0(1 + \delta)} \left( \frac{\pi^2}{h_0(1 + \delta)} - \beta \delta h_0 \right) - \beta - 1 & > 0, \quad (5.1) \\
\frac{1}{h_0(1 + \delta)} \left( \frac{d \pi^2}{h_0(1 + \delta)} - \beta \delta h_0 \right) - \beta - c & > 0. \quad (5.2)
\end{align*}
\]

For such fixed \( \delta \) and \( \beta \), let

\[
\sigma(t) = h_0 \left( 1 + \delta - \frac{\delta}{2} e^{-\beta t} \right), \quad t \geq 0; \quad W(y) = \sin \pi y, \quad 0 \leq y \leq 1,
\]

\[
\bar{u}(t, x) = M e^{-\beta t} W \left( \frac{x}{\sigma(t)} \right), \quad \bar{v}(t, x) = b M e^{-\beta t} W \left( \frac{x}{\sigma(t)} \right), \quad 0 \leq x \leq \sigma(t),
\]
where $M$ is a large positive constant. It is obvious that

$$\bar{u}(t, 0) = \bar{v}(t, 0) = \ddot{u}(t, \sigma(t)) = \ddot{v}(t, \sigma(t)) = 0, \; \forall \; t \geq 0, \quad (5.3)$$

and

$$\ddot{u}(0, x) \geq u_0(x), \; \ddot{v}(0, x) \geq v_0(x), \; \forall \; 0 \leq x \leq \sigma_0(1 + \delta/2) \quad (5.4)$$

provided that $M \gg 1$. Moreover, for such fixed constants $\delta, \beta$ and $M$, since $\sigma(t) \geq h_0(1 + \delta/2)$, it is easy to see that there exists $0 < \mu_0 \ll 1$ such that

$$\sigma'(t) + \mu(\ddot{u}_x + \rho \ddot{v}_x)|_{x=\sigma(t)} = e^{-\beta t} \left( \frac{\beta \delta h_0}{2} - \mu M \pi (1 + bp) \frac{1}{\sigma(t)} \right) > 0, \; \forall \; t \geq 0 \quad (5.5)$$

provided that $0 < \mu \leq \mu_0$.

Denote $y = x/\sigma(t)$. The direct calculation yields,

$$\ddot{u}_t - \ddot{u}_{xx} - \ddot{u}(1 - \ddot{u}) = \ddot{u} \left( \frac{\pi^2}{\sigma^2(t)} - \beta - 1 - e^{-\beta t} \frac{\beta \delta h_0}{\sigma(t)} \times \frac{\pi y}{2 \sin \pi y} \cos \pi y + \ddot{u} \right)$$

$$\geq \ddot{u} \left( \frac{\pi^2}{\sigma^2(t)} - \beta - 1 - e^{-\beta t} \frac{\beta \delta h_0}{\sigma(t)} \times \frac{\pi y}{2 \sin \pi y} \cos \pi y \right).$$

Since $\cos \pi y \leq 0$ for $1/2 \leq y \leq 1$, and $\sigma(t)$ is increasing, we have

$$\ddot{u}_t - \ddot{u}_{xx} - \ddot{u}(1 - \ddot{u}) \geq \ddot{u} \left( \frac{\pi^2}{\sigma^2(t)} - \beta - 1 \right)$$

$$\geq \ddot{u} \left( \frac{\pi^2}{h_0^2(1 + \delta)^2} - \beta - 1 \right)$$

$$> 0, \; \forall \; \sigma(t)/2 \leq x \leq \sigma(t)$$

by (5.1). Remember that $0 \leq \cos \pi y \leq 1$, $y \leq \frac{2}{\pi} \sin \pi y$ for all $0 \leq y \leq 1/2$, and $e^{-\beta t} \leq 1$ for all $t \geq 0$. We have that, for all $t > 0$ and $0 \leq x \leq \sigma(t)/2$,

$$e^{-\beta t} \frac{\beta \delta h_0}{\sigma(t)} \times \frac{\pi y}{2 \sin \pi y} \cos \pi y \leq \frac{\beta \delta h_0}{\sigma(t)}.$$

It follows that, for all $t > 0$ and $0 \leq x \leq \frac{1}{2} \sigma(t)$,

$$\ddot{u}_t - \ddot{u}_{xx} - \ddot{u}(1 - \ddot{u}) \geq \ddot{u} \left[ \frac{1}{\sigma(t)} \left( \frac{\pi^2}{\sigma(t)} - \beta - 1 - \frac{\beta \delta h_0}{\sigma(t)} \right) \right]$$

$$\geq \ddot{u} \left[ \frac{1}{h_0(1 + \delta)} \left( \frac{\pi^2}{h_0(1 + \delta)} - \beta - 1 \right) \right]$$

$$> 0$$

by (5.1). In conclusion, we have

$$\ddot{u}_t - \ddot{u}_{xx} - \ddot{u}(1 - \ddot{u}) > 0, \; \forall \; t > 0, \; 0 \leq x \leq \sigma(t). \quad (5.6)$$

Remember $\ddot{v}(t, x) = b\ddot{u}(t, x)$; in view of (5.2), similar to the above, we can verify that

$$\ddot{v}_t - d\ddot{v}_{xx} - \ddot{v}(c - \ddot{v} + b\ddot{u}) > 0, \; \forall \; t > 0, \; 0 \leq x \leq \sigma(t). \quad (5.7)$$

Notice that (5.3)–(5.7), by virtue of Lemma 4.1 we have $\sigma(t) \geq h(t)$. Taking $t \to \infty$ we have $h_\infty \leq \sigma(\infty) = h_0(1 + \delta) < \infty$. The proof is complete. \(\square\)
Theorem 5.2 Suppose that $h_0 < \Lambda$. Then there exist $\mu^* \geq \mu_0 > 0$, depending on $u_0(x)$, $v_0(x)$ and $h_0$, such that $h_\infty = \infty$ if $\mu > \mu^*$, and $h_\infty \leq \Lambda$ if $\mu \leq \mu_0$ or $\mu = \mu^*$.

**Proof.** The proof is similar to that of [11, Theorem 3.9] and [29, Theorem 5.4]. For the convenience to the reader we shall give the details because this is a main theorem in this section. We will write $(u_\mu, v_\mu, h_\mu)$ in place of $(u, v, h)$ to clarify the dependence of the solution of (DFB) and/or (NFB) on $\mu$. Define

$$\Sigma^* = \{ \mu > 0 : h_{\mu, \infty} \leq \Lambda \}.$$  

By Lemma 5.2, $(0, \mu_0) \subset \Sigma^*$. In view of Lemma 5.1, $\Sigma^* \cap [\mu_0, \infty) = \emptyset$. Therefore, $\mu^* := \sup \Sigma^* \in [\mu_0, \mu^0]$. By this definition and Theorem 5.1 we find that $h_{\mu, \infty} = \infty$ when $\mu > \mu^*$. Hence, $\Sigma^* \subset (0, \mu^*)$.

We will show that $\mu^* \in \Sigma^*$. Otherwise, $h_{\mu^*, \infty} = \infty$. There exists $T > 0$ such that $h_{\mu^*}(T) > \Lambda$. By the continuous dependence of $(u_\mu, v_\mu, h_\mu)$ on $\mu$, there is $\varepsilon > 0$ such that $h_{\mu}(T) > \Lambda$ for $\mu \in (\mu^* - \varepsilon, \mu^* + \varepsilon)$. It follows that for all such $\mu$,

$$\lim_{t \to \infty} h_{\mu}(t) \geq h_{\mu}(T) > \Lambda.$$ 

Therefore, $(\mu^* - \varepsilon, \mu^* + \varepsilon) \cap \Sigma^* = \emptyset$, and $\sup \Sigma^* \leq \mu^* - \varepsilon$. This contradicts the definition of $\mu^*$.

Define

$$\Sigma_* = \{ \nu : \nu \geq \mu_0 \text{ such that } h_{\mu, \infty} \leq \Lambda \text{ for all } 0 < \mu \leq \nu \},$$

where $\mu_0$ is given by Lemma 5.2. Then $\mu_* := \sup \Sigma_* \leq \mu^*$ and $(0, \mu_*) \subset \Sigma_*$. Similarly to the above, we can prove that $\mu_* \in \Sigma_*$. The proof is completed. \[ \square \]

6 Asymptotic spreading speed

In this section we provide an upper bound for $\limsup_{t \to \infty} \frac{h(t)}{t}$, which shows that the asymptotic spreading speed (if exists) for both problems (DFB) and (NFB) cannot be faster than $2 \max \{ \sqrt{cd}, 1 \}$ under some suitable conditions. The number $2 \max \{ \sqrt{cd}, 1 \}$ seems to be the minimal speed of traveling wave fronts of the prey-predator system

$$\begin{cases}
    u_t - u_{xx} = u(1 - u - av), & t > 0, \ x \in \mathbb{R}, \\
    v_t - dv_{xx} = v(c - v + bu), & t > 0, \ x \in \mathbb{R},
\end{cases} \quad (6.1)$$

please refer to [19].

It is easy to see that Theorem 5.17 of [20] still holds for the traveling wave fronts of the following system

$$\begin{cases}
    u_t - u_{xx} = u(1 - u), & t > 0, \ x \in \mathbb{R}, \\
    v_t - dv_{xx} = v(c - v + bu), & t > 0, \ x \in \mathbb{R}.
\end{cases}$$

Thus, for any given $s > 2 \max \{ \sqrt{cd}, 1 \}$, the following problem

$$\begin{cases}
    s\phi'' + \phi'' + \phi(1 - \phi) = 0, & s\psi'' + d\psi'' + \psi(c - \psi + b\phi) = 0 \text{ in } \mathbb{R}, \\
    \phi(-\infty) = 1, \ \psi(-\infty) = b + c, \ (\phi, \psi)(\infty) = (0, 0), \ \phi(0) = 1/2, \\
    \phi' < 0, \ \psi' < 0 \text{ in } \mathbb{R},
\end{cases} \quad (6.2)$$

...
has a solution \((\phi(\xi), \psi(\xi))\), with \(\xi = x - st\). Moreover, \((\phi(\xi), \psi(\xi))\) satisfies
\[
\lim_{\xi \to \infty} \phi(\xi)e^{\lambda_1 \xi} = \lim_{\xi \to \infty} \psi(\xi)e^{\lambda_2 \xi} = 1,
\]
where
\[
\lambda_1 = \frac{s + \sqrt{s^2 - 4}}{2} > 0, \quad \lambda_2 = \frac{s + \sqrt{s^2 - 4cd}}{2d} > 0.
\]

**Theorem 6.1** Let \((u, v, h)\) be the solution of the problem (DFB) or (NFB) and \(h_\infty = \infty\). If for any given \(s > 2 \max \{\sqrt{cd}, 1\}\), the problem (6.2) has a solution \((\phi(\xi), \psi(\xi))\) satisfying
\[
\psi(\xi) \geq \beta \phi(\xi), \quad \forall \, \xi \in \mathbb{R}
\]
for some positive constant \(\beta\) (which may depend on \(s\)). Then we have
\[
\limsup_{t \to \infty} \frac{h(t)}{t} \leq 2 \max \{\sqrt{cd}, 1\}.
\]

Before giving the proof of Theorem 6.1, we state one remark to guarantee the condition (6.5).

**Remark 6.1** For any given \(s > 2 \max \{\sqrt{cd}, 1\}\), let \(\lambda_1\) and \(\lambda_2\) be given by (6.4). By the carefully calculations we have that if one of the following holds:
(a) \(d \geq 1, cd \geq 1\);
(b) \(d \geq 1, cd < 1\) and \(c + d \geq 2\);
(c) \(d < 1, cd \geq 1\) and \(2cd \geq c + d\),
then \(\lambda_2 \geq \lambda_1\). In view of (6.3) and the limits:
\[
\lim_{\xi \to -\infty} \phi(\xi) = 1, \quad \lim_{\xi \to -\infty} \psi(\xi) = b + c,
\]
it can be seen that there exists a positive constant \(\beta\) such that (6.5) holds.

**Proof of Theorem 6.1** The idea of this proof comes from [14]. For any given \(s > 2 \max \{\sqrt{cd}, 1\}\), let \((\phi(\xi), \psi(\xi))\) be the solution of (6.2) satisfying (6.5).

Choose \(m > k \gg 1\) such that
\[
m\beta \geq 2bk, \quad \text{which implies} \quad (m - 1)\beta \geq 2b(k - 1),
\]
\[
k\phi(\xi) > \|u_0\|_\infty, \quad m\psi(\xi) > \|v_0\|_\infty, \quad \forall \, \xi \in [0, h_0].
\]
For such fixed \(m\) and \(k\), recall that \((\phi(\xi), \psi(\xi)) \to 0\) and \((\phi'(\xi), \psi'(\xi)) \to 0\) as \(\xi \to \infty\), there exists \(\sigma_0 > h_0\) such that
\[
\phi(\sigma_0) < \min_{0 \leq x \leq h_0} \left( \phi(x) - \frac{u_0(x)}{k} \right), \quad \psi(\sigma_0) < \min_{0 \leq x \leq h_0} \left( \psi(x) - \frac{v_0(x)}{m} \right),
\]
\[
\phi(\sigma_0) < 1 - 1/k, \quad \psi(\sigma_0) < 2c(m - 1)/(m + 3m^2),
\]
\[
-\mu[k\phi'(\sigma_0) + m\psi'(\sigma_0)] < s.
\]
Set $\sigma(t) = \sigma_0 + st$ and

$$\bar{u}(t, x) = k\phi(x - st) - k\phi(\sigma_0), \quad \bar{v}(t, x) = m\psi(x - st) - m\psi(\sigma_0).$$

It is obvious that

$$\bar{u}(t, \sigma(t)) = \bar{v}(t, \sigma(t)) = 0, \quad \forall \, t \geq 0.$$ 

Since $\phi' < 0$, $\psi' < 0$, we see that

$$\bar{u}(t, 0) > 0, \quad \bar{v}(t, 0) > 0, \quad \bar{u}_x(t, 0) < 0, \quad \bar{v}_x(t, 0) < 0, \quad \forall \, t \geq 0.$$ 

It is deduced from (6.7) that

$$\bar{u}(0, x) > u_0(x), \quad \bar{v}(0, x) > v_0(x), \quad \forall \, 0 \leq x \leq h_0.$$ 

By the first inequality of (6.8),

$$\bar{u}_t - \bar{u}_{xx} - \bar{u}(1 - \bar{u}) = k \left[ (k - 1) \left( \phi - \frac{k\phi(\sigma_0)}{k - 1} \right)^2 + \phi(\sigma_0) \frac{k - 1 - k\phi(\sigma_0)}{k - 1} \right] \geq 0.$$ 

In order to save space, we denote $\phi(\sigma_0)$ and $\psi(\sigma_0)$ by $\phi_0$ and $\psi_0$, respectively. Applying the second inequality of (6.8), (6.5) and (6.6) we have

$$\frac{1}{m}[\bar{v}_t - d\bar{v}_{xx} - \bar{v}(c - \bar{v} + b\bar{u})] = \frac{(m - 1)}{2} \left( \psi - \frac{2m}{m - 1} \psi_0 \right)^2 + \psi_0 \left( c - \frac{m(1 + 3m)}{2(m - 1)} \psi_0 \right) + \frac{1}{2} \psi [(m - 1)\psi - 2b(k - 1)]$$

$$+ \frac{1}{2} \psi_0 (m\psi_0 - 2bk\phi_0) + bk(\phi\psi_0 + \phi_0\psi)$$

$$> \frac{1}{2} \psi[(m - 1)\beta - 2b(k - 1)] + \frac{1}{2} \psi_0\psi_0(m\beta - 2bk)$$

$$\geq 0.$$ 

It follows from (6.9) that

$$\sigma'(t) = s > -\mu \left[ k\phi'(\sigma_0) + \rho m\psi'(\sigma_0) \right] = -\mu[\bar{u}_x(t, \sigma(t)) + \rho\bar{v}_x(t, \sigma(t))] .$$

Recall $\sigma_0 > h_0$; so we have verified the conditions of Lemma 4.1.

In view of Lemma 4.1 $\sigma(t) \geq h(t)$. Therefore

$$\limsup_{t \to \infty} \frac{h(t)}{t} \leq \lim_{t \to \infty} \frac{\sigma(t)}{t} = s.$$ 

By the arbitrariness of $s > 2 \max \{ \sqrt{cd}, 1 \}$ we get

$$\limsup_{t \to \infty} \frac{h(t)}{t} \leq 2 \max \{ \sqrt{cd}, 1 \} .$$

The proof is complete. \qed
Remark 6.2 Let \((w, g)\) and \((z, p)\) be solutions of the free boundary problems

\[
\begin{cases}
    w_t - w_{xx} = w(1 - w), & t > 0, \quad 0 < x < g(t), \\
    w_x(t, 0) = w(t, g(t)) = 0, & t > 0, \\
    g'(t) = -\eta w_x(t, g(t)), & t > 0, \\
    w(0, x) = w_0(x), & x \in [0, g_0], \\
    g(0) = g_0 \geq h_0
\end{cases}
\]

and

\[
\begin{cases}
    z_t - dz_{xx} = z(c - z), & t > 0, \quad 0 < x < p(t), \\
    z_x(t, 0) = z(t, p(t)) = 0, & t > 0, \\
    \eta'(t) = -\zeta z_x(t, p(t)), & t > 0, \\
    z(0, x) = z_0(x), & x \in [0, p_0], \\
    p(0) = p_0 \leq h_0.
\end{cases}
\]

respectively. Assume that \(g(\infty) = \infty\) and \(p(\infty) = \infty\) (these will be true under the suitable conditions on the parameters \(g_0, p_0, \eta\) and \(\zeta\), refer to \([11]\)). It follows from the result of \([11]\) that there are positive constants \(g^*\) and \(p^*\), such that

\[
\lim_{t \to \infty} \frac{g(t)}{t} = g^*, \quad \lim_{t \to \infty} \frac{p(t)}{t} = p^*.
\]

Suppose that \(\eta \geq \mu, \zeta \leq \mu\rho\), and

\[
u_0(x) \leq w_0(x) \quad \text{in} \quad [0, h_0], \quad v_0(x) \geq z_0(x) > 0 \quad \text{in} \quad [0, p_0].
\]

For the solution \((u, v, h)\) of the problem (NFB), by the comparison principle (Lemma 4.2 and its similar version) we have that \(p(t) \leq h(t) \leq g(t)\). Hence,

\[
p^* \leq \liminf_{t \to \infty} \frac{h(t)}{t}, \quad \limsup_{t \to \infty} \frac{h(t)}{t} \leq g^*.
\]

7 Discussion

In this paper, we have examined a Lotka-Volterra type prey-predator model with free boundary \(x = h(t)\) for both prey and predator, which describes the movement process through the free boundary. We envision that the two species initially occupy the region \([0, h_0]\) and have a tendency to expand their territory together. Then we extend some results of \([11]\) and \([17]\) for one species case and \([14]\) for two-species weak competition system case to the prey-predator system. The dynamic behavior are discussed. Let \(\Lambda = \pi \min \{\sqrt{d/c}, 1\}\) for the problem (DFB), and \(\Lambda = \frac{\pi}{2} \min \{\sqrt{d/c}, 1\}\) for the problem (NFB). It was proved that:

(i) If the size of initial habitat is not less than \(\Lambda\), or it is less than \(\Lambda\) but the moving parameter/coefficient \(\mu\) of the free boundary is greater than \(\mu^*\) (it depends on the initial data \((u_0, v_0)\) and \(h_0\)), then the two species will spread successfully. Moreover,
(ia) To the problem (NFB), as \( t \to \infty \), both \( u(t, x) \) and \( v(t, x) \) go to positive constants for the weakly hunting case: \( ac < 1 \) and \( ab < 1 \); while \( u(t, x) \to 0 \) and \( v(t, x) \to c \) for the strongly hunting case: \( ac > 1 \);

(ib) To the problem (DFB), if \( a(b + c) < 1 \), then \( u(t, x) \) and \( v(t, x) \) satisfy

\[
\liminf_{t \to \infty} u(t, x) \geq \bar{u}(x), \quad \limsup_{t \to \infty} u(t, x) \leq \bar{u}(x), \quad \liminf_{t \to \infty} v(t, x) \geq \bar{v}(x), \quad \limsup_{t \to \infty} v(t, x) \leq \bar{v}(x)
\]

uniformly in any compact subset of \([0, \infty)\), where \( \bar{u}, \bar{v}, \underline{u} \) and \( \underline{v} \) are positive functions given by Theorem \( \text{[2.3]} \).

(ii) While if the size of initial habitat is less than \( \Lambda \) and the moving parameter/coefficient \( \mu \) of the free boundary is less than \( \mu^* \), then \( \lim_{t \to \infty} h(t) < \infty \), and \( \|u(t, x), v(t, x)\|_{C([g(t), h(t)])} \to 0 \) as \( t \to \infty \). That is, the two species will disappear eventually.

The above conclusions not only provide the spreading-vanishing dichotomy and criteria governing spreading and vanishing, but also provided the long time behavior of \((u(t, x), v(t, x))\). If the size of initial habitat is small, and the moving parameter is small enough, it causes no population can survive eventually, while they can coexist if the size of habitat or the moving parameter is large enough, regardless of initial population size. This phenomenon suggests that the size of the initial habitat and the moving parameter are important to the survival for the two species. It is well-known that the effect of habitat size to the survival for species with Dirichlet boundary problem is quite important (see, for example, [1]).

Finally, Theorem \( \text{[6.1]} \) reveals that the asymptotic spreading speed (if exists) cannot be faster than the minimal speed for the traveling wave fronts corresponding to the model \( \text{(6.1)} \). It would be very interesting if one can realize how the asymptotic spreading speed depends on these parameters. In [19] and [25], Lin and Pan, respectively, have obtained some interesting results for the asymptotic spreading speeds of the model \( \text{(6.1)} \) by constructing appropriate and elaborate upper and/or lower solutions. Their conclusion seems to show that the prey and predator may have different asymptotic spreading speeds.

A great deal of previous mathematical investigation on the spreading of population has been based on the traveling wave fronts of prey-predator system \( \text{(6.1)} \). A striking difference between our present problems and \( \text{(6.1)} \) is that the spreading front in our present problems is given explicitly by a function \( x = h(t) \), beyond which the population densities of both prey and predator are 0, while in \( \text{(6.1)} \), the two species become positive for all \( x \) once \( t \) is positive. Secondly, \( \text{(6.1)} \) guarantees successful spreading of the two species for any nontrivial initial populations \((u(0, x)\) and \(v(0, x)\), regardless of their initial sizes and supporting area, but the dynamics of our present problems exhibit the spreading-vanishing dichotomy. The phenomenon exhibited by this dichotomy seems closer to the reality.

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References

[1] G. Bunting, Y. Du and K. Krakowski, *Spreading speed revisited: Analysis of a free boundary model*, Networks and Heterogeneous Media (special issue dedicated to H. Matano), 7(2012), 583-603.

[2] L. Caffarelli and S. Salsa, *A Geometric Approach to Free Boundary Problems*, Grad. Stud. Math., Vol. 68, Amer. Math. Soc., Providence, RI, 2005.

[3] X. F. Chen and A. Friedman, *A free boundary problem arising in a model of wound healing*, SIAM J. Math. Anal., 32(4)(2000), 778-800.

[4] X. F. Chen and A. Friedman, *A free boundary problem for an elliptic-hyperbolic system: an application to tumor growth*, SIAM J. Math. Anal., 35(2003), 974-986.

[5] J. Crank, *Free and Moving Boundary Problem*, Clarendon Press, Oxford, 1984.

[6] Y. H. Du, *Order Structure and Topological Methods in Nonlinear Partial Differential Equations*, Vol. 1: Maximum Principles and Applications, World Scientific, Singapore, 2005.

[7] Y. H. Du and Z. Guo, *Spreading-vanishing dichotomy in a diffusive logistic model with a free boundary*, II, J. Differential Equations, 250(2011), 4336-4366.

[8] Y. H. Du and Z. Guo, *The Stefan problem for the Fisher-KPP equation*, J. Differential Equations, 253(3)(2012), 996-1035.

[9] Y. H. Du, Z. M. Guo and R. Peng, *A diffusive logistic model with a free boundary in time-periodic environment*, J. Funct. Anal., 265(2013), 2089-2142.

[10] Y. H. Du and X. Liang, *Pulsating semi-waves in periodic media and spreading speed determined by a free boundary model*. Preprint, 2012.

[11] Y. H. Du and Z. G. Lin, *Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary*, SIAM J. Math. Anal., 42(2010), 377-405. Erratum; SIAM J. Math. Anal., 45(2013), 1995-1996.

[12] Y. H. Du and B. D. Lou, *Spreading and vanishing in nonlinear diffusion problems with free boundaries*, J. Eur. Math. Soc., to appear.

[13] Y. H. Du, H. Matsuzawa and M. L. Zhou, *Sharp estimate of the spreading speed determined by nonlinear free boundary problems*. Preprint, 2012.

[14] J. S. Guo and C. H. Wu, *On a free boundary problem for a two-species weak competition system*, J. Dyn. Diff. Equat., 24(2012), 873-895.

[15] D. Hilhorst, M. Iida, M. Mimura and H. Ninomiya, *A competition-diffusion system approximation to the classical two-phase Stefan problem*, Japan J. Indust. Appl. Math., 18(2)(2001), 161-180.

[16] D. Hilhorst, M. Mimura and R. Schätzle, *Vanishing latent heat limit in a Stefan-like problem arising in biology*, Nonlinear Anal. Real World Appl., 4(2003), 261-285.

[17] Y. Kaneko and Y. Yamada, *A free boundary problem for a reaction-diffusion equation appearing in ecology*, Adv. Math. Sci. Appl., 21(2)(2011), 467-492.

[18] H. L. Li, P. Y. H. Pang and M. X. Wang, *Boundary blow-up solutions for logistic-type porous media equations with nonregular source*, J. London Math. Soc., 80(2009), 273-294.

[19] G. Lin, *Spreading speeds of a Lotka-Volterra predator-prey system: the role of the predator*, Nonlinear Analysis, 74(2011), 2448-2461.
[20] G. Lin, W. T. Li and M. J. Ma, Traveling wave solutions in delayed reaction diffusion systems with applications to multi-species models, Discrete Cont. Dyn. Syst. B, 13(2)(2010), 393-414.

[21] Z. G. Lin, A free boundary problem for a predator-prey model, Nonlinearity, 20(2007), 1883-1892.

[22] M. Mimura, Y. Yamada and S. Yotsutani, A free boundary problem in ecology, Japan J. Appl. Math., 2(1985), 151-186.

[23] M. Mimura, Y. Yamada and S. Yotsutani, Stability analysis for free boundary problems in ecology, Hiroshima Math. J., 16(1986), 477-498.

[24] M. Mimura, Y. Yamada and S. Yotsutani, Free boundary problems for some reaction diffusion equations, Hiroshima Math. J., 17(1987), 241-280.

[25] S. X. Pan, Asymptotic spreading in a Lotka-Volterra predator-prey System. Preprint, 2012.

[26] R. Peng and X. Q. Zhao, The diffusive logistic model with a free boundary and seasonal succession, Discrete Cont. Dyn. Syst. A, 33(5)(2013), 2007-2031.

[27] R. Ricci and D. A. Tarzia, Asymptotic behavior of the solutions of the dead-core problem, Nonlinear Analysis, 13(1989), 405-411.

[28] L. I. Rubinstein, The Stefan Problem, American Mathematical Society, Providence, RI, 1971.

[29] M. X. Wang and J. F. Zhao, A free boundary problem for a predator-prey model with double free boundaries. Preprint, 2012.

[30] J. F. Zhao and M. X. Wang, A free boundary problem of a predator-prey model with higher dimension and heterogeneous environment, Nonlinear Anal.: Real World Appl., 16(2014), 250-263.