Constraining Almost Degenerate Three-Flavor Neutrinos

Hisakazu Minakata\(^1,2\) and Osamu Yasuda\(^1\)

\(^1\)Department of Physics, Tokyo Metropolitan University
Minami-Osawa, Hachioji, Tokyo 192-03, Japan

\(^2\)Institute for Nuclear Theory, University of Washington
Seattle, Washington 98195-1550, USA

(September, 1996)

Abstract

We discuss constraints on a scenario of almost degenerate three-flavor neutrinos imposed by the solar and the atmospheric neutrino anomalies, hot dark matter, and neutrinoless double $\beta$ decays. It is found that in the Majorana version of the model the region with relatively large $\theta_{13}$ is favored and a constraint on the CP violating phases is obtained.

14.60.Pq, 26.65.+t, 23.40.-s, 95.35.+d
There exist several experimental hints which indicate that most probably neutrinos have tiny masses and flavor mixings. The first is the solar neutrino deficit observed in four different experiments, the chlorine, the Kamiokande II-III, GALLEX and SAGE [1–4]. It became highly unlikely that the data of various experiments can be reconciled with any sensible modifications of the standard solar model. The second is the atmospheric neutrino anomaly, the large deviation in the observed ratio $\nu_\mu/\nu_e$ from the expectation of the Monte-Carlo simulations [5,6]. While the anomaly was not observed within the statistics of the NUSEX and the Frejus experiments [7,8], the evidences in the Kamiokande and IMB detectors are so impressive that they force us to consider seriously about the anomaly. The presence of the anomaly is also supported by the newest tracking detector, Soudan 2 [9].

The possible third hint for neutrino masses comes from the cosmological model with cold and hot dark matter (CHDM). The neutrinos is the only known candidate for the hot component. They could be responsible for the large-scale structure formation in a way consistent with the COBE observation of anisotropy of cosmic microwave background [10–12]. While less direct compared with the first and the second hints, it provides a good motivation for examining the possibility of neutrino masses of a few eV range.

It has been pointed out by various authors that if at least one of the neutrino states has mass of the dark matter scale and if there is a hierarchy in two $\Delta m^2$, the difference in squared masses, the accelerator and the reactor experiments put powerful constraints on mixing angles [13–15]. It is very remarkable that the mixing pattern of neutrinos is determined to be essentially unique [13] if one imposes the additional constraints that come from the requirement of solving either the solar neutrino problem or the atmospheric neutrino anomaly, together with that from neutrinoless double $\beta$ decays [16].

The problem with the above framework with only three-flavor neutrinos (i.e., without sterile neutrinos) is that one cannot account for the solar neutrino deficit, the atmospheric neutrino anomaly, and the hot dark matter simultaneously. The only known possibility that can accommodate these two phenomena as well as supplying neutrino masses appropriate for hot dark matter within the standard three-flavor framework is the case of almost degenerate
neutrinos (ADN). An incomplete list of earlier references on ADN is in [12][19].

In this paper we discuss the constraints that can be imposed on such almost degenerate neutrino scenario from the solar and the atmospheric neutrino observations as well as the terrestrial neutrino experiments. We will point out that, in the case of Majorana neutrinos, the neutrinoless double $\beta$ decay experiment [16] is of key importance. In particular, the solar neutrino and the double $\beta$ decay experiments constrain the mixing angle $\theta_{13}$ to be large.

Let us start by defining more precisely what we mean by the almost degenerate neutrinos. Due to the requirement of solving the solar and the atmospheric neutrino problems the two $\Delta m^2$ should have values $\lesssim 10^{-5}\text{eV}^2$ and $\sim 10^{-2}\text{eV}^2$, respectively. This implies that three neutrino states are degenerate up to the accuracy of 0.1 eV. Then, the requirement from the hot dark matter hypothesis implies that they must have masses of the order of a few to several eV [10][12]. Then, the degeneracy in masses is smaller than 0.01eV, hence the name of almost degenerate neutrinos (ADN).

For definiteness, we assign the smaller $\Delta m^2$ to $\Delta m^2_{12} \equiv m_2^2 - m_1^2$ and the larger to $\Delta m^2_{13}$. It should be noticed that this can be done without loss of generality. Despite the almost degeneracy in neutrino masses there is a hierarchy in $\Delta m^2$; $\Delta m^2_{13} \simeq \Delta m^2_{23} \gg \Delta m^2_{12}$. It allows us to simplify greatly formulae for the oscillation probabilities. With neutrino mixing matrix $U_{\alpha i}$ they read,

$$P(\nu_\beta \to \nu_\alpha) = 4|U_{\alpha 3}|^2 |U_{\beta 3}|^2 \sin^2 \left( \frac{\Delta m^2_{13} L}{4E} \right)$$

$$1 - P(\nu_\alpha \to \nu_\alpha) = 4|U_{\alpha 3}|^2 (1 - |U_{\alpha 3}|^2) \sin^2 \left( \frac{\Delta m^2_{13} L}{4E} \right),$$

where the CP violating terms have been dropped in the approximation with the mass hierarchy, i.e., they are obtained under the so called one mass scale dominance approximation.

We use the standard form of Cabibbo-Kobayashi-Maskawa quark mixing matrix

$$U = \begin{pmatrix}
    c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
    -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\
    s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}
\end{pmatrix},$$

3
for the neutrino mixing matrix. From (1) and (2) one can see that the accelerator and the reactor experiments probe the mixing angles $\theta_{13}$ and $\theta_{23}$.

We first summarize the constraints from the accelerator and the reactor experiments. Unlike the case of dark-matter-mass neutrinos with hierarchy the constraints form these terrestrial experiments are very mild. With $\Delta m^2 \lesssim 0.01\text{eV}^2$ only the relevant channel is $\nu_e$ disappearance experiments whose most extensive runs were done at Bugey [17] and at Krasnoyarsk [18]. We note that there is no constraint on $s_{213}$ for $\Delta m^2_{13} < 7 \times 10^{-3}\text{eV}^2$, where we have made the substitutions of the variables in [17] and [18] $\theta \rightarrow \theta_{13}$, $\Delta m^2 \rightarrow \Delta m^2_{13}$, which follow from the present approximation with the mass hierarchy. Notice that there is an allowed region at large $s_{213}^2$, whose dominant part will be excluded by the solar neutrino constraint as we will see below.

Let us now address the constraint from the solar neutrino experiments. While extensive analyses have been done within two-flavor mixing scheme the full three-flavor analysis of the solar neutrino experiments is very rare. To our knowledge it has been carried out quite recently for the MSW mechanism [20] by Fogli, Lisi, and Montanino [21]. We do not know corresponding analyses for the vacuum oscillation solution, while there was an attempt [22]. For this reason let us focus on the MSW solution in this paper.

In their extensive analysis in the three-flavor framework with mass-squared hierarchy they observed amazingly that the MSW solution still exists at large $s_{13}^2$ [21], contrary to what was naively thought. The well-known small-$s_{12}$ and large-$s_{12}$ solutions fuse into a single one at around $s_{13}^2 = 0.33$ and this large-$s_{13}$ solution extends up to $s_{13}^2 \simeq 0.6$. The large-$s_{13}$ solution is interesting because the “two-flavor” parameters $\Delta m^2_{12}$ and $s_{12}^2$ differ from that obtained by the two-flavor analysis. At the largest value of $s_{13}$, $s_{13}^2 = 0.6$, $s_{12}^2 \simeq 2 \times 10^{-2}$ and $\Delta m^2_{12} \simeq 4 \times 10^{-6}\text{eV}^2$. In contrast, the best fit values of the two-flavor analysis are $s_{12}^2 \simeq 2 \times 10^{-3}$ and $\Delta m^2_{12} \simeq 5.2 \times 10^{-6}\text{eV}^2$. We will see below that the constraint from neutrinoless double $\beta$ decays does indeed prefer the large-$s_{13}$ solution.

We discuss the constraint from neutrinoless double $\beta$ decays, which applies only to the Majorana neutrinos. We will see that it gives rise to the strongest constraint. Observation
of no neutrinoless double $\beta$ decay implies the constraint on $< m_{ve} >$, which can be written in our notation of the mixing matrix as

$$< m_{ve} > = \left| c_{12}^2 c_{13}^2 m_{1} e^{-i(\beta+\gamma)} + s_{12}^2 c_{13}^2 m_{2} e^{i(\beta-\gamma)} + s_{13}^2 m_{3} e^{2i(\gamma-\delta)} \right|,$$

(4)

where $\beta$ and $\gamma$ are the extra CP-violating phases characteristic to Majorana neutrinos \[23, 24\].

Let us first discuss the CP-invariant case ($e^{2i\beta}, e^{i(\beta+3\gamma-2\delta)} = \pm 1$) because it is easier to understand. In this case the phase factors in (4) can be reduced to the CP parities $\eta_j$ of mass eigenstates $j$ with masses $m_j$ \[25\]. Under the circumstance of almost degeneracy with which we are working we can approximate the expressions of $< m_{ve} >$ by ignoring the mass differences. Then, it further simplifies depending upon the pattern of the CP parities of three neutrinos. Let us take the convention that $\eta_1 = +$ and denote them collectively as $(\eta_1, \eta_2, \eta_3) \equiv (1, e^{2i\beta}, e^{i(\beta+3\gamma-2\delta)}) = (+, -)$ etc. Then,

$$r \equiv \frac{< m_{ve} >}{m} = \begin{cases} 
1 & \text{for } (+ + +) \\
|1 - 2s_{13}^2| & \text{for } (+ + -) \\
|1 - 2s_{12}^2 c_{13}^2| & \text{for } (+ - +) \\
|1 - 2c_{12}^2 c_{13}^2| & \text{for } (+ - -) 
\end{cases}$$

(5)

Let us refer to the ratio $< m_{ve} > / m$ as $r$ hereafter.

We take the mixed dark matter model with $\Omega_{total} = 1$ to estimate the masses of neutrinos. The CHDM model with $\Omega_{total} = 1$ might have problems with age of the universe. The measurement by the Hubble Space Telescope \[24\] gave a value of $h = 0.8 \pm 0.17$, where the Hubble constant $H_0$ is given by $h$ as $H_0 = 100 \ h \ km/s \cdot Mpc$. The value of $H_0$ suggests that the total contribution $\Omega_{total}$ by matter to the density parameter should be smaller than 1 in order to have the age of the universe greater than 10 Gyr. Our attitude to this problem is that we must take at least 2 $\sigma$ uncertainty in the observed value of the Hubble constant seriously because the systematic errors in various methods of measuring the Hubble constant do not appear to be well understood. (For a recent status of measurement of the Hubble constant, see, e.g., \[27\]).
We assume that 20-30% of the universe is shared by the hot dark matter. We note that the neutrino contribution to the $\Omega$ parameter is $\Omega_\nu = \left(\sum m_i / 91.5\text{eV}\right)h^{-2}$ [28]. The CHDM model with three kinds of neutrinos has been analyzed by Pogosyan and Starobinsky [11] and they concluded that the allowed region is given by $0.55 \lesssim h \lesssim 0.7$, $0.25 \lesssim \Omega_\nu \lesssim 0.3$. If we take these values, we obtain $2.3\text{ eV} \lesssim m_j \simeq m \lesssim 4.5\text{eV}$ as masses of almost degenerate neutrinos. We will use $m=2.3\text{ eV}$, and $4.5\text{ eV}$ for neutrino mass as reference values in the following analysis.

We impose the experimental bound on $<m_{\nu e}>$ obtained by negative results of neutrinoless double $\beta$ decays. The most stringent one to date is $<m_{\nu e}> \lesssim 0.68\text{eV}$ derived in the $^{76}\text{Ge}$ experiment by Klapdor-Kleingrothaus et al. as quoted in [19]. It implies the bound on the $r$ parameter $r \leq 0.29$, and 0.15 for neutrino masses $m=2.3\text{ eV}$, and $4.5\text{ eV}$, respectively.

The immediate consequence of the constraint from neutrinoless double $\beta$ decays is that the first pattern of the CP parity, $(+++)$, is excluded. Other patterns are not immediately excluded but their parameters are subject to the constraint. In Figs. 1 and 2 we have plotted the allowed regions of the neutrinoless double $\beta$ decay constraint for each respective pattern of the CP parities for $r \leq 0.15$ and $r \leq 0.29$, respectively. The upper rectangular region is for the CP parity $(+ - -)$, the lower-left band for $(+ - -)$, and the lower-right band for $(+ - +)$. Also plotted in Figs. 1 and 2 as darker shaded parts are the allowed regions with 90% CL for the three-flavor MSW solution to the solar neutrino problem obtained by Fogli et al. [21]. It is actually the superposition of the allowed regions with the mass squared difference $\Delta m_{12}^2$ from $10^{-6}\text{eV}^2$ to $1.0 \times 10^{-4}\text{eV}^2$.

From this figure one can draw several conclusions: The CP parity pattern of $(+-+)$ is excluded since the two allowed regions do not overlap. The patterns $(+++)$ and $(+--)$ are allowed and they prefer the large-$s_{13}$ solution of solar neutrino problem. The “two-flavor” large angle solution is also marginally allowed. The small angle MSW solutions, which are drawn almost on the axis of $s_{12}^2 = 0$ in Figs. 1 and 2, are not compatible with the double $\beta$ decay constraint for $m=2.3\text{ eV}$ and $4.5\text{ eV}$. In closer detail, with neutrino mass of $4.5\text{eV}$...
(r ≤ 0.15) the solution exists for 3.2×10^{-6}eV^2 < \Delta m_{12}^2 < 6.8×10^{-5}eV^2 only if s_{13}^2 > 0.3. With m=2.3eV (r ≤ 0.29) it exists for 3.2×10^{-6}eV^2 < \Delta m_{12}^2 < 1.0 × 10^{-4}eV^2 only if s_{13}^2 > 0.02.

In general CP-noninvariant cases, we have to keep the two CP violating phases $\beta$ and $\gamma$ in (4). Namely, we have

$$< m_{\nu e} > = m \left| c_{13}^2 (e^{-i\beta} c_{12}^2 + e^{i\beta} s_{12}^2) + e^{i(3\gamma - 2\delta)} s_{13}^2 \right|$$

$$\geq m \left| c_{13}^2 (1 - \sin^2 \beta \sin^2 2\theta_{12})^{1/2} - s_{13}^2 \right|,$$

(6)

where we have ignored the mass differences and the equality in the second line holds when

$$\arg(e^{-i\beta} c_{12}^2 + e^{i\beta} s_{12}^2) = 3\gamma - 2\delta + (2n + 1)\pi,$$

(7)

where $n$ is an integer. Note that the constraint from neutrinoless double $\beta$ decays becomes even more stringent if the CP violating phases $\beta$, $\gamma$, $\delta$ do not satisfy the relation (4). Then, our task is to look for the region which satisfies $r \leq 0.29$, 0.15, respectively, with $\beta$, $\gamma$, and $\delta$ unconstrained. The resulting bounds coincide with those obtained in the CP-conserving cases and are presented in Figs. 3 and 4. For $r \leq 0.15$ which is obtained from $h=0.7$ and $\Omega_{\nu}=0.3$, the solution exists only if $s_{13}^2 > 0.3$, and for $r \leq 0.29$ only if $s_{13}^2 > 0.02$. It should be emphasized that, irrespective of whether CP is violated or not, the small-$s_{12}$ MSW solution, which is favored by theorists most, is disfavored in the CHDM model with almost degenerate neutrino masses.

On the other hand, we can get a condition for the CP violating phases $\beta$, $\gamma$, and $\delta$ by imposing both constraints from neutrinoless double $\beta$ decays and from the solar neutrino deficits with $s_{12}^2$ and $s_{13}^2$ constrained by the three-flavor analysis of [21]. The results are shown in Figs. 5 and 6, where the allowed regions are located in the neighborhood of the line $\beta + 3\gamma - 2\delta = \pm\pi$. In these plots the CP-conserving cases with the patterns $(+++)$, $(+++)$, $(+-*)$ and $(--*)$ correspond to the points $(\beta, 3\gamma - 2\delta) = (0,0)$, $\pm(\pi, \pi)$, $\pm(\pi, -\pi)$; $(0,\pi), (\pi,0)$; $\pm(\pi/2, -\pi/2)$; $\pm(\pi/2, \pi/2)$, respectively. From this we can verify in Figs. 5 and 6 that the CP-conserving cases with the CP parities $(+++)$ and $(+-*)$ are indeed excluded both for $r \leq 0.15$ and $r \leq 0.29$. 

7
Finally, let us briefly discuss the constraints from the atmospheric neutrino anomaly. There have been several three-flavor analyses of the atmospheric neutrino anomaly [29–31]. Among them, the most recent and the most detailed are the ones done by one of the authors [30], and by Fogli, Lisi, Montanino, and Scioscia [31]. The analyses by these two groups give rise to slightly different $2\sigma$ allowed regions on the $s_{13}^2 - s_{23}^2$ parameter plane. The difference stems from their treatments of the data of the NUSEX [7] and Frejus [8] experiments which are included in [31] and are not in [30]. As far as the constraint for $\theta_{23}$ is concerned, it is concluded in either analysis [30,31] that the allowed region with 90% CL for $\Delta m_{13}^2 \sim 5 \times 10^{-3}$ eV$^2$ has to satisfy $s_{23}^2 \gtrsim 1/4$. However, there is a difference between the two analyses on the allowed region for $\theta_{13}$. If one includes the data of all the experiments of atmospheric neutrinos [31], then the solution with small-$s_{13}$ is allowed. On the other hand, if one considers only the multi-GeV Kamiokande data [30], the solution with $s_{13}^2 \lesssim 0.1$ is excluded at 90% confidence level. As we have seen above, the allowed region for $r \leq 0.15$ exists for rather large values of $s_{13}$, so the difference of the two analyses [30,31] turns out to be irrelevant for $m \gtrsim 3$eV.

To summarize, we have discussed the almost degenerate three-flavor neutrino scenario as a simultaneous solution to the solar, the atmospheric and the dark matter problems. We have shown, using the constraints from neutrinoless double beta decays as well as these observational data of the solar and the atmospheric neutrinos, that large value of $s_{13}^2$ is favored, leaving a little room for solutions with small $s_{13}^2$ and large $s_{12}^2$. The neutrinoless double $\beta$ decay constraint imposed in ADN makes the small angle MSW solution untenable in this scenario. If three neutrinos turn out to be degenerate in masses and if precise values of $s_{12}^2$ and $s_{13}^2$ are both determined experimentally, then we get information on the relation among the CP violating phases $\beta$, $\gamma$ and $\delta$.

The authors would like to thank N. Okada and S. Sasaki for discussions on astrophysical aspects of dark matter scenarios. One of us (H.M.) is grateful to George Fuller for illuminating conversations on CHDM cosmology, and to Institute for Nuclear Theory, University of Washington for partial support during the completion of this work. We have been supported in part by Grant-in-Aid for Scientific Research of the Ministry of Education, Science and
Culture under #0560355, and H.M. is also supported by Grant-in-Aid for Scientific Research on Priority Areas under #08237214.
REFERENCES

[1] B. T. Cleveland et al., Nucl. Phys. B (Proc. Suppl.) 38 (1995) 47.

[2] K. S. Hirata et al., Phys. Rev. D44 (1991) 2241; Y. Suzuki, Nucl. Phys. B (Proc. Suppl.) 38 (1995) 54.

[3] P. Anselmann et al., Phys. Lett. B285 (1992) 376; ibid B314 (1993) 445; ibid 327, (1994) 377; ibid B342, (1995) 440.

[4] A. I. Abazov et al., Phys. Rev. Lett. 67 (1991) 3332; J. N. Abdurashitov et al., Nucl. Phys. B (Proc. Suppl.) 38 (1995) 60.

[5] K. S. Hirata et al., Phys. Lett. B205 (1988) 416; ibid B280 (1992) 146; Y. Fukuda et al., Phys. Lett. B335 (1994) 237;

[6] D. Casper et al., Phys. Rev. Lett. 66 (1991) 2561; R. Becker-Szendy et al., Phys. Rev. D46 (1992) 3720.

[7] M. Aglietta et al., Europhys. Lett. 8(7) (1989) 611;

[8] K. Daum et al., Z. Phys. C66 (1995) 417.

[9] M. C. Goodman, Nucl. Phys. B (Proc. Suppl.) 38 (1995) 337.

[10] J. A. Holtzman, Astrophys. J. Suppl. 71 (1989) 1; J. A. Holtzman and J. R. Primack, Astrophys. J. 405 (1993) 428; J. R. Primack, J. Holtzman, A. Klypin, and D. O. Caldwell, Phys. Rev. Lett. 74 (1995) 2160.

[11] D. Pogosyan and A. Starobinsky, astro-ph/9502019.

[12] K. S. Babu, R. K. Schaefer, and Q. Shafi, Phys. Rev. D53 (1996) 606.

[13] H. Minakata, Phys. Rev. D52 (1995) 6630; Phys. Lett. B356 (1995) 61.

[14] S. M. Bilenky, A. Bottino, C. Giunti, and C. W. Kim, Phys. Lett. B356 (1995) 273.

[15] G. L. Fogli, E. Lisi, and G. Scioscia, Phys. Rev. D52 (1995) 5334.
[16] M. K. Moe, Nucl. Phys. B (Proc. Suppl.) 38 (1995) 36.

[17] B. Achkar et al., Nucl. Phys. B434 (1995) 503.

[18] G. S. Vidyakin et al., JETP Lett. 59 (1994) 390.

[19] D. Caldwell and R. N. Mohapatra, Phys. Rev. D48 (1993) 3259; D50 (1994) 3477; S. T. Petcov and A. Yu. Smirnov, Phys. Lett. B322 (1994) 109; A. S. Joshipura, Z. Phys. C64 (1994) 31; A. Ioannissyan and J. W. F. Valle, Phys. Lett. B332 (1994) 93; B. Kayser, Talk at IIInd Rencontres du Vietnam, “Physics at the Frontiers of the Standard Model”, Ho Chi Minh, October 21-28, 1995.

[20] S. P. Mikheyev and A. Smirnov, Nuovo Cim. 9C (1986) 17; L. Wolfenstein, Phys. Rev. D17 (1978) 2369.

[21] G. L. Fogli, E. Lisi, and D. Montanino, INSSNS-AST 96/21, hep-ph/9605273.

[22] Z. G. Berezhiani and A. Rossi, Phys. Lett. B367 (1996) 219.

[23] J. Schechter and J. W. F. Valle, Phys. Rev. D22 (1980) 2227; S. M. Bilenky, J. Hosek, and S. T. Petcov, Phys. Lett. B94 (1980) 495; M. Doi et al., Phys. Lett. B102 (1981) 323.

[24] M. Fukugita and T. Yanagida, in Physics and Astrophysics of Neutrinos (Springer-Verlag, Tokyo, 1994)

[25] L. Wolfenstein, Phys. Lett. B107 (1981) 77.

[26] W.L. Freedman et al., Nature 371 (1994) 385.

[27] D. N. Schramm, Talk presented at Inauguration Conference of Asia Pacific Center for Theoretical Physics, June 4-10, 1996, Seoul, Korea.

[28] E. W. Kolb and M. S. Turner, The Early Universe (Addison-Wesley Publishing Co., California, 1990).
[29] J. G. Learned, S. Pakvasa, and T. J. Weiler, Phys. Lett. B 207, 79 (1988); V. Barger and K. Whisnant, Phys. Lett. B 209, 365 (1988); K. Hidaka, M. Honda, and S. Midorikawa, Phys. Rev. Lett. 61, 1537 (1988); S. Midorikawa, M. Honda, and K. Kasahara, Phys. Rev. D 44, R3379 (1991); J. G. Learned, S. Pakvasa, and T. J. Weiler, Phys. Lett. B 298, 149 (1993); A. Acker, A. B. Balantekin, and F. Loreti, Phys. Rev. D 49, 328 (1994); J. Pantaleone, Phys. Rev. D 49, R2152 (1994); G. L. Fogli, E. Lisi, and G. Scioscia, Phys. Rev. D 52, 5334 (1995); S. M. Bilenky, C. Giunti, and C. W. Kim, Astropart. Phys. 4, 241 (1996); J. J. Gomez-Cadenas and M. C. Gonzalez-Garcia, CERN-TH-95-80, hep-ph/9504246; M. Narayan, M. V. N. Murthy, G. Rajasekaran, and S. Uma Sankar, Phys. Rev. D 53, 2809 (1996); S. Goswami, K. Kar and A. Raychaudhuri, CUPP-95-3, hep-ph/9505395.

[30] O. Yasuda, TMUP-HEL-9603, hep-ph/9602342.

[31] G. L. Fogli, E. Lisi, D. Montanino, and G. Scioscia, INSSNS-AST 96/41, hep-ph/9607251.
FIGURE CAPTIONS

Fig.1,2. The Lighter shaded areas bounded by the thicker lines are the allowed regions of the neutrinoless double $\beta$ decay constraints for each respective pattern of the CP parities in the CP-conserving case with $r < 0.15$ ($m=4.5\text{eV}$) and $r < 0.29$ ($m=2.3\text{eV}$), respectively. The darker shaded areas bounded by the thinner lines are the allowed regions with 90% CL for the three-flavor MSW solution of the solar neutrino problem obtained by Fogli et al. \cite{21}.

Fig.3,4. The Lighter shaded areas are the allowed regions of the neutrinoless double $\beta$ decay constraints for the general CP violating case with $r < 0.15$ ($m=4.5\text{eV}$) and $r < 0.29$ ($m=2.3\text{eV}$), respectively, where the CP violating phases $\beta$, $\gamma$ and $\delta$ are unconstrained. The darker shaded areas are the same as in Figs. 1 and 2.

Fig.5,6. The shaded areas are the allowed regions obtained from the neutrinoless double $\beta$ decay experiments and the solar neutrino analysis with 90% CL by Fogli et al. \cite{21} for $r < 0.15$ ($m=4.5\text{eV}$) and $r < 0.29$ ($m=2.3\text{eV}$), respectively, where $s_{12}^2$ and $s_{13}^2$ take all possible values within the constraint of $|\Delta m^2_{21}|<4\times10^{-5}\text{eV}^2$. 

Fig. 1

$r < 0.15$

(+ + -)

(+ - -)

(+ - +)

$S_{13}^2$

$s_{12}^2$
Fig. 2
Fig. 3

$r < 0.15$
Fig. 4

$r < 0.29$
$3\gamma - 2\delta$

$r < 0.15$

Fig. 5
Fig. 6

$3\gamma - 2\delta$

$r < 0.29$