Optimal dispatch scheme for DSO and prosumers by implementing three-phase distribution locational marginal prices

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Abstract: Since the distribution system operator (DSO) cannot directly control prosumers with controllable resources, this study proposes an optimal dispatch method of using three-phase distribution locational marginal prices (DLMPs) as effective economic signals to incentivise prosumers’ behaviours. The three-phase DLMP model, DLMPs for active power demand, active power output and reactive power output are calculated. To alleviate the imbalance, congestions and voltage violations in active distribution networks (ADNs), the DSO and prosumers should be coordinated. The authors develop such a coordinated control scheme for the DSO and prosumers, in which the DSO generates and broadcasts three-phase DLMPs as price signals to induce prosumers’ behaviours. They prove that given the DLMPs as settlement prices, the optimal dispatch of the ADN will also maximise the surplus of prosumers. Therefore, the power output of rational prosumers will match the optimal dispatch, resulting in better operational conditions of ADNs. Then the three-phase imbalance, congestions and voltage violations will be well reduced. Numerical tests demonstrate the effectiveness of the proposed approach.

1 Introduction

1.1 Background

With the increasing amount of distributed generations (DGs), power end-users are transforming from traditional passive consumers to prosumers that also actively provide energy services to the grid [1]. The operation of active distribution networks (ADNs) has encountered significant challenges because of the large penetration of DGs and flexible loads (FLs) in prosumers. These active resources may make the state of the distribution network more volatile, and may even lead to power overflows on distribution lines or transformer windings and voltage over-limits. Moreover, ADNs are typically three-phase systems. If its degree of three-phase imbalance is too high, the ADN may suffer from additional power loss, congestions on distribution lines and transformers and voltage deviations of neutral points [2]. To handle these operational issues adequately, the distribution system operator (DSO) should coordinate with proactive prosumers in its territory to make use of their controllable resources. To better integrate the DGs and FLs, a new energy dispatch framework for ADNs should be established, with a proper market design that can maximise the social welfare and ensure the legal interests of market participants [3].

1.2 Literature review

There are many direct control methods such as shedding loads directly by system operators to address the operational issues [4, 5], but these methods neglect the impact of proactive customers, which is inadequate for the operation of ADNs. The distribution locational marginal price (DLMP) is developed in papers [6–8], reflecting the marginal cost of supplying the next incremental loads at different locations. The DLMP can be used as economic signals to incentivise prosumers to optimally adjust their power consumption and generations according to its physical significance. Papers [9–11] use DLMPs in day-ahead markets to alleviate congestions in future distribution networks with high penetrations of EVs. To lift congestions with demand response, a distribution congestion price based on the DLMP is proposed in [12]. In [13], DLMPs are calculated for both active power and reactive power to motivate distributed energy resources to contribute to congestion management and voltage support, based on a mixed-integer second-order-cone programming (SOCP) model for DSO optimisation considering network losses. The above applications of DLMPs are proven to be effective in congestion and voltage management. However, they are still based on single-phase models. Since the single-phase model is built assuming the three phases are balanced and does not consider the mutual inductance and interphase capacitance among different phases, it cannot reflect the three-phase imbalance and will involve errors. To take the imbalance characteristic in ADNs into account, a three-phase DLMP based on the AC optimal power flow (ACOPF) model is presented in [14]. Therein, the ACOPF problem is modelled as the marginal cost to serve an incremental unit of demand at a specific phase at a certain node and is calculated using the Lagrangian multipliers in the three-phase ACOPF model. The optimal Lagrangian multipliers can accurately reflect the sensitivity of the optimal cost with respect to changes in bounds of the constraints on demands at the global optimal solution. However, the ACOPF problem is non-convex and it is hard to solve for the global optimal solution. To transform the ACOPF problem into a convex problem, SOCP relaxation is presented in [15–18] and is used to make the single-phase ACOPF problem convex to calculate DLMP [13]. The proposed SOCP relaxation is exact for radial networks but may not be exact for mesh networks [15, 16]. Due to the mutual inductance and interphase capacitance among different phases, the three-phase ADNs model is essentially a mesh network. So, the SOCP relaxation cannot be applied in the three-phase ACOPF model and thus the three-phase DLMP cannot be achieved by solving the three-phase ACOPF model using SOCP relaxation. To solve this problem, paper [19] applies a sparse moment relaxation approach to solve the three-phase ACOPF problem and to calculate the three-phase DLMP. However, the optimal solution to the moment relaxation problem may not be feasible to the original ACOPF, especially for low-order moment relaxations. In addition, this method may suffer from poor computation efficiencies so it may not satisfy real-time DLMP calculation in large-scale ADNs. Therefore, it is necessary to find a method to calculate three-phase DLMP efficiently and accurately. Moreover, since the real and reactive powers are closely correlated in ADNs, it will involve considerable error if the impact of reactive power is not considered in the dispatch problem. To reflect the impact of the reactive power
on the price as well as induce the reactive output of the DGs, the DLMP for reactive power should be considered. In addition, the above papers [6–13, 19] derive the DLMP with respect to the load demand. The difference between load demand and the generator's output is not considered. The incremental costs for the power demand and generator's output are different because of the different characteristics of their active and reactive power. To derive the precise price to induce the prosumer's behaviour, the DLMP for power demand and power output should be determined, respectively.

1.3 Contributions

In order to reduce three-phase imbalance, line congestions and nodal voltage violations in ADNs, a market based optimal dispatch method by implementing three-phase DLMP is proposed in this paper. Comparing to existing literature, the optimal dispatch model of ADNs adopted in this paper has the following characteristics: (i) a new constraint on the level of three-phase imbalance has been added, which is measured by the NEMA (National Electric Manufacturers Associations of the USA) Std; (ii) the constraints of the three-phase ACOPF model are linearised by employing linearisation approximations and a linearised power flow model method in [20] to calculate three-phase DLMPs efficiently; (iii) DLMPs for active power demand (PD-DLMP), DG's active power output (PG-DLMP) and reactive power output (QG-DLMP) have been derived, respectively. By solving such an optimal dispatch problem, the DSO obtains the optimal quantities and prices.

However, the DSO cannot really mandate the power output of DGs owned by prosumers. Instead, the DSO broadcasts three-phase prices to prosumers and expects them to respond accordingly. Thus, the crux of our problem is whether prosumers will respond 'reasonably' or not. This paper partly answers this question by proving that if the DSO broadcasts three-phase DLMPs to prosumers, then the prosumers' optimal responses that maximise their surpluses are generating/consuming as per the optimal quantities solved from the optimal dispatch problem. Therefore, if all prosumers are rational and follow the optimal dispatch results, the security and efficiency of ADNs will be guaranteed.

The remainder of this paper is organised as follows. Section 2 describes the optimisation models for DSO and prosumers and determines the three-phase PD-DLMP, PG-DLMP and QG-DLMP. Section 3 introduces the method that uses DLMPs to alleviate three-phase imbalance, congestions and voltage violations in ADNs, and gives the proof process. Section 4 uses numerical simulations to analyse the effectiveness of the proposed method, and conclusions are drawn in Section 5.

2 Problem modelling

2.1 Assumptions

Before proceeding to the mathematical formulations, prerequisite assumptions and the reasons for the assumptions are first summarised as follows.

For the OPF model of DSO, we assume:

(1) The root node is the power supply point (PSP), which can be regarded as a generator with infinite capacity to balance the demand and supply. Since in ADNs, the root node is connected to the transmission network and obtains power from the main grid according to the power demand of the ADNs.

(2) To the transmission grid, the ADN can be regarded as the price taker [21]. The price to buy electricity from the main grid is the locational marginal price (LMP) $\pi_{LMP}$ at the root node, which is constant. So, the corresponding cost function $C(p_{LMP}^\varphi)$ for root node at phase $\varphi$ can be expressed as

$$C(p_{LMP}^\varphi) = \pi_{LMP} \cdot p_{LMP}^\varphi,$$

where $p_{LMP}^\varphi$ is the active power from the main grid.

(3) For each prosumer and generator $i$ in phase $\varphi$, its utility function $U_i(p_{\varphi}^i)$ and cost function $C_i(p_{\varphi}^i)$ are quadratic [22–24], which are written as

$$U_i(p_{\varphi}^i) = c_i^u(p_{\varphi}^i)^2 + c_i^p p_{\varphi}^i + c_i^q,$$

$$C_i(p_{\varphi}^i) = c_i^c(p_{\varphi}^i)^2 + c_i^p p_{\varphi}^i + c_i^q,$$

where $p_{\varphi}^i$ is the prosumer's active power demand; $p_{\varphi}^g$ is the DG's active power output; parameters $c_i^u$ to $c_i^q$ are coefficients of the corresponding functions;

(4) The power factor of the prosumer's demand is a constant, which is a general assumption for modelling the load demand [25]. So, the corresponding cost function

$C(p_{\varphi}^i) = \pi \cdot p_{\varphi}^i - \eta \cdot q_{\varphi}^i,$

(5) where $\eta = q_{\varphi}^i / p_{\varphi}^i$.

\begin{align*}
\eta &\leq u \\
p_{\varphi}^i \leq &\bar{p}_{\varphi} \leq \bar{p}_{\varphi}^i \\
q_{\varphi}^i \leq &\bar{q}_{\varphi} \leq \bar{q}_{\varphi}^i
\end{align*}
\[ P \leq P \leq \bar{P}_d \]  \hspace{1cm} (15) \\
\[ P \leq P \leq \bar{P}_d \]  \hspace{1cm} (16) \\
\[ q \leq q \leq \bar{q}_q \]  \hspace{1cm} (17) \\
\[ \delta \leq \delta, \forall i \in \mathcal{N} \]  \hspace{1cm} (18)

where the objective function (5) is to maximise the social welfare as the difference between the total utility of prosumers \( U(p_d) \) and the total cost of DGs and electricity purchased from the main grid \( C(p_d) \). Vectors \( p_i, p_i \) are the active power demand and generators' active power output at all nodes, respectively.

Constraints (6)–(10) represent branch power flow constraints, which are derived from the linearised distribution power flow model (LBPF-2) [20]. Constraints (9) and (10) are power balance constraints at the root node, where \( \mathcal{N}_{\text{root}} \) is the set of branches that connected to the root node. Since the power balance relationship at the other nodes is involved in the branch power constraints (6) and (7), there is no need for the power balance equation of the other nodes. Vectors \( p_i, q_i \) are three-phase branch active and reactive power flows, respectively, and vectors \( q_i, q_i \) are three-phase reactive power demand and generators' reactive power output, \( u_i \) is the square of three-phase voltage magnitude, coefficient matrices \( K_i \) to \( K_i \) are constant. More details about the three-phase linear power flow model can be found in paper [20]. A numerical test is provided in Appendix 1 to illustrate the linear power flow model (LBPF-2) is sufficiently accurate for engineering applications. The root node is connected to the transmission network with the voltage constraint (11), where \( u_{\text{out}} \) is squared voltage magnitude at the root node and \( u_{\text{ref}} \) is the square of reference voltage magnitude. Constraint (12) is the constant power factor constraint that the proportion of the reactive power demand to active power demand is a constant \( \mu \) [25]. The circular constraint (13) is the branch capacity constraint, where \( s_{\text{b}, \text{max}} \) is the maximum apparent power capacity of the branch. The circular constraint can be linearised using the quadratic constraint linearisation method in [27] by using several square constraints, shown as Fig. 1. In general, the use of more square constraints will increase the accuracy of the approximation. In this paper, two square constraints (19) are employed to substitute for (13), which is sufficiently accurate for engineering applications

\[
\begin{align*}
-\bar{s}_{\text{b}, \text{max}} & \leq p_b \leq \bar{s}_{\text{b}, \text{max}}, \\
-\bar{s}_{\text{b}, \text{max}} & \leq q_b \leq \bar{s}_{\text{b}, \text{max}}, \\
-\sqrt{3} \cdot s_{\text{b}, \text{max}} & \leq p_b + q_b \leq \sqrt{3} \cdot s_{\text{b}, \text{max}}, \\
-\sqrt{2} \cdot s_{\text{b}, \text{max}} & \leq p_b - q_b \leq \sqrt{2} \cdot s_{\text{b}, \text{max}}.
\end{align*}
\]  \hspace{1cm} (19)

Constraints (14)–(17) represent, respectively, lower and upper limits of voltage magnitude \( u \), \( u \), generators' active power output \( p_i, p_i \), active power demand \( p_i \), and generators' reactive power output \( q_i, q_i \). We also incorporate the limit on three-phase imbalance (18) into the optimal dispatch model. Constraint (18) represents the imbalance index \( \delta \) should not exceed its limit \( \delta \), and \( \mathcal{N} \) is the set of all nodes. To maintain the convexity of the problem, we approximate (18) using (20), where parameter \( u_{\text{ref}}^i \) is squared voltage magnitude at node \( i \) at phase \( \varphi \). The derivation of the approximate process is introduced in the Appendix. A numerical test is provided in Appendix 2 to illustrate the accuracy of the approximation

\[
(1 - \delta^2) \sum u_{\text{ref}}^i \leq 3u_{\text{ref}}^i \leq (\delta + 1) \sum u_{\text{ref}}^i, \quad \varphi = a, b, c. \]  \hspace{1cm} (20)

In this way, the optimal dispatch problem of DSO is a convex QP problem with linear constraints, which can be summarised into the following compact form:

\[
\begin{align*}
\text{max} & \quad U(p_d) - C(p_d), \\
\text{s.t.} & \quad M \cdot x + m \leq 0, (\mu q_d), \\
N \cdot x + n = 0, (\lambda s_d), \\
\bar{x} \leq x \leq \tilde{x}, (\mu^*_d, \mu^*_q),
\end{align*}
\]  \hspace{1cm} (21)

Inequality constraint (22) is derived from constraints (6)–(10), (12)–(14) and (18), equality constraint (23) is derived from constraints (6)–(12), inequality constraint (24) represents constraints (15)–(17). The vector \( x \) consists of three-phase power demand \( p_d \), real power generation \( p_d \) and reactive power generation \( q_d \), with the lower and upper limit \( \bar{x}, \tilde{x} \). Coefficient matrices \( M, N \) include submatrices \( M_{\text{pd}}, M_{\text{pg}}, M_{\text{ql}} \) and \( N_{\text{pd}}, N_{\text{pg}}, N_{\text{ql}} \) corresponding to parameters \( p_i, p_i, q_i \), respectively. We can derive that the value of parameter \( M_{\text{pd}}, N_{\text{pd}} \) is larger than the value of the parameter \( M_{\text{pg}}, N_{\text{pg}} \) when the \( \mu \) is larger than 0, which is satisfied when the demand is the pure load. Parameter vectors \( \mu, \mu \) are constant. Vectors \( \mu_d, \lambda_d, \mu_q, \mu_q \) are dual variables of corresponding constraints. The superscript T denotes transpose.

2.3 Derivation of three-phase DLMP

Different from the transmission system, reactive power flows may have significant impacts on the dispatch and operation of ADNs. To reflect this in DLMPs, three types of prices are proposed as follows:

(1) Prices for the active power demand \( p_d \) (PD-DLMP) \( p_d^{\text{PD}} \):

Under the constant power factor assumption adopted in this paper, the change of active power demand \( p_d \) will also affect its reactive power demand \( q_d \) proportionally, which is not priced separately in the proposed framework. Therefore, PD-DLMP actually represents the increase of total cost with one unit more active power demand \( p_d \) and \( \eta \) unit more reactive power demand \( q_d \) at a specific phase at a certain bus.

(2) Prices for the DG's active power output \( p_i \) (PG-DLMP) \( p_i^{\text{PG}} \):

Note that the DG's active power output \( p_i \) and reactive power output \( q_i \) are both decision variables in (P2), the change of DG's active power output \( p_i \) does not affect its reactive power output \( q_i \). Therefore, the PG-DLMP represents the increase of the total cost of other generators with one unit less active power output \( p_i \) of DG \( i \), which should be different from PD-DLMP.
(3) Prices for the DG's reactive power output $q_g$ (QG-DLMP) $\pi_{QG}$. Since the DG's reactive power output is not proportional to the active output, we propose the QG-DLMP for the DG's reactive power output, which represents the increase of total cost of other generators with one unit less reactive power output $q_g$ of DG $i$ at a specific phase at a certain bus.

Since the optimal dispatch model of DSO is convex with the zero duality gap, we can construct the three-phase DLMP via Lagrangian multipliers of corresponding constraints. The Lagrangian of the optimal dispatch problem P2 is expressed as follows:

$$\begin{align*}
L(x, \mu_s, \lambda_N, \mu_i, \mu_i^r) &= -U(p_d) + C(p_g) \\
&+ \mu_M^T \cdot (M_{pd} \cdot p_d + M_{pg} \cdot p_g + M_{qg} \cdot q_g + m) \\
&+ \lambda_N^T \cdot (N_{pd} \cdot p_d + N_{pg} \cdot p_g + N_{qg} \cdot q_g + n) \\
&+ \mu_i^T \cdot (x - \bar{x}) - \mu_i^* \cdot (x - \bar{x}).
\end{align*}$$

(26)

Specifically, to price demand at node $i$, we first change the active power demand $p_d$ in (P2) from a decision variable to a parameter, i.e. remove the utility function of $p_d$ from the objective function and its quantity limits (24) then fix $p_d$ at its optimal value. Second, we apply the envelop theorem to analyse the increase of total cost with one unit more active power demand $p_d$. Denote the Lagrangian with $p_d$ as a parameter as $L_{pd}$. Assume the considered OPF problem has an optimal solution $(x^*, \lambda_N^*, (\mu_s^*, \lambda_N^*, \mu_i^*, \mu_i^r))$ the corresponding Lagrangian multipliers, the PD-DLMP can be derived as

$$\begin{align*}
\pi_{PD}^{DLMP} &= \frac{\partial(-f_{pd})}{\partial p_d} \bigg|_{x^*} \\
&= \frac{\partial L_{pd}}{\partial p_d} \bigg|_{x^*, \mu_s^*, \lambda_N^*, \mu_i^*, \mu_i^r} \\
&= M_{pd}^T \cdot \mu_s^* + N_{pg}^T \cdot \lambda_N^* \\
\end{align*}$$

(27)

where $f_{pd}$ represents the objective function (21) of P2.

Similarly, the PG-DLMP can be derived with respect to the inelastic active power generation as follows:

$$\begin{align*}
\pi_{PG}^{DLMP} &= \frac{\partial(-f_{pg})}{\partial p_d} \bigg|_{x^*} \\
&= \frac{\partial L_{pg}}{\partial p_d} \bigg|_{x^*, \mu_s^*, \lambda_N^*, \mu_i^*, \mu_i^r} \\
&= -M_{pg}^T \cdot \mu_s^* - N_{pg}^T \cdot \lambda_N^*.
\end{align*}$$

(28)

Similarly, for the reactive power, the QG-DLMP can be derived with respect to the inelastic reactive power generation as follows:

$$\begin{align*}
\pi_{QG}^{DLMP} &= \frac{\partial(-f_{qg})}{\partial q_g} \bigg|_{x^*} \\
&= \frac{\partial L_{qg}}{\partial q_g} \bigg|_{x^*, \mu_s^*, \lambda_N^*, \mu_i^*, \mu_i^r} \\
&= -M_{qg}^T \cdot \mu_s^* - N_{qg}^T \cdot \lambda_N^*.
\end{align*}$$

(29)

Although we propose different prices for consumers and DGs, we have the following theorem.

Theorem 1: The PD-DLMP, PG-DLMP and QG-DLMP determined by (27)–(29) are non-discriminative.

The proof of Theorem 1 is as follows.

Proof: From the determination of PD-DLMP, PG-DLMP and QG-DLMP determined by (27)–(29), we can derive that

$$\begin{align*}
\pi_{PD}^{DLMP} &= \pi_{PD}^{NLMP} + \eta \pi_{QG}^{DLMP},
\end{align*}$$

(30)

If the power factor of DGs and demand are the same, i.e.

$$\begin{align*}
\frac{q_g}{p_d} &= \frac{q_g}{p_d} = \eta,
\end{align*}$$

(31)

we have

$$\begin{align*}
\frac{\pi_{PD}^{DLMP}p_d}{\pi_{QG}^{DLMP}q_g} &= \frac{p_d}{q_g},
\end{align*}$$

(32)

which indicates the total payment of the demand and DG are proportional to their active power injections. So, the proposed DLMPs are non-discriminative. □

In the optimal dispatch model for DSO, we consider the power loss, line flow constraints, voltage magnitude constraints and three-phase voltage imbalance constraints, so the value of the proposed three-phase PD-DLMP, PG-DLMP and QG-DLMP also consist of components reflecting the power loss, congestions, as well as voltage and imbalance level limits. In what follows, we demonstrate that taking the proposed three-phase DLMP as price signals, the branch power overflows, voltage violations and three-phase imbalance will be alleviated.

2.4 Response of prosumers

Since the prosumers are assumed to be economically rational, they will make the response to the price signal to maximise their own surplus. The optimisation model for prosumer $i$ at phase $\phi$ is as follows:

$$\begin{align*}
\text{P3: } &\max \ U_i(p_{d,i}^\phi, q_{g,i}^\phi, \mu_{pd,i}^\phi, \mu_{pg,i}^\phi, \mu_{qg,i}^\phi) \\
&\text{subject to: } \sum_\phi x_i^\phi \leq c_i^\phi, (\mu_{pd,i}^\phi, \mu_{pg,i}^\phi, \mu_{qg,i}^\phi),
\end{align*}$$

(33)

where

$$\begin{align*}
x_i^\phi &= [p_{d,i}^\phi, q_{g,i}^\phi, \mu_{pd,i}^\phi, \mu_{pg,i}^\phi, \mu_{qg,i}^\phi],
\end{align*}$$

(34)

The objective of prosumers is to maximise their individual surplus, which is expressed as the demand utility $U_i(p_{d,i}^\phi)$ plus the benefit to sell the DGs’ output $\mu_{pg,i}^\phi$, $\mu_{qg,i}^\phi$ minus DGs’ cost $C_i(p_{d,i}^\phi)$ and the payment to buy electricity from main grid $\mu_{pd,i}^\phi$. Parameter $\pi_{PD}^{\phi,i}$ is the price for the active power that the producer is charged; parameters $\pi_{PD}^{\phi,i}$, $\pi_{PG}^{\phi,i}$, $\pi_{QG}^{\phi,i}$ are prices for the prosumer $i$ at phase $\phi$ to sell DG's active and reactive power output; parameter $q_{g,i}^\phi$ is generators' reactive output at node $i$ at phase $\phi$; parameter $x_i^\phi$ represents the variables, include active power demand $p_{d,i}^\phi$, generators' active power output $p_{d,i}^\phi$ and generators' reactive power output $q_{g,i}^\phi$. Prosumers only knows the price and their own information, so the constraints are the lower and upper bounds $x_i^\phi$, $c_i^\phi$ of the demand and generators' output (34), without any network constraints and operational constraints. Parameters $\mu_{pd,i}^\phi$, $\mu_{pg,i}^\phi$, $\mu_{qg,i}^\phi$ are dual variables of corresponding constraints.

3 Three-phase DLMP based optimal dispatch method

In a real electricity market, DSO cannot really control prosumers and their DGs directly, so releasing price signals and anticipating prosumers to react accordingly is the best possible way to manage the system. In the current electricity market, different prosumers are charged a given flat tariff which is often set as the LMP at the root node. Charging different prosumers the same price is not fair, and the given flat tariff will not reflect the difference of three phases, power loss and congestions in ADNs, or control the operation of the system, so we use the proposed three-phase PD-DLMP...
DLMP, PG-DLMP and QG-DLMP to guide the prosumers’ behaviour relating to the active power demand, active and reactive power output, respectively, and then the branch power overflows, voltage violations and three-phase imbalance will be alleviated.

We have shown in the following theorem and corollary that taking the proposed three-phase DLMP as price signals, the active demand power as well as the generators’ active and reactive output power will obey the dispatch of DSO.

**Theorem 2:** Assume that the settlement prices $x_i^{PD}$, $x_i^{PG}$ and $x_i^{QG}$ are determined by (27)–(29) as PD-DLMP, PG-DLMP and QG-DLMP, then the optimal power consumption $p_{pd,i}$ and the output of DGs $q_{g,i}$ of an arbitrary prosumer $i$ in the optimal dispatch model P2 will also be optimal for the surplus maximisation model P3 for the same prosumer.

**Proof:** The Karush-Kuhn-Tucker (KKT) conditions of the DSO’s optimal dispatch problem P2 are

$$-U(p_d) + M_{pd}^T \mu_{pd} + N_{pd}^T \lambda_\phi - \mu_{pd} - M_{pd} = 0,$$

$$C_i(p_d) + M_{pg}^T \mu_{pg} + N_{pg}^T \lambda_\phi - \mu_{pg} - M_{pg} = 0,$$

$$M_{qg}^T \mu_{qg} + N_{qg}^T \lambda_\phi - \mu_{qg} - \mu_{qg} = 0,$$

$$N_i \cdot x + n_i = 0,$$

$$\mu_{pd} \cdot (-M_{pd} \cdot x - m) = 0,$$

$$\mu_{pg} \cdot (x - x_i) = 0,$$

$$\mu_{qg} \cdot (-x + \bar{x}) = 0,$$

$$\mu_{pd} \geq 0,$$

$$\mu_{pg} \geq 0,$$

$$\mu_{qg} \geq 0,$$

$$x - \bar{x} \geq 0,$$

$$x + \bar{x} \geq 0,$$

where parameter $U(p_d)$ is the derivative of utility function with respect to prosumers’ active demand $p_d$, parameter $C(p_d)$ is the derivative of cost function with respect to generators’ active output $p_g$.

The KKT conditions of the prosumers’ problem P3 are

$$-U_i(p_{d,i}) + x_i^{PD} + \mu_{pd,i}^+, -\mu_{pd,i}^-, = 0,$$

$$C_i(p_{d,i}) - x_i^{PD} + \mu_{pd,i}^+, -\mu_{pd,i}^-, = 0,$$

$$-x_i^{QG} + \mu_{qg,i}^+, -\mu_{qg,i}^-, = 0,$$

$$\mu_{pd,i}^+ \cdot (x_i^d - \bar{x}) = 0,$$

$$\mu_{qg,i}^+ \cdot (x_i^q - \bar{x}) = 0,$$

$$\mu_{pd,i}^+ \geq 0,$$

$$\mu_{qg,i}^+ \geq 0,$$

$$\mu_{pd,i}^- \geq 0,$$

$$\mu_{qg,i}^- \geq 0,$$

where parameters $U_i(p_{d,i})$, $C_i(p_{d,i})$ are the derivative of corresponding functions for prosumer $i$ at phase $\phi$.

In the proposed method, the prices $x_i^{PD}$, $x_i^{PG}$ and $x_i^{QG}$ sent to prosumers are PD-DLMP $x_{PD}^D$, PG-DLMP $x_{PG}^D$ and QG-DLMP $x_{QG}^D$, of which the values are determined in (27)–(29). Assuming that $(p_{d}^*, p_{g}^*, q_{g}^*, \mu_{pd}^*, \lambda_\phi, \mu_{qg}^*, \mu_{pg}^*)$ is a solution to the KKT conditions of the DSO’s problem P2, (37) will be transformed into

$$-U_i(p_{d,i}) + M_{pd}^T \mu_{pd,i}^+ + N_{pd}^T \lambda_\phi = \mu_{pd,i}^+, -\mu_{pd,i}^- - M_{pd} = 0,$$

$$C_i(p_{d,i}) + M_{pg}^T \mu_{pg,i}^+ + N_{pg}^T \lambda_\phi = \mu_{pg,i}^+, -\mu_{pg,i}^- - M_{pg} = 0,$$

$$M_{qg}^T \mu_{qg,i}^+ + N_{qg}^T \lambda_\phi = \mu_{qg,i}^+, -\mu_{qg,i}^- = 0,$$

where $(\cdot)^T$ represents the element at node $i$ at phase $\phi$.

By comparing the KKT conditions, it is evident that $(p_{d,i}^*, p_{g,i}^*, q_{g,i}^*, \mu_{pd,i}^+, \lambda_\phi, \mu_{qg,i}^+, \mu_{pg,i}^*)$, which is the solution to the KKT conditions of the DSO’s problem P2 at node $i$ of phase $\phi$, satisfies the KKT conditions (37) and (38) of the prosumers’ problem P3. This means $(p_{d,i}^*, p_{g,i}^*, q_{g,i}^*, \mu_{pd,i}^+, \lambda_\phi, \mu_{qg,i}^+, \mu_{pg,i}^*)$, which is the optimal solution to DSO problem P2 at node $i$ at phase $\phi$, is also an optimal solution to the prosumer’s problem P3.

**Corollary 1:** Under the same assumption as in Theorem 2 and when QG-DLMP is not equal to zero, the optimal solution to the prosumer in P3 is unique and it is equal to that solved from the optimal dispatch problem P2.

**Proof:** The objective function (33) of P3 has quadratic functions with respect to active power $p_{d,i}^*$, $p_{g,i}^*$ with positive definite Hessian matrix, and the constraints (34) are affine functions, so the prosumers’ optimisation problem is strictly convex QP problem with respect to active power. So, the active power optimal solution is unique. As the prosumers’ objective function (33) has an affine form with respect to reactive power, the reactive power optimal solution will reach the upper bound of reactive power limit if price $x_i^{QG}$ is positive, and will reach the lower bound if price $x_i^{QG}$ is negative. When we use PD-DLMP, PG-DLMP and QG-DLMP as price signals and QG-DLMP is not equal to zero, the solution to prosumers’ problem is unique. According to Theorem 2 that the optimal solution to DSO problem P2 at node $i$ at phase $\phi$ is also a solution to the prosumer’s problem P3, and because of the uniqueness of the optimal solution to the prosumers’ problem P3, any optimal solution to the prosumers’ problem must also be the optimal solution to the DSO problem at node $i$ at phase $\phi$. Based on the above conclusions, the DSO problem P2 and the prosumers’ problem P3 are equivalent.

Consequently, prosumers will make demand response to DLMPs to maximise their surplus, at the same time social welfare will be maximised. When the demand is the pure load, due to the value of parameter $M_{pd}^T, N_{pd}^T$ is always larger than the value of the parameter $-M_{pq}^T, -N_{pq}^T$ the PG-DLMP is always less than the PD-DLMP. Therefore, for a prosumer with DGs at one specific node, the active output of the DG will satisfy its own demand first and then supply the demand of the other prosumers if there is surplus output left. In addition, in practice, DGs’ reactive outputs will help to reduce active power loss in the system, because the prosumers’ DGs’ output range is small compared with whole reactive power demands, the reactive outputs of DGs will usually reach their bounds. In this case, the corresponding dual variables $\mu_{pq,i}^+$ or $\mu_{pq,i}^-$ are non-zero. According to (40), the QG-DLMPs will usually not equal to zero, so the proposed method can be applied in most cases. If the QG-DLMPs are equal to zero, the dispatch of prosumers will remain unchanged

$$M_{pd}^T \mu_{pd}^+ + N_{pd}^T \lambda_\phi = \mu_{pd}^+, -\mu_{pd}^- = 0.$$

In particular, this proved equivalence is not a trivial extension of the results of the LMP in wholesale market in that: (i) the prosumer’s optimal behaviour under non-zero QG-DLMP in our proposed method is unique; (ii) both reactive power and voltage constraints are taken into account in our method, but the wholesale market and LMP only focus on the active power; (iii) we price the active power of demand and DGs’ output separately while the wholesale market only considers the LMP for the active power demand.

In the proposed scheme, DSO solves the optimal dispatch problem P2 at the start of each price interval, then calculates the three-phase DLMPs and broadcasts the price. Prosumers will solve their own optimisations P3, and under given assumptions, they will voluntarily follow the instructions of the DSO. The framework of the proposed method is illustrated in Fig. 2.

As the two optimal problems P2, P3 are equivalent, no iteration is needed in the process. Note that in the optimal dispatch problem
demonstrated this in our simulations in the next section.

concentrations and voltage better. The more accurate the forecast and the root node, the proposed method will control imbalance, three-phase imbalance, branch power flow and voltage will be caused. The branch power overflow, voltage violations and three-phase imbalance although we use the proposed method. But still, the effectiveness of the proposed method with the current electricity market mechanism that charged different prosumers the same flat tariff. In response to flat tariff, each prosumer solved its own optimal problem and determined its demand and DGs' output. Then the power flows and nodal voltages in the system were obtained.

Scenario A1 was a base scenario that no branch power overflow, voltage violation and imbalance violation occur, with the voltage magnitude limit [0.95, 1.05] p.u., imbalance index limit 0.03 and branch apparent power limit [−3, 3] MVA. The PD-DLMP, PG-DLMP and flat tariff for Scenario A1 were shown in Fig. 4. It can be observed that the three-phase PD-DLMP and PG-DLMP was different at different nodes and different phases, due to the different characteristic of phases. When there were no violations, the differences of price were caused by the power loss. The PG-DLMP is lower than the PD-DLMP at the same node. In this case, the DG's output of the prosumer will satisfy its own demand first.

In Scenario B1, we tested the effectiveness of the proposed method in congestion management by setting the apparent power limit on branch 3 (from node 3 to node 4) as 1 MVA based on Scenario A1. Fig. 5 showed the three-phase PD-DLMP, PG-DLMP and flat tariff in Scenario B1. When the congestion on branch 3 at phase c occurred, the PD-DLMP of its downstream nodes increased drastically, which resulted in the decrease in the demand of corresponding nodes as shown in Fig. 6. Table 2 showed the active outputs of DG4 and DG6 in Scenario A1 and B1 under DLMP. It can be observed that the DGs' active outputs were different for the change of DLMPs. As a result, the branch power overflow was alleviated. Fig. 7 showed the branch active power at phase c under DLMP and flat tariff in Scenario B1. The dashed line represented the transmission capacity. It can be observed that, in Scenario B1, the branch apparent power of branch 3 at phase c was limited within its upper bound under DLMP, in this way we achieved the congestion management. We can also observe from Fig. 7 that in Scenario B1, if the prosumer made the response to the flat tariff, the branch power exceeded limit, but when we used our proposed method, the violation of branch power limit was alleviated.

### 4.2 Three-phase DLMP and its effect on congestions, voltage and imbalance management

Table 1 Coefficients of generation cost function

| DG    | c_{l,i} | c_{v,i} | c_{\phi,i} |
|-------|---------|---------|------------|
| DG1-DG3 | 0.04    | 0.2     | 0          |
| DG4-DG6 | 31      | 31      | 0          |

Fig. 2 Illustration of framework of the proposed optimal dispatch method

Fig. 3 Modified IEEE 33-bus distribution system

Fig. 4 Three-phase PD-DLMP, PG-DLMP and flat tariff in Scenario A1

P1, voltages and branch power flows are in acceptable ranges and the degree of three-phase imbalance is less than the limit δ. Therefore, if prosumers follow their dispatch instructions, the three-phase imbalance, branch power flow and voltage will be controlled within the expected limitations.

In practice, forecast error and uncertainty of prosumers may cause the branch power overflows, voltage violations and three-phase imbalance although we use the proposed method. But still, compared with the current electricity market mechanism that sends prosumers the flat tariff signals, which is often set as the LMP at the root node, the proposed method will control imbalance, congestions and voltage better. The more accurate the forecast and the more stable the prosumers, the more significant the three-phase DLMP will manage the operation of the system. We have also demonstrated this in our simulations in the next section.

### 4 Numerical tests

#### 4.1 Simulation setup

We tested the effectiveness of three-phase DLMP in alleviating branch power overflows, voltage violations and three-phase imbalance on a modified three-phase IEEE 33-bus distribution system shown as Fig. 3. More information about branch parameters and load profiles is available online [28]. Different prosumers at different phases and nodes had different utility functions, which made the system imbalanced. Node 1 was the PSP, which was regarded as a conventional generator with infinite capacity. Six different DGs (DG1, DG2, DG3, DG4, DG5, DG6) were connected to buses 3, 6, 12, 18, 22 and 33 separately, and the active output limit of DGs was [0, 0.1] MW. The power factor of the prosumer's demand is randomly selected from [0.8, 0.9]. The LMP at PSP was $30/MWh. The coefficients of DGs' cost functions were listed in Table 1. The output of DGs was assumed phase independent.

Fig. 5 showed the three-phase PD-DLMP and verified the effectiveness in congestions, voltage and imbalance management in ideal conditions without forecast error and uncertainty. Scenario A1 is a base scenario that no branch power overflow, voltage violation and imbalance violation occur. Scenarios B1, C1 and D1 are derived from Scenario A1 by changing the limits in the constraints to make the corresponding constraints active to verify the effectiveness in congestions, voltage and imbalance management. Second, we tested the effectiveness of reactive output of DGs using Scenario D2 by changing the reactive output limit of the DGs in Scenario A1. Third, we used the Monte Carlo method to simulate the practice with forecast error and the uncertainty of prosumers' behaviour using Scenarios B3, C3 and D3 by adding the forecast error and the uncertainty in Scenarios B1, C1 and D1. In each scenario, we used the LMP at the root node as flat tariff to compare the effectiveness of the proposed method with the current electricity market mechanism that charged different prosumers the same flat tariff. In response to flat tariff, each prosumer solved its own optimal problem and determined its demand and DGs' output. Then the power flows and nodal voltages in the system were obtained.

Scenario A1 was a base scenario that no branch power overflow, voltage violation and imbalance violation occur, with the voltage magnitude limit [0.95, 1.05] p.u., imbalance index limit 0.03 and branch apparent power limit [−3, 3] MVA. The PD-DLMP, PG-DLMP and flat tariff for Scenario A1 were shown in Fig. 4. It can be observed that the three-phase PD-DLMP and PG-DLMP was different at different nodes and different phases, due to the different characteristics of phases. When there were no violations, the differences of price were caused by the power loss. The PG-DLMP is lower than the PD-DLMP at the same node. In this case, the DG's output of the prosumer will satisfy its own demand first.

In Scenario B1, we tested the effectiveness of the proposed method in congestion management by setting the apparent power limit on branch 3 (from node 3 to node 4) as 1 MVA based on Scenario A1. Fig. 5 showed the three-phase PD-DLMP, PG-DLMP and flat tariff in Scenario B1. When the congestion on branch 3 at phase c occurred, the PD-DLMP of its downstream nodes increased drastically, which resulted in the decrease in the demand of corresponding nodes as shown in Fig. 6. Table 2 showed the active outputs of DG4 and DG6 in Scenario A1 and B1 under DLMP. It can be observed that the DGs' active outputs were different for the change of DLMPs. As a result, the branch power overflow was alleviated. Fig. 7 showed the branch active power at phase c under DLMP and flat tariff in Scenario B1. The dashed line represented the transmission capacity. It can be observed that, in Scenario B1, the branch apparent power of branch 3 at phase c was limited within its upper bound under DLMP, in this way we achieved the congestion management. We can also observe from Fig. 7 that in Scenario B1, if the prosumer made the response to the flat tariff, the branch power exceeded limit, but when we used our proposed method, the violation of branch power limit was alleviated.

Table 2 showed the active outputs of DG4 and DG6 in Scenario A1 and B1 under DLMP. It can be observed that the DGs' active outputs were different for the change of DLMPs. As a result, the branch power overflow was alleviated. Fig. 7 showed the branch active power at phase c under DLMP and flat tariff in Scenario B1. The dashed line represented the transmission capacity. It can be observed that, in Scenario B1, the branch apparent power of branch 3 at phase c was limited within its upper bound under DLMP, in this way we achieved the congestion management. We can also observe from Fig. 7 that in Scenario B1, if the prosumer made the response to the flat tariff, the branch power exceeded limit, but when we used our proposed method, the violation of branch power limit was alleviated.
In Scenario C1, we changed the voltage magnitude limit in Scenario A1 to [0.97, 1.03] p.u. to test the effectiveness of the proposed method in voltage management. Fig. 8 showed the three-phase PD-DLMP, PG-DLMP and flat tariff in Scenario C1. It can be observed that, after changing the voltage limit, to maintain the voltage at phase c within the new limitation, the PD-DLMP at phase c increased, inducing prosumers to decrease their demands, which was shown in Fig. 6. Fig. 9 illustrated the node voltage magnitude in Scenario C1. The dashed line represented the voltage magnitude limitation. It can be observed that, the voltage exceeded limit under flat tariff, but was well maintained within the voltage limits under DLMP, then the voltage was controlled.

In Scenario D1, the imbalance index limit $\delta$ in Scenario A1 was changed to 0.015 to validate the effectiveness of DLMP on the three-phase imbalance. Fig. 10 showed the three-phase PD-DLMP and flat tariff in Scenario D1. It can be observed that, to control imbalance violation, DLMPs on phases with more load increased, inducing the change of the distribution of prosumers' demands, which was shown in Fig. 6, and reduced the degree of imbalance. Fig. 11 illustrated the voltage imbalance index in Scenarios A1 and D1. The dashed line represented the imbalance limit. It can be observed that the imbalance index was kept within its limits under DLMP but exceeded its limits under flat tariff in Scenario D1.

### 4.3 Effectiveness of three-phase QG-DLMP and reactive outputs of DGs

In Scenario D2, we changed the reactive output limit of DG2, DG4, DG6 to [0, 0.1] MVar compared with Scenario D1 to test the effectiveness of three-phase QG-DLMP and reactive output of DGs. Table 3 illustrated the three-phase QG-DLMP and the reactive outputs for DG2, DG4, DG6 in Scenario D2. It can be observed that QG-DLMPs induced the reactive output of DGs. By comparing the PD-DLMPs in Scenarios D1 and D2 illustrated in Fig. 10, it can be observed that when DGs dispatched reactive power, the PD-DLMPs changed. Table 4 was the social welfare of Scenarios D1 and D2, and it showed that the social welfare of Scenarios D1 and D2 was higher than that of Scenarios A1 and D1.

---

**Table 2** Active outputs of DG4 and DG6 in Scenarios A1 and B1 under DLMP

| Scenario  | Output of DG under DLMP, MW |
|-----------|----------------------------|
| DG4 phase c | 0.0156 | 0.0104 |
| DG6 phase c | 0.0577 | 0.0558 |

---

**Fig. 5** Three-phase PD-DLMP, PG-DLMP and flat tariff in Scenario B1

**Fig. 8** Three-phase PD-DLMP and flat tariff in Scenario C1

**Fig. 9** Node voltage magnitude in Scenario C1 under DLMP and flat tariff

(a) Under DLMP in Scenario C1, (b) Under flat tariff in Scenario C1

**Fig. 10** Three-phase PD-DLMP and flat tariff in Scenarios D1 and D2

**Fig. 11** Imbalance index in Scenarios A1 and D1 under DLMP and flat tariff

(a) Scenario A1, (b) Scenario D1

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Table 3 Three-phase QG-DLMP and reactive outputs of DG2, DG4, DG6 in Scenario D2

|                    | Phase a     | Phase b     | Phase c     |
|--------------------|-------------|-------------|-------------|
| DLMP for DG2, $/MVarh | 4.3142      | 3.6388      | 7.6805      |
| reactive output of DG2, MVar | 0.1         | 0.1         | 0.1         |
| DLMP for DG4, $/MVarh | 8.1932      | 6.2816      | 16.7693     |
| reactive output of DG4, MVar | 0.1         | 0.1         | 0.1         |
| DLMP for DG6, $/MVarh | 7.0566      | 7.1289      | 9.5178      |
| reactive output of DG6, MVar | 0.1         | 0.1         | 0.1         |

Table 4 Social welfare of Scenarios D1 and D2

| Scenario     | Social welfare, $ |
|--------------|-------------------|
| D1           | 82.3326           |
| D2           | 83.1628           |

Table 5 Average over-limit branch apparent power, voltage magnitude, imbalance index in Scenarios B3, C3 and D3 under DLMP and flat tariff

| Standard deviation of the prosumers' utility function coefficients, % | Average over-limit branch apparent power in scenario B3, MVA | Average over-limit voltage magnitude in Scenario C3, p.u. | Average over-limit imbalance index in Scenario D3 |
|---------------------------------------------------------------|----------------------------------------------------------|----------------------------------------------------------|--------------------------------------------------|
|                  | DLMP | Flat tariff | DLMP | Flat tariff | DLMP | Flat tariff |
| 0                | 1.376 × 10⁻⁵ | 0.357      | 1.53 × 10⁻⁸ | 0.0940      | 2.65 × 10⁻⁸ | 0.0228      |
| 1                | 7.505 × 10⁻⁴ | 0.357      | 9.54 × 10⁻⁸ | 0.0940      | 8.36 × 10⁻⁸ | 0.0228      |
| 2                | 0.0021   | 0.357      | 6.32 × 10⁻⁷ | 0.0941      | 1.90 × 10⁻⁵ | 0.0230      |
| 3                | 0.0045   | 0.357      | 4.74 × 10⁻⁶ | 0.0942      | 9.31 × 10⁻⁵ | 0.0230      |
| 4                | 0.0087   | 0.362      | 4.82 × 10⁻⁵ | 0.0943      | 2.44 × 10⁻⁴ | 0.0232      |
| 5                | 0.0094   | 0.368      | 2.21 × 10⁻⁴ | 0.0943      | 3.57 × 10⁻⁴ | 0.0232      |

welfare increased in Scenario D2. It was because DGs' reactive outputs contributed to the voltage control process, then the active power reached a better optimal solution with less power loss, and a higher social welfare was achieved accordingly.

4.4 Analysis of the proposed method considering forecast error and uncertainty

In practice, the forecast error and the uncertainty of prosumers' behaviour are always involved. To simulate the practice, we formulated the Scenarios B3, C3 and D3 based on Scenarios B1, C1 and D1. We assumed that the forecast error was normally distributed. The mean value was 0, and the standard deviation was 1% of the predicted value. The uncertainty of prosumers’ utilities was simulated as the utility function coefficient obeyed normal distribution. The mean value was the original coefficient, and the standard deviation was 0, 1, 2, 3, 4 and 5% of the original coefficient, respectively. The Monte Carlo method was used with 500 tests to achieve the result.

Table 5 showed the average over-limit branch apparent power, average over-limit voltage magnitude and average over-limit imbalance index in Scenarios B3, C3 and D3 under DLMP and flat tariff, respectively. It can be observed that when there was only a forecast error in the system, which meant the utility uncertainty standard deviation was 0, the violation in branch power, voltage and imbalance under the proposed DLMP method was much lower than that under the flat tariff. With the increase of prosumers’ uncertainty in the utility function, the violation under DLMP increased, but the corresponding parameters were still much lower than those under the flat tariff, which indicated the proposed method achieved better management in congestions, voltage and imbalance, although there were forecast errors and uncertainty in the system. It can be concluded that by using the proposed method, serious violations of the operation limits will not occur.

4.5 Computational platform and efficiency

All simulations were implemented on a personal laptop with an Intel Core i7-7500M 2.70 GHz processor and 16 GB of RAM. The proposed models were programmed in Matlab 2016a and solved by an embedded IBM CPLEX 12.5 solver with the YALMIP interface. In our case studies, regarding the modified IEEE 33-bus system, the CPU time for solving the DSOs' dispatch problem is 0.19 s on average. This illustrates that the DSO can calculate the price signals in a short time. The computational efficiency of our proposed method is acceptable for practical application.

5 Conclusion

This paper proposes an optimal dispatch and pricing method to alleviate imbalance, congestions and voltage violations based on three-phase DLMP. A three-phase convex OPF model considering the limits on the degree of three-phase imbalances, voltage magnitudes and branch power flows is presented for the DSO. The three-phase PD-DLMP, PG-DLMP and QG-DLMP are derived with respect to the active power demand, active and reactive power output. The prosumers' response process is analysed and the equivalence of the DSO's and prosumer's model is proved. The proposed method was tested on a modified three-phase imbalanced IEEE 33-bus distribution system. Simulation results indicate the three-phase DLMPs are different at different locations and phases. Under the incentive of the different DLMP prices, prosumers will adjust their demands and DGs' outputs, and then the imbalance, congestions and voltage violations will be alleviated. In addition, DGs' reactive output will contribute to the management process to make social welfare increase. Meanwhile, the forecast errors and uncertainty are considered using Monte Carlo simulations. The test results demonstrate that the proposed method will achieve better management compared with the current mechanism using a flat tariff signal. Moreover, the computational time to calculate the DLMPs is short, so our proposed method is acceptable for practical application.

Since we regarded the prosumer's demand with a united form, the proposed model may have limitations when the demands have different characteristics. Future research may involve modelling the demand precisely and driving the corresponding DLMPs.
8 Appendix

8.1 Appendix 1: accuracy analysis of the linearised distribution power flow model (LBPF-2)

We test the accuracy of the linearised three-phase branch flow model (LBPF-2) [20] model using an IEEE 33-bus distribution system. We applied the LBPF-2 to calculate power flow for the test system, and analysed their errors under different loading conditions. The percentage of basic loads is ranged from 20 to 180%. The backward-forward sweep method was used to provide the accurate power flow for comparison. The results of the maximum and mean absolute errors of the voltage magnitude are shown as Fig. 12. It can be observed that the maximum and mean absolute errors of the voltage magnitude of LBPF-2 are all less than $1 \times 10^{-9}$, which is very small. So, the linearised three-phase branch flow (LBPF-2) is sufficiently accurate for engineering applications.

8.2 Appendix 2: derivation of (20) and accuracy analysis of corresponding approximation

We elaborate on the derivation processes of (20) and provide a numerical test to illustrate the accuracy of the corresponding approximation.

According to (42) and (43), we have

$$
(V_i^a)^2 + (V_i^b)^2 + (V_i^c)^2 = \frac{(V_i^a)^2}{V_i^a + V_i^b + V_i^c} 
$$

From (4) and (18), we have

$$
(V_i^a)^2 + (V_i^b)^2 + (V_i^c)^2 = \frac{(V_i^a)^2}{V_i^a + V_i^b + V_i^c} 
$$

According to (1) and (4), we have

$$
1 - \delta \leq \frac{V_i^p}{\sum V_i^p} \leq \frac{1 + \delta}{3}, \quad \delta = a, b, c. 
$$

According to (42) and (43), we get (20).

Next, we provide a numerical test to illustrate the accuracy of the approximation in (42).

Since the voltage magnitudes are around 1 in ADNs, we changed the voltage magnitudes $V_i^a, V_i^b, V_i^c$ from 0.9 to 1.1, respectively. The mean and maximum absolute percentage errors with the unit of 100% of the approximation in (42) are illustrated in Table 6. It can be observed that the errors are small, so the approximation in (42) is sufficiently accurate for engineering applications.
**Fig. 12** Maximum and mean absolute errors of the voltage magnitude

**Table 6** Mean and maximum absolute percentage error of the approximation in (42)

| error (100%) | Mean   | Max   |
|--------------|--------|-------|
|              | 0.27   | 0.94  |