On three-dimensional flows of pore pressure activated Bingham fluids

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To Professor Vsevolod Alekseevich Solonnikov on the occasion of his 85th birthday

Abstract

We are concerned with a system of partial differential equations describing internal flows of homogeneous incompressible fluids of Bingham type in which the value of activation (the so-called yield) stress depends on the internal pore pressure governed by an advection-diffusion equation. This model seems to be suitable for the description of important complex processes, such as liquefaction, occurring in granular water-saturated materials. After providing the physical background of the considered model, paying attention to the assumptions involved in its derivation, we focus on the PDE analysis of the initial and boundary value problems that are interesting from geophysical point of view. We give several equivalent descriptions for the considered class of fluids of Bingham type. In particular, we exploit the possibility to write such a response as an implicit tensorial constitutive equation, involving the pore pressure, the deviatoric part of the Cauchy stress and the velocity gradient. Interestingly, this tensorial response can be characterized by two scalar constraints. We employ a similar approach to treat stick-slip boundary conditions that include no-slip, Navier’s slip and slip as special cases. Within such a setting we prove long time and large data existence of weak solutions to the evolutionary problem in three dimensions.

1 Introduction

The mechanical behavior of water saturated geological materials such as soils or sands is known to involve the notion of the so-called effective stress (or effective pressure), introduced in 1920 by Terzaghi [37]. The effective pressure is defined as the difference between the total pressure in the medium and the pressure of the interstitial fluid - pore pressure. Water saturated geological materials are mixtures composed of a solid unconsolidated matrix and an interstitial pore space occupied by a fluid, see Fig. 1. With this picture in mind the total stress exerted on any control surface in such a medium comprises two contributions, namely the stress transmitted by the fluid and the stress transmitted by the solid matrix. Mechanical loading or unloading of such a saturated material due to external forcing leads to redistribution of the stresses between the two phases, which in general can be a rather complex process. Despite its complexity,
several general observations can be made. First, if during the process the stress in the matrix increases, for example by reducing the pore pressure while keeping the total loading constant, the solid structure compactifies and becomes more rigid. A textbook example of this process is the beach sandcastle stabilization, when the fluid flowing out of the wet sand stabilizes the sand by “sticking” the sand grains closer together (here also capillary phenomena play a significant role). Second, as an opposite extreme, it may happen that during some processes the pressurized interstitial fluid bears almost the whole mechanical load exerted on the system, which leads to effective mechanical decoupling of the matrix grains and the so called “liquefaction” can occur. Liquefaction is an almost complete loss of mechanical strength of the granular material followed by its flow. This can happen if, for example, the mechanical loading of the granular material takes place on a sufficiently short time scale (and often periodically) as it may happen during an excitation of the material by seismic waves propagating through the granular structure during an earthquake. This may lead to a gradual build-up of the pore pressure which cannot relax rapidly enough and consequently, to the liquefaction. Results of such process can be rather dramatic, in particular if the discussed material is for example a saturated sediment on which civil structures are built, as was the case during the 1964 Niigata earthquake or 1964 Alaska earthquake.

In this article, our first aim is to derive a model for a material that behaves as rigid until the “liquefaction” criterion is reached and the whole material starts to flow. Such a material response is articulated in terms of the so called incompressible Bingham fluid. The activation (liquefaction) criterion is controlled by an evolving scalar quantity, the effective pressure, for which an advection-diffusion type of evolution equation is traditionally used in porous medium hydrology [14]. In Section 2, we develop the model from the principles within the framework of mixture theory (see [38, 3, 33, 31, 15, 26]) through a number of physically reasonable approximations. In particular, we compare our final system with the one studied recently by Chupin and Mathé [11]; they differ by the structure of the right-hand side in the equation for the (fluid/effective) pressure. In Section 3, we reformulate the response as an implicit equation involving the constitutively determined part of the stress, the symmetric part of the velocity gradient and the pore pressure. We also give an alternative characterization of such implicit constitutive equation in terms of two scalar constraints (extending here an interesting observation from [11]). Next, we focus on the mathematical analysis of the initial and boundary value problem in three-dimensional domains. Note that Chupin and Mathé analyzed only two-dimensional flows in [11], which is easier as the energy equality holds for a weak solution. We consider internal flows when the whole boundary is impermeable and we study the problem with stick-slip boundary conditions (that can be also equivalently written as an implicit constitutive equation on the boundary and characterized by two inequalities). Stick-slip (or threshold slip) states that the velocity does not slip until the amplitude of the tangent part of the normal traction on the boundary exceeds a certain critical value. This boundary condition, which is physically relevant to the pore pressure activated fluids considered in the bulk, includes Navier’s slip and (perfect) slip boundary conditions as special cases. We establish the long-time and large-data existence of the corresponding weak solutions; see Section 4 for the formulation of the main result and Section 6 for its proof. We exploit the characterization of the implicit constitutive equations by two scalar constraints, both in the bulk and on the boundary, as a tool to show that the limit object of suitable approximative sequences fulfills these constitutive
equations as well, see Proposition 5.3 proved in Section 5, where we also introduce the approximations and study their properties. Finally, we comment on possible results for no-slip boundary conditions and further extensions in the concluding section.

2 Derivation of the model in the framework of multiphase theory

We consider a medium that is composed, at sufficiently small meso-scale, of two types of materials: a solid unconsolidated matrix and the interstitial pore space occupied by a fluid, see in Fig. 1. These two materials are called phases and various quantities related to them will be distinguished by subscripts "s" and "f", respectively. Based on the theory of multi-component materials (see e.g. [15] if one starts with the multiphase media description or [26] if one uses the theory of mixtures based on the assumption of co-occupancy), we will formulate the individual mass and momentum balances for both components. Restraining ourselves to a purely mechanical setting, for simplicity, we do not need to formulate the balance equations for energy and entropy.

The balance equations for mass read as follows:

\[
\frac{\partial (\phi \rho_f^m v_f)}{\partial t} + \text{div}(\phi \rho_f^m v_f) = 0 ,
\]

\[
\frac{\partial ((1 - \phi) \rho_s^m v_s)}{\partial t} + \text{div}((1 - \phi) \rho_s^m v_s) = 0 .
\]

Here \(\phi \rho_f^m\) and \(\phi \rho_s^m\) denote the material (true) densities of the fluid and the solid, \(\phi\) denotes the volume fraction of the liquid (equal to matrix porosity in the saturated case considered here) and \(v_f\) and \(v_s\) denote the velocities of the fluid and the solid phase, respectively. The zero on the right-hand side of equations (2.1) expresses the fact that we do not consider any mass transfer between the liquid and solid phases.

The balance equations for linear momentum for these two phases take the form

\[
\frac{\partial (\phi \rho_f^m v_f)}{\partial t} + \text{div}(\phi \rho_f^m v_f \otimes v_f) = \text{div} T_f + \phi \rho_f^m b + I ,
\]

\[
\frac{\partial ((1 - \phi) \rho_s^m v_s)}{\partial t} + \text{div}((1 - \phi) \rho_s^m v_s \otimes v_s) = \text{div} T_s + (1 - \phi) \rho_s^m b - I ,
\]

where, for \(u, w \in \mathbb{R}^3\) the symbol \(u \otimes w\) denotes the tensor of components \((u \otimes w)_{ij} := u_i w_j\) with \(i, j = 1, 2, 3\), while \(T_f\) and \(T_s\) stand for the fluid and solid Cauchy stresses, respectively, both of which are assumed to be symmetric (i.e. \(T_f = T_f^T\), \(T_s = T_s^T\)). The quantity \(I\) represents the interaction force between the phases. The interaction nature of this force is reflected by the fact that it appears with plus sign in one equation and with minus in the other. Finally \(b\) is the body force (same for both phases, typically this is the gravity acceleration vector). In reality, both phases are compressible, i.e. both material densities \(\rho_s^m\) and \(\rho_f^m\) must be specified by a corresponding state equation. In the isothermal setting considered here, such relation would take the form of dependence
on the material stress state of the particular phase. Since the dominant compressibility effect in the context of real-world geological materials is not related to the changes of material densities, but rather to the changes in porosity in reaction to the applied loading (see [14], chapter 4), we neglect the former effect by setting
\[
\rho^m_f = \text{const}_f \quad \text{and} \quad \rho^m_s = \text{const}_s. \tag{2.3}
\]
Dividing now (2.1a) by \( \rho^m_f \) and (2.1b) by \( \rho^m_s \) and summing the resulting equations, we obtain
\[
\text{div} \, \mathbf{v}_s = -\text{div}(\phi(\mathbf{v}_f - \mathbf{v}_s)) \tag{2.4a}
\]
Inserting this relation into (2.1b) (divided by \( \rho^m_s \)) yields the evolution equation for porosity
\[
\frac{\partial \phi}{\partial t} + \mathbf{v}_s \cdot \nabla \phi = -(1-\phi) \text{div}(\phi(\mathbf{v}_f - \mathbf{v}_s)) \tag{2.4b}
\]
Under the assumptions (2.3), the system of equations (2.4) is equivalent to the system (2.1).

The balance equations for linear momentum (2.2) are reformulated in terms of an equivalent system, where the balance equation for linear momentum of the solid is replaced by the balance equation for linear momentum of the mixture as a whole. Thus, using (2.3), we get
\[
\rho^m_f \left( \frac{\partial (\phi \mathbf{v}_f)}{\partial t} + \text{div}(\phi \mathbf{v}_f \otimes \mathbf{v}_f) \right) = \text{div} \, \mathbf{T}_f + \phi \rho^m_f \mathbf{b} + \mathbf{I}, \tag{2.5a}
\]
\[
\rho^m_s \left( \frac{\partial \mathbf{v}_s}{\partial t} + \text{div}(\mathbf{v}_s \otimes \mathbf{v}_s) \right) = \text{div} \, \mathbf{T} + \rho \mathbf{b} - \frac{\partial}{\partial t} (\phi (\rho^m_f \mathbf{v}_f - \rho^m_s \mathbf{v}_s)) \\
- \text{div}(\phi (\rho^m_f \mathbf{v}_f \otimes \mathbf{v}_f - \rho^m_s \mathbf{v}_s \otimes \mathbf{v}_s)), \tag{2.5b}
\]
where, in the second equation, we introduced the total Cauchy stress \( \mathbf{T} \) and the total density \( \rho \) by
\[
\mathbf{T} := \mathbf{T}_s + \mathbf{T}_f, \quad \rho := \phi \rho^m_f + (1-\phi) \rho^m_s = \rho^m_s + \phi (\rho^m_f - \rho^m_s). \tag{2.6}
\]
Next we introduce several simplifications.

- In the balance of linear momentum for the fluid (2.5a), we ignore the inertial forces, i.e. the whole left-hand side of (2.5a) is set to zero. We further consider \( \mathbf{T}_f \) of the form
\[
\mathbf{T}_f = -p^I_f \mathbf{I}, \tag{2.7}
\]
where \( p^I_f \) is the true pressure in the interstitial fluid (pore pressure). Finally, the interaction force \( \mathbf{I} \) takes a simple form corresponding to the linear drag
\[
\mathbf{I} = -\alpha (\mathbf{v}_f - \mathbf{v}_s) + p^I_f \nabla \phi, \tag{2.8}
\]
where \( \alpha \) is the drag coefficient of the form
\[
\alpha := \frac{\phi^2 \mu_f}{k(\phi)}, \tag{2.9}
\]
\( \mu_t \) being the dynamic viscosity of the fluid and \( k(\phi) \) the permeability of the matrix. The presence of the second term on the right-hand side of (2.8) is known from multiphase continuum theory as an artefact of the volume averaging technique [15], which must be present to cancel out in the fluid momentum balance with a corresponding term coming from the divergence of eq. (2.7). See also [20] where such terms occur from the derivation directly.

- In the balance equation (2.5b), we keep the inertial term only on the left-hand side and neglect the last two terms on the right-hand side using a rough scaling argument stating that the scale of these terms is at most the scale of the left-hand side, multiplied by the scale of porosity, which, in the considered applications, typically does not exceed a few percent. Furthermore, since \( \phi \) is typically below 0.1, we conclude from (2.6) that \( \rho \) is approximately equal to \( \rho_m^s \), so \( \rho = \rho_m^s \) in what follows.

With these simplifications, the balance equations (2.4) and (2.5) take the form

\[
\begin{align*}
\frac{\partial \phi}{\partial t} + \mathbf{v}_s \cdot \nabla \phi &= -(1 - \phi) \div (\phi \mathbf{v}_f - \mathbf{v}_s), \\
\div \mathbf{v}_s &= -\div(\phi(\mathbf{v}_f - \mathbf{v}_s)), \\
\phi \nabla p_t &= \phi \rho_m^f b - \alpha(\mathbf{v}_f - \mathbf{v}_s), \\
\rho_m^s \left( \frac{\partial \mathbf{v}_s}{\partial t} + \div(\mathbf{v}_s \otimes \mathbf{v}_s) \right) &= \div \mathbf{T} + \rho b. 
\end{align*}
\]

Next, we multiply (2.10c) by \( \phi \), use (2.9) and apply divergence to the result. After inserting the outcome of these computations in (2.10b) we obtain

\[
\div \mathbf{v}_s = \div \left( k(\phi) \frac{\nabla p_t}{\mu_t} - \rho_m^m b \right).
\]

As a consequence, (2.11) replaces (2.10b) and (2.10c) can be eliminated from (2.10).

The next assumption, based on the observations of hydrological phenomena in granular materials (see [14]), states that the porosity of the matrix can be described by a constitutive relation of the form

\[
\phi = \tilde{\phi}(p_{\text{eff}}) \quad \text{where} \quad p_{\text{eff}} := p - p_c.
\]

The quantity \( p_{\text{eff}} \), called effective pressure as introduced by Terzaghi [37], is defined as the difference between the total mixture pressure and the fluid (pore) pressure. The quantity \( p_{\text{eff}} \) is assumed to reflect the part of the loading borne by the solid matrix. Inserting the constitutive assumption (2.12) in (2.10a) and using (2.11) we obtain the following evolution equation for the effective pressure

\[
\frac{d\tilde{\phi}}{dp_{\text{eff}}} \left( \frac{\partial p_{\text{eff}}}{\partial t} + \mathbf{v}_s \cdot \nabla p_{\text{eff}} \right) = (1 - \phi) \div \left( k(\phi) \frac{\nabla p_t}{\mu_t} - \rho_m^m b \right).
\]

Setting

\[
-\frac{1}{\beta} := \frac{d\tilde{\phi}}{dp_{\text{eff}}} \quad \text{with} \beta > 0,
\]

\[
\frac{\partial p_{\text{eff}}}{\partial t} + \mathbf{v}_s \cdot \nabla p_{\text{eff}} = (1 - \phi) \div \left( k(\phi) \frac{\nabla p_t}{\mu_t} - \rho_m^m b \right).
\]
replacing \( p^t \) in (2.11) and (2.13) by \( p - p^{\text{eff}} \) and splitting the total Cauchy stress \( T \) as
\[
T = -p I + S,
\]
we arrive at the following set of governing equations
\[
\text{div} \, v_s = \text{div} \left( \frac{k(\phi)}{\mu_f} \left( \nabla p - \nabla p^{\text{eff}} - \rho_m f b \right) \right), \tag{2.16a}
\]
\[
\rho_m^s \left( \frac{\partial v_s}{\partial t} + \text{div} \, (v_s \otimes v_s) \right) = -\nabla p + \text{div} \, S + \rho_m^s b, \tag{2.16b}
\]
\[
\frac{\partial p^{\text{eff}}}{\partial t} + v_s \cdot \nabla p^{\text{eff}} = -(1 - \phi) \beta \text{div} \left( \frac{k(\phi)}{\mu_f} \left( \nabla p - \nabla p^{\text{eff}} - \rho_m f b \right) \right). \tag{2.16c}
\]

The rheology of the granular material needs to be specified by providing a constitutive relation for the stress \( S \). We shall assume that the material of the granular phase behaves as very stiff until the yield (liquefaction) threshold is reached at which moment the granular phase starts to flow as a liquid. A simplest constitutive relation for this type of response is characterized by the so-called Bingham fluid [2], where the solid part responses as a perfectly rigid body until the magnitude of the stress exceeds the threshold when the solid flows as a Newtonian fluid. This type of response is usually written in the following way
\[
\begin{cases}
|S| \leq \tau(p^{\text{eff}}) & \text{if and only if } D = 0, \\
|S| > \tau(p^{\text{eff}}) & \text{if and only if } S = \tau(p^{\text{eff}}) \frac{D}{|D|} + 2
\nu_s D.
\end{cases} \tag{2.17}
\]

Here \( D \) is the symmetric part of the velocity gradient
\[
D := \frac{1}{2} \left( \nabla v_s + (\nabla v_s)^T \right),
\]
\( \nu_s > 0 \) is the viscosity and \( \tau(p^{\text{eff}}) \) is the yield (liquefaction) threshold, depending on the effective pressure. Typically,
\[
\tau(p^{\text{eff}}) = q_0 (p^{\text{eff}})^+, \tag{2.18}
\]
where \( q_0 \) is a constant and the symbol \((\cdot)^+\) denotes the positive part of a quantity, i.e. \((\psi)^+ := \max(\psi, 0)\).

The system (2.16) – (2.18) seems to be a meaningful, physically justified and relatively simple model worth of studying. We however do not further investigate this system here, as our goal is to identify the assumptions that lead to the model analyzed in [11]. Towards this aim, we introduce the following additional assumptions:

- **Pore pressure evolution approximation.** Using the relation \( p^t = p - p^{\text{eff}} \) we rewrite (2.16c) as an evolution equation for \( p^t \)
\[
\frac{\partial p^t}{\partial t} + v_s \cdot \nabla p^t = \frac{\partial p}{\partial t} + v_s \cdot \nabla p + (1 - \phi) \beta \text{div} \left( \frac{k(\phi)}{\mu_f} \left( \nabla p^t - \rho_m^s b \right) \right). \tag{2.19}
\]

Next we assume that the dominant contribution to the total pressure \( p \) in eq. (2.19) comes from the hydrostatic part, which may in general depend explicitly on time to include problems with evolving boundary. Consequently, we replace \( p(x, t) \) by \( p_s(x, t) \) in eq. (2.19), where \( p_s \) is a given function. Also, we replace \((1 - \phi) \) by \( 1 \) on the
right-hand side of \((2.19)\) since, as set above, we are interested in situations where \(\phi < 0.1\) and finally, we assume that both the permeability \(k\) and the compressibility parameter \(\beta\) are constant. Setting thus

\[ K := \frac{\beta k}{\mu_f} \geq 0 , \]

the equation \((2.19)\) simplifies to the form

\[ \frac{\partial p^i_t}{\partial t} + v_s \cdot \nabla p^i_t = K \Delta p^i_t - \text{div}(K \rho_m b) + v_s \cdot \nabla p_s . \]

(2.21)

- **Yield criterion approximation.** Also in the yield criterion, we replace the pressure \(p\) in the definition of the effective pressure \((2.12)\) by \(p_s\), i.e. instead of \((2.18)\), we have

\[ \tau(p^i_t) = q_0(p_s - p^i_t)^+ . \]

(2.22)

- **Incompressibility.** We ignore the effect of porosity changes in \((2.16a)\) by replacing \((2.16a)\) with the incompressibility constraint

\[ \text{div} v_s = 0 . \]

(2.23)

With the above set of simplifying assumptions, the final reduced system of governing equations reads as follows

\[ \text{div} v_s = 0 , \]

(2.24a)

\[ \rho_m^\prime \left( \frac{\partial v_s}{\partial t} + \text{div} (v_s \otimes v_s) \right) = -\nabla p + \text{div} S + \rho_m b , \]

(2.24b)

\[ \frac{\partial p^i_t}{\partial t} + v_s \cdot \nabla p^i_t = K \Delta p^i_t - \text{div}(K \rho_m b) + v_s \cdot \nabla p_s , \]

(2.24c)

where \(S\) satisfies

\[
\begin{cases}
|S| \leq \tau(p^i_t) & \text{if and only if } D = \emptyset , \\
|S| > \tau(p^i_t) & \text{if and only if } S = \tau(p^i_t) \frac{D}{|D|} + 2 \nu_s D , \text{ with } \tau(p^i_t) = q_\ast (p_s - p^i_t)^+ .
\end{cases}
\]

(2.25)

### 3 \textbf{(Re)-formulations of the Problem}

Let \(T\) be a positive real number and \(\Omega \subset \mathbb{R}^3\) a bounded domain with the boundary \(\partial \Omega\). We set \(Q_T := (0, T) \times \Omega\) and \(\Sigma_T := (0, T) \times \partial \Omega\). The symbol \(n : \partial \Omega \rightarrow \mathbb{R}^3\) denotes the outer unit normal vector while for any vector \(z\) defined on \(\partial \Omega\) we set \(z_T := z - (z \cdot n)n\) representing the projection of \(z\) to the tangent plane.

We consider unsteady flows of a homogeneous incompressible non-Newtonian fluid of a Bingham type with a variable threshold, described in the previous section, see \((2.24)\)-(2.25). In what follows, we slightly change the notation and write \(v\) instead of \(v_s\), \(g_\ast\) instead of \(\rho_m^\prime\) and \(p_\ast\) instead of \(p^i_t\). We also set \(g := -\text{div}(K \rho_m \mathbf{b})\).

Following a recent observation in \[5\] (see also \[29\], \[6\], \[7\]) the rheological behaviour \((2.25)\) can be equivalently written as

\[ 2 \nu_s D = \frac{(|S| - \tau(p_\ast))^+}{|S|} S \text{ where } \tau(p_\ast) = q_\ast (p_s - p_\ast)^+ . \]

(3.1)
We are thus interested in solving the following problem. For given \( g_*, \nu_*, q_* \in (0, \infty) \), \( b : Q_T \to \mathbb{R}^3 \), \( g : Q_T \to \mathbb{R} \), \( p_* : Q_T \to \mathbb{R} \), and \( \sigma_* : Q_T \to \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R} \) satisfying

\[
\begin{align*}
\text{div} \, v &= 0 \text{ in } Q_T, \\
ge_* (\partial_t v + \text{div}(v \otimes v)) - \text{div} \mathcal{S} + \nabla p &= g_* b \text{ in } Q_T, \\
\mathcal{G}(\mathcal{S}, \mathcal{D}, p_t) &= 0 \text{ in } Q_T, \\
\tau_t + v \cdot \nabla p_t - K \Delta p_t &= \mathcal{G} + v \cdot \nabla p_s \text{ in } Q_T,
\end{align*}
\]

(3.2)

where \( \mathcal{G} \) is a continuous function defined on \( \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \) through

\[
\mathcal{G}(\mathcal{S}, \mathcal{D}, p_t) = \frac{(|\mathcal{S}| - q_* (p_s - p_t)^+)^-}{|\mathcal{S}|} \mathcal{S} - 2\nu_* \mathcal{D}.
\]

(3.3)

In addition, the unknown functions \( (v, p, p_t, \mathcal{S}) \) are required, for given \( \sigma_*, \gamma_* \in [0, \infty) \) and \( \nu_0 : \Omega \to \mathbb{R}^3 \) (such that \( \text{div} \nu_0 = 0 \) in \( \Omega \)) and \( p_0 : \Omega \to \mathbb{R} \), to fulfill the following initial and boundary conditions:

\[
\begin{align*}
v(0, \cdot) &= \nu_0 \text{ and } p_t(0, \cdot) = p_0 \text{ in } \Omega, \\
v \cdot n &= 0 \text{ and } \nabla p_t \cdot n = 0 \text{ on } \Sigma_T, \\
\gamma_* v_\tau &= \frac{(|s| - s_*)^+}{|s|} \nu \text{ on } \Sigma_T.
\end{align*}
\]

(3.4), (3.5), (3.6)

The conditions in (3.5) state that the boundary is impermeable, while (3.6) characterizes the result of the interaction of the fluid and the boundary along the boundary. Here

\[
s := -\lfloor T \nu \rfloor = -\lfloor \mathcal{S} \nu \rfloor.
\]

Note that (3.6) usually written as

\[
\begin{cases}
|s| \leq s_* & \text{if and only if } v_\tau = 0, \\
|s| \geq s_* & \text{if and only if } s = s_* \frac{v_\tau}{|v_\tau|} + \gamma_* v_\tau,
\end{cases}
\]

(3.7)

describes the stick-slip (or threshold slip) and includes, as special cases, Navier's slip condition by taking \( s_* = 0 \) and \( \gamma_* > 0 \), and perfect slip condition if in addition \( \gamma_* = 0 \). Note that the no-slip condition is obtained by letting either \( s_* \to +\infty \) or \( \gamma_* \to +\infty \).

One of the motivations for this work is a recent paper by Chupin and Mathé [11] where the authors characterize the tensorial response (3.1) through two scalar constraints:

\[
|\mathcal{Z}| \leq \tau(p_t) \quad \text{and} \quad \mathcal{Z} : \mathcal{D} \geq \tau(p_t)|\mathcal{D}| \quad \text{where} \quad \mathcal{Z} := \mathcal{S} - 2\nu_* \mathcal{D} \quad \text{and} \quad \tau(p_t) = q_* (p_s - p_t)^+.
\]

(3.8)

In fact, Chupin and Mathé [11] considered the second constraint with the equality sign in their existence result concerning planar flows, but then they incorrectly argue when performing the limit in the constitutive equation (see Step 2 (a) in [11]). This difficulty can be overcome easily if the inequality is used here instead of the equality, as shown in the proof of Theorem 4.2 in Section 6 below.
Before we prove that (3.1) and (3.8) are equivalent, we provide analogously a condition that characterizes (3.6). It takes the form
\[ |z| \leq s_* \quad \text{and} \quad z \cdot v_{\tau} \geq s_*|v_{\tau}| \]
where \( z := s - \gamma_* v_{\tau} \).

Next, we prove the following statement.

**Proposition 3.1.** The following equivalences hold:

(a) \( (\ref{2.25}) \iff (\ref{3.1}) \iff (\ref{3.8}) \);

(b) \( (\ref{3.7}) \iff (\ref{3.6}) \iff (\ref{3.9}) \).

**Proof.** The equivalence \( (\ref{2.25}) \iff (\ref{3.1}) \) is simple. We prove that \( (\ref{3.1}) \) is equivalent to \( (\ref{3.8}) \). Let us first assume that \( (S, D, p_f) \) fulfil \( (\ref{3.1}) \). If \( D = O \) then \( |S| = \tau(p_f) \) and \( Z = S \) and \( (\ref{3.8}) \) holds. If \( D \neq O \), then \( |S| > \tau(p_f) \), and the formula \( (\ref{3.1}) \) implies
\[ S = 2\nu_* D = \tau(p_f) \frac{S}{|S|}. \] (3.10)
Hence \( Z := S - 2\nu_* D \) fulfils \( |Z| = \tau(p_f) \). Next, by taking the modulus of \( (3.10) \) it follows
\[ |S| - \tau(p_f) = 2\nu_* |D|. \]
Inserting this back to \( (\ref{3.1}) \), we get
\[ \frac{S}{|S|} = \frac{D}{|D|}. \]
Employing this in \( (3.10) \), we obtain first
\[ Z = S - 2\nu_* D = \tau(p_f) \frac{D}{|D|} \]
and then, after taking the scalar product with \( D \),
\[ Z : D = \tau(p_f)|D|, \]
which is the second assertion in \( (\ref{3.8}) \).

Next, we assume that \( (S, D, p_f) \) fulfil \( (\ref{3.8}) \). Then, if \( D \neq O \),
\[ \tau(p_f)|D| \leq Z : D \leq |Z||D| \leq \tau(p_f)|D|, \]
which implies
\[ Z : D = \tau(p_f)|D| \] (3.11)
as well as the equality in the Cauchy-Schwarz inequality. Then necessarily
\[ Z = aD. \]
Inserting this structure in \( (3.11) \) we obtain
\[ \tau(p_f)|D| = a|D|^2. \]
Hence \( Z = \tau(p_f) \frac{D}{|D|} \) and
\[ S = 2\nu_* D + \tau(p_f) \frac{D}{|D|}. \] (3.12)
Also, we have
\[ |S| = \left( \frac{2\nu_s|D| + \tau(p_1)}{|D|} \right) |D| = 2\nu_s|D| + \tau(p_1) \]
which implies (as \( D \neq \emptyset \))
\[ |S| - \tau(p_1) > 0, \text{ and also } \frac{S}{|S|} = \frac{D}{|D|}. \]
This together with (3.12) implies (3.1) for \( D \neq \emptyset \). If \( D = \emptyset \) then, by (3.8), \( S = \mathbb{Z} \) and \( |S| \leq \tau(p_1) \) and (3.1) holds. The proof of the equivalence of (3.1) and (3.8) is complete.

The proof of the statement (b) is done in the same manner.

\[ \square \]

4 Definition of weak solution and Main Result

In order to define the weak solution to the considered problem and to formulate the result, we need to fix the notation. For any \( q \in [1, \infty] \) the symbol \( \| \cdot \|_q \) stands for the \( L^q \)-norm in the usual Lebesgue space \( L^q(\Omega) \) while \( \| \cdot \|_{1,q} \) for the norm in the usual Sobolev space \( W^{1,q}(\Omega) \). If \( X \) is a Banach space of scalar functions then \( X^3 \) denotes the space of vector-valued functions having three components, each of them belonging to \( X \). Similarly \( X^{3 \times 3} \) denotes the space of tensor-valued functions, with each component belonging to \( X \). For a Banach space \( X \) we denote the relevant Bochner space by \( L^q(0,T; X) \).

Let us introduce the notation for spaces of solenoidal functions and for spaces of functions which have zero normal component on the boundary, for the domain \( \Omega \). We set for any \( q \in [1, \infty) \)
\[ L^q_{n,\text{div}} := \left\{ \nu \in C_0^\infty(\Omega)^3; \text{div } \nu = 0 \right\}. \]

Next, we define
\[ W^{1,2}_{n,\text{div}} := \left\{ \nu \in W^{1,2}(\Omega)^3; \nu \cdot n = 0 \text{ on } \partial \Omega \right\}, \]
\[ W^{1,2} := (W^{1,2}_{n,\text{div}})^*, \quad W^{-1,2} := (W^{-1,2}_{n,\text{div}})^*, \]
\[ W^{1,\infty}_{n,\text{div}} := \left\{ \nu \in W^{1,\infty}(\Omega)^3; \nu \cdot n = 0 \text{ on } \partial \Omega \right\}, \]
\[ W^{1,\infty} := (W^{1,\infty}_{n,\text{div}})^*, \quad W^{-1,\infty} := (W^{-1,\infty}_{n,\text{div}})^*. \]

By the Helmholtz decomposition (in the case that \( \Omega \in C^{1,1} \)) it holds
\[ W^{1,2} = W^{1,2}_{n,\text{div}} \oplus \{ \nabla \varphi; \varphi \in W^{2,2}(\Omega), \nabla \varphi \cdot n = 0 \text{ on } \partial \Omega \}. \]

Note that such a decomposition is not valid for \((W^{1,2}_0(\Omega))^3\).

The symbol \( \mathbb{D} \varphi \) stands for the symmetric part of the gradient of a vector-valued function \( \varphi \), i.e. \( \mathbb{D} \varphi := \frac{1}{2}(\nabla \varphi + (\nabla \varphi)^T) \).

In what follows, we also set for simplicity and without loss of any generality
\[ q_* = 2\nu_* = \gamma_* = K = q_* = 1. \]

**Definition 4.1** (Definition of weak solution). Let \( s_* > 0 \),
\[ v_0 \in L^2_{n,\text{div}}, \quad p_0 \in L^\infty(\Omega), \quad p_s \in L^\infty(Q_T), \quad p_s(0) \in L^\infty(\Omega), \quad b \in L^2(0,T; W^{-1,2}), \quad (4.1) \]
Remark weak solution to the problem fulfilling (4.1) Theorem 4.2 the foundation for a direct application of mixed finite element (or spectral) methods. In avoiding tools such as multivalued calculus or variational inequalities [1] in our analysis, example [19] and the references therein). On the other hand, we intentionally aim at incorporate the tools of calculus of variations suitable for Stokes-type problems (see for fluids and in the analysis of steady flows in three dimensions in [27]. Neither can we in [23] and [34], also in the analysis of two-dimensional unsteady flows of the Bingham dependent flows in Chupin and Mathé [11] or the higher differentiability techniques used in our analysis such tools as the energy equality, used in the analysis of planar time-dependent flows in Chapin and Mathé [11] or the higher differentiability techniques used in [23] and [34], also in the analysis of two-dimensional unsteady flows of the Bingham fluids and in the analysis of steady flows in three dimensions in [27]. Neither can we incorporate the tools of calculus of variations suitable for Stokes-type problems (see for example [19] and the references therein). On the other hand, we intentionally aim at avoiding tools such as multivalued calculus or variational inequalities [1] in our analysis, see [19], or [31] (which is considered however in a different context). In our opinion, the concept of solution (expressed in terms of identities) considered here is stronger, its large-data existence can be proved and has some other advantages. For example, it forms the foundation for a direct application of mixed finite element (or spectral) methods. In

and one of the following requirements be satisfied

$$g \in L^q(Q_T)$$ and $$\partial_t p_s - \Delta p_s \in L^q(Q_T)$$ with $$q > \frac{5}{2}.$$ (4.2)

$$p_s \in L^q(0,T;W^{1,q}(\Omega))$$ with $$q > 10$$ and $$g \in L^q(Q_T)$$ with $$q > \frac{5}{2}.$$ (4.3)

We say that $$(v, p_t, p, S, s)$$ is a weak solution to the problem (3.2)-(3.6) if

$$v \in L^\infty(0, T; L^2_{n,\text{div}}) \cap L^2(0, T; W^{1,2}_{n,\text{div}}), \partial_t v \in (L^2(0, T; W^{1,2}_n(\Omega)) \cap L^5(Q_T)^3)^*,$$

$$p_t \in L^\infty(Q_T) \cap L^2(0, T; W^{1,2}(\Omega)), \partial_t p_t \in (L^2(0, T; W^{1,2}(\Omega))^*),$$

$$p = p_t + p_2$$ where $$p_t \in L^2(Q_T)$$ and $$p_2 \in L^\frac{2}{3}(0, T; W^{1,\frac{2}{3}}(\Omega)),$$

$$S \in L^2(Q_T)^{3x3}, s \in L^2(\Sigma_T)^3,$$

$$\langle \partial_t v, w \rangle + (S, Dw) - (v \otimes v, Dw) + (s, w)_{\partial \Omega} = \langle b, w \rangle + (p_t, \text{div } w) - (\nabla p, w)$$
for all $$w \in W^{1,2}_{n,\text{div}}$$ and a.e. in $$(0, T),$$

$$\langle \partial_t p_t, z \rangle - (p_t v, \nabla z) + (\nabla p_t, \nabla z) = (g, z) - (p_s v, \nabla z)$$
for all $$z \in W^{1,2}$$ and a.e. in $$(0, T),$$

$$Dv = \frac{(|S| - \tau(p_t))^+ |S|}{|S|} S$$ where $$\tau(p_t) = (p_s - p_t)^+$$ for a.a. $$(t, x) \in Q_T,$$

$$v_T = \frac{(|s| - s_s)^+ |s|}{|s|} s$$ for a.a. $$(t, x) \in \Sigma_T,$$

$$\lim_{t \to 0^+} \|v(t) - v_0\|_2 = 0 \quad \text{and} \quad \lim_{t \to 0^+} \|p(t) - p_0\|_2 = 0.$$

**Theorem 4.2 (Main Theorem).** For any $$\Omega \in C^{1,1}, T > 0$$ and for arbitrary $$v_0, p_0, p_s, b$$ fulfilling (4.1) and for arbitrary $$g$$ and $$p_s$$ fulfilling either (4.2) or (4.3), there exists a weak solution to the problem (3.2) in the sense of Definition 4.1.

**Remark 4.3.** We wish to emphasize that due to Proposition 3.1 the tensorial constitutive equation $$Dv = \frac{(|S| - \tau(p_t))^+ |S|}{|S|} S$$ in $$Q_T$$ as well as the vectorial equation $$v_T = \frac{(|s| - s_s)^+ |s|}{|s|} s$$ on $$\Sigma_T$$ can be replaced by any of its equivalent forms. It is in particular interesting that the tensorial equations can be characterized by two (scalar) inequalities.

Note that Theorem 4.2 presents the existence result to a supercritical problem; this is a problem where the solution itself is not an admissible test function in the weak formulation of the governing equations. Indeed, in our case $$v$$ belongs to $$L^\frac{5}{3}(Q_T)^3,$$ however, admissible test functions have to be from $$L^5(Q_T)^3$$ due to the fact that $$\text{div}(v \otimes v) = \sum_{r=1}^3 v_r \frac{\partial v_r}{\partial x_r}$$ and $$\nabla p_2$$ belong to $$L^\frac{2}{3}(Q_T)^3.$$ This is the reason why we cannot involve in our analysis such tools as the energy equality, used in the analysis of planar time-dependent flows in Chapin and Mathé [11] or the higher differentiability techniques used in [23] and [34], also in the analysis of two-dimensional unsteady flows of the Bingham fluids and in the analysis of steady flows in three dimensions in [27]. Neither can we incorporate the tools of calculus of variations suitable for Stokes-type problems (see for example [19] and the references therein). On the other hand, we intentionally aim at avoiding tools such as multivalued calculus or variational inequalities [1] in our analysis, see [19], or [31] (which is considered however in a different context). In our opinion, the concept of solution (expressed in terms of identities) considered here is stronger, its large-data existence can be proved and has some other advantages. For example, it forms the foundation for a direct application of mixed finite element (or spectral) methods. In
order to identify the non-linear constitutive equation pointwise in the considered domain
\((0, T) \times \Omega\) when taking the limit from the approximative problem to the original one, and
in order to overcome difficulties connected with the low integrability of \(v\), we incorporate
the so-called \(L^\infty\)-truncation method. This method replaces \(v^n - v\), where \(\{v^n\}_{n=1}^{+\infty}\)
is solution of a suitably constructed approximative problem, by a truncated function that
coincides with \(v^n - v\) on a large set and the measure of the complementary set can be
made arbitrarily small uniformly with respect to \(n\). Although the origin of the method
goes back to elliptic problems with an \(L^1\)-right-hand side (see [13], [17] and [32]), we refer
here mainly to its development for evolutionary problems in fluid mechanics, see [18], [9],
[39]. The result by Wolf [39] similarly as those by Solonnikov (see [35] and [36]) and
Koch and Solonnikov [21] concerning the properties of evolutionary Stokes-like systems
with no-slip boundary conditions indicate the difficulties connected with the impossibility
to establish the integrability of the pressure \(p\) for generalizations of the Navier-Stokes
equations (with variable viscosity) in three-dimensions. This is why we treat the stick-slip
boundary conditions in this study. It reveals that the analysis of the three-dimensional
evolutionary supercritical problems associated with the stick-slip boundary conditions
differs remarkably from the analysis of analogous problems connected with the no-slip
boundary conditions. We refer to [6] for a detailed discussion of this issue noting that
Theorem 4.2 guarantees that \(p \in L^1(Q_T)\). We remark that the integrability of the
pressure is important in the analysis of problems with the viscosity dependent on the
temperature (see [8], [7] or [28]) or the viscosity dependent on the pressure (see [9] or
[10]), but it is also an interesting mathematical question itself.

5 Approximations

Before introducing the approximations, we recall that in Section 3 we set

\[ Z = S - Dv, \]

and analogously we can also define

\[ z := s - v_T. \]

For any \(n \in \mathbb{N}\), let \(G_n : \mathbb{R} \to \mathbb{R}\) be a smooth function such that \(G_n(u) = 1\) if \(|u| \leq n\),
\(G_n(u) = 0\) if \(|u| \geq 2n\) and \(|G'_n| \leq \frac{2}{n}\). We consider the following approximative system:

\[
\begin{align*}
\text{div } v &= 0 \quad \text{in } Q_T, \\
\partial_t v + \text{div}(v \otimes v)G_n(|v|^2) - \text{div } Dv - \text{div } Z + \nabla p &= b \quad \text{in } Q_T, \\
\partial_t p_T + v \cdot \nabla p_T - \Delta p_T &= g + v \cdot \nabla p_s \quad \text{in } Q_T, \\
Z &= Z_n(p_T, Dv) := (p_T - p_s)' + \frac{\text{div } v}{||Dv|| + \frac{n}{2}} \quad \text{in } Q_T, \\
z &= \zeta_n(v_T) := s_\ast \frac{v_T}{|v_T| + \frac{n}{2}} \quad \text{on } \Sigma_T, \\
v \cdot n &= 0 \quad \text{and } \nabla p_T \cdot n = 0 \quad \text{on } \Sigma_T, \\
v(0) &= v_0 \quad \text{and } p_T(0) = p_0 \quad \text{in } \Omega.
\end{align*}
\]

(5.1)
It is not difficult to check that if \( Z = Z_n(p_f, D) \) and \( \hat{Z} = Z_n(p_f, \hat{D}) \), then
\[
(Z - \hat{Z}) : (D - \hat{D}) \geq \frac{(p_f - p_s)^+}{n} \left( \frac{|D| - |\hat{D}|}{n} \right)^2 \geq 0.
\]

(5.2)

A similar monotone property holds for \( z = \zeta_n(v_r) \).

**Proposition 5.1.** Let \( n \in \mathbb{N} \) be fixed and \( s_+ > 0 \). Let \( v_0 \in L^2_{n, \text{div}}, \ p_0 \in L^2(\Omega), \ b \in L^2(0, T; W^{1,2}_n), \ g \in L^2(Q_T) \) and \( p_s \in L^\infty(Q_T) \), then there exists a weak solution to the problem [5.1], i.e. a quadruple \((v^n, p^n, Z^n, z^n)\) such that

\[
v^n \in L^\infty(0, T; L^2_{n, \text{div}}) \cap L^2(0, T; W^{1,2}_{n, \text{div}}), \quad \partial_t v^n \in (L^2(0, T; W^{1,2}_n(\Omega)))^*,
\]

(5.3)

\[
p^n_0 \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega)), \quad \partial_t p^n_0 \in (L^2(0, T; W^{1,2}(\Omega)))^*,
\]

(5.4)

\[Z^n \in L^\infty(Q_T)^3 \times 3, \quad Z^n \in L^\infty(\Sigma_T)^3,\]

(5.5)

\[
\int_0^T (\partial_t v^n, w) dt + \frac{1}{2} \int_\Omega \nabla v^n : \nabla w + Z^n : \nabla w + G_n(|v^n|^2) \text{div}(v^n \otimes v^n) \cdot w dx dt
\]

\[
+ \int_{\Sigma_T} z^n \cdot \tau \cdot w + p^n_0 \cdot \tau \cdot w \text{d}\sigma dt = \int_0^T (b, w) dt \quad \text{for all } w \in L^2(0, T; W^{1,2}_{n, \text{div}}),
\]

(5.6)

\[
\int_0^T (\partial_t p^n, z) dt + \int_{Q_T} p^n_0 \cdot \nabla z + \nabla p^n_0 \cdot \nabla z dx dt = \int_{Q_T} g z - p_s v^n \cdot \nabla z dx dt
\]

(5.7)

for all \( z \in L^4(0, T; W^{1,2}(\Omega)) \).

\[Z^n = Z_n(p^n_0, \nabla v^n) \quad \text{a.e. in } Q_T,\]

(5.8)

\[z^n = \zeta_n(v^n) \quad \text{a.e. in } \Sigma_T,\]

(5.9)

\[
\lim_{t \to 0^+} \|v^n(t) - v_0\|_2 = 0 \quad \text{and} \quad \lim_{t \to 0^+} \|p^n(t) - p_0\|_2 = 0.
\]

(5.10)

**Proof.** Due to the presence of \( G_n \) that truncates the convective term and the properties of the approximations \( Z_n \) and \( \zeta_n \) introduced above, the proof of the existence of weak solutions to the problem [5.1] is a variant of the standard monotone operator technique (see [24], [22] or [9]). To be more specific, we briefly outline the proof using the Galerkin method. Since \( n \) is fixed, we write \((v, p_f, Z, z)\) instead of \((v^n, p^n, Z^n, z^n)\) in the proof.

**Step 1. Galerkin system.** Let \( \{w^i\}_{i \in \mathbb{N}} \) be an orthogonal basis in \( W^{1,2}_{n, \text{div}} \) consisting of eigenfunctions of the Stokes operator subject to \( v \cdot n = 0 \) and \( (\nabla v)n \cdot n = 0 \) on \( \Sigma_T \). Let analogously \( \{z^j\}_{j \in \mathbb{N}} \) be an orthogonal basis in \( W^{1,2}(\Omega) \) consisting of eigenfunctions of the Laplace operator subject to the relevant homogeneous boundary conditions. Then the local in time existence of

\[
v^n(t, x) := \sum_{r=1}^m c^n_r(t) w^r(x), \quad p^n_0(t, x) := \sum_{r=1}^m d^n_r(t) z^r(x)
\]

(5.11)

satisfying

\[
\frac{d v^n}{dt} + (\nabla v^n, \nabla w) + (Z(p^n_0, \nabla v^n), \nabla w) + (\text{div}(v^n \otimes v^n) G(|v^n|^2), w) + (v^n, w)_{\Omega} + (\zeta_n(v^n), w)_{\Omega} = (b, w), \quad r = 1, \ldots, m,
\]

(5.12)

and

\[
(\partial_t p^n, z^r) - (p^n_0 v^n, \nabla z^r) + (\nabla p^n_0, \nabla z^r) = (g, z^r) - (p_s v^n, \nabla z^r), \quad r = 1, \ldots, m,
\]

(5.13)
together with the corresponding initial conditions $v_0^m$ and $p_0^m$, obtained by projecting $v_0 \in L^2_{n,\text{div}}$ onto the span of $[w^1, \ldots, w^m]$ and $p_0 \in L^2(\Omega)$ onto the span of $[z^1, \ldots, z^m]$, follows from the Caratheodory theory for systems of ordinary differential equations.

Global in time existence is, as usual, a consequence of the uniform estimates which we show next.

**Step 2. Uniform estimates.** Multiplying (5.12) by $c_r^m(t)$ and (5.13) by $d_r^m(t)$ and taking the sum over $r$ from 1 to $m$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|v^m(t)\|_2^2 + \|Dv^m(t)\|_2^2 + \int_{\Omega} Z_n(p_t^m, Dv^m) : Dv^m \, dx + \|v^m_r(t)\|_{2,\partial\Omega}^2$$

$$+ \int_{\partial\Omega} \zeta_n(v_r^m) \cdot v^m_r \, d\sigma = \langle b, v^m \rangle,$$

(5.14)

$$\frac{1}{2} \frac{d}{dt} \|p_t^m(t)\|_2^2 + \|\nabla p_t^m(t)\|_2^2 = (g, p_t^m) - (p_s v^m, \nabla p_t^m),$$

(5.15)

By Korn’s and Young’s inequalities (see for example [9, Lemma 1.11] for details), using also the fact that the last two terms at the right-hand side of (5.14) are non-negative, one concludes from (5.14) that

$$\sup_{t \in [0,T]} \|v^m(t)\|_2^2 + \int_{Q_T} |Dv^m|^2 \, dx \, dt + \int_{\Sigma_T} |v^m_r|^2 \, d\sigma$$

$$\leq C \|b\|_{L^2(0,T,W^{1,2}_n)} + \|v_0\|_2^2 =: C(b, v_0).$$

(5.16)

Similarly, using also (5.16), one obtains from (5.15) that

$$\sup_{t \in [0,T]} \|p_t^m(t)\|_2^2 + \int_{Q_T} |\nabla p_t^m|^2 \, dx \, dt \leq C \|g\|_{L^2(Q_T)} + C(b, v_0) \|p_s\|_{L^\infty(\Omega)} + \|p_0\|_2^2.$$  

(5.17)

By the interpolation inequalities

$$\|z\|_{\frac{6}{5}} \leq \|z\|_2^{\frac{3}{5}} \|z\|_6 \leq C \|z\|_2 \|z\|_{1,2},$$

(5.18)

$$\|z\|_4 \leq \|z\|_2 \|z\|_6 \leq C \|z\|_2 \|z\|_{1,2},$$

(5.19)

and the trace inequalities (see [9, Lemma 1.12]), we obtain

$$\sup_m \left( \|v^m\|_{\frac{6}{5},Q_T} + \|p_t^m\|_{\frac{6}{5},Q_T} + \|v^m_r\|_{\frac{5}{3},\Sigma_T} \right) < +\infty$$

(5.20)

and also

$$\sup_m \left( \int_0^T \left( \|v^m(t)\|_{\frac{6}{5}}^2 + \|p_t^m\|_{\frac{6}{5}}^2 \right) \, dt \right) < +\infty.$$  

(5.21)

It then follows from the explicit formulas for $Z_n$ and $\zeta_n$ that $Z^m := Z_n(p_t^m, Dv^m)$ and $z^m := \zeta_n(v_r^m)$ fulfill

$$\sup \left( \|Z^m\|_{\frac{6}{5},Q_T} + \|z^m\|_{\infty,\Sigma_T} \right) < +\infty.$$  

(5.22)

Finally, the fact that the projectors

$$W^{1,2}_{n,\text{div}} \hookrightarrow [w^1, \ldots, w^m], \quad W^{-1,2}(\Omega) \hookrightarrow [z^1, \ldots, z^m]$$
are continuous and (5.21) imply that
\[
\sup_m \left( \| \partial_t v^m \|_{L^2(0,T;W^{-1,2})} + \| \partial_t p^m_t \|_{L^2(0,T;W^{-1,2})} \right) < +\infty. \tag{5.23}
\]

**Step 3. Limit.** The above uniform estimates imply the existence of \( v, p_t, Z \) and \( z \) and subsequences of \( \{ v^m \}, \{ p^m_t \}, \{ Z^m \} \) and \( \{ z^m \} \) converging weakly (or *-weakly) to \( v, p_t, Z \) and \( z \) in the function spaces indicated in Proposition 5.1 and fulfilling the following strong convergences (due to Aubin-Lions compactness lemma and its variant, see [9, Lemma 1.12], involving the trace theorem):

\[
v^m \to v \text{ a.e. in } QT \text{ and strongly in } L^2(QT)^3 \text{ for any } q \in \left[ 1, \frac{10}{3} \right), \tag{5.24}
\]

\[
p^m_t \to p_t \text{ a.e. in } QT \text{ and strongly in } L^3(QT)^3 \text{ for any } q \in \left[ 1, \frac{10}{3} \right), \tag{5.25}
\]

\[
v^m_r \to v_r \text{ a.e. in } \Sigma_T \text{ and strongly in } L^9(\Sigma_T)^3 \text{ for any } q \in \left[ 1, \frac{8}{3} \right). \tag{5.26}
\]

These weak and strong convergences suffice to show that \( v, p_t, Z \) and \( z \) fulfil the weak formulations stated in Proposition 5.1.

Since the proof of the attainment of the initial conditions is standard, see e.g. [25], it remains to show that \( Z = Z_n(p_t, Dv) \) and \( z = \zeta_n(v_r) \).

**Step 4. Attainment of the constitutive equations.** We first notice that (5.26) together with (5.22) imply, by Lebesgue’s theorem that

\[
z^m = \zeta_n(v^m) \to \zeta_n(v_r) \text{ weakly in } L^2(\Sigma_T)^3.
\]

It implies that

\[
z = \zeta_n(v_r) \text{ a.e. in } \Sigma_T \tag{5.27}
\]

and

\[
\lim_{m \to +\infty} \int_{\Sigma_T} z^m \cdot w^m \, d\sigma dt = \int_{\Sigma_T} \zeta_n(v_r) \cdot w_r \, d\sigma dt. \tag{5.28}
\]

Next, integrating (5.14) over \((0,T)\) and taking limsup of the resulting identity, we obtain, using the above convergences and the weak lower semicontinuity of the \( L^2 \)-norm, that

\[
\frac{1}{2} \| v(t) \|^2_2 + \int_{Q_T} |Dv|^2 \, dx \, dt + \int_{\Sigma_T} |v_r|^2 \, d\sigma dt + \int_{\Sigma_T} \zeta_n(v_r) \cdot v_r \, d\sigma dt
\]

\[
+ \limsup_{m \to +\infty} \int_{Q_T} \tilde{z}^m : Dv \, dx \, dt \le \int_0^T (b, v) \, dt + \frac{1}{2} \| v_0 \|^2_2. \tag{5.29}
\]

On the other hand, taking \( w = v \) in the established weak formulation of the equation for \( v \), we get, using also (5.27),

\[
\frac{1}{2} \| v(t) \|^2_2 + \int_{Q_T} |Dv|^2 \, dx \, dt + \int_{\Sigma_T} |v_r|^2 \, d\sigma dt + \int_{\Sigma_T} \zeta_n(v_r) \cdot v_r \, d\sigma dt
\]

\[
+ \int_{Q_T} Z : Dv \, dx \, dt = \int_0^T (b, v) \, dt + \frac{1}{2} \| v_0 \|^2_2. \tag{5.30}
\]

Comparing (5.29) with (5.30), we conclude that

\[
\limsup_{m \to +\infty} \int_{Q_T} \tilde{z}^m : Dv^m \, dx \, dt \le \int_{Q_T} Z : Dv \, dx \, dt. \tag{5.31}
\]
Finally, it follows from (5.2) that
\[ 0 \leq \int_{Q_T} (Z_n(p_f^n, \nabla v^n) - Z_n(p_f^n, \nabla)) : (\nabla v^n - \nabla) \, dx \, dt \quad \text{for all } \nabla \in L^2(Q_T). \] (5.32)

Since, by (5.25),
\[ Z_n(p_f^n, \nabla) := (p_f^n - p_s)^+ \frac{\nabla}{|\nabla| + \frac{1}{n}} \to (p_f - p_s)^+ \frac{\nabla}{|\nabla| + \frac{1}{n}} =: Z_n(p_f, \nabla) \text{ strongly in } L^2(Q_T) \]
and
\[ \nabla v^n \rightharpoonup \nabla v \text{ weakly in } L^2(Q_T), \]
we conclude from (5.32) and (5.31) that
\[ 0 \leq \int_{Q_T} (Z - Z_n(p_f, \nabla)) : (\nabla v - \nabla) \, dx \, dt \quad \text{for all } \nabla \in L^2(Q_T). \] (5.33)

The choice \( A = \nabla v \pm \lambda B \) for \( B \in L^2(Q_T) \) arbitrary and \( \lambda > 0 \), leads to
\[ 0 \leq \pm \int_{Q_T} (Z - Z_n(p_f, \nabla v \pm \lambda B)) : B \, dx \, dt \quad \text{for all } B \in L^2(Q_T). \]

Letting \( \lambda \to 0^+ \), we obtain
\[ 0 = \int_{Q_T} (Z - Z_n(p_f, \nabla)) : B \, dx \, dt \quad \text{for all } B \in L^2(Q_T), \]
which implies \( Z = Z_n(p_f, \nabla) \) a.e. in \( Q_T \).

The proof of Proposition 5.1 is complete.

**Proposition 5.2.** Let all the assumptions in Proposition 5.1 be satisfied. In addition, assume that one of the following requirements holds:
\[ p_s(0) \in L^\infty(\Omega) \quad \text{and} \quad g, \partial_t p_s - \Delta p_s \in L^q(Q_T) \quad \text{with } q > \frac{5}{2}, \] (5.34)
\[ p_s \in L^q(0,T;W^{1,q}(\Omega)) \quad \text{with } q > 10 \quad \text{and} \quad g \in L^q(Q_T) \quad \text{with } q > \frac{5}{2}, \] (5.35)
then, for each \( n \in \mathbb{N} \), there exists a weak solution to the problem (5.1) in the sense of Proposition 5.1 satisfying \( p^n_f \in L^\infty(Q_T) \). In fact,
\[ \sup_n \| p^n_f \|_{L^\infty(Q_T)} < +\infty. \]

Consequently, \( \partial_t p^n_f \in L^2(0,T;W^{-1,2}(\Omega)) \) and (5.7) holds for all \( z \in L^2(0,T;W^{1,2}(\Omega)) \).

**Proof.** In what follows we shall prove explicitly that \( p_f \in L^\infty(Q_T) \) using the Moser iteration technique. By the interpolation inequality (5.18), it follows that
\[ \int_0^T \| z \|_{L^\infty}^{\frac{10}{3}} \leq C \left( \sup_{t \in [0,T]} \| z \|_{L^2} \right)^{\frac{4}{3}} \int_0^T \| z \|_{L^{1,2}}^2. \] (5.36)
Consequently,
\[ p_f \in L^{\frac{10}{3}}(Q_T) \quad \text{and} \quad v \in L^{\frac{10}{3}}(Q_T)^3. \] (5.37)
Let us first consider the case given by (5.34). Then, once we set \( h := -(\partial_t p_s - \Delta p_s) \), \( G := g + h \) and \( P := p_t - p_s \), we can rewrite the third equation in (5.1) as

\[
\partial_t P + v \cdot \nabla P - \Delta P = G \quad \text{with} \quad G \in L^q(Q_T) \quad \text{and} \quad q > \frac{5}{2}.
\] (5.38)

For \( s > 2 \) and \( m \in \mathbb{N} \) consider \( |P_m|^{s-2}P_m \) with \( P_m := T_m(P) \) as test function in the weak formulation of (5.38). Here \( T_m : \mathbb{R} \rightarrow \mathbb{R} \) is defined through \( T_m(z) = z \) if \( |z| \leq m \) and \( T_m(z) = \text{m} \text{sign} z \) if \( |z| > m \). Note that \( |P_m|^{s-2}P_m \) is an admissible test function.

After integrating by parts and employing \( 4 \), and (5.42) then leads to

\[
\frac{1}{s} \frac{d}{dt} \|P_m\|^s_s + (s-1) \int_\Omega |\nabla P_m|^2|P_m|^{s-2} \leq \int_\Omega |G||P_m|^{s-1}. \quad \text{(5.39)}
\]

Next, integrating with respect to the time, straightforward computations imply

\[
\|P_m(t)\|^s_s \leq \frac{4s(s-1)}{s^2} \int_0^t \|\nabla P_m\|^2 \, dt \leq s \int_{Q_T} |G||P_m|^{s-1} + \|P(0)\|^s_s =: A. \quad \text{(5.40)}
\]

Since \( \frac{4s(s-1)}{s^2} > 1 \), it follows from (5.40) that

\[
\sup_{t \in [0,T]} \|P_m(t)\|^s_s \leq A^{\frac{1}{s}}, \quad \int_0^T \|\nabla P_m(t)\|^2 \, dt \leq A. \quad \text{(5.41)}
\]

Using (5.18) and (5.36) with \( z = |P_m|^\frac{1}{s} \), and combining the result with (5.41), we obtain

\[
\int_0^T \|P_m(t)\|^\frac{2q}{q-1} \, dt = \int_0^T \|P_m(t)\|^\frac{2q}{q-1} \, dt \leq C \left( \sup_{t \in [0,T]} \|P_m(t)\|^\frac{1}{s} \right)^\frac{q}{q-1} \int_0^T \|\nabla P_m(t)\|^\frac{q}{q-1} \, dt \leq CA^\frac{q}{q-1}. \quad \text{(5.42)}
\]

The definition of \( A \) and (5.42) then leads to

\[
\|P_m\|_{\frac{2q}{q-1},Q_T} \leq s^\frac{1}{2} C \|G\|^\frac{1}{q,Q_T} \|P_m\|_{q,Q_T}^{\frac{q-1}{q-1},Q_T} + C\frac{q}{q-1} \|P(0)\|_\infty. \quad \text{(5.43)}
\]

We can introduce the following iteration scheme. Setting

\[
s_0 := \frac{10}{3} \frac{q}{q-1}, \quad (\bar{s}_i - 1) := s_i \quad \text{and} \quad s_{i+1} := \frac{5}{3} \bar{s}_i, \quad \text{(5.44)}
\]

which leads to \( \bar{s}_i = \frac{q-1}{q} s_i + 1 \) and hence

\[
s_{i+1} = \frac{5}{3} \frac{q - 1}{q} s_i + \frac{5}{3} \bar{s}_i + 1, \quad \bar{s}_{i+1} = \frac{5}{3} \frac{q - 1}{q} s_i + 1,
\]

we obtain

\[
\|P_m\|_{s_{i+1},Q_T} \leq s_i^\frac{1}{2} C \|G\|^\frac{1}{q,Q_T} \|P_m\|_{s_i,Q_T}^{\frac{q-1}{q-1},Q_T} + C\frac{q}{q-1} \|P(0)\|_\infty.
\]

Noticing that

\[
\frac{5}{3} \frac{q - 1}{q} > 1 \iff q > \frac{5}{2},
\]

we can rewrite the third equation in (5.1) as
we observe that \( s_i \to +\infty \) as \( i \to +\infty \). By iteration, we get
\[
\|P_m\|_{s_i+1, Q_T} \leq C \sum_{j=0}^{i} \frac{s_{j+1}}{s_j} \prod_{k=j+1}^{i} \left( \frac{s_{k+1}}{s_k} \right) \|G\|_{q, Q_T} \max \left\{ 1, \|P\|_{s_0} \right\}
\]
\[
\|P_0\|_{0, Q_T} \leq C_1 \quad \text{where} \quad C_1 := \max \{1, \|G\|_{q, Q_T}\},
\]
\[
\|P(0)\|_{\infty} \leq C_2 \quad \text{where} \quad C_2 := \max \{1, \|P(0)\|_{\infty}\}.
\]
Since \( \frac{s_{k+1}}{s_k} \leq 1 \), we notice that all products can be bounded by 1. Consequently, (5.46) leads to (assuming that \( C \geq 1 \))
\[
\|P_m\|_{s_i+1, Q_T} \leq C \sum_{j=0}^{i} \frac{s_{j+1}}{s_j} \prod_{k=j+1}^{i} \left( \frac{s_{k+1}}{s_k} \right) C_1 \max \left\{ 1, \|P_m\|_{s_0} \right\}
\]
\[
\|P_0\|_{0, Q_T} \leq C_1 \quad \text{where} \quad C_1 := \max \{1, \|G\|_{q, Q_T}\},
\]
\[
\|P(0)\|_{\infty} \leq C_2 \quad \text{where} \quad C_2 := \max \{1, \|P(0)\|_{\infty}\}.
\]
Note that the right-hand side is independent of \( m \) as well as \( n \). Taking the limit as \( i \to +\infty \), since \( s_{i+1} \to +\infty \), by the convergence of the sums due to the d’Alambert criterion, we conclude that
\[
P_m \in L^{\infty}(Q_T) \quad \text{for all} \quad m \in \mathbb{N} \quad \Rightarrow \quad P \in L^{\infty}(Q_T).
\]
From the relation \( p_t = P + p_s \) and since \( p_s \in L^{\infty}(Q_T) \), it finally follows that
\[
p_t \in L^{\infty}(Q_T).
\]
On the other hand, assuming (5.35) we first observe that if \( p_s \in L^q(0,T;W^{1,q}(\Omega)) \) with \( q > 10 \) and \( v \in L^{10}(Q_T) \), then \( v \cdot \nabla p_s \in L^1(Q_T) \) with \( \ell > \frac{5}{2} \). Consequently, (5.35) implies that \( g + v \cdot \nabla p_s \in L^1(Q_T) \) with \( \ell > \frac{5}{2} \). Then, we conclude exactly as in the case given by (5.34) that
\[
p_t \in L^{\infty}(Q_T).
\]

The following lemma regards the attainment of the constitutive equations.

**Proposition 5.3** (Convergence Lemma). Let \( U \subset Q_T \) be an arbitrary measurable bounded
set and let \( \{Z^n\}_{n=1}^{\infty}, \{D^n\}_{n=1}^{\infty} \) and \( \{p^n_i\}_{i=1}^{\infty} \) be such that
\[
Z^n = \tau(p^n_i) \frac{D^n}{|D^n| + \frac{1}{n}} \text{ with } \tau(p^n_i) = q_\ast(p^n_i - p_\ast)^+, \tag{5.48}
\]
\[
\sup_{n \in \mathbb{N}} \|p^n_i\|_\infty \leq C < \infty, \tag{5.49}
\]
\[
Z^n \to Z \text{ weakly in } L^2(U)^{3 \times 3}, \tag{5.50}
\]
\[
D^n \to D \text{ weakly in } L^2(U)^{3 \times 3}, \tag{5.51}
\]
\[
p^n_i \to p_i \text{ strongly in } L^2(U) \text{ and a.e. in } U, \tag{5.52}
\]
\[
\limsup_{n \to \infty} \int_U Z^n : D^n \leq \int_U Z : D, \tag{5.53}
\]
then, setting \( S = Z + D, \)
\[
D = \left( \frac{|S|}{|S|} - \tau(p_i) \right)^+ \text{ a.e. in } U. \tag{5.54}
\]

**Proof.** We split the proof into three steps. Using the fact that (5.54) is, by Proposition 3.1, equivalent to (5.8) (with \( z_\ast = q_\ast = 1 \)), we first show that \( |Z^n| \leq \tau(p^n_i) \) and thus \( |Z^n| \leq \tau(p^n_i) \). For any subset \( \omega \subset U \) it holds
\[
\int_\omega |Z^n| \leq \int_\omega \tau(p^n_i). \tag{5.55}
\]
Since \( \tau(\cdot) \) is Lipschitz, (5.52) implies that
\[
\tau(p^n_i) \to \tau(p_i) \text{ strongly in } L^2(U) \tag{5.56}
\]
and
\[
\tau(p^n_i) \to \tau(p_i) \text{ a.e. in } U. \tag{5.57}
\]
By virtue of (5.55), (5.57) and the lower semicontinuity of \( \int_\omega |Z^n| \) with respect to the weak convergence in \( L^1(\omega) \) (which follows from (5.50) since \( U \) is bounded), we get
\[
|Z|_{L^1(\omega)} \leq \|\tau(p_i)\|_{L^1(\omega)} \text{ for all } \omega \subset U. \tag{5.58}
\]
Lebesgue’s Differentiation Theorem then implies
\[
|Z| \leq \tau(p_i) \text{ a.e. in } U. \tag{5.59}
\]

**Step 2.** In order to establish that
\[
Z^n : D^n \to Z : D \text{ weakly in } L^1(U) \tag{5.60}
\]
we set
\[
\hat{Z}^n := \tau(p^n) \frac{D}{|D| + \frac{1}{n}}, \tag{5.61}
\]
and
\[
\hat{Z} := \begin{cases} 
\tau(p_i) \frac{D}{|D|} & \text{if } D \neq \emptyset, \\
\emptyset & \text{otherwise}. 
\end{cases} \tag{5.62}
\]
Thanks to (5.57) we have that \( \hat{Z}^n \rightarrow \hat{Z} \) almost everywhere in \( Q_T \), and since \( \hat{Z}^n \) is essentially bounded (because of (5.70)), Lebesgue’s Convergence Theorem yields
\[
\hat{Z}^n \rightarrow \hat{Z} \quad \text{strongly in } L^2(U). \tag{5.63}
\]
Employing (5.50) and (5.53) and the convergences (5.63) and (5.51), we get
\[
\limsup_{n \to \infty} \int_U (Z^n - \hat{Z}^n) : (D^n - D) \leq 0. \tag{5.64}
\]
But since (5.61), the monotone property (5.2) yields
\[
(Z^n - \hat{Z}^n) : (D^n - D) \geq 0 \quad \text{a.e. in } U. \tag{5.64}
\]
This together with (5.64) implies that
\[
(Z^n - \hat{Z}^n) : (D^n - D) \rightarrow 0 \quad \text{strongly in } L^1(U),
\]
and thus surely
\[
(Z^n - \hat{Z}^n) : (D^n - D) \rightarrow 0 \quad \text{weakly in } L^1(U). \tag{5.65}
\]
Since the strong convergence (5.63) and weak convergence (5.51) imply that
\[
\hat{Z}^n : (D^n - D) \rightarrow 0 \quad \text{weakly in } L^1(U), \tag{5.66}
\]
so (5.65) yields
\[
Z^n : (D^n - D) \rightarrow 0 \quad \text{weakly in } L^1(U). \tag{5.67}
\]
Finally employing (5.50) in (5.67) we conclude
\[
Z^n : D^n \rightarrow Z : D \quad \text{weakly in } L^1(U). \tag{5.68}
\]

**Step 3.** It remains to show \( Z : D \geq \tau(p_t)|D| \). First we note that
\[
|\tau(p^n_t)|D^n| - Z^n : D^n| = \tau(p^n_t) \frac{1}{n} \left\| D^n \right\| + \frac{1}{n}. \tag{5.69}
\]
Since \( \tau \) is a Lipschitz function, (5.49) gives
\[
\|\tau(p^n_t)\|_{\infty} \leq C \quad \text{uniformly in } n. \tag{5.70}
\]
Then the right hand side in (5.69) is essentially bounded by \( \frac{C}{n} \) and thus
\[
|\tau(p^n_t)|D^n| - Z^n : D^n| \rightarrow 0 \quad \text{in } L^\infty(U), \tag{5.71}
\]
which implies that
\[
\lim_{n \to +\infty} \int_U \varphi(\tau(p^n_t)|D^n| - Z^n : D^n) = 0 \quad \text{for all } \varphi \in L^\infty(U). \tag{5.72}
\]
Moreover, from (5.51) and (5.56) we get
\[
\varphi \tau(p^n_t)|D^n| \rightarrow \varphi \tau(p_t)|D| \quad \text{in } L^1(U) \text{ for all } \varphi \in L^1(U) \tag{5.73}
\]
and the weak lower semicontinuity of the \( L^1 \)-norm implies that, for all \( \varphi \in L^\infty(U) \) such that \( \varphi \geq 0, \)
\[
\int_U \varphi \tau(p_t)|D| \leq \liminf_{n \to +\infty} \int_U \varphi \tau(p^n_t)|D^n| \, dx. \tag{5.74}
\]
Using (5.60) together with (5.72), (5.74) and (5.60), we obtain

\[
\int_U \varphi(p(t))|D| - Z : D \leq \liminf_{n \to +\infty} \int_U \varphi(p^n(t))|D^n| - Z^n : D^n = 0
\]

(5.75)

for any non-negative \(\varphi \in L^\infty(U)\). Hence

\[
Z : D \geq \tau(p_f)|D| a.e. in U,
\]

which is (3.8). □

6 Proof of the Main Theorem

The proof is split in the following steps.

Step 1. Approximations. From Proposition 5.1 and Proposition 5.2 we get, for each \(n \in \mathbb{N}\), the existence of \((u^n, p^n, z^n, s^n)\) satisfying

\[
\int_0^T \langle \partial_t u^n, w \rangle dt + \int_{Q_T} G_n(|v^n|^2) v^n \otimes v^n \cdot \nabla w dxdt + \int_{Q_T} (Dv^n + Z^n) : Dw dxdt
\]

\[+ \int_{\Sigma_T} (v^n + z^n) \cdot w_n \Gamma dt - \int_0^T \langle b, w \rangle dt = 0 \text{ for all } w \in L^2(0,T;W^{1,2}_{n,div}),
\]

(6.1)

\[
\int_0^T \langle \partial_t p^n_t, z \rangle dt + \int_{Q_T} \nabla p^n_t \cdot v^n z + \nabla p^n \cdot \nabla z dx dt = \int_{Q_T} g z - v^n \nabla z p_s dxdt
\]

(6.2)

for all \(z \in L^2(0,T;W^{1,2}(\Omega))\),

and

\[
Z^n = (p^n - p_s)^+ \frac{Dp^n}{|Dv^n| + \frac{1}{n}} \text{ a.e. in } Q_T \text{ and } z^n = s_* \frac{v^n}{|v^n| + \frac{1}{n}} \text{ a.e. in } \Sigma_T.
\]

(6.3)

Step 2. Reconstruction of the pressure. We set

\[
p^n := (-\Delta_N)^{-1} \text{div } h^n \text{ with } \int_{\Omega} p^n(t,\cdot) = 0,
\]

(6.4)

where \(-\Delta_N\) denotes the Laplace operator associated with the homogeneous Neumann boundary conditions and

\[
h^n := \text{div} (Dv^n + Z^n) + \text{div}(v^n \otimes v^n)G_n(|v^n|) - b,
\]

(6.5)

associated with the boundary conditions \(v \cdot n = 0\) and \(z^n = s_* \frac{v^n}{|v^n| + \frac{1}{n}}\) on \(\Sigma_T\). It means that \(p^n\) solves (for a.a. \(t \in [0,T]\))

\[
(p^n, \Delta \varphi) = \langle h^n, \nabla \varphi \rangle \text{ for all } \varphi \in W^{2,2}(\Omega) \text{ with } \nabla \varphi \cdot n = 0 \text{ on } \partial \Omega,
\]

(6.6)

whereas

\[
h^n \in L^2(0,T;W^{1,2}_{n,div}).
\]

(6.7)
Step 3. Uniform estimates with respect to (5.16) and (5.20) using also Korn’s inequality, we obtain

we observe that due to (6.1) and (6.6) we get

\[ \langle \partial_t v_n, \tilde{w} \rangle + (p^n, \Delta \varphi) = -\langle \partial_t v_n, \tilde{w} + \nabla \varphi \rangle + (p^n, \text{div} (\tilde{w} + \nabla \varphi)) . \]

Hence

\[ \langle \partial_t v_n, w \rangle + (Dv_n, Dw) + (Z^n, Dw) + (v_n, w_t)_{\partial \Omega} + (z^n, w_t)_{\partial \Omega} \]

\[ + \text{div}(v_n \otimes v_n) G_n(|v_n|^2), w \rangle = (p^n, \text{div} w) + \langle b, w \rangle \]

for all \( w \in W_n^{1,2} \).

### Step 3. Uniform estimates with respect to \( n \) and limit as \( n \to +\infty \).

Taking \( v^n \) as test function in the (6.1) and \( p^n_i \) in the (6.2), and proceeding similarly as in the derivation of (6.16) and (5.20), using also Korn’s inequality, we obtain

\[
\sup_n \left( \| v^n \|_{L^\infty(Q_T)} + \| Dv^n \|_{L^2(Q_T)} + \| \nabla v^n \|_{L^2(Q_T)} + \| p^n_i \|_{L^2(\Sigma_T)} \right) < +\infty, \tag{6.10}
\]

\[
\sup_n \left( \| p^n_i \|_{L^\infty(0,T;L^2(\Omega))} + \| \nabla p^n_i \|_{L^2(Q_T)} \right) < +\infty, \tag{6.11}
\]

\[
\sup_n \left( \| v^n \|_{W^{1,2}_0, Q_T} + \| v^n \|_{\Sigma_T} \right) < +\infty. \tag{6.12}
\]

It follows directly from the proof of Proposition 5.2 that

\[
\sup_n \| p^n_i \|_{\infty, Q_T} < +\infty. \tag{6.13}
\]

It then follows from (6.3) that

\[
\sup_n \left( \| Z^n \|_{L^\infty(0,T;L^\infty(\Omega))} \right) < +\infty. \tag{6.14}
\]

Since

\[
G_n(|v^n|^2) \text{div}(v^n \otimes v^n) = \sum_{h=1}^{3} v^n_h \frac{\partial v^n}{\partial x_h} G_n(|v^n|^2),
\]

and \( \sup_n \| G_n(|v^n|^2) \|_{L^\infty(Q_T)} \leq 1 \), it follows from (6.10), (6.13) and Hölder’s inequality that

\[
\sup_n \left( \| G_n(|v^n|^2) \text{div}(v^n \otimes v^n) \|_{L^\frac{4}{3}(Q_T)} \right) < +\infty. \tag{6.15}
\]

For further analysis it is suitable to perform the following decomposition of the pressure \( p^n \). Setting

\[
h_2^n := G_n(|v^n|^2) \text{div}(v^n \otimes v^n)
\]

and

\[
p_2^n := (-\Delta) \text{div} h_2^n,
\]

we conclude from (6.15) that

\[
\sup_n \| \nabla p_2^n \|_{L^2(Q_T)} < +\infty. \tag{6.16}
\]
Furthermore, \( h^n := h - h^n \) fulfills \( \sup_n \| h^n \|_{L^2(0,T;W^{-1,2})} < +\infty \), consequently \( p^n := p^n - p^n \) satisfies
\[
\sup_n \| p^n \|_{L^2(Q_T)} < +\infty,
\]
and it follows from (6.9) that
\[
(\partial_t v^n, w) = \int_{\Omega} (Z^n - \nabla v^n + p^n) : \nabla w \, dx + (b, w) - \int_{\partial \Omega} (v^n + z^n) \cdot w \, d\sigma_x - \int_{\Omega} G_n(|v^n|^2) \, \text{div}(v^n \otimes v^n) + \nabla p^n : w \, dx \text{ for all } w \in L^2(0,T;W^{1,2}) \cap L^\infty(Q_T)^3.
\]
The above uniform estimates then imply that
\[
\sup_n \| \partial_t v^n \|_{L^2(0,T;W^{1,2}(\Omega) \cap L^\infty(Q_T)^3)} < +\infty,
\]
and similarly
\[
\sup_n \| \partial_t p^n \|_{L^2(0,T;W^{1,2})} < +\infty.
\]
Due to uniform estimates (6.10), (6.13), (6.14), (6.15), (6.16), (6.17), (6.19), (6.20), the Aubin-Lions compactness lemma and the compact embedding of the Sobolev spaces into the space of traces, we get the following convergences for subsequences that we do not relabel:
\[
v^n \rightarrow v \text{ weakly in } L^2(0,T;W^{1,2}),
\]
\[
p^n \rightarrow p_t \text{ weakly in } L^2(0,T;W^{1,2}),
\]
\[
p^n \rightarrow p_t \text{ strongly in } L^q(Q_T) \text{ for all } q \in \left[1, \frac{10}{3}\right),
\]
\[
\partial_t p^n \rightarrow \partial_t p_t \text{ weakly in } (L^2(0,T;W^{1,2}))^*,
\]
\[
p^n \rightarrow p_t \text{ weakly}^* \text{ in } L^\infty(Q_T),
\]
\[
Z^n \rightarrow Z \text{ weakly}^* \text{ in } L^\infty(Q_T)^{3 \times 3},
\]
\[
z^n \rightarrow z \text{ weakly}^* \text{ in } L^\infty(0,T;L^\infty(\partial \Omega)^3),
\]
\[
v^n \rightarrow v \text{ a.e. in } Q_T \text{ and strongly in } L^q(Q_T)^3 \text{ for all } q \in \left[1, \frac{10}{3}\right),
\]
\[
v^n \rightarrow v_t \text{ a.e. in } \Sigma_T \text{ and strongly in } L^q(0,T;L^q(\partial \Omega)^3) \text{ for all } q \in \left[1, \frac{8}{3}\right),
\]
\[
p^n \rightarrow p_t \text{ weakly in } L^2(Q_T),
\]
\[
p^n \rightarrow p_2 \text{ weakly in } L^{\frac{4}{3}}(0,T;W^{1,\frac{4}{3}}(\Omega)),
\]
\[
\partial_t v^n \rightarrow \partial_t v \text{ weakly in } (L^2(0,T;W^{1,2}_n(\Omega))) \cap L^\infty(Q_T)^3, \quad G_n(|v^n|^2) \div(v^n \otimes v^n) \rightarrow g \text{ weakly in } L^{\frac{8}{3}}(0,T;W^{1,\frac{4}{3}}(\Omega)).
\]

It is not difficult to observe that due to the fact that
\[
\| G_n(|v^n|^2) \|_{L^\infty(Q_T)} \leq 1 \text{ and } G_n(|v^n|^2) \rightarrow 1 \text{ strongly in } L^q(Q_T) \text{ for all } q \in [1, +\infty),
\]
and due to (6.21) and (6.28) we have
\[
g = \text{div}(v \otimes v).
Integrating then \(6.9\) with respect the time between 0 and \(T\) and taking the limit as \(n \to \infty\) we get

\[
\hat{T} \langle \partial_t v, w \rangle dt + \int_{Q_T} \mathbb{D}v : \mathbb{D}w \, dx dt + \int_{Q_T} \mathcal{Z} : \mathbb{D}w \, dx dt - \int_{Q_T} \langle b, w \rangle dt + \int_{\Sigma_T} (v_r + z) \cdot w_r \, d\sigma dt - \int_{Q_T} p_1 \mathbf{div} \, w \, dx dt + \int_{Q_T} \mathcal{Q}_T \mathbf{div} (v \otimes v) \cdot w \, dx dt + \int_{Q_T} \mathcal{Q}_T Dv : Dw \, dx dt + \int_{Q_T} \mathcal{Q}_T Z : Dw \, dx dt - \int_{Q_T} \langle b, w \rangle dt + \int_{Q_T} \mathcal{Q}_T p_1 \mathbf{div} \, w \, dx dt = 0
\]

for all \(w \in L^2(0, T; W^{1,2}_{n,\mathbf{div}}(\Omega)) \cap L^5(Q_T)^3\).

Integrating \(6.2\) with respect the time between 0 and \(T\) and taking the limit as \(n \to \infty\) we get

\[
\hat{T} \langle \partial_t p, z \rangle dt + \int_{Q_T} p_1 \mathbf{div} \, v \cdot \nabla z + \nabla p_1 \cdot \nabla z \, dx dt = \int_{Q_T} g z \, dx dt + \int_{Q_T} p_2 \mathbf{div} \, v \cdot \nabla z \, dx dt \quad \text{for all } z \in L^2(0, T; W^{1,2}(\Omega)).
\]

**Step 4.** Attainment of the constitutive equation on the boundary. Since the structure of the constitutive equation on the boundary \(6.3\) is simpler than that used in Proposition 5.3, we can apply this assertion to this case as well. Indeed, we know that not only

\[
z^n = s_* \frac{v^n_r}{|v^n_r|} + \frac{1}{n} \quad \text{a.e. on } \Sigma_T,
\]

\[
z^n \rightharpoonup z \quad \text{weakly in } L^q(\Sigma_T) \quad \text{for all } q \in [1, +\infty),
\]

\[
v^n_r \rightharpoonup v_r \quad \text{weakly in } L^s(\Sigma_T),
\]

but also

\[
v^n_r \rightarrow v_r \quad \text{strongly in } L^q(\Sigma_T) \quad \text{for all } q \in \left[1, \frac{8}{3}\right).
\]

Consequently,

\[
\lim_{n \to +\infty} \int_{\Sigma_T} z^n \cdot v^n_r \, d\sigma_x \, dt = \int_{\Sigma_T} z \cdot v_r \, d\sigma_x \, dt,
\]

and by Proposition 5.3 we get for \(s = z + v_r\)

\[
v_r = \frac{|s| - s_*}{|s|} s \quad \text{a.e. in } \Sigma_T.
\]

**Step 5.** Attainment of the constitutive equation in the bulk. We wish to use Proposition 5.3 and we can notice that all of its assumptions \(5.48\)–\(5.52\) are all fulfilled except \(5.53\). To prove it, we have to overcome the difficulty that \(v\) is not an admissible test function in \(6.34\). This is why we employ the so-called \(L^\infty\)-truncation method applied to \(v^n - v\). Let \(\{\lambda^n\}, A, B\) such that \(0 < A \leq \lambda^n \leq B < \infty\), where \(A, B\) are independent of \(n\) (but sufficiently large) and together with \(\lambda^n\) will be specified later. Consider the truncated velocity difference

\[
w^n := T_{\lambda^n} (v^n - v) := (v^n - v) \min \left\{1, \frac{\lambda^n}{|v^n - v|}\right\},
\]

\[
\hat{T} \|
w^n\|_{L^\infty(\Omega_T)} \leq B, \quad \text{and } w^n \rightarrow 0 \quad \text{a.e. in } Q_T, \quad \text{Lebesgue’s Theorem implies that}
\]

\[
w^n \to 0 \quad \text{strongly in } L^s(Q_T)^3 \quad \text{for every } s \geq 1,
\]
and similarly, as \( \sup_n \| w^n \|_{\infty, Q_T} \leq B \), and \( (6.29) \).

\[ w^n_T \rightarrow 0 \text{ strongly in } L^2(0, T; L^2(\partial \Omega)^3). \] (6.38)

Since

\[
\nabla w^n = \begin{cases} 
\nabla v^n - \nabla v & \text{if } |v^n - v| < \lambda_n, \\
\frac{\lambda_n}{|v^n - v|}(\nabla v^n - \nabla v) - \lambda_n(v^n - v) \odot \frac{(\nabla v^n - \nabla v)(v^n - v)}{|v^n - v|^3} & \text{otherwise,}
\end{cases}
\] (6.39)

we observe that

\[
|\text{div } w^n| \leq \begin{cases} 
0 & \text{if } |v^n - v| < \lambda_n, \\
\frac{2\lambda_n(|\nabla v^n| + |\nabla v|)}{|v^n - v|} & \text{otherwise}
\end{cases}
\] (6.40)

and

\[
|\nabla w^n| \leq 2|\nabla v^n - \nabla v|.
\] (6.41)

Then, due to \( (6.10) \) and \( (6.41) \), \( \nabla w^n \) is uniformly bounded in \( L^2(Q_T)^{3 \times 3} \) and, up to a subsequence, it converges weakly in \( L^2(Q_T)^{3 \times 3} \). But employing \( (6.37) \) it follows that the weak limit has to be zero, i.e.

\[
\nabla w^n \rightharpoonup 0 \text{ weakly in } L^2(Q_T)^{3 \times 3} \text{ and } D w^n \rightharpoonup 0 \text{ weakly in } L^2(Q_T)^{3 \times 3}.
\] (6.42)

Inserting \( w^n \) in \( (6.18) \), we get

\[
\limsup_{n \to \infty} \int_{Q_T} Z^n : \nabla w^n - p^n_1 \text{div } w^n + \mathbb{D}v^n : \mathbb{D}w^n \, dx \, dt
\]

\[
= \limsup_{n \to \infty} \left[ -\int_0^T \langle \partial_t v^n, w^n \rangle \, dt - \int_{Q_T} G(|v^n|^2) \text{div} (v^n \otimes v^n) \cdot w^n \, dx \, dt \\
+ \int_{\Sigma_T} (w^n + z^n) \cdot w^n_\sigma \, d\sigma \, dt - \int_{Q_T} \nabla p^n_2 \cdot w^n \, dx \, dt - \int_0^T \langle b, w^n \rangle \, dt \right].
\] (6.43)

Now, by virtue of \( (6.37) \), \( (6.33) \) and \( (6.34) \), we observe that

\[
\lim_{n \to \infty} \int_{Q_T} G_n(|v^n|^2) \text{div}(v^n \otimes v^n) + \nabla p^n_2 \cdot w^n \, dx \, dt + \int_0^T \langle b, w^n \rangle \, dt = 0,
\] (6.44)

and by virtue of \( (6.10) \) and \( (6.38) \), it holds

\[
\lim_{n \to \infty} \int_{\Sigma_T} (w^n + z^n) \cdot w^n_\sigma \, d\sigma \, dt = 0.
\] (6.45)

Since \( w^n \rightharpoonup 0 \) weakly in \( L^2(0, T; W^{1,2}(\Omega)^3) \cap L^2(Q_T)^3 \) by \( (6.37) \) then

\[
\lim_{n \to +\infty} \int_0^T \langle \partial_t v^n, w^n \rangle \, dt = 0,
\]

thus

\[
\liminf_{n \to +\infty} \int_0^T \langle \partial_t v^n, w^n \rangle \, dt = \liminf_{n \to +\infty} \int_0^T \langle \partial_t (v^n - v), w^n \rangle \, dt.
\] (6.46)
Moreover,
\[
\int_0^T \langle \partial_t (v^n - v), w^n \rangle \ dt = \int_0^T \partial_t \left( \frac{|v^n - v|^2}{2} \right) \min \left\{ 1, \frac{\lambda^n}{|v^n - v|} \right\} \ dx \ dt \\
= \int_{Q_T} \partial_t F^n(x,t) \ dx,
\]
(6.47)
where
\[
F^n(x,t) := \begin{cases} 
\frac{|v^n - v|^2}{2} & \text{if } |v^n - v| \leq \lambda^n, \\
\lambda^n |v^n - v| - \frac{\lambda^n}{2} & \text{if } |v^n - v| > \lambda^n.
\end{cases}
\]
Thus from (6.47) and since \( F^n(x,0) = 0 \) we get
\[
\int_0^T \langle \partial_t (v^n - v), w^n \rangle \ dt = \int_{\Omega} F^n(x,T) \ dx,
\]
(6.48)
taking the liminf we finally arrive at
\[
\lim \inf_{n \to +\infty} \int_0^T \langle \partial_t (v^n - v), w^n \rangle \ dt = \lim \inf_{n \to +\infty} \int_{\Omega} F^n(x,T) \ dx \geq 0,
\]
(6.49)
but this is equivalent to
\[
\lim \sup_{n \to \infty} \left[ - \int_0^T \langle \partial_t v^n, w^n \rangle \ dt \right] \leq 0.
\]
(6.50)
Collecting (6.44), (6.45), (6.50), it follows from (6.43) that
\[
\lim \sup_{n \to \infty} \int_{Q_T} Z^n : D w^n - (p^n_1 \div w^n) + D v^n : D w^n \ dx \ dt \leq 0.
\]
(6.51)
Since \((D v^n - D v) : D w^n \geq 0\) and \(\lim_{n \to +\infty} \int_{Q_T} D v : D w^n \ dx \ dt = 0\), (6.40) and (6.51) imply that
\[
\lim \sup_{n \to \infty} \int_{Q_T} Z^n : D w^n + D v^n : D w^n \ dx \ dt \\
\leq \lim \sup_{n \to \infty} \int_{Q_T} |p^n_1| |\div w^n| \ dx \ dt \\
\leq \lim \sup_{n \to \infty} \int_{\{|w^n - v| \geq \lambda^n\}} \frac{\lambda^n}{|v^n - v|} |p^n_1| (|\nabla u^n| + |\nabla v|).
\]
(6.52)
Let \(Z \in L^{3\times 3}_{\infty}(Q_T)\) be such that
\[
Z = \begin{cases} 
\partial & \text{if } D v = \emptyset, \\
\tau(p) & \text{if } D v \neq \emptyset.
\end{cases}
\]
(6.53)
Since \(\lim_{n \to +\infty} \int_{Q_T} Z : D w^n \ dx \ dt = 0\) thanks to (6.42), we arrive at
\[
\lim \sup_{n \to \infty} \int_{Q_T} (Z^n - Z) : D w^n \ dx \ dt \leq \lim \sup_{n \to \infty} \int_{\{|w^n - v| \geq \lambda^n\}} \frac{\lambda^n}{|v^n - v|} |p^n_1| (|\nabla u^n| + |\nabla v|).
\]
(6.54)
Splitting the integral on the left-hand side of (6.54) into two parts, one integrated over \{\|\mathbf{v}^n - \mathbf{v}\| \leq \lambda^n\} the other over \{\|\mathbf{v}^n - \mathbf{v}\| > \lambda^n\}, using (6.39), and moving the latter to the right-hand side and estimating it by (6.41), we get

\[
\limsup_{n \to \infty} \int_{\{\|\mathbf{v}^n - \mathbf{v}\| \leq \lambda^n\}} (Z^n - \mathcal{Z}) : \mathcal{D}(\mathbf{v}^n - \mathbf{v}) \leq C \limsup_{n \to \infty} \int_{\{\|\mathbf{v}^n - \mathbf{v}\| > \lambda^n\}} \frac{\lambda^n}{\|\mathbf{v}^n - \mathbf{v}\|} I^n, \quad (6.55)
\]

where

\[
I^n := (|p|^2 + |\nabla \mathbf{v}|^2 + |\nabla \mathbf{v}|^2 + |Z^n|^2 + |\mathcal{Z}|^2) \quad \text{and} \quad \sup_n \int_Q I^n \, dx \, dt < +\infty.
\]

Let \(N \in \mathbb{N}\) be arbitrary. We fix \(A = N\) and \(B = N^{N+1}\) and define

\[
Q^n_i := \{(t, x) \in Q_T; N^i \leq \|\mathbf{v}^n - \mathbf{v}\| \leq N^{i+1}\}, \quad i = 1, \ldots, N.
\]

Since

\[
\sum_{i=1}^N \int_{Q^n_i} I^n \leq C_* \quad (6.56)
\]

there is, for each \(n \in \mathbb{N}\), an index \(i_n \in \{1, \ldots, N\}\) such that

\[
\int_{Q^n_{i_n}} I^n \leq \frac{C_*}{N}. \quad (6.57)
\]

Setting \(\lambda^n = N^{i_n}\), the right-hand side of (6.55) can be estimated as follows using (6.57) and the fact that \(I^n\) is uniformly bounded in \(L^1(Q_T)\)

\[
\int_{\{\|\mathbf{v}^n - \mathbf{v}\| \geq N^{i_n}\}} \frac{N^{i_n}}{\|\mathbf{v}^n - \mathbf{v}\|} I^n
\]

\[
= \int_{\{N^i \leq \|\mathbf{v}^n - \mathbf{v}\| \leq N^{i+1}\}} \frac{N^{i_n}}{\|\mathbf{v}^n - \mathbf{v}\|} I^n + \int_{\{\|\mathbf{v}^n - \mathbf{v}\| \geq N^{i+1}\}} \frac{N^{i_n}}{\|\mathbf{v}^n - \mathbf{v}\|} I^n \quad (6.58)
\]

\[
\leq \int_{Q^n_{i_n}} I^n + \frac{1}{N} \int_{\{\|\mathbf{v}^n - \mathbf{v}\| \geq N^{i+1}\}} I^n \leq \frac{C_*}{N}.
\]

Let

\[
W^n := (Z^n - \mathcal{Z}) : (D\mathbf{v}^n - D\mathbf{v}).
\]

Then (6.55) and (6.58) imply that

\[
\limsup_{n \to \infty} \int_{\|\mathbf{v}^n - \mathbf{v}\| \leq \lambda^n} W^n \leq \frac{C_*}{N} \quad (6.59)
\]

Now we show that

\[
(W^n)^- \to 0 \text{ strongly in } L^1(Q_T). \quad (6.60)
\]

Recalling that \(Z^n = Z^n(p^n, \mathcal{D}\mathbf{v}^n)\) and incorporating (5.2), we get

\[
W^n = (Z^n - Z^n(p^n, \mathcal{D}\mathbf{v})) : (D\mathbf{v}^n - D\mathbf{v}) + (Z^n(p^n, \mathcal{D}\mathbf{v}) - Z^n) : (D\mathbf{v}^n - D\mathbf{v}) \geq (Z^n(D\mathbf{v}, \tau(p^n)) - Z^n) : (D\mathbf{v}^n - D\mathbf{v}). \quad (6.61)
\]
Splitting $Q_T = \{|Dv| = 0\} \cup \{|Dv| > 0\}$, thanks to the definitions of $Z^n$ and $Z$ and since $p^n$ converges pointwise, we get

$$Z^n(Dv, \tau(p^n)) \to Z \text{ a.e. in } Q_T.$$ 

Also, independently of $n$,

$$|Z^n(Dv, \tau(p^n)) - Z| \leq C|Dv|.$$ 

By the Dominated Convergence Theorem and since $(Dv^n - Dv)$ is bounded in $L^2(Q_T)$ yield (6.60).

Combining (6.59), (6.60) and recalling that $A = N \leq \lambda^n$,

$$\limsup_{n \to \infty} \int_{|u^n - v| \leq N} |W^n| \leq \frac{C^*}{N}.$$ 

With the help of the Hölder and Chebyshev inequalities, we observe that

$$\int_{Q_T} \sqrt{|W^n|} \leq \int_{|u^n - v| \leq N} \sqrt{|W^n|} + \int_{|u^n - v| > N} \sqrt{|W^n|} \leq |Q_T|^{\frac{1}{2}} \left( \int_{|u^n - v| \leq N} |W^n| + \|W^n\|_{L^2(Q_T)} \sqrt{|\{|u^n - v| > N\}|} \right) \leq \frac{C}{\sqrt{N}}$$ 

which implies that for a suitable subsequence,

$$W^n \to 0 \quad \text{a.e. in } Q_T.$$ 

Applying Egorov Theorem, one concludes that

$$W^n \to 0 \quad \text{strongly in } L^1(Q_T \setminus E_j),$$ 

where $E_j \subset Q_T$ are such that $\lim_{j \to \infty} |E_j| = 0$. It follows from the definition of $W^n$ and the weak convergences (6.21), (6.26) that

$$\limsup_{n \to \infty} \int_{Q_T \setminus E_j} Z^n : Dv^n dxdt = \limsup_{n \to \infty} \int_{Q_T \setminus E_j} Z : (Dv^n - Dv) + Z^n : Dv dxdt = \int_{Q_T \setminus E_j} Z : Dv dxdt.$$ 

Thus, the assumptions (5.48)-(5.53) of Proposition 5.3 are verified with $U = Q_T \setminus E_j$, for all $j \in \mathbb{N}$. Due to the properties of $E_j$, we finally conclude, using (5.54), that

$$Dv = \frac{(|S| - \tau(p_t))^+}{|S|} S.$$ 

The proof of Theorem 4.2 is complete.

**Conclusion**

This study has been inspired by recent research concerning implicitly constituted materials on one hand and by a recent interesting paper by Chupin and Mathé [11] on the other hand. This study extends the results presented in [11] in several directions. First, we have studied slightly different system of PDEs, namely the one we were able to derive from the basic governing equations of the theory of mixtures, under the cascade of several
justified simplifications. Second, the activated system contains in comparison to [11], a non-trivial right-hand side in the equation for the fluid pressure $p_f$. Consequently, we had to use a different approach to get $L^\infty$-estimates for $p_f$. Third, inspired by [11] we provide characterization of the constitutive equation in Proposition 5.1. Using one of these equivalent descriptions, one can correct the proof in [11] and get a useful tool exploited in the proof of Proposition 5.3 here. Fourth, we considered stick-slip boundary conditions that are not only physically relevant but, on contrary to no-slip boundary condition, guarantees the integrability of the pressure up to the boundary. Finally, we use $L^\infty$-truncation method to analyze three-dimensional flows (while the result in [11] concerns planar flows). We wish to remark that it is possible to use, instead of $L^\infty$-truncation, a solenoidal Lipschitz truncation (introduced in [4]) and consider the formulation free of the pressure. From the application point of view, the system of equations analyzed here seems to be suitable to describe the phenomena of liquefaction. Of course, as one may conclude from Section 2, this topic provides several questions for further research.

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