Asymptotics of discrete $q$-Freud II
orthogonal polynomials from the
$q$-Riemann Hilbert problem

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Abstract
We investigate a Riemann–Hilbert problem (RHP), whose solution corresponds
to a group of $q$-orthogonal polynomials studied earlier by Ismail et al. Using
RHP theory we determine new asymptotic results in the limit as the degree
of the polynomials approach infinity. The RHP formulation also enables us to
obtain further properties. In particular, we consider how the class of polynomi-
als and their asymptotic behaviours change under translations of the $q$-discrete
lattice and determine the asymptotics of a related $q$-discrete Painlevé equation.

Keywords: Riemann–Hilbert problem, $q$-orthogonal polynomials,
$q$-difference calculus
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1. Introduction

Orthogonal polynomials are a key component of a wide array of mathematical problems. They
provide the basis of solutions of Sturm–Liouville problems [1], their zeros are related to the
eigenvalue distribution of random matrices [2], they describe transition probabilities in birth-
death models [3] and are used in numerical spectral approximation methods, just to name a
few examples. Their importance in describing physical phenomena was recognised over two

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hundred years ago (Legendre, Laplace) and they continue to be pivotal in describing mathematical and physical problems.

In this paper we study a class of \( q \)-orthogonal polynomials and deduce new results concerning their asymptotic behaviour as the degree tends to infinity. The orthogonality measure of these polynomials is supported on the discrete lattice, \( q^k \), for \( k \in \mathbb{Z} \). We also determine some properties of a \( q \)-Painlevé equation associated with these \( q \)-orthogonal polynomials.

1.1. Notation

For completeness, we recall some well known definitions and notations from the calculus of \( q \)-differences. These definitions can be found in [4]. Throughout the paper we will assume \( q \) is real and \( 0 < q < 1 \).

**Definition 1.1.** We define the Pochhammer symbol, \( q \)-derivative and Jackson integral as follows:

1. The Pochhammer symbol \( (x; q)_\infty \) is
   \[
   (x; q)_\infty = \prod_{j=0}^{\infty} (1 - xq^j).
   \]
   We denote product of Pochhammer symbols in multiple variables by \( (x_1, x_2; q)_\infty \),
   \[
   (x_1, x_2; q)_\infty = \prod_{j=0}^{\infty} (1 - x_1q^j)(1 - x_2q^j).
   \]

2. The \( q \)-derivative is defined by
   \[
   D_q f(x) = \frac{f(qx) - f(x)}{x(q - 1)}.
   \] (1.1)
   Note that
   \[
   D_q x^n = [n]_q x^{n-1},
   \]
   where
   \[
   [n]_q = \frac{q^n - 1}{q - 1}.
   \]

3. The Jackson integral from \( q^i \) to \( q^j \) for some integers \( i < j \) is given by
   \[
   \int_{q^i}^{q^j} f(x) d_q x = \sum_{k=i}^{j} f(q^k)q^k.
   \]
   The Jackson integral from \( -q^i \) to \( q^i \) for some integers \( i, j \) is given by
   \[
   \int_{-q^i}^{q^j} f(x) d_q x = \sum_{k=j}^{\infty} f(-q^k)q^k + \sum_{k=i}^{\infty} f(q^k)q^k.
   \]
**Definition 1.2.** In this paper \( \mathbb{N} \) will denote the set of natural numbers including zero (i.e. 0,1,2,3,...), unless otherwise stated.

We recall the definition of an appropriate Jordan curve and admissible weight function given in [5, definition 1.2] (with slight modification).

**Definition 1.3.** A positively oriented Jordan curve \( \Gamma \) in \( \mathbb{C} \) with interior \( \mathcal{D}_- \subset \mathbb{C} \) and exterior \( \mathcal{D}_+ \subset \mathbb{C} \) is called appropriate if

\[
\pm q^k \in \begin{cases} \\
\mathcal{D}_- & \text{if } k \geq 0, \ (k \in \mathbb{Z}), \\
\mathcal{D}_+ & \text{if } k < 0, \ (k \in \mathbb{Z}),
\end{cases}
\]

and,

\[
e^{\frac{2\pi i(n+ka)}{q}} q^{-k} \in \mathcal{D}_+, \ (n \in 0, 1, 2, 3 \text{ and } k \in \mathbb{N}).
\]

**Definition 1.4 ([5]).** Define \( h_q : \mathbb{C} \setminus (\{0\} \cup \{\pm q^k\}_{k=-\infty}^{\infty}) \to \mathbb{C} \) by

\[
h_q(z) = \sum_{k=-\infty}^{\infty} \frac{2zq^k}{z^2 - q^{2k}} = \sum_{k=-\infty}^{\infty} \left( \frac{q^k}{z-q^k} + \frac{q^k}{z+q^k} \right).
\]  

Note that \( h_q(z) \) satisfies the \( q \)-difference equation

\[
h_q(qz) = h_q(z).
\]

In appendix \( A \), we show that \( h_q(z) \) has certain unique properties.

### 1.2. Background

Let \( \{P_n(x)\}_{n=0}^{\infty} \) be a class of monic polynomials which satisfy the orthogonality relation

\[
\int_{\mathbb{R}} P_n(x)P_m(x)w(x)dx = \gamma_n \delta_{n,m},
\]

for some weight function \( w(x) \). Equation (1.4) gives rise to the three term recurrence relation

\[
xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \alpha_n P_{n-1}(x),
\]

where the recurrence coefficients are given by

\[
\alpha_n = \frac{\gamma_n}{\gamma_{n-1}}, \ \beta_n = \frac{\int_{\mathbb{R}} xP_n(x)^2dx}{\gamma_n}.
\]

For even weight functions, \( w(-x) = w(x) \), we find \( \beta_n = 0 \) for all \( n \).

Following the pioneering work of Freud and others, questions arising about the asymptotic locations of the zeros of \( P_n(x) \) and behaviour of \( \alpha_n \) as \( n \to \infty \) have led to many developments in orthogonal polynomials and approximation theory. Motivated by the work of Deift et al [6] we use the setting of the Riemann Hilbert Problem (RHP) to answer such questions (in theorems 1.5, 1.7 and 1.8) for a class of \( q \)-orthogonal polynomials. In particular, we study polynomials which satisfy the orthogonality condition

\[
\int_{-\infty}^{\infty} P_n(x)P_m(x) \frac{dx}{(-x^2; q^2)^\infty} = \gamma_n \delta_{n,m}.
\]

Throughout this paper we will label these polynomials as \( q \)-Freud II polynomials to be consistent with the nomenclature of the digital library of mathematical functions (DLMF) [7, chapter 18]. \( q \)-Freud II polynomials were studied earlier by Ismail and Mansour [8], however Ismail et al considered orthogonality on a continuous measure over \( \mathbb{R} \). We will show in section 1.3 that
this continuous class of polynomials can readily be extended to those with a discrete measure, which will be the focus of this paper.

A significant development in the theory of orthogonal polynomials is the observation that the recurrence coefficients \( \alpha_n \) often give rise to discrete Painlevé equations \([9]\). For example, the recurrence coefficients of orthogonal polynomials with weight function \( e^{-x^4/4} \) satisfy the discrete equation \([10]\), which is a case of the first discrete Painlevé equation, or dP\(_I\) \([11]\).

The recurrence coefficients of \( q \)-Freud II polynomials also satisfy a discrete Painlevé equation, where the non-autonomous term in the equation is now iterated on a multiplicative lattice. (For the terminology distinguishing types of discrete Painlevé equations, we refer to Sakai \([12]\).) As detailed by Ismail and Mansour \([8]\) the recurrence coefficients of \( q \)-Freud II polynomials satisfy

\[
\alpha_n (\alpha_n + 1 + q \alpha_n - 1 \alpha_n + q - 2 \alpha_n^{-1} \alpha_n + q^{-2} \alpha_n^{-1} \alpha_n^{-1} - q^{2n-3} \alpha_n^{-1} \alpha_n^{-1}) = (q^{-n} - 1) q^{1-n}. \tag{1.7}
\]

In Sections 1.3 and 8 we extend this result initially determined by Ismail et al for a continuous measure to a larger class of discrete \( q \)-Freud II polynomials, using techniques similar to those found in [13].

Through the connection to RHP theory developed in this paper we obtain new insights into the solutions of equation (1.7). For example, theorem 1.8 shows that there exists more than one real positive solution of equation (1.7), and we notice that the asymptotic behaviour as \( n \to \infty \) varies between solutions.

There is an important feature of \( q \)-difference equations that affects our discussion of \( q \)-Freud II polynomials. The distinguishing feature of \( q \)-Freud II polynomials is that their weight function satisfies a \( q \)-difference equation. However, there are infinitely many weights involving a free \( q \)-periodic function \( C(x) \), where \( C(qx) = C(x) \), which satisfy a given \( q \)-difference equation. If a weight with \( C \neq 1 \) were to be chosen, the resulting family of orthogonal polynomials may have properties that differ from the class we consider. We expand on this point below.

### 1.3. Defining \( q \)-Freud II polynomials

We define the family of weight functions \( u_m : \mathbb{C} \setminus \{ e^{\frac{\pi i (1 + 2k)}{2m}} q^{-k} \}_{k=0}^{\infty} \to \mathbb{C} \), where \( n = 0, 1, \ldots, 2m - 1 \), as

\[
u_m(x) = \frac{1}{(-x^{2m}; q^{2m})_{\infty}},
\]

where \( 1 \leq m \in \mathbb{N} \) [8]. They satisfy the \( q \)-difference equation

\[
D_q u_m(x) = \frac{-x^{2m-1}}{1-q} u_m(x).
\tag{1.8}
\]

This is analogous to classical Freudian weights \( v_m(x) = e^{-x^m} \), which satisfy the differential equation

\[
\frac{d}{dx} v_m(x) = -2mx^{2m-1} v_m(x).
\tag{1.9}
\]
However, a key difference between these two relations is that equation \((1.8)\) is a discrete relation. In particular if \(u_m(x)\) satisfies equation \((1.8)\) then \(u_m(x)C(x)\) also does, for any function \(C(x)\) satisfying \(C(qx) = C(x)\). Consider the case

\[
u_1(x) = \frac{1}{(-x^2; q^2)_{\infty}}.
\]

\((1.10)\)

This weight gives rise to discrete \(q\)-Hermite II polynomials \([7, chapter 18.27]\). The sequence of discrete \(q\)-Hermite II polynomials, \([H_n(x)]_{n=0}^{\infty}\), are orthogonal with respect to any measure \(u_1(x)C(x)\). In particular they satisfy the continuous orthogonality condition

\[
\int_{-\infty}^{\infty} H_n(x)H_m(x)u_1(x)dx = \gamma_n\delta_{n,m},
\]

on the real line, and also satisfy the discrete orthogonality condition

\[
\int_{-\infty}^{\infty} H_n(cx)H_m(cx)u_1(cx)d_qx = \gamma_n^{(c)}\delta_{n,m},
\]

for any constant \(c\). In contrast, as we will show in section 8 this is not true for the weight,

\[
w(x) = u_2(x) = \frac{1}{(-x^4; q^4)_{\infty}},
\]

\((1.11)\)

which is the focus of this paper. Thus, when describing \(q\)-Freud II orthogonal polynomials, one also has to specify their orthogonality weight.

We will call these polynomials \(qF_{II}\) polynomials. In section 8, we discuss the implications of our results to polynomials orthogonal with respect to the weights of the form

\[
\int_{-\infty}^{\infty} P_n(x)P_m(x)w(x)d_qx = \gamma_n^{(c)}\delta_{n,m},
\]

\((1.12)\)

for any constant \(q < c \leq 1\). We will call these polynomials \(qF_{II}^{(c)}\) polynomials.

### 1.4. Main results

We are now in a position to state the main results of this paper, which are listed as theorems 1.5, 1.7 and 1.8 below. The first main result concerns the asymptotic behaviour of orthogonal polynomials as their degree approaches infinity.

**Theorem 1.5.** Suppose that \([P_n(z)]_{n=0}^{\infty}\) is a family of monic polynomials, orthogonal with respect to the weight \(w(z)d_qz\). Define \(t = zq^{n/2}\). Then, as \(n \to \infty\), for even \(n \in \mathbb{N}\):

\[
P_n(z) = \begin{cases} 
(-1)^{n/2}q^{n/2}(z^{-1})a(z)\left(\frac{\mu_n}{\mu_0} + O(q^{n/4})\right) & \text{for } |z| \leq q^{-n/4}, \\
q^{n}a_{\infty}(t)\left(1 + O(q^{n/4})\right) & \text{for } |z| > q^{-n/4},
\end{cases}
\]
where \( a(z) \) is a solution of equation (4.2), defined in definition 4.4, and, \( a_\infty(t) \) is a solution of equation (5.2), defined in lemma 5.2 (they are both independent of \( n \)). Similarly \( P_n(z) \) gives us detailed knowledge about the zeros of 

\[
\alpha_n = q^{\frac{3}{2}(1-n)} \left( A + O(q^{n/2}) \right), \\
\alpha_{n-1} = q^{\frac{5}{2}(3-n)} \left( B - 1 + O(q^{n/2}) \right), \\
\alpha_n = q^{-n} \left( q + O(q^{n/2}) \right),
\]

for some non-zero constants \( A \) and \( B \), where \( \gamma_n, \alpha_n \), and \( \alpha_{n-1} \) are defined by equations (1.4) and (1.6) respectively.

Our second main result concerns the asymptotic behaviour of recurrence coefficients and \( L_2 \) norm of \( P_n \) as \( n \) approaches infinity.

**Theorem 1.7.** Under the same hypotheses as theorem 1.5, we have for even \( n \in \mathbb{N} \):

\[
\gamma_n = q^{\frac{3}{2}(1-n)} \left( A + O(q^{n/2}) \right), \\
\gamma_{n-1} = q^{\frac{5}{2}(3-n)} \left( B - 1 + O(q^{n/2}) \right), \\
\alpha_n = q^{-n} \left( q + O(q^{n/2}) \right),
\]

for some non-zero constants \( A \) and \( B \), where \( \gamma_n, \alpha_n \), and \( \alpha_{n-1} \) are defined by equations (1.4) and (1.6) respectively.

Like the constants \( \mu_2 \) etc used in the statement of theorem 1.5, the authors could not determine a closed form expression for \( A \) and \( B \). They can however be evaluated numerically using the arguments in section 5.

Our third main theorem answers the question of uniqueness posed by Ismail and Mansour [8, remark 6.4].

**Theorem 1.8.** There exist infinitely many real positive solutions of the discrete equation

\[
\alpha_n(\alpha_{n+1} + q^{n-1} \alpha_n + q^{-2} \alpha_{n-1} - q^{2n-3} \alpha_{n+1} \alpha_n \alpha_{n-1}) = (q^{n-1} - 1)q^{1-n}. \tag{1.13}
\]

In particular, the recurrence coefficients \( \{\alpha_n^{(c)}\}_{n=0}^\infty \), which correspond to polynomials with orthogonality condition given by equation (1.12), satisfy equation (1.13). Furthermore, for \( c \neq 1, q^{1/2} \).
\[
\lim_{n \to \infty} \frac{\alpha_n^{(c)} - \alpha_n^{(1)}}{q^{-n} - \alpha_n^{(1)}} \neq 0. \tag{1.14}
\]

For \( c = q^{1/2} \), the sequences \( \{\alpha_n^{(q^{1/2})}\}_{n=0}^{\infty} \) and \( \{\alpha_n^{(1)}\}_{n=0}^{\infty} \) agree for all \( n \).

Theorem 1.8 immediately follows from theorem 8.3 proved in section 8.

1.5. Outline

This paper is structured as follows. In section 2 we state and solve a RHP (definition 2.1) whose solution is given in terms of \( qF_2 \) polynomials. We then make a series of transformations to this RHP in section 3. By taking the limit \( n \to \infty \) this motivates the form of a near-field and far-field RHP, whose solutions are determined in sections 4 and 5 respectively. In section 6 we glue together the near and far-field solutions to approximate our initial RHP in the limit \( n \to \infty \). Consequently we prove theorems 1.5 and 1.7 in section 7. In section 8 we discuss the implications of our results to the recurrence coefficients of \( qF_2 \) polynomials and prove theorem 1.8. In appendix A we prove some important properties of the function \( h_n(z) \) which are used in solving the near and far-field RHPs. Finally, for completeness we prove a well-known result concerning RHPs whose jump approaches the identity in appendix B.

2. Statement of RHP

We begin the main arguments of this paper by introducing and solving a RHP (definition 2.1) whose solution is given in terms of \( qF_2 \) orthogonal polynomials.

Definition 2.1 (\( qF_2 \) RHP). Let \( \Gamma \) be an appropriate curve (see definition 1.3) with interior \( D_- \) and exterior \( D_+ \). A 2 \( \times \) 2 complex matrix function \( Y_n(z) \), \( z \in \mathbb{C} \), is a solution of the \( qF_2 \) RHP if it satisfies the following conditions:

(i) \( Y_n(z) \) is meromorphic in \( \mathbb{C} \setminus \Gamma \), with simple poles at \( z = \pm q^{-k} \) for \( 1 \leq k \in \mathbb{N} \).

(ii) \( Y_n(z) \) has continuous boundary values \( Y^{-}_n(s) \) and \( Y^{+}_n(s) \) as \( z \) approaches \( s \in \Gamma \) from \( D_- \) and \( D_+ \) respectively, where

\[
Y_n^+(s) = Y_n^-(s) \begin{bmatrix} 1 & h_q(s)w(s) \\ 0 & 1 \end{bmatrix}, \quad s \in \Gamma, \tag{2.1a}
\]

and \( w(s) \) is defined in equation (1.11).

(iii) The residue at each pole \( z = \pm q^{-k} \) for \( 1 \leq k \in \mathbb{N} \) is given by

\[
\text{Res}(Y_n(\pm q^{-k})) = \lim_{z \to \pm q^{-k}} Y_n(z) \begin{bmatrix} 0 & (z \mp q^{-k})h_q(z)w(z) \\ 0 & 0 \end{bmatrix} \tag{2.1b}
\]

(iv) \( Y_n(z) \) satisfies

\[
Y_n(z) \begin{bmatrix} z^{-n} & 0 \\ 0 & z^* \end{bmatrix} = I + \mathcal{O} \left( \frac{1}{z} \right), \quad \text{as} \ |z| \to \infty, \tag{2.1c}
\]

for \( z \) such that \( |z \pm q^{-k}| > r \), for all \( 1 \leq k \in \mathbb{N} \), for fixed \( r > 0 \).

Remark 2.2. Note that the matrix \( Y_n(\pm q^{-k}) \) has poles in its second column for \( 1 \leq k \in \mathbb{N} \). Thus, the asymptotic decay does not hold near these poles. This is why, following
equation (2.1c), we require the added condition \( z \) must be such that \( |z + q^{-k}| > r \), for all \( 1 \leq k \in \mathbb{N} \), for fixed \( r > 0 \).

We now determine the solution of the \( q \)-RHP.

**Lemma 2.3.** The unique solution of the \( q \)-RHP given by definition 2.1 is

\[
Y_n(z) = \left[ \frac{P_n(z)}{\gamma_{n-1} P_{n-1}(z)} \right] \frac{P_n(z)w(s)h_q(s)}{z-s} \frac{ds}{s} = \gamma_{n-1} \frac{P_n(z)w(s)h_q(s)}{s} \frac{ds}{s},
\]

where \( \{P_n(z)\}_{n=0}^{\infty} \) is a sequence of monic polynomials satisfying the orthogonality condition

\[
\int_{-\infty}^{\infty} P_n(s)P_m(s)w(s)\,ds = \gamma_n \delta_{n,m}.
\]

**Proof.** The proof follows along similar lines to [5, section 2(a)], with adjustments needed for the current case where the orthogonality weight is not contained in a compact set in \( \mathbb{R} \).

We show that the second row of \( Y_n(z) \) must be given by equation (2.2). A similar argument can be carried out for the first row. To declutter notation we will label \( Y_n(z) \) as \( Y(z) \) for the rest of this proof.

It follows from the asymptotic condition, equation (2.1c), that the \((1, 1)\) entry of \( Y \) must have leading order \( e^z \) as \( z \to \infty \). As \( Y_{(1,1)}(z) \) is analytic and its jump condition, equation (2.1a), is given by the identity we immediately conclude that \( Y_{(1,1)}(z) \) is a monic polynomial of degree \( n \). Similarly, it follows that \( Y_{(2,1)}(z) \) is a polynomial of degree at most \( n-1 \). We denote \( Y_{(2,1)}(z) \) by \( Q_{n-1}(z) \).

Consider the bottom right entry of equation (2.2). By the jump condition, equation (2.1a), we have

\[
Y_{(2,2)}(s) = Q_{n-1}(s)w(s)h_q(s) + Y_{(2,2)}(s) - \gamma_{n-1} \delta_{n,n}.
\]

If there was no residue condition, equation (2.1b), then this scalar equation would be solved by the Cauchy transform

\[
Y_{(2,2)}(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{Q_{n-1}(s)w(s)h_q(s)}{z-s} \frac{ds}{s},
\]

which is analytic in \( \mathbb{C} \setminus \Gamma \) and satisfies equation (2.3). The residue condition can be readily resolved by letting

\[
Y_{(2,2)}(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{Q_{n-1}(s)w(s)h_q(s)}{z-s} \frac{ds}{s} = \int_{q^{-1}}^{\infty} \frac{Q_{n-1}(s)w(s)h_q(s)}{z-s} \frac{ds}{s},
\]

which satisfies both equations (2.1a) and (2.1b). The only step remaining is to prove the asymptotic condition, equation (2.1c), for \( Y_{(2,2)}(z) \). Substituting our expression for \( h_q(s) \) into equation (2.5), we find

\[
Y_{(2,2)}(z) = \frac{1}{2\pi i} \oint_{\Gamma} \sum_{k=-\infty}^{\infty} \left( \frac{Q_{n-1}(s)w(s)}{z-s} + \frac{Q_{n-1}(s)w(s)}{z-s} \right) \frac{ds}{s}.
\]
which by Cauchy’s integral formula, for \( z \in \text{ext}(\Gamma) \), becomes

\[
Y_{(2,2)}(z) = \sum_{k=0}^{\infty} \left( q^k \frac{Q_{n-1}(q^k)w(q^k)}{z - q^k} + q^k \frac{Q_{n-1}(-q^k)w(-q^k)}{z + q^k} \right) + \int_{q^{-1}}^{\infty} \frac{Q_{n-1}(\pm s)w(\pm s)}{(z \mp s)} \, dq s,
\]

where the sum to infinity is well defined on \( \Gamma \), as \( h_q(s) \) converges as \( k \to \infty \), and the Jackson integral of an analytic function is well defined. Using the geometric series with remainder

\[
\frac{1}{z - x} = \sum_{k=0}^{l} \left( \frac{x^k}{z^{k+1}} \right) + \frac{x^{l+1}}{z^{l+1}(z - x)}, \quad \text{for } x \neq z,
\]

we find

\[
Y_{(2,2)}(z) = \int_{-\infty}^{\infty} \frac{Q_{n-1}(x)w(x)x^n}{z^n(z-x)} \, dx + \sum_{k=0}^{n-1} \frac{1}{z^{k+1}} \int_{-\infty}^{\infty} Q_{n-1}(x)w(x)x^k \, dq x.
\]

Note that the asymptotic condition, equation (2.1c), holds when the last term on the rhs is zero for \( k = 0, 1, 2, \ldots, n - 2 \). This is true iff

\[
\int_{-\infty}^{\infty} Q_{n-1}(x)w(x)x^k \, dq x = 0, \quad \text{for } k \leq n - 2,
\]

which is satisfied when \( Q_{n-1} \) is an orthogonal polynomial of degree \( n - 1 \) on the \( q \)-lattice with respect to the weight \( w(x) \). This is the class of \( qF_1 \) polynomials. We conclude that the solution of \( Y_{(2,2)}(z) \) is given by

\[
Y_{(2,2)}(z) = \begin{cases} 
\frac{1}{Y_{n-1}} \int_{-\infty}^{\infty} \frac{P_{n-1}(x)w(x)}{z^k - x} \, dq x - \frac{1}{Y_{n-1}} P_{n-1}(z)w(z)h_q(z), & \text{for } z \in \text{int}(\Gamma), \\
\frac{1}{Y_{n-1}} \int_{-\infty}^{\infty} \frac{P_{n-1}(x)w(x)}{z^k - x} \, dq x, & \text{for } z \in \text{ext}(\Gamma).
\end{cases}
\]

After appropriate scaling, and repeating the same arguments for the first row, it follows that equation (2.2) is a solution of the \( q \)-RHP given by definition 2.1.

Uniqueness of this solution follows from consideration of the determinant. Observe that the jump matrix \( J = Y^{-1} Y_+ \) satisfies \( \text{det}(J) = 1 \). It immediately follows that \( \text{det}(Y^+) = \text{det}(Y^-) \) on \( \Gamma \). Furthermore, by the residue condition, equation (2.1b), \( \text{det}(Y) \) has no poles. Thus, \( \text{det}(Y) \) is an entire function. By the asymptotic condition, equation (2.1c), \( \text{det}(Y) \to 1 \), and so by Liouville’s theorem, it follows that \( \text{det}(Y) = 1 \). This implies \( Y^{-1} \) exists and is meromorphic in \( \mathbb{C}/\Gamma \).

Now suppose that there exists a second solution of the \( q \)-RHP and denote this solution by \( \tilde{Y} \). If we define \( M = \tilde{Y} Y^{-1} \), it follows that the jump conditions (and residue conditions) effectively cancel and \( M_+ = M_- \). Thus, \( M \) is entire and \( M \to I \) as \( z \to \infty \). Hence, by Liouville’s theorem \( M = I \). We conclude that \( \tilde{Y} = Y \) and, therefore, there is a single unique solution of the \( q \)-RHP. \( \square \)

**Remark 2.4.** It can be shown that solution of the RHP given by definition 2.1 satisfies a Lax pair, namely a \( q \)-difference equation in \( z \) and a recurrence relation for the parameter \( n \). The proof follows using the same arguments as in [5].

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Remark 2.5. One can repeat the arguments above to show that there is a unique solution of definition 2.1 with a modified function
\[ h_q(z) \rightarrow c^{-1} h_q(z/c), \]
for some real constant c, and that it is given by
\[ Y_n^{(c)}(z) = \begin{bmatrix} P_n^{(c)}(z) \\ (-1)^{n-1} P_{n-1}^{(c)}(z) \end{bmatrix} = \begin{bmatrix} \frac{\gamma_n^{(c)}}{\Gamma(z-c)} \int \frac{p_n^{(1)}(s)^2 h_q(s/c)}{2\pi i (s-z)} ds + \int_{-\infty}^{\infty} \frac{p_n^{(1)}(s)^2 w(s/c)}{(z-c)^{1/2}} ds q \gamma_n^{(c)} d_y s \\ \gamma_n^{(c)} \int \frac{p_n^{(1)}(s)^2 h_q(s/c)}{2\pi i (s-z)} ds + \int_{-\infty}^{\infty} \frac{p_n^{(1)}(s)^2 w(s/c)}{(z-c)^{1/2}} ds q \gamma_n^{(c)} d_y s \end{bmatrix}, \]
where \( \{P_n^{(c)}(z)\}_{n=0}^{\infty} \) satisfies the orthogonality condition
\[ \int_{-\infty}^{\infty} P_n^{(c)}(cs) P_m^{(c)}(cs) w(cs) d_y s = \gamma_n^{(c)} \delta_{n,m}. \]

3. Transformations of RHP

In this section we will transform the RHP given in definition 2.1 to more easily determine the asymptotics of \( Y_n(z) \) as \( n \rightarrow \infty \). Throughout this section we will assume that \( n \) is even. First, we introduce some new functions which will be used when transforming the RHP. We show that these functions satisfy certain difference equations.

Definition 3.1. Define \( g: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C} \) as
\[ g(z) = (z^2; q^2)_{\infty} (q^2 z^{-2}; q^2)_{\infty}. \]

Lemma 3.2. \( g(z) \) satisfies the difference equation
\[ g(qz) = -z^{-2} g(z). \]

Proof. Substituting \( qz \) into equation (3.1) we find
\[ g(qz) = (q^2 z^2; q^2)_{\infty} (z^{-2}; q^2)_{\infty}, \]
\[ = \prod_{j=0}^{\infty} (1 - q^2 z^2 q^{2j}) \prod_{j=0}^{\infty} (1 - z^{-2} q^{2j}), \]
\[ = \frac{1 - z^{-2}}{1 - z} \prod_{j=0}^{\infty} (1 - z^{-2} q^{2j}) \prod_{j=0}^{\infty} (1 - q^2 z^{-2} q^{2j}), \]
\[ = \frac{1 - z^{-2}}{1 - z^2} g(z), \]
\[ = -z^{-2} g(z). \]

Remark 3.3. By induction using equation (3.2) we find that for even \( n \)
\[ g(q^{-n/2} z) = (-1)^n q^{-n^2/4 - n/2} z^n g(z). \]

Definition 3.4. Define \( \omega: \mathbb{C} \setminus \{0\} \cup \{e^{\pi i (1 + 2k)} q^{2k}\}_{k=-\infty}^{\infty} \rightarrow \mathbb{C} \), where \( n = 0, 1, 2, 3, \) as
\[ \omega(z) = 1 / (-z^4, -q^2 z^{-4}; q^2)_{\infty}. \]

Lemma 3.5. \( \omega(z) \) satisfies the difference equation
\[ \omega(qz) = z^4 \omega(z). \]
Proof. The proof follows from the definition of $\omega(z)$ (by applying the same arguments as in lemma 3.2).

Lemma 3.6. The function $w(z)$ defined in equation (1.11) satisfies the difference equation

$$w(qz) = (1 + z^4)w(z).$$

(3.6)

Proof. The proof follows from the definition of $w(z)$ (by applying the same arguments as in lemma 3.2).

Remark 3.7. By induction using equation (3.6) we find that for even $n$

$$w(q^{-n/2}z) = z^{-2n}q^n(\frac{3}{4} + 1)\left( -q^{2n+4}z^{-4}\right)\omega(z).$$

(3.7)

Taking $z \to zq^{n/2}$, equation (3.7) gives

$$w(z) = z^{-2n}q^n\left( -\frac{3}{4}z^{-4}\right)\omega(zq^{n/2}).$$

(3.8)

3.1. RHP transformations

Before proceeding we introduce some notation. Consider an appropriate curve $\Gamma$ (see definition 1.3), scaled such that the modulus of the points on it are multiplied by $q^{-n/2}$. We denote this new curve by $\Gamma_{q^{-n/2}}$. If $\Gamma$ were the unit circle, $\Gamma_{q^{-n/2}}$ would be a circle with radius $q^{-n/2}$.

We define $D_{-q^{-n/2}}$ and $D_{+q^{-n/2}}$ similarly.

Consider the following transformation to the RHP given by definition 2.1:

$$U_n(z) = \begin{cases} Y_n(z), & \text{for } z \in D_+, \\ Y_n(z) \begin{bmatrix} g(z)^{-1} & 0 \\ 0 & g(z) \end{bmatrix}, & \text{for } z \in D_{-q^{-n/2}} \setminus D_-, \\ Y_n(z) \begin{bmatrix} z^{-n} & 0 \\ 0 & z^n \end{bmatrix}, & \text{for } z \in D_{+q^{-n/2}}. \end{cases}$$

This gives a new RHP for $U_n$ with two jumps (and also some more poles which will be discussed shortly). At $z \in \Gamma_{q^{-n/2}}$ we apply equation (3.3) to determine

$$g(q^{-n/2}e^{i\theta})(q^{-n/2}e^{i\theta})^{-n} = g(e^{i\theta})(-1)^{n/2}q^{n^2/4-n/2}.$$

(3.9)

Motivated by equation (3.9) we make the transformation

$$W_n(z) = \begin{cases} \begin{bmatrix} c_n & 0 \\ 0 & c_n^{-n} \end{bmatrix} U_n(z) \begin{bmatrix} c_n^{-n} & 0 \\ 0 & c_n \end{bmatrix}, & \text{for } z \in D_{+q^{-n/2}}, \\ \begin{bmatrix} c_n & 0 \\ 0 & c_n \end{bmatrix} U_n(z), & \text{for } z \in D_{-q^{-n/2}}, \end{cases}$$

(3.10)

where

$$c_n = (-1)^{1/2}q^{n/4-1/2}.$$
(i) \( W_n(z) \) is meromorphic in \( \mathbb{C} \setminus (\Gamma \cup \Gamma_{q^{-n/2}}) \), with simple poles at \( z = \pm q^{-k} \) for \( 1 \leq k \in \mathbb{N} \).

(ii) \( W_n(z) \) has continuous boundary values \( W_n^- (s) \) and \( W_n^+ (s) \) as \( z \) approaches \( s \in \Gamma \) from \( D_- \) and \( D_+ \) respectively, where

\[
W_n^+ (s) = W_n^- (s) \begin{bmatrix} g(s)^{-1} & g(s)w(s)h_q(s) \\ 0 & g(s) \end{bmatrix}, \quad s \in \Gamma.
\] (3.11a)

(iii) \( W_n(z) \) has continuous boundary values \( W_n^- (s) \) and \( W_n^+ (s) \) as \( z \) approaches \( s \in \Gamma_{q^{-n/2}} \) from \( D_{-q^{-n/2}} \) and \( D_{+q^{-n/2}} \) respectively, where

\[
W_n^+ (s) = W_n^- (s) \begin{bmatrix} g(sq^{n/2}) & 0 \\ 0 & g(sq^{n/2})^{-1} \end{bmatrix}, \quad s \in \Gamma_{q^{-n/2}}.
\] (3.11b)

(iv) \( W_n(z) \) satisfies

\[
W_n(z) = I + O \left( \frac{1}{z} \right), \quad \text{as } |z| \to \infty.
\] (3.11c)

Note that \( W_n(\pm q^{-k}) \) has poles in the second column for \( 1 \leq k \in \mathbb{N} \). Thus, the decay condition does not hold near these poles. For example: the decay condition holds for \( z \) such that \( |z \pm q^{-k}| > r \), for \( k \in \mathbb{N} \), for fixed \( r > 0 \).

(v) The residue at the poles \( z = \pm q^{-k} \) for \( 1 \leq k \leq n/2 \) is given by

\[
\text{Res}(W_n(\pm q^{-k})) = \lim_{z \to \pm q^{-k}} W_n(z) \begin{bmatrix} 0 & 0 \\ (z \mp q^{-k})^{-1}h_q(z)^{-1}w(z)^{-1} & 0 \end{bmatrix}.
\] (3.11d)

(vi) The residue at the poles \( z = \pm q^{-k} \) for \( k > n/2 \) is given by

\[
\text{Res}(W_n(\pm q^{-k})) = \lim_{z \to \pm q^{-k}} W_n(z) \begin{bmatrix} 0 & 0 \\ 0 & (z \mp q^{-k})^{-2n}h_q(z)w(z)^{2n} \end{bmatrix}.
\] (3.11e)

4. Near-field RHP

We will show that the solution \( W_n(z) \) of the RHP given in definition 3.8 approaches a limiting solution \( G(z) \). To do this, we are going to solve two separate RHPs, which we will call the near-field RHP and the far-field RHP. These RHPs will be chosen to mimic the two jump conditions satisfied by \( W_n(z) \) at \( \Gamma \) and \( \Gamma_{q^{-n/2}} \) respectively. This section is devoted to the solution of the near-field RHP.

Motivated by the form of equation (3.11a), we first introduce the following RHP.

**Definition 4.1 (\( \mathcal{W} \)-RHP).** Let \( \Gamma \) be an appropriate curve (see definition 1.3) with interior \( D_- \) and exterior \( D_+ \). A \( 2 \times 2 \) complex matrix function \( \mathcal{W}(z), z \in \mathbb{C} \), is a solution of the \( \mathcal{W} \)-RHP if it satisfies the following conditions:

(i) \( \mathcal{W}(z) \) is meromorphic in \( \mathbb{C} \setminus \Gamma \), with simple poles in the first column at \( z = \pm q^{-k} \) for \( 1 \leq k \in \mathbb{N} \).

(ii) \( \mathcal{W}(z) \) has continuous boundary values \( \mathcal{W}^- (s) \) and \( \mathcal{W}^+ (s) \) as \( z \) approaches \( s \in \Gamma \) from \( D_- \) and \( D_+ \), and
The residue at the poles follows immediately.

\[ \mathfrak{M}^+(s) = \mathfrak{M}^-(s) \begin{bmatrix} g(s)^{-1} & g(s)w(s)h_q(s) \\ 0 & g(s) \end{bmatrix}, \ s \in \Gamma, \]  
(4.1a)

where \( w(s) \) and \( g(s) \) are defined in equations (1.11) and (3.1) respectively.

(iii) \( \mathfrak{M}(z) \) satisfies

\[ \mathfrak{M}(z) = \begin{bmatrix} 1 & 0 \\ \dfrac{h_q(z)}{H} & 1 \end{bmatrix} + \mathcal{O}\left(\frac{1}{z}\right), \text{ as } |z| \to \infty, \]  
(4.1b)

where,

\[ H = \lim_{z \to \infty} \omega(z)g(z)^2h_q(z)^2, \]  
(4.1c)

is a non-zero constant (and \( \omega(z) \) is defined in equation (3.4)). Due to the simple poles in the first column of \( \mathfrak{M}(z) \), the asymptotic decay condition only holds for \(|z| > q^{-k}/R\) for any \( R > 0 \).

(iv) The residue at the poles \( z = \pm q^{-k} \) for \( 1 \leq k \in \mathbb{N} \) is given by

\[ \text{Res}(\mathfrak{M}(\pm q^{-k})) = \lim_{z \to \pm q^{-k}} \mathfrak{M}(z) \begin{bmatrix} 0 \\ (z \mp q^{-k})g(z)^{-2}h_q(z)^{-1}w(z)^{-1} \end{bmatrix}. \]  
(4.1d)

**Remark 4.2.** Note that for any fixed constant \( q < c < 1 \) the function \( h_q(z) \) is \( \mathcal{O}(1) \) for \(|z| = cq^k\), as \( k \to -\infty \). This is because \( h_q(z) \) is \( q \)-periodic (see equation (1.3)). Thus, away from the poles of \( h_q(z) \) (at \( z = q^k \)) the \((2, 1)\) entry in the asymptotic condition for \( \mathfrak{M}(z) \), equation (4.1b), is \( \mathcal{O}(1) \).

To solve this RHP, a series of lemmas are required. We first solve a \( q \)-difference equation whose solution will be used to construct the first column of \( \mathfrak{M}(z) \) for \( z \in \mathcal{D}^- \).

**Lemma 4.3.** Consider the difference equation

\[ y(q^{-2}z) + (q^3z^2(1 + q^{-1}) - (1 + q^{-1}))y(q^{-1}z) + q^{-1}(1 + q^{-4}z^4)y(z) = 0. \]  
(4.2)

There exist two entire solutions of equation (4.2), one even and one odd.

**Proof.** Let

\[ y(z) = \sum_{i=0}^{\infty} y_i z^i. \]

Substituting this into equation (4.2) and comparing coefficients of \( z \), we find that \( y(z) \) is a solution iff

\[ y_i = \frac{(1 + q^{-1})q^{-i}y_{i-2} + q^{-4}y_{i-4}}{q^{2i+1} - (1 + q)q^{-i} + 1}. \]

Lemma 4.3 follows immediately.  

**Definition 4.4.** Following lemma 4.3 we define

\[ a(z) = \sum_{i=0}^{\infty} a_{2i} z^{2i}, \]
as the even entire solution to equation (4.2), normalised such that \( a_0 = 1 \). Similarly, we define

\[
b(z) = \sum_{i=0}^{\infty} b_{2i+1} z^{2i+1},
\]
as the odd entire solution, normalised such that \( b_1 = 1 \).

Motivated by the \((1, 2)\) entry of the right side of equation (4.1a), we consider the properties of the product \( y(z) g(z) w(z) \).

**Lemma 4.5.** Define

\[
v(z) = y(z) g(z) w(z),
\]
then \( v(z) \) is a solution of the difference equation

\[
(q^6 z^{-4} + q^{-2}) v(q^{-2} z) + (z^{-2} q (1 + q) - q^{-1} (1 + q^{-1})) v(q^{-1} z) + q^{-1} v(z) = 0.
\] (4.3)

Furthermore, there exist two solutions to equation \((4.3)\) analytic in \( \mathbb{C} \setminus \{0\} \) which can be represented by an even and odd power series at infinity.

**Proof.** From the definition of \( v(z) \) and lemmas 3.2 and 3.6 we find that

\[
y(q^{-1} z) = \frac{v(q^{-1} z)}{g(q^{-1} z) w(q^{-1} z)},
\]

\[
= -\frac{q^2 z^{-2} (1 + q^{-4} z^4) v(q^{-1} z)}{g(z) w(z)}.
\]

Substituting the above into equation (4.2) we determine that \( v(z) \) satisfies the difference equation

\[
(q^6 z^{-4} + q^{-2}) v(q^{-2} z) - (q^{-1} (1 + q^{-1}) - (q^2 + q) z^{-2}) v(q^{-1} z) + q^{-1} v(z) = 0.
\]

Let \( v(z) = \sum_{i=0}^{\infty} v_i z^{-i} \). Such a power series is a solution of equation (4.3) iff

\[
v_i = -\frac{q (1 + q) q^4 v_{i-2} + q^2 v_{i-4}}{q - (1 + q) q^4 + q^3}.
\]

Lemma 4.5 follows immediately (note that the power series for \( v(z) \) converges everywhere).

**Definition 4.6.** Following lemma 4.5 we define

\[
\phi_{\text{even}}(z) = \sum_{i=0}^{\infty} \phi_{\text{even},2i} z^{-2i},
\]
as the even solution to equation (4.3), normalised such that \( \phi_{\text{even},0} = 1 \). Similarly, we define

\[
\phi_{\text{odd}}(z) = \sum_{i=0}^{\infty} \phi_{\text{odd},2i+1} z^{-2i-1},
\]
as the odd solution, normalised such that \( \phi_{\text{odd},1} = 1 \).

Motivated by the \((1, 1)\) entry of the right side of equation (4.1a) for \( z \in \mathcal{D}_+ \), we consider the properties of the product \( y(z) g(z)^{-1} \).
Lemma 4.7. Define
\[ u(z) = y(z)g(z)^{-1}, \]
then \( u(z) \) is a solution of the difference equation
\[ q^{-5}u(q^{-2}z) + q^{-1}(z^{-2}(1 + q^{-1}) - q^{-3}(1 + q^{-1}))u(q^{-1}z) + (z^{-4} + q^{-4})u(z) = 0. \] (4.4)

Furthermore, there exist two solutions to equation (4.4) holomorphic for \( |z| > q \) which can be represented by an even and odd power series at infinity.

Proof. From the definition of \( u(z) \) and equation (3.2) we find that
\[
y(q^{-1}z) = u(q^{-1}z)g(q^{-1}z), \\
= -q^{-2}z^2u(q^{-1}z)g(z).
\]
Substituting the above into equation (4.2) we determine that \( u(z) \) satisfies the difference equation
\[ q^{-6}z^4u(q^{-2}z) - q^{-2}z^2(q^{-3}z^2(1 + q^{-1}) - (1 + q^{-1}))u(q^{-1}z) + q^{-1}(1 + q^{-4}z^4)u(z) = 0, \]
which one can readily show is equivalent to equation (4.4). Let \( u(z) = \sum_{i=0}^{\infty} u_i z^{-i} \). Such a power series is a solution of equation (4.4) iff
\[ u_i = -\frac{(1 + q)q^{i+1}u_{i-2} + q^5u_{i-4}}{q - (1 + q)q^4 + q^{4i}}. \] (4.5)
For large index \( i \), we can deduce from equation (4.5) that
\[
\max(\{u_i, |u_{i+2}\}) = (q^4 + O(q^6)) \max(\{|u_{i-4}|, |u_{i-2}|\}).
\]
Taking a telescopic product we conclude that the sum \( \sum_{i=0}^{\infty} u_i z^{-i} \) converges if \( |z| > q \). Lemma 4.5 follows immediately.

Definition 4.8. Following lemma 4.7 we define
\[
\varphi_{\text{even}}(z) = \sum_{i=0}^{\infty} \varphi_{\text{even}, 2i}z^{-2i},
\]
as the even solution to equation (4.4), analytic in \( |z| > q \), normalised such that \( \varphi_{\text{even}, 0} = 1 \). Similarly, we define
\[
\varphi_{\text{odd}}(z) = \sum_{i=0}^{\infty} \varphi_{\text{odd}, 2i+1}z^{-2i-1},
\]
as the odd solution, analytic in \( |z| > q \), normalised such that \( \varphi_{\text{odd}, 1} = 1 \).

We now look at how the functions defined in definitions (4.4)-(4.8) piece together.

Lemma 4.9. The function \( a(z)g(z)^{-1} \), where \( a(z) \) is defined in definition 4.4, can be written as
\[ a(z)g(z)^{-1} = \eta_1 h_q(z)\varphi_{\text{odd}}(z) + \eta_2 \varphi_{\text{even}}(z), \]
where \( \eta_1, \eta_2 \) are constants. Similarly \( b(z)g(z)^{-1} \) can be written as
\[ b(z)g(z)^{-1} = \eta_3 \varphi_{\text{odd}}(z) + \eta_4 h_q(z)\varphi_{\text{even}}(z). \]
Proof. From lemma 4.7, we conclude that \( a(z) \) satisfies the same difference equation as \( g(z) \varphi_{\text{even}}(z) \) and \( g(z) \varphi_{\text{odd}}(z) \). It follows that
\[
a(z) = g(z)(C_1(z) \varphi_{\text{even}}(z) + C_2(z) \varphi_{\text{odd}}(z)),
\]
for some functions \( C_i(z) \) which satisfy \( C_i(qz) = C_i(z) \). As \( a(z) \) is analytic everywhere and \( \varphi_{\text{even}}(z) \) and \( \varphi_{\text{odd}}(z) \) are holomorphic for \( |z| > q \) we conclude that \( C_i(z) \) is constant or has simple poles at the zeros of \( g(z) \), which occur at \( \pm q^k \) for \( k \in \mathbb{Z} \). Applying corollary A.2 to \( C_1(z) \) and \( C_2(z) \), and comparing even and odd terms we conclude that
\[
a(z) = g(z)(\eta_1 h_1(z) \varphi_{\text{odd}}(z) + \eta_2 \varphi_{\text{even}}(z)),
\]
and the first part of lemma 4.9 follows immediately. The equation for \( b(z)/g(z) \) also follows using similar arguments.

Remark 4.10. It follows from the above lemma that \( \eta_1, \eta_2, \eta_3 \) and \( \eta_4 \) are all non-zero. We know that \( a(z) \) is a non-zero analytic function and thus \( a(q^k) \neq 0 \) for large enough \( k \). However, as shown in corollary A.4, \( g(q^k) = h_1(q^{k+1/2}) = 0 \) for \( k \in \mathbb{Z} \). Thus, if \( \eta_1 = 0 \) or \( \eta_2 = 0 \) then we arrive at a contradiction. (Note that by equation (4.4) \( \varphi(z) \) cannot have poles along the real axis.)

Lemma 4.11. \( \phi_{\text{even}}(z) \) as defined in lemma 4.5 can be written as
\[
\phi_{\text{even}}(z) = g(z)w(z)(\lambda_3 h_2(z)b(z) + \lambda_4 a(z)),
\]
where are \( \lambda_3, \lambda_4 \) are non-zero constants. Similarly
\[
\phi_{\text{odd}}(z) = g(z)w(z)(\lambda_1 h_2(z)a(z) + \lambda_2 b(z)),
\]

Proof. From lemma 4.5 we conclude that \( \phi_{\text{even}}(z) \) satisfies the same difference equation (equation (4.3)) as \( a(z)g(z)w(z) \) and \( b(z)g(z)w(z) \). It follows that
\[
\phi_{\text{even}}(z) = g(z)w(z)(C_1(z)a(z) + C_2(z)b(z)),
\]
for some functions \( C_i(z) \) that satisfy \( C_i(qz) = C_i(z) \) [14, section 18]. As \( a(z) \) and \( b(z) \) are analytic everywhere and \( \phi_{\text{even}}(z) \) is holomorphic in \( \mathbb{C} \setminus \{0\} \) we conclude that \( C_i(z) \) is constant or has simple poles at the zeros of \( g(z) \), which occur at \( \pm q^k \) for \( k \in \mathbb{Z} \). Applying corollary A.2 to \( C_1(z) \) and \( C_2(z) \), and comparing even and odd terms we conclude that
\[
\phi_{\text{even}}(z) = g(z)w(z)(\lambda_4 a(z) + \lambda_3 h_2(z)b(z)),
\]
and the first part of lemma 4.9 follows immediately. The equation for \( \phi_{\text{odd}}(z) \) also follows using similar arguments. Repeating the arguments in remark 4.10 we conclude \( \lambda_1, \lambda_2, \lambda_3 \) and \( \lambda_4 \) are non-zero.

Remark 4.9. We note that \( w(z) \) has poles at \( e^{\frac{i(n+2m)\pi}{4}}q^{-k} \) for \( k \in \mathbb{N} \), and \( n \in 0, 1, 2, 3 \). As \( \phi_{\text{even}} \) is analytic at these locations we conclude from lemma 4.11 that
\[
\lambda_4 a \left( e^{\frac{i(n+2m)\pi}{4}}q^{-k} \right) + \lambda_3 h_2 \left( e^{\frac{i(n+2m)\pi}{4}}q^{-k} \right) b \left( e^{\frac{i(n+2m)\pi}{4}}q^{-k} \right) = 0,
\]
and,
\[
\lambda_2 b \left( e^{\frac{i(n+2m)\pi}{4}}q^{-k} \right) + \lambda_1 h_2 \left( e^{\frac{i(n+2m)\pi}{4}}q^{-k} \right) a \left( e^{\frac{i(n+2m)\pi}{4}}q^{-k} \right) = 0.
\]
Thus,
\[
\lambda_4 = -\lambda_3 \frac{h_2(e^{\frac{3\pi}{4}})b(e^{\frac{\pi}{4}})}{a(e^{\frac{\pi}{4}})},
\]
\[
\lambda_2 = -\lambda_1 \frac{h_2(e^{\frac{3\pi}{4}})a(e^{\frac{\pi}{4}})}{b(e^{\frac{\pi}{4}})}.
\]
and,
\[ \lambda_2 = -\lambda_1 \frac{h_q(e^{\pi i})a(e^{\pi i})}{b(e^{\pi i})}. \]

We are now in a position to solve the RHP given in definition 4.1.

**Lemma 4.13.** The solution of the \( \mathcal{M} \)-RHP (definition 4.1) is given by:
\[
\mathcal{M}(z) = \begin{cases} 
\eta_2^{-1}a(z) & \lambda_2\eta_2^{-1}\lambda_1^{-1}w(z)b(z) \\
\lambda_3b(z) & \lambda_4w(z)a(z) \\
\eta_2^{-1}a(z) & \lambda_2\eta_2^{-1}\lambda_1^{-1}w(z)b(z) \\
\lambda_3b(z) & \lambda_4w(z)a(z)
\end{cases} \begin{bmatrix} g(z)^{-1} w(z)g(z)h_q(z) \\ 0 & g(z) \end{bmatrix},
\]
for \( z \in D_+ \),

where \( \lambda_i \) is defined in lemma 4.11.

**Proof.** First we show that condition (i) (meromorphy) is satisfied. With the choice of \( \lambda_i \) from lemma 4.11 it is clear that \( \mathcal{M}(z) \) has analytic entries in the second column for \( z \in D_+ \). In particular the second column has entries equal to \( \eta_2^{-1}\lambda_1^{-1}\phi_{\text{odd}}(z) \) and \( \phi_{\text{even}}(z) \) for \( z \in D_+ \). Meromorphicity of the lhs column follows immediately from the definition of \( a(z) \) and \( b(z) \).

Conditions (ii) and (iv) follow immediately from the definition of \( \mathcal{M}(z) \). It is left to show condition (iii) is satisfied. Taking the limit \( z \to \infty \) of equation (4.6) and applying lemma 4.9 we find
\[ \lambda_3\eta_4 h_q(z)^2 + \lambda_4 \eta_2 \sim \frac{1}{g(z)^2 w(z)}. \]

Thus,
\[ \lambda_3 = \lim_{k \to \infty} \frac{1}{\eta_4 g(q^{-k})^2 w(q^{-k})h_q(q^{-k})^2}. \]

From equations (1.11) and (3.4) one can readily determine that \( w(z) \sim \omega(z) \) as \( z \to \infty \). We also note that \( \omega(z)g(z)^2 h_q(z)^2 \) is a \( q \)-periodic function (see equations (3.2) and (3.5)). Thus,
\[ \lim_{k \to \infty} g(q^{-k})^2 w(q^{-k})h_q(q^{-k})^2 = \mathcal{H}, \]
where \( \mathcal{H} \) is defined in equation (4.1e). Hence it follows that
\[ \lambda_3 = \frac{1}{\eta_4 \mathcal{H}}. \]

Applying lemma 4.9 again we find that as \( z \to \infty \)
\[ \lambda_3 b(z)g(z)^{-1} \sim \frac{h_q(z)}{\mathcal{H}}. \]

We conclude that as \( z \to \infty \) the matrix \( \mathcal{M}(z) \) behaves like
\[
\mathcal{M}(z) = \begin{bmatrix} \eta_2^{-1}a(z) & \lambda_2\eta_2^{-1}\lambda_1^{-1}w(z)b(z) \\
\lambda_3b(z) & \lambda_4w(z)a(z) \\
\eta_2^{-1}a(z) & \lambda_2\eta_2^{-1}\lambda_1^{-1}w(z)b(z) \\
\lambda_3b(z) & \lambda_4w(z)a(z)
\end{bmatrix} \begin{bmatrix} g(z)^{-1} w(z)g(z)h_q(z) \\ 0 & g(z) \end{bmatrix},
\]
\[ = \begin{bmatrix} 1 & 0 \\
h(z)^{-1} & \mathcal{O}(1/z) \end{bmatrix} + \mathcal{O}(1/z). \quad (4.7) \]

\[ \square \]
5. Far-field RHP

In this section, we solve the far-field RHP, which we denote by \( \mathcal{W} \)-RHP (see definition 5.1). The independent variable in the far-field and near-field RHPs are related through a scaling transformation. To distinguish the two, we use \( t \) instead of \( z \) to denote a complex variable in this section. Motivated by the form of equation (3.11b) we introduce the following RHP.

**Definition 5.1 (\( \mathcal{W} \)-RHP).** Let \( \Gamma \) be an appropriate curve (see definition 1.3) with interior \( D_- \) and exterior \( D_+ \). A \( 2 \times 2 \) complex matrix function \( \mathcal{W}(t) \), \( t \in \mathbb{C} \), is a solution of the RHP if it satisfies the following conditions:

(i) \( \mathcal{W}(t) \) is meromorphic in \( \mathbb{C} \setminus \Gamma \), with simple poles at \( t = \pm q^k \) for \( k \in \mathbb{Z} \).

(ii) \( \mathcal{W}(t) \) has continuous boundary values \( \mathcal{W}^-(s) \) and \( \mathcal{W}^+(s) \) as \( t \) approaches \( s \) from \( D_- \) and \( D_+ \) respectively, where

\[
\mathcal{W}^+(s) = \mathcal{W}^-(s) \begin{bmatrix} g(s) & 0 \\ 0 & g(s)^{-1} \end{bmatrix}, \quad s \in \Gamma.
\]  

(iii) \( \mathcal{W}(t) \) satisfies

\[
\mathcal{W}(t) = I + \mathcal{O} \left( \frac{1}{t} \right), \quad \text{as } |t| \to \infty.
\]  

Due to the simple poles in the second column of \( \mathcal{W}(t) \), the asymptotic decay condition only holds for \( |z \pm q^{-k}| > R \) for any \( R > 0 \).

(iv) The residue at the poles \( t = \pm q^k \) for \( k \in \mathbb{N} \) is given by

\[
\text{Res}(\mathcal{W}(\pm q^k)) = \lim_{t \to \pm q^k} \mathcal{W}(t) \begin{bmatrix} 0 & 0 \\ (t \mp q^{-1})g(t)^{-2}h_q(t)^{-1}\omega(t)^{-1} & 0 \end{bmatrix},
\]  

where \( \omega(t) \) is defined in equation (3.4).

(v) The residue at the poles \( t = \pm q^{-k} \) for \( 1 \leq k \in \mathbb{N} \) is given by

\[
\text{Res}(\mathcal{W}(\pm q^{-k})) = \lim_{t \to \pm q^{-k}} \mathcal{W}(t) \begin{bmatrix} 0 & (t \mp q^{-1})h_q(t)(\omega(t)^{-1})^{-1} \\ 0 & 0 \end{bmatrix}.
\]  

We will explicitly solve this RHP, using a similar approach to section 4. To do so, we prove a sequence of lemmas. We first solve a \( q \)-difference equation whose solution will be used to construct the first column of \( \mathcal{W}(t) \) for \( t \in D_+ \).

**Lemma 5.2.** Consider the difference equation

\[
y_1(q^{-2}t)/7^4t^{-2} - (1 - q^2(q + 1)t^{-2})y_1(q^{-1}t) + y_1(t) = 0.
\]  

There exists a solution of equation (5.2), analytic in \( \mathbb{C} \setminus \{0\} \), which can be represented by the even power series

\[
a_\infty(t) = \sum_{j=0}^{\infty} a_{2j}t^{-2j},
\]

where we take \( a_0 = 1 \) (w.l.o.g.).

Similarly, there exists a solution analytic in \( \mathbb{C} \setminus \{0\} \), of the difference equation

\[
y_2(q^{-2}t)/7^4t^{-2} - (q^{-1} - q^2(q + 1)t^{-2})y_2(q^{-1}t) + y_2(t) = 0,
\]  

where we take \( a_0 = 1 \) (w.l.o.g.).
which can be represented by the odd power series

\[ b_\infty(t) = \sum_{j=0}^{\infty} b_{2j+1} r^{-2j-1}, \]

where without loss of generality we let \( b_1 = 1 \).

**Proof.** Let

\[ y_1(z) = \sum_{i=0}^{\infty} y_i t^{-i}. \]

Substituting this into equation (5.2) and comparing coefficients of \( t \), we find that \( y(t) \) is a solution iff

\[ y_i = - \frac{(1+q)q^i y_{i-2} + q^{2i-1} y_{i-4}}{q^i - 1}. \]

The first part of lemma 5.2 follows immediately. The second part follows using similar arguments.

As in section 4 we now define the different functions that we will use to construct \( W(t) \). These functions are solutions to various \( q \)-difference equations motivated by the form of the jump condition, equation (5.1a), and the residue condition, equation (5.1c).

**Lemma 5.3.** Define

\[ \alpha(t) = y_1(t)g(t)\omega(t), \]

where \( y_1(t) \) satisfies the difference equation given by equation (5.2) and \( \omega(t) \) is defined in equation (3.4). Then \( \alpha(t) \) is a solution of the difference equation

\[ q\alpha(q^{-2}t) + (q^{-2}t^2 - (1+q))\alpha(q^{-1}t) + \alpha(t) = 0. \]  

(5.4)

Furthermore, there exist two entire solutions of equation (5.4) which can be represented by an even and odd power series.

Similarly, define

\[ \beta(t) = y_2(t)g(t)\omega(t), \]

where \( y_2(t) \) satisfies the difference equation given by equation (5.3), then \( \beta(t) \) satisfies the difference equation

\[ q\beta(q^{-2}t) + (q^{-2}t^2 - (1+q))\beta(q^{-1}t) + \beta(t) = 0. \]  

(5.5)

Furthermore, there exist two entire solutions which can be represented by an even and odd power series.

**Proof.** From the definition of \( \alpha(t) \), equations (3.2) and (3.5) we find that

\[ y_1(q^{-1}t) = \frac{\alpha(q^{-1}t)}{g(q^{-1}t)\omega(q^{-1}t)}, \]

\[ = - \frac{q^{-2}t^2 \alpha(q^{-1}t)}{g(t)\omega(t)}. \]

Substituting the above into equation (5.2) we determine that \( \alpha(t) \) satisfies the difference equation

\[ \alpha(q^{-2}t)q + (q^{-2}t^2 - (q + 1))\alpha(q^{-1}t) + \alpha(z) = 0. \]
Let $\alpha(t) = \sum_{i=0}^{\infty} \alpha_i t^i$. Such a power series is a solution of equation (5.4) iff

$$\alpha_i = -\frac{q^{-1-\iota} \alpha_{\iota-2}}{1 - (1 + q)q^{-i} + q^{-1-i}}.$$ 

The first part of lemma 5.3 follows immediately. The second part follows using similar arguments. □

**Definition 5.4.** Following lemma 5.3 we define

$$\Psi^{(1)}_{\text{even}}(t) = \sum_{i=0}^{\infty} \Psi^{(1)}_{\text{even,2i}} t^{2i},$$

as the even entire solution to equation (5.4), normalised such that $\Psi^{(1)}_{\text{even,0}} = 1$. Similarly we define

$$\Psi^{(1)}_{\text{odd}}(t) = \sum_{i=0}^{\infty} \Psi^{(1)}_{\text{odd,2i+1}} t^{2i+1},$$

as the odd entire solution to equation (5.4), normalised such that $\Psi^{(1)}_{\text{odd,1}} = 1$.

Furthermore, we define

$$\Psi^{(2)}_{\text{even}}(t) = \sum_{i=0}^{\infty} \Psi^{(2)}_{\text{even,2i}} t^{2i},$$

as the even entire solution to equation (5.5), normalised such that $\Psi^{(2)}_{\text{even,0}} = 1$. Similarly we define

$$\Psi^{(2)}_{\text{odd}}(t) = \sum_{i=0}^{\infty} \Psi^{(2)}_{\text{odd,2i+1}} t^{2i+1},$$

as the odd entire solution to equation (5.5), normalised such that $\Psi^{(2)}_{\text{odd,1}} = 1$.

**Lemma 5.5.** Define

$$\Theta_1(t) = y_1(t)g(t)^{-1},$$

where $y_1(t)$ is a solution of equation (5.2). Then $\Theta_1(t)$ is a solution of equation (5.4).

Similarly, $\Theta_2(t) = y_2(t)g(t)^{-1}$ is a solution of equation (5.5).

**Proof.** From the definition of $\Theta_1(t)$ we find that

$$\Theta_1(q^{-1}t) = \frac{y_1(q^{-1}t)}{g(q^{-1}t)},$$

$$= \frac{\alpha(q^{-1}t)}{g(q^{-1}t)^2 w(q^{-1}t)},$$

$$= \frac{\alpha(q^{-1}t)}{g(t)^2 w(t)},$$

where we have used equations (3.2) and (3.5) to arrive at the final line. The first part lemma 5.5 follows immediately from equation (5.6). The proof for $\Theta_2(t)$ follows from the same arguments. □

**Lemma 5.6.** The function $a_\infty(t)g(t)^{-1}$, where $a_\infty(t)$ is defined in lemma 5.2, can be written as

$$a_\infty(t)g(t)^{-1} = \mu_1 h_q(t) \Psi^{(1)}_{\text{odd}}(t) + \mu_2 \Psi^{(1)}_{\text{even}}(t).$$
where \( \mu_1, \mu_2 \) are non-zero constants. Similarly \( b_\infty(t)g(t)^{-1} \) can be written as
\[
b_\infty(t)g(t)^{-1} = \mu_3 \Psi_{\text{odd}}^{(1)}(t) + \mu_4 h_q(t) \Psi_{\text{even}}^{(1)}(t).
\] (5.7)

**Proof.** From lemma 5.5 we conclude that \( a_\infty(t) \) satisfies the same difference equation (equation (5.5)) as \( g(t)\Psi_{\text{even}}^{(1)}(t) \) and \( g(t)\Psi_{\text{odd}}^{(1)}(t) \). It follows that
\[
a_\infty(t) = g(t)(C_1(t)\Psi_{\text{even}}^{(1)}(t) + C_2(t)\Psi_{\text{odd}}^{(1)}(t)),
\]
for some functions \( C_1(t) \) which satisfy \( C_1(qt) = C_1(t) \). As \( a_\infty(t) \) is analytic in \( \mathbb{C} \setminus \{0\} \) and \( \Psi_{\text{even}}^{(1)}(t) \) and \( \Psi_{\text{odd}}^{(1)}(t) \) are entire we conclude that \( C_1(t) \) is constant or has simple poles at the zeros of \( g(t) \), which occur at \( \pm q^k \) for \( k \in \mathbb{Z} \). Applying corollary A.2 and comparing even and odd terms we conclude that
\[
a_\infty(t) = g(t)(\mu_1 h_q(t) \Psi_{\text{odd}}^{(1)}(t) + \mu_2 \Psi_{\text{even}}^{(1)}(t))
\]
and the first part of lemma 5.6 follows immediately. The equation for \( b_\infty(t)g(t)^{-1} \) also follows using similar arguments. Repeating the arguments in remark 4.10 we conclude \( \mu_1, \mu_2, \mu_3 \) and \( \mu_4 \) are non-zero.

**Lemma 5.7.** Let \( b_\infty(t), \mu_4 \) and \( \Psi_{\text{even}}^{(1)}(t) \) be defined as in lemma 5.6. Then,
\[
\frac{1}{\mu_4 \mathcal{H}} b_\infty(t)g(t)^{-1} \sim \frac{h_q(t)}{\mathcal{H}},
\]
(5.8)
as \( t \to 0 \), where \( \mathcal{H} \) is defined in equation (4.1c). It is also true that
\[
\text{Res} \left( \frac{1}{\mu_4 \mathcal{H}} b_\infty(q^k)\omega(q^k)h_q(q^k) \right) = \text{Res}(\Psi_{\text{even}}^{(1)}(q^k)g(q^k)^{-1}),
\]
(5.9)
for \( k \in \mathbb{Z} \).

**Proof.** Equation (5.8) follows immediately from taking the limit \( t \to 0 \) in equation (5.7). Multiplying equation (5.7) by \( \omega(t)g(t)h_q(t) \) we find
\[
b_\infty(t)\omega(t)h_q(t) = \omega(t)g(t)h_q(t)\mu_3 \Psi_{\text{odd}}^{(1)}(t) + \omega(t)g(t)^2h_q(t)^2\mu_4 (\Psi_{\text{even}}^{(1)}(t)g(t)^{-1}).
\]
Studying the residue at \( t = q^k \) for \( k \in \mathbb{Z} \) we find that
\[
\text{Res}(b_\infty(q^k)\omega(q^k)h_q(q^k)) = \text{Res} \left( \omega(q^k)g(q^k)^2h_q(q^k)^2\mu_4 (\Psi_{\text{even}}^{(1)}(q^k)g(q^k)^{-1}) \right),
\]
\[= \text{Res} \left( \mathcal{H}\mu_4 \Psi_{\text{even}}^{(1)}(q^k)g(q^k)^{-1} \right).\]
\[\square\]

**Remark 5.8.** Repeating the arguments of lemma 5.7 one can readily show
\[
\text{Res}(a_\infty(q^k)\omega(q^k)h_q(q^k)) = \text{Res} \left( \mathcal{H}\mu_1 \Psi_{\text{odd}}^{(1)}(q^k)g(q^k)^{-1} \right).
\]
(5.10)

We require one last lemma before determining the solution of the far-field RHP.

**Lemma 5.9.** Let \( \Psi_{\text{even}}^{(1)}(t) \) be defined as in lemma 5.3, then
\[
\Psi_{\text{even}}^{(1)}(t)g(t)^{-1} = c_\Psi + \mathcal{O}(t^{-1}), \text{ as } t \to \infty,
\]
where \( c_\Psi \) is a non-zero constant and this limit clearly does not hold near the poles of \( \Psi_{\text{even}}^{(1)}(t)g(t)^{-1} \), but holds for \( t \) satisfying \( |t - q^k| > r \), for some fixed \( r > 0 \) and all \( k \in \mathbb{Z} \).
We first show that the residue of the poles of \( \Psi_{\text{even}}^{(2)}(t)g(t)^{-1} \) are vanishing faster than \( q^{k/2} \) as \( |k| \to \infty \).

Consider the case \( k \to +\infty \) (\( t \to 0 \)), from equation (3.3) we find that
\[
g(q^{k/2}t) = (-1)^{n/2}q^{-n/4}g(z).
\]
By lemma 5.3 we know that \( \Psi_{\text{even}}^{(2)}(q^{k}) \sim 1 \) as \( k \to \infty \). Thus, we conclude
\[
\text{Res}(\Psi_{\text{even}}^{(2)}(q^{k})g(q^{k})^{-1}) < O(q^{k/2}), \quad \text{as} \quad k \to +\infty.
\]
Note that the above statement is true for \( O(q^{k^2}) \), with \( c < 1 \).

Consider the case \( k \to -\infty \). From lemma 5.7 it is clear that a bound on \( \text{Res}(b_{\infty}(q^{k})\omega(q^{k})h_{q}(q^{k})) \) as \( k \to -\infty \) is equivalent to a bound on \( \text{Res}(\Psi_{\text{even}}^{(2)}(q^{k})g(q^{k})^{-1}) \).

By definition in lemma 5.2 we determine that \( b_{\infty}(q^{k}) = O(q^{-k}) \). Furthermore, using induction on equation (3.5) we find
\[
\omega(q^{k}t) = q^{2k(k-1)}t^{k}w(t).
\]
It follows
\[
\text{Res}(b_{\infty}(q^{k})\omega(q^{k})h_{q}(q^{k})) < O(q^{k^2}), \quad \text{as} \quad k \to -\infty.
\]
Let \( R_{k} \) be the residue of \( \Psi_{\text{even}}^{(2)}(t)g(t)^{-1} \) at \( t = q^{k} \). Define the function
\[
F(t) = \Psi_{\text{even}}^{(2)}(t)g(t)^{-1} - \sum_{k=-\infty}^{\infty} \frac{R_{k}}{t-q^{k}},
\]
where this sum is well defined for all \( t \) because we have just shown \( R_{k} < O(q^{k/2}) \) as \( |k| \to \infty \).

It follows \( F(t) \) is holomorphic in \( \mathbb{C} \setminus \{0\} \) and can be represented by a Laurent series which converges everywhere. We will show that \( F(t) = \sum_{j=0}^{\infty} F_{j}t^{-j} \) (i.e. there are no positive powers of \( t \)). Applying equations (3.2) and (5.5) we find that \( v(t) = \Psi_{\text{even}}^{(2)}(t)g(t)^{-1} \) satisfies the difference equation
\[
v(q^{-2}t) + ((1 + q)q^{3}r^{-2} - 1)v(q^{-1}t) + q^{3}r^{-4}v(t) = 0.
\]
(5.11)

Writing the above in matrix form we have
\[
\begin{bmatrix}
v(q^{-2}t) \\
v(q^{-1}t)
\end{bmatrix} =
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} - q^{3}r^{-2}
\begin{bmatrix}
-(1 + q) & -q^{3}r^{-4} \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
v(q^{-1}t) \\
v(t)
\end{bmatrix}.
\]
(5.12)

Observe that the eigenvalues of the first matrix on the right in the above equation are 1 and 0. Hence, repeatedly applying equation (5.12) to determine the behaviour of \( v(t) \) as \( t \to \infty \) is essentially a Pochhammer symbol with matrix entries. Thus, \( v(t) \), and consequently \( F(t) \) are bounded by a constant as \( t \to \infty \). We now show that this constant is non-zero. From lemma 5.3 we know that \( \Psi_{\text{even}}^{(2)}(t) \) is an entire function (which is not the constant function), hence \( \Psi_{\text{even}}^{(2)}(t) \) must grow in some direction. \( \Psi_{\text{even}}^{(2)}(t) \) satisfies equation (5.5),
\[
q^{3}\Psi_{\text{even}}^{(2)}(q^{-2}t) + (r^{2}q^{-3} - (1 + q))\Psi_{\text{even}}^{(2)}(q^{-1}t) + \Psi_{\text{even}}^{(2)}(t) = 0.
\]

It follows that as \( t \) becomes large there must exist a ray where
\[
r^{2}q^{-3}\Psi_{\text{even}}^{(2)}(q^{-1}t) \gg \Psi_{\text{even}}^{(2)}(t).
\]
Thus, along this ray
\[
\Psi_{\text{even}}^{(2)}(q^{-2}t) = -r^{2}q^{-3}\Psi_{\text{even}}^{(2)}(q^{-1}t)(1 + O(r^{-2})).
\]
and applying equation (3.2) we conclude $\psi_{\text{even}}^{(2)}(t)g(t)\rightarrow^{-1}$ approaches a constant along this ray. Thus, $F(t) = \sum_{j=0}^{\infty} F_j t^{-j}$ and $F_0 \neq 0$. Lemma 5.9 follows immediately.

**Remark 5.10.** Repeating the same arguments as in lemma 5.9 we can conclude
\[
\psi_{\text{odd}}^{(1)}(t)g(t)\rightarrow^{-1} \sim O(t^{-1}), \text{ as } t \rightarrow \infty,
\] (5.13)
where again this limit clearly does not hold near the poles of $\psi_{\text{odd}}^{(1)}(t)g(t)\rightarrow^{-1}$, but holds for $t$ satisfying $|t - d_k| > r$, for some fixed $r > 0$ and all $k \in \mathbb{Z}$.

Now we are in a position to solve the far-field RHP given in definition 5.1. Let $\mathcal{W}(t)$ be given by
\[
\mathcal{W}(t) = \begin{bmatrix}
    a_\infty(t) & \mu_i \mathcal{H} \psi_{\text{odd}}^{(1)}(t) \\
    b_\infty(t) & \mu_i \mathcal{H} \psi_{\text{even}}^{(2)}(t)
\end{bmatrix} / \mu_4 \mathcal{H} c_{\psi}^{-1}, \\
\quad \quad \text{for } t \in \mathcal{D}_-,
\]
(5.14)
\[
= \begin{bmatrix}
    a_\bigodds(t) & \mu_i \mathcal{H} \psi_{\text{even}}^{(2)}(t)g(t)\rightarrow^{-1} \\
    b_\bigodds(t) & \mu_i \mathcal{H} \psi_{\text{odd}}^{(1)}(t)g(t)\rightarrow^{-1}c_{\psi}^{-1}
\end{bmatrix} , \quad \text{for } t \in \mathcal{D}_+.
\]

Consider the conditions for this function to solve the far-field RHP. First, we note that condition (i), i.e. meromorphicity, is satisfied as by definition $a_\infty$, $b_\infty$, $\psi_{\text{odd}}^{(1)}$ and $\psi_{\text{even}}^{(2)}$ are all analytic in $\mathbb{C} \setminus \{0\}$ (see lemmas 5.2 and 5.3). Second, note that condition (ii), the jump condition, holds by direct calculation. Third, to show condition (iii), i.e. asymptotic decay, observe that $\mathcal{W}_{(1,1)}(t) = 1 + O(1/t^r)$ as $t \rightarrow \infty$ by the definition of $a_\infty(t)$, similarly $\mathcal{W}_{(1,2)}(t) = O(1/t)$ as $t \rightarrow \infty$ by the definition of $b_\infty(t)$. From lemma 5.9 we conclude that $\mathcal{W}_{(2,2)}(t) = 1 + O(1/t)$ as $t \rightarrow \infty$. Similarly from remark 5.10 we conclude $\mathcal{W}_{(2,1)}(t) = O(1/t)$ as $t \rightarrow \infty$. The remaining conditions (iv) and (v), the residue conditions, follow from lemma 5.7 and remark 5.8. Furthermore, from lemmas 5.6, 5.7 and 5.9 we find
\[
\mathcal{W}(t) = \begin{bmatrix}
    \mu_2 & 0 \\
    h_g(t) & c_{\psi}^{-1}
\end{bmatrix} + O(t), \quad \text{as } t \rightarrow 0.
\] (5.15)

**Remark 5.11.** Note that for any fixed constant $q < c < 1$ the function $h_g(t)$ is $O(1)$ for $|t| = cq^k$, as $k \rightarrow +\infty$. This is because $h_g(t)$ is $q$-periodic (see equation (1.3)). Thus, the $(2, 1)$ entry in equation (5.15) is $O(1)$ for $|t| = cq^k$.

### 6. Gluing together near- and far-field RHPs

We will now glue together the near- and far-field RHPs to approximate the RHP for $W_n(z)$ as $n \rightarrow \infty$. The near- and far-field variables in sections 4 and 5 are related by the linear transformation $t = zq^{n/2}$.

We first make a linear transformation to the near-field RHP solution. Let
\[
\mathfrak{W}(z) = \begin{bmatrix}
    \mu_2 & 0 \\
    0 & c_{\psi}^{-1}
\end{bmatrix} \mathfrak{W}(z).
\] (6.1)
Applying equations (4.7) and (5.15) we immediately deduce that
\[
\lim_{z \rightarrow \infty} \mathfrak{W}(z) = \lim_{t \rightarrow 0} \mathfrak{W}(t).
\]
We next make a slight modification to $\mathfrak{W}(t)$ given by equation (5.14). Note that the residue condition for $W_n(z)$ given in equation (3.11e) is different to that for $\mathfrak{W}(t)$ given in

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equation (5.1d). This is not a significant issue however, as we will show these two residue conditions differ by an error of order $O(q^{n/2})$. To resolve the issue we define the new function

$$\tilde{\mathcal{W}}(t) = \mathcal{W}(t) - \left(1 - \frac{w(z)z^{2n}e^{2n}}{\omega(t)}\right)\mathcal{W}(t) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$  \hspace{1cm} (6.2)

For $t \in \mathcal{D}_+$ the residue condition for $\mathcal{W}(t)$ is given by equation (5.1d). Thus, after applying equation (6.2) we find the residue for $\tilde{\mathcal{W}}(t)$ is given by

$$\text{Res}(\tilde{\mathcal{W}}(\pm q^{-k})) = \lim_{t \to \pm q^{-k}} \tilde{\mathcal{W}}(t) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$  \hspace{1cm} (6.3)

This now matches the residue condition for $W_n(z)$ (equation (3.11e)). Note that the relationship between $t$ and $z$, $\mathcal{W}(t)$ and $W_n(z)$ will become clear when we define $G(z)$ shortly. Substituting in equation (3.8) into equation (6.2) we find that

$$\tilde{\mathcal{W}}(t) = \mathcal{W}(t) - (1 - (-z^{-2};q^4)_\infty)\mathcal{W}(t) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$  \hspace{1cm} (6.4)

Thus, the difference between $\tilde{\mathcal{W}}(t)$ and $\mathcal{W}(t)$ is bounded by $O(q^{n/2})$ for $z > q^{-n/4}$ ($t > q^{n/4}$).

We are now in a position to determine the asymptotic behaviour of $W_n(z)$.

Define

$$G(z) = \begin{cases} \tilde{\mathcal{W}}(z), & \text{for } z \in \mathcal{D}_{-q^{-n/4}}, \\ \tilde{\mathcal{W}}(zq^{n/4}), & \text{for } z \in \mathcal{D}_{+q^{-n/4}}, \end{cases}$$  \hspace{1cm} (6.5)

and, furthermore let

$$R(\zeta) = W_n(\zeta q^{-n/4})G(\zeta q^{-n/4})^{-1}.$$  \hspace{1cm} (6.6)

Note that by construction the poles and jump conditions of $W_n(\zeta q^{-n/4})$ cancel with those of $G(\zeta q^{-n/4})$, leaving only the jump in $G(\zeta q^{-n/4})$ at $|\zeta| \sim 1$. Thus, $R(\zeta)$ satisfies the following RHP.

**Definition 6.1 (R(ζ) RHP).** A $2 \times 2$ complex matrix function $R(\zeta)$, $\zeta \in \mathbb{C}$, is a solution of the $R(\zeta)$ RHP if it satisfies the following conditions:

(i) $R(\zeta)$ is analytic in $\mathbb{C} \setminus \Gamma$.

(ii) $R(\zeta)$ has continuous boundary values $R^-(s)$ and $R^+(s)$ as $\zeta$ approaches $s \in \Gamma$ from $\mathcal{D}_-$ and $\mathcal{D}_+$ respectively, where

$$R^+(s) = R^-(s)\tilde{\mathcal{W}}(sq^{-n/4})^{-1}\tilde{\mathcal{W}}(sq^{n/4}) s \in \Gamma.$$  \hspace{1cm} (6.7a)

(iii) $R(\zeta)$ satisfies

$$R(\zeta) = I + O\left(\frac{1}{\zeta}\right), \text{as } |\zeta| \to \infty.$$  \hspace{1cm} (6.7b)

From equations (4.7) and (5.15) we find that

$$\|\tilde{\mathcal{W}}(sq^{-n/4})^{-1}\tilde{\mathcal{W}}(sq^{n/4}) - I\|_\Gamma = \|\left(I + O(q^{n/4})\right)^{-1} \left(I + O(q^{n/4})\right) - I\|_\Gamma = O(q^{n/4}).$$

Thus, we can apply theorem B.2 to conclude

$$|R(\zeta) - I| = O(q^{n/4}).$$  \hspace{1cm} (6.8)
Remark 6.2. We note that there are many eligible choices for \(|z|\) where we could have defined the jump \(\mathcal{W}([z])^{-1}\mathcal{W}([q^{n/2}])\). However, from equations (4.7) and (5.15) we observe that the choice of \(|z| = q^{-n/2}\) (where \(|t| = 1/|z|\)) minimises the error of \(R(\zeta)\).

7. Proofs of main theorems

Having proved equation (6.8), we are now in a position to prove the first two main theorems of this paper.

Proof of theorem 1.5. By the definition of \(R(\zeta)\), we find that

\[
W_n(z)G(z)^{-1} = R(q^{n/4}),
\]
\[
= I + O(q^{n/4}),
\]
\[
W_n(z) = (I + O(q^{n/4}))G(z).
\]

Looking at the \((1,1)\)-entry of \(W_n(z)\) we find that for \(z \leq q^{-n/4}\),

\[
c_n^0 P_n(z) = G_{(1,1)}(z) + O(q^{n/4})G_{(2,1)}(z).
\]

Thus, applying equation (4.13) we find

\[
(-1)^{n/2} q^{3(n-1)/2} P_n(z) = \frac{\lambda_2}{\eta_2} a(z) + O(q^{n/4})\frac{\lambda_3}{c_\psi} b(z).
\]

Repeating the arguments above for each matrix entry, theorem 1.5 follows immediately.

Proof of theorem 1.7. Using the transformations detailed in section 3.1 we find that

\[
W_n(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{t} \begin{bmatrix} 0 & \gamma_{n-1}^{-1} e^{-2n} \\ \gamma_{n-1}^{-1} e^{-2n} & 0 \end{bmatrix} + O(t^{-2}).
\]

Let,

\[
\mathcal{W}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{t} \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} + O(t^{-2}).
\]

Note that there is no difference in the \(O(1/t)\) term between \(\mathcal{W}(t)\) and \(\mathcal{W}(t)\). Using the definition of \(R(\zeta)\) given in equations (6.6) and (6.8) we find

\[
\gamma_n ((-1)^{n/2} q^{3(n-1)/2}) = A(1 + O(q^{n/2})) q^{-n/2},
\]
\[
\gamma_n = A(1 + O(q^{n/2})) q^{-n(n-1)/2},
\]
\[
= A(1 + O(q^{n/2})) q^{-n(n-1)/2}.
\]

Similarly in the bottom left term we find in the limit \(n \to \infty:\)

\[
\gamma_{n-1} = q^{3(3-n)} \left( B^{-1} + O(q^{n/2}) \right).
\]

Remark 7.1. By equation (5.14), remark 5.10 and the definition of \(b_\infty(t)\) we conclude that \(A\) and \(B\) are non-zero. Furthermore, these definitions also allow one to numerically compute \(A\) and \(B\).

Taking the ratio of \(\gamma_n\) and \(\gamma_{n-1}\) we find that

\[
\alpha_n = \gamma_n / \gamma_{n-1} = ABq^{-n}(1 + O(q^{n/2})).
\]
However, we can determine that $AB = q$ by considering the arguments presented in theorem 8.3. In particular, we know that in the limit $n \to \infty$ the $qP_n$ polynomials satisfy equations (8.22) and (8.23). □

8. Recurrence coefficients and $q$-discrete Painlevé

As discussed in remark 2.5, the class of monic polynomials $\{P^{(c)}_n(x)\}_{n=0}^\infty$ satisfying the orthogonality condition

$$\int_{-\infty}^{\infty} P^{(c)}_n(cx)P^{(c)}_m(cx)w(cx)dx = \gamma^{(c)}_n \delta_{n,m}, \tag{8.1}$$

where $q < c \leq 1$, satisfy a corresponding RHP. In this section we discuss the connection between the RHP, the asymptotic behaviour of $P^{(c)}_n(z)$ and uniqueness results concerning their recurrence coefficients. First, we use the RHP to show that in general $P^{(c)}_n(z) \neq P^{(1)}_n(z)$.

**Lemma 8.1.** Let $\{P^{(c)}_n(z)\}_{n=0}^\infty$ be the class of monic polynomials with orthogonality condition given by equation (8.1). Then, the two classes of orthogonal polynomials corresponding to the cases $c = 1$ and $c = q^{1/2}$ are the same. Furthermore,

$$\frac{\gamma^{(c)}_n}{\gamma^{(q^{1/2})}_n} = \frac{q^{1/2}h_q(q^{\pi/4})}{h_q(q^{1/2}w^{\pi/4})}.$$

Moreover, if $c \neq 1, q^{1/2}$ then $\{P^{(c)}_n(z)\}_{n=0}^\infty \neq \{P^{(1)}_n(z)\}_{n=0}^\infty$.

**Proof.** Let

$$\tilde{Y}^{(1,2)}_n(z) = \int_{-\infty}^{\infty} \frac{P^{(c)}_n(x)w(x)}{z-x}dx - P^{(1)}_n(z)w(z)h_q(z).$$

From the arguments in section 2 it follows that $\tilde{Y}^{(1,2)}_n(z)$ is meromorphic with simple poles at location of the poles of $w(z)$. At these locations

$$\text{Res}(\tilde{Y}^{(1,2)}_n(z)) = -\text{Res}(P^{(1)}_n(z)w(z)h_q(z)). \tag{8.2}$$

Considering the function

$$F_n(z) = \frac{h_q(q^{1/2}w^{\pi/4})}{h_q(q^{\pi/4})} \tilde{Y}^{(1)}_n(z) + P^{(1)}_n(z)w(z)h_q(q^{1/2}z),$$

from equation (8.2), lemma A.3 and remark A.6 we conclude that $F_n(z)$ is meromorphic with simple poles at $z = q^{k+1/2}$ for $k \in \mathbb{Z}$. In particular, $F_n(z)$ does not have poles at the location of the poles of $w(z)$. The residue of $F_n(z)$ at $z = q^{k+1/2}$ is given by

$$\text{Res}(F_n(q^{k+1/2})) = P^{(1)}_n(q^{k+1/2})w(q^{k+1/2})q^{k+1/2}.$$

Note that $w(z)$ decays much faster than inverse polynomial decay and thus

$$\lim_{z \to \infty} F_n(z) = \frac{h_q(q^{1/2}w^{\pi/4})}{h_q(q^{\pi/4})} \lim_{z \to \infty} \tilde{Y}^{(1)}_n(z).$$
Hence, the solution for the RHP corresponding to \( c = q^{1/2} \) (see remark 2.5) can also be written as

\[
Y_n^{(q^{1/2})}(z) = \begin{cases} 
\left[ \frac{P_n^{(1)}(z)}{P_n^{(1)}(z)} \frac{F_n^{(1)}(z)}{F_n^{(1)}(z)} \frac{q^{-1/2}F_n(z)}{q^{1/2}F_n(z)} \right], & \text{for } z \in \mathcal{D}_+,
\left[ \frac{P_n^{(1)}(z)}{P_n^{(1)}(z)} \frac{F_n^{(1)}(z)}{F_n^{(1)}(z)} \frac{q^{-1/2}(F_n(z) - P_n^{(1)}(z)w(z)h_q(q^{1/2}z))}{q^{1/2}(q^{1/2}z)} \right], & \text{for } z \in \mathcal{D}_-.
\end{cases}
\]

Thus, we have shown that \( P_n^{(1)}(z) = Y_n^{(q^{1/2})}(z) \). One can readily deduce from section 2 that

\[
\lim_{z \to \infty} \gamma_n^{(1)}(z) = \frac{\gamma_n^{(1)}}{e^{\pi/4}},
\]

it follows from remark 2.5,

\[
\gamma_n^{(q^{1/2})}q^{1/2} = \frac{h_q(q^{1/2}e^{\pi/4})}{h_q(e^{\pi/4})} \gamma_n^{(1)}.
\]

Finally, we prove that \( P_n^{(c)}(z) \neq P_n^{(1)}(z) \) if \( c \neq q^{1/2} \). Assume to the contrary that \( \{P_n^{(c)}(z)\}_{c=0}^{\infty} = \{P_n^{(1)}(z)\}_{c=0}^{\infty} \). Let, \( Y_n^{(c)} \) be the solution of the corresponding RHP given in remark 2.5. Note that the first column of \( Y_n^{(c)} \) is the same as that in \( Y_n^{(1)} \).

By remark 2.4, we know that the second column of \( Y_n^{(c)} \) must satisfy the same \( q \)-difference equation as the second column of \( Y_n^{(1)} \), where \( \hat{Y}_n = Y_n \) restricted to \( z \in \mathcal{D}_- \). If \( \{P_n^{(c)}(z)\}_{c=0}^{\infty} = \{P_n^{(1)}(z)\}_{c=0}^{\infty} \) then by the analyticity of \( \hat{Y}_n \) and comparing even and odd terms we conclude the second column of \( Y_n^{(c)} \) must satisfy

\[
\hat{Y}_n^{(c)}(z) = \hat{Y}_n^{(1)}(z) \begin{bmatrix} 1 & 0 \\
0 & C_0 \end{bmatrix}, \quad \text{for } z \in \mathcal{D}_-,
\]

where \( C_0 \) is a constant. Denoting the \((1,2)\)-entry of \( \hat{Y}_n^{(c)}(z) \), for \( z \in \mathcal{D}_- \), by \( \hat{Y}_n^{(c)}(1,2) \), we conclude that \( \hat{Y}_n^{(c)}(1,2) = C_0 \hat{Y}_n^{(1)}(1,2) \).

By the analyticity of \( Y_n^{(c)}(z) \), we deduce that at the poles of \( w(z) \), in particular at \( z = e^{i\pi/4}, e^{3i\pi/4}, e^{-i\pi/4}, e^{-3i\pi/4} \), we have

\[
C_0 \hat{Y}_n^{(1)}(z) + w(z)P_n^{(c)}(z)h(z/c) = 0.
\]

Furthermore,

\[
\hat{Y}_n^{(1)}(z) + w(z)P_n^{(1)}(z)h(z) = 0.
\]

Equations (8.3) and (8.4) can only be satisfied if the ratio \( h(z/c)/h(z) \) is equal at the four points \( z = e^{i\pi/4}, e^{3i\pi/4}, e^{-i\pi/4}, e^{-3i\pi/4} \). By lemma A.5 and remark A.6, it follows that the equality can only hold at these four points if \( c = 1, q^{1/2} \), which is the desired result. \( \square \)
Theorem 8.2. Suppose that the sequence of monic polynomials \( \{P_n^{(c)}(x)\}_{n=0}^{\infty} \) satisfies the orthogonality condition
\[
\int_{-\infty}^{\infty} P_n^{(c)}(cx)P_m^{(c)}(cx)w(cx)dx = \delta_n,m,
\]
where \( w(x) \) is given by equation (1.11) and \( q < c \leq 1 \). Then, the recurrence coefficients \( \{\alpha_n^{(c)}\}_{n=0}^{\infty} \), which occur in the recurrence relation
\[
xP_n^{(c)}(x) = P_{n+1}^{(c)}(x) + \alpha_n^{(c)}P_{n-1}^{(c)}(x),
\]
solve the equation:
\[
\alpha_n(\alpha_{n+1} + q^{n+1} - q^n - q^{n-1} - q^{n-2}) = (q^{-n} - 1)q^{1-n},
\]
with initial conditions \( \alpha_n = 0 \) for \( n \leq 0 \). Furthermore,
\[
D_q^{-1}P_n^{(c)} = [n]_q^{-1}P_n^{(c)} - \frac{q^n - 1}{q - 1}\alpha_n^{(c)}\alpha_{n-2}^{(c)}P_n^{(c)}.
\]

Proof. In order to show equation (8.7), we observe that
\[
\sum_{k=-\infty}^{\infty} D_q^{-1}(P_n^{(c)}(cq^k))P_m^{(c)}(cq^k)w(cq^k)q^k
\]
\[
= \sum_{k=-\infty}^{\infty} P_n^{(c)}(cq^k)(P_n^{(c)}(cq^{k+1}) - P_n^{(c)}(cq^k))w(cq^k)q^k
\]
\[
+ \sum_{k=-\infty}^{\infty} c^k q^{2k}P_n^{(c)}(cq^k)P_m^{(c)}(cq^{k+1})w(cq^k)
\]
\[
= \sum_{k=-\infty}^{\infty} P_n^{(c)}(cq^k)P_m^{(c)}(cq^k)w(cq^k)q^k,
\]
where we have shifted the index of summation and used equations (1.1) and (3.6) to obtain the result. Applying the orthogonality condition to equation (8.8), we find
\[
D_q^{-1}P_n^{(c)} = A_n^{(c)}P_{n-1}^{(c)} + B_n^{(c)}P_{n-3}^{(c)},
\]
where \( A_n^{(c)}, B_n^{(c)} \) are constants depending on \( n \). We can immediately find \( A_n^{(c)} \) explicitly by using the identity
\[
D_q^{-1}(x^q) = [n]_q^{-1}x^{q-1}.
\]
Since the sequence \( \{P_n^{(c)}(x)\}_{n=0}^{\infty} \) consists of monic polynomials, we obtain
\[
A_n^{(c)} = [n]_q^{-1}.
\]
In order to derive \( B_n^{(c)} \), we first note that:
\[
D_q^{-1}(xP_n^{(c)}) = xP_n^{(c)}(x).
\]
We now take the \( q^{-1} \)-derivative of both sides of equation (8.5). After gathering linearly independent terms in equation (8.10) we find
\[
(q^{-1}[n]_q^{-1} + 1 - [n + 1]_q^{-1})P_n^{(c)}(x) + (q^{-1}[n]_q^{-1}\alpha_n^{(c)} + q^{-1}B_n^{(c)} - B_{n+1}^{(c)} - \alpha_n^{(c)}[n - 1]_q^{-1})P_{n-2}^{(c)}(x) + (q^{-1}B_n^{(c)}\alpha_{n-3}^{(c)} - \alpha_n^{(c)}B_{n-1}^{(c)})P_{n-4}(x) = 0,
\]
which leads to two equations for \( B_i \) and \( \alpha_{i}^{(c)} \):

\[
q^{-1}[n]_{q^{-1}} \alpha_{n-1}^{(c)} + q^{-1} B_n^{(c)} = B_{n+1}^{(c)} + \alpha_n^{(c)} [n - 1]_{q^{-1}},
\]

\[
q^{-1} B_n^{(c)} \alpha_{n-1}^{(c)} = \alpha_n^{(c)} B_{n+1}^{(c)}.
\]

Equation (8.12) implies \( B_n^{(c)} = \tilde{c}q^n \alpha_n^{(c)} \alpha_{n-1}^{(c)} \), for some constant \( \tilde{c} \).

We now proceed to show that \( \alpha_n^{(c)} \) satisfies equation (8.6) and in the process we will determine \( \tilde{c} \). Substituting \( B_n^{(c)} \) into equation (8.11) we find

\[
q^{-n-2}[n]_{q^{-1}} \alpha_n^{(c)} = q^{-n-1}[n - 1]_{q^{-1}} \alpha_n^{(c)} + \tilde{c} \left( \alpha_{n+1}^{(c)} - q^{-2} \alpha_{n-2}^{(c)} \right).
\]

We will rearrange equation (8.13) with the goal of obtaining a telescoping sum. Multiplying equation (8.13) by \( 1 + d\alpha_n^{(c)} \alpha_{n-1}^{(c)} q^{2n-3} \), for some constant \( d \), we have

\[
q^{-n-2}[n]_{q^{-1}} \alpha_n^{(c)} = q^{-n-1}[n - 1]_{q^{-1}} \alpha_n^{(c)} + d\alpha_n^{(c)} q^{-3} \alpha_n^{(c)} q^{-4} [n - 1]_{q^{-1}} - \tilde{c} \left( \alpha_{n+1}^{(c)} - q^{-2} \alpha_{n-2}^{(c)} \right).
\]

Therefore, we find

\[
q^{-n-2}[n]_{q^{-1}} \alpha_n^{(c)} = q^{-n-1}[n - 1]_{q^{-1}} \alpha_n^{(c)} + \tilde{d} \left( \alpha_{n+1}^{(c)} - q^{-2} \alpha_{n-2}^{(c)} \right).
\]

Letting \( \tilde{d} = q^{-2} \), we are led to

\[
q^{-n-2}[n]_{q^{-1}} \alpha_n^{(c)} = q^{-n-1}[n - 1]_{q^{-1}} \alpha_n^{(c)} + \tilde{d} \left( \alpha_{n+1}^{(c)} - q^{-2} \alpha_{n-2}^{(c)} \right).
\]

Thus, if we choose \( d = \tilde{d} \) and take the sum from 2 to \( n \) we find

\[
q^{-n}[1 - q^{-n}] \alpha_n^{(c)} = q^{-n} - q^{-1} \alpha_n^{(c)} + \tilde{d} \left( \alpha_{n+1}^{(c)} - q^{-2} \alpha_{n-2}^{(c)} \right).
\]

It remains to determine \( \tilde{c} \) and \( (1 - q^{-1})/\alpha_1^{(c)} - \tilde{c} (\alpha_2^{(c)} + \alpha_1^{(c)}) \). Observe that

\[
\sum_{k=-\infty}^{\infty} D_{q^{-1}}((cq^k)^p)w(cq^k)q^k = \sum_{k=-\infty}^{\infty} \frac{(cq^k)^{p+3}w(cq^k)q^k}{q^{-1} - 1},
\]
where we have used equations (1.1) and (3.6) to determine equation (8.14). Without loss of
generality assume that \( \int_{-\infty}^{\infty} w(cx)dx = 1 \). Applying equation (8.14) we find that
\[
\int_{-\infty}^{\infty} (cx)^3 w(cx)dx = (q^{-1} - 1), \tag{8.15}
\]
\[
\int_{-\infty}^{\infty} (cx)^6 w(cx)dx = (q^{-3} - 1) \int_{-\infty}^{\infty} (cx)^2 w(cx)dx. \tag{8.16}
\]
Using equation (8.5) we determine
\[
\alpha_1^{(c)} = \int_{-\infty}^{\infty} (cx)^2 w(cx)dx, \tag{8.17}
\]
\[
\alpha_3^{(c)} = \frac{\int_{-\infty}^{\infty} P_3^{(c)}(cx)^2 w(cx)dx}{\int_{-\infty}^{\infty} P_2^{(c)}(cx)^2 w(cx)dx}, \tag{8.18}
\]
\[
= \frac{\int_{-\infty}^{\infty} \left( (cx)^6 - 2 \left( \alpha_1^{(c)} + \alpha_2^{(c)} \right) (cx)^2 + \left( \alpha_1^{(c)} + \alpha_2^{(c)} \right)^2 \right) w(cx)dx}{\int_{-\infty}^{\infty} (cx)^4 - 2 \alpha_1^{(c)} (cx)^2 + (\alpha_1^{(c)})^2 w(cx)dx}.
\]
Furthermore, by the orthogonality of \( \{P^n(c)(x)\}_{n=0}^{\infty} \) we find
\[
\int_{-\infty}^{\infty} P_1^{(c)}(cx)P_3^{(c)}(cx)w(cx)dx = \int_{-\infty}^{\infty} (cx)^4 - \left( \alpha_1^{(c)} + \alpha_2^{(c)} \right) (cx)^2 w(cx)dx,
\]
\[
= 0, \tag{8.19}
\]
where we have used the recurrence relation, equation (8.5), to determine \( P_3^{(c)}(x) \). We also observe that
\[
D_{q^{-1}} P_3^{(c)} = A_n^{(c)} P_2^{(c)} + B_3^{(c)} P_0^{(c)},
\]
\[
= [3]_{q^{-1}} P_2^{(c)} + \tilde{c} q^3 \alpha_3^{(c)} \alpha_2^{(c)} \alpha_1^{(c)}. \tag{8.20}
\]
Using equations (8.15)–(8.20) we deduce that
\[
\tilde{c} = -1.
\]
Consequently, equation (8.19) implies that
\[
q^{-1} - 1 = \alpha_1^{(c)} (\alpha_1^{(c)} + \alpha_2^{(c)}).
\]
These values give equations (8.6) and (8.7) and theorem 8.2 follows immediately. \( \square \)

As a consequence of theorem 8.2 the sequences \( \{\alpha_n^{(c)}\}_{n=1}^{\infty} \) all provide positive solutions of equation (8.6). However, we will show that the limits of these sequences, as \( n \to \infty \), are in general not the same. In particular, we show that the asymptotic limit as \( n \to \infty \) of \( \alpha_n^{(1)} \) does not equal the limit of \( \alpha_n^{(c)} \) if \( c \neq 1, q^{1/2} \).

**Theorem 8.3.** Suppose the sequence of orthogonal polynomials \( \{P_n^{(c)}(z)\}_{n=0}^{\infty} \) and recurrence coefficients \( \{\alpha_n^{(c)}\}_{n=1}^{\infty} \) are defined as in theorem 8.2. Then, it follows that
\[
\lim_{n \to \infty} \frac{\alpha_n^{(c)} - \alpha_n^{(1)}}{q^{1-n} - \alpha_n^{(1)}} \neq 0, \tag{8.21}
\]
if \( c \neq 1, q^{1/2} \).
Proof. From lemma 8.1, we have \( \{\alpha_n^{(1)}\}_{n=0}^{\infty} = \{\alpha_n^{(q/2)}\}_{n=0}^{\infty} \). It remains to study the case \( c \neq 1, q^{1/2} \). Let \( y_1(z) \) and \( y_2(z) \) be two solutions of equation (4.2). Then, \( Y(z) = [y_1(z), y_2(z)]^T \) is a solution of the matrix equation

\[
Y(q^{-1}z) = \begin{bmatrix} 1 - q^{-2}z^2 & q^{-2}z^2 \\ -q^{-2}z & 1 - q^{-2}z^2 \end{bmatrix} Y(z).
\]  

(8.22)

Using the definition of the \( q \)-derivative, and equation (8.5) to write \( P_{n-3}(x) \) in terms of \( P_n(x) \) and \( P_{n-1}(x) \), equation (8.7) can be re-written as

\[
P_n^{(c)}(q^{-1}z) = (1 - q^{-n-2}z^2) P_n^{(c)}(z) + (q^{-n-1}z - z\alpha_n^{(c)} q^{-n-3} + z^2\alpha_n^{(c)} q^{-n-3}) P_{n-1}^{(c)}(z).
\]  

(8.23)

Taking \( n \to n - 1 \) and again using equation (8.5), equation (8.23) also allows us to express \( P_{n-1}^{(c)}(q^{-1}z) \) in terms of \( P_n^{(c)}(z) \) and \( P_{n-1}^{(c)}(z) \). This results in the matrix difference equation

\[
\begin{bmatrix} P_n^{(c)}(q^{-1}z) \\ q^{-n/2} P_{n-1}^{(c)}(q^{-1}z) \end{bmatrix} = \begin{bmatrix} 1 - q^{-n-2}z^2 & l_n z \\ k_n z & 1 - q^{-n-3}z^2 - l_n z \end{bmatrix} \begin{bmatrix} P_n^{(c)}(z) \\ q^{-n/2} P_{n-1}^{(c)}(z) \end{bmatrix} + r(z),
\]  

where

\[
l_n = q^{-n/2} - \alpha_n^{(c)} \alpha_{n-1}^{(c)} q^{-n/2},
\]

\[
k_n = q^{-n/2} / \alpha_{n-1}^{(c)} - \alpha_n^{(c)} q^{-n/2},
\]

and \( r(z) \) is given by

\[
r(z) = z \begin{bmatrix} 0 & z^2 a^q q^{-n/2} - q^{-n/2} \\ z^2 a^q q^{-n/2} - q^{-n/2} (\alpha_{n-1}^{(c)})^{-1} & z q^{-n/2} / \alpha_{n-1}^{(c)} - z a^q q^{-n/2} + z^2 q^{-n/2} \end{bmatrix}.
\]  

(8.24)

For the case \( c = 1 \) we know that as \( n \to \infty \) equation (8.24) approaches equation (8.22) (this follows from the the proof that \( W_n(z) \sim G(z) \) as \( n \to \infty \), see section 7). Assume that \( \alpha_n^{(c)} \) has the same asymptotic behaviour as \( \alpha_n^{(1)} \). Then equation (8.24) must similarly approach equation (8.22). In particular, \( r(z) = o(1) \) as \( n \to \infty \) and, \( l_n, k_n \) approach non-zero constants. Note that the condition \( r(z) = o(1) \) as \( n \to \infty \) follows from the first order asymptotic behaviour i.e. \( \alpha_n^{(c)} \sim q^{1-n} \) as \( n \to \infty \). In order for \( l_n \) and \( k_n \) to approach the same non-zero constant as the case \( c = 1 \) we require the asymptotic behaviour to match to second order.

We conclude using remarks 2.4 and 2.5, and repeating the same arguments used in the proof of lemma 4.13, that if \( \alpha_n^{(c)} \) has the same asymptotic behaviour as \( \alpha_n^{(1)} \), then for \( z \in D^+, Y_{2n}^{(c)}(z) \) approaches the matrix

\[
Y_{2n}^{(c)}(z) \sim \begin{bmatrix} a(z) & w(z)(h_q(z/c)a(z) - \lambda_1 b(z)) \\ b(z) & w(z)(h_q(z/c)b(z) - \lambda_2 a(z)) \end{bmatrix}, \text{ as } n \to \infty,
\]  

(8.25)

for some real constants \( \lambda_1, \lambda_2 \) (recall \( a(z) \) and \( b(z) \) are defined in definition 4.4). By the meromorphicity of \( Y_{2n}^{(c)}(z) \) at the poles of \( w(z) \) we find

\[
h_q(z/c)a(z) - \lambda_1 b(z) = h_q(z/c)b(z) - \lambda_2 a(z) = 0.
\]  

(8.26)

Thus,

\[
h_q(z/c)^2 = \lambda_1 \lambda_2.
\]  

(8.27)
Hence, $h_q(z/c)$ is either real or imaginary. We conclude from lemma A.5 and remark A.6 that equation (8.27) is satisfied iff $c = 1, q^{1/2}$. It follows that $\alpha_n^{(c)}$ has the same asymptotic behaviour as $\alpha_n^{(1)}$ (i.e. equation (8.21) is zero) iff $c = 1, q^{1/2}$.

\[\Box\]

9. Conclusion

In this paper, we described the asymptotic behaviour of a class of $qF_\Pi$ polynomials, defined in section 1.3, by using the $q$-RHP setting [5]. Our main results are theorems 1.5, 1.7 and 1.8. In theorems 1.5 and 1.7, we provided detailed asymptotic results for $qF_\Pi$ polynomials. In theorem 1.8, we detailed the implications of our analysis for the $q$-Painlevé equation satisfied by the recurrence coefficients, $\{\alpha_n\}_{n=1}^\infty$, of $qF_\Pi$ polynomials (see equation (1.5)).

Perhaps the most unexpected results of this paper concern the effect of the properties of $h_q(z)$ on the class of orthogonal polynomials that arise when the lattice is scaled. The values of $h_q(z)$ at the poles of the weight function $w(z)$ play an important role in determining the behaviour of orthogonal polynomials supported on the scaled lattice $cq^k, k \in \mathbb{Z}$. This observation enabled us to determine whether the orthogonal polynomials were invariant as $c$ varies. Furthermore, we were able to compare the asymptotic behaviours of polynomials when the lattice was scaled.

This paper focused on the weight $(-x^4; q^4)_\infty^{-1}$. But, the methodology can readily be extended to describe discrete $q$-Hermite II polynomials [7, chapter 18.27] with weight $(-x^2; q^2)_\infty^{-1}$. However, generalising the results to higher order weights i.e. $(-x^{2m}; q^{2m})_{\infty}^{-1}$, for $m > 2$, remains an open problem. The key difficulty is describing and solving the near-field RHP for higher order weights. One important aspect of this problem is accounting for the increased number of poles when dealing with higher order weights.

Another possible direction of future research could be determining all of the positive solutions of equation (1.7) and the asymptotic behaviour of different solutions as $n \to \infty$. In this paper we described the asymptotic behaviour of one particular solution, which satisfies the limit

\[\lim_{n \to \infty} q^n \alpha_n^{(1)} = q. \tag{9.1}\]

Numerical evidence suggests that the rhs of equation (9.1) oscillates for other positive solutions of equation (1.7). It would be interesting to see if the $q$-RHP formalism is able to accurately capture this behaviour. One avenue to achieve this would be to extend the detailed asymptotic results obtained in this paper to $qF_\Pi^{(c)}$ polynomials, defined in section 1.3.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Appendix A. Properties of $h_q(z)$

In this section we prove some properties of the function $h_q(z)$ defined in equation (1.2), which are used in this paper. Before discussing $h_q(z)$ we first prove a necessary lemma.

**Lemma A.1.** Let $C(z)$ be a function defined on $\mathbb{C} \setminus \{0\}$, which is analytic everywhere except for simple poles at $q^k$ for $k \in \mathbb{Z}$. Then, $C(qz) \neq C(z)$ for all $z \in \mathbb{C} \setminus \{q^k\}_{k=\pm \infty}$.

**Proof.** We prove the result by contradiction. Assume $C(qz) = C(z)$. Define

$$G(z) = (-z, -qz^{-1}; q)_\infty.$$  

By direct calculation one can show $G(qz) = z^{-1}G(z)$. Furthermore, by definition, $G(z)$ is zero on the $q$-lattice $q^k$, $k \in \mathbb{Z}$. Let $F(z) = C(z)G(z)$, then it follows $F(z)$ is analytic in $\mathbb{C} \setminus \{0\}$ and satisfies the difference equation

$$F(qz) = z^{-1}F(z).$$  

(A.1)

As $F(z)$ is analytic in $\mathbb{C} \setminus \{0\}$ we can write $F(z)$ as the Laurent series

$$F(z) = \sum_{k=-\infty}^{\infty} F_k q^k.$$  

Comparing the coefficients of $z$ in equation (A.1), one can readily determine

$$F_k = c_0 q^{k(k-1)/2}.  \quad (A.2)$$

However, there is only one solution with $z$ coefficients given by equation (A.2) (up to scaling by a constant) and it follows that $F(z) = c_0 G(z)$. Thus, if $C(qz) = C(z)$, then $C(z) = c_0$, and $C(z)$ has no poles.

**Corollary A.2.** Let $C(z)$ be a function defined on $\mathbb{C} \setminus \{0\}$, which is analytic everywhere except for simple poles at $\pm q^k$ for $k \in \mathbb{Z}$. Furthermore, suppose $C(z)$ satisfies $C(qz) = C(z)$ for all $z \in \mathbb{C} \setminus \{q^k\}_{k=\pm \infty}$. Then, $C(z) = c_1 h_q(z) + c_0$, where $c_0$ and $c_1$ are constants and $h_q(z)$ is as defined in definition 1.4.

**Proof.** As both $C(z)$ and $h_q(z)$ have simple poles at $z = -1$ we conclude that there exists a $c_1 \neq 0$ such that

$$\text{Res}(C(-1)) = c_1 \text{Res}(h_q(-1)).$$

Furthermore, both $C(z)$ and $h_q(z)$ are invariant under the transformation $z \to qz$, hence for all $k \in \mathbb{Z}$

$$\text{Res}(C(-q^k)) = c_1 \text{Res}(h_q(-q^k)).$$

Thus, the function

$$D(z) = C(z) - c_1 h_q(z),$$

is meromorphic in $\mathbb{C} \setminus \{0\}$, with possible simple poles at $q^k$ for $k \in \mathbb{Z}$, and satisfies $D(qz) = D(z)$. However, by lemma A.1, $D(z)$ can not have simple poles at $q^k$ for $k \in \mathbb{Z}$. Hence, $D(z)$
is analytic in $\mathbb{C} \setminus \{0\}$ and it follows that $D(z)$ can be written as a convergent Laurent series. Thus,

$$D(z) = \sum_{j=-\infty}^{\infty} d_j z^j.$$  

Substituting this into the $q$-difference equation $D(qz) = D(z)$, we conclude $D(z) = d_0 (= c_0)$ and corollary A.2 follows immediately.

**Lemma A.3.** Re$(h_0(z)) = 0$ along the circles $|z| = 1$ and $|z| = q^{1/2}$.

**Proof.** Let $z = re^{i\theta}$. Substituting this into equation (1.2) and determining the real part we find

$$\text{Re}(h_0(re^{i\theta})) = \sum_{k=-\infty}^{\infty} \frac{2rq^k \cos(\theta) (r^2 - q^{2k})}{r^4 + q^{4k} - 2r^2 q^{2k} \cos(2\theta)}. \quad (A.3)$$

First we consider the case $r = 1$. The rhs of equation (A.3) becomes

$$\sum_{k=-\infty}^{\infty} \frac{2q^k \cos(\theta) (1 - q^{2k})}{1 + q^{4k} - 2q^{2k} \cos(2\theta)} = \sum_{k=1}^{\infty} \frac{2q^k \cos(\theta) (1 - q^{2k})}{1 + q^{4k} - 2q^{2k} \cos(2\theta)} + \sum_{k=-\infty}^{-1} \frac{2q^k \cos(\theta) (1 - q^{2k})}{1 + q^{4k} - 2q^{2k} \cos(2\theta)} = \sum_{k=1}^{\infty} \frac{2q^k \cos(\theta) (1 - q^{2k})}{1 + q^{4k} - 2q^{2k} \cos(2\theta)} = 0.$$  

Next we consider the case $r = q^{1/2}$. The rhs of equation (A.3) can be written as

$$\sum_{k=-\infty}^{\infty} \frac{2q^{k+1/2} \cos(\theta) (q - q^{2k})}{q^2 + q^{4k} - 2q^{2k+1} \cos(2\theta)} = \sum_{k=1}^{\infty} \frac{2q^{k+1/2} \cos(\theta) (q - q^{2k})}{q^2 + q^{4k} - 2q^{2k+1} \cos(2\theta)} + \sum_{k=-\infty}^{0} \frac{2q^{k+1/2} \cos(\theta) (q - q^{2k})}{q^2 + q^{4k} - 2q^{2k+1} \cos(2\theta)}$$

$$= \sum_{k=1}^{\infty} \frac{2q^{k+1/2} \cos(\theta) (q - q^{2k})}{q^2 + q^{4k} - 2q^{2k+1} \cos(2\theta)} + \sum_{k=0}^{\infty} \frac{2q^{-k+1/2} \cos(\theta) (q^{2k} - q^{-2k})}{q^2 + q^{4k} - 2q^{-2k+1} \cos(2\theta)}$$

$$= \sum_{k=1}^{\infty} \frac{2q^{k+1/2} \cos(\theta) (q - q^{2k})}{q^2 + q^{4k} - 2q^{2k+1} \cos(2\theta)} + \sum_{j=1}^{\infty} \frac{2q^{j+1/2} \cos(\theta) (q^j - q)}{q^2 + q^{4j} - 2q^{j+1} \cos(2\theta)} = 0.$$
Corollary A.4. The function \( h_q(z) \) defined in equation (1.2) can also be written as

\[
    h_q(z) = c_1 \frac{z(q^2, qz^{-2}; q^4)_{\infty}}{(z^2, q^2 z^{-2}; q^4)_{\infty}},
\]

for some constant \( c_1 \).

Proof. We observe that the function

\[
    h(z) = \frac{z(q^2, qz^{-2}; q^4)_{\infty}}{(z^2, q^2 z^{-2}; q^4)_{\infty}},
\]

is meromorphic with simple poles at \( \pm q^k \) for \( k \in \mathbb{Z} \), and satisfies \( h(qz) = h(z) \). By corollary A.2 we conclude \( h(z) = c_1 h_q(z) + c_0 \). By definition, \( h(z) \) has zeros at \( z = q^{1/2+k} \), for \( k \in \mathbb{Z} \), and by lemma A.3, \( h_q(z) \) also has zeros at \( z = q^{1/2+k} \), for \( k \in \mathbb{Z} \). Thus, we conclude

\[
    h_q(z) = c_1 \frac{z(q^2, qz^{-2}; q^4)_{\infty}}{(z^2, q^2 z^{-2}; q^4)_{\infty}}
\]

for some constant \( c_1 \).

Lemma A.5. Along the ray \( z = re^{i\pi/4} (r \in \mathbb{R}_{\geq 0}) \), \( \text{Re}(h_q(z)) = 0 \), except at \( r = q^{k/2} \), for \( k \in \mathbb{Z} \), where \( h_q(z) \) is complete imaginary.

Proof. From equation (A.3) we determine that the real part of \( h_q(re^{i\pi/4}) \) is given by

\[
    \text{Re}(h_q(re^{i\pi/4})) = \sum_{k=-\infty}^{\infty} \frac{\sqrt{2}rq^k(r^2 - q^{2k})}{r^4 + q^{4k}}.
\]

Define the function

\[
    F(u) = \sum_{k=-\infty}^{\infty} \frac{uq^k(u^2 - q^{2k})}{u^4 + q^{4k}},
\]

where \( u \) is a complex variable. We note that the sum is well defined and converges for all \( u \neq q^k e^{i(\pi/2 + 2n\pi)/4} \), where \( k \in \mathbb{Z} \) and \( n = 0, 1, 2, 3 \). Furthermore, \( F(u) \) satisfies the \( q \)-difference equation \( F(qu) = F(u) \). We now multiply \( F(u) \) by a function with zeros at the poles of \( F(u) \) to give us a function analytic in \( \mathbb{C} \setminus \{0\} \). Define

\[
    g(u) = F(u)(-u^4, q^4 u^{-4}; q^4)_{\infty}.
\]

Note that \( (-u^4, q^4 u^{-4}; q^4)_{\infty} \) is an even function and satisfies \( -(qu)^4, q^4(qu)^{-4}; q^4)_{\infty} = u^{-4}(-u^4, q^4 u^{-4}; q^4)_{\infty} \). Thus, \( g(u) \) is analytic in \( \mathbb{C} \setminus \{0\} \), \( g(u) \) satisfies the \( q \)-difference equation \( g(qu) = u^{-4} g(u) \), and \( g(u) \) is an odd function. Let us represent \( g(u) \) with the convergent Laurent series

\[
    g(u) = \sum_{j=-\infty}^{\infty} g_{4j} u^{4j}.
\]

As \( g(u) \) satisfies the difference equation \( g(qu) = u^{-4} g(u) \) we find

\[
    g_{4j+4} = q^{4j} g_j.
\]
Hence, we conclude that there are four linearly independent solutions, which can be chosen such that two are odd and two are even. As \( g(u) \) is odd we conclude it is the sum of two linearly independent odd solutions. Consider the two functions

\[
G_1(u) = u(u^2, q^2u^{-2}; q^2)\infty (qu^2, qu^{-2}; q^2)\infty,
\]

\[
G_2(u) = u(-u^2, -q^2u^{-2}; q^2)\infty (-qu^2, -qu^{-2}; q^2)\infty.
\]

We note that \( G_1(u) \) has zeros at \( u = \pm q^{k/2} \), for \( k \in \mathbb{Z} \) and \( G_2(u) \) has zeros at \( u = \pm i q^{k/2} \), for \( k \in \mathbb{Z} \). Furthermore, both \( G_1(u) \) and \( G_2(u) \) are odd and satisfy the difference equation \( G(qu) = u^{-4}G(u) \). Thus,

\[
g(u) = c_1 G_1(u) + c_2 G_2(u),
\]

for some constants \( c_1 \) and \( c_2 \). By lemma \( \text{A.3} \) we conclude that \( c_2 = 0 \) and \( g(u) \) only has zeros at the zeros of \( G_1(u) \) which occur at \( u = \pm q^{k/2} \), for \( k \in \mathbb{Z} \). Hence, this is where the zeros of \( F(u) \) are and lemma \( \text{A.5} \) follows immediately.

**Remark A.6.** From equation (A.3) we deduce that

\[
\text{Re}(h_q(\text{re}^{i\pi/4})) = -\text{Re}(h_q(\text{re}^{3i\pi/4})) = -\text{Re}(h_q(\text{re}^{-i\pi/4})) = \text{Re}(h_q(\text{re}^{-3i\pi/4})).
\]

Using an analogous expression to equation (A.3), for the imaginary part of \( h_q(z) \), one can also show by direct calculation that

\[
\text{Im}(h_q(\text{re}^{i\pi/4})) = \text{Im}(h_q(\text{re}^{3i\pi/4})) = -\text{Im}(h_q(\text{re}^{-i\pi/4})) = -\text{Im}(h_q(\text{re}^{-3i\pi/4})) \neq 0.
\]

**Appendix B. RHP theory**

For completeness we recall a well known result from RHP theory and prove it below. Let \( R(z) \) be a solution of the following RHP:

**Definition B.1.** Let \( \Gamma \) be an appropriate curve (see definition 1.3) with interior \( \mathcal{D}_- \) and exterior \( \mathcal{D}_+ \). A \( 2 \times 2 \) complex matrix function \( R(z) \), \( z \in \mathbb{C} \), is a solution of the RHP (B.1) if it satisfies the following conditions:

(i) \( R(z) \) is analytic in \( \mathbb{C} \setminus \Gamma \).

(ii) \( R(z) \) has continuous boundary values \( R^- (s) \) and \( R^+ (s) \) as \( z \) approaches \( s \in \Gamma \) from \( \mathcal{D}_- \) and \( \mathcal{D}_+ \) respectively, where

\[
R^+ (s) = R^- (s)J(s), \quad s \in \Gamma,
\]

for a \( 2 \times 2 \) matrix \( J(s) \).

(iii) \( R(z) \) satisfies

\[
R(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathcal{O} \left( \frac{1}{z} \right) \text{ as } |z| \to \infty.
\]

**Theorem B.2.** Suppose that \( \Gamma \) is analytic and \( \mathcal{D}_- \) is a neighbourhood of \( \Gamma \). Furthermore, for a given \( 0 < \epsilon < 1 \) suppose that

\[
||J(s) - I|| < \epsilon,
\]

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in this neighbourhood (where $||.||$ is the matrix norm). Then, the solution of the RHP given by definition B.1 satisfies

$$||R(z) - I|| < O(\epsilon),$$

for all $z \in \mathbb{C}$.

**Proof.** Multiple sources give a proof of theorem B.2 with various conditions on the jump matrix $J$ [6, 15]. For our setting the jump matrix is quite well behaved and satisfies all these constraints. We include a brief proof for completeness. Let

$$\Delta(z) = \frac{1}{\epsilon}(J(z) - I),$$

substituting $\Delta(z)$ into equation (B.1a) gives

$$R^+(z) = R^-(z)(I + \epsilon \Delta(z)).$$

By the asymptotic condition, equation (B.1b), we conclude that

$$R(z) = I + \frac{\epsilon}{2\pi i} \oint_{\Gamma} \frac{R_-(s)\Delta(s)}{z - s} \, ds.$$  \hspace{1cm} (B.2)

Let $L$ be defined as $L = \sup_{z \in \mathbb{C}_-} (||R(z)||)$, and let the maximum be at $z_L$. As $R(z)$ is analytic in $\mathbb{D}_-$, it follows $|R(z)|$ achieves its maximum on the boundary (i.e. on $\Gamma$). By assumption $R(z)$ and $\Delta(z)$ are also analytic for some fixed distance $r$ from $\Gamma$, let us call this curve $\Gamma_i$. Therefore,

$$R(z_L) = \left( I + \frac{\epsilon}{2\pi i} \oint_{\Gamma_i} \frac{R_-(s)\Delta(s)}{z_L - s} \, ds \right) (I + \epsilon \Delta(z_L))^{-1},$$  \hspace{1cm} (B.3)

where $(I + \epsilon \Delta(z_L))^{-1}$ can be determined using the Neumann series

$$(I + \epsilon \Delta(z_L))^{-1} = \sum_{j=0}^{\infty} (-\epsilon \Delta(z_L))^j.$$  \hspace{1cm} (B.4)

We conclude from equation (B.3)

$$L < \left| I + \frac{\epsilon \|\Delta\|_{\Gamma_i} \text{len}(\Gamma_i)}{2\pi r} \left( \sum_{j=0}^{\infty} (-\epsilon \|\Delta\|_{\Gamma_i})^j \right) \right|,$$

and hence $|L - 1| = O(\epsilon)$. Thus we find that,

$$|R(z) - I| < c_\Gamma O(\epsilon),$$  \hspace{1cm} (B.5)

for some constant

$$c_\Gamma = \frac{\|\Delta\|_{\Gamma_i} \text{len}(\Gamma_i)}{2\pi r} + \|\Delta\|_{\Gamma},$$

which is independent of $\epsilon$. \hfill \Box

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