On perturbations of dynamical semigroups defined by covariant completely positive measures on the semi-axis

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Abstract

We consider perturbations of dynamical semigroups on the algebra of all bounded operators in a Hilbert space generated by covariant completely positive measures on the semi-axis. The construction is based upon unbounded linear perturbations of generators of the preadjoint semigroups on the space of nuclear operators. As an application we construct a perturbation of the semigroup of non-unital *-endomorphisms on the algebra of canonical anticommutation relations resulting in the flow of shifts.

Keywords: perturbations of dynamical semigroups, covariant completely positive measures on the semi-axis, the flow of shifts on the algebra of canonical anticommutation relations

1 Introduction

The theory of one-parameter $C_0$-semigroups (strong continuous) of linear transformations $T_t : X \rightarrow X$, $t \geq 0$, on the Banach space $X$ introduced in the pioneering papers [1][2] states the conditions for the closed linear operator $\mathcal{L}$ with a dense domains $D(\mathcal{L}) \subset X$ to be a generator of $T = \{T_t, t \geq 0\}$
such that \( T_t = \exp(tL) \), \( t \geq 0 \). If \( X = \mathcal{S}_1(H) \) is the Banach space of nuclear operators in a Hilbert space \( H \) the claim of strong continuity for orbits of \( T \) possessing the property of non-increasing a trace is equivalent to weak continuity that is \( Tr(T_t(\rho)x) \to Tr(\rho x) \) if \( t \to 0 \) for all \( \rho \in \mathcal{S}_1(H) \) and \( x \in B(H) \) (the algebra of all bounded operators in \( H \)). In \([4]\) it was shown that the perturbation of the generator \( L \) by a linear map \( \Delta \) satisfying some additional conditions can be represented in the form of integral equation including the operator-valued measure generated by \( \Delta \). Together with a perturbation of \( T \) it is naturally to consider the corresponding perturbation of the adjoint semigroup \( T^* = \{ T_t^* : t \geq 0 \} \) on the algebra \( B(H) \). We realize this construction starting directly with the measure. As an example we construct a perturbation of the semigroup of non-unital *-endomorphisms on the algebra of canonical anticommutation relations (CAR). As a result of this perturbation we obtain the flow of shifts on the CAR algebra \([5]\). Earlier we have announce our result for the CAR algebra in \([6]\). Note that the perturbations of a semigroup on \( B(H) \) generated by the perturbation of the generator \( L \) of the corresponding preadjoint semigroup on \( \mathcal{S}_1(H) \) by a linear map with the domain containing \( D(L) \) form the basis for the construction of non-standard quantum dynamical semigroups \([7, 8]\).

2 Preliminaries

Let \( \Psi : \mathcal{S}_1(H) \to \mathcal{S}_1(H) \) be a linear bounded map on the Banach space of nuclear operators \( \mathcal{S}_1(H) \) in a Hilbert space \( H \). Since \( (\mathcal{S}_1(H))^* = B(H) \) (the algebra of all bounded operators in \( H \)) the adjoint map \( \Phi = \Psi^* : B(H) \to B(H) \) is weak* continuous. \( \Psi \) is said to be preadjoint of \( \Phi \) and denoted \( \Psi = \Phi_* \). A one-parameter family of linear *-maps \( \Phi_t : B(H) \to B(H), t \geq 0 \), is said to be a dynamical semigroup if

(i) \( \Phi_{t+s} = \Phi_t \circ \Phi_s \), \( t, s \geq 0 \), \( \Phi_0 = Id; \)

(ii) each \( \Phi_t \) is completely positive and \( \Phi_t(I) \leq I, t \geq 0; \)

(iii) \( Tr(\rho \Phi_t(x)) \) is continuous in \( t \) for all \( x \in B(H), \rho \in \mathcal{S}_1(H) \).

In the case, \( \{ \Psi_t = (\Phi_t)_*, t \geq 0 \} \) is a \( C_0 \)-semigroup on \( \mathcal{S}_1(H) \) with the property \( Tr(\Psi_t(\rho)) \leq Tr(\rho) \) for all \( \rho > 0, \rho \in \mathcal{S}_1(H) \). Thus, there is a generator \( L \) with the dense domain \( D(L) \subset \mathcal{S}_1(H) \) such that \( \rho_t = \Phi_t(\rho), \rho \in \mathcal{S}_1(H) \) is a solution to the Cauchy problem

\[
\frac{d\Psi_t(\rho_t)}{dt} = L(\rho_t), \ t > 0, \\
\rho_0 = \rho.
\]
Following to [4] let us define a perturbation of $\mathcal{L}$ of the form

$$\hat{\mathcal{L}} = \mathcal{L} + \Delta,$$

where the linear map $\Delta : \text{dom}\mathcal{L} \to \mathcal{S}_1(H)$ satisfies the properties

(i) for any positive definite matrix $||\rho_{jk}||$, $\rho_{jk} \in \text{dom} \mathcal{L}$ the matrix $||\Delta(\rho_{jk})||$ is positive definite;

(ii) $\text{Tr}(\Delta(\rho)) \leq -\text{Tr}(\mathcal{L}(\rho))$, $\rho \in \text{dom} \mathcal{L}$, $\rho > 0$.

Consider the measure $\mathcal{M}_*$ with values in the set of complete positive maps on $\mathcal{S}_1(H)$ determined by the formula

$$\mathcal{M}_*([],s) = \int_t^s \Delta \circ \Psi_r dr. \quad (1)$$

By a construction the measure $\mathcal{M}$ satisfies the relation

$$\mathcal{M}_*([],s) \circ \Psi_r = \mathcal{M}_*([],t+r,s+r), \ s, t, r \geq 0. \quad (2)$$

Then [4] the equation

$$\frac{d}{dt}\text{Tr}(\rho \hat{\Phi}_t(x)) = \text{Tr}((\mathcal{L} + \Delta)(\rho)\hat{\Phi}_t(x)), \ \rho \in \text{dom}\mathcal{L}, \ x \in B(H), \quad (3)$$

is equivalent to the integral equation

$$\hat{\Phi}_t - \int_0^t \mathcal{M}(dt) \circ \hat{\Phi}_{t-s} = \Phi_t, \ t \geq 0, \quad (4)$$

where the measure $\mathcal{M}$ consists of maps on $B(H)$ adjoint to $\mathcal{M}_*$. Due to (2) the measure $\mathcal{M}$ has the covariant property

$$\Phi_r \circ \mathcal{M}([],s) = \mathcal{M}(][t+r,s+r]), \ t, s, r \geq 0. \quad (5)$$

Given two completely positive maps $\Theta_1$ and $\Theta_2$ on $B(H)$ we shall use the notation

$$\Theta_1 \succ \Theta_2$$

iff

$$\Theta_1 - \Theta_2$$

is completely positive. In [4] it was considered the measures satisfying (5) generated by the formula

$$\mathcal{M}([],s) = \Phi_t \circ \Theta - \Phi_s \circ \Theta, \quad (6)$$

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where \( \Theta \) is the excessive completely positive map in the sense that

\[
\Theta > \Phi_t \circ \Theta, \ t > 0.
\]

In the case, it was shown that (3) has a unique minimal solution \( \bar{\Phi}_t^\infty \) possessing the property that any other solution \( \hat{\Phi}_t \) satisfies

\[
\bar{\Phi}_t > \bar{\Phi}_t^\infty, \ t \geq 0.
\]

3 Perturbations generated by measures

We consider measures \( \mathcal{M} \) on Borel subsets of the semi-axis \( \mathbb{R}_+ \) such that given \( 0 \leq t \leq s \leq +\infty \)

(i) \( \mathcal{M}([t,s]) : B(H) \to B(H) \) is a completely positive linear map;

(ii) \( \mathcal{M}([t,s])(I) \leq I; \)

(iii) \( \mu_{\rho,x}([t,s]) = Tr(\rho \mathcal{M}([t,s])(x)) \) is a \( \sigma \)-additive measure on \( \mathbb{R}_+ \) for any fixed \( \rho \in \mathcal{S}_1(H) \) and \( x \in B(H) \).

Together with \( \mathcal{M}([t,s]) \) it is naturally to examine a preadjoint map \( \mathcal{M}_*([t,s]) : \mathcal{S}_1(H) \to \mathcal{S}_1(H) \) possessing the property \( Tr(\mathcal{M}_*([t,s])(\rho)) \leq Tr(\rho) \) for all positive \( \rho \in \mathcal{S}_1(H) \).

Notice that if additionally \( \mathcal{M}(\mathbb{R})(I) = I \), then \( \mathcal{M} \) is said to be a completely positive instrument \([9]\). Nevertheless this property should not take place for our purposes.

The measure \( \mathcal{M} \) is said to be covariant with respect to dynamical semigroup \( \Phi \) if (5) holds true. It is not clear whether each covariant measure \( \mathcal{M} \) satisfying (5) can be obtained from some excessive map \( \Theta \) by means of (1). Developing the techniques of \([4]\) we suggest replacing (6) to an arbitrary measure \( \mathcal{M} \) with values in the set of completely positive maps satisfying (5).

**Proposition 1.** Given a measure \( \mathcal{M} \) covariant with respect to dynamical semigroup \( \Phi \) there exists a minimal solution to (4).

**Proof.** Following to \([4]\) let us consider the iteration process

\[
\Phi^{n+1}_t = \Phi^n_t + \int_0^t \mathcal{M}(dr) \circ \Phi^n_r.
\]

Due to \( \Phi_t(I) \leq I \) and \( \mathcal{M}([t,s])(I) \leq I, \ 0 \leq t \leq s \leq +\infty \) we get

\[
\Phi^{n+1}_t > \Phi^n_t, \ \Phi^n_t(I) \leq I.
\]

It results in \( \Phi^n_t \) tends to the minimal solution \( \bar{\Phi}_t \) of (4). \( \square \)
4 Perturbations of no-event semigroups

Following to [7] \( \Psi_0 = \{ \Psi_0^t : \mathcal{S}_1(H) \rightarrow \mathcal{S}_1(H), t \geq 0 \} \) is said to be a no-event semigroup if every pure state \( \rho = |\psi\rangle \langle \psi| \) is mapped to a multiple of a pure state. Such a semigroup necessary has the form

\[
\Psi_t(\rho) = T_t \rho T_t^*, \quad t \geq 0,
\]

where \( T_t = \exp(tK) \) is a \( C_0 \)-semigroups of contractions. Hence the generator \( K \) is a maximum dissipative operator due to [10]. The generator \( \mathcal{L} \) of \( \Psi \) is acting by the formula

\[
\mathcal{L}(\rho) = K \rho + \rho K, \quad \rho \in D(\mathcal{L}),
\]

where the domain \( D(\mathcal{L}) \) includes rank one operators \( |\psi\rangle \langle \xi| \), \( \psi, \xi \in D(K) \). Take linear operators \( L_j : D(K) \rightarrow H \) possessing the property

\[
\sum_j ||L_j \psi||^2 \leq -2 \text{Re} \langle \psi, K\psi \rangle
\]

and define a linear map \( \Delta : D(\mathcal{L}) \rightarrow \mathcal{S}_1(H) \) by the formula

\[
\Delta(|\psi\rangle \langle \xi|) = \sum_j |L_j \psi \rangle \langle L_j \xi|,
\]

\( \psi, \xi \in D(K) \). Consider the measure determined by (11). Then, it has the form (3), where the excessive completely positive map \( \Theta \) is given by the formula (11, Lemma 2)

\[
\langle \psi, \Theta(x) \psi \rangle = \int_0^{+\infty} \sum_j \langle L_j T_t \psi, x L_j T_t \psi \rangle dt, \quad \psi \in D(K), \quad x \in B(H).
\]

Below we shall give an example of perturbation for a no-event semigroup on the algebra of canonical anticommutation relations.

5 Algebra of canonical anticommutation relations \( \mathcal{A}(H) \)

Here we record the basic concepts about the algebra of canonical anticommutation relations [11]. Let \( H \) be a separable infinite dimensional Hilbert space. Fix the orthonormal basis \( \{|j\rangle\}_{j=1}^{+\infty} \) in \( H \). Then, the antisymmetric
Fock space $F(H)$ over one-particle Hilbert space $H$ is a Hilbert space with the orthonormal basis $|0\rangle$, $|j_1 \ldots j_n\rangle$, where the indices $j_1 < j_2 < \cdots < j_n$ and $n$ run over the set $\{1, 2, 3, \ldots\}$. The vector $|0\rangle$ is said to be vacuum. Let us define the ladder operators $a_k^\dagger$, $a_k$ by the formula

$$a_k^\dagger |j_1 \ldots j_n\rangle = \begin{cases} (-1)^s |j_1 \ldots j_s k j_{s+1} \ldots j_n\rangle & \text{if } j_s < k < j_{s+1}, \\ 0 & \text{if } k \in \{j_1, \ldots, j_n\} \end{cases},$$

$$a_k |j_1 \ldots j_n\rangle = \begin{cases} (-1)^{s+1} |j_1 \ldots j_{s-1} j_{s+1} \ldots j_n\rangle & \text{if } j_s = k, \\ 0 & \text{if } k \notin \{j_1, \ldots, j_n\}. \end{cases}$$

$$a_k^\dagger |0\rangle = |k\rangle, \quad a_k |0\rangle = 0, \quad k = 1, 2, 3, \ldots$$

It follows that

$$a_k a_k^\dagger + a_k^\dagger a_k = I, \quad (a_k)^2 = (a_k^\dagger) = 0,$$

$$a_k a_j = -a_j a_k, \quad a_k^\dagger a_j^\dagger = -a_j^\dagger a_k^\dagger.$$

The $C^*$-algebra $\mathfrak{A}(H)$ generated by the ladder operators is said to be the algebra of canonical anticommutation relations (CAR) in the Fock representation. The CAR algebra $\mathfrak{A}(H)$ is generated by monomials $x_{j_1} \ldots x_{j_n}$, where $j_1 < j_2 < \cdots < j_n$ and $x_{j_s} \in \{a_{j_s}^\dagger, a_{j_s}, a_{j_s} a_{j_s}^\dagger\}$.

Let us define a linear map on rank one operators by the formula

$$\Xi_s(|j_1 \ldots j_n\rangle \langle r_1 \ldots r_m|) = \sum_{s,k} (-1)^{s+k} \delta_{j_s r_k} |j_1 \ldots j_{s-1} j_{s+1} \ldots j_n\rangle \langle r_1 \ldots r_{k-1} r_{k+1} \ldots r_m|, \quad (7)$$

$$\Xi_s(|0\rangle \langle r_1 \ldots r_m|) = \Xi_s(|j_1 \ldots j_n\rangle \langle 0|) = \Xi_s(|0\rangle \langle 0|) = 0.$$

Notice that (7) is the sum of partial traces over minimal subsystems [12].

**Proposition 2.** Formula (7) correctly determines a linear map on $\mathfrak{S}_1(F(H))$ which can be uniquely extended to the completely positive map on $B(F(H))$. This map doesn’t have the property of non-increasing a trace.

**Proof.** It is straightforward to check that

$$\Xi_s(|j_1 \ldots j_n\rangle \langle r_1 \ldots r_m|) = \sum_k a_k |j_1 \ldots j_n\rangle \langle r_1 \ldots r_m| a_k^\dagger.$$

It follows that $\Xi_s$ can be uniquely extended to a completely positive map on $B(F(H))$. Denote

$$Q = \sum_k a_k^\dagger a_k.$$
Since
\[ a_k^\dagger a_k \langle j_1 \ldots j_n \rangle = \sum_s \delta_{k,j_s} \langle j_1 \ldots j_n \rangle, \quad a_k^\dagger a_k \langle 0 \rangle = 0 \]
for any \( j_1 < j_2 < \cdots < j_n \) we get
\[ Q \langle j_1 \ldots j_n \rangle = n \langle j_1 \ldots j_n \rangle. \]
Hence \( Tr(\Xi_* (\langle j_1 \ldots j_n \rangle \langle j_1 \ldots j_n \rangle)) = Tr(Q \langle j_1 \ldots j_n \rangle \langle j_1 \ldots j_n \rangle) = n \) and \( \Xi_* \) has not the property of non-increasing a trace.

Given \( f = \sum_j c_j |j\rangle, \quad \sum_j |c_j|^2 < +\infty \), let us define the ladder operators \( a^\dagger(f), a(f) \) over \( f \in H \) by the formula
\[ a(f) = \sum_j c_j^* a_j, \quad a^\dagger(f) = \sum_j c_j a_j^\dagger \]
satisfying the relations
\[ a(f)a^\dagger(g) + a^\dagger(g)a(f) = \langle f, g \rangle I, \]
\[ a(f)a(g) + a(g)a(f) = a^\dagger(f)a^\dagger(g) + a^\dagger(g)a^\dagger(f) = 0. \]
It follows from the definition that the ladder operators
\[ ||a(f)|| = ||a^\dagger(f)|| = ||f|| \]
in the \( C^* \)-algebra \( \mathfrak{A}(H) \).

Let us introduce the outer multiplication \( \Lambda \) over indexes \( j_1 \ldots j_n \) such that if \( j_1 < j_2 < \cdots < j_n \), then
\[ j_1 \Lambda j_2 \Lambda \ldots \Lambda j_n = |j_1 \ldots j_n \rangle \]
and
\[ j_s \Lambda j_k = -j_k \Lambda j_s. \]
Following this way, for \( f_j = \sum_k c_{jk} |k\rangle, \quad \sum_k |c_{jk}|^2 < +\infty \), we can put
\[ f_1 \Lambda \ldots \Lambda f_n = \sum_{k_1, \ldots, k_n} c_{1k_1} \ldots c_{nk_n} j_{k_1} \Lambda \ldots \Lambda j_{k_n} \]
Hence, given \( f_j, g_k \in H \) we can define vectors \( |f\rangle = f_1 \Lambda \ldots \Lambda f_n, |g\rangle = g_1 \Lambda \ldots \Lambda g_n \in H^{\otimes_n} \). Fix \( n \) and denote \( H^{\otimes_n} \) the closed linear envelope of
vectors $|f\rangle$ in the Fock space $F(H)$. Then, a restriction of the inner product in $F(H)$ to $H^{\otimes n}$ reads

$$\langle f|g\rangle_{H^{\otimes n}} = det|| \langle f_j|g_k\rangle||,$$

where the outer multiplication $\Lambda$ satisfies the rule

$$f \Lambda g = -g \Lambda f, \ f, g \in H.$$

Alternatively we can define the orthogonal projection $P_a$ in a tensor product $H^{\otimes n}$ as follows

$$P_a(f_1 \otimes \cdots \otimes f_n) = \frac{1}{n!} \sum_{\epsilon \in S_n} (-1)^{\mid \epsilon \mid} f_{\epsilon(1)} \otimes \cdots \otimes f_{\epsilon(n)},$$

where the sum is taken over the set of all permutations $S_n$ and $|\epsilon|$ is a signature of permutation $\epsilon \in S_n$. By this way,

$$H^{\otimes n} = P_a(H^{\otimes n})$$

and is said to be an n-th antisymmetric tensor product of $H$.

Given $f \in H^{\otimes n}$ we denote

$$a(f) = a(f_1) \cdots a(f_n), \ a^\dagger(f) = a^\dagger(f_1) \cdots a^\dagger(f_n).$$

It follows that

$$a^\dagger(f)g = f \Lambda g$$

and

$$a(f)g = \sum_k (-1)^{k+1} \langle f, g_k \rangle g_1 \Lambda \cdots \Lambda g_{k-1} \Lambda g_k \Lambda \cdots \Lambda g_n,$$

where $f \in H$, $g \in H^{\otimes n}$.

6 The semigroup of shifts on $\mathfrak{A}(H)$

Put $H = L^2(\mathbb{R}_+)$ and define the semigroup of shifts in $H$ by the formula

$$(S_t f)(x) = \begin{cases} f(x-t), & x > t; \\ 0, & 0 \leq x \leq t, \end{cases}$$

$t \geq 0, \ f \in H$. The conjugate semigroup of contractions $S_t^* = e^{td}$ has a generator

$$df = \frac{df}{dx}, \ f \in D(d) = \{ f \mid f' \in L^2(\mathbb{R}_+) \}.$$
We also need the semigroup of shifts in $F(H)$ obtained by lifting $(S_t)$ as follows
\[ \hat{S}_t(f_1 \Lambda \ldots \Lambda f_n) = S_t f_1 \Lambda \ldots \Lambda S_t f_n, \quad \hat{S}_t|0\rangle = |0\rangle, \tag{8} \]
$t \geq 0$, $f_j \in H$. The generator $\hat{d}$ of its conjugate semigroup of contractions $\hat{S}_t^* = e^{itd}$ is given by the formula
\[ \hat{d}|f\rangle = \sum_{j=1}^{n} f_1 \Lambda \ldots \Lambda f_{j-1} \Lambda df_j \Lambda f_{j+1} \Lambda \ldots \Lambda f_n, \quad \hat{d}|0\rangle = 0, \tag{9} \]
$f_j \in D(d)$. Along (9) we need a preconjugate operator acting by the formula
\[ \hat{d}^*|f\rangle = -\sum_{j=1}^{n} f_1 \Lambda \ldots \Lambda f_{j-1} \Lambda df_j \Lambda f_{j+1} \Lambda \ldots \Lambda f_n, \quad \hat{d}^*|0\rangle = 0, \tag{9} \]
$f_j \in D(d^*) = \{ f \mid f' \in L^2(\mathbb{R}_+), \ f(0) = 0 \}$. Using (8) it is possible to determine the dynamical semigroup on $B(F(H))$ as follows
\[ \Phi_t(x) = \hat{S}_t x \hat{S}_t^*, \tag{10} \]
t $\geq 0$, $x \in B(F(H))$. The preadjoint semigroup $\Psi_t : \mathcal{S}_1(F(H)) \to \mathcal{S}_1(F(H))$ is given by the formula
\[ \Psi_t(\rho) = \hat{S}_t^* \rho \hat{S}_t, \tag{11} \]
t $\geq 0$, $\rho \in \mathcal{S}_1(F(H))$. Note that (11) can be directly extended to the semigroup of non-unital *-endomorphisms on $B(F(H))$. The generator $L$ of (11) is determined by the formula
\[ L(\rho) = [\hat{d}, \rho] = \hat{d}\rho - \rho \hat{d}, \]
where $\rho$ belongs to the domain $D(L)$ which is dense in $\mathcal{S}_1(F(H))$. It is straightforward to see that $D(L)$ contains rank one operators
\[ |f\rangle \langle g|, \quad |f\rangle \langle 0|, \quad |0\rangle \langle g|, \quad |0\rangle \langle 0|, \]
where $f_j, g_k \in D(d)$.

The semigroup of unital *-endomorphisms $\tilde{\Phi}_t$ on $\mathfrak{A}(H)$ defined by the relation
\[ \tilde{\Phi}_t(a(f)) = a(S_t f), \quad t \geq 0, \tag{12} \]
is said to be the flow of shifts on the CAR algebra [5]. Denote $f_{i,j} = f_1 \Lambda \ldots \Lambda f_{j-1} \Lambda f_j \ldots \Lambda f_n$ and define a linear *-map on $D(L)$ by the formula
\[ \Delta(|f\rangle \langle g|) = \sum_{j,k} (-1)^{j+k} f_j(0) \overline{g}_k(0) \langle f_{i,j} \rangle \langle g_{i,k}|, \tag{13} \]
f $j, k \in D(d)$. 

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Theorem 1.

\[ \text{Tr}(\mathcal{L}(\ket{f}\bra{g}) + \Delta(\ket{f}\bra{g}))a(h)a^\dagger(e)) = \bra{f}(a(\hat{d}_e)h)a^\dagger(e) + a(h)a^\dagger(\hat{d}_e))\ket{g}. \]

\( f_j, g_k \in D(d), \ h_j, e_k \in D(d^\ast). \)

**Proof.** The trace

\[ \text{Tr}(\mathcal{L}(\ket{f}\bra{g}))a(h)a^\dagger(e)) \]

can be represented as a sum of elements given as a multiplication of inner products of \( h_l, e_m \) and \( f_j, g_k \) as well as the derivatives of \( f_j, g_k \) such that only at least one of them could contain a derivative of the following possible forms

\[ \bra{f'_j}e_k, \bra{f'_j}h_k, \bra{e_k}g'_j, \bra{h_k}g'_j \]

or

\[ \bra{f'_j}g_k, \bra{f'_j}g'_k. \]

If (14) is implemented, then the derivative can be passed to the other side because \( e_j, h_k \in D(d^\ast) \) resulting in \( e_j(0) = h_k(0) = 0, \) e.g. \( \bra{f'_j}e_k = -\bra{f_j}e'_k. \)

Taking integration by parts in (15) we obtain the term outside the integral of the form \( f_j(0)g_k(0) \) but it is self destructing with the corresponding term in

\[ \text{Tr}(\Delta(\ket{f}\bra{g}))a(h)a^\dagger(e)). \]

More formally,

\[ \text{Tr}(\mathcal{L}(\ket{f}\bra{g}))a(h)a^\dagger(e)) = \bra{a^\dagger(h)g, a^\dagger(e)d^\ast f} + \bra{a^\dagger(h)g, a^\dagger(e)f} \]

The first term in (16) can be rewritten as

\[ \bra{\hat{a}(h)g, a^\dagger(e)d^\ast f} = \sum_{j=1}^{n} \bra{h\Lambda f_1 \Lambda \cdots f_{j-1} \Lambda df_j \Lambda f_{j+1} \Lambda \cdots f_n} = \]

\[ \bra{h\Lambda g, \hat{d}(e\Lambda f)} - \bra{h\Lambda g, (\hat{d} e)\Lambda f} = \]

\[ \bra{h\Lambda g, \hat{d}(e\Lambda f)} + \bra{a^\dagger(h)g, a^\dagger(\hat{d}_e)\Lambda f} \]

because \( \hat{d}_e = -\hat{d} e \) if \( e \in D(\hat{d}^\ast). \) Integrating by parts the first term in (17) we obtain

\[ \bra{h\Lambda g, \hat{d}(e\Lambda f)} = \bra{(\hat{d}_e)h\Lambda g, e\Lambda f} - \bra{h\Lambda(\hat{d}g), e\Lambda f} - \]

\[ \sum_{j,k} (-1)^{j+k} f_j(0)g_k(0) \bra{h\Lambda g_{j,k}, e\Lambda f_{j,k}} \]

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due to $e_j(0) = h_k(0) = 0$ in virtue of $e, h \in D(\hat{d}_*)$. Substituting (18) to (17) we get
\[
\langle a^\dagger(h)g, a^\dagger(e)\hat{d}f \rangle = \langle a^\dagger(\hat{d}_* h)g, a^\dagger(e)\hat{d}f \rangle
\]
\[
+ \langle a^\dagger(h)g, a^\dagger(\hat{d}_* e)\hat{d}f \rangle
\]  
Comparing (16) and (19) completes the proof.

Let us define the map $\mathcal{M}_*$ on measurable sets on $\mathbb{R}_+$ with values in the set of linear maps defined on rank one operators $|f\rangle \langle g|$ by the formula
\[
\mathcal{M}_*([t, s))(|f\rangle \langle g|) = \sum_{j,k} (-1)^{j+k} \int_t^s dr f_j(r) \overline{g_k(r)} \Psi_r(|f_j\rangle \langle g_k|),
\]  
\[
\mathcal{M}_*([t, s))(|0\rangle \langle f|) = \mathcal{M}_*([t, s))(|f\rangle \langle 0|) = \mathcal{M}_*([t, s))(|0\rangle \langle 0|) = 0.
\]  
Notice that formally
\[
\mathcal{M}_*([t, s]) = \int_t^s dr \Delta \circ \Psi_r,
\]  
where $\Delta$ is defined by (13). Moreover,
\[
\lim_{t \to 0} \mathcal{M}_*([0, t))(|f\rangle \langle g|) = \Delta(|f\rangle \langle g|)
\]  
for any choice of $f_j, g_k \in D(d)$.

**Proposition 3.** The map $\mathcal{M}$ conjugate to (20) is the measure covariant with respect to $\Phi$.

**Proof.** Analogously to the proof of Proposition 2 let us define a completely positive map by the formula
\[
\Xi_* (\delta)(\rho) = a(\chi_{[0, \delta]}(\rho) a^\dagger(\chi_{[0, \delta]}), \ \rho \in \mathcal{S}_1(H),
\]  
where $\chi_{[0, \delta]}$ is a characteristic function of the segment $[0, \delta]$ and $\delta > 0$. Given $0 \leq t < s$ and integer $n$ consider the auxiliary completely positive map
\[
\Sigma_n = \sum_{j=1}^n \Xi_* \left( \frac{s-t}{n} \right) \circ \Psi_{(\frac{s-t)}{n}}.
\]  

For the Kraus operators $V_j = a(\chi_{[0, \frac{2t}{n}]}(t^n)) \hat{S}_{(s-t)}$ of $\Sigma_n$ let us examine the sum

$$Q_n = \sum_{j=1}^{n} V_j^* V_j = \sum_{j=1}^{n} \hat{S}_{(s-t)}(t^n) a^\dagger(\chi_{[0, \frac{2t}{n}]}(t^n)) \hat{S}_{(s-t)}.$$ 

Taking into account

$$||a^\dagger(\chi_{[0, \frac{2t}{n}]}(t^n))|| = \frac{s-t}{n}$$

we obtain

$$Q_n < (s-t)I.$$ 

It follows that $\Sigma_n$ is non-increasing a trace. Hence

$$M_s([t, s)) = \lim_{n \to +\infty} \Sigma_n$$

is a completely positive map and $Tr(M_s([t, s)))(\rho) \leq Tr(\rho)$ for all positive $\rho \in \mathcal{S}_1(H)$.

The flow of shifts $\hat{\Phi}_t$ determined by (12) has the generator $\hat{\mathcal{L}}$ acting by

$$\hat{\mathcal{L}}(a(h)a^\dagger(e)) = a(\hat{d}, h)a^\dagger(e) + a(h)a^\dagger(\hat{d}, e)$$

for any $h, e \in D(d^n)$. On the other hand, it follows from Theorem 1 that the generator $\hat{\mathcal{L}}_s$ of the preadjoint semigroup $\hat{\Psi}_t = (\hat{\Phi}_t)_s$ is equal to $\mathcal{L} + \Delta$. In the next theorem we show that $\hat{\Phi}_t$ satisfies the integral equation.

**Theorem 2.** The flow of shifts (12) is a solution to the integral equation

$$\hat{\Phi}_t - \int_0^t M(ds) \circ \hat{\Phi}_{t-s} = \hat{\Phi}_t, \quad t \geq 0,$$

where $M([t, s)) : B(F(H)) \to B(F(H))$ is the measure determined in Proposition 3.

**Proof.** The solution to (26) exists and defines a dynamical semigroup due to Proposition 4. Apply the left hand side of (26) to $a(h)a^\dagger(e)$, multiply to $|f\rangle \langle g|$ and take a trace, then

$$Tr(|f\rangle \langle g| \hat{\Phi}_t(a(h)a^\dagger(e))) - \int_0^t Tr(|f\rangle \langle g| M(ds) \circ \hat{\Phi}_{t-s}(a(h)a^\dagger(e))) =$$
\[ Tr(\langle g | a(h) a^{\dagger}(e) \rangle) - \int_0^t Tr(\bar{\Phi}_{t-s} \circ \mathcal{M}_s(ds)(\langle f | a(h) a^{\dagger}(e) \rangle) \] (27)

Suppose that \( f_j, g_k \in D(d) \) and \( h_j, e_k \in D(d^*) \). Taking the derivative at zero from (27) we obtain

\[ Tr(\langle g | (a(\hat{d}_s) a^{\dagger}(e) + a(h) a^{\dagger}(\hat{d}_s) e)) \rangle - Tr(\Delta(\langle f | a(h) a^{\dagger}(e) \rangle) \]

due to (21) and (25). Now the result follows from Theorem 1. 

\[ \Box \]

Taking into account (22), (23) and (24) we can conclude that for getting \( \bar{\Phi}_t \) to be unital the measure \( \mathcal{M} \) creates a particle with the wave function \( \chi_{r,r+dr} \) at each time moment \( r \) preceding \( t \).

7 Conclusion

We consider perturbations of a dynamical semigroup \( \Phi \) on the algebra of all bounded operators determined by solutions of integral equations with respect to measures \( \mathcal{M} \) on the semi-axis \( \mathbb{R}_+ \) with values in the set of completely positive maps which is covariant with respect to \( \Phi \) such that \( \Phi_{r} \circ \mathcal{M}([t,s]) = \mathcal{M}([t+r,s+r]), s,t,r \geq 0 \). As an example we construct the perturbation of the semigroup of non-unital *-endomorphisms on the algebra of canonical anticommutation relations resulting in the flow of shifts.

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