DECOUPLING FOR PERTURBED CONES AND MEAN SQUARE OF $|\zeta(\frac{1}{2} + it)|$

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Abstract. An improved estimate is obtained for the mean square of the modulus of the zeta function on the critical line. It is based on the decoupling techniques in harmonic analysis developed in [B-D].

1. Introduction

The aim of this paper is to establish improved bounds for the mean square of $|\zeta(\frac{1}{2} + it)|$ on short intervals.

More precisely, we will prove

**Theorem 3.** Let the function $I : [0, \infty) \times (0, \infty) \rightarrow \mathbb{R}$ be given by

$$I(t, U) = \frac{1}{2U} \int_{t-U}^{t+U} \left| \zeta\left( \frac{1}{2} + i\tau \right) \right|^2 d\tau.$$  

Then, for all $\varepsilon > 0$, one has

$$I\left(t, t^{1515/4816 + \varepsilon}\right) = O(\log t) \text{ as } t \to \infty.$$  

and

**Theorem 4.** Define $E(T)$ by

$$E(T) = \int_0^T \left| \zeta\left( \frac{1}{2} + it \right) \right|^2 dt - \left( \log \left( \frac{T}{2\pi} \right) + 2\gamma - 1 \right) T \quad (T \geq 1)$$

($\gamma =$ Euler-Mascheroni constant). Then for all $\varepsilon > 0$

$$E(T) = O\left( T^{131/416 + \varepsilon} \right) \text{ as } T \to \infty.$$  

Theorem 4 improves upon the estimate $E(T) = O(T^{131/416} (\log T)^{131/416})$ obtained in [W10], noting that

$$\frac{131}{416} = 0.314903... \text{ while } \frac{1515}{4816} = 0.314576...$$

(see the Remarks in Section 13 for a more detailed dimension).

The basic approach is the Bombieri-Iwaniec method and its further developments, in particular the contributions of M. Huxley and the second author.

Recall that there are two main parts to the Bombieri-Iwaniec approach, referred to as the First and Second Spacing Problem. Roughly speaking, the content of the first spacing problem are
certain moment inequalities while the second spacing problem is a distributional issue. These two components are then combined by an application of the large sieve. See [H96]. A novelty in this work is a different treatment of the first spacing problem using recent developments around the ‘decoupling principle’ in harmonic analysis (see [B-D]). An earlier application of these results on bounding ζ(\(\frac{1}{2} + it\)), i.e. towards the Lindelöf hypothesis, appears in the first author’s paper [B].

While the original Bombieri-Iwaniec method deals with one-variable exponential sums, the present context involves exponential sums with two variables (see Theorem 2 in Section 6). As a consequence, the mean value estimates in the first spacing problem involves points on surfaces rather than curves. These surfaces turn out to be perturbed cones. Our improvement in treating the first spacing problem relies on exploiting the additional curvature in the radial direction and moment inequalities for \(q > 4\).

Next some more details.

Let \(k \sim K, \ell \sim L, K > L\) and
\[
\omega(k, \ell) = \frac{(k + \ell)^{3/2} - (k - \ell)^{3/2}}{3} = k^{\frac{3}{2}}\ell + c k^{\frac{-3}{2}}\ell^3 + \cdots \quad (c \neq 0).
\]
(1.1)

Let \(\eta > 0\) be a small parameter. Motivated by the first spacing problem, we are interested in bounding moments
\[
\left\| \sum_{k \sim K, \ell \sim L} a_{k, \ell} e(\ell x_1 + k\ell x_2 + \omega(k, \ell) x_3) \right\|_{L^q_\#(|x_1| < 1, |x_2| < 1, |x_3| < \frac{1}{\eta L \sqrt{K}})}
\]
(1.2)

where \(|a_{k, \ell}| \leq 1\) and \(L^q_\#\) refers to the averaged \(L^q\)-norm. The case \(q = 4\) corresponds to the ‘classical’ treatment (cf. [H96]) and we aim at \(q > 4\) (not necessarily an integer) in order to achieve better estimates when exploiting the large sieve (which basically amounts to an application of Hölder’s inequality). It is important to point out that this improvement uses essentially the perturbative terms \(c k^{-3/2}\ell^3 + \cdots\) in (1.1) and that we are unable to establish a similar result for the case \(\omega(k, \ell) = k^{3/2}\ell\) of the unperturbed cone.

Setting \(s = \frac{k}{K}, t = \frac{\ell}{L}\), we consider the surface
\[
\tilde{C} : \begin{cases} 
  z = t \\
  y = st \\
  x = s^{\frac{3}{2}}t - c\varepsilon^2 s^{-\frac{3}{2}}t^2 + \cdots
\end{cases}
\]
(1.3)

with \(\varepsilon = \frac{k}{K}, s, t \sim 1\).

Thus \(\tilde{C}\) parametrizes a perturbed cone and our approach to (1.2) consists in invoking as initial step the so-called ‘decoupling theory’ from [B-D]. The role of this step is to achieve a variable restriction, after which we again exploit further arithmetical considerations as in earlier treatments. But because of restrictions of the variables \(k\) and \(\ell\) to suitably small intervals this arithmetical component becomes more straightforward.

In the next four sections, we prove the main analytical inequalities (see Propositions 10 and 10') needed for our enhanced treatment of the first spacing problem. We will use without much explanation several techniques and results from modern harmonic analysis in the presence of curvature. The decoupling result for curved (hyper) surfaces (applied here to surfaces in \(\mathbb{R}^3\)) will be an essential ingredient.
2. Decoupling inequalities for the cone

Consider the truncated cone $C$ and let $C_{\frac{1}{N}}$ be a $\frac{1}{N}$-neighborhood of $C$.

Here $N$ is some scale (will be taken to be $\sim \frac{1}{\epsilon^2}$ in (1.3) later on).

Partition $C_{\frac{1}{N}}$ into plates $\{\sigma\}$ as indicated above.

Thus each $\sigma$ has dimensions $\sim \frac{1}{\sqrt{N}} \times \frac{2}{N} \times 1$.

Denoting $B_N \subset \mathbb{R}^3$ a ball (not necessarily centered at 0) of size $N$, the following decoupling inequality is proven in [B-D].

**Proposition 1.** Assume $\text{supp } \hat{f} \subset C_{\frac{1}{N}}$ and denote $f_\sigma = (\hat{f}|_\sigma)^\vee$ the Fourier restriction of $f$ to $\sigma$. Then

$$
\|f\|_{L^6(B_N)} \ll N^{0^+} \left( \sum_\sigma \|f_\sigma\|_{L^6(B_N)}^2 \right)^{\frac{1}{2}}
$$

(2.1)

where $\ll N^{0^+}$ means $\leq C \epsilon N^\epsilon$ for any $\epsilon > 0$.

The next inequality we need is a consequence of the multilinear theory from [BCT].

Given functions $f_1, f_2, f_3$, supp $\hat{f}_i \subset C_{\frac{1}{N}}$, we say that $f_1, f_2, f_3$ are separated provided supp $\hat{f}_i$ are contained in slabs that are angularly $O(1)$-separated.

**Proposition 2.** (trilinear inequality)

Let $f_1, f_2, f_3$, supp $\hat{f}_i \subset C_{\frac{1}{N}}$ be separated. Then

$$
\| |f_1f_2f_3|^{\frac{4}{3}}\|_{L^3(B_N)} \ll N^{0^+} \prod_{i=1}^3 \|f_i\|_{L^2(B_N)}^{1/3}.
$$

(2.2)

More generally, assume supp $\hat{f}_i \subset C_{\frac{1}{N}} \cap R_\delta$, where $R_\delta$ is an angular sector of size $\delta$ and that moreover supp $\hat{f}_i$ are $O(\delta)$-separated ($\delta > \frac{1}{\sqrt{N}}$)
We obtain then

**Proposition 2’**. Under the above assumptions on \(f_1, f_2, f_3\), one has the inequality

\[
\| |f_1f_2f_3|^\frac{1}{4} \|_{L^2_\sigma(B_N)} \ll \delta^{-\frac{1}{4}} N^{\theta} \prod_{i=1}^3 \| f_i \|_{L^2_\sigma(B_{\theta N})}^{1/3}.
\] (2.3)

We will not explain the deduction of Proposition 2’ from Proposition 2 in detail, but just point out that it is based on the rescaling map

\[
L^\circ : C \to C : (\xi_1, \xi_2, \xi_3) \mapsto \left( \frac{\xi_1 - \xi_2}{2\sigma} + \frac{\xi_1 + \xi_3}{2}, \frac{\xi_2}{\sqrt{\sigma}}, \frac{\xi_3 - \xi_1}{2\sigma} + \frac{\xi_1 + \xi_3}{2} \right).
\] (2.4)

mapping \(C \cap R_\delta\) to \(C \cap R_{\frac{\delta}{2}}\).

Note that obviously \(\| f_i \|_{L^2_\sigma(B_N)} \sim \left( \sum_{\sigma} \| f_i,\sigma \|_{L^2_\sigma(B_N)}^2 \right)^{\frac{1}{2}}\) by orthogonality.

Next, we perform an interpolation between (2.1) and (2.3), setting

\[
\begin{align*}
\frac{1}{4} & = \frac{1-\theta}{2} + \frac{\theta}{6} \\
\frac{1}{q} & = \frac{1-\theta}{3} + \frac{\theta}{6}
\end{align*}
\]

with \(\theta = \frac{3}{4}, q = \frac{25}{4}\).

We obtain

**Proposition 3**. Let \(f_1, f_2, f_3\) be as in Proposition 2’. Then

\[
\| |f_1f_2f_3|^\frac{1}{4} \|_{L^2_\sigma(B_N)} \ll \delta^{-\frac{1}{4}} (1-\theta) N^{\theta} \prod_{i=1}^3 \left( \sum_{\sigma} \| f_i,\sigma \|_{L^2_\sigma(B_N)}^2 \right)^{\frac{1}{2}}.
\]

Note that this interpolation is not trivial and requires the ‘balanced wave packet decompositions’ introduced in [B-D].
Our final step is to derive from the multi-linear inequalities (2.5) a linear inequality. This is a relatively easy multiscale argument going back to [B-G] and which will not be repeated here. The upshot of this argument is that we recover \( \|f\|_{L^q(B_N)} \) from the contributions

\[
\delta^{-\frac{1}{q}} N^{\alpha+} \left( \sum_{\alpha} \left( \sum_{\sigma \subseteq \alpha} \|f_\sigma\|_{L^q(B_N)} \right)^{\frac{2}{q}} \right)^{\frac{1}{q}} \\
N^{\alpha+} \delta^{-\frac{1}{q}} (\delta \sqrt{N})^{\frac{1}{q}} \left( \sum_{\sigma} \left( \sum_{\sigma \subseteq \alpha} \|f_\sigma\|_{L^q(B_N)} \right)^{\frac{2}{q}} \right)^{\frac{1}{q}} \\
N^{\frac{1}{q}+} \left( \sum_{\sigma} \|f_\sigma\|_{L^q(B_N)} \right)^{\frac{1}{q}}
\]

where \( \{\alpha\} \) refers to a partition of \( C_{\frac{1}{\sqrt{N}}} \) in \( \delta \)-slabs with \( \delta \) taking dyadic values between \( \frac{1}{\sqrt{N}} \) and 1.

The final statement in this section is following Proposition 4. If \( \text{supp} \hat{f} \subset C_{\frac{1}{\sqrt{N}}} \), then for \( q = \frac{25}{4} \)

\[
\|f\|_{L^q(B_N)} \ll N^{\frac{1}{q}+} \left( \sum_{\sigma} \|f_\sigma\|_{L^q(B_N)} \right)^{\frac{1}{q}}.
\]

(2.6)

3. Decoupling for the perturbed cone

We are now returning to the perturbed cone \( \tilde{C} \) defined by (1.3).

Note that at scale \( N \sim \frac{1}{\varepsilon^2}, \) \( \tilde{C} \) may be identified with the cone

\[
C : \begin{cases} 
  z = t \\
  y = st \\
  x = s^{\frac{1}{2}}t
\end{cases}
\]

and hence at this scale (2.6) remains applicable to \( \tilde{C} \) as well.

Assuming \( \eta < \varepsilon^2 \), it follows that if \( \text{supp} \hat{f} \subset \tilde{C}_\eta \), then

\[
\|f\|_{L^q(B_{\frac{1}{\sqrt{n}}})} \ll \left( \frac{\varepsilon}{\varepsilon} \right)^{\frac{1}{q}+} \left( \sum_{\sigma} \|f_\sigma\|_{L^q(B_{\frac{1}{\sqrt{n}}})} \right)^{\frac{1}{q}}.
\]

(3.1)

Here the left side stands for

\[
\left( \frac{1}{|B_{\frac{1}{\sqrt{n}}}|} \int_{B_{\frac{1}{\sqrt{n}}}} \|f\|_{L^q(B_{\frac{1}{\sqrt{n}}})}^4 dx \right)^{\frac{1}{4}}
\]

and (3.1) is deduced from (2.6) just by partitioning \( B_{\frac{1}{\sqrt{n}}} \) in balls of size \( N = \frac{1}{\varepsilon^2} \). The slabs \( \sigma \) have angular width \( \varepsilon \).

Exploiting the perturbative term \( \varepsilon^2 s^{-\frac{1}{2}} t^2 + O(\varepsilon^4) \) in (1.3), we will perform a further decoupling of \( f_\sigma \).
Let us first rewrite (1.3) as
\[ x = y^{\frac{1}{4}} z^{\frac{1}{2}} - c \varepsilon^2 \frac{z^{7/2}}{y^{3/2}} + O(\varepsilon^4) \]  
(3.2)
and making the substitution
\[
\begin{align*}
  z &= z_1 + y_1 \\
  y &= z_1 - y_1 \sim 1
\end{align*}
\]
\[ x^2 = z_1^2 - y_1^2 - 2c\varepsilon^2 \frac{(z_1 + y_1)^4}{z_1 - y_1} + O(\varepsilon^4). \]  
(3.3)
Fixing \( \sigma \), perform a rotation in \((x_1, y_1)\)-plane
\[
\begin{align*}
  x &= (\cos \theta) x_2 - (\sin \theta) y_2 \\
  y_1 &= (\sin \theta) x_2 + (\cos \theta) y_2
\end{align*}
\]
to put \( \sigma \) in position \(|y_2| < \varepsilon \). Writing \( y_2 = \varepsilon y_3 \), (3.3) becomes then
\[ z_1^2 = x_2^2 + \varepsilon^2 y_3^2 + 2c\varepsilon^2 \frac{(z_1 + (\sin \theta)x_2 + \varepsilon(\cos \theta)y_3)^4}{z_1 - (\sin \theta)x_2 - \varepsilon(\cos \theta)y_3} + O(\varepsilon^4) \]
\[ = x_2^2 + \varepsilon^2 y_3^2 + 2c\varepsilon^2 \frac{(z_1 + (\sin \theta)x_2)^4}{z_1 - (\sin \theta)x_2} + O(\varepsilon^3). \]
Hence
\[ z_1 = x_2 + \frac{1}{2} \varepsilon^2 \frac{y_3^2}{x_2} + c\varepsilon^2 \frac{(1 + \sin \theta)^4}{1 - \sin \theta} x_2 + O(\varepsilon^3). \]  
(3.4)
Set \( z_1 - x_2 = \varepsilon^2 z_2 \) to obtain
\[ z_2 = \frac{1}{2} \frac{y_3^2}{x_2} + C \frac{(1 + \sin \theta)^4}{1 - \sin \theta} x_2 + O(\varepsilon). \]  
(3.5)
Note that the Hessian equals
\[
\left| \begin{array}{cc}
  \frac{y_3^2}{x_2} + 2c \varepsilon^2 \frac{(1 + \sin \theta)^4}{1 - \sin \theta} & \frac{y_3}{x_2} \\
  -\frac{y_3}{x_2} & \frac{1}{x_2} \\
\end{array} \right| = 2c \frac{(1 + \sin \theta)^4}{1 - \sin \theta} \frac{1}{x_2}. \]  
(3.6)
Since \( z_1 - y_1 \sim 1, z_1 - x_2 \sin \theta \sim 1 \) and, by (3.4), \( (1 - \sin \theta)x_2 \sim 1 \). Thus there is a further decoupling in \((x_2, y_3)\) at scale \( \frac{\sqrt{\varepsilon}}{\varepsilon} \), hence in \((x_2, y_2)\) at scale \( \frac{\sqrt{\varepsilon}}{\varepsilon} \), \( \sqrt{\eta} \)-angular, \( \frac{\sqrt{\varepsilon}}{\varepsilon} \)-radial in \((x_1, y_1)\)-space. Since \( t = z \), there is a decomposition in \( t \) at scale \( \sqrt{\eta} \). Since \( \tan \theta = \frac{y_3}{x_2} = \frac{z - y}{2z} = \frac{1 - s}{2(z - c\varepsilon^2 s - \eta)} \), the angular decomposion at scale \( \sqrt{\eta} \) corresponds to a decomposition in \( s \) at scale \( \sqrt{\eta} \).

Returning to (3.1), the preceding leads to the further decoupling
\[
\|f\|_{L^4(B_\frac{1}{\eta})} \ll \left( \frac{1}{\varepsilon} \right)^{\frac{1}{2} +} \left( \frac{\varepsilon}{\sqrt{\eta}} \right)^{\frac{1}{2}} \left( \sum_{\tau} \|f_s\|_{L^4(B_\frac{1}{\eta})} \right)^{\frac{1}{t}} \]  
(3.7)
with \( \{\tau\} \) a partition of \( \tilde{C} \) at scale \( \sqrt{\eta} \) in \( s \) and scale \( \frac{\sqrt{\eta}}{\varepsilon} \) in \( t \). Hence, with \( q = \frac{24}{5} \) and assuming \( \varepsilon > \sqrt{\eta} \), we get

**Proposition 5.** The following decoupling inequality holds for \( \tilde{C} \). Let \( \text{supp} \tilde{f} \subset \tilde{C}_\eta \). Then
\[
\|f\|_{L^4(B_\frac{1}{\eta})} \ll \varepsilon^{\frac{1}{4} -} \eta^{-\frac{1}{4}} \left( \sum_{\tau} \|f_s\|_{L^4(B_\frac{1}{\eta})} \right)^{\frac{1}{t}} \]  
(3.8)
with \( \{ \tau \} \) a partition in \((\sqrt{\eta}, \frac{\sqrt{2}}{e})\) rectangles in \((s, t)\).

Recall that \( s = \frac{k}{K}, t = \frac{4}{L}, \varepsilon = \frac{L}{K}, \omega(k, \ell) = k^{\frac{1}{2}} + c\varepsilon^{1-\frac{3}{4}} \ldots \) and we assumed \( \eta < \frac{L^2}{e^2} \). In future applications, \( \eta \) will moreover satisfy

\[
\frac{1}{KL} < \eta < \frac{1}{K} < \frac{1}{L}.
\]

In particular, \( K < L^3 \).

Obviously the ball \( B_{\frac{\eta}{x}} \) in (3.8) may be replaced by any larger domain of the form

\[
[x] < X_1 \times [x_2] < X_2 \times [x_3] < X_3
\]

where \( X_1, X_2, X_3 \geq \frac{1}{\eta} \).

Returning to (1.2), it follows that

\[
\left\| \sum_{k=K, \ell=L} a_{k\ell} e\left( \frac{L}{K} x_1 + \frac{k}{K} L x_2 + \frac{\omega(k, \ell)}{\sqrt{KL}} x_3 \right) \right\|_{L^p([x_1] < x_2 < K L, [x_3] < \frac{1}{\eta} L^2)} \lesssim \left( \frac{L}{K} \right)^{\frac{1}{2} - \eta} \left( \sum_{\alpha, \beta} \left\| \sum_{k=I_\alpha, \ell=J_\beta} a_{k\ell} e\left( \frac{L}{K} x_1 + \frac{k}{K} L x_2 + \frac{\omega(k, \ell)}{\sqrt{KL}} x_3 \right) \right\|_{L^4([x_1] < x_2 < K L, [x_3] < \frac{1}{\eta} L^2)}^4 \right)^{\frac{1}{2}}
\]

with \( \{ I_\alpha \} \) a partition of \([k \sim K] \) in \( \sqrt{\eta} K \)-intervals and \( \{ J_\beta \} \) a partition of \([\ell \sim L] \) in \( \sqrt{\eta} L = \sqrt{\eta} K \)-intervals.

Note that the function \( \sum_{k=K, \ell=L} a_{k\ell} e\left( \ell x_1 + k \ell x_2 + \omega(k, \ell) x_3 \right) \) in (1.2) is 1-periodic in \( x_1, x_2 \). Thus the previous inequality may be reformulated as

**Proposition 6.**

\[
\left\| \sum_{k=K, \ell=L} a_{k\ell} e(\ell x_1 + k \ell x_2 + \omega(k, \ell) x_3) \right\|_{L^p([x_1] < x_2 < K L, [x_3] < \frac{1}{\eta} L^2)} \lesssim \left( \frac{L}{K} \right)^{\frac{1}{2} - \eta} \left( \sum_{\alpha, \beta} \left\| \sum_{k=I_\alpha, \ell=J_\beta} e(\ell x_1 + k \ell x_2 + \omega(k, \ell) x_3) \right\|_{L^4([x_1] < x_2 < K L, [x_3] < \frac{1}{\eta} L^2)}^4 \right)^{\frac{1}{2}}.
\]

Clearly the expression

\[
\sum_{\alpha, \beta} \left\| \sum_{k=I_\alpha, \ell=J_\beta} e(\ell x_1 + k \ell x_2 + \omega(k, \ell) x_3) \right\|_{L^4([x_1] < x_2 < K L, [x_3] < \frac{1}{\eta} L^2)}^4
\]

amounts to the number of integral solutions of the system

\[
\begin{align*}
\ell_1 + \ell_2 &= \ell_3 + \ell_4 \\
\ell_1 k_1 + \ell_2 k_2 &= \ell_3 k_3 + \ell_4 k_4 \\
\omega(k_1, \ell_1) + \cdots - \omega(k_4, \ell_4) &= O(\eta \sqrt{KL})
\end{align*}
\]

(3.10) (3.11) (3.12)
with

\[ k \sim K, \ell \sim L \]
\[ \text{diam } (k_1, k_2, k_3, k_4) \leq \sqrt{\eta K} \]
\[ \text{diam } (\ell_1, \ell_2, \ell_3, \ell_4) < \frac{\sqrt{\eta}}{\varepsilon} L = \sqrt{\eta K} \] (3.13) (3.14)

**Proposition 7.** The number of solutions of (3.10)-(3.14) is bounded by

\[ \eta^2 K^{5+} + \eta K^3 L. \] (3.15)

**Proof.** We discard (3.12) which in fact is easily seen to be redundant. In what follows, we ignore the effect of divisor functions, which introduce an extra factor \( K^{O+} \).

Set \( \Delta k_i = k_i - k_4 \) \( (i = 1, 2, 3) \). Thus \( |\Delta k_i| \leq \sqrt{\eta K} \).

Since

\[ \ell_1 \Delta k_1 + \ell_2 \Delta k_2 = \ell_3 \Delta k_3 \]
\[ (\ell_1 - \ell_3) \Delta k_1 + (\ell_2 - \ell_3) \Delta k_2 = \ell_3(\Delta k_3 - \Delta k_1 - \Delta k_2). \] (3.16) (3.17)

Assume \( \Delta k_1 + \Delta k_2 \neq \Delta k_3 \). Choose \( \ell_1 - \ell_3, \ell_2 - \ell_3, \Delta k_1, \Delta k_2 \) \( (\eta K^2 \frac{\sqrt{\eta}}{\varepsilon} L^2 \)-possibilities) \( (3.17) \Rightarrow \ell_3, \Delta k_3 \).

Since there are \( K \) possibilities for \( k_4 \), this gives

\[ \frac{\eta^2}{\varepsilon^2} K^3 L^2 \leq \eta^2 K^5. \]

If \( \Delta k_1 + \Delta k_2 = \Delta k_3, (\ell_1 - \ell_3) \Delta k_1 + (\ell_2 - \ell_3) \Delta k_2 = 0 \).

If \( (\ell_1 - \ell_3) \Delta k_1 \neq 0 \), choose \( \ell_1, \ell_1 - \ell_3, \Delta k_1 \) \( (\frac{\sqrt{\eta}}{\varepsilon} L^2 \sqrt{\eta K}\)-possibilities)

\[ \Rightarrow \ell_2, \Delta k_2, \Delta k_3 \]

which gives the contribution \( \eta L K^3 \).

Case \( (\ell_1 - \ell_3) \Delta k_1 = (\ell_2 - \ell_3) \Delta k_2 = 0 \).

- \( \ell_1 = \ell_2 = \ell_3 = \ell_4 \) \( \Rightarrow L \eta K^2, K \leq L \eta K^3 \)
- \( \ell_1 = \ell_3, \ell_2 = \ell_4, \Delta k_2 = 0 \) \( \Rightarrow L \frac{\sqrt{\eta}}{\varepsilon} L \sqrt{\eta K} K = \eta K^3 L \) by (3.11)
- \( \Delta k_1 = \Delta k_2 = \Delta k_3 = 0 \Rightarrow k_1 = k_2 = k_3 = k_4 \Rightarrow K, L, \left(\frac{L \sqrt{\eta}}{\varepsilon}\right)^2 = \eta K^3 L. \)

This proves Proposition 7. \( \blacksquare \)

The final statement of this section becomes then

**Proposition 8.** Let \( |a_{k, \ell}| \leq 1 \) \( \text{(arbitrary)} \) and \( q = \frac{24}{5} \). Then

\[ \left\| \sum_{k \sim K, \ell \sim L} a_{k, \ell} e(\ell x_1 + k \ell x_2 + \omega(k, \ell)x_3) \right\|_{L^6_{\theta}([|x_1| < K, |x_2| < 1, |x_3| < \frac{1}{\sqrt{\eta K L^2}}])}
\[ L^6_{L^2}([|x_1| < K^2, |x_2| < K, |x_3| < K^{3/2} \cdot |x|])}
\[ \ll K^{5+} L^\frac{1}{2} \left(1 + \eta \frac{K^2}{L}\right)^\frac{1}{2} \] (3.18)

under assumption (3.9) on \( \eta \).
4. The basic moment inequalities

The main results from this section are Propositions 10 and 10'.

In order to get an estimate on (1.2) for some \( q_\nu > 4 \), we interpolate (3.18) with a bound on

\[
\left\| \sum_{k \sim K, \ell \sim L} a_{k\ell} e(\ell x_1 + k \ell x_2 + \omega(k, \ell) x_3) \right\|_{L_\nu^2([x_1|<K, |x_2|<1, |x_3|<\frac{L^2}{K^2}])} \leq \left\| \sum_{k \sim K, \ell \sim L} a_{k\ell} e(\ell x_1 + k \ell x_2 + \omega(k, \ell) x_3) \right\|_{L_\nu^2([x_1|<\frac{L^2}{K^2}, |x_2|<\frac{L^2}{K^2}, |x_3|<\frac{L^2}{K^2}])} + \nu
\]

with \( \nu \in \mathbb{Z}, \nu \geq 3 \).

Note that if \( \Delta \ell < \frac{L^2}{K^2}, \Delta(k\ell) < \frac{L^2}{K^2} \), then \( \Delta k < \frac{L^2}{K^2} \).

Denoting \( F(x_1, x_2, x_3) = \sum_{k \sim K, \ell \sim L} a_{k\ell} e(\ell x_1 + k \ell x_2 + \omega(k, \ell) x_3) \), it follows that (4.1) is bounded by

\[
\left\| \left( \sum |F_{\tau}|^2 \right)^{\frac{1}{2}} \right\|_{L_\nu^2([x_1|<\frac{L^2}{K^2}, |x_2|<1, |x_3|<\frac{L^2}{K^2}])} = \left\| \left( \sum |F_{\tau}|^2 \right)^{\frac{1}{2}} \right\|_{L_\nu^2([x_1|<1, |x_2|<1, |x_3|<\frac{L^2}{K^2}])}
\]

with \( \{\tau\} \) a partition of \( [k \sim K] \times [\ell \sim L] \) in intervals \( I_{\alpha} \times J_{\beta} \) with \( |I_{\alpha}| < \frac{L^2}{K^2}, |J_{\beta}| < \frac{L^2}{K^2} \).

We used here again periodicity of \( F \) in \( x_1 \). If necessary, we refine the partition further as to restrict \( \ell \) to intervals of size 1. Thus

\[
(4.2) < \left( 1 + \frac{L^2}{K^2} \right)^{\frac{1}{2}} \left\| \left( \sum |F_{\tau'}|^2 \right)^{\frac{1}{2}} \right\|_{L_\nu^2([x_1|<1, |x_2|<1, |x_3|<\frac{L^2}{K^2}])}
\]

with \( \{\tau'\} \) a partition in intervals \( I_{\alpha} \times J_{\beta} \), \( |I_{\alpha}| = \frac{L^2}{K^2}, |J_{\beta}| < \frac{L^2}{K^2} \).

Evaluation of

\[
\left\| \left( \sum |F_{\tau'}|^2 \right)^{\frac{1}{2}} \right\|_{L_\nu^2([x_1|<1, |x_2|<1, |x_3|<\frac{L^2}{K^2}])}^{2\nu}
\]

amounts to the number of integral solutions of

\[
\begin{align*}
(k_1 - k_2)\ell_1 + (k_3 - k_4)\ell_3 + \cdots + (k_{2\nu-1} - k_{2\nu})\ell_{2\nu-1} = 0 \\
\omega(k_1, \ell_1) - \omega(k_2, \ell_1) + \cdots + \omega(k_{2\nu-1}, \ell_{2\nu-1}) - \omega(k_{2\nu}, \ell_{2\nu-1}) = O(\sqrt{KL}\eta)
\end{align*}
\]

in \( k_1, \ldots, k_{2\nu} \sim K, \ell_1, \ell_3, \ldots, \ell_{2\nu-1} \sim L \) and with

\[
|k_1 - k_2|, \ldots, |k_{2\nu-1} - k_{2\nu}| < \frac{L^2}{K}.
\]

Equivalently, consider the system

\[
\begin{align*}
u_1 \ell_1 + u_2 \ell_2 + \cdots + u_{\nu} \ell_{\nu} = 0 \\
\omega(k_1 + u_1, \ell_1) - \omega(k_2 + u_2, \ell_2) - \cdots + \omega(k_{2\nu} + u_{\nu}, \ell_{\nu}) - \omega(k_{2\nu}, \ell_{\nu}) = O(\sqrt{KL}\eta)
\end{align*}
\]

with \( k_i \sim K, \ell_i \sim L \) and \( u_i = O(\frac{K^2}{U}) \).

Assume \( |u_1| \geq |u_2|, \ldots, |u_{\nu}| \). The contribution of \( u_1 = 0 \) is \( L^\nu K^\nu \). Next, consider the contribution of \( |u_1| \sim \frac{1}{U} \neq 0, U \leq L^2 K^{-1} > 1 \). Fix \( k_2, \ldots, k_\nu, \ell_2, \ldots, \ell_\nu, u_2, \ldots, u_{\nu} \). From (4.7) we retrieve \( \ell_1, u_1 \). Considering (4.8) as an equation in \( k_1 \), we obtain the bound

\[
K^{\nu - 1}L^{\nu - 1}U^{\nu - 1} \leq \frac{1}{U} + \eta K^2 \leq K^{\nu - 1}L^{\nu - 1} \left( \frac{L^2}{K} \right)^{\nu - 1} + \eta K^\nu + L^{\nu - 1} \left( \frac{L^2}{K} \right)^{\nu - 2} \leq \frac{L^{3\nu - 3} + \eta K^3 L^{3\nu - 5}}{L^{3\nu - 3} + \eta K^3 L^{3\nu - 5}}.
\]
Thus the number of solutions of (8.8)-(8.10) is at most

$$K^{\nu+\varepsilon}L^\nu \left(1 + \frac{L^{2\nu-3}}{K^\nu} + (\eta KL)\frac{L^{2\nu-6}}{K^{\nu-2}}\right).$$

Hence

**Proposition 9.**

$$(4.1) \ll \left(1 + \frac{L^3}{K^2}\right)^{\frac{1}{2} - \frac{1}{2\nu}} \left(1 + \frac{L^{2\nu-3}}{K^\nu} + (\eta KL)\frac{L^{2\nu-6}}{K^{\nu-2}}\right)^{\frac{1}{2\nu}} K^\frac{\nu}{2} L^{\frac{1}{2}}.$$  (4.10)

It remains to interpolate between (3.18) and (4.11).

We obtain the following

**Proposition 10.** For $\nu \geq 3$, take $q_\nu = \frac{13\nu-12}{3\nu-2} > 4$.

We have, assuming $\eta < \frac{L^2}{K^2}$ and $\frac{1}{KL} < \eta < \frac{1}{K} < \frac{1}{L}$

$$\| \sum_{k \sim K, \ell \sim L} a_{k\ell} e(\ell x_1 + k\ell x_3 + \nu(k,\ell)x_3) \|_{L^{4\nu}([|x_1|<1,|x_2|<1,|x_3|<\sqrt{KL\eta}])} \ll \left(1 + \eta K^2 \right)^{\frac{3(\nu-1)}{2(2\nu-1)}} \left(1 + \frac{L^3}{K^2}\right)^{\frac{\nu-1}{2(2\nu-1)}} \left(1 + \frac{L^{2\nu-3}}{K^\nu} + (\eta KL)\frac{L^{2\nu-6}}{K^{\nu-2}}\right)^{\frac{1}{2(2\nu-1)}} K^\frac{\nu}{2} L^{\frac{1}{2}}.$$  (4.12)

Note that if $\eta > \frac{L^2}{K^2}$, we may ignore the $\varepsilon$-terms in (1.3) i.e. we are in the pure conical situation. We get the inequality

$$\|f\|_{L^{4\nu}(B^\frac{1}{4})} \ll \left(\frac{1}{\eta}\right)^{\frac{1}{4} + \varepsilon} \left(\sum_{\sigma} \|f_\sigma\|_{L^{4\nu}(B^\frac{1}{4})}^4\right)^{\frac{1}{8}}.$$  (4.13)

instead of (2.6), with $\{\sigma\}$ a partition in $\sqrt{\eta}$-plates.

Hence instead of Proposition 6, we obtain, with $F$ defined as above,

$$\|F\|_{L^{4\nu}([|x_1|<K,|x_2|<1,|x_3|<\sqrt{KL\eta}]) L^\frac{24}{\eta} (x + \left[|x_1| < \frac{1}{L\eta}, |x_2| < \frac{1}{KL\eta}, |x_3| < \frac{1}{\sqrt{KL\eta}}\right])} \ll \left(\frac{1}{\eta}\right)^{\frac{1}{4} + \varepsilon} \left(\sum_{\alpha} \|F_\alpha\|_{L^{4\nu}([|x_1|<1,|x_2|<1,|x_3|<\sqrt{KL\eta}])}^4\right)^{\frac{1}{8}}.$$  (4.14)

with $\{I_\alpha\}$ a partition of $|k \sim K|$ in $\sqrt{\eta}K$-intervals.

The expression $\sum_\alpha \|F_\alpha\|_{L^{4\nu}}^4$ amounts to the number of integral solutions of (3.10)-(3.12) under the only restriction (3.13) and from the analysis in Proposition 7, we get the bound

$$\left(\frac{1}{\eta}\right)^{\frac{1}{4}} K^\varepsilon (L^2(\sqrt{\eta}K)^2 K + \sqrt{\eta}K^2L^2 + KL^3)^{\frac{1}{4}}.$$  (4.15)
Hence, since $\eta \gtrsim L^2/K^2$

$$\| \sum_{k \sim K, \ell \sim L} a_{k\ell} e(\ell x_1 + k_1 \ell x_2 + \omega(k, \ell) x_3) \|_{L^2_{\#}} \ll (1 + \eta K L^3)^{\frac{1}{2}} K^{\frac{1}{2}} + L^{\frac{1}{2}}$$

(4.16)

which is the same as the r.h.s. of (3.18).

In the l.h.s. of (3.18), $K^2 L^2$ is replaced by $\frac{1}{\eta}$ and, using only the $L^2$-norm, there is by (3.9) the trivial bound $K^{\frac{1}{2}} L^{\frac{1}{2}}$ on (4.1).

It follows that Proposition 10 remains valid without the assumption $\eta < \frac{L^2}{K^2}$. Hence

**Theorem 1.** For $\nu \geq 3$, $q_\nu = \frac{13\nu - 12}{3\nu - 2}$ and $\eta$ satisfying $\frac{1}{K L} < \eta < \frac{1}{K L} < \frac{K^2}{L^2}$, inequality (4.12) holds.

Note that for $\eta = \frac{1}{K L}$, the first factor in (4.12) becomes $1 + \frac{K^2}{L^2}$. For $k > L^2$, we establish an alternative bound.

Assume

$$K \geq L^2.$$  

(4.17)

We may then replace Proposition 10 by

**Proposition 10’**.

$$\ll (1 + \eta K L^{\frac{1}{2}})^{\frac{1}{2}} K^{\frac{1}{2}} + L^{\frac{1}{2}}$$

(4.18)

where $q = \frac{48}{11}$.

**Sketch of the Argument**

Instead of (4.1), we will bound

$$\| \sum_{k \sim K, \ell \sim L} \cdots \|_{L^2_{\#}}^{\frac{1}{2}}$$

appealing to inequalities (2.1), (2.3) derived from the multi-linear theory in order to bound $L^3_{\#}$.

(Note that in (4.1) the $L^2_{\#}$ was bounded by a simple orthogonality argument which does not exploit the geometric structure).

In the separated case, Proposition 2 provides a bound on the inner $L^3_{\#}$-norm in (4.19) by

$$\left\| \sum_{\tau} |F_\tau|^2 \right\|_{L^3_{\#}}^{\frac{1}{2}}$$

(4.20)

with $\{\tau\}$ a partition of $[k \sim K] \times [\ell \sim L]$ in intervals $I_\alpha \times J_\beta$ with $|I_\alpha| = L, |J_\beta| = 1$ (we use here the fact that $L^2 \leq K$).

According to Proposition 2’, the $\delta$-separated case ($\frac{L}{K} < \delta < 1$) involves an additional factor $\delta^{-\frac{1}{2}}$. 


Next, we bound
\[ \left\| \left( \sum_{\tau} |F_{\tau}|^2 \right)^{\frac{1}{2}} \right\|_{L^{\frac{2\nu}{\nu}}} \] (4.21)
similarly to (4.2). Thus \((4.21)^{2\nu}\) amounts to the number of solutions of (4.4), (4.5) where now
\[ |k_1 - k_2|, \ldots, |k_{2\nu} - k_{2\nu}| < L \] (4.22)
instead of (4.6). Thus a similar calculation as leading to (4.10) gives the bound (using the same notation)
\[ K^{\nu}L^{\nu} + K^{\nu-1}L^{\nu-1}U^{\nu-1} + \eta K^{\nu+1}L^{\nu-1}U^{\nu-2} \leq K^{\nu}L^{\nu} + K^{\nu-1}L^{2\nu-2} + \eta K^{\nu+1}L^{2\nu-3}. \] (4.23)
The natural choice is \(\nu = 4\), leading to
\[ K^{\frac{1}{2} + \frac{1}{4}}L^{\frac{1}{4}}(1 + \eta KL)^{\frac{1}{8}} \] (4.24)
as bound for the transverse contribution to (4.19).

This contribution of the ‘\(\delta\)-separated’ case is bounded by
\[ \delta^{-\frac{1}{2}}K^{\frac{1}{2} + \frac{1}{4}}L^{\frac{1}{4}}(1 + \eta KL)^{\frac{1}{8}}. \] (4.25)
As before, we are interpolating (4.19) with \(\| \|_{L^{\frac{1}{4}} L^{\frac{1}{2}}}\) bounded by (3.18), i.e.
\[ K^{\frac{1}{2} + \frac{1}{4}}L^{\frac{1}{4}}\left(1 + \eta \frac{K^2}{L}\right)^{\frac{1}{8}}. \] (4.26)
Note that for the \(\delta\)-separated contribution in Proposition 4 analyzed below Proposition 3, there is an extra factor \(\delta^{\frac{1}{8}}\) that was dropped. This factor needs to be added to the r.h.s. of (3.18) so that instead of (4.26), one gets in fact
\[ \delta^{\frac{1}{8}}K^{\frac{1}{2} + \frac{1}{4}}L^{\frac{1}{4}}\left(1 + \eta \frac{K^2}{L}\right)^{\frac{1}{8}}. \] (4.27)
Interpolating (4.25), (4.27) with
\[ \frac{1}{q} = \frac{1 - \theta}{8} + \frac{\theta}{4} = \frac{1 - \theta}{3} + \theta \frac{5}{24} \]
\[ \theta = \frac{5}{6}, q = \frac{48}{11} \]
leads to (4.18).

Above Propositions 10 and 10' form the basis of our treatment of the first spacing problem.
5. A variant of the double large sieve

Let $X, Y \subset \mathbb{R}^d$ be bounded sets, with

\[
| x_i | < U_i \text{ for } x \in X \quad | y_i | < V_i \text{ for } y \in Y. \tag{5.1}
\]

Estimate

\[
\sum_{x \in X, y \in Y} e(x.y) = (5.2)
\]

Let $\nu$ be the discrete measure on $\mathbb{R}^d$ defined by

\[
\nu = \sum_{x \in X} \delta_x.
\]

By (5.1),

\[
(5.2) = \int_{U_1 \times \cdots \times U_d} \left[ \sum_{y \in Y} e(x,y) \right] \nu(dx) \sim
\]

\[
|(10.2)| \lesssim \int_{U_1 \times \cdots \times U_d} \left| \sum_{y \in Y} e(x,y) \right| \|\mathbb{E}[\nu](x)\| dx
\]

with $\mathbb{E}[\nu]$ the conditional expectation of $\nu$ at scale $\min(U_1, \frac{1}{V_1}) \times \cdots \times \min(U_d, \frac{1}{V_d})$.

Take $2 < q < \infty$. By Hölder’s inequality $\left( \frac{1}{p} + \frac{1}{q} = 1 \right)$

\[
(5.3) \leq \left[ \int_{U_1 \times \cdots \times U_d} \left| \sum_{y \in Y} e(x,y) \right|^q dx \right]^{\frac{1}{q}} \|\mathbb{E}[\nu]\|_p
\]

and by interpolation

\[
\|\mathbb{E}[\nu]\|_p \leq |X|^{1 - \frac{2}{q}} \|\mathbb{E}[\nu]\|_2^{\frac{2}{q}}.
\]

Clearly

\[
\|\mathbb{E}[\nu]\|_2^2 = \left( \frac{1}{U_1} + V_1 \right) \cdots \left( \frac{1}{U_d} + V_d \right). \tag{5.4}
\]

with

\[
(5.4) = \{|(x, x') \in X \times X; |x_i - x'_i| < \frac{1}{V_i} (1 \leq i \leq d)\}|.
\]

Hence we arrive at

\[
|(5.2)| \lesssim (1 + U_1 V_1)^{\frac{1}{q}} \cdots (1 + U_d V_d)^{\frac{1}{q}} |X|^{1 - \frac{2}{q}}
\]

\[
\left\{ (x, x') \in X \times X; |x_i - x'_i| < \frac{1}{V_i} (1 \leq i \leq d) \right\} \right\}^{\frac{1}{q}}
\]

\[
\left[ \int_{U_1 \times \cdots \times U_d} \left| \sum_{y \in Y} e(x,y) \right|^q dx \right]^{\frac{1}{q}}. \tag{5.5}
\]
6. An application: bounds for exponential sums with a difference

Proposition 10 and Proposition 10′ supply new information concerning the ‘First Spacing Problem’ of the Bombieri-Iwaniec method for the estimation of exponential sums (see [H96] or [G&K91] for descriptions of the Bombieri-Iwaniec method, and [H96, Part III] and [W10, Section 3] for relevant previous results on the First Spacing Problem). With the aid of the variant of the Bombieri-Iwaniec ‘Double Large Sieve’ developed in Section 5 we are able to exploit this new information, and so achieve a small but significant advance in the application of the Bombieri-Iwaniec method to a certain class of exponential sums that is of some significance in the analytic theory of numbers. Our results in this direction are contained in the following theorem, the proof of which forms the subject of both the remainder of the present section and the whole of the next six sections.

**Theorem 2.** Let \( \varepsilon > 0 \) and \( C_2, C_3, \ldots, C_6 \geq 2 \) be real constants. Let \( \nu \geq 6 \) be an integer constant, and let

\[
q_\nu = \frac{2(13\nu - 12)}{6\nu - 5}.
\]

Let \( F(x) \) be a real function that is five times continuously differentiable for \( \frac{1}{3} \leq x \leq 3 \), and let \( g(x), G(x) \) be bounded functions of bounded variation on \( \frac{1}{2} \leq x \leq 1 \). Let \( M \) and \( T \) be large positive parameters, let \( H \geq 1 \), and let

\[
S = \sum_{H/2 < h \leq H} g \left( \frac{h}{H} \right) \sum_{M/2 < m \leq M} G \left( \frac{m}{M} \right) e \left( TF \left( \frac{m + h}{M} \right) - TF \left( \frac{m - h}{M} \right) \right).
\]

Suppose moreover that, on the interval \( \left[ \frac{1}{3}, 3 \right] \), the derivatives \( F^{(2)}(x), \ldots, F^{(5)}(x) \) satisfy:

\[
\left| F^{(r)}(x) \right| \leq C_r \quad (r = 2, 3, 4, 5),
\]

\[
\left| F^{(r)}(x) \right| \geq C_r^{-1} \quad (r = 2, 3, 4),
\]

and

\[
\left| F^{(2)}(x)F^{(4)}(x) - 3F^{(3)}(x)^2 \right| \geq C_5^{-1}.
\]

Then one has the following, in which \( B_5 \) and \( B_4 \) are small positive constants constructed from \( C_2, \ldots, C_6 \).

(A) If \( H, M \) and \( T \) satisfy the three conditions

\[
H \geq M^{-9}T^4(\log T)^{171/648} \quad \text{if} \quad M \leq T^{1/10}(\log T)^{57/140},
\]

\[
H \geq M^{11}T^{-6}(\log T)^{171/648} \quad \text{if} \quad M \geq T^{2/11}(\log T)^{-27/140},
\]

\[
H \leq B_3MT^{- \frac{14956 - 400}{76800}}(\log T)^{- \frac{5680}{76800}},
\]

then either

\[
H \ll MT^{- \frac{4956}{7680}}(\log T)^{\frac{5680}{7680}},
\]

and

\[
S \ll T^{\varepsilon}H \min \left\{ \left( \frac{M}{H} \right)^{\frac{4956}{7680}} T^{\frac{5680}{76800}}, \left( \frac{H}{M} \right)^{\frac{5680}{76800}} T^{\frac{4956}{7680}}, \left( \frac{H}{M} \right) T^{\frac{14956 - 400}{76800}} \right\},
\]
or else
\[
S \ll H \left( \frac{H}{M} \right)^{\frac{(14)}{(14)}} T(\frac{33}{33}) q_{r}^{-1} + \frac{33}{33} + \epsilon .
\] (6.9)

(B) If \( H, M \) and \( T \) satisfy the two conditions
\[
M \leq C_6 T^{\frac{1}{2}} , \tag{6.10}
\]
\[
H \leq B_4 \min \left\{ \left( \frac{M^{155 \nu - 480} (\log T)^{328}}{T^{46 \nu - 459}} \right)^{\frac{1}{310 - 310}} \left( \frac{M^{5 \nu - 12}}{T} \right)^{\frac{1}{310 - 310}}, \left( \frac{M^{3 \nu - 12}}{T} \right)^{\frac{1}{310 - 310}} \right\} , \tag{6.11}
\]
then either
\[
H \ll \min \left\{ M^{155 \nu - 480} (\log T)^{328}, M^{\frac{1}{2}} T^{-\frac{1}{140}} \right\} \tag{6.12}
\]
and
\[
S \ll T^{\frac{1}{2}} \min \left\{ T^{\frac{17}{55}} M^{\frac{17}{55}} H^{\frac{17}{55}} + T^{\frac{17}{55}} M^{\frac{17}{55}} H^{\frac{17}{55}} + T^{\frac{17}{55}} M^{-\frac{17}{55}} H^{\frac{17}{55}} + T^{\frac{17}{55}} M^{-\frac{17}{55}} H^{\frac{17}{55}} , \right. \right.
\]
\[
\left. \left. T^{\frac{17}{55}} M^{\frac{17}{55}} H^{\frac{17}{55}} + T^{\frac{17}{55}} M^{\frac{17}{55}} H^{\frac{17}{55}} \right\} , \tag{6.13}
\]
or else
\[
S \ll T^{\frac{17}{17}} q_{r}^{-1} + \frac{17}{17} + \epsilon M^{\frac{17}{17}} H^{\frac{17}{17}} q_{r}^{-1} H^{\frac{17}{17}} q_{r}^{-1} + \frac{17}{17} + \epsilon M^{\frac{17}{17}} q_{r}^{-1} H^{\frac{17}{17}} q_{r}^{-1} + \frac{17}{17} \tag{6.14}
\]

Remarks.

(i) This theorem is not quite all that one can prove. We have omitted to include in it a ‘Part (C)’ that might be obtained by using ‘Case (C)’ of Huxley’s results (Lemma 10.1 and Lemma 10.2, below) concerning the Second Spacing Problem of the Bombieri-Iwaniec method. Moreover, in (6.6) and (6.11), we have chosen to impose an upper bound on \( H \) that is slightly stronger than our method requires: it would otherwise have been necessary to include in the upper bounds for \( S \) certain extra terms associated with the perturbing effect of the first three factors of the bound given in (4.12), above. Our insistence on the conditions (6.6) and (6.11) may be considered harmless: for it is not one of the factors limiting what we are able to achieve through our applications, in Section 13, of Theorem 2. Our work in Section 13 is similarly unaffected by the omission of a ‘Part (C)’ from Theorem 2 (it being only cases with \( M \ll T^{1/2} \) that are relevant for the applications considered in that section).

(ii) A preference for simplicity has also led us to simplify the hypotheses on \( F(x) \) by strengthening them beyond what is strictly necessary. We mention here that the condition that (6.2) hold for \( r = 4 \) can be omitted when \( M > T^{147/328} (\log T)^{-2907/45920} \), and that one can omit the condition (6.3) when \( M < T^{181/328} (\log T)^{2907/45920} \). These restrictions on the enforcement of (6.2) (for \( r = 4 \)) and (6.3) are analogous to what occurs in [H03, (1.11) and (1.12)], and they have the same origin (in the works [H04] and [H05] of Huxley). Despite what has just been noted, we shall work with Theorem 2 as it is stated: this creates one slight complication in our proof of Lemma 13.1, where we are obliged to make certain that (6.3) holds.

(iii) Although the bound on \( S \) in (6.9) becomes stronger as \( \nu \) increases, the extent to which this can be exploited is limited due to the fact that the upper bound on \( H \) in (6.6) also strengthens
as $\nu$ is increased. This condition (6.6) arises from the assumption that we make in (11.13), below. When $\nu \geq 8$, the condition (6.6) requires that we have $H < MT^{-99/314(\log T)^{969/49860}}$, and so, since $99/314 = 0.315286... > 131/416 = 0.314903...$, we are prevented from using the corresponding case of (6.9) to improve upon the bound $E(T) \leq T^{131/416}(\log T)^{32587/8320}$ obtained in [W10]. When one has instead $\nu = 7$ the condition (6.6) becomes $H \leq B_5MT^{-643/(\log T)^{969/40960}}$, with $643/2048 = 0.313964... < 131/416$, and so does not prevent us from using the corresponding case of (6.9) to improve upon the above mentioned bound for $E(T)$.

The assumption (11.13) is made for convenience (it simplifies many calculations). It could be replaced by the weaker assumption that $H \ll N^{2/3}R^{1/3}$. This would have the effect of replacing the condition (6.6) with the condition $H \ll MT^{-247/792(\log T)^{323/12320}}$ (which, if one takes the implicit constant to be $B_5$, is precisely the case $\nu = 6$ of (6.6)). Since $247/792 = 0.31186$, this last upper bound on $H$ is weak in comparison to those mentioned above. This relaxation of our assumption (11.13) would, at the same time, lead to the bound (6.9) being weakened through the appearance of an extra term arising from the factor $(1 + K^{-\nu}L^{24\nu-3} + (\eta KL)K^{2-\nu}L^{2\nu-6})^{1/(26\nu-24)}$ that occurs in (4.12). That is, we would have

$$S \ll \left( \frac{H}{M} \left( \frac{q_{\nu}}{4} \right)^{-\frac{1}{\nu}} T^{(\frac{4}{3}b)q_{\nu}^{-1} + \frac{49}{7}} + \left( \frac{H}{M} \right)^{\frac{49}{7}(\frac{4}{3}b)q_{\nu}^{-1} - \frac{5}{2}} T^{-\frac{2441}{2600} - \frac{59}{720} q_{\nu}^{-1}} \right)^{\nu} H$$

in place of (6.9). Then, in order that we could obtain the estimate $S \ll M/\log T$ (as we do in the result (13.8) of Lemma 13.1, below), we would need the parameter $H$ to satisfy

$$H \ll MT^{-\max\{a(q_{\nu}), b(q_{\nu})\}},$$

with

$$a(q) = \frac{49q + 66}{4(41q + 44)} \quad \text{and} \quad b(q) = \frac{2341q - 5900}{8(897q - 2200)}.$$ 

Note that $q_{\nu}$ (defined in Theorem 2) is an increasing function of $\nu$ on $[6, \infty)$, and that $a(q)$ and $b(q)$ are, respectively, decreasing and increasing on $[q_6, \infty)$. A calculation shows that $a(q_7) = 1273/4053 = 0.3140883... > b(q_7) = 0.3140809...$, whereas $a(q_8) = 0.31406... < b(q_8) = 6323/20128 = 0.31413...$. Therefore, even if we had not made the assumption (11.13), we would still have had to have $c > 1273/4053$ in the hypothesis (13.1) of Lemma 13.1: note, in particular, that the size of the extra term that would appear in (6.9) could not be reduced by some adjustment of the parameter $N$ (for (10.14) would continue to give the optimal choice of $N$ to use with ‘Case (A)’ of Lemma 10.1).

(iv) Our proof of Theorem 2 splits naturally into two cases, which are (roughly speaking) that in which $H \ll MT^{-1/3}$, and that in which $H \gg MT^{-1/3}$. It is only in the latter case that Proposition 10 and Proposition 10’ yield new information concerning the first spacing problem of the Bombieri-Iwaniec method. In our treatment of the case ‘$H \ll MT^{-1/3}$’ we use nothing more than some of the bounds for the exponential sum $S$ that were already obtained in [W10]. It is convenient to get this case out of the way before beginning any work on the proof of the case ‘$H \gg MT^{-1/3}$’. Therefore we include in this section the following lemma, from which (via a sequence of straightforward corollaries) we obtain a proof of the case ‘$H \ll MT^{-1/3}$’ of Theorem 2.

Lemma 6.1. Let the hypotheses of Theorem 2 concerning $\varepsilon, C_2, \ldots, C_5, F(x), g(x), G(x), M, T$ and $H$ be satisfied. Then one has the following, in which $B_0$ is a small positive constant constructed from $C_2, \ldots, C_5$.

(A) If $H, M$ and $T$ satisfy (6.4), (6.5) and the condition

$$H \leq B_0MT^{-\frac{48}{471}(\log T)^{29695}},$$

(6.15)
then one has

\[ S \ll_\varepsilon H \left( \frac{H}{M} \right)^{\frac{1}{2}} T^{\frac{4}{3} + \varepsilon}. \]  

(B) If \( H, M \) and \( T \) satisfy the two conditions

\[ T^{\frac{4}{3}} \leq M \leq T^{\frac{7}{10} \left( \log T \right)^{\frac{2}{11}}} , \]

\[ H \leq \min \left\{ B_0 M^{\frac{9}{10} T^{-\frac{2}{3}} \left( \log T \right)^{\frac{2}{11}}} , B_0 M^{\frac{1}{10} T^{-\frac{2}{3}}}, M^{-9} T^4 \left( \log T \right)^{\frac{1/11}{2}} \right\} , \]

then one has

\[ S \ll_\varepsilon T^{\frac{4}{3} + \varepsilon} M^{\frac{9}{10}} H^{\frac{1}{11} + \varepsilon} + T^{\frac{7}{10} + \varepsilon} M^{-\frac{1}{10}} H^{\frac{2}{11}} . \]

**Proof.** What is stated in this lemma is a slightly weakened and specialized version of what follows immediately from [W10, Proposition 1, Parts (A) and (B)] if one assumes the case \((\kappa, \lambda) = (3/10, 57/140)\) of a certain ‘Hypothesis \(H(\kappa, \lambda)\)’ (formulated in [H03, Section 1]): note, in particular, that we may assume \(B_0 \leq 1\), so that the conditions (6.4), (6.5) and (6.15) will imply that one has

\[ T^{141/328} \left( \log T \right)^{1083/9184} \leq M \leq T^{187/328} \left( \log T \right)^{1083/9184} , \]

which is the case \(\kappa = 3/10, \lambda = 57/140, C_6 = 1\) of [W10, Condition (1.7)]. The lemma therefore follows by virtue of it having been shown, in [W10, Theorem 1], that [W10, Proposition 1] remains valid if the first of its hypotheses (to the effect that one has \((\kappa, \lambda) \in \{(K, L) \in [1/4, 1/3] \times [0, \infty) : \text{Hypothesis } H(K, L) \text{ is valid}\}) is replaced by the hypotheses that one has \(\kappa = 3/10\) and \(\lambda = 57/140\). 

**Corollary 6.1.1.** Let the hypotheses of Theorem 2, up to and including the condition (6.3), be satisfied. Suppose moreover that \(H, M, T\) satisfy the conditions (6.4), (6.5) and (6.6) of Part (A) of that theorem (in which \(B_5\) is a certain small positive constant constructed from \(C_2, \ldots , C_5\)). Then the bound (6.16) holds.

**Proof.** Since \(\nu \geq 6\), we have both

\[ \frac{149\nu - 400}{16(29\nu - 75)} \geq \frac{247}{792} = 0.31186 \geq 0.29878 \ldots = \frac{49}{164} \quad \text{and} \quad \frac{969\nu}{2240(29\nu - 75)} \leq \frac{969}{36960} < \frac{969}{22960} , \]

and so (assuming, as we may, that \(\log T \geq 1\), and that \(B_5\) in Theorem 2 is not greater than the constant \(B_0\) in Lemma 6.1) it follows from (6.6) that the condition (6.15) is satisfied. Therefore (given the hypotheses of the corollary) it follows from Part (A) of Lemma 6.1 that we obtain the bound (6.16) for \(S\). 

**Corollary 6.1.2.** Let the hypotheses of Theorem 2, up to and including the condition (6.3), be satisfied. Suppose moreover that \(H, M, T\) satisfy the conditions (6.10) and (6.11) in Part (B) of that theorem (in which \(B_4\) is a certain small positive constant constructed from \(C_2, \ldots , C_6\)). Then the bound (6.19) holds.

**Proof.** Since \(\nu \geq 6\), we have \(\frac{5}{13} < \frac{5\nu - 12}{13\nu - 36} \leq \frac{5}{11}\), and so (given that \(H \geq 1 \geq B_4\)) the condition (6.11) implies that we have both

\[ M \geq B_4^{\frac{1}{3}} T^\frac{2}{3} \geq T^\frac{2}{3} \]

and

\[ H \leq B_4 \left( \frac{M^{\frac{1}{3}}}{T} \right)^{\frac{2}{3}} . \]

Furthermore, assuming (as we may) that \(\log T \geq 1\), it follows by a calculation that if \(M \leq T^{1/2}\) then the term \(\left(M^{155\nu - 480} \left( \log T \right)^{969\nu/140} / T^{46\nu - 160} \right)^{1/(189\nu - 480)}\) that occurs in (6.11) is monotonic
decreasing, as a function of $\nu \in [6, \infty)$. By this, (6.10), and the point noted in (6.21), it follows from (6.11) that we have:

$$H \leq B_4 \min \left\{ C_6^{\frac{149}{160}} M \frac{\nu^6}{70} T^{-\frac{58}{171}} (\log T)^{\frac{684}{155}} \cdot \left( \frac{M^3}{T} \right)^{\frac{1}{4}}, \right\}$$

$$= B_4 \left( \frac{M^3}{T} \right)^{\frac{1}{4}} \min \left\{ C_6^{\frac{149}{160}} (\log T)^{\frac{684}{155}} \left( \frac{T}{M^2} \right)^{\frac{969}{350}} \cdot \left( \frac{M^3}{T} \right)^{\frac{1}{4}} \right\}$$

$$\leq B_4 MT^{-\frac{1}{4}} C_6^{\frac{149}{160}} (\log T)^{\frac{171}{550}} T^{\frac{17}{684}}.$$  \hfill (6.22)

Since $T$ is large, since $\frac{17}{684} - \frac{1}{2} = -\frac{211}{684} = -0.308 \ldots < -0.298 \ldots = -\frac{49}{164}$, and since we may assume that $B_4 \leq B_0 / C_6$ (where $B_0$ is the constant in Lemma 6.1), the above shows that

the condition (6.15) is satisfied. \hfill (6.23)

Given that $M^{3/2} T^{-1/2} = (M^3 / T)^{1/2}$, that $M^{35/69} T^{-6/69} = (T / M^2)^{17/69} (M^3 / T)^{1/3}$, and that $1/2 > 3/7$ and $17/69 > 17/109$, we are similarly able to deduce from (6.10), (6.21) and (6.22) that

the condition (6.18) is satisfied if $H \leq M^{-9} T^4 (\log T)^{\frac{171}{550}}$. \hfill (6.24)

We observe that if $M \leq T^{7/16} (\log T)^{57/448}$ and $H \leq M^{-9} T^4 (\log T)^{171/140}$ then it follows by (6.20) and (6.24) that the conditions (6.17) and (6.18) are satisfied, so that Part (B) of Lemma 6.1 yields the bound (6.19). If instead $M > T^{7/16} (\log T)^{57/448}$ and $H \leq M^{-9} T^4 (\log T)^{171/140}$ then, by (6.10) and the corollary of results of Kusmin (or Landau) and Van der Corput that is noted in [W10, Equation (4.8)], one has

$$S \ll H \left( HTM^{-2} \right)^{\frac{1}{2}} M^{\frac{7}{2}} = T^{\frac{1}{2}} M^{-\frac{7}{2}} H^{\frac{7}{2}} < T^{\frac{1}{3}} M^{\frac{7}{2}} H^{\frac{7}{2}}$$

(as a short calculation shows), and so it is again the case that the bound (6.19) holds. These observations show that (6.19) holds whenever $H \leq M^{-9} T^4 (\log T)^{171/140}$. Therefore we may assume (for the remainder of this proof) that one has

$$H > M^{-9} T^4 (\log T)^{\frac{171}{550}}.$$ \hfill (6.25)

By (6.25) and (6.10) it follows that the conditions (6.4) and (6.5) are satisfied (here we assume, as we may, that one has $T^{1/16} (\log T)^{-57/448} > C_6$). By (6.23), we have also (6.15). Furthermore (assuming, again, that $T$ is sufficiently large in terms of $C_6$) it follows from (6.10) that the upper bound on $M$ in (6.17) is satisfied, while by (6.25) and (6.15) (in which we assume $B_0 \leq 1$), it follows that we have

$$M^{-9} T^4 (\log T)^{\frac{171}{550}} < MT^{-\frac{1}{40}} (\log T)^{\frac{684}{155}},$$

which enables us to deduce that the lower bound on $M$ in (6.17) is satisfied, and so to conclude that both of the bounds on $M$ in (6.17) hold. Since each one of the conditions (6.4), (6.5) and (6.15) is satisfied, it therefore follows by Part (A) of Lemma 6.1 that we obtain the bound (6.16) for $S$. A calculation shows that (6.16) and (6.25) imply the bound $S \ll_x T^{7/10} M^{7/32} H^{171/160}$, which (in turn) implies (6.19). \hfill \blacksquare

**Corollary 6.1.3.** Let $C$ be a positive constant. Suppose that the hypotheses of Theorem 2, up to and including (6.3), be satisfied. Then there exist small positive constants $B_5$ and $B_4$ (constructed
from $C_2, \ldots, C_6$ such that the results stated in Parts (A) and (B) of Theorem 2 are valid whenever $H, M$ and $T$ satisfy

$$TH^3 \leq CM^3.$$  \hfill (6.26)

**Proof.** Let the constants $B_5$ and $B_4$ be the same as in Corollary 6.1.1 and Corollary 6.1.2, respectively.

In considering Part (A) of Theorem 2, we may assume that the conditions (6.4)-(6.6) are all satisfied (otherwise there is nothing to prove). By Corollary 6.1.1 it follows that the bound (6.16) holds. Our hypothesis (6.26) implies that $(H/M)^{1/3}T^{131/400} \ll (M/H)^{277/600}T^{397/2400}$. By this and (6.16) it follows that we obtain the bound (6.8) for $S$. Since $\frac{149}{464} = 0.32112 \ldots < \frac{1}{4}$, and since $T$ is assumed to be large, the hypothesis (6.26) also implies that the condition (6.7) is satisfied. Both (6.7) and (6.8) have been shown to hold. We therefore find that the result stated in Part (A) of Theorem 2 is valid subject to the condition in (6.26) and the stated hypotheses.

We now have only to consider Part (B) of Theorem 2. We assume that the relevant conditions, (6.10) and (6.11), are satisfied. It follows by Corollary 6.1.2 that the bound (6.19) holds. Since $\nu \geq 6$, it is moreover the case that the remarks concerning the implications of (6.11) that were made at the beginning of the proof of Corollary 6.1.2 are still valid in the present context, and so, by (6.10) and (6.20), we have also:

$$M^2 \ll T \leq M^3.$$  \hfill (6.27)

In order to complete this proof we need only show that the bounds for $H$ and $S$ in (6.12) and (6.13) hold. Since (6.26) implies $H \ll (M^3/T)^{1/3}$, and since we have already established that (6.19) holds, we are able to verify that (6.12) and (6.13) hold by observing that (6.26) and (6.27) imply that one has:

$$M^4 \frac{T}{M^2} H \frac{T}{M^3} T^{-\frac{4}{11}} = \left( \frac{M^3}{T} \right)^{\frac{1}{3}} \left( \frac{T}{M^2} \right)^{\frac{17}{37}} \geq \left( \frac{M^3}{T} \right)^{\frac{1}{3}} H, \quad M^4 \frac{T}{M^2} H \frac{T}{M^3} T^{-\frac{4}{11}} = \left( \frac{M^3}{T} \right)^{\frac{1}{3}} \geq \left( \frac{M^3}{T} \right)^{\frac{1}{3}} H.$$  

$$\frac{T}{M^2} M \frac{H}{M^3} T^{-\frac{4}{11}} = \left( \frac{M^3}{T} \right)^{\frac{17}{37}} \left( \frac{M^3}{T} \right)^{\frac{17}{37}} \ll 1.$$  

**Remark.** Note that it is only in Corollary 6.1.3 that we have assumed that $H \ll MT^{-1/3}$ (that being what the condition (6.26) effectively states). Indeed, we shall later make use of Lemma 6.1.1 and Lemma 6.1.2 in dealing with certain cases in which one does not have $H \ll MT^{-1/3}$ (see below (12.46) for where this occurs).
7. Initial steps in the application of the Bombieri-Iwaniec method

The only cases that need concern us, in completing our proof Theorem 2, are those in which $H$, $M$ and $T$ satisfy the condition

\[ H > 32C_3^{1/3}MT^{-1/4}. \]  

(7.1)

Indeed, whenever this condition is not satisfied we obtain the results of Theorem 2 by virtue of the case $C = 2^{15}C_3$ of Corollary 6.1.3. The factor $32C_3^{1/3}$ that occurs in (7.1) is put there in order to ensure that we get the final upper bound seen in (11.8), below. We shall assume henceforth that (7.1) holds.

We shall bound $S$ by applying the Bombieri-Iwaniec method; we follow [W10] in using results of Huxley [H03] on the ‘Second Spacing Problem’ associated with this method, but shall modify the approach taken in [W10] in order to make use of new results on the ‘First Spacing Problem’ obtained in Proposition 10 and Proposition 10’. For the sake of brevity we shall have occasion to refer to steps and intermediate results from the proofs given in [H03], [W04] and [W10] (this seems preferable to repeating the relevant calculations).

By partial summation it shall suffice to consider the case in which one has

\[ S = S_F(T;H,H_1;M,M_1) = \sum_{H_1 < h \leq H} \sum_{M_1 < m \leq M} e\left(TF\left(\frac{m+h}{M}\right) - TF\left(\frac{m-h}{M}\right)\right), \]

(7.2)

with some $H_1 \in [H/2,H)$ and some $M_1 \in [M/2,M)$. If $S$ is substituted for $S$, then the function $F$ is effectively replaced by $-F$. By this device we are freed from having to consider any case in which $F^3(x)$ is negative valued on the interval $[1/3,3]$. In the cases where one has $F^3(x) > 0 > F^2(x)$ for $1/3 \leq x \leq 3$, we may divide the sum $S$ up into 13 similar sums, $S_1, \ldots, S_{13}$ (say), and can do this in such a way that, within the sum $S_j$, the variable of summation $m$ is constrained to lie in an interval $(M_j',M_j)$ of length not exceeding $M/26$. We may then rewrite $S_j$ by means of a substitution of the form $m = M_j' - m'$. The effect of this is that $F(x)$ is replaced by the function $F_j(x) = -F((M_j' - M_j'x)/M)$, where $M_j' = M_j' - M_j' + O(1)$. Provided that $M_j' \in Z$ is suitably chosen, we will then have both $F^3_j(x) > 0$ and $F^2_j(x) > 0$ for $1/3 \leq x \leq 3$. It is moreover possible to ensure that each of the sums $S_1, \ldots, S_{13}$, when rewritten in the way just indicated, will satisfy all of the same conditions as are attached to the sum $S$ in the statement of Theorem 2 (albeit with each $C_r$ there possibly having to be increased by a certain factor $\Phi_r \in (1,24^{1/4})$). If one or more of the conditions (6.4), (6.6) or (6.11) should cease to be satisfied when $M$ is replaced by $M_j'$, then this can be remedied by means of the substitutions $F_j = \delta_j F_j', T = \delta_j^{-1}T_j'$, where $\delta_j$ is a suitable constant satisfying $1 > \delta_j \geq 24^{-4}$. Therefore the only cases of Theorem 2 requiring further attention are those in which both $F^3(x)$ and $F^2(x)$ are positive valued on the interval $[1/3,3]$, and so we may assume this henceforth.

In applying the Bombieri-Iwaniec method to $S$ we repeat, with one exception (that being the utilization of [W04, Equations (2.32) and (2.33)]), the steps described in [W04, Sections 2-5]. These steps assume (from the outset) a fixed choice of parameters $N,R \in \mathbb{N}$ satisfying:

\[ \frac{1}{(R-1)^2} > \frac{2NT}{C_3M^3} \geq \frac{1}{R^2}, \]

(7.3)

\[ \frac{HN^2}{MR^2} = O(1) \]

(7.4)

and

\[ \max\left\{ \frac{H}{R^2}, \frac{R}{H}, \frac{H}{N}, \frac{N}{M} \right\} \leq B_1, \]

(7.5)
where the constant $B_1 \in (0, 1)$ is assumed to be sufficiently small in terms of $C_3$.

The initial step is a partitioning of the range of the variable of summation $m$ that is achieved through a covering of the interval $[M_1 - 2N, M]$ by a minimal set of disjoint intervals $I_0, I_1, \ldots, I_t$, each of length $N$. To each interval $I_i$, there corresponds an ‘arc’ $J_i \subset \mathbb{R}$ that is the image of $I_i$ under the mapping $x \mapsto 2TM^{-2}F^{(3)}(x/M)$. These arcs are classified as ‘major’ or ‘minor’, according to the case $r = 1$ of the rules set out in [W04]. Some arcs are then fused, so that some minor arcs (and all major arcs) become parts of ‘long major arcs’. For each $i \geq 2$ such that $J_i$ is not part of any long major arc we choose $a/q \notin J_{i-2} \cap \mathbb{Q}$ such that $q$ is minimized, subject to the constraints $q \geq R$ and $(a, q) = 1$ (the arc $J_{i-2}$ in these cases being minor, though it may, at the same time, be part of a long major arc) and we put $I(a/q) = I_i$. Each such $I(a/q)$ can then be classified as either ‘bad’, or else ‘good’, according to how well $a/q$ can be approximated by rationals of a smaller denominator (see [H03, Page 600] or [W10, Section 2] for details). As a consequence of the results of [W04, Section 3] concerning long major arcs, the results of [W04, Equations (2.16) and (6.21)] concerning $q'$, the bound [W04, Equation (2.30)] and the case $\eta = 0$ of [W04, Equation (4.5), (4.22) and (5.1)], one has

$$|S| \leq O\left(\frac{MR \log N}{B_2 H^{\frac{1}{2}}N^{\frac{1}{2}}} + \sum_{c} \sum_{R \leq Q \leq B_2 H} S(C_Q), \right) \tag{7.6}$$

where each $C$ is a subset of the set $I_\pi = \{ I_i : 2 \leq i \leq \ell \text{ and } J_i \text{ is not part of a long major arc} \}$ and each $Q$ is an integer of the form $2^{b-1}R$ (with $b \in \mathbb{N}$), while

$$S(C_Q) = \sum_{I(a/q) \in C_Q} \left| \sum_{H_1 < h \leq H} \sum_{k \in I(a/q) \cap (M_1, M]} e\left( TF\left(\frac{k + h}{M}\right) - TF\left(\frac{k - h}{M}\right) \right) \right| \tag{7.7}$$

and $B_2$ denotes a positive constant that is chosen to be sufficiently small (in terms of $C_3$). To clarify this we remark that the variable of summation $C$ in (7.6) is subject to the condition

$$C \in \{ G^{[A(j)]} : j \in \mathbb{N} \} \cup \{ \mathcal{B} \}, \tag{7.8}$$

where

$$A(j) = 2^{j-1} \log N,$$

$$G^{[A]} = \{ I(a/q) \in I_\pi : I(a/q) \text{ is good and } A > \alpha(a/q; Q') \geq A \min\{1/2, A - \log N\} \}$$

and

$$\mathcal{B} = \{ I(a/q) \in I_\pi : I(a/q) \text{ is bad} \},$$

while $Q'$ is a certain parameter (to be specified later) and $\alpha(a/q; Q')$ is given by [W10, Equation (2.1)]. Note that, by the relevant definitions, the sets $\mathcal{B}$ and $G^{[A(j)]}$ ($j = 1, 2, \ldots$) are pairwise disjoint, and so, in light of our remarks preceding (10.3) (below), it follows by [H03, Lemma 2.3] that one has

$$|B_Q| + \sum_{j=1}^{\infty} \left| G^{[A(j)]}_Q \right| = |\{ I(a/q) \in I_\pi : Q \leq q \leq 2Q \}| \ll \frac{MR^2}{NQ^2} \quad (Q \geq R). \tag{7.9}$$

In [W04, Equation (2.30)] it is found (by partial summation) that for each $I(a/q) \in I_\pi$ one has a bound of the form

$$\sum_{H_1 < h \leq H} \sum_{k \in I(a/q)} e\left( TF\left(\frac{k + h}{M}\right) - TF\left(\frac{k - h}{M}\right) \right) \ll \sum_{h = H_2}^{H_3} \sum_{a = N_2}^{N_3} e\left( \frac{(an + b + \kappa)h}{q} + \mu n^2 h + \mu h^3 \right) \tag{7.9},$$
in which $H_2$, $H_3$, $N_2$ and $N_3$ are certain positive integers (dependent on $I(a/q)$) satisfying
\[
\frac{H}{2} \leq H_2 \leq H_3 \leq H \quad \text{and} \quad N \leq N_2 \leq N_3 \leq 3N,
\]
while $b = b(a/q) \in \mathbb{Z}$, $\kappa = \kappa(a/q) \in [0, 1)$ and $\mu = \mu(a/q) > 0$ are given by:
\[
b + \kappa = 2qTM^{-1}F^{(1)}(m/M) \quad \text{and} \quad \mu = TM^{-3}F^{(3)}(m/M),
\]
with $m$ being a nearest integer to the number $x \in (M_1 - 2N, M]$ satisfying $2TM^{-2}F^{(2)}(x/M) = a/q$. By this, (7.7), (7.9) and the case $r = 1$, $\chi = \chi_0$, $\eta = 0$ of [W04, Equations (4.5), Equation (4.22) and Lemma 5.4] we may conclude that, for $Q \in [R, B_2H]$ and $C$ satisfying (7.8), one has
\[
S(Q) \ll \left(\frac{R}{Q}\right) \left(\frac{H}{N}\right)^{\frac{1}{2}} M \log N + \sum_{I(a/q) \in C} |\sigma(a/q)|, \quad (7.10)
\]
where
\[
\sigma(a/q) = \sum_{L_2 \leq \ell < L_3} \sum_{K_4(\ell) \leq \phi(\kappa-\ell)} e^{\left(\frac{\chi(a/q) \cdot y(k,\ell)(\kappa)}{4\mu q}\right)^{1/2} \left(\frac{((k-\kappa)^2 - \ell^2)^{1/4}}{4\mu q}\right)} , \quad (7.11)
\]
with:
\[
L_j = 2\mu q H_j N_j \quad (j = 2, 3), \quad K_4(\ell) = 2\mu q \max \left\{H_2^2, (\ell/2\mu q N_3)^2\right\}, \quad K_5(\ell) = 2\mu q \min \left\{H_3^2, (\ell/2\mu q N_2)^2\right\},
\]
\[
y(k,\ell)(k) = \left(\frac{k\ell, \omega(k-\ell)}{3} \right) = (k\ell, \omega(k-\ell)) \quad \text{(see (1.1))}
\]
and
\[
\chi^{(a/q)} = \left(\left\{\frac{\pi}{q}\right\}, \left\{\frac{\pi b}{q}\right\}, \left\{\frac{1}{\sqrt{\mu q}}\right\}\right),
\]
where we have, in the last line above, $\{\beta\} = \beta - \max\{j \in \mathbb{Z} : j \leq \beta\}$ (the ‘fractional part’ of $\beta \in \mathbb{R}$), and take $\pi$ to be any integer satisfying $a\pi \equiv 1 \pmod{q}$. Note that we have here implicitly corrected an erroneous statement made in [W04, (4.12)], but not propagated to any subsequent part of [W04]. For future reference note that, by (6.1), (6.2), (7.3) and (7.5), one has
\[
\frac{1}{2NR^2} \leq \mu < \frac{2C_3}{NR^2}. \quad (7.12)
\]
8. Preparations for the modified double large sieve

The sum over $I(a/q)$ in (7.10)-(7.11) is not suitable for an immediate application of the Bombieri-Iwaniec double large sieve (for which see [H96, Lemma 5.6.6], for example). The principal reason for this is the dependence of the ranges of summation of both $k$ and $\ell$ upon the interval $I(a/q)$. The same problem occurs in [W04, Section 6], and we shall make use of one part of the solution given there. However, since we do not indulge here in the averaging over $\eta \in (-1/2, 1/2)$ that was found useful in [W04], and since it is the modified form of the double large sieve from Section 5 that we seek to apply, the preparations we shall make for its application have to differ in certain other respects from the preparatory steps undertaken in [W04, Section 6]. In particular we shall deal in a different way with terms depending on the variable $\kappa = \kappa(a/q)$.

Our first concern is with the dependence of the condition $K_4(\ell) \leq \phi(k - \kappa, \ell) \leq K_5(\ell)$ upon $\kappa$. Given any $\ell \in (L_2, L_3)$, the set \( \{ u \in \mathbb{R} : K_4(\ell) \leq \phi(u, \ell) \leq K_5(\ell) \} \) is one subinterval of $(0, \infty)$, and so, for $0 \leq \kappa < 1$, the sums $\sigma(a/q)$ and

\[
\sigma_1(a/q) = \sum_{L_2 < \ell < L_3} \sum_{K_4(\ell) \leq \phi(k, \ell) \leq K_5(\ell)} \frac{e \left( \frac{x(a/q) \cdot y(k, \ell)(\kappa)}{4\mu q} \right)}{(4\mu q)^{\frac{1}{2}} ((k - \kappa)^2 - \ell^2)^{\frac{1}{2}}}
\]

contain less than $2L_3$ summands that are not common to both. One can show furthermore that within either one of the sums $\sigma(a/q)$, $\sigma_1(a/q)$ one has

\[
HQ \ll \ell \ll \frac{C^2 HQ}{R^2} \quad \text{and} \quad \frac{N}{H} \ll \frac{k}{\ell} \ll \frac{N}{H}
\]

(where the implicit constants are absolute). Therefore, given (7.5) (where one may take $B_1$ to be arbitrarily small) and (7.12), the summands of $\sigma(a/q)$ or $\sigma_1(a/q)$ have absolute values that are bounded above by

\[
O \left( \frac{1}{\sqrt{\mu q k}} \right) \ll \frac{R^2}{Q}
\]

and so we have:

\[
\sigma(a/q) - \sigma_1(a/q) \ll L_3 R^2 Q^{-1} \ll H.
\]

By elementary calculus, we have also

\[
(k - \kappa)^2 - \ell^2)^{-\frac{1}{2}} = (k^2 - \ell^2)^{-\frac{1}{2}} (1 + O (k^{-1})) \quad (0 \leq \kappa < 1)
\]

within the sum $\sigma_1(a/q)$. We therefore have:

\[
\sigma_1(a/q) - \sigma_2(a/q) \ll L_3 \sum_{NQ/R^2 \ll k \ll NQ/R^2} \frac{R^2}{Qk} \ll H,
\]

where

\[
\sigma_2(a/q) = \sum_{L_2 < \ell < L_3} \sum_{K_4(\ell) \leq \phi(k, \ell) \leq K_5(\ell)} \frac{e \left( \frac{x(a/q) \cdot y(k, \ell)(\kappa)}{4\mu q} \right)}{(4\mu q)^{\frac{1}{2}} (k^2 - \ell^2)^{\frac{1}{2}}}.
\]

Next we work to replace $y^{(k, \ell)}(\kappa) = y^{(k, \ell)}(\kappa(a/q))$ by a (higher dimensional) vector that is independent of the interval $I(a/q)$. Using the binomial theorem and some elementary estimates, we find that within the sum $\sigma_2(a/q)$ one has

\[
y^{(k, \ell)}_3(\kappa) = \omega(k - \kappa, \ell) = O \left( \ell k^{-\frac{3}{2}} \right) + \frac{3}{2} \sum_{j=0}^{2} \left( \frac{3/2}{j} \right) (-\kappa)^j \left( (k + \ell)^{\frac{3}{2}} - (k - \ell)^{\frac{3}{2}} \right),
\]
so that
\[ x_3^{(a/q)}y_3^{(k,\ell)}(\kappa) = O \left( (\mu q^3)^{-\frac{1}{2}} \ell k^{-\frac{3}{2}} \right) + \sum_{j=0}^{2} \frac{(-\kappa)^j}{j! (\mu q^3)^{\frac{j}{2}}} \omega_j(k, \ell), \]
where
\[ \omega_j(u, \ell) = \frac{\partial^j}{\partial u^j} \omega(u, \ell) \quad (u > \ell > 0). \]

After noting that we will have here
\[ (\mu Q^3)^{-\frac{1}{2}} \ell k^{-\frac{3}{2}} \ll (N/R)^{\frac{1}{2}} (H/N) (N/R)^{-\frac{1}{2}} = HR/N^2 \leq B_1^2, \]
we are able to deduce that
\[ \sigma_2(a/q) - \sigma_3(a/q) \ll \sum_{0<\ell(HQ/R^2)} \sum_{k\geq NQ/R^2} \mu^{-1} Q^{-2} \ell k^{-3} \ll NR^2 Q^{-2} (H/N)^2 \ll H^2/N < H, \quad (8.3) \]
where
\[ \sigma_3(a/q) = \sum_{L_2<\ell<L_3} \sum_{K_4(\ell) \leq \phi(k, \ell) \leq K_5(\ell)} e \left( \frac{\bar{x}(a/q) \cdot \bar{y}^{(k,\ell)}}{(4\mu q)^{\frac{1}{2}} (k^2 - \ell^2)^{\frac{1}{2}}} \right) \]
with
\[ \bar{x}^{(a/q)} = \left( \left\{ \frac{\pi a}{q} \right\} \cdot \left\{ \frac{\pi b}{q} \right\}, \frac{1}{\sqrt{\mu q^3}}, \frac{-\kappa}{\sqrt{\mu q^3}}, \frac{\kappa^2}{2\sqrt{\mu q^3}} \right) \]
and
\[ \bar{y}^{(k,\ell)} = (k\ell, \ell, \omega(k, \ell), \omega_1(k, \ell), \omega_2(k, \ell)). \]

The sum \( \sigma_3(a/q) \) is now almost suitable for the application of the double large sieve: the sole remaining problem is the dependence upon \( I(a/q) \) of the ranges of summation for \( k \) and \( \ell \). By virtue of our elimination of \( \kappa \) from the conditions of summation, the problem just mentioned is essentially the special case \( \kappa = 0 \) of the problem that is addressed in the first two thirds of [W04, Section 6], and can therefore be dealt with by employing exactly the same method as is described there. We begin the process by observing that the conditions on the pair \((k, \ell) \in \mathbb{Z}^2 \) in the sum \( \sigma_3(a/q) \) are satisfied if and only if one has both
\[ 2H_2\sqrt{\mu q} \leq \sqrt{k+\ell} - \sqrt{k-\ell} \leq 2H_3\sqrt{\mu q} \quad (8.4) \]
and
\[ 2N_2\sqrt{\mu q} \leq \sqrt{k+\ell} + \sqrt{k-\ell} \leq 2N_3\sqrt{\mu q}. \quad (8.5) \]

Similarly to what is found in [W04, Equations (6.8)-(6.10)], we have now
\[ \sigma_3(a/q) = \sum_{K_0} \sum_{L_0} \sigma_4(a/q; K_0, L_0), \quad (8.6) \]
where
\[ \sigma_4(a/q; K_0, L_0) = \sum_{L_0<\ell \leq 2L_0} \sum_{K_0<k \leq 2K_0} e \left( \frac{\bar{x}^{(a/q)} \cdot \bar{y}^{(k,\ell)}}{(4\mu q)^{\frac{1}{2}} (k^2 - \ell^2)^{\frac{1}{2}}} \right). \]
(with the superfix \((a/q)\) attached to the inner summation indicating that \(k\) and \(\ell\) are constrained to satisfy (8.4) and (8.5)), while \((K_0, L_0)\) runs over the pairs of integer powers of 2 that satisfy

\[
\frac{K_0}{K}, \frac{L}{L} \in \left[ \frac{1}{8} \cdot 144C_3^2 \right] \quad \text{and} \quad \frac{12N}{H} \geq \frac{K_0}{L_0} \geq \frac{N}{4H} \geq 16 , \tag{8.7}
\]

with

\[
K = \frac{NQ}{R^2} \quad \text{and} \quad L = \frac{HQ}{R^2} .
\]

We may deal with the sum \(\sigma_4(a/q; K_0, L_0)\) in the same way that the sum \(B_{1,1}^*(K_0, L_0; \kappa)\), defined in [W04, Equation (6.11)], is dealt with in [W04, Pages 342-344]. In particular, by means of an application of [W04, Lemma 6.1], it can be shown that one has

\[
\sigma_4(a/q; K_0, L_0) \ll \int_{-1}^{1} \int_{-1}^{1} \frac{|\sigma_5(a/q; K_0, L_0, w)|}{\Delta(K_0, w)} dw_1 dw_2 + H \log N , \tag{8.8}
\]

where

\[
w = (w_1, w_2) , \quad \Delta(K_0, w) = (K_0^{-2} + |w_1|) (K_0^{-2} + |w_2|) > 0
\]

and

\[
\sigma_5(a/q; K_0, L_0, w) = \sum_{L_0 < \ell \leq 2L_0} \sum_{K_0 < k \leq 2K_0} \frac{e^{\left(\frac{\sqrt{(a/q) \cdot (k, \ell)} - K_0^{3/2} w \cdot c^{(k, \ell)}}{(4\mu q)^{1/2} (k^2 - \ell^2)^{1/4}}\right)}}{(4\mu q)^{1/2} (k^2 - \ell^2)^{1/4}} ,
\]

with

\[
c^{(k, \ell)} = (\sqrt{k + \ell} - \sqrt{k - \ell}, \sqrt{k + \ell} + \sqrt{k - \ell}) .
\]

By (7.10), (8.1)-(8.3), (8.6), (8.8) and (7.9) and (7.5), it follows that for \(Q \in [R, B_2H]\) and \(C\) satisfying (7.8) one has either

\[
S(C_Q) \ll \left( \frac{R}{Q} \right) \left( \frac{H}{N} \right)^\frac{1}{2} M \log N \leq \left( \frac{H}{N} \right)^\frac{1}{2} M \log N , \tag{8.9}
\]

or else

\[
S(C_Q) \ll \sum_{K_0} \sum_{L_0} \int_{-1}^{1} \int_{-1}^{1} \left( \sum_{I(a/q) \in C_Q} \left| \sigma_5(a/q; K_0, L_0, w) \right| \right) (\Delta(K_0, w))^{-1} dw_1 dw_2 ,
\]

where \((K_0, L_0)\) runs over the pairs of integer powers of 2 that satisfy (8.7). In the latter case one should observe that the integral \(\int_{-1}^{1} \int_{-1}^{1} (\Delta(K_0, w))^{-1} dw_1 dw_2\) is equal to \((2 \log (1 + K_0^2))^2\), and that the relevant number of pairs \((K_0, L_0)\) does not exceed \(O(\log C_3)\). It may therefore be deduced that in that latter case one will have

\[
S(C_Q) \ll (\log K_0)^2 \sum_{I(a/q) \in C_Q} |\sigma_5(a/q; K_0, L_0, w)| ,
\]

for some \(w \in \mathbb{R}^2\) and some pair \((K_0, L_0)\) satisfying (8.7).

In order to present our conclusions (just reached) in a form slightly more convenient for the work in the next section, we remark that they trivially imply that, if \(Q \in [R, B_2H]\) and \(C\) satisfies (7.8),
then either (8.9) holds, or else there exists some pair \((K_0, L_0)\) satisfying (8.7) and some \(W \in \mathbb{R}^2\) such that
\[
S(C_Q) \ll (R \log N)^2 Q^{-1} \sum_{I(a/q) \in C_Q} |\tilde{\sigma}_6(a/q; K_0, L_0, W)| = (R \log N)^2 Q^{-1} \tilde{S}(C_Q) \quad \text{(say), (8.10)}
\]
where
\[
\tilde{\sigma}_6(a/q; K_0, L_0, W) = \sum_{L_0 < \ell \leq 2L_0} \sum_{K_0 < k \leq 2K_0} \psi_{k,\ell}(W)e \left( \frac{x^{(a/q)} \cdot \bar{y}^{(k,\ell)}}{k^2 - \ell^2} \right),
\]
with
\[
\psi_{k,\ell}(W) = \left( \frac{K_0^2 - 4L_0^2}{k^2 - \ell^2} \right)^\frac{1}{2} e \left( -W \cdot e(k,\ell) \right)
\]
(so that \(\psi_{k,\ell}(W)\) is here independent of \(I(a/q)\) and is such that \(|\psi_{k,\ell}(W)| < 1\) when \(k/K_0, \ell/L_0 \in (1, 2)\)).

It is worth noting that, by (7.5) and (11.11) and (11.12) (below), the rightmost bound in (8.9) is stronger than the bounds for \(S(C_Q)\) that we shall ultimately obtain in (12.11) and (12.14) (below). Therefore, in the course of our proof of (12.11) and (12.14) (spanning Sections 9-12, below) we may suppose it to be the case that (8.9) does not hold, and, on the basis of that supposition, may infer from the preceding paragraph that one does have the bound (8.10) (with \(K_0, L_0\) and \(W\) as just described above).

9. The application of the modified double large sieve

Let
\[
\mathcal{X} = \mathcal{X}(C, Q) = \left\{ \tilde{x}^{(a/q)} : I(a/q) \in C_Q \right\}
\]
and
\[
\mathcal{Y} = \mathcal{Y}(K_0, L_0) = \left\{ \tilde{y}^{(k,\ell)} : K_0 < k \leq 2K_0, L_0 < \ell \leq 2L_0 \text{ and } k, \ell \in \mathbb{Z} \right\}.
\]
By virtue of the definitions of \(\tilde{x}_1^{(a/q)}, \tilde{x}_2^{(a/q)}, \tilde{y}_1^{(k,\ell)}, \tilde{y}_2^{(k,\ell)}\), the mappings \(I(a/q) \mapsto \tilde{x}^{(a/q)} \in \mathcal{X}\) and \((k, \ell) \mapsto \tilde{y}^{(k,\ell)} \in \mathcal{Y}\) are injective functions on the domains \(C_Q\) and \((\{K_0, 2K_0\} \times \{L_0, 2L_0\}) \cap \mathbb{Z}^2\), respectively. Therefore we may deduce from (8.10) that, for a certain pair of functions \(\alpha : \mathcal{X} \to \mathbb{C}\) and \(\beta : \mathcal{Y} \to \mathbb{C}\), determined by \(\mathcal{X}, \mathcal{Y}, K_0, L_0\) and \(W\), and satisfying
\[
|\beta(y)| < 1 = |\alpha(x)| \quad (x \in \mathcal{X}, y \in \mathcal{Y}), \quad \text{(9.1)}
\]
one has:
\[
0 \leq \tilde{S}(C_Q) = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} e(x \cdot y) \alpha(x) \beta(y). \quad \text{(9.2)}
\]

In Section 5 it is shown how a modified form of the double large sieve may be be used to obtain useful upper bounds for the absolute value of a sum similar to the above sum over \((x, y) \in \mathcal{X} \times \mathcal{Y}\). Only the case in which \(\alpha(x)\) and \(\beta(y)\) both have range \(\{1\}\) is treated there, so our next task is to show that this restriction does not prevent us from using the large sieve of Section 5 to get upper bounds for \(\tilde{S}(C_Q)\).

A helpful observation is that one has:
\[
\alpha(x) = \sum_{m=1}^{4} (-i)^m \max \{0, \text{Re} \ (i^m \alpha(x))\} \quad (x \in \mathcal{X}),
\]

where \(\tilde{\sigma}_6(a/q; K_0, L_0, W)\) is here independent of \(I(a/q)\) and is such that \(|\tilde{\sigma}_6(a/q; K_0, L_0, W)| < 1\) when \(k/K_0, \ell/L_0 \in (1, 2)\)).
and (of course) a similar formula for each \( \beta(y) \) occurring in (9.2). By using these formulae, and a change in the order of summation, one rewrite the sum on the right-hand side of (9.2) as a sum of sixteen sums that are each similar to the original sum, but have a product of the form 
\[-i^{m+n} \max\{0, \Re(i^m \alpha(x))\} \max\{0, \Re(i^n \beta(y))\}\]  
in place of \( \alpha(x)\beta(y) \). From this and (9.1) we may deduce that there exist functions 
\[\alpha_1 : \mathcal{X} \to [0, 1], \quad \beta_1 : \mathcal{Y} \to [0, 1]\]
such that 
\[0 \leq \widetilde{S}(C_Q) \leq 16 \left| \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} e(x \cdot y)\alpha_1(x)\beta_1(y) \right|.
\]
Within the last sum over \((x, y) \in \mathcal{X} \times \mathcal{Y}\) we may apply the substitutions 
\[\alpha_1(x) = \int_0^{\alpha_1(x)} d\theta \quad \text{and} \quad \beta_1(x) = \int_0^{\beta_1(x)} d\phi.
\]
Then, via a change in the order of summation and integration, we are able to deduce that 
\[0 \leq \widetilde{S}(C_Q) \leq 16 \int_0^1 \int_0^1 \left| \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} e(x \cdot y) \right| d\theta d\phi.
\]
Since the integral here does not exceed the maximum value attained by its integrand, it follows that there exist subsets 
\[\mathcal{X}_1 \subseteq \mathcal{X} \quad \text{and} \quad \mathcal{Y}_1 \subseteq \mathcal{Y}\]
such that one has 
\[0 \leq \widetilde{S}(C_Q) \leq 16 |S^*(\mathcal{X}_1, \mathcal{Y}_1)|,
\]
with 
\[S^*(\mathcal{X}_1, \mathcal{Y}_1) = \sum_{x \in \mathcal{X}_1} \sum_{y \in \mathcal{Y}_1} e(x \cdot y).
\]

Let \( V \) be chosen so that one has 
\[V \geq 1\]
(this choice will ultimately be determined by our use of Lemma 10.1 and Lemma 10.2, below, and so \( V \) will depend on the set \( C \), and on which of (10.14) or (10.15) is taken as the definition of \( N^* \)).
Then, by (7.12), (8.7), (9.3) and the definitions of the sets \( C_Q, \mathcal{X}, \mathcal{Y} \) and their elements, we have: 
\(\mathcal{X}_1, \mathcal{Y}_1 \subseteq \mathbb{R}^3 \) and 
\[|x_i| < D_i \quad \text{and} \quad |y_i| < E_i^V \quad (x \in \mathcal{X}_1, y \in \mathcal{Y}_1, i = 1, \ldots, 5),\]
where 
\[D = \left(1, 1, \sqrt{\frac{2NR^2}{Q^3}}, \sqrt{\frac{2NR^2}{Q^3}}, \sqrt{\frac{NR^2}{2Q^3}}\right) \quad \text{and} \quad E^V = \left(5K_0L_0V, 3L_0, 3L_0\sqrt{K_0}, \frac{2L_0}{\sqrt{K_0}}, \frac{L_0}{\sqrt{K_0}}\right).
\]
Given we assume \( R \leq Q \leq B_2H \), it follows by (7.5), (8.7), (9.5) and (9.7) that 
\[\prod_{i=1}^5 (1 + D_iE_i^V) \ll \left(\frac{HNQ^2V}{R^4}\right) \left(\frac{HQ}{R^2}\right) \left(\frac{H}{R^2}\right) \left(\frac{H}{Q}\right) (1) = \frac{H^4N^2Q^2V}{R^8}.
\]
By this and (9.6), an application of the large sieve inequality (5.5) shows that we have

\[ S^* \left( \mathcal{X}_1, \mathcal{Y}_1 \right) \ll \left( A_p \left( \mathcal{Y}_1, \mathcal{D} \right) \tilde{b} \left( \mathcal{X}_1, \mathcal{E}^{[V]} \right) H^4 N^2 Q^2 R^{-8} \right)^{\frac{1}{p}} |\mathcal{X}_1|^{1 - \frac{1}{d}} \quad (2 < p < \infty), \]

where

\[ A_p \left( \mathcal{Y}_1, \mathcal{D} \right) = \frac{1}{2^5 D_1 D_2 \cdots D_5} \int_{-\frac{D_1}{2}}^{\frac{D_1}{2}} \cdots \int_{-\frac{D_5}{2}}^{\frac{D_5}{2}} \left| \sum_{y \in \mathcal{Y}_1} e(x \cdot y) \right|^p dx_5 \cdots dx_1 \]

and

\[ \tilde{b} \left( \mathcal{X}_1, \mathcal{E}^{[V]} \right) = \left| \left\{ (x, x') \in \mathcal{X} \times \mathcal{X} : |x_j - x'_j| < 1/E_j^{[V]} \ (j = 1, \ldots, 5) \right\} \right|. \]

Now, by (9.3) and the definition of the set \( \mathcal{X} \), we have \( |\mathcal{X}_1| \leq |\mathcal{X}| = |\mathcal{C}_Q| \). Therefore, by the bound just obtained for \( S^* \left( \mathcal{X}_1, \mathcal{Y}_1 \right) \), in combination with (8.10) and (9.4), one has:

\[ S \left( \mathcal{C}_Q \right) \ll \left( A_p \left( \mathcal{Y}_1, \mathcal{D} \right) \tilde{b} \left( \mathcal{X}_1, \mathcal{E}^{[V]} \right) V H^4 N^2 R^{-6} \right)^{\frac{1}{p}} \left( |\mathcal{C}_Q| R/Q \right)^{1 - \frac{1}{d}} R (\log N)^2 \quad (2 < p < \infty). \quad (9.8) \]

10. The second spacing problem

Since there is a one-to-one correspondence between the elements of \( \mathcal{C}_Q \) and those of \( \mathcal{X} \), it therefore follows by (9.3) and the definitions of both \( \mathcal{X} \) and \( \tilde{b} \left( \mathcal{X}_1, \mathcal{E}^{[V]} \right) \) that one has

\[ \tilde{b} \left( \mathcal{X}_1, \mathcal{E}^{[V]} \right) \leq \left| \left\{ (x, x') \in \mathcal{X} \times \mathcal{X} : |x_j - x'_j| < 1/E_j^{[V]} \ (j = 1, \ldots, 4) \right\} \right| \]

\[ = \left| \left\{ (I(a/q), I(a'/q')) \in \mathcal{C}_Q \times \mathcal{C}_Q : \left| \frac{x_j(a/q)}{q} - \frac{x'_j(a'/q')}{q'} \right| < 1/E_j^{[V]} \ (j = 1, \ldots, 4) \right\} \right| \]

\[ \leq B \left( \mathcal{C}_Q ; V \right), \quad (10.1) \]

where (in light of (9.7), (9.5), (8.7), (7.12) and (7.5)) one may take \( B \left( \mathcal{C}_Q ; V \right) \) to be the number of pairs of intervals \( I(a/q), I(a'/q') \in \mathcal{C}_Q \) satisfying a system of inequalities of the form

\[ \left\| \frac{x - \alpha}{q} \right\| \leq \Delta_1, \quad \left\| \frac{\mu \cdot q}{q} - 1 \right\| \leq \Delta_2, \quad \left\| \frac{\tau b}{q} - \frac{\sigma b'}{q'} \right\| \leq \Delta_3 \quad \text{and} \quad |\kappa - \kappa'| \leq \Delta_4, \]

in which \( \left\| \alpha \right\| = \min \{ |\alpha - n| : n \in \mathbb{Z} \} \), while the numbers \( \Delta_1, \ldots, \Delta_4 \) are determined by \( V, K_0, L_0, R, N \) and \( C_5 \), and satisfy:

\[ \Delta_1 \asymp \frac{R^4}{H N Q^2 V}, \quad \Delta_2 \asymp \frac{R^2}{H N}, \quad \Delta_3 \asymp \frac{R^2}{H Q} \quad \text{and} \quad \Delta_4 \asymp \frac{Q}{H} \quad (10.2) \]

(with the notation \( X \asymp Y \) signifying that one has both \( X \ll Y \) and \( Y \ll X \)).

The problem of obtaining good upper bounds for \( B \left( \mathcal{C}_Q ; V \right) \) is essentially the same 'Second Spacing Problem' as that referred to in [H03, Section 3] (see also [H05, Section 3] for a somewhat generalized definition of this problem). Indeed, the only difference between the two that is of any significance is that, whereas the function \( 2F^{(1)}(x) \) and its derivatives play a certain part in determining the second spacing problem in this present paper (i.e. they play their part in determining the set \( \mathcal{X} \)), the corresponding part in [H03] is played instead by the function there named \( F(x) \), and its derivatives. The only consequence of this difference is that, where a condition of (for example) the form \( C \geq |F^{(r)}(x)| \geq 1/C \) is assumed in [H03], we shall instead need only an assumption implying...
that one has $C' \geq 2|F^{(r+1)}(x)| \geq 1/C'$, for some constant $C' \geq 1$. Therefore each of the results on the second spacing problem that are stated in [H03, Section 3] implies a similar result for $B(C_Q; V)$ (differing only in that the hypotheses concerning derivatives of $F(x)$ are modified in the way that our preceding remark indicates). For the same reason we are able to infer from [H03, Lemmas 2.3, 2.4 and 2.5] certain bounds for the number of elements in each set $C_Q$ that we need consider. One of these bounds is (7.9) (above). The other two assume more about how the classification of elements of $I_{\pi}$ (as being either good or bad) is done. That classification is dependent on a pair of chosen parameters, $\eta$ and $Q'$. If these chosen parameters satisfy

$$1 + \frac{M^2}{T} \leq Q' < \eta R < R$$

(10.3)

then one may infer from [H03, Lemma 2.3, Lemma 2.4 and Lemma 2.5] that one has

$$|B_Q| \ll \frac{\eta MR^2}{NQ^2} + \frac{MQ'}{NQ} \quad (R \leq Q \leq B_2H)$$

(10.4)

and

$$|G_Q^{[A]}| \ll \frac{MR^2 \log N}{ANQ^2} \quad (R \leq Q \leq B_2H \text{ and } A \geq \log N).$$

(10.5)

Huxley’s unconditional results in [H03, Lemma 3.4] are an outcome of his work in [H05] on ‘resonance curves’. It is assumed in [H03, Lemma 3.4] that one has

$$V_0^2 \ll N \sqrt{V_1V_2},$$

(10.6)

with

$$V_0 = \left( \frac{H}{R} \right)^{\frac{1}{119}}, \quad V_1 = \frac{R^4}{HN} \quad \text{and} \quad V_2 = \frac{M^2}{HN^3}. \quad (10.7)$$

Note that (10.6) is [H03, Condition (3.12)]. The parameters $V_1$ and $V_2$ have a significance that is explained below [H03, Equation (3.5)]. There is also another parameter $\Delta'$ that plays a part within certain calculations of [H05]. In [H03, Lemma 3.4] it is assumed that one may assign $\Delta'$ a value such that

$$\left( \frac{R}{H} \right)^{\frac{1}{17}} \ll \frac{\Delta'}{\Delta_2} \ll \left( \frac{R}{H} \right)^{\frac{1}{44}},$$

(10.8)

$$\frac{Q'}{R} \ll \left( \frac{\Delta' R}{\Delta_2 H} \right)^{\frac{1}{44}} \eta,$$

(10.9)

$$V_0 \gg \frac{\Delta_2 R^2 \eta}{\Delta' Q'^2}$$

(10.10)

and

$$\left( \frac{\Delta'}{\Delta_2} \right) \left( \frac{Q'}{R} \right)^6 \gg \left( \frac{R}{H} \right)^{\frac{6}{17}}.$$ \quad (10.11)

From [H03, Lemma 3.4] we infer, as an immediate corollary, the following lemma.

**Lemma 10.1. (Huxley).** Let those of the hypotheses of Theorem 2 that concern $F(x)$ be satisfied. Suppose also that (7.3)-(7.5), (10.2), (10.3) and (10.6)-(10.11) hold. Put

$$Q_2 = \frac{R^{60} H^{33}}{(\log(H/R))^{17}}.$$ \quad (10.12)
Then one has
\[ B(G_Q^{[A]}; V) \ll \frac{M}{N} + \left( 1 + \frac{Q}{Q_2} \right) \left( \frac{M^2 R^4 (\log N)^{\frac{3}{7} A^{\frac{11}{70}}}}{H^2 N^4 V_0^{\frac{2}{7}}} \right) \quad (R \leq Q \leq B_2 H \text{ and } A \geq \log N) \] (10.13)
in the following three independent cases:

- (A) when one has \( V = V_0 \ll \min \{V_1, V_2\} \);
- (B) when \( M^2 \ll T \), with \( V = \max \{V_1, 1\} \);
- (C) when \( M^2 \gg T \), with \( V = \max \{V_2, 1\} \).

**Remark.** In the present work (where our main concern is with bounds for exponential sums that are of use in estimating \( E(T) \)) we shall employ (10.13) only in the cases (A) and (B) (hence the lack of any ‘Case (C)’ in our statement of Theorem 2).

The choice of \( N \) must (of course) be made prior to the application of Lemma 10.1, and the case of the lemma that applies will depend on that choice. Nevertheless one can take account of the form of the bound (10.13), and the definitions of the cases of the lemma in optimizing that choice. We shall restrict our choice of \( N \in \mathbb{N} \) so as to have either
\[ N \asymp H \left( \frac{M}{H} \right)^{\frac{11}{70}} T^{-\frac{49}{100}} (\log T)^{\frac{969}{1400}}, \] (10.14)
or else
\[ N \asymp \min \left\{ \frac{M^2 (\log T)^{\frac{560}{560}}}{T H^{\frac{1}{70}}} , \frac{B_2 M^2}{T H^{\frac{1}{70}}} \right\}, \] (10.15)
where \( B_2 \) is a sufficiently small positive constant (constructed from \( C_2, \ldots, C_6 \)). These are essentially the same choices for \( N \) that are described in in [H03, Equations (3.19)-(3.21)] (except that we specialize to the case \( \kappa = 3/10, \lambda = (1/4) + (11/70) = 57/140 \) of what is stated there). We allow the choice (10.14) only if that choice results in Case (A) of Lemma 10.1 being applicable; the choice (10.15) is similarly associated with Case (B) of the lemma (and so is permissible only when \( M^2 \ll T \)).

For certain combinations of values of \( T, M \) and \( H \), both the options (10.14) and (10.15) may be available (we shall then consider what is the outcome from each of the two choices of \( N \)).

These choices for \( N \) (and the associated restrictions on the use of Lemma 10.1) are exactly what is required in order to ensure that we never find the term \( M/N \) on the right-hand side of (10.13) dominating the other term there. For this reason we obtain, every time, bounds of the form
\[ B(G_Q^{[A]}; V) \ll \left( 1 + \frac{Q}{Q_2} \right) \left( \frac{M^2 R^4 (\log N)^{\frac{3}{7} A^{\frac{11}{70}}}}{H^2 N^4 V_0^{\frac{2}{7}}} \right), \] (10.16)
with
\[ V = V(G_Q^{[A]}) = \begin{cases} V_0 & \text{if (10.14) is assumed;} \\ \max \{V_1, 1\} & \text{if instead (10.15) is assumed.} \end{cases} \] (10.17)

The association of (10.15) with Case (B) of Lemma 10.1 also has the effect of ensuring that we have
\[ \min \{V_1, V_2\} \gg 1. \] (10.18)
This bound is immediate when Case (A) of the lemma applies (for (7.5) and (10.7) imply that one has \( V_0 > 1 \)); when Case (B) of the lemma applies one obtains (10.18) by virtue of (10.7), (10.15), (7.3) and (7.5).
We require, in addition to the bound (10.13), some sufficiently strong bounds for the numbers $B(B_Q;V)$ ($R \leq Q \leq B_2H$). For this we fall back on the following immediate corollary of [H03, Lemma 3.2].

**Lemma 10.2. (Huxley).** Let those of the hypotheses of Theorem 2 that concern $F(x)$ be satisfied. Suppose also that (7.3)-(7.5), (10.3) and (10.7) hold. Put

$$Q_3 = \frac{R \cdot H^{\frac{1}{2}}}{\log(H/R)}.$$  \hfill (10.19)

Then one has

$$B(B_Q;V) \ll \frac{M}{N} + \left(1 + \frac{Q}{Q_3}\right)\left(\frac{M^2R^4}{H^2N^4(H/R)^2V}\right) \quad (R \leq Q \leq B_2H)$$  \hfill (10.20)

in the following three independent cases:

(A) when one has $V = H/R \ll \min\{V_1, V_2\}$;
(B) when $M^2 \ll T$, with $V = \max\{V_1, 1\}$;
(C) when $M^2 \gg T$, with $V = \max\{V_2, 1\}$.

Note that the conditions defining Case (B) in this lemma are the same as those defining Case (B) in Lemma 10.1. Since $V_0$ is greater than $H/R$ it may also be observed that, whenever the choice $V = V_0$ would cause the conditions of Case (A) of Lemma 10.1 to be satisfied, the alternate choice $V = H/R$ would ensure (instead) that the conditions of Case (A) of Lemma 10.2 are satisfied. Consequently, whenever a bound of the form (10.16)-(10.17) is obtained (in the manner indicated above) it will follow from Lemma 10.2 that one also obtains the bound

$$B(B_Q;V) \ll \left(1 + \frac{Q}{Q_3}\right)\left(\frac{M^2R^4(\log N)^{\frac{27}{14}}}{H^2N^4(H/R)^2V}\right),$$  \hfill (10.21)

with

$$V = V(B) = \begin{cases} \frac{H}{R} & \text{if (10.14) is assumed;} \\ \max\{V_1, 1\} & \text{if instead (10.15) is assumed.} \end{cases}$$  \hfill (10.22)
11. The first spacing problem: bounds from decoupling for perturbed cones

From (9.7) and the definition of \( A_p(\mathcal{Y}_1, D) \) in Section 9 it may be deduced that, for some point \((\xi_3, \xi_4, \xi_5) \in \mathbb{R}^3\) satisfying \(-D_j \leq \xi_j \leq D_j\) \((j = 3, 4, 5)\), one has

\[
0 \leq A_p(\mathcal{Y}_1, D) \leq \frac{1}{D_3} \int_{-1}^{1} \int_{-1}^{1} \int_{\xi_3}^{\xi_3+4} \sum_{y \in \mathcal{Y}_1} e \left( \left( x_1, x_2, x_3, \xi_4, \xi_5 \right) \cdot y \right) \right| \right|_p dx_3 dx_2 dx_1.
\]

By this, (9.3) and the definitions relating to the set \(\mathcal{Y}_1\), one obtains bounds of the form

\[
\frac{1}{D_3} \int_{-1}^{1} \int_{-1}^{1} \int_{\xi_3}^{\xi_3+4} \sum_{k_0<k \leq 2K_0} \sum_{L_0<\ell \leq 2L_0} a_{k, \ell} e \left( x_1k \ell + x_2 \ell + x_3 \omega(k, \ell) \right) \left| \right|_p dx_3 dx_2 dx_1 
\geq A_p(\mathcal{Y}_1, D) \geq 0,
\]

with certain complex coefficients \(a_{k, \ell}\) that are independent of \((x_1, x_2, x_3) \in \mathbb{R}^3\), and that satisfy

\[
|a_{k, \ell}| \leq 1 \quad (K_0 < k \leq 2K_0 \text{ and } L_0 < \ell \leq 2L_0).
\]

Therefore, upon recalling the notation from Section 4 (and renumbering \(x_1\) and \(x_2\)), we arrive at

\[
\left( A_p(\mathcal{Y}_1, D) \right)^{\frac{1}{2}} \leq \left\| \sum_{k \sim K_0, \ell \sim L_0} a_{k, \ell} e \left( x_1k \ell + x_2 \ell + x_3 \omega(k, \ell) \right) \right\|_{L^p_\mathbb{R} \left[ |x_1|<1, |x_2|<1, |x_3|<\frac{4}{5}D_3 \right]}.
\]

By (9.7) one has

\[
D_3 = \left( \frac{NR^2}{32Q^3} \right)^{\frac{1}{2}} = \frac{1}{\sqrt{K_0L_0}} \eta \quad (\text{say}),
\]

where, by (8.7),

\[
\eta = \left( \frac{32Q^3}{NR^2} \right)^{\frac{1}{2}} K_0^{-\frac{1}{2}} L_0^{-1} = \frac{(Q/R)^2}{(KK_0/32)^{\frac{1}{2}} L_0}.
\]

By (8.7) (again) it follows that

\[
\eta \asymp \frac{(Q/R)^2}{K_0L_0} \quad (11.4)
\]

and, in particular, that one has:

\[
\eta \geq \frac{1}{K_0L_0} \quad (Q \geq R), \quad (11.5)
\]

\[
\eta < \frac{1}{K_0} \quad (Q \leq B_2H \leq H/64, \text{ say}) \quad (11.6)
\]

and

\[
\frac{1}{K_0} \leq \frac{1}{8L_0}. \quad (11.7)
\]

Moreover, by (8.7), (7.3), (7.5) and (7.1), one has also:

\[
(K_0/L_0)^2 \eta = 2^{\frac{1}{2}} (Q/R)^2 K^{-\frac{1}{2}} (K_0/L_0)^{\frac{1}{2}} L_0^{-\frac{3}{2}} \leq 2^{\frac{1}{2}} (Q/R)^2 K^{-\frac{1}{2}} (12K/L)^{\frac{1}{2}} (8/L)^{\frac{3}{2}} \leq 3072\sqrt{3}NR^2/H^3 < 6144\sqrt{3}C_3(M/H)^3 T^{-1} < 1. \quad (11.8)
\]
Assuming that we have
\[ R \leq Q \leq B_2 H \leq H/64 , \tag{11.9} \]
it follows by (11.1) and (11.5)-(11.8) that an application of Proposition 10 yields the upper bound
\[
\left\| \sum_{k \sim K_0, \ell \sim L_0} a_{k,\ell} e(x_1 \ell + x_2 k \ell + x_3 \omega(k, \ell)) \right\|_{L^p_{\mu} \left[ |x_1| < 1, |x_2| < 1, \left| x_3 \right| < \sqrt{\nu K_0 L_0} \right]} 
\ll_{\nu, \varepsilon} \left( 1 + \frac{\eta K_0^3}{L_0} \right)^{\frac{3(\nu - 1)}{4 - \nu}} \left( 1 + \frac{L_0^3}{K_0^2} \right)^{\frac{1}{13 - \nu}} \left( 1 + \frac{L_0^{2\nu - 3}}{K_0^{\nu - 2}} + \frac{(\eta K_0 L_0) L_0^{2\nu - 6}}{K_0^{\nu - 2}} \right)^{\frac{1}{13 - \nu}} K_0^{\varepsilon + \frac{2}{3} L_0^2} \tag{11.10}
\]
for all \( \varepsilon > 0 \) and all pairs \((\nu, q_\nu) \in \mathbb{R}^2 \) such that
\[
\nu \in \mathbb{Z} , \quad \nu \geq 3 \quad \text{and} \quad q_\nu = \frac{2(13\nu - 12)}{6\nu - 5} . \tag{11.11}
\]

We assume henceforth that \( \nu \) and \( q_\nu \) are as in (11.11), and that one also has both
\[ \nu \geq 6 \tag{11.12} \]
and
\[ H \ll N^{\frac{\nu - 2}{2 - \nu}} R^{\frac{\nu - 4}{2 - \nu}} . \tag{11.13} \]

Note that, by our assumptions in (7.5), the right-hand side of (11.13) is a monotonic decreasing function of \( \nu \), and so (11.12) and (11.13) certainly imply that one has \( H \ll N^{2/3} R^{1/3} \). By (11.4), (8.7) and (11.9) we have also
\[
1 + \frac{\eta K_0^3}{L_0} \approx 1 + \frac{(Q/R)^2 K}{L^2} = 1 + \frac{N Q}{H^2} , \\
1 + \frac{L_0^3}{K_0^2} \approx 1 + \frac{L^3}{K^2} = 1 + \frac{H^3 Q}{N^2 R^2}
\]
and, since \( H^3 \ll N^2 R \),
\[
1 + \frac{L_0^{2\nu - 3}}{K_0^{\nu - 2}} + \frac{(\eta K_0 L_0) L_0^{2\nu - 6}}{K_0^{\nu - 2}} \ll 1 + \left( \frac{(Q/R)^2 L^{2\nu - 6}}{K^{\nu - 2}} \right) \left( 1 + \frac{(R/Q)^2 L^3}{K^2} \right) \leq 1 + \left( \frac{H^{2\nu - 6} Q^{\nu - 2}}{R^{2\nu - 6} N^{\nu - 2}} \right) \left( 1 + \frac{H^3}{N^2 R} \right) \ll 1 + \frac{H^{2\nu - 6} Q^{\nu - 2}}{R^{2\nu - 6} N^{\nu - 2}} .
\]

Therefore, subject to the assumptions made, the bound (11.10) implies that one has, for \( \varepsilon > 0 \),
\[
\left\| \sum_{k \sim K_0, \ell \sim L_0} a_{k,\ell} e(x_1 \ell + x_2 k \ell + x_3 \omega(k, \ell)) \right\|_{L^p_{\mu} \left[ |x_1| < 1, |x_2| < 1, \left| x_3 \right| < \sqrt{\nu K_0 L_0} \right]} 
\ll_{\nu, \varepsilon} \left( 1 + \frac{N Q}{H^2} \right)^{\frac{3(\nu - 1)}{12 - \nu}} \left( 1 + \frac{H^3 Q}{N^2 R^2} \right)^{\frac{1}{13 - \nu}} \left( 1 + \frac{H^{2\nu - 6} Q^{\nu - 2}}{R^{2\nu - 6} N^{\nu - 2}} \right)^{\frac{1}{13 - \nu}} K_0^{\varepsilon + \frac{2}{3} L_0^2} . \tag{11.14}
\]

In order to describe our use of (11.14) (and of the different bound (11.23), below) it is helpful to distinguish between certain cases. We shall find it convenient to consider two main cases:
Case I, in which (11.9) holds and one has
\[ 24C_3H > \sqrt{NR} ; \]  
(11.15)

Case II, in which (11.9) holds and (11.15) does not hold.

We shall also find it useful to split the latter case up into two (more specialized) cases:

Case II(i), in which one has
\[ 1 \leq \frac{NR}{(24C_3H)^2} \ll \frac{Q}{R} \ll \frac{B_2H}{R} ; \]  
(11.16)

Case II(ii), in which one has
\[ 1 \leq \frac{Q}{R} \leq \min \left\{ \frac{NR}{(24C_3H)^2} , \frac{B_2H}{R} \right\} . \]  
(11.17)

Note that Case I, Case II(i) and Case II(ii) are mutually exclusive cases. Note also that if (11.9) holds then one of the above three conditions (i.e. one of (11.15), (11.16), or (11.17)) will be satisfied. Therefore we may complete the work of this section by obtaining, in each one of the cases (i.e. in Case I, in Case II(i) and in Case II(ii)), a sufficiently strong upper bound for some \( A_p(Y_1, D) \) with \( p > 2 \) (in fact we shall always have \( p \geq q_6 = 132/31 > 4 \), since it is in that range that the bounds on \( A_p(Y_1, D) \) are optimal, for our purposes).

In Case I we note that the conditions (11.9), (11.15) and assumptions (7.5), (11.12) and (11.13) imply:
\[ 1 + \frac{NQ}{H^2} \ll 1 + \frac{Q}{R} \ll \frac{Q}{R} , \quad 1 + \frac{H^3Q}{N^2R^2} \ll 1 + \frac{Q}{R} \quad \text{and} \quad 1 + \frac{H^{2\nu - 6}Q^{\nu - 2}}{R^{2\nu - 6}N^{\nu - 2}} \ll 1 + \frac{Q^{\nu - 2}}{R^{\nu - 2}} \ll \nu \left( \frac{Q}{R} \right)^{\nu - 2} . \]

Therefore we find that, by (11.2), (11.3), (11.14) and (11.11), one has
\[ (A_{q,v}(Y_1, D))^{1/q_v} \ll_{\nu, \varepsilon} \left( \frac{Q}{R} \right)^{\frac{8\nu - 9}{17(13\nu - 17)}} K_0^{\frac{\nu}{2} + \frac{1}{2} \varepsilon} L_0^{\frac{1}{2} \varepsilon} = \left( \frac{Q}{R} \right)^{1 - 3\varepsilon - \frac{1}{2}} K_0^{\nu + \frac{1}{2} \varepsilon} L_0^{\frac{1}{2} \varepsilon} \quad \text{(in Case I)} . \]  
(11.18)

With regard to Case II(i), we may note that (11.16) implies \( H < N^{1/2}R^{1/2} \), and so (given (7.5)) it follows that (11.13) will hold for all \( \nu \geq 6 \). We also have (in Case II(i)):
\[ R < \frac{NR^2}{H^2} \ll Q < H , \]
and so:
\[ 1 + \frac{NQ}{H^2} < \frac{NR}{H^2} + \frac{NQ}{H^2} \ll \frac{NQ}{H^2} , \quad 1 + \frac{H^3Q}{N^2R^2} < 1 + \left( \frac{H^2}{NR} \right)^2 < 2 \]
and
\[ 1 + \frac{H^{2\nu - 6}Q^{\nu - 2}}{R^{2\nu - 6}N^{\nu - 2}} = 1 + \left( \frac{H^2Q}{R^2N} \right)^{\nu - 2} \left( \frac{R^2}{H^2} \right)^{\nu - 2} \ll_{\nu} \left( \frac{H^2Q}{R^2N} \right)^{\nu - 2} . \]
Therefore, bearing in mind that (11.12) implies $1/(13\nu - 12) < 1/(13\nu - 13)$, we may deduce from (11.14) that in Case II(i) one has

$$
\left\| \sum_{k \sim K_0, \ell \sim L_0} a_{k,\ell} e(x_1\ell + x_2k\ell + x_3\omega(k,\ell)) \right\|_{L^p_{\nu,p} \left[ |x_1| < 1, |x_2| < 1, |x_3| < \frac{1}{\sqrt{K_0 L_0}} \right]} \lesssim_{\nu,\varepsilon} \left( \frac{Q}{H^2} \right)^{\frac{1}{3p}} \left( \frac{NR^2/H^2}{2} \right)^{\frac{1}{p}} K_0^{\frac{1}{2}} L_0^{\frac{1}{2}}
$$

(11.19)

for all integers $\nu \geq 6$.

In order to simplify the application of (11.19) we observe now that, by (11.11) and (11.12),

$$
q_\nu = \left( \frac{13}{3} \right) (1 - \delta_\nu) \quad \text{with} \quad \delta_\nu = \frac{7}{3(6\nu - 5)} \in \left( 0, \frac{7}{403} \right) \subset \left( 0, \frac{1}{13} \right).
$$

Therefore, given (11.1), a trivial bound on the relevant sums over $k$ and $\ell$ is enough to show that one has

$$
\left\| \sum_{k \sim K_0, \ell \sim L_0} a_{k,\ell} e(x_1\ell + x_2k\ell + x_3\omega(k,\ell)) \right\|_{L^p_{\nu,p} \left[ |x_1| < 1, |x_2| < 1, |x_3| < \frac{1}{\sqrt{K_0 L_0}} \right]} \lesssim_{\nu,\varepsilon} (K_0 L_0)^{\delta_\nu} \left( \frac{Q}{NR^2/H^2} \right)^{\frac{1}{3p}} \left( \frac{NR^2/H^2}{2} \right)^{\frac{1}{p}} K_0^{\frac{1}{2} + \frac{\delta_\nu}{2}} L_0^{\frac{1}{2}}.
$$

By this and (11.19) we are able to conclude that, for $\varepsilon > 0$ and all integers $\nu \geq 6$, one has:

$$
\left\| \sum_{k \sim K_0, \ell \sim L_0} a_{k,\ell} e(x_1\ell + x_2k\ell + x_3\omega(k,\ell)) \right\|_{L^p_{\nu,p} \left[ |x_1| < 1, |x_2| < 1, |x_3| < \frac{1}{\sqrt{K_0 L_0}} \right]} \lesssim_{\nu,\varepsilon} (K_0 L_0)^{\delta_\nu} \left( \frac{Q}{NR^2/H^2} \right)^{\frac{1}{3p}} \left( \frac{NR^2/H^2}{2} \right)^{\frac{1}{p}} K_0^{\frac{1}{2} + \frac{\delta_\nu}{2}} L_0^{\frac{1}{2}}.
$$

(11.20)

We now choose to put $\nu = 6 + \lfloor 1/\varepsilon \rfloor$ (where $\lfloor x \rfloor = \max\{j \in \mathbb{Z} : j \leq x\}$). This, given (8.7), (7.5) and the definition of $\delta_\nu$, is easily enough to ensure that one has

$$(K_0 L_0)^{\delta_\nu} \ll K_0^{\delta_\nu} \leq K_0^{\delta}.$$

Therefore it follows by (11.2), (11.3) and the case $\nu = 6 + \lfloor 1/\varepsilon \rfloor$ of (11.20) that we have, for $\varepsilon > 0$,

$$
\left( A^{\frac{1}{2}}(Y_1,D) \right)^{\frac{1}{3p}} \ll_{\varepsilon} \left( \frac{Q}{NR^2/H^2} \right)^{\frac{1}{3p}} \left( \frac{NR^2/H^2}{2} \right)^{\frac{1}{p}} K_0^{\frac{1}{2} + \frac{\delta_\nu}{2}} L_0^{\frac{1}{2}} \quad \text{(in Case II(i)).}
$$

(11.21)

In Case II(ii) it follows by (8.7) and (11.17) that, in addition to (11.1) and (11.5)-(11.8), one has

$$
\frac{L_0^3}{K_0} \leq \frac{(24C_3L_0)^2}{K} = \frac{(24C_3)^2 H^2 Q}{R^2 N} \leq 1,
$$

(11.22)
and so it follows from Proposition 10' that we have, for $\varepsilon > 0$,

$$
\left\| \sum_{k \sim K_0, \ell \sim L_0} a_{k, \ell} e(x_1 \ell + x_2 k \ell + x_3 \omega(k, \ell)) \right\|_{L^4_{\xi}} \lesssim \varepsilon \left( 1 + \eta K_0 L_0 \right)^{\frac{1}{8}} \left( 1 + \frac{\eta K_0^2}{L_0^2} \right) K_0^{\varepsilon + \frac{1}{2}} L_0^{\frac{3}{2}}.
$$

By (11.4), (11.17), (11.22) and (8.7), we have here

$$
1 + \eta K_0 L_0 \approx 1 + \frac{Q^2}{R^2} \quad \text{and} \quad 1 + \frac{\eta K_0^2}{L_0^2} \approx 1 + \frac{Q^2 K_0}{R^2 L_0^2} \lesssim \frac{Q^2 K_0}{R^2 L_0^2} = \frac{NQ}{H^2}.
$$

It therefore follows from (11.2), (11.3) and (11.23) that, for $\varepsilon > 0$, one has:

$$
\left( A_{\pm} \right) (\mathcal{Y}, D) \lesssim \varepsilon \left( \frac{Q}{R} \right)^{\frac{1}{3}} \left( \frac{N Q}{H^2} \right)^{\frac{1}{3}} K_0^{\varepsilon + \frac{1}{2}} L_0^{\frac{3}{2}} \quad \text{(in Case II(i)).}
$$

12. Results from the application of the Bombieri-Iwaniec method

Let $N$ be given either by (10.14), or else by (10.15). We seek a bound for the sum, over $\mathcal{C}$ and $Q$, one the right-hand side of (7.6). The bound (10.5) implies that one has

$$
\mathcal{G}_Q^{[A]} \neq \emptyset \quad \text{only if} \quad A \ll \frac{MR^2 \log N}{NQ^2}.
$$

Therefore the sum on the right-hand side of (7.6) has (given (7.5)) no more than $O((\log M)^2)$ terms. For this reason it will be enough that we obtain bounds for $S(C_Q)$ that are uniform, in the sense of being independent of the indices of summation, $\mathcal{C}$ and $Q$.

Suppose now that $S(C_Q)$ is one of the terms occurring in the sum on the right-hand side of (7.6). It follows that we have $R \leq Q \leq \varepsilon H/2$ and either $\mathcal{C} = B$ (the set of ‘bad’ intervals $I(a/q)$), or else $\mathcal{C} = \mathcal{G}^{[A]}$ for some $A \geq \log N$. In the latter case Huxley’s bounds (10.5) and (10.16)-(10.17) imply:

$$
\left( B(\mathcal{G}_Q^{[A]} ; V) \right)^{\frac{1}{2}} \left( |\mathcal{G}_Q^{[A]} | R/Q \right)^{1 - \frac{1}{p}} \lesssim \left( 1 + \frac{Q}{Q_2} \right)^{\frac{1}{p}} \left( \frac{M^2 R^4 (\log N)^{2/3} A M Q}{H^2 N^4 V_0} \right)^{\frac{1}{p}} \left( \frac{M R^3 \log N}{N Q^3} \right)^{1 - \frac{1}{p}},
$$

when $p > 2$. The right-hand side here is a decreasing function of $A$ for each fixed choice of $p$ satisfying $p > 2 + \frac{11}{90}$. Therefore, if $p > 4$ (say), then we will certainly have

$$
\left( B(\mathcal{G}_Q^{[A]} ; V) \right)^{\frac{1}{2}} \left( |\mathcal{G}_Q^{[A]} | R/Q \right)^{1 - \frac{1}{p}} \lesssim \left( 1 + \frac{Q}{Q_2} \right)^{\frac{1}{p}} \left( \frac{M^2 R^4 (\log N)^{2/3} A M Q}{H^2 N^4 V_0} \right)^{\frac{1}{p}} \left( \frac{M R^3 \log N}{N Q^3} \right)^{1 - \frac{1}{p}} = \left( 1 + \frac{Q}{Q_2} \right)^{\frac{1}{p}} \left( \frac{R^4 (\log N)^{2/3} A M Q}{H^2 N^4 V_0^2} \right)^{\frac{1}{p}} \left( \frac{R^4}{Q} \right)^{3 - \frac{1}{p}} M N^{-1}.
$$

If we have instead $\mathcal{C} = B$, then we choose to observe that (10.4), (10.8) and (10.9) imply the bound

$$
|B_Q| \lesssim \frac{\eta M R^2}{N Q^2} \left( 1 + \frac{Q}{Q^*} \right),
$$
where
\[ Q^* = \frac{\eta R^2}{Q'} \gg \left( \frac{\Delta' R}{\Delta_2 H} \right)^{-\frac{3}{4}} R \gg \left( \frac{H}{R} \right)^{\frac{1}{140}} R = Q_4 \text{ (say)}. \] (12.2)

By this and (10.21)-(10.22), we obtain:
\[
(B(B_Q;V)V)^{\frac{1}{p}} \left( |B_Q|R/Q \right)^{1-\frac{2}{p}}
\ll \left( 1 + \frac{Q}{Q_3} \right) \left( \frac{M^2 R^4 (\log N)^{\frac{27}{140}}}{H^2 N^4(H/R)^{\frac{2}{p}}} \right)^{\frac{1}{p}} \left( 1 + \frac{Q}{Q_4} \right)^{1-\frac{2}{p}} \left( \frac{\eta M R^3}{N Q_4} \right)^{1-\frac{2}{p}}
= \left( 1 + \frac{Q}{Q_4} \right)^{1-\frac{2}{p}} \left( 1 + \frac{Q}{Q_3} \right)^{\frac{1}{p}} \left( \frac{\eta^p - 2 R^4 (\log N)^{\frac{27}{140}}}{H^2 N^2(H/R)^{\frac{2}{p}}} \right)^{\frac{1}{p}} \left( \frac{R}{Q} \right)^{3-\frac{2}{p}} M N^{-1}. \] (12.3)

By (10.12), (10.19), (12.2) and (7.5), we have
\[ Q_2 \ll Q_3 < Q_4. \]

Therefore, subject to the condition that \( p \) and \( \eta \) satisfy
\[ \frac{\eta^{p-2}}{(H/R)^{\frac{2}{p}}} \ll \frac{1}{V_0^2}, \] (12.4)
we may conclude from (12.1) and (12.3) that one has
\[
(B(C_Q;V)V)^{\frac{1}{p}} \left( \frac{|C_Q|R}{Q} \right)^{1-\frac{2}{p}} \ll \left( 1 + \frac{Q}{Q_2} \right)^{\frac{1}{p}} \left( \frac{R^4 (\log N)^{\frac{27}{140}}}{H^2 N^2 V_0^{\frac{2}{p}}} \right)^{1-\frac{2}{p}} \left( \frac{R}{Q} \right)^{3-\frac{2}{p}} \left( \frac{M N}{N} \right) \text{ if } Q \leq Q_4. \] (12.5)

Assuming that \( p > 4 \), it follows from (10.7) that the condition (12.4) will be satisfied if and only if one has \( \eta \ll (H/R)^{-2/(51(p-2))} \). Therefore, given (7.5), we certainly obtain (12.5) subject to the conditions that
\[ 4 < p < \infty \text{ and } \eta \ll \left( \frac{H}{R} \right)^{-\frac{2}{51}}. \] (12.6)

If however, we have \( Q \geq Q_4 \) and \( C = B \), then we choose not to use the bound (10.4) for \( |B_Q| \), and instead simply recall (7.9). Since \( Q_4 > Q_3 \), this single change enables us to replace (12.3) with the alternative bound:
\[
(B(B_Q;V)V)^{\frac{1}{p}} \left( |B_Q|R/Q \right)^{1-\frac{2}{p}} \ll \left( \frac{Q}{Q_3} \right)^{\frac{1}{p}} \left( \frac{R^4 (\log N)^{\frac{27}{140}}}{H^2 N^2(H/R)^{\frac{2}{p}}} \right)^{\frac{1}{p}} \left( \frac{R}{Q} \right)^{3-\frac{2}{p}} M N^{-1},
\]
when \( Q \geq Q_4 \). This last upper bound exceeds that on the right-hand side of (12.5) by a factor \( \Phi^{1/p} \) (say), where, by (10.7), (10.12) and (10.19), \( \Phi \) satisfies
\[ \Phi < \left( \frac{Q_2}{Q_3} \right) \left( \frac{V_0}{H/R} \right)^{\frac{1}{p}} \left( \frac{H}{R} \right)^{\frac{1}{140}} \left( \log \frac{H}{R} \right)^{\frac{1}{p}}. \] (12.7)

By this, (12.1) and the trivial conditional inequality
\[ 1 \leq (Q/Q_4)^{\theta} \quad (Q \geq Q_4 \text{ and } \theta \geq 0), \]
we may conclude that if one chooses to put
\[ \theta_p = \begin{cases} 
  p - 4 & \text{if } 4 < p < 13/3, \\
  1/2 & \text{if } p = 13/3, \\
  7/11 & \text{if } 13/3 < p < \infty
\end{cases} \] (12.8)

(for example), then the condition \( Q \leq Q_4 \) can be omitted from (12.5) if the bound appearing there is weakened through multiplication by \( (1 + (Q/Q_4)^{\theta_p} \Phi)^{1/p} \). That is, we have (given (12.7)):
\[
(B(C_Q; V) V)^{\frac{\theta}{p}} (|C_Q R/Q|^{1 - \frac{\theta}{p}} \leq \left( 1 + (\frac{Q}{Q_4})^{\theta_p} \left( \frac{H}{R} \right)^{\frac{\theta}{p}} \right)^{\theta_p} (1 + \frac{Q}{Q_2})^{\frac{\theta}{p}} \left( \frac{R}{Q} \right)^{3 - \frac{\theta}{p}} \left( \frac{MR}{N^{1 - \frac{\theta}{p}}} \right) (A_p(Y_1, D) \frac{\theta}{p} ,
\]
subject to (12.6) and (12.8) holding.

By the bound just obtained, in combination with (9.8), (10.1), (7.5) and (10.7), we find that
\[
S(C_Q) \leq \left( 1 + \frac{Q}{Q_4} \left( \frac{H}{R} \right)^{\theta_p} \right)^{\frac{\theta}{p}} (1 + \frac{Q}{Q_2})^{\frac{\theta}{p}} \left( \frac{R}{Q} \right)^{3 - \frac{\theta}{p}} \left( \frac{MR}{N^{1 - \frac{\theta}{p}}} \right) (A_p(Y_1, D) \frac{\theta}{p} ,
\]
subject to (12.6), (12.8) and the condition \( \varepsilon > 0 \). We shall bound the factor \( (A_p(Y_1, D))^{1/p} \) in (12.9) through an appeal to the results of the previous section. Our choice of \( p \) therefore depends, in each individual case, on which one of the results (11.18), (11.21), (11.24) is applied, and will (in all cases) satisfy
\[
p \in \{ q_\nu : \nu \in \mathbb{Z} \text{ and } \nu \geq 6 \} \cup \left\{ \frac{13}{3}, \frac{48}{11} \right\} \subset \mathbb{Q} \cap \left( \frac{132}{31}, \frac{48}{11} \right) \subset (4.258, 4.36) . \tag{12.10}
\]

It is helpful to note that, by (8.7) and (7.5), we have
\[
K_0^{-\frac{\theta}{p}} L_0^{\frac{\theta}{p}} \leq (Q/N)^{\frac{\theta}{p}} H^{\frac{\theta}{p}} \leq \frac{QH^{\frac{\theta}{p}}N^{\frac{\theta}{p}}}{R^2} \leq \frac{QH^{\frac{\theta}{p}}N^{\frac{\theta}{p}}}{R^2} (R \leq Q \leq B_2 H)
\]
in (11.18), (11.21) and (11.24). Using this we deduce from (11.18), (11.21) and (11.24) three corresponding upper bounds for \((A_p(Y_1, D))^{1/p} Q^{(\theta_p+7)/(\theta_p-3)}\) that, by virtue of (12.8), are each independent of \( Q \). By these bounds, combined with (12.9), we obtain upper bounds for \( S(C_Q) \) that are, in each case, monotonic decreasing functions of \( Q \). In particular, by (12.9), (11.18) and the lower bound on \( Q \) in (11.9), we obtain (for \( \varepsilon > 0 \)):
\[
S(C_Q) \leq \varepsilon, \nu \left( \frac{H}{R} \right)^{\frac{\theta}{p}} \left( \frac{MR}{N^{1 - \frac{\theta}{p}}} \right) \left( \frac{H^2 N^{\frac{\theta}{p}}}{R^2} \right) = \left( \frac{H}{R} \right)^{\frac{\theta}{p}} \left( \frac{MR}{N^{1 - \frac{\theta}{p}}} \right) MN^\varepsilon \quad \text{(in Case I)}, \tag{12.11}
\]
subject to the final part of (12.6) holding (and with \( \nu, q_\nu \) and ‘Case I’ being as described in Section 11). Note that the expression on the right-hand side of (12.11) involves fewer factors than that in (12.9). This is owing to the fact that, by (10.12), (11.11), (11.12), (12.2), (12.8) and (7.5), one has both \( R/Q_2 < 1 \) and
\[
\left( \frac{R}{Q_1} \right)^{\theta_p} \left( \frac{H}{R} \right)^{\frac{\theta}{p}} \leq \left( \frac{H}{R} \right)^{\frac{\theta}{p}} \left( \frac{R}{Q_1} \right)^{\theta_p} \leq \left( \frac{H}{R} \right)^{\frac{\theta}{p}} \left( \frac{R}{Q_1} \right)^{\theta_p} < 1 ,
\]
when $p = q_r$. Since $8/31 < 1/2 < 7/11$, it follows from (12.8) that one has an even stronger bound for

$(R/Q_4)^{6r}(H/R)^{4/119}$ when $p \in \{13/3, 48/11\}$.

We postpone discussion of Case II(i) until after dealing with Case II(ii). In that latter case we may note that (11.17) implies $Q \geq R$. Therefore, reasoning similar to that which produced (12.11) enables it to be deduced from (12.9) and (11.24) that one has, for $\varepsilon > 0$,

$$S(C_Q) \ll_{\varepsilon} \left( \frac{N R}{H^2} \right)^{\frac{5}{117}} \left( \frac{H}{R} \right)^{\left( \frac{22}{117} \right)} \left( \frac{H}{N} \right)^{\frac{1}{7}} M N^\varepsilon = \left( \frac{H}{N} \right)^{\frac{3}{117}} \left( \frac{H}{N} \right)^{\frac{1}{7}} M N^\varepsilon \quad \text{(in Case II(ii))},$$

subject to the final part of (12.6) holding.

In Case II(i) of Section 11 we apply (12.9) in combination with (11.21). The conditions (11.17) defining Case II(i) certainly imply $Q > R$, and so, for reasons given in the paragraph containing (12.11), we find that

$$S(C_Q) \ll_{\varepsilon} \left( \frac{H}{R} \right)^{\left( \frac{22}{117} \right)} \left( \frac{M R}{N^{1-\varepsilon}} \right) \left( \frac{R}{N R^2/H^2} \right)^{\frac{5}{117}} \left( \frac{N R}{H^2} \right)^{\frac{1}{7}} \left( \frac{H}{R} \right)^{\frac{\varepsilon}{2}} \left( \frac{H \frac{2}{N} N^{\frac{4}{117} - \varepsilon}}{R} \right)$$

$$= \left( \frac{N R}{H^2} \right)^{\frac{5}{117}} \left( \frac{H}{R} \right)^{\frac{22}{117}} \left( \frac{H}{N} \right)^{\frac{1}{7}} M N^\varepsilon$$

$$= \left( \frac{H}{R} \right)^{\frac{22}{117}} \left( \frac{H}{N} \right)^{\frac{1}{7}} M N^\varepsilon,$$

in Case II(i).

We observe that this last upper bound for $S(C_Q)$ will exceed the corresponding Case II(ii) upper bound (given in (12.12)) only when one has $(H/R)^{1/221}(H/N)^{5/312} > 1$, and so only when

$$\frac{N}{H} < \left( \frac{H}{R} \right)^{\frac{22}{117}} \quad \text{(12.13)}.$$

Subject to (12.13) holding, one has, by (10.12),

$$\frac{N R^2/H^2}{Q_2} = \left( \frac{N}{H} \right) \left( \frac{R}{H} \right)^{\frac{22}{117}} \left( \log \frac{H}{R} \right)^{\frac{1}{7}} < \left( \frac{H}{R} \right)^{-\frac{22}{117}} \left( \log \frac{H}{R} \right)^{\frac{1}{7}} \ll 1$$

and, by (12.2) and (12.8),

$$\left( \frac{N R^2/H^2}{Q_4} \right)^{\frac{1}{3}} \left( \frac{H}{R} \right)^{\frac{22}{117}} = \left( \frac{N}{H} \right) \left( \frac{R}{H} \right)^{\frac{22}{117}} \left( \frac{H}{R} \right)^{\frac{1}{7}} < \left( \frac{H}{R} \right)^{-\frac{22}{117}} \left( \frac{H}{R} \right)^{-\frac{1}{7}} = \left( \frac{H}{R} \right)^{-\frac{22}{117}} < 1.$$

It therefore follows from (12.9), (11.21) and the lower bound estimate $Q \gg N R^2/H^2$ (implied by (11.16)) that, when the conditions for Case II(i) are satisfied and (12.13) holds, one will have (for $\varepsilon > 0$):

$$S(C_Q) \ll_{\varepsilon} \left( \frac{R}{N R^2/H^2} \right)^{2-\frac{16}{117}} \left( \frac{H}{R} \right)^{\frac{22}{117}} \left( \frac{M R}{N^{1-\varepsilon}} \right) \left( \frac{N R}{H^2} \right)^{\frac{1}{7}} \left( \frac{H \frac{2}{N} N^{\frac{4}{117} - \varepsilon}}{R} \right)$$

$$= \left( \frac{H^2}{N R} \right)^{\frac{22}{117}} \left( \frac{H}{N} \right)^{\frac{1}{7}} M N^\varepsilon$$

$$= \left( \frac{H}{R} \right)^{\frac{22}{117}} \left( \frac{H}{N} \right)^{\frac{1}{7}} M N^\varepsilon,$$
subject to the final condition of (12.6) being satisfied.

By this last finding, allied with both (12.12) and the observation made at the start of the preceding paragraph, one has (for $\varepsilon > 0$):

$$S(C_Q) \ll \varepsilon \left( \left( \frac{H}{R} \right)^{\frac{5}{32}} \left( \frac{H}{N} \right)^{\frac{7}{32}} + \left( \frac{H}{R} \right)^{\frac{109}{220}} \left( \frac{H}{N} \right)^{\frac{17}{220}} \right) MN^\varepsilon \quad \text{(in Case II),}$$

subject to the final condition of (12.6) being satisfied.

Given the definitions (in Section 11) of Cases I and II, and bearing in mind the point noted in the first paragraph of this section, it follows from (7.5), (7.6), (12.11) and (12.14) that we have, for each $\varepsilon > 0$, a bound of the form

$$|S| \leq \Psi_{\varepsilon,\nu}(H/R, H/N)M^{1+\varepsilon},$$

where

$$\Psi_{\varepsilon,\nu}(\Delta, \delta) = \begin{cases} C_{II}(\varepsilon) \left( \Delta^{\frac{5}{32}} \delta^{\frac{7}{32}} + \Delta^{\frac{109}{220}} \delta^{\frac{17}{220}} \right) & \text{if } 576C_3^2 \Delta \delta \leq 1, \\
C(\varepsilon, \nu)\Delta^{\frac{5}{22}} \delta^{-1} & \text{otherwise,} \end{cases}$$

with $C_{II}(\varepsilon)$ denoting a positive constant constructed from $C_2, \ldots, C_5$ and $\varepsilon$, while $C(\varepsilon, \nu)$ denotes a positive constant constructed from $C_2, \ldots, C_6, \varepsilon$ and $\nu$. This, of course, assumes that the values of $T$, $H$ and $M$, and that of our chosen of parameter $N$ (an integer satisfying either (10.14), or else (10.15)), are consistent with being able to satisfy all of the conditions (7.3), (7.4), (7.5), (10.3), (10.6), (10.8)-(10.11), (11.12), (11.13) and (12.6), as well as the condition attached to the relevant case, ‘(A)’ or ‘(B)’, of Lemma 10.1. We devote the remainder of the section to obtaining a resolution of this issue that will complete our proof of Theorem 2. Therefore we shall no longer be assuming that all of the conditions just mentioned are satisfied (for our goal, in what follows, is a concise description of the circumstances in which certain, quite specific, choices of $N \in \mathbb{N}$ and $R \in \mathbb{N}$ will satisfy all those conditions). We shall, however, find it convenient to assume that the positive constants $C(I, \nu)$, $C_{II}(\varepsilon)$ that occur in (12.16) satisfy

$$C(\varepsilon, \nu)/C_{II}(\varepsilon) \geq 24C_3.$$

This causes no loss of generality, for we are effectively able to ensure that (12.17) will hold by substituting $\max\{C(\varepsilon, \nu), 24C_3C_{II}(\varepsilon)\}$ for $C(\varepsilon, \nu)$ in (12.16).

As a first step, we specify the parameters $\eta$, $Q'$ and $\Delta'$ (occurring in (10.3), (10.8)-(10.11) and (12.6)) by putting:

$$\eta = \left( \frac{R}{H} \right)^{\frac{1}{2} + \frac{1}{32}}, \quad \Delta' = \left( \frac{R}{H} \right)^{\frac{1}{2}} \Delta_2 \quad \text{and} \quad Q' = \left( \frac{\Delta'R}{\Delta_2H} \right)^{\frac{1}{2}} R_\eta = \left( \frac{R}{H} \right)^{\frac{1}{2}} \Delta_2 \quad \text{and} \quad Q' = \left( \frac{R}{H} \right)^{\frac{1}{2}} R_\eta = \left( \frac{R}{H} \right)^{\frac{1}{2}} R .$$

Assuming that we have $0 < R/H < 1$ (as (7.5) would imply), these specifications, along with that of $V_0$ in (10.7), can be shown to ensure that the conditions (10.8)-(10.11) and the condition on $\eta$ in (12.6) are satisfied, and that $Q'$ and $\eta$ satisfy $Q' < \eta R < R$ (as stated in (10.3)). We are therefore able to reduce the set of conditions on $\eta$, $Q'$ and $\Delta'$ to the combination of (12.18) and the single condition $Q' \geq 1 + M^2/T$ (seen in (10.3)); given (12.18), this single condition (on $Q'$) will be satisfied if one has

$$R \left( \frac{R}{H} \right)^{\frac{59}{350}} \geq 1 + \frac{M^2}{T} .$$
Lemma 12.1. Let $E_1, E_2 \in [1/16, 16]$. Let $\tilde{N}, \tilde{R} \in (0, \infty)$ and $\nu \in [6, \infty)$ be such that one has both
\[
E_1 \tilde{N} \tilde{R}^2 = \frac{C_3 M^3}{2T} \tag{12.20}
\]
and
\[
E_2 B_1 \tilde{N}^{\frac{\nu-2}{\nu}} \tilde{R}^{\frac{\nu-4}{\nu}} \geq H \tag{12.21}
\]
Let $\tilde{V}_0, \tilde{V}_1, \tilde{V}_2 \in (0, \infty)$ be determined by the constraint that the equalities in (10.7) should hold if $\tilde{V}_0, \tilde{V}_1, \tilde{V}_2, \tilde{N}$ and $\tilde{R}$ are substituted for $V_0, V_1, V_2, N$ and $R$, respectively. Suppose that one has either
\[
E_2 \min\{\tilde{V}_1, \tilde{V}_2\} \geq \tilde{V}_0 \tag{12.22}
\]
or
\[
M \leq C_6 T \quad \text{and} \quad \tilde{N}^3 \leq \frac{E_2 B_3^3 M^6}{T^2 H}. \tag{12.23}
\]
Then the conditions (7.4), (7.5), (10.6) and (12.19) will hold if $\tilde{V}_0, \tilde{V}_1, \tilde{V}_2, \tilde{N}$ and $\tilde{R}$ are substituted for $V_0, V_1, V_2, N$ and $R$, respectively.

Proof. Let (7.4)', (7.5)', (10.6)' and (12.19)' denote the conditions that (7.4), (7.5), (10.6) and (12.19) (respectively) become when $\tilde{N}, \tilde{R}, \tilde{V}_0, \tilde{V}_1$ and $\tilde{V}_2$ are substituted for $N, R, V_0, V_1$ and $V_2$ (respectively). We are required to show that it follows from the hypotheses of the lemma that the conditions (7.4)', (7.5)', (10.6)' and (12.19)' are satisfied.

Since $E_1^{-1} \leq 16$ and $E_2 \leq 16$, it follows from (7.1), (12.20) and (12.21) that we have
\[
\tilde{N} \tilde{R}^2 \leq 8C_3 M^3 T^{-1} < 2^{-12} H^3 \leq B_1^3 \tilde{N}^{\frac{3\nu-6}{\nu}} \tilde{R}^{\frac{3\nu-12}{\nu}} \tag{12.24}
\]
and so
\[
\left(\frac{\tilde{R}}{N}\right)^2 < B_1^3 \left(\frac{\tilde{R}}{N}\right)^{\frac{3\nu-12}{\nu}} \quad \text{and} \quad \frac{H}{N} \leq 16B_1 \left(\frac{\tilde{R}}{N}\right)^{\frac{\nu-4}{\nu}}. \tag{12.25}
\]
Since $\nu \geq 6$, and since we may assume here that $0 < B_1 < 1/16$, the last two inequalities above imply that one has both
\[
\frac{\tilde{R}}{N} < B_1^{4(\nu-3)} \leq B_1^3 \leq 2^{-12} \quad \text{and} \quad \frac{H}{N} \leq 16B_1^{\frac{4(\nu-3)}{\nu}} \leq 16B_1^3 \leq B_1. \tag{12.25}
\]
Note that the first two inequalities of (12.24) imply that one has $H^3 > 2^{12} \tilde{N} \tilde{R}^2$, and so $H/\tilde{N} > 2^{12} \tilde{R}^2/H^2$. By this and the final two inequalities of (12.25), one can deduce that
\[
\frac{\tilde{R}}{H} < \frac{B_1}{16} \leq 2^{-8}. \tag{12.26}
\]

Next we observe that (12.22) and (12.26) would imply:
\[
\min\{\tilde{V}_1, \tilde{V}_2\} \geq E_2^{-1} \tilde{V}_0 = E_2^{-1} \left(\frac{H}{R}\right)^{1/2} > 16^{-1} \left(\frac{16}{B_1}\right)^{1/2} > B_1^{-1} \geq 16.
\]
If (12.22) does not hold, then (by hypothesis) we have instead the inequalities in (12.23), and can combine these with (12.20) so as to obtain:
\[
\tilde{V}_1 = \frac{\tilde{R}^4}{HN} = \frac{(E_1 \tilde{N} \tilde{R}^2)^2}{E_1^2 H N^3} \geq \frac{(E_1 \tilde{N} \tilde{R}^2)^2}{E_1^2 E_2 B_3 M^6 T^{-2}} = \frac{C_3^2}{4E_1^2 E_2 B_3} \geq 2^{-14} B_3^{-3}
\]
and
\[ \tilde{V}_2 = \frac{M^2}{HN^3} \geq \frac{T^2}{E_2B_3^4M^4} \geq \frac{1}{16C_6^4B_3^4}. \]

Since we may assume (for example) that \( 0 < B_3 \leq (32C_6)^{-4/3} < 2^{-20/3} \), it follows from (12.26) and
the points just noted that, regardless of whether it is (12.22) or (12.23) that holds, we are certain
to have
\[ \tilde{V}_j \geq 16 \quad (j = 0, 1, 2). \] (12.27)

The inequality \( \tilde{V}_1 \geq 16 \) implies \( H\tilde{N} \leq \tilde{R}^4/16 \). This, together with (12.25) gives:
\[ \frac{H}{R^2} \leq \frac{4B_1(H\tilde{N})^{\frac{2}{3}}}{R^2} \leq B_1. \] (12.28)

The inequality \( \tilde{V}_2 \geq 16 \) implies \( H\tilde{N}^3 \leq M^2/16 \), and so \( H\tilde{N}^2 < M^2/\tilde{N} \). This, together with (12.25),
(12.26), (12.28) and the hypothesis \( H \geq 1 \), gives \( \tilde{N}^2 \leq H\tilde{N}^2 \leq M^2/\tilde{N} = M^2(\tilde{R}/\tilde{N})(\tilde{R}/H)H/R^2 < M^2B_1^4 \), so that one has
\[ \frac{\tilde{N}}{M} < B_1^{\frac{2}{3}} \leq B_1. \] (12.29)

By (12.25), (12.26), (12.28) and (12.29), we conclude that the condition (7.5)’ is satisfied.

By (12.27), one has
\[ 1 > \frac{1}{(\tilde{V}_1\tilde{V}_2)^{\frac{1}{2}}} = \frac{H\tilde{N}^2}{MR^2}, \]
which implies (7.4)’. With regard to (10.6), we note that (12.27) and (12.28) imply:
\[ \frac{\tilde{V}_0^2}{N(\tilde{V}_1\tilde{V}_2)^{\frac{1}{2}}} < \frac{\tilde{V}_0^2}{N} < \frac{\tilde{V}_0^{\frac{2}{3}}}{N} = \frac{H^{\frac{2}{3}}}{N^{\frac{2}{3}}R} \leq B_1 \left( \frac{H^{\frac{1}{3}}}{N^{\frac{2}{3}}R} \right)^{\frac{1}{2}}. \]

Since it moreover follows from (12.21), the hypothesis \( \nu \geq 6 \) and the first three inequalities of (12.25)
that one has \( H \leq E_2B_1\tilde{N}^{2/3}\tilde{R}^{1/3} \leq 16B_1(\tilde{N}\tilde{R})^{1/3} \), we may deduce that
\[ \frac{\tilde{V}_0^2}{N(\tilde{V}_1\tilde{V}_2)^{\frac{1}{2}}} < B_1 \left( (16B_1)^{\frac{1}{3}} \right)^{\frac{1}{2}} < (16B_1)^{\frac{1}{3}} \leq 1, \]
so that (10.6)’ is satisfied.

Finally, with regard to the condition (12.19)’, we note that (12.20) implies
\[ \left( \frac{\tilde{V}_1}{\tilde{V}_2} \right)^{\frac{1}{2}} = \frac{\tilde{R}^2\tilde{N}}{M} = \frac{C_3M^2}{2E_1\tilde{T}} \geq \frac{M^2}{32\tilde{T}}, \]
so that, by (12.27), one has:
\[ 1 + \frac{M^2}{\tilde{T}} \leq 1 + 32 \left( \frac{\tilde{V}_1}{\tilde{V}_2} \right)^{\frac{1}{2}} \leq 9\tilde{V}_1^{\frac{1}{2}}. \]

This proof may therefore be completed by observing that (12.26) and the first three inequalities of
(12.25) imply that one has
\[ \frac{9\tilde{V}_1^{\frac{1}{2}}}{R(R/H)^{\frac{2m}{4}}} = \frac{9\tilde{R}(H/R)^{\frac{m+1}{4}}}{(HN)^{\frac{1}{2}}} = 9 \left( \frac{\tilde{R}}{R} \right)^{\frac{m}{2}} \left( \frac{\tilde{R}}{H} \right)^{\frac{1}{2}} \left( \frac{\tilde{R}}{N} \right)^{\frac{1}{2}} < 1. \]
Corollary 12.1.1. Suppose that the hypotheses of Lemma 12.1 are satisfied, and that $E_1 = E_2 = 1$. Put

$$N = \lceil \tilde{N} \rceil \quad (12.30)$$

and

$$R = \left\lceil \frac{C_3 M^3}{2NT} \right\rceil ^{\frac{1}{2}} \quad (12.31)$$

(where $\lceil x \rceil = \min\{n \in \mathbb{Z} : n \geq x\}$, the ‘ceiling’ function). Then the conditions (7.3), (7.4), (7.5), (10.6)-(10.7), (11.13) and (12.19) are satisfied, and one has:

$$\tilde{N} \leq N < 2\tilde{N} \quad (12.32)$$

and

$$\frac{\bar{R}}{\sqrt{2}} < R < 2\bar{R} \quad (12.33)$$

Proof. Assume (12.30) and (12.31). Then, by Lemma 12.1,

$$\tilde{N} + 1 > N \geq \tilde{N} \geq B_1^{-1}H \geq B_1^{-1} \geq 1 ,$$

and so we obtain (12.32). By (12.20) and Lemma 12.1, we have also:

$$\frac{C_3 M^3}{2NT} = \frac{\bar{R}^2}{H} \geq B_1^{-1}H \geq B_1^{-1} \geq 16 .$$

By this, (12.31) and (12.32), we have

$$1 < \left( \frac{C_3 M^3}{2NT} \right)^{\frac{1}{2}} \leq R < \left( \frac{C_3 M^3}{2NT} \right)^{\frac{1}{2}} + 1 < 2 \left( \frac{C_3 M^3}{2NT} \right)^{\frac{1}{2}} , \quad (12.34)$$

and so (with the help of (12.32) and (12.20)) the result (12.33) follows.

By (12.34), we have $ENR^2 = C_3 M^3/(2T)$ for some $E \in [1, 4)$. Given that $\nu \geq 6$, it moreover follows from (12.32), (12.33) and the hypotheses of Lemma 12.1 stated between (12.21) and (12.23) that one has $H \leq 2^{1/4}B_1^{\nu/(\nu-2)}(2^{\nu-6})R^{(\nu-4)/(2\nu-6)}$ and either $M \leq C_6 T^{1/2}$ and $N^3 < 8B_3^2 M^6/(T^2 H)$, or else $V_0 < 16 \min\{V_1, V_2\}$. Therefore we have (11.13), and it follows by the case $(E_1, E_2) = (E, 16)$ of Lemma 12.1 that the conditions (7.4), (7.5), (10.6)-(10.7) and (12.19) are satisfied. By (12.34), we have also the inequalities stated in (7.3). ■

Lemma 12.2. Let $E' \in [\sqrt{2}, \infty)$. Let the hypotheses of Theorem 2, up to and including (6.3), be satisfied. Let the function $\Psi_{\varepsilon, \nu}(\Delta, \delta)$ be as described in (12.16), and let (10.14)' and (10.15)' denote the conditions that (10.14) and (10.15) (respectively) become when, in both of those two conditions, one substitutes $\tilde{N}$ and the relation of equality for $N$ and the relation ‘$\simeq$’ (respectively). Suppose moreover that the hypotheses of Lemma 12.1 are satisfied, and that $E_1 = E_2 = 1$. Then one has

$$|S| \leq \Psi_{\varepsilon, \nu}(E' H/\bar{R}, H/\tilde{N}) M^{1+\varepsilon} \quad (12.35)$$

in each of the following two independent cases:

(A)' when the conditions (12.22) and (10.14)' are satisfied;

(B)' when $M \leq C_6 T^{1/2}$ and the condition (10.15)' is satisfied.
**Proof.** Let $N$ and $R$ be given by (12.30) and (12.31). Since $\tilde{N} > 0$, it follows that we have $N, R \in \mathbb{N}$. By Corollary 12.1.1, it is moreover the case that the conditions (7.3), (7.4), (7.5), (10.6)-(10.7), (11.13) and (12.19) are satisfied, and that the inequalities in (12.32) and (12.33) hold. By hypothesis, we have (11.12). By choosing $\eta$, $Q'$ and $\Delta'$ to be as stated in (12.18), we are able to ensure that the conditions (10.3), (10.8)-(10.11) and (12.6) are satisfied (regarding this point see the remarks preceding (12.10), and those preceding (12.19)). Therefore we obtain the result stated in (12.15)-(12.16) if both (10.14) and the conditions attached to Case (A) of Lemma 10.1 are satisfied, or if both (10.15) and the conditions attached to Case (B) of Lemma 10.1 are satisfied. This occurs in Case (A)': for (10.14)' and (12.32) imply (10.14), while (12.22), (12.32) and (12.33) imply the bound $V_0 \ll \min\{V_1, V_2\}$. It also occurs in Case (B)', for (10.15)' and (12.32) imply (10.15), while the inequality $M \leq C_6 T^{1/2}$ implies $M^2 \ll T$. Therefore we may complete this proof by showing that one has $\Psi(E' H/R, H/\tilde{N}) \geq \Psi(H/R, H/N)$.

By (12.32), (12.33) and our hypothesis concerning $E'$, we have $H/\tilde{N} \geq H/N$ and $(E' H/R)/(H/R) \geq E'/\sqrt{T} \geq 1$. By Lemma 12.1 we have, moreover, $H/\tilde{N} \leq B_1 \leq 1$, and by (7.5) we have $H/R \geq B_1^{-1} \geq 1$. Given the points just noted and the definition of $\Psi(\Delta, \delta)$ in (12.16), we find that the desired inequality $\Psi(E' H/R, H/\tilde{N}) \geq \Psi(H/R, H/N)$ is a consequence of the observations that, for $\sigma = \pm 1$, one has both $(\partial/\partial \Delta) \Psi(\Delta, \delta) > 0$ and $(\partial/\partial \delta) \Psi(\Delta, \delta) > 0$ on the subset $\mathcal{R}(\sigma) = \{ (\Delta, \delta) \in [1, \infty) \times (0, 1) : 576 C_3^2 \Delta \delta \sigma < \sigma \}$. Moreover, subject to (12.37) holding, the conditions (12.38) and (12.39) are satisfied if and only if one has both

\[
H \leq B_1 \left( \frac{C_3 M^3}{2NT} \right)^{\frac{\sigma - 4}{2(\sigma - 3)}} \tilde{N}^{\frac{\sigma - 4}{2(\sigma - 3)}}.
\]
and
\[
\min \left\{ \left( \frac{C_3 M^3}{2T} \right)^{\frac{3}{16}}, \left( \frac{C_3 M^3}{2T} \right)^{\frac{5}{32}} M^{\frac{11}{32}} \right\} \geq H^{\frac{171}{140}} \tilde{N}. \tag{12.41}
\]

A calculation shows that, when \( C_3 = 2 \) and \( \tilde{N} \) is given by (12.36), the conditions (12.40) and (12.41) become conditions on \( T, M \) and \( H \) that are effectively equivalent to the conditions (6.4)-(6.6) of Theorem 2 (albeit with \( B_1^{25/(\nu-3)/(29\nu-75)} \) in place of the constant \( B_2 \)): note in particular that, although neither (6.4) nor (6.5) applies when \( M \) lies in the interval \( \mathcal{E} = \left( T^{7/16} (\log T)^{57/448}, T^{9/16} (\log T)^{-57/448} \right) \), it does nevertheless follow directly from (7.1) that we have \( H \geq (\log T)^{171/140} \max \{ T^3 M^{-9}, T^{-6} M^{11} \} \) when \( M \) lies in the interval \( (T^{13/30} (\log T)^{171/1400}, T^{17/30} (\log T)^{-171/1400}) \supset \mathcal{E} \). Therefore, given that we have \( B_1 \in (0, 1), C_3 \geq 2 \) and \( \nu \geq 6 \), it follows that the conditions (6.4)-(6.6), with \( B_5 = B_1^{25/29} \) (say), are sufficient to ensure that if one puts \( E' = 16/9 \) and chooses \( \tilde{N} \) and \( \tilde{R} \) to be as stated in (12.36) and (12.37) then the hypotheses of Case (A)' of Lemma 12.2 will be satisfied. Consequently it follows from Lemma 12.2 that, if \( B_5 \leq B_1 \) and the conditions in (6.4)-(6.6) are satisfied, then one has
\[
\frac{|S|}{HM^\varepsilon} \leq \begin{cases} 
\left( \frac{10}{9} \right)^{\frac{309}{100}} C_{11}(\varepsilon) \left( \frac{M}{R} \right)^{\frac{27T}{110}} T^{\frac{399}{100}} + \left( \frac{H}{M} \right)^{\frac{39}{10}} T^{\frac{143}{100}} & \text{if } H \leq (\log T)^{1968} M^{\frac{1}{120}} T^{\frac{9}{10}}, \\
\left( \frac{18}{5} \right)^{\frac{317}{302}} C_{11}(\varepsilon, \nu) \left( \frac{H}{M} \right)^{\frac{12}{29}} T^{\frac{29}{448}} & \text{otherwise.}
\end{cases}
\tag{12.42}
\]

We observe also that, in order for the hypotheses of Case (B)' of Lemma 12.2 to be satisfied, it is enough that one have (12.37), (12.38), (6.10) and
\[
\tilde{N} = \min \left\{ \left( \frac{M^\varepsilon (\log T)^{\frac{969}{2900}} M}{T^{13/30} H^{\frac{11}{32}}} \right)^{\frac{1}{4}}, B_3 M^2 \right\} \tag{12.43}
\]
(note, in particular, that (6.10) and (12.43) imply that the case \( E_2 = 1 \) of the condition (12.23) is satisfied). Subject to \( \tilde{R} \) being given by (12.37), the condition (12.38) becomes equivalent to the inequality in (12.40), and so (given that \( C_3 \geq 2 \) and \( \nu \geq 6 \)) we may deduce that, when \( \tilde{N} \) and \( \tilde{R} \) are as stated in (12.37) and (12.43), the condition (12.38) will hold if one has
\[
\left( \frac{H}{B_1} \right)^{\frac{4(\nu-3)}{\nu-4}} \leq \left( \frac{M^3}{T} \right)^{\frac{\nu-4}{4}} \min \left\{ \left( \frac{M^\varepsilon (\log T)^{\frac{969}{2900}} M}{T^{13/30} H^{\frac{11}{32}}} \right)^{\frac{1}{4}}, B_3 M^2 \right\}.
\]

A calculation shows that this last inequality is satisfied if one has the upper bound on \( H \) in (6.11), with \( B_4 = B_1^{12/13} B_3^{3/7}, \) (say). Therefore, subject to \( \tilde{R} \) and (6.11) holding (with \( B_4 \) as just stated), we find that by applying Lemma 12.2, with \( \tilde{N} \) and \( \tilde{R} \) given by (12.43) and (12.37), and with \( E' = 16/9 \), one is able to obtain the bounds
\[
\frac{|S|}{M^{1+\varepsilon}} \leq \begin{cases}
\left( \frac{10}{9} \right)^{\frac{309}{100}} C_{11}(\varepsilon) \left( \frac{H}{R} \right)^{\frac{27T}{110}} T^{\frac{399}{100}} + \left( \frac{H}{R} \right)^{\frac{39}{10}} T^{\frac{143}{100}} & \text{if } \frac{2^{10} C_2^2 H^2}{RN^2} \leq 1,
\left( \frac{18}{5} \right)^{\frac{317}{302}} C_{11}(\varepsilon, \nu) \left( \frac{H}{R} \right)^{\frac{12}{29}} T^{\frac{29}{448}} & \text{otherwise},
\end{cases}
\]
\[
\leq \begin{cases}
\left( \frac{3}{4} \right) C_{11}(\varepsilon) \left( \frac{H^3 T}{M^3} \right)^{\frac{1}{16}} T^{\frac{100}{110}} + \left( \frac{H^3 T}{M^3} \right)^{\frac{39}{10}} T^{\frac{143}{100}} & \text{if } \frac{2^{21} C_3^3 H^2 T}{M^3} \leq \tilde{N},
\left( \frac{3}{4} \right) C_{11}(\varepsilon, \nu) \left( \frac{H^3 T}{M^3} \right)^{\frac{1}{16}} T^{\frac{29}{448}} & \text{otherwise}.
\end{cases}
\]
By these bounds, in which \( \frac{1}{\nu} - (\frac{1}{\nu})q_0^{-1} > \frac{1}{\nu} - \frac{1}{\nu_0} > 0 \), while \( \tilde{N} \) is as stated in (12.43), we are able to conclude that, subject to the conditions (6.10) and (6.11) both being satisfied, one will have

\[
S \ll_{\varepsilon} \left( T^{\frac{15\nu}{200}} M^{\frac{11\nu}{200}} H^{\frac{15\nu}{200}} + T^{\frac{25\nu}{200}} M^{\frac{11\nu}{200}} H^{\frac{15\nu}{200}} + T^{\frac{35\nu}{200}} M^{-\frac{15\nu}{200}} H^{\frac{15\nu}{200}} + T^{\frac{45\nu}{200}} M^{-\frac{15\nu}{200}} H^{\frac{15\nu}{200}} \right) B_3^{-\frac{5}{3}} M^{\varepsilon} \quad (12.44)
\]

if

\[
H \leq \min \left\{ (128C_3)^{-\frac{6}{10}} M^{\frac{15\nu}{100}} T^{-\frac{4}{80}} (\log T)^{\frac{3}{80}}, \ (128C_3)^{-\frac{6}{10}} M^{\frac{15\nu}{100}} T^{-\frac{4}{80}} \right\}, \quad (12.45)
\]

and will otherwise have

\[
S \ll_{\varepsilon, \nu} \left( T^{\frac{11\nu}{200}} M^{\frac{11\nu}{200}} H^{\frac{15\nu}{200}} + T^{\frac{21\nu}{200}} M^{-\frac{15\nu}{200}} H^{\frac{15\nu}{200}} + T^{\frac{31\nu}{200}} M^{-\frac{15\nu}{200}} H^{\frac{15\nu}{200}} \right) B_3^{-\frac{5}{3}} M^{\varepsilon} . \quad (12.46)
\]

Since \( B_1 \) and \( B_3 \) are both positive constants constructed from \( C_2, \ldots, C_6 \), the results of Theorem 2 are an immediate consequence of the combination of Corollary 6.1.1, Corollary 6.1.2 and our conclusions reached in (12.42) and (12.44)-(12.46): note, in particular, that (6.4)-(6.6) imply

\[
M \leq B_5^{\frac{1}{16}} T^{\frac{57(55\nu - 144)}{55(55\nu - 144)}} (\log T)^{-\frac{57(55\nu - 144)}{55(55\nu - 144)}},
\]

while (6.10) and (6.11) imply \( M \leq C_6 T^{1/2} \), and so (given that \( \nu \geq 6 \), that \( 0 < B_5, B_4 \leq 1 \leq C_6 \), and that \( T \) is large) it follows that neither Part (A) nor Part (B) of Theorem 2 will apply unless one has \( M \leq C_6 T^{901/1584} \), so that \( M^{\varepsilon} \ll_{\varepsilon} T^{\varepsilon} \). This completes our proof that Theorem 2 is valid when the condition (7.1) is satisfied; given what was noted below (7.1), it has therefore been shown that Theorem 2 is valid in all cases.

13. Applications to the mean square of \( |\zeta(\frac{1}{2} + it)| \)

**Theorem 3.** Let the function \( I : [0, \infty) \times (0, \infty) \to \mathbb{R} \) be given by:

\[
I(t,U) = \frac{1}{2\pi} \int_{t-U}^{t+U} |\zeta(\frac{1}{2} + it)|^2 \, dt .
\]

Suppose that \( \varepsilon \) is a positive constant. Then one has

\[
I \left( t, t^{\frac{15\nu}{200} + \varepsilon} \right) = O(\log t) \quad \text{as} \quad t \to \infty .
\]

**Theorem 4.** Let the function \( E(T) \) be defined on the interval \([1, \infty)\) by:

\[
E(T) = \int_0^T |\zeta(\frac{1}{2} + it)|^2 \, dt - \left( \log \left( \frac{T}{2\pi} \right) + 2\gamma - 1 \right) T \quad (T \geq 1),
\]

where \( \gamma \) denotes the Euler-Mascheroni constant. Suppose that \( \varepsilon \) is a positive constant. Then one has

\[
E(T) = O \left( T^{\frac{15\nu}{200} + \varepsilon} \right) \quad \text{as} \quad T \to \infty .
\]

Remarks.
The cases within the proof of Lemma 13.1 that are crucial in determining the limit $1515$ $U$ satisfies the more restrictive condition exponents, $\epsilon$ (with some constant
follow from the validity of the hypothesis $H$ done no work on that.

Theorem 3 might be replaced by $2811$ $\nu$ (with by the application of Proposition 10
section): they lie along the boundary that separates those cases within Case II that are best dealt
The corresponding cases in Section 11 fall within the scope of Case I I (which is defined in that

To have included, in this paper, a proper discussion of the (conditional) consequences of Huxley’s
Hypothesis $H(\kappa, \lambda)$ would have led to an unwanted degree of complexity in our results and their
proofs: we have (in any case) nothing certain to report regarding progress on this matter. It may
nevertheless be worth mentioning that, on the basis of certain calculations, we do expect that, subject
to the validity of the hypothesis $H(133/457, 0)$, the number $1273/4053 = 0.314088...$ occurring in
Theorem 3 might be replaced by $2811/8951 = 0.314043...$ (this would require using also the methods
of the present paper). It is more complicated to determine what consequences of this sort would
follow from the validity of the hypothesis $H(\kappa, \lambda)$ in cases where $133/457 > \kappa \geq 1/4$, and we have
done no work on that.

In Lemma 13.1 (below) the bound $S \ll M/\log T$ is obtained whenever one has $U \geq T^{\epsilon+1273/4053}$
(with some constant $\epsilon > 0$), whereas the stronger bound $S \ll U H/\log T$ is obtained only when
$U$ satisfies the more restrictive condition $U \geq T^{\epsilon+1515/4816}$. This is the reason for the differing
exponents, $\epsilon + 1273/4053$ and $\epsilon + 1515/4816$, that occur in Theorem 3 and Theorem 4, respectively.
The cases within the proof of Lemma 13.1 that are crucial in determining the limit $1515/4816$ (in
(13.8)) are, perhaps surprisingly, not those in which $H = M T^{O(\epsilon)-1515/4816}$: they are instead those
cases in which one has either

$$T^{O(\epsilon)+\frac{641}{1630}} \leq M \ll T^\frac{4}{2} \quad \text{and} \quad H = M T^{O(\epsilon)-\frac{489}{8951}},$$

or

$$T^{O(\epsilon)+\frac{1041}{2084}} \geq M \gg T^\frac{4053}{8951} \quad \text{and} \quad H = M^{\frac{425}{8951}} T^{O(\epsilon)-\frac{42}{8951}}.$$  

The corresponding cases in Section 11 fall within the scope of Case II (which is defined in that
section): they lie along the boundary that separates those cases within Case II that are best dealt
with by the application of Proposition 10’ from those cases in which a stronger bound on $S$ is
obtained by appealing instead to either Corollary 6.1.1 or Corollary 6.1.2. Since these crucial cases
are quite far from being in Case I, which is the only case in which one is left with a free choice of
$\nu \in \{6, 7, 8, \ldots \}$, it seems likely that, in our proof of the case $c > 1515/4816$ of Lemma 13.1, we might
have been able to put $\nu$ equal to an arbitrary element of the set $\{6, 7, 8, \ldots \}$, instead of making the
specific choice $\nu = 7$ indicated in (13.9). However (as we hope is made clear by Remark (iii) below
Theorem 2) we really do need to put $\nu = 7$ in our proof of the case $c \leq 1515/4816$ of Lemma 13.1.

Given that $\zeta(\pi) = \bar{\zeta(s)}$ for all $s \in \mathbb{C} - \{1\}$, it is a direct consequence of the definitions in
Theorem 3 and Theorem 4 that, when $U = t^c$, one has:

$$I(t, U) = \begin{cases} \frac{E(t+U) - E(t-U)}{2U} + \frac{1}{2U} \int_{t-U}^{t+U} \left( \log \left( \frac{x}{2\pi} \right) + 2\gamma \right) dx & \text{if } 0 < c < 1 \text{ and } t \geq 2^1/(1-c), \\ \frac{E(2U)}{2U} + \log \left( \frac{U}{\pi} \right) + 2\gamma - 1 & \text{if } c = 1 \text{ and } t \geq 1/2; \\ \frac{E(U+t) + E(U-t)}{2U} + \log \left( \frac{U}{2\pi} \right) + 2\gamma - 1 + O \left( \frac{t^2}{U^2} \right) & \text{if } c > 1 \text{ and } t \geq 2^1/(c-1). \end{cases}$$
By this it is readily be seen that Theorem 4 contains (i.e. implies immediately) those cases of Theorem 3 in which \( \varepsilon \) exceeds the difference between \( \frac{1273}{4053} = 0.314088... \) and \( \frac{1515}{4816} = 0.314576... \), which is \( 0.00048808... < 2^{-11} \).

**Lemma 13.1.** Let \( \delta_0 \) and \( c \) be constants satisfying \( 0 < \delta_0 < 1 \) and
\[
\frac{1273}{4053} < c < \frac{1}{3}.
\]

Suppose that \( T \) is a large positive parameter, that \( U \in \mathbb{R} \) satisfies
\[
T^c \leq U \leq 3T^c,
\]
and that \( H, H_1, M, M_1 \in (0, \infty) \) satisfy:
\[
\frac{H}{2} \leq H_1 \leq H, \quad \frac{M}{2} \leq M_1 \leq M,
\]
and either
\[
UH \leq M \leq 4T^\frac{7}{4},
\]
or else
\[
H > \frac{U^\frac{7}{2}}{T}.
\]

Suppose moreover that \( b \in \mathbb{Z} \), that \( F : [1/3, 3] \to \mathbb{R} \) is the function given by
\[
F(x) = \log(x) - \frac{bM^2x^2}{8T} \quad (1/3 \leq x \leq 3),
\]
and that \( S = S_F(T; H, H_1; M, M_1) \) is the exponential sum defined in (7.2). Then one has:
\[
S \ll \begin{cases} 
\frac{UH}{\log T} & \text{if } c > \frac{1515}{4816}, \\
\frac{M}{\log T} & \text{otherwise.}
\end{cases}
\]

**Proof.** Since \( S = 0 \) if \( H < 1 \), we may assume throughout that \( H \geq 1 \).

We shall complete this proof by showing that the bound (13.8) is a corollary of the case \( \nu = 7 \) of the results of Theorem 2. As a first step towards this we verify that the sum \( S \) is such that the relevant hypotheses of Theorem 2 are satisfied. Given (7.2) and (13.3), the present sum \( S \) is similar in form to the sum \( S \) occurring in Theorem 2: it corresponds to the special case in which the functions \( g(x) \) and \( G(x) \) of the theorem are the step functions defined on the interval \([1/2, 1]\) by \( g(x) = |(H_1/H, \infty) \cap \{x\}| \) and \( G(x) = |(M_1/M, \infty) \cap \{x\}| \). By (13.7), we have the cases \( r = 3, 4, 5 \) of (6.1) and the cases \( r = 3, 4 \) of (6.2) for any choice of \( C_3, C_4 \) and \( C_5 \) satisfying \( C_r \geq (r - 1)!3^r \) \((r = 3, 4, 5)\). Before considering (6.3) and the case \( r = 2 \) of the conditions (6.1) and (6.2) it should be noted that the definitions (7.2) and (13.7) imply that the sum \( S \) depends on the integer \( b \) only insofar as it depends on whether \( b \) is even or odd (indeed, each term of the sum \( S \) is of the form \( \phi(h, m)e(-bmh/2) \), where \( \phi(h, m) \) is a factor that is independent of \( b \)). The integers 1 and 325 are both odd, whereas 0 is an even number. Therefore we may assume that either it is the case that
M ≤ T^{1/2}/3 and b ∈ \{0,1\} or else it is the case that M > T^{1/2}/3 and b ∈ \{0,325\}. In either of these two cases one has

\[ F^{(2)}(x) = -\left( \frac{1}{x^2} + \frac{bM^2}{4T} \right) \]

and

\[ F^{(2)}(x)F^{(4)}(x) - 3F^{(3)}(x)^2 = \left( \frac{1}{x^2} + \frac{bM^2}{4T} \right) \left( \frac{6}{x^4} \right) - 3 \left( \frac{2}{x^3} \right)^2 = \frac{3bM^2}{2T} - \frac{6}{x^6}, \]

for all x lying in the interval [1/3, 3]. Given that we have 0 ≤ b ≤ 325 and (by (13.4)) M^2/T ≤ 16, it follows that we have the case r = 2 of both (6.1) and (6.2) for any choice of C_2 satisfying C_2 ≥ 1309. If b = 0 then we have (6.3) for any choice of C_5 ≥ 3^5/2. If b = 1 and M ≤ T^{1/2}/3, then we have (6.3) for any choice of C_5 ≥ (4/3)(3^5/2) = 162. In the remaining cases, where b = 325 and M > T^{1/2}/3, we have (6.3) for any choice of C_5 ≥ 54T/M^2, and so for any C_5 ≥ 486. We conclude that, in all the cases under consideration, the conditions (6.1)-(6.3) of Theorem 2 will hold if one puts \(C_r = (r - 1)3^r\) (r = 3, 4, 5) and \(C_2 = 1309\). It follows that the results of Part (A) and Part (B) of Theorem 2 will be applicable to the sum \(S = S_F(T; H, H_1; M, M_1)\), provided only that it can be shown that the relevant additional conditions (i.e. (6.4)-(6.6) for Part (A); (6.10)-(6.11) for Part (B)) are satisfied.

Given the upper bound on M in (13.4), we may assume that the condition (6.5) is satisfied (for we shall have \(T^{9/16}(\log T)^{-57/448} > 4T^{1/2}\), provided only that T is large enough); the same bound on M trivially implies that the condition (6.10) will be satisfied if we put \(C_6 = 4\). We choose now to put:

\[ \nu = 7 \]

and \(q = q_\nu\), with \(q_\nu\) as defined in Theorem 2, so that

\[ q = q_7 \leq \frac{158}{37} = 4 + \frac{10}{37} = 4.270. \]

By (13.9), the condition (6.6) will be satisfied if and only if

\[ H ≤ B_3MT^{-\frac{641}{960}}(\log T)^{\frac{609}{600}}. \]

Since our hypotheses in (13.1), (13.2) and (13.4) imply that we have \(H/M ≤ 1/U ≤ T^{-c}\), where c is a constant satisfying \(c > 1273/4053 = 0.31408... > 643/2048 = 0.31396...\), it follows that the condition (13.11) will be satisfied if T is large enough (in terms of the small positive constant \(B_3\)). We may therefore assume that (13.11) does hold, so that the case \(\nu = 7\) of the condition (6.6) is satisfied.

To complete the data concerning our application of Theorem 2 (the implications of which are discussed below) we now specify \(\varepsilon\) by putting \(\varepsilon = \varepsilon(c)\), where \(\varepsilon(c)\) is equal to \((c - (1515/4816))/200\) if \(c > 1515/4816\), and is otherwise equal to \((c - (1273/4053))/5\). Given (13.1), this ensures that \(\varepsilon\) is a constant satisfying the following three conditions:

\[ 0 < \varepsilon \leq \left( \frac{4}{5} \right) (c - \frac{1515}{4816}) \quad \text{if} \quad c > \frac{1515}{4816}, \quad (13.12) \]

\[ 0 < \varepsilon \leq \left( \frac{4}{5} \right) (c - \frac{1273}{4053}), \quad (13.13) \]

\[ 0 < \varepsilon \leq 0.0001. \quad (13.14) \]

Our next steps depend on whether or not it is the case that the first inequality occurring in (6.4) is satisfied. Suppose, firstly, that one does have

\[ H ≥ M^{-9}T^{4}(\log T)^{\frac{461}{600}}. \]
Then, recalling the points noted in the previous paragraph, we are able to conclude that all three of the conditions ((6.4), (6.5) and (6.6)) attached to Part (A) of Theorem 2 are satisfied, and so it follows by (13.9), (13.10) and Part (A) of that theorem that either

\[ S \ll H \left( \frac{H}{M} \right) \frac{323}{300} T^{\frac{1273}{300} + \epsilon}, \tag{13.16} \]

or else one has the bounds stated in (6.7) and (6.8). By (13.2), (13.4), (13.13) and (13.4) (again), the bound (13.16) would imply

\[ S \ll \frac{HT^{\frac{1273}{300} + \epsilon}}{U \frac{323}{300}} = \left( T^{\frac{1273+3950\epsilon}{4053}} \frac{4053}{300} \right) U H \leq \frac{U H}{T^{\epsilon}} \left( \frac{4053}{300} \right) \left( \frac{1273+3950\epsilon}{4053} \right) < \frac{U H}{T^{2\epsilon}} \leq \frac{M}{T^{2\epsilon}}, \]

and so would yield the result (13.8) of the lemma. If we do not have (13.16), then we have instead (6.7) and (6.8), which imply that one has

\[ S \ll HT^{\epsilon} \min\{X, Z\} + T^{\epsilon} Y, \]

with

\[ X = \left( \frac{M}{H} \right) \left( \frac{H}{M} \right) \frac{323}{300} T^{\frac{1273}{300} + \epsilon} \ll \left( \frac{M}{H} \right) \left( \frac{\log T^{\frac{1969}{300}}}{T^{\frac{4053}{300}}} \right) \frac{323}{300} T^{\frac{1273}{300} + \epsilon} = \frac{M (\log T)}{HT^{\frac{1969}{300}}} \ll \frac{M}{T^{2\epsilon} H}, \]

\[ Y = \left( \frac{H}{M} \right) \frac{1969}{300} T^{\frac{1133}{300}} \ll \left( \frac{\log T}{T^{\frac{4053}{300}}} \right) \frac{1969}{300} T^{\frac{1133}{300}} = T^{\frac{3785}{300} + \epsilon} (\log T) \frac{1969}{300} \]

and

\[ Z = (H/M)^{1/25} T^{131/400}, \]

so that one obtains:

\[ \min\{X, Z\} \leq X^{\frac{323}{300}} Z^{\frac{1273}{300} + \epsilon} = T^{\frac{1515}{4816}}. \]

These bounds would imply, firstly, that

\[ S \ll HT^{\epsilon} X + HT^{\epsilon} Y \ll \frac{M}{T^{\epsilon}} + HT^{\frac{3785}{12064} + 2\epsilon} < \frac{M + Hu}{T^{\epsilon}} \leq \frac{2M}{T^{\epsilon}}, \]

(with the latter part of this following by virtue of (13.14), (13.1), (13.2) and (13.4), given that 3785/12064 < 0.31375, whereas 1273/4053 > 0.31408) and, secondly, that one has:

\[ \frac{S}{H} \ll T^{\frac{1515}{4816} + \epsilon} + T^{\frac{3785}{12064} + 2\epsilon} \leq T^{\epsilon} < \frac{U}{T^{\epsilon}} \text{ if } c > \frac{1515}{4816}, \]

(with the inequality in the middle following by (13.1), (13.12) and (13.14)). Therefore, in the event that (6.7) and (6.8) hold, we obtain (13.8). Since we have found that (13.8) is obtained whether or not the bound (13.16) holds, this completes our proof in respect of the cases in which the condition (13.15) is satisfied.

Suppose now that (13.15) does not hold, so that one has

\[ H < M^{-9} T^4 (\log T)^\frac{1313}{1969}. \tag{13.17} \]
This implies \( M < H^{-1/9} T^{4/9} (\log T)^{19/140} \leq T^{4/9} (\log T)^{19/140} \). Therefore, provided that \( T \) is sufficiently large in terms of the small positive constant \( \delta_0 \), it will be the case that the inequality (13.5) does not hold; this is, by hypothesis, incompatible with it simultaneously being the case that (13.6) does not hold, and so we may henceforth assume that the inequality (13.6) is satisfied.

The remainder of this proof rests on the application of Part (B) of Theorem 2. Our first task, therefore, is to verify that the condition (6.11) is satisfied (it already having been noted that (13.4) gives (6.10)). We recall that the inequality (13.11) was found to hold (and that this was subject only to the hypotheses of the lemma). Therefore (13.11) holds in the present case, and by it and (13.17), we may deduce that

\[
H < \left( B_3 M (\log T) \frac{4096}{T^{4215}} \right)^{\frac{1056}{4053}} \left( \frac{T^4 (\log T)^{171}}{M^9} \right)^{\frac{119}{219}} = B_5^{4096} \left( \frac{M^{4096-1071} (\log T)^{606} + 2007}{T^{1260 + 475}} \right)^{\frac{1}{4275}} = B_5^{4096} \left( \frac{M^{605} (\log T)^{606}}{T^{162}} \right)^{\frac{119}{219}}.
\]

Now Theorem 2 would remain valid if one substituted the constant \( B_5' = \min\{B_5, B_4^{4215/4096}\} \) in place of the constant \( B_5 \) in (6.6) (indeed, this would either have no effect, or would slightly weaken the content of the theorem). We may therefore assume that the constants \( B_5 \) and \( B_4 \) in Theorem 2 satisfy \( B_5^{4096/4215} \leq B_4 \). By this, and the upper bound on \( H \) that was just obtained (above), we find that the case \( \nu = 7 \) of the condition (6.11) will be satisfied if it is the case that one has:

\[
H \leq B_4 \left( \frac{M^3}{T} \right)^{\frac{23}{5}}.
\]

To see that this does hold, we recall that (13.6) was shown to hold (subject to our assumption (13.17)), and note that, by the combination of the first inequality in (13.4) with (13.6), (13.2) and (13.1), one has

\[
\left( \frac{1}{H} \right) \left( \frac{M^3}{T} \right)^{\frac{23}{5}} \geq \left( \frac{1}{H} \right) \left( \frac{U^3 H^3}{T} \right)^{\frac{23}{5}} = U^{\frac{26}{5} H^{\frac{14}{5}} T^{-\frac{11}{5}} \left( T^{\frac{11}{5}} \right)^{\frac{11}{5}}} = U^{\frac{118}{5} T^{-\frac{11}{5}}} \geq T^{\left( \frac{118}{5} \right) \left( \frac{37}{175} \right) H} > \frac{1}{B_4}
\]

(with the final inequality following from the relations \( c > 1273/4053 = 0.3140... > 37/118 = 0.3135... \), provided that \( T \) is sufficiently large in terms of the small positive constant \( B_4 \)). This completes our verification of the condition (6.11).

Since both (6.10) and (6.11) are satisfied, it follows by (13.9), (13.10) and Part (B) of Theorem 2 that either

\[
S \ll T^{\frac{1053}{4053} + \varepsilon} M^{\frac{1053}{4053}} H^{\frac{117}{4053}} + T^{\frac{1053}{4053} + \varepsilon} M^{-\frac{417}{2048}} H^{\frac{417}{2048}},
\]

or else one has the bounds stated in (6.12) and (6.13). Here we observe that one has:

\[
\frac{T^{164 H^{\frac{19}{19}} H^{\frac{417}{2048}}}}{H(H/M)^{\frac{1053}{4053} T^{\frac{159}{2048}} + \varepsilon}} = \left( \frac{M^{1053} H^{\frac{117}{4053}}}{T^{1068}} \right)^{\frac{1053}{4053}} = \left( \frac{M^9 H^{\frac{117}{4053}}}{T^4} \right)^{\frac{1053}{4053}} \ll T^\varepsilon
\]

(the last inequality following by virtue of our assumption (13.17)). By this and the first calculation appearing below (13.16), we find that, given the hypothesis (13.1), it must follow from (13.18) that one has:

\[
\frac{S}{H} \ll \frac{U}{T^\varepsilon} + T^{\frac{1053}{4053} + \varepsilon} M^{-\frac{417}{2048}} H^{-\frac{417}{2048}}.
\]
Recall now that the assumption (13.17) ensured that (13.6) must hold. By (13.4) and (13.6), we have \( H > U^{7/2} T^{-1} \) and \( M \geq UH > U^{9/2} T^{-1} \). By these inequalities, the above bound for \( S/H \) and the hypothesis (13.2), we find that

\[
S/H \ll \frac{U}{T^\varepsilon} + \left( \frac{T}{U^{9/2}} \right)^{\frac{407}{204}} \left( \frac{T}{U^{7/2}} \right)^{\frac{417}{203}} T^{\frac{401}{204} + \varepsilon} = \frac{U}{T^\varepsilon} + \left( \frac{T^{1655}}{U} \right)^{\frac{5277}{2035}} T^c U \ll \frac{U}{T^\min\{\varepsilon, (\frac{5277}{2035})(c^{13}\frac{469}{469} - \varepsilon)\}}.
\]

Therefore, given that \( 1655/5277 < 0.3137 \), whereas \( 1273/4053 > 0.314 \), the hypothesis (13.1) and inequality (13.14) are enough to ensure that we obtain the bound \( S \ll UHT^{-\varepsilon} \). By this and the first inequality of (13.4), we may conclude that the bound in (13.8) holds when one has both (13.17) and (13.18).

The only cases that remain to be considered are those in which one has (13.17) and (instead of (13.18)) the bounds (6.12) and (6.13). By (6.12), we have \( M \gg T^{1/3} H^{13/15} \). This leads to bounds for a couple of the terms occurring on the right-hand side of (6.13):

\[
T^{\frac{113}{137}} M^{\frac{111}{137}} H^{\frac{470}{137}} \ll T^\frac{2}{137} H^{\frac{56}{137}} \quad \text{and} \quad T^{\frac{79}{137}} M^{\frac{151}{137}} H^{\frac{101}{137}} \ll T^\frac{2}{137} H^{\frac{203}{137}} = T^\frac{2}{137} H^{\frac{529}{137}}. \tag{13.19}
\]

Note the greater magnitude of the latter bound: regarding it, we observe that, since the inequality (13.6) holds, one has

\[
T^\frac{2}{137} H^{\frac{529}{137}} = T^\frac{2}{137} H^{\approx \frac{32}{37}} < T^\frac{2}{2686} \left( \frac{T}{U^\frac{1}{2}} \right)^{\frac{437}{137}} = T^\frac{2}{2686} \left( \frac{T}{U^\frac{1}{2}} \right)^{\frac{437}{137}} \leq \frac{U}{T^{\min\{\varepsilon, (\frac{5237}{2686})(c^{13}/137 - \varepsilon)\}}} \leq \frac{U}{T^\varepsilon}, \tag{13.20}
\]

(the last two inequalities following by virtue of (13.2), (13.1) and (13.14), given that one has \( 137/437 < 0.3136 \), whereas \( 1273/4053 > 0.314 \)).

By (13.17) and (6.12), we have also \( M^9 H < T^4 (\log T)^{17/46} \) and \( M^{-155} H^{189} \ll T^{-46} (\log T)^{969/46} \), and so:

\[
M^{-\frac{114}{137}} H^{\frac{433}{137}} = (M^9 H)^{\frac{2109}{2686}} \left( \frac{H^{189}}{M^{155}} \right)^{\frac{439}{2686}} \ll \left( T^4 (\log T)^{17/46} \right)^{\frac{2109}{2686}} \left( \frac{(\log T)^{969/46}}{T^{46}} \right)^{\frac{439}{2686}}.
\]

By this bound, together with (13.1), (13.2) and (13.14), it follows that we have

\[
T^{\frac{113}{137}} M^{-\frac{114}{137}} H^{\frac{433}{137}} \ll T^{\frac{375/137}{2686}} T^{\varepsilon} < T^{c^{-\varepsilon}} \leq \frac{U}{T^\varepsilon} \tag{13.21}
\]

(note what this has in common with the bound at the end of the paragraph containing (13.15)).

By (13.19), (13.20) and (13.21), the bound (6.13) for \( S \) implies:

\[
S/H \ll \frac{U}{T^\varepsilon} + T^\varepsilon \min\{X_1, Z_1\} + T^\varepsilon \min\{Y_1, Z_1\}, \tag{13.22}
\]

where

\[
X_1 = T^{\frac{144}{137}} M^{\frac{144}{137}} H^{-\frac{144}{2686}} , \quad Y_1 = T^{\frac{46}{2686}} M^{\frac{46}{2686}} H^{-\frac{46}{2686}} \quad \text{and} \quad Z_1 = T^{\frac{144}{2686}} M^{\frac{144}{2686}} H^{-\frac{144}{2686}}. \tag{13.23}
\]

We note, firstly, that it follows from (13.22) and the first inequality in (13.4) that one has

\[
S/M \ll \left( \frac{1}{T^\varepsilon} \right) \left( 1 + \left( \frac{H}{M} \right) X_1 T^{2\varepsilon} + \left( \frac{H}{M} \right) Y_1 T^{2\varepsilon} \right). \tag{13.24}
\]
By (13.23), one has

\[
\left( \frac{H}{M} \right) X_1 = \frac{T^{\frac{13}{20}} H^{\frac{17}{20}}}{M^{\frac{19}{20}}} = \left( \frac{HT^{\frac{77}{200}}}{M^{\frac{19}{20}}} \right)^{\frac{537}{880}} .
\]

Moreover, the bound (6.12) and assumption (13.17) imply that one has here

\[
H \ll \left( \frac{M^{\frac{155}{199}} (\log T)^{\frac{779}{820}}}{T^{\frac{155}{199}}} \right)^{\frac{20149}{40196}} \left( \frac{T^4 (\log T)^{\frac{121}{140}}}{M^9} \right)^{\frac{415}{3996}} = \frac{M^{\frac{839}{1326}} (\log T)^{\frac{121}{140}}}{T^{\frac{415}{3996}}} ,
\]

and so (given (13.14)):

\[
\left( \frac{H}{M} \right) X_1 T^{2\varepsilon} \ll \left( \frac{(\log T)^{\frac{121}{140}}}{T^{\frac{38}{49}}} \right)^{\frac{537}{880}} T^{2\varepsilon} \ll T^{-3\varepsilon - \frac{83}{1326}} < 1 .
\] (13.25)

Regarding the final term in (13.24), we note that, since (6.12) implies \( M \gg T^{1/3} H^{13/15} \), it follows from (13.23) that one has

\[
\left( \frac{H}{M} \right) Y_1 T^{2\varepsilon} = \frac{T^{\frac{37}{20}} H^{\frac{37}{20}}}{M^{\frac{37}{20}}} \ll \frac{T^{\frac{37}{20}} + 2\varepsilon H^{\frac{37}{20}}}{M^{\frac{37}{20}}} \left( T^4 H^{\frac{37}{20}} \right)^{\frac{283}{3120}} = \frac{T^{2\varepsilon}}{H^{\frac{37}{20}}} ,
\] (13.26)

where, since (13.6) holds, one has \( H > U^{\frac{7}{2}} T^{-1} \geq T^{(7/2)c - 1} \), with \( (7/2)c - 1 > 115/1158 > 153/2490 \) (by (13.1)). By (13.24), (13.25), (13.26) and (13.14), we find that

\[ S \ll \frac{M}{T^{\varepsilon}} . \] (13.27)

Note that (13.22) also implies the bound

\[
\frac{S}{H} \ll \frac{U + X_2 T^{2\varepsilon} + Y_2 T^{2\varepsilon}}{T^{\varepsilon}} ,
\] (13.28)

where

\[ X_2 = X_1^{\frac{37}{57}} Z_1^{\frac{37}{57}} \text{ and } Y_2 = Y_1^{\frac{37}{57}} Z_1^{\frac{37}{57}} , \]

so that, by (13.23) and (13.17), one has:

\[
X_2 = \left( T^{\frac{92}{100}} M^{\frac{2409}{3200}} H^{\frac{321}{620}} \right)^{\frac{37}{57}} = \left( M^9 H \right)^{\frac{37}{57}} T^{\frac{139}{880}} \ll \left( T^4 (\log T)^{\frac{174}{140}} \right)^{\frac{37}{57}} T^{\frac{139}{880}} \ll T^{\frac{1415}{4816} + \varepsilon}
\]

and, similarly,

\[
Y_2 = \left( T^{\frac{96}{639}} M^{\frac{3409}{6800}} H^{\frac{1402}{6800}} \right)^{\frac{37}{57}} \ll \left( T^4 (\log T)^{\frac{174}{140}} \right)^{\frac{41523}{48160}} T^{\frac{549408}{68000}} \ll T^{\frac{26749}{140} + \varepsilon} = T^{\frac{1515}{4816} + \varepsilon} .
\]

By (13.28), the above bounds for \( X_2 \) and \( Y_2 \), and (13.2) and (13.12), we have

\[
S \ll \left( \frac{U H}{T^{\varepsilon}} \right) \left( 1 + \frac{X_2 T^{2\varepsilon} + Y_2 T^{2\varepsilon}}{T^{\varepsilon}} \right) \ll \left( \frac{U H}{T^{\varepsilon}} \right) \left( 1 + \frac{T^{3\varepsilon}}{T^{\varepsilon - \frac{1515}{4816}}} \right) \ll \frac{U H}{T^{\varepsilon}} \text{ if } \varepsilon > \frac{1515}{4816} .
\]

This bound on \( S \), together with that in (13.27), imply what is stated in (13.8), and so complete the proof. ■
Lemma 13.2. Let $t, U > 0$ satisfy $(\pi t)^{1/2} \leq U^2 \leq t/(2\pi)$. Let $I(t, U)$ be as defined in Theorem 3, above. Then either it is the case that
\[
I(t, U) \ll \log t ,
\] (13.29)
or else there exists some
\[
T \in \left[ \frac{t}{4\pi}, \frac{3t}{4\pi} \right] ,
\] (13.30)
some
\[
M \in \left[ 1, (2T)^{\frac{1}{2}} \right] ,
\] (13.31)
and some
\[
M_1 \in \left[ \frac{M}{2}, M \right] ,
\] (13.32)
such that the sum
\[
S_\ast (T; U; M, M_1) = \sum_{0 < h \leq (e^{1/U - 1})M/2} \sum_{M_1 < m \leq M} \left( \frac{m + h}{m - h} \right)^{2\pi iT} ,
\] (13.33)
satisfies
\[
\frac{|S_\ast (T; U; M, M_1)|}{M} \gg \frac{I(t, U)}{\log t} .
\] (13.34)

Proof. By the case $r = 1$ of the results contained in [W04, Lemma 10.2 and Lemma 10.3] it follows that, for some $\tau$ lying in the interval $[t/2, 3t/2]$, one has
\[
I(t, U) \ll \left( \sum_{0 < m \leq K_0} \frac{1}{m} \right) + t^{\frac{1}{2}} U^{-2} + U^2 t^{-\frac{3}{2}} + \sum_{j=0}^{\infty} |E(U; \tau; K_j)| ,
\]
where
\[
K_j = 2^{-j} \left( \frac{t}{2\pi} \right)^{\frac{1}{2}} \quad \text{and} \quad E(U; \tau; K) = \sum_{0 < k \leq K} \frac{1}{k} \sum_{1 \leq d \leq (e^{1/U - 1})K/2} \left( \frac{k + d}{k - d} \right)^{-i\tau} .
\]

Given the hypotheses of the lemma concerning $t$ and $U$, it follows that we have the bound
\[
\frac{I(t, U)}{\log t} \ll 1 + \frac{1}{|J|} \sum_{j \in J} |E(U; \tau; K_j)| ,
\]
where $J = \{ j \in \mathbb{Z} : 1 \leq 2^j \leq K_0 \}$. Therefore, either it is the case that the relation (13.29) holds, or else we must have
\[
\frac{I(t, U)}{\log t} \ll |E(U; \tau; K_j)| \quad \text{for some} \quad j \in J .
\]

With regard to the latter of these two cases, we observe that, by partial summation and the invariance of the absolute value under complex conjugation, it follows that if $K > 0$ then one has
\[
E(U; \tau; K) \ll \frac{1}{K} \left| \sum_{K' < k \leq K} \sum_{1 \leq d \leq (e^{1/U - 1})K/2} \left( \frac{k + d}{k - d} \right)^{-i\tau} \right| \quad \text{for some} \quad K' \in \left[ \frac{K}{2}, K \right] .
\]
By applying this with \( K = K_j \), and then substituting \( 2\pi T, M, M_1, m \) and \( h \) for \( \tau, K_{j*}, K_{j*}', k \) and \( d \) (respectively), we obtain what is described in (13.30)-(13.34). ■

**Lemma 13.3.** Let \( U \geq 1 \). Suppose that \( T, M, M_1 > 0 \) satisfy the conditions (13.31) and (13.32). Put

\[
D(T, U, M) = \min \left\{ \left( \frac{e^{1/U} - 1}{2} \right) M, \frac{U^2}{T} \right\}. \tag{13.35}
\]

Then one has

\[
\sum_{0 \leq h \leq D(T, U, M)} \sum_{M_1 < m \leq M} \left( \frac{m + h}{m - h} \right)^{2\pi i T} \ll M. \tag{13.36}
\]

**Proof.** Note firstly that, for \( h \in \mathbb{N} \), one has

\[
\sum_{M_1 < m \leq M} \left( \frac{m + h}{m - h} \right)^{2\pi i T} = \sum_{M_1 < m \leq M} e(f_h(m)) = W_h \quad \text{(say),}
\]

where \( f_h(x) = T \log(x + h) - T \log(x - h) \). One can show moreover that, when \( h/M \) is sufficiently small (in absolute terms), the exponential sum \( W_h \) may be estimated through an application of the theory of exponent pairs: one will then obtain, in particular, the bounds

\[
W_h \ll \left( \frac{hT}{M^2} \right) \frac{\hat{M}}{hT} + \frac{M^2}{hT} = (hT)^{\hat{\theta}} + O \left( \frac{1}{h} \right) \ll (hT)^{\hat{\theta}}, \tag{13.37}
\]

which derive from the exponent pair \( BAAB(0,1) = (2/7,4/7) \) (i.e. we are here applying the case \( k = 2/7, l = 4/7, s = 2, y = 2hT, N = M/2 \) of the result stated in [G&K91, Equation (3.3.4)]). Since (13.35) implies \( D(T, U, M) \leq (e^{1/U} - 1)M/2 \), where we have \( 0 < e^{1/U} - 1 = e^{1/U} - e^0 < ((1/U) - 0)e^{1/U} \leq e/U \), it follows that if \( U \) is sufficiently large (in absolute terms) then, by (13.37) and (13.35) (again), one will have

\[
\sum_{0 \leq h \leq D(T, U, M)} |W_h| \ll \sum_{0 \leq h \leq D(T, U, M)} (hT)^{\hat{\theta}} \ll (D(T, U, M))^{\hat{\theta}} T^{\hat{\theta}} \leq \left( \frac{e^{1/U} - 1}{2} \right) M \left( \frac{U^2}{T} \right)^{\hat{\theta}} \leq \frac{eM}{2},
\]

and so will obtain the result stated in (13.36).

The only cases of the lemma requiring further proof are those in which one has \( U \leq U_0 \), with \( U_0 \) equal to a certain positive absolute constant. We note that, by (13.35) and the hypotheses concerning \( U, T \) and \( M \), one has \( D(T, U, M) \leq U^{1/2}/T \leq 2U^{7/2} \). Since this trivially implies the bound

\[
\sum_{0 \leq h \leq D(T, U, M)} |W_h| \leq D(T, U, M)M \ll U^{7/2}M,
\]

we therefore find that (13.36) holds when \( U \) is less than or equal to the absolute constant \( U_0 \). ■

**Lemma 13.4.** Let \( c \) be a constant satisfying

\[
\frac{1515}{4816} < c < \frac{1}{3}, \tag{13.38}
\]

The, for all \( t, \Delta \geq 1 \) such that

\[
t^c \log t \leq \Delta \leq (2t)^c \log(2t), \tag{13.39}
\]
the sum
\[ G^+(t, \Delta) = \sum_{0 < \Delta \log(m) - \Delta \log(n) \leq \log t} \frac{(m/n)^t}{\sqrt{mn} \log(m/n)} \exp \left(- \left( \frac{1}{2} \Delta \log(m/n) \right)^2 \right) \] (13.40)
satisfies
\[ G^+(t, \Delta) \ll (\log t)^2 \Delta. \] (13.41)

**Proof.** It may be assumed that \( t \) satisfies a condition of the form \( T \geq T_0 \), where \( T_0 \) denotes an arbitrarily large positive constant: for the bound (13.41) is trivial when one has \( 1 \leq t \ll 1 \). In particular we may assume that \( t \geq T_0 > \left( \frac{8\pi}{3} \right)^7 \) and, given (13.39) and (13.38) (in which \( \frac{1515}{4816} > 2/7 \)), may also assume that
\[ \left( \frac{t}{4} \right)^\frac{1}{2} \geq \Delta \log t \quad \text{and} \quad \frac{\Delta \log T}{\log t} \geq t^{\frac{1}{2}} \geq \log t \geq \log T_0 > 7 \log \left( \frac{8\pi}{3} \right). \] (13.42)
This justifies the application of the bound for \( G^+(t, \Delta) \) that is noted in [W10, Equation (6.7)]. From that bound it follows that either (13.41) holds, or else one has a bound of the form
\[ G^+(t, \Delta) \ll (\log t)^2 H^{-1} |S_F(T; H, H_1; M, M_1)| = (\log t)^2 H^{-1} |S| \quad \text{(say),} \] (13.43)
where \( S = S_F(T; H, H_1; M, M_1) \) is as defined in (7.2), while \( F : [1/3, 3] \to \mathbb{R} \) is the function given by (13.7), with \( b \) some constant that is equal to either 0 or 1, and \((T, H, H_1, M, M_1)\) is some point of \( \mathbb{R}^5 \) such that \( T, H, H_1, M \) and \( M_1 \) satisfy:
\[ \left| T - \frac{t}{2\pi} \right| \leq \frac{\Delta \log t}{2\pi} \leq \frac{t}{8\pi} \] (13.44)
and
\[ \frac{\Delta H}{\log T} \leq M \leq 4T^{\frac{1}{2}}, \] (13.45)
as well as the conditions in (13.3) and either the inequality
\[ H > \frac{(\Delta \log T)^{\frac{1}{2}}}{T}, \] (13.46)
or else a condition of the form (13.5) in which \( \delta_0 \) is equal to a certain positive absolute constant. In cases where (13.41) holds there is nothing further to prove. Therefore it may henceforth be assumed that the function \( F \) and real parameters \( T, H, H_1, M \) and \( M_1 \) fit the description just given, and are, moreover, such that the relation (13.43) holds.

With the application of Lemma 13.1 in mind, we put
\[ U = \frac{\Delta}{\log T}, \]
so that, by (13.45), the condition (13.4) is satisfied. The condition (13.1) is implied by (13.38), and since \( t > (8\pi/3)^7 \), it follows by (13.38), (13.39) and (13.44) that the condition (13.2) is satisfied also. By the assumptions we have made, the inequalities in (13.3) are satisfied, the function \( F(x) \) is as
stated in (13.7), and either (13.5) holds, or else we have (13.46). Moreover, if (13.46) holds, then (given we have (13.44) and \( t > (8\pi/3)^7 \)) it implies

\[
H > \left( \frac{U^2}{T} \right) (\log T)^7 > \left( \frac{U^2}{T} \right) \left( \frac{6 \log t}{7} \right)^7 > \left( \frac{U^2}{T} \right) \left( \frac{6 \log 8\pi}{3} \right)^7,
\]

and so gives the inequality (13.6). Therefore we may conclude that Lemma 13.1 applies, so that the bound (13.8) is obtained. By (13.8), (13.38) and (13.44), we have

\[
S_F(T; H, H_1; M, M_1) \leq \frac{U H}{\log T} = \frac{\Delta H}{(\log T)^2} \leq \frac{\Delta H}{(\log t)^2}.
\]

By this, (13.43) and (13.42), we find that one has \( G^+(t, \Delta) \leq \Delta \leq (\log t)^2 \Delta \).

**The proof of Theorem 3.** We put \( c = \frac{1273}{4053} + \varepsilon \) and \( U = t^c \). In view of Remark (iv) following the statements of Theorem 3 and Theorem 4 (at the beginning of this section), it will be enough to consider only those cases in which one has \( 0 < \varepsilon \leq 2^{-11} \), and so (given that \( 1273/4053 < 3^{-1} - 2^{-6} \)) we may certainly assume that \( c \) satisfies the inequalities in (13.1). Since we have only to bound \( I(t, t^c) \) for all sufficiently large positive values of \( t \), we may certainly assume also that \( t \geq (2\pi)^3 \), that \( U \geq (2\pi)^3c > 1 \), and that any \( T \) satisfying (13.30) will (by virtue of the implied inequality \( T > t/4\pi \)) certainly be large enough to permit the application of Lemma 13.1 (should all the other hypotheses of that lemma happen to be satisfied). Then, given that \( 1273/4053 = 0.3140\ldots > 5/16 = 0.3125 \), it follows by (13.1), Lemma 13.2 and Lemma 13.3 that either it is the case that

\[
I(t, t^c) = I(t, U) \ll \log t,
\]

or else, for some \((T, M, M_1) \in \mathbb{R}^3\) satisfying (13.30), (13.31) and (13.32), one has

\[
\frac{I(t, t^c)}{\log t} = \frac{I(t, U)}{\log t} \ll \frac{1}{M} \left| \sum_{U^{7/2}T^{-1} < h \leq (e^{1/U} - 1)M/2} \sum_{M_1 < m \leq M} \left( m + h \right)^{2\pi i T} \right| \leq \frac{1}{M} \left| \sum_{U^{7/2}T^{-1} < h \leq (e^{1/U} - 1)M/2} \sum_{M_1 < m \leq M} \left( m + h \right)^{2\pi i T} \right| \leq \frac{1}{M} \left| \sum_{U^{7/2}T^{-1} < h \leq (e^{1/U} - 1)M/2} \sum_{M_1 < m \leq M} \left( m + h \right)^{2\pi i T} \right| \]  

(13.48)

(which would imply also that \( M > 2(e^{1/U} - 1)^{-1}U^{7/2}T^{-1} \)). Only the latter of these two cases requires further consideration: for the validity of the bound in (13.47) is what we are seeking to establish in this proof. Accordingly, we note that by splitting the sum in (13.48) at points where \( h \in \{2^{-j}(e^{1/U} - 1)M : j \in \mathbb{N} \} \), and then applying the triangle inequality and the principle that the arithmetic mean of \( N \) real numbers will not exceed the greatest of those numbers, it may be deduced that, for some \((H, H_1) \in \mathbb{R}^2\) satisfying both

\[
\frac{U^2}{T} < H \leq \left( \frac{e^{1/U} - 1}{2} \right) M,
\]

\[
\frac{H}{2} \leq H_1 \leq H,
\]

one has:

\[
\frac{|S_L|}{M} \gg \frac{I(t, U)}{(\log t)^2} = \frac{I(t, t^c)}{(\log t)^2},
\]

(13.51)

where \( L \) denotes the function \( L(x) = \log(x) \) \((1/3 \leq x \leq 3) \) and \( S_L \) is the sum \( S_L(T; H, H_1; M, M_1) \) that is given by the case \( F(x) = L(x) \) of (7.2).
Based on observations made earlier, we have \( U \geq (2\pi)^{3c} \geq (2\pi)^{15/16} > 2 > 1/\log 2 \), and so the rightmost inequality if (13.49) will imply \( H \leq (1/U) \exp((1/U) - \log 2) M < M/U \). This, together with the fact that \( T, M, M_1, H \) and \( H_1 \) satisfy (13.31), (13.32), (13.49) and (13.50), is enough to ensure that (13.3), (13.4) and (13.6) hold. Since \( c \) satisfies (13.1) and \( T \) satisfies (13.30), we have also \( T^{-c}U = (t/T)^c \in [(4\pi/3)^c, (4\pi)^c] \), in which \( (4\pi/3)^c > 1 \) and \( (4\pi)^c < (4\pi)^{1/3} < 3 \). Consequently we find that (13.2) holds as well (as does (13.7), when \( F(x) = L(x) = \log(x) \) and \( b = 0 \)). Since we are assured of having \( T \) be sufficiently large for Lemma 13.1 to apply, it therefore follows by that lemma that the bound (13.8) is obtained when \( S = S_L \), so that we must have \( S_L \ll (\log T)^{-1} \max\{UH,M\} \ll M/(\log T) \) (with the final inequality holding by virtue of it being the case that the condition (13.4) is satisfied). Since (13.30) holds, and since we assume that \( t \geq (2\pi)^3 \), we have also \( \log T > \log(t) - 2\log(2\pi) \geq (1/3) \log(t) \), and so may deduce from the bound just obtained for \( S_L \) that one has \( |S_L|/M \ll 1/\log t \). By this and (13.51), we obtain the desired estimate (13.47).

The proof of Theorem 4. We put \( c = \frac{1515}{3} + \frac{\pi}{2} \). It will suffice to consider only cases in which \( \varepsilon \) lies in the interval \((0, 1/28)\) (say), and so we may certainly assume that the constant \( c \) satisfies the inequalities in (13.38).

Suppose now that \( T \) satisfies \( T > (8/(1-3c))^{12/(1-3c)} \) (for example). We then put \( \Delta = T^{-c}\log T \). By (13.38) and our supposition concerning \( T \) it follows that we have \( 1 \leq \Delta \leq (T/4)^{1/3} \), and so, as an immediate corollary of the estimates contained in [H&H90, Lemma 8.1], we find that one has

\[
E(T) - E(T/2) = O\left(\Delta(\log T)^2 \right) + G(T) - G(T/2),
\]

(13.52)

where \( G(t) \) denotes a certain sum that is defined in [H&H90, Lemma 8.1]: one can show, in particular, that

\[
G(t) = 4\text{Im}\left(G^+(t, \Delta) \right) + O(1) \quad (t \in \{T, T/2\}),
\]

(13.53)

where \( G^+(t, \Delta) \) is the sum defined in (13.40) (to show this requires essentially nothing more than the properties of complex conjugation and elementary bounds for certain of the terms occurring in the sum \( G^+(t, \Delta) \)). By our choice of \( \Delta \), the condition (13.39) is satisfied for \( t = T \), and also for \( t = T/2 \). Therefore (given that the condition (13.38) is also satisfied) we obtain from Lemma 13.4 the upper bound (13.41) for \( t = T \), and also for \( t = T/2 \), and so it follows, by (13.52) and (13.53), that we have:

\[
E(T) - E(T/2) \ll \Delta(\log T)^2 = T^{-c}(\log T)^3 \ll T^{\frac{1515}{1515 + \frac{1}{2}} + \frac{\pi}{2}}(\log T)^3 \ll T^{\frac{1515}{1515 + \frac{1}{2}} + \varepsilon}.
\]

(13.54)

From this estimate, and the formula for the sum of a geometric series, we may infer that one has

\[
E(T) - E\left(\frac{T}{2}\right) \ll T^{\frac{1515}{1515 + \varepsilon}}.
\]

(13.55)

where \( j > 0 \) is the least integer such that \( T/2^j \leq (8/(1-3c))^{12/(1-3c)} \). To complete the proof we have only to observe that one has \( E(U) \ll 1 \ll U^{\frac{1515}{1515 + \varepsilon}} \) for all \( U \) satisfying \( 1 \leq U \leq (8/(1-3c))^{12/(1-3c)} \) (this being a trivial corollary of the elementary fact that, since \( |z(1/2 + it)|^2 \geq 0 \) for all real \( t \), one must have \( |E(U)| \leq |E(V)| + 2(\log(2\pi V) + 2\gamma - 1)V \) whenever \( 1 \leq U \leq V \).)
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