FINITE DEFORMATIONS OF CFT
AND SPACETIME GEOMETRY

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Abstract

We demonstrate in detail how the space of two-dimensional quantum field theories can be parametrized by off-shell states of free closed string moving in a flat background. The dynamic equation corresponding to the condition of conformal invariance includes an infinite number of higher order terms, and we give an explicit procedure for their calculation. The symmetries corresponding to equivalence relations of theories are described. In this framework we show how to perform a nonperturbative analysis in the low-energy limit and prove that it corresponds to the Brans-Dicke theory of gravity interacting with a skew symmetric tensor field.

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1 Introduction

String theory has been formulated in an unusual way. Usually, physicists start from a description of symmetries, find a symmetrical classical action, then quantize it; i.e., define the Feynman rules. In string theory it is quite the opposite. The Feynman rules (also called Polyakov rules) are known, and physicists are trying to restore from them a classical theory and symmetries. This is important for decoupling of nonphysical states, understanding the nonperturbative structure and establishing connection with spacetime geometry. There is a belief that classical closed string states can be associated with quantum conformal field theories in two dimensions (CFT), which are usually defined as theories of the single string moving in some nontrivial spacetime background. The condition of anomaly cancellation leads to the so-called $\beta$-function equation on the background fields. The main advantage of this approach is its more or less explicit connection to spacetime geometry, and the main drawback is that it usually focuses only on massless fields. Treatment of massive fields is problematic, and, therefore, characterization of dynamical degrees of freedom is obscure. Studying of symmetries is also complicated by the fact that classically equivalent CFT may correspond to inequivalent quantum theories. In addition, it is not clear how to derive the Polyakov diagram technique in this approach.

Alternative approaches [1, 2, 3, 4, 5] are based on the operator formalism [6]. In [1, 2] a direct connection has been found between (1,1)-primary fields of arbitrary mass level and infinitesimal deformations of the Virasoro algebra representation. Later, in [3] it was shown how this work can be related to deformations of CFT in the operator formalism. However, there was a serious problem related to ambiguity of the vertex operator commutators, which prevents calculations of deformations beyond the first order and of the symmetry algebra structure constants. The reason for this problem is contact singularities of $T$-products of vertex operators inserted close to each other. In [7] a correct regularization was outlined for integrals of such $T$-products. The condition for the deformed theory to be conformally symmetrical induced a dynamic equation for off-shell vertex operator functions. This equation is nonlinear because the regularization is not conformally invariant. Primary fields represent its solutions in the linear approximation. Thus, existence of contact singularities makes, in fact, the theory nontrivial.

From our point of view, the ambiguity of vertex operator commutators is
one of principal and should not be resolved. It deals with a fact that vertex operators are not exactly operators, but some other objects (the definition will be given later in this paper). They can be related to some operators. However, such operators act between different spaces; i.e., are not automorphisms, and commutators make sense only for automorphic operators. In this approach we cannot formulate the string theory in the language of some universal algebra of symmetries. However, this does not make the theory incorrect or less attractive. Absence of an algebra is compensated by the structure relating algebraic and geometrical objects (see Section 2). In fact, what we do is the following. First we define a space $\mathcal{X}$ of 2D quantum field theories which includes all equivalence classes and can be explicitly parametrized. Then, symmetries are transformations of $\mathcal{X}$ within equivalence classes of theories. Such symmetries do not make a closed algebra and, in fact, depend on how we parametrized the theory or, more exactly, how we defined $\mathcal{X}$. Applying this approach we give an explicit procedure for calculating all the higher order terms of the equation of motion, which is a condition for the theories to be conformally invariant, and also nonperturbatively prove that low-energy deformations of CFT can be described by Brans-Dicke gravity interacting with a skew symmetric tensor field. This generalizes the analogous result [8, 9] found in the linear approximation.

Many other physicists still believe that a universal string symmetry algebra exists. They are trying to find a right theory space for which the symmetry algebra closes. It is too early to evaluate their results as they did not proceed far enough to calculate all the commutators and, therefore, cannot check whether the symmetry algebra is closed or not. We doubt that it can be done in principle, at least, without excessive number of variables. However, some global symmetry groups have been found (see, for example, [10]).

2 Axiomatic Conformal Field Theory

Here we will give an axiomatic definition of CFT together with its motivation and analysis. In particular, we will show how to introduce vertex operators and representation of the Virasoro algebra in the suggested axiomatic framework. This approach to CFT is similar to the operator formalisms [8] and [11].

Let us consider a Euclidean (1+1)-quantum field theory defined on a
tubelike world sheet $S^1 \times \mathbb{R}^1$. States of the theory in different moments of time are connected by the propagator

$$\Psi_{t_2} = S_{t_2,t_1} \Psi_{t_1}, \quad S_{t_2,t_1} = \exp[-H(t_2 - t_1)].$$

(2.1)

Here $H$ is a Hamiltonian. From a quantum-mechanical point of view, the propagator satisfying

$$S_{t_3,t_1} = S_{t_3,t_2} S_{t_2,t_1}$$

(2.2)

completely defines the theory. However, it is insufficient for world-sheet transformation properties of the theory. In order to define such properties we have to assign states to any closed oriented contour $\Gamma$ in $\Sigma$ (not necessarily of a fixed time). As there is no preferred parametrization for such contours, the states associated with differently parametrized contours should be related by some representation of the contour transformation group. Spaces $\mathcal{H}_\Gamma$ associated with different contours $\Gamma$ cannot be covariantly identified as they are spaces of representation of different transformation groups and, therefore, should be considered independent. Instead of identifying spaces $\mathcal{H}_{\Gamma_1}, \mathcal{H}_{\Gamma_2}$ we should covariantly assign a unitarian operator between them,

$$\mathcal{H}_{\Gamma_1} \xrightarrow{\hat{\nu}} \mathcal{H}_{\Gamma_2},$$

(2.3)

to each even (keeping orientation) isomorphic map between contours $\Gamma_1 \xrightarrow{\nu} \Gamma_2$. It must be done in such a way that a superposition of two maps $\nu = \nu_1 \circ \nu_2$ would be represented by an operator product $\hat{\nu} = \hat{\nu}_1 \hat{\nu}_2$. This structure is called a functor. Thus, we can formulate the following statement.

**Axiom 1** There is defined a functor $\mathcal{H}$ from the category of oriented closed contours to the category of Hilbert spaces.

We can naturally extend this functor to include multicomponent contours, which will correspond to multiparticle states.

**Axiom 2** If the contour $\Gamma$ consists of $N$ components $\Gamma_1, \ldots, \Gamma_N$, then

$$\mathcal{H}_\Gamma = \bigotimes_{i=1}^{N} \mathcal{H}_{\Gamma_i}.$$
Let $\Gamma_{\text{in}}$, $\Gamma_{\text{out}}$ be two nonintersecting contours encircling the world sheet counterclockwise with $\Gamma_{\text{out}}$ following $\Gamma_{\text{in}}$ in time. Then, the states in $\mathcal{H}^{\Gamma_{\text{out}}}$ should be expressed through the states in $\mathcal{H}^{\Gamma_{\text{in}}}$ by means of a linear map generalizing the propagator (2.1). It makes us treat contours $\Gamma_{\text{in}}$, $\Gamma_{\text{out}}$ differently, which violates the covariance of the formalism under time-inverting transformations. It can be avoided, if we require the following:

**Axiom 3** The spaces corresponding to the contrary-oriented contours are conjugated.

Then the propagator can be considered as an element of $\mathcal{H}^{0\Sigma}$, where $\Sigma$ is part of the world sheet enclosed between the contours.

Now let the world sheet be endowed with a conformal structure. For oriented two-dimensional surfaces it is the same as a complex structure. The propagator of a conformally symmetrical theory must be conformally invariant. Therefore, to define a propagator, we only need to know the conformal structure of the part of the word tube corresponding to the process and do not need to refer to its particular position in the tubelike world-sheet. Moreover, using the sewing procedure (see Axiom 7 below) we can assign an element of $\mathcal{H}^{0\Sigma}$ to Riemann surfaces which cannot be mapped to the tube. They may have an arbitrary number of handles and boundary components. If the boundary of $\Sigma$ consists of more then two components, it will be a scattering amplitude with the number of handles equal to the number of the diagram loops. A Riemann surface $D$ of the disk topology (no handles, one boundary component) can be considered as the compactified past (the part of the world-sheet consisting of points with $t \leq t_0$) and we can assign to it a vacuum state. Altogether, it can be formulated in the following way.

**Axiom 4** To any bordered Riemann surface $\Sigma$ there corresponds in a conformal invariant way a specific element $\langle 0 \rangle_\Sigma$ of $\mathcal{H}^{0\Sigma}$.

A multicomponent surface represents a set of independent scattering processes. The corresponding amplitude is a product of amplitudes of these processes.

**Axiom 5** If the surface $\Sigma$ consists of $N$ components $\Sigma_1 \ldots \Sigma_N$, then

$$\langle 0 \rangle_{\Sigma} = \bigotimes_{i=1}^{N} \langle 0 \rangle_{\Sigma_i}.$$
To the anticonformal bijection between Riemann surfaces there corresponds an odd map between their boundaries and, therefore, according to Axiom 3 an antilinear operator between associated spaces. For a CP-invariant theory, the amplitude must be invariant under such antilinear operators. This requirement can be formulated as follows.

**Axiom 6**  
Amplitudes corresponding to Riemann surfaces with conjugated complex structure are conjugated to each other.

The propagator must establish a transitive relation between state spaces associated with different moments of time, and the amplitudes must be compatible with this relation. This property can be covariantly formulated as follows.

**Axiom 7 (sewing)**  
The amplitude corresponding to the surface $\Sigma$ can be expressed through the amplitude corresponding to the surface $\Sigma_\Gamma$, resulting from the cutting $\Sigma$ along the closed contour $\Gamma$, by the formula

$$\langle 0 \rangle_\Sigma = Sp_\Gamma \langle 0 \rangle_{\Sigma_\Gamma}. \tag{2.4}$$

Here $Sp_\Gamma$ is an operator contracting components of $H^{\partial \Sigma}$ corresponding to two copies of the contour $\Gamma$ with opposite orientations.

This axiom generalizes the property (2.2) of a conventional quantum mechanical propagator.

The set of axioms above can be thought of as a definition of CFT. We will show that all the important attributes of CFT, such as vertex operators and a representation of Virasoro algebra, can be derived from them. In order to describe a CFT with a central charge we should relax the axioms above, changing them to their projective analogues. Then, instead of the functor to the category of Hilbert spaces, we will have a functor to the category of projective Hilbert spaces, and amplitudes, vacuum spaces and propagators will be defined only up to a constant multiplier.

For a chirally symmetrical CFT the spaces corresponding to contrary-oriented contours can be identified. Therefore, instead of the functor from the category of oriented contours, we will have a functor from the category of nonoriented contours; i.e., maps changing contour orientation will be also
represented. Then, according to axiom 3, spaces $\mathcal{H}^\Gamma$ must be selfconjugated; i.e. they must be endowed with antilinear automorphism $C$, satisfying

$$C^2 = \chi, \quad (C\eta, \xi) = \chi(C\xi, \eta) \quad (\chi = \pm 1).$$

Here delimiters ($\ast, \ast$) denote a Hilbert product. The form $(c\ast, \ast)$ is bilinear. It is symmetric if $\chi = 1$ and skew symmetric otherwise. The skew symmetric option is available only in the case of nontrivial central charge. It can be shown that in the chiral symmetrical case, an anomaly can be ruled out from the functor which can be defined in nonprojective way. However, a constant multiplier still may appear in (2.4), which may induce nontrivial central charge in this case.

2.1 Vertex operators

We will say that an element $\langle \Psi \rangle_\Sigma$ of $\mathcal{H}^{\partial \Sigma}$ has support in a closed set $S \subset \Sigma$, if for any contour $\Gamma$ surrounding $S$ counterclockwise (or a set of contours if $S$ consists of more then one component) it can be represented as

$$\langle \Psi \rangle_\Sigma = \text{Sp}_\Gamma \langle 0 \rangle_{\Sigma_{\text{ext}}} \otimes \langle \Psi \rangle_{\Sigma_{\text{in}}}.$$  

Here $\Sigma_{\text{in}}$ and $\Sigma_{\text{ext}}$ are, respectively, the internal and external parts of $\Sigma$ divided by the contour $\Gamma$, and $\langle \Psi \rangle_{\Sigma_{\text{in}}}$ is an element of $\mathcal{H}^\Gamma$. Considering the map $\langle \Psi \rangle_\Sigma \to \langle \Psi \rangle_{\Sigma_{\text{in}}}$ as an equivalence relation, we can identify the states corresponding to different areas of the Riemann surface but having the same support. Let us denote the space resulting from such identification as $\mathcal{H}_S$. For its elements we will use the capital Greek letter and for their images in $\mathcal{H}^{\partial \Sigma}$ the same letters enclosed in angular brackets with a superscript index referring to the related Riemann surface; for example,

$$\Psi \rightarrow \langle \Psi \rangle_\Sigma \quad (\Psi \in \mathcal{H}_S, \quad \langle \Psi \rangle_\Sigma \in \mathcal{H}^{\partial \Sigma}). \quad (2.5)$$

To endow $\mathcal{H}_S$ with topology, we will call a sequence in $\mathcal{H}_S$ vanishing, if for any $\Sigma_{\text{in}}$ such as $\Sigma_{\text{in}} \setminus \partial \Sigma_{\text{in}} \supset S$ it projects to a vanishing sequence in $\mathcal{H}^{\partial \Sigma_{\text{in}}}$.

It is easy to see that $\mathcal{H}_S$ then will be a Banach space. Normally, images of propagators and, therefore, images of projections are dense in $\mathcal{H}^{\partial \Sigma}$. Thus, the difference between the spaces $\mathcal{H}_S$ and $\mathcal{H}^{\partial \Sigma}$ is, in some sense, topological. As we will see later in this section, states with support in a single point can
be interpreted as vertex operators. More exactly, it can be thought of as an exact definition of what is usually called vertex operators. The spaces \( H_z \equiv H_{\{z\}} \) (\( z \in \Sigma \)) form a bundle on \( \Sigma \), sections of which we will call vertex operator functions. To show that elements of this spaces are, indeed, vertex operators, we should first define their \( T \)-product. It can be done as follows:

\[
\langle \Psi_0 \cdots \Psi_n \rangle_{\Sigma} = \text{Sp}_{\partial \Sigma_{\text{ext}}} \bigotimes_{i=0}^n \langle \Psi_i \rangle_{D_i} \otimes \langle 0 \rangle_{\Sigma_{\text{ext}}} \quad (\Psi_i \in H_{z_i}).
\]  

(2.6)

Here \( D_i \) (\( i = 0, N \)) are subsets of \( \Sigma \) such as

\[
z_i \in D_i, \quad D_i \bigcap D_j = \emptyset \quad (i \neq j),
\]

and \( \Sigma_{\text{ext}} \) is their complement in \( \Sigma \). This \( T \)-product is a multilinear map from \( H_{z_0} \times \cdots \times H_{z_N} \) to \( H_{\partial \Sigma} \). Acting analogously to (2.6), we can put into correspondence to element of \( H_z \) operators from \( H_S \) to \( H_{S'} \) if \( z \in (S' \setminus \partial S') \setminus S \). Note that \( S \subset S' \). Therefore, such operators are never automorphisms, and their commutators should not be defined.

The Virasoro algebra does not have a bounded natural representation in \( H_{\partial \Sigma_{\text{in}}} \). The conformal transformations deform the boundary of \( \Sigma_{\text{in}} \) and, therefore, corresponding to them linear operators are not automorphisms, as they act between Hilbert spaces associated with different contours. However, we can define such a representation in the space \( H_z \), which is independent of the position of the boundary.

2.2 Local multipliers

Let \( \Gamma \) be a closed contour, which divides \( \Sigma \) into \( \Sigma_1 \) and \( \Sigma_2 \). We will say that an operator \( \mathcal{O} \) in \( \mathcal{H}^\Gamma \) has support in \( S \subset \Gamma \), if the state \( \text{Sp}_\Gamma \langle 0 \rangle_{\Sigma_2} \otimes \mathcal{O} \langle 0 \rangle_{\Sigma_1} \) has support in \( S \), and denote the algebra of such operators as \( \mathcal{L}_{\Gamma,S} \). We will call an invertible operator \( U : \mathcal{H}^\Gamma \longrightarrow \mathcal{H}^\Gamma \) a local multiplier, if a similarity transformation generated by \( U \) does not enlarge the support of any operator; i.e.,

\[
\forall S \in \Gamma : \quad U \mathcal{L}_{\Gamma,S} U^{-1} \subseteq \mathcal{L}_{\Gamma,S}.
\]  

(2.7)

It can be shown that the group of local multiplier does not depend on how the contour is placed on the Riemann surface and is not affected by CFT deformations. It makes this group an important tool in the description of deformed theories and symmetries between them.
2.3 Residuelike operations

We will call a linear operator from the space of functions on $\Sigma^{N+1} \setminus \Delta(\Sigma^{N+1})$ to the space of functions on $\Sigma$ a residuelike operator of rank $N$, if the value of its image function at the point $z$ in $\Sigma$ is determined by the behavior of its argument function in an arbitrarily small environment of the point in $\Sigma^{N+1}$ with all the coordinates equal to $z$. We will denote such operations in one of the following ways

$$G(z_0) = \mathcal{R}_{z_N = \ldots = z_0} F(z_1, \ldots, z_0)$$

or

$$G(z_0) = \mathcal{R}_{\bar{z}_N = \ldots = \bar{z}_0} F(z_1, \ldots, z_0).$$

The zero rank residuelike operations are simply local operators in the space of functions, for example, differential operators. The conventional residue is an example of a rank 1 residuelike operation defined for holomorphic functions.

Using $T$-product (2.6) we can define representation of residuelike operations by multilinear products in the space of vertex operator functions:

$$\{\Psi_i\}_{i=1}^N \rightarrow \Upsilon = \mathcal{R}_{z_N = \ldots = z_0} \Psi_0 \cdots \Psi_N,$$

$$\langle \Upsilon(z_0) \rangle^\Sigma \overset{\text{def}}{=} \mathcal{R}_{z_N = \ldots = z_0} \langle \Psi_0 \cdots \Psi_N \rangle^\Sigma. \quad (2.8)$$

This representation applied to the differential operators $\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}$ produces a flat nonintegrable connection for the vertex operator bundle.

2.4 Energy-momentum tensor

An infinitesimal deformation of the conformal structure can be given as

$$\delta(\bar{z}) = \varepsilon_{\bar{z}}^z dz, \quad \delta(z) = \varepsilon_{z}^z d\bar{z},$$

where $\varepsilon_{\bar{z}}^z, \varepsilon_{z}^z$ are so-called Beltrami differentials. Applying the sewing property (axiom [4]), one can show that infinitesimal deformation of the amplitudes

\footnote{The symbol $\Delta(\Sigma^{N+1})$ denotes diagonal subset of Dekart product $\Sigma^{N+1}$, elements of which contain at least one pair of equal points.}

\footnote{The symbols $\partial_z, \partial_{\bar{z}}$ denote holomorphic and antiholomorphic derivatives. If the variable of differentiation should not be specified, we will also denote them as $\partial, \bar{\partial}$.}
are additive, i.e.,
\[ \delta(0)_\Sigma = \frac{1}{\pi} \int_\Sigma (\varepsilon^z \langle T_{zz} \rangle_\Sigma + \varepsilon^z \langle T_{\bar{z}\bar{z}} \rangle_\Sigma) d^2z. \]

Here \( T_{zz}, T_{\bar{z}\bar{z}} \) are some vertex operator functions. The Beltrami differentials
\[ \varepsilon^z = \partial_z v^z, \quad \varepsilon^{\bar{z}} = \partial_{\bar{z}} v^{\bar{z}} \]
can be produced by the infinitesimal world sheet transformation
\[ \delta z = v^z, \quad \delta \bar{z} = v^{\bar{z}}. \]

Therefore, the corresponding deformation of amplitude must be trivial; i.e.,
\[
0 = \frac{1}{\pi} \int_\Sigma (\partial_z v^z \langle T_{zz} \rangle_\Sigma + \partial_{\bar{z}} v^{\bar{z}} \langle T_{\bar{z}\bar{z}} \rangle_\Sigma) d^2z
\]
\[
= -\frac{1}{\pi} \int_\Sigma (v^z \partial_z v^z \langle T_{zz} \rangle_\Sigma + v^{\bar{z}} \partial_{\bar{z}} v^{\bar{z}} \langle T_{\bar{z}\bar{z}} \rangle_\Sigma) d^2z
\]
\[
= -\frac{1}{\pi} \int_\Sigma (v^z \partial_z T_{zz} \langle T_{zz} \rangle_\Sigma + v^{\bar{z}} \partial_{\bar{z}} T_{\bar{z}\bar{z}} \langle T_{\bar{z}\bar{z}} \rangle_\Sigma) d^2z.
\]

This condition will be satisfied for any \( v^z, v^{\bar{z}} \) only if
\[ \partial_z T_{zz} = \partial_{\bar{z}} T_{\bar{z}\bar{z}} = 0. \quad (2.9) \]

It can be shown that \( T_{zz}, T_{\bar{z}\bar{z}} \) are left and right components of the conformal energy-momentum tensor; i.e., the representation of the left and right Virasoro algebra in \( H_{z_0} \) can be expressed through them
\[ L_{v^z} \Upsilon(z_0) = \text{Res}_{z=z_0} v^z T_{zz}(z) \Upsilon(z_0), \]
\[ \bar{L}_{v^{\bar{z}}} \Upsilon(z_0) = \text{Res}_{\bar{z}=\bar{z}_0} v^{\bar{z}} T_{\bar{z}\bar{z}}(z) \Upsilon(z_0). \quad (2.10) \]

Here \( v^z \) and \( v^{\bar{z}} \) are, respectively, holomorphic and antiholomorphic tangent fields, and \( L_{v^z} \) and \( \bar{L}_{v^{\bar{z}}} \) are the corresponding generators of the representation. For generators corresponding to the basis elements of the Virasoro algebra
\[ v^z_n = (z - z_0)^{n+1}, \quad v^{\bar{z}}_n = (\bar{z} - \bar{z}_0)^{n+1} \quad (n \in \mathbb{Z}) \]
we will use the canonical notations
\[ L_n \equiv L_{v^z_n}, \quad \bar{L}_n \equiv \bar{L}_{v^{\bar{z}}_n}. \]
3 Infinitesimal Deformation of CFT

As is known, the closed string states can be described as elements of the space of the Virasoro representation $H_{z_0}$ satisfying the set of equations

$$(L_k + \delta_{k,1})\Psi(z_0) = (\bar{L}_k + \delta_{k,0})\Psi(z_0) = 0 \quad (k \geq 0).$$

(3.1)

Using parallel transportation $z \rightarrow z + c$ we can transform $\Psi(z_0) \in H_{z_0}$ to the state $\Psi(z_0 + c) \in H_{z_0+c}$ satisfying the analogous set of equations. In this way, we can define $\Psi$ as a translationally invariant vertex operator function. The formula (3.1) together with the condition of translation invariance

$$(d + L_{-1})\Psi = (\bar{d} + \bar{L}_{-1})\Psi = 0$$

(3.2)

can be written as

$$L_v \psi + \partial_z v^z \Psi = \bar{L}_v \psi + \partial_{\bar{z}} v^{\bar{z}} \Psi = 0 \quad (\partial_{\bar{z}} v^z = \partial_z v^{\bar{z}} = 0).$$

(3.3)

This means that $\Psi$ is a $(1,1)$-primary field; i.e., a conformally invariant vertex operator function of the conformal dimension $(1,1)$. Such primary fields parametrize infinitesimal deformations of CFT:

$$\langle 0 \rangle_{\Sigma} \longrightarrow \langle 0 \rangle_{\Sigma} + \delta \langle 0 \rangle_{\Sigma}, \quad \delta \langle 0 \rangle_{\Sigma} = \frac{1}{\pi} \int_{\Sigma} \langle \Psi \rangle_{\Sigma} d^2z.$$

(3.4)

The conformal invariance of the amplitudes deformed in this way will be retained if we deform the Virasoro representation as

$$L_v \psi \rightarrow \bar{L}_v \psi + \delta L_v \psi,$$ $$\delta L_v \psi = \frac{1}{2\pi i} \oint_{\Gamma} \Psi v^z \, d\bar{z};$$ $$\bar{L}_v \bar{\psi} \rightarrow \bar{L}_v \bar{\psi} + \delta \bar{L}_v \bar{\psi},$$ $$\delta \bar{L}_v \bar{\psi} = \frac{1}{2\pi i} \oint_{\Gamma} \Psi v^{\bar{z}} \, dz.$$ (3.5)

This is equivalent to the formula for infinitesimal deformations of the Virasoro representation proposed in [1, 2].

3.1 Deformation of vertex operators

In order to understand how to proceed deformations beyond the first order we must describe the space of vertex operators of infinitesimally deformed
theory. Let us identify spaces of the vertex operators of the infinitesimally deformed and initial theories. In order to do it, we should deform the map (2.5) making its image consist of the states having point support with respect to the deformed propagator (3.4). Formally, it can be done as follows:

\[ \langle \Upsilon \rangle_\Sigma \rightarrow \langle \Upsilon \rangle_\Sigma + \delta \langle \Upsilon \rangle_\Sigma, \quad \delta \langle \Upsilon(z_0) \rangle_\Sigma = \frac{1}{\pi} \int_{\Sigma} \langle \Psi(z) \Upsilon(z_0) \rangle_\Sigma d^2z. \] (3.6)

However, in general, the integral on the right-hand side of this formula may be divergent because of the contact singularity of the $T$-product. This reflects the absence of conformally invariant connection for the vertex operator bundle on the theory space. Otherwise, all deformed theories would be equivalent to the initial one. If we use in (3.6) the simple cutoff regularization

\[ \delta_R \langle \Upsilon(z_0) \rangle_\Sigma = \frac{1}{\pi} \int_{\Sigma \setminus D_{z_0,R}} \langle \Psi(z) \Upsilon(z_0) \rangle_\Sigma d^2z \]

\[ (D_{z_0,R} = \{ z \in \Sigma, |z - z_0| \leq R \}), \]

the deformed vertex operators will have support in the disk $D_{z_0,R}$ rather than in the single point $z_0$. States with support in the disk can be also produced by an average of the cutoff regularizations with smaller radii

\[ \delta \langle \Upsilon(z_0) \rangle_\Sigma = \int_0^\infty \delta_r \langle \Upsilon(z_0) \rangle_\Sigma d\mu(r). \] (3.7)

Here $d\mu$ is a generalized measure in $\mathbb{R}_+$ having support in $[0, R]$ and integrable in a product with all the functions having a finite degree singularity at $r = 0$. Diminishing support of the measure we can diminish the support of the state. If we use the measure with support in zero, states will have one-point support. Such a measure exists and can be defined by the formula

\[ \int_0^\infty r^{2\alpha} \mu(r) \, dr = \Lambda(\alpha) \quad (\alpha \in \mathbb{R}). \] (3.8)

Here, $\Lambda$ is a smooth function on $\mathbb{R}$ satisfying

\[ \Lambda(0) = 1, \quad \Lambda(\alpha) = 0 \quad (\alpha > A) \] (3.9)

for some positive $A$. A related proposal for regularization corresponding to specific stepfunction $\Lambda = \Theta(-\alpha)d^\alpha$ was independently made in [3]. Discontinuity of this $\Lambda$ at $r = 0$ creates ambiguity for calculation, especially in the low-energy limit. For convenience we will impose on $\Lambda$ the condition

\[ \Lambda(k) = 0 \quad (k \in \mathbb{Z}, k \geq 1). \]
Then the regularization will not affect integrals of regular functions. The deformation (3.6) of embedding (2.5) induces an analogous deformation of the $T$-product (2.6):

$$
\delta \langle \Upsilon_0(z_0) \cdots \Upsilon_n(z_n) \rangle_\Sigma = \frac{1}{\pi} \int_\Sigma \langle \Psi(z) \Upsilon_0(z_0) \cdots \Upsilon_n(z_n) \rangle_\Sigma d^2z.
$$

(3.10)

Here the regularization (3.7) must be applied separately for each of the vertices.

### 3.2 Deformation of representation of residuelike operations

The action (2.8) of residuelike operations on vertex operator functions depends on the $T$-product and, therefore, should be deformed together with it. For the infinitesimal deformation of the $T$-product (3.10), the corresponding deformation of residuelike operations will be

$$
\delta R_{z_n=\cdots=z_0} \Upsilon_0(z_0) \cdots \Upsilon_n(z_n) = R_{z_n=\cdots=z_0} \Upsilon_0(z_0) \cdots \Upsilon_n(z_n) \Psi(z).
$$

(3.11)

Here $R_{z_n=\cdots=z_0}$ is a next rank residuelike operation defined as

$$
R_{z_n=\cdots=z_0} F = R_{z_n=\cdots=z_0} \frac{1}{\pi} \int_\Sigma F d^2z - \frac{1}{\pi} \int_\Sigma R_{z_n=\cdots=z_0} F d^2z.
$$

(3.12)

It is, indeed, a residuelike operation, because the right-hand side of (3.12) does not depend on the area of integration as far as it includes $z_0$. We will call this operation a successor of $R_{z_n=\cdots=z_0}$.

The residue theorem can be generalized for contour integrals of nonholomorphic functions having a finite number of point singularities

$$
\frac{1}{2\pi i} \oint_{\partial \Sigma} G \, dz = \frac{1}{\pi} \int_\Sigma \partial_z G \, d^2z + \sum_{i=1}^{n} \text{Res}_{z=z_i} G.
$$

(3.13)

Here $\text{Res}_{z=z_i}$ is a generalized residue operation defined for nonholomorphic functions as an average of contour integrals around the circles $C_{z_i,r} = \partial D_{z_i,r}$ with the measure (3.8)

$$
\text{Res}_{z=z_i} F \overset{\text{def}}{=} \int_0^{\infty} d\mu(r) \frac{1}{2\pi i} \oint_{C_{z_i,r}} F \, dz.
$$

(3.14)
More explicitly, this residue can be given as

\[ \text{Res}_{z=z_0} \frac{g(z)}{|z-z_0|^{2\alpha}} = \sum_{i=0}^{\infty} \frac{\Lambda(i-\alpha)}{i!(i+1)!} \partial_z^i \partial_{\bar{z}}^{i+1} g_{|z=z_0} \quad (\alpha \in \mathbb{R}). \] (3.15)

Here \( g \) is a function defined and regular in some environment of \( z_0 \). Note that the sum on the right-hand side of (3.15) always has a finite number of nontrivial terms due to the property (3.9) of \( \Lambda \). Using the fact that a contour integral of a full differential is trivial, it can be shown that

\[ \text{Res}_{z=z_0} \partial_z F = \text{Res}_{\bar{z}=z_0} \partial_{\bar{z}} F. \] (3.16)

Applying consequently (3.13), we can rewrite the first term on the right-hand side of (3.12) as

\[ \mathcal{R}_{z_n=\cdots=z_0} \frac{1}{\pi} \int_{\Sigma} F \, d^{2}z = \mathcal{R}_{z_n=\cdots=z_0} \left( \frac{1}{2\pi i} \oint_{\partial \Sigma} \partial_{\bar{z}}^{-1} F \, dz - \sum_{i=0}^{N} \text{Res}_{z=z_i} \partial_{\bar{z}}^{-1} F \right) \]

\[ = \frac{1}{2\pi i} \oint_{\partial \Sigma} \mathcal{R}_{z_n=\cdots=z_0} \partial_{\bar{z}}^{-1} F \, dz \]

\[ - \sum_{i=0}^{N} \mathcal{R}_{z_n=\cdots=z_0} \text{Res}_{z=z_i} \partial_{\bar{z}}^{-1} F \]

\[ = \frac{1}{\pi} \int_{\Sigma} \mathcal{R}_{z_n=\cdots=z_0} F \, d^{2}z + \text{Res}_{z=z_0} \mathcal{R}_{z_n=\cdots=z_0} \partial_{\bar{z}}^{-1} F \]

\[ - \sum_{i=1}^{N} \mathcal{R}_{z_n=\cdots=z_0} \text{Res}_{z=z_i} \partial_{\bar{z}}^{-1} F \]

and use for successors the formula

\[ \mathcal{R}_{z_n=\cdots=z_0} F = \text{Res}_{z=z_0} \mathcal{R}_{z_n=\cdots=z_0} \partial_{\bar{z}}^{-1} F - \sum_{i=0}^{N} \mathcal{R}_{z_n=\cdots=z_0} \text{Res}_{z=z_i} \partial_{\bar{z}}^{-1} F. \] (3.17)

Here \( \partial_{\bar{z}}^{-1} F \) is a function on \( \Sigma^{n+2} \setminus \Delta(\Sigma^{n+2}) \) such as \( \partial_{\bar{z}} \partial_{\bar{z}}^{-1} F = F \). In fact, a function \( \partial_{\bar{z}}^{-1} F \) defined in some open environment of the main diagonal \( z = z_n = \cdots z_0 \) can be used in (3.17) as well. We can add to \( \partial_{\bar{z}}^{-1} F \) any function meromorphic with respect to \( z \), and it will not affect the right-hand side of this formula. Let us apply (3.17) to calculate a successor of the
antiholomorphic derivative. Then taking into account translation invariance of the generalized residue we will have
\[
\partial_{\bar{z} = z_0} F = \text{Res}_{z = z_0} \bar{\partial}_\bar{z} \bar{\partial}_z^{-1} F - \partial_{\bar{z} = z_0} \text{Res}_{\bar{z} = \bar{z}_0} \bar{\partial}_\bar{z}^{-1} F \\
= \text{Res}_{z = z_0} \partial_{\bar{z} = z_0} \bar{\partial}_\bar{z}^{-1} F - \text{Res}_{\bar{z} = \bar{z}_0} (\partial_{\bar{z}_0} + \partial_{\bar{z}}) \partial_{\bar{z}}^{-1} F \\
= -\text{Res}_{z = z_0} F.
\]
(3.18)

Therefore, the antiholomorphic derivative is deformed as
\[
\delta \partial_{\bar{z}} \Upsilon = -\text{Res}_{z' = z} \Psi(z') \Upsilon(z).
\]
(3.19)

In the case of the cutoff regularization, this result can be easily understood as an effect of the deformation of the integration area. A more detailed technique of the successor calculation can be found in Appendix A.

4 Finite Deformations

Let us use, for deformed amplitudes the following formula
\[
\langle 0 \rangle^\Psi_S = \left\langle \exp \frac{1}{\pi} \int_S \Psi d^2z \right\rangle_S.
\]
(4.1)

Here \( \Psi \) is some vertex operator function (not necessarily primary), and the contact divergences are regularized by the method (3.7). Then the sewing property (axiom 7) will be automatically satisfied, and only the condition of conformal invariance will remain to be implemented. Here and afterward we mark all the deformed objects with the superscript symbol of the vertex operator function parametrizing the deformation. We will identify vertex operators of the deformed and initial theories, by means of the formula
\[
\langle \Upsilon(z_0) \rangle_S^\Psi = \left\langle \Upsilon(z_0) \exp \frac{1}{\pi} \int_S \Psi(z) d^2z \right\rangle_S.
\]
(4.2)

Then the T-product (2.6) for the deformed theory will be
\[
\langle \Upsilon_0(z_0) \cdots \Upsilon_n(z_n) \rangle_S^\Psi = \left\langle \Upsilon_0(z_0) \cdots \Upsilon_n(z_n) \exp \frac{1}{\pi} \int_S \Psi(z) d^2z \right\rangle_S.
\]
(4.3)
The formulas (4.1), (4.2) and (4.3) generalize the analogous formulas (3.4), (3.6) and (3.10) for infinitesimal deformations.

Let us consider a family of deformed theories associated with a scale \( \tau \) vertex operator function \( \tau \Psi (\tau) \in \mathbb{R} \). It is easy to see that the derivative of the corresponding \( T \)-product (4.3) with respect to \( \tau \) can be given as

\[
\frac{d}{d\tau} \langle 0 \rangle \tau \Psi \Sigma = \frac{1}{\pi} \int_\Sigma \langle \Psi(z) \rangle \tau \Psi \Sigma d^2z. \quad (4.4)
\]

Therefore, we can use the formulas (3.11) to calculate derivatives and higher derivatives of the deformed residuelike operations with respect to \( \tau \). Substituting them to the Taylor expansion

\[
R^\Psi = \sum_{i=0}^{\infty} \frac{1}{i!} \left( \frac{\partial}{\partial \tau} \right)^i R^\tau \Psi \bigg|_{\tau=0},
\]

we will come to the following perturbative formula for the finite deformation:

\[
R^\Psi_{z_0, \ldots, z_n} \Psi_0 (z_0) \cdots \Psi_n (z_n) = \sum_{i=0}^{\infty} \frac{1}{i!} R^{z_0, \ldots, z_i, \bar{z}_1, \ldots, \bar{z}_i} \Psi_0 (z_0) \cdots \Psi_n (z_n) \times \Psi(z_{n+1}) \cdots \Psi(z_{n+i}). \quad (4.5)
\]

In particular, for the deformed holomorphic and antiholomorphic differentials we will have

\[
\partial^\Psi z \Psi = \partial_z \Psi - \sum_{i=1}^{\infty} \frac{1}{i!} \text{Res}_{\bar{z}_1, \ldots, \bar{z}_i} \Psi(z_1) \cdots \Psi(z_i),
\]

\[
\partial^\Psi \bar{z} \Psi = \partial_{\bar{z}} \Psi - \sum_{i=1}^{\infty} \frac{1}{i!} \text{Res}_{z_1, \ldots, z_i} \Psi(z_{n+1}) \cdots \Psi(z_{n+i}). \quad (4.6)
\]

Note that their commutator is kept to be trivial

\[
[\partial^\Psi z, \partial^\Psi \bar{z}] = 0.
\]

Therefore, this also can be used as a method to construct the Lax pair.

After we regularized the contact divergences in the \( T \)-exponent (4.3), the boundary divergence still remained and even became stronger in the higher orders of the approximation. Of course, we can regularize the boundary divergence analogously as we did the contact divergences, but then it is difficult
to implement the condition of conformal invariance and, what is more im-
portant, it becomes impossible to satisfy it. Conformally invariant in-
finitesimal deformations corresponding to primary fields, with the only ex-
ceptions of some specific cases, cannot be corrected to restore conformal invariance even in the second approximation. However, the condition of conformal invari-
ance with respect to the fixed functor is, in fact, too strong, as the functor itself may be deformed. If the boundary regularization is not specified, the field $\Psi$ defines amplitudes only modulo the group of local multipliers (2.7). Therefore, the condition of conformal invariance can be applied only modulo this group, i.e., modulo boundary term.

Let us fix regularization for one of the boundary components of some
specific Riemann surfaces. Then we can resolve the ambiguity of amplitudes
for all the rest of Riemann surfaces requiring the sewing property (axiom
[1]) to be satisfied. The ambiguity will still remain if the deformed theory has symmetries. Otherwise, all the amplitudes will be projectively defined. Then, we can define the deformed functor, requiring that the operators (2.3) representing maps between contours are deformed by means of a product with
local multipliers (2.7), thus making the deformed amplitudes conformally invariant. The energy-momentum tensor for such a deformed functor can be shown to be

$$T^{\Phi}_{z\bar{z}} = T_{z\bar{z}} + \Phi_{z\bar{z}}, \quad T^{\Phi}_{\bar{z}z} = T_{\bar{z}z} + \Phi_{\bar{z}z}. \quad (4.7)$$

Here $\Phi_{z\bar{z}}, \Phi_{\bar{z}z}$ are some normalizable vertex operator functions. Note, that the components of the energy-momentum tensor themselves are not normal-
izable. The parameters $\Phi_{z\bar{z}}, \Phi_{\bar{z}z}$ depend on the deformation of the functor and the method of regularization.

Applying (2.9) to the deformed energy-momentum tensor (4.7)

$$\partial^\Psi_z T^{\Phi}_{z\bar{z}} = \partial^\Psi_{\bar{z}} T^{\Phi}_{\bar{z}z} = 0 \quad (4.8)$$

and using the formula (1.6) for the deformed derivatives, we will come to an
equation on $\Psi, \Phi_{z\bar{z}}, \Phi_{\bar{z}z}$. These equations and vertex operator functions $\Psi, \Phi_{z\bar{z}}, \Phi_{\bar{z}z}$ can be interpreted as equations of motion and dynamic fields. The existence of $\Phi_{z\bar{z}}$ and $\Phi_{\bar{z}z}$ satisfying this equation is, in fact, a criteria for the
deformed theory to be conformally symmetrical.

We can calculate the deformed representation of the Virasoro algebra
substituting the deformed energy-momentum tensor (4.7) and the deformed
residue (4.3) to the formula (2.10):

\[
L^\psi_\nu \Upsilon(z_0) = \text{Res} z = z_0 v^z(z) T^\Phi_{z z}(z) \Upsilon(z_0) = \sum_{i=0}^{\infty} \frac{1}{i!} \text{Res} z = z_0 v^z(z) T^\Phi_{z z}(z) \Upsilon(z_0) \Psi(z_i) \cdots \Psi(z_1),
\]

\[
\bar{L}^\psi_\nu \Upsilon(z_0) = \text{Res} z = z_0 v^{\bar{z}}(z) T^\Phi_{\bar{z} \bar{z}}(z) \Upsilon(z_0) = \sum_{i=0}^{\infty} \frac{1}{i!} \text{Res} z = z_0 v^{\bar{z}}(z) T^\Phi_{\bar{z} \bar{z}}(z) \Upsilon(z_0) \Psi(z_i) \cdots \Psi(z_1).
\]

The commutative relations for the deformed Virasoro operators

\[
[L^\psi_n, L^\psi_k] = (k - n) L^\psi_{k+n} + \frac{D}{12} \delta_{k+n,0} n(n^2 - 1),
\]

\[
[\bar{L}^\psi_n, \bar{L}^\psi_k] = (k - n) \bar{L}^\psi_{k+n} + \frac{D}{12} \delta_{k+n,0} n(n^2 - 1)
\]

\[
[L^\psi_n, \bar{L}^\psi_k] = 0
\]

are equivalent to

\[
0 = \text{Res} z = z' z' T^\Phi_{z z}(z') T^\Phi_{z z}(z)(z' - z)^k + \delta_{k,0} \partial_z T^\Phi_{z z} + \delta_{k,1} T^\Phi_{z z} - \frac{D}{2} \delta_{k,3}
\]

\[
= \text{Res} \bar{z} = \bar{z}' \bar{z}' T^\Phi_{\bar{z} \bar{z}}(\bar{z}') T^\Phi_{\bar{z} \bar{z}}(\bar{z})(\bar{z}' - \bar{z})^k + \delta_{k,0} \partial_{\bar{z}} T^\Phi_{\bar{z} \bar{z}} + \delta_{k,1} T^\Phi_{\bar{z} \bar{z}} - \frac{D}{2} \delta_{k,3}
\]

\[
= \text{Res} \bar{z} = \bar{z}' \bar{z}' T^\Phi_{\bar{z} \bar{z}}(\bar{z}') T^\Phi_{\bar{z} \bar{z}}(\bar{z})(\bar{z}' - \bar{z})^k = \text{Res} z = z T^\Phi_{z z}(z) T^\Phi_{z z}(z)(z' - z)^k (k \geq 0).
\]

Let us show that these equations are satisfied. Indeed, the vertex operator functions in their left parts are (anti)holomorphic, and, as a consequence of (4.7), normalizable. Nontrivial normalizable (anti)holomorphic modes correspond to symmetries of the deformed theory, which form a Kac-Moody algebra, as in the Wess-Zumino-Witten model. Presence of such symmetries is a specific, in some sense, degenerate case. In the usual situation holomorphic modes do not exist and, therefore, the equations are automatically satisfied. For symmetric phase we can satisfy them adding to \(T^\Phi_{z z}\) some holomorphic vertex operator function.
4.1 Symmetries

The vertex operator functions $\Psi, \Psi'$ parametrize equivalent theories if corresponding amplitudes \((4.1)\) are similar modulo the group of local multipliers \((2.7)\), or, what is the same, modulo the boundary term

$$\langle 0 \rangle_{\Sigma}^{\Psi'} \sim \langle 0 \rangle_{\Sigma}^{\Psi}.$$  

The similarity transformations of amplitudes induce a covariant transformation of vertex operators

$$\Upsilon \rightarrow \Upsilon' : \langle \Upsilon' \rangle_{\Sigma}^{\Psi'} \sim \langle \Upsilon \rangle_{\Sigma}^{\Psi}.$$  

Infinitesimal symmetry transformations of the initial theory can be described as

$$\Psi \rightarrow \Psi + \delta_{\xi} \Psi, \quad \Upsilon \rightarrow \Upsilon + \hat{\xi} \Upsilon :$$

$$\hat{\xi} \Upsilon(z_0) = \text{Res}_{z = z_0} \xi_z(z) \Upsilon(z_0) + \text{Res}_{z = z_0} \xi_{\bar{z}}(z) \Upsilon(z_0),$$

$$\delta_{\xi} \Psi = \partial_{z} \xi_{z} + \partial_{\bar{z}} \xi_{\bar{z}} \quad (\Psi = 0). \quad (4.10)$$

Here $\xi = (\xi_z, \xi_{\bar{z}})$ is a pair of vertex operator functions, parametrizing the symmetries. The increment of the $T$-exponent \((4.3)\) under the transformation \((4.10)\) in a linear approximation can be given as

$$\delta_{\xi} \left( \Upsilon(z_0) \exp \frac{1}{\pi} \int \Psi(z) \, d^2 z \right) \approx \frac{1}{\pi} \int d^2 z_1 \Psi(z_1) \text{Res}_{z = z_0} \xi_z(z_2) \Upsilon(z_0)$$

$$+ \frac{1}{2\pi} \int d^2 z_1 \int d^2 z_2 \left( \Psi(z_1) \partial_{z} \xi_{z}(z_2) + \partial_{\bar{z}} \xi_{z}(z_1) \Psi(z_2) \right) + z \leftrightarrow \bar{z}$$

$$= \frac{1}{\pi} \int d^2 z_1 \left( \frac{1}{2} \Psi(z_1) \text{Res}_{z = z_0} \xi_z(z_2) \Upsilon(z_0) - \Upsilon(z_0) \text{Res}_{z = z_1} \xi_z(z_2) \Psi(z_1) \right)$$

$$- \partial_{z_1} \frac{1}{\pi} \int d^2 z_2 \left( \Upsilon(z_0) \xi_{z}(z_1) \Psi(z_2) + \Upsilon(z_0) \text{Res}_{z = z_1} \Psi(z_2) \xi_{z}(z_1) \right) + z \leftrightarrow \bar{z}$$

$$= \frac{1}{\pi} \int d^2 z_1 \left( \frac{1}{2} \Psi(z_1) \text{Res}_{z = z_0} \xi_z(z_2) \Upsilon(z_0) - \Upsilon(z_0) \text{Res}_{z = z_1} \xi_z(z_2) \Psi(z_1) \right)$$

$$\text{asym}_{\Upsilon(z_0) \xi_{z}(z_1) \Psi(z_2) \, d^2 z_2} + z \leftrightarrow \bar{z}$$

$$- \frac{1}{2} \text{Res}_{z = z_0} \int \Upsilon(z_0) \xi_{z}(z_1) \Psi(z_2) \, d^2 z_2 + z \leftrightarrow \bar{z}$$

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$$= - \int \Upsilon(z_0) \text{Res}_{z_2 = z_1} \xi_z(z_2) \Psi(z_1) \, d^2z_1 - \frac{1}{2} \text{Res}_{z_2 = z_1 = z_0} \Psi(z_2) \xi_z(z_1) \Upsilon(z_0)$$

$$+ z \leftrightarrow \bar{z}.$$

Here we used formulas (3.13) and (3.12) and disregarded the boundary term. For brevity we omitted the angular brackets symbolizing the $T$-product. This increment is trivial if $\Psi = 0$; i.e., for deformations of the initial theory. If $\Psi \neq 0$, it can be trivialized by the following first order correction to the symmetry transformation (4.10):

$$\delta_\xi \Psi(z) = \partial_\xi \xi_z(z) + \text{Res}_{z_1 = z_0} \xi_z(z) \Psi(z) + z \leftrightarrow \bar{z} + O(\Psi^2)$$

$$\hat{\xi} \Upsilon(z_0) = \text{Res}_{z = z_0} \xi_z(z) \Upsilon(z_0) + \frac{1}{2} \text{Res}_{z_2 = z_1 = z_0} \Psi(z_2) \xi_z(z_1) \Upsilon(z_0)$$

$$+ z \leftrightarrow \bar{z} + O(\Psi^2). \quad (4.11)$$

The higher order corrections to this transformation can be calculated analogously. However, we cannot write an explicit formula or give an explicit procedure for their calculations yet.

Note that the symmetries described above do not correspond directly to the similarity transformations, which also depend on $\Psi$ and the boundary regularization. Therefore, the commutator of such symmetries may be field dependent. In other words, the symmetry algebra should not be closed. In some sense, it is similar to a gauge-fixed Yang-Mills theory.

It might be asked what relationship the symmetries elucidated here have to the suggestion of Banks and Martinec [12] that renorm group redundancies may contribute to the symmetry algebra. In our formalism, such redundancies can be interpreted as a simultaneous changing of regularization parameters $\Lambda(\alpha)$ together with the vertex operator function $\Psi$ in such a way that it does not affect an equivalence class of deformed theory. In fact, using such changing regularizations symmetries may help to close the symmetry algebra. Then we could combine vertex operator function $\Psi$ and regularization parameters to unique covariant object parametrizing two-dimensional field theories. However, our attempts to do it resulted in an excessive number of auxiliary fields, that made the symmetry algebra as wide as it would be if we simply extended it with field dependent transformations.

We don’t think that it is a physical problem. It just means that a classification of physical states as elements of a quotient space for some closed
symmetry group does not work in the string theory.

4.2 Translationally invariant theories

The regularization of contact divergences (3.7) is not conformally invariant, which complicates the implementation of the conformally invariance for deformed amplitudes. However, this regularization is still translationally invariant. Therefore, if $\Psi$ is a translationally invariant vertex operator function; i.e., obeys (3.2), it will correspond to a translationally symmetrical theory. This let us reduce the space of dynamic fields to the space of translationally invariant vertex operator functions.

Then the deformed translation operators $L_{-1}^\Psi$, $\bar{L}_{-1}^\Psi$ will be always defined, whether $\Psi$ obeys the equation of motion (4.8) or not. They can be given as

$$
L_{-1}^\Psi \Upsilon(z) = \text{Res}_{z'=z} T^\Psi_{zz}(z') \Upsilon(z) + \text{Res}_{z'=z} T^\Psi_{zz\bar{z}}(z) \Upsilon(z)
$$

$$
\bar{L}_{-1}^\Psi \Upsilon(z) = \text{Res}_{z'=\bar{z}} T^\Psi_{z\bar{z}}(z') \Upsilon(z) + \text{Res}_{z'=\bar{z}} T^\Psi_{z\bar{z}\bar{z}}(z) \Upsilon(z).
$$

Here $T^\Psi_{zz\bar{z}}$ is a nonconformal energy-momentum tensor, satisfying

$$
\partial_z T^\Psi_{zz}\bar{z} + \partial_{\bar{z}} T^\Psi_{z\bar{z}} = \partial_z T^\Psi_{zz\bar{z}} + \partial_{\bar{z}} T^\Psi_{z\bar{z}\bar{z}} = 0,
$$

$$
T^\Psi_{zz\bar{z}} \simeq T_{zz\bar{z}}, \quad T^\Psi_{z\bar{z}} \simeq T_{z\bar{z}}, \quad T^\Psi_{z\bar{z}\bar{z}} \simeq T_{z\bar{z}\bar{z}} \approx 0.
$$

Here the symbol $\simeq$ indicates that the difference between expressions at its left and right is a normalizable vertex operator function.

A vertex operator function translationally invariant for initial theory remains translationally invariant for deformed theory. Therefore,

$$
\partial_z^\Psi + L_{-1}^\Psi = \partial_z + L_{-1}, \quad \partial_{\bar{z}}^\Psi + \bar{L}_{-1}^\Psi = \partial_{\bar{z}} + \bar{L}_{-1}.
$$

Using here (4.6) we will come to the following expression for the deformed translation operators:

$$
L_{-1}^\Psi \Upsilon = L_{-1} \Upsilon + \sum_{i=1}^{\infty} \frac{1}{i!} \text{Res}_{z_1=\ldots=z_i=\bar{z}} \Psi(z_1) \ldots \Psi(z_i) \Upsilon(z),
$$

$$
\bar{L}_{-1}^\Psi \Upsilon = \bar{L}_{-1} \Upsilon + \sum_{i=1}^{\infty} \frac{1}{i!} \text{Res}_{z_1=\ldots=z_i=\bar{z}} \Psi(z_1) \ldots \Psi(z_i) \Upsilon(z).
$$
Let us substitute Ψ in the first equation of (4.13) by τΨ and then differentiate it with respect to τ:

\[ 0 = -\text{Res}_{\tau z_1} \Psi(z_1)T_{\tau z_2} + \partial_{\tau z} \partial_{\tau} T_{\tau z_2} + \partial_{\tau z} \partial_{\tau} T_{\tau z_2}(z). \] (4.16)

Using translation invariance of Ψ here in the form

\[ 0 = \partial_{\tau z} \Psi(z) + \text{Res}_{\tau z_1} T_{\tau z_2}(z_1)\Psi(z) + \text{Res}_{\tau z_1} T_{\tau z_2}(z_1)\Psi(z), \]

we will come to the equation:

\[ 0 = \partial_{\tau z} \Psi(z) + \text{Res}_{\tau z_1} T_{\tau z_2}(z_1)\Psi(z) + \text{Res}_{\tau z_1} T_{\tau z_2}(z_1)\Psi(z). \]

This equation will be satisfied if

\[ \frac{\partial}{\partial \tau} T_{\tau z_2}(z) = B_{\tau z_1} \Psi(z_1)T_{\tau z_2} + A_{\tau z_1} \Psi(z_1)T_{\tau z_2}(z), \]

\[ \frac{\partial}{\partial \tau} T_{\tau z_2}(z) = A_{\tau z_1} \Psi(z_1)T_{\tau z_2} + B_{\tau z_1} \Psi(z_1)T_{\tau z_2}(z). \] (4.17)

Here \( A_{\tau z_1} \), \( B_{\tau z_1} \) are residuelike operations satisfying

\[ \partial_{\tau} A_{\tau z_1} + \partial_{\tau} B_{\tau z_1} = \text{Res}_{\tau z_1} + \text{Res}_{z_2 z_1}. \] (4.18)

Such residuelike operations exist and can be defined as

\[ A_{\tau z_1} = \sum_{i=0}^{\infty} \frac{2l-i-1}{i!(i+k)!} \Lambda \left( i-k \right) \partial_{\tau}^{i-k-1} F(z) \quad (k \notin 2\mathbb{Z}, \ k \geq 1) \]

\[ 0, \quad \text{otherwise}, \]

\[ B_{\tau z_1} = \sum_{i=0}^{\infty} \frac{2l-i-1}{i!(i-k)!} \Lambda \left( i-k \right) \partial_{\tau}^{i-k-1} F \quad (k \notin 2\mathbb{Z}, \ k \leq -1) \]

\[ 0, \quad \text{otherwise}. \]

\[ G = F \left( \frac{z + z_1}{2} \right) |z_1 - z|^{-2\alpha} (z_1 - z)^{-k-1} \] (4.19)
Differentiating (4.17) with respect to $\tau$ and using initial conditions

$$\tau = 0 : \quad T^{\psi}_{zz} = T^{zz}, \quad T^{\psi}_{z\bar{z}} = 0,$$

we can recurrently calculate all the higher derivatives of the energy-momentum tensor and then substitute them to the Taylor expansion:

$$T^{\psi} = \sum_{i=0}^{\infty} \frac{1}{i!} \left( \frac{\partial}{\partial \tau} \right)^{i} T^{\psi}_{\tau=0}.$$

For example, in the second order approximation we have

$$T^{\psi}_{zz}(z) = T^{zz}_{zz}(z) + B_{z_{1}=z_{1}=z} \Psi(z_{1}) T^{zz}_{zz}(z)$$
$$+ \frac{1}{2} B_{z_{2}=z_{2}=z} \Psi(z_{2}) \Psi(z_{1}) T^{zz}_{zz}(z)$$
$$+ \frac{1}{2} A_{z_{2}=\bar{z}} A_{z_{1}=z} \Psi(z_{2}) \Psi(z_{1}) T^{zz}_{zz}(z)$$
$$+ \frac{1}{2} A_{\bar{z}_{1}=\bar{z}} A_{z_{1}=z} \Psi(z_{1}) \Psi(z) + 0(\Psi^{3}),$$

$$T^{\psi}_{z\bar{z}}(z) = \Psi(z) + A_{z_{1}=z} \Psi(z_{1}) T^{zz}_{zz}(z)$$
$$+ \frac{1}{2} A_{z_{2}=z_{1}=z} \Psi(z_{2}) \Psi(z_{1}) T^{zz}_{zz}(z)$$
$$+ \frac{1}{2} B_{z_{2}=\bar{z}} A_{z_{2}=z} \Psi(z_{2}) \Psi(z_{1}) T^{zz}_{zz}(z)$$
$$+ \frac{1}{2} A_{\bar{z}_{1}=z} \Psi(z_{1}) \Psi(z) + 0(\Psi^{3}).$$

The analogous formulas for $T^{\bar{z}}_{z\bar{z}}$, $T^{\bar{z}}_{zz}$ can be obtained by exchanging the symbols $z$ and $\bar{z}$.

The symmetries (4.11) remaining for translationally invariant theories can be parametrized by the translationally invariant vertex operator functions $\xi_{z}$, $\xi_{\bar{z}}$. The choice of the energy-momentum tensor is not unique, as the adding to its components full differentials of translationally invariant vertex operator functions

$$T^{\psi} \rightarrow T^{\psi, \Phi} : \quad T^{\psi, \Phi}_{zz} = T^{zz}_{zz} - \partial_{z}^{\psi} \Phi_{z}, \quad T^{\psi, \Phi}_{z\bar{z}} = T^{zz}_{z\bar{z}} + \partial_{z}^{\psi} \Phi_{\bar{z}},$$
$$T^{\psi, \Phi}_{z\bar{z}} = T^{zz}_{z\bar{z}} - \partial_{\bar{z}}^{\psi} \Phi_{z}, \quad T^{\psi, \Phi}_{\bar{z}z} = T^{zz}_{\bar{z}z} + \partial_{\bar{z}}^{\psi} \Phi_{z} \quad (4.21)$$
will not affect the corresponding translation operators (4.12). Therefore, its transformation under (4.11) may differ from covariant by such full differentials; i.e.,

$$\delta_\xi T^\Psi \equiv \frac{\delta T^\Psi}{\delta \Psi} \delta_\xi \Psi = \dot{\xi} T^\Psi - H(\xi),$$

where

$$H(\xi)_{zz} = -\partial^\Psi_z A(\xi)_z, \quad H(\xi)_{z\bar{z}} = \partial^\Psi_{\bar{z}} A(\xi)_{\bar{z}},$$

$$H(\xi)_{z\bar{z}} = -\partial^\Psi_{\bar{z}} A(\xi)_z, \quad H(\xi)_{\bar{z}z} = \partial^\Psi_z A(\xi)_{\bar{z}}.$$

However, the modified energy-momentum tensor $T^\Psi, \Phi$ will transform covariantly, if we define a transformation law for the fields $\Phi_z, \Phi_{\bar{z}}$ by the formula

$$\Phi \rightarrow \Phi + \dot{\xi} \Phi + A(\xi). \quad (4.22)$$

As the vertex operator functions $\Phi_z, \Phi_{\bar{z}}$ are also translationally invariant, we can instead of (4.21) use for the covariant energy-momentum tensor an expression

$$T^\Psi, \Phi_{zz} = T^\Psi_{zz} + L^\Psi_{-1} \Phi_z, \quad T^\Psi, \Phi_{z\bar{z}} = T^\Psi_{z\bar{z}} - \bar{L}^\Psi_{-1} \Phi_{\bar{z}},$$

$$T^\Psi, \Phi_{\bar{z}z} = T^\Psi_{\bar{z}z} + \bar{L}^\Psi_{-1} \Phi_z, \quad T^\Psi, \Phi_{\bar{z}\bar{z}} = T^\Psi_{\bar{z}\bar{z}} - L^\Psi_{-1} \Phi_{\bar{z}}. \quad (4.23)$$

If the theory is conformally symmetrical, there exists $\Phi_z, \Phi_{\bar{z}}$ trivializing the contradiagonal components of $T^\Psi, \Phi$

$$T^\Psi, \Phi_{\bar{z}z} = T^\Psi, \Phi_{z\bar{z}} = 0. \quad (4.24)$$

Then the diagonal components of $T^\Psi, \Phi$ will form a conformal energy-momentum tensor. They will be (anti)holomorphic in a consequence of (4.13). Therefore, the equation (4.24) and the translationally invariant vertex operator functions $\Psi, \Phi_z$ and $\Phi_{\bar{z}}$ can be considered as an equation of motion and dynamic fields for the closed string field theory. This equation is equivalent to (4.8) for theories without symmetries and may be a stronger requirement otherwise.

### 4.3 Linear approximation

In the terms of (4.20) corresponding to the first order deformations of the energy-momentum tensor, the residuallike operations (4.19) are applied to
the functions holomorphic with respect to $z$. By using for such functions the Loran expansion

$$F = \sum_i (z' - z)^{-i-1} \text{Res}_{z'=z} (z' - z)^i F,$$

it can be shown that

$$\mathcal{A}_{z'=z} F = \frac{1}{2} \sum_{i=1}^{\infty} \frac{1}{k!} \partial_z^{i-1} \text{Res}_{z'=z} (z' - z)^i F, \quad \mathcal{B}_{z'=z} F = 0. \quad (4.25)$$

Therefore, in the linear approximations components of the deformed energy-momentum tensor are equal to

$$T^{\Psi}_{zz} = T_{zz}, \quad T^{\Psi}_{\bar{z}\bar{z}} = \mathcal{O}_0 \Psi \quad T^{\Psi}_{\bar{z}z} = T_{\bar{z}z}, \quad T^{\Psi}_{zz} = \overline{\mathcal{O}_0} \Psi. \quad (4.26)$$

Hereafter

$$\mathcal{O}_k = \delta_{k,0} + \sum_{j=0}^{\infty} \frac{(L_{-1})^j L_{k+j}}{(k+j+1)!}, \quad \overline{\mathcal{O}_k} = \delta_{k,0} + \sum_{j=0}^{\infty} \frac{\overline{L}_{-1})^j \overline{L}_{k+j}}{(k+j+1)!}. \quad (4.27)$$

States with bounded energy obey the condition

$$L_i \Psi = 0 \quad (i \geq l), \quad (4.28)$$

where $l$ is a level of string excitation. For such states a number of nontrivial term in the sum in (4.27) is always finite.

In the linear approximation we can substitute the deformed translation operators $L_{-1}^\Psi, \overline{L}_{-1}^\Psi$ in the formula (4.23) for the covariant energy-momentum tensor by the initial ones

$$T_{zz}^{\Psi,\Phi} = \mathcal{O}_0 \Psi - \overline{L}_{-1} \Phi_z, \quad T_{\bar{z}\bar{z}}^{\Psi,\Phi} = \overline{\mathcal{O}_0} \Psi - \overline{L}_{-1} \Phi_{\bar{z}},$$

$$T_{zz}^{\Psi} = T_{zz} + L_{-1} \Phi_z, \quad T_{\bar{z}\bar{z}}^{\Psi} = T_{\bar{z}\bar{z}} + \overline{L}_{-1} \Phi_{\bar{z}}. \quad (4.29)$$

Then, the linearized equation of motion (4.24) will be

$$\mathcal{O}_0 \Psi = \overline{L}_{-1} \Phi_z, \quad \overline{\mathcal{O}_0} \Psi = L_{-1} \Phi_z. \quad (4.30)$$
Let us calculate the covariant transformation of the energy-momentum tensor under linearized symmetries (4.10)

\[ \hat{T}_{zz}(z) = \text{Res}_{z'=z} \xi_z (z') T_{zz}(z) + \text{Res}_{z'=z} \xi_{z'} (z') T_{zz}(z) + \text{Res}_{z_1=z} \sum_i (z - z')^{i-1} L_{1-i} \xi_{z'} (z') \]

\[ + \text{Res}_{z_1=z} \sum_i (z - z_1)^{i-1} L_{1-i} \xi_{z_1} (z_1) = L_{-1} O_0 \xi_z (z). \]

Analogously,

\[ \hat{T}_{\bar{z}\bar{z}} = L_{-1} \bar{O}_0 \xi_{\bar{z}}. \]

Substituting it together with (4.26) in (4.22) we will come to the following formula for the transformation of \( \Phi \) under these symmetries:

\[ \Phi_z \rightarrow \Phi_z + O_0 \xi_z, \quad \Phi_{\bar{z}} \rightarrow \Phi_{\bar{z}} + \bar{O}_0 \xi_{\bar{z}}. \]

(4.31)

Applying (A.5) in (4.9) and disregarding the higher order terms, we will come to the following formula for deformation of the left Virasoro representation in \( H_{z_0} \):

\[ \delta L_{v^z} = \text{Res}_{z=z_1=z_0} \Psi (z) v^z T_{zz}(z_1) - \text{Res}_{z=z_0} v^z \partial_z \Phi_z (z) \]

\[ = \text{Res}_{z=z_0} J[v^z]_z (z) + \text{Res}_{z=z_0} J[v^z]_{\bar{z}} (z). \]

(4.32)

Here

\[ J[v^z]_z = \Phi_z \partial_z v^z \]

and

\[ J[v^z]_{\bar{z}} = \sum_{k=1}^{\infty} \frac{(-)^k}{k!} \partial_z^{k-1} \text{Res}_{z_1=z} (z_1 - z)^k \Psi (z) v^z T_{zz}(z_1) - v^z \partial_z \Phi_z \]

\[ = - \sum_{k=1}^{\infty} \sum_{i=0}^{\infty} \frac{1}{i!k!} L_{k+i-1} \partial_z^i v^z \Psi (z) T_{zz}(z_1) - v^z \partial_z \Phi_z \]

\[ = - \sum_{i=0}^{\infty} \frac{1}{i!} \partial_z^i v^z O_i \Psi + v^z (\Psi + \bar{L}_{-1} \Phi_\bar{z}). \]
Using the linearized equation of motion (4.30), we can also write it as

$$J[v^z]_z = v^z\Psi - \sum_{i=1}^{\infty} \frac{1}{i!} O_i \partial^i_z v^z.$$  

Analogously, for the deformation of the right Virasoro representation we have

$$\delta L[v^z] = \text{Res}_{z=z_0} J[v^z]_z(z) + \text{Res}_{z\equiv z_0} J[v^z]_z(z), \quad (4.33)$$

where

$$J[v^z]_z = v^z\Psi - \sum_{i=1}^{\infty} \frac{1}{i!} \partial^i_z v^z \partial^i_z \Psi, \quad \bar{J}[v^z]_\bar{z} = \partial_\bar{z} v^z \Phi_z.$$  

Because of (4.28), the operators $J_z$, $\bar{J}_z$, $\bar{J}_\bar{z}$ act on tangent fields $v^z$, $v^{\bar{z}}$ as differential operators of the order $l$. For the constant fields $v_{z-1}^z = v_{-1}^{\bar{z}} \equiv 1$, they are

$$J_\bar{z} = \bar{J}_z = \Psi, \quad J_z = \bar{J}_\bar{z} = 0$$

and, therefore,

$$\delta L_{-1} = \text{Res}_{z=z_0} \Psi(z), \quad \delta \bar{L}_{-1} = \text{Res}_{z\equiv z_0} \Psi(z).$$

Note that the formula (4.13) taken in the linear approximations gives the same result.

According to the definition of nonholomorphic residue (3.14), the deformation (4.32), (4.33) of the Virasoro operators is an average of the deformations

$$\delta L[v^z] = \frac{1}{2\pi i} \oint \Gamma J[v^z]_z d\bar{z} + \frac{1}{2\pi i} \oint \Gamma J[v^z]_\bar{z} dz,$$

$$\delta \bar{L}[v^z] = \frac{1}{2\pi i} \oint \Gamma J[v^z]_\bar{z} dz + \frac{1}{2\pi i} \oint \Gamma J[v^z]_z d\bar{z}, \quad (4.34)$$

taken over circular contours $\Gamma$. It can be shown that solutions of (4.30) satisfy

$$L[v^z]_z = \partial_z J[v^z]_z + \partial_{\bar{z}} J[v^z]_{\bar{z}}, \quad \bar{L}[v^z]_{\bar{z}} = \partial_{\bar{z}} \bar{J}[v^z]_{\bar{z}} + \partial_z \bar{J}[v^z]_z,$$

what can be interpreted as conformal invariance of the deformed propagator (3.4) under the deformed representation of the Virasoro algebra (4.32). Such a simple interpretation of conformal invariance cannot be applied beyond the linear approximation because of the boundary divergence.
Equations (4.30) are satisfied if $\Phi$ is trivial and $\Psi$ obeys the conventional closed string equation (3.1). Then

$$J[v^z]z = v^z\Psi, \quad J[v^z]z = v^z\Psi, \quad J[v^z]z = J[v^z]z = 0,$$

and the deformation of the Virasoro representation (4.34) will be the same as (3.3). The relaxation of the equations of motion, what we have here, is compensated by the symmetries (4.10) and (4.31) and does not create additional physical degrees of freedom. Analogous to (4.32) infinitesimal deformations of the Virasoro representation corresponding to vertex operator functions, which are not primary fields, have been first found in [8, 9] in the low-energy limit.

5 Spacetime Interpretation

In this section we will show how the definition of CFT given in Section 2 deals with the conventional path integral approach and define the initial (vacuum) CFT. Then we will formulate a method for nonperturbative analysis in the low-energy limit and show how it corresponds to Brans-Dicke theory of gravity interacting with a skew symmetric tensor field.

5.1 Path integral approach

Let us denote infinite dimensional manifolds of continuous maps from the contour $\Gamma$ and the surface $\Sigma$ to the $D$-dimensional manifold $\mathcal{M}$ (spacetime) as $\mathcal{M}_\Gamma$ and $\mathcal{M}_\Sigma$, respectively. We can define $\mathcal{H}_\Gamma$ as a space of continuous functions in $\mathcal{M}_\Gamma$, endowed with the Hilbert product

$$(\Psi, \Phi) = \int_{\mathcal{M}_\Gamma} \bar{\Psi}(x^\Gamma) \Phi(x^\Gamma) d^Dx^\Gamma.$$

Then the conformally symmetrical amplitudes can be formally defined through the path integrals

$$\langle 0 \rangle_\Sigma \left[ x_{0^\Sigma} \right] = \int_{x_{i=0}^\Sigma = x_0} \exp \left( -S \left[ x^\Sigma \right] \right) dx^\Sigma. \tag{5.1}$$

Here $S$ is some conformally invariant action. However, in general, this method to construct CFT’s is not quite correct. First of all, the path integral procedure is rather ambiguous; in addition, it can violate conformal
invariance and be divergent. An attempt to describe CFT in such a way leads to the well-known $\beta$-function approach, the shortcoming of which is mentioned in the Introduction. However, in the case of flat background; i.e., for the action

$$S = \frac{1}{\pi} \int_{\Sigma} G_{\nu\mu} \partial x^\nu \bar{\partial} x^\mu \, d^2z$$

(5.2)

with constant $G_{\nu\mu}$, the path integral is Gaussian and can be explicitly calculated up to a constant multiplier. States having one-point support can be formally expressed through the path integral

$$\Phi(z_0)[x^\Gamma] = \int_{x|_{x_0}=x_0} \phi(z_0) \exp \left( -S \left[ x^{\Sigma} \right] \right) \, dx^{\Sigma}.$$  

(5.3)

In the case of linear function $\phi = x^\nu$, this integral can be reduced to Gaussian and also explicitly calculated. Corresponding vertex operators $X^\nu$ are operators of local string coordinates. For the nonlinear $\phi$ integral (5.3) is divergent. We will put into correspondence to such nonlinear functions normal ordered operators $\phi_\circ$, defined by means of the Wick formula with the Green function

$$G(x^\nu(z), x^\mu(u)) = -2G^{\nu\mu} \ln |z - u|.$$  

(5.4)

Note that such normal ordering is not conformally invariant.

5.2 Global deformations of spacetime metric

Let us consider deformations corresponding to the vertex operator function of the type

$$\Psi = H_{\nu\mu} \partial X^\nu \bar{\partial} X^\mu(z').$$  

(5.5)

Here $H_{\nu\mu}$ is a matrix with constant coefficients. CP-invariant deformed theories (theories obeying Axiom 3) correspond to Hermitian matrices

$$H_{\nu\mu} = \bar{H}_{\mu\nu}.$$  

Such matrices can be given as

$$H_{\nu\mu} = B_{\nu\mu} + iA_{\nu\mu},$$

where $B$ and $A$ are, respectively, symmetric and skew symmetric real matrices. Let us apply formula (3.19) to calculate the deformation of the coordinate differentials:

$$\delta \bar{\partial} X^\eta = -\text{Res}_{z'=z} H_{\nu\mu} \partial X^\nu(z') \bar{\partial} X^\mu(z') X^\eta(z).$$
Here the T-product of the vertex operators under the residue has the contact singularity
\[ \langle \partial X^\nu(z') \bar{\partial} X^\mu(z') X^\eta(z) \rangle_\Sigma \approx -\frac{1}{z' - z} G^{\nu\eta}\langle \bar{\partial} X^\mu(z') \rangle_\Sigma - \frac{1}{z' - \bar{z}} G^{\mu\eta}\langle \partial X^\nu(z') \rangle_\Sigma \]
and, therefore,
\[ \delta \partial X^\eta = G^{\mu\eta} H_{\nu\mu} \partial X^\nu. \] (5.6)
The symbol "\( \approx \)" indicates that the difference between expressions at its right and left is a regular function.

It can be shown that deformation of the T-product (3.10) corresponding to the vertex operator function (5.5) does not affect contact singularities of the coordinate differentials. Therefore, deformation of such singularities is determined by the deformations (5.6)
\[ \delta(\partial X^\alpha(z_1)\partial X^\beta(z_2))_\Sigma \approx \langle \delta \partial X^\alpha(z_1)\partial X^\beta(z_2) \rangle_\Sigma + \langle \partial X^\alpha(z_1)\delta \partial X^\beta(z_2) \rangle_\Sigma \approx G^{\mu\alpha} H_{\nu\mu}(\partial X^\nu(z_1)\partial X^\beta(z_2))_\Sigma + G^{\mu\beta} H_{\nu\mu}(\partial X^\alpha(z_1)\partial X^\nu(z_2))_\Sigma \approx \frac{2}{(z_1 - z_2)^2} G^{\mu\alpha} G^{\nu\beta} B_{\nu\mu}(0)_\Sigma. \]

Analogously, for antiholomorphic coordinate differentials we have
\[ \delta(\bar{\partial} X^\eta(z_1)\bar{\partial} X^\eta(z_2))_\Sigma = \frac{2}{(z_1 - z_2)^2} G^{\mu\alpha} G^{\nu\alpha} B_{\nu\mu}(0)_\Sigma. \] (5.7)

Thus, such deformed theory is still a theory in the flat background, with a modified metric
\[ g^{\alpha\beta} = G^{\alpha\beta} + \delta g^{\alpha\beta}, \quad \delta g^{\alpha\beta} = 2 B_{\nu\mu} G^{\mu\alpha} G^{\nu\beta}. \] (5.8)

Therefore, the family of scaled fields \( \tau H_{\nu\mu} \partial X^\nu \bar{\partial} X^\mu \) parametrize a family of CFT’s corresponding to different flat metrics \( g = g(\tau) \). The deformations (5.6) and (5.7) of the coordinated differentials are linear. Therefore, the deformed coordinate differentials can be expressed through the initial coordinate differentials,
\[ \partial^{\tau \psi} X^\eta = U(\tau)_{\xi}^\eta \partial X^\xi, \quad \bar{\partial}^{\tau \psi} X^\eta = \bar{U}(\tau)_{\xi}^\eta \partial X^\xi, \]
or, in the vector form,
\[ \partial^{\tau} \Psi \mathbf{X} = U(\tau) \partial \mathbf{X}, \quad \bar{\partial}^{\tau} \Psi \mathbf{X} = \bar{U}(\tau) \partial \mathbf{X}^{\xi}. \] (5.9)

Here \( U, \bar{U} \) are some matrices. Each of them can be used to calculate the modified contravariant metric:
\[ g(\tau) = U(\tau) G U^{T}(\tau) = \bar{U}(\tau) G \bar{U}^{T}(\tau). \] (5.10)

Let us express the field \( \Psi \) through deformed coordinate differentials,
\[ \Psi = h(\tau)_{\nu \mu} \partial^{\tau} \Psi X^{\nu} \bar{\partial}^{\tau} \Psi X^{\mu}, \quad h(\tau) = (U^{-1})^{T}(\tau) H U^{-1}(\tau), \] (5.11)
and apply to the deform theory formulas (5.6) and (5.7). Then we will come to the differential equations
\[ \frac{\partial}{\partial \tau} \partial^{\tau} \Psi \mathbf{X} = g(\tau) h^{T}(\tau) \partial^{\tau} \Psi \mathbf{X} = \bar{U} G H^{T} \partial \mathbf{X}, \]
\[ \frac{\partial}{\partial \tau} \bar{\partial}^{\tau} \Psi \mathbf{X} = g(\tau) h(\tau) \bar{\partial}^{\tau} \Psi \mathbf{X} = U G H \partial \mathbf{X}, \]
which can also be written as
\[ \frac{\partial}{\partial \tau} U(\tau) = \bar{U}(\tau) G H^{T}, \quad \frac{\partial}{\partial \tau} \bar{U}(\tau) = U(\tau) G H. \] (5.12)

Solving these equations with the initial conditions
\[ U(\tau)|_{\tau=0} = \bar{U}(\tau)|_{\tau=0} = 1 \]
and putting then \( \tau = 1 \) we can calculate \( U \) and \( \bar{U} \),
\[ U = \cosh \left( \sqrt{G H G H^{T}} \right) + H^{T} \frac{\sinh \left( \sqrt{G H G H^{T}} \right)}{\sqrt{G H G H^{T}}}, \]
\[ \bar{U} = \cosh(\sqrt{G H^{T} G H}) + H \frac{\sinh \left( \sqrt{G H^{T} G H} \right)}{\sqrt{G H^{T} G H}}, \]
and then substitute it to (5.10). It gives the following formula for the deformed metric:
\[ g = \frac{1}{2} \cosh \left( \sqrt{H H^{T}} \right) + \frac{1}{2} \cosh \left( \sqrt{H^{T} H} \right) \]
\[ + H^{T} \frac{\sinh 2 \left( \sqrt{H H^{T}} \right)}{2 \sqrt{H H^{T}}} + H \frac{\sinh 2 \left( \sqrt{H^{T} H} \right)}{2 \sqrt{H^{T} H}}. \] (5.13)
Here we put for simplicity $G = 1$. In particular, if $H$ is symmetric, we will have
\[ U = \tilde{U} = \exp(H), \quad g = \exp(2H). \]

5.3 Low-energy limit

We will call a spacetime function or a tensor field $\psi$ a slowly varying field of
the order $k$ if it satisfies
\[ \frac{\partial^n}{\partial^n \eta} \psi = 0(\epsilon^{n+k}). \]

Here $\epsilon$ is an infinitesimally small parameter characterizing the scale of energy.
If the value of $k$ is not specified, we will assume that it is trivial. For a
deformation corresponding to a vertex operator function $\Psi = H_{\nu\mu} \partial \partial X^\mu$
with a slowly varying field $H_{\nu\mu}$, the asymptotic behavior of the coordinate
$T$-product can be shown to be approximately the same as for deformation
with a constant field; i.e.,
\[ \langle X^\nu(z') X^\mu(z) \rangle^\Psi_\Sigma \approx -2 \langle \psi(\mu) \phi(\nu) \rangle^\Psi_\Sigma \ln |z' - z| + O(\epsilon). \]

Here $g^{\nu\mu}$ is a contravariant metric tensor defined in (5.13). In a more covariant
way this formula can be written as
\[ \langle \psi(z_1) : \phi(z_2) : \rangle^\Psi_\Sigma \approx -2 \langle \psi^{\nu\mu} \phi_{\nu\mu} : \rangle^\Psi_\Sigma \ln |z' - z| + O(\epsilon^3). \]

Here $\psi, \phi$ are slowly varying space time functions. As usual, superscript
indices following a semicolon denote derivatives of spacetime functions or
covariant (Christoffel) derivatives of spacetime tensor fields, and the covariant
and contravariant metric tensors are used to to raise and lower indices. As a
consequence of (5.14) we have
\[ \langle \psi^{\nu\mu} \phi_{\nu\mu} (z) \rangle := -\text{Res}_{\Sigma\zeta=\zeta} \partial^\Psi \psi(z') : \phi(z) : + O(\epsilon^4). \]

Let us take the antiholomorphic derivative of both sides of this equation
\[ \bar{\partial}^\Psi \psi^{\nu\mu} \phi_{\nu\mu} (z) := -\text{Res}_{\Sigma\zeta=\zeta} \bar{\partial}^\Psi \psi(z') : \phi(z) : \]
\[ -\text{Res}_{\Sigma\zeta=\zeta} \bar{\partial}^\Psi : \phi(z) : + O(\epsilon^5). \]

In the most general case, the action of the deformed conformal Laplacian
here in the low-energy limit can be given as
\[ \bar{\partial}^\Psi \partial^\Psi \psi := (\psi_{\nu\mu} + i \psi^{\nu\eta} C_{\eta\nu\mu}) \partial^\Psi X^\nu \bar{\partial}^\Psi X^\mu : + O(\epsilon^4). \]
Here $C_{\nu\mu}$ is the first order slowly varying tensor field. This field, if it is not trivial, violates chiral invariance, and, in some sense, makes the theory heterotic. Let us substitute (5.17) to (5.16),

$$\text{Res}_{z'=z} \partial^\nu \psi(z') : \partial^\nu : \phi(z) : = - : (\phi_{\mu\eta} - i C_{\eta\mu\nu}) \psi_{\eta} \partial^\nu X^\mu : + O(\epsilon^5) \quad (5.18)$$

and then apply (3.16) to the right-hand side:

$$\text{Res}_{z'=z} \partial^\nu \psi(z') : \partial^\nu : \phi(z) : = - : (\phi_{\mu\eta} - i C_{\nu\eta\mu}) \psi_{\eta} \partial^\nu X^\mu : + O(\epsilon^5). \quad (5.19)$$

The action of the residuelike operations (4.19) on vertex operator functions with slowly varying coefficients can be given as

$$A^\nu_{z'=z} \partial^\nu \psi(z') : \partial^\nu : \phi(z) : = - : \psi_{\mu} \phi_{\nu} : + O(\epsilon^4)$$

$$B^\nu_{z'=z} \partial^\nu \psi(z') : \partial^\nu : \phi(z) : = O(\epsilon^4). \quad (5.20)$$

Substituting it to (4.18) and then applying (5.19), we will have

$$C_{\nu\mu} \left( \psi_{\eta} \phi_{\nu} + \phi_{\eta} \psi_{\nu} \right) = O(\epsilon^5).$$

Therefore,

$$C_{\nu\mu} + C_{\nu\mu} = O(\epsilon^5).$$

Analogously, using conjugated equations it can be shown that

$$C_{\nu\mu} + C_{\mu\nu} = O(\epsilon^5).$$

Thus, $C_{\nu\mu}$ is a completely skew symmetric tensor field. As a consequence of it, this field is real for CP-invariant deformation. We can generalize Eqs. (5.17), (5.18) and (5.19) substituting the fields $\partial^\nu \psi : \partial^\nu : \phi$ by the vertex operator functions of a more general type $: k_{\nu} \partial^\nu X^\nu : , : u_{\nu} \partial^\nu X^\nu :$:

$$\text{Res}_{z'=z} : k_{\nu} \partial^\nu X^\nu(z') : \psi(z) : = - k^{\nu} \psi_{\nu} + O(\epsilon^3), \quad (5.21)$$

$$\text{Res}_{z'=z} : k_{\mu} \partial^\mu X^\mu(z') : \psi(z) : = - k^{\eta} \psi_{\eta} + O(\epsilon^3), \quad (5.22)$$

$$\partial^\nu : k_{\nu} \partial^\nu X^\nu : = \left( k_{\nu\mu} + i k^{\eta} C_{\eta\mu\nu} \right) \partial X^\nu \bar{X}^\mu : + O(\epsilon^3), \quad (5.23)$$

$$\partial^\nu : k_{\mu} \partial^\nu X^\mu : = \left( k_{\mu\nu} + i k^{\eta} C_{\eta\mu\nu} \right) \partial X^\nu \bar{X}^\mu : + O(\epsilon^3), \quad (5.24)$$

$$\text{Res}_{z'=z} : u_{\mu} \partial^\mu X^\mu(z') : \psi(z) : = - \left( u_{\mu\eta} - i C_{\nu\eta\mu} u_{\nu} \right) \partial^\mu X^\mu : + O(\epsilon^3), \quad (5.25)$$

$$\text{Res}_{z'=z} : k_{\eta} \partial^\eta X^\eta(z') : u_{\mu} \partial^\mu X^\mu(z) : = - \left( k^{\eta} \left( u_{\mu\eta} - i C_{\nu\mu\eta} u_{\nu} \right) + d k_{\mu\eta} U^{\eta} \right) \times \partial^\nu X^\mu : + O(\epsilon^3). \quad (5.26)$$
The term in the last equation including external differential \(dk_{\eta\mu} = k_{\eta\mu} - k_{\mu\eta}\) is intended to satisfy formula 1.18. It is trivial for a gradient tangent field \(k_\nu = \psi_\nu\).

We can use the formula (5.26) and the formula conjugated to (5.25) to calculate the residue

\[
\text{Res}_{z'=z}^\Psi : k_\eta \bar{\partial}^\Psi X^\eta(z') : m_{\nu\mu} \bar{\partial}^\Psi X^\nu \bar{\partial}^\Psi X^\mu(z) :
\]

\[
= - k_\eta (m_{\nu\mu} - iC_{\eta\mu\nu} m_{\nu}) \bar{\partial}^\Psi X^\nu \bar{\partial}^\Psi X^\mu :
\]

\[
- : dk_{\eta\mu} m_{\nu} \bar{\partial}^\Psi X^\nu \bar{\partial}^\Psi X^\mu : + O(\epsilon^3).
\]

(5.27)

Applying here 1.19 and taking into account that

\[
\mathcal{A}_{z'=z}^\Psi : k_\eta \bar{\partial}^\Psi X^\eta(z') : m_{\nu\mu} \bar{\partial}^\Psi X^\nu \bar{\partial}^\Psi X^\mu(z) :
\]

\[
= - k_\eta m_{\nu\mu} \bar{\partial}^\Psi X^\nu + O(\epsilon^2),
\]

\[
\mathcal{A}_{z=\bar{z}}^\Psi : k_\eta \bar{\partial}^\Psi X^\eta(\bar{z}') : m_{\nu\mu} \bar{\partial}^\Psi X^\nu \bar{\partial}^\Psi X^\mu(\bar{z}') :
\]

we will also have

\[
\text{Res}_{z'=z}^\Psi : m_{\nu\mu} \bar{\partial}^\Psi X^\nu \bar{\partial}^\Psi X^\mu(z') :
\]

\[
= - k_\eta (m_{\nu\mu} - m_{\mu\nu}) + iC_{\eta\mu\nu} m_{\nu} \bar{\partial}^\Psi X^\nu \bar{\partial}^\Psi X^\mu :
\]

\[
- : m_\nu k_{\mu\sigma} \bar{\partial}^\Psi X^\nu \bar{\partial}^\Psi X^\mu : + O(\epsilon^3).
\]

(5.28)

Now, let us calculate a holomorphic derivative of both sides of (5.19):

\[
\text{Res}_{z'=z}^\Psi : (\psi_{\nu\mu} + i\psi_\nu C_{\eta\mu\nu}) \bar{\partial}^\Psi X^\nu \bar{\partial}^\Psi X^\mu(z') : \bar{\partial}^\Psi : \phi(z):
\]

\[
+ \text{Res}_{z'=\bar{z}}^\Psi : \psi(z') : (\phi_{\nu\mu} + i\phi^\sigma C_{\sigma\nu\mu}) \bar{\partial}^\Psi X^\nu \bar{\partial}^\Psi X^\mu(z') :
\]

\[
= - (\psi_{\nu\mu} \phi_{\mu\nu} + \psi_{\nu\sigma} C_{\eta\mu\nu} \phi_{\eta\sigma}) \bar{\partial}^\Psi X^\nu \bar{\partial}^\Psi X^\mu :
\]

\[
- i : (C_{\sigma\mu\nu} \psi_{\eta\sigma} \phi_{\eta\nu} - C_{\eta\mu} \psi_{\eta\sigma} \phi_{\eta\nu} - C_{\eta\mu\nu} \psi_{\eta\sigma} \phi_{\eta\nu}) \bar{\partial}^\Psi X^\nu \bar{\partial}^\Psi X^\mu : + O(\epsilon^6).
\]

Applying here 5.28 and (5.27), we will have

\[
C_{\sigma\nu\mu\eta} + C_{\eta\nu\sigma\mu} - C_{\eta\mu\sigma\nu} + C_{\eta\mu\nu} = O(\epsilon^4)
\]

(5.29)

or, in a geometric form,

\[
dC = O(\epsilon^4).
\]

This means that \(C_{\eta\mu\nu}\) is a cocycle differential three-form. If topology of the spacetime is trivial, all cocycles are exact and, therefore, the field \(C_{\eta\mu\nu}\) can be represented as

\[
C = d\omega + O(\epsilon^3).
\]

(5.30)
Here $\omega$ is a slowly varying differential two-form; i.e., a skew symmetric rank 2 tensor field. In order to express $\omega_{\nu\mu}$ through the deformation parameters $H_{\nu\mu}$, we will again consider a family of deformation theories corresponding to scaled fields $\tau H_{\nu\mu}$ and calculate a derivative of the deformed Laplacian (5.17) with respect to $\tau$

$$\frac{\partial}{\partial \tau} \partial^\nu \partial^\mu = \left( \frac{\partial}{\partial \tau} \psi_{\nu\mu} + i\psi_{\eta} C_{\sigma\nu\mu} \frac{\partial}{\partial \tau} g^{\sigma\eta} + i\psi_{\eta} \frac{\partial}{\partial \tau} C_{\nu\mu} \right) \partial^\nu X^\nu \partial^\mu X^\mu :$$

$$+ \left( \psi_{\nu\mu} + i\psi_{\eta} C_{\eta\nu\mu} \right) \left( \partial^\nu \partial^\mu \partial^\nu X^\nu + \partial^\nu \partial^\nu \partial^\mu \partial^\nu X^\mu \right). \quad (5.31)$$

Using formulas (3.19) and (5.28) on the right-hand side of this equation, we will have

$$\frac{\partial}{\partial \tau} \partial^\nu \partial^\mu \psi = \partial^\nu \left( h_{\nu\mu} \psi_{\mu} \partial^\mu X^\mu \right) - \text{Res}_{z=1} : h_{\nu\mu} \partial^\nu \partial^\mu X^\nu \partial^\mu X^\mu (z^\prime) : : \psi_{\eta} \partial^\eta X^\eta (z) :$$

$$= \left( (\psi;_{\nu\sigma} + i\psi_{\eta} C_{\eta\nu\sigma}) h^\sigma_{\mu} + (\psi;_{\sigma\mu} + i\psi_{\eta} C_{\eta\nu\sigma}) h^\sigma_{\nu} \partial^\nu X^\nu \partial^\mu X^\mu \right)$$

$$+ : (\psi_{\eta} (h_{\nu\eta;\mu} - h_{\nu\mu;\eta} + h_{\eta\mu;\nu}) + 2iC_{\nu\mu\eta} b^\eta) \partial^\nu X^\nu \partial^\mu X^\mu : + O(\epsilon^4). \quad (5.32)$$

Here $b_{\nu\mu}$ and $ia_{\nu\mu}$ are, respectively, symmetric and skew symmetric parts of $h_{\nu\mu}$; i.e.,

$$h_{\nu\mu} = b_{\nu\mu} + ia_{\nu\mu}, \quad b_{\nu\mu} = b_{\mu\nu}, \quad a_{\nu\mu} = -a_{\mu\nu}.$$ 

According to (5.3), the increment of the metric can be given as

$$\frac{\partial}{\partial \tau} g^{\nu\mu} = 2b^{\nu\mu}, \quad \frac{\partial}{\partial \tau} g_{\nu\mu} = -2b_{\nu\mu}. \quad (5.33)$$

The corresponding increments of the Christoffel symbols are

$$\frac{\partial}{\partial \tau} \Gamma^\eta_{\nu\mu} = -(b^\eta_{\nu;\mu} + b^\eta_{\mu;\nu} - b_{\nu;\mu;\eta})$$

and, therefore,

$$\frac{\partial}{\partial \tau} \psi_{\nu\mu} = \psi_{\eta} \left( b_{\eta\nu;\mu} + b_{\eta\mu;\nu} - b_{\nu;\mu;\eta} \right). \quad (5.34)$$
Substituting (5.32), (5.33), (5.34) to (5.31), we will come to the following formula for the increment of $C$:

$$
\frac{\partial}{\partial \tau} C_{\eta \mu} = a_{\nu \eta ; \mu} - a_{\nu \mu ; \eta} + a_{\eta \mu ; \nu} + O(\epsilon^3),
$$
or, in a geometric form,

$$
\frac{\partial}{\partial \tau} C = -da + O(\epsilon^3).
$$

This corresponds to the following increment of the field $\omega$:

$$
\frac{\partial}{\partial \tau} \omega = -a = -\frac{1}{2i}(h - h^T).
$$

Integrating this equation we will finally have

$$
\omega = -\frac{1}{2i} \int_0^1 (h - h^T) \, d\tau.
$$

According to (5.11) and (5.12), $h(\tau)$ can be calculated in one of the following ways:

$$
h = (U^{-1})^T H U^{-1}, -\frac{\partial}{\partial \tau} U^{-1} (U^{-1})^T = -(\tilde{U}^{-1})^T \frac{\partial}{\partial \tau} \tilde{U}^{-1}.
$$

Local space-time transformations

$$
\delta g_{\nu \mu} = \epsilon_{\nu ; \mu} + \epsilon_{\mu ; \nu}, \quad \delta \omega_{\nu \mu} = \epsilon^\eta \omega_{\nu \mu ; \eta} + \epsilon^{\nu \eta} \omega_{\eta \mu} + \epsilon^{\mu \eta} \omega_{\nu \eta},
$$

change the vertex-operator relations established above to their equivalents. In addition, these relations will not change at all under transformation

$$
\delta \omega = d\varsigma.
$$

Here $\varsigma$ is a differential one-form; i.e., cotangent field. The transformations (5.36) and (5.37) are a particular case of more general symmetries (4.11) with the following choice of the vertex operator functions parametrizing them:

$$
\xi_z =: (\epsilon_\nu + i\varsigma_\nu) \partial X^\nu :; \quad \xi_z =: (\epsilon_\mu - i\varsigma_\mu) \bar{\partial} X^\mu :.
$$

In string theory, the off-shell theory and symmetries depend, as we have seen, on regularization of the contact divergence. Therefore, changing regularization parameters $\Lambda(\alpha)$ we will also change the symmetry algebra. Not
all such algebras can be closed because the Lie algebraic structure is usually rigid. Of course, one can try to find a specific regularization method for which the symmetry algebra is closed. However, there is no indication that it can be done. In the low-energy limit the picture is almost independent of regularization, which actually allowed the symmetry algebra to be closed.

5.4 Equation of motion

Let us consider an action of the deformed Virasoro operator

\[ L_1^\Psi = \text{Res}_{z'=z}^\Psi (z' - z)^2 T_{zz}^\Psi (z') \]  

(5.38)
on the vertex \( : k_\nu \partial^\Psi X^\nu : \). In the locally Galilean system of coordinates we can use for \( T_{zz}^\Psi \) the formula

\[ T_{zz}^\Psi = \frac{1}{2} \lim_{z' \to z} \Psi \left( \partial^\Psi X^\eta (z') \partial^\Psi X^\eta (z) + \frac{D}{(z' - z)^2} \right) + O(\epsilon^2). \]

Substituting it to (5.38) and applying (5.21) and (5.25) we will have

\[ L_1^\Psi : k_\nu \partial^\Psi X^\nu : = -\text{Res}_{z'=z}^\Psi \partial^\Psi X^\nu (z') : k_\nu (z) : = : k_\nu : + O(\epsilon^3), \]

\[ L_1^\Psi : f_{\nu\mu} \partial^\Psi X^\nu \partial^\Psi X^\mu : = -\text{Res}_{z'=z}^\Psi \partial^\Psi X^\nu (z') : f_{\nu\mu} \partial^\Psi X^\mu (z) : = : (f_{\nu\mu} - i f^\nu_{\eta\mu} C_{\eta\mu}) \partial^\Psi X^\mu : + O(\epsilon^3). \]

Using this formula together with (5.24) one can show that

\[ \bar{\partial}^\Psi L_1^\Psi : k_\nu \partial^\Psi X^\nu : = : k^{\eta\mu} \partial^\Psi X^\mu : + O(\epsilon^4) \]

\[ L_1^\Psi \bar{\partial}^\Psi : k_\nu \partial^\Psi X^\nu : = : (k^{\nu\mu} + k^{\eta\nu}(C^\sigma^\nu C_{\mu\sigma} - i C^\eta_{\nu\mu})) \bar{\partial}^\Psi X^\mu : + O(\epsilon^4), \]

and, therefore,

\[ [\bar{\partial}^\Psi, L_1^\Psi] : k_\nu \partial^\Psi X^\nu : = : (R_{\nu\mu} - C^\nu_{\sigma\rho} C_{\mu\sigma\rho} + i C^\eta_{\nu\mu}) k_\nu \partial^\Psi X^\mu : + O(\epsilon^4). \]  

(5.39)

Here we use the formula

\[ k^{\nu}_{\nu\mu} - k^{\nu}_{\mu\nu} = R_{\nu\mu} k^{\nu}, \]

(5.40)

where \( R_{\nu\mu} \) is the Ricci curvature. Taking antiholomorphic derivatives of (5.38) and applying consequently formulas (4.13) and (3.16), we can express the
commutator $[\bar{\partial}^\Psi, L^\Psi_1]$ through the contradiagonal components of the energy-momentum tensor:

$$
\left[\bar{\partial}^\Psi, L^\Psi_1\right] = \text{Res}_{z' = z}(z' - z)^2\bar{\partial}^\Psi T^\Psi_{zz}(z') - \text{Res}_{z' = z}(z' - z)^2\partial^\Psi T^\Psi_{zz}(z')
\quad = 2\text{Res}_{z' = z}(z' - z)\bar{\partial}^\Psi T^\Psi_{zz}(z') - \text{Res}_{z' = z}\bar{\partial}^\Psi((z' - z)^2T^\Psi_{zz}(z'))
\quad = 2\text{Res}_{z' = z}(z' - z)\partial^\Psi T^\Psi_{zz}(z') - \text{Res}_{z' = z}(z' - z)^2\bar{\partial}^\Psi T^\Psi_{zz}(z').
$$

Comparing this with (5.39) we will have

$$
T^\Psi_{zz} = -\frac{1}{2}(R_{\nu\mu} - C^\sigma_{\nu\rho}C_{\mu\sigma\rho} + iC_{\eta\nu\mu\eta})\bar{\partial}^\Psi X^\nu\bar{\partial}^\Psi X^\mu + O(\epsilon^4). \quad (5.41)
$$

Analogously, we can derive the same formula for $T^\Psi_{\bar{z}z}$; i.e.,

$$
T^\Psi_{\bar{z}z} = T^\Psi_{zz} + O(\epsilon^4). \quad (5.42)
$$

Therefore, the equations of motion (4.24) can be written as

$$
T^\Psi_{zz} = -\bar{\partial}^\Psi \Phi_z = -\partial^\Psi \Phi_{\bar{z}} + O(\epsilon^4). \quad (5.43)
$$

If the background fields have no symmetries, this is equivalent to

$$
T^\Psi_{zz} + \partial^\Psi \bar{\partial}^\Psi \Phi = O(\epsilon^4).
$$

Here $\Phi$ is some vertex operator function. In order to make this equation solvable with $T^\Psi_{zz}$ given by formula (5.41) we shall put

$$
\Phi = :\phi:,
$$

where $\phi$ is some space-time function. Then we will come to the following equation of motion in the low-energy limit

$$
R_{\nu\mu} + iC_{\eta\nu\mu\eta} = C^\sigma_{\nu\rho}C_{\mu\sigma\rho} + 2(\phi_{,\nu\mu} + i\phi^\eta C_{\eta\nu\mu}).
$$

This equation decouples to symmetric and skew symmetric parts

$$
R_{\nu\mu} = C^\sigma_{\nu\rho}C_{\mu\sigma\rho} + 2\phi_{,\nu\mu},
\quad C_{\eta\nu\mu\eta} = 2\phi^\eta C_{\eta\nu\mu}. \quad (5.44)
$$
The skew symmetric part can be also written as
\[ \left( \frac{1}{\Theta} C_{\eta \nu \mu} \right) :_{\eta} = 0, \]
or, in a geometric form, as
\[ d \left( \frac{1}{\Theta} C^{\nu} \right) = 0. \]
Here \( \Theta = \exp 2\phi \) is a so-called dilaton field. If the field \( C_{\eta \nu \mu} \) is trivial, the first equation in (5.44) will be a conventional Brans-Dicke equation derived earlier in the \( \beta \)-function approach. Note that the transformations
\[ \phi \rightarrow \phi + \text{const} \]
do not change the corresponding CFT and, therefore, only the derivatives of \( \phi \) are physically important. These derivatives must be bounded and slowly varying. However, the function \( \phi \) itself should not necessarily comply with either of the conditions.

5.5 Deformation of central charge

Let us differentiate the first equation in (5.44) and then contract indices
\[ R_{\nu \mu} = C^{\nu \sigma \rho} C_{\mu \sigma \rho} + C^{\nu \sigma \rho} C_{\mu \sigma \rho \nu} + 2 \phi_{\mu \nu} \nu. \]
Using here the formulas (5.29) and (5.40) and the identity
\[ R_{\nu \mu} = \frac{1}{2} R_{\mu} \quad (R \equiv R_{\nu} \nu), \]
respectively, on the right and left hand sides we will have
\[ \frac{1}{2} R_{\mu} = -4 \phi_{\nu \mu} \phi_{\nu} + \frac{1}{3} C^{\nu \sigma \rho} C_{\nu \sigma \rho \mu}. \]
This equation can be easily integrated
\[ R + 4 \phi_{\nu} \phi_{\nu} - \frac{1}{3} C^{\nu \sigma \rho} C_{\nu \sigma \rho} = \text{const} \equiv m^2. \quad (5.45) \]
Contracting indices in the first equation in (5.44) we will also have

\[ R = C^{\nu\sigma\rho}C_{\nu\sigma\rho} + 2\Box\phi. \]

Using this formula we can exclude the curvature from the (5.45)

\[ 2\Box\phi + 4\phi^{\nu\mu}\phi_{\nu} = m^2 - \frac{2}{3}C^{\nu\sigma\rho}C_{\nu\sigma\rho}. \]

This is equivalent to the following equation on the dilaton field

\[ \Box\Theta = \left( m^2 - \frac{2}{3}C^{\nu\sigma\rho}C_{\nu\sigma\rho} \right)\Theta. \]

Therefore, \( m \) can be interpreted as a dilaton mass. It is the only topological characteristic of the space-time dynamics. Therefore, it must be related to the only topological characteristic of CFT – a central charge. We can easily find this relation using the flat solution

\[ g_{\nu\mu} = \text{const}, \quad C_{\nu\rho\mu} = 0, \quad \phi = k_{\nu}x^{\nu}, \quad m^2 = 4k^{\nu}k_{\nu}, \]

corresponding to the trivial parameter \( \Psi \). The deformed energy-momentum tensor (4.21), in this case, can be given as

\[ T_{\Phi}^{zz} = T_{zz} - \partial\Phi_{z} = -\frac{1}{2} :\partial X^{\nu}\partial X_{\nu} : -k_{\nu}(\partial)^2 X^{\nu}. \]

Its two-point correlation function has the contact singularity

\[ \langle T_{zz}^{\Phi}(z_{1})T_{zz}^{\Phi}(z_{2})\rangle_{\Sigma} \approx \frac{D + \frac{1}{2}m^2}{(z_{1} - z_{2})^4}\langle 0 \rangle_{\Sigma}. \]

This corresponds to the central charge equal to

\[ c = D + \frac{1}{2}m^2. \quad (5.46) \]

In particular, the conventional Brans-Dicke theory with the massless dilaton describe critical deformations of CFT.
6 Problems and Perspectives

In this paper we formulated the equation of motion for the closed string field theory and described its symmetries. In order to do this we introduced a specific regularization of the contact singularities of the $T$-product of vertex operators. Despite the fact that classes of gauge equivalent solutions correspond to equivalence classes of CFT and, therefore, do not depend on this regularization, the off-shell states, the equation of motion and the symmetries depend on it. In addition, the symmetries as well as equivalence relations between solutions in different regularizations are very nontransparent. It makes it difficult to formulate a classical action corresponding to the equation we found here and, therefore, to quantize the theory. Some ideas on how this problem may be solved by means of auxiliary fields were suggested in [10]. However, we are not sure that the string theory can be, in principle, covariantly quantized in the canonical path integral approach, which requires the action to be covariantly defined, and hope that there is a more explicit approach to describe the quantum string field theory.

Solving the equation of motion derived here we may try to find string vacuum which is crucial for obtaining results observable in the low-energy experiments. Because of the infinite number of the higher order terms in this equation, it is not an easy task.

The formalism suggested here works equally well for the string theory with all possible central charges. Moreover, it gives a mechanism for central charge deformation (5.46). As a negative consequence of this, the massive symmetries corresponding to critical string are not understood. These symmetries may not be related to equivalence relations of CFT. It is a very important principle drawback of interpretation of the closed string field theory as the theory of CFT’s. More recently we have found that CFT induces certain noncommutative functor from the category of two dimensional surfaces and suggest that such functors, rather then CFT’s, must be associated with classical string states. This might explain the origin of the additional critical symmetries and help to close symmetry algebra. It is especially encouraging to see that deformations of such functor can be associated with certain $BRST$-like cohomologies.
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A Successor Calculation

Here we will present a method for calculation of successors defined in Section 3.2. In order to use the formula (3.17) we should first learn how to calculate $\partial^{-1}_{\overline{z}} F$ for an arbitrary smooth function $F$, having diagonal singularities. Such a function can always be considered as a linear combination of the following samples:

$$\prod_{i=0}^{n} |z - z_i|^{-2\alpha_i} \quad (\alpha_i \in \mathbb{R}) \quad (A.1)$$

with meromorphic coefficients. Therefore, the problem can be reduced to calculation of $\partial^{-1}_{\overline{z}}$ for these samples. Let us use for these samples the integral representation

$$|z - z_i|^{-2\alpha_i} = \frac{1}{\Gamma(\alpha_i)} \int_{0}^{\infty} \exp \left( -s_i |z - z_i|^2 \right) s_i^{\alpha_i - 1} ds_i. \quad (A.2)$$

Then (A.1) will be expressed through integrals over $ds_i$ of the functions

$$E_{s_0, \ldots, s_n} = \exp \left( -\sum_{i=0}^{n} s_i |z - z_i|^2 \right),$$

which can be explicitly integrated over $\overline{z}$

$$\partial^{-1}_{\overline{z}} E_{s_0, \ldots, s_n} = \left( \sum_{k=0}^{n} s_k (z - z_k) \right)^{-1} \left( \exp \left( -\sum_{i=0}^{n} s_i |z - z_i|^2 \right) - \exp \left( -\left( \sum_{k=0}^{n} s_k \right)^{-1} \sum_{0 \leq i \leq j} s_i s_j |z_i - z_j|^2 \right) \right). \quad (A.3)$$

The second holomorphic term here is meant to cancel the singularity at $z = (\sum_{k=0}^{n} s_k)^{-1} \sum_{k=0}^{n} s_k z_k$. Using this formula we found an explicit result for successors of rank 2 resuduelike operations. For example,

$$\text{Res}_{\overline{z} = \overline{z}_1 = \overline{z}_0} |z - z_1|^{-2\alpha} |z - z_0|^{-2\beta} |z_1 - z_0|^{-2\gamma} F = \sum_{\overline{i} + \overline{j} = i + j + 1} \frac{\Gamma(\alpha + \beta - \overline{i}) \Gamma(\overline{i} + 1 - \beta) \Gamma(1 - \alpha)}{\Gamma(\alpha) \Gamma(\beta - \overline{i}) \Gamma(\overline{i} + 2 - \alpha - \beta)} \Delta(\overline{i} + \overline{j} + 1 - \alpha - \beta - \gamma)$$

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Here $F$ is a function of $z, z_1, z_0$ regular in some environment of the main diagonal $z = z_1 = z_0$. The generalized number of composition is defined to be

$$
C_k^\alpha \equiv \begin{cases} 
\frac{\alpha(\alpha-1)\cdots(\alpha-k)}{k!} & (k \geq 0) \\
0 & (k < 0)
\end{cases}
$$

The poles on the right-hand side of (A.4) completely cancel each other, so that the successor depends on parameters $\alpha, \beta$ and $\gamma$ regularly.

The formula for this successor can be simplified if it acts on the function, which depends on $z_1$ holomorphically. Such a function can be always represented as

$$
F = \sum_{k \in \mathbb{Z}} A_k(z, z_0)(z_1 - z)^{-1-k} + B_k(z_1 - z_0)^{-1-k}
$$

and, therefore, its antiderivative can be given as

$$
\frac{\partial}{\partial z}^{-1} F = \sum_{k \in \mathbb{Z}} \frac{\partial}{\partial z}^{-1} A_k(z, z_0)(z_1 - z)^{-1-k} + \frac{\partial}{\partial z}^{-1} B_k(z, z_0)(z_1 - z_0)^{-1-k}.
$$

Substituting this to (3.17), and using (3.16) we will have

$$
\text{Res}_{z = z_1 = z_0} F = \sum_{k = 0}^{\infty} \frac{(-)^k}{k!} \text{Res}_{z_1 = z_0} \partial_{z_1}^k A_k = \sum_{k = 1}^{\infty} \frac{(-)^k}{k!} \partial_{z_1}^{k-1} A_k.
$$

Noticing here, that for positive $k$

$$
A_k(z, z_0) = \text{Res}_{z_1 = z} (z_1 - z)^k F(z, z_1, z_0),
$$
we will come to the following formula for the successor

\[
\text{Res}_{z = z_0} F = \text{Res}_{z = z_0} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \partial_z^{k-1} \text{Res}_{z = z_0} (z_1 - z)^k F(z, z_1, z_0). \quad (A.5)
\]

In this way, we can always express the higher rank residue successor applied to functions meromorphic with respect to one of the variables through the lower rank residue successors.

The calculation of successors can be also based on their analytical properties. Let \( \Sigma \) in (3.12) be a Riemann sphere \( \bar{\mathbb{C}} \). The sample functions (A.1) may have singularities at \( z = \infty \). For regularization of corresponding divergence of the integral we can use the formula

\[
\int_{\bar{\mathbb{C}}} = \int_0^\infty d\mu(r) \int_{D(0, \cdot)}.
\]

Difference between the regularizations corresponding to parameters \( \Lambda \) and \( \Lambda' \) can be given as

\[
\int_{\bar{\mathbb{C}}} - \int_{\bar{\mathbb{C}}} = \sum_{i=0}^n I_{z = z_i} + I_{z = \infty} \left( \xi(\alpha) = \frac{\Lambda(\alpha) - \Lambda'(\alpha)}{\alpha} \right).
\]

Here \( I_{z = z_0} \) is a residuelike operation defined as

\[
I_{z = z_0} |z - z_0|^{2(\alpha - 1)} z^k = \xi(\alpha) \delta_{k,0}, \quad I_{z = \infty} |z - z_0|^{2(\alpha - 1)} z^k = \xi(-\alpha) \delta_{k,0}.
\]

Therefore, the combination

\[
\mathcal{I}(F) = \int_{\bar{\mathbb{C}}} F - \sum_{i=0}^n I_{z = z_i} F - I_{z = \infty} F \left( \xi = -\frac{\Lambda}{\alpha} \right)
\]

does not depend on \( \Lambda \). In fact, it is the analytical continuation of the non-regularized integral from the divergence-free area of the parameters. All the poles of \( \mathcal{I} \) are determined by the singularities of the residuelike operations resulting from singular behavior of \( \xi(\alpha) \) at \( \alpha = 0 \).
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