Universal power-law exponents in differential tunneling conductance for planar insulators near Mott criticality at low temperatures

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Abstract

We consider the low-temperature differential tunneling conductance $G$ for interfaces between a planar insulating material in the Mott-class and a metal. For values of the applied potential difference $V$ that are not very small, there is an experimentally observed universal regime in which $G \sim V^m$, where $m$ is a universal exponent. We consider the theoretical prediction of the values of $m$ by using the method of Effective Field Theory (EFT), which is appropriate for discussing universal phenomena. We describe the Mott material by the EFT pertaining the long-distance behavior of a spinless Hubbard-like model with nearest neighbors interactions previously considered. At the Mott transition, the EFT is known to be given by a double Abelian Chern-Simons theory. The simplest realization of this theory at the tunneling interface yields a Conformal Field Theory with central charges $(c, \tilde{c}) = (1, 1)$ and Jain filling fraction $\nu = 2/3$ describing a pair of independent counter-propagating chiral bosons (one charged and one neutral). Tunneling from the material into the metal is, therefore, described by this EFT at the Mott critical point. The resulting tunneling conductance behaves as $G \sim V^{(1/\nu - 1)}$, yielding the prediction $m = 1/2$, which compares well (within a 10% deviation) with the results for this exponent in two experimental studies considered here.

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Tunneling spectroscopy is a powerful experimental tool to probe response functions of materials, such as the differential conductance $G$, which is defined as $G = dI/dV$, where $I$ is the resulting tunneling current flowing through the junction for the applied potential difference $V$ across it. For values of $V$ that are smaller than a characteristic scale $V_0$, $G$ encodes a non-universal behavior, but for $V > V_0$ there is a universal regime characterized by a relation in the form of a 'non-linear Ohm’s Law', $I \sim V^{m+1}$, where $m$ is a universal exponent. This relation leads to $G \sim V^m$, characteristic of this type of experiments.

In this letter, we shall focus on the universal exponent $m$ that characterizes the quantum tunneling between a planar insulator (e.g., doped semiconductors or semiconductor oxides such as $SrTiO_3$) near the Mott-like critical point and a conductor, which is the situation relevant to two series of experimental results that we will address here [2] [3]. The universal regime of (moderately) large $V$ can be successfully described by Effective Field Theories ($EFT$). One such an effective theory pertaining the class of materials considered in the experimental studies [2] [3] was proposed in [5]. It was obtained as the long-range limit of a discrete planar lattice $AF$ model, near the critical Mott point. The field content of this theory is given in terms of two Chern-Simons Abelian fields. These fields give rise to a Conformal Field Theory ($CFT$) at the boundary of the sample. For the experimental setups considered here, there is no external magnetic field that would break the chiral symmetry of the $CFT$, as in the case of the Quantum Hall Effect ($QHE$). Nevertheless, the results of [5] indicate a factorized chiral structure on this $CFT$. Even beyond the critical point, the idea of a factorized structure could be pursued. In the following, we shall assume that the temperature is low enough so that its energy scale could be disregarded.

For the sake of completeness, we shall briefly review here the key aspects of this theory [5]. We first consider the $EFT$ pertaining a planar Mott material in the bulk and in the tunneling boundary. We start by considering the following Hamiltonian model on a 2D spatial lattice:

$$H_{2D} = -\frac{t}{2} \sum_{x,\mu} \left[ \psi^\dagger(x+ae_\mu) e^{ia_\mu} \psi(x) + \text{h.c.} \right] + U \sum_{x,\mu} \rho(x) \rho(x+ae_\mu), \quad (1)$$

where $\psi(x)$ is the fermion field, $x$ labels the lattice sites and $e_\mu$ are the unit lattice vectors pointing to the nearest neighbors of a given site, $a$ is the lattice spacing, $t$ is the hopping parameter, $U$ is the (constant) Coulomb potential, $\rho(x)$ is the charge density (normal-ordered with respect to the half-filling ground state), $\rho(x) = [:\psi^\dagger(x)\psi(x): -1/2]$ and $a_\mu$ is an Abelian statistical gauge field defined on the links of the lattice. This model is well
suited for describing electrons in systems with ‘narrow pocket Fermi surfaces’ (see, e.g., [5]), relevant for the experimental setups considered here. In the continuum (thermodynamic) limit the above model can be mapped onto a two-dimensional anisotropic Heisenberg (XXZ-spin model) by means of a two-dimensional Jordan-Wigner transformation. The low-energy, long-range \( EFT \) of this model yields an Abelian gauge theory with two bosonic statistical Chern-Simons fields.

For the sake of clarity of the exposition, let us first consider an Abelian Chern-Simons gauge theory, based on a single statistical field \( A_\mu(x) \) defined on (2 + 1)-dimensional space-time (\( x \) denotes now a (2 + 1)-dimensional coordinate \( x^\mu \), with \( \mu = 0, 1, 2 \)). Its action is given by:

\[
S_{CS} = \frac{k}{4\pi} \int d^3x \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda , \tag{2}
\]

where \( k \) is the self-coupling constant. This a topological gauge field theory, meaning that there is no bulk dynamics since its hamiltonian vanishes identically (\( H = 0 \)). Subtleties regarding the signature of space-time are overlooked in this simple discussion (see [7]). The natural observables of this theory are Wilson Loops, or flux tubes:

\[
W_n[\Gamma] = P \exp \left( in \oint_\Gamma A_\mu dx^\mu \right) , \quad n \in \mathbb{Z} \; , \tag{3}
\]

where \( P \) denotes path-ordering and the exponents may be thought of as statistical fluxes subtended by the Wilson loop \( \Gamma \). Notice that the switching of the sign of the coupling constant \( k \) in the action could be absorbed by a sign flip of the gauge field, leaving the action unmodified but changing the sign of the statistical fluxes [7].

The \( EFT \) for the model (1) slightly away from the Mott critical point is given by the double lattice Chern-Simons action:

\[
S_{DCSL} = \frac{k}{4\pi} \int d^3x A_\mu^{(1)} K_{\mu\nu} A_\nu^{(1)} - \frac{k}{4\pi} \int d^3x A_\mu^{(2)} K_{\mu\nu} A_\nu^{(2)} , \tag{4}
\]

with coupling constants \( k > 0 \) and \(-k\), and where \( A^{(1)} \) and \( A^{(2)} \) are two Abelian statistical gauge fields. Here we consider the coordinates \( x \) on a cubic lattice of spacing \( a \), with forward difference operators \( d_\mu f(x) = [f(x + a\epsilon_\mu) - f(x)]/a \), forward shift operators \( S_\mu f(x) = f(x + a\epsilon_\mu) \), and the kernel \( K_{\mu\nu} = S_\mu \epsilon_{\mu\nu\alpha} d_\alpha \) (no summation is implied over equal indices \( \mu \) and \( \nu \)) [10].

Notice that the fields \( A^{(1)} \) and \( A^{(2)} \) may be thought of as having opposite orientation in the Wilson loop circulation, or, equivalently, as having statistical fluxes of opposite sign (consider \( n = 1 \) in (3)). This theory has
Quantum Group Symmetry $U_q\left(\hat{sl}(2)\right) \otimes U_q\left(\hat{sl}(2)\right)$ with deformation parameter $q = e^{i\pi/k}$. At the critical Mott point, $q = -1$ and $k = 1$ \[7\] \[8\].

In the rest of our analysis, we shall consider that we are close enough to the Mott critical point so that the EFT may be further simplified with action:

$$S_{EFT} = \frac{1}{4\pi} \int d^3 x e^{\mu\nu\lambda} A^{(1)}_{\mu} \partial_{\nu} A^{(1)}_{\lambda} - \frac{1}{4\pi} \int d^3 x e^{\mu\nu\lambda} A^{(2)}_{\mu} \partial_{\nu} A^{(2)}_{\lambda},$$

where the the continuum limit of (4) has been taken. This is a theory of two Abelian Chern-Simons statistical gauge fields with two types of vortices, given by the opposite spatial Wilson loop orientations (defined, e.g., with respect to the spatial plane, but consistent definitions in Euclidean 3D space can be achieved nonetheless \[9\]). As any topological theory, it may give rise to dynamical phenomena only at the boundaries of the spatial region. This region could be a line or segment. For the sake of simplicity, we consider it to be a circle of radius $R$ (any boundary will provide a proper length scale). We shall assume that the difference in statistical magnetic flux originated in both fields gives rise to a pair of chiral bosonic fields with different propagating senses along this line. That is, we consider that the difference in Wilson loop vorticity translates into two chiralities at the boundary (further discussion on the boundary projection of Wilson loops may be found in \[9\]). In this sense, the dynamics becomes identical to that of the edge states in a sample displaying the Quantum Hall Effect (QHE).

In the following, we consider the action (5) to be defined on a space-time manifold $\mathcal{M}$ with the topology of a cylinder made from a spatial disk, parametrized by an angle $\theta$ and a radius $R$, and time $t$. In this geometry, the Chern-Simons action induces a boundary field theory with dynamics given by the choice of boundary conditions \[11\], which may be identified with a CFT \[7\]. The Gauss Law constraint derived from (5) imposes flat field strength tensors $F^{(i)}_{\mu\nu} = 0$, where $F^{(i)}_{\mu\nu} = \partial_{\mu} A^{(i)}_{\nu} - \partial_{\nu} A^{(i)}_{\mu}$, $(i = 1, 2)$. The ‘pure gauge’ fields that solve this constraint are expressed in terms of bosonic Abelian ones $\varphi^{(i)}$ such that

$$A^{(i)}_{\mu} = -\frac{1}{2\pi} \partial_{\mu} \varphi^{(i)},$$

where $i = 1, 2$. The bosonic fields can be taken as propagating boundary fields, provided adequate boundary conditions are chosen (see, e.g. \[11\]). We note that global parity is preserved in the action (5), and therefore we take the bosonic fields as counter-propagating modes at the spatial boundary of $\mathcal{M}$. We choose, without loss of generality, $\varphi^{(1)}$ ($\varphi^{(2)}$) to be right (left) handed, respectively, i.e, $\varphi^{(i)}(R\theta \mp v_it)$, where $v_i$ are positive dimensionless
Fermi velocity parameters, introduced by the boundary conditions. It is well-known that the free bosonic chiral (anti-chiral) theory of $\phi^{(1)} (\phi^{(2)})$ defines a $c = 1$ ($\tilde{c} = 1$ CFT). The latter theory is usually defined on the complex plane obtained by mapping of the cylinder onto it by $z = \exp(\tau + i\theta)$, where $\tau = it$ is the Euclidean time.

In order to make the CFT explicit, we consider the Abelian (chiral and anti-chiral) currents:

$$J^{(i)} = -\frac{1}{2\pi} \frac{\partial \phi^{(i)}}{\partial \theta} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \alpha^{(i)}_n e^{in(\theta + v_it/R)} , \quad i = 1, 2 ,$$ (7)

whose Fourier modes $\alpha^{(i)}_n$ satisfy the two-component Abelian current algebra,

$$[ \alpha^{(i)}_n, \alpha^{(j)}_m ] = \delta^{ij} n \delta_{n+m,0} .$$ (8)

The corresponding generators of the Virasoro algebra are obtained by the familiar Sugawara construction $L^{(i)} =: (j^{(i)})^2 : /2$, and read:

$$L_0^{(i)} = \frac{1}{2} \alpha_0^{(i)} + \sum_{n=0}^{\infty} \alpha^{(i)}_n \alpha^{(i)}_{-n} , \quad L_m^{(i)} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha^{(i)}_{m-n} \alpha^{(i)}_n \quad i = 1, 2 .$$ (9)

They satisfy

$$[ L_n^{(i)}, L_m^{(j)} ] = \delta^{ij} \left\{ (n - m)L_{n+m}^{(i)} + \frac{c}{12} n(n^2 - 1)\delta_{n+m,0} \right\} , \quad c = 1 .$$ (10)

The generators of conformal transformations are thus given by $L_n = L_0^{(1)} + L_0^{(2)}$. The highest weight representations of the Abelian algebra (8) are labeled by the eigenvalues of $J_0^{(1)}$ and $L_0^{(i)}$; the eigenvalue of $(J_0^{(1)} + J_0^{(2)})$ is proportional to the quasi-particle charge $Q$ and $2 (L_0^{(1)} + L_0^{(2)})$ is their fractional statistics $\theta/\pi$. The Hilbert space of the two-component Abelian CFT is made of a consistent set of these representations, which is complete with respect to the fusion rules, which are the selection rules for the composition of quasi-particle excitations in the theory (the so-called bootstrap self-consistent conditions) [6]. In the Abelian theory, the fusion rules simply require the addition of the two-component charges of the quasi-particles; as a consequence, the allowed charge values correspond to the points of a two-dimensional lattice. Each lattice specifies a theory: the adjacency matrix of the lattice contains some parameters which are partially determined by the physical conditions. These determine the quantization of charges and conformal dimensions. Special lattices allow for an extended symmetry [12]: the
extension from $\hat{U}(1) \times \hat{U}(1)$ to $\hat{U}(1)_{\text{diagonal}} \times SU(2)$ is relevant for the present CFT, as we shall consider in a moment. The bosonic field corresponding to the $\hat{U}(1)_{\text{diagonal}}$ is assumed to be compactified on a circle, which introduces the compactification radius as a parameter that defines the unit of charge. The Hamiltonian of the Abelian theory which assigns a linear spectrum to the edge excitations can be written in terms of the currents (7) as follows:

$$H_R = \frac{\pi}{R^2} \int_0^{2\pi R} dx : \left(v_1 J^{(1)}(1) - v_2 J^{(2)}(2)\right) := \frac{1}{R} \left[v_1 L_0^{(1)} - v_2 L_0^{(2)} - \frac{1}{12}\right].$$

We emphasize here, however, that the spectrum that is relevant for discussing the Hilbert space of the CFT is that of the charge and Virasoro operators rather than that of the edge Hamiltonian.

The coupling of the Abelian CFT to the physical electric field should be addressed when considering tunneling experiments. In terms of the individual currents (7), the electric current should be taken as the symmetric combination of both, given that the labels $i = 1, 2$ are arbitrary. We therefore introduce the current and Virasoro generators in the basis that factorizes the theory into charged and neutral sectors [15]:

$$J = J^{(1)} + J^{(2)}, \quad J^3 = \frac{1}{2} \left(J^{(1)} - J^{(2)}\right), \quad L = L^{(1)} + L^{(2)} = L^Q + L^S,$$

$$L^Q = \frac{1}{4} : (J)^2 : , \quad L^S = : (J^3)^2 : ,$$

We have introduced the charge scale parameter $1/\sqrt{2s - 1}$ which is given by the compactification radius of the charged bosonic field. The corresponding spectrum of (12) may be determined from the bosonic Fock space structure and was found to be quantized according to [12, 15]: $s = 2, 4, \ldots$, and that it splits in two cases: (I): $l \in \mathbb{Z}, n \in \mathbb{Z}$, and (II): $l \in \mathbb{Z} + 1/2, n \in \mathbb{Z} + 1/2$. The spectrum yields the values of charge $Q$ and fractional statistics $2h = \theta/\pi$ of the quasi-particle excitations ($h$ is the conformal dimension, i.e., the eigenvalue of $L_0$ and $\theta$ the quantum statistical phase). The discretization of the parameters in (12) is the consequence of imposing upon the spectrum the condition of the existence of excitations with the quantum numbers of physical electrons, i.e., $Q = 1, 2, \ldots$ and $2h = 1, 3, \ldots$. A posteriori, this spectrum is identified with that of the Jain states [13], for opposite moving chiral bosons [12] and filling fractions $\nu = 2/(2s - 1)$, with $s = 2, 4, \ldots$. Each value in the spectrum (12) is the highest weight of a pair of representations of the Abelian current algebra, which describe an infinite tower of edge excitations, generated by the bosonic Fock-space operators $\alpha_n^{(i)}$. 

A posteriori
n < 0, i = 1, 2 \[6\], corresponding to Fermionic particle-hole transitions, or to their anyonic generalizations.

The neutral sector of the spectrum displays an extended symmetry \(SU(2)_1\), which is apparent in this basis. We observe that there are two highest weights with dimension one, \(i.e. \ n = \pm 1\), which correspond to the additional currents:

\[
J^{\pm} = \exp \left( \pm i \sqrt{2} \varphi \right) ; \quad \varphi = \frac{1}{2} \left( \varphi^{(1)} - \varphi^{(2)} \right) .
\]

The two fields \(J^{\pm}\), together with \(J^3\) in \(12\), form the \(SU(2)_1\) current algebra of level \(k = 1\); their Fourier modes satisfy:

\[
\begin{align*}
\left[ J_n^a , J_m^b \right] &= i \epsilon^{abc} J_{n+m}^c + \frac{k}{2} \delta^{ab} \delta_{n+m,0} , \quad k = 1, \ a, b, c = 1, 2, 3, \\
\left[ L_n^z , J_m^a \right] &= -m J_{n+m}^a ;
\end{align*}
\]

note that the operators \(J_n^a\) commute with the generators of the charged sector \(\left( L_n^Q, J_m^a \right)\). As is well known, there are two highest-weight representations of the \(SU(2)_1\) algebra, which are labeled by the isospin \(\sigma = \alpha/2\), with \(\alpha = 0, 1\).

The Jain spectrum of the theory defined by \(12\) and \(11\) can now be written in this basis that makes apparent the decomposition into \(U(1)\) and \(SU(2)_1\) sectors \(15\), so that both cases \(I\) and \(II\) may be displayed together:

\[
\begin{align*}
\nu &= \frac{2}{2s-1} , \quad Q &= \frac{2l - \alpha}{2s - 1} , \\
L_0 &= \frac{(2l - \alpha)^2}{4(2s - 1)} - \frac{\alpha(2 - \alpha)}{4} + r , \quad r \in \mathbb{Z} .
\end{align*}
\]

Here \(l \in \mathbb{Z}\). We verify that, for \(\alpha = 1\) and \(l = s\) we obtain \(Q = 1\) and \(2h = s - 1 + 2r\), which are the correct quantum numbers for electrons.

We now turn our attention to the tunneling between the Mott material and a metal. The characteristics of the tunneling current \(I\) versus the applied voltage \(V\) have been studied in the context of Luttinger models, which are Abelian CFTs, in \(16\) \(17\). For the sake of completeness, we provide here an independent derivation of the simplest of their results based on a scaling argument \(4\) (we thank Andrea Cappelli for suggesting this to us).

The tunneling of a CFT may be described in standard fashion (see \(14\)) at low temperatures: in the experiments it is assumed that the thermal energy is much smaller than any other relevant energy scale. We consider the real coupling of a localized tunneling interaction to the effective CFT
action describing the Mott material as in \[17\] and \[14\]:

\[
S_T = g \int \Psi^\dagger \Psi \delta(x) dx dt, \tag{16}
\]

where \(\Psi(t, x)\) is the free fermion field describing the electrons. Without loss of generality, we assume the tunneling to be localized on the 1D boundary at \(x = 0\).

Only physical electrons tunnel between the material described by the \(CFT\) and the metal, so that \(\Psi\) has electric charge \(Q = 1\) and conformal dimension \(h = p/2\), where \(p\) is an odd integer. We first consider the case of the Laughlin states, with filling fraction \(\nu = 1/p\), in which all the quasi-particles in the spectrum are electrically charged. Therefore, the scaling dimension of the coupling constant \(g\) is \((1 - \nu)\). To first perturbative order in \(g\), this yields \(I \sim g V^p\) because both \(V\) and \(I\) have scaling dimension 1 (\(I\) is the time derivative of the charge, or, equivalently, electrical resistance is dimensionless). Therefore, we conclude that \(dI/dV \sim V^{(p-1)}\). The result for the conductance, is, therefore:

\[
G = dI/dV \sim V^{(1/\nu - 1)}, \tag{17}
\]

which predicts \(m = (1/\nu - 1)\) and that was obtained in \[16\] \[17\] using alternative methods. Following \[14\], we generalize this result to other \(CFTs\) describing interacting electrons such that \(\nu\) may take more general values. Those are theories in which the quasi-particle excitations may have one electrically charged and neutral modes. Nevertheless, there is always an excitation with the quantum numbers of the electron among those, which is the only one that may tunnel into the metal. In these more general scenarios, the tunneling interaction is always irrelevant in the sense of the Renormalization Group \[4\] since (minus) the scaling dimension of the coupling constant \(g\) satisfies \((1/\nu - 1) > 0\), which holds for any values of \(\nu < 1\), i.e., all the ones considered here. It is, however, useful to note that the interaction is increasingly irrelevant as \(\nu\) takes smaller values.

We now consider the \(CFT\) describing the insulator and its relation to the tunneling experiments. As discussed before, the \((c, \bar{c}) = (1, 1)\) \(CFT\) has symmetry \(U(1) \times SU(2)_1\), and \(\nu = 2/(2p - 1), p = 2, 4, \ldots\). The lowest possible values are \(\nu = 2/3, 2/7, \ldots\). As noted before, all of these values yield irrelevant interactions, but \(\nu = 2/3\) is the least irrelevant among them, becoming therefore the favored value. A similar discussion applies to the Fermi interaction for the Beta-decay \[4\]. For this filling fraction, the tunneling exponent yields \(m = (1/\nu - 1) = 1/2\), and this is therefore the prediction of our \(EFT\). This value compares well with the relevant experimental data of \[2\] \[3\]. In
the latter, we consider the experiments done for Sm only, which are the only ones that fit in our assumptions for the EFT, i.e., AF order, insulator. In the former, it is reported that $m \approx 0.43 - 0.47$, whereas in the second it was found that $m \approx 0.44$. The deviation from the theoretical prediction values of 0.5 is of the order of 10%. The magnitude of it is compatible with the corresponding deviation in the quasi-particle tunneling experiments in the QHE quoted by Gattli [14], $(2.7 - 2.65$ vs $3)$. 

We therefore conclude that the low-lying EFT for the Mott materials in the tunneling experiments considered here yields a consistent $(2 + 1)$-dimensional structure composed of two Chern-Simons fields. For each Chern-Simons theory, the excitations are described by two classes of Wilson loops (vortices): charged and neutral modes, of opposite circulation. We speculate that in the low temperature phase, these vortices array themselves into an ordered (AF) lattice [5].

It is worthwhile to mention that the EFT [5] is only the simplest that may be proposed to describe the Mott-class materials. Actually, it simply generalizes to a topological $(2 + 1)$-dimensional theory the underlying $SU(2)$ symmetry of the original fermionic model. The EFT contains the minimal number of degrees of freedom at large distance scales consistent with the original symmetry of the small distance degrees of freedom. By considering microscopic lattice fermion models with additional degrees of freedom as a starting point, it is natural to expected that further EFTs with enlarged symmetries and field content may be obtained.

We would like to conclude with some remarks. The assumption of planarity is not fundamental, since the EFT is a Chern-Simons topological field theory that may be defined in 3D Euclidean space and quantized choosing any time surface [7] [9]. The EFT is the precise field theory describing the universality class of the fermionic model [11] at the critical point, including in particular all the states in the Hilbert space and may be, therefore, regarded as a good starting point for further studies along the lines discussed here. We stress that the relation between the EFT and specific models or characteristics relevant to specific materials pertains only their universal, long-distances emerging universal data [4]. We also remark that the prediction of the universal tunneling exponent $m$ was obtained by a scaling argument, which is a first-order result in Renormalization Group perturbation theory. By the same reason, the zero-width form-factor for the tunneling barrier assumed in [16] does not modify the first-order prediction of $m$, since finite barrier widths would modify [16] by higher order corrections in the transferred momentum, and the EFT considers the limit of vanishing momenta.
References

[1] See, *e.g.*, G. Binnig and H. Rohrer, *Rev. Mod. Phys.*, **59**, 615-625, (1987); E. L. Wolf, *Principles of Electron Tunneling Spectroscopy*, Oxford Science Publications (1985).

[2] Marke Lee and J. G. Massey, V. L. Nguyen and B. I. Shklovskii, *Phys. Rev. B* **60**, 1582-1591,1999.

[3] Patrick B. Marshall, Evgeny Mikheev, Santosh Raghavan, and Susanne Stemmer, *Phys. Rev. Lett.* **117**, 046402,2016.

[4] J. Polchinski, *Effective Field Theory and the Fermi Surface*, Lectures presented at TASI 1992. [arXiv:hep-th/9210046](https://arxiv.org/abs/hep-th/9210046).

[5] F.L. Bottesi and G. R. Zemba, *Ann. Phys.* **326**, 1916-1940,2011.

[6] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, *Nuclear. Phys.B* **241** (1984) 333; for a modern textbook, see: P. Di Francesco, P. Mathieu and D. Senechal, *Conformal Field Theories*, Springer-Verlag, (1996).

[7] E. Witten *Comm. Math. Phys.* **121** (1989) 351.

[8] G. Grensing, *Phys. Lett.B* **419** 258 (1998).

[9] G. R. Zemba, *Int. J. Mod. Phys. A* **5** 559-569,1990.

[10] C. A. Trugengerber *Topics in planar Gauge Theories*, Lectures given at Lausanne U., Winter 1994-1995.

[11] For a review, see: X. G. Wen, *Int. J. Mod. Phys.B* **6** (1992) 1711; *Adv. in Phys.* **44** (1995) 405.

[12] J. Fröhlich and A. Zee, *Nuclear. Phys.* **364 B** (1991) 517; X.-G. Wen and A. Zee, *Phys. Rev.* **46 B** (1993) 2290. J. Fröhlich and E. Thiran, *J. Stat. Phys.* **76** (1994) 209; J. Fröhlich, T. Kerler, U. M. Studer and E. Thiran, *Nuclear. Phys.B* **453** (1995) 670.

[13] For a review see: J. K. Jain, *Adv. in Phys.* **44** (1992) 105, *Science* **266** (1994) 1199.

[14] D. Christian Glattli, *Séminaire Poincaré* **2**, 75-100,2004.

[15] A. Cappelli, G. R. Zemba, *Nucl. Phys. B* **540** 610-638,1999.
[16] C. L. Kane and Matthew P. A. Fisher, *Phys. Rev.* B 46, 15233-15262, 1992.

[17] Claudio de C. Chamon and Eduardo Fradkin, *Phys. Rev.* B 56, 2012-2025, 1997.