**ReaS: Combining Numerical Optimization with SAT Solving**

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**Abstract**

In this paper, we present ReaS, a technique that combines numerical optimization with SAT solving to synthesize unknowns in a program that involves discrete and floating-point computation. ReaS makes the program end-to-end differentiable by smoothing any Boolean expression that introduces discontinuity such as conditionals and relaxing the Boolean unknowns so that numerical optimization can be performed. On top of this, ReaS uses a SAT solver to help the numerical search overcome local solutions by incrementally fixing values to the Boolean expressions. We evaluated the approach on 5 case studies involving hybrid systems and show that ReaS can synthesize programs that could not be solved by previous SMT approaches.

**1 Introduction**

Gradient-based numerical techniques are becoming a popular mechanism for solving program synthesis problems. Neural networks have shown that by doing automatic differentiation over complex computational structures (deep networks), they can solve many complex, real world problems [19, 22, 33]. There is also a growing body of work on using neural networks to learn programs with discrete control structure from examples, such as neural Turing machines [18] and others that follow similar ideas [21, 23, 27, 31]. For many problems with discrete structure, however, recent work [15] has shown that neural networks are not as effective as state-of-the-art program synthesis tools like Sketch [34, 35] that is based on SAT and SMT solving.

This paper shows that a combination of SAT and gradient-based numerical optimization can be effective in solving synthesis problems that involve both discrete and floating-point computation. The combination of numerical techniques and SAT is not itself new; SMT solvers such as Z3 [9], dReal [13] and the more recent work on satisfiability modulo convex optimization (SMC) [32] have also explored problems involving the combination of discrete structure and continuous functions using the DPLL(T) framework [12]. However, we show that the approach followed by SMT solvers, while effective for many applications, is sub-optimal for program synthesis problems. The key problem with prior approaches is the way in which they separate the discrete and the continuous parts of the problem, which loses high-level structure that could be exploited by gradient descent. A consequence of this loss of structure is that for some problems the SMT solver requires an exponential number of calls to the numerical solver; SMC mitigates this by focusing on a special class of problems called monotone SMC formulas but does not generalize to arbitrary problems.

Our technique, called ReaS (for Real Synthesis), exploits the full program structure by making the program end-to-end differentiable. It leverages automatic differentiation [26] to perform numerical optimization over a smooth approximation of the full program. At the same time, ReaS uses a SAT solver to both deal with constraints on discrete variables and constrain the search space for numerical optimization by fixing values of Boolean expressions in the program. This allows ReaS to explore the numerical search space based on the structure of the program.

End-to-end differentiability is achieved by smoothing the Boolean structure such as conditionals using a technique similar to [6] and relaxing Boolean unknowns to reals in the range [0, 1] similar to mixed integer programming. The smoothing algorithm we use is a simplified version to what is used in [6], replacing sharp transitions in conditionals with smooth transition functions such as sigmoid. Despite using a simpler smoothing approach, our technique works better for two reasons. First, the simpler smoothing approach allows us to use automatic/algorithmic differentiation, unlike [6] which had to rely on gradient free optimization techniques (Nelder-Mead) which are not as effective as the gradient-based methods. Most importantly, though, ReaS’s use of a SAT solver to fix the values of the Boolean expressions allows it to better tolerate the inevitable approximation error at branches, allowing us to get better results while using less precise (and more efficient) approximations compared to [6].

**The full system** We implemented the ReaS technique with the Sketch system as the front-end. Similar to Sketch, in ReaS the programmer writes a high-level implementation...
with unknowns. In our case, these unknowns can be either reals or Booleans. In addition, the programmer can also introduce assertions to specify the intended program behavior. The synthesis problem is to find values for these unknowns such that all the assertions in the program are satisfied. The front-end language of Sketch is very expressive which includes support for arrays, ADTs, heap-allocated structures, etc, and hence, all these features can be used to specify the problem in ReaS as well.

**Results** We use ReaS to solve some interesting synthesis problems that are more complex than anything that has been solved by prior work. In particular, we focus on synthesizing parametric controllers for hybrid systems. This is a good fit for ReaS because of the combination of continuous and discrete reasoning, and because the goal is to find satisfying assignments corresponding to the unknown parameters, as opposed to proving unsatisfiability, which our system is unable to do. We show that ReaS can solve these problems in 3-11 minutes whereas previous SMT solvers such as dReal and Z3 cannot solve them.

**Summary of Contributions**

- We present ReaS, a novel technique to combine numerical optimization with SAT reasoning that allows us to do efficient reasoning on programs involving both discrete and continuous functions.
- ReaS achieves end-to-end differentiability in the presence of discrete structure using smoothing and relaxing techniques and it uses a SAT solver to make discrete decisions to overcome local solutions from numerical search.
- We evaluated the system by synthesizing parametric controllers for several interesting hybrid system scenarios that previous systems cannot solve.

2 Overview

In this section, we first present a stylized synthesis task to illustrate the kinds of problems ReaS can handle and then use a synthetic example to illustrate how ReaS works.

2.1 Illustrative Example

**Example 2.1 (Lane change controller).** Consider the scenario shown in Figure 1. The task is to synthesize a controller program that can move the car 1 from lane 1 to lane 2 within $T$ time-steps without colliding with any of the other cars.

The program with unknowns for the controller is shown in Figure 2. The controller is composed of 5 modes. Each mode has rules for setting the two controls of the car—the acceleration and the steering angle. The synthesizer must discover a sequence of steps to perform the lane change; first accelerating until it is in the correct position to do the lane change, then starting the lane change by turning the wheels to the left, then turning right once it is in the new lane, then adjusting its velocity to match the other cars, and finally maintaining a stable velocity. The unknowns in the program are the switching conditions for changing from one mode to another, and the exact values for the acceleration and the steering angle in each mode.

The space of switching conditions is described by the function genSwitchExp() which encodes a discrete choice between four different inequalities on the relevant state variables. Note that the genSwitchExp function is marked as a generator. A generator in ReaS is treated as a macro that gets inlined at every call, and the solver can choose different values for the unknowns for each different instantiation.

Figure 3 shows the specification for this synthesis task. The specification simulates $T = 50$ time steps where at each time step, it calls the controller code that sets the control values based on the current state of the cars and then, moves the car and the world by one time step ($dt = 0.1s$). In this problem, we assume that the other vehicles around Car 1 in the world have a constant velocity. Finally, the specification asserts that there is no collision at any time step and the goal is reached at the end of the simulation.

Solving this problem is a significant challenge for three reasons. First, the five-mode controller program is simulated 50 times in the specification, so there are $8 \times 10^{34}$ paths in the synthesized program. Moreover at each timestep, DetectCollision has to check for collisions against each of 5 other cars, and each of these checks has to consider 8 separate conditions that can indicate a collision between two cars, so there are a total of 2000 checks. Second, this sketch has 12 Boolean unknowns and 22 real unknowns, so there is a very large space to search. Finally, we use a bicycle-model for the dynamics of the car. Even though this model is simpler than dynamics of a real car, it is still fairly complex and involves non-linear functions such as sine, cosine, and square root.

Because of theese challenges, existing approaches do not perform well on this synthesis task. Many SMT solvers such as Z3 do not provide full support for non-linear real arithmetic and hence, cannot synthesize this program. We also found that dReal, an SMT solver for reals, is unable to solve this problem (see Section 5). On the other hand, smoothing approaches also fail because of the amount of boolean structure in the problem. For comparison, the most complex problem that was solved by the system in [6] was similar to this one, but involved only 1 obstacle (not 5), with 2 collision conditions (instead of 8), and only scaled to 35 time steps.

![Figure 1. Lane change initial scenario](image-url)
null
**Numerical optimization with Smoothing.** The first important feature of ReAS is the ability to use numerical optimization by performing automatic differentiation on a smooth approximation of the program. Figure 4(a) shows the graph for $a$ as function of $x_1$ and Figure 4(b) shows its smooth approximation. What jumps out from the graph is that while the branches do introduce discontinuities, the function is still very amenable to numerical optimization. However, numerical optimization can introduce its own problems. For example, even with the ability to smooth away discontinuities and automatically compute derivatives for the whole program, it is clear from the figure that gradient-based optimization will only succeed if it starts at $x_1 > 0$. Otherwise, the algorithm will be stuck on a local minima. In this example, initializing numerical search uniformly at random on the allowed range of $x_1$ would give a 77% probability of failing with a local minima, which is already bad, but it is not hard to see how a small change to the program could make this probability arbitrarily close to 100%. Smoothing can ameliorate the problems inherent in numerical search, but cannot eliminate them altogether.

**Using SAT solver to eliminate local solutions.** The ReAS approach is to turn the SMT paradigm on its head. In SMT, the SAT solver always has an abstraction of the complete program, the numerical solver checks that the current solution corresponds to that solution; in this case $\overline{y_1}, \overline{y_2}, \overline{y_3}, y_4, \overline{y_5}$. As the SAT solver sets these variables and checks them against the boolean constraints, the numerical solver checks that the current solution is still valid when the respective branches have been fixed in the program. As a result, the SAT solver helps refine the solution provided by the numerical solver, until a precise solution to the overall problem has been produced.

3 The ReAS Intermediate Representation

The core language (L) of ReAS is shown below. The language consists of real-valued expressions $E$, Boolean expressions $B$ and Boolean unknowns $H$. The real-valued expressions can either be a real unknown $x$, a constant $c$, a real-valued operation $op$ such as addition, multiplication, sine, cosine, etc, or a if-then-else expression $ite$. The Boolean expressions can either be a comparison $E \geq 0$, a conjunction, or a negation. There are two kinds of $ite$ expressions in the language—one where the conditional is a Boolean expression and the other where the conditional is a Boolean unknown because they are treated differently for the numerical problem. The language does not allow Boolean unknowns to appear anywhere other than as a conditional. However, this does not decrease the expressive power of the language since those cases can be reduced to the core language easily. The imperative programs shown in Section 2 can be converted into this core language using straight-forward transformation passes and loop unrolling.

\[
\begin{align*}
E & := x \mid c \mid op(\{E_1, E_2\}) \mid ite(B, E_1, E_2) \mid ite(H, E_1, E_2) \\
B & := E \geq 0 \mid B_1 \wedge B_2 \mid \neg B \\
H & := y
\end{align*}
\]

The semantics of some features of the language is shown below. It is defined in terms of a mapping $\sigma$ from the unknowns to actual values.

\[
\begin{align*}
[x]_\sigma & := \sigma[x] \\
[y]_\sigma & := \sigma[y] \\
\llbracket E_1 + E_2 \rrbracket_\sigma & := \llbracket E_1 \rrbracket_\sigma + \llbracket E_2 \rrbracket_\sigma \\
\llbracket ite(B, E_1, E_2) \rrbracket_\sigma & := \llbracket B \rrbracket_\sigma \cdot \llbracket E_1 \rrbracket_\sigma + (\neg \llbracket B \rrbracket_\sigma) \cdot \llbracket E_2 \rrbracket_\sigma \\
\llbracket E \geq 0 \rrbracket_\sigma & := \llbracket E \rrbracket_\sigma \geq 0
\end{align*}
\]

3.1 Synthesis Problem

The synthesis problem is given a program $P = \text{assert}(B_1, \cdots, B_k)$, find $\sigma$ such that $\forall k. \llbracket B_k \rrbracket_\sigma = 1$. For now on, we will use $\psi(\sigma)$
to represent this predicate. So, the synthesis problem is to solve the formula: $\exists \sigma. \psi(\sigma)$.

ReAS divides the synthesis problem into a SAT part ($\psi_{B}$) and a numerical part ($\psi_{N}$). We, first, describe the abstraction process for producing $\psi_{B}$ and $\psi_{N}$ and Section 4 describes the algorithm to solve $\psi$ by repetitively solving $\psi_{B}$ and $\psi_{N}$.

### 3.2 Boolean Abstraction ($\alpha_{B}$)

The Boolean abstraction is obtained similar to the process used in SMT solvers. Concretely, the language for Boolean constraints ($L_{B}$) is:

$$
\overline{F}_{B} := \text{assert}(\overline{B}_{1}, \ldots, \overline{B}_{k})
$$

$$
\overline{B} := y | \overline{B}_{1} \land \overline{B}_{2} | \neg \overline{B}
$$

The language is almost same as the $B$ expressions in the original language. The only exception is that the term $\overline{E} \geq 0$ is now replaced by Boolean unknowns.

The Boolean abstraction, $\alpha_{B}$, is a function from $L$ to $L_{B}$ and is defined as below:

$$
\alpha_{B}(y) = y
$$

$$
\alpha_{B}(E \geq 0) = \text{Create new } y
$$

$$
\alpha_{B}(B_{1} \land B_{2}) = \alpha_{B}(B_{1}) \land \alpha_{B}(B_{2})
$$

$$
\alpha_{B}(\neg B) = \neg \alpha_{B}(B)
$$

### 3.3 Numerical Abstraction ($\alpha_{N}$)

We now describe the abstraction process to produce $\psi_{N}$ from $\psi$. The abstraction is defined with respect to a function $I$, called interface mapping, that maps every Boolean expression in $\psi$ to one of $\{0, 1, \perp\}$. $I(B_{i}) = v_{i} \neq \perp$ means that the Boolean expression $B_{i}$ should have the value $v_{i}$. If $v_{i} = \perp$, then the value is not yet set. The interface mapping specializes a synthesis problem defined by $\psi$ to

$$
\exists \sigma. \psi[\{B_{k}/v_{k}\}_{k}]|\sigma) \land I[B_{i}]|\{B_{k}/v_{k}\}_{k \neq i}]|\sigma = v_{i}
$$

where the spec $\psi$ is first simplified by substituting the expressions $B_{k}$ with the values $v_{k} = I(B_{k})$ when $v_{k} \neq \perp$, and additional constraints are added to ensure that $\sigma$ satisfies the assignments in $I$.

The goal of the numerical abstraction ($\alpha_{N}$) is to produce a smooth approximation $\psi_{N}$ of the specialized problem above that can be fed to an off-the-shelf numerical solver to find an assignment to $\sigma$ if one exists. Note that unlike the SMC approach, ReAS does not require $\psi_{N}$ to be composed of linear/convex functions. However, we want $\psi_{N}$ to be smooth and continuous because numerical algorithms perform poorly in the presence of discontinuities. The main source of discontinuity arises when the Boolean expression in an $\land$ or a conjunction is not yet set by $I$. ReAS eliminates these discontinuities by performing a program transformation that replaces the sharp transitions with smooth transition functions such as sigmoid as described in the next subsection.

#### 3.3.1 Abstraction rules

The language for numerical constraints ($L_{N}$) in ReAS is shown below. At the top level, we have a conjunction of numerical inequalities ($\overline{E} \geq 0$) where $\overline{E}$ is again similar to $E$ in the original language minus the $\land$ expression.

$$
\overline{F}_{E} := \overline{E}_{1} \geq 0 \land \cdots \land \overline{E}_{n} \geq 0
$$

$$
\overline{E} := x | c \cdot \text{op}((\overline{E}_{1}), \ldots, \overline{E}_{n})
$$

Note that there are numerical algorithms (such as sequential quadratic programming) that take in a conjunction of inequalities and perform constrained optimization on them directly. Even for numerical algorithms that can only perform unconstrained optimization (such as plain gradient descent), it is easy to transform a conjunction of inequalities into a smooth objective function.

Given a program $\overline{F} = \text{assert}(B_{1}, \ldots, B_{k})$ in $L$, the goal of the abstraction is to produce a program $\overline{F}_{E}$ in $L_{N}$. In order to do that, we define a transformation rule $\overline{e} \rightarrow (\overline{e}, \overline{p})$ where $\overline{e}$ is an expression in $L$ (either an $\land$ or a $B$ expression), $\overline{e}$ is the corresponding expression in $L_{N}$, and $\overline{p}$ is the conjunction of numerical constraints obtained so far. This formulation allows us to collect numerical constraints from intermediate expressions if necessary. The transformation is defined in terms of two parameters: $(\beta, \epsilon)$. $\beta$ is the smoothing parameter that controls how smooth the approximation should be. Higher values for $\beta$ mean less smoothing. $\epsilon$, a small positive constant, is the precision of numerical calculation that arises due to im-precise floating point computation. Because of this, the expression $\overline{E} \geq 0$ in $L_{N}$ actually means $\overline{E} \geq -\epsilon$.

The rules for performing the abstraction for expressions in the language are shown in Figure 5. We first focus on the rules for expressions other than $\land$ and conjunction expressions. For an $\land$ expression, the abstraction produces an $\overline{E}$ that smoothly approximates the original expression. For simple cases such as a real unknown and constant, the abstraction just returns the same expression as shown in the RHOLE and CONST rules. In this case, there are no intermediate constraints and hence, $p$ is just true (T). For an $\text{op}$ expression, the OP rule recursively smooths the expressions in its arguments and then creates a new $\text{op}$ expression with these smoothed replacements and concatenates the intermediate constraints obtained from abstracting the arguments. ReAS assumes that $\text{op}$ is itself a smooth and continuous operation. For operations like division or $\sqrt{t}$ that are not defined for all inputs, we replace them with continuous approximations, and rely on the frontend to introduce assertions that ensure their inputs stay away from the regions where the approximations differ significantly from the true operations.

In order to understand the abstraction rules for the $B$ expressions, we first define a function called $P$-distance.

**Definition 3.1 (P-distance).** The $P$-distance (short for positive-distance) for a Boolean expression $\overline{b}$ in $L$ is a function $P_{d}$
that takes in an assignment to the unknowns $\sigma$ and produces a real value such that $\forall x ((P_d(\sigma) \geq c \implies [b_1]_\sigma = 1) \land (P_d(\sigma) \leq -c \implies [b_2]_\sigma = 0))$. Thus, for numerical purposes, we can assume $b \approx P_d \geq 0$. There can be more than one $P$-distance function for the same Boolean expression.

The abstraction rule for a $B$ expression in $L$ produces an $E$ expression that is a smooth approximation to a $P$-distance function for $B$. For example, for $e \geq 0$, the natural choice for its $P$-distance function is $e$ itself. So, the rule $GE$ first computes the smooth approximation of $e$. For Boolean expressions, the rules also need to take into account whether there is a value assigned to them in the mapping. This is done using the $\text{MatchNUpdate}$ function which takes in the smoothed expression $\bar{e}$ and the value $v$ from $I$. Then, it checks for one of the three cases: 1. if $v = \perp$, it means that the value is not yet set and hence, there is no update and no new constraints, 2. if $v = 1$, then the expression $\bar{e}$ is replaced with a large positive constant ($K = 100$) and a new constraint $\bar{e} \geq 0$ is added, and 3. if $v = 0$, then similar to case 2, the expression $\bar{e}$ is replaced with a large negative constant and the condition $-\bar{e} \geq 0$ is added. The reason for replacing $\bar{e}$ with constants in cases 2 and 3 is so that other expressions that depend on $\bar{e}$ can infer the Boolean value of the $B$ expression it represents without any ambiguity (because of the large magnitude of these constants). The NOT rule for the $\neg b$ expression, similarly, first computes the abstraction for $b$ and negates the expression obtained to get a smooth approximation to a $P$-distance for $\neg b$.

Finally, the abstraction for the assert expression iteratively smooths each of its $b$ arguments and creates the final set of numerical constraints $P_E$ to form the numerical problem $\psi_E$. This final set of constraints is the conjunction of the constraints obtained after transforming each Boolean argument ($p_i$) and as well as constraints to ensure that the abstraction of each Boolean argument ($\bar{e}_i$) is greater than 0.

**Example 3.2.** Consider the program $P =$ assert($\text{ite}(x_1 \geq 0, x_2, x_3)$) where $x_1, x_2, x_3$ are real unknowns and let $I(x_1 \geq 0) = 0$. The abstraction for $x_1 \geq 0$ results in $(-100, -x_1 \geq 0)$. So, the $\text{ite}$ expression can be thought as $\text{ite}(-100 \geq 0, x_2, x_3)$. Even though we have not yet discussed the rule for abstracting $\text{ite}$ expressions, in this case, it is clear that the abstraction should just produce $x_3$. Overall, abstracting $P$, in this example, will result in $P_E = x_3 \geq 0 \land -x_1 \geq 0$.

Now, we can look at the rules for expressions that introduce discontinuities i.e. if-then-else and conjunctions.

**If-then-else** Let us first consider the $\text{ite}$ expression of the form $\text{ite}(b, e_1, e_2)$ that has a Boolean expression as the condition. The rule $\text{ITE1}$ describes the abstraction for these expressions. The rule first recursively smooths the expressions $e_1$ and $e_2$ resulting in expressions $\bar{e}_1$ and $\bar{e}_2$. Then,

\[ \text{ITE1} \]

\[
\begin{align*}
\text{ITE1} & \quad \bar{e} = \bar{e}_1 \land \bar{e}_2, (1 - \bar{e}_2), t = \mathcal{F}_s(\bar{e}_2 - \bar{e}_1) \\
& \quad \text{ite}(b, e_1, e_2) \Rightarrow (\bar{e}_1, p_1 \land p_2 \land p_c) \\
\end{align*}
\]

\[ \text{ITE2} \]

\[
\begin{align*}
\text{ITE2} & \quad \bar{e} = \bar{e}_1 \land \bar{e}_2, (1 - \bar{e}_2), t = \mathcal{F}_s(\bar{e}_2 - \bar{e}_1) \\
& \quad \text{ite}(y, e_1, e_2) \Rightarrow (\bar{e}_1, p_1 \land p_2 \land p_c) \\
\end{align*}
\]

\[ \text{MATCHNUPDATE}(\bar{e}, v) = \begin{cases} 
(K, (\bar{e} \geq 0)), & \text{if } v = \perp \\
(K, (-\bar{e} \geq 0)), & \text{if } v = 1 \\
(\neg K, (\bar{e} \geq 0)), & \text{if } v = 0 
\end{cases} \]

Figure 5. Numerical abstraction rules. $T$ stands for true, $K$ is a large positive constant (100) and $\delta$ is a small positive constant. Note that expressions such as $e_1 + e_2$ are symbolic expressions. $\mathcal{F}_s$ is a smooth transition function.
the condition \( b \) is also transformed to produce the approximation for its P-distance function \( \tilde{e}_c \). If we were to use the actual semantics of the ite expression, we would get \( \tilde{e}_1 + (\tilde{e}_c \geq 0) + \tilde{e}_2 * (1 - (\tilde{e}_c \geq 0)) \). Note that the operations +, * in the above expression are actually symbolic operations. Clearly, this function \( \tilde{e}_c \geq 0 \) has a discontinuity at \( \tilde{e}_c = 0 \). To overcome this discontinuity, the ITE1 rule replaces \( \tilde{e}_c \geq 0 \) with \( F_\tilde{e}(\tilde{e}) \) where \( F_\tilde{e} \) is a smooth transition function:

\[
F_\tilde{e}(x) = \text{sigmoid}_\beta(x) = \frac{1}{1 + e^{-\beta x}}
\]

The rule ITE2 describes the abstraction for expressions of the form \( \text{ite}(y, e_1, e_2) \). In this case, \( y \) is abstracted using the BHOLE1 or the BHOLE2 rule. In the case where \( I \) already fixes the value of \( y \) to 1 or 0, the ite expression will be simplified to just \( \tilde{e}_1 \) or \( \tilde{e}_2 \) respectively. If the value of \( y \) is not yet set, then the rule BHOLE2 creates a new real unknown \( x \) corresponding to \( y \). This new variable is constrained to be in the interval \([0, 1]\). In addition, the constraint \( x(1 - x) \leq \delta \) where \( \delta = 0.1/\beta \) is added to enforce that \( x \) is either close to 0 or 1. The ite expression is, then, abstracted by a linear combination of \( \tilde{e}_1 \) and \( \tilde{e}_2 \) such that when \( x = 1 \), the abstraction will result in \( \tilde{e}_1 \) and when \( x = 0 \), the result will be \( \tilde{e}_2 \).

**Conjunctions** The rule for abstracting a conjunction of two Boolean expressions is based on the following lemma:

**Lemma 3.3.** Let \( P_{d_1} \) and \( P_{d_2} \) be the P-distance functions for Boolean expressions \( b_1 \) and \( b_2 \). Then, \( P_d = \min(P_{d_1}, P_{d_2}) \) is a P-distance function for \( b_1 \land b_2 \).

Based on the lemma, the AND rule first gets the abstractions of \( b_1 \) and \( b_2 \) and then smooths the min of the results. The smoothing is done by rewriting \( \min(\tilde{e}_1, \tilde{e}_2) \) as \( \text{ite}(\tilde{e}_2 - \tilde{e}_1 \geq 0, \tilde{e}_1, \tilde{e}_2) \) and applying the ITE1 rule.

### 3.3.2 Computing gradients

One of the advantages of our numerical abstraction algorithm is that once the smoothed program \( \tilde{P}_E \) is produced, we can use automatic differentiation [26] to symbolically compute the gradients necessary to perform gradient-based numerical search. For example, consider the expression \( e = e_1 + e_2 + e_3 \) with two unknowns \( x = [x_1, x_2] \) and let \( \sigma \) be an assignment to the unknowns. Since there are two unknowns, each sub-expression will have two gradients, i.e.

\[
\Gamma(e, \sigma) = \left[ \frac{\partial e}{\partial x_1}, \frac{\partial e}{\partial x_2} \right] \]

where \( \Gamma \) is the notation used to get the gradients for any sub-expression \( e \) at the assignment \( \sigma \). Automatic differentiation applies the chain rule repeatedly to each elementary expression i.e. in the above example

\[
\Gamma(e, \sigma) = [e_3 | \sigma] + (\Gamma(e_1, \sigma) + [e_2 | \sigma] + \Gamma(e_2, \sigma) + \Gamma(e_3, \sigma)
\]

This process allows us to calculate the gradients accurately and in time that is proportional to the time it takes to evaluate an expression.

![Figure 6. Overview of the solver interface.](image-url)

#### 3.4 Properties of Numerical Abstraction

Let \( e \) be the original expression, \( \tilde{e} \) be the result of the abstraction and let \( x \) be the set of unknown reals in \( \tilde{e} \), then the following theorems hold for any \( I \).

**Theorem 3.4 (Continuity).** \( \tilde{e} \) is continuous with respect to \( x \) under the assumption that all op operations in \( e \) are continuous with respect to their operands.

**Theorem 3.5 (Differentiability).** \( \frac{\partial \tilde{e}}{\partial x} \) is continuous with respect to \( x \) under the assumption that all op operations in \( e \) are differentiable with respect to their operands.

**Theorem 3.6 (Closeness).** \( \lim_{\beta \to \infty} e \approx \tilde{e} \)

This theorem states that when \( \beta \to \infty \), both \( e \) and its abstraction \( \tilde{e} \) agree on almost all assignments to the unknowns. We say almost because \( e \) and \( \tilde{e} \) may not agree at the branching points. For example, consider \( \text{ite}(x \geq 0, 1, 0) \). In this case when \( x = 0 \), the abstraction will have undefined behavior.

#### 4 Numerical Solver + SAT Solver

Given a synthesis problem \( \exists \sigma, \psi(\sigma) \), the last section described the algorithms to produce the Boolean abstraction \( \psi_B \) and the numerical abstraction \( \psi_N \) when given an interface mapping \( I \). In this section, we will describe the algorithm to combine the abstractions so that the original problem can be solved.

#### 4.1 Architecture

The high-level architecture showing the interaction between the different components of the system is shown in Figure 6. First, there is a REAS core that acts as an interface between the numerical solver and the SAT solver. The core is responsible for creating the appropriate abstractions for the numerical solver and the SAT solver and invoking them when necessary to solve \( \psi \). The core also maintains the interface mapping \((I)\). The core will be discussed in detail in Section 4.2.

The numerical solver component can be any black box gradient-based algorithm that can take in a set of constraints
ψ_N^β and perform numerical search to produce SAT/UNSAT along with the satisfying assignment to the real unknowns. There can be two issues with the result produced by the numerical solver. First, if the result is SAT, it is possible that the satisfying assignment is not a true solution because of the approximations introduced by smoothing. Second, when the result is UNSAT, it is possible that the numerical algorithm ran into a local solution since gradient-based algorithms are incomplete in the presence of non-convex functions.

To overcome these issues with the numerical solver, we have a SAT solver component that gradually fixes values to the Boolean expressions in ψ. Fixing values to Boolean expressions will eliminate the approximations introduced by smoothing because, now, we do not have to smooth for any discontinuities these Boolean expressions may cause. Moreover, it allows the numerical solver to focus the search in the region where the assignments to the Boolean expressions are satisfied.

In order to do this, we had to make several changes to the SAT solver. A typical SAT solver interface takes in a problem, converts to CNF constraints, solves it and finally produces SAT/UNSAT. But, for the interaction with the numerical solver, the SAT solver needs to be incremental similar to the DPLL(T) solvers. However, there are two main differences between our approach and the DPLL(T) approach. First, in DPLL(T), the theory solver throws conflicts whenever the current assignment to the theory atoms is unsatisfiable. However, in our case, because the numerical solver can run into local solutions, the conflicts are not strong. Therefore, in ReAS, the numerical solver throws soft conflicts instead. Second, when the current assignment is satisfiable, the theory solver in DPLL(T) approach also returns a list of consequences for any new variables that are forced by current assignment. In ReAS, when a satisfiable solution is found by the numerical solver, this solution can be used to set values for the Boolean expressions and because of the closeness property of the numerical abstraction, most of these assignments can contribute to producing the final correct solution. However, it is not possible to mark these assignments as strong consequences because a different solution will produce different assignments to the Boolean expressions. Hence, in ReAS, we treat these assignments as suggestions that the SAT solver can use. These suggestions are treated as new decisions in the SAT solver, and can be retracted if it resulted in a conflict later.

Thus, the SAT solver in ReAS has two new data-structures.

- **Soft Learnts.** This data-structure is similar to the learnts data-structure used for normal conflicts. Soft learnts data-structure keeps the list of all learnt clauses arising from soft-conflicts as well as any other conflict that arises due to these soft learnts.

- **Suggestions.** Suggestions data-structure is an ordered list of assignments to the Boolean expressions suggested by the numerical solver.

The SAT solver provides the following operations that the ReAS core can call:

- **Init(ψ_B).** Initialize the problem with the constraints from ψ_B.
- **SolveIncremental().** Does incremental SAT solving by setting values to new variables and doing unit propagations. When picking a new variable to set, the SAT solver first tries the assignments in the suggestions data-structure. The method stops once an interface variable is set or if the problem is unsatisfiable. The output is the list of new interface mapping along with SAT / SOFT_UNSAT / UNSAT. A SOFT_UNSAT is an UNSAT that is caused due to soft learnts.
- **AddSoftConflict(conflict clause).** Performs conflict analysis and adds the learnt clause to the soft learnts data-structure.
- **RemoveSoftLearnts().** Clears soft learnts.
- **SetSuggestions(list of suggestions).** Populates the suggestions data-structure.
- **RemoveSuggestions().** Clears suggestions.
- **Restart().** Backtracks all assignments made by the SAT solver so far.

With these components, we can, now, describe the actual interaction algorithm used by the ReAS core.

### 4.2 ReAS Core

The algorithm that ReAS uses to solve ψ is shown in Figure 7. We use S to denote the SAT solver and N to denote the numerical solver. The algorithm starts by initializing S with the Boolean abstraction ψ_B. The interface mapping I is initialized to Empty meaning all Boolean expressions are assigned to ⊥. Then, there are three phases that are performed repeatedly in a loop until a solution is found or the problem is detected to be unsatisfiable or the resource limits are reached. At a high-level, the first phase runs N on the numerical abstraction ψ_N for the current I, the second phase updates the SAT solver data-structures based on the result from N, and the third phase does more SAT solving to create a new I.

In the first phase, similar to [6], ReAS runs the numerical solver multiple times for various values of the smoothing parameter β. It starts with a small value for β i.e. more smoothing and gradually increases the value of β until a certain limit is reached, each time the numerical solver starting its search from the assignment found for the previous β value. The numerical solver returns the solution found in the final iteration.

In the second phase, if the result returned by N is SAT, the core uses the satisfying assignment to generate suggestions for the SAT solver. Figure 8 shows the algorithm ReAS uses
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**SOLVE**(ψ):
1: \( \psi_B = \alpha_B(\psi) \)
2: \( S.\text{Init}(\psi_B) \)
3: \( I = \text{Empty} \)
4: \textbf{while} true \textbf{do}
5: \( \psi_N = \alpha_N(\psi, I) \) \text{ \textit{Phase 1}}
6: \( (\text{res}, \sigma_x) = N.\text{Solve}(\psi_N) \)
7: \textbf{if} res = SAT then \( S.\text{SetSuggestions}(\sigma_x) \)
8: \( s = \text{GenSuggestions}(\sigma_x) \)
9: \( S.\text{SetSuggestions}(s) \)
10: \textbf{else}
11: \( S.\text{RemoveSuggestions}() \)
12: \textbf{if} IsConflict(I) then
13: \( c = \text{GenConflict}(\sigma_x) \)
14: \( S.\text{AddSoftConflict}(c) \)
15: \( (\text{res}, I') = S.\text{SolveIncremental}) \) \text{ \textit{Phase 3}}
16: \textbf{if} res = SAT then
17: \( I = I' \) \textbf{then}
18: \textbf{return} SAT
19: \textbf{else}
20: \( I \leftarrow I' \)
21: \textbf{if} res = UNSAT then \textbf{return} UNSAT
22: \textbf{if} res = SOFT_UNSAT then
23: \textbf{if} num_restarts < RESTART_LIMIT then
24: \( S.\text{RemoveSoftLearnts}() \)
25: \( S.\text{Restart}() \)
26: \( I = \text{Empty} \)
27: \textbf{else}
28: \textbf{return} UNSAT
29: \textbf{end}

**GenSuggestions**(\( \sigma_x, I \)):
1: \( s = \text{Priority List} \)
2: \textbf{for} \( b \in I \) such that \( I(b) = \bot \) \textbf{do}
3: \( d = ||\alpha_N(b)||_\sigma_x \)
4: \textbf{Add} \( (\psi, d \geq 0) \) to \( s \) with cost = |\( d \)|
5: \textbf{for} \( y \in I \) such that \( I(y) = \bot \) \textbf{do}
6: \( d = ||\alpha_N(y)||_\sigma_x \)
7: \textbf{Add} \( (\psi, d \geq 1/2) \) to \( s \) with cost = |\( d - 1/2 \)|
8: \textbf{return} \( s \)

Figure 7. ReaS algorithm.

Figure 8. Algorithm to generate suggestions.

to generate suggestions. The algorithm iterates over each Boolean expression in \( I \) whose value is not yet set and evaluates its numerical abstraction on the current assignment producing \( d \) (Line 3). Then, the algorithm suggests the value of this Boolean expression to be \( d \geq 0 \) because recall that \( d \) actually represents the evaluation of a smooth-approximation to the P-distance function. For a Boolean unknown, the algorithm similarly evaluates its abstraction on the current assignment. However, in this case, \( d \) is a value in \([0, 1]\) and the algorithm suggests that the Boolean unknown is True if \( d \geq 1/2 \). Each suggestion is also associated with a cost and the SAT solver tries the suggestions in the increasing order of cost. Our algorithm gives lower cost to Boolean expressions/unknowns that are most uncertain (i.e. close to a branch) as they are more likely reasons for introducing inaccuracies in the numerical abstraction and hence, the SAT solver tries them first.

If the result returned by \( N \) is UNSAT, then there are two possible options: 1. the SAT solver can continue to set values for more Boolean expressions assuming that the numerical solver might have run into a local solution or 2. the core can generate a soft conflict that would make the SAT solver come up with a different setting to \( I \), hence, temporarily directing the search to a different region. The heuristic IsConflict decides whether to do option 1 or 2. ReaS uses a simple heuristic that checks if the number of Boolean expressions that have values \( \neq \bot \) is greater than a conflict threshold (\( \eta = 5 \)) to generate a soft conflict.

In phase 3, \( S.\text{SolveIncremental} \) is called which tries to set more Boolean variables. If the outcome of this call is SAT and there is no change to \( I \), then it means that a correct solution is found and the recent state of the SAT solver and the numerical solver will provide the values for \( \sigma_y \) and \( \sigma_x \). On the other hand, if the result is UNSAT (due to normal conflicts), then it means that \( \psi_B \) is unsatisfiable which implies \( \psi \) is also unsatisfiable. If the result is SOFT_UNSAT, this means that the soft conflicts led the SAT solver to detect an un satisfiability. Since the soft conflicts are just temporary conflicts, the algorithm clears the soft conflicts and restarts the SAT solver so that the SAT solver can make different decisions this time. When the number of restarts exceeds the RESTART_LIMIT, a SOFT_UNSAT is treated as an UNSAT. In all other cases, the algorithm continues the while loop with the new \( I \) generated by the SAT solver.

### 4.3 Properties of ReaS

If ReaS produces a solution \( \sigma \) to \( \psi \), then \( \psi(\sigma) \) is true. Note, only smoothing approaches such as [6] do not have this property because of the approximations introduced during smoothing.

However, there are situations where ReaS may not find a solution even if there exists a solution. In practice, however, a right balance between the conflict threshold \( \eta \) and the RESTART_LIMIT allows ReaS to solve some of the complex synthesis problems as shown in Section 5.

### 5 Evaluation

In this section, we evaluate ReaS on 5 case-studies involving hybrid systems. In particular, we focus on answering the following questions: (1) Can ReaS solve complex synthesis problems that involve a combination of discrete and
continuous reasoning? (2) How does it compare with only smoothing techniques? (3) How does it compare with existing SMT solvers and mixed integer approaches?

**Experimental Setup** ReaS uses the SNOPT software [16] for performing numerical optimization. SNOPT uses sequential quadratic programming and can handle constrained optimization better than standard gradient descent techniques. For the SAT solver, we took the MiniSAT solver and modified it as described in Section 4. All experiments are run on a machine using 2.4GHz Intel i5 core with 8GB RAM.

### 5.1 Benchmarks

Apart from the lane changing example in Section 2, we used ReaS to synthesize two benchmarks involving a 1-dimensional quad-copter, the parallel parking benchmark from [7], and the thermostat benchmark from [20]. Figure 9 lists the 5 benchmarks together with some statistics such as the number of Boolean and real unknowns, the number of iterations the controller (to be synthesized) is simulated in the specification, and the total number of Boolean expressions and assertions. Full benchmark problems along with the demos of the synthesized solutions can be found in the supplementary material.

**Quadcopter obstacle avoidance** The task is to synthesize a controller to perform the maneuver shown in Figure 10 without colliding with the obstacle. The program with unknowns for this controller is shown in Figure 11. This controller has three modes where each mode uses a proportional-derivative (PD) controller to set the forces that need to be generated by the two rotors of a simplified 1-D quadcopter. The synthesizer is required to find the switching conditions (which are based on the position of the copter similar to the lane change benchmark) as well as the parameters of the PD controllers for the different modes. Note that, in this case, a single PD controller is not sufficient to perform the task and hence, it is necessary to compose multiple PD controllers as shown in the template. The synthesizer is also required to find the values of the intermediate desired states for the PD controllers.

**Quadcopter landing** Using the same template shown in Figure 11, but with different specifications, it is possible to synthesize controllers for achieving other goals. In this benchmark, we synthesize a controller for landing a quadcopter gracefully. The copter starts at a position above the ground with an initial thrust that imbalances the copter and target is to reach a position on the ground without crashing. There are no obstacles (other than the ground) in this case, but the synthesizer still needs to figure out how to compose the different PD controllers to achieve the goal.

**Parallel parking** This benchmark synthesizes a controller to parallel park a car as described in [7] (a tool based on the smooth interpretation work [6]). The template for this benchmark is similar to the template for the lane changing benchmark. Our template is different from [7] in two aspects. [7] uses switching conditions based on time; we replaced them with conditions based on the state of the car since it leads to more robust controllers. [7] only uses 10 time-steps to do the simulation, but in our template, we decrease the step size and increased the number of simulation steps to 100. This reinforces the fact that our technique scales much better than the smoothing technique that [7] uses.

**Thermostat** The final benchmark is to synthesize the thermostat controller described in [20]. This thermostat is a state machine with four states: OFF, HEATING, ON and COOLING. The switching conditions for transitioning from HEATING to ON and COOLING to OFF are fixed by the constraints of the thermostat’s heater. The other two switching conditions should be figured out by the synthesizer such that the temperature of the room is maintained between 18°C and 20°C. In addition to the safety conditions, this benchmark encodes some performance metrics, in particular, it adds minimum dwell time constraints for the OFF and ON phases. The minimum dwell time constraint states that the thermostat should at least be in the state for T seconds before transitioning to the next state. The system in [20] takes in these universal constraints (that should be true in every state) and uses a fix-point computation based algorithm to find the switching conditions. In ReaS, we instead specify the problem by simulating the thermostat for 500 time steps with δt = 2s and asserting that the constraints are satisfied at every time step. Most of the SMT solvers choke when given a problem of this magnitude, but ReaS is still able to synthesize it. Figure 12 shows how the room temperature and the state of the thermostat change with a controller that is synthesized with minimum dwell time constraint of 200s for the OFF and ON phases.

### 5.2 Results

Figure 9 shows the evaluation results. The Time column lists the time taken in seconds (20th percentile, median, and 80th percentile) to synthesize the benchmarks in ReaS. This experiment is run with a conflict threshold η = 5 and unlimited RESTART_LIMIT, but with a timeout of 30 minutes. We ran each benchmark 10 times and ReaS is able to synthesize all the benchmarks for all 10 runs within 25 minutes. The #N and #R columns show the median number of numerical iterations and SAT solver restarts taken by these benchmarks.

Next, we performed an experiment to compare ReaS with an only smoothing approach. For each benchmark, we took the numerical abstraction when interface mapping I is empty and ran our numerical solver on this abstraction with a random initial assignment to the unknowns. We ran each benchmark 300 times and collected the number of times the numerical solver is able to converge to a valid solution for the abstraction (shown in #S column in Figure 9). Since, some
null
structure of the benchmark. However, we can see that for any of the small conflict thresholds \(3 \leq \eta \leq 7\), the system can synthesize the benchmarks in reasonable time.

### 5.3 Comparison to SMT solvers

We compared ReaS with dReal, a state-of-the-art SMT solver for reals. In order to do this comparison, we wrote a script to translate the benchmarks written in ReaS into the SMT-lib format and fed them to the dReal solver. However, dReal could not solve any of our benchmarks with a timeout of 60 minutes. We also ran Z3 on the thermostat benchmark (the only benchmark in our suite that does not contain non-linear functions) and found that even Z3 could not solve with a timeout of 60 minutes. These results support our claim that the traditional SMT approaches do not scale for these kinds of synthesis problems.

### 5.4 Comparison to mixed integer approaches

In the final experiment, we evaluate how the mixed integer approach compares to ReaS. Since most of the mixed integer solvers only handle linear and convex constraints, we created a toy version of the lane change benchmark. We replaced the complex dynamics of the car with a simple point car model that can only either move along the x-direction or the y-direction. To deal with conditionals and disjunctions, we manually translated them to linear constraints using the big-M method. This translation, however, introduces one 0-1 integer variable for each Boolean expression in the original problem and a continuous variable for every conditional. We verified that the translation to the mixed integer format is correct by fixing the values to the unknowns with a solution found by ReaS. We ran this translated version using Gurobi, a state of the art mixed integer solver. However, Gurobi was not able to find a solution with a timeout of 60 minutes, whereas ReaS was able to solve the benchmark in just 3 seconds. This shows that even the mixed integer solvers are not suitable for handling these benchmarks.

### 6 Related Work

To the best of our knowledge, this paper is the first that achieves both end-to-end differentiability of a program and uses a SAT solver to help perform numerical optimization to find unknowns in a program. However, there are several related works for each of the two pieces.

**End-to-end differentiability:** The idea of achieving end-to-end differentiability in new in the neural networks community. The entire back-propagation algorithm is based on this principle. Our idea of smoothing using sigmoids is inspired by the Neural networks’ use of sigmoids in-place of step functions to make the network differentiable. For neural networks, there are libraries such as TensorFlow [1] where users can write their networks in a high-level language and the library can automatically compute the required gradients. In ReaS, we use similar ideas to programmatically smooth conditionals and conjunctions. The recent works in neural Turing machines [18] and neural program interpreters [21, 23, 27, 31] achieve end-to-end differentiability in the presence of discrete structure. They accomplish this by turning the discrete variables into continuous variables for encoding the probabilities of the different discrete options. On the other hand, in this paper, we relax Boolean unknowns to real unknowns in the range \([0, 1]\) and then, use a SAT solver to fix their values. This approach allows us to benefit from the fact that SAT solvers are inherently better at handling discrete problems.

Smoothing in the context of programs is introduced in the smooth interpretation work [6] and subsequently used in [5] to solve synthesis problems involving Boolean and quantitative objectives. Smoothing using sigmoids that we use in this paper is a simplified version of the smoothing algorithm used in [6]. However, our approach allows us to use automatic differentiation techniques to compute gradients necessary for the numerical optimization.

**SAT/SMT solvers:** There have been many approaches to SMT solving over real numbers that incorporate numerical methods. Examples include convex optimization algorithms [3, 28, 32], interval-based algorithms [11, 14], Bernstein polynomials [24], and linearization algorithms [8]. However, these approaches strictly partition the problem into a Boolean part and many of numerical parts that loses the structural dependencies between the numerical parts and hence, they are not able to leverage the benefits of doing numerical optimization on the entire problem.

ReaS’s modifications to the SAT solver to support soft learns and suggestions is similar to the idea of assumptions in [2, 25]. These works use assumptions to support incremental SAT solving when there are constraints that only hold for one invocation. In our approach, we use soft learns and suggestions to inform the SAT solver that they can be revoked any time if a conflict is detected.

**Hybrid Systems:** The hybrid systems community have also looked at problems involving discrete and continuous components. Some of the recent works in this area include [17, 29, 30]. These approaches use a discrete abstraction of the continuous components and perform purely discrete reasoning. On the other hand, our approach shows that by leveraging
the numerical structure of the problem, it is possible to scale hybrid systems synthesis to very complex problems.

Control Optimization: The control optimization community usually uses mixed integer programming to solve these kinds of problems. For example, [10] uses the mixed integer approach for motion planning in UAVs. Similarly, mixed integer programming is used extensively in model predictive control [4]. In these approaches the task is to directly learn the actions for every time step rather than a program for the controller. Learning a program introduces more discreteness and as shown in the evaluation, mixed integer approaches do not work well for the benchmarks in this paper.

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