Evidence for a hard equation of state in the cores of neutron stars

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Abstract

The equation of state for matter with energy density above
$2 \times 10^{14}$ gm/cm$^3$ is parametrized by $P = kN^\Gamma$, where $N$ is the number density, $\Gamma$ is the adiabatic index, and $k$ a constant. Using this scheme to generate thousands of models, together with data on neutron star masses, it is found, for a core region with constant adiabatic index, that the central density must satisfy $10^{15}$ gm/cm$^3 < \rho_c < 10^{16}$ gm/cm$^3$, with $\Gamma > 2.2$. Further preliminary results indicate, based on the observed neutrino flux from supernova 1987a, that this number must be considerably higher, on the order of 3.5. These results provide evidence for a hard equation of state in the cores of neutron stars.

Author’s note: This paper was originally submitted to the Astrophysical Journal circa 1995. It is uploaded here by request of new authors who are currently exploring the use of a similar parameterization scheme for similar purposes. The figures and tables referred to in the paper are no longer available.

1 Introduction

There have been a number of attempts to derive the equation of state (EOS) of matter at high density, but all of these attempts have suffered under the constraint of a lack of experimental data. The physics of matter above about $2\times10^{14}$ grams per cubic centimeter is essentially unknown, and there is considerable controversy as to whether or not the EOS in this region is hard or soft.

In view of this, a mathematical parametrization of the equation of state for highly compressed matter was developed which allows a general study of all possible equations of state. The process of parametrization involves several simple steps: (1) the choice of the mathematical relationship(s)
with which to model the EOS; (2) the regions in which these mathematical relationships hold; (3) the use of continuity conditions at the boundaries of these regions to fix various constants.

The parameters chosen here consist of the adiabatic index (indices) for the matter, and the number densities at which phase transitions occur, if any. Alternately, energy densities may be chosen instead of number densities. Constraining data obtained from these models by physical observations may then shed some light on the physics in the core region.

Using this mathematical scheme, approximately twenty-five thousand one-parameter models were generated for static neutron stars, each corresponding to a different adiabatic index in the core region. In addition, several derived equations of state were mapped onto parameter spaces of one and three dimensions, and the overall viability of the scheme tested by finding whether or not the parameter models agreed with the originals in such basic computations as mass and radius.

The generic parametrization is given by

\[ P = \kappa N^\Gamma \]  

where \( P \) is the pressure, \( N \) the number density, \( \Gamma \) is the adiabatic index, and \( k \) a constant which is determined by continuity. The first law of thermodynamics in the absence of heat can be written

\[ \frac{dE}{E + \frac{P}{c^2}} = \frac{dN}{N} \]  

This equation can be integrated by noticing that

\[ d\left(\frac{E}{N}\right) = \frac{dE}{N} - \frac{E}{N^2} dN \]  

can be suitably rearranged. Together with Equation 1 and 2 it is straightforward to find that

\[ E = \frac{P}{c^2 (\Gamma - 1)} + DN \]  

where \( D \) is a constant of integration which, like \( k \), is established by continuity. (It will come as no surprise that \( D \) has approximately the same magnitude as one AMU). The first number density will be chosen to be \( N_0 = 1.182 \times 10^{18} \) baryons per cubic centimeter, which corresponds to an energy density of \( E_0 = 2.004 \times 10^{14} \) grams per cubic centimeter in the BPS/BBP equation of state, at the threshold of the nuclear regime. These numbers were obtained from tables provided by Jim Ipser of the University of Florida. Additional parameters can also be used if desired to model more complex cores involving phase transitions. Using more parameters also allows a better approximation to existing equations of state. As different regions are added, continuity with the previous region determines the constants in the parametrization for the new region.

Once the equation of state is specified, in this case Baym-Bethe-Pethick with the parametrization scheme implemented in the high-density region, it’s a simple matter to integrate the Oppenheimer-Volkoff equations to obtain the mass and radius of the stellar model.
2 Parameter Matching for Derived Equations of State

The next step is to map existing derived equations of state onto the parameter space. If this is possible and yields a good approximation, then it can be concluded that the simple mathematical model can give meaningful answers to physical questions.

There are various possible choices for the optimization of the fit, and each choice will, in general, yield slightly different answers. A natural choice, for example, is to derive the equation of the best-fit plane in P-E-N space, subject to continuity with the actual equation of state at the onset of the nuclear regime. While this results in a fairly good mapping, a better fit can be obtained by ignoring the energy density $E$ altogether and finding the best fit line of $\ln(P)$ to $\ln(N)$, subject to continuity; that is, the line must contain the point $(\ln(N_0), \ln(P_0))$, where $N_0$ and $P_0$ are the first fiducial density and the pressure corresponding to it, respectively.

With the data points ranging from 0 to $L$, this is found to be

$$\Gamma = \frac{\sum_{i=1}^{L}(\ln(N_0) - \ln(N_i))(\ln(P_0) - \ln(P_i))}{\sum_{i=1}^{L}(\ln(N_0) - \ln(N_i))^2}$$

(5)

One, two, three or more regions can be strung together in tandem. The table shows the results for several representative equations of state when using either one adiabatic index parameter or two adiabatic parameters together with a transition density parameter. In the case of one parameter, the variational calculation immediately gives the best fit. In the three-parameter case, the transition number density was changed by small steps through the domain of definition, and the lowest sum of squares deviation of the natural log of the pressures was the criterion for selection of the best fit, with the pressures chosen directly off the tables provided by Ipser. An average percent difference was calculated by means of

$$\Delta P_{av} = \frac{1}{L} \sum_{i=1}^{L} \frac{\vert P_i - kP_i^f \vert}{P_i}$$

(6)

Here, $P_i$ is the pressure from the table. The strength of this average deviation condition is its simplicity. The weakness is that larger pressures, where deviations may result in more serious perturbations of the numerical calculation, are treated on the same footing as smaller pressures.

An excellent test of this fitting procedure is to attempt to reproduce a table of masses and radii for various central densities obtained from a derived equation of state, such as Moskowski, or Baym-Bethe-Pethick. This was done for several different equations of state. Most of the values differed from the actual values by only a few percent at most. Three different Pandharipande models were exceptional, with deviations exceeding twenty percent in some cases. This was due to artificial discontinuities in the tables where the BBP equation of state was joined to Pandharipande’s EOS. The discontinuity was evidenced by the appearance of a zone with high adiabatic index ($\Gamma > 5$) between the top of the BBP EOS and the first entry in Pandharipande’s EOS. According to John Friedmann [3], this
can be fixed by adjusting an arbitrary constant in the potential of Pandharipande’s theory. Because these tables are widely circulated, users of Pandharipande’s equation of state should be aware that some adjustment in this potential may be necessary.

3 The Equation of State Inside Neutron Stars

Twenty-five thousand single-parameter models were generated with adiabatic indices ranging between 4/3 and 10/3 and central density between $2 \times 10^{14}$ and $1 \times 10^{16}$ grams per cubic centimeter. The stars were taken to be static, with only a single phase in the core region, described by a single adiabatic parameter. In each case, the mass and radius of the star was calculated.

The radii of neutron stars are only poorly known, though it is generally thought they lie between fifteen and thirty kilometers. Masses, on the other hand, are quite well known from measurements made on binary systems. Therefore, a graph of the mass of the star against adiabatic index of the core region and log of the central density was created. The results are presented in Figure 1, which displays several mass contours projected onto the plane of the independent variables. Most neutron star masses fall between 1.27 and 1.65 solar masses [5]; therefore contour curves for these masses are plotted. In addition, the curve where the adiabatic speed of sound reaches the speed of light is shown, along with a stability curve. Models falling to the right of either one of these latter two lines can be rejected on physical grounds.

Altogether, these curves provide constraints to the possible models. Under the given assumptions, it appears that neutron stars for which measurements are available must have central densities between $1 \times 10^{15}$ and $1 \times 10^{16}$ grams/cubic centimeter, and that the adiabatic index must be in excess of about 2.2.

Because neutrino flux measurements from the 1987 supernova in the Large Magellanic Cloud favor a soft equation of state, at least up to four times nuclear density, we began study of a three-parameter case consisting of two regions in the core, where the index in the lower density region is soft. Preliminary results of this study, still in progress, indicate that the core region must have a much stiffer equation of state—about 3.5—in order to explain the observed masses. Generally, the softer the outer core region, the harder the inner core.

A final interesting and somewhat puzzling result was obtained by mapping the radius of the stellar models against adiabatic index for constant central energy density. As might be expected, the radius decreases as the equation of state becomes stiffer, since the higher pressure effectively raises the gravitational mass of the star. Paradoxically, this curve comes to a minimum and swings back up again, an effect which becomes more pronounced with increasing central density. It would be interesting to find an explanation for this behavior.
4 Concluding Remarks

The most important result obtained from this study is the apparent hardness of the equation of state at high densities under the two assumptions that the core has a single phase and that rotation does not dramatically affect the mass of the star. This supports a previous and very different analysis [2]. Derived equations of state, however, are better modeled by three parameters—two indices and a transition number density at the point of a phase transition. The neutrino flux of supernova 1987a supports a soft equation of state up to about four times nuclear density, but preliminary results indicate that, in this case, there must be an even harder core region beyond that, with adiabatic index around 3.5. Rotation will probably not change the results very much, but this, too, should be studied. This work is currently in progress; it is expected that the bounds on the central density and adiabatic index will be narrowed somewhat. Regardless, it appears that the observed masses of neutron stars point to a hard equation of state in the core region, with a central density between $10^{15}$ and $10^{16}$ grams per cubic centimeter.

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