FULTON-HANSEN AND BARTH-LEFSCHETZ THEOREMS FOR SUBVARIETIES OF ABELIAN VARIETIES

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Sommese showed that a large part of the geometry of a smooth subvariety of a complex abelian variety depends on "how ample" its normal bundle is (see § 1 for more details). Unfortunately, the only known way of measuring this ampleness uses rather strong properties of the ambient abelian variety.

We show that a notion of non-degeneracy due to Ran is a good substitute for ampleness of the normal bundle. It can be defined as follows: an irreducible subvariety V of an abelian variety X is geometrically non-degenerate if for any abelian variety Y quotient of X, the image of V in Y either is Y or has same dimension as V. This property does not require V to be smooth; for smooth subvarieties, it is (strictly) weaker than ampleness of the normal bundle.

Our main result is a Fulton-Hansen type theorem for an irreducible subvariety V of an abelian variety: the dimension of the "secant variety" of V along a subvariety S (defined as V − S), and that of its "tangential variety" along S (defined in the smooth case as the union of the projectivized tangent spaces to V at points of S, translated at the origin) differ by 1. Corollaries include a new proof of the finiteness of the Gauss map and an estimate on the ampleness of the normal bundle of a smooth geometrically non-degenerate subvariety.

We also complement Sommese’s work with a new Barth-Lefschetz theorem for subvarieties of abelian varieties whose proof is based on an idea of Schneider and Zintl. Let C be a smooth curve in an abelian variety X; we apply this result to give an estimate on the dimension of the singular locus of C + · · · + C in X.

We work over the field of complex numbers.

1. Geometrically non-degenerate subvarieties

Recall ([S1]) that a line bundle L on an irreducible projective variety V is k–ample if, for some m > 0, the line bundle L^m is generated by its global sections and the fibers of the associated map \( \phi_{L^m}: V \to \mathbb{P}^N \) are all of dimension \( \leq k \). A vector bundle E on V is k–ample if the line bundle \( \mathcal{O}_{E^*}(1) \) is k–ample. Ordinary ampleness coincide with 0–ampleness.

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Let $X$ be an abelian variety and let $V$ be an irreducible variety with a morphism $f : V \to X$. Let $V^0$ be the open set of smooth points of $V$ at which $f$ is unramified. Define the normal bundle to $f$ as the vector bundle on $V^0$ quotient of $f^*(TX)|_{V^0}$ by $TV^0$.

For any $x \in X$, let $\tau_x$ be the translation by $x$. For any $v \in V$, the differential of the map $\tau_{-f(v)}f$ at $v$ is a linear map $T_vV \to T_0X$, which we will simply denote by $f^*$.

**Proposition 1.1.** Under the above assumptions, let $S$ be a complete irreducible subvariety of $V^0$. The following properties are equivalent

(i) the restriction to $S$ of the normal bundle to $f$ is $k$–ample;

(ii) for any hyperplane $H$ in $T_0X$, the set $\{ s \in S \mid f_*(T_sV) \subseteq H \}$ has dimension $\leq k$.

**Proof.** Let $N$ be the restriction to $S$ of the normal bundle to $f$ and let $\iota : PN^* \to Pf^*(T^*X)|_S$ be the canonical injection. The morphism

$$\phi : PN^* \xrightarrow{\iota} Pf^*(T^*X)|_S \simeq PT_0^*X \times S \xrightarrow{pr_1} PT_0^*X$$

satisfies

$$\phi^*\mathcal{O}_{PT_0^*X}(1) = \iota^*\mathcal{O}_{Pf^*(T^*X)|_S}(1) = \mathcal{O}_{PN^*}(1).$$

It follows that $N$ is $k$–ample if and only if the fibers of $\phi$ have dimension $\leq k$ ([S1], prop. 1.7). The proposition follows, since the restriction of the projection $PN^* \to S$ to any fiber of $\phi$ is injective. ■

When $X$ is simple, the normal bundle to any smooth subvariety of $X$ is ample ([H]). More generally, the normal bundle to any smooth subvariety of $X$ is $k$–ample, where $k$ is the maximum dimension of a proper abelian subvariety of $X$ ([S1], prop. 1.20).

Following Ran, we will say that a $d$-dimensional irreducible subvariety $V$ of $X$ is **geometrically non-degenerate** if the kernel of the restriction $H^0(X, \Omega_X^d) \to H^0(V_{reg}, \Omega_{V_{reg}}^d)$ contains no non-zero decomposable forms. This property holds if and only if for any abelian variety $Y$ quotient of $X$, the image of $V$ in $Y$ either is $Y$ or has same dimension as $V$ ([R1], lemma II.12).

**Examples.** 1) A divisor is geometrically non-degenerate if and only if it is ample; a curve is geometrically non-degenerate if and only if it generates $X$. Any geometrically non-degenerate subvariety of positive dimension generates $X$, but the converse is false in general. However, any irreducible subvariety of a *simple* abelian variety is geometrically non-degenerate.

2) If $\ell$ is a polarization on $X$ and $V$ is an irreducible subvariety of $X$ with class a rational multiple of $\ell^C$, it follows from [R1], cor. II.2 and II.3 that $V$ is non-degenerate in the sense of [R1], II, hence geometrically non-degenerate. In particular, the subvarieties $W_d(C)$ of the Jacobian of a curve $C$ are geometrically non-degenerate; it can be checked that their normal bundle is ample when they are smooth (use prop. 1.1).
We generalize this notion as follows. Let $k$ be a non-negative integer.

**Definition 1.2.** An irreducible subvariety $V$ of an abelian variety $X$ is $k$-geometrically non-degenerate if and only if for any abelian variety $Y$ quotient of $X$, the image of $V$ in $Y$ either is $Y$ or has dimension $\geq \dim(V) - k$.

**Proposition 1.3.** In an abelian variety, any smooth irreducible subvariety with $k$-ample normal bundle is $k$-geometrically non-degenerate.

**Proof.** Let $\pi : X \to Y$ be a quotient of $X$ such that $\pi(V) \neq Y$. The tangent spaces to $V$ along a smooth fiber of $\pi|_V$ are all contained in a fixed hyperplane, hence general fibers of $\pi|_V$ have dimension $\leq k$ by prop. 1.1.

The converse is not true, as the construction sketched below shows, but a partial converse will be obtained in 2.3. Roughly speaking, if $Y$ is a quotient of $X$, and if the image $W$ of $V$ in $Y$ is not $Y$, $k$-geometrical nondegeneracy requires that the general fibers of $V \to W$ be of dimension $\leq k$, whereas $k$-ampleness of the normal bundle requires that every fiber of $V \to W$ be of dimension $\leq k$.

Let $L_E$ be an ample line bundle on an elliptic curve $E$, with linearly independent sections $s_1, s_2$ defining a morphism $E \to \mathbb{P}^1$ with ramification points $(e_1, 1), \ldots, (e_4, 1) \in \mathbb{P}^1$. Let $L_Y$ be an ample line bundle on a simple abelian variety $Y$ of dimension $\geq 3$, with linearly independent sections $t_1, t_2, t_3$ such that $\text{div}(t_3)$, $F = \text{div}(t_1) \cap \text{div}(t_2) \cap \text{div}(t_3)$ and $\text{div}(e_it_1 + t_2) \cap \text{div}(t_3)$ are smooth for $i = 1, \ldots, 4$ (such a configuration can be constructed using results from [D2]). Set $X = E \times Y$ and define a subvariety $V$ of $X$ by the equations $s_1t_1 + s_2t_2 = t_3 = 0$; then $V$ is smooth of codimension 2, geometrically non-degenerate, but its normal bundle is not ample (for all $e \in E$ and $f \in F$, one has $T_{(e,f)} V \subset T_f(\text{div}(t_3))$, only 1-ample (cor. 2.3).

**Proposition 1.4.** Let $X$ be an abelian variety and let $V$ and $W$ be irreducible subvarieties of $X$. Define a morphism $\phi : V^r \times W \to X^r$ by $\phi(v_1, \ldots, v_r, w) = (v_1 - w, \ldots, v_r - w)$. If $V$ is $k$-geometrically non-degenerate,

$$\dim \phi(V^r \times W) \geq \min(r \dim(X), r \dim(V) + \dim(W) - k) .$$

**Proof.** Assume first $r \dim(V) + \dim(W) - k \geq r \dim(X)$. Let $\pi : X \to X/K$ be a quotient of $X$. I claim that $r \dim \pi(V) + \dim \pi(W) \geq r \dim(X/K)$. If $\pi(V) = X/K$, this is obvious; otherwise, we have $\dim \pi(V) \geq \dim(V) - k_0$, where $k_0 = \min(k, \dim(K))$, hence

$$r \dim \pi(V) + \dim \pi(W) \geq r(\dim(V) - k_0) + \dim(W) - \dim(K) \geq r \dim(X) + k - rk_0 - \dim(K) \geq r \dim(X/K) .$$

It follows that $(V, \ldots, V, W)$ (where $V$ is repeated $r$ times) fills up $X$ in the sense of [D1], (1.10); th. 2.1 of loc.cit. then implies that $\phi$ is onto.
Assume now \( s = r \text{codim}(V) - \text{dim}(W) + k > 0 \); let \( C \) be an irreducible curve in \( X \) that generates \( X \). Let \( W' \) be the sum of \( W \) and \( s \) copies of \( C \); then \( r \dim(V) + \dim(W') - k = r \dim(X) \) and the first case shows that the sum of the image of \( \phi \) and \( s \) curves is \( X' \). The proposition follows. \( \blacksquare \)

We obtain a nice characterization of \( k \)-geometrically non-degenerate varieties.

**Corollary 1.5.** An irreducible subvariety \( V \) of an abelian variety \( X \) is \( k \)-geometrically non-degenerate if and only if it meets any subvariety of \( X \) of dimension \( \geq \text{codim}(V) + k \).

**Proof.** Assume that \( V \) meets any subvariety of \( X \) of dimension \( \geq \text{codim}(V) + k \) and let \( \pi : X \to Y \) be a quotient of \( X \). If \( \pi(V) \neq Y \), there exists a subvariety \( W \) of \( Y \) of dimension \( \dim(Y) - \dim \pi(V) - 1 \) that does not meet \( \pi(V) \). Since \( V \) does not meet \( \pi^{-1}(W) \),

\[
\text{codim}(V) + k > \dim \pi^{-1}(W) = \dim(X) - \dim \pi(V) - 1
\]

hence \( \dim \pi(V) \geq \dim(V) - k \) and \( V \) is \( k \)-geometrically non-degenerate. Conversely, assume \( V \) is \( k \)-geometrically non-degenerate; let \( W \) be an irreducible subvariety of \( X \) of dimension \( \geq \text{codim}(V) + k \). Proposition 1.4 shows that \( V - W = X \), hence \( V \) meets \( W \). \( \blacksquare \)

2. A Fulton-Hansen-type result

Fulton and Hansen proved in [FH] (cf. also [FL1], [Z1], [Z2]) a beautiful result that relates the dimension of the tangent variety and that of the secant variety of a subvariety of a projective space. We prove an analogous result for a subvariety of an abelian variety.

Let \( X \) be an abelian variety and let \( V \) be a variety with a morphism \( f : V \to X \). Recall that \( f \) is unramified along a subvariety \( S \) of \( V \) if \( \Delta_S = \{(v, s) \in V \times S \mid v = s\} \) is an open subscheme of \( V \times_X S \). Following [FL1], we will say that \( f \) is weakly unramified along \( S \) if \( \Delta_S \) is a connected component of \( V \times_X S \), ignoring scheme structures. In that case, if \( p : V \times S \to X \) is the morphism defined by \( p(v, s) = f(v) - f(s) \) and \( \epsilon : \tilde{X} \to X \) is the blow-up of the origin, there exists a commutative diagram

\[
\begin{array}{ccc}
\tilde{Y} & \overset{\tilde{p}}{\longrightarrow} & \tilde{X} \\
\alpha \downarrow & & \downarrow \epsilon \\
V \times S & \overset{p}{\longrightarrow} & X
\end{array}
\]

where \( \alpha \) is the blow-up of \( V \times_X S \). Let \( E \) be the exceptional divisor above \( \Delta_S \subset V \times S \) and set \( T(V, S) = \tilde{p}(E) \). It is a subscheme of \( \text{PT}_0X \) contained in \( \bigcup_{s \in S} \text{PT}_s V \) and equal
to the latter when \( V \) is smooth along \( S \) and \( f \) is unramified along \( S \). Loosely speaking, \( T(V, S) \) is the set of limits in \( \hat{X} \) of \( (f(v) - f(s)) \), as \( v \in V \) and \( s \in S \) converge to the same point. Obviously, \( \dim T(V, S) < \dim (f(V) - f(S)) \).

**Theorem 2.1.**—Let \( X \) be an abelian variety and let \( V \) be an irreducible projective variety with a morphism \( f : V \to X \). Let \( S \) be a complete irreducible subvariety of \( V \) along which \( f \) is weakly unramified. Then \( \dim T(V, S) = \dim (f(V) - f(S)) - 1 \).

We begin with a lemma.

**Lemma 2.2.**—Let \( C \) be an irreducible projective curve with a morphism \( g : C \to X \) such that \( g(C) \) is a smooth curve through the origin. Assume that \( g \) is unramified at some point \( c_0 \in C \) with \( g(c_0) = 0 \) and that \( \text{PT}_0 g(C) \notin T(V, S) \). The morphism \( h : V \times C \to X \) defined by \( h(v, c) = f(v) - g(c) \) is weakly unramified along \( S \times \{c_0\} \) and \( T(V \times C, S \times \{c_0\}) \) is contained in the cone over \( T(V, S) \) with vertex \( \text{PT}_0 g(C) \).

One can prove that \( T(V \times C, S \times \{c_0\}) \) is actually equal to the cone.

**Proof.** Let \( \Gamma \) be a smooth irreducible curve, let \( \gamma_0 \) be a point on \( \Gamma \) and let \( q = (q_1, q_2, q_3) : \Gamma \to (V \times C) \times_X S \) be a morphism with \( q(\gamma_0) = (s_0, c_0, s_0) \). We need to prove that \( q(\Gamma) \subset \Delta'_S \), where \( \Delta'_S = \{(s, c_0, s) | s \in S\} \); since \( \Delta_S \) is a connected component of \( V \times_X S \), it suffices to show that \( q_2 \) is constant. Suppose the contrary; then \( (q_1, q_3) \) lifts to a morphism \( \tilde{q}_{13} : \Gamma \to \tilde{Y} \) and \( g \) to a morphism \( \tilde{g} : C \to \hat{X} \). Since \( p(q_1, q_3) = qg_2 \), one has \( \tilde{p} \tilde{q}_{13} = \tilde{g} q_2 \) and \( g(q_0) = \tilde{p}(\tilde{q}_{13}(\gamma_0)) \in T(V, S) \). This contradicts the hypothesis since \( g(c_0) \) is the point \( \text{PT}_0 g(C) \) of \( \text{PT}_0 X \). This proves the first part of the lemma.

The second part is similar: let \( \tilde{Z} \to (V \times C) \times_X S \) be the blow-up of \( (V \times C) \times_X S \), let \( \Gamma \) be a smooth irreducible curve with a point \( \gamma_0 \in \Gamma \) and let \( \tilde{q} : \Gamma \to \tilde{Z} \) be a morphism such that \( \tilde{g}(\gamma_0) \) is in the exceptional divisor above \( \Delta'_S \). Write \( q = \alpha \tilde{q} = (q_1, q_2, q_3) \) and keep the same notation as above. Then \( pq(\Gamma) \) is contained in the surface \( pq_{13}(\Gamma) - g(C) \) hence \( \tilde{p} \tilde{q}(\gamma_0) \) belongs to the line in \( \text{PT}_0 X \) through \( \tilde{p}(\tilde{q}_{13}(\gamma_0)) \) and \( g(c_0) = \text{PT}_0 g(C) \). This proves the lemma.

**Proof of the theorem.** We proceed by induction on the codimension of \( f(V) - f(S) \). Assume \( f(V) - f(S) = X \); if \( T(V, S) \neq \text{PT}_0 X \), pick a point \( u \notin T(V, S) \) and a smooth projective curve \( C' \) in \( X \) tangent to \( u \) at 0, and such that the restriction induces an injection \( \text{Pic}^0(X) \to \text{Pic}^0(C') \). Let \( C \) be a smooth curve with a connected ramified double cover \( g : C \to C' \) unramified at a point \( c_0 \) above 0; the map \( \text{Pic}^0(C') \to \text{Pic}^0(C) \) induced by \( g \) is injective.

Since \( p \) is surjective, \( C' \) generates \( X \) and \( C \) is smooth, th. 3.6 of [D1] implies that \( (V \times S) \times_X C \) is connected. If \( h : V \times C \to X \) is defined by \( h(v, c) = f(v) - g(c) \), it follows that \( (V \times C) \times_X S \) is also connected. On the other hand, the lemma implies that the set \( \{(s, c_0, s) \} \) is a connected component of, hence is equal to, \( (V \times C) \times_X S \). It follows that \( h^{-1}(f(S)) = S \times \{c_0\} \). Since \( g^{-1}(0) \) consists of 2 distinct points, this is absurd, hence \( T(V, S) = \text{PT}_0 X \) and the theorem holds in this case.
Assume now $f(V) - f(S) \neq X$. Take a curve $C'$ as above; by the lemma, the morphism $f': V \times C' \to X$ defined by $f'(v, c') = f(v) + c'$ is weakly unramified along $S \times \{0\}$, and $\dim T(V \times C', S \times \{0\}) \leq \dim T(V, S) + 1$. It follows from the induction hypothesis that

$$\dim T(V, S) \geq \dim (f(V) + C' - f(S)) - 2 = \dim (f(V) - f(S)) - 1,$$

which proves the theorem. ■

The following corollary provides a partial converse to prop. 1.3.

**Corollary 2.3.** Let $X$ be an abelian variety of dimension $n$ and let $V$ be an irreducible projective variety of dimension $d$ with a morphism $f: V \to X$ such that $f(V)$ is $k$–geometrically non-degenerate. Let $V^0$ be the open set of smooth points of $V$ at which $f$ is unramified. The restriction of the normal bundle to $f$ to any complete irreducible subvariety $S$ of $V^0$ is $(n - d - 1 + k)$–ample.

**Proof.** By prop. 1.1, we must show that for any hyperplane $H$ in $T_0X$, any irreducible component $S_H$ of $\{ s \in S | f_*(T_s V) \subset H \}$ has dimension $\leq n - d - 1 + k$. But $T(V, S_H)$ is contained in $H$ and the theorem gives $f(V) - f(S_H) \neq X$. Since $f$ is unramified along $S_H$ and $f(V)$ is $k$–geometrically non-degenerate, prop. 1.4 implies that $f(V) - f(S_H)$ has dimension $\geq d + \dim(S_H) - k$; this proves the corollary. ■

It should be noted that the corollary also follows from the main result of [Z3] (cor. 1), whose proof is unfortunately so sketchy (to say the least) that I could not understand it.

**Corollary 2.4.** Let $X$ be an abelian variety and let $V$ be an irreducible projective variety with a morphism $f: V \to X$. Let $L$ be a linear subspace of $T_0X$ and let $S$ be a complete irreducible subvariety of $V$ along which $f$ is unramified. Assume that $\dim(f_*(T_s V) \cap L) < m$ for all $s \in S$, and let $\Delta_{f(S)}$ be the small diagonal in $f(S)^m$. Then

$$\dim(f(V)^m - \Delta_{f(S)}) < m \dim(X) - \dim(L) + m.$$

In particular, if $m \leq \dim(L)$ and $f(V)$ is $k$–geometrically non-degenerate,

$$m \dim(V) + \dim(S) < m \dim(X) - \dim(L) + m + k.$$

**Proof.** Let $r = \dim(L)$; the variety $N = \{ [t_1, \ldots, t_m] \in P(L^m) | t_1 \wedge \cdots \wedge t_m = 0 \}$ has codimension $r - m + 1$ in $P(L^m)$. Consider the morphism $f^m: V^m \to X^m$ and the subvariety $\Delta_S$ of $V^m$. The hypothesis imply that in $P(T_0X^m)$, the intersection of $T(V^m, \Delta_S)$ and $P(L^m)$ is contained in $N$. It follows that

$$\dim T(V^m, \Delta_S) \leq \dim(N) + \dim P(T_0X^m) - \dim P(L^m)$$

$$= \dim P(T_0X^m) - (r - m + 1).$$

The first inequality of the corollary follows from th. 2.1, and the second from prop. 1.4. ■
3. Applications to the Gauss map

We keep the same setting: X is an abelian variety and V an irreducible projective variety of dimension d with a morphism \( f : V \to X \). Let \( V^0 \) be the open set of smooth points of V at which \( f \) is unramified; define the Gauss map \( \gamma : V^0 \to G(d, T_0X) \) by \( \gamma(v) = f_*(T_vV) \). The following result was first proved by Ran ([R2]), and by Abramovich ([A]) in all characteristics.

**Proposition 3.1.** Let X be an abelian variety and let V be an irreducible projective variety with a morphism \( f : V \to X \). If \( S \) is a complete irreducible variety contained in a fiber of the Gauss map, \( f(V) \) is stable by translation by the abelian variety generated by \( f(S) \). In particular, the Gauss map of a smooth projective subvariety of X invariant by translation by no non-zero abelian subvariety of X is finite.

**Proof.** Under the hypothesis of the proposition, \( T(V, S) \) has dimension \( \dim(V) - 1 \); th. 2.1 implies \( f(V) - f(S) = f(V) \), hence the proposition. □

For any linear subspace \( L \) of \( T_0X \) and any integer \( m \leq \dim(L) \), let \( \Sigma_{L,m} \) be the Schubert variety \( \{ M \in G(d, T_0X) \mid \dim(L \cap M) \geq m \} \); its codimension in \( G(d, T_0X) \) is \( m(\text{codim}(L) - d + m) \).

**Proposition 3.2.** Let X be an abelian variety and let V be an irreducible projective variety of dimension d with a morphism \( f : V \to X \) such that \( f(V) \) is k-geometrically non-degenerate. Let \( \gamma : V^0 \to G(d, T_0X) \) be the Gauss map, let \( L \) be a linear subspace of \( T_0X \) and let \( m \) be an integer \( \leq \dim(L) \). Any complete subvariety \( S \) of \( V^0 \) of dimension \( \geq \text{codim} \Sigma_{L,m} + (m - 1)(\dim(L) - m) + k \) meets \( \gamma^{-1}(\Sigma_{L,m}) \).

**Proof.** Apply cor. 2.4. □

The hypothesis could probably be weakened to \( \dim(S) \geq \text{codim} \Sigma_{L,m} + k \) (see next proposition); the proposition gives that for \( m = 1 \) or \( \dim(L) \). The corresponding Schubert varieties are \( \Sigma_{L,1} = \{ M \in G(d, T_0X) \mid L \cap M \neq 0 \} \) and \( \Sigma_{L,\dim(L)} = \{ M \in G(d, T_0X) \mid L \subset M \} \).

More generally, a result of Fulton and Lazarsfeld imposes strong restrictions on the image of the Gauss map of smooth subvarieties with ample normal bundle which I believe should also hold for geometrically non-degenerate subvarieties.

**Proposition 3.3.** Let X be an abelian variety and let V be a smooth irreducible projective variety of dimension d with an unramified morphism \( f : V \to X \) and Gauss map \( \gamma : V \to G(d, T_0X) \). Assume that the normal bundle to \( f \) is ample; any subvariety \( S \) of \( \gamma(V) \) meets any subvariety of \( G(d, T_0X) \) of codimension \( \leq \dim(S) \).

**Proof.** If \( Q \) is the universal quotient bundle on \( G(d, T_0X) \), the pull-back \( \gamma^*(Q) \) is isomorphic to the normal bundle \( f^*TX/TV \), hence is ample. It follows from [FL2] that for each Schubert variety \( \Sigma_\lambda \) of codimension \( m \) in \( G(d, T_0X) \) and each irreducible subvariety \( S \) of V of dimension \( m \), one has \( \int_S \gamma^*[\Sigma_\lambda] > 0 \). Now the class of any irreducible subvariety
Z of \(G(d,T_0X)\) of codimension \(m\) is a linear combination with non-negative coefficients (not all zero) of the Schubert classes; this implies \(\int_S \gamma^*[Z] > 0\), hence \(S \cap \gamma^{-1}(Z) \neq \emptyset\). 

Regarding the Gauss map of a smooth subvariety of an abelian variety, Sommese and Van de Ven also proved in [SV] a strong result for higher relative homotopy groups of pull-backs of smooth subvarieties of the Grassmannian.

4. A Barth-Lefschetz-type result

Sommese has obtained very complete results on the homotopy groups of smooth subvarieties of an abelian variety. For example, he proved in [S2] that if \(V\) is a smooth subvariety of dimension \(d\) of an abelian variety \(X\), with \(k\)-ample normal bundle, \(\pi_q(X,V) = 0\) for \(q \leq 2d - n - k + 1\). For arbitrary subvarieties, we have the following:

**Theorem 4.1.** Let \(X\) be an abelian variety and let \(V\) be a \(k\)-geometrically non-degenerate normal subvariety of \(X\) of dimension \(> \frac{1}{2}(\dim(X) + k)\). Then \(\pi_1^{\text{alg}}(V) \simeq \pi_1^{\text{alg}}(X)\).

**Proof.** The case \(k = 0\) is cor. 4.2 of [D1]. The general case is similar, since the hypothesis implies that the pair \((V,V)\) satisfies condition \((*)\) of [D1].

Going back to smooth subvarieties, I will give an elementary proof of (a slight improvement of) the cohomological version of Sommese’s theorem, based on the following vanishing theorem ([LP]) and the ideas of [SZ].

**Vanishing Theorem 4.2 (Le Potier, Sommese).** Let \(E\) be a \(k\)-ample rank \(r\) vector bundle on a smooth irreducible projective variety \(V\) of dimension \(d\). Then

\[
H^q(V, E^* \otimes \Omega^p_V) = 0 \quad \text{for} \quad p + q \leq d - r - k.
\]

Recall also the following elementary lemma from [SZ]:

**Lemma 4.3.** Let \(0 \to F \to E_0 \to E_1 \to \ldots \to E_k \to 0\) be an exact sequence of sheaves on a scheme \(V\). Assume \(H^s(V, E_i) = 0\) for \(0 \leq i < k\) and \(s \leq q\); then \(H^q(V,F) \simeq H^{q-k}(V,E_k)\).

**Theorem 4.4.** Let \(V\) be a smooth irreducible subvariety of dimension \(d\) of an abelian \(n\)-fold \(X\) and let \(L\) be a nef line bundle on \(V\). Assume that the normal bundle \(N\) of \(V\) in \(X\) is a direct sum \(\oplus N_i\), where \(N_i\) is \(k_i\)-ample of rank \(r_i\). For \(j > 0\),

\[
H^j(V, S^jN^* \otimes L^{-1}) = 0 \quad \text{for} \quad q \leq d - \max(r_i + k_i).
\]

**Proof.** Since \(S^jN^*\) is a direct summand of \(S^{j-1}N^* \otimes N^*\), it is enough to show, by induction on \(j\), that \(H^j(V, S^jN^* \otimes N_i^* \otimes L^{-1})\) vanishes for \(j > 0\) and \(q \leq d - r_i - k_i\). Since \(N_i \otimes L\) is \(k_i\)-ample, the case \(j = 0\) follows from Le Potier’s theorem. For \(j > 0\), tensor the exact sequence

\[
0 \to S^jN^* \to S^{j-1}N^* \otimes \Omega^1_{X|V} \to \ldots \to \Omega^j_{X|V} \to \Omega^j_V \to 0
\]

by \(N_i^* \otimes L^{-1}\). Since \(\Omega^1_X\) is trivial, the induction hypothesis and the lemma give \(H^q(V, S^jN^* \otimes N_i^* \otimes L^{-1}) \simeq H^{q-j}(V, \Omega^j_V \otimes N_i^* \otimes L^{-1})\), and this group vanishes for \(q \leq d - r_i - k_i\) by Le Potier’s theorem.
We are now ready to prove our version of Sommese’s result.

**Theorem 4.5.**— *Let $V$ be a smooth irreducible subvariety of dimension $d$ of an abelian $n$-fold $X$. Assume that its normal bundle a direct sum $\oplus N_i$, where $N_i$ is $k_i$-ample of rank $r_i$. Then*

a) $H^q(X, V; C) = 0$ for $q \leq d - \max(r_i + k_i) + 1$;

b) for all nonzero elements $P$ of $\text{Pic}^0(V)$, the cohomology groups $H^q(V, P)$ vanish for $q \leq d - \max(r_i + k_i)$.

**Remarks 4.6.** 1) It is likely that a) should hold for cohomology with integral coefficients.

2) If the normal bundle is $k$-ample, we get $H^q(X, V; C) = 0$ for $q \leq 2d - n - k + 1$. If the normal bundle is a sum of ample line bundles, $H^q(X, V; C) = 0$ for $q \leq d$; in particular, the restriction $H^0(X, \Omega_X^d) \rightarrow H^0(V, \Omega_V^d)$ is injective and $V$ is non-degenerate in the sense of [R1], II, hence also geometrically non-degenerate.

3) By [GL], $H^q(V, P) = 0$ for $P$ outside of a subset of codimension $\geq d - q$ of $\text{Pic}^0(V)$. By [S], this subset is a union of translates of abelian subvarieties of $X$ by torsion points.

**Proof of the theorem.** For a), it is enough by Hodge theory to study the maps

\[ H^i(X, \Omega_X^j) \rightarrow H^i(V, \Omega_{X|V}^j) \rightarrow \psi^* H^i(V, \Omega_V^j). \]

Since $\Omega_X^j$ is trivial, we only need look at $\phi : H^i(X, \mathcal{O}_X) \rightarrow H^i(V, \mathcal{O}_V)$ and $\psi$. We begin with $\psi$. We may assume $j > 0$. Let $M_j$ be the kernel of the surjection $\Omega_{X|V}^j \rightarrow \Omega_V^j \rightarrow 0$. The long exact sequence of th. 4.4 gives

\[ 0 \rightarrow S^j \mathcal{N}^* \rightarrow S^{j-1} \mathcal{N}^* \otimes \Omega_{X|V}^1 \rightarrow \ldots \rightarrow \mathcal{N}^* \otimes \Omega_{X|V}^{i-1} \rightarrow M_j \rightarrow 0. \]

The lemma and the theorem then yield

\[ H^i(V, M_j) \simeq H^{i+j-1}(V, S^j \mathcal{N}^*) = 0 \]

for $i + j - 1 \leq d - \max(k_i + r_i)$, since $j > 0$. This implies that $\psi$ has the required properties.

For $i = 0$, the map $\psi$ is $H^0(X, \Omega_X^j) \rightarrow H^0(V, \Omega_V^j)$. By Hodge symmetry, this proves that $\phi$ also has the required properties, hence the first point.

For b), we may assume $d - \max(k_i + r_i) \geq 1$, in which case the first point implies $\text{Pic}^0(X) \simeq \text{Pic}^0(V)$. Let $P \in \text{Pic}^0(X)$ be nonzero; the same proof as above yields $H^0(V, \Omega_V^j \otimes P_{|V}) = 0$ for $q \leq d - \max(k_i + r_i)$. The theorem follows from the existence of an anti-linear isomorphism $H^0(V, \Omega_V^j \otimes P_{|V}) \simeq H^q(V, P^*_{|V})$ ([GL]).
I will end this section with an amusing consequence of th. 4.5. If \( C \) is a curve in an abelian variety \( X \), write \( C_d \) for the subvariety \( C + \cdots + C \) (\( d \) times) of \( X \). Recall that if \( C \) is general of genus \( n \) and \( d < n \), the singular locus of \( C_d = W_d(C) \) in the Jacobian \( J_C \) has dimension \( 2d - n - 2 \).

**Proposition 4.7.** Let \( X \) be an abelian variety of dimension \( n \) and let \( C \) be a smooth irreducible curve in \( X \). Assume that \( C \) generates \( X \) and that its Gauss map is birational onto its image. Then, for \( d < n \), the singular locus of \( C_d \) has dimension \( \geq 2d - n - 1 \) unless \( X \) is isomorphic to the Jacobian of \( C \) and \( C \) is canonically embedded in \( X \).

**Proof.** Let \( \gamma : C_{\text{reg}} \to \mathbf{P}T_0X \) be the Gauss map and let \( \pi : C^{(d)} \to C_d \) be the sum map. The image of the differential of \( \pi \) at the point \((c_1, \ldots, c_d)\) is the linear subspace of \( T_0X \) generated by \( \gamma(c_1), \ldots, \gamma(c_d) \). Since \( C \) generates \( X \), the curve \( \gamma(C) \) is non-degenerate; it follows that for \( c_1, \ldots, c_d \) general, the points \( \gamma(c_1), \ldots, \gamma(c_d) \) span a \((d - 1)\)-plane whose intersection with the curve \( \gamma(C) \) consists only of these points. Thus \( \pi \) is birational. Moreover, if \( x = c_1 + \cdots + c_d \) is smooth on \( C_d \), then \( \gamma(c_i) \in T_x^\ast(T_xC_d) \cap \gamma(C) \) hence \( \pi^{-1}(x) \) is finite. By Zariski’s Main Theorem, \( \pi \) induces an isomorphism between \( \pi^{-1}((C_d)_{\text{reg}}) \) and \((C_d)_{\text{reg}} \).

Let \( s \) be the dimension of the singular locus of \( C_d \) and assume \(-1 \leq s \leq 2d - n - 2 \). Let \( L \) be a very ample line bundle on \( X \); the intersection \( W \) of \( C_d \) with \((s + 1)\) general elements of \(|L|\) is smooth of dimension \( \geq 2 \) and contained in \((C_d)_{\text{reg}} \). If \( H \) is a hyperplane in \( T_0X \) and \( x = c_1 + \cdots + c_d \in W \), the inclusion \( T_xC_d \subset H \) implies \( \gamma(c_i) \in PH \cap \gamma(C) \); the restriction of \( N_{C_d/X} \) to \( W \) is ample by prop. 1.1. Since \( N_{W/X} \) is the direct sum of this restriction and of \((s + 1)\) copies of \( L \), the restriction \( H^1(X, \mathcal{O}_X) \to H^1(W, \mathcal{O}_W) \) is bijective by th. 4.5. On the other hand, the line bundle \( \pi^\ast L \) is nef and big on \( C^{(d)} \), hence the Kawamata-Viehweg vanishing theorem ([K], [V]) implies

\[
H^1(C^{(d)}, \mathcal{O}_{C^{(d)}}) \subset H^1(\pi^{-1}(W), \mathcal{O}_{\pi^{-1}(W)}) \simeq H^1(W, \mathcal{O}_W).
\]

Since \( H^1(C, \mathcal{O}_C) \simeq H^1(C^{(d)}, \mathcal{O}_{C^{(d)}}) \) ([M]), we get \( h^1(C, \mathcal{O}_C) \leq h^1(X, \mathcal{O}_X) \) and there must be equality because \( C \) generates \( X \). Thus, the inclusion \( C \subset X \) factors through an isogeny \( \phi : J_C \to X \). Since \( \pi \) is birational, the inverse image \( \phi^{-1}(C_d) \) is the union of \( \text{deg}(\phi) \) translates of \( W_d(C) \). But any two translates of \( W_d(C) \) meet along a locus of dimension \( \geq 2d - n > s \), hence \( \phi \) is an isomorphism. ■

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Abstract: We prove the following Fulton-Hansen type result for an irreducible subvariety \( V \) of an abelian variety \( X \): the dimension of the “secant variety” of \( V \) along a subvariety \( S \) (defined as \( V - S \)), and that of its “tangential variety” along \( S \) (defined in the smooth case as the union of the projectivized tangent spaces to \( V \) at points of \( S \), translated at the origin) differ by 1. Corollaries include a new proof of the finiteness of the Gauss map and an estimate on the ampleness of the normal bundle, for certain smooth subvarieties of \( X \). We also prove, using ideas of Schneider and Zintl, a new Barth-Lefschetz theorem for smooth subvarieties of \( X \). Let \( C \) be a smooth curve in \( X \); we apply this result to give an estimate on the dimension of the singular locus of \( C + \cdots + C \) in \( X \).