Abstract

We introduce a set-valued generalization of Nash equilibrium, called $M$ equilibrium, which is based on ordinal monotonicity – players’ choice probabilities are ranked the same as the expected payoffs based on their beliefs – and ordinal consistency – players’ beliefs yield the same ranking of expected payoffs as their choices. Using results from semi-algebraic geometry, we prove there exist a finite number of $M$ equilibria, each consisting of a finite number of connected components. Generically, $M$-equilibria can be “color coded” by their ranks in the sense that choices and beliefs belonging to the same $M$ equilibrium have the same color. We show that colorable $M$-equilibria are behaviorally stable, a concept that strengthens strategic stability. Furthermore, set-valued and parameter-free $M$-equilibrium envelopes various parametric models based on fixed-points, including QRE as well as a new and computationally simpler class of models called $\mu$ Equilibrium.

We report the results of several experiments designed to contrast $M$-equilibrium predictions with those of existing behavioral game-theory models. A first experiment considers five variations of an asymmetric-matching pennies game that leave the predictions of Nash, various versions of QRE, and level-$k$ unaltered. However, observed choice frequencies differ substantially and significantly across games as do players’ beliefs. Moreover, beliefs and choices are heterogeneous and beliefs do match choices in any of the games. These findings contradict existing behavioral game-theory models but accord well with the unique $M$ equilibrium. Follow up experiments employ $3 \times 3$ games with a unique pure-strategy Nash equilibrium and multiple $M$ equilibria. The belief and choice data exhibit coordination problems that could not be anticipated through the lens of existing behavioral game-theory models.

Keywords: $M$ Equilibrium, $\mu$ Equilibrium, Behavioral Stability
1. Introduction

The central solution concept of game theory, the Nash equilibrium, rests on the assumption of mutually consistent behavior: each player’s choice is optimal given others’ choices. While the Nash equilibrium is defined in terms of a fixed-point condition on choices (Nash, 1950), it is quiet about how they come about. Presumably, players form beliefs about others’ behavior and use these to optimize their choices. Seen this way, the Nash assumption of mutually consistent behavior is equivalent to the following two conditions: players’ choices are best responses to their beliefs and players’ beliefs are correct.

Observed choices conform to Nash-equilibrium predictions in some settings, but a large body of work in experimental game theory has documented systematic departures from Nash equilibrium (e.g. Goeree and Holt, 2001). Nasar (1998) describes how the lack of experimental support for his equilibrium concept caused Nash to lose confidence in the relevance of game theory after which he turned to pure mathematics in his later research. Selten, who shared the 1994 Nobel Prize with Nash, likewise concluded that “game theory is for proving theorems, not for playing games.”

This paper is motivated by the desire for an empirically relevant game theory. We introduce a novel solution concept, $M$ equilibrium, which replaces the assumption of perfect maximization (“no mistakes”) with an ordinal monotonicity condition – players’ choice probabilities are ranked the same as their associated expected payoffs – and the assumption of perfect beliefs (“no surprises”) with an ordinal consistency condition – players’ beliefs yield the same ranking of expected payoffs as their choices. If the latter condition holds, which does not require beliefs to be homogeneous or correct, we say that players’ beliefs support their choices.

$M$ equilibrium is a set-valued solution concept that puts beliefs and choices on an equal footing. Specifically, $M$ equilibrium considers all beliefs that support a certain equilibrium choice and, dually, it considers all choices supported by a certain equilibrium belief. An $M$
equilibrium consists of the largest belief set and the largest choice set such that choices satisfy ordinal monotonicity and beliefs support choices.

We prove there exists an $M$ equilibrium for any normal-form game. We explore the geometry of $M$ equilibrium sets, which are examples of semi-algebraic sets (see e.g. Coste, 2002). Borrowing results from semi-algebraic geometry, we show there are finite number of $M$ equilibria, each consisting of a finite number of connected components. Generically, there can be an even or odd number of $M$ equilibria, unlike fixed-point theories, such as Nash equilibrium, that generically yield an odd number of equilibria (e.g. Harsanyi, 1973). We show that Nash equilibria arise as limit points of the $M$-equilibrium sets. There may be fewer, as many, or more $M$ equilibria than Nash equilibria. Moreover, an $M$ equilibrium may contain zero, one, or more Nash equilibria. Importantly, we show the measure of the $M$-equilibrium choice set falls quickly with the number of players and the number of possible choices.

Generically, any $M$ equilibrium can be “color coded” by the ranks of the equilibrium-choice profiles in the sense that all choices and beliefs that belong to the same $M$ equilibrium must have the same color. To illustrate, consider the symmetric two-player game in Table 1 where choice and belief profiles are compositions of red ($R$), yellow ($Y$), and blue ($B$). This “Mondrian game” has three symmetric $M$ equilibria: the colored “planes” in the left panel of Figure 1 show the $M$-equilibrium choice sets, and the right panel shows the corresponding $M$-equilibrium belief sets. Beliefs of a certain color support actions with the same color. The thick interior lines reflecting payoff indifferences form the boundaries of the colored planes\(^2\) Conform Mondrian’s quote, drawing these lines is somewhat superfluous as all relevant information is encoded by the planes, including the Nash equilibria. For example, since all three colored planes include a vertex of the choice simplex, each color corresponds to a pure-strategy Nash equilibrium. Furthermore, the point on the edge of the choice simplex where the red (or yellow) plane borders the white plane corresponds to a degenerate mixed-strategy Nash equilibrium\(^3\) These are the five symmetric Nash equilibria of the Mondrian game.

We show that, generically, the interior of the $M$ equilibrium sets consists of choices and beliefs that are behaviorally stable. Roughly speaking, an $M$-equilibrium profile is behaviorally stable when small errors in implementation or perception do not destroy its equilibrium nature. In other words, an $M$-equilibrium profile is behaviorally stable when the profile is also an $M$-

\(^2\)Together with the three lines (not shown) that divide the simplex in six equal parts, i.e. the three lines where two of the three choice probabilities are equal.

\(^3\)If $(p_R, p_B, p_Y)$ denotes the symmetric equilibrium profile then the red boundary point is $(\frac{2}{3}, \frac{1}{3}, 0)$ and the yellow boundary point is $(\frac{1}{3}, 0, \frac{2}{3})$.\n
Table 1: Mondrian’s “composition of Red, Yellow, and Blue” game.

|   | R  | B  | Y  |
|---|----|----|----|
| R | 9,9| 6,8| 4,4|
| B | 8,6| 8,8| 2,4|
| Y | 4,4| 4,2| 5,5|

Figure 1: The symmetric $M$-equilibrium choice sets (left) and belief sets (right) for the Mondrian game in Table 1.

...equilibrium profile for all nearby games. Obviously, the concept of behavioral stability builds on ideas from the literature on refinements of Nash equilibrium (e.g. Van Damme, 1991), in particular, Kohlberg and Mertens’ (1986) “strategic stability.” The latter requires that nearby games have a Nash-equilibrium profile that is close to, but not necessarily equal to, the Nash-equilibrium profile of the original game. Behavioral stability strengthens this requirement by insisting that perturbations of the game do not change the set of $M$-equilibrium profiles. We believe this stronger requirement is needed for $M$ equilibrium to be empirically relevant.

We establish $M$ equilibrium as a “meta theory” of various parametric models that rely on fixed-point conditions. In particular, we introduce a class of $\mu$-Equilibrium models in which choice probabilities are parametric functions of the ranks of their associated expected payoffs. $\mu$-Equilibrium choices follow from a fixed-point condition and $\mu$-equilibrium beliefs support the fixed-point choice profile. We show that $\mu$-equilibrium choices are easy to compute and typically...
supported by a continuous set of beliefs. Moreover, we prove that an $M$-equilibrium of a given color contains all different $\mu$-equilibrium fixed points supported by beliefs of the same color. Conversely, by varying $\mu$, the different $\mu$-equilibrium fixed-points “fill out” the $M$-equilibrium choice set.

We report the results of a series of laboratory experiments that test $M$ equilibrium. The first experiment considers five variations of an asymmetric-matching pennies game that leave the predictions of various behavioral game-theory models (Nash, QRE, and level-$k$) unaltered. However, observed choice frequencies differ substantially and significantly across games as do players’ beliefs. Moreover, beliefs do not match choices, and beliefs and choices are heterogeneous in any of the games. These findings contradict the behavioral game-theory models but accord well with the unique $M$ equilibrium.

Follow up experiments exploit the fact that there can be multiple $M$ equilibria in games with a unique pure-strategy Nash equilibrium. In particular, the experiment employs variations of $3 \times 3$ games with a unique pure-strategy Nash equilibrium but several $M$ equilibria. The belief and choice data reveal the resulting coordination problems that play no role in traditional fixed-point theories such as Nash and QRE (which both yield unique predictions in these games).

1.1. Prior Approaches

A distinct feature of $M$ equilibrium is that beliefs and choices play a dual role. Choices satisfy an ordinal monotonicity condition, which determines the largest set of choices supported by a certain belief, and beliefs satisfy an ordinal consistency condition, which determines the largest set of beliefs that support a certain choice. In particular, $M$-equilibrium beliefs may differ across players and may differ from $M$-equilibrium choices. In other words, $M$ equilibrium allows for “surprises” unlike Nash and QRE, which assume correct beliefs.

1.1.1. Quantal Response Equilibrium

McKelvey and Palfrey’s (1995) Quantal Response Equilibrium (QRE) incorporates the possibility of errors into an equilibrium framework. In particular, the Nash best-response correspondences are replaced by smooth and increasing response functions, known as the “quantal response” or “better response” functions. This is the “QR” part. In addition, QRE retains the Nash-equilibrium assumption that beliefs are correct so that choice probabilities are derived relative to the true expected payoffs taking into account others’ error-prone behavior. This is the “E” part.
A prominent behavioral economist once quipped “I like the Q and the R but not the E.” To illustrate why QRE’s equilibrium assumption is problematic, consider the dominance-solvable game in Table 2. If $p (q)$ denotes the probability with which Column (Row) chooses $L (U)$ then $p^* = q^* = 0$ in the unique Nash equilibrium. Logit QRE choice and belief probabilities, in contrast, follow from

\[
q = \frac{e^{\lambda_R \pi_R(U)}}{e^{\lambda_R \pi_R(U)} + e^{\lambda_R \pi_R(D)}} = \frac{1}{1 + e^{\lambda_R (1+2p)}}
\]

\[
p = \frac{e^{\lambda_C \pi_C(L)}}{e^{\lambda_C \pi_C(L)} + e^{\lambda_C \pi_C(R)}} = \frac{1}{1 + e^{\lambda_C (3-2q)}}
\]

with $\lambda_R$ and $\lambda_C$ non-negative “precision” or “rationality” parameters determining the sensitivity of choices with respect to expected payoffs. Note that the choice probabilities on the left are the same as those used to compute expected payoffs on the right. In other words, QRE beliefs (and choices) are fixed-points of a set of non-transcendental equations.

If QRE is seen as a model of boundedly-rational players who make mistakes it seems inconsistent to assume they are able to solve the above fixed-point equations. Indeed, not even fully rational players can solve these equations, which admit approximate numerical solutions only (and only after picking specific values for $\lambda_R$ and $\lambda_C$). Second, while players’ choices may differ in equilibrium, players must have identical beliefs about the likelihood of each choice, i.e. all players’ beliefs must coincide with a single (fixed) point in the probability simplex. This level of homogeneity in beliefs is unrealistic and will be falsified by any experiment that elicits beliefs as elements of the simplex, see Section 5.

Importantly, the “E” in QRE is far more demanding than the equilibrium assumption underlying Nash equilibrium. Not only does it involve solving transcendental equations rather than simply checking for consistency (of the type “if Row chooses $D$ then Column chooses $R$, and if Column chooses $R$ then Row chooses $D$”), beliefs also have to be extremely precise. Any small change in beliefs results in a different choice profile, causing the QRE fixed-point conditions to be violated. In contrast, the Nash equilibrium is often robust to small, and
possibly heterogenous, variations in beliefs. For the game in Table 2, for instance, Row’s and Column’s best responses coincide with Nash-equilibrium choices for any beliefs they hold. In other words, even when Row and Column have different and incorrect beliefs about what choices may transpire, their best responses to their beliefs support the Nash equilibrium in this game.

We are not suggesting that the Nash equilibrium is a more robust or a more reliable predictor of behavior than QRE. For that, its degenerate predictions (in case of a pure-strategy Nash equilibrium) are too easily falsified by a single deviant choice. QRE avoids such zero-likelihood problems by offering a statistical theory of behavior in games. But QRE is too demanding about the “E” part by insisting that beliefs are homogeneous – even when players have different rationality parameters – and correct – even when that requires solving transcendental equations. While the “QR” part avoids zero-likelihood problems when QRE is applied to choice data, its “E” part will surely be rejected by any data on beliefs (see Section 5).

Approaches that do allow for surprises can generally be divided into “equilibrium” or “non-equilibrium” models.

1.1.2. Equilibrium Belief-Based Models

An early example of an equilibrium model of surprises is Random Belief Equilibrium (Friedman and Mezzetti, 2005). In this model, players’ beliefs are draws from a distribution around a central strategy profile, called the “focus.” Players best respond to their beliefs and the equilibrium condition is that expected choices coincide with the focus of the belief distributions.

Rogers, Palfrey, and Camerer (2008) introduce a family of equilibrium models where player i’s choice probabilities follow from a logit quantal response function with rationality parameter λ_i. In the most general formulation, called Subjective Quantal Response Equilibrium (SQRE), players have subjective beliefs about the distributions from which others’ rationality parameters are drawn. The (Bayes-Nash) equilibrium condition is on players’ strategies, i.e. how rationality parameters map into choices, rather than on the choices themselves.

This general model can be restricted in two possible ways to establish a connection with related models. First, in Truncated Quantal Response Equilibrium (TQRE), players have downward-looking beliefs about others’ rationality parameters. Rogers, Palfrey, and Camerer (2008) show that the (non-equilibrium) Cognitive Hierarchy model can be seen as a limit case of discretized TQRE. Second, in Heterogeneous Quantal Response Equilibrium (HQRE), players have common and correct beliefs about the distributions of the rationality parameters. In this case, the equilibrium condition is on choices as in standard QRE (which arises as a limit case
when the rationality-parameter distributions are degenerate).

1.1.3. Non-Equilibrium Belief-Based Models

Prominent examples include the level-\(k\) model (Stahl and Wilson, 1994, 1995; Nagel, 1995) and the related Cognitive Hierarchy model (Camerer, Ho and Chong, 2004). In these models, players differ in skill and their beliefs about others’ skill levels are “downward looking.” In the level-\(k\) model, level-0 randomizes or makes a “non-strategic” choice (if such a choice is easily identified). Level-1 players best respond to level-0 choices, level-2 players best respond to level-1 choices, etc. In the Cognitive Hierarchy model, level-0 randomizes while level-\(k\) players, with \(k \geq 1\), assume others’ skill levels follow a truncated Poisson distribution over \(0, \ldots, k - 1\) and best respond to the implied distribution of choices.

Noisy Introspection (Goeree and Holt, 2004) is also a non-equilibrium model, but it does not assume downward-looking beliefs. Instead, it is based on the idea that forming higher-order beliefs is increasingly difficult.

1.1.4. Differences with \(M\) Equilibrium

\(M\) equilibrium differs from these prior approaches in several ways. First, \(M\) equilibrium is a set-valued, rather than a fixed-point, solution concept. As a result, it is typically simpler to compute. Second, \(M\) equilibrium is a parameter-free theory, unlike the aforementioned models that require specific parametric assumptions for the quantal responses (e.g. logit-based SQRE), the belief distributions (e.g. Dirichlet-based RBE), the distribution of levels (e.g. Poisson-based CH), etc. As a result, its predictions can be confronted with (experimental) data without the need to estimate parameters. Third, \(M\) equilibrium treats beliefs in a way that does not neatly fit the “equilibrium” versus “non-equilibrium” classification. Unlike level-\(k\) type models, \(M\) equilibrium does not make ad hoc assumptions about the belief-formation process to arrive at a specific model for disequilibrium beliefs (whether it be in terms of others’ “levels of strategic thinking,” as in level-\(k\) and Cognitive Hierarchy, or in terms of others’ “rationality parameters,” as in SQRE and its descendants). And unlike RBE or QRE type models, \(M\) equilibrium does not assume correct beliefs. Rather \(M\) equilibrium includes all beliefs that yield the same ranking of expected payoffs. The rationale is that all those beliefs support the same set of choices, and, in this sense, they sustain an equilibrium situation in which there is “no need for change.”

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4Goeree and Holt (2004) show that Noisy Introspection can be interpreted as a stochastic version of rationalizability (Bernheim, 1984; Pearce, 1984).
1.2. Organization

The next section introduces the rank correspondence, which replaces the best-response correspondence and is used to implement the ordinal monotonicity condition that options with higher expected payoffs are chosen more likely *without specifying by how much*. Stated differently, choice probabilities are ranked the same as expected payoffs, a requirement sometimes referred to as “stochastic rationality.” Section 3 pairs the ordinal monotonicity condition for choices with the ordinal consistency condition for beliefs to define an $M$ equilibrium. Existence for general normal-form games is proven and several properties and examples are discussed. Section 4 introduces a parametric class of $\mu$-Equilibrium models where choices follow from fixed-point conditions. We show that $M$ equilibrium “minimally envelopes” these parametric models. Section 5 reports results from various experiments designed to contrast $M$-equilibrium predictions with those of the existing behavioral game-theory models. Section 6 offers a summary of our results, and the concluding Section 7 discusses several extensions of the basic $M$-equilibrium approach and discusses the value of $M$ equilibrium when its predictions accord well with the data and when they do not. The Appendices contain the proofs not shown in the main text, results from statistical tests based on the experimental data, as well as additional details about the methods used in the data analysis.

2. Preliminaries

Consider a finite normal-form game $G = (N, \{C_i\}_{i=1}^n, \{\Pi_i\}_{i=1}^n)$, where $N = \{1, \ldots, n\}$ is the set of players, and for $i \in N$, $C_i = \{c_{i1}, \ldots, c_{iK_i}\}$ is player $i$’s choice set and $\Pi_i : C \rightarrow \mathbb{R}$, with $C = \prod_{j=1}^n C_j$, is player $i$’s payoff function. Let $\Sigma_i$ denote the set of probability distributions over $C_i$. An element $\sigma_i^c \in \Sigma_i$ is player $i$’s choice profile, which is a mapping from $C_i$ to $\Sigma_i$, where $\sigma_i^c(c_i)$ is the probability that player $i$ chooses $c_i \in C_i$. Player $i$’s beliefs about player $j \in N$ are represented by $\sigma_i^b_{ij}$, which is a mapping from $C_j$ to $\Sigma_j$, where $\sigma_i^b_{ij}(c_j)$ is the probability that $i$ assigns to player $j$ choosing $c_j \in C_j$.

The concatenation of the $\sigma_i^b_{ij}$ is player $i$’s belief profile, $\sigma_i^b$, which is a mapping from $C$ to $\Sigma = \prod_{j=1}^n \Sigma_j$. Player $i$’s expected payoff given belief $\sigma_i^b$ is $\pi_i(\sigma_i^b) = \sum_{c \in C} p_i(c)\Pi_i(c)$ where $p_i(c) = \prod_{j=1}^n \sigma_i^b_{ij}(c_j)$. Choosing $c_{ik}$ for sure is represented by the choice profile $e_{ik} = (0, \ldots, 0, 1, 0, \ldots, 0) \in \Sigma_i$ and the associated expected payoff, given belief $\sigma_i^c$ (with $\sigma_i^b_{ii} = c_{ik}$), is denoted $\pi_{ik}(\sigma_i^c)$. Finally, $E_i = \{e_{ik}\}_{k=1}^{K_i}$ represents all of player $i$’s

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5 Including player $i$’s beliefs about her own choices in $\sigma_i^b$ is done for notational convenience only. Throughout we assume player $i$’s belief about her own choices are correct, i.e. $\sigma_i^b_{ii} = \sigma_i^c$ for $i \in N$. 

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“pure choices” and \( \pi_i = (\pi_{i1}, \ldots, \pi_{iK_i}) \in \mathbb{R}^{K_i} \) denotes the vector of associated expected payoffs.

A player’s best-response correspondence maps the vector of expected payoffs to a player’s choice profile. This mapping may be point or set valued, e.g. the best-response correspondence assigns probability 1 to the choice with the strictly highest payoff but it allows for mixtures when there is a tie for the highest payoff. Player \( i \)’s best-response correspondence can generally be defined as the convex hull of the elements in \( E_i \) that yield the highest expected payoff.

\[
BR_i(\pi_i) = \text{Conv}\{\sigma_i \in E_i \mid \forall \ k, \ell \text{ such that } \sigma_{ik} > \sigma_{i\ell} \Rightarrow \pi_{ik} \geq \pi_{i\ell}\} \tag{1}
\]

For the case of \( K_i = 3 \) possible actions, the best-response correspondence is illustrated in the top panel of Figure 2. The left-most simplex pertains to the case when there is a strict highest payoff in which case the best-response is one of the vertices of the simplex. In the middle panel, two options tie for the highest payoff and now the best-response correspondence produces an edge of the simplex. Finally, in the right panel all three options have the same payoff and the best-response correspondence is the entire simplex.

To allow for the possibility of suboptimal choices, we soften the payoff maximization rule that underlies the best-response correspondence (as in QRE) but in a manner that retains its ordinal nature ( unlike QRE). Let \( r_{i1} = (1, 2, \ldots, K_i)/{(1/2)K_i(K_i + 1)} \in \Sigma_i \) and let \( R_i = \{r_{ik}\}_{k=1}^{K_i} \) denote the set of vectors that result by permuting all elements of \( r_{i1} \). Then player \( i \)’s rank correspondence is defined as:

\[
\text{rank}_i(\pi_i) = \text{Conv}\{\sigma_i \in R_i \mid \forall \ k, \ell \text{ such that } \sigma_{ik} > \sigma_{i\ell} \Rightarrow \pi_{ik} \geq \pi_{i\ell}\} \tag{2}
\]

The bottom panel of Figure 2 illustrates the rank correspondence for the case \( K_i = 3 \). If there is no tie the rank correspondence yields one of the six points in the left panel, when two payoffs tie the rank correspondence produces one of the six line segments in the middle panel, and when all three payoffs tie the rank correspondence produces the hexagon in the right panel.

The best-response and rank correspondences share some important features. First, their images are closed and convex sets (see e.g. Figure 2), i.e. both are upper-hemicontinuous correspondences. Second, both define idempotent mappings in the sense that \( BR_i \circ BR_i = BR_i \) and \( \text{rank}_i \circ \text{rank}_i = \text{rank}_i \). Consider, for instance, the two-dimensional case: \( BR_i(x, y) = (1, 0) \) when \( x > y \), \( BR_i(x, y) = (0, 1) \) when \( x < y \), and \( BR_i(x, y) = \cup_{0 \leq p \leq 1}(p, 1 - p) \) when \( x = y \).
Figure 2: The image of the $BR$ correspondence (top panel) and the $rank$ correspondence (bottom panel) for different $\pi \in \mathbb{R}^3$. If there are no ties between elements in $\pi$ then both $BR$ and $rank$ return a single point in the simplex. The three (six) possible such cases are depicted in the left panels. In case of exactly one tie, $BR$ yields one of the three edges of the simplex while $rank$ yields one of six line segments in the simplex, as show by the middle panels. Finally, if all elements in $\pi$ are equal, then $BR$ equals the entire simplex while $rank$ is equal to a hexagon in the simplex’ interior returns, see the right panels.
Note that \( BR_i(1, 0) = (1, 0), \) \( BR_i(0, 1) = (0, 1), \) and
\[
\bigcup_{0 \leq p \leq 1} BR_i(p, 1 - p) = (1, 0) \cup \bigcup_{0 \leq q \leq 1} (q, 1 - q) \cup (0, 1)
\]
where the middle term occurs when \( p = \frac{1}{2}. \) Hence, \( BR_i(BR_i(x, y)) = BR_i(x, y) \) for all \( (x, y) \in \mathbb{R}^2. \) A similar argument establishes \( rank_i(rank_i(x, y)) = rank_i(x, y) \) for all \( (x, y) \in \mathbb{R}^2. \) This result is intuitive: ranking alternatives that were already ranked results in the same outcome.

The best-response and rank correspondences also differ in important ways. First, the image of the rank correspondence is contained in the interior of the simplex, i.e. all options are assigned strictly positive probability, while the best-response correspondence may assign zero probability to one or more options. Second, non-optimal options matter for the rank correspondence but not for the best-response correspondence (which is why there are only three vertices in the top-left simplex compared to six vertices in the lower-left simplex of Figure 2). Stated differently, ranking options retains all ordinal information about their expected payoffs, while some information is lost when picking only the best. As a result, \( BR \circ rank = BR, \) but not necessarily \( rank \circ BR = rank. \)

3. \( M \) Equilibrium

Let \( \sigma^c \) and \( \sigma^b \) denote the concatenations of players’ choice and belief profiles respectively, and let \( rank \) and \( \pi \) denote the concatenations of players’ rank correspondences and profit functions. We write \( \pi(\sigma^b) \) for the profile of expected payoffs based on players’ beliefs and \( \pi(\sigma^c) \) for the profile of expected payoffs when beliefs are correct, i.e. \( \sigma_i^b = \sigma_i^c \) for \( i \in N. \) The set of possible choice profiles is \( \Sigma = \prod_{i \in N} \Sigma_i \) and the set of possible belief profiles is \( \Sigma^n. \)

**Definition 1** We say \( (M^c, M^b) \subset \Sigma \times \Sigma^n \) form an M Equilibrium if they are the closures of the largest non-empty sets \( M^c \) and \( M^b \) that satisfy
\[
rank(\sigma^c) \subseteq rank(\pi(\sigma^b)) = rank(\pi(\sigma^c)) \tag{3}
\]
for all \( \sigma^c \in M^c, \sigma^b \in M^b. \) The set of \( M \) equilibria of \( G \) is denoted \( \mathcal{M}(G) = (M^c(G), M^b(G)). \)

The characterization of \( M \) equilibrium in (3) provides an intuitive generalization of Nash equi-
librium. The assumption of perfect maximization, \( \sigma^c \in BR(\pi^b) \), is replaced with an ordinal monotonicity condition, \( \text{rank}(\sigma^c) \subseteq \text{rank}(\pi^b) \), and the assumption of perfect beliefs, \( \sigma^b = \sigma^c \), is replaced with an ordinal consistency condition, \( \text{rank}(\pi^b) = \text{rank}(\pi^c) \).

**Proposition 1** \( \mathcal{M}(G) \) is non-empty for any normal-form game.

**Proof.** Recall from Section 2 that the rank correspondence is upper-hemicontinuous and idempotent. Kakutani’s (1941) fixed-point theorem implies existence of a profile, \( \sigma \), that satisfies \( \sigma \in \text{rank}(\pi(\sigma)) \), and since rank is idempotent we have \( \text{rank}(\sigma) \subseteq \text{rank}(\pi(\sigma)) \). Hence, \( \mathcal{M}^c \supseteq \{\sigma\} \) and \( \mathcal{M}^b \supseteq \{\sigma\} \times \cdots \times \{\sigma\} \).

**Example 1.** Consider a matching-pennies game where a Row and Column player choose Heads or Tails. Row receives 1 if their choices match and loses 1 if they don’t. Column’s payoffs are the negative of Row’s payoffs. Let \( \sigma^u \) denote the profile where each player randomizes uniformly over Heads and Tails. Players’ expected payoffs are the same when evaluated at \( \sigma^u \), which is thus a Nash-equilibrium profile. Moreover, \( \text{rank}(BR(\pi(\sigma^u))) = \text{rank}(\pi(\sigma^u)) \), so the Nash-equilibrium condition, \( \sigma^u \in BR(\pi(\sigma^u)) \), implies the \( M \)-equilibrium condition, \( \text{rank}(\sigma^u) \subseteq \text{rank}(\pi(\sigma^u)) \). It is readily verified that \( \sigma^u \) is the only profile such that \( \text{rank}(\sigma^u) \subseteq \text{rank}(\pi(\sigma^u)) \). Hence, \( \mathcal{M}(G) \) consists of a single Nash-equilibrium profile in this (non-generic) example.

In general, \( \text{rank} \circ BR \neq \text{rank} \), so the argument in Example 1 cannot be applied to show that any Nash equilibrium is an \( M \) equilibrium. For instance, for a pure-strategy Nash-equilibrium profile \( \sigma \) in a game with three or more options, \( \text{rank}(\sigma) \) is multi-valued while \( \text{rank}(\pi(\sigma)) \) is single-valued when the expected payoffs of non-optimal choices are unequal. Hence, \( \text{rank}(\sigma) \neq BR(\text{rank}(\sigma)) \) and \( \text{rank}(\sigma) \nsubseteq \text{rank}(\pi(\sigma)) \), so pure-strategy Nash-equilibrium profiles are generally not examples of \( M \) equilibria. Instead, they arise as boundary points of the \( M \)-equilibrium sets. The same is true for degenerate mixed-strategy Nash-equilibrium profiles that lie on the boundary of the simplex.\(^6\) We next present a slightly relaxed definition of \( M \) equilibrium that allows for the inclusion of these boundary cases more directly.

### 3.1. Semi-Algebraic Geometry of \( M \) Equilibrium

Here we present an alternative definition of \( M \) equilibrium that highlights its connection with semi-algebraic geometry. Recall that a semi-algebraic set is defined by a finite number of poly-

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\(^6\)Note that \( \text{rank}(BR(\pi(\sigma^c))) = \text{rank}(\pi(\sigma^c)) \) for any non-degenerate mixed-strategy Nash-equilibrium profile \( \sigma^c \), which are thus always part of \( M \) equilibrium (see also Example 1).
nominal equalities and inequalities. Let $R = \prod_{i=1}^n R_i$ denote the set of all possible permutations of players’ rank vectors. For $r = (r_{1k_1}, \ldots, r_{nk_n}) \in R$ let $\Sigma_r = \{\sigma \in \Sigma \mid \text{rank}(\sigma) \supseteq r\}$.\footnote{Each of the $\Sigma_r$ corresponds to one of the $|R| = \prod_{i \in N} K_i!$ equally-sized parts of $\Sigma$ and $\Sigma = \cup_{r \in R} \Sigma_r$.}

**Definition 2** Fix $r \in R$. We say $(M^c_r, M^b_r) \subset \Sigma_r \times \Sigma^n$ form an $M_r$ Equilibrium if they are the closures of the largest non-empty sets $M^c_r$ and $M^b_r$ that satisfy

$$
\begin{align*}
&\left(\sigma^c_{ik} - \sigma^c_{i\ell}\right)\left(\pi_{ik}(\sigma^c) - \pi_{i\ell}(\sigma^c)\right) > 0 \quad \text{or} \quad \sigma^c_{ik}\left(\pi_{ik}(\sigma^c) - \pi_{i\ell}(\sigma^c)\right) = 0 \\
&\left(\pi_{ik}(\sigma^c) - \pi_{i\ell}(\sigma^c)\right)\left(\pi_{ik}(\sigma^b) - \pi_{i\ell}(\sigma^b)\right) > 0 \quad \text{or} \quad \sigma^b_{ik}\left(\pi_{ik}(\sigma^b) - \pi_{i\ell}(\sigma^b)\right) = 0
\end{align*}
$$

for all $k \neq \ell \in \{1, \ldots, K_i\}$, $i \in N$, $\sigma^c \in M^c_r$, and $\sigma^b \in M^b_r$. The set of $M$ equilibria of $G$ is $\mathcal{M}(G) = \bigcup_{r \in R} (\overline{M^c_r}(G), \overline{M^b_r}(G))$.

**Remark 1.** Definitions 1 and 2 are equivalent except for profiles that lie on the simplex’ boundary. For these boundary profiles, Definition 2 relaxes the constraint that $\sigma^c$ cannot have more equal elements than $\pi(\sigma^c)$ as implied by $\text{rank}(\sigma^c) \subseteq \text{rank}(\pi(\sigma^c))$ in Definition 1. As a result, also degenerate Nash profiles satisfy (4) while they generally do not satisfy (3). Non-emptiness of $\mathcal{M}(G)$ under Definition 2 thus follows from existence of Nash equilibrium.

Definition 2 shows that $M$ equilibrium is generally characterized by a mixture of inequalities and equalities. If only inequalities suffice to describe the $M$ equilibrium then its choice and belief sets have full dimension. Because if inequalities hold for some choice and belief profiles then, by continuity of expected payoffs, they hold for choice and belief profiles that are sufficiently close. In general, however, semi-algebraic sets can have components of various dimensions.

**Example 2.** The top-left panel of Figure 3 shows the different components of the symmetric $M$-equilibrium choice set for the game in the left panel of Table 3. The colored lines in Figure 3 correspond to cases where an equality in (4) holds, either because two expected payoffs are equal

| R   | B   | Y   |
|-----|-----|-----|
| R   | 6, 6| 2, 6| 1, 1 |
| B   | 6, 2| 3, 3| 1, 8 |
| Y   | 1, 1| 8, 1| 9, 9 |

| R   | B   | Y   |
|-----|-----|-----|
| R   | 3, 3| 6, 6| 9, 5 |
| B   | 6, 6| 3, 3| 9, 2 |
| Y   | 5, 9| 2, 9| 10, 10 |

Table 3: Examples of non-generic symmetric two-player games with three possible choices. For the game in the left panel, a payoff-indifference line lies on the simplex’ boundary. For the game in the right panel, a payoff-indifference line lies on one of the simplex’ diagonals.
Figure 3: The left panel shows the symmetric $M$-equilibrium choices and the right panel shows $M$-equilibrium beliefs. The top and middle panels correspond to the non-generic games in Table 3 and the bottom panel to the generic game in Table 4. The green curves indicate $M$-equilibrium profiles where two expected payoffs are equal, the red lines indicate $M$-equilibrium profiles contained in the simplex’ boundary, and the grey lines indicate $M$-equilibrium profiles contained in one of the simplex’ diagonals. The blue shaded areas correspond to interior $M$-equilibrium profiles.
Table 4: A generic symmetric three-player game with three possible choices.

|   | R   | B   | Y   |
|---|-----|-----|-----|
| R | 14  | 12  | 30  |
| B | 12  | 4   | 32  |
| Y | 30  | 32  | 10  |

Table 4: A generic symmetric three-player game with three possible choices.

(green) or because the profile lies on the simplex’ boundary (red). The grey lines correspond to the simplex’ diagonals. For this non-generic game, there are four $M$ equilibria: two have dimension zero (the Nash-equilibrium profiles $\sigma^c = (1, 0, 0)$ and $\sigma^c = (\frac{8}{13}, 0, \frac{5}{13})$), one has dimension one (the closure of the set of profiles $\sigma^c = (p, 0, 1-p)$ for $0 < p < \frac{1}{2}$), and one has dimension two (the closure of the set of profiles $\sigma^c = (\sigma^c_1, \sigma^c_2, \sigma^c_3)$ that satisfy $0 < \sigma^c_1 < \sigma^c_2 < \sigma^c_3$ and $\sigma^c_1 + \sigma^c_2 + \sigma^c_3 = 1$). Except for the Nash-equilibrium profile $\sigma^c = (1, 0, 0)$, these are also $M$-equilibria under Definition $\Box$. The middle-panel shows a case where the symmetric $M$-equilibrium choice and belief sets are one dimensional and have measure zero. The bottom panel shows that the $M$-equilibrium sets can, generically, have multiple connected components.

**Proposition 2** For any normal-game $G$, there is at least one and at most a finite number of $M$ equilibria, each of which consist of a finite number of components.

**Proof.** That there is at least one $M$ equilibrium was shown in Proposition $\Box$. That there are finitely many $M$ equilibria follows from basic results in semi-algebraic geometry, see e.g. Coste (2002).

### 3.2. Coloring of $M$ Equilibrium

The lower-dimensional $M$-equilibrium components shown in the top and middle panels of Figure $\Box$ arise when a special condition is met: a payoff-indifference line coincides with the boundary of the simplex or one of its diagonals. Games for which this occurs are non-generic in the sense
that if the game is perturbed slightly then these lower-dimensional components disappear and only the sets of full dimension remain. The reason that full-dimensional sets do not disappear is that the rank correspondence is single-valued on the interior of these sets and, hence, the expected payoffs can be strictly ranked. By continuity of expected payoffs, this strict ranking remains the same if the game is slightly perturbed. Also, the expected payoffs based on the supporting beliefs must have the same unique ranks, which allows us to “color” the \( M \)-equilibrium choice and belief sets by their rank vector.

**Proposition 3** Generically,

(i) an \( M \) equilibrium is a set of positive measure in \( \Sigma \times \Sigma^n \),

(ii) an \( M \) equilibrium is characterized or “colored” by the rank vector \( r = (r_{1k_1}, \ldots, r_{nk_n}) \in \mathbb{R} \),

(iii) Nash-equilibrium profiles and \( \sigma^u \) are boundary points of some \( M \)-equilibrium choice set.

**Example 3.** To illustrate, consider the four \( 2 \times 2 \) games in Table 5. Let \( \sigma_R = (q, 1 - q) \) and \( \sigma_C = (p, 1 - p) \) denote Row’s and Column’s choice profiles, where \( q \) and \( p \) are the probabilities that Row and Column choose \( A \) respectively. Then, for instance, \( \text{rank}(\sigma_R) = (\frac{2}{3}, \frac{2}{3}) \) when \( q < \frac{1}{2} \) and \( \text{rank}(\sigma_R) = (\frac{2}{3}, \frac{1}{3}) \) when \( q > \frac{1}{2} \). Since the entries in the rank vectors add up to 1, we can characterize any \( M \) equilibrium by the first entries of the players’ rank vectors. That leaves four possible \( M \) equilibria that can be color coded: \((\frac{1}{3}, \frac{1}{3})\) (red), \((\frac{1}{3}, \frac{2}{3})\) (grey), \((\frac{2}{3}, \frac{1}{3})\) (blue), and \((\frac{2}{3}, \frac{2}{3})\) (yellow).

In Figure 4, the left panels show the actions sets, \( \Sigma^c \), and the right panels show the belief sets, \( \Sigma^b \), for the four games in Table 5. In the left panels, the square at \((\frac{1}{2}, \frac{1}{2})\) indicates the uniform randomization profile, \( \sigma^u \), and the disks indicate Nash equilibria. For each of the four games, the beliefs sets are shown in the same color as the actions they support. For example, in the symmetric coordination game, \( A \) has a higher expected payoff only when a player believes the chance the other plays \( A \) exceeds \( \frac{2}{3} \). The requirement \( \text{rank}(\sigma^c) = \text{rank}(\pi(\sigma^b)) = \text{rank}(\pi(\sigma^c)) \) then implies \( \Sigma^c = \Sigma^b = [\frac{2}{3}, 1]^2 \) as shown by the same-sized yellow squares in the top panels of Figure 4. In contrast, \( B \) has a higher expected payoff when a player believes the chance the other plays \( A \) does not exceed \( \frac{2}{3} \). Now, the requirement \( \text{rank}(\sigma^c) = \text{rank}(\pi(\sigma^b)) = \text{rank}(\pi(\sigma^c)) \) implies \( \Sigma^c = [0, \frac{1}{2}]^2 \) and \( \Sigma^b = [0, \frac{2}{3}]^2 \) as shown by the red squares in the top-left and top-right panels of Figure 4 respectively.

\(^8\)To be precise, the choice sets are based on only the first entries of \( \sigma_R \) and \( \sigma_C \), i.e. \( q \) for Row and \( p \) for Column (as the two entries of the \( \sigma \)’s add up to 1). Correspondingly, the belief sets are based on Row’s belief about \( p \) and Column’s belief about \( q \). This allows us to depict \( \Sigma^c \) and \( \Sigma^b \) in two-dimensional graphs.
Table 5: A symmetric coordination game, an asymmetric game of chicken, an asymmetric matching-pennies game, and a non-generic game with a continuum of Nash equilibria.

|   | A  | B  |
|---|----|----|
| A | 2, 2 | 2, 1 |
| B | 1, 2 | 4, 4 |

Table 6: A symmetric $3 \times 3$ game with a unique symmetric mixed-strategy Nash equilibrium (left panel), a unique symmetric pure-strategy equilibrium (middle panel), and the maximum number of symmetric equilibria (right panel).

|   | R  | B  | Y  |
|---|----|----|----|
| R | 6, 6 | 7, 1 | 2, 7 |
| B | 1, 7 | 6, 6 | 7, 0 |
| Y | 7, 2 | 0, 7 | 6, 6 |

The $M$ equilibria for the asymmetric game of chicken and the asymmetric matching-pennies game can be worked out similarly. In the game on the far-right in Table 5, $B$ is the weakly dominant choice for Row and $A$ is the dominant choice for Column. The Nash equilibria are $p = 1$ and $q \in [0, 1]$ and the trembling-hand-perfect equilibrium is $p = 1$ and $q = 0$. In this non-generic example, there is an $M$-equilibrium of full dimension, i.e. $\Sigma_c = [\frac{1}{2}, 1] \times [0, \frac{1}{2}]$ and $\Sigma_b = [0, 1]^2$, and a non-colorable $M$-equilibrium of lower dimension, i.e. $\Sigma_c = \Sigma_b = \{1\} \times [0, 1]$.

**Example 4.** Next, consider the three symmetric $3 \times 3$ games in Table 6. The left-most game has a unique mixed-strategy Nash equilibrium, the middle game is dominance solvable and has a unique pure-strategy Nash equilibrium, while the right-most game has the maximal number (7) of symmetric Nash equilibria (three pure strategy and four mixed-strategy equilibria). The corresponding $M$-equilibrium sets are shown in Figure 5.

Different from the $2 \times 2$ case, the symmetric $M$-equilibrium choice sets for the $3 \times 3$ case may contain neither a Nash-equilibrium profile nor the the random-behavior profile. For example, the bottom panel shows a case of a completely disconnected set for which this is the case. The middle panels show a case where both players choosing “yellow” is the unique pure-strategy Nash equilibrium. Most of the belief set is “blue,” however, and these beliefs support choice profiles where “blue” is the most likely choice.

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9We will discuss this possibility in more detail in the next section where we compare $M$ equilibrium with parametric theories such as QRE.
Figure 4: The left panels show the $M$-equilibrium choice sets and the right panels show the $M$-equilibrium belief sets for the four games in Table 5. The belief sets are shown in the same color as the actions they support. In the left panels, the square at the center indicates the uniform randomization profile and the disks / line indicate Nash equilibria.
Figure 5: The left panels show the $M$-equilibrium choice sets and the right panels show the $M$-equilibrium belief sets for the three games in Table 6. The belief sets are shown in the same color as the actions they support. In the left panels, the square at the center indicates the uniform randomization profile and the disks indicate Nash equilibria.
Proposition 4 Generically,

(i) $|\mathcal{M}(G)|$ may be even or odd, and may be less than, equal, or greater than the number of Nash equilibria.

(ii) An $M$-equilibrium may contain zero, one, or multiple Nash equilibria.

(iii) The measure of an $M$-equilibrium choice set is bounded by $\prod_{i=1}^{n} 1/|C_i|!$

(iv) In contrast, an $M$-equilibrium belief set may have full measure.

Proof. Properties (i)-(ii) are demonstrated by Figures 4 and 5. Property (iv) holds, for instance, for any dominance-solvable game. Property (iii) follows since each $C_i$ can be partitioned into $|C_i|!$ equally-sized subsets indexed by the ranks of the entries of the choice profiles it contains. Since $\text{rank}_i(\sigma_i^c)$ must be constant on an $M$-equilibrium choice set for all $i \in N$, the $M$-equilibrium choice set must be contained in the Cartesian product of a single such subset for each player. Hence, its size cannot be larger than $\prod_{i=1}^{n} 1/|C_i|!$.

A priori, the set-valued nature of $M$ equilibrium might have been considered a drawback as it might render its predictions non-falsifiable. Proposition 4 shows this is not the case. The size of an $M$-equilibrium choice set falls quickly (in fact, factorially or exponentially fast) with the number of players and the number of possible choices.

Our interest in colorable $M$-equilibria is based on the intuition that they are the empirically relevant ones. To make this precise, we define a new stability notion called behavioral stability. For $\varepsilon > 0$, let $G(\varepsilon)$ denote the set of normal-form games that result from $G$ by perturbing any of its payoff numbers by at most $\varepsilon$.

Definition 3 We say that $(\sigma^c, \sigma^b) \in \Sigma \times \Sigma^n$ is a behaviorally stable profile of the normal-form game $G$ if there exists an $\varepsilon > 0$ such that $(\sigma^c, \sigma^b) \in \mathcal{M}(G')$ for all $G' \in G(\varepsilon)$. Let $B(G)$ denote the closure of the set of behaviorally stable profiles of $G$.

In words, a choice-belief profile is behaviorally stable for the game $G$ if it is $M$-equilibrium profile for $G$ as well as for all nearby games. Note that behavioral stability is a sharpening of the “strategic stability” criterion introduced in Kohlberg and Mertens (1986). The latter requires a strategically-stable Nash profile to be “close to” the Nash equilibrium of the perturbed game. Behavioral stability, in contrast, requires the $M$-equilibrium profile to be an $M$-equilibrium of the perturbed game.
Proposition 5  If $M \in \mathcal{M}(G)$ is colorable then $M \subseteq \mathcal{B}(G)$.

Proof. Profiles in the interior of a colorable $M$-equilibrium are those for which the rank correspondence is single valued, i.e. that can be characterized by a single vector $r \in R$. This means that at those profiles, expected payoffs can be strictly ranked. Since expected payoffs are continuous in the payoff numbers, they will be ranked the same for games that are sufficiently close. Hence, the interior of $M$ is contained in $\mathcal{B}(G)$, and since $\mathcal{B}(G)$ is closed, $M \subseteq \mathcal{B}(G)$.

Remark 2. Generically, only colorable $M$-equilibrium profiles are behaviorally stable. For instance, for the game in the left panel of Table 3 only the profiles in the full-dimensional set in the top panel of Figure 3 are behaviorally stable. However, it is readily verified that for the matching-pennies game of Example 1, the behaviorally stable set consists of a single non-colorable $M$-equilibrium profile.

4. Parametric Models of Stochastic Choice

Like $M$ equilibrium, the parametric models introduced in this section obey the ordinal monotonicity condition that choice probabilities are ordered the same as their associated expected payoffs. Unlike set-valued $M$ equilibrium, however, their predictions are based on fixed-points. For $i \in N$, let $\mu_i \in \Sigma_i$ and let $R_i(\mu_i) = \{\mu_{ik}\}_{k=1}^{K_i}$ denote the set of vectors that result by permuting the elements of $\mu_i$. For $i \in N$, we define player $i$’s $\text{rank}_{i}^{\mu_i}$ correspondence as follows

$$\text{rank}_{i}^{\mu_i}(\pi_i) = \text{Conv}(\{\sigma_i \in R_i(\mu_i) \mid \forall k, \ell \text{ such that } \sigma_{ik} > \sigma_{i\ell} \Rightarrow \pi_{ik} \geq \pi_{i\ell}\})$$  \hspace{1cm} (5)

The $\text{rank}_{i}^{\mu_i}$ correspondence includes random behavior and best response behavior as special cases and satisfies a generalized idempotence condition.

Proposition 6  For all $i \in N$,

(i) $\text{rank}_{i}^{\mu_i}(\pi_i) = \sigma_i^u$ for all $\pi_i \in \mathbb{R}^{K_i}$ when $\mu_i = \sigma_i^u$.

(ii) $\text{rank}_{i}^{\mu_i}(\pi_i) = \text{rank}_i(\pi_i)$ for any $\mu_i \in R_i$.

(iii) $\text{rank}_{i}^{\mu_i}(\pi_i) = BR_i(\pi_i)$ for any $\mu_i \in E_i$.

(iv) $\text{rank}_{i}^{\mu_i} \circ \text{rank}_{i}^{\mu_i} = \text{rank}_{i}^{\mu_i}$ for any $\mu_i, \mu_i' \in \Sigma_i$ such that $\text{rank}_i(\mu_i')$ is single valued.
\[ \text{rank}^\mu_i(\mu_i) = \mu_i \text{ for any } \mu_i \in \Sigma_i. \]

For \( \mu = (\mu_1, \ldots, \mu_n) \in \Sigma \), let \( \text{rank}^\mu \) denote the concatenation of the \( \text{rank}^\mu_i \) for \( i \in N \).

**Definition 4** We say \((\sigma^c, \Sigma^b)\) is a \( \mu \)-Equilibrium of the normal-form game \( G \) if, for all \( \sigma^b \in \Sigma^b \),

\[ \sigma^c \in \text{rank}^\mu(\pi(\sigma^b)) \tag{6} \]

and \( \Sigma^b \) is the closure of the largest set, \( \Sigma^b \), such that \( \text{rank}^\mu(\pi(\sigma^b)) = \text{rank}^\mu(\pi(\sigma^c)) \) for all \( \sigma^b \in \Sigma^b \). The set of all \( \mu \)-equilibria of \( G \) is denoted \( E^\mu(G) \).

The image of the \( \text{rank}^\mu \) correspondence is a closed and convex set (see e.g. Figure 2 for the case of \( \mu \in \mathbb{R} \) and \( \mu \in E \)). Existence of an \( \mu \)-equilibrium thus follows from Kakutani’s (1941) fixed-point theorem. This also implies that \( \Sigma^b \) is non-empty as \( \Sigma^b \ni \sigma^c \) if \( \sigma^c \in \text{rank}^\mu(\pi(\sigma^c)) \).

**Proposition 7** \( E^\mu(G) \) is non-empty for any \( \mu \in \Sigma \) and any normal-form game \( G \).

While \( \mu \)-equilibrium choice profiles are defined as fixed-points, they are easy to compute. The reason is that the right side of (6) does not vary continuously with players’ beliefs but, instead, is a piecewise-constant function over the simplex that takes on only finitely many values.

**Example 5.** To illustrate the simplicity of \( \mu \)-equilibrium computations, consider the asymmetric game of chicken in the second panel of Table 5. Let \( \sigma_C = (p, 1-p) \) and \( \sigma_R = (q, 1-q) \) where \( p \) \((q)\) denotes the probability with which Column (Row) chooses \( A \). To obtain a parsimonious model, let \( \mu_R = \mu_C = (1^\rho, 2^\rho)/(1^\rho + 2^\rho) \) where \( \rho \geq 0 \) is a “rationality parameter.” Then the \( \text{rank}^\mu \) responses are

\[
\text{rank}^\mu_R(p) = \begin{cases} 
\frac{1}{1 + 2^\rho} & \text{if } p < \frac{4}{5} \\
\frac{1}{1 + 2^\rho}, \frac{1}{1 + 2^\rho} & \text{if } p = \frac{4}{5} \\
\frac{1}{1 + 2^\rho} & \text{if } p > \frac{4}{5}
\end{cases} \quad \text{and} \quad
\text{rank}^\mu_C(q) = \begin{cases} 
\frac{1}{1 + 2^\rho} & \text{if } q < \frac{8}{9} \\
\frac{1}{1 + 2^\rho}, \frac{1}{1 + 2^\rho} & \text{if } q = \frac{8}{9} \\
\frac{1}{1 + 2^\rho} & \text{if } q > \frac{8}{9}
\end{cases}
\]

The \( \text{rank}^\mu \) responses are “flat,” i.e. \( \text{rank}^\mu_R(p) = \text{rank}^\mu_C(q) = \frac{1}{2} \), when \( \rho = 0 \), and they limit to standard best responses when \( \rho = \infty \). Figure 5 shows Row’s and Column’s \( \text{rank}^\mu \) responses when \( \rho \) increases from 0 in the top-left panel to 5 in the bottom-right panel. The intersection of the \( \text{rank}^\mu \) correspondences typically consists of an odd number of points (1 or 3) except at \( \rho = 2 \), in which case it contains \( p = \frac{4}{5} \) and any \( q \in [\frac{1}{5}, \frac{4}{5}] \), and at \( \rho = 3 \), when a bifurcation occurs and the intersection contains \( q = \frac{8}{9} \) and any \( p \in [\frac{1}{9}, \frac{4}{5}] \).
Figure 6: Row’s (solid) and Column’s (dashed) ordinal responses for the asymmetric game of chicken in the second panel of Table 5 when $\rho \in \{0, 1, 2, \frac{5}{2}, 3, 5\}$ as indicated in the panels.

Figure 7: In the left panel, the colored (dashed) curves show the $\mu$-equilibrium (logit-QRE) choice correspondence for the asymmetric game of chicken in the second panel of Table 5 for $0 \leq \rho \leq \infty$. The square at the center corresponds to random behavior ($\rho = 0$) and the disks correspond to Nash equilibria ($\rho = \infty$). In the right panel, the colored areas correspond to the supporting beliefs, $\Sigma^b$, for the different ranges of $\rho$. 
More generally, let $\Gamma_\mu(\rho)$ denote the $\mu$-equilibrium correspondence, which consists of a choice part and an associated supporting belief part: $\Gamma_\mu(\rho) = (\Gamma_\mu^c(\rho), \Gamma_\mu^b(\rho))$. The colored curves in the left panel of Figure 7 show the $\mu$-equilibrium correspondence while the dashed curves show the logit-QRE correspondence. There are some similarities. Each have a “principal branch” that starts at the center when $\rho = 0$ and ends at the pure-strategy Nash equilibria ($p^* = 1, q^* = 0$) when $\rho = \infty$. And each have an additional branch that connects the other pure-strategy Nash equilibrium, ($p^* = 0, q^* = 1$), with the mixed equilibrium, ($p^* = \frac{4}{5}, q^* = \frac{8}{9}$).

There are also some differences. First, the $\mu$-equilibrium correspondence can be computed easily and characterized analytically. More importantly, $\mu$-equilibrium choices are generally supported by a range of beliefs. In the right panel of Figure 7, the three differently colored rectangles show the supporting beliefs that correspond to the part of the $\mu$-equilibrium correspondence with the same color in the left panel. In contrast, the beliefs that support logit-equilibrium choices are forced to be correct and, hence, have zero measure, see the dashed curves in the right panel of Figure 7.

We next generalize the findings in Example 5. For this we need to define the two possible limit cases, i.e. $\rho = 0$ and $\rho = \infty$, more generally. The former case corresponds to random behavior, which leads us to define: $\text{RAND} = (\sigma^u, \{\sigma^b | \text{rank}(\pi(\sigma^b)) = \text{rank}(\pi(\sigma^u))\})$. The latter case corresponds to best-response behavior, which leads to the following definition.

**Definition 5** We say $(\sigma^c, \Sigma^b)$ form a belief-augmented Nash equilibrium (BEAUNE) if $\sigma^c$ is a best response to any belief in $\Sigma^b$, i.e. for all $\sigma^b \in \Sigma^b$,

$$\sigma^c \in BR(\pi(\sigma^b))$$

and $\Sigma^b$ is the closure of the largest set, $\Sigma^b$, such that $BR(\pi(\sigma^b)) = BR(\pi(\sigma^c))$ for all $\sigma^b \in \Sigma^b$.

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10 The full description of the $\mu$-equilibrium correspondence for the asymmetric game of chicken in the second panel of Table 5 is:

$$\Gamma_\mu(\rho) = \begin{cases} 
\left\{ \left( \left[ \frac{1}{1+2\rho}, \frac{1}{1+2\rho} \right], \left[ 0, \frac{2}{3} \right] \times \left[ 0, \frac{2}{3} \right] \right) \right\} & \text{if } 0 \leq \rho < 2 \\
\left\{ \left( \frac{1}{3}, \frac{2}{3} \times [0, \frac{2}{3}] \right) \times \left( \frac{2}{3}, 1 \right) \right\} & \text{if } \rho = 2 \\
\left\{ \left( \left[ \frac{1}{1+2\rho}, \frac{1}{1+2\rho} \right], \left[ 0, \frac{2}{3} \right] \times \left[ 0, \frac{2}{3} \right] \right) \right\} & \text{if } 2 < \rho < 3 \\
\left\{ \left( \frac{2}{3}, \frac{2}{3} \times [0, \frac{2}{3}] \right) \cup \left( \left[ \frac{2}{3}, \frac{2}{3} \right] \times \left[ \frac{2}{3}, 1 \right] \right) \right\} & \text{if } \rho = 3 \\
\left\{ \left( \left[ \frac{1}{1+2\rho}, \frac{1}{1+2\rho} \right], \left[ 0, \frac{2}{3} \right] \times \left[ 0, \frac{2}{3} \right] \right) \right\} & \text{if } \rho > 3 \\
\end{cases}$$

where the first part corresponds to actions and the second part to beliefs about the other player. In the above expressions, $\{x, y\}$ denotes a point and $[x, y]$ an interval.
For instance, for the game in Table 2 of the Introduction, \( \sum^b = [0, 1]^2 \) contains all possible beliefs, reflecting the dominant-strategy nature of the game. In contrast, the non-degenerate mixed-strategy Nash-equilibrium profile for the asymmetric matching-pennies game in Table 3 is supported only by the profile itself.

**Proposition 8** For \( i \in N \), define \( \mu_i(\rho) = (1^\rho, 2^\rho, \ldots, K_i^\rho) / \sum_k k^\rho \) where \( \rho \geq 0 \), and let \( \mu(\rho) \) denote their concatenation. The \( \mu \)-equilibrium correspondence

\[
\Gamma_\mu : \rho \mapsto E_{\mu(\rho)}(G)
\]

has the following properties for generic games:

(i) \( \Gamma_\mu(\rho) \) is upper hemicontinuous.

(ii) \( |\Gamma_\mu^c(\rho)| \) is odd and \( \Gamma_\mu^b(\rho) \) has strictly positive measure for almost all \( \rho \).

(iii) \( \Gamma_\mu(\rho) \) limits to a BEAUNE when \( \rho \to \infty \) and to RAND when \( \rho \to 0 \).

(iv) \( \Gamma_\mu(\rho) \) has a principal branch that connects RAND to exactly one BEAUNE. The other BEAUNE are connected as pairs.

**Remark 3.** These properties are standard and mimic those of the logit-QRE correspondence, see McKelvey and Palfrey (1995, 1996). They point out that not all Nash equilibria arise as limit points of QRE, and they call those that do “approachable.” Likewise, not all BEAUNE arise as limit points of \( \Gamma_\mu(\rho) \) when \( \rho \to \infty \). We will see, however, that colorable BEAUNE, i.e. those that are boundary points of a colorable \( M \)-equilibrium, are approachable.

The main interest in Proposition 8 is that it offers a single-parameter alternative to logit-QRE, *which does not restrict beliefs to be correct.* It would be more realistic, however, to assume that players’ rationality parameters differ, which begs the question what choices and beliefs occur under a heterogeneous \( \mu \)-equilibrium. We next show that by varying \( \mu \), the \( \mu \)-equilibrium models “fill out” the set of \( M \) equilibria.

### 4.1. \( M \) Equilibrium as a Meta Theory

We first compare \( M \) equilibrium with the Luce-QRE model for which analytical solutions exist for simple games.
Example 6. To glean some intuition, consider the asymmetric matching-pennies game in the third panel of Table 5. Suppose players’ choice probabilities follow from expected payoffs using a Luce choice rule, e.g. the chance that Column and Row choose $A$ is

$$p = \frac{\pi_{A}^{RC}}{\pi_{A}^{RC} + \pi_{B}^{RC}}$$

$$q = \frac{\pi_{A}^{PR}}{\pi_{A}^{PR} + \pi_{B}^{PR}}$$

where $\rho_{R} \geq 0$ and $\rho_{C} \geq 0$ are rationality parameters. These equations define the Luce-QRE fixed-point equations, which can readily be solved for the asymmetric matching-pennies game:

$$p^{Luce}(\rho_{R}, \rho_{C}) = \frac{1}{1 + 5 \frac{\rho_{C}(1-\rho_{R})}{\rho_{C}\rho_{R}+1}}$$

$$q^{Luce}(\rho_{R}, \rho_{C}) = \frac{1}{1 + 5 \frac{\rho_{R}(1+\rho_{C})}{\rho_{C}\rho_{R}+1}}$$

And given these expressions it is straightforward to show that

$$\bigcup_{\rho_{R} \geq 0, \rho_{C} \geq 0} \left\{ p^{Luce}(\rho_{R}, \rho_{C}), q^{Luce}(\rho_{R}, \rho_{C}) \right\} = [0, \frac{1}{2}] \times \left[ \frac{1}{6}, \frac{1}{2} \right] \cup \left[ \frac{1}{2}, \frac{5}{6} \right] \times [0, \frac{1}{6}]$$

i.e. the closure of the set of all Luce-QRE fixed-points equals the union of the “red” and “blue” $M$-equilibrium choice sets in Figure 4.

While the $M$-equilibrium choice set envelopes all Luce-QRE profiles, the modeling of beliefs differs sharply in the two models. $M$-equilibrium choices in the “red” set are supported by “red” beliefs while “blue” actions are supported by “blue” beliefs. In other words, beliefs and choices do not necessarily match, they just have the same color. Also, players may hold different beliefs as long as the “color” of their choices matches that of their beliefs. In contrast, the “colorblind” Luce-QRE model assumes that beliefs are homogeneous across players and correct. The same is true more generally for QRE models: if $\sigma^{c}$ denotes a QRE action profile then $\sigma_{i}^{b} = \sigma^{c}$ for all $i \in N$. As a result, the collection of all QRE beliefs will have measure zero in $\Sigma^{n}$ and collection of all QRE choices can only be compared with the union of $M$-equilibrium choice sets irrespective of their color.

In contrast, $\mu$-equilibrium choices are colorable in the sense that they are supported by a set of beliefs of the same color. We next show that $\mu$-equilibrium choices of a certain color fill
Proposition 9 Let $M \in \mathcal{M}(G)$ be colorable, i.e. characterized by a single rank vector $r \in R$, then

$$M = \bigcup_{\text{rank}(\mu) = r} E_\mu(G) \quad (8)$$

BEAUNE that are boundary points of a colorable $M$-equilibrium are approachable as is RAND.

Proof. Suppose $(\sigma^c, \sigma^b) \in E_\mu(G)$ for some $\mu$ with $\text{rank}(\mu) = r$ then $\sigma^c \in \text{rank}^u(\pi(\sigma^b)) = \text{rank}^u(\pi(\sigma^c))$. Applying $\text{rank}$ and using the properties of Proposition 6 yields $\text{rank}(\sigma^c) \subseteq \text{rank}(\pi(\sigma^b)) = \text{rank}(\pi(\sigma^c))$, i.e. $(\sigma^c, \sigma^b) \in M$. This shows that $M$ contains the union of the $E_\mu(G)$ and since $M$ is closed it follows that

$$M \supseteq \bigcup_{\text{rank}(\mu) = r} E_\mu(G)$$

Conversely, if $(\sigma^c, \sigma^b)$ is in the interior of $M$ then we have $\text{rank}(\sigma^c) = r$ and $\text{rank}(\sigma^c) \subseteq \text{rank}(\pi(\sigma^b)) = \text{rank}(\pi(\sigma^c))$. Applying $\text{rank}^e$ and using the properties of Proposition 6 yields $\sigma^c \in \text{rank}^e(\pi(\sigma^b)) = \text{rank}^e(\pi(\sigma^c))$, i.e. $(\sigma^c, \sigma^b) \in E_\sigma^e(G)$. This shows that the interior of $M$ is contained in the union of the $E_\mu(G)$. Since $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$, it follows that

$$M \subseteq \bigcup_{\text{rank}(\mu) = r} E_\mu(G)$$

which completes the proof of (8). Any BEAUNE $(\sigma^c, \Sigma^b)$ that is a boundary point of $M$ can thus be obtained as the limit of $E_\mu(G)$ when $\mu \to \sigma^c$, while RAND follows when $\mu \to \sigma^u$. ■

Remark 4. The proof uses property (iv) of Proposition 6 which only applies if the $\text{rank}^u$ correspondences does not “lose any information,” which explains the restriction to $\text{rank}(\mu)$ being single valued. We conjecture that non-colorable $M$ equilibria follow by considering $\mu$ such that $\text{rank}(\mu)$ is multi-valued. For instance, for the matching-pennies game of Example 1, the unique $M$-equilibrium is given by $M = E_{\sigma^u}(G)$. More generally, we conjecture that

$$\mathcal{M}(G) = \bigcup_{\mu \in \Sigma} E_\mu(G)$$

But our main interest is in the colorable $M$ equilibria since, generically, these are the only ones that are behaviorally stable.
A similar comparison of $M$ equilibrium to the union QRE, i.e. for different formulations of the quantal response functions, is complicated by the fact that QRE assumes correct point beliefs. As a result, the set of beliefs traced out by different QRE models has the same dimension as the set of choice profiles, $\Sigma$, which has zero measure in the set of belief profiles, $\Sigma^n$. Varying over all possible QRE models will, therefore, not “fill out” the $M$-equilibrium belief sets.

However, we can establish an equivalence result for the union of $M$-equilibrium choice sets. Imposing correct beliefs, $\sigma^b = \sigma^c$ for $i \in N$, reduces (3) to $\text{rank}(\sigma^c) = \text{rank}(\pi(\sigma^c))$. Let $\mathcal{R}$ denote the infinitely-dimensional space of regular quantal responses (see e.g. Goeree, Holt, and Palfrey, 2016) with typical element $R \in \mathcal{R}$, and let $QRE_R(G)$ denote the set of quantal response equilibria of $G$ with respect to $R$. Moreover, let $\tilde{\mathcal{M}}^c(G)$ denote the closure of the intersection of $\mathcal{M}^c(G)$, i.e. the union of all $M$-equilibrium choice sets, with the simplex interior. Goeree et al. (2018) show that

$$\tilde{\mathcal{M}}^c(G) = \bigcup_{R \in \mathcal{R}} QRE_R(G)$$

In other words, by varying over the infinite-dimensional space of all possible quantal response functions, the QRE fixed-points essentially “fill out” the union of $M$-equilibrium choice sets. However, unlike $\mu$-equilibrium (see Proposition 9), it is not possible to do this “color by color,” i.e. to fill out individual $M$-equilibrium choice sets. Further care must be taken in interpreting the QRE “equivalence” result as the next example shows.

**Example 7.** Consider the middle game in Table 6, in which $R$ is dominated by a fifty-fifty combination of $Y$ and $B$. Using the logit formulation, the probability a player chooses $R$ is given by

$$p_R = \frac{1}{1 + e^{\lambda(\pi_Y - \pi_R)}} + e^{\lambda(\pi_B - \pi_R)} \leq \frac{1}{1 + 2 e^{\lambda((\pi_Y + \pi_B)/2 - \pi_R)}} \leq \frac{1}{3}$$

where the first inequality follows from convexity of the exponential function and the second inequality follows since choosing $R$ is worse than randomizing uniformly over $B$ and $Y$. Note that the $1/3$ bound holds irrespective of a player’s beliefs (in particular, beliefs don’t have to be correct) and irrespective of the rationality parameter. As a result, there can be no logit QRE, and no SQRE for that matter, with $p_R > \frac{1}{3}$.

However, there is an $M$-equilibrium choice set with $p_B > p_R > \frac{1}{3}$, see the blue set in the middle-left panel of Figure 5. While this finding does not invalidate the result that the

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11 The logit QRE correspondence consists of a single curve that starts at the random choice profile and crosses only the cyan and green sets to end at the unique Nash equilibrium.

12 Expected payoffs are $\pi_R = \pi_B + 2 - 16p_R$ and $\pi_R = \pi_Y + 9p_R - 3$, so $\text{rank}(\sigma^c) = \text{rank}(\pi(\sigma^c))$ for $\sigma^c = (p_R, p_B, p_Y)$ with $p_B > p_R > \frac{1}{3}$. 

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union of all QRE fixed-point equals the union of all $M$-equilibrium choice sets (it just means other-than-logit quantal responses are needed to produce blue QRE choices), it underscores the impossibility for QRE to fill out individual $M$-equilibrium sets “color by color.” By varying the two players’ rationality parameters, logit QRE fills out the green and cyan $M$-equilibrium choice sets but it does not produce any choice profiles in the blue $M$-equilibrium choice set.

Given the size of its belief set (see the middle-right panel of Figure 5), the blue $M$ equilibrium is likely to be relevant empirically even though it cannot be reached by logit QRE. We confirm this conjecture in the next section using a variant of the game in Table 6 with an even larger blue choice and belief set. The main reason QRE is falsified, however, is because it assumes homogeneous and correct beliefs. In the experiments reported below, beliefs generally differ across subjects and beliefs differ from choices.

5. An Experiment

We report the results from a series of experiments to illustrate how $M$ equilibrium provides a lens through which to better understand strategic behavior in games. The experiments shared some common features. Sixteen participants joined each experimental session. Subjects were first given instructions in a power-point presentation that was read aloud. Subjects played two-player matrix games with either two or three possible choices. Each game was played for either 8 rounds (the $2 \times 2$ games) or 15 rounds (the $3 \times 3$ games). In each round, participants were randomly rematched with a different participant in a perfect stranger protocol, i.e. ensuring that two players never played the same game together more than once. This feature was made explicitly clear to subjects.

On the screen that showed the payoff table and that let subjects choose a row of the matrix, subjects were also asked to submit their beliefs about their opponent’s choice in terms of “percentage chances.” Belief elicitation was incentivized using a generalization of a method proposed by Wilson and Vespa (2016), which is an implementation of Hossain and Okuis’ (2013) binarized scoring rule (BSR). The BSR is incentive compatible for general risk-preferences and, hence, avoids issues of risk-aversion that plague other scoring rules (e.g. quadratic scoring rule). Wilson and Vespa’s (2016) method operationalizes the BSR for binary-choice settings.
in a way that is simple to explain. Subjects submit the “percentage chance” with which they believe their opponent chooses each action by moving a single slider between endpoints labeled $A$ and $B$. Any point on the slider corresponds to a unique chance of $A$ being played (and $B$ with complementary chance) with the $A$ endpoint ($B$ endpoint) corresponding to the belief that the opponent chooses $A$ ($B$) for sure. The point chosen by the subject is then compared with two computer-generated random points on the slider. If the chosen point is closer to the opponent’s actual choice (one of the endpoints) than at least one of the two randomly-drawn points then the subject receives a fixed prize. The next proposition generalizes this method so that it can be used for general normal-form games.

**Proposition 10** Consider the elicitation mechanism where player $i \in N$ uses $S_i = \sum_{j \neq i} K_j$ sliders, labeled $S_{jk}$ for $j \neq i$ and $k = 1, \ldots, K_j$, to report beliefs $q_{jk}$ with $q_{jk} \geq 0$ and $\sum_{k=1}^{K_j} q_{jk} = 1$. For each slider, two (uniform) random numbers are drawn and player $i$’s belief for that slider is “correct” if the reported belief is closer to the actual outcome (0 or 1) than at least one of the random draws. If players are risk-neutral then the elicitation mechanism is incentive compatible when a prize, $P \geq 0$, is paid for all correct beliefs for any randomly selected subset of sliders. If players are not risk neutral then the elicitation mechanism is incentive compatible if a prize is paid when the stated belief is correct for a single randomly selected slider.

**Proof.** Let $u_i(x)$ denote player $i$’s utility of being paid a prize mount $x$ (with $u_i(0) = 0$) and let $p$ and $q$ denote the concatenations of player $i$’s true and reported beliefs respectively. Player $i$ wins a prize $P$ for slider $S_{jk}$ with chance

$$P_{jk} = p_{jk}(1 - (1 - q_{jk})^2) + (1 - p_{jk})(1 - q_{jk}^2)$$

and gets 0 with complementary probability. If all correct beliefs for a random subset $S \subseteq S_i$ of sliders pay a prize $P$ then player $i$’s expected utility of reporting $q$ when her true beliefs are $p$ is

$$U_i(p, q) = \sum_{W \subseteq S} u_i(P|W|) \left( \frac{|S|}{|W|} \right) \prod_{S_{jk} \in W} P_{jk} \prod_{S_{jk} \notin W} (1 - P_{jk})$$

(9)

where $W$ is the subset of selected sliders for which player $i$ wins the prize $P$. If player $i$ is risk neutral, i.e. $u_i(x) = x$, then this reduces to the expected number of wins

$$U_i(p, q) = \frac{P|S|}{|S_i|} \sum_{S_{jk} \in S_i} P_{jk}$$
and optimizing with respect to \( q \) yields

\[
\nabla_q U_i(p, q) = 2 \frac{P|S|}{|S_i|} (p - q)
\]

so truthful reporting is optimal. If player \( i \) is not risk neutral then there can only be two possible payoff outcomes, 0 and \( P \), for the elicitation mechanism to be incentive compatible. In other words, \(|S| = 1\) and (9) reduces to

\[
U_i(p, q) = \frac{u_i(P)}{|S_i|} \sum_{s_{jk} \in S_i} P_{jk}
\]

and truthful reporting is again optimal. ■

After all subjects had submitted their choices and beliefs, they would be shown their opponent’s choice, the points on the belief slider chosen by the computer, and whether their submitted beliefs were close enough to their opponent’s actual play to win the prize. Their payoff in each round was randomly selected to be either their payoff from the game or their payoff from the belief elicitation task. This was done to avoid subjects hedging between the two tasks. At the end of the experiment, subjects were informed of their total earnings and paid.

5.1. Heterogeneous and Incorrect Beliefs

First, consider the \( 2 \times 2 \) asymmetric matching pennies (AMP) games in Table 7, which are all derived from the same parametric form shown in the top-left panel. The use of a common parametric form guarantees that the best response structure is identical across games. Hence, all games in this family have the same unique mixed-strategy Nash equilibrium. Let \( p \) and \( q \) denote the probability with which Column and Row choose \( A \) respectively. The Nash equilibrium for any such game is \((p^*, q^*) = (\frac{1}{2}, \frac{1}{6})\).\(^{15}\) The \( M \)-equilibrium action set shown as the red colored area in the left panel of Figure 8. The red colored area in the right panel shows the \( M \)-equilibrium belief set.

The numbered points in the top-left (top-right) panel of Figure 8 show the average choices (beliefs) in games 1-5. The ellipsis around each point represents the 95% confidence interval for the samples means.

\(^{15}\)The study of such games has been instrumental in the development of alternative models of strategic behavior. See, for instance, Ochs (1995), Erev and Roth (1998), Goeree, Holt, and Palfrey (2003), Selten and Chmura (2008), as well as the comment by Brunner et al. (2010) and the reply by Selten et al. (2010).
Result 1 In the AMP games:

(i) Subjects’ choices differ across games.

(ii) Subjects’ beliefs differ across games.

(iii) Subjects’ choices differ from their beliefs in any of the games.

(iv) Subjects’ choices are heterogeneous.

(v) Subjects’ beliefs are heterogeneous.

Support for these findings can be found in the Appendix. They are obviously at odds with Nash and logit-QRE, which both predict that choices and beliefs are identical and homogeneous across all five games and that beliefs match choices. (In addition, the Nash-equilibrium prediction, \((p^*, q^*) = (\frac{1}{2}, \frac{1}{2})\) is far from observed choice averages.) To allow for heterogeneity within the QRE framework, Rogers, Palfrey and Camerer (2009) propose several generalizations. Heterogeneous Quantal Response Equilibrium (HQRE) allows for heterogeneous choices by assuming that players’ rationality parameters, which determine the sensitivity of their logit quantal responses with respect to expected payoffs, are draws from commonly-known distributions. In HQRE, players’ beliefs are thus assumed to be correct. The HQRE model is at odds with findings (i)-(iii) and (v). The most general QRE model that allows for heterogeneity in
choices and beliefs is Subjective Quantal Response Equilibrium. SQRE assumes that players’
have subjective beliefs about the distributions that others’ rationality parameters are drawn
from. SQRE is therefore not at odds with findings (iii)-(iv), but since it is based on logit
quantal responses, it predicts no change in choices and beliefs across games, which is refuted
by findings (i) and (ii). Similarly, the level-\(k\) and Cognitive Hierarchy (CH) models, which
are based on best responses, yield identical predictions for choices and beliefs across games,
contradicting findings (i) and (ii).

Result 2 The findings in Result 1 contradict the predictions of Nash equilibrium, QRE, HQRE,
SQRE, level-\(k\), and Cognitive Hierarchy, but accord well with \(M\)-equilibrium predictions.

Support: Set-valued \(M\) equilibrium easily accommodates the variations in choices and beliefs
across games as well as the fact that beliefs differ from choices. As can be seen from Figure 8,
average choices and beliefs fall within the \(M\)-equilibrium sets.

\(M\) equilibrium relies on the assumption of stochastic rationality. This posits that subjects
will choose the alternative that is best, given their beliefs, more often. In Table 8 we report the
fraction of best responses given stated beliefs for the five games. As can be seen, these range
between .55 and .75, which is in accordance with stochastic rationality.
| AMP game | 1    | 2    | 3    | 4    | 5    | average |
|----------|------|------|------|------|------|---------|
| Row      | .61  | .56  | .55  | .60  | .58  | .58     |
| Column   | .75  | .73  | .66  | .72  | .75  | .72     |
| average  | .68  | .65  | .61  | .66  | .67  | .65     |

Table 8: Fraction of best responses for each role in each of the five AMP games.

| DS1     | A     | B     | C     |     |     |     |
|---------|-------|-------|-------|-----|-----|-----|
| A       | 80, 80| 30, 160| 20, 10|     |     |     |
| B       | 160, 30| 30, 30| 10, 40|     |     |     |
| C       | 10, 20| 40, 10| 30, 30|     |     |     |

| DS2     | A     | B     | C     |     |     |     |
|---------|-------|-------|-------|-----|-----|-----|
| A       | 75, 75| 5, 155| 190, 5|     |     |     |
| B       | 155, 5| 5, 5 | 180, 15|     |     |     |
| C       | 5, 190| 15, 180| 200, 200|     |     |     |

| NL      | A     | B     | C     |     |     |     |
|---------|-------|-------|-------|-----|-----|-----|
| A       | 70, 70| 60, 500| 10, 50|     |     |     |
| B       | 500, 60| 40, 40| 0, 61 |     |     |     |
| C       | 50, 10| 61, 0 | 30, 30|     |     |     |

| KM      | A     | B     | C     |     |     |     |
|---------|-------|-------|-------|-----|-----|-----|
| A       | 120, 120| 90, 60| 60, 120|     |     |     |
| B       | 60, 90| 90, 90| 60, 90|     |     |     |
| C       | 120, 60| 90, 60| 30, 30|     |     |     |

Table 9: A symmetric 3 × 3 game with a unique symmetric mixed-strategy Nash equilibrium (left panel), a unique symmetric pure-strategy equilibrium (middle panel), and the maximum number of symmetric equilibria (right panel).

### 5.2. $M$ Equilibrium Multiplicity

Next consider the two symmetric 3 × 3 games in the first row of Table 9. As in the AMP experiments, these two games share the same underlying parametric structure. As a result, both have the same best-response structure and a unique Nash equilibrium in pure strategies: \{C, C\}. In fact, note that these games are dominance solvable. One might reasonably expect Nash to predict well and that behavior would be homogeneous across the two games.

Subjects’ behavior does not support these predictions. The top-left (top-right) panel of Figure 9 shows average choices (beliefs) in the two experiments. Like in the 2 × 2 AMP experiments, observed choices and beliefs are far from Nash and from each other. Our theory and, in particular, the multiplicity of $M$ equilibria for these games allows for a better understanding of these results.

The $M$-equilibrium choice and belief sets for these games are shown in Figure 9. Notice

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In particular, one game can be obtained by the other by adding a constant to each column (row) of Row’s (Column’s) payoffs.
that there are four $M$ equilibrium sets. The Nash equilibrium is part of the yellow set, which is supported by a relatively small $M$-equilibrium belief set. The blue choice set is the furthest from Nash but is supported by the largest belief set. Multiplicity introduces the issue of strategic coordination. Two players may be both playing $M$-equilibrium strategies supported by the corresponding $M$ equilibrium beliefs, but the two could be different. In DS1, subjects seem to coordinate on the blue equilibrium, as both average choices and beliefs fall within the corresponding $M$ equilibrium sets. This is not the case for DS2, where average choices fall outside of any $M$-equilibrium choice set.

Our explanation is that even though the games share the same best response structure, the differences in the payoff matrices induce different beliefs in the two games. In DS2, predicting the opponent’s play is harder, leading to a higher degree of strategic mis-coordination. To see why, notice that in DS1, options $A$ and $B$ offer an attractive “upside” and the same “downside” as $C$. This makes the upper left sub-matrix focal. In DS2, the high payoff at $\{C, C\}$ makes the Nash equilibrium more salient but the upper-left sub-matrix also remains focal because it provides safer options.

To examine these assertions, we provide a more detailed analysis of the beliefs and their nexus to choices in the experiment. We separate elicited beliefs in each of the two games in different clusters using the $k$-means algorithm (MacQueen, 1967). We then take the average of the choices corresponding to beliefs in each cluster. The middle and bottom panels of Figure 9 show the results of this exercise.

As one can see from the middle panels of Figure 9, in DS1 elicited beliefs are mainly concentrated in the lower side of the blue belief set. The corresponding average choices are also in the blue choice set, indicating that subjects in this game are mostly playing the blue $M$ equilibrium. Interestingly, in the two small clusters were beliefs are outside of the blue set, the corresponding average choices are very close to the set of the same color, indicating that in some instances some subjects did play a different $M$ equilibrium.

The bottom panels of Figure 9 correspond to DS2. As we expected, elicited beliefs here are more spread out, with a substantial number lying outside of the blue set. Nevertheless, except for two clusters (depicted by the cyan and magenta colored circles) in all other cases average choices lie within or very close to the set of the same color of the corresponding beliefs. In contrast to DS1, the blue set contains the average choices of only two of these clusters, while one of the clusters is essentially playing the $M$ equilibrium that includes Nash.
Figure 9: The colored sets indicate the four $M$ equilibria for the DS1 and DS2 games and the black circle at the top indicates the unique Nash equilibrium. In the top panels, the colored circles indicate the average choices (left) and beliefs (right) for DS1 (red) and DS2 (green). The middle (DS1) and bottom (DS2) panels show a $k$-means analysis for the DS games. The $k$-means algorithm was performed on the elicited belief data. Each colored circle corresponds to the average choices (left) and elicited beliefs (right) within each cluster. The size of the circles is proportional to the number of observations belonging to the particular cluster.
5.3. No Logit

We obtain similar results for the symmetric $3 \times 3$ game in the lower left of Table 9, labeled “NL” for “no Logit.” For this game $\{C, C\}$ is also the unique Nash equilibrium and there are again four $M$ equilibria. These are depicted in the graphs of Figure 10.

The two with the largest belief sets also have the largest choice sets and are therefore expected to be empirically most relevant. An interesting feature in this game is that the profiles in the blue equilibrium cannot be part of any logit-QRE (see also Example 7 above). Nevertheless, as can be seen from the experimental results depicted in Figure 10, both the blue and the cyan $M$ equilibria are empirically relevant. A $k$-means clustering analysis again shows strong evidence of strategic mis-coordination and reveals that the majority of choices (62%) are in the “no Logit” blue region.

As pointed out above, the issue of equilibrium-multiplicity only arises in the context of $M$ equilibrium – the “no Logit” game has a unique Nash equilibrium as well as a unique QRE. The strategic coordination problems uncovered by the choice and belief data cannot be explained through the lens of existing behavioral game-theory models.

5.4. Stability

Next we take up the issue of stability. Definitions of stability abound, but generally the concept tries to capture the robustness of some equilibria to perturbations of game form perception (“strategic stability”). It follows that that stable equilibria are the empirically valid ones. As mentioned before, profiles in the interior of $M$ equilibrium sets are behaviorally stable, which is a stronger notion than strategic stability stability.

When multiple $M$ equilibria exists, as is the case for the two DS games, stability cannot help predict the empirical relevance of any of them. Here instead, we look at a game with multiple Nash equilibria (bottom left of Table 9), identified by McLennan (2016) as one where “observed behavior will not be characterized by repetition of any one of the equilibria.” This prediction follows the application of his index $+1$ principle, introduced in that paper, and because the six pure strategy Nash equilibria can be arranged in a circle, where mixtures of any adjacent pure equilibria are mixed equilibria.\(^{17}\) The KM game is originally introduced in Kohlberg and Mertens (1986). In stark contrast to this prediction of instability we find the existence of a

\(^{17}\)As McLennan (2016) points out, a more precise (albeit less catchy) version of the “index $+1$ principle” is the “Euler characteristic equals index principle.” Demichelis and Ritzberger (2003) show that the latter condition is necessary for any “natural” dynamic process of adjustment to converge. For the KM game, the Euler characteristic of the set of Nash equilibria is zero, while its index is $+1$.\]
Figure 10: Behavior in the NL game. The colored sets correspond to the four $M$ equilibrium sets for this game. The two graphs on the top show average choices (left) and elicited beliefs (right). The two bottom graphs show the data organized in clusters obtained using the $k$-means algorithm on the elicited beliefs data. Each colored circle corresponds to the average choices (left) and elicited beliefs (right) within each cluster. The size of the circles is proportional to the number of observations belonging to the particular cluster.
unique symmetric and behaviorally stable $M$ equilibrium.

Figure 11 shows the $M$ equilibrium for the KM game. The data from the experiment is shown here already broken down in clusters, using the same procedure as before. It is clear from the picture that the unique symmetric $M$ equilibrium captures almost the entirety of subjects’ behavior, both in terms of choices and beliefs. This provides further support for $M$ equilibrium, and behavioral stability, as an empirically relevant theory.

6. Summary

This paper introduces a novel set-valued solution concept, $M$ equilibrium, which is based on an ordinal monotonicity condition – players’ choice probabilities are ranked the same as their associated expected payoffs – and an ordinal consistency condition – players’ beliefs result in the same ranking of expected payoffs as their choices.

The first condition, also known as stochastic rationality, captures the idea that unbiased mistakes dilute choice probability away from better to worse options but not to the extent that they overturn their ranking. The rationale behind the second condition is that there is no reason to improve beliefs when doing so leaves choices unaffected. For instance, in the dominance-solvable game of the Introduction (see Table 2), all beliefs result in the same ranking of choices.
and there is no need to correctly anticipate the other’s behavior. The ordinal consistency condition underlying $M$ equilibrium captures this “no-need-for-change” intuition.

We prove existence of a finite number of $M$ equilibria, each with a finite number of components, for any normal-form game (Propositions 1 and 2). We show that Nash equilibria are typically not examples of $M$ equilibria. Rather, Nash equilibria arise as “abstract limit points of the $M$-equilibrium planes,” to paraphrase Mondrian. There may be an even or odd number of $M$ equilibria, fewer or more $M$ equilibria than Nash equilibria, and an $M$ equilibrium may contain any number of Nash equilibria. Importantly, the measure of any $M$-equilibrium choice set falls quickly with the number of players and the number of possible choices (Proposition 4). We show that, generically, $M$ equilibria can be “color coded” by a single rank vector in the sense that choices and supporting beliefs are of the same color. We introduce the concept of behavioral stability, which strengthens Kohlberg and Merten’s (1986) strategic stability, and show that colorable $M$ equilibria are the behaviorally stable ones (Proposition 5).

We introduce a new class of parametric $\mu$-equilibrium models, which replace players’ best-response correspondences with rank-based correspondences. We prove existence of $\mu$ equilibrium for any normal-form game (Proposition 7) and show that, unlike QRE, the $\mu$-equilibrium choices can be easily and analytically determined. Importantly, $\mu$-equilibrium choices are supported by a set of beliefs. As a result, $\mu$-equilibrium beliefs may be heterogeneous and incorrect. In other words, $\mu$-equilibrium allows for “mistakes” as well as “surprises,” again unlike QRE, which assumes homogeneous and correct beliefs.

A common criticism of parametric models like $\mu$ equilibrium and QRE is that they introducing “degrees of freedom” into the theory. QRE, for instance, requires the specification of players’ quantal responses, which are elements of an infinite-dimensional space. This begs the question whether QRE can be falsified. Haile, Hortacsu, and Kosenok (2008) suggest the answer is “no,” although their proof uses quantal responses that violate stochastic monotonicity. In contrast, most of the QRE literature considers regular quantal responses that obey stochastic monotonicity, see e.g. Goeree, Holt, and Palfrey (2016). Nonetheless, also the set of regular quantal responses is infinite dimensional, so the question remains whether QRE is falsifiable.

We answer this question through the lens of $M$ equilibrium. In particular, we show that $M$ equilibrium, a parameter-free and set-valued solution concept, is a meta theory that minimally envelopes various parametric fixed-point models, including $\mu$-equilibrium and QRE (see Proposition 9 and subsequent discussion) and other models obeying stochastic rationality. Proposition 4 shows that the measure of any $M$-equilibrium choice set falls factorially fast with the number
of players and possible choices, confirming that the parametric models can (easily) be falsified.  

We test $M$ equilibrium in a series of experiments. Five versions of an asymmetric matching-pennies game, which differ only by additive constants to the payoffs, show significantly different choices and beliefs. This finding is problematic for all existing behavioral game-theory models (level-$k$, SQRE and its HQRE/TQRE/QRE descendants, Cognitive Hierarchy), but is easily accommodated by set-valued $M$ equilibrium.

Further experiments based on dominance-solvable $3 \times 3$ games with a unique Nash equilibrium and four $M$ equilibria raise the possibility of mis-coordination and highlight the role of beliefs. In one version of a dominance-solvable game, elicited beliefs predict that choices are mainly in the $M$-equilibrium set furthest away from the unique Nash equilibrium – and they are. In an “equivalent” version, beliefs predict that choices are scattered over the four $M$ equilibria – and they are. See Figure [9]. In another experiment, beliefs predict that choices are mostly in the region that logit-QRE cannot attain – and they are. See Figure [10]. A final experiment, predicted to result in unstable actions and beliefs, shows the empirical relevance of behavioral stability. The KM game has a unique $M$ equilibrium to which both choice and belief data conform. See Figure [11]. Combined these findings confirm the potential of $M$ equilibrium as an empirically relevant game theory.

7. Outlook

Nasar’s (1998) book “A Beautiful Mind” details how Nash was disappointed by the lack of empirical support for his solution concept, which led him to return to doing research in pure mathematics. Interestingly, some of the machinery Nash developed underlies the alternative approach pursued in this paper. For example, an $M$ equilibrium is a semi-algebraic set with finitely many components of different dimensions (see Section 3.1, in particular, Figure 3), which are now known as “Nash cells” (e.g. Coste, 2005).

In retrospect, von Neumann’s reaction when Nash introduced him to his solution concept – “that’s just a fixed-point” – may have foreshadowed its empirical weakness. The Nash equilibrium predicts certain choice profiles without detailing how they come about – they are simply solutions to some fixed-point equations. Quantal Response Equilibrium (QRE), developed almost

\[18\text{Furthermore, Example 7 shows there exist } M\text{-equilibrium sets that contain no logit-QRE at all. The “no Logit” experiment was inspired by this example. We find that the majority of the choice data (62\%) is in the “no Logit” region.}\]
half a century later (McKelvey and Palfrey, 1995), takes the role of fixed-points even further by requiring that both choices and beliefs derive from them. Even in a simple dominant-strategy solvable game, the resulting non-transcendental equations can only be solved numerically, when Nash-equilibrium choices are supported irrespective of players’ beliefs.19 As the experimental results reported in this paper highlight, and as von Neumann intuited, the assumption of homogeneous and correct beliefs that drives fixed-points theories like Nash and QRE, is untenable. In none of the experiments reported in this paper are beliefs homogeneous or correct.

Yet choices and beliefs are more likely to be “right” than “wrong.” This is the simple premise underlying the $M$-equilibrium: players’ choice probabilities are ranked the same as the expected payoffs based on their beliefs, and players’ beliefs yield the same ranking of expected payoffs as their choices. The mathematical consequences of this simple premise are governed by semi-algebraic geometry, a field in mathematics that Nash made distinguished contributions to. Importantly, the empirical consequences of this simple premise are corroborated by belief and choice data from several of our experiments.

Of course, $M$ equilibrium will not be universally correct. There are well-documented cases where behavioral factors (e.g. other-regarding preferences, risk aversion, etc.) play an important role. These factors can be incorporated in an extension of the theory by replacing expected payoffs with expected utilities. But even when accounting for behavioral elements, $M$ equilibrium is unlikely to always be correct. Given the minimal assumptions that $M$ equilibrium imposes, the reason it fails then offers important insights about behavior. Is stochastic rationality violated or is it that beliefs do not satisfy a minimal consistency condition? Whether correct or not, $M$ equilibrium offers a novel and promising approach toward an empirically relevant game theory.

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19In Crawford’s (2018) words “QRE is a fixed point in a high-dimensional space of distributions, making its thinking justification cognitively far more demanding than for Nash equilibrium.”
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A. Support for Result 1

Support: (i) As can be seen in the top left graph of figure 8, there is very little overlap between the confidence intervals for the average choice frequencies in the five games. Formal statistical testing confirms this. The left panel of table 10 reports the p-values from a Hotteling’s t-test in pairwise comparisons of the average choice frequency in different games. In most cases, pairwise comparisons show differences that are significant at the 5% level. The null hypothesis that average choice frequencies in all games are jointly equal can be rejected at the 1% level. This remains true even when excluding game 4 from the test, as it is the single most different game.

(ii) Again, one can see in the top right graph of figure 8 that there is little overlap of the confidence intervals for average beliefs for the five games. The results of Hotteling’s t-tests for the comparison of means confirms what is observed in the graph. The right panel of table 10 reports the p-values from the test. Average beliefs are statistically different across games for any reasonable significance level.

(iii) A careful inspection of the two top graphs in figure 8 reveals that for all five games, average choice frequencies lie far from the corresponding average reported beliefs. Again, using Hotelling’s t-test we confirm this finding. The p-values from comparing actions to beliefs in each of the games are very close to zero in all five cases.

(iv) Figure 12 shows the empirical cumulative density functions (cdf’s) for the average choice by Row and Column in each game. If actions were homogeneous, then these should look like the cdf for the observed success rate in a binomial distribution with 8 trials and a true success rate in each trial equal to the average choice across all individuals. We use a Monte Carlo simulation with 1000 repetitions to produce this cdf and compare the two. Using a Kolmogorov-Smirnov test we can reject that the two are the same for all cases at the 1% level for all games except game 2. For game 2 we can reject at the 5% level for Row (p-val = .031) but not for Column (p-val = .117). This result is further supported by a Cochran’s Q test comparing individual choices in each game to one another. The test rejects those being the same for both Row and Column in all games for any level of significance.

(v) We use Friedman’s test to compare reported beliefs across individuals in each game,

\footnote{For the joint test we use the Bonferroni correction. The null of simultaneous equality is rejected at a level of significance \( \alpha \) by comparing the lowest p-value in all \( m \) pairwise comparisons to \( \frac{\alpha}{m} \) and rejecting whenever it is lower. This ensures that the family-wise error rate (FWER), which is the probability that at least one of the averages is not equal to the others, remains at the desired level \( \alpha \). This is the most conservative test based on the FWER, in the sense that it rejects the null less often.}
Table 10: Experiment 1: p-values from pairwise comparisons between games.

separately for Row and Column. In all cases we find there is heterogeneity that is significant at any reasonable significance level.

Figure 12: Heterogeneity of actions in Experiment 1. Empirical cdf’s for subjects’ actions (blue) and cdf’s of the observed success rate in MC simulations of binomial distributions with 8 trials and true success rate equal to subjects’ overall average choice (orange).

B. Cluster analysis with the k-means algorithm.

As mentioned in the main text, we classify our data of elicited beliefs into different clusters. This is done in a theory-free way using the k-means algorithm (MacQueen, 1967). The way it
works is as follows. Given \( k \), the number of clusters one wants to use, \( k \) random ‘centroids’ are chosen, which are points in the same space as that of the data. In the next step, each observation is matched to its closest centroid. To determine closeness we use the euclidean distance that is the standard option in this class of problems. Next, new centroids are calculated by taking the mean of the observations in each cluster. The algorithm proceeds iteratively until it converges to a stable set of clusters.

While convergence is guaranteed, the resulting clusters will depend on the random initialisation of the algorithm. To address the issue we take the standard approach which is to repeat the analysis for a large number of times (5000 in our case) and choose the outcome with the smallest sum of errors (distance of each observation from its cluster centroid).

The final issue to address is the number of clusters, \( k \). We approach the problem using what is dubbed the ‘elbow method’. We run the analysis for \( k \in \{2, \ldots, 15\} \) and calculate the sum of errors in each case. This is then plotted against \( k \), giving a convex, decreasing curve. For each game we choose the \( k \) that is at the ‘elbow’ of the curve. Figure 13 shows the curves for the case of the two DS games. The In the analysis in the main text we use \( k = 7 \) for these games.

![Figure 13: K means in the DS games. The curves show the sum of distances for different values of \( k \) for DS1 (blue line) and DS2 (red line).](image)