Exploring Defective Eigenvalue Problems with the Method of Lifting

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Abstract

Consider an $N \times N$ matrix $A$ for which zero is a defective eigenvalue. In this case, the algebraic multiplicity of the zero eigenvalue is greater than the geometric multiplicity. We show how an inflated $(N+1) \times (N+1)$ matrix $L$ can be constructed as a rank one perturbation to $A$, such that $L$ is singular but no longer defective, and the nullvectors of $L$ can be easily related to the nullvectors of $A$. The motivation for this construction comes from linear stability analysis of an experimental reaction-diffusion system which exhibits the Turing instability. The utility of this scheme is accurate numerical computation of nullvector(s) corresponding to a defective zero eigenvalue. We show that numerical computations on $L$ yield more accurate eigenvectors than direct computation on $A$.

Key words: Turing patterns, defective eigenvalue problems

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1 Introduction

In this paper, we present a method for improving the accuracy of computed eigenvalues and eigenvectors of defective matrices [1]. This algorithm, which

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we refer to as lifting, involves construction of a nondefective matrix in a higher
dimensional space. Standard linear algebra procedures are shown to yield more
accurate numerical results for the lifted matrix. Our scheme formally resembles
the construction of an augmented system in pseudo-arclength continuation [2,3,4,5].
Its formulation was motivated by the structure of the linear stability analysis of a particular chemical reaction-diffusion system, known as
the chlorine-dioxide–iodine–malonic-acid system [6,7,8,9] (henceforth referred
to as CDIMA), which exhibits the Turing instability.

First, we present the Lifting Theorem. We demonstrate its numerical implementation on “small” and “large” defective eigenproblems. We describe the
connection with the CDIMA linear stability problem, and discuss the general applicability of this scheme.

2 Lifting

An eigenvalue is defective if its algebraic multiplicity is greater than its geometric multiplicity. (The geometric multiplicity of an eigenvalue equals the
dimension of the space spanned by the eigenvectors corresponding to that
eigenvalue.) Consider an eigenvalue with geometric multiplicity equal to one: It is nondefective if the corresponding left and right eigenvectors are nonorthog-
onal. For a defective eigenvalue, lifting corresponds to a geometric procedure
of casting the corresponding eigenvectors of the matrix and the adjoint matrix
into a higher dimensional space in which these vectors are no longer orthogo-
nal. In this section, we show how to construct such a well-conditioned matrix
in the higher dimensional space from a defective matrix.

For simplicity, we define the notation:

\[
\text{algebraic multiplicity of eigenvalue } \mu \text{ of matrix } M \equiv AM(M; \mu)
\]
\[
\text{geometric multiplicity of eigenvalue } \mu \text{ of matrix } M \equiv GM(M; \mu).
\]

Although in the following we consider zero eigenvalues and corresponding nullvectors, the discussion holds generally for nonzero eigenvalues, with \( A = M - \mu I_N \), where \( I_N \) is the appropriate \( N \times N \) identity matrix.

**Theorem:** Consider an \( N \times N \) matrix \( A \), such that zero is a defective eigenvalue of \( A \) with \( AM(A; 0) = n \) and \( GM(A; 0) = 1 \), \( n > 1 \). The right and left unit nullvectors of \( A \) are given by \( \phi \) and \( \psi^T \), respectively. An \( (N+1) \times (N+1) \) matrix
matrix $\mathcal{L}$ constructed according to

$$
\mathcal{L} = \begin{pmatrix}
A & 0 \\
0^T & 0
\end{pmatrix} + vw^T.
$$

(1)

where

$$
v = \begin{pmatrix}
v \\
\eta
\end{pmatrix}, \quad w = \begin{pmatrix}
w \\
\omega
\end{pmatrix},
$$

(2)

will have a simple zero eigenvalue if

(i) $w^\top \phi \neq 0,
(ii) \psi^\top v \neq 0,
(iii) \eta \omega \neq 0.

(3)(4)(5)

$v$ and $w$ are $N$-dimensional vectors, and $\eta$ and $\omega$ are scalars.

**Proof:** First, we show there exists a nontrivial vector $\Phi$, given by

$$
\Phi = \begin{pmatrix}
x \\
\xi
\end{pmatrix},
$$

(6)

such that

$$
\mathcal{L} \Phi = 0.
$$

(7)

For $\Phi$ to be a right nullvector of $\mathcal{L}$, the $N$ components of $(\mathcal{L} \Phi)$ in the original space, as well as the $(N + 1)^{th}$ component in the inflated space must be zero:

$$
A x + \alpha v = 0
$$

$\alpha \eta = 0,

(8)(9)

where

$$
\alpha = w^\top x + \omega \xi.
$$

(10)
Since \( \eta \neq 0 \), for Eq. 9 to be satisfied, we must have \( \alpha = 0 \). Then, Eq. 8 implies \( x = r \phi \), where \( r \) is a constant, and \( \xi \) must be given by

\[
\xi = -r \frac{w^T \phi}{\omega}.
\tag{11}
\]

Hence, the pointwise projection of the right nullvector of \( L \) onto the original \( N \)-dimensional space is the right nullvector of \( A \), up to a multiplicative constant. Additionally, its \((N + 1)^{th}\) component, \( \xi \), is nonzero.

Likewise, we can show that \( L \) has a nontrivial left nullvector \( \Psi^T \) given by

\[
\Psi = \begin{pmatrix} y \\ \zeta \end{pmatrix},
\tag{12}
\]

where

\[
y = s \psi, \quad \zeta = -s \frac{\psi^T v}{\eta}.
\tag{13}
\]

Again, \( s \) is a constant, and we have \( \zeta \neq 0 \).

Next, we must show that zero is a simple eigenvalue of \( L \). This will be done in two steps:

\[
(i) \quad GM(L; 0) = 1, \\
(ii) \quad \Psi^T \Phi = \psi^T \phi + \zeta \xi \neq 0.
\tag{14}
\]

First, we prove \( GM(L; 0) = 1 \) by contradiction. Assume \( GM(L; 0) > 1 \). Then, there exists \( \Phi' \neq \Phi \) such that

\[
L \Phi' = 0, \quad \Phi' = \begin{pmatrix} x' \\ \xi' \end{pmatrix}.
\tag{15}
\]

Eqs. 8 and 9 imply \( A x' = 0 \). We distinguish two cases:

a. If \( x' \neq r \phi \), then \( x' \) is a right nullvector of \( A \), different from \( r \phi \). This violates the assumption \( GM(A; 0)=1 \).

b. If \( x' = r \phi \), we must have \( \xi' \neq \xi \) for \( \Phi' \neq \Phi \). However, from Eqs. 9 and 10, \( \xi' = -r w^T \phi / \omega = \xi \). Therefore, \( \Phi' \) is not different from \( \Phi \).

\[\text{4}\] If \( \Phi \) is normalized to give the unit right nullvector, then \( r^2 = (1 + w^T \phi / \omega)^{-1} \).
Hence, we have shown that \( GM(L; 0) = GM(A; 0) = 1 \). (This proof is easily generalized to \( GM(L; 0) = GM(A; 0) = \nu \).) To show (ii), we note that zero being a defective eigenvalue of \( A \) with \( GM(A; 0) = 1 \) implies \( \psi^T \phi = 0 \). However, the inner product given by Eq. 14 will still be nonzero, since \( \zeta \xi \neq 0 \). This completes the proof that zero is a simple eigenvalue of \( L \).

It is important to consider the generic case in numerical applications, where \( A \) is not exactly defective but almost-defective, \( \psi^T \phi \approx 0 \). For zero to be a nondefective eigenvalue of \( L \), the inner product given by Eq. 14 must be nonzero:

\[
\psi^T \phi + \left( \frac{\psi^T v}{\eta} \right) \left( \frac{w^T \phi}{\omega} \right) \neq 0.
\] (16)

For example, constructing \( L \) with \( v \) and \( w \) such that \( v = \phi \), \( \eta = -1 \), and \( \omega = w^T \phi \), would violate the condition above. Therefore, when \( A \) is almost-defective, Eq. 16 must be additionally satisfied to guarantee that \( L \) will not be defective. Below, we give a general prescription for choosing the lifting vectors, which unlike the pathological special case just presented, should avoid construction of a defective lifted matrix.

Consider choosing (for simplicity) \( \eta = \omega = 1 \), and \((v, w) = (\beta v_r, \gamma w_r)\):

\[
v_{r_i} = \text{rand}(i)/N_v, \quad w_{r_j} = \text{rand}(j)/N_w,
\] (17)

where \text{rand}(i) are random numbers on \([-1, 1]\), \(N_v\) and \(N_w\) are normalizing factors leading to unit random vectors, and \( \gamma \) and \( \beta \) are constants. This construction should guarantee that when \( A \) is singular and exactly defective (with \( AM(A; 0) = n, GM(A; 0) = 1 \)), or close to defective, then zero will be a simple eigenvalue of \( L \). We will refer to the constants \( \gamma \) and \( \beta \) as lifting parameters, since the \((N + 1)^{th}\) components of the left/right nullvectors of \( L \) are proportional to these constants, respectively, and measure the projections of these vectors onto the new, inflated direction.

To extend the above theorem and proof to the case where \( A \) is defective and \( AM(A; 0) > GM(A; 0) > 1 \), we must show that \( L \) can be constructed such that zero is a nondefective, multiple eigenvalue with \( AM(L; 0) = GM(L; 0) = GM(A; 0) \). With multiple zero eigenvalues, \( L \) will be defective if there exists a vector \( g \) such that

\[
Lg = \Phi_i,
\] (18)

for some \( \Phi_i \in \mathcal{N}(L) \), where \( i = 1 \ldots \nu \), and \( \mathcal{N}(L) \) is the nullspace of \( L \). (If such a vector \( g \) existed, then it would be a generalized eigenvector of \( L \).)
associated with an \( m \times m \) Jordan block, where \( m > 1 \). Taking the inner product of Eq. 18 with \( \Psi_j^T \), where \( \Psi_j^T \in \mathcal{N}(L^T) \), \( j = 1 \ldots \nu \), gives

\[
\Psi_j^T \Phi_i = 0. \tag{19}
\]

Therefore, for \( L \) to be nondefective, it must be constructed such that no vector in the right nullspace of \( L \) is orthogonal to its entire left nullspace.

### 3 Numerical Results

#### 3.1 2 × 2 Defective Eigenproblem

Consider the following matrix, \( M \):

\[
M = \begin{pmatrix}
\pi & 1 \\
-\pi^2/4 & \epsilon
\end{pmatrix}. \tag{20}
\]

The eigenvalues of \( M \) are given by

\[
\mu_\pm = \left[ \pi + \epsilon \pm \sqrt{(\epsilon^2 - 2\pi \epsilon)} \right] / 2. \tag{21}
\]

The two components of an eigenvector \( \phi^T = (\phi_1, \phi_2) \) satisfy

\[
\phi_2/\phi_1 = \mu - \pi. \tag{22}
\]

where \( \mu = \mu_\pm \). For \( \epsilon = 0 \), \( M \) is defective. We use the lifting algorithm to explore the error in computing the eigenvector associated with \( \mu_+ \) when \( M \) is close to being defective, \( \epsilon \ll 1 \). The lifted matrix \( L \) is constructed with \( A = M - \mu_+ I_2 \) according to

\[
\begin{align*}
v &= \begin{pmatrix} \beta v_r \\ 1 \end{pmatrix}, \\
w &= \begin{pmatrix} \beta w_r \\ 1 \end{pmatrix}
\end{align*} \tag{23}
\]

where \((v_r, w_r)\) are two-dimensional random unit vectors, and we set \( \gamma = \beta \) without loss of generality.
The right unit nullvector $\Phi$ of $L$ is computed using MATLAB’s `eig()` function [10,11]. We explore the lifting error, constructed according to

$$E(\epsilon, \beta) = |\Phi_2/\Phi_1 - (\mu - \pi)|.$$  \hspace{1cm} (24)

as a function of parameters $\epsilon$ and $\beta$.

Figure 1 shows $E(\epsilon, \beta)$ as a function of the lifting parameter $\beta$ for different values of $\epsilon$. The basic trend is that of decreasing error as a function of increasing $\beta$ for small to moderate values of $\beta$. In this figure, each point corresponds to the mean error computed using 1000 unit random lifting vectors, $(v_r, w_r)$. The root-mean-square (rms) of the distribution of errors is of the order of the mean error for small values of the lifting parameter. For $\beta \gtrsim O(1)$, the rms is at most an order of magnitude greater than the mean error for values of $\epsilon$ close to machine precision; for larger values of $\epsilon$, the rms is equal to a few times the mean error. We have verified that the larger values of rms are a result of a few outliers. In Figure 2, for lifting parameter $\beta = 1.0$, we compare the mean error with lifting (open circles) and without lifting (open triangles) as a function of $\epsilon$. In this figure, we also show the error using lifting vectors given by $(v, w) = (\psi, \phi)$, such that the nonorthogonality conditions Eqs. 3 and 4 are satisfied by definition (open squares). We note that the lifting error is within an order of magnitude of machine precision for all values of $\epsilon$.

The condition number of the simple singular value of the lifted matrix is given by the reciprocal of

$$s(0) = |\Psi^\dagger \Phi|,$$  \hspace{1cm} (25)

where $\Psi$ and $\Phi$ are the left and right unit nullvectors of $L$. In Figure 3, we show the condition number as a function of the lifting parameter $\beta$ for different values of $\epsilon$, using a single pair of unit random lifting vectors: For $\epsilon \ll 1$, the condition number clearly improves with increasing lifting parameter.

3.2 $N \times N$ Defective Eigenproblem

For applications in which the matrix is a discrete approximation to a continuous operator, $A$ will have dimension $N \gg 1$. To explore the method of lifting in computing the defective eigenvector of larger matrices, we constructed such an example from the $2 \times 2$ matrix above: with $M$ in the top left block, along with the $(N - 2) \times (N - 2)$ block tridiagonal matrix from Poisson’s equation in the bottom right, we applied a similarity transformation to the resulting $N \times N$ matrix to obtain a general matrix with no special properties (such
as symmetry or bandedness). The eigenvalue, $\mu_+$, of this matrix is exactly known, and the lifted matrix $\mathbf{L}$ is constructed as in the $2 \times 2$ case.

Once the right nullvector of $\mathbf{L}$, given by $\Phi^T = (x, \xi)$ is numerically computed, the lifting error is constructed as before:

$$
\mathcal{E}(\epsilon, \beta) = \left| \left( Sx \right)_2 / \left( Sx \right)_1 - (\mu_+ - \pi) \right|,
$$

(26)

(where $S$ is the similarity transformation). In Figure 4, we show the mean error and its rms computed using 50 unit random lifting vectors, $(v_r, w_r)$. We find that for small values of $\beta$, the lifting error is dominated by the error in computing the “zero” eigenvalue, $\lambda_0$, of $\mathbf{L}$. For $\beta \gtrsim O(1)$, the magnitude of this eigenvalue remains at machine precision, while the lifting error is larger. For values of $\beta$ such that the magnitude of the “zero” eigenvalue is at machine precision, the lifting error is dominated by the error in computing the associated eigenvector. We note that there is an “optimal” value of $\beta$ giving the smallest lifting error, which is of order unity. For values of $\beta$ much larger than unity, the error increases as a function of increasing $\beta$.

4 CDIMA Eigenproblem

The symmetry-breaking instability of a system from a homogeneous steady state to a patterned state, predicted in 1952 by Alan Turing [12], was observed for the first time in the chlorite-iodide-malonic acid $\text{ClO}_2^-\text{I}^-\text{MA}$ (CIMA) reaction-diffusion system [13,14]. In practice, due to boundary chemical feeds, the steady state is not homogeneous, but rather depends on the single space coordinate along the feed gradient. A realistic model of the simpler chlorine dioxide-iodine-malonic acid $\text{ClO}_2\text{I}_2\text{MA}$ (CDIMA) reaction, which is similar to CIMA in terms of its stationary pattern-forming and dynamical behavior, was put forth by Lengyel, Rabai, and Epstein (henceforth referred to as LRE) [15,16]. In particular, they demonstrated that the fortuitous choice of starch as the color indicator for visualizing the patterns in this system provides the necessary rescaling of the diffusion coefficient of the activator (iodide) relative to the inhibitor (chlorite) species required for this instability to occur. Starch is a large molecule that binds to the gel matrix of the reactor and is thus effectively immobile. The reversible binding of iodide with starch results in the purple starch-triiodide complex and an effective slowing down of the diffusion rate of iodide.

We begin with the general formulation of an $(N + 1)$-component reaction-diffusion system with $N$ mobile and a single immobile species (as in the
CDIMA system), following [7]:
\[
\frac{\partial C}{\partial t} = D \cdot \nabla^2 C + \mathcal{F}(U) + \zeta g.
\] (27)

In the evolution equation above,
\[
\begin{align*}
C &= \begin{pmatrix} U \\ W \end{pmatrix}, &
D &= \begin{pmatrix} D & 0 \\ 0^\top & 0 \end{pmatrix}, &
\mathcal{F} &= \begin{pmatrix} f \\ 0 \end{pmatrix}, &
\zeta &= \begin{pmatrix} R \\ 1 \end{pmatrix},
\end{align*}
\] (28)

\(U\) represents the concentrations of mobile species and \(W\) that of the immobile species, \(f = f(U; \alpha)\) describes the reactions among the mobile species with control parameters \(\alpha\), and \(g = g(U, W; \beta)\) describes the reaction between the mobile species and the immobile species with control parameters \(\beta\). \(D\) is the diffusion matrix for the mobile subsystem, and \(R\) is a constant stiochiometry vector. We refer the interested reader to previous works for the description of the LRE model of the CDIMA system.

The steady state of this system in the presence of externally imposed boundary feed gradients satisfies:
\[
\begin{align*}
&f(U_s; \alpha) + D \cdot \frac{\partial^2 U_s}{\partial z^2} = 0, \\
g(U_s, W_s; \beta) = 0
\end{align*}
\] (29) (30)

with boundary conditions \(C_s(0) = C_0\) and \(C_s(\ell) = C_\ell\). We will focus on the physically relevant special case of scalar diffusion, \(D = D I\), where \(I\) is the identity matrix, as the diffusion constants of ions in solution are comparable. It is clear from the formulation given by Eq. 27, that the steady state of the mobile subsystem is independent of the immobile reactions. However, as illustrated in previous works [6,7,8,9], the immobile subsystem affects the stability of the mobile system. The stability of the steady state to a mode with transverse wavenumber \(k\) (\(\hat{k} \perp \hat{z}\)) leads to the following linear operator:
\[
\mathcal{L}_k = \begin{pmatrix} df & 0 \\ 0^\top & 0 \end{pmatrix} - D \left( k^2 + \partial^2_z \right) \begin{pmatrix} I & 0 \\ 0^\top & 0 \end{pmatrix} + \zeta dg^\top,
\] (31)

where \(df = f_U\) is the Jacobian derivative of \(f\) with respect to \(U\), and \(dg^\top = (g_U, g_W)\) is the Jacobian derivative of \(g\) with respect to \((U, W)\), evaluated at the steady state. The eigenvalue problem determining the stability of \(C_s(z)\) to the mode with wavenumber \(k\) is given by
\[
\mathcal{L}_k \cdot \Psi_k = \lambda_k \Psi_k
\] (32)
where $\Psi = (\Psi^M, \Psi^I)$. The space $S$ on which the linear operator acts can be written as $S = S_M \oplus S_I$, where $S_M$ and $S_I$ are mobile and immobile subspaces, respectively. In gradient systems, the basis set spanning $S_M$ is given by the Cartesian product of $N$ orthogonal unit vectors $\{\hat{e}_i\}$, $i = 1, \ldots, N$, corresponding to the $N$ mobile species, and $N_z$ basis functions, defined on the interval in the $z$-direction, for example given by $\{\sin m\pi z/L_z\}$ for Dirichlet boundary conditions, where $m = 1, \ldots, N_z$. Similarly, $S_I$ would be spanned by $\hat{e}_{N+1} \otimes \{\sin m\pi z/L_z\}$. We make the physical assumption that no instability occurs in the immobile subsystem, $\Psi^M \neq 0$.

A Turing bifurcation occurs when a single $\lambda_k$ goes through zero and becomes positive as the parameter $\alpha$ goes through the critical value $\alpha_c$. These conditions

$$\lambda_{k_c} = \frac{d\lambda_k}{dk} \bigg|_{k=k_c} = 0$$

are satisfied at the onset of the instability, and define the critical wavenumber $k_c$. Note that in the case of scalar diffusion, no such instability can occur in the absence of reaction with the immobile species. From the above,

$$\mathbf{L}_{k_c} \cdot \Psi_{k_c} = 0,$$  \hspace{1cm} (34)

and

$$\left. \frac{\partial \mathbf{L}}{\partial k} \right|_{k=k_c} \cdot \Psi_{k_c} + \mathbf{L}_{k_c} \cdot \left. \frac{\partial \Psi}{\partial k} \right|_{k=k_c} = 0.$$  \hspace{1cm} (35)

Taking the inner product of Eq. 35 on the left by $\Psi^\dagger_{k_c}$, the nullvector of $\mathbf{L}^\dagger_{k_c}$, we find

$$\Psi^M_{k_c} \cdot \Psi^M_{k_c} = 0,$$  \hspace{1cm} (36)

for $k_c \neq 0$. In the absence of the immobile reaction, this implies that zero is a defective eigenvalue of $\mathbf{L}_{k_c}$ at the Turing bifurcation. However, it follows that

$$\Psi^I_{k_c} \cdot \Psi_{k_c} = \Psi^I_{k_c} \Psi^I_{k_c} \neq 0$$  \hspace{1cm} (37)

since there must exist a nontrivial solution $\Psi^I$ to

$$g_U \cdot \Psi^M_{k_c} + g_W \Psi^I_{k_c} = 0.$$  \hspace{1cm} (38)
(Otherwise a nontrivial null vector $\Psi_{k_c}$ would not exist.) Hence, zero is not a defective eigenvalue of $L_{k_c}$. Furthermore, it can be easily shown that the singularity of $L_k$ as a function of parameters $\alpha$ and wavenumber $k$ is independent of the immobile reaction [7].

In this way, the structure of the linear stability of the $(N + 1)$-component reaction-diffusion system at the Turing bifurcation motivated the formulation of a general scheme for appropriately lifting a defective eigenvalue problem to a higher dimensional space in which it is no longer defective. Despite this connection, our numerical examples of the lifting algorithm in the previous section did not include linear stability analysis of the CDIMA system. This is because the improvement in accurate calculation of the eigenvector associated with the defective eigenvalue is best illustrated when the eigenvalue is exactly known. For the gradient CDIMA system, the defective eigenvalue $\mu_c \equiv Dk_c^2$ is not known apriori.

5 Concluding Remarks

The stability analysis of the experimental CDIMA reaction-diffusion system has led to the formulation of a new scheme in computational linear algebra for more accurate numerical solution of defective eigenvectors. We have demonstrated improvement in numerical accuracy by several orders of magnitude using illustrative toy examples. This scheme should be of utility in physical applications where accurate eigenvectors of defective matrices are sought, as well as of general use in improving accuracy of matrix computations involving such matrices.

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Fig. 1. The lifting error, $\mathcal{E}(\epsilon, \beta)$, is plotted as a function of lifting parameter, $\beta$, for different values of $\epsilon$. The lifting vectors are given by: $v = (\beta \mathbf{v}_r, 1)$ and $w = (\beta \mathbf{w}_r, 1)$, where $\mathbf{v}_r$ and $\mathbf{w}_r$ are two-dimensional unit random vectors. Each point corresponds to the mean error obtained using 1000 unit random lifting vectors, $(\mathbf{v}_r, \mathbf{w}_r)$. 
Fig. 2. Error as a function of $\epsilon$ with lifting parameter $\beta = 1.0$: Open circles show the error using lifting vectors given by $v = (\beta v_r, 1)$ and $w = (\beta w_r, 1)$, where $v_r$ and $w_r$ are two-dimensional unit random vectors. Each point corresponds to the mean error obtained using 1000 unit random lifting vectors, $(v_r, w_r)$. Open squares show the error using lifting vectors given by $v = (\beta \psi, 1)$ and $w = (\beta \phi, 1)$, where $\psi$ and $\phi$ are the left and right unit nullvectors of $A$, respectively. The error without lifting is given by the long-dashed line (open triangles).
Fig. 3. The condition number for the singular value of the $3 \times 3$ lifted matrix, $L$, is plotted as a function of lifting parameter, $\beta$, for different values of $\epsilon$. A single, unit random lifting vector, $(v_r, w_r)$, was used to construct the lifted matrix.
Fig. 4. The lifting error (open circles) and the magnitude of the “zero” eigenvalue, $|\lambda_0|$, of $L$ (filled squares) are plotted as a function of lifting parameter, $\beta$, with $\epsilon = 10^{-12}$ and $N = 500$. The error without lifting is shown for comparison (long-dashed line). The lifting vectors are given by: $v = (\beta v_r, 1)$ and $w = (\beta w_r, 1)$, where $v_r$ and $w_r$ are $(N - 1)$-dimensional unit random vectors. Each point corresponds to the average quantity obtained using 50 unit random lifting vectors, $(v_r, w_r)$, and the error bars give the rms of the distribution of errors.