Radiated momentum in gravitational two-body scattering including time-asymmetric effects

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We compute to high post-Newtonian accuracy the 4-momentum (linear momentum, and energy), radiated as gravitational waves in a two-body system undergoing gravitational scattering. We include, for the first time, all the relevant time-asymmetric effects that arise when consistently going two and a half post Newtonian orders beyond the leading post Newtonian order. We find that the inclusion of time-asymmetric radiative effects (both in tails and in the radiation-reacted hyperbolic motion) is crucial to ensure the mass-polynomiality of the post-Minkowskian expansion ($G$ expansion) of the radiated 4-momentum. Imposing the mass-polynomiality of the corresponding individual impulses yields strong constraints on conservative-like radiative contributions (which start at the fourth post-Minkowskian, and fifth post-Newtonian, order).

I. INTRODUCTION

Gravitational scattering has attracted a renewed interest in recent years, both for conservative and dissipative (i.e., gravitational-radiation-related) effects. Various approximation methods (post-Newtonian, post-Minkowskian, quantum perturbation theory, effective field theory, string theory) have been applied to this problem. For a sample of results on (classical or quantum) post-Minkowskian (PM) gravitational scattering, see, e.g., Refs. [1–19]. For recent PM results on radiative losses during gravitational scattering and related results, see, e.g., Refs. [20–25].

In the spinless case, the state of the art for PM scattering (of spinless bodies) is $O(G^3)$ for radiation-reacted scattering [22, 24, 26, 28], and $O(G^4)$ for the conservative case [10, 14]. The state of the art for radiative losses during gravitational scattering is $O(G^3)$ for radiated angular momentum [24], and $O(G^3)$ for radiated 4-momentum [9, 20, 21, 23]. While finalizing this work a $O(G^3)$-accurate computation of the (radiation-reacted) individual 4-momentum changes (or “impulses”), $\Delta p_{\text{rad}}$, and of the loss of 4-momentum of the system, appeared on arXiv [25].

The relation between radiative losses of energy, linear momentum, and angular momentum, and the radiation-reaction contribution to scattering has been worked out, to linear order in radiation-reaction, in Refs. [25, 30]. One of the aims of the present work is to go beyond the purely linear-in-radiation-reaction treatment of Refs. [25, 30]. This will be done by focussing on the various time-asymmetric effects arising in the radiative losses of energy and linear momentum during hyperbolic encounters.

The post-Newtonian (PN) approximation method has also recently played a useful role in tackling gravitational scattering. The state of the art for the PN scattering (of spinless bodies) in the conservative case is the fourth post-Newtonian (4PN) accuracy [31]. This was generalized in Refs. [32, 34] to the 5PN, and 6PN, accuracies (modulo the knowledge of a few, yet undetermined, Hamiltonian coefficients). The state of the art for the PN-expanded computation of the radiative losses (to gravitational waves) of energy, angular momentum and linear momentum is as follows: the radiated energy and angular momentum (for spinless bodies) have been computed at the absolute 4.5PN order (corresponding to a 2PN fractional accuracy) in Refs. [30, 35, 38]. Higher-order terms (corresponding to, at least, the 3PN fractional accuracy) have been computed in Refs. [35, 38]. The radiated linear momentum is currently known to the (absolute) 5.5PN order [31, 40] (corresponding to a 2PN fractional accuracy). The state of the art for the PN-expanded computation of the scattering of spinless bodies is the 5PN level, at which an inconsistency with the mass-polynomiality of the $G^4$ (4PM) contribution was highlighted in [38] (for the conservative sector), and remains unresolved despite recent work on the additional radiative contributions [41].

The aims of the present paper are:

1. to complete the PN knowledge of the radiated energy by including both the fractional 2.5PN contribution (linked to the 2.5PN radiation-reaction modification of the hyperbolic motion) which was incorrectly argued to vanish in Ref. [37], and the “instantaneous” 3PN-level contribution first derived in Ref. [37], and rederived here;

2. to improve the knowledge of the radiated angular momentum by including both the fractional 2.5PN contribution (computed here for the first time) and the 3PN-level contribution (obtained here by

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1 We recall that the leading PN orders of radiative losses is the 2.5PN order for energy and angular momentum, while it is the 3.5PN order for linear momentum.
adding instantaneous 3PN terms \(^3\) and higher-order tails \(^2\);

3. to raise the knowledge of the radiated linear momentum to the fractional 3PN accuracy (corresponding to the absolute 6.5PN order).

4. to bring new light on the apparent mass-violation in the scattering at the 4PM and 5PN orders.

The accuracy increase (from 2PN to 3PN fractional accuracy) in the radiated linear momentum requires that many new physical effects be taken into account: indeed, we will need to take into account: (i) 2.5PN radiation-reaction effects in the hyperbolic motion; (ii) 2.5PN “instantaneous” contributions to the radiative multipole moments \(^3\), (iii) the 1PN fractional correction to the leading-order tail\(^2\) contribution to the radiated linear momentum (which was first computed in Ref. \[30\]); (iv) 3PN accuracy in several multipoles and in the hyperbolic motion; and (v) higher-order tails in the momentum loss \[40\].

To complete the so-obtained increased PN-expanded knowledge of the radiated 4-momentum \(P^\mu_{rad} = (\mathbf{E}_{rad}, \mathbf{P}_{rad})\) we will re-express it in terms of Lorentz-invariant form factors by decomposing it on the basis \(u_{1\mu}^\mu, u_{2\mu}^\mu, b_{12}^\mu\), defined by the initial four velocities of the bodies and the direction of the vectorial impact parameter \(b_{12}^\mu = b_{12}/b\), with \(b_{12} = b_1^\mu - b_2^\mu\). More precisely, it will be useful to decompose it as

\[
P^\mu_{rad} = P^\mu_{1+2}(u_{1\mu}^\mu + u_{2\mu}^\mu) + P^\mu_{1-2}(u_{1\mu}^\mu - u_{2\mu}^\mu) + P^\mu_{b_{12}\mu} b_{12}^\mu.
\]

We will show below (generalizing considerations introduced in Refs. \[30, 43\]) that, at each order in \(G\), the PM expansion of the form factors, \(P^\mu_{1+2}, P^\mu_{1-2}, P^\mu_{b_{12}}\) (expressed as functions of \(b\) and of the relative Lorentz factor \(\gamma \equiv -u_{1\mu}^\mu u_{2\mu}^\mu\)), have a polynomial structure in the two masses \(m_1, m_2\), e.g.

\[
P^\mu_{1+2} = \frac{G^3}{b^3} m_1^2 m_2^2 F^\mu_{1+2},
\]

with

\[
F^\mu_{1+2} = \sum_{n \geq 1} \frac{G^{n-3}}{b^{n-3}} S^N_{n-3}(m_1, m_2; \gamma).
\]

Here, and in the following, the notation \(S^N_{n-3}(m_1, m_2; \gamma)\) denotes a homogeneous symmetric polynomial of order \(N\) in the two masses, with coefficients depending on the Lorentz factor \(\gamma \equiv -u_{1\mu}^\mu u_{2\mu}^\mu\).

At the 3PM level \((O(G^3))\), only one form factor of \(P^\mu_{rad}\) is non vanishing, namely \(P^\mu_{1+2}\), with

\[
P^\mu_{1+2} = \frac{G^3}{b^3} m_1^2 m_2^2 \mathcal{E}(\gamma)\frac{G}{\gamma^3 + 1}.
\]

The exact value of the function \(\mathcal{E}(\gamma)\) has been computed in Refs. \[9, 23, 24, 46, 47\], while its PN expansion was computed to order \(v^{15}\) included in \[30\], see Eq. (5.19) there. For illustration, let us display the beginning of the PN expansion of \(\mathcal{E}(\gamma)\), when expressed in terms of \(p^\mu_{\infty}\) as

\[
\mathcal{E}(\gamma) = \pi \left( \frac{37}{15} p^\mu_{\infty} + \frac{1357}{840} p^\mu_3 + \frac{27953}{10080} p^\mu_5 - \frac{676273}{354816} p^\mu_7 + O(p^\mu_9) \right).
\]

Using our newly acquired PN-expanded knowledge on the values of \(E_{rad}\) and \(P_{rad}\) [computed in the center-of-mass (c.m.) frame], we will be able both to check the mass-polynomiality structure of the form factors \(P^\mu_{rad}\) entering the decomposition \([16]\), and to compute their expansions in powers of \(p^\mu_{\infty}\) at the following fractional accuracies: 3PN for \(F^\mu_{1+2}\), and 2.5PN for \(F^\mu_{1-2}\), and \(F^\mu_{b_{12}}\). Finally, we will use the so-acquired improved knowledge of \(P^\mu_{rad}\) to constrain the radiation-reaction induced contributions to the individual changes \(\Delta p^\mu_{\nu}\) (also called “impulses”) of the 4-momenta of the two bodies. As we will recall in more detail below, Refs. \[29, 30\] have derived the effect of radiation reaction on the individual momentum changes \(\Delta p^\mu_{\nu}\) only to linear order in radiation reaction, and within a restricted set of assumptions. Namely, writing the equations of motion of each particle as a perturbed “conservative” (Hamiltonian) system involving an additional “radiation-reaction force” \(F^\mu_{rr}\) Refs. \[29, 30\] worked only to linear order in \(F^\mu_{rr}\), and, furthermore, often assumed that the latter radiation-reaction force was time-antisymmetric\(^4\). Under these assumptions, Refs. \[29, 30\] derived an expression for \(\Delta p^\mu_{\nu}\) of the form

\[
\Delta p^\mu_{\nu} = \Delta p^\mu_{\nu}^{cons} + \Delta p^\mu_{\nu}^{lin} + \Delta p^\mu_{\nu}^{nonlin}.
\]

Here the term \(\Delta p^\mu_{\nu}^{lin}\) denotes the contribution linear in the radiation reaction derived in \[30\], while the term \(\Delta p^\mu_{\nu}^{nonlin}\) denotes the missing remainder, due to non-linear effects in \(F^\mu_{rr}\). Ref. \[30\] had illustrated the existence of non-linear effects in \(F^\mu_{rr}\) by computing (within the standard PN approach) a contribution to \(\Delta p^\mu_{\nu}\) quadratic in \(F^\mu_{rr}\). In addition, it has been known for

\[^2\] We recall that tail contributions to gravitational radiation start at the fractional 1.5PN order \([12, 14]\).

\[^3\] We use a mostly plus signature.

\[^4\] The time reversal operation is taken around the moment of closest approach of the time-symmetric unperturbed conservative dynamics, considered in the center-of-mass frame.
a long time \(45\) \(46\) that there are hereditary, tail-related contributions to the equations of motion. These contributions are time-asymmetric, i.e., neither time-even, nor time-odd. At the 4PN level, they can be uniquely decomposed in a time-even conservative piece (contributing to the Hamiltonian), and a time-odd piece giving a nonlocal-in-time contribution to \(F^\mu_{\text{rr}}\) (see, Section VI of \(44\)). However, as we shall discuss below, this simple decomposition becomes more tricky at the 5PN level. This is indeed the PN level where quadratic effects in \(\delta \text{ decomposition becomes more tricky at the 5PN level.}
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II. FRAMEWORK

To set up the stage for our computations below, let us recall that the general expressions for the radiative fluxes (at infinity), linear momentum and angular momentum in terms of the radiative multipole moments \(U_L\) and \(V_L\) (defined at future null infinity) read \(48\) \(53\) \(54\)

\[
\frac{dE^\text{rad}}{dt_{\text{ret}}} \equiv \mathcal{F}_E = \sum_{l=2}^{\infty} \frac{G}{c^{2l+1}} \left[ \frac{(l+1)(l+2)}{(l-1)!l!(2l+1)!} U_L U_{L+1} \right] + \frac{4l(l+2)}{c^2(l-1)(l+1)!l!(2l+1)!} V_L V_{L+1}^\text{rad},
\]

(2.1)

\[
\frac{dP^\text{rad}}{dt_{\text{ret}}} \equiv \mathcal{F}_{Pi} = \sum_{l=2}^{\infty} \frac{G}{c^{2l+1}} \left[ \frac{2(l+2)(l+3)}{(l+1)!l!(2l+3)!} U_L U_{L+1} \right] + \frac{G}{c^{2l+5}} \left[ \frac{8(l+3)}{(l+1)!l!(2l+3)!} V_L V_{L+1} \right] + \frac{8l(l+2)}{c^{2l+3}} \left[ \frac{U_L U_{L+1}}{l!(l+1)!l!(2l+1)!} \epsilon_{abcd} U_{aL} U_{bL} \right],
\]

(2.2)

and

\[
\frac{dJ^\text{rad}}{dt_{\text{ret}}} \equiv \mathcal{F}_{Ji} = \mathcal{F}_{Ji} = \epsilon_{abcd} \frac{G}{c^{2l+1}} \sum_{l=2}^{\infty} \left[ \frac{l+1}{l-1} l!l!(2l+1)! \right] U_{aL} U_{bL} V_{cL} V_{dL}^\text{rad}.
\]

(2.3)

Here, \(t_{\text{ret}} = t - \frac{\tau}{c} \equiv \frac{2GM}{c^3} \ln \left( \frac{\tau}{\tau_0} \right)\) is the retarded time (with \(M\) denoting the total Arnowitt-Deser-Misner (ADM) mass of the spacetime, and \(\tau_0\) a constant length scale), while \(U_{\text{rr}}\) and \(V_{\text{rr}}\) are the mass-type and current-type radiative multipole moments, respectively (with \(L = i_1 i_2 \cdots i_l\) being a multi-index consisting of \(l\) spatial indices). They are related to the source multipole moments \(I_L\) and \(J_L\) by relations having the structure\(42\)

\[
U_{L}(t) = I_{L}^{(1)}(t) + \frac{G}{c^{2l}} (\text{tail + semi-hered. + instantaneous})
\]

+ \(G/c^3\) (semi-hered. + instantaneous) + higher-order tails,

\[
V_{L}(t) = J_{L}^{(1)}(t) + \frac{G}{c^3} (\text{tail + instantaneous})
\]

+ \(G/c^3\) (semi-hered. + instantaneous) + higher-order tails.\n
(2.4)

Here the “tail terms” are given by integrals extending over the full past history of the source of the type

\[
+ \frac{2GM}{c^3} \int_0^{\tau} dt_{\text{ret}} I_{L+2}(t-\tau) \left( \ln \frac{\tau}{\tau_0} + \text{const.} \right),
\]

(2.5)

with \(\tau_0 = r_0/c\). The semi-hereditary (semi-hered.) terms (also known as memory terms) are time anti-derivatives of products of multipole moments, whereas the instantaneous terms are polynomials in (time-derivatives) of the source multipole moments. Notice that there are no semi-hereditary contributions to the radiative current moments. In the case of radiative mass moments, instead, the \(O(\frac{G}{c^3})\) semi-hereditary terms first appear for \(l = 4\), while at the next order \(O(\frac{G}{c^5})\) they are already present for \(l = 2\). Furthermore, both the energy and the

\(5\) Henceforth we replace the argument \(t_{\text{ret}}\) of the radiative multipole moments simply by the dynamical time variable \(t = t_{\text{ret}} + \text{cst}\) describing the binary motion (in the center-of-mass system).
linear momentum fluxes [22] only contain time derivatives of the radiative moments [24], so that all the semi-hereditary terms give instantaneous contributions to both $F_E(t)$ and $F_P(t)$.

The higher-order tail contributions (tail-squared, tails-of-tails, etc.) start at fractional order $\left(\frac{25M}{3}\right)^2$, i.e., 3PN. We will take into account these fractional 3PN contributions in all radiated quantities: energy, angular momentum and linear momentum. To reach the 3PN accuracy, we also need to take into account all semi-hereditary and instantaneous terms that contribute at the fractional 2.5PN level. Among the 2.5PN effects, an important, and subtle one, comes from the 2.5PN-level correction to the hyperbolic motion induced by the leading-order radiation-reaction force. It is the subject of the next Section.

III. 2.5PN CORRECTION TO THE QUASI-KEPLERIAN PARAMETRIZATION FOR HYPERBOLICLIKE ORBITS

In order to explicitly compute the 2.5PN correction to hyperbolic motion\(^6\) caused by the leading-order radiation-reaction force (considered as a first-order perturbation of the 2PN equations of motion), it is convenient to follow Ref. [55] in using Lagrange’s method of variation of constants. This is done by rewriting the hyperbolic version\(^7\) of the solution of the 2PN-level equations of motion\(^6\) (which depends on four integration constants, say $c_1, c_2, c_3, c_4$) in terms of four time-dependent versions of the integration constants, say $c_1(t), c_2(t), c_1(t), c_2(t)$. Namely, one writes

\[
\begin{align*}
    r &= S(l, c_1(t), c_2(t)), \\
    \dot{r} &= \dot{n}(c_1(t), c_2(t)) \frac{\partial S(l, c_1(t), c_2(t))}{\partial l}, \\
    \phi &= c_\phi(t) + \tilde{W}(l, c_1(t), c_2(t)), \\
    \dot{\phi} &= \dot{n}(c_1(t), c_2(t)) \frac{\partial \tilde{W}(l, c_1(t), c_2(t))}{\partial l}.
\end{align*}
\]

(3.1)

Here the functions $S(l, c_1, c_2)$ and $\tilde{W}(l, c_1, c_2)$ are defined by eliminating the auxiliary variables $v$ and $V$ (by expressing them as functions of $l, c_1$ and $c_2$) from the four equations

\[
\begin{align*}
    S &= \bar{a}_e (e_r \cosh v - 1), \\
    \tilde{W} &= K[V + f_\phi \sin 2V + g_\phi \sin 3V], \\
    l &= c_1 \sinh v - v + f_1 V + g_1 \sin V, \\
    V &= 2 \arctan \left( \frac{e_\phi + 1}{e_\phi - 1} \tanh \frac{v}{2} \right).
\end{align*}
\]

(3.2)

In these equations, the Quasi-Keplerian orbital parameters $a_r, c_r, c_1, c_2, K \equiv 1 + k; f_\phi, g_\phi, f_1, g_1$ are functions of the two (2PN) integrals of motion $c_1, c_2$. Similarly to $S$ and $\tilde{W}$, the auxiliary variable $v$ can be considered as a function of $l, c_1, c_2$: $v = v(l, c_1, c_2)$. One could choose as basic 2PN constants, $c_1, c_2$, the energy $E$ of the system (or the specific binding energy $\tilde{E} \equiv (E - (MC^2)/(\mu c^5))$ and the angular momentum $J$ of the system (or the dimensionless angular momentum $j = J/(GM\mu)$) (see, e.g., Table VIII of Ref. [33], for the harmonic-coordinates-case expressions of the orbital parameters). In the following, we find more convenient to use $c_1 = \bar{a}_e$ and $c_2 = e_r$. The harmonic-coordinates expressions of the Quasi-Keplerian orbital parameters, as functions of $\bar{a}_e$ and $e_r$, will be presented below when discussing the generalization of this representation at the 3PN level. The auxiliary variable\(^8\) $v$ is then considered as a function of the form $v = v(l, \bar{a}_e, e_r)$, with the dependence on $\bar{a}_e$ entering only beyond the leading order (LO).

The perturbed motion is then expressed, besides allowing $c_1, c_2$ and $c_\phi$ to be functions of time, by describing the time dependence of the basic angle $l$ of the hyperbolic-planar motion in the following way:

\[
l(t) = \int_{t_0}^t \dot{n}(c_1(t), c_2(t)) dt + c_1(t).
\]

(3.3)

Here, $t_0$ is an arbitrary reference time, and the four former “constants” $c_1(t), c_2(t), c_1(t), c_2(t)$ are now time varying. Inserting Eqs. (3.1), (3.3) in the perturbed equations of motion determines the system of four first-order evolutions equations that must be satisfied by the four quantities $c_1(t), c_2(t), c_1(t), c_2(t)$, say

\[
\frac{dc_\alpha}{dt} = F_\alpha(l, c_\beta), \quad \alpha, \beta = 1, 2, l, \phi,
\]

(3.4)

where the functions $F_\alpha$ are linear in the perturbing (rel-
ative) acceleration. They generally read

\[ \frac{dc_1}{dt} = \frac{\partial c_1(x, v)}{\partial v} \cdot A_{rr}, \]
\[ \frac{dc_2}{dt} = \frac{\partial c_2(x, v)}{\partial v} \cdot A_{rr}, \]
\[ \frac{dc_0}{dt} = -\left( \frac{\partial S}{\partial t} \right)^{-1} \left[ \frac{\partial S \, dc_1}{\partial c_1} + \frac{\partial S \, dc_2}{\partial c_2} \right], \]
\[ \frac{dc_{\phi}}{dt} = -\frac{\partial W \, dc_1}{\partial c_1} \cdot \frac{\partial W \, dc_2}{\partial c_2}, \quad (3.5) \]

where \( A_{rr} \) denotes the (relative) radiation-reaction acceleration (which starts at 2.5 PN). When choosing \( c_1 = E \) and \( c_2 = J \), and when working in the Hamiltonian formalism, the first two varying-constant equations read:

\[ \frac{dE}{dt} = \frac{\partial H}{\partial p_1} F^r_{rr}, \quad \frac{dJ}{dt} = \frac{\partial H}{\partial p_2} F^r_{\phi}, \quad (3.6) \]

where \( F^r_{rr} \) denotes the relative radiation-reaction force.

When choosing \( c_1 = \tilde{a}_r \), and \( c_2 = e_r \), and when working at the leading PN order, these two equations read

\[ \frac{d\tilde{a}_r}{dt} = -2\tilde{a}_r v \cdot A_{rr}, \]
\[ \frac{de_r}{dt} = \left( \frac{e_r^2 - 1}{e_r} \right) \tilde{a}_r v \cdot A_{rr} + \frac{\sqrt{e_r^2 - 1} \left[ x \times A_{rr} \right]_z}{e_r \sqrt{\tilde{a}_r}}. \quad (3.7) \]

As we need to compute the time dependence of the source multipole moments expressed in harmonic coordinates, we shall use here the (leading order) value of \( A_{rr} \) in harmonic coordinates, namely

\[ A_{rr} = -\frac{8}{5} \nu \frac{G^2 M^2}{c^3 r} \left[ \left( 3v^2 + \frac{17 GM}{3 r} \right) i n + \left( v^2 + \frac{3GM}{r} \right) v \right]. \quad (3.8) \]

Working at the leading 2.5PN order, denoting

\[ \chi \equiv e_r \cosh v - 1, \quad (3.9) \]

(where the auxiliary variable \( v \) is the same as in Eqs. (3.2)), and decomposing the four varying constants \( c_\alpha(t) \) as

\[ c_\alpha(t) = c^0_\alpha + \delta^r c_\alpha(t), \quad (3.10) \]

with constants \( c^0_\alpha \),\(^9\) one finds the following explicit (2.5

PN-accurate) evolution system\(^{10}\) for the four \( \delta^r c_\alpha(t) \)'s:

\[ \frac{d\delta^r a_r}{dt} = \frac{\nu}{a^3} \left[ \frac{32}{5 \Lambda^3} - \frac{512}{15 \Lambda^4} + \frac{16(-49 + 9e_r^2)}{15 \Lambda^5} \right], \]
\[ \frac{d\delta^r e_r}{dt} = \frac{\nu (e_r^2 - 1)}{a^3 e_r} \left[ \frac{56(e_r^2 - 1)^2}{3 \Lambda^6} - \frac{8(9e_r^2 - 49)}{15 \Lambda^5} \right] + \frac{136}{15 \Lambda^4} + \frac{8}{5 \Lambda^3}, \]
\[ \frac{d\delta^r c_1}{dt} = \nu \sinh v \left[ \frac{56(e_r^2 - 1)^2}{3 \Lambda^6} - \frac{8(9e_r^2 - 14)}{15 \Lambda^5} + \frac{8(43e_r^2 - 3)}{15 \Lambda^4} + \frac{32 e_r^2}{5 \Lambda^3} \right], \]
\[ \frac{d\delta^r c_{\phi}}{dt} = \nu \sinh v \sqrt{e_r^2 - 1} \left[ \frac{8}{a^3} - \frac{9e_r^2 - 14}{15 \Lambda^5} \right]. \quad (3.11) \]

Let us note in passing that we have checked these results on the 2.5PN-level variation of the 2PN quasi-Keplerian parameters of hyperbolic-like motions by relating them to the results of Ref. [55] on the 2.5PN-level, radiation-reaction correction to the quasi-Keplerian parametrization of elliptic-like motions. In order to relate the two types of results we used the fact that the latter 2.5PN-level, radiation-reaction correction only depends on the Newtonian-level Keplerian parametrization (which admits a smooth analytic continuation when changing the sign of the binding energy). We then had to go through two different steps: (i) to relate the elliptic and hyperbolic quasi-Keplerian parametrizations by a simple analytic continuation (as used, e.g., at the IPN level in Ref. [60]); and (ii) to take into account the fact that Ref. [55] worked in a different coordinate system [namely ADM coordinates], corresponding to a different explicit expression for the radiation-reaction force. Some partial results on the comparison to the results of Ref. [55] are given in Appendix B.

It is convenient to integrate perturbed quantities with respect to the auxiliary variable \( v \) by using the unperturbed relation \( \frac{dv}{dt} = \tilde{a}_r^{3/2} \chi \). The explicit solution of the

\(^9\) Below, we ease the notation by denoting \( c^0_\alpha \) simply as \( c_\alpha \), while the full quantity \( c_\alpha(t) \) is always indicated with its time dependence.

\(^{10}\) Following usual practice, we often use scaled variables (factoring out some appropriate powers of \( M \) or \( \mu \)) when studying the relative motion.
above evolution system then reads

\[
\delta^{\tau\tau}\bar{a}_r(t) = \frac{\nu}{\bar{a}_r^{3/2}} \left\{ \frac{4(37\varepsilon_r^4 + 292\varepsilon_r^2 + 96)}{15(\varepsilon_r^2 - 1)^{3/2}} A(t(v) + \sinh v \left( \frac{28\varepsilon_r}{3\chi^2} + \frac{4\varepsilon_r(36\varepsilon_r^2 + 49)}{45(\varepsilon_r^2 - 1)^3\chi^2} + \frac{2\varepsilon_r(111\varepsilon_r^2 + 314)}{45(\varepsilon_r^2 - 1)^2\chi^2} + \frac{2\varepsilon_r(673\varepsilon_r^2 + 602)}{45(\varepsilon_r^2 - 1)^3\chi^2} \right) + \frac{2(673\varepsilon_r^2 + 602)}{45(\varepsilon_r^2 - 1)^3\chi^2} \right\},
\]

\[
\delta^{\tau\tau}e_r(t) = \frac{\nu}{\bar{a}_r^{3/2}} \left\{ \frac{2\varepsilon_r(121\varepsilon_r^2 + 304)}{15(\varepsilon_r^2 - 1)^{3/2}} A(t(v) + \sinh v \left( -\frac{14(\varepsilon_r^2 - 1)}{3\chi^2} - \frac{2(36\varepsilon_r^2 + 49)}{45\chi^3} \right) - \frac{(291\varepsilon_r^2 + 134)}{45\chi^2(\varepsilon_r^2 - 1)} + \frac{(72\varepsilon_r^2 + 1069\varepsilon_r^2 + 134)}{45(\varepsilon_r^2 - 1)^2\chi^2} \right) - \frac{(72\varepsilon_r^2 + 1069\varepsilon_r^2 + 134)}{45(\varepsilon_r^2 - 1)^2\chi^2} \right\},
\]

\[
\delta^{\tau\tau}c_l(t) = \frac{\nu}{\bar{a}_r^{3/2}} \left\{ \frac{14(\varepsilon_r^2 - 1)^2}{3e_r^2\chi^2} + \frac{8(\varepsilon_r^2 - 1)(9\varepsilon_r^2 - 14)}{45\varepsilon_r^2\chi^3} \right) - \frac{4(-3 + 43e_r^2)}{15e_r^2\chi^2} - \frac{32}{5\chi^2} \right\},
\]

\[
\delta^{\tau\tau}c_\phi = \frac{\nu\sqrt{\varepsilon_r^2 - 1}}{\bar{a}_r^{3/2}e_r^2} \left\{ \frac{14(\varepsilon_r^2 - 1)}{3\chi^2} + \frac{8(\varepsilon_r^2 - 14)}{45\chi^3} \right\} - \frac{4}{5\chi^2},
\]

where

\[
A(t(v)) \equiv \arctan \left[ \alpha \tanh \left( \frac{v}{2} \right) \right] + \arctan \alpha,
\]

with

\[
\alpha \equiv \sqrt{\frac{\varepsilon_r + 1}{\varepsilon_r - 1}},
\]

and where the dependence of \( v \) on \( t \) is the unperturbed one. Here we have assumed the boundary conditions \( \lim_{t \to -\infty} \delta c_\phi(t) = 0 \).

By looking at this solution, one sees that \( \delta^{\tau\tau}c_l(t) \) and \( \delta^{\tau\tau}c_\phi(t) \) are even functions of \( t \), so that they tend to the same value (here chosen to be zero) both at \( t = -\infty \) and at \( t = +\infty \). By contrast, the two other quantities \( \delta^{\tau\tau}\bar{a}_r(t) \) and \( \delta^{\tau\tau}e_r(t) \) vary between \( t = -\infty \) and \( t = +\infty \). More precisely, one gets total variations \( [f] \equiv f(+\infty) - f(-\infty) \) given by

\[
[\delta^{\tau\tau}\bar{a}_r] = \frac{4}{15} \frac{\nu}{\bar{a}_r^{3/2}(\varepsilon_r^2 - 1)^3} \left[ \frac{673\varepsilon_r^2 + 602}{3} + \frac{37\varepsilon_r^2 + 292\varepsilon_r^2 + 96}{(\varepsilon_r^2 - 1)^{1/2}} \arccos \left( \frac{-1}{e_r} \right) \right],
\]

\[
[\delta^{\tau\tau}e_r] = -\frac{2}{15} \frac{\nu}{\bar{a}_r^{3/2}(\varepsilon_r^2 - 1)^2} e_r \left[ \frac{72\varepsilon_r^2 + 1069\varepsilon_r^2 + 134}{3} + \frac{e_r^2(121\varepsilon_r^2 + 304)}{(\varepsilon_r^2 - 1)^{1/2}} \arccos \left( \frac{-1}{e_r} \right) \right],
\]

with \( [\delta^{\tau\tau}\bar{a}_r] = (2\Delta^2/\nu) \delta^{\tau\tau}\bar{E}^N \), as from Eqs. (C7)-(C9) of Ref. [30]. These total variations agree with the total scattering changes in \( a_r \) and \( e_r \) obtained in Eqs. (6.1) and (6.2) of Ref. [30] by assuming (to leading PN order) balance equations for energy and angular momentum, between the system and radiation.

To complete the solution of the radiation-reacted motion one needs to inject the results, Eqs. (3.12), in the definitions of \( l(t) \) and \( \phi(t) \). In other words, one must now evaluate the functions \( l(t) = l^0(t) + \delta^{\tau\tau}l(t) \) and \( \phi(t) = \phi^0(t) + \delta^{\tau\tau}\phi(t) \), where \( l^0(t) = l^0(t - t_0) \), \( \phi^0(t) = e_{\phi} + W(l^0(t), c_{l1}, c_{l2}) \), and where

\[
\delta^{\tau\tau}l(t) = \int_{t_0}^{t} \delta^{\tau\tau}\bar{a}(t)dt + \delta^{\tau\tau}c_l(t),
\]

\[
\delta^{\tau\tau}\phi(t) = \delta^{\tau\tau}c_\phi + \frac{\partial W}{\partial t} \delta^{\tau\tau}l + \frac{\partial W}{\partial c_1} \delta^{\tau\tau}c_1 + \frac{\partial W}{\partial c_2} \delta^{\tau\tau}c_2.
\]

If we work only to the leading PN order (i.e. the 2.5PN order) we can (in the radiation-reacted contributions) use the Newtonian-level approximation (notably \( \bar{a} \approx (a_r)^{-3/2}, K \approx 1 \)) so as to get

\[
\delta^{\tau\tau}l(t) = \frac{\nu}{15a_r^{3/2}(\varepsilon_r^2 - 1)^3} \left[ -(673\varepsilon_r^2 + 602) \times \left[ \chi + 1 + e_r \sinh v - v \right] - (111\varepsilon_r^2 + 314)(\varepsilon_r^2 - 1) \ln \chi \right. + \frac{2(36\varepsilon_r^2 + 49)(\varepsilon_r^2 - 1)^2}{15\chi^3} + \frac{105(\varepsilon_r^2 - 1)^3}{\chi^2} \times \left. \frac{6(37\varepsilon_r^2 + 292\varepsilon_r^2 + 96)}{\sqrt{\varepsilon_r^2 - 1}} \times \right. \left[ \mathcal{L} + (e_r \sinh v - v) \arctan \alpha \right] + \text{const.},
\]

where

\[
\mathcal{L} \equiv \int_{\nu}^{v} d\nu \chi(v) \arctan \left( \alpha \tanh \left( \frac{v}{2} \right) \right),
\]

and where the integration constant can be chosen at will (e.g., to make \( \delta^{\tau\tau}l(t) \) vanish when \( v = 0 \), i.e., at the
moment of closest approach in the hyperbolic motion. Changing the integration variable as \(v = 2 \arctan h T\) in the above integral yields

\[
\mathcal{L} = \int dT \left[ -\frac{1}{1 + T} - \frac{1}{1 - T} + e_r \left( \frac{1}{(1 + T)^2} + \frac{1}{(1 - T)^2} \right) \right] \arctan (\alpha T), \tag{3.19}
\]

which can be solved in terms of dilogarithms. Explicitly,

\[
\int dT \frac{\arctan (\alpha T)}{1 + T} = \frac{\alpha}{2(1 + \alpha^2)} \left[ 2 \ln(1 + T) - (1 + i\alpha) \ln(1 + i\alpha T) - (1 - i\alpha) \ln(1 - i\alpha T) \right], \tag{3.20}
\]

and

\[
\int dT \frac{\arctan (\alpha T)}{1 + T} = \frac{i}{2} \left[ \ln \left( \frac{\alpha(1 + T)}{\mp i + \alpha} \right) \ln(1 - i\alpha T) + \ln \left( \frac{\alpha(1 + T)}{\pm i + \alpha} \right) \ln(1 + i\alpha T) - \ln \left( \frac{i - \alpha T}{i + \alpha} \right) \right] + L_{12} \left( \frac{i - \alpha T}{i + \alpha} \right). \tag{3.21}
\]

Finally, the solution for the orbit \(x^i(t) = x^i_{<2PN}(t) + \delta^{rr} x^i(t)\), obtained by varying \(l, c_1, c_2, \phi_0\) in the functions \(r(l, c_1, c_2)\) and \(\phi(l, c_1, c_2, \phi_0)\) defined by Eqs. (3.1), (3.2), reads

\[
\delta^{rr} r(t) = \frac{1}{\alpha} \hat{a}_r e_r, \sinh v \delta^{rl} l(t) + \chi \delta^{rr} \hat{a}_r(t) + \frac{\alpha}{e_r} \left( -1 + \frac{e_r^2 - 1}{\chi^2} \right) \delta^{rr} e_r(t),
\]

\[
\delta^{rr} \phi(t) = \frac{\sqrt{e_r^2 - 1}}{\chi^2} \delta^{rl} l(t) + \delta^{rr} \phi_0 - \frac{1}{\alpha} \left( \frac{\sqrt{e_r^2 - 1}}{\chi} + \frac{1}{\sqrt{e_r^2 - 1}} \right) \delta^{rr} e_r(t). \tag{3.22}
\]

Taking into account the time-even character of \(\delta^{rr} c_1(t)\) and \(\delta^{rr} c_2(t)\), the total change, \([\delta^{rr} \phi]\), between \(-\infty\) and \(+\infty\), of the value of \(\delta^{rr} \phi(t)\) is then easily seen\(^{11}\) to be

\[
[\delta^{rr} \phi] = -\frac{[\delta^{rr} e_r]}{e_r \sqrt{e_r^2 - 1}}. \tag{3.23}
\]

This agrees with the leading PN order result obtained in Ref. [29] for the radiation-reaction contribution to the (relative) scattering angle: \(\chi_{rr}^{2PN} = [\delta^{rr} \phi]\). As already mentioned in Ref. [29], the general linear-response formula, Eq. (5.99) there, for \(\chi_{rr}(E, j)\) is generally valid (to linear order in radiation reaction) beyond the leading PN order, under the two conditions that the unperturbed conservative motion be time-symmetric, and that the radiation-reaction force be time-antisymmetric. [These two conditions generally ensure that \(\frac{d\delta^{rr} \phi}{dt}\) and \(\frac{d\delta^{rr} \phi}{dt}\) will be time-symmetric, while \(\frac{d\delta^{rr} e_r}{dt}\) and \(\frac{d\delta^{rr} e_r}{dt}\) will be time-antisymmetric, so that \([c_1] = 0\) and \([c_2] = 0\). For completeness, we present in Appendix A the explicit expressions of the 2.5PN, 3.5PN and 4.5PN contributions to the function \(\chi_{rr}(E, j)\).

## IV. CONTRIBUTION TO THE RADIATED LINEAR MOMENTUM COMING FROM THE RADIATION-REACTION CORRECTION TO HYPERBOLIC MOTION

Having in hands the radiation-reaction correction to hyperbolic motion we can now come back to the analytical determination of the linear-momentum loss at the fractional 3PN accuracy.

Inserting in Eq. (2.2) the expressions (2.3) for the radiative moments in terms of the source moments, and taking into account all instantaneous, semi-hereditary and hereditary terms contributing at the 3PN level, we get a radiative linear-momentum flux of the form

\[
\mathcal{F}_P^i = \mathcal{F}_P^i \text{inst} + \Delta \mathcal{F}_P^i \text{inst} + \mathcal{F}_P^i \text{tail} + \mathcal{F}_P^i \text{higher-order tails}, \tag{4.1}
\]

Here: the “leading-order instantaneous” term \(\mathcal{F}_P^i \text{inst} \leq 3PN\) is defined by replacing in Eq. (2.2) the radiative moments \(U_L, V_L\) by the source ones, \(I_L, J_L\); the “supplementary instantaneous” contribution \(\Delta \mathcal{F}_P^i \text{inst}\) combines contributions bilinear in (the derivatives of) \(I_L, J_L\) coming both from the instantaneous terms and the semi-hereditary ones in Eq. (2.3); finally the “tail” terms (both linear tails, and higher-order tails) \(\mathcal{F}_P^i \text{tail} + \mathcal{F}_P^i \text{higher-order tails}\) denote the contribution bilinear in \(I_L, J_L\) and in the various hereditary contributions to \(U_L, V_L\).

The complete expression for the linear momentum flux at the 2.5PN fractional accuracy level is given in Eqs. (2.3)-(2.5) of Ref. [13].

Note that the notation used there is

\[
\mathcal{F}_P^i \text{inst} = \mathcal{F}_P^i \text{inst} + \Delta \mathcal{F}_P^i \text{inst}, \tag{4.2}
\]

In order to reach the 3PN accuracy, we need: (i) to insert in these expressions the 3PN-accurate expressions of the source moments \(I_L(t), J_L(t)\) considered as functions of dynamical time \(t\); and (ii) to add the higher-order tail contribution to the hereditary term \(\mathcal{F}_P^i \text{hered}\).

When evaluating 3PN-accurate values of the relevant \(k\)th time-derivatives, \(I_L^{(k)} \leq 3PN(t), J_L^{(k)} \leq 3PN(t)\), of the source moments one needs to use the 3PN-level equations of

---

\(^{11}\) The term proportional to \(\delta^{rr} l(t)\) in \(\delta^{rr} \phi(t)\) vanishes at infinity.
motion (including the 2.5PN radiation-reaction contribution), and then to express these time-differentiated moments along radiation-reacted hyperbolic-like solutions of the equations of motion. The latter are obtained by adding the 2.5PN-level, radiation-reaction effects discussed in the previous Section to the conservative 3PN hyperbolic-like solutions (which will be discussed below).

Let us symbolically write the motions as

\[ x^{\leq 3\text{PN}}(t) = x^{3\text{PN,cons}}(t) + \delta r x(t), \]
\[ v^{\leq 3\text{PN}}(t) = v^{3\text{PN,cons}}(t) + \delta r v(t). \]  

(4.3)

As a consequence, the first contribution, \( F_{P_i \text{inst}}^{\text{inst}, I, J \leq 3\text{PN}}(t) \), to the linear-momentum flux is naturally decomposed as a sum of two terms:

\[ F_{P_i \text{inst}}^{\text{inst}, I, J \leq 3\text{PN}}(t) = F_{P_i \text{inst}}^{\text{inst}, I, J 3\text{PN,cons}}(t) + \delta r F_{P_i \text{inst}}^{\text{inst}, I, J}(t). \]  

(4.4)

In these expressions, and below, the symbol \( \delta r \) will be used to denote the 2.5PN-level radiation-reaction-generated contribution to some physical quantity, \( Q(t) = Q(x(t), v(t)) \), considered as a function of dynamical time \( t \). In the previous section, we obtained (at leading order) the various needed radiation-reaction contributions, \( \delta r x(t), \delta r v(t), \delta r^2 Q(t) \) by using Lagrange’s method of variation of constants.

Finally, integrating \( F_{P_i} \) over \( t \) (from \(-\infty \) to \(+\infty \)) we get the total linear momentum radiated in gravitational waves during a full hyperbolic encounter:

\[ P_{r i}^{\text{rad}} = \int_{-\infty}^{+\infty} dt \, F_{P_i}(t). \]  

(4.5)

The 6.5PN-accurate value of \( P_{r i}^{\text{rad}} \) is then obtained as a sum of various contributions, say:

\[ P_{r i}^{\text{rad}} = P_{r i}^{\text{rad inst}, I, J 3\text{PN,cons}} + \delta r P_{r i}^{\text{rad inst}, I, J} + \Delta P_{r i}^{\text{rad inst}, I, J} + P_{r i}^{\text{rad higher-order tails}}. \]  

(4.6)

The resulting vectorial contributions will be projected on an orthonormal basis \( e_x, e_y, e_z \) defined in terms of the vectorial impact parameter \( \bar{b}_1 = \bar{b}_1^x \) of the initial four velocities \( u_1^x, u_1^y, u_1^z \) of the two bodies, and of the conservative\(^\text{12} \) part of the scattering angle, \( \chi_{\text{cons}} \) (see e.g., Table X of Ref. 30, also recalled at 2PN in Eqs. (3.35)-(3.36) below for convenience). The basis \( e_x, e_y \) was already used in Ref. 30 (see Eq. (3.49) there). Its definition is recalled in Appendix A. Let us only mention here that \( e_z \) is in the direction of the major axis of the hyperbolic-like relative orbit (direction of closest approach).

The 2PN-accurate value of the instantaneous contribution to linear-momentum loss has been evaluated in Ref. 30, see Eqs. (G6)-(G9) there. We have extended this result by including both the higher-order tail effects (which were computed in Ref. 10), and the 3PN-level conservative effects. The technology (including a 3PN-accurate quasi-Keplerian representation of hyperbolic-like motions) needed for computing 3PN-level conservative instantaneous contribution will be discussed below.

Let us discuss here the evaluation of the radiation-reaction-related contribution \( \delta r F_{P_i \text{inst}, I, J} \). To obtain it, it is enough to evaluate the Newtonian flux

\[ F_{P_i \text{inst}, I, J N} = \frac{64}{105} \frac{G^3 M^4 m_2 - m_1}{r^4 c^2} \lambda^2 \times (A^N u_i + B^N v_i), \]  

(4.7)

with

\[ A^N = \dot{r} \left( \frac{55}{8} \lambda^2 - \frac{45}{8} \lambda^2 + \frac{3 GM}{r} \right), \]
\[ B^N = -\left( \frac{25}{4} \lambda^2 - \frac{19}{4} \lambda^2 + \frac{GM}{r} \right), \]  

(4.8)

along the radiation-reaction-perturbed orbit, i.e., by substituting in it

\[ r(t) = r_N(t) + \delta r r(t), \quad \dot{r}(t) = \dot{r}_N(t) + \delta r \dot{r}(t), \quad \phi(t) = \phi_N(t) + \delta r \phi(t), \quad \dot{\phi}(t) = \dot{\phi}_N(t) + \delta r \dot{\phi}_N(t), \]  

(4.9)

taking then the time integral, retaining only linear corrections. [The magnitude of the relative velocity in Eq. (4.5) above should not be confused with the auxiliary variable used to parametrize the orbit, denoted by the same letter \( v \).]

The variations \( \delta r r(t) \) and \( \delta r \phi(t) \) are given by Eq. (3.22). The related variations \( \delta r^2 r(t) \) and \( \delta r^2 \phi(t) \) are obtained either by taking the time derivatives of \( \delta r r(t) \) and \( \delta r \phi(t) \) or by varying the functions \( \dot{r}(l, c_1, c_2) \) and \( \phi(l, c_1, c_2, e) \) in Eqs. (3.1) and (3.2). This yields

\[ \delta r^2 \dot{r}(l) = \frac{e_r}{a_r^{1/2} \chi^4} (e_r - \cos \chi) \delta r^2 \dot{l}(l) - \frac{1}{2} \frac{\sinh \nu}{a_r^{1/2} \chi^4} \delta r^2 \dot{a}_r(l), \]
\[ - \frac{(e_r^2 - 1) \sinh \nu}{a_r^{1/2} \chi^4} \delta r^2 e_r(l), \]
\[ \delta r^2 \dot{\phi}(l) = -\frac{e_r \sqrt{e_r^2 - 1}}{a_r^{3/2} \chi^4} \sinh \nu \delta r^2 \dot{l}(l) - \frac{3}{2} \frac{e_r}{a_r^{3/2} \chi^2} \delta r^2 \dot{a}_r(l), \]
\[ + \frac{e_r}{a_r^{3/2} \chi^2} \left[ 1 + \frac{2(e_r^2 - 1)}{e_r^2 \chi^2} \right] \delta r^2 e_r(l) \times \delta r^2 \dot{e}_r(l). \]  

(4.10)

We have checked that they satisfy \( \frac{d\delta r^2 \dot{r}(l)}{dt} = \delta r^2 \dot{r}(l) \) and \( \frac{d\delta r^2 \dot{\phi}(l)}{dt} = \delta r^2 \dot{\phi}(l) \).

We finally get the 2.5PN correction to the Newtonian flux

\[ F_{P_i}^{\text{inst}, I, J N}|_{x=x_N^p} + \delta r F_{P_i}^{\text{inst}, I, J}, \]  

(4.11)
which has to be integrated along the orbit to yield

$$\delta^{\text{rad inst}} P_{i}^{\text{inst}, I, J} = \int dt \delta^{\text{rad inst}} F_{P}^{\text{inst}, I, J}. \quad (4.12)$$

The final exact results are given by the following functions of $\bar{a}_r$ and $e_r$ (here, and below, $\eta$ is a placeholder to indicate a half PN order $\frac{1}{12}$):

$$\delta^{\text{rad inst}} P_{x}^{\text{inst}, I, J} = - (Mc)^{m_2 - m_1 M} \nu^3 \eta^5 \frac{1}{e_r \left[ \bar{a}_r(e_r^2 - 1) \right]^{3/2}} \left[ \arccos \left( \frac{1}{e_r} \right) \left( 110146 \frac{\bar{r}}{675} + 132304 \frac{e_r^2}{135} e_r^2 + 5134544 \frac{e_r^4}{4725} + 1365802 \frac{e_r^6}{4725} \right. \right. \right.$$

$$\left. \left. + \left( e_r^2 - 1 \right)^{1/2} \left( 8576 + 1164474 \frac{e_r^2}{2025} e_r^2 + 22762729 \frac{e_r^4}{18375} e_r^4 + 1623904259 \frac{e_r^6}{1984500} e_r^6 + 159585499 \frac{e_r^8}{1323000} e_r^8 + 15872 \frac{e_r^{10}}{6125} e_r^{10} \right) \right), \quad (4.13)$$

$$\delta^{\text{rad inst}} P_{y}^{\text{inst}, I, J} = (Mc)^{m_2 - m_1 M} \nu^3 \eta^5 \frac{1}{e_r \left[ \bar{a}_r(e_r^2 - 1) \right]^{3/2}} \left[ e_r \arccos \left( \frac{1}{e_r} \right) \left( 2479 \frac{e_r^6}{225} \right. \right.$$

$$\left. \left. + 22616 \frac{e_r^4}{45} e_r^4 + 35416 \frac{e_r^2}{75} e_r^2 + 48256 \right), \quad (4.13)$$

The first terms of their expansions in inverse powers of $j$ (equivalent, remembering $j \propto G^{-1}$ to a PM expansion) read

$$\delta^{\text{rad inst}} P_{x}^{\text{inst}, I, J} = - (Mc)^{m_2 - m_1 M} \nu^3 \eta^5 \left[ \frac{15872 \bar{r}^8}{6125 j^6} \right.$$

$$\left. + 30331 \frac{\bar{r}^8}{360 j^6} \right.$$

$$\left. + \frac{2423475 \nu^6}{165375 \bar{r}^6 j^6} + O \left( \frac{1}{j^8} \right) \right]. \quad (4.14)$$

$$\delta^{\text{rad inst}} P_{y}^{\text{inst}, I, J} = (Mc)^{m_2 - m_1 M} \nu^3 \eta^5 \left[ \frac{148 \bar{r}^8}{25 j^6} \right.$$

$$\left. + \frac{2048 \bar{r}^8}{15 j^6} \right.$$

$$\left. + \frac{160406 \nu^6}{675 \bar{r}^6 j^6} + O \left( \frac{1}{j^8} \right) \right]. \quad (4.14)$$

V. 2.5PN INSTANTANEOUS CONTRIBUTIONS TO THE RADIATED LINEAR MOMENTUM

Let us now evaluate the third contribution to $P^{\text{rad inst}}$ denoted $\Delta F_{P}^{\text{inst}}$ in Eq. \ref{14}. This contribution is obtained by integrating over time the (2.5PN level) "instantaneous" part of the linear momentum flux in terms of the source multipole moments. Using the results of Ref. \ref{61}, Ref. \ref{43} has explicated this instantaneous part as the $O(\frac{1}{j})$ term in Eq. (2.3) there. Recently Ref. \ref{62} has provided an explicit expression for this 2.5PN instantaneous part of the linear momentum flux as a function of the (relative) position and velocity along the orbit, see Eq. (4.1) there. For clarity, we reproduce here this explicit expression:

$$\Delta F_{P_{i}}^{\text{inst}, I, J} = - \frac{64}{105} \frac{G^3 M^4}{r c^7} \frac{m_2 - m_1 M}{M} \nu^2 \times \left( A^{2.5PN} \eta_i + B^{2.5PN} \eta_i \right), \quad (5.1)$$

where

$$A^{2.5PN} = \frac{GM}{rc^5 \nu^3} \left[ \frac{701}{90} \nu^6 \frac{91137}{96} \nu^4 r^2 + \frac{41611}{40} \nu^2 r^4 \right.$$

$$\left. - \frac{49219}{96} \nu^6 \frac{4 G^3 M^3}{r^3} \right.$$

$$\left. - \frac{1237 \nu^6}{180} \frac{6607}{90} r^2 \right) \right]$$

$$B^{2.5PN} = \frac{GM}{rc^5 \nu^3} \left[ \frac{157787}{480} \nu^6 - \frac{39869}{60} \nu^2 r^2 + \frac{31913}{96} \nu^6 \right.$$

$$\left. + \frac{10773 \nu^6}{360} \frac{737 G^2 M^2}{r^2} \right). \quad (5.2)$$

The integral along a hyperboliclike orbit of $F_{P}^{\text{inst}, 2.5PN}$ can be explicitly evaluated. After projection on the $x$ and $y$ axes defined in Eq. \ref{13}, one finds
\[
\Delta p_{x}^{\text{rad inst } I,J} = (Mc)\frac{m_{2} - m_{1}}{M} \nu^{3} \eta^{5} \frac{e_{r}}{a_{r}(e_{r}^{2} - 1)^{3/2}} \times \\
\left[ \arccos \left( -\frac{1}{e_{r}} \right) \frac{491447}{33600} e_{r}^{8} + \frac{123798}{175} e_{r}^{6} + \frac{30800977}{4950} e_{r}^{4} + \frac{13714844}{4725} e_{r}^{2} + \frac{2125082}{4725} \right] \\
+ \left( e_{r}^{2} - 1 \right)^{1/2} \arccos \left( -\frac{1}{e_{r}} \right) \frac{797850313}{3528000} e_{r}^{8} + \frac{7556008631}{3175200} e_{r}^{6} + \frac{4935155857}{1323000} e_{r}^{4} + \frac{652923197}{661500} e_{r}^{2} + \frac{5266216}{496125} \right] \\
= \left( Mc \right) \frac{m_{2} - m_{1}}{M} \nu^{3} \eta^{5} \frac{e_{r}}{a_{r}(e_{r}^{2} - 1)^{3/2}} \times \\
\left[ \arccos \left( -\frac{1}{e_{r}} \right) \frac{491447 \pi p_{\infty}^{9}}{67200} + \frac{13272832 p_{\infty}^{8}}{55125} + \frac{494871 \pi p_{\infty}^{7}}{1280} + \frac{1954525568 p_{\infty}^{6}}{496125} + O \left( \frac{1}{j^{3}} \right) \right], \\
\Delta p_{y}^{\text{rad inst } I,J} = 0 .
\]

(5.3)

In the last line of the first equation, we have also given the first few terms of its large-\( j \) expansion. Let us note that this contribution is (contrary to the other 2.5PN contribution discussed in the previous Section) purely oriented along the \( x \) axis, i.e. along the vectorial distance of closest approach.

**VI. NEW CONTRIBUTIONS TO THE RADIATED ENERGY**

Let us repeat for the radiated energy the above treatment for the radiated linear momentum, namely

\[
E^{\text{rad}} = E^{\text{rad inst } I,J, 3\text{PN}} + E^{\text{rad tail}} + E^{\text{rad higher-order tails}} + \Delta E^{\text{rad inst } I,J} .
\]

(6.1)

Here, we have indicated the (fractional) 3PN level of accuracy for the instantaneous term \( E^{\text{rad inst } I,J, 3\text{PN}} \). The 2PN-accurate instantaneous energy loss \( E^{\text{rad inst } I,J, 2\text{PN}} \) was first obtained in \[50\] (see Eqs. (C7)-(C13)); its extension at the 3PN level was obtained in \[57\]. We have redone an independent 3PN-accurate computation of the energy loss and found agreement with the final results of Ref. \[57\] (after correcting several typos in the 3PN quasi-Keplerian expressions of Ref. \[50\], see Appendix \[D\]). The leading-PN-order contribution to the linear-tail \( E^{\text{rad tail}} \) has been obtained in \[50\] (see Eq. (D26)), while its 1PN correction is given in Eq. (5.20) of Ref. \[39\]; see also Ref. \[35\] for a Fourier space analysis. The higher-order tail contribution \( E^{\text{rad higher-order tails}} \) has been derived in Refs. \[39, 58\]. As discussed in the text below Eq. (3.1) of Ref. \[39\], the last contribution \( \Delta E^{\text{rad inst } I,J} \) vanishes (because of the time-odd character of its integrand):

\[
\Delta E^{\text{rad inst } I,J} = 0 .
\]

(6.2)

Ref. \[57\] claimed (see below Eq. (42) there) that, because of the time-odd character of radiation reaction, the term \( \delta^{rr} E^{\text{rad inst } I,J} \) was similarly vanishing. We found that this was not correct because of the time-asymmetric character of the motion perturbation \( \delta^{rr} x(t) \), \( \delta^{rr} v(t) \). We got a non-zero result for \( \delta^{rr} E^{\text{rad inst } I,J} \). We further found that this non-vanishing contribution plays a crucial role in obtaining a correct mass-polynomiality behavior for the radiated (four) momentum.

The exact expression of \( \delta^{rr} E^{\text{rad inst } I,J} \) in terms of \( \bar{a}_{r} \) and \( e_{r} \) reads

\[
\delta^{rr} E^{\text{rad inst } I,J} = (Mc^{2}) \nu^{3} \eta^{5} \frac{1}{\bar{a}_{r}(e_{r}^{2} - 1)^{3/2}} \left[ \arccos \left( -\frac{1}{e_{r}} \right) \frac{7696}{225} e_{r}^{8} + \frac{53936}{75} e_{r}^{6} + \frac{150272}{225} e_{r}^{4} + \frac{14336}{25} e_{r}^{2} \right. \\
\left. + \left( e_{r}^{2} - 1 \right)^{1/2} \arccos \left( -\frac{1}{e_{r}} \right) \frac{592}{25} e_{r}^{6} + \frac{46592}{675} e_{r}^{4} + \frac{44176}{27} e_{r}^{2} + \frac{1106464}{675} \right] \\
+ \frac{e_{r}^{2} - 1}{e_{r}^{2}} \left( \frac{44848}{225} e_{r}^{6} + \frac{11056}{25} e_{r}^{4} + \frac{313024}{225} e_{r}^{2} - \frac{8576}{225} \right) .
\]

(6.3)

The beginning of its expansion in \( \frac{1}{j} \) (i.e., of its PM expansion in powers of \( G \)) reads

\[
\delta^{rr} E^{\text{rad inst } I,J} = (Mc^{2}) \nu^{3} \eta^{5} \left[ \frac{296}{25} \frac{p_{\infty}^{2}}{j^{3}} + \left( \frac{50176}{225} + \frac{1924}{225} \right)^{2} \right] \frac{p_{\infty}^{4}}{j^{6}} + \frac{5008}{135} \frac{p_{\infty}^{6}}{j^{9}} + O \left( \frac{p_{\infty}^{8}}{j^{12}} \right) .
\]

(6.4)

Adding this term to the 1PN corrections to the LO tails \[33, 38, 51\] then gives the following complete expression for
the 2.5PN radiated energy

\[
E_{2.5PN}^{rad} = (Mc^2)\nu^3 \left[ \left( \frac{1216}{105} - \frac{2848\nu}{15} \right) \frac{p^6_\infty}{j^4} + \left( \frac{296}{25} - \frac{15291\pi^2}{280} \nu - \frac{24993\pi^2}{1120} + \frac{9216}{35} \right) \pi \frac{p^5_\infty}{j^5} + \left( \frac{71488}{75} - \frac{2974508\pi^2}{4725} \nu + \frac{2898\zeta(3)}{5} + \frac{1024\pi^2}{135} + \frac{56708}{105} \right) \frac{p^6_\infty}{j^6} + \left( \frac{56008}{135} - \frac{23514\pi^2}{7} + \frac{30285\pi^4}{112} - \frac{689958\pi^4}{3584} - \frac{13135915\pi^4}{7392} + \frac{210176}{225} \right) \pi \frac{p^7_\infty}{j^7} + O \left( \frac{p^8_\infty}{j^8} \right) \right]. \tag{6.5}
\]

Let us also exhibit the \( \frac{1}{j} \) expansion of the full 3PN-level contribution to the energy loss, which combines terms from several sources: the (exact) instantaneous contribution linked to 3PN-level multipole moments [33], and the higher-order tails (tails-of-tails and tail squared) [36, 38, 39].

\[
E_{3PN}^{rad} = (Mc^2)\nu^3 \left[ \left( -\frac{148\nu^3}{15} + \frac{321\nu^2}{280} - \frac{2699\nu}{504} - \frac{676273}{354816} \right) \pi \frac{p^1_\infty}{j^1} + \left( \frac{2366\nu}{9} - \frac{164\nu^2}{3} + \frac{1223594\nu}{33075} - \frac{151854}{13475} \right) \pi \frac{p^2_\infty}{j^2} + \left( -\frac{1823\nu}{5} + \frac{12269\nu^2}{80} + \frac{7697\nu}{480} - \frac{4059\pi^2}{640} \right) \nu - \frac{10593}{350} \ln \left( \frac{p_\infty}{2} \right) + \frac{99\pi^2}{10} - \frac{29573617463}{310464000} \right) \pi \frac{p^3_\infty}{j^3} + \left( \frac{150892\nu}{45} - \frac{4201976\nu^2}{1575} + \frac{875970284}{297675} - \frac{212216\pi^2}{1575} \right) \nu - \frac{18955264}{23625} \ln(2p_\infty) + \left( \frac{177152\pi^2}{675} + \frac{36589282372}{11694375} \right) \pi \frac{p^4_\infty}{j^4} + \left( -\frac{13955\nu^3}{6} + \frac{1419153\nu^2}{448} + \frac{68898691}{36288} - \frac{51947\pi^2}{384} \right) \nu - \frac{337906}{315} \ln \left( \frac{p_\infty}{2} \right) + \frac{58957\zeta(3)}{32} + \frac{3158\pi^2}{9} + \frac{375465709757}{8467200} \right) \pi \frac{p^5_\infty}{j^5} + O \left( \frac{p^6_\infty}{j^6} \right). \tag{6.6}
\]

An equivalent expression (and extended up to \( 1/j^{15} \)), can be found in Ref. [38]. [Note that Eq. (B3) of the published version (and of the arxiv version 1) uses a different parametrization, \( p \neq p_\infty \), while Eq. (C3) of the arxiv version 2 has been updated with the notation \( p = p_\infty \).]

VII. NEW CONTRIBUTIONS TO THE RADIATED ANGULAR MOMENTUM

Similarly, for the angular momentum, we have

\[
\begin{align*}
J^{rad} &= J^{rad \ inst \ I, J\ 3PN} + J^{rad \ tail} \\
&+ J^{rad \ higher-order \ tails} + \delta^{\nu r} J^{rad \ inst \ I, J} \\
&+ \Delta J^{rad \ inst \ I, J}.
\end{align*}
\tag{7.1}
\]

The 2.5PN instantaneous term is vanishing also in this case [39]. Therefore, the only contributions at that order come from the 1PN corrections to the LO tails, a memory term [36, 39], and the radiation-reaction correction to hyperbolic motion. The latter turns out to be

\[
\delta^{\nu r} J^{rad \ inst \ I, J} = \frac{GM^2}{c^3} \nu^3 \eta^5 \left[ \frac{1}{[\alpha_r(c_r^2 - 1)]^{9/2}} \left[ \arccos^2 \left( -\frac{1}{c_r} \right) \left( \frac{5264}{75} c_r^1 + \frac{1792}{15} c_r^2 + \frac{8192}{25} \right) \\
+ (c_r^2 - 1)^{1/2} \arcsin \left( \frac{1}{c_r} \left( \frac{752}{25} c_r^4 + \frac{59792}{225} c_r^2 + \frac{33248}{45} \right) \right) \\
+ \frac{c_r^2 - 1}{c_r^2} \left( \frac{128}{25} c_r^6 + \frac{1328}{75} c_r^4 + \frac{23968}{45} c_r^2 + \frac{8576}{225} \right) \right]. \tag{7.2}
\]
as an exact expression in terms of $\bar{a}_r$ and $c_r$. The beginning of its $\frac{1}{7}$ expansion is:

$$g_{rr} \, j_{\text{inst} \, 1, J} = \frac{GM^2}{c^4} \nu^3 \hat{y}^5 \left[ \frac{128}{25} p_5^{\infty} + \frac{376}{25} \frac{p_5^{\infty}}{j^3} + \left( \frac{4352}{75} + \frac{1316}{75} \pi^2 \right) \frac{p_5^{\infty}}{j^5} + \frac{52456}{225} \frac{p_5^{\infty}}{j^6} + O \left( \frac{p_5^{\infty}}{j^7} \right) \right].$$

Adding all terms leads to the final result:

$$J_{2,3 PN}^{\text{rad}} = \frac{GM^2}{c^4} \nu^3 \left[ \left( \frac{1184}{21} - \frac{431936\nu}{1575} \right) \frac{p_5^{\infty}}{j^3} + \left( \frac{7816}{525} - \frac{2232\pi^2}{35} \nu - \frac{1305\pi^2}{112} + \frac{7488}{25} \pi \right) \frac{p_5^{\infty}}{j^4} + \left( \frac{225536}{525} - \frac{201724\pi^2}{33075} \nu + \frac{4116\zeta(3)}{5} - \frac{130668\pi^2}{6615} + \frac{147064}{315} \right) \frac{p_5^{\infty}}{j^5} + \left( \frac{365392}{1575} - \frac{57037\pi^2}{21} + \frac{102619\pi^4}{448} \right) \nu + \frac{163083\pi^4}{1792} - \frac{18227\pi^2}{28} + \frac{32}{15} \pi \right) \frac{p_5^{\infty}}{j^6} + O \left( \frac{p_5^{\infty}}{j^7} \right) \right].$$

VIII. 1PN-ACCURATE TAIL CONTRIBUTION TO THE RADIATED LINEAR MOMENTUM

Let us now tackle the technically challenging (fractionally 1PN) tail contribution to the radiated linear momentum, namely the term $P_1^{\text{rad tail}}$ in Eq. 11. It is the time integral of the following linear-momentum-flux integrand (see Eq. 2.5 of Ref. 43)

$$F_1^{p, \text{tail}} = \left\{ \frac{4}{63} \left( F_1^{i(4)} I_2^{(5)} + F_1^{i(3)} I_3^{(6)} \right) + \frac{32}{45} \left( * F_1^{i(4)} J_2^{(5)} + * F_1^{i(3)} I_3^{(6)} \right) + \frac{1}{c^2} \left[ \frac{1}{567} \left( F_1^{i(5)} I_3^{(6)} + F_1^{i(4)} I_4^{(7)} \right) + \frac{1}{63} \left( * F_1^{i(4)} J_3^{(6)} + * F_1^{i(3)} I_4^{(6)} \right) + \frac{8}{63} \left( F_1^{i(3)} J_3^{(6)} + F_1^{i(2)} J_3^{(6)} \right) \right] \right\},$$

where $M = M(1 + \nu \dot{E})$ is the total ADM mass of the system, and the definitions of the quantities $F_1^{i(n)}$, $Y_m$ in terms of the source multipole moments are given in Table II.
Introducing the shorthand notation

$$\langle f \rangle = \int_{-\infty}^{\infty} dt f(t), \quad (8.2)$$

for the total time-integral of an arbitrary function $f(t)$ over the full scattering process, we need to evaluate

$$P_{\text{rad \ tail}} \equiv \langle P_{\text{P \ tail}} \rangle. \quad (8.3)$$

We found useful to evaluate this integral in the frequency domain by using a quasi-Keplerian parametrization of the motion in harmonic coordinates. We refer to previous works for a review of all necessary tools (see, e.g., Ref. 35).

Expanding the various multipole moments as Fourier integrals

$$X_L(t) \equiv \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \hat{X}_L(\omega),$$

$$\hat{X}_L(\omega) \equiv \int_{-\infty}^{\infty} \hat{X}_L(t) e^{i\omega t}, \quad (8.4)$$

leads to (denoting $\int_{0}^{\infty} \frac{d\omega}{2\pi}$),

$$\langle F_{l_3(4)}^{(5)} + F_{l_4(3)}^{(5)} + F_{l_2(4)}^{(5)} \rangle = \int_{0}^{\infty} \omega^8 \left( i\pi S_{i} - \frac{7}{10} S_{i}^+ \right),$$

$$\langle F_{l_3(4)}^{(5)} + F_{l_2(4)}^{(5)} \rangle = \int_{0}^{\infty} \omega^7 \left( \pi R_{i}^+ - \frac{i}{4} R_{i}^- \right),$$

$$\langle F_{l_3(4)}^{(5)} + F_{l_4(3)}^{(5)} \rangle = \int_{0}^{\infty} \omega^7 \left( \pi U_{i}^+ - \frac{7}{20} U_{i}^- \right),$$

$$\langle F_{l_3(4)}^{(5)} + F_{l_2(4)}^{(5)} \rangle = \int_{0}^{\infty} \omega^9 \left( \pi V_{i}^+ - \frac{i}{20} V_{i}^- \right),$$

$$\langle F_{l_3(4)}^{(5)} + F_{l_2(4)}^{(5)} + F_{l_3(5)}^{(5)} \rangle = \int_{0}^{\infty} \omega^8 \left( i\pi Z_{i} - \frac{1}{2} Z_{i}^+ \right), \quad (8.5)$$

where

$$S_{i}(\omega) = \hat{I}_{ijk}(\omega)\hat{I}_{ijk}(\omega) + \hat{I}_{ijk}(\omega)\hat{I}_{ijk}(\omega),$$

$$R_{i}^{\pm}(\omega) = \epsilon_{ijk} \left[ \hat{I}_{ijk}(\omega)\hat{J}_{kab}(\omega) \mp \hat{I}_{ijk}(\omega)\hat{J}_{kab}(\omega) \right],$$

$$U_{i}^{\pm}(\omega) = \hat{I}_{ijk}(\omega)\hat{J}_{kab}(\omega) \mp \hat{I}_{ijk}(\omega)\hat{J}_{kab}(\omega),$$

$$V_{i}^{\pm}(\omega) = \epsilon_{ijk} \left[ \hat{I}_{ijk}(\omega)\hat{J}_{kab}(\omega) \mp \hat{I}_{ijk}(\omega)\hat{J}_{kab}(\omega) \right],$$

$$Z_{i}^{\pm}(\omega) = \hat{J}_{ijk}(\omega)\hat{J}_{ijk}(\omega) \mp \hat{J}_{ijk}(\omega)\hat{J}_{ijk}(\omega). \quad (8.6)$$

The leading PN order tail contribution 35 (i.e., the first two lines in Eqs. (8.5)) has been already computed in Ref. 30 (see also Ref. 40). We focus here on the next-to-leading order (fractionally 1PN) tail contribution. We need to take into account the fractional 1PN corrections to the first two lines in Eqs. (8.5), whereas the leading PN order is enough for the remaining three lines in Eqs. (8.5). The final results for the large-$\gamma$ expansions of the (nonvanishing) components $P_{x}^{\text{rad \ tail}}$ and $P_{y}^{\text{rad \ tail}}$ are
\[ p_{\text{rad tail}} = \frac{m_2 - m_1}{M} v^2 \eta \left[ \frac{191}{100} \pi \eta^2 + \pi \eta(\frac{9529}{1000} \nu - \frac{2657}{2000} \nu) p_{\infty}^5 + \frac{2068}{5} \eta^{1.5} + \eta^2 \left( \frac{2}{15} \right)^{1.5} \right] \]

\[ \frac{267583}{24000} \eta^{1.5} + \eta^2 \left( \frac{121123}{24000} \nu - \frac{1256643}{48000} \nu \right) p_{\infty}^5 + \frac{64576}{25} \eta^{1.5} + \eta^2 \left( \frac{2924112}{2810} \nu - \frac{9900672}{2810} \right) \frac{\eta^{1.5}}{j^2} + O \left( \frac{1}{j^5} \right), \]

\[ p_{\text{rad y}} = \frac{m_2 - m_1}{M} v^2 \eta \left[ \frac{128}{4} \pi \eta^{1.5} + \pi \eta^2 \left( \frac{192}{1000} \nu - \frac{121123}{24000} \nu \right) p_{\infty}^5 + \pi \left( \frac{267583}{24000} \eta^{1.5} + \eta^2 \left( \frac{121123}{24000} \nu - \frac{1256643}{48000} \nu \right) p_{\infty}^5 + \frac{64576}{25} \eta^{1.5} + \eta^2 \left( \frac{2924112}{2810} \nu - \frac{9900672}{2810} \right) \frac{\eta^{1.5}}{j^2} + O \left( \frac{1}{j^5} \right) \right] \]

These tail contributions take into account the physical retarded-tail interaction between the bodies, so that they are asymmetric under time-reversal (they were called “past tails” in Refs. [30, 36]). Let us note in passing that replacing the retarded-kernel in the time-domain tail integral by its time-symmetric projection, would lead to the following integral:

\[ p_{\text{rad sym tail}} = \frac{G^2 M}{c^4} \left\{ \frac{32}{35} \pi \int_0^\infty \frac{d\omega}{2\pi} \omega^2 R_+^i(\omega) + \frac{4}{63} \pi \int_0^\infty \frac{d\omega}{2\pi} \omega^2 S_+^i(\omega) + \frac{1}{c^2} \left[ \frac{1}{567} \pi \int_0^\infty \frac{d\omega}{2\pi} \omega^{10} U_+^i(\omega) + \frac{1}{63} \pi \int_0^\infty \frac{d\omega}{2\pi} \omega^8 V_+^i(\omega) + \frac{8}{63} \pi \int_0^\infty \frac{d\omega}{2\pi} \omega^8 Z_+^i(\omega) \right] \right\}, \]

implying

\[ p_{x,\text{sym tail}} = 0, \quad p_{y,\text{sym tail}} = p_{\text{rad tail}}. \]

**IX. 3PN-LEVEL CONTRIBUTION TO THE RADIATED LINEAR MOMENTUM**

The radiated instantaneous linear momentum at the fractional 3PN accuracy can be obtained by integrating the 3PN instantaneous linear momentum flux,

\[ F_{Pt,\text{inst} I,J \leq 3PN} = \frac{G}{c^2} \left( \frac{f_0}{c^2} + \frac{1}{c^3} f_1^1 + \frac{1}{c^4} f_1^2 + \frac{1}{c^5} f_1^{2.5} + \frac{1}{c^6} f_1^3 \right), \]

with

\[ f_0 = \frac{2}{63} \delta_{ij} \delta_{jk} + \frac{16}{45} \epsilon_{ijk} \delta_{jk} \delta_{kc}, \]

\[ f_1 = \frac{4}{63} \delta_{ij} \delta_{jk} + \frac{1}{1134} \delta_{ij} \delta_{jk} \delta_{lk} + \frac{1}{126} \delta_{ij} \delta_{jk} \delta_{kl}, \]

\[ f_2 = \frac{2}{4} \delta_{jklm} \delta_{ijkl} + \frac{1}{59400} \delta_{ijkl} \delta_{jklm}, \]

\[ f_3 = \frac{1}{22275} \delta_{jklm} \delta_{ijkl} + \frac{1}{4343625} \delta_{ijklm} \delta_{ijklm} + \frac{1}{514600} \delta_{ijklm} \delta_{ijklm}, \]

where

1. \( I_{ij}, \) needed at 3PN, see Eqs. (3.1)-(3.2) of Ref. [64]; see also Eqs. (3.19)-(3.20) of Ref. [63];
2. \( I_{ijk}, \) needed at 3PN, see Eqs. (4.9)-(4.10) of Ref. [65] for the expression in standard harmonic coordinates;
3. \( I_{ijkl}, \) needed at 2PN, see Eq. (3.23a) of Ref. [63];
4. \( I_{ijklm}, \) needed at 1PN, see Eq. (3.23b) of Ref. [65].
5. $I_{ijklmn}$, needed at N, see Eq. (3.23c) of Ref. 63;
6. $J_{ij}$, needed at 3PN, see Eqs. (3.6)-(3.7) of Ref. 64 for the expression in standard harmonic coordinates;
7. $J_{ijk}$, needed at 2PN, see Eq. (3.26a) of Ref. 63;
8. $J_{ijkl}$, needed at 1PN, see Eq. (3.26b) of Ref. 63;
9. $J_{ijklm}$, needed at N, see Eq. (3.26c) of Ref. 63.

The final 3PN instantaneous term for a generic orbit reads

$$F_{P_i}^{\text{inst}1J3PN} = \frac{G^3M^3\nu^2}{r_4^{1/7}}(m_2 - m_1)\eta^4(A^{3PN}I^n_i + B^{3PN}v^i),$$

(9.3)

with

$$A^{3PN} = \left(\frac{50647}{4095} + \frac{6891347}{45045} \nu + \frac{378098}{715} \nu^2 - \frac{17700712}{45045} \nu^3\right) v^8$$

$$+ \left[\left(\frac{1486192}{50150} - \frac{38702087}{45045} \nu - \frac{124611538}{45045} \nu^2 + \frac{7420632}{5005} \nu^3\right) v^2 + \frac{5742794}{345} \nu + \frac{10668793}{9009} \nu^2 + \frac{3851017}{6435} \nu^3\right] \frac{GM}{r} v^6$$

$$+ \left[\frac{2039066}{5005} - \frac{9529066}{45045} \nu + \frac{18801898}{4095} \nu^2 - \frac{9617408}{5005} \nu^3\right] v^4$$

$$+ \frac{925151368}{945945} \nu + \frac{682213787}{135135} \nu^2 - \frac{865924949}{135135} \nu^3 - \frac{958050097}{135135} \nu^4 \right] \frac{GMr^2}{r} v^4$$

$$+ \left[\frac{173961042956}{165540375} - \frac{486464}{3675} \ln \left(\frac{r}{r_0}\right) + \frac{1845}{28} r^2 - \frac{312503921}{675675} \nu + \frac{33465314}{27027} \nu^2 - \frac{16820996}{45045} \nu^3\right] \frac{G^2M^2}{r^2} v^2$$

$$+ \left[\frac{683368}{1287} - \frac{20252888}{9009} \nu - \frac{27439754}{9009} \nu^2 + \frac{9326388}{9009} \nu^3\right] v^6$$

$$+ \left[\frac{836163341}{10010} + \frac{88323799}{3003} \nu - \frac{21042751}{2310} \nu^2 + \frac{103758968}{45045} \nu^3\right] \frac{G^5r^4}{r} v^4$$

$$+ \left[\frac{742542259516}{165540375} + \frac{1647104}{3675} \ln \left(\frac{r}{r_0}\right) + \frac{6642078}{25025} + \frac{3075}{14} \pi^2 \nu - \frac{50439274}{10395} \nu^2 + \frac{13610134}{12285} \nu^3\right] \frac{G^2M^2r^2}{r^2} v^2$$

$$+ \left[\frac{69490090246}{55180125} + \frac{60092}{3675} \ln \left(\frac{r}{r_0}\right) + \frac{56867}{840} r^2 - \frac{1668895}{2457} \nu - \frac{993904}{2457} \nu^2 + \frac{891622}{731} \nu^3\right] \frac{G^3M^3}{r^3} v^2$$

$$+ \left[\frac{58349}{273} + \frac{2486416}{3003} \nu + \frac{2067616}{1001} \nu^2 - \frac{195680}{1001} \nu^3\right] \frac{G^4M^4}{r^4} v^2$$

$$+ \left[\frac{1228477436}{157675} - \frac{408167713}{90090} \nu + \frac{69532495}{18018} \nu^2 - \frac{1022414}{1287} \nu^3\right] \frac{G^5M^5}{r^5} v^2$$

$$+ \left[\frac{111525644752}{33108075} - \frac{42496}{147} \ln \left(\frac{r}{r_0}\right) + \frac{615}{4} r^2 - \frac{67591807}{27027} \nu + \frac{153157904}{45045} \nu^2 - \frac{4165558}{6435} \nu^3\right] \frac{G^2M^2r^4}{r^2} v^2$$

$$+ \left[\frac{4786484054}{5016375} + \frac{174592}{3675} \ln \left(\frac{r}{r_0}\right) - \frac{20869}{280} r^2 + \frac{1498465697}{4729725} \nu + \frac{15571532}{27027} \nu^2 - \frac{9925382}{135135} \nu^3\right] \frac{G^3M^3}{r^3} v^2$$

$$+ \left[\frac{21844644124}{49662125} + \frac{11904}{1225} \ln \left(\frac{r}{r_0}\right) - \frac{41}{70} r^2 + \frac{2315974202}{4729725} \nu + \frac{179768}{3003} \nu^2 - \frac{40396}{6237} \nu^3\right] \frac{G^4M^4}{r^4} v^2$$

(9.4)
\[ P_{3\text{PN}} = \left( -\frac{438226}{15015} + \frac{40657}{231}\nu^2 + \frac{1010414}{3003}\nu^3 + \frac{3600536}{15015}\nu^4 \right) \nu^8 \]
\[ + \left[ \frac{9523744}{45045} - \frac{395467}{315}\nu^2 + \frac{7045306}{3465}\nu^3 - \frac{3058024}{3465}\nu^4 \right] \nu^2 \]
\[ + \left( -\frac{143914678}{1576575} - \frac{3249853}{6435}\nu^2 + \frac{1744231}{3003}\nu^3 - \frac{2536901}{9009}\nu^4 \frac{GM}{r} \right) \nu^6 \]
\[ + \left[ \frac{1756652}{3003} + \frac{5370304}{5005}\nu^2 - \frac{15979258}{45045}\nu^3 + \frac{19699546}{6435}\nu^4 \right] \nu^4 \]
\[ + \left( -\frac{230962616}{315315} + \frac{2612939}{715}\nu^2 - \frac{24270509}{6435}\nu^3 + \frac{44195497}{45045}\nu^4 \frac{GM\nu^2}{r} \right) \nu^6 \]
\[ + \left( -\frac{429821166328}{496621125} + \frac{337504}{3675}\ln \left( \frac{r}{r_0} \right) + \frac{6510}{280}\pi^2 + \frac{80267816}{405405}\nu - \frac{1917296}{4095}\nu^2 + \frac{43333016}{405405}\nu^3 \right) \frac{G^2M^2}{r^2} \nu^4 \]
\[ + \left[ \frac{3974752}{6435} - \frac{27078616}{9009}\nu + \frac{1667210}{693}\nu^2 - \frac{2625664}{5005}\nu^3 \right] \frac{GM\nu^4}{r} \nu^6 \]
\[ + \left( \frac{46881654}{35035} - \frac{124322753}{18018}\nu^2 + \frac{115944133}{20790}\nu^3 - \frac{6226289}{6435}\nu^4 \right) \frac{GM\nu^4}{r} \nu^6 \]
\[ + \left( \frac{205470694976}{55180125} - \frac{1286752}{3675}\ln \left( \frac{r}{r_0} \right) + \frac{47299991}{45045} - \frac{3813}{35}\pi^2 \nu + \frac{8684332}{3861}\nu^2 - \frac{7223842}{19305}\nu^3 \right) \frac{G^2M^2\nu^2}{r^2} \nu^6 \]
\[ + \left( \frac{25042228006}{70945875} + \frac{704}{175}\ln \left( \frac{r}{r_0} \right) + \frac{21607}{840}\pi^2 + \frac{783374999}{1289925}\nu + \frac{411716}{10395}\nu^2 - \frac{8327414}{405405}\nu^3 \right) \frac{G^3M^3}{r^3} \nu^2 \]
\[ + \left( -\frac{1974958}{9009} + \frac{440666}{429}\nu - \frac{4954028}{9009}\nu^2 + \frac{767248}{9009}\nu^3 \right) \nu^8 \]
\[ + \left( -\frac{219642376}{315315} + \frac{47166701}{12870}\nu - \frac{3852731}{1638}\nu^2 + \frac{629897}{2145}\nu^3 \right) \frac{GM\nu^6}{r} \nu^6 \]
\[ + \left( -\frac{875401568}{315315} + \frac{1600}{7}\ln \left( \frac{r}{r_0} \right) + \frac{24149}{280}\pi^2 + \frac{58379561}{45045}\nu - \frac{31792976}{19305}\nu^2 + \frac{23299432}{135135}\nu^3 \right) \frac{G^2M^2\nu^2}{r^2} \nu^6 \]
\[ + \left( -\frac{1458600746}{165540375} + \frac{716608}{11025}\ln \left( \frac{r}{r_0} \right) + \frac{9389}{280}\pi^2 - \frac{517735453}{14189175}\nu - \frac{34654208}{135135}\nu^2 + \frac{596782}{81081}\nu^3 \right) \frac{G^3M^3\nu^2}{r^3} \nu^2 \]
\[ + \left( \frac{5316518908}{45147375} - \frac{44032}{11025}\ln \left( \frac{r}{r_0} \right) + \frac{41}{14}\pi^2 + \frac{5234464748}{14189175}\nu - \frac{140216}{12285}\nu^2 + \frac{1524088}{405405}\nu^3 \right) \frac{G^4M^4}{r^4}. \tag{9.5} \]

The integration along hyperbolic-like orbits (see Appendix D) can be carried on exactly and the sought for 3PN contribution reads

\[ P_{x\text{rad inst, } I,J, 3\text{PN}} = 0, \]
\[ P_{y\text{rad inst, } I,J, 3\text{PN}} = (Mc)^m_2 - m_1 \frac{\nu^2\nu^6}{c^2} \frac{1}{c^2_{1,2}(c^2_{1,2} - 1)} \left[ \frac{Q^4_{y^6} + Q^4_{y^6} A + Q^4_{y^6} A^2 + Q^A_{y^6} A^3}{(Mc)^m_2} \right]^2. \tag{9.6} \]
with \( A = \arccos(-1/c_r) \) and

\[
Q_{z^3}^a = e_r^8 \left( \frac{37\nu}{20} - \frac{375}{112} \right) + e_r^9 \left( \frac{72427}{420} - \frac{7\nu}{2} \right) + e_r^{10} \left( \frac{1104\nu}{5} + 482 \right) + e_r^{11} \left( \frac{804\nu}{5} + 8798 \right) + \frac{32}{5},
\]

\[
Q_{q^2}^a = \sqrt{e_r^2 - 1} \left[ e_r^6 \left( \frac{315117}{2800} - \frac{849\nu}{20} \right) + e_r^{10} \left( \frac{1831\nu}{5} + 6287443 \right) + e_r^{12} \left( \frac{3861\nu}{5} + 61543 \right) + \frac{164\nu}{5} + 15473 \right],
\]

\[
Q_{q^4}^a = e_r^2 \left( \frac{41053\nu^3}{1225} + \frac{3180022\nu^6}{2205} + \frac{4558096\nu^9}{735} - \frac{19581152\nu^{12}}{3675} - \frac{9002752}{11025} \right) \ln \left( \frac{2e_r^2}{e_r^2 - 1} \right) - \frac{32\nu}{e_r^2},
\]

where

\[
\text{Cl}_2(x) = \frac{i}{2} \left[ \text{Li}_2(e^{-ix}) - \text{Li}_2(e^{ix}) \right],
\]

is the Clausen function of order 2.

As expected, these terms involve the arbitrary length scale \( r_0 \) (entering the retarded time as well as the relation connecting harmonic to modified harmonic coordinates) which disappears in the complete expression when all 3PN hereditary terms are included. Indeed, this is exactly the case when using the results of Ref. [40] for the higher-order tail contributions. We list below the final large-\( j \) expansion (including terms from \( 1/j^3 \) up to terms \( 1/j^7 \)) of both \( P_z \) and \( P_y \).
\[ P_{x}^{3\text{PN}} = -\nu^{2}(m_{2} - m_{1}) \left[ \frac{196096\nu_{\infty}^{2}}{945j^{3}} + \frac{20719\pi^{3}\nu_{\infty}^{8}}{320j^{6}} + \frac{42739712\pi^{2}}{53125j^{7}} + \frac{9226496\nu_{\infty}^{7}}{3725j^{8}} + O \left( \nu_{\infty}^{6}j^{8} \right) \right], \]
\[ P_{y}^{3\text{PN}} = \nu^{2}(m_{2} - m_{1}) \left\{ \pi \left[ \frac{1531643}{1182720} - \frac{25781}{10080} \nu - \frac{197}{560} \nu^{2} - \frac{74}{15} \nu^{3} \right] \frac{p_{11}}{j^{3}} + \left( \frac{1211876}{72765} - \frac{118676}{6615} \nu - \frac{76}{15} \nu^{2} - 140\nu^{3} \right) \frac{p_{10}}{j^{3}} + \pi \left[ \frac{37806320227}{790272000} + \frac{503\nu^{2}}{70} + \frac{41053}{2450} \nu_{\infty} + \left( \frac{945563}{8064} - \frac{4059\nu}{1280} \right) \nu + \left( \frac{17617\nu}{840} - \frac{6199\nu^{3}}{30} \right) \frac{p_{9}}{j^{6}} \right] \right. \]
\[ \left. + \left[ \frac{39351925056}{191008125} + \frac{1042432\nu^{2}}{4725} - \frac{85434368\ln(2\nu_{\infty})}{165375} + \left( \frac{815056834}{297675} - \frac{8528\nu^{2}}{105} \right) \nu + \left( \frac{174074\nu^{2}}{225} - \frac{30422\nu^{3}}{15} \right) \frac{p_{8}}{j^{6}} \right] \right. \]
\[ \left. + \left[ \frac{1006741665001549}{312947712000} - \frac{907691\nu^{2}}{2688} - \frac{303491\zeta(3)}{224} - \frac{35125513\ln(2\nu_{\infty})}{44100} + \left( \frac{2124695071}{725760} - \frac{3017083\nu^{2}}{30720} \right) \nu \right] \right) \}
\[ + \left. \frac{1209467\nu^{2}}{960} - \frac{30181\nu^{3}}{20} \right\} \frac{p_{2}}{j^{3}} + O \left( \frac{p_{6}}{j^{8}} \right). \quad (9.9) \]

X. SUMMARY OF RESULTS FOR THE ENERGY, ANGULAR MOMENTUM AND LINEAR MOMENTUM LOSSES IN THE C.M. FRAME

For the convenience of the reader, let us summarize here the new results derived in this work concerning the losses of energy, angular momentum, and linear momentum (radiated as gravitational waves), as recorded in the (initial) c.m. frame. In this section we use the notation of our previous work [30] for parametrizing the PM expansions of the radiative losses by the coefficients of their power expansion in \( \frac{1}{j} \), namely

\[
\frac{E_{\text{rad}}}{M} = +\nu^{2} \sum_{n=4}^{\infty} \frac{E_{n}}{j^{n}}, \quad \frac{J_{\text{rad}}}{J_{\text{c.m.}}} = +\nu \sum_{n=2}^{\infty} \frac{J_{n}}{j^{n}}, \quad \frac{P_{x}^{\text{rad}}}{M} = +\frac{m_{2} - m_{1}}{M} \nu^{2} \sum_{n=4}^{\infty} \frac{P_{x,n}}{j^{n}}, \quad \frac{P_{y}^{\text{rad}}}{M} = +\frac{m_{2} - m_{1}}{M} \nu^{2} \sum_{n=3}^{\infty} \frac{P_{y,n}}{j^{n}}. \quad (10.1)
\]

Here the left-hand sides have been adimensionalized, and we pulled out some powers of \( \nu \) on the right-hand sides, to ensure that the expansion coefficients \( E_{n}, J_{n}, P_{x,n}, P_{y,n} \) are dimensionless, and that their LO PN contribution is \( \nu \)-independent. [We recall that \( J_{\text{c.m.}} = b\nu_{\infty}/h = GM^{2}v_{IJ} \).] Note that in Ref. [30] we focussed on the PM expansion of \( P_{y}^{\text{rad}} \), because \( P_{x}^{\text{rad}} \) was subdominant, and linked to time-asymmetric hereditary tail effects. See Eq. (H3) there, giving the LO contribution to \( P_{x}^{\text{rad}} \).

A. Energy loss in the c.m. frame

The radiated c.m. energy \( E_{\text{rad}} \) has been evaluated at the 2PN fractional accuracy in our previous work Ref. [30]. The corresponding \( \frac{1}{j} \)-expansion PM coefficients were given (up to \( \frac{1}{j^{3}} \)) in the first five lines of Table IX there. In the present work, we have computed the heretofore unexamined fractional 2.5PN instantaneous contribution due to the radiation-reaction correction to hyperbolic motion (incorrectly argued to vanish in [37]), and we have used the results of [36–38] when computing the fractional 3PN contribution in the form of a \( \frac{1}{j} \)-expansion (see Eqs. (6.5) and (6.6)). In order to confirm the value of the fractional 3PN contribution to the radiated energy, we have done an independent computation of the instantaneous, 3PN-level contribution. The technically most challenging part of the latter computation comes from inserting the 3PN-accurate hyperbolic motion in the 3PN-accurate quadrupole moment. Following Ref. [56], the computation uses a 3PN-level, hyperbolic version of the quasi-Keplerian representation of binary motion. In redoing the computation of the latter hyperbolic quasi-Keplerian representation, we found that there were several typos in the results displayed in Ref. [56]. For the convenience of the reader, we give the corresponding corrected results in Appendix D.

Our results are displayed in Table II. Many of the \( \nu \)-dependent terms can be directly checked by using the polynomiality rule satisfied by the coefficients \( E_{n} \), namely

\[
h^{n+1}E_{n} = P_{n}^{(\nu-2)/2}(\nu), \quad (10.2)
\]
where \( P_{n}^{(\nu-2)/2}(\nu) \) denotes a polynomial of order \( N \) in \( \nu \), having \( \gamma \)-dependent coefficients. This rule was pointed out in
Ref. [33] (see also Eq. (7.7) in Ref. [30]). We shall give below another simple proof of this polynomiality rule. Our results on the coefficients $E_n$ satisfy this polynomiality rule after adding all separate contributions. For instance, at the 4PM order ($n = 4$), if one would consider separately the 3PN contribution (§ term on the second line of Eq. (10.2) it would violate the polynomiality rule because of the terms $-(2\eta^6 p^2 + 16\eta^4 x^2)$. In fact, these terms precisely cancel rule-violating terms in $h^5E_4$, coming from lower PN contributions in $E_4$.

While writing up our results, a PN-exact computation of the $G^4$ energy coefficient $E_4$ was made public [23]. Our (fractionally 3PN accurate) PN-expanded result listed in Eq. (D27) of Ref. [30], and Table III here agrees (when expressed in terms of $E_4^\mu \equiv h^5E_4$) with the 3PN expansion of the curly bracket on the right-hand side of Eq. (8) in Ref. [23].

Let us also note that we have included in the Table the PN-acquired knowledge of the 3PM-level contribution $E_3$, though $E_3$ has been determined as an exact function of $p_\infty$ [3, 23]. It agrees with the corresponding term in [23], and thereby provides an additional check of our PN calculations.

B. Angular momentum loss in the c.m. frame

The fractionally 2PN-accurate expansion of the PM coefficients $J_n$ of the radiated c.m. angular momentum $J^{\text{rad}}$ can also be found in Table IX of Ref. [30], up to $n = 7$. In the present work we have raised their accuracy to the 3PN order, by computing the missing term in the instantaneous part of the radiated angular momentum at the 2.5PN level due to the radiation-reaction correction to hyperbolic motion, thereby completing partial results available in the literature for the various contributions through the 3PN order [30, 32]. The final result is given by Eqs. (4.2) and (4.3) as an expansion in inverse angular momentum. The post-2PN coefficients are listed in Table XI. The 2PM and 3PM coefficients $J_2$ and $J_3$ are known exactly (see Refs. [28] and [29], respectively), but are also shown for completeness.

Concerning the $\nu$-structure of the coefficients $J_n$, they satisfy the polynomiality rule [33]

$$h^n J_n + h^{n-1}\nu E_n = P_n^{\gamma_{(n-3)/2}}(\nu),$$

with $n \geq 3$, whereas $h^2J_2$ is independent of $\nu$.

C. Linear momentum loss in the c.m. frame

Table IX of Ref. [30] listed the PN expansion of the coefficients $J_n$ of the PM expansion of the $y$-component of the radiated linear momentum $J^{\text{rad}}$ in the c.m. frame, accurate to the 2PN fractional order. The corresponding post-2PN contributions up to the 3PN order are listed in Table III.

As pointed out in [30] (and as is further discussed below) the coefficients $P_{yn}$ must satisfy the polynomiality property

$$h^{n+1} P_{yn} = P_{n-3}(\nu)\,.$$  
(10.4)

Our results on the coefficients $P_{yn}$ satisfy this polynomiality rule after adding all separate contributions. For instance, at order $n = 4$ the term proportional to $+\eta^5 \frac{3\eta^2 \eta^2}{4\eta^2 \eta^2}$ in the fractionally 1PN tail term (8.7) would separately violate the rule (10.3), but is needed to cancel corresponding rule-violating terms in $h^5P_{y3}$.

We recall that $P_{y3}$ is exactly known in PM sense, being related to $E_3$ by

$$P_{y3} = \frac{\sqrt{\gamma - 1}}{\gamma + 1} E_3.$$  
(10.5)

The PN expansion of the coefficients $P_{xn}$ are instead listed in Table III. These expansions include the leading-order (past-tail) contribution computed in [30], and complete them by two further terms in the PN expansion (fractionally 2.5PN and 3PN).

The coefficients $P_{xn}$ satisfy (see below) the polynomiality property

$$h^n P_{xn} = P_{n-4}(\nu).$$  
(10.6)

Our results on the coefficients $P_{xn}$ were found to satisfy this polynomiality rule after adding all separate contributions, and notably the one linked to radiation-reaction modifications of the orbital motion. E.g., at order $G^4$ the term proportional to $-\eta^4 \frac{3\eta^2 \eta^2}{4\eta^2 \eta^2}$ in the fractionally 1PN tail term (8.7) would separately violate the rule (10.6), but is needed to cancel corresponding rule-violating terms in $h^4P_{x4}$, while, at order $G^5$, the term $-(Mc^2)^{2\eta^2 \eta^2} \nu^{2\eta^1 \eta^1} \eta^4 \frac{3\eta^2 \eta^2}{4\eta^2 \eta^2}$ in $\delta^{\text{inst}} \text{rad}$, Eq. (4.14), is non-polynomial by itself, but corrects the non-polynomiality of other contributions.

XI. LORENTZ-INVARIENT FORM FACTORS FOR $P^{\mu}_{\text{rad}}$, AND MASS-POLYNOMIALITY RULES

In the sections above, we have discussed the values of the losses of energy, angular momentum and linear momentum in the c.m. frame. This was motivated by the fact that the multipolar-post-Minkowskian approach [50, 52] to gravitational radiation is conveniently applied within the c.m. frame of the binary system. Let us now re-express these c.m.-based, and PN-expanded, results in a Lorentz-invariant way.

As was pointed out in previous works (e.g. [30, 45]), if one expresses the individual momentum changes (or impulses), $\Delta \mathbf{p}^\mu_1$, $\Delta \mathbf{p}^\mu_2$, during gravitational scattering, and therefore also the radiated 4-momentum $\mathbf{P}^{\mu}_{\text{rad}} = -(\Delta \mathbf{p}^\mu_1 + \Delta \mathbf{p}^\mu_2)$, in terms of the incoming 4-velocities, $u_1^\mu$, $u_2^\mu$, and
TABLE II: New terms at the 2.5PN and 3PN level of fractional accuracy improving the PN expansion given in Table IX of Ref. [21] of the coefficients $E_n$, $J_n,$ and $P_n$, entering the PM expansion (10.1) of the radiated energy, angular momentum, and $y$-component of the linear momentum, respectively.

\[
\begin{align*}
E_2^{2\text{PN}} &= \left[ \left( \frac{376273}{2948} + \frac{6675}{604} \right) p_{\infty}^8 + \left( \frac{2911}{240} - \frac{145}{24} \right) p_{\infty}^3 \right] + O(p_{\infty}^{10}) \\
E_3^{2\text{PN}} &= \left[ \left( \frac{1216}{15} \right) \nu p_{\infty}^4 + \left( \frac{15184}{1120} \right) \nu^2 - \frac{266}{9} \nu^3 \right] p_{\infty} + O(p_{\infty}^{10}) \\
E_2^{3\text{PN}} &= \left[ \left( \frac{20042}{12} \right) \nu p_{\infty}^3 \right] + O(p_{\infty}^{10}) \\
E_3^{2\text{PN}} &= \left[ \left( \frac{295376}{1043} \right) \nu^2 + \left( \frac{67840}{200} \right) \nu^3 \right] p_{\infty} + O(p_{\infty}^{10}) \\
E_2^{3\text{PN}} &= \left[ \left( \frac{7576}{2432} \right) \nu \right] + O(p_{\infty}^{10}) \\
E_3^{3\text{PN}} &= \left[ \left( \frac{3512513}{4494} \right) \nu^3 \right] + O(p_{\infty}^{10}) \\
J_2^{2\text{PN}} &= \left[ \left( \frac{1733}{4} \right) \nu^2 \right] + O(p_{\infty}^{10}) \\
J_3^{2\text{PN}} &= \left[ \left( \frac{1537844}{25} \right) \nu^3 \right] + O(p_{\infty}^{10}) \\
J_2^{3\text{PN}} &= \left[ \left( \frac{365992}{7007} \right) \nu^2 + \left( \frac{102619}{2448} \right) \nu^3 \right] + O(p_{\infty}^{10}) \\
J_3^{3\text{PN}} &= \left[ \left( \frac{23614}{p_{\infty}} \right) \nu \right] + O(p_{\infty}^{10}) \\
\end{align*}
\]

TABLE III: PN-expansion of the coefficients $P_{2n}$ of the x-component of the radiated linear momentum through the 3PN fractional accuracy.

\[
\begin{align*}
P_4^{2\text{PN}} &= \left[ \left( \frac{1494}{9} \right) \nu \right] + O(p_{\infty}^{10}) \\
P_5^{2\text{PN}} &= \left[ \left( \frac{20608}{9} \right) \nu + \left( \frac{11604}{145} \right) \nu^2 \right] + O(p_{\infty}^{10}) \\
P_6^{2\text{PN}} &= \left[ \left( \frac{26788}{360} \right) \nu^2 + \left( \frac{26788}{240} \right) \nu^3 \right] + O(p_{\infty}^{10}) \\
P_7^{2\text{PN}} &= \left[ \left( \frac{64576}{16} \right) \nu + \left( \frac{308976}{9876} \right) \nu^2 \right] + O(p_{\infty}^{10}) \\
\end{align*}
\]
of the vectorial impact parameter $b_{12}^\mu \equiv b_{1}^\mu - b_{2}^\mu$, their expansion coefficients in powers of $G$ must be polynomials in the two masses $m_1, m_2$. Let us show here what information we can thereby get from such mass-polynomiality.

We can decompose $P_{\text{rad}}^\mu$ as follows

$$P_{\text{rad}}^\mu = P_{\text{rad}}^{1+2}(m_1, m_2, \gamma, b)(u_{1-} + u_{2-}^\mu) + P_{\text{rad}}^{1-2}(m_1, m_2, \gamma, b)(u_{1-} - u_{2-}^\mu) + P_{b_{12}}(m_1, m_2, \gamma, b)\hat{b}_{12}^\mu. \quad (11.1)$$

Taking into account the symmetry of $P_{\text{rad}}^\mu$ under the 1 ↔ 2 exchange, and the (anti-)symmetry of $u_{1-} + u_{2-}^\mu - (u_{1-} - u_{2-}^\mu)$, we see that the first form factor $P_{\text{rad}}^{1+2}(m_1, m_2, \gamma, b)$ must be 1 ↔ 2-symmetric, while $P_{\text{rad}}^{1-2}(m_1, m_2, \gamma, b)$ and $P_{b_{12}}(m_1, m_2, \gamma, b)$ must be 1 ↔ 2-antisymmetric. We can then use the further facts that: (i) radiative losses of energy and linear momentum being quadratic in the retarded-time derivative of the waveform must contain a factor $(m_1 m_2)^2$; and (ii) $P_{\text{rad}}^{1+2}(m_1, m_2, \gamma, b)$ starts at order $G^3$, while $P_{\text{rad}}^{1-2}(m_1, m_2, \gamma, b)$ and $P_{b_{12}}(m_1, m_2, \gamma, b)$ start at order $G^4$. The mass-polynomiality of the PM expansion coefficients of $P_{\text{rad}}^\mu$ then allows us to see that we can write

$$P_{1+2}^{\text{rad}}(m_1, m_2, \gamma, b) = \frac{G^3}{b^3} m_2^2 m_2^2 S P_{1+2}^{\text{rad}}(m_1, m_2),$$

$$P_{1-2}^{\text{rad}}(m_1, m_2, \gamma, b) = \frac{G^4}{b^4} m_2^2 m_2^2 (m_2 - m_1) P_{1-2}^{\text{rad}},$$

$$P_{b_{12}}^{\text{rad}}(m_1, m_2, \gamma, b) = \frac{G^4}{b^4} m_2^2 m_2^2 (m_2 - m_1) P_{b_{12}}^{\text{rad}}, \quad (11.2)$$

where the dimensionless factors $\hat{P}_{1+2}^{\text{rad}}, \hat{P}_{1-2}^{\text{rad}}, \hat{P}_{b_{12}}^{\text{rad}}$ have PM expansions of the form

$$\hat{P}_{1+2}^{\text{rad}} = \sum_{n \geq 3} \frac{G^{n-3} M^{n-3}}{b^{n-3}} S P_{n-3}^{1+2}(m_1, m_2),$$

$$\hat{P}_{1-2}^{\text{rad}} = \sum_{n \geq 4} \frac{G^{n-4} M^{n-4}}{b^{n-4}} S P_{n-4}^{1-2}(m_1, m_2),$$

$$\hat{P}_{b_{12}}^{\text{rad}} = \sum_{n \geq 4} \frac{G^{n-4} M^{n-4}}{b^{n-4}} S P_{b_{12}}^{1+2}(m_1, m_2), \quad (11.3)$$

Here, $SP_{N}^{X}(m_1, m_2)$ denotes a symmetric polynomial of order $N$ in the two masses. By scaling out the total mass $M = m_1 + m_2$, each such polynomial can be rewritten as

$$SP_{N}^{X}(m_1, m_2) = M^{N} P_{\frac{X}{2}}^{1+2, G^{n}}(\gamma, \nu), \quad (11.4)$$

where $P_{\frac{X}{2}}^{1+2, G^{n}}(\gamma, \nu)$ is a polynomial in $\nu$ of order $\lfloor \frac{N}{2} \rfloor$ (the integer part of $\frac{N}{2}$), with $\gamma$-dependent coefficients. In order to keep track of the PM order $n$, we add a label $G^n$, and we also sometimes keep the notation $\lfloor \frac{N}{2} \rfloor$ with $N = n - 3$ or $N = n - 4$ (e.g., we write $\lfloor \frac{1}{2} \rfloor$ instead replacing it by its numerical value $0$).

We thereby see that while at order $G^3$ (3PM order), $P_{\text{rad}}^\mu$ was described by only one function of $\gamma$, namely (see Eq. (14))

$$SP_{0}^{1+2}(m_1, m_2) = p_{\frac{+}{2}}^{1+2, G^{3}}(\gamma) = \frac{\mathcal{E}(\gamma)}{\gamma + 1}, \quad (11.5)$$

it will involve three functions of $\gamma$ at order $G^4$, namely

$$P_{1+2, G^{4}} = \frac{G^4}{b^4} m_2^2 m_2^2 S P_{1+2}^{1+2}(m_1, m_2),$$

$$P_{1-2, G^{4}} = \frac{G^4}{b^4} m_2^2 m_2^2 (m_2 - m_1) S P_{1-2}^{1+2}(m_1, m_2),$$

$$P_{b_{12}, G^{4}} = \frac{G^4}{b^4} m_2^2 m_2^2 (m_2 - m_1) S P_{b_{12}}^{1+2}(m_1, m_2), \quad (11.6)$$

At order $G^5$, we have four functions of $\gamma$:

$$P_{1+2, G^{5}} = \frac{G^5}{b^5} m_2^2 m_2^2 S P_{1+2}^{1+2}(m_1, m_2),$$

$$P_{1-2, G^{5}} = \frac{G^5}{b^5} m_2^2 m_2^2 (m_2 - m_1) S P_{1-2}^{1+2}(m_1, m_2),$$

$$P_{b_{12}, G^{5}} = \frac{G^5}{b^5} m_2^2 m_2^2 (m_2 - m_1) S P_{b_{12}}^{1+2}(m_1, m_2), \quad (11.7)$$

where $P_{\frac{+}{2}}^{1+2, G^{5}}(\gamma, \nu)$, being linear in $\nu$, involves two independent functions of $\gamma$. At order $G^n$, $P_{\text{rad}}^\mu$ generally involves

$$N_{G^n}^{\text{rad}} = \left[ \frac{n - 1}{2} \right] + 2 \times \left[ \frac{n - 2}{2} \right], \quad (11.8)$$

functions of $\gamma$.

Let us now discuss how to relate the Lorentz-invariant building blocks $p_{\frac{+}{2}}^{1+2, G^{n}}(\gamma, \nu), p_{\frac{+}{2}}^{1-2, G^{n}}(\gamma, \nu), p_{b_{12}}^{1+2, G^{n}}(\gamma, \nu)$ parametrizing the PM expansion of $P_{\text{rad}}^\mu$ to our previous c.m.-frame, PN-expanded, results on $E_{\text{rad}}^\mu, \mathcal{F}_{\text{rad}}^\mu, \mathcal{P}_{\text{rad}}^\mu$. 
A first step in this direction consists in computing the projections of $P^\mu_{rad}$ on the three unit vectors $U^\mu$, $n^\mu$, and $\hat{b}_{12}$, where $U^\mu$ is the c.m. time axis, such that

$$MhU^\mu = m_1u^\mu_{1-} + m_2u^\mu_{2-}, \quad (11.9)$$

and where $n^\mu$ is the unit vector in the c.m.-frame direction of $u^\mu_{1-}$, such that

$$MhP \equiv n^\mu = (m_2 + \gamma m_1)u^\mu_{1-} - (m_1 + \gamma m_2)u^\mu_{2-}. \quad (11.10)$$

The definition of $E^{rad}$, namely $E^{rad} = -u^\mu \mu_{rad}$ then yields

$$MhE^{rad} = (m_1u^\mu_{1-} + m_2u^\mu_{2-})P^\mu_{rad}. \quad (11.11)$$

From the definition Eq. (11.1) of $e_x$ and $e_y$, we deduce that

$$P^\mu_{rad} \equiv n^\mu \mu_{rad} = \sin \frac{x_{cons}}{2} P^\mu_x + \cos \frac{x_{cons}}{2} P^\mu_y, \quad (11.12)$$

while

$$P^\mu_{rad} \equiv \hat{b}_{12} \mu_{rad} = \cos \frac{x_{cons}}{2} P^\mu_x - \sin \frac{x_{cons}}{2} P^\mu_y. \quad (11.13)$$

Inserting the parametrization (11.11) into these results then yields the following links between $E^{rad}$, $P^\mu_x$, $P^\mu_y$ (remembering the definitions (11.12), (11.13)) and the form factors of $P^\mu_{rad}$:

$$MhE^{rad} = M(\gamma + 1)P^\mu_{1+2} + (m_2 - m_1)(\gamma - 1)P^\mu_{1-2}, \quad (11.14)$$

$$MhP^\mu_{rad} = (m_2 - m_1)p^{\infty}_x P^\mu_{1+2} + M p^{\infty}_x P^\mu_{1-2}, \quad (11.15)$$

$$P^\mu_{rad} = \hat{b}_{12} P^\mu_{rad}. \quad (11.16)$$

These simple links can be easily inverted to express $P^\mu_{1+2}$ and $P^\mu_{1-2}$ as linear combinations of $hE^{rad}$ and $hP^\mu_{rad}$, and we have used them to extract the values of $P^\mu_{1+2}$ and $P^\mu_{1-2}$. Before exhibiting our results, several remarks are in order.

Let us first note that while the mass-polynomiality of the form factor $P^\mu_{rad}$ immediately implies the mass-polynomiality of $P^\mu_{rad} \equiv \hat{b}_{12} P^\mu_{rad}$, the mass-polynomiality of the two other form factors, $P^\mu_{1+2}$ and $P^\mu_{1-2}$, implies the mass-polynomiality of the combinations $MhE^{rad}$ and $MhP^\mu_{rad}$. In these combinations it is crucial to include the factor $Mh = M \sqrt{1 + 2\mu(\gamma - 1)} = E^x_{\text{cm}}$. (including the extra mass factor $M$, which cannot be, generally, factored out on the right-hand sides).

In more detail, we have

$$MhE^{rad} = \frac{G^3}{b^4} m_1^2 m_2^2 M(\gamma + 1) \hat{P}^{rad}_{1+2} + \frac{G^4}{b^4} m_1^2 m_2^2 (m_2 - m_1)^2 (\gamma - 1) \hat{P}^{rad}_{1-2},$$

$$MhP^\mu_{rad} = \frac{G^3}{b^3} (m_2 - m_1) p^{\infty}_{\infty} \hat{P}^{rad}_{1+2} + \frac{G^4}{b^3} m_1^2 m_2^2 (m_2 - m_1) M p^{\infty}_{\infty} \hat{P}^{rad}_{1-2},$$

where we recall that the various dimensionless factors $P^\mu_{rad}$ have the more explicit structure

$$P^\mu_{rad} = \sum_{\nu \geq 0} \frac{G N^\nu}{b N^\nu} S\nu^\mu(m_1, m_2) = \sum_{\nu \geq 0} \left( \frac{GM}{b} \right)^N P^\mu_{\nu}(\nu). \quad (11.17)$$

These structures give a direct proof of the $\nu$-structures pointed out in our previous works, notably:

$$\left( \frac{hE^{rad}}{M} \right)^\nu = \left( \frac{GM}{b} \right)^n \nu^2 P^\nu_{[(n-2)/2]}(\nu),$$

and also

$$\left( \frac{hP^\mu_{rad}}{M} \right)^\nu = \left( \frac{GM}{b} \right)^n \nu^2 m_2 - m_1 M P^\nu_{[(n-3)/2]}(\nu).$$

Note also that, while in $MhE^{rad}$ the dimensionless form factor $\hat{P}^{rad}_{1+2}$ is multiplied by the small PN factor $\gamma - 1 = O(\nu^2_{\infty})$, in $MhP^\mu_{rad}$ the two form factors $\hat{P}^{rad}_{1+2}$ and $\hat{P}^{rad}_{1-2}$ contribute with the same PN weight (at any given order in $G$).

Inserting the mass-polynomiality structures of $P^\mu_{rad}$ and $MhP^\mu_{rad}$ in the expressions of $P^\mu_{1+2}$ and $P^\mu_{1-2}$, and using the mass-polynomiality of the magnitude of the conservative momentum transfer

$$Q = P_{\text{c.m.}} \sin \frac{x_{cons}}{2} = \frac{G m_1 m_2}{b} \left[ 2\gamma^2 - 1 \right] \gamma^2 - 1 + \frac{G}{b} S_1(m_1, m_2) + \frac{G^2}{b^2} S_2(m_1, m_2) + \cdots$$

(11.19)

which yields

$$\sin \frac{x_{cons}}{2} = \frac{Gh}{b} \left[ 2\gamma^2 - 1 \right] \gamma^2 - 1 + \frac{G}{b} S_1(m_1, m_2) + \frac{G^2}{b^2} S_2(m_1, m_2) + \cdots$$

(11.20)

one can easily derive the following mass-polynomiality structures

$$P^\mu_{rad} = \frac{G^4}{b^4} m_1^2 m_2^2 (m_2 - m_1) [S_0(m_1, m_2) + \frac{G}{b} S_1(m_1, m_2) + \cdots]$$

(11.21)

\[13\] Here we use the expansion in powers of $\hat{P}^{rad}_{\nu}$. When using the expansion in $\frac{1}{\nu}$, one must add an extra factor $h^{\nu}$ at order $\frac{1}{\nu}$, as used in Eq. (10.2).
and

\[ M \dot{P}_y^{\text{rad}} = \frac{G^3}{b^2} m_1^2 m_2^2 (m_2 - m_1) [SP_0(m_1, m_2) + \frac{G^2}{b^2} SP_2(m_1, m_2) + \ldots] . \]  

(11.22)

As above, each such mass-polynomiality structure leads, after scaling out the appropriate power of \( \frac{GM}{b^2} \), a polynomial structure in the symmetric mass ratio \( \nu \) (with \( \gamma \)-dependent coefficients, namely

\[ \frac{G^N}{b^N} SP_N^X(m_1, m_2) = \left( \frac{GM}{b} \right)^N P^X_N(\nu) . \]  

(11.23)

We have already mentioned above that our c.m.-based, and PN-based, results on \( P_0^{\text{rad}}, P_2^{\text{rad}} \) and \( P_4^{\text{rad}} \) were all in agreement (after adding all separate contributions, and notably the one linked to reaction-reaction modifications of the orbital motion) with the \( \nu \)-polynomial rules rederived here. We can therefore encapsulate the full, current PN-expanded information on \( P_\mu^{\text{rad}} \) in the values of the \( \gamma \)-dependent \( \nu \)-polynomials \( P_N^X(\gamma, \nu) \) parametrizing the form factors, see Eqs. (11.2)–(11.7).

At order \( G^3 \) our results yield

\[ P^{1,2,3,4}_N(\gamma) = \pi \left( \frac{37}{30} \mu_\infty + \frac{839}{1680} \beta_\infty + \frac{544}{2016} \eta_\infty \right) + \frac{1531643}{1187200} \mu_\infty + O(\mu_\infty^4) \), \]  

(11.24)

which agrees with the fractionally 3PN-level expansion of the exact result

\[ P^{1,2,3,4}_N(\gamma) = \pi \hat{E}(\gamma) + O(\mu_\infty^4) \]. \]  

(11.25)

At order \( G^4 \) we find

\[ P^{1,2,3,4}_N(\gamma) = \frac{784}{45 \mu_\infty} + \frac{2168}{175} \beta_\infty + \frac{1568}{45} \eta_\infty + \frac{98666}{11025} \mu_\infty^3 \]

\[ = \frac{352}{105} \beta_\infty^3 - \frac{2702747}{363825} \beta_\infty + O(\beta_\infty^4) \],  

(11.27)

While writing up our results, a PN-exact computation of the 4PM contribution to \( P_\mu^{\text{rad}} \) and notably, its \( b_0^2 \) projection, appeared on arXiv [23]. Our (fractionally 3PN accurate) result for the corresponding form factor \( P^{1,2,3,4}_N(\gamma) \) just listed agrees with the corresponding terms in Eq. (10) of [23].

Similarly at \( O(G^5) \) we have

\[ P^{1,2,3,4}_N(\gamma, \nu) = \pi \left[ \frac{61}{5 \beta_\infty^2} + \frac{34073}{1680} \beta_\infty + \frac{297}{40} \eta_\infty - \frac{33923}{2880} \mu_\infty + \left( \frac{31029}{2240} \pi_\infty^2 + \frac{1489497}{11200} \right) \mu_\infty \right] \]

\[ + \frac{99}{20} \pi_\infty^2 + \frac{3469506843}{620928000} \ln \left( \frac{\beta_\infty^2}{2} \right) \mu_\infty^3 \]

\[ + \nu \pi \left[ - \frac{55}{12 \beta_\infty^3} + \frac{6427}{10080} \beta_\infty^3 + \frac{877}{400} \eta_\infty^2 - \frac{939}{560} \beta_\infty^2 \right] \mu_\infty^2 \]

\[ + \frac{55}{12 \beta_\infty^3} \beta_\infty^3 + \frac{1491}{400} \eta_\infty^2 - \frac{939}{560} \beta_\infty^2 \mu_\infty^2 + \left( \frac{255491}{10080} \beta_\infty^3 + \frac{4059}{1280} \pi_\infty^2 \right) \beta_\infty^2 + O(\beta_\infty^4) \],  

(11.27)

XII. INFORMATION ON THE INDIVIDUAL IMPULSES \( \Delta p_\mu^\gamma \) DERivable FROM \( P^{\mu \gamma}_N \)

Let us now discuss what information on the individual momentum changes (or impulses), \( \Delta p_1^\mu, \Delta p_2^\mu \), can be extracted from our results on \( P^{\mu \gamma}_N \) by using four facts:

First, the coefficients of the PM expansion of \( \Delta p_1^\mu, \Delta p_2^\mu \) in terms of the incoming 4-velocities, \( u_1^\mu, u_2^\mu \) and of
the vectorial impact parameter $b^\mu \equiv b_1^\mu - b_2^\mu$, must be polynomials in the two masses $m_1, m_2$. More precisely, one has (for the first particle)

$$\Delta p_{1\mu} = -2Gm_1m_2 \frac{2\gamma^2 - 1}{\gamma^2 - 1} \frac{b_{12\mu}}{b^2} + \sum_{n \geq 2} \Delta p_{1\mu}^{nPOM},$$  \hspace{1cm} (12.1)$$

where each term $\Delta p_{1\mu}^{nPOM}$ is a combination of the three vectors $b_{12\mu}/h, u_{1\mu}^{\perp}$, and $u_{2\mu}^{\perp}$, with coefficients that are, at each order in $G$, homogeneous polynomials in $m_1$ and $m_2$, containing the product $m_1m_2$ as an overall factor. Symbolically

$$\Delta p_{1\mu}^{nPOM} \sim \frac{Gm_1m_2}{b^n} [(Gm_1)^{n-1}$$

$$+ (Gm_1)^{n-2}Gm_2 + \cdots + (Gm_2)^{n-1}],$$  \hspace{1cm} (12.2)$$

where each term is a combination of the three vectors $b^{\mu}/h, u_1^{\mu \perp}$, and $u_2^{\mu \perp}$, with coefficients that are functions of $\gamma$. [Note that contrary to the case of $p_{rad}^{\mu}$, $\Delta p_{1\mu}^{nPOM}$ is not symmetric under particle exchange.]

Second, linear momentum conservation implies that the radiated momentum is equal to

$$\Delta p_1^{\mu} + \Delta p_2^{\mu} = -P^{\mu}_{rad}. \hspace{1cm} (12.3)$$

Third, we have the decomposition

$$\Delta p_{a\mu} = \Delta p_{a\mu}^{\text{cons}} + \Delta p_{a\mu}^{\text{rr rel}} + \Delta p_{a\mu}^{\text{rr nonlin}}. \hspace{1cm} (12.4)$$

Here: (i) the conservative part $\Delta p_{a\mu}^{\text{cons}}$ is known up to the sixth PN order (modulo 6 still unknown parameters, $32 \ 33 \ 67$), while its $G$ expansion is known exactly up to order $G^4$ included $10 \ 23$; (ii) the linear-response contribution $\Delta p_{a\mu}^{\text{rr rel}}$ is known (modulo some linear, time-even radiation-reaction effects discussed below) from our previous work $30$; while (iii) the remainder term $\Delta p_{a\mu}^{\text{rr nonlin}}$ can be described as containing the contributions that are higher-order in radiation-reaction (starting with the quadratic order $O(F_{\nu \mu}^{r2})$).

And fourth, as we are going to show, the linear-response contribution happens to satisfy, by itself, the momentum conservation law $12.3$, namely

$$\Delta p_{1\mu}^{\text{rr rel}} + \Delta p_{2\mu}^{\text{rr rel}} = -P^{\mu}_{rad}. \hspace{1cm} (12.5)$$

The linear-response contribution $\Delta p_{a\mu}^{\text{rr rel}}$ was obtained in $30$ as the sum of two terms: a relative motion term $\Delta p_{a\mu}^{\text{rel}}$, and a recoil term $\Delta p_{a\mu}^{\text{rec}}$,

$$\Delta p_{a\mu}^{\text{rr rel}} = \Delta p_{a\mu}^{\text{rel}} + \Delta p_{a\mu}^{\text{rec}}. \hspace{1cm} (12.6)$$

From Eqs. (3.32) and (3.33) in $30$, we have

$$\Delta p_{a\mu}^{\text{rr rel}} = \chi^{rr rel} \frac{d}{d\chi_{\text{cons}}} \Delta p_{a\mu}^{\text{cons}} + \frac{\Delta P_\mu}{P_{\text{c.m.}}} p_{a\mu}^{\text{rad}},$$

$$- \frac{m_1^2 \Delta P_\mu}{E_1 P_{\text{c.m.}}} U_\mu, \hspace{1cm} (12.7)$$

and

$$\Delta p_{a\mu}^{\text{rec}} = \frac{E_a}{E_{\text{c.m.}}} p_{rad}^{\mu} - \frac{(p_{a\mu}^{\text{rad}})}{E_{\text{c.m.}}} U_\mu. \hspace{1cm} (12.8)$$

Here,

$$\Delta P_{\text{c.m.}} = - \frac{E_1 E_2}{E_{\text{c.m.}}} P_{\text{c.m.}} P_\mu^{\text{rad}}, \hspace{1cm} (12.9)$$

and the quantities $E_a, E_{\text{c.m.}} = E_1 + E_2 = M\hbar, P_{\text{c.m.}} = m_1 p_{a\mu}^{\text{rad}}/E_{\text{c.m.}}, p_{a\mu}^{\text{rad}}$ (outgoing momenta), $U_\mu \equiv (p_{1\mu} + p_{2\mu})/E_{\text{c.m.}}$ are all taken along the unperturbed conservative motion.

When summing over the particle label $a$, taking into account the fact that $\sum_a \Delta p_{a\mu}^{\text{cons}} = 0$ and $\sum_a p_{a\mu}^{\text{rad}} = \sum_a p_{a\mu}^{\text{rad}} = E_{\text{c.m.}} U_\mu$, one easily finds that Eq. (12.5) is (exactly) satisfied. This identity (together with the fact that $\sum_a \Delta p_{a\mu}^{\text{cons}} = 0$) implies the somewhat remarkable identity that the remainder (nonlinear) term in the linear-response formula $12.4$ must separately satisfy the identity

$$\Delta p_{1\mu}^{\text{rr nonlin}} + \Delta p_{2\mu}^{\text{rr nonlin}} = 0. \hspace{1cm} (12.10)$$

In other words, the remainder contribution $\Delta p_{a\mu}^{\text{rr nonlin}}$ must be antisymmetric under particle exchange.

At this point we need to complete one result derived in Ref. $30$, namely Eq. (3.25) there, giving the value of the radiation-reaction contribution, $\chi^{rr rel}$, to the relative scattering angle. Note first that the actual value of $\chi^{rr rel}$ did matter in the proof we have just given of the validity of Eq. (12.5). Indeed, after summing over $a$, the coefficient of $\chi^{rr rel}$ is

$$\frac{d}{d\chi_{\text{cons}}} \left[ \sum_a \Delta p_{a\mu}^{\text{cons}} \right], \hspace{1cm} (12.11)$$

which vanishes because $\sum_a \Delta p_{a\mu}^{\text{cons}}$ vanishes, independently of the value of $\chi_{\text{cons}}$.

The only place were the assumption of time-antisymmetry of the radiation reaction force was crucial in the derivation of the linear-response formula in Ref. $30$ was in the derivation of the value of $\chi^{rr rel}$ (leading to Eq. (3.25) there). Going back to the previous derivation of $\chi^{rr rel}$ in Ref. $29$, it was explained, around Eq. (5.98) there, that one could (when using Lagrange’s method of variation of constants) directly relate $\chi^{rr rel}$ to the radiative losses of (c.m.) energy and angular momentum if the time-derivatives of $dE(t)/dt$ and $d\Delta E(t)/dt$ were odd functions of time (around the moment of closest approach in the conservative motion). As $dE(t)/dt$ and $d\Delta E(t)/dt$ are linear expressions in the radiation-reaction force, their time-odd character is directly linked to the time-odd character of $F_{rr}$ (as was discussed at the end of section III above), while working with the LO, 2.5PN radiation-reaction force). As we were aware of this limitation in Ref. $30$, and limited our study of radiation-reaction effects to the 4.5PN level, because we had shown there (see Eq. (H3) there),
that, at the 5PN level there arose a non-zero value of $P_{x}^{\text{rad}}$ (while a time-odd $F_{rr}$ implies a vanishing value for $P_{x}^{\text{rad}}$).

One of the aims of the present paper is to go beyond the limitations of Ref. 30, and to discuss the physical effects present in $P_{\mu\nu}^{\text{rad}}$ and in $\Delta p_{\mu
u}$ that are related to time-asymmetric (rather than simply time-odd) radiative processes. Time-asymmetric effects in the equations of motion first enter at the 4PN level via tail-transported hereditary processes 48. However, at the 4PN level one can still uniquely decompose these contributions to the dynamics into a nonlocal-in-time conservative (time-symmetric) contribution, and a nonlocal-in-time dissipative (time-antisymmetric) one 49. This postpones the presence of genuinely time-asymmetric effects to the 5PN level. In addition, when considering the small mass-ratio limit (say $m_1 \ll m_2$, which implies $\nu \ll 1$), one can also uniquely decompose the dynamics in a conservative (time-symmetric) contribution, and a nonlocal-in-time dissipative (time-antisymmetric) one, if one works at the first-order self-force approximation, i.e. if one keeps only terms of order $m_1$ in the acceleration of particle 1, i.e. terms of order $m_1^2$ in the force acting on particle 1 23. The level where the separation time-even versus time-odd becomes ambiguous is the second-order self-force approximation, corresponding to terms of order $m_1^3$ in the force acting on particle 1. When scaling out the total mass, such terms contain a factor $\nu^3$.

When staying at the level of linear effects in $F_{rr}$, a re-examination of the proof of the linear-response formula in Ref. 30 shows that the only $O(F_{rr})$ modification to take into account is the presence of an extra contribution in Eq. (3.25) there. One gets an explicit expression for the latter extra contribution by using the varying-constant version of the quasi-Keplerian representation, Eqs. (3.5). From the equation parametrizing $\phi(t)$, and the link $\chi = [\phi]^+_{-\infty} - \pi + [c_{\phi}(t)]$, the variation of $\phi$, we get (using $c_1 = E$ and $c_2 = J$)

$$\chi = [W(l, E(t), J(t))] - \pi + [c_{\phi}(t)] .$$

(12.12)

The first term yields (when separating out the conservative contribution and linearly expanding in the radiative losses of energy and angular momentum) our usual linear-response formula for the radiative contribution to the c.m. relative scattering angle. The second contribution is new (and exists only when $F_{rr}(t)$ is time-asymmetric, rather than time-odd). This yields the result

$$\chi^{\text{rel}}_{rr} = -\left(1\frac{\partial \chi^{\text{cons}}_{rr}}{\partial E} F_{x}^{\text{rad}} + 1\frac{\partial \chi^{\text{cons}}_{rr}}{\partial J} J_{x}^{\text{rad}}\right) + \Delta c_{\phi} ,$$

(12.13)

where the first contribution has been evaluated at the $O(F_{rr})$ accuracy, and where a formal, but explicit, expression for the additional contribution $\Delta c_{\phi} = [c_{\phi}] = \int_{-\infty}^{+\infty} dt \frac{d\phi_{\text{rel}}(t)}{dt}$ is obtained from the last equation in Eqs.

3.5, and reads

$$\Delta c_{\phi} = \int_{-\infty}^{+\infty} dt \left[ \frac{\partial W}{\partial \phi} \left( \frac{\partial S}{\partial \phi} \right)^{-1} \left[ \frac{\partial S}{\partial E} \frac{dE}{dt} + \frac{\partial S}{\partial J} \frac{dJ}{dt} \right] - \frac{\partial W}{\partial E} \frac{dE}{dt} - \frac{\partial W}{\partial J} \frac{dJ}{dt} \right] .$$

(12.14)

Here $dE/dt$ and $dJ/dt$ are linear expressions in $F_{rr}(t)$, defined by the first two equations in Eqs. 3.5 (or, explicitly, Eqs. 3.6 in the Hamiltonian formalism).

We leave to future work the use of this result to directly estimate the additional term (starting at the 5PN level), $\Delta c_{\phi}$, in $\chi^{\text{rel}}_{rr}$, linked to time-asymmetric radiation-reaction effects. For the time being, let us rewrite the radiation-reaction contribution $\Delta p_{\mu\nu}^{\text{rad}}$ to the impulses in a slightly new form, namely

$$\Delta p_{\mu\nu}^{\text{rad}} = \Delta p_{\mu\nu}^{\text{rad,odd}} + \Delta p_{\mu\nu}^{\text{rad,nonrad}}$$

(12.15)

Here: $\Delta p_{\mu\nu}^{\text{rad,odd}}$ denotes the part of our linear-response formula obtained when assuming that $F_{rr}$ is time-odd (so that $\Delta c_{\phi} = 0$, and $p_{x}^{\text{rad}} = 0$);

$$\Delta p_{\mu\nu}^{\text{rad,nonrad}} = \Delta c_{\phi} \frac{d}{dx_{\text{cons}}^\mu} p_{\mu\nu}^{\text{cons}} ,$$

(12.16)

is the additional term linked to a non-zero $\Delta c_{\phi}$ (note that $\Delta c_{\phi}$ is symmetric under particle exchange), and

$$\Delta p_{\mu\nu}^{\text{rad}} = - \frac{E_{0}}{E_{\text{c.m.}}} p_{x}^{\text{rad}} c_{x}^{\mu} - \frac{(p_{x}^{\text{rad}})^{\mu}}{E_{\text{c.m.}}} U_{\mu} ,$$

(12.17)

is the additional term linked to a non-zero value of $p_{x}^{\text{rad}}$ deduced from our general linear-response formula. The last term, $\Delta p_{\mu\nu}^{\text{rad,nonrad}}$ is the same remainder term as in our previous decomposition (nonlinear in radiation-reaction and satisfying the antisymmetry constraint Eq. 12.10).

From our previous work, and from the considerations above, we know that both $\Delta c_{\phi}$ and $p_{x}^{\text{rad}}$ start at order $\frac{\nu}{m_1}$, i.e. at 4PM and 5PN. One useful source of information on the various contributions to $\Delta p_{\mu\nu}^{\text{rad}}$ is that they should combine to ensure the mass-polynomiality of $\Delta p_{\mu\nu}^{\text{rad}}$ [We assume here, consistently with previous works, that $\Delta p_{\mu\nu}^{\text{rad}}$ has been defined so as to be mass-polynomial.] We leave to future work a discussion of the constraints on the other projections of $\Delta p_{\mu\nu}^{\text{rad}}$. Here we shall focus on the 4PM-level value of the projection of $\Delta p_{1\mu}^{\text{rad}}$ on the vectorial impact parameter:

$$\Delta p_{1\mu}^{\text{rad}} \equiv \Delta p_{1\mu}^{\text{rad}} b_{1\mu}^{\text{rad}} .$$

(12.18)

By projecting Eq. 12.15 on $b_{1\mu}^{\text{rad}}$, and considering the 4PM-level term, we have

$$\Delta p_{1\mu}^{\text{rad}} = \Delta p_{1\mu}^{\text{rad,odd}} - \frac{E_{1}}{E_{\text{c.m.}}} p_{x}^{\text{rad}} G_{1} + \Delta p_{1\mu}^{\text{rad,nonrad}} + \Delta p_{1\mu}^{\text{rad,nonrad}} .$$

(12.19)
We have the following knowledge on the various terms in this result. The first term, $\Delta^{\text{rr lin-odd}}_{1bG^4}$ was derived in Eq. (7.16) of Ref. [30] as a function of $E^{G^4}$, $P^{\text{rad}}_{xG}$ and $J^{\text{rad}}_{G^4}$. At the time, only $E^{G^4}_{\gamma}$ [8, 23] and $J^{\text{rad}}_{G^4}$ [28] were known (in a PN-exact sense). Since then, the exact value of $J^{\text{rad}}_{G^4}$ has been obtained in Ref. [24]. This leads to the following exact value of $\Delta^{\text{rr lin-odd}}_{1bG^4}$:

$$
\Delta^{\text{rr lin-odd}}_{1bG^4} = \frac{G^4}{b^4}m_1^2m_2^2 \left( C_{bM}^{4\text{PM}}(\gamma)M + C_{bM1}^{4\text{PM}}(\gamma)m_1 \right),
$$

(12.20)
with coefficients (see Eq. (7.31) of Ref. [30], and Eq. (19) of Ref. [24])

$$
C_{bM}^{4\text{PM}}(\gamma) = \pi \tilde{E}^2(\gamma^2 - 5) - \frac{3}{4} \tilde{J}_2(5\gamma^2 - 1)
$$

$$
\tilde{J}_2 = \frac{(2\gamma^2 - 1)}{(\gamma^2 - 1)^{1/2}},
$$

(12.21)

$$
C_{bM1}^{4\text{PM}}(\gamma) = -\pi \tilde{E} \frac{2\gamma^2 - 1}{(\gamma + 1)^{1/2}}.
$$

Here, $\tilde{E} = E/\pi$, $\tilde{J}_2 = 2(2\gamma^2 - 1)(\gamma^2 - 1)^{1/2}I$ (with $I$ defined in [28]), and $\tilde{J}_2 = (\gamma^2 - 1)(C + 2D)$ (with $C$ and $D$ defined in [24]).

This contribution to $\Delta^{\text{rr}}_{1bG^4}$ is polynomial in the masses, therefore the three other terms on the right-hand side of Eq. (12.19) should also define a fifth-order polynomial in $m_1$ and $m_2$. This gives useful information on the terms $\Delta^{\text{rr}}_{1bG^4}$ and $\Delta^{\text{rr nonlin}}_{1bG^4}$ because the term $-E_1 = \frac{E_{\gamma}}{E_{\gamma_{\text{cm}}}}P^{\text{rad}}_{xG}$, see Eq. (11.21), is non-polynomial in the masses. This raises the following issue: given the non-polynomial term

$$
-\frac{E_1}{E_{\gamma_{\text{cm}}}}P^{\text{rad}}_{xG} = \frac{m_1(m_1 + \gamma m_2)}{M M^2 h^2} P^{\text{rad}}_{xG},
$$

(12.22)

with a mass-polynomial factor of the type (see Eq. (11.21))

$$
P^{\text{rad}}_{xG} = \frac{G^4}{b^4}m_1^2m_2^2(m_2 - m_1) p_x^{G^4}(\gamma),
$$

(12.23)

where $p_x^{G^4}(\gamma)$ depends only on $\gamma$, what type of extra contribution of the type

$$
\Delta^{\text{rr nonlin}}_{1bG^4} = \Delta^{\text{rr nonlin}}_{1bG^4} + \Delta^{\text{rr nonlin}}_{1bG^4}.
$$

(12.24)

can be added to it to guarantee that the sum becomes polynomial in the masses. The answer to this question depends on the information we have on the structure of the term $\Delta^{\text{rr nonlin}}_{1bG^4}$. We know two things about the latter term. First, it is symmetric under particle exchange (because $\Delta c_\phi$ is symmetric, while $\Delta^{\text{cons}}_{\phi G^4}$, $\Delta^{\text{nonlin}}_{\phi G^4}$ and $b^\mu_{12}$ are all antisymmetric under $1 \leftrightarrow 2$ exchange). Second, as explained above, when working in the gravitational self-force approach (with $m_1 \ll m_2$), the ambiguities in separating the force acting on $m_1$ in a time-even, conservative part, and a time-odd, radiation-reaction part, only arise at the second-order self-force approximation, i.e. for terms in the acceleration that are $O(m_1^2)$, or terms in the force acting on $m_1$ that are $O(m_1^3)$. When considering the $1 \leftrightarrow 2$-symmetric impulse $\Delta^{\text{rr nonlin}}_{1bG^4}$, this means that $\Delta^{\text{rr nonlin}}_{1bG^4}$ contains a factor $\nu^3$ (i.e. one extra $\nu$ factor than $P^{\text{rad}}_{xG}$, which contains a factor $m_1^2 m_2^2 = O(\nu^2)$).

The unique solution satisfying all these constraints is

$$
\Delta^{\text{rr nonlin}}_{1bG^4} = \frac{G^4}{b^4}m_1^2m_2^2(m_2 - m_1) p_x^{G^4}(\gamma) = \frac{G^4}{b^4}m_1^2m_2^2 \left( \frac{m_2 - m_1}{m_2 - m_1} \right) P^{\text{rad}}_{xG^4},
$$

(12.25)

leading to

$$
-\frac{E_1}{E_{\gamma_{\text{cm}}}}P^{\text{rad}}_{xG^4} + \Delta^{\text{rr nonlin}}_{1bG^4} = \frac{G^4}{b^4}m_1^2m_2^2 p_x^{G^4}(\gamma).
$$

(12.26)

However, the addition of such a term would correspond (after dividing by $m_1$) to an additional scattering angle for particle 1 proportional to the first power of $m_1$, i.e. at the first-order self-force level, we argued above that the physical effects should arise at the second-order self-force level and involve (for particle 1) a factor $m_1^3$.

In other words, the unique solution to our constraints yields

$$
\Delta^{\text{rr}}_{1bG^4} = \Delta^{\text{rr lin-odd}}_{1bG^4} + \frac{G^4}{b^4}m_1^3m_2^2 P^{G^4}_{x}(\gamma),
$$

(12.28)

where $\Delta^{\text{rr lin-odd}}_{1bG^4}$ is defined in Eq. (12.20), and where $p_x(\gamma)$ is defined by Eq. (11.21). In other words, the full, 4PM-level, transverse impulse of the first particle reads

$$
\Delta^{\text{rr}}_{1bG^4} = \Delta^{\text{cons}}_{1bG^4} + \Delta^{\text{rr nonlin}}_{1bG^4} + \frac{G^4}{b^4}m_1^3m_2^2 P^{G^4}_{x}(\gamma).
$$

(12.29)

The latter equation corresponds to Eq. (18) in Ref. [24], with the value $\frac{G^4}{b^4}m_1^3m_2^2 P^{G^4}_{x}(\gamma)$ for the (undefined) term denoted $\frac{G^4}{b^4}M^{G^4}_{\chih_{\text{even}}}$ there. Note that our reasoning has given a direct relation between this term and the value of $P^{\text{rad}}_{xG^4}$, namely

$$
\frac{G^4}{b^4}m_1^3m_2^2 P^{G^4}_{x}(\gamma) = \frac{m_1}{m_2 - m_1} P^{\text{rad}}_{xG^4}.
$$

(12.30)

At this stage, we only know the beginning of the PN
expansion of the function $\rho_x^G(\gamma)$, namely

$$
\rho_x^{G^4}(\gamma) = \frac{h^4 P_x}{(\gamma^2 - 1)^2} = \pi \left( \frac{-1491}{400} \rho_x^b + \frac{26757}{5600} \rho_x^b + O(\rho_x^b) \right). 
$$

When separating out the 4PM conservative contribution $\Delta p_{1b}^{G^4}$ from the $h_{12}^b$-projected impulse in our Eq. (12.29), the term $\Delta G_{\text{lin}}^\text{odd}$ (first derived in Eq. (7.16) of [25]) coincides with the term $c_2^{(4)\text{diss}}$ in Eq. (15) of [25], while the remaining term $\frac{e^4}{\mu} m_1 m_2 \rho_x^{G^4}(\gamma)$ has the same mass structure as the term $c_4^{(4)\text{diss}}$ in Eq. (15) of [25], and agrees (within our limited PN accuracy) with the PN-exact value given there.

Let us finally remark that, if one considers the solution Eq. (12.29), the mass-scaling of the term $\frac{e^4}{\mu} m_1 m_2 \rho_x^{G^4}(\gamma)$ makes it impossible to tame the high-energy behavior of $\Delta p_{1b}^{G^4}$.

When considering the high-energy (HE) limit $\gamma \to \infty$ for a fixed value of the scattering angle $\chi_1 \sim \frac{c r_{\text{lin}}^a}{\mu}$, with $E_{\text{cm.}} = M h \propto \gamma^\frac{3}{2}$, one would expect, in this limit, (suitably scaled) scattering observables to admit a finite limit. If the formal $G \to 0$ limit commuted with the HE limit, this would imply, in particular, that each term in the PM expansion of the impulse would admit a finite HE limit (at fixed $\chi_1 \sim \frac{c r_{\text{lin}}^a}{\mu}$). This is the case at orders $G^1$ and $G^2$. At the $G^3$ level, the conservative contribution $\Delta p_{1b}^{G^3} / P_{\text{cm.}}$ is logarithmically larger than its expected contribution $\sim \gamma^3$. However, it was found [27, 28] that this logarithmic divergence is tamed when completing the conservative impulse by the radiative correction $\Delta G_{\text{lin}}$. This raises the hope that a similar taming might occur at order $G^4$.

At order $G^4$ the ratio $\Delta p_{1b}^{G^4} / (P_{\text{cm.}} \chi_1^3)$ is power-law divergent, being proportional to $\gamma^\frac{3}{2}$. In terms of the un-rescaled impulse this divergence is $\Delta p_{1b}^{G^4} \propto \gamma^3$. Parametrizing the various contributions to the HE limit of the impulse according to

$$
\Delta p_{1b}^{G^4} \approx \frac{G_0^4 m_1^2 m_2}{b^4} \pi C \gamma^3, 
$$

the coefficient entering the conservative contribution $\Delta p_{1b}^{G^4}$ is

$$
C^{G^4} = -105 \frac{8}{27} (4 \ln(2) - 1 + 4 \ln(2)^2) (m_1 + m_2). 
$$

As pointed out in [24], the linear-response radiative contribution $\Delta p_{1b}^{G^4 \text{lin-odd}}$ is similarly $\propto \gamma^3$. However, the corresponding coefficient is

$$
C^{G^4 \text{lin-odd}} = \frac{35}{4} [m_1 (1 + 8 \ln(2)) + 2 m_2 (1 + 5 \ln(2))], 
$$

which has the correct sign, but not the correct value to cancel the “bad” high-energy behavior of the conservative contribution. If we assume that the function $\rho_x^{G^4}(\gamma)$ entering our additional contribution has a HE behavior of the type $\rho_x^{G^4}(\gamma) \approx \pi r_x \gamma^3$, it will contribute another term of order $\gamma^3$, with a coefficient

$$
C^{G^4 \text{lin-odd}} = c_x m_1. 
$$

It is, however, easy to see that, whatever the value of $c_x$, such an additional term (proportional only to $m_1$) cannot tame the contribution proportional to $m_2$, i.e. cannot yield a vanishing total coefficient $C^{G^4} = C^{G^4 \text{cons}} + C^{G^4 \text{lin-odd}} + C^{G^4 \text{lin-odd}}$. Indeed, the latter turns out to be

$$
C^{G^4 \text{lin-odd}} = \left( \frac{35}{2} \ln(2) - \frac{175}{8} \ln(2)^2 + c_x \right) m_1 + \left( \frac{35}{2} \ln(2) - \frac{105}{2} \ln(2)^2 \right) m_2. 
$$

In order to tame the HE behavior of $\Delta p_{1b}^{G^4}$, i.e. reduce it from $\gamma^3$ to, say, $\gamma^2$, or $\gamma^2 \ln \gamma$ [16] one would need to add a suitable extra contribution of the (disallowed) type (12.27).

We do not view the inability of our minimal additional term to tame the HE behavior of the $G$-expanded impulse as a blemish. It seems indeed probable that the $G \to 0$ limit does not commute with the HE limit $\gamma \to \infty$. This is notably indicated by the studies of the HE limit of the total gravitational-wave energy emitted during the collision of massless particles [68–70]. While the HE limit of the $O(G^3)$ leading-order radiative energy loss exceeds the energy $E_{\text{cm.}}$, available in the system by a factor $\propto \gamma^\frac{3}{2}$, the works [68–70] suggest that (due to coherence effects in the beamed radiation) the HE limit of radiative losses is finite, and of order $\chi_1^3 \ln \frac{1}{\chi_1}$.

To end our discussion of the radiative contributions to the impulse of the first particle $\Delta p_{1b}^{G^4}$, let us give the linear-response radiative contribution to the coefficients of its expansion on the basis $u_{1-}, u_{2-}, u_{12}$. Namely:

$$
\Delta p_{1b}^{G^4 \text{lin-odd} + P_x} = c_{1r}^{\text{lin-odd} + P_x} u_{1-}^b + c_{2r}^{\text{lin-odd} + P_x} u_{2-}^b + c_b^{\text{lin-odd} + P_x} b_{12}^b. 
$$

15 The recent result of Ref. [22] happens to lead to a coefficient $c_x$, which precisely annuls the coefficient of $m_1$ in $C^{G^4 \text{lin-odd}}$.

16 Such a reduction, would ensure the HE vanishing of the ratio $\Delta p_{1b}^{G^4} / (P_{\text{cm.}} \chi_1^3)$, as expected from the structure of the massless scattering discussed in Ref. [16].
Here the notation $\Delta p_1^{rr \text{ lin} - \text{odd} + P_x}$ denotes the sum of all the currently known radiative contributions to $\Delta p_1^{rr}$, namely, following the notation of Eq. (12.15), the sum of the first and third contributions to $\Delta p_1^{rr}$.

$$\Delta p_1^{rr \text{ lin} - \text{odd} + P_x} \equiv \Delta p_1^{rr \text{ lin} - \text{odd}} + \Delta p_1^{rr P_x}. \quad (12.38)$$

In terms of this notation, the full radiative result would be

$$\Delta p_{1 \mu}^{rr} = \Delta p_1^{rr \text{ lin} - \text{odd} + P_x} + \Delta p_1^{rr \Delta c_{\text{eo}}} + \Delta p_1^{rr \text{ nonlin}}. \quad (12.39)$$

As discussed above the missing contribution $\Delta p_1^{rr \Delta c_{\text{eo}}} + \Delta p_1^{rr \text{ nonlin}}$ is not currently known, but it is constrained by two properties: it is antisymmetric under particle exchange, and it should ensure that the full result $\Delta p_{1 \mu}^{rr}$ is polynomial in masses.

We have seen above that the latter constraints led to useful information when considering the 4PM-level contributions to the coefficient $c_b^{rr \text{ lin} - \text{odd} + P_x} = b_{12} \Delta p_1^{rr}$.

In Ref. [30], we consistently remained below the 5PN level (i.e., the 2.5PN fractional level) in computing the coefficients $c_1^{rr}$, $c_2^{rr}$, which means that we had not included the contribution from $P_x$, which starts at the 5PN (and 4PM) level. Our results were presented in Eqs. (8.6)–(8.11) there. Here, we improve the latter results by adding the fractional 2.5PN and 3PN contributions to the scattering coefficients of particle 1, at orders $O(G^4)$ and $O(G^5)$, when they are expanded in powers of $\hat{b}$. Contrary to Eqs. (8.6)–(8.11) in [30], where the scattering coefficients were presented in a rescaled version, we present them here in their non-rescaled form. At $O(G^4)$ we find
At $O(G^5)$ we find

$$c_{\mu \nu}^{1 \text{rr, lin-odd} + P_4 \text{PM}} = \frac{G^5 m_1^2 m_2^2}{b^5} \left\{ \begin{array}{l}
\frac{416}{45 p_\infty^2} (m_1 + m_2)^2 \\
+ \frac{16964}{1575} - \frac{47\pi^2}{5} m_1^2 + \frac{49988}{1575} - \frac{94\pi^2}{5} m_1 m_2 + \frac{203264}{1575} - \frac{47\pi^2}{5} m_2^2 \\
+ \frac{159232}{3675} - \frac{1243\pi^2}{56} m_1^2 + \frac{2283544}{11025} - \frac{1489\pi^2}{35} m_1 m_2 + \frac{116692}{1225} - \frac{5697\pi^2}{280} m_2^2 \\
+ \frac{896}{45 p_\infty} (m_1 + m_2)^2 - \frac{2368}{25} m_1^2 + \frac{279424}{525} m_1 m_2 + \frac{10304}{45} m_2^2 \\
+ \left[ -\frac{2294592}{36825} + \frac{241}{120} \pi^2 m_1^2 + \frac{6786972}{363825} + \frac{23783}{3360} \pi^2 m_1 m_2 + \left( \frac{9728}{275} + \frac{3407}{672} \pi^2 m_2^2 \right) p_\infty \right] \right\} + O(p_\infty^2),
\end{array} \right.$$
energy: see Eqs. (6.3)–(6.4) for the needed terms to complete the 3PN results of Ref. [37]; angular momentum: see Eqs. (7.2)–(7.3) for the needed terms to complete the 3PN results of Refs. [34, 37]; linear momentum: see Eqs. (4.13), (6.3), (6.4), and (9.9) for the needed terms to improve the results of Ref. [34] to the 3PN order; (ii) we clarified the validity of the linear-response computation of the radiative contribution to the individual impulses (see Sec. XII); (iii) we showed the compatibility of our c.m.-computed, PN-expanded results on the total radiated 4-momentum with the mass-polynomiality of its Lorentz-invariant form factors (see Eqs. (11.2)–(11.4)); and (iv) we proposed an explicit way of completing the linear-response formula so as to satisfy the mass-polynomiality of the transverse impulse at order $G^4/b^4$ (see Eq. (12.29)). Exploring other possibilities to complete the linear-response formula will be considered in future works, either by comparing our results with 4PM computations or by improving the PN accuracy of the linear momentum loss.

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Appendix A: Notation and useful formulas

We list below some useful formulas which one often needs to have at hand. The incoming c.m. Lorentz factor $\gamma = -u^- \cdot u^-_c$ and its associated momentumlike variable $p^-_c$ are related by

$$p^-_c \equiv \sqrt{\gamma^2 - 1}. \quad (A1)$$

The dimensionless angular momentum $j$ is related to the original c.m. angular momentum $J$ by

$$j \equiv \frac{cJ}{Gm_1m_2}. \quad (A2)$$

The vectorial impact parameter (orthogonal to $u^-_c$ and $u^-_2$) $b_{12} = b_1 - b_2 = b_0$ together with the conservative scattering angle $\chi_{cons}$ defines the entrance of the Cartesian-like basis vectors $e_x$ and $e_y$ as follows (see Eq. (3.49) of Ref. [30]):

$$e_x = \cos \frac{\chi_{cons}}{2} \hat{b} + \sin \frac{\chi_{cons}}{2} n^- ,$$

$$e_y = - \sin \frac{\chi_{cons}}{2} \hat{b} + \cos \frac{\chi_{cons}}{2} n^- . \quad (A3)$$

where

$$n^- = \frac{m_1m_2}{p^-_{cm}E^-_{cm}} (E^-_2 - \frac{E^-_1}{m_1} u^-_1 - \frac{E^-_2}{m_2} u^-_2) , \quad (A4)$$

and (see Eqs. (A4) and (A5) of Ref. [30])

$$P^-_{cm} = \frac{m_1m_2}{E^-_{cm}} \sqrt{\gamma^2 - 1} ,$$

$$E^-_{cm} = Mc^2h = Mc^2\sqrt{1 + 2\nu(\gamma - 1)} . \quad (A5)$$

An equivalent expression for $n^-$ is the following

$$n^- = \frac{(u^-_2 \wedge u^-_1)U^-}{\sqrt{\gamma^2 - 1}} , \quad (A6)$$

where the wedge product of two vectors $A$ and $B$ is standardly defined as

$$A \wedge B = A \otimes B - B \otimes A , \quad (A7)$$

so that the contraction with a third vector $C$ is given by $[A \wedge B]|^C = A(B \cdot C) - B(A \cdot C)$. Here we have used the decomposition $p^-_a = m_a u^-_a = E^-_a U^- + p^-_a$ (where $p^-_a$ is orthogonal to $U^-$, and $p^-_1 = -p^-_2 = p^-$), with

$$U^- = \frac{p^-_1 + p^-_2}{|p^-_1 + p^-_2|} = \frac{1}{E^-_{cm}} (m_1 u^-_1 + m_2 u^-_2) , \quad (A8)$$

and $E^-_{cm} = E^-_1 + E^-_2$.

In the present paper, to ease the notation, we often remove the “c.m.” label from both energy and linear momentum, namely

$$E^-_{c.m.} \rightarrow E^- , \quad P^-_{c.m.} \rightarrow p^- . \quad (A9)$$

Let us also recall the following expressions (see Eqs. (A9) of Ref. [30]) for the incoming c.m. energy of each particle

$$E^-_1 = \frac{m_1(m_2\gamma + m_2)}{E^-} , \quad E^-_2 = \frac{m_2(m_1\gamma + m_2)}{E^-} . \quad (A10)$$

as well as the relation between the dimensionless angular momentum and the impact parameter

$$\frac{1}{j} = \frac{GMh}{bp^-_c} = \frac{GE^-_{cm}}{bp^-_c} . \quad (A11)$$

When describing the conservative scattering it is useful to introduce the c.m. direction of the outgoing momenta, $n^+_c$, as well as its associated orthogonal direction $B_c$, namely

$$\hat{B}_c = \cos(\chi_{cons}) \hat{b} + \sin(\chi_{cons}) n^-_c , \quad n^+_c = - \sin(\chi_{cons}) \hat{b} + \cos(\chi_{cons}) n^-_c . \quad (A12)$$

In the text we used the relation

$$\hat{B}_c = \frac{d}{d\chi_{cons}} n^+_c \quad (A13)$$
The dyad \((\hat{\mathbf{B}}, \mathbf{n}^+_{\text{cons}})\) differs from the incoming dyad \((\hat{\mathbf{b}}, \mathbf{n}_-\)) by a rotation of angle \(\chi_{\text{cons}}\). The dyad \((\mathbf{e}_x, \mathbf{e}_y)\) is midway between the latter two dyads, being obtained from the incoming dyad by a rotation of angle \(\frac{1}{2} \chi_{\text{cons}}\).

The conservative scattering of the particle 1 corresponds then to the change \(p_1 \to p_1^{\text{cons}}\) of its linear momentum

\[
p_1^- = E_1 U + p \cdot \mathbf{n}_-, \quad p_1^{+\text{cons}} = E_1 U + p \cdot \mathbf{n}_+^{\text{cons}}, \quad (A14)
\]

such that

\[
\Delta p_1^{\text{cons}} = p_1^{+\text{cons}} - p_1^- = p - (\mathbf{n}_+^{\text{cons}} - \mathbf{n}_-).
\]

The following representation

\[
\Delta p_1^{\text{cons}} = c_b^{\text{cons}} \hat{\mathbf{b}} + c_{u_1}^{\text{cons}} u_1 + c_{u_2}^{\text{cons}} u_2,
\]

with

\[
c_b^{\text{cons}} = -p \sin \chi_{\text{cons}}, \quad c_{u_1}^{\text{cons}} = \frac{m_1 E_2}{E} (\cos \chi_{\text{cons}} - 1), \quad c_{u_2}^{\text{cons}} = \frac{m E_1}{E} (\cos \chi_{\text{cons}} - 1),
\]

is also used.

For particle 2 we have instead

\[
p_2^- = E_2 U - p \cdot \mathbf{n}_-, \quad p_2^{+\text{cons}} = E_2 U - p \cdot \mathbf{n}_+^{\text{cons}}, \quad (A18)
\]

with

\[
\Delta p_2^{\text{cons}} = p_2^{+\text{cons}} - p_2^- = -p - (\mathbf{n}_+^{\text{cons}} - \mathbf{n}_-).
\]

Therefore \(\Delta p_1^{\text{cons}} + \Delta p_2^{\text{cons}} = 0\). Note that

\[
\frac{d}{d \chi_{\text{cons}}} \Delta p_1^{\text{cons}} = -p \cdot \hat{\mathbf{B}}, \quad \frac{d}{d \chi_{\text{cons}}} \Delta p_2^{\text{cons}} = p \cdot \hat{\mathbf{B}}.
\]

**Appendix B: Relating hyperbolic-motion results to elliptic-motion ones by analytic continuation**

As a check on our computation, in Section III of the 2.5PN, radiation-reaction correction to the quasi-Keplerian parametrization of hyperbolic-like motions, we have (successfully) related it to the corresponding 2.5PN, radiation-reaction correction to the quasi-Keplerian parametrization of elliptic-like motions derived in Ref. [55] (by using the elliptic version of Lagrange’s method of variation of constants). As already mentioned in the text, this comparison used two different ingredients: (i) analytic continuation between elliptic and hyperbolic quasi-Keplerian parametrizations (at the Newtonian order); and (ii) the use of a different expression for the radiation-reaction force, because of a difference in coordinates (ADM versus harmonic).

Let us only mention a few technical steps of this comparison. The analytic continuation relating the elliptic eccentric anomaly, \(u\), to the hyperbolic one, \(v\) is simply \(u \to iv\). This has to be taken together with the replacement \(a_r \to -a_r\). Concerning the gauge dependence of the radiation-reaction force, let us recall that, in a general coordinate system, the 2.5PN-level radiation-reaction acceleration depends on two gauge parameters, \(\alpha\) and \(\beta\), and reads [74, 72]

\[
\mathbf{A}^{rr} = -\frac{8}{5} \frac{G^2 M^2}{c^5} \left[-A_{2.5PN} \dot{\mathbf{v}} + B_{2.5PN} \mathbf{v} \right], \quad (B1)
\]

where

\[
A_{2.5PN} = 3(1 + \beta) v^2 + \frac{1}{3} (23 + 6 \alpha - 9 \beta) \frac{GM}{r} - 5 \beta r^2,
\]

\[
B_{2.5PN} = (2 + \alpha) v^2 + (2 - \alpha) \frac{GM}{r} - 3(1 + \alpha) r^2. \quad (B2)
\]

For example, in harmonic coordinates \(\alpha = -1\) and \(\beta = 0\),

\[
A_{2.5PN, h} = 3v^2 + \frac{17 GM}{3 r},
\]

\[
B_{2.5PN, h} = v^2 + \frac{GM}{r}. \quad (B3)
\]

Other useful gauge-choices correspond to the Burke-Thorne reactive potential \((\alpha = 4, \beta = 5)\), and to ADM coordinates \((\alpha = 5, \beta = 3)\).

One can then easily derive the variation of constants in a general gauge. For example the \((\alpha, \beta)\)-dependent equation for \(\delta \tau^\alpha e_\tau\) reads

\[
\frac{d}{dt} \delta \tau^\alpha e_\tau = \frac{8r(1 - e_\tau^2)}{15a_\tau^4 e_\tau} \left\{ \frac{12 \alpha - 6 \beta + 15}{\chi^3} + \frac{-48 \alpha + 33 \beta - 65}{\chi^4} + \frac{21(e_\tau^2 - 3) \beta - 9(2\alpha + 3) e_\tau^2 + 60 \alpha + 109}{\chi^5} + \frac{24(e_\tau^2 - 1)(\alpha - \frac{15}{2} \beta + \frac{30}{22})}{\chi^6} - \frac{15 \beta (e_\tau^2 - 1)^2}{\chi^7} \right\}.
\]

In the ADM case this equation becomes

\[
\frac{d}{dt} \delta \tau^\alpha e_\tau = \frac{8r(1 - e_\tau^2)}{15a_\tau^4 e_\tau} \left\{ \frac{17}{\chi^3} - \frac{46}{\chi^2} + \frac{6e_\tau^2 + 20}{\chi^3} - \frac{54(e_\tau^2 - 1)}{\chi^6} - \frac{45(e_\tau^2 - 1)^2}{\chi^7} \right\}, \quad (B5)
\]

as in Eq. (56.b) of Ref. [55], while in the harmonic case we find

\[
\frac{d}{dt} \delta \tau^\alpha e_\tau = \frac{8r(1 - e_\tau^2)}{15a_\tau^4 e_\tau} \left\{ \frac{3}{\chi^3} - \frac{17}{\chi^2} + \frac{9e_\tau^2 - 49}{\chi^3} + \frac{35(e_\tau^2 - 1)}{\chi^6} \right\}. \quad (B6)
\]
Appendix C: Radiation-reaction contribution to the relative scattering angle up to the 4.5 PN accuracy

Ref. [29] (see Eq. (5.99) there) has shown that, to linear order in radiation-reaction, and under the assumption of a time-odd radiation-reaction force, the radiation-reaction contribution to the relative scattering angle (in the c.m. frame), $\chi_{\text{rr,rel}}$ can be computed through a linear-response formula involving the radiative losses of energy and angular momentum. We have generalized this linear-response formula above, see Eq. (12.13), by including the term $\Delta \phi$ that is non-zero when the radiation-reaction force contains a time-even piece. As discussed above, such a correction in $\chi_{\text{rr,rel}}$ starts at the 2.5PN level. In other words, the first two terms on the right-hand side of Eq. (12.13) suffice to evaluate $\chi_{\text{rr,rel}}$ up to the 4.5PN level, by using the known radiative losses at the 4.5PN accuracy (as the radiative losses start at the 2.5PN level, this corresponds to a fractional 2PN accuracy).

At the leading-order, 2.5PN level, we have given in the text a direct rederivation of the value of $\chi_{\text{rr,rel}}$, see Eq. (8.23). The explicit expression of $\chi_{\text{rr,rel}}^{2.5\text{PN}} = [\delta^N_{\text{rr}}]^{2.5\text{PN}}$ in terms of $a_r$ and $e_r$ reads

$$\chi_{\text{rr,rel}}^{2.5\text{PN}}(a_r, e_r) = \frac{2\nu}{15a_r^{3/2}(e_r^2 - 1)^{5/2}} \times \left[ \frac{72e_r^4 + 1069e_r^2 + 134}{3e_r^2} + \frac{121e_r^2 + 304}{\sqrt{e_r^2 - 1}} \arccos \left( -\frac{1}{e_r} \right) \right], \quad (C1)$$

which, when expressed in terms of the conserved energy and angular momentum, becomes

$$\chi_{\text{rr}}^{2.5\text{PN}}(p_\infty, j) = \frac{2\nu}{15j^5} \times \left[ \frac{72p_\infty^4 j^4 + 1213p_\infty^2 j^2 + 1275}{3(1 + p_\infty^2 j^2)} + \frac{121p_\infty^2 j^2 + 425}{p_\infty j} \mathcal{A}(p_\infty, j) \right], \quad (C2)$$

where

$$\mathcal{A}(p_\infty, j) = \arccos \left( -\frac{1}{\sqrt{1 + p_\infty^2 j^2}} \right). \quad (C3)$$

The large-$j$ expansion of the latter expression reproduces the leading PN order of the PM expansion of $\chi_{\text{rr}}$, the first terms of which (up to $O(G^7)$) are listed in Table XI of Ref. [30].

When going to higher PN levels in the radiative losses (still keeping below the absolute 5PN level) we must take into account that the radiative losses contain fractional corrections at the following levels: 1PN, 1.5PN and 2PN. The 1.5PN correction to the losses is the leading-order tail effect (which is still described by a time-odd radiation reaction). Let us first discuss the 1PN and 2PN fractional corrections, leading to contributions to $\chi_{\text{rr,rel}}$ at the 3.5PN and 4.5PN levels.

The expressions of $\chi_{\text{rr,rel}}$ at the $(n + \frac{1}{2})$-PN levels (for $n = 3, 4$) have the general structure

$$\chi_{\text{rr}}(p_\infty, j)^{(n+\frac{1}{2})\text{PN}} = A_n^{(\text{PN})}(p_\infty, j; \nu)A^2(p_\infty, j) + A_1^{(\text{PN})}(p_\infty, j; \nu)A(p_\infty, j) + A_0^{(\text{PN})}(p_\infty, j; \nu). \quad (C4)$$

Using the 2PN conservative scattering angle, Eq. (45) of Ref. [31],

$$\frac{\chi_{\text{cons}}}{2} = \frac{\chi_{\text{cons}}^{N}}{2} + \frac{\chi_{\text{cons}}^{1\text{PN}}}{2} \eta^2 + \frac{\chi_{\text{cons}}^{2\text{PN}}}{2} \eta^4 + O(\eta^6), \quad (C5)$$

where

$$\frac{\chi_{\text{cons}}^{N}}{2} = A(p_\infty, j) - \frac{\pi}{2}, \quad \frac{\chi_{\text{cons}}^{1\text{PN}}}{2} = \frac{3}{j^2} A(p_\infty, j) + \frac{p_\infty (3 + 2j^2 p_\infty^2)}{j(1 + j^2 p_\infty^2)},$$

$$\frac{\chi_{\text{cons}}^{2\text{PN}}}{2} = -\frac{p_\infty}{4j^4 (1 + j^2 p_\infty^2)^2} j^4 p_\infty^2 (-181 + 26\nu) + 2j^2 p_\infty^2 (-95 + 28\nu) + 30\nu - 105, \quad (C6)$$

and the fractionally 2PN-accurate expressions (when excluding tails) for the radiated energy and angular momentum given in Ref. [31], Eqs. (C10)–(C13) and (E4)–(E10), we get the following explicit results:...
\[ \chi_{rr}^{3.5\text{PN}}(p_\infty, j) = \frac{2\nu}{7 \pi} \left\{ \left( \frac{168(p_\infty j)^2}{5} + 72 \right) A^2(p_\infty, j) \right. \\
+ \left[ (p_\infty j)^3 \left( \frac{2311}{840} - \frac{437\nu}{30} \right) + (p_\infty j) \left( \frac{11647}{60} - \frac{424\nu}{3} \right) + \frac{13447}{40} - \frac{1127\nu}{6} \right] A(p_\infty, j) \\
+ \frac{1}{((p_\infty j)^2 + 1)^2} \left( (p_\infty j)^6 \left( \frac{40}{7} - \frac{8\nu}{5} \right) + (p_\infty j)^6 \frac{92639}{1400} - \frac{7681\nu}{90} \right) + (p_\infty j)^4 \left( \frac{5049251}{12600} - \frac{3503\nu}{10} \right) \\
\left. \right\}, \]

\[ \chi_{rr}^{4.5\text{PN}}(p_\infty, j) = \frac{2\nu}{9 \pi} \left\{ \left( (p_\infty j)^4 \left( \frac{534}{7} - \frac{373\nu}{5} \right) + (p_\infty j)^2 \left( \frac{816 - \frac{2808\nu}{5}}{5} \right) - \frac{745\nu + 1586}{5} \right) A^2(p_\infty, j) \\
+ \frac{1}{((p_\infty j)^2 + 1)^3} \left( (p_\infty j)^6 \left( \frac{511\nu^2}{24} - \frac{75253\nu}{1680} + \frac{44759}{1120} \right) + (p_\infty j)^6 \left( \frac{366\nu^2}{24} - \frac{136789\nu}{168} + \frac{1020745}{1512} \right) \\
\right. \\
\left. \right\}, \]

For completeness, the corresponding PN-expansion coefficients when considering \( \chi_{rr} \) as a function of \( \dot{a}_r \) and \( e_r \) are

\[ \chi_{rr}^{3.5\text{PN}}(a_r, e_r) = \frac{\nu}{\dot{a}_r^{7/2}} \left( \frac{e_r^2 - 1}{7/2} \right) \left( C^3_{2.5\text{PN}} \arccos^2 \left( \frac{1}{e_r} \right) + C^4_{3.5\text{PN}} \frac{e_r}{\sqrt{e_r^2 - 1}} \arccos \left( \frac{1}{e_r} \right) + C^5_{0} \right); \]

\[ \chi_{rr}^{4.5\text{PN}}(a_r, e_r) = \frac{\nu}{\dot{a}_r^{9/2}} \left( \frac{e_r^2 - 1}{9/2} \right) \left( C^3_{2.5\text{PN}} \arccos^2 \left( \frac{1}{e_r} \right) + C^4_{4.5\text{PN}} \frac{e_r}{\sqrt{e_r^2 - 1}} \arccos \left( \frac{1}{e_r} \right) + C^5_{0} \right); \]

where

\[ C^3_{2.5\text{PN}} = \frac{336}{5} e_r^2 + \frac{384}{5} e_r^2; \]

\[ C^3_{1.5\text{PN}} = \left( \frac{2783}{420} + \frac{47\nu}{15} \right) e_r^4 + \left( \frac{260}{3} \frac{1507}{7} \right) e_r^2 - \frac{1832}{15} \frac{1594}{105}; \]

\[ C^3_{0.5\text{PN}} = \left( \frac{8}{5} + \frac{288}{35} \right) e_r^4 + \left( \frac{1253}{45} \frac{1396049}{6300} \right) e_r^2 - \frac{7498}{45} \frac{71683}{450} + \left( \frac{64}{5} \frac{39394}{1575} \right) \frac{1}{e_r^2}; \]
and

\[ C_2^{4.5\text{PN}} = \left( -\frac{1716}{35} + \frac{94}{5} \nu \right) e_r^4 + \left( -\frac{10008}{35} - \frac{2624}{5} \nu \right) e_r^2 - 480 \nu + \frac{16904}{35}, \]

\[ C_1^{4.5\text{PN}} = \left( \frac{9}{20} \nu^2 + \frac{7783}{840} \nu + \frac{82489}{1680} \right) e_r^6 + \left( \frac{49}{3} \nu^2 + \frac{48821}{84} \nu - \frac{417001}{3780} \right) e_r^4 + \left( \frac{514}{5} \nu^2 + \frac{427622}{105} \nu - \frac{1607}{63} \right) e_r^2 + 88 \nu^2 + \frac{19066}{15} \nu - 19882, \]

\[ C_0^{4.5\text{PN}} = \left( -\frac{2}{5} \nu^2 + \frac{242}{35} \nu + \frac{808}{45} \right) e_r^6 + \left( \frac{1367}{180} \nu^2 + \frac{72587}{2520} \nu + \frac{28987039}{176400} \right) e_r^4 + \left( \frac{365}{6} \nu^2 + \frac{72257}{18} \nu - \frac{147017953}{793800} \right) e_r^2 + \frac{5956}{45} \nu^2 - \frac{98228321}{630} \nu + \left( \frac{36}{5} \nu^2 - \frac{56108}{315} \nu + \frac{16847071}{99225} \right) \frac{1}{e_r^2}. \]  

(C10)

Let us finally discuss the tail-related contribution to \( \chi_{\text{rr,rel}} \). The leading-order, 4PN tail contribution is obtained by inserting in the linear-response formula the \((j\text{-expanded})\) Eqs. (D26) and (F2) of Ref. [30]. The result is the following

\[ \chi_{\text{rr,rel}}^{4\text{PN}}(p_\infty, j) = \nu \left[ \frac{7168}{45} \frac{p_\infty^5}{j^5} + \frac{53}{20} \frac{p_\infty^2}{j^2} + \left( \frac{512}{9} + \frac{153856}{675} \pi^2 \right) \frac{p_\infty}{j^3} + O \left( \frac{1}{j^4} \right) \right]. \]  

(C11)

If we formally insert also the fractional 1PN correction to the tail, we get (by using the 2.5PN accurate expressions for \( E^{\text{rad}} \) and \( J^{\text{rad}} \) derived above in Eqs. (6.5) and (7.4), respectively) the following 5PN-level contribution to \( \chi_{\text{rr,rel}} \):

\[ \chi_{\text{rr,rel}}^{5\text{PN from tail in losses}}(p_\infty, j) = \nu \left[ \frac{4992}{35} - \frac{676096}{1575} \nu \right] \frac{p_\infty^5}{j^5} + \left( \frac{32079}{1120} \pi^2 - \frac{145536}{175} \nu \pi^2 - \frac{7767}{70} \nu^2 + \frac{14032}{525} \nu \right) \frac{p_\infty^4}{j^4} + \left( \frac{7014}{5} \zeta(3) - \frac{515456}{33075} \pi^2 + \frac{206188}{105} \nu + \frac{207}{5} \pi^4 - \frac{89216}{105} \nu - \frac{18853168}{33075} \nu \pi^2 \right) \frac{p_\infty^3}{j^3} + O \left( \frac{p_\infty^2}{j^2} \right). \]  

(C12)

Note, however, that, at this level, there are several other contributions that should be added to this result.

**Appendix D: 3PN-accurate quasi-Keplerian parametrization of the hyperbolic motion**

The 3PN-accurate quasi-Keplerian parametrization of the hyperboliclike motion is

\[ r = \bar{a}_r (\epsilon_r \cosh v - 1), \]

\[ \bar{a}_r = \epsilon_r \sinh w - f_r V + g_t \sin V + h_t \sin 2V + i_t \sin 3V, \]

\[ \phi = K[V + f_\phi \sin 2V + g_\phi \sin 3V + h_\phi \sin 4V + i_\phi \sin 5V], \]  

(D1)

with

\[ V(v) = 2 \arctan \left[ \sqrt{\frac{\epsilon_\phi + 1}{\epsilon_\phi - 1} \tanh \frac{v}{2}} \right]. \]  

(D2)

The 3PN orbital parameters in modified harmonic coordinates along hyperboliclike orbits were obtained in Ref. [56]. However, their expressions are affected by typos, which we discovered when rederriving the 3PN-accurate quasi-Keplerian parametrization of hyperboliclike motions. We list below these typos.

1. Eq. 2.36b. Third line: the term \( 4\eta^3 \) should be replaced by \( 2E_j 4\eta^3 - 195\eta^2 + 1120\eta - 1488 \) for \( 430080 \).
We find

\[ E = \frac{1}{2\nu} + \left( \frac{7}{8} - \frac{1}{8} \right) \frac{\eta^2}{\bar{a}_r^2} + \left[ \frac{25}{16} - \frac{7}{16} \right] \nu + \frac{1}{e_r^2} \right] \eta^4 \bar{a}_r^4 + \]

\[ + \frac{363}{128} - \frac{149}{128} \nu + \frac{21}{64} \nu^2 - \frac{5}{128} \nu^3 + \left[ 5 + \left( \frac{41}{126} \eta^2 - \frac{17033}{840} \right) \frac{\nu + \frac{7}{4} \nu^2}{(e_r^2 - 1)^2} \right] \eta^6 \bar{a}_r^6; \]

\[ j = \sqrt{a_r \sqrt{e_r^2 - 1}} + \left[ \frac{1 - \frac{1}{2} \nu}{e_r^2 - 1} + \left( 3 + \frac{1}{4} \nu \right) \sqrt{e_r^2 - 1} \right] \eta^4 \bar{a}_r^4. \]

We find

\[ \bar{n} = \frac{1}{a_r^{3/2}} - \frac{1}{2} (-9 + \nu) \frac{\eta^2}{a_r^2} + \left[ \frac{3}{8} \nu^2 - \frac{25}{8} \nu + \frac{147}{8} \right] \frac{\eta^4}{a_r^2}; \]

\[ + \left[ \frac{1181}{16} - \frac{235}{16} \nu + 3 \nu^2 - \frac{5}{16} \nu^3 + \left( 39 + \left( \frac{123}{128} \eta^2 - \frac{29351}{280} \right) \nu + \frac{35}{4} \nu^2 \right) \right] \frac{\eta^6}{a_r^2}; \]

\[ K = 1 + \frac{3}{a_r^2} \frac{\eta^2}{(e_r^2 - 1) \bar{a}_r} + \left[ \frac{\nu - \frac{7}{4}}{e_r^2 - 1} + \left( \frac{\nu + \frac{5}{4}}{(e_r^2 - 1)^2} \right) \right] \frac{\eta^4}{a_r^4}; \]

\[ + \left[ \frac{\nu - \frac{7}{4}}{e_r^2 - 1} + \left( \frac{\nu + \frac{5}{4}}{(e_r^2 - 1)^2} \right) \right] \frac{\eta^4}{a_r^4}; \]

\[ e_r = 1 + \left( \frac{3}{2} \nu \right) \frac{\eta^2}{a_r} + \left[ 16 - \frac{67}{8} \nu + \frac{15}{8} \nu^2 + \frac{4 - 7 \nu}{e_r^2 - 1} \right] \frac{\eta^4}{a_r^4}; \]

\[ + \left( \frac{64 - \frac{599}{16} \nu + \frac{219}{16} \nu^2 - 35 \nu^3}{(e_r^2 - 1)^2} \right) \frac{\eta^6}{a_r^6}; \]

\[ e_r = 1 + \frac{1}{2} \frac{\eta^2}{a_r} + \left[ \frac{29}{32} \nu + \frac{15}{8} \nu^2 + \left( \frac{5 - \frac{35}{16} \nu + \frac{15}{16} \nu^2}{e_r^2 - 1} \right) \right] \frac{\eta^4}{a_r^4}; \]

\[ + \left( \frac{4 + \frac{205}{256} \eta^2 - \frac{10443}{448} \nu + \frac{799}{448} \nu^2 - \frac{15}{4} \nu^3}{(e_r^2 - 1)^2} \right) \frac{\eta^6}{a_r^6}; \]

The remaining 3PN orbital parameters still expressed as functions of \( \bar{a}_r \) and \( e_r \) are listed in Table IV.

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[1] T. Damour, “High-energy gravitational scattering and the general relativistic two-body problem,” Phys. Rev. D 97, no.4, 044038 (2018) [arXiv:1710.10599 [gr-qc]].
| $f_1$ | $\frac{3(p-2a)}{2a} \eta^4 + \frac{144(4c^2-19c+40)}{2a} \eta^2 - \frac{1}{2} \pi \sqrt{c^2 - (c+1)^2} \eta^2$ |
| $f_2$ | $\frac{s^2}{8c}(c+1) \eta^4 - \frac{s^2}{192}(c+1)^2 \eta^2 - \frac{2840}{3}(c+1) \eta^2 + 26880 \eta^2 + 66900 \eta^4 + (\frac{107104}{5} + 30135 \eta^2) \eta^4$ |
| $g_1$ | $\frac{s^2}{8c}(c+1) \eta^4 - \frac{s^2}{192}(c+1)^2 \eta^2 - \frac{2840}{3}(c+1) \eta^2 + 26880 \eta^2 + 66900 \eta^4 + (\frac{107104}{5} + 30135 \eta^2) \eta^4$ |
| $g_2$ | $\frac{s^2}{8c}(c+1) \eta^4 - \frac{s^2}{192}(c+1)^2 \eta^2 - \frac{2840}{3}(c+1) \eta^2 + 26880 \eta^2 + 66900 \eta^4 + (\frac{107104}{5} + 30135 \eta^2) \eta^4$ |
| $h_1$ | $\frac{1}{2}(c^2 - 40c + 110) \eta^2 + \frac{1}{2}(c^2 - 40c + 110) \eta^2$ |
| $h_2$ | $\frac{1}{2}(c^2 - 40c + 110) \eta^2 + \frac{1}{2}(c^2 - 40c + 110) \eta^2$ |
| $i_1$ | $\frac{1}{2}(c^2 - 40c + 110) \eta^2 + \frac{1}{2}(c^2 - 40c + 110) \eta^2$ |
| $i_2$ | $\frac{1}{2}(c^2 - 40c + 110) \eta^2 + \frac{1}{2}(c^2 - 40c + 110) \eta^2$ |

TABLE IV: The orbital parameters of the 3PN quasi-Keplerian hyperbolic representation in modified harmonic coordinates, expressed as functions of $\bar{a}$, and $e_\nu$. The corresponding (equivalent) expressions in terms of $\bar{E}$ and $j$ have been given in Ref. [5d].

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