On tight bounds for the Lasso
Sara van de Geer
April 3, 2018

Abstract We present upper and lower bounds for the prediction error of the Lasso. For the case of random Gaussian design, we show that under mild conditions the prediction error of the Lasso is up to smaller order terms dominated by the prediction error of its noiseless counterpart. We then provide exact expressions for the prediction error of the latter, in terms of compatibility constants. Here, we assume the active components of the underlying regression function satisfy some “betamin” condition. For the case of fixed design, we provide upper and lower bounds, again in terms of compatibility constants. As an example, we give an up to a logarithmic term tight bound for the least squares estimator with total variation penalty.

Keywords and phrases. compatibility, Lasso, linear model, lower bound

MSC 2010 Subject classifications. 62J05, 62J07

1 Introduction

Let $X \in \mathbb{R}^{n \times p}$ be an input matrix and $\beta^0 \in \mathbb{R}^p$ a vector of unknown coefficients. Consider an $n$-vector of noisy observations

$$Y = X\beta^0 + \epsilon$$

where the noise $\epsilon \in \mathbb{R}^n$ is a vector of i.i.d. standard Gaussians independent of $X$. The Lasso estimator $\hat{\beta}$ is

$$\hat{\beta} \in \arg \min_{b \in \mathbb{R}^p} \left\{ \|Y - Xb\|_2^2 + 2\lambda\|b\|_1 \right\}$$

with $\lambda > 0$ a regularization parameter (Tibshirani [1996]). Its prediction error is $\|X(\hat{\beta} - \beta^0)\|_2^2$. Main aim of this paper is to provide lower bounds for this prediction error, bounds which show that compatibility constants necessarily enter into the picture.

The results of this paper can be summarized as follows. Firstly, suppose the design is random and that $\Sigma_0 := \mathbb{E}X^TX/n$ exists. Let $\beta^*$ be the noiseless Lasso for random design

$$\beta^* \in \arg \min_{b \in \mathbb{R}^p} \left\{ n\|\Sigma_0^{1/2}(b - \beta^0)\|_2^2 + 2\lambda\|b\|_1 \right\}.$$  (2)

For the case where the rows of $X$ are i.i.d $\mathcal{N}(0, \Sigma_0)$, we show in Theorem 4.3 that $\|X(\hat{\beta} - \beta^0)\|_2$ is up to lower order terms equal to $\sqrt{n}\|\Sigma_0^{1/2}(\beta^* - \beta^0)\|_2$.

---

1Research supported by Isaac Newton Institute for Mathematical Sciences, program Statistical Scalability, EPSRC Grant Number LNAG/036 RG91310.
This result is true under the condition that (after normalizing the co-variance
matrix $\Sigma_0$ to having bounded entries) the largest eigenvalue $\Lambda_{\max}^2$ of $\Sigma_0$ is of
small order $\log n$, and under some mild condition on the growth of the com-
patibility constants as $n$ increases. Secondly, we provide in Theorem 5.1 exact
expressions for the prediction error of the noiseless Lasso in terms of compatibil-
ity constants. We require here “betamin” conditions, which roughly say that
the non-zero coefficients of $\beta^0$ should have the appropriate signs and remain
above the noise level in absolute value. Thirdly, for the case of fixed design, we
present upper and lower bounds for the prediction error $\|X(\hat{\beta} - \beta^0)\|_2^2$ in terms
of weighted compatibility constants. Theorem 7.1 states the lower bounds, as-
suming again certain betamin conditions. The upper bounds we present are
similar to those obtained the literature and presented for completeness. They
are in Corollary 8.1. As an illustration we consider least squares estimation with a (one-dimensional) total variation penalty. We arrive in Corollary 9.1 at
lower and upper bounds that are the same up to a logarithmic term.

There are general upper bounds in the literature, in particular sharp oracle
bounds as in Koltchinskii et al. [2011] (see also Giraud [2014], Theorem 4.1 or
van de Geer [2010], Theorem 2.2). The oracle bounds involve a compatibil-
ity constant, and an improved version of this constant has been developed in
Sun and Zhang [2012], Belloni and Wang [2014] and Dalalyan et al. [2017].

Main theme of this paper is to gain further insight into the role of the com-
patibility constant when applying the Lasso and to see how it occurs in lower
bounds. In Zhang et al. [2014] it is shown that for a given sparsity level, there
is a design and a lower bound for the mean prediction error in the noisy case,
that holds for any polynomial time algorithm. This lower bound is close to the
known upper bounds and in particular shows that compatibility conditions or
restricted eigenvalue conditions cannot be avoided. This has also been shown
by Bellec [2017], where a choice of the particular vector of regression coefficients
$\beta^0$ leads to a lower bound matching the upper bound. We further elaborate on
this issue, and provide lower bounds that hold for a large class of vectors $\beta^0$.

To get an idea of the flavour of the type of bounds we are after, we present in
Theorem 1.1 the case of random design. Details of its proof can be found in
Subsection 11.9. We provide more explicit statements in Theorem 4.3.

Throughout the paper, the active set of $\beta^0$ is denoted by $S_0 := \{j : \beta^0_j \neq 0\}.$
Its size is denoted by $s_0 := |S_0|.$ Our betamin condition is as follows (its meaning
should become more clear after looking at Section 3 where compatibility
constants are defined).

**Condition 1.1** Let

$$b^* \in \arg \min \left\{ \|\Sigma_0^{1/2} b\|_2 : \sum_{j \in S_0} |b_j| - \sum_{j \notin S_0} |b_j| = 1 \right\}$$

and for $j \in S_0$ let $z^*_j$ be the sign of $b^*_j.$ We say that $\beta^0$ satisfies the betamin
condition for the noiseless case with random design if
\[ z_j^* \beta_0^* > \frac{z_j^* b_j^*}{\| \Sigma_0^{1/2} b \|^2_2} \frac{\lambda}{n}, \quad \forall j \in S_0. \]

**Theorem 1.1** Let the rows of \( X \) be i.i.d. \( \mathcal{N}(0, \Sigma_0) \), let \( \| \Sigma_0 \|_\infty \) be the maximal entry in the co-variance matrix \( \Sigma_0 \) and \( \Lambda_{\text{max}}^2 \) be its largest eigenvalue. For \( S \subset \{1, \ldots, p\} \), let \( \kappa^2(S) \) be the compatibility constant defined in Definition 3.1. Suppose that
\[ \Lambda_{\text{max}}^2 / \| \Sigma_0 \|_\infty = o(\log(2p)), \]
and
\[ \max \left\{ \left( \frac{\| \Sigma_0 \|_\infty}{\kappa^2(S)} \right) \log(2p) \frac{|S|}{n} : S \subset \{1, \ldots, p\}, \quad |S| \leq \left( \frac{\Lambda_{\text{max}}^2}{\kappa^2(S_0)} \right) 4s_0 \right\} = o(1). \]
For some \( t > 0 \), take the tuning parameter \( \lambda \) to satisfy
\[ 3\| \Sigma_0 \|_{1/2} \left( \sqrt{2n(\log(2p) + t)} + 2(\log(2p) + t) \right) \leq \lambda = O \left( \sqrt{\| \Sigma_0 \|_{1/2} \log(2p)} \right). \]
Then, under Condition 1.1 (the betamin condition for the noiseless case with random design), we have
\[ \| X(\hat{\beta} - \beta^0) \|_2^2 = \frac{\lambda^2/n}{\| \Sigma_0^{1/2} b^* \|^2_2} (1 + o_P(1)) + O_P(1) \]
(where in fact \( s_0 \| \Sigma_0^{1/2} b^* \|^2_2 = \kappa^2(S_0) \)).

2 Organization of the paper

In Section 3 the definition of compatibility constants is given and also some of their properties are discussed. Section 4 shows that for the case of random design the squared “bias” of the Lasso dominates its “variance”. Section 5 then gives expressions for this “bias”, i.e. for the noiseless Lasso. Here, we examine fixed design but the results carry over immediately to random design. In Section 6 the result of Section 5 is illustrated with the total variation penalty (in one dimension). Section 7 presents lower bounds for the case of fixed design, and Section 8 presents some upper bounds. Corollary 8.1 is essentially as in the papers Sun and Zhang [2012], Belloni and Wang [2014] and Dalalyan et al. [2017], albeit that do not consider the approximately sparse case. Section 9 has upper and lower bounds for the least squares estimator with total variation penalty in the noisy case. Section 10 concludes. Section 11 contains the proofs.

3 Compatibility constants

We introduce some notation in order to be able to define the compatibility constants. This notation will also be helpful at other places. For \( S \subset \{1, \ldots, p\} \)
and a vector \( b \in \mathbb{R}^p \) let \( b_S \in \mathbb{R}^p \) be the vector with entries \( b_{j,S} := b_j \mathbf{1}\{j \in S\}, \)
\( j = 1, \ldots, p \). We apply the same notation for the \(|S|\)-dimensional vector \( \{b_j\}_{j \in S} \).
We moreover write \( b_{-S} := b_{S^c} \) where \( S^c \) is the the complement of the set \( S \).

3.1 Theoretical compatibility constants

The population version of the compatibility constant will be used for the case of random design \( X \). We call the population version the theoretical compatibility constant.

**Definition 3.1** Let \( \Sigma_0 := \mathbf{E}X^TX/n \) (assumed to exist). Let \( S \subset \{1, \ldots, p\} \) be a set of indices and \( u \geq 0 \) be a constant. The theoretical compatibility constant is
\[
\kappa^2(u, S) := \min \left\{ \frac{|S|}{n} \frac{\|\Sigma_0^{1/2} b\|_2^2}{\|b\|_1 - u \|b_{-S}\|_1} : \frac{\|b\|_1 - \|b_{-S}\|_1 = 1}{n} \right\}.
\]
For \( u = 1 \) we write \( \kappa(1, S) =: \kappa_S \).

3.2 Empirical compatibility constants

For a vector \( w \) we let \( W := \text{diag}(w) \) be the diagonal matrix with \( w \) on the diagonal.

**Definition 3.2** ([Belloni and Wang [2014], Dalalyan et al. [2017]]) Let \( S \subset \{1, \ldots, p\} \) be a set of indices and \( w \in \mathbb{R}^{p-|S|} \) be a vector of non-negative weights. The (empirical) compatibility constant is is
\[
\hat{\kappa}^2(w, S) := \min \left\{ \frac{|S|}{n} \frac{\|Xb\|_2^2}{\|b\|_1 - \|Wb_{-S}\|_1 = 1} : \frac{\|b\|_1 - \|b_{-S}\|_1 = 1}{n} \right\}.
\]
For the case where \( w = 1 \) where \( 1 \) denotes a vector with all entries equal to one, put \( \hat{\kappa}^2(S) := \hat{\kappa}^2(1, S) \).

3.3 Some properties of compatibility constants

One readily sees that the theoretical and empirical compatibility constants differ only in terms of the matrix used in the quadratic form (which is \( \Sigma_0 \) in the theoretical case and the Gram matrix \( \hat{\Sigma} := XX/n \) in the empirical case). Thus, when discussing their basic properties it suffices to deal with only one of the two. In this section, we therefore restrict attention to the empirical version \( \hat{\kappa}(w, S) \). Note that we have generalized the empirical version as compared to the theoretical one, by considering general weight vectors, not just constant vectors. With some abuse of notation, we write \( \hat{\kappa}(u, S) = \hat{\kappa}(u1, S) \) when the weights are the constant vector \( u1 \) (it should be clear from the context what is meant).

The empirical compatibility constant as given in Definition 3.2 is from [Belloni and Wang [2014] or Dalalyan et al. [2017]]. Another version, from for instance van de Geer.
Definition 3.3 Let $S \subset \{1, \ldots, p\}$ be a set of indices and $u > 0$ be a constant. The (older) compatibility constant is
\[
\hat{\phi}^2(u, S) := \min \left\{ |S| \|Xb\|_2^2 / n : \|b_S\|_1 = 1, \|b_{-S}\|_1 \leq 1/u \right\}.
\]

Let $\hat{\kappa}^2(S) := \hat{\phi}^2(1, S)$ be the compatibility constant for the case $u = 1$.

The constant $\hat{\phi}^2(u, S)$ compares, for $b$'s satisfying a “cone condition” $\|b_{-S}\|_1 \leq \|b_S\|_1 / u$, the $\ell_2$-norm $\|Xb\|_2$ with the $\ell_1$-norm $\|b_S\|_1$. The constant $\hat{\kappa}(u, S)$ is similar, but takes in the comparison more advantage of a “cone condition” $\|b_S\|_1 - u\|b_{-S}\|_1 > 0$. When $\hat{\kappa}^2(S) > 0$ the null space property holds (Donoho and Tanner [2005]). We will need throughout that the compatibility constant is strictly positive at $S_0$ (if it is zero our results cease to be of any interest). This means that we implicitly require throughout Condition 3.1

Condition 3.1 The matrix of $X^T_{S_0} X_{S_0}$ is invertible.

Here, for any $S \subset \{1, \ldots, p\}$ the matrix $X_S = \{X_j\}_{j \in S}$ is the $n \times |S|$ matrix consisting of the columns of $X$ corresponding to the set $S$.

The newer version $\hat{\kappa}(u, S)$ is an improvement over $\hat{\phi}(u, S)$ in the sense that $\hat{\kappa}(u, S)$ is the larger of the two.

Lemma 3.1 For all $u > 0$ it is true that
\[
\hat{\kappa}^2(u, S) \geq \hat{\phi}^2(u, S).
\]

Let now for some $v > 0$
\[
b^* \in \arg\min \left\{ \|Xb\|_2^2 : \|b_S\|_1 - v\|b_{-S}\|_1 = 1 \right\}.
\]

Then by definition
\[
\hat{\kappa}^2(v, S) = |S| \|Xb^*\|_2^2 / n.
\]

The restriction $\|b_S\|_1 - v\|b_{-S}\|_1 = 1$ does not put any bound on the $\ell_1$-norm of $b^*_S$. However, if there is a little room to spare, its $\ell_1$-norm is bounded. This will be useful to understand the betamin conditions (Conditions 1.1 and 5.1). For simplicity we examine only the value $v = 1$.

Lemma 3.2 Let
\[
b^* \in \arg\min \left\{ \|Xb\|_2^2 / n : \|b_S\|_1 - \|b_{-S}\|_1 = 1 \right\}.
\]

Then for $0 \leq u < 1$
\[
\|b^*_S\|_1 \leq \frac{\hat{\kappa}(S) - u\hat{\kappa}(u, S)}{(1 - u)\hat{\kappa}(u, S)}.
\]
3.4 Comparing empirical and theoretical and compatibility

Having random quadratic forms in mind, the fact that $\|b_S\|_1 - \|b_{-S}\|_1 = 1$ gives no bound on the $\ell_1$-norm can be a problem. Again, if there is a little room to spare in the value of $u$ in the compatibility constant, one does get a bound on the $\ell_1$-norm. We show this in Lemma 3.3 and with this tool in hand we lower bound the empirical compatibility constant in terms of the theoretical one in Lemma 3.4.

Lemma 3.3 Let $v > u > 0$. Then

$$\hat{\kappa}^2(v, S) \geq \min \left\{ |S| \frac{\|Xb\|^2}{2} / n : \|b_S\|_1 - u \|b_{-S}\|_1 = 1, \|b\|_1 \leq 1 + (1+u)/(v-u) \right\}.$$ 

The following lemma will be applied when bounding the prediction error of $\hat{\beta}$ in terms of that of the noiseless Lasso $\beta^*$. The lemma may also be of interest in itself with applications elsewhere.

Lemma 3.4 Suppose the rows of $X$ are i.i.d. $\mathcal{N}(0, \Sigma_0)$. Let $\|\Sigma_0\|_{\infty}$ be the largest entry in the matrix $\Sigma_0$. For $v > u$, $(1 + u)/(v-u) = O(1)$ and

$$\left( \frac{\|\Sigma_0\|_{\infty}}{\hat{\kappa}^2(u, S)} \right) s \log(2p) \frac{n}{\kappa^2(u, S)} = o(1),$$

it is true with probability tending to one that

$$\hat{\kappa}^2(v, S) \geq (1 - \eta)^2 \kappa^2(u, S),$$

where $\eta = o(1)$.

4 Comparison with the noiseless Lasso when the design is random

In this section we assume that the rows of $X$ are i.i.d. copies of a Gaussian row vector with mean zero and co-variance matrix $\Sigma_0$. We denote the largest eigenvalue of $\Sigma_0$ by $\Lambda_{\text{max}}^2$ and let $\|\Sigma_0\|_{\infty}$ be its largest entry. We define a noiseless version $\beta^*$ of the Lasso where also the random design is replaced by its population counterpart:

$$\beta^* \in \arg\min_{b \in \mathbb{R}^p} \left\{ n \|\Sigma_0^{1/2}(b - \beta^0)\|^2_2 + 2\lambda \|b\|_1 \right\}.$$ 

The normalization with $n$ is to put things on the scale of the empirical version, as $EX^T X = n \Sigma_0$. One may think of $\|X(\beta^* - \beta^0)\|_2$ as “bias” and $\|X(\hat{\beta} - \beta^*)\|_2$ as “variance”. We first investigate in some detail the “variance” part in Theorems 4.1 and 4.2. Then we apply the triangle inequality as a way to establish that the squared “bias” dominates the “variance”, see Theorem 4.3.
Theorem 4.1 Suppose that
\[ \rho^2 := \max \left\{ \left( \frac{\| \Sigma_0 \|_{\infty}}{\kappa^2(S)} \right) \log(2p)|S| \right\} : S \subset \{ 1, \ldots, p \}, |S| \leq \left( \frac{\Lambda_{\max}^2}{\kappa^2(S_0)} \right) 4s_0 = o(1). \]
Take for some \( t > 0 \)
\[ \lambda \geq 3\| \Sigma_0 \|_{\infty}^{1/2} \left( \sqrt{2n(\log(2p) + t)} + 2(\log(2p) + t) \right) \]
and define
\[ \gamma := (2\Lambda_{\max}) \sqrt{n}/\lambda + (2/\| \Sigma_0 \|_{\infty}^{1/2}) \rho \lambda / \sqrt{n \log(2p)}. \]
Then we have for all \( x > 0 \) with probability at least \( 1 - 4 \exp[-t] - \exp[-x] - o(1) \) that
\[ \| X(\hat{\beta} - \beta^*) \|_2 \leq \gamma \sqrt{n} \| \Sigma_0^{1/2} (\beta^* - \beta^0) \|_2 + \sqrt{2x}. \]
Using concentration of measure, one can remove the dependency of the confidence level on the value of \( t \). This value appears in the choice of the tuning parameter \( \lambda \). We make some rather arbitrary choices for the constants.

Theorem 4.2 With the conditions and notations of Theorem 4.1 and assuming in addition that \( 4 \exp[-t] < 1/8 \) (say), for \( n \) large enough and for all \( x > 0 \), with probability at least \( 1 - 2 \exp[-x] \),
\[ \| X(\hat{\beta} - \beta^*) \|_2 \leq \gamma \sqrt{n} \| \Sigma_0^{1/2} (\beta^* - \beta^0) \|_2 + 4 \sqrt{\log 2} + \sqrt{2x}. \]
We can now make a type of bias-variance decomposition. The triangle inequality tells us that
\[ \left| \| X(\hat{\beta} - \beta^0) \|_2 - \| X(\beta^* - \beta^0) \|_2 \right| \leq \| X(\hat{\beta} - \beta^*) \|_2. \]
We then approximate the empirical “bias” \( \| X(\beta^* - \beta^0) \|_2 \) by the theoretical “bias” \( \sqrt{n} \| \Sigma_0^{1/2} (\beta^* - \beta_0) \|_2 \) (which is easy as \( \beta^* \) and \( \beta^0 \) are non-random vectors), and use Theorem 4.1 or 4.2 to bound the “variance” \( \| X(\beta - \beta^*) \|_2 \).

Theorem 4.3 With the conditions and notations of Theorem 4.2, we have for \( n \) sufficiently large, for all \( x > 0 \) with probability at least \( 1 - 2 \exp[-x] \)
\[ \left| \| X(\hat{\beta} - \beta^0) \|_2 - \sqrt{n} \| \Sigma_0^{1/2} (\beta^* - \beta^0) \|_2 \right| \]
\[ \leq (\gamma + o(1)) \sqrt{n} \| \Sigma_0^{1/2} (\beta^* - \beta^0) \|_2 + 4 \sqrt{\log 2} + \sqrt{2x}. \]

Corollary 4.1 Recall that we defined \( \gamma \) as
\[ \gamma := (2\Lambda_{\max}) \sqrt{n}/\lambda + (2/\| \Sigma_0 \|_{\infty}^{1/2}) \rho \lambda / \sqrt{n \log(2p)}. \]
Therefore, with the conditions and notations of Theorem 4.3 and assuming in addition
- \( \Lambda^2_{\text{max}}/\|\Sigma_0\|_\infty = o(\log(2p)) \),
and
- \( \lambda = o(\sqrt{\|\Sigma_0\|_\infty n \log(2p)})/\rho \),
we get with probability at least \( 1 - 2 \exp[-x] \)
\[
\|X(\hat{\beta} - \beta^0)\|_2 - \sqrt{n}\|\Sigma_0^{1/2}(\beta^* - \beta^0)\|_2 = o(\sqrt{n}\|\Sigma_0^{1/2}(\beta^* - \beta^0)\|_2) + 4\sqrt{\log 2 + 2x}.
\]
In words: the squared “bias” dominates the “variance”.

**Remark 4.1** With the help of Lemma 12.5, one may also prove bounds for \( \sqrt{n}\|\Sigma_0(\hat{\beta} - \beta^0)\|_2 \) to complete those for \( \|X(\hat{\beta} - \beta^0)\|_2 \). We refrain from doing this here to avoid digressions.

## 5 The noiseless case with fixed design

In this section we study fixed design \( X \) and the noiseless Lasso
\[
\beta^* \in \arg\min_{b \in \mathbb{R}^p} \left\{ \|X(b - \beta^0)\|_2^2 + 2\lambda^*\|b\|_1 \right\}. \tag{3}
\]

In principle the noiseless Lasso considered here differs from (2), although one can say that for fixed design \( \hat{\Sigma} = \mathbb{E}\Sigma_0 =: \Sigma_0 \), with \( \hat{\Sigma} := X^TX/n \) being the Gram matrix. In what follows in this section, we do not use any specific properties of \( \hat{\Sigma} \) and the theory goes through for any positive semi-definite matrix, \( \Sigma \) say. In the upcoming illustration on functions of bounded variation, the fixed design setup is the natural one.

Note that we supplied the tuning parameter \( \lambda^* \) with a subscript \( * \). This is because in Theorem 8.1 we consider a case with different tuning parameters for the noisy and the noiseless case, say \( \lambda \) and \( \lambda^* \).

The Karush-Kuhn-Tucker (KKT) conditions for the noiseless Lasso read
\[
X^TX(\beta^* - \beta^0) + \lambda^*\zeta^* = 0, \quad \zeta^* \in \partial\|\beta^*\|_1, \tag{4}
\]
where \( \partial\|b\|_1 \) denotes the sub-differential of \( b \mapsto \|b\|_1 \):
\[
\partial\|b\|_1 = \left\{ z \in \mathbb{R}^p : z^Tb = \|b\|_1, \|z\|_\infty \leq 1 \right\}.
\]

Recall that
\[
\hat{\kappa}^2(S) = |S|\|Xb^*\|_2^2/n
\]
where
\[
b^* \in \arg\min_{b \in \mathbb{R}^p} \left\{ \|Xb\|_2 : \|bs\|_1 - \|b_{-S}\|_1 = 1 \right\}. \tag{5}
\]
Note that \( b^* \) given in (5) is not unique, for example we can flip the signs of \( b^* \) (i.e., replace \( b^* \) by \(-b^*)

8
In Theorem 5.1 below we give a tight result for the noiseless case under the condition that the active coefficients in $\beta^0$ are sufficiently large in absolute value: Condition 5.1. Here sufficiently large depends on the magnitude of the entries of a solution $b^*$ of (5) with $S = S_0$. Therefore, it is of interest to know how large $b^*$ is. Lemma 3.2 considers its $\ell_1$-norm, and in view of this lemma we conclude that if there is a little room to spare, the $\ell_1$-norm of $\|b^*_S\|_1$ is bounded, or - in other words - $\{b^*_j|S|\}_{j \in S}$ is bounded “on average”.

For the next condition it is useful to know that we show in Lemma 11.6 that for $b^*$ given in (5), each coefficient $b^*_j$ with $j \in S$ is nonzero (provided $\hat{\kappa}^2(S_0) > 0$).

**Condition 5.1** Suppose $\hat{\kappa}^2(S_0) > 0$. Let $b^*$ satisfy (5) with $S = S_0$. Denote, for $j \in S_0$, the sign of $b^*_j$ as $z^*_j$. We say that $\beta^0$ satisfies the betamin condition for the noiseless case with fixed design if

$$z^*_j \beta^0_j > \frac{z^*_j b^*_j s_0}{\hat{\kappa}^2(S_0)} \frac{\lambda^*}{n} \quad \forall \quad j \in S_0.$$ 

Here is the main theorem for the noiseless case.

**Theorem 5.1** Suppose $\hat{\kappa}^2(S_0) > 0$. Let $b^*$ satisfy (5) with $S = S_0$. If $\beta^0$ satisfies Condition 5.1 (the betamin condition for the noiseless case with fixed design), then there exists a solution $\beta^*$ of the KKT conditions (4) such that

$$\|X(\beta^* - \beta^0)\|_2^2 = \frac{s_0}{\hat{\kappa}^2(S_0)} \frac{\lambda^{*2}}{n}.$$ 

### 6 The total variation penalty in the noiseless case

In this section Theorem 5.1 is illustrated with the total variation penalty. For a vector $f \in \mathbb{R}^n$, its total variation is defined as

$$TV(f) := \sum_{i=2}^{n} |f_i - f_{i-1}|.$$ 

Fix a vector $f^0 \in \mathbb{R}^n$ and let $f^* \in \mathbb{R}^n$ is the least squares approximation of $f^0$ with total variation penalty

$$f^* \in \arg \min_{f \in \mathbb{R}^n} \left\{ \|f - f^0\|_2^2 + 2\lambda^* TV(f) \right\}.$$ 

Theorem 6.1 presents an explicit expression for the compatibility constant $\hat{\kappa}^2(S_0)$ where $S_0$ is the set consisting of the locations of the jumps of $f^0$. Invoking Theorem 5.1 one then arrives at an explicit expression for $\|f^* - f^0\|_2^2$ provided the jumps of $f^0$ are sufficiently large, see Corollary 6.1.

First, we need to rewrite problem (6) as a (noiseless) Lasso problem. Indeed, for $j = 1 \ldots, n$,

$$f_j = \sum_{i=1}^{n} (f_i - f_{i-1})l\{j \geq i\} =: (Xb)_j.$$
where \( X_{j,i} = 1 \{ j \geq i \} \) and \( b_i = f_i - f_{i-1} \), with \( f_0 := 0 \). Hence we can say that
\[
f^0 = X\beta^0 \quad \text{and} \quad f^* = X\beta^* \quad \text{with}
\]
\[
\beta^* := \arg \min_{b \in \mathbb{R}^n} \left\{ \|X(b - \beta^0)\|_2^2 + 2\lambda^* \sum_{i=2}^n |b_i| \right\}.
\]

Note that the first coefficient \( b_1 \) is not penalized. It is therefore typically active, and we consider the active set as the location of the jumps augmented with the index \( \{1\} \). We slightly adjust the definition of the compatibility constant to deal with the a coefficient without penalty: we set for \( S \subset \{2, \ldots, n\} \)
\[
\kappa^2(S) := \min \left\{ |S \cup \{1\}| \|Xb\|_2^2 : \|b_S\|_1 - \|b_{-(S \cup \{1\})}\|_1 = 1 \right\}.
\] (7)

Let now \( S := \{d_1 + 1, d_1 + d_2 + 1, \ldots, d_1 + \cdots + d_s + 1\} \) for some \( \{d_j\}_{j=1}^s \subset \{2, \ldots, n\} \) satisfying \( \sum_{j=1}^s d_j + 2 < n \). The set \( S \) represents locations of jumps, \( d_1 \) is the location of the first jump and \( \{d_j\}_{j=2}^s \) are the distances between jumps. Let \( d_{s+1} := n - \sum_{j=1}^s d_j \), the distance between the last jump and the end point. For simplicity we assume that \( d_j \) is even for all \( j \in \{2, \ldots, s\} \).

**Theorem 6.1** The compatibility constant \( \hat{\kappa}^2(S) \) is, up the constant 4 and the scaling by \( 1/n \), the harmonic mean of of the distances between jumps, including the distance between starting point and first jump and last jump and endpoint:
\[
\hat{\kappa}^2(S) = \frac{s + 1}{n} \left( \frac{2n}{d_s} + \sum_{j=1}^s \frac{2n}{d_j} \right).
\]

In fact
\[
\hat{\kappa}^2(S) = (s + 1)\|Xb^*\|_2^2/n
\]
where \( b^*_j = 0 \) for all \( j \notin S \) and \( b^* = \tilde{b}/\|\tilde{b}\|_1 \) with
\[
\tilde{b}_{d_{s+1}} = \left( \frac{n}{d_1} + \frac{2n}{d_2} \right),
\]
\[
\tilde{b}_{d_{s+1}} = \left( \frac{2n}{d_2} + \frac{2n}{d_3} \right),
\]
\[
\vdots
\]
\[
\tilde{b}_{d_s} = (-1)^{s+1} \left( \frac{2n}{d_s} + \frac{n}{d_{s+1}} \right).
\]

**Corollary 6.1** Suppose \( f^0 \) jumps at \( S_0 := S = \{d_1 + 1, d_1 + d_2 + 1, \ldots, d_1 + \cdots + d_s + 1\} \), with \( s = s_0 \). Assume \( f^0 \) alternates between jumps up and jumps down. Suppose moreover that
\[
|f^0_{d_{s+1}} - f^0_{d_{s+1}}| \geq \left( \frac{n}{d_s} + \frac{2n}{d_{s+1}} \right) \frac{\lambda^*}{n},
\]
\[
|f^0_{d_{s+1}} - f^0_{d_{s+1}}| \geq \left( \frac{2n}{d_2} + \frac{2n}{d_3} \right) \frac{\lambda^*}{n},
\]
\[
\vdots
\]
\[
|f^0_{d_{s+1}} - f^0_{d_{s+1}}| \geq \left( \frac{2n}{d_{s_0}} + \frac{n}{d_{s_0+1}} \right) \frac{\lambda^*}{n}.
\]
Then by Theorem 5.1 combined with Theorem 6.1
\[ \|f^* - f^0\|_2^2 = \left(\frac{n}{d_1} + \sum_{j=2}^{s_0} \frac{4n}{d_j} + \frac{n}{d_{s_0+1}}\right) \lambda^* \]  
\[ \frac{1}{n}. \]

At this point it may be helpful to look how this normalizes. Say we choose \( \lambda^* = \sqrt{n \log n} \). Suppose \( \max_{1 \leq j \leq s_0 + 1} n/d_j = \mathcal{O}(s_0 + 1) \). Then the jumps of \( f^0 \) are required to be of order at least \( (s_0 + 1)\sqrt{\log n/n} \). We then obtain
\[ \|f^* - f^0\|_2^2 \leq \mathcal{O}\left( (s_0 + 1)^2 \log n \right). \]

7 A lower bound in the noisy case with fixed design

We now turn to the Lasso \( \hat{\beta} \) in the noisy case, given by
\[ \hat{\beta} \in \arg \min_{b \in \mathbb{R}^p} \left\{ \|Y - Xb\|_2^2 + 2\lambda \|b\|_1 \right\} \]
where
\[ Y = X\beta^0 + \epsilon. \]

We investigate the case of fixed design \( X \). Recall that we assume throughout i.i.d. standard Gaussian noise.

7.1 Towards betamin conditions

Consider some vector \( \bar{v} \in \mathbb{R}^{p-s_0} \) with \( 0 < \bar{v}_j < 1 \) for all \( j \). This vector represents the “noise” that is to be overruled by the penalty. Define the collection of weights
\[ \mathcal{W}(ar{v}) := \left\{ w \in \mathbb{R}^{p-s_0} : 1 - \bar{v}_j \leq w_j \leq 1 + \bar{v}_j \; \forall \; j \right\}. \]

Let for \( \bar{W} := \text{diag}(1 + \bar{v}) \)
\[ b^*(\bar{v}) \in \arg \min \left\{ \|Xb\|_2^2 : \|b_{S_0}\|_1 - \|\bar{W}b_{-S_0}\|_1 = 1 \right\}, \]
\[ z^*_j(\bar{v}) := \text{sign}(b^*_j(\bar{v})), \; j \in S_0. \]

Then by definition \( \bar{\kappa}^2(1 + \bar{v}, S_0) = s_0 \|Xb^*(\bar{v})\|_2^2/n \). We remark here that by a slight adjustment of Lemma 11.6 the assumption \( \bar{\kappa}(1 + \bar{v}, S_0) > 0 \) ensures that \( b^*_j(\bar{v}) \neq 0 \) for all \( j \in S_0 \).

For \( w \in \mathcal{W}(\bar{v}) \) we define the convex problem with linear and convex constraints
\[ b(w) \in \arg \min \left\{ \|Xb\|_2^2 : z^*_{S_0}(\bar{v})b_{S_0} - \|\bar{W}b_{-S_0}\|_1 \geq 1 \right\}. \]

Finally, define
\[ b_j(\bar{v}) := \max_{w \in \mathcal{W}(\bar{v})} |b_j(w)|/\|Xb(w)\|_2^2, \; j \in S_0. \]
7.2 Projections

We denote the projection of $X_{-S_0}$ on the space spanned by the columns of $X_{S_0}$ by $X_{-S_0}PX_{S_0}$. The projection is always defined but as it is implicitly assumed that $X_{S_0}^TX_{S_0}$ is invertible (Condition 3.1), we can clarify what we mean by projection by writing

$$X_{-S_0}PX_{S_0} := X_{S_0}(X_{S_0}^TX_{S_0})^{-1}X_{S_0}^TX_{-S_0}.$$  

The anti-projection is denoted by

$$X_{-S_0}AX_{S_0} = X_{-S_0} - X_{-S_0}PX_{S_0}.$$  

We define the matrix

$$V_{-S_0,-S_0} := \left( X_{-S_0}AX_{S_0} \right)^T \left( X_{-S_0}AX_{S_0} \right) = X_{-S_0}^T(X - X_{S_0}(X_{S_0}^TX_{S_0})^{-1}X_{S_0}^T)X_{-S_0},$$

and let $\{v^2_j\}_{j \notin S_0}$ be the diagonal elements of this matrix.

7.3 A lower bound

The main result for the noisy case is presented in the next theorem. Here, we use the notations and definitions of the previous two subsections.

**Theorem 7.1** Take for some $t > 0$,

$$\lambda > \|v_{-S_0}\|_\infty \sqrt{2(\log(2p) + t)}. \quad (8)$$

Define

$$\bar{v}_j := v_j \sqrt{2(\log(2p) + t)}/\lambda, \quad j \notin S_0$$

and

$$\bar{u}_j := u_j \sqrt{2(\log(2p) + t)}/\lambda, \quad j \in S_0.$$  

where $\{u_j\}_{j \in S_0}$ are the diagonal elements of the matrix $(X_{S_0}^TX_{S_0})^{-1}$. Assume that $\bar{\kappa}(1 + \bar{v}, S_0) > 0$ and that the following betamin condition holds:

$$|\beta_j^0| > \lambda(b_j(\bar{v}) + \bar{u}_j), \quad \text{sign}(\beta_j^0) = z_j^*(\bar{v}) \forall j \in S_0.$$  

Then for all $x > 0$ with probability at least $1 - \exp[-t] - \exp[-x]$ there is a solution $\hat{\beta}$ of the KKT conditions such that

$$\|X(\hat{\beta} - \beta^0)\|_2 \geq \sqrt{\frac{s_0}{\kappa^2(1 + \bar{v}, S_0)}} \left( \frac{\chi^2}{n} - \sqrt{s_0} - \sqrt{2x} \right). \quad (9)$$

Note that for $j \in S_0$, the quantity $u_j$ is the variance of the ordinary least squares estimator of $\beta_j^0$ for the case $S_0$ is known. Thus the betamin condition of Theorem 7.1 needs that the magnitude of the active coefficients should exceed the noise level of the ordinary least squares estimator for known $S_0$.  


8 Comparison with the noiseless Lasso when the design is fixed

This section studies the case of fixed design and compares the noisy Lasso
\[ \hat{\beta} := \arg \min_{b \in \mathbb{R}^p} \left\{ \| Y - Xb \|_2^2 + 2\lambda \| b \|_1 \right\} \]
with the noiseless Lasso
\[ \beta^* := \arg \min_{b \in \mathbb{R}^p} \left\{ \| X(b - \beta^0) \|_2^2 + 2\lambda^* \| b \|_1 \right\} \]
where \( \lambda^* \leq \lambda \). We let \( S^* \) be active set of \( \beta^* \) and its cardinality \( s^* := |S^*| \).
We investigate the error \( \| X(\hat{\beta} - \beta^0) \|_2 \) in Theorem 8.1. For \( \lambda^* = 0 \) we see that \( \beta^* = \beta^0 \) and then Theorem 8.1 gives a bound for \( \| X(\hat{\beta} - \beta^0) \|_2 \). This is elaborated upon in Corollary 8.1. The case \( \lambda^* = \lambda \) is detailed in Corollary 8.2. The error \( \| X(\hat{\beta} - \beta^*) \|_2 \) can then seen as “variance” and \( \| X(\beta^* - \beta^0) \|_2 \) as “bias”.

8.1 Projections

We now introduce some notations and definitions similar to the ones in Subsections 7.2, now for general \( S \) instead of just \( S = S_0 \). The projection of \( X_S \) on the space spanned by the columns of \( X_S \) is denoted by \( X_S P X_S \). Recall that such projections are defined, also if \( X_S \) does not have full column rank. The anti-projection is
\[ X_{-S} A X_S := X_{-S} - X_{-S} P X_S. \]
Define the matrix
\[ V_{-S,-S}^S := \left( X_{-S} A X_S \right)^T \left( X_{-S} A X_S \right) \]
and let \( \{(v_j^S)^2\}_{j \notin S} \) be the diagonal elements of this matrix.

8.2 Upper bound

Recall the KKT conditions for \( \beta^* \) as given in (4), involving the vector \( \zeta^* \) in the sub-differential \( \partial \| \beta^* \|_1 \).

**Theorem 8.1** Fix a set \( S \) with cardinality \( |S| = s \). Assume that that for some \( t > 0 \)
\[ \lambda > \| v_{-S}^S \|_\infty \sqrt{2(\log(2p) + t)} \] (10)
and write
\[ v_j^S := v_j^S \sqrt{2(\log(2p) + t)}/\lambda, \ j \notin S. \] (11)
Suppose that
\[ \lambda^* |\zeta_j^*| / \lambda < 1 - \bar{v}_j^S \quad \forall \ j \notin S. \]
Define
\[ \bar{w}_j^S := \frac{1 - \bar{v}_j^S - \lambda^* |\zeta_j^*| / \lambda}{1 - \lambda^* / \lambda}, \ j \notin S. \]
We have for all \( x \) with probability at least \( 1 - \exp[-t] - \exp[-x] \)
\[ \|X(\hat{\beta} - \beta^*)\|_2 \leq \sqrt{\frac{s}{\hat{\kappa}^2(\bar{w}_j^S, S)}} \sqrt{\frac{(\lambda - \lambda^*)^2}{n}} + \sqrt{s + \sqrt{2x}}. \tag{12} \]

**Corollary 8.1** If we take the tuning parameter \( \lambda^* \) of the noiseless Lasso equal to zero, Theorem 8.1 gives the following: with probability at least \( 1 - \exp[-t] - \exp[-x] \)
\[ \|X(\hat{\beta} - \beta^0)\|_2 \leq \sqrt{s_0 / \hat{\kappa}^2(1 - \bar{v}, S_0)} \sqrt{\lambda^2 / n} + \sqrt{s_0 + \sqrt{2x}}. \]

This result is comparable to results in Sun and Zhang [2012], Belloni and Wang [2014], and Dalalyan et al. [2017], albeit that we do not deal with the extension to the approximately sparse case. One may check that the combined conclusions of this corollary with that of Theorem 7.7 also hold with probability at least \( 1 - \exp[-t] - \exp[-x] \).

**Corollary 8.2** We can also take \( \lambda^* = \lambda \) in Theorem 8.1. We then formally put \( \bar{w}_j^S = \infty \) for all \( j \notin S \) and we put \( \bar{\kappa}(\bar{w}) = \infty \) as well. Let \( S \) with \( |S| = s \). Assume that
\[ |\zeta_j^*| < 1 - \bar{v}_j^S \quad \forall \ j \notin S \tag{13} \]
(this implies \( S \supset S_\ast \)). We have with probability at least \( 1 - \exp[-t] - \exp[-x] \)
\[ \|X(\hat{\beta} - \beta^*)\|_2 \leq \sqrt{s + \sqrt{2x}}. \]

Corollary 8.2 is of interest only when \( \sqrt{s} \) is small enough. This is the case if \( \hat{\Sigma} := X^TX/n \) has a well behaved maximal eigenvalue \( \hat{\lambda}_{\max}^2 \). Indeed, one can show in the same way as in Lemma 11.2 (where \( \hat{\Sigma} \) is replaced by \( \Sigma_0 \)) that
\[ s \leq \left( \frac{\hat{\lambda}_{\max}^2}{(1 - \|\bar{v}^S\|_\infty)^2} \right) n \lambda_X \|X(\beta^* - \beta^0)\|^2_2. \]

Thus if \( \hat{\lambda}_{\max}^2 / (\|\hat{\Sigma}\|_\infty(1 - \|\bar{v}^S\|_\infty)^2) = o(\log(2p)) \), then \( s = o(\|X(\beta^* - \beta^0)\|^2_2) \).

However, for the case of fixed design, one might not want to impose such eigenvalue conditions. Alternatively, one may want to resort to irrepresentable conditions. To this end, fix a set \( S \supset S_0 \). Let for \( j \notin S \), the projection of the \( j \)th column \( X_j \) on \( X_S \) be denoted by
\[ X_j P X_S := X_S \gamma_{S,j}. \]

Then it is not difficult to see that for \( j \notin S \), \( |\zeta_j^*| \leq \|\gamma_{S,j}\|_1 \). In other words, a sufficient condition for (13) to hold is the irrepresentable condition
\[ \|\gamma_{S,j}\|_1 \leq 1 - \bar{v}_j^S, \ \forall \ j \notin S. \]

We conclude that under irrepresentable conditions the squared “bias” \( \|X(\beta^* - \beta^0)\|^2_2 \) dominates the “variance” \( \|X(\hat{\beta} - \beta^*)\|^2_2 \).
The total variation penalty in the noisy case

We continue with the total variation penalty of Section 6, but now in a noisy setting:

\[ Y = f^0 + \epsilon, \]

where \( f^0 \in \mathbb{R}^n \) is an unknown vector. The least squares estimator with total variation penalty is

\[
\hat{f} \in \arg \min_{f \in \mathbb{R}^n} \left\{ \| Y - f \|^2_2 + 2\lambda \text{TV}(f) \right\}. \tag{14}
\]

As has become clear from the previous sections, to assess the prediction error in the noisy case one needs to evaluate the compatibility constant \( \hat{\kappa}(w, S) \) with weights \( w_j \neq 1 \) for \( j \notin S \). For the upper bound on the prediction error, we need lower bounds on \( \hat{\kappa}(w, S) \). These are derived in Dalalyan et al. [2017], Proposition 2. We re-derive (and slightly improve) their result using a different proof (the proof in Dalalyan et al. [2017] applies a probabilistic argument).

Suppose as in Section 6 that the locations of the jumps are

\[ S := \left\{ d_1 + 1, d_1 + d_2 + 1, \ldots, d_1 + \cdots + d_s + 1 \right\} \]

for some \( \{ d_j \}_{j=1}^s \subset \{ 2, \ldots, n \} \) satisfying \( \sum_{j=1}^s d_j + 2 < n \). Let \( d_{s+1} := n - \sum_{j=1}^s d_j \). Assume again for simplicity that \( d_j \) is even for all \( j \in \{ 2, \ldots, s \} \).

Lemma 9.1 Let \( w_1, \ldots, w_n \) be non-negative weights. We have

\[
\sqrt{s + 1} \frac{\hat{\kappa}(w, S)}{\hat{\kappa}(S)} \leq \| w \|_\infty \frac{\sqrt{s + 1}}{\hat{\kappa}(S)} + \sqrt{n \sum_{i=2}^n (w_i - w_{i-1})^2},
\]

where as in Theorem 6.1

\[
\frac{s + 1}{\hat{\kappa}^2(S)} = \frac{n}{t_d} + \sum_{j=2}^s \frac{4n}{d_j} + \frac{n}{d_{s+1}}.
\]

Corollary 9.1 Using the notation of Section 8 suppose that \( \lambda \) satisfies (10) with and let \( \bar{v} = \bar{v}^{S_0} \) be given in (11), both with \( S := S_0 \). Define \( \bar{v}_i = 0 \) for all \( i \in S_0 \). We then have with \( w_i := 1 - \bar{v}_i, j \notin S_0 \cup \{ 1 \}, w_1 = w_2 \) and \( w_i = 1, i \in S_0 \) that

\[
| w_i - w_{i-1} | \leq \| v_i - v_{i-1} \|_\infty, \quad i = \{ 2, \ldots, n \}.
\]

In Dalalyan et al. [2017] it is shown in their Proposition 3 that

\[
\sum_{i=2}^n (v_i - v_{i-1})^2 / \| v \|^2_\infty \leq (s_0 + 1) \log n.
\]

Hence one obtains from Lemma 9.1 with \( S = S_0 \), combined with Corollary 8.7

\[
\frac{\sqrt{s_0 + 1}}{\hat{\kappa}(1 - \bar{v}, S_0)} \leq \sqrt{\frac{s_0 + 1}{\hat{\kappa}(S_0)}} + \sqrt{\frac{(s_0 + 1) \log n}{n}}
\]

15
where as before

\[
\frac{s_0 + 1}{\hat{\kappa}^2(S_0)} = \frac{n}{d_1} + \sum_{j=2}^{s_0} \frac{4n}{d_j} + \frac{n}{d_{s_0+1}}.
\]

Thus, with probability at least \(1 - \exp[-t] - \exp[-x]\)

\[
\|\hat{f} - f^0\|_2 \leq \lambda \left( \sqrt{\frac{(s_0 + 1)}{n\hat{\kappa}^2(S_0)}} + \sqrt{\frac{(s_0 + 1) \log n}{n}} \right) + \sqrt{s_0} + \sqrt{2x}.
\]

Theorem 6.1 implies that

\[
\hat{\kappa}(1 + \bar{v}, S_0) \leq \hat{\kappa}(S_0).
\]

Recall that for the combined conclusion of Theorem 7.1 and Corollary 8.1 we do not have to change the confidence level (which is \(1 - \exp[-t] - \exp[-x]\)). We therefore obtain that if the jumps of \(f^0\) are sufficiently large in absolute value, as given in Theorem 7.1, then with probability at least \(1 - \exp[-t] - \exp[-x]\)

\[
\lambda \sqrt{\frac{s_0 + 1}{n\hat{\kappa}^2(S_0)}} - \sqrt{s_0} - \sqrt{2x} \leq \|\hat{f} - f^0\|_2 \leq \lambda \sqrt{\frac{s_0 + 1}{n\hat{\kappa}^2(S_0)}} + \sqrt{s_0} + \sqrt{2x}
\]

\[
+ \lambda \sqrt{\frac{(s_0 + 1) \log n}{n}}.
\]

10 Conclusion

This paper establishes that in a sense the squared “bias” of the Lasso dominates the “variance”. Moreover, lower bounds for the prediction error are given. These lower often match up to constants or logarithmic factors the upper bounds, or are in fact tight up to smaller order terms. The bounds show that compatibility constants necessarily enter into the picture. The lower bounds require “betamin” conditions, and - for the case of random design - also certain sparsity conditions. It is as yet unclear what can be said when betamin conditions to hold. In combination with this, it would also be of great interest to know what happens when the regression coefficients are not (approximately) sparse. As far as we know the question to what extend the Lasso will have large prediction error when sparseness assumptions are violated (i.e. when the Lasso is used in a scenario not meant for it) is still open.

Acknowledgements: We thank Rico Zenklusen from the Institute of Operations Research, ETH Zürich, and Hamza Fawzi from Department of Applied Mathematics and Theoretical Physics at the University of Cambridge, for very helpful discussions.
11 Proofs

11.1 Proofs of the lemmas in Section 3

Proof of Lemma 3.1. We have to show that $\hat{\kappa}^2(u, S) \geq \hat{\phi}^2(u, S)$. Write

$$A := \left\{ b : \|b - S\|_1 \leq \|b_S\|_1/u, \|b_S\|_1 > 0 \right\}$$

and

$$B := \left\{ b : \|b_S\|_1 - u\|b - S\|_1 > 0 \right\}.$$ 

Then $B \subset A$.

Thus

$$\hat{\phi}^2(u, S) = \min \left\{ |S|\|Xb\|_2^2/n : b \in A \right\} \leq \min \left\{ |S|\|Xb\|_2^2/n : b \in B \right\} = \hat{\kappa}^2(u, S).$$

Proof of Lemma 3.2. This lemma bounds the $\ell_1$-norm of the minimizer $b^*$ if there is a little room to spare. We have

$$\|b^*_S\|_1 - u\|b^*_S - S\|_1 \leq \sqrt{|S|/n}\|Xb^*\|_2/\hat{\kappa}(u, S) = \hat{\kappa}(S)/\hat{\kappa}(u, S).$$

On the other hand

$$\|b^*_S\|_1 - u\|b^*_S - S\|_1 = \|b^*_S\|_1 - \|b^*_{-S}\|_1 + (1 - u)\|b^*_{-S}\|_1 = 1 + (1 - u)\|b^*_{-S}\|_1.$$

Thus

$$\|b^*_{-S}\|_1 \leq \frac{\hat{\kappa}(S) - \hat{\kappa}(u, S)}{(1 - u)\hat{\kappa}(u, S)},$$

yielding

$$\|b^*_S\|_1 = 1 + \|b^*_{-S}\|_1 \leq \frac{\hat{\kappa}(S) - u\hat{\kappa}(u, S)}{(1 - u)\hat{\kappa}(u, S)}.$$

Proof of Lemma 3.3. This lemma shows that one has a bound for the $\ell_1$-norm in the “cone condition” if there is a little room to spare. Consider a vector $b \in \mathbb{R}^p$ satisfying

$$\|b_S\|_1 - u\|b - S\|_1 = 1.$$
Since
\[ \|b_S\|_1 - v\|b_{-S}\|_1 = \|b_S\|_1 - u\|b_{-S}\|_1 - (v - u)\|b_{-S}\|_1 \]
we obtain
\[ (v - u)\|b_{-S}\|_1 = \|b_S\|_1 - u\|b_{-S}\|_1 - 1 \leq \|b_S\|_1 - u\|b_{-S}\|_1. \]
Moreover, clearly
\[ \|b_S\|_1 - u\|b_{-S}\|_1 = (v - u)\|b_{-S}\|_1 + 1 \geq 1. \]
It follows that
\[
\min \left\{ \|Xb\|_2 : \|b_S\|_1 - v\|b_{-S}\|_1 = 1 \right\} \\
\geq \min \left\{ \|Xb\|_2 : (v - u)\|b_{-S}\|_1 \leq \|b_S\|_1 - u\|b_{-S}\|_1, \|b_S\|_1 - u\|b_{-S}\|_1 \geq 1 \right\}.
\]
Suppose now that for some \( c > 1 \)
\[ (v - u)\|b_{-S}\|_1 \leq \|b_S\|_1 - u\|b_{-S}\|_1, \|b_S\|_1 - u\|b_{-S}\|_1 = c. \]
Define
\[ \tilde{b} := b/c. \]
Then
\[ (v - u)\|\tilde{b}_{-S}\|_1 \leq 1, \|\tilde{b}_S\|_1 - u\|\tilde{b}_{-S}\|_1 = 1. \]
Moreover
\[ \|Xb\|_2 = c\|X\tilde{b}\|_2 > \|X\tilde{b}\|_2. \]
Therefore
\[
\min \left\{ \|Xb\|_2 : (v - u)\|b_{-S}\|_1 \leq \|b_S\|_1 - u\|b_{-S}\|_1, \|b_S\|_1 - u\|b_{-S}\|_1 \geq 1 \right\} \\
= \min \left\{ \|Xb\|_2 : (v - u)\|b_{-S}\|_1 \leq 1, \|b_S\|_1 - u\|b_{-S}\|_1 = 1 \right\}.
\]
But if \( (v - u)\|b_{-S}\|_1 \leq 1 \) and \( \|b_S\|_1 - u\|b_{-S}\|_1 = 1 \) we see that
\[
\|b\|_1 \leq \|b_S\|_1 + \|b_{-S}\|_1 = 1 + (1 + u)\|b_{-S}\|_1 \\
\leq 1 + (1 + u)/(v - u).
\]
\[ \square \]

**Proof of Lemma 3.4.** This lemma lower bounds the empirical compatibility constant by the theoretical one. Here is a proof. If \( \|b_S\|_1 - u\|b_{-S}\|_1 = 1 \) we know that
\[ 1 \leq \|\Sigma_0^{1/2}b\|_2 \sqrt{\tau}/\kappa(u, S). \]
It therefore follows from Lemma 3.3 that
\[ \hat{\kappa}^2(v, S) \geq \left\{ |S|\|Xb\|_2^2/n : \|b_S\|_1 - u\|b_{-S}\|_1 = 1, \|b\|_1 \leq M(u, v)\|\Sigma_0^{1/2}b\|_2 \right\} \]

18
where
\[ M(u, v) := (1 + (1 + u)/(v - u))\sqrt{s/\kappa(u, S)} = o(\sqrt{n/(\|\Sigma_0\|_\infty \log(2p))}) \]

In view of Lemma 12.5 we know that when \( M = o(\sqrt{n/(\|\Sigma_0\|_\infty \log(2p))}) \), then with probability tending to one
\[
\inf_{\|b\|_1 \leq M\|\Sigma_0^{1/2}b\|_2} \frac{\|Xb\|_2^2/n}{\|\Sigma_0^{1/2}b\|_2^2} \geq (1 - \eta_M)^2
\]
for suitable \( \eta_M = o(1) \). Hence with probability tending to one
\[
\min \left\{ \frac{\|Xb\|_2^2/n}{\|b\|_1} : \|b_S\|_1 - u\|b_{-S}\|_1 = 1, \|b\|_1 \leq M(u, v)\|\Sigma_0^{1/2}b\|_2 \right\}
\geq (1 - \eta_{M(u,v)})^2 \min \left\{ \|\Sigma_0^{1/2}b\|_2^2 : \|b_S\|_1 - u\|b_{-S}\|_1 = 1 \right\} = (1 - \eta_{M(u,v)})^2 \kappa^2(u, S).
\]

11.2 Proof of Theorem 4.1.
The proof is organized as follows. We first present a bound for \( \|\Sigma_0(\beta^* - \beta_0)\|_2 \) in Lemma 11.1. This will be used to bound later the number of active variables \( s_* \) of \( \beta^* \), or rather some extended version of it involving sub-differential calculus, see Lemma 11.2. We then establish in Lemma 11.3 a deterministic bound assuming we are on some subset of the underlying probability space. Then in Lemma 11.4 we show that this subset has large probability.

The noiseless Lasso \( \beta^* \) given in (2) satisfies the KKT conditions
\[
n\Sigma_0(\beta^* - \beta_0) + \lambda \zeta^* = 0, \; \zeta^* \in \partial \|\beta^*\|_1,
\]
where \( \partial \|b\|_1 \) is the sub-differential of \( b \mapsto \|b\|_1 \):
\[
\partial \|b\|_1 := \left\{ z : \|z\|_\infty \leq 1, \; z^T b = \|b\|_1 \right\}.
\]
This will be used in Lemma 11.2 and again in Lemma 11.3. In the latter we also invoke the KKT conditions for \( \hat{\beta} \)
\[
X^T X(\hat{\beta} - \beta_0) + \lambda \hat{\zeta} = X^T \epsilon, \; \hat{\zeta} \in \partial \|\hat{\beta}\|_1.
\]

11.2.1 A bound for the number of active variables of \( \beta^* \)
First we bound the prediction error of \( \beta^* \).

Lemma 11.1 Suppose \( \kappa^2(S_0) > 0 \). Then
\[
n\|\Sigma_0^{1/2}(\beta^* - \beta_0)\|_2^2 \leq \frac{s_0}{\kappa^2(S_0)} \frac{\lambda^2}{n}.
\]
Proof of Theorem 11.1. This follows from a slight adjustment of Theorem 8.1 in this paper. This is a big detour however, so let us present a self-contained proof as well. By the KKT conditions (15)

\[-(\beta^* - \beta^0)^T \mathbf{z}^* \leq \|\beta^0\|_1 - \|\beta^*\|_1 \leq \|\beta^*_S - \beta^0\|_1 - \|\beta^*_S\|_1.\]

So if \(|\Sigma_0^{1/2} (\beta^* - \beta^0)\|_2^2 > 0\) we obtain by the definition of the compatibility constant \(\kappa^2(S_0)\) that

\[n \|\Sigma_0^{1/2} (\beta^* - \beta^0)\|_2^2 \leq \lambda \sqrt{s_0} \|\Sigma_0^{1/2} (\beta^* - \beta^0)\|_2 \kappa(S_0).\]

This yields the result of the lemma. \(\square\)

Consider the set \(S_* := \{\beta_j^* \neq 0\}\) of active coefficients of \(\beta^*\). We bound the size of this set. In fact we look at bound for the size of a potentially larger set, namely the set \(S_*^{(\nu)} := \{j : |\mathbf{z}_j^*| \geq 1 - \nu\}\) where \(0 \leq \nu < 1\) is arbitrary.

Note that indeed \(S_* \subset S_*^{(\nu)}\). We pin down the value of \(\nu\) to \(\nu = 1/2\) but the argument goes through for other values if one adjusts the constants accordingly. We still keep the symbol \(\nu\) at places to facilitate tracking the constants.

Lemma 11.2 We have that

\[|S_*^{(\nu)}| \leq \frac{\Lambda_\max^2}{(1 - \nu)^2} \frac{n^2}{\lambda^2} \|\Sigma_0^{1/2} (\beta^* - \beta^0)\|_2^2 \leq \frac{\Lambda_\max^2}{(1 - \nu)^2} \frac{s_0}{\kappa^2(S_0)}\]

Proof of Lemma 11.2. Since

\[\|\mathbf{z}^*\|_2^2 \geq \|\mathbf{z}^*_{S_*^{(\nu)}}\|_2^2 \geq (1 - \nu)^2 |S_*^{(\nu)}|\]

it follows from the KKT conditions (15) that

\[(1 - \nu)^2 |S_*^{(\nu)}| \leq \|\Sigma_0^{(\beta^* - \beta^0)}\|_2^2 \frac{n^2}{\lambda^2} \leq \Lambda_\max^2 \|\Sigma_0^{1/2} (\beta^* - \beta^0)\|_2^2 \frac{n^2}{\lambda^2}.\]

The proof is completed by applying the upper bound of Lemma 11.1

\[\|\Sigma_0^{1/2} (\beta^* - \beta^0)\|_2^2 \leq \frac{s_0}{\kappa^2(S_0)} \frac{\lambda^2}{n^2}.\]

\(\square\)

11.2.2 Projections

Let \(S := S_*^{(\nu)}\), \(s := |S|\) (where \(\nu = 1/2\)). Set

\[U(S) := \|\epsilon X_S\|_2\]

where \(\epsilon X_S\) is the projection of \(\epsilon\) on the space spanned by the columns of \(X_S\). Denote the anti-projection of \(X_{-S}\) on this space by

\[X_{-S}A X_S := X_{-S} - X_{-S}P X_S.\]
11.2.3 Choice of \(\lambda\)

Recall we take for some \(t > 0\)

\[
\lambda \geq 3\|\Sigma_0\|_1^{1/2}\left( \sqrt{2(\log(2p) + t)} + 2(\log(2p) + t) \right).
\]

11.2.4 The sets \(T_1, T_2\) and \(T_3\)

Write

\[
v_0 := \|\Sigma_0\|_1^{1/2}\left( \sqrt{2n(\log(2p) + t)} + 2(\log(2p) + t) \right)/\lambda.
\]

We now define a suitable subset of the underlying probability space, on which we can derive the searched for inequality. This subset will be the intersection of the following sets:

\[
T_1 := \left\{ \|(X - S)^T \epsilon\|_\infty \leq \lambda v_0, \ U(S) \leq \sqrt{s} + \sqrt{2x} \right\},
\]

\[
T_2 := \left\{ \|(X^T X - n\Sigma_0)(\beta^* - \beta_0)\|_\infty \leq \lambda \delta \right\},
\]

\[
T_3 := \left\{ \kappa^2((v - v_0 - \delta)/\delta, S) \geq (1 - \eta)^2 \kappa^2(S) \right\},
\]

where \(x > 0\) is arbitrary, \(\delta := \|\Sigma_0^{1/2}(\beta^* - \beta_0)\|_2\), and where \(\eta \in (0, 1)\) is arbitrary. We pin down \(\eta\) to \(\eta = 1/2\) like we did with \(\nu\). We require that \(\nu - v_0 - 2\delta > 0\). Since \(\nu = 1/2\) and \(v_0 \leq 1/3\) this is the case for \(\delta \leq 1/(12)\). In view of Lemma 11.1, Theorem 4.1 is about the case \(\delta = o(1)\), so \(\delta \leq 1/(12)\) will be true for \(n\) sufficiently large.

11.2.5 Deterministic part

Lemma 11.3 On \(T_1 \cap T_2 \cap T_3\) it holds that

\[
\|X(\hat{\beta} - \beta^*)\|_2 \leq \left( \frac{\Lambda_{\text{max}}}{1 - \nu} \frac{\sqrt{n}}{\lambda} + \sqrt{\frac{s}{\kappa^2(S) (1 - \eta)n}} \right) \sqrt{n\delta} + \sqrt{2x}.
\]

Proof of Lemma 11.3. The KKT conditions (15) and (16), for \(\beta^*\) and \(\hat{\beta}\) respectively, are

\[
X^T X(\beta^* - \beta_0) + \lambda \zeta^* = Z,
\]

with \(Z := (X^T X - n\Sigma_0)(\beta^* - \beta_0)\), and

\[
X^T X(\hat{\beta} - \beta_0) + \lambda \hat{\zeta} = X^T \epsilon.
\]

So subtracting the first from the second

\[
X^T X(\hat{\beta} - \beta^*) + \lambda \hat{\zeta} - \lambda \zeta^* = X^T \epsilon - Z.
\]
Multiplying with $\hat{\beta} - \beta^*$ yields
\[ \|X(\hat{\beta} - \beta^*)\|_2^2 + \lambda(\hat{\beta} - \beta^*)^T(\hat{\zeta} - \zeta^*) = (\hat{\beta} - \beta^*)^T(X^T\epsilon - Z). \]  \hspace{1cm} (17)

We write (as in the proof of Theorem 8.1) with $S := S_\nu$, $s := |S|$
\[ X_S\hat{b}_S := X_S(\hat{\beta}_S - \beta^*_S) + (X_SPX_S)\hat{\beta}_S. \]

Since $|\zeta_j^*| \leq 1 - \nu < 1$ for all $j \notin S$, it must be true that $\beta^*_S = 0$. Therefore
\[ X(\hat{\beta} - \beta^*) = X_S\hat{b}_S + (X_SAX_S)\hat{\beta}_S. \]

So
\[ (\hat{\beta} - \beta^*)^TX^T\epsilon = \hat{b}_S^TX_S^T\epsilon + \hat{\beta}_S^T(X_SAX_S)^T\epsilon. \]

We use that (on $T_1$)
\[ \hat{b}_S^TX_S^T\epsilon \leq U(S)\|X_S\hat{b}_S\|_2 \leq U(S)\|X(\hat{\beta} - \beta^*)\|_2 \leq (\sqrt{s} + \sqrt{2x})\|X(\hat{\beta} - \beta^*)\|_2 \]

and
\[ \hat{\beta}_S^T(X_SAX_S)^T\epsilon \leq \|\hat{\beta}_S\|_1\|\|X_SAX_S\|\|_\infty \leq \lambda\|\epsilon\|\hat{\beta}_S\|_1. \]

Moreover (on $T_2$)
\[ -(\hat{\beta} - \beta^*)^TZ \leq \|\hat{\beta} - \beta^*\|_1\|Z\|_\infty \leq \lambda\|\epsilon\|\hat{\beta}_S\|_1. \]

Then
\[ (\hat{\beta} - \beta^*)^T(\zeta^* - \hat{\zeta}) = \hat{\beta}_S^T\zeta_S^* - \beta^T\zeta^* + \beta^T\hat{\zeta} - \hat{\beta}_S^T\hat{\zeta} \]
\[ = \hat{\beta}_S^T\zeta_S^* - \|\beta^*\|_1 + \|\beta^*\|_1 - \|\hat{\beta}_S\|_1 \]
\[ \leq \|\hat{\beta}_S\|_1 - \|\beta^*\|_1 + \|\beta^*\|_1 - \|\hat{\beta}_S\|_1 \]
\[ + \|\hat{\beta}_S^T\zeta_S^* - \|\hat{\beta}_S\|_1 \]
\[ = \|\hat{\beta}_S\|_1 - \|\hat{\beta}_S\|_1 \]
\[ \leq (1 - \nu)|\hat{\beta}_S|_1 - |\hat{\beta}_S|_1 \]
\[ = -\nu|\hat{\beta}_S|_1. \]

Inserting these bounds in (17) gives
\[ \|X(\hat{\beta} - \beta^*)\|_2^2 + \lambda(\nu - v_0 - \delta)|\hat{\beta}_S|_1 \leq (\sqrt{s} + \sqrt{2x})\|X(\hat{\beta} - \beta^*)\|_2 + \lambda\|\hat{\beta}_S - \beta^*_S\|_1. \]

If
\[ \|X(\hat{\beta} - \beta^*)\|_2 \leq (\sqrt{s} + \sqrt{2x}) \]

we are done as by Lemma 11.2 $\sqrt{s} \leq \Lambda_{\max}\delta n / ((1 - \nu)\lambda)$. If
\[ \|X(\hat{\beta} - \beta^*)\|_2 > (\sqrt{s} + \sqrt{2x}) \]
we get
\[(\nu - v_0 - \delta)\|\hat{\beta}_S\|_1 < \delta \|\hat{\beta}_S - \beta^*_S\|_1\]
or
\[\|\hat{\beta}_S - \beta^*_S\|_1 - ((\nu - v_0 - \delta)/\delta)\|\hat{\beta}_S\|_1 > 0.\]
But (on \(T_3\))
\[
\|\hat{\beta}_S - \beta^*_S\|_1 - ((\nu - v_0 - \delta)/\delta)\|\hat{\beta}_S\|_1 \\
\leq \frac{\sqrt{s}\|X(\hat{\beta} - \beta^*)\|_2}{\sqrt{n}\kappa((\nu - v_0 - \delta)/\delta), S) } \\
\leq \frac{\sqrt{s}\|X(\hat{\beta} - \beta^*)\|_2}{\sqrt{n}\kappa(S)(1 - \eta)}.
\]
This gives
\[
\|X(\hat{\beta} - \beta^*)\|_2 \leq \sqrt{s} + \sqrt{2x} + \lambda\sqrt{s}/(\sqrt{n}\kappa(S)(1 - \eta)).
\]
Again, by Lemma 11.4, \(\sqrt{s} \leq \Lambda_{\max}\delta n/((1 - \nu)\lambda).\) We see that
\[
\|X(\hat{\beta} - \beta^*)\|_2 \leq \left(\frac{\Lambda_{\max}\sqrt{n}}{(1 - \nu)\lambda} + \frac{\sqrt{s}}{\kappa(S)(1 - \eta)}\frac{\lambda}{(1 - \eta)n}\right)\sqrt{n}\delta + \sqrt{2x}.
\]
\[\square\]

11.2.6 Random part

We apply the tools of Section 12.

Lemma 11.4 It holds that
\[
\Pr(T_1 \cap T_2 \cap T_3) \geq 1 - 4 \exp[-t] - \exp[-x] - o(1).
\]

Proof of Lemma 11.4. We first show that \(\Pr(T_1) \geq 1 - 2 \exp[-t] - \exp[-x].\) One component of this is to show that with probability at least \(1 - 2 \exp[-t]\)
\[
\|(X_SAX_S)^T\|_{\infty} \leq \lambda v_0.
\]
For a square matrix \(B,\) let \(\text{diag}(B)\) be its diagonal. By Lemma 12.1 we know that with probability at least \(1 - \exp[-t]\)
\[
\|(X_SAX_S)^T\|_{\infty} \leq \|\text{diag}((X_SAX_S)^T(X_SAX_S))\|_{\infty}^{1/2} \sqrt{2 \log(2p) + t}.
\]
But
\[
\|\text{diag}((X_SAX_S)^T(X_SAX_S))\|_{\infty} \leq \|\text{diag}(X^TX)\|_{\infty}.
\]
Moreover in view of Lemma 12.2 and using the union bound, with probability at least \(1 - \exp[-t]\)
\[
\left|\text{diag}(X^TX)\|_{\infty}^{1/2} - \sqrt{n}\|\text{diag}((\Sigma_0)\|_{\infty}^{1/2}\right| \leq \|\Sigma_0\|_{\infty}^{1/2} \sqrt{2 \log(2p) + t}.
\]
So with probability at least \(1 - 2 \exp[-t]\)
\[
\|(X - SAX)\epsilon\|_\infty \leq \|\Sigma_0\|^{1/2}_\infty \left(\sqrt{2n(\log(2p) + t)} + 2(\log(2p) + t)\right) \leq \lambda v_0.
\]

The second component is to show that
\[
P(U(S) \leq \sqrt{s + \sqrt{2x}}) \leq \exp[-x],
\]
but this follows immediately from Lemma 12.2.

Next we show that \(P(T_2) \leq 2 \exp[-t]\). Set \(Z := (X^T X - n \Sigma_0)(\beta^* - \beta^0)\).

Clearly \(X(\beta^* - \beta^0)\) is a Gaussian vector with i.i.d. entries with mean zero and variance \(\|\Sigma_0^{1/2}(\beta^* - \beta^0)\|^2_2\). Hence, applying Lemma 12.3 with \(\sigma_u^2 \leq \|\Sigma_0\|_\infty, \sigma_v^2 = \|\Sigma_0^{1/2}(\beta^* - \beta^0)\|^2_2\) and using the union bound, we obtain that with probability at least \(1 - 2 \exp[-t]\)
\[
\|Z\|_\infty \leq 3\|\Sigma_0\|^{1/2}_\infty \|\Sigma_0^{1/2}(\beta^* - \beta^0)\|_2 \left(\sqrt{2n(\log(2p) + t)} + \log(2p) + t\right).
\]

Finally, the result \(P(T_3) = 1 - o(1)\) follows from Lemma 3.4.

11.2.7 Collecting the pieces

Combining Lemma 11.3 with Lemma 11.4 completes the proof of Theorem 4.1.

11.3 Proof of Theorems 4.2 and 4.3

We use the concentration of measure, Lemma 12.4.

Proof of Theorem 4.2. Let \(m^* := E(\|X(\hat{\beta} - \beta^*)\|_2|X)\). Then we have (by Lemma 12.4) that with probability at least \(1 - 1/8 - 3/4 - o(1)\)
\[
\|X(\hat{\beta} - \beta^*)\|_2 \geq m^* - 2\sqrt{\log 2}
\]
as well as (by Theorem 4.1),
\[
\|X(\hat{\beta} - \beta^*)\|_2 \leq \gamma \sqrt{n}\|\Sigma_0^{1/2}(\beta^* - \beta^0)\|_2 + 2\sqrt{\log 2}.
\]

Thus
\[
m^* \leq \gamma \sqrt{n}\|\Sigma_0^{1/2}(\beta^* - \beta^0)\|_2 + 4\sqrt{\log 2}.
\]

Applying again Lemma 12.4 we see that
\[
P\left(\|X(\hat{\beta} - \beta^*)\| \geq \gamma \sqrt{n}\|\Sigma_0^{1/2}(\beta^* - \beta^0)\|_2 + 4\sqrt{\log 2 + \sqrt{2x}}\right)
\]
\[
\leq P\left(\|X(\hat{\beta} - \beta^*)\| \geq m^* + \sqrt{2x}\right) \leq 2 \exp[-x].
\]
Proof of Theorem 4.3. By the triangle inequality
\[ \|X(\hat{\beta} - \beta^0)\|_2 - \|X(\beta^* - \beta^0)\|_2 \leq \|X(\hat{\beta} - \beta^*)\|_2. \]

By Lemma 12.2 with probability at least \(1 - 2/n\)
\[ \|X(\beta^* - \beta^0)\|_2 - \sqrt{n}\|\Sigma_0^{1/2}(\beta^* - \beta^0)\|_2 \leq (\sqrt{2\log n})\|\Sigma_0^{1/2}(\beta^* - \beta^0)\|_2. \]

So with probability at least \(1 - 4\exp[-t] - \exp[-x] - o(1) - 2/n\) (subtracting the term \(2/n\) to follow the argument, as of course it can be included in the \(o(1)\) term)
\[ \|X(\hat{\beta} - \beta^0)\|_2 - \|\Sigma_0^{1/2}(\beta^* - \beta^0)\|_2 \leq (\gamma + \sqrt{2\log n/n})\sqrt{n}\|\Sigma_0^{1/2}(\beta^* - \beta^0)\|_2 + \sqrt{2x}. \]

Let \(m^0 := \mathbb{E}(\|X(\hat{\beta} - \beta^0)\|_2 | X)\). Using the same arguments as in Theorem 4.2, we arrive at
\[ m^0 - 2\sqrt{\log 2} \leq (1 + \gamma + \sqrt{2\log n/n})\sqrt{n}\|\Sigma_0^{1/2}(\beta^* - \beta^0)\|_2 + 2\sqrt{\log 2} \]
and
\[ (1 - \gamma - \sqrt{2\log n/n})\sqrt{n}\|\Sigma_0^{1/2}(\beta^* - \beta^0)\|_2 - 2\sqrt{\log 2} \leq m^0 + 2\sqrt{\log 2}, \]
or
\[ m^0 - \sqrt{n}\|\Sigma_0^{1/2}(\beta^* - \beta^0)\|_2 \leq (\gamma + \sqrt{2\log n/n})\sqrt{n}\|\Sigma_0^{1/2}(\beta^* - \beta^0)\|_2 + 4\sqrt{\log 2}. \]

Thus, inserting the triangle inequality
\[
\begin{align*}
&\|X(\hat{\beta} - \beta^0)\|_2 - \sqrt{n}\|\Sigma_0^{1/2}(\beta^* - \beta^0)\|_2 \\
\leq &\|X(\hat{\beta} - \beta^0)\|_2 - m^0 + \|m^0 - \sqrt{n}\|\Sigma_0^{1/2}(\beta^* - \beta^0)\|_2 \\
\leq &\|X(\hat{\beta} - \beta^0)\|_2 - m^0 + (\gamma + \sqrt{2\log n/n})\sqrt{n}\|\Sigma_0^{1/2}(\beta^* - \beta^0)\|_2 + 4\sqrt{\log 2}.
\end{align*}
\]
Apply Lemma 12.4 again to finalize the result. \(\square\)

11.4 Proof of Theorem 5.1

To establish Theorem 5.1, we first need to study the minimizer \(b^*\) in (5). The minimization
\[ \min \left\{ \|Xb\|_2^2 : \|b_S\|_1 - \|b_{-S}\|_1 = 1 \right\} \]
has non-convex constraints. If we fix the signs within $S$ of a possible solution $b$, one can reformulate it as a convex problem with convex constraints. This is done in Lemma 11.5. We then show that $b^*_j \neq 0$ for all $j \in S$ in Lemma 11.6. This is important because given the signs within $S$ of a potential solution $b$, we want the restrictions on these signs to be non-active so that the Lagrangian formulation is of a similar form as the KKT conditions (4) for the noiseless Lasso. This Lagrangian form is then given in Lemma 11.8 with Lemma 11.7 serving as a preparation. The Lagrangian form of Lemma 11.8 with $S = S_0$ in a sense resembles the KKT conditions (4) when the active coefficients in the vector $\beta^0_S$ have appropriate signs and $|\beta^0_j|$ is for $j \in S_0$ large enough. This allows one to find a solution $\beta^*$ of the KKT conditions (4) with the prescribed prediction error.

11.4.1 Non-sparseness within $S$

Our first step is to ascertain that a solution $b^* \in \arg \min_{b \in \mathbb{R}^p} \left\{ \|Xb\|_2^2 : \|b_S\|_1 - \|b_{-S}\|_1 = 1 \right\}$ can be found by searching over (at most) $2^{|S|}$ convex problems with convex constraints. This is done in the next lemma, where we also show that the equality constraint $\|b_S\|_1 - \|b_{-S}\|_1 = 1$ can be replaced by an inequality constraint $\|b_S\|_1 - \|b_{-S}\|_1 \geq 1$.

Lemma 11.5 We have

$$\min \left\{ \|Xb\|_2^2 : \|b_S\|_1 - \|b_{-S}\|_1 = 1 \right\} = \min \left\{ \|Xb\|_2^2 : \|b_S\|_1 - \|b_{-S}\|_1 \geq 1 \right\} = \min_{z_S \in \{\pm 1\}^{|S|}} \min_b \left\{ \|Xb\|_2^2 : z_j b_S - \|b_{-S}\|_1 \geq 1, z_j b_j \geq 0 \forall j \in S \right\}.$$ 

Proof of Lemma 11.5. To show that the equality constraint can be turned into an inequality constraint let us consider some $b \in \mathbb{R}^p$ for which it holds that $\|b_S\|_1 - \|b_{-S}\|_1 = c$, where $c$ is a constant bigger than 1. Let $\tilde{b} := b/c$. Then

$$\|\tilde{b}_S\|_1 - \|\tilde{b}_{-S}\|_1 = \left( \|b_S\|_1 - \|b_{-S}\|_1 \right) / c = 1.$$ 

Moreover

$$\|X\tilde{b}\|_2 = \|Xb\|_2 / c < \|Xb\|_2.$$ 

Thus the first equality of the lemma must be true.

We now show the second equality of the lemma. If for some $z_S \in \{\pm 1\}$ it holds that $z_j b_j \geq 0$ for all $j \in S$, we have $z_S^T b_S = \|b_S\|_1$. Conversely, if we define
for \( j \in S \) with \( b_j \neq 0 \), \( z_j := b_j/|b_j| \) as the sign of \( b_j \), and define \( z_j \in \{\pm 1\} \) arbitrarily for \( j \in S \) with \( b_j = 0 \), then we have \( z_j b_j \geq 0 \) for all \( j \in S \). Thus

\[
\left\{ b : \|b_S\|_1 - \|b_{-S}\|_1 \geq 1 \right\} = \bigcup_{z_S \in \{\pm 1\}^{|S|}} \left\{ b : z_S^T b_S - \|b_{-S}\|_1 \geq 1, \ z_j b_j \geq 0 \right\}.
\]

We establish in the next lemma that sign constraints on \( b_S^* \) are not active: \( b_S^* \) is so to speak maximally non-sparse. We assume that \( \kappa^2(S) > 0 \), so for \( S = S_0 \) we implicitly assume Condition 3.1.

**Lemma 11.6** Suppose that \( \kappa(S) \neq 0 \). Then for any minimizer \( b^* \) of the problem

\[
\min \left\{ \|Xb\|_2 : \|b_S\|_1 - \|b_{-S}\|_1 = 1 \right\}
\]

it holds that \( b_j^* \neq 0 \) for all \( j \in S \).

**Remark 11.1** A (very) special case of Lemma 11.6 is the minimization problem

\[
b_S^* \in \arg\min \left\{ |b_S|^2 : \|b_S\|_1 = 1 \right\}.
\]

Clearly the solution has \( |b_S^*| = 1/|S| \neq 0 \) for all \( j \in S \). More generally, for the case without “b_{-S}-part” one can apply a geometric argument to show that whenever \( X_S^TX_S \) is non-singular

\[
b_S^* \in \arg\min \left\{ \|Xb_S\|_2 : \|b_S\|_1 = 1 \right\}
\]

must have all its components in \( S \) nonzero. Indeed, let \( r := \|Xb_S^*\|_2 \). Then \( r > 0 \) by the non-singularity of \( X_S^TX_S \). Let \( \mathcal{E} \) be the ellipsoid \( \mathcal{E} := \{b_S : \|Xb_S\|_2 \leq r\} \) and \( \mathcal{B} := \{b_S : \|b_S\|_1 \leq 1\} \). It is easy to see that \( \mathcal{E} \) must be included in \( \mathcal{B} \). Now \( b_S^* \) is a point on the boundary of both \( \mathcal{E} \) and \( \mathcal{B} \), so any supporting hyperplane to \( \mathcal{E} \) must also be supporting to \( \mathcal{B} \). The key observation is that any point on the boundary of \( \mathcal{E} \) has a unique supporting hyperplane (given by the gradient of the quadratic form); and that points on the boundary of \( \mathcal{B} \) that have a unique supporting hyperplane are exactly those points with no zero entry.

**Proof of Lemma 11.6.** We use the representation of Lemma 11.5. Let \( z_S^* \in \{\pm 1\}^{|S|} \) satisfy \( z_S^T b_S^* = \|b_S^*\|_1 \) and \( z_j^* b_j^* \geq 0 \) for all \( j \in S \). Then \( b^* \) is a solution of the convex minimization problem with (linear and) convex constraints

\[
\min \left\{ \|Xb\|_2^2 : z_S^T b_S - \|b_{-S}\|_1 \geq 1, z_j^* b_j \geq 0, \ \forall \ j \in S \right\}.
\]

Note that in the minimization, one may replace the inequality constraint \( z_S^T b_S - \|b_{-S}\|_1 \geq 1 \) by an inequality constraint \( z_S^T b_S - \|b_{-S}\|_1 = 1 \). This follows from the same arguments as used in the proof of Lemma 11.5. A reason to replace the equality constraint by an inequality constraint is that the restrictions become convex.
The solution of the convex problem with convex constraints can be found using Lagrange multipliers \( \tilde{\lambda} \) and \( \mu_S \), where \( \tilde{\lambda} \geq 0 \) and where \( \mu_S \) is an \(|S|\)-vector with non-negative entries. The Lagrangian formulation is

\[
\min \left\{ \|Xb\|_2^2 + 2\tilde{\lambda} \left( \|b-S\|_1 - z_*^S b_S - 1 \right) - 2 \sum_{j \in S} \mu_{j,S} z_*^j b_j \right\}.
\]

Because the inequality constraint can be replaced by an equality constraint, we know that in fact \( \tilde{\lambda} > 0 \). The Lagrangian formulation has KKT conditions

\[
X^T X b^* = \tilde{\lambda} z^* + \text{diag}(\mu_S) z_*^S,
\]

where \( z_*^s \) is an element of the sub-differential

\[
-\partial\|b_*\|_1 = \left\{ z_\cdot - \|z_\cdot\|_1 \leq 1, \ z_*^T b_* = -\|b_*\|_1 \right\}.
\]

It follows that for \( j \in S \)

\[
b_j^* \neq 0 \Rightarrow \mu_{j,S} = 0.
\]

Let \( \mathcal{N} := \{ j \in S : b_j^* = 0 \} \). Then we have by the above argument

\[
(X^T X b^*)_{-\mathcal{N}} = \tilde{\lambda} z^*_{-\mathcal{N}}
\]

\[
(X^T X b^*)_{\mathcal{N}} = \tilde{\lambda} z^*_{\mathcal{N}} + \text{diag}(\mu_{\mathcal{N}}) z_*^\mathcal{N}.
\]

The tangent plane of \( \{ b : \|Xb\|_2 = \|Xb^*\|_2 \} \) at \( b^* \) is

\[
\mathcal{U} := \{ u = b^* + v : v^T X^T X b^* = 0 \}.
\]

The idea of the proof is now to take an element \( u = b^* + tv \) in this tangent plane with \( t > 0 \) and with \( v_j \neq 0 \) for at least one \( j \in \mathcal{N} \) and such that \( v_j \neq 0 \) has the same sign as \( b_j^* \) for all \( j \in S \setminus \mathcal{N} \). For \( j \notin S \) we take \( v_j = 0 \). Then \( b := b^* + tv \) has \( \|b_S\|_1 - \|b_{-S}\|_1 > 1 \) and this leads for a suitable scale \( t \) to

\[
\frac{\|Xb\|_2}{\|b_S\|_1 - \|b_{-S}\|_1} < \|Xb^*\|_2.
\]

Let us now work out this idea. It cannot be true that \( b_j^* = 0 \) for all \( j \in S \) as \( \|b_S^*\|_1 \geq 1 \). Hence \( S \setminus \mathcal{N} \neq \emptyset \). Take (for example) \( v_j = z_j^* \) for all \( j \in S \setminus \mathcal{N} \). Then

\[
v^T_{S \setminus \mathcal{N}} z^*_{S \setminus \mathcal{N}} = z^*_{S \setminus \mathcal{N}} z^*_{S \setminus \mathcal{N}} = |S \setminus \mathcal{N}|.
\]

Now \( \tilde{\lambda} > 0 \) and the entries of \( \mu_{\mathcal{N}} \) are all positive as well (since \( \mu_j = 0 \) for some \( j \in \mathcal{N} \) would imply \( b_j^* = 0 \) for this \( j \), which is not possible by the definition of \( \mathcal{N} \)). Therefore we can choose

\[
v^T_{\mathcal{N}} (\tilde{\lambda} z^*_{\mathcal{N}} + \text{diag}(\mu_{\mathcal{N}}) z_*^{\mathcal{N}}) = -\tilde{\lambda} |S \setminus \mathcal{N}|.
\]
Then at least one entry of \(v_{\mathcal{N}}\) has to be non-zero and moreover

\[
v^T X^T X b^* = \lambda v_{S \setminus \mathcal{N}}^T z_{S \setminus \mathcal{N}}^* + v_{\mathcal{N}}^T (\lambda z_{\mathcal{N}} + \text{diag}(\mu_{\mathcal{N}}) z_{\mathcal{N}}^*)
\]

\[
= \lambda |S \setminus \mathcal{N}| - \lambda |S \setminus \mathcal{N}|
\]

\[= 0.
\]

We thus have for all \(t > 0\)

\[
\|X(b^* + tv)\|_2^2 = \|Xb^*\|_2^2 + t^2 \|Xv\|_2^2.
\]

Moreover

\[
\|b^*_S + tv_S\|_1 = \|b^*_S|_1 + t\|v_{S \setminus \mathcal{N}}\|_1 + t\|v_{\mathcal{N}}\|_1
\]

\[= \|b^*_S\|_1 + t\|v\|_1.
\]

Therefore

\[
\|b^*_S + tv_S\|_1 - \|b^*_S\|_1 = \|b^*_S\|_1 - \|b^*_S\|_1 + t\|v\|_1
\]

\[= 1 + t\|v\|_1.
\]

It follows that

\[
\frac{\|X(b^* + tv)\|_2^2}{(\|b^*_S + tv_S\|_1 - \|b^*_S\|_1)^2}
\]

\[= \frac{\|Xb^*\|_2^2 + t^2 \|Xv\|_2^2}{(1 + t\|v\|_1)^2}.
\]

Define

\[
A := \|Xb^*\|_2^2 + t^2 \|Xv\|_2^2 - \|Xb^*\|_2^2(1 + t\|v\|_1)^2
\]

\[= t^2 \|Xv\|_2^2 - 2t\|Xb^*\|_2^2\|v\|_1 - t^2 \|Xb^*\|_2^2\|v\|_1^2
\]

\[= t^2(\|Xv\|_2^2 - \|Xb^*\|_2^2\|v\|_1^2) - 2t\|Xb^*\|_2^2\|v\|_1^2.
\]

We will show that for suitable \(t > 0\) the constant \(A\) is strictly negative. This means

\[
\|X(b^* + tv)\|_2^2 < \|Xb^*\|_2^2(\|b^*_S + tv_S\|_1 - \|b^*_S\|_1)^2
\]

and so we arrive at a contradiction. To show \(A < 0\) we distinguish two cases. If

\[
\|Xv\|_2^2 \leq \|Xb^*\|_2^2\|v\|_1^2
\]

then \(A < 0\) for all \(t > 0\). If

\[
\|Xv\|_2^2 > \|Xb^*\|_2^2\|v\|_1^2
\]

then \(A < 0\) for all \(t\) satisfying

\[
0 < t < \frac{2\|Xb^*\|_2^2\|v\|_1^2}{\|Xv\|_2^2 - \|Xb^*\|_2^2\|v\|_1^2}.
\]

Here we used the assumption that \(\|Xb^*\|_2^2 > 0\) so that the above right hand side is indeed strictly positive.
11.4.2 Lagrangian form

We now present the Lagrangian form given the signs within the set $S$ and given that within the set $S$ the solution has non-zero entries. Let for each $z_S \in \{\pm 1\}^{|S|} \backslash S$ and given $b^*_{-S}(z_S)$ be the solution that minimizes

$$\min \left\{ \|Xb\|_2^2 : z_S^T b_S - \|b_S\|_1 \geq 1, z_j b_j \geq 0, \forall j \in S \right\}.$$ 

Define

$$Z_S := \left\{ z_S \in \{-1,1\}^{|S|} : z_j b^*_j(z_S) > 0 \forall j \in S \right\}.$$ 

Lemma 11.7 We have for all $z_S \in Z_S$

$$X^T X b^*(z_S) = z^*(z_S) \|X b^*(z_S)\|_2^2$$

where $z^*_S(z_S) = z_S$ and $z^*_{-S}(z_S) \in -\partial \|b^*_{-S}(z_S)\|_1$.

Proof of Lemma 11.7. To prove this result it is useful to repeat some arguments of the proof of Lemma 11.6. The convex minimization problem with (linear and) convex constraints

$$\min \left\{ \|Xb\|_2^2 : z_S^T b_S - \|b_S\|_1 \geq 1, z_j b_j \geq 0, \forall j \in S \right\}$$

can be solved using Lagrange multipliers $\tilde{\lambda}$ and $\mu_S$, where $\tilde{\lambda} > 0$ and $\mu_S$ is an $|S|$-vector with non-negative entries. The Lagrangian formulation is

$$\min \left\{ \|Xb\|_2^2 + 2\tilde{\lambda} \left( \|b_S\|_1 - z_S^T b_S - 1 \right) - 2 \sum_{j \in S} \mu_j z_j b_j \right\}.$$ 

This has KKT conditions

$$X^T X b^*(z_S) = \tilde{\lambda} z^* + \text{diag}(\mu_S) z_S,$$

where $z^*_S = z_S$ and $z^*_{-S}(z_S) \in z^*_{-S}(z_S)$ depends on $z_S$ and is an element of the sub-differential

$$-\partial \|b^*_{-S}(z_S)\|_1 = \left\{ z_{-S} : \|z_{-S}\|_\infty \leq 1, z_S^T b^*_{-S}(z_S) = -\|b^*_{-S}\|_1 \right\}.$$ 

It follows that for $j \in S$

$$b^*_j(z_S) \neq 0 \implies \mu_j S = 0.$$ 

The assumption that $z_S \in Z_S$ thus gives $\mu_S = 0$. The KKT conditions then read

$$X^T X b^*(z_S) = \tilde{\lambda} z^*.$$ 

One sees that

$$1 = z^{*T} b^*(z_S) = b^{*T}(z_S) X^T X b^*(z_S) / \tilde{\lambda} = \|X b^*(z_S)\|_2^2 / \tilde{\lambda}.$$ 

This gives

$$\tilde{\lambda} = \|X b^*(z_S)\|_2^2.$$ 

$$\Box$$

We apply the above lemma with $z_S := \partial \|b^*_S\|_1$. This gives the following result.
Lemma 11.8 Suppose \( \hat{\kappa}(S) \neq 0 \). Let
\[
b^* \in \arg \min \left\{ \|Xb\|_2^2 : \|b_S\|_1 - \|b_{-S}\|_1 = 1 \right\}
\]
Then
\[
X^TXb^* = z^*\|Xb^*\|_2^2.
\]
where \( z^*_S = \partial \|b^*_S\|_1 \) and \( z^*_{-S} \in -\partial \|b^*_{-S}\|_1 \).

Proof of Lemma 11.8. By Lemma 11.6 for each
\[
b^* \in \arg \min \left\{ \|Xb\|_2^2 : \|b_S\|_1 - \|b_{-S}\|_1 = 1 \right\}
\]
it holds that \( b^*_j \neq 0 \) for all \( j \in S \). We can therefore define \( z^*_j := b^*_j/|b^*_j| \) for all \( j \in S \) and then \( z^*_S = \partial \|b^*_S\|_1 \in \mathcal{Z}_S \). The result now follows from Lemma 11.7.

11.4.3 Finalizing the proof of Theorem 5.1.

With the help of Lemma 11.8 we are now in the position to prove Theorem 5.1.

Proof of Theorem 5.1. Let \( b^* \) and \( z^* \) be as in Lemma 11.8 with \( S = S_0 \). Define
\[
\beta' = \beta^0 - \frac{b^*s_0}{\hat{\kappa}^2(S_0)} \frac{\lambda^*}{n}.
\]
Then
\[
X^TX(\beta' - \beta^0) = -\lambda^*X^TXb^*s_0 \frac{\mu}{n\hat{\kappa}^2(S_0)} = -\lambda^*X^TXb^* \frac{\lambda^*}{\|Xb^*\|_2^2} = -\lambda^*z^*.
\]
Let \( S_* := \{j : b^*_j \neq 0\} \). Then by Lemma 11.6 \( S_0 \subset S_* \). Furthermore
\[
z^*_j\beta'_j = \begin{cases} 
z^*_j\beta'^0_j - \lambda z^*_j b^*_j/\|Xb^*\|_2 > 0 & j \in S_0 \\
-\lambda^*z^*_j b^*_j/\|Xb^*\|_2 > 0 & j \in S_* \setminus S_0 \\
0 & j \notin S_*
\end{cases}
\]
It follows that \( z^* \in \partial \|\beta'\|_1 \). Thus, \( \beta' = \beta^* \) is a solution of the KKT conditions (11) with \( \zeta^* = z^* \). It holds moreover that
\[
\|X(\beta^* - \beta^0)\|_2^2 = \frac{\lambda^*s_0}{\|Xb^*\|_2^4} \frac{\lambda^*}{n\hat{\kappa}^2(S_0)}.
\]
\( \square \)
11.5 Proof of Theorem 6.1

The proof of Theorem 6.1 consists of several steps. First we note that, given the sizes of its jumps, the total variation of a function is the smallest when this function is decreasing or increasing. This is stated in Lemma 11.9 as a trivial fact. As a consequence, if one subtracts from an arbitrary function value - or minus this value - the total variation, the result will be at most the average of the absolute values. This is shown in Lemma 11.10. Lemma 11.10 is then applied at each jump separately, as \( \|b_S\|_1 - \|b_{-S}\|_1 \) in this example amounts to subtracting at each jump some total variation to the left or to the right of this jump. Lemma 11.11 shows how this works for one jump. Then Theorem 6.1 is in part proved by applying this lemma to each jump. This leads to a lower bound for \( \hat{\kappa}^2(S) \). The proof is completed by showing that this lower bound is achieved by the vector \( b^* \) as given in Theorem 6.1.

For \( f \in \mathbb{R}^n \) we define the ordered vector

\[
 f_{(1)} \leq \cdots \leq f_{(n)},
\]

with arbitrary ordering within ties.

**Lemma 11.9** It holds that

\[
 TV(f) \geq f_{(n)} - f_{(1)}
\]

with equality if \( f \) is increasing or decreasing.

**Proof of Lemma 11.9.** Trivial. \( \Box \)

**Lemma 11.10** It holds for any \( j \in \{1, \ldots, n\} \) that

\[
 f_j - TV(f) \leq f_{(1)} - \frac{1}{n} \sum_{i=1}^{n} |f_i|, \quad \text{and} \quad -f_j - TV(f) \leq -f_{(n)} - \frac{1}{n} \sum_{i=1}^{n} |f_i|.
\]

**Proof of Lemma 11.10.** We have from Lemma 11.9 that \( TV(f) \geq f_{(n)} - f_{(1)} \). Moreover, \( f_j \leq f_{(n)} \). Thus

\[
 f_j - TV(f) \leq f_j - (f_{(n)} - f_{(1)}) \leq f_{(n)} - (f_{(n)} - f_{(1)}) = f_{(1)}. \]

**Case 1:** if \( f_{(1)} < 0 \) obviously \( f_{(1)} < \frac{1}{n} \sum_{i=1}^{n} |f_i| \).

**Case 2:** if \( f_{(1)} \geq 0 \) then \( f_i \geq 0 \) for all \( i \) and then

\[
 f_{(1)} \leq \sum_{i=1}^{n} f_i/n = \sum_{i=1}^{n} |f_i|/n.
\]
In the same way
\[-f_j - TV(f) \leq -f_j - (f_{(n)} - f_{(1)}) \leq -f_{(1)} - (f_{(n)} - f_{(1)}) = -f_{(n)}.\]

**Case 1:** if \(f_{(n)} > 0\) then \(-f_{(n)} < \frac{1}{n} \sum_{i=1}^{n} |f_i|\).

**Case 2:** if \(f_{(n)} \leq 0\) then \(f_i \leq 0\) for all \(i\) and then
\[-f_{(n)} \leq - \sum_{i=1}^{n} f_i/n = \sum_{i=1}^{n} |f_i|/n.\]

\[\square\]

**Lemma 11.11** Let \(f \in \mathbb{R}^n\) with total variation \(TV(f) = \sum_{i=2}^{n} |f_i - f_{i-1}|\) and \(g \in \mathbb{R}^m\) with total variation \(TV(g) = \sum_{i=2}^{m} |g_i - g_{i-1}|\). Then for any \(j \in \{1, \ldots, n\}\) and \(k \in \{1, \ldots, m\}\)
\[|f_j - g_k| - TV(f) - TV(g) \leq \frac{1}{n} \sum_{i=1}^{n} |f_i| + \frac{1}{m} \sum_{i=1}^{m} |g_i|.

**Proof of Lemma 11.11** Suppose without loss of generality that \(f_j \geq g_k\). Then by Lemma 11.10
\[|f_j - g_k| - TV(f) - TV(g) = (f_j - TV(f)) + (-g_k - TV(g)) \leq \sum_{i=1}^{n} |f_i|/n + \sum_{i=1}^{m} |g_i|/m \leq \frac{1}{n} \sum_{i=1}^{n} |f_i| + \frac{1}{m} \sum_{i=1}^{m} |g_i|.

\[\square\]

**Proof of Theorem 6.1** Let for \(j = 2, \ldots, s\), \(u_j \in \mathbb{N}\) satisfy \(1 \leq u_j \leq d_j - 1\).

We may write for \(f = Xb\),
\[||b_S||_1 - ||b_{-(S \cup \{1\})}||_1 = |f_{d_1+1} - f_{d_1}| - \sum_{i=2}^{d_1} |f_i - f_{i-1}| - \sum_{i=d_1+2}^{d_1+u_2} |f_i - f_{i-1}|
+ |f_{d_1+d_2+1} - f_{d_1+d_2}| - \sum_{i=d_1+u_2+1}^{d_1+d_2} |f_i - f_{i-1}| - \sum_{i=d_1+d_2+1}^{d_1+d_2+u_3} |f_i - f_{i-1}|
\ldots
+ |f_{d_1+\ldots+d_s-1+1} - f_{d_1+\ldots+d_s-1}| - \sum_{i=d_1+\ldots+d_{s-2}+u_{s-1}+1}^{d_1+\ldots+d_{s-1}} |f_i - f_{i-1}| - \sum_{i=d_1+\ldots+d_{s-1}+1}^{d_1+\ldots+d_{s-1}+u_s} |f_i - f_{i-1}|
+ |f_{d_1+\ldots+d_s+1} - f_{d_1+\ldots+d_s}| - \sum_{i=d_1+\ldots+d_s+1+u_{s+1}+1}^{d_1+\ldots+d_s+2} |f_i - f_{i-1}| - \sum_{i=d_1+\ldots+d_s+2}^{n} |f_i - f_{i-1}|\]
\[
\leq \frac{1}{d_1} \sum_{i=1}^{d_1} |f_i| + \frac{1}{u_2} \sum_{i=d_1+1}^{d_1+u_2} |f_i|
+ \frac{1}{d_2 - u_2} \sum_{i=d_1+u_2+1}^{d_1+d_2} |f_i| + \frac{1}{u_3} \sum_{i=d_1+d_2+1}^{d_1+d_2+u_3} |f_i|
\]
\[
\vdots
\]
\[
+ \frac{1}{d_s-1 - u_{s-1}} \sum_{i=d_1+\ldots+d_{s-2}+u_{s-1}+1}^{d_1+\ldots+d_{s-1}+u_s} |f_i| + \frac{1}{u_s} \sum_{i=d_1+\ldots+d_{s-1}+1}^{d_1+\ldots+d_s} |f_i|
+ \frac{1}{d_s - u_s} \sum_{i=d_1+\ldots+d_{s-1}+u_s+1}^{d_1+\ldots+d_s} |f_i| + \frac{1}{d_{s+1}} \sum_{i=d_1+\ldots+d_s+1}^{n} |f_i|
\]
\[
\leq \sqrt{\frac{1}{d_1} + \frac{1}{u_2} + \frac{1}{d_2 - u_2} + \ldots + \frac{1}{d_s-1 - u_{s-1}} + \frac{1}{u_s} + \frac{1}{d_s - u_s} + \frac{1}{d_{s+1}}}
\times \sqrt{\sum_{i=1}^{n} |f_i|^2},
\]

where in the first inequality we applied Lemma [11.11] and the second one follows from the Cauchy-Schwarz inequality. The assumption that for all \(j \in \{2, \ldots, s\}\) that \(d_j\) is even allows us to take \(u_j = d_j/2\) to arrive at

\[
\kappa^2(S) \geq \frac{s + 1}{\frac{n}{d_1} + \sum_{j=2}^{s} \frac{2n}{d_j} + \frac{n}{d_{s+1}}}.\]

Now for the reverse inequality, let \(\tilde{b}\) be given as in the theorem and and \(\tilde{f} := X\tilde{b}\). Then \(\tilde{f}\) is equal to

\[
\tilde{f}_i = \begin{cases} 
-\frac{n}{d_1} & i = 1, \ldots, d_1 \\
\frac{2n}{d_2} & i = d_1 + 1, \ldots, d_1 + d_2 \\
\vdots & \\
(-1)^s \frac{2n}{d_s} & i = \sum_{j=1}^{s-1} d_j + 1, \ldots, \sum_{j=1}^{s} d_j \\
(-1)^{s+1} \frac{n}{d_{s+1}} & i = \sum_{j=1}^{s} d_j + 1, \ldots, n 
\end{cases}
\]
By the definition of $\tilde{f} = X\tilde{b}$,

$$\|\tilde{b}_S\|_1 = \sum_{j=1}^n |\tilde{f}_{d_j+1} - \tilde{f}_{d_j}| = \frac{n}{d_1} + \frac{2n}{d_2} + \frac{2n}{d_3} + \cdots + \frac{2n}{d_{s-1}} + \frac{2n}{d_s} + \frac{n}{d_{s+1}}$$

and also

$$\sum_{i=1}^n \tilde{f}_i^2 = d_1\tilde{f}_{i_1}^2 + \cdots + d_{s+1}\tilde{f}_{i_{s+1}}^2$$

$$= \frac{n^2}{d_1} + 4\sum_{j=2}^s \frac{n^2}{d_j} + \frac{n^2}{d_{s+1}}.$$ 

Note also that

$$\|\tilde{b}_{-(S \cup \{1\})}\|_1$$

$$= \sum_{i=2}^{d_1} |\tilde{f}_i - \tilde{f}_{i-1}| + \sum_{i=d_1+2}^{d_2} |\tilde{f}_i - \tilde{f}_{i-1}| + \cdots + \sum_{i=d_1+\cdots+d_s+2}^n |\tilde{f}_i - \tilde{f}_{i-1}|$$

$$= 0$$

It follows that

$$\frac{(s + 1)\|X\tilde{b}\|_2^2/n}{(\|\tilde{b}_S\|_1 - \|\tilde{b}_{-(S \cup \{1\})}\|_1)^2} = \frac{\sum_{i=1}^n \tilde{f}_i^2/n}{\left(\sum_{j=1}^s |\tilde{f}_{d_j+1} - \tilde{f}_{d_j}|\right)^2}$$

$$= \frac{s + 1}{\frac{n}{d_1} + \sum_{j=2}^s \frac{4n}{d_j} + \frac{n}{d_{s+1}}}.$$ 

11.6 Proof of Theorem 7.1

To prove Theorem 7.1, we first establish the Lagrangian form of the minimization problem where we have the convex constraint $z^*_S(\bar{v})b_S - \|Wb_S\|_1 \geq 1$. Then we recall the projections and we introduce a subset $\mathcal{T}$ of the underlying probability space where the lower bound of Theorem 7.1 holds. The latter is shown in Lemma 11.13. Finally, we show that the subset $\mathcal{T}$ has large probability.
11.6.1 Lagrangian form

Recall for \( w \in W(\bar{v}) \) the convex problem with linear and convex constraints

\[
\begin{aligned}
b(w) \in \arg \min \{ \| Xb \|_2^2 : \ z_{S_0}^T(\bar{v})b_{S_0} - \| Wb_{-S_0} \|_1 \geq 1 \}.
\end{aligned}
\]

Note that here we do not require the positivity constraint \( z_j^*(\bar{v})b_j \geq 0 \) for all \( j \in S_0 \). The next lemma gives its Lagrangian form. This form plays in the proof of Theorem 7.1 the same role as in the proof of Theorem 5.1 for the noiseless version. We also show that for \( w \in W(\bar{v}) \) the minimum \( \| Xb(w) \|_2^2 \) is not larger than \( \| Xb^*(\bar{v}) \|_2^2 \) (recall that by definition \( \hat{\kappa}^2(1 + \bar{v}, S_0) = s_0 \| Xb^*(\bar{v}) \|_2^2/n \)).

**Lemma 11.12** We have

\[
X^T Xb(w) = \| Xb(w) \|_2^2 Wz(w),
\]

with

\[
z_{S_0}(w) = z_{S_0}^*(\bar{v}), \ z_{-S_0}(w) \in -\partial \| b_{-S_0}(w) \|_1.
\]

Moreover, for \( w \in W(\bar{v}) \)

\[
s_0 \| Xb(w) \|_2^2/n \leq \hat{\kappa}^2(1 + \bar{v}, S_0).
\]

**Proof of Lemma 11.12** The problem

\[
\min \{ \| Xb \|_2^2 : \ z_{S_0}^T(\bar{v})b_{S_0} - \| Wb_{-S_0} \|_1 \geq 1 \}
\]

has Lagrangian

\[
X^T Xb(w) = \tilde{\lambda} Wz(w)
\]

with \( z_{S_0}(w) = z_{S_0}^*(\bar{v}) \) and \( z_{-S_0}(w) \in -\partial \| b_{-S_0}(w) \|_1 \). Moreover

\[
\| Xb(w) \|_2^2 = \tilde{\lambda} b(w)^T Wz(w) = z_{S_0}^*(\bar{v})b_{S_0} - \| Wb_{-S_0} \|_1 = 1
\]

because the minimum is reached at the boundary. So

\[
\tilde{\lambda} = \| Xb(w) \|_2^2.
\]

To obtain the second statement of the lemma, we use similar arguments as in the proof of Lemma 3.1. We have

\[
\| Xb(w) \|_2 = \min \left\{ \frac{\| Xb \|_2}{z_{S_0}^*(\bar{v})b_{S_0} - \| W_{-S_0}b_{-S_0} \|_1} : z_{S_0}^*(\bar{v})b_{S_0} - \| W_{-S_0}b_{-S_0} \|_1 > 0 \right\}
\]

But for \( w \in W \) and \( \bar{w} := 1 + \bar{v} \), we know

\[
\| Wb_{-S_0} \|_1 \leq \| \bar{W}b_{-S_0} \|_1
\]

and so

\[
z_{S_0}^*(\bar{v})b_{S_0} - \| Wb_{-S_0} \|_1 > z_{S_0}^*(\bar{v})b_{S_0} - \| \bar{W}b_{-S_0} \|_1.
\]
Let
\[ A := \left\{ b : z_{S_0}^T(\bar{v})b_{S_0} - \|Wb_{-S_0}\|_1 > 0 \right\} \]
and
\[ B := \left\{ b : z_{S_0}^T(\bar{v})b_{S_0} - \|\bar{W}b_{-S_0}\|_1 > 0 \right\}. \]
Then \( B \subset A \). Hence
\[ \|Xb(w)\|_2 = \min_{b \in A} \frac{\|Xb\|_2}{z_{S_0}^T(\bar{v})b_{S_0} - \|Wb_{-S_0}\|_1} \leq \min_{b \in B} \frac{\|Xb\|_2}{z_{S_0}^T(\bar{v})b_{S_0} - \|Wb_{-S_0}\|_1} \leq \min_{b \in B} \frac{\|Xb^*(\bar{v})\|_2}{z_{S_0}^T(\bar{v})b_{S_0} - \|Wb_{-S_0}\|_1} = \|Xb^*(\bar{v})\|_2 = \sqrt{n\kappa(1 + \bar{v}, S_0)}/\sqrt{s_0}. \]
\[ \square \]

11.6.2 Projections

Recall the notation of Subsection 7.2 and that moreover the diagonal elements of the matrix \((X_{S_0}^TX_{S_0})^{-1}\) are denoted by \(\{u^2_j\}_{j \in S_0}\). We write
\[ \hat{u}_{S_0} := (X_{S_0}^TX_{S_0})^{-1}X_{S_0}^T\epsilon. \]
We denote the projection of \(\epsilon\) on the space spanned by the columns of \(X_{S_0}\) by
\[ \epsilon P_{X_{S_0}} := X_{S_0}(X_{S_0}^TX_{S_0})^{-1}X_{S_0}^T\epsilon = X_{S_0}\hat{u}_{S_0}, \]
and write
\[ U(S_0) := \|\epsilon P_{X_{S_0}}\|_2. \]

11.6.3 Choice of \(\lambda\)

Recall that we require that for some \(t > 0\)
\[ \lambda > \|v_{-S_0}\|_\infty \sqrt{2(\log(2p) + t)}. \]

11.6.4 The set \(\mathcal{T}\)

Recall
\[ \bar{u}_j := u_j \sqrt{2(\log(2p) + t)/\lambda}, \ j \in S_0, \ \bar{v}_j := v_j \sqrt{2(\log(2p) + t)/\lambda}, \ j \notin S_0. \ (18) \]
Let $\mathcal{T}$ be the set
\[
\mathcal{T} := \left\{ |\hat{u}_j| \leq \lambda \bar{u}_j \quad \forall j \in S_0 \right\} 
\cap \left\{ |\hat{v}_j| \leq \lambda \bar{v}_j \quad \forall j \notin S_0 \right\} \cap \left\{ U(S_0) \leq \sqrt{s_0} + \sqrt{2x} \right\}.
\]

We show in Subsection 11.6.6 that $\mathbf{P}(\mathcal{T}) \geq 1 - \exp[-t] - \exp[-x]$.

### 11.6.5 Deterministic part

The idea is now to incorporate the noisy part of the KKT conditions for the noisy Lasso into a weighted sub-differential, creating in that way KKT conditions of the same form as the noiseless KKT conditions (see (19) in the proof). To do so, we first put part of the noise in the vector $\beta^0$ without adding additional non-zeros. This makes it possible not to change the sub-differential at $S_0$. The rewriting of the KKT conditions make them resemble the Lagrangian form of Lemma 11.12.

We will use the KKT conditions (16) for $\hat{\beta}$:
\[
-X^T(Y - X\hat{\beta}) = -\lambda \hat{\zeta}, \quad \hat{\zeta} \in \partial \|\hat{\beta}\|_1.
\]

**Lemma 11.13** Suppose we are on the set $\mathcal{T}$ defined in Subsection 11.6.4. Then under the conditions of Theorem 7.1
\[
\|X(\hat{\beta} - \beta^0)\|_n \geq \frac{\lambda \sqrt{s_0}}{\sqrt{n} \kappa(1 + \bar{v}, S)} + \sqrt{2x}
\]

**Proof of Lemma 11.13** Set
\[
\hat{\beta}^0_{S_0} := \beta^0 + \hat{u}_{S_0}, \quad \hat{\beta}^0_{-S_0} := 0.
\]

Then
\[
Y = X\beta^0 + \epsilon = X_{S_0}\hat{\beta}^0_{S_0} + X_{S_0}\hat{u}_{S_0} + \epsilon AX_{S_0} = X\hat{\beta}^0 + \epsilon AX_{S_0}.
\]

The KKT conditions (16) are
\[
-X^T(Y - X\hat{\beta}) = -\lambda \hat{\zeta}.
\]

We have
\[
Y - X\hat{\beta} = -X(\hat{\beta} - \hat{\beta}^0) - \epsilon AX_{S_0}.
\]

Therefore
\[
-X^T(Y - X\hat{\beta}) = X^T X(\hat{\beta} - \hat{\beta}^0) - X^T(\epsilon AX_{S_0}).
\]
But
\[ X^T_{S_0}(\epsilon AX_{S_0}) = 0, \]
and
\[ X^T_{-S_0}(\epsilon AX_{S_0}) = X^T_{-S_0} - X^T_{-S_0}X_{S_0}(X^T_{S_0}X_{S_0})^{-1}X^T_{S_0}\epsilon \]
\[ = (X_{-S_0}AX_{S_0})^T\epsilon. \]
Hence the KKT conditions read
\[ X^TX(\hat{\beta} - \hat{\beta}^0) = -\lambda\hat{\zeta} + \hat{v}, \]
where
\[ \hat{v}_{S_0} = 0, \quad \hat{v}_{-S_0} = (X_{-S_0}AX_{S_0})^T\epsilon. \]
Set \( \hat{S} := \{ j : \hat{\beta}_j \neq 0 \} \) and define for all \( j \in \hat{S}\backslash S_0 \)
\[ \hat{w}_j := 1 + \hat{v}_j/(\lambda\hat{\zeta}_j). \]
By assumption (since we are on \( T \) \( |\hat{v}_j| < \lambda\bar{v}_j \), so \( \hat{w}_j \geq 1 - \bar{v}_j \) for all \( j \in \hat{S}\backslash S_0 \).
For \( j \notin \hat{S} \cup S_0 \) we define
\[ \hat{w}_j := \max\{|1 + \hat{v}_j/\lambda|, 1 - \bar{v}_j\}. \]
Then for \( j \notin \hat{S} \cup S_0 \)
\[ \lambda\hat{\zeta}_j + \hat{v}_j = \lambda|\hat{\zeta}_j + \hat{v}_j/\lambda|\text{sign}(\hat{\zeta}_j + \hat{v}_j/\lambda) \]
\[ = \begin{cases} \hat{w}_j\text{sign}(\hat{\zeta}_j + \hat{v}_j/\lambda), & |\hat{\zeta}_j + \hat{v}_j/\lambda| \geq 1 - \bar{v}_j \\ \hat{w}_j\frac{|\hat{\zeta}_j + \hat{v}_j/\lambda|}{1 - \bar{v}_j}\text{sign}(\hat{\zeta}_j + \hat{v}_j/\lambda), & |\hat{\zeta}_j + \hat{v}_j/\lambda| \leq 1 - \bar{v}_j \end{cases} \]
\[ = \hat{w}_j\hat{\zeta}_j, \]
where
\[ \hat{\zeta}_j := \begin{cases} \text{sign}(\hat{\zeta}_j + \hat{v}_j/\lambda), & |\hat{\zeta}_j + \hat{v}_j/\lambda| \geq 1 - \bar{v}_j \\ \frac{|\hat{\zeta}_j + \hat{v}_j/\lambda|}{1 - \bar{v}_j}\text{sign}(\hat{\zeta}_j + \hat{v}_j/\lambda), & |\hat{\zeta}_j + \hat{v}_j/\lambda| \leq 1 - \bar{v}_j \end{cases}. \]
One readily verifies that (on \( T \) \( \hat{w}_j \leq 1 + \bar{v}_j \) for all \( j \notin S_0 \). Taking \( \tilde{\zeta}_j = \hat{\zeta}_j \) for \( j \in S \cup S_0 \) we arrive at the KKT conditions
\[ X^TX(\hat{\beta} - \hat{\beta}^0) = -\lambda\hat{W}\tilde{\zeta}, \quad \tilde{\zeta} \in \partial\|\hat{\beta}\|_1 \tag{19} \]
and where \( \hat{W} = \text{diag}(\hat{w}) \) with \( \hat{w} \in \mathcal{W}(\hat{v}) \). Let now \( S_0^+ := \{ j \in S_0 : z_j^*(\hat{v})b_j(\hat{w}) > 0 \} \) and \( S_0^- := \{ j \in S_0 : z_j^*(\hat{v})b_j(\hat{w}) \leq 0 \} \). Take
\[ \beta' = \hat{\beta}^0 - \lambda b_j(\hat{w})/\|Xb(\hat{w})\|_2^2. \]

**Case 1** Let \( j \in S_0 \). By our condition on \( \beta^0 \) we know that for \( j \in S_0 \), \( |\beta_j^0| > \lambda|b_j(\hat{w})|/\|Xb(\hat{w})\|_2^2 + |\hat{u}_{S_0}| \), so \( |\beta_j^0| \geq |\beta_j^0| - |\hat{u}_{S_0}| > \lambda|b_j(\hat{w})|/\|Xb(\hat{w})\|_2^2 \). If \( z_j^*(\hat{v}) = 1 \) and \( b_j(\hat{w}) > 0 \), then \( \beta_j^0 > 0 \) and
\[ \beta_j' = |\hat{\beta}_j^0| - \lambda|b_j(\hat{w})|/\|Xb(\hat{w})\|_2^2 > 0. \]
If $z^*_j(\bar{v}) = 1$ and $b_j(\hat{w}) \leq 0$, then $\hat{\beta}^0_j > 0$ and we have
$$\beta'_j = |\hat{\beta}^0_j| + \lambda |b_j(\hat{w})|/\|Xb(\hat{w})\|_2^2 > 0.$$  
If $z^*_j(\bar{v}) = -1$ and $b_j(\hat{w}) < 0$, then $\hat{\beta}^0_j < 0$ and
$$\beta'_j = -|\hat{\beta}^0_j| + \lambda |b_j(\hat{w})|/\|Xb(\hat{w})\|_2^2 < 0.$$  
If $z^*_j(\bar{v}) = -1$ and $b_j(\hat{w}) \geq 0$, then $\hat{\beta}^0_j < 0$ and
$$\beta'_j = -|\hat{\beta}^0_j| - \lambda |b_j(\hat{w})|/\|Xb(\hat{w})\|_2^2 < 0.$$ 

**Case 2** Let now $j \notin S_0$. Then
$$\beta'_j = -\lambda b_j(\hat{w})/\|Xb(\hat{w})\|_2^2,$$
so
$$z_j(\hat{w})\beta'_j = -\lambda z_j(\hat{w})b_j(\hat{w})/\|Xb(\hat{w})\|_2^2 > 0.$$ 
Thus
$$z(\hat{w}) \in \partial \|\beta'\|_1.$$ 
Furthermore, by the first part of Lemma 11.12
$$X^T X (\beta' - \hat{\beta}^0) = -\lambda X^T Xb(\hat{w})/\|Xb(\hat{w})\|_2 = \text{satisfies} - \lambda \bar{W} z(\hat{w}).$$ 
So $\beta' = \hat{\beta}$ satisfies the KKT conditions with $\tilde{\zeta} = z(\hat{w})$. We further have
$$\|X(\hat{\beta} - \hat{\beta}^0)\|_2^2 = \lambda^2 b^T(\hat{w})\bar{W} z(\hat{w})/\|Xb(\hat{w})\|_2^2 = \lambda^2/\|Xb(\hat{w})\|_2^2 \geq \lambda^2 s_0/(n\hat{\kappa}^2(1 + \bar{v}, S_0))$$
where in the last step we used the second part of Lemma 11.12. Finally, by the triangle inequality
$$\|X(\hat{\beta} - \beta^0)\|_2 \geq \|X(\hat{\beta} - \hat{\beta}^0)\|_2 - U(S_0) \geq \frac{\lambda \sqrt{s_0}}{\sqrt{n\hat{\kappa}(1 + \bar{v}, S_0)}} - U(S_0) \geq \frac{\lambda \sqrt{s_0}}{\sqrt{n\hat{\kappa}(1 + \bar{v}, S_0)}} - \sqrt{s_0} - \sqrt{2}x.$$ 

**11.6.6 Random part**

In Lemma 11.13 we showed that the conclusion (9) of Theorem 7.1 holds on the set $\mathcal{T}$. This subsection obtains that $P(\mathcal{T}) \geq 1 - \exp[-t] + \exp[-x].$
Lemma 11.14 It holds that
\[ P(T) \geq 1 - \exp[-t] - \exp[-x]. \]

Proof of Lemma 11.14. Apply Lemma 12.1 with \( Z_j = \hat{u}_j/u_j \) for \( j \in S_0 \) and \( Z_j = \hat{v}_j/v_j \) for \( j \notin S_0 \) to find that with probability at least \( 1 - \exp[-t] \)
\[ |\hat{u}_j| \leq \lambda \bar{u}_j \forall j \in S_0, \quad |\hat{v}_j| \leq \lambda \bar{v}_j \forall j \notin S_0. \]
Furthermore, the random variable \( U^2(S_0) \) has a chi-squared distribution with \( s_0 \) degrees of freedom. Lemma 12.2 gives that with probability at least \( 1 - \exp[-x] \),
\[ U(S_0) \leq \sqrt{s_0 + 2x}. \]
\[ \square \]

11.6.7 Collecting the pieces
Combining Lemma 11.13 with Lemma 11.14 completes the proof of Theorem 7.1.

11.7 Proof of Theorem 8.1.
The proof is along the lines of Theorem 4.1.

11.7.1 Comparing the KKT conditions
We compare the KKT conditions for the noisy Lasso with those for the noiseless Lasso.

Lemma 11.15 It holds that
\[ \|X(\hat{\beta} - \beta^*)\|_2^2 + \lambda\|\hat{\beta}\|_1 - \lambda^*\hat{\beta}^T z^* \leq (\hat{\beta} - \beta^*)^TX^T \epsilon + (\lambda - \lambda^*)\|\beta^*\|_1. \]

Proof of Lemma 11.15. The KKT conditions (16) for \( \hat{\beta} \) can be written as
\[ X^T X(\hat{\beta} - \beta^0) + \lambda \hat{\zeta} = X^T \epsilon. \]
where \( \hat{\zeta} \in \partial\|\hat{\beta}\|_1 \). By the KKT conditions (4) for \( \beta^* \)
\[ X^T X(\beta^* - \beta^0) + \lambda^* \zeta^* = 0. \]
Hence, taking the difference
\[ X^T X(\hat{\beta} - \beta^*) + \lambda \hat{\zeta} - \lambda^* \zeta^* = X^T \epsilon. \]
Multiply by \( (\hat{\beta} - \beta^*)^T \) to find
\[ \|X(\hat{\beta} - \beta^*)\|_2^2 + \lambda(\hat{\beta} - \beta^*)^T \hat{\zeta} - \lambda^*(\hat{\beta} - \beta^*)^T \zeta^* = (\hat{\beta} - \beta^*)^T X^T \epsilon. \]
But
\[
\lambda(\hat{\beta} - \beta^*)^T \hat{\zeta} - \lambda^*(\hat{\beta} - \beta^*)^T \zeta^*
\]
\[
= \lambda \| \hat{\beta} \|_1 - \lambda^* \hat{\beta}^T \zeta^* + \lambda^* \| \beta^* \|_1 - \lambda \beta^T \hat{\zeta}
\]
\[
= \lambda \| \hat{\beta} \|_1 - \lambda^* \hat{\beta}^T \zeta^* + \lambda \| \beta^* \|_1 - \lambda \beta^T \hat{\zeta} - (\lambda - \lambda^*) \| \beta^* \|_1
\]
\[
\geq \lambda \| \hat{\beta} \|_1 - \lambda^* \hat{\beta}^T \zeta^* - (\lambda - \lambda^*) \| \beta^* \|_1
\]
where we used that
\[
\| \beta^* \|_1 - \beta^T \hat{\zeta} \geq 0.
\]
Therefore
\[
\| X(\hat{\beta} - \beta^*) \|^2_2 + \lambda \| \hat{\beta} \|_1 - \lambda^* \hat{\beta}^T z^* \leq (\hat{\beta} - \beta^*)^T X^T \epsilon + (\lambda - \lambda^*) \| \beta^* \|_1.
\]

11.7.2 Projections

Recall the notation of Subsection 8.1. We let moreover \( \hat{v}_{-S_0} \) be the vector
\[
\hat{v}_{S}^* : = (X_{-S} A X_{S})^T \epsilon.
\]
As before, we denote the projection of \( \epsilon \) on the space spanned by the columns of \( X_{S} \) by \( \epsilon P X_{S} \) and write
\[
U(S) := \| \epsilon P X_{S} \|_2.
\]

11.7.3 Choice of \( \lambda \)

Recall that we require that for some \( t > 0 \)
\[
\lambda > \| v_{S}^* \|_\infty \sqrt{2 \log(2p) + t}.
\]

11.7.4 The set \( T^S \)

Recall
\[
v^S := v_j^S \sqrt{2 \log(2p) + t}/\lambda, \ j \notin S.
\]
Let
\[
T^S := \{ |\hat{v}_j| \leq \lambda \hat{v}_j \ \forall \ j \notin S \} \cap \{ U(S) \leq \sqrt{s} + \sqrt{2x} \}.
\]

11.7.5 Deterministic part

Lemma 11.16 On the set \( T^S \) it holds that
\[
\| X(\hat{\beta} - \beta^*) \|_2 \leq \sqrt{s} + \sqrt{2x} + (\lambda - \lambda^*) \sqrt{s/n} / \kappa(\hat{w}^S, S).
\]

42
Proof of Lemma 11.16. Since $S_* \subset S$

$$X(\hat{\beta} - \beta^*) = X_S \hat{b}_S + X_{-S}AX_S\hat{\beta}_{-S}$$

where

$$X_S \hat{b}_S = X_S(\hat{\beta}_S - \beta_S^*) + (X_SP X_S)\hat{\beta}_{-S}.$$ 

In view of Lemma 11.15,

$$||X(\hat{\beta} - \beta^*)||_2^2 + \lambda||\hat{\beta}||_1 - \lambda^* \beta^T z^*$$

$$\leq \hat{b}_S^TX_S^T \epsilon + \left[X_{-S}AX_S\hat{\beta}_{-S}\right]^T \epsilon + (\lambda - \lambda^*)\|\beta^*\|_1$$

By the Cauchy-Schwarz inequality and since we are on $T^S$

$$\hat{b}_S^TX_S^T \epsilon \leq U(S)||X \hat{b}_S||_2 \leq (\sqrt{s} + \sqrt{2x})||X \hat{b}_S||_2 \leq (\sqrt{s} + \sqrt{2x})||X(\hat{\beta} - \beta^*)||_2$$

where in the last inequality we used Pythagoras rule. Moreover, by the definition of $\hat{v}_{-S}$ and since we are on the set $T^S$

$$\left[X_{-S}AX_S\hat{\beta}_{-S}\right]^T \epsilon = \hat{\beta}_{-S}^T \hat{v}_{-S} \leq \lambda \sum_{j \notin S} \hat{v}_{-S}^j |\hat{\beta}_j|.$$ 

On the other hand,

$$\lambda||\hat{\beta}_{-S}||_1 - \lambda^* \zeta_s^T \hat{\beta}_{-S} \geq \lambda \sum_{j \notin S} (1 - \lambda^*)|\zeta_s^j|/\lambda)|\hat{\beta}_j|$$

and

$$(\lambda - \lambda^*)||\beta^*||_1 - \lambda||\hat{\beta}_S||_1 + \lambda^* z^* \hat{\beta}_S \leq (\lambda - \lambda^*)||\hat{\beta}_S - \beta_S^*||_1.$$ 

If $||X(\hat{\beta} - \beta^*)||_2 \leq \sqrt{s} + \sqrt{2x}$ we are done. Suppose therefore that $||X(\hat{\beta} - \beta^*)||_2 > \sqrt{s} + \sqrt{2x}$. Then we see that

$$||X(\hat{\beta} - \beta^*)||_2^2 - (\sqrt{s} + \sqrt{2x})||X(\hat{\beta} - \beta^*)||_2$$

$$= ||X(\hat{\beta} - \beta^*)||_2 \left(||X(\hat{\beta} - \beta^*)||_2 - \sqrt{s} - \sqrt{2x}\right)$$

$$> 0.$$ 

But then

$$\lambda \sum_{j \notin S} (1 - \hat{v}_{-S}^j - \lambda^*|\zeta_s^j|/\lambda)|\hat{\beta}_j| < (\lambda - \lambda^*)||\hat{\beta}_S - \beta_*^*||_1.$$ 

or

$$||\hat{\beta}_S - \beta_S^*||_1 - ||W^S \hat{\beta}_{-S}||_1 > 0.$$ 

Then

$$||\hat{\beta}_S - \beta_S^*||_1 - ||W^S \hat{\beta}_{-S}||_1 \leq (\sqrt{s/n})||X(\hat{\beta} - \beta^*)||_2 / \kappa(w^S, S).$$ 

43
We thus arrive at
\[ \| X(\hat{\beta} - \beta^*) \|_2^2 \leq \left( \sqrt{s} + \sqrt{2x} + (\lambda - \lambda^*) \sqrt{s/n/\hat{\kappa}(\bar{w}_S, S)} \right) \| X(\hat{\beta} - \beta^*) \|_2 \]
or
\[ \| X(\hat{\beta} - \beta^*) \|_2 \leq \sqrt{s} + \sqrt{2x} + (\lambda - \lambda^*) \sqrt{s/n/\hat{\kappa}(\bar{w}_S, S)}. \]
\[\Box\]

11.7.6 Random part

**Lemma 11.17** We have
\[ P(T^S) \geq 1 - \exp[-t] - \exp[-x]. \]

**Proof of Lemma [11.17]** This follows from Lemma [12.1] and Lemma [12.2]. \[\Box\]

11.7.7 Finalizing the proof of Theorem 8.1

Combine Lemma [11.16] with Lemma [11.17].

11.8 Proof of the lemma in Section 9

**Proof of Lemma 9.1**. Write \( g_i := w_i f_i, i = 1, \ldots, n \) and \( u_j := d_j/2, j = 2, \ldots, s \). Then we have
\[
\sum_{j=1}^s |g_{d_j+1} - g_{d_j}| - \sum_{i=2}^{d_1} |g_i - g_{i-1}| - \sum_{j=2}^{d_j+1} \sum_{i=d_j+1}^{d_{j+1}-1} |g_i - g_{i-1}| - \sum_{i=d_s+1}^n |g_i - g_{i-1}| \\
\leq \frac{1}{d_1} \sum_{i=1}^{d_1} |g_i| + \frac{1}{u_2} \sum_{i=d_1+1}^{d_1+u_2} |g_i| \\
+ \frac{1}{d_2 - u_2} \sum_{i=d_1+u_2+1}^{d_1+d_2} |g_i| + \frac{1}{u_3} \sum_{i=d_1+d_2+1}^{d_1+d_2+u_3} |g_i| \\
\ldots \\
+ \frac{1}{d_{s-1} - u_{s-1}} \sum_{i=d_1+\ldots+d_{s-2}+u_{s-1}+1}^{d_1+\ldots+d_{s-1}+u_s} |g_i| + \frac{1}{u_s} \sum_{i=d_1+\ldots+d_{s-1}+1}^{d_1+\ldots+d_{s-1}+u_s} |g_i| \\
+ \frac{1}{d_s - u_s} \sum_{i=d_1+\ldots+d_{s-1}+u_s+1}^{d_1+\ldots+d_s} |g_i| + \frac{1}{d_{s+1}} \sum_{i=d_1+\ldots+d_s+1}^n |g_i| \\
\]
\[ \leq \left( \frac{1}{d_1^2} \sum_{i=1}^{d_1} w_i^2 + \frac{1}{u_2^2} \sum_{i=d_1+1}^{d_1+u_2} w_i^2 \right) + \frac{1}{(d_2 - u_2)^2} \sum_{i=d_1+u_2+1}^{d_1+d_2} w_i^2 + \frac{1}{u_3^2} \sum_{i=d_1+d_2+1}^{d_1+d_2+u_3} w_i^2 + \frac{1}{(d_{s-1} - u_{s-1})^2} \sum_{i=d_1+\ldots+d_{s-1}+u_{s-1}+1}^{d_1+\ldots+d_{s-1}+u_s} w_i^2 + \frac{1}{u_{s}^2} \sum_{i=d_1+\ldots+d_{s-1}+u_{s}+1}^{d_1+\ldots+d_{s-1}+u_{s}+1} w_i^2 \]
\[ \times \left( \sum_{i=1}^{n} f_i^2 \right)^{1/2} \]
\[ \leq \sqrt{\frac{n}{d_1} + \frac{n}{u_2} + \frac{n}{d_2 - u_2} + \ldots + \frac{n}{d_{s-1} - u_{s-1}} + \frac{n}{u_{s}} + \frac{n}{d_{s} - u_{s}} + \frac{n}{d_{s+1}}} \times \sqrt{\sum_{i=1}^{n} |f_i|^2 / n} \times ||w||_{\infty}. \]

Moreover
\[ \sum_{j=1}^{s} w_{d_j+1} |f_{d_j+1} - f_d| - \sum_{i=2}^{d_1} w_i |f_i - f_{i-1}| \]
\[ - \sum_{j=2}^{s-1} \sum_{i=d_j+1}^{d_{j+1}} w_i |f_i - f_{i-1}| - \sum_{i=d_s+1}^{n} w_i |f_i - f_{i-1}| \]
\[ \leq \sum_{j=1}^{s} |g_{d_j+1} - g_d| - \sum_{i=2}^{d_1} |g_i - g_{i-1}| - \sum_{j=2}^{s-1} \sum_{i=d_j+1}^{d_{j+1}} |g_i - g_{i-1}| - \sum_{i=d_s+1}^{n} |g_i - g_{i-1}| \]
\[ + \sum_{i=2}^{n} |w_i - w_{i-1}| |f_{i-1}|, \]

and
\[ \sum_{i=2}^{n} |w_i - w_{i-1}| |f_{i-1}| \leq \sqrt{\sum_{i=2}^{n} (w_i - w_{i-1})^2} \sqrt{\sum_{i=2}^{n} f_{i-1}^2} \]
\[ \leq \sqrt{\sum_{i=2}^{n} (w_i - w_{i-1})^2} \sqrt{\sum_{i=1}^{n} f_i^2} \]
Thus we conclude
\[
\sum_{j=1}^{s} w_{d_j+1} |f_{d_j+1} - f_{d_j}| \\
- \sum_{i=2}^{d_1} w_i |f_i - f_{i-1}| - \sum_{j=1}^{d_j} \sum_{i=d_j+1}^{d_{j+1}} w_i |f_i - f_{i-1}| - \sum_{i=d_s+1}^{n} w_i |f_i - f_{i-1}|
\leq \left( \|w\|_\infty \sqrt{\frac{1}{d_1} + \sum_{j=2}^{s} \frac{4}{d_j} + \frac{1}{d_{s+1}}} + \sqrt{n \sum_{i=2}^{n} (w_i - w_{i-1})^2} \right) \sqrt{n \sum_{i=1}^{n} f_i^2 / n}.
\]

\[\square\]

11.9 Proof of Theorem 1.1

This follows from Corollary 4.1 combined with Theorem 5.1, where in the latter we replace \(\hat{\Sigma} := X^T X/n\) by the population version \(\Sigma_0\). This works because we replaced Condition 5.1 by its population counterpart Condition 1.1.

12 Tools from probability theory

We first present three standard lemmas for Gaussian random variables, Lemmas 12.1, 12.2 and 12.3. These three lemmas are followed by a concentration of measure result and a result for Gaussian quadratic forms.

**Lemma 12.1** Let \(Z_1, \ldots, Z_p\) be standard normal random variables. Then it holds for all \(t > 0\) that
\[
P\left( \max_{1 \leq j \leq p} |Z_j| \geq \sqrt{2(\log(2p) + t)} \right) \leq \exp[-t].
\]

**Proof of Lemma 12.1** For each \(t > 0\)
\[
P(|Z_1| \geq \sqrt{2t}) \leq 2 \exp[-t].
\]

So by the union bound, for any \(t > 0\),
\[
P\left( \max_{1 \leq j \leq p} |Z_j| > \sqrt{2(\log(2p) + t)} \right) \leq p P(|Z_1| \geq \sqrt{2(\log(2p) + t)}) \leq 2p \exp[-(\log(2p + t)] = \exp[-t].
\]

\[\square\]

**Lemma 12.2** Let \(Z := (Z_1, \ldots, Z_T)^T\) be a vector with i.i.d. standard Gaussian entries. Then it holds for all \(x > 0\) that
\[
P\left( \|Z\|_2 \geq \sqrt{T} + \sqrt{2x} \right) \leq \exp[-x]
\]

46
\[ \mathbb{P}\left(||Z||_2 - \sqrt{T} \geq \sqrt{2x}\right) \leq 2 \exp[-x]. \]

**Proof of Lemma 12.2.** This follows from concentration of measure \cite[Borell 1975, Giné and Nickl 2015, Theorem 2.5.7]{Borell1975, GineNickl2015} because the map \( Z \mapsto ||Z||_2 \) is Lipschitz. Alternatively, one may apply Lemma 1 in \cite[Laurent and Massart 2000]{LaurentMassart2000}.

**Lemma 12.3** Let \( (U, V) \in \mathbb{R}^{n \times 2} \) have i.i.d Gaussian rows with mean zero and covariance matrix
\[
\begin{pmatrix}
\sigma_u^2 & \sigma_{uv} \\
\sigma_{uv} & \sigma_v^2
\end{pmatrix}.
\]

Then for all \( t > 0 \), with probability at least \( 1 - 4 \exp[-t] \)
\[ |U^T V - n\sigma_{uv}| \leq 3\sigma_u\sigma_v \left( \sqrt{2nt} + t \right). \]

**Proof of Lemma 12.3.** By standard arguments (see \cite[van de Geer 2017]{VanDeGeer2017} for tracking down some constants) one can derive that with probability at least \( 1 - 4 \exp[-t] \)
\[ |U^T V - n\sigma_{uv}| \leq (\sigma_u\sigma_v + 2|\sigma_{u,v}|)\sqrt{2nt} + (\sigma_u\sigma_v + 2|\sigma_{u,v}|)t. \]

We simplify this to: with probability at least \( 1 - 4 \exp[-t] \)
\[ |U^T V - n\sigma_{uv}| \leq 3\sigma_u\sigma_v \left( \sqrt{2nt} + t \right). \]

This is the concentration of measure lemma that we use in Section 4.

**Lemma 12.4** For any \( b \in \mathbb{R}^p \) and all \( x > 0 \), we have
\[ \mathbb{P}\left(||X(\hat{\beta} - b)||_2 \geq m_b + \sqrt{2x}\right) \leq \exp[-x] \]
and
\[ \mathbb{P}\left(||X(\hat{\beta} - b)||_2 - m_b \geq \sqrt{2x}\right) \leq 2 \exp[-x] \]
where \( m_b := \mathbb{E}(||X(\hat{\beta} - b)||_2|X) \).

**Proof of Lemma 12.4.** This follows from concentration of measure see e.g. \cite[Borell 1975, or Giné and Nickl 2015, Theorem 2.5.7]{Borell1975, GineNickl2015}, as the map \( \epsilon \mapsto ||X(\hat{\beta} - b)||^2 \) is Lipschitz, see also \cite[van de Geer and Wainwright 2017]{VanDeGeerWainwright2017}.

Finally, we give a result for Gaussian quadratic forms.
Lemma 12.5 Let $X$ have i.i.d. $N(0, \Sigma_0)$-distributed rows and let $M$ be a (sequence of) constant(s) such that

$$M^2 = o\left(n/(\|\Sigma_0\|_\infty \log(2p))\right).$$

Then, for a suitable sequence $\eta_M = o(1)$, with probability tending to one

$$\inf_{\|b\|_1 \leq M\|\Sigma_0^{1/2}b\|_2} \frac{\|Xb\|_2^2/n}{\|\Sigma_0^{1/2}b\|_2^2} \geq (1 - \eta_M)^2.$$

Proof of Lemma 12.5. See for example Chapter 16 in van de Geer [2016] and its references, or van de Geer and Muro [2014]. \qed

References

P. C. Bellec. Optimistic lower bounds for convex regularized least-squares. \textit{arXiv preprint arXiv:1703.01332}, 2017.

V. Belloni, A. and Chernozhukov and L. Wang. Pivotal estimation via square-root lasso in nonparametric regression. \textit{Annals of Statistics}, 42(2):757–788, 2014.

C. Borell. The Brunn-Minkowski inequality in Gauss space. \textit{Inventiones Mathematicae}, 30(2):207–216, 1975.

A. S. Dalalyan, M. Hebiri, and J. Lederer. On the prediction performance of the lasso. \textit{Bernoulli}, 23(1):552–581, 2017.

D.L. Donoho and J. Tanner. Neighborliness of randomly projected simplices in high dimensions. \textit{Proceedings of the National Academy of Sciences of the United States of America}, 102(27):9452–9457, 2005.

E. Giné and R. Nickl. \textit{Mathematical Foundations of Infinite-Dimensional Models}. Cambridge University Press, 2015.

C. Giraud. \textit{Introduction to High-Dimensional Statistics}, volume 138. CRC Press, 2014.

V. Koltchinskii, K. Lounici, and A.B. Tsybakov. Nuclear-norm penalization and optimal rates for noisy low-rank matrix completion. \textit{Annals of Statistics}, 39(5):2302–2329, 2011.

B. Laurent and P. Massart. Adaptive estimation of a quadratic functional by model selection. \textit{Annals of Statistics}, pages 1302–1338, 2000.

T. Sun and C.-H. Zhang. Scaled sparse linear regression. \textit{Biometrika}, 99:879–898, 2012.

R. Tibshirani. Regression analysis and selection via the Lasso. \textit{Journal of the Royal Statistical Society Series B}, 58:267–288, 1996.
S. van de Geer. *Estimation and Testing Under Sparsity: École d’Été de Probabilités de Saint-Flour XLV-2016*. Springer Science & Business Media, 2016.

S. van de Geer. On the efficiency of the de-biased lasso, 2017. arXiv:1708.07986.

S. van de Geer and A. Muro. On higher order isotropy conditions and lower bounds for sparse quadratic forms. *Electronic Journal of Statistics*, 8:3031–3061, 2014.

S. van de Geer and M. Wainwright. On concentration for (regularized) empirical risk minimization. *Sankhyā*, 79-A:159–200, 2017.

S.A. van de Geer. The deterministic Lasso. In *JSM proceedings, 2007*, 140. American Statistical Association, 2007.

Y. Zhang, M. Wainwright, and M. Jordan. Lower bounds on the performance of polynomial-time algorithms for sparse linear regression. In *COLT*, pages 921–948, 2014.