On the uncertainty relations for an electron in a constant magnetic field

Francisco M. Fernández

INIFTA (UNLP, CCT La Plata-CONICET), División Química Teórica, Blvd. 113 S/N, Sucursal 4, Casilla de Correo 16, 1900 La Plata, Argentina

E-mail: fernande@quimica.unlp.edu.ar
Abstract. We discuss the uncertainty relation for the azimuthal angle \( \phi \) and the \( z \)-component of the angular momentum \( L_z \). To this end we derive the uncertainty relation for an arbitrary pair of observables and discuss the conditions for its validity. By means of a simple parameter-dependent state we illustrate the well-known fact that the standard uncertainty relation for the coordinate and its conjugate momentum does not apply the pair of observables \( \phi - L_z \). This analysis is motivated by a discussion of the motion of an electron in a constant magnetic field appeared recently in this journal (Eur. J. Phys. 33 (2012) 1147) where the author assumed the validity of the standard uncertainty relation for the pair \( \phi - L_z \).

1. Introduction

In a recent paper Strange [1] discussed some quantum-mechanical properties of an electron in a constant magnetic field. Since the system is axially symmetric along the field direction (chosen to be the \( z \) axis) then the projection of the angular momentum along that axis is a constant of the motion. The motion of the electron is free along the \( z \) axis and bounded on the plane \( x - y \). Restricting the motion of the electron to this plane Strange discussed the uncertainty relation for the azimuthal angle \( \phi \) and the \( z \)-component of the angular momentum \( L_z \) that he assumed to be \( \Delta \phi \Delta L_z \geq \hbar \). However, he did not take into account some of the subtleties of this uncertainty relation that make it quite different from that for a cartesian coordinate and its conjugate linear momentum. The \( \phi - L_z \) uncertainty relation was discussed by several authors in the past [2–7]. There is even an interesting series of pedagogical articles on the subject [4–7], not without some controversy [4,5]. According to those papers the uncertainty relation invoked by Strange is incorrect. For this reason we deem it worthwhile to carry out a more detailed analysis of the results derived by this author, particularly because the \( \phi - L_z \) uncertainty relation is suitable for an undergraduate course on quantum mechanics [4–7].

In section 2 we derive the uncertainty relation for an arbitrary pair of observables following Chisolm [7]. In section 3 we first outline Strange’s results based on the incorrect
On the uncertainty relations for an electron in a constant magnetic field

φ − Lz uncertainty relation and then derive an exact one following Kraus [3, 5] and Chisolm [7]. We also contrast the exact uncertainty relation with the incorrect one by means of a state that is somewhat more general than the one chosen by Strange. Finally, in section 4 we summarize the main results of this paper and draw conclusions.

2. The uncertainty relations

In order to make this paper sufficiently self-contained and facilitate the discussion of the uncertainty relation for the electron in a constant magnetic field [1] in what follows we derive the uncertainty relation for an arbitrary pair of observables. There are different ways of deriving it [6, 7] and in what follows we resort to the well known Schwarz inequality [7]. To this end consider the usual complex inner product in quantum mechanics in terms of the bra-ket notation:

\[ \langle f | g \rangle = \langle g | f \rangle^* \]

The Schwarz inequality states that

\[ |\langle f | g \rangle|^2 \leq \langle f | f \rangle \langle g | g \rangle \]

for any two vectors |f⟩ and |g⟩ in the state vector space. Chisolm [7] derived a somewhat more general uncertainty relation from the obvious expression

\[ |\langle f | g \rangle|^2 = \frac{1}{4} (\langle f | g \rangle + \langle g | f \rangle)^2 + \frac{1}{4} |\langle f | g \rangle - \langle g | f \rangle|^2 \]

However, for present purposes it is sufficient to take into account that

\[ |\langle f | g \rangle| \geq \frac{1}{2} |\langle f | g \rangle - \langle g | f \rangle| \]

(3)

(that is to say |\langle f | g \rangle| \geq |\text{Im} \langle f | g \rangle|)) that leads to

\[ \sqrt{\langle f | f \rangle \langle g | g \rangle} \geq \frac{1}{2} |\langle f | g \rangle - \langle g | f \rangle| \]

(4)

Let |ψ⟩ be the state of the system normalized to unity (⟨ψ | ψ⟩ = 1) and A and B the Hermitean operators for two quantum-mechanical observables. We define

|f⟩ = (A − ⟨A⟩) |ψ⟩ and |g⟩ = (B − ⟨B⟩) |ψ⟩, where ⟨Q⟩ = ⟨ψ | Q | ψ⟩, so that

\[ \langle f | f \rangle = \langle A^2 \rangle - \langle A \rangle^2 = (\Delta A)^2 \]

\[ \langle g | g \rangle = \langle B^2 \rangle - \langle B \rangle^2 = (\Delta B)^2 \]

(5)
On the uncertainty relations for an electron in a constant magnetic field

Since \( \langle f | g \rangle = \langle \hat{A} \psi | \hat{B} \psi \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle \) then it follows from equation (4) that

\[
\Delta A \Delta B \geq \frac{1}{2} |\langle \hat{A} \psi | \hat{B} \psi \rangle - \langle \hat{B} \psi | \hat{A} \psi \rangle| \tag{6}
\]

If \( \hat{B} \psi \rangle \) belongs to the domain of \( \hat{A} \) and \( \hat{A} \psi \rangle \) to the domain of \( \hat{B} \) then we can write

\[
\langle \hat{A} \psi | \hat{B} \psi \rangle = \langle \psi | \hat{A} \hat{B} | \psi \rangle \\
\langle \hat{B} \psi | \hat{A} \psi \rangle = \langle \psi | \hat{B} \hat{A} | \psi \rangle \tag{7}
\]

and thus obtain the standard uncertainty relation

\[
\Delta A \Delta B \geq \frac{1}{2} |\langle [\hat{A}, \hat{B}] \rangle| \tag{8}
\]

where \( [\hat{A}, \hat{B}] = \hat{A} \hat{B} - \hat{B} \hat{A} \) is the well known commutator. The interested reader will find a more detailed discussion of the domains and ranges of operators in the literature already cited \([2–7]\).

Before applying the results of this section to a particular model in the next one it is worth stressing the fact that equation (8) is valid provided that the root-mean-square deviations \( \Delta A \) and \( \Delta B \) are calculated according to equation (5) and that equations (7) hold for the chosen state \( |\psi \rangle \). If the chosen state and operators do not satisfy the latter conditions we can still use the more general inequality (6).

3. Uncertainty relation for the azimuthal angle and angular momentum

Strange \([1]\) described the motion of the electron in the \( x - y \) plane in polar coordinates \( x = r \cos \phi, y = r \sin \phi \), where \( 0 \leq r = \sqrt{x^2 + y^2} < \infty \) and \( 0 \leq \phi < 2\pi \). For simplicity we omit the variable \( r \) that is not relevant to the discussion of the uncertainty relation for \( \hat{\phi} \) and \( \hat{L}_z \) that commutes with the Hamiltonian operator of the system. In the coordinate representation we define these operators as follows:

\[
\hat{\phi} \psi (\phi) = \phi \psi (\phi) \\
\hat{L}_z \psi (\phi) = -i\hbar \frac{\partial}{\partial \phi} \psi (\phi) \tag{9}
\]

where \( \psi (\phi) = \langle \phi | \psi \rangle \). Although it has been argued that this definition of the quantum-mechanical operator for the azimuthal angle may not be correct \([2,4,5]\) we keep it here.
because it is relevant to the discussion of the results obtained by Strange \[1\]. Besides, Chisolm \[7\] already chose this definition of \( \hat{\phi} \) in his discussion of the uncertainty relations. We assume the state vectors to be periodic functions of period \( 2\pi \) \((f(\phi + 2\pi) = f(\phi))\) and choose the standard inner product

\[
\langle f | g \rangle = \int_{0}^{2\pi} f(\phi) g(\phi) \, d\phi
\]  

Strange \[1\] stated that “The azimuthal angle-angular momentum uncertainty relation is \( \Delta \phi \Delta L \geq \hbar \).” The origin of this uncertainty relation is unclear as it differs from the standard one \( \Delta x \Delta p \geq \hbar / 2 \) for the coordinate \( x \) and its conjugate momentum \( p \). In order to verify this uncertainty relation he later chose “an equally weighted sum of the \( m = 0 \) and \( m = 1 \) state”. Since he did not write the state explicitly we suppose that it was of the form

\[
\psi_S(\phi) = \frac{1}{2\sqrt{\pi}} \left( 1 + e^{i\phi} \right)
\]  

from which we obtain \( \langle \hat{L}_z \rangle = \hbar / 2, \langle \hat{L}_z^2 \rangle = \hbar^2 / 2 \) and \( \Delta L_z = \hbar / 2 \) in agreement with his results. Arguing that “the uncertainty in angle arises directly from the fact that the origin of the angular coordinate is arbitrary” he chose \( (\Delta \phi)_S = \pi \) and obtained \( (\Delta \phi)_S \Delta L_z = \pi \hbar / 2 \). However, in section \[2\] we showed that the uncertainty relation \[8\] is valid if the root-mean-square deviations are calculated as in equation \[5\]. In the present case the inequality holds for \( \Delta \phi = \sqrt{2 + \pi^2 / 3} \) and, therefore, also for \( (\Delta \phi)_S > \Delta \phi \).

The results just discussed are valid for the particular state \[11\]. It is convenient to derive the \( \phi - L_z \) uncertainty relation for an arbitrary wave function \( \psi(\phi) \) of period \( 2\pi \). If we integrate \( \langle \hat{L}_z \psi | \phi \psi \rangle \) by parts we obtain \[7\]

\[
\langle \hat{L}_z \psi | \phi \psi \rangle = \langle \psi | \hat{L}_z \phi \psi \rangle + i\hbar 2\pi |\psi(2\pi)|^2
\]  

and equation \[6\] leads to the exact inequality

\[
\Delta \phi \Delta L_z \geq \frac{\hbar}{2} \left| 2\pi |\psi(2\pi)|^2 - 1 \right|
\]  

already derive earlier by other authors \[3,5,7\]. The reason why \( \langle \hat{L}_z \psi | \phi \psi \rangle \neq \langle \psi | \hat{L}_z \phi \psi \rangle \) is that \( \phi \psi(\phi) \), unlike \( \psi(\phi) \), is not a periodic function of period \( 2\pi \) and, consequently,
On the uncertainty relations for an electron in a constant magnetic field

does not belong to the domain of $\hat{L}_z$ (a more detailed discussion of this issue is available in the articles already cited [2–7]). However, note that when $|\psi\rangle = |\psi_S\rangle$ the right-hand-side of equation (13) is exactly $\hbar/2$ because $|\psi_S(2\pi)|^2 = 1/\pi$. In other words, the ‘standard’ uncertainty relation $\Delta \phi \Delta L_z \geq \hbar/2$ is valid for the particular wave function $\psi_S(\phi)$ chosen by Strange as an illustrative example.

Since the right-hand side of equation (13) may be smaller than $\hbar/2$ the standard uncertainty relation $\Delta \phi \Delta L_z \geq \hbar/2$ is not guaranteed. We think that it is a worthy pedagogical experiment to test its validity on other state functions. For example, we can try a more general linear combination of the same two states with $m = 0$ and $m = 1$:

$$\psi(a, \phi) = \frac{1}{\sqrt{2\pi}} \left( a + \sqrt{1-a^2} e^{i\phi} \right)$$

(14)

where $-1 \leq a \leq 1$, which reduces to $\psi_S(\phi)$ when $a = 1/\sqrt{2}$. With this simple function we easily obtain

$$R(a) = \frac{\hbar}{2} \left| 2\pi |\psi(a, 2\pi)|^2 - 1 \right| = \hbar |a| \sqrt{1-a^2}$$

$$\Delta L_z = \hbar |a| \sqrt{1-a^2} = R(a)$$

$$\Delta \phi = \left( 4a \sqrt{1-a^2} + \frac{\pi^2}{3} \right)^{1/2}$$

(15)

Note that $\Delta L_z = 0$ when $a = 0$ or $a = 1$ because $\psi(0, \phi)$ and $\psi(1, \phi)$ are eigenfunctions of $\hat{L}_z$, and that in both cases $\Delta \phi = \pi/\sqrt{3}$. Besides, it follows from $\Delta \phi \Delta L_z \geq R(a)$ that $\Delta \phi \geq 1$ for all $-1 \leq a \leq 1$.

Fig. 1 shows that $\Delta \phi \Delta L_z = \hbar/2$ at four points: $a_1 \approx -0.91$, $a_2 \approx -0.41$, $a_3 \approx 0.25$ and $a_4 \approx 0.97$. The standard uncertainty relation $\Delta \phi \Delta L_z \geq \hbar/2$ holds only for $a_1 \leq a \leq a_2$ and $a_3 \leq a \leq a_4$, while, on the other hand, the exact one $\Delta \phi \Delta L_z \geq R(a)$ is valid for all $a$. In addition to it, $R(a) = \hbar/2$ only for $a = \pm 1/\sqrt{2}$, that is to say, for an equally weighted sum of the states with $m = 0$ and $m = 1$.

Fig. 2 shows that $\pi > \Delta \phi > 1$ for all $-1 \leq a \leq 1$ so that if the uncertainty relation holds for the root-mean-square deviation $\Delta \phi$ then it also holds for $(\Delta \phi)_S = \pi$ as argued above.
Fig. 3 shows that $\pi \Delta L_z = \hbar/2$ at four points $a'_1 = -a'_4 \approx -0.99$ and $a'_2 = -a'_3 \approx -0.16$ and that $\pi \Delta L_z \geq \hbar/2$ for $a'_1 \leq a \leq a'_2$ and $a'_3 \leq a \leq a'_4$. This uncertainty relation fails for $a$ outside those intervals. We appreciate that the inequality invoked by Strange [1] (which he arbitrarily chose to be $\Delta \phi \Delta L_z \geq \hbar$) is not valid for all possible states of the system.

For simplicity we have restricted the discussion of the uncertainty relation to states that depend only on the azimuthal angle. By no means does such restriction invalidate the conclusions drawn from the state (14) that are illustrated in figures 1, 2 and 3. However, as a further pedagogical exercise it is worth taking into account the actual motion of the electron on the $x-y$ plane. If we repeat the calculation for states $f(r, \phi) = \langle r, \phi | f \rangle$ and the inner product

$$\langle f | g \rangle = \int_0^\infty \int_0^{2\pi} f(r, \phi)^* g(r, \phi) r \, d\phi \, dr$$

(16)

we obtain the exact uncertainty relation

$$\Delta \phi \Delta L_z \geq \frac{\hbar}{2} |2\pi \rho(2\pi) - 1|$$

(17)

where

$$\rho(\phi) = \int_0^\infty |\psi(r, \phi)|^2 \, dr$$

(18)

Equation (17) is a generalization of the uncertainty relation (13) that was derived earlier by Kraus [3,5]. Note that equation (17) is suitable for the $(r, \phi)$-dependent states chosen by Strange [1] to illustrate the probability backflow. For example, using Strange’s three-term wavefunction (his equation (11) properly normalized) [1] we obtain $\Delta \phi \Delta L_z \approx 1.99\hbar$ and $\frac{\hbar}{2} |2\pi \rho(2\pi) - 1| \approx 0.844\hbar$ that satisfy the uncertainty relation (17). Exactly in the same way we can easily generalize the uncertainty relations derived by Chisolm [7] that provide tighter lower bounds to the products of square-root-mean deviations.

4. Conclusions

In this paper we have shown that the uncertainty relation invoked by Strange [1] in his discussion of the probability backflow is only valid for a particular set of wave functions.
The electron in a constant magnetic field is a suitable example for showing that the $\phi - L_z$ uncertainty relation should be applied carefully because it is different from the $x - p$ one. In order to keep the discussion as simple as possible we have avoided more complicated issues like the correct form of the operator for the azimuthal angle and of its square-root-mean deviation [2–5]. Instead, we have kept the most straightforward definitions of both the operator $\hat{\phi}$ and its square-root-mean deviation $\Delta \phi$ [7] that proved suitable for the analysis of the results obtained by Strange [1].

Finally, we point out that in the case of the motion of a particle in three dimensions one can easily derive uncertainty relations similar to equation (17) that generalize those derived earlier by other authors [3, 5, 7].

[1] Strange P 2012 Eur. J. Phys. 33 1147.
[2] Judge D 1964 Nuovo Cim. 31 332.
[3] Kraus K 1965 Z. Physik 188 374.
[4] Perlman H S and Troup G J 1969 Am. J. Phys. 37 1060.
[5] Kraus K 1970 Am. J. Phys. 38 1489.
[6] Peslak Jr. J 1979 Am. J. Phys. 47 39.
[7] Chisolm E D 2001 Am. J. Phys. 69 368.
Figure 1. $\Delta \phi \Delta L_z / \hbar$ and $R(a) / \hbar$ vs. $a$

Figure 2. $\Delta \phi$ vs. $a$
Figure 3. $\pi \Delta L_z / \hbar$ vs. $a$