ON EXTENSION OF OVERCONVERGENT LOG ISOCRISTALS ON LOG SMOOTH VARIETIES

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Abstract. By works of Kedlaya and Shiho, it is known that, for a smooth variety $\mathcal{X}$ over a field of positive characteristic and its simple normal crossing divisor $Z$, an overconvergent isocrystal on the compliment of $Z$ satisfying a certain monodromy condition can be extended to a convergent log isocrystal on $(\mathcal{X}, \mathcal{M}_Z)$, where $\mathcal{M}_Z$ is the log structure associated to $Z$. We prove a generalization of this result: for a log smooth variety $(\mathcal{X}, \mathcal{M})$ satisfying some conditions, an overconvergent log isocrystal on the trivial locus of a direct summand of $\mathcal{M}$ satisfying a certain monodromy condition can be extended to a convergent log isocrystal on $(\mathcal{X}, \mathcal{M})$.

0. Introduction

Let $K$ be a field of characteristic 0 complete with respect to a non-Archimedean valuation whose residue field $k$ has a positive characteristic $p$. Let $\mathcal{V}$ be the ring of integer of $K$.

Kedlaya proved in [13] that, for a smooth variety $\mathcal{X}$ and an open subset $\mathcal{X}$ of $\mathcal{X}$ such that $Z := \mathcal{X} \setminus \mathcal{X}$ is a simple normal crossing divisor, an overconvergent isocrystal on $(\mathcal{X}, \mathcal{M}) / K$ with unipotent monodromy can be extended to a convergent log isocrystal on $(\mathcal{X}, \mathcal{M}) / K$ where $\mathcal{M}$ is the log structure on $\mathcal{X}$ associated to $Z$. Shiho extended this result to the case of more general monodromy ($\Sigma$-unipotence) in [21]. These results are $p$-adic analogue of the theory of canonical extension of regular singular integrable connections on algebraic varieties over $C$ which is developed by Deligne in [5].

We give an overview of Shiho’s result. Let $Z = \bigcup_{i=1}^{r} Z_i$ be the decomposition of $Z$ to irreducible components. Let $Z'_i$ be the smooth locus of $Z_i$. Take a sufficiently local lift $\mathcal{X} \hookrightarrow P$ into a smooth $p$-adic formal scheme $P$ over $\mathcal{V}$. Then the tube $|Z'_i|_{P}$ is isomorphic to a product of a disk $\mathbb{A}_K^1 [0,1)$ and the rigid space $Q_K$ associated to a locally closed smooth $p$-adic formal subscheme $Q$ of $P$ which lifts $Z'_i$. An overconvergent isocrystal $\mathcal{E}$ on $(\mathcal{X}, \mathcal{M}) / K$ is represented by a $\nabla$-module on some strict neighborhood $W$ of $|X|_{P}$. The intersection of $W$ and $|Z'_i|_{P}$ contains a product of an annulus $\mathbb{A}_K^1 [\lambda, 1)$ and $Q_K$ for some $0 < \lambda < 1$. Let $\Sigma_i$ be a subset of $Z_p$ satisfying some non-Liouville hypothesis. $\mathcal{E}$ is $\Sigma_i$-unipotent along $Z_i$ if $\mathcal{E}$, restricted to the above product, has a filtration whose successive quotients are pullbacks of $\nabla$-modules on $Q_K$ twisted by $\nabla$-modules $(\mathcal{O}_{\mathbb{A}_K^1 [\lambda, 1)}, d + \xi \cdot \text{id})$ on the annulus for some element $\xi$ of $\Sigma_i$. For $\Sigma = \prod_{i=1}^{r} \Sigma_i$, if $\mathcal{E}$ has $\Sigma$-unipotent along $Z_i$ for each

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i, then $E$ can be extended to a convergent log isocrystal on $(\mathcal{X},\mathcal{M})/K$ whose
exponents are contained in $\Sigma$.

Di Proietto extended this theory in [7] to the case where $\mathcal{X}$ is the log special
fiber of a proper semistable variety over $V$.

In this paper, we extend the above results to the case when $\mathcal{X}$ is not necessarily
smooth but log smooth with some log structure on $\mathcal{X}$ and $X$ is the trivial locus
of a part of the log structure, under some assumptions: Let $N \to V$ be a log
structure on $\text{Spf} V$ which is trivial on $\text{Spm} K$. Let $(\mathcal{X},\mathcal{M}_0 \oplus \mathcal{M})$ be a log smooth
variety over $(\text{Spec} k,N)$ satisfying some conditions. (In particular, the morphism
$N \to \mathcal{M}_0 \oplus \mathcal{M}$ must factor through $\mathcal{M}_0 \to \mathcal{M}_0 \oplus \mathcal{M}$.) Let $X$ be the trivial locus
of $\mathcal{M}$. Let $\mathcal{S}$ be a subsheaf of $\mathcal{M}^p \otimes_{\mathbb{Z}} \mathbb{Z}_p$ satisfying some non-Liouville hypothesis.
We prove that an overconvergent log isocrystal on $(X,\mathcal{X},\mathcal{M}_0 \oplus \mathcal{M})/(\text{Spf} V,N)$ with
"$\mathcal{S}$-unipotent" monodromy can be extended to a convergent log isocrystal on
$(\mathcal{X},\mathcal{M}_0 \oplus \mathcal{M})/(\text{Spf} V,N)$.

For the definition of $\mathcal{S}$-unipotence and the proof of the above result, a generalized
polyannulus $\mathcal{A}_{M,K}(I)$ associated to a fine monoid $M$ which is a polyannulus in
$\text{Spm} K\langle M \rangle$ plays the role of the polyannulus $\mathcal{A}(I)$ defined in [13].

Our result is an extension of the results of Kedlaya [13], Shiho [21] and Di Proi-
etto [7], except that our definition of unipotence of overconvergent log isocrystals
is stronger than theirs. In the situation of [13], [21] and [7], we can check the
unipotence at each codimension one component. (See Proposition 4.4.4 in [13].)
However in our situation, we must assume the unipotence at every point. (See
Remark 3.5.8.) So some arguments in [13] or in [21] can be omitted in our paper.

We explain the contents of each section. In Section 1, we prepare the basic
notions and results about monoid theory and log structures. Especially, we define
the notion of semi-saturatedness of monoids in §1.1, which is a weaker version of
saturatedness. In §1.2, we define the notions of log structure and log smoothness
on schemes, formal schemes and rigid spaces.

In Section 2, we prove some properties of log $\nabla$-modules on log rigid spaces.
Roughly speaking, the contents of this section is a generalization of those of Section
1 and Section 2 of [21]. In §2.1, we define the notion of polyannuli associated to
fine weighted monoids and prove their basic properties. We introduce the notion
of log $\nabla$-modules with exponents in $\Sigma$ in §2.2 and the notion of $\Sigma$-unipotent log
$\nabla$-modules in §2.3. Next, we adapt key propositions in Section 2 of [21] to our
situation. Generalization proposition is proven in §2.4 and transfer theorem is proven
in §2.5.

In Section 3, we develop the theory of log isocrystals and prove the main re-
result. In §3.1, we generalize the equivalence between the category of convergent log
isocrystals and the category of convergent log $\nabla$-modules (Theorem 6.4.1 in [13])
to our situation. We introduce the notion of log isocrystals with exponents in $\mathcal{S}$ in
§3.2 and the notion of $\mathcal{S}$-unipotent log isocrystals in §3.3. But the global definition
of $\mathcal{S}$-unipotence is postponed to §3.5. In §3.4, we adapt overconvergent generalization
proposition (Proposition 2.7 in [21]) to our situation. In §3.5, we define the notion
of $\mathcal{S}$-unipotence of log isocrystals and prove the main result.

0.1. Conventions.

(1) Throughout this paper, $K$ is a field of characteristic 0 complete with respect
to a non-Archimedean valuation $|\cdot|$. $k$ is the residue field of $K$ and it is
assumed to be a field of characteristic $p > 0$. $\mathcal{V}$ is the ring of integers of
1. Preliminaries

1.1. Monoids. Let $M$ be a monoid. $M$ is called \textit{finitely generated} if there exists a surjective monoid homomorphism $\mathbb{N}^r \to M$ for some $r \in \mathbb{N}$. $M$ is called \textit{integral} if the natural map $M \to M^{\text{gp}}$ is injective, i.e., for any $a, b, c \in M$ with $a + c = b + c$, $a = b$. $M$ is called \textit{fine} if it is finitely generated and integral. $M$ is called \textit{sharp} if $M^\ast = \{0\}$. An integral monoid $M$ is called \textit{saturated} if for any $a \in M^{\text{gp}}$ and $n \in \mathbb{N}_{>0}$ with $na \in M$, $a \in M$.

Moreover, we define a property of monoids which is weaker than saturatedness.

**Definition 1.1.1.** A fine monoid $M$ is called \textit{semi-saturated} if for any $a \in M^{\text{gp}}$ and $n \in \mathbb{N}_{>0}$ the exists $m \in \mathbb{N}$ such that $(nm + 1)a \in M$.

A \textit{submonoid} of $M$ is a subset $N$ which is stable under the operation $+$ and contains the identity element 0. For a submonoid $N \subseteq M$, we define a monoid $M/N$ by dividing $M$ by the equivalence relation $\equiv \cdot (\mod N)$ which is defined as follows: $a \equiv b \ (\mod N)$ if there exist $c, d \in N$ such that $a + c = b + d$. If $M$ is fine, $(M/N)^{\text{gp}} = M^{\text{gp}}/N^{\text{gp}}$ and $M/N$ is the image of $M$ under the canonical map $M \to M^{\text{gp}} \to M^{\text{gp}}/N^{\text{gp}}$. $N^{-1}M$ denotes the submonoid of $M^{\text{gp}}$ generated by $M$. 

$K^\ast$ is the subset $\sqrt{|K^\times|} \cup \{0\}$ of $\mathbb{R}_{>0}$. In Section 3, we will assume moreover that the valuation of $K$ is discrete.

(2) For a field extension $K \subseteq K'$, we denote the scalar extension $- \otimes_K K'$ by $(-)^{K'}$.

(3) In this paper, a \textit{monoid} means a commutative monoid with an identity element. The operation of a monoid is written additively and the identity element of a monoid is denoted by 0. The set of units of a monoid $M$ is written as $M^\ast$ and $\overline{M}$ means $M/M^\ast$. The Grothendieck group of a monoid $M$ is written as $M^{\text{gp}}$.

(4) In this paper, for a fine monoid $M$ and $m, m' \in M$, we write $m' \leq m$ if there exists $m'' \in M$ such that $m' + m'' = m$ and we write $m' < m$ if $m' \leq m$ and $m' \neq m$. If $M$ is sharp, with this binary relation $\leq$, $(M, \leq)$ is a poset.

(5) For a finitely generated abelian group $G$, the torsion subgroup of $G$ is written as $G^{\text{tor}}$. We write $G/G^{\text{tor}}$ as $G^{\text{free}}$.

(6) In this paper, $\mathbb{N}$ denotes the set of non-negative integers (hence $0 \in \mathbb{N}$) and it is regarded as a monoid with the additive operation $\cdot$. The set of positive integers is denoted by $\mathbb{N}_{>0}$.

(7) For a complete topological ring $A$ with respect to a norm $|\cdot|$, $A\langle X_1, \ldots, X_n \rangle$ denotes the ring of series $\sum_{\mathbf{c} \in \mathbb{N}^n} c_{\mathbf{c}} X_1^{c_1} \cdots X_n^{c_n}$ such that the set

$$\left\{ \mathbf{c} \in \mathbb{N}^n \mid |c_{\mathbf{c}}| > a \right\}$$

is finite for any $a \in \mathbb{R}_{>0}$. More generally, for a monoid $M$, $A(M)$ denotes the ring of series $\sum_{m \in M} c_m t^m$ such that the set $\{ m \in M \mid |c_m| > a \}$ is finite for any $a \in \mathbb{R}_{>0}$. Here, $t^m$ is the element associated to $m \in M$ and $t^{m_1 + m_2} = t^{m_1} \cdot t^{m_2}$ for any $m_1, m_2 \in M$.

(8) In this paper, a point of a rigid space means a closed point of it. A geometric point of a rigid space means a geometric point lying over a closed point of it. 


Proposition 1.1.4 in [10]. Assume that The results except the last equality is a consequence of Lemma 3.1.3 and Proof. Moreover, if \( M \) is semi-saturated, any \( m \in M \) exists no face \( F \) such that \( F \subseteq F' \subseteq M \).

Proposition 1.1.2.

1. If \( M \) is saturated, it is semi-saturated.
2. If \( M \) is semi-saturated and sharp, \( M^{sp} \) is torsion-free.
3. If \( M \) is semi-saturated, for any submonoid \( N, M/N \) is semi-saturated.
4. A fine monoid \( M \) is semi-saturated if and only if for any face \( F \) of \( M \), \((M/F)^{sp}\) is torsion-free.

Proof.

1. It is clear.
2. Assume that for \( a \in M^{sp} \) and \( n \in \mathbb{N}_{>0}, na = 0 \). Then there exists \( m \in \mathbb{N} \) such that \((nm + 1)a = a \in M\). Then, since \( a + (n - 1)a = 0 \) and \( M \) is sharp, \( a = 0 \).
3. Take \( a \in M^{sp} \) and \( n \in \mathbb{N}_{>0} \) such that \( n\overline{a} \in M/N \) where \( \overline{a} \) is the image of \( a \) in \((M/N)^{sp}\). There exists \( b \in N \) such that \( na + b \in M \), so \( n(a + b) \in M \). Then there exists \( m \in \mathbb{N} \) such that \((mn + 1)(a + b) \in M \). This implies \((mn + 1)\overline{a} \in M/N \).
4. The former condition implies the latter condition as a consequence of 2 and 3. Conversely, assume that for any face \( F \), \((M/F)^{sp}\) is torsion-free. Take \( a \in M^{sp}, n \in \mathbb{N}_{>0} \) such that \( na \in M \). Let \( F := \{ b \in M | \exists m \in \mathbb{N}, b \leq nma \} \). This is a face of \( M \). Let \( \overline{a} \) be the image of \( a \) in \((M/F)^{sp}\). Then \( n\overline{a} = 0 \) by the definition of \( F \). Since \((M/F)^{sp}\) is torsion-free by assumption, \( \overline{a} = 0 \). Then there exists \( b \in F \) such that \( a + b \in F \) so we can take \( m \in \mathbb{N}, c \in M \) such that \( b + c = nma \) by the definition of \( F \). Therefore \((nm + 1)a = (a + b) + c \in M \).

\( \square \)

Remark 1.1.3. One of the simplest examples of semi-saturated but not saturated monoid is \( \mathbb{N} \setminus \{1\} \). Moreover, any submonoid \( M \subseteq \mathbb{N} \) such that \( \mathbb{N} \setminus M \) is finite is semi-saturated.

The first result of the following lemma is almost the same as Corollary 3.1.5 of [10] but the assumption is a bit generalized.

Lemma 1.1.4. Let \( f : N \to M \) be a surjective morphism of fine monoids such that \( M^{sp} \) is torsion-free. Put \( \tilde{N} := (f^{sp})^{-1}(M) \subseteq N^{sp} \). Then the induced morphism \( \tilde{f} : \tilde{N} \to M \) has a section \( M \to \tilde{N} \) and it gives a splitting \( \tilde{N} \cong M \oplus \text{Ker}(f^{sp}) \). Moreover, if \( M \) is sharp, \((\text{Im}(s) + N) \cap \text{Ker}(f^{sp}) = \text{Ker}(f) \).

Proof. The results except the last equality is a consequence of Lemma 3.1.3 and Proposition 3.1.4 in [10]. Assume that \( M \) is sharp. Take \( a \in M \) and \( b \in N \) such that \( s(a) + b \in \text{Ker}(f^{sp}) \). Then \( a + f(b) = 0 \), so \( a = 0 \) and \( f(b) = 0 \). Thus \( s(a) + b = b \in \text{Ker}(f) \).

\( \square \)

We also have to define the notion of vertical homomorphisms.

Definition 1.1.5. A homomorphism \( f : N \to M \) of fine monoids is vertical if for any \( m \in M \) there exists \( n \in N \) such that \( m \leq f(n) \).
1.2. **Log structures.** In this section, let the category of spaces $\text{Esp}$ be one of the followings:

1. The category of schemes over $k$.
2. The category of $p$-adic formal schemes over $V$.
3. The category of rigid spaces over $K$.

Let $q$ be $p$ in the first or the second case, $0$ in the third case.

**Remark 1.2.1.** In [9], Gillam and Molcho developed a general theory of log structures on categories of spaces satisfying some axioms. But they consider only classical topologies, so we cannot use their theory. Talpo and Vistoli defined the notion of log structures on more general topos in [23], but their formalism is too general for our purpose. It seems to be possible to find appropriate axioms of spaces to explicate log structure theories on them, but we do not do it and consider only the categories which appear in this paper.

In this section, we call an object of $\text{Esp}$ a **space**. Let $X$ be a space. Let $X_{\text{ét}}$ be the étale site over $X$ and let $O_X$ be the structure sheaf on $X_{\text{ét}}$ (for the proof of the sheaf property in the case of rigid spaces, see Appendix A. of [6]). A **pre-log structure** on $X$ is a pair of a sheaf of monoids $M$ on $X_{\text{ét}}$ and a morphism of sheaves of monoids $\alpha : M \to O_X$ where $O_X$ is the structure sheaf which is regarded as a sheaf of monoids by the multiplication. A pre-log structure $(M, \alpha)$ is called a **log structure** if $\alpha^{-1}(O_X^*) \cong O_X^*$ via $\alpha$. For a pre-log structure $(M, \alpha)$, we define its **associated log structure** as the push-out of $M$ and $O_X^*$ over $\alpha^{-1}(O_X^*)$ in the category of sheaves of monoids on $X_{\text{ét}}$.

A **log space** is a pair of a space $X$ and a log structure $(M, \alpha)$ on $X_{\text{ét}}$. $M$ denotes $M/O_X^*$. For a rigid space $X$, $O_X^+$ denotes a subsheaf of power-bounded elements of $O_X$. For a scheme or a formal scheme $X$, $O_X^+$ denotes $O_X$. A chart of $(M, \alpha)$ is a pair of a monoid $M$ and a monoid morphism $M \to O_X^+$ such that, when it is regarded as a pre-log structure on $X$, its associated log structure is isomorphic to $M$. A chart $M \to O_X^+(X)$ is **fine** if $M$ is fine. $M$ is **fine** if there exists a fine chart étale locally on $X$.

The category of fine spaces has fiber products.

Let $x$ be a geometric point of $X$. A chart $M \to O_X(x)$ of $(M, \alpha)$ is **good at $x$** if the canonical homomorphism $M \to \overline{M}_{x}$ is an isomorphism.

Let $X$ be a space and $M_1, M_2$ be two log structures. We define the **direct sum** $M_1 \oplus M_2$ of $M_1$ and $M_2$ as the push-out $M_1 \oplus_{O_X} M_2$ in the category of sheaves of monoids on $X$. It is the direct sum in the category of log structures on $X$.

A morphism $f : (X, M) \to (Y, N)$ of log spaces is a pair of a morphism $f : X \to Y$ of spaces and a morphism $f^# : f^{-1}(N) \to M$ of sheaves of monoids such that the following diagram commutes:

$$
\begin{array}{c}
\downarrow & \downarrow \\
\text{f^{-1}(N)} & \rightarrow & M \\
\text{f^{-1}(O_Y)} & \rightarrow & O_X.
\end{array}
$$

$f$ is **strict** if the log structure associated to $f^{-1}(N)$ is isomorphic to $M$ via $f^#$. 


A chart of $f$ is a triplet of a chart $M \to \mathcal{O}_X^+(X)$ of $\mathcal{M}$, a chart $N \to \mathcal{O}_Y^+(Y)$ of $\mathcal{N}$ and a monoid morphism $h : N \to M$ such that the diagram commutes:

$$
\begin{array}{ccc}
N & \xrightarrow{f^{-1}(N)} & M \\
\downarrow{h} & & \downarrow{f} \\
M & \xrightarrow{f} & \mathcal{M}.
\end{array}
$$

A chart of $f$ is fine if $N$ and $M$ are fine. A chart of $f$ is good at a geometric point $\mathfrak{p}$ of $X$ if $M \to \mathcal{O}_X^+(X)$ is good at $\mathfrak{p}$.

For a monoid $M$, $\mathbb{A}_M$ denotes

- $\mathbb{A}_{M,k} := \text{Spec } k[M]$ with log structure associated to $M \to k[M]$ if $\text{Esp}$ is the category of schemes over $k$,
- $\mathbb{A}_{M,V} := \text{Spf } \mathcal{V}(M)$ with log structure associated to $M \to \mathcal{V}(M)$ if $\text{Esp}$ is the category of $p$-adic formal schemes over $\mathcal{V}$,
- $\mathbb{A}_{M,K}[0,1] := \text{Spm } K(M)$ with log structure associated to $M \to K(M)$ if $\text{Esp}$ is the category of rigid spaces over $K$.

For a monoid $M$, giving a chart of a log space $(X, \mathcal{M})$ of the form $M \to \mathcal{O}_X^+(X)$ is equivalent to giving a strict morphism $(X, \mathcal{M}) \to \mathbb{A}_M$. Thus, when a morphism $X \to \mathbb{A}_M$ of spaces is given, we can associate to it a log structure $\mathcal{M}$ on $X$ endowed with a chart the form $M \to \mathcal{O}_X^+(X)$, by defining $\mathcal{M}$ as the pullback of the log structure of $\mathbb{A}_M$ and considering the resulting strict morphism $(X, \mathcal{M}) \to \mathbb{A}_M$.

Next, we fix a chart $N \to \mathcal{O}_Y^+(Y)$ of a log space $(Y, \mathcal{N})$ and a morphism $N \to M$ of monoids. Then, giving a chart of a morphism of log spaces $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ which contains the chart $N \to \mathcal{O}_Y^+(Y)$ and the morphism $N \to M$ as a part of data is equivalent to giving a strict morphism $(X, \mathcal{M}) \to \mathbb{A}_M \times_{\mathbb{A}_N} (Y, \mathcal{N})$ such that the composition of it and the projection $\mathbb{A}_M \times_{\mathbb{A}_N} (Y, \mathcal{N}) \to (Y, \mathcal{N})$ is equal to $f$. Thus, when a morphism $X \to \mathbb{A}_M \times_{\mathbb{A}_N} Y$ of spaces is given, we can associate to it a log structure $\mathcal{M}$ on $X$, a morphism of log spaces $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ whose underlying morphism of spaces is equal to the composition $X \to \mathbb{A}_M \times_{\mathbb{A}_N} Y \to Y$ and a chart of $f$ which contains the chart $N \to \mathcal{O}_Y^+(Y)$ and the morphism $N \to M$ above as a part of data, by defining $\mathcal{M}$ as the pullback of the log structure of $\mathbb{A}_M \times_{\mathbb{A}_N} (Y, \mathcal{N})$ and considering the resulting strict morphism $(X, \mathcal{M}) \to \mathbb{A}_M \times_{\mathbb{A}_N} (Y, \mathcal{N})$.

**Definition 1.2.2.** Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a morphism of fine log spaces.

Assume there exists a chart $N \to \mathcal{O}_Y(Y)$ of $\mathcal{N}$. $f$ is log smooth (resp. log étale) if, étale locally on $X$ and $Y$, there exists a fine chart $h : N \to M$ of $f$ such that $h$ is injective, the torsion part of the cokernel of $h^{\text{gp}}$ (resp. the cokernel of $h^{\text{gp}}$) is finite group whose order is prime to $q$ and the induced morphism $X \to Y \times_{\mathbb{A}_N} \mathbb{A}_M$ is étale.

In general, $f$ is log smooth (resp. log étale) if there exists an (admissible) étale covering $(Y_\lambda)_{\lambda \in \Lambda}$ such that $\mathcal{N}|_{Y_\lambda}$ has a chart and $f|_{f^{-1}(Y_\lambda)}$ is log smooth (resp. log étale).

**Remark 1.2.3.** This definition is equivalent to the classical one in the case of schemes by Theorem 3.5 of [11]. In the case of rigid spaces, this definition is the same with the one in [6].

**Proposition 1.2.4.** A morphism $f = (\underline{f}, f^\#) : (X, \mathcal{M}) \to (Y, \mathcal{N})$ of fine log spaces is log smooth (resp. log étale) and strict, then $\underline{f}$ is smooth (resp. étale).
Proof. This proof is the same as the proof of Theorem 6.5.6 in [9]. By Definition, we may assume that there exists a fine chart

\[
\begin{array}{ccc}
N & \xrightarrow{b} & \mathcal{N}(Y) \\
\downarrow{h} & & \downarrow{f^*} \\
M & \xrightarrow{a} & \mathcal{M}(X)
\end{array}
\]

satisfying the condition in the definition of log smoothness (resp. log étaleness).

Take a geometric point \( \varpi \) of \( X \) arbitrarily. Let \( \varpi := f(\varpi) \). Let \( F := a^{-1}(O^*_{X, \varpi}) \), \( G := b^{-1}(O^*_{Y, \varpi}) \). \( F \) is a face of \( M \) and \( G \) is a face of \( N \). By Theorem 2.1.9 of [16], \( F \) and \( G \) are finitely generated. \( A_{M}^{F} \) is an open subspace of \( A_{M} \) and \( A_{N}^{G} \) is an open subspace of \( A_{N} \). Shrinking \( X \) to some neighborhood of \( x \) and \( Y \) to some neighborhood of \( y \), we can replace \( M \) by \( M/F \) and \( N \) by \( N/G \). So we may assume that \( F = M^* \) and \( G = N^* \). Since \( f \) is strict, \( M/F = N/G \). Thus, \( F/G \) is a subgroup of \( M_{gp}/N_{gp} \), so \( A_{F} \rightarrow A_{G} \) is smooth (resp. étale) by the assumption of \( h \). Also, since \( M \) is the pushout of \( F \leftarrow G \rightarrow N \), we see that \( A_{M} \rightarrow A_{N} \) is also smooth (resp. étale). Since \( X \rightarrow Y \times_{A_{N}} A_{M} \) is étale by assumption, \( X \rightarrow Y \) is smooth (resp. étale). \( \square \)

Lemma 1.2.5. (cf. Lemma 3.1.1 of [10]) Let \( f : (X, \mathcal{M}) \rightarrow (Y, \mathcal{N}) \) be a morphism of fine log spaces. Let \( \varpi := f(\varpi) \). Put \( M := M_{\varpi}, \ N := N_{\varpi} \) and let \( h : N \rightarrow M \) be the induced morphism. Assume \( h \) is injective and the cokernel of \( h^{bp} \) is torsion-free. Take an arbitrary chart \( N \rightarrow N' \) on a neighborhood of \( \varpi \) which is good at \( \varpi \). Then there exists a good chart of \( f \) at \( \varpi \)

\[
\begin{array}{ccc}
N & \xrightarrow{h} & N' \\
\downarrow{h'} & & \downarrow{} \\
M & \xrightarrow{a} & M
\end{array}
\]

on some neighborhood of \( \varpi \).

Proof. It can be proven as in Lemma 3.1.1 of [10].

First, we can take a fine chart chart of \( f \)

\[
\begin{array}{ccc}
N & \xrightarrow{h'} & N' \\
\downarrow{h} & & \downarrow{} \\
M & \xrightarrow{a} & M
\end{array}
\]

on some neighborhood of \( \varpi \). Indeed, we can take a fine chart \( M'' \rightarrow M \) on some neighborhood of \( \varpi \). Let \( g : (M'' \oplus N')^{bp} \rightarrow M'_{\varpi}^{bp} \rightarrow M^{bp} \) be the natural surjective morphism and let \( M' := g^{-1}(M) \). Then, by the same argument as the proof of Lemma 2.10 of [11], there exists a chart \( M' \rightarrow M \) on some neighborhood of \( \varpi \) and we can take \( h' \) as the natural map \( N \rightarrow M' \).

By the assumption that \( h^{bp} \) is injective and its cokernel is torsion-free, \( M^{bp} \cong N^{bp} \oplus (M/N)^{bp} \) and we can take a section of the natural surjective morphism \( (M')^{bp} \rightarrow M^{bp} \rightarrow (M/N)^{bp} \). So we can take a section \( s : M^{bp} \rightarrow (M')^{bp} \) of \( (M')^{bp} \rightarrow M^{bp} \). Let \( p : M' \rightarrow M_{\varpi} \rightarrow M \) be the natural morphism and let \( M := (p^{bp})^{-1}(M) \). Then the image of \( M \) under \( s \) is contained by \( M \) and \( M \rightarrow M_{\varpi} \rightarrow M_{\varpi}^{bp} \rightarrow M^{bp} \) is the inclusion \( M \hookrightarrow M^{bp} \). Thus, by the same
argument as the proof of Lemma 2.10 of [11], there exists a chart $M \to M$ on some neighborhood of $\overline{\tau}$.

\[\square\]

Lemma 1.2.6. (cf. Lemma 3.1.1 of [10]) Let $f : (X, M) \to (Y, N)$ be a log smooth morphism of fine log spaces. Let $\overline{\tau}$ be a geometric point of $X$ and $\overline{\eta} := f(\overline{\tau})$. Put $M := \overline{M}_{\overline{\tau}}$, $N := \overline{N}_{\overline{\eta}}$ and let $h : N \to M$ be the induced morphism. Assume $h$ is injective and $M^{\text{sp}}$ is torsion-free. Take an arbitrary chart $N \to N$ on a neighborhood of $\overline{\eta}$ which is good at $\overline{\eta}$. Then there exists a good chart of $f$ at $\overline{\tau}$ such that the induced morphism $X \to Y \times_{\overline{\eta}} \mathbb{A}_M$ is smooth on a neighborhood of $\overline{\tau}$.

Proof. Take a chart

\[
\begin{array}{ccc}
N & \longrightarrow & N \\
\downarrow h & & \downarrow \\
M & \longrightarrow & M
\end{array}
\]

on a neighborhood $U$ of $\overline{\tau}$ such that $h'$ is injective, the order of $\text{Coker}(h')^{\text{tor}}$ is finite and prime to $q$ and $U \to Y \times_{\overline{\eta}} \mathbb{A}_L$ is étale. There is a natural surjective morphism $s : L \to \overline{M}_{\overline{\tau}} \cong M$. Let $\hat{L} := (s^{\text{sp}})^{-1}(M)$. There exists a homomorphism $\hat{L} \to \overline{M}_{\overline{\tau}}$, and we can take a chart $\hat{L} \to M$ on some neighborhood of $\overline{\tau}$ by augment in the proof of Lemma 2.10 of [11]. The morphism $\mathbb{A}_{\hat{L}} \to \mathbb{A}_L$ is log-étale and strict on some neighborhood of the image of $\overline{\tau}$, so étale there. Hence the morphism $X \to Y \times_{\overline{\eta}} \mathbb{A}_{\hat{L}}$ is étale on the neighborhood of $\overline{\tau}$. By Lemma 1.1.4, $\hat{L} \cong M \oplus \text{Ker}(s^{\text{sp}})$. Then

\[
\begin{array}{ccc}
N & \longrightarrow & N \\
\downarrow h & & \downarrow \\
M & \longrightarrow & \hat{L} \longrightarrow M
\end{array}
\]

is a good chart of $f$ at $\overline{\tau}$. Since $s^{\text{sp}} \circ (h')^{\text{sp}} = h^{\text{sp}}$ and $h^{\text{sp}}$ is injective, $\text{Ker}(s^{\text{sp}}) \to \text{Coker}(h')^{\text{sp}}$ is injective, so $\text{Ker}(s^{\text{sp}})^{\text{tor}}$ is finite and its order is prime to $q$. Thus the map $\mathbb{A}_{\hat{L}} \to \mathbb{A}_M$ induced by the inclusion $M \hookrightarrow M \oplus \text{Ker}(s^{\text{sp}}) \cong \hat{L}$ is log-smooth and strict on some neighborhood of the image of $\overline{\tau}$, so smooth there. Therefore $X \to Y \times_{\overline{\eta}} \mathbb{A}_M$ is smooth on some neighborhood of $\overline{\tau}$. \[\square\]

We define the notion of differential modules on log spaces.

Definition 1.2.7. Let $f : (X, M, \alpha) \to (Y, N, \beta)$ be a morphism of log spaces. The log differential module is defined as follows:

\[
\Omega_{X, M/(Y, N)}^{\log, 1} := \left( \Omega_{X/Y}^{1} \oplus \left( O_X \otimes M^{\text{sp}} \right) \right) / R
\]

where $R$ is the sub-$O_X$-module locally generated by

- $(\alpha(a), 0) - (0, \alpha(a) \otimes a)$ for each section $a \in M$,
- $(0, 1 \otimes a)$ for each section $a \in \text{Im}(f^\#)$.
We denote the element \((0, 1 \otimes a)\) by \(d \log a\) or \(d \log \alpha(a)\) for each \(a \in M^{\text{sp}}\).

When \(Y = \text{Spec}\ k, \text{Spf}\ V\) or \(\text{Spm}\ K\) and \(N\) is trivial, \(\Omega_{(X,M)}/(Y,N)\) may be simply denoted by \(\Omega_{(X,M)}^{1}\).

2. Log \(\nabla\)-modules

2.1. Polyannuli. Let \(M\) be a fine monoid. The rigid analytic space \(\mathbb{A}_{M,K}\) is defined as \((\text{Spec} K[M])^\text{an}\). We regard \(\mathbb{A}_{M,K}\) as a log rigid space by the log structure associated to \(M \to \Gamma(\mathbb{A}_{M,K}, \mathcal{O}_{\mathbb{A}_{M,K}})\). For \(m \in M\), we write \(t^m\) as the image of \(m\) by map \(M \to \Gamma(\mathbb{A}_{M,K}, \mathcal{O}_{\mathbb{A}_{M,K}})\). If \(M\) is sharp, the point of \(\mathbb{A}_{M,K}\) defined by \(t^m = 0\) for all \(m \in M \setminus \{0\}\) is called the vertex and denoted by \(0\).

Definition 2.1.1. A weighted monoid is a monoid \(M\) equipped with a monoid homomorphism \(h : M \to \mathbb{N}\) (which is called weighting of \(M\)) such that \(h^{-1}(0) = M^*\).

Remark 2.1.2. By Lemma 2.2.2 of [10], for any fine monoid \(M\), there always exists a monoid homomorphism \(h\) as in Definition 2.1.1.

Note that for any weighted monoid \(M\), \(h\) can be extended to the group homomorphism \(M^{\text{sp}} \to \mathbb{Z}\) which is also denoted by \(h\).

Definition 2.1.3. (Definition 3.1.1 of [13]) A subinterval of \([0, +\infty)\) is called aligned if any endpoint at which it is closed is contained in \(\Gamma^*\). A subinterval of \([0, +\infty)\) is called quasi-open if it is of the form \([0, a)\) with \(a > 0\) or of the form \((a, b)\) with \(0 < a < b\).

Definition 2.1.4. (a generalization of 3.1.2 of [13]) Let \(M\) be a fine weighted monoid. For an aligned subinterval \(I\) of \([0, \infty)\) we define the polyannuli \(\mathbb{A}_{M,K}(I)\) as follows:

1. If \(I = [a, b]\) for \(a, b \in \Gamma^*\) with \(0 \leq a \leq b\),
   \[
   \mathbb{A}_{M,K}(I) = \left\{ x \in \mathbb{A}_{M,K} \mid \forall m \in M, a^b m \leq |t^m(x)| \leq b^b m \right\}.
   \]
2. In general,
   \[
   \mathbb{A}_{M,K}(I) = \bigcup_{[a,b] \subseteq I} \mathbb{A}_{M,K}[a,b].
   \]

We write \(\mathbb{A}_{M,K}[a,b]\) or \(\mathbb{A}_{M,K}(a,b)\) etc. instead of \(\mathbb{A}_{M,K}([a,b])\) or \(\mathbb{A}_{M,K}((a,b))\) etc.

Remark 2.1.5. It is enough to check the condition \(a^b m \leq |t^m(x)| \leq b^b m\) in the above definition of \(\mathbb{A}_{M,K}[a,b]\) for a set of (finite) generators of \(M\) instead of all \(m \in M\). So \(\mathbb{A}_{M,K}[a,b]\) is an affinoid open subset of \(\mathbb{A}_{M,K}\) for any \(a \leq b, 0 < b\) and \(\mathbb{A}_{M,K}(I)\) is an admissible open subset for any \(I \neq [0,0]\).

Remark 2.1.6. For \(I = [0,0]\), if \(M\) is sharp, \(\mathbb{A}_{M,K}[0,0]\) is a single point (the vertex) \(\text{Spm}\ K\) with a log structure defined by \(\alpha : M \to K\) such that \(\alpha(m) = 1\) if \(m = 0\) and 0 otherwise. Thus
\[
\Omega_{\mathbb{A}_{M,K}[0,0]}^{1} = M^{\text{sp}} \otimes \mathbb{Z}\.
\]

Remark 2.1.7. If \(M = \mathbb{N}^n\) and \(h : \mathbb{N}^n \to \mathbb{N}\) is the morphism which takes the sum of all entries, \(\mathbb{A}_{K}(I)\) is equal to \(\mathbb{A}_{h}^n(I)\) in [13].

Proposition 2.1.8. Let \(M\) be a fine sharp weighted monoid.
(1) For $0 < b \in \Gamma^*$,

$$\Gamma(\mathbb{A}_M,K[0,b],\mathcal{O}_{\mathbb{A}_M,K[0,b]}) = \left\{ \sum_{m \in M} c_m t^m \left| c_m \in K, \lim_{h(m) \to \infty} |c_m| b^{h(m)} = 0 \right. \right\}. $$

(2) For $m \in M^g$, we write

$$h^+(m) = \min \left\{ h(m^+), m^+ \in M, m = m^+ - m^- \right\},$$

$$h^-(m) = \min \left\{ h(m^-), m^+ \in M, m = m^+ - m^- \right\} = h^+(m) - h(m),$$

$$|h|(m) = h^+(m) + h^-(m).$$

Then for $a,b \in \Gamma^*$, $0 < a \leq b$,

$$\Gamma(\mathbb{A}_M[K[a,b],\mathcal{O}_{\mathbb{A}_M,K[a,b]}) = \left\{ \sum_{m \in M^{g_p}} c_m t^m \left| c_m \in K, \lim_{h(m) \to \infty} |c_m| a^{-h^-(m)} b^{h^+(m)} = 0 \right. \right\}. $$

**Proof.** Let $m_1, \ldots, m_n$ be a set of generators of $M$.

(1) The region $\mathbb{A}_M,K[0,b]$ is defined by $\frac{|t^m|}{b^{h(m)}} \leq 1$ in $\mathbb{A}_M,K$. Hence

$$\Gamma(\mathbb{A}_M,K[0,b],\mathcal{O}_{\mathbb{A}_M,K[0,b]}) = K \left\{ \frac{t^{m_1}}{b^{h(m_1)}}, \ldots, \frac{t^{m_n}}{b^{h(m_n)}} \right\} = \left\{ \sum_{k \in \mathbb{N}^n} c_k t^{k_1 m_1 + \cdots + k_n m_n} \left| c_k \in K, \lim_{|k| \to \infty} b^{|k|} c_k = 0 \right. \right\}.$$

At the last equality, we take $m = k_1 m_1 + \cdots + k_n m_n$ and $c_m = \sum_{m=k_1 m_1 + \cdots + k_n m_n} c_k$. Note that when $|k| \to \infty$, $h(k_1 m_1 + \cdots + k_n m_n) \to \infty$ and vice versa.

(2) The region $\mathbb{A}_M,K[a,b]$ is defined by $\frac{|t^m|}{a^{h(m)}} \geq 1$, $\frac{|t^m|}{a^{h(m)}} \leq 1$ in $\mathbb{A}_M,K$. Hence

$$\Gamma(\mathbb{A}_M,K[a,b],\mathcal{O}_{\mathbb{A}_M,K[a,b]}) = K \left\{ \frac{a^{h(m)}}{t^{m_1}}, \ldots, \frac{a^{h(m_n)}}{t^{m_n}} \right\} = \left\{ \sum_{k,k' \in \mathbb{N}^n} c'_{k,k'} t^{\sum_{i=1}^{n} k_i m_i - \sum_{i=1}^{n} k'_i m_i} \left| c'_{k,k'} \in K, \lim_{|k|+|k'| \to \infty} a^{-\sum_{i=1}^{n} k'_i h(m_i)} b^{h(m)} |c'_{k,k'}| = 0 \right. \right\}.$$

At the third equality, we take $m^+ = \sum_{i=1}^{n} k_i m_i$, $m^- = \sum_{i=1}^{n} k'_i m_i$, and $c''_{m^+,m^-} = \sum_{m^+ = k_1 m_1 + \cdots + k_n m_n} c'_{k,k'}$. At the last
equality, we take $c_m = \sum_{m' + m'' = m} c_{m''}^m$. Note that $a^{-h(m)}b^{h(m'+m)} \geq a^{-h(m)}b^{h(m')} + h(m)K$ for any $m = m' + m''$.

\[\square\]

**Corollary 2.1.9.** $\mathbb{A}_{M,K}[0,1] = \text{Spm} \ K\langle M \rangle$ and $\mathbb{A}_{M,K}[1,1] = \text{Spm} \ K\langle M^{gp} \rangle$ for any fine sharp weighted monoid $M$.

**Remark 2.1.10.** Let $M$ be a fine sharp weighted monoid. Let $M^{sat} := \{ m \in M^{gp} \mid \exists n \in \mathbb{N}_{>0}, nm \in M \}$. Then $h$ can be extended to a map $M^{sat} \to \mathbb{N}$ which is also denoted by $h$ and $M^{sat}$ with $h$ is also a weighted monoid. We prove that, if $0 \not\in I$, $\mathbb{A}_{M,K}(I) = \mathbb{A}_{M^{sat},K}(I)$. Let $m_1, \ldots, m_n$ be a set of generators of $M^{sat}$ ($M^{sat}$ is finite by Corollary 2.2.5 of [10]). For each $m_i$, take $m_i' \in M$ such that $m_i + m_i' \in M$ and $m_i \in \mathbb{N}_{>0}$ such that $nm_i \in M$. Let $s := (n_1 - 1)m_1' + \cdots + (n_g - 1)m_g'$. Since $nm_i + (n_i - 1)m_i' \in M$ for all $n_i \in \mathbb{N}$, $m_i + s \in M$ for all $m \in M^{sat}$. Let $h_{sat,+}, h_{sat,-}$ be the $h^+, h^-$ in Proposition 2.1.8 with $M$ replaced by $M^{sat}$. Then $h_{sat,+}(m) \leq h^+(m) \leq h_{sat,+}(m + h(s))$ and $h_{sat,-}(m) \leq h^-(m) \leq h_{sat,-}(m + h(s))$. So by Proposition 2.1.8, $\mathbb{A}_{M,K}[a,b] = \mathbb{A}_{M^{sat},K}[a,b]$ for any $a, b \in \Gamma^*$ with $0 < a \leq b$. By definition, for any aligned interval $f \leq \langle 0, \infty \rangle$, $\mathbb{A}_{M,K}(I) = \mathbb{A}_{M^{sat},K}(I)$.

**Proposition 2.1.11.** Let $M$ be a fine sharp weighted monoid. For any admissible open subset $U \subseteq \mathbb{A}_{M,K}$ containing the vertex, $\mathbb{A}_{M,K}[0,a] \subseteq U$ for some $0 < a \in \Gamma^*$.

**Proof.** We may assume that $U = \{ x \in \mathbb{A}_{M,K}[0,1] \mid |f_1(x)| \leq |g(x)|, \ldots, |f_n(x)| \leq |g(x)| \}$ for some $f_1, \ldots, f_n, g \in K\langle M \rangle$ such that $(f_1, \ldots, f_n, g) = K\langle M \rangle$.

Let $f_i = \sum_{m \in M} c_{i,m} m^m$ for $1 \leq i \leq n$ and $g = \sum_{m \in M} d_m m^m$ which converge on $\mathbb{A}_{M,K}[0,1]$. By the assumption that the vertex is contained in $U$, $|c_{i,0}| \leq |d_0|$ for all $1 \leq i \leq n$. If $|d_0| = 0$, $|c_{i,0}| = 0$ for all $1 \leq i \leq n$ and it contradicts the assumption of $f_1, \ldots, f_n, g$. Thus $|d_0| > 0$. By the convergence of $f_1, \ldots, f_n$ and $g$, we can take $0 < a \in \Gamma^*$ such that, for all $m \in M \setminus \{0\}$, $|c_{i,m} a^k(m)| \leq |d_0|$ for all $1 \leq i \leq n$ and $|d_m a^h(m)| < |d_0|$. Then for any point $x$ in $\mathbb{A}_{M,K}[0,a]$, $|f_i(x)| \leq |d_0| = |g(x)|$ for all $1 \leq i \leq n$, so $x \in U$. \[\square\]

2.2. Log $\nabla$-modules and exponents. In this subsection, we define the notion of exponents of log $\nabla$-modules and prove some basic properties.

**Lemma 2.2.1.** Let $E$ be a finite dimensional vector space over $K$ and $\Omega$ a finitely generated abelian group. Let $\nabla : E \to E \otimes_\mathbb{Z} \Omega$ be a $K$-linear morphism such that the composition

$$E \xrightarrow{\nabla} E \otimes_\mathbb{Z} \Omega \xrightarrow{\nabla \otimes id} E \otimes_\mathbb{Z} \Omega \otimes_\mathbb{Z} \Omega \xrightarrow{\nabla \otimes \nabla \otimes \ldots} E \otimes_\mathbb{Z} \Omega \otimes_\mathbb{Z} \Omega \otimes_\mathbb{Z} \Omega \ldots$$

is zero. Then there exist $\xi_1, \ldots, \xi_n \in \Omega \otimes_\mathbb{Z} K$ and a decomposition $E = \bigoplus_{i=1}^n E_i$ such that $E_i \neq 0$, $\nabla(E_i) \subseteq E_i \otimes_\mathbb{Z} \Omega$ and $\nabla - \xi_i \cdot id$ is nilpotent on $E_i$, which means that for any map $\phi : \Omega \to \mathbb{Z}$ of abelian groups, the composition map

$$E_i \xrightarrow{\nabla - \xi_i \cdot id} E_i \otimes_\mathbb{Z} \Omega \xrightarrow{id \otimes \phi} E_i$$

is nilpotent.
Proof: Take some $\Omega^\text{free} \cong \mathbb{Z}^r$ and let $\text{pr}_i : \Omega \to \mathbb{Z}$ be the composition of the natural map $\Omega \to \Omega^\text{free}$ and the $i$-th projection. Put $\partial_i := (\text{id}_E \otimes \text{pr}_i) \circ \nabla$. By the assumption, $\partial_1, \ldots, \partial_r$ commute with each other. So, when we take the Jordan decomposition $E = \bigoplus_{i=1}^r E_{1,i}$ with respect to $\partial_1$, then $\partial_2, \ldots, \partial_r$ act each $E_{1,i}$ and we can take the Jordan decomposition of each $E_{1,i}$ with respect to $\partial_2$ and so on. If we repeat this process up to $\partial_r$, we reach the decomposition satisfying the above condition. \hfill $\square$

For a fine log rigid space $(X, \mathcal{M})$, the \textit{trivial locus of $\mathcal{M}$} is $\{ x \in X \mid \mathcal{M}_x = 0 \}$. A log $\nabla$-module on $(X, \mathcal{M})$ is a locally free $\mathcal{O}_X$-module $E$ equipped with an integrable connection $\nabla : E \to E \otimes \Omega^\text{log,1}_{(X,\mathcal{M})}$. We define residues and exponents of log $\nabla$-modules on log rigid spaces in the same way as [15].

\textbf{Definition 2.2.2.} (cf. Definition 2.1.1 of [15]) Let $(X, \mathcal{M})$ be a fine log rigid space. Let $(E, \nabla)$ be a log $\nabla$-module on $(X, \mathcal{M})$.

1. Let $x$ be a geometric point lying over $x \in X$. Let $M := \mathcal{M}_x$. Let $\Omega^\text{log,1}_{(X,\mathcal{M})} = M^{\text{gp}} \otimes_\mathbb{Z} \overline{K}$ be the canonical surjective map.

Let $E(x) := E_{\mathfrak{m}}/m_{\mathfrak{m}}E_{\mathfrak{m}}$ where $m_{\mathfrak{m}}$ is the maximal ideal at $x$. The \textit{residue of $E$ at $x$} is the unique $\overline{K}$-linear map $\rho_x$ such that the following diagram commutes:

$$
\begin{array}{ccc}
E & \xrightarrow{\nabla} & E \otimes \Omega^\text{log,1}_{(X,\mathcal{M})} \\
\downarrow & & \downarrow \\
E(x) & \xrightarrow{\rho_x} & E(x) \otimes_\mathbb{Z} M^{\text{gp}}
\end{array}
$$

Since $E(x)$ is finite dimensional over $\overline{K}$, for some $\xi_1, \ldots, \xi_n \in M^{\text{gp}} \otimes_\mathbb{Z} \overline{K}$, $(E(x), \rho_x)$ has a decomposition as in Lemma 2.2.1. $\xi_1, \ldots, \xi_n$ are called \textit{exponents of $E$ at $x$}.

2. Let $S \subseteq \mathcal{M}^{\text{gp}} \otimes_\mathbb{Z} \overline{K}$ be a subsheaf. If for any geometric point $x$, the exponents of $E$ at $x$ are contained in $S_x$, it is said that the all exponents of $E$ are contained in $S$.

\textbf{Remark 2.2.3.} It is clear by linear algebra that, in the situation of Definition 2.2.2, exponents of any quotient of $E$ is a subset of the exponents of $E$ at any geometric point.

For a subsheaf $S \subseteq \mathcal{M}^{\text{gp}} \otimes_\mathbb{Z} \overline{K}$, we define $\text{LNM}_{(X,\mathcal{M})}^S$ as the category of log $\nabla$-modules on $(X, \mathcal{M})$ such that all the exponents are contained in $S$. We will prove that $\text{LNM}_{(X,\mathcal{M})}^S$ is an abelian category under some assumptions in this subsection. For a rigid space $Y$ and a weighted monoid $M$, $Y \times \mathbb{M},(I)$ naturally has a log structure induced by $M$. For a subset $\Sigma \subseteq M^{\text{gp}} \otimes_\mathbb{Z} \overline{K}$, we denote the image of $\Sigma$ under the natural map $M^{\text{gp}} \otimes_\mathbb{Z} \overline{K} \to \mathcal{M}^{\text{gp}} \otimes_\mathbb{Z} \overline{K}$, where $\mathcal{M}$ is the log structure of $Y \times \mathbb{M},(I)$, also by $\Sigma$. $\text{LNM}_{Y \times \mathbb{M},(I)}^{\Sigma}$ is the category of log $\nabla$-modules on $Y \times \mathbb{M},(I)$ such that all the exponents are contained in $\Sigma$.

For $Y \times \mathbb{M},(I)$,

$$
\Omega^\text{log,1}_{(Y \times \mathbb{M},(I))} = \left( \Omega^1_Y \otimes \mathcal{O}_{\mathbb{M},(I)} \right) \oplus \left( \Omega^\text{log,1}_{\mathbb{M},(I)}(I) \otimes \mathcal{O}_Y \right).
$$
We write \( d = d^Y + d_{\log} \) corresponding to this decomposition where \( d \) is the derivation on \( Y \times \mathbb{A}_{M,K}(I) \). For a log \( \nabla \)-module \((E, \nabla)\) on \( Y \times \mathbb{A}_{M,K}(I) \), we write \( \nabla = \nabla^Y + \nabla_{\log} \) corresponding to this decomposition.

We relate resiude and exponents in the above definition with those in the sense of \([21]\).

The next lemma and its proof are essentially the same as Proposition-Definition 1.24 in \([22]\).

**Lemma 2.2.4.** Let \( Y \) be a smooth connected rigid space and \( M \) a fine sharp weighted monoid. Let \((E, \nabla)\) be a log \( \nabla \)-module on \( Y \times \mathbb{A}_{M,K}[0,0] \). Then there exist \( q_1, \ldots, q_n \in M^{gp} \otimes_{\mathbb{Z}} K \) and a decomposition \( E_K = \bigoplus_{i=1}^n E_i \) such that \( E_i \neq 0 \), \( \rho_{E_i}(E_i) \subset E_i \otimes_{\mathbb{Z}} M^{gp} \) and \( \rho_{E_i} \mid \nabla = \xi_i \cdot \text{id} \) is nilpotent on \( E_i \).

Moreover, for any finite extension \( K' \) of \( K \) such that \( \xi_1, \ldots, \xi_n \in M^{gp} \otimes_{\mathbb{Z}} K' \), \( E_{K'} \) admits such a decomposition.

For any geometric point \( \overline{y} \in Y \), the exponents of \( E \) at \( \overline{y} \) in the sense of \([22]\) are \( \xi_1, \ldots, \xi_n \). Especially, if \( E \) is an object of \( \text{LNM}_{Y \times \mathbb{A}_{M,K}[0,0]} \), then \( \xi_1, \ldots, \xi_n \in \Sigma \).

**Proof.** Let \( e := \text{rk} E \). Take some \( (M^{gp})^{\text{free}} \cong \mathbb{Z}^e \) and let \( \text{res}_j := (\text{id} \otimes \text{pr}_j) \circ \rho : E \rightarrow E \) where \( \text{pr}_j \) is the projection \( \mathbb{Z}^e \rightarrow \mathbb{Z} \) to the \( j \)-th factor.

We decompose \( E_K \) with respect to \( \text{res}_1 \). For \( \xi \in K \), we define a subspace \( H(\xi) \) of \( E_K \) by

\[
\Gamma(V, H(\xi)) = \left\{ v \in \Gamma(V, E_K) \mid \exists k > 0, ((\text{res}_1)_{K} - \xi \cdot \text{id})^k(v) = 0 \right\}.
\]

Then there exists a natural map

\[
\bigoplus_{\xi \in K} H(\xi) \rightarrow E_K
\]

which is injective by the definition.

Take an affinoid connected subspace \( U = \text{Spm} R \subseteq Y \) such that \( E|_U \) is free. Let \( Q_1(t) \in R[t] \) be the characteristic polynomial of \( \text{res}_1 \). Let \( F := \mathbb{A}^e E|_U \otimes_K K[t] \) and \( \psi := \mathbb{A}^e (t - \text{res}_1) : F \rightarrow F \). Then \( \psi \) is the multiplication by \( Q_1(t) \). Let \( \nabla_F : F \rightarrow F \otimes \Omega^1_Y \) be the morphism induced by \( \nabla \). By the integrability of \( \nabla \), the following diagram commutes:

\[
\begin{array}{ccc}
F & \xrightarrow{\psi} & F \\
\downarrow{\nabla_F} & & \downarrow{\nabla_F} \\
F \otimes \Omega^1_Y & \xrightarrow{\psi \otimes \text{id}} & F \otimes \Omega^1_Y.
\end{array}
\]

Thus the coefficients of \( Q_1(t) \) are killed by \( d_Y \), the derivation on \( Y \).

Take a point \( y \in U \) and let \( K(y) \) be the residue field at \( y \) which is a finite extension of \( K \). Then \( \text{Ker}(d_y) \cong K(y) \) where \( d_y \) is the derivation on \( \mathcal{O}_{Y,y} \). Since \( U \) is integral, \( R \rightarrow \mathcal{O}_{Y,y} \) is injective and so we can regard \( Q_1(t) \) as an element of \( K(y)[t] \subseteq K(t) \).

Let \( \xi_{1,1}, \ldots, \xi_{1,n_1} \in K \) be the roots of \( Q_1(t) \). Since \( Q_1((\text{res}_1)_{K}) = 0 \) on \( U \), \( (E|_U)_{K} = \bigoplus_{i=1}^{n_1} H(\xi_{1,i})|_U \). So the map \([22,41]\) is surjective on \( U \). Because \( U \) is arbitrary, \([22,41]\) is surjective on \( X \).

Since any \( H(\xi) \) is a direct summand of \( E_K \), it is locally free and \( \text{rk} H(\xi) \) is constant on \( Y \) because \( Y \) is connected. Especially, \( H(\xi) \neq 0 \) if and only if \( \xi \in \frac{\partial Y}{\partial Y} \).
Remark 2.2.7. Let \( \{\xi_1, \ldots, \xi_{n_1}\} \), which means that this set is independent of the choice of \( U \). Let \( E_{1,i} := H(\xi_{1,i}) \), then \( E_{r}(\nabla) = \bigoplus_{i=1}^{n_1} E_{1,i} \) on \( Y \). By the integrability of \( \nabla \), each \( E_{1,i} \) is a log \( \nabla \)-module on \( Y \times \mathbb{A}_{M,K}[[0,0]] \).

We can take such decomposition of each \( E_{1,i} \) with respect to \( \text{res}_2 \), and so on. If we repeat this process to \( \text{res}_r \), we reach the decomposition satisfying the required condition.

If \( \xi_1, \ldots, \xi_n \in M^{sp} \otimes \mathbb{Z} K' \), then we get elements of \( K' \) as the roots of the characteristic polynomial on each step, so we can decompose \( E_{K'} \) instead of \( E_{K} \).

For any geometric point \( \overline{\eta} \in Y \), let \( E_i(\overline{\eta}) := (E_i)_{/\overline{\eta}}/m_{\overline{\eta}}(E_i)_{/\overline{\eta}} \) where \( m_{\overline{\eta}} \) is the maximal ideal at \( \overline{\eta} \), then we have \( E(\overline{\eta}) = \bigoplus_{i=1}^{n} E_i(\overline{\eta}) \) and \( \rho \otimes k(\overline{\eta}) - \xi_i \cdot \text{id} \) acts on \( E_i(\overline{\eta}) \) nilpotently. Since any \( E_i \) is a direct summand of \( E_{K'} \), it is locally free and \( \text{rk} E_i \) is constant on \( Y \), so \( E_i(\overline{\eta}) \neq 0 \). This means the last assertion. \( \square \)

**Definition 2.2.5.** \( \rho \) (resp. \( \xi_1, \ldots, \xi_n \)) in Lemma 2.2.4 is called the residue of \((E, \nabla)\) (resp. exponents of \((E, \nabla)\)).

Let \( \phi : (M_{sp})_{\text{free}} \rightarrow \mathbb{Z}^r \) be an injective homomorphism. \( \phi \) induces \( \phi \otimes \text{id}_K : M_{sp} \otimes \mathbb{Z} K \rightarrow K^r \). Then we have an inclusion

\[
\Omega_{K, M, K(l)}^{\log, 1} = \mathcal{O}_{K, M, K(l)} \otimes_{\mathbb{Z}} M^{sp} \subseteq \bigoplus_{i=1}^{r} \mathcal{O}_{K, M, K(l)} d \log t_i
\]

induced by \( \phi \otimes \text{id}_K \), where \( \{d \log t_i\} \) is the canonical basis of \( K^r \).

**Definition 2.2.6.** Let \( S \subseteq \overline{K} \) be a subset. \( \Sigma \subseteq M_{sp} \otimes \mathbb{Z} K \) is called locally \((S-D)\) if there exists a homomorphism \( \phi : (M_{sp})_{\text{free}} \rightarrow \mathbb{Z}^r \) such that for any two elements \((\alpha_i), (\beta_i) \in (\phi \otimes \text{id}_K)(\Sigma)\) and for any \( 1 \leq i \leq r, \alpha_i - \beta_i \in S \).

**Remark 2.2.7.** In Definition 2.2.6, we can also assume that \( \phi \otimes \text{id}_K \) is isomorphic, hence the inclusion \((2.2.5.1)\) is an equality. Indeed, let \( e_1, \ldots, e_r \) be the standard basis of \( \mathbb{Z}^r \). If \( \mathbb{Z} e_i \cap \Im \phi = \{0\} \) for some \( i \), we can take \( \text{pr}_i \circ \phi \) instead of \( \phi \), where \( \text{pr}_i : \mathbb{Z}^r \rightarrow \mathbb{Z}^r \) is the projection removing the \( i \)-th entry. Repeating this process, we arrive at the situation where \( \mathbb{Z} e_i \cap \Im \phi = \{0\} \) for any \( i \) or \( r = 0 \). Then the rank of \((M_{sp})_{\text{free}}\) is equal to \( r \) and \( \phi \otimes \text{id}_K \) is isomorphic.

**Definition 2.2.8.** For \( \alpha \in \overline{K} \), we define its type \( \text{type}(\alpha) \) as the convergent radius of the series \( \sum_{s \in \mathbb{N}, s \neq \alpha} (s-\alpha)^{-1} t^s \). If \( \text{type}(\alpha) < 1 \) or \( \text{type}(\alpha) < 1 \), \( \alpha \) is called a \( p \)-adic Liouville number. We put:

\[
\text{NI} = \overline{K} \setminus (\mathbb{Z} \setminus \{0\}),
\]

\[
\text{PT} = \{ \alpha \in \overline{K} \mid \text{type}(\alpha) > 0 \},
\]

\[
\text{NL} = \{ \alpha \in \overline{K} \mid \alpha \text{ is not a } p \text{-adic Liouville number} \}.
\]

We omit the hyphen in the notation when we write \((\text{NI-D}), (\text{PT-D})\) or \((\text{NL-D})\).

**Definition 2.2.9.** Let \( Y \) be a rigid space, \( M \) a fine sharp weighted monoid, \( \Sigma \subseteq M_{sp} \otimes \mathbb{Z} K \) a subset and \( I \subseteq [0, \infty) \) an aligned subinterval. Since we have the equalities

\[
\Omega_{Y \times \mathbb{A}_{M,K}, M,l}^{\log, 1} = \left( \Omega_{K, M, K(l)}^{\log, 1} \otimes_{K} \mathcal{O}_{Y} \right) \oplus \left( \Omega_{K, M, K(l)}^{\log, 1} \otimes \mathcal{O}_{Y} \right)
\]

\[
= \left( \Omega_{Y, K}^{\log, 1} \otimes \mathcal{O}_{Y \times \mathbb{A}_{M,K}, M,l} \right) \oplus \left( M_{sp} \otimes \mathcal{O}_{Y \times \mathbb{A}_{M,K}, M,l} \right)
\]

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there exists a natural inclusion
\[ \Omega_{Y \times h_{M,K}[0,0]}^{1} \rightarrow \Omega_{Y \times h_{M,K}(I)}^{1}. \]

The functor \( \mathcal{U}_I : \text{LNM}_{Y \times h_{M,K}[0,0],\Sigma} \rightarrow \text{LNM}_{Y \times h_{M,K}[0,0],\Sigma} \) is defined as follows: \( \mathcal{U}_I \) sends \((E, \nabla) \in \text{LNM}_{Y \times h_{M,K}[0,0],\Sigma}\) to \(E \otimes_K \mathcal{O}_{h_{M,K}(I)}\) endowed with the connection
\[ \mathbf{v} \otimes t^m \mapsto (\nabla(\mathbf{v}) + \mathbf{v} \log t^{m}) \otimes t^m \]
where \( \mathbf{v} \) is a section of \( \mathcal{O}_Y \) and \( m \in M^{\text{gp}} \).

The following lemma is a key proposition of this subsection.

**Lemma 2.2.10.** (cf. Lemma 1.9 of [21]) Let \( Y = \text{Spm} R \) be a smooth connected affinoid space, \( M \) a fine sharp weighted monoid and \( \Sigma \subseteq M^{\text{gp}} \otimes_{\mathbb{Z}} \overline{K} \) be an locally \((\text{NI} \cap \text{PT})\)-D subset. Let \( a \in (0, \infty) \cap \Gamma^* \) and \((E, \nabla)\) an object of \( \text{LNM}_{Y \times h_{M,K}[0,a],\Sigma} \) such that \( E|_{|Y \times 0} \) is free. Then there exists \( b \in (0, a] \cap \Gamma^* \) such that \( E|_{|Y \times h_{M,K}[0,b]} \) is in the essential image of \( \mathcal{U}_{[0,b]} : \text{LNM}_{Y \times h_{M,K}[0,0],\Sigma} \rightarrow \text{LNM}_{Y \times h_{M,K}[0,b],\Sigma}. \)

**Proof.** As the proof of Lemma 1.9 of [21], we can take \( a \) and assume that \( E \) is free. Let \( \mathbf{e} = (e_1, \ldots, e_n) \) be a basis of \( E \).

Take \( \phi : M \rightarrow \mathbb{Z}^r \) such that \( \phi \otimes \text{id}_K \) is isomorphic with respect to which the condition of \((\text{NI} \cap \text{PT})\)-D is satisfied. Let \( \partial \log t_1, \ldots, \partial \log t_r \) be the induced basis of \( \Omega_{h_{M,K}[0,a]}^{\text{log}} \).

Write \( \nabla^{\text{log}}(\mathbf{v}) = \sum_{i=1}^{r} \partial_i(\mathbf{v}) \partial \log t_i \) and let \( A^i = \sum_{m \in M} A^i_m t^m \in \text{Mat}_n(R \otimes_K \mathcal{O}_{h_{M,K}[0,a]}) \) be the matrix representation of \( \partial_i \) with respect to \( \mathbf{e} \).

We construct matrices \( B = \sum_{m \in M} B_m t^m \) and \( B' = \sum_{m \in M} B'_m t^m \) satisfying the following equations for all \( 1 \leq i \leq r. \)

\[ A^i B + \partial_i B = BA^i_0, \]
\[ BB' = I. \]

These are equivalent to the equations for all \( m \in M \)

\[ A^i_0 B_m - B_m A^i_0 + m_i B_m = - \sum_{m' + m'' = m \atop m', m'' \neq 0} A_{m'} B_{m''}, \tag{2.2.10.1} \]

\[ \sum_{m' + m'' = m} B_{m'} B'_{m''} = \begin{cases} I & m = 0, \\ O & m \neq 0. \end{cases} \tag{2.2.10.2} \]

Here, \( m_i \) is the \( i \)-th entry of \( \phi(m) \).

By Lemma 2.2.4, we can take the exponents of \( E|_{Y \times h_{M,K}[0,0]} \), which we denote by \( \xi_1, \ldots, \xi_r \in M^{\text{gp}} \otimes_{\mathbb{Z}} \overline{K} \). For \( 1 \leq i \leq r, \) we denote the \( i \)-th entry of \((\phi \otimes \text{id}_K)(\xi_j)\) by \( \xi_{ij} \in \overline{K} \). By the proof of Lemma 2.2.4, \( \{\xi_{ij}\} \) are the eigenvalues of \( A^i_0 \).

To construct \( B \), we first show the claim which is proven in the proof of Lemma 1.9 in [21].
Claim 2.2.10.3. Let \( g : \text{Mat}_n(R) \to \text{Mat}_n(R) \) be the map defined as \( X \mapsto A_0^i X - X A_0^i \). Then for any \( s \in \mathbb{Z} \setminus \{0\} \), \( g + s \cdot \text{id} \) is invertible and there exist constants \( C \gg 0, e \gg 0 \) independent of \( s \) such that

\[
\left| (g + s \cdot \text{id})^{-1} \right| \leq C \left( \max_{j,j'} \left\{ \left| \xi_{i,j} - \xi_{i,j'} + s \right|^{-1}, 1 \right\} \right)^e,
\]

where \( |\cdot| \) is the operator norm.

Proof. To prove this, we may enlarge \( K \) so that \( \{\xi_{i,j}\}_{i,j} \subseteq K \).

Let \( R^n = \bigoplus_{j=1}^l E_j \) be the decomposition of \( R^n \), where \( E_j \) is the \( R \)-submodule on which \( A_0^i - \xi_{i,j} \cdot \text{id} \) is nilpotent. Then

\[
\text{Claim 2.2.10.3.} \quad \text{Let } g \text{ acts on each } \text{Hom}(E_{j'}, E_j) \text{ because } A_0^i \text{ acts on each } E_j.
\]

Define \( g_1 \) and \( g_2 \) as follows:

\[
g_1 : \bigoplus_{j,j'=1}^l \text{Hom}(E_{j'}, E_j) \to \bigoplus_{j,j'=1}^l \text{Hom}(E_{j'}, E_j),
\]

\[
\psi (X_{j,j'})_{j,j'} \mapsto ((\xi_{i,j} - \xi_{i,j'}) X_{j,j'})_{j,j'},
\]

\[
g_2 : \bigoplus_{j,j'=1}^l \text{Hom}(E_{j'}, E_j) \to \bigoplus_{j,j'=1}^l \text{Hom}(E_{j'}, E_j),
\]

\[
\psi (X_{j,j'})_{j,j'} \mapsto ((A_0^i - \xi_{i,j}) X_{j,j'} - X_{j,j'} (A_0^i - \xi_{i,j'}))_{j,j'}.
\]

Then \( g = g_1 + g_2 \). \( g_1 + s \cdot \text{id} \) is invertible for any \( s \in \mathbb{Z} \setminus \{0\} \) because \( \xi_{i,j} - \xi_{i,j'} + s \neq 0 \) for any \( j, j' \) by the assumption. \( g_2 \) is nilpotent because \( A_0^i - \xi_{i,j} \) acts \( E_j \) nilpotently. So \( g + s \cdot \text{id} \) is invertible.

We can take a constant \( c \gg 0 \) independently of \( s \) such that \( \left| (g_1 + s \cdot \text{id})^{-1} \right| \leq c \max_{j,j'} \left| \xi_{i,j} - \xi_{i,j'} + s \right|^{-1}. |g_2| \leq c. \)

Take \( e \gg 0 \) such that \( g_2^e = 0 \), then \( (g + s \cdot \text{id})^{-1} = \sum_{k=0}^{e-1} (-1)^k (g_1 + s \cdot \text{id})^{-k-1} g_2^k \). So

\[
\left| (g + s \cdot \text{id})^{-1} \right| \leq \max_{0 \leq k < e} |g_1 + s \cdot \text{id}|^{-k-1} |g_2|^k \leq c^{2e-1} \left( \max_{j,j'} \left\{ \left| \xi_{i,j} - \xi_{i,j'} + s \right|^{-1}, 1 \right\} \right)^e.
\]

Hence we obtain the required inequality by taking \( C = c^{2e-1}. \) \( \square \)

We also prove the following claim.

Claim 2.2.10.4. Fix \( m^0 \in M, 1 \leq i_0, i_1 \leq r. \) Assume:

- Claim 2.2.10.1 and 2.2.10.2 are true when \( i = i_0, m \leq m^0. \)
- 2.2.10.1 is true when \( i = i_1, m < m^0. \)
- \( m_{i_0}^0 \neq 0. \)

Then 2.2.10.1 is true when \( i = i_1, m = m^0. \)
Proof. Let $J$ be the $R$-submodule of $R[[M]]$ generated by $\{ t^m \mid m \not\in m^0 \}$. It is an ideal. The first assumption means:

$$A^i B + \partial_{i_0} B \equiv BA_0^i \pmod{J},$$

$$BB' \equiv I \pmod{J}.$$  

By the integrability of $(E, \nabla)$

$$\partial_{i_0} \partial_i (eB) = \partial_i \partial_{i_0} (eB).$$

The left hand side is

$$\partial_i, \partial_{i_0} (eB) = \partial_{i_0} \left( e \left( A^i B + \partial_{i_0} B \right) \right) \equiv \partial_{i_0} \left( eBA_0^i \right) \pmod{J}$$

$$= e \left( A^i B + \partial_{i_0} B \right) A_0^i \equiv eB \left( B' A^i B + B' \partial_i B \right) A_0^i \pmod{J}.$$  

The right hand side is

$$\partial_{i_0} \partial_i (eB) = \partial_{i_0} \left( e \left( A^i B + \partial_i B \right) \right) \equiv \partial_{i_0} \left( eB \left( B' A^i B + B' \partial_i B \right) \right) \pmod{J}$$

$$\equiv eB \left( A_0^i + \partial_{i_0} \right) \left( B' A^i B + B' \partial_i B \right) \pmod{J}.$$  

Let $A' = B' A^i B + B' \partial_i B$, then

(2.2.10.5) $$A_0^i A' + \partial_{i_0} A' \equiv A' A_0^i \pmod{J}.$$  

By the comparison of the coefficients of $t^m$ in (2.2.10.5)

(2.2.10.6) $$A_0^i A_{m_0} - A'_{m_0} A_0^i + m_{i_0} A_{m_0} = O.$$  

Here, $A'_{m_0}$ is the coefficient of $t^m$ in $A'$.  

By (2.2.10.3), Claim 2.2.10.3 and the assumption that $m_{i_0} \not= 0$, $A'_{m_0} = 0$. By this and the second assumption, $A' \equiv A_0^i \pmod{J}$, so $A^i B + \partial_{i_0} B \equiv BA_0^i \pmod{J}$. Therefore (2.2.10.1) is true when $i = i_0, m = m_0$.  

Let $h$ be the weighting of $M$. We construct $B_m$ and $B'_m$ inductively with respect to $h(m)$.

First, we let $B_0 = B'_0 = I$.  

Take $m \in M, m \not= 0$. Take $1 \leq i_0 \leq r$ such that $m_{i_0} \not= 0$ (this is always possible because $M \cap (M^{gp})^{tor} = \{0\}$ by the assumption of $M$.) We assume that $B_{m'}, B'_{m'}$ are constructed for all $m'$ with $h(m') < h(m)$, especially for all $m' < m$.  

By Claim 2.2.10.3 we can take $B_m$ satisfying (2.2.10.4) for $i = i_0$, and can easily take $B'_m$ satisfying (2.2.10.2). Then by Claim 2.2.10.4 (2.2.10.1) is true for any $i$.  

Next, we prove that $B = \sum m \in M B_m t^m$ converges on $A_{m, k}[0, b]$ for some $b > 0$.  

For all $m \in M$ and $i$ such that $m_i \not= 0$, let

$$z_{m, i} = \max \left\{ \max_{j, j'} |\xi_{i, j} - \xi_{i, j'} - m_i|^{-1}, 1 \right\}.$$  

For $m \in M$, let
\[
Z_m = \max_{m^1 < m^2 < \cdots < m^s = m} \prod_{k=1}^{s} \min_{i,m^k \neq 0} z_{m^k,i}.
\]

Take \( C > 0, \varepsilon > 0 \) which satisfy the inequality of Claim 2.2.10.3. We can also assume that \( |A'_m| \leq C a^{-h(m)} \) for all \( i \) and \( m \in M \) because all \( A_i' \)'s converge in \( K_{M,K}[0,a] \).

We prove inductively that

\[(2.2.10.7) \quad |B_m| \leq Z_m^e C^{2h(m)} a^{-h(m)}.\]

The case \( m = 0 \) is clear.

Assume this is true for all \( m' < m \). By Claim 2.2.10.3 and (2.2.10.1)

\[
|B_m| \leq \min_{i,m, \neq 0} C_{z,m,i}^e \left| \sum_{m',m''=m} A_{m'} B_{m''} \right|
\]
\[
\leq \min_{i,m, \neq 0} C_{z,m,i}^e \max_{m',m''=m} |A_{m'}| |B_{m''}|
\]
\[
\leq \min_{i,m, \neq 0} C_{z,m,i}^e \max_{m',m''=m} C a^{-h(m')} Z_m^e C^{2h(m'')} a^{-h(m'')}
\]
\[
= \max_{m',m''=m} \left( \min_{i,m, \neq 0} z_{m,i} Z_{m''} \right)^e C^{2h(m'')} a^{-h(m)}
\]
\[
\leq Z_m^e C^{2h(m)} a^{-h(m)}.
\]

We must estimate \( Z_m \).

Claim 2.2.10.8. There exist constants \( C_1, C_2 > 0 \) such that for any \( m^1 < m^2 < \cdots < m^s \) and \( x > 0 \) with \( \left\{ 1 \leq k \leq s \mid \log \min_{i,m^k \neq 0} z_{m^k,i} \geq x \right\} > rn^2, h(m^s) - h(m^1) \geq C_1 e^{C_2 x} \).

Proof. By the pigeonhole principle, there exist \( m^k \) and \( i, j' \) such that \( m^j = m^j' \neq 0 \) and that \( |\xi_{i,j} - \xi_{i,j'} - m^k_i|, |\xi_{i,j} - \xi_{i,j'} - m^k_i| \leq e^{-x} \). There exists \( m' \in M \) such that \( m^k + m' = m^k' \). Because \( m'_i = m^k_i \), \( m'_i \leq e^{-x} \).

On the other hand, there exists a constant \( c_1 > 0 \) which depends only on the valuation of \( K \) such that for any \( a \in \mathbb{Z}, \log |a| \geq -c_1 \log |a|, \) where \( |x| \) is the usual absolute value of \( \mathbb{R} \). There also exists a constant \( c_2 > 0 \) which depends only on \( h \) such that for any \( 1 \leq i \leq r \) and \( m \in M, |m_i| \leq c_2 h(m) \).

Then \( x \leq -\log |m'_i| \leq c_1 \log |m'_i| \leq c_1 (\log c_2 + \log h(m')) \).

Therefore, if we put \( C_1 := c_2^{-1}, C_2 := c_1^{-1} \), then \( h(m^s) - h(m^1) \geq h(m') \geq C_1 e^{C_2 x} \).

By Claim 2.2.10.8 for \( m^1 < m^2 < \cdots < m^s = m \) and \( x > 0 \),
\[
\left\{ \begin{array}{c}
1 \leq k \leq s \\
\log \min_{i,m^i \neq 0} z_{m^i,i} \geq x
\end{array} \right\} \leq (rn^2 + 1) \left( 1 + \frac{h(m)}{C_1} e^{-C_2 x} \right).
\]

Therefore, if we put
\[
v_m := \max_{m' < m} \log \left( \min_{i,m^i \neq 0} z_{m',i} \right),
\]
log $Z_m$ is estimated as follows:
\[
\log Z_m = \max_{m^1 < \ldots < m^s = m} \sum_{j=1}^{s} \log \left( \min_{i,m^j \neq 0} z_{m^j,i} \right)
\leq \int_0^{v_m} (rn^2 + 1) \left( 1 + \frac{h(m)}{C_1} e^{-C_2 x} \right) dx
\leq (rn^2 + 1) \left( v_m + \int_0^{\infty} \frac{h(m)}{C_1} e^{-C_2 x} dx \right)
= (rn^2 + 1) \left( v_m + \frac{h(m)}{C_1 C_2} \right).
\]

By the assumption about exponents, $\xi_{i,j} - \xi_{i,j'}$ has positive type for any $i, j, j'$. Hence
\[
\log \left( \min_{i,m^i \neq 0} z_{m^i,i} \right) \in O\left( \max_i |m_i|_0 \right) \subseteq O(h(m)),
\]
and so $v_m \in O(h(m))$.

Therefore log $Z_m \in O(h(m))$. Thus, by (2.2.10.7), $B$ converges on $Y \times A_M, K[0, b]$ for some $0 < b \ll a$.

Let $f = eB$ as a basis of $E|_{Y \times A_M, K[0, b]}$. Then the matrix representation of $\partial_i$ with respect to $f$ is $A_i^0 \in \text{Mat}_{n_i}(R)$. Thus the $R$-module generated by $f$ is stable under $\partial_i$.

Finally, we show that the $R$-module generated by $f$ is stable under $\nabla^Y$. Let $\omega_1, \ldots, \omega_r$ be a local basis of $\Omega^1_Y$. We write $\nabla^Y = \sum_{i=1}^{r} \partial_i^\prime \omega_i^\prime$ and let $D^\prime = \sum_{m \in M} D_m^\prime t^m$ be the matrix representation of $\partial_i^\prime$ with respect to $f$.

Let $m \in M \setminus \{0\}$. Take $1 \leq i \leq r$ such that $m_i \neq 0$. By the integrability of $\nabla$, $\partial_i^\prime \partial_i(f) = \partial_i^\prime \partial_i(f)$. So we have $A_i^0 D^\prime_m + \partial_i^\prime D^\prime = D^\prime A_i^0$, which means
\[
A_i^0 D^\prime_m + m_i D^\prime_m - D^\prime_m A_i^0 = 0.
\]
By Claim 2.2.10(3), $D^\prime_m = 0$.

Therefore $D^\prime = D^\prime_0$ and $\nabla^Y$ is also defined on the $R$-module generated by $f$.

So the $R$-module generated by $f$ equipped with $\nabla^Y$ and $\partial_i$ for $1 \leq i \leq r$ defines an object of $\text{LNM}_{Y \times A_M, K[0, b], \Sigma}$, whose image under $\mathcal{U}_{[0, b]}$ is equal to $E_{Y \times A_M, K[0, b]}$.  

Next, we define a global version of the notion of $(S\text{-}D)$.

**Definition 2.2.11.** Let $S$ be a subset of $K$ and $M$ a fine monoid. A subset $\Sigma \subseteq M^{\text{op}} \otimes_{\mathbb{Z}} K$ is called $(S\text{-}D)$ if, for all face $F$ of $M$, the image of $\Sigma$ in $(M/F)^{\text{op}} \otimes_{\mathbb{Z}} K$ is locally $(S\text{-}D)$.

**Lemma 2.2.12.** Let $M$ be a semi-saturated monoid. For any facet $F$ of $M$, $(M/F)^{\text{op}} \cong \mathbb{Z}$. 

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Proof. By the definition of facet, the dimension of $M/F$ in the sense of (5.4) of [12] is 1. By Proposition (5.5) of [12], the rank of $(M/F)^{\gp}$ is 1. Since $M$ is semi-saturated, $(M/F)^{\gp}$ is torsion-free. So $(M/F)^{\gp} \cong \mathbb{Z}$. \qed

Proposition 2.2.13. In the situation of Definition 2.2.11 assume that $M$ is semi-saturated and that, for all $s \in M$ and $n \in \mathbb{N}_{>0}$ such that $ns \in S$, $s \in S$. Then $\Sigma$ is (S-D) if and only if for all facet $F$ of $M$ and for any two elements $\alpha, \beta$ in the image of $\Sigma$ in $(M/F)^{\gp} \otimes_{\mathbb{Z}} K = \mathcal{F}$, $\alpha - \beta$ is contained in $S$.

Proof. If $\Sigma$ is (S-D), for all facet $F$ of $M$, there exists a map $\phi : (M/F)^{\gp} = \mathbb{Z} \hookrightarrow \mathbb{Z}$ satisfying condition in Definition 2.2.6. By retaking $-\phi$ as $\phi$ if necessary, we may assume that $\phi(1) > 0$. By the assumption about $\Sigma$, we may assume that $\phi = id$. Then $\Sigma$ satisfies the latter condition.

Conversely, assume that $\Sigma$ satisfies the latter condition. For any face $F$ of $M$, the map

$$(M/F)^{\gp} \to \prod_{F' \subseteq F'} (M/F')^{\gp},$$

where $F'$ runs through all facets of $M$ containing $F$, satisfies the condition of being locally (S-D). \qed

Remark 2.2.14. By Proposition 2.2.13 and Remark 2.2.7 if $S$ satisfies the condition of 2.2.10 $M$ is semi-saturated and $\Sigma$ is (S-D), we can take $\phi$ in Definition 2.2.6 such that $\phi(M) \subseteq \mathbb{N}^r$ and $\phi \otimes id$ is isomorphic.

Remark 2.2.15. NI, PT and NL satisfy the assumption of Proposition 2.2.13 because for any $\alpha \in K$, $n \in \mathbb{N}_{>0}$

$$\text{type}(n\alpha) \leq \text{type}(\alpha)^{\frac{1}{r}}.$$ 

Definition 2.2.16. Let $S$ be a subset of $\mathcal{K}$, $(X, \mathcal{M})$ a fine log rigid space and $S \subseteq \mathcal{M}^{\gp} \otimes_{\mathbb{Z}} K$ a subsheaf. $S$ is called (S-D) if, for any geometric point $\mathfrak{p}$ of $X$, it admits a good chart $M \to \mathcal{M}$ on some neighborhood of $\mathfrak{p}$ and an (S-D) subset $\Sigma \subseteq M^{\gp} \otimes_{\mathbb{Z}} K$ such that $S$ is the image of $\Sigma$ under the induced map $M^{\gp} \otimes_{\mathbb{Z}} K \to \mathcal{M}^{\gp} \otimes_{\mathbb{Z}} K$.

Remark 2.2.17. The definition of (S-D)-ness is independent of the choice of a good chart. Indeed, let $\alpha : M \to \mathcal{M}$ and $\beta : M \to \mathcal{M}$ be two good charts at $\mathfrak{p}$ on some neighborhood of $\mathfrak{p}$, $M \xrightarrow{\alpha} \mathcal{M} \to \mathcal{M} = M$ and $M \xrightarrow{\beta} \mathcal{M} \to \mathcal{M} = M$ coincide, so $M \xrightarrow{\alpha} \mathcal{M} \to \mathcal{M}$ and $M \xrightarrow{\beta} \mathcal{M} \to \mathcal{M}$ coincide after shrinking the neighborhood of $\mathfrak{p}$. Then the two maps $M^{\gp} \otimes_{\mathbb{Z}} K \to \mathcal{M}^{\gp} \otimes_{\mathbb{Z}} K$ induced by $\alpha$ and $\beta$ coincide.

Remark 2.2.18. Let $S$ be a subset of $\mathcal{K}$. For any rigid space $Y$, a semi-saturated weighted monoid $M$, an aligned subinterval $I \subseteq [0, \infty)$ and an (S-D) subset $\Sigma \subseteq M^{\gp} \otimes_{\mathbb{Z}} K$, the subsheaf on $Y \times \mathcal{A}_{M,K}(I)$ defined by $\Sigma$ is (S-D). Indeed, for any geometric point $(\mathfrak{y}, \mathfrak{p})$ of $Y \times \mathcal{A}_{M,K}(I)$, put $F := \{ m \in M \mid t^{m}(\mathfrak{p}) \neq 0 \}$. Then $M/F = \overline{\mathcal{M}((\mathfrak{y}, \mathfrak{p}))}$ where $\mathcal{M}$ is the log structure induced by $M$. $\mathcal{A}_{M,K}(I) \cap \mathcal{A}_{F^{-1}M,K}$ is an open subset of $\mathcal{A}_{M,K}(I)$ containing the image of $\mathfrak{p}$. By Lemma 1.1.4 the map $F^{-1}M \to M/F$ has a section $M/F \to F^{-1}M$, which gives a good chart at $(\mathfrak{y}, \mathfrak{p})$ on $Y \times (\mathcal{A}_{M,K}(I) \cap \mathcal{A}_{F^{-1}M,K})$. Let $\Sigma_F$ be the image of $\Sigma$ under the map $M^{\gp} \otimes_{\mathbb{Z}} K \to (M/F)^{\gp} \otimes_{\mathbb{Z}} K$, which is also (S-D) by definition. The subsheaf
defined by $\Sigma$ is the image of $\Sigma_F$ under the map $(M/F)^{gp} \otimes_Z K \to \mathcal{M}^{gp} \otimes_Z K$ on $Y \times (\mathcal{A}_{M,K}(I) \cap \mathcal{A}_{F^{-1},M,K})$.

**Proposition 2.2.19.** (cf. Proposition 1.11 in [21]) Let $(X,M)$ be a log smooth rigid space over $K$ such that $\mathcal{M}^{gp}$ is torsion-free at any geometric point $\mathfrak{p}$ of $X$. Let $\mathcal{S} \subseteq \mathcal{M}^{gp} \otimes_Z K$ be an $((NI \cap PT)$-D) subsheaf. Then $\mathcal{LNM}_{(X,M),\mathcal{S}}$ is an abelian category.

**Proof.** The proof is almost the same as the proof of Proposition 1.11 in [21].

Let $f : (E, \nabla_E) \to (F, \nabla_F)$ be a morphism in $\mathcal{LNM}_{(X,M),\mathcal{S}}$. It is easy to show that $\text{Ker}(f)$ and $\text{Coker}(f)$ are locally free because of Remark [22.1.7].

By Lemma 3.2.14 of [13], it is enough to show very locally, i.e., it is enough to show local freeness for some open neighborhood of any point. Take $x \in X$ and a geometric point $\mathfrak{p}$ lying over $x$. Let $M := \mathcal{M}_{\mathfrak{p}}$. By Lemma 1.2.6 there exist an étale neighborhood $U \to X$ of $\mathfrak{p}$ and a good chart $M \to M|_U$ at $\mathfrak{p}$ such that the induced morphism $U \to \mathcal{A}_{M,K}[0,1]$ is smooth. We may assume that this morphism has a decomposition $U \to \mathcal{A}_K^n \times \mathcal{A}_{M,K}[0,1] \to \mathcal{A}_{M,K}[0,1]$ for some $n$ where the first morphism is étale and the second morphism is the second projection. By the assumption about $\mathcal{S}$ and Remark [22.1.7] we may also assume that there exists an $((NI \cap PT)$-D) subset $\Sigma \subseteq M^{gp} \otimes_Z K$ such that $\mathcal{S}|_U$ is induced by it.

We may replace $K$ by its finite extension and assume that $\mathfrak{p}$ maps to a $K$-rational point of $U$. By Lemma 3.1.5 of [4], on some open neighborhood of the image of $\mathfrak{p}$ in $U$, both $U \to X$ and $U \to \mathcal{A}_K^n \times \mathcal{A}_{M,K}[0,1]$ are open immersions. Take some $h : M \to \mathbb{N}$ and regarded $M$ as a weighted monoid by $h$, then there exists an open neighborhood of $\mathfrak{p}$ which is isomorphic to $Y \times \mathcal{A}_{M,K}[0,a]$ for some smooth affinoid space $Y$ and $a > 0$ by Proposition 2.1.11. So we may assume that $(X,M) = Y \times \mathcal{A}_{M,K}[0,a]$ and that $\mathcal{S} = \Sigma$.

We may assume that $E$ and $F$ are free. Then, by Lemma 2.2.10 we may assume that $E$ and $F$ are in the image of $\mathcal{U}_{[0,a]}$.

Take some morphism $M \to \mathbb{N}^r$ as in Remark 2.2.14 and denote by $\partial_i$ the composition of $\nabla^{log}$ and the map induced by the projection $M^{gp} \to \mathbb{Z}^r \to \mathbb{Z}$ to the $i$-th factor. Let $E', F'$ be the objects of $\mathcal{LNM}_{Y \times \mathcal{A}_{M,K}[0,0],\Sigma}$ such that $\mathcal{U}_{[0,a]}(E') = E, \mathcal{U}_{[0,a]}(F') = F$. After we replace $K$ by its finite extension if necessary, we can take a basis $\mathbf{e} = (\mathbf{e}_1, \ldots, \mathbf{e}_l)$ of $E'$ such that $\mathbf{e}_1$ is an eigenvector of all $\partial_i$'s.

It is enough to show that $f(\mathbf{e}_1) \in F'$ because of the induction of the ranks of $E$ and $F$. (See the proof of Proposition 3.2.14 in [13] or the proof of Proposition 1.11 in [21].)

There exists $(\xi_1, \ldots, \xi_r) \in \Sigma$ such that $\partial_i(\mathbf{e}_1) = \xi_i \mathbf{e}_1$. Write $f(\mathbf{e}_1) = \sum_{m \in M} a_m \mathbf{f}_m$ where $a_m \in F'$.

We will prove that for any $1 \leq i \leq r$, if $m_i \neq 0$ then $a_m = 0$.

We may assume that $i = 1$. After we replace $K$ by its finite extension if necessary, we can take a basis $\mathbf{f} = (\mathbf{f}_1, \ldots, \mathbf{f}_n)$ of $F'$ such that the matrix representation of $\partial_1$ with respect to it is the Jordan standard form. Then there exist $(\eta_1, \ldots, \eta_r) \in \Sigma$ and $0 = n_0 < n_1 < \cdots < n_k = n$ such that

$$\partial_1 \mathbf{f}_{n_i} = \eta_i \mathbf{f}_{n_i} \quad (1 \leq i \leq k),$$

$$\partial_1 \mathbf{f}_j = \eta_i \mathbf{f}_j + \mathbf{f}_{j+1} \quad (1 \leq i < k, \ n_i < j < n_{i+1}).$$
Let $a_m = \sum_{i=1}^r a_m f_i$. Then by the commutativity of $f$ and the connections
\[
(f \otimes \text{id}_{\Omega^1_{\log, \mathcal{X}}(X, \mathcal{M})}) (\nabla_E(e)) = \nabla_E(f(e))
\]
and comparing the coefficients of $d \log t_1$ on both hand sides, we obtain the equalities:
\[
(\xi_i - \eta_i) a_{m,n,i+1} = m_i a_{m,n,+1} (0 \leq i < k), \quad (\xi_i - \eta_i) a_{m,n,j} = m_i a_{m,j} + a_{m,j-1} (0 \leq i < k, \ m_i + 2 \leq j \leq m_i + 1).
\]

By the assumption for $\Sigma$, $\xi_i - \eta_i + 1$ is not a non-zero integer. So if $m_1 \neq 0$, $a_{m,n,+1} = 0$ because of the first equality. Then by the second equality, we see that $a_{m,j} = 0$ for all $j$. \hfill \square

**Corollary 2.2.20.** For any smooth rigid space $Y$, any fine sharp semi-saturated monoid $M$, a subset $\Sigma \subseteq M^{sp} \otimes \mathbb{Z} \overline{K}$ which is $((\text{NI} \cap \text{NL})$-$D)$ and any aligned subinterval $I \subseteq [0, \infty)$, $\text{LNM}_{Y \times \mathbb{A}_{M,K}(I), \Sigma}$ is an abelian category.

**Proof.** By Remark 2.2.18 and Proposition 2.2.19 \hfill \square

### 2.3. Unipotence

In this subsection, we define the notion of $\Sigma$-unipotence of log $\nabla$-modules and prove some basic properties.

**Definition 2.3.1.** (cf. 1.3 of [21]) Let $Y$ be a smooth rigid space, $M$ a fine sharp semi-saturated monoid, $I \subseteq [0, \infty)$ an aligned subinterval, and $\Sigma \subseteq M^{sp} \otimes \mathbb{Z} \overline{K}$ be an $((\text{NI} \cap \text{NL})$-$D)$ subset. Let $\pi_1$ and $\pi_2$ be the first and second projections of $Y \times \mathbb{A}_{M,K}(I)$.

1. An object $(E, \nabla)$ of $\text{LNM}_{Y \times \mathbb{A}_{M,K}(I), \Sigma}$ is $\Sigma$-constant if it is isomorphic to $\pi_1^* E_Y \otimes \pi_2^* C_\xi$ for some $\nabla$-module $E_Y$ on $Y$ and $\xi \in \Sigma \cap (M^{sp} \otimes \mathbb{Z} \overline{K})$, where $C_\xi = (\mathcal{O}_{\mathbb{A}_{M,K}(I)}, d + \xi \cdot \text{id})$. Note that $M^{sp} \otimes \mathbb{Z} \overline{K}$ is injected to $\Omega^{\log, 1}_{\mathbb{A}_{M,K}(I)}$ and $d + \xi \cdot \text{id}$ is the map $f \in \mathcal{O}_{\mathbb{A}_{M,K}(I)} \mapsto df + f\xi$.

2. An object $(E, \nabla)$ of $\text{LNM}_{Y \times \mathbb{A}_{M,K}(I), \Sigma}$ is $\Sigma$-unipotent if it admits a filtration by subobjects $0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_n = E$ in $\text{LNM}_{Y \times \mathbb{A}_{M,K}(I), \Sigma}$ such that any successive quotient is $\Sigma$-constant. We denote by $\text{ULNM}_{Y \times \mathbb{A}_{M,K}(I), \Sigma}$ the full subcategory of $\text{LNM}_{Y \times \mathbb{A}_{M,K}(I), \Sigma}$ whose objects are $\Sigma$-unipotent objects.

3. An object $(E, \nabla)$ of $\text{LNM}_{Y \times \mathbb{A}_{M,K}(I), \Sigma}$ is potentially $\Sigma$-constant (resp. potentially $\Sigma$-unipotent) if there exists a finite extension $K'$ of $K$ such that $E_{K'}$ is $\Sigma$-constant (resp. $\Sigma$-unipotent). We denote by $\text{ULNM}'_{Y \times \mathbb{A}_{M,K}(I), \Sigma}$ the full subcategory of $\text{LNM}_{Y \times \mathbb{A}_{M,K}(I), \Sigma}$ whose objects are potentially $\Sigma$-unipotent objects.

**Remark 2.3.2.** In the case $I = [0, 0]$, an object of $\text{LNM}_{Y \times \mathbb{A}_{M,K}[0,0], \Sigma}$ is $\Sigma$-unipotent if and only if its all exponents are contained in $(M^{sp} \otimes \mathbb{Z} \overline{K})$, because of Lemma 2.2.3. In particular, $\text{LNM}_{Y \times \mathbb{A}_{M,K}[0,0], \Sigma} = \text{ULNM}'_{Y \times \mathbb{A}_{M,K}[0,0], \Sigma}$ (cf. Remark 1.13 of [21]).

**Remark 2.3.3.** In the case $0 \notin I$, for any $\xi_1, \xi_2 \in M^{sp} \otimes \mathbb{Z} \overline{K}$ such that $\xi_1 - \xi_2 \in M^{sp}$, $C_{\xi_1} \cong C_{\xi_2}$ on $\mathbb{A}_{M,K}(I)$. Indeed, let $m := \xi_1 - \xi_2 \in M^{sp}$, then the morphism $f \in \mathcal{O}_{\mathbb{A}_{M,K}(I)} \mapsto ft^m \in \mathcal{O}_{\mathbb{A}_{M,K}(I)}$ gives an isomorphism $C_{\xi_1} \cong C_{\xi_2}$ as log $\nabla$-modules. So $\Sigma$-constantness and $\Sigma$-unipotence of log $\nabla$-modules are only dependent on the image of $\Sigma$ in $M^{sp} \otimes \mathbb{Z} \overline{K}/\mathbb{Z}$.

The next proposition and its proof are analogues of 3.3.2 of [13] and 1.14 of [21].
Proposition 2.3.4. Let $Y$ be a smooth rigid space, $M$ a fine sharp semi-saturated monoid, $I \subseteq [0, \infty)$ a quasi-open subinterval of positive length, and $\Sigma \subseteq M^{gp} \otimes_{\mathbb{Z}} \mathbb{K}$ an $((\mathcal{N}\cap \mathcal{N})\mathcal{L})$-subgroup. Then the morphisms

$$\text{Ext}^i(E, E') \to \text{Ext}^i(U_\ell(E), U_\ell(E'))$$

induced by $U_\ell : \text{LNM}_{\mathcal{Y} \times \mathcal{A}_M,K[0,0],\Sigma} \to \text{LNM}_{\mathcal{Y} \times \mathcal{A}_M,K(1),\Sigma}$ are isomorphic for all $E, E' \in \text{LNM}_{\mathcal{Y} \times \mathcal{A}_M,K[0,0],\Sigma}$ and $i \geq 0$.

Proof. Let $F = E' \otimes E'$. By 3.3.1 of [13], we have isomorphisms

$$\text{Ext}^i(E, E') = H^i\left(Y, F \otimes \Omega_{\mathcal{Y} \times \mathcal{A}_M,K[0,0]}^{\log,\bullet}\right),$$

$$\text{Ext}^i(U_\ell(E), U_\ell(E')) = H^i\left(Y \times \mathcal{A}_M,K(I), U_\ell(F) \otimes \Omega_{\mathcal{Y} \times \mathcal{A}_M,K(I)}^{\log,\bullet}\right)$$

for any $i \geq 0$.

Note that

$$U_\ell(F) \otimes \Omega_{\mathcal{Y} \times \mathcal{A}_M,K(I)}^{\log,1} = F \otimes \Omega_{\mathcal{Y} \times \mathcal{A}_M,K[0,0]}^{\log,1} \otimes \mathcal{O}_{\mathcal{A}_M,K(I)}.$$"
write $E = E_Y \otimes C_{\xi}, E' = E'_Y \otimes C_{\xi'}$ for some $\xi, \xi' \in \Sigma$ and then $F = E'_Y \otimes E_Y \otimes C_{\xi - \xi'}$.

Let
\[
g_2 : \Gamma \left( Y \times A_{M,K}(I) , U_I(F) \otimes \Omega^{\log \bullet}_{Y \times A_{M,K}(I)/Y} \right) \to \Gamma \left( Y, F \otimes \Omega^{\log \bullet}_{Y \times A_{M,K}(I)/Y} \right)
\]
be the map induced by the ‘taking constant coefficient’ map
\[
\Gamma \left( A_{M,K}(I) , \mathcal{O}_{A_{M,K}(I)} \right) \to K.
\]

Then $g_2 \circ g_1$ is the identity. We show that $g_1 \circ g_2$ is homotopic to the identity by constructing a homotopy $\varphi$.

Take $\varphi : M \to \mathbb{N}$ as in Remark 2.2.14 which induces
\[
\Omega^{\log \cdot}_{A_{M,K}(I)} = \bigoplus_{i=1}^r \mathcal{O}_{A_{M,K}(I)} d \log t_i.
\]

For $m \in M_{sp} \setminus \{0\}$ and $1 \leq i_1 < \cdots < i_k \leq r$, let $l = l(m)$ be the least integer such that $m_l \neq 0$, and we define
\[
\varphi \left( t^m \prod_{j=1}^k d \log t_{i_j} \right) = \begin{cases} (-1)^{s-1} m_l + \xi_l' - \xi_l t^m \prod_{1 \leq j \leq k \atop j \neq s} d \log t_{i_j} & \text{if } i_s = l \text{ for some } s, \\ 0 & \text{otherwise.} \end{cases}
\]

Note that by the assumption for $\Sigma$, $m_l + \xi_l' - \xi_l \neq 0$. We also define
\[
\varphi \left( t^0 \prod_{j=1}^k d \log t_{i_j} \right) = 0.
\]

We will show that $\varphi$ can be extended naturally to the map of graded modules,
\[
\varphi : \Gamma \left( Y \times A_{M,K}(I) , U_I(F) \otimes \Omega^{\log \bullet}_{Y \times A_{M,K}(I)} \right) \to \Gamma \left( Y \times A_{M,K}(I) , U_I(F) \otimes \Omega^{\log \bullet}_{Y \times A_{M,K}(I)} \right)
\]

to show this, it is enough to show that, if \( \sum_{m \in M_{sp}} c_m t^m \) converges on $Y \times A_{M,K}(I)$,
\[
\sum_{m \in M_{sp} \setminus \{0\}} c_m m_l + \xi_l' - \xi_l t^m
\]
also converges on $Y \times A_{M,L}(I)$ for any $l$.

Let $[a,b] \subseteq I$ be an aligned closed subinterval. By the assumption, $|c_m| a^{-h(m)} b^{h'(m)} \to 0$ when $|h|(m) \to \infty$. Since $\xi_l' - \xi_l$ is not a $p$-adic Liouville number, for any $s < 1$,
\[
\frac{c_m |m_l|_a}{m_l + \xi_l' - \xi_l} \to 0
\]
when $|m_l|_R \to \infty$. Take a constant $C$ such that $|m_l|_R \leq C |h|(m)$ for any $m \in M_{sp}$. Then
\[
\left| \frac{c_m}{m_l + \xi_l' - \xi_l} \right| a^{-h(m)} b^{h'(m)} s^{C|h|(m)} \to 0.
\]

So the series (2.3.4.2) converges on $A_{M,K} [a/sC, bsC]$. Since $I$ is quasi-open, (2.3.4.2) converges on $A_{M,K}(I)$. 

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To finish the proof, we show that \( \varphi \) gives a homotopy between \( g_1 \circ g_2 \) and the identity.

Take \( m \neq 0 \) and \( 1 \leq i_1 < \cdots < i_k \leq r \), and let \( l = l(m) \). If \( i_s = l \) for some \( s \),

\[
\nabla_F \varphi \left( \prod_{j=1}^{k} d \log t_{i_j} \right) = \nabla_F \left( \frac{(-1)^{s-1}}{m_i + \xi_{i} - \xi_l} \prod_{j \neq s} d \log t_{i_j} \right)
\]

\[
= t^{m} \prod_{j=1}^{k} d \log t_{i_j} + (-1)^{s-1} \sum_{i' \notin \{i_r\}} \frac{m_{i'} + \xi_{i'} - \xi_l}{m_i + \xi_i - \xi_l} t^{m} d \log t_{i'} \wedge \prod_{j \neq s} d \log t_{i_j},
\]

\[
\varphi \nabla_F \left( \prod_{j=1}^{k} d \log t_{i_j} \right) = \varphi \left( \sum_{i' \notin \{i_r\}} \left( m_{i'} + \xi_{i'} - \xi_l \right) t^{m} d \log t_{i'} \wedge \prod_{j=1}^{k} d \log t_{i_j} \right)
\]

\[
= (-1)^{s} \sum_{i' \notin \{i_r\}} \frac{m_{i'} + \xi_{i'} - \xi_l}{m_i + \xi_i - \xi_l} t^{m} d \log t_{i'} \wedge \prod_{j \neq s} d \log t_{i_j}.
\]

Otherwise, \( \varphi \left( \prod_{j=1}^{k} d \log t_{i_j} \right) = 0 \) and

\[
\varphi \nabla_F \left( \prod_{j=1}^{k} d \log t_{i_j} \right) = \varphi \left( \sum_{i' \notin \{i_r\}} \left( m_{i'} + \xi_{i'} - \xi_l \right) t^{m} d \log t_{i'} \wedge \prod_{j=1}^{k} d \log t_{i_j} \right)
\]

\[
= t^{m} \prod_{j=1}^{k} d \log t_{i_j}.
\]

Also, for \( m = 0 \), \( \nabla_F \varphi \left( \prod_{j=1}^{k} d \log t_{i_j} \right) = \varphi \nabla_F \left( \prod_{j=1}^{k} d \log t_{i_j} \right) = 0.
\]

So \( \nabla_F \varphi + \varphi \nabla_F = \text{id} - g_1 \circ g_2 \).

The following corollary is the analogue of Corollary 1.15 and Corollary 1.16 in [21].

**Corollary 2.3.5.** Let \( Y, M, \Sigma \) be as in [2.3.4] and \( I \subseteq [0, \infty) \) an aligned subinterval of positive length. Then the functors

\[
U_t : \text{ULNM}_{Y \times \mathbb{A}_M, K[0, \infty), \Sigma} \rightarrow \text{ULNM}_{Y \times \mathbb{A}_M, K(I), \Sigma}
\]
\[
U_t : \text{ULNM}'_{Y \times \mathbb{A}_M, K[0, \infty), \Sigma} \rightarrow \text{ULNM}'_{Y \times \mathbb{A}_M, K(I), \Sigma}
\]

are fully-faithful. Moreover, if \( I \) is quasi-open, these functors are equivalences of categories.

**Proof.** The proof is the same as the proofs of Corollary 1.15 and Corollary 1.16 in [21].

To prove Proposition 2.3.7 below, we show the following lemma, which is a generalization of Lemma 3.2:19 in [13].

**Lemma 2.3.6.** Let \( Y \) be an affinoid space, \( M \) a fine sharp monoid such that \( M^{\text{gp}} \) is torsion-free, \( I \subseteq [0, \infty) \) an aligned closed subinterval and \( E \) a log \( \nabla \)-module on \( Y \times \mathbb{A}_M, K(I) \). Take \( e_1, \ldots, e_n \in \Gamma(Y \times \mathbb{A}_M, K(I), E) \) which are linearly independent
over $\mathcal{O}_Y$. Assume that there exists $\xi \in M^{sp} \otimes \mathbb{Z} \overline{K}$ such that $\nabla_{E}^{log}(e_i) = \xi e_i$ for each $i$. Then $e_1, \ldots, e_n$ are linearly independent over $\mathcal{O}_{Y \times A_{M,K}(1)}$.

**Proof.** The case when $I = [0, 0]$ is clear. Assume $I \neq [0, 0]$. Making $I$ smaller if necessary, we may assume that $0 \notin I$. Take some $M^{sp} \cong \mathbb{Z}^r$. Write $\{d \log t_i\}$ as the induced basis of $\Omega^{log} \lambda_{M,K}(1)$ and $d^{log} = \sum_{i=1}^{r} \partial_i d \log t_i$. Let $l > 0$, define an operator $D_{l}$ on $\mathcal{O}_{Y \times A_{M,K}(1)}$ by

$$D_{l} = \prod_{i=1}^{r} \prod_{j \neq 0} \frac{1}{l}(\partial_i - j)$$

(cf. the proof of 3.4.1 in [13]).

We prove that for all $f = \sum_{m \in M^{sp}} f_m t^m \in \mathcal{O}_{Y \times A_{M,K}(1)}$, $D_{l}(f)$ converges to $f_0$ when $l \to \infty$. Let

$$M^{sp}_{\Sigma} = \{ m \in M^{sp} \mid \{ m_i \} \mid m_i < -l \text{ or } m_i = 0 \text{ or } m_i > l \text{ for any } 1 \leq i \leq r \}.$$  

Then

$$D_{l}(f) - f_0 = \sum_{m \in M^{sp}} \prod_{i=1}^{r} \prod_{j \neq 0} \frac{m_i - j}{l} f_m t^m.$$  

For any $m$, $\prod_{i=1}^{r} \prod_{j \neq 0} \frac{m_i - j}{l}$ is an integer because it is a product of binomial coefficients. Since $\min_{m \in M^{sp}} |h_i(m)| \to \infty$ when $l \to \infty$, $\sup_{m \in M^{sp}} |f_m t^m| \to 0$ on $A_{M,K}(1)$. Therefore $(D_{l}(f) - f_0) \to 0$ on $A_{M,K}(1)$.

Assume $\sum_{k=1}^{n} c_k e_k = 0$ for some $c_1, \ldots, c_n \in \mathcal{O}_{Y \times A_{M,K}(1)}$. Applying $\nabla_{E}$, we have

$$\sum_{k=1}^{n} (\partial_i(c_k) e_k + c_k \xi \epsilon e_k) = 0,$$

so $\sum_{k=1}^{n} \partial_i(c_k) e_k = 0$ for each $i$. Therefore $D_{l}(c_k) e_k = 0$ for any $l$.

If $c_k \neq 0$ for some $k$, we may assume $c_{k,0} \neq 0$, where $c_{k,0}$ is the constant term of $c_k$, by multiplying $t^m$ for some $m \in M^{sp}$ to all $c_k$. $D_{l}(c_k)$ converges to $c_{k,0}$, so $\sum_{k=1}^{n} c_{k,0} e_k = 0$. This contradicts the assumption that $e_1, \ldots, e_n$ are linearly independent over $\mathcal{O}_Y$.  

□

**Proposition 2.3.7.** (cf. Proposition 1.17 of [21]) Let $Y$, $M$, $\Sigma$ be as in Proposition 2.3.6, and $I \subseteq [0, \infty)$ be a quasi-open subinterval or a closed aligned subinterval of positive length. Any subquotient of an object of $\text{ULNM}_{Y \times A_{M,K}(1), \Sigma}$ (resp. $\text{ULNM}_{Y \times A_{M,K}(1), \Sigma}$) is an object of $\text{ULNM}_{Y \times A_{M,K}(1), \Sigma}$ (resp. $\text{ULNM}_{Y \times A_{M,K}(1), \Sigma}$). In particular, $\text{ULNM}_{Y \times A_{M,K}(1), \Sigma}$ and $\text{ULNM}_{Y \times A_{M,K}(1), \Sigma}$ are abelian subcategories of $\text{LNMY}_{Y \times A_{M,K}(1), \Sigma}$.

**Proof.** It is enough to show the case of $\text{ULNM}_{Y \times A_{M,K}(1), \Sigma}$.

As in the proof of 1.17 in [21], we may assume that $I$ is a closed aligned subinterval and it is enough to show that any quotient of a $\Sigma$-constant module is also $\Sigma$-constant or that any subobject of a $\Sigma$-constant module is also $\Sigma$-constant.

First, we prove this in the case when $Y = \text{Sp} \mathbb{M}$. Let $E$ be a $\Sigma$-constant log $\nabla$-module on $Y \times A_{M,K}(1)$ and $f : E \to E'$ be a surjection in $\text{LNMY}_{Y \times A_{M,K}(1), \Sigma}$. There exists an element $\xi \in \Sigma$ such that $E = F \otimes K \mathcal{O}_{M,K}(1)$, where $F = \{ e \in E | \nabla_{E}^{log}(e) = \xi e \}$. Let $F'$ be the image of $F$ under $f$. The map
\[ F' \otimes_K \mathcal{O}_{\mathbb{A}_{M,K}(I)} \to E' \] is surjective by the surjectivity of \( f \) and injective by Lemma \ref{lem:extension}

So \( E' \) is \( \Sigma \)-constant.

Next we prove it for general \( Y \). Let \( E \) be a \( \Sigma \)-constant log \( \nabla \)-module on \( Y \times \mathbb{A}_{M,K}(I) \) and \( f : E' \to E \) be an injection in \( \mathrm{LNM}_{Y \times \mathbb{A}_{M,K}(I), \Sigma} \).

It is enough to show that \( E' \) is \( \Sigma \)-constant after replacing \( K \) by its finite Galois extension \( K' \). Indeed, if \( E'_{K'} \) is \( \Sigma \)-constant, there exists a \( \Sigma \)-constant object \( E'_{K',0} \) of \( \mathrm{LNM}_{Y \times \mathbb{A}_{M,K}(0,0)} \) such that \( \mathcal{U}_I(E'_{K',0}) = E'_{K'} \). For each \( \sigma \in \mathrm{Gal}(K'/K) \), there exists a natural isomorphism \( \iota_{\sigma} : \sigma^* E'_{K',0} \xrightarrow{\sim} E'_{K'} \). By Proposition\ref{prop:extension}, there exists a unique morphism \( \iota_{\sigma,0} : \sigma^* E'_{K',0} \xrightarrow{\sim} E'_{K',0} \) such that \( \mathcal{U}_I(\iota_{\sigma,0}) = \iota_{\sigma} \). So there exists a \( \Sigma \)-constant object \( E'_{0} \) of \( \mathrm{LNM}_{Y \times \mathbb{A}_{M,K}(0,0)} \) such that \( (E'_{0})_{K'} = E'_{K',0} \) by Galois descent. Since \( \mathcal{U}_I(E'_{0}) = E'_0 \), \( E' \) is \( \Sigma \)-constant.

Thus we may assume that \( \mathbb{A}_{M,K}(I) \) has some \( K \)-rational point \( x : \mathrm{Spm} K \hookrightarrow \mathbb{A}_{M,K}(I) \). We denote the morphism \( Y \to Y \times \mathbb{A}_{M,K}(I) \) induced by \( x \) also by \( x \). Let \( \pi_1 \) and \( \pi_2 \) be the first and second projections of \( Y \times \mathbb{A}_{M,K}(I) \). Take \( \xi \in \Sigma \) such that \( E \otimes \pi_2^* C_{-\xi} \) is \( \{0\} \)-constant. Put

\[ E'' := \pi_1^* \mathrm{Im} \left( x^* \left( E' \otimes \pi_2^* C_{-\xi} \right) \to x^* \left( E \otimes \pi_2^* C_{-\xi} \right) \right), \]

then \( E'' \) is a \( \{0\} \)-constant log \( \nabla \)-module and \( E'' \otimes \pi_2^* C_\xi \) is a subobject of \( E \). We have to show \( E'' \otimes \pi_2^* C_\xi \to E' \). Since it is enough to show it very locally on \( Y \), we may assume that \( Y = \mathbb{A}_K^n[0,p^{-m}] \). Let \( L \) be the completion of fraction field of \( \mathcal{O}_Y \) with respect to the spectrum norm. It is enough to show \( E'' \otimes \pi_2^* C_\xi \to E' \) after scalar extension to \( L \). The equality \( E''_L \otimes \pi_2^* C_\xi = E'_L \) is true on \( \mathbb{A}_{M,L}(I) \) since we have already proved that \( E'_L \) is \( \Sigma \)-constant. So \( E' \) is \( \Sigma \)-constant.

\[ \square \]

### 2.4. Generalization propositions

In this section, we adapt the Proposition 2.5 of [21] called generalization to our situation.

Let \( Y \) be a smooth rigid space, \( M \) a fine sharp weighted monoid, \( I \subseteq [0, \infty) \) an aligned subinterval. For a log \( \nabla \)-module \( E \) on \( Y \times \mathbb{A}_{M,K}(I) \) and \( \xi \in M^{gp} \otimes_{\mathbb{Z}} K \), we put:

\[ H^0_\xi(Y \times \mathbb{A}_{M,K}(I), E) := \{ v \in \Gamma(Y \times \mathbb{A}_{M,K}(I), E) \mid \nabla^\log (v) = \xi \cdot v \}. \]

**Lemma 2.4.1.** (cf. Lemma 2.3 of [21]) Let \( Y = \mathrm{Spm} R \) be a smooth affinoid space, \( \Sigma \subseteq M^\text{gp} \otimes_{\mathbb{Z}} \mathbb{Z}_p \) an \( (\mathbb{N} \cap \mathbb{N})\)-D subset, \( I \subseteq [0, \infty) \) a quasi-open subinterval of positive length. Let \( L \supseteq R \) be one of the following:

1. an affinoid algebra over \( K \) such that \( \mathrm{Spm} L \) is smooth and the supremum norm of \( L \) restricts to the supremum norm of \( R \).
2. a complete field with respect to a multiplicative norm which restricts to the supremum norm on \( R \).

For the first case, we put \( \mathbb{A}_{M,L}(I) := \mathrm{Spm} L \times \mathbb{A}_{M,K}(I) \). For the second case \( \mathbb{A}_{M,L}(I) \) means the polyannulus over \( L \) as usual.

Let \( E \) be an object of \( \mathrm{LNM}_{Y \times \mathbb{A}_{M,K}(I), \Sigma} \) such that the induced object \( F := E_L \) in \( \mathrm{LNM}_{\mathbb{A}_{M,L}(I), \Sigma} \) is \( \Sigma \)-unipotent. Then for any aligned closed subinterval \([b,c] \subseteq I\), there exists an element \( \xi \in \Sigma \) such that valued \( H^0_\xi(Y \times \mathbb{A}_{M,K}[b,c], E) \neq 0 \).

**Proof.** This proof is almost same as the proof of Lemma 2.3 of [21].

By Proposition 2.3.5, we can take \( W \in \mathrm{ULNM}_{\mathbb{A}_{M,L}(0,0), \Sigma} \) such that \( F = \mathcal{U}_I(W) \).
Let \( \rho \) be the residue of \( W \) and \( \xi_1, \ldots, \xi_n \) the exponents of \( W \). Let \( W = \bigoplus_{k=1}^{n} W_k \) be the decomposition such that \( \rho - \xi_k \cdot \text{id} \) acts on \( W_i \) nilpotently. Let \( \psi_k : W \to W_k \) be the projection. We also put
\[
W'_k := \{ w \in W \mid \rho(w) = \xi_k w \}
\]

Take a homomorphism \( M \to \mathbb{N}^r \) as in Remark \[2.2.14\] Let \( \partial_i \) be composition of \( \nabla_{\log} \) and the \( i \)-th projection of \( \mathbb{Z}^r \). Let \( \text{res}_i \) be the decomposition of \( \rho \) and the \( i \)-th projection of \( \mathbb{Z}^r \). We denote the \( i \)-th entry of \( \xi_k \) by \( \xi_{i,k} \). We can take polynomials \( Q_1, \ldots, Q_r \) such that \( Q_i \) divides the minimal polynomial of \( \text{res}_i \) and the image of \( \prod_{i=1}^r Q_i(\text{res}_i) \) is not zero and contained in \( W'_1 \).

By the definition of \( W_k \), we can take \( q \in \mathbb{N} \) such that \( (\text{res}_i - \xi_{i,k} \cdot \text{id})^q(W_k) = 0 \) for all \( 1 \leq i \leq r \) and \( 1 \leq k \leq n \).

For \( l \in \mathbb{N} \), we define the operator \( D_l \) on \( E \) as follows:
\[
D_l := \prod_{i=1}^{r} Q_i(\partial_i) \left( \prod_{k=1}^{n} \prod_{j=1}^{l} \left( \frac{j - (\partial_i - \xi_{i,k})}{j - (\xi_{i,1} - \xi_{i,k})} \right) \right)^{q_i}.
\]

Take two intervals \([b, c] \subseteq [d, e] \subseteq I\) such that \( c < e \) and if \( b > 0 \), then \( d < b \) (if \( b = 0 \), then \( d = 0 \)). We prove that for any \( v \in \Gamma(Y \times k_{M,K}[d, e]) \), \( D_l(v) \) converges to an element of \( H^0_{\xi_i}(Y \times k_{M,K}[b, c], E) \) when \( l \to \infty \).

Put \( v = \sum_{m \in M^{\text{sp}}} v_m t^m \) where \( v_m \in W \). Let \( v_{m,k} := \psi_k(v_m) \).
\[
D_l(v) = \sum_{k=1}^{n} D_l \left( \sum_{m \in M^{\text{sp}}} v_{m,k} t^m \right) = \sum_{k=1}^{n} \sum_{m \in M^{\text{sp}}} t^m \left( \prod_{i=1}^{r} Q_i(\text{res}_i + m_i) \prod_{k=1}^{n} \prod_{j=1}^{l} \left( \frac{j - (\text{res}_i + m_i - \xi_{i,k})}{j - (\xi_{i,1} - \xi_{i,k})} \right) \right)^{q_i} (v_{m,k})
\]

For the term of \( t^0 \), since \( \prod_{i=1}^{r} Q_i(\text{res}_i)(v) \in W'_1 \),
\[
\prod_{i=1}^{r} Q_i(\text{res}_i) \left( \prod_{k=1}^{n} \prod_{j=1}^{l} \left( \frac{j - (\text{res}_i - \xi_{i,k})}{j - (\xi_{i,1} - \xi_{i,k})} \right) \right)^{q_i} (v_{0,k}) = \prod_{i=1}^{r} Q_i(\text{res}_i)(v_{0,k})
\]

For the term of \( t^m \) such that \( 0 < |m_i| \leq l \) for some \( i \), it is equal to zero since this expression contains \( (\text{res}_i - \xi_{i,k})^q(v_{m,k}) \) for any \( k \).

For other terms, the coefficient of \( t^m \) is written as
\[
\sum_{\alpha_i > 0} \prod_{i=1}^{r} c_{i,\alpha_i}^{m_i, l} (\text{res}_i - \xi_{i,k})^{\alpha_i} (v_{m,k})
\]
where \( c_{i,\alpha_i}^{m_i, l} \) are some constants. Let \( M_i^{\text{sp}} := \{ m \in M^{\text{sp}} \mid \{ m_i < l \text{ or } m_i = 0 \text{ or } m_i > l \text{ for any } 1 \leq i \leq r \} \} \).

We write
\[
D_l(v) - \prod_{i=1}^{r} Q_i(\text{res}_i)(v_0) = \sum_{k=1}^{n} \sum_{\alpha_i > 0} \left( \sum_{m \in M_i^{\text{sp}}} \prod_{i=1}^{r} c_{i,\alpha_i}^{m_i, l} m (\text{res}_i - \xi_{i,k})^{\alpha_i} (v_{m,k}) \right)
\]

\[28\]
By Claim 3 in the proof of Lemma 2.3 in [21], it is proven that for any \( \delta > 1 \), \( |c_{i,a}^{j,b}| \leq (\text{const}) \delta^{ij}a \).

We prove that \( \sum_{m \in M^{sp}} \prod_{i=1}^{n} c_{i,a_i}^{m_i,l_i} t^m (\text{res}_i - \xi_{i,k})^{\alpha_i}(v_{m,k}) \) converges in \( Y \times \mathbb{A}_{M,K}[b,c] \) for any \( k, \alpha \) which means that \( D_t(v) - \prod_{i=1}^{r} Q_i(\text{res}_i)(v_0) \) converges to zero.

Take \( C > 0 \) such that \( \sum_{i=1}^{r} |m_i| \leq C |h|(m) \). Take \( \delta > 1 \) such that \( d \leq \delta^{-C}b \), \( \delta^C \leq c \). Then

\[
\left| \prod_{i=1}^{r} c_{i,a_i}^{m_i,l_i} (\text{res}_i - \xi_{i,k})^{\alpha_i}(v_{m,k}) \right| b^{-h^{-1}(m)} c^{h^+(m)} \\
\leq (\text{const}) \delta^{\sum |m_i|} b^{-h^{-1}(m)} c^{h^+(m)} |v_{m,k}|
\]

By the assumption that \( v = \sum_{m \in M^{sp}} v_m t^m \) converges in \( Y \times \mathbb{A}_{M,K}[d,e] \), this converges to zero when \( |h|(m) \to \infty \).

So \( D_t(v) \) converges to

\[
f(v) := \prod_{i=1}^{r} Q_i(\text{res}_i)(v_0) \in H^0_{\xi_1}(Y \times \mathbb{A}_{M,K}[b,c], E).
\]

Because \( F = U_t(W) \), the image of \( \Gamma(Y \times \mathbb{A}_{M,L}[b,c], F) \) under the map

\[
v = \sum_{m \in M^{sp}} v_m t^m \to \prod_{i=1}^{r} Q_i(\text{res}_i)(v_0)
\]

is \( \prod_{i=1}^{r} Q_i(\text{res}_i)(W) \). Since \( \Gamma(Y \times \mathbb{A}_{M,K}[d,e], E) \otimes_R L \) is dense in \( \Gamma(Y \times \mathbb{A}_{M,L}[b,c], F) \), \( \text{Im}(f) \otimes_R L \) is dense in \( \prod_{i=1}^{r} Q_i(\text{res}_i)(W) \), so \( f \neq 0 \). Therefore \( H^0_{\xi_1}(Y \times \mathbb{A}_{M,K}[b,c], E) \neq 0 \).

Proposition 2.4.2. (generalization, cf. Proposition 2.4 in [21]) In the condition of Lemma 2.4.1 we also assume that \( 0 \notin I \). If \( F := E_L \in \text{LNM}_{\mathbb{A}_{M,L}[I], \Sigma} \) is \( \Sigma \)-unipotent, \( E \) is also \( \Sigma \)-unipotent.

Proof. Let \( [b,c] \subseteq I \) be a closed aligned subinterval of positive length. By Lemma 2.4.1 there exists \( \xi \in \Sigma \) such that \( H_E := H^0_{\xi_1}(Y \times \mathbb{A}_{M,K}[b,c], E) \neq 0 \).

Let \( \pi_Y : Y \times \mathbb{A}_{M,K}[I] \to Y \) be the projection. \( H_E \) is a finitely generated \( R \)-module and \( \pi_Y^*(H_E) \to E \) is injective on \( Y \times \mathbb{A}_{M,K}[b,c] \) by the proof of Claim 1 of Proposition 2.4 in [21]. Also, by the proof of Claim 2 of Proposition 2.4 in [21], \( H_E \) is a \( \nabla \)-module on \( Y \), so it is locally free. Moreover, \( H_E \) admits a map \( H_E \to H_E \otimes_{\mathbb{A}_{M,L}} M^{sp} \) induced by \( \nabla^{\text{log}} \) of \( E \). So \( H_E \) can be regarded as an object of \( \text{LNM}_{Y \times \mathbb{A}_{M,K}, [0,0]_L, \Sigma} \). Put \( G := U_{[b,c]}(H_E) \). Then \( G \) is a \{\xi\}-constant object of \( \text{LNM}_{Y \times \mathbb{A}_{M,K}, [b,c], \Sigma} \) and \( G = \pi_Y^*(H_E) \to E \) is injective. Then, by the induction on the rank of \( E \), we see that \( E \) is \( \Sigma \)-unipotent on \( Y \times \mathbb{A}_{M,K}[I] \) for any aligned closed subinterval \( [b,c] \subseteq I \). So \( E \) is \( \Sigma \)-unipotent on \( Y \times \mathbb{A}_{M,K}[I] \).
2.5. Transfer theorem. In this section, we adapt the Proposition 2.5 of [21] called the transfer theorem to our situation. To do this, we define the notion of log-convergence.

**Definition 2.5.1.** (Definition 2.4.1 of [13]) Let \( X \) be an affinoid space and \( E \) a coherent \( \mathcal{O}_X \)-module. For \( \eta > 0 \), multi-indexed series \( \{v_k\}_{k \in \mathbb{N}^n} \) of elements of \( \Gamma(X,E) \) is \( \eta \)-null if for any multi-indexed series \( \{c_k\}_{k \in \mathbb{N}^n} \) of elements of \( K \) such that \( |c_k| \leq \eta^{|k|} \), \( \{c_kv_k\}_{k \in \mathbb{N}^n} \) converges to zero when \( |k| \to \infty \).

**Definition 2.5.2.** (cf. Definition 2.9 in [21]) Let \( Y = \text{Spm} R \) be a smooth connected affinoid space, \( M \) a fine sharp semi-saturated weighted monoid and \( a \) an element of \( (0,1) \cap \Gamma^* \). Let \( \phi : M \to \mathbb{Z}^r \) be a homomorphism which induces an isomorphism \( M^{gp} \otimes_\mathbb{Z} K \cong K' \). Let \( \partial_1, \ldots, \partial_r \) be the dual basis of the basis of \( \Omega^{\log,1}_{X,M,K} \) induced by \( \phi \). A log \( \nabla \)-module \( E \) on \( Y \times \mathbb{A}_{M,K}[0,a] \) is called log-convergent with respect to \( \phi \) if for any \( a' \in (0,a) \cap \Gamma^* \), \( \eta \in (0,1) \) and \( v \in \Gamma(Y \times \mathbb{A}_{M,K}[0,a'],E) \)

\[
(2.5.2.1) \quad \left\{ \frac{1}{k_1!k_2!\cdots k_r!} \left( \prod_{i=1}^{r} \prod_{j=0}^{k_i-1} (\partial_i - j) \right) (v) \right\}_{k_1,k_2,\ldots,k_r \in \mathbb{N}}
\]

is \( \eta \)-null on \( Y \times \mathbb{A}_{M,K}[0,a'] \).

**Remark 2.5.3.** Any subquotient of a log-convergent object is also log-convergent.

**Remark 2.5.4.** By the same calculation as that after Remark 2.10 of [21], to check log-convergence, it is enough to check the \( \eta \)-nullity of \( (2.5.2.1) \) for a set of generators \( v \) of \( \Gamma(Y \times \mathbb{A}_{M,K}[0,a'],E) \).

Indeed, by induction of \( k \), one can check the following equation:

\[
\prod_{j=0}^{k_i-1} (\partial_i - j) (f v) = \sum_{k' = 0}^{k} \binom{k}{k'} \prod_{j=0}^{k'-1} (\partial_i - j) (f) \prod_{j=0}^{k-k'-1} (\partial_i - j) (v)
\]

for \( 1 \leq i \leq r \), \( v \in \Gamma(Y \times \mathbb{A}_{M,K}[0,a'],E) \) and \( f \in \Gamma(Y \times \mathbb{A}_{M,K}[0,a'],\mathcal{O}) \). Let \( P_k := (\prod_{i=1}^{r} \prod_{j=0}^{k_i-1} (\partial_i - j)) \) for \( k = (k_1, \ldots, k_r) \), then

\[
P_k(fv) = \sum_{k' \leq k} P_{k'}(f) P_{k-k'}(v).
\]

For \( m \in M \), \( P_k(t^m) = \prod_{i=1}^{r} \binom{m}{k_i} t^{m_i} \) where \( \phi(m) = (m_1, \ldots, m_r) \). So \( |P_k(f)| \leq |f| \). Thus,

\[
|P_k(fv)| \leq \max_{k' \leq k} |f| |P_{k-k'}(v)|.
\]

Therefore, when \( v = \sum_{s=0}^{g} f_s v_s \)

\[
|P_k(v)| \eta^{|k|} \leq \max_{s,k' \leq k} |f_s| |P_{k-k'}(v_s)| \eta^{|k|}
\]

\[
\leq \max_{s,k' \leq k} \left( |f_s| \eta^{|k'|} \right) \left( |P_{k-k'}(v_s)| \eta^{|k-k'|} \right)
\]

for \( \eta < 1 \).

To prove the transfer theorem (Proposition 2.5.6), we prove the following lemma which is an analogue of Lemma 3.1.6 of [13].
Lemma 2.5.5. Let \( R \) be an affinoid algebra endowed with a norm \( |\cdot| \) and let \( M \) be a fine weighted monoid. For \( a \in \Gamma^* \cap \mathbb{R}_{>0} \) and a formal power series \( f = \sum_{m \in M} c_m t^m \) with \( c_m \in R \), we put

\[
|f|_a := \sup_{m \in M} \left\{ |c_m| a^{b(m)} \right\}.
\]

Then for \( a, b \in \Gamma^* \cap \mathbb{R}_{>0} \) and \( c \in [0,1] \),

\[
|f|_{a \cdot b^{1-c}} \leq |f|^c_a |f|^{1-c}_b.
\]

Proof.

\[
|f|_{a \cdot b^{1-c}} = \sup_{m \in M} \left\{ |c_m| a^{b(m)} b^{(1-c) b(m)} \right\}
\]

\[
= \sup_{m \in M} \left\{ \left( |c_m| a^{b(m)} \right)^c \left( |c_m| b^{b(m)} \right)^{1-c} \right\}
\]

\[
\leq \sup_{m \in M} \left\{ |c_m| a^{b(m)} \right\} \sup_{m \in M} \left\{ |c_m| b^{b(m)} \right\}^{1-c}
\]

\[
= |f|^c_a |f|^{1-c}_b.
\]

\( \square \)

Proposition 2.5.6. (transfer theorem, cf. Proposition 2.12 in [21]) Let \( Y = \text{Sp}_R R \) be a smooth connected affinoid space, \( M \) a fine sharp semi-saturated weighted monoid. Let \( \Sigma \subseteq M^\text{gp} \otimes_{\mathbb{Z}} \mathbb{Z}_p \) be an ((NI \cap NL)-D) subset. Take \( \phi : M \rightarrow \mathbb{Z}^r \) as in Definition 2.5.2. For any an object \( E \) of \( \text{LNM}_{Y \times \mathbb{A}_M,K[0,1],\Sigma} \), if \( E \) is log-convergent with respect to \( \phi \), then \( E \) is \( \Sigma \)-unipotent.

Proof. By induction on the rank of \( E \), it is enough to show that for any log-convergent object \( E \) of \( \text{LNM}_{Y \times \mathbb{A}_M,K[0,a],\Sigma} \) there exists a \( \Sigma \)-constant subobject of \( E \) in \( \text{LNM}_{Y \times \mathbb{A}_M,K[0,a'],\Sigma} \) for any \( 0 < a' < a < 1 \).

Let \( W := E|_{Y \times \mathbb{A}_M,K[0,0]} \). We define \( \xi_i \in \mathbb{Z}_p \), \( \psi_k \in \mathbb{N}, Q_i, D_i \) as in the proof of Proposition 2.4.4 from this \( W \).

As in the proof of Proposition 2.12 in [21], we want to show that for any \( 0 < a' < a < 1 \) and \( \mathbf{v} \in \Gamma(Y \times \mathbb{A}_M,K[0,a],E) \), \( D_i(\mathbf{v}) \) converges on \( \mathbb{A}_M,K[0,a'] \) when \( l \to \infty \).

First, we show that \( D_i(\mathbf{v}) \) is \( \rho \)-null on \( \mathbb{A}_{M,L}[0,a] \) for any \( \rho \in (0,1) \). Put

\[
P_l(y) := \prod_{j=1}^l \frac{j - 1 - y}{j}
\]

and

\[
c_{ijkl} := \prod_{j=1}^l \frac{j}{j - (\xi_{1,k} - \xi_{i,k})} \cdot \frac{j}{j + (\xi_{1,k} - \xi_{i,k})}.
\]

Note that \( P_l(\xi) \in \mathbb{Z}_p \) for any \( \xi \in \mathbb{Z}_p \). Then,

\[
\prod_{j=1}^l \frac{j - (\partial_i - \xi_{i,k})}{j - (\xi_{1,k} - \xi_{i,k})} \cdot \frac{j + (\partial_i - \xi_{i,k})}{j + (\xi_{1,k} - \xi_{i,k})}
\]

\[
= c_{ijkl} P_l(-\partial_i + \xi_{i,k} - 1) P_l(-\partial_i + \xi_{i,k} - 1)
\]

\[
= c_{ijkl} \left( \sum_{q=0}^l P_{l-q}(-\xi_{i,k} - 1) P_q(\partial_i) \right) \left( \sum_{q=0}^l P_{l-q}(\xi_{i,k} - 1) P_q(-\partial_i) \right).
\]
$P_q(-y)$ is written as a linear combination of $P_q(y), P_{q-1}(y), \ldots, P_0(y)$ with integer coefficients. $P_q(y)$ $P_{q_2}(y)$ is written as a linear combination of $P_{q_1+q_2}(y)$, $P_{q_1+q_2-1}(y), \ldots, P_0(y)$ with integer coefficients. So we can write

$$D_l = \left( \prod_{i=1}^r Q_l(\partial_l) \prod_{k=1}^n c_{ikl} \right) \left( \sum_{q_1, \ldots, q_r=0}^{2nl} \text{(const.)} P_{q_1}(\partial_1) \cdots P_{q_r}(\partial_r) \right),$$

where the constants are contained in $\mathbb{Z}_p$. Thus

$$|D_l(v)| \leq \left( \prod_{i=1}^r \prod_{k=1}^n |c_{ikl}| \right) \max_{0 \leq q_1, \ldots, q_r \leq 2nl} \left| P_{q_1}(\partial_1) \cdots P_{q_r}(\partial_r) \left( \prod_{i=1}^r Q_l(\partial_l)(v) \right) \right|.$$

By the assumption on $\Sigma$ and log-convergence, this is $\rho$-null for any $\rho \in (0, 1)$.

We can take $b \in (0, a]$ such that $E|_{Y \times \mathbb{A}_{M,K}[0,b]} = U_{[0,b]}(W)$. Indeed, if we take a finite admissible affine covering $Y = \bigsqcup U_i$ such that $W|_{U_i}$ is free, by Lemma 2.2.10 we may assume $E|_{U_i \times \mathbb{A}_{M,K}[0,b]} = U_{[0,b]}(W|_{U_i})$, so $E|_{Y \times \mathbb{A}_{M,K}[0,b]} = U_{[0,b]}(W)$ by Corollary 2.3.5. Then $(D_{l+1}(v) - D_l(v))$ is $\eta\prime$-null on $Y \times \mathbb{A}_{M,K}[0,b]$ for some $\eta > 1$ by the claim 2 in the proof of Lemma 2.3 of [21].

Because $E$ is locally free, it is a direct summand of a free module $F$. We fix a basis of $F$. Applying Lemma 2.5.4 to each component with respect to the basis of $F$, $(D_{l+1}(v) - D_l(v))$ is $\rho^{1-c_\eta} \eta'$-null on $Y \times \mathbb{A}_{M,K}[0,a']$ where $c, c_\eta \in (0, 1)$ and $a'' := a^{1-c_\eta}$. Take $c$ such that $\rho^{1-c_\eta} \eta = 1$, then $D_l(v)$ converges on $Y \times \mathbb{A}_{M,K}[0, a'']$. If we take $\rho < 1$ arbitrarily close to 1, $c$ can be arbitrarily close to 0 and $a''$ can be arbitrarily close to $a$.

So, $f(v) := \lim_l D_l(v)$ exists on $Y \times \mathbb{A}_{M,K}[0,a']$ for any $0 < a' < a$ and it is equal to $\prod_{i=1}^r Q_i(\text{res}_i)(v)$ on $Y \times \mathbb{A}_{M,K}[0,b]$ by the calculation in the proof of Proposition 2.4.1. Thus $f(v)$ is an element of $H^0_e(Y \times \mathbb{A}_{M,K}(0,a'), E)$ and $f \neq 0$ by the argument similar to the proof of Proposition 2.4.1. Hence $H_E := H^0_e(Y \times \mathbb{A}_{M,K}(0,a'), E)$ is non-zero.

By the same argument as the proof of Proposition 2.12 in [21], we see that $\pi^\ast_{\mathcal{O}'} H_E$ is $\Sigma$-constant subobject of $E$ where $\pi_{\mathcal{O}'} : Y \times \mathbb{A}_{M,K}[0, a'] \to Y$ is the projection. \hfill $\square$

## 3. Log isocrystals

In this section, we assume that $\mathcal{V}$ is a discrete valuation ring.

Let $\mathcal{N}$ be a finite monoid and $\alpha : \mathcal{N} \to \mathcal{V}$ a monoid homomorphism. $(\text{Spf} \mathcal{V}, \mathcal{N})$ denotes $\text{Spf} \mathcal{V}$ with the log structure induced by $\alpha$. We assume that $\alpha$ is a good chart at the unique point. We also assume that the image of $\alpha$ does not contain 0, i.e., $\text{Spf} \mathcal{V}, \mathcal{N}$ induces a trivial log structure on $\text{Spec} \mathcal{N}$. $(\text{Spec} \mathcal{N}, \mathcal{N})$ denotes $\text{Spec} \mathcal{N}$ with the log structure induced by $\alpha$ such that the underlying scheme is separated and of finite type over $\mathcal{k}$.

For a monoid $\mathcal{M}$, $\mathbb{A}_{M,K}$ denotes $\text{Spec} \mathcal{N}[\mathcal{M}]$ as a log scheme over $\mathcal{k}$ with the log structure defined by the natural map $\mathcal{M} \to k[\mathcal{M}]$. $\mathbb{A}_{M,\mathcal{V}}$ denotes $\text{Spf} \mathcal{V}[\mathcal{M}]$ as a log formal scheme over $\mathcal{V}$ with the log structure defined by the natural map $\mathcal{M} \to \mathcal{V}[\mathcal{M}]$.

### 3.1. Convergent log isocrystals and convergent log $\nabla$-modules

Let $(\mathcal{X}, \mathcal{M})$ be a finite log variety over $(\text{Spec} \mathcal{N}, \mathcal{N})$ and $\mathcal{X}$ an open subscheme of $\mathcal{X}$. In this subsection, we prove that, locally on $\mathcal{X}$, convergent log isocrystals correspond to convergent log $\nabla$-modules.

First, we recall the definition of log tubes and sheaves of overconvergent sections.
Definition 3.1.1. Let $\overline{(X, \mathcal{M})}$ be a fine log variety over $(\text{Spec } k, N)$ and $i : (\overline{(X, \mathcal{M})}) \hookrightarrow (P, \mathcal{L})$ a closed immersion into a fine log formal scheme which is separated and topologically of finite type over $(\text{Spf } \mathcal{V}, N)$. By Proposition-Definition 2.10. of 20, there exists a unique homeomorphic exact closed immersion $\overline{i^\text{ex}} : (\overline{(X, \mathcal{M})}) \hookrightarrow (P^\text{ex}, \mathcal{L}^\text{ex})$ such that $i$ factors through $\overline{i^\text{ex}}$ and $i^\text{ex}$ is universal among such immersions, i.e., for any homeomorphic exact closed immersion $i' : (\overline{(X, \mathcal{M})}) \hookrightarrow (P', \mathcal{L}')$ such that $i$ factors through $i'$, it is uniquely decomposed as $i' = j \circ i^\text{ex}$ where $j : (P', \mathcal{L}') \rightarrow (P^\text{ex}, \mathcal{L}^\text{ex})$.

Let $\text{sp} : P^\text{ex}_K \rightarrow \overline{X}$ be the specialization map. For an open subscheme $X \subseteq \overline{X}$, the log tube $|X|^\text{log}$ is defined to be the inverse image of $X$ under $\text{sp}$. In particular, $|X|^\text{log}_p = P^\text{ex}_K$.

Remark 3.1.2. (cf. Definition 6.1.4 of 13) Let $\overline{(X, \mathcal{M})}$ and $i : (\overline{(X, \mathcal{M})}) \hookrightarrow (P, \mathcal{L})$ be as in Definition 3.1.1. Assume $i$ has a factorization

$$(\overline{(X, \mathcal{M})}) \xrightarrow{i'} (P', \mathcal{L}') \xrightarrow{f'} (P, \mathcal{L})$$

such that $i'$ is an exact closed immersion and $f'$ is formally log étale. Then $P^\text{ex}$ is the completion of $P'$ along $\overline{X}$. So $|X|^\text{log}_p = |X|^\text{log}_p$ for any open subscheme $X \subseteq \overline{X}$.

Definition 3.1.3. Let $\overline{(X, \mathcal{M})}$ and $i : (\overline{(X, \mathcal{M})}) \hookrightarrow (P, \mathcal{L})$ be as in Definition 3.1.1. Let $X \subseteq \overline{X}$ be an open subscheme. We put

$$j^! \mathcal{O}_{|X|^\text{log}} := \lim_{\text{V}} i_V^{-1} \mathcal{O}_{|X|^\text{log}}$$

where $V$ runs through strict neighborhoods of $|X|^\text{log}_p$ in $|X|^\text{log}_p$ and $i_V : V \hookrightarrow |X|^\text{log}_p$ is the open immersion, and call this sheaf the sheaf of overconvergent sections.

We recall the definition of overconvergent log isocrystals in §4 of 19 or §10 of 7.

Definition 3.1.4. (cf. Definition 6.1.7 of 13) Let $\overline{(X, \mathcal{M})}$ be a fine log variety over $(\text{Spec } k, N)$ and $X \subseteq \overline{X}$ an open subscheme. Assume there exists a commutative diagram:

$$\begin{array}{ccc}
(\overline{(X, \mathcal{M})}) & \xrightarrow{i} & (P, \mathcal{L}) \\
\downarrow & & \downarrow \\
(\text{Spec } k, N) & \xrightarrow{\text{sp}} & (\text{Spf } \mathcal{V}, N)
\end{array}$$

where $i$ is a closed immersion and $(P, \mathcal{L})$ is a $p$-adic fine log formal scheme separated, topologically of finite type and formally log smooth over $(\text{Spf } \mathcal{V}, N)$.

Let $(P(i), \mathcal{L}(i))$ be the $(i + 1)$-fold fiber product of $(P, \mathcal{L})$ over $(\text{Spf } \mathcal{V}, N)$. We denote the projections by $\pi_0, \pi_1 : (P(1), \mathcal{L}(1)) \rightarrow (P, \mathcal{L})$, $\pi_{0,1}, \pi_{1,2}, \pi_{0,2} : (P(2), \mathcal{L}(2)) \rightarrow (P(1), \mathcal{L}(1))$ and the diagonal map by $\Delta : (P, \mathcal{L}) \rightarrow (P(1), \mathcal{L}(1))$.

An overconvergent log isocrystal on $(X, \overline{(X, \mathcal{M})})$ over $(\text{Spf } \mathcal{V}, N)$ is a coherent $j^! \mathcal{O}_{|X|^\text{log}}$-module $\mathcal{E}$ equipped with a $j^! \mathcal{O}_{|X|^\text{log}}$-module isomorphism $\epsilon : \pi_1^* (\mathcal{E}) \rightarrow \pi_0^* (\mathcal{E})$ such that $\Delta^*(\epsilon) = \text{id}_\mathcal{E}$ and the cocycle condition $\pi_{0,1}^* (\epsilon) \circ \pi_{1,2}^* (\epsilon) = \pi_{0,2}^* (\epsilon)$ holds on $|X|^\text{log}_p$. $\mathcal{E}$ is locally free if $\mathcal{E}$ is locally free as a $j^! \mathcal{O}_{|X|^\text{log}}$-module.

If $X = \overline{X}$, we call them convergent log isocrystals on $(\overline{(X, \mathcal{M})})$ over $(\text{Spf } \mathcal{V}, N)$.
A locally free overconvergent log isocrystal naturally induces a locally free \( j^! \mathcal{O}[X^{\log}_P] \)-module with an integrable connection. So it defines a log \( \nabla \)-module on some strict neighborhood of \( X^{\log}_P \) in \( X^{\log}_P \).

We can always take \((P, \mathcal{L})\) as in Proposition 3.1.4 étale locally on \( X \). Hence, for general \((X, \mathcal{M})\), we can define the category of overconvergent log isocrystals on \((X, X, \mathcal{M})\) using an appropriate étale hypercovering.

**Definition 3.1.5.** Let \((X, \mathcal{M})\) be a fine log variety over \((\text{Spec} \, k, N)\) and \(X \subseteq \overline{X}\) an open subscheme. We take an étale covering \((X_0, \mathcal{M}_0) \rightarrow (X, \mathcal{M})\) such that \((X_0, \mathcal{M}_0)\) has an immersion \((X_0, \mathcal{M}_0) \hookrightarrow (P^0, \mathcal{L}^0)\) satisfying the condition of Definition 3.1.4. For \(n=1,2\), let \((X_n, \mathcal{M}_n)\) be the \((n+1)\)-fold fiber product of \((X_0, \mathcal{M}_0)\) over \((X, \mathcal{M})\), let \((P^n, \mathcal{L}^n)\) be the \((n+1)\)-fold fiber product of \((P^0, \mathcal{L}^0)\) over \((\text{Spf} \, V, N)\) and let \(X^n := X \times_{\overline{X}} \overline{X}^n\). For \(n = 0, 1, 2\), let \(I^{1,n}\) be the category of overconvergent log isocrystals on \((X^n, \overline{X}^n, \mathcal{M}^n)\) over \((\text{Spf} \, V, N)\) defined by \((X^n, \overline{X}^n, \mathcal{M}^n) \rightarrow (P^n, \mathcal{L}^n)\). The category of overconvergent log isocrystals on \((X, X, \mathcal{M})\) over \((\text{Spf} \, V, N)\) is defined as the category of descent data with respect to \(I^{1,0} \rightarrow I^{1,1} \rightarrow I^{1,2}\).

**Remark 3.1.6.** The above definition does not depend on the choice of an étale covering \((X_0, \mathcal{M}_0) \rightarrow (X, \mathcal{M})\) and an immersion \((X_0, \mathcal{M}_0) \hookrightarrow (P^0, \mathcal{L}^0)\) by Lemma 4.5 of [19].

To define the notion of convergent log \( \nabla \)-modules, we introduce the notion of charted standard small frame.

**Definition 3.1.7.** (cf. Definition 3.3 of [21]) A charted standard small frame of \((X, \mathcal{M})\) is the following data:

- an exact closed immersion \((X, \mathcal{M}) \hookrightarrow (P, \mathcal{L})\) over \((\text{Spf} \, V, N)\) where \(P\) is an affine formally log smooth \( p \)-adic formal scheme over \((\text{Spf} \, V, N)\),
- a chart

\[
\begin{array}{ccc}
N & \xrightarrow{\alpha} & V \\
\downarrow{g} & & \downarrow \\
M & \xrightarrow{\beta} & \mathcal{O}_P
\end{array}
\]

of the structure morphism \((P, \mathcal{L}) \rightarrow (\text{Spf} \, V, N)\),

- \(\gamma_1, \ldots, \gamma_d \in \mathcal{O}_P\),

such that \(P_k = \overline{X}\) and the morphism \(P \rightarrow \mathbb{A}^d_P \times \mathbb{A}_{M, V} \times \mathbb{A}_{N, V} \text{Spf} \, V\) induced by \(\gamma_1, \ldots, \gamma_d\) and the chart is formally étale.

A charted standard small frame is good at \(\pi\) if \(M = \mathcal{M}_\pi\).

**Remark 3.1.8.** Assume that \((X, \mathcal{M})\) is log smooth over \((\text{Spf} \, V, N)\). Then, étale locally on \(\overline{X}\), there exists a charted standard small frame. Indeed, étale locally we
can take a chart

\[
\begin{array}{ccc}
N & \xrightarrow{a} & k \\
\downarrow & & \downarrow \\
M & \longrightarrow & \mathcal{O}_X
\end{array}
\]

such that the morphism \( \overline{X} \rightarrow \mathbb{A}_{M,k} \times_{\mathbb{A}_{N,k}} \operatorname{Spec} k \) is the composition of an étale morphism \( \overline{X} \rightarrow \mathbb{A}^d_k \times_{\mathbb{A}_{M,k}} \mathbb{A}_{N,k} \operatorname{Spec} k \) and the projection \( \mathbb{A}^d_k \times_{\mathbb{A}_{M,k}} \mathbb{A}_{N,k} \operatorname{Spec} k \rightarrow \mathbb{A}_{M,k} \times_{\mathbb{A}_{N,k}} \operatorname{Spec} k \) for some \( d \in \mathbb{N} \). We can take a lift \( P \) of \( \overline{X} \) such that the following diagram is Cartesian

\[
\begin{array}{ccc}
\overline{X} & \longrightarrow & \mathbb{A}^d_k \times_{\mathbb{A}_{M,k}} \mathbb{A}_{N,k} \operatorname{Spec} k \\
\downarrow & & \downarrow \\
P & \longrightarrow & \mathbb{A}^d_k \times_{\mathbb{A}_{M,V}} \mathbb{A}_{N,V} \operatorname{Spf} V
\end{array}
\]

and the bottom horizontal morphism is formally étale. We define the log structure \( L \) on \( P \) over \( (\operatorname{Spf} V, N) \) by \( P \rightarrow \mathbb{A}^d_k \times_{\mathbb{A}_{M,V}} \mathbb{A}_{N,V} \operatorname{Spf} V \). The exact closed immersion \( (\overline{X}, M) \rightarrow (P, L) \) forms a charted standard small frame with \( \gamma_1, \ldots, \gamma_d \in \mathcal{O}_P \) and the chart associated to the above diagram.

Moreover, for any geometric point \( \overline{x} \) of \( \overline{X} \), if \( N \rightarrow M_{\overline{x}} \) is injective and \( (M_{\overline{x}}/N)^{gp} \) is torsion-free, there exists a charted standard small frame which is good at \( \overline{x} \). Indeed, by Lemma 1.2.5 we can take a good one as the above chart.

In the rest of this subsection, we assume the existence of a charted standard small frame \( (P, M, \gamma_1, \ldots, \gamma_d) \). Let \( X \) be an open subscheme of \( \overline{X} \).

Let \( (P(i), L(i)) \) be the \((i + 1)\)-st fiber product of \((P, L)\) over \((\operatorname{Spf} V, N)\).

**Definition 3.1.9.** (cf. Definition 6.3.1 of [13], Definition 3.4 of [21]) A log \( \nabla \)-module \( E \) on \( P_K \) is convergent if it the restriction of \( E \) to some strict neighborhood of \( \overline{X} \) in \( P_K \) comes from a locally free overconvergent log isocrystal on \( (X, \mathcal{X}, M) / (\operatorname{Spf} V, N) \).

**Proposition 3.1.10.** (cf. Theorem 6.4.1 of [13]) If \( X \) is dense in \( \overline{X} \), there is an equivalence between the category \( I_{\operatorname{conv}}((\overline{X}, M)/(\operatorname{Spf} V, N))^{lf} \) of locally free convergent log isocrystals on \( (X, \mathcal{X}, M) / (\operatorname{Spf} V, N) \) and the category \( \widehat{\operatorname{MIC}}_{\operatorname{conv}}((\overline{X}, M)/(\operatorname{Spf} V, N)) \) of convergent log \( \nabla \)-modules on \( P_K \).

**Proof.** Let \( I_{\operatorname{inf}}((\overline{X}, M)/(\operatorname{Spf} V, N)) \) be the category of isocrystals on the infinitesimal site \((\mathcal{M}, M)/(\operatorname{Spf} V, N))_{\operatorname{inf}} \) which is defined in Definition 8 of [7]. By §3 of [7], the category \( \widehat{\operatorname{MIC}}((\overline{X}, M)/(\operatorname{Spf} V, N)) \) of log \( \nabla \)-modules on \( P_K \) equivalent to \( I_{\operatorname{inf}}((\overline{X}, M)/(\operatorname{Spf} V, N)) \). By Theorem 3 of [7], the natural functor from \( I_{\operatorname{conv}}((\overline{X}, M)/(\operatorname{Spf} V, N))^H \) to \( I_{\operatorname{inf}}((\overline{X}, M)/(\operatorname{Spf} V, N)) \) is fully faithful, so the natural functor from \( I_{\operatorname{conv}}((\overline{X}, M)/(\operatorname{Spf} V, N))^H \) to \( \widehat{\operatorname{MIC}}((\overline{X}, M)/(\operatorname{Spf} V, N)) \) is fully faithful. We have to show that the essential image of this functor is equal to \( \widehat{\operatorname{MIC}}_{\operatorname{conv}}((\overline{X}, M)/(\operatorname{Spf} V, N)) \).
It is clear that the image is contained by \( \hat{\operatorname{MIC}}_{\text{conv}} \left( (\overline{X}, \mathcal{M}) / (\text{Spf} V, N) \right) \). For \( E \in \hat{\operatorname{MIC}}_{\text{conv}} \left( (\overline{X}, \mathcal{M}) / (\text{Spf} V, N) \right) \), the restriction of \( E \) to \( X \) comes from a convergent log isocrystal on \((X, \mathcal{M}|_X) / (\text{Spf} V, N)\). So by Proposition 8 of [2], \( E \) is contained in the image of \( I_{\text{conv}} \left( (\overline{X}, \mathcal{M}) / (\text{Spf} V, N) \right) \).

\[ \square \]

### 3.2. Exponents of convergent log isocrystals.

**Definition 3.2.1.** Let \((\overline{X}, \mathcal{M})\) be a log smooth variety over \((\text{Spec} \, k, N)\). Let \( \mathfrak{p} \) be a geometric point of \( \overline{X} \). Put \( M := \overline{M}_{\mathfrak{p}} \). Assume that \( N \to M \) is injective and \((M/N)^{\text{gp}}\) is torsion-free. Let \( \Sigma \) be a subset of \( M^\text{gp} \otimes_{\mathbb{Z}} \overline{K} \).

Let \( \mathcal{E} \) be a locally free convergent log isocrystal on \((\overline{X}, \mathcal{M}) / (\text{Spf} V, N)\). Take a charted standard small frame \( ((U, \mathcal{M}|_U) \hookrightarrow (P, \mathcal{L}), M \to \mathcal{O}_P, \gamma_1, \ldots, \gamma_d) \) of some neighborhood \( U \) of \( \mathfrak{p} \) in \( \overline{X} \) which is good at \( \mathfrak{p} \). Let \( E_P \) be the log \( \mathcal{V} \)-module on \( P_K \) induced by \( \mathcal{E} \). We say that \( \mathcal{E} \) has exponents in \( \Sigma \) at \( \mathfrak{p} \) if the exponents of \( E_P \) are contained in \( \Sigma \).

**Proposition 3.2.2.** The above definition is independent of the choice of charted standard small frames.

**Proof.** We assume that \( \overline{X} = U \) has two charted standard small frames

\[
((\overline{X}, \mathcal{M}) \hookrightarrow (P_1, \mathcal{L}_1), \beta_1 : M \to \mathcal{O}_{P_1}, \gamma_{1,1}, \ldots, \gamma_{1,d})
\]

and

\[
((\overline{X}, \mathcal{M}) \hookrightarrow (P_2, \mathcal{L}_2), \beta_2 : M \to \mathcal{O}_{P_2}, \gamma_{2,1}, \ldots, \gamma_{2,d})
\]

which are good at \( \mathfrak{p} \).

Let \( (P_{1,2}, \mathcal{L}_{1,2}) := (P_1, \mathcal{L}_1) \times_{(\text{Spf} V, N)} (P_2, \mathcal{L}_2) \). Let \( \pi_1 : (P_{1,2}, \mathcal{L}_{1,2}) \to (P_1, \mathcal{L}_1) \) and \( \pi_2 : (P_{1,2}, \mathcal{L}_{1,2}) \to (P_2, \mathcal{L}_2) \) be the projections. Let \( m_1, \ldots, m_r \in M^{\text{gp}} \) be a set of lifts of free generators of \((M/N)^{\text{gp}}\). Put \( \beta(m)_i := \pi_i^*(\beta_i(m)) \) for \( i = 1, 2 \).

Let \( M(1) \) be the amalgamated sum of 2 copies of \( M \) over \( N \) in the category of fine monoids. Let \( M(1)' \) be the submonoid of \( M(1)^{\text{gp}} \) generated by \( M(1) \) and the kernel of the codiagonal homomorphism \( M(1)^{\text{gp}} \to M^{\text{gp}} \). There exists a chart \( M(1) \to \mathcal{O}_{P_{1,2}} \) of \( \mathcal{L}_{1,2} \) induced by \( \beta_1 \) and \( \beta_2 \).

Let \( (P_3, \mathcal{L}_3) := (P_{1,2}, \mathcal{L}_{1,2}) \times_{\hat{\mathcal{M}}_{M(1)'}, \mathcal{V}} \hat{\mathcal{M}}_{M(1)}, \mathcal{V} \).

We denote the natural map \( (P_3, \mathcal{L}_3) \to (P_{1,2}, \mathcal{L}_{1,2}) \) by \( \iota \). The following diagram commutes:

\[
\begin{array}{ccc}
(X, \mathcal{M}) & \xrightarrow{\iota} & \hat{\mathcal{M}}_{M, \mathcal{V}} \\
| & | & | \\
(P_{1,2}, \mathcal{L}(j)) & \rightarrow & \hat{\mathcal{M}}_{M(1)}, \mathcal{V}
\end{array}
\]

where the left vertical morphism is the diagonal map. Thus there exists the diagonal map \( (X, \mathcal{M}) \to (P_3, \mathcal{L}_3) \). Since \( \iota^*(\hat{\mathcal{M}}_{M(1)}) \) is invertible on \( P_3 \) for any \( m \in M \), the map \( M \ni m \mapsto \iota^*(\beta(m)) \) is a fine chart of \( \mathcal{L}_3 \). Thus \( (\overline{X}, \mathcal{M}) \to (P_3, \mathcal{L}_3) \) is an exact closed immersion. So

\[
\overline{X}_{\text{log}}^{P_{1,2}} = \overline{X}_{P_3}
\]

On the other hand, we define the morphism

\[
f : (P_3, \mathcal{L}_3) \to \hat{\mathcal{V}}^{\gamma_{1,d}} \times (P_2, \mathcal{L}_2)
\]
by \( \beta(m_1) = 1, \ldots, \beta(m_r) = 1, \pi_1(\gamma_1) - \pi_2(\gamma_1), \ldots, \pi_1(\gamma_d) - \pi_2(\gamma_d) \in \mathcal{O}_{P_2} \), regarded as \( P_3 \rightarrow \mathbb{A}^1_v \), and the projection \( \pi_2 \circ \iota \). This is log étale and strict, so étale. So,

\[
|X|_{P_3} = |X|_{\mathbb{A}_v^{r+4} \times P_3} = \mathbb{A}^{r+4}\times(0,1) \times |X|_{P_2}
\]

by Proposition 1.3.1 of [2].

The following diagram commutes:

\[
\begin{array}{ccc}
(P_1, \mathcal{L}_1) & \overset{\pi_{1 o r}}{\longrightarrow} & (P_3, \mathcal{L}_3) \\
\downarrow & & \downarrow \\
(P_2, \mathcal{L}_2) & \overset{pr_2}{\longrightarrow} & (P_2, \mathcal{L}_2),
\end{array}
\]

where \( pr_2 \) is the projection and \( s \) is the section to \( \{0\} \times P_2 \). We denote the morphism \( |X|_{P_2} \rightarrow |X|_{\mathbb{A}_v^{r+4} \times P_2} \) induced by \( s \) also by \( s \) and denote the morphism \( |X|_{\mathbb{A}_v^{r+4} \times P_2} \rightarrow |X|_{P_3} \rightarrow |X|_{P_1} \) induced by \( \pi_1 \circ \iota \) also by \( \pi_1 \). Then \( E_{P_2} = (\pi_1 o s)^*(E_{P_1}) \). Since

\[
(\pi_1 o s)^*(\beta_1(m)) = s^*(\beta(m)) = \beta_2(m)
\]
on \( |X|_{P_2} \) for any \( m \in M \), the pullback of the sheaf induced by \( \Sigma \) on \( P_1 \) by \( \pi_1 \circ \iota \circ s \) is equal to the sheaf induced by \( \Sigma \) on \( P_2 \). Therefore if the exponents of \( E_{P_1} \) are contained in \( \Sigma \), the exponents of \( E_{P_2} \) are contained \( \Sigma \).

**Definition 3.2.3.** Let \( (X, \mathcal{M}) \) be a log smooth variety over \( \text{Spec} k, N \) such that \( N \rightarrow \mathcal{M}_\mathfrak{p} \) is injective and \( (\mathcal{M}_\mathfrak{p}/N)^{gp} \) is torsion-free at any geometric point \( \mathfrak{p} \) of \( X \). Let \( S \subseteq \mathcal{M}^{gp} \otimes \mathbb{Z} \mathcal{R} \) be a subsheaf. A locally free convergent log isocrystal \( E \) on \( (X, \mathcal{M}) / (\text{Spf} \mathcal{V}, N) \) has exponents in \( S \) if for any geometric point \( \mathfrak{p} \), \( E \) has exponents in \( S_{\mathfrak{p}} \) at \( \mathfrak{p} \).

**3.3. Unipotence of overconvergent log isocrystals.** We consider the following situation:

**Situation 3.3.1.** Let \( (X, \mathcal{M}_0 \oplus \mathcal{M}) \) be a log variety over \( \text{Spec} k, N \) satisfying the following conditions:

- \( \mathcal{M} \) and \( \mathcal{M}_0 \) are fine.
- The map \( N \rightarrow \mathcal{M}_0 \oplus \mathcal{M} \) factors as \( N \rightarrow \mathcal{M}_0 \rightarrow \mathcal{M}_0 \oplus \mathcal{M} \).
- \( (X, \mathcal{M}_0 \oplus \mathcal{M}) \) is log smooth over \( \text{Spec} k, N \).
- \( N \rightarrow (\mathcal{M}_0)_\mathfrak{p} \) is injective and \( ((\mathcal{M}_0)_\mathfrak{p}/N)^{gp} \) is torsion-free at any geometric point \( \mathfrak{p} \) of \( X \).
- \( (\mathcal{M}_0)^{gp} \) is torsion-free at any geometric point \( \mathfrak{p} \) of \( X \).

Let \( X \) be the trivial locus of \( \mathcal{M} \) which is an open dense subset of \( X \) as shown below.

Let \( \mathfrak{p} \) be a geometric point of \( X \). Let \( M_0 := (\mathcal{M}_0)_\mathfrak{p} \) and \( M := \mathcal{M}_\mathfrak{p} \). Take a charted standard small frame \( (X, \mathcal{M}) \leftrightarrow (P, \mathcal{L}), \beta : M_0 \oplus M \rightarrow \mathcal{O}_P, \gamma_1, \ldots, \gamma_d \) which is good at \( \mathfrak{p} \). (We assume that it can be taken globally.) Then there exists a natural étale morphism

\[
f : (P, \mathcal{L}) \rightarrow \hat{\mathbb{A}}^{d}_v \times \left( \hat{\mathbb{A}}^{M_0, \mathcal{V}} \times (\text{Spf} \mathcal{V}, N) \right) \times \hat{\mathbb{A}}^{M, \mathcal{V}}
\]
for some $d$. Let $Q$ be the inverse image of the vertex of $\hat{A}_{M,V}$ under the map $P \to \hat{A}_{M,V}$ and $Z := Q_k \subseteq \overline{X}$. Then $\overline{X} \supseteq Z$. Note that $Q \to \hat{A}_d^V \times (\hat{A}_{M_0,V} \times \hat{A}_{N,V} \times \text{Spf } V)$ is étale. Let $\mathcal{L}_0$ be the log structure on $Q$ over $(\text{Spf } V, N)$ induced by $Q \to \hat{A}_{M_0,V} \times \hat{A}_{N,V} \times \text{Spf } V$.

$X$ is dense in $\overline{X}$ since it is the inverse image of a dense subset $\hat{A}_d^V \times (\hat{A}_{M_0,V} \times \hat{A}_{N,V} \times \text{Spf } V)$ under the étale morphism $f$.

Note that $\tau |_{\overline{Q}} = Q_K$. Note also that $\overline{M}$ is semi-saturated. Indeed, for any face $F$ of $M$, there exists a geometric point $\overline{y}$ of $\overline{X}$ such that $\overline{X}_{\overline{y}} = M/F$. By the assumption, $(M/F)^{\text{sp}}$ is torsion-free. So $\overline{M}$ is semi-saturated by Proposition 13.2.

**Lemma 3.3.2.** In this situation, 

$$|Z|_P \cong |Z|_Q \times \hat{A}_{M,K}[0,1).$$

**Proof.** Let $(P \times Q, \mathcal{L}_{P \times Q}) := (P, \mathcal{L}) \times_{(\text{Spf } V, N)} (Q, \mathcal{L}_0)$. Let $\pi_1: (P \times Q, \mathcal{L}_{P \times Q}) \to (P, \mathcal{L})$ and $\pi_2: (P \times Q, \mathcal{L}_{P \times Q}) \to (Q, \mathcal{L}_0)$ be the first and second projections. $(\cdot)_i$ denotes $\pi^{\ast}_i(\cdot)$ for $i = 1, 2$. Let $m_1, \ldots, m_r \in M_0^{\text{sp}} / N^\ast$ be a set of free generators of $M_1^{\text{sp}} / N^\ast$ and let $m_1', \ldots, m_r' \in M_0$ be their lifts. Let $m_1, \ldots, m_r \in M^{\text{sp}}$ be a set of free generators of $M^{\text{sp}}$. Let $M_0(1) := M_0 \oplus N$ and $M_0(1)'$ the submonoid of $M_0(1)^{\text{sp}}$ generated by $M_0(1)$ and the kernel of the codiagonal homomorphism $M_0(1)^{\text{sp}} \to M_0^{\text{sp}}$. Let $(P', \mathcal{L}') := (P \times Q, \mathcal{L}_{P \times Q}) \times_{\hat{A}_{M_0(1),V}} \hat{A}_{M_0(1),V}$. Let $\iota: (P', \mathcal{L}') \to (P \times Q, \mathcal{L}_{P \times Q})$ be the projection. There exists a natural closed immersion $(Z, \mathcal{M}|_Z) \to (P', \mathcal{L}')$.

Since $\iota^\ast \left( \frac{\beta(m')}{\beta(m_1')} \right)$ is an invertible element of $\mathcal{O}_{P'}$ for any $m' \in M_0$, $M_0 \oplus M \ni m' + m \mapsto \beta(m' + m)_1$ is a fine chart of $\mathcal{L}'$. So $(Z, \mathcal{M}|_Z) \to (P', \mathcal{L}')$ is an exact closed immersion.

We define a morphism 

$$(P', \mathcal{L}') \to (P, \mathcal{L}) \times \hat{A}_V^{d + r'}$$

by the projection $\pi_1 \circ \iota$ and elements $\iota^\ast((\gamma_1)_2 - (\gamma_1)_1), \ldots, \iota^\ast((\gamma_d)_2 - (\gamma_d)_1)$,

$$\iota^\ast \left( \frac{\beta(m')}{\beta(m_1')} \right) - 1, \ldots, \iota^\ast \left( \frac{\beta(m')}{\beta(m_1')} \right) - 1$$

of $\mathcal{O}_{P'}$ regarded as morphisms $P' \to \hat{A}_V$. Then this is étale (since log étale and strict) and $(Z, \mathcal{M}|_Z) \to (P', \mathcal{L}') \to (P, \mathcal{L}) \times \hat{A}_V^{d + r'}$ is the closed immersion to $(Z, \mathcal{M}|_Z) \times_0 \{0\}$ where $0$ is the vertex of $\hat{A}_V^{d + r'}$. So,

$$|Z|_{P'} \cong |Z|_{P \times \hat{A}_V^{d + r'}} = |Z|_P \times \hat{A}_V^{d + r'}[0,1).$$

On the other hand, we define a morphism 

$$(P', \mathcal{L}') \to \hat{A}_{M,V} \times \hat{A}_V^{d + r'} \times (Q, \mathcal{L}_0)$$

by $(P', \mathcal{L}') \xrightarrow{\pi_1 \circ \iota} (P, \mathcal{L}) \to \hat{A}_{M,V}$, elements $\iota^\ast((\gamma_1)_2 - (\gamma_1)_1), \ldots, \iota^\ast((\gamma_d)_2 - (\gamma_d)_1)$,

$$\iota^\ast \left( \frac{\beta(m')}{\beta(m_1')} \right) - 1, \ldots, \iota^\ast \left( \frac{\beta(m')}{\beta(m_1')} \right) - 1$$

of $\mathcal{O}_{P'}$ regarded as morphisms $P' \to \hat{A}_V$ and the projection $\pi_2 \circ \iota$. Then this is also étale and $(Z, \mathcal{M}|_Z) \to (P', \mathcal{L}') \to \hat{A}_{M,V} \times \hat{A}_V^{d + r'} \times (Q, \mathcal{L}_0)$ is the closed immersion to $0 \times 0 \times (Z, \mathcal{L}_0|_Z)$ where the first
Let $d_{\pi}^d \epsilon_{\gamma} M, V$ and the second $0$ is the vertex of $d_{\pi}^d + r'$. So,

$$Z_{\pi} \cong Z_{d_{\pi}^d + r', Q} = \hat{A}_{M, K} [0, 1] \times A_{d_{\pi}^d + r', [0, 1] \times Z_{Q}}.$$

Comparing two isomorphisms, we have $Z_{\pi} \cong Z_{Q} \times \hat{A}_{M, K} [0, 1]$. \(\square\)

**Lemma 3.3.3.** (cf. Proposition 2.2.13 of [2] or Lemma 6.3.4 of [13]) Let $E$ be a convergent log $\nabla$-module on $P_{K}$. Then the restriction of $E$ to $Z_{\pi} \cong Z_{Q} \times \hat{A}_{M, K} [0, 1]$ is log-convergent.

**Proof.** Let $(P(1), \mathcal{L}(1))$ be the fiber product of 2 copies of $(P, \mathcal{L})$ over $(\text{Spf } \mathcal{V}, N)$. Let $\pi_1, \pi_2 : (P(1), \mathcal{L}(1)) \rightarrow (P, \mathcal{L})$ be the two projections. Put $(\cdot)^\gamma := \pi_i^\gamma (\cdot)$ for $i = 1, 2$.

Let $(M + M_0) \otimes (M + M_0)$ the summand of $(M + M_0) \otimes (M + M_0)$ generated by $(M + M_0)$ and the kernel of the co-diagonal homomorphism $(M + M_0) \otimes (M + M_0) \rightarrow (M + M_0)^{op}$. Let $(P(1), \mathcal{L}(1))' := (P(1), \mathcal{L}(1)) \times A_{M + M_0}$, $\hat{A}_{M + M_0}(\mathcal{L}(1), V)$. Then $\hat{X}_{\mathcal{L}(1)}^\log = \hat{X}_{\mathcal{L}(1)}$ as in the proof of Proposition 3.2.2.

Let $m_1, \ldots, m_r$ and $m'_1, \ldots, m'_r$ be as in the previous proof and let $m_{r+i} := m'_i$ for $1 \leq i \leq r'$.

By the proof of Proposition 3.2.2, $\Omega_{(P(1), \mathcal{L}(1)/\text{Spf } \mathcal{V}, N)}$ is generated by $d (\gamma_1)_1, \ldots, d (\gamma_d)$, $d (\gamma_1)_2, \ldots, d (\gamma_d)$, $d (\mathcal{L}(1))$, $d (\mathcal{L}(1))_2$, and $d (\mathcal{L}(1))_2$.

Let $\alpha_k := \frac{\pi_k^{m_k}}{\pi_k^{m_k}}$ for $1 \leq k \leq r + r'$ and $\tau := (\gamma)_{2} - (\gamma)_{1}$ for $1 \leq l \leq d$.

By the definition of convergent log $\nabla$-modules, there exists a locally free convergent log isocrystal $\mathcal{E}$ on $(X, \hat{X}, M)$ over $(\text{Spf } \mathcal{V}, N)$ which coincides with $E$ on some strict neighborhood of $X_{(1)}$. We can regard $\mathcal{E}$ as a locally free module on some strict neighborhood $V$ of $X_{\pi}$ equipped with an isomorphism $\epsilon : \pi_2^\gamma (X) \rightarrow \pi_1^\gamma (X)$ on some strict neighborhood $W$ of $X_{(1)}$ contained in $\pi_1^{-1}(V) \cap \pi_2^{-1}(V)$.

**Claim 3.3.3.1.** The isomorphism $\epsilon$ is written by

$$\epsilon (1 \otimes v) = \sum_{i_1, \ldots, i_{r+r'+d} = 0}^\infty \prod_{k=1}^{r+r'} \left( \frac{u_k - 1}{i_k!} \right)^{i_k} \prod_{l=1}^{i_k} \frac{\partial_{\gamma_l}}{i_{r+r'+d}!} \left( \prod_{k=1}^{r+r'} \prod_{x=0}^{i_k} \left( \frac{\partial_{\gamma_l}}{i_{r+r'+d}!} \right)^{i_{r+r'+d}} \right)(v)$$

for any section $v$ of $E$.

**Proof.** Let $I := \text{Ker}(\mathcal{O}_{(1)} \rightarrow \mathcal{O}_{P})$ and define the $n$-th log infinitesimal neighborhood $P^n(1)$ of $P$ by $\mathcal{O}_{P^n(1)} := \mathcal{O}_{(1)} / I^{n+1}$ for $n \in \mathbb{N}$. Let $\epsilon^n$ be the restriction of $\epsilon$ to $W \cap P^n(1)$. We define $\epsilon^n : \pi_2^\gamma (X)_{W \cap P^n(1)} \rightarrow \pi_1^\gamma (X)_{W \cap P^n(1)}$ as

$$\epsilon^n (1 \otimes v) = \sum_{i_1, \ldots, i_{r+r'+d} = 0}^\infty \prod_{k=1}^{r+r'} \left( \frac{u_k - 1}{i_k!} \right)^{i_k} \prod_{l=1}^{i_k} \frac{\partial_{\gamma_l}}{i_{r+r'+d}!} \left( \prod_{k=1}^{r+r'} \prod_{x=0}^{i_k} \left( \frac{\partial_{\gamma_l}}{i_{r+r'+d}!} \right)^{i_{r+r'+d}} \right)(v).$$

Then $\epsilon^1$ and $\epsilon^1$ coincide on $W \cap P^1(1)$.
By the cocycle condition, for \( n \in \mathbb{N} \), the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{O}_{\mathcal{P}_{2n}(1)} \otimes \mathcal{O}_{\mathcal{P}} & \xrightarrow{\text{id} \otimes \varepsilon^n} & \mathcal{O}_{\mathcal{P}_{2n}(1)} \otimes \mathcal{O}_p \otimes \mathcal{E} \\
\downarrow & & \downarrow \\
\mathcal{E} \otimes \mathcal{O}_{\mathcal{P}} & \xrightarrow{\varepsilon^n \otimes \text{id}} & \mathcal{O}_{\mathcal{P}_{2n}(1)} \otimes \mathcal{O}_{\mathcal{P}} \otimes \mathcal{E} \\
\mathcal{O}_{\mathcal{P}_{2n}(1)} \otimes \mathcal{O}_{\mathcal{P}} & \xrightarrow{\text{id} \otimes \varepsilon^n} & \mathcal{O}_{\mathcal{P}_{2n}(1)} \otimes \mathcal{O}_{\mathcal{P}} \otimes \mathcal{E} \\
\end{array}
\]

where \( \mathcal{O}_{\mathcal{P}_{2n}(1)} \to \mathcal{O}_{\mathcal{P}_{2n}(1)} \otimes \mathcal{O}_{\mathcal{P}} \otimes \mathcal{E} \) is the natural map. This map is injective.

The above diagram also commutes even if we replace \( \varepsilon^n, \varepsilon^m \) by \( \varepsilon^{2n}, \varepsilon^{2m} \), respectively. Thus if \( \varepsilon^n \) and \( \varepsilon^m \) coincide, \( \varepsilon^{2n} \) and \( \varepsilon^{2m} \) coincide and so \( \varepsilon^n \) and \( \varepsilon^m \) coincide for \( m \leq 2n \). By induction, \( \varepsilon^n \) and \( \varepsilon^m \) coincide for all \( n \in \mathbb{N} \).

By the proof of Proposition 3.2.2,

\[
\mathcal{X}_{(\mathcal{P},1)} = A_K^{r+1} \times [0,1] \times \mathcal{X}_p
\]

where the coordinates of \( A_K^{r+1} \times [0,1] \) are \( u_1, \ldots, u_r, 1, \tau_1, \ldots, \tau_d \). Thus for any \( \eta < 1 \) and any \( \mathbf{v} \in \Gamma(\mathcal{X}_p, \mathcal{E}) \), the multi-indexed series

\[
\left( \frac{1}{i_1! \cdots i_r!} \prod_{k=1}^{r} \prod_{x=0}^{i_k-1} (\partial_x - x) \right) \mathbf{v}
\]

is \( \eta \)-null on \( \mathcal{X}_p \). Since \( \mathcal{X} \) is dense \( \mathcal{X} \), the spectral seminorm on \( \Gamma(\mathcal{X}_p, \mathcal{O}) \) restricts to the spectral seminorm on \( \Gamma(\mathcal{X}_p, \mathcal{O}) \). Thus this series is \( \eta \)-null on \( \mathcal{X}_p \).

For any \( a < 1 \), sections of \( E \) on \( \mathcal{X}_p \) generate all sections of \( E \) on \( Z_{(\mathcal{P},1)} \). By Remark 2.5.4, the restriction of \( E \) to \( Z_{(\mathcal{P},1)} \) is log-convergent.

Put \( R := A_M^d \times \left( \hat{A}_{\mathcal{M},\mathcal{Y}} \times \hat{A}_{\mathcal{N},\mathcal{Y}} \right) \) and let \( f_M : (\mathcal{P}, \mathcal{L}) \to \hat{A}_{\mathcal{M},\mathcal{Y}} \) be the composition of \( f \) and the projection \( R \to \hat{A}_{\mathcal{M},\mathcal{Y}} \).

Let \( F \subseteq M \) be a face. There exists a natural closed immersion \( \hat{A}_{\mathcal{F},\mathcal{Y}} \hookrightarrow \hat{A}_{\mathcal{M},\mathcal{Y}} \) of underlying formal schemes defined by \( \mathcal{V}(M) \ni \sum_{m \in M} c_m t^m \mapsto \sum_{m \in F} c_m t^m \in \mathcal{V}(F) \). Put

\[
Q_F := f_M^{-1}(\hat{A}_{\mathcal{F},\mathcal{Y}}) = \{ \mathbf{p} \in P \mid \forall m \in M \setminus F : \beta(m)(\mathbf{p}) = 0 \},
\]

\[
\tilde{Q}_F := f_M^{-1}(\hat{A}_{\mathcal{F},-1,M,\mathcal{Y}}) = \{ \mathbf{p} \in P \mid \forall m \in F : \beta(m)(\mathbf{p}) \neq 0 \},
\]

\[
\tilde{Q}_F := Q_F \cap \tilde{P}_F
\]

\( Q_F \) is closed and \( \tilde{Q}_F \) is open in \( P \). Let \( \mathcal{L}_F \) be the log structure on \( Q_F \) over \( (\mathcal{P}, \mathcal{Y}) \) induced by \( Q_F \to \left( \hat{A}_{\mathcal{M},\mathcal{Y}} \times \hat{A}_{\mathcal{N},\mathcal{Y}} \right) \) \( \mathcal{L}_{(\mathcal{P},1)} \). Note that \( P = \bigcup_F \tilde{Q}_F \) where \( F \) runs through all faces of \( M \). Put \( Z_F := (Q_F)_k, \mathcal{X}_F := (\tilde{P}_F)_k, \mathcal{Z}_F := (\tilde{Q}_F)_k \). Note that \( X = \mathcal{X}_M \).
Applying the Lemma 1.1.2 to $M \to M/F$, we have an isomorphism $F^{-1}M \cong M/F \oplus F^\text{sp}$. Take an isomorphism $F^\text{sp} \cong \mathbb{Z}^{r_F}$ where $r_F$ is the rank of $F^\text{sp}$. Then the morphism $f$ and these isomorphisms induce an étale morphism

\[(\tilde{P}_F, \mathcal{L}|_{\tilde{P}_F}) \to R \times \hat{\mathbb{A}}_V^{r_F} \times \hat{\mathbb{A}}_{M/F,V} = \hat{\mathbb{A}}_V^{d+r_F} \times \left(\hat{\mathbb{A}}_{M_0,V} \times (\text{Spf} \mathcal{V}, N)\right) \times \hat{\mathbb{A}}_{M/F,V}.
\]

The lifting $(\tilde{X}_F, (M_0 \oplus M)|_{\tilde{X}_F}) \to (\tilde{P}_F, \mathcal{L}|_{\tilde{P}_F})$, the chart induced by $(\tilde{X}_F, (M_0 \oplus M)|_{\tilde{X}_F}) \to \left(\hat{\mathbb{A}}_{M_0,V} \times \hat{\mathbb{A}}_{N,V} (\text{Spf} \mathcal{V}, N)\right) \times \hat{\mathbb{A}}_{M/F,V}$ and the sections of $\mathcal{O}_{\tilde{P}_F}$ inducing $\tilde{P}_F \to \hat{\mathbb{A}}_V^{d+r_F}$ forms a charted standard small frame which is good at points of $Z_F$. Thus by Lemma 3.3.2

\[(3.3.3.3) \quad \left[Z_F\right]_{Q_F} \cong \left(\hat{Q}_F\right)_K.
\]

An overconvergent log isocrystal $\mathcal{E}$ on $(X, \overline{X}, (M_0 \oplus M)|_{\mathcal{L}})/(\text{Spf} \mathcal{V}, N)$ defines a log $\nabla$-module $E_F$ on $\left[Z_F\right]_{Q_F} \times \hat{\mathbb{A}}_{M/F,K}(a, 1)$ for some $a \in (0, 1) \cap \Gamma^\ast$.

**Definition 3.3.4.** Let $\Sigma \subseteq M^\text{sp} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ be an ((NI ∩ NL)-D) subset. $\mathcal{E}$ is $\Sigma$-unipotent with respect to $(P, \mathcal{L})$, if any face $F$ of $M$ of the induced log $\nabla$-module on $\left[Z_F\right]_{Q_F} \times \hat{\mathbb{A}}_{M/F,K}(a, 1)$ is $\Sigma_F$-unipotent, where $\Sigma_F$ is the image of $\Sigma$ under the projection $M^\text{sp} \otimes_{\mathbb{Z}} \mathbb{Z}_p \to (M^\text{sp}/F^\text{sp}) \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

**Remark 3.3.5.** By Proposition 2.3.5, this definition is independent of the choice of $a$.

**Remark 3.3.6.** If $\mathcal{E}$ is $\Sigma$-unipotent with respect to a charted standard small frame $(P, \mathcal{L})$, then $\mathcal{E}$ is also $\Sigma_\mathcal{L}$-unipotent with respect to the charted standard small frame of $(\tilde{X}_F, (M_0 \oplus M)|_{\tilde{X}_F})$ induced by (3.3.3.2) for any face $F$ of $M$. Indeed, any face $\mathcal{F}$ of $M/F$ corresponds with a face $F'$ of $M$ containing $F$ and $\left[Z_{\mathcal{F}'}\right]_{\tilde{F}'} = \left[Z_F\right]_{\tilde{F}}$.

**Remark 3.3.7.** By Remark 2.3.3, $\Sigma$-unipotence of overconvergent log isocrystals only depends on the image of $\Sigma$ in $M^\text{sp} \otimes_{\mathbb{Z}} (\mathbb{Z}_p/\mathbb{Z})$.

**Remark 3.3.8.** In the subsection 3.5, we will prove that the definition is independent of the choice of charted standard small frames good at $\mathfrak{F}$ under some assumptions.

### 3.4. Overconvergent generization

In this subsection, we adapt Proposition 3.5.3 of [13] or Proposition 2.7 of [21] to our situation.

**Proposition 3.4.1.** (overconvergent generization, cf. Proposition 2.7 in [21]) Let $(P, L_0 \oplus \mathcal{L})$ be an affine connected $p$-adic fine log formal scheme log smooth over $(\text{Spf} \mathcal{V}, N)$ such that the structure morphism $N \to L_0 \oplus \mathcal{L}$ factors through $L_0 \to L_0 \oplus \mathcal{L}$. Let $M' \to \mathcal{O}_P$ be a fine chart of $\mathcal{L}$. Assume that $L_0'$ is trivial on $P_K$. Let $X$ be the trivial locus of $L|_{P_K}$. Let $M$ be a fine sharp semi-saturated monoid. Let $\Sigma \subseteq M^\text{sp} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ be an ((NI ∩ NL)-D) subset and $J \subseteq (0, 1)$ a quasi-open subinterval of positive length. Let $V$ be a strict neighborhood of $[X]_P$ in $P_K$ and $E$ an object
of \( \text{LNM}_{V \times \mathbb{A}_{M,K}(I)} \), whose restriction to \( |X|_{\rho} \times \mathbb{A}_{M,K}(I) \) is \( \Sigma \)-unipotent. Then, for any closed aligned subinterval \([b, c] \subseteq I\) of positive length, there exists a strict neighborhood \( V' \) of \( |X|_{\rho} \) in \( P_K \) contained in \( V \), such that the restriction of \( E \) to \( V' \times \mathbb{A}_{M,K}[b, c] \) is \( \Sigma \)-unipotent.

**Proof.** This proof is essentially the same as the proof of Proposition 2.7 in [21].

We may assume that \( V \) is affinoid. Assume that \( E \) is \( \Sigma \)-unipotent on \( |X|_{\rho} \times \mathbb{A}_{M,K}(I) \). Let \([b', c'] \subseteq [d, e] \subseteq I\) be aligned closed subintervals such that \( d < b' < b, c < c' < e\). We define \( D_l \) as in the proof of Lemma 2.4.1 As shown in the proof of Proposition 2.4.2 and in the proof of Proposition 2.5.6 for some \( \xi \in \Sigma \) and some \( \eta > 1\), for any \( v \in \Gamma(V \times \mathbb{A}_{M,K}[d, e], E) \), \( \|(D_{i+1} - D_i)(v)\| \) is \( \eta \)-null on \( |X|_{\rho} \times \mathbb{A}_{M,K}[b', c'] \) and \( D_l(v) \) converges to an element \( f(v) \) of \( H^0_{\xi}(|X|_{\rho} \times \mathbb{A}_{M,K}[b', c'], E) \). Moreover, for some \( v \in \Gamma(V \times \mathbb{A}_{M,K}[d, e], E) \), \( f(v) \neq 0 \).

Let \( m_1', \ldots, m'_g \in M' \) be a set of generators of \( M' \). For \( \lambda \in (0, 1) \cap \Gamma^* \), put \( V_\lambda := \left\{ x \in P_K \left| \sum_{i=1}^g m_i'(x) \geq \lambda \right. \right\} \). Let \( W \subseteq V \times \mathbb{A}_{M,K}[b', c'] \) be an affinoid subspace such that \( E \) is free on \( W \). We can show that \( \{(D_{i+1} - D_i)(v)\} \) is \( \rho \)-null for some \( \rho > 0 \) on \( W \cap (V_\lambda \times \mathbb{A}_{M,K}[b', c']) \) for some \( \lambda \in (0, 1) \cap \Gamma^* \) by the same calculation as in the proof of Proposition 2.7 in [21]. If \( W \cap (\cup_{\lambda} V_\lambda \times \mathbb{A}_{M,K}[b', c']) = \emptyset \), by the maximum modulus principle, \( W \cap (V_\lambda \times \mathbb{A}_{M,K}[b', c']) = \emptyset \) for some \( \lambda \in (0, 1) \cap \Gamma^* \). Otherwise, by Proposition 3.5.2 of [13], there exists \( \lambda \in (0, 1) \cap \Gamma^* \) such that \( \{(D_{i+1} - D_i)(v)\} \) is 1-null on \( W \cap (V_\lambda \times \mathbb{A}_{M,K}[b', c']) \).

So we can take \( \lambda \in (0, 1) \cap \Gamma^* \) such that \( D_l(v) \) converges over \( V_\lambda \times \mathbb{A}_{M,K}[b', c'] \). Put \( H_E := H^0_{\xi}(V_\lambda \times \mathbb{A}_{M,K}[b', c'], E) \neq 0 \). Then as shown in the proof of Proposition 2.4.2, \( H_E \) can be regarded as an object of \( \text{LNM}_{\mathbb{A}_{M,K} (0,0)} \). Let \( F := \mathcal{U}_{[b', c']}(H_E) \). Then \( F \) is a \( \Sigma \)-constant subobject of \( \mathbb{A}_{M,K}[b', c'] \) as proven in the proof of Proposition 2.4.2. By induction of the rank of \( E \), we can assume that \( E/F \) is \( \Sigma \)-unipotent on \( V_\lambda \times \mathbb{A}_{M,K}[b, c] \) for some \( \lambda \leq \lambda \). Hence \( E \) is \( \Sigma \)-unipotent on \( V_\lambda \times \mathbb{A}_{M,K}[b, c] \). □

### 3.5. Extension of overconvergent log isocrystals

In this section, we prove that unipotent overconvergent log isocrystals can be extended to convergent log isocrystals. The well-definedness of unipotence is proven by this extension property.

Let \( \mathop{\text{Spec}}(X, M_0 \oplus M) \) and \( X \) be as in Situation 3.3.1. Let \( \mathfrak{p} \) be some geometric point of \( \overline{X} \). Put \( M_0 := (M_0)_{\mathfrak{p}} \) and \( M := \overline{M}_{\mathfrak{p}} \). We assume that there exists a charted standard small frame

\[
\left( (P, \mathcal{L}), M_0 \oplus M \rightarrow \mathcal{O}_P, \gamma_1, \ldots, \gamma_d \right)
\]

which is good at \( \mathfrak{p} \). We continue to use the symbols defined in [33]. Note that \( P_K = \bigcup_F \mathbb{Z}_F[p] \) by definition, where \( F \) runs through all faces of \( M \). We show that this covering can be enlarged to an admissible covering.

**Lemma 3.5.1.** For each \( F \), there exists a strict neighborhood \( V_F \) of \( \mathbb{Z}_F[p] \) in \( |Z_F|_p \) and an isomorphism \( V_F \cong V'_F \), which is an extension of \( \mathbb{Z}_{M/F,K} \), where \( V'_F \) is some strict neighborhood of \( \mathbb{Z}_F[q] \times \mathbb{A}_{M/F,K}[0, 1] \) in \( (Q_F)_K \times \mathbb{A}_{M/F,K}[0, 1] \). Moreover, \( P_K = \bigcup_F V_F \) is an admissible covering.
Proof. If $V_F$ is a strict neighborhood of $\bar{Z}_F \subset Z_F|_P$, the covering
\[ |Z_F|_P = V_F \cup \bigcup_{F' \subseteq F} |Z_{F'}|_P \]
is admissible, where $F'$ runs through proper faces of $F$. Thus, if we take $V_F$ for each face $F$ of $M$, the covering $P_K = \bigcup_F V_F$ is admissible.

Let $F$ be a face of $M$. Take a section $s : M/F \to F^{-1}M$ of the natural projection $M \to M/F$ as Lemma 1.1.4 and let $M_s := M + \text{Im } s \subseteq F^{-1}M$. The morphism $\hat{k}_{M_s,V} \to \hat{k}_{F,V} \times \hat{k}_{M/F,V}$ which is defined by the inclusion $F \to M$ and $s$ is log étale since $F^{\text{gp}} \otimes (M/F)^{\text{gp}} \cong M^{\text{gp}}$. By Lemma 1.1.4, $M_s \cap F^{\text{gp}} = F$. $F$ is face of $M_s$. Indeed, if $a + b \in F$ for $a, b \in M_s$, then $a, b \in F^{\text{gp}}$ because $F^{\text{gp}}$ is a face of $F^{-1}M$, so $a, b \in M_s$. Thus there exists a natural closed immersion $\hat{k}_{F,V} \to \hat{k}_{M_s,V}$ of underlying formal schemes and $\hat{k}_{F,V} \to \hat{k}_{M_s,V} \to \hat{k}_{M/F,V}$ is the morphism to the vertex of $\hat{k}_{M/F,V}$. Let $(P_s, \mathcal{L}_s) := (P, \mathcal{L}) \times \hat{k}_{M_s,V}$ which is log étale over $(P, \mathcal{L})$ and contains $Q_F$.

Let
\[ (P^{'F}_s, \mathcal{L}'_s) := (P_s, \mathcal{L}_s) \times_{R \times \hat{k}_{F,V} \times \hat{k}_{M_s,V}} ((Q_F, \mathcal{L}_F) \times \hat{k}_{M_s,V}). \]

Note that $F^{-1}M = F^{-1}M_s \cong F^{\text{gp}} \otimes M/F$. Then the following diagram commutes:

\[
\begin{array}{ccc}
(Q_F, \mathcal{L}|_{Q_F}) & \longrightarrow & (P, \mathcal{L}) \\
\downarrow & & \downarrow \quad f \\
(Q_F, \mathcal{L}|_{Q_F}) & \longrightarrow & (P_s, \mathcal{L}_s)
\end{array}
\]

\[
\begin{array}{ccc}
(Q_F, \mathcal{L}|_{Q_F}) & \longrightarrow & (P_s, \mathcal{L}_s) \\
\downarrow & & \downarrow \quad f \\
(Q_F, \mathcal{L}|_{Q_F}) & \longrightarrow & (P'_s, \mathcal{L}'_s)
\end{array}
\]

\[
\begin{array}{ccc}
(Q_F, \mathcal{L}_{F \times M/F}) & \longrightarrow & (Q_F, \mathcal{L}_F) \times \hat{k}_{M_s,V} \\
\downarrow & & \downarrow \quad (f, \text{id}) \\
(Q_F, \mathcal{L}_{F \times M/F}) & \longrightarrow & (Q_F, \mathcal{L}_F) \times \hat{k}_{M_s,V} \times \hat{k}_{M/F,V}
\end{array}
\]

where $Q_F \rightrightarrows Q_F \times \hat{k}_{M/F,V}$ is the closed immersion to the vertex of $\hat{k}_{M/F,V}$ and $\mathcal{L}_{F \times M/F}$ is the log structure on $Q_F$ which is the pullback by this morphism.

$(P'_s, \mathcal{L}'_s) \to (P, \mathcal{L})$ and $(P'_s, \mathcal{L}'_s) \to (Q_s, \mathcal{L}_s) \times \hat{k}_{M/F,V}$ are log étale, so étale on some neighborhood of $\hat{Q}_F$. Thus, by the strong fibration lemma, there exists an isomorphism between some strict neighborhood of $\bar{Z}_F \subset |Z_F|_Q$ and some strict neighborhood of $\bar{Z}_F \subset |Z_F|_Q$.

Proposition 3.5.2. We also assume that $N \to (\mathcal{M}_0)$ is vertical for any geometric point $\bar{x}$ of $\overline{X}$. Let $\Sigma \subseteq M^{\text{gp}} \otimes \mathbb{Z}_p$ be an $(\{N \cap NL\} - D)$ subset. The category of overconvergent log isocrystals over $(X, \overline{X}, \mathcal{M}_0 \oplus \mathcal{M}) / (\text{Spf } V, N)$ which is $\Sigma$-unipotent with respect to $(P, \mathcal{L})$ is equivalent to the category of convergent log $\nabla$-modules on $P_K$ which has exponents in $\Sigma$.  

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Indeed, if the following claim (3.5.2.1) is true, for any pairs of faces $F, F'$, $V$ can take a strict neighborhood of $\tilde{V}$ because the number of faces of a strict neighborhood $W$ for any two faces $F, F'$.

Claim 3.5.2.1. For any two faces $F_1, F_2$ of $M$, for any sufficiently small strict neighborhood $V_1$ of $\tilde{F}_1$, $\tilde{F}_1 \cap F_2$ and any sufficiently small strict neighborhood $V_2$ of $\tilde{F}_2$, $\tilde{F}_2 \cap F_1$ such that the restriction of $E_F$ to $V_1 \cup V_2$ can be uniquely extended to $W_1 \cup W_2$.

Proof. It is enough to show the claim in the case of $F_1$ and $F_1 \cap F_2$ and in the case of $F_2$ and $F_1 \cap F_2$. Indeed, $F_1 \cap F_2$ is also face and $Z_{F_1 \cap F_2} = Z_{F_1} \cap Z_{F_2}$. For any strict neighborhood $V_1 \cup V_2$ of $\tilde{F}_1 \cap F_2$ and any sufficiently small strict neighborhood $V_2$ of $\tilde{F}_2$, $\tilde{F}_2 \cap F_1$ such that $V_1 \cup V_2 \subseteq V_{1, 2}$. For any sufficiently small strict neighborhood $V_1$ of $\tilde{F}_1 \cap F_2$ and any sufficiently small strict neighborhood $V_2$ of $\tilde{F}_2 \cap F_1$ such that $V_1 \cap V_2$. So for any sufficiently small strict neighborhood $V_1$ of $\tilde{F}_1 \cap F_2$ and any sufficiently small strict neighborhood $V_2$ of $\tilde{F}_2 \cap F_1$ such that $V_1 \cap V_2$ is a sufficiently small strict neighborhood of $\tilde{F}_1 \cap F_2$ and in the case of $F_2$ and $F_1 \cap F_2$. For any geometric point $\mathfrak{F}$ of $\mathcal{M}$, $\mathcal{M}_0$ is trivial on $(Q_F)_K$. By overconvergent generalization (Proposition 3.3.1), for any $b \in (a, 1) \cap \Gamma^*$, there exists a strict neighborhood $V_b \subseteq V$ of $(\tilde{Q}_F)_K$ such that the restriction of $E_F$ to $V_b \times \mathbb{A}_{M/F,K} \times (Q_F)_K \times \mathbb{A}_{M/F,K}(0, 1)$ is $\Sigma$-unipotent. By Proposition 3.3.3, this can be uniquely extended to a $\Sigma$-unipotent $\nabla$-module on $V_b \times \mathbb{A}_{M/F,K}(0, b)$. Take an increasing sequence $\tilde{b} = (b_i)_{i \in \mathbb{N}}$ such that $b_i \to 1$ when $i \to \infty$. Let $V_{\tilde{b}} := \bigcup V_{b_i} \times \mathbb{A}_{M/F,K}(0, b_i)$. Then $E_F$ can be uniquely extended to $V_{\tilde{b}}$. So $\mathcal{E}$ can be uniquely extended to some strict neighborhood $V_F$ of $\tilde{Z}_F[p]$ in $\tilde{Z}_F[p]$ and a strict neighborhood $V_F'$. So $\mathcal{E}$ can be uniquely extended to $W_F$ for all faces $F$ of $M$ can be glued together.
can be uniquely extended to $V_1 \cup (V_1 \cap V_2)$ and to $V_2 \cup (V_1 \cap V_2)$, so it can be also uniquely extended to $V_1 \cup V_2$. Hence we may assume that $F_1 \supseteq F_2$.

Take a section $s_2 : M/F_2 \to F_2^{-1}M$ of $F_2^{-1}M_2 \to M/F_2$ and a section $\pi_1 : M/F_1 \to F_1^{-1}M/F_2$ of $F_1^{-1}M/F_2 \to M/F_1$. $s_2$ can be extended to $F_1^{-1}s_2 : F_1^{-1}M/F_2 \to F_1^{-1}M$. Let $s_1 := (F_1^{-1}s_2) \circ \pi_1 : M/F_1 \to F_1^{-1}M$ which is a section of $F_1^{-1}M \to M/F_1$.

Let $M_{s_i} = \text{Im}(s_i) + M$.

$$(P_{s_i}', \mathcal{L}_{s_i}') = \left( (P, \mathcal{L}) \times \hat{\mathcal{L}}_{M_i, \mathcal{V}} \right)_{\mathcal{F}_1, \mathcal{L}_{s_i} (\mathcal{L}_{F_i})} \times \left( \left( \mathcal{L}_{F_i} \times \hat{\mathcal{L}}_{M_i} \right) \times \hat{\mathcal{L}}_{M_i} \right)$$

as above for $i = 1, 2$.

Let $s_2 := s_2|_{F_1/F_2} : F_1/F_2 \to F_2^{-1}F_1$. Let $F_1, s_2 := \text{Im}(s_2) + F_1/F_2$.

$$(Q'_{F_1, s_2}, \mathcal{L}'_{F_1, s_2}) := \left( \left( \mathcal{L}_{F_1} \times \hat{\mathcal{L}}_{F_1, s_2} \right) \times \hat{\mathcal{L}}_{F_1, s_2} \right) \times \left( \left( \mathcal{L}_{F_2} \times \hat{\mathcal{L}}_{F_2} \right) \times \hat{\mathcal{L}}_{F_2, s_2} \right).$$

Let $(M/F_2)_{s_1} = \text{Im}(\pi_1) + M/F_2$.

Then the following diagram commutes:

where $\mathcal{L}_{F_1, M/F_1}, \mathcal{L}_{F_1, s_2, M/F_1}, \mathcal{L}_{F_2, F_2/F_2, M/F_1}, \mathcal{L}_{F_2, (M/F_2)_{s_1}}$, and $\mathcal{L}_{F_2, M/F_2}$ are log structures which are defined as the pullbacks by the horizontal arrows in the above
diagram. Any sufficiently small strict neighborhood $V_1$ of $\tilde{Z}_{F_1} \left[ p \right]_{\mathbb{P}}$ in $\left| Z_{F_1} \right|_{\mathbb{P}}$ is isomorphic to some strict neighborhood $V'_1$ of $\left( \tilde{Q}_{F_1} \right)_K \times \mathbb{A}_{M/F_1,K}[0,1]$ in $\left( Q_{F_1} \right)_K \times \mathbb{A}_{M/F_1,K}[0,1]$. We may assume $V'_1$ is of the form of $\bigcup_{i \in \mathbb{N}} V_{1,i} \times \mathbb{A}_{M/F_1,K}[0,b_i]$ where $V_{1,i}$’s are strict neighborhoods of $\left( \tilde{Q}_{F_1} \right)_K$ in $\left( Q_{F_1} \right)_K$ and $b_i \to 1$ because the set of strict neighborhoods of this form forms a fundamental system of strict neighborhoods.

On the other hand, any sufficiently small strict neighborhood $V_2$ of $\tilde{Z}_{F_2} \left[ p \right]_{\mathbb{P}}$ in $\left| Z_{F_2} \right|_{\mathbb{P}}$ is isomorphic to some strict neighborhood $V'_2$ of $\left( \tilde{Q}_{F_2} \right)_K \times \mathbb{A}_{F_1/F_2,K}[0,1]$ in $\left( Q_{F_2} \right)_K \times \mathbb{A}_{M/F_1,K}[0,1]$. Likewise, we may assume $V'_2$ is of the form of $\bigcup_{i \in \mathbb{N}} V_{2,i} \times \mathbb{A}_{F_1/F_2,K}[0,b_i] \times \mathbb{A}_{M/F_1,K}[0,b_i]$ where $V_{2,i}$’s are strict neighborhoods of $\left( \tilde{Q}_{F_2} \right)_K$ in $\left( Q_{F_2} \right)_K$ and $b_i \to 1$.

Any sufficiently small strict neighborhood $V_3$ of $\tilde{Z}_{F_3} \left[ q_{F_1} \right]_{\mathbb{P}}$ in $\left| Z_{F_3} \right|_{q_{F_1}}$ is isomorphic to some strict neighborhood $V'_3$ of $\left( \tilde{Q}_{F_3} \right)_K \times \mathbb{A}_{F_1/F_2,K}[0,1]$ in $\left( Q_{F_3} \right)_K \times \mathbb{A}_{F_1/F_2,K}[0,1]$. Let $g : V_3 \xrightarrow{\sim} V'_3$ be the isomorphism. We may also assume that $V_{2,i} \times \mathbb{A}_{F_1/F_2,K}[0,b_i] \subseteq V'_3$ for each $i \in \mathbb{N}$. We have to prove that the extensions of $\mathcal{E}$ coincide on $\left( V_{1,i} \cap g^{-1}(V_{2,i} \times \mathbb{A}_{F_1/F_2,K}[0,b_i]) \right) \times \mathbb{A}_{M/F_1,K}[0,b_i]$. Fix $i$. If $b_i$ is close enough to 1, $\mathcal{E}$ defines a log $\nabla$-module on $V_{2,i} \times \mathbb{A}_{F_1/F_2,K}(a,b_i) \times \mathbb{A}_{M/F_1,K}(a,b_i)$ for some $a \in (0,b_i)$. We may assume that it is $\Sigma$-unipotent by overconvergent generization. We may assume that $g(V_{1,i} \cap g^{-1}(V_{2,i} \times \mathbb{A}_{F_1/F_2,K}[0,b_i])) \subseteq V_{2,i} \times \mathbb{A}_{F_1/F_2,K}(a,b_i)$.

By Proposition 2.5.6 the extension of $\Sigma$-unipotent $\nabla$-module on $g(V_{1,i} \cap g^{-1}(V_{2,i} \times \mathbb{A}_{F_1/F_2,K}[0,b_i])) \times \mathbb{A}_{M/F_1,K}(a,b_i)$ to $g(V_{1,i} \cap g^{-1}(V_{2,i} \times \mathbb{A}_{F_1/F_2,K}[0,b_i])) \times \mathbb{A}_{M/F_1,K}[0,b_i]$ is unique. So the extensions of $\mathcal{E}$ coincide.

Conversely, let $E_F$ be a convergent log $\nabla$-module on $P_K$ which has the exponents in $\Sigma$. Let $F$ be a face of $M$. The restriction $E_{\tilde{F}}$ of $E_F$ to $\tilde{F}$ is also a convergent log $\nabla$-module. Since $N \to (\mathcal{M}_0)_\mathbb{P}$ is vertical for any geometric point $\mathfrak{p}$ of $\mathfrak{X}$, the assumption of Lemma 3.3.3 holds. By Lemma 3.3.3 the restriction of $E_{\tilde{F}}$ to $\tilde{Z}_{F} \left[ q_{F_1} \right]_{\mathbb{P}} = \tilde{Z}_{F} \left[ q_{F_1} \right]_{\mathbb{P}} \times \mathbb{A}_{M/F,K}[0,1]$ is log-convergent and has the exponents in $\Sigma_{\tilde{F}}$. By Proposition 2.5.6 this is $\Sigma_{\tilde{F}}$-unipotent. So a convergent log $\nabla$-module on $P_K$ which has the exponents in $\Sigma$ is restricted to a $\Sigma$-unipotent overconvergent log isocrystal.

By this proposition, we have the well-definedness of $\Sigma$-unipotence.

**Proposition 3.5.3.** In Situation 3.3.1 we also assume that $N \to (\mathcal{M}_0)_\mathbb{P}$ is vertical at a geometric point $\mathfrak{p}$. Let $\Sigma \subseteq M^{sp} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ be an $((\mathfrak{N} \cap \mathfrak{NL})$-D) subset. The $\Sigma$-unipotence does not depend the choice of good charted standard small frames, i.e., for two charted standard small frames

\[ ((P_1, \mathcal{L}_1), M_0 \oplus M \to \mathcal{O}_{P_1}, \gamma_{1,1}, \ldots, \gamma_{1,d}) \]
and
\[ ((P_2, \mathcal{L}_2), M_0 \oplus M \to \mathcal{O}_{P_2}, \gamma_2, \ldots, \gamma_d) \]
which are good at \( \pi \), if an overconvergent isocrystal \( E \) on
\( (X, \overline{X}, M_0 \oplus M) / (\text{Spf } \mathcal{V}, N) \) satisfies the condition of \( \Sigma \)-unipotence with respect to the first frame, then \( E \) also satisfies the condition of \( \Sigma \)-unipotence with respect to the second frame.

**Proof.** Take \( P_3 \) and \( s \) as in the proof of Proposition \( \[3.2.2\] \). If \( E \) is \( \Sigma \)-unipotent with respect to the first frame, \( E \) is extended to a convergent log \( \nabla \)-module \( E_{P_3} \) on \( (P_1)_{\mathcal{K}} \) by Proposition \( \[3.5.2\] \). Let \( E_{P_3} = (\pi_1 \circ s)^*(E_{P_1}) \). Then it is a convergent log \( \nabla \)-module which has exponents in \( \Sigma \). Take a face \( F \) of \( M \). Let \( \Sigma_F \) be as in the definition of \( \Sigma \)-unipotence. Put
\[ Q_{2,F} := \{ p \in P_2 \mid \forall m \in M \setminus F : \beta(m)(p) = 0 \}, \]
\[ \tilde{P}_{2,F} := \{ p \in P_2 \mid \forall m \in F : \beta(m)(p) \neq 0 \}, \]
\[ \tilde{Q}_{2,F} := Q_{2,F} \cap \tilde{P}_{2,F}. \]
and \( Z_{2,F} := (Q_{2,F})_k, \tilde{Z}_{2,F} := \left( \tilde{Q}_{2,F} \right)_k \). The restriction of \( E_{P_3} \) to \( \tilde{Z}_{2,F} \) satisfies the condition of \( \Sigma \)-unipotence.

Finally, we consider the global case.

**Definition 3.5.4.** Let \( (X, M) \) be a log variety over \( (\text{Spec } k, N) \). Let \( S \subseteq \overline{M}_{gp} \otimes_k K \) be a subsheaf and \( S \subseteq K \) a subset. \( S \) is called \( (S-D) \) if for any geometric point \( \pi \) of \( X \), \( S_{\pi} \subseteq \overline{M}_{wp} \otimes_k K \) is \( (S-D) \) and there exists a good chart \( \overline{M}_{\pi} \to M \) on some neighborhood \( U \) of \( \pi \) such that \( S \) is equal to the image of \( S_{\pi} \) under the map \( \overline{M}_{\pi} \otimes_k K \to \overline{M}_{wp} \otimes_k K \) on \( U \).

**Definition 3.5.5.** In Situation \( \[3.3.1\] \) we also assume that \( N \to (M_0)_{\pi} \) is vertical at any geometric point \( \pi \). For a geometric point \( \pi \) of \( X \) and an \( ((N\cap \text{NL})-D) \) subset \( \Sigma \subseteq \overline{M}_{wp} \otimes_k Z_p \), an overconvergent log isocrystal \( E \) on \( (X, \overline{X}, M_0 \oplus M) \) is \( \Sigma \)-unipotent at \( \pi \) if \( E \) is \( \Sigma \)-unipotent with respect to charted standard small frames which are good at \( \pi \). Let \( S \subseteq \overline{M}_{wp} \otimes_k K \) be an \( ((N\cap \text{NL})-D) \) subsheaf. \( E \) is \( S \)-unipotent if, for any geometric point \( \pi \) of \( X \), \( E \) is \( S_{\pi} \)-unipotent at \( \pi \).

**Remark 3.5.6.** By Remark \( \[3.3.6\] \) if \( E \) is \( S_{\pi} \)-unipotent at \( \pi \), there exists some neighborhood \( U \) of \( \pi \) such that \( E \) is \( S_{\pi} \)-unipotent at any geometric point \( \pi \) of \( U \).

The following theorem is the main result of this paper.

**Theorem 3.5.7.** In Situation \( \[3.3.1\] \) we also assume that \( N \to (M_0)_{\pi} \) is vertical at any geometric point \( \pi \). Let \( S \subseteq \overline{M}_{wp} \otimes_k Z_p \) be an \( ((N\cap \text{NL})-D) \) subsheaf. There exists an equivalence between the category of \( S \)-unipotent overconvergent log isocrystals on \( (X, \overline{X}, M_0 \oplus M) / (\text{Spf } V, N) \) and the category of locally free convergent log isocrystals on \( (\overline{X}, M_0 \oplus M) / (\text{Spf } V, N) \) which have exponents in \( S \).
Proof. It is enough to show it étale locally, so we can assume the existence of a charted standard small frame. Then it is the consequence of Proposition 3.5.2 and Proposition 3.1.10. □

Remark 3.5.8. The definition of unipotence in this paper is stronger than that of [21], because only monodromies at codimension one components are considered in [21]. In our situation, the consideration to codimension one components is not enough.

For example, assume $p \neq 2$ and let $M := \{(a_1, a_2) \in \mathbb{N}^2 \mid a_1 + a_2 \equiv 0 \pmod{2}\}$. Then $M$ is a fine sharp semi-saturated monoid. Put $(\overline{X}, M) := \hat{a}_{M,k}$ and $(P, \mathcal{L}) := \hat{a}_{M,V}$. Let $x := t^{(2,0)}$, $y := t^{(0,2)}$ and $z := t^{(1,1)}$. The trivial locus $X$ of $M$ is $\overline{X} \setminus \{(x = 0) \cup \{y = 0\}\}$. Let $(E, \nabla)$ be a free $\nabla$-module on $P_K \setminus \{(x = 0) \cup \{y = 0\}\}$ of rank 1 such that

$$\nabla(e) = e \frac{dx}{2x},$$

where $e$ is a basis of $E$. Let $\mathcal{E}$ be an overconvergent log isocrystal on $(X, \overline{X}, M) / K$ associated to $(E, \nabla)$. Clearly $\mathcal{E}$ is $\{0\}$-constant at the points of $\{x \neq 0, y = 0\}$. Moreover, $M$ is also $\{0\}$-constant at the points of $\{x = 0, y \neq 0\}$. Indeed,

$$\nabla(z^{-1}e) = -z^{-1}e \frac{dy}{2y},$$

and $z^{-1}e$ is also a basis of $E$. But $\mathcal{E}$ is not $\{0\}$-unipotent at the vertex of $\overline{X}$.

This phenomenon is caused by the following fact on monoids: Let $F_1$ and $F_2$ be the two facets of $M$. Then $M \to M/F_1 \oplus M/F_2$ is injective. But $\Sigma$-unipotence is only dependent on the image of $\Sigma$ in $M^{gp} \otimes_{\mathbb{Z}} (\mathbb{Z}_p / \mathbb{Z})$ by Remark 3.3.7 and $M^{gp} \otimes_{\mathbb{Z}} (\mathbb{Z}_p / \mathbb{Z}) \to ((M/F_1)^{gp} \otimes_{\mathbb{Z}} (\mathbb{Z}_p / \mathbb{Z})) \oplus ((M/F_2)^{gp} \otimes_{\mathbb{Z}} (\mathbb{Z}_p / \mathbb{Z}))$ is not injective. Indeed, $(2, 0) \otimes \frac{1}{2} \in M^{gp} \otimes_{\mathbb{Z}} (\mathbb{Z}_p / \mathbb{Z})$ is a non-zero element of the kernel of this map. So the $\{0\}$-unipotence at $\hat{Z}_{F_1}$ and $\hat{Z}_{F_2}$ does not imply the $\{0\}$-unipotence at $\hat{Z}_{F_1 \cap F_2}$.

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