ON THE ŁOJASIEWICZ EXPONENT, SPECIAL DIRECTION AND MAXIMAL POLAR QUOTIENT

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Abstract

For a local singular plane curve germ \( f(X, Y) = 0 \) we characterize all nonsingular \( \lambda \in \mathbb{C}\{X, Y\} \) such that the Łojasiewicz exponent of \( \text{grad} \ f \) is not attained on the polar curve \( J(\lambda, f) = 0 \). When \( f \) is not Morse we prove that for the same \( \lambda \)'s the maximal polar quotient \( q_0(f, \lambda) \) is strictly less than its generic value \( q_0(f) \). Our main tool is the Eggers tree of singularity constructed as a decorated graph of relations between balls in the space of branches defined by using a logarithmic distance.

1 Introduction, main results

Let \( \mathbb{C}\{X, Y\} \) be the ring of convergent power series in two variables. If \( f = f_1^{m_1} \cdots f_r^{m_r} \) is a decomposition of \( f \) into irreducible pairwise coprime factors in \( \mathbb{C}\{X, Y\} \) then we put \( f_{\text{red}} = f_1 \cdots f_r \). We call \( f \) reduced if \( f = f_{\text{red}} \).

For a nonzero series \( f = \sum c_{\alpha\beta}X^\alpha Y^\beta \in \mathbb{C}\{X, Y\} \) we define the order \( \text{ord} f \) as the minimum of \( \alpha + \beta \) corresponding to nonzero \( c_{\alpha\beta} \) and the initial form \( \text{in} f = \sum_{\alpha + \beta = \text{ord} f} c_{\alpha\beta}X^\alpha Y^\beta \). We put \( \text{ord} 0 = \infty \) by convention. We call \( f \) singular if \( 2 \leq \text{ord} f < \infty \), nonsingular if \( \text{ord} f = 1 \) and a unit if \( \text{ord} f = 0 \).

For \( f, g \in \mathbb{C}\{X, Y\} \) of positive orders we say that \( f \) and \( g \) are transverse if the system \( \text{in} f = \text{in} g = 0 \) has no solutions in \( \mathbb{C}^2 \setminus \{0\} \). Otherwise we call \( f \) and \( g \) tangent. By \( t(f) = \text{ord}(\text{in} f)_{\text{red}} \) we denote the number of different tangents of \( f \). We call \( f \) unitangent if \( t(f) = 1 \) and multitangent if \( t(f) > 1 \).

Let \( f \in \mathbb{C}\{X, Y\} \) be a nonzero series without constant term. The series \( f \) defines the curve germ \( f = 0 \) at \( 0 \in \mathbb{C}^2 \). We extend the term: singular (nonsingular, unitangent, multitangent) for germs and the term: transverse (tangent) for pairs of
germs. The singularity \( f = 0 \) is isolated if and only if \( f \) is reduced. Whenever we write a “singularity” in this article we mean an “isolated singularity”.

Assume that \( f \) is reduced (\( \text{ord} f \geq 1 \)). The Lojasiewicz exponent of \( f \) with respect to a subset \( A \subset \mathbb{C}^2, 0 \in A \setminus \{0\} \), is defined to be

\[
\mathcal{L}_0(f|A) = \inf \{ \theta \geq 0 : |\text{grad} \ f(z)| \geq c|z|^\theta \text{ for } z \in A \text{ near zero in } \mathbb{C}^2, c > 0 \}. \tag{1}
\]

We write \( \mathcal{L}_0(f) \) for \( \mathcal{L}_0(f|\mathbb{C}^2) \). For nonsingular \( f \) we have \( \mathcal{L}_0(f) = 0 \). When \( \mathcal{L}_0(f) = \mathcal{L}_0(f|A) \) we say that the Lojasiewicz exponent \( \mathcal{L}_0(f) \) is attained on \( A \).

Let \( \lambda \in \mathbb{C}\{X,Y\} \) be a regular parameter (i.e. \( \lambda(0) = 0, \lambda \) nonsingular). Consider the germ \( \Gamma_{f,\lambda} \) of polar curve

\[
J(\lambda,f) = \frac{\partial \lambda}{\partial X} \frac{\partial f}{\partial Y} - \frac{\partial \lambda}{\partial Y} \frac{\partial f}{\partial X} = 0.
\]

**Definition 1.1**

(a) We define \( \lambda \) to be a special parameter for \( f \) if the Lojasiewicz exponent \( \mathcal{L}_0(f) \) is not attained on \( \Gamma_{f,\lambda} \).

(b) A direction \( w \in \mathbb{P}^1(\mathbb{C}) \) is defined to be a special direction of \( f \) if there exists a special parameter \( \lambda \) tangent to \( w \).

One of the goals of this paper is to describe all special parameters as well as all special directions of \( f \). After M. Lejeune-Jalabert and B. Teissier [26] we know that for the generic direction \( (a:b) \in \mathbb{P}^1(\mathbb{C}) \) the parameter \( \lambda = bX - aY \) is not special for \( f \). For a mapping \( (f_1,f_2): (\mathbb{C}^2,0) \to (\mathbb{C}^2,0), f_1, f_2 \in \mathbb{C}\{X,Y\} \), with isolated zero, the Lojasiewicz exponent can be defined analogously to [11]. Chądzyński and Krasiński [5] proved that this exponent is attained on \( \{f_1 = 0\} \) or on \( \{f_2 = 0\} \). This result applied to the gradient of singularity \( f = 0 \) after coordinate change can be written as

**Theorem 1.2** ([5], Main Theorem). Let \( \lambda, \mu \) be two transversal regular parameters. Then the Lojasiewicz exponent \( \mathcal{L}_0(f) \) is attained on \( \Gamma_{f,\lambda} \) or on \( \Gamma_{f,\mu} \).

**Corollary 1.3** A singularity \( f = 0 \) has at most one special direction.

The following result was obtained independently by Boguslawska [2] and by Kuo and Parusiński [21]. After coordinate change it can be written as

**Theorem 1.4** ([2], Theorem 2 and [21], Theorem 3.1). Let \( \lambda \) be a regular parameter transversal to the singularity \( f = 0 \). Then the Lojasiewicz exponent \( \mathcal{L}_0(f) \) is attained on \( \Gamma_{f,\lambda} \).

**Corollary 1.5** If the special direction of \( f = 0 \) exists it is tangent to \( f = 0 \).

We are going to consider the following problems: (1) to find the conditions for the existence of the special direction for singularity \( f = 0 \); (2) if this direction exists, to determine its position for multitangent \( f \); (3) to decide: whether or not every regular parameter tangent to the special direction is special for \( f \)? Theorem 1.6 explains (1) and (2) as well as gives a positive answer to (3). We call \( f = f^{(1)} \ldots f^{(t)} \) a tangential decomposition of \( f \) if the components \( f^{(1)}, \ldots, f^{(t)} \) are unitangent and pairwise transverse.
**Theorem 1.6** (Main Result A)

Let $f = 0$ be a singularity and let $f = f^{(1)} \ldots f^{(t)}$ be a tangential decomposition of $f$ ($t \geq 1$). Then

(i) $\mathcal{L}_0(f) = \max_{i=1,\ldots,t} (\mathcal{L}_0(f^{(i)}) + \text{ord } f - \text{ord } f^{(i)})$.

(ii) Let $\lambda$ be a regular parameter. If the maximum in (i) is realized for exactly one index $i_0 \in \{1,\ldots,t\}$ then $\lambda$ is special for $f$ if and only if $\lambda$ is tangent to $f^{(i_0)}$.

(iii) If the maximum in (i) is realized for two or more indices from $\{1,\ldots,t\}$ then there are no special parameters for $f$.

We prove (i) of Theorem 1.6 in Section 2 and (ii), (iii) in Section 4.

When all tangential components of $f$ are nonsingular, we call $f = 0$ an ordinary singularity. If additionally $\text{ord } f = 2$ then we call $f = 0$ a Morse singularity.

**Corollary 1.7** Assume that $f = 0$ is an ordinary singularity. Then for every local parameter $\lambda$

$$\mathcal{L}_0(f) = \mathcal{L}_0(f|_{\Gamma_{f,\lambda}}) = \text{ord } f - 1.$$  

**Corollary 1.8** The tangent direction of any unitangent singularity is special.

**Example 1.9** Let $f = f^{(1)} f^{(2)}$ where $f^{(1)} = Y^5 + X^2$ and $f^{(2)} = Y(Y^2 - X^4)$. By direct computation (or for example by using [27]) we obtain $\mathcal{L}_0(f^{(1)}) = 4$ and $\mathcal{L}_0(f^{(2)}) = 5$. We have $\mathcal{L}_0(f^{(i)}) + \text{ord } f - \text{ord } f^{(i)} = 7$ for $i = 1, 2$. By Theorem 1.6 $\mathcal{L}_0(f) = 7$ and the special direction does not exist.

**Remark 1.10** An interesting family of examples of singularities without special direction was proposed by Gwoździewicz (oral communication). Let $f \in \mathbb{C}\{X, Y\}$ be such that $f(X, Y) = f(Y, X)$ with the only tangents $X = 0$ and $Y = 0$. By symmetry of $f$ and both Theorems 1.2 and 1.4 we conclude that there are no special parameters. We do not need Theorem 1.6.

Let us recall some facts concerning the Łojasiewicz exponent $\mathcal{L}_0(f)$ of a holomorphic function defined by $f \in \mathbb{C}\{X_1, \ldots, X_n\}$ with an isolated singularity at zero. Let $[x]$ stand for the integer part of $x$. Lu and Chang [30] (developing the results of Kuo [19], Kuiper [18], Bochnak and Łojasiewicz [11]) proved that adding to $f$ monomials of order greater than $[\mathcal{L}_0(f)] + 1$ does not change the topological type of singularity $f = 0$. The minimal integer with this property is called the $C^0$-sufficiency degree of $f$. Teissier [33] showed that this degree equals $[\mathcal{L}_0(f)] + 1$ (Kucharz [17] found an example that the analogous equality is not true in the real case). In the same paper Teissier found a relation between the Łojasiewicz exponent and the maximal polar invariant. References to papers concerning the different kinds of the Łojasiewicz exponents can be found in [37].

In dimension two Kuo and Lu [20] described $\mathcal{L}_0(f)$ in terms of a tree model constructed on the basis of Puiseux roots of $f = 0$. Following Teissier’s result,
authors focused their attention on polar invariants and so called polar quotients. A survey of results concerning this subject in dimension two is given in [11]. We explain the notions of polar quotients and polar invariants for curve germs. For any \( f, g \in \mathbb{C}\{X, Y\} \) the intersection multiplicity \((f, g)_{0}\) is defined to be the \( \mathbb{C}\)-codimension of the ideal generated by \( f \) and \( g \) in \( \mathbb{C}\{X, Y\} \). Take an irreducible \( h \in \mathbb{C}\{X, Y\} \). We call \( h \), as well as the corresponding germ \( h = 0 \), a branch (smooth branch if \( h \) is nonsingular). The semigroup of \( h \) is

\[
\Gamma(h) = \{(h, g)_{0} : g \in \mathbb{C}\{X, Y\}, h \text{ does not divide } g\}.
\]

Now, let \( f, g \in \mathbb{C}\{X, Y\} \) be reduced series. We call two germs \( f = 0 \) and \( g = 0 \) equisingular if there exist factorizations \( f = f_{1} \ldots f_{r} \) and \( g = g_{1} \ldots g_{s} \) into branches such that \( r = s \), \( \Gamma(f_{i}) = \Gamma(g_{i}) \) for \( i = 1, \ldots, r \) and \((f_{i}, f_{j})_{0} = (g_{i}, g_{j})_{0}\) for \( i, j = 1, \ldots, r \). Equisingularity relation defines equisingularity classes in the set of germs. By an equisingularity invariant we mean a function constant in every equisingularity class.

For \( f, \lambda \in \mathbb{C}\{X, Y\} \) (\( f \) singular, \( \lambda \) nonsingular) let us consider the set of polar quotients of \( f \) with respect to parameter \( \lambda \):

\[
Q(f, \lambda) = \left\{ \frac{(f, g)_{0}}{(\lambda, g)_{0}} : g \text{ irreducible factor of } J(\lambda, f), g \neq \lambda \right\}.
\]

We define the maximal polar quotient \( q_{0}(f, \lambda) \) as \( \text{max} Q(f, \lambda) \) if \( Q(f, \lambda) \neq \emptyset \) and as \( -\infty \), otherwise (for the case \( Q(f, \lambda) = \emptyset \) see Example [12] and Remark [33]). Teissier proved that the set \( Q(f) := Q(f, \lambda) \) does not depend on sufficiently generic \( \lambda \) and that it is an equisingularity invariant of \( f \). We call \( Q(f) \) the set of polar invariants. It is always nonempty for singular \( f \). Then \( q_{0}(f) = \text{max} Q(f) \) is called the maximal polar invariant. Teissier [38] proved that

\[
\mathcal{L}_{0}(f) = q_{0}(f) - 1.
\]

Analogously, as we did for the germs, we define the equisingularity of pairs \((f, \lambda), (g, \mu)\) [16]. We consider equisingularity classes and equisingularity invariants for pairs. According to [15] we know that the set \( Q(f, \lambda) \) is an invariant in this sense. Assume that \( f, \lambda \) are transverse. In this case Ploski [36] showed \( q_{0}(f) = q_{0}(f, \lambda) \) (the equality \( Q(f) = Q(f, \lambda) \) was shown in [15]); by Theorem 1.4 and (1) we obtain \( \mathcal{L}_{0}(f) = \mathcal{L}_{0}(f|_{\Gamma f, \lambda}) = q_{0}(f, \lambda) - 1 \). In the following theorem we explain relations between these numbers for an arbitrary \( \lambda \).

**Theorem 1.11** (Main Result B)

Let \( f = 0 \) be a singularity and let \( \lambda \in \mathbb{C}\{X, Y\} \) be a regular parameter. Then:

(a) \( \mathcal{L}_{0}(f) \geq \mathcal{L}_{0}(f|_{\Gamma f, \lambda}) \geq q_{0}(f, \lambda) - 1 \).

(b) Moreover, if \( f = 0 \) is not Morse then the equalities \( \mathcal{L}_{0}(f) = \mathcal{L}_{0}(f|_{\Gamma f, \lambda}) \) and \( q_{0}(f) = q_{0}(f, \lambda) \) are satisfied for exactly the same \( \lambda \)'s.

We prove this theorem in Section 4.

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Example 1.12 Assume that \( f = XY \) and \( \lambda = X \). By direct computation we obtain \( \mathcal{L}_0(f) = \mathcal{L}_0(f|_{\Gamma_{f,\lambda}}) = 1 \) and \( q_0(f) = 2 \). But \( J(\lambda, f) = (\partial f/\partial Y) = X \). Hence \( Q(f, \lambda) = \emptyset \) and \( q_0(f) > q_0(f, \lambda) = -\infty \). This explains the assumption "\( f \) is not Morse" in Theorem 1.11(b) and later in Corollary 1.14.

Let us observe two corollaries of Theorem 1.11. The first one is straightforward.

**Corollary 1.13** (see [36], Corollary 1.4)
For a singularity \( f = 0 \) and a regular parameter \( \lambda \) we have \( q_0(f) \geq q_0(f, \lambda) \).

As a consequence we obtain

**Corollary 1.14** Let \( \lambda, \mu \) be two transversal regular parameters. Then if \( f \) is not Morse then
\[
q_0(f) = \max\{q_0(f, \lambda), q_0(f, \mu)\}.
\]

Proof. From the quoted result of Chądzyński and Krasiński we can assume that \( \mathcal{L}_0(f) = \mathcal{L}_0(f|_{\Gamma_{f,\lambda}}) \). Hence by Theorem 1.11(b) \( q_0(f) = q_0(f, \lambda) \). We finish the proof by using Corollary 1.13.

Our main tool is the Eggers tree [7, 8, 39, 40, 32] which is a decorated graph that represents the equisingularity class of a germ \( f = 0 \).

In Section 2 we propose a new construction of the Eggers tree of \( f = 0 \) by using the order of contact of Ploski [34]. We do not need Puiseux series which were used in the original construction [7, 8]. Ploski proved that the order of contact of every two branches satisfies the axioms of logarithmic distance. This distance allows us to define characteristic contacts (5) for every singular branch. We can also consider balls (every branch inside the ball is a center of this ball). We assign to the germ \( f = 0 \) the set of balls called Eggers collection (Definition 2.2). In this collection we have the balls that come from intersections of branches and the balls that come from singular branches and their characteristic contacts. The Eggers tree is a graph determined by the Eggers collection (Definition 2.6). The balls correspond to vertices of the graph. The edges correspond to inclusions of successive balls.

It is recently proved [10] that the order of contact satisfies the axioms of logarithmic distance also in positive characteristic. This suggests an application of this new construction for singularities over an arbitrary field.

As an application of the Eggers tree technique we give a recursive version of Eggers formula for polar invariants \( Q(f) \) (14). The formula for \( Q(f) \) together with (11) suffices to prove Theorem 1.6(i).

In order to describe a position of an arbitrary branch \( h \) with respect to the germ \( f = 0 \) we consider the ball \( B_f(h) \) with \( h \) as a center. The radius of \( B_f(h) \) equals the maximal order of contact of \( h \) with branches of \( f = 0 \) (see: a definition before Property 3.1).

Let \( \lambda \) be an arbitrary regular parameter (possibly a branch of the germ \( f = 0 \)). In Section 3 we give formulas for the Lojasiewicz exponent \( \mathcal{L}_0(f|_{\Gamma_{f,\lambda}}) \) (Proposition 3.5, Corollary 3.6). These formulas involve the position of \( \lambda \) as well as the positions of branches of the polar \( J(\lambda, f) \) with respect to \( f = 0 \). We show (Example 3.9) equisingular pairs \((f, \lambda), (f', \lambda')\) such that \( \mathcal{L}_0(f|_{\Gamma_{f,\lambda}}) \neq \mathcal{L}_0(f'|_{\Gamma_{f',\lambda'}}) \). Hence \( \mathcal{L}_0(f|_{\Gamma_{f,\lambda}}) \) is
not in general an equisingularity invariant of the pair \((f, \lambda)\). This example concerns
the very specific equisingularity class when \(f = 0\) is unitangent and \(B_f(\lambda)\) coincides
with the unique ball of the Eggers collection. For each different class the Lojasiewicz
exponent \(\mathcal{L}_0(f|\Gamma_{f,\lambda})\) is an invariant (see: Lemma 4.6).

In Section 4 we propose Theorem 4.3 to factorize the polar \(J(\lambda, f)\) involving
only the equisingularity information of the pair \((f, \lambda)\). If \(\lambda\) is tranversal to \(f = 0\)
then for every factor \(g\) of \(J(\lambda, f)\) the ball \(B_f(g)\) belongs to the Eggers collection.
When \(\lambda\) is tangent to \(f = 0\) the position of \(B_f(g)\) in the Eggers collection is not in
general determined by the eqisingularity class of the pair \((f, \lambda)\). This phenomenon
was observed by Kuo and Parusiński ([22], Example 8.1) for Kuo-Lu trees [20]. In
this case we assign \(g\) to the nearest successive ball in the Eggers collection. Finally,
every factor \(g\) of \(J(\lambda, f)\) (different from \(\lambda\)) is assigned to a ball \(B\) of the Eggers tree
or to the ball \(B = B_f(\lambda)\). The “packages” of \(g\)’s form factors \(h_B\) of \(J(\lambda, f)\).
In Theorem 4.3 (ii) we describe the contacts of \(g\)’s with \(\lambda\). In Theorem 4.3 (iii) we give
two formulas for \((h_B, \lambda)_0\). The first one is analogous to that from [7, 8]; the second
concerns the ball \(B_f(\lambda)\) and the balls from the Eggers collection which have \(\lambda\) as
their centre. As a consequence of Theorem 4.3 we obtain a version of the result of
Eggers (Corollary 4.5). For typical equisingularity classes, different from the class
of Example 3.9 we describe \(\mathcal{L}_0(f|\Gamma_{f,\lambda})\) in Lemma 4.6. This lemma allows us to
prove Theorem 4.6 (ii, iii). We obtain formulas for polar quotients (Proposition 4.10),
for their multiplicities (Remark 4.11) and for the maximal polar quotient \(q_0(f, \lambda)\)
(Lemma 4.12). Applying Lemmas 4.6 and 4.12 we prove Theorem 1.11.

Theorem 4.3 is a version of known results ([7, 23, 24, 8, 15, 31, 22]). We gener-
alize [7, 23, 24, 8, 15]. In [7, 8] (resp. in [23, 24]) \(\lambda\) is generic (resp. \(\lambda\) is transversal
to \(f\)) whereas in Theorem 4.3 \(\lambda\) is an arbitrary regular parameter. In comparison
to [15], where \(Q(f, \lambda)\) is described in terms of the equisingularity class of the pair
\((f, \lambda)\), we give formulas for the multiplicities of polar quotients. The paper of Maugendre [31]
covers a more general situation of jacobian quotients. For nonzero series
\(f, g \in \mathbb{C}\{X, Y\}\) without constant terms the jacobian curve \(J(f, g) = 0\) is considered.
Every branch \(h\) of \(J(f, g)\) which is not a branch of \(fg\) defines a jacobian quotient
\((f, h)_0/(g, h)_0\). Maugendre described the set of jacobian quotients in terms of the
minimal resolution of \(f \cdot g\). Applying this result with smooth \(g\) we can obtain the
set of polar quotients but without multiplicities. Kuo and Parusiński [22] consid-
ered the case when the Puiseux roots of \(fg\) are different. They constructed a tree
model \(T(f, g)\) similar to that of [20]. They described how the Puiseux roots of \(J(f, g)\)
“leave” \(T(f, g)\). This construction depends on the choice of the coordinate system.
It is possible to apply this result to prove Theorem 4.3 but it requires effort to move
from Puiseux roots to branches and to eliminate an influence of the coordinate sys-
tem. Finally, we decided to present in Section 5 a self-contained proof based on the
technique of paths of the Newton algorithm from [28].

From Theorem 4.3 it follows that the polar quotients together with their multi-
plicities are equisingularity invariants of the pair: germ, regular parameter (see [13]).
The analogous fact for jacobian pairs was recently proved in [33, 12].
2 The Eggers tree

In this section we construct the Eggers tree by using the order of contact of Ploski. We propose a recursive version of the Eggers formula for the polar invariants (17). By using the formula we prove Theorem 1.6(i).

Let us denote by $\mathcal{B}$ the set of all branches. From Ploski [34, 6] we know that for branches $f, g \in \mathcal{B}$ the order of contact

$$d(f, g) = \frac{(f, g)_{0}}{(\text{ord } f)(\text{ord } g)}$$

satisfies the axioms of logarithmic distance:

(D1) $d(f, g) = \infty$ if and only if the germs $f = 0$ and $g = 0$ coincide,

(D2) $d(f, g) = d(g, f),$

(D3) $d(f, g) \geq \min\{d(f, h), d(g, h)\}$.

Since $(f, g)_{0} \geq (\text{ord } f)(\text{ord } g)$ we have $d(f, g) \geq 1$. Moreover $d(f, g) = 1$ if and only if $f, g$ are transverse. A simple consequence of (D3) is

(D3') If $d(f, h) \neq d(g, h)$ then $d(f, g) = \min\{d(f, h), d(g, h)\}$.

Characteristic contacts

Recall that the semigroup of a branch $f$ can be written as $\Gamma(f) = \mathbb{N}\bar{\beta}_0 + \ldots + \mathbb{N}\bar{\beta}_g$ where $\bar{\beta}_0 < \ldots < \bar{\beta}_g$ is the minimal sequence of semigroup generators. We call $g = g(f)$ the number of characteristic pairs of $f$. For smooth branches we have $g = 0, \bar{\beta}_0 = 1$. For $k = 1, 2, \ldots$ we define the characteristic contacts $\bar{\beta}_k$

$$d_k(f) = \sup\{d(f, h) : h \in \mathcal{B}, g(h) < k\}.$$  \hspace{1cm} (5)

For $k > g(f)$ we have $d_k(f) = \infty$. For singular branch $f$ we have

$$d_k(f) = \frac{\text{GCD}(\bar{\beta}_0, \ldots, \bar{\beta}_{k-1})\bar{\beta}_k}{(\bar{\beta}_0)^2} \text{ for } k = 1, \ldots, g(f).$$  \hspace{1cm} (6)

We have $d_1 < \ldots < d_g$ which is equivalent to $n_k\bar{\beta}_k < \bar{\beta}_{k+1}$ ($k = 1, \ldots, g(f) - 1$). We write $\text{char}(f) = \{d_1, \ldots, d_g\}$. By $(n_1, \ldots, n_g)$ we denote the corresponding sequence $n_k := \text{GCD}(\bar{\beta}_0, \ldots, \bar{\beta}_{k-1})/\text{GCD}(\bar{\beta}_0, \ldots, \bar{\beta}_k), k = 1, \ldots, g$. We have (compare [31], Proposition 3.2)

$$d_1 \in \frac{\mathbb{N}}{n_1} \setminus \mathbb{N} \text{ and } d_k \in \left\{ \frac{\mathbb{N}}{(n_1 \ldots n_{k-1})^2n_k} \setminus \frac{\mathbb{N}}{(n_1 \ldots n_{k-1})^2} \right\}, \text{ for } k = 2, \ldots, g.$$  \hspace{1cm} (7)

Let us denote $\nu_0 = 1, \nu_1 = n_1, \ldots, \nu_g = n_1 \ldots n_g$ ($\nu_g = \text{ord } f$). The formula

$$n_k = \min\left\{ n \geq 1 : d_k \in \frac{\mathbb{N}}{\nu_k^2\nu_{k-1}^2} \right\}, \text{ for } k = 1, \ldots, g$$  \hspace{1cm} (8)

enables us to reconstruct the sequence $(n_1, \ldots, n_g)$ from $(d_1, \ldots, d_g)$.

The following classical facts are useful.
Property 2.1

1) For \( f \in \mathcal{B} \), \( R \in \mathbb{Q} \cap (1, \infty) \) there exists \( g \in \mathcal{B} \) such that \( d(f, g) = R \).

2) For \( f \in \mathcal{B} \), \( g = g(f) > 0 \) there exists a sequence of branches \( f_0, \ldots, f_{g-1} \) such that \( g(f_k) = k \) and \( d(f, f_k) = d_{k+1}(f) \) for \( k = 0, \ldots, g - 1 \).

For singular \( f \) in 2) \( f_0 \) is the classical maximal contact of Hironaka.

Balls and trees
Let \( f \in \mathcal{B} \) and let \( R \in (1, \infty) \). The set \( \mathcal{B}(f, R) = \{ g \in \mathcal{B} : d(f, g) \geq R \} \) will be referred to as the \emph{ball} with center \( f \) and radius \( R \). By using (\( D_3 \)) we can prove that every element of the ball is a center of this ball. Clearly \( \mathcal{B} = \mathcal{B}(f, 1) \) for \( f \in \mathcal{B} \).

For \( f, g \in \mathcal{B} \) we put \( \mathcal{B}(f, g) = \mathcal{B}(f, d(f, g)) \). For each ball \( \mathcal{B} \) we define the \emph{diameter} \( d(\mathcal{B}) = \inf \{ d(f, g) : f, g \in \mathcal{B} \} \) which is equal to the radius. For any two balls \( \mathcal{B}, \mathcal{B}' \) if \( \mathcal{B} \cap \mathcal{B}' \neq \emptyset \) then \( \mathcal{B} \subset \mathcal{B}' \) or \( \mathcal{B} \supset \mathcal{B}' \). We define \( \mathcal{B} \leq \mathcal{B}' \) if \( \mathcal{B} \supset \mathcal{B}' \) and \( \mathcal{B} < \mathcal{B}' \) if \( \mathcal{B} \leq \mathcal{B}' \) and \( \mathcal{B} \neq \mathcal{B}' \). Let \( f \in \mathcal{B} \) and let \( R, R' \geq 1 \). By Property 2.1 we obtain

\[ \mathcal{B}(f, R) = \mathcal{B}(f, R') \iff R = R'. \tag{9} \]

Now we want to define the \emph{Eggers} collection of a singularity. It is a finite set of balls. Let us consider a germ \( f = 0 \) and the factorization \( f = f_1 \ldots f_r \) into branches; \( r = r(f) \) is the number of branches (\( r \geq 1 \)).

**Definition 2.2 (Eggers collection)**

By the \emph{Eggers collection} of the germ \( f = 0 \) we mean the collection of balls

\[ \tilde{\mathcal{E}}(f) = \{ \mathcal{B}(f_i, f_j) \}_{i,j=1, \ldots, r} \cup \bigcup_{i=1, \ldots, r} \mathcal{B}(f_i, d_{i,k})_{k=1, \ldots, g(f_i)} \]

where \( \{ d_{i,1}, \ldots, d_{i,g(f_i)} \} \) are the characteristic contacts of singular branches.

Let us observe that the balls \( \mathcal{B}(f_i, f_i), i = 1, \ldots, r, \) of infinite diameters are in the collection. Balls of finite diameters form the \emph{truncated Eggers collection} \( \mathcal{E}(f) \). For a smooth branch \( f \) we have

\[ \tilde{\mathcal{E}}(f) = \{ \mathcal{B}(f, f) \} \text{ and } \mathcal{E}(f) = \emptyset. \]

The following proposition is a consequence of Property 2.1.

**Proposition 2.3** Let us consider a ball \( \mathcal{B} \).

(a) Then the characteristic contacts strictly less than \( d(\mathcal{B}) \) of every branch of \( \mathcal{B} \) are the same.

(b) For every \( d \geq d(\mathcal{B}) \) there exists \( f \in \mathcal{B} \) such that \( d \notin \text{char}(f) \).

This allows us to define the \emph{characteristic} of a ball \( \mathcal{B} \) as

\[ \text{char}(\mathcal{B}) = \bigcap_{f \in \mathcal{B}} \text{char}(f). \tag{10} \]
Corollary 2.4 All the elements of char$(B)$ are strictly less than $d(B)$.

Corollary 2.5 If $B$ has a smooth centre then char$(B) = \emptyset$.

Now, we want to define the number $\nu(B)$ for every ball $B$. If char$(B) = \emptyset$ then we put $\nu(B) = 1$. If char$(B) = (d_1, \ldots, d_k)$ with the corresponding sequence $(n_1, \ldots, n_k)$ then we put $\nu(B) = n_1 \ldots n_k$. In analogy to (8) we define

$$n(B) = \min \left\{ n \geq 1 : d(B) \in \frac{\mathbb{N}}{\nu(B)^2n} \right\}.$$ \hspace{1cm} (11)

We call $B$ a characteristic ball if $n(B) > 1$ and a noncharacteristic ball if $n(B) = 1$. Let $B$ be a ball and let $Z$ be a set of balls. We call $B' \in Z$ a direct successor of $B$ in $Z$ if $B < B'$ and from $B < B_1 \leq B'$, $B_1 \in Z$, it follows that $B_1 = B'$. If additionally $B \in Z$ then we call $(B, B')$ a pair of successive balls in $Z$.

Definition 2.6 (Eggers tree)

We define the Eggers tree of the germ $f = 0$ as a graph whose vertices are the balls of the Eggers collection $\bar{E}(f)$ and the edges are the pairs of successive balls $(B, B')$ in $\bar{E}(f)$. From the axioms $(D_1-D_3)$ it follows that this graph is a rooted tree where the root is the ball with the minimal diameter. Black (white) vertices are the balls of finite (infinite) diameters. We call an edge $(B, B')$ discontinuous when $d(B) \in \text{char}(B')$ and solid when $d(B) \in \text{char}(B')$. Moreover, we assign $d(B)$ as a decoration to every black vertex. By the truncated Eggers tree we mean the analogous graph constructed for the truncated Eggers collection $\bar{E}(f)$.

In what follows we denote an edge $(B, B')$ as $B < B'$. By $B_{\text{min}}(f)$ we denote the ball with the minimal diameter in $\bar{E}(f)$.

Remark 2.7 Eggers originally assigned contact exponents $c.ex.(B)$ to every black vertex of the tree. We can obtain these exponents by the following classical computation. If char$(B) = \emptyset$ then $c.ex.(B) = d(B)$. If char$(B) = (d_1, \ldots, d_k)$ with the corresponding sequence $(n_1, \ldots, n_k)$ then we have

$$c.ex.(B) = d_1 + n_1(d_2 - d_1) + \ldots + n_1 \ldots n_k(d(B) - d_k).$$ \hspace{1cm} (12)

The following simple property is useful for constructing the tree.

Property 2.8 Let $B_1 = B(f_1, R_1)$ and $B_2 = B(f_2, R_2)$. Then

$$B_1 \cap B_2 = \emptyset \iff \min(R_1, R_2) > d(f_1, f_2).$$

Corollary 2.9 For $B_1, B_2$ as above:

(a) $B_1 < B_2 \iff \left( R_1 < R_2 \text{ and } R_1 \leq d(f_1, f_2) \right),$ 

(b) $B_1 = B_2 \iff R_1 = R_2 \leq d(f_1, f_2).$
Example 2.10 Let us consider $f = (Y^5 + X^2)Y(Y^2 - X^4)$ as in Example 1.9 with four irreducible branches $f_1 = Y^5 + X^2$, $f_2 = Y$, $f_3 = Y - X^2$, $f_4 = Y + X^2$. The contacts $d(f_i, f_j)$ $(i, j = 1, \ldots, 4)$ of different branches are presented in the table

\[
\begin{array}{cccc}
1 & 2 & 2 & 1 \\
1 & 2 & 2 & 1
\end{array}
\]

By using Property 2.8 and Corollary 2.9 we recognize that the Eggers collection $\mathcal{E}(f)$ has three balls with finite diameters $B = B(f_1, f_2) = B(f_1, f_3) = B(f_1, f_4)$, $B_1 = B(f_1, 5/2)$, $B_2 = B(f_2, f_3) = B(f_2, f_4) = B(f_3, f_4)$ and four balls with infinite diameters $\{f_1\}, \{f_2\}, \{f_3\}, \{f_4\}$ that can be identified with the branches. Let us notice that only $f_1$ is singular with $\text{char}(f_1) = \{5/2\}$. Other branches are smooth. There is only one solid edge $B_1 \subseteq \{f_1\}$. All the other edges are discontinuous. The ball $B_1$ is characteristic whereas $B_2$ and $B$ are noncharacteristic.

\[
\begin{array}{c}
\{f_1\} \quad \{f_2\} \quad \{f_3\} \quad \{f_4\} \\
\{\infty\}
\end{array}
\]\

Now, let us consider an arbitrary ball $B$ and the set of branches $\mathcal{B}_f = \{f_1, \ldots, f_r\}$ of the germ $f = 0$. By $t_f(B)$ we denote the number of direct successors $\{B_1, \ldots, B_t\}$ of $B$ in $\mathcal{E}(f)$. By $t_f^{(1)}(B)$ (resp. $t_f^{(2)}(B)$) we denote the number of direct successors $B_l$, $l \in \{1, \ldots, t_f(B)\}$, that $d(B) \notin \text{char}(B_l)$ (resp. $d(B) \in \text{char}(B_l)$). Clearly, $t_f(B) = t_f^{(1)}(B) + t_f^{(2)}(B)$. If $B \in \mathcal{E}(f)$ then $t_f^{(1)}(B)$ (resp. $t_f^{(2)}(B)$) equals the number of discontinuous (resp. solid) edges that leave $B$. We have

**Proposition 2.11**

Let $f_i \sim f_j \Leftrightarrow d(f_i, f_j) > d(B)$ be a relation in the set $\mathcal{B}_f \cap B$. Then

(a) it is an equivalency relation,

(b) $t_f(B)$ equals the number of equivalency classes of the relation in $\mathcal{B}_f \cap B$,

(c) if $B$ is characteristic then $t_f^{(1)}(B) \in \{0, 1\}$.

Proof. (a) is a direct consequence of the axioms ($D_1$-$D_3$). To prove (b) we first observe that the number of direct successors of $B$ does not change when we substitute $\mathcal{E}(f)$ by $\mathcal{E}_{\text{int}}(f) := \{B(f_i, f_j)\}_{i,j=1,\ldots,r}$. Then we use the axioms. To prove (c) let us consider a characteristic ball $B$. It suffices to show that if $f, g \in B$ and $d(B) \notin \text{char}(f)$, $d(B) \notin \text{char}(g)$ then $d(f, g) > d(B)$. By Property 2.11 we choose $f', g' \in \mathcal{B}$ such that
\( \text{char}(f') = \text{char}(g') \) and \( d(f, f') > d(B) \), \( d(g, g') > d(B) \). Clearly \( \text{ord} f' = \text{ord} g' = \nu(B) \). Since \( d(f, g) \geq d(B) \) then \( d(f', g') \geq d(B) \). We have \( d(f', g') \in \mathbb{N}/\nu(B)^2 \) whereas \( d(B) \in \mathbb{N}/(\nu(B)^2n(B)) \setminus \mathbb{N}/\nu(B)^2 \) by (7). Hence \( d(f', g') > d(B) \) and \( d(f, g) \geq \min\{d(f, f'), d(f', g'), d(g', g)\} > d(B) \) ■

Let us observe that if \( B < B' \) then by (10) \( \text{char}(B) \subset \text{char}(B') \).

**Property 2.12** Let \( B < B' \) be an edge. Then

(a) the edge is discontinuous if and only if \( \text{char}(B') = \text{char}(B) \).

(b) For solid edge \( \text{char}(B') = \text{char}(B) \cup \{d(B)\} \).

By a *chain* in the Eggers collection (tree) we mean an increasing sequence of successive balls (vertices).

**Remark 2.13** The equisingularity class of a singularity can be reconstructed from its Eggers tree. The branches correspond to white vertices. In order to recognize the characteristic of a branch we consider the chain that joins the minimal vertex with the corresponding white vertex and we apply Property 2.12. The contact between branches \( f_i, f_j \) is

\[
d(f_i, f_j) = \max\{d(B) : B \in \mathcal{E}(f), B \leq \{f_i\}, B \leq \{f_j\}\}.\]

**Tangential decomposition**

Consider the germ \( f = 0 \) with branches \( \mathcal{B}_f = \{f_1, \ldots, f_r\} \). Applying Proposition 2.11(a,b) with \( B = \mathcal{B} \) we divide \( \mathcal{B}_f \) due to the equivalency relation \( d(f_i, f_j) > 1 \).

When we multiply the branches inside each class we obtain a tangential decomposition \( f = f^{(1)} \ldots f^{(t)} \) (as in Introduction) where \( t = t(f) \) is the number of tangents of the germ. The following property follows directly from Definition 2.2.

**Property 2.14** If \( t = t(f) > 1 \) then \( \mathcal{E}(f) = \mathcal{B} \cup \mathcal{E}(f^{(1)}) \cup \ldots \cup \mathcal{E}(f^{(t)}) \).

**Orders, polar invariants and multiplicities of balls**

For an arbitrary ball \( B \) and the germ \( f = 0 \) with branches \( \mathcal{B}_f = \{f_1, \ldots, f_r\} \) we define the *order* \( O_f(B) = \sum_i \text{ord} f_i \) where the summation runs over \( f_i \in B \). It is convenient to define the *family of balls determined by* \( f \):

\[
\mathcal{T}(f) = \{B \text{ ball } : O_f(B) > 0\}. \tag{13}
\]

We have \( \mathcal{E}(f) \subset \mathcal{T}(f) \). We write \( \mathcal{T}(f) \) when we omit the balls with infinite diameters. We say that a ball \( B \in \mathcal{T}(f) \) *lies on the edge* \( B_1 < B_2 \) (\( B_1, B_2 \in \mathcal{E}(f) \)) if \( B_1 < B \leq B_2 \).

We define the pair \((\mathcal{B}, B_{\text{min}}(f))\) to be the *trunk* of \( f \). We say that \( B \) lies on the trunk if \( \mathcal{B} \leq B \leq B_{\text{min}}(f) \). The family \( \mathcal{T}(f) \) contains exactly these balls that lie on the edges or on the trunk. Let us observe that the function \( B \mapsto O_f(B) \) is constant for balls \( B \) lying on the one edge or on the trunk. This function has “jumps” only for \( B \in \mathcal{E}_{\text{int}}(f) \). Let us observe that \( O_f(B) = \text{ord} f \) for balls lying on the trunk.

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For every ball $B \in \mathcal{T}(f)$ we define the number $q_f(B)$. First we define $q_f(B)$ for $B \in \mathcal{E}(f)$. We consider the unique chain $B_{\min}(f) = B_1 < B_2 < \ldots < B_t = B$ and we put

$$q_f(B_k) = \begin{cases} O_f(B_1)d(B_1), & k = 1 \\ q_f(B_{k-1}) + O_f(B_k)(d(B_k) - d(B_{k-1})), & 1 < k \leq t. \end{cases} \quad (14)$$

Then we use a linear interpolation due to $d(B)$ to define $q_f(B)$ for every $B \in \mathcal{T}(f)$ (i.e. if $B$ lies on the edge $B_1 < B_2$ then $q_f(B) = q_f(B_1) + O_f(B)(d(B) - d(B_1))$, clearly $O_f(B) = O_f(B_2)$). We always have $q_f(B) = \text{ord } f$. Let us observe that

$$\text{for } B, B' \in \mathcal{T}(f) \text{ if } B < B' \text{ then } q_f(B) < q_f(B'). \quad (15)$$

To every ball $B \in \mathcal{T}(f)$ we assign the number

$$m_f(B) = \nu(B)(t_f(1)(B) + n(B)t_f(2)(B) - 1) \quad (16)$$

Let us observe that $m_f(B)$ is positive if and only if $B \in \mathcal{E}(f)$.

Eggers [7, 8] proved that

$$Q(f) = \{q_f(B) : B \in \mathcal{E}(f)\} . \quad (17)$$

Because of a difference in approach the analogous formulas of Eggers have a different form. We reprove the result of Eggers in Corollary 4.5. Eggers also obtained the “multiplicities” of $q \in Q(f)$ as the sum of $m_f(B)$ over balls $B \in \mathcal{E}(f)$ that lead to the same value of $q$ (see: Remark 4.11).

Returning to Example 2.10 we obtain $O_f(B) = \text{ord } f = 5$, $O_f(B_1) = 2$, $O_f(B_2) = 3$, $qf(B) = O_f(B)d(B) = 5$, $q_f(B_1) = q_f(B) + (d(B_1) - d(B))O_f(B_1) = 8$, $q_f(B_2) = q(B) + (d(B_2) - d(B))O_f(B_2) = 8$, $m_f(B) = 1$, $m_f(B_1) = 1$, $m_f(B_2) = 2$.

We can apply the formulas of $Q(f)$ to compute the Lojasiewicz exponent. From (11) we have

$$L_0(f) = \max_{B \in \mathcal{E}(f)} (q_f(B) - 1) . \quad (18)$$

In Example 2.10 we obtain $L_0(f) = 7$ (as earlier). Now, we can prove part (i) of the Main Result.

**Proof of Theorem 1.6 (i)**

For any unitangent $f$, the formula is obvious. Assume that $f$ is multitangent. Then $B \in \mathcal{E}(f)$. If $f = 0$ is an ordinary singularity then $\mathcal{E}(f) = \{B\}$ and the formula is straightforward. Let $f^{(1)}, \ldots, f^{(s)}$ be all the singular unitangent components ($s \geq 1$).

We have $\mathcal{E}(f) = \{B\} \cup \mathcal{E}(f^{(1)}) \cup \ldots \cup \mathcal{E}(f^{(s)})$ by Property 2.14. Let us observe that $q_f(B) = \text{ord } f$ and $q_f(B) > \text{ord } f$ for $B \in \mathcal{E}(f) \setminus \{B\}$ by (15). Therefore, we can omit $B$ in (13). Observing that $q_f(B) = q_{f^{(i)}}(B) + \text{ord } f - \text{ord } f^{(i)}$ for every $B \in \mathcal{E}(f^{(i)})$, $i = 1, \ldots, s$, we obtain

$$L_0(f) = \max_{i=1}^s \max_{B \in \mathcal{E}(f^{(i)})} (q_{f^{(i)}}(B) - 1 + \text{ord } f - \text{ord } f^{(i)}) \quad (19)$$

$$= \max_{i=1}^s (L_0(f^{(i)}) + \text{ord } f - \text{ord } f^{(i)}) \quad (20)$$

and clearly we can take the last maximum over $i = 1, \ldots, t$. 

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3 Polar quotients and Łojasiewicz exponent

Let us consider a singularity \( f = 0 \) and a regular parameter \( \lambda \). In this section we give a formula for the maximal polar quotient \( q_0(f, \lambda) \) (Corollary 3.4) and the formula for the Łojasiewicz exponent \( \ell_0(f|\Gamma_{f,\lambda}) \) (Corollary 3.6). In both formulas we use the Eggers collection extended by the balls encoding the positions of branches of the polar \( J(\lambda, f) = 0 \) with respect to the singularity \( f = 0 \). We give three examples concerning the inequalities from Theorem 1.11 (a). The most important is Example 3.9. It shows that there exists a specific equisingularity class of the pair \((f, \lambda)\) such that the Łojasiewicz exponent \( \ell_0(f|\Gamma_{f,\lambda}) \) with respect to the polar curve is not an equisingularity invariant inside this class (Remark 3.10).

Position of a branch with respect to a germ

We need to describe a position of a branch \( h \in B \) with respect to a germ \( f = 0 \) by using equisingularity information of the pair \((f, h)\). To this end we consider the chain \( K_f(h) = \{ B(f_1, h), \ldots, B(f_r, h) \} \) (we write \( K_f(h) \) when we omit the ball with infinite diameter). Let us denote \( B_f(h) = \max K_f(h) \). We have \( d(B_f(h)) < \infty \) if and only if \( h \) is not a branch of \( f \). By using \((D_3)\) we obtain

**Property 3.1** \( \bar{E}(f) \cup K_f(h) = \bar{E}(f) \cup \{ B_f(h) \} \).

**Property 3.2** If \( h \) is not a branch of \( f \) then

\[
\frac{(f, h)_0}{\ord h} = q_f(B_f(h)).
\]

Polar quotients

Now, let us consider the factorizations

\[ f = \lambda^\delta \bar{f} \quad \text{and} \quad J(\lambda, f) = \lambda^\delta g_1 \ldots g_u \quad (21) \]

where \( \lambda \) does not divide \( \bar{f} \) and \( g_1, \ldots, g_u \) are irreducible factors of \( J(\lambda, f) \) different from \( \lambda \). It is important that in both formulas we have the same \( \delta \). We will denote this number by \( \delta_\lambda(f) \). Since \( f \) is reduced \( \delta_\lambda(f) \in \{0, 1\} \).

**Remark 3.3** Since

\[
\sum_{j=1}^u (g_j, \lambda)_0 = (\bar{f}, \lambda)_0 - 1, \quad (22)
\]

the condition \( Q(f, \lambda) = \emptyset \) is equivalent to \( (\bar{f}, \lambda)_0 = 1 \). This means that \( \bar{f} \) is a smooth branch which is transverse to \( \lambda \). Hence \( f = 0 \) is a Morse singularity with \( \lambda \) as a branch.

**Corollary 3.4**

\[ (a) \quad Q(f, \lambda) = \left\{ \frac{q_f(B_f(g_j))}{d(g_j, \lambda)} : j = 1, \ldots, u \right\}. \]
(b) If \((\tilde{f}, \lambda)_0 > 1\) then \(q_0(f, \lambda) = \max_{j=1, \ldots, u} \frac{q_f(B_f(g_j))}{d(g_j, \lambda)}\).

Proof. We apply Property 3.2 to (3) and we use
\[
\frac{(f, g_j)_0}{(\lambda, g_j)_0} = \frac{q_f(B_f(g_j))}{d(g_j, \lambda)}, \quad j = 1, \ldots, u.
\]

In Corollary 3.6 we obtain analogous formulas for \(\mathcal{L}_0(f|\Gamma_{f, \lambda})\).

The Łojasiewicz exponent with respect to the polar curve

In the following proposition we use a natural extension of the intersection multiplicity to quotients of series.

**Proposition 3.5** Let us consider an isolated singularity \(f = 0\) and a regular parameter \(\lambda\). Then
\[
\mathcal{L}_0(f|\Gamma_{f, \lambda}) = \max_h \frac{(\frac{\partial f}{\partial X}, h)_0}{\text{ord } h}
\]
where \(h\) runs over irreducible factors of the polar \(J(\lambda, f)\).

Proof. We apply the formula from [26]:
\[
\mathcal{L}_0(f|\Gamma_{f, \lambda}) = \max_{\gamma} \frac{\text{ord } ((\text{grad } f) \circ \gamma)}{\text{ord } \gamma}
\]
where \(\gamma(T) \in \mathbb{C}\{T\}^2, \gamma(0) = 0 \in \mathbb{C}^2\), runs over a finite set of analytic arcs that parametrize the branches of \(\Gamma_{f, \lambda}\). We can assume that \(\lambda = X\). Then \(J(\lambda, f) = \partial f/\partial Y\). By (23) we have
\[
\mathcal{L}_0(f|\partial f/\partial Y = 0) = \max_h \min \left\{ \frac{(\frac{\partial f}{\partial X}, h)_0}{\text{ord } h}, \frac{(\frac{\partial f}{\partial Y}, h)_0}{\text{ord } h} \right\} = \max_h \frac{(\frac{\partial f}{\partial X}, h)_0}{\text{ord } h}
\]
where \(h\) runs over all branches of \(\partial f/\partial Y\). Let us write \(f = X^{\delta} \tilde{f}\) where \(\delta = \delta_X(f)\). If \(\delta = 1\) then \(h = X\) is a branch of \(\partial f/\partial Y\). In this case
\[
\frac{(\frac{\partial f}{\partial X}, h)_0}{\text{ord } h} = \frac{\left( \tilde{f} + X \frac{\partial \tilde{f}}{\partial X}, X \right)_0}{\text{ord } X} = \frac{(\frac{f}{X}, h)_0}{\text{ord } h}.
\]

When \(h \neq X\) we finish by using a parametrization of \(h\) of the type \(\gamma(T) = (T^N, z(T)) \in \mathbb{C}\{T\}^2, \gamma(0) = 0\) (see [35]).

When \(\lambda\) divides \(f\) we define \(\tilde{B} = \max\{B(\lambda, f_i) : i = 1, \ldots, r, f_i \neq \lambda\}\).

**Corollary 3.6** With notation from [21] the number \(\mathcal{L}_0(f|\Gamma_{f, \lambda})\) equals

1. \(\max_{j=1, \ldots, u} (q_f(B_f(g_j)) - d(g_j, \lambda))\) if \(\delta_\lambda(f) = 0\),
2. \(\max \left\{ q_f(\tilde{B}) - d(\tilde{B}), \max_{j=1, \ldots, u} (q_f(B_f(g_j)) - d(g_j, \lambda)) \right\}\) if \(\delta_\lambda(f) = 1\) and \((\tilde{f}, \lambda)_0 > 1\),
(3) $q_f(\tilde{B}) - d(\tilde{B})$ if $\delta_\lambda(f) = 1$ and $(\bar{f}, \lambda)_0 = 1$.

Proof. By Proposition 3.5 and Property 3.2 for $h = g_j$, $j \in \{1, \ldots, u\}$ we obtain

$$\frac{(f, g_j)_0}{\text{ord } g_j} = (f, g_j)_0 - (\lambda, g_j)_0 = q_f(B_f(g_j)) - d(g_j, \lambda).$$

If $h = \lambda$ then

$$\frac{(f, \lambda)_0}{\text{ord } \lambda} = (\bar{f}, \lambda)_0 = q_f(\tilde{B}) - d(\tilde{B}).$$

Examples

In the first example we illustrate formulas from Corollaries 3.4 and 3.6. We consider irreducible factors of the type

$$aX^p + bY^q + \sum_{\alpha \beta} c_{\alpha\beta}X^\alpha Y^\beta, \quad ab \neq 0, \quad \text{GCD}(p, q) = 1.$$  

We write shorter $aX^p + bY^q + \ldots$.

Example 3.7 Let

$$f = Y^7 + XY^4 + X^2Y^2 - 2X^3 = (Y^2 - X + \ldots)(Y^2 + 2X + \ldots)(Y^3 + X + \ldots)$$

and $\lambda = X$. Then

$$J(\lambda, f) = \frac{\partial f}{\partial Y} = 7Y^6 + 4XY^3 + 2X^2Y = Y(Y^2 + \frac{1}{2}X + \ldots)(7Y^3 + 4X + \ldots).$$

Let us denote by $f_1, f_2, f_3$ the branches of $f = 0$ and by $g_1, g_2, g_3$ the branches of $J(\lambda, f) = 0$, respectively. The collection $\mathcal{E}(f)$ has the only one ball $B_1 = B(f_1, f_2) = B(f_1, f_3) = B(f_2, f_3)$. From (18) we have $\mathcal{L}_0(f) = q_f(B_1) - 1 = 5$. We consider the extended collection

$$\mathcal{\tilde{E}}(f) \cup \{B_f(\lambda), B_f(g_1), B_f(g_2), B_f(g_3)\}$$

and its graphical representation.

We denote the position of $B_f(\lambda)$ by an arrow and the positions of $B_f(g_j)$ by coils. We have $q_f(B_f(g_1)) = 3$, $q_f(B_f(g_2)) = 6$, $q_f(B_f(g_3)) = 7$, $d(g_1, \lambda) = 1$, $d(g_2, \lambda) = 2$, $d(g_3, \lambda) = 3$. By Corollary 3.6 $\mathcal{L}_0(f|\Gamma_{f,\lambda}) = 4$ and by Corollary 3.3 $q_0(f, \lambda) = 3$. Both inequalities from Theorem 1.11 (a) are strict.
In the following example \( \lambda \) is a branch of \( f \).

**Example 3.8** Let us consider \( f = f_1 f_2 = X(Y^2 + X), \lambda = X \). We have \( \mathcal{E}(f) = \{ \overline{B} \} \) where \( \overline{B} = B(f_1, f_2), q_f(\overline{B}) = 4 \). Hence \( \mathcal{L}_0(f) = q_f(\overline{B}) - 1 = 3 \). We have \( J(\lambda, f) = \partial f/\partial Y = 2XY = \lambda g \). Since \( \lambda \) is a branch of \( J(\lambda, f) \) we denote it by a coil arrow.

We have \( q_f(B_f(g)) = 2, d(g, \lambda) = 1, d(\overline{B}) = 2 \). By Corollary 3.6 (2) \( \mathcal{L}_0(f|\Gamma_{f,\lambda}) = \max\{q_f(\overline{B}) - d(\overline{B}), q_f(B_f(g)) - d(g, \lambda)\} = \max\{2, 1\} = 2 \) and by Corollary 3.4 \( q_0(f, \lambda) = 2 \). Let us notice that here \( \mathcal{L}_0(f|\Gamma_{f,\lambda}) = \mathcal{L}_0(f|\{\lambda = 0\}) \).

The following example shows that the position of \( B_f(g) \) is not determined (in general) by the equisingularity class of \((f, \lambda)\) (compare [22], Example 8.1). This phenomenon enable us to find equisingular pairs \((f, \lambda), (f', \lambda')\) such that \( \mathcal{L}_0(f|\Gamma_{f,\lambda}) \neq \mathcal{L}_0(f'|\Gamma_{f',\lambda'}) \).

**Example 3.9** Let us consider \( f = f_1 f_2 = Y^4 - X^2 \) and \( f' = f'_1 f'_2 = Y^4 - X^2 + X^2Y \). We put \( \lambda = \lambda' = X \). We have \( J(\lambda, f) = \partial f/\partial Y = 4Y^3 \) and \( J(\lambda', f') = \partial f'/\partial Y = 4Y^3 + X^2 \). By Corollary 3.6 \( \mathcal{L}_0(f|\Gamma_{f,\lambda}) = 1 \) whereas \( \mathcal{L}_0(f'|\Gamma_{f',\lambda'}) = 3/2 \).

We have \( \mathcal{L}_0(f) = \mathcal{L}_0(f') = 3 \) and \( q_0(f, \lambda) = q_0(f', \lambda') = 2 \).

**Remark 3.10** The equisingularity class in the above example is very specific. For the pair \((f, \lambda)\) it can be written as \( t(f) = 1 \) and \( \mathcal{E}(f) = \{B_f(\lambda)\} \). As we will see in Lemma 4.6 for every different equisingularity class the Łojasiewicz exponent \( \mathcal{L}_0(f|\Gamma_{f,\lambda}) \) is an invariant.
4 Factorization of polar curve

We consider a singular germ $f = 0$, a regular parameter $\lambda$ and a factorization $J(\lambda, f) = \lambda^\delta g_1 \ldots g_u$, $\delta = \delta_\lambda(f)$, as in $[21]$. In this section we present Theorem 4.3 in which every $g_j$ ($j = 1, \ldots, u$) is assigned to a ball $B \in \mathcal{E}(f) \cup \{B_f(\lambda)\}$ of finite diameter. This assignment corresponds to a partition $1, \ldots, u = \bigcup_B J_B$. In this section we present Theorem 4.3 in which every $g_j$ ($j = 1, \ldots, u$) is assigned to a ball $B \in \mathcal{E}(f) \cup \{B_f(\lambda)\}$ of finite diameter. This assignment corresponds to a partition $1, \ldots, u = \bigcup_B J_B$. By putting $h_B = \prod_{j \in J_B} g_j$ we result in the factorization $J(\lambda, f) = \lambda^\delta \prod_B h_B$. For $\lambda$ transversal to $f$ we obtain a version of the result of Eggers (Corollary 4.5). Then we describe $\mathcal{E}_0(f|\Gamma_f, \lambda)$ in terms of the equisingularity class of pair $f, \lambda$ (Lemma 4.6) and we prove parts (ii,iii) of Main Result A (Theorem 1.6). Next, we compute the polar quotients $Q(f, \lambda)$ (Proposition 4.10) and their multiplicities (Remark 4.11). We describe the maximal polar quotient $q_0(f, \lambda)$ (Lemma 4.12). By using Lemmas 4.6 and 4.12 we prove Main Result B (Theorem 1.11).

Contact of two branches with respect to a germ

Let $f_1, \ldots, f_r \in \mathcal{B}$ be branches of the germ $f = 0$. For any $g, h \in \mathcal{B}$ by $(D_3)$ we obtain

$$d(g, h) \geq \max_{i=1,\ldots,r} \min \{d(f_i, g), d(f_i, h)\}.$$  \hspace{1cm} (24)

We say that the contact between branches $g$ and $h$ is determined by their positions with respect to $f = 0$ when we have the equality in (24). We denote the right side of (24) by $d_f(g, h)$.

For a ball $B \subset \mathcal{B}$ and a branch $h \in \mathcal{B}$ we define

$$d(B, h) = \inf \{d(h', h) : h' \in B\}.$$ \hspace{1cm} (25)

**Property 4.1** If $B = B(g, R)$ then $d(B, h) = \min \{d(g, h), R\}$.

By using $(D_3)$ we obtain

**Proposition 4.2** $d_f(g, h) = d(B_f(g), h) = d(B_f(h), g)$.

**Factorization theorem**

The role of balls from the chain $\overline{K}_f(\lambda)$ is specific. In order to recognize the minimal and the maximal ball in this chain we define characteristic functions

$$\sigma_{f,\lambda}^{\min}(B) = \begin{cases} 1 & \text{if } B = \min \overline{K}_f(\lambda) \\ 0 & \text{otherwise} \end{cases} \hspace{1cm} \sigma_{f,\lambda}^{\max}(B) = \begin{cases} 1 & \text{if } B = B_f(\lambda) \\ 0 & \text{otherwise} \end{cases}.$$  

**Theorem 4.3** Let $f = 0$ be a singular germ and let $\lambda$ be a regular parameter. There exists a factorization $J(\lambda, f) = \lambda^\delta \prod_B h_B$ where $\delta = \delta_\lambda(f)$ and $B$ runs over all the balls from $\mathcal{E}(f) \cup \{B_f(\lambda)\}$ with finite diameters such that

$$d(B_f(\lambda)) = \infty \iff \delta_\lambda(f) > 0$$  

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(i) if $g$ is a branch of $h_B$ then $B_f(g) = B$ for $B \neq \min K_f(\lambda)$ and $B_f(g) \leq B$ for $B = \min K_f(\lambda)$,

(ii) $d(g, \lambda) = d_f(g, \lambda)$,

(iii) the number $(h_B, \lambda)_0$ equals:

1. $d(B, \lambda)\nu(B)[t_f^{(1)}(B) + n(B)t_f^{(2)}(B) - 1]$ for $B \in \mathcal{E}(f) \setminus K_f(\lambda)$;

2. $d(B)n(B)[t_f(B) - 1 + \sigma_{f,\lambda}^{\max}(B)] - \sigma_{f,\lambda}^{\min}(B)$ for $B \in K_f(\lambda)$.

We prove this theorem in Section 5. We denote the number described in (iii) by $m_{f,\lambda}(B)$. From part (iii) of Theorem 4.3 we obtain

**Corollary 4.4** Let $B \in \mathcal{E}(f) \cup \{B_f(\lambda)\}$, $d(B) < \infty$. Then $m_{f,\lambda}(B) = 0$ if and only if $d(B) = 1$ and one of the following conditions holds

(a) $\lambda$ is transversal to $f$ and $t(f) = 1$,

(b) $\lambda$ is tangent to $f$ and $t(f) = 2$.

For $\lambda$ transversal to $f$ we obtain

**Corollary 4.5** Let $f = 0$ be a singular germ and let $\lambda$ be a regular parameter transversal to $f$. Then there exists a factorization $J(\lambda, f) = \prod_{B \in \mathcal{E}(f)} h_B$, such that if $g$ is a branch of $h_B$ then $d(g, \lambda) = 1$ and $(f, g)_0/(\text{ord } g) = q_f(B)$. Moreover

$$\text{ord } h_B = \nu(B)[t_f^{(1)}(B) + n(B)t_f^{(2)}(B) - 1].$$  \hspace{1cm} (26)

Proof. We have $\min K_f(\lambda) = B_f(\lambda) = B$ and $\delta_f(f) = 0$. The factorization from Theorem 4.3 has the form $J(\lambda, f) = \prod_{B \in \mathcal{E}(f)} h_B$. We can omit the ball $B_f(\lambda)$: if $t(f) = 1$ then $m_{f,\lambda}(B_f(\lambda)) = 0$ by Corollary 4.4 if $t(f) > 1$ then $B_f(\lambda) \in \mathcal{E}(f)$. Let $B \in \mathcal{E}(f)$ and let $g$ be an irreducible factor of $h_B$. Since the ball $\min K_f(\lambda)$ has the minimal possible diameter, we have $B_f(g) = B$. Let us write $B = B(f_i, R), i \in \{1, \ldots, r\}$. By Property 4.1 we have $d(B, \lambda) = \min\{d(f_i, \lambda), R\} = 1$. From Theorem 4.3 (ii) and Proposition 4.2 we obtain $d(g, \lambda) = d_f(g, \lambda) = d(B_f(g), \lambda) = d(B, \lambda) = 1$. Hence $(h_B, \lambda) = \text{ord } h_B$. The equality $(f, g)_0/(\text{ord } g) = q_f(B)$ follows directly from Property 3.2. To check (26) let us observe that $K_f(\lambda) = \{B_f(\lambda)\} = \{B\}$. Therefore, if $B \in K_f(\lambda)$ then $\sigma_{f,\lambda}^{\max}(B) = \sigma_{f,\lambda}^{\min}(B) = 1$, $d(B, \lambda) = d(B) = 1$ and $\nu(B) = n(B) = 1$. In this case both formulas from Theorem 4.3 (iii) coincide.

**Consequences for the Łojasiewicz exponent**

Let us observe that for $B_1, B_2 \in \mathcal{E}(f) \cup \{B_f(\lambda)\}$ of finite diameters such that $B_1 < B_2$ we have

$$q_f(B_1) - d(B_1, \lambda) \leq q_f(B_2) - d(B_2, \lambda).$$  \hspace{1cm} (27)
Lemma 4.6 Let $f = 0$ be a singular germ and let $\lambda$ be a regular parameter. Then

$$L_0(f|\Gamma_{f,\lambda}) \leq \max_{B \in \mathcal{E}(f)} (q_f(B) - d(B, \lambda)).$$

If $t(f) \neq 1$ or $\mathcal{E}(f) \neq \{B_f(\lambda)\}$ then the equality holds.

Proof. Let us assume first that $\lambda$ is not a branch of $f$. Let us denote by $L_1$ the number from Corollary 3.6 (1). We want to show

$$L_1 \leq \max_{B \in \mathcal{E}(f)} (q_f(B) - d(B, \lambda)). \quad (28)$$

with equality when $t(f) \neq 1$ or $\mathcal{E}(f) \neq \{B_f(\lambda)\}$. In order to prove (28) let us choose a branch $g_j$ of $J(\lambda, f)$ as in the beginning of this section. It suffices to find a ball $B' \in \mathcal{E}(f)$ such that

$$q_f(B_f(g_j)) - d(g_j, \lambda) \leq q_f(B') - d(B', \lambda). \quad (29)$$

Let us choose $B \in \mathcal{E}(f) \cup \{B_f(\lambda)\}$ such that $g_j$ is a factor of $h_B$ from Theorem 4.3. If $B \in \mathcal{E}(f)$ then we put $B' = B$ and we show (29) by using parts (i),(ii) of Theorem 4.3, Proposition 1.2 and (27). If $B = B_f(\lambda) \notin \mathcal{E}(f)$ then we define $\mathcal{Z}^* = \{B \in \mathcal{E}(f) : B_f(\lambda) < B\}$. When $\mathcal{Z}^*$ is nonempty we choose $B' \in \mathcal{Z}^*$ and we obtain (29) as earlier.

When $\mathcal{Z}^* = \emptyset$ we define $\mathcal{Z}_s = \{B \in \mathcal{E}(f) : B < B_f(\lambda)\}$. Since $\mathcal{E}(f) \neq \emptyset$ therefore $\mathcal{Z}_s \neq \emptyset$. We put $B' = \max \mathcal{Z}_s$. In this case $B \neq \min \overline{K}_f(\lambda)$. Therefore $B_f(g_j) = B$.

Since $\text{char}(B) \subset \text{char}(\lambda) = \emptyset$ and $\mathcal{Z}^* = \emptyset$ we have $O_f(B') = O_f(B) = 1$. Therefore, in this case we even obtain the equality in (29).

In order to prove “$\geq$” in (28) let us assume that $t(f) \neq 1$ or $\mathcal{E}(f) \neq \{B_f(\lambda)\}$. For any $B \in \mathcal{E}(f)$ it suffices to find a branch $g_j$ of $J(\lambda, f)$ such that

$$q_f(B_f(g_j)) - d(g_j, \lambda) \geq q_f(B) - d(B, \lambda). \quad (30)$$

If $B \neq \min \overline{K}_f(\lambda)$ then $d(B) > 1$ and by Corollary 4.4 we have $(h_B, \lambda)_0 > 0$. We choose a branch $g_j$ of $h_B$ and we obtain (30) (even equality) as earlier. Assume that $B = \min \overline{K}_f(\lambda)$. If $t(f) > 2$ or $\lambda$ is transversal to $f$ then $B = \min \overline{K}_f(\lambda) = B$. Since $(h_B, \lambda)_0 > 0$ in this case, we choose a branch $g_j$ of $h_B$. We have $B_f(g_j) = B$. When $t(f) \leq 2$ and $\lambda$ is tangent to $f$ we consider two cases. If $\min \overline{K}_f(\lambda) < B_f(\lambda)$ then we choose $g_j$ as a branch of $h_{B_f(\lambda)}$. If $\min \overline{K}_f(\lambda) = B_f(\lambda)$ then $f$ must be unitangent and therefore $\mathcal{E}(f) \neq \{B_f(\lambda)\}$. We take a ball $B' \in \mathcal{E}(f) \setminus \{B_f(\lambda)\}$ and we choose $g_j$ as a branch of $h_{B'}$. In all these cases we obtain (30). Hence we showed equality in (28).

Now, let us assume that $\lambda$ is a branch of $f$. Let $L_2$ be the number from Corollary 3.6 (2). We want to show

$$L_2 \leq \max_{B \in \mathcal{E}(f)} (q_f(B) - d(B, \lambda)). \quad (31)$$
with equality when $t(f) \neq 1$ or $\mathcal{E}(f) \neq \{B_f(\lambda)\}$. We have $\tilde{B} \in \mathcal{E}(f)$. Then the term $q_f(\tilde{B}) - d(\tilde{B})$ of $L_2$ is less than or equal to the right side of $(31)$. Now, let us consider a branch $g_i$ of $J(\lambda, f)$ different from $\lambda$. By Theorem 4.3 we choose $B \in \mathcal{E}(f)$ such that $g_i$ is a branch of $h_B$ (we omit $B_f(\lambda)$ because $d(B_f(\lambda)) = \infty$). This $B$ gives us the expected estimation.

Now let us assume that $t(f) \neq 1$ or $\mathcal{E}(f) \neq \{B_f(\lambda)\}$. In order to prove “$\geq$” in $(31)$ we choose $B \in \mathcal{E}(f)$. When $B = \min \mathcal{K}_f(\lambda)$ then $B \leq \tilde{B}$, hence $q_f(\tilde{B}) - d(B) \geq q_f(B) - d(B, \lambda)$. If $B \neq \min \mathcal{K}_f(\lambda)$ we consider as previously.

**Remark 4.7** Let us denote by $\tilde{L}_0(f, \lambda)$ the number that stands on the right side of the inequality in Lemma 4.6. Clearly, it is an equisingularity invariant of the pair $(f, \lambda)$.

By Lemma 4.6 we have $L_0(f|\Gamma_f, \lambda) \leq \tilde{L}_0(f, \lambda)$ with equality when $t(f) \neq 1$ or $\mathcal{E}(f) \neq \{B_f(\lambda)\}$. In Example 3.9 we have $\tilde{L}_0(f, \lambda) = 2$.

By using Lemma 4.6 we can finish the proof of the Main Result. First, we prove the following

**Proposition 4.8.** Let $f = 0$ be a singular unitangent germ and let $\lambda$ be a regular parameter. Then

(a) if $\lambda$ is tangent to $f$ then $L_0(f|\Gamma_f, \lambda) \leq \tilde{L}_0(f, \lambda) < L_0(f)$,

(b) if $\lambda$ is transversal to $f$ then $L_0(f|\Gamma_f, \lambda) = \tilde{L}_0(f, \lambda) = L_0(f)$.

**Proof.** (a) Since $f$ is singular we have $\mathcal{E}(f) \neq \emptyset$. Let $B_1 = \min \mathcal{E}(f)$ and let $B_2 = \min \mathcal{K}_f(\lambda)$. We have $d(B_1) > 1$ and $d(B_2) > 1$. For any $B \in \mathcal{E}(f)$ we obtain $d(B, \lambda) \geq \min\{d(B_1), d(B_2)\} > 1$. By Lemma 4.6

$$L_0(f|\Gamma_f, \lambda) \leq \tilde{L}_0(f, \lambda) = \max_{B \in \mathcal{E}(f)} (q_f(B) - d(B, \lambda)) < L_0(f).$$

(b) If $\lambda$ is transversal to $f$ then $\mathcal{E}(f) \neq \{B_f(\lambda)\}$. Moreover $d(B, \lambda) = 1$ for every $B \in \mathcal{E}(f)$. We apply Lemma 4.6.

**Proof of Theorem 1.6 (ii),(iii)**

Let $f^{(1)}, \ldots, f^{(t)}$, $t = t(f)$, be unitangent components of $f$ and let us denote

$$M_i = L_0(f^{(i)}) + \text{ord } f - \text{ord } f^{(i)} , \quad i = 1, \ldots, t.$$  (32)

We have $M_i > \text{ord } f - 1$ if and only if $f^{(i)}$ is singular $(i = 1, \ldots, t)$. We may assume that $M_1 \geq \ldots \geq M_t$. From part (i) we have $L_0(f) = M_1$. Let $s$ be the number of singular components $0 \leq s \leq t$.

Proof of (ii). We claim that $M_1 > \text{ord } f - 1$. For $t(f) = 1$ it follows from the fact that $f$ is singular. For $t(f) > 1$ it is a consequence of the assumption that the maximum $M_1$ is realized for exactly one index from $(1, \ldots, t)$. Hence, the corresponding component $f^{(1)}$ is singular and therefore $s \geq 1$. Let $\lambda$ be a regular parameter. As in the proof of part (i) of the theorem we obtain

$$\tilde{L}_0(f, \lambda) = \max_{i=1,\ldots,s} \{\tilde{L}_0(f^{(i)}, \lambda) + \text{ord } f - \text{ord } f^{(i)}\}.$$ (33)
Assume that $\lambda$ is tangent to $f^{(1)}$. If $s = 1$ then by Proposition 4.8 (a)
\[
\mathcal{L}_0(f|\Gamma_{f,\lambda}) \leq \tilde{\mathcal{L}}_0(f, \lambda) = \tilde{\mathcal{L}}_0(f^{(1)}, \lambda) + \text{ord} f - \text{ord} f^{(1)} < \mathcal{L}_0(f^{(1)}) + \text{ord} f - \text{ord} f^{(1)} = \mathcal{L}_0(f).
\]
If $s > 1$ then $M_1 > M_2$ and we have
\[
\mathcal{L}_0(f|\Gamma_{f,\lambda}) \leq \tilde{\mathcal{L}}_0(f, \lambda) = \max\{\tilde{\mathcal{L}}_0(f^{(1)}, \lambda) + \text{ord} f - \text{ord} f^{(1)}, M_2\} < \mathcal{L}_0(f).
\]
In order to prove the opposite implication in (ii) suppose that $\lambda$ is transversal to $f^{(1)}$. In this case $B \in \mathcal{E}(f)$, therefore the condition $\mathcal{E}(f) \neq \{B_f(\lambda)\}$ is satisfied. According to Lemma 4.6 we have
\[
\mathcal{L}_0(f|\Gamma_{f,\lambda}) = \tilde{\mathcal{L}}_0(f, \lambda) = \mathcal{L}_0(f^{(1)}) + \text{ord} f - \text{ord} f^{(1)} = \mathcal{L}_0(f).
\]
Proof of (iii). We have $t = t(f) \geq 2$ and $M_1 = M_2 = \mathcal{L}_0(f)$. If $M_1 = \text{ord} f - 1$, then all the tangential components of $f$ are nonsingular (ordinary singularity). In this case $\mathcal{E}(f) = \{B\}$. By Lemma 4.6 $\mathcal{L}_0(f, \lambda) = q_f(B) - d(B, \lambda) = \text{ord} f - 1 = \mathcal{L}_0(f)$ for every regular parameter $\lambda$. If $M_1 > \text{ord} f - 1$ then $f^{(1)}$ and $f^{(2)}$ are singular ($s \geq 2$). Since every regular parameter $\lambda$ is transversal to $f^{(1)}$ or to $f^{(2)}$ we obtain $\mathcal{L}_0(f|\Gamma_{f,\lambda}) = \mathcal{L}_0(f)$ as earlier.

**Consequences for polar quotients**

Below, we apply Theorem 4.3 to polar quotients. We use notation of this theorem.

**Proposition 4.9** If $B = \min \overline{K}_f(\lambda)$ and $g$ is a branch of $h_B$ then $d(g, \lambda) = d(B_f(g))$.

Proof. By (i) of the theorem we have $B_f(g) \leq B = \min \overline{K}_f(\lambda)$. Since $d(\min \overline{K}_f(\lambda)) \leq \min_{i,j} d(f_i, f_j)$, then $d(f_i, g) = \ldots = d(f_r, g) = d(B_f(g))$. By using (ii) we obtain $d(g, \lambda) = d_f(g, \lambda) = \max, \min\{d(f_i, g), d(f_i, \lambda)\} = d(B_f(g))$.

From Corollary 3.4 and Theorem 4.3 we obtain

**Proposition 4.10**
\[
Q(f, \lambda) = \left\{ \frac{q_f(B)}{d(B, \lambda)} : B \in \mathcal{E}(f) \cup \{B_f(\lambda)\}, d(B) < \infty, m_{f, \lambda}(B) > 0 \right\}.
\]

Proof. Let us choose a factor $g_j$ of $J(\lambda, f)$ ($j \in \{1, \ldots, u\}$) as in Corollary 3.4. By Theorem 4.3 (i) $g_j$ is a factor of $h_B$ for $B \in \mathcal{E}(f) \cup \{B_f(\lambda)\}$, $d(B) < \infty$, $m_{f, \lambda}(B) > 0$. If $B \neq \min \overline{K}_f(\lambda)$ then $B_f(g_j) = B$. We finish by using (ii) and Proposition 4.2 $d(g_j, \lambda) = d_f(g_j, \lambda) = d(B_f(g_j), \lambda) = d(B, \lambda)$.

If $B = \min \overline{K}_f(\lambda)$ then $B_f(g_j) \leq B$. We have $q_f(B_f(g_j)) = (\text{ord} f)d(B_f(g_j))$ and $q_f(B) = (\text{ord} f)d(B)$. We finish by using Proposition 4.9 and Property 4.4
\[
\frac{q_f(B_f(g_j))}{d(g_j, \lambda)} = \frac{q_f(B_f(g_j))}{d(B_f(g_j))} = \text{ord} f = \frac{q_f(B)}{d(B)} = \frac{q_f(B)}{d(B, \lambda)}.
\]
**Remark 4.11** (multiplicities of polar quotients)
To every $q \in Q(f, \lambda)$ we can assign a multiplicity $m_q = \sum_B m_{f, \lambda}(B)$ where $B$ runs over all balls from $\mathcal{E}(f) \cup \{B_f(\lambda)\}$ with finite diameters such that $q_f(B)/d(B, \lambda) = q$. By (22) we have $\sum_{q \in Q(f, \lambda)} m_q = (f, \lambda)_0 - 1$.

It is important in the following lemma that we can omit the ball $B_f(\lambda)$ in the cases (i) and (ii).

**Lemma 4.12** (description of the maximal polar quotient)

(i) Assume that $t(f) \neq 2$. Then

$$q_0(f, \lambda) = \max_{B \in \mathcal{E}(f)} \frac{q_f(B)}{d(B, \lambda)}, \quad (34)$$

(ii) If $t(f) = 2$ and $\# \mathcal{E}(f) \geq 2$ then

$$q_0(f, \lambda) = \max_{B \in \mathcal{E}(f) \setminus \{B\}} \frac{q_f(B)}{d(B, \lambda)}, \quad (35)$$

(iii) Assume that $t(f) = 2$ and $\# \mathcal{E}(f) = 1$ (Morse case). If $\lambda$ is not a branch of $f$ then

$$q_0(f, \lambda) = \frac{q_f(B_f(\lambda))}{d(B_f(\lambda))} = \frac{(f, \lambda)_0}{(f, \lambda)_0 - 1} \quad (36)$$

and if $\lambda$ is a branch of $f$ then $q_0(f, \lambda) = -\infty$.

**Proof.** Let us consider the set of balls from Proposition 4.10

$$\mathcal{Z} = \{B \in \mathcal{E}(f) \cup \{B_f(\lambda)\} : d(B) < \infty, m_{f, \lambda}(B) > 0\}.$$ 

(i) In order to prove $(\leq)$ in (34) we choose $B \in \mathcal{Z}$. It suffices to find $B' \in \mathcal{E}(f)$ such that $q_f(B)/d(B, \lambda) \leq q_f(B')/d(B', \lambda)$. If $B \in \mathcal{E}(f)$ we put $B' = B$. Suppose that $B = B_f(\lambda)$. Since $\mathcal{E}(f)$ is nonempty, at least one of the following conditions holds:

(a) there exists $B_1 \in \mathcal{E}(f)$ such that $B_1 < B$ and $B_1$ is the direct predecessor of $B$,

(b) there exists $B_2 \in \mathcal{E}(f)$ such that $B < B_2$ and $B_2$ is the direct successor of $B$. In case (a) we have $d(B_1, \lambda) = d(B_1)$, $d(B, \lambda) = d(B)$. Moreover, $O_f(B) = 1$ because $\lambda$ is smooth. Hence

$$\frac{q_f(B_1)}{d(B_1, \lambda)} - \frac{q_f(B)}{d(B, \lambda)} = \frac{q_f(B_1) - d(B_1)(d(B) - d(B_1))}{d(B_1)d(B)} > 0.$$ 

In case (b) we have $d(B_2, \lambda) = d(B, \lambda) = d(B)$. Therefore,

$$\frac{q_f(B_2)}{d(B_2, \lambda)} - \frac{q_f(B)}{d(B, \lambda)} = \frac{O_f(B_2)(d(B_2) - d(B))}{d(B)} > 0.$$ 

As $B'$ we choose $B_1$ or $B_2$. In order to prove inequality $(\geq)$ it suffices to show $\mathcal{Z} \supset \mathcal{E}(f)$. Let $B \in \mathcal{E}(f)$ and suppose that $m_{f, \lambda}(B) = 0$. Since $t(f) \neq 2$, by
Corollary 4.13 we obtain \( d(B) = 1 \) and \( f \) is unitangent. Therefore, \( d(B) > 1 \) for every \( B \in \mathcal{E}(f) \), which is a contradiction. Hence \( m_{f,\lambda}(B) > 0 \) and inclusion is proved.

(ii) Since \( t(f) = 2 \) we have \( B \in \mathcal{E}(f) \). Let us notice that \( m_{f,\lambda}(B) = \sigma_{f,\lambda}^{\text{max}}(B) \). If \( \lambda \) is tangent to \( f \) then \( \sigma_{f,\lambda}^{\text{max}}(B) = 0 \). Hence \( B \notin \mathcal{Z} \). By the assumption \( \# \mathcal{E}(f) \geq 2 \) the set \( \mathcal{E}(f) \setminus \{B\} \) is nonempty. We prove \((\leq)\) as in (i). The inequality \((\geq)\) is a consequence of \( \mathcal{Z} \supset \mathcal{E}(f) \setminus \{B\} \). If \( \lambda \) is transversal to \( f \) then \( B_f(\lambda) = B \) and \( \mathcal{Z} = \mathcal{E}(f) \). Since \( \mathcal{E}(f) \setminus \{B\} \) is nonempty, the inequality \((\leq)\) follows from the fact that \( q_f(B)/d(B, \lambda) \) is now the minimal possible polar quotient. The inequality \((\geq)\) follows from the obvious inclusion as earlier.

(iii) We have \( \mathcal{E}(f) = \{B\} \). If \( \lambda \) is transversal to \( f \) then \( B_f(\lambda) = B \), \( m_{f,\lambda}(B) = \sigma_{f,\lambda}^{\text{max}}(B) = 1 \), \( q_f(B)/d(B, \lambda) = 2 \). If \( \lambda \) is tangent to \( f \) then \( m_{f,\lambda}(B) = 0 \). If \( \lambda \) is not a branch of \( f \) then \( m_{f,\lambda}(B_f(\lambda)) = d(B_f(\lambda)) \geq 1 \). Then \( q_f(B_f(\lambda))/d(B_f(\lambda)) = (f, \lambda_0/((f, \lambda_0 - 1) \). If \( \lambda \) is a branch of \( f \) then \( d(B_f(\lambda)) = \infty \) and \( Q(f, \lambda) = \emptyset \).}

**Corollary 4.13** In all the cases \( \max_{B \in \mathcal{E}(f)} \frac{q_f(B)}{d(B, \lambda)} \geq q_0(f, \lambda) \).

**Example 4.14** Let \( f = f_1 f_2 f_3 = Y(Y^2 - X)(Y^2 + X) \), \( \lambda = X \). We have \( \mathcal{E}(f) = \{B_0, B_1\} \), where \( B_0 = B(f_1, f_2) = B(f_1, f_3) \) and \( B_1 = B(f_2, f_3) \) with \( q_f(B_0) = 3 \), \( q_f(B_1) = 5 \), \( d(B_0, \lambda) = 1 \) and \( d(B_1, \lambda) = 2 \).

\[
\begin{array}{ccc}
\infty & {f_1} & {f_2} & {f_3} \\
2 & B_0 & B_1 \\
1 & & & \\
\end{array}
\]

Although \( q_f(B_0)/d(B_0, \lambda) > q_f(B_1)/d(B_1, \lambda) \) we have \( q_0(f, \lambda) = q_f(B_1)/d(B_1, \lambda) = 5/2 \) since we omit \( B_0 = B \) by Lemma 4.12(ii).

By using both Lemmas 4.6 and 4.12 we can prove Theorem 1.11

**Proof of Theorem 1.11**

(a) For any \( B \in \mathcal{E}(f) \) we have

\[
q_f(B) - 1 \geq q_f(B) - d(B, \lambda) \geq \frac{q_f(B)}{d(B, \lambda)} - 1.
\]

If \( t(f) \neq 1 \) or \( \mathcal{E}(f) \neq \{B_f(\lambda)\} \) then we finish by using (18), Lemma 4.6 and Corollary 4.13. When \( t(f) = 1 \) and \( \mathcal{E}(f) = \{B_f(\lambda)\} \) let us denote \( B_0 = B_f(\lambda) \). We have \( B_0 = \min \mathcal{K}_f(\lambda) \). By Theorem 4.3 \( f(\lambda) = h_{B_0} \), \( (h_{B_0}, \lambda_0) = d(B_0)n(B_0)t_f(B_0) - 1 > 0 \). By using Corollary 3.9 and Proposition 4.9 we obtain

\[
\mathcal{L}_0(f; \Gamma_{f,\lambda}) = \max_{g} (q_f(B_f(g)) - d(g, \lambda)) = \max (\text{ord } f - 1) d(B_f(g)) ,
\]
where $g$ runs over irreducible factors of $h_{B_0}$. As in the proof of Proposition 4.10 we show that $q_0(f, \lambda) = \text{ord } f$. Moreover $\mathcal{L}_0(f) = (\text{ord } f) d(B_0) - 1$, which gives desired inequalities.

(b) Let us consider three cases

(I) $t(f) > 2$ or $(t(f) = 1$ and $\# \mathcal{E}(f) > 1)$ or $(t(f) = 1$ and $\mathcal{E}(f) \neq \{B_f(\lambda)\})$

(II) $t(f) = 1$ and $\mathcal{E}(f) = \{B_f(\lambda)\}$

(III) $t(f) = 2$ and $\# \mathcal{E}(f) > 1$

(I) In this case we obtain the desired equivalency from the formulas

$$\mathcal{L}_0(f|\Gamma_f,\lambda) = \max_{B \in \mathcal{E}(f)} (q_f(B) - d(g, \lambda)),$$

$$q_0(f, \lambda) = \max_{B \in \mathcal{E}(f)} \frac{q_f(B)}{d(B, \lambda)}.$$  

(II) We prove in this case that none of the equalities from the statement of the theorem can not be satisfied. Let us denote $B_0 = B_f(\lambda)$. Since the singularity is unitangent, $d(B_0) > 1$. By using the formulas as in the proof of (a) we obtain $q_0(f) - q_0(f, \lambda) \geq (\text{ord } f)(d(B_0) - 1)$ and $\mathcal{L}_0(f) - \mathcal{L}_0(f|\Gamma_f,\lambda) \geq d(B_0) - 1$.

(III) Now, the following formulas are true

$$\mathcal{L}_0(f|\Gamma_f,\lambda) = \max_{B \in \mathcal{E}(f)} (q_f(B) - d(g, \lambda)),$$

$$q_0(f, \lambda) = \max_{B \in \mathcal{E}(f) \setminus \{B\}} \frac{q_f(B)}{d(B, \lambda)}.$$  

($\Rightarrow$) Assume that $\mathcal{L}_0(f) = \mathcal{L}_0(f|\Gamma_f,\lambda)$. Hence, there exists $B_1 \in \mathcal{E}(f)$ such that $\mathcal{L}_0(f) = q_f(B_1) - d(B_1, \lambda)$, therefore $d(B_1, \lambda) = 1$. Since $\# \mathcal{E}(f) > 1$, and taking into consideration (28) we can assume that $B_1 \neq B$. For $B_1$ we obtain $q_0(f) = q_f(B_1) = q_0(f, \lambda)$.

($\Leftarrow$) Let us assume that $q_0(f) = q_0(f, \lambda)$. Hence, there exists $B_2 \in \mathcal{E}(f) \setminus \{B\}$ such that $q_f(B_2)/d(B_2, \lambda) = q_0(f)$. Therefore, $q_f(B_2) = q_0(f)$ and $d(B_2, \lambda) = 1$. We obtain $\mathcal{L}_0(f) = \mathcal{L}_0(f|\Gamma_f,\lambda)$.  

\[ \blacksquare \]
5 Proof of factorization theorem

In this section we prove Theorem 4.3. We can consider the derivative as the polar curve. We apply the main result of [28], where a version of the Newton algorithm [4] provides a description of all polar quotients including multiplicities ([28],Theorem 2.1).

Here we reformulate this result to describe the roots of the derivative (Theorem 5.2). The number of these roots, described by Proposition 5.17, gives us “multiplicities” of branches of the derivative assigned to the balls (Lemma 5.15). Next, we study the characteristics of branches and we describe all possible balls within the fixed coordinate system. By using this description we assign the roots of the derivative to the balls (Theorem 5.2).

The roots of derivative

We need some preliminaries. We consider the ring \( \mathbb{C}\{X\}^* = \bigcup_{N \geq 1} \mathbb{C}\{X^{1/N}\} \) of Puiseux series. For every nonzero \( y(X) \in \mathbb{C}\{X\}^* \) the order \( \text{ord} y \) stands for the minimal power with nonzero coefficient and in \( y \) is the corresponding monomial. We put \( \text{ord} 0 = \infty \) and in \( 0 = 0 \). It is convenient to consider the ring \( \mathbb{C}\{X^*,Y\} = \bigcup_{N \geq 1} \mathbb{C}\{X^{1/N},Y\} \). Take \( f = \sum c_{\alpha \beta} X^\alpha Y^\beta \in \mathbb{C}\{X^*,Y\} \). As usual, we define the support \( \text{supp} f = \{(\alpha, \beta) : c_{\alpha \beta} \neq 0\} \), the Newton diagram \( \Delta(f) \) as \( \text{conv} (\text{supp} f + \mathbb{R}_2^2) \), and the Newton polygon \( \mathcal{N}_f = \mathcal{N}(f) \) as the set of compact faces of \( \Delta(f) \) (we use the term “face” in the meaning of “1-dimensional face”). By \( \delta_Y(f) \) (resp. \( \delta_X(f) \)) we denote the distance between \( \Delta(f) \) and the horizontal axis (resp. vertical axis).

For \( S \in \mathcal{N}_f \), by \( |S|_1 \) and \( |S|_2 \) we denote the lengths of projections of \( S \) onto the horizontal and vertical axes, respectively. We call the ratio \( |S|_1/|S|_2 \) the inclination of \( S \). We denote it by \( \text{incl}(S) \). We define \( \text{incl}(\mathcal{N}_f) = \{\text{incl}(S) : S \in \mathcal{N}_f\} \) if \( \delta_Y(f) = 0 \) or \( \text{incl}(\mathcal{N}_f) = \{\text{incl}(S) : S \in \mathcal{N}_f\} \cup \{\infty\} \) if \( \delta_Y(f) > 0 \). For \( \theta > 0 \) (or \( \theta = -\infty \)) it is useful to consider the polygon \( \mathcal{N}_f^\theta \) which consists of all \( S \in \mathcal{N}_f \) with \( \text{incl}(S) > \theta \).

We have \( \text{incl}(\mathcal{N}_f^\theta) = \text{incl}(\mathcal{N}_f) \cap (\theta, \infty) \). We define the initial form of \( f \) with respect to \( S \) as \( \text{in}(f,S) = \sum c_{\alpha \beta} X^\alpha Y^\beta \) where \( (\alpha, \beta) \in S \cap \text{supp} f \). By \( t(f,S) \) we denote the number of different roots of the polynomial \( \text{in}(f,S)(1,Y) \in \mathbb{C}[Y] \). The number \( \varepsilon(S) \in \{-1,0\} \) is defined as \( -1 \) when \( S \) touches the horizontal axis and as \( 0 \), otherwise. The function \( d(f,S) = |S|_2 + \varepsilon(S) - t(f,S) + 1 \) denotes that \( d(f,S) = 0 \) if and only if every nonzero root of \( \text{in}(f,S) \) in \( \mathbb{C}\{X\}^* \) is of multiplicity 1. Then, we call the series \( f \) nondegenerate on \( S \).

For any \( \varphi \in \mathbb{C}\{X\}^* \), ord \( \varphi > 0 \) one can apply the substitution \( f_\varphi(X,Y) = f(X,\varphi + Y) \in \mathbb{C}\{X^*,Y\} \) ([1], [14], [21]). Clearly, \( f_\varphi = f \) for \( \varphi = 0 \). Consider the ring \( \mathbb{C}[X]^* = \bigcup_{N \geq 1} \mathbb{C}[X^{1/N}] \) of Puiseux polynomials. For \( \varphi \in \mathbb{C}[X]^* \), deg \( \varphi < \infty \). Put \( \text{deg} 0 = -\infty \). We define the set \( \text{Track}(f) \subset \mathbb{C}[X]^* \) of tracks (of the Newton algorithm) for \( f \) as the minimal set satisfying two properties: (I) \( 0 \in \text{Track}(f) \), (II) for every \( \varphi \in \text{Track}(f) \), if there exists \( S \in \mathcal{N}^\text{deg } \varphi(f_\varphi) \), then for every nonzero root \( aX^\theta \) of \( \text{in}(f_\varphi,S) \), \( \varphi + aX^\theta \in \text{Track}(f) \). In [28] (Proposition 3.11) we give three different characterizations of the set \( \text{Track}(f) \). We will write \( \mathcal{N}_\varphi \) instead of \( \mathcal{N}^\text{deg } \varphi(f_\varphi) \) when \( f \) is fixed. We call a series \( \psi \in \mathbb{C}\{X\}^* \) a continuation of \( \varphi \in \mathbb{C}[X]^* \) if ord \( (\varphi - \psi) > \text{deg } \varphi \). Then we write \( \psi = \varphi + \ldots \). By \( \text{Track}_\varphi(f) \) we denote the set of all tracks from \( \text{Track}(f) \) that are continuations of \( \varphi \).
In order to deal with multiple roots we use the notion of symmetric power [41]. For elements \( a_1, \ldots, a_s \) of a given set we define the system \( \mathbf{A} = \langle a_1, \ldots, a_s \rangle \) as the sequence \( a_1, \ldots, a_s \) treated as unordered. Put \( \deg \mathbf{A} = s \). For \( \mathbf{A} = \langle a_1, \ldots, a_s \rangle \) and \( \mathbf{B} = \langle b_1, \ldots, b_t \rangle \) we have a natural addition \( \mathbf{A} \oplus \mathbf{B} = \langle a_1, \ldots, a_s, b_1, \ldots, b_t \rangle \) with the neutral element \( \langle \rangle \) (empty system). Instead of \( \langle a : m \rangle \) we write \( a : m \) with m times

\[ \text{convention } \langle a : 0 \rangle = \langle \rangle. \]

If \( a \) appears in \( \mathbf{A} \) at least one time then we write \( a \in \mathbf{A} \).

Now, assume that \( \text{ord} f(0, Y) = p > 0 \). We consider the system \( \text{Zer} f = \langle y_1, \ldots, y_p \rangle \) of all solutions of \( f = 0 \) in \( \mathbb{C}\{X\}^* \). Let \( \varphi \in \text{Track}(f) \). By \( \text{Zer}_\varphi f \) we denote the system of all solutions from \( \text{Zer} f \) that are continuations of \( \varphi \). Our aim is to describe the system \( \text{Zer}(\partial f/\partial Y) = \langle z_1, \ldots, z_{p-1} \rangle \). We define a solution \( z(X) \in \text{Zer}_\varphi(\partial f/\partial Y) \) to be of the \( \varphi \)-first kind if \( \text{ord}(z - \varphi) \in \text{incl}(\mathcal{N}_\varphi) \) and of the \( \varphi \)-second kind otherwise. We control the “kind” by the following proposition (see: Proposition 3.4, [29]). For \( S \in \mathcal{N}_\varphi \) we put \( w_{\varphi,S}(Y) = \text{in}(f_\varphi, S)(1, Y) \).

**Proposition 5.1** Let \( z(X) \in \text{Zer}_\varphi(\partial f/\partial Y) \). Then

(i) If \( z(X) \) is of the \( \varphi \)-first kind then:

(a) if \( \text{ord}(z - \varphi) = \infty \) (i.e. \( z = \varphi \)) then \( \delta_Y(f_\varphi) > 1 \),

(b) if \( \text{ord}(z - \varphi) < \infty \) then there exists \( S \in \mathcal{N}(f_\varphi) \) such that \( z(X) = \varphi + aX^{\text{incl}(S)} + \ldots \) \( (a \neq 0) \) and \( \delta_Y(f_\varphi) = 1 \).

(ii) Solutions of the \( \varphi \)-second kind exist if and only if both conditions hold:

- the lowest face \( S = L \) of \( \mathcal{N}_\varphi \) touches the horizontal axis (i.e. \( w_{\varphi,L}(0) \neq 0 \)),

- \( \text{ord}(w_{\varphi,L}(Y) - w_{\varphi,L}(0)) \geq 2 \).

(iii) If \( z(X) \) is of the \( \varphi \)-second kind then \( \text{ord}(z - \varphi) > \max \text{incl}(\mathcal{N}_\varphi) \).

Let \( \varphi \in \text{Track}(f) \). We define the system \( \text{Zer}_{f_{\infty}}(\partial f/\partial Y) \) (resp. \( \text{Zer}_{f_{\infty}}(\partial f/\partial Y) \)) which consists of those \( z(X) \in \text{Zer}_\varphi(\partial f/\partial Y) \) that \( \text{ord} f(X, z(X)) < \infty \) (resp. \( \text{ord} f(X, z(X)) = \infty \)). We have \( \text{Zer}_\varphi(\partial f/\partial Y) = \text{Zer}_{f_{\infty}}(\partial f/\partial Y) \oplus \text{Zer}_{f_{\infty}}(\partial f/\partial Y) \). We put \( \mathbf{C}_\varphi = \langle \varphi : \delta_Y(f_\varphi) > 1 \rangle \) if \( \delta_Y(f_\varphi) > 1 \) and \( \mathbf{C}_\varphi = \langle \rangle \) if \( \delta_Y(f_\varphi) \in \{0, 1\} \). For \( S \in \mathcal{N}_\varphi \) we define the system \( \mathbf{B}_{\varphi,S} \) (resp. \( \mathbf{A}_{\varphi,S}^I \)) of the \( \varphi \)-first kind solutions \( z(X) \) such that \( \text{ord}(z - \varphi) = \text{incl}(S) \) and \( \text{in}(z - \varphi) \) is a root (resp. is not a root) of \( \text{in}(f_\varphi, S) \). By \( \mathbf{A}_{\varphi,S}^II \) we denote the system of all \( \varphi \)-second kind solutions. We put

\[
\mathbf{A}_{\varphi,S} = \begin{cases} 
\mathbf{A}_{\varphi,S}^I & \text{if } S \text{ does not touch the horizontal axis} \\
\mathbf{A}_{\varphi,S}^I \oplus \mathbf{A}_{\varphi}^II & \text{if } S \text{ touches the horizontal axis}
\end{cases}
\]

The following theorem is a reformulation of Theorem 2.1 from [28]. The proof is analogous.

**Theorem 5.2** Let \( \varphi \in \text{Track}(f) \).
(a) \[ \text{Zer}_\varphi(\partial f/\partial Y) = \bigoplus_{S \in \mathcal{N}_\varphi} (A_{\varphi,S} \oplus B_{\varphi,S}) \bigoplus C_\varphi \]
with \( \deg A_{\varphi,S} = t(f, S) - 1 \), \( \deg B_{\varphi,S} = d(f, S) \).

(b) Let \( S \in \mathcal{N}_\varphi, B_{\varphi,S} = \bigoplus_{aX^\vartheta} \text{Zer}_{\varphi+aX^\vartheta}(\partial f/\partial Y) \),
where \( aX^\vartheta \) runs over all multiple nonzero roots of \( \text{in}(f, S) \).

(c) \[ \text{Zer}^\text{fin}_\varphi(\partial f/\partial Y) = \bigoplus_{\psi \in \text{Track}(f)} \bigoplus_{S \in \mathcal{N}_\psi} A_{\psi,S} \]

Now, assume that \( f \) is reduced. Then for every \( \varphi \in \text{Track}(f) \) we have \( \text{Zer}_\varphi(\partial f/\partial Y) = \text{Zer}_\varphi^\text{fin}(\partial f/\partial Y) \) and part “C” disappears. For \( \varphi = 0 \) we obtain the following two corollaries. We write \( A_S, B_S \) instead of \( A_{0,S}, B_{0,S} \).

**Corollary 5.3** (see Corollary 2.5(a), [28])

\[ \text{Zer}(\partial f/\partial Y) = \bigoplus_{S \in \mathcal{N}_f} (A_S \oplus B_S) . \]

For every \( S \in \mathcal{N}_f \)

(a) \( \deg A_S = t(f, S) - 1 \),

(b) \( \deg B_S = d(f, S) \).

**Corollary 5.4** For \( S \in \mathcal{N}_f \)

\[ B_S = \bigoplus_{\varphi \in \text{Track}_{aX^\vartheta}(f)} \bigoplus_{S \in \mathcal{N}_\varphi} A_{\varphi,S} , \]
where \( aX^\vartheta \) runs over all multiple nonzero roots of \( \text{in}(f, S) \).

The information presented in Corollary 5.3 corresponds to the first step of the Newton algorithm. The information presented in Corollary 5.4 corresponds to the following steps.

**Characteristic Newton diagram of a branch**

Recall a notion of the cycle generated by a Puiseux series \( y(X) \in \mathbb{C}\{X\}^* \). Let \( N(y) \) be the minimal possible \( N \) such that \( y \in \mathbb{C}\{X^{1/N}\} \). Suppose that \( 0 < \text{ord} y < \infty \). We write

\[ y(X) = a_1X^{v_1/N} + a_2X^{v_2/N} + \ldots \quad a_1, a_2, \ldots \neq 0 , \]

\( 0 < v_1 < v_2 < \) integers, \( \text{GCD}(N, v_1, v_2, \ldots) = 1 \). We put \( \text{cycle}(y) = \langle y_0, \ldots, y_{N-1} \rangle \),
where

\[ y_i(X) = a_1^{v_1i}X^{v_1/N} + a_2^{v_2i}X^{v_2/N} + \ldots , \ i = 0, \ldots, N - 1 , \]

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\( \varepsilon \) is a primitive root of 1 of degree \( N \). For \( y = 0 \) we put \( \text{cycle}(y) = \langle 0 \rangle \). The product
\[
[y] := \prod_{i=0}^{N-1} (Y - y_i(X)) \in \mathbb{C}\{X,Y\}
\] (37)
defines a branch. We have \([0] = Y\). On the other hand, every branch coprime with \( X \) can be written in the form of (37) up to an invertible factor from \( \mathbb{C}\{X,Y\} \).

To every \( y \in \mathbb{C}\{X\}^* \), \( \text{ord } y > 0 \), we assign a generalized characteristic sequence \((b_0, \ldots, b_h)\) such that \( b_0 = N(y) \). If \( N(y) = 1 \) then \( h(y) = 0 \) and \((b_0, \ldots, b_h) = (1)\). If \( N(y) > 1 \) then we define characteristic positions \((j_1, \ldots, j_h)\) as
\[
j_k = \min\{j > j_{k-1} : \gcd(v_0, \ldots, v_{j-1}) > \gcd(v_0, \ldots, v_{j})\}, \quad k = 1, 2, \ldots
\]
\[
j_h = \min\{j : \gcd(v_0, \ldots, v_{j}) = 1\}.
\]
with conventions \( j_0 = 0 \) and \( v_0 = N(y) \). Then \((b_0, \ldots, b_h) = (N(y), v_{j_1}, \ldots, v_{j_h})\).

Let \( f = [y] \). We can reconstruct the contact \( d(f,X) \) by using \((b_0, \ldots, b_h)\). We have \( b_0 = (f,X)_0 \). If \( h(y) = 0 \) the \( d(f,X) = b_0 \). If \( h(y) > 1 \) then \( \text{ord } f = \min\{b_0, b_1\} \).

Hence \( d(f,X) = b_0/\gcd(b_0, b_1) \). Therefore, if \( b_0 < b_1 \) then \( d(f,X) = 1 \) (\( f, X \) are transverse). If \( b_0 > b_1 \) then \( d(f,X) = b_0/b_1 \) (\( f, X \) are tangent). If \( d(f,X) \not\in \mathbb{Z} \) then \( f, X \) are in the maximal contact.

Let \( f_y = f(X, y(X) + Y) \in \mathbb{C}\{X,Y\} \). The Newton diagram \( \Delta(f_y) \) can be described in terms of characteristics \((b_0, \ldots, b_h)\). Let \( e_k := \gcd(b_0, \ldots, b_k), k = 0, \ldots, h \).

**Property 5.5** (see [14], Property 3.1 or [28], Section 5)
\[
\Delta(f_y) = \sum_{k=1}^{h} \left\{ \frac{(b_k/b_0)(e_{k-1} - e_k)}{e_{k-1} - e_k} \right\} + \left\{ \frac{\infty}{1} \right\}.
\]
The sum over the empty set equals the zero element \( \{0\} \). The diagram \( \Delta(f_y) \) does not depend on the choice of \( y_i \in \text{cycle}(y) \). For \( \Delta \subset \mathbb{R}_+^2, c > 0 \) let \( c\Delta = \{ca : a \in \Delta\} \).

We define the characteristic Newton diagram of \( f \) (with respect to \( X \)) as \( \Delta_{\text{char}}^X f := (1/\text{ord } f)\Delta(f_y) \).

The diagram \( \Delta_{\text{char}}^X f \) has a vertex \( (0, d) \) on the vertical axis; \( d = d(f,X) \). The distance between the diagram and the horizontal axis equals \( 1/(\text{ord } f) \). The inclinations of
successive faces are $b_1/b_0,\ldots,b_h/b_0$. Let us denote by $\alpha_1,\ldots,\alpha_h$ the abscissae of points where the lines determined by successive faces intersect the horizontal axis. We restore the characteristic $\text{char}(f) = \{d_1,\ldots,d_g\}$ by the following formula, which is a consequence of the Abhyankar inverse rule [3]. The characteristic $\{d_1,\ldots,d_g\}$ equals

\[
\begin{align*}
\{\alpha_1,\ldots,\alpha_h\} & \text{ if } d(f,X) = 1, \ g = h, \\
\{d\alpha_1,\ldots,d\alpha_h\} & \text{ if } d(f,X) > 1 \text{ the contact is maximal, } g = h, \\
\{d\alpha_2,\ldots,d\alpha_h\} & \text{ if } d(f,X) > 1 \text{ the contact is not maximal, } g = h - 1.
\end{align*}
\]

The sequence $(n_1,\ldots,n_g)$ can be restored by using (38). We check that

\[
N(y) = d(f,X)n_1\ldots n_g.
\]

For a Newton diagram $\Delta$ we define the number $\alpha = \alpha(\kappa,\Delta)$ which equals the abscissa of the point where the line of inclination $\kappa > 0$, supporting $\Delta$, intersects the horizontal axis.

![Newton diagram](attachment:image.png)

We also need the inverse operation. We define the number $\kappa = \kappa(\alpha,\Delta)$ as the inclination of the line supporting $\Delta$ which intersects the horizontal axis at the point $(\alpha,0)$.

Let us consider $z(X) \in \mathbb{C}\{X\}^*$, ord $z > 0$. Following [14] let us put

\[
o_f(z) = \max_{y_i \in \text{cycle}(y)} \text{ord}(y_i - z).
\]

The number $o_f(z_j)$ does not depend on the choice of $z_j \in \text{cycle}(z)$. Let $g = [z]$. We have

\[
o_f(z) = o_g(y) = \max_{i,j} \text{ord}(y_i - z_j).
\]

Let us denote (39) by $\kappa(f,g,X)$. Let us consider the ball $B = B(f,g)$, let c.ex.$(B)$ be the number defined in (12) and let $d_X(f,g) := \min\{d(f,X),d(g,X)\}$. By using the inverse rule of Abhyankar we check

**Proposition 5.6**

(a) If $d(f,g) = d_X(f,g)$ then $\kappa(f,g,X) = \min\left\{\frac{1}{d(f,X)},\frac{1}{d(g,X)}\right\}$,

(b) if $d(f,g) > d_X(f,g)$ then $\kappa(f,g,X) = \frac{\text{c.ex.}(B) - d_X(f,g) + 1}{d_X(f,g)}$.
The following property is crucial for our purposes.

**Property 5.7** (compare [14], Property 3.3)
With previous notation let \( \kappa = \kappa(f, g, X) \). Then
\[
d(f, g) = d(X, g) \alpha(\kappa, \Delta_X^\text{char} f) = d(f, X) \alpha(\kappa, \Delta_X^\text{char} g)
\]

**Description of balls in coordinates** \( X, Y \)
Below we characterize an arbitrary ball \( B \subset B \) in the fixed coordinates \( X, Y \). First, we need a fact concerning Puiseux polynomials. Suppose that \( \varphi \in \mathbb{C}[X]^* \setminus \{0\} \), \( \text{ord} \varphi > 0 \). Let \( R_1(\varphi) := d([\varphi], X)\alpha(\deg \varphi, \Delta_X^\text{char} [\varphi]) \).

**Property 5.8** If \( f \in B \) satisfies \( d(f, [\varphi]) > R_1(\varphi) \) then there exists \( y \in \text{Zer} f \) that is a continuation of \( \varphi \).

**Corollary 5.9** With the above assumptions \( \text{char}([\varphi]) \subset \text{char}(f) \).

Let \( B \subset B \) be an arbitrary ball. We measure positions of \( B \) with respect to the axes \( X \) and \( Y \) by the numbers \( d(B, X) \) and \( d(B, Y) \) (see (25)). In any case we have \( d(B, X) \leq d(B) \) and \( d(B, Y) \leq d(B) \).

**Property 5.10** (classification of balls)
Let \( R = d(B) \). Then:

(I) If \( R = 1 \) then \( B = B \).

(II) If \( 1 < R = d(B, X) \) then \( B = B(X, R) \)

(III) If \( 1 < R = d(B, Y) \) then \( B = B(Y, R) \).

(IV) If \( R > \max \{d(B, X), d(B, Y)\} \) then there exists \( \varphi \in \mathbb{C}[X]^* \setminus \{0\} \), \( \text{ord} \varphi > 0 \) such that \( R_1(\varphi) < R \) and \( B = B([\varphi], R) \).

Moreover, the classes of balls described above are disjoint.

A ball \( B \) belongs to the joined class (III)+(IV) if and only if \( d(B, X) < d(B) \). Since for \( \varphi = 0 \) we have \([\varphi] = Y\) it is convenient to put \( R_1(0) = 1 \). After that in the joined class (III)+(IV) we have the following

**Property 5.11** Let \( B \) be a ball satifying \( d(B, X) < d(B) \). Then if \( B = B([\varphi], R) = B([\tilde{\varphi}], R) \) where \( \varphi, \tilde{\varphi} \in \mathbb{C}[X]^* \), \( \text{ord} \varphi > 0 \), \( \text{ord} \tilde{\varphi} > 0 \), \( R_1(\varphi) < R, R_1(\tilde{\varphi}) < R \) then \([\varphi] = [\tilde{\varphi}] \) (i.e. \( \varphi \) and \( \tilde{\varphi} \) are in the same cycle).

First, we compute the invariants \( \text{char}(B), \nu(B), n(B) \) which depend only on the ball. Let \( B \) be a ball from Property 5.10 with the radius \( R = d(B) \) and let \( d_X = d(B, X) \). We assign to \( B \) the numbers

\[
\kappa = \begin{cases} 
1 & \text{in case (I)} \\
1/R & \text{in case (II)} \\
R & \text{in case (III)} \\
\kappa(R/d_X, \Delta_X^\text{char} [\varphi]) & \text{in case (IV)} 
\end{cases}
\]
\[ N = \begin{cases} 
1 & \text{in cases (I), (II), (III)} \\
N(\varphi) & \text{in case (IV)}. 
\end{cases} \tag{41} \]

Let us write \( \varkappa = m/(N\bar{n}) \), \( \gcd(\bar{n}, m) = 1 \).

**Proposition 5.12** Let \( B \) be a ball from Property 5.10. Then with the previous notation we have

| case | \( \text{char}(B) \) | \( \nu(B) \) | \( n(B) \) |
|------|----------------|-------------|-------------|
| (I)  | \( \emptyset \) | 1           | 1           |
| (II) | \( \emptyset \) | 1           | \( m \)    |
| (III)| \( \emptyset \) | 1           | \( \bar{n} \) |
| (IV) | \( \text{char}([\varphi]) \) | \( N/d_X \) | \( \bar{n} \) |

Proof. (I), (II), (III) follow directly from the definition.

(IV). Since \( [\varphi] \) is a center of \( B \) we have \( \text{char}([\varphi]) \supset \text{char}(B) \). Let \( f \in B \). We have \( d(f, [\varphi]) \geq R > R_1(\varphi) \). Then we use Corollary 5.9. We obtain \( \nu(B) = N/d_X \) from (38). To show \( n(B) = \bar{n} \) we use (11) and (12). We check that

\[ \left\{ n \geq 1 : d(B) \in \frac{N}{\nu(B)^2 n} \right\} = \left\{ n \geq 1 : \text{c.ex.}(B) \in \frac{N}{\nu(B)n} \right\}. \]

We finish by using Proposition 5.6.

Now, for a ball from family (13) determined by the germ we want to compute the numbers \( t^{(1)} \) and \( t^{(2)} \) of direct successors in the Eggers tree. These numbers depend not only on the ball but also on the germ. For \( f \in \mathbb{C}\{X^{1/N}, Y\} \) by \( r^{(N)}(f) \) we denote the number of pairwise coprime factors of \( f \) in \( \mathbb{C}\{X^{1/N}, Y\} \); \( r_0^{(N)}(f) \) stands for the number of factors different from \( X \) and \( Y \). For \( N = 1 \) we write \( r(f) \) and \( r_0(f) \), respectively. We put \( \varepsilon_X(f) = 1 \) if \( X \) appears as a factor of \( f \) and \( \varepsilon_X(f) = 0 \), otherwise. Analogously we define \( \varepsilon_Y(f) \). Let \( (a, b) \) be a vector \( (a, b > 0) \) and let \( f = \sum c_{\alpha} X^\alpha Y^\beta \). We define the initial form \( \text{in}_{(a, b)} f \) of \( f \) with respect to \( (a, b) \) as \( \sum c_{\alpha} X^\alpha Y^\beta \) where \( (\alpha, \beta) \in \text{supp} f \) and \( \alpha a + \beta b = \inf\{\alpha a + \beta b : (\alpha, \beta) \in \text{supp} f\} \).

For \( S \in \mathcal{N}_f \) we have \( \text{in}(f, S) = \text{in}(\text{in}_{(s, s)}, f) \). For a generic \((a, b)\) \( \text{in}_{(a, b)} f \) is a monomial.

For a pair \((\varphi, \varkappa), \varphi \in \mathbb{C}[X]^*, \text{ord} \varphi > 0, \varkappa > \text{deg} \varphi \) we consider two Puiseux series \( y = \varphi + aX^\varkappa + \ldots, z = \varphi + bX^\varkappa + \ldots \), where \( a, b \in \mathbb{C} \). Let \( N = N(\varphi), \varkappa = m/(N\bar{n}) \), \( \gcd(\bar{n}, m) = 1 \). Let \( f = [y] \) and \( g = [z] \). We need a tool for estimating the contact \( d(f, g) \). Let

\[ R_{\varphi, \varkappa} = \begin{cases} 
\max\{\varkappa, 1/\varkappa\} & \text{if } \varphi = 0 \\
d([\varphi], X) \alpha(\varkappa, \Delta_X^\text{char}([\varphi])) & \text{if } \varphi \neq 0. 
\end{cases} \tag{42} \]

**Property 5.13** Assume that if \( \varphi = 0 \) and \( \varkappa < 1 \) then \( a, b \) are nonzero. Otherwise, \( a, b \) are arbitrary. Then \( d(f, g) \geq R_{\varphi, \varkappa} \) and the following conditions are equivalent:

1. \( d(f, g) > R_{\varphi, \varkappa} \),

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(2) \( a^\lambda = b^\lambda \),

(3) \( aX^\kappa \) is a root of \( \text{in}_{(1,\kappa)} g_\varphi \) in \( \mathbb{C}\{X\}^* \),

(4) \( bX^\kappa \) is a root of \( \text{in}_{(1,\kappa)} f_\varphi \) in \( \mathbb{C}\{X\}^* \),

(5) \( \text{in}_{(1,\kappa)} f_\varphi \) and \( \text{in}_{(1,\kappa)} g_\varphi \) can be written as \( X^{\zeta_i} (Y^\eta_i - cX^{m_i/N})^\mu_i \) up to nonzero constants with \( \zeta_i \geq 0, \eta_i > 0 \) \((i = 1, 2)\) and \( c = a^\lambda = b^\lambda \) (possibly zero).

Now, let us consider a germ \( f = 0, f \in \mathbb{C}\{X,Y\} \) reduced, and an arbitrary ball \( B \) from Property 5.10. The formulas for computing the numbers \( t_f^{(1)}(B) \) and \( t_f^{(2)}(B) \) are presented in the following proposition which is a consequence of Property 5.13. If \( B \) is characteristic \( (n(B) > 1) \) then in order to determine \( t_f^{(1)}(B) \) we consider the equivalency class of a branch \( h \in B \) such that \( d(B) \notin \text{char}(h) \). (see proof of Proposition 2.11(c)). In cases (II), (III), (IV) as \( h \) we choose \( X, Y \) and \([\varphi]\), respectively.

**Proposition 5.14**

| case | \( t_f^{(1)}(B) \) | \( t_f^{(2)}(B) \) |
|------|-------------------|-------------------|
| (I)  | \( r(\text{in} f) \) | 0 |
| (II) | \( n(B) = 1 \)  \( r_0(\text{in}_{(1,R)} f) + \varepsilon_X(\text{in}_{(1,R)} f) \) | 0 |
|      | \( n(B) > 1 \)   \( \varepsilon_X(\text{in}_{(1,R)} f) \) | \( r_0(\text{in}_{(1,R)} f) \) |
| (III)| \( n(B) = 1 \)  \( r_0(\text{in}_{(1,R)} f) + \varepsilon_Y(\text{in}_{(1,R)} f) \) | 0 |
|      | \( n(B) > 1 \)   \( \varepsilon_Y(\text{in}_{(1,R)} f) \) | \( r_0(\text{in}_{(1,R)} f) \) |
| (IV) | \( n(B) = 1 \)  \( r_0^{(N)}(\text{in}_{(1,\kappa)} f_\varphi) + \varepsilon_Y(\text{in}_{(1,\kappa)} f_\varphi) \) | 0 |
|      | \( n(B) > 1 \)   \( \varepsilon_Y(\text{in}_{(1,\kappa)} f_\varphi) \) | \( r_0^{(N)}(\text{in}_{(1,\kappa)} f_\varphi) \) |

**Proof of Theorem 4.3**

We consider a singularity \( f = 0, f \in \mathbb{C}\{X,Y\} \) reduced, and a regular parameter \( \lambda \). Without loss of generality we can assume that \( \lambda = X \). Then \( \mathcal{J}(\lambda, f) = \partial f/\partial Y \). For every \( \varphi \in \text{Track}(f) \) we consider the Newton polygon \( \mathcal{N}_\varphi := \mathcal{N}^{\text{deg} \varphi}(f_\varphi) \). For \( \varphi = 0 \) we obtain the classical Newton polygon \( \mathcal{N}_f \). Let \( \mathcal{B}_f = \{ f_1, \ldots, f_r \} \) be the set of branches of \( f \). We can write \( f = X^{d_X(f)} \tilde{f} \) where \((\tilde{f}, X)_0 = p > 0 \). Clearly \( \text{Zer}(f) = \text{Zer}(\tilde{f}) \) and \( \text{Zer}(\partial f/\partial Y) = \text{Zer}(\partial \tilde{f}/\partial Y) \). We apply Theorem 5.2 to \( \tilde{f} \). As in (21) we consider the factorization \( \partial f/\partial Y = X^{d_X(f)} g_1 \cdots g_u \) where the branches \( g_1, \ldots, g_u \) are coprime with \( X \). We have \( \text{Zer}(\partial f/\partial Y) = \text{Zer} g_1 \oplus \cdots \oplus \text{Zer} g_u \). With the notation of Theorem 5.2 we state the following

**Lemma 5.15** Let \( z(X) \in \text{Zer}(\partial f/\partial Y) \). Then

(I) if \( z \in \mathbf{A}_S, S \in \mathcal{N}_f, \text{incl}(S) = 1 \) then \( \mathcal{B}_f([z]) = \mathcal{B} \);

(II) if \( z \in \mathbf{A}_S, S \in \mathcal{N}_f, \text{incl}(S) < 1 \) then

(\cdot) if \( z \) is of the first kind then \( \mathcal{B}_f([z]) = \mathcal{B}(X, 1/\text{incl}(S)) \);

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if $z$ is of the second kind then $B_f([z]) < B(X, 1/\text{incl}(S))$;

(III) if $z \in A_S$, $S \in \mathcal{N}_f$, $\text{incl}(S) > 1$ then $B_f([z]) = B(Y, \text{incl}(S))$;

(IV) if $z \in A_{\varphi,S}$, $\varphi \neq 0$, $S \in \mathcal{N}_{\varphi}$ then $B_f([z]) = B([\varphi], R_{\varphi,S})$ where $R_{\varphi,S} := R_{\varphi,\text{incl}(S)}$.

Proof. By Corollary 2.9(b) we have

$$B_f([z]) = B(h, R) \iff \left( \max\{d(f_1, [z]), \ldots, d(f_r, [z])\} = R \right).$$

Recall that

$$\text{ord} z = \frac{d([z], Y)}{d([z], X)}. \quad (44)$$

(I). Let $z \in A_S$, $\text{incl}(S) = 1$. Let $I_S = \{i \in \{1, \ldots, r\} : d(f_i, X) = d(f_i, Y) = 1\}$. Assume that $z$ is of the first kind (0-first kind). That is $\text{ord} z = \text{incl}(S) = 1$ (the styles of edges are not expressed in the sketches).

Hence $d([z], X) = d([z], Y) = 1$ and by the definition of $A_S$ in $z$ is not a root of $\text{in}(f, S) = \text{in} f = \prod_{i=1}^{r} f_i$. For $i \in I_S$ there exists $S_i \in \mathcal{N}(f_i)$ parallel to $S$ and $\text{in} f_i = \text{in}(f_i, S_i)$. Therefore, $d(f_i, [z]) = 1$ by Property 5.13. For $i \notin I_S$ we have $d(f_i, X) > 1$ or $d(f_i, Y) > 1$. Hence $d(f_i, [z]) = 1$ by (D′ 3) and we have $B_f([z]) = B$ by (43).

If $z$ is of the second kind then $S$ touches the horizontal axis (Proposition 5.1(ii)).

We have $d(f_i, Y) = 1$ for $i \in \{1, \ldots, r\}$ and $\text{ord} z > \text{incl}(S) = 1$ by Proposition 5.1(iii). Hence $d([z], Y) > 1$ by (44). From (D′ 3) we obtain $d(f_i, [z]) = 1$ for $i \in \{1, \ldots, r\}$ which gives $B_f([z]) = B$ by (43).
(II). Let \( z \in A_S \), \( \text{incl}(S) < 1 \). Let \( I_S = \{ i \in \{1, \ldots, r\} : d(f_i, X) = 1/\text{incl}(S) \} \). Assume that \( z \) is of the first kind.

Hence \( d([z], X) = 1/\text{incl}(S) \) and \( z \) is not a root of \( \text{in}(f, S) = \prod_{i=1}^{r} \text{in}(f_i) \) with \( \vec{v} = (1, \text{incl}(S)) \). Analogously, as earlier by using Property 5.13 and \((D_3')\) we conclude that \( B_f([z]) = B(X, 1/\text{incl}(S)) \).

If \( z \) is of the second kind, then \( S \) touches the horizontal axis and \( \text{ord} z > \text{incl}(S) \) by Proposition 5.1(iii).

(III)+(IV). Let \( z \in A_{\varphi,S} \), \( S \in \mathcal{N}_\varphi \) (if \( \varphi = 0 \) then \( \text{incl}(S) > 1 \)). Let \( I_{\varphi,S} = \{ i \in \{1, \ldots, r\} : d(f_i, [\varphi]) = R_{\varphi,S} \} \). Assume that \( z \) is of the \( \varphi \)-first kind.

Then \( \text{ord}(z - \varphi) = \text{incl}(S) \) and \( d([z], [\varphi]) = R_{\varphi,S} \) by Property 5.13. By the definition of \( A_{\varphi,S} \) \( \text{in}(z - \varphi) \) is not a root of \( \text{in}(f_\varphi, S) = \prod_{i=1}^{r} \text{in}(f_i)_\varphi \) with \( \vec{v} = (1, \text{incl}(S)) \). As in (I) by using Property 5.13 and \((D_3')\) we conclude that \( B_f([z]) = B([\varphi], R_{\varphi,S}) \).

If \( z \) is of the \( \varphi \)-second kind, then \( S \) touches the horizontal axis, \( \text{ord}(z - \varphi) > \text{incl}(S) \)
and \( d(f_i, [\varphi]) \leq R_{\varphi,S} \) for \( i \in \{1, \ldots, r\} \) by Proposition 5.13.

By Proposition 5.13 \( d([z], [\varphi]) > R_{\varphi,S} \). Therefore, by \((D'_3)\) \( d(f_i, [z]) = d(f_i, [\varphi]) \). Hence \( d(f_i, [z]) \leq R_{\varphi,S} \) with equality for \( i \in I_{\varphi,S} \). By using (13) we obtain \( B_f([z]) = B([\varphi], R_{\varphi,S}) \).

Let \( \text{Track}(f) = \text{Track}(f) \setminus \{0\} \). Following Lemma 5.15 let us define three sets of balls. We denote by \( \mathcal{T}_1 \) the set of balls \( B(X, 1/\max(\mathcal{N}_f)) \) where \( S \in \mathcal{N}_f \) with \( \max(\mathcal{N}_f) < 1 \), by \( \mathcal{T}_2 \) the set of balls \( B(Y, \max(\mathcal{N})) \) where \( S \in \mathcal{N}_f \) with \( \max(\mathcal{N}) \geq 1 \) and by \( \mathcal{T}_3 \) the set of balls \( B([\varphi], R_{\varphi,S}) \) where \( \varphi \in \text{Track}(f)^* \), \( S \in \mathcal{N}_f \).

The chain \( \overline{\mathcal{K}}_f(X) = \{B(f_1, X), \ldots, B(f_r, X)\} \) plays an important role in our approach (see Property 5.1 and the text before).

**Property 5.16** (description of the chain \( \overline{\mathcal{K}}_f(X) \))

(a) \( \mathcal{T}_1 \subset \overline{\mathcal{K}}_f(X) \subset \{B\} \cup \mathcal{T}_1 \cup \{B(X, X)\} \) with

- \( B \in \overline{\mathcal{K}}_f(X) \iff \max(\mathcal{N}_f) \geq 1 \),
- \( B(X, X) \in \overline{\mathcal{K}}_f(X) \iff \delta_X(f) > 0 \).

(b) \( \min \overline{\mathcal{K}}_f(X) = \begin{cases} B(X, 1/\max(\mathcal{N}_f)) & \text{if } \max(\mathcal{N}_f) < 1 \\ B & \text{if } \max(\mathcal{N}_f) \geq 1 \end{cases} \)

(c) \( B_f(X) = \begin{cases} B(X, X) & \text{if } \delta_X(f) > 0 \\ B(X, 1/\min(\mathcal{N}_f)) & \text{if } \delta_X(f) = 0 \text{ and } \min(\mathcal{N}_f) < 1 \\ B & \text{if } \delta_X(f) = 0 \text{ and } \min(\mathcal{N}_f) \geq 1 \end{cases} \)

Lemma 5.15 and Property 5.16 allow us to finish the proof of part (ii) of the factorization theorem. We want to show the equality \( d(g, X) = d_f(g, X) \) for every branch \( g \) of \( \partial f / \partial Y \) where \( d_f(g, X) \) is defined by (24). If \( B_f(g) \neq B_f(X) \) then the equality is a direct consequence of \((D_1 - D_3)\). Assume that \( B_f(g) = B_f(X) \). To finish the proof it suffices to find a branch \( f_{\text{io}} \in B_f \) such that

\[
d(g, X) = \min\{d(f_{\text{io}}, g), d(f_{\text{io}}, X)\}. \tag{45}
\]

Let us consider the cases determined for \( B_f(X) \) by Property 5.16.

If \( \delta_X(f) > 0 \) then \( B_f(X) = B(X, X) \). The equality \( B_f(g) = B_f(X) \) leads to \( g = X \). With \( f_{\text{io}} = X \) we obtain \( \infty \) on both sides of (45).
If $\delta_X(f) = 0$ and $\min \text{incl}(N_f) < 1$ then $B_f(X) = B(X, 1/\text{incl}(S))$ where $S$ has the minimal inclination. Now, every branch of $\partial f/\partial Y$ has the form $g = [z]$ where $z \in \text{Zer}(\partial f/\partial Y)$. From $B_f([z]) = B(X, 1/\text{incl}(S))$ and Lemma 5.15 it follows that $z \in A_S$ and $z$ is of the first kind. We obtain (45) for every $i_0 \in I$ where $I$ is defined in the proof of Lemma 5.10.

If $\delta_X(f) = 0$ and $\min \text{incl}(N_f) \geq 1$ then $B_f(X) = B$. The case of Lemma 5.15(II)(i) is impossible. The only possibility is that $z \in A_S$, $S \in N_f$, incl($S$) = 1. As in the proof of Lemma 5.15 we show that (45) is satisfied for every $i_0 \in \{1, \ldots, r\}$.

We are in a good position to finish the proof of parts (i) and (iii) of Theorem 4.3. Let $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$ and let $\mathcal{T}' := \mathcal{T} \cup \{B\}$. By Property 5.16 $\min \overline{\mathcal{K}}_f(X) \in \mathcal{T}'$. Let $\text{Zer}(\partial f/\partial Y) = \{z_1, \ldots, z_{p-1}\}$ as earlier. Let $B \in \mathcal{T}'$. For $B \neq \min \overline{\mathcal{K}}_f(X)$ we put $J_B = \{j \in \{1, \ldots, p-1\} : B_f([z]) = B\}$ and for $B = \min \overline{\mathcal{K}}_f(X)$ we put $J_B = \{j \in \{1, \ldots, p-1\} : B_f([z]) \leq B\}$. For every $B \in \mathcal{T}'$ let us define $h_B = \prod_{j \in J_B}(Y - z_j)$ with convention $\prod_{\emptyset} = 1$. Clearly $h_B \in \mathbb{C}\{X, Y\}$. By Lemma 5.15 and Theorem 5.2 for $B \in \mathcal{T}$ the number $(h_B, X)_0$ equals

$t(f, S) - 1$ if $B = B(X, 1/\text{incl}(S))$, $S \in N_f$, incl($S$) < 1

$t(f, S) - 1$ if $B = B(Y, \text{incl}(S))$, $S \in N_f$, incl($S$) \geq 1

$N(\varphi)(t(f, S) - 1)$ if $\varphi \in \text{Track}(f)^*$, $S \in N_\varphi$

For every $B \in \mathcal{T}'$ we put

$m'(B) = \begin{cases} (h_B, X)_0 & \text{if } B \in \mathcal{T} \\ 0 & \text{if } B \notin \mathcal{T}. \end{cases}$

The second case is possible only if $B = B$ and $B \notin \mathcal{T}$.

To finish the proof of Theorem 4.3(iii) we need the following

**Proposition 5.17** For $B \in \mathcal{T}'$ the number $m'(B)$ equals:

\[d(B, X)\nu(B)[t_f^{(1)}(B) + n(B)t_f^{(2)}(B) - 1] \text{ for } B \in \mathcal{T}' \setminus \overline{\mathcal{K}}_f(X);\]

\[d(B)n(B)[t_f(B) - 1 + \sigma_{f,X}^{\text{max}}(B)] - \sigma_{f,X}^{\text{min}}(B) \text{ for } B \in \overline{\mathcal{K}}_f(X).\]

Proof. Assume first that $B \in \mathcal{T}_3$ (hence $B \notin \overline{\mathcal{K}}_f(X)$ by Property 5.10). There exist $\varphi \in \text{Track}(f)^*$ and $S \in N_\varphi$ such that $B = B([\varphi], R_\varphi, S)$. We have incl($S$) = $m/(N\bar{n})$, GCD($\bar{n}, m$) = 1, $N = N(\varphi)$, $\bar{n} = n(B)$ (Proposition 5.12). Let us observe that $t(f, S) = n(B)r_0^{(N)}(\text{incl}(f, S)) + \varepsilon_Y(\text{incl}(f, S))$. We finish by using (38), Property 4.1 and Property 5.14.

If $B \in \mathcal{T}_2 \setminus \{B\}$ (hence $B \notin \overline{\mathcal{K}}_f(X)$) we put $\varphi = 0$ above.

If $B \in \mathcal{T}_1$ then $B \in \overline{\mathcal{K}}_f(X)$. We have $B = B(X, 1/\text{incl}(S))$, $S \in N_f$, incl($S$) < 1. In this case $\sigma_{f,X}^{\text{max}}(B) = 1 - \varepsilon_X(\text{incl}(f, S))$ and $\sigma_{f,X}^{\text{min}}(B) = 1 - \varepsilon_Y(\text{incl}(f, S))$. We have $t(f, S) = d(B)n(B)r_0^{(N)}(\text{incl}(f, S)) + \varepsilon_Y(\text{incl}(f, S))$. We finish by using Proposition 5.14.

Now, consider the case when $B \in \mathcal{T}$. Hence there exists $S \in N_f$ with incl($S$) = 1 and $B \in \overline{\mathcal{K}}_f(X)$. In this case $\sigma_{f,X}^{\text{max}}(B) = 1 - \varepsilon_X(\text{incl}(f, S))$ and $\sigma_{f,X}^{\text{min}}(B) = 1$. We finish by using Proposition 5.14.
If $B \notin T$ then there does not exist a face of inclination 1 in $N_f$. We check the appropriate formulas directly.

To end the proof of Theorem 4.3 (i) and (iii) we observe that for $B \in T'$ if $m'(B) > 0$ then $B \in E(f) \cup \{B_f(X)\}$.

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