MULTI-PRIME RSA OVER GALOIS APPROACH

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Many variants of RSA cryptosystem exist in the literature. One of them is RSA over polynomials based on Galois approach. In standard RSA modulus of product of two large primes whereas in the Galois approach author consider product of two irreducible polynomials as modulus. We use this idea and extend Multi-prime RSA over polynomials.

1. INTRODUCTION

RSA Cryptosystem is the first practical realization of the public-key system invented by Rivest, Shamir and Aldeman [3]. One of the variant of RSA is obtained by modifying the RSA modulus i.e. Multiprime RSA [1] and the other variant is extension of RSA over polynomials [2]. We extended the Multiprime RSA over polynomials.

1.1. RSA Cryptosystem. The best known public-key cryptosystem is the RSA, named after its inventors Rivest, Shamir and Adleman [3]. The RSA cryptosystem is defined as below:

**Key Generation:** We generate two randomly and large primes \( p \) and \( q \) and computes the product

\[
    n = pq.
\]

Then choose an integer \( e \) with \( 1 < e < \phi(n) \) and \( \gcd(e, \phi(n)) = 1 \).

Then compute the integer \( d \) with \( 1 < d < \phi(n) \) and \( de = 1 \mod \phi(n) \).

Since \( \gcd(e, \phi(n)) = 1 \), such a number \( d \) exists. It can be computed by Extended Euclidean Algorithm.

Public key is pair \((n,e)\) and Private key is \((d,p,q)\). \( n \) is called the RSA modulus, \( e \) is encryption exponent and \( d \) is decryption exponent.

**Encryption:** Let \( x \in Z_n \) be the plaintext then \( x \) can be encrypted as

\[
    e_k(x) = x^e \mod n
\]

\( e_k(x) = y \) is the ciphertext.

**Decryption:** If \( y \in Z_n \) is the ciphertext, then \( x \) can be computed as

\[
    x = d_k(y) = y^d \mod n.
\]

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1.2. Multi-Prime RSA. As the name suggests in Multi-prime RSA uses the modulus of the form $N = p_1 p_2 \ldots p_r$, product of more than two primes introduced by Collins, Hopkins, Longford and Sabin [1]. We first describe the key generation, encryption and decryption as below:

**Key generation:** The key generation algorithm takes as input a security parameter $n$ and an additional parameter $b$. It generates an RSA public/private key pair as follows:

1. Generate $b$ distinct primes $p_1, ..., p_b$ each $\lfloor n/b \rfloor$-bits long. Set $N \leftarrow \prod_{i=1}^{b} p_i$. For a 1024-bit modulus we can use at most $b = 3$ (i.e., $N = pqr$).
2. Pick the same $e$ used in standard RSA public keys, namely $e = 65537$. Then compute $d = e^{-1} \mod \phi(N)$ . As usual, we must ensure that $e$ is relatively prime to $\phi(N) = \prod_{i=1}^{b} (p_i - 1)$.

The public key is $(N, e)$; the private key is $d$.

**Encryption:** Given a public key $(N, e)$, the encrypter encrypts exactly as in standard RSA.

**Decryption:** Decryption is done using the Chinese Remainder Theorem (CRT). Let $r_i = d \mod (p_i - 1)$. To decrypt a ciphertext $C$ first compute, $M_i = C^{r_i \mod p_i}$ for each $i, 1 \leq i \leq b$. Then combines the $M_i$’s using the CRT to obtain $M = C^d \mod N$.

1.3. Extension of RSA Over Polynomials. This cryptosystem was proposed by Karvitz and Reeds in 1982 [2]. In standard RSA cryptosystem the modulus $n$ is the product of two primes, and the security of RSA depends on factoring the modulus $n$. Karvitz and Reeds took two irreducible polynomials and then considered the product of two polynomials as the RSA modulus. The key generation, encryption and decryption is defined as below:

**Key generation:** Let $F$ be a finite field and choose two irreducibles polynomials $p_1(x)$ and $p_2(x)$ of higher order of say $n_1$ and $n_2$ and then compute $f(x) = p_1(x)p_2(x)$ of degree $n = n_1 + n_2$.

Choose a $d$ such that it is relatively prime to $(|F|^{n_1} - 1)(|F|^{n_2} - 1)$ and then compute $e$ such that $ed = 1 \mod (|F|^{n_1} - 1)(|F|^{n_2} - 1)$.

Then public key will be $e$ and the secret key will be $d$.

**Encryption:** Let $m$ be message, put the message into $|F|$-array representation and break it into block each of size $n$.

Each block is associated with a polynomial over $F$ of degree less than $n$.

The plaintext is encrypted by $C(x) = (M(x)^e \mod f(x))$, then $c(x)$ will be the ciphertext.

**Decryption:** If $C(x)$ is the ciphertext then using the decryption exponent $d$ then the plaintext can be computed as $M(x) = C(x)^d \mod f(x)$.
2. Proposed Scheme Based On Galois Approach

We will introduce a new scheme based on Galois approach by modifying the RSA modulus as we have discussed in multiprime RSA. Before considering that scheme we introduce the notion of Chinese remainder theorem over polynomials.

**Theorem 2.1. Chinese Remainder Theorem Over Polynomials:**
Suppose \( m_1(x), \ldots, m_r(x) \) be polynomials that are relatively prime to each other. Let \( a_1(x), \ldots, a_n(x) \) be polynomials and then the following system of congruences:
\[
p(x) \equiv a_1(x)(mod m_1(x)) \\
p(x) \equiv a_2(x)(mod m_2(x)) \\
\vdots \\
p(x) \equiv a_r(x)(mod m_r(x))
\]
has a unique solution modulo \( M(x) = m_1(x) \times m_2(x) \ldots m_r(x) \), which is given by
\[
\sum_{i=1}^{r} a_i(x)M_i(x)y_i(x)mod M(x),
\]
where \( M_i(x) = M(x)/m_i(x) \) and \( y_i(x) = M_i(x)^{-1}mod m_i(x) \), for \( 1 \leq i \leq r(x) \).

Using the above stated theorem we can extent the concept of RSA-CRT over polynomials also. Let us consider the key generation, encryption and decryption as following:

**Key Generation:** Same as RSA over Galois Approach.

**Encryption:** Same as we have discussed in RSA over Galois Approach.

**Decryption:**
If \( c(x) \) is the ciphertext and \( d \) is the private key then, first Compute,
\[
d_p = dmod|F|^n_1 - 1 \quad \text{and} \quad d_q = dmod|F|^n_2 - 1
\]
then Compute
\[
M_p(x) = C^{d_p}mod p_1(x) \quad \text{and} \quad M_q(x) = C^{d_q}mod p_2(x)
\]
Now, using Chinese remainder theorem we can find the plaintext \( m \) as,
\[
M(x) = y_{p_1}(x)M_{p_1}(x)p_1(x) + y_{p_2}(x)M_{p_2}(x)p_2(x), \quad \text{where}
\]
\[
y_{p_1}(x) = M_{p_1}^{-1}(x)mod p_2(x) \quad \text{and} \quad y_{p_2}(x) = M_{p_2}^{-1}(x)mod p_1(x).
\]
Now, we will consider the Multiprime RSA over polynomial as here we will modify the RSA modulus by considering it of the type \( f(x) = p_1(x)p_2(x)\ldots p_b(x) \), product of more than two irreducible polynomials of large degree.
Let us consider its key generation, encryption and decryption:
Key generation: The key generation algorithm takes as input a security parameter $n$ and an additional parameter $b$. It generates an RSA public/private key pair as follows:

1. Generate $b$ distinct irreducible polynomials $p_1(x),...,p_b(x)$ each of degree $n/b$ over the finite field $F$. Set $f(x) = \Phi(x)$.
2. Choose a $d$ such that it is relatively prime to $\phi(f(x)) = (|F|^{n_1} - 1)(|F|^{n_2} - 1),...,(|F|^{n_b} - 1)$ and compute $e$ such that $e = d^{-1} \mod \phi(f(x))$.

The public key is $(f(x), e)$; the private key is $d$.

Encryption: Same as Standard RSA over polynomials as we have discussed before.

Decryption: Decryption is performed using the Chinese Remainder Theorem (CRT). Let $r_i = d \mod (|F|^{n_i} - 1)$. To decrypt a ciphertext $C(x)$, first computes

$$M_i(x) = C(x)^{r_i} \mod p_i(x)$$

for each $i, 1 \leq i \leq b$.

Then combines the $M_i(x)$’s using the CRT to obtain $M(x) = C(x)^d \mod N$.

The advantage of the proposed scheme over [1] is same as the advantage of multi-prime RSA [2] and RSA with CRT [3] over standard RSA [3].

Example 2.2. Let us consider an example of above scheme we use $b = 3$, consider the three irreducible polynomials $p_1(x) = x^3 + x + 1$, $p_2(x) = x^3 + 2 + x$ and $p_3(x) = x^2 + x + 1$ over $Z_2$.

We compute $f(x) = p_1(x) * p_2(x) * p_3(x) = x^8 + x^6 + x^5 + x^4 + x^3 + x^2 + 1$

then $\phi(f(x)) = (2^3 - 1)(2^3 - 1)(2^2 - 1) = 147$. We choose $e = 34$ randomly such that $\gcd(e, \phi(f(x))) = 1$, now compute $d = e^{-1} \mod \phi(f(x)) = 34^{-1} \mod 147 = 13$.

Let $m(x) = x^4 + x^3 + 1$ be the plaintext the on encryption we get,

$$c(x) = x^4 + x^3 + x^2 + 1 \mod f(x) = x^5 + x^4 + x^3 + x^2 + 1.$$ 

and now for decryption, compute $r_1 = d \mod (|F|^{n_1} - 1) = 6$, $r_2 = d \mod (|F|^{n_2} - 1) = 6$ and $r_3 = d \mod (|F|^{n_3} - 1) = 6$. Then,

$$m(x) = c(x)^{r_1} \mod p_1(x) = x + 1$$

$$m(x) = c(x)^{r_2} \mod p_2(x) = x^2$$

$$m(x) = c(x)^{r_3} \mod p_3(x) = 1$$

then using Chinese remainder theorem plaintext will be

$$m(x) = c(x)^d \mod (x^8 + x^6 + x^5 + x^4 + x^3 + x^2 + 1) = x^4 + x^3 + 1$$

The above example can be easily implemented using MATLAB to avoid long calculations.

References

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