Ioannis Papageorgiou  
IME, Universidade de Sao Paulo

MODIFIED LOG-SOBOLEV INEQUALITY FOR A COMPACT PURE JUMP MARKOV PROCESS WITH DEGENERATE JUMPS.

Abstract. We study the modified log-Sobolev inequality for a class of pure jump Markov processes that describe the interactions between brain neurons. In particular, we focus on a finite and compact process with degenerate jumps inspired by the model introduced by Galves and Löcherbach in [16]. As a result, we obtain concentration properties for empirical approximations of the process.

1. Introduction

We study properties of the model introduced by Galves and Löcherbach in [16], in order to describe the interaction activity occurring between brain neurons. We focus in particular on finite networks of compact neurons taking values in the domain of the invariant measure. What is in particular interesting about this jump process is the degenerated character of the jumps, in the sense that after a particle spikes, it then jumps to zero and therefore loses its memory. In addition, the spike probability of a specific neuron at any time depends on its actual position at that time and so depends on the past of the whole neural system since the last time this neuron had a spike.

The aim of the paper is to show the modified logarithmic Sobolev inequality for the model and consequently obtain empirical concentration properties. In [21] Poincaré type inequalities were proven. There were two separate cases that were examined. At first, the initial configuration was a general one, and then the initial configuration belonged to the domain of the invariant measure. In the current paper where we are restricted exclusively to the case where the initial configuration

---

2010 Mathematics Subject Classification.  60K35, 26D10, 60G99.

Key words and phrases. modified log-Sobolev inequality, brain neuron networks, Pure Jump Markov Processes, concentration, empirical approximations.

Address: Neuromat, Instituto de Matematica e Estatistica, Universidade de Sao Paulo, rua do Matao 1010, Cidade Universitaria, Sao Paulo - SP- Brasil - CEP 05508-090.

Email: ipapageo@ime.usp.br, papyannis@yahoo.com

This article was produced as part of the activities of FAPESP Research, Innovation and Dissemination Center for Neuromathematics (grant 2013/07699-0, S.Paulo Research Foundation); This article is supported by FAPESP grant (2017/15587-8).
belongs to the domain of the invariant measure we will obtain the stronger modified log-Sobolev inequality, of the following form

\[ P_t \left( f \log \frac{f}{P_t f} \right) \leq C(t) P_t \left( \frac{\Gamma(f, f)}{f} \right), \]

for the associated semigroup \( P_t \). In that way, we show that when the process enters at the invariant domain, despite the degenerate nature of its jumps, it behaves similar to a non degenerate jump process.

As a result of the modified log-Sobolev inequality, for any neuron \( j \) and times \( t_1 < \ldots < t_n < T \) for some \( T > 0 \), we can obtain concentration inequalities for empirical approximations of the process as the ones shown bellow

\[ P \left( \left| \frac{\sum_{k=1}^{n} f(X_{t_k}^j)}{n} - \sum_{k=1}^{n} \mathbb{E}[f(X_{t_k}^j)] \right| \geq \epsilon \right) \leq D e^{-\epsilon n}. \]

These imply that the empirical approximation converges exponential fast to the process as the number of the observables \( n \) goes to infinity.

In the next section, we present the model that describes the neuroscience framework of the problem.

1.1. The model. We want to model the action potential, called spike, of the membrane potential of a neuron. In relation to this spiking activity, there are two important features. The first is the degenerate nature of the spiking, that relates to the fact that whenever a neuron spikes its membrane potential resets to zero. The second characteristic is that the probability of a neuron \( j \) with membrane potential on an actual position \( x_{t_k}^j \) to spike at a given time \( t \), depends on its position at this time, through its intensity \( \phi(x_{t_k}^j) \), where \( \phi : \mathbb{R}_+ \to \mathbb{R}_+ \) is a given intensity function. The interaction between the neurons occurs by all the post-synaptic neurons \( i \) receiving an additional amount of membrane potential \( W_{j \to i} \) from the neuron \( j \) that spiked. In the current work we consider the case of pure jump Markov process, abbreviated as PJMP, where inactivity occurs, e.g. lack of a drift, between two consecutive spikes.

The spiking activity of an individual neuron, can be modeled by a single point process as in [9], [14], [15], [16], [18] and [20]. In these papers the emphasis is put on describing the spiking time. Here however, we focus on modelling the interactions occurring between the neurons in the network through spikes, as was done in [1], [19], [11], [12], [10], [24], [27] and [3].

To do this, for a network comprising of \( N > 1 \) neurons, we consider the Markov process \( X_t = (X_t^1, \ldots, X_t^N) \), representing the membrane potential of each neuron at time \( t \in \mathbb{R}_+ \). Then, for every \( t \geq 0 \) and \( i = 1, \ldots, N \), \( X_t \) solves the following
equation

\[ X^i_t = X^i_0 - \int_0^t \int_0^\infty X^i_{s-1} 1_{\{z \leq \phi(X^i_{s-1})\}} N^i(ds, dz) \]

\[ + \sum_{j \neq i} W_{j \rightarrow i} \int_0^t \int_0^\infty 1_{\{z \leq \phi(X^j_{s-1})\}} 1_{\{X^i_{s-1} \leq m - W_{j \rightarrow i}\}} N^j(ds, dz), \]

where \( (N^i(ds, dz))_{i=1,\ldots,N} \) is a family of i.i.d. Poisson random measures on \( \mathbb{R}_+ \times \mathbb{R}_+ \) with intensity measure \( dsdz \).

For any test function \( f : \mathbb{R}_+^N \rightarrow \mathbb{R} \) and \( x \in [0, m]^N \) the generator of the process is given by

\[ (1.2) \quad \mathcal{L}f(x) = \sum_{j=1}^N \phi(x^j) [f(\Delta_j(x)) - f(x)] \]

where we have denoted

\[ (1.3) \quad (\Delta_j(x))_i = \begin{cases} x^i + W_{j \rightarrow i} & i \neq j \text{ and } x^i + W_{j \rightarrow i} \leq m \\ x^i & i \neq j \text{ and } x^i + W_{j \rightarrow i} > m \\ 0 & i = j \end{cases} \]

for some \( m > 0 \) and weights \( W_{j \rightarrow ij} > 0 \). Furthermore, we also assume that for some strictly positive constant \( \delta \), the intensity function satisfies the following conditions:

\[ (1.4) \quad \phi(x) \geq \delta. \]

1.2. Main results. For simplicity, we will widely use the following convention. For the expectation of a function \( f \) with respect to a measure \( \nu \) we will write

\[ \nu(f) = \int f dv. \]

We consider a Markov process \( (X_t)_{t \geq 0} \) which is described by the Markov semigroup \( P_t f(x) = E^x(f(X_t)) \) and \( \mathcal{L} \) the associated infinitesimal generator.

We define \( \mu \) to be the invariant measure for the semigroup \( (P_t)_{t \geq 0} \) if and only if

\[ \mu P_t = \mu. \]

Define \( D \) the domain of the invariant measure \( \mu \), that is

\[ D = \left\{ x \in (\mathbb{Z} \cap [0, m])^N : \mu(x) > 0 \right\}. \]

Furthermore, we define the so called ”carré du champ” operator by:

\[ \Gamma(f, g) := \frac{1}{2}(\mathcal{L}(fg) - f \mathcal{L}g - g \mathcal{L}f). \]
For the PJMP process defined as in (1.2)-(1.3) we then have
\[ \Gamma(f, f) = \frac{1}{2} \left( \sum_{i=1}^{N} \phi(x_i) \left[ f(\Delta_i(x)) - f(x) \right]^2 \right). \]

We are interested in studying the modified log-Sobolev inequality for the semigroup \( P_t \) on a discreet setting (see [28], [13], [31], [2] and [8]). In [21] a Poincaré type inequality was shown for the semigroup \( P_t \) of the bounded process (1.2)-(1.3) for general initial configurations. In the current paper, we study again bounded neurons (that is \( m < \infty \)), but this time we focus exclusively on the case where the initial configuration belongs on the domain of the invariant measure. Restricting the domain of the process, allows to strengthen the results for the semigroup. The method that we use, is based on the so called semigroup method which is used to prove log-Sobolev and Poincaré inequalities for the semigroup \( P_t \) (see [2] and [31]), usually with a constant that depends on time \( t \).

Then, we study concentration inequalities for empirical approximations of the model. Although, in principle we use technics that relate modified log-Sobolev inequalities to concentration properties (see [29] and [30], [6], [22] and [23]), in order to obtain the empirical concentration inequalities, we actually need to extend the results of the modified log-Sobolev inequality to cylindrical functions. Although, neither the initial modified inequality obtained, nor the one for the cylindrical functions has the standard form (they both include the carré du champ beyond one jump), we still manage to obtain the desired concentration properties.

Before we proceed with the presentation of the results we will clarify a distinction on the dual nature of the initial configuration from which the process may start. This classification is based on the return probability to the initial configuration. We recall that the main mechanism of the dynamics dictates that the membrane potential of every neuron lies within some positive compact set and that whenever a neuron spikes, every other neuron jumps some length up, while the only movement downwards that it can do is to fall to zero when and only it spikes. Furthermore, in between spikes the neurons stay still. That implies that there is a finite number of possible configurations to which the membrane potential of the neural system can return after every neuron has spiked at least one time. This is the domain of the invariant measure \( \mu \) of the semigroup \( P_t \). As a result, whenever an initial configuration does not belong to the domain of the invariant measure, after the process has entered the invariant domain it can never return back to the initial configuration. In [21] the focus was on general initial configurations that were allowed not to belong to the domain of the invariant measure. Examples of such configurations are easy to construct. One can for instance think of configurations where \( x^i = x \) for all \( i = 1, \ldots, N \), or \( x^i \) not being an analogue of \( W_{j \rightarrow i} \). In the current work however, we restrict ourselves exclusively to the cases where the
initial configuration, and so, every configuration, belongs in the domain of the invariant measure.

The first result of the paper related to the modified log-Sobolev inequality follows.

**Theorem 1.1.** Assume the PJMP as described in (1.2)-(1.4). Then, for any \( x \in D \), we have the following type of modified log-Sobolev inequality

\[
Pt \left( f(x) \log \frac{f(x)}{Pt f(x)} \right) \leq \delta(t)Pt \left( \frac{\Gamma(f, f)(x)}{f(x)} \right) + \delta(t) \sum_{j=1}^{N} Pt \left( \frac{\Gamma(f, f)(\Delta_j(x))}{f(x)} \right) + \delta(t) \sum_{i,j=1}^{N} Pt \left( \frac{\Gamma(f, f)(\Delta_i \Delta_j(x))}{f(x)} \right)
\]

where \( \delta(t) \) is an increasing polynomial of degree three, for which \( \delta(0) = 0 \).

The modified log-Sobolev inequality presented on the theorem, is not very different from the standard form (1.1) of the modified log-Sobolev inequality. At first it should be noticed that it implies the same concentration properties, as presented in Proposition 1.3. Furthermore, for a special class of functions it is equivalent to the standard modified log-Sobolev inequality studied for example in [8] and [2], as shown on the following corollary.

**Corollary 1.2.** For any \( i \in \{1, ..., N\} \), define \( f_i : [0, m]^N \rightarrow \mathbb{R} \) a function that depends only on \( x^i \). Assume the PJMP as described in (1.2)-(1.4), with equal weights \( w_{ij} = w, \forall 1 \leq i, j \leq N \), for some \( w \in (0, +\infty) \). If for every \( i \in \{1, ..., N\} \), \( f_i \) is either decreasing and convex, or increasing and concave, then for every \( x \in D \),

\[
P_t \left( f_i(x) \log \frac{f_i(x)}{Pt f_i(x)} \right) \leq \zeta(t)Pt \left( \frac{\Gamma(f_i, f_i)(x)}{f_i(x)} \right)
\]

where \( \zeta(t) \) is a polynomial of degree three.

This modified log-Sobolev inequality describes one neuron in the context of the whole system.

It should be noted that the main hindrance in obtaining a log-Sobolev inequality, is down to the degenerate character of the jump process under study, since the loss of memory of the spiking neuron does not allow the translation property

\[
\mathbb{E}^{x+y}f(X_t) = \mathbb{E}^{x}f(X_t + y)
\]

used in [31] and [2] to show the relevant inequalities. The absence of the translation property implies that the inequalities \( \Gamma(P_t f, P_t f) \leq P_t \Gamma(f, f) \) and \( \sqrt{\Gamma(P_t f, P_t f)} \leq P_t \sqrt{\Gamma(f, f)} \) that are used to show Poincaré and log-Sobolev inequalities respectively do not hold. This is directly related with the \( \Gamma_2 \) criterion (see [4] and [5]) which provides log-Sobolev and Poincaré inequalities (see also [2]). Still, a weaker property shown here proves the modified log-Sobolev inequality of the theorem.
In that way, we see that despite the degenerate character of the process due to it’s discontinuous jumps, when it enters the domain of the invariant measure, it does not behave very different from a non degenerate process which satisfies the typical sweeping out relation $\Gamma(P_t f, P_t f) \leq P_t \Gamma(f, f)$ and the $\Gamma_2$ criterion. This is further testified by the concentration properties obtained below, as well as by the classical modified log-Sobolev inequality of Corollary 1.2. Still, both the higher than order one time constant and the additional two jump terms, highlight how much complicated this process is.

As a direct result of the modified log-Sobolev inequality of Theorem 1.1 we obtain the following concentration inequality.

**Proposition 1.3.** Assume the PJMP as described in (1.2)-(1.4) and that $f_i$ is Lipschitz continuous functions that depend only on $x^i$, with Lipschitz constant 1. Consider $T > 0$. Then, for the function $f(x) = \sum_{i=1}^N f_i(x^i)$, $x \in D$, the following concentration inequality holds

$$P\left(\left|f(X_t) - \mathbb{E}^x f(X_t)\right| \geq r\right) \leq Q e^{-r^2},$$

for any $t \leq T$ and a positive constant $Q$ that depends on $T$.

It should be noted that the above concentration inequality is the concentration property that is typically derived from the standard modified log-Sobolev inequality.

Concentration properties like this one are of course more interesting in the context of non-bounded random variables, and less in the context of finite many bounded valued random variables as in Proposition 1.3, although in the context of the current paper they highlight the closeness between the modified inequality of Theorem 1.1 and the standard inequality of Corollary 1.2.

More interesting are concentration inequalities for empirical measures, as shown in the following theorem. These concentration estimates show that the empirical estimates stabilize exponentially fast as the number of observables goes to infinity.

**Theorem 1.4.** Consider some $T > 0$ and $i \in \{1, ..., N\}$. Assume $f : \mathbb{R} \to \mathbb{R}$ a Lipschitz function, with Lipschitz constant 1, and a sequence of times $0 = t_1 < t_2 < ... < t_n \leq T$, such that

$$\sum_{k=1}^n \delta(t_k - t_{k-1}) 3^k \sum_{r=1}^{k+1} (Nd)^{r+4} < \infty,$$

for $\delta(t)$ as in Theorem 1.1 and a constant $d = \frac{M(\max_{ij} w_{ij})^2}{2} e^{2m}$. Then, for any $x \in D$, we have

$$P\left(\left|\frac{\sum_{k=1}^n f(X^i_{t_k})}{n} - \frac{\sum_{k=1}^n \mathbb{E}^x [f(X^i_{t_k})]}{n}\right| \geq \epsilon\right) \leq Ge^{-\epsilon n}$$

where $G$ a strictly positive constant.
One should notice that the concentration approximation of the theorem, provides a measure of intrinsic proximity to equilibrium, since as \( n \) goes to infinity, the \( \frac{\sum_{k=1}^{n} \mathbb{E}[f(X^t_k)]}{n} \) converges a.s, by the Statistical Ergodic Theorem (see [32]), to the expectation of \( f \) with respect to the invariant measure \( \mu(f) \). Since \( \delta(t) \) is an increasing polynomial of order three with \( \delta(0) = 0 \), in order to have condition (1.5) satisfied, we just need the distance of consequential times to decrease in such a way that

\[
\delta(t_k - t_{k-1}) \leq \left( \frac{2}{3} \right)^k \left( \sum_{r=1}^{k+1} (Nd)^{r+4} \right)^{-1}.
\]

From a technical point of view, contrary to the concentration inequalities of Proposition 1.3, the empirical concentration inequalities present the challenge of involving non bounded quantities, since they involve concentrations of sums \( \sum_{k=1}^{n} f(X^t_k) \), for \( n \) going to infinity. One should observe that the empirical concentration property does not follow directly from Proposition 1.3, as will be discussed in section 3 but from an extension of the modified inequality for cylindrical functions.

A few words about the structure of the paper. As already mentioned, the absence of the translation property, poses difficulties in obtaining the appropriate sweeping out relations of the form \( \Gamma(P_t f, P_t f) \leq P_t \Gamma(f, f) \), to prove modified log-Sobolev inequalities. Thus, in the first section 2 of the paper where we show the modified log-Sobolev inequalities, we focus on obtaining an alternative weaker sweeping out inequality (Lemma 2.7). Since, the process we study is characterized by degenerate jumps, the inequality we obtain involves additional terms that include the carré du champ after the first jump. As a result, the modified inequality we obtain in Theorem 1.1 also includes some additional terms that involve the carré du champ after the first jump as well.

These additional terms however, do not alter in essence the inequality, since as we show in Proposition 1.3 it implies the same concentration properties with the typical modified log-Sobolev inequality (1.1). The proof of these concentration properties are presented at the very end of the paper in section 3.2. Furthermore, we see that for a class of functions the inequality presented in Theorem 1.1 can be reduced to that of the typical form (1.1) as presented in Corollary 1.2. The proof of this result follows the proof of the main modified log-Sobolev inequality, at section 2.4.

The empirical concentration inequality of Theorem 1.4 is presented in 3.1. Since the property refers to cylindrical multi-times functions, to show the concentration properties we first extend some of the properties obtained in 2 to cylindrical functions, as for instance is a generalised sweeping out relation shown in Lemma 3.2.
2. PROOF OF THE MODIFIED LOG-SOBOLEV INEQUALITY.

We start by showing some technical results.

2.1. Technical results. We start by showing properties of the jump probabilities of the degenerate PJMP processes. Our process is restricted on the compact domain \( D' := \{ x \in \mathbb{R}_+^N : x^i \leq m, \ 1 \leq i \leq N \} \). Since we exclusively study configurations on the domain of the invariant measure \( \mu \), that is \( D = \{ x \in D' : \mu(x) > 0 \} \), we write \( D \) for the elements of \( D' \) that belong to the domain of the invariant measure. Since each of the finite many neurons can visit only a finite number of positions, by standard arguments of finite dimensional compact discrete Markov Chains following from the Perron-Frobenius theorem (see for instance [28] and [26]), we conclude that the invariant measure \( \mu \) exists and is unique. Following the same argumentation, if we denote the probability the process starting from \( x \) to be at \( y \) after time \( t \) by \[ \pi_t(x, y) := P_x(X_t = y), \] then \[ \lim_{t \to \infty} \pi_t(x, y) = \mu(y) \geq e > 0, \] for some \( e > 0 \) uniformly on \( x \) and \( y \). If we define the set of reachable positions of the process starting from \( x \) after time \( t \) as \( D_x := \{ y \in D, \pi_t(x, y) > 0 \} \), then one should observe that since there is not movement between two consecutive spikes, \( D_x \) is finite.

For any time \( s \in \mathbb{R}_+ \) and \( x \in D \), we denote by \( p_s(x) \) the probability that starting at time 0 from position \( x \), the process has no jump in the interval \([0, s]\). Then, if we denote \( \varphi(x) = \sum_{j \in I} \varphi(x^j) \), we have

\[ p_s(x) = e^{\varphi(x)}. \]

Furthermore, for a given neuron \( i \in I \) denote by \( p^i_s(x) \) the probability that in the interval \([0, s]\) only the neuron \( i \) spikes, and it does exactly one time. Then, for every \( x \in D \) s.t. \( \varphi(\Delta^i(x)) \neq \varphi(x) \), we compute

\[ p^i_s(x) = \int_0^s \phi(x^i)e^{-u\varphi(x)}e^{-(s-u)\varphi(\Delta^i(x))}du = \frac{\phi(x^i)}{\phi(x) - \varphi(\Delta^i(x))} \left( e^{-\varphi(\Delta^i(x))} - e^{-\varphi(x)} \right), \]

while

\[ p^i_s(x) = s\phi(x^i)e^{-\varphi(x)} \]

when \( \varphi(\Delta^i(x)) = \varphi(x) \). One should observe that \( p^i_s(x) \) is continuous, strictly increasing on \((0, t_0(i, x))\) and strictly decreasing on \((t_0(i, x), +\infty)\), for \( t_0(i, x) = \frac{\ln(\varphi(x) - \varphi(\Delta^i(x)))}{\ln(\varphi(x) - \varphi(\Delta^i(x)))} \) when \( \varphi(\Delta^i(x)) \neq \varphi(x) \) and \( t_0(i, x) = \frac{1}{\varphi(x)} \) when \( \varphi(\Delta^i(x)) = \varphi(x) \). The following two lemmata follow partly technics applied in [21] to show
similar bounds, only that in the current paper, taking advantage of the restriction to configurations on the domain of the invariant measure, we obtain stronger results. 
Since by the construction of the process, both the number of neurons and the cardinality of $D$ are bounded, we can define
\[
t_0 := \min_{x \in D, i \in \{1, \ldots, N\}} t_0(i, x),
\]
which is strictly positive.

**Lemma 2.1.** Assume the PJMP as described in (1.2)-(1.4). There exists a positive constant $C_{1.1}$ such that for every $u \leq t$
\[
\frac{\pi_u(x, y)}{\pi_t(x, y)} \leq C_{1.1}.
\]

**Proof.** Since $D$ is finite, there exists a constant $e > 0$, such that for every $x \in D$, one has $\mu(x) > e > 0$. Since, $\lim_{t \to \infty} \pi_t(x, y) = \mu(y)$ for every $x, y \in D$ we conclude that
\[
(2.1) \quad \exists \hat{t} > 0 : \forall t \geq \hat{t}, \pi_t(x, y) > e, \forall x, y \in D.
\]
We can then write
\[
\frac{\pi_u(x, y)}{\pi_t(x, y)} \leq \frac{1}{e},
\]
which proves the bound for every $t \geq \hat{t}$. It remains to show the same result for the case $t \leq \hat{t}$. Since $u \leq t$ we can write
\[
\pi_t(x, y) \geq \pi_u(x, y)p_{t-u}(y) = \pi_u(x, y)e^{-(t-u)\hat{\phi}(y)} \geq \pi_u(x, y)e^{-N\hat{\phi}},
\]
where above we have denoted $\hat{\phi} := \sum_{x \in \{0, m\}} \hat{\phi}(x)$. This implies
\[
\frac{\pi_u(x, y)}{\pi_t(x, y)} \leq e^{N\hat{\phi}}
\]
for every $t \leq \hat{t}$. 

\[
□
\]

**Lemma 2.2.** Assume the PJMP as described in (1.2)-(1.4). There exists a positive constant $C_{1.2}$ such that for every $u \leq t$
\[
\frac{\pi^2_u(\Delta^i(x), y)}{\pi_t(x, y)} \leq C_{1.2}
\]
for every $t \geq t_0$, as well as, for every $t \leq t_0 \forall y \in D_x \setminus \{\Delta^i(x)\}$. 

Proof. For $\hat{t}$ as in (2.1), we distinguish three separate cases:

(A) $t \geq \hat{t}$, (B) $t_0 < t \leq \hat{t}$ and (C) $t \leq t_0$.

A) At first we examine the case $t \geq \hat{t}$. As in the previous lemma, for every $t \geq \hat{t}$, we have $\pi_t(x, y) > e$, which directly leads to the following bound

$$\frac{\pi^2_u(\Delta'(x), y)}{\pi_t(x, y)} \leq \frac{1}{e}$$

for every $t \geq \hat{t}$.

B) We now study the case $t_0 < t \leq \hat{t}$. Here we also distinguish over separate subcases (B1) and (B2).

B1) If $p^i_{t_0}(x) \leq \pi_{t-t_0}(\Delta'(x), y)$, we can then write

$$\pi_t(x, y) \geq p^i_{t_0}(x)\pi_{t-t_0}(\Delta'(x), y) \geq (p^i_{t_0}(x))^2,$$

in order to bound the denominator. The numerator can be bounded by $\pi_u(\Delta'(x), y) \leq 1$. This gives the bound

$$\frac{\pi^2_u(\Delta'(x), y)}{\pi_t(x, y)} \leq \frac{1}{(p^i_{t_0}(x))^2}.$$

B2) Now consider $\pi_{t-t_0}(\Delta'(x), y) < p^i_{t_0}(x)$ and recall that $p^i_x(x)$ as a function of $s$ is continuous, strictly increasing on $(0, t_0)$ with $p^i_0(x) = 0$. Also, $\pi_{t-s}(\Delta'(x), y)$ as a function of $s$ is continuous and takes value $\pi_t(\Delta'(x), y) > 0$ for $s = 0$. We conclude that there exists $s^*_x \in (0, t_0)$ such that $p^i_{s^*_x}(x) = \pi_{t-s}(\Delta'(x), y)$.

Once more we will consider two subcases.

B2.1) At first assume that $u \leq t - s^*_x$. Then

$$\frac{\pi^2_u(\Delta'(x), y)}{\pi_t(x, y)} \leq \frac{\pi^2_u(\Delta'(x), y)}{p^i_{s^*_x}(x)\pi_{t-s^*_x}(\Delta'(x), y)} = \frac{\pi^2_u(\Delta'(x), y)}{\pi^2_{t-s^*_x}(\Delta'(x), y)} \leq C_{1,1}^2$$

from Lemma [2.1]

B2.2) Now we consider the case where $u \geq t - s^*_x$. We can write

$$\frac{\pi^2_u(\Delta'(x), y)}{\pi_t(x, y)} \leq \frac{(\pi_{u-s^*}(\Delta'(x), y)p_{s^*}(y) + \sup_{z \in D}(1-p_{s^*}(z))^2}{p^i_{s^*_x}(x)\pi_{t-s^*_x}(\Delta'(x), y)}$$

for any $s^* \in (0, u)$ and $s < (0, t)$. If we choose $s = s^*_x$ and $s^* \geq 0$ s.t. $u - s^* = t - s^*_x$ we get

$$\frac{\pi^2_u(\Delta'(x), y)}{\pi_t(x, y)} \leq (p_{s^*}(y))^2 + 2p_{s^*}(y)\frac{1 - e^{-s^*_xN\phi}}{p^i_{s^*_x}(x)^2} + \left(\frac{1 - e^{-s^*_xN\phi}}{p^i_{s^*_x}(x)}\right)^2,$$

where above we also use that since $u \leq t$ we have $s^* \leq s^*_x$ and so $1 - e^{-s^*_xN\phi} \leq 1 - e^{-s^*_xN\phi}$. To bound the right hand side, we need to bound $\frac{1 - e^{-s^*_xN\phi}}{p^i_{s^*_x}(x)}$. Since
where above we also used the lower bound \( \phi(x) \geq \delta \) from condition (1.4). One notices that when \( s_\ast \) goes to zero we obtain a bound that depends on \( t_0 \). Since the right hand side is bounded uniformly for every \( s_\ast \leq t_0 \) we obtain the desirable bound.

C) To finish the proof, it remains to consider the case where \( t \leq t_0 \) and \( y \neq \Delta^i(x) \). We will use (2.2) again. Since \( \pi_t-s(\Delta^i(x), y) \) is continuous as a function of \( s \) and takes values \( \pi_t(\Delta^i(x), y) > 0 \) and \( \pi_0(\Delta^i(x), y) = 0 \) respectively for \( s = 0 \) and \( s = t \), we deduce that there exists \( s_\ast \in (0, t) \subset (0, t_0) \) such that \( p^t_{s_\ast}(x) = \pi_{t-s_\ast}(\Delta^i(x), y) \) and we are back in the previous case, and so the desirable bound follows similarly to (B2.2). \( \square \)

2.2. modified log-Sobolev inequality. We start by showing some useful lemmas that will be used to bound the entropy of the semigroup.

**Lemma 2.3.** Assume the PJMP as described in (1.2)-(1.4). If \( t - s \geq t_0 \), then for every \( x \in D \)

\[
\Pi_1 := \left( \int_0^{t-s} \left( \mathbb{E}^{\Delta^i(x)} - \mathbb{E}^x \right)(\mathcal{L}f(X_u))du \right)^2 \leq 2(t-s)^2MC_1P_t-s\left( \frac{\Gamma(f, f)(y)}{f(y)} \right) P_{t-s}(f(x)).
\]

**Proof.** For \( \pi_t(x, y) \) being the probability kernel of \( \mathbb{E}^x \), we have

\[
P_tf(x) = \mathbb{E}^x(f(X_t)) = \sum_y \pi_t(x, y)f(y).
\]

Then we can write

\[
\Pi_1 = \left( \int_0^{t-s} \sum_{y \in D} \left( \pi_u(\Delta^i(x), y) - \pi_u(x, y) \right) \mathcal{L}f(y)du \right)^2 \leq 2(t-s) \int_0^{t-s} \left( \sum_{y \in D} \pi_u(\Delta^i(x), y) \mathcal{L}f(y) \right)^2 du := \Psi_1 + \left( \sum_{y \in D} \pi_u(x, y) \mathcal{L}f(y) \right)^2 du := \Psi_2
\]

where in the last bound we used Jensen’s inequality to pass the square inside the integral. Since for every \( z \in D \) the number of sites that can be visited are finite,
define \( d = \max_{x \in D} |D_x| \). If we use the Cauchy-Schwarz inequality to bound the square of the sum we obtain

\[
\Psi_1 = \left( \sum_{y \in D} \pi_u(\Delta_i(x), y) \frac{\mathcal{L} f(y)}{f(y)} f(y)^\frac{1}{2} \right)^2 \leq d^2 \left( \sum_{y \in D} \pi_u^2(\Delta_i(x), y) \left( \frac{\mathcal{L} f(y)}{f(y)} \right)^2 \right) \left( \sum_{y \in D} \pi_u^2(\Delta_i(x), y) f(y) \right) = d^2 \left( \sum_{y \in D} \pi_{t-s}(x, y) \frac{\pi_u^2(\Delta_i(x), y)}{\pi_{t-s}(x, y)} \left( \frac{\mathcal{L} f(y)}{f(y)} \right)^2 f(y) \right).
\]

Since \( u \leq t - s \) and \( t - s \geq t_0 \), we can now use Lemma 2.2 to bound the two fractions

\[
\Psi_1 \leq C_{1,2}^2 d^2 \left( \sum_{y \in D} \pi_{t-s}(x, y) \frac{\mathcal{L} f(y)}{f(y)} \right) \left( \sum_{y \in D} \pi_{t-s}(x, y) f(y) \right).
\]

Denote \( M := \sup_{x \in D} \left( \sum_{i=1}^{N} \phi(x^i) + 1 \right)^2 \). About the square of the generator \( \mathcal{L}(f) \) of \( f \), we can write

\[
(\mathcal{L}(f)(y))^2 = \left( \sum_{i=1}^{N} \phi(y^i) \right)^2 \left( \sum_{i=1}^{N} \frac{\phi(y^i)}{\sum_{i=1}^{N} \phi(y^i)} [f(\Delta_i(y)) - f(y)] \right)^2 \leq M \Gamma(f, f)(y)
\]

where above we first divided with the normalisation constant \( \sum_{i=1}^{N} \phi(\Delta_i(x)^i) \), since \( \phi(x) \geq \delta \), and then used Jensen’s inequality to pass the square inside the sum. Putting everything together, we get

\[
\Psi_1 \leq C_{1,2}^2 Md^2 P_{t-s} \left( \frac{\Gamma(f, f)(y)}{f(y)} \right) P_{t-s}(f(y)).
\]

We now compute \( \Psi_2 \). We will use again Cauchy-Schwarz, but this time for the measure \( P_{t-s} \). For this we will write

\[
\Psi_2 = \left( \sum_{y \in D} \pi_u(x, y) \mathcal{L} f(y) \right)^2 \leq \left( \sum_{y \in D} \pi_{t-s}(x, y) \frac{\pi_u(x, y)}{\pi_{t-s}(x, y)} \frac{\mathcal{L} f(y)}{f(y)} f(y)^\frac{1}{2} \right)^2 \leq \left( \sum_{y \in D} \pi_{t-s}(x, y) \left( \frac{\pi_u(x, y)}{\pi_{t-s}(x, y)} \right)^2 \left( \frac{\mathcal{L} f(y)}{f(y)} \right)^2 \right) \left( \sum_{y \in D} \pi_{t-s}(x, y) f(y) \right).
\]
We can bound \((\mathcal{L}(f))^2\) as we did in the computation of \(\Psi_1\) and bound the fraction from Lemma 2.2 to get

\[
\Psi_2 \leq C_{1,1}^2 M d^2 P_{t-s} \left( \frac{\Gamma(f,f)(y)}{f(y)} \right) P_{t-s}(f(y)).
\]

One should notice that the upper bounds of \(\Psi_1\) and \(\Psi_2\) do not depend on the integration variable \(u\) appearing in \(\Pi_1\). So, if we put everything together we finally obtain

\[
\Pi_1 \leq 2(t-s)^2 MC_1 P_{t-s} \left( \frac{\Gamma(f,f)(y)}{f(y)} \right) P_{t-s}(f(x))
\]

where \(C_1 = d^2(C_{1,1}^2 + C_{1,2}^2)\).

Lemma 2.4. Assume the PJMP as described in (1.2)-(1.4). Then, for \(t-s < t_0\),

\[
\Pi_2 := \left( \int_0^{t-s} \left( \mathbb{E} \Delta_i(x) - \mathbb{E}^x \mathcal{L} f(X_u) du \right)^2 \right) \leq 8t_0^2 M \Gamma(f,f)(\Delta_i(x)) + 4t_0^2 M C_1 P_{t-s} \left( \frac{\Gamma(f,f)(y)}{f(y)} \right) P_{t-s}(f(x)).
\]

Proof. We will work as in the previous lemma. Since \(t-s < t_0\), the bounds from Lemma 2.2 do not hold for all \(y \in D\) and so we will break the sum in two parts as shown below.

\[
\Pi_2 \leq 2 \left( \int_0^{t-s} \left( \pi_u(\Delta_i(x), \Delta_i(x)) - \pi_u(x, \Delta_i(x)) \right) \mathcal{L} f(\Delta_i(x)) du \right)^2 + \sum_{y \in D, y \neq \Delta_i(x)} \left( \pi_u(\Delta_i(x), y) - \pi_u(x, y) \right) \mathcal{L} f(y) du \]

\[
(2.3) \quad + 2 \left( \int_0^{t-s} \left( \sum_{y \in D, y \neq \Delta_i(x)} \pi_u(\Delta_i(x), y) - \pi_u(x, y) \right) \mathcal{L} f(y) du \right)^2.
\]

We first calculate the first summand of (2.3). We can write

\[
\Pi_{11} \leq 4 \left( \sum_{j=1}^N \phi(\Delta_i(x)^j) |f(\Delta_j(\Delta_i(x))) - f(\Delta_i(x))| du \right)^2
\]

\[
\leq 4t_0^2 \left( \sum_{j=1}^N \phi(\Delta_i(x)^j) \right)^2 \left( \sum_{j=1}^N \frac{\phi(\Delta_i(x)^j)}{\sum_{j=1}^N \phi(\Delta_i(x)^j)} |f(\Delta_j(\Delta_i(x))) - f(\Delta_i(x))| \right)^2
\]
where above we divided with the normalisation constant \( \sum_{j=1}^{N} \phi(\Delta_i(x))^j \), since \( \phi(x) \geq \delta \). We can now apply the Holder inequality on the sum, so that

\[
\sum_{j=1}^{N} \phi(\Delta_i(x))^j (f(\Delta_j(\Delta_i(x))) - f(\Delta_i(x))^2) = 4t_0^2 M \Gamma(f, f)(\Delta_i(x)).
\]

We now calculate the second summand of (2.3). For this term we will work similar to Lemma 2.3.

\[
\int_0^{t-s} \left( \sum_{y \in D, y \neq \Delta_i(x)} \pi_u(\Delta_i(x), y) \mathcal{L}f(y) \right)^2 du.
\]

Since when \( y \in D, y \neq \Delta_i(x) \) the bounds from lemmata 2.1 and 2.2 still hold even when \( t \leq t_0 \), we can bound \( \Theta_1 \) and \( \Theta_2 \) exactly as we did in the previous lemma for \( \Psi_1 \) and \( \Psi_2 \) respectively, and so we eventually obtain

\[
\sum_{j=1}^{N} \phi(\Delta_i(x))^j (f(\Delta_j(\Delta_i(x))) - f(\Delta_i(x))^2) = 4t_0^2 MC_1 \Gamma(f, f)(\Delta_i(x)).
\]

Combining the bounds for \( \sum_{j=1}^{N} \phi(\Delta_i(x))^j (f(\Delta_j(\Delta_i(x))) - f(\Delta_i(x))^2) \) proves the lemma.

Combining together Lemma 2.3 and Lemma 2.4 we get

**Corollary 2.5.** For the PJMP as described in (1.2)-(1.4), we have

\[
\left( \int_0^{t-s} (E_{\Delta_i(x)}(\mathcal{L}f(x_u)) - E^x(\mathcal{L}f(x_u))) \right)^2 du \leq c \Gamma(f, f)(\Delta_i(x)) + c(t-s) P_{t-s}(f(x)) + c(t-s) P_{t-s}(\frac{\Gamma(f, f)(x)}{f(x)}) P_{t-s}(f(x))
\]

where \( c = 8t_0^2 M \) and \( c(t) = 4t_0^2 MC_1 + 2t^2 MC_1 \).

Next we show an additive property for the semigroup, when the semigroup is on the denominator.

**Lemma 2.6.** For the PJMP as described in (1.2)-(1.4), we have

\[
P_s \left( \frac{g(x)}{P_{t-s}f(x)} \right) \leq d(t-s) P_t \left( \frac{g(x)}{f(x)} \right) + d(t-s) \sum_{j=1}^{N} P_t \left( \frac{g(\Delta_j(x))}{f(x)} \right)
\]

where \( d(t) = 2 + 8t^2 M^2 C_{1.1} \).
Proof. By Dynkin’s formula
\[ P_t g^\frac{1}{2}(x) = \mathbb{E}^x g^\frac{1}{2}(x_t) = g^\frac{1}{2}(x) + \int_0^t \mathbb{E}^x (L g^\frac{1}{2}(x_u)) du, \]
we have
\[ \frac{g(x)}{P_{t-s} f(x)} \leq 2 \left( \frac{\mathbb{E}^x g^\frac{1}{2}(x_{t-s})}{P_{t-s} f(x)} \right)^2 + 2 \left( \int_0^{t-s} \frac{\mathbb{E}^x (L g^\frac{1}{2}(x_u)) du}{P_{t-s} f(x)} \right)^2. \]  
(2.4)

For the first term on the right hand side of (2.4), if we use the Cauchy-Schwarz inequality we have
\[ \left( \frac{\mathbb{E}^x g^\frac{1}{2}(x_{t-s})}{P_{t-s} f(x)} \right)^2 \leq \frac{P_{t-s} \left( \frac{g(x)}{f(x)} \right)}{P_{t-s} f(x)} = P_{t-s} \left( \frac{g(x)}{f(x)} \right). \]  
(2.5)

We will now compute the second term in the right hand side of (2.4). From Jensen’s inequality we have
\[ \left( \int_0^{t-s} \mathbb{E}^x (L g^\frac{1}{2}(x_u)) du \right)^2 \leq (t-s) \int_0^{t-s} \left( \mathbb{E}^x (L g^\frac{1}{2}(x_u)) \right)^2 du. \]  
(2.6)

If we write
\[ \left( \mathbb{E}^x (L g^\frac{1}{2}(x_u)) \right)^2 = \left( \sum_{y \in D} \pi_{u}(x, y) L(g^\frac{1}{2}(y)) \right)^2 \]
\[ = \left( \sum_{y \in D} \pi_{t-s}(x, y) \pi_{u}(x, y) \frac{L(g^\frac{1}{2}(y)) f^\frac{1}{2}(y)}{f^\frac{1}{2}(y)} \right)^2, \]
from Cauchy-Schwarz inequality and Lemma 2.1 we bound
\[ \left( \mathbb{E}^x (L g^\frac{1}{2}(x_u)) \right)^2 \leq \left( \sum_{y \in D} \pi_{t-s}(x, y) f(y) \right) \left( \sum_{y \in D} \pi_{t-s}(x, y) \frac{\pi^2_{t-s}(x, y) (L(g^\frac{1}{2}(y))^2)}{f(y)} \right) \]
\[ \leq C_{1,1}^2 (P_{t-s} f(x)) \left( \sum_{y \in D} \pi_{t-s}(x, y) \frac{(L(g^\frac{1}{2}(y))^2}{f(y)} \right). \]
Furthermore, if we use once more Cauchy-Schwarz inequality and the bound 
\[ \sum_{j=1}^{N} \phi(y^j) \leq M, \] 
we have 
\[ \left( \mathcal{L}(g^\frac{1}{2}(y)) \right)^2 = \left( \sum_{j=1}^{N} \phi(y^j)(g^\frac{1}{2}(\Delta_j(y)) - g^\frac{1}{2}(y)) \right)^2 \leq 2M^2 \sum_{j=1}^{N} g(\Delta_j(y)) + 2M^2 g(y). \]

So we can bound 
\[ \left( \mathbb{E}^x \mathcal{L}(g^\frac{1}{2}(x_u)) \right)^2 \leq 2M^2 C_{1,1}^2 (P_{t-s}f(x)) \sum_{j=1}^{N} P_{t-s} \left( \frac{g(\Delta_j(x))}{f(x)} \right) + + 2M^2 C_{1,1}^2 (P_{t-s}f(x)) P_{t-s} \left( \frac{g(x)}{f(x)} \right). \]

From this and (2.6), we obtain the following bound for the second term on the right of (2.4)
\[ P_s \left( \frac{\int_0^{t-s} \mathbb{E}^x (\mathcal{L}g^\frac{1}{2}(x_u)) du}{P_{t-s}f(x)} \right)^2 \leq (t-s)^2 2M^2 C_{1,1}^2 \sum_{j=1}^{N} P_t \left( \frac{g(\Delta_j(x))}{f(x)} \right) + + 2(t-s)^2 M^2 C_{1,1}^2 P_t \left( \frac{g(x)}{f(x)} \right) \]
where once more we used that \( P_s P_{t-s} = P_t \). From the last bound together with (2.5) and (2.6) we finally get
\[ P_s \left( \frac{g(x)}{P_{t-s}f(x)} \right) \leq (2 + 4(t-s)^2 M^2 C_{1,1}^2) P_t \left( \frac{g(x)}{f(x)} \right) + + (t-s)^2 4M^2 C_{1,1}^2 \sum_{j=1}^{N} P_t \left( \frac{g(\Delta_j(x))}{f(x)} \right). \]

Before we present the proof of the Theorem 1.1, we show a sweeping out relationship for the carré du champ.

**Lemma 2.7.** For the PJMP as described in (1.2)-(1.4), we have
\[ \Gamma(P_{t-s}f, P_{t-s}f)(x) \leq 2\Gamma(f, f)(x) + 2c \sum_{i=1}^{N} \phi(x^i)\Gamma(f, f)(\Delta_i(x)) + + 2Mc(t-s)P_{t-s} \left( \frac{\Gamma(f, f)(x)}{f(x)} \right) P_{t-s}(f(x)). \]
for \( c \) and \( c(t) \) as defined in Corollary 2.5.

**Proof.** From the definition of the carré du champ

\[
\Gamma(P_{t-s}f, P_{t-s}f)(x) = \sum_{i=1}^{N} \phi(x_i)(E^{\Delta_i(x)}f(x_{t-s}) - E^{x}f(x_{t-s}))^2.
\]

If we use the Dynkin’s formula

\[
E^{x}f(x_t) = f(x) + \int_{0}^{t} E^{x}(\mathcal{L}f(x_u))du
\]

we get

\[
(E^{\Delta_i(x)}f(x_{t-s}) - E^{x}f(x_{t-s}))^2 \leq 2\left(f(\Delta_i(x)) - f(x)\right)^2 + 2c(t-s)P_{t-s}(f)P_{t-s}(f).
\]

In order to bound the second term above we will use the bound shown in Corollary 2.5. This together with (2.7) gives

\[
\Gamma(P_{t-s}g, P_{t-s}g)(x) \leq 2\Gamma(f, f)(x) + 2c\sum_{i=1}^{N} \phi(x_i)\Gamma(f, f)(\Delta_i(x)) + 2Mc(t-s)P_{t-s}(f).
\]

We have obtained all the technical results that we need to prove Theorem 1.1.

2.3. **Proof of Theorem 1.1**

We will work similar to [2]. Denote \( P_t f(x) = E^x f(x_t) \). If we define \( \phi(s) = P_s(P_{t-s}f \log P_{t-s}f) \) then, for every \( f \geq 0 \)

\[
\phi'(s) = \frac{1}{2} P_s(\mathcal{L}(P_{t-s}f \log P_{t-s}f) - (1 + \log P_{t-s}f)\mathcal{L}(P_{t-s}f))
\]

where above we used that for a semigroup and its associated infinitesimal generator the following well know relationships: \( \frac{d}{ds} P_s = \mathcal{L}P_s = P_s\mathcal{L} \) (see for example [17]).
integrate, we will finally obtain 
\[ P \] as well as the semigroup property 
\[ c \] since 
\[ \delta \] for 
\[ \alpha \] where 
\[ \log \]

If we use Lemma 2.7 to bound the carré du champ of the semigroup we get
\[ \phi'(s) \leq P_s \left( \frac{1}{P_{t-s} f} \Gamma(P_{t-s} f, P_{t-s} f) \right). \]

If we use Lemma 2.7 to bound the carré du champ of the semigroup we get
\[ \phi'(s) \leq 2P_s \left( \frac{\Gamma(f, f)(x)}{P_{t-s} f(x)} \right) + 2cM \sum_{i=1}^{N} P_s \left( \frac{\Gamma(f, f)(\Delta_i(x))}{P_{t-s} f(x)} \right) + 
+ 2Mc(t)P_sP_{t-s} \left( \frac{\Gamma(f, f)(x)}{f(x)} \right), \]
since \( c(t - s) \leq c(t). \) We can use Lemma 2.6 to bound the first and second term as well as the the semigroup property \( P_sP_{t-s} = P_t. \) We will then get
\[ \phi'(s) \leq \alpha'(t)P_t \left( \frac{\Gamma(f, f)(x)}{f(x)} \right) + \beta(t) \sum_{j=1}^{N} P_t \left( \frac{\Gamma(f, f)(\Delta_j(x))}{f(x)} \right) + 
+ \gamma'(t) \sum_{i=1}^{N} \sum_{j=1}^{N} P_t \left( \frac{\Gamma(f, f)(\Delta_i(\Delta_j(x)))}{f(x)} \right) \]
for \( \alpha'(t) = 2Mc(t) + 2d(t), \beta'(t) = 2(cM + 1)d(t) \) and \( \gamma(t) = 2cMd(t). \) If we integrate, we will finally obtain
\[ \phi(t) - \phi(0) = P_t(f \log f) - P_t f \log f \]
\[ \leq \alpha(t)P_t \left( \frac{\Gamma(f, f)(x)}{f(x)} \right) + \beta(t) \sum_{j=1}^{N} P_t \left( \frac{\Gamma(f, f)(\Delta_j(x))}{f(x)} \right) + 
+ \gamma(t) \sum_{i=1}^{N} \sum_{j=1}^{N} P_t \left( \frac{\Gamma(f, f)(\Delta_i(\Delta_j(x)))}{f(x)} \right) \]
where \( \alpha(t) = t\alpha'(t), \beta(t) = t\beta'(t) \) and \( \gamma(t) = t\gamma'(t). \) Then the proof is completed for \( \delta(t) = \max\{\alpha(t), \beta(t), \gamma(t)\}. \)

2.4. proof of Corollary 1.2

In order to prove the corollary, it is sufficient, to bound the carré du champ operators \( \Gamma(f_k, f_k)(\Delta_i(x)) \) and \( \Gamma(f_k, f_k)(\Delta_i(\Delta_j(x))) \), by \( \Gamma(f_k, f_k)(x), \) for all \( k = 1, \ldots, N. \)

A) Consider the case where \( f_k \) is decreasing and convex. Then by the convexity we have \( f_k(\frac{x+y}{2}) \leq \frac{f_k(x) + f_k(y)}{2} \) for every \( x, y \) on the domain of \( f_k. \)
At first assume that $k \neq i$. We compute

$$\Gamma(f_k, f_k)(\Delta_i(x)) = \frac{1}{2} \sum_{j=1}^{N} \phi(\Delta_i(x)^j)(f_k(\Delta_j(\Delta_i(x)))) - f_k(\Delta_i(x))^2$$

$$\leq \frac{M}{\delta} \sum_{j=1,j\neq k}^{N} \phi(x^j) (f_k(x + 2w) - f_k(x + w))^2$$

$$+ \frac{M}{\delta} \frac{1}{2} \phi(x^k) (f_k(0) - f_k(x + w))^2$$

where above we used (1.4) and $\phi \leq M$ and that the weights are all equal $w_{ji} = w$.

To bound the first term on the right hand side of (2.8) we choose $z = x$ and $y = 2x + w$. We then have

$$f_k(x + w) \leq \frac{f_k(x) + f_k(x + 2w)}{2} \uparrow$$

$$(f_k(x + w) - f_k(x + 2w))^2 \leq (f_k(x) - f_k(x + w))^2$$

since $f_k$ is decreasing and the weight $w$ is positive. This implies

$$\frac{M}{\delta} \frac{1}{2} \sum_{j=1,j\neq k}^{N} \phi(x^j) (f_k(x + 2w) - f_k(x + w))^2 \leq \frac{M}{\delta} \Gamma(f_k, f_k)(x).$$

In order to bound the second term on the right hand side of (2.8), we choose $z = 0$ and $y = x + w$, for $x > 0$. Then, we have

$$f_k\left(\frac{x + w}{2}\right) \leq \frac{f_k(x + w) + f_k(0)}{2}.$$ 

But in the domain of the invariant measure $D$, the membrane potential of a neuron can be either bigger and equal to $w$, or equal to zero. This is because when a neuron $k$ spikes, its membrane potential is set to 0. Then, it can only leave from zero when another neuron spikes, in which case it goes to $w$. In this way, we have that either $x_k = 0$ or $x_k \geq w$. As a result, $x > 0$, implies that

$$x \geq w \iff \frac{x + w}{2} \leq x \iff f_k(x) \leq f_k\left(\frac{x + w}{2}\right)$$

since $f_k$ is decreasing. Then (2.11) becomes

$$f_k(x) \leq \frac{f_k(x + w) + f_k(0)}{2} \iff f_k(0) - f_k(x + w) \leq 2(f_k(0) - f_k(x))$$

which, since $f_k$ is decreasing, readily implies

$$\frac{M}{\delta} \frac{1}{2} \phi(x^k) (f_k(0) - f_k(x + w))^2 \leq \frac{2M}{\delta} \frac{1}{2} \phi(x^k) (f_k(0) - f_k(x))^2$$

(2.12)
for every $x > 0$. In the case where $x = 0$, we observe that, for any $j \neq k$

$$f_k(0) - f_k(w) = f_k(0) - f_k(\Delta_j(0)),$$

and so

$$\frac{M}{\delta} \frac{1}{2} \phi(0) (f_k(0) - f_k(w))^2 \leq \frac{M}{\delta} \Gamma(f_k, f_k)(0).$$

Combining this last bound together with (2.12), we obtain

$$\frac{M}{\delta} \frac{1}{2} \phi(x^k) (f_k(0) - f_k(x + w))^2 \leq \frac{2M}{\delta} \Gamma(f_k, f_k)(x)$$

for every $x \in D$. Finally, if we use (2.13) to bound (2.8), we get

$$\Gamma(f_k, f_k)(\Delta_i(x)) \leq \frac{2M}{\delta} \Gamma(f_k, f_k)(x).$$

A2) Now assume $k = i$. Since, for every $x > 0$, $\Delta_k(x^k) = 0$, we can compute

$$\Gamma(f_k, f_k)(\Delta_k(x)) = \frac{1}{2} \sum_{j=1}^{N} \phi(0)(f_k(\Delta_j(0)) - f_k(0))^2 \leq \frac{N - 1}{2} \phi(0) (f_k(w) - f_k(0))^2.$$

But since $f_k$ is a decreasing function, one gets that for every $x > 0$, $f_k(x) \leq f_k(w)$, since every $x \in D$, that is $x > 0$, satisfies $x \geq w$. As a result we have

$$(f_k(0) - f_k(w))^2 \leq (f_k(0) - f_k(x))^2$$

$$= (f_k(\Delta_k(x) - f_k(x))^2$$

$$\leq \frac{1}{\delta} \phi(x^k) (f_k(\Delta_k(x) - f_k(x))^2$$

$$\leq \frac{2}{\delta} \Gamma(f_k, f_k)(x).$$

Putting this in (2.15) we get

$$\Gamma(f_k, f_k)(\Delta_k(x)) \leq \frac{(N - 1)\phi(0)}{\delta} \Gamma(f_k, f_k)(x),$$

for every $x > 0$. In the case where $x = 0$, the same result follows trivially.

B) Now, consider the case where $f_k$ is increasing and concave. Since this case follows similar to the decreasing and convex case studied in (A), we just highlight the differences. By the concavity we have $f_k(\frac{x + y}{2}) \geq \frac{f_k(x) + f_k(y)}{2}$ for every $x, y$ on the domain of $f_k$. To bound the first term on the right hand side of (2.8) if we choose $z = x$ and $y = 2x + w$, since $f_k$ is increasing, we obtain $(f_k(x + w) - f_k(x + w))^2 \leq (f_k(x + w) - f_k(x))^2$, which implies (2.10).
To bound the second term on the right hand side of (2.8) choose \( z = 0 \) and \( y = x + w \), for \( x > 0 \). Then,

\[
f_k(x) \geq f_k\left(\frac{x + w}{2}\right) \geq \frac{f_k(x + w) + f_k(0)}{2},
\]

which leads to

\[
f_k(x + w) - f_k(0) \leq 2(f_k(x) - f_k(0))
\]

and so (2.12) for every \( x > 0 \) is again satisfied. The case \( x = 0 \) follows identically to (A) and so (2.13) is verified for every \( x \in D \), and so (2.14) follows for \( k \neq i \).

Now assume \( k = i \). Since for every \( x > 0 \), \( \Delta_k(x^k) = 0 \), we can compute

\[
\Gamma(f_k, f_k)(\Delta_k(x)) = \frac{1}{2} \sum_{j=1}^{N} \phi(0)(f_k(\Delta_j(0)) - f_k(0))^2
\]

\[
\leq \frac{N - 1}{2} \phi(0) (f_k(w) - f_k(0))^2.
\]

But since \( f_k \) is a decreasing function, one gets that for every \( x > 0 \), \( f_k(x) \leq f_k(w) \), since every strictly positive \( x \in D \) satisfies \( x \geq w \). As a result we have

\[
(f_k(0) - f_k(w))^2 \leq (f_k(0) - f_k(x))^2 \leq (f_k(\Delta_k(x) - f_k(x))^2 \leq \frac{1}{\delta} \phi(x^k) (f_k(\Delta_k(x) - f_k(x))^2 \leq \frac{2}{\delta} \Gamma(f_k, f_k)(x).
\]

Putting this in (2.16) we get

\[
\Gamma(f_k, f_k)(\Delta_k(x)) \leq \frac{N - 1}{\delta} \phi(0) \Gamma(f_k, f_k)(x),
\]

for every \( x > 0 \). In the case where \( x = 0 \), the same result as in (A) follows trivially.

Now assume \( k = i \neq j \). Since, for every \( x > 0 \), \( \Delta_k(x^k) = 0 \), as before we have

\[
\Gamma(f_k, f_k)(\Delta_k(x)) \leq \frac{N - 1}{2} \phi(0) (f_k(w) - f_k(0))^2.
\]

But since \( f_k \) is an increasing function, one gets that for every \( x > 0 \), \( f_k(x) \geq f_k(w) \), and so

\[
(f_k(w) - f_k(0))^2 \leq (f_k(x) - f_k(0))^2 = (f_k(x) - f_k(\Delta_k(x))^2 \leq \frac{2}{\delta} \Gamma(f_k, f_k)(x).
\]

Putting this in (2.16) we get

\[
\Gamma(f_k, f_k)(\Delta_k(x)) \leq \frac{N - 1}{\delta} \phi(0) \Gamma(f_k, f_k)(x),
\]

for every \( x > 0 \). In the case where \( x = 0 \), the same result follows trivially.
Combining together (A) and (B), we conclude that there exists a \( R > 0 \) such that
\[
\Gamma(f_k, f_k)(\Delta_i(x)) \leq R \Gamma(f_k, f_k)(x).
\]
Then, by induction we further get
\[
\Gamma(f_k, f_k)(\Delta_i(\Delta_j(x))) \leq R \Gamma(f_k, f_k)(\Delta_j(x)) \leq R^2 \Gamma(f_k, f_k)(x),
\]
and the corollary is proven.

3. CONCENTRATION AND EMPIRICAL APPROXIMATION.

Concentration inequalities
\[
\mu(\{|f - \mu(f)| > r\}) \leq ce^{-\lambda r^p}, \quad p \geq 1,
\]
for a probability measure \( \mu \), have been associated by Talagrand (see [29] and [30]) to Poincaré and log-Sobolev inequalities (see also [4], [22] and [23]). In the current section we prove the typical concentration inequality of Proposition 1.3 and the concentration empirical approximation of Theorem 1.4. We begin with the proof of Theorem 1.4.

3.1. proof of the concentration empirical approximation of Theorem 1.4

The concentration empirical approximations, as in Theorem 1.4, demonstrate how sharp are the approximations of the process as the number of observables goes to infinity (see [25] and [7] for strong concentration results). As briefly mentioned in the introduction, these do not follow from the concentration inequalities of Proposition 1.3, since, the empirical concentration inequalities involve non bounded quantities, as is the sum \( \sum_{k=1}^{n} f(X^i_{t_k}) \) for an increasing to infinity \( n \). If one tries to obtain Theorem 1.4 from Proposition 1.3 by applying the proposition to the bounded quantity \( \frac{\sum_{k=1}^{n} f(X^i_{t_k})}{n} \), will then obtain
\[
P\left( \left| \frac{\sum_{k=1}^{n} f(X^i_{t_k})}{n} - \frac{\sum_{k=1}^{n} E[X^i_{t_k}]}{n} \right| \geq \epsilon \right) \leq Q e^{-\epsilon},
\]
which is a concentration property whose right hand side does not depend on \( n \). In general, to obtain the concentration inequalities of Theorem 1.4, one needs to first derive a modified log-Sobolev inequality for a set of cylindrical functions, independent of the size of input \( n \). Such an inequality is obtained in (3.2). Then, from this inequality, the concentration property for the multi-times function is obtained.

We start with some technical results.

Lemma 3.1. Assume \( f_i : [0, m] \to [0, \infty) \) is a 1-Lipschitz function depending only on \( x^i \). Then, for any \( x \in D \) and any \( k \in \mathbb{N} \),
\[
e^{-\lambda f_i(\Delta_j(\ldots\Delta_k(x)))} \leq d^k e^{-\lambda f_i(x)}
\]
If $j$ for any $\Gamma$ for some constant $d$

But $f$ is Lipschitz continuous, with Lipschitz constant one, which means that $|f_i(x) - f_i(y)| \leq |x - y|$, and that for every $y \in \{0, w_{ji}\}$: $f_i(y) = f_i(x) + f_i(x) \leq |y - x| + f_i(x) \leq m + f_i(x)$, since $x \leq m$. So,

$$\Gamma(e^{\lambda f_i}, e^{\lambda f_i}(\Delta_j(x))) = \frac{1}{2} \sum_{k=1}^{N} \phi(x^k) \left[ e^{\lambda f_i(x^k)} - e^{\lambda f_i(0)} \right]^2 \leq \frac{\lambda^2}{2} \sum_{k=1}^{N} \phi(x^k) \left[ f_i(x^k) - f_i(0) \right]^2 e^{2\lambda \max\{f_i(0), f_i(w_{ji})\}}.$$ 

But $f_i$ is Lipschitz continuous, with Lipschitz constant one, which means that $|f_i(x^k) - f_i(0)|^2 \leq w_{ji}^2$, and that for every $y \in \{0, w_{ji}\}$: $f_i(y) = f_i(x) + f_i(x) \leq |y - x| + f_i(x) \leq m + f_i(x)$, since $x \leq m$. So,

$$\Gamma(e^{\lambda f_i}, e^{\lambda f_i}(\Delta_j(x))) = \frac{1}{2} \sum_{k=1}^{N} \phi(x^k) \left[ e^{\lambda f_i(x^k + w_{ji})} - e^{\lambda f_i(x^k + w_{ki})} \right]^2 \leq \frac{\lambda^2}{2} \sum_{k=1}^{N} \phi(x^k) \left[ f_i(x^k + w_{ji} + w_{ki}) - f_i(x^k + w_{ji}) \right]^2 e^{2\lambda \max\{f_i(x^k + w_{ji} + w_{ki}), f_i(x^k + w_{ji})\}}.$$ 

$$\Gamma(e^{\lambda f_i}, e^{\lambda f_i}(\Delta_j(x))) = \frac{1}{2} \sum_{k=1}^{N} \phi(x^k) \left[ f_i(x^k + w_{ji} + w_{ki}) - f_i(x^k + w_{ji}) \right]^2 e^{2\lambda \max\{f_i(x^k + w_{ji} + w_{ki}), f_i(x^k + w_{ji})\}}.$$ 

But $\max\{f_i(x^k + w_{ji} + w_{ki}), f_i(x^k + w_{ji})\} \leq |w_{ji} + w_{ki}| + f_i(x)$, which leads to

$$\Gamma(e^{\lambda f_i}, e^{\lambda f_i}(\Delta_j(x))) = \frac{1}{2} \sum_{k=1}^{N} \phi(x^k) \left[ f_i(x^k + w_{ji} + w_{ki}) - f_i(x^k + w_{ji}) \right]^2 e^{2\lambda \max\{f_i(x^k + w_{ji} + w_{ki}), f_i(x^k + w_{ji})\}}.$$
Since in this section we will be concerned with cylindrical functions, in order to ease the notation, for any sequence of times \(0 = t_1 < t_2 < \ldots < t_n \leq T\) and a function \(F : \mathbb{R}^n \rightarrow \mathbb{R}\), depending on \(\{X_{t_k}, k = 1, \ldots, n\}\), we will write \(\hat{P}_{t_n}F(x) := \mathbb{E}^x[F(X_{t_1}, \ldots, X_{t_n})]\). When \(n = 1\), this is of course the Markov semigroup, \(\hat{P}_tF(x) = \mathbb{E}^x[F(X_t)]\).

The following lemma will be the main iteration tool that will be used in order to obtain the modified log-Sobolev inequality for cylindrical functions shown in (3.2).

**Lemma 3.2.** Assume the PJMP as described in (1.2)-(1.4). Assume \(\xi : D^n \rightarrow D^n\) and \(f : \mathbb{R}^n \rightarrow \mathbb{R}\) are such that \(\frac{f((y))}{f(x)} \leq d^2\) for any \(y \in D\), for some \(d > 1\). Then, for any sequence of times \(0 = t_1 < t_2 < \ldots < t_n \leq T\), we have

\[
P_{t_k+1-t_k}\left( \frac{\Gamma(\hat{P}_{t_k}f, \hat{P}_{t_k}f)(\xi(x))}{\hat{P}_{t_k}f(x)} \right) \leq \sum_{r=1}^{k+2} \left( \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \sum_{i_3=1}^{N} \hat{P}_{t_k+1} \left( \frac{\Gamma(f,f)(\Delta_{i_3}\Delta_{i_2}\Delta_{i_1}(\xi(x)))}{f(x)} \right) \right)
\]

where above we have denoted \(b_k = 3^k b_0(k)\), for \(b_0(k) = \max\{2 + 2d^2Mc(t_k - t_{k-1}), 2cM + 2d(t_k - t_{k-1}), 2cMd(t_k - t_{k-1})\}\).

**Proof.** We can write

\[
\Gamma(\hat{P}_{t_k}f, \hat{P}_{t_k}f)(\xi(x)) = \Gamma(P_{t_k-t_{k-1}}\hat{P}_{t_{k-1}}f, P_{t_k-t_{k-1}}\hat{P}_{t_{k-1}}f)(\xi(x)),
\]

where above \(\hat{P}_{t_k}\) and \(\hat{P}_{t_{k-1}}\) refer to the expectation with respect to \(\{X_{t_1}, \ldots, X_{t_k}\}\) and \(\{X_{t_1}, \ldots, X_{t_{k-1}}\}\) respectively. One should notice that \(P_{t_k-t_{k-1}}\) is the semigroup related to \(X_{t_k}\) (which appears after time \(t_k - t_{k-1}\) from \(X_{t_{k-1}}\)) and so we can apply the tools we have already obtained about the Markov semigroups in previous sections. We can start by applying the sweeping out relationship of Lemma 2.7 to get the semigroup out of the carré du champ

\[
\Gamma(\hat{P}_{t_k}f, \hat{P}_{t_k}f)(\xi(x)) \leq 2\Gamma(\hat{P}_{t_{k-1}}f, \hat{P}_{t_{k-1}}f)(\xi(x)) + 2c \sum_{i=1}^{N} \phi(\xi(x))\Gamma(\hat{P}_{t_{k-1}}f, \hat{P}_{t_{k-1}}f)(\Delta_i(\xi(x))) + 2Mc(t_k - t_{k-1})P_{t_k-t_{k-1}}\left( \frac{\Gamma(\hat{P}_{t_{k-1}}f, \hat{P}_{t_{k-1}}f)(\xi(x))}{\hat{P}_{t_{k-1}}f(x)} \right) \hat{P}_{t_k}(f(\xi(x)));
\]
since $P_{t_k-t_{k-1}} \hat{P}_{t_{k-1}} = \hat{P}_t$. From this we obtain

$$P_{t_{k+1}-t_k} \left( \frac{\Gamma(\hat{P}_t f, \hat{P}_t f)(\xi(x))}{\hat{P}_t f(x)} \right) \leq 2P_{t_{k+1}-t_k} \left( \frac{\Gamma(\hat{P}_{t_k f}, \hat{P}_{t_k f})(\xi(x))}{\hat{P}_{t_k} f(x)} \right) +$$

$$+ 2cM \sum_{i=1}^{N} P_{t_{k+1}-t_k} \left( \frac{\Gamma(\hat{P}_{t_k f}, \hat{P}_{t_k f})(\Delta_i(x))}{\hat{P}_{t_k} f(x)} \right) +$$

$$+ 2d^2Mc(t_k - t_{k-1})P_{t_{k+1}-t_k} \left( \frac{\Gamma(\hat{P}_{t_k f}, \hat{P}_{t_k f})(\xi(x))}{\hat{P}_{t_k} f(x)} \right),$$

where above we also used that $\hat{P}_t(f\xi(x)) \leq d^2$. This follows from the hypothesis $f(\xi(y)) \leq d^2 f(y), \forall y \in D$, after taking the expectation on both sides with respect to the $\hat{P}_t$, that is, $\hat{P}_t(f\xi(x)) \leq d^2 \hat{P}_t(f(x))$. We can write the denominator of the first two terms on the right hand side as $\hat{P}_t = P_{t_k-t_{k-1}} \hat{P}_{t_{k-1}}$, and then apply Lemma 2.6 to reduce the denominator from $\hat{P}_t$ to $\hat{P}_{t_{k-1}}$, as shown below.

$$P_{t_{k+1}-t_k} \left( \frac{\Gamma(\hat{P}_t f, \hat{P}_t f)(\xi(x))}{\hat{P}_t f(x)} \right) \leq$$

$$(2 + 2d^2Mc(t_k - t_{k-1}))P_{t_{k+1}-t_k} \left( \frac{\Gamma(\hat{P}_{t_k f}, \hat{P}_{t_k f})(\xi(x))}{\hat{P}_{t_k} f(x)} \right) +$$

$$+(2cM + 2d(t_k - t_{k-1})) \sum_{i=1}^{N} P_{t_{k+1}-t_k} \left( \frac{\Gamma(\hat{P}_{t_k f}, \hat{P}_{t_k f})(\Delta_i(x))}{\hat{P}_{t_k} f(x)} \right) +$$

$$+ 2cMd(t_k - t_{k-1}) \sum_{i=1}^{N} \sum_{i=1}^{N} P_{t_{k+1}-t_k} \left( \frac{\Gamma(\hat{P}_{t_k f}, \hat{P}_{t_k f})(\Delta_i(\Delta_{i_j}(\xi(x))))}{\hat{P}_{t_k} f(x)} \right).$$

If we define $b_0(k) = \max\{2 + 2d^2Mc(t_k - t_{k-1}), 2cM + 2d(t_k - t_{k-1}), 2cMd(t_k - t_{k-1})\}$, then by induction we finally get

$$P_{t_{k+1}-t_k} \left( \frac{\Gamma(\hat{P}_t f, \hat{P}_t f)(\xi(x))}{\hat{P}_t f(x)} \right) \leq$$

$$b_k \sum_{r=1}^{k+2} \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \sum_{i_1=1}^{N} \hat{P}_{t_k} f(x) \left( \frac{\Gamma(f, f)(\Delta_{i_1 + 2}(\Delta_{i_1}(\xi(x))))}{\hat{P}_t f(x)} \right),$$

for $b_k = 3^k b_0(k)$, where above we used that $t_0 = 0$ and $\hat{P}_0 f(x) = P_0 f(x) = f(x)$. \(\square\)
In the next proposition a bound of the entropy of multi-times functions is presented. Furthermore, in the process of proving this bound a modified log-Sobolev inequality is also established, in \[3.2\], for multi-times functions.

**Proposition 3.3.** Consider some \(T > 0\) and any sequence of times \(0 = t_1 < t_2 < \ldots < t_n \leq T\), such that

\[
D(T) = 3 \sum_{k=1}^{\infty} \delta(t_k - t_{k-1})b_{k-1} \sum_{r=1}^{k+1} (Nd)^{r+1} < \infty.
\]

For \(i = 1, \ldots, N\), assume \(f_i : [0, m] \to [0, \infty)\) is a 1-Lipschitz function depending only on \(x^i\) and \(f(X_{t_1}^i, \ldots, X_{t_n}^i) := \sum_{k=1}^{n} f_i(X_{t_k}^i)\). Then, for \(\lambda \leq 1\),

\[
\hat{P}_t \left( e^{\lambda f(X_{t_1}^i, \ldots, X_{t_n}^i)} \log \frac{e^{\lambda f(x)}}{\hat{P}_t e^{\lambda f(x)}} \right) \leq \lambda^2 D(T) \hat{P}_t \left( e^{\lambda f(X_{t_1}^i, \ldots, X_{t_n}^i)} \right).
\]

**Proof.** To prove the multi-times modified log-Sobolev inequality of the proposition, we will take advantage of the modified inequality shown on Theorem 1.1 and then use iteration, as was done in [31] and \[8\] to prove coercive inequalities for cylindrical functions. To do so, we will first form the entropies for the successive times. To ease the notation, we will write \(f(x)\) for \(f(X_{t_1}^i, \ldots, X_{t_n}^i)\). We have

\[
\hat{P}_t \left( e^{\lambda f(x)} \log \frac{e^{\lambda f(x)}}{\hat{P}_t e^{\lambda f(x)}} \right) = \sum_{k=1}^{n} P_{t_{n-k}t_{k-1}} \left( \hat{P}_{k-1} e^{\lambda f(x)} \log \frac{\hat{P}_{k-1} e^{\lambda f(x)}}{P_{t_{k-1}t-1} e^{\lambda f(x)}} \right),
\]

where above we used that \(P_t \hat{P}_s = \hat{P}_{t+s}\). We can now apply the modified log-Sobolev inequality of Theorem 1.1 to bound the entropies of \(\hat{P}_t f\) with respect to the measure \(P_{t_{k-1}t_{k-1}}\) involved in the sum. This gives

\[
P_{t_{k-1}t_{k-1}} \left( \hat{P}_{k-1} e^{\lambda f(x)} \log \frac{\hat{P}_{k-1} e^{\lambda f(x)}}{P_{t_{k-1}t_{k-1}} e^{\lambda f(x)}} \right) \leq \\
\delta(t_k - t_{k-1}) P_{t_{k-1}t_{k-1}} \left( \frac{\Gamma(\hat{P}_{k-1} e^{\lambda f}, \hat{P}_{k-1} e^{\lambda f})(\Delta_j(x))}{P_{t_{k-1}t_{k-1}} e^{\lambda f(x)}} \right) + \\
\delta(t_k - t_{k-1}) \sum_{j=1}^{N} P_{t_{k-1}t_{k-1}} \left( \frac{\Gamma(\hat{P}_{k-1} f, \hat{P}_{k-1} f)(\Delta_j(x))}{P_{t_{k-1}t_{k-1}} f(x)} \right) + \\
\delta(t_k - t_{k-1}) \sum_{1,j=1}^{N} P_{t_{k-1}t_{k-1}} \left( \frac{\Gamma(\hat{P}_{k-1} f, \hat{P}_{k-1} f)(\Delta_j(x))}{P_{t_{k-1}t_{k-1}} f(x)} \right).
\]
If we use the last inequality to bound the right hand side of (3.1) we get
\[
\hat{P}_n (e^{\lambda f(x)} \log \frac{e^{\lambda f(x)}}{\hat{P}_n e^{\lambda f(x)}}) \leq \\
\sum_{k=1}^{n} \delta (t_k - t_{k-1}) P_{t_k - t_{k-1}} \left( \sum_{r=1}^{k+1} \hat{P}_n (e^{\lambda f(x)} (\Gamma(\hat{P}_{t_k-1} e^{\lambda f}, \hat{P}_{t_k-1} e^{\lambda f})(x) \right) \\
+ \sum_{k=1}^{n} \delta (t_k - t_{k-1}) \sum_{j=1}^{N} P_{t_k - t_{k-1}} \left( \sum_{i=1}^{N} \hat{P}_n (\Gamma(\hat{P}_{t_k-1} f, \hat{P}_{t_k-1} f)(\Delta_j(x)) \right) \\
+ \sum_{k=1}^{n} \delta (t_k - t_{k-1}) \sum_{i,j=1}^{N} P_{t_k - t_{k-1}} \left( \sum_{r=1}^{k+1} \hat{P}_n (\Gamma(\hat{P}_{t_k-1} f, \hat{P}_{t_k-1} f)(\Delta_i(\Delta_j(x)))) \right) \right) \\
+ \sum_{k=1}^{n} \delta (t_k - t_{k-1}) \sum_{i,j=1}^{N} b_{k-1} \sum_{r=1}^{k+1} \hat{P}_n \left( \Gamma(e^{\lambda f}, e^{\lambda f})(\Delta_{i+2}(\Delta_{i+1}(\Delta_i(x)))) \right) \\
+ \sum_{k=1}^{n} \delta (t_k - t_{k-1}) \sum_{i,j=1}^{N} b_{k-1} \sum_{r=1}^{k+1} \hat{P}_n \left( \Gamma(e^{\lambda f}, e^{\lambda f})(\Delta_{i+2}(\Delta_{i+1}(\Delta_i(\Delta_j(x))))) \right) \\
+ \sum_{k=1}^{n} \delta (t_k - t_{k-1}) \sum_{i,j=1}^{N} b_{k-1} \sum_{r=1}^{k+1} \hat{P}_n \left( \Gamma(e^{\lambda f}, e^{\lambda f})(\Delta_{i+2}(\Delta_{i+1}(\Delta_i(\Delta_j(x))))) \right) \right),
\]
where above we used that \( P_{t_k - t_{k-1}} P_{t_k - t_{k-1}} \hat{P}_{t_k-1} = \hat{P}_n \). From Lemma 3.1 we can further bound
\[
\Gamma(e^{\lambda f}, e^{\lambda f})(\Delta_{j_1}(\Delta_{j_{r-1}} \cdots \Delta_{j_1}(x))) \leq \lambda^2 d e^{2\lambda f(\Delta_{j_{r-1}} \cdots \Delta_{j_1}(x))} \leq \lambda^2 d e^{2\lambda f(x)},
\]
where at first inequality we used the second assertion of the lemma to bound the carré du champ and then the first assertion. From this we finally obtain
\[
\hat{P}_n (e^{\lambda f(x)} \log \frac{e^{\lambda f(x)}}{\hat{P}_n e^{\lambda f(x)}}) \leq 3\lambda^2 \sum_{k=1}^{n} \delta (t_k - t_{k-1}) b_{k-1} \sum_{r=1}^{k+1} (Nd)^{r+4} \hat{P}_n (e^{\lambda f(x)}),
\]
and so the proposition follows for \( D(T) = 3 \sum_{k=1}^{\infty} \delta (t_k - t_{k-1}) b_{k-1} \sum_{r=1}^{k+1} (Nd)^{r+4}. \)
proof of Theorem 1.4

We can now prove Theorem 1.4. Consider \( f(X_{t_1}, \ldots, X_{t_n}) = \sum_{k=1}^{n} f(X_{t_k}) \). For economy we will write \( f(x) \) for \( f(X_{t_1}, \ldots, X_{t_n}) \) and \( X_{t_1} = x \). Then, for \( \lambda \leq 1 \), Proposition 3.3 gives

\[
\hat{P}_t \left( e^{\lambda f(x)} \log \frac{e^{\lambda f(x)}}{\hat{P}_t e^{\lambda f(x)}} \right) \leq D(T) \hat{P}_t \left( e^{\lambda f(x)} \right).
\]

Denote \( \Psi(\lambda) = \hat{P}_t \left( e^{\lambda f(x)} \right) \). Then, the last inequality can be written as

\[
\lambda \Psi' (\lambda) \Psi (\lambda) - \log \Psi (\lambda) \leq \lambda^2 D(T) e^{\lambda}.
\]

If we now divide with \( \lambda^2 \) we then get

\[
\frac{d}{d\lambda} \left( \frac{\log \Psi (\lambda)}{\lambda} \right) \leq D(T) e^{\lambda}.
\]

Since \( \lim_{\lambda \to 0} \frac{\log \Psi (\lambda)}{\lambda} = \hat{P}_t (f(x)) \), by integration we obtain

\[
\hat{P}_t \left( e^{\lambda(f(x) - \hat{P}_t f(x))} \right) \leq e^{D(T)\lambda} \int_0^\lambda e^{as} ds
\]

and so

\[
\hat{P}_t \left( f(x) - \hat{P}_t f(x) \geq \epsilon \right) \leq e^{-\epsilon \lambda} \hat{P}_t \left( e^{\lambda(f(x) - \hat{P}_t f(x))} \right) \leq e^{-\epsilon \lambda} e^{D(T)\lambda} \int_0^\lambda e^{as} ds.
\]

As a result, if we consider \( \lambda = 1 \) we have

\[
\hat{P}_t \left( \frac{\sum_{k=1}^{n} f(X_{t_k})}{n} - \hat{P}_t \left( \frac{\sum_{k=1}^{n} f(X_{t_k})}{n} \right) \geq \epsilon \right) = \hat{P}_t \left( \sum_{k=1}^{n} X_{t_k} - \hat{P}_t \left( \sum_{k=1}^{n} X_{t_k} \right) \geq n\epsilon \right) \leq e^{-\epsilon} \hat{P}_t \left( e^{(f(x) - \hat{P}_t f(x))} \right) \leq e^{-\epsilon} e^{D(T)\lambda} \int_0^\lambda e^{as} ds.
\]

where \( G = e^{D(T)\lambda} \int_0^\lambda e^{as} ds \). Next, if we repeat the same for \(-f\) in the place of \( f \), we get

\[
\hat{P}_t \left( \frac{\sum_{k=1}^{n} f(X_{t_k})}{n} - \sum_{k=1}^{n} f(X_{t_k}) \geq \epsilon \right) \leq e^{-\epsilon} G.
\]

To finish the proof, it is insufficient to observe that \( D(T) < \infty \) implies that \( \delta(t_k - t_{k-1}) \) is uniformly bounded for any \( k \), which implies the same for \( t_k - t_{k-1} \). From this we conclude that \( b_0(k) \) is also uniformly bounded on \( k \). As a result, for \( D(T) < \infty \), it is sufficient to assume condition (1.5) of the statement of the theorem. \( \square \)
3.2. proof of Proposition 1.3: Consider $f(x) = \sum_{i=1}^{N} f_i(x)$, for some $x = (x^1, ..., x^N)$ and $0 < \lambda \leq 1$. Then, from Theorem 1.1 we directly obtain

\[
P_t(e^{\lambda f(x)} \log \frac{e^{\lambda f(x)}}{P_t e^{\lambda f(x)})} \leq \delta(t) P_t \left( \frac{\Gamma(e^{\lambda f}, e^{\lambda f})(x)}{e^{\lambda f(x)}} \right) + \delta(t) \sum_{j=1}^{N} P_t \left( \frac{\Gamma(e^{\lambda f}, e^{\lambda f})(\Delta_j(x))}{e^{\lambda f(x)}} \right).
\]

(3.3)

For the carré du champ on the first term on the right hand side, we can compute

\[
\Gamma(f, f)(x) = \frac{1}{2} \sum_{j=1}^{N} \phi(x^j)(e^{\lambda f}(\Delta_j(x)) - e^{\lambda f(x)})^2
\]

\[
\leq \lambda^2 2^{N-1} \sum_{j=1}^{N} \phi(x^j) \sum_{i=1}^{N} (f_i(\Delta_j(x)) - f_i(x))^2 e^{\lambda f} \sum_{i=1}^{N} \max_{i=1} (f_i(x), f_i(\Delta_j(x))).
\]

But, since $f_i$ is Lipschitz continuous, with Lipschitz constant 1, we can bound $|f_i(\Delta_j(x)) - f_i(x)| \leq m + \max_{i,j} \{w_{ij}\}$ and $f_i(\Delta_j(x)) \leq f_i(x) + m + \max_{i,j} \{w_{ij}\}$, and so the last can be bounded as follows

(3.4)

\[
\Gamma(f, f)(x) \leq \lambda^2 D_1 e^{\lambda f(x)},
\]

where $D_1 = 2^{N-1} MN \max_{i,j} \{w_{ij}\} e^{2N(m + \max_{i,j} \{w_{ij}\})}$. Similarly, for the second term of (3.3) we compute

(3.5)

\[
\Gamma(f, f)(\Delta_i(x)) \leq \frac{1}{2} \sum_{j=1}^{N} \phi(\Delta_i(x)^j)(e^{\lambda f}(\Delta_j(\Delta_i(x))) - e^{\lambda f}(\Delta_i(x)))^2
\]

\[
\leq \lambda^2 D_2 e^{\lambda f(x)},
\]

for $D_2 = 2^{N-1} M(m + \max_{i,j} \{w_{ij}\})^2 e^{2N(m + 2\max_{i,j} \{w_{ij}\})}$, where above we used that $|f_i(\Delta_j(\Delta_i(x))) - f_i(\Delta_i(x))| \leq m + \max_{i,j} \{w_{ij}\}$ and $\max_{i} f_i(\Delta_j(\Delta_i(x))) \leq f_i(x) + m + 2\max_{i,j} \{w_{ij}\}$. If we work as in the first two terms, the third term on the right hand side of (3.3) can be bounded by

(3.6)

\[
\Gamma(f, f)(\Delta_j(\Delta_i(x))) \leq \lambda^2 D_3 e^{\lambda f(x)},
\]

where now $D_3 = 2^{N-1} M(m + 2\max_{i,j} \{w_{ij}\})^2 e^{2N(m + \max_{i,j} \{w_{ij}\})}$. Combining together the bounds (3.4) - (3.6) to bound the right hand side of (3.3) leads to

\[
P_t(e^{\lambda f(x)} \log \frac{e^{\lambda f(x)}}{P_t e^{\lambda f(x)})} \leq \lambda^2 \delta(t) \left( D_1 + ND_2 + N^2 D_3 \right) P_t \left( e^{\lambda f(x)} \right).
\]

The rest of the proof of Proposition 1.3 follows on the same lines of the proof of Theorem 1.4 for $\Psi(\lambda) = P_t \left( e^{\lambda f(x)} \right)$. □
References

[1] M. André, A result of metastability for an infinite system of spiking neurons. J Stat Phys, 177, 984-1008 (2019).

[2] C. Ane and M. Ledoux, Rate of convergence for ergodic continuous Markov processes: Lyapunov versus Poincaré. Probab. Theory Relat. Fields 116, 573-602 (2000).

[3] R. Azaïs, J.B. Bardet, A. Genadot, N. Krell and P.A. Zitt, Piecewise deterministic Markov process (pdm DPS). Recent results. Proceedings 44, 276-290 (2014).

[4] D. Bakry, L’hypercontractivité et son utilisation en théorie des semigroupes. Ecole d’Eté de Probabilités de St-Flour. Lecture Notes in Math., 1581, 1-114, Springer (1994).

[5] D. Bakry, On Sobolev and logarithmic Sobolev inequalities for Markov semigroups. New trends in Stochastic Analysis, 43-75, World Scientific (1997).

[6] S. Bobkov and M. Ledoux, Poincaré’s inequalities and Talagrand’s concentration phenomena for the exponential distribution. Probab Theory Relat Fields 107, 383-400 (1997).

[7] F. Bolley, A. Guillin and C. Villani, Quantitative Concentration Inequalities for Empirical Measures on Non-compact Spaces. C. Probab. Theory Relat. Fields 137, 541593 (2007).

[8] D. Chafai, Entropies, convexity, and functional inequalities. J. Math. Kyoto Univ. 44 (2), 325 - 363 (2004).

[9] J. Chevalier, Mean-field limit of generalized Hawkes processes. Stochastic Processes and their Applications 127 (12), 3870 - 3912 (2017).

[10] A. Crudu, A. Debussche, A. Muller and O. Radulescu, Convergence of stochastic gene networks to hybrid piecewise deterministic processes. The Annals of Applied Probability 22, 1822-1859, (2012).

[11] M.H.A. Davis, Piecewise-deterministic Markov processes: a general class off nondiffusion stochastic models. J. Roy. Statist. Soc. Ser. B, 46(3) 353 - 388 (1984).

[12] M.H.A. Davis, Markov models and optimization. Monographs on Statistics and Applied Probability, vol. 49 Chapman & Hall, London. (1993)

[13] P. Diaconis and L. Saloff-Coste, Logarithmic Sobolev inequalities for finite Markov Chains. The Annals of Applied Probability 6, 695-750, (1996).

[14] A. Duarte, E. Löcherbach and G. Ost, Stability, convergence to equilibrium and simulation of non-linear Hawkes Processes with memory kernels given by the sum of Erlang kernels. ESAIM: PS 23, 770-796 (2019).

[15] A. Duarte and G. Ost, A model for neural activity in the absence of external stimuli. Markov Processes and Related Fields 22, 37-52 (2016).

[16] A. Galves and E. Löcherbach, Infinite Systems of Interacting Chains with Memory of Variable Length-A Stochastic Model for Biomedical Neural Nets. J Stat Phys 151, 896-921 (2013).

[17] A. Guionnet and B. Zegarlinski, Lectures on Logarithmic Sobolev Inequalities, IHP Course 98, 1-134 in Seminaire de Probabilite XXVI, Lecture Notes in Mathematics 1801, Springer (2003).

[18] N. Hansen, P. Reynaud-Bouret and V. Rivoirard, Lasso and probabilistic inequalities for multivariate point processes. Bernoulli, 21(1) 83-143 (2015).

[19] P. Hodara, N. Krell and E. Löcherbach, Non-parametric estimation of the spiking rate in systems of interacting neurons. E. Stat Inference Stoch Process, 1-16 (2016).

[20] P. Hodara and E. Löcherbach, Hawkes processes with variable length memory and an infinite number of components. Adv. Appl. Probab 49, 84-107 (2017).

[21] P. Hodara and I. Papageorgiou, Poincaré type inequalities for compact degenerate pure jump Markov processes. Mathematics 7(6), 518 (2019).

[22] M. Ledoux, The concentration of measure phenomenon. Mathematical Surveys and monographs 89, AMS (2001).
[23] M. Ledoux, *Concentration of measure and logarithmic Sobolev inequalities*. Séminaire de Probabilités XXXV. Lecture notes in Math. 1709, 120-216, Springer (1999).

[24] E. Löcherbach, *Absolute continuity of the invariant measure in piecewise deterministic Markov Processes having degenerate jumps*. Stoch. Process. Their Appl. 128, 1797-1829 (2018).

[25] F. Malrieu, *Logarithmic Sobolev inequalities for some nonlinear PDEs*. Stochastic Process. Appl. 95, 109-132 (2001).

[26] J. R. Norris, *Markov Chains*. Vol 2 Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, (1998).

[27] K. Pakdaman, M. Thieullen and G. Wainrib, *Fluid limit theorems for stochastic hybrid systems with application to neuron models*. Adv. Appl. Probab 42, 761-794, (2010).

[28] L. Saloff-Coste, *Lectures on finite Markov chains*. IHP Course 98, École d’ Éte de Probabilités de Saint-Flour XXVI, Lecture Notes in Math. 1665, 301-413, Springer (1996).

[29] M. Talagrand, *Concentration of measure and isoperimetric inequalities in product spaces*. Publ. Math. I.H.E.S. 81, 73-205, Springer (1995).

[30] M. Talagrand, *A new isoperimetric inequality and concentration of measure phenomenon*. In: Lindenstrauss, J., Milman, V.D. (eds.) Geometric Aspects of Functional Analysis, Lecture notes in Math. 1469, 94-124, Springer-Verlag, Berlin (1991).

[31] F.-Y. Wang and C. Yuan, *Poincaré inequality on the path space of Poisson point processes*. J Theor Probab 23 (3), 824-833 (2010).

[32] K. Yosida, *Functional Analysis*. Springer-Verlag (1980).