Absence of Quantum Time Crystals in Ground States

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Recently, Wilczek proposed a fascinating new concept of time crystals, which spontaneously break the continuous time translation symmetry, in analogy with ordinary crystals which break the continuous spatial translation symmetry. Li et al. soon followed with a concrete proposal for an experimental realization and observation of a (space-)time crystal using trapped ions in a ring threaded by an Aharonov-Bohm flux. However, the very existence, even as a matter of principle, of time crystals is rather controversial. Bruno proved a no-go theorem for time crystals for a specific class of Hamiltonians, which covers the original proposal. A similar observation was made for a charge density wave in a superconducting ring by Nozières. However, since these arguments were not fully general, several new realizations of time crystals, which avoid these no-go arguments, were proposed.

In fact, a part of the confusion can be attributed to the lack of a precise mathematical definition of time crystals. Here, we first propose a definition of time crystals in the equilibrium, which is a natural generalization of that of ordinary crystals and can be formulated precisely also for time crystals. We then prove generally the absence of time crystals defined as such, in the ground state of an arbitrary Hamiltonian which consists only of short-range interactions.

While the nature of the time crystal is less understood, it is natural to expect that time crystals would be most easily realized, if at all, at zero temperature, as in the case of ordinary crystals. Thus, we focus on the question whether time crystals can exist as the ground state, as in Wilczek’s original proposal.

Naively, time crystals would be defined in terms of the ground-state expectation value of an observable $\hat{O}(t)$. If it exhibits a periodic time dependence, the system may be regarded as a time crystal. However, the very definition of the ground state $\hat{H}|0\rangle = E_0|0\rangle$ immediately implies that the expectation value of any Heisenberg operator $\hat{O}(t) \equiv e^{iHt}\hat{O}(0)e^{-iHt}$ is time-independent.

Yet it is too early to reject the idea of time crystals just from this observation, since a similar argument would preclude ordinary (spatial) crystals. One might naively define crystals from a spatially modulating expectation value of the density operator $\hat{\rho}(\vec{x}) = e^{-i\hat{P}\vec{x}}\hat{\rho}(0)e^{i\hat{P}\vec{x}}$. The unique ground state of the Hamiltonian in a finite box is nevertheless symmetric and hence $\hat{P}|0\rangle = 0$, implying that $\langle 0|\hat{\rho}(\vec{x})|0\rangle$ is constant over space.

More generally, the ground-state expectation value of any order parameter vanishes in a finite-size system, except when the order parameter commutes with the Hamiltonian (e.g. ferromagnets). “Symmetry-breaking states” with a finite expectation value of the order parameter are generally not eigenstates of the Hamiltonian but are superpositions of eigenstates with energy eigenvalues of $O(1/V)$, where $V$ is the volume of the system (see also Supplementary Information). Here and in the following we set $E_0 = 0$ for brevity.

A convenient and frequently used prescription to avoid this difficulty is to apply a symmetry-breaking field. For example, in the case of antiferromagnets on a cubic lattice, we apply a staggered magnetic field $h_s(\vec{R}) = h\cos(\vec{Q} \cdot \vec{R})$ [$\vec{Q} \equiv (\pi/a)(1,\ldots,1)$] by adding a term $-\sum_{\vec{R}} h_s(\vec{R})\hat{s}_z^{\vec{R}}$ to the Hamiltonian, where $\vec{R}$’s are lattice sites and $\hat{s}_z^{\vec{R}}$ is the spin on the site $\vec{R}$. The ground state under the field has a well-defined Néel order. One computes physical quantities of interest using this state and then take the limit $V \to \infty$ and $h \to 0$ in this order. In the case of crystals, we apply a potential $v(\vec{x}) = h\sum_{\vec{G}} v_{\vec{G}}\cos(\vec{G} \cdot \vec{x})$ with a periodic position dependence. Here, $\vec{G}$’s are the reciprocal lattice of the postulated crystalline order.

This prescription is quite useful but unfortunately is not straightforwardly applicable to time crystals. The symmetry-breaking field for time crystals has to have a periodic time dependence. In the presence of such a field, the “energy” becomes ambiguous and is defined only modulo the frequency of the periodic field, making it difficult to select states or to take statistical ensembles based on energy eigenvalues. Bruno discussed a no-go theorem of time crystals using this type of symmetry-breaking fields, but the class of Hamiltonian studied in that paper is very special in that there exists a rotating frame where the Hamiltonian restores the continuous time translation. In this rotating frame, energy eigenvalues are well defined and one can therefore discuss energies in the lab frame by rotating back to the original frame. However, such a rotating frame does not exist in general.

As a result, the argument can rule out only a particular type of time crystals realized as spontaneous rotation in a ring. Therefore an alternative definition of time crystals is called for. Indeed, we will propose a definition of time crystals which is applicable to very general Hamiltonians.
Before doing so, however, let us point out a crucial difference of time crystals from other spontaneous symmetry breakings. As explained above, symmetry-breaking states usually consist of low-lying states with energies proportional to $V^{-1}$, which are asymptotically degenerate with the ground state in the large volume limit. However, in order to produce a periodic time dependence with a finite period $T$, symmetry-breaking states for time crystals must be superpositions of eigenstates with the energy difference $\Delta E$ integral multiples of $\Omega_0 = 2\pi/T$. Since the energy is bounded from below as opposed to the momentum, the energy expectation value of such superposition states must be always higher than the ground state energy by an $O(1)$ amount. Thus it seems difficult to realize the symmetry-breaking state for time crystal in the ground state. In fact, in the following, we will demonstrate the difficulty in a precise manner.

In order to circumvent the problems mentioned earlier in defining time crystals using a time-dependent symmetry-breaking field, here we define time crystals based on the long-range behavior of correlation functions. In fact, all conventional symmetry breakings can be defined in terms of correlation functions, without introducing any symmetry-breaking field. That is, we say the system has a long-range order if the equal-time correlation function of the local order parameter $\hat{\phi}(\vec{x}, t)$ satisfies

$$
\lim_{|\vec{x} - \vec{x}'| \to \infty} \lim_{V \to \infty} \langle 0 \vert \hat{\phi}(\vec{x}, 0) \hat{\phi}(\vec{x}', 0) \vert 0 \rangle \equiv \sigma^2 \neq 0. \quad (1)
$$

One can equivalently use the integrated order parameter $\hat{\Phi}(t) \equiv \int_V d^3 x \hat{\phi}(\vec{x}, t)$, for which the long-range order is defined as $\lim_{V \to \infty} \langle 0 \vert \hat{\Phi}(0)^2 \vert 0 \rangle / V^2 = \sigma^2$. For example, in the case of antiferromagnets, $\hat{\phi} (\vec{R}, t) = \cos(\vec{Q} \cdot \vec{R}) \hat{s}_z^R(t)$ and $\sigma$ is related to the staggered magnetization density. It has been proven quite generally that the long-range order $\sigma$ guarantees a nonzero expectation value of the order parameter when a symmetry breaking field is applied.\textsuperscript{14,15} One can therefore characterize spontaneous symmetry breaking using the symmetric ground state $\vert 0 \rangle$, which itself does not have a finite expectation value of the order parameter.\textsuperscript{16}

Spontaneous breaking of the space translation symmetry into a discrete subgroup can also be defined by the correlation function. Namely, if the long-range correlation approaches to a periodic function $\hat{\Phi}(t)$

$$
\lim_{V \to \infty} \langle 0 \vert \hat{\phi}(\vec{x}, 0) \hat{\phi}(\vec{x}', 0) \vert 0 \rangle \equiv f(\vec{x} - \vec{x}') \neq 0 \quad (2)
$$

for sufficiently large $|\vec{x} - \vec{x}'|$, the system exhibits a spontaneous crystalline order. Note again that $\langle 0 \vert \hat{\phi}(\vec{x}, t) \vert 0 \rangle$ itself is a constant over space. For instance, we set $\hat{\phi} = \hat{\rho}$ for ordinary crystals, and $\hat{\phi} = \hat{s}_a$ for spin-density waves.

Let us now define time crystals in an analogous manner to these-well known cases. If the system spontaneously breaks the continuous time translation into a discrete one, we expect that the correlation function exhibits a periodic time dependence. Therefore, generalizing equations (1) and (2), we say the system realizes a time crystal if the integrated order parameter satisfies

$$
\lim_{V \to \infty} \lim_{\vec{R} \to \infty} \langle 0 \vert \hat{\Phi}(t) \hat{\Phi}(t') \vert 0 \rangle / V^2 = f(t - t'), \quad (3)
$$

where $f(t)$ is a periodic function of $t$. In terms of the local order parameter $\hat{\phi}(\vec{x}, t)$, the condition reads

$$
\lim_{V \to \infty} \lim_{|\vec{x} - \vec{x}'| \to \infty} \langle 0 \vert \hat{\phi}(\vec{x}, t) \hat{\phi}(\vec{x}', t') \vert 0 \rangle \equiv f(t - t').
$$

If $f$ is a periodic function of both space and time, we call it a space-time crystal. For example, Li et al.\textsuperscript{4} investigated a Wigner crystal in a ring threaded by an Aharonov-Bohm flux and predicted its spontaneous rotation, which would be a realization of space-time crystal. If this were indeed the case, the density at $(x_1, t_1)$ and $(x_2, t_2)$ would be correlated as illustrated in Fig. 1.

One might think that we could define time crystals based the time dependence of equal-position correlation functions. Should we adopt this definition, however, rather trivial systems would qualify as time crystals. For example, consider a two-level system $\hat{H} = -\sigma_0 \sigma_z / 2$ and set $\hat{\phi}(t) \equiv \sigma_z(t) = e^{i\Omega_0 t} \sigma_z e^{-i\Omega_0 t} = \sigma_z \cos \Omega_0 t + \sigma_y \sin \Omega_0 t$. The correlation function $\langle 0 \vert \hat{\phi}(t) \hat{\phi}(0) \vert 0 \rangle$ of the ground state $\vert 0 \rangle = (1, 0)^T$ exhibits a periodic time dependence $e^{-i\Omega_0 t}$. The same applies to the equal-position correlation function in independent two-level systems spread over the space. Clearly we do not want to classify such a trivial, uncorrelated system as a time crystal. “Crystal” should be reserved for systems exhibit correlated, coherent behaviors, which are captured by long-distance correlation functions, be it an ordinary crystal or a time crystal.

We now prove quite generally that time crystals defined above are not possible in ground states. Our argument is a straightforward application of the theorem by Horsch and von der Linden.\textsuperscript{17,18} The only assumption used in the proof is the locality of the Hamiltonian and commutation relations. Namely, the Hamiltonian $\hat{H}$ is a

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{**Time-dependent correlation.** a, Wigner crystal of ions in a ring threaded by an Aharonov-Bohm flux, as proposed in Ref. 4 as a possible realization of a time crystal. b, Illustration of the time dependent correlation function, should the time crystal is indeed realized as a spontaneous rotation of the density wave (crystal) in the ground state, as proposed. The density-density correlation function between $(x_1, t_1)$ and $(x_2, t_2)$ must exhibit an oscillatory behavior as a function of $t_1 - t_2$ for fixed $x_1$ and $x_2$.}
\end{figure}
spatial integral of the local Hamiltonian density and the equal time commutation relation of any two operators \(\hat{\phi}(\vec{x}, t)\) and \(\hat{\phi}(\vec{x}', t)\) may be nonzero only near \(\vec{x} = \vec{x}'\).

As the first step, we show that the nonzero \(f(t - t')\) implies an equal-time long-range order in equations (1). To that end, we insert a complete set \(\sum_n |n\rangle\langle n|\) to equation (3). After some manipulations, one finds

\[
\lim_{V \to \infty} \int_0^\infty d\Omega \rho(\Omega) e^{-\mathsf{d}\Omega t} = f(t),
\]

where \(\rho(\Omega) \equiv V^{-2} \sum_n |\langle 0|\hat{\Phi}(0)|n\rangle|^2 \delta(\Omega - E_n)\). Since \(\rho(\Omega) \geq 0\) for all \(\Omega\), equation (4) with the help of the triangle inequality tells us that \(f(0) \geq |f(t)| \geq 0\). Moreover, \(f(0)\) must be positive, since \(f(t)\) should not be completely zero. Hence, setting \(t' = t\) in equation (3), we see that \(f(0) > 0\) represents a long-range order.

Next, we show that the energy expectation value of the state \(|\Phi\rangle \equiv \hat{\Phi}(0)|0\rangle/\langle \hat{\Phi}(0)|0\rangle\) is \(O(V^{-1})\), meaning that \(|\Phi\rangle\) is one of the low-lying excited states discussed above. First of all, this state is well defined since \(|\langle 0|\hat{\Phi}(0)|\Phi\rangle|^2 \geq |\langle 0|\hat{\Phi}(0)|\Phi\rangle|^2 |\langle 0|\hat{\Phi}(0)\rangle|^2 > 0\). It is also orthogonal to the ground state because of the absence of the expectation value \(\langle 0|\hat{\Phi}(0)|0\rangle = 0\). To estimate the energy expectation value, we rewrite \(\langle \Phi|\hat{H}|\Phi\rangle\) using a double commutator\(^{17,18}\):

\[
\langle \Phi|\hat{H}|\Phi\rangle = \frac{\langle 0|\langle \hat{\Phi}(0)|\hat{H}|\hat{\Phi}(0)|0\rangle\rangle}{\langle 0|\hat{\Phi}(0)|0\rangle^2}.
\]

Both the Hamiltonian \(\hat{H}\) and the operator \(\hat{\Phi}(0)\) involve a spatial integration and each of them introduces a factor of \(V\), while each commutation relation reduces a factor of \(V\) thanks to the assumed locality of commutation relations. Hence, the numerator is at most the order of \(V^3V^{-2} = V\), while the denominator \(\langle 0|\hat{\Phi}(0)|0\rangle^2\). Therefore, we conclude that \(\langle \Phi|\hat{H}|\Phi\rangle = O(V^{-1})\). Inserting the complete set, we have

\[
\int_0^\infty d\Omega \rho(\Omega)\Omega = O(V^{-1}).
\]

Equations (4) and (5) are indeed contradicting each other. Equation (5) suggests that, for any given \(\Omega > 0\) fixed, \(\lim_{V \to \infty} \rho(\Omega) = 0\) (recall that \(\rho(\Omega) \geq 0\)), but equation (4) needs nonzero \(\rho(\Omega)\) for a \(\Omega > 0\). Therefore the initial assumption, that the system is a time crystal, must be false. We note that the present proof does not exclude a possible periodic oscillation in the correlation function of the order parameter in a finite-size system, with a small amplitude of \(O(1/V)\). In fact, such an oscillation can be shown to exist in simple models including a collection of independent two-level systems.

It is straightforward to extend this proof to space-time crystals characterized by a periodic function \(f(t - t', \vec{x} - \vec{x}')\) of space and time and we present the proof in our Supplementary Information. There we also discuss the relation of the time-dependent long-range order \(f(t - t')\) to the divergence of the linear response to a dynamical field. On the other hand, it would be desirable to extend our proof to finite temperatures in the future, for the sake of completeness.

As a final remark, let us clarify the relation between our result and spontaneous oscillations of quantum states which are known to exist. As a characteristic example, a Josephson junction exhibits a periodically oscillating superconducting current driven by the applied dc voltage across the link (ac Josephson effect). However, this is rather a direct consequence of the choice of the initial state, which is not in the equilibrium. In fact, as we discuss details in Supplementary Information, this is essentially equivalent to a spin Larmor precession in a magnetic field. It requires a non-vanishing transverse polarization in the initial state of the spin, which means that the system is not in the equilibrium. Our result, which is valid on ground states, of course does not exclude such spontaneous oscillations of nonequilibrium quantum states. The latter, however, are well known and should not be called time crystals.
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Supplementary Information

Anderson’s tower of states

In this section, we review Anderson’s tower of states for reader’s convenience, since this topic is covered only briefly in the existing literature.1–3

Let us take an antiferromagnet as an example. It breaks the $G = \text{SO}(3)$ spin rotation into a $H = \text{SO}(2)$ subgroup. Its low-energy physics can be captured by a nonlinear sigma model with the target space $G/H = S^2$,

$$
\mathcal{L} = \frac{\rho}{2v^2} \vec{n}^2 - \frac{\rho}{2} (\partial_i \vec{n})^2,
$$

where $\vec{n}(\vec{x}, t) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ is a unit vector that represents the direction of the Néel order parameter. We work in a finite volume $V = L^d$ and take the periodic boundary condition. Using unconstrained variables $\theta$ and $\phi$, it is straightforward to derive the Hamiltonian $H = \int d^d x \left[ (\partial \phi/\partial \theta) \dot{\theta} + (\partial \phi/\partial \phi) \dot{\phi} - \mathcal{L} \right]$.

$$
H = \int d^d x \left[ \frac{\mu^2}{2\rho} \rho \nabla s^2 + \frac{\rho}{2} (\partial_i \vec{n})^2 \right].
$$

(6)

Here, $\vec{s}(\vec{x}, t) = (\rho/v^2) \vec{n} \times \vec{n}$ represents the spin density with commutation relations (or Poisson brackets)

$$
[s_\alpha(\vec{x}, t), s_\beta(\vec{x}', t)] = i\delta^\alpha_\beta(\vec{x} - \vec{x}')\epsilon_{\alpha\beta\gamma} s_\gamma(\vec{x}, t),
$$

(7)

$$
[s_\alpha(\vec{x}, t), n_\beta(\vec{x}', t)] = i\delta^\alpha_\beta(\vec{x} - \vec{x}')\epsilon_{\alpha\beta\gamma} n_\gamma(\vec{x}, t),
$$

$$
n_\alpha(\vec{x}, t), n_\beta(\vec{x}', t) = 0.
$$

The model in Eq. (6) with these algebras is called the quantum rotor model. After Fourier transformation $f(\vec{x}, t) = V^{-1/2} \sum_\vec{k} e^{i\vec{k} \cdot \vec{x}} f_\vec{k}(t)$, the Hamiltonian becomes

$$
H = \frac{\mu^2}{2\rho} V S^2 + \sum_{\vec{k} \neq 0} \left( \frac{\mu^2}{2\rho} \vec{s}_\vec{k} \cdot \vec{s}_\vec{k} + \frac{\rho k^2}{2} \vec{n}_\vec{k} \cdot \vec{n}_\vec{k} \right).
$$

(8)

where $\vec{S} = \int d^d x \vec{s}(\vec{x}, t)$ is the conserved total spin.

The first term of Eq. (8) describes the tradeoff between lowering the energy and reducing the fluctuation of the order parameter.2,3 Namely, the ground state of the Hamiltonian is an $S = 0$ eigenstate of $\vec{S}^2 = S(S + 1)$, as expected from the Marshall-Lieb-Mattis theorem. However, even if $\vec{S}^2$ does not commute with $\vec{n}$, the $S = 0$ state has a divergent fluctuation of the order parameter $\vec{n}$. Therefore, in order to get a well-defined expectation value of the order parameter, we have to take a superposition between different $S$ states, paying the energy cost $\Delta E = v^2 S(S + 1)/(2\rho V)$.

The second term of Eq. (8) denotes the Goldstone excitation with the energy gap $\Delta E' = 2\pi v/L = 2\pi v/V^{1/2}$. These two energy scales are well-separated $\Delta E/\Delta E' \propto V^{-\frac{d-1}{2}} \ll 1$, except in one spatial dimension where continuous symmetries can never be broken unless the order parameter commutes with the Hamiltonian.

It is widely accepted that symmetry-breaking states become degenerate with the ground state in the large volume limit. However, we found it a little bit subtle. If one includes a large enough $S$ of the order $O(\sqrt{V})$ into the superposition (as in the case of the coherent state of superfluids), the energy does not vanish even in the limit $V \to \infty$. One possible resolution might be introducing two separate limits.4 Namely, one defines a symmetry-breaking state by taking a superposition including up to $S_0 < \infty$, and calculate various expectation values using the state, and take the $V \to \infty$ limit with $S_0$ fixed. Since $S_0$ is kept finite, the state successfully becomes degenerate with the ground state. One then takes the limit $S_0 \to \infty$.

Let us briefly discuss a ferromagnet for comparison. The order parameter this time is the total spin operator, which commutes with the Hamiltonian. Hence, no quantum fluctuation is expected. Indeed, its effective Lagrangian reads

$$
\mathcal{L} = m \cos \theta \dot{\phi} - \frac{\rho}{2} (\partial_i \vec{n})^2,
$$

and the Hamiltonian $H = \int d^d x \left[ (\partial \phi/\partial \theta) \dot{\theta} + (\partial \phi/\partial \phi) \dot{\phi} - \mathcal{L} \right]$ is

$$
H = \int d^d x \frac{\mu^2}{2s^2} (\partial_i \vec{s})^2 = \sum_{\vec{k}} \frac{\rho k^2}{m^2} \vec{s}_\vec{k} \cdot \vec{s}_\vec{k}.
$$

Here, $\vec{s} = m\vec{n}$ is the spin density that satisfies the commutation relation in Eq. (7). In this case, the $\vec{k} = 0$ term is absent as opposed to the above antiferromagnet case. Therefore, symmetry breaking states can be taken as a simultaneous eigenstate of $\vec{H}$, $\vec{S}^2$, and $S_z$. As a result, symmetry breaking of this type can happen even in a finite volume and/or in one spatial dimension.

One can easily carry out the the same analysis for crystals and superfluids. The effective Lagrangian describing the phonon vibration is given by

$$
\mathcal{L} = -\frac{1}{2} mn_0 \dot{u}^2 - \mathcal{E}(\partial_i \vec{u}) = \frac{1}{2} \int d^d x \vec{p}^2 + \sum_{\vec{k} \neq 0} \frac{1}{2mn_0} p_\vec{k}^2 \cdot \vec{p}_\vec{k} + \mathcal{E}(k_i \vec{u}_\vec{k}),
$$

where $\vec{u}$ is the displacement field and $\mathcal{E}(\partial_i \vec{u})$ is the elastic energy of the lattice deformation. Introducing the canonical momentum $\vec{p} = mn_0 \vec{u}$ with $[u^i, p^j] = i\delta^{ij}\delta^d(\vec{x} - \vec{x}')$, one gets

$$
H = \frac{\vec{p}^2}{2mn_0 V} + \sum_{\vec{k} \neq 0} \left( \frac{1}{2mn_0} p_\vec{k}^2 \cdot \vec{p}_\vec{k} + \mathcal{E}(k_i \vec{u}_\vec{k}) \right),
$$

where $\vec{P} = \int d^d x \vec{p}$. Again there is a tradeoff between the fluctuation and the energy: the fluctuation of the center of mass must be much smaller than the lattice constant to be a well-defined crystal, but forming a superposition of different momentum states cost energy $\Delta E = G^2/(2mn_0 V)$. 
For superfluids, one may start from the Gross Pitaevskii model:

\[ \mathcal{L} = i \psi \mathbf{i} \psi - \frac{1}{2m} \nabla \psi \mathbf{x} \nabla \psi - \frac{g}{2} \psi \mathbf{i} \psi \psi \]

Introducing the variables \( n \) and \( \theta \) by \( \psi = \sqrt{n} e^{-i\mu t - i\theta} \),

\[ H - \mu N = \int d^d x \left( \frac{\partial \mathcal{L}}{\partial \theta} \theta - \mathcal{L} \right) \]
\[ = \frac{g}{2} V (N - n_0 V)^2 \]
\[ + \sum_{\mathbf{k} \neq 0} \left[ \frac{\mu k^2}{2m g} \theta_\mathbf{k} \theta_\mathbf{k} + \frac{g}{2} \left( 1 + \frac{k^2}{4m \mu} \right) \delta n_\mathbf{k} \delta n_\mathbf{k} \right], \]

where \( \delta n(\mathbf{x}, t) \equiv n(\mathbf{x}, t) - n_0, \) \( n_0 \equiv \mu / g, \) \( N \equiv \int d^d x n(\mathbf{x}, t), \) and \( [\theta(\mathbf{x}, t), n(\mathbf{x}', t)] = i \delta(\mathbf{x} - \mathbf{x}') \). One sometimes uses a coherent state of \( \psi_{\mathbf{k}=0} \) as an approximate symmetry-breaking state, but such a state has a Poisson distribution of the particle number and the order parameter may remain finite.

Divergence of susceptibility

Here we discuss the relation between the long-range order and the divergence of the susceptibility. In the case of an ordinary symmetry breaking, it suffice to consider a static perturbation \( H' = -h \Phi \) in the first-order (non-degenerate) perturbation theory:

\[ \frac{\langle 0 | \Phi | 0 \rangle^{(1)}}{V} = \chi_0 h, \quad \chi_0 \equiv \frac{2}{V} \sum_n \frac{\langle 0 | \Phi | n \rangle^2}{E_n}. \]

Using the Cauchy-Schwarz, we can show \(^5\)

\[ \chi_0 \geq \frac{2V \sigma^2}{E_*}. \]

Recalling that \( E_* = O(1/V) \), we see \( \chi_0 \geq C V^2 \to \infty \) as \( V \to \infty \). Hence, if we take the thermodynamic limit first and then turn off the external field, the expectation value of the order parameter may remain finite.

In the case of time crystals, we have to consider a time-dependent perturbation \( H' = -\theta(t) h(t) \Phi \). The expectation value of the order parameter at the leading order in the perturbation is given by

\[ \langle \Phi \rangle(t)/V = f_0^t dt' \chi(t - t') h(t'), \]

where

\[ \chi(t) = \frac{i \theta(t) \langle 0 | \Phi(t) \Phi(0) | 0 \rangle}{V}, \]
\[ \chi(\omega) = \sum_n \frac{1}{V} \left( \frac{\langle 0 | \Phi | n \rangle^2}{\omega + i \delta + E_n} - \frac{\langle 0 | \Phi | n \rangle^2}{\omega + i \delta - E_n} \right). \]

In particular, for \( h(t) = h \cos \Omega t \),

\[ \frac{\langle 0 | \Phi | 0 \rangle^{(1)}}{V} = -h \sum_{n|E_n \neq \Omega} \frac{\langle 0 | \Phi | n \rangle^2 \cos E_n t - \cos \Omega t}{E_n^2 - \Omega^2} \]
\[ + h \sum_n \frac{|\langle 0 | \Phi | n \rangle|^2 \delta E_n, \Omega t \sin \Omega t}{V}. \]

The second term of right-hand side represents the resonant response, whose amplitude becomes bigger and bigger in proportion to \( t \), just like the forced oscillation problem in a classical mechanics.

Although we proved the no-go theorem in the main text, for now, let us assume a time-dependent long-range order \( f(t - t') \), whose period is \( 2\pi / \Omega_0 \). It implies the existence of a nonzero Fourier component \( f_m | f(t) = \sum_{m=-\infty}^{\infty} f_m e^{-im\Omega_0 t} \). Then, if we set the frequency of the external field \( \Omega \) to be \( m \Omega_0 \), equation (4) of the main text suggests that the coefficient of the resonant response \( t \sin m \Omega_0 t \) is proportional to \( h/V \):

\[ h \sum_n \frac{|\langle 0 | \Phi | n \rangle|^2 \delta E_n, m \Omega_0}{V} \approx hV f_m. \]

Hence, if we apply the dynamical field for a finite time interval \( 0 \leq t \leq T \) and if take the limit \( V \to \infty \) first, the resonant response diverges.

Grand-canonical ensemble

So far we have focused on ground states of the Hamiltonian \( \hat{H} \). This is motivated by the zero temperature limit of the canonical ensemble \( \langle \hat{O}(t) \rangle_{\beta} \equiv \text{Tr}[\hat{O}(t) e^{-\beta \hat{H}}]/\text{Tr}[e^{-\beta \hat{H}}] \). However, in the ground-canonical ensemble

\[ \langle \hat{O}(t) \rangle_{(\beta, \mu)} \equiv \text{Tr}[\hat{O}(t) e^{-\beta (\hat{H} - \mu \hat{N})}]/\text{Tr}[e^{-\beta (\hat{H} - \mu \hat{N})}], \]

those selected in the zero temperature limit are ground states of \( \hat{H} - \mu \hat{N} \), rather than \( \hat{H} \). Nevertheless, the time evolution of the Heisenberg operator \( \hat{O}(t) \) is still defined by \( \hat{H} \), i.e., \( \hat{O}(t) \equiv e^{i \hat{H} t} \hat{O}(0) e^{-i \hat{H} t} \). This mismatch can produce a time dependence of expectation values as we shall see now. If we define \( \hat{O}'(t) \equiv e^{i \hat{H}(t - \hat{N} \mu)^t} \hat{O}(0) e^{-i \hat{H}(t - \hat{N} \mu)^t} \) and assume \( \hat{N}, \hat{O}(0) \) with \( q \) a real number, then

\[ \hat{O}(t) = e^{-i(\hat{H} - \mu \hat{N}) t} \hat{O}(0) e^{-i(\hat{H} - \mu \hat{N}) t} e^{i(\hat{H} - \mu \hat{N}) t} = \hat{O}'(t) e^{-i q t}. \]

Therefore, even if \( \langle \hat{O}(t) \rangle \) is time-independent, \( \langle \hat{O}(t) \rangle \) has a time dependence \( e^{-i q t} \). For example, the order parameter for a Bose-Einstein condensate has a time dependence \( \langle \hat{\psi}(\mathbf{x}, t) \rangle = \psi_0 e^{-i\omega t^{6,7}} \).

It would be natural to expect that this kind of time dependence may be used to realize a time crystal. However, it cannot be measured by any physical operator as long as
the particle number is exactly conserved: $[\hat{H}, \hat{N}] = 0$. In order to extract the time-dependence of the condensate order parameter, the system has to be attached to another system to allow change of the number of particles. As a simplest setup, we may prepare two condensates with different chemical potentials $\mu_1, \mu_2$ and measure their time-dependent interference pattern $\propto e^{-i(\mu_1 - \mu_2)t}$ in terms of the current between the condensates, or equivalently the change of the number of particles in each condensate. This is nothing but the ac Josephson effect. In fact, in Ref. 9, a proposal of time crystal based on this effect was made.

However, in order to observe the ac Josephson effect, the initial state simply must not be in the equilibrium. In order to see this, it is helpful to use the mapping of the ac Josephson effect in two coupled condensates to a quantum spin in a magnetic field. For simplicity, let us consider condensates of bosons without any internal degree of freedom, and suppose there is only one single-particle state in each condensate. Then the system can be described by the two set of bosonic annihilation/creation operators, $a, a^\dagger$ and $b, b^\dagger$. The Hamiltonian of the system, in the limit of zero coupling between the two condensates, is given as

$$H = \mu_1 a^\dagger a + \mu_2 b^\dagger b = (\mu_1 - \mu_2)\frac{a^\dagger a - b^\dagger b}{2} + \frac{\mu_1 + \mu_2}{2} N,$$

where $N = a^\dagger a + b^\dagger b$ is the total number of particles in the coupled system. Without a coupling to the outside environment, $N$ is exactly conserved and can be regarded as a constant. As a consequence, the second term in the Hamiltonian proportional to $N$ can be ignored.

With $N$ being exactly conserved, this system of coupled condensates can be mapped to a quantum spin model by identifying the bosons as Schwinger bosons. The Hamiltonian now reads

$$H = BS^z + \text{const.},$$

where $B = \mu_1 - \mu_2$ and $S^z$ is the $z$-component of the quantum spin with the spin quantum number $S = (N - 1)/2$. Similarly, the current operator between the two condensates is given by

$$J = \frac{\eta}{i}(a^\dagger b - b^\dagger a) = 2\eta S^y.$$

The ac Josephson effect, in the quantum spin language, is just a Larmor precession about the magnetic field. The oscillatory behavior of the current in the ac Josephson effect just corresponds to the oscillation of the expectation value of $S^y$ in the Larmor precession.

In order to observe the Larmor precession, the initial state must have a non-vanishing expectation value of the transverse component ($S^x$ or $S^y$). This excludes the ground state, in which the spin is fully polarized along the magnetic field in $z$ direction, as well as thermodynamic equilibrium at arbitrary temperature.

In Ref. 9 it was argued that, by taking the limit of weak coupling, the dissipation can be made arbitrarily small. While this is certainly true, the lack of dissipation does not mean that the system is in an equilibrium, as it is clear by considering the spin Larmor precession in a magnetic field.

Absence of space-time crystals

Here we extend our argument for time crystals to space-time crystals. The basic idea remains unchanged.

By the definition of space-time crystals (see the main text), for any $\epsilon > 0$, there exists $\bar{R} > 0$ such that for all $|\vec{x} - \vec{x}'| > \bar{R}$,

$$\lim_{V \to \infty} \left| \frac{\langle 0|\hat{\phi}(\vec{x}, t)\hat{\phi}(\vec{x}', t')|0 \rangle - f(t - t', \vec{x} - \vec{x}')}{\epsilon} \right| < 1.$$  

Here $f(t - t', \vec{x} - \vec{x}')$ is a periodic function of $t - t'$ and $\vec{x} - \vec{x}'$. Suppose that $\hat{G}$ is one of the Fourier components of $f$:

$$f_\hat{G}(t) = \lim_{V \to \infty} \frac{1}{V} \int_V d^d x f(t, \vec{x}) \cos(\hat{G} \cdot \vec{x}).$$

We then introduce a Hermitian operator

$$\hat{\Phi}_\hat{G}(t) \equiv \int_V d^d x \hat{\phi}(\vec{x}, t) \cos(\hat{G} \cdot \vec{x}),$$

We could use $\sin(\hat{G} \cdot \vec{x})$, but for the current purpose this choice is sufficient. Using these definitions, one can show

$$\lim_{V \to \infty} \langle 0|\hat{\Phi}_\hat{G}(t)\hat{\Phi}_\hat{G}(t')|0 \rangle / V^2 = \frac{1}{2} f_\hat{G}(t). \quad (9)$$

All one has to do is to replace equation (3) of the main text with equation (9) here. Namely, one can show that (i) $f_\hat{G}(0) > 0$ represents the equal-time long-range crystalline order and that (ii) the energy expectation value of the state defined by

$$|\Phi_\hat{G}\rangle \equiv \frac{\hat{\Phi}_\hat{G}(0)|0\rangle}{|\langle 0|\hat{\Phi}_\hat{G}(0)|0 \rangle|}$$

is $O(V^{-1})$ exactly in the same way as in the main text. These result in contradicting relations of the spectral function $\rho_\hat{G}(\Omega) \equiv V^{-2} \sum_n |\langle 0|\hat{\Phi}_\hat{G}(0)|n\rangle|^2 \delta(\Omega - E_n)$, analogously to equations (4) and (5).

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