Stability of entire solutions to supercritical elliptic problems involving advection

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Abstract
We examine the equation given by

\[-\Delta u + a(x) \cdot \nabla u = u^p \quad \text{in } \mathbb{R}^N, \tag{1}\]

where \( p > 1 \) and \( a(x) \) is a smooth vector field satisfying some decay conditions. We show that for \( p < p_c \), the Joseph-Lundgren exponent, that there is no positive stable solution of (1) provided one imposes a smallness condition on \( a \) along with a divergence free condition. In the other direction we show that for \( N \geq 4 \) and \( p > \frac{N-1}{N-3} \) there exists a positive solution of (1) provided \( a \) satisfies a smallness condition. For \( p > p_c \) we show the existence of a positive stable solution of (1) provided \( a \) is divergence free and satisfies a smallness condition.

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1 Introduction and results
In this article we are interested the existence versus nonexistence of positive stable solutions of

\[-\Delta u + a(x) \cdot \nabla u = u^p \quad \text{in } \mathbb{R}^N, \tag{2}\]
where \( p > 1 \) and \( a(x) \) is a smooth vector field satisfying some decay conditions. We now define the notion of stability and for this we prefer to work on a general domain.

**Definition 1.** Let \( u \) denote a nonnegative smooth solution of (2) in an open set \( \Omega \subset \mathbb{R}^N \). We say \( u \) is a stable solution of (2) in \( \Omega \) provided there is some smooth positive function \( E \) such that

\[
-\Delta E + a(x) \cdot \nabla E \geq \frac{p}{p-1} u E \quad \text{in } \Omega.
\]

(3)

We begin by recalling some facts in the case where \( a(x) = 0 \). There has been much work done on the existence and nonexistence of positive classical solutions of

\[
-\Delta u = u^p, \quad \text{in } \mathbb{R}^N.
\]

(4)

For \( N \geq 3 \) there exists a critical value of \( p \), given by \( p_S = \frac{N+2}{N-2} \), such that for \( 1 < p < p_S \) there is no positive classical solution of (4) and for \( p > p_S \) there exist positive classical solutions, see [2, 4, 10, 15]. By definition we call a nonnegative solution \( u \) of (4) stable if

\[
\int pu^{p-1} \phi^2 \leq \int |\nabla \phi|^2 \quad \forall \phi \in C_c^\infty(\mathbb{R}^N),
\]

(5)

which is nothing more than the stability of \( u \) using (3), after using a variational principle. The additional requirement that the solution be stable drastically alters the existence versus nonexistence results. It is known that there is a new critical exponent, the so called Joseph-Lundgren exponent \( p_c \), such that for all \( 1 < p < p_c \) there is no positive stable solution of (4) and for \( p > p_c \) there exists positive stable solutions of (4). The value of the \( p_c \) is given by

\[
p_c = \begin{cases}
\frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)} & N \geq 11 \\
\infty & 3 \leq N \leq 10.
\end{cases}
\]

The first implicit appearance of \( p_c \) was in the work [18] where they examined \(-\Delta u = \lambda (u + 1)^p \) on the unit ball in \( \mathbb{R}^N \) with zero Dirichlet boundary conditions. The exponent \( p_c \) first explicitly appeared in the works [20, 17] where they examined the stability of radial solutions to a parabolic version of (4). Their results easily imply the existence of a positive radial stable solution of (4) when \( p > p_c \) and the nonexistence of positive radial stable solutions in the case of \( p < p_c \). More recently there has been interest in finite Morse index solutions of either (4) and the generalized version given by

\[
-\Delta u = |u|^{p-1} u, \quad \text{in } \mathbb{R}^N.
\]

(6)
In [14] they completely classified the finite Morse index solutions of (6) and again the critical exponent $p_c$ was involved. For results regarding singular nonlinearities, general nonlinearities, or quasilinear equation see [3, 10, 11, 12, 13, 1].

In the work [6] the nonexistence of nontrivial solutions of

$$-\text{div}(\omega_1 \nabla u) = \omega_2 u^p \quad \text{in } \mathbb{R}^N,$$

was examined where $\omega_i$ are some nonnegative functions. In the special case where $\omega_1 = \omega_2$ this equation reduces to

$$-\Delta u + \nabla \gamma(x) \cdot \nabla u = u^p \quad \text{in } \mathbb{R}^N,$$

(7)

where $\gamma$ is a scalar function. Even though (7) and (2) are similar a major difference is that (7) is variational in nature; critical points of

$$E(u) = \frac{1}{2} \int e^{-\gamma} |\nabla u|^2 - \frac{1}{p+1} \int e^{-\gamma} |u|^{p+1},$$

are solutions of (7). This variational structure of (7) allows one to prove various nonexistence results for (7) by slightly modifying the nonexistence proofs used in proving similar results for $-\Delta u = u^p$ in $\mathbb{R}^N$. This approach will generally not work for (2) since in general there will not be a variational structure.

In [7] the regularity of the extremal solution, $u^*$, associated with problems of the form

$$\begin{cases}
-\Delta u + a(x) \cdot \nabla u &= \lambda f(u) & \text{in } \Omega \\
u &= 0 & \text{on } \partial \Omega,
\end{cases}$$

was examined for various nonlinearities $f$. Here $a(x)$ was an arbitrary smooth advection and the main difficulty was to to utilize the stability of $u^*$ in a meaningful way. As mentioned earlier, this is not a problem when $a(x)$ is the gradient of a scalar function. The main tool used was the generalized Hardy inequality from [5]. This same approach was extended to more general nonlinearities in [20].

We now list our results.

**Theorem 1.** Suppose $3 \leq N \leq 10$ or $N \geq 11$ and $1 < p < p_c$. Suppose $a(x)$ is a smooth divergence free vector field satisfying $|a(x)| \leq \frac{C}{|x|+1}$ with $0 < C$ sufficiently small. Then there is no positive stable solution of (2).
The next result gives a decay estimate in the case of $p < p_c$. We are including this result since it may allow one to use a Lane-Emden type of change of variables to obtain a nonexistence result without a smallness condition on the advection.

**Theorem 2.** Suppose $\frac{N+2}{N-2} < p < p_c$, $a(x)$ is a smooth divergence free vector field with $|a(x)| \leq \frac{C}{|x|^{N-1}}$ and $|a| \in L^N(\mathbb{R}^N)$. Then any positive stable solution $u$ of (2) satisfies
\[
\lim_{|x| \to \infty} |x|^{\frac{2}{p-1}} u(x) = 0.
\] (8)

The approach to solve Theorem 1 will be to combine the methods used in [14] with the techniques from [7] which relied on generalized Hardy inequalities from [5]. The same approach will be used in the proof of Theorem 2 with an added scaling argument.

Our final result gives an existence result.

**Theorem 3.**
1. Suppose $N \geq 4$, $p > \frac{N+1}{N-3}$ and $a(x)$ is some smooth vector field with $|a(x)| \leq \frac{C}{|x|^{N-1}}$. If $0 < C$ is sufficiently small there exists a positive solution of (2).

2. Suppose $N \geq 11$, $p > p_c$ and let $a(x)$ denote some smooth divergence free vector field with $|a(x)| \leq \frac{C}{|x|^{N-1}}$. For $0 < C$ sufficiently small (2) has a positive stable solution.

The idea of the proof will be to look for a solution $u$ as a perturbation of the positive radial solution $w$ of $-\Delta w = w^p$ in $\mathbb{R}^N$ with $w(0) = 1$. See the beginning of Section 3 for details on $w$. The framework we will use to prove the existence of a positive solution will be the approach developed in [8]. Their interest was in the existence of positive solutions of $-\Delta u = u^p$ in $\Omega \subset \mathbb{R}^N$ an exterior domain with zero Dirichlet boundary conditions.

**Open Problem.** It would be interesting to see if these smallness conditions on $a(x)$ can be removed, possibly at the expense of adding some additional decay requirements.

## 2 Nonexistence proofs

**Remark 1.** A computation shows that $p < p_c$ is equivalent to the condition
\[
\frac{N}{2} < 1 + \frac{2p}{p-1} + \frac{2}{p-1} \sqrt{p^2 - p}.
\] (9)
For our nonexistence results it will be easier to deal with \( (9) \).

Theorem 1 and Theorem 2 will depend on the following energy estimate, which we state for a general domain.

**Proposition 1.** Suppose \( u \) is a smooth positive stable solution of \( (2) \) and \( a(x) \) is smooth divergence free vector field. Then for all \( 1 \leq T, 0 < \beta < 1, 0 < \varepsilon, 0 < \delta, \frac{1}{2} < t \) and \( 0 \leq \psi \in C^\infty_c(\Omega) \) we have

\[
\left( \beta p - \frac{Tt^2}{2t-1} \right) \int u^{2t+p-1} \psi^2 + \beta (1 - \beta - \varepsilon) \int \frac{\nabla E|^2}{E^2} u^{2t} \psi^2 \\
+ (T - 1) \int |\nabla (u^t \psi)|^2 \\
\leq \left( \frac{\beta}{4\varepsilon} + \frac{Tt\delta}{2t-1} \right) \int |a|^2 u^{2t} \psi^2 \\
+ \left( T + \frac{Tt}{4\delta(2t-1)} \right) \int u^{2t} |\nabla \psi|^2 \\
+ \frac{T|t-1|}{2(2t-1)} \int u^{2t} |\Delta \psi|^2. \tag{10}
\]

Define the following parameters

\[
t_-(p) = p - \sqrt{p^2 - p} \quad \text{and} \quad t_+(p) = p + \sqrt{p^2 - p}.
\]

A computation shows that for \( t_-(p) < t < t_+(p) \) we have \( p - \frac{t^2}{2t-1} > 0 \). This restriction on \( t \) will be related to the restrictions on \( t \) we must impose if one wants to obtain an estimate from Proposition 1.

**Proof of Proposition 1.** Suppose \( u \) is a smooth positive stable solution of \( (2) \) in \( \Omega \) and let \( E > 0 \) satisfy \( (3) \). From \( \ref{5} \) we have the following generalized Hardy inequality

\[
\beta \int \frac{-\Delta E}{E} \phi^2 + (\beta - \beta^2) \int \frac{\nabla E|^2}{E^2} \phi^2 \leq \int |\nabla \phi|^2, \quad \forall \phi \in C^\infty_c(\Omega), \tag{11}
\]

for all \( \beta \mathbb{R} \). Adding \( T \int |\nabla \phi|^2 \) to both sides of the inequality, using the fact that \( E \) satisfies \( \ref{3} \) and taking \( \phi = u^t \psi \) where \( \psi \in C^\infty_c(\Omega) \) gives

\[
\beta p \int u^{p-1} u^{2t+p-1} \psi^2 - \beta \int \frac{a \cdot \nabla E}{E} u^{2t} \psi^2 \\
+ (\beta - \beta^2) \int \frac{\nabla E|^2}{E^2} u^{2t} \psi^2 + (T - 1) \int |\nabla (u^t \psi)|^2 \\
\leq T \int |\nabla (u^t \psi)|^2.
\]
Note that the right side expands as

\[ Tt^2 \int u^{2t-2} \nabla u^2 \psi^2 + 2tT \int u^{2t-1} \nabla u \cdot \nabla \psi + T \int u^{2t} |\nabla \psi|^2. \]

We now wish to eliminate the term \( \int u^{2t-2} |\nabla u|^2 \psi^2 \) from the inequality. To do this we multiply (2) by \( u^{2t-1} \psi^2 \) and integrate over \( \Omega \) to arrive at

\[(2t - 1) \int u^{2t-2} |\nabla u|^2 \psi^2 = \int u^{p+2t-1} \psi^2 - \int a \cdot \nabla u u^{2t-1} \psi^2 - 2 \int \nabla u \cdot \nabla \psi u^{2t-1} \psi.\]

Using this equality we replace the desired term in the inequality to arrive at an inequality of the form

\[
\left( \beta p - \frac{Tl^2}{2t - 1} \right) \int u^{2t+p-1} \psi^2 + \beta(1 - \beta) \int \frac{|\nabla E|^2}{E^2} u^{2t} \psi^2
+ (T - 1) \int |\nabla (u^t \psi)|^2 \leq T \int u^{2t} |\nabla \psi|^2
+ \sum_{k=1}^{3} I_k \quad (12)
\]

where

\[
I_1 = \left( 2Tt - \frac{2T l^2}{2t - 1} \right) \int u^{2t-1} \psi \nabla u \cdot \nabla \psi,
\]

\[
I_2 = \frac{Tl^2}{2t - 1} \int a(x) \cdot \nabla uu^{2t-1} \psi^2,
\]

\[
I_3 = \beta \int a(x) \cdot \nabla E E u^{2t} \psi^2.
\]

An integration by parts shows that

\[
I_1 = \frac{T(1 - t)}{2(2t - 1)} \int u^{2t} \Delta (\psi^2).
\]

An integration by parts shows that

\[
|I_2| \leq \frac{Tt}{2t - 1} \int |a| \psi \nabla \psi |u^{2t},
\]

and an application of Young’s inequality shows this is less than or equal to

\[
\frac{Tt \delta}{2t - 1} \int |a|^2 \psi^2 u^{2t} + \frac{Tt}{(2t - 1)4\delta} |\nabla \psi|^2 u^{2t}.
\]
An application of Young’s inequality shows that
\[ |I_3| \leq \beta \varepsilon \int \frac{|
abla E|^2}{E^2} u^{2t} \psi^2 + \frac{\beta}{4\varepsilon} \int |a|^2 u^{2t} \psi^2. \]
Using these upper bounds in (12) and regrouping gives the desired result.

**Proof of Theorem 1.** We assume that \( u \) is a positive stable solution of (2). Firstly note that
\[ \int |a|^2 u^{2t} \psi^2 \leq C \int u^{2t} \psi^2 \left| \frac{x}{|x|^2} \right|, \]
after considering the conditions on \( a \). Also note by Hardy’s inequality we have
\[ \int |\nabla (u^t \psi)|^2 \geq C_N \int u^{2t} |x|^2, \]
where \( C_N = \frac{(N-2)^2}{4} \). Putting these into (11) gives
\[
\left( \beta p - \frac{T t^2}{2t - 1} \right) \int u^{2t+p-1} \psi^2 \\
+ \beta (1 - \beta - \varepsilon) \int \frac{|
abla E|^2}{E^2} u^{2t} \psi^2 \\
+ C_1 \int \frac{u^{2t} \psi^2}{|x|^2} \leq C_2 \int u^{2t} \left( |\nabla \psi|^2 + |\Delta (\psi^2)| \right) \quad (13)
\]
where
\[ C_1 = (T - 1)C_N - C^2 \left( \frac{\beta}{4\varepsilon} + \frac{T t \delta}{2t - 1} \right), \]
and \( C_2 = C_2(T, t, \delta) \). Note that for each \( t_-(p) < t < t_+(p) \) we have \( \beta p - \frac{T t^2}{2t - 1} > 0 \) provided \( \beta < 1 \) and \( T > 1 \) are chosen sufficiently close to 1. We now pick \( \varepsilon > 0 \) small enough such that \( 1 - \beta - \varepsilon > 0 \). We now assume \( C > 0 \) is sufficiently small such that \( C_1 \geq 0 \). We then arrive at an estimate of the form
\[
\left( \beta p - \frac{T t^2}{2t - 1} \right) \int u^{2t+p-1} \psi^2 \leq C_2 \int u^{2t} \left( |\nabla \psi|^2 + |\Delta (\psi^2)| \right), \quad (14)
\]
for all \( \psi \in C_c^\infty(\mathbb{R}^N) \). We now assume that \( \phi \) is a smooth cut-off function with, \( 0 \leq \phi \leq 1 \), \( \phi = 1 \) in \( B_R \) and compactly supported in \( B_{2R} \) such that
\[ \left| \nabla \phi \right| \leq \frac{C}{R} \text{ and } \left| \Delta \phi \right| \leq \frac{C}{R^2} \text{ where } C \text{ is independent of } R. \] Putting \( \psi = \phi^m \) where \( m \) is a large integer into (14) gives
\[ \left( \beta p - \frac{T t^2}{2t - 1} \right) \int u^{2t+p-1} \phi^{2m} \leq C_2 C_m \int u^{2t} \phi^{2m-2} \left( \left| \nabla \phi \right|^2 + \left| \Delta \phi \right| \right), \]
where \( C_m \) depends only on \( m \). We now apply Hölder’s inequality to see the right hand side of this inequality is bounded above by
\[ C_2 C_m \left( \int u^{2t+p-1} \frac{\phi^{(m-1)(2t+p-1)}}{t} \, dx \right)^{\frac{2t}{2t+p-1}} \left( \int \left( \left| \nabla \phi \right|^2 + \left| \Delta \phi \right| \right)^{\frac{2t+p-1}{p-1}} \, dx \right)^{\frac{p-1}{2t+p-1}}. \]
Now note that for sufficiently large \( m \) we have that \( \frac{(m-1)(2t+p-1)}{t} > 2m \) and hence we can replace the first term on the right hand side of the inequality with
\[ \left( \int u^{2t+p-1} \phi^{2m} \, dx \right)^{\frac{2t}{2t+p-1}}, \]
which allows one to cancel terms to arrive at
\[ \left( \beta p - \frac{T t^2}{2t - 1} \right)^{\frac{2t+p-1}{p-1}} \int u^{2t+p-1} \phi^{2m} \leq \tilde{C}_m \int \left( \left| \nabla \phi \right|^2 + \left| \Delta \phi \right| \right)^{\frac{2t+p-1}{p-1}}. \]
We now take into account the support of \( \phi \) and how \( \phi \) scales to arrive at
\[ \int_{B_R} u^{2t+p-1} \leq C_0 R^{N-2-\frac{2(2t+p-1)}{p-1}}, \]
where \( C_0 \) depends on the various parameters but is independent of \( R \). Now provided \( N - 2 - \frac{2(2t+p-1)}{p-1} < 0 \) we can send \( R \to \infty \) to arrive at a contradiction. Now note we can pick a \( t \in (t_-(p), t_+(p)) \) such that this exponent is negative provided
\[ \frac{N(p-1)}{2} < 2 \left( p + \sqrt{p^2 - p} \right) + p - 1, \]
which is precisely (13).

\textbf{Proof of Theorem 2.} Suppose \( 0 < u \) is a smooth stable solution of (2) and \( E > 0 \) solves (3). Let \( |x_k| \to \infty \) and set \( r_k := \frac{|x_k|}{k} \). By passing to a subsequence we can assume that \( \{ B(x_k, r_k) : k \geq 1 \} \) is a disjoint family of balls. We now define the rescaled functions
\[ u_k(x) = \frac{x}{r_k^2} u(x_k + r_k x), \quad a_k(x) = r_k a(x_k + r_k x), \quad E_k(x) = E(x_k + r_k x), \]
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and we restrict $|x| < 2$. Then equation (2) and (3) are satisfied on $B_2$ with $u_k, a_k, E_k$ replacing $u, a, E$. Note that $a_k(x)$ is a sequence of smooth divergence free vector fields which satisfy the bound $|a_k(x)| \leq C$ for all $|x| < 2$. From this we see the term involving $a_k$ in (10) will be a lower order term as far as powers of $u$ are concerned and hence will cause no issues. With the conditions on $N$ and $p$ there is some $t - p < t < t + p$ such that $2t + p - 1 > \frac{N}{2}(p - 1) > 0$ and by taking $T = 1$ (we can take $T = 1$ since the advection term is lower order) and $\beta < 1$ sufficiently close to 1 we can assume $\beta p - \frac{t^2}{t^2 + 1} > 0$. Let $0 \leq \phi \leq 1$ be compactly supported in $B_2$ with $\phi = 1$ on $B_1$ and put $\psi = \phi^m$, where $m$ a large integer, into (10) where now $u, a, E$ are given by $u_k, a_k, E_k$. Arguing as in the proof of Theorem 1 one can obtain a bound of the form

$$\int_{B_1} u_k^{2t + p - 1} \leq C_0,$$

where $C_0$ depends on the various parameters but is independent of $k$. Now note that $u_k > 0$ is a sequence of smooth positive solutions of

$$-\Delta u_k + a_k(x) \cdot \nabla u_k = C_k(x) u_k \quad \text{in } B_2,$$

where $C_k(x) = u_k^{p-1}$. The above integral estimate shows that $C_k$ is bounded in $L^q(B_1)$ for some $q > \frac{N}{2}$. We can now apply a Harnack inequality from [19] to see that

$$\sup_{B_{\frac{1}{2}}} u_k \leq C \inf_{B_{\frac{1}{2}}} u_k. \quad (15)$$

If we can show that $\inf_{B_{\frac{1}{2}}} u_k \to 0$ then one has $\sup_{B_{\frac{1}{2}}} \to 0$ and in particular this gives

$$|x_k|^{\frac{2t}{p - 1}} u(x_k) \leq 4^{\frac{2}{p - 1}} \sup_{B_{\frac{1}{2}}} u_k \to 0$$

which gives us the desired decay estimate. To show $\inf_{B_{\frac{1}{2}}} u_k \to 0$ we will show

$$\int_{B_1} u_k^{\frac{(p - 1)N}{2}} \to 0.$$
and if we show that \( u \in L^{\frac{(p-1)N}{p-2}}(\mathbb{R}^N) \) then we’d have the desired result since

\[
\int_{\mathbb{R}^N} u^{\frac{(p-1)N}{p-2}} \geq \sum_{k=1}^{\infty} \int_{B(x_k,r_k)} u^{\frac{(p-1)N}{p-2}}.
\]

Towards this we now set \( t = \frac{(p-1)(N-2)}{4} \) and note that the condition on \( N \) and \( p \) imply that \( t_-(p) < t < t_+(p) \). We now pick \( \beta < 1 \) but sufficiently close such that \( \beta p - \frac{t^2}{2t-1} > 0 \) and pick \( \varepsilon > 0 \) sufficiently small such that \( 1 - \beta - \varepsilon > 0 \). Let \( \phi \) be the smooth cut-off function from the proof of Theorem 1, which is equal to 1 in \( B_R \) and compactly supported in \( B_{2R} \). We now put \( \psi = \phi^m \), where \( m \) is a large integer, into (10) taking \( T = 1 \), to arrive at inequality of the form

\[
\int u^{2t+p-1} \phi^{2m} \leq C_0 \int |a|^2 u^{2t} \phi^{2m} + C_0 \int u^{2t} \phi^{2m-2} (|\nabla \phi|^2 + |\Delta \phi|) \tag{16}
\]

We now let \( \tau \) be such that \( 2t\tau = 2t + p - 1 \) and let \( \tau' \) denote the conjugate index of \( \tau \). Applying Hölder’s inequality to the right hand side of (16) and arguing as in the proof of Theorem 1 we arrive at an inequality, for sufficiently large \( m \), of the form

\[
\int u^{2t+p-1} \phi^{2m} \leq C_0 \int_{B_{2R}} |a|^{2\tau'} + C_0 \int_{B_{2R}} (|\nabla \phi|^2 + |\Delta \phi|)^{\tau'},
\]

where \( C_0 \) is a constant which depends on the various parameters but is independent of \( R \). A computation shows that \( \tau' = \frac{N}{2} \) and \( 2t + p - 1 = \frac{N}{2}(p - 1) \). Using these explicit values and the scaling of \( \phi \) we arrive at

\[
\int_{B_R} u^{\frac{N(p-1)}{2}} \leq C_0 \int_{B_{2R}} |a|^N + C_0,
\]

and from this we obtain the desired bound on \( u \) after recalling that \( |a| \in L^N(\mathbb{R}^N) \).

\[
\square
\]

3 \ Existence proofs

The positive radial solution.

For \( p > \frac{N+2}{N-2} \) let \( w = w(r) \) denote the positive radial decreasing solution of

\(-\Delta w = w^p \) in \( \mathbb{R}^N \) with \( w(0) = 1 \). Asymptotics of \( w \) as \( r \to \infty \) are given by

\[
w(r) = \beta^\frac{1}{p-1} r^{\frac{p-2}{p-1}} (1 + o(1)),
\]

10
where

$$\beta = \beta(p, N) = \frac{2}{p-1} \left( N - 2 - \frac{2}{p-1} \right).$$

In the case where $p > p_c$ the refined asymptotics are given by

$$w(r) = \beta \frac{1}{p-1} r^{\frac{2}{p-1}} + \frac{a_1}{r^{\mu_0}} + o \left( \frac{1}{r^{\mu_0}} \right),$$

where $a_1 < 0$ and $\mu_0 > \frac{2}{p-1}$; see [17].

We begin by analysing the radial solution $w$ as defined above. Let $v(r) = \beta \frac{1}{p-1} r^{\frac{2}{p-1}}$ where $\beta$ is defined as above.

**Lemma 1.** Suppose $p > p_c$, $v(r) = \beta \frac{1}{p-1} r^{\frac{2}{p-1}}$ and $\beta$ is defined as in the definition of $w$.

1. Then $v \geq w$ in $\mathbb{R}^N$.

2. There is some $\varepsilon > 0$ such that

$$\int (p + \varepsilon) w^{p-1} \phi^2 \leq \int |\nabla \phi|^2 \quad \forall \phi \in C_0^\infty(\mathbb{R}^N). \quad (17)$$

**Proof.** 1) Note that $v(r) > w(r)$ for large $r$ and small $r$. Towards a contradiction we assume that there is $0 < r_0 < r_1$ such that $w(r) > v(r)$ for all $r_0 < r < r_1$ with $w = v$ at $r = r_0, r_1$. A computation shows that for $p > p_c$ there is some $\varepsilon > 0$ such that $(p + \varepsilon) \beta \leq \frac{(N-2)^2}{4}$ and then from Hardy’s inequality we obtain

$$\int (p + \varepsilon) v^{p-1} \phi^2 \leq \int |\nabla \phi|^2 \quad \forall \phi \in C_0^\infty(\mathbb{R}^N). \quad (18)$$

From this we see that $v$ is a stable singular solution of $-\Delta v = v^p$ in $\mathbb{R}^N$ and in particular its a stable solution of

$$-\Delta v = v^p \text{ in } r_0 < r < r_1 \quad \text{with } v = w \text{ on } r = r_0, r_1.$$

It is possible to use the stability of $v$ to show that $v$ is the minimal solution of this equation with the given prescribed boundary conditions. This fact relies on the strict convexity of the nonlinearity. Noting that $w$ satisfies the same equation with the prescribed boundary conditions one must have $v \leq w$ on $r_0 < r < r_1$ since $v$ is a minimal solution. This gives us the desired contradiction.

2) The result is immediate after combining the pointwise comparison between $w$ and $v$ and using (18).
For the remainder we always refers to the above radial solution and $L$ to the linear operator $L(\phi) = -\Delta \phi - pw^{p-1}\phi$.

We now define the various function spaces. For $\sigma > 0$ but small, define
\[
\|\phi\|_{\tilde{X}_\sigma} := \sup_{|x| \leq 1} |x|^\sigma |\phi(x)| + \sup_{|x| \geq 1} |x|^{\frac{2}{p-1}} |\phi(x)|,
\]
and
\[
\|f\|_{Y_\sigma} := \sup_{|x| \leq 1} |x|^{|\sigma+2}|f(x)| + \sup_{|x| \geq 1} |x|^{\frac{2}{p-1}+2} |f(x)|.
\]
Let $\tilde{X}_\sigma$ and $Y_\sigma$ denote the completions of $C^\infty_c(\mathbb{R}^N \setminus \{0\})$ under the appropriate norms.

The following linear estimate is from [8] and is a key starting point for their work. They also obtain results in the case of $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$ in [8] and also in another of their works [9]. This case is harder to deal with but luckily are main interest is in the case of $p > p_c$ which allows us to avoid the harder case.

**Theorem A.** [8] Suppose $N \geq 4$ and $p > \frac{N+1}{N-3}$. There exists some small $\sigma > 0$ such that for any $f \in Y_\sigma$ there exists some $\phi \in \tilde{X}_\sigma$ such that $L(\phi) = f$ in $\mathbb{R}^N$. Moreover the linear map $T : Y_\sigma \to \tilde{X}_\sigma$, given by $T(f) = \phi$, is continuous.

For our approach we won’t work directly with $\tilde{X}_\sigma$ but instead work with a slight variant that allows us to handle the advection term. So towards this define the norm
\[
\|\phi\|_{X_\sigma} := \sup_{|x| \leq 1} (|x|^\sigma |\phi(x)| + |x|^{\sigma+1} |\nabla \phi(x)|) \\
+ \sup_{|x| \geq 1} \left(|x|^{\frac{2}{p-1}} |\phi(x)| + |x|^{\frac{2}{p-1}+1} |\nabla \phi(x)|\right)
\]
and let $X_\sigma$ denote the completion of $C^\infty_c(\mathbb{R}^N \setminus \{0\})$ with respect to this norm.

**Lemma 2.** Suppose $N \geq 4$ and $p > \frac{N+1}{N-3}$. For sufficiently small $\sigma > 0$ and for all $f \in Y_\sigma$ there exists some $\phi \in X_\sigma$ such that $L(\phi) = f$ in $\mathbb{R}^N$. Moreover the linear map $T : Y_\sigma \to X_\sigma$ defined by $T(f) = \phi$ is continuous.

**Proof of Lemma 2.** Suppose $f \in Y_\sigma$ and let $\phi \in \tilde{X}_\sigma$ be such that $L(\phi) = f$ in $\mathbb{R}^N$. Then there exists some $C > 0$, independent of $f$ and $\phi$, such that $\|\phi\|_{\tilde{X}_\sigma} \leq C\|f\|_{Y_\sigma}$. Our goal is to now show there is some $C_1 > 0$,
independent of $f$ and $\phi$, such that $\|\phi\|_{X_\sigma} \leq C_1 \|f\|_{Y_\sigma}$ and this will complete
the proof. Define the re-scaled functions $\phi_m(x) = \phi(x_m + r_m x)$ where $|x_m| > 0$, $r_m = |x_m|/4$ for $|x| < 1$. Note that

$$-\Delta \phi_m(x) = pr_m^2 w(x_m + r_m x)^{p-1} \phi(x_m + r_m x) + r_m^2 f(x_m + r_m x) =: g_m(x),$$

for all $x \in B_1$. We now obtain some estimates on $\phi_m$ using the following
result, which is just an elliptic regularity result coupled with the Sobolev
imbedding theorem: for $t > N$ there is some $C_t$ such that

$$\sup_{B_{3/4}} |\nabla \phi_m(x)| \leq C_t \left( \int_{|x| < 1} |\Delta \phi_m(x)|^{1/t} dx \right)^{1/t} + C_t \int_{|x| < 1} |\phi_m(x)| dx. \quad (19)$$

We now assume we are in the case of $|x_m| \geq 1$. Using the fact that $f \in Y_\sigma$ and $\phi \in \tilde{X}_\sigma$ one sees that $|x_m|^{p-1} |g_m(x)| \leq C$ for all $|x| < 1$ and $m$. Putting
these estimates into (19) gives $\sup_{B_{3/4}} |\nabla \phi_m(x)| \leq C |x_m|^{p-1}$ and from this
we see that

$$|x_m|^{p-1} |\nabla \phi(x_m)| \leq C_1.$$ 

The case of $|x_m| \leq 1$ is handled as above. Combining these results gives us
the desired bound.

**Proof of Theorem 3.1.** To solve (2) we first consider solving the
related problem given by

$$-\Delta u + a(x) \cdot \nabla u = |u|^p \quad \text{in } \mathbb{R}^N. \quad (20)$$

To do this we perturb off the radial solution $w$ of the advection free problem.
So we look for a solution of the form $u = \phi + w$. So we need to find a solution $\phi$ of

$$L(\phi) = -a \cdot \nabla w - a \cdot \nabla \phi + |w + \phi|^p - pw^{p-1}\phi - w^p \quad \text{in } \mathbb{R}^N, \quad (21)$$

where $L(\phi) = -\Delta \phi - pw^{p-1}\phi$. Letting $T$ be defined as in Lemma 2 we are
looking for a $\phi \in X_\sigma$ such that

$$\phi = -T(a \cdot \nabla w) - T(a \cdot \nabla \phi) + T(|w + \phi|^p - pw^{p-1}\phi - w^p). \quad (22)$$

To find such a $\phi$ we define $J(\phi)$ to be the mapping on the right hand side
of (22) and we will now show that for a suitable $R$ that $J$ is a contraction
mapping on the closed ball $B_R$, centered at the origin, in $X_\sigma$. We will then argue that $u = w + \phi$ is positive. We begin by showing $J$ is into $B_R$. In what follows $C$ can depend on $p,a,w$ but not on $x,R,\phi$ and $\sigma$ provided $\sigma$ is small. Let $R > 0$ and let $\phi \in B_R$. Then note that that there is some $C > 0$ such that
\[ ||J(\phi)||_{X_\sigma} \leq C ||a \cdot \nabla w||_{Y_\sigma} + C ||a \cdot \nabla \phi||_{Y_\sigma} + C ||w + \phi|^p - pw^{p-1}\phi - w^p||_{Y_\sigma}. \] (23)

We now estimate the terms on the right hand side.
\[
\|a \cdot \nabla w\|_{Y_\sigma} \leq \sup_{|x| \leq 1} |a(x)||x| \sup_{|x| \leq 1} |x|^{1+c}\|\nabla w(x)\|
+ \sup_{|x| \geq 1} |a(x)| \sup_{|x| \geq 1} |x|^\frac{c}{p-1} \|\nabla w(x)\|
\leq (\sup_{x} |a(x)||x|) \|w\|_{X_\sigma}
\]

The same argument shows that
\[
\|a \cdot \nabla \phi\|_{Y_\sigma} \leq (\sup_{x} |a(x)|) \|\phi\|_{X_\sigma}
\]

We now approximate the last term in (23). For this we need the following real analysis result. There exists some $C = C_p$ such that for all numbers $w > 0$ and $\phi \in \mathbb{R}$ we have
\[
||w + \phi|^p - pw^{p-1}\phi - w^p| \leq C (w^{p-2}\phi^2 + |\phi|^p).
\]

Set $\Gamma = |w + \phi|^p - pw^{p-1}\phi - w^p$. Then one sees
\[
\|\Gamma\|_{Y_\sigma} \leq C \sup_{|x| \leq 1} |x|^\sigma (w^{p-2}\phi^2 + |\phi|^p)
+ C \sup_{|x| \geq 1} |x|^\frac{\sigma}{p-1} (w^{p-2}\phi^2 + |\phi|^p)
:= CI_1 + CI_2.
\]

Then note that
\[
I_1 = \sup_{|x| \leq 1} \left( |x|^{2-\sigma}w^{p-2} (|x|^\sigma \phi(x))^2 + |x|^\sigma (|\phi(x)||x|^\sigma)^p \right)
\leq \sup_{|x| \leq 1} \left( |x|^{2-\sigma}w^{p-2}\|\phi\|^2_{X_\sigma} + |x|^\sigma (\|\phi\|^p_{X_\sigma}) \right)
\leq C\|\phi\|^2_{X_\sigma} + C\|\phi\|^p_{X_\sigma}
\]
for sufficiently small \( \sigma > 0 \). One can similarly show that

\[
I_2 \leq \sup_{|x| \geq 1} \left( |x|^{2-p} w \right)^{p-2} \|\phi\|_{X_\sigma}^p + \|\phi\|_{X_\sigma}^p
\]

\[
\leq C \|\phi\|_{X_\sigma}^2 + \|\phi\|_{X_\sigma}^p.
\]

So combining these results we arrive at

\[
\|J(\phi)\|_{X_\sigma} \leq C \sup_x |x| a(x) + C \sup_x |x| a(x) \|\phi\|_{X_\sigma}
\]

\[
+ C \|\phi\|_{X_\sigma}^2 + C \|\phi\|_{X_\sigma}^p.
\]

(24)

Before choosing \( R \) we examine the condition on \( J \) to be a contraction on \( B_R \). First note there is some \( C = C_p \) such that for all numbers \( w > 0 \) and \( \hat{\phi}, \phi \in \mathbb{R} \) one has

\[
|\hat{\phi} + w|^p - |\phi + w|^p - pw^{p-1}(\hat{\phi} - \phi)| \leq CM|\hat{\phi} - \phi|
\]

(25)

where

\[
M = w^{p-2} \left( |\hat{\phi}| + |\phi| + |\hat{\phi}|^{p-1} + |\phi|^{p-1} \right).
\]

Let \( \hat{\phi}, \phi \in B_R \). Then

\[
J(\hat{\phi}) - J(\phi) = -T(a \cdot \nabla (\hat{\phi} - \phi)) + T(|w + \hat{\phi}|^p - |w + \phi|^p - pw^{p-2}(\hat{\phi} - \phi)),
\]

and so

\[
\|J(\hat{\phi}) - J(\phi)\|_{X_\sigma} \leq C \|a \cdot \nabla (\hat{\phi} - \phi)\|_{Y_\sigma}
\]

\[
+ C \|w + \hat{\phi}|^p - |w + \phi|^p - pw^{p-2}(\hat{\phi} - \phi)\|_{Y_\sigma}
\]

\[= C J_1 + C J_2
\]

Arguing as above one easily sees that \( J_1 \leq \sup_{x} (|x||a(x)||\hat{\phi} - \phi|_{X_\sigma} \). Using (25) we see that

\[
J_2 \leq C \sup_{|x| \leq 1} |x|^{2+\sigma} M|\hat{\phi} - \phi| + C \sup_{|x| \geq 1} |x|^{2+\sigma} |M|\hat{\phi} - \phi| \leq: C J_3 + C J_4.
\]

We now compute the various terms in \( J_3 \) and \( J_4 \).

\[
\sup_{|x| \leq 1} |x|^{2+\sigma} w^{p-1}|\hat{\phi}||\hat{\phi} - \phi| \leq \sup_{|x| \leq 1} (|x|^{2-\sigma} w^{p-2}) \|\hat{\phi}\|_{X_\sigma} \|\hat{\phi} - \phi\|_{X_\sigma}.
\]
Also we have
\[
\sup_{|x| \leq 1} |x|^{2+\sigma} |\hat{\phi}^{p-1}| \hat{\phi} - \phi | \leq \sup_{|x| \leq 1} |x|^{2-\sigma-(p-1)} \|\hat{\phi}\|^{p-1} \|\hat{\phi} - \phi\|_{X_{\sigma}},
\]
\[
\leq \|\hat{\phi}\|^{p-1} \|\hat{\phi} - \phi\|_{X_{\sigma}},
\]
for small enough \(\sigma > 0\). Combining these results we obtain
\[
J_3 \leq \left( \sup_{|x| \leq 1} |x|^{2-\sigma} w^{p-2} R + 2R^{p-1} \right) \|\hat{\phi} - \phi\|_{X_{\sigma}}
\]
\[
\leq \left( CR + 2R^{p-1} \right) \|\hat{\phi} - \phi\|_{X_{\sigma}}.
\]
One can argue in a similar fashion to show
\[
J_4 \leq \sup_{|x| \geq 1} \left( |x|^{\frac{2}{p-1}} w \right)^{p-2} \left( \|\hat{\phi}\|_{X_{\sigma}} + \|\phi\|_{X_{\sigma}} \right) \|\hat{\phi} - \phi\|_{X_{\sigma}}
\]
\[
+ \left( \|\hat{\phi}\|_{X_{\sigma}}^{p-1} + \|\phi\|_{X_{\sigma}}^{p-1} \right) \|\hat{\phi} - \phi\|_{X_{\sigma}}
\]
\[
\leq \left( CR + 2R^{p-1} \right) \|\hat{\phi} - \phi\|_{X_{\sigma}}.
\]
Combining the results we obtain an inequality of the form
\[
\|J(\hat{\phi}) - J(\phi)\|_{X_{\sigma}} \leq C \left( \sup_{|x|} |x||a(x)| + R + R^{p-1} \right) \|\hat{\phi} - \phi\|_{X_{\sigma}}. \tag{26}
\]
We now pick \(R\) and put conditions on \(a\). Fix \(R\) sufficiently small such that \(CR^2 + CR^p \leq \frac{R}{10}\) and such that \(CR + CR^{p-1} < \frac{1}{2}\). Now impose a smallness condition on \(a\) such that \(C \sup_{x} |x||a(x)| + C \sup_{x} |x||a(x)||R \leq \frac{R}{10}\) and \(C \sup_{x} |x||a(x)| < \frac{1}{4}\). These conditions are sufficient to show that \(J\) is a contraction mapping on \(B_R\) in \(X_{\sigma}\) and hence by the Contraction Mapping Principle there is some \(\phi \in B_R\) such that \(J(\phi) = \phi\), which was the desired result. So we have \(\phi \in B_R\) such that
\[
-\Delta(w + \phi) + a \cdot \nabla(w + \phi) = |w + \phi|^p \quad \text{in } \mathbb{R}^N. \tag{27}
\]
By taking \(R > 0\) smaller, which imposes a further smallness condition on \(a\), we can assume that
\[
\sup_{|x| \geq 1} |x|^\frac{2}{p-1} |\phi(x)| \leq \frac{1}{10} \inf_{|y| \geq 1} |y|^\frac{2}{p-1} w(y). \tag{28}
\]
Using this one sees that \(\phi + w > 0\) on \(|x| \geq 1\). Note there are some possible regularity issues for \(\phi\) near the origin. But taking \(\sigma > 0\) small enough and
applying elliptic regularity theory, along with a bootstrap, one sees that $\phi$ is at least $C^{2,\alpha}$ in a ball around the origin for some $\alpha > 0$. One can now apply the maximum principle to see that $u = w + \phi$ is a positive solution of (2).

\textbf{Proof of Theorem 3.} \textit{First note that a computation shows that $p_c > \frac{N+1}{N-1}$. For $R > 0$ sufficiently small there is some $u_R > 0$ which satisfies (2) and as $R$ gets small one imposes smallness conditions on $a$. For $m \geq 2$ an integer let $E = E_{m,R} > 0$ denote the first eigenfunction of $L(E) := -\Delta E + a \cdot \nabla E - pu^{p-1}_R E$ on the ball $B_m$ with $E = 0$ on $\partial B_m$ and let $\mu_{m,R}$ denote the first eigenvalue. We now multiply the equation for $E$ by $E$ and integrate over $B_m$. Using the fact that $a$ is divergence free (this is only spot we utilize this fact) one sees, after a suitable $L^2$ normalization of $E$, that}

$$\int |\nabla E|^2 = \int pu^{p-1}_R E^2 + \mu_{m,R}.$$

We now extend $E$ outside $B_m$ by setting it to be zero and we use the fact that $w$ satisfies (17) to arrive at

$$(p + \varepsilon) \int w^{p-1} E^2 \leq \int pu^{p-1}_R E^2 + \mu_{m,R},$$

for some fixed $\varepsilon > 0$. Note that $u_R \to w$ in $X_\sigma$ as $R \to 0$ and so we can argue as in (28), that for any $\delta > 0$ we can pick $R$ small enough such that $u_R(x) \leq (\delta + 1)w(x)$ for all $|x| \geq 1$. Using elliptic regularity and Sobolev imbedding one sees that the restriction of $u_R$ to the unit ball converges to the restriction of $w$ uniformly. And so we can assume that $u^{p-1}_R \leq w^{p-1} + \delta$ for $|x| \leq 1$ for small enough $R$. Using this estimates and breaking the integrals into the regions $|x| \geq 1$ and $|x| \leq 1$ one arrives at

$$((p + \varepsilon) - p(1 + \delta)^{p-1}) \int_{|x| \geq 1} w^{p-1} E^2 + (\varepsilon - p\delta) \int_{|x| \leq 1} w^{p-1} E^2 \leq \mu_{m,R},$$

for sufficiently small $R$. Now by taking $\delta > 0$ small enough one sees that for fixed $R$ small enough we have $\mu_{m,R} \geq 0$ for all $m \geq 2$. We now fix this $R$ and let $u = u_R$, $E_m = E_{m,R}$ and $\mu_m = \mu_{m,R}$. So we have that $E_m > 0$ satisfies

$$\begin{cases}
-\Delta E_m + a(x) \cdot \nabla E_m = pu^{p-1}_R E_m + \mu_m E_m & \text{in } B_m \\
E_m = 0 & \text{on } \partial B_m.
\end{cases}$$

Lets assume that $\mu_m \to 0$. By suitably scaling $E_m$ we can assume that $E_m(0) = 1$. Now fix $k \geq 1$ an integer and let $m \geq k + 2$. Now note that $E_m$
satisfies the same equation on $B_{k+1}$ and hence by Harnack’s inequality there is some $C_k > 0$ such that

$$\sup_{B_k} E_m \leq C_k \inf_{B_k} E_m \leq C_k,$$

for all $m \geq k + 2$. Using elliptic regularity one can show that $E_m$ is bounded in $C^{1,\alpha}(B_k)$ and by a diagonal argument there is some subsequence of $E_m$, which we still denote by $E_m$, which converges to some $E \geq 0$ locally in $C^{1,\beta}$ for some $\beta > 0$ and $E(0) = 1$. One can then argue that $E$ satisfies

$$-\Delta E + a(x) \cdot \nabla E = pu^{p-1}E \quad \text{in } \mathbb{R}^N,$$

and then we can apply the strong maximum principle to see that $E > 0$. This shows that $u$ is a stable solution of (2) which was the desired result.

We now show $\mu_m \to 0$. We begin by putting $E_m$, which we $L^2$ normalize, into (11) with $\beta = \frac{1}{2}$ to arrive at

$$\mu_m \int \phi^2 + \frac{1}{2} \int \frac{|\nabla E_m|^2}{E_m^2} \phi^2 \leq 2 \int |\nabla \phi|^2 + \int \frac{a \cdot \nabla E_m}{E_m} \phi^2,$$

for all $\phi \in C_c^\infty(B_m)$. We now use Young’s inequality to arrive at

$$\mu_m \int \phi^2 + \frac{1}{2} \int \frac{|\nabla E_m|^2}{E_m^2} \phi^2 \leq 2 \int |\nabla \phi|^2 + \varepsilon \int \frac{|\nabla E_m|^2}{E_m^2} \phi^2 + \frac{1}{4\varepsilon} \int |a|^2 \phi^2.$$

By taking $\varepsilon > 0$ small enough and re-grouping terms and by using the fact that $|a(x)| \leq \frac{C^2}{|x|^2}$ along with Hardy’s inequality, one can obtain

$$\mu_m \int \phi^2 \leq C \int |\nabla \phi|^2 \quad \forall \phi \in C_c^\infty(B_m),$$

where $C$ is independent of $m$. From this we can conclude that $\lim \sup_m \mu_m \leq 0$ but we already have $\mu_m \geq 0$ and hence we have the desired result.

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