Generating Functions, Weighted and Non-Weighted Sums for Powers of Second-Order Recurrence Sequences

Pantelimon Stănică *
Auburn University Montgomery, Department of Mathematics
Montgomery, AL 36117, USA
e-mail: stanpan@strudel.aum.edu

Abstract

In this paper we find closed forms of the generating function \( \sum_{k=0}^{\infty} U_r^n x^n \), for powers of any non-degenerate second-order recurrence sequence, \( U_{n+1} = aU_n + bU_{n-1} \), \( a^2+4b \neq 0 \), completing a study began by Carlitz [1] and Riordan [4] in 1962. Moreover, we generalize a theorem of Horadam [3] on partial sums involving such sequences. Also, we find closed forms for weighted (by binomial coefficients) partial sums of powers of any non-degenerate second-order recurrence sequences. As corollaries we give some known and seemingly unknown identities and derive some very interesting congruence relations involving Fibonacci and Lucas sequences.

1 Introduction

DeMoivre (1718) used the generating function (found by using the recurrence) for the Fibonacci sequence \( \sum_{i=0}^{\infty} F_i x^i = \frac{x}{1-x-x^2} \), to obtain the identities \( F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \), \( L_n = \alpha^n + \beta^n \) (Lucas numbers) with \( \alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2} \), called Binet formulas, in honor of Binet who in fact rediscovered them more than one hundred years later, in 1843 (see [3]).

*Also associated with the Institute of Mathematics of Romanian Academy, Bucharest, Romania
Reciprocally, using the Binet formulas, we can find the generating function easily

$$\sum_{i=0}^{\infty} F_i x^i = \frac{1}{\sqrt{5}} \sum_{i=0}^{\infty} (\alpha^i - \beta^i) x^i = \frac{1}{\sqrt{5}} \left( \frac{1}{1 - \alpha x} - \frac{1}{1 - \beta x} \right) = \frac{x}{1 - x - x^2},$$

since \(\alpha \beta = -1, \alpha + \beta = 1\).

The question that arises is whether we can find a closed form for the generating function for powers of Fibonacci numbers, or better yet, for powers of any second-order recurrence sequences. Carlitz \[1\] and Riordan \[4\] were unable to find the closed form for the generating functions \(F(r, x)\) of \(F_n^r\), but found a recurrence relation among them, namely

$$(1 - L_r x + (-1)^r x^2) F(r, x) = 1 + rx \sum_{j=1}^{[\frac{r}{2}]} (-1)^j A_{rj} F(r - 2j, (-1)^j x),$$

with \(A_{rj}\) having a complicated structure (see also \[2\]). We are able to complete the study began by them and find a closed form for the generating function for powers of any non-degenerate second-order recurrence sequence. We would like to point out, that this ”forgotten” technique we employ can be used to attack successfully other sums or series involving any second-order recurrence sequence. In this paper we also find closed forms for non-weighted partial sums for non-degenerate second-order recurrence sequences, generalizing a theorem of Horadam \[3\] and also weighted (by the binomial coefficients) partial sums for such sequences.

2 Generating Functions

We consider the general non-degenerate second-order recurrences \(U_{n+1} = aU_n + bU_{n-1}\), \(a^2 + 4b \neq 0\). We intend to find the generating function \(U(r, x) = \sum_{i=0}^{\infty} U_i^r x^i\). It is known that the Binet formula for the sequence \(U_n\) is \(U_n = A\alpha^n - B\beta^n\), where \(\alpha = \frac{1}{2}(a + \sqrt{a^2 + 4b})\), \(\beta = \frac{1}{2}(a - \sqrt{a^2 + 4b})\) and \(A = \frac{U_1 - U_0 \beta}{\alpha - \beta}, B = \frac{U_1 - U_0 \alpha}{\alpha - \beta}\). We associate the sequence \(V_n = \alpha^n + \beta^n\), which satisfies the same recurrence, with the initial conditions \(V_0 = 2, V_1 = a\).
\textbf{Theorem 1.} We have

\[ U(r, x) = \sum_{k=0}^{r-1} (-1)^k A^k B^k \binom{r}{k} \frac{A^{r-2k} - B^{r-2k} + (-b)^k (B^{r-2k} \alpha^r - 2k - A^{r-2k} \beta^r - 2k)x}{1 - (-b)^k V_{r-2k} - x^2}, \]

if \( r \) odd, and

\[ U(r, x) = \sum_{k=0}^{r-1} (-1)^k A^k B^k \binom{r}{k} \frac{B^{r-2k} + A^{r-2k} - (-b)^k (B^{r-2k} \alpha^r - 2k + A^{r-2k} \beta^r - 2k)x}{1 - (-b)^k V_{r-2k} + x^2} + \left( \frac{r}{\tau} \right) A^\tau (-B)^\tau \frac{1}{1 - (-1)^\tau x}, \]

if \( r \) even.

\textbf{Proof.} We evaluate

\[ U(r, x) = \sum_{i=0}^{\infty} \left( \sum_{k=0}^{r} \binom{r}{k} (A \alpha^i B \beta^i)^{r-k} \right) x^i 
= \sum_{k=0}^{r} \binom{r}{k} A^k (-B)^{r-k} \sum_{i=0}^{\infty} (\alpha^k \beta^{r-k} x)^i 
= \sum_{k=0}^{r} \binom{r}{k} A^k (-B)^{r-k} \frac{1}{1 - \alpha^k \beta^{r-k} x}. \]

If \( r \) odd, then associating \( k \leftrightarrow r - k \), we get

\[ U(r, x) = \sum_{k=0}^{r-1} \binom{r}{k} \left( \frac{A^k (-B)^{r-k}}{1 - \alpha^k \beta^{r-k} x} + \frac{A^{r-k} (-B)^k}{1 - \alpha^{r-k} \beta^k x} \right) 
= \sum_{k=0}^{r-1} (-1)^k \binom{r}{k} \left( \frac{A^{r-k} B^k}{1 - \alpha^{r-k} \beta^k x} - \frac{A^k B^{r-k}}{1 - \alpha^k \beta^{r-k} x} \right) 
= \sum_{k=0}^{r-1} (-1)^k \binom{r}{k} \frac{A^{r-k} B^k - A^k B^{r-k} + (A^k B^{r-k} \alpha^{r-k} \beta^k - A^{r-k} B^k \alpha^k \beta^{r-k})x}{1 - (\alpha^k \beta^{r-k} + \alpha^{r-k} \beta^k)x + \alpha^r \beta^r x^2} 
= \sum_{k=0}^{r-1} (-1)^k \binom{r}{k} \frac{A^{r-k} B^k - A^k B^{r-k} + (-1)^k b^k (A^k B^{r-k} \alpha^{r-2k} - A^{r-k} B^k \beta^{r-2k})x}{1 - (-1)^k b^k V_{r-2k} - x^2}. \]

If \( r \) even, then then associating \( k \leftrightarrow r - k \), except for the middle term, we get

\[ U(r, x) = \sum_{k=0}^{r-1} \binom{r}{k} \left( \frac{A^k (-B)^{r-k}}{1 - \alpha^k \beta^{r-k} x} + \frac{A^{r-k} (-B)^k}{1 - \alpha^{r-k} \beta^k x} \right) + \left( \frac{r}{\tau} \right) A^\tau (-B)^\tau \frac{1}{1 - (-1)^\tau x} 
= \sum_{k=0}^{r-1} (-1)^k \binom{r}{k} \left( \frac{A^{r-k} B^k}{1 - \alpha^{r-k} \beta^k x} + \frac{A^k B^{r-k}}{1 - \alpha^k \beta^{r-k} x} \right) + \left( \frac{r}{\tau} \right) A^\tau (-B)^\tau \frac{1}{1 - (-1)^\tau x}. \]
We can derive the following beautiful identities

**Corollary 2.** If $U_0 = 0$, then $A = B = \frac{U_1}{a-\beta}$ and

$$U(r, x) = A^{r-1} \sum_{k=0}^{r-1} \binom{r}{k} \frac{b^k U_{r-2k} x}{1 - (-b)^k V_{r-2k} x - x^2}, \text{ if } r \text{ odd}$$

$$U(r, x) = A^r \sum_{k=0}^{r-1} (-1)^k \binom{r}{k} \frac{2 - (-b)^k V_{r-2k} x + x^2 + \binom{r}{2} \frac{(-1)^r A^r}{1 - (-1)^2 x}}{1 - (-1)^2 x}, \text{ if } r \text{ even.}$$

**Corollary 3.** If $\{U_n\}_n$ is a non-degenerate second-order recurrence sequence and $U_0 = 0$, then

$$U(1, x) = \frac{A^2 U_1 x}{1 - V_1 x - x^2} \quad (1)$$

$$U(2, x) = \frac{-A^2 (V_2 + 2x(x-1))}{(x+1)(x^2 - V_2 x + 1)} \quad (2)$$

$$U(3, x) = \frac{A^4 U_1 x ((a^2 + 2b) - 2a^2 b x - (a^2 + 2b) x^2)}{(1 - V_3 x - x^2)(1 + b V_4 x - x^2)} \quad (3)$$

**Proof.** We use Corollary 2. The first two identities are straightforward. Now,

$$U(3, x) = A^4 \left( \frac{U_3 x}{1 - V_3 x - x^2} + \frac{b U_1 x}{1 + b V_1 x - x^2} \right)$$

$$= A^4 x U_3 + b U_1 + b (U_3 V_1 - U_1 V_3) x - (U_3 + b U_1) x^2$$

$$= A^4 U_1 x ((a^2 + 2b) - 2a^2 b x - (a^2 + 2b) x^2)$$

$$= \frac{A^4 U_1 x ((a^2 + 2b) - 2a^2 b x - (a^2 + 2b) x^2)}{(1 - V_3 x - x^2)(1 + b V_4 x - x^2)}.$$
since $U_3 + bU_1 = (a^2 + 2b)U_1$ and $U_3V_1 - U_1V_3 = -2a^2U_1$. \hfill \square

**Remark 4.** If $U_n$ is the Fibonacci sequence, then $a = b = 1$, and if $U_n$ is the Pell sequence, then $a = 2, b = 1$.

### 3 Horadam’s Theorem

Horadam \[3\] found some closed forms for partial sums $S_n = \sum_{i=1}^{n} P_i$, $S_{-n} = \sum_{i=1}^{n} P_{-i}$, where $P_n$ is the generalized Pell sequence, $P_{n+1} = 2P_n + P_{n-1}$, $P_1 = p, P_2 = q$. Let $p_n$ be the ordinary Pell sequence, with $p = 1, q = 2$, and $q_n$ be the sequence satisfying the same recurrence, with $p = 1, q = 3$. He proved

**Theorem 5 (Horadam).** For any $n$,

$S_{4n} = q_2n(pq_{2n-1} + qq_{2n}) + p - q; \quad S_{4n-2} = q_{2n-1}(pq_{2n-2} + qq_{2n-1})$

$S_{4n+1} = q_2n(pq_{2n} + qq_{2n+1}) - q; \quad S_{4n-1} = q_2n(pq_{2n-2} + qq_{2n-1}) - q$

$S_{-4n} = q_2n(-pq_{2n+2} + qq_{2n+1}) + 3p - q; \quad S_{-4n+2} = q_2n(-pq_{2n} + qq_{2n-1}) + 2p$

$S_{-4n+1} = q_2n(pq_{2n+1} - qq_{2n}) + p; \quad S_{-4n-1} = q_{2n+1}(pq_{2n+2} - qq_{2n+1}) + 2p - q.$

We observe that Horadam’s theorem is a particular case of the partial sum for a non-degenerate second-order recurrence sequence $U_n$. In fact, we find $S_{n,r}^U(x) = \sum_{i=0}^{n} U_i^r x^i$. For simplicity, we let $U_0 = 0$. Thus, $U_n = A(\alpha^n - \beta^n)$ and $V_n = \alpha^n + \beta^n$. We prove

**Theorem 6.** We have

$$S_{n,r}^U(x) = A^r \sum_{k=0}^{\frac{r-1}{2}} \binom{r}{k} \frac{U_{r-2k}x - (-1)^{kn}U_{(r-2k)(n+1)}x^{n+1} - (-1)^{k(n+1)}U_{(r-2k)n}x^{n+2}}{1 - (-1)^{kV_{r-2k}x - x^2}}$$

(4)
if \( r \) odd, and

\[
S^U_{n,r}(x) = A^r \sum_{k=0}^{\frac{r-1}{2}} \binom{r}{k} \frac{V_{r-2k}x - (-1)^k V_{(r-2k)(n+1)} x^{n+1} - (-1)^{k(n+1)} V_{(r-2k)n} x^{n+2}}{1 - (-1)^k V_{r-2k}x + x^2} \\
+ A^r \left( \frac{r}{2} \right) \frac{(-1)^{\frac{r}{2}(n+1)} x^{n+1} - 1}{(-1)^{\frac{r}{2}} x - 1} \tag{5}
\]

if \( r \) even.

**Proof.** We evaluate

\[
S^U_{n,r}(x) = \sum_{i=0}^{n} \sum_{k=0}^{r} \left( \frac{r}{k} \right) (A\alpha)^k (-A\beta)^{r-k} x^i
\]

\[
= A^r \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} \sum_{i=0}^{n} (\alpha^k \beta^{r-k} x)^i
\]

\[
= A^r \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} (\alpha^k \beta^{r-k} x)^{n+1} - 1
\]

Assume \( r \) odd. Then, associating \( k \leftrightarrow r - k \), we get

\[
S^U_{n,r}(x) = A^r \sum_{k=0}^{\frac{r-1}{2}} (-1)^k \binom{r}{k} \frac{\left( \alpha^{r-k} \beta^k x \right)^{n+1} - 1 - \left( \alpha^k \beta^{r-k} x \right)^{n+1} - 1}{\alpha^k \beta^{r-k} x - 1}
\]

\[
= A^r \sum_{k=0}^{\frac{r-1}{2}} (-1)^k \binom{r}{k} \left( \frac{\alpha^{r-k} \beta^k x - 1}{\alpha^k \beta^{r-k} x - 1} \right) \left( \alpha^{k} \beta^{r-k} x - 1 \right) \left( \alpha^{r-k} \beta^k x - 1 \right) - \frac{\left( \alpha^{r-k} \beta^k x - 1 \right) \left( \alpha^{k} \beta^{r-k} x - 1 \right)}{\left( \alpha^{k} \beta^{r-k} x - 1 \right) \left( \alpha^{r-k} \beta^k x - 1 \right)}
\]

\[
= A^r \sum_{k=0}^{\frac{r-1}{2}} (-1)^k \binom{r}{k} \frac{\alpha^{k(n+1) - kn} \beta^{r-k} x^{n+2}}{-\alpha^{(r-k)(n+1)} \beta^{k(n+1)} x^{n+1} - \alpha^k \beta^{r-k} x}
\]

\[
- \alpha^{r-k(n+1)} \beta^{k(n+1)} x^{n+1} + \alpha^k \beta^{r-k} x
\]

\[
- \alpha^{r+k(n+1) - kn} x^{n+2} + \alpha^{r-k} \beta^k x
\]

\[
- \alpha^{k(n+1)} \beta^{(r-k)(n+1)} \beta^{(r-k)(n+1)} x^{n+1}
\]

\[
1 - (-1)^k \left( \alpha^{r-2k} + \beta^{r-2k} \right) + \alpha^r \beta^{r-2k} x^2
\]
\[
\begin{align*}
A^r \sum_{k=0}^{r-1} (-1)^k & \binom{r}{k} (-1)^k (\alpha^{r-2k} - \beta^{r-2k})x - (-1)^{k(n+1)}(\alpha^{(r-2k)(n+1)} - \\
&- \beta^{(r-2k)(n+1)})x^{n+1} + (-1)^{r+kn}(\alpha^{(r-2k)n} - \beta^{(r-2k)n})x^{n+2} \\
&= A^{r-1} \sum_{k=0}^{r-1} \binom{r}{k} \frac{U_{r-2k}x - (-1)^{kn}U_{(r-2k)(n+1)}x^{n+1} - (-1)^{(k+1)n}U_{(r-2k)n}x^{n+2}}{1 - (-1)^kV_{r-2k}x - x^2}.
\end{align*}
\]

Assume \( r \) even. Then, as before, associating \( k \leftrightarrow r - k \), except for the middle term, we get

\[
S_{n,r}^U(x) = A^r \sum_{k=0}^{r-1} (-1)^k \binom{r}{k} (-1)^k (\alpha^{r-2k} + \beta^{r-2k})x - (-1)^{k(n+1)}(\alpha^{(r-2k)(n+1)} + \\\n+ \beta^{(r-2k)(n+1)})x^{n+1} + (-1)^{r+kn}(\alpha^{(r-2k)n} + \beta^{(r-2k)n})x^{n+2} \\
+ A^r \binom{r}{r/2} \frac{(-1)^{r/2}x^{n+1} - 1}{(-1)^{r/2}x - 1}
\]

\[
= A^r \sum_{k=0}^{r-1} \binom{r}{k} \frac{V_{r-2k}x - (-1)^{kn}V_{(r-2k)(n+1)}x^{n+1} - (-1)^{(k+1)n}V_{(r-2k)n}x^{n+2}}{1 - (-1)^kV_{r-2k}x + x^2} \\
+ A^r \binom{r}{r/2} \frac{(-1)^{r/2}x^{n+1} - 1}{(-1)^{r/2}x - 1}.
\]

Taking \( r = 1 \), we get the partial sum for any non-degenerate second-order recurrence sequence, with \( U_0 = 0 \),

**Corollary 7.** \( S_{n,1}^U(x) = \frac{x(U_1 - U_{n+1}x^n - U_nx^{n+2})}{1 - V_1x - x^2} \)

**Remark 8.** Horadam’s theorem follows easily, since \( S_n = S_{n,1}^P(1) \). Also \( S_{-n} \) can be found without difficulty, by observing that \( P_{-n} = pp_{-n-2} + qp_{-n-1} = -p(-1)^{n+2}p_{n+2} - q(-1)^{n+1}p_{n+1} \), and using \( S_{n,1}^P(-1) \).
4 Weighted Combinatorial Sums

In \[6\] there are quite a few identities of the form $\sum_{i=0}^{n} \binom{n}{i} F_i = F_{2n}$, or $\sum_{i=0}^{n} \binom{n}{i} F^2_i$, which is $5^{\lfloor \frac{n-1}{2} \rfloor} L_n$ if $n$ even, and $5^{\lfloor \frac{n-1}{2} \rfloor} F_n$, if $n$ odd. A natural question is: for fixed $r$, what is the closed form for the weighted sum $\sum_{i=0}^{n} \binom{n}{i} F^r_i$ (if it exists)? We are able to answer the previous question, not only for the Fibonacci sequence, but also for any second-order recurrence sequences. Let $S_{r,n}(x) = \sum_{i=0}^{n} \binom{n}{i} U^r_i x^i$.

**Theorem 9.** We have

$$S_{r,n}(x) = \sum_{k=0}^{r} \binom{r}{k} A^k (-B)^{r-k}(1 + \alpha^k \beta^{r-k} x)^n.$$  

Moreover, if $U_0 = 0$, then $S_{r,n}(x) = A^r \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} (1 + \alpha^k \beta^{r-k} x)^n$.

**Proof.** Let

$$S_{r,n}(x) = \sum_{i=0}^{n} \binom{n}{i} \sum_{k=0}^{r} \binom{r}{k} (A \alpha^i)^k (-B \beta^i)^{r-k} x^i$$

$$= \sum_{k=0}^{r} \binom{r}{k} A^k (-B)^{r-k} \sum_{i=0}^{n} \binom{n}{i} (\alpha^k \beta^{r-k} x)^i$$

$$= \sum_{k=0}^{r} \binom{r}{k} A^k (-B)^{r-k} (1 + \alpha^k \beta^{r-k} x)^n$$

If $U_0 = 0$, then $A = B$, and $S_{r,n}(x) = A^r \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} (1 + \alpha^k \beta^{r-k} x)^n$.

Studying Theorem 9, we observe that we get nice sums involving the Fibonacci and Lucas sequences (or any such sequence, for that matter), if we are able to express 1 plus/minus a power of $\alpha, \beta$ as the same multiple of a power of $\alpha$, respectively $\beta$. The following lemma turns out to be very useful.
Lemma 10. The following identities are true

\[ \alpha^{2s} - (-1)^s = \sqrt{5} \alpha^s F_s \]
\[ \beta^{2s} - (-1)^s = -\sqrt{5} \beta^s F_s \]
\[ \alpha^{2s} + (-1)^s = L_s \alpha^s \]
\[ \beta^{2s} + (-1)^s = L_s \beta^s . \]  

(6)

Proof. Straightforward using the Binet formula for \( F_s \) and \( L_s \).

Theorem 11. We have

\[ S_{4r,n}(1) = 5^{-2r} \left( \sum_{k=0}^{2r-1} (-1)^k \binom{4r}{k} F_{2r-k} L_{(2r-k)n} + \binom{4r}{2r} 2^n \right) \]  

(7)

\[ S_{4r+2,n}(1) = 5^{\frac{2r+1}{2}-(2r+1)} \sum_{k=0}^{2r} \binom{4r+2}{k} F_{2r+1-k} F_{n(2r+1-k)}, \text{ if } n \text{ odd} \]  

(8)

\[ S_{4r+2,n}(1) = 5^{\frac{2r+1}{2}-(2r+1)} \sum_{k=0}^{2r} (-1)^k \binom{4r+2}{k} F_{2r+1-k} L_{n(2r+1-k)} \text{ if } n \text{ even.} \]  

(9)

Proof. We use Theorem 9. Associating \( k \leftrightarrow 4r + 2 - k \), except for the middle term in \( S_{4r+2,n}(1) \), we obtain

\[ S_{4r+2,n}(1) = 5^{-(2r+1)} \sum_{k=0}^{2r} (-1)^k \binom{4r+2}{k} \left( (1 + \alpha^k \beta^{4r+2-k})^n + (1 + \alpha^{4r+2-k} \beta^k)^n \right) \]
\[ = 5^{-(2r+1)} \sum_{k=0}^{2r} (-1)^k \binom{4r+2}{k} \left( (1 + (-1)^k \beta^{4r+2-2k})^n + (1 + (-1)^k \alpha^{4r+2-2k})^n \right) \]
\[ = 5^{-(2r+1)} \sum_{k=0}^{2r} (-1)^{k(n+1)} \binom{4r+2}{k} \left( ((-1)^k + \beta^{2(2r+1-k)})^n + ((-1)^k + \alpha^{2(2r+1-k)})^n \right) . \]  

(10)

We did not insert the middle term, since it is equal to

\[ 5^{-(2r+1)} (-1)^{2r+1} \binom{4r+2}{2r+1} (1 + \alpha^{2r+1} \beta^{2r+1})^n \]
\[ = 5^{-(2r+1)} (-1)^{2r+1} \binom{4r+2}{2r+1} (1 + (-1)^{2r+1})^n = 0 . \]
Assume first that \( n \) is odd. Using (6) into (10), and observing that 
\[
(-1)^{2r+1-k} = \alpha^{2(2r+1-k)} + (-1)^k,
\]
we get
\[
S_{4r+2,n}(1) = 5^{-2r+1} \sum_{k=0}^{2r} (-1)^{(n+1)k} \binom{4r}{k} \frac{n^{k+1} F_{2r+1-k}^n}{n^{2r+1-k}} L_{n(2r+1-k)}
\]
Assume \( n \) even. As before,
\[
S_{4r+2,n}(1) = 5^{-2r+1} \sum_{k=0}^{2r} (-1)^{(n+1)k} \binom{4r}{k} \frac{n^{k+1} F_{2r+1-k}^n}{n^{2r+1-k}} L_{n(2r+1-k)}
\]
In the same way, associating \( k \leftrightarrow 4r - k \), except for the middle term,
\[
S_{4r,n}(1) = 5^{-2r} \sum_{k=0}^{2r-1} (-1)^{k+1(n+1)} \binom{4r}{k} \left( \binom{(-1)^k + \beta^{2(2r-k)}}{2r} + \binom{(-1)^k + \alpha^{2(2r-k)}}{2r} \right) 2^n
\]
Remark 12. In the same manner we can find 
\[
\sum_{i=0}^{n} \binom{n}{i} U_{p_i} x^i.
\]
As a consequence of the previous theorem, for the even cases, and working out the details
for the odd cases we get
Corollary 13. We have

\[
\sum_{k=0}^{n} \binom{n}{i} F_i = F_{2n}
\]
\[
\sum_{k=0}^{2n} \binom{2n}{i} F_i^2 = 5^{n-1} L_{2n}
\]
\[
\sum_{k=0}^{2n+1} \binom{2n+1}{i} F_i^2 = 5^n F_{2n+1}
\]
\[
\sum_{k=0}^{n} \binom{n}{i} F_i^3 = \frac{1}{5} (2^n F_{2n} + 3F_n)
\]
\[
\sum_{k=0}^{n} \binom{n}{i} F_i^4 = \frac{1}{25} (3^n L_{2n} - 4(-1)^n L_n + 6 \cdot 2^n).
\]

Proof. The second, third and fifth identities follow from the previous theorem. Now, using Theorem 9, with \( A = \frac{1}{\sqrt{5}} \), we get

\[
S_{1,n}(1) = \frac{1}{\sqrt{5}} \sum_{k=0}^{1} (-1)^{1-k} \binom{1}{k} (1 + \alpha^k \beta^{1-k})^n
\]
\[
= \frac{1}{\sqrt{5}} (- (1 + \beta)^n + (1 + \alpha)^n) = \frac{1}{\sqrt{5}} (\alpha^{2n} - \beta^{2n}) = F_{2n}.
\]

Furthermore, the fourth identity follows from

\[
S_{3,n}(1) = \frac{1}{5\sqrt{5}} \sum_{k=0}^{3} (-1)^{3-k} \binom{3}{k} (1 + \alpha^k \beta^{3-k})^n
\]
\[
= \frac{1}{5\sqrt{5}} (- (1 + \beta^3)^n + 3(1 + \alpha \beta^2)^n - 3(1 + \alpha^2 \beta)^n + (1 + \alpha^3)^n)
\]
\[
= \frac{1}{5\sqrt{5}} (- (2\beta^2)^n + 3\alpha^n - 3\beta^n + (2\alpha^2)^n)
\]
\[
= \frac{1}{5} (2^n F_{2n} + 3F_n),
\]

since \( 1 + \beta^3 = 2\beta^2 \), \( 1 + \alpha^3 = 2\alpha^2 \). \( \square \)

We remark the following

Corollary 14. We have, for any \( n \),
(i) \(2^n F_{2n} + 3F_n \equiv 0 \pmod{5}\)

(ii) \(3^n L_{2n} - 4(-1)^n L_n + 6 \cdot 2^n \equiv 0 \pmod{5^2}\)

(iii) \[\sum_{k=0}^{2r} \binom{4r+2}{k} F_{2r+1-k}^n F_{n(2r+1-k)} \equiv 0 \pmod{5^{4r+2-\frac{2k}{2}}}, \text{ if } n \text{ is odd, } n \leq 8r + 3.\]

(iv) \[\sum_{k=0}^{2r} (-1)^k \binom{4r+2}{k} F_{2r+1-k}^n L_{n(2r+1-k)} \equiv 0 \pmod{5^{4r+2-\frac{n}{2}}}, \text{ if } n \text{ is even, } n \leq 8r + 2.\]

(v) \[\sum_{k=0}^{2r-1} (-1)^{k(n+1)} \binom{4r}{k} L_{2r-k}^n L_{(2r-k)n} + \binom{4r}{2r} 2^n \equiv 0 \pmod{5^{2r}}.\]

Taking other values for \(x\) (as desired) in Theorem 9, for instance, \(x = -1\) and working out the details, we get the following

**Theorem 15.** We have

\[S_{4r,n}(-1) = 5^{-2r} \sum_{k=0}^{2r-1} (-1)^k F_{2r-k}^n L_{(4r-2k)n} \binom{4r}{k}, \text{ if } n \text{ even}\]

\[S_{4r,n}(-1) = -5^{-2r} \sum_{k=0}^{2r-1} F_{2r-k}^n F_{(4r-2k)n} \binom{4r}{k}, \text{ if } n \text{ odd}\]

\[S_{4r+2,n}(-1) = 5^{-2r+1} \left( \sum_{k=0}^{2r} (-1)^{k(n+1)+n} \binom{4r+2}{k} L_{2r+1-k}^n L_{(2r+1-k)n} - 2^n \binom{4r+2}{2r+1} \right).\]

**Proof.** We use \(x = -1\) in Theorem 9. Associating \(k \leftrightarrow 4r - 2 - k\) in \(S_{4r+2,n}(-1)\), we obtain

\[S_{4r+2,n}(-1) = 5^{-2r+1} \sum_{k=0}^{2r} (-1)^k \binom{4r+2}{k} \left( (1 - \alpha^k \beta^{4r+2-k})^n + (1 - \alpha^{4r+2-k} \beta^k)^n \right)\]

\[- 5^{-2r+1} 2^n \binom{4r+2}{2r+1}\]

\[= 5^{-2r+1} \sum_{k=0}^{2r} (-1)^k \binom{4r+2}{k} \left( (1 - (-1)^k \beta^{4r+2-2k})^n + (1 - (-1)^k \alpha^{4r+2-2k})^n \right)\]

\[- 5^{-2r+1} 2^n \binom{4r+2}{2r+1}\]
Therefore, for any \( n \), \( k \) and for even powers and a similar idea for odd powers produces

\[
-5^{-(2r+1)} \sum_{k=0}^{2r} (-1)^{k(n+1)} \binom{4r+2}{k} \left( ((-1)^k - \beta^{2(2r+1-k)})^n + ((-1)^k - \alpha^{2(2r+1-k)})^n \right)
\]

\[
-5^{-(2r+1)} 2^n \binom{4r+2}{2r+1}
\]

\[
= 5^{-(2r+1)} \sum_{k=0}^{2r} (-1)^{k(n+1)+n} \binom{4r+2}{k} L_{2r+1-k}^n L((2r+1-k)n) - 5^{-(2r+1)} 2^n \binom{4r+2}{2r+1},
\]

since \((-1)^k - \beta^{2r+2-2k} = -L_{2r+1-k}\beta^{2r+1-k}\) and \((-1)^k - \alpha^{4r+2-2k} = -L_{2r+1-k}\alpha^{2r+1-k}\), by Lemma \([10]\).

In the same way, associating \( k \leftrightarrow 4r - k \), with the middle term zero,

\[
S_{4r,n}(1) = 5^{-2r} \sum_{k=0}^{2r-1} (-1)^k \binom{4r}{k} \left( (1 - \alpha^k \beta^{4r-k})^n + (1 - \alpha^{4r-k} \beta^k)^n \right)
\]

\[
= 5^{-2r} \sum_{k=0}^{2r-1} (-1)^{k(n+1)} \binom{4r}{k} \left( ((-1)^k - \beta^{2(2r-k)})^n + ((-1)^k - \alpha^{2(2r-k)})^n \right)
\]

\[
= 5^{-2r} \sum_{k=0}^{2r-1} (-1)^{k(n+1)} \binom{4r}{k} \left( 5^n F_{2r-k}^n \beta^{(2r-k)n} + 5^n (-1)^n F_{2r-k}^n \alpha^{(2r-k)n} \right)
\]

\[
= 5^{n+1} 2^{-2r} \sum_{k=0}^{2r-1} (-1)^{k(n+1)+n} F_{2r-k}^n (4r+2k) \binom{4r}{k} (\alpha^{(2r-k)n} + (-1)^n \beta^{(2r-k)n}),
\]

since \((-1)^k - \beta^{4r-2k} = \sqrt{5} F_{2r-k} \beta^{2r-k}\) and \((-1)^k - \alpha^{4r-2k} = -\sqrt{5} F_{2r-k} \alpha^{2r-k}\), by Lemma \([10]\).

Therefore, for \( n \) even, \( S_{4r,n}(1) = 5^{n+1} 2^{-2r} \sum_{k=0}^{2r-1} (-1)^{k(n+1)+n} F_{2r-k}^n L((4r-2k)n) \binom{4r}{k} \), and for \( n \) odd, \( S_{4r,n}(1) = 5^{n+1} 2^{-2r} \sum_{k=0}^{2r-1} (-1)^{k(n+1)+n} F_{2r-k}^n F_{(4r-2k)n} \binom{4r}{k} \).

A consequence for even powers and a similar idea for odd powers produces
Corollary 16. We have

\[ \sum_{i=0}^{n} (-1)^i \binom{n}{i} F_i = -F_n \]

\[ \sum_{i=0}^{n} (-1)^i \binom{n}{i} F_i^2 = \frac{1}{5} \left( (-1)^n L_n - 2^{n+1} \right) \]

\[ \sum_{i=0}^{n} (-1)^i \binom{n}{i} F_i^3 = \frac{1}{5} \left( (-2)^n F_n - 3F_2n \right) \]

\[ \sum_{i=0}^{n} (-1)^i \binom{n}{i} F_i^4 = \frac{5}{2} (L_{2n} - L_n), \text{ if } n \text{ even} \]

\[ \sum_{i=0}^{n} (-1)^i \binom{n}{i} F_i^4 = -5^{\frac{n+1}{2}} (F_{2n} + 4F_n), \text{ if } n \text{ odd}. \]

Proof. The first identity is simple application of Theorem 9. The identities for even powers are consequences of Theorem 13. Now, using Theorem 14 we get

\[ S_{3,n}(-1) = \frac{1}{5\sqrt{5}} \left( -(1 - \beta^3)^n + 3(1 - \alpha^2 \beta)^n - 3(1 - \alpha^3)^n \right) \]

\[ = \frac{1}{5\sqrt{5}} \left( (-2)^n \beta^n + 3\beta^{2n} - 3\alpha^{2n} + (-2)^n \alpha^n \right) = \frac{1}{5} \left( (-2)^n F_n - 3F_2n \right), \]

since \( 1 - \beta^3 = -2\beta, 1 - \alpha^3 = -2\alpha. \)

We remark the following

Corollary 17. We have, for any \( n \), \( (-1)^n L_n - 2^{n+1} \equiv 0 \) (mod 5) and \( (-2)^n F_n - 3F_2n \equiv 0 \) (mod 5).

References

[1] L. Carlitz, Generating Functions for Powers of Certain Sequences of Numbers, Duke Math. J. 29 (1962), pp. 521-537.

[2] A.F. Horadam, Generating functions for powers of a certain generalized sequence of numbers, Duke Math. J. 32 (1965), pp. 437-446.
[3] A.F. Horadam, Partial Sums for Second-Order Recurrence Sequences, *Fibonacci Quarterly*, Nov. 1994, pp. 429-440.

[4] J. Riordan, Generating functions for powers of Fibonacci numbers, *Duke Math. J.* 29 (1962), pp. 5-12.

[5] M. Rumney, E.J.F. Primrose, Relations between a Sequence of Fibonacci Type and a Sequence of its Partial Sums, *The Fibonacci Quarterly*, 9.3 (1971), pp. 296-298.

[6] S. Vajda, Fibonacci & Lucas Number and the Golden Section - Theory and Applications, John Wiley & Sons, 1989.

AMS Classification Numbers: 05A10, 05A19, 11B37, 11B39, 11B65, 11B83