Extended finite operator calculus - an example of algebraization of analysis

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Abstract

“A Calculus of Sequences” started in 1936 by Ward constitutes the general scheme for extensions of classical operator calculus of Rota - Mullin considered by many afterwards and after Ward. Because of the notation we shall call the Ward’s calculus of sequences in its afterwards elaborated form - a $\psi$-calculus.

The $\psi$-calculus in parts appears to be almost automatic, natural extension of classical operator calculus of Rota - Mullin or equivalently - of umbral calculus of Roman and Rota.

At the same time this calculus is an example of the algebraization of the analysis - here restricted to the algebra of polynomials. Many of the results of $\psi$-calculus may be extended to Markowsky $Q$-umbral calculus where $Q$ stands for a generalized difference operator, i.e. the one lowering the degree of any polynomial by one.

This is a review article based on the recent first author contributions [1]. As the survey article it is supplemented by the short indicative glossaries of notation and terms used by Ward [2], Viskov [7,8]
Markowsky [12], Roman [28-32] on one side and the Rota-oriented notation on the other side [9-11, 1, 3, 4, 35] (see also [33]).

KEY WORDS: extended umbral calculus, Graves-Heisenberg-Weyl algebra
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"The modern evolution... has on the whole been marked by a trend of algebraization."

Herman Weyl

1 Introduction

We shall call the Wards calculus of sequences [2] in its afterwards last century elaborated form - a \(\psi\)-calculus because of the Viskov's efficient notation [3]-[8] - adopted from Boas and Buck. The efficiency of the Rota oriented language and our notation used has been already exemplified by easy proving of \(\psi\)-extended counterparts of all representation independent statements of \(\psi\)-calculus [2]. Here these are \(\psi\)-labelled representations of Graves-Heisenberg-Weyl (GHW) [3], [1], [16], [17] algebra of linear operators acting on the algebra \(P\) of polynomials.

As a matter of fact \(\psi\)-calculus becomes in parts almost automatic extension of Rota - Mullin calculus [9] or equivalently - of umbral calculus of Roman and Rota [9, 10, 11]. The \(\psi\)-extension relies on the notion of \(\partial_\psi\)-shift invariance of operators with \(\psi\)-derivatives \(\partial_\psi\) staying for equivalence classes representatives of special differential operators lowering degree of polynomials by one [7, 8, 12]. Many of the results of \(\psi\)-calculus may be extended to Markowsky \(Q\)-umbral calculus [12] where \(Q\) stands for arbitrary generalized difference operator, i.e. the one lowering the degree of any polynomial by one. \(Q\)-umbral calculus [12] - as we call it - includes also those generalized difference operators, which are not series in \(\psi\)-derivative \(\partial_\psi\) whatever an admissible \(\psi\) sequence would be (for - "admissible" - see next section).

The survey proposed here reviews the operator formulation of “A Calculus of Sequences” started in 1936 by Ward [2] with the indication of the decisive role the \(\psi\)-representations of Graves-Heisenberg-Weyl (GHW) algebra account for formulation and derivation of principal statements of the \(\psi\)-extension of finite operator calculus of Rota and its extensions.

Restating what was said above let us underline that all statements of standard finite operator calculus of Rota are valid also in the case of \(\psi\)-extension under the almost mnemonic, automatic replacement of \(\{D, \hat{x}, id\}\) generators of GHW by their \(\psi\)-representation correspondents \(\{\partial_\psi, \hat{x}_\psi, id\}\) - see definitions 2.1 and 2.5. Naturally any specification of admissible \(\psi\) - for example the famous one defining...
q-calculus - has its own characteristic properties not pertaining to the standard case of Rota calculus realization. Nevertheless the overall picture and system of statements depending only on GHW algebra is the same modulo some automatic replacements in formulas demonstrated in the sequel. The large part of that kind of job was already done in [1, 3, 35].

The aim of this presentation is to give a general picture (see: Section 3) of the algebra of linear operators on polynomial algebra. The picture that emerges discloses the fact that any ψ-representation of finite operator calculus or equivalently - any ψ-representation of GHW algebra makes up an example of the algebraization of the analysis with generalized differential operators [12] acting on the algebra of polynomials.

We shall delimit all our considerations to the algebra $P$ of polynomials or sometimes to the algebra of formal series. Therefore the distinction between difference and differentiation operators disappears. All linear operators on $P$ are both difference and differentiation operators if the degree of differentiation or difference operator is unlimited.

If all this is extended to Markowsky $Q$-umbral calculus [12] then many of the results of ψ-calculus may be extended to $Q$-umbral calculus [12]. This is achieved under the almost automatic replacement of $\{D, \hat{x}, id\}$ generators of GHW or their ψ-representation $\{\partial_\psi, \hat{x}_\psi, id\}$ by their $Q$-representation correspondents $\{Q, \hat{x}_Q, id\}$ - see definition 2.5.

The article is supplemented by the short indicatory glossaries of notation and terms used by Ward [1], Viskov [7, 8], Markowsky [12], Roman [28-31] on one side and the Rota-oriented [9-11] notation on the other side [3, 4, 35, 1].

2 Primary definitions, notation and general observations

In the following we shall consider the algebra $P$ of polynomials $P = \mathbb{F}[x]$ over the field $\mathbb{F}$ of characteristic zero. All operators or functionals studied here are to be understood as linear operators on $P$. It shall be easy to see that they are always well defined.

Throughout the note while saying “polynomial sequence $\{p_n\}_0^\infty$” we mean $\deg p_n = n; n \geq 0$ and we adopt also the convention that $\deg p_n < 0$ iff $p_n \equiv 0$.

Consider $\mathfrak{F}$ - the family of functions’ sequences (in conformity with Viskov [7, 8, 3] notation ) such that:

$\mathfrak{F} = \{ \psi; R \supseteq [a, b] ; q \in [a, b] ; \psi(q) : \mathbb{Z} \rightarrow \mathbb{F} ; \psi_0(q) = 1 ; \psi_n(q) \neq 0; \psi_{-n}(q) = 0; n \in \mathbb{N} \}$.

We shall call $\psi = \{\psi_n(q)\}_{n \geq 0} ; \psi_n(q) \neq 0; n \geq 0$ and $\psi_0(q) = 1$ an admissible
Remark 2.1.

a) For any rational function $R$ the corresponding factorial $R(q^n)!$ of the sequence $R(q^n)$ is defined naturally as it is defined for $n_\psi$ sequence, i.e.: $R(q^n)! = R(q^n)R(q^{n-1})...R(q^1)$ The choice $\psi_n(q) = [R(q^n)]^{-1}$ and $R(x) = \frac{1-x}{1-q}$ results in the well known $q$-factorial $n_q! = n_q(n-1)_q!; \ n_q! = 0! = 1$ while the $\psi$-derivative $\partial_\psi$ becomes now $(n_\psi = n_q)$ the Jackson’s derivative.

$$\partial_\psi^n \exp_y = \sum_{k=0}^{\infty} \frac{y^k}{k!}$$

b) Note also that if $\psi = \{\psi_n(q)\}_{n \geq 0}$ and $\phi = \{\varphi_n(q)\}_{n \geq 0}$ are two admissible sequences then $[\partial_\psi, \partial_\phi] = 0$ iff $\psi = \varphi$. Here $[,]$ denotes the commutator of operators.

Definition 2.1. Let $\psi$ be admissible. Let $\partial_\psi$ be the linear operator lowering degree of polynomials by one defined according to $\partial_\psi x^n = n_\psi x^{n-1}; \ n \geq 0$. Then $\partial_\psi$ is called the $\psi$-derivative.

Remark 2.1. a) For any rational function $R$ the corresponding factorial $R(q^n)!$ of the sequence $R(q^n)$ is defined naturally as it is defined for $n_\psi$ sequence, i.e. $R(q^n)! = R(q^n)R(q^{n-1})...R(q^1)$ The choice $\psi_n(q) = [R(q^n)]^{-1}$ and $R(x) = \frac{1-x}{1-q}$ results in the well known $q$-factorial $n_q! = n_q(n-1)_q!; \ n_q! = 0! = 1$ while the $\psi$-derivative $\partial_\psi$ becomes now $(n_\psi = n_q)$ the Jackson’s derivative $[25, 26, 27, 2, 3]$

$$\partial_\psi^n \exp_y = \sum_{k=0}^{\infty} \frac{y^k}{k!}$$

Definition 2.2. Let $E^y(\partial_\psi) \equiv \exp_y \{y \partial_\psi\} = \sum_{k=0}^{\infty} \frac{y^k}{k!} \partial_\psi.$ $E^y(\partial_\psi)$ is called the generalized translation operator.

Note 2.1. $[3, 4, 11]$

$E^a(\partial_\psi) f(x) \equiv f(x + \psi a)$; $(x + \psi a)^n \equiv E^a(\partial_\psi) x^n$; $E^a(\partial_\psi) f = \sum_{n \geq 0} \frac{a^n}{n!} \partial_\psi^n f$;

and in general $(x + \psi a)^n \neq (x + \psi a)^{n-1}(x + \psi a)$. Note also $[11]$ that in general $(1 + \psi(-1))^{2n+1} \neq 0; \ n \geq 0$ though $(1 + \psi(-1))^{2n} = 0; \ n \geq 1$.

Note 2.2. $[11]$ \exp_y (x + \psi y) \equiv E^x(\partial_\psi) \exp_y \{y\}$ - while in general $\exp_y \{x+y\} \neq \exp_y \{x\} \exp_y \{y\}$

Possible consequent use of the identity $\exp_y (x + \psi y) \equiv \exp_y \{x\} \exp_y \{y\}$ is quite encouraging. It leads among others to “$\psi$-trigonometry” either $\psi$-elliptic or
ψ-hyperbolic via introducing \( \cos \psi, \sin \psi, \cosh \psi, \sinh \psi \) or in general ψ-hyperbolic functions of m-th order
\[
\left\{ h_j^{(\psi)} (\alpha) \right\}_{j \in \mathbb{Z}_m}
\]
defined according to [13]
\[
R \ni \alpha \to h_j^{(\psi)} (\alpha) = \frac{1}{m} \sum_{k \in \mathbb{Z}_m} \omega^{-kj} \exp_\psi \left\{ \omega^k \alpha \right\} ; \quad \omega = \exp \left\{ \frac{2\pi i}{m} \right\} .
\]
where \( 1 < m \in \mathbb{N} \) and \( \mathbb{Z}_m = \{0, 1, ..., m - 1\} \).

**Definition 2.3.** A polynomial sequence \( \{p_n\}_\infty \) is of ψ-binomial type if it satisfies the recurrence
\[
E^y (\partial_\psi) p_n (x) \equiv p_n (x + \psi y) \equiv \sum_{k \geq 0} \binom{n}{k} \psi b_k, k \geq 1; b_{n,1} \neq 0 \text{ of course. Without loose of generality take } b_{1,1} = 1. \text{ Then } \exists \{q_k\}_{k \geq 2} \subset \mathbb{F} \text{ and there exists admissible } \psi \text{ such that }
\]
\[
Q = \partial_\psi + \sum_{k \geq 2} q_k \partial_\psi^k
\]
if and only if
\[
b_{n,k} = \binom{n}{k} \psi b_{k,k}; \quad n \geq k \geq 1; b_{n,1} \neq 0; b_{1,1} = 1.
\]
If \( \{q_k\}_{k \geq 2} \) and an admissible ψ exist then these are unique.
Notation 2.1. In the case (2.2) is true we shall write : \( Q = Q ( \partial_\psi ) \) because then and only then the generalized differential operator \( Q \) is a series in powers of \( \partial_\psi \).

Remark 2.2. Note that operators of the (2.1) form constitute a group under superposition of formal power series (compare with the formula (S) in [13]). Of course not all generalized differential operators satisfy (2.1) i.e. are series just only in corresponding \( \psi \)-derivative \( \partial_\psi \) (see Proposition 3.1). For example \( \hat{x} = \frac{1}{2} D \hat{x} D - \frac{1}{3} D^3 \). Then \( Q x^n = \frac{1}{2} n^2 x^{n-1} - \frac{1}{3} n^3 x^{n-3} \) so according to Observation 2.1 \( n_\psi = \frac{1}{2} n^2 \) and there exists no admissible \( \psi \) such that \( Q = Q ( \partial_\psi ) \). Here \( \hat{x} \) denotes the operator of multiplication by \( x \) while \( n_\psi \) is a special case of \( n_\psi \) for the choice \( n_\psi = n \).

Observation 2.2. From theorem 3.1 in [12] we infer that generalized differential operators give rise to subalgebras \( \sum Q \) of linear maps (plus zero map of course) commuting with a given generalized differential operator \( Q \). The intersection of two different algebras \( \sum Q_1 \) and \( \sum Q_2 \) is just zero map added.

The importance of the above Observation 2.2 as well as the definition below may be further fully appreciated in the context of the Theorem 2.1 and the Proposition 3.1 to come.

Definition 2.5. Let \( \{ p_n \}_{n \geq 0} \) be the normal polynomial sequence [12], i.e. \( p_n (0) = 0 \); \( n \geq 1 \). Then we call it the \( \psi \)-basic sequence of the generalized differential operator \( Q \) if in addition \( Q p_n = n_\psi p_{n-1} \). In parallel we define a linear map \( \hat{x} Q : P \rightarrow P \) such that \( \hat{x} Q p_n = \frac{1}{(n+1)_\psi} p_{n+1} \); \( n \geq 0 \). We call the operator \( \hat{x} Q \) the dual to \( Q \) operator.

When \( Q = Q ( \partial_\psi ) = \partial_\psi \) we write for short: \( \hat{x} Q ( \partial_\psi ) \equiv \hat{x} \partial_\psi \equiv \hat{\partial}_\psi \) (see: Definition 2.3).

Of course \( [Q, \hat{x} Q] = \text{id} \) therefore \( \{ Q, \hat{x} Q, \text{id} \} \) provide us with a continuous family of generators of GHW in - as we call it - \( Q \)-representation of Graves-Heisenberg-Weyl algebra.

In the following we shall restrict to special case of generalized differential operators \( Q \), namely to those \( Q = Q ( \partial_\psi ) \) which are \( \partial_\psi \)-shift invariant operators [3, 4, 1] (see: Definition 2.6).

At first let us start with appropriate \( \psi \)-Leibnitz rules for corresponding \( \psi \)-derivatives.

**\( \psi \)-Leibnitz rules:**

It is easy to see that the following hold for any formal series \( f \) and \( g \):

for \( \partial_q \): \( \partial_q ( f \cdot g ) = ( \partial_q f ) \cdot g + ( \hat{Q} f ) \cdot ( \partial_q g ) \), where \( ( \hat{Q} f ) (x) = f (qx) \);  
for \( \partial_R = R (q \hat{Q}) \partial_0 \): \( \partial_R (f \cdot g) (z) = R (q \hat{Q}) \{ (\partial_0 f) (z) \cdot g(z) + f(0) (\partial_0 g) (z) \} \)
where - note - $R(q) x^{n-1} = n_R x^{n-1} ; (n_\psi = n_R = n_{R(q)} = R(q^n))$ and finally for $\partial_\psi = \hat{n}_\psi \partial_0$:

$$\partial_\psi (f \cdot g)(z) = \hat{n}_\psi \{(\partial_0 f)(z) \cdot g(z) + f(0)(\partial_0 g)(z)\}$$

where $\hat{n}_\psi x^{n-1} = n_\psi x^{n-1} ; n \geq 1$.

**Example 2.1.** Let $Q(\partial_\psi) = D \hat{D}$, where $\hat{D} f(x) = x f(x)$ and $D = \frac{d}{dx}$. Then $\psi = \left\{\left[\frac{1}{(n^2)!}\right]\right\}_{n \geq 0}$ and $Q = \partial_\psi$. Let $Q(\partial_\psi) R(q \hat{Q}) \partial_0 \equiv \partial_R$. Then $\psi = \left\{\left[\frac{1}{(n^2)!}\right]\right\}_{n \geq 0}$ and $Q = \partial_\psi \equiv \partial_R$. Here $R(z)$ is any formal Laurent series; $Q f(x) = f(qx)$ and $n_\psi = R(q^n)$. $\partial_0$ is $q = 0$ Jackson derivative which as a matter of fact - being a difference operator is the differential operator of infinite order at the same time:

$$\partial_0 = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{n-1}}{n!} \frac{d^n}{dx^n}. \quad (2.3)$$

Naturally with the choice $\psi_{R_n}(q) = \left[\frac{1}{(n^2)!}\right]^{-1}$ and $R(x) = \frac{1-x}{1-q}$ the $\psi$-derivative becomes the Jackson’s derivative [25, 26, 27, 2, 3] $\partial_q$:

$$(\partial_q \varphi)(x) = \frac{1-q \hat{Q}}{(1-q)} \partial_0 \varphi(x).$$

The equivalent to [25, 26, 27] form of Bernoulli-Taylor expansion one may find [16] in *Acta Eruditorum* from November 1694 under the name “series universalissima”.

(Taylor’s expansion was presented in his “Methodus incrementorum directa et inversa” in 1715 - edited in London).

**Definition 2.6.** Let us denote by $\text{End}(P)$ the algebra of all linear operators acting on the algebra $P$ of polynomials. Let

$$\sum_{\psi} = \{T \in \text{End}(P); \forall \alpha \in F; [T, E^\alpha (\partial_\psi)] = 0\}.$$

Then $\sum_{\psi}$ is a commutative subalgebra of $\text{End}(P)$ of $F$-linear operators. We shall call these operators $T : \partial_\psi$-shift invariant operators.

We are now in a position to define further basic objects of “$\psi$-umbral calculus” [3, 4, 1].

**Definition 2.7.** Let $Q(\partial_\psi) : P \rightarrow P$; the linear operator $Q(\partial_\psi)$ is a $\partial_\psi$-delta operator iff

a) $Q(\partial_\psi)$ is $\partial_\psi$ - shift invariant;
b) \( Q(\partial_{\psi})(id) = const \neq 0 \) where \( id(x) = x \).

The strictly related notion is that of the \( \partial_{\psi} \)-basic polynomial sequence:

**Definition 2.8.** Let \( Q(\partial_{\psi}) : P \to P \) be the \( \partial_{\psi} \)-delta operator. A polynomial sequence \( \{p_n\}_{n \geq 0} \) with degree \( p_n = n \) such that:

1) \( p_0(x) = 1 \);
2) \( p_n(0) = 0 \); \( n > 0 \);
3) \( Q(\partial_{\psi})p_n = n\psi p_{n-1} \) \( \partial_{\psi} \)-delta operator \( Q(\partial_{\psi}) \) is called the \( \partial_{\psi} \)-basic polynomial sequence of the \( \partial_{\psi} \)-delta operator.

**Identification 2.1.** It is easy to see that the following identification takes place:

\( \partial_{\psi} \)-delta operator \( Q(\partial_{\psi}) = \partial_{\psi} \)-shift invariant generalized differential operator \( Q \).

Of course not every generalized differential operator might be considered to be such.

**Note 2.3.** Let \( \Phi(x; \lambda) = \sum_{n \geq 0} \frac{\lambda^n}{n!}p_n(x) \) denotes the \( \psi \)-exponential generating function of the \( \partial_{\psi} \)-basic polynomial sequence \( \{p_n\}_{n \geq 0} \) of the \( \partial_{\psi} \)-delta operator \( Q \equiv Q(\partial_{\psi}) \) and let \( \Phi(0; \lambda) = 1 \). Then \( Q\Phi(x; \lambda) = \lambda\Phi(x; \lambda) \) and \( \Phi \) is the unique solution of this eigenvalue problem. If in addition (2.2) is satisfied then there exists such an admissible sequence \( \varphi \) that \( \Phi(x; \lambda) = \exp_{\varphi}\{\lambda x\} \) (see Example 3.1).

The notation and naming established by Definitions 2.7 and 2.8 serve the target to preserve and to broaden simplicity of Rota’s finite operator calculus also in its extended “\( \psi \)-umbral calculus” case. As a matter of illustration of such notation efficiency let us quote after [3] the important Theorem 2.1 which might be proved using the fact that \( \forall Q(\partial_{\psi}) \exists! \) invertible \( S \in \Sigma_{\psi} \) such that \( Q(\partial_{\psi}) = \partial_{\psi}S \). (For Theorem 2.1 see also Theorem 4.3. in [12], which holds for operators, introduced by the Definition 2.5). Let us define at first what follows.

**Definition 2.9.** (compare with (17) in [8])

The Pincherle \( \psi \)-derivative is the linear map \( ^T : \Sigma_{\psi} \to \Sigma_{\psi} \):

\[
T^T = T \hat{x}_{\psi} - \hat{x}_{\psi}T \equiv [T, \hat{x}_{\psi}]
\]

where the linear map \( \hat{x}_{\psi} : P \to P \) is defined in the basis \( \{x^n\}_{n \geq 0} \) as follows

\[
\hat{x}_{\psi}x^n = \frac{\psi_{n+1}(q)(n + 1)}{\psi_n(q)}x^{n+1} = \frac{(n + 1)}{(n + 1)\psi}x^{n+1}, \quad n \geq 0.
\]

Then the following theorem is true [3]
Theorem 2.1. (ψ-Lagrange and ψ-Rodrigues formulas [34, 11, 12, 23, 3])

Let \( \{ p_n(x) \}_{n=0}^{\infty} \) be \( \partial_\psi \)-basic polynomial sequence of the \( \partial_\psi \)-delta operator \( Q(\partial_\psi) \).

Let \( Q(\partial_\psi) = \partial_\psi S \). Then for \( n > 0 \):

1. \( p_n(x) = Q(\partial_\psi)' S^{-n-1} x^n \);
2. \( p_n(x) = S^{-n} x^n - \frac{n\psi}{n} (S^{-n})' x^{n-1} \);
3. \( p_n(x) = \frac{n\psi}{n} S^{-n-1} x^n \);
4. \( p_n(x) = \frac{n\psi}{n} (Q(\partial_\psi))^{-1} p_{n-1}(x) \) (ψ-Rodrigues ψ-formula).

For the proof one uses typical properties of the Pincherle ψ-derivative [3]. Because \( \partial_\psi' = id \) we arrive at the simple and crucial observation.

Observation 2.3. [3,35]
The triples \( \{ \partial_\psi, \hat{x}_\psi, id \} \) for any admissible ψ-constitute the set of generators of the ψ-labelled representations of Graves-Heisenberg-Weyl (GHW) algebra [17, 18, 19, 35, 1]. Namely, as easily seen \( [\partial_\psi, \hat{x}_\psi] = id \). (compare with Definition 2.5)

Observation 2.4. In view of the Observation 2.3 the general Leibnitz rule in ψ-representation of Graves-Heisenberg-Weyl algebra may be written (compare with 2.2.2 Proposition in [18]) as follows

\[
\partial_\psi^n \hat{x}_\psi^m = \sum_{k=0}^{\infty} \binom{n}{k} \binom{m}{k} k! \hat{x}_\psi^{m-k} \partial_\psi^{n-k}.
\] (2.4)

One derives the above ψ-Leibnitz rule from ψ-Heisenberg-Weyl exponential commutation rules exactly the same way as in \( \{ D, \hat{x}, id \} \) GHW representation - (compare with 2.2.1 Proposition in [18]). ψ-Heisenberg-Weyl exponential commutation relations read:

\[
\exp\{t\partial_\psi\} \exp\{a\hat{x}_\psi\} = \exp\{at\} \exp\{a\hat{x}_\psi\} \exp\{t\partial_\psi\}.
\] (2.5)

To this end let us introduce a pertinent ψ-multiplication \( *_\psi \) of functions as specified below.

Notation 2.2.

\[
x *_\psi x^n = \hat{x}_\psi(x^n) = \frac{(n+1)!}{(n+1)} x^{n+1}; \quad n \geq 0 \quad \text{hence} \quad x *_\psi 1 = 1^{-1}_\psi x \neq x
\]

\[
x^n *_\psi x = \hat{x}_\psi^n(x) = \frac{1}{(n+1)!} x^{n+1}; \quad n \geq 0 \quad \text{hence} \quad 1 *_\psi x = x \text{ therefore}
\]

\[
x *_\psi \alpha 1 = x *_\psi \alpha = \alpha 1^{-1}_\psi x \text{ and} \quad \alpha 1 *_\psi x = \alpha *_\psi x = \alpha x \quad \text{and}
\]

\[
\forall x, \alpha \in F; \quad f(x) *_\psi x^n = f(\hat{x}_\psi) x^n.
\]
For \( k \neq n \) \( x^n * \psi x^k \neq x^k * \psi x^n \) as well as \( x^n * \psi x^k \neq x^{n+k} \) - in general i.e. for arbitrary admissible \( \psi \); compare this with \((x + \psi a)^n \neq (x + \psi a)^{n-1}(x + \psi a)\).

In order to facilitate in the future formulation of observations accounted for on the basis of \( \psi \)-calculus representation of GHW algebra we shall use what follows.

**Definition 2.10.** With Notation 2.2 adopted let us define the *\( \psi \) powers* of \( x \) according to

\[
x^{n*\psi} \equiv x * \psi x^{(n-1)*\psi} = \hat{x}_\psi(x^{(n-1)*\psi}) = x * \psi x * \psi ... * \psi x = \frac{n!}{n!} x^n; \quad n \geq 0.
\]

Note that \( x^{n*\psi} * \psi x^{k*\psi} = \frac{n!}{n!} x^{(n+k)*\psi} \neq x^{k*\psi} * \psi x^{n*\psi} = \frac{k!}{k!} x^{(n+k)*\psi} \) for \( k \neq n \) and \( x^{0*\psi} = 1 \).

This noncommutative \( \psi \)-product \( * \psi \) is devised so as to ensure the following observations.

**Observation 2.5.**

(a) \( \partial_\psi x^{n*\psi} = nx^{(n-1)*\psi}; \quad n \geq 0 \)

(b) \( \exp_\psi[\alpha x] \equiv \exp\{\alpha \hat{x}_\psi\}1 \)

(c) \( \exp[\alpha x] * \psi (\exp_\psi\{\beta \hat{x}_\psi\}1) = (\exp_\psi\{[\alpha + \beta] \hat{x}_\psi\})1 \)

(d) \( \partial_\psi(x^k * \psi x^{n*\psi}) = (Dx^k) * \psi x^{n*\psi} + x^k * \psi (\partial_\psi x^{n*\psi}) \) hence

(e) \( \partial_\psi(f * \psi g) = (Df) * \psi g + f * \psi (\partial_\psi g); f, g \) - formal series

(f) \( f(\hat{x}_\psi)g(\hat{x}_\psi) 1 = f(x) * \psi g(x); g(x) = g(\hat{x}_\psi)1 \).

Now the consequences of Leibniz rule (e) for difference-ization of the product are easily feasible. For example the Poisson \( \psi \)-process distribution \( \pi_m(x) = \frac{(\lambda x)^m}{m!} * \psi \exp_\psi[-\lambda x] \) (2.6)

which is the unique solution (up to a constant factor) of the \( \partial_\psi \)-difference equations systems

\[
\partial_\psi p_m(x) + \lambda p_m(x) = \lambda p_{m-1}(x) \quad m > 0; \quad \partial_\psi p_0(x) = -\lambda p_0(x)
\]

(2.7)

Naturally \( N(\lambda, x) = \exp[\lambda x] * \psi \exp_\psi[-\lambda x] \).

As announced - the rules of \( \psi \)-product \( * \psi \) are accounted for on the basis of \( \psi \)-calculus representation of GHW algebra. Indeed, it is enough to consult Observation 2.3 and to introduce \( \psi \)-Pincherle derivation \( \hat{\partial}_\psi \) of series in powers of the symbol \( \hat{x}_\psi \) as below. Then the correspondence between generic relative formulas turns out evident.
Observation 2.6. Let $\hat{\partial}_\psi \equiv \frac{\partial}{\partial \hat{x}_\psi}$ be defined according to $\hat{\partial}_\psi f(\hat{x}_\psi) = [\partial_\psi, f(\hat{x}_\psi)]$. Then $\hat{\partial}_\psi \hat{x}_\psi^n = n\hat{x}_\psi^{n-1}$; $n \geq 0$ and $\hat{\partial}_\psi \hat{x}_\psi^1 = \partial_\psi x^{* \psi}$ hence $[\hat{\partial}_\psi f(\hat{x}_\psi)] 1 = \partial_\psi f(x)$ where $f$ is a formal series in powers of $\hat{x}_\psi$ or equivalently in $\star$ powers of $x$.

As an example of application note how the solution of 2.7 is obtained from the obvious solution $p_m(\hat{x}_\psi)$ of the $\hat{\partial}_\psi$-Pincherle differential equation 2.8 formulated within G-H-W algebra generated by $\{ \partial_\psi, \hat{x}_\psi, id \}$

$$\hat{\partial}_\psi p_m(\hat{x}_\psi) + \lambda p_m(\hat{x}_\psi) = \lambda p_{m-1}(\hat{x}_\psi) m > 0 \ ; \ \partial_\psi p_0(\hat{x}_\psi) = -\lambda p_0(\hat{x}_\psi). \quad (2.8)$$

Namely: due to Observation 2.5 (f) $p_m(x) = p_m(\hat{x}_\psi)1$, where

$$p_m(\hat{x}_\psi) = \frac{(\lambda \hat{x}_\psi)^m}{m!} \exp[-\lambda \hat{x}_\psi]. \quad (2.9)$$

3 The general picture of the algebra $\text{End}(P)$ from GHW algebra point of view

The general picture from the title above relates to the general picture of the algebra $\text{End}(P)$ of operators on $P$ as in the following we shall consider the algebra $P$ of polynomials $P = F[x]$ over the field $F$ of characteristic zero.

With series of Propositions from [1,3,35,21] we shall draw an over view picture of the situation distinguished by possibility to develop further umbral calculus in its operator form for any polynomial sequences $\{p_n\}_{0}^{\infty}$ [12] instead of those of traditional binomial type only.

In 1901 it was proved [20] that every linear operator mapping $P$ into $P$ may be represented as infinite series in operators $\hat{x}$ and $D$. In 1986 the authors of [21] supplied the explicit expression for such series in most general case of polynomials in one variable (for many variables see: [22]). Thus according to Proposition 1 from [21] one has:

**Proposition 3.1.** Let $Q$ be a linear operator that reduces by one the degree of each polynomial. Let $\{q_n(\hat{x})\}_{n \geq 0}$ be an arbitrary sequence of polynomials in the operator $\hat{x}$. Then $\hat{T} = \sum_{n \geq 0} q_n(\hat{x})Q^n$ defines a linear operator that maps polynomials into polynomials. Conversely, if $\hat{T}$ is linear operator that maps polynomials into polynomials then there exists a unique expansion of the form

$$\hat{T} = \sum_{n \geq 0} q_n(\hat{x})Q^n.$$
It is also a rather matter of an easy exercise to prove the Proposition 2 from [21].

**Proposition 3.2.** Let $Q$ be a linear operator that reduces by one the degree of each polynomial. Let $\{q_n(\hat{x})\}_{n \geq 0}$ be an arbitrary sequence of polynomials in the operator $\hat{x}$. Let a linear operator that maps polynomials into polynomials be given by

$$\hat{T} = \sum_{n \geq 0} q_n(\hat{x})Q^n.$$  

Let $P(x; \lambda) = \sum_{n \geq 0} q_n(x)\lambda^n$ denotes indicator of $\hat{T}$. Then there exists a unique formal series $\Phi(x; \lambda); \Phi(0; \lambda) = 1$ such that:

$$Q\Phi(x; \lambda) = \lambda \Phi(x; \lambda).$$

Then also $P(x; \lambda) = \Phi(x; \lambda)^{-1} \hat{T} \Phi(x; \lambda)$.

**Example 3.1.** Note that $\partial_\psi \exp_\psi\{\lambda x\} = \lambda \exp_\psi\{\lambda x\}; \exp_\psi[x]_{|x=0} = 1$. (*)  

Hence for indicator of $\hat{T}; \hat{T} = \sum_{n \geq 0} q_n(\hat{x})\partial_\psi^n$ we have:

$$P(x; \lambda) = \left[\exp_\psi\{\lambda x\}\right]^{-1} \hat{T} \exp_\psi\{\lambda x\}. (**)$$

After choosing $\psi_n(q) = [n_q!]^{-1}$ we get $\exp_\psi\{x\} = \exp_q\{x\}$. In this connection note that $\exp_0(x) = \frac{1}{1-x}$ and $\exp(x)$ are mutual limit deformations for $|x| < 1$ due to:

$$\frac{\exp_0(z)-1}{z} = \exp_0(z) \Rightarrow \exp_0(z) = \frac{1}{1-z} = \sum_{k=0}^\infty z^k; |z| < 1,$$

i.e.

$$\exp(x) \leftarrow_q \exp_q(x) = \sum_{n=0}^\infty \frac{x^n}{n_q!} \rightarrow_{q-0} \frac{1}{1-x}.$$  

Therefore corresponding specifications of (*) such as $\exp_0(\lambda x) = \frac{1}{1-\lambda x}$ or $\exp(\lambda x)$ lead to corresponding specifications of (**) for divided difference operator $\partial_0$ and $D$ operator including special cases from [21].

To be complete let us still introduce an important operator $\hat{x}Q(\partial_\psi)$ dual to $Q(\partial_\psi)$.

**Definition 3.1.** (see Definition 2.5)  

Let $\{p_n\}_{n \geq 0}$ be the $\partial_\psi$-basic polynomial sequence of the $\partial_\psi$-delta operator $Q(\partial_\psi)$.

A linear map $\hat{x}Q(\partial_\psi): P \rightarrow P; \hat{x}Q(\partial_\psi) p_n = \frac{(n+1)}{n+1} p_{n+1}; n \geq 0$ is called the operator dual to $Q(\partial_\psi)$.  

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Comment 3.1. Dual in the above sense corresponds to adjoint in $\psi$-umbral calculus language of linear functionals’ umbral algebra (compare with Proposition 1.1.21 in [23]).

It is now obvious that the following holds.

Proposition 3.3. Let \( \{q_n \left( \hat{x}_Q(\partial_\psi) \right) \}_{n \geq 0} \) be an arbitrary sequence of polynomials in the operator $\hat{x}_Q(\partial_\psi)$. Then $T = \sum_{n \geq 0} q_n \left( \hat{x}_Q(\partial_\psi) \right) Q (\partial_\psi)^n$ defines a linear operator that maps polynomials into polynomials. Conversely, if $T$ is linear operator that maps polynomials into polynomials then there exists a unique expansion of the form

$$T = \sum_{n \geq 0} q_n \left( \hat{x}_Q(\partial_\psi) \right) Q (\partial_\psi)^n.$$ (3.1)

Comment 3.2. The pair $Q(\partial_\psi), \hat{x}_Q(\partial_\psi)$ of dual operators is expected to play a role in the description of quantum-like processes apart from the $q$-case now vastly exploited [3, 4].

Naturally the Proposition 3.2 for $Q(\partial_\psi)$ and $\hat{x}_Q(\partial_\psi)$ dual operators is also valid.

Summing up: we have the following picture for $\text{End}(P)$ - the algebra of all linear operators acting on the algebra $P$ of polynomials.

$$Q(P) \equiv \bigcup_Q \sum_Q \subset \text{End}(P)$$

and of course $Q(P) \neq \text{End}(P)$ where the subfamily $Q(P)$ (with zero map) breaks up into sum of subalgebras $\sum_Q$ according to commutativity of these generalized difference-tial operators $Q$ (see Definition 2.4 and Observation 2.2). Also to each subalgebra $\sum_\psi$ i.e. to each $Q(\partial_\psi)$ operator there corresponds its dual operator $\hat{x}_Q(\partial_\psi)$

$$\hat{x}_Q(\partial_\psi) \notin \sum_\psi$$

and both $Q(\partial_\psi)$ & $\hat{x}_Q(\partial_\psi)$ operators are sufficient to build up the whole algebra $\text{End}(P)$ according to unique representation given by (3.1) including the $\partial_\psi$ and $\hat{x}_\psi$ case. Summarising: for any admissible $\psi$ we have the following general statement.

General statement:

$$\text{End}(P) = \{\partial_\psi, \hat{x}_\psi\} = \{Q(\partial_\psi), \hat{x}_Q(\partial_\psi)\} = \{Q, \hat{x}_Q\}$$
i.e. the algebra $\text{End}(P)$ is generated by any dual pair $\{Q, \hat{x}_Q\}$ including any dual pair $\{Q(\partial_\psi), \hat{x}_Q(\partial_\psi)\}$ or specifically by $\{\partial_\psi, \hat{x}_\psi\}$ which in turn is determined by a choice of any admissible sequence $\psi$.

As a matter of fact and in another words: we have bijective correspondences between different commutation classes of $\partial_\psi$-shift invariant operators from $\text{End}(P)$, different abelian subalgebras $\sum_\psi$, different $\psi$-representations of the reduced incidence algebra $\text{R}(L(S))$ - isomorphic to the algebra $\Phi_\psi$ of $\psi$-exponential formal power series [8] and finally - distinct $\psi$-umbral calculi [8, 11, 21, 34, 35]. These bijective correspondences may be naturally extended to encompass also $Q$-umbral calculi[12,1], $Q$-representations of GHW algebra [1] and abelian subalgebras $\sum_Q$.

(Recall: $\text{R}(L(S))$ is the reduced incidence algebra of $L(S)$ where $L(S) = \{A; A \subset S; |A| < \infty\}; S$ is countable and $(L(S); \subseteq)$ is partially ordered set ordered by inclusion [11, 3]).

This is the way the Rota’s devise has been carried into effect. The devise “much is the iteration of the few” [11] - much of the properties of literally all polynomial sequences - as well as GHW algebra representations - is the application of few basic principles of the $\psi$-umbral difference operator calculus [3, 35, 1].

$\psi$-Integration Remark:
Recall $\partial_q x^n = x^{n-1}$. $\partial_q$ is identical with divided difference operator. $\partial_0$ is identical with $\partial_\psi$ for $\psi = \{\psi(q)_n\}_{n \geq 0}$; $\psi(q)_n = 1$; $n \geq 0$. Let $\hat{Q} f(x) f(qx)$.
Recall also that there corresponds to the “$\partial_q$ difference-ization” the q-integration [25, 26, 27] which is a right inverse operation to “$q$-difference-ization” [35, 1].

Namely

$$F(z) \equiv \left( \int_q \varphi \right)(z) := (1 - q) z \sum_{k=0}^{\infty} \varphi \left( q^k z \right) q^k$$ (3.2)

i.e.

$$F(z) \equiv \left( \int_q \varphi \right)(z) = (1 - q) z \left( \sum_{k=0}^{\infty} q^k \hat{Q}^k \varphi \right)(z) =$$

$$= \left( (1 - q) z \frac{1}{1 - q\hat{Q}} \varphi \right)(z).$$ (3.3)

Of course

$$\partial_q \circ \int_q = \text{id}$$ (3.4)

as

$$\frac{1 - q\hat{Q}}{(1 - q) \partial_0} \left( (1 - q) \hat{z} \frac{1}{1 - q\hat{Q}} \right) = \text{id}. \quad (3.5)$$
Naturally (3.5) might serve to define a right inverse operation to “q-difference-ization”

\[(\partial_q \varphi)(x) = \frac{1-q^\hat{Q}}{1-q^0} \partial_0 \varphi(x)\]

and consequently the “q-integration” as represented by (3.2) and (3.3). As it is well known the definite q-integral is a numerical approximation of the definite integral obtained in the \(q \to 1\) limit. Following the q-case example we introduce now an \(R\)-integration (consult Remark 2.1).

\[
\int_R x^n = \left( \hat{x} \frac{1}{R(q^\hat{Q})} \right) x^n = \frac{1}{R(q^{n+1})} x^{n+1}; \quad n \geq 0 \tag{3.6}
\]

Of course \(\partial_R \circ \int_R = \text{id}\) as

\[
R(q^\hat{Q}) \partial_o \left( \hat{x} \frac{1}{R(q^\hat{Q})} \right) = \text{id}. \tag{3.7}
\]

Let us then finally introduce the analogous representation for \(\partial_\psi\) difference-ization

\[
\partial_\psi = \hat{n}_\psi \partial_o; \quad \hat{n}_\psi x^{n-1} = n_\psi x^{n-1}; \quad n \geq 1. \tag{3.8}
\]

Then

\[
\int_\psi x^n = \left( \hat{x} \frac{1}{\hat{n}_\psi} \right) x^n = \frac{1}{(n+1)_\psi} x^{n+1}; \quad n \geq 0 \tag{3.9}
\]

and of course

\[
\partial_\psi \circ \int_\psi = \text{id} \tag{3.10}
\]

**Closing Remark:**

The picture that emerges discloses the fact that any \(\psi\)-representation of finite operator calculus or equivalently - any \(\psi\)-representation of GHW algebra makes up an example of the algebraization of the analysis - naturally when constrained to the algebra of polynomials. We did restricted all our considerations to the algebra \(P\) of polynomials. Therefore the distinction in-between difference and differentiation operators disappears. All linear operators on \(P\) are both difference and differentiation operators if the degree of differentiation or difference operator is unlimited. For example \(\frac{d}{dx} = \sum_{k \geq 1} d_k \Delta^k\) where \(d_k = [\frac{d}{dx} x^k]_{x=0} = (-1)^{k-1}(k-1)!\)

or \(\Delta = \sum_{n \geq 1} \frac{\delta_n}{n!} \frac{dx^n}{dx}\) where \(\delta_n = [\Delta x^n]_{x=0} = 1\). Thus the difference and differential operators and equations are treated on the same footing. For new applications -
due to the first author see [4,1,36-41]. Our goal here was to deliver the general scheme of "ψ-umbral" algebraization of the analysis of general differential operators [12]. Most of the general features presented here are known to be pertinent to the $Q$ representation of finite operator calculus (Viskov, Markowsky, Roman) where $Q$ is any linear operator lowering degree of any polynomial by one. So it is most general example of the algebraization of the analysis for general differential operators [12].

4 Glossary

In order to facilitate the reader a simultaneous access to quoted references of classic Masters of umbral calculus - here now follow short indicatory glossaries of notation used by Ward [2], Viskov [7,8], Markowsky [11], Roman [28]-[32] on one side and the Rota-oriented notation on the other side. See also [33].

| Ward                          | Rota - oriented (this note) |
|-------------------------------|-------------------------------|
| $[n] ; [n]!$                  | $n_{\psi}; n_{\psi}!$         |
| basic binomial coefficient $[n, r] = \frac{[n]!}{[r]![n-r]!}$ | $\psi$-binomial coefficient $\binom{n}{k}_{\psi} \equiv \frac{n^k_{\psi}}{k!_{\psi}}$. |
| $D = D_x$ - the operator $D$  | $\partial_\psi$ - the $\psi$-derivative |
| $D x^n = [n] x^{n-1}$         | $\partial_\psi x^n = n_{\psi} x^{n-1}$ |
| $(x + y)^n$                   | $(x +_\psi y)^n$              |
| $(x + y)^n \equiv \sum_{r=0}^{n} [n, r] x^{n-r} y^r$ | $(x +_\psi y)^n = \sum_{k=0}^{n} \binom{n}{k}_{\psi} x^k y^{n-k}$ |
| Ward | Rota - oriented (this note) |
|------|----------------------------|
| basic displacement symbol | generalized shift operator |
| $E^t; \ t \in \mathbb{Z}$ | $E^y (\partial \psi) \equiv \exp_{\psi} \{y \partial \psi\}; \ y \in \mathbb{F}$ |
| $E \varphi(x) = \varphi(x + 1)$ | $E(\partial \psi) \varphi(x) = \varphi(x + \psi 1)$ |
| $E^t \varphi(x) = \varphi(x + t)$ | $E^y(\partial \psi)x^n \equiv (x + \psi y)^n$ |
| basic difference operator | $\psi$-difference delta operator |
| $\Delta = E - id$ | $\Delta_{\psi} = E^y(\partial \psi) - id$ |
| $\Delta = \varepsilon(D) - id = \sum_{n=0}^{\infty} \frac{D^n}{n!} - id$ | |

| Roman | Rota - oriented (this note) |
|-------|----------------------------|
| $t; \ tx^n = nx^{n-1}$ | $\partial_{\psi}$ - the $\psi$-derivative |
| $\langle t^k|p(x) \rangle = p^{(k)}(0)$ | $\partial_{\psi}x^n = n_{\psi}x^{n-1}$ |
| | $[\partial_{\psi}^k p(x)]|_{x=0}$ |
| Roman                                      | Rota - oriented (this note)                                                                 |
|-------------------------------------------|---------------------------------------------------------------------------------------------|
| evaluation functional \( \epsilon_y(t) = \exp \{ yt \} \) | generalized shift operator \( E^y(\partial_\psi) = \exp_\psi \{ y \partial_\psi \} \) |
| \( \langle t^k | x^n \rangle = n! \delta_{n,k} \)   | \( [E^y(\partial_\psi)p_n(x)]|_{x=0} = p_n(y) \)                                             |
| \( \langle \epsilon_y(t) | p(x) \rangle = p(y) \)                         | \( E^y(\partial_\psi)p_n(x) = \sum_{k \geq 0} \binom{n}{k}_\psi p_k(x) p_{n-k}(y) \)     |
| \( \epsilon_y(t) x^n = \sum_{k \geq 0} \binom{n}{k}_\psi x^k y^{n-k} \) |                                                                                             |
| formal derivative \( f'(t) \equiv \frac{d}{dt} f(t) \)   | Pincherle derivative \( [Q(\partial_\psi)]' \equiv \frac{d}{d \partial_\psi} Q(\partial_\psi) \) |
| \( \mathcal{F}(t) \) compositional inverse of formal power series \( f(t) \) | \( Q^{-1}(\partial_\psi) \) compositional inverse of formal power series \( Q(\partial_\psi) \) |
| \( \theta_t; \ \theta_t x^n = x^{n+1}; \ n \geq 0 \) |                                                                                             |
| \( \theta_t f(t) = \hat{x} D \)                     | \( \hat{x}_\psi; \ \hat{x}_\psi x^n = \frac{n+1}{(n+1)_\psi} x^{n+1}; \ n \geq 0 \) |
| \( \sum_{k \geq 0} \frac{s_k(x)}{k_\psi!} t^k = \)   |                                                                                             |
| \[ g(\mathcal{F}(z)) \]^{-1} \exp \{ x \mathcal{F}(t) \} |                                                                                             |
| \{ s_n(x) \}_{n \geq 0} - Sheffer sequence \( g(t), f(t) \) |                                                                                             |
| for \( (g(t), f(t)) \)                             |                                                                                             |
| \( \sum_{k \geq 0} \frac{s_k(x)}{k_\psi!} z^k = \) |                                                                                             |
| \( s(q^{-1}(z)) \exp_\psi \{ xq^{-1}(z) \} \)       |                                                                                             |
| \( g(t), s(t) \) indicators of \( Q(\partial_\psi) \) and \( S_{\partial_\psi} \) |                                                                                             |
| Roman                                           | Rota - oriented (this note)                                      |
|------------------------------------------------|-----------------------------------------------------------------|
| $g(t) s_n(x) = q_n(x)$ - sequence associated for $f(t)$ | $s_n(x) = S_{\partial_{\psi}}^{-1} q_n(x) - \partial_{\psi}$ - basic sequence of $Q(\partial_{\psi})$ |
| The expansion theorem:                           | The First Expansion Theorem:                                    |
| $h(t) = \sum_{k=0}^{\infty} \frac{[h(t)|p_k(x)]}{k!} f(t)^k$ | $T = \sum_{n \geq 0} \frac{[T p_n(z)]|z=0}{n_{\psi}} Q(\partial_{\psi})^n$ |
| $p_n(x)$ - sequence associated for $f(t)$        | $\partial_{\psi}$ - basic polynomial sequence $\{p_n\}_{0}^{\infty}$ |
| $\exp\{y f(t)\} = \sum_{k=0}^{\infty} \frac{p_k(y)}{k!} t^k$ | $\exp_{\psi} \{x Q^{-1}(x)\} = \sum_{k \geq 0} \frac{p_k(y)}{k!} z^k$ |
| The Sheffer Identity:                            | The Sheffer $\psi$-Binomial Theorem:                           |
| $s_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} p_k(y) s_{n-k}(x)$ | $s_n(x + \psi y) = \sum_{k \geq 0} \binom{n}{k} \psi s_k(x) q_{n-k}(y)$ |

| Viskov                                          | Rota - oriented (this note)                                     |
|------------------------------------------------|----------------------------------------------------------------|
| $\theta_{\psi}$ - the $\psi$-derivative         | $\partial_{\psi}$ - the $\psi$-derivative                      |
| $\theta_{\psi} x^n = \frac{\psi_{n-1}}{\psi_n} x^{n-1}$ | $\partial_{\psi} x^n = n_{\psi} x^{n-1}$                       |
| Viskov                                           | Rota - oriented (this note) |
|-------------------------------------------------|-----------------------------|
| $A_p (p = \{p_n\}_0^\infty)$                    | $Q$                         |
| $A_p p_n = p_{n-1}$                             | $Q p_n = n_\psi p_{n-1}$    |
| $B_p (p = \{p_n\}_0^\infty)$                    | $\hat{x} Q$                |
| $B_p p_n = (n+1) p_{n+1}$                       | $\hat{x} Q p_n = \frac{n+1}{(n+1)_\psi} p_{n+1}$ |
| $E^y_p (p = \{p_n\}_0^\infty)$                  | $E^y (\partial_\psi) \equiv \exp_{\psi} \{y \partial_\psi\}$ |
| $E^y_p p_n(x) = \sum_{k=0}^{n} p_{n-k}(x)p_k(y)$ | $E^y (\partial_\psi) p_n(x) = \sum_{k\geq0} \binom{n}{k}_\psi p_k(x)p_{n-k}(y)$ |
| $T - \varepsilon_p$-operator:                   | $E^y$ - shift operator:     |
| $T A_p = A_p T$                                 | $E^y \varphi(x) = \varphi(x + \psi y)$ |
| $\forall y \in F TE^y_p = E^y_p T$             | $\forall \alpha \in F [T, E^\alpha (\partial_\psi)] = 0$ |
| $Q - \delta_\psi$-operator:                     | $Q(\partial_\psi)$ - $\partial_\psi$-delta-operator: |
| $Q - \epsilon_p$-operator and                   | $Q(\partial_\psi)$ - $\partial_\psi$-shift-invariant and |
| $Q x = \text{const} \neq 0$                     | $Q(\partial_\psi)(id) = \text{const} \neq 0$ |
| Viskov | Rota - oriented (this note) |
|--------|---------------------------|
| \( \{p_n(x), n \geq 0\} \) - \((Q, \psi)\)-basic polynomial sequence of the \( \delta_\psi \)-operator \( Q \) | \( \{p_n\}_{n \geq 0} \) - \( \partial_\psi \)-basic polynomial sequence of the \( \delta_\psi \)-delta-operator \( Q(\partial_\psi) \) |
| \( \psi \)-binomiality property | \( \psi \)-binomiality property |
| \( \Psi_y s_n(x) = \) | \( E^y(\partial_\psi)p_n(x) = \) |
| \( = \sum_{m=0}^{n} \frac{\psi_n \psi_{n-m}}{\psi_n} s_m(x)p_{n-m}(y) \) | \( = \sum_{k \geq 0} \binom{n}{k}_\psi p_k(x)p_{n-k}(y) \) |
| \( T = \sum_{n \geq 0} \psi_n [VTp_n(x)]Q^n \) | \( T = \sum_{n \geq 0} \frac{|Tp_n(z)|_{z=0}}{n!} Q(\partial_\psi)^n \) |
| \( T \Psi_y p(x) = \) | \( Tp(x + \psi, y) = \) |
| \( \sum_{n \geq 0} \psi_n s_n(y)Q^n STp(x) \) | \( \sum_{k \geq 0} \frac{s_k(y)}{k!} Q(\partial_\psi)^k STp(x) \) |
| Markowsky                                                                 | Rota - oriented |
|--------------------------------------------------------------------------|-----------------|
| \( L \) - the differential operator                                      | \( Q \)         |
| \( L p_n = p_{n-1} \)                                                    | \( Q p_n = n_\psi p_{n-1} \) |
| \( M \)                                                                  | \( \hat{x} Q \) |
| \( M p_n = p_{n+1} \)                                                    | \( \hat{x} Q p_n = \frac{n+1}{(n+1)_\psi} p_{n+1} \) |
| \( L_y \)                                                                | \( E^y (Q) = \sum_{k \geq 0} \frac{p_k(y)}{k_\psi^k} Q^k \) |
| \( L_y p_n(x) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y) \) | \( E^y (Q) p_n(x) = \sum_{k \geq 0} \binom{n}{k}_\psi p_k(x) p_{n-k}(y) \) |
| \( E^a \) - shift-operator:                                               | \( E^y \) - \( \partial_\psi \)-shift operator: |
| \( E^a f(x) = f(x+a) \)                                                  | \( E^y \varphi(x) = \varphi(x+\psi y) \) |
| \( G \) - shift-invariant operator:                                      | \( T \) - \( \partial_\psi \)-shift invariant operator: |
| \( EG = GE \)                                                            | \( \forall \alpha \in F \ [T, E(Q)] = 0 \) |
| \( G \) - delta-operator:                                                | \( L = L(Q) - Q_\psi \)-delta operator: |
| \( Gx = \text{const} \neq 0 \)                                           | \([L, Q] = 0 \) and |
| \( L(id) = \text{const} \neq 0 \)                                        | \( L(id) = \text{const} \neq 0 \) |
| Markowsky | Rota - oriented |
|-----------|----------------|
| \( D_L(G) \) | \( G' = [G(Q), \hat{x}_Q] \) |
| \( L \)- Pincherle derivative of \( G \) | \( Q \)- Pincherle derivative |
| \( D_L(G) = [G, M] \) | \( \{p_n\}_{n \geq 0} \)-\( \psi \)-basic polynomial sequence of the generalized difference operator \( Q \) |
| \( \{Q_0, Q_1, \ldots\} \)- basic family for differential operator \( L \) | \( \{p_n\}_{n \geq 0} \)-\( \psi \)-basic |
| Binomiality property | \( Q \)-\( \psi \)-binomiality property |
| \( P_n(x + y) = \sum_{i=0}^{n} \binom{n}{i} P_i(x) P_{n-i}(y) \) | \( E^y(Q)p_n(x) = \sum_{k \geq 0} \binom{n}{k} \psi_p k(x)p_{n-k}(y) \) |

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