PROXIMAL ESTIMATION AND INference

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Abstract

We develop a unifying convex analysis framework characterizing the statistical properties of a large class of penalized estimators. Our framework interprets penalized estimators as proximal estimators, defined by a proximal operator applied to a corresponding initial estimator. We characterize the asymptotic distribution of proximal estimators with a closed-form formula depending only on (i) the asymptotic distribution of the initial estimator, (ii) the estimator’s limit penalty subgradient and (iii) the inner product defining the associated proximal operator. In parallel, we characterize the Oracle features of proximal estimators from the properties of their penalty subgradients. Using our approach, we systematically cover linear regression settings with a regular or irregular design, and we develop new $\sqrt{\alpha}$-consistent, asymptotically normal proximal estimators featuring the Oracle property.

Keywords: Penalized estimator, Ridgeless estimator, asymptotic distribution, oracle property, nearly-singular design, proximal operator, Moreau decomposition, subgradient, convex conjugation, epigraph convergence.
1 Introduction

This paper builds a unifying convex analysis framework for studying with a single approach the statistical properties of a large class of penalized estimators. Our theory borrows from a reinterpretation of penalized estimators as proximal estimators, which are defined by a proximal operator applied to a suitable initial estimator.

We characterize the asymptotic behaviour of proximal estimators, based on closed-form asymptotic distributions that only depend on (i) the asymptotic distribution of the initial estimator, (ii) the estimator's limit penalty subgradient and (iii) the inner product defining the underlying proximal operator. In parallel, we determine the Oracle (Fan and Li (2001)) properties of proximal estimators, via the properties of the sequence of their penalties’ subgradients. We then exploit our general proximal estimation theory to systematically cover linear regression settings with a regular, singular or nearly-singular design. For the latter setting, we build new $\sqrt{n}$-consistent, asymptotically normal proximal estimators, which possess the Oracle property and perform satisfactorily in practically relevant Monte Carlo simulations.

To motivate our approach in the simplest setting, consider a parameter of interest $\beta_0 \in \mathbb{R}^p$ and a linear regression model of the form:

$$Y = X\beta_0 + \varepsilon,$$  \hspace{1cm} (1)

where random variables $X$ and $\varepsilon$, defined on probability space $(\Omega, \mathcal{F}, \mathbb{P})$, take values in $\mathbb{R}^{n \times p}$ and $\mathbb{R}^n$, respectively. A Penalized Least Squares Estimator (PLSE) for parameter $\beta_0$ is defined by the solution of a penalized Least Squares problem of the form:

$$\hat{\beta}_n := \arg\min_{\beta \in \mathbb{R}^p} \left\{ \frac{1}{2n} \| Y - X\beta \|^2_2 + \lambda_n f_n(\beta) \right\},$$  \hspace{1cm} (2)

where $\| \cdot \|_2$ is the $l_2$–norm in $\mathbb{R}^n$ induced with Euclidean inner product $(\cdot, \cdot)$, $f_n$ a (possibly stochastic) penalty function and $\lambda_n > 0$ a penalty parameter. Obviously, the well-known Least Squares Estimator (LSE) arises in problem (2) when $\lambda_n f_n = 0$. Examples of estimators of the form (2) defined by a non-stochastic penalty $\lambda_n f_n$ are Ridge (Hoerl and Kennard (1970)), Lasso (Tibshirani (1996)), Elastic Net (Zou and Hastie (2005)) and Group Lasso (Yuan and Lin (2006)). Examples of PLSEs defined by a stochastic penalty are Adaptive Lasso (Zou (2006)) and Adaptive Elastic Net (Zou and Zhang (2009)).

Under a regular design, sample design matrix $X'X/n$ is almost surely positive definite and it is well-known that the LSE is the best unbiased linear estimator of parameter $\beta_0$. Likewise, under a regular design PLSEs are well-defined for a wide class of penalties $f_n$. They can be applied, e.g., to improve on the bias-variance tradeoff of LSEs or to perform Oracle variable selection. Under a singular or nearly-singular design, the sample design matrix is either almost surely singular, or still almost surely positive definite but converging in probability to a singular population design matrix as the sample size grows. In either case, the LSE may either not even exist or not give rise to a well-behaved asymptotic distribution, due to a lack of parameter identification.\footnote{Early works on (a non-asymptotic view of) nearly-singular designs can be found, e.g., in Vinod (1976) Stewart (1987), Stone and Brooks (1990) and Sundberg (1993). More recent applications involving nearly-singular design can be found, e.g., in Phillips (2001), Gabaix and Ibragimov (2011), Aswani et al. (2011) and Phillips (2016).}

Conversely, the PLSE’s asymptotic distribution may either not exist or imply a slower rate of convergence than $1/\sqrt{n}$, as highlighted, e.g., in Knight and Fu (2000) and Knight (2008) for the family of Bridge–type estimators under a deterministic nearly-singular design. \footnote{Caner (2008) shows that similar slow rate of convergences apply in nearly-singular settings to GMM and GEL estimators.} Therefore, the
construction of well-behaved \(\sqrt{n}\)-consistent, asymptotically normal LS–type estimators and corresponding PLSEs under an irregular design is a key open problem in this literature.

To introduce our proximal estimation approach, let \(\beta^*_n\) be an initial estimator of the given parameter of interest \(\beta_0\), \(W_n\) a (possibly stochastic) symmetric positive definite weighting matrix, and \(\|\cdot\|_{W_n}\), the norm in \(\mathbb{R}^p\) induced by inner product \(\langle \cdot, \cdot \rangle_{W_n} := \langle W_n \cdot, \cdot \rangle\).

A proximal estimator is then defined as follows:

\[
\hat{\beta}_n := \text{prox}_{\lambda_n f_n}^{W_n} (\beta^*_n) := \arg\min_{\beta \in \mathbb{R}^p} \left\{ \frac{1}{2} \| \beta^*_n - \beta \|^2_{W_n} + \lambda_n f_n(\beta) \right\}.
\]  

Here, \(\text{prox}_{\lambda_n f_n}^{W_n}(\beta^*_n)\) is the proximal operator associated with proximal estimator \(\hat{\beta}_n\). The range of possible values of this proximal operator is given by the solutions of a family of penalized minimum distance problems, while its valuation at the initial estimator \(\beta^*_n\) yields the proximal estimator \(\hat{\beta}_n = \text{prox}_{\lambda_n f_n}^{W_n} (\beta^*_n)\).

By design, estimation approach (3) covers a large class of penalized estimators, which are induced by corresponding choices of initial estimator \(\beta^*_n\) for linear or nonlinear models, such as M- or GMM-type estimators. We show in Section 2 that PLSEs of the form (2) are proximal estimators, depending on the LSE as initial estimator and the sample design matrix as weighting matrix. Therefore, proximal estimators offer a natural nesting framework including PLSEs and many other important estimators. Crucially, the tight link between proximal estimators and proximal operators gives rise to a unifying theoretical framework, which covers both regular and irregular design settings, for studying the properties of all these estimators with powerful convex analysis tools. It also highlights the key sources of failure of PLSEs under an irregular design, which we show can be naturally overcome with appropriate choices of initial linear estimators and weighting matrices in optimization problem (3).

Our unifying theoretical framework covers proximal estimators defined by penalties in the class \(\Gamma(\mathbb{R}^p)\) of convex, lower-semicontinuous and proper functions taking values in \((-\infty, \infty)\).\footnote{We allow penalty function \(f_n\) in equation (3) to be extended real-valued, in order to (i) naturally accommodate adaptive penalties that may converge to an extended real-valued limit penalty as \(n \to \infty\) and (ii) embed penalties modelling convex parameter constraints with the characteristic function of a set in constrained Least Squares estimation problems (Liew (1976)). While part of our asymptotic results can be derived also for non convex penalties (e.g., quasi convex penalties), part of our asymptotic characterizations intrinsically rely on Moreau’s decomposition (see Bauschke et al. (2016, Thm. 14.3)) and convex duality arguments, thus requiring convexity.} For this class of penalties, we characterize the asymptotic distribution of proximal estimators using suitable functional operations applied to the estimator’s limit penalties. A key such functional operation is convex conjugation, which for functions \(f \in \Gamma(\mathbb{R}^p)\) and inner product \(\langle \cdot, \cdot \rangle_{W_n}\) defines convex conjugate \(f^* \in \Gamma(\mathbb{R}^p)\) as:

\[
f^*(\theta) := \sup_{\beta \in \mathbb{R}^p} \{ \langle \theta, \beta \rangle_{W_n} - f(\beta) \}.
\]

We find that convex conjugation directly provides important insights into the properties of a proximal estimator. We show in Section 2, using Moreau’s decomposition (see Bauschke et al. (2016, Thm. 14.3)), that any proximal estimator is equivalently given by a dual proximal estimator defined with conjugate penalty \((\lambda_n f_n)^*\). For proximal estimators defined by a sublinear penalty, the dual proximal estimator is simply the projection on a corresponding closed convex set. This result extends well-known soft-thresholding formulas holding for Lasso and similar PLSEs under an orthonormal design, where they can be computed in closed-form as residual projections of the LSE on particular polyhedra, to any regular design.

Concerning the asymptotic properties of proximal estimators, we obtain in Section 3 two main functional characterizations of their asymptotic distribution. In each of these
dual characterizations, the asymptotic distribution follows by applying a limit proximal operator to a random variable reproducing the asymptotic distribution of the given initial estimator. The first limit operator is the proximal operator of the directional derivative of the limit penalty at the parameter of interest. The second limit proximal operator computes the residual of a projection on the limit penalty subgradient at the parameter of interest. These findings hold for penalties of class $\Gamma(\mathbb{R}^p)$ under fairly simple and weak high-level conditions, and, when applied to linear regression models, they are applicable to both regular and irregular design settings.\footnote{These conditions ensure (i) existence of a well-behaved asymptotic distribution for the given initial estimator, (ii) a positive definite probability limit for the sequence of weighting matrices $W_n$, and (iii) a well-defined limit in epigraph for the sequences of penalties $\lambda_n f_n$; See, e.g., Salinetti and Wets (1981) and Salinetti and Wets (1986) for definitions and properties of epigraph convergence in distribution and in probability.}

Regarding the Oracle properties of proximal estimators, in Section 3 we further provide high-level necessary and sufficient conditions using our convex analysis framework, which are applicable to both regular and irregular design settings.\footnote{The Oracle efficient asymptotic distribution property of any penalized estimator holds pointwise and not uniformly over the data generating process; see Leeb and Pötscher (2008) and references thereof.} The necessary condition for a consistent variable selection implies an unbounded subgradient of the associated limit penalty in the directions of the inactive components of the parameter of interest. It is violated, e.g., by the Lasso penalty. The sufficient condition for a consistent variable selection ensures a uniform separation from the origin of the active penalties’ subgradient components that are associated with incorrect selections of the parameter of interest. It is satisfied, e.g., by Adaptive Lasso and Adaptive Elastic net-type penalties. Finally, the sufficient condition for an Oracle asymptotic distribution is phrased in terms of simple constraints between a proximal estimator’s limit weighting matrix and the asymptotic covariance matrix of an underlying efficient initial estimator.

Using our framework based on proximal estimators, we systematically build in Section 4 a widely applicable Oracle estimation method for linear regression models that is robust to a potential design irregularity. We overcome the lack of identification of these settings, by considering as a parameter of interest the Ridgeless Least Squares population parameter, i.e., the minimum norm Least Squares population parameter. We further build a $\sqrt{n}$-consistent, asymptotically normal initial estimator of this parameter of interest, which is robust to design irregularities and asymptotically equivalent to the LSE under a regular design. Here, it may appear natural to consider as an initial estimator the Ridgeless Least Squares estimator, which recently gained considerable new attention because of its generalization properties in high dimensional linear regression settings.\footnote{This estimator is called the Intrinsic Estimator in applications coping with inherently singular linear settings in Age-Period-Cohort problems; see, e.g., Fu (2000) and Yang et al. (2008). Its generalization properties in high-dimensional settings are recently studied in Hastie et al. (2019), Bartlett et al. (2020) and Richards et al. (2021).} However, this estimator is not $\sqrt{n}$-consistent under a nearly-singular design. Therefore, we introduce a modified Ridgeless estimator, which we show is asymptotically normal with a standard $1/\sqrt{n}$ convergence rate irrespective of the underlying design regularity. Oracle proximal estimators robust to irregular designs are then built by applying to our modified Ridgeless estimator proximal operators with a penalty satisfying the sufficient high-level Oracle conditions derived in Section 3.

Finally, all proofs of the mathematical results are collected in the Appendix, while additional auxiliary findings and a simulation analysis of proximal estimators are collected in the Supplementary Material.
2 Basic properties of proximal estimators

For the large family of penalties $f_n \in \Gamma(\mathbb{R}^p)$, proximal estimators in definition (3) are well-defined. Moreover, their associated proximal operator is almost everywhere differentiable and it satisfies an equivalent dual representation that is particularly insightful for the subfamily of sublinear penalties. These properties are collected in the next proposition.

**Proposition 2.1** (Basic properties of proximal estimators). Let initial estimator $\hat{\beta}^*_n$ in definition (3) be well-defined and weighting matrix $W_n$ be positive definite. Let further penalty $f_n$ be of class $\Gamma(\mathbb{R}^p)$. Then, proximal estimator $\text{prox}_{\lambda_n f_n} W_n \hat{\beta}^*_n$ exists and is unique. Its associated proximal operator $\text{prox}_{\lambda_n f_n} W_n$ is Lebesgue almost surely differentiable on a set of $\mathbb{P}$ probability one, and such that:

$$\text{prox}_{\lambda_n f_n} W_n = \text{Id} - \text{prox}_{\lambda_n f_n} W_n^* .$$

(4)

If furthermore penalty $f_n$ is sublinear, then identity (4) reads:

$$\text{prox}_{\lambda_n f_n} W_n = \text{Id} - P_{W_n C_n} ,$$

(5)

with closed convex set

$$C_n = \bigcap_{\beta \in \mathbb{R}^p} \{ \theta \in \mathbb{R}^p : \langle \theta, \beta \rangle W_n \leq \lambda_n f_n(\beta) \} ,$$

(6)

and projection operator $P_{W_n C_n} : \mathbb{R}^p \to \mathbb{R}^p$ defined by:

$$P_{W_n C_n}(\theta) := \arg \min_{\theta' \in \mathbb{R}^p} \left\{ \| \theta - \theta' \|_{W_n} : \theta' \in C_n \right\} .$$

(7)

According to Proposition 2.1, proximal estimators are well-defined for a large class of penalties and they arise from a differentiable transformation of initial estimator $\hat{\beta}^*_n$. Using identity (4), such differentiable transformation can be equivalently written as a proximal operator of penalty $\lambda_n f_n$ or as a residual of a proximal operator of conjugate penalty $(\lambda_n f_n)^*$. The latter representation is particularly transparent for the subfamily of sublinear penalties, because it reproduces a proximal estimator by the residual of a projection of the initial estimator on set $C_n$ in equation (6).

2.1 Benchmark sublinear penalties

Table 1 in the Supplementary Material collects closed-form examples of conjugate penalties $(\lambda_n f_n)^*$ for many well-known penalties in the literature, including the Lasso, Adaptive Lasso, Group Lasso, and Elastic Net penalties, as well as for penalties defining constrained proximal estimators. Among these penalties, the Lasso, Group Lasso and Adaptive Lasso penalties are sublinear. Hence, they give rise to a projection formula with a corresponding closed-form set $C_n$ in equation (7), which is in both cases a polyhedron.

**Proposition 2.2** (Lasso and Adaptive Lasso penalties). For the Lasso and Adaptive Lasso penalties, set $C_n$ in equation (7) is explicitly given as detailed below.

(i) **Lasso penalty.** Let penalty $f_n$ be defined by $f_n(\beta) = \| \beta \|_1$. It then follows:

$$C_n = \{ \theta : \| W_n \theta \|_{\infty} \leq \lambda_n \} = \bigcap_{j=1}^p \{ \theta : \langle e_j, \theta \rangle W_n \leq \lambda_n \} ,$$

(8)

where $e_j$ denotes the $j$-th unit vector in $\mathbb{R}^p$.

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Note: Convex conjugate penalty $(\lambda_n f_n)^*$ in equation (4) is defined under inner product $\langle \cdot, \cdot \rangle_{W_n}$. It is equivalently given by a penalty of the form $\lambda_n f_n^*(W_n \cdot / \lambda_n)$, where $f_n^*$ is the convex conjugate of penalty $f_n$ under the Euclidean inner product.
(ii) **Adaptive Lasso penalty.** Let penalty \( f_n \) be defined by \( f_n(\beta) = \sum_{j=1}^p |\beta_j|/|\hat{\beta}_{nj}| \), for some auxiliary estimator \( \hat{\beta}_n \). It then follows:

\[
C_n = \left\{ \theta : \left\| \hat{\beta}_n \circ W_n \theta \right\|_{\infty} \leq \lambda_n \right\} = \bigcap_{j=1}^p \left\{ \theta : |\langle e_j, \theta \rangle W_n| \leq \frac{\lambda_n}{|\hat{\beta}_{nj}|} \right\},
\]

where \( \circ \) denotes the Hadamar product.

Polyhedral set \( C_n \) in equations (8)–(9) is just an intersection of half-spaces defined with inner product \( \langle \cdot, \cdot \rangle_{W_n} \). Therefore, the special case of an identity weighting matrix \( W_n = I_n \) gives rise to established soft-thresholding formulas in the literature for Lasso and Adaptive Lasso estimators.

### 2.2 Ordinary LS–type initial estimators

In addition to penalty \( f_n \) and weighting matrix \( W_n \), the third building block of a proximal estimator is the initial estimator \( \hat{\beta}_n^I \). Naturally, our framework is applicable to a wide range of well-behaved initial estimators, such as several linear estimators of the parameter of interest in linear regression model (1), or M– and GMM–type estimators in more general moment condition models. When focusing on linear estimators for linear model (1), it turns out that any PLSE of the form (2) can be written as a proximal estimator defined with a linear initial estimator and a corresponding inner product. We make this statement precise in the next proposition, which covers both regular and irregular designs.

**Proposition 2.3.** In linear model (1), let \( Q_n := X'X/n \) and \( \hat{\beta}_n^I := A_n^+ X'Y/n \), where \( A_n^+ \) is the Moore-Penrose inverse of a \( X \)–measurable, symmetric and positive semi-definite \( p \times p \) random matrix \( A_n \). Let further \( \text{prox}_{\lambda_n f_n} (\beta_n^I) \) be a proximal estimator defined with positive definite weighting matrix \( \bar{A}_n := A_n + (I_n - A_n A_n^+) \). It then follows, for any penalty \( f_n \in \Gamma(\mathbb{R}^p) \):

\[
\text{prox}_{\lambda_n f_n} (\beta_n^I) = \arg\min_{\beta \in \mathbb{R}^p} \left\{ \frac{1}{2n} \| Y - X \beta \|_2^2 + \lambda_n \bar{f}_n(\beta) \right\},
\]

with an extended penalty \( \bar{f}_n \in \Gamma(\mathbb{R}^p) \) defined by \( \bar{f}_n(\beta) := f_n(\beta) + \frac{1}{2n} \beta'(\bar{A}_n - Q_n)\beta \), if and only if:

\[
\text{Kernel}(A_n) \subset \text{Kernel}(Q_n)
\]

and matrix \( \bar{A}_n - Q_n \) is positive semi-definite.

In Proposition 2.3, setting \( A_n \) equal to the sample design matrix \( Q_n \) yields as an initial estimator the Ridgeless LSE

\[
\beta_n^I := Q_n^+ X'Y/n = \arg\min_{\beta \in \mathbb{R}^p} \{ \| \beta \|_2 : Q_n \beta = X'Y/n \},
\]

i.e., the minimum norm solution of the Least Squares sample optimality conditions. Under a regular or a nearly-singular design, regularity of the sample design matrix implies \( Q_n^+ = Q_n^{-1} \), \( Q_n = Q_n \) and \( \bar{f}_n = f_n \). Hence, the Ridgeless LSE is the Ordinary LSE and any PLSE (2) is equivalently defined by proximal estimator \( \text{prox}_{\lambda_n f_n} (\beta_n^I) \). Under a singular design, proximal estimator \( \text{prox}_{\lambda_n f_n} (\beta_n^I) \) equals a PLSE defined with the extended penalty \( \bar{f}_n \neq f_n \) in equation (2). This extended penalty ensures that the corresponding proximal estimator is well-defined for any penalty \( f_n \in \Gamma(\mathbb{R}^p) \), by additionally penalizing parameter vectors in the kernel of matrix \( Q_n \).

\[\text{Proposition 2.3 gives rise to further useful links between PLSEs and other established linear initial estimators, which imply corresponding projection formulas for PLSEs such as the Elastic Net and Adaptive Elastic Net estimators. See Section 2 of the Supplementary Material for a detailed derivation of these links.}\]
Remark 1. While proximal estimator \( \text{prox}_{\lambda_n f_n}(\hat{\beta}_n^s) \) is always well-defined, its asymptotic properties are inherently different under a regular or irregular design. Under a regular or singular design, the asymptotic distribution follows with a standard \( 1/\sqrt{n} \) convergence rate from the (normal) asymptotic distribution of the Ridgeless LSE. Under a nearly-singular design, both estimator \( \hat{\beta}_n^s \) and proximal estimators built on it do not converge with a standard \( 1/\sqrt{n} \) convergence rate. We overcome this issue in Section 4, with a new family of proximal estimators implied by corresponding choices of matrix \( A_n \neq Q_n \) and initial estimator \( \hat{\beta}_n^s \neq \hat{\beta}_n^f \). Under a regular and a singular design, these estimators are asymptotically equivalent to proximal estimators built with initial estimator \( \hat{\beta}_n^s \) and weighting matrix \( Q_n \). In addition, they give rise to a well-defined asymptotic distribution with \( 1/\sqrt{n} \) convergence rate under a nearly-singular design, thus producing asymptotic properties and convergence rates robust to potential design irregularities.

3 Asymptotic properties of proximal estimators

We characterize with a functional convex analysis approach the asymptotic properties of proximal estimators under a fixed \( p \) and large \( n \) asymptotics.

3.1 Preliminaries

Consistency of a proximal estimator naturally follows from corresponding high-level assumptions on the consistency of initial estimator \( \hat{\beta}_n^s \) and weighting matrix \( W_n \) for a corresponding parameter of interest and a well-behaved limit weighting matrix, respectively.

Assumption 1. There exists \( \beta_0 \in \mathbb{R}^p \) and positive definite matrix \( W_0 \) such that \( \hat{\beta}_n^s \rightarrow_{\text{Pr}} \beta_0 \) and \( W_n \rightarrow_{\text{Pr}} W_0 \).

Further, penalties \( \lambda_n f_n \) are required to converge to a limit penalty, under a notion of convergence ensuring convergence of the sequences of minimizers of objective functions of class \( \Gamma(\mathbb{R}^p) \), i.e., epigraph convergence.\(^9\)

Assumption 2.

(i) There exists proper function \( f_0 : \mathbb{R}^p \rightarrow (-\infty, \infty] \) such that \( f_n \rightarrow_{\text{Pr}} f_0 \) in epigraph;

(ii) \( \beta_0 \in \bigcap_{n \in \mathbb{N}} (\text{dom}(f_0) \cap \text{dom}(f_n)) \).

Assumption 1 is a weak one, which can be satisfied by appropriate choices of initial estimators and weighting matrices, e.g., in a nearly-singular linear regression setting with the modified Ridgeless LSE of Section 4. Assumption 2 is weak as well, and satisfied by all benchmark penalties in Table 1 of the Supplementary Material. For instance, the limit penalty of the Adaptive Lasso penalties from Proposition 2.2, based on a consistent auxiliary estimator \( \tilde{\beta}_n^s \) of parameter \( \beta_0 \), is given in closed-form by:

\[
f_0(\beta) = \sum_{j=1}^{p} \left[ \frac{|\beta_j|}{|\beta_0|} I(\beta_0 j \neq 0) + \iota_0(\beta_j) I(\beta_0 j = 0) \right],
\]

with indicator function \( I(\cdot) \) and where \( \iota_0(\beta_j) = 0 \) if \( \beta_j = 0 \) and \( \iota_0(\beta_j) = \infty \) else. This extended real-valued penalty in class \( \Gamma(\mathbb{R}^p) \) has domain:

\[
\text{dom}(f_0) = \text{span}\{e_j : \beta_0 j \neq 0\}.
\]

\(^9\)See, e.g., Rockafellar and Wets (2009) for the deterministic case, and Salinetti and Wets (1981), Salinetti and Wets (1986), Geyer (1994) and Knight (1999) for the stochastic case. Assumption 2 also implies epigraph convergence in probability of conjugate penalty \( f_n^* \) to limit penalty \( f_0^* \), since epigraph convergence is preserved under convex conjugation Mosco (1971, Thm. 1) and \( f = f^{**} \) for any \( f \in \Gamma(\mathbb{R}^p) \) Bauschke et al. (2016, Cor. 13.38).
3.2 Asymptotic distribution

From conjugate formula (4) in Proposition 2.1, we characterize the joint asymptotic distribution of proximal estimator \( \text{prox}_{(\lambda_n f_n)}^W (\beta_n^*) \) and its conjugate proximal estimator.\(^\text{10}\) For the first of these distributions, we make use of the directional derivative of limit penalty \( f_0 \) at \( \beta_0 \), defined for \( b \in \mathbb{R}^p \) by:

\[
\rho_{\beta_0}(b) := \lim_{\alpha \downarrow 0} \frac{f_0(\beta_0 + \alpha b) - f_0(\beta_0)}{\alpha}.
\]

For the second distribution, we use the subgradient of \( f_0 \) at \( \beta_0 \), defined under the limit inner product \( \langle \cdot, \cdot \rangle_W \):

\[
\partial f_0(\beta_0) := \bigcap_{\beta \in \mathbb{R}^p} \{ t \in \mathbb{R}^p : f_0(\beta) - f_0(\beta_0) - \langle \beta - \beta_0, t \rangle_W \geq 0 \}.
\]

**Remark 2.** Under Assumption 2, \( \rho_{\beta_0} \) and \( \partial f_0(\beta_0) \) are well-defined, i.e., \( \rho_{\beta_0} \) exists in the extended-real line and \( \partial f_0(\beta_0) \) is nonempty (see Bauschke et al. (2016, Thm. 17.2 and Remark 16.2)). These two objects are related by the identity (see Bauschke et al. (2016, Prop. 17.17)) \( \rho_{\beta_0} = (\partial f_0(\beta_0))^* = \sigma_{\partial f_0(\beta_0)} \), i.e., the directional derivative of \( f_0 \) at \( \beta_0 \) is the support function of set \( \partial f_0(\beta_0) \). Moreover, for any proper closed convex set \( C \), the directional derivative at \( \beta_0 \) of characteristic function \( f_0 = c_C \) is the support function of the normal cone of set \( C \) at \( \beta_0 \), defined by:\(^\text{11}\)

\[
N_C(\beta_0) := \left\{ \theta : \sup_{z \in C} \langle \theta, z - \beta_0 \rangle_{W_0} \leq 0 \right\} \quad \beta_0 \in C,
\]

otherwise.

Tables 2 and 3 in the Supplementary Material report the closed-form directional derivative and subgradient expressions for benchmark limit penalties.

The third high-level assumption needed to derive the asymptotic distribution of proximal estimators ensures existence of an asymptotic distribution of initial estimator \( \hat{\beta}_n^* \) under a corresponding convergence rate.

**Assumption 3.** For some some random vector \( \eta \) and rate \( r_n \) such that \( r_n \to \infty \) as \( n \to \infty \), \( r_n \left( \hat{\beta}_n^* - \beta_0 \right) \to d \eta \).

The next theorem characterizes the asymptotic distribution of proximal estimators and their conjugate estimators under the above assumptions.

**Theorem 3.1.** Let Assumptions 1–3 hold. It then follows:

(i) If \( \lambda_n r_n \to \lambda_0 > 0 \), then:

\[
r_n \left( \text{prox}_{\lambda_n f_n}^W (\hat{\beta}_n^*) - \beta_0 \right) \to d \text{prox}_{\lambda_0 \rho_{\beta_0}} W_0 (\eta),
\]

with

\[
\text{prox}_{\lambda_0 \rho_{\beta_0}} W_0 (\eta) = \left( I_d - P_{\lambda_0 \partial f_0(\beta_0)} W_0(\eta) \right),
\]

(ii) If \( \lambda_n r_n \to \lambda_0 = 0 \) and \( \lambda_n r_n f_n \to \nu_{\text{dom}(f_0)} \) in epigraph, the above limits hold with \( \lambda_0 \rho_{\beta_0} \) and \( \lambda_0 \partial f_0(\beta_0) \) replaced by \( \sigma_{N_{\text{dom}(f_0)}(\beta_0)} \) and \( N_{\text{dom}(f_0)}(\beta_0) \), respectively.

\(^{10}\)Recall that we use the convention of defining the convex conjugate of a finite sample penalty \( \lambda_n f_n \) with the sample inner product \( \langle \cdot, \cdot \rangle_W \).

\(^{11}\)This directional derivative expression follows from Bauschke et al. (2016, Example 16.13) and Bauschke et al. (2016, Prop. 17.17), using the identities \( \rho_{\beta_0} = \sigma_{\partial C(\beta_0)} = \sigma_{N_C(\beta_0)} \).
The asymptotic distribution in the first row of limit (16) is the distribution of a proximal operator applied to random vector \( \eta \), with penalty \( \lambda_0 \rho \beta_0 \) or \( \sigma_{\text{dom}(f_0)}(\beta_0) \) under asymptotic regimes (i) or (ii), respectively. Equivalently, in equation (17) it is the distribution of the residual of a projection of random vector \( \eta \) onto the subgradient \( \lambda_0 \partial f(\beta_0) \) or \( N_{\text{dom}(f_0)}(\beta_0) \).

For instance, for the adaptive Lasso penalty (13) the asymptotic distribution under asymptotic regime (ii) with \( r_n = \sqrt{n} \) follows using Bauschke et al. (2016, Prop. 6.22) as the distribution of the Gaussian projection:

\[
\text{prox}_{\sigma_{N_{\text{dom}(f_0)}(\beta_0)}}(\eta) = \bigg( (1 - P_{W_0,\beta_0}^\perp) \bigg) \eta = P_{W_0,\beta_0}^{\perp} \eta,
\]

with \( S^\perp \) the orthogonal complement of \( S \subset \mathbb{R}^p \) under inner product \( \langle \cdot, \cdot \rangle_{W_0} \).

### 3.3 Oracle property

The Oracle property (Fan and Li (2001)) concerns the ability of an estimator to consistently select and efficiently estimate the nonzero components of parameter vector \( \beta_0 \). In our setting, the Oracle selection property can be studied under high-level Assumptions 1–3. To study the Oracle efficient distribution property, we introduce following asymptotic Gaussian distribution assumption.

**Assumption 4.** \( r_n (\hat{\beta}_n - \beta_0) \rightarrow_d M_0^+ Z \), for some \( p \times p \) positive semi-definite matrix \( M_0 \) and a normally distributed random vector \( Z \sim N(0, \Omega_0) \) with positive semi-definite covariance matrix \( \Omega_0 \).

Assumption 4 covers irregular designs, since matrices \( M_0 \) and \( \Omega_0 \) are not required to be regular. It is satisfied under standard regularity conditions by, e.g., many M– and GMM–like initial estimators.\(^{12}\)

Using the notation \( \mathcal{A} = \{ j : \beta_{0j} \neq 0 \} \) to denote the active set and \( (x)_{\mathcal{A}} \) \((M)_{\mathcal{A}}\) to denote the subvector [submatrix] of vector \( x \) [matrix \( M \)] with components [rows and columns] indexed by the active set, the Oracle selection property can be formulated in our framework as follows.

**Definition 3.2** (Oracle property). Proximal estimator (3) satisfies the Oracle property if:

1. It yields consistent variable selection: \( \lim_{n \to \infty} \mathbb{P}(\hat{\mathcal{A}}_n = \mathcal{A}) = 1 \), where:
   \[
   \hat{\mathcal{A}}_n := \left\{ j : \left( \text{prox}_{\lambda_0,\beta_0} (\hat{\beta}_n^*) \right)_j \neq 0 \right\}.  
   \]

2. The proximal estimator’s asymptotic distribution for estimating the active components is given by:
   \[
   (Id - P_{B_0}^\perp)(\eta)_{\mathcal{A}} = [(M_0)_{\mathcal{A}}]^+ (Z)_{\mathcal{A}},
   \]
   where \( B_0 = \lambda_0 \partial f(\beta_0) \) or \( B_0 = N_{\text{dom}(f_0)}(\beta_0) \), under asymptotic regimes (i) or (ii) in Theorem 3.1, respectively.

The next proposition provides an easily verifiable necessary condition for Oracle property 1.

\(^{12}\)For the linear regression setting (1), under a regular or a singular design, Assumption 3 is satisfied in stationary environments by Ridgeless LSE \( \hat{\beta}_n^* = Q_n^{\perp} X' Y / n \), using positive semi-definite matrices \( M_0 = E[X' X / n] \) and \( \Omega_0 = \text{Var}(X' \epsilon / \sqrt{n}) \). However, Assumption 4 is not satisfied by the Ridgeless LSE under a nearly-singular design.
Proposition 3.1 (Necessary condition for Oracle property 1). Let Assumptions 1–3 be satisfied and asymptotic regimes (i) or (ii) in Theorem 3.1 hold. If Oracle Property 1 holds then:

\[
P ( \eta )_{\mathcal{A}^c} = \left( \begin{array}{c} \mathbf{p}^{W_0} (\eta ) \end{array} \right)_{\mathcal{A}^c} = 1 . \tag{21}
\]

Condition (21) is connected to the subgradient of the limit penalty of a proximal estimator. Whenever random vector \( \eta \) has support containing at least a nontrivial subspace of \( \mathbb{R}^p \), it shows that proximal estimators giving rise to limit penalties with bounded subgradients, as it is the case for the Lasso estimator, do not perform consistent variable selection if \( \lambda_n r_n \rightarrow \lambda_0 \geq 0 \) as \( n \rightarrow \infty \).

Sufficient conditions for Oracle property 1 impose constraints on the sequence of penalties of a proximal estimator, which enable an asymptotic identification of the zero components in vector \( \beta_0 \). They restrict the final sample behaviour of the subgradient of the proximal estimator’s penalty under the Euclidean inner product, defined by:

\[
\partial^e f_n (\beta) := \bigcap_{\beta \in \mathbb{R}^p} \left\{ t \in \mathbb{R}^p : f_n (\beta) - f_n (\beta_0) - (\beta - \beta_0, t) \geq 0 \right\} . \tag{22}
\]

Proposition 3.2 (Sufficient condition for Oracle property 1). Let \( \hat{\beta}_n := \text{prox}_{\lambda_n f_n} (\tilde{\beta}_n^*) \) where \( \tilde{\beta}_n^* - \beta_0 = O_P (1 / r_n) \) for positive rate \( r_n \) such that \( r_n \rightarrow \infty \) as \( n \rightarrow \infty \), and suppose that Assumption 1 holds. Let further \( v_n^{\text{opt}} \in \lambda_n \partial^e f_n (\hat{\beta}_n) \) be the subgradient random vector such that \( v_n^{\text{opt}} = W_n (\hat{\beta}_n^* - \beta_0) \). Then, Oracle property 1 holds if as \( n \rightarrow \infty \):

\[
r_n \left\| (v_n^{\text{opt}})_{\mathcal{A}^c} \right\|_1 \rightarrow P_r \infty , \tag{23}
\]

whenever \( \mathcal{A}^c \neq \emptyset \).

Condition (23) enables a correct asymptotic identification of the active components in vector \( \beta_0 \). In the proof we show that it essentially does so by ensuring that the sequence of optimal subgradient subvectors \( (v_n^{\text{opt}})_{\mathcal{A}^c} \) of a proximal estimator is asymptotically incompatible with an incorrect selection of the nonzero elements of \( \beta_0 \). For instance, using the closed-form Euclidean subgradient of the Adaptive Lasso penalty in Proposition 2.2, we obtain:

\[
\partial^e f_n (\beta) = \bigcap_{j=1}^p P_{nj} (\beta) , \tag{24}
\]

with set \( P_{nj} (\beta) \subset \mathbb{R}^p \) given by:

\[
P_{nj} (\beta) = \left\{ \begin{array}{c} \{ t : t_j = \frac{\text{sign}(\beta_j)}{|\beta_j|} \} ; \beta_j \neq 0 \\ \{ t : t_j \in \left[ -\frac{1}{|\beta_n j|}, \frac{1}{|\beta_n j|} \right] \} ; \beta_j = 0 \end{array} \right. . \tag{25}
\]

This gives as \( n \rightarrow \infty \), whenever \( \mathcal{A}^c \neq \emptyset \):

\[
r_n \left\| (v_n^{\text{opt}})_{\mathcal{A}^c} \right\|_1 = \lambda_n r_n \sum_{j \in \mathcal{A}^c} (1 / |\hat{\beta}_n j|) \rightarrow P_r \infty , \tag{26}
\]

if:

\[
(\hat{\beta}_n)_{\mathcal{A}^c} = O_P (1 / r_n) \quad \text{and} \quad \lambda_n r_n^2 \rightarrow \infty . \tag{27}
\]

Therefore, Oracle property 1 holds for the Adaptive Lasso penalty under asymptotic regime (27) if \( \hat{\beta}_n^* - \beta_0 = O_P (1 / r_n) \).

Finally, easily verifiable conditions for Oracle property 2 are provided in the next proposition.
Proposition 3.3. Let Assumptions 1–2 and 4 be satisfied and asymptotic regime (ii) in Theorem 3.1 hold for limit penalty (19). Let further the distribution of random vector \([M_0]_A^\top(Z)_A\) be the efficient Gaussian distribution for estimating the subvector \((\beta_0)_A\) of active components. Then Oracle property 2 holds if and only if:

\[
(W_0M^+_0\Omega_0M^+_0W_0)_A = (W_0)_A[(M_0)_A]^\top(\Omega_0)_A[(M_0)_A]^\top(W_0)_A.
\]

(28)

The latter condition holds when \(\Omega_0 = \sigma^2M_0\) for some \(\sigma \in \mathbb{R}\) and \(W_0 = M_0\).

The sufficient condition in Proposition 3.3 is applicable to both regular and irregular linear regression designs. In such settings, condition \(\Omega_0 = \sigma^2M_0\) for some \(\sigma \in \mathbb{R}\) and \(W_0 = M_0\) corresponds to the standard homoskedasticity assumption on the regression residuals in linear model (1).

4 Proximal estimators for irregular designs

We build \(\sqrt{n}\)-consistent, asymptotically normal, Oracle proximal estimators in linear regression model (1), which are robust to a design irregularity, starting from the following assumption.

Assumption 5. Random matrices \((X, \epsilon) \in \mathbb{R}^{n \times p} \times \mathbb{R}^n\) are such that:

(i) There exists \(p \times p\) matrix \(Q_0\) such that \(Q_n := X^\top X/n \rightarrow_p Q_0\).

(ii) \(X^\top \epsilon/\sqrt{n} \rightarrow_d Z \sim N(0, \Omega_0)\), for a symmetric positive semi-definite matrix \(\Omega_0\).

Assumption 5 does not require regularity of matrices \(Q_0\) or \(\Omega_0\), allowing for a design irregularity. It can be satisfied by invoking well-known Laws of Large Numbers and Central Limit Theorems for the underlying processes.

4.1 Definition of singular and nearly-singular designs

We define irregular designs as follows.

Definition 4.1. In the context of Assumption 5:

(i) A singular design is such that \(Q_{0n} := \mathbb{E}[Q_n] = Q_0\) for a singular matrix \(Q_0\).

(ii) A nearly-singular design is such that \(\text{Range}(Q_{0n}) \supseteq \text{Range}(Q_0)\) for every \(n \in \mathbb{N}\) and \(Q_{0n} \rightarrow Q_0\) as \(n \rightarrow \infty\).

A singular design arises, e.g., when observations \(X_1, \ldots, X_n\) in the columns of matrix \(X^\top\) are generated by a stationary process \(\{X_i : i \in \mathbb{N}\}\) with a singular second moment matrix \(Q_0 = \mathbb{E}[X_1X_1^\top]\). Various nearly-singular designs are studied in Knight and Fu (2000), Phillips (2001), Knight (2008), Gabaix and Ibragimov (2011), Aswani et al. (2011) and Phillips (2016). The first three papers cover deterministic designs, while Phillips (2016) explicitly considers stochastic designs. Here, near-singularities appear, e.g., when matrix \(X^\top\) is spanned with a stationary stochastic matrix \(U^\top := [U_1, \ldots, U_n] \in \mathbb{R}^{p \times n}\) of rank \(r_U < p\) and a second stationary stochastic matrix \(V^\top := [V_1, \ldots, V_n] \in \mathbb{R}^{p \times n}\), which is scaled with a vector \(a := (a_1, \ldots, a_n)^\top\) of non negative constants such that \(a_n \rightarrow 0\) as \(n \rightarrow \infty\):

\[
X^\top = U^\top + V^\top \text{diag}(a).
\]

(29)

The associated finite sample population design matrix reads:

\[
Q_{0n} = \mathbb{E}[U_iU_i^\top] + \frac{\|a\|^2}{n} \mathbb{E}[U_iV_i^\top + V_iU_i^\top] + \frac{\|a\|^2}{n} \mathbb{E}[V_iV_i^\top],
\]

11
with $Q_0 = \mathbb{E}[U_1 U_1']$. Here, Assumption 5 (ii) holds under weak conditions, e.g., when $\frac{1}{n} \text{Var}(V'ae) \rightarrow 0$ as $n \rightarrow \infty$ and $U'e/\sqrt{n}$ satisfies a central limit theorem.\textsuperscript{13} From Definition 4.1, a near-singularity appears if and only if $U_1$ and $V_1$ are not linearly dependent, in which case the projection of $V_1$ on the kernel of $Q_0$ is not trivial $P$—almost surely. We denote by $P_0$ ($P_0^\perp$) the orthogonal projection matrix on the kernel (range) of $Q_0$ and make use in the sequel of following parametrization of a regular or irregular design.

**Assumption 6.** There exists symmetric matrix $\Delta$ and a sequence of positive scalars $\tau_n$ with $\tau_n \rightarrow \infty$, such that:

$$Q_{0n} = Q_0 + \frac{1}{\tau_n} \Delta + o(1/\tau_n).$$

When $Q_0$ is singular and $P_0 \Delta \neq 0$ ($\Delta = 0$), parametrization (30) covers a nearly-singular design of rate $1/\tau_n$ (a singular design). When $Q_0$ is regular and $\Delta = 0$ a regular design is induced. For instance, in model (29) we have $\tau_n = n/\|a\|_1$ in general, and $\tau_n = n/\|a\|_2^2$ when $\mathbb{E}[U_1 V_1'] = 0$.

### 4.2 Selected properties of irregular designs

Under Assumption 5 (i), sample design matrix $Q_n$ converges to matrix $Q_0$. However, singular and nearly-singular designs imply different convergence properties for the generalized inverses of these matrices.

**Lemma 4.2.** Under Assumption 5 (i) following statements hold:

(i) For a singular design: $Q_n^+ \rightarrow_{P_0} Q_0^+$ as $n \rightarrow \infty$.

(ii) For a nearly-singular design: $Q_n^+ \not\rightarrow_{P_0} Q_0^+$ as $n \rightarrow \infty$.

From Lemma 4.2, Moore Penrose inverses of sample design matrices are consistent estimators of the Moore Penrose inverse of the population design matrix under a singular design, but not a nearly-singular design. Therefore, Ridgeless LSE (12) is consistent in the former, but not the latter case.

Above features are related to the properties of the set of parameters of interest that can be estimated consistently under a nearly-singular design. Let $P_{0n}$ be the orthogonal projection matrix on the space spanned by the eigenvectors of $Q_{0n}$ associated with its $k_0 := p - \text{Rank}(Q_0)$ smallest eigenvalues, and consider following matrix factorization:

$$Q_{0n} := \hat{Q}_{0n} + Q_{1n} := P_{0n}^+ Q_{0n} P_{0n}^+ + P_{0n} Q_{0n} P_{0n}^+, \tag{31}$$

where $P_{0n}^\perp = I_n - P_{0n}$. Since $\text{Rank}(\hat{Q}_{0n}) = \text{Rank}(Q_0)$ and $\hat{Q}_{0n} \rightarrow Q_0$ as $n \rightarrow \infty$ under Assumption 6, $Q_{0n}^+ \rightarrow Q_0^+$. Furthermore, for any parameter $\beta_0$ in model (1) it follows as $n \rightarrow \infty$, when $\frac{1}{n} \mathbb{E}[X'e] = 0$:

$$\hat{\beta}_{0n} := Q_{0n}^+ \mathbb{E} \left[ \frac{X'Y}{n} \right] = \hat{Q}_{0n}^+ Q_{0n} \beta_0 = P_{0n}^+ \beta_0 \rightarrow P_0^+ \beta_0, \tag{32}$$

i.e., $\beta_{0n}$ is a smooth finite sample parameterization of the unique parameter of interest $\beta_0$ in model (1) such that $\beta_0 = P_0^+ \beta_0$. The local behavior of $\hat{\beta}_{0n}$ with respect to this parameter of interest is clarified in the next lemma.

\textsuperscript{13}For instance, for a stationary process $(V_i)_{i \in \mathbb{N}}$ such that $\mathbb{E}[e|V] = 0$, it follows as $n \rightarrow \infty$:

$$\frac{1}{n} \mathbb{E}[V' \text{diag}(a)ee' \text{diag}(a)V] = \frac{\|a\|_2^2}{n} \mathbb{E}[\epsilon_1^2 V_1 V_1'] \rightarrow 0.$$
Lemma 4.3. Given local parameterization (32) and a population parameter of interest such that \( \beta_0 = P_0^\perp \beta_0 \), it follows:
\[ \tau_n (\beta_{0n} - \beta_0) = P_0 \Delta Q_0^+ \beta_0 + o(1). \]

Lemma 4.3 shows that \( \beta_{0n} \) parametrizes a set of local alternatives of size \( 1/\tau_n \) to population parameter \( \beta_0 = P_0^\perp \beta_0 \), which are induced by the given near-singularity. When \( \Delta \) has range contained in the range or the kernel of \( Q_0^+ \), then \( P_0 \Delta Q_0^+ = 0 \) and \( \beta_{0n} - \beta_0 = o(1/\tau_n) \), i.e., the local alternatives vanishes faster than the design near-singularity. In all other situations, they are of the same order as the near-singularity.

**Remark 3.** It is instructive to compare the properties of local parametrization (32) with the parameterization:
\[ \beta_{0n} : = Q_0^+ E X' Y_n = Q_0^+ Q_0 \beta_0 = P_0^\perp \beta_0. \]

(33)

It follows that \( \beta_{0n} \) converges to a well-defined population parameter only if \( \text{Rank}(Q_0^+) \) converges. If such convergence holds, then \( \text{Rank}(Q_{0n}) \rightarrow \text{Rank}(Q_0) \) and \( \beta_{0n} \rightarrow P_0^\perp \beta_0 \) as \( n \rightarrow \infty \). Therefore, parametrization (33) is not robust to near singularities implying a non converging rank of \( Q_{0n} \).

### 4.3 Rank consistent estimation of population design matrix

Section 4.4 introduces a \( \sqrt{n} \)-consistent, asymptotically normal estimator of the unique Least Squares population parameter such that \( \beta_0 = P_0^\perp \beta_0 \). Such estimator is based on a consistent estimator of \( Q_0 \) that ensures a consistent rank estimation under a nearly-singular design. Let:
\[ Q_n = E_n \text{diag}(\sigma_n) E_n' , \]

(34)

be the spectral decomposition of \( Q_n \) and \( \sigma_n \) its (ordered) spectrum. Next, consider following proximal estimator for the spectrum \( \sigma_0 \) of \( Q_0 \):
\[ \hat{\sigma}_n : = \text{prox}_{\nu_n f_n}(\sigma_n) = \arg\min_{\sigma \in \mathbb{R}^p} \left\{ \frac{1}{2} \| \sigma_n - \sigma \|^2 + \nu_n f_n(\sigma) \right\} , \]

(35)

for a sequence of stochastic penalties \( f_n \in \Gamma(\mathbb{R}^p) \) and \( \nu_n > 0 \). Finally, let:
\[ \hat{Q}_n := E_n \text{diag}(\hat{\sigma}_n) E_n' , \]

(36)

where for any \( j = 1, \ldots, p \):
\[ \hat{\sigma}_{nj} := \begin{cases} 0 & \text{if } \sigma_{nj} = 0 \\ \sigma_{nj} & \text{else} \end{cases} . \]

(37)

Intuitively, \( \hat{Q}_n \) consistently estimates \( Q_0 \) and its rank under a singular and a nearly-singular design, if spectrum estimator (35) consistently selects the active components in spectrum \( \sigma_0 \). This is feasible using established penalties in the literature under the following high-level assumption.

**Assumption 7.** Following boundedness in probability condition holds:
\[ Q_n - Q_{0n} = O_P(1/\sqrt{n}) . \]

(38)

Assumption 7 is stronger than Assumption 5 (i) but it is still fairly general and can be ensured, e.g., by a suitable Central Limit Theorem. It delivers a consistent spectrum estimator detailed in the next proposition.
Proposition 4.1. Let Assumptions 6–7 hold and positive sequence \(\{\rho_n\}\) be such that \(\rho_n \to \infty, \rho_n = o(\min\{\sqrt{n}, \tau_n\})\) and \(v_n \rho_n \to \kappa > 0\) as \(n \to \infty\). Let further penalty \(f_n\) in equation (35) be the Lasso penalty:

\[
f_n(\sigma) = ||\sigma||_1.
\]

It then follows, as \(n \to \infty\): \(Q_n \to \Pr Q_0, \Pr (\text{Rank}(Q_n) = \text{Rank}(Q_0)) \to 1\) and \(Q_n^+ \to \Pr Q_0^+\).

4.4 Minimum norm parameter of interest and modified Ridgeless estimator

When \(Q_0\) is singular, parameter \(\beta_0\) in model (1) is not identified. Therefore, let \(\delta_0 := \lim_{n \to \infty} E[\frac{X'Y}{n}]\) and consider the set of population LS solutions:

\[
B_0 := \arg\min_{\beta \in \mathbb{R}^p} \lim_{n \to \infty} \frac{1}{n} E[\|Y - X\beta\|^2] = \{\beta \in \mathbb{R}^p : Q_0\beta = \delta_0\} = \{Q_0^+\delta_0\} + \text{Kernel}(Q_0).
\]

In this set, the minimum Euclidean norm parameter is given by:

\[
\beta_0 := \arg\min_{\beta \in B_0} \|\beta\|_2 = Q_0^+\delta_0.
\]

It is the unique element of \(B_0\) in the range of matrix \(Q_0\), i.e., the only population Least Squares solution such that \(\beta_0 = P_0^+\beta_0\). Hence, it is also the only element of \(B_0\) that can be smoothly parametrized using local alternatives (32) from Section 4.2. Furthermore, since the kernel of \(Q_0\) does not depend on variable \(Y\), the minimum norm LS parameter is also the only truly intrinsic parameter of interest in set \(B_0\), which does not depend on a component in the kernel of \(Q_0\). In view of these desirable properties, we work with minimum norm LS parameter \(\beta_0\) as our parameter of interest.

While Ridgeless estimator (12) is inconsistent under a nearly-singular design, Proposition 4.1 suggests following modified Ridgeless LSE:

\[
\hat{\beta}_{ls} := \hat{Q}_n^+ X'Y/n = \arg\min_{\beta \in \mathbb{R}^p} \{\|\beta\|_2 : \hat{Q}_n\beta = \hat{Q}_n\hat{Q}_n^+ X'Y/n\}.
\]

(40)

Equivalently, this estimator is a proximal estimator built with the Euclidean projection of the Ridgeless estimator on the range of \(\hat{Q}_n\):

\[
\hat{\beta}_{ls} = \hat{Q}_n^+ \hat{Q}_n \hat{\beta}_{ls} = \arg\min_{\beta \in \mathbb{R}^p} \left\{ \frac{1}{2} \|\hat{\beta}_{ls} - \beta\|^2 : \beta \in \text{Range}(\hat{Q}_n) \right\}.
\]

(41)

To obtain the asymptotic distribution of estimator (40) under a regular and irregular design, we finally rely on following high-level assumptions.

Assumption 8.

(i) \(\sqrt{n}(Q_n - Q_{0n}) \to_d \Theta\), for a normally distributed random matrix \(\Theta\).

(ii) \(P_0(Q_n - Q_{0n})P_0^+ = o_P(1/\sqrt{n})\).

Assumption 8 (i) states a central limit theorem for sample design matrix \(Q_n\) and implies Assumption 7. It is needed to control the uncertainty generated by the estimation of matrix \(Q_0^+\) in the modified Ridgeless estimator, whenever this uncertainty is not asymptotically negligible.

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When comparing Ridgeless and modified Ridgeless estimators (12) and (40), the additional orthogonal projection matrix \(Q_n Q_n^+\) in optimization problem (40) ensures existence of a Least Squares estimator in the range of matrix \(Q_n\).
Assumption 8 (ii) is a weak condition on the rate at which the estimated cross-second moments of projections of observations on the kernel and the range of matrix $Q_0$ vanish under a nearly-singular design, causing the uncertainty of the estimation of matrix $Q_0^+$ in the modified Ridgeless estimator to become negligible asymptotically. This weak assumption is satisfied when data are geometrically mixing and following moment inequality holds:

$$\tau = \sum_{i=1}^{n} b_i \to 0 \text{ as } n \to \infty.$$ 

for some $\delta > 0$ and scalars $b_i \geq 0$ such that $\frac{1}{n} \sum_{i=1}^{n} b_i \to 0$ as $n \to \infty$. The latter inequality holds, e.g., for data generating process (29) when components $U_i, V_i$ have finite moments of order $4 + \delta'$ for some $\delta' > 0$.

The next theorem provides the asymptotic distribution of modified Ridgeless LSE (40) for both regular and irregular designs.

**Theorem 4.4.** Let Assumption 5–7 be satisfied. If $\Delta = 0$, then:

$$\sqrt{n} (\hat{\beta} - \beta_0) \to_d Q_0^+ Z.$$ 

Let further $\Delta \neq 0$ and $\sqrt{n}/n \to c \geq 0$ as $n \to \infty$. If Assumption 8 (i) holds, then:

$$\sqrt{n} (\hat{\beta} - \beta_0) \to_d P_0 (\Theta + c\Delta) Q_0^+ \beta_0 + Q_0^+ Z.$$ 

If Assumption 8 (ii) holds, then:

$$\sqrt{n} (\hat{\beta} - \beta_0) \to_d c P_0 \Delta Q_0^+ \beta_0 + Q_0^+ Z.$$ 

The asymptotic distribution of the modified Ridgeless LSE in Theorem 4.4 is Gaussian, features a standard $1/\sqrt{n}$ convergence rate, and is independent of near-singularities that either (i) vanish sufficiently fast as the sample size grows ($\sqrt{n} = o(\tau_n)$) or (ii) vanish at rate $1/\sqrt{n}$ but are such that $P_0 \Delta P_0^+ = 0$. When $\sqrt{n} = O(\tau_n)$ and $P_0 \Delta P_0^+ \neq 0$, the asymptotic distribution is shifted by noncentrality parameter $P_0 \Delta Q_0^+ \beta_0$, reflecting the asymptotic power of this distribution for detecting local alternatives $\beta_{0 \delta}$ from Section 4.2. Furthermore, even though in general the estimation of the second moment matrix $Q_0$ may impact asymptotic distribution (44), such uncertainty is asymptotically negligible when weak Assumption 8 (ii) holds, giving rise to the simpler asymptotic distribution expression (45).

Finally, note that for fast vanishing near-singularities satisfying Assumption 8 (ii), the asymptotic distribution of the modified Ridgeless estimator has support inside the range of matrix $Q_0$. It does not only when near-singularities are such that the uncertainty deriving from the estimation of second moment matrix $Q_0$ affects the resulting asymptotic distributions. These are interesting differences to the asymptotic distributions derived in Knight and Fu (2000) and Knight (2008) for Bridge-type estimators, which all have support inside the kernel of $Q_0$.

### 4.5 Oracle proximal estimation for irregular designs

Under a nearly-singular design, several PLSEs display non standard asymptotic properties; see again Knight and Fu (2000). However, Theorem 4.4 and the simple composition of

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16For instance, in the context of a data generating process (29) with orthogonal regular and nearly-singular components ($\mathbb{E}[U_i V_i^T] = 0$), this asymptotic distribution is unaffected by near-singularities such that $a_n = O(1/n^{1/2})$.

17For instance, Theorem 6 of Knight and Fu (2000) gives the asymptotic distribution of PLSEs with penalty functions in the Bridge Family (Frank and Friedman (1993)), under a nearly-singular design with rate $\tau_n = \sqrt{n}$. In this case, Bridge PLSEs feature a non-Gaussian asymptotic distribution with a rate of convergence of $n^{1/4}$.
different proximal operators allowed by our framework easily produce Oracle estimators with a Gaussian asymptotic distribution and a standard convergence rate, in a way that is robust to a design irregularity. Indeed, given modified Ridgeless LSE (40) and penalty $f_n \in \Gamma(\mathbb{R}^p)$, the following proximal estimator is well-defined under a regular, a singular and a nearly-singular design:

$$
\text{prox}_{\lambda_n f_n}^Q_n(\hat{\beta}^n) = \arg\min_{\beta \in \mathbb{R}^p} \left\{ \frac{1}{2} \left\| \hat{\beta}^n - \beta \right\|^2_{Q_n^{-1}} + \lambda_n f_n(\beta) \right\},
$$

where $Q_n = Q_n + I - Q_n Q_n^+$.  \(^{17}\) Whenever Assumption 2 holds, the asymptotic distribution of proximal estimator (46) then follows with Theorem 3.1.

**Corollary 4.5.** Let Assumptions 5–7 and Assumption 8 (ii) hold. Further, assume that $\sqrt{n} = o(\tau_n)$ or $P_0 \Delta P_0^+ = 0$. It then follows:

(i) If $\lambda_n \sqrt{n} \rightarrow 0 > 0$, then:

$$
\sqrt{n} \left( \begin{array}{c}
\text{prox}_{\lambda_n f_n}^Q_n(\hat{\beta}^n) - \beta_0 \\
\text{prox}_{\lambda_n f_n}^Q_n(\hat{\beta}^n) - \beta_0
\end{array} \right) \rightarrow_d \left( \begin{array}{c}
\text{prox}_{\lambda_0 \rho \beta_0}^Q_0(Q_0^+ Z) \\
P Q_0^+ (Q_0^+ Z)
\end{array} \right),
$$

with

$$
\text{prox}_{\lambda_0 \rho \beta_0}^Q_0(Q_0^+ Z) = \left( I_d - P Q_0^+ (Q_0^+ Z) \right) (Q_0^+ Z).
$$

(ii) If $\lambda_n \sqrt{n} \rightarrow 0 = 0$ and $\lambda_n \sqrt{n} f_n \rightarrow_{Pr} \text{dom}(f_0)$ in epigraph, the above limits hold with $\lambda_0 \rho \beta_0^+$ and $\lambda_0 \partial f_0(\beta_0)$ replaced by $\sigma_{N_{\text{dom}(f_0)}}(\beta_0)$ and $N_{\text{dom}(f_0)}(\beta_0^+)$, respectively.

To satisfy the Oracle property in Definition 3.2, we can finally adopt asymptotic regime (ii) in Corollary 4.5 with an adaptive penalty satisfying the conditions of Proposition 3.2 and Proposition 3.3. The next corollary builds such Oracle estimator with the Adaptive Lasso-type penalty. \(^{18}\)

**Corollary 4.6.** Let Assumptions 5–7 and Assumption 8 (ii) hold, where in Assumption 5 (ii) $\Omega_0 = \sigma^2 Q_0$ for some $\sigma \in \mathbb{R}$, and let either $\sqrt{n} = o(\tau_n)$ or $P_0 \Delta P_0^+ = 0$. Furthermore, let penalty $f_n$ in proximal estimator (46) be given for any $\beta \in \mathbb{R}^p$ by following Adaptive Lasso–type penalty:

$$
f_n(\beta) = \sum_{i=1}^P \frac{|\beta_i|}{|\beta_{i0}|}.
$$

If $\lambda_n \sqrt{n} \rightarrow 0$ and $\lambda_n n \rightarrow \infty$, then:

$$
\sqrt{n} \left( \begin{array}{c}
(\text{prox}_{\lambda_n f_n}^Q_n(\hat{\beta}^n) - \beta_0)_{A} \\
(\text{prox}_{\lambda_n f_n}^Q_n(\hat{\beta}^n) - \beta_0)_{A^c}
\end{array} \right) \rightarrow_d \left( \begin{array}{c}
[(Q_0^+ Z)_A]^+ (Z)_A \\
0
\end{array} \right).
$$

Moreover, Oracle property 1 in Definition 3.2 holds.

\(^{17}\)Under Assumption 7, Proposition 4.1 implies $Q_n \rightarrow_{Pr} Q_0 + I - Q_0 Q_0^+$, with a positive definite limit matrix by construction. Using Theorem 4.4, we are therefore in the general framework of Section 3.2, in which we have satisfied Assumptions 1 and 3.

\(^{18}\)Explicit Oracle proximal estimators for regular and irregular designs can be built with our approach using other established adaptive penalties in the literature, such as, e.g., Adaptive Elastic Net–type penalties.
5 Concluding remarks

By its nature, the unifying proximal estimation approach in this paper is extendable to study further relevant questions in the literature. These include, e.g., the properties of Least Absolute Deviations proximal estimators under an irregular design and those of modified Ridgeless-type instrumental variable estimators under weak identification. The differentiability of proximal estimators with respect to their initial estimator is also ideally suited for building robust penalized estimators with bounded influence function. Such estimators can be easily built with a corresponding proximal operator applied to existing bounded-influence estimators for linear or nonlinear models.
Supplementary Material for:

PROXIMAL ESTIMATION AND INFERENCE

This supplementary file is organized as follows. Section 1 compiles the proofs of all mathematical results outlined in the main text. Section 2 establishes the consistency of proximal estimators under general assumptions. In Section 3, we discuss a selection of benchmark convex penalty functions in the context of the main results in the paper. Moreover, we derive the closed-form expression of their convex conjugate, directional derivatives and subgradients. Section 4 devises a Monte Carlo simulation analysis for regular, singular and nearly-singular linear regression settings. In these settings, we compare the performance of the Ridgeless and our modified Ridgeless estimator, as well as corresponding proximal estimators. Finally, all figures and tables are collected in Section 5.

1 Proofs of the mathematical statements in the main text

Statements involving random variables are understood $\mathbb{P}$–almost surely, unless stated otherwise.

Proof of Proposition 2.1. If $f_n \in \Gamma(\mathbb{R}^p)$, then $\text{prox}_{\lambda_n f_n}$ is the proximal operator of $\lambda_n f_n$ under the inner product induced by $W_n$ — see, e.g., Bauschke et al. (2016, Def. 12.23) — and $\text{prox}_{\lambda_n f_n}(\beta_n)$ uniquely exists. Bauschke et al. (2016, Prop. 12.28) implies that $\text{prox}_{\lambda_n f_n}$ is Lipschitz continuous with Lipschitz constant 1, hence it is Lebesgue $\mathbb{P}$–almost surely differentiable, while identity (4) follows from Moreau’s decomposition (see Bauschke et al. (2016, Thm. 14.3)). Finally, any sublinear function $\lambda_n f_n$ in $\Gamma(\mathbb{R}^p)$ is the support function of a corresponding closed, convex and nonempty set $C_n = \{\theta \in \mathbb{R}^p : \langle \theta, \beta_n \rangle_{W_n} \leq \lambda_n f_n(\beta_n) \text{ for all } \beta_n \in \mathbb{R}^p\}$, see Hiriart-Urruty and Lemaréchal (2004, Thm. 3.1.1). Hence, the convex cojnjugate of $\lambda_n f_n$ under inner product $\langle \cdot, \cdot \rangle_{W_n}$ is the indicator function of set $C_n$, $(\lambda_n f_n)^\ast = \iota_{C_n}$, see Bauschke et al. (2016, Example 13.3(i)), and identity (4) specializes to identity (5). □

Proof of Proposition 2.2. The Lasso and the Adaptive Lasso penalties, given respectively by $f_n(\beta) = \|\beta\|_1$ and $f_n(\beta) = \sum_{j=1}^p |\beta_j|/|\hat{\beta}_{nj}|$, are sublinear functions. Therefore, their associated proximal estimators satisfy projection formula (5) with an associated closed convex set (6). Using standard duality relations between norms, set $C_n$ for the Lasso is given by following polyhedron:

$$C_n = \{\theta \in \mathbb{R}^p : \sup_{\beta \neq 0} \frac{\langle W_n \theta, \beta \rangle}{\|\beta\|_1} \leq \lambda_n \} = \{\theta \in \mathbb{R}^p : \|W_n \theta\|_\infty \leq \lambda_n \}.$$

For the Adaptive Lasso, we similarly get using the change of variable $v = (v_1, \ldots, v_p)'$ with $v_j := \beta_j/\hat{\beta}_{nj}$:

$$C_n = \left\{\theta \in \mathbb{R}^p : \sup_{v \neq 0} \frac{\langle \hat{\beta}_n \circ W_n \theta, v \rangle}{\|v\|_1} \leq \lambda_n \right\} = \left\{\theta \in \mathbb{R}^p : \|\hat{\beta}_n \circ W_n \theta\|_\infty \leq \lambda_n \right\}.$$

This concludes the proof. □
Proof of Proposition 2.3. We first have, for any weighting matrix $W_n$ and penalty $f_n \in \Gamma(\mathbb{R}^p)$:

$$\text{prox}_{\lambda f_n} (\hat{\beta}^*_n) = \arg\min_{\beta \in \mathbb{R}^p} \left\{ \frac{1}{2} \| \hat{\beta}^* - \beta \|^2_{W_n} + \lambda f_n(\beta) \right\} = \arg\min_{\beta \in \mathbb{R}^p} \left\{ \frac{1}{2} \beta' W_n \beta - \beta' W_n A_n^+ X' Y / n + \lambda_n \bar{f}(\beta) \right\} = \arg\min_{\beta \in \mathbb{R}^p} \left\{ \frac{1}{2} \beta' Q_n \beta - \beta' W_n A_n^+ X' Y / n + \lambda_n \bar{f}(\beta) \right\},$$

where $\bar{f}(\beta) := f_n(\beta) + \frac{1}{n} \beta' (W_n - Q_n) \beta$ and $\bar{f} \in \Gamma(\mathbb{R}^p)$ if and only if $W_n - Q_n$ is positive semi-definite. On the other hand,

$$\hat{\beta}_n := \arg\min_{\beta \in \mathbb{R}^p} \left\{ \frac{1}{2n} \| Y - X \beta \|^2_2 + \lambda_n \bar{f}(\beta) \right\} = \arg\min_{\beta \in \mathbb{R}^p} \left\{ \frac{1}{2} \beta' Q_n \beta - \beta' X' Y / n + \lambda_n \bar{f}(\beta) \right\}.$$

Therefore:

$$\text{prox}_{\lambda f_n} (\hat{\beta}^*_n) = \hat{\beta}_n \iff W_n A_n^+ X' Y / n = X' Y / n.$$

For weighting matrices of the form $W_n = A_n$, the latter condition reads:

$$A_n A_n^+ X' Y / n = X' Y / n.$$

Since $A_n A_n^+$ is an orthogonal projection matrix on the columns of $A_n$, this condition holds if and only if $\text{Range}(A_n) \supseteq \text{Range}(X') = \text{Range}(Q_n)$ or, equivalently, if $\text{Kernel}(A_n) \subset \text{Kernel}(Q_n)$. This concludes the proof. \(\square\)

Proof of Theorem 3.1. For any $b \in \mathbb{R}^p$, let $P_n(b) := g_n(b) + h_n(b)$, where:

$$g_n(b) := \frac{1}{2} \left\| r_n(\hat{\beta}^*_n - \beta_0) - b \right\|^2_{W_n} ; h_n(b) := r_n^2 \lambda_n \left[ f_n(\beta_0 + b/r_n^2) - f_n(\beta_0) \right].$$

Similarly, $P_0(b) := g_0(b) + h_0(b)$ with $g_0(b) := \frac{1}{2} \left\| \eta - b \right\|^2_{W_0}$ and $h_0(b) := \lambda_0 r_0^2 (\beta_0)$, under asymptotic regime (i), or $h_0(b) := \sigma_{\text{dom}(f_0)}(\beta_0)(b)$, under asymptotic regime (ii), respectively. Under Assumptions 1–3, $P_n$ and $P_0$ are strictly convex and coercive (Bauschke et al. (2016, Prop. 11.14)).\(^{19}\) Moreover, the minimum of $P_n$ is uniquely attained at

$$\hat{b}_n = r_n \left( \text{prox}_{\lambda f_n} (\hat{\beta}^*_n) - \beta_0 \right).$$

Analogously, the minimum of $P_0$ is uniquely attained at $\text{prox}_{\lambda_0 r_0^2} (\eta)$, under asymptotic regime (i), or $\text{prox}_{\sigma_{\text{dom}(f_0)}(\beta_0)} (\eta)$, under asymptotic regime (ii). Therefore, we only need to prove that $P_n \to_d P_0$ in epigraph, in order to imply the convergence:

$$\text{prox}_{h_n} (\hat{b}_n) = \arg\min_{b \in \mathbb{R}^p} P_n(b) \to_d \arg\min_{b \in \mathbb{R}^p} P_0(b) = \text{prox}_{h_0} (\eta);$$

see Knight (1999, Thm. 5). To prove that $P_n \to_d P_0$ in epigraph, note first that under Assumption 3, we have $g_n(b) \to_d g_0(b)$, uniformly on compact sets. Therefore, using Attouch (1984, Thm. 2.15) and Geyer (1994) we are left to prove that $h_n \to_{Pr} h_0$ in

\(^{19}\)Notice that, under Assumption 2, $f_0 \in \Gamma(\mathbb{R}^p)$; see Rockafellar and Wets (2009, Ch. 7).
epigraph. Consider first asymptotic regime (i). By Assumption 2 and Attouch (1984, Thm. 3.66), subgradient \( \partial f_n(\beta_0) \) converges in probability to subgradient \( \partial f_0(\beta_0) \) in the Painlevé-Kuratowski sense, and, equivalently, \( \iota_{\partial f_n(\beta_0)} \to \Pr \iota_{\partial f_0(\beta_0)} \), in epigraph. Moreover, by Bauschke et al. (2016, Prop. 17.17), the convex conjugates of \( \iota_{\partial f_n(\beta_0)} \) and \( \iota_{\partial f_0(\beta_0)} \) equal the directional derivatives of \( f_n \) and \( f_0 \) at \( \beta_0 \), respectively. Thus, under Assumption 2(ii), continuity of convex conjugation with respect to epigraph convergence implies that the directional derivative of \( f_n \) at \( \beta_0 \) epiconverges in probability to \( \rho_{\beta_0} \), the directional derivative of \( f_0 \) at \( \beta_0 \). From the definition of directional derivative, we therefore obtain \( h_n \to \Pr h_0 = \lambda_0 \rho_{\beta_0} \), in epigraph, as required. Consider next asymptotic regime (ii). Using the same arguments as for item (i), we obtain that \( h_n \) converges in epigraph to the directional derivative of \( \iota_{\text{dom}(f_0)} \) at \( \beta_0 \). By Bauschke et al. (2016, Prop. 17.17), the directional derivative of \( \iota_{\text{dom}(f_0)} \) at \( \beta_0 \) is the convex conjugate of \( \iota_{\partial \text{dom}(f_0)}(\beta_0) = \iota_{\text{Ndom}(f_0)}(\beta_0) \), which in turn is given by \( \sigma_{\text{Ndom}(f_0)}(\beta_0) \), the support function of \( \text{Ndom}(f_0)(\beta_0) \); see also Bauschke et al. (2016, Example 13.3). Therefore, we obtain \( h_n \to \Pr h_0 = \sigma_{\text{Ndom}(f_0)}(\beta_0) \), in epigraph, as required. Finally, the conjugate asymptotic distribution characterization follows with Moreau’s decomposition (Bauschke et al. 2016, Thm. 14.3)):

\[
\text{prox}_{h_0}^W(\eta) = \left( I_d - \text{prox}_{h_0}^W \right)(\eta),
\]

where \( h_0^* \) is the convex conjugate of the directional derivative at \( \beta_0 \) of function \( \lambda_0 f_0 \) and function \( \iota_{\text{dom}(f_0)} \), under asymptotic regime (i) and (ii), respectively. Using again Bauschke et al. (2016, Prop. 17.17), we thus obtain under asymptotic regime (i): \( \text{prox}_{h_0}^W = P_{\lambda_0 \partial f_0(\beta_0)}^W \). Similarly, under asymptotic regimes (ii): \( \text{prox}_{h_0}^W = P_{\lambda_0 \partial f_0(\beta_0)}^W \). This concludes the proof.

**Proof of Proposition 3.1.** Condition \( \mathcal{A} = \hat{\mathcal{A}}_n \) holds if and only if \( \mathcal{A}^c = \hat{\mathcal{A}}_n^c \), which implies

\[
\hat{\beta}^c_n = \left( \text{prox}_{\iota_{\lambda_n f_n}(\cdot)}(\hat{\beta}_n^c) \right)_j
\]

for all \( j \in \mathcal{A}^c \). Therefore, if Oracle property 1 holds, we obtain

\[
1 = \limsup_{n \to \infty} \Pr \left( \hat{\mathcal{A}}_n = \mathcal{A} \right) = \limsup_{n \to \infty} \Pr \left( \hat{\mathcal{A}}^c_n = \mathcal{A}^c \right)
\leq \limsup_{n \to \infty} \Pr \left( r_n(\hat{\beta}_n^c)_{\mathcal{A}^c} = r_n \left( \text{prox}_{\iota_{\lambda_n f_n}(\cdot)}(\hat{\beta}_n^c) \right)_{\mathcal{A}^c} \right)
\leq \Pr \left( \left( \eta \right)_{\mathcal{A}^c} = \left( P_{\lambda_0 \partial f_0}^W(\eta) \right)_{\mathcal{A}^c} \right),
\]

where the last inequality follows from Theorem 3.1. This concludes the proof.

**Proof of Proposition 3.2.** We first have, using the definition of a proximal operator:

\[
\hat{\beta}_n - \beta_0 = \arg\min_{u \in \mathbb{R}^p} \left\{ \frac{1}{2n} \left\| \hat{\beta}_n^c - \beta_0 - u \right\|_{\lambda_n f_n(\beta_0 + u)}^2 + \lambda_n f_n(\beta_0 + u) \right\}
= \text{prox}_{\lambda_n f_n(\hat{\beta}_n^c - \beta_0)}^W(\hat{\beta}_n^c - \beta_0),
\]

with penalty \( \tilde{f}_n \in \Gamma(\mathbb{R}^p) \) defined by \( \tilde{f}_n(u) = f_n(\beta_0 + u) \). This gives, using \( \hat{\beta}_n - \beta_0 = O_{\Pr}(1/r_n) \), Assumption 1 and the fact that \( \text{prox}_{\lambda_n f_n} \) is Lipschitz continuous with Lipschitz constant 1 (Bauschke et al. 2016, Prop. 12.30)):

\[
\left\| \hat{\beta}_n - \beta_0 \right\| \leq \left\| W_n^{-1/2} \right\| \left\| W_n^{1/2} \right\| \left\| \hat{\beta}_n^c - \beta_0 \right\| = O_{\Pr}(1/r_n),
\]
where \( \|A\| \) denotes the operator norm of a matrix \( A \). Hence, \( \hat{\beta}_n \rightarrow P \beta_0 \) and \( v_n^{opt} = O_{P}(1/r_n) \), using again Assumption 1. Consistency of \( \hat{\beta}_n \) further yields:

\[
P(\hat{A}_n \subset A) \rightarrow_{n \to \infty} 1,
\]

where \( \hat{A}_n := \{j : \hat{\beta}_{nj} \neq 0\} \). If \( A^c = \emptyset \) then \( P(\hat{A}_n = A) \rightarrow_{n \to \infty} 1 \). Therefore, let \( A^c \neq \emptyset \) and consider the inequality:

\[
r_n \|v_n^{opt}\|_1 \geq 1_{(A_n \cap A^c = \emptyset)} r_n \|v_n^{opt}\|_{A^c} \|_1 ,
\]

where \( 1_A \) denotes the indicator function of an event \( A \in \mathcal{F} \). Since the left hand side of this inequality is bounded in probability and condition (23) holds, we must have:

\[
\lim \sup_{n \to \infty} P(\hat{A}_n \cap A^c \neq \emptyset) = 0.
\]

Hence:

\[
P(\hat{A}_n = A) = P((\hat{A}_n \subset A) \cap (\hat{A}_n \cap A^c = \emptyset)) \rightarrow_{n \to \infty} 1.
\]

This concludes the proof.

\[\square\]

**Proof of Proposition 3.3.** Under the given conditions, the proximal estimator’s asymptotic distribution is given by the distribution of following Gaussian projection, using the same arguments as in the derivation of equation (18):

\[
\text{prox}_{\mathcal{S}_{\text{dom}(f_0)}(\beta_0)}(\eta) = P_{\text{span}\{e_j : j \in A\}}(M_0^+ Z).
\]

By definition,

\[
\left(P_{\text{span}\{e_j : j \in A\}}(M_0^+ Z)\right)_{A^c} = 0.
\]

Moreover, explicit computations yield:

\[
\left(P_{\text{span}\{e_j : j \in A\}}(M_0^+ Z)\right)_{A^c} = \left[(W_0)_{A}\right]^{-1}(W_0 M_0^+ Z)_{A^c}.
\]

The asymptotic covariance matrix of this last estimator is

\[
\left[(W_0)_{A}\right]^{-1}(W_0 M_0^+ \Omega_0 M_0^+ W_0)_{A} \left[(W_0)_{A}\right]^{-1}.
\]

This covariance matrix equals the covariance matrix of Gaussian random vector \([(M_0)_{A}]^+(Z)_{A}\) if and only if condition (28) holds. Furthermore, if \( \Omega_0 = \sigma^2 M_0 \) for some \( \sigma \in \mathbb{R} \) and \( W_0 = \overline{M_0} \) we obtain:

\[
(W_0 M_0^+ \Omega_0 M_0^+ W_0)_{A} = \sigma^2 (M_0)_{A},
\]

and

\[
(W_0)_{A}[(M_0)_{A}]^+(\Omega_0)_{A}[(M_0)_{A}]^+W_0)_{A} = \sigma^2 (M_0)_{A} (M_0)_{A} (\overline{M_0})_{A} = \sigma^2 (M_0)_{A},
\]

because \( (\overline{M_0})_{A} (M_0)_{A} = (M_0)_{A} (M_0)_{A}^+ \), i.e., condition (28) is satisfied. This concludes the proof. \[\square\]
Proof of Lemma 4.2. Under a singular design, $Q_n$ is an unbiased estimator of $Q_0$ and $Q_n \rightarrow P_r Q_0$ by Assumption 5 (i). Therefore, $\mathbb{P} (\text{Rank}(Q_n) = \text{Rank}(Q_0)) \rightarrow 1$ using the results in the Appendix of Madan et al. (1984). This convergence further implies $Q_n^+ \rightarrow P_r Q_0^+$ using Madan et al. (1984, Cor. 8). The statement for nearly-singular designs directly follows from the fact that $\text{Rank}(Q_n) > \text{Rank}(Q_0)$ for $n$ large enough, which is incompatible with the convergence of $Q_n^+$ to $Q_0^+$, using again Madan et al. (1984, Cor. 8). This concludes the proof.

Proof of Lemma 4.3. For a population parameter of interest such that $\beta_0 = P_0^\perp \beta_0$, we have:

$$
\tau_n (\hat{\beta}_0 n - \beta_0) = \frac{\tau_n (\hat{P}_0 n - P_0^\perp) \beta_0}{\tau_n (\hat{P}_0 n - P_0^\perp) \beta_0}
$$

$$
= -\tau_n (\hat{P}_0 n - P_0^\perp) \beta_0 + \tau_n P_0^\perp (Q_n Q_0 - Q_0 \beta_0 + \tau_n P_0^\perp (Q_n Q_0 - Q_0) Q_0^+ \beta_0 + o(1)
$$

$$
= \tau_n P_0^\perp (Q_n Q_0 - Q_0) P_0^\perp \beta_0 + \tau_n P_0^\perp (Q_n Q_0 - Q_0) Q_0^+ \beta_0 + o(1)
$$

This concludes the proof.

Proof of Proposition 4.1. Under Assumption 7, we have:

$$
\sigma(Q_n) - \sigma(Q_0) = O_P(1/\min\{\sqrt{n}, \tau_n\}),
$$

because $\sigma$ is Lipschitz continuous by the Hoffman-Wielandt Theorem Hoffman and Wielandt (1953). Moreover,

$$
\hat{\sigma}_n - \sigma(Q_0) = O_P(1/\min\{\sqrt{n}, \tau_n\}),
$$

using as in the proof of Proposition 3.2 the fact that proximal operators are Lipschitz continuous (see Bauschke et al. (2016, Prop. 12.30)). We next show that:

$$
\mathbb{P}(A = \hat{A}_n) \rightarrow 1,
$$

as $n \rightarrow \infty$, where $A := \{ j : \sigma_j(Q_0) \neq 0 \}$ and $\hat{A}_n := \{ j : \hat{\sigma}_{nj} \neq 0 \}$. Since

$$
\hat{\sigma}_n - \sigma(Q_0) = O_P(1/\min\{\sqrt{n}, \tau_n\}),
$$

we have $\mathbb{P}(A \subset \hat{A}_n) \rightarrow 1$, as $n \rightarrow \infty$. Hence, we are left to show that $\mathbb{P}(A^c \cap \hat{A}_n) \rightarrow 0$ as $n \rightarrow \infty$ when $A^c \neq \emptyset$. To this end, note that for penalty (39) proximal estimator (35) is given in closed-form for any $j = 1, \ldots, p$ by:

$$
\hat{\sigma}_{nj} = \max\{\sigma_j(Q_n) - \nu_n, 0\}.
$$

Therefore,

$$
\mathbb{P}(A^c \cap \hat{A}_n) \leq \sum_{j \in A^c} \mathbb{P} (\hat{\sigma}_{nj} \neq 0) = \sum_{j \in A^c} \mathbb{P} (\rho_n \sigma_j(Q_n) > \rho_n \nu_n) \rightarrow 0,
$$

as $n \rightarrow \infty$, because $\sigma_j(Q_n) = o(1/\rho_n)$ for any $j \in A^c$ and $\nu_n \rho_n \rightarrow \kappa > 0$. Summarizing, we have shown that proximal estimator (35) induced by a Lasso-type penalty consistently selects the active components in parameter vector $\sigma(Q_0)$:

$$
\mathbb{P}( (\hat{\sigma}_n)_{A^c} = (\sigma(Q_0))_{A^c} ) \rightarrow 1,
$$

In other words, we show that proximal estimator $\hat{\sigma}_n$ performs a consistent selection of the nonzero eigenvalues in spectrum $\sigma(Q_0)$. 

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21 In other words, we show that proximal estimator $\hat{\sigma}_n$ performs a consistent selection of the nonzero eigenvalues in spectrum $\sigma(Q_0)$. 

Proposition 4.1. To continue the proof, we first write the following identity:

\[ \Delta = E_n \text{diag}(\sigma_n) E'_n \to_{Pr} E_0 \text{diag}(\sigma(Q_0)) E'_0 = Q_0 \, , \]

and

\[ \mathbb{P}(\text{Rank}(\hat{Q}_n) = \text{Rank}(Q_0)) = \mathbb{P}(A_n = A) \to 1 \, , \]

as \( n \to \infty \). This convergence also implies \( \hat{Q}_n^+ \to_{Pr} Q_0^+ \) by Madan et al. (1984, Cor. 8), concluding the proof.

Proof of Theorem 4.4. To prove the theorem, denote by \( P_n \) the orthogonal projection matrix on the eigenspace generated by the eigenvectors of \( Q_n \) associated with its smallest \( k_0 = p - \text{Rank}(Q_0) \) eigenvalues, and by \( \hat{P}_n \) the orthogonal projection matrix on the kernel of \( Q_n \). Note that under the given assumptions, \( \mathbb{P}(\hat{P}_n = P_n) \to 1 \) as \( n \to \infty \) from Proposition 4.1. To continue the proof, we first write the following identity:

\[ \sqrt{n}(\hat{\beta}_n - \beta_0) = -\sqrt{n}(I - \hat{Q}_n^+ Q_n) \beta_0 + \hat{Q}_n^+ X' \epsilon / \sqrt{n} = -\sqrt{n}(P_n - P_0) \beta_0 + \hat{Q}_n^+ X' \epsilon / \sqrt{n} \, , \quad (51) \]

since \( \hat{Q}_n^+ Q_n = I - \hat{P}_n \) and \( P_0 \beta_0 = 0 \). We next make use of the following expansion, which holds for any consistent estimator \( A_n^+ \) of \( Q_0^+ \) (such as \( \hat{Q}_n^+ \)):

\[ P_n - P_0 = -Q_0^+ (Q_n - Q_0) P_n - P_0 (Q_n - Q_0) A_n^+ + o(1/\sqrt{n}) \]

where in the last equation we exploited the fact that \( Q_n - Q_0 = O_{Pr}(1/\sqrt{n}) \). Since \( \mathbb{P}(\hat{P}_n = P_n) \to 1 \) as \( n \to \infty \), identity (51) yields:

\[ \sqrt{n}(\hat{\beta}_n - \beta_0) = \sqrt{n}P_0(Q_n - Q_0)Q_0^+ \beta_0 + \hat{Q}_n^+ X' \epsilon / \sqrt{n} + o_{Pr}(1) \]

\[ = \sqrt{n}P_0(Q_n - Q_0)P_0^+ Q_0^+ \beta_0 + \hat{Q}_n^+ X' \epsilon / \sqrt{n} + o_{Pr}(1) \, . \]

If \( \Delta = 0 \), then \( Q_{0m} = Q_0 \) and \( P_0 Q_n = 0 \), \( \mathbb{P} \)-almost surely, i.e.:

\[ \sqrt{n}P_0(Q_n - Q_{0m}) = 0 \, . \]

If \( \Delta \neq 0 \), Assumption 8 (i) yields:

\[ \sqrt{n}P_0(Q_n - Q_{0m}) \to_{d} P_0 \Theta \, , \]

while, under Assumption 8 (ii) we have:

\[ \sqrt{n}P_0(Q_n - Q_{0m})P_0^+ = o_p(1) \, . \]

Furthermore, using Assumption 5 and Proposition 4.1, we obtain following convergence in distribution as \( \sqrt{n}/\tau_n \to c \) when \( n \to \infty \):

\[ \sqrt{n}P_0(Q_{0m} - Q_0)Q_0^+ \beta_0 + Q_n^+ X' \epsilon / \sqrt{n} \to_{d} cP_0 \Delta Q_0^+ \beta_0 + Q_0^+ Z \, . \]

This concludes the proof.

Proof of Corollary 4.6. Under Assumptions 5–7, where \( \Omega_0 = \sigma^2 Q_0 \), Assumption 8 (ii) and either \( \sqrt{n} = o(\tau_n) \) or \( P_0 \Delta P_0^+ = 0 \) holding, the modified Ridgeless estimator (40) for estimating sub parameter vector \((\beta_0)_A\) is asymptotically normally distributed with standard \( 1/\sqrt{n} \) convergence rate and an efficient covariance matrix \( \sigma^2 [(Q_0)_A]^+ \). Furthermore,

\[ \lambda_n \sqrt{n} f_n \to_{Pr} l_{dom(f_0)} \, , \]

as \( n \to \infty \). Therefore:

\[ \hat{Q}_n = E_n \text{diag}(\sigma_n) E'_n \to_{Pr} E_0 \text{diag}(\sigma(Q_0)) E'_0 = Q_0 \, , \]
in epigraph, where \( \text{dom}(f_0) = \text{span}\{e_j : \beta_{0j} \neq 0\} \). Therefore, we are in the setting of Proposition 3.3 with \( W_0 = M_0, \Omega_0 = \sigma^2 M_0 \) and \( M_0 := Q_0 \). As a consequence, the sufficient condition in Proposition 3.3 for Oracle property 2 in Definition 3.2 is satisfied. Oracle property 1 in Definition 3.2 is satisfied as well, because the sufficient Oracle condition in Proposition 3.2 holds under the given assumptions:

\[
\sqrt{n} \left\| (v_{n,pt}^{opt})_{A^c} \right\| = \lambda_n \sqrt{n} \sum_{j \in A^c} (1/|\beta_{nj}|) \rightarrow_P \infty .
\]

This concludes the proof. \qed

2 Consistency of proximal estimators

This section provides the proof of the consistency of proximal estimators under Assumptions 1 and 2, which ensure, under an appropriate asymptotic regime for tuning parameter \( \lambda_n \), that the objective function minimized by a proximal estimator converges in probability in epigraph to a deterministic strictly convex and coercive function.

Proposition 2.1 (Proximal estimators’ limit in probability). Let Assumptions 1 and 2 be satisfied.

(i) If \( \lambda_n \rightarrow \lambda_0 > 0 \), then:

\[
\left( \begin{array}{c} \text{prox}_{\lambda_n f_n}^W(\beta_n^*) \\ \text{prox}_{(\lambda_n f_n)^*}^W(\beta_n^*) \end{array} \right) \rightarrow_P \left( \begin{array}{c} \text{prox}_{\lambda_0 f_0}^W(\beta_0) \\ \text{prox}_{(\lambda_0 f_0)^*}^W(\beta_0) \end{array} \right)
\]

with convex conjugate \( f_0^* \) under inner product \( \langle \cdot, \cdot \rangle^W_0 \). Moreover,

\[
\text{prox}_{\lambda_0 f_0}(\beta_0) = \left( \text{Id} - \text{prox}_{(\lambda_0 f_0)^*}^W \right)(\beta_0).
\]

(ii) If \( \lambda_n \rightarrow \lambda_0 = 0 \) and \( \lambda_n f_n \rightarrow_P \text{Id}_{\text{dom}(f_0)} \) in epigraph, above limits hold with \( \text{prox}_{\lambda_0 f_0}^W(\beta_0) \) replaced by \( P_{\text{dom}(f_0)}^W(\beta_0) = \beta_0 \) and \( \text{prox}_{(\lambda_0 f_0)^*}^W(\beta_0) \) replaced by \( 0 \).

Proof. To prove the proposition, we first let \( g_n(\beta) := \frac{1}{2} \left\| \beta_n^* - \beta \right\|^2 W_n \) and \( g_0(\beta) := \frac{1}{2} \left\| \beta_0 - \beta \right\|^2 W_0 \). Moreover, let \( P_n(\beta) := g_n(\beta) + \lambda_n f_n(\beta) \) and \( P_0(\beta) := g_0(\beta) + h_0(\beta) \), where \( h_0 := \lambda_0 f_0 \) and \( h_0 := \text{Id}_{\text{dom}(f_0)} \), respectively, under asymptotic regime (i) and (ii). Given the strict convexity and coercivity Bauschke et al. (2016, Prop. 11.14) of functions \( P_n \) and \( P_0 \), we only need to prove that \( P_n \rightarrow_P P_0 \) in epigraph, in order to imply the convergence:

\[
\text{prox}_{\lambda_n f_n}^W(\beta_n^*) = \arg\min_{\beta \in \mathbb{R}^p} P_n(\beta) \rightarrow_P \arg\min_{\beta \in \mathbb{R}^p} P_0(\beta) = \text{prox}_{\lambda_0 f_0}^W(\beta_0) .
\]

(52)

To show that \( P_n \rightarrow_P P_0 \) in epigraph, note that \( \lambda_n f_n \rightarrow_P h_0 \) in epigraph by assumption. Moreover, \( g_n \rightarrow_P g_0 \) uniformly on compact subsets of \( \mathbb{R}^p \), because \( g_n \) is continuous in parameters \( W_n \) and \( \beta_n^* \), which have limit in probability \( W_0 \) and \( \beta_0 \), respectively, under Assumption 1. Therefore, from Attouch (1984, Thm. 2.15) \( P_n \rightarrow_P P_0 \) in epigraph, and convergence (52) holds. The identity \( \text{prox}_{\lambda_0 f_0}^W = \text{Id} - \text{prox}_{h_0}^W \), with \( h_0 = (\lambda_0 f_0)^* \) follows under asymptotic regime (i) with Moreau’s decomposition Bauschke et al. (2016, Thm. 14.3). Similarly, under asymptotic regime (ii) \( h_0 = \text{Id}_{\text{dom}(f_0)} \) and \( \beta_0 = \text{prox}_{(\lambda_0 f_0)^*}^W(\beta_0) = 0 \) from Moreau’s decomposition Bauschke et al. (2016, Thm. 14.3). This concludes the proof. \[\square\]
3 Benchmark convex penalty functions

3.1 Proximal estimators induced with linear initial estimators

While in the main text we explored useful links between PLSEs and linear ordinary LS–type estimators, further such links can be established. For instance, since from Proposition 2.1 any proximal estimator depending on a sublinear penalty can always be defined and computed with an extended soft-thresholding formula, it may be convenient where possible to work with proximal estimators defined with a sublinear penalty and a linear initial estimator different from the LSE.

Let for instance $A_n = Q_n(\lambda_{2n})$ in Proposition 3, where $\lambda_{2n} > 0$ and $Q_n(\lambda_{2n}) := \lambda_{2n} I_n + Q_n$. Then, $\hat{\beta}_n = \beta_n := Q_n(\lambda_{2n})^{-1} X'Y / n$ is the Ridge estimator. Since $\tilde{A}_n = Q_n(\lambda_{2n})$ by construction, it then follows, for any $\lambda_{1n} \geq 0$ and penalty $f_n \in \Gamma(R^p)$:

$$\operatorname{prox}_{\tilde{A}_n}^{\lambda_{1n}, f_n}(\hat{\beta}_n) = \arg\min_{\beta \in \mathbb{R}^p} \left\{ \frac{1}{2n} \| Y - X \beta \|^2 + \frac{\lambda_{2n}}{2} \| \beta \|^2 + \lambda_{1n} f_n(\beta) \right\}.$$  \hspace{1cm} (53)

Therefore, any PLSE with penalty given by a weighted sum of the Ridge penalty and penalty $f_n \in \Gamma(R^p)$ is a proximal estimator obtained by an application of proximal operator $\operatorname{prox}_{\tilde{A}_n}^{\lambda_{1n}, f_n}$ to the Ridge estimator.

For instance, when $f_n = \| \cdot \|_1$ is the Lasso penalty, then $\operatorname{prox}_{\tilde{A}_n}^{Q_n(\lambda_{2n})}(\hat{\beta}_n)$ is the so-called naive Elastic Net estimator in Zou and Hastie (2005), which with Proposition 1 and Proposition 2 satisfies the closed-form projection formula:

$$\operatorname{prox}_{\tilde{A}_n}^{Q_n(\lambda_{2n})}(\hat{\beta}_n) = \left( I_d - P_{\tilde{C}(\lambda_{1n}, \lambda_{2n})} \right) (\hat{\beta}_n),$$

with polyhedron:

$$\tilde{C}(\lambda_{1n}, \lambda_{2n}) := \bigcap_{j=1}^p \{ \theta : |(e_j, \theta)| Q_n(\lambda_{2n}) | \leq \lambda_{1n} \}.$$  \hspace{1cm} (54)

The naive Elastic Net estimator is known to produce an excessive amount of shrinkage, which is why Zou and Hastie (2005) propose to rescale it by scaling factor $1 + \lambda_{2n}$, giving rise to the so-called Elastic Net estimator. The Elastic Net estimator is a proximal estimator $\operatorname{prox}_{\tilde{A}_n}^{Q_n(\lambda_{2n})}(\hat{\beta}_n)$, which depends on a rescaled weighting matrix

$$Q_n(\lambda_{2n}) := \frac{1}{1 + \lambda_{2n}} Q_n(\lambda_{2n}),$$

and a rescaled initial Ridge estimator:

$$\tilde{\beta}_n := (Q_n(\lambda_{2n}))^{-1} (X'Y / n).$$

Therefore, it satisfies the projection formula:

$$\operatorname{prox}_{\tilde{A}_n}^{Q_n(\lambda_{2n})}(\hat{\beta}_n) = \left( I_d - P_{\tilde{C}(\lambda_{1n}, \lambda_{2n})} \right) (\tilde{\beta}_n),$$  \hspace{1cm} (55)

with polyhedron:

$$\tilde{C}(\lambda_{1n}, \lambda_{2n}) := \bigcap_{j=1}^p \{ \theta : |(e_j, \theta)| Q_n(\lambda_{2n}) | \leq \lambda_{1n} \}.$$  \hspace{1cm} (56)

Clearly, analogous projection formulas hold for the Adaptive Elastic Net Estimator in Zou and Zhang (2009). This estimator is a proximal estimator $\operatorname{prox}_{\tilde{A}_n}^{Q_n(\lambda_{2n})}(\hat{\beta}_n)$, which is defined using the Adaptive Lasso penalty $f_n$ in Proposition 2 of the main text.
3.2 Benchmark penalties and their convex transforms

Our general convex analysis framework naturally covers proximal estimation settings induces by several benchmark penalties $f_n$ in the literature. Examples of such non-stochastic penalties are the Ridge (Hoerl and Kennard (1970)), the Lasso (Tibshirani (1996)), the Elastic Net (Zou and Hastie (2005)) and the Group Lasso (Yuan and Lin (2006)). Examples of stochastic penalties are the Adaptive Lasso (Zou (2006)) and the Adaptive Elastic Net (Zou and Zhang (2009)). Some of these penalties are collected for easier reference in Table 1 of this Online Appendix.

Penalties in Table 1 may depend on (i) $l_r$-norms, defined by $\|\beta\|_r := \left(\sum_{j=1}^p |\beta_j|^r\right)^{1/r}$ for $r \geq 1$ and by $\|\beta\|_\infty := \max\{|\beta_j| : j = 1,\ldots,p\}$ for $r = \infty$, (ii) a $n^\gamma$-consistent estimator $\hat{\beta}_n (\gamma > 0)$ of $\beta_0$ and (iii) an indicator function of a closed convex set $S$, defined by $\iota_S(\theta) = 0$ for $\theta \in S$ and $\iota_S(\theta) = \infty$ else. In the Group lasso, the parameter of interest is partitioned as $\beta = (\beta_1',\ldots,\beta_K')'$, using corresponding subvectors $\beta_1,\ldots,\beta_K$ of varying dimension. In the constrained Least Squares setting, $C \subset \mathbb{R}^p$ is a nonempty closed convex set defining a family of convex constraints on parameter $\theta$.

Convex conjugates $f_n^*$ of these penalties in Table 1 may depend on (i) dual norms of some $l_r$-norm, (ii) the Euclidean distance from a closed convex set $S$, defined by $d_S(\theta) := \inf_{\beta \in S} \|\theta - \beta\|_2$, and (iii) the support function of a set $S \subset \mathbb{R}^p$, defined as $\sigma_S(\theta) := \sup_{\beta \in S} \{\langle \theta, \beta \rangle \}$. Note that while convex conjugate penalty $(\lambda_n f_n)^*$ is defined under inner product $\langle \cdot, \cdot \rangle_W$, it is equivalently given by a conjugate penalty of the form $\lambda_n f_n^*(W_n, \cdot)$, where $f_n^*$ is the convex conjugate of penalty $f_n$ under the Euclidean inner product.

Table 2 of this Online Appendix reports the directional derivatives $\partial_\beta f_n(\beta)$ for the limit penalties $f_0$ associated with benchmark penalties $f_n$ in Table 1. For the Group Lasso, the notation $b_k^{(j)}$ denotes the $j^{th}$ component of subvector $b_k$, where $b = (b_1',\ldots,b_K')' \in \mathbb{R}^p$. For the constrained Least Squares, the directional derivative equals the support function $\sigma_{N_C}(\beta_0)$ of the normal cone of set $C$ at $\beta_0$; see Bauschke et al. (2016, Example 16.13) and Bauschke et al. (2016, Prop. 17.17). The directional derivative of the Elastic Net penalty is not given in Table 2, since it is readily obtained as a convex combination with weights $w \in (0,1)$ and $1-w$ of the directional derivatives of the Lasso and the Ridge. Table 3 reports instead the subgradients $\partial f_0(\beta_0)$ of the limit penalties $f_0$ associated with benchmark penalties $f_n$ from Table 1. The subgradient of the Elastic Net penalty, which is not reported, is given by the Cartesian product of the subgradient of the Lasso penalty scaled by $w$ and the subgradient of the Ridge penalty scaled by $1-w$, respectively.

4 Monte Carlo evidence

We document via Monte Carlo simulation the finite sample properties of our proximal estimators robust to irregular designs. To this end, we consider the regular Monte Carlo setting in Tibshirani (1996, Example 1) and extend it to jointly cover a regular, a singular and a nearly-singular design. We generate 5000 realizations of random vector $Y$ from linear model (1) using parameter vector:

$$ (3, 1.5, 0, 0, 2, 0, 0, 0)' \; , $$

an IID normally distributed error term $\epsilon \sim N(0,\sigma_0^2 I_n)$ with $\sigma_0^2 = 2$. To generate the predictors, we consider a setting with a regular, a singular and a nearly-singular design. In the regular design, we simulate the predictors as $X_i \sim N(0,Q_r)$ with a regular matrix $Q_r$. In the singular design, we simulate the predictors as $X_i \sim N(0,Q_s)$ with a singular matrix $Q_s$. Finally, in the nearly-singular design, we simulate the predictors as $X_i \sim N(0,Q_{0n})$.
with nearly-singular matrix:

\[ Q_{0:n} = Q_s + (Q_r - Q_s)/\sqrt{n}, \]

which corresponds to a parametrization \( \tau_n = \sqrt{n} \) of near singularity in Assumption 7 with \( \Delta = Q_r - Q_s \). Matrix \( Q_r \) has \( jk \)-components equal to \( 0.5^{j-k} \) as in Tibshirani (1996, Example 1). Singular matrix \( Q_s \) is identical to \( Q_r \), but with the fifth row (column) replaced by a fix linear combination of the second and third rows (columns) of \( Q_r \). A visual representation of the covariance structure under \( Q_r \) and \( Q_s \) is depicted in Figure 1.

With these parametrizations, we build the relevant parameter of interest given by the minimum norm least squares solution in each of the above settings. By construction, we have \( \beta_0 = (3, 1.5, 0, 0, 2, 0, 0, 0)' \) under a regular design, while we obtain \( \beta_0 = (3, 1.893, 0.393, 0, 1.32, 0, 0, 0)' \) under an irregular design. We build our modified Ridgeless estimator using a tuning parameter \( \nu_n = n^{-\alpha} \) such that \( \alpha \in [0.1, 0.5] \), which is covered by the admissible asymptotic tuning parameter regimes in Proposition 4.1.

Figure 2 compares the box plots of sample squared estimation errors for Ridgeless (RL) \( \hat{\beta}_n^{ls} \) and Modified Ridgeless (MRL) \( \hat{\beta}_n^{ls} \) initial estimators using sample sizes \( n = 100, 200 \). We find that, under a regular or singular design, the two estimators produce a similar distribution of sample squared estimation errors. In contrast, a notable difference emerges in a nearly-singular design scenario, where sample squared estimation errors of the RL estimator are approximately an order of magnitude larger. To assess the robustness of these results to the choice of the MRL hyper-parameter, we show in Figure 3 that using \( \nu_n = n^{-\alpha} \) for any \( \alpha \in [0.3, 0.5] \) delivers essentially identical Monte Carlo results. Figure 4 further demonstrates that the RL estimator is not \( \sqrt{n} \)-consistent under a nearly-singular design, while the MRL estimator is. Indeed, the Monte Carlo quartiles of the distribution of \( \sqrt{n} (\hat{\beta}_n^{ls} - \beta_0) \) all diverge as the sample size increases, while those of \( \sqrt{n} (\hat{\beta}_n^{ls} - \beta_0) \) converge quite rapidly.

The above features of RL and MRL estimators clearly impact those of proximal estimators built on them. Figure 5 reports the quartiles of sample squared estimation errors of Adaptive Lasso proximal estimators built with a RL (RLAL) and a MRL (MRAL) initial estimator, based on various choices of tuning parameter \( \lambda_n = n^{-\gamma} \) with \( \gamma \in (0.5, 1) \), consistently with the admissible asymptotic regimes in Corollary 4.6.\(^{22}\) We find that, under a regular or a singular design, both estimators produce similar quartiles of sample squared estimation errors. In contrast, under a nearly-singular design the quartiles of sample squared estimation errors for the RLAL proximal estimator are dramatically larger. Moreover, the quartiles of sample squared estimation errors of the MRAL proximal estimator systematically improve on those of the MRL estimator across all designs for essentially all tuning parameter choices such that \( \gamma \in (0.5, 1) \).

Finally, Figure 6 reports the surfaces of Monte Carlo variable selection probabilities \( \mathbb{P}(\hat{A}_n = A) \) of RLAL and MRLAL proximal estimators in dependence of tuning parameter \( \gamma \in (0.5, 1) \).\(^{23}\) Under a regular and singular design, both estimators produce essentially identical variable selection probabilities that monotonically increase (decrease) with the sample size (tuning parameter \( \gamma \)), as theoretically expected. Under a nearly-singular design, the exact variable selection probabilities of MRLAL proximal estimators are substantially larger and still increase with the sample size, as theoretically expected. In contrast, the exact variable selection probabilities of RLAL proximal estimators are not even increasing with the sample size.

---

\(^{22}\)By construction, under a regular design the RLAL estimator is the standard Adaptive Lasso estimator.

\(^{23}\)Figure 7 of the Online Appendix reports for completeness also the Monte Carlo inclusion probabilities \( \mathbb{P}(\hat{A}_n \supset A) \) of estimators RLAL and MRLAL.
5 Figures and tables

Figure 1: Heatmap of population design matrices $Q_r$ and $Q_s$ in our Monte Carlo simulation settings.
Figure 2: Monte Carlo quartiles of sample squared errors $\| \hat{\beta}_n - \beta_0 \|_2^2$ for Ridgeless (RL, black solid lines and areas) and modified Ridgeless (MRL, blue dashed lines and areas), using a tuning parameter $\nu_n = n^{-3/8}$ and sample sizes $n = 100, 200$, under a regular, singular and nearly-singular design, respectively.
Figure 3: Monte Carlo quartiles of sample squared errors $\left\| \hat{b}_n^n - \beta_0 \right\|_2^2$ for Ridgeless (RL, black solid lines and areas) and modified Ridgeless (MRL, blue dashed lines and areas), using a tuning parameter $\nu_n = n^{-\alpha}$ [$\alpha \in [0.1, 0.5]$] and sample sizes $n = 100, 200$, under a regular, singular and nearly-singular design, respectively.

Figure 4: Monte Carlo quartiles of normalized sample squared errors $n \left\| \hat{b}_n^n - \beta_0 \right\|_2^2$ for Ridgeless (RL, black line and area) and modified Ridgeless (MRL, blue line and area) proximal estimators, using a tuning parameter $\nu_n = n^{-3/8}$ and sample sizes $n \in [100, 1000]$, under a nearly singular design.
Figure 5: Monte Carlo quartiles of sample squared errors $\left\| \hat{\beta}_n - \beta_0 \right\|^2_2$ for modified Ridgeless (MRL, black dotted lines) using tuning parameter $\nu_n = n^{-3/8}$, Ridgeless Adaptive Lasso (RLAL, blue solid lines and areas) and modified Ridgeless Adaptive Lasso (MR-LAL, red dashed lines and areas) proximal estimators, using tuning parameters $\lambda_n = n^{-\gamma}$ [$\gamma \in (0.5, 1)$] and sample sizes $n = 100, 200$, under a regular, singular and nearly singular design, respectively.
Figure 6: Monte Carlo detection probabilities $\mathbb{P}(\hat{A}_n = A)$ for Ridgeless Adaptive Lasso (RLAL, blue solid line) and modified Ridgeless Adaptive Lasso (MRLAL, red dashed line) proximal estimators, using tuning parameters $\lambda_n = n^{-\gamma}$ [$\gamma \in (0.5, 1)$] and sample sizes $n = 100, 200$, under a regular, singular and nearly singular design, respectively.
Figure 7: Monte Carlo inclusion probabilities $\mathbb{P}(\hat{A}_n \supset A)$ for Ridgeless Adaptive Lasso (RLAL, blue solid line) and modified Ridgeless Adaptive Lasso (MRLAL, red dashed line) proximal estimators, using tuning parameters $\lambda_n = n^{-\gamma}$ [$\gamma \in (0.5, 1)$] and sample sizes $n = 100, 200$, under a regular, singular and nearly-singular design, respectively.
| Penalty            | \( f_n(\beta) \)                                      | \( \lambda_n f_n^*(\theta/\lambda_n) \) |
|-------------------|------------------------------------------------------|------------------------------------------|
| Ridge             | \( \frac{1}{2} \| \beta \|_2^2 \)                  | \( \frac{1}{2\lambda_n} \| \theta \|_2^2 \) |
| Lasso             | \( \| \beta \|_1 \)                                 | \( \iota_{B_n}(\theta), \ B_n := \bigcap_{i=1}^p \{ \theta : |\theta_j| \leq \lambda_n \} \) |
| Adaptive Lasso    | \( \sum_{j=1}^p |\beta_j|/|\tilde{\beta}_{nj}| \) | \( \iota_{\tilde{B}_n}(\theta), \ \tilde{B}_n := \bigcap_{i=1}^p \{ \theta : |\theta_j| \leq \lambda_n/|\tilde{\beta}_{nj}| \} \) |
| Group Lasso       | \( \sum_{k=1}^K \| \beta_k \|_2 \)                | \( \iota_{G_n}(\theta), \ G_n := \bigcap_{k=1}^K \{ \theta : \| \theta_k \|_2 \leq \lambda_n \} \) |
| Elastic Net       | \( \| \beta \|_1 + \frac{1-w}{2} \| \beta \|_2^2 \), \ w \in (0, 1) | \( \frac{1}{2\lambda_n(1-w)} \sigma_C(\theta) \) |
| constrained LS    | \( \iota_C(\beta) \)                                | \( \frac{1}{2\lambda_n(1-w)} \sigma_C(\theta) \) |

**Table 1:** Penalty functions and convex conjugates under the Euclidean inner product
### Table 2: Penalty functions and directional derivatives $\rho_{\beta_0}(b)$

| Penalty $f_n(\beta)$ | Directional derivative $\rho_{\beta_0}(b)$ |
|-----------------------|------------------------------------------|
| $\frac{1}{2} \|\beta\|^2_2$ | $\langle b_i, \beta_0 \rangle$ |
| $\|\beta\|_1$ | $\sum_j b_j \text{sign}(\beta_{0j}) I_{\{\beta_{0j} \neq 0\}} + |b_j| I_{\{\beta_{0j} = 0\}}$ |
| $\sum_{j=1}^p |\beta_j|/|\tilde{\beta}_{nj}|$ | $\sum_j \left[ \frac{b_j}{\beta_{0j}} I_{\{\beta_{0j} \neq 0\}} + \ell_{0j} I_{\{\beta_{0j} = 0\}} \right]$ |
| $\sum_{k=1}^K \|\beta_k\|_2$ | $\sum_{k=1}^K \left[ \frac{\|b_k\|_2}{\|\beta_{0k}\|_2} I_{\{\beta_{0k} \neq 0\}} + \|b_k\|_2 I_{\{\beta_{0k} = 0\}} \right]$ |
| $\iota_C(\beta)$ | $\sigma_{N_C(\beta_0)}(b)$ |

### Table 3: Penalty functions and subgradients $\partial f_0(\beta_0)$

| Penalty $f_n(\beta)$ | Subgradient $\partial f_0(\beta_0)$ |
|-----------------------|-----------------------------------|
| $\frac{1}{2} \|\beta\|^2_2$ | $\{\beta_0\}$ |
| $\|\beta\|_1$ | $(\bigcap_{j: \beta_{0j} \neq 0} \{t_j : t_j = \text{sign}(\beta_{0j})\}) \cap (\bigcap_{j: \beta_{0j} = 0} \{t_j : t_j \in [-1, 1]\})$ |
| $\sum_{j=1}^p |\beta_j|/|\tilde{\beta}_{nj}|$ | $\bigcap_{j: \beta_{0j} \neq 0} \{t_j : t_j = 1/\beta_{0j}\}$ |
| $\sum_{k=1}^K \|\beta_k\|_2$ | $(\bigcap_{k: \beta_{0k} \neq 0} \{t_k : t_k^{(j)} = \text{sign}(\beta_{0k}^{(j)})/\|b_k\|_2^2\}) \cap (\bigcap_{k: \beta_{0k} = 0} \{t_k^{(j)} : t_k^{(j)} \in [-1, 1]\})$ |
| $\iota_C(\beta)$ | $N_C(\beta_0)$ |
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