On hamiltonian colorings of trees

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Abstract. A hamiltonian coloring $c$ of a graph $G$ of order $n$ is a mapping $c : V(G) \to \{0, 1, 2, \ldots\}$ such that $D(u, v) + |c(u) - c(v)| \geq n - 1$, for every two distinct vertices $u$ and $v$ of $G$, where $D(u, v)$ denotes the detour distance between $u$ and $v$ which is the length of a longest $u,v$-path in $G$. The value $hc(c)$ of a hamiltonian coloring $c$ is the maximum color assigned to a vertex of $G$. The hamiltonian chromatic number, denoted by $hc(G)$, is the $\min\{hc(c)\}$ taken over all hamiltonian coloring $c$ of $G$. In this paper, we present a lower bound for the hamiltonian chromatic number of trees and give a sufficient condition to achieve this lower bound. Using this condition we determine the hamiltonian chromatic number of symmetric trees, firecracker trees and a special class of caterpillars.

Keywords: Hamiltonian coloring, hamiltonian chromatic number, symmetric tree, firecracker, caterpillar.

1 Introduction

A hamiltonian coloring $c$ of a graph $G$ of order $n$ is a mapping $c : V(G) \to \{0, 1, 2, \ldots\}$ such that $D(u, v) + |c(u) - c(v)| \geq n - 1$, for every two distinct vertices $u$ and $v$ of $G$, where $D(u, v)$ denotes the detour distance between $u$ and $v$ which is the length of a longest $u,v$-path in $G$. The value $hc(c)$ of a hamiltonian coloring $c$ is the maximum color assigned to a vertex of $G$. The hamiltonian chromatic number $hc(G)$ of $G$ is $\min\{hc(c)\}$ taken over all hamiltonian coloring $c$ of $G$. It is clear from definition that two vertices $u$ and $v$ can be assigned the same color only if $G$ contains a hamiltonian $u,v$-path, and hence a graph $G$ can be colored by a single color if and only if $G$ is hamiltonian-connected. Thus the hamiltonian chromatic number of a connected graph $G$ measures how close $G$ is to being hamiltonian-connected. The concept of hamiltonian coloring was introduced by Chartrand et al.\cite{2} which is a variation of radio $k$-coloring of graphs.

At present, the hamiltonian chromatic number is known only for handful of graph families. Chartrand et al.\cite{23} determined the hamiltonian chromatic number for complete graph $K_n$, cycle $C_n$, star $K_{1,k}$, complete bipartite graph $K_{r,s}$ and presented upper bound for the hamiltonian chromatic number of paths and trees. The exact value of hamiltonian chromatic number of paths which is equal to the radio antipodal number $ac(P_n)$ was given by Khennoufa and Togni
in [5]. Shen et al. [7] discussed the hamiltonian chromatic number for graphs $G$ with $\max\{D(u, v) : u, v \in V(G), u \neq v\} \leq n/2$, where $n$ is the order of graph $G$; such graphs are called graphs with maximum distance bound $n/2$ or $DB(n/2)$ graphs for short and they determined the hamiltonian chromatic number for double stars and a special class of caterpillars.

In this paper, we present a lower bound for the hamiltonian chromatic number of trees (Theorem 4) and give a sufficient condition to achieve this lower bound (Theorem 5). Using this condition we determine the hamiltonian chromatic number of symmetric trees, firecracker trees and a special class of caterpillars. We use an approach similar to the one used in [1] to derive a lower bound of the hamiltonian chromatic number of trees. We remark that our proof for the hamiltonian chromatic number of a special class of caterpillars is simpler than one given in [7] by different approach. We also inform the readers that the hamiltonian chromatic number obtain in this paper is one less than that defined in [2, 3, 4, 5, 7] as we allowed 0 for coloring while they do not.

2 Preliminaries

A tree is a connected graph that contains no cycle. The diameter of $T$, denoted by $diam(T)$ or simply $d$, is the maximum distance among all pairs of vertices in $T$. The eccentricity of a vertex in a graph is the maximum distance from it to other vertices in the graph, and the center of a graph is the set of vertices with minimum eccentricity. It is well known that the center of a tree $T$, denoted by $C(T)$, consists of a single vertex or two adjacent vertices, called the central vertex/vertices of $T$. We view $T$ as rooted at its central vertex/vertices; if $T$ has only one central vertex $w$ then $T$ is rooted at $w$ and if $T$ has two adjacent central vertices $w$ and $w'$ then $T$ is rooted at $w$ and $w'$ in the sense that both $w$ and $w'$ are at level 0. If $u$ is on the path joining another vertex $v$ and central vertex $w$, then $u$ is called ancestor of $v$, and $v$ is a descendant of $u$. Let $u \notin C(T)$ be adjacent to a central vertex. The subtree induced by $u$ and all its descendant is called a branch at $u$. Two branches are called different if they are at two vertices adjacent to the same central vertex, and opposite if they are at two vertices adjacent to different central vertices. Define the detour level of a vertex $u$ from the center of graph by

$$L(u) := \min\{D(u, w) : w \in C(T)\}, u \in V(T).$$

Define the total detour level of $T$ as

$$L(T) := \sum_{u \in V(T)} L(u).$$

For any $u, v \in V(T)$, define $\phi(u, v) := \max\{L(t) : t$ is a common ancestor of $u$ and $v\}$, and

$$\delta(u, v) := \begin{cases} 
1, & \text{if } C(T) = \{w, w'\} \text{ and path } P_{uv} \text{ contains an edge } ww', \\
0, & \text{otherwise.}
\end{cases}$$
Lemma 1. Let $T$ be a tree with diameter $d \geq 2$. Then for any $u, v \in V(T)$ the following holds:

1. $\phi(u, v) \geq 0$;
2. $\phi(u, v) = 0$ if and only if $u$ and $v$ are in different or opposite branches;
3. $\delta(u, v) = 1$ if and only if $T$ has two central vertices and $u$ and $v$ are in opposite branches;
4. the detour distance $D(u, v)$ in $T$ between $u$ and $v$ can be expressed as

$$D(u, v) = \mathcal{L}(u) + \mathcal{L}(v) - 2\phi(u, v) + \delta(u, v).$$

(1)

Note that for a tree $T$ the detour distance $D(u, v)$ is same as the ordinary distance $d(u, v)$ as there is unique path between any two vertices $u$ and $v$ of $T$. Thus, one can use expression (1) for ordinary distance $d(u, v)$ which can also be used for other purpose.

Define

$$\varepsilon(T) := \begin{cases} 0, & \text{if } C(T) = \{w\}, \\ 1, & \text{if } C(T) = \{w, w'\}. \end{cases}$$

$$\varepsilon'(T) := 1 - \varepsilon(T).$$

3 On Hamiltonian colorings of trees

For a connected graph $G$ of order $n \geq 5$, by defining $D(\sigma) = \sum_{i=1}^{n-1} D(v_i, v_{i+1})$ for an ordering $\sigma : v_1, v_2, \ldots, v_n$ and $\overline{D}(G) = \max\{D(\sigma) : \sigma \text{ is an ordering of } V(G)\}$, Chartrand et al.\cite{4} established the following lower bound for the hamiltonian chromatic number of a connected graph $G$.

Theorem 1. (\cite{4}) If $G$ is a connected graph of order $n \geq 5$, then $hc(G) \geq (n - 1)^2 + 1 - \overline{D}(G)$.

For an ordering $\sigma : v_1, v_2, \ldots, v_n$ of the vertices of $G$, define $c_\sigma$ to be an assignment of positive integers to $V(G)$: $c_\sigma(v_1) = 1$ and $c_\sigma(v_{i+1}) - c_\sigma(v_i) = (n-1) - D(v_i, v_{i+1})$ for each $1 \leq i \leq n-1$. If $\max\{D(u, v) : u,v \in V(G), u \neq v\} \leq n/2$ for a connected graph $G$ of order $n$ then such a graph $G$ is called a graph with maximum distance bound $n/2$ or $DB(n/2)$ graph for short. Shen et al.\cite{7} proved the following Theorems about $DB(n/2)$ graphs and using it determined the hamiltonian chromatic number for double stars and a special class of caterpillars.

Theorem 2. (\cite{7}) Let $G$ be a $DB(n/2)$ graph of order $n \geq 4$. Then for any $\sigma$, $c_\sigma$ is a hamiltonian coloring for $G$ with $hc(c_\sigma) = (n - 1)^2 + 1 - D(\sigma)$. 
Theorem 3. ([7]) If \( G \) is DB\((n/2)\) graph of order \( n \geq 5 \), then \( hc(G) = (n - 1)^2 + 1 - D(G) \), and for any \( \sigma \) such that \( D(\sigma) = D(G) \), \( hc(\sigma) = hc(G) \). Namely, \( c_\sigma \) is a minimum hamiltonian coloring for \( G \).

Now, let \( T \) be a tree with maximum degree \( \Delta \). Note that a hamiltonian coloring \( c \) of \( T \) is injective for \( \Delta(T) \geq 3 \) as in this case no two vertices of \( T \) contain hamiltonian path. Throughout this section we consider \( T \) with \( \Delta(T) \geq 3 \) then \( c \) induces a linear order of the vertices of \( T \), namely \( V(T) = \{u_0, u_1, ..., u_{n-1}\} \) (where \( n = |V(T)| \)) such that

\[
0 = c(u_0) < c(u_1) < ... < c(u_{n-1}) = \text{span}(c).
\]

Theorem 4. Let \( T \) be a tree of order \( n \geq 4 \) and \( \Delta(T) \geq 3 \). Then

\[
hc(T) \geq (n - 1)(n - 1 - \varepsilon(T)) + \varepsilon'(T) - 2\mathcal{L}(T).
\]

Proof. It is enough to prove that any hamiltonian coloring of \( T \) has span not less than the right-hand side of (2). Suppose \( c \) is any hamiltonian coloring of \( T \) then \( c \) order the vertices of \( T \) into a linear order \( u_0, u_1, ..., u_{n-1} \) such that \( 0 = c(u_0) < c(u_1) < ... < c(u_{n-1}) \). By definition of \( c \), we have \( c(u_{i+1}) - c(u_i) \geq n - 1 - D(u_i, u_{i+1}) \) for \( 0 \leq i \leq n - 1 \). Summing up these \( n - 1 \) inequalities, we obtain

\[
\text{span}(c) = c(u_{n-1}) - c(u_0) \geq (n - 1)^2 - \sum_{i=0}^{n-1} D(u_i, u_{i+1})
\]

(3)

Case-1: \( T \) has one central vertex. In this case, we have \( \phi(u_i, u_{i+1}) \geq 0 \) and \( \delta(u_i, u_{i+1}) = 0 \) for \( 0 \leq i \leq n - 2 \) by the definition of the function \( \phi \) and \( \delta \). Since \( T \) has only one central vertex, \( u_0 \) and \( u_{n-1} \) cannot be the central vertex of \( T \) simultaneously. Hence \( \mathcal{L}(u_0) + \mathcal{L}(u_{n-1}) \geq 1 \). Thus, by substituting (1) in (3),

\[
\text{span}(c) \geq (n - 1)^2 - \sum_{i=0}^{n-1} \left[ \mathcal{L}(u_i) + \mathcal{L}(u_{i+1}) - 2\phi(u_i, u_{i+1}) - \delta(u_i, u_{i+1}) \right]
\]

\[
= (n - 1)^2 - 2\sum_{i=0}^{n-1} \mathcal{L}(u_i) + \mathcal{L}(u_0) + \mathcal{L}(u_{n-1}) - 2\sum_{i=0}^{n-1} \phi(u_i, u_{i+1})
\]

\[
\geq (n - 1)^2 + 1 - 2\mathcal{L}(T)
\]

\[
= (n - 1)(n - 1 - \varepsilon(T)) + \varepsilon'(T) - 2\mathcal{L}(T).
\]

Case-2: \( T \) has two central vertices. In this case, we have \( \phi(u_i, u_{i+1}) \geq 0 \) and \( \delta(u_i, u_{i+1}) \leq 1 \) for \( 0 \leq i \leq n - 2 \) by the definition of the function \( \phi \) and \( \delta \). Since \( T \) has two central vertices, we can set \( \{u_0, u_{n-1}\} = \{w, w'\} \). Thus, by substituting (1) in (3),

\[
\text{span}(c) \geq (n - 1)^2 - \sum_{i=0}^{n-1} [\mathcal{L}(u_i) + \mathcal{L}(u_{i+1}) - 2\phi(u_i, u_{i+1}) + \delta(u_i, u_{i+1})]
\]
\begin{align*}
(n-1)^2 - 2 \left( \sum_{i=0}^{n-1} |L(u_i) + L(u_{i+1})| - 2 \sum_{i=0}^{n-1} \phi(u_i, u_{i+1}) + \sum_{i=0}^{n-1} \delta(u_i, u_{i+1}) \right) \\
= (n-1)^2 - 2 \sum_{i=0}^{n-1} L(u_i) + L(u_0) + L(u_{n-1}) + \sum_{i=0}^{n-1} \delta(u_i, u_{i+1}) \\
\geq (n-1)^2 - 2 \sum_{u \in V(T)} L(u_i) + (n-1) \\
= (n-1)(n-2) - 2\mathcal{L}(T) \\
= (n-1)(n-1-\varepsilon(T)) + \varepsilon(T) - 2\mathcal{L}(T).
\end{align*}

**Theorem 5.** Let $T$ be a tree of order $n \geq 4$ and $\Delta(T) \geq 3$. Then

\[bc(T) = (n-1)(n-1-\varepsilon(T)) + \varepsilon(T) - 2\mathcal{L}(T) \quad (4)\]

holds if there exists a linear order $u_0, u_1, \ldots, u_{n-1}$ with $\theta = c(u_0) < c(u_1) < \ldots < c(u_{n-1})$ of the vertices of $T$ such that

1. $u_0 = w$, $u_{n-1} \in N(w)$ when $C(T) = \{w\}$ and $\{u_0, u_{n-1}\} = \{w, w'\}$ when $C(T) = \{w, w'\}$.
2. $u_i$ and $u_{i+1}$ are in different branches when $C(T) = \{w\}$ and opposite branches when $C(T) = \{w, w'\}$.
3. $D(u_i, u_{i+1}) \leq n/2$, for $0 \leq i \leq n-2$.

Moreover, under these conditions the mapping $c$ defined by

\[c(u_0) = 0 \quad (5)\]

\[c(u_{i+1}) = c(u_i) + n - 1 - L(u_i) - L(u_{i+1}) - \varepsilon(T), 0 \leq i \leq n-2 \quad (6)\]

is an optimal hamiltonian coloring of $T$.

**Proof.** Suppose that a linear order $u_0, u_1, \ldots, u_{n-1}$ of the vertices of $T$ satisfies the conditions (1), (2) and (3) of hypothesis, and $c$ is defined by (5) and (6). By Theorem 4 it is enough to prove that $c$ is a hamiltonian coloring whose span is equal to $c(u_{n-1}) = (n-1)(n-1-\varepsilon(T)) + \varepsilon(T) - 2\mathcal{L}(T)$.

Let $c$ is defined by (5) and (6). Without loss of generality we assume that $j - i \geq 2$. Then

\[c(u_j) - c(u_i) = \sum_{t=i}^{j-1} [c(u_{t+1}) - c(u_t)] \]

\[= \sum_{t=i}^{j-1} [n - 1 - L(u_t) - L(u_{t+1}) - \varepsilon(T)] \]

\[= \sum_{t=i}^{j-1} [n - 1 - D(u_t, u_{t+1})] \]
Note that $D(u_i, u_j) \geq 1$; it follows that $|c(u_j) - c(u_i)| + D(u_i, u_j) \geq n - 1$. Hence, $c$ is a hamiltonian coloring for $T$. The span of $c$ is given by

$$\text{span}(c) = c(u_{n-1}) - c(u_0)$$

$$= \sum_{i=0}^{n-2} [c(u_{i+1}) - c(u_i)]$$

$$= \sum_{i=0}^{n-2} [n - 1 - \mathcal{L}(u_i) - \mathcal{L}(u_{i+1}) - \varepsilon(T)]$$

$$= (n - 1)^2 - \sum_{i=0}^{n-2} [\mathcal{L}(u_i) + \mathcal{L}(u_{i+1})] - (n - 1)\varepsilon(T)$$

$$= (n - 1)(n - 1 - \varepsilon(T)) - 2\mathcal{L}(T) + \mathcal{L}(u_0) + \mathcal{L}(u_{n-1})$$

$$= (n - 1)(n - 1 - \varepsilon(T)) + \varepsilon(T) - 2\mathcal{L}(T)$$

Therefore, $hc(T) \leq (n - 1)(n - 1 - \varepsilon(T)) + \varepsilon(T) - 2\mathcal{L}(T)$. This together with (8) implies (9) and that $c$ is an optimal hamiltonian coloring.

**Corollary 1.** Let $T$ be a DB$(n/2)$ tree (or $d \leq n/2$) of order $n \geq 4$ and $\Delta(T) \geq 3$, where $d$ is diameter of $T$. Then

$$hc(T) = (n - 1)(n - 1 - \varepsilon(T)) + \varepsilon(T) - 2\mathcal{L}(T)$$

(7)

holds if there exists a linear order $u_0, u_1, \ldots, u_{n-1}$ with $0 = c(u_0) < c(u_1) < \ldots < c(u_{n-1})$ of the vertices of $T$ such that

1. $u_0 = w$, $u_{n-1} \in N(w)$ when $C(T) = \{w\}$ and $\{u_0, u_{n-1}\} = \{w, w'\}$ when $C(T) = \{w, w\}$,
2. $u_i$ and $u_{i+1}$ are in different branches when $C(T) = \{w\}$ and opposite branches when $C(T) = \{w, w'\}$.

Moreover, under these conditions the mapping $c$ defined by

$$c(u_0) = 0$$

$$c(u_{i+1}) = c(u_i) + n - 1 - \mathcal{L}(u_i) - \mathcal{L}(u_{i+1}) - \varepsilon(T), 0 \leq i \leq n - 2$$

(8)

is an optimal hamiltonian coloring of $T$.

**Proof.** The proof is straight forward by Theorem 3 as for any tree $T$, $\max\{D(u, v) : u, v \in V(G), u \neq v\} \leq d \leq n/2$. 

4 Hamiltonian coloring of some families of tree

In this section, we determine the hamiltonian chromatic number for three families of tree using Corollary \[\ref{corollary} \] We continue to use terminology and notation defined in the previous section.

A symmetric tree is a tree in which all vertices other than leaves (degree-one vertices) have the same degree and all leaves have the same eccentricity. Let \(k, d \geq 2\) be integers. We denote the symmetric tree with diameter \(d\) and non-leaf vertices having degree \(k+1\) by \(T_{k+1}(d)\). A \(k\)-star is a tree consisting of \(k\) leaves and another vertex joined to all leaves by edges. We define the \((n, k)\)-firecracker trees, denoted by \(F(n, k)\), to be the tree obtained by taking \(n\) copies of a \((k-1)\)-star and identifying a leaf of each of them to a different vertex of a path of length \(n-1\). A tree is said to be a caterpillar \(C\) if it consists of a path \(v_1v_2...v_m(m \geq 3)\), called the spine of \(C\), with some hanging edges known as legs, which are incident to the inner vertices \(v_2, v_3, ..., v_m-1\). If \(d(v_i) = k\) for \(i = 2, 3, ..., m-1\), then we denote the caterpillar by \(C(m, k)\), where \(d(v_i)\) denotes the degree of \(v_i\). For all above defined trees it is easy to verify that \(d \leq n/2\), and hence \(DB(n/2)\) trees as \(\max\{D(u,v):u,v \in V(T)\}\) \(\leq d \leq n/2\).

Now we determine the hamiltonian chromatic number for above defined trees using Corollary \[\ref{corollary} \] Note that for this purpose it is enough to give a linear order \(u_0, u_1, ..., u_{n-1}\) of vertices of \(T\) which satisfies conditions of Corollary \[\ref{corollary} \].

**Theorem 6.** Let \(k, d \geq 2\) be integers. Then \(hc(T_{k+1}(d))\)

\[
\begin{align*}
\text{if } d &\text{ is even,} \\
\text{if } d &\text{ is odd.}
\end{align*}
\]

\[
\text{(10)}
\]

**Proof.** Note that \(T_{k+1}(d)\) has one or two central vertex/vertices depending on \(d\) and hence we consider the following two cases.

**Case 1:** \(d\) is even.

In this case \(T_{k+1}(d)\) has a unique central vertex, denoted by \(w\). Denote the children of the central vertex \(w\) by \(w^1, w^2, ..., w^{k+1}\). Denote the \(k\) children of each \(w^t\) by \(w_{i0}^t, w_{i1}^t, ..., w_{i(k-1)}^t\), \(0 \leq i \leq k-1, 1 \leq t \leq k+1\). Inductively, denote the \(k\) children of \(w_{i_1,i_2,..,i_t}^t\) (where \(0 \leq i_1, i_2, ..., i_t \leq k-1, 1 \leq t \leq k+1\)) by \(w_{i_1,i_2,..,i_t}^1, w_{i_1,i_2,..,i_t}^2, ..., w_{i_1,i_2,..,i_t}^{k+1}\), \(0 \leq i_1, i_2, ..., i_t \leq k-1, \sum_{t+1 \leq t \leq [d/2]} k^t\).
We give a linear order \( u_0, u_1, \ldots, u_{n-1} \) of the vertices of \( T_{k+1}(d) \) as follows. We first set \( u_0 \equiv w \). Next, for \( 1 \leq j \leq n-k-2 \), let

\[
u_j := \begin{cases} 
  e_s, & \text{if } j \equiv t \pmod{(k+1)} \text{ for } t \text{ with } 1 \leq t \leq k, \\
  v_s^{k+1}, & \text{if } j \equiv 0 \pmod{(k+1)}. 
\end{cases}
\]

Finally, let

\[
u_j := w^{j-n+k+2}, \quad n-k-1 \leq j \leq n-1.
\]

Note that \( u_{n-1} \equiv w^{k+1} \) is adjacent to \( w \), and for \( 1 \leq i \leq n-2 \), \( u_i \) and \( u_{i+1} \) are in different branches so that \( \phi(u_i, u_{i+1}) = 0 \).

Case 2: \( d \) is odd.

In this case \( T_{k+1}(d) \) has two (adjacent) central vertices, denoted by \( w \) and \( w' \). Denote the neighbours of \( w \) other than \( w' \) by \( w_0, w_1, \ldots, w_{k-1} \) and the neighbours of \( w' \) other than \( w \) by \( w'_0, w'_1, \ldots, w'_{k-1} \). For \( 0 \leq i \leq k-1 \), denote the \( k \) children of each \( w_i \) (respectively, \( w'_i \)) by \( w_{i0}, w_{i1}, \ldots, w_{i(k-1)} \) (respectively, \( w'_{i0}, w'_{i1}, \ldots, w'_{i(k-1)} \)). Inductively, for \( 0 \leq i_1, i_2, \ldots, i_l \leq k-1 \), denote the \( k \) children of \( w_{i_1,i_2,\ldots,i_l} \) (respectively, \( w'_{i_1,i_2,\ldots,i_l} \)) by \( w_{i_1,i_2,\ldots,i_l,0}, \ldots, w_{i_1,i_2,\ldots,i_l,k-1} \) (respectively, \( w'_{i_1,i_2,\ldots,i_l,0}, \ldots, w'_{i_1,i_2,\ldots,i_l,k-1} \)), where \( 0 \leq i_{l+1} \leq k-1 \). We rename

\[
u_j := u_{i_1,i_2,\ldots,i_l}, \quad v'_j := w'_{i_1,i_2,\ldots,i_l}, \quad \text{where } j = 1 + i_1 + i_2 k + \cdots + i_l k^{l-1} + \sum_{t+1 \leq t \leq |d/2|} k^t.
\]

We give a linear order \( u_0, u_1, \ldots, u_{n-1} \) of the vertices of \( T_{k+1}(d) \) as follows. We first set

\[
u_0 := w, \quad u_{n-1} := w',
\]

and for \( 1 \leq j \leq n-2 \), let

\[
u_j := \begin{cases} 
  e_s, & \text{if } j \equiv 0 \pmod{2} \\
  v_s, & \text{if } j \equiv 1 \pmod{2}.
\end{cases}
\]

Then \( u_i \) and \( u_{i+1} \) are in opposite branches for \( 1 \leq i \leq n-2 \), and \( u_{i+2j}, j=0,1,\ldots,(k-1) \) are in different branches for \( 1 \leq i \leq n-2k+1 \), so that \( \phi(u_i, u_{i+1}) = 0 \) and \( \delta(u_i, u_{i+1}) = 1 \).

Therefore, in each case above, a defined linear order of vertices satisfies the conditions of Corollary 1. The Hamiltonian coloring defined by (9) is an optimal Hamiltonian coloring whose span equal to the right-hand side of (7). But it is straightforward to verify that the order of \( T_{k+1}(d) \) is given by

\[
n := \begin{cases} 
  1 + \frac{k+1}{k-1}(k^{\frac{d}{2}} - 1), & \text{if } d \text{ is even}, \\
  2 \left( 1 + \frac{k+1}{k-1}(k^{\frac{d+1}{2}} - 1) \right), & \text{if } d \text{ is odd}.
\end{cases}
\]
With the help of formula 1 + 2x + 3x^2 + ... + px^{p-1} = \frac{px^p}{x-1} - \frac{x^{p-1} - 1}{x-1}$, one can verify that the total level of $T_{k+1}(d)$ is given by

$$L(T_{k+1}(d)) := \begin{cases} (k + 1) \left( \frac{d_k^2}{2(k-1)} - \frac{k - 1}{(k-1)^2} \right), & \text{if } d \text{ is even} \\ 2k \left( \frac{d_k^2}{2(k-1)} - \frac{k - 1}{(k-1)^2} \right), & \text{if } d \text{ is odd}. \end{cases} \tag{12}$$

By substituting (11) and (12) into (7), we obtain the right-hand side of (10) is the hamiltonian chromatic number of $T_{k+1}(d)$.

**Theorem 7.** For $m \geq 3$ and $k \geq 4$,

$$hc(F(m, k)) = \begin{cases} m^2k^2 - 6m(k - 1) - \frac{k}{2}(m^2 - 1) + 2, & \text{if } m \text{ is odd}; \\ m^2k^2 - 6m(k - 1) - \frac{k}{2}m^2 + 2, & \text{if } m \text{ is even}. \end{cases} \tag{13}$$

**Proof.** Let $w_1^1, w_2^1, \ldots, w_k^1$ denote the vertices of the $i^{th}$ copy of the $(k - 1)$-star in $F(m, k)$, where $w_1^1$ is the apex vertex (center) and $w_2^1, \ldots, w_k^1$ are the leaves. Without loss of generality we assume that $w_1^1, w_2^1, \ldots, w_k^1$ are identified to the vertices in the path of length $m - 1$ in the definition of $F(m, k)$. Note that $F(m, k)$ has one or two central vertex/vertices depending on $m$ and hence we consider the following two cases.

**Case-1:** $m$ is odd.

In this case $F(m, k)$ has only one central vertex $w$ which is $w_k^{1+1}$. We give a linear order $u_0, u_1, \ldots, u_{n-1}$ of the vertices of $F(m, k)$ as follows. We first set $u_0 = w = w_k^{1+1}$. Next, for $1 \leq t \leq n - m$, let

$$u_t := w_t^j, \text{ where } t = \begin{cases} (j - 1)m + (i - \lfloor \frac{m}{2} \rfloor), & \text{if } i = \lfloor \frac{m}{2} \rfloor \\ (j - 1)m + 2i, & \text{if } i < \lfloor \frac{m}{2} \rfloor \\ (j - 1)m + 2(i - \lfloor \frac{m}{2} \rfloor) + 1, & \text{if } i > \lfloor \frac{m}{2} \rfloor. \end{cases}$$

Finally, for $n - m + 1 \leq t \leq n - 1$, let

$$u_t := w_t^j, \text{ where } t = \begin{cases} (j - 1)m - 2(i - \lfloor \frac{m}{2} \rfloor) + 1, & \text{if } i < \lfloor \frac{m}{2} \rfloor \\ (j - 1)m + 2(m - i + 1), & \text{if } i > \lfloor \frac{m}{2} \rfloor. \end{cases}$$

**Case-2:** $m$ is even.

In this case $F(m, k)$ has two central vertices $w$ and $w'$ which are $w_k^2$ and $w_k^{2+1}$ respectively. We give a linear order $u_0, u_1, \ldots, u_{n-1}$ of the vertices of $F(m, k)$ as follows. We first set $u_0 = w' = w_k^{2+1}$ and $u_{n-1} = w = w_k^2$. Next, for $1 \leq t \leq n - m + 1$, let

$$u_t := w_t^i, \text{ where } t = \begin{cases} (j - 1)m + 2i - 1, & \text{if } i \leq \frac{m}{2} \\ (j - 1)m + 2(i - \frac{m}{2}), & \text{if } i > \frac{m}{2}. \end{cases}$$
Finally, for \( n - m + 2 \leq t \leq n - 2 \), let
\[
u_t := w^j_i, \quad \text{where } t = \begin{cases} 
(j - 1)m + 2i - 1, & \text{if } i < \frac{m}{2} \\
(j - 1)m + 2(i - 1 - \frac{m}{2}), & \text{if } i > \frac{m}{2} + 1.
\end{cases}
\]

Therefore, in each case above, a defined linear order of vertices satisfies conditions of Corollary 1. The hamiltonian coloring defined by (8) and (9) is an optimal hamiltonian coloring whose span equal to the right-hand side of (7).

But the order and total level of firecrackers \( F(m, k) \) are given by
\[
u = mk
\]
(14)
\[
\mathcal{L}(F(m, k)) := \begin{cases}
\frac{km^2 + (8k - 12)m - k}{4}, & \text{if } m \text{ is odd}, \\
\frac{km^2 + 6m(k - 2)}{4}, & \text{if } m \text{ is even}.
\end{cases}
\]
(15)

By substituting (14) and (15) into (7), we obtain the right-hand side of (13) is the hamiltonian chromatic number of \( F(m, k) \).

Theorem 8. Let \( m, k \geq 3 \). Then \( hc(C(m, k)) \) is the hamiltonian chromatic number of \( F(m, k) \).

Proof. Let \( v_1, v_2, \ldots, v_m \) be the vertices of spine and \( v^j_i, 1 \leq j \leq k - 2 \) are pendent vertices at \( s^i \), \( 2 \leq i \leq m - 1 \) vertex of spine. Note that \( C(m, k) \) has one or two central vertex/vertices depending on \( m \) and hence we consider the following two cases.

Case-1: \( m \) is odd.

In this case \( C(m, k) \) has only one central vertex which is \( v_{\lceil \frac{m}{2} \rceil} = w \). We first set \( u_0 = v_{\lceil \frac{m}{2} \rceil + 1}, u_{n-1} = w \) and other vertices as follows.

For \( 1 \leq t \leq m - 2 \),
\[
u_t := v_i, \quad \text{where } t = \begin{cases} 
2i - 1, & \text{if } i < \lfloor \frac{m}{2} \rfloor, \\
2(i - \lfloor \frac{m}{2} \rfloor), & \text{if } i > \lfloor \frac{m}{2} \rfloor + 1.
\end{cases}
\]

For \( m - 1 \leq t \leq n - 1 \),
\[
u_t := v^j_i, \quad \text{where } t = \begin{cases} 
(m - 2)j + 2(i - 1), & \text{if } i < \lfloor \frac{m}{2} \rfloor, \\
(m - 2)j + 1, & \text{if } i = \lfloor \frac{m}{2} \rfloor, \\
(m - 2)j + 2(i - \lfloor \frac{m}{2} \rfloor) + 1, & \text{if } i > \lfloor \frac{m}{2} \rfloor.
\end{cases}
\]

Case-2: \( m \) is even.

In this case \( C(m, k) \) has two central vertices which are \( v_{\frac{m}{2}} = w \) and \( v_{\frac{m}{2} + 1} = w' \). We first set \( u_0 = v_{\frac{m}{2} + 1}, u_{n-1} = \frac{m}{2} \) and other vertices as follows.
For $1 \leq t \leq m - 2$,

$$u_t := v_i, \text{ where } t = \begin{cases} 
2i - 1, & \text{if } i < \frac{m}{2} - 1, \\
2(i - \frac{m}{2}), & \text{if } i > \frac{m}{2} + 1.
\end{cases}$$

For $m - 1 \leq t \leq n - 1$,

$$u_t := v_i^j, \text{ where } t = \begin{cases} 
(m - 2)j + 2(i - 2) + 1, & \text{if } i \leq \frac{m}{2}, \\
(m - 2)j + 2(i - \frac{m}{2}), & \text{if } i > \frac{m}{2}.
\end{cases}$$

Therefore, in each case above, a defined linear order of vertices satisfies conditions of Corollary 1. The hamiltonian coloring defined by (8) and (9) is an optimal hamiltonian coloring whose span equal to the right-hand side of (7).

But the order and total level of caterpillars $C(m, k)$ are given by

$$n := m(k - 1) - 2(k - 2) \quad (17)$$

$$L(C(m, k)) := \begin{cases} 
\frac{(m^2 - 5)(k - 1)}{4} + 1, & \text{if } m \text{ is odd}, \\
\frac{m(m - 2)(k - 1)}{4}, & \text{if } m \text{ is even}.
\end{cases} \quad (18)$$

By substituting (17) and (18) into (7), we obtain the right-hand side of (16) is the hamiltonian chromatic number of $C(m, k)$.

We remark that Theorem 5 is also useful to determine hamiltonian chromatic number of non $DB(n/2)$ trees. See the following result.

**Theorem 9.** Let $P'_m$ be a tree obtained by attaching a pendant vertex to central vertex/vertices of path $P_m$. Then

$$hc(P'_m) := \begin{cases} 
\frac{1}{2}(m^2 - 1), & \text{if } m \text{ is odd}, \\
\frac{m^2}{2} + 2m - 4, & \text{if } m \text{ is even}.
\end{cases} \quad (19)$$

**Proof.** The order and total level of $P'_m$ are given by

$$n := \begin{cases} 
m + 1, & \text{if } m \text{ is odd}, \\
m + 2, & \text{if } m \text{ is even}.
\end{cases} \quad (20)$$

$$L(P'_m) := \begin{cases} 
\frac{m^2 + 1}{4}, & \text{if } m \text{ is odd}, \\
\frac{m^2 - 2m + 8}{4}, & \text{if } m \text{ is even}.
\end{cases} \quad (21)$$

Substituting (20) and (21) into (2) we obtain that the right-hand side of (19) is a lower bound for $hc(P'_m)$. Now we give a linear ordering of vertices of $P'_m$ which satisfies conditions of $hc(P'_m)$. Note that $P'_m$ has one central vertex when $m$ is odd and two adjacent central vertices when $m$ is even. Hence we consider the following two cases.
Case-1: $m$ is odd.
Let $v_1v_2...v_m$ be the vertices of path and $v'$ be the vertex attached to central vertex $v_{(m+1)/2}$ then we order the vertices as follows:

$$v_{(m+1)/2}, v_1, v_{(m+3)/2}, v_2, v_{(m+5)/2}, v_3, v_{(m+7)/2}, ..., v_{(m-1)/2}, v_m, v'.$$

Rename the vertices of $P'_m$ in the above ordering by $u_0, u_1, ..., u_{n-1}$. Namely, let $u_0 = v_{(m+1)/2}$, $u_1 = v_1, ..., u_{n-1} = v'$ then it satisfies conditions of Theorem 5.

Case-2: $m$ is even.
Let $v_1v_2...v_m$ be the vertices of path and $v'$ and $v''$ are attached to central vertices $v_{m/2}$ and $v_{m/2+1}$ then we order the vertices as follows:

$$v_{m/2+1}, v_1, v_{m/2+2}, v_2, v_{m/2+3}, v_3, ..., v_{m/2+1}, v_m, v', v'', v_{m/2}.$$  

Rename the vertices of $P'_m$ in the above ordering by $u_0, u_1, ..., u_{n-1}$. Namely, let $u_0 = v_{m/2+1}$, $u_1 = v_1, ..., u_{n-1} = v_{m/2}$ then it satisfies conditions of Theorem 5.

Therefore, in each case above, a defined linear order of vertices of $P'_m$ satisfies conditions of Theorem 5 and hence the hamiltonian coloring defined by (5) and (6) is an optimal hamiltonian coloring whose span is (4) which is (19) for the current case.

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