Local invariants of braiding quantum gates—associated link polynomials and entangling power

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Abstract

For a generic $n$-qubit system, local invariants under the action of $SL(2,\mathbb{C})^\otimes n$ characterize non-local properties of entanglement. In general, such properties are not immediately apparent and hard to construct. Here we consider two-qubit Yang–Baxter operators and show that their eigenvalues completely determine the non-local properties of the system. Moreover, we apply the Turaev procedure to these operators and obtain their associated link/knot polynomials. We also compute their entangling power and compare it with that of a generic two-qubit operator.

Keywords: braiding quantum gates, local invariants, Yang–Baxter operators, link polynomials, entangling power

1. Introduction

Entanglement, perhaps the most bizarre feature of the quantum world [1, 2], plays a crucial role in quantum information processing and quantum computation [3, 4]. Its non-local nature goes against our classical intuition, but it can be used to analyze a quantum system in a systematic manner, via group theory and classical invariant theory [5, 6]. The parameters appearing in

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quantum states and quantum operators can in fact be organized by their response under the local action of $SL(2, \mathbb{C})^\otimes n$, for an $n$-qubit system defined on $(\mathbb{C}^2)^\otimes n$. This action defines an orbit space of equivalence classes such that the states or operators in a given orbit have the same non-local properties. This analysis has been performed in [7–13] for local unitaries of pure and mixed states on finite-dimensional Hilbert spaces and in [14] for two-qubit gates. As is well known from these early works, the systematic computation of these local invariants is a tedious task that gets harder as one increases the number of qubits. Nevertheless, this is important to understand all possible entanglement in a finite quantum system and hence must be tackled.

In this work we explore the possibility of simplifying this task by ‘creating’ quantum systems with braid operators built from Yang–Baxter operators (YBOs), i.e. operators that solve the (spectral parameter-independent) Yang–Baxter equation. In recent years it has been understood that such operators can also act as quantum gates [15–23], leading to the speculation of a broad connection between topological and quantumentanglement. We work on the two-qubit space ($n = 2$), though we expect the properties we find to generalize to higher $n$.

For a generic two-qubit operator, an obvious set of independent local invariants under $SL(2, \mathbb{C})^\otimes 2$ are class functions of the operator or functions of its independent eigenvalues, since $SL(2, \mathbb{C})^\otimes 2$ acts on the operator as a similarity transformation. However, this is not necessarily the whole story and there may be more independent local invariants. We expect the number of independent local invariants to reduce and get closer to the number of independent eigenvalues if some constraints on the operator are imposed. We explicitly see that all the local invariants are solely functions of the eigenvalues for two-qubit braid operators of the form

$$
\begin{pmatrix}
\star & 0 & 0 & \star \\
0 & \star & \star & 0 \\
0 & \star & \star & 0 \\
\star & 0 & 0 & \star
\end{pmatrix}. 
$$

(1.1)

These matrices generate entangled two-qubit states and we denote them X-type operators, for obvious reasons. We find twelve such classes of YBOs that can be both unitary and non-unitary 7.

We organize our results as follows. In section 2 we find one linear and five independent quadratic invariants for an arbitrary two-qubit operator under the action of $SL(2, \mathbb{C})^\otimes 2$. The same procedure can also be carried out for more than two qubits. For the special case of an arbitrary X-type two-qubit operator, we show in section 3 that independent invariants are exhausted by one linear and five quadratic invariants. In section 4 we restrict the X-type operators to braid operators and observe that all the local invariants are expressed solely as functions of the eigenvalues and that the number of independent local invariants coincides with the number of independent eigenvalues in each of twelve possible classes. In section 5 we enhance the X-type braid operators using the procedure outlined in [25] and compute their associated link/knot polynomials. It turns out that the polynomials are not always local invariant, although they can be expressed in terms of the eigenvalues of the braid operators. In section 6 we also consider the entangling powers [26] of the X-type braid operators and compare them with the entangling power of an arbitrary X-type operator. We end with an outlook and discussion in section 7. In appendix A we investigate the relation between our classification of X-type YBOs

7 In [24], two-qubit braid operators are completely classified in ten forms, as discussed in appendix A. Among these ten forms, $R_{i1,3}$ and $R_{i2,3}$ in (A.9) are not of the form (1.1). We check in appendix B that $R_{i1,3}$ and $R_{i2,3}$ have analogous properties to the X-type YBOs.
and Hietarinta’s classification [24]. For completeness, an analogous computation is presented in appendix B for families of braid operators that are not of the form (1.1).

2. \( SL(2, \mathbb{C})^{\otimes 2} \) invariants of general two-qubit operators

We consider an operator \( R \) acting on two qubits \( |i_1 i_2 \rangle \) as a \( 4 \times 4 \) matrix (its row and column are labeled by \((i_1, i_2)\) and \((\bar{i}_1, \bar{i}_2)\), respectively):

\[
R |i_1 i_2 \rangle = \sum_{\bar{i}_1 \bar{i}_2 = 0}^{1} R_{i_1 \bar{i}_1, i_2 \bar{i}_2} |\bar{i}_1 \bar{i}_2 \rangle.
\]  

(2.1)

When an invertible local operator (ILO)

\[
Q = Q_1 \otimes Q_2 \in SL(2, \mathbb{C})^{\otimes 2}
\]

acts on two-qubit states \((Q |i_1 i_2 \rangle, Q |\bar{i}_1 \bar{i}_2 \rangle)\), we can interpret that \( R \) is transformed as \( QRQ^{-1} \). More precisely

\[
R_{i_1 \bar{i}_1, i_2 \bar{i}_2} \rightarrow \sum_{\bar{i}_1 \bar{i}_2 = 0}^{1} (Q_1)_{i_1 \bar{a}_1} (Q_2)_{i_2 \bar{a}_2} R_{\bar{a}_1 \bar{a}_1, \bar{a}_2 \bar{a}_2} (Q_1^{-1})_{\bar{a}_1 i_1} (Q_2^{-1})_{\bar{a}_2 i_2},
\]

(2.2)

where untilded (tilded) indices with \( a = 1, 2 \), say \( i_0 (\bar{i}_0) \), are transformed by \( Q_a \) \((Q_a^{-1})\). From this transformation property, one can see that invariants under the action of the ILO can be constructed from a set of 8s by contracting their indices with the four invariant tensors \( \epsilon_{i_0 j_0}, \epsilon_{i_1 j_1}, \delta_{i_0 j_0}, \delta_{i_1 j_1} \), with \( \epsilon_{01} = -\epsilon_{10} = 1, \epsilon_{00} = \epsilon_{11} = 0 \). Note that the resulting expressions would also be invariant under the action of a general ILO belonging to \( \mathbb{C}^{*} \cdot SL(2, \mathbb{C})^{\otimes 2} \), namely stochastic local operations and classical communication (SLOCC) [27]. The factor \( \mathbb{C}^{*} \) represents the multiplication by a nonzero complex number and does not affect \( R \), since these are similarity transformations.

In the following, we present the invariants at linear and quadratic orders in \( R \). Einstein’s convention—repeated indices are understood to be summed over—is used for notational simplicity.

**Linear invariant.** The invariant at linear order is only one and given by

\[
I_1 = R_{i_1 \bar{i}_1, i_2 \bar{i}_2} \delta_{i_1 \bar{i}_1} \delta_{i_2 \bar{i}_2} = \text{Tr} \; R,
\]

(2.4)

where \( \text{Tr} \) denotes the trace taken on the whole Hilbert space of the two qubits. In this case there is no possibility of contraction by \( \epsilon_{i_0 j_0} \) or \( \epsilon_{i_1 j_1} \). In order to use \( \epsilon_{i_1 j_1} \) for example, we need two untilded indices with the suffix 1, but \( R_{\bar{i}_1 \bar{i}_1, \bar{i}_2 \bar{i}_2} \) has only one.

**Quadratic invariants.** By exhausting all possible index contractions of \( R_{\bar{i}_1 \bar{i}_1, \bar{i}_2 \bar{i}_2} \), by the invariant tensors, we first list eight invariants which are independent of \( I_1^2 \):

\[
I_{2,1} = R_{i_1 \bar{i}_1, i_2 \bar{i}_2} R_{j_1 \bar{j}_1, j_2 \bar{j}_2} \delta_{i_1 j_1} \delta_{i_2 j_2} = \text{Tr} \; R^2,
\]

(2.5)

\[
I_{2,2} = R_{i_1 \bar{i}_1, i_2 \bar{i}_2} R_{j_1 \bar{j}_1, j_2 \bar{j}_2} \delta_{i_1 \bar{i}_1} \delta_{i_2 \bar{i}_2} = \text{tr} \; [\text{tr}_1 R^2],
\]

(2.6)

\[
I_{2,3} = R_{i_1 \bar{i}_1, i_2 \bar{i}_2} R_{j_1 \bar{j}_1, j_2 \bar{j}_2} \delta_{i_1 \bar{i}_1} \delta_{i_2 \bar{i}_2} = \text{tr} \; [\text{tr}_2 R^2].
\]

(2.7)
where \( \text{tr}_a \) \((a = 1, 2)\) denotes the partial trace taken on the local Hilbert space at the \( a \)th qubit;

\[
I_{2.4} = R_{i_1 j_1 t_1} R_{j_2 h_2} R_{h_3 \ldots} \epsilon_{i_1 j_1 h_1} e_{i_2 h_2} \delta_{j_2 h_2} = \text{Tr}[Y_I(\Theta_1 R)Y_I R], \tag{2.8}
\]

\[
I_{2.5} = R_{i_1 j_1 t_1} R_{j_2 h_2} R_{h_3 \ldots} \delta_{i_1 h_1} e_{i_2 h_2} e_{j_2 h_2} = \text{Tr}[R Y_I(\Theta_2 R) Y_I], \tag{2.9}
\]

where \( Y_a \) is the Pauli \( y \)-matrix acting on the \( a \)th qubit, and \( \Theta_a \) represents the partial transpose with respect to indices on the \( a \)th qubit;

\[
I_{2.6} = R_{i_1 j_1 t_1} R_{j_2 h_2} R_{h_3 \ldots} \epsilon_{i_1 j_1 h_1} e_{i_2 h_2} \delta_{j_2 h_2} = \text{Tr}[Y_I(\Theta_1 R)Y_I(\text{tr}_2 R)], \tag{2.10}
\]

\[
I_{2.7} = R_{i_1 j_1 t_1} R_{j_2 h_2} R_{h_3 \ldots} \delta_{i_1 h_1} e_{i_2 h_2} e_{j_2 h_2} = \text{Tr}[(\text{tr}_1 R) Y_2 (\text{tr}_1 \Theta_2 R) Y_2], \tag{2.11}
\]

\[
I_{2.8} = R_{i_1 j_1 t_1} R_{j_2 h_2} R_{h_3 \ldots} \epsilon_{i_1 j_1 h_1} e_{i_2 h_2} e_{j_2 h_2} = \text{Tr}[R^T Y_1 Y_2 R Y_1 Y_2], \tag{2.12}
\]

In addition to the eight quadratic invariants above, there are two more quadratic invariants constructed from one \( R \) acting on the qubits 1 and 2 (denoted by \( R_{12} \)), and the other \( R \) acting on a qubit outside of this space, e.g. acting on the qubits 2 and 3 (labeled by the indices \( j_1, j_3 \) and denoted by \( R_{23} \)):

\[
I_{2.9} = R_{i_1 j_1 t_1} R_{j_2 h_2} R_{h_3 \ldots} \delta_{i_1 h_1} \delta_{j_2 h_2} = \text{Tr}[(\text{tr}_1 R_{12})(\text{tr}_3 R_{23})], \tag{2.13}
\]

\[
I_{2.10} = R_{i_1 j_1 t_1} R_{j_2 h_2} R_{h_3 \ldots} \delta_{i_1 h_1} \delta_{j_2 h_2} e_{i_2 h_2} e_{j_2 h_2} = \text{Tr}[Y_2(\text{tr}_1 \Theta_2 R_{12}) Y_2(\text{tr}_3 R_{23})]. \tag{2.14}
\]

We should notice that \( I_{2.1}^r, I_{2.1}^r \) \((r = 1, \ldots, 10)\) are not all linearly independent. In fact, the equation

\[
a_1 I_{1} + \sum_{r=1}^{10} a_{2.r} I_{2.r} = 0 \tag{2.15}
\]

for arbitrary \( R \) has the nontrivial solution

\[
a_{2.6} = a_{2.1} + a_{2.2} - a_{2.4},
\]

\[
a_{2.7} = a_{2.1} + a_{2.3} - a_{2.5},
\]

\[
a_{2.8} = -a_{2.1} + a_{2.4} + a_{2.5},
\]

\[
a_{2.9} = a_{2.10} = -a_{1} - a_{2.1} - a_{2.2} - a_{2.3}. \tag{2.16}
\]

Plugging this into (2.15) yields \( a_1 f_1(\{I\}) + \sum_{s=1}^{5} a_{2.s} f_{2.s}(\{I\}) = 0 \), where \( f_1(\{I\}) \) and \( f_{2.s}(\{I\}) \) \((s = 1, \ldots, 5)\) denote linear combinations of the quadratic invariants. Since this equality holds for arbitrary \( a_1 \) and \( a_{2.s} \) \((s = 1, \ldots, 5)\), we obtain the relations \( f_1(\{I\}) = 0 \) and
In this section, we consider the case that the $f_{2,s}(I)$ is defined as

\[ f_{2,s}(I) = 0 \quad (s = 1, \ldots, 5) \]

whose explicit form is

\[
\begin{align*}
I_1^2 - I_{2,9} - I_{2,10} &= 0, \\
I_{2,1} + I_{2,6} + I_{2,7} - I_{2,8} - I_{2,9} - I_{2,10} &= 0, \\
I_{2,2} + I_{2,6} - I_{2,8} - I_{2,10} &= 0, \\
I_{2,3} + I_{2,7} - I_{2,9} - I_{2,10} &= 0, \\
I_{2,4} - I_{2,6} + I_{2,8} &= 0, \\
I_{2,5} - I_{2,7} + I_{2,8} &= 0.
\end{align*}
\]  

(2.17)

From (2.17), we see that only five of the quadratic invariants (e.g., $I_{2,4}$, $I_{2,5}$, $I_{2,8}$, $I_{2,9}$, $I_{2,10}$) are independent.

3. **$SL(2, \mathbb{C})^2$ invariants of X-type two-qubit operators**

In this section, we consider the case that the $4 \times 4$ matrix $R$ in the previous section takes the X-type form:

\[
R = \begin{pmatrix}
  h_1 & 0 & 0 & h_2 \\
  0 & h_3 & h_4 & 0 \\
  0 & h_5 & h_6 & 0 \\
  h_7 & 0 & 0 & h_8
\end{pmatrix},
\]  

(3.1)

which is relevant to generate entangled states. $h_i (i = 1, \ldots, 8)$ are complex parameters and the matrix eigenvalues are given by

\[
\begin{align*}
\lambda_{1\pm} &= \frac{1}{2} \left[ h_1 + h_8 \pm \sqrt{(h_1 - h_8)^2 + 4h_2h_7} \right], \\
\lambda_{2\pm} &= \frac{1}{2} \left[ h_3 + h_6 \pm \sqrt{(h_3 - h_6)^2 + 4h_4h_5} \right].
\end{align*}
\]  

(3.2)

Since the eigenvalues do not change under general similarity transformations for $R$, they are obviously $SL(2, \mathbb{C})^2$-invariant combinations of the parameters.

It can be seen that the invariants presented in the previous section are not simply a function of these eigenvalues, but they contain other terms, that have to be invariant combinations on their own. To check this, let us specialize the linear and quadratic invariants to (3.1)

\[
\begin{align*}
I_1 &= h_1 + h_3 + h_6 + h_8 = \lambda_{1+} + \lambda_{1-} + \lambda_{2+} + \lambda_{2-}, \\
I_{2,4} &= 2(h_1h_6 - h_3h_5 - h_2h_7 + h_3h_8) \\
&= 2\{\lambda_{1+}\lambda_{1-} + \lambda_{2+}\lambda_{2-} + (\lambda_{1+} + \lambda_{1-})(\lambda_{2+} + \lambda_{2-})\} \\
&\quad - 2(h_1 + h_8)(h_3 + h_8), \\
I_{2,5} &= 2(h_1h_3 - h_2h_5 - h_2h_7 + h_2h_8) \\
&= 2\{\lambda_{1+}\lambda_{1-} + \lambda_{2+}\lambda_{2-} + (\lambda_{1+} + \lambda_{1-})(\lambda_{2+} + \lambda_{2-})\} \\
&\quad - 2(h_1 + h_3)(h_6 + h_8), \\
I_{2,8} &= 2(h_3h_5 + h_3h_6 + h_2h_7 + h_1h_8) \\
&= -2\{\lambda_{1+}\lambda_{1-} + \lambda_{2+}\lambda_{2-} + (\lambda_{1+} + \lambda_{1-})(\lambda_{2+} + \lambda_{2-})\} \\
&\quad + 2(h_1 + h_1)(h_6 + h_8) + 2(h_1 + h_8)(h_3 + h_8), \\
I_{2,9} &= h_1^2 + h_2^2 + (h_1 + h_8)(h_3 + h_6) + 2h_3h_6.
\end{align*}
\]
From the rhs of each formula of (3.3), we can identify the part not expressed by the eigenvalues as an operator of the invariants, because there might appear more independent invariants when we consider \( R \) invariants containing higher powers of \( h \). Hence, (3.2) we can conclude that the six combinations

\[
I_{2,10} = h_1^2 + h_2^2 + (h_1 + h_3)(h_6 + h_8) + (h_1 + h_6)(h_3 + h_8),
\]

where \( I_{2,1}, I_{2,2}, I_{2,3}, I_{2,6}, \) and \( I_{2,7} \) are obtained from the above through the identities (2.17). On the rhs in each formula of (3.3), we can identify the part not expressed by the eigenvalues as an additional invariant combination\(^8\). From \( I_{2,4}, I_{2,5}, I_{2,9} \) and \( I_{2,10} \), we see that \( h_1 - h_8 \) and \( h_3 - h_6 \) are invariants. Since \( \lambda_{1+} + \lambda_{1-} = h_1 + h_8 \) and \( \lambda_{2+} + \lambda_{2-} = h_3 + h_6 \) are also invariant, it is seen that \( h_1, h_3, h_6, \) and \( h_8 \) are invariant themselves. Looking at (3.2), we can conclude that the six combinations

\[
h_1, h_3, h_6, h_8, h_2 h_7, h_4 h_5, \tag{3.4}
\]

are independent \( SL(2, \mathbb{C}) \) invariants. These can also be expressed as

\[
h_1 = \frac{1}{2} \left[ \lambda_{1+} + \lambda_{1-} + \sqrt{I_{2,9} - I_{2,8} - \frac{I_{2,4} + I_{2,5}}{2}} \right],
\]

\[
h_3 = \frac{1}{2} \left[ \lambda_{2+} + \lambda_{2-} + \sqrt{I_{2,10} - I_{2,8} - \frac{I_{2,4} + I_{2,5}}{2}} \right],
\]

\[
h_6 = \frac{1}{2} \left[ \lambda_{2+} + \lambda_{2-} - \sqrt{I_{2,10} - I_{2,8} - \frac{I_{2,4} + I_{2,5}}{2}} \right],
\]

\[
h_8 = \frac{1}{2} \left[ \lambda_{1+} + \lambda_{1-} - \sqrt{I_{2,9} - I_{2,8} - \frac{I_{2,4} + I_{2,5}}{2}} \right],
\]

\[
h_2 h_7 = \frac{1}{4} \left( \lambda_{1+} - \lambda_{1-} \right)^2 - I_{2,9} + I_{2,8} + \frac{I_{2,4} + I_{2,5}}{2},
\]

\[
h_4 h_5 = \frac{1}{4} \left( \lambda_{2+} - \lambda_{2-} \right)^2 - I_{2,10} + I_{2,8} + \frac{I_{2,4} + I_{2,5}}{2}, \tag{3.5}
\]

which are clearly functions of the eigenvalues and quadratic local invariants \( I_{2,4}, I_{2,5}, I_{2,8}, I_{2,9} \) and \( I_{2,10} \).

As far as the number of independent invariants is concerned, we have two additional invariants other than the eigenvalues. Note that, in principle, six is the lower bound of the number of the invariants, because there might appear more independent invariants when we consider invariants containing higher powers of \( R \). However, by studying the dimension of orbits of the operator \( R \) in (3.1) under the action of \( SL(2, \mathbb{C}) \) as follows, one can show that the number of independent invariants is precisely six.

---

\(^8\) For example, we can show that \((h_1 + h_8)(h_6 + h_8)\) cannot be written in terms of the eigenvalues as follows. Suppose it is a function of the eigenvalues: \((h_1 + h_8)(h_6 + h_8) = f(h_{1+}, h_{1-}, h_{2-}, h_{2+}).\) Taking derivatives with respect to \( h_2, h_8 \), we see that \( f \) depends on the eigenvalues only through the combination \( \lambda_{1+} + \lambda_{1-} = h_1 + h_8 \). Hence, \((h_1 + h_8)(h_6 + h_8) = f(h_1 + h_8, h_3 + h_6).\) Derivatives on the rhs with respect to \( h_1 \) and \( h_8 \) should give the same result, whereas this is not the case on the lhs. This inconsistency proves the statement. A similar proof goes for \((h_1 + h_8)(h_3 + h_8).\)
The operator $R$ acting on the two qubits $i$ and $i+1$ can be expanded in terms of the Pauli-matrix basis as

$$R = lI_{i+1} + a_3Z_{i+1} + a_6I_{i+1} + b_0Z_{i+1}$$
$$+ b_1XX_{i+1} + b_2XY_{i+1} + b_4YX_{i+1} + b_5Y_{i+1},$$

(3.6)

where $I$ is the $2 \times 2$ unit matrix, and $X$, $Y$ and $Z$ are the Pauli matrices. $l, a_3, a_6, b_0$ are functions of $h_1, h_3, h_6$ and $h_8$, whereas $b_1, b_2, b_4, b_5$ depend on $h_2, h_4, h_5, h_7$ as

$$b_1 = \frac{1}{4} (h_2 + h_4 + h_5 + h_7), \quad b_2 = \frac{i}{4} (h_2 - h_4 + h_5 - h_7), \quad b_4 = \frac{i}{4} (h_2 + h_4 - h_5 - h_7), \quad b_5 = \frac{1}{4} (-h_2 + h_4 + h_5 - h_7).$$

(3.7)

To study the orbits we consider the Lie algebra generators of $SL(2, \mathbb{C})^{\otimes 2}$ and their commutators with $R$. Among six such commutators, namely $[X, I_{i+1}, R]$, $[X, I_{i+1}, R]$, $[Y, I_{i+1}, R]$, $[I, I_{i+1}, R]$, $[Z, I_{i+1}, R]$ and $[I, Z_{1+1}, R]$, the first four generate terms which are not present in the original operator $R$. For example,

$$[X, I_{i+1}, R] = -\frac{i}{2} (h_1 + h_3 - h_5 - h_6) Y_{i+1} - \frac{i}{2} (h_1 + h_3 - h_6 + h_8) Y_{i+1}$$
$$+ \frac{i}{2} (h_2 + h_4 + h_5 - h_7) Z_{i+1} + \frac{i}{2} (-h_2 + h_4 + h_5 - h_7) Z_{i+1},$$

(3.8)

provides terms proportional to $Y_{i+1}$, $Z_{i+1}$, $Z_{i+1}$ and $Z_{i+1}$ which are not contained in (3.6). In what follows, we do not consider such commutators that do not preserve the X-type form. On the other hand, the last two commutators $[Z, I_{i+1}, R]$ and $[I, Z_{1+1}, R]$ preserve the X-type form and modify the coefficients $b_1, b_2, b_4$ and $b_5$ as

$$[Z, I_{i+1}, R] = b_1' X_{i+1} + b_2' Y_{i+1} + b_4' Y_{i+1} + b_5' Y_{i+1},$$
$$[I, Z_{1+1}, R] = b_1'' X_{i+1} + b_2'' Y_{i+1} + b_4'' Y_{i+1} + b_5'' Y_{i+1},$$

(3.9)

where

$$b_1' = \frac{1}{2} (h_2 - h_4 + h_5 - h_7), \quad b_2' = \frac{i}{2} (h_2 + h_4 + h_5 + h_7), \quad b_4' = \frac{i}{2} (h_2 - h_4 - h_5 + h_7), \quad b_5' = \frac{1}{2} (-h_2 - h_4 + h_5 + h_7),$$

(3.10)

and

$$b_1'' = \frac{1}{2} (h_2 + h_4 - h_5 - h_7), \quad b_2'' = \frac{i}{2} (h_2 - h_4 - h_5 + h_7), \quad b_4'' = \frac{i}{2} (h_2 + h_4 + h_5 + h_7), \quad b_5'' = \frac{1}{2} (-h_2 + h_4 - h_5 + h_7).$$

(3.11)

These actions do not change the coefficients $l, a_3, a_6$ and $b_0$, consistently to $h_1, h_3, h_6$ and $h_8$ being invariants. Regarding the operator $R$ as a vector, the elements of the Lie algebra of $SL(2, \mathbb{C})^{\otimes 2}$ generate six independent directions in which this vector changes, implying that the dimension of the orbit is six.

Let us now consider only the two-dimensional orbit generated by $Z, I_{i+1}$ and $I, Z_{1+1}$. The orbit forms a two-dimensional surface in a four-dimensional space spanned by $b_1, b_2, b_4$ and $b_5$, or
equivalently by \( h_2, h_4, h_5 \) and \( h_7 \). Since two directions perpendicular to the surface correspond to invariants under the actions, there should exist two invariant combinations made of \( h_2, h_4, h_5 \) and \( h_7 \). Thus, we identify the parameters \( h_1, h_3, h_6, h_8 \) and two combinations of the parameters \( h_2, h_4, h_5, h_7 \) as independent invariants, for a total of six elements. Note that this time six is the upper bound of the number of the independent invariants. These are invariants under the action of only two of the generators of \( SL(2, \mathbb{C}) \). When considering the action of the full generators, more constraints for the invariants may arise and the number could possibly decrease.

Combining this with the previous assertion that six is the lower bound of the number of invariants, we conclude that there are precisely six \( SL(2, \mathbb{C}) \) invariants (3.4) that one can construct out of the X-type operators. These six independent local invariants are spanned by the single linear invariant \((I_1)\) and the five quadratic invariants \((I_{2,4}, I_{2,5}, I_{2,8}, I_{2,9}, I_{2,10})\) of (3.3). These can also be viewed as ‘coordinates’ which label the different orbits of \( R \) under the action of \( SL(2, \mathbb{C}) \).

4. \( SL(2, \mathbb{C}) \) invariants for X-type YBOs

In this section we consider the invariants for X-type matrices (3.1) that are YBOs, i.e. invertible solutions to the Yang–Baxter equation:

\[
(R \otimes I_2)(I_2 \otimes R)(R \otimes I_2) = (I_2 \otimes R)(R \otimes I_2)(I_2 \otimes R).
\]

(4.1)

In contrast to the additional \( SL(2, \mathbb{C}) \) invariants (different from the eigenvalues) found for general X-type matrices in the previous section, we find by direct inspection that for these operators here all the quadratic invariants depend only on the eigenvalues.

We list the YBOs (except the trivial one \( R \propto I_2 \)) into the following twelve classes and include the corresponding results for the quadratic invariants \( I_{2,4}, I_{2,5}, I_{2,8}, I_{2,9}, I_{2,10} \) with the help of Mathematica:

- **Class 1:** \( h_2 = h_3 = h_6 = h_7 = 0 \)

  The eigenvalues are \( \lambda_{1,+} = h_1, \lambda_{1,-} = h_8, \lambda_{2,\pm} = \pm \sqrt{h_3 h_5} (\mp \lambda_2) \) with the quadratic invariants

\[
\begin{align*}
I_{2,4} &= I_{2,5} = -2\lambda_1^2, & I_{2,8} &= 2(\lambda_{1,+} \lambda_{1,-} + \lambda_2^2), \\
I_{2,9} &= \lambda_{1,+}^2 + \lambda_{1,-}^2, & I_{2,10} &= 2\lambda_{1,+} \lambda_{1,-}.
\end{align*}
\]

(4.2)

Note that \( I_{2,8} = I_{2,10} - I_{2,4} \) and hence there are only three independent local invariants. This coincides with the number of independent eigenvalues: \( h_1, h_8 \) and \( \sqrt{h_3 h_5} \).

- **Class 2:** \( h_1 = h_4 = h_5 = h_8 = 0, h_6 = h_3 \)

  The eigenvalues are \( \lambda_{1,\pm} = \pm \sqrt{h_2 h_7} (\mp \lambda_4), \lambda_{2,\pm} = h_3 (\mp \lambda_2) \) with the quadratic invariants

\[
\begin{align*}
I_{2,4} &= I_{2,5} = -2\lambda_4^2, & I_{2,8} &= 2(\lambda_{1,+} \lambda_{1,-} + \lambda_2^2), \\
I_{2,9} &= I_{2,10} = 2\lambda_4^2.
\end{align*}
\]

(4.3)

In this case also we have the number of independent eigenvalues, namely \( h_3, \sqrt{h_2 h_7} \), equal to the number of independent local invariants: \( I_{2,4}, I_{2,10} \) (note in fact that \( I_{2,8} = I_{2,10} - I_{2,4} \)).

- **Class 3:** \( h_2 = h_3 = 0, h_4 = -h_1, h_5 = h_8, h_6 = h_1 + h_8 \)

  The eigenvalues are \( \lambda_{1,+} = \lambda_{2,\pm} = h_1 (\mp \lambda_4), \lambda_{1,-} = \lambda_2 = h_8 (\mp \lambda_2) \) with the quadratic invariants
\[ I_{2.4} = 2\lambda_+(\lambda_+ + 2\lambda_-), \quad I_{2.5} = 2\lambda_-(2\lambda_+ + \lambda_-), \quad I_{2.8} = 0, \]
\[ I_{2.9} = 2(\lambda_+^2 + \lambda_-^2 + \lambda_+\lambda_-), \quad I_{2.10} = 2(\lambda_+^2 + \lambda_-^2 + 3\lambda_+\lambda_-). \] 

We have two independent eigenvalues in this case, \( h_1 \) and \( h_8 \), and two independent local invariants as
\[ \lambda_+\lambda_- = \frac{I_{2.10} - I_{2.9}}{4}, \quad 2\lambda_+^2 = I_{2.4} - I_{2.10} + I_{2.9}, \quad 2\lambda_-^2 = I_{2.5} - I_{2.10} + I_{2.9}, \]

which helps solve for \( I_{2.10} - I_{2.9} \) in terms of \( I_{2.4} \) and \( I_{2.5} \) as
\[ I_{2.10} - I_{2.9} = \frac{2}{3} \left[ I_{2.4} + I_{2.5} \pm \sqrt{(I_{2.4} + I_{2.5})^2 - 3I_{2.4}I_{2.5}} \right]. \] 

There are seven other solutions belonging to this class:

- Class 4: \( h_2 = h_3 = h_7 = 0, h_4 = -h_8, h_5 = -h_6, h_6 = h_1 + h_8 \)
- Class 5: \( h_2 = h_3 = h_7 = 0, h_4 = -h_8, h_5 = h_1, h_6 = h_1 + h_8 \)
- Class 6: \( h_2 = h_3 = h_7 = 0, h_4 = h_5 = -h_6, h_5 = h_1 + h_8 \)
- Class 7: \( h_2 = h_3 = h_7 = 0, h_4 = h_5 = h_6, h_5 = h_1 + h_8 \)
- Class 8: \( h_2 = h_3 = h_7 = 0, h_4 = h_5 = -h_6, h_5 = h_1 + h_8 \)
- Class 9: \( h_2 = h_3 = h_7 = 0, h_4 = h_5 = h_6, h_5 = h_1 + h_8 \)
- Class 10: \( h_2 = h_3 = h_7 = 0, h_4 = h_5 = h_6, h_5 = h_1 + h_8 \)

The eigenvalues are \( \lambda_{1.2} = \lambda_{2.1} = \lambda_1(= \lambda_1), \lambda_{2.2} = -h_1 + h_6(= \lambda_1) \) with the quadratic invariants
\[ I_{2.4} = I_{2.5} = 2\lambda_1(\lambda_1 + 2\lambda_2), \quad I_{2.8} = 2\lambda_1(\lambda_1 - \lambda_2), \quad I_{2.9} = 2\lambda_1(2\lambda_1 + \lambda_2), \quad I_{2.10} = 5\lambda_1^2 + 4\lambda_1\lambda_2 + \lambda_2^2. \] 

Here we have two independent eigenvalues, \( h_1 \) and \(-h_1 + h_6 \). We can verify that only two of the four local invariants are independent by the expressions
\[ \lambda_1^2 = \frac{I_{2.8} + I_{2.9}}{6}, \quad \lambda_2^2 = \frac{(I_{2.9} - 2I_{2.8})^2}{6(I_{2.8} + I_{2.9})^3}. \]

implying that \( I_{2.4} \) and \( I_{2.10} \) depend on \( I_{2.8} \) and \( I_{2.9} \). Thus we again see that the number of independent eigenvalues is the same as the number of independent local invariants.

Another solution \( h_2 = h_3 = h_7 = 0, h_8 = h_1, h_5 = \frac{h_1}{h_6}(h_1 - h_8), h_8 = -h_1 + h_6 \) belongs to this class.

Class 5: \( h_2 = h_3 = h_7 = 0, h_4 = h_5 = h_6, h_5 = h_1 + h_6 \)

The eigenvalues are \( \lambda_{1.1} = \lambda_{2.1} = h_1(= \lambda_1), \lambda_{1.2} = -h_1 + h_6(= \lambda_1) \) with the quadratic invariants
\[ I_{2.4} = 2\lambda_+(\lambda_+ + 2\lambda_-), \quad I_{2.5} = 2\lambda_-(2\lambda_+ + \lambda_-), \quad I_{2.8} = 0, \quad I_{2.9} = 2(\lambda_+^2 + \lambda_-^2 + \lambda_+\lambda_-), \quad I_{2.10} = 2(\lambda_+^2 + \lambda_-^2 + 3\lambda_+\lambda_-). \] 

Once again we have two independent eigenvalues, \( h_1 \) and \(-h_1 + h_6 \). We see that only two of the four local invariants are independent from the expressions,
\[ \lambda_+ \lambda_- = \frac{I_{2,10} - I_2}{4}, \quad \lambda_+^2 = \frac{I_{2,4} - I_{2,10} + I_2}{2}, \]
\[ \lambda_-^2 = \frac{I_{2,5} - I_{2,10} + I_2}{2}, \]
where we can solve for \( I_{2,9} - I_{2,10} \) in terms of \( I_{2,4} \) and \( I_{2,5} \),
\[ I_{2,9} - I_{2,10} = \frac{-4(I_{2,4} + I_{2,5}) \pm \sqrt{16(I_{2,4} + I_{2,5})^2 - 48I_{2,4}I_{2,5}}}{6}, \]
which in turn implies that the two eigenvalues, \( \lambda_+ \) and \( \lambda_- \) are functions of \( I_{2,4} \) and \( I_{2,5} \). Thus we have the same number of independent local invariants and independent eigenvalues.

Another solution \( h_2 = h_6 = h_7 = 0, h_5 = \frac{h_4}{h_2}(h_1 - h_3), h_8 = -h_1 + h_3 \) belongs to this class.

- **Class 6**: \( h_3 = h_6 = \frac{h_1 + h_5}{2}, h_4 = h_5 = -\frac{\sqrt{h_1^2 + h_5^2}}{2}, h_7 = \frac{(h_1 + h_5)^2}{4h_2} \)

  The eigenvalues are \( \lambda_{1\pm} = \lambda_{2\pm} = \frac{1}{2} \left[ h_1 + h_8 \pm \sqrt{2(h_1^2 + h_5^2)} \right] \) (\( \equiv \pm \lambda \)) with the quadratic invariants
  \[ I_{2,4} = I_{2,5} = 2\lambda_+ \lambda_-, \quad I_{2,8} = 2(\lambda_+ + \lambda_-)^2, \]
  \[ I_{2,9} = 2(\lambda_+^2 + \lambda_-^2 + \lambda_+ \lambda_-), \quad I_{2,10} = 2(\lambda_+^2 + \lambda_-^2 + 3\lambda_+ \lambda_-). \]

  We can show that only two of these four local invariants are independent, \( I_{2,4} \) and \( I_{2,8} \) as can be seen from the expressions
  \[ \lambda_+ \lambda_- = \frac{I_{2,4}}{2}, \quad \lambda_+^2 + \lambda_-^2 = \frac{I_{2,8}}{2} - I_{2,4}, \]
  which helps solve for the independent eigenvalues, \( \lambda_+ \) and \( \lambda_- \) in terms of \( I_{2,4} \) and \( I_{2,8} \).

- **Class 7**: \( h_4 = h_5 = -h_1, h_6 = h_3, h_7 = \frac{h_5}{h_2} \)

  The eigenvalues are \( \lambda_{1+} = \lambda_{2+} = h_1 + h_3 (\equiv \lambda_+), \quad \lambda_{1-} = -\lambda_{2-} = h_1 - h_3 (\equiv \lambda_-) \)
  with the quadratic invariants
  \[ I_{2,4} = I_{2,5} = -2\lambda_+^2, \quad I_{2,8} = 2(\lambda_+^2 + \lambda_-^2), \quad I_{2,9} = I_{2,10} = 2\lambda_-^2. \]

  Clearly in this case we have two independent eigenvalues, \( h_1 + h_3 \) and \( h_1 - h_3 \) and two independent local invariants, \( I_{2,4} \) and \( I_{2,9} \) as \( I_{2,8} = I_{2,9} - I_{2,4} \).

  Another solution \( h_4 = h_5 = h_1, h_6 = h_3, h_8 = h_1, h_7 = \frac{h_1}{h_2} \) belongs to this class.

- **Class 8**: \( h_3 = h_5 = h_6 = h_8 = h_1, h_4 = -h_1, h_7 = -\frac{h_1}{h_2} \)

  The eigenvalues are \( \lambda_{1+} = \lambda_{2+} = (1 \pm i)h_1 (\equiv \lambda_+) \) with the quadratic invariants
  \[ I_{2,4} = I_{2,5} = I_{2,9} = I_{2,10} = 2(\lambda_+ + \lambda_-)^2, \quad I_{2,8} = 0. \]

  Here we only have one local invariant which is consistent with the number of independent eigenvalues, depending on \( h_1 \).

  Another solution \( h_3 = h_4 = h_6 = h_8 = h_1, h_5 = -h_1, h_7 = -\frac{h_1}{h_2} \) belongs to this class.
Class 9: \( h_2 = h_3 = h_6 = 0, h_8 = h_1, h_4 = h_5 = -h_1 \)

The eigenvalues are \( \lambda_{1\pm} = \lambda_{2\pm} = h_1(\pm \lambda), \lambda_{2-} = -h_1(= -\lambda) \) with the quadratic invariants

\[
I_{2,4} = I_{2,5} = -2\lambda^2, \quad I_{2,8} = 4\lambda^2, \quad I_{2,9} = I_{2,10} = 2\lambda^2. \tag{4.16}
\]

Once again we have a single local invariant and a single independent eigenvalue.

There are three other solutions belonging to this class:

* \( h_3 = h_6 = h_7 = 0, h_8 = h_1, h_4 = h_5 = -h_1 \)
* \( h_2 = h_3 = h_6 = 0, h_4 = h_5 = h_8 = h_1 \)
* \( h_3 = h_6 = h_7 = 0, h_4 = h_5 = h_8 = h_1 \)

Class 10: \( h_2 = h_3 = h_6 = 0, h_4 = h_5 = h_8 = h_1 \)

The eigenvalues are \( \lambda_{1\pm} = \lambda_{2\pm} = \pm h_1(\pm \lambda) \) with the quadratic invariants

\[
I_{2,4} = I_{2,5} = I_{2,10} = -2\lambda^2, \quad I_{2,8} = 0, \quad I_{2,9} = 2\lambda^2. \tag{4.17}
\]

There is a single local invariant and a single independent eigenvalue depending on \( h_1 \).

There are three other solutions in this class:

* \( h_3 = h_6 = h_7 = 0, h_4 = h_5 = h_8 = -h_1 \)
* \( h_2 = h_3 = h_6 = 0, h_4 = h_5 = h_8 = -h_1 \)
* \( h_3 = h_6 = h_7 = 0, h_4 = h_5 = h_8 = -h_1 \)

Class 11: \( h_2 = h_6 = 0, h_1 = h_5 = h_8, h_4 = -h_8, h_3 = 2h_8 \)

The eigenvalues are \( \lambda_{1\pm} = \lambda_{2\pm} = h_8(\pm \lambda) \) with the quadratic invariants

\[
I_{2,4} = I_{2,5} = I_{2,9} = 6\lambda^2, \quad I_{2,8} = 0, \quad I_{2,10} = 10\lambda^2. \tag{4.18}
\]

The number of local invariants coincides with the number of independent eigenvalues.

There are seven other solutions belonging to this class:

* \( h_6 = h_7 = 0, h_1 = h_5 = h_8, h_4 = -h_8, h_3 = 2h_8 \)
* \( h_2 = h_6 = 0, h_1 = h_5 = h_8, h_3 = 2h_8 \)
* \( h_6 = h_7 = 0, h_1 = h_5 = h_8, h_3 = 2h_8 \)
* \( h_2 = h_3 = 0, h_5 = h_8 = h_1, h_4 = -h_1, h_6 = 2h_1 \)
* \( h_3 = h_7 = 0, h_5 = h_8 = h_1, h_4 = -h_1, h_6 = 2h_1 \)
* \( h_2 = h_3 = 0, h_4 = h_5 = h_8 = h_1, h_5 = -h_1, h_6 = 2h_1 \)
* \( h_3 = h_7 = 0, h_4 = h_5 = h_8 = h_1, h_5 = -h_1, h_6 = 2h_1 \)

Class 12: \( h_4 = h_5 = 0, h_3 = h_6 = \frac{i+1}{2}h_1, h_9 = -ih_1, h_7 = -\frac{i}{2}\frac{h_1}{h_5} \)

The eigenvalues are \( \lambda_{1\pm} = \lambda_{2\pm} = \frac{i+1}{2}h_1(\pm \lambda) \) with the quadratic invariants

\[
I_{2,4} = I_{2,5} = 2\lambda^2, \quad I_{2,8} = 8\lambda^2, \quad I_{2,9} = 6\lambda^2, \quad I_{2,10} = 10\lambda^2. \tag{4.19}
\]

The number of local invariants coincides with the number of independent eigenvalues.

Another solution \( h_4 = h_5 = 0, h_3 = h_6 = \frac{i+1}{2}h_1, h_8 = ih_1, h_7 = \frac{i}{2}\frac{h_1}{h_5} \) belongs to this class.

This classification is based on the pattern of eigenvalues and quadratic invariants, which is different from the criterium used by Hietarinta [24]. The relation between the two classifications is detailed in appendix A.
5. Link invariants for X-type YBOs

A theorem due to Alexander [28] states that every knot/link embedded in $S^3$ can be obtained as a closure of a braid group element. In order for this to be valid, the braid group generators must satisfy two additional moves, apart from the three usual Reidemeister moves, called the Markov moves. This leads to the enhancement procedure of Turaev and the subsequent computation of knot/link polynomials [25], which we perform in the following.

**Definition.** An enhanced YBO is a quadruple $(R, \mu, x, y)$, with $R : V \otimes V \to V \otimes V$ (a braid operator), $\mu : V \to V$ and $x, y \in \mathbb{C}^*$ such that the following conditions hold

\[(a) \quad [R, \mu \otimes \mu] = 0, \quad (5.1)\]
\[(b) \quad \text{tr}_2 [R(\mu \otimes \mu)] = xy \mu, \quad (5.2)\]
\[(c) \quad \text{tr}_2 [R^{-1}(\mu \otimes \mu)] = x^{-1} y \mu, \quad (5.3)\]

where, as above, $\text{tr}_2$ denotes the partial trace on the second qubit space. Let $B_n$ be the $n$-strand braid group generated by $\sigma_1, \ldots, \sigma_{n-1}$. Link polynomials for a braid group element $\xi \in B_n$ are then obtained as

\[L_R(\xi) = x^{-w(\xi)} y^{-n} \text{Tr} \left[ \rho_R(\xi) \mu^\otimes n \right], \quad (5.4)\]

where $w(\xi)$ is the writhe of the link, and $\rho$ is a representation of $B_n$ constructed from the YBO $R$ as

\[\rho_R(\sigma_i) = I^\otimes i-1 \otimes R_{i,i+1} \otimes I^\otimes n-i+1. \quad (5.5)\]

We take $V = (\mathbb{C}^2)^\otimes n$ for qubit systems and $n = 2$ for two-qubit systems. $I = I_2$ and $R_{i,i+1}$ denotes $R$ acting on the $i$th and $(i+1)$th qubits.

Note that the polynomials obtained from (5.4) are not always invariant under the local action of $SL(2, \mathbb{C})^\otimes n$ due to the presence of $\mu^\otimes n$. In the case when $\mu = I$, the link polynomials are local invariants. As we explicitly see in (5.11) and (5.12), the link polynomials (5.4) with $\mu \neq I$ are not expected to be local invariants even if they are expressible only in terms of the eigenvalues. We can say that any local invariant constructed from an X-type YBO is expressed as a function depending only on eigenvalues of the YBO. However, the converse is not true in general.

We now enhance the twelve classes of X-type braid operators obtained in section 4 and compute the associated link invariants for two- or three-strand cases as examples. All of these are expressed solely in terms of the eigenvalues. Since it turns out that in each case the Skein relation for the braid operators depends only on the eigenvalues, we can say that all the other link invariants generated via the Skein relation are also functions only of the eigenvalues. Enhancement of Hietarinta’s solutions [24] and associated link invariants are investigated in [29]. Cases for unitary solutions are also discussed in [30]. Although our results overlap with the results obtained there, we present them from our viewpoint in order to make this paper self-contained.

---

9 Recall that the second and third Reidemeister moves represent the relations, $\sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = 1$ and $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$, respectively.

10 YBOs automatically become braid operators since the far-commutativity conditions $\sigma_i \sigma_j = \sigma_j \sigma_i$ ($|i - j| > 1$) are trivially satisfied.
• Class 1: \( R_1 = \begin{pmatrix} h_1 & 0 & 0 & 0 \\ 0 & h_4 & 0 & 0 \\ 0 & 0 & h_5 & 0 \\ 0 & 0 & 0 & h_8 \end{pmatrix} \)

In this case we can enhance the braid operator when \( \mu = I, \mu = Z, \mu = I \pm Z \).

* \( \mu = I, x = \pm h_1, y = \pm 1 \) and \( h_8 = h_1 \). For example the link invariants corresponding to a two-strand braid group element \( \xi = \sigma^k (k \in \mathbb{Z}) \) are given by

\[
L_R (\sigma_1^k) = \begin{cases} 
2 + 2 \left( \frac{\sqrt{h_4 h_5}}{h_1} \right)^k & (k \text{ even}) \\
\pm 2 & (k \text{ odd}) 
\end{cases}
\]  
(5.6)

that distinguish links with even linking numbers. At \( h_8 = h_1 \) the braid operator has three eigenvalues \( \{\lambda_1 = h_1, \pm \lambda_2 = \pm \sqrt{h_4 h_5} \} \) implying that a scaled version of this braid operator, \( g_i = \pm \sqrt{\lambda_1 \lambda_2} R_i \), realizes the Birman–Murakami–Wenzl (BMW) algebra \( C_\mu (l, m) \) [31, 32]:

\[
e_i = \frac{1}{m} \left( g_i + g_i^{-1} \right) - 1, \quad e_i^2 = \left[ \frac{1}{m} \left( \frac{l + 1}{l} \right) - 1 \right] e_i, \quad (5.7)
\]

\[
e_i g_i \pm e_i = l \mp e_i, \quad e_i g_i = g_i e_i = l^{-1} e_i, \quad (5.8)
\]

with \( l = \pm i \sqrt{\lambda_1 \lambda_2} \) and \( m = \mp i \frac{\lambda_1 - \lambda_2}{\sqrt{\lambda_1 \lambda_2}} \). From (5.7), we obtain

\[
g_i^2 = \left( m + \frac{1}{l} \right) g_i + \left( 1 + \frac{m}{l} \right) \cdot 1 - \frac{1}{l} g_i^{-1} = 0. \quad (5.9)
\]

The Skein relation for the braid operator in this case can be read off from (5.9) as \( g_i \) and \( g_i^{-1} \) can be thought of as positive and negative crossings respectively. This helps us to obtain other link invariants in a combinatorial manner, which are also expressed in terms of the eigenvalues. Although the BMW algebra underlies the Kauffman polynomial of two variables [33] in general, we see that (5.9) with (5.6) generates link invariants in the single variable \( \frac{\sqrt{h_4 h_5}}{h_1} \).

* \( \mu = Z, x = \pm h_1, y = \pm 1 \) and \( h_8 = -h_1 \). The link invariants corresponding to a two-strand braid group element \( \xi = \sigma^k (k \in \mathbb{Z}) \) become

\[
L_R (\sigma_1^k) = \left[ 1 - \left( \frac{\sqrt{h_4 h_5}}{h_1} \right)^k \right] \left[ 1 + (-1)^k \right], \quad (5.10)
\]

that distinguish links with even linking numbers. In this case, the braid operator has four eigenvalues \( \{\pm \lambda_1 = \pm h_1, \pm \lambda_2 = \pm \sqrt{h_4 h_5} \} \) and satisfies the identity \( R_1^2 - (h_1^2 + h_4 h_5) R_1 + h_1^2 h_4 h_5 R_1^{-1} = 0 \). At a glance it seems to lead to nontrivial \( G_2 \)-link invariants [34], but actually it does not as discussed in [29, 30].

Note that (5.10) is not a local invariant although it can be expressed in terms of the eigenvalues. Actually, under the transformation (2.3) with \( Q_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \) satisfying
\(a_d j - b_j c_j = 1 \ (j = 1, 2), \) (5.10) changes as
\[
2 \left( \prod_{j=1}^{2} (a_d j + b_j c_j) \right) \left[ 1 - \left( \frac{\sqrt{h_2 h_3}}{h_1} \right)^k \right]
\]
for \(k\) even, and
\[
\mp 4 \frac{(h_3 h_5) \lambda_2}{h_1} \mu (a_1 c_1 b_2 d_2 h_4 + b_1 d_1 a_2 c_2 h_3)
\]
for \(k\) odd. \(R\) can be diagonalized as \(\Omega \text{diag}(\lambda_1, \lambda_2, -\lambda_2, -\lambda_1)\Omega^{-1}\), where \(\Omega = 1 \oplus \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{h_2} & \sqrt{h_5} \\ \sqrt{h_5} & -\sqrt{h_2} \end{pmatrix} \oplus 1\) depends on \(\sqrt{h_4/h_5}\) besides the eigenvalue \(\lambda_2\). In general, when \(\mu \neq I\), such dependence in \(\Omega\) will appear in the link invariants (5.4), and the result will not be expressed only in terms of the eigenvalues. Here we see that \(\Omega^{-1}(\mu \otimes \mu) \Omega = \text{diag}(1, -1, -1, 1)\) and that the dependence in \(\Omega\) accidentally disappear. For this reason, the expression (5.10) is expressed only in terms of the eigenvalues, which however does not mean the local invariance as seen in (5.11) and (5.12).

* \(\mu = I + Z, x = \pm h_1, y = \pm 2\), there is no relation between \(h_1\) and \(h_8\) in this case. The link invariants obtained in this case are just constants: \(L_R(\sigma_1^k) = 1\) for \(k\) even and \(\pm 1\) for \(k\) odd.

* \(\mu = I - Z, x = \pm h_8, y = \pm 2\), there is no relation between \(h_1\) and \(h_8\) in this case. The link invariants obtained in this case are the same constants as above.

For the third and fourth cases, the braid operators have four different eigenvalues, and the identity \(R_1^1 + (h_1 + h_8)R_2^2 + (h_1 h_8 - h_2 h_3)R_3^3 + (h_1 + h_8)h_2 h_3 R_1^1 - h_1 h_2 h_3 h_4 R_4^1 = 0\) holds. Despite this relation, nontrivial \(G_2\)-link invariants cannot be obtained. As discussed in [29], any link invariant turns out to be 1 or -1 due to the property \(R_1^{\pm 1}(\mu \otimes \mu) = h_8^{\pm 1}(\mu \otimes \mu)\) for the third and \(R_1^{\pm 1}(\mu \otimes \mu) = h_8^{\pm 1}(\mu \otimes \mu)\) for the fourth.

- **Class 2:** \(R_2 = \begin{pmatrix} 0 & 0 & 0 & h_2 \\ 0 & h_3 & 0 & 0 \\ 0 & 0 & h_3 & 0 \\ h_7 & 0 & 0 & 0 \end{pmatrix}\)

In this case the braid operator can be enhanced using only \(\mu = I\). We then have \(x = \pm h_3, y = \pm 1\). The link invariants obtained are similar to the class 1 counterpart as seen for an element of the two-strand braid group, \(\xi = \sigma_1^k (k \in \mathbb{Z})\):
\[
L_R(\sigma_1^k) = \begin{cases} 2 + 2 \left( \frac{\sqrt{h_2 h_3}}{h_3} \right)^k & (k \text{ even}) \\ \pm 2 & (k \text{ odd}) \end{cases}
\]
that distinguish links with even linking numbers. The braid operator has three eigenvalues \(\{\lambda_2 = h_3, \pm \lambda_1 = \pm \sqrt{h_2 h_3}\}\) implying that a scaled version of this braid operator, \(g_t = \mp \frac{1}{\sqrt{\lambda_1 \lambda_2}} R_2\), realizes the BMW algebra \(\mathcal{C}_n(l, m)\) at \(l = \pm i \sqrt{\frac{3 \lambda_2}{\lambda_1}}\) and \(m = \mp \frac{\lambda_1 - \lambda_2}{\sqrt{\lambda_1 \lambda_2}}\).
Here enhancement occurs when $\mu = Z$ and $\mu = \mp \sqrt{\frac{b_8-b_5}{h_7}} I + X - iY \mp \sqrt{\frac{b_8-b_5}{h_7}} Z$.

* $\mu = Z$ and $x = \pm i\sqrt{h_1 h_8}$, $y = \mp i \frac{b_8}{h_8}$. The link invariants $L_R(\sigma_1^k)$ vanish in this case.

* $\mu = -\sqrt{\frac{b_8-b_5}{h_7}} I + X - iY + \sqrt{\frac{b_8-b_5}{h_7}} Z$, and $x = \pm h_5$, $y = \mp 2 \sqrt{\frac{b_8-b_5}{h_7}}$. The link invariants are constants in this case: $L_R(\sigma_1^1) = (\pm 1)^k$.

* $\mu = \sqrt{\frac{b_8-b_5}{h_7}} I + X - iY - \sqrt{\frac{b_8-b_5}{h_7}} Z$, and $x = \pm h_8$, $y = \pm 2 \sqrt{\frac{b_8-b_5}{h_7}}$. The link invariants are the same constants as above.

As there are two distinct eigenvalues in these cases, $\{h_1, h_8\}$, each with multiplicity two, we expect to realize the Hecke algebra, $H_n(q)$, generated by invertible $\sigma_i$,

$$\sigma_i^2 = (q-1) \sigma_i + q, \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad (5.14)$$

using this braid operator [35]. This happens either for $\sigma_i = -\frac{1}{h_1} R_3$ at $q = -\frac{b_8}{h_1}$ or $\sigma_i = -\frac{1}{h_8} R_3$ at $q = -\frac{b_5}{h_8}$. The Skein relation is read off from the first equation of (5.14).

* Class 4: $R_4 = \begin{pmatrix} h_1 & 0 & 0 & 0 \\ 0 & h_1 & h_6 & 0 \\ 0 & h_1 & h_6 & 0 \\ h_7 & 0 & 0 & h_8 \end{pmatrix}$

In this case enhancement is possible when $\mu = I$, $\mu = Z$, $\mu = I \pm Z$ and when $\mu = I + \frac{b_8-b_5}{2h_1-h_6} Z$.

* $\mu = I$, $x = \pm h_1$, $y = \pm 1$ and $h_6 = 0$. We obtain constant link invariants: $L_R(\sigma_1^k) = 4$ for $k$ even and $\pm 2$ for $k$ odd.

* $\mu = Z$, $x = \pm i h_1$, $y = \mp i$ and $h_6 = 2h_1$. The link invariants $L_R(\sigma_1^k)$ vanish.

* $\mu = I + Z$, $x = \pm h_1$, $y = \pm 2$. We obtain constant link invariants $L_R(\sigma_1^k) = (\pm 1)^k$.

* $\mu = I - Z$, $x = \pm h_1$, $y = \pm 2$. We obtain the same constant link invariants as above.

* $\mu = I + \frac{b_8-b_5}{2h_1-h_6} Z$ and $x = \pm \frac{h_8^2}{\sqrt{h_1-h_6}}$, $y = \mp 2 \sqrt{\frac{b_8-b_5}{h_7}} Z$. In this case we obtain non-trivial link polynomials as seen in a two-strand braid group element $\xi = \sigma_1^{k_1} (k \in \mathbb{Z})$:

$$L_R(\sigma_1^1) = \left[ \frac{\pm h_1^2}{\sqrt{h_1-h_6}} \right]^{-k} \frac{-h_1(-h_1+h_6)^{k} + h_1^2 (3h_1^2 - 3h_1h_6 + h_6^2)}{h_1(h_1-h_6)}, \quad (5.15)$$

and in a three-strand braid group element $\xi = \sigma_1^{k_1} \sigma_2^{k_2} (k_1, k_2 \in \mathbb{Z})$:

$$L_R(\sigma_1^{k_1} \sigma_2^{k_2}) = \left[ \frac{\pm h_1^2}{\sqrt{h_1-h_6}} \right]^{-k_1-k_2+1} \frac{1}{h_1^2 (2h_1^2 - 3h_1h_6 + h_6^2)} \times \prod_{a=1}^{2} \left\{ -h_1(-h_1+h_6)^{k_a+1} + h_1^2 (3h_1^2 - 3h_1h_6 + h_6^2) \right\}. \quad (5.16)$$
Since there are two distinct eigenvalues in these cases, \(\{h_1, -h_1 + h_6\}\) with multiplicities three and one respectively, we expect to realize the Hecke algebra \((5.14), H_n(q)\), with this braid operator and this indeed happens for either \(\sigma_i = \frac{1}{h_1 - h_6} R_4\) at \(q = \frac{h_1 - h_6}{h_1}\) or \(\sigma_i = -\frac{1}{h_1} R_4\) at \(q = \frac{h_1 - h_6}{h_1}\). We can show that \((5.15)\) and \((5.16)\) depend on the eigenvalues only through their ratio \((-h_1 + h_6)/h_1\).

- **Class 5:** \(R_5 = \begin{pmatrix}
    h_1 & 0 & 0 & 0 \\
    0 & 0 & h_4 & 0 \\
    0 & h_4 (h_1 - h_6) & h_6 & 0 \\
    0 & 0 & 0 & -h_1 + h_6
\end{pmatrix}
\)

Here enhancement occurs for \(\mu = Z\) and \(\mu = I \pm Z\).

* \(\mu = Z, x = \pm \sqrt{h_1 (h_1 - h_6)}, y = \pm \sqrt{h_1 - h_6}\). In this case the two-strand braid group elements give vanishing link invariants. We can also see that elements of the three-strand braid group vanish \(^{11}\): \(L_R (\sigma_1^3 \sigma_2^k) = L_R (\sigma_1^3 \sigma_1^m \sigma_2^n) = 0\) for \(k, l, m, n \in \mathbb{Z}\).

* \(\mu = I + Z, x = \pm h_1, y = \pm 2\). In this case, we obtain constant link invariants: \(L_R (\sigma_1^3) = (\pm 1)^k\).

* \(\mu = I - Z, x = \pm (h_1 - h_6), y = \mp 2\). In this case we also obtain constant link invariants: \(L_R (\sigma_1^3) = (\mp 1)^k\).

For these cases, \(R_5\) has two different eigenvalues \(\{h_1, -h_1 + h_6\}\) with multiplicity two for each. We see that the Hecke algebra \((5.14), H_n(q)\), is realized by \(\sigma_i = -\frac{1}{h_1} R_5\) with \(q = \frac{h_1 - h_6}{h_1}\) or \(\sigma = \frac{1}{h_1 - h_6} R_5\) with \(q = \frac{h_1}{h_1 - h_6}\).

- **Class 6:** \(R_6 = \begin{pmatrix}
    h_1 & 0 & -\sqrt{h_1^2 + h_2^2} & 0 \\
    0 & h_1 + h_8 & \frac{2}{2} & 0 \\
    -\sqrt{h_1^2 + h_2^2} & \frac{2}{2} & h_1 + h_8 & 0 \\
    \frac{(h_1 + h_8)^2}{4h_2} & 0 & 0 & h_8
\end{pmatrix}
\)

Enhancement is possible for the following five cases (\(\lambda_\pm = \frac{1}{2} \left( h_1 + h_8 \pm \sqrt{2(h_1^2 + h_2^2)} \right)\) denote the eigenvalues of \(R\):

* \(\mu = Z, x = \pm \frac{h_1 - h_8}{h_1}, y = \pm 1\). The link invariants \(L_R (\sigma_1^k)\) vanish.

* \(\mu = I + \frac{1}{2} \frac{h_1 + h_8}{2h_1^2 + h_2^2} X + \frac{1}{2} \frac{h_1 - h_8}{2h_1^2 + h_2^2} Y - \frac{2h_1}{2h_1^2 - h_8} Z, x = \pm \lambda, y = \pm 2\).

* \(\mu = I - \frac{1}{2} \frac{h_1 + h_8}{2h_1^2 + h_2^2} X + \frac{1}{2} \frac{h_1 - h_8}{2h_1^2 + h_2^2} Y - \frac{2h_1}{2h_1^2 - h_8} Z, x = \pm \lambda, y = \pm 2\).

* \(\mu = I + \frac{1}{2} \frac{h_1 + h_8}{2h_1^2 + h_2^2} X + \frac{1}{2} \frac{h_1 - h_8}{2h_1^2 + h_2^2} Y + \frac{2h_1}{2h_1^2 - h_8} Z, x = \pm \lambda, y = \pm 2\).

* \(\mu = I - \frac{1}{2} \frac{h_1 + h_8}{2h_1^2 + h_2^2} X + \frac{1}{2} \frac{h_1 - h_8}{2h_1^2 + h_2^2} Y + \frac{2h_1}{2h_1^2 - h_8} Z, x = \pm \lambda, y = \pm 2\).

For the last four cases, we obtain the same result for the link invariants: \(L_R (\sigma_1^k) = (\pm 1)^k\). Each of the eigenvalues \(\lambda_\pm\) has multiplicity two. The braid operator can be used to realize the Hecke algebra by \(\sigma_i = -\frac{1}{\chi} R_6\) with \(q = -\frac{1}{\chi}\) or \(\sigma_i = -\frac{1}{\chi} R_6\) with \(q = \frac{1}{\chi}\).

\(^{11}\) Note that \(L_R (\sigma_1^k \sigma_1^\ell)\) and \(L_R (\sigma_1^m \sigma_2^n)\) reduce to \(L_R (\sigma_1^k \sigma_2^m)\) and \(L_R (\sigma_1^\ell \sigma_2^n)\) respectively, because \(R\) commutes with \(\mu \otimes \mu\).
The case $\mu = 1$ alone enhances this operator when $x = \pm (h_1 + h_3), y = \pm 1$. We obtain non-trivial link invariants in this case as seen for two-strand and three-strand braid group elements:

$$L_R (\sigma^1_1) = (-1)^{k^2} 2 \left[ 1 + \frac{1 + (-1)^k (h_1 - h_3) \pm \sqrt{h_1 + h_3}}{2} \right],$$

(5.17)

$$L_R (\sigma^1_1 \sigma^1_2) = (-1)^{k^2+1} 2 \left[ 1 + \frac{1 + (-1)^k (h_1 - h_3) \pm \sqrt{h_1 + h_3}}{2} \right]$$

$$\times \left[ 1 + \frac{1 + (-1)^k (h_1 - h_3) \pm \sqrt{h_1 + h_3}}{2} \right].$$

(5.18)

These distinguish only links with even linking numbers\(^{12}\). There are three distinct eigenvalues in this case, $\{h_1 + h_3, \pm (h_1 - h_3)\}$ with multiplicities two, one and one respectively. The operator $g_i = \pm \frac{x}{\sqrt{h_1 - h_3}} R_7$ realizes the BMW algebra (5.7) and (5.8) at $l = \mp i \sqrt{\frac{1}{2} + \frac{1}{2} \pm \sqrt{\pm \frac{1}{2} \mp \sqrt{\pm \frac{1}{2} - \frac{1}{2}}}}$ and $m = \pm i \sqrt{\frac{1}{2} - \frac{1}{2} \mp \sqrt{\pm \frac{1}{2} \pm \sqrt{\pm \frac{1}{2} - \frac{1}{2}}}}$

- **Class 7: $R_7$**

$$\begin{pmatrix}
  h_1 & 0 & 0 & h_2 \\
  0 & h_3 & -h_1 & 0 \\
  0 & -h_1 & h_3 & 0 \\
  h_1^2 & 0 & 0 & h_1 \\
\end{pmatrix}$$

- **Class 8: $R_8$**

$$\begin{pmatrix}
  h_1 & 0 & 0 & h_2 \\
  0 & h_1 & -h_1 & 0 \\
  0 & h_1 & h_1 & 0 \\
  \bar{h}_1^2 & 0 & 0 & \bar{h}_1 \\
\end{pmatrix}$$

Enhancement occurs for $\mu = 1$ at $x = \pm \sqrt{2} h_1, y = \pm \sqrt{2}$. We obtain just constant link invariants: $L_R (\sigma^1_1) = (-1)^{k^2} 2 \cos (\sqrt{2} k)$. There are two distinct eigenvalues, $(1 \pm i)h_1$ leading to a realization of the Hecke algebra either when $\sigma_i = -\frac{1}{2 h_1} R_8$ at $q = i$ or when $\sigma_i = -\frac{1}{2 h_1} R_8$ at $q = -i$.

- **Class 9: $R_9$**

$$\begin{pmatrix}
  h_1 & 0 & 0 & 0 \\
  0 & 0 & -h_1 & 0 \\
  0 & -h_1 & 0 & 0 \\
  h_1^2 & 0 & 0 & h_1 \\
\end{pmatrix}$$

In this case enhancement is only possible with $\mu = 1$ and $x = \pm h_1, y = \pm 1$. The link invariants obtained are just constant: $L_R (\sigma^1_1) = 4$ for $k$ even and $\pm 2$ for $k$ odd. There are two distinct eigenvalues in this case, $\{h_1, -h_1\}$ with multiplicities of three and one respectively. The Jordan decomposition of $R_9$ leads to the identity $R_9^2 - h_1 R_9 - h_1^2 1 + h_1^2 R_9^{-1} = 0$ instead of the Hecke algebra realization.

\(^{12}\) Since the linking number is well-defined for two-component links, this statement has a meaning for $k$ even in $L_R (\sigma_1^1)$ and for $k/l$ even/odd or odd/even in $L_R (\sigma_1^1 \sigma_2^1)$. Links obtained by taking closure of the braids have two components in the cases. As other cases, they have one component for $k$ odd in $L_R (\sigma_1^1)$ and for both $k$ and $l$ odd in $L_R (\sigma_1^2 \sigma_2^2)$. Three components for both $k$ and $l$ even in $L_R (\sigma_1^1 \sigma_2^1)$. 17
However, $\sigma_i$ is not equivalent to the permutation operator (see (A.3) for its matrix form), because patterns of the eigenvalues are different. The permutation operator has the eigenvalues 1 and $-1$ with multiplicities three and one, respectively.
the Hecke algebra at \( q = -1 \) after scaling it with a factor, \( -\frac{1+i}{h_1} \). Again, the relation reduces to \((\sigma_i + 1)^2 = 0\).

For all the cases, link invariants directly computed or generated via the Skein relations are functions of a single variable that is a combination of the eigenvalues of the braid operator. This seems to match the claim in [29]—the best invariant of links obtained from the enhanced YBOs is the Jones polynomial.

6. Entangling power

We have seen in section 4 that the independent local invariants for the X-type YBOs are functions of just their independent eigenvalues, implying that in these systems the quantum entanglement and its non-local properties are obtained in terms of the eigenvalues of the ‘entanglers’. A subtle feature, as we observed, is that this is not true for an entangler that is not a YBO. As a further check of this, we compute here the entangling powers [26] of the X-type YBOs and compare it with the entangling power of an arbitrary X-type entangler.

The entangling power for an operator \( U \) is defined as

\[
e_P(U) = \overline{E(U|\psi_1\rangle\otimes|\psi_2\rangle)},
\]

where the overline denotes an average over some distribution of the product states, \(|\psi_1\rangle\otimes|\psi_2\rangle\) and \(E\) denotes an entanglement measure for two-qubit states. To determine the entanglement measure in a two-qubit space, we look for independent local invariants under the action of \( SL(2, \mathbb{C})^\otimes 2\). The entanglement measure we choose to compute the entangling power is expected to be a function of only these local invariants.

6.1. Invariant of two-qubit states under \( SL(2, \mathbb{C})^\otimes 2\)

A two-qubit state

\[
|\psi\rangle = \sum_{i_1, i_2 = 0}^1 t_{i_1 i_2} |i_1 i_2\rangle
\]

with coefficients \( t_{i_1 i_2} \) is changed by an ILO \( Q = Q_1 \otimes Q_2 \in SL(2, \mathbb{C})^\otimes 2 \) as

\[
Q |\psi\rangle = \sum_{i_1, i_2, j_1, j_2 = 0}^1 t_{i_1 i_2} (Q_1)_{i_1 j_1} (Q_2)_{i_2 j_2} |j_1 j_2\rangle,
\]

which amounts to the change of the coefficients:

\[
t_{i_1 i_2} \rightarrow \sum_{j_1, j_2 = 0}^1 t_{j_1 j_2} (Q_1)_{i_1 j_1} (Q_2)_{i_2 j_2}.
\]

Invariant quantities under the change (6.4) can be constructed by contracting indices of the coefficients by invariant tensors \( \epsilon_{i_1 j_1} (a = 1, 2) \) for \( SL(2, \mathbb{C}) \) at the \( a \)th qubit. The invariant of the lowest order is quadratic in \( t \):

\[
J_2 = t_{i_1 i_2} t_{j_1 j_2} \epsilon_{i_1 j_1} \epsilon_{i_2 j_2} = 2 \det t.
\]
One can show that there is no independent invariant at higher orders in $t$ as follows. It is easy to see that we cannot construct invariants of odd orders in $t$. An invariant to the $2N$th order in $t$ can be expressed as

$$J_{2N} = t_{i_1} t_{j_1} j_2 \epsilon_{i_1 j_1} (K_{2N-2})_{i_2 j_2}, \tag{6.6}$$

where $(K_{2N-2})_{i_2 j_2}$ denotes a polynomial of the $(2N - 2)$th order in $t$ with indices other than $i_2$ and $j_2$ contracted. We assume that invariants up to the order less than $2N$ are functions of $J_2$.

Due to the identity

$$t_{i_1} t_{j_1} j_2 \epsilon_{i_1 j_1} = (\det t) \epsilon_{i_2 j_2} = \frac{1}{2} J_2 \epsilon_{i_2 j_2}, \tag{6.7}$$

we obtain

$$J_{2N} = \frac{1}{2} J_2 \epsilon_{i_2 j_2} (K_{2N-2})_{i_2 j_2}. \tag{6.8}$$

Note that $\epsilon_{i_2 j_2} (K_{2N-2})_{i_2 j_2}$ is an invariant of the $(2N - 2)$th order and thus a function of $J_2$ by the assumption. Hence, $J_{2N}$ is also a function of $J_2$, which completes a proof by the induction.

As another proof, we show that there is just a single local invariant for a two-qubit state, by considering the infinitesimal action of $SL(2, C) \otimes 2$ on an arbitrary two-qubit state, as we mentioned below (3.6). This is obtained from the expressions

$$X_{i}I_{i+1} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} \alpha_3 \\ \alpha_4 \\ \alpha_1 \\ \alpha_2 \end{pmatrix}, \quad Y_{i}I_{i+1} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} -i \alpha_3 \\ -i \alpha_4 \\ i \alpha_1 \\ i \alpha_2 \end{pmatrix}, \tag{6.9}$$

and

$$Z_{i}I_{i+1} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ -\alpha_3 \\ -\alpha_4 \end{pmatrix},$$

and

$$I_{i}X_{i+1} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} \alpha_2 \\ \alpha_1 \\ \alpha_4 \\ \alpha_3 \end{pmatrix}, \quad I_{i}Y_{i+1} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} -i \alpha_2 \\ i \alpha_1 \\ -i \alpha_4 \\ i \alpha_3 \end{pmatrix}, \tag{6.10}$$

with $i$ and $i + 1$ denoting the first and second qubits respectively. It can be checked that only three of these six vectors are linearly independent. The three vectors generate a three-dimensional hypersurface in the four dimensions spanned by $\alpha_1, \ldots, \alpha_4$. A single direction perpendicular to the hypersurface corresponds to a single local invariant.

Note that a general ILO belongs to $\mathbb{C}^* \cdot SL(2, \mathbb{C})^{\otimes 2}$ rather than $SL(2, \mathbb{C})^{\otimes 2}$. Due to the overall factor $\mathbb{C}^*$ (multiplication by a nonzero complex number), only the value of $J_2$ being zero or nonzero has an SLOCC-invariant meaning and labels SLOCC classes. For instance, $J_2 \neq 0 (= 0)$ indicates the Bell-state class (the product-state class).
6.2. Entangling power for a general X-type two-qubit operator

Consider a general two-qubit product state $|P\rangle = (a_1|0\rangle + b_1|1\rangle) \otimes (a_2|0\rangle + b_2|1\rangle)$ with unit norm. The X-type two-qubit operator in (3.1) acts on $|P\rangle$ to give

$$R |P\rangle = (a_1 a_2 h_1 + b_1 b_2 h_2) |00\rangle + (a_1 b_2 h_3 + b_1 a_2 h_4) |01\rangle + (a_1 b_2 h_5 + b_1 a_2 h_6) |10\rangle + (a_1 a_2 h_7 + b_1 b_2 h_8) |11\rangle.$$  \hfill (6.11)

The local invariant under $\text{SL}(2, \mathbb{C})^{\otimes 2}$ for this state is given by

$$\det t = h_1 h_7 a_1^2 a_2^2 + h_2 h_8 b_1^2 b_2^2 - h_3 h_6 a_1^2 b_2^2 - h_4 h_5 b_1^2 a_2^2$$

$$\quad + (h_1 h_8 + h_2 h_7 - h_3 h_5) a_1 a_2 b_1 b_2.$$  \hfill (6.12)

We choose $|\det t|^2$ as our entanglement measure\(^{14}\), and use the parametrization

$$a_1 = e^{i\phi_1} \cos \theta_1, \quad b_1 = e^{-i\phi_1} \sin \theta_1,$$

$$a_2 = e^{i\phi_2} \cos \theta_2, \quad b_2 = e^{-i\phi_2} \sin \theta_2,$$  \hfill (6.13)

which fixes the overall phase of each of the one-qubit states $a_i|0\rangle + b_i|1\rangle$ ($i = 1, 2$). Each one qubit corresponds to a point on the unit sphere (Bloch sphere) as

$$r_i^{(1)} \equiv \left( a_i^* |0\rangle + b_i^* |1\rangle \right) X (a_i|0\rangle + b_i|1\rangle) = \sin(2\theta_i) \cos(-2\phi_i),$$

$$r_i^{(2)} \equiv \left( a_i^* |0\rangle + b_i^* |1\rangle \right) Y (a_i|0\rangle + b_i|1\rangle) = \sin(2\theta_i) \sin(-2\phi_i),$$

$$r_i^{(3)} \equiv \left( a_i^* |0\rangle + b_i^* |1\rangle \right) Z (a_i|0\rangle + b_i|1\rangle) = \cos(2\theta_i).$$  \hfill (6.14)

where we see that $(2\theta_i, -2\phi_i)$ parametrizes the unit sphere for each $i = 1, 2$. Under a uniform distribution on the Bloch spheres, namely averaging as

$$\bar{\varphi} \equiv \frac{4}{\pi^2} \int_{-\pi}^0 d\phi_1 \int_{-\pi}^0 d\phi_2 \int_{0}^{\pi/2} d\theta_1 \sin \theta_1 \cos \theta_1$$

$$\times \int_{0}^{\pi/2} d\theta_2 \sin \theta_2 \cos \theta_2 \cos(\phi_1, \phi_2, \theta_1, \theta_2),$$  \hfill (6.15)

we find the entangling power as

$$e_\rho(R) = \frac{1}{3} \left[ |h_1 h_7|^2 + |h_2 h_8|^2 + |h_3 h_6|^2 + |h_4 h_5|^2 \right]$$

$$\quad + \frac{1}{6} |h_1 h_8 + h_2 h_7 - h_3 h_5 - h_4 h_6|^2.$$  \hfill (6.16)

Whereas the term on the second line consists only of $\text{SL}(2, \mathbb{C})^{\otimes 2}$ invariant combinations (3.4), the terms on the first line do not.

\(^{14}\) In [26], the linear entropy $1 - \text{tr}_\rho \rho^2$ with $\rho$ being the reduced density matrix of a two-qubit pure state $\rho \equiv \frac{1}{\text{tr}_\rho |\Psi\rangle \langle \Psi|}$ is used as entanglement measure. The linear entropy of $|\Psi\rangle = \sum_{i,j=0}^1 h_{1i} |i\rangle |j\rangle$ is computed as $2 |\text{det} t|^2$. When the state $|\Psi\rangle$ is normalized, the denominator is 1 and the expression coincides with $|\det t|^2$ up to the numerical factor 2.
where $R$ in (3.1) becomes unitary when
\[
\begin{align*}
  h_1 &= r_1 e^{i \varphi_1}, & h_2 &= \sqrt{1 - r_1^2} e^{i \varphi_2}, & h_3 &= r_3 e^{i \varphi_3}, \\
  h_4 &= \sqrt{1 - r_3^2} e^{i \varphi_4}, & h_5 &= -\sqrt{1 - r_3^2} e^{i(\varphi_3 + \varphi_6 - \varphi_4)}, \\
  h_6 &= r_3 e^{i \varphi_6}, & h_7 &= -\sqrt{1 - r_3^2} e^{i(\varphi_3 + \varphi_6 - \varphi_7)}, & h_8 &= r_1 e^{i \varphi_8},
\end{align*}
\]
with $r_1, r_2, r_3 \in [0, 1]$ and $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_6, \varphi_8 \in [0, 2\pi]$. Corresponding to (3.4), we see that $r_1, r_3, \varphi_1, \varphi_3, \varphi_6, \varphi_8$ are $SL(2, \mathbb{C})$-invariant parameters. Then it is easy to verify that $e_p(R)$ depends only on such local invariants. This is consistent with known results about the entangling power of unitary quantum gates [36].

### 6.3. Entangling power of X-type YBOs

We discuss the entangling power of the twelve classes of X-type YBOs separately. We see that although the entangling power is not always a function only of eigenvalues for general YBOs, it is always so for unitary YBOs.

- **Class 1:**
  The YBO $R_1$ has four free parameters, $h_1, h_4, h_5, h_8$, and its eigenvalues are $\lambda_{1+} = h_1$, $\lambda_{1-} = h_6, \pm \lambda_2 = \pm \sqrt{h_2 h_3}$. The entangling power (6.16) reads

\[
  e_p(R_1) = \frac{1}{6} |h_1 h_6 - h_4 h_5|^2 = \frac{1}{6} |\lambda_{1+} \lambda_{1-} - \lambda_2^2|^2,
\]

which is a function of only the local invariants, as expected, and can be expressed only by the eigenvalues. The enhancement procedure possibly imposes a relation $h_1 = h_8$ or $h_1 = -h_8$, which however does not affect the above properties. The unitary YBO with $|h_1| = |h_4| = |h_5| = |h_6| = 1$ also preserves the properties.

- **Class 2:**
  Free parameters of the YBO $R_2$ are $h_2, h_3, h_7$, and its eigenvalues are $\pm \lambda_1 = \pm \sqrt{h_2 h_7}, \lambda_2 = h_3$. The entangling power

\[
  e_p(R_2) = \frac{1}{6} |h_2 h_7 - h_3 h_7|^2 = \frac{1}{6} |\lambda_1^2 - \lambda_2^2|^2,
\]

is a function of only the local invariants, expressed only in terms of the eigenvalues. These properties are not changed by enhancement or by imposing the unitary condition $|h_2| = |h_3| = |h_7| = 1$.

- **Class 3:**
  The YBO $R_3$ is a function of $h_1, h_7, h_8$, and its eigenvalues are $\lambda_+ = h_1, \lambda_- = h_8$, which are not changed by the enhancement. The entangling power is computed to be

\[
  e_p(R_3) = \frac{1}{9} \left[ |h_1 h_7|^2 + |h_1 (h_1 + h_8)|^2 \right] + \frac{2}{3} |h_1 h_8|^2
  = \frac{1}{9} \left[ |\lambda_+ h_7|^2 + |\lambda_+ (\lambda_+ + \lambda_-)|^2 \right] + \frac{2}{3} |\lambda_+ \lambda_-|^2,
\]
which is now dependent on $h_7$, a parameter that changes under the local action of $SL(2, \mathbb{C})^{\otimes 2}$. $R_3$ is unitary for $h_1 = -h_8 = e^{i\phi_3}$ and $h_7 = 0$, turning it into a special case of class 1. Then (6.21) becomes a constant $\frac{2}{3}$, which is a trivial function of the eigenvalues.

- **Class 4:**
  The YBO $R_4$ has three parameters $h_1, h_4$ and $h_6$, with its eigenvalues $\lambda_1 = h_1$ and $\lambda_2 = -h_1 + h_6$, which is kept intact by enhancement. The entangling power is computed to be
  \[
e_p(R_4) = \frac{1}{9} |h_4 h_6|^2 + \frac{1}{6} |h_1 h_6|^2 = \frac{1}{9} (\lambda_1 + \lambda_2) h_4^2 + \frac{1}{6} |\lambda_1 (\lambda_1 + \lambda_2)|^2,
  \]
  which depends on $h_4$, a parameter that changes under the local action of $SL(2, \mathbb{C})^{\otimes 2}$. $R_4$ becomes unitary when $|h_1| = |h_4| = 1$ and $h_6 = 0$. Then the entangling power vanishes, which implies that the unitary $R_4$ is not an entangler.

- **Class 5:**
  The YBO $R_5$ is again a function of $h_1, h_4$ and $h_6$, with its eigenvalues $\lambda_+ = h_1$ and $\lambda_- = -h_1 + h_6$, before and after enhancement. The entangling power becomes
  \[
e_p(R_5) = \frac{1}{9} |h_4 h_6|^2 + \frac{2}{3} |h_1 (h_1 - h_6)|^2 = \frac{1}{9} (\lambda_+ + \lambda_-) h_4^2 + \frac{2}{3} |\lambda_+ \lambda_-|^2,
  \]
  which contains $h_4$, a parameter that changes under the $SL(2, \mathbb{C})^{\otimes 2}$. $R_5$ becomes unitary when $|h_1| = |h_4| = 1$ and $h_6 = 0$, making it a special case of class 1. Then, the entangling power becomes the constant $\frac{2}{3}$.

- **Class 6:**
  The YBO $R_6$ has three parameters $h_1, h_2, h_8$, and its eigenvalues are given by $\lambda_\pm = \pm \left[ h_1 + h_8 \pm \sqrt{2(h_1^2 + h_8^2)} \right]$, before and after the enhancement. The entangling power becomes
  \[
e_p(R_6) = \frac{1}{9} [ h_2 h_8]^2 + \frac{1}{16} \left( \frac{h_1 (h_1 + h_8)^2}{h_2^2} \right)^2 + \frac{1}{4} (h_1 + h_8) \sqrt{h_1^2 + h_8^2}^2 + \frac{1}{24} |h_1 - h_8|^4,
  \]
  which is now dependent on $h_2$, a parameter that changes under the $SL(2, \mathbb{C})^{\otimes 2}$. $h_1$ and $h_8$ are expressed by the eigenvalues as $h_1 = \frac{\lambda_+ + \lambda_-}{2} \pm \sqrt{-\lambda_+ \lambda_-}$ and $h_8 = \frac{\lambda_+ + \lambda_-}{2} \mp \sqrt{-\lambda_+ \lambda_-}$. In this case $R_6$ cannot be unitary for any choice of the parameters.

- **Class 7:**
  The YBO $R_7$ is a function of $h_1, h_2$ and $h_3$, with its eigenvalues $\lambda_+ = h_1 + h_3$ and $\pm \lambda_- = \pm (h_1 - h_3)$, which is not changed by enhancement. The entangling power is computed to be
  \[
e_p(R_7) = \frac{1}{9} \left[ |h_1 h_2|^2 + \frac{|h_1 h_3|^2}{|h_2|^2} + 2|h_1 h_3|^2 \right],
  \]
  where $h_2$ changes under the local action of $SL(2, \mathbb{C})^{\otimes 2}$. $h_1$ and $h_3$ are expressed by the eigenvalues: $h_1 = \frac{1}{2} (\lambda_+ + \lambda_-)$ and $h_3 = \frac{1}{2} (\lambda_+ - \lambda_-)$. Note that the $SL(2, \mathbb{C})^{\otimes 2}$-invariant part of the second line in (6.16) vanishes in this case.

  $R_7$ is unitary when $h_1 = r_1 e^{i\phi_1}$, $h_2 = \sqrt{1 - r_1^2} e^{i\phi_2}$ and $h_3 = -i \sqrt{1 - r_1^2} e^{i\phi_3}$. Then (6.25) is dependent on just $r_1$ that is a local invariant from (6.18), and can be written in terms of the eigenvalues.
• Class 8:
  This time $R_8$ is a function of $h_1$ and $h_2$ with its eigenvalues $(1 \pm i)h_1$, which is preserved by enhancement. The entangling power becomes
  \[
  e_{\rho}(R_8) = \frac{1}{9} \left[ |h_1 h_2|^2 + \frac{|h_1|^2}{|h_2|^2} + 2|h_1|^2 \right],
  \]  
  where $h_2$ changes under the $SL(2, \mathbb{C})^{\otimes 2}$. Note that the $SL(2, \mathbb{C})^{\otimes 2}$-invariant part of the second line in (6.16) vanishes.
  $R_8$ is unitary for $h_1 = \frac{1}{\sqrt{2}} e^{i\varphi_1}$ and $h_2 = \frac{1}{\sqrt{2}} e^{i\varphi_2}$. Then the entangling power (6.26) becomes a constant $\frac{1}{3}$, which is a trivial function of the eigenvalues.

• Class 9:
  $R_9$ is a function of $h_1$ and $h_7$ with its eigenvalues $\pm h_1$, which is not changed by enhancement. The entangling power is computed to be
  \[
  e_{\rho}(R_9) = \frac{1}{9} |h_1 h_7|^2,
  \]  
  which is now dependent on $h_7$, a parameter that changes under the local action of $SL(2, \mathbb{C})^{\otimes 2}$. Again the second line in (6.16) vanishes.
  $R_9$ becomes unitary when $h_1 = e^{i\varphi_1}$ and $h_7 = 0$, making it a special case of class 1. Then the entangling power vanishes, implying that the unitary $R_9$ is not an entangler.

• Class 10:
  Again, $R_{10}$ is a function of just $h_1$ and $h_7$, with its eigenvalues $\pm h_1$, before and after enhancement. The entangling power is given by
  \[
  e_{\rho}(R_{10}) = \frac{1}{9} |h_1 h_7|^2 + \frac{2}{9} |h_1|^4,
  \]  
  where $h_7$ changes under the $SL(2, \mathbb{C})^{\otimes 2}$.
  $R_{10}$ is unitary for $h_1 = e^{i\varphi_1}$ and $h_7 = 0$, making it a special case of class 1. Then $e_{\rho}(R_{10}) = \frac{2}{3}$, a trivial function of the eigenvalues.

• Class 11:
  $R_{11}$ has free parameters $h_7$ and $h_8$, and its eigenvalue is $h_8$, which is not affected by enhancement. The entangling power is
  \[
  e_{\rho}(R_{11}) = \frac{1}{9} |h_7 h_8|^2 + \frac{10}{9} |h_8|^4,
  \]  
  where $h_7$ changes under the $SL(2, \mathbb{C})^{\otimes 2}$ in this case. $R_{11}$ cannot be unitary.

• Class 12:
  $R_{12}$ is a function of $h_1$ and $h_2$, with its eigenvalue $\frac{1}{\sqrt{2}} h_1$, before and after enhancement. The entangling power is
  \[
  e_{\rho}(R_{12}) = \frac{1}{9} \left[ |h_1 h_2|^2 + \frac{|h_1|^6}{4|h_2|^2} \right] + \frac{1}{6} |h_1|^4,
  \]  
  where $h_2$ changes under the $SL(2, \mathbb{C})^{\otimes 2}$. $R_{12}$ cannot be unitary.

The Bell matrix (3.1) with $h_1 = h_2 = h_3 = h_4 = h_6 = h_8 = \frac{1}{\sqrt{2}}$ and $h_5 = h_7 = -\frac{1}{\sqrt{2}}$ gives the entangling power $\frac{5}{8}$ that is not the largest in a two-qubit system. For the unitary case, classes 1, 2, 3, 5 and 10 can give the maximum value $\frac{1}{3}$. The Bell matrix gives the largest entanglement when it acts to $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$. But, it does not when it acts to a general product state.
7. Outlook

Quantum gates realized using braid operators are expected to create a robust entangled state from a product state. The entangled states thus obtained depend on parameters forming local invariants and are insensitive to local perturbations. Such parameters should characterize non-local properties of quantum entanglement. This criterion can be used to exclude braid operators that do not possess this property. To achieve this, it is essential to identify the complete set of parameters of local invariants for a braiding quantum gate that would determine the quantum entanglement of these systems. For the twelve classes of the X-type two-qubit braid operators considered in this paper, we found that the complete set is fixed by the independent eigenvalues of these operators. This is in marked contrast with the case of a generic two-qubit operator, whose eigenvalues alone are not sufficient to determine the entanglement measures of the system.

One of possible future directions would be to analyze robustness of entanglement [37] for braiding quantum gates and to understand how topological properties coming from the braid contribute to the robustness of the quantum entanglement. In addition, it would be crucial to check these features for multi-qubit braid operators that can be constructed using the generalized Yang–Baxter equation [38, 39] for which several solutions have been found [40–44].

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Data availability statement

No new data were created or analysed in this study.

Appendix A. Relation to the classification by Hietarinta

This rather technical appendix is devoted to a comparison between our results and the ones obtained by Hietarinta in [24].

A.1 Classification by Hietarinta

We start by summarizing Hietarinta’s classification. In [24], all solutions to the constant algebraic Yang–Baxter equation:

\[
R_{j_1, j_2, k_1, k_2} R_{j_3, j_4, k_3, k_4} R_{j_5, j_6, k_5, k_6} = R_{j_5, j_6, k_5, k_6} R_{j_1, j_2, k_1, k_2} R_{j_3, j_4, k_3, k_4} R_{j_5, j_6, k_5, k_6}
\]  

(A.1)
are presented. All the indices of $R$ take value 0 or 1. Here we represent $R$ in $4 \times 4$-matrix form as
\[
R = \begin{pmatrix}
R_{0,0,0} & R_{0,0,1} & R_{0,1,0} & R_{0,1,1} \\
R_{0,1,0} & R_{0,1,1} & R_{1,0,0} & R_{1,0,1} \\
R_{1,0,0} & R_{1,0,1} & R_{1,1,0} & R_{1,1,1} \\
R_{1,1,0} & R_{1,1,1} & R_{1,1,1} & R_{1,1,1}
\end{pmatrix}.
\] (A.2)

Via the replacement $R \rightarrow PR$ with $P$ being the permutation matrix
\[
P = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\] (A.3)

(A.1) is transcribed as the braided Yang–Baxter equation:
\[
R_{i_1 j_1 k_1, l_1} R_{i_2 j_2 k_2, l_2} = R_{i_1 j_2 k_2, l_1} R_{i_2 j_1 k_1, l_2} R_{i_3 j_2 k_2, l_1} R_{i_3 j_1 k_1, l_2}
\] (A.4)

that is identical to (4.1).

Relevant results in [24] are summarized for solutions to (A.4) as follows. The continuous transformations
\[
R \rightarrow \kappa(Q \otimes Q)R(Q \otimes Q)^{-1},
\] (A.5)

with $\kappa$ a complex factor and $Q$ an invertible $2 \times 2$ matrix, map a solution to a solution. Each of the following discrete transformations
\[
R_{i,j,k,l} \rightarrow R_{\bar{i},\bar{j},\bar{k},\bar{l}},
\] (A.6)
\[
R_{i,j,k,l} \rightarrow R_{\bar{j},\bar{i},\bar{k},\bar{l}},
\] (A.7)
\[
R_{i,j,k,l} \rightarrow R_{j,i,k,l}
\] (A.8)

also maps a solution to a solution, where (A.6) means the matrix transpose taken in (A.2), and $\bar{i}$ is the negation of $i$, i.e., $\bar{0} \equiv 1$ and $\bar{1} \equiv 0$ in (A.7). Up to the transformations (A.5)–(A.8), all the invertible solutions to (A.4), except the trivial solution $R \propto 1$, are classified by the ten matrices:
\[
R_{\text{H3,1}} = \begin{pmatrix}
k & 0 & 0 & 0 \\
0 & 0 & p & 0 \\
0 & q & 0 & 0 \\
0 & 0 & s & 0
\end{pmatrix},
\]
\[
R_{\text{H2,1}} = \begin{pmatrix}
k^2 & 0 & 0 & 0 \\
0 & k^2 - pq & kp & 0 \\
0 & kq & 0 & 0 \\
0 & 0 & 0 & k^2
\end{pmatrix},
\]
\[
R_{\text{H2,2}} = \begin{pmatrix}
k^2 & 0 & 0 & 0 \\
0 & k^2 - pq & kp & 0 \\
0 & kq & 0 & 0 \\
0 & 0 & 0 & -pq
\end{pmatrix},
\]

\[15\text{Note that, to identify the matrix (A.2) with the expression in [24] [see equation (4) there], the indices 0 and 1 here should be identified with 1 and 2 in [24], respectively. Pairs of indices 01 and 10 are swapped in [24].}\]
\[ \mathbf{R}_{H2,3} = \begin{pmatrix} k & p & q & s \\ 0 & 0 & k & p \\ 0 & k & 0 & q \\ 0 & 0 & 0 & k \end{pmatrix}, \]

\[ \mathbf{R}_{H1,1} = \begin{pmatrix} p^2 + 2pq - q^2 & 0 & 0 & p^2 - q^2 \\ 0 & p^2 - q^2 & p^2 + q^2 & 0 \\ 0 & p^2 + q^2 & p^2 - q^2 & 0 \\ 0 & 0 & 0 & 2pq - q^2 \end{pmatrix}, \]

\[ \mathbf{R}_{H1,2} = \begin{pmatrix} p & 0 & 0 & k \\ 0 & p - q & p & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 0 & -q \end{pmatrix}, \quad \mathbf{R}_{H1,3} = \begin{pmatrix} k^2 & -kp & kp & pq \\ 0 & 0 & k^2 & kq \\ 0 & k^2 & 0 & -kq \\ 0 & 0 & 0 & k^2 \end{pmatrix}, \]

\[ \mathbf{R}_{H1,4} = \begin{pmatrix} 0 & 0 & 0 & p \\ 0 & k & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & q & 0 & 0 \end{pmatrix}, \]

\[ \mathbf{R}_{H0,1} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{R}_{H0,2} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}. \quad (A.9) \]

For X-type solutions that we consider in the text, the classification is valid with removing \( \mathbf{R}_{H1,3} \) and setting \( p = q = 0 \) in \( \mathbf{R}_{H2,3} \):

\[ \mathbf{R}'_{H2,3} = \begin{pmatrix} k & 0 & 0 & s \\ 0 & 0 & k & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & 0 & k \end{pmatrix}. \quad (A.10) \]

### A.2. Our solutions

Our solutions in classes 1–12 presented in section 4 are classified by eigenvalues and quadratic invariants. Here we classify all the nontrivial solutions in section 4 to the nine families (\( \mathbf{R}_{H3,1}, \mathbf{R}_{H2,1}, \mathbf{R}_{H2,2}, \mathbf{R}'_{H2,3}, \mathbf{R}_{H1,1}, \mathbf{R}_{H1,2}, \mathbf{R}_{H1,3}, \mathbf{R}_{H0,1}, \mathbf{R}_{H0,2} \)).

- **Class 1**: \( \mathbf{R}_1 = \begin{pmatrix} h_1 & 0 & 0 & 0 \\ 0 & 0 & h_4 & 0 \\ 0 & h_5 & 0 & 0 \\ 0 & 0 & 0 & h_6 \end{pmatrix} \)

  This falls into \( \mathbf{R}_{H3,1} \) with \( k = h_1, p = h_4, q = h_5 \) and \( s = h_6 \).

- **Class 2**: \( \mathbf{R}_2 = \begin{pmatrix} 0 & 0 & 0 & h_2 \\ 0 & h_3 & 0 & 0 \\ 0 & 0 & h_3 & 0 \\ h_7 & 0 & 0 & 0 \end{pmatrix} \)

  This falls into \( \mathbf{R}_{H1,4} \) with \( k = h_3, p = h_2 \) and \( q = h_7 \).
Class 3: \( R_3 = \begin{pmatrix}
  h_1 & 0 & 0 & 0 \\
  0 & 0 & -h_1 & 0 \\
  0 & h_8 & h_1 + h_8 & 0 \\
  h_7 & 0 & 0 & h_8
\end{pmatrix} \)

This falls into \( RH_{1,2} \) with \( k = h_7, p = h_8 \) and \( q = -h_1 \) by the transformation (A.7).

The seven other solutions given in the text are equivalent to the representative as

\[ * R_{3-1} = \begin{pmatrix}
  h_1 & 0 & 0 & 0 \\
  0 & 0 & h_1 & 0 \\
  0 & -h_8 & h_1 + h_8 & 0 \\
  h_7 & 0 & 0 & h_8
\end{pmatrix} \] becomes \( R_3 \) after the transformations (A.7), (A.8) and (A.6) with the redefinition \( h_1 \leftrightarrow h_8 \).

\[ * R_{3-2} = \begin{pmatrix}
  h_1 & 0 & 0 & 0 \\
  0 & 0 & -h_8 & 0 \\
  0 & h_1 & h_1 + h_8 & 0 \\
  0 & 0 & 0 & h_8
\end{pmatrix} \] becomes \( R_3 \) after the transformations (A.7) and (A.8) with the redefinition \( h_1 \leftrightarrow h_8 \) and \( h_2 \rightarrow h_7 \).

\[ * R_{3-3} = \begin{pmatrix}
  h_1 & 0 & 0 & h_2 \\
  0 & 0 & h_6 & 0 \\
  0 & -h_1 & h_1 + h_8 & 0 \\
  0 & 0 & 0 & h_8
\end{pmatrix} \] becomes \( R_3 \) after (A.6) with \( h_2 \rightarrow h_7 \).

\[ * R_{3-4} = \begin{pmatrix}
  h_1 & 0 & 0 & h_2 \\
  0 & 0 & h_6 & 0 \\
  0 & h_1 + h_8 & -h_1 & 0 \\
  0 & 0 & h_8 & h_1 + h_8
\end{pmatrix} \] becomes \( R_{3-3} \) after (A.7).

\[ * R_{3-5} = \begin{pmatrix}
  h_1 & 0 & 0 & h_2 \\
  0 & h_1 + h_8 & h_1 & 0 \\
  0 & -h_8 & 0 & 0 \\
  0 & 0 & 0 & h_8
\end{pmatrix} \] becomes \( R_{3-1} \) after (A.6) and (A.8) with \( h_2 \rightarrow h_7 \).

\[ * R_{3-6} = \begin{pmatrix}
  h_1 & 0 & 0 & 0 \\
  0 & h_1 + h_8 & h_1 & 0 \\
  0 & 0 & 0 & h_8 \\
  h_7 & 0 & 0 & h_8
\end{pmatrix} \] becomes \( R_{3-1} \) after (A.8).

\[ * R_{3-7} = \begin{pmatrix}
  h_1 & 0 & 0 & 0 \\
  0 & h_1 + h_8 & h_8 & 0 \\
  0 & 0 & 0 & h_8 \\
  h_7 & 0 & 0 & h_8
\end{pmatrix} \] becomes \( R_3 \) after (A.8).

Class 4: \( R_4 = \begin{pmatrix}
  h_1 & 0 & 0 & 0 \\
  0 & 0 & h_4 & 0 \\
  0 & h_4 (h_1 - h_6) & h_6 & 0 \\
  0 & 0 & 0 & h_1
\end{pmatrix} \)
This falls into \( R_{1/2.1} \) with \( k = \sqrt{h_1}, \ p = \sqrt{h_1}(h_1 - h_3) \) and \( q = \frac{h_3}{\sqrt{h_1}} \) by the transformation (A.8).

The other solution \( R_{5.1} = \left( \begin{array}{cccc}
     h_1 & 0 & 0 & 0 \\
     0 & h_3 & h_4 & 0 \\
     0 & h_4(h_1 - h_5) & 0 & 0 \\
     0 & 0 & 0 & h_1
   \end{array} \right) \) becomes \( R_4 \) after (A.6) and (A.8) with \( h_3 \rightarrow h_6 \).

- **Class 5:** \( R_5 = \left( \begin{array}{cccc}
     h_1 & 0 & 0 & 0 \\
     0 & h_3 & h_4 & 0 \\
     0 & \frac{h_4}{h_3}(h_1 - h_6) & h_6 & 0 \\
     0 & 0 & 0 & -h_1 + h_6
   \end{array} \right) \)

  This falls into \( R_{1/2.2} \) with \( k = \sqrt{h_1}, \ p = \sqrt{h_1}(h_1 - h_5) \) and \( q = \frac{h_5}{\sqrt{h_1}} \) by the transformation (A.8).

  The other solution \( R_{5.2} = \left( \begin{array}{cccc}
     h_1 & 0 & 0 & 0 \\
     0 & h_3 & h_4 & 0 \\
     0 & h_4(h_1 - h_6) & 0 & 0 \\
     0 & 0 & 0 & -h_1 + h_3
   \end{array} \right) \) becomes \( R_5 \) after (A.6) and (A.8) with \( h_1 \rightarrow h_6 \).

- **Class 6:** \( R_6 = \left( \begin{array}{cccc}
     h_1 & 0 & 0 & 0 \\
     0 & \frac{h_1 + h_8}{2} & -\sqrt{\frac{h_1^2 + h_8^2}{2}} & h_2 \\
     0 & -\sqrt{\frac{h_1^2 + h_8^2}{2}} & \frac{h_1 + h_8}{2} & 0 \\
     \frac{(h_1 + h_8)^2}{4h_2} & 0 & 0 & h_8
   \end{array} \right) \)

  This falls into \( R_{1/1.1} \) with \( p = \frac{h_1 + h_8}{2} \) and \( q = -\frac{1}{2} \left[ h_1 + h_8 + \sqrt{2(h_1^2 + h_8^2)} \right] = -\lambda_+ \) by the transformation (A.5). It can be seen that \( s(\mathcal{Q} \otimes \mathcal{Q})R_{1/1.1}(\mathcal{Q} \otimes \mathcal{Q})^{-1} = R_6 \) with \( \kappa = -\frac{1}{2\lambda_+} \) and \( \mathcal{Q} = \left( \begin{array}{c}
     \sqrt{2h_2} \\
     0
   \end{array} \right) \sqrt{h_1 + h_8} \).

  The other solution \( R_{6.1} = \left( \begin{array}{cccc}
     h_1 & 0 & 0 & 0 \\
     0 & \frac{h_1 + h_8}{2} & \sqrt{\frac{h_1^2 + h_8^2}{2}} & h_2 \\
     0 & \sqrt{\frac{h_1^2 + h_8^2}{2}} & \frac{h_1 + h_8}{2} & 0 \\
     \frac{(h_1 + h_8)^2}{4h_2} & 0 & 0 & h_8
   \end{array} \right) \) becomes \(-R_6 \) with the redefinition \( h_a \rightarrow -h_a \) (\( a = 1, 2, 8 \)).

- **Class 7:** \( R_7 = \left( \begin{array}{ccc}
     h_1 & 0 & 0 \\
     0 & h_3 & -h_1 \\
     \frac{h_3^2}{h_2} & 0 & h_1
   \end{array} \right) \)
This falls into $R_{H1,4}$ with $k = h_1 + h_3$ and $p = q = h_1 - h_3$ by the transformation (A.5):

$$\kappa (Q \otimes \bar{Q}) R_{H1,4} (Q \otimes \bar{Q})^{-1} = R_7 \text{ with } \kappa = 1 \text{ and } Q = \begin{pmatrix} i \sqrt{h_2} & -i \sqrt{h_2} \\ \sqrt{h_3} & \sqrt{h_3} \end{pmatrix}.$$  

The other solution $R_{7-1} = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_3 & h_1 \\ 0 & h_1 & h_3 \end{pmatrix}$ is not equivalent to $R_7$. Actually, we can see that $R_{7-1}$ falls into $R_{H3,1}$ with $p = q = h_1 - h_3$ and $k = s = h_1 + h_3$ by $\kappa (Q \otimes \bar{Q}) R_{H3,1} (Q \otimes \bar{Q})^{-1} = R_{7-1}$ with $\kappa = 1$ and $Q = \begin{pmatrix} \sqrt{h_2} & 0 \\ \sqrt{h_3} & \sqrt{h_3} \end{pmatrix}$. However, $R_7$ and $R_{7-1}$ belong to the same class in our classification, since they have the same eigenvalues and quadratic invariants. We explicitly see that they are $SL(2, \mathbb{C})$-equivalent: $(Q_1 \otimes Q_2) R_7 (Q_1 \otimes Q_2)^{-1} = R_{7-1}$ with

$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}. \quad (A.11)$$

• Class 8: $R_8 = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_1 & h_3 \\ h_3^2 & h_2 & 0 \end{pmatrix}$

This falls into $R_{H0,2}$ by the transformation (A.5): $\kappa (Q \otimes \bar{Q}) R_{H0,2} (Q \otimes \bar{Q})^{-1} = R_8$ with $\kappa = h_1$ and $Q = \begin{pmatrix} 0 & i \sqrt{h_2} \\ \sqrt{h_1} & 0 \end{pmatrix}$.

The other solution $R_{8-1} = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_1 & h_3 \\ -h_1 & h_1 & 0 \end{pmatrix}$ becomes $R_8$ by (A.8).

• Class 9: $R_9 = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & 0 & -h_3 \\ 0 & -h_3 & 0 \end{pmatrix}$

This falls into $R_{H0,1}$ by the successive transformations (A.5) and (A.6): $\kappa (Q \otimes \bar{Q}) R_{H0,1} (Q \otimes \bar{Q})^{-1}$ with $\kappa = h_1$ and $Q = \begin{pmatrix} \sqrt{h_1} & 0 \\ 0 & \sqrt{h_1} \end{pmatrix}$ followed by (A.6) gives $R_9$.

Among the other three solutions

$$R_{9-1} = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & 0 & -h_3 \\ 0 & -h_3 & 0 \end{pmatrix},$$
\( R_{9-2} = \begin{pmatrix} h_1 & 0 & 0 & 0 \\ 0 & 0 & h_1 & 0 \\ 0 & h_1 & 0 & 0 \\ h_7 & 0 & 0 & h_1 \end{pmatrix} \),  \\
\( R_{9-3} = \begin{pmatrix} h_1 & 0 & 0 & h_2 \\ 0 & 0 & h_1 & 0 \\ 0 & h_1 & 0 & 0 \\ 0 & 0 & 0 & h_1 \end{pmatrix} \).

\( R_{9-1} \) becomes \( R_9 \) by (A.6) with \( h_2 \to h_7 \), whereas \( R_{9-2} \) and \( R_{9-3} \) are not equivalent to the representative \( R_9 \). Actually, \( R_{9-2} \) falls into \( R'_{H2,3} \) with \( k = h_1 \) and \( s = h_7 \) by the transformation (A.6), and \( R_{9-3} \) becomes \( R_{9-2} \) by (A.6) with \( h_2 \to h_7 \). However, these two groups are \( SL(2, \mathbb{C}) \otimes \mathbb{C}^2 \) equivalent: \((Q_1 \otimes Q_2)R_9(Q_1 \otimes Q_2)^{-1} = R_{9-2} \) with (A.11).

**Class 10:** \( R_{10} = \begin{pmatrix} h_1 & 0 & 0 & 0 \\ 0 & 0 & -h_1 & 0 \\ 0 & -h_1 & 0 & 0 \\ h_7 & 0 & 0 & -h_1 \end{pmatrix} \)

This falls into \( R_{H1,2} \) with \( k = h_7 \) and \( p = q = -h_1 \) by the transformation (A.7).

The other three solutions are equivalent to the representative \( R_{10} \) as

- \( R_{10-1} = \begin{pmatrix} h_1 & 0 & 0 & h_2 \\ 0 & 0 & -h_1 & 0 \\ 0 & -h_1 & 0 & 0 \\ h_7 & 0 & 0 & -h_1 \end{pmatrix} \) becomes \( R_{10} \) by (A.6) with \( h_2 \to h_7 \).
- \( R_{10-2} = \begin{pmatrix} h_1 & 0 & 0 & 0 \\ 0 & 0 & h_1 & 0 \\ 0 & h_1 & 0 & 0 \\ h_7 & 0 & 0 & -h_1 \end{pmatrix} \) becomes \( R_{10} \) by (A.7) and (A.6) with \( h_1 \to -h_1 \).
- \( R_{10-3} = \begin{pmatrix} h_1 & 0 & 0 & h_2 \\ 0 & 0 & h_1 & 0 \\ 0 & h_1 & 0 & 0 \\ 0 & 0 & 0 & -h_1 \end{pmatrix} \) becomes \( R_{10-2} \) by (A.6) with \( h_2 \to h_7 \).

**Class 11:** \( R_{11} = \begin{pmatrix} h_8 & 0 & 0 & 0 \\ 0 & 2h_8 & -h_8 & 0 \\ 0 & h_8 & 0 & 0 \\ h_7 & 0 & 0 & h_8 \end{pmatrix} \)

This falls into \( R_{H1,2} \) with \( k = h_7, p = h_8 \) and \( q = -h_8 \) by the transformation (A.6). The other seven solutions are equivalent to the representative \( R_{11} \) as

- \( R_{11-1} = \begin{pmatrix} h_8 & 0 & 0 & h_2 \\ 0 & 2h_8 & -h_8 & 0 \\ 0 & h_8 & 0 & 0 \\ 0 & 0 & 0 & h_8 \end{pmatrix} \) becomes \( R_{11} \) by (A.7) and (A.8) with \( h_2 \to h_7 \).
- \( R_{11-2} = \begin{pmatrix} h_8 & 0 & 0 & 0 \\ 0 & 2h_8 & h_8 & 0 \\ 0 & -h_8 & 0 & 0 \\ h_7 & 0 & 0 & h_8 \end{pmatrix} \) becomes \( R_{11-1} \) by (A.6) with \( h_7 \to h_2 \).
\[ R_{11-3} = \begin{pmatrix} h_8 & 0 & 0 & h_2 \\ 0 & 2h_8 & h_8 & 0 \\ 0 & -h_8 & 0 & 0 \\ 0 & 0 & 0 & h_8 \end{pmatrix} \]
becomes \( R_{11} \) by \((A.6)\) with \( h_2 \to h_7 \).

\[ R_{11-4} = \begin{pmatrix} h_1 & 0 & 0 & 0 \\ 0 & 0 & -h_1 & 0 \\ 0 & h_1 & 2h_1 & 0 \\ h_7 & 0 & 0 & h_1 \end{pmatrix} \]
becomes \( R_{11-2} \) by \((A.8)\) with \( h_1 \to h_8 \).

\[ R_{11-5} = \begin{pmatrix} h_1 & 0 & 0 & h_2 \\ 0 & 0 & -h_1 & 0 \\ 0 & h_1 & 2h_1 & 0 \\ 0 & 0 & 0 & h_1 \end{pmatrix} \]
becomes \( R_{11-3} \) by \((A.8)\) with \( h_1 \to h_8 \).

\[ R_{11-6} = \begin{pmatrix} h_1 & 0 & 0 & 0 \\ 0 & 0 & h_1 & 0 \\ 0 & h_1 & 2h_1 & 0 \\ h_7 & 0 & 0 & h_1 \end{pmatrix} \]
becomes \( R_{11} \) by \((A.8)\) with \( h_1 \to h_8 \).

\[ R_{11-7} = \begin{pmatrix} h_1 & 0 & 0 & h_2 \\ 0 & 0 & h_1 & 0 \\ 0 & -h_1 & 2h_1 & 0 \\ 0 & 0 & 0 & h_1 \end{pmatrix} \]
becomes \( R_{11-1} \) by \((A.8)\) with \( h_1 \to h_8 \).

\[ \begin{align*} \text{Class 12: } R_{12} &= \begin{pmatrix} h_1 & 0 & 0 & h_2 \\ 0 & \frac{1 - i}{\sqrt{2}}h_1 & 0 & 0 \\ 0 & 0 & 1 - i \sqrt{2}h_1 & 0 \\ -i \sqrt{2}h_1 & 0 & (4,3) & (4,4) \end{pmatrix} \\ \end{align*} \]

This falls into \( R_{\text{H1}} \) with \( p = 1 \) and \( q = i \) by the transformation \((A.5)\): \( \kappa(Q \otimes Q)R_{\text{H1}}(Q \otimes Q)^{-1} = R \) with \( \kappa = \frac{h_1}{2(1 + i)} \) and \( Q = \begin{pmatrix} \sqrt{(1 + i)h_2} & 0 \\ 0 & -\sqrt{h_1} \end{pmatrix} \).

The other solution \( R_{12-1} = \begin{pmatrix} h_1 & 0 & 0 & h_2 \\ 0 & 1 + i \sqrt{2}h_1 & 0 & 0 \\ 0 & 0 & \frac{1 + i}{2}h_1 & 0 \\ i \sqrt{2}h_1 & 0 & 0 & i h_1 \end{pmatrix} \) becomes \( R_{12} \) by \((A.6)\) and \((A.7)\) with \( h_1 \to -ih_1 \).

### Appendix B. Local invariants, link polynomials and entangling power of \( R_{\text{H1},3} \) and \( R_{\text{H2},3} \)

Among all the two-qubit braid operators in \((A.9)\), the X-type braid operators analyzed in this paper do not fully capture solutions of the form \( R_{\text{H1},3} \) and \( R_{\text{H2},3} \). For completeness we analyze those cases here.
The enhancement discussed in section 5 is possible with \( \mu = I - \frac{\mu_{R} + \mu_{H}}{2} \). The entangling power is obtained as \( e_p(R_{H13}) = \frac{1}{4} |k| \). The unitary solutions are found at \( p = q = s = 0 \) and \( |k| = 1 \), in which case the operator is no longer an entangler.

**References**

[1] Bell J S 1964 On the Einstein–Podolsky–Rosen paradox *Physics* 1 195

[2] Schrödinger E 1983 *The Present Situation in Quantum Mechanics: a Translation of Schrödinger’s Cat Paradox Paper, Quantum Theory and Measurement* ed J A Wheeler and W H Zurek (Princeton, NJ: Princeton University Press) p 152

[3] Lo H K, Popescu S and Spiller T 1998 *Introduction to Quantum Computing and Information* (Singapore: World Scientific)

[4] Nielsen M A and Chuang I L 2010 *Quantum Computation and Quantum Information: 10th Anniversary Edition* (Cambridge: Cambridge University Press)

[5] Weyl H 1939 *The Classical Groups* (Princeton: Princeton University Press)

[6] Springer T A 1977 *Invariant Theory* (Lecture Notes in Mathematics vol 585) (Berlin: Springer)

[7] Linden N, Popescu S and Rudbery A 1999 Non-local properties of multi-particle density matrices *Phys. Rev. Lett.* 83 243–7

[8] Grassl M, Rötteler M and Beth T 1998 Computing local invariants of quantum-bit systems *Phys. Rev. A* 58 1833–9

[9] Carteret H A and Rudbery A 2000 Local symmetry properties of pure three-qubit states *J. Phys. A: Math. Gen.* 33 4981–5002

[10] Sudbery A 2001 On local invariants of pure three-qubit states *J. Phys. A: Math. Gen.* 34 643–52

[11] Luoque I-C and Thibon J-Y 2003 The polynomial invariants of four qubits *Phys. Rev. A* 67 042303

[12] Walter M, Gross D and Eisert J 2016 Multi-partite entanglement *Lectures on quantum information* (D. Bruss and G. Leuchs Eds. (Wiley-VCH, Weinheim, 2006) arXiv: 1612.02437 [quant-ph])

[13] Makhlin Y 2002 Nonlocal properties of two-qubit gates and mixed states and optimization of quantum computations *Quantum Inf. Process.* 1 243–52
[15] Aravind P K 1997 Borromean entanglement of the GHZ state *Potentiality, Entanglement and Passion-at-a-Distance* (Boston Studies in the Philosophy of Science vol 194) ed R S Cohen, M Horne and J Stachel (Berlin: Springer)

[16] Kauffman L H and Lomonaco S J Jr 2004 Braiding operators are universal quantum gates *New J. Phys.* **6** 134

[17] Zhang Y, Kauffman L H and Ge M-L 2005 Universal quantum gate, Yang–Baxterization and Hamiltonian *Int. J. Quantum Inf.* **3** 669–78

[18] Hietarinta J 1992 All solutions to the constant quantum Yang–Baxter equation in two-dimensions *Phys. Lett. A* **165** 245

[19] Vidal G and Tarrach R 1999 Robustness of entanglement *Phys. Rev. A* **59** 141–55

[20] Franko J M, Rowell E C and Wang Z 2006 Extraspecial 2-groups and braiding quantum gates *Quantum Inf. Comput.* **10** 685–702