Two-dimensional Riemannian and Lorentzian geometries from second order ODEs

Emanuel Gallo *

FaMAF, Universidad Nacional de Córdoba, 5000 Córdoba, Argentina

Abstract

In this note we give an alternative geometrical derivation of the results recently presented by García-Godínez, Newman and Silva-Ortigoza in [1] on the class of all two-dimensional riemannian and lorentzian metrics from 2nd order ODEs which are in duality with the two dimensional Hamilton-Jacobi equation. We show that, as it happens in the Null Surfaces Formulation of General Relativity, the Wünschmann-like condition can be obtained as a requirement of a vanishing torsion tensor. Furthermore, from these second order ODEs we obtain the associated Cartan connections.

*egallo@fis.uncor.edu
1 Introduction

In a couple of recent works, García-Godínez, Newman and Silva-Ortigoza (GNS) presented the (pseudo)-riemannian geometries which are hidden in certain class of differential equations (those which satisfy a Wünschmann-like condition, \( I_{GNS} = 0 \)). In the first of these works \[1\], they studied how to obtain all two-dimensional riemannian and lorentzian metrics from certain class of 2nd order ODEs. Furthermore, in \[2\], they extended their work and showed how to get all three-dimensional metrics from certain class of three second-order PDEs and also from a certain class of third order ODEs. From now on, we will say that these equations are in the GNS-class.

The special status of these ODEs and PDEs is that they are in duality with the Hamilton-Jacobi equation. For example, if we have a 2nd order ODE in the GNS-class,

\[
 u'' = \Lambda(u, u', s),
\]

(1)

and if we know a solution \( u = Z(x^a, s) \), with \( x^a = (x^1, x^2) \) integration constants, then this solution automatically satisfies the two-dimensional Hamilton-Jacobi equation:

\[
 g^{ab} \nabla_a Z \nabla_b Z = 1,
\]

(2)

where \( \nabla_a \) means a differentiation with respect to \( x^a \), and \( g^{ab} \) is a (pseudo)-riemannian metric constructed from \( \Lambda \) and its derivatives.

All these problems share similar characteristics with the problem of the Null Surface Formulation (NSF) of General Relativity in three and four dimension \([3]-[6]\). In NSF, from certain class of differential equations, known as the Wünschmann class, one can construct all three and four dimensional conformal lorentzian metrics. The 3-dim conformal metrics are obtained from a class of third order ODE,

\[
 u''' = F(u, u', u'', s),
\]

(3)

with \( F \) satisfying the so called Wünschmann condition \( I[F] = 0 \). Likewise, the 4-dim metrics are obtained from a pair of second order of PDEs,

\[
 u_{ss} = \Lambda(s, s^*, u, u_s, u_{s*}, u_{ss*}),
\]

\[
 u_{s^*s^*} = \Lambda^*(s, s^*, u, u_s, u_{s*}, u_{ss*}).
\]

(4)

where \( s \) and \( s^* \) are complex variables, and \( \Lambda \) and \( \Lambda^* \) satisfy the generalized Wünschmann condition \( W[\Lambda, \Lambda^*] = 0 \) and its complex conjugate. It can
be shown that $I[F]$, and $W[Λ, Λ^*]$, are invariant under contact transformations [9, 10, 11]. Again, these equations are in duality with another equation, namely the eikonal equation,

$$g^{ab}∇_a Z ∇_b Z = 0. \quad (5)$$

and the level surfaces of the solution $u = Z(x^a, s)$ to (3), or $u = Z(x^a, s, s^*)$ to (4) are null surfaces of the respective metrics that they generate.

In NSF, the Wünschmann condition can be obtained in several ways [3, 7, 8, 13]. Two of these were used by GNS to obtain the Wünschmann-like condition for differential equations in duality with the Hamilton-Jacobi equation. There exist a third method which is used in NSF, the torsion-free method, and from which one can get not only the Wünschmann class and its respective metrics, but also more geometrical structures associated to the equations, in particular all normal Cartan conformal connections [12, 13]. In this note we show that this method can also be applied to the problem of (pseudo)-riemannian metrics discussed by GNS. In particular, we show how the torsion free condition restrict the class of second order ODEs to those belong to the GNS-class and such that we get all two-dimensional riemannian and lorentzian metrics, and respective Cartan connections.

In section 2 we briefly present the notation and basic concepts about the geometry of 2nd order ODEs. In section 3, we show how to get the GNS-class from the torsion free condition, and construct the associated Cartan connections. Finally, in the conclusions, we discuss the extension of this approach to the problem of the GNS class of third order ODE’s.

## 2 Notation and basics notions

Let the second order ODE be

$$u'' = Λ(u, u', s) \quad (6)$$

where $s ∈ \mathbb{R}$ is the independent variable, and the primes denote derivative of the dependent variable $u$ with respect to $s$.

On the jet-space $J^1$ with local coordinates $(s, u, u')$ we consider the Pfaffian system $P$

$$\begin{align*}
ω^1 &= du - u' ds, \\
ω^2 &= du' - Λ ds.
\end{align*} \quad (7) \quad (8)$$

2
Local solutions of (6) are in one to one correspondence with integral curves 
\( \gamma : \mathbb{R} \to J^1 \) of \( \mathcal{P} \) satisfying \( \gamma^* ds \neq 0 \). These curves are generated by the vector field on \( J^1 \) given by
\[
e_s \equiv D = \frac{\partial}{\partial s} + u' \frac{\partial}{\partial u} + \Lambda \frac{\partial}{\partial u'}.
\] (9)

We will restrict the domain of \( \Lambda \) to a open neighborhood \( U \) of \( J^1 \) where \( \Lambda \) is \( C^\infty \) and the Cauchy problem is well posed. Then, it follows from Frobenius theorem that the solution space \( M \) is a two-dimensional \( C^\infty \) manifold, and we will denote a given local coordinates system on it by \( x^a = (x^1, x^2) \). It means that we can construct a map \( Z : M \times \mathbb{R} \to \mathbb{R}, \ u = Z(x^a, s) \), such that by a given \( x^a_0 \in M \) the map \( u = Z(x^a_0, s) \) is a solution of (6).

Then, if on \( M \times \mathbb{R} \) we define a pfaffian system \( \mathcal{S} \) generated by
\[
\beta^1 = Z_a dx^a,
\]
\[
\beta^2 = Z'_a dx^a,
\]

(10)

where primes means derivatives on \( s \), and \( Z_a = \partial_a Z \), it follows that there exist a diffeomorphism \( \zeta : J^1 \to M \times \mathbb{R} \) which pulls back the pfaffian system \( \mathcal{S} \) on the system \( \mathcal{P} \), i.e.
\[
\zeta^* \mathcal{S} = \mathcal{P}.
\]

We will make use of this diffeomorphism later.

### 3 Riemannian and lorentzian geometries from second order ODEs

From \( \omega^1, \omega^2 \) which generate the pfaffian system \( \mathcal{P} \), we construct the following one-forms:
\[
\theta^1 = \frac{1}{\sqrt{2}} (\omega^1 + a \omega^2),
\] (11)
\[
\theta^2 = \frac{1}{\sqrt{2}} (\omega^1 - a \omega^2),
\] (12)

where \( a = a(s, u, u') \) is a non-vanishing function to be determined. Next, we construct a degenerate metric on \( J^1 \),
\[
h(u, u', s) = 2\theta^1 \otimes \theta^2 = \eta_{ij} \theta^i \otimes \theta^j,
\] (13)
where

\[ \eta_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

Note that if \( a^2 > 0 \), then \( \theta^1 \) and \( \theta^2 \) behave as null real vectors, and if \( a^2 < 0 \), they are complex null vectors.

Let be \( \omega^i_j \) a connection such that:

A. The connection is skew-symmetric

\[ \omega_{ij} = \omega_{[ij]}, \quad (14) \]

where \( \omega_{ij} = \eta_{kj} \omega^{k \cdot j} \).

B. The one-forms \( \theta^1 \) and \( \theta^2 \) satisfy the Cartan’s torsion-free first structure equations

\[ T^i \equiv d\theta^i + \omega^i_j \wedge \theta^j = 0. \quad (15) \]

Now, we state and prove the following theorem:

**Theorem:** The Torsion-free condition on the skew-symmetric connection:

1. Uniquely determines the connection, with the only non vanishing component given by

\[ \omega_{[12]} = -\frac{1}{\sqrt{2}}(\ln a)_u \theta^1 + \frac{1}{\sqrt{2}}(\ln a)_u \theta^2 + \frac{1}{a} ds. \quad (16) \]

2. Uniquely determines the function \( a \) in terms of \( \Lambda \):

\[ a^2 = \frac{1}{\Lambda u}. \quad (17) \]

3. Impose a Wünschmann-like condition on \( \Lambda \):

\[ I_{\text{GNS}} = Da + a \Lambda u' = 0. \quad (18) \]
Proof: From (11) and (12) we have:

\[
dθ^1 = -\frac{1}{\sqrt{2a}}a_1 θ^1 \wedge θ^2 - \frac{1}{2a} (1 + Da + a^2 Λ_u + aΛ_u') \theta^1 \wedge ds
\]
\[
-\frac{1}{2a} (-1 - Da + a^2 Λ_u - aΛ_u') \theta^2 \wedge ds,
\]

\[
dθ^2 = \frac{1}{\sqrt{2a}}a_2 θ^1 \wedge θ^2 + \frac{1}{2a} (-1 + Da + a^2 Λ_u + aΛ_u') \theta^1 \wedge ds
\]
\[
+ \frac{1}{2a} (1 - Da + a^2 Λ_u - aΛ_u') \theta^2 \wedge ds.
\]

(19)

The condition of free-torsion (15) reads

\[
dθ^1 - ω_{[12]} \wedge θ^1 = 0,
\]
\[
dθ^2 + ω_{[12]} \wedge θ^2 = 0,
\]

(21)

(22)

and by solving these equations we get:

\[
ω_{[12]} = -\frac{1}{\sqrt{2}}(\ln a)_u θ^1 + \frac{1}{\sqrt{2}}(\ln a)_v θ^2 + \frac{1}{2a} (1 + Da + a^2 Λ_u + aΛ_u') ds,
\]

(23)

Together to the three conditions:

\[
(-1 - Da + a^2 Λ_u - aΛ_u') = 0,
\]
\[
(-1 + Da + a^2 Λ_u + aΛ_u') = 0,
\]
\[
(Da + aΛ_u') = 0.
\]

(24)

(25)

(26)

Finally, from (23), and the conditions (24), (25), (26) we get the results stated in the theorem. Q.E.D.

Note now, that with the map \(ζ : J^1 \rightarrow M × R\) discussed in section 2, we have a 1-parameter family of (pseudo)-riemannian metrics in the solution space \(M\) (riemannian metrics if \(Λ_u < 0\), and lorentzian metrics if \(Λ_u > 0\)), i.e: we have the following family of metrics:

\[
g(x^a, s) = (ζ^{-1})^* h,
\]

(27)
or written in coordinates:

\[
g(x^a, s) = \beta^1 \otimes \beta^1 - \frac{1}{\Lambda u} \beta^2 \otimes \beta^2 = \left[ Z_a Z_b - \frac{1}{\Lambda u} Z'_a Z'_b \right] dx^a dx^b. \tag{28}
\]

in fact, all they are equivalents, because it is easy to show that \( h \) satisfy:

\[
\mathcal{L}_{e_i} h = 0. \tag{29}
\]

Finally, let us collect the one-forms \( \theta^i \) and \( \omega^i_j \) into the matrix-valued one-form

\[
\omega_c = \begin{pmatrix}
0 & 0 & 0 \\
\tilde{\theta}^1 & -\omega_{[12]} & 0 \\
\tilde{\theta}^2 & 0 & \omega_{[12]}
\end{pmatrix},
\]

and let us study two cases:

a). \( \Lambda u > 0 \) : In this case we have a lorentzian metric, and \( \omega_c \) takes its values in the lie algebra of \( \text{SO}(1, 1) \times \mathbb{R}^2 \).

This matrix valued one-form can be regarded as a \( \text{SO}(1, 1) \times \mathbb{R}^2 \) Cartan connection \[14\] on the principal bundle \( \text{SO}(1, 1) \to P \to M \) with associated curvature \( \Omega_c = d\omega_c + \omega_c \wedge \omega_c \) given by

\[
\Omega_c = \begin{pmatrix}
0 & 0 & 0 \\
T^1 & \Omega^1_1 & \Omega^1_2 \\
T^2 & \Omega^2_1 & \Omega^2_2
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 \\
0 & -R & 0 \\
0 & 0 & R
\end{pmatrix}, \tag{30}
\]

where \( \Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j \) is the standard curvature, and

\[
R = -\frac{1}{\alpha} a_{uu} \tilde{\theta}^1 \wedge \tilde{\theta}^2. \tag{31}
\]

b). \( \Lambda u < 0 \) : In this case we have a riemannian metric, and \( \omega_c \) takes its values in the lie algebra of \( \text{SO}(2) \times \mathbb{R}^2 \).

This construction gives a \( \text{SO}(2) \times \mathbb{R}^2 \) Cartan connection on the principal bundle \( \text{SO}(2) \to P \to M \) with associated curvature \( \Omega_c = d\omega_c + \omega_c \wedge \omega_c \) given by a similar formula to (30).
4 Conclusions

In this note we show that, as it happens in NSF, all two-dimensional riemannian and lorenzian metrics can be obtained from the geometrical condition of a torsion-free connection. Furthermore, we construct all associated Cartan connections to these equations. This approach can be extended to the study of third order ODE and a pair of 2nd order PDEs, but the choice of the differential quadratic form $h$, is not \textit{a priori} so clear. In fact, all these problems should be studied with the Cartan’s equivalence method \cite{15} applied to differential equation under a subgroup of contact transformations: the canonical transformations. With this algorithmic method, the group arising from these equations can be naturally obtained as the group of allowed transformations in the study of the equivalence problem. Work in this area has begun.

5 Acknowledgments

The author thanks Carlos Kozameh for the reading and revision of this paper. The author is supported by CONICET.

References

[1] P. García-Godínez, E. T. Newman, G. Silva-Ortigoza, \textit{2-geometries and the Hamilton-Jacobi equation}, J. Math. Phys. \textbf{45}, 725 (2004).

[2] P. García-Godínez, E. T. Newman, G. Silva-Ortigoza, \textit{3-geometries and the Hamilton-Jacobi equation}, J. Math. Phys. \textbf{45}, 2543 (2004).

[3] C. N. Kozameh, E.T. Newman, \textit{Theory of Light Cone Cuts of Null Infinity}, J. Math. Phys., \textbf{24}, 2481 (1983).

[4] S. Frittelli, C. Kozameh, E.T. Newman, \textit{Dynamics of Light-Cone Cuts of Null Infinity}, Phys. Rev. D. \textbf{56}, 4729 (1997).

[5] S. Frittelli, C. N. Kozameh, E.T. Newman, \textit{GR via Characteristic Surfaces}, J. Math. Phys. \textbf{36}, 4984 (1995).

[6] D. Forni, C. Kozameh, M. Iriondo, \textit{Null surfaces formulation in 3D}, J. Math. Phys. \textbf{41}, 5517 (2000).
[7] S. Frittelli, C. Kozameh, E.T. Newman, *Differential Geometry from Differential Equations*, Communications in Mathematical Physics, **223**, p.383 (2001).

[8] S. Frittelli, E.T. Newman, P. Nurowski, *Conformal Lorentzian metrics on the spaces of curves and 2-surfaces*, Class. Quantum Grav. **20** 3649, (2003).

[9] S-S. Chern, *The Geometry of the Differential Equation $y'''' = F(x, y, y', y'')$*, in Selected Papers, Springer-Verlag, (1978), original (1940).

[10] P. Nurowski, *Conformal connection and equivalence problem for third order ODEs*, Int. J. Mod. Phys. A, **17**, N20, 2770 (2002).

[11] S. Frittelli, N. Kamran, E.T. Newman, *Differential Equations and Conformal Geometry*, Journal of Geometry and Physics, **43**, 133 (2002).

[12] S. Frittelli, C. Kozameh, E.T. Newman, P. Nurowski, *Cartan Normal Conformal Connections from Differential Equations*, Class. Quantum Grav. **19**, 5235 (2002).

[13] E. Gallo, C. Kozameh, T. Newman, K. Perkins, *Cartan Normal Conformal Connections from Pairs of 2nd Order PDEs*, to appear in Class. Quantum Grav. [gr-qc/0404072].

[14] R.W. Sharpe, *Differential Geometry: Cartan’s generalization of Klein’s Erlangen program*, Graduate Texts in Mathematics, Springer, (2000).

[15] P. Olver, *Equivalence, Invariants and Symmetry*, Cambridge University Press, Cambridge, (1995).