A Fibration of the Hipercubic Lattice in Bethe Ansatz

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Abstract. We show that the configuration space in the Bethe Ansatz is a locally hipercubic lattice with some boundary. The lattice admits a fibre structure. The base of the fibration has a meaning of the mass centre space (with a suitable scale; indeed, we use the sum of position of the deviations) of configurations and the fibre is the relative space of configurations. The structure of the fibre depends on the point in the base (it means that on the position of the mass centre).

1. Introduction
In the paper we study the problem of the configuration space in Bethe Ansatz [1], [2] for the closed Heisenberg chain of $N$ nodes with $r$ deviations [3]-[5]. This space is embedded in the so-called model manifold [6], which depends only on the number of deviations $r$ and does not depend on the number of nodes $N$. The model manifold admits a fibre bundle structure [7]-[9]. The base of the bundle is a circle and can be treated as a space of all positions of the geometric mass centre of deviations. The fibres are interiors of $(r-1)$-dimensional simplexes [10]-[12] and have a meaning of the space of relative positions of deviations. The configuration space of Bethe Ansatz is a locally hipercubic lattice with respect to the natural metric induced from the Euclidean metric. The fibration of the model manifolds follows a fibration of the hipercubic lattice. In order to avoid fractions as the base points we take the sum of positions of deviations instead of the mass centre. So we avoid nonintegral numbers.

The paper is organised as follows. In the section 2 we introduce some notation used in this work. In the section 3 we consider the universal covering of the model manifold and its fibration. In the section 4 we show that the configuration space in Bethe Ansatz is a locally $r$-dimensional hipercubic lattice which admits a fibre bundle structure. The standard fibre of the bundle decomposes into some partial fibres depending on the base point modulo $r$. In the section 5 we present two examples – the configuration spaces of Bethe Ansatz for the cases $N = 6$, $r = 3$ and $N = 7$, $r = 3$.

2. Some notation
By $S$ we denote the quotient group $S = \mathbb{R}/\mathbb{Z}$, where $\mathbb{R}$ is the additive group of real numbers and $\mathbb{Z}$ - of integers. From the topological point of view the space $S$ is a circle, and the canonical projection $p : \mathbb{R} \longrightarrow S$, $p(x) = [x]$, $[x] := x + \mathbb{Z}$, is a universal covering.
Further, for any natural number $n \in \mathbb{N}$ ($\mathbb{N} := \{1, 2, \ldots\}$) by $\mathbb{Z}_n$ we denote the $n$-element additive cyclic group $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$, where $n\mathbb{Z} = \{\ldots, -2n, -n, 0, n, 2n, \ldots\}$. So we have $\mathbb{Z}_n = \{[0]_n, [1]_n, \ldots, [n-1]_n\}$, where $[k]_n := k + n\mathbb{Z}$.

The classical configuration space of $r$ identical hard core particles on a manifold $M$ can be defined in two equivalent ways. The first definition states that the space is the generic stratum of the action $l$ $Q$ of identical indistinguishable hard core particles on the circle $S$. The second – that the space is the set of all $r$-element subsets of the manifold $M$. We denote by $Q_r(M)$ the configuration space.

3. The model manifold

Now let us briefly consider the configuration space $Q_r(S)$ of $r$ identical indistinguishable hard core particles on the circle $S$. The space $Q_r(S)$ is the model manifold for the Bethe Ansatz [6]. The universal covering space of $Q_r(S)$ is the configuration space of all orbits of the action $l$ $Q$ $P$ $S$ of $r$ identical indistinguishable hard core particles on the circle $S$, so we identify these spaces

$$Q_r(S) = \{O_l(x_1, \ldots, x_r) : (x_1, \ldots, x_r) \in P_r\}.$$ (4)

The canonical projection

$$p_r : P_r \rightarrow Q_r(S)$$ (5)

is a universal covering of the configuration space $Q_r(S)$. The fibres of the covering (5) are orbits of the action $l$. A different approach to similar problems is presented in the work [13].

The covering space $P_r$ of the model manifold $Q_r(S)$ admits also a fibre bundle structure

$$\tilde{p}_r : P_r \rightarrow \mathbb{R} , \quad \tilde{p}_r(x_1, \ldots, x_r) = x_1 + \ldots + x_r ,$$ (6)

where the projection $\tilde{p}_r$ assigns to a given configuration the sum of coordinates. The fibre $F_s$ over $s \in \mathbb{R}$ is the intersection of the hyperplane of equation $\sum_{k=1}^r x_k = s$ with the space $P_r$. The fibre $F_s$ of the bundle is the interior of the $(r-1)$-dimensional simplex $F_s$ with vertices $x^1(s), \ldots, x^r(s)$. Coordinates of the $k$-th vertex are given by the following formula

$$x^k_1(s) = \ldots = x^k_r(s) = \frac{s+k}{r} - 1 , \quad x^k_{k+1}(s) = \ldots = x^k_r(s) = \frac{s+k}{r} .$$

The relative coordinates defined by the sequence of equations $a_1 = x_2 - x_1$, $a_2 = x_3 - x_2$, $\ldots$, $a_{r-1} = x_{r} - x_{r-1}$, $a_r = 1 + x_1 - x_r$ are the baricentric coordinates at fibre $F_s$. It means, that any point $x$ with given $s$ and $a = (a_1, \ldots, a_r)$ can be expressed in the form

$$x(s, a) = a_1 x^1(s) + \ldots + a_r x^r(s) .$$ (7)
These coordinates are not independent because they satisfy the relation $a_1 + \ldots + a_r = 1$, $a_k > 0$ for $k = 1, \ldots, r$. The formula (7) gives the diffeomorphism $R \times \Delta_{r-1} \rightarrow P_r$, $(s, a) \mapsto x(s, a)$, where $\Delta_{r-1}$ is the interior of the standard $(r-1)$-dimensional simplex in $R^r$: 

$$\Delta_{r-1} = \{a \in R^r : a_k \geq 0, a_1 + \ldots + a_r = 1\}.$$

The action $l$ in the coordinates $(s, a)$ is expressed by the formula $l(x(s, a)) = x(s + 1, \text{cycle}(a))$, where $\text{cycle}(a_1, a_2, \ldots, a_r) = (a_2, \ldots, a_r, a_1)$. Moreover, the model manifold $Q_r(S)$ is the total space of the bundle 

$$\pi_r : Q_r(S) \rightarrow S, \quad \pi_r(O_l(x(s, a))) = [s].$$

The model manifold is diffeomorphic and isometric to the quotient manifold

$$\left(\left(\pi_r\right)^{-1}(\langle 0, 1 \rangle)/\sim, x \sim lx \text{ for } x \in F_0\right).$$

In other words $x(0, a) \sim x(1, \text{cycle}(a)), a \in \Delta$.

4. The configuration space in Bethe Ansatz

The space of classical configurations of $r$ Bethe pseudoparticles on a magnetic ring of $N$ nodes is $Q_r(Z_N)$. It consists of $\text{binom} (N, r)$ points. The space can be considered as the set of orbits of the action $L$ of $Z$ on the set

$$P_{N,r} := \{(j_1, \ldots, j_r) \in Z^r : j_1 < j_2 < \ldots < j_r < N + j_1\},$$

where $L(j_1, \ldots, j_r) = (j_2, \ldots, j_r, N + j_1)$. The metric on $Q_r(Z_N)$ is induced as a quotient one from $P_r$. We see that $P_{N,r} = d_N(P_r) \cap Z^r$ and $L = d_N \circ l \circ d_1|_{P_{N,r}}$, where $d_n$ is the operator defined by the following formula $d_n(x_1, \ldots, x_r) = (ax_1, \ldots, ax_r)$.

The set $P_{N,r}$ is a locally $r$-dimensional hypercubic lattice, with respect to the standard Euclidean metric in $R^r$, with some boundary because it is the intersection of the open set $d_N(P_r)$ in $R^r$ and $Z^r$. The locally hypercubic lattice $P_{N,r}$ can be fibred in analogous way to $P_r$, eq. (6)

$$\pi_{N,r} : P_{N,r} \rightarrow Z, \quad \pi_{N,r}(j_1, \ldots, j_r) = j_1 + \ldots + j_r.$$  

The locally hypercubic lattice can be embedded into the prism $P_r$ by $\tilde{i}_{N,r} : P_{N,r} \rightarrow P_r$, $\tilde{i}_{N,r}(j_1, \ldots, j_r) = \left(\frac{j_1}{N}, \ldots, \frac{j_r}{N}\right)$. The image of $\tilde{i}_{N,r}$ is an $l$-invariant subset of $P_r$. So the embedding $\tilde{i}_{N,r}$ implies the embedding

$$i_{N,r} : Q_r(Z_N) \rightarrow Q_r(S), \quad i_{N,r}(O_l(j_1, \ldots, j_r)) = O_l\left(\frac{j_1}{N}, \ldots, \frac{j_r}{N}\right).$$

So for any $N > r$ the formula (13) defines the embedding of the configuration space $Q_r(Z_N)$ into the model manifold $Q_r(S)$.

The space $P_{N,r}$ has two structures:

1) the locally hypercubic lattice structure inherited from the $P_{N,r}$ with the lattice constant equal to one;
2) the fibre bundle structure inherited from $Q_r(S)$ with the bundle projection defined by the way

$$\pi_{N,r} : Q_r(Z_N) \rightarrow Z_N, \quad \pi_{N,r}(O_l(j_1, \ldots, j_r)) = [j_1 + \ldots + j_r]_N.$$

For given coordinates $s = j_1 + \ldots + j_r$ and $t = (j_2 - j_1, j_3 - j_2, \ldots, j_r - j_{r-1}, N + j_1 - j_r)$ we can calculated coordinates $j$ by the following formula

$$j_i(s, t) = \frac{1}{r}(s + \sum_{k=1}^{r} kt_k) - \sum_{k=i}^{r} t_k.$$
The standard fibre
\[ \Delta_{N,r} = \{(t_1, \ldots, t_r) : t_1 + \ldots + t_r = N, \ t_k \in \mathbb{N}, \ k = 1, \ldots, r\} \] (16)
consists of \(\binom{N-1}{r-1}\) elements and decomposes into partial fibres
\[ \Delta_{N,r,[s]} = \{(t_1, \ldots, t_r) \in \Delta_{N,r} : t_1 + 2t_2 + \ldots + (r-1)t_{r-1} + s = 0 \mod r\} \] (17)
so the partial fibre depends on the point \(s\) in the base. The index \(s\) is taken modulo \(r\) on the left side, because the equation defining the set of relative configurations is modulo \(r\). The set
\[ F_{N,r,s} = \{j(s,t) : t \in \Delta_{N,r,[s]}\} \] (18)
is the fibre of the bundle over \(s\), and it is the set of all configurations in \(P_{N,r}\) whose sum of coordinates is \(j_1 + j_2 + \ldots + j_r = s\).

**Remark.** The sense of the decomposition (17) is the following. The partial fibre \(\Delta_{N,r,[s]}\) is the set of all relative configurations \(t = (t_1, \ldots, t_r)\) such that \(j(s,t)\) given by means of eq. (15) is the lattice point. For \(t \notin \Delta_{N,r,[s]}\), the position \(j(s,t)\) has fractional coordinates.

The locally hipercubic lattice in Bethe Ansatz can be presented as the set
\[ \left( j\left( \bigcup_{s \in \{0,N\}} \{s\} \times \Delta_{N,r,s} \right) \right) / \sim, \text{ where } j(0,t) \sim j(N,\text{cycle}(t)), t \in \Delta_{N,r,0}. \] (19)
The bundle projection takes the form \(\pi_{N,r}([j(s,t_1,\ldots,t_r)]_{\sim}) = \{s\}_N\).

### 5. Examples

In this section we consider two examples.

1. For the case \(N = 6\) and \(r = 3\) the standard fibre is the set of all lattice points of the triangle with vertices \((1,1,4),(1,4,1),(4,1,1)\):

\[ \Delta_{6,3} = \] (20)
\[ \{ (1,1,4), (1,2,3), (1,3,2), (1,4,1), \]
\[ (2,1,3), (2,2,2), (2,3,1), \]
\[ (3,1,2), (3,2,1), \]
\[ (4,1,1) \}. \]

The standard fibre decomposes into \(r = 3\) partial fibres \(\Delta_{6,3,0}, \Delta_{6,3,1}, \Delta_{6,3,2}\):

\[ \Delta_{6,3,0} = \{(1,1,4),(1,4,1),(4,1,1)\} \cup \{(2,2,2)\}, \] (21)
\[ \Delta_{6,3,1} = \{(1,2,3),(2,3,1),(3,1,2)\}, \] (22)
\[ \Delta_{6,3,2} = \{(1,3,2),(2,1,3),(3,2,1)\}. \] (23)

The partial fibres are invariant for the action of the group \(\mathbb{Z}_3\) because \(r = 3\) is a divisor of \(N = 6\). The set of configurations over base points \(s = 0,1,\ldots,6\) is the union of the fibres \(F_{6,3,0},\ldots,F_{6,3,6}\) given by (24–30):

\[ F_{6,3,0} = \{(-1,0,1),(-2,-1,3),(-3,1,2)\} \cup \{(-2,0,2)\}, \] (24)
Let us observe that the fibres $F_{6,3,0}$, $s = 1, \ldots, 6$ are not equinumerous. Finally, the configuration space for Bethe Ansatz for $N = 6$, $r = 3$ is

$$Q_6(\mathbb{Z}_3) = \{[[j_1, j_2, j_3]] : (j_1, j_2, j_3) \in \bigcup_{s \in \{1,6\}} F_{6,3,s}\}.$$  

2. For the case $N = 7$ and $r = 3$ the standard fibre

$$\Delta_{7,3} =$$

$$(1,1,5), (1,2,4), (1,3,3), (1,4,2), (1,5,1),$$

$$(2,1,4), (2,2,3), (2,3,2), (2,4,2),$$

$$(3,1,3), (3,2,2), (3,3,1),$$

$$(4,1,2), (4,2,1),$$

$$(5,1,1)$$

decomposes also into three partial fibres:

$$\Delta_{7,3,0} = \{(1,1,5), (1,4,2), (2,2,3), (3,3,1), (4,1,2)\}.$$  

$$\Delta_{7,3,1} = \{(1,2,4), (1,5,1)(2,3,2), (3,1,3), (4,2,1)\}.$$  

$$\Delta_{7,3,2} = \{(1,3,3), (2,1,4), (2,4,1), (3,2,2), (5,1,1)\}.$$  

The partial fibres are not invariant for the cyclic action of the group $\mathbb{Z}_3$ because $r = 3$ is not a divisor of $N = 7$. Indeed, we have

$$\text{cycle}(\Delta_{7,3,0}) = \Delta_{7,3,1}, \text{ cycle}(\Delta_{7,3,1}) = \Delta_{7,3,2}, \text{ cycle}(\Delta_{7,3,2}) = \Delta_{7,3,0}.$$  

The set of configurations over base points $s = 0, 1, \ldots, 7$ is the union of the fibres $F_{7,3,0}, \ldots, F_{7,3,7}$:

$$F_{7,3,0} = \{(-1,0,1), (-2,-1,3), (-2,0,2), (-3,0,3), (-3,1,2)\}.$$  

$$F_{7,3,1} = \{(-1,0,2), (-2,-1,4), (-2,0,3), (-2,1,2), (-3,1,3)\}.$$  

$$F_{7,3,2} = \{(-1,0,3), (-1,1,2), (-2,0,4), (-2,1,3), (-3,2,3)\}.$$  

$$F_{7,3,3} = \{(0,1,2), (-1,0,4), (-1,1,3), (-2,1,4), (-2,2,3)\}.$$  

$$F_{7,3,4} = \{(0,1,3), (-1,0,5), (-1,1,4), (-2,0,6), (-2,1,5), (-3,0,5), (-3,1,4)\}.$$  

$$F_{7,3,5} = \{(0,2,3), (-1,1,0), (-1,2,1), (-2,1,2), (-3,0,3), (-3,1,2)\}.$$  

$$F_{7,3,6} = \{(1,0,2), (-1,1,3), (-2,0,4), (-2,1,5), (-3,0,5), (-3,1,4)\}.$$  

$$F_{7,3,7} = \{(1,1,4), (-1,2,0), (-2,1,1), (-3,0,2), (-3,1,1)\}.$$  

$$F_{7,3,8} = \{(1,2,1), (-1,3,0), (-2,2,0), (-3,1,0), (-3,2,1)\}.$$  

$$F_{7,3,9} = \{(2,0,3), (-2,1,2), (-3,2,3), (-3,3,2)\}.$$  

$$F_{7,3,10} = \{(2,1,3), (-3,0,2), (-3,1,1), (-3,2,1)\}.$$  

$$F_{7,3,11} = \{(2,2,0), (-3,1,0), (-3,2,0)\}.$$  

$$F_{7,3,12} = \{(3,0,3), (-3,1,1), (-3,2,1)\}.$$  

$$F_{7,3,13} = \{(3,1,2), (-3,2,2)\}.$$  

$$F_{7,3,14} = \{(3,2,3)\}.$$  

$$F_{7,3,15} = \emptyset.$$
The fibres $F_{7,3,r}$ are equinumerous because 7 and 3 are coprime. In order to obtain the configuration space $Q_N(\mathbb{Z}_r)$ we have to glue points from the fibre $F_{7,3,0}$ with appropriate points from $F_{7,3,7}$:

\[ (-1,0,1) \sim (0,1,6), (-2,-1,3) \sim (-1,3,5), (-3,1,2) \sim (1,2,4), \]

\[ (-2,0,2) \sim (0,2,5), (-3,0,3) \sim (0,3,4). \]

So the configuration space for Bethe Ansatz for $N = 7$, $r = 3$ is

\[ Q_7(\mathbb{Z}_3) = \{([j_1,j_2,j_3])_s : (j_1,j_2,j_3) \in \bigcup_{s \in \{1,7\}} F_{7,3,s} \}. \]

6. Concluding remarks

The fibration of the hypercubic lattice is compatible with the embedding of the lattice into the model manifold which has also the fibre bundle structure. The partial fibres are cyclic invariant in the case when $r$ is a divisor of $N$ what was showed in the first example in the section 5. This example also shows that the partial fibres can be not equinumerous, but in the case when $r$ and $N$ are coprime they are equinumerous what is showed in the second example.

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