A Note on Warm Standby System

Nil Kamal Hazra and Asok K. Nanda *
Department of Mathematics and Statistics
IISER Kolkata, India
August, 2014

Abstract

In this note, we investigate, under what circumstances, a warm standby system (formed by $n$ active components and $m$ warm standby components) has more number of surviving warm standby components than another similar system at the time of $k$th failure of the active component of the respective system. The number of such components being random, the comparison has been done with respect to different stochastic orders, viz. usual stochastic order, hazard rate and reversed hazard rate orders, and likelihood ratio order.

Key Words and Phrases: Order statistics, permanent of a matrix, RR$_2$ function, stochastic orders

1 Introduction

The failure of a system could happen at any time, and we have absolutely no control on that. However, we can enhance the lifetime of a system by incorporating standby (or redundant) components into the system. Standby components are mostly of three types – hot (or active) standby, warm standby and cold standby. Here we study the system with warm standby components, called warm standby system. For this system, a redundant component undergoes two operational environments. Initially, it functions in a milder environment (in which a redundant component has non-zero failure rate which is less than its actual failure rate), thereafter it switches over to the usual environment (in which the system is running) after the original component fails. It might happen that

*e-mail: asok.k.nanda@gmail.com; asok@iiserkol.ac.in, corresponding author.
the redundant component fails before switching over to the usual environment. Warm standby system is well studied in the literature by different researchers, namely, Cha et al. [5], Li et al. [9], Eryilmaz [6], Hazra and Nanda [7] and the references therein.

For an absolutely continuous component life $Z$, we denote the probability density function by $f_Z(\cdot)$, the cumulative distribution function by $F_Z(\cdot)$ given by $F_Z(t) = P(Z \leq t)$. The survival or reliability function of the random variable $Z$ is written as $\bar{F}_Z(\cdot) = 1 - F_Z(\cdot)$. Further, the indicator function $I_{[a>b]}$ is defined as

$$I_{[a>b]} = \begin{cases} 1, & \text{if } a > b \\ 0, & \text{if } a \leq b. \end{cases}$$

For a collection $\{X_1, X_2, \ldots, X_n\}$ of random variables, the corresponding order statistics are denoted as $X_{1,n} \leq X_{2,n} \leq \cdots \leq X_{n,n}$.

In order to compare the lifetimes of two systems stochastic orders are very useful tool. In the literature many different types of stochastic orders have been developed. The following well known definitions may be obtained in Shaked and Shanthikumar [11].

**Definition 1.1** Let $X$ and $Y$ be two absolutely continuous random variables with respective supports $(l_X, u_X)$ and $(l_Y, u_Y)$, where $u_X$ and $u_Y$ may be positive infinity, and $l_X$ and $l_Y$ may be negative infinity. Then, $X$ is said to be smaller than $Y$ in

(a) likelihood ratio (lr) order, denoted as $X \leq_{lr} Y$, if

$$\frac{f_Y(t)}{f_X(t)} \text{ is increasing in } t \in (l_X, u_X) \cup (l_Y, u_Y);$$

(b) hazard rate (hr) order, denoted as $X \leq_{hr} Y$, if

$$\frac{\bar{F}_Y(t)}{\bar{F}_X(t)} \text{ is increasing in } t \in (-\infty, \max(u_X, u_Y));$$

(c) reversed hazard rate (rhr) order, denoted as $X \leq_{rhr} Y$, if

$$\frac{F_Y(t)}{F_X(t)} \text{ is increasing in } t \in (\min(l_X, l_Y), \infty);$$

(d) usual stochastic (st) order, denoted as $X \leq_{st} Y$, if $\bar{F}_X(t) \leq \bar{F}_Y(t)$ for all $t \in (-\infty, \infty)$. \[\square\]
In the following diagram we present a chain of implications of the stochastic orders (cf. Shaked and Shanthikumar [11]):

\[
\begin{align*}
X \leq_{hr} Y \\
\uparrow & \quad \downarrow \\
X \leq_{hr} Y & \rightarrow X \leq_{st} Y. \\
\downarrow & \quad \uparrow \\
X \leq_{r_{hr}} Y
\end{align*}
\]

For the sake of completeness, Below we give the definition of an RR\(_2\) function, which has been borrowed from Karlin [8].

**Definition 1.2** Let \( X \) and \( Y \) be two linearly ordered sets. Then, a real-valued function \( \kappa(\cdot, \cdot) \) defined on \( X \times Y \), is said to be reverse regular of order 2 (written as RR\(_2\)) if

\[
\kappa(x_1, y_1)\kappa(x_2, y_2) \leq \kappa(x_1, y_2)\kappa(x_2, y_1),
\]

for all \( x_1 < x_2 \) and \( y_1 < y_2 \).

Throughout the paper, increasing and decreasing properties are not used in strict sense. For any differentiable function \( k(\cdot) \), we write \( k'(t) \) to denote the first derivative of \( k(t) \) with respect to \( t \). By \( a \overset{\text{def}}{=} b \) we mean that \( a \) is defined as \( b \).

The rest of the paper is organized as follows. In Section 2 we consider two warm standby systems where each system is formed by \( n \) active components and \( m \) warm standby components. We assume that one system has stochastically stronger active components than those of the other system. Then we show that the total number of surviving warm standby components at the time of \( k \)th failure of the active component of a warm standby system dominates that of another warm standby system with respect to different stochastic orders, viz. usual stochastic order, hazard rate order, reversed hazard rate order and likelihood ratio order. In Section 3 we study a similar kind of comparison result with respect to the usual stochastic order. Here we assume that two warm standby systems have different sets of warm standby components, and one set of warm standby components is superior to that of the other set with respect to the usual stochastic order.
2 Comparison Based on Single Set of Standby Components

Consider a warm standby system formed by \( n \) active components having lifetimes \( X = (X_1, X_2, \ldots, X_n) \) and \( m \) warm standby components having lifetimes \( Y = (Y_1, Y_2, \ldots, Y_m) \). We denote \( N_k(X, Y) \) as the total number of surviving warm standby components at the time when the \( k \)th active component of the system fails.

Below we compare two warm standby systems where the active components of one system dominate those of the other system with respect to the usual stochastic order. We show that the total number of surviving warm standby components at the time of \( k \)th failure of the active component of a warm standby system (formed by stochastically stronger active components) is less than that of another warm standby system (formed by stochastically weaker active components) with respect to the usual stochastic order.

**Theorem 2.1** Let \( X = (X_1, X_2, \ldots, X_n) \) and \( X^* = (X_1^*, X_2^*, \ldots, X_n^*) \) be the lifetimes of the two groups of active components, and \( Y = (Y_1, Y_2, \ldots, Y_m) \) be those of a group of standby components. Further assume that all \( X_i, X_i^* \) and \( Y_i \) are independent. If \( X_i \leq_{st} X_i^* \) for all \( i = 1, 2, \ldots, n \), then \( N_k(X^*, Y) \leq_{st} N_k(X, Y) \).

**Proof:** Note that

\[
\bar{F}_{N_k(X,Y)}(r) = P(N_k(X, Y) > r)
= \int_0^\infty P(N_k(X, Y) > r | X_{k,n} = t) dF_{X_{k,n}}(t)
= \int_0^\infty P\left( \sum_{j=1}^m I_{[Y_j > t]} > r \right) dF_{X_{k,n}}(t)
= \int_0^\infty \bar{F}_{Y_{m-r,m}}(t) dF_{X_{k,n}}(t) \tag{2.1}
= \int_0^\infty F_{X_{k,n}}(t) dF_{Y_{m-r,m}}(t). \tag{2.2}
\]

Similarly,

\[
\bar{F}_{N_k(X^*,Y)}(r) = \int_0^\infty F_{X^*_{k,n}}(t) dF_{Y_{m-r,m}}(t). \tag{2.3}
\]
Since, $X_i \leq_{st} X^*_i$ for all $i = 1, 2, \ldots, n$, by Corollary 3.2 of Belzunce et al. [3], we have $F_{X_{k,n}}(t) \geq F_{X^*_{k,n}}(t)$ for all $t \in (0, \infty)$. Hence $N_k(X^*, Y) \leq_{st} N_k(X, Y)$.

The following counterexample shows that the condition $X_i \leq_{st} X^*_i$ given in the above theorem cannot be relaxed.

**Counterexample 2.1** Let $X = (X_1, X_2, X_3)$ and $X^* = (X^*_1, X^*_2, X^*_3)$ be the lifetimes of the two groups of active components with hazard rates $(4, 5, 6)$ and $(7, 8, 2)$, respectively. Further, let $Y = (Y_1, Y_2, Y_3)$ be those of a group of standby components with hazard rates $(1, 3, 4)$. Assume that all $X_i, X^*_i$ and $Y_i$ are independent. Clearly, $X_1 \geq_{st} X^*_1, X_2 \geq_{st} X^*_2$ and $X_3 \leq_{st} X^*_3$. Thus, $X_i \leq_{st} X^*_i$ does not hold for all $i = 1, 2, 3$. Now,

$$F_{N_2(X, Y)}(1) = \int_{0}^{\infty} \left( e^{-4t} + e^{-7t} + e^{-5t} - 2e^{-8t} \right) \left( 9e^{-9t} + 11e^{-11t} + 10e^{-10t} - 30e^{-15t} \right) dt$$

$$= 0.737$$

and

$$F_{N_2(X^*, Y)}(1) = \int_{0}^{\infty} \left( e^{-4t} + e^{-7t} + e^{-5t} - 2e^{-8t} \right) \left( 15e^{-15t} + 10e^{-10t} + 9e^{-9t} - 34e^{-17t} \right) dt$$

$$= 0.753.$$ 

Thus, $F_{N_2(X^*, Y)}(1) \geq F_{N_2(X, Y)}(1)$, and hence $N_2(X^*, Y) \not\leq_{st} N_2(X, Y)$.

In the following theorem we extend the above result to the hazard rate order. But before that we state the following lemma which may be obtained in Capéraà [4].

**Lemma 2.1** Let $\alpha_1(\cdot)$ and $\beta_1(\cdot)$ be two nonnegative real-valued functions such that $\beta_1(\cdot)$ and $\alpha_1(\cdot)/\beta_1(\cdot)$ are increasing. Further, let $U_1$ and $U_2$ be two continuous nonnegative random variables. Then

$$\int_{0}^{\infty} \frac{\alpha_1(t)dF_{U_1}(t)}{\beta_1(t)dF_{U_1}(t)} \leq \int_{0}^{\infty} \frac{\alpha_1(t)dF_{U_2}(t)}{\beta_1(t)dF_{U_2}(t)}$$

if, and only if, $U_1 \leq_{hr} U_2$.

**Theorem 2.2** Let $X = (X_1, X_2, \ldots, X_n)$ and $X^* = (X^*_1, X^*_2, \ldots, X^*_n)$ be the lifetimes of the two groups of active components, and $Y = (Y_1, Y_2, \ldots, Y_m)$ be those of a group of standby components. Further assume that all $X_i, X^*_i$ and $Y_i$ are independent. If $X_i \leq_{hr} X^*_j$ for all $i, j = 1, 2, \ldots, n$, then $N_k(X^*, Y) \leq_{hr} N_k(X, Y)$. 

5
Proof: Note from (2.2) and (2.3), that 
\[ N_k(X^*, Y) \leq_{hr} N_k(X, Y) \] if, and only if,
\[ \int_0^\infty \frac{F_{X_{k,n}}(t)dF_{Y_{m-r,m}}(t)}{F_{X_{k,n}}^*(t)dF_{Y_{m-r,m}}(t)} \text{ is increasing in } r. \]

This is equivalent to the fact that, for \( r \leq s \),
\[ \int_0^\infty \frac{F_{X_{k,n}}(t)dF_{Y_{m-s,m}}(t)}{F_{X_{k,n}}^*(t)dF_{Y_{m-s,m}}(t)} \leq \int_0^\infty \frac{F_{X_{k,n}}(t)dF_{Y_{m-r,m}}(t)}{F_{X_{k,n}}^*(t)dF_{Y_{m-r,m}}(t)}, \]
or equivalently,
\[ \int_0^\infty \frac{\alpha_1(t)dF_{U_1}(t)}{\beta_1(t)dF_{U_1}(t)} \leq \int_0^\infty \frac{\alpha_1(t)dF_{U_2}(t)}{\beta_1(t)dF_{U_2}(t)}, \tag{2.4} \]
where \( \alpha_1(t) = F_{X_{k,n}}^*(t), \beta_1(t) = F_{X_{k,n}}(t), F_{U_1}(t) = F_{Y_{m-s,m}}(t), \) and \( F_{U_2}(t) = F_{Y_{m-r,m}}(t). \)

Now, by Theorem 1.B.26 of Shaked and Shanthikumar [11] we have
\[ U_1 \leq_{hr} U_2. \tag{2.5} \]

Further, since \( X_i \leq_{rhr} X_j^* \) for all \( i, j \), by Theorem 1.B.61 of Shaked and Shanthikumar [11] we have \( X_{k,n} \leq_{rhr} X_{k,n}^* \), which is equivalent to the fact that
\[ \frac{\alpha_1(t)}{\beta_1(t)} \text{ is increasing in } t. \tag{2.6} \]

Thus, on using (2.5) and (2.6), Lemma 2.1 gives (2.4), and hence \( N_k(X^*, Y) \leq_{hr} N_k(X, Y). \)

Remark 2.1 Counterexample [27] can be used to show that the condition \( X_i \leq_{rhr} X_j^* \) for all \( i, j \), given in Theorem 2.2 cannot be removed. \( \square \)

The following lemma may be obtained in Shaked and Shanthikumar [11].

Lemma 2.2 Let \( \alpha_2(\cdot) \) and \( \beta_2(\cdot) \) be two nonnegative real-valued functions such that \( \beta_2(\cdot) \) and \( \alpha_2(\cdot)/\beta_2(\cdot) \) are decreasing. Further, let \( W_1 \) and \( W_2 \) be two continuous nonnegative random variables. Then
\[ \int_0^\infty \frac{\alpha_2(t)dF_{W_1}(t)}{\beta_2(t)dF_{W_1}(t)} \leq \int_0^\infty \frac{\alpha_2(t)dF_{W_2}(t)}{\beta_2(t)dF_{W_2}(t)} \]

if, and only if, \( W_1 \leq_{rhr} W_2. \) \( \square \)
The next theorem extends the result discussed in Theorem 2.1 to the reversed hazard rate order.

**Theorem 2.3** Let \( X = (X_1, X_2, \ldots, X_n) \) and \( X^* = (X_1^*, X_2^*, \ldots, X_n^*) \) be the lifetimes of the two groups of active components, and \( Y = (Y_1, Y_2, \ldots, Y_m) \) be those of a group of standby components. Further assume that all \( X_i, X_i^* \) and \( Y_i \) are independent. If \( X_i \leq_{hr} X_j^* \) for all \( i, j = 1, 2, \ldots, n \), then \( N_k(X^*, Y) \leq_{rhr} N_k(X, Y) \).

**Proof:** From (2.2) and (2.3) we have

\[
F_{N_k(X,Y)}(r) = 1 - \int_0^\infty F_{X_k,n}(t)dF_{Y_{m-r,m}}(t)
\]

\[
= \int_0^\infty \bar{F}_{X_k,n}(t)dF_{Y_{m-r,m}}(t),
\]

and

\[
F_{N_k(X^*,Y)}(r) = \int_0^\infty \bar{F}_{X_k^*,n}(t)dF_{Y_{m-r,m}}(t).
\]

Hence, \( N_k(X^*, Y) \leq_{rhr} N_k(X, Y) \) if, and only if,

\[
\int_0^\infty \bar{F}_{X_k,n}(t)dF_{Y_{m-r,m}}(t)
\]

is increasing in \( r \).

This is equivalent to the fact that, for \( r \leq s \),

\[
\frac{\int_0^\infty \bar{F}_{X_k,n}(t)dF_{Y_{m-r,m}}(t)}{\int_0^\infty \bar{F}_{X_k^*,n}(t)dF_{Y_{m-r,m}}(t)} \leq \frac{\int_0^\infty \bar{F}_{X_k,n}(t)dF_{Y_{m-s,m}}(t)}{\int_0^\infty \bar{F}_{X_k^*,n}(t)dF_{Y_{m-s,m}}(t)}
\]

or equivalently,

\[
\frac{\int_0^\infty \alpha_2(t)dF_{W_1}(t)}{\int_0^\infty \beta_2(t)dF_{W_1}(t)} \geq \frac{\int_0^\infty \alpha_2(t)dF_{W_2}(t)}{\int_0^\infty \beta_2(t)dF_{W_2}(t)}.
\]

(2.7)

where \( \alpha_2(t) = \bar{F}_{X_k,n}(t), \beta_2(t) = \bar{F}_{X_k^*,n}(t), F_{W_1}(t) = F_{Y_{m-s,m}}(t), \) and \( F_{W_2}(t) = F_{Y_{m-r,m}}(t) \).

Now, by Theorem 1.B.56 of Shaked and Shanthikumar [11] we have

\[ W_1 \leq_{rhr} W_2. \]

(2.8)
Further, since $X_i \leq_{hr} X_j^*$ for all $i, j$, by Theorem 1.B.36 of Shaked and Shanthikumar \[11\] we have $X_{k,n} \leq_{hr} X_{k,n}^*$, which is equivalent to the fact that

$$\frac{\alpha_2(t)}{\beta_2(t)} \text{ is decreasing in } t.$$  \[(2.9)\]

Thus, on using (2.8) and (2.9), Lemma 2.2 gives (2.7), and hence $N_k(X^*, Y) \leq_{rhr} N_k(X, Y)$.

\[\square\]

**Remark 2.2** The condition $X_i \leq_{hr} X_j^*$ for all $i, j$, given in Theorem 2.3 cannot be relaxed as Counterexample 2.1 shows.

\[\square\]

In order to extend the above discussed results to the likelihood ratio order we shall take help of the concept of permanent of a matrix, which will be used to prove a few lemmas that are required to establish the desired result.

Let $A = ((a_{i,j}))$ be an $n \times n$ matrix. Then the permanent of $A$ is defined as

$$\text{Per } A = \sum_S \prod_{j=1}^{n} a_{j,i_j},$$

where $\sum_S$ denotes the sum over all $n!$ permutations $(i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$. If $a_1, a_2, \ldots, a_n$ are column vectors of $A$, then the permanent of $A$ can be written as

$$\text{Per } A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ r_1 & r_2 & \cdots & r_n \end{bmatrix},$$

where the matrix is formed by $r_1$ copies of $a_1$, $r_2$ copies of $a_2$ and so on. For more discussion on permanent of a matrix we may refer the reader to Minc \[10\], Bapat \[2\], and Balakrishnan \[1\].

The following lemma is borrowed from Bapat \[2\], which is used to prove Lemma 2.4.

**Lemma 2.3** Let $\mathbf{z, b, c, d}$ be nonnegative vectors of order $m$. Then, for $1 \leq r \leq m$,

$$r \begin{bmatrix} \mathbf{z} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\ r-1 & m-r & 1 & 1 \end{bmatrix} \geq (r-1) \begin{bmatrix} \mathbf{z} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\ r & m-r & 1 & 1 \end{bmatrix}.$$

**Lemma 2.4** Let $D(r, t) = \sum_{\{A: |A|=r\}} \left( \prod_{i \in A} \tilde{F}_{Y_i}(t) \prod_{j \in A^C} F_{Y_j}(t) \right)$, for $r = 0, 1, \ldots, m$, and $0 < t < \infty$, where $A \subseteq \{1, 2, \ldots, m\}$, $A^C$ is the complement of $A$ and $|A|$ is the cardinality of $A$. Then $D(r, t)$ is RR$_2$ in $(r, t)$.  \[8\]
Proof: To prove that $D(r, t)$ is RR$_2$, it suffices to show that, for $0 \leq r \leq m - 1$, and for $0 < t < s < \infty$, 

$$D(r, t)D(r + 1, s) \leq D(r, s)D(r + 1, t),$$ 

or equivalently, 

$$\frac{D(r, t)}{D(r + 1, t)} \text{ is increasing in } t \in (0, \infty).$$ 

This is equivalent to the fact that 

$$\alpha_r(t) \overset{\text{def.}}{=} \begin{bmatrix} \tilde{F}(t), F(t) \\ r \quad m-r \end{bmatrix} \text{ is increasing in } t \in (0, \infty),$$ 

where $f(t), F(t)$ and $\tilde{F}(t)$ are the column vectors $(f_{Y_1}(t), f_{Y_2}(t), \ldots, f_{Y_m}(t))'$, $(F_{Y_1}(t), F_{Y_2}(t), \ldots, F_{Y_m}(t))'$ and $(\tilde{F}_{Y_1}(t), \tilde{F}_{Y_2}(t), \ldots, \tilde{F}_{Y_m}(t))'$, respectively. Because the permanent is a multilinear function of its columns, $\alpha_r(t)$ can be differentiated by taking the derivative of one column at a time, keeping the rest fixed, and then adding up the permanent of the resulting matrices. Thus, we have 

$$\alpha'_r(t) = \Delta'_1(t) + \Delta'_2(t),$$ 

where 

$$\Delta'_1(t) = (r + 1) \begin{bmatrix} \tilde{F}(t), F(t) \\ r \quad m-r \end{bmatrix} \begin{bmatrix} \tilde{F}(t), F(t), f(t) \\ r \quad m-r-1 \end{bmatrix} - r \begin{bmatrix} \tilde{F}(t), F(t) \\ r+1 \quad m-r \end{bmatrix} \begin{bmatrix} \tilde{F}(t), F(t), f(t) \\ r+1 \quad m-r \end{bmatrix},$$ 

and 

$$\Delta'_2(t) = (m - r) \begin{bmatrix} \tilde{F}(t), F(t) \\ r+1 \quad m-r \end{bmatrix} \begin{bmatrix} \tilde{F}(t), F(t), f(t) \\ r \quad m-r-1 \end{bmatrix}$$ 

$$- (m - r - 1) \begin{bmatrix} F(t), F(t) \\ r \quad m-r \end{bmatrix} \begin{bmatrix} \tilde{F}(t), F(t), f(t) \\ r+1 \quad m-r-2 \end{bmatrix}. $$

By taking $z = \tilde{F}(t), b = F(t) = c$ and $d = f(t)$, we have, from Lemma 2.3, $\Delta'_1(t) \geq 0$. Further, by taking $z = F(t), b = \tilde{F}(t) = d$ and $c = f(t)$, we have, from Lemma 2.3, $\Delta'_2(t) \geq 0$. Hence $\alpha_r(t)$ is increasing in $t$. 

The following lemma will be used in the next theorem. The idea of the proof is due to Karlin [8].
Lemma 2.5 Let \( \kappa(x, y) > 0 \), defined on \( \mathcal{X} \times \mathcal{Y} \) be RR\(_2\), where \( \mathcal{X} \) and \( \mathcal{Y} \) are subsets of real line. Assume that a function \( f(\cdot, \cdot) \) defined on \( \mathcal{X} \times \mathcal{Y} \) is such that

(i) for each \( x \), \( f(x, y) \) changes sign at most once, and if the change of sign does occur, it is from positive to negative, as \( y \) traverses \( \mathcal{Y} \);

(ii) for each \( y \), \( f(x, y) \) is increasing in \( x \);

(iii) \( \omega(x) = \int_{\mathcal{Y}} \kappa(x, y)f(x, y)d\mu(y) \) exists absolutely and defines a continuous function of \( x \), where \( \mu \) is a sigma-finite measure.

Then \( \omega(x) \) changes sign at most once, and if the change of sign does occur, it is from negative to positive.

Proof: Let \( x_0 \) be a point where \( \omega(x) \) changes its sign, as \( x \) traverses \( \mathcal{X} \). Then to prove the result, it suffices to show that \( \omega(x) \geq 0 \) for all \( x > x_0 \). Since \( \omega(x_0) = 0 \), corresponding to \( x_0 \), there exists a point \( y_0 \) such that \( f(x_0, y) \leq 0 \) for all \( y > y_0 \) and \( f(x_0, y) \geq 0 \) for all \( y < y_0 \). The existence of such a \( y_0 \) is guaranteed because of the assumption that \( \omega(x_0) = 0 \) and (i) above. Write

\[
\frac{\omega(x)}{\kappa(x, y_0)} = \frac{\omega(x)}{\kappa(x, y_0)} - \frac{\omega(x_0)}{\kappa(x_0, y_0)} = \int_{\mathcal{Y}} \left[ \frac{\kappa(x, y) - \kappa(x_0, y)}{\kappa(x, y_0) - \kappa(x_0, y_0)} \right] f(x_0, y)d\mu(y) + \int_{\mathcal{Y}} \left[ f(x, y) - f(x_0, y) \right] \frac{\kappa(x, y)}{\kappa(x_0, y_0)} d\mu(y).
\]

Consider the following two cases.

Case I: Let \( x > x_0 \) and \( y > y_0 \). Then the first integral is positive because \( \kappa(x, y) \) is RR\(_2\) and \( f(x_0, y) \leq 0 \) for all \( y > y_0 \). Further, the second integral is positive because of (ii). Thus \( \omega(x) \geq 0 \).

Case II: Let \( x > x_0 \) and \( y < y_0 \). Then the first integral is positive because \( \kappa(x, y) \) is RR\(_2\) and \( f(x_0, y) \geq 0 \) for all \( y < y_0 \). Further, the second integral is positive because of (ii). Thus \( \omega(x) \geq 0 \). Hence the result is proved.

In the next theorem we show that the result discussed in Theorem 2.1 also holds for the likelihood ratio order.

Theorem 2.4 Let \( X = (X_1, X_2, \ldots, X_n) \) and \( X^* = (X_1^*, X_2^*, \ldots, X_n^*) \) be the lifetimes of the two groups of active components, and \( Y = (Y_1, Y_2, \ldots, Y_m) \) be those of a group of standby components. Further assume that all \( X_i, X_i^* \) and \( Y_i \) are independent. If \( X_i \leq_{lr} X_j^* \) for all \( i, j = 1, 2, \ldots, n \), then \( N_k(X^*, Y) \leq_{lr} N_k(X, Y) \).
Proof: Note that, for \( r = 0, 1, \ldots, m, \)
\[
P(N_k(X, Y) = r) = \int_0^\infty P(N_k(X, Y) = r | X_{k,n} = t) dF_{X_{k,n}}(t)
\]
\[
= \int_0^\infty P\left( \sum_{j=1}^m I_{Y_j > t} = r \right) dF_{X_{k,n}}(t)
\]
\[
= \int_0^\infty D(r, t) dF_{X_{k,n}}(t),
\]
where \( D(r, t) \) is as defined in Lemma 2.4. Similarly,
\[
P(N_k(X^*, Y) = r) = \int_0^\infty D(r, t) dF_{X^*_{k,n}}(t).
\]

Let \( v \) be any real number. Consider the relation
\[
P(N_k(X, Y) = r) - vP(N_k(X^*, Y) = r) = \int_0^\infty D(r, t) \left( f_{X_{k,n}}(t) - vf_{X^*_{k,n}}(t) \right) dt.
\]

Since \( X_i \leq_{lr} X_j^* \) for all \( i, j = 1, 2, \ldots, n \), by Theorem 1.C.33 of Shaked and Shanthikumar, we have \( X_{k,n} \leq_{lr} X^*_{k,n} \), which gives that
\[
\frac{f_{X_{k,n}}(t)}{f_{X^*_{k,n}}(t)} \text{ is decreasing in } t.
\]
Thus \( f_{X_{k,n}}(t) - vf_{X^*_{k,n}}(t) \) changes sign at most once, and if the change of sign does occur, it is from positive to negative, as \( t \) goes from 0 to \( \infty \). Further, by Lemma 2.4 we have that
\[
D(r, t) \text{ is RR}_2 \text{ in } (r, t).
\]
Therefore, on using Lemma 2.5 we have that \( P(N_k(X, Y) = r) - vP(N_k(X^*, Y) = r) \) changes sign at most once, and if the change of sign does occur, it is from negative to positive, as \( r \) goes from 0 to \( m \). Thus, \( P(N_k(X, Y) = r)/P(N_k(X^*, Y) = r) \) is increasing in \( r \), and hence \( N_k(X^*, Y) \leq_{lr} N_k(X, Y) \).

\( \square \)

Remark 2.3 Counterexample 2.1 can be used to show that the condition \( X_i \leq_{lr} X_j^* \) for all \( i, j \), given in Theorem 2.4 cannot be removed.
3 Comparison Based on Two Sets of Standby Components

Here we consider two different batches of warm standby components instead of two different batches of active components. Below we show that the total number of surviving warm standby components at the time of \( k \)th failure of the active component of a system is less than that of another system with respect to the usual stochastic order, provided the standby components of one batch is smaller than those of the other batch with respect to the usual stochastic order.

**Theorem 3.1** Let \( X = (X_1, X_2, \ldots, X_n) \) be the lifetimes of a group of active components, and \( Y = (Y_1, Y_2, \ldots, Y_m) \) and \( Y^* = (Y_1^*, Y_2^*, \ldots, Y_n^*) \) be those of two groups of standby components. Further assume that all \( X_i, Y_i \) and \( Y_i^* \) are independent. If \( Y_i \leq_{st} Y_i^* \) for all \( i = 1, 2, \ldots, n \), then \( N_k(X, Y) \leq_{st} N_k(X, Y^*) \).

**Proof:** From (2.1) we have

\[
\bar{F}_{N_k(X,Y)}(r) = \int_0^\infty \bar{F}_{Y_{m-r,m}}(t) dF_{X_{k,n}}(t)
\]

and

\[
\bar{F}_{N_k(X,Y^*)}(r) = \int_0^\infty \bar{F}_{Y_{m-r,m}^*}(t) dF_{X_{k,n}}(t).
\]

Because, \( Y_i \leq_{st} Y_i^* \) for all \( i = 1, 2, \ldots, n \), by Corollary 3.2 of Belzunce et al. [3], we have \( \bar{F}_{Y_{m-r,m}}(t) \leq \bar{F}_{Y_{m-r,m}^*}(t) \) for all \( t \in (0, \infty) \). Hence \( N_k(X, Y) \leq_{st} N_k(X, Y^*) \). \( \square \)

The following counterexample shows that the condition \( Y_i \leq_{st} Y_i^* \) given in the above theorem cannot be relaxed.

**Counterexample 3.1** Let \( X = (X_1, X_2, X_3) \) be the lifetimes of a group of active components with hazard rates \((4, 5, 6)\). Further, let \( Y = (Y_1, Y_2, Y_3) \) and \( Y^* = (Y_1^*, Y_2^*, Y_3^*) \) be those of two groups of standby components with hazard rates \((1, 3, 4)\) and \((5, 2, 6)\), respectively. Assume that all \( X_i, Y_i \) and \( Y_i^* \) are independent. Clearly, \( Y_1 \geq_{st} Y_1^* \), \( Y_2 \leq_{st} Y_2^* \) and \( Y_3 \geq_{st} Y_3^* \). Thus, \( Y_i \leq_{st} Y_i^* \) does not hold for all \( i = 1, 2, 3 \). Now,

\[
\bar{F}_{N_2(X,Y)}(1) = \int_0^\infty \left( e^{-4t} + e^{-7t} + e^{-5t} - 2e^{-8t} \right) \left( 9e^{-9t} + 11e^{-11t} + 10e^{-10t} - 30e^{-15t} \right) dt
\]

\[
= 0.737
\]
and

\[
\tilde{F}_{N_2(X,Y^\ast)}(1) = \int_0^\infty (e^{-7t} + e^{-8t} + e^{-11t} - 2e^{-13t}) (9e^{-9t} + 11e^{-11t} + 10e^{-10t} - 30e^{-15t}) \, dt
\]

= 0.529.

Thus, \( \tilde{F}_{N_2(X,Y)}(1) \geq \tilde{F}_{N_2(X,Y^\ast)}(1) \), and hence \( N_2(X,Y) \not\leq_{st} N_2(X,Y^\ast) \).

4 Conclusion

When any component of a system, upon failure, is replaced by a warm standby component is called warm standby system. Clearly, the number of warm standby components available for use at the time of \( k \)th component failure (for any fixed number \( k \)) is a random variable. In this article we compare such random variables for two warm standby systems. Two separate cases have been studied in this paper — (i) Two systems having a single set of warm standby components, (ii) Single system having two separate warm standby components. The comparison for the first case is done with respect to usual stochastic order, hazard rate order, reversed hazard rate order and likelihood ratio order, whereas for the second case, the comparison is done only for usual stochastic order. In all the above cases under (i) it is observed that the number of warm standby components is smaller corresponding to the system having stronger active components, whereas in case of (ii), this number is smaller when the warm standby components are weaker.

Acknowledgements

Financial support from Council of Scientific and Industrial Research, New Delhi (Grant No. 09/921(0060)2011-EMR-I) is sincerely acknowledged by Nil Kamal Hazra.

References

[1] Balakrishnan, N. (2007). Permanents, order statistics, outliers, and robustness. Revista Matemática Complutense, 20, 7-107.

[2] Bapat, R.B. (1993). Symmetric function means and permanents. Linear Algebra and Its Applications, 182, 101-108.
[3] Belzunce, F., Franco, M., Ruiz, J.M. and Ruiz, M.C. (2001). On partial orderings between coherent systems with different structures. *Probability in the Engineering and Informational Sciences, 15*, 273-293.

[4] Capéra, P. (1988). Tail ordering and asymptotic efficiency of rank tests. *The Annals of Statistics, 16*, 470-478.

[5] Cha, J.H., Mi, J. and Yun, W.Y. (2008). Modelling a general standby system and evaluation of its performance. *Applied Stochastic Models in Business and Industry, 24*, 159-169.

[6] Eryilmaz, S. (2013). Reliability of a $k$-out-of-$n$ system equipped with a single warm standby component. *IEEE Transactions on Reliability, 62*, 499-503.

[7] Hazra, N.K. and Nanda, A.K. (2014). General standby component allocation in series and parallel systems. [arXiv:1401.0132v1](https://arxiv.org/abs/1401.0132).

[8] Karlin, S. (1968). *Total Positivity*. Stanford University Press, Stanford, California.

[9] Li, X., Yan, R. and Zuo, M. (2009). Evaluating a warm standby system with components having proportional hazard rates. *Operations Research Letters, 37*, 56-60.

[10] Minc, H. (1978). *Permanents, Encyclopedia of Mathematics and its Applications*. Addison-Wesley, Reading, Massachusetts.

[11] Shaked, M. and Shanthikumar, J.G. (2007). *Stochastic Orders*. Springer, New York.