The analysis of the dynamics of the electorate system by using q-distribution-a case study.

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In this work we study the system of the votes, the mechanism of the electoral support formation, and also the elements of its dynamics, by analyzing the data from several election processes in Albania. Firstly, we evidence the specific features and the characteristics of the distributions of votes through a descriptive approach, and next we use those findings to identify the nature of the elementary processes of the agreement, the defects of the system and dynamical issues. The distributions of the votes for the majority or majority-like election as by polling stations reference results a two-parts function. The part of the distribution located in the small vote fraction fits to a power law or to a q-exponential function, therefore the foremost factor of the electoral support for the subjects populating this zone is based in the preferential attachment rule, with some modification. Consequently, the small subjects or independent candidates, realize their electoral attractiveness based on the individual performance. Also, their voters act rationally and usually gather sufficient information before deciding to support them. The bell-shaped part of the distribution which describes the votes of the candidates of the main parties, fits better to the q-gaussian functions. In this case, electoral support is affected strongly by the political activists (militants) which harvest local influences to convict people producing an extra support for the candidates of big parties, regardless of their performance and electoral values. This physiognomy is characteristic for all legislative and administrative majority voting or other majority-like elections as practically behave the closed-lists elections of 2009, 2013, 2017 and also the semi-opened list of the 2021. The distributions of the closed-list votes in the administrative elections are mostly of the exponential or q-exponential type. Also, the distributions based on the data from electoral constituencies which include many polling stations resulted q-exponentials for all types of elections. We connected the q-exponential form of the distribution with the electoral network failures, system deficiencies and heterogeneity effects. In 2021, the distributions of the votes for subjects is obtained similar to the typical recent majority voting distribution, a mix of the power law and q-gaussian functions. The distribution of the votes for the candidates on the semi-open list for those elections resulted a mix of two q-exponentials. We associated this last with the difficulties of the voters to understand new electoral rules and additional other causes of the non-electoral nature. Also, the electorate network might have suffered extra irregularity issues due to the inadequate sizes of elections units, etc. The distributions of the votes for the two main parties are found q-gaussians with $q \sim 1.32$ and $q \sim 1.57$ for the right and the left wing respectively. Based on the non-stationarity level measured by the q-value, significant redistribution events are expected for the left-wing network, whereas the right-wing network would experience...
fewer changes in *ceteris paribus* socio-electoral conditions. Interestingly, the mix of the votes for two main political parties has produced a q-gaussian with q=1.004, and subsequently, the joint system is found in a more relaxed state. Therefore, the compound network including two main parties is likely to not undergo significant redistribution of the votes in the near future. This means that the small subjects or the fresh-born ones are not likely to cause changes on the system. Based on the deductions for electoral agreement formation, we used our recently introduced q-opinion approach to model the electoral opinion formation. In this model, the q-opinion produces an additional term that multiplies the modified preferential attachment probability for the link establishment. Herein, the q-parameter is calculated by using an ad-hoc formula involving the performance of the candidate as utility function, which associates the agreement behavior as the response, with the candidate performance as the offer or the cause factor. The quantity q henceforth acts as activation-inhibition switch of the extra utility involved in the q-opinion model, and particularly it provides a nonzero voter’s support for the high-performance opponent candidates. The model has reproduced the distributions analyzed in this study. It resulted that many voters in this electorate system act rationally, despite their affiliations.

**Key word.** Distributions, socio-physics, opinion formation, election

1. Introduction

The voting processes is a very important social activity from the sociometric point of view, because of the large part of population involved in it, the specifics of the mechanisms conditioning the supportive agreement for a subject and also for the time dimension expressed by their periodicity. The electoral system is a typical opinion system with a network structure provided from the way or routes of the contacts, and the set of rules which define the agreement mechanism. The structure of the voting system has been considered as starting point for the mathematical voter’s model introduced by Liggett, which becomes very popular in the quantitative studies of the human interactions and social systems, as seen in [1],[2]. The physical extensions of the voter’s model have been numerous, interesting and successful, as detailed in references [3], [4] and others. From the social science’s point of view, the voting process is a decision-making process anticipated by the preference formation events in a given social environment. So, the conceptual entity encompassing the social electoral network and the rules (mechanisms) of the agreement between the constitutive opinion nodes, can be considered methodologically as a unique object which we will call herein “electorate system”. The electorate system is a particular social network in the network science terminology. For similar system, the process of the link establishment has been analyzed thoroughly [4], [5],[8], [11] etc. Typical physical phenomenon as the condensation and clustering, the segregation, the phase transition, critical behavior etc., have been also analyzed for those systems in [5], [8] [16] etc. In particular, the identification of the distribution on social networks can help to associate the process outcomes with their causes, and vice versa, quite similarly with the statistical physics approach. Physically, the opinion node (i) attracts another node with some specific “force” based on a given utility of the link $u_{ij}$ which finally concludes in the connecting probability between the nodes. So, in the preferential attachment model, the attractiveness of the nodes (i) exerted toward the incoming node (j) is proportional to the actual grade of the nodes, and therefore the linking probability is $p(i,j) \sim k_i$ where $k_i$ is the grade of the node (the number of the links). Under this rule, the distribution of the nodes with k links is a power law (PL) $p(k) \sim k^{-\gamma}$ with exponent $\gamma \sim 3, -2$, see [4], [11] etc. The power law distributions are so common in the human-activity based systems. For election case, the PL distribution has been confirmed firstly in the reference [6]. In the paper [5], the emergence of PL has been argued as result of multiplicative processes that initially give rise to a lognormal distribution, which for large variance, practically is transformed in a power law. The discussions in the references [5] and [11] and other consideration, reveal the necessity for the complexity
approach on the study social networks and their characteristic distributions, but also the descriptive analysis can extract important information from them. Practically, the study of the distributions in this framework would bring together physical methods, mathematics modeling and sociometric interpretation, completing a big part of the system’s depiction. In the reference [7] the formation of the political agreement is approximated with magnetic orientation due to the local field created by militants (zealots), and consequently a Gaussian distribution emerges. Theoretical analysis based on the network theory presented in the references [8], [11], [12] etc., includes also exponentials to describe the distributions for human behavior-based systems. Next, based on the generalized q-entropy consideration, the q-functions appear as interesting distribution functions for the (complex) systems based on the human activity, see [9], [10] and references therein.

Note that from the physical point of view, a specific distribution as a macro-scale object is related directly with the mechanism of the interaction in the micro scale level. By using the information collected from the descriptive analysis of the distributions, one can proceed with identification of the interactions between opinion nodes. This last deduction can be tested by a backward analysis which starts from a model based on the presumed interactions followed by the check of the distribution emerging from the links established by the proposed mechanisms. We will this idea in the closing part of the study.

2. The q-functions as candidate distribution for human activity-based systems
An interesting group of the distributions based on the q-functions has become popular last decades [14],[15], [21] etc. They are particularly important to be used for the out-of-equilibrium systems. Mathematically, the q-functions category includes a large set of parametrical forms encompassed in the hypergeometric functions, see [21], [22]. In particular, the q-exponential functions have been considered as physical object in the role of the distributions arising from the q-entropy optimization as detailed in the references [9], [10], [14], [15] etc. Theoretical arguments have been provided in [10] by the q-extension of the central limit theorem called q-CLT clarifying the role of the q-Gaussians (a specific q-exponential case) as the attractor of the non-stationary distributions. Also, interesting conjectures with multifractal properties and non-equilibrium thermodynamics have been discussed in [15], [14] and the references therein. We will use the q-exponentials and q-gaussians in the succeeding analysis, so for the easiness of the readers we are showing herein a very short introduction of them. Consider the q-entropy that is the analogue of the classical one for the correlated microstates \( p_i \) in the canonical ensemble. For the discrete case it reads

\[
S_q = \frac{1}{q-1} \left(1 - \sum_{i=1}^{w} p_i^q \right)
\]

and the correspondent form for the continuous case is

\[
S_q = \frac{1}{q-1} \left(1 - \int_0^\infty \rho^q dx \right)
\]

as proposed by C. Tsallis, see [9]. According to [9], [14], [15] etc., the nonadditive q-entropy (1) or (2) is related with the nonlinearity, the correlation of the microstates, and non-equilibrium processes. By imposing the q-mean \( \mu_q = \int_0^\infty \rho^q dx \) as constraint, the optimization of the q-entropy results in the q-exponential distribution of the general form

\[
\rho(x) = \frac{2-q}{\mu_q} \left(1 - (1-q) \frac{x}{\mu_q} \right)^\frac{1}{1-q}
\]

where the subscript + denotes “positive argument”. Next if the q-variance is taken as constraint, we obtain the q-Gaussian which has the general pdf formula
\[ G_q(x) = \frac{1}{\sqrt{1-q}} \frac{\sigma_q}{Z_q} \left( 1 - (1-q) \left( \frac{x-\mu_q}{2\sigma_q} \right)^2 \right)^{\frac{1}{1-q}} \]  
(4)

where the partition function \( Z_q \) is given by following forms

\[ Z_q = \begin{cases} 
\frac{\pi}{\sqrt{1-q}} \frac{r^{(2-q)}}{r^{(1-q)}} & q < 1 \\
\sqrt{\pi} q = 1 \\
\frac{\pi}{q-1} \frac{r^{(2-q)}}{r^{(1-q)}} & 1 < q < 3 
\end{cases} \]  
(5)

The q-mean \( \mu_q \), q-variance \( \sigma_q \) appearing in (4) recover the classical values in the limit \( q \to 1 \), where also the q-Gaussian converges to the classical gaussian. The q-parameter in (4) is known as the \( q_{\text{entropy}} \) or the \( q_{\text{statistical}} \) Tsallis’ index and the difference \( q - 1 \) measures the distance of the state under scrutiny from the stationarity. It is known also as the complexity parameter. Notice that there are other important q-parameters in this formalism, as the one that measures the sensitivity to the initial conditions (the \( q_{\text{sensitive}} \) or the rate of relaxation of the state toward the stationary state (the \( q_{\text{relaxation}} \)), which are practically inseparable to the \( q_{\text{entropy}} \) in the q-analysis. It is worth to mention that also, when dealing with those objects we should make use of q-algebra where algebraic operations and also q-functions are defined specifically and straightforwardly see [9], [10] etc., for details. In the following we will limit our analysis in the stationarity issues, hence only \( q_{\text{entropy}} \) would be considered. So, if \( q \)-value in (4) has the value \( 1 < q < \frac{5}{3} \) the q-distribution is stationary and the variance is finite given by \( \sigma_q = \frac{1}{\beta(5-3q)} \) for \( \frac{5}{3} < q < 2 \) the variance is infinite, the mean is finite and the distribution is un-stationary. For \( 2 < q < 3 \) the mean and the variance are undefined, and for \( q > 3 \) there is no distribution object related with the q-gaussian (4).

3. The empiric analyses of the electorate system

Herein we will start with descriptive analyses based on the thorough investigation of the distributions of the votes. The distribution object is identified straightforwardly. For every election units (s) we read the votes gathered by each subject as the initial variable \( v_{j,s} \) so we have the initial statistics. If we consider each voting units as an entity of the ensemble, we should normalize the variable to minimize the effect of the heterogenous size of election units on the numbers \( v_{j,s} \), so the new variable is \( V_{j,s} = \frac{v_{j,s}}{\sum v_{j,s}} \), which varies from 0 to 1. Now, as a routine of the distribution analysis we ensue with discretization or binning process by introducing the frequencies \( f \) of the events \"the subject (i) has gathered \( v_i \) votes.\" The support of the fraction of votes is partitioned in N classes \( N = \frac{1}{h} \) where \( h \) is the bin width which has to be optimized by an appropriate mathematical procedure (say Scott, Freedman-Diaconic etc.). The discrete support now is \( \{ V_i; (n-1)h \leq V_i < nh \} \) so we evaluate the discrete frequencies \( f_n = f(V_n) \) as the number of subjects that have their fraction of the vote in the \( n^{th} \) interval. Finally, the densities of the occurrences \( V_n \) are \( \rho(V_n) = \frac{f_n}{N_{\text{subjects}}} \). The Polling Station (the voting centers-VC), or electoral zone (EC-the election constituency) can be considered as elements of the ensemble because the campaign is organized and realized based on those electoral structures. Note that VC have been kept unchanged for successive election held in the country and therefore the candidates and subjects apply door to door campaign based on the voting centers strategy. From the other side, the election constituencies (EC) are considered also as physical reference because the
electoral outcome (the mandate) comes from the votes assembled on their basis. They constitute social networks in this view. Notice that after constating the non-bell form of the empirical distribution, we used the Freedman-Djacoic rule for the optimal bin-width in the discretization process mentioned above. Note also that q-functions are very sensitive to the parameter’s changes, so we employed a step by step procedure of the fitting starting from the nonlinear least square fit. We tried the q-gaussian items in the full analytical form (4) and (5), and if the fit was not appropriate, we validated the less restrictive empirical form $G_{q(x)} = a\left(1 - (1 - q)\left(\frac{x-m}{b}\right)\right)^{\frac{1}{1-q}}$. In this last case, the fit has been qualified as only “statistically optimal”, because only the physical forms (4) and (5) provide the rigorous interpretation of the parameters. In both cases the q-parameter has its full theoretical meaning, and we used them to estimate the stationarity of the (electoral) network states represented by the distribution function fitted. Next, by the identification of the exact distribution we continue with the investigation of the guessed mechanism of the one by one interactions in the network. General analysis provided in [11], [8], etc., has shown that the mechanisms governing the link establishment in a social network are complex, and therefore the distribution are expected to be complicated functions. Note that by our recent evidences we have observed this property expressed in the multiparty distributions. We associate the multiparty nature of the distributions in our system with mixed types of the mechanisms responsible for the formation of the political support. Next, according to the arguments of the reference [5], the PL distribution is the high-variance limit of the lognormal shape which itself echoes the multiplicativity of the elementary processes. Henceforth, from the PL evidences as part of the composite distributions we assumed that the additivity and multiplicativity coexist in our system, and therefore a more generalized distribution based on the q-functions (3) or (4) should be used for the fit.

3.1. Review of the distributions based on small electoral network-the Polling Stations reference

In the parliamentary election in 2001 and 2005, 100 seats were elected directly by the majority voting whereas 40 other seats have been completed from the results in the proportional voting by a formula that emphasizes the proportional nature of the system, but in reality, has opened the doors for tactical voting and other irregularities [13]. However, the majority and proportional voting components have been clearly distinguishable for those elections. For the majority voting and PS reference we obtained a multiparty distribution, Figure 1. It has a power law nature in the low fractions <12% and a near to gaussian form for the fraction [12%.95%] succeeded by disturbances in the high values limit >0.95, see reference [13] and [23] for details. Those evidences indicate the presence of the two separate subnetworks, characterized by different electoral agreement mechanisms, say, the preferential attachment mechanism responsible of the PL part of the distribution and the zealotry effects or local magnetism created by the majority rule responsible for the gaussian-like distribution. By direct evidences we observed that the small subjects usually own small fraction of votes, and thereby they gathered the votes based on the preferential attachment mechanisms. The big parties and their candidates are the owners of the high fraction of the votes, contributing for the gaussian-like part of the distribution. By employing arguments of literature [7] and by also by proposing some extension, we assumed that the big electoral subjects’ militants (zealot) impose permanent attractiveness on the voter’s local environment, which acts like a local field in a magnetic medium. Next, the distributions related to the proportional voting result in deformed exponentials. After 2009, the legislative voting has become fully proportional. Now, the distribution of legislative proportional votes becomes a two-part function, the same as the one observed in the previous majority voting. The distributions for the proportional administrative voting are found mixed q-exponentials for all historical references. Those findings confirm the fact that the distributions are usually not-unique functions and the non-stationarity of the corresponding state. For illustration purpose we have presented in the figure 2, the graphs of the distributions for some districts in the administrative election of 2015, where the bipartite q-
exponential form is evident. The characteristic q-exponential form indicates the mixed nature of effects as the heterogeneities and defects, the correlation of the microstate or other nonlinear properties.

\[ \text{Figure 1.} \text{ The two-part distribution for the majority and majority-like voting (as by logarithmic presentations). Left picture: the distributions according to the Polling Station results. 2005 (black curves), 2009 (red line), 2013 (blue line); Right picture: The distribution of votes by Election Zone (more than 100 PS comprised). 2001(blue line), 2005(red line), 2009(black line), 2013(green line)} \]

In this context, the electoral support in the administrative voting comprises various electoral mechanisms and also non-electoral elements as the tactical for example.

\[ \text{Figure 2.} \text{ Illustration of the not unique form of the distribution. The proportional part of the votes in administrative elections of 2015, for some constituencies (districts). The solid lines represent the sum of the two q-exponentials} \]

It worked this way: The members of an electoral coalition have agreed to accumulate all the votes in the support of the candidate for mayor which usually is a big-party nominee. In exchange, the voters of the big parties are oriented to support the small allies’ list, but it happens that those votes overcome the votes they gave for the mayor. This mechanism redistributed disproportionally the proportional votes effectuating a mechanical increase of the votes for small parties. Other effects could be present also, but again, there are not product of electoral attractiveness and contribute in the complex nature of the final distribution. Next,
referring to the results of the 2013 for particular subjects we have identified q-gaussians for each political wing votes. Based on the q-parameters we have estimated the incoming behavior for the subnetworks. So, for the right-wing subjects we obtained $q \approx 1.47$ and for the left we obtained $q \approx 1.3$, [23]. Noting this finding under the q-analysis framework, for the next elections (say at 2017) the right-wing network would have undergone intensive dynamics in its pathway toward relaxation, in ceteris paribus conditions. It happened really that the right wing lost those elections as result of an unprecedent displacement of the votes, which confirmed its network changes and the vote redistribution, as predicted. Just to mention, we realized good prediction for the outcomes of proportional votes by employing the q-entropy features for subsystems represented by partial distributions (illustrated in the figure 2), [23], which are not reproducing herein.

3.2. Some specific features for the political wings’ votes in the elections of 2021

For the elections of the 2021, there has been enacted a new election rule that allows preferential votes for the candidates. There were 45 political parties in the race, but in the ballot paper, there appeared 13 in total, because many of subjects have been grouped in four electoral coalitions. However, the ballot paper permitted the identification of the candidates’ affiliation, assuring the opportunity for the voters to address the preferential vote. The voters list for a voting station usually has 500-1000 voters, whereas the election constituency contained 50000-15000 voters with one exception, the Tirana district, that has more than 500000 voters. Hereto, we analyzed the distributions for votes of the electoral subjects and their candidates referring to the results by PS and EC references. Now, the distributions of the votes of the electoral subjects (political parties) as by PS reference has fitted well with q-gaussians whereas the distribution of votes of the candidates has fitted better to the q-exponentials. This reversed behavior compared with other elections’ results is the first interesting particularity evidenced. We observe that the distribution of the votes of the two main parties differs from the distribution of the votes of the small subjects. Note that the application of the analytic version of q-functions (3) and (4) would warrant the non-occasional fitting, but due to the relatively small number of points in some references, the form (6) is used instead in many cases.

![Figure 3. Log-Log view of the q-gaussians fitted to the empirical distributions for two leading parties. Election of 2021, reference units are Polling Stations. The fit is realized by a routine based on the NLLS MATLAB](image)

The best fitted distributions for the votes of the left and right wings are q-gaussians with q-values at 1.32 and 1.57 respectively. Both systems are highly non-stationary as $q \gg 1$ therefore both networks are characterized by heterogeneities that push the distributions far from the stationarity. Also, the electoral results of the subject differ significantly from one polling station to another, indicating the presence of the
network size effects as the heterogeneity element. Interestingly, this behavior is observed for all election district, emphasizing that the heterogeneity has the origin in the polling station. This could be related to the size effect of PS, the interferences and overlapping of the subjects and list performances, but also the network failures and non-electoral factors are likely to be contributors too. The left-wing network system is found in the edge of the infinite variance zone \( (q > \frac{5}{3}) \), given that \( q=1.57 \) has a non-neglectable uncertainty. Nevertheless, \( q < \frac{5}{3} \) in both cases and hence, in \( \text{ceteris paribus} \) conditions, those networks are expected to undergo the relaxation dynamics. This process would redistribute the votes among different election zones toward a gaussian shape. Therefore, the electorate system of the left-wing party is expected to be more dynamical compared with the right-wing’s system. Another issue is the examination of the deviances in the support. To assess the homogeneity of the fit we used the ‘relative distance’ \( 2 \frac{\sqrt{\text{fitted} - \text{data}}}{\sqrt{\text{fitted} + \text{data}}} \).

So far, the fit is better in the intermediate range \([0.25, 0.7]\) in both cases, and the differences from the fitted curves become considerable in the edges, Figure 3. It resulted that for the major part of the votes (encompassed in the interval \([0.25, 0.7]\)) the dominant mechanism of electoral support for the main political subject (parties) is the activity of the militants (zealot) and the political attractions (a quasi-field) effect on the electoral body, responsible for the near to-gaussian shape. However, the observed deviations from the q-gaussians evidenced that those effects are not overriding for all the voting body (in the full support interval). In the low fraction’s zones, a portion of the votes escaped the strong political fields. For the right-wing, the deviations from the q-gaussian are higher in the high fractions’ zone whereas the points of empirical distribution for the left-wing deviate from the q-gaussian in both extremes. The extremal points are usually located out of a whatsoever smooth curve for both cases, regardless of our fit, so we associate them with disturbances or irregularities in the corresponding networks. For the right-wing, the one side departure of the left-side points (small fractions) from the q-Gaussian indicates “homogenous” irregularities, which can be related to the split-off vote’s effect due to the daughter party’s competition, abnormal low performance in particular zones etc. For the left-wing, this behavior is observed in right hand side of the graph. Note that in this case, the overall fit with the q-gaussian is improved significantly and the also the q-value decreases if we exclude the two first points of the distribution (low fractions). Therefore, the high deviations from the q-gaussian for high fractions in this case are not substantial. They seem to be result of the cumulative vote of all leftist groups that have been dissolved in the main left-wing party. Other irregularities also may be present but not discoverable by the analysis of the distributions separately. It is interesting to remark that the deviations are asymmetric in their own, but they look compensatory in a conjointly consideration. Henceforth, we obtained a relaxed q-gaussian distribution with \( q \approx 1.004 \) when considering the votes of both wings together. It indicates that the main parties’ voters are well-fitted in the conjoint network, and in principle there are no expectations for significant development in it. In some extend these features suggests that competitors as small or new-born parties are not risky for the main block because the relaxed steady state won’t be exposed to the imperative redistribution of the votes. In particular, the inward or outward flows of the votes for this system are not likely to occur in the \( \text{ceteris paribus} \) (c.p.) conditions. Henceforth, the main parties may exchange votes among each other, and probably they will do, but no need to open the “envelope” which is closed in the relaxed q-gaussian. Note that c.p., herein means “no wide-ranging changes” in the actual political configuration, which might not be the case in the future.

3.3. The two-part distribution for electoral subjects and candidates for the election of 2021

The distribution of votes for all parties by the polling station reference looks a scoop-down shape in the logarithmic presentation. It is quite similar with the one obtained for the majority voting discussed above. The low fractions of votes belong to the small parties, quite similarly with the distribution for candidates in the elections of 2001 and 2005 analysed in [13]. The distribution of the votes for small electoral subjects is
PL-shape in the full support interval, without the first point. The votes of the all not small subjects contribute for the q-gaussian part of the bipartite distribution, figure 4. If taking the separation point at 15%, the q-gaussian fitted to this part has the q values at around 1.9-2.1 which decrease to 1.79 when we refined the FD histogram optimization by excluding the first point (near to zero fractions). Considering the distribution of the conjoint main party’s votes, it resulted that the effect of the middle-size subject’s votes is significantly perturbative, bringing the q-values from 1.004 to 1.79. It reveals the realistic contribution of the parties of the intermediate size in the system. Their role may break the c.p., conditions mentioned above. Next, we observed that the distributions are essentially multiparty functions that cannot be fitted by any unique distribution function. Though, an empiric fit can be made with the function \( f(V) \sim \exp_q(V - V_0)^3 \). Both findings indicate that the state of the electorate system is highly nonstationary. By a refined view of the distribution portion for the small parties (the apparently PL branch) we obtained that a q-exponential curve fitted better. It is not contradictory with PL behavior, if we remind the PL limit-nature of the q-exponentials. This new fit offers also the q-measure from the attractor of the distribution. We observe also that the small parties’ fraction of votes is noisy in the right-side limit, suggesting that the separation point with the q-gaussian distribution is not clearly defined. The relaxation processes could push it in the right if the PL nature would be accented or pull it at the left if the (q-) exponential nature would be promoted by the dynamical processes occurring in the system. Again, the deviations in this branch can be related to the subnetwork heterogeneities, and the exponential nature is interrelated with defects on the corresponding network. Finally, it’s worth to have a look on the distribution of votes for the candidates.

![Distribution for all electoral subjects](image)

For the PS reference, the densities of the fraction of votes do not converge to a distribution function object. This property remains for all districts and also for the political-wings grouped candidates. Due to the big number of candidates, the corresponding network (PS) is considered very young (or alternatively, the size of the voting center is too small), therefore this is a normal behavior. The distribution of the votes for the election constituencies’ reference, is q-exponential with a noisy queue. The q-exponential nature of the distribution indicates the presence of the multiple factors and the failures’ effect on the electoral candidates’ subnetworks. We qualify as possible causes the voter’ perception of the list composition, the disproportional campaign of candidates by their ranks in the lists, due to the semi-closed nature of the lists. Also, the implementation of the semi-opened proportional system probably has not been assimilated by the voters. It seemed also, that the individual efforts of a given candidate to attract voters by the preferential attachment rule [resulting in power law] has triggered the attention of group of voters which behave as un-decided or un-loyal, causing failures on the links establishment.

**Figure 4.** The double regime identified from the two-part distribution of the votes for all electoral subjects. Election of 2021, Albania. The calculation unit references are Polling Station. Bin optimised by Freedman-Diaconic rule. The fit realized by an ad hoc routine based on NLLS, MATLAB.
Other factors can be contributors for the departure from the expected PL shape, too.

\[ \text{Figure 5.} \] The Distribution of the votes for candidates according to the Elections Constituencies results. Details are shown in the legend. Election of 2021, Albania

3.4. An extra view for the spatial heterogeneity for the results of the electoral subjects

Here we have checked that the non-stationarity feature which was verified by the \( q \)-parameter in the discussion above is related also with the disproportion of the shares of votes among classes (fractions) in different election districts. It can be revealed by using the concept of dis-equality embedded in the Lorenz curve and largely used in econometrics. With careful extension, it can be used herein also. The “pay-off electoral response” in the vote shares can be estimated mathematically by the Gini coefficient similarly with econometrics applications, despite of its original definition. In this case we considered the proportion of votes gathered by a given subject in each electoral constituency as the variable to be analysed.

\[ \text{Figure 6.} \] The adopted Lorentz curves. Exhibition of the high heterogeneity among election counties for the elections of 2021. The data used are the votes of the electoral subjects.

We observed that Gini indicator changes significantly from one district to the other ranging from 0.45 to 0.94. This fact supports the scrutiny that the most contributing factor for the typical high non-stationarity of the output electoral states is related to the significant differences among districts. The heterogeneity related to the size of the electoral zone and sub-zones may affect the result in some degree, and there are also claiming on this issue based on the political consideration. From the scientific point of view, those claims
are not baseless, and in some extent, the distortions in the final result can be related to the un-appropriate size of the election constituencies. The Lorentz curves for each electoral constituency shown in the Figure 6 displays the significance of this heterogeneity. In the socio-political framework, those results advocate for a better organization of the electoral zones to diminish the artificial and mechanical distortions observed.

4. The q-opinion modelling for the electorate system

Based on the analysis of the q-distributions presented above, we can go further in the modelling of the electorate system. Putting aside for a while the presence of the network failure and heterogeneities, we can admit that electoral support depends appropriately on the individual preferences, militancy, subject performance, mouth-to-mouth discussing etc. We observe that for small network or young ones (PS reference), our old formula proposed in [13] works and the two-part distribution is reproduced. The distributions based on the EC reference have a near to PL shape with a weak gaussian nodulation in its right edge. The distributions obtained for the votes of candidates in 2021 also reveal the perturbated PL shape with nodulation in its edge, figure 7. Therefore, we should reconsider the old model presented in [13] to meet more requirements. In the following subsection we will present this modification and comments.

![Figure 7](image)

**Figure 7.** The near to PL limit distribution by EC references. Left picture, the semi opened list of the candidates, 2021. Right frame, the distribution by EC references in the full proportional voting in legislative election of 2013

4.1. The modified preferential attachment model

The observations confirm that the distribution for majority-like voting based on the PS reference for all elections has two parts, a PL curve in the fractions up to 15%-20%, and a near to-gaussian shape for the remaining part. For the PL part of the distribution the exponent is found in the range [-1.7, -2.8], depending on the election considered. Therefore, for the subsystem related to this part, the mechanism of the political support is based on the preferential attachment rule. The probability for the establishment of the link between the voter (i) with candidate (j) is given by the rule $p_{i,j} \sim \frac{k_j}{\text{sum}(k_j)}$ introduced in [5], where $k_j$ is the number of voters that already have agreed to vote for this candidate. For the big parties, the role of the militants, or the zealot effect according to [7] is substantial. They are present everywhere, so the voter (i) would see the candidate (j) always accompanied with some them, so the attractiveness of the candidate is multiplied with a term $h_j$, proportional with the number of his average supporters, [13]. Assuming the
homogeneity of all election’s zones, the formula becomes \( p_{i,j} \sim h \frac{k_j}{\text{sum}(k_j)} \). Small subjects have \( h = 1 \), so the original preferential attachment rule is recovered. In [13] we considered also the network size effects by a further (empirical) correction of the connecting probability at \( p_{i,j} \sim h \frac{k_j^2}{\text{sum}(h_{jk}^2)} \). This model reproduced well the distribution obtained for the distribution of votes by PS reference for several elections, figure 8.

The model has worked qualitatively for the distribution of the votes in at least one election process on each four countries chosen for testing purposes. In short, it would be a good start for our new model. Next, we remark that the votes outside the bell-like distribution has changed from around to 8% in 2001, at 15% at 2005 and reached 22% in 2013. We connected this with the increasing tendency for rational voting which supposedly would contribute in the PL part of the distribution. Note that for administrative election of 2017 we estimated that around 28% of the voters have acted as non-militant. However, the very intriguing feature of the electorate have been observed in 2013 when an important faction of the voters has overcome their prior bias and also at 2017 where around 10% of the traditional voters has chosen to absentee. So, this shifting behaviour should be considered in the model, or simply we should model the voter as rational.

\[ \text{Figure 8. The testing of the modified preferential attachment model for the distribution of votes. Up-left graph: the election of 2010 in UK; the red squares are from real data; the blue pentagons are from the simulation. Right side, election on Canada. Green dots are from real points, blue diamond, are from our simulation. The figure below, in the left side: simulation for the distribution of votes for Albania. The red marks, real data (Polling Station, 2005). Black stars, the model' estimates.} \]

4.2. Modelling the rational electorate system by using q-opinion approach
According to the requirements of the sought model described above, we propose to use our q-opinion model introduced in [25] and [26] which is capable to produce the alternation of the opinion outcome. To make
easier for the reader we are briefing herein the outline of the q-opinion model [26]. We started from the q-
utility function based on the Hamiltonian used in the reference [27] and references therein for the dimer XY
magnets. This utility has an extra term in similarity with the q-summation as follow
\[ U = \sum \vec{O}_i \cdot \vec{O}_j - F \sum \vec{O}_i + q \left( \sum \vec{O}_i \cdot \vec{O}_j \right) - F \sum \vec{O}_i + q \left( \frac{1}{2} \left( \vec{O}^2 - 2 \right) - F \vec{O} + q \left( \frac{1}{2} \left( \vec{O}^2 - 2 \right) \vec{F} \vec{O} \right) \right) \] (6)

In (7) the vector \( \vec{O} = \vec{O}_1 + \vec{O}_2 \) represents the opinion on the couple, \( \vec{F} \) is the exterior factor acting as the
exterior field in the magnetic system, \( J \) is the intensity of the interaction which is assumed constant, and can
take the ± sign to account for collaborative or reactive initial preferences in the opinion couple. The vectors \( \vec{O}_i \) have magnitude 1 in the model. The agreement level toward the issue \( \vec{F} \) is given by the value \( O_x = \cos(q) \), and the couple is considered as strongly interconnected (or the weak field limit). Based on the
statistical mechanics calculation the average opinion for this case is obtained in the analytic form [25], [26]
\[ \langle O \rangle_F = \frac{\int_0^\beta e^{-J q \vec{O}^2 / 2} \left( \beta \vec{F} \vec{O} \left( 1 - \frac{q}{2} \vec{O}^2 - 2 \right) \right) \cdot \vec{O} \left( 1 - \frac{q}{2} \vec{O}^2 - 2 \right) d\vec{O}}{\int_0^\beta e^{-J q \vec{O}^2 / 2} \left( \beta \vec{F} \vec{O} \left( 1 - \frac{q}{2} \vec{O}^2 - 2 \right) \right) d\vec{O}} \] (7)

Here \( \beta \) is the thermodynamic temperature, \( I(.) \) are the incomplete Bessel functions of the first kind. By using
(8) we have calculated the thermodynamic agreement in the couple which depend directly on the q-value.
For \( q=0 \), the XY magnet system is recovered, whereas for \( q \neq 0 \) various scenarios develop, see [26].
Particularly we ask that the q-parameter has to be assigned by a deterministic procedure introduced below.
The thermodynamic temperature of the system which in theory measures all the disturbing effects, see [28], [29] etc., can be taken \( T=1 \) without losing the generality. Following the idea presented in [13], for the main
political subjects we assign the intensity of the electoral field at \( F=3 \); for the other parties we take \( F=2 \) and
for small subjects or their candidates we pose \( F=1 \), but they can be assigned according to phenomenological
observation for different electoral environments. Now we introduce the electoral performance function \( H = A \ast B \) where A are coefficients and B some socio-political performances. H can take values from zero to
infinity in principle. The q-parameter value is proposed by the following formula
\[ q = \frac{J - \frac{2f}{1+\exp(A+\beta)}}{1+\exp(A+\beta)} \] (8)

Remember that according to the formula (8) and numerical calculation performed in [26], the average
thermodynamic opinion is higher for high value of \( q \ast f \). So, if the electoral Hamiltonian (electoral
performance) H is high and the voter contacted is a potential supporter (\( f = 1 \)), we get \( q = 1 - \frac{2}{1+\ln f} = 1 \) so the q-XY utility (7) has the extra negative term which favorize a higher agreement level compared to
the classical XY approach. If the electoral performance of the subject is too low, we get \( q = 1 - \frac{2}{1+\exp(0)} = 0 \) and the calculation using (7) reproduces the classical magnetic system. The interesting scenario is the case
when the voter and the candidate are politically opponents (\( f = -1 \)) and supposedly the agreement is high.
We get \( q = -1 + \frac{2}{1+\ln f} = -1 \), so \( q \ast f = 1 \). According to the calculation provided in [25] and [26], the
agreement level is not neglectable in those conditions. The model says that for very high performance of the
candidates, there would be always a level of support: It is very high (practically 1) for normal temperature
if the candidate belongs to the preferred parties, but it is always nonzero even for the candidates of
the opponent side. If the performance is low (\( H = A \ast B < \)), we obtain \( q = -1 + \frac{2}{1+\exp(0)} = 0 \), and the
system produces a very low agreement level, practically zero for normal temperature (not too high). It can
be clarified if we remember that like the antiferromagnetic case in the T=0 limit, the opinions in the couple
should be aligned antiparallel giving the total opinion zero, \( \vec{\theta} = \vec{\theta}_1 - \vec{\theta}_2 = 0 \). So, the support for the opponent subject would be zero if its electoral performance is zero. Therefore, the expected militant behavior would be realized for the low electoral performance only. The model has considered therefore a rational behavior for all voters. Now, the campaign would run in this way (the pseudo-algorithm):

- A subject (i) randomly meets a voter (j) and start talking with him by offering an electoral bundle \( A = \{a, b, c, d, \ldots\} \). At this stage, the voters identify the [candidate, subject] affiliation, so we apply the 50:50 rule, by the condition if \( \text{rand}(\cdot) > 0.5 \) then \( f = 1 \) and \( f = -1 \) otherwise. Next, the voter realizes his initial belief as the average opinion on the temporal couple. So [using the rational voter’s logic] we evaluate the q-parameter by using (9) which works as the parameter in the opinion couple: the candidate and voter effectively behave as a q-opinion pair. Next, the prior (bias) factor of the agreement \( O_{ij} \) is evaluated by using the formula (8).

- The voter looks around if others have agreed upon the offers, that if there are other links established so far \( k_f \), and adjusts his decision based on the new generalized preferential attachment rule

\[
 p_{i,f} \sim O_{ij} h_f \frac{k_f}{\text{sum}(k_f)}
\]  

(9)

- The process continue until all voters are contacted by the candidates. The process could be realized in a given (known) topology network, but we used herein the simplest system of the gaseous nodes. We have used the data from pre-election survey in the assignment of the vector \( A \) of performances coefficients for calculating the \( H \) quantity. The better practice would be the expert measurement of \( H \).

During the campaign of the 2021, the public perception of the averaged government performance has been reported by media as follows: 0.53 quotes for the leadership, 0.63 on the management of the earthquake and Covid-19 crisis, 0.42 for economical comportment. For the oppositions, it has been reported: 0.35 in the leadership, 0.42 on the management of the earthquake and Covid-19 crisis, 0.38 for economical comportment. Based on the above arguments, the vector of the performances by political groups is evaluated \( A_{\text{incumbent}} = [0.53, 0.63, 0.45]; A_{\text{opposition}} = [0.35, 0.42, 0.38] \) and we used herein the all B-coefficients at unity for application purposes, so the \( H \) quantity is replaced each step appropriately in the algorithm. So far, by using the integrated formula (10) we have reproduced the distribution that we obtained in the reference of the polling station results, figure 8. By using the model for the simulation as by EC references, we also observed that the reproduction of the distribution has been acceptable. Although, in this case there have been
observed some deviances which can be reduced by the improvement of the electoral utility measurement. However, the distribution obtained by summed results has changed the shape toward a mostly PL form, with weak nodulation in the right-hand side. As a conclusion remark for this paragraph, this model suggests that the electorate is not polarized definitively and it acted rationally. It seems that the final decision of each voter is based on the electoral performance of the subjects. If this last has a significantly high value, many voters are likely to consider supporting the list of candidates of the opponent political wing. Next, by increasing the size of the election network, we obtained that the gaussian nodulation become weaker but does not disappear, which suggests that for big network, the modified preferential attachment rule works as classically expected, and therefore the voters would decide to support a certain subject or candidate based mostly on the information about the other attitudes and preferences.

5. Conclusions
The distribution for the votes in the Albanian election are multi-part functions where each component has been fitted very good with q-exponentials or q-gaussians. For small voting units we obtained a scoop-down distribution in the logarithmic representation, which indicates the presence of the two principal mechanisms of the political agreement. Usually the small subjects gather the votes based on the performance whereas big subjects use their militants and activists which create local fields that influence the voters during the campaign. The state of the electorate system is usually nonstationary due to the effect of the electoral and non-electoral factors. The network failures or other heterogeneities have affected significantly the distributions in the low fractions of votes which belong to the small parties. Next, by the analysis of the distributions obtained for the election of 2021, we expect that the processes of the redistributions would be more intensive in the left-wing electorate compared to the right-wing electorate. By analyzing the joint system that contains the two main parties together, the distribution of the votes resulted a relaxed q-gaussian, and therefore no significant alteration or redistribution are expected to occur in ceteris paribus conditions for this compound system. In particular, the effect of other subjects would be non-significant, and the system would remain in this metastable state, meaning that the most probable event is the exchange on the votes between the two main parties, rather than the exchange with the outside groups. Next, the votes of the intermediate size’ subjects act as destabilization factor for the distribution of the votes in the two main party’s jointed system. Brining all the findings of the descriptive analyses for the distribution of the votes, we realized a working model for the electorate system under scrutiny. It is based on the modified preferential attachment and includes an additional term calculated by the q-opinion approach. In this case, a nonzero agreement would be obtained for every couple voter-candidate, regardless to their mutual affiliation. Using this model by employing a simple electoral utility in the calculation of its q-parameter, we have reproduced the distributions observed empirically for both small and big electoral units’ references. Therefore, it resulted that rational voters constitute a considerable part of the voting body for the electorate system considered in this study.

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