In scattering by singular potentials $g^2U(s;r)$, the coupling constant $g^2$ is continuously decreased to zero while the stage $s$ of singularity raised simultaneously beyond all limits by some functional relation $F(g^2; s) = 0$. In the extreme situation of this double limit, even the mere existence of a nontrivial physical scattering problem is questionable. By iterating a pair of integral equations, the relevant solution is developed here in terms of wave functions into a pair of convergent series, each of which reduces in the double limit $\{g^2 \to 0; s \to \infty\}$ to a single term calculable by quadrature.
Just as in the regular case, also in the cases of repulsive singular potentials $g^2 U(s; r)$ the problem of scattering at extreme values of the parameters involved is expected to become solvable by some simple, asymptotically exact expressions. In this paper a review will be given over the main cases of varying the potential parameters contained either in linear ($g^2$) or nonlinear ($s$) positions. The simplest example is the increase, at invariable form factor, of the linear parameter beyond all limits. This is the strong coupling limit, i.e. \( \{g^2 \to \infty; s = \text{fixed}\} \). For singular potentials this problem has already been solved by a smooth version of the semiclassical approach [1,2]. The complementary case is manifested by the variation of the nonlinear parameter $s$. The notation to be applied will ensure that the stage of singularity should increase, at fixed value of the coordinate $r$, with increasing values of $s$ to infinity. The asymptotical situation \( \{g^2 = \text{fixed}; s \to \infty\} \) will be referred to here as the 'supersingularity' limit. This extreme scattering problem was also attacked by a semiclassical procedure, which furnished correct and simple results in the limit mentioned for a variety of singular potentials [3]. Nevertheless, within our scheme of extreme scattering problems, the most interesting point seems to be the simultaneous variation of linear and nonlinear potential parameters. A particular example is the double limit \( \{g^2 \to 0; s \to \infty\} \). Such a problem is, of course, uniquely specified only by adding to the expression $g^2 U(s; r)$ as input information a functional of the form $F(g^2; s) = 0$, which governs the interdependence of the parameters themselves.

We are going to scrutinize cases of such double limits via combining various classes of potentials with different types of interdependence between linear and nonlinear parameters. The underlying formalism is supplied by the approach developed for solving the supersingularity problem in [3]. That argument will be briefly outlined below. A spinless particle is scattered by a central singular potential at the energy $k^2$ in the channel of index $l$. To start the discussion with, we introduce a triad of auxiliary orbital angular momenta such as

\[
\lambda^2_{\epsilon}(l) = (l + \frac{1}{2})^2; \quad \lambda^2_{\phi}(l) = l(l + 1); \quad \lambda^2(l) = \frac{1}{2}[\lambda^2_{\epsilon}(l) + \lambda^2_{\phi}(l)].
\]  

(1)
The concept of the ‘matching distance’ $R$, into which the variable parameters $g^2$ and $s$ will be lumped together, is introduced by the ‘Master equation’ as

$$k^2 R^2 - g^2 R^2 U(s; R) - \lambda^2 = 0, \quad \text{whence } R = R(g^2, s).$$  \hspace{1cm} (2)$$

The dimensionless radial coordinate $t$, understood as

$$t = \frac{r}{R}; \quad t < 1 \text{ if } r < R,$$

works as an essential technical means. The exact radial Schrödinger equation is recast in the new variable as

$$\left\{ \frac{d^2}{dt^2} + k^2 R^2 - g^2 R^2 U(s; Rt) - \frac{l(l + 1)}{t^2} \right\} u^\pm(t) = 0, \quad [0 \leq t].$$  \hspace{1cm} (4)$$

The regular solution $u^+(t)$ will be represented by a pair of series, one for the ‘exponential’ (or $\epsilon$) and another one for the ‘trigonometric’ (or $\tau$) region as follows:

$$u^+(t) = u^+_\epsilon(t) = \sum_{n=0}^{\infty} w^+_\epsilon_n(t), \quad [t < 1];$$  \hspace{1cm} (5)$$

and

$$u^+(t) = u^+_\tau(t) = \sum_{m=0}^{\infty} w^+_\tau_m(t), \quad [t > 1].$$  \hspace{1cm} (6)$$

The notation used here is resolved by the following set of identities:

$$w^+_\epsilon_0(t) = \left( \frac{k^2}{R^2(\epsilon)} \right)^{\frac{1}{2}} e^{\pm R \int_1^t dt' |K_\epsilon(t')|};$$  \hspace{1cm} (7)$$

$$w^+_\tau_0(t) = \left( \frac{k^2}{R^2(\tau)} \right)^{\frac{1}{2}} \{ C^\pm \cos[R \int_1^t dt' |K_\tau(t')|] + S^\pm \sin[R \int_1^t dt' |K_\tau(t')|] \},$$  \hspace{1cm} (8)$$

and the iteration scheme

$$w^+_\epsilon_n(t) = \int_0^t dt' G^+_\epsilon(t, t') \Delta_\epsilon(t') w^+_\epsilon_{n-1}(t'), \quad [t < 1],$$  \hspace{1cm} (9)$$

$$w^+_\tau_m(t) = \int_1^t dt' G^+_\tau(t, t') \Delta_\tau(t') w^+_\tau_{m-1}(t'), \quad [t > 1].$$  \hspace{1cm} (10)$$
where \( n, m = 1, 2, 3, \ldots \). The constants \( C^+ \) and \( S^+ \) are specified by requiring smooth matching at \( t = 1 \), in principle of the infinite series (5) and (6) themselves, in practice, however, of their higher order, [N,M], cut-off approximations [3]. As regards \( C^- \) and \( S^- \), they can be freely chosen but for the condition \( C^+ S^- \neq C^- S^+ \). Furthermore, the local wave number squares have been understood as

\[
K^2_{\gamma}(t) = \mp \{ k^2 - g^2 U(s; Rt) - \frac{\lambda^2_{\gamma}}{R^2 t^2} \}, \quad [\gamma = (\epsilon, \ t \leq 1],
\]

while the resolvents are given by the definitions

\[
G^+_{\gamma}(t, t') = \frac{1}{d^+_{\gamma}} [w^+_{\gamma}(t) w^-_{\gamma}(t') - w^-_{\gamma}(t) w^+_{\gamma}(t')], \quad [\gamma = \epsilon, \tau],
\]

with the Wronskians

\[
d^+_{\epsilon} = -2kR, \quad d^+_{\tau} = C^+ S^- - C^- S^+.
\]

Finally, the residual potentials are given in the respective regions by

\[
\Delta_{\gamma}(t) = -\frac{5}{16} \left( \frac{1}{K^2_{\gamma}(t)} \right)^2 + \frac{1}{4} \frac{1}{K^2_{\gamma}(t)} \frac{d^2 K^2_{\gamma}(t)}{dt^2} - \frac{\lambda^2_{\gamma} - l(l + 1)}{t^2}.
\]

The series expansions (5)-(6) had been found [1] absolutely convergent whenever two integrals, \( P_{\gamma}(t), \quad [\gamma = \epsilon, \tau], \) are bounded. That is to say, the convergence criteria read

\[
P_{\gamma}(t) \equiv R \int_{t_{\gamma}}^{t} dt'[p_{\gamma}(t')] < c_{\gamma} < \infty, \quad [\gamma = (\epsilon, \ t_{\gamma} = (0, 1)], \quad t \leq 1],
\]

where the concepts of the ‘discriminants’, namely

\[
p_{\gamma}(t) \equiv \frac{\Delta_{\gamma}(t)}{RK_{\gamma}(t)}, \quad [\gamma = \epsilon, \tau]
\]

were introduced.

The potentials to be included in the discussion will occur as products of 3 factors, such as a coupling constant \( g^2 \), a core factor \( V_{\epsilon}(s; r) \) and a tail factor \( V_{\tau}(r) \). The
function $V_e(s; r)$ will be singular at $r = 0$ either exponentially or powerlaw in $r$, while $V_\tau(r)$ should decrease for $r \to \infty$ exponentially or powerlike. Owing to the simultaneous variation of the parameters $g^2$ and $s$, an everywhere extreme and a locally extreme effect will face each other. Increasing values of $s$ should correspond, by definition, to raising stages of the singularity. The interparameter relationship $F(g^2; s) = 0$ is to appear decomposed into a pair of $R$-functions. Out of them, the function $g^2(R)$ will be supplied as input information while $s(R)$ introduced via the Master equation (2). In this way, the explicit presence of both the linear and the nonlinear parameters can be eliminated from the asymptotical formulas. The limit $R \to \infty$ will be checked for each potential class separately to recover the double limit $g^2(R) \to 0; s(R) \to \infty$.

Our choice of both the $R$-function $g^2(R)$ and of the $r$-functions $V_e(s; r)$ and $V_\tau(r)$ is either an exponential (E) or a powerlaw (P) dependence. It is therefore convenient to refer to each of our potential classes by a triad of the capitals $E$ or $P$, e.g. EEE, EPP, PEE etc., in the order of $g^2(R), V_e(r), V_\tau(r)$.

**Case EEE**

This is, perhaps, the most interesting potential class in our discussions. A rapidly decreasing coupling constant will compete with a rapidly raising stage of the $r = 0$ point singularity of the interaction. The set of formulae below leads from the definition of the fixed parameter form of the potential up to proving fulfilment of the criteria for convergence of the series (5)-(6) in the double limit considered. Accordingly, we write

$$g^2 U(s; r) = \frac{1}{r_0^2} e^{-\frac{R}{r_0}} e^{\frac{r_1 s}{r_2}} e^{-\frac{r}{r_2}}, \quad (17)$$

with the fixed positive constants $r_0, r_1, r_2$ and the variable $s$. The variation of the singularity parameter $s(R)$ is governed by the Master equation (2), the exact form of which reads in this potential class

$$e^{\frac{r_1 s(R)}{R}} = (k^2 - \frac{\lambda^2}{R^2}) r_0^2 e^{R(\frac{1}{r_0} + \frac{1}{r_2})}. \quad (18)$$
Considered as the definition of the function \( s(R) \), this equation implies for \( R \to \infty \) the order relationship \( O\{ s(R) \} = O\{ \frac{R^2}{r_0 r_2} \} \). This verifies our expectation is equivalent to the double limit \( \{ g^2 \to 0; s \to \infty \} \). Upon incorporating Eq. (18) into the definition (11) one obtains

\[
K^2_s(t) \to \mp k^2 \{ 1 - \left( k^2 r_0^2 \right)^{\frac{1}{2}} - 1 \} e^{R[(\frac{1}{r_0} + \frac{1}{r_2})^{\frac{1}{2}} - (\frac{1}{r_0} + \frac{1}{r_2} t)]} - \frac{\lambda^2}{k^2 R^2 t^2} \}
\]

(19)

for \( R \to \infty \). The notation is the same as the one used in Eq.(11). Hence the discriminants \( p_s(t) \) of the definition (16) are extracted, first for the region \( \epsilon \), as

\[
p_\epsilon(t) \to -\frac{R}{16 k} \left[ \left( \frac{1}{r_0} + \frac{1}{r_2} \right) \frac{1}{t^2} + \frac{1}{r_2} \right]^2 e^{-\frac{R}{2}[(\frac{1}{r_0} + \frac{1}{r_2}) - (\frac{1}{r_0} + \frac{1}{r_2} t)]}, \quad [R \to \infty].
\]

(20)

As to the region \( \tau \), the local wave number square (19) contains an exponentially vanishing term, which greatly simplifies the formalism. Indeed, one simply gets in the long run

\[
p_\tau(t) \to -\frac{3\lambda^2}{2k^2 R^2 t^4}, \quad [R \to \infty].
\]

(21)

Returning to the definitions (15)-(16), one concludes from the relationship (20) that the function \( P_\epsilon(t) \) is majorized in \( t = (0, 1) \) for the case EEE by \( \Gamma(2) = 1 \). As to the integral \( P_\tau(t) \), it does not involve in \( t = (1, \infty) \) any singularity and so it also remains finite.

The local wave number square (11) contains in the region \( \tau \) for \( R \to \infty \) in each of our potential classes exponentially vanishing interaction contributions only. These become asymptotically negligible in comparison to the energy and the centrifugal term. As a consequence, expression (11) reduces for \( \gamma = \tau \) to the very same formula (21), independently of the actual potential. Thereby, the convergence of the series (6) is guaranteed in every case. The further discussions can be therefore restricted to the respective \( \epsilon \) regions.

**Case EEP**

Within the potential classes to be included in the present discussions, the potential tail exerts virtually no influence on the conflict between the vanishing coupling
constant and the increasing singularity of the core factor. Therefore, no essential
difference is expected between the cases EEE and EEP. The present class of po-
tentials is introduced as

\[ g^2 U(s; r) = \frac{1}{r_0^2} e^{-\frac{R}{r_0}} e^{\frac{r_1}{r_2}} \left( \frac{r_2}{r_2 + r} \right)^{\sigma}, \quad [\sigma > 8], \quad (22) \]

where the quantities \( r_0, r_1, r_2 \) all are positive and fixed against variation. The
asymptotical Master equation (2), which specifies the function \( s(R) \) for large values
of \( R \), becomes in our double limit

\[ e^{\frac{r_1 s(R)}{R}} \to (k^2 - \frac{\lambda^2}{R^2}) r_0^2 e^{\frac{R}{r_2}} \left( \frac{r_2}{r_2} \right)^{\sigma}, \quad [R \to \infty], \quad (23) \]

whence one extracts \( O\{s(R)\} > O\{\frac{R^2}{r_0 r_1}\} \) for \( R \to \infty \). The formulae (11) and (23)
combine then into

\[ K^2_\epsilon(t) \to k^2 \left( \frac{1}{k^2 r_0^2} \right)^{\frac{1}{t} - 1} \quad (24) \]

Hence one concludes for the exponential region that by the identity (16)

\[ p_\epsilon(t) \to -\frac{1}{16} (kr_0)^{-\left(\epsilon - \frac{1}{2} - 1\right)} t^\frac{\epsilon}{2} - 4 \frac{1}{kR} \left( \frac{r_2}{R} \right)^{\frac{\epsilon}{2} - 3} e^{-\frac{R}{r_0} (\epsilon - 1)}. \quad (25) \]

The integrability of this expression in \( t = (0, 1) \) is by Eq.(22) obvious.

**Case PEE**

This class should lie near the pure supersingular case, where \( g^2 = \) fixed. The
potential is introduced as

\[ g^2 U(s; r) = \frac{1}{r_0^2} \left( \frac{r_0}{r} \right)^{\sigma_0} e^{\frac{r_1}{r}} e^{-\frac{r_2}{r}} \quad (26) \]

The large-\( R \) form of the Master equation (2) reads so

\[ e^{\frac{r_1 s(R)}{R}} \to k^2 r_0^2 \left( \frac{R}{r_0} \right)^{\sigma_0} e^{\frac{R}{r_2}}, \quad [R \to \infty], \quad (27) \]

which suggests the relationship \( O\{s(R)\} > O\{\frac{R^2}{r_1 r_2}\} \). Incorporation of the definition
(27) into Eq.(11) yields in the region \( \epsilon \)

\[ K^2_\epsilon(t) \to k^2 \left( k^2 r_0^2 \right)^{\frac{1}{t} - 1} \left( \frac{R}{r_0} \right)^{\frac{1}{t} - 1} e^{-\frac{R}{r_2} (\epsilon - t)}, \quad [R \to \infty]. \quad (28) \]
This expression furnishes at \( R \to \infty \) the discriminant (16) as
\[
p_{\epsilon}(t) \to -\frac{1}{16} \left( \frac{1}{t^2} + 1 \right)^2 \frac{1}{kR} \left( \frac{R}{r_0} \right)^2 (kr_0)^{1-\frac{1}{t}} \left( \frac{r_0}{R} \right)^{\frac{1}{2}\sigma_0(\frac{1}{t}-1)} e^{-\frac{R}{2\sigma_2}(\frac{1}{t}-t)}. \tag{29}
\]
The exponential factor involved guarantees the existence of \( P_{\epsilon}(t) \) in \( t = (0, 1) \).

**Case PEP**

We shall now treat the scattering potentials
\[
g^2 U(s; r) = \frac{1}{r_0^2} \left( \frac{r_0}{R} \right)^{\sigma_0} e^{\frac{r_1 + r}{r_2 + r}} \left( \frac{r}{r_2 + r} \right)^{\sigma_2}, \quad [\sigma_2 > 8]. \tag{30}
\]
The corresponding Master equation (2) becomes asymptotically
\[
e^{-\frac{r_1 + r}{R}} \to k^2 \frac{R}{r_0} \left( \frac{R}{r_2} \right)^{\sigma_0} \left( \frac{R}{r_2} \right)^{\sigma_2}, \quad [R \to \infty]. \tag{31}
\]
The order relationship between the parameters is extracted now as \( O\{s(R)\} \gg O\{\frac{R}{r_1}\} \). The local wavenumber square (11) and the discriminant (16) are therefore obtained in the region \( \epsilon \) as
\[
K^2_\epsilon(t) \to k^2 \left[ k^2 \frac{R^{\sigma_0 + \sigma_2}}{r_0 r_2} \right]^{\frac{1}{t} - 1} \frac{1}{t^{\sigma_2}}, \quad [R \to \infty], \tag{32}
\]
and
\[
p_{\epsilon} \to -\frac{1}{16} t^{\sigma_2 - 4} \frac{1}{kR} \ln \left( \frac{R^{\sigma_0 + \sigma_2}}{r_0 r_2} \right) \left[ \frac{R^{\sigma_0 + \sigma_2}}{r_0 r_2} \right]^{-\frac{1}{t}(\frac{1}{t}-1)}. \tag{33}
\]
The integrability of the last expression is by analysis obvious both at fixed and increasing values of \( R \) as well as near and off the origin \( t = 0 \).

**Case EPE**

Off the singularity region this potential class develops in our double limit the strongest suppression. Indeed,
\[
g^2 U(s; r) = \frac{1}{r_0^2} e^{-\frac{r_1 + r}{r_2}} \left( \frac{r_1}{r_2} \right)^s e^{-\frac{r}{r_2}}. \tag{34}
\]
The Master equation (2) can now be recast for asymptotical parameter values as
\[
e^{-\frac{r_1 + r}{R}} \to k^2 \frac{r_0^2}{r_2} e^{R \left( \frac{r_1}{r_0} + \frac{1}{r_2} \right)}, \quad [R \to \infty]. \tag{35}
\]
The singularity parameter $s(R)$ increases thus proportionally to $R^2$. Incorporation of the expression (35) into Eq. (11) yields for the exponential region

$$K^2_e(t) \rightarrow \frac{1}{r_0^2} \left[ k^2 r_0^2 e^{R \left( \frac{r_1}{r_0} + \frac{r_2}{r_2} \right)} \right] ^ \frac{1}{2} e^{-R \left( \frac{r_1}{r_0} + \frac{r_2}{r_2} \right)}, \quad [R \rightarrow \infty].$$

(36)

The discriminant (16) is extracted hence as

$$p_e(t) \rightarrow -\frac{1}{16} \left( \frac{1}{r_0} + \frac{1}{r_2} \right)^2 \left( \frac{r_0 R}{2} \right) e^{-\frac{1}{4} R \left( \frac{r_1}{r_0} + \frac{r_2}{r_2} \right)}, \quad [R \rightarrow \infty].$$

(37)

Near the singularity point $t = 0$, the exponential decrease of the potential at $t = 0$ dominates for $R \rightarrow \infty$ the powerlaw increase there. The quantity $P_e(t)$ is thus finite.

**Case EPP**

This interaction is very similar to the previous one within the region near the singularity point. The potential reads now

$$g^2 U(s; r) = \frac{1}{r_0^2} e^{-\frac{r}{r_0} \left( \frac{r_1}{r} + \frac{r_2}{r} \right)} \left( \frac{r_2}{r_2 + r} \right)^\sigma, \quad [\sigma > 2].$$

(38)

The Master equation (2) governs the large-$R$ dependence of the singularity parameter $s$ as

$$e^\frac{r_1 s(R)}{r} \rightarrow k^2 r_0^2 \left( \frac{R}{r_2} \right)^\sigma, \quad [R \rightarrow \infty].$$

(39)

Hence one extracts the order relationship $O\{s(R)\} > O\{\frac{R}{r_2}\}$ for $R \rightarrow \infty$. On the other hand, the asymptotical forms of (11) and (16) follow from relationship (39) as

$$K^2_e(t) \rightarrow k^2 \frac{1}{r_0^2} \left[ k^2 r_0^2 e^{\frac{r}{r_0} \left( \frac{R}{r_2} \right)^\sigma} \right] ^ \frac{1}{2} - 1, \quad [R \rightarrow \infty],$$

(40)

as well as the simultaneously and exponentially decreasing discriminant

$$p_e(t) \rightarrow -\frac{1}{16} \frac{R}{r_0^2 k^2 t^4} e^{-\frac{r}{r_0} (\frac{1}{2} - 1)}, \quad [R \rightarrow \infty].$$

(41)

This again means convergence of the corresponding series (5).

**Case PPP**
The interaction behind this symbol is written in the variable $r$ as

$$g^2 U(s; r) = \frac{1}{r_0^2} \left( \frac{r_0}{R} \right)^{\sigma_0} \left( \frac{r_1 + r}{r} \right)^s \left( \frac{r_2}{r_2 + r} \right)^{\sigma_2}, \quad [\sigma_2 > 4]. \quad (42)$$

The asymptotical Master equation is now extracted as

$$e^{\frac{r_0 s(R)}{R}} \to k^2 r_0^2 \left( \frac{R}{r_0} \right)^{\sigma_0} \left( \frac{R}{r_2} \right)^{\sigma_2}, \quad [R \to \infty]. \quad (43)$$

The increase of $s(R)$ for large values of the matching distance is slightly more rapid than that of $\frac{R}{r_1}$. The local wave number square becomes in our double limit

$$K^2_\epsilon(t) \to k^2 \frac{1}{t^{\frac{1}{\sigma_2}}} \left[ k^2 r_0^2 \left( \frac{R}{r_0} \right)^{\sigma_0} \left( \frac{R}{r_2} \right)^{\sigma_2} \right]^\frac{1}{t - 1}, \quad (44)$$

while the discriminant (16) develops then the asymptotical form

$$p_\epsilon(t) \to -\frac{1}{16} \ln \left( \frac{R^{\sigma_0 + \sigma_2}}{r_0^{\sigma_0} r_2^{\sigma_2}} \right) - \frac{1}{k R t^\left(\frac{1}{\sigma_2} - 4\right)} \left[ k^2 r_0^2 \frac{R^{\sigma_0 + \sigma_2}}{r_0^{\sigma_0} r_2^{\sigma_2}} \right]^\left(-\frac{1}{t - 1}\right). \quad (45)$$

The exponential decay involved in the limit $R \to \infty$ for $t < 1$ ensures the existence of $P_\epsilon(t)$ in region $\epsilon$.

**Case PPE**

The core factor implies, in fact, for the double limit we are interested in, again a hidden exponential dependence on the singularity parameter $s$. Indeed,

$$g^2 U(s; r) = \frac{1}{r_0^2} \left( \frac{r_0}{R} \right)^{\sigma_0} \left( \frac{r_1 + r}{r} \right)^s e^{-\frac{r}{r_2}}. \quad (46)$$

By analysis, one obtains the Master equation in the form

$$e^{\frac{r_1 s(R)}{R}} \to k^2 r_0^2 \left( \frac{R}{r_0} \right)^{\sigma_0} e^{\frac{R}{r_2}}, \quad [R \to \infty]. \quad (47)$$

This result implies $O\{s(R)\} > O\{\frac{R^2}{r_1 r_2}\}$. One also concludes from the asymptotical expression (47) that

$$K^2_\epsilon(t) \to k^2 \left[ k^2 r_0^2 \left( \frac{R}{r_0} \right)^{\sigma_0} \right]^\frac{1}{t - 1} e^{\frac{R}{r_2}(\frac{1}{t - 1})}, \quad [R \to \infty], \quad (48)$$
and
\[ p_\epsilon(t) \to -\frac{1}{16} \frac{1}{kR} \left( \frac{R}{r_2} \right)^2 \left( \frac{1}{t^2} + 1 \right)^2 e^{-\frac{R}{r_2}(\frac{1}{t^2} - t)}, \quad [R \to \infty]. \tag{49} \]

Except for the single point \( t = 1 \), this expression exponentially vanishes in the limit \( R \to \infty \) within the region \( \epsilon \). Consequently, the function \( P_\epsilon(t) \) of the identity (15) does exist also in this case.

The behaviour of the functions \( p_\gamma(t) \), \( [\gamma = \epsilon, \tau] \), of (16) have been studied for eight classes of repulsive singular potentials along both of the complementary regions \( \epsilon \) and \( \tau \). Owing to the existence and boundedness of the functions \( P_\gamma(t) \) of (15), both series (5) and (6) are absolutely convergent and reduce in the limit we are interested in to the respective leading terms [1,2]. Accordingly, the scattering wave functions develop for \( R \to \infty \) the following asymptotical forms
\[ u^+(t) \to \left( \frac{k^2}{K^2_\epsilon(t)} \right)^{\frac{1}{4}} e^{R \int_1^t dt' |K_\epsilon(t')|}, \quad [t < 1], \tag{50} \]
as well as
\[ u^+(t) \to \left( \frac{k^2}{K^2_\tau(t)} \right)^{\frac{1}{4}} \left\{ C_0^+ \cos[R \int_1^t dt' K_\tau(t')] + S_0^+ \sin[R \int_1^t dt' K_\tau(t')] \right\}, \quad [t > 1]. \tag{51} \]

The constants \( C_0^+ \) and \( S_0^+ \) involved are, in fact, \( R \)-dependent and fixed uniquely by postulating smooth matching of the external wave function (51) to (50), the internal one, at \( t = 1 \). The functions \( K_\gamma(t) \) are supplied for inclusion into (50)-(51) by the large-\( R \) expressions (19), (24), (28), (32), (36), (40), (44) and (48) for the respective potential classes.

The overall conclusions extracted can be lumped into the following three points:
(a) The double limits of combining vanishing linear and diverging nonlinear potential parameters may give rise, at different types of interdependence of these variable constants, to reasonable scattering problems, (b) These are solved by pairs of absolutely convergent series, (c) the lengths of which get reduced, just in the limit scrutinized, to single terms calculable by quadrature.

Finally, only a slight hint at philosophy. The coupling constant \( g^2 \) may be regarded as the \textit{quantity} inherent in the singularity of the scattering potential, while the
stage $s$ of singularity could be a measure of its quality. The above argument may thus yield an example for treating the relationship between 'quality' and 'quantity' under extreme circumstances.

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