SECOND HANKEL DETERMINANT FOR BI-STARLIKE AND BI-CONVEX FUNCTIONS OF ORDER $\beta$

ERHAN DENIZ, MURAT ÇAĞLAR, AND HALIT ORHAN

Abstract. In the present investigation the authors obtain upper bounds for the second Hankel determinant $H_{2}(2)$ of the classes bi-starlike and bi-convex functions of order $\beta$, represented by $S^{\ast}_{\beta}(\beta)$ and $K_{\sigma}(\beta)$, respectively. In particular, the estimates for the second Hankel determinant $H_{2}(2)$ of bi-starlike and bi-convex functions which are important subclasses of bi-univalent functions are pointed out.

1. Introduction and definitions

Let $A$ denote the family of functions $f$ analytic in the open unit disk $U = \{z \in C : |z| < 1\}$ of the form

\begin{equation}
 f(z) = z + \sum_{n=2}^{\infty} a_{n}z^{n}.
\end{equation}

Let $S$ denote the class of all functions in $A$ which are univalent in $U$. The Koebe one-quarter theorem (see [7]) ensures that the image of every $f \in S$ contains a disk of radius $1/4$. So, every $f \in S$ has an inverse function $f^{-1}$ satisfying $f^{-1}(f(z)) = z$ ($z \in U$) and

\begin{equation}
 f(f^{-1}(w)) = w \quad (|w| < r_{0}(f) \quad r_{0}(f) \geq 1/4)
\end{equation}

where $f^{-1}(w) = w - a_{2}w^{2} + (2a_{2}^{2} - a_{3})w^{3} - (5a_{2}^{3} - 5a_{2}a_{3} + a_{4})w^{4} + ...$.

A function $f \in A$ is said to be bi-univalent in $U$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $U$. Let $\sigma$ denote the class of bi-univalent functions in $U$ given by [14].

Two of the most famous subclasses of univalent functions are the class $S^{\ast}(\beta)$ of starlike functions of order $\beta$ and the class $K(\beta)$ of convex functions of order $\beta$. By definition, we have

\begin{equation}
 S^{\ast}(\beta) = \left\{ f \in S : \Re \left( \frac{zf'(z)}{f(z)} \right) > \beta; z \in U; 0 \leq \beta < 1 \right\}
\end{equation}

and

\begin{equation}
 K(\beta) = \left\{ f \in S : \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \beta; z \in U; 0 \leq \beta < 1 \right\}.
\end{equation}

The classes consisting of starlike and convex functions are usually denoted by $S^{\ast} = S^{\ast}(0)$ and $K = K(0)$, respectively.

For $0 \leq \beta < 1$, a function $f \in \sigma$ is in the class $S^{\ast}_{\sigma}(\beta)$ of bi-starlike functions of order $\beta$, or $K_{\sigma}(\beta)$ of bi-convex functions of order $\beta$ if both $f$ and its inverse map $f^{-1}$ are, respectively, starlike or convex of order $\beta$. These classes were introduced by Brannan and Taha [2] in 1985. Especially the classes $S^{\ast}_{\sigma}(0) = S^{\ast}_{\sigma}$ and $K_{\sigma}(0) = K_{\sigma}$ are bi-starlike and bi-convex functions, respectively. In 1967, Lewin [17] showed that for every functions $f \in \sigma$ of the form (1.1), the second coefficient of $f$ satisfy the inequality $|a_{2}| < 1.51$. In 1967, Brannan and Clunie [1] conjectured that $|a_{2}| \leq \sqrt{2}$ for $f \in \sigma$. Later, Netanyahu [18] proved that $\max_{f \in \sigma}|a_{2}| = 4/\sqrt{3}$. In 1985, Kedziorawski [13] proved Brannan and Clunie’s conjecture for $f \in S^{\ast}_{\sigma}$. In 1985, Tan [25] obtained the bound for $a_{2}$ namely $|a_{2}| < 1.485$ which is the best known estimate for functions in the class $\sigma$. Brannan and Taha [2] obtained estimates on the initial coefficients $|a_{2}|$ and $|a_{3}|$ for functions in the classes $S^{\ast}_{\beta}(\beta)$ and $K_{\sigma}(\beta)$. Recently, Deniz [6] and Kumar et al. [13] both extended and improved the results of Brannan and Taha [2] by generalizing their classes using subordination. The problem of estimating coefficients $|a_{n}|, n \geq 2$ is still open. However, a lot of results for $|a_{2}|$, $|a_{3}|$ and $|a_{4}|$ were proved for

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Corresponding author: edeniz36@gmail.com (Erhan Deniz).
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Lemma 1.2. On the other hand, very recently Zaprawa [28], [29] have studied on Fekete-Szegő problem further generalized functional

\[ H_{q}(n) = \begin{bmatrix}
    a_{n} & a_{n+1} & \ldots & a_{n+q-1} \\
    a_{n+1} & a_{n+2} & \ldots & a_{n+q} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n+q-1} & a_{n+q} & \ldots & a_{n+2q-2}
\end{bmatrix} (a_{1} = 1). \]

This determinant was discussed by several authors with \( q = 2 \). For example, we can know that the functional \( H_{2}(1) = a_{3} - a_{2}^{2} \) is known as the Fekete-Szegő functional and they consider the further generalized functional \( a_{3} - \mu a_{2}^{2} \) where \( \mu \) is some real number (see, [3]). In 1969, Keogh and Merkes [14] proved the Fekete-Szegő problem for the classes \( S^{\ast} \) and \( K \). Someone can see the Fekete-Szegő problem for the classes \( S^{\ast}(\beta) \) and \( K(\beta) \) at special cases in the paper of Orhan et al. [20]. On the other hand, very recently Zaprawa [28], [29] have studied on Fekete-Szegő problem for some classes of bi-univalent functions. In special cases, he gave Fekete-Szegő problem for the classes \( S^{\ast}(\beta) \) and \( K_{\sigma}(\beta) \). In 2014, Zaprawa [28] proved the following results for \( \mu \in \mathbb{R} \),

\[
f \in S_{\mu}^{\ast}(\beta) \Rightarrow |a_{2} - \mu a_{2}^{2}| \leq \begin{cases}
    1 - \beta; & 2(1 - \beta)|\mu - 1|; \\
    \frac{1 - \beta}{2(1 - \beta)} & \mu \geq \frac{1}{2}
\end{cases}
\]

and

\[
f \in K_{\sigma}(\beta) \Rightarrow |a_{2} - \mu a_{2}^{2}| \leq \begin{cases}
    1 - \beta; & (1 - \beta)|\mu - 1|; \\
    \frac{1 - \beta}{(1 - \beta)} & \mu \geq \frac{1}{2}
\end{cases}
\]

The second Hankel determinant \( H_{2}(2) \) is given by \( H_{2}(2) = a_{2}a_{4} - a_{2}^{2} \). The bounds for the second Hankel determinant \( H_{2}(2) \) obtained for the classes \( S^{\ast} \) and \( K \) in [12]. Recently, Lee et al. [10] established the sharp bound to \( |H_{2}(2)| \) by generalizing their classes using subordination. In their paper, one can find the sharp bound to \( |H_{2}(2)| \) for the functions in the classes \( S^{\ast}(\beta) \) and \( K(\beta) \).

In this paper, we seek upper bound for the functional \( H_{2}(2) = a_{2}a_{4} - a_{2}^{2} \) for functions \( f \) belonging to the classes \( S_{\mu}^{\ast}(\beta) \) and \( K_{\sigma}(\beta) \).

Let \( \mathcal{P} \) be the class of functions with positive real part consisting of all analytic functions \( \mathcal{P} : \mathcal{U} \to \mathbb{C} \) satisfying \( p(0) = 1 \) and \( \Re p(z) > 0 \).

To establish our main results, we shall require the following lemmas.

**Lemma 1.1.** [22] If the function \( p \in \mathcal{P} \) is given by the series

\[ p(z) = 1 + c_{1}z + c_{2}z^{2} + \ldots \]

then the sharp estimate \( |c_{k}| \leq 2 \) \((k = 1, 2, \ldots)\) holds.

**Lemma 1.2.** [10] If the function \( p \in \mathcal{P} \) is given by the series \( c_{2}x^{2} + c_{3}z^{3} \), then

\[ 2c_{2} = c_{2}^{2} + x(4 - c_{2}^{2}) \]

and

\[ 4c_{3} = c_{3}^{2} + 2(4 - c_{2}^{2})c_{1}x - c_{1}(4 - c_{2}^{2})x^{2} + 2(4 - c_{2}^{2}) \left( 1 - |x|^{2} \right) z, \]

for some \( x, z \) with \( |x| \leq 1 \) and \( |z| \leq 1 \).

2. **Main results**

Our first main result for the class \( S_{\mu}^{\ast}(\beta) \) as follows:

**Theorem 2.1.** Let \( f(z) \) given by \( f(z) = \frac{c_{2}}{z} + c_{3}z^{2} + \ldots \) be in the class \( S_{\mu}^{\ast}(\beta), 0 \leq \beta < 1 \). Then

\[ |a_{2}a_{4} - a_{2}^{2}| \leq \begin{cases}
    \frac{1}{2} (1 - \beta)^{2} (4\beta^{2} - 8\beta + 5), & \beta \in \left[ 0, \frac{2 - \sqrt{13}}{4} \right], \\
    \frac{1}{2(1 - \beta)} (13\beta^{2} - 14\beta + 2), & \beta \in \left( \frac{2 - \sqrt{13}}{4}, \frac{1}{2} \right].
\end{cases} \]
Thus, we can easily establish that

\[ \frac{zf'(z)}{f(z)} = \beta + (1 - \beta)p(z) \quad \text{and} \quad \frac{wg'(w)}{g(w)} = \beta + (1 - \beta)q(w) \]

where \( p(z) = 1 + c_1z + c_2z^2 + \ldots \) and \( q(w) = 1 + d_1w + d_2w^2 + \ldots \) in \( \mathcal{P} \).

Comparing coefficients in (2.2), we have

\begin{align*}
(2.3) & \quad a_2 = (1 - \beta)c_1, \\
(2.4) & \quad 2a_3 - a_2^2 = (1 - \beta)c_2, \\
(2.5) & \quad 3a_4 - 3a_3a_2 + a_2^3 = (1 - \beta)c_3
\end{align*}

and

\begin{align*}
(2.6) & \quad -a_2 = (1 - \beta)d_1, \\
(2.7) & \quad 3a_2^2 - 2a_3 = (1 - \beta)d_2, \\
(2.8) & \quad -10a_2^3 + 12a_3a_2 - 3a_4 = (1 - \beta)d_3.
\end{align*}

From (2.8) and (2.10), we arrive at

\[ c_1 = -d_1 \]

and

\[ a_2 = (1 - \beta)c_1. \]

Now, from (2.4), (2.7) and (2.10), we get that

\[ a_3 = (1 - \beta)^2 c_1^2 + \frac{(1 - \beta)}{4} (c_2 - d_2). \]

Also, from (2.5) and (2.8), we find that

\[ a_4 = \frac{2}{3} (1 - \beta)^3 c_1^3 + \frac{5}{8} (1 - \beta)^2 c_1 (c_2 - d_2) + \frac{1}{6} (1 - \beta) (c_3 - d_3). \]

Thus, we can easily establish that

\[ |a_2a_4 - a_3^2| = \left| -\frac{1}{3} (1 - \beta)^4 c_1^4 + \frac{1}{8} (1 - \beta)^3 c_1^2 (c_2 - d_2) \\
+ \frac{1}{6} (1 - \beta)^2 c_1 (c_3 - d_3) - \frac{1}{16} (1 - \beta)^2 (c_2 - d_2)^2 \right|. \]

According to Lemma 1.2 and (2.9), we write

\[ \begin{cases} 
2c_2 = c_1^2 + x(4 - c_1^2) \\
2d_2 = d_1^2 + x(4 - d_1^2)
\end{cases} \implies c_2 - d_2 = \frac{4 - c_1^2}{2}(x - y) \]

and

\begin{align*}
4c_1 &= c_1^4 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2) \left( 1 - |x|^2 \right) z, \\
4d_1 &= d_1^4 + 2(4 - d_1^2)d_1y - d_1(4 - d_1^2)y^2 + 2(4 - d_1^2) \left( 1 - |y|^2 \right) w,
\end{align*}

\[ c_3 - d_3 = \frac{c_3^3}{2} + \frac{c_1(4 - c_1^2)}{2}(x + y) - \frac{c_1(4 - c_1^2)}{2}(x^2 + y^2) + \frac{4 - c_1^2}{2} \left( (1 - |x|^2) z - (1 - |y|^2) w \right). \]
for some $x, y, z, w$ with $|x| \leq 1, |y| \leq 1, |z| \leq 1$ and $|w| \leq 1$. Using (2.14) and (2.15) in (2.13), and applying the triangle inequality we have

\[
|a_2a_4 - a_3^2| = \left| \frac{1}{3} (1 - \beta)^4 c_1^4 + \frac{1}{16} (1 - \beta)^3 (4 - c_1^2)(x - y) + \frac{1}{6} (1 - \beta)^2 c_1 (4 - c_1^3)(x + y) \right|
\]

\[
\geq \frac{1}{6} (1 - \beta)^3 c_1^2 (4 - c_1^2) (|x| + |y|)
\]

\[
+ \frac{1}{12} (1 - \beta)^2 c_1 (4 - c_1^2) (x^2 + y^2) + \frac{1}{64} (1 - \beta)^2 (4 - c_1^2)^2 (|x| + |y|)^2.
\]

Since $p \in \mathcal{P}$, so $|c_1| \leq 2$. Letting $c_1 = c$, we may assume without restriction that $c \in [0, 2]$. Thus, for $\lambda = |x| \leq 1$ and $\mu = |y| \leq 1$ we obtain

\[
|a_2a_4 - a_3^2| \leq T_1 + T_2(\lambda + \mu) + T_3(\lambda^2 + \mu^2) + T_4(\lambda + \mu)^2 = F(\lambda, \mu)
\]

where

\[
T_1 = T_1(c) = \frac{(1 - \beta)^3}{12} \left[ (1 + 4(1 - \beta)^2) c^4 - 2c^2 + 8c \right] \geq 0,
\]

\[
T_2 = T_2(c) = \frac{1}{48} (1 - \beta)^2 c^2 (4 - c^2)(7 - 3\beta) \geq 0,
\]

\[
T_3 = T_3(c) = \frac{1}{24} (1 - \beta)^2 c(4 - c^2)(c - 2) \leq 0,
\]

\[
T_4 = T_4(c) = \frac{1}{64} (1 - \beta)^2 (4 - c^2)^2 \geq 0.
\]

Now we need to maximize $F(\lambda, \mu)$ in the closed square $S = \{(\lambda, \mu) : 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1\}$. Since $T_3 < 0$ and $T_3 + 2T_4 > 0$ for $c \in [0, 2)$, we conclude that

\[
F(\lambda, \mu) = G(\mu) = (T_3 + T_4) \mu^2 + T_2 \mu + T_1.
\]

Thus the function $F$ cannot have a local maximum in the interior of the square $S$. Now, we investigate the maximum of $F$ on the boundary of the square $S$.

For $\lambda = 0$ and $0 \leq \mu \leq 1$ (similarly $\mu = 0$ and $0 \leq \lambda \leq 1$), we obtain

\[
F(0, \mu) = G(\mu) = (T_3 + T_4) \mu^2 + T_2 \mu + T_1.
\]

i. The case $T_3 + T_4 \geq 0$ : In this case for $0 < \mu < 1$ and any fixed $c$ with $0 \leq c < 2$, it is clear that $G'(\mu) = 2(T_3 + T_4) \mu + T_2 > 0$, that is, $G(\mu)$ is an increasing function. Hence, for fixed $c \in [0, 2)$, the maximum of $G(\mu)$ occurs at $\mu = 1$, and

\[
\max G(\mu) = G(1) = T_1 + T_2 + T_3 + T_4.
\]

ii. The case $T_3 + T_4 < 0$ : Since $T_2 + 2(T_3 + T_4) \geq 0$ for $0 < \mu < 1$ and any fixed $c$ with $0 \leq c < 2$, it is clear that $T_2 + 2(T_3 + T_4) < 2(T_3 + T_4) \mu + T_2 < T_2$ and so $G'(\mu) > 0$. Hence for fixed $c \in [0, 2)$, the maximum of $G(\mu)$ occurs at $\mu = 1$.

Also for $c = 2$ we obtain

\[
F(\lambda, \mu) = \frac{4}{3} (1 - \beta)^2 (4\beta^2 - 8\beta + 5).
\]

Taking into account the value (2.16), and the cases i and ii, for $0 \leq \mu \leq 1$ and any fixed $c$ with $0 \leq c \leq 2$,

\[
\max G(\mu) = G(1) = T_1 + T_2 + T_3 + T_4.
\]

For $\lambda = 1$ and $0 \leq \mu \leq 1$ (similarly $\mu = 1$ and $0 \leq \lambda \leq 1$), we obtain

\[
F(1, \mu) = H(\mu) = (T_3 + T_4) \mu^2 + (T_2 + 2T_4) \mu + T_1 + T_2 + T_3 + T_4.
\]

Similarly to the above cases of $T_3 + T_4$, we get that

\[
\max H(\mu) = H(1) = T_1 + 2T_2 + 2T_3 + 4T_4.
\]
Since \( G(1) \leq H(1) \) for \( c \in [0,2] \), max \( F(\lambda, \mu) = F(1,1) \) on the boundary of the square \( S \). Thus the maximum of \( F \) occurs at \( \lambda = 1 \) and \( \mu = 1 \) in the closed square \( \overline{S} \).

Let \( K : [0,2] \to \mathbb{R} \) \( \quad (2.17) \)

Let \( K(c) = \max F(\lambda, \mu) = F(1,1) = T_1 + 2T_2 + 2T_3 + 4T_4 \).

Substituting the values of \( T_1, T_2, T_3 \) and \( T_4 \) in the function \( K \) defined by (2.17), yield
\[
K(c) = \frac{(1 - \beta)^2}{48} \left[ (16 \beta^2 - 26 \beta + 5) c^4 + 24(2 - \beta)c^2 + 48 \right].
\]

Assume that \( K(c) \) has a maximum value in an interior of \( c \in [0,2] \), by elementary calculation we find
\[
K'(c) = \frac{(1 - \beta)^2}{12} \left[ (16 \beta^2 - 26 \beta + 5) c^4 + 12(2 - \beta)c \right].
\]

As a result of some calculations we can do the following examine:

**Case 1:** Let \( 16 \beta^2 - 26 \beta + 5 \geq 0 \), that is, \( \beta \in \left[0, \frac{13 - \sqrt{481}}{16}\right] \). Therefore \( K'(c) > 0 \) for \( c \in (0,2) \). Since \( K \) is an increasing function in the interval \( (0,2) \), maximum point of \( K \) must be on the boundary of \( c \in [0,2] \), that is, \( c = 2 \). Thus, we have
\[
\max_{0 \leq c \leq 2} K(c) = K(2) = \frac{4}{3} (1 - \beta)^2 (4 \beta^2 - 8 \beta + 5).
\]

**Case 2:** Let \( 16 \beta^2 - 26 \beta + 5 < 0 \), that is, \( \beta \in \left(\frac{13 - \sqrt{481}}{16}, 1\right) \). Then \( K'(c) = 0 \) implies the real critical point \( c_0 = 0 \) or \( c_0 = \sqrt{\frac{-12(2 - \beta)}{16 \beta^2 - 26 \beta + 5}} \). When \( \beta \in \left(\frac{29 - \sqrt{577}}{32}, \frac{29 + \sqrt{577}}{32}\right) \), we observe that \( c_0 \geq 2 \), that is, \( c_0 \) is out of the interval \( (0,2) \). Therefore the maximum value of \( K(c) \) occurs at \( c_0 = 0 \) or \( c = c_0 \) which contradicts our assumption of having the maximum value at the interior point of \( c \in (0,2) \). Since \( K \) is an increasing function in the interval \( (0,2) \), maximum point of \( K \) must be on the boundary of \( c \in [0,2] \), that is, \( c = 2 \). Thus, we have
\[
\max_{0 \leq c \leq 2} K(c) = K(2) = \frac{4}{3} (1 - \beta)^2 (4 \beta^2 - 8 \beta + 5).
\]

When \( \beta \in \left(\frac{29 - \sqrt{577}}{32}, 1\right) \) we observe that \( c_0 < 2 \), that is, \( c_0 \) is interior of the interval \([0,2]\). Since \( K''(c_0) < 0 \), the maximum value of \( K(c) \) occurs at \( c = c_0 \). Thus, we have
\[
\max_{0 \leq c \leq 2} K(c) = K(c_0) = K\left(\sqrt{\frac{-12(2 - \beta)}{16 \beta^2 - 26 \beta + 5}}\right) = (1 - \beta)^2 \left(\frac{13 \beta^2 - 14 \beta - 7}{16 \beta^2 - 26 \beta + 5}\right).
\]

This completes the proof of the Theorem 2.1. \( \square \)

For \( \beta = 0 \), Theorem 2.1 readily yields the following coefficient estimates for bi-starlike functions.

**Corollary 2.2.** Let \( f(z) \) given by (1.1) be in the class \( S^*_2 \). Then
\[
|a_{2}a_{4} - a_{3}^2| \leq \frac{20}{3}.
\]

Our second main result for the class \( K_0(\beta) \) is following:

**Theorem 2.3.** Let \( f(z) \) given by (1.1) be in the class \( K_\sigma(\beta) \), \( 0 \leq \beta < 1 \). Then
\[
|a_{2}a_{4} - a_{3}^2| \leq \frac{(1 - \beta)^2}{24} \left(\frac{5 \beta^2 + 8 \beta - 32}{3 \beta^2 - 3 \beta - 4}\right).
\]

**Proof.** Let \( f \in K_\sigma(\beta) \) and \( g = f^{-1} \). Then
\[
1 + \frac{zf''(z)}{f'(z)} = \beta + (1 - \beta)p(z) + 1 + \frac{ug''(w)}{g'(w)} = \beta + (1 - \beta)q(w)
\]

where \( p(z) = 1 + c_1 z + c_2 z^2 + \ldots \) and \( q(w) = 1 + d_1 w + d_2 w^2 + \ldots \) in \( P \).

Now, equating the coefficients in (2.20), we have
\[
\begin{align*}
2a_2 & = (1 - \beta)c_1, \\
6a_4 - 4a_2^2 & = (1 - \beta)c_2, \\
12a_4 - 18a_3a_2 + 8a_2^3 & = (1 - \beta)c_3
\end{align*}
\]
and
\begin{align*}
(2.24) & \quad -2a_2 = (1 - \beta)d_1, \\
(2.25) & \quad 8a_2^2 - 6a_3 = (1 - \beta)d_2, \\
(2.26) & \quad -32a_3^3 + 42a_3a_2 - 12a_4 = (1 - \beta)d_3. 
\end{align*}
From (2.21) and (2.24), we arrive at
\begin{equation}
(2.27) \quad c_1 = -d_1 
\end{equation}
and
\begin{equation}
(2.28) \quad a_2 = \frac{1}{2} (1 - \beta)c_1. 
\end{equation}
Now, from (2.22), (2.25) and (2.28), we get that
\begin{equation}
(2.29) \quad a_3 = \frac{1}{4} (1 - \beta)^2 c_1^2 + \frac{1}{12} (1 - \beta) (c_2 - d_2). 
\end{equation}
Also, from (2.23) and (2.26), we find that
\begin{equation}
(2.30) \quad a_4 = \frac{5}{48} (1 - \beta)^3 c_1^4 + \frac{5}{48} (1 - \beta)^2 c_1 (c_2 - d_2) + \frac{1}{24} (1 - \beta) (c_3 - d_3). 
\end{equation}
Thus, we can easily establish that
\begin{equation}
|a_2 a_4 - a_3^2| = \left| -\frac{1}{96} (1 - \beta)^4 c_1^4 + \frac{1}{192} (1 - \beta)^3 c_1^2 (4 - c_1^2) (x - y) \\
+ \frac{1}{48} (1 - \beta)^2 c_1 \left[ c_1^3 \left( \frac{1}{2} + \frac{4 - c_1^2}{4} \right)(x + y) - \frac{4 - c_1^2}{4} c_1 (x^2 + y^2) + \frac{(1 - \beta)c_1^2}{2} \right] (\beta x z - \beta y w) \right| \\
- \frac{1}{288} (1 - \beta)^2 (4 - c_1^2)^2 (x - y)^2 \\
\leq \frac{1}{96} (1 - \beta)^4 c_1^4 + \frac{1}{96} (1 - \beta)^2 c_1^4 + \frac{1}{48} (1 - \beta)^2 c_1 (4 - c_1^2) \\
+ \left[ \frac{1}{192} (1 - \beta)^3 c_1^2 (4 - c_1^2) + \frac{1}{96} (1 - \beta)^2 c_1^2 (4 - c_1^2) \right] (|x| + |y|) \\
+ \left[ \frac{1}{192} (1 - \beta)^2 c_1^2 (4 - c_1^2) - \frac{1}{96} (1 - \beta)^2 c_1 (4 - c_1^2) \right] (|x|^2 + |y|^2) + \frac{1}{576} (1 - \beta)^2 (4 - c_1^2)^2 (|x| + |y|)^2. 
\end{equation}
Since \( p \in \mathcal{P} \), so \(|c_1| \leq 2\). Taking \( c_1 = c \), we may assume without restriction that \( c \in [0, 2] \). Thus, for \( \lambda = |x| \leq 1 \) and \( \mu = |y| \leq 1 \) we obtain
\begin{equation}
|a_2 a_4 - a_3^2| \leq M_1 + M_2(\lambda + \mu) + M_3(\lambda^2 + \mu^2) + M_4(\lambda + \mu)^2 = \Psi(\lambda, \mu) 
\end{equation}
where
\begin{align*}
M_1 &= M_1(c) = \frac{1 - \beta}{96}\left[(1 + (1 - \beta)^2)^c e^4 - 2c^3 + 8c\right] \geq 0, \\
M_2 &= M_2(c) = \frac{1}{192} (1 - \beta)^2 c_1^2 (4 - c_1^2)(3 - \beta) \geq 0, \\
M_3 &= M_3(c) = \frac{1}{192} (1 - \beta)^2 c_1 (4 - c_1^2) (c - 2) \leq 0, \\
M_4 &= M_4(c) = \frac{1}{576} (1 - \beta)^2 (4 - c_1^2)^2 \geq 0. 
\end{align*}
Therefore we need to maximize \( \Psi(\lambda, \mu) \) in the closed square \( \mathcal{S} = \{(\lambda, \mu) : 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1 \} \).
To show that the maximum of \( \Psi \) we can follow the maximum of \( F \) in the Theorem 2.1. Thus the maximum of \( \Psi \) occurs at \( \lambda = 1 \) and \( \mu = 1 \) in the closed square \( \mathcal{S} \). Let \( \Phi : [0, 2] \to \mathbb{R} \) defined by
\begin{equation}
\Phi(c) = \max \Psi(\lambda, \mu) = \Psi(1, 1) = M_1 + 2M_2 + 2M_3 + 4M_4. 
\end{equation}
Substituting the values of $M_1, M_2, M_3$ and $M_4$ in the function $\Phi$ given by (2.32), yield

$$\Phi(c) = \frac{(1 - \beta)^2}{288} \left[ (3\beta^2 - 3\beta - 4) c^4 + 4(8 - 3\beta)c^2 + 32 \right].$$

Assume that $\Phi(c)$ has a maximum value in an interior of $c \in [0, 2]$, by elementary calculation we find

$$\Phi'(c) = \frac{(1 - \beta)^2}{72} \left[ (3\beta^2 - 3\beta - 4) c^3 + 2(8 - 3\beta)c \right].$$

Setting $\Phi'(c) = 0$, since $0 < c < 2$, and $3\beta^2 - 3\beta - 4 < 0$ and $8 - 3\beta > 0$ for every $\beta \in [0, 1)$, we have the real critical point $c_0 = \sqrt{\frac{2(3\beta - 8)}{3\beta^2 - 3\beta - 4}}$. Since $c_0 \leq 2$ for every $\beta \in [0, 1)$ and so $\Phi''(c_0) < 0$, the maximum value of $\Phi(c)$ corresponds to $c = c_0$, that is,

$$\max_{0 < c < 2} \Phi(c) = \Phi(c_0) = \Phi \left( \sqrt{\frac{2(3\beta - 8)}{3\beta^2 - 3\beta - 4}} \right) = \frac{(1 - \beta)^2}{24} \left( \frac{5\beta^2 + 8\beta - 32}{3\beta^2 - 3\beta - 4} \right).$$

On the other hand,

$$\Phi(0) = \frac{(1 - \beta)^2}{9} \quad \text{and} \quad \Phi(2) = \frac{(1 - \beta)^2}{6} \left( \beta^2 - 2\beta + 2 \right).$$

Consequently, since $\Phi(0) < \Phi(2) \leq \Phi(c_0)$ we obtain $\max_{0 < c < 2} \Phi(c) = \Phi(c_0)$.

This completes the proof of the Theorem 2.3. \( \square \)

For $\beta = 0$, Theorem 2.3 readily yields the following coefficient estimates for bi-convex functions.

**Corollary 2.4.** Let $f(z)$ given by (1.1) be in the class $\mathcal{K}_\sigma$. Then

$$|a_2a_4 - a_3^2| \leq \frac{1}{3}.$$

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Department of Mathematics, Faculty of Science and Letters, Kafkas University, Kars, Turkey.
E-mail address: edeniz36@gmail.com (Erhan Deniz), mcaglar25@gmail.com (Murat Çağlar)

Department of Mathematics, Faculty of Science, Ataturk University, Erzurum, 25240, Turkey.
E-mail address: orhanhalit607@gmail.com (Halit Orhan)