AN ITERATIVE ENERGY ESTIMATE FOR DEGENERATE EINSTEIN MODEL OF BROWNIAN MOTION

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Abstract. We consider the degenerate Einstein’s Brownian motion model when the time interval $\tau$ of free jumps (particle-jumps before the collisions), reciprocals to the number of particles per unit volume $u(x, t) \geq 0$, at the point of observation $x$ at time $t$. The parameter $\tau \in (0, C]$, which controls the characteristics of the fluid, "almost decreases", with respect to $u$, and converges to $\infty$ as $u \to 0$. This degeneration leads to the localization of the particle-distribution in the media. In the paper, we present a structural condition of the time interval and the frequency of these free jumps as functions of $u$ which guarantees the finite speed of propagation of $u$.

1. Introduction

Consider the thought experiment of the fluid which occupy a bounded domain $\Omega \subset \mathbb{R}^N$ by the particles for $N = 1, 2, \ldots$. Let $\vec{\Delta} \triangleq \Delta_1, \Delta_2, \cdots, \Delta_N >$ be the $N$ dimensional vector of free jump, which we define as the particles’ jumps without collisions. In definition of free jump we follow the classical Einstein paradigm in his famous thesis \cite{4}, where in literature the term "free pass" is used instead (see \cite{13}). From this point of view events of free jump and "free pass" are synonyms.

Einstein assumed that process of random motion of the particles is characterized by events of free jumps. He proposed two main parameters which characterize free jumps: the time
interval \( \tau \) within which the free jumps occur, and frequency \( \varphi(\Delta) \) of the occurrences of free jumps of the length \( \Delta \). Einstein assumed that diffusion: \[ \left[ \int_{-\infty}^{\infty} \Delta^2 \varphi(\Delta) \, d\Delta \right] / \tau \] is to be constants, which allowed him to reduce his thought experiment to the classical heat equation with constant coefficients, where that exhibits the so-called effect of infinite speed of the perturbation, namely, if \( u(x_0, t_0) \) be the concentration of particles at some moment of time \( t_0 \) and at any point \( x_0 \), is positive , then \( u(x, t) > 0 \) for all \( x \) and \( t \geq t_0 \). This property of \( u(x, t) \) is very unrealistic. The question which we asked in this article is as follows: Can Einstein’s paradigm of free jumps be generalised in such way that one sees finite speed of propagation of the particles, i.e., if \( u(x, 0) = 0 \) at some point \( x_0 \) then \( u(x_0, t) = 0 \) during time interval \([0, T]\) for some \( T > 0 \)?

In this article, we will present that if time interval of free jumps \( \tau \) is inverse proportional to the concentration \( u \), and/or \( u \) is proportional to the variance \( \sigma^2 \equiv \left[ \int_{-\infty}^{\infty} \Delta^2 \varphi(\Delta) \, d\Delta \right] \), then under some assumptions of the proportionality, our solution of degenerate Einstein equation in IBVP(4.5) will exhibit the finite speed of propagation, which is closely relate to so-called Barenblatt solutions for degenerate porous medium equation (see [2]). The origin of the porous medium equation is differ from thought experiment of the Einstein paradigm. Namely, the first continuity equation hypothesised in the form:

\[
L(\rho, \vec{J}) \triangleq \rho_t + \nabla \cdot \left( \rho \vec{J} \right) = 0,
\]

where density \( \rho \in \mathbb{R} \), flux \( \vec{J} \in \mathbb{R}^N \) in porous media that equals to the velocity \( \vec{v} \) of the flow (see [1]). It is assumed that \( \rho \) is function of the pressure \( p \) and can be approximated based on thermodynamic experiment. For example, \( \rho = p^\lambda \) for some gasses by thermodynamic laws and \( \vec{v} = -\frac{k}{\mu} \nabla p \), where \( k \) - permeability of porous media and \( \mu \) is viscosity subject to the experimental Darcy equation (see [10]). Combining these experimental relations in (1.1) one can get the following divergent equation for scalar pressure function:

\[
L(w) = (w)_t - \frac{k}{\mu} \frac{1}{\lambda + 1} \Delta \left( w^{\frac{\lambda + 1}{\lambda}} \right) = 0,
\]

where \( w = p^\lambda \) and \( \lambda > 0 \). In the pioneering work by G. I. Barenblat, A.S. Kompaneetz and Ya.B. Zeldovich (see [5]), it was shown that under specific initial and boundary conditions there exists a self-similar solution of the equation (1.2), which exhibits finite speed of propagation. Evidently, the Einstein operator \( L_E \) in (3.7) is in nondivergent form that is based
on the thought experiment, which allows to interpret the results on observable data in term of the length of free jumps, and then adjust parameters $\tau$ and $\varphi$ of the model to execute further analysis. In this paper, we will follows Einstein approach but will use technique for divergent equation.

The article is organized as follows. In §2, we consider generalization of classical Einstein model of Brownian motion to $\mathbb{R}^N$, when the major parameters of the system, i.e., time interval $\tau$ of the free jumps and the frequency $\varphi$ of the free jumps depend on the concentration of the number of particle per unite volume $u$. We use the generic mass conservation law with absorption-reaction term and basic stochastic principles to derive a partial differential inequality (PDI) under the Einstein’s axioms in §3. By introducing local forces and nonlocality processes, we conclude the deterministic PDI 3.7 which governs the dynamics of the generalized Einstein paradigm. Using Hypothesis 1 in §4, we define $u$ as the weakly approximated solution to the nonlinear initial boundary value problem 4.5 such that holds the limit in (4.7). We Introduce the assumption on functions $H, F$ and $G$ in §5 to explicitly structures coefficients as in the functions $H$ in (5.6) and $F$ in (5.8) in a way that $u$ conserves the finite speed of propagation in §7. In §6, we establish collaborating Lemmas and introduce Ladyzhenskaya Iterative scheme which apply in §7 to prove the localization property of $u$ based on De-Georgi’s construction, if the initial energy functional $Y_0[T]$ preserves certain boundednes with respect to some $T' > 0$. §8 is focused on models of the functions that attest all constrains on the functions $F, G$ and $H$, which consequently guarantees the localization theorem 7. We present auxiliary results of functional spaces in §10 to prove the uniform boundednesses regarding the classical solution $u^\varepsilon$ which ensures its limit as in (4.7).

2. Generalized Einstein paradigm

Let $u(x,t)$ be the function which represent the number of particles per unit volume, suspended in the medium of interest at point $x \in \mathbb{R}^N$ and at time $t > 0$. Denote $\mathcal{J}(\tau)$ to be the set of vectors with noncolliding jumps corresponding to the time interval $\tau$. We call $\vec{\Delta} \in \mathbb{R}^N$ to be a vector of free jump of particles if $\vec{\Delta} \in \mathcal{J}(\tau)$. Assume the following extension of the axioms in classical Einstein Brownian motion.
Assumption 1.  
(i) Time interval of free jumps $\tau$, expected vector $\vec{\Delta}$ of a free jump $\vec{\Delta}$ and probability density function of free jump $\varphi(\vec{\Delta})$ are the only parameters which characterize process of free jumps. Note that in a view of the definition of the set $\mathcal{J}(\tau)$, if $\vec{\Delta} / \in \mathcal{J}(\tau)$ then $\varphi(\vec{\Delta}) = 0$.

(ii) The key parameters $\tau$ and $\varphi$ can depend on the concentration of the particles and also the space-time coordinate.

Axiom 1. Whole universe axiom:

(2.1) \[ \int_{\mathcal{J}(\tau)} \varphi(\vec{\Delta}) d\vec{\Delta} = 1. \]

Assumption 2. Symmetry of free jumps:

(2.2) \[ \varphi(\vec{\Delta}) = \varphi(-\vec{\Delta}). \]

Existence of second moments:

(2.3) \[ \int_{\mathcal{F}(\tau)} |\vec{\Delta}|^n \varphi(\vec{\Delta}) d\vec{\Delta} < \infty. \]

Note that it follows from the preceding assumption that the expectation of $\vec{\Delta}$ equals zero, and there exists a covariance matrix $[\sigma_{ij}^2]$ of free jumps, where

(2.4) \[ \sigma_{ij}^2 \equiv \int_{\mathcal{J}(\tau)} \Delta_i \Delta_j \varphi(\vec{\Delta}) d\vec{\Delta}, \quad \text{for } i, j = 1, 2, \ldots, N. \]

Evidently $\sigma_{ij}^2(x, t)$ depend on space $x$ and time $t$. We postulate generalized Einstein’s axiom for the number of particles found at point $x$ at time $t + \tau$ a, in the control volume $dv$ by:

Axiom of Mass Conservation.

(2.5) \[ u(x, t + \tau) \cdot dv = \int_{\mathcal{J}(\tau)} u(x + \vec{\Delta}, t) \varphi(\vec{\Delta}) d\vec{\Delta} \cdot dv + \tau \cdot \int_{\mathcal{J}(\tau)} M(x + \vec{\Delta}, t) \varphi(\vec{\Delta}) d\vec{\Delta} \cdot dv. \]

The axiom intuitively expressed that at any given point in space $x$ at time $t + \tau$, we will observe density of all particles with free jumps from the point $x$ at time $t$, "+ density" of particles which "produced" and "− density" of particles which "consumed" during time interval $[t, t + \tau]$. For comparison, see the first formula with integral on page 14 in [4]. We modeled (2.5) by adding the term $M(\cdot)$ which reports the process of rate of absorption-consumption during the time interval $[t, t + \tau]$ due to particles interaction along of free jumps.
We consider the scenario, when the consumption dominates over the production by setting $M(\cdot) \leq 0$.

**Remark 1.** Einstein definition of density of particles in [4] differs from the fundamental definition of the density of fluid, rather means the concentration of the volume of interest.

### 3. Derivation of Partial differential Inequality

In this section, we derive partial differential inequality whose solution exhibits the feature of finite speed of propagation. Let $\zeta = (\zeta_1, \zeta_2, \cdots, \zeta_N) \in \mathbb{R}^N$. Assume that $u(x,t) \in C^{2,1}_{x,t}$. By Taylor’s Expansion [7] and using (2.1) - (2.4), we get

\begin{equation}
\int_{J(\tau)} u(x + \Delta, t) \varphi(\Delta) d\Delta = u(x, t) + \frac{1}{2} \sum_{i,j=1}^{N} \sigma_{ij}^2 u_{x_i x_j}(x, t) + R_\zeta,
\end{equation}

where

\begin{equation}
R_\zeta \triangleq \int_{J(\tau)} \sum_{|\zeta| = 2} H_\zeta(x, \Delta, t)(\Delta)^{\zeta} \varphi(\Delta) d\Delta
\end{equation}

with locally bounded function $H_\zeta$ such that $\lim_{\Delta \to 0} H_\zeta(x, \Delta, t) = 0$. Using (3.1) in (2.5) yields

\begin{equation}
u(x, t + \tau) - u(x, t) = \frac{1}{2} \sum_{i,j=1}^{N} \sigma_{ij}^2 u_{x_i x_j}(x, t) + R_\zeta + \tau \int_{J(\tau)} M(x + \Delta, t) \varphi(\Delta) d\Delta.
\end{equation}

Observe that (3.3) is defined at different points in space and time. We eliminate this ambiguity and derive the equation at the same point, by using Carathéodory’s differential criterion: there exists a function $\psi^t$ such that

\begin{equation}
u(x, t + \tau) - u(x, t) = \tau \psi^t(x, t, \tau),
\end{equation}

where $\lim_{s \to 0} \psi^t(x, t, s) = u_t(x, t)$. Using (3.4) in (3.3) we get

\begin{equation}
\tau u_t(x, t) = \frac{1}{2} \sum_{i,j=1}^{N} \sigma_{ij}^2 u_{x_i x_j}(x, t) + R_\zeta + \tau \int_{J(\tau)} M(x + \Delta, t) \varphi(\Delta) d\Delta.
\end{equation}

For the forthcoming iterative procedure, we assume that

**Assumption 3.**

\begin{equation}
R_\zeta + \tau \int_{J(\tau)} M(x + \Delta, t) \varphi(\Delta) d\Delta \leq 0.
\end{equation}
First term $R_\zeta$ is responsible for nonlocality of the process. $R_\xi$ reflects concentration jumps in time, for instance, due to birth and death in biological system. The last term in (3.6) can be interpreted as a local force.

For example if $M$ noise during time of observation and therefore it can be stochastic. Using (3.6) in (3.5) and approximating the first order terms, it follows that the function $u(x,t)$ can be estimated by the following partial differential inequality:

\[ L_E u = \tau u_t - \sum_{i,j=1}^{N} \sigma_{ij}^2 u_{x_i x_j} \leq 0. \]  

\[ (3.7) \]

Observe that the partial differential inequality (3.7) is deterministic, while the stochastic nature is modeled by functions $\tau$ and $\varphi$ (the latter appears in (3.7) in the form of the covariance $\sigma_{ij}^2$ in (2.4)), which are key characteristics of the process dynamics. In general, they can be functions of spatial $x$ and time $t$ variables, concentration $u$ and its gradient $\nabla u$.

In this report, we present $\tau$ and $\sigma_{ij}^2$ satisfy the following Hypothesis in addition to (2.1) and (2.2).

**Hypotheses 1.** Let $P \in C[0, \infty)$ be a function such that

\[ c_1 P(u)|\xi|^2 \leq \sum_{i,j=1}^{N} \frac{1}{\tau} \sigma_{ij}^2 \xi_i \xi_j \leq c_2 P(u)|\xi|^2, \quad \xi \in \mathbb{R}^N, \]

for some $0 < c_1 < c_2$. Here $P(0) = 0$ and $0 < P(s) < c_3 < \infty$ when $s > 0$.

**Remark 2.** In (3.8) $P(u)$, is a composite parameter of stochastic processes of free jumps, which characterizes the relative covariance with respect to the time interval of free jumps.

In our scenario, $P(u)$ degenerate at $u = 0$ which mean that the growth due to dispersion is slower than the time interval of dispersion, as the concentration vanishes.

4. **Nonlinear Degenerate IBVP**

For the sake of simplicity, we analyse the case when the covariance matrix is diagonal and positively defined. For $i, j = 1, \ldots, N$, let $\sigma_{ij}^2 = [\sigma^2(u)] \delta_{ij}$ for some $[\sigma^2(u)] > 0$. Consequently $P = [\sigma^2(u)]/[\tau(u)]$, and (3.7) takes form

\[ u_t \leq P(u) \Delta u. \]

(4.1)

In order to derive the nonlinear degenerate IBVP, and to study structural forms of the coefficients, we define the functions $H$, $F$ and $G$ as follows.
Definition 1. (D-1) Let \( h > 0 \) such that \( h \in C(0, \infty) \) and integrable at 0. Then
\[
H(u) \triangleq \int_0^u h(s) \, ds.
\]

(D-2) Let \( F \triangleq hP \) and \( h, P \) be such that \( F(0) = 0 \), and \( F \) is differentiable on \((0, \infty)\) with a locally bounded derivative \( F' > 0 \).

(D-3) Let \( G \) be such that \( \sqrt{F'(u)} \triangleq G'(u) \) and \( G(0) = 0 \). Then \( G(s) \triangleq \int_0^s \sqrt{F'(s)} \, ds \).

Remark 3. Note that \( 0 \leq G(u) = \int_0^u G'(s)ds \leq \sqrt{u \int_0^u F'(s)ds} = \sqrt{uF(u)} \), and all functions \( F, G \) and \( H \) are increasing on closed interval.

We multiply (4.1) by \( h(u) \) and obtain
\[
Lu \triangleq [H(u)]_t - F(u)\Delta u \quad \text{on} \quad \Omega \times (0, T],
\]
for some \( T > 0 \). Here \( u \) is positive measurable function such that \( u(\cdot, t) \to u(\cdot, 0) \) as \( t \to 0 \) in local measure, with \( u \in L^\infty_{\text{loc}}(\Omega \times [0, T]) \), \( \nabla u \in L^2_{\text{loc}}(\Omega \times (0, T]) \) and \( u_t \in L^1_{\text{loc}}(\Omega \times (0, T]) \).

Hence \( F(u)\Delta u \) is understood in the weak sense, i.e., for all \( \theta \in \text{Lip}_c(\Omega) \) we write
\[
-\int_\Omega \theta F(u)\Delta u \, dx = \int_\Omega \left[ F(u)(\nabla u)(\nabla \theta) + F'(u)|\nabla u|^2 \theta \right] \, dx
\]
\[
= \int_\Omega \left[ F(u)(\nabla u)(\nabla \theta) + |\nabla G(u)|^2 \theta \right] \, dx.
\]

Then under the Hypotheses 1, we defined \( u(x, t) \) as a nonnegative solution of the following partial differential inequality
\[
\text{IBVP} = \begin{cases}
[H(u)]_t - F(u)\Delta u \leq 0 & \text{in} \quad \Omega \times (0, T], \\
u(x, 0) = 0 & \text{in} \quad \Omega' \Subset \Omega, \\
u(x, t) = 0 & \text{on} \quad \partial \Omega \times (0, T],
\end{cases}
\]
where \( u(x, 0) \geq 0 \), is continuous on \( \Omega \). The condition on the boundary \( \partial \Omega \times (0, T] \) is not essential to prove the finite speed of propagation, namely, for every ball \( B_R(x_0) \Subset \Omega' \) and \( c < 1 \), there exists \( T' = T'(x_0, c) \in (0, T] \) such that \( u(x, t) = 0 \) for all \( (x, t) \in B_{cR}(x_0) \times [0, T] \) in Theorem 1. Observe that \( u \) in IBVP (4.5) is degenerates when \( u \to 0 \). Therefore its solution \( u(x, t) \notin C^{2,1}_{x,t}(\Omega \times (0, T]) \). Our result is qualitative and does not address the existence of the
solutions, but the obtained property of the solution is applicable for a weakly approximated solution \( u \) (see [9], [11]), which is defined in the following regularized problem for some \( u^\varepsilon \in C^{2,1}_{x,t}(\Omega \times (0,T]) \cap C_0(\Omega \times (0,T]) \) as follows.

\[
\text{IBVP}_\varepsilon = \begin{cases} 
[H(u^\varepsilon)]_t - (F(u^\varepsilon) + \varepsilon) \Delta u^\varepsilon &= 0 \quad \text{in } \Omega \times (0,T], \\
u^\varepsilon(x,0) &= \varepsilon \quad \text{in } \Omega' \subset \Omega, \\
u^\varepsilon(x,t) &= \varepsilon \psi(x) \quad \text{on } \partial\Omega \times (0,T],
\end{cases}
\]

where \( u^\varepsilon(x,0) \) is continuous in \( \Omega \). In this article, the constants in all estimates for \( u^\varepsilon \) do not depend on \( \varepsilon \). This observation allows us to pass to the limit in the final estimates, and conclude the localization property for the limiting function

\[
u(x,t) = \lim_{\varepsilon \to 0} u^\varepsilon(x,t),
\]

which is considered as a weak passage to the limit (see [3]). The obtained function \( \nu(x,t) \) is called a weakly approximated solution of the IBVP (4.5), when the first differential inequality is replaced by the differential equation \( Lu = 0 \), which will exhibit localisation property. Further details on weakly approximated solution \( \nu \) of IBVP (4.5), will be presented in §10. For in detail discussions on the existence of weakly approximated solutions, see [9] and [11]. Hence further we will assume that \( \Omega \) is a domain with Lipschitz boundary.

5. Assumptions on Functions and Interpretation

Let us state main properties and additional assumption on the functions \( H \) and \( F \) on some finite domain: \([0,M]\).

Assumption 4.

(A-1) Exists \( C_1 > 0 \) such that \( F(s) \leq C_1 G'(s)G(s) \), where \( s \in [0,M] \).

(A-2) Exists \( C_2 > 0 \) such that \( (\sqrt{sF(s)})^\lambda \leq C_2 H(s) \), where \( 0 < \lambda < 2 \) and \( s \in [0,M] \).

We will choose \( H \) and \( P \) such that Assumption 4 holds.

Proposition 1. Let \( \Lambda + 1 = \frac{2}{\lambda} \). Assume \( F \) and \( H \) be as in (A-2). Then

\[
H(s) \geq \left( H^{-\Lambda}(M) + \Lambda C_2^{1+\Lambda} \int_s^M \frac{1}{\tau P(\tau)} d\tau \right)^{-\frac{1}{\Lambda}}.
\]
Proof. By raising the estimate in remark 3 to the power \( \lambda \) and using (A-2), we get

\[
G^\lambda(s) \leq \left( \sqrt{sF(s)} \right)^\lambda \leq C_2 H(s) \tag{5.2}
\]

\[
F(s) \leq C_2^{\frac{\lambda}{2}} \frac{H^\frac{\lambda}{2}(s)}{s}. \tag{5.3}
\]

By (D-2), we write (5.2) becomes

\[
\frac{h(s)}{H^{\Lambda+1}(s)} \leq \frac{C_2^{1+\Lambda}}{sP(s)} \tag{5.4}
\]

Then by integrating (5.4) over \((s, M]\), we obtain the estimate (5.1). \(\Box\)

The preceding proposition implies the following choice of function \(H\).

**Definition 2.** Let \(P\) in Hypotheses 1 be such that \(0 < \int_M^\infty \frac{ds}{sP(s)} < \infty\), for some finite \(M\). Define

\[
I(s) \triangleq \int_s^\infty \frac{d\sigma}{\sigma P(\sigma)} ; \quad M > s > 0. \tag{5.5}
\]

Consequently

\[
H(s) \triangleq [\Lambda I(s)]^{-\frac{1}{\Lambda}} = \left( \Lambda \int_s^\infty \frac{1}{\tau P(\tau)} d\tau \right)^{-\frac{1}{\Lambda}}, \quad s > 0. \tag{5.6}
\]

Function \(P(s)\), and consequently \(H(s)\) are defined only on bounded interval. In order notation to be simple, we extended \(I(s)\) on whole axis in such order that \(I(s) \geq c_4 > 0\) on \([M, \infty)\).

**Remark 4.** \(H\) in (5.6) has \(H(0) = 0\) since \(P(0) = 0\). By substituting (5.6) in (4.2),

\[
h(s) = \frac{1}{sP(s)} [\Lambda I(s)]^{-\frac{1}{\Lambda} - 1} = \frac{1}{sP(s)} H^{(\Lambda+1)}(s), \quad s > 0. \tag{5.7}
\]

Using (5.7) in (D-2), we get

\[
F(s) = h(s)P(s) = \frac{1}{s} H^{(\Lambda+1)}(s) = \left( \Lambda s^{\frac{1}{\Lambda}} I(s) \right)^{-\frac{1}{\Lambda} - 1}. \tag{5.8}
\]

Consequently, \([sF(s)]^{\frac{1}{\Lambda}} = H^{(\Lambda+1)}(s) = H(s)\), since \((\Lambda + 1)\frac{1}{\Lambda} = 1\). Thus, (A-2) holds with \(C_2 = 1\).

The next proposition imposes conditions on \(P\) in such a way that \(F\) is nonnegatively increasing on \((0, M]\).
Proposition 2. Let \( F = \left[ \Lambda s^{\frac{\Lambda}{\Lambda + 1}} I(s) \right]^{-\frac{1}{\Lambda + 1}} \). Assume \( \exists \) constants \( A, B \) such that

\[
\sup_{0 < s < M} P(s) I(s) = A, \tag{5.9}
\]

\[
\limsup_{s \to 0} P(s) I(s) = B < A. \tag{5.10}
\]

If \( \frac{\Lambda + 1}{\Lambda} > A \) then \( F'(s) > 0 \) on \( (0, M] \), and if \( \frac{\Lambda + 1}{\Lambda} > a \) then \( \lim_{s \to 0} F(s) = 0 \).

Proof. First we prove that function \( F \) is increasing. Observe that it suffices to prove that

\[
s \mapsto s^{\frac{\Lambda}{\Lambda + 1}} I(s), \tag{5.11}
\]

decreases on \( (0, M] \). Note that

\[
\frac{d}{ds} \left( s^{\frac{\Lambda}{\Lambda + 1}} I(s) \right) = \left( \frac{\Lambda}{\Lambda + 1} \frac{1}{P(s)} \right) \left( P(s) I(s) - \frac{\Lambda + 1}{\Lambda} \right) s^{-\frac{1}{\Lambda + 1}}. \tag{5.12}
\]

Thus, together with \( [\Lambda + 1]/[\Lambda] > A \), one has \( F'(s) > 0 \) for \( s \in (0, M] \). Next, we show that \( s^{\frac{\Lambda}{\Lambda + 1}} I(s) \to \infty \) when \( s \to 0 \), which implies \( \lim F(s) = 0 \). By (5.10), for every \( \epsilon > 0 \) there exists \( s_\epsilon \in (0, M] \) such that \( I(s) P(s) < a + \epsilon \) for \( s \in (0, s_\epsilon) \), which yields the following inequalities.

\[
I(s) < (a + \epsilon) \frac{1}{P(s)} = -(a + \epsilon) s I'(s),
\]

\[
\frac{d}{ds} \ln I(s) < -\frac{1}{a + \epsilon} \cdot \frac{1}{s},
\]

\[
I(s) > I(s_\epsilon) \left( \frac{s_\epsilon}{s} \right)^{\frac{1}{a + \epsilon}} \text{ for } 0 < s < s_\epsilon,
\]

\[
s^{\frac{\Lambda}{\Lambda + 1}} I(s) \geq I(s_\epsilon) s_\epsilon^{-\frac{1}{a + \epsilon}} \times s^{\frac{\Lambda}{\Lambda + 1}} s_\epsilon^{-\frac{1}{a + \epsilon}}.
\]

Since \( [\Lambda + 1]/[\Lambda] > B \), one can choose \( \epsilon \) such that \( \frac{\Lambda + 1}{\Lambda} > a + \epsilon \implies \frac{\Lambda}{\Lambda + 1} - \frac{1}{a + \epsilon} < 0 \). Hence \( s^{\frac{\Lambda}{\Lambda + 1}} I(s) \to \infty \) when \( s \to 0 \). \( \square \)

Remark 5. If \( A \) and \( B \) exist then indeed \( [\Lambda + 1]/[\Lambda] > A > B \) in Proposition 2. However, \( A \) is finite only if \( B \) is finite, due to \( P(s) I(s) \in C(0, M] \).

Next, we define the equivalence of functions and almost monotone functions.

Definition 3. (i) Functions \( f \) and \( g \) on a set \( E \) are equivalent and write \( f(x) \asymp g(x) \) on \( E \), if there exists a constant \( c \geq 1 \) such that \( c^{-1} g(x) \leq f(x) \leq c g(x) \) for all \( x \in E \).
A function $f$ on an interval $I \subset \mathbb{R}$ is called *almost decreasing (almost increasing)* if it is equivalent to a nonincreasing (nondecreasing) function of an interval. Equivalently, there exists $c > 0$ such that $f(t) \geq cf(s)$, where $t, s \in I$ and $t < s$.

In the following proposition, we provide the sufficient conditions for the assumption (A-1).

**Proposition 3.** Let $\exists \mu > 0$, such that $s \mapsto P(s)I^\mu(s)$ is an almost decreasing function. Then (A-1) holds.

**Proof.** By direct computation, it follows from (5.8) that

\begin{align}
F(s) &\simeq s^{-1}I^{-\frac{1}{2}}(s) \\
F'(s) &\simeq \frac{F(s)}{s}[P(s)I(s)]^{-1}.
\end{align}

Using (5.13), (5.14) and (D-3)

\begin{align}
\frac{F(s)}{[G(s)G'(s)]} = \frac{F(s)}{\left(\int_{0}^{s}\sqrt{F'(t)}\,dt\right)\left(\sqrt{F'(s)}\right)}
\end{align}

\begin{align}
&\leq \frac{F(s)}{\left(\int_{0}^{s}\sqrt{\frac{F(t)}{t}}[P(t)I(t)]^{-\frac{1}{2}}\,dt\right)\left(\sqrt{\frac{F(s)}{s}[P(s)I(s)]^{-1}}\right)}
&= \frac{[I(s)]^{-\frac{1}{2}}[P(s)I(s)]^{\frac{1}{2}}}{\int_{0}^{s}t^{-1}[I(t)]^{-\frac{1}{2}}[P(t)]^{-\frac{1}{2}}\,dt}
&= \frac{[I(s)]^{-\frac{1}{2}}[P(s)I^\mu(s)]^{\frac{1}{2}}}{\int_{0}^{s}t^{-1}[I(t)]^{-\frac{1}{2}}[P(t)]^{-\frac{1}{2}}\,dt}.
\end{align}

By the Cauchy’s mean value theorem, there exists $t \in (0, s)$ such that

\begin{align}
\frac{[I(s)]^{-\frac{1}{2}}[P(s)I(t)]^{-\frac{1}{2}}}{\int_{0}^{s}t^{-1}[I(t)]^{-\frac{1}{2}}[P(t)]^{-\frac{1}{2}}\,dt} = \left(\frac{1}{2\Lambda} + \frac{\mu}{2}\right) \frac{[I(t)]^{-\frac{1}{2}}[P(t)]^{-\frac{1}{2}}}{t^{-1}[I(t)]^{-\frac{1}{2}}[P(t)]^{-\frac{1}{2}}} \geq [P(t)I^\mu(t)]^{-\frac{1}{2}}.
\end{align}

Since $t \mapsto P(t)I^\mu(t)$ is almost decreasing, one has

\begin{align}
\frac{F(s)}{G(s)G'(s)} \simeq \left[\frac{P(s)I^\mu(s)}{P(t)I^\mu(t)}\right]^{\frac{1}{2}} \leq C; \quad 0 < t < s.
\end{align}

The sufficient condition for the existence of $\mu > 0$ for almost decreasing function $t \mapsto P(t)I^\mu(t)$ is given by the next remark.
Remark 6. Let \( \exists \tilde{P} \in C^1(0, \infty) \) such that \( P(s) \asymp \tilde{P}(s) \) on \( s \in (0, M] \), and

\[
\limsup_{s \to 0} s \tilde{I}(s) \tilde{P}'(s) < \infty,
\]

where \( \tilde{I}(s) \triangleq \int_s^\infty \frac{dt}{t \tilde{P}(t)} \). Consequently, \( s \mapsto s \tilde{I}(s) \tilde{P}'(s) \) is bounded on \( (0, M] \). Let

\[
B = \sup_{0 < s < M} s \tilde{I}(s) \tilde{P}'(s).
\]

Fix \( \mu \geq B \). Then the function \( Q(s) = \tilde{P}(s) \tilde{I}^\mu(s) \) is nonincreasing since

\[
Q'(s) = \tilde{P}'(s) \tilde{I}^\mu(s) - \mu s^{-1} \tilde{I}^{\mu-1}(s) = s^{-1} \tilde{I}^{\mu-1}(s) \left[ s \tilde{I}(s) \tilde{P}'(s) - \mu \right] \leq 0.
\]

Finally, note that \( P(s) \tilde{I}^\mu(s) \asymp \tilde{P}(s) \tilde{I}^\mu(s) \).

6. Auxiliary integral estimates

We will start with auxiliary generic estimates for function \( u \), and cutoff function \( \theta \) w.r.t. the nonlinear functions \( F \), and \( G \).

Lemma 1. Let \( u, \nabla u \) be measurable functions, and let \( \theta \in \text{Lip}_c(\Omega) \). Let \( F, G \in C^1(0, M) \cup C[0, M] \) satisfy (A-1). Then

\[
\nabla u \cdot \nabla (\theta^2 F(u)) \geq \frac{1}{2} |\nabla (\theta G(u))|^2 - (2C_1^2 + 1)G^2(u)|\nabla \theta|^2.
\]

Proof. We compute

\[
\nabla u \cdot \nabla (\theta^2 F(u)) = \theta^2 F'(u)|\nabla u|^2 + 2F(u) \nabla u \cdot \theta \nabla \theta
\]

(6.1)

\[
= \theta^2 F'(u)|\nabla u|^2 + 2 \frac{F(u)}{G'(u)} G'(u) \nabla u \cdot \nabla \theta.
\]

Using (A-1) in assumption in right-hand side of above yields

\[
= \theta^2 |G'(u)|^2 |\nabla u|^2 + 2 \frac{F(u)}{G'(u)} \left( G'(u) \nabla u \cdot \theta + G(u) \nabla \theta \right) \cdot \nabla \theta - 2 \frac{F(u)}{G'(u)} G(u) |\nabla \theta|^2
\]

(6.2)

\[
= |\nabla (\theta G(u)) - G(u) \nabla \theta|^2 + 2 \frac{F(u)}{G'(u)} \nabla (\theta G(u)) \cdot \nabla \theta - 2 \frac{F(u)}{G'(u)} G(u) |\nabla \theta|^2
\]

\[
= |\nabla (\theta G(u))|^2 + 2 \left[ \frac{F(u)}{G'(u)} - G(u) \right] \cdot \nabla \theta \cdot \nabla (\theta G(u)) - 2 \left[ \frac{F(u)}{G'(u)} - G(u) \right] |\nabla \theta|^2 G(u).
\]

(6.3)

\[
\geq |\nabla (\theta G(u))|^2 - 2 \left[ \frac{F(u)}{G'(u)} - G(u) \right] \cdot \nabla \theta \cdot \nabla (\theta G(u)) - 2 \left[ \frac{F(u)}{G'(u)} - G(u) \right] |\nabla \theta|^2 G(u).
\]
By Cauchy’s Inequality,
\[ 2 \left[ \frac{F(u)}{G(u)} - G(u) \right] \nabla \theta \cdot \nabla (\theta G(u)) \leq 2 \left[ \frac{F(u)}{G(u)} - G(u) \right]^2 |\nabla \theta|^2 + \frac{1}{2} |\nabla (\theta G(u))|^2. \]

Then (6.3) becomes
\[ \nabla u \cdot \nabla [\theta^2 F(u)] \geq \frac{1}{2} |\nabla (\theta G(u))|^2 - 2 \left[ \frac{F(u)}{G'(u)} - G(u) \right]^2 |\nabla \theta|^2 - \left[ 2 \frac{F(u)}{G'(u)} - G(u) \right] |\nabla \theta|^2 G(u) \]
\[ = \frac{1}{2} |\nabla (\theta G(u))|^2 - 2 \left[ \frac{F(u)}{G'(u)} \right]^2 - 2 \frac{F(u)}{G'(u)} G(u) + G^2(u) \] |\nabla \theta|^2 \]
\[ \geq \frac{1}{2} |\nabla (\theta G(u))|^2 - (2C_1^2 + 1)G^2(u)|\nabla \theta|^2. \]

\[ \square \]

**Lemma 2.** Assume (A-2) holds. Let \( u \) be a measurable function on \( \Omega \) such that \( u \in (0, M] \), \( \nabla u \in L^2_{\text{loc}}(\Omega \times (0, T)) \) and \( \nabla G(u) \in L^2_{\text{loc}}(\Omega \times [0, T]) \). Let \( j = 2/[N - 2] > 0 \) for Gagliardo – Nirenberg – Sobolev inequality

\[ \|\psi\|_{L^{2+2j}}^2 \leq S|\nabla \psi|_{L^2}^2. \]

Let \( K \in \Omega \), and \( \theta_n \in \text{Lip}_c(\Omega) \) be such that \( \theta_n = 1 \) on \( K \). Then

\[ \int_0^t \int_K G^2(u) dx dt \leq c_1 t^{1-(1+j)k} \left[ \sup_{0 \leq \tau \leq T} \int_\Omega \theta_n^2 H(u(\tau)) dx + \int_0^t \int_\Omega |\nabla (\theta_n u)|^2 dx d\tau \right]^{1+jk}, \]

where \( k = [2 - \lambda]/[2 + 2j - \lambda] \), \( c_1 = C_2^{1-k}S^{k(1+j)} \) with \( \lambda \) and \( C_2 \) as in (A-2).

**Proof.** Note that \( \lambda = [2 - (2 + 2j)k]/[1 - k] \). By (5.2), recall that \( G(s)^\lambda \leq C_2 H(s) \). Then

\[ G^2(u) \leq C_2^{1-k} G^{2(1+j)k}(u) H^{1-k}(u). \]

Integrate both side of (6.7) over \( K \times (0, t) \), we have
\[ \int_0^t \int_K G^2(u) dx dt \leq C_2^{1-k} \int_0^t \int_K G^{2(1+j)k}(u) H^{1-k}(u) dx d\tau \]
\[ \leq C_2^{1-k} \int_0^t \int_K (|\theta_n G(u)|^{2(1+j)})^k (\theta_n^2 H(u))^{1-k} d\tau \]
\[ \leq C_2^{1-k} \int_0^t \left[ \int_\Omega |\theta_n G(u)|^{2(1+j)} dx \right]^k \left[ \int_\Omega \theta_n^2 H(u) dx \right]^{1-k} d\tau \]
\[ \leq C_2^{1-k} S^{k(1+j)} \int_0^t \left[ \int_\Omega |\nabla (\theta_n G(u))|^2 dx \right]^{(1+j)k} d\tau \left[ \sup_{0 \leq \tau \leq T} \int_\Omega \theta_n^2 H(u) dx \right]^{1-k}, \]
by (6.5). We apply the Holder inequality for time integral and then using the estimate 
\(x^y y^w \leq (x + y)^{\nu + w}; x, y, v, w > 0\) to get the following
\[
\int_0^t \int K G^2(u)dxdt \leq C_2^{1-k}S^{k(1+j)} \left[ \sup_{0 \leq \tau \leq T} \int_{\Omega} \theta_2^n H(u)dx \right]^{1-k} t^{1-k(1+j)} \left[ \int_0^t \int_{\Omega} |\nabla(\theta_2^n G(u))|^2 dx d\tau \right]^{(1+j)k}
\]
\[
\leq C_2^{1-k} S^{k(1+j)} t^{1-k(1+j)} \left[ \sup_{0 \leq \tau \leq T} \int \theta_2^n H(u)dx + \int_0^t \int_{\Omega} |\nabla(\theta_2^n G(u))|^2 dx d\tau \right]^{1+jk}.
\]
In the proof of the main theorem 1 on localisation, we used the following iterative inequality in [8].

**Ladyzhenskaya-Uraltceva iterative Lemma 1.** Let sequence \(y_n\) for \(n = 0, 1, 2, \ldots\), be nonnegative sequence satisfying the recursion inequality, \(y_{n+1} \leq c \, b^n \, y_n^{1+\delta}\) with some constants \(c, \delta > 0\) and \(b \geq 1\). Then
\[
y_n \leq c \, \frac{(1+\delta)^{n-1}}{\delta^2} \, b^{\frac{n-1}{\delta}} \, y_0^{(1+\delta)^n}.
\]
In particular if \(y_0 \leq \theta_L = c^{-\frac{1}{\delta}} \, b^{\frac{1}{\delta^2}}\) and \(b > 1\), then \(y_n \leq \theta \, b^{-\frac{n}{\delta}}\) and consequently,
\[
y_n \to 0 \quad \text{when} \quad n \to \infty.
\]

7. **Localization of the solution of degenerate Einstein Equation**

We prove the main theorem of localization property by constructing De-Georgi’s machinery to establish the corresponding iterative energy inequality.

**Definition 4.** \(R > 0\) and \(b > 2\). Let \(R_n\) be a decreasing sequence with
\[
(7.1) \quad R_n \triangleq R_{n-1} - Rb^{-n} = R \left[ \frac{b - 2 + b^{-n}}{b - 1} \right]; n = 1, 2, \ldots.
\]
Then \(R_0 \triangleq R\) and \(\lim_{n \to \infty} R_n = R[b - 2]/[b - 1]\). Let \(\theta_n(x) \in \text{Lip}_c(\Omega)\) such that
\[
(7.2) \quad \theta_n(x) \triangleq \left[ \frac{(R_n - ||x - x_0||_\infty)}{R_n - R_{n+1}} \wedge 1 \right] = \begin{cases} 
0; & x \notin B_n(x_0), \\
1; & x \in B_n(x_0),
\end{cases}
\]
where \(B_n \triangleq B_{R_n}(x_0) \subset \Omega\). Then one can show that \(||\nabla \theta_n(x)||_\infty \leq |b^{n+1}|/R\).

**Theorem 1.** Let \(u\) be a positive solution of IBVP (4.5). Let \(\Omega' \subset \Omega\) be such that \(u(x, 0) = 0\) for \(x \in \Omega'\). Then for every ball \(B_R(x_0) \subset \Omega'\) and every \(R' \in (0, R)\), there exists \(T' > 0\) such that \(u(x, t) = 0\) for \((x, t) \in B_{R'}(x_0) \times [0, T']\).
Proof. We set \([b-2]/[b-1]=R'/R\). Multiplying inequality in IBVP (4.5) by \(\theta_n^2\) and integrate over \(\Omega \times (0,t)\), we find that

\[
\int_{\Omega} \theta_n^2 H(u) \, dx + \int_{0}^{t} \int_{\Omega} \nabla u \nabla (\theta_n^2 F(u)) \, dx \, d\tau \leq 0.
\]

Using Lemma 1 in (7.3) we get

\[
\int_{\Omega} \theta_n^2 H(u) \, dx + \frac{1}{2} \int_{0}^{t} \int_{\Omega} |\nabla (\theta_n G(u))|^2 \, dx \, d\tau \leq (2C_1^2 + 1) \int_{0}^{t} \int_{B_n} G^2(u) |\nabla \theta_n|^2 \, dx \, d\tau.
\]

In particular, \(\nabla G(u) \in L^2_{\text{loc}}(\Omega \times [0,T])\). Using Lemma 2 in (7.4), we obtain

\[
\int_{\Omega} \theta_n^2 H(u) \, dx + \frac{1}{2} \int_{0}^{t} \int_{\Omega} |\nabla (\theta_n G(u))|^2 \, dx \, d\tau \leq \beta \sup_{0 \leq \tau \leq t} \int_{B_n} \theta_n^2 H(u) \, dx + \int_{0}^{t} \int_{B_n} |\nabla (\theta_n G(u))|^2 \, dx \, d\tau
\]

where \(D \triangleq [b^4(2C_1^2 + 1)C_2^{(1+k)}]/R^2\). As \(0 < t \leq T'\), by taking the supremum over \(t\)

\[
\sup_{0 \leq \tau \leq T'} \int_{B_n} \theta_n^2 H(u) \, dx + \int_{0}^{T'} \int_{B_n} |\nabla (\theta_n G(u))|^2 \, dx \, d\tau \leq \beta \sup_{0 \leq \tau \leq T'} \int_{B_n} \theta_n^2 H(u) \, dx + \int_{0}^{T'} \int_{B_n} |\nabla (\theta_n G(u))|^2 \, dx \, d\tau
\]

Let \(\beta \triangleq [1 - (1+j)k]/kj\). Note that \(\beta > 0\) and \(\beta + 1 - (1+j)k = \beta(1+kj)\). Multiply both sides of (7.6) by \([T']^\beta\), we get

\[
[T']^\beta \sup_{0 \leq \tau \leq T'} \int_{\Omega} \theta_n^2(x) H(u) \, dx + [T']^\beta \int_{0}^{T'} \int_{\Omega} |\nabla (\theta_n(x) G(u))|^2 \, dx \, d\tau \leq D \cdot (b^2)^{n-1} \left[ [T']^\beta \sup_{0 \leq \tau \leq T'} \int_{B_n} \theta_n^2(x) H(u) \, dx + [T']^\beta \int_{0}^{T'} \int_{B_n} |\nabla (\theta_n(x) G(u))|^2 \, dx \, d\tau \right]^{1+jk}
\]

Observe that \(\text{supp} \, \theta_n = B_n\). Then above becomes

\[
[T']^\beta \sup_{0 \leq \tau \leq T'} \int_{B_n} \theta_n^2(x) H(u) \, dx + [T']^\beta \int_{0}^{T'} \int_{B_n} |\nabla (\theta_n(x) G(u))|^2 \, dx \, d\tau \leq D \cdot (b^2)^{n-1} \left[ [T']^\beta \sup_{0 \leq \tau \leq T'} \int_{B_{n-1}} \theta_{n-1}^2(x) H(u) \, dx + [T']^\beta \int_{0}^{T'} \int_{B_{n-1}} |\nabla (\theta_{n-1}(x) G(u))|^2 \, dx \, d\tau \right]^{1+jk}
\]

Define

\[
Y_n[T'] \triangleq [T']^\beta \sup_{0 \leq \tau \leq T'} \int_{B_n} \theta_n^2(x) H(u) \, dx + [T']^\beta \int_{0}^{T'} \int_{B_n} |\nabla (\theta_n(x) G(u))|^2 \, dx \, d\tau.
\]
Then (7.7) yields the iterative inequality

\[(7.9) \quad Y_n[T'] \leq D \cdot (b^2)^{n-1} Y_{n-1}^{1+kj}[T'] .\]

Let \( T' \) in (7.8) be such that \( Y_0[T'] \leq D - \frac{1}{kj b} - \frac{2}{k^2 j^2} \). Then by Ladyzhenskaya-Uraltceva iterative Lemma \([8]\), \( Y_n[T'] \to 0 \) whenever \( n \to \infty \).

**Remark 7.** In fact, it is sufficient to assume that \( u \) has certain bounds and positivity on \( B_R(x_0) \times [0,T] \). It is enough to assume that nonnegative solution of Cauchy problem, belong to the class of bounded functions in \( R^N \times [0,\infty) \), and then use the maximum principle to prove that, \( u \) is bounded by initial data at \( R^N \times \{0\} \).

8. Models for Degeneracy

Without loss of generality, in this section, we will assume that \( u \in (0,1] \), and illustrate some generic examples of the function \( P \), for which hold all constrains on the functions \( F \), \( G \) and \( H \), with the following summarized remark.

**Remark 8.** Let \( P \) and \( I \in C^1(0,\infty) \) be as in Definition 2. Assume that

\[(8.1) \quad \lim sup_{s \to 0} P(s)I(s) < \infty , \]

\[(8.2) \quad \lim sup_{s \to 0} sP'(s)I(s) < \infty . \]

Then Propositions 2, Propositions 3 and remark 6 hold. Consequently, Assumption 4 justifies and, thus Theorem 1 on finite speed of propagation asserts.

Next, we provide examples of the function \( P \) in remark 8, where \( P(0) = 0 \) and \( P \in C[0,\infty) \).

**Example 1.** \( P(s) = s^\beta \), where \( s \in [0,\infty) \) and \( \beta > 0 \).

\[(8.3) \quad I(s) = \int_s^\infty t^{-\beta-1} dt = \beta^{-1} s^{-\beta} . \]

\[(8.4) \quad P(s)I(s) = \beta^{-1} . \]

\[(8.5) \quad sP'(s)I(s) = 1 . \]
Example 2. $P(s) = \exp\left(-\frac{1}{s^\beta}\right), s \in [0, 1), \beta > 0.$

(8.6) \quad I(s) = \int_s^1 t^{-1} \exp\left(\frac{1}{t^\beta}\right) dt.

(8.7) \quad P(s)I(s) = \exp\left(-\frac{1}{s^\beta}\right) \left[ \int_s^1 t^{-1} \exp\left(\frac{1}{t^\beta}\right) dt \right].

By L'Hôpital's rule

(8.8) \quad \lim_{s \to 0} P(s)I(s) \equiv \lim_{s \to 0} s^\beta \equiv 0,

verifies that there exists $\Lambda$ such that $P(s)I(s) < \frac{\Lambda + 1}{\Lambda}, s \in [0, 1).$ Then

(8.9) \quad sI(s)P'(s) = s \left[ \int_s^1 t^{-1} \exp\left(\frac{1}{t^\beta}\right) dt \right] \exp\left(-\frac{1}{s^\beta}\right) s^{-\beta - 1},

(8.10) \quad \lim_{s \to 0} sI(s)P'(s) = \lim_{s \to 0} \frac{\left[ \int_s^1 t^{-1} \exp\left(\frac{1}{t^\beta}\right) dt \right] s^{-\beta}}{\beta \exp\left(\frac{1}{s^\beta}\right)}.

By L'Hôpital's rule

(8.11) \quad \lim_{s \to 0} sI(s)P'(s) \equiv 1 + \lim_{s \to 0} \frac{\left[ \int_s^1 t^{-1} \exp\left(\frac{1}{t^\beta}\right) dt \right]}{\exp\left(\frac{1}{s^\beta}\right)}

(8.12) \quad \equiv 1 + \lim_{s \to 0} s^\beta

(8.13) \quad = 1.

Then $\lim_{s \to 0} F(s)[G(s)G'(s)]^{-1} \equiv 1$, which verifies the existence of $C_1 > 0$ in Assumption (A-1).

Example 3. $P(s) = \exp\left(-\int_s^1 \frac{\zeta(\tau)}{\tau} d\tau\right), s \in (0, 1]$ and $0 < k_1 < \zeta(s) < k_2.$

(8.14) \quad I(s) = \int_s^1 t^{-1} \exp\left(\int_t^1 \frac{\zeta(\tau)}{\tau} d\tau\right) dt.

Then

(8.15) \quad P(s)I(s) = \frac{\left[ \int_s^1 t^{-1} \exp\left(\int_t^1 \frac{\zeta(\tau)}{\tau} d\tau\right) dt \right]}{\left[ \exp\left(\int_s^1 \frac{\zeta(\tau)}{\tau} d\tau\right)\right]^2}.
By L'Hôpital's rule

\[ \lim_{s \to 0} P(s)I(s) \equiv \lim_{s \to 0} \frac{1}{\zeta(s)} \equiv 1, \]  

which verifies that there exists $\Lambda$ such that $P(s)I(s) < \frac{\Lambda + 1}{\Lambda}$ $\forall$ $s \in [0, 1)$. Note that $\zeta(s) \equiv 1$. Then

\[ sI(s)P'(s) = \left[ \int_{s}^{1} t^{-1} \exp \left( \int_{t}^{1} \frac{\zeta(\tau)}{\tau} d\tau \right) dt \right] \exp \left( -\int_{t}^{1} \frac{\zeta(\tau)}{\tau} d\tau \right) \zeta(s), \]  

\[ \lim_{s \to 0} sI(s)P'(s) = \lim_{s \to 0} \frac{1}{\zeta(s)} \equiv 1. \]  

We apply L'Hôpital's rule to get

\[ \lim_{s \to 0} sI(s)P'(s) = \lim_{s \to 0} \frac{1}{\zeta(s)} \equiv 1. \]  

Hence $\lim_{s \to 0} F(s)[G(s)G'(s)]^{-1} \equiv 1$, which verifies that there exists $C_1 > 0$ in Assumption (A-1).

**Example 4.** Let $\zeta$ be such that $0 < k_3 \leq \frac{\zeta}{\zeta_0} \leq k_4$ for some $\zeta_0' \leq 0$ and $\sup_{0<s<1} |\zeta'_0| = c_0 < \infty$. Here $\lim_{\tau \to 0} \zeta(\tau) = \infty$. Provided $P(s) = \exp \left( -\int_{s}^{1} \frac{\zeta(\tau)}{\tau} d\tau \right)$, $s \in (0, 1]$.

Similarly in (8.16) we get

\[ \lim_{s \to 0} P(s)I(s) \equiv \lim_{s \to 0} \frac{1}{\zeta(s)} = 0, \]  

verifies the existence of $\Lambda$ such that $P(s)I(s) < \frac{\Lambda + 1}{\Lambda}$ $\forall$ $s \in [0, 1)$. Note that $\zeta(s) \equiv \zeta_0(s)$. Then

\[ sI(s)P'(s) = s \left[ \int_{s}^{1} t^{-1} \exp \left( \int_{t}^{1} \frac{\zeta(\tau)}{\tau} d\tau \right) dt \right] \exp \left( -\int_{t}^{1} \frac{\zeta(\tau)}{\tau} d\tau \right) \zeta(s). \]  

\[ \lim_{s \to 0} sI(s)P'(s) = \lim_{s \to 0} \frac{1}{\zeta(s)} \equiv 1. \]
By L'Hôpital's rule

\[
\lim_{s \to 0} sI(s)P'(s) \equiv 1 + \lim_{s \to 0} \left[ \int_s^1 t^{-1} \exp \left( \int_t^1 \frac{\zeta(\tau)}{\tau} d\tau \right) dt \right] \frac{s\zeta'(s)}{\zeta(s)} \exp \left( \int_s^1 \frac{\zeta(\tau)}{\tau} d\tau \right) dt \equiv 1 + \lim_{s \to 0} \frac{C_0}{\zeta(s)} = 1.
\]

(8.23)

Hence \( \lim_{s \to 0} F(s)[G(s)G'(s)]^{-1} \equiv 1 \), which verifies existence of \( C_1 > 0 \) in Assumption (A-1).

Note that one can consider \( P \) functions in a way \( P \approx \tilde{P} \) as in remark 6, for more general examples.

9. Auxiliary properties of Functional spaces, and a priori estimates for the solution

**Theorem 2.** Let \( X, Y \) be Banach spaces such that \( X \subset Y \). Then \( Y^* \subset X^* \), where * denotes the dual space.

**Proof.** Let \( f \in Y^* \). Then \( f(x) \) is well defined for all \( x \in X \subset Y \). Moreover,

\[
\sup_{||x||_X = 1} |f(x)| \leq \sup_{||x||_X = 1} ||f||_{Y^*} ||x||_Y \leq ||f||_{Y^*} \quad \Rightarrow \quad ||f||_{X^*} \leq ||f||_{Y^*}. \]

\( \square \)

**Theorem 3.** Let \( X, Y \) be Banach spaces and \( Z \) be a topological vector space: \( X, Y \subset Z \). Let \( X + Y \triangleq \{ z = x + y \mid x \in X, y \in Y \} \), where \( ||z||_{X+Y} = \inf \{ ||x||_X + ||y||_Y \} \). Then

(B1) \( X + Y \) is a Banach space.

(B2) \( X, Y \subset [X + Y] \).

(B3) \( [X + Y]^* = X^* \cap Y^* \)

**Proof.** Let \( \sum z_n \in X + Y \) be such that \( \sum ||z_n||_{X+Y} < \infty \). Let \( x_n \in X, y_n \in Y \) be such that \( x_n + y_n = z_n \) and, \( ||x_n||_X + ||y_n||_Y < ||z_n||_{X+Y} + 2^{-n} \), where \( n \in \mathbb{N} \). Then \( \sum ||x_n||_X + \sum ||y_n||_Y < \sum ||z_n||_{X+Y} + 2^{-n} \). Since \( X \) and \( Y \) are Banach spaces, there exist \( x \in X \) and \( y \in Y \) such that \( z_n = x + y \).

\( \square \)
Proof. Let \( x \in X \) then \( x \in [X + Y] \). Then \( \|x\|_{X+Y} \leq \|x\|_X \), and \( \|y\|_{X+Y} \leq \|y\|_Y \). \(\Box\)

Proof. Since \( X \subset X+Y \) and \( y \subset X+Y \), it follows from Theorem 2, \( [X+Y]^* \subset X^* \) and \( [X+Y]^* \subset Y^* \). Hence \( [X+Y]^* \subset [X^* \cap Y^*] \). Next we establish the converse. Let \( f \in [X^* \cap Y^*] \), \( z \in [X + Y] \). Let \( x_n \in X \), \( y_n \in Y \) be such that \( z = x_n + y_n \) and
\[
\sum_n \|x_n\|_X + \sum_n \|y_n\|_Y < \sum_n \|z_n\|_{X+Y} + \frac{1}{n}, \quad \text{for } n \in \mathbb{N}.
\]
Then \( f(z) = f(x) + f(y) \) is well-defined. Then
\[
|f(z)| \leq \|f\|_{X^*} \|x_n\|_X + \|f\|_{Y^*} \|y_n\|_Y
\]
\[
\leq \max \left[ \|f\|_{X^*}, \|f\|_{Y^*} \right] \left[ \|x_n\|_X + \|y_n\|_Y \right]
\]
\[
\leq \|f\|_{X^* \cap Y^*} \left[ \|z\|_{X+Y} + \frac{1}{n} \right]
\]
\[
\leq \|f\|_{X^* \cap Y^*} \|z\|_{X+Y},
\]
when \( n \to \infty \). Therefore \( f \in [X + Y]^* \) and \( \|f\|_{[X+Y]^*} \leq \|f\|_{[X^* \cap Y^*]} \). \(\Box\)

Theorem 4. (see [6]). Let \( B_0, B_1 \) and \( B_2 \) be Banach Spaces such that \( B_0 \subseteq B_1 \subseteq B_2 \). Let \( \mathbb{F} \subset L^p(I, B_0) ; I \subset \mathbb{R} \). If \( \|f\|_{L^p(I, B_0)} < \infty \) and \( \|f\|_{L^1(I, B_2)} < \infty \) for \( f \in \mathbb{F} \), then \( \mathbb{F} \) is compact in \( L^p(I, B_1) \).

10. WEAKLY APPROXIMATED SOLUTION OF DEGENERATE EINSTEIN EQUATION.

In this section we will prove roundness of the regularised solution \( u^\epsilon(x, t) \) in the space defined by LHS in (10.3), and compactness in \( L^q_{loc}(\Omega \times I) \) (see Corollary 1).

Theorem 5. Let \( u^\epsilon(x, t) ; 0 < \epsilon \leq u^\epsilon \leq K < \infty \), be a classical solution of the problem
\[
IBVP-G = \begin{cases}
H_t(u^\epsilon) - [F(u^\epsilon) + \epsilon] \Delta u^\epsilon = 0 & \text{in } \Omega \times (0, T),
\end{cases}
\]
\[
\begin{align*}
\text{IBVP-G} &= \left\{ 
\begin{array}{ll}
\quad & u^\epsilon(x, 0) = \epsilon + g(x) \quad \text{in } \Omega, \\
\quad & u^\epsilon(x, t) = \epsilon \psi(x) \quad \text{on } \partial \Omega \times (0, T),
\end{array}
\right.
\end{align*}
\]
where \( g(x) \geq 0 \), \( g(x) \in W^{1,2}(\Omega) \) and \( \psi(x) \geq 0 \). Let \( \bar{H}(u^\epsilon) = \int_0^{u^\epsilon} \sqrt{h(s)/[F(s) + \epsilon]} \, ds \). Then for any \( 0 < \tau \leq T \),
\[
\int_0^\tau \int_\Omega [\bar{H}(u^\epsilon)]^2 dx dt + \int_\Omega |\nabla u^\epsilon(x, \tau)|^2 dx = \int_\Omega |\nabla g(x)|^2 dx.
\]
Furthermore,

\[ \int_0^T \int_\Omega [\tilde{H}_t(u^\epsilon)]^2 \, dx \, dt + \int_0^T \int_\Omega |\nabla u^\epsilon(x, t)|^2 \, dx \, dt + \int_0^T \int_\Omega |u^\epsilon(x, t)|^2 \, dx \, dt \leq C(\Omega, T) \left( \int_\Omega |\nabla g(x)|^2 \, dx + K^2 \int_{\partial\Omega} |\psi(x)|^2 \, ds \right). \]

**Proof.** By definition of \( H \) in (4.2), we get

\[ H_t(u^\epsilon) = h(u^\epsilon)u^\epsilon_t, \quad [\tilde{H}_t(u^\epsilon)]^2 = \frac{h(u^\epsilon)}{F(u^\epsilon) + \epsilon} \cdot [u^\epsilon_t]^2, \]

therefore \( H_t(u^\epsilon) - (F(u^\epsilon) + \epsilon)\Delta u^\epsilon = 0 \equiv u^\epsilon_t \cdot h(u^\epsilon)/[F(u^\epsilon) + u^\epsilon] - \Delta u^\epsilon = 0. \)

Multiply first equation in (10.1) by \( (u_t) \) we get

\[ \begin{cases} 
[\tilde{H}_t(u^\epsilon)]^2 - \Delta u^\epsilon u^\epsilon_t = 0, & \text{in } \Omega \times (0, T), \\
\epsilon^\prime(x, 0) = \epsilon + g(x) & \text{in } \Omega, \\
\epsilon^\prime(x, t) = \epsilon \cdot \psi(x) & \text{on } \partial \Omega \times (0, T).
\end{cases} \]

Integrate over \( \Omega \times (0, \zeta_1) \) for \( 0 < \zeta_1 \leq T \),

\[ \int_0^{\zeta_1} \int_\Omega [\tilde{H}_t(u^\epsilon)]^2 \, dx \, dt = \int_0^{\zeta_1} \int_\Omega \Delta u^\epsilon u^\epsilon_t \, dx \, dt \]

\[ = - \frac{1}{2} \int_\Omega \int_0^{\zeta_1} \int_\Omega (|\nabla u^\epsilon|^2)_t \, dx \, dt. \]

Integrate the right-hand side over time, we get

\[ 2 \int_0^{\zeta_1} \int_\Omega [\tilde{H}_t(u^\epsilon)]^2 \, dx \, dt + \int_\Omega \int_0^{\zeta_1} \int_\Omega |\nabla u^\epsilon(x, \zeta_1)|^2 \, dx = \int_\Omega |\nabla g(x)|^2 \, dx ; \quad 0 < \zeta_1 \leq T. \]

By (10.9), we write following two estimates:

\[ 2 \int_0^{\zeta_1} \int_\Omega [\tilde{H}_t(u^\epsilon)]^2 \, dx \, dt \leq \int_\Omega |\nabla g(x)|^2 \, dx, \]

\[ \int_0^{\zeta_1} \int_\Omega |\nabla u^\epsilon(x, \zeta_1)|^2 \, dx \, d\zeta_1 \leq \zeta_2 \int_\Omega |\nabla g(x)|^2 \, dx ; \quad 0 < \zeta_2 \leq T. \]

By adding (10.10) and (10.11) we get

\[ 2 \int_0^{\zeta_1} \int_\Omega [\tilde{H}_t(u^\epsilon)]^2 \, dx \, dt + \int_0^{\zeta_2} \int_\Omega |\nabla u^\epsilon(x, \zeta_1)|^2 \, dx \, d\zeta_1 \leq (1 + \zeta_2) \int_\Omega |\nabla g(x)|^2 \, dx. \]

By Friedrich’s inequality \([12]\), for \( u \in W^{1,2}(\Omega) \), one can write

\[ C_{\zeta_1}^{-1} \int_0^{\zeta_1} \int_\Omega [u^\epsilon(x, \zeta_1)]^2 \, dx \, d\zeta_1 - \zeta_2 \cdot c^2 \int_{\partial\Omega} |\psi|^2 \, ds \leq \int_0^{\zeta_1} \int_\Omega |\nabla u^\epsilon(x, \zeta_1)|^2 \, dx \, dt. \]
Let $\epsilon_0: \frac{1}{2} < \epsilon_0 < 1$ be fixed. Then we use (10.13) in (10.12) yields

$$2 \int_0^{\zeta_2} \int_\Omega |\tilde{H}_t(u^\epsilon)|^2 \, dx \, dt + \epsilon_0 \int_0^{\zeta_2} \int_\Omega |\nabla u^\epsilon(x, \zeta_1)|^2 \, dx \, \zeta_1$$

$$+ (1 - \epsilon_0)C_F^{-1} \int_0^{\zeta_2} \int_\Omega |u^\epsilon(x, \zeta_1)|^2 \, dx \, \zeta_1$$

$$\leq (1 + \zeta_2) \int_\Omega |\nabla g(x)|^2 \, dx + (1 - \epsilon_0)\zeta_2 \cdot \epsilon_2 \int_{\partial \Omega} |\psi(x)|^2 \, ds,$$

where $\zeta_2 \in (0, T]$. Setting $\zeta_2 = T$,

$$\int_0^T \int_\Omega |\tilde{H}_t(u^\epsilon)|^2 \, dx \, dt + \int_0^T \int_\Omega |\nabla u^\epsilon(x, t)|^2 \, dx \, dt + \int_0^T \int_\Omega |u^\epsilon(x, t)|^2 \, dx \, dt$$

$$\leq C(\Omega, T) \left[ \int_\Omega |\nabla g(x)|^2 \, dx + \epsilon^2 \int_{\partial \Omega} |\psi(x)|^2 \, ds \right],$$

where $C(\Omega, T) \triangleq \max[1 + T, (1 - \epsilon_0)T]/\min[2, \epsilon_0, (1 - \epsilon_0)C_F^{-1}] = [1 + T]/\min[\epsilon_0, (1 - \epsilon_0)C_F^{-1}]$.

Thus we obtain (10.3) by replacing $\epsilon$ by $K$ on the right-hand side of (10.14).

**Remark 9.** Below in the Theorem 6 we will prove that $u^\epsilon$ is uniformly bounded in $L^2(I, W^{1,2}(\Omega))$. $\tilde{H}_t(u^\epsilon)$ is uniformly bounded in $L^2(\Omega \times (0, T))$.

**Theorem 6.** Let $u^\epsilon$ be a family of strong solutions to problem

$$[H(u^\epsilon)]_t - [F(u^\epsilon) + \epsilon]u^\epsilon = 0 \quad \text{in } \Omega \times (0, T),$$

$$u^\epsilon(x, 0) = \epsilon \quad \text{in } \Omega' \subseteq \Omega,$$

$$u^\epsilon(x, t) = \epsilon \psi(x) \quad \text{on } \partial \Omega \times (0, T),$$

such that $[H(u^\epsilon)]_t, u^\epsilon \in L^2_{\text{loc}}(\Omega \times (0, T))$, satisfying the estimate $0 < \epsilon \leq u^\epsilon \leq K$. Then

$$\int_{\Omega \times (0, T)} |\nabla(\theta G(u^\epsilon))|^2 \, dx \, dt$$

$$\leq C_4 \left[ \int_\Omega \theta^2 |G(u^\epsilon(x, 0))|^2 \, dx + \int_{\Omega \times (0, T)} |\nabla \theta|^2 (G(u^\epsilon))^2 \, dx \, dt + K_\epsilon \right],$$

for every $\theta \in C^1_c(\Omega)$. Then $G(u^\epsilon)$ is uniformly bounded in $L^2(I, W^{1,2}_{\text{loc}}(\Omega))$ for each $\epsilon > 0$.

**Proof.** Multiply both sides of the equality in (10.15) by $\theta^2$, and integrate over $\Omega \times (0, T)$:

$$\int_\Omega [H[u^\epsilon(x, T)] - H[u^\epsilon(x, 0)]\theta^2 \, dx$$

$$= -\int_0^t \int_\Omega [\nabla u^\epsilon]^2 F'(u^\epsilon)\theta^2 - [F(u^\epsilon) + \epsilon] \nabla u^\epsilon \cdot \nabla(\theta^2) \, dx \, dt$$
Rearrange above to the following inequality:

\[(10.18) \quad \int_{\Omega} [H(u^\epsilon(x,0))\theta^2 \, dx + \epsilon \int_0^t \int_{\Omega} |\nabla u_\epsilon \nabla (\theta^2)| \, dxdt \geq \int_0^t \int_{\Omega} |\nabla u^\epsilon|^2 F'(u^\epsilon) \theta^2 \, dxdt - \int_0^t \int_{\Omega} |F(u^\epsilon) \nabla u^\epsilon \cdot \nabla (\theta^2)| \, dxdt.\]

Next, by (D-2), we get \(|\nabla u^\epsilon|^2 F'(u^\epsilon) \cdot \theta^2 = |\nabla G(u^\epsilon)|^2 \theta^2\). Applying Cauchy’s Inequality

\[(10.19) \quad |\nabla G(u^\epsilon)|^2 \theta^2 \geq |\nabla (\theta G(u^\epsilon))| - |G(u^\epsilon) \nabla \theta| \quad \text{for some fixed } 0 < \epsilon_1 < \frac{1}{2}, \text{ and}\]

\[(10.20) \quad \geq (1 - 2\epsilon_1)|\nabla (\theta G(u^\epsilon))|^2 - \left( \frac{1}{2\epsilon_1} - 1 \right) |G(u^\epsilon) \nabla \theta|^2,\]

for some fixed \(0 < \epsilon_1 < \frac{1}{2}\), and

\[(10.21) \quad \epsilon \int_0^t \int_{\Omega} |\nabla u_\epsilon \nabla (\theta^2)| \, dxdt \leq \epsilon \left[ \int_0^t \int_{\Omega} |\nabla u_\epsilon|^2 \, dxdt + 2 \int_0^t \int_{\Omega} |\nabla (\theta)|^2 \, dxdt \right] \leq K_2(\epsilon) \quad \text{Next, using (A-1) we compute}\]

\[\int_{\Omega} H[u^\epsilon(x,0)] \cdot \theta^2 \, dx + K_2(\epsilon) + \left( \frac{1}{2\epsilon_1} - 1 \right) \int_0^t \int_{\Omega} |G(u^\epsilon) \nabla \theta|^2 \, dxdt \geq (1 - 2\epsilon_1) \int_0^t \int_{\Omega} |\nabla (\theta G(u^\epsilon))|^2 \, dxdt - \frac{C_1}{2} \int_0^t \int_{\Omega} |\nabla (G^2(u^\epsilon)) \cdot \nabla (\theta^2)| \, dxdt.\]

Once again applying Cauchy’s Inequality, We compute the following

\[(10.24) \quad \geq \left[ 1 - 2\epsilon_1 - 2\epsilon_2(1 - 2\epsilon_1 + C_1) \right] |\theta \nabla G(u^\epsilon)|^2 - \frac{1}{2\epsilon_2} |(1 + C_1) + 2\epsilon_1|^2 |G(u^\epsilon) \nabla \theta|^2,\]

where \([1 - 2\epsilon_1]/[2(1 - 2\epsilon_1 + C_1)] > \epsilon_2\). Using (10.24) in (10.23), we obtain

\[(10.25) \quad \int_{\Omega} H[u^\epsilon(x,0)] \cdot \theta^2 \, dx + K_2(\epsilon) + \left[ \frac{1}{2\epsilon_1} + \frac{1}{2\epsilon_2} (1 + C_1) + 2\epsilon_1 \right] \int_0^t \int_{\Omega} |G(u^\epsilon) \nabla \theta|^2 \, dxdt \geq \int_0^t \int_{\Omega} \left[ 1 - 2\epsilon_1 - 2\epsilon_2(1 - 2\epsilon_1 + C_1) \right] |\theta \nabla G(u^\epsilon)|^2 \quad \text{Note that } H(u^\epsilon) \text{ is bounded for } 0 < u^\epsilon < K. \text{ By selecting}\]

\[\quad C_4 = \max \left[ 1, \left( \frac{1}{2\epsilon_1} + \frac{1}{2\epsilon_2} (1 + C_1) + 2\epsilon_1 \right) \right] \left[ 1 - 2\epsilon_1 - 2\epsilon_2(1 - 2\epsilon_1 + C_1) \right],\]

gives (10.16).
**Proposition 4.** Assume all conditions in Proposition 1 and Proposition 2 hold. If

$$\liminf_{s \to 0} P(s) \left[ \int_0^M \frac{1}{\sigma P(\sigma)} \, d\sigma \right]^{\frac{1}{\lambda}} > 0,$$

then

$$\sup_{c > 0} \left[ \int_0^t \int_{\Omega} \theta |\nabla H(u^c)|^2 \, dxdt \right] \leq C_\theta < \infty,$$

for every $\theta \in C^1_c(\Omega)$.

**Proof.** Note that $|\nabla H(u^c)| \leq \frac{H'(u^c)}{G'(u^c)} |\nabla G(u^c)|$. By Theorem 6, it suffices to prove that

$$\sup_{0 < s < K} \frac{H'(s)}{G'(s)} < \infty.$$ 

For this end, it is enough to verify $\limsup_{s \to 0} \frac{H'(s)}{G'(s)} < \infty$. By remark 4

(10.27) $$F(s) = \Lambda^{-\frac{1}{\lambda}} s^{-\frac{1}{\lambda}} (I(s))^{-\frac{1}{\lambda}},$$

(10.28) $$F'(s) = B_1 s^{-2}(I(s))^{-\frac{1}{\lambda}} (P(s))^{-1} \left( \frac{1 + \Lambda}{\Lambda} - P(s)I(s) \right); \quad B_1 = \Lambda^{-\frac{1}{\lambda}}.$$

By (D-2) in Definition (D-3)

(10.29) $$G'(s) = \sqrt{B_1} s^{-1}(I(s))^{-\frac{1}{\lambda}} (P(s))^{-\frac{1}{2}} \sqrt{\frac{1 + \Lambda}{\Lambda} - P(s)I(s)}.$$ 

We write $H(s)$ in term of $I(s)$ in (5.6)

(10.30) $$H(s) = (\Lambda I(s))^{-\frac{1}{\lambda}},$$

(10.31) $$H'(s) = B_1 (I(s))^{-\frac{1}{\lambda}} s^{-1}(P(s))^{-1}.$$

Applying (10.31) and (10.29) we get

(10.32) $$\frac{H'(s)}{G'(s)} = \frac{1}{\sqrt{B_1}} \frac{1}{\sqrt{(I(s))^{-\frac{1}{\lambda}} P(s)}} \sqrt{\frac{1 + \Lambda}{\Lambda} - I(s)P(s)}.$$ 

By Proposition 2 we have $\frac{1 + \Lambda}{\Lambda} > I(s)P(s) > 0$; $s \in [0, M)$. Thus, by (10.32), it follows that

$$\limsup_{s \to 0} \frac{H'(s)}{G'(s)} < \infty,$$

whenever $\liminf_{s \to 0} P(s)(I(s))^\frac{1}{\lambda} > 0$. 

**Theorem 7.** Let $u^c$ satisfies Theorem 6. Then $u^c$ holds the following uniform estimates

(H1) $$\int_{\Omega' \times I} |\Psi H_t(u^c)| \, dxdt \leq C(\Omega', K) \left[ ||\Psi||_{L^\infty(\Omega' \times I)} + ||\nabla \Psi||_{L^2(\Omega' \times I)} \right];$$

(H2) $$\int_{\Omega' \times I} |\nabla H(u^c)|^2 \, dxdt \leq C(\Omega', K),$$

on $\Omega' \times I$, for each $\Psi \in C^1_c(\Omega' \times I)$ and for any $\Omega' \subset \Omega$. 

Proof. Multiply both sides of the inequality in (4.6) by $\Psi$, and integrate over $\Omega'_t \equiv \Omega' \times I$:

\begin{align}
\int_{\Omega'_t} |\Psi H_t(u^\epsilon)| \, dx \, dt & \leq \int_{\Omega'_t} |\nabla u^\epsilon \cdot \nabla [F(u^\epsilon)\Psi]| \, dx \, dt \\
& \leq \int_{\Omega'_t} |F(u^\epsilon)\nabla u^\epsilon \cdot \nabla \Psi| + |\Psi F'(u^\epsilon)(\nabla u^\epsilon)^2| \, dx \, dt.
\end{align}

Using (A-1) on the right-hand side of (10.33), we write

\begin{align}
\int_{\Omega'_t} |\Psi H_t(u^\epsilon)| \, dx \, dt & \leq C_2 \int_{\Omega'_t} |G(u^\epsilon)G'(u^\epsilon)\nabla u^\epsilon \cdot \nabla \Psi| \, dx \, dt + \int_{\Omega'_t} |\Psi G'(u^\epsilon)^2(\nabla u^\epsilon)^2| \, dx \, dt \\
& \leq \frac{C_2}{2} \int_{\Omega'_t} |\nabla G^2(u^\epsilon)||\nabla \Psi| \, dx \, dt + \|\Psi\|_{L^\infty(\Omega'_t)} \int_{\Omega'_t} |\nabla G(u^\epsilon)|^2 \, dx \, dt.
\end{align}

Rearrange the right-hand side of (10.34), we get

$$\leq \max \left[ \frac{C_2}{2}, 1 \right] \left[ \|\nabla \Psi\|_{L^2(\Omega'_t)} + \|\Psi\|_{L^\infty(\Omega'_t)} \right] \cdot \left[ \int_{\Omega'_t} |\nabla G^2(u^\epsilon)| \, dx \, dt + \int_{\Omega'_t} |\nabla G(u^\epsilon)|^2 \, dx \, dt \right].$$

Note that $0 < \epsilon \leq u^\epsilon \leq K$, and $G(u^\epsilon)$ is uniformly bounded in $L^2(I,W^{1,2}_{loc}(\Omega))$ by Theorem 6. Thus we obtain (H1).

From above theorem follows

**Corollary 1.** If we obtain an estimate analogous to (10.16), replacing $\nabla G(u^\epsilon)$ with $\nabla H(u^\epsilon)$, then we can apply Theorem (4) to conclude compactness of $\{H(u^\epsilon)\}$ in $L^2(I,L^q_{loc}(\Omega))$ with $q < \frac{2N}{N-2}$. Since $H: \mathbb{R} \to \mathbb{R}$ is a homeomorphism, and $u_\epsilon$ is uniformly bounded, it follows that $\{u^\epsilon\}$ is compact in $L^q_{loc}(\Omega \times I)$, $q \geq 1$.

**References**

[1] D. G. Aronson. The porous medium equation. In: Fasano A., Primicerio M. (eds) Nonlinear Diffusion Problems, volume 1224 of Lecture Notes in Mathematics. Springer, Berlin,Heidelberg, 1986. https://doi.org/10.1007/BFb0072687.

[2] G. I. Barenblatt. Scaling, Self-similarity, and Intermediate Asymptotics. Cambridge University Press, 1996. 10.1017/CBO9781107050242.

[3] M. G. Crandall, H. Ishii, and P.-L. Lions. User’s guide to viscosity solutions of second order partial differential equations. Bull. Amer. Math. Soc., 27:1–67, 1992. 10.1090/S0273-0979-1992-00266-5.

[4] A. Einstein. Investigations on the Theory of the Brownian Movement. 1956. 1, 4, 5

[5] L. C. Evans. Partial Differential Equations, volume 19. American Mathematical Society, 2010.
[6] J. Simon. Compact sets in the space $L^p(O,T;B)$. *Annali di Matematica Pura ed Applicata*, 146:65–96, 1986. [https://doi.org/10.1007/BF01762360](https://doi.org/10.1007/BF01762360), (p.84-86).

[7] K. Königsberger.  *Analysis 2*, volume 2 of *Springer-Lehrbuch*. Springer Berlin Heidelberg, 2006. [https://books.google.lk/books?id=V3crjPiI-mMC](https://books.google.lk/books?id=V3crjPiI-mMC).

[8] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural’ceva. *Linear and Quasi-linear Equations of Parabolic Type*, volume 23 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1968. 14, 16

[9] G. M. Lieberman. *Second Order Parabolic Differential Equations*. World Scientific, 1996. 8

[10] M. Muskat. *The Flow of Homogeneous Fluids Through Porous Media*. UMI Books on Demand. Ann Arbor (Michigan), UMI, 2004. 2

[11] N.V.Krylov. *Nonlinear Elliptic and Parabolic Equations of the Second Order*. 1987. 8

[12] V.G.Maz’ja. *Sobolev Spaces*. Springer Berlin, Heidelberg, 1985. [https://doi.org/10.1007/978-3-662-09922-3](https://doi.org/10.1007/978-3-662-09922-3).

[13] W. Vincenti and C. Kruger. *Introduction to physical gas dynamics*. Krieger Publishing Company, 1965.