The dependence of the speed of sound in the Earth’s atmosphere on its density and the correction of Mach’s number

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Abstract. According to the modern theory of sound, its speed in the Earth’s atmosphere depends only on temperature and does not depend on its density (height). The report shows a significant dependence of the speed of sound on the density of the atmosphere and, consequently, the requirement to adjust the Mach number, which plays an important role in determining the aerodynamic parameters of an aircraft.

1. Introduction
The Mach number plays a key role in determining the aerodynamic parameters for supersonic and hypersonic airplanes. As is known, the Mach number is equal to the ratio of speed of the aircraft relative to the medium, to the speed of sound in this medium. The importance of the Mach number is explained by the fact, that it determines whether the velocity of the flow of a gas medium (or the movement of a body in the gas) exceeds the speed of sound in it or not. Supersonic and subsonic flight regimes have fundamental differences in aviation. This difference is expressed in the fact that in supersonic regimes, narrow layers of rapid and significant changes in flow parameters (shock waves) emerge which lead to an increase in the resistance of bodies while in motion, the concentration of heat flows at their surface, and the possibility of burning through the hull of the bodies, and so on. Based on this, it is clear how important it is to correctly determine the speed of Sound at different altitudes of the Earth’s atmosphere.

Currently, the speed of sound is understood as the velocity of propagation of the adiabatic density perturbation in the air under the pressure perturbation and is determined by the formula , where the index s indicates that the derivative is taken at constant entropy, i.e. sound is considered as adiabatic. This expression is used to calculate the speed of sound in the whole atmosphere. Theoretical studies conducted at the Ilia Vekua Sukhumi Institute of Physics and Technology showed, that such an approach is valid only for a homogeneous medium in which the density depends only on pressure . In a heterogeneous (non-isentropic) medium, such as the Earth’s atmosphere, as a result of the impact on it of the gravitational field of the Earth, the density also depends on the entropy .

This circumstance leads to the existence of an isobaric speed of sound along with the adiabatic speed of sound, which clearly depends on the altitude (density) of the atmosphere. The true value of the speed of sound in the troposphere is a combination of these two sounds and it significantly differs from the adiabatic speed of sound, especially in the upper part of the troposphere, which must be taken into account when determining the Mach number.

2. The speed of sound in the troposphere
Figure1 shows the experimental distribution of temperature over through altitude in the atmosphere of the earth [1]. We see that up to a height of approximately 11 km, the temperature drops according to a strictly linear law. This part of the atmosphere is called the troposphere, which lacks a heat source and therefore the entropy is constant, i.e. The adiabatic equation can be used:
\[
\frac{ds}{dt} = \frac{\partial s}{\partial t} + (\nabla s) \cdot \mathbf{V} = 0, 
\]

(1)

Where \( \mathbf{V} \) is the velocity of the air particle. At the upper boundary of the troposphere, the drop of the temperature stops sharply, remaining constant up to the height of about 20 km (tropopause), and then in the stratosphere it rises. In [2], the author suggests that an exothermic reaction of ozone synthesis \( \text{O}_2 + \text{O} \rightarrow \text{O}_3 + 24 \text{kcal/mol} \) occurs at this altitude, which is the reason for such temperature distribution dynamics. Below we show that our theory fully confirms this hypothesis.

![Figure 1. Temperature as a function of geometrical altitude](image)

It is assumed in modern theory, that the gravity of the Earth does not affect high-frequency oscillations in a sound wave and, therefore, the atmosphere represents a homogeneous medium for it (see, for example, [3], [4]). Based on this, it is assumed that the atmospheric density depends only on pressure \( \rho = \rho(P) \) and then, the propagation velocity of a small perturbation of density is expressed by the following formula:

\[
C_s = \sqrt{\frac{\partial p}{\partial \rho}},
\]

(2)

and is defined as the speed of adiabatic sound. Considering air to be an ideal gas \( p = nkT \), where \( n = \rho/m \) - concentration of gas molecules, \( k \) - Boltzmann constant, \( T \) - absolute temperature), the connection between pressure and density under an adiabatic process is determined by the following relation:

\[
\frac{p}{p_0} = \left( \frac{\rho}{\rho_0} \right)^\gamma.
\]

(3)

Here, \( p_0 \) and \( \rho_0 \) are the initial values of pressure and density, while an adiabatic index \( \gamma = c_p/c_v \) = 1.4 is the ratio of thermal capacities at constant pressure and volume. Based on these ratios, from (2) the following is found for speed of sound:
Here \( R = 8.314 \text{J/Kmol} \) is a gas constant, \( m_0 = 4.81 \times 10^{-26} \text{kg} \) is mass of a single air molecule and \( M = 29 \times 10^{-3} \text{kg/mol} \) — mass of one mole of air. Formula (4) is used to calculate the speed of sound in the Earth’s atmosphere at any altitude, which means that it depends only on temperature and does not depend on the altitude (density) of the atmosphere. In other words, the speed of sound at the north pole at \(-40^\circ \text{C}\) is the same as in the stratosphere at an altitude of 65 km. This conclusion contradicts the fundamental concepts of physics: firstly, the speed of sound represents a thermodynamic parameter of any medium, and in the event of its heterogeneity, it must clearly depend on the spatial coordinate; and secondly, it is impossible for the propagation velocity of the perturbation of the density to not depend on the density itself.

The reason for these contradictions is the assumption, that the atmosphere is homogeneous with respect to sound waves. In fact, the atmospheric density depends not only on pressure but also on entropy, i.e. \( \rho = \rho(p,s) \) and then the density perturbation will be equal to the following:

\[
\rho' = \left( \frac{\partial \rho_0}{\partial p_0} \right)_p p' + \left( \frac{\partial \rho_0}{\partial s_0} \right)_p s'.
\]  

The first term in (5) determines the adiabatic change in the air density due to a change in mass in the constant volume, while the second term determines an isobaric change of density due to a change in volume of the constant air mass due to a fluctuation of temperature caused by an entropy fluctuation, which occurs due to the pressure perturbation, i.e.

\[
s' = \left( \frac{\partial S_0}{\partial P_0} \right)_T p'.
\]  

It is believed here that at a given height the temperature is constant, and the process is isothermal. Substituting (6) into (5) we easily find the following:

\[
\rho' = \frac{1}{C^2} p',
\]  

where

\[
C^2 = \frac{C_s^2 C_p^2}{C_s^2 + C_p^2},
\]

\[
C_s^2 = \left( \frac{\partial p_0}{\partial \rho_0} \right)_s = \gamma \frac{kT}{m_0},
\]

\[
C_p^2 = \left[ \left( \frac{\partial p_0}{\partial s_0} \right)_T \left( \frac{\partial s_0}{\partial p_0} \right)_T \right]^{-1}.
\]  

Obviously, \( C_s \), we can be called the isobaric speed of sound or the speed of isobaric sound. Using well-known thermodynamic expressions ([3], §13.)

\[
\left( \frac{\partial p_0}{\partial s_0} \right)_T = \frac{T}{c_p} \left( \frac{\partial p_0}{\partial T} \right)_p, \quad \text{and} \quad \left( \frac{\partial s_0}{\partial p_0} \right)_T = \frac{1}{\rho_0} \left( \frac{\partial \rho_0}{\partial T} \right)_p,
\]

for \( C_p^2 \) we get
Substituting in (11) the Laplace’s barometric formula \( \rho_0 = \rho_0^0 \exp(-m_0 g z / kT) \), where \( \rho_0^0 \) is the air density at sea level, we finally find

\[
C_p^2 = \frac{c_p \rho_0^2}{T(\frac{\partial \rho_0}{\partial T})} .
\] (11)

Using (9) and (12), from (8) we will discover the true value of the speed of sound in the troposphere, which depends on the altitude and the temperature:

\[
C(z, T) = \sqrt{\frac{\gamma kT}{m_0(1 + \gamma m_0 g^2 z^2 / c_p^2 kT^2)}} = 20.05 \sqrt{T} \frac{1}{\sqrt{1 + 4.69 \times 10^{-4} z^2 / T^2}} .
\] (13)

Expressing \( z \) from the Laplace barometric formula and substituting it in (13) we find the dependence of the speed of sound in the troposphere on the pressure and the temperature:

\[
C(p, T) = \sqrt{\frac{\gamma kT}{m_0(1 + \gamma kT / \rho_0 m_0 \ln p / p_0)}} = 20.05 \sqrt{T} \frac{1}{\sqrt{1 + 0.40(\ln 351,557T / p)^2}}.
\] (14)

Here \( p_0 = 1.013 \times 10^5 \, Pa \) and \( T_0 = 288.15^6 \, K \) are pressure and temperature at a sea level.

We believe that formulas (13) and (14) are valid only in the troposphere, on the altitude of up to 11km, since above this altitude the process of the release of heat takes place and the law of conservation of entropy, used in our theory, does not hold. It can be seen from (13) that the condition \( z = 0 \) is an equivalent to the condition \( g = 0 \), i.e. at the sea level, the atmosphere is uniform for a sound wave and \( C = C_s(T) \).

3. The discussion of the results

In fig. 2.Graphs represent the distribution of the adiabatic and true velocities of sounds along the altitude of the troposphere. We see that as the altitude is rising the influence of the Earth’s gravitational field on the speed of sound increases.
Figure 2. Dependence of the isobaric \(C_s(T)\) (the green curve) and true \(C(z,T)\) (the blue curve) speeds of sounds on the altitude in the troposphere.

Table 1 represents the altitude distribution of the values of the adiabatic \(C_s(T) = C_{int}(T)\) and true \(C(z,T) = C(p,T)\) velocities of sounds in the stratosphere. Relative errors between the values of the true and adiabatic speeds of sound \((\Delta C/C)\% = [(C_s(T) - C(z,T))/C(z,T)]100\%\) and the corresponding Mach numbers \((\Delta M/M)\% = [(V/C(z,T) - V/C_s(T))]/[V/C(z,T)]100\%\) are also presented in the table respectively. As seen, the relative error in determining the Mach number at an altitude of 11 km is 110 times greater than at an altitude of 1 km. For example, for an airplane flying at the speed of 1000 m/s, at an altitude of 11 km, the Mach number calculated by formula \(M_1 = 1000/295.07 = 3.4\), and by formula \(M_2 = 1000/198.26 = 5.0\). It is apparent, that neglect of such a large error during the calculation of the aircraft aerodynamic parameters is unacceptable.

| \(z(m)\) | \(T^0(K)\) | \(C_s(m/sec)\) | \(C(m/sec)\) | \((\Delta C/C)\%)\% | \((\Delta M/M)\%)\% |
|---|---|---|---|---|---|
| 1000.00 | 281.65 | 336.43 | 335.45 | 0.29 | 0.29 |
| 2000.00 | 275.15 | 332.53 | 328.47 | 1.22 | 1.22 |
| 3000.00 | 268.65 | 328.58 | 319.34 | 2.89 | 2.81 |
| 4000.00 | 262.15 | 324.58 | 308.12 | 5.31 | 5.07 |
| 5000.00 | 255.65 | 320.53 | 295.04 | 8.62 | 7.95 |
| 6000.00 | 249.15 | 316.43 | 280.42 | 12.79 | 11.38 |
| 7000.00 | 242.65 | 312.27 | 264.66 | 18.02 | 15.25 |
| 8000.00 | 236.15 | 308.06 | 248.18 | 24.1 | 19.44 |
| 9000.00 | 229.65 | 303.79 | 231.38 | 31.35 | 23.84 |
| 10000.00 | 223.15 | 299.46 | 214.65 | 39.49 | 28.32 |
| 11000.00 | 216.65 | 295.07 | 198.26 | 48.83 | 32.81 |

Table 1. The altitude distribution of the values of the adiabatic \(C_s(T) = C_{int}(T)\) and true \(C(z,T) = C(p,T)\) velocities of sound in the troposphere and the relative errors between the true and adiabatic velocities of sound and the corresponding Mach numbers.

The logical proof of the validity of our theory is the graphs of the dependencies on the altitude of the adiabatic \(C_s(T)\) and isobaric \(C_p(z,T)\) sound velocities shown in Figure 3.
Figure 3. The dependences of adiabatic $C_s(T)$ (Green curve) and isobaric $C_p(z,T)$ (Red curve) sound velocities from the altitude in the troposphere.

It can be seen that at the altitude of $z \approx 10200\text{m}$ these graphs intersect. Equating $C_s^2(T)$ and $C_p^2(T)$ from formulas (9) and (12), we obtain the following:

$$\frac{\gamma k T}{m_0} = \frac{c_p k^2 T^3}{m_0 g^2 z^2} \Rightarrow z = \sqrt[3]{\frac{c_p k T}{m_0 g}}.$$

(15)

Substituting in (15) $T = \alpha + \beta z, \alpha = 288,150\text{K}, \beta = -6.52\text{K/m}$ we will find

$$z = \sqrt[3]{\frac{c_p k \alpha^2 / m_0 g^2}{1 + \sqrt{c_p k \beta^2 / m_0 g^2}}} = 10230\text{m}.$$

(16)

This height almost exactly coincides with the upper boundary of the troposphere, defined in Fig. 1. Thus, we can say with great certainty that expressions (15) and/or (16) are equations of the upper boundary of the troposphere, where the adiabatic and isobaric sound velocities are equated, i.e. resonance $\omega_s = \omega_p$ occurs. As is known, the sharp change in the dynamics of the process, which is observed in Fig. 1, is usually associated with resonance. Thus, it can be assumed, that the resonance of the adiabatic and isobaric frequencies of sound is a trigger for the exothermic reaction of the ozone synthesis and, as a consequence, the release of a large amount of heat. As we see, our hypothesis fully coincides with the hypothesis of the author of [2].

4. Conclusion

The dependence of the speed of sound in the troposphere on altitude and temperature was theoretically proved by us in 2012 and was published in the work [5]. This work is recognized by the world scientific community and is available on several sites on the Internet (see, for example, [6]). Despite this, the index of its citation is insignificant and the Internet to this day gives the values of sound velocities at different altitudes of the atmosphere, calculated by the formula (4) [7]. Obviously, this can be explained by the fact that our theoretical results have not been experimentally confirmed. We hope that this conference will be
able to contribute to the popularization of this problem and in the relevant scientific community will show a desire to conduct the necessary experiments. The importance of such experiments lies in the fact that if the laboratories succeed in creating conditions corresponding to the upper boundary of the troposphere and confirm the fact of heat generation, we will get an alternative energy source.

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