THE $\epsilon$-MAXIMAL OPERATOR AND HAAR MULTIPLIERS ON VARIABLE LEBESGUE SPACES

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Abstract. C. Stockdale, P. Villarroya, and B. Wick introduced the $\epsilon$-maximal operator to prove the Haar multiplier is bounded on the weighted spaces $L^p(w)$ for a class of weights larger than $A_p$. We prove the $\epsilon$-maximal operator and Haar multiplier are bounded on variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ for a larger collection of exponent functions than the log-Holder continuous functions used to prove the boundedness of the maximal operator on $L^p(\mathbb{R}^n)$. We also prove that the Haar multiplier is compact when restricted to a dyadic cube $Q_0$.

1. Introduction

In [2], it was proved that the dyadic maximal operator $M^d$ is bounded on variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ for log-Holder continuous exponent functions $p(\cdot) \in LH(\mathbb{R}^n)$ with $1 < p_- \leq p_+ < \infty$. In [7], the $\epsilon$-maximal operator and $\epsilon$-sparse operator were introduced to establish boundedness for the Haar multiplier on $L^p(w)$ for a class of weights larger than $A_p$. Motivated by these two results, we prove the $\epsilon$-maximal operator and the Haar multiplier are bounded on variable Lebesgue spaces for a collection of exponent functions larger than $LH(\mathbb{R}^n)$. In addition, we prove a local compactness result for the Haar multiplier similar to the result in [7].

Before stating our results precisely, we briefly explain some of the definitions involved. These are stated in more detail in Section 2. An exponent function $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ is a Lebesgue measurable function. We denote the essential infimum and supremum of $p(\cdot)$ by $p_-$ and $p_+$. In this paper, we only consider exponent functions where $1 < p_- \leq p_+ < \infty$. Given such an exponent function $p(\cdot)$, we can define the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ as the collection of Lebesgue measurable functions satisfying $\|f\|_{p(\cdot)} < \infty$.

We denote the set of all dyadic cubes in $\mathbb{R}^n$ by $\mathcal{D}$. In all our definitions and results, $\epsilon = \{\epsilon_Q\}_{Q \in \mathcal{D}}$ is a bounded collection of real numbers indexed by dyadic cubes $Q \in \mathcal{D}$ such that for any $P, Q \in \mathcal{D}$, if $P \subseteq Q$, then $\epsilon_P \leq \epsilon_Q$. We refer to this assumption as the domination property of $\epsilon$. Given such a collection, we define the Haar multiplier $T_\epsilon$ for all $f \in L_{\text{loc}}^1(\mathbb{R}^n)$ by

$$T_\epsilon f = \sum_{Q \in \mathcal{D}} \epsilon_Q \langle f, h_Q \rangle h_Q,$$

(1)

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Here, \( \langle f, h_Q \rangle = \int_Q f(y) h_Q(y) dy \), and \( h_Q \) is the Haar function adapted to \( Q \) defined by

\[
h_Q = |Q|^{-1/2} \left( \chi_Q - \frac{1}{2^n} \chi_{\hat{Q}} \right),
\]

where \( \hat{Q} \) is the dyadic parent of \( Q \). We will prove that the Haar multiplier is bounded on \( L^{p(\cdot)}(\mathbb{R}^n) \) under certain assumptions on \( p(\cdot) \). To do so, we will use the \( \epsilon \)-maximal operator as a tool to control the Haar multiplier. The \( \epsilon \)-maximal operator \( M_\epsilon \) is defined by

\[
M_\epsilon f(x) = \sup_{Q \in \mathcal{D}} \epsilon_Q \int_Q |f(y)| dy \chi_Q(x).
\]

It is known that the dyadic maximal operator \( M^d \) is bounded on \( L^{p(\cdot)}(\mathbb{R}^n) \) when \( p(\cdot) \in LH_0(\mathbb{R}^n) \cap LH_\infty(\mathbb{R}^n) \). The set \( LH_\infty(\mathbb{R}^n) \) denotes the collection of exponent functions \( p(\cdot) \) that are log-Holder continuous at infinity, i.e., there exists constants \( C_\infty \) and \( p_\infty \) such that for all \( x \in \mathbb{R}^n \),

\[
|p(x) - p_\infty| \leq \frac{C_\infty}{\log(e + |x|)}.
\]

The set \( LH_0(\mathbb{R}^n) \) consists of exponent functions that are locally log-Holder continuous, meaning there exists a constant \( C_0 \) such that for all \( x, y \in \mathbb{R}^n \) with \( |x - y| < 1/2 \), we have

\[
|p(x) - p(y)| \leq \frac{C_0}{\log(|x - y|)}.
\]

When \( p_+ < \infty \), \( LH_0(\mathbb{R}^n) \) is equivalent to the Diening condition: there exists a constant \( C_0 \) such that for any cube \( Q \),

\[
|Q|^{p_-(Q) - p_+(Q)} \leq C.
\]

The Diening condition can be used to give a local condition on exponent functions that is adapted to dyadic operators. In the study of Hardy martingale spaces with variable exponents, the authors of [5] used the following condition to prove a strong-type Doob maximal inequality:

\[
\mathbb{P}(A)^{p_-(A) - p_+(A)} \leq K,
\]

where \( K \) is a constant depending on the exponent function \( p(\cdot) \), \( \mathbb{P} \) is a probability measure, and \( A \) is a measurable set. In [8], a similar inequality given by

\[
|I|^{p_-(I) - p_+(I)} \leq C,
\]

where \( I \) is a dyadic interval in \([0, 1]\), was used to prove boundedness of some other types of maximal operators for variable exponent spaces.

Note that \( M_\epsilon f(x) \leq \|\epsilon\|_\infty M^d f(x) \) where \( \|\epsilon\|_\infty = \sup_{Q \in \mathcal{D}} \epsilon_Q \). Consequently, if \( p(\cdot) \in LH_0(\mathbb{R}^n) \cap LH_\infty(\mathbb{R}^n) \), then \( M_\epsilon \) is bounded on \( L^{p(\cdot)}(\mathbb{R}^n) \). However, we can replace \( LH_0(\mathbb{R}^n) \) with the weaker local condition \( \epsilon LH_0(\mathbb{R}^n) \).

The set \( \epsilon LH_0(\mathbb{R}^n) \) consists of exponent functions \( p(\cdot) \) satisfying the \( \epsilon \)-Diening condition

\[
\left( \frac{|Q|}{\epsilon_Q} \right)^{p_-(Q) - p_+(Q)} \leq C,
\]

for all \( Q \in \mathcal{D} \), where \( C \) depends only on \( n \) and \( p(\cdot) \). Note that if \( \epsilon_Q = 1 \) for all \( Q \in \mathcal{D} \), we obtain the Diening condition (5), discussed earlier. However, the \( \epsilon \)-Diening condition (7) is
a dyadic condition that is also adapted to the collection $\epsilon$. This makes $\epsilon LH_0(\mathbb{R}^n)$ a natural condition to use to prove the $\epsilon$-maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

**Theorem 1.1.** Given $\epsilon = \{\epsilon_Q\}_{Q \in \mathcal{D}}$, let $p(\cdot) \in LH_\infty(\mathbb{R}^n) \cap \epsilon LH_0(\mathbb{R}^n)$ with $1 < p_- \leq p_+ < \infty$. Then $M_\epsilon$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$: i.e. there exists a constant $C = C(n, p(\cdot), \epsilon)$ such that for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$
\|M_\epsilon f\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}.
$$

Theorem 1.1 can then be used as a tool to prove the Haar multiplier is bounded on $L^{p(\cdot)}$.

**Theorem 1.2.** Given $\epsilon = \{\epsilon_Q\}_{Q \in \mathcal{D}}$, let $\sqrt[\epsilon]{} = \{\sqrt[\epsilon]{} Q\}_{Q \in \mathcal{D}}$. If $p(\cdot) \in \sqrt[\epsilon]{} LH_0(\mathbb{R}^n) \cap LH_\infty(\mathbb{R}^n)$ with $1 < p_- \leq p_+ < \infty$, then the Haar multiplier $T_\epsilon$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

If we restrict ourselves to $L^{p(\cdot)}(Q_0)$ for some dyadic cube $Q_0$, we can improve the properties of the Haar multiplier to get compactness. Let $\mathcal{D}(Q_0)$ be the collection of dyadic cubes contained in $Q_0$, and $\epsilon LH_0(Q_0)$ be the same as (7), but for $Q \in \mathcal{D}(Q_0)$.

**Theorem 1.3.** Given $Q_0 \in \mathcal{D}$, $\epsilon = \{\epsilon_Q\}_{Q \in \mathcal{D}(Q_0)}$ and $0 < \alpha < 1/2$, let $\epsilon^\alpha = \{\epsilon^\alpha_Q\}_{Q \in \mathcal{D}(Q_0)}$. Suppose

$$
(8) \quad \lim_{N \to \infty} \sup \{\epsilon_Q : \ell(Q) < 2^{-N}\} = 0
$$

and $p(\cdot) \in \epsilon^\alpha LH_0(Q_0)$ with $1 < p_- \leq p_+ < \infty$. Then the Haar multiplier is compact on $L^{p(\cdot)}(Q_0)$.

**Remark 1.4.** In [7, Section 2.4], the authors give a compactness result for weighted spaces on all of $\mathbb{R}^n$. Their result requires a condition on the collection $\epsilon$ that implies that $\epsilon_Q \to 0$ as $\ell(Q) \to \infty$. However, this is impossible if $\epsilon$ has the domination property. Their proof, however, gives a local compactness property, and our proof is modeled on theirs.

The compactness result for weighted spaces $L^p(w)$ on $\mathbb{R}^n$ was proved using extrapolation in [4, Section 5], but our local compactness result on variable Lebesgue spaces is new and cannot be proved using the extrapolation results in [4].

The remainder of this paper is organized as follows. In Section 2, we state the necessary definitions and lemmas for variable Lebesgue spaces. We prove Theorem 1.1 in Section 3. In Section 4, we prove Theorems 1.2 and 1.3. Lastly, in Section 5, we show that the $\epsilon LH_0(\mathbb{R}^n)$ hypothesis of Theorem 1.1 is weaker than the local log-Hölder continuity condition defined in inequality (4). We do this by showing there are exponent functions that are not in $LH_0(\mathbb{R}^n)$, but are in $\epsilon LH_0(\mathbb{R}^n)$ for some $\epsilon$.

Throughout this paper, $C$ will denote a constant that may vary in value from line to line and which will depend on underlying parameters. If we want to specify the dependence, we will write, for instance, $C(n, \epsilon)$. If the value of the constant is not important, we will often write $A \lesssim B$ instead of $A \leq cB$ for some constant $c$. We will also use the convention that $1/\infty = 0$.

2. Preliminaries

We begin with the necessary definitions related to variable Lebesgue spaces. We refer the reader to [2] for more information.
Definition 2.1. An exponent function on a set $\Omega$ is a Lebesgue measurable function $p(\cdot) : \Omega \to [1, \infty)$. Denote the collection of exponent functions on $\Omega$ by $P(\Omega)$. Denote the essential infimum and essential supremum of $p(\cdot)$ on a set $E$ by $p_-(E)$ and $p_+(E)$, respectively. Denote $p_+(\Omega)$ by $p_+$ and $p_-(\Omega)$ by $p_-$.

Definition 2.2. Given $p(\cdot) \in P(\Omega)$ with $p_+ < \infty$, and a Lebesgue measurable function $f$, define the modular associated with $p(\cdot)$ by

$$\rho_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)}dx.$$  

If $f(\cdot)^{p(\cdot)} \notin L^1(\Omega)$, define $\rho_{p(\cdot)} = +\infty$. In situations where there is no ambiguity we will simply write $\rho(f)$.

Definition 2.3. Given $p(\cdot) \in P(\Omega)$, define the space $L^p(\Omega)$ as the set of Lebesgue measurable functions $f$ satisfying $\|f\|_{L^p(\Omega)} < \infty$, where the norm $\| \cdot \|_{L^p(\Omega)}$ is defined as

$$\|f\|_{L^p(\Omega)} = \inf \{ \lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \leq 1 \}.$$  

In situations where there is no ambiguity, we will write $\|f\|_{p(\cdot)}$ instead of $\|f\|_{L^p(\Omega)}$.

The following propositions relate the modular and the norm and will be used to prove Theorem [1.1]. The first proposition allows us to conclude a norm is finite when the modular is finite.

Proposition 2.4. [2, Proposition 2.12] Given $p(\cdot) \in P(\Omega)$ with $p_+ < \infty$, $f \in L^p(\Omega)$ if and only if $\rho(f) < \infty$.

Proposition 2.5. [2, Corollary 2.22] Let $p(\cdot) \in P(\Omega)$. If $\|f\|_{p(\cdot)} \leq 1$, then $\rho(f) \leq \|f\|_{p(\cdot)}$.

In [2, Theorem 3.16], it is proven that the Hardy-Littlewood maximal operator $M$ is bounded on $L^p(\mathbb{R}^n)$ when $p(\cdot) \in LH_0(\mathbb{R}^n) \cap LH_\infty(\mathbb{R}^n)$ with $p_+ < \infty$. Since the dyadic maximal operator $M^d$ is bounded pointwise by the Hardy-Littlewood maximal operator, i.e. $M^d f(x) \leq M f(x)$, the same assumptions on $p(\cdot)$ ensure that $M^d$ is bounded on $L^p(\mathbb{R}^n)$. We can weaken the local assumption $LH_0(\mathbb{R}^n)$ by incorporating the collection $\epsilon$ into the definition of $\epsilon LH_0(\mathbb{R}^n)$ stated in inequality [7] from Section [1].

Unfortunately, we cannot incorporate the collection $\epsilon$ into the $LH_\infty(\mathbb{R}^n)$ definition in any way to weaken it. This is due to the fact that the $\epsilon$-maximal operator is pointwise equivalent to the dyadic maximal operator near infinity. More precisely, given a function $f$ that is bounded and supported on a dyadic cube $Q_0$, we have that

$$\epsilon_{Q_0} M^d f(x) \leq M f(x) \leq \|\epsilon\|_\infty M^d f(x),$$

for almost every $x \notin Q_0$. Since the constants $\epsilon_{Q_0}$ and $\|\epsilon\|_\infty$ do not depend on any information about $\epsilon_Q$ for $Q \neq Q_0$, any condition near infinity that we use to bound $M f$ outside of $Q_0$ will have to be the same condition we use to bound $M^d f$ outside of $Q_0$, and not a condition based on the properties of the collection $\epsilon$.

Since we are assuming that $p(\cdot) \in LH_\infty(\mathbb{R}^n)$, we can use following lemma when proving Theorem [1.1].

Lemma 2.6. [2, Lemma 3.26] Let $p(\cdot) \in LH_\infty(\mathbb{R}^n)$ with $1 < p_- \leq p_+ < \infty$. Let $R(x) = (e + |x|)^{-n}$. Then there exists a constant $C$, depending on $n$ and the $LH_\infty$ constants of $p(\cdot)$,
such that given any set $E$ and any function $F$ with $0 \leq F(x) \leq 1$, for $x \in E$,

\begin{align}
(9) \quad \int_E F(x)^p(x) \, dx &\leq C \int_E F(x)^p \, dx + \int_E R(x)^p \, dx,
(10) \quad \int_E F(x)^p \, dx &\leq C \int_E F(x)^p(x) \, dx + \int_E R(x)^p \, dx.
\end{align}

We now provide the definition and properties of dyadic cubes. These properties are well-known and can be found in [2, Section 3.2].

**Definition 2.7.** Let $Q_0 = [0,1)^n$, and let $\mathcal{D}_0$ be the set of all translates of $Q_0$ whose vertices are on the lattice $\mathbb{Z}^n$. More generally, for each $k \in \mathbb{Z}$, let $Q_k = 2^{-k}Q_0 = [0,2^{-k})^n$, and let $\mathcal{D}_k$ be the set of all translates of $Q_k$ whose vertices are on the lattice $2^{-k}\mathbb{Z}^n$. Define the set of dyadic cubes $\mathcal{D}$ by

$$\mathcal{D} = \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k.$$ 

**Proposition 2.8.** Dyadic cubes have the following properties:

1. For each $k \in \mathbb{Z}$, if $Q \in \mathcal{D}_k$, then $\ell(Q) = 2^{-k}$, where $\ell(Q)$ is the side length of $Q$.
2. For each $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, there exists a unique cube $Q \in \mathcal{D}_k$ such that $x \in Q$.
3. Given any two cubes $Q_1, Q_2 \in \mathcal{D}$, either $Q_1 \cap Q_2 = \emptyset$, $Q_1 \subset Q_2$, or $Q_2 \subset Q_1$.
4. For each $k \in \mathbb{Z}$, if $Q \in \mathcal{D}_k$, then there exists a unique cube $\widehat{Q} \in \mathcal{D}_{k-1}$ such that $Q \subset \widehat{Q}$. ($\widehat{Q}$ is referred to as the dyadic parent of $Q$.)
5. For each $k \in \mathbb{Z}$, if $Q \in \mathcal{D}_k$, then there exist $2^n$ cubes $P_i \in \mathcal{D}_{k+1}$ such that $P_i \subset Q$.

The following proposition presents an equivalent characterization of $\epsilon LH_0(\mathbb{R}^n)$ which will be used in the proof of Theorem [11].

**Proposition 2.9.** Given $\epsilon = \{\epsilon_Q\}_{Q \in \mathcal{D}}$, $p(\cdot) \in \epsilon LH_0(\mathbb{R}^n)$ if and only if there exists $C > 0$ such that for all $Q \in \mathcal{D}$ and $x \in Q$,

\begin{align}
(11) \quad \left(\frac{|Q|}{\epsilon_Q}\right)^{p-(Q)-p(x)} &\leq C.
\end{align}

Proof. Assume $p(\cdot) \in \epsilon LH_0(\mathbb{R}^n)$. Fix $Q \in \mathcal{D}$. Observe that if $|Q|/\epsilon_Q > 1$, then for any $x \in Q$, (11) holds with $C = 1$. Suppose $|Q|/\epsilon_Q \leq 1$. Then for any $x \in Q$, we have

$$\left(\frac{|Q|}{\epsilon_Q}\right)^{p-(Q)-p(x)} \leq \left(\frac{|Q|}{\epsilon_Q}\right)^{p-(Q)-p_+(Q)} \leq C.$$ 

To prove the reverse direction, observe that if $|Q|/\epsilon_Q > 1$, then (11) holds with $C = 1$. Suppose $|Q|/\epsilon_Q \leq 1$. Let $\delta > 0$ be arbitrarily small and choose $x_0 \in Q$ such that $p(x_0) + \delta > p_+(Q)$. Then by the definition of $\epsilon LH_0(\mathbb{R}^n)$, we have

$$\left(\frac{|Q|}{\epsilon_Q}\right)^{p-(Q)-p_+(Q)} \leq \left(\frac{|Q|}{\epsilon_Q}\right)^{p-(Q)-p_+(Q)+\delta} \leq C \left(\frac{|Q|}{\epsilon_Q}\right)^{-\delta}.$$ 

Letting $\delta$ tend to 0, we see that $p(\cdot) \in \epsilon LH_0(\mathbb{R}^n)$. \qed

In order to prove Theorem [11] we need the following Calderon-Zygmund decomposition for the $\epsilon$-maximal operator. This is very similar to the classical Calderon-Zygmund decomposition for the dyadic maximal operator [2, Lemma 3.9]. For the convenience of the reader we include the short proof.
Lemma 2.10. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ be such that $\int_{Q} |f(y)|dy \to 0$ as $|Q| \to \infty$. Given $\lambda > 0$, there exists a (possibly empty) collection of disjoint dyadic cubes $\{Q_j^\lambda\}_j$ such that

(12) \quad \Omega_\lambda = \{x \in \mathbb{R}^n : M_\epsilon f(x) > \lambda\} = \bigcup_j Q_j^\lambda,

and for each $Q_j^\lambda$,

(13) \quad \lambda < \epsilon_{Q_j^\lambda} \int_{Q_j^\lambda} |f(y)|dy \leq 2^n \lambda.

Proof. If $\Omega_\lambda$ is empty, then we choose an empty collection and the conclusions hold trivially. Suppose $\Omega_\lambda$ is nonempty and let $x \in \Omega_\lambda$. Then there exists $Q \in \mathcal{D}$ containing $x$ such that

$$\epsilon_{Q_x} \int_{Q} |f(y)|dy > \lambda.$$ 

Since $\{\epsilon_Q\}_{Q \in \mathcal{D}}$ is bounded and $\int_{Q} |f(y)|dy \to 0$ as $|Q| \to \infty$, there is a maximal dyadic cube with this property. Denote it by $Q_x$. Clearly, $\Omega_\lambda \subseteq \bigcup_{x \in \Omega_\lambda} Q_x$. The reverse inclusion holds as well. To see this, consider any $Q_x$ and let $z \in Q_x$. Then

$$M_\epsilon f(z) \geq \epsilon_{Q_x} \int_{Q_x} |f(y)|dy \chi_{Q_x}(z) > \lambda,$$

and so $z \in \Omega_\lambda$. By the nature of dyadic cubes, the cubes $\{Q_x\}_{x \in \Omega_\lambda}$ are all equal or disjoint. Also, note that since each $D_k$ is a countable collection, and $\mathcal{D}$ is the countable union of all $D_k$, we have that $\mathcal{D}$ is countable. Consequently, there are at most countably many such cubes $Q_x$. Enumerate these cubes by $\{Q_j^\lambda\}_j$. Clearly these cubes satisfy (12).

The lower bound in (13) is immediate by our choice of $\{Q_j^\lambda\}_j$. To show the upper bound, observe that by the domination property of the collection $\epsilon$, we have $\epsilon_{Q_j^\lambda} \geq \epsilon_{Q_j^\lambda}$. Combining this with the maximality of our choice of $Q_j^\lambda$,

$$\lambda \geq \epsilon_{Q_j^\lambda} \int_{Q_j^\lambda} |f(y)|dy \geq \epsilon_{Q_j^\lambda} \int_{Q_j^\lambda} |f(y)|dy \geq 2^{-n} \epsilon_{Q_j^\lambda} \int_{Q_j^\lambda} |f(y)|dy,$$

Multiplying by $2^n$ gives the desired upper bound. \hfill \Box

In order to prove Theorem 1.3, we need a local version of Lemma 2.10. We state the local version and briefly outline the adaptations to the proof of Lemma 2.10 needed to prove it.

Lemma 2.11. Given $Q_0 \in \mathcal{D}$, $\epsilon = \{\epsilon_Q\}_{Q \in \mathcal{D}(Q_0)}$, and $f \in L^1_{\text{loc}}(Q_0)$, for any $\lambda > \epsilon_{Q_0} \int_{Q_0} |f(y)|dy$, there exists a (possibly empty) collection of disjoint cubes $\{Q_j^\lambda\}_j$ such that

(14) \quad \Omega_\lambda = \{x \in Q_0 : M_\epsilon f(x) > \lambda\} = \bigcup_j Q_j^\lambda,

and for each $Q_j^\lambda$,

(15) \quad \lambda < \epsilon_{Q_j^\lambda} \int_{Q_j^\lambda} |f(y)|dy \leq 2^n \lambda.
Proof. Choose the collection \( \{Q_j^\lambda\}_j \) as in the proof of Lemma 2.10. The lower bound in inequality (15) is immediate. The proof of the upper bound depends on every \( Q_j^\lambda \) having a dyadic parent \( \hat{Q}_j^\lambda \) in \( Q_0 \), which will hold if and only if \( Q_0 \) is not in the collection \( \{Q_j^\lambda\}_j \). Recall that we chose the cubes \( Q_j^\lambda \) as the maximal cubes satisfying \( \epsilon Q_j^\lambda \int_{Q_j^\lambda} |f(y)|dy > \lambda \).

Since we only consider \( \lambda > \epsilon Q_j^\lambda \int_{Q_j^\lambda} |f(y)|dy \), we have that \( Q_0 \) is not in \( \{Q_j^\lambda\}_j \). Hence, every cube in \( \{Q_j^\lambda\}_j \) has a dyadic parent in \( Q_0 \), and so the proof of the upper bound in inequality (15) is the same as in the proof of Lemma 2.10.

The following lemma allows us to apply the Calderon-Zygmund decomposition to any function in \( L^{p(\cdot)}(\mathbb{R}^n) \) when \( p_+ < \infty \).

Lemma 2.12. [2, Lemma 3.29] Given \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \), suppose \( p_+ < \infty \). Then for all \( f \in \mathcal{L}^{p(\cdot)}(\mathbb{R}^n) \), \( \int_Q |f(y)|dy \to 0 \) as \( |Q| \to \infty \).

In order to prove Theorem 1.2, we will need some lemmas about the conjugate exponent function \( p'(\cdot) \) defined pointwise by

\[
\frac{1}{p'(x)} = 1 - \frac{1}{p(x)}.
\]

The first two lemmas will allow us to transfer properties of \( p(\cdot) \) to \( p'(\cdot) \). The first lemma is well-known and is an immediate consequence of the definition. See [2].

Lemma 2.13. Let \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \) with \( 1 < p_- \leq p_+ < \infty \). Then \( p(\cdot) \in L^{H_\infty}(\mathbb{R}^n) \) if and only if \( p'(\cdot) \in L^{H_\infty}(\mathbb{R}^n) \).

Lemma 2.14. \( p(\cdot) \in \epsilon L^{H_0}(\mathbb{R}^n) \) if and only if \( p'(\cdot) \in \epsilon L^{H_0}(\mathbb{R}^n) \).

Proof. Assume \( p(\cdot) \in \epsilon L^{H_0}(\mathbb{R}^n) \). Let \( Q \in \mathcal{D} \). Observe that since \( (p_+)'(Q) - (p_-)'(Q) \leq 0 \), we have that if \( |Q|/\epsilon Q > 1 \), then inequality (7) holds with \( C = 1 \). Suppose that \( (|Q|/\epsilon Q) \leq 1 \). To show \( (p_+)'(Q) \in \epsilon L^{H_0}(\mathbb{R}^n) \), it suffices to show that there is a constant \( C_1 > 0 \) depending only on \( p(\cdot) \) such that for any \( Q \in \mathcal{D} \), we have

\[
(p_+)'(Q) - (p_-)'(Q) \geq C_1(p_+(Q) - p_-(Q)).
\]

To see why, observe that since \( |Q|/\epsilon Q \leq 1 \) and \( p(\cdot) \in \epsilon L^{H_0}(\mathbb{R}^n) \), if (16) holds, then we have

\[
\left( \frac{\epsilon Q}{|Q|} \right)^{(p_+)'(Q) - (p_-)'(Q)} \geq C_1(p_+(Q) - p_-(Q)).
\]

Flipping both sides, we have that

\[
\left( \frac{|Q|}{\epsilon Q} \right)^{(p_-)'(Q) - (p_+)'(Q)} \leq C_1(p_-(Q) - p_+(Q)).
\]

Since \( p(\cdot) \in \epsilon L^{H_0}(\mathbb{R}^n) \), the right hand side is bounded by a constant depending only on \( n \) and \( p(\cdot) \). Hence, \( (p_+)'(\cdot) \in \epsilon L^{H_0}(\mathbb{R}^n) \).

We now prove that inequality (16) holds. First, recall that by the definition of conjugate exponent functions, we have

\[
\frac{1}{(p_+)'(Q)} = 1 - \frac{1}{p_-(Q)}, \quad \text{and} \quad \frac{1}{(p_-)'(Q)} = 1 - \frac{1}{p_+(Q)}.
\]
Using these properties, we have that
\[(p')^+(Q) - (p')^-(Q) = (p')^+(Q)(p')^-(Q) \left[ \frac{1}{(p')^-(Q)} - \frac{1}{(p')^+(Q)} \right] \]
\[= (p')^+(Q)(p')^-(Q) \left[ \frac{1}{p^-(Q)} - \frac{1}{p^+(Q)} \right] \]
\[= \frac{(p')^+(Q)(p')^-(Q)}{p^-(Q)p^+(Q)}[p^+(Q) - p^-(Q)] \]
\[\geq \frac{(p')^2}{(p^2)^2}[p^+(Q) - p^-(Q)] \]

This proves inequality (16), and so \(p'() \in \epsilon LH_0(\mathbb{R}^n)\). The proof of the converse is the same, except we interchange the roles of \(p()\) and \(p'()\).

The following lemma allows us to apply the previous two lemmas in our estimates when proving Theorem 1.2 and Theorem 1.3.

**Lemma 2.15.** [2, Proposition 2.37] Given \(p() \in \mathcal{P}(\Omega)\) with \(1 < p^- \leq p^+ < \infty\), define the associate norm \(\| \cdot \|_{p()}\) by
\[
\|f\|_{p()} = \sup \left\{ \int_{\Omega} f(x)g(x)dx : g \in L^{p'}(\Omega), \|g\|_{p'} \leq 1 \right\}.
\]
Then for any \(f \in L^{p}(\Omega)\), we have
\[
\|f\|_{p()} \leq \|f\|_{p()}'\cdot
\]

The next lemma is the variable exponent version of Holder’s inequality.

**Lemma 2.16.** [2, Theorem 2.26] Given \(p() \in \mathcal{P}(\Omega)\) with \(1 < p^- \leq p^+ < \infty\), for all \(f \in L^{p}(\Omega)\) and \(g \in L^{p'}(\Omega)\), \(fg \in L^{1}(\Omega)\) and
\[
\int_{\Omega} |f(x)g(x)|dx \leq \|f\|_{p()}\|g\|_{p'}.
\]

### 3. Boundedness of the \(\epsilon\)-Maximal Operator

The proof of Theorem 1.4 is adapted from [2, Theorem 3.16]. We begin the proof by making some reductions. We may assume \(f()\) is nonegative since \(M_{\epsilon}(f) = M_{\epsilon}(|f|)\). By homogeneity, we may further assume that \(\|f\|_{p()} = 1\). From Proposition 2.5, we get that \(\rho(f) \leq 1\). Decompose \(f\) as \(f_1 + f_2\), where
\[
f_1 = f\chi_{\{x:f(x)>1\}}, \quad f_2 = f\chi_{\{x:f(x)\leq 1\}}.
\]

Then \(\rho(f_i) \leq \|f_i\|_{p()} \leq 1\) for \(i = 1, 2\). Further, since \(M_{\epsilon}f \leq M_{\epsilon}f_1 + M_{\epsilon}f_2\), it will suffice to show for \(i = 1, 2\) that \(\|M_{\epsilon}f_i\|_{p()} \leq C(n, p(), \epsilon)\). Since \(p_+ < \infty\), by Proposition 2.4 it will in turn suffice to show that for \(i = 1, 2\),
\[
\rho(M_{\epsilon}f_i) = \int_{\mathbb{R}^n} M_{\epsilon}f_i(x)^{p}(x)dx \leq C(n, p(), \epsilon).
\]

First we consider the estimate for \(f_1\). Let \(A = 2^n\). For each \(k \in \mathbb{Z}\), define
\[
\Omega_k = \{x \in \mathbb{R}^n : M_{\epsilon}f_1(x) > A^k\}.
\]
Observe that, up to a set of measure zero, $\mathbb{R}^n = \bigcup_{k \in \mathbb{Z}} \Omega_k \setminus \Omega_{k+1}$. Since $p_+ < \infty$, by Lemma 2.12 $f$ satisfies the hypotheses of Lemma 2.10. Thus, for each $k$ we may form a collection of pairwise disjoint cubes $\{Q^k_j\}_j$ such that (12) and (13) hold. For each $k$, define the sets $E^k_j = Q^k_j \cap (\Omega_k \setminus \Omega_{k+1})$. Then for each $k$, $\{E^k_j\}_j$ forms a pairwise disjoint collection such that $\Omega_k \setminus \Omega_{k+1} = \bigcup_j E^k_j$.

Now observe that
\[
\rho(M_\varepsilon f_1) = \sum_k \int_{\Omega_k \setminus \Omega_{k+1}} M_\varepsilon f_1(x)^p(x) \, dx \\
\leq \sum_k \int_{\Omega_k \setminus \Omega_{k+1}} (A^{k+1})^p(x) \, dx \\
\leq A^{p+} \sum_{k,j} \int_{E^k_j} \left( \frac{\epsilon_{Q^k_j}}{|Q^k_j|} \right)^{p(x)} \left( \int_{Q^k_j} f_1(y) \, dy \right)^{p(x)} \, dx.
\]

For each $k$ and $j$, define $p_{jk} = p_-(Q^k_j)$. Since for any $x \in \mathbb{R}^n$, $f_1(x) > 1$ or $f_1(x) = 0$, we then have
\[
(17) \quad \int_{Q^k_j} f_1(y) \, dy \leq \int_{Q^k_j} f_1(y)^{p(y)/p_{jk}} \, dy \leq \int_{Q^k_j} f_1(y)^{p(y)} \, dy \leq 1.
\]

Now observe that by Proposition 2.9, inequality (17), and Holder’s inequality we have
\[
\sum_{k,j} \int_{E^k_j} \left( \frac{\epsilon_{Q^k_j}}{|Q^k_j|} \right)^{p(x)} \left( \int_{Q^k_j} f_1(y) \, dy \right)^{p(x)} \, dx \\
\leq \sum_{k,j} \int_{E^k_j} \left( \frac{\epsilon_{Q^k_j}}{|Q^k_j|} \right)^{p_{jk}} \left( \int_{Q^k_j} f_1(y) \, dy \right)^{p(x)} \, dx \\
\leq (1 + \|\varepsilon\|_{\infty})^{p+} \sum_{k,j} \int_{E^k_j} |Q^k_j|^{-p_{jk}} \left( \int_{Q^k_j} f_1(y) \, dy \right)^{p(x)} \, dx \\
\leq (1 + \|\varepsilon\|_{\infty})^{p+} \sum_{k,j} \int_{E^k_j} |Q^k_j|^{-p_{jk}} \left( \int_{Q^k_j} f_1(y)^{p(y)/p_{jk}} \, dy \right)^{p(x)} \, dx \\
\leq (1 + \|\varepsilon\|_{\infty})^{p+} \sum_{k,j} \int_{E^k_j} \left( \int_{Q^k_j} f_1(y)^{p(y)} \, dy \right)^{p_{jk}} \, dx \\
\leq (1 + \|\varepsilon\|_{\infty})^{p+} \sum_{k,j} \int_{E^k_j} M^d[f_1^{p(+)}] (x)^{p_+} \, dx \\
\leq C(p(\cdot), \varepsilon) \int_{\mathbb{R}^n} M^d[f_1^{p(+)}] (x)^{p_+} \, dx.
\]
Since \( p_- > 1 \), we have \( \|M^d f_i\|_{L^p_\infty} \leq (p_-)^\alpha \|f_i\|_{L^p_\infty} \) (see [6] Theorem 2.3, [3] Exercise 2.1.12). Combining this with the fact that \( \rho(f_1) \leq 1 \), we have that

\[
\rho(M_i f_1) \leq C(n, p(\cdot), \epsilon) \rho(f_1) \leq C(n, p(\cdot), \epsilon).
\]

To estimate \( \rho(M_i f_2) \) observe that since \( f_2 \leq 1 \), we have \( \int_Q f_2(y) dy \leq 1 \) for all \( Q \in \mathcal{D} \). Thus,

\[
\frac{\epsilon_Q}{\|\epsilon\|_\infty} \int_Q f_2(y) dy \chi_Q(x) \leq 1,
\]

for all \( x \in \mathbb{R}^n \). Hence, \( 0 \leq \|\epsilon\|_\infty^{-1} M_i f_2 \leq 1 \). Let \( R(x) = (e + |x|)^{-n} \). Note that since \( p_- > 1 \), we have \( p_\infty > 1 \), and so \( \int_{\mathbb{R}^n} M_i^d f_2(x)^p dx \leq ((p_\infty)^{\alpha}) \int_{\mathbb{R}^n} f(x)^p dx \). Combining this with inequalities (9), (10), and the pointwise bound \( M_i f_2(x) \leq \|\epsilon\|_\infty M_i^d f_2(x) \), we have that

\[
\int_{\mathbb{R}^n} M_i f_2(x)^p dx \leq \|\epsilon\|_\infty^p \int_{\mathbb{R}^n} \|\epsilon\|_\infty^{-1} M_i f_2(x)^p dx + \|\epsilon\|_\infty \int_{\mathbb{R}^n} R(x)^p dx
\]

\[
\leq C \|\epsilon\|_\infty^p \int_{\mathbb{R}^n} \|\epsilon\|_\infty^{-1} M_i f_2(x)^p dx + \|\epsilon\|_\infty \int_{\mathbb{R}^n} R(x)^p dx
\]

\[
\leq C \|\epsilon\|_\infty^p \int_{\mathbb{R}^n} \|\epsilon\|_\infty^{-1} M_i^d f_2(x)^p dx + \|\epsilon\|_\infty \int_{\mathbb{R}^n} R(x)^p dx
\]

\[
\leq C ((p_\infty)^{\alpha}) \|\epsilon\|_\infty^p \int_{\mathbb{R}^n} f_2(x)^p dx + \|\epsilon\|_\infty \int_{\mathbb{R}^n} R(x)^p dx
\]

\[
\leq C(n, p(\cdot), \epsilon) \int_{\mathbb{R}^n} f_2(x)^p dx + C(n, p(\cdot), \epsilon) \int_{\mathbb{R}^n} R(x)^p dx.
\]

Since \( \rho(f_2) \leq 1 \) and \( \int_{\mathbb{R}^n} R(x)^p dx \) is finite, we have that \( \int_{\mathbb{R}^n} M_i f_2(x)^p dx \leq C(n, p(\cdot), \epsilon) \). This completes the proof of Theorem 1.1.

We will need a local version of Theorem 1.1 to prove Theorem 1.3. We state the local version and outline the modifications to the proof. Note that the necessary lemmas and propositions used to prove Theorem 1.1 still hold when replacing \( \mathbb{R}^n \) with \( Q_0 \).

**Lemma 3.1.** Given \( Q_0 \in \mathcal{D} \), \( \epsilon = \{\epsilon_Q\}_{Q \in \mathcal{D}(Q_0)} \), if \( p(\cdot) \in \mathcal{H}_0(Q_0) \) with \( 1 < p_- < p_+ < \infty \), then there exists a constant \( C = C(n, p(\cdot), \epsilon, Q_0) \) such that

\[
\|M_i f\|_{L^{p(\cdot)}} \leq C\|f\|_{L^{p(\cdot)}}
\]

for all \( f \in L^{p(\cdot)}(Q_0) \).

**Proof.** Using the same reductions as in the proof of Theorem 1.1, we must show that \( \rho(M_i f_1) \leq C \) for \( i = 1, 2 \). Since \( |Q_0| \) is finite and \( M_i f_2 \leq 1 \), we immediately have that

\[
\rho(M_i f_2) \leq |Q_0|.
\]

To estimate \( \rho(M_i f_1) \), let \( A = 2^n \) and \( k_0 \) be the smallest integer such that

\[
2^{k_0} > \epsilon_Q \int_{Q_0} |f_1(y)| dy.
\]
For $k \geq k_0$, define $\Omega_k$ the same as in Theorem 1.1 and define $\Omega = Q_0 \backslash \Omega_{k_0}$. Then $Q_0 = \Omega \cup \bigcup_{k \geq k_0} \Omega_k \backslash \Omega_{k+1}$. We now write $f(M_{\epsilon}f_1)$ as

$$\rho(M_{\epsilon}f_1) = \int_{\Omega} M_{\epsilon}f_1(x) p(x) dx + \sum_{k \geq k_0} \int_{\Omega_k \backslash \Omega_{k+1}} M_{\epsilon}f_1(x) p(x) dx.$$  

The second term is estimated using $\epsilon LH_{0}(Q_0)$, inequality (17), and Holder’s inequality the same way as in Theorem 1.1. To estimate the first term, observe that for $x \in \Omega$, we have $M_{\epsilon}f_1(x) \leq A_{k_0}$. Thus

$$\int_{\Omega} M_{\epsilon}f_1(x) p(x) dx \leq A_{k_0} p + \Omega_0 |Q_0|.$$  

This completes the proof.  

4. HAAR MULTIPLIERS

To prove the Haar multiplier defined in (1) is bounded on $L^p(w)$, in [7] they proved it was dominated by a sparse operator. To state their result, we need two definitions.

**Definition 4.1.** We say $S \subset D$ is a sparse collection of dyadic cubes if for every $Q \in S$,

$$\sum_{P \in \eta_S(Q)} |P| \leq \frac{1}{2} |Q|,$$

where $\eta_S(Q)$ is the set of maximal elements of $S$ that are strictly contained in $Q$.

**Remark 4.2.** Given a collection of cubes $S$, for each $Q \in S$, define $E_Q = \left( \bigcup_{P \in \eta_S(Q)} P \right)^c \cap Q$. Then Definition 4.1 is equivalent to the condition that $|Q| \leq 2|E_Q|$, for all $Q \in S$. Furthermore, $\{E_Q\}_{Q \in S}$ is a pairwise disjoint collection.

**Definition 4.3.** Given a sparse collection $S$ and $\epsilon = \{\epsilon_Q\}_{Q \in D}$, for all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, define the $\epsilon$-sparse operator $S_\epsilon$ by

$$S_\epsilon f(x) = \sum_{Q \in S} \epsilon_Q \int_{Q} f(y) dy \chi_Q(x).$$

The following theorem allows us to reduce the proofs about $T_\epsilon$ to proofs about $\epsilon$-sparse operators.

**Theorem 4.4.** [7, Theorem 1.2] Given $\epsilon = \{\epsilon_Q\}_{Q \in D}$, if $f$ is bounded with compact support, then there exists a sparse collection $S$ such that the associated $\epsilon$-sparse operator $S_\epsilon$ satisfies

$$|T_\epsilon f(x)| \lesssim S_\epsilon |f|(x)$$

for almost every $x \in \text{supp}(f)$.

We now prove Theorem 1.2.

**Proof.** We will first prove that $\|T_\epsilon f\|_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}$ for any $f \in L^\infty_c(\mathbb{R}^n)$. Fix such an $f$. We need only show that for the sparse collection $S$ from Theorem 4.4, the associated $\epsilon$-sparse operator $S_\epsilon$ satisfies

$$\|S_\epsilon f\|_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}.$$  

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By Lemma 2.15, there exists \( g \in L^{p'}(\mathbb{R}^n) \) with \( \|g\|_{p'} \leq 1 \) such that
\[
\|S_\epsilon f\|_{p(\cdot)} \leq \|S_\epsilon f\|'_{p(\cdot)} \leq 2 \int_{\mathbb{R}^n} S_\epsilon f(x) g(x) dx.
\]
By Remark 4.2 and Lemma 2.16 we have that
\[
\int_{\mathbb{R}^n} S_\epsilon f(x) g(x) dx = \sum_{Q \in S} \epsilon_Q \int_Q f(y) dy \int_Q g(x) dx
\leq 2 \sum_{Q \in S} \sqrt{\epsilon_Q} \int_Q f(y) dy \sqrt{\epsilon_Q} \int_Q g(x) dx |E_Q|
\leq 2 \sum_{Q \in S} \|M_{\sqrt{\tau}} f(t) M_{\sqrt{\tau}} g(t) dt
\leq 2 \int_{\mathbb{R}^n} \|M_{\sqrt{\tau}} f(t) M_{\sqrt{\tau}} g(t) dt
\leq 4 \|M_{\sqrt{\tau}} f\|_{p(\cdot)} \|M_{\sqrt{\tau}} g\|_{p'(\cdot)}.
\]
Since \( p(\cdot) \in \sqrt{\tau} L H_0(\mathbb{R}^n) \), by Lemma 2.14 we have \( p' (\cdot) \in \sqrt{\tau} L H_0(\mathbb{R}^n) \). By Lemma 2.13 since \( p(\cdot) \in L H_\infty(\mathbb{R}^n) \), we have \( p' (\cdot) \in L H_\infty(\mathbb{R}^n) \). Hence, by Theorem 1.4 we have
\[
\|M_{\sqrt{\tau}} f\|_{p(\cdot)} \|M_{\sqrt{\tau}} g\|_{p'(\cdot)} \leq C \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)} \leq C \|f\|_{p(\cdot)}.
\]
Hence, \( \|T_c f\|_{p(\cdot)} \leq \|f\|_{p(\cdot)} \) for \( f \in L^\infty_c(\mathbb{R}^n) \).

Now given any \( f \in L^{p(\cdot)}(\mathbb{R}^n) \), since \( L^\infty_c(\mathbb{R}^n) \) is dense in \( L^{p(\cdot)}(\mathbb{R}^n) \) (See [2, Theorem 2.72]) and \( T_c \) is linear, the desired inequality follows by a standard approximation argument.

We now prove Theorem 1.3.

Proof. It suffices to show that for any sparse collection \( S \subset D(Q_0) \), the associated \( \epsilon \)-sparse operator \( S_\epsilon \) is compact on \( L^{p(\cdot)}(Q_0) \). Fix \( S \subset D(Q_0) \). For each \( N \in \mathbb{N} \), define the set \( D_N \) by
\[
D_N = \{Q \in D(Q_0) : 2^{-N} \leq \ell(Q) \leq 2^N\},
\]
and define the operator \( S_{\epsilon,N} \) by
\[
S_{\epsilon,N} f = \sum_{Q \in D_N \cap S} \epsilon_Q \int_Q f(y) dy \chi_Q,
\]
for \( f \in L^{p(\cdot)}(Q_0) \). Since \( Q_0 \) is bounded, \( D_N \) is a finite collection for all \( N \). Hence \( S_{\epsilon,N} \) is a finite rank operator for all \( N \). We will show that \( S_{\epsilon,N} \) converges to \( S_\epsilon \) in operator norm, i.e. \( S_{\epsilon,N} f \to S_\epsilon f \) uniformly for all \( f \) in the unit ball of \( L^{p(\cdot)}(Q_0) \). Fix such an \( f \). Observe that
\[
S_\epsilon f - S_{\epsilon,N} f = \sum_{Q \in D_N \cap S} \epsilon_Q \int_Q f(y) dy \chi_Q.
\]
By Lemma 2.15 we may choose a \( g \in L^{p'(\cdot)}(Q_0) \) with \( \|g\|_{p'(\cdot)} \leq 1 \) such that
\[
\left\| \sum_{Q \in D_N \cap S} \epsilon_Q \int_Q f(y) dy \chi_Q \right\|_{p(\cdot)} \leq 2 \int_{Q_0} \left( \sum_{Q \in D_N \cap S} \epsilon_Q \int_Q f(y) dy \chi_Q(x) \right) g(x) dx.
\]
We next use Remark 4.2 in the same way as in the proof of Theorem 1.2, but we split \( \epsilon \) into one factor of \( \epsilon^{1-2a} \) and two factors of \( \epsilon^a \) before using Lemma 2.16. This gives

\[
\int_{Q_0} \sum_{D_N \cap S} \epsilon f(y) dy \chi_Q(x) g(x) dx \leq 2 \sum_{Q \in D_N^c \cap S} \epsilon f(y) dy \int_Q g(x) dx dz
\]

\[
\leq 2 \sup_{Q \in D_N^c} \epsilon^{1-2a} \sum_{Q \in D_N^c \cap S} \int_{E_Q} M_{\epsilon^a} f(z) M_{\epsilon^a} g(z) dz
\]

\[
\leq 2 \sup_{Q \in D_N^c} \epsilon^{1-2a} \int_{Q_0} M_{\epsilon^a} f(z) M_{\epsilon^a} g(z) dz
\]

\[
\leq 4 \sup_{Q \in D_N^c} \epsilon^{1-2a} \|M_{\epsilon^a} f\|_{p(\cdot)} \|M_{\epsilon^a} g\|_{p'(\cdot)}
\]

Since \( p(\cdot) \in \epsilon^a LH_0(Q_0) \), by Lemma 2.14 we have that \( p'(\cdot) \in \epsilon^a LH_0(Q_0) \). Thus, by Lemma 3.1 we have that

\[
\|M_{\epsilon^a} f\|_{p(\cdot)} \|M_{\epsilon^a} g\|_{p'(\cdot)} \leq \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)} \leq 1,
\]

and so we need only show that \( \sup_{Q \in D_N^c} \epsilon^{1-2a} \rightarrow 0 \) as \( N \rightarrow \infty \).

Choose \( N_0 \) such that \( 2^{N_0} = \ell(Q_0) \). Then for all \( N \geq N_0 \), there are no cubes \( Q \in \mathcal{D}(Q_0) \) such that \( \ell(Q) > 2^N \). Hence \( D_N^c = \{Q \in \mathcal{D}(Q_0) : \ell(Q) < 2^{-N}\} \). Since we assume \( \lim_{N \rightarrow \infty} \sup \{\epsilon_Q : \ell(Q) < 2^{-N}\} = 0 \), we have that \( \sup_{Q \in D_N^c} \epsilon^{1-2a} \rightarrow 0 \) as \( N \rightarrow \infty \). Thus, \( S_{\epsilon,N} \rightarrow S_{\epsilon} \), and so \( S_{\epsilon} \) is a limit of finite rank operators. Hence, \( S_{\epsilon} \) is compact on \( L^{p(\cdot)}(Q_0) \) (See [1] p. 174). Consequently, Theorem 4.4 gives us that the Haar multiplier \( T_\epsilon \) is also compact on \( L^{p(\cdot)}(Q_0) \).

\[\Box\]

5. EXAMPLES

In this section, we give sufficient conditions on the collection \( \epsilon \) so that a specific exponent function \( p(\cdot) \) is not locally log-Holder continuous, but is in \( \epsilon LH_0(\mathbb{R}) \). Let \( 0 < a < 1 \) and define

\[
p(x) = \begin{cases} 
2 & x \leq 0 \\
2 + (\log_2 \frac{2}{x})^{-a} & 0 < x < 1 \\
3 & x \geq 1
\end{cases}
\]

(19)

This exponent function is not in \( LH_0(\mathbb{R}) \), as proved in [2] Section 4.4]. We next define the conditions on \( \epsilon \) for which \( p(\cdot) \in \epsilon LH_0(\mathbb{R}) \). Let \( C \geq 1 \). For each \( n \in \mathbb{Z} \) with \( n \geq 0 \), let \( Q_n = [0, 2^{-n}) \) and \( Q'_n = [2^{-(n+1)}, 2^{-n}) \). Let \( \epsilon = \{\epsilon_Q\}_{Q \in \mathcal{D}} \) be any collection satisfying

\[
\epsilon_{Q_n} \leq 2^{-n} C^{(n+1)a} \text{ for all } n \geq 0,
\]

(20)

\[
\epsilon_{Q'_n} = \epsilon_{Q_n} \text{ for all } n \geq 0,
\]

(21)

\[
\epsilon_Q = C \text{ for all other cubes } Q
\]

(22)

For such a collection \( \epsilon \), we have \( p(\cdot) \in \epsilon LH_0(\mathbb{R}) \). Let \( n \geq 0 \). First observe that since \( (\log_2(2/x))^{-a} \) is an increasing function, it attains its infimum at the left endpoint, and its
supremum on the right endpoint of any cube. Consequently, we have
\begin{align}
   (23) & \quad p_-(Q_n) - p_+(Q_n) = -(n+1)^{-a} \\
   (24) & \quad p_-(Q'_n) - p_+(Q'_n) = (n+2)^{-a} - (n+1)^{-a}
\end{align}

For the cube $Q_n$, by inequalities (23) and (20), we have
\[
\left( \frac{|Q_n|}{\epsilon_{Q_n}} \right)^{p_-(Q_n) - p_+(Q_n)} = \left( \frac{2^{-n}}{\epsilon_{Q_n}} \right)^{-(n+1)^{-a}} = (2^n \epsilon_{Q_n})^{(n+1)^{-a}} \leq (C^{(n+1)^a})^{(n+1)^{-a}} = C.
\]

For the cube $Q'_n$, by (21) and inequality (24), we have
\[
\left( \frac{|Q'_n|}{\epsilon_{Q'_n}} \right)^{p_-(Q'_n) - p_+(Q'_n)} = \left( \frac{2^{-n-1}}{\epsilon_{Q'_n}} \right)^{(n+2)^{-a} - (n+1)^{-a}} = (2^{n+1} \epsilon_{Q'_n})^{(n+1)^{-a} - (n+2)^{-a}} \leq (2C^{(n+1)^a})^{(n+1)^{-a} - (n+2)^{-a}} = 2^{(n+1)^{-a} - (n+2)^{-a}} C^{1 - \frac{2}{n+1}} \alpha^{-a}
\]

This expression is bounded, since both $2^{(n+1)^{-a} - (n+2)^{-a}}$ and $C^{1 - \frac{2}{n+1}} \alpha^{-a}$ are decreasing and converge to 1 as $n \to \infty$. Consequently, we obtain the upper bound at $n = 0$, which is $(2C)^{1-2^{-a}}$.

Now consider cubes of the form $Q = [0, 2^k)$ for any $k \in \mathbb{N}$. Then
\[
\left( \frac{|Q|}{\epsilon_{Q}} \right)^{p_-(Q) - p_+(Q)} = \left( \frac{2^k}{\epsilon_{Q}} \right)^{2-3} = \frac{C}{2^k} \leq \frac{C}{2}.
\]

Lastly, consider any cube $Q$ that is neither $Q_n$ nor $Q'_n$ for any $n \geq 0$ and is not of the form $[0, 2^k)$ for $k \in \mathbb{N}$. These cubes do not intersect $[0, 1)$ and so $p(\cdot)$ is constant on these cubes. Hence $p_-(Q) - p_+(Q) = 0$ and so
\[
\left( \frac{|Q|}{\epsilon_{Q}} \right)^{p_-(Q) - p_+(Q)} = 1.
\]

Thus, $p(\cdot) \in \epsilon LH_0(\mathbb{R})$.

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