Abstract

The Heisenberg Oscillator Algebra admits irreducible representations both on the ring $B$ of polynomials in infinitely many indeterminates (the bosonic representation) and on a graded-by-charge vector space, the semi-infinite exterior power of an infinite-dimensional $\mathbb{Q}$-vector space $V$ (the fermionic representation). Our main observation is that $V$ can be realized as the $\mathbb{Q}$-vector space generated by the solutions to a generic linear ODE of infinite order. Within this framework, the well known boson-fermion correspondence for the zero charge fermionic space is a consequence of the formula expressing each solution to a linear ODE as a linear combination of the elements of the universal basis of solutions. In this paper we extend the picture for linear ODEs of finite order. Vertex operators are defined and fully described in this case.

Keywords and Phrases. Generic Linear ODEs, Boson-Fermion Correspondence, Vertex Operators.

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1 Introduction

1.1 Statement of the Results. This paper is about vertex operators related to solutions to generic linear ODE of order $r \in \mathbb{N} \cup \{\infty\}$. If $r < \infty$, the main characters of the play are:

1. a ring $B_r := \mathbb{Q}[e_1, \ldots, e_r]$ of polynomials in the indeterminates $(e_1, \ldots, e_r)$ with rational coefficients, the $r$-th bosonic Fock space;

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2. the polynomial $E_r(t) = 1 - e_1 t + \cdots + (-1)^r e_r t^r \in B_r[t]$ and the sequence $H_r := (h_j)_{j \in \mathbb{Z}}$ defined by the equality $E_r(t) \sum_{n \in \mathbb{Z}} h_n t^n = 1$, holding in the ring of formal power series $B_r[[t]]$;

3. a vector space $V_r := \text{Span}_\mathbb{Q}(u_i)_{i \in \mathbb{Z}}$, where $u_i = \sum_{n \geq 0} h_{n+i} t^n / n!$;

4. The $r$-th fermionic Fock space of total charge $i \in \mathbb{Z}$ defined by $F^r_i := \text{Span}_\mathbb{Q}(\Phi^r_{i,\lambda})_{\lambda \in P_r}$, where $P_r$ is the set of all partitions $\lambda := (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq 0)$ of length at most $r$ and

$$\Phi^r_{i,\lambda} := u_{i+\lambda_1} \land u_{i+1+\lambda_2} \land \cdots \land u_{i-r+1+\lambda_r} \land u_{i-r} \land u_{i-r-1} \land \cdots$$  \hspace{1cm} (1)

5. The kernel $K_r \subseteq B_r[[t]]$ of the generic linear Ordinary Differential Operator

$$D^r - e_1 D^{r-1} + \cdots + (-1)^r e_r : B_r[[t]] \to B_r[[t]],$$  \hspace{1cm} (2)

where $D$ is the usual formal derivative on formal power series.

It is easily checked (Cf. [7]) that $(u_0, u_{-1}, \ldots, u_{-r})$ is a $B_r$-basis of $K_r$, i.e. $u_{-i} \in K_r$ for $0 \leq i \leq r - 1$, and if $\phi \in K_r$ there are unique $U_i(\phi) \in B_r$ such that $\phi = \sum_{i=0}^{r-1} U_i(\phi) u_{-i}$ (Cf. 4.3). This suffices to prove:

**Proposition 5.3 (Boson-Fermion correspondence).** The fermionic Fock space $F^r_0$ is a free $B_r$-module of rank 1 generated by the vacuum vector $\Phi_0^r := u_0 \land u_{-1} \land u_{-2} \land \cdots$. More precisely $\Phi^r_{i,\lambda} = \Delta_\lambda(H_r) \Phi^r_0$, where $\Delta_\lambda(H_r) := \det(h_{\lambda_i+j+i})_{i,j \leq r}$ is the Schur determinant associated to the partition $\lambda$ and the sequence $H_r$.

Each $\alpha \in F^r_0$ is a finite $\mathbb{Q}$-linear combination of typical semi-infinite exterior monomials $\Phi^r_{i,\lambda}$ (Cf. (1)): the boson-fermion correspondence $F^r_0 \to B_r$ maps $\alpha \mapsto \alpha / \Phi^r_0$, with obvious meaning of the notation (Section 5).

Since $B_r = \bigoplus_{\lambda \in P_r} \mathbb{Q} \Delta_\lambda(H_r)$, the canonical $B_r$-module structure of $F^r_0$ prescribed by Proposition 5.3 allows to define vertex operators $\Gamma_r(z), \Gamma^\vee_r(z) : B_r \to B_r[[z^{-1}, z]]$ through the equalities:

$$\Gamma_r(z) \Delta_\lambda(H_r) = \left( \sum_{i \in \mathbb{Z}} z^i u_i \land \Phi^r_{i,\lambda} \right) \otimes \otimes 1_{B_r} / \Phi^r_0,$$

$$\Gamma^\vee_r(z) \Delta_\lambda(H_r) = \left( \sum_{i \in \mathbb{Z}} z^{-i} u_i \land \Phi^r_{i,\lambda} \right) \otimes \otimes 1_{B_r} / \Phi^r_0.$$  

Consider the $\mathbb{Q}$-homomorphisms $G_r(z), G^\vee_r(z) : B_r \to B_r[[z^{-1}, z]]$ given by:

$$G_r(z) \Delta_\lambda(H_r) := E_r(z)(\Gamma_r(z) \Delta_\lambda(H_r)) \quad \text{and} \quad G^\vee_r(z) \Delta_\lambda(H_r) := \frac{z \Gamma^\vee_r(z) \Delta_\lambda(H_r)}{E_r(z)}.$$  

One has (Corollaries 6.6 and 7.1):

$$G_r(z) h_n = h_n - \frac{h_{n-1}}{z} \quad \text{and} \quad G_r^\vee(z) h_n = \sum_{i \geq 0} \frac{h_{n+i}}{z^i},$$

so that, in fact, $G_r(z) h_n$ and $G_r^\vee(z) h_n$ belong to $B_r[z^{-1}]$. By extending by $\mathbb{Q}[z^{-1}]$-linearity, an easy check proves that $G_r^\vee(z) G_r(z) h_n = G_r(z) G_r(z) h_n = h_n$. Indeed, it turns out that $G_r(z) \Delta_\lambda(H_r)$ and $G_r^\vee(z) \Delta_\lambda(H_r)$ both belong to $B_r[z^{-1}]$ and that $G_r(z)$ and $G_r^\vee(z)$ are one inverse of the other once they are extended by $\mathbb{Q}[z^{-1}]$-linearity. This is a consequence of the main result of this paper, which is the explicit description of $\Gamma_r(z)$ and $\Gamma^\vee_r(z)$.

**Theorems 6.7 and 7.3.** The operators $G_r(z), G^\vee_r(z)$ commute with taking the Schur determinants, i.e.

$$G_r(z) \Delta_\lambda(H_r) = \Delta_\lambda(G_r(z) H_r) \quad \text{and} \quad G_r^\vee(z) \Delta_\lambda(H_r) = \Delta_\lambda(G_r^\vee(z) H_r),$$

2
where \( G_r(z)H_r := (G_r(z)h_n)_{n \in \mathbb{Z}} \) and \( G_\nu^\vee(z)H_r := (G_\nu^\vee(z)h_n)_{n \in \mathbb{Z}} \). Therefore:

\[
\Gamma_r(z)\Delta_\lambda(H_r) = \frac{1}{E_r(z)}\Delta_\lambda(G_r(z)H_r),
\]

(3)

\[
\Gamma_\nu^\vee(z)\Delta_\lambda(H_r) = \frac{E_r(z)}{z}\Delta_\lambda(G_\nu^\vee(z)H_r).
\]

(4)

The proof of formula (3) is based on the following expression holding in the fermionic picture (Proposition 6.3):

\[
\left(\sum_{i \in \mathbb{Z}} z^i u_i \wedge \Phi_{r-1,\lambda}^\nu\right) \otimes \mathbb{Q} 1_{B_r} = \frac{1}{E_r(z)} \exp \left(\frac{t}{z}\right) \wedge \Phi_{r-1,\lambda}.
\]

(5)

while the proof of (4) is based on the equality

\[
\left(\sum_{i \in \mathbb{Z}} z^i u_i \wedge \Phi_{r-1,\lambda}^\nu\right) \otimes \mathbb{Q} 1_{B_r} = \sum_{\lambda \in \mathbb{Z}} \frac{z^{-\lambda_1} z^{1-\lambda_2} \ldots z^{r-1-\lambda_r}}{h_{\lambda_1+1} \cdot \cdot \cdot h_{\lambda_2+1} \cdot \cdot \cdot h_{\lambda_r+1}} \Delta_{\lambda_1+1, \ldots, \lambda_r+1}(H_r).
\]

(6)

stated and proven in Lemma 7.2.

Let us stress that the first summand of (6) is obtained from \( \Delta_\lambda(H_r) \) by substituting each entry \( h_{\lambda_i+i} \) of its first row by the monomial \( z^{i-1-\lambda_i} \) (\( 1 \leq i \leq r \)).

Formulas (3)–(4) and (5)–(6) are new in shape and perspective as they involve in an essential way the use of solutions to a linear ODE of order \( r \). They generalize the classical framework of the boson-fermion correspondence, as explained e.g. in [9, 10], that arises in the representation theory of the Heisenberg algebra. This last picture is in fact obtained by letting \( r \) going to \( \infty \).

Our Section 8 supplies a new transparent proof of the classical expression of the vertex operators as displayed, e.g., in [10, p. 54] or [1, Section 4]. The proof is based on the fact that \( G_\infty(z), G_\nu^\vee(z) \) are ring homomorphisms (contrarily to \( G_r(z), G_\nu^\vee(z) \), which are not for finite \( r \)) and then can be taken as exponentials of a first order differential operator. It is also probably worth to mention that for \( r = \infty \), each \( u_{-i} \) (\( i \in \mathbb{Z} \)) remarkably satisfies (in many different ways) the Kadomtsev-Petviashvili (KP) equation in the Hirota bilinear form (Cf. Remark 4.5).

1.2 Relationships with Schubert Calculus. Mainly inspired by mathematical physics (see e.g. [1, 9, 10, 11, 17]), the boson-fermion correspondence has a number of nice geometrical consequences. In relatively recent times its connection with the geometry of the Hilbert schemes of points in the affine plane has been investigated by Nakajima [16]. More classically, a system of PDEs encoded in the tensor product of the vertex operators \( \Gamma_\infty(z), \Gamma_\nu^\vee(z) \) allows to embed an (infinite dimensional) Universal Grassmann Manifold ([10, p. 76]) into the ring of polynomials in infinitely many indeterminates with complex coefficients. For different applications of vertex operators see also [18].

The guiding idea of our investigation was to look at the boson-fermion correspondence as an infinite dimensional manifestation of the classical Schubert Calculus for the Grassmann varieties \( G := G(k, C^n)_k \) of \( k \)-planes in \( C^n \), as formulated in [4] through a derivation on the exterior algebra \( \wedge M \) of a free module of rank \( n \). Within this framework, Schubert cycles in the Chow ring of \( G \) can be seen as a kind of bosonic counterpart of generalized wronskians associated to a linear ODE having as coefficients the Chern classes of the tautological bundle over \( G \) (see [7]).

The explicit bridge between generic linear ODEs and Schubert Calculus was established in [7] (see also [8]), where a Giambelli (or Jacobi-Trudy) formula for generalized wronskians, associated to a fundamental system of solutions, is proven. It generalizes the proof of the classical theorem, due to Abel and Liouville, according which if the wronskian determinant associated to a fundamental system of solutions to a linear ODE does not vanish at a point, then it vanishes nowhere.
The purely algebraic treatment of [7] suggests in a irresistible way to define generic linear ODEs of infinite (countable) order, with indeterminate coefficients \((e_1, e_2, \ldots)\) (see also [6]). The basis elements of the fermionic space \(F^\infty_n\) can be seen as generalized wronskians associated to a basis of solutions, which is what allows to rephrase the classical framework of the boson-fermion correspondence.

Equipping the exterior algebra of a free module \(M\) with a derivation, as in [4], is the same as endowing \(\bigwedge M\) with a sequence of endomorphisms satisfying certain Leibniz like rules with respect to the wedge product. Their restriction to a fixed \(k\)-th exterior power has been studied purely in terms of symmetric functions by Laksov and Thorup in [12, 13]. They show that the formalism of Schubert Calculus for Grassmann varieties is entirely encoded by the canonical symmetric structure of the \(k\)-th exterior power of a polynomial ring. It equips the latter with a structure of a free module of rank \(1\) over the ring of symmetric functions in \(k\) indeterminates.

Remarkably, at the end of the introduction of [12], the authors claim that “In the work of E. Date, M. Jimbo, M. Kashiwara and T. Miwa [2], Schur functions appear in connection with exterior products (but) in another context. See also the work of V. G. Kac and A. K. Raina [10].” The present paper shows, on the contrary, that the context is pretty much the same and is based on the classical elegant interplay (described e.g. in [14]) between the complete symmetric functions, the elementary symmetric functions and those which are sum of powers.

1.3 The paper is organized as follows. Section 2 collects a few preliminaries, Section 3 defines the \(r\)-th bosonic Fock space. It will be interpreted as the \(\mathbb{Q}\)-polynomial ring generated by the indeterminate coefficients of a generic linear ODE (Section 4).

Fermionic-Fock spaces generated by formal power series related with linear ODEs are introduced in Section 5. The boson-fermion correspondence for \(F^\infty_n\) is proven, through the application of the universal Cauchy formula (20). Still in this section the vertex operators \(\Gamma^r(z)\) are defined. They are discussed in detail in Sections 6 and 7. The final Section 8 (re-)computes the shape of the vertex operators associated to linear ODEs of infinite order, re-obtaining classical formulas.

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2 Preliminaries and Notation

2.1 Throughout the paper \(\mathbb{N}\) will denote the non negative integers and \(\mathbb{N}^*\) the positive integers. We denote by \(\mathcal{P}\) the set of all partitions, i.e. the set of all monotonic non increasing sequences of non negative integers such that all but finitely many terms are zero. The length \(\ell(\lambda)\) of the partition is the number of its non zero terms, called parts. An arbitrary partition \(\lambda\) of length at most \(r\) will be written as \(\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r)\). Let \(\mathcal{P}_r(\lambda)\) be the set of all partitions of length at most \(r\) and \(a := (a_i)_{i \in \mathbb{Z}}\) any bilateral sequence of elements of some ring \(A\). Then \(\Delta_\lambda(a) := \det(a_{\lambda_i-j+i})_{1 \leq i, j \leq r} \in A\) is the Schur determinant associated to \(\lambda\) and to \(a\).

2.2 Let \(A\) be any \(\mathbb{Q}\)-algebra and let \(A[t] \subseteq A[[t]]\) be the standard inclusion of the polynomial ring into the formal power series. A monic polynomial \(P \in A[t]\) of degree \(r\) will be written as:

\[
P := t^r - e_1(P)t^{r-1} + \ldots + (-1)^re_r(P), \quad e_i(P) \in A.
\]  (7)

where \((-1)^je_j(P)\) denotes the coefficient of \(t^j\). If \(P\) splits in \(A\) as a product of \(r\) distinct linear factors, \(e_j(P)\) is precisely the \(j\)-th elementary symmetric polynomial function in the roots of \(P\).
As $A$ contains the rational numbers, each $\phi \in A[[t]]$ can be written in the form
\[
\phi = \sum_{n \geq 0} a_n \frac{t^n}{n!}, \quad (a_n \in A).
\] (8)

If $a_n = a^n$, for some $a \in A$, then $\exp(at)$ stands for the associated exponential formal power series $\exp(at) = \sum_{n \geq 0} a^n t^n / n!$.

2.3 The map $D^j : A[[t]] \to A[[t]]$, $j \in \mathbb{N}$, defined by:
\[
D^j \sum_{n \geq n} a_n \frac{t^n}{n!} = \sum_{n \geq 0} a_{n+j} \frac{t^n}{n!},
\]
is the identity of $A[[t]]$ for $j = 0$ while for $j > 0$ is the $j$th iterated of $D := D^1$, the formal derivative on $A[[t]]$, the unique (infinite) linear extension of the map $Dt^n = nt^{n-1}$. The commutative $A$-subalgebra $A[D]$ of $\text{End}_A(A[[t]])$ of linear ordinary differential operators (ODO) with $A$-coefficients is gotten by evaluating each $P \in A[t]$ (monic or not) at $D$. In particular
\[
\ker P(D) := \{ y \in A[[t]] \mid D^j y - e_1(P) D^{j-1} y + \ldots + (-1)^j e_j(P) y = 0 \},
\]
is the $A$-submodule of $A[[t]]$ of the solutions of the linear homogeneous ODE $P(D)y = 0$.

3 Bosonic Fock Spaces

3.1 Let $r \in \mathbb{N} \cup \{\infty\}$ and $E_r := (e_i)_{i \in [1, r] \cap \mathbb{N}}$ be a sequence of $r$ indeterminates over $\mathbb{Q}$. For instance $E_1 = (e_1), E_2 = (e_1, e_2), \ldots, E_\infty := (e_1, e_2, \ldots)$. We write
\[
E_r(t) = 1 + \sum_{i \in [1, r] \cap \mathbb{N}} (-1)^i e_i t^i \in B_r[[t]]
\] (9)
for the formal power series having as coefficients the terms of the sequence $E_r$. If $r < \infty$ then $E_r(t) = 1 - e_1 t + e_2 t^2 + \ldots + (-1)^r e_r t^r$. Imitating [10], we call $r$th bosonic Fock space the polynomial $\mathbb{Q}$-algebra $B_r := \mathbb{Q}[E_r]$. For $s \geq r$ there are obvious $\mathbb{Q}$-algebra epimorphisms $B_s \to B_r$ mapping $e_j \mapsto e_j$ if $j \leq r$ and $e_j$ to 0 if $j > r$.

3.2 Let $H_r = (h_i)_{i \in \mathbb{Z}}$ and $X_r := (x_i)_{i \in \mathbb{N}}$ be the sequences of coefficients of the formal power series $H_r(t) := \sum_{n \in \mathbb{Z}} h_n t^n$ and $X_r(t) = \sum_{n \geq 0} x_n t^n$ defined through the equalities in $B_r[[t]]$:
\[
H_r(t) = \frac{1}{E_r(t)} = \exp(X_r(t)).
\] (10)
where $E_r$ is defined by (9). The first equality (10) gives:
\[
\sum_{n \in \mathbb{Z}} h_n t^n = \frac{1}{1 + \sum_{1 \leq i \leq r} (-1)^i e_i t^i} = 1 + \sum_{n \geq 1} \left( \sum_{1 \leq i \leq r} (-1)^i e_i t^i \right)^n,
\]
showing that $h_j = 0$ if $j < 0$, $h_0 = 1$, $h_1 = e_1$, $h_2 = e_1^2 - e_2$, \ldots Moreover the equality $H_r(t) E_r(t) = 1$ implies the relation
\[
h_n + \sum_{1 \leq i \leq r} (-1)^i e_i h_{n-i} = 0, \quad (n \in \mathbb{Z}),
\] (11)
which gives for $(k \in \mathbb{N}^*)$ the well known relation $h_k = \det(e_{j+i+1})_{1 \leq i, j \leq k}$, under the convention $e_0 = 1$ and $e_j = 0$ if $j > r$, [14]. The first few terms of $H_r$ in terms of the $x_i$s are (Cf. [10] where the $h_i$s are called $S_i$)
\[
h_1 = x_1, \quad h_2 = \frac{x_1^2}{2} + x_2, \quad h_3 = \frac{x_1^3}{3!} + x_1 x_2 + x_3, \ldots
\]
Let $\mathcal{I}_{H_r}$ be the ideal of $B_r$ generated by the relations (11). Then

$$B_r = \mathbb{Q}[H_r] = \frac{\mathbb{Q}[H_{\infty}]}{\mathcal{I}_{H_r}} \cong \mathbb{Q}[h_1, \ldots, h_r].$$

In other words, if $r < \infty$, each $h_{r+1+j}$ is an explicit polynomial expression in $(h_1, \ldots, h_r)$. Similarly the equality $E_r(t)^{-1} = \exp(X_r(t))$ implies

$$(r + 1 + j)x_{r+1+j} - (r + j)e_1x_{r+1} + \ldots + (-1)^r j e_r x_{r+1} = 0, \quad (j \geq 0)$$

i.e. each $x_{r+1+j}$ can be expressed as a $B_r$-linear combination of $x_{j+1}, \ldots, x_{r+j}$. By induction each $x_{r+1+j}$ is a polynomial in $x_1, \ldots, x_r$ with $\mathbb{Q}$-coefficients. If $\mathcal{I}_{X_r}$ is the ideal of the relations (12), then

$$B_r = \mathbb{Q}[X_r] = \frac{\mathbb{Q}[X_{\infty}]}{\mathcal{I}_{X_r}} \cong \mathbb{Q}[x_1, \ldots, x_r].$$

Writing $h_n, x_n \in B_r$ means, respectively, the unique polynomial $h_n \mod \mathcal{I}_{H_r} \in \mathbb{Q}[h_1, \ldots, h_r]$ and the unique polynomial $x_n \mod \mathcal{I}_{X_r} \in \mathbb{Q}[x_1, \ldots, x_r]$, where $x_n, h_n$ have been regarded as elements of $B_r$ with $s > n$. For instance if $r = 2$, then $h_3 = h_1^2 - 2h_1 h_2 \in \mathbb{Q}[h_1, h_2].$

### 3.3 Proposition.

If $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathcal{P}_r$ and $H_r$ is as in 3.2, let

$$\Delta_\lambda(H_r) := \det(h_{\lambda_j-j+i})_{1 \leq i,j \leq r} \mod \mathcal{I}_{H_r}.$$ 

It is well known that [14, p. 41]:

$$\mathbb{Q}[h_1, \ldots, h_r] \cong B_r = \bigoplus_{\lambda \in \mathcal{P}_r} \mathbb{Q} \cdot \Delta_\lambda(H_r).$$

### 3.4 From now on let $A$ be any $B_r$-algebra, fixed once and for all. For $\phi = \sum_{n \geq 0} a_n t^n/n!$ define linear forms $U_k : A[[t]] \to A$ by:

$$U_k(\phi) = a_k + \sum_{i \geq 1} (-1)^i e_i a_{k-i}, \quad (k \in \mathbb{N}),$$

agreeing that $e_j = 0$ if $j \geq r + 1$ and $a_j = 0$ if $j < 0$. In particular $U_0(\phi) = a_0, U_1(\phi) = a_1 - e_1 a_0, U_2(\phi) = a_2 - e_1 a_1 + e_2 a_0$, ... Let

$$u_j := \sum_{n \geq 0} h_{n+j} \frac{t^n}{n!} \in B_r[[t]], \quad (j \in \mathbb{Z}).$$

One easily checks that $D^i u_j = u_{i+j}$ for $i \geq 0$ and $j \in \mathbb{Z}$. By abuse of notation, we shall write $u_j \in A[[t]]$ instead of $u_j \otimes_{B_r} 1_A \in A[[t]]$. Notice that for $j \geq 0$

$$u_{-j} = \frac{t^j}{j!} + \frac{h_1}{(j + 1)!} \frac{t^{j+1}}{(j+1)!} + \ldots$$

and

$$u_j = h_j + h_{j+1} t + h_{j+2} \frac{t^2}{2!} + \ldots$$

Obviously, all the $u_j$s are linearly independent over $\mathbb{Q}$. Moreover:

### 3.5 Proposition.

If $\phi \in A[[t]]$ then:

$$\phi = \sum_{i \geq 0} U_i(\phi) u_{-i}. $$

(17)
Proof. Because of (15), it is obvious that each \( \phi \in A[[t]] \) can be written as an infinite linear combination \( \phi = \sum_{i \geq 0} a_i u_{-i} \), for some \( a_i \in A \). For each \( j \geq 0 \), the linearity of \( U_j \) yields

\[
U_j(\phi) = \sum_{i \geq 0} a_i U_j(u_{-i}).
\]

By (15) and Definition (13), \( U_j(u_{-j}) = 1 \). If \( j \neq -i \), instead

\[
U_j(u_{-i}) = h_{n-i+j} + \sum_{r \leq k \leq r} (-1)^r e_k h_{n-i+j-k} = 0,
\]

which is (11) for \( n - i + j \). Thus \( U_j(\phi) = a_j \) and each \( \phi \in A[[t]] \) is the sum (17) of its “projections” along \( u_{-i} \), for \( i \in \mathbb{N} \).

The Proposition implies:

3.6 Corollary. For each \( r \in \mathbb{N} \cup \{ \infty \} \) and each \( j \geq 0 \):

\[
\frac{t^j}{j!} u_{-j} + \sum_{i \in \{1, \ldots, r\} \cap \mathbb{N}} (-1)^i e_i u_{-j-i}.
\]

(18)

For example, if \( r = 3 \), one has \( 1 = u_0 - e_1 u_{-1} + e_2 u_{-2} + e_3 u_{-3} \) and for \( j \geq 1 \)

\[
\frac{t^j}{j!} = u_{-j} - e_1 u_{-j-1} + e_2 u_{-j-2} + e_3 u_{-j-3}.
\]

Notice that if \( r = \infty \) the r.h.s. of (18) is an infinite sum.

3.7 Remark. Relation (10) shows that each \( \phi \in B_r[[t]] \) can be regarded as a function of \( (x_i)_{i \in \{1, r\} \cap \mathbb{N}}. \)

4 Generic linear ODEs

4.1 Let \( t, z \) be indeterminates over \( B_r \) and \( e^{Dz} := \exp(Dz) \in B_r[[z]] \). Clearly \( e^{Dz} : B_r[[t]] \rightarrow B_r[[t, z]] \) and

\[
U_r(e^{Dz}) = D^r - e_1 D^{r-1} + \ldots + (-1)^r e_r \in B_r[D]
\]

(19)

is the generic linear ordinary differential operator of order \( r \). Denote \( K_r := \ker U_r(e^{Dz}) \), the submodule of \( B_r[[t]] \) whose elements are solutions of \( U_r(e^{Dz})y = 0 \), the generic linear homogeneous ODE of order \( r \).

4.2 Proposition. We have \( \phi \in K_r \otimes B_r \ A \iff U_{n+r}(\phi) = 0 \) for all \( n \geq 0 \).

Proof. Write

\[
U_r(e^{Dz}) \phi = (D^r - e_1 D^{r-1} + \ldots + (-1)^r e_r) \sum_{n \geq 0} a_n \frac{t^n}{n!} = \sum_{n \geq 0} a_{n+r} - e_1 a_{n+r-1} + \ldots + (-1)^r e_r a_n \frac{t^n}{n!}.
\]

Thus \( \phi \in K_r \otimes B_r \ A \) if and only if \( U_{n+r}(\phi) = 0 \) for all \( n \geq 0 \).

4.3 Proposition. If \( r < \infty \) the \( r \)-tuple \((u_0, u_{-1}, \ldots, u_{-r+1})\) is a \( B_r \)-basis of \( K_r \).

Proof. Clearly \((u_0, u_1, \ldots, u_{-r+1})\) are \( B_r \)-linear independent by (15). They belong to \( K_r \) because \( U_{r+n}(u_{-j}) = 0 \) (Proposition 3.5), and are solutions to the generic linear ODE by Proposition 4.2. Then each \( \phi \in K_r \subseteq A[[t]] \) can be written as

\[
\phi = U_0(\phi) u_0 + U_1(\phi) u_{-1} + \ldots + U_{r-1}(\phi) u_{-r+1},
\]

(20)

applying once again Proposition 3.5 and Proposition 4.2.

Formula (20) is called in [7] Universal Cauchy formula.
4.4 Remark. The solutions \((u_j)_{j \in \{0,1,\ldots\}}\) of the generic linear ODE of order \(r\) are universal in the following sense. For any associative commutative \(\mathbb{Q}\)-algebra \(A\) and each monic \(P \in A[t]\) of degree \(r\), then \(\ker P(D) = K_r \otimes_{B_r} A\), where \(A\) is regarded as a \(B_r\)-algebra via the unique morphism that maps \(e_i \mapsto e_i(P)\).

4.5 Remark. If \(r = \infty\) we say that \((u_0, u_1, \ldots)\) is a fundamental system of solutions to the linear ODE of infinite order having \(E_\infty := (e_1, e_2, \ldots)\) as sequence of coefficients. In this sense, each element \(\phi \in B_\infty[[t]]\) is a solution to the linear ODE of infinite order. The relation \(\sum_{n \in \mathbb{Z}} h_n t^n = \exp(\sum x_i t^i)\) easily implies that

\[
\frac{\partial^i h_n}{\partial x_i^n} = \frac{\partial h_n}{\partial x_j} = h_{n-j}.
\] (21)

More generally:

\[
\frac{\partial h_j}{\partial x_i} = \sum_{n \geq 0} \frac{\partial h_{n+j}}{\partial x_i} \frac{t^n}{n!} = \sum_{n \geq 0} h_{n+j-i} \frac{t^n}{n!} = u_{j-i}, \quad (j \in \mathbb{Z}).
\]

Thus for \(n \geq 1\), \(u_j\) is the PDE

\[
\frac{\partial^4 \phi}{\partial x_1^4} - 4 \frac{\partial \phi}{\partial x_3} \frac{\partial \phi}{\partial x_4} + 3 \phi \left( \frac{\partial^2 \phi}{\partial x_2^2} \right)^2 - 3 \phi \left( \frac{\partial \phi}{\partial x_2} \right)^2 + 3 \frac{\partial \phi}{\partial x_2} \frac{\partial \phi}{\partial x_3} - 4 \frac{\partial \phi}{\partial x_3} \frac{\partial^2 \phi}{\partial x_4^2} = 0,
\]

which is the bilinear form of the Kadomtsev-Petviashvili (KP) equation

\[
\frac{3}{4} \frac{\partial^2 f}{\partial y^2} - \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial t} - \frac{3}{2} \frac{\partial f}{\partial x} - \frac{1}{4} \frac{\partial^3 f}{\partial x^3} \right) = 0,
\] (22)

up to the substitution \(x_n = x, x_{2n} = y, x_{3n} = t\) and putting \(f(x, y, t) = 2 \frac{\partial^2 \phi}{\partial x^2} (\log \phi)\). This is a nice way to phrase the known fact that the complete symmetric polynomials in infinitely many indeterminates are solutions to the KP equation (22) (in the Hirota bilinear form—see [10, p. 75]).

5 Fermionic Fock spaces

5.1 Given \(r \in \mathbb{N} \cup \{\infty\}\), let \(V_r := \bigoplus_{j \in \mathbb{Z}} Q \cdot u_j\), with \(u_j \in B_r[[t]]\) as in (14). For each \(i \in \mathbb{Z}\), let

\[
\Phi_i := u_i \wedge u_{i-1} \wedge \ldots \wedge u_{i-r+1} \wedge u_{i-r} \wedge u_{i-r-1} \wedge \ldots
\]

and for each \(\lambda \in \mathcal{P}_r\), let

\[
\Phi^{\lambda}_{i,0} := u_{i+\lambda_1} \wedge u_{i+1+\lambda_2} \wedge \ldots \wedge u_{i-r+1+\lambda_r} \wedge \Phi_{i-r}
\]

so that \(\Phi^{\lambda}_{i,0} = \Phi^{\lambda}_{i}\). Let \(F^\infty_i := \bigoplus_{\lambda \in \mathcal{P}_r} \mathbb{Q} \cdot \Phi^{\lambda}_{i,0}\). Notice that \(i \neq j\) implies that \(F^\infty_i \cap F^\infty_j = 0\).

The direct sum \(\bigoplus_{j \in \mathbb{Z}} F^\infty_j\) is often denoted by \(\bigwedge_{\mathbb{Q}}^{\infty/2} V_r\) in the literature, and is called semi-infinite exterior power of \(V_r\).

5.2 Remark. If \(r = \infty\), we write \(\Phi_i\) instead of \(\Phi^{\infty}_i\) and \(F_i\) instead of \(F^\infty_i\). In particular \(F_i = \bigoplus_{\lambda \in \mathcal{P}_r} \mathbb{Q} \cdot \Phi^{\lambda}_{i,0}\), and a typical monomial of \(F_i\) is

\[
\wedge u_{i+\lambda_1} \wedge \ldots \wedge u_{i-k+1+\lambda_k} \wedge \Phi_{i-k}
\]

where \(\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k)\) is a partition of length at most \(k\), \(k \geq 1\).
It is a trivial remark that for each $i$ and each $r$ there is a natural, although in principle not canonical, isomorphism $\sigma_r^i : F_r^i \to B_r$ given by $\Phi_{r,\lambda} \mapsto \Delta_{\lambda}(H_r)$. Using that isomorphism each $F_r^i$ gets a structure of $B_r$-module of rank 1 generated by $\Phi_r^i$. On the other hand $K_r = \ker U_r(e^{Dz})$ has a canonical structure of $B_r$-module of rank $r$ [7, Theorem 2.1]. It turns out that $\wedge^r K_r$ itself is a free $B_r$-module of rank 1 and thus $F_0^r := \wedge^r K_r \wedge \Phi_r^r$ inherits a structure of rank 1 free $B_r$-module. Our first remark is that such a structure coincides with that induced by $\sigma_0^r$, via the Universal Cauchy Formula (20).

5.3 Proposition (The Boson-Fermion Correspondence for $F_0^r$). The vector space $F_0^r$ has a canonical structure of free $B_r$-module of rank 1 generated by $\Phi_0^r$.

**Proof.** We shall construct a $B_r$-module morphism $F_0^r \to B_r$ mapping $\Phi_0^r$ to 1. We distinguish two cases. If $r < \infty$, one sees that

$$F_0^r = \bigwedge^r K_r \wedge \Phi_r^r := \text{Span}_0 \{v_0 \wedge \ldots \wedge v_{r-1} \wedge \Phi_r^r \mid v_i \in K_r\}. $$

By the universal Cauchy formula (20), $v_i = \sum_{0 \leq j \leq r-1} U_j(v_i) u_{-j}$. For each typical monomial of $F_0^r$ one has

$$(v_0 \wedge \ldots \wedge v_{r-1} \wedge u_{-1} \wedge \ldots ) \otimes \Phi_r^r = \sum_{0 \leq i_0 \leq r-1} U_{i_0}(v_0) u_{-i_0} \wedge \sum_{0 \leq i_1 \leq r-1} U_{i_1}(v_1) u_{-i_1} \wedge \ldots \wedge \sum_{0 \leq i_{r-1} \leq r-1} U_{i_{r-1}}(v_{r-1}) \wedge \Phi_r^r =

$$

$$\begin{vmatrix}
U_0(v_0) & \ldots & U_0(v_{r-1}) \\
\vdots & \ddots & \vdots \\
U_{r-1}(v_0) & \ldots & U_{r-1}(v_{r-1})
\end{vmatrix} u_0 \wedge u_{-1} \wedge \ldots \wedge u_{-r+1} \wedge \Phi_r^r = \det((U_i(v_j))_{0 \leq i,j \leq r-1}) \Phi_0^r, \quad (23)
$$

and $\det((U_i(v_j))_{0 \leq i,j \leq r-1}) \in B_r$. Define $F_0^r \to B_r$ by

$$v_0 \wedge v_1 \wedge \ldots \wedge v_{r-1} \wedge u_{-1} \wedge \ldots \wedge u_{-r+1} \wedge \Phi_r^r = \det((U_i(v_j))_{0 \leq i,j \leq r-1}) \Phi_0^r, \quad (24)
$$

It is clearly an isomorphism. In fact $\det((U_i(v_j))_{0 \leq i,j \leq r-1}) = 0$ implies $v_0, \ldots , v_{r-1}$ are linearly dependent over $B_r$ and then $v_0 \wedge \ldots \wedge v_{r-1} \wedge \Phi_r^r = 0$. Moreover $\Phi_0^r/\Phi_0^r = 1$.

If $r = \infty$ and $v_0, \ldots , v_{k-1} \in B_\infty$, then for each $0 \leq i \leq k - 1$

$$v_i = \sum_{j \geq 0} U_j(v_i) u_{-j}, \quad (24)$$

which is in general an infinite linear combination. However substituting the expression (24) of $v_i$ into $v_0 \wedge v_1 \wedge \ldots \wedge v_{r-1} \wedge u_{-1} \wedge \ldots$ one sees that all the summands involving $u_{-j}$, with $j \geq k$, vanish due to skew-symmetry of the wedge product. Arguing as in the case $r < \infty$, one easily obtains the desired formula:

$$v_0 \wedge v_1 \wedge \ldots \wedge v_{k-1} \wedge u_{-k} \wedge \ldots = \det((U_i(v_j))_{0 \leq i,j \leq k}) \Phi_0^r. \quad \blacksquare$$

5.4 Corollary. One has:

$$\frac{\Phi_{r,\lambda}^0}{\Phi_0^r} = \sigma_r^i(\Phi_{r,\lambda}^0) = \Delta_{\lambda}(H_r).$$

**Proof.** If $r < \infty$ and $v_i = u_{-i-\lambda_{i+1}}, 0 \leq i \leq r - 1$ one has

$$\frac{\Phi_{r,\lambda}^0}{\Phi_0^r} = \det((U_i(u_{-j-\lambda_j}))_{0 \leq i,j \leq r-1}).$$
By writing explicitly the determinant and using the definition of the $u_j$s and of the linear maps $U_i$, one obtains:

\[
\begin{vmatrix}
U_0(u_{0+\lambda_1}) & \ldots & U_0(u_{-r+1+\lambda_r}) \\
\vdots & \ddots & \vdots \\
U_{r-1}(u_{0+\lambda_1}) & \ldots & U_{r-1}(u_{-r+1+\lambda_r})
\end{vmatrix}
= \begin{vmatrix}
\h_{\lambda_1} & \h_{\lambda_2-1} & \ldots & \h_{\lambda_r-1} \\
\h_{\lambda_1+1} & \h_{\lambda_2-2} & \ldots & \h_{\lambda_r+1} \\
\vdots & \vdots & \ddots & \vdots \\
\h_{\lambda_1+r-1} & \h_{\lambda_2+r-2} & \ldots & \h_{\lambda_r}
\end{vmatrix}.
\] (25)

Using the multilinearity and skew symmetry, the determinant (25) simplifies into:

\[
\begin{vmatrix}
\h_{\lambda_1} & \h_{\lambda_2-1} & \ldots & \h_{\lambda_r-1} \\
\h_{\lambda_1+1} & \h_{\lambda_2-2} & \ldots & \h_{\lambda_r+1} \\
\vdots & \vdots & \ddots & \vdots \\
\h_{\lambda_1+r-1} & \h_{\lambda_2+r-2} & \ldots & \h_{\lambda_r}
\end{vmatrix} = \det(h_{\lambda_j-j+i}|_{0 \leq i, j \leq r-1}) = \Delta_\lambda(H_r),
\]
as desired.

\section{5.5} The module structure $B_r \otimes F_0^r \to F_0^r$ induced by the Boson-Fermion correspondence is then defined by imposing the equalities

\[
\begin{cases}
P\Phi_0^r,\lambda(H_r) &= (P \cdot \Delta_\lambda(H_r))\Phi_0^r, \\
\Delta_\lambda(H_r)\Phi_0^r &= \Phi_0^r,\lambda.
\end{cases}
\]

Since each $P \in B_r$ is a polynomial in $(e_i)_{i \in [1,r] \cap \mathbb{N}}$ it suffices to know how to expand the product $e_i \Delta_\lambda(H_r)$ as a $\mathbb{Q}$-linear combinations of Schur polynomials $\Delta_\lambda(H_r)$. This is prescribed by the following version of Pieri’s formula:

\[
e_i \Delta_\lambda(H_r) = \Delta_{\lambda+i}(H_r)
\] (26)

where for each integer $i \in [0, r] \cap \mathbb{N}$ we set

\[
\Delta_{\lambda+i}(H_r) = \sum \Delta(\lambda_{\pm i_1, \ldots, \lambda_r \pm i_r})(H_r)
\]

the sum being over all $r$-tuples $(i_1, i_2, \ldots, i_r)$ such that $0 \leq i_j \leq 1$, $\sum i_j = i$ and

\[
\lambda_1 \pm i_1 \geq \ldots \geq \lambda_r \pm i_r.
\]

(i.e. $\lambda \pm i$ is a partition). For example:

\[
\Delta_{(32)+2}(H_3) = \Delta_{(43)}(H_3) + \Delta_{(331)}(H_3) + \Delta_{(421)}(H_3)
\]

\[
\Delta_{(32)-1}(H_3) = \Delta_{(22)}(H_3) + \Delta_{(31)}(H_3)
\]

A minute of reflection shows that the action of $e_i$ can be described directly on $F_0^r$. Let

\[
u_{i_0} \wedge u_{i-1} \wedge \ldots \wedge u_{i-r+1} \wedge \Phi_{-r}^r \in F_0^r,
\]

where the indices $i_0, i_1, \ldots, i_{-r}$ are not necessarily in decreasing order. Then:

\[
e_j u_{i_0} \wedge u_{i-1} \wedge \ldots \wedge u_{i-r+1} \wedge \Phi_{-r}^r = \sum_{(j_0, \ldots, j_{-r-1}) \in m(j)} u_{i_0+j_0} \wedge u_{i-1+j_1} \wedge \ldots \wedge u_{i-r+1+j_{r-1}} \wedge \Phi_{-r}^r
\] (27)

where $m(j) := \{j_0, \ldots, j_{r-1}\} \in \mathbb{N}^r \mid 0 \leq j_i \leq 1, \sum j_i = j \}$. Clearly on the right hand side of (27) some summands can vanish, whence the surviving partitions in formula (26).
5.6 Let $V^{r'}_r = \text{Hom}_Q(V_r, Q)$. If $\alpha \in V^{r'}_r$, the contraction map $\alpha \in \text{Hom}_Q(F^r_{+1}, F^r_r)$ is defined by:
\[ \alpha \cdot (u_k \cdot z \cdot u_{k-1} \cdot \ldots) = \alpha(u_k) u_{k-1} \cdot z \cdot u_{k-2} \cdot \ldots - \alpha(u_k) u_{k-1} \cdot z \cdot u_{k-2} \cdot \ldots + \alpha(u_k) u_{k-1} \cdot z \cdot u_{k-2} \cdot \ldots. \]
For each $i \in \mathbb{Z}$, let $u^{r'}_i \in V^{r'}_r$ defined by $u^{r'}_i(u_j) = \delta_{ij}$. Following [10], define the formal Laurent series
\[ X(z) = \sum_{i \in \mathbb{Z}} u^{r'}_i \cdot z^i \in V_r[[z, z^{-1}]] \text{ and } X^{r'}(z) = \sum_{i \in \mathbb{Z}} u^{r'}_i \cdot z^{-i} \in V^{r'}_r[[z, z^{-1}]]. \]

Consider
\[ \begin{cases} \ X(z) \wedge : F^r_{+1} & \longrightarrow F^r_0 \ [z, z^{-1}] \\ \Phi^r_{-1, \lambda} & \longrightarrow X(z) \wedge \Phi^r_{-1, \lambda} := \sum_{i \in \mathbb{Z}} u_i \wedge \Phi^r_{-1, \lambda} \cdot z^i \end{cases} \]
and
\[ \begin{cases} \ X^{r'}(z) \wedge : F^r_{+1} & \longrightarrow F^r_0 \ [z, z^{-1}] \\ \Phi^r_{1, \lambda} & \longrightarrow X^{r'}(z) \wedge \Phi^r_{1, \lambda} := \sum_{i \in \mathbb{Z}} u_i \wedge \Phi^r_{1, \lambda} \cdot z^{-i} \end{cases} \]

5.7 Remark. In the sequel we shall disregard the fermionic Fock spaces $F^r_i$ for $i \neq 0, 1, -1$, meaning irrelevant for the purposes of the present exposition. Most of the formulas deduced in the sequel, obtained for $F^r_1, F^r_0$ and $F^r_{-1}$ only, can be easily generalized for any triple $F^r_{+1}, F^r_r$ and $F^r_{-1}$ with no substantial change. This will be discussed in a forthcoming paper.

5.8 The boson counterparts of the operators $X(z)$ and $X^{r'}(z)$ are the vertex operators $\Gamma_r(z) : B_r \rightarrow B_r[[z, z^{-1}]]$ and $\Gamma^{r'}(z) : B_r \rightarrow B_r[[z, z^{-1}]]$, defined as:
\[ \Gamma_r(z) \Delta_{\lambda}(H_r) = \frac{(X(z) \wedge \Phi^r_{-1, \lambda}) \otimes_Q 1_{B_r}}{\Phi^r_0}, \]
\[ \Gamma^{r'}(z) \Delta_{\lambda}(H_r) = \frac{(X^{r'}(z) \wedge \Phi^r_{1, \lambda}) \otimes_Q 1_{B_r}}{\Phi^r_0}. \]
Notice that if one considers on $F^r_1$ the same $B_r$-module structure of $F^r_0$, it vanishes. This is why in (29) the wedge is considered with respect to the $\mathbb{Q}$-vector space structure. The expression of $\Gamma_r(z)$ and $\Gamma^{r'}(z)$ are very well known in the case when $r = \infty$: see [10] and Section 8 where they are deduced in an alternative way. Next two sections are devoted to determine the shape and the properties of $\Gamma_r(z)$ and $\Gamma^{r'}(z)$.

6 The Vertex Operator $\Gamma_r(z)$

6.1 To describe the vertex operator $\Gamma_r(z) : B_r \rightarrow B_r[[z, z^{-1}]]$ we determine the action on each element of the distinguished basis $(\Delta_{\lambda}(H_r) \mid \lambda \in \mathcal{P}_r)$ of $B_r$.

6.2 Lemma. For each $r \in \mathbb{N}^* \cup \{\infty\}$:
\[ (X_r(z) \wedge \Phi^r_{-1, \lambda}) \otimes_Q 1_{B_r} := \frac{1}{E_r(z)} \sum_{i \in [0, r] \cap \mathbb{N}} E_i(z) z^{-n} u_i \wedge \Phi^r_{-1, \lambda}. \]

Proof. The Lemma will be proven for $r < \infty$ in a way that obviously extends to the case $r = \infty$. Let $u^r_{-1, \lambda} = u_{-1} \wedge \ldots \wedge u_{-r}$. Then
\[ X(z) \wedge \Phi^r_{-1, \lambda} = X(z) \wedge u^r_{-1, \lambda} \wedge \Phi^r_{-r-1} = \sum_{i \geq -r} z^i u_i \wedge u^r_{-1, \lambda} \wedge \Phi^r_{-r-1} \]
If \( r < \infty \), the summation index of (31) runs over all integers \( \geq -r \) because \( u_i \wedge \Phi_{-r-1}' = 0 \) for all \( i \leq -r - 1 \). Now:

\[
\sum_{i \geq -r} u_i z^i = \frac{u_{-r}}{z^r} + \sum_{0 \leq i \leq r-1} \frac{u_{-i}}{z^i} + \sum_{j \geq 1} u_j z^j = \frac{u_{-r}}{z^r} + \sum_{0 \leq i \leq r-1} \frac{u_{-i}}{z^i} + \sum_{j \geq 1} \sum_{0 \leq i \leq r-1} U_i(u_j) u_{-i} z^j,
\]

where we wrote \( u_j = \sum_{0 \leq i \leq r-1} U_i(u_j) u_{-i} \) using Proposition 3.5 and the fact that \( U_i(u_j) = 0 \) if \( i \geq r \), because \( u_i \) is solution to the generic linear ODE of order \( r \). By suitably grouping the summands one obtains:

\[
\sum_{i \geq -r} u_i z^i = \frac{u_{-r}}{z^r} + \sum_{0 \leq i \leq r-1} \left( \frac{1}{z^i} + \sum_{j \geq 1} U_i(u_j) z^j \right) u_{-i}.
\]  

(32)

Observes now that, since \( U_{i+j}(u_0) = 0 \),

\[
U_i(u_j) = h_{i+j} - c_1 h_{i+j+1} + \cdots + c_r h_j = - \sum_{i+1 \leq k \leq r-1} (-1)^k e_k h_{j+i-k}
\]

and then (32) can be written as

\[
\sum_{i \geq -r} u_i z^i = \frac{u_{-r}}{z^r} + \sum_{0 \leq i \leq r-1} \left( \frac{1}{z^i} - \sum_{j \geq 1 i+1 \leq k \leq r-1} (-1)^k e_k h_{j+i-k} z^j \right) u_{-i} =
\]

\[
= \frac{u_{-r}}{z^r} + \sum_{0 \leq i \leq r-1} \left( \frac{1}{z^i} - \sum_{i+1 \leq k \leq r-1} (-1)^k e_k \sum_{j \geq 1} h_{j+i-k} z^j \right) u_{-i} =
\]

\[
= \frac{u_{-r}}{z^r} + \sum_{0 \leq i \leq r-1} \left( \frac{1}{z^i} - \sum_{i+1 \leq k \leq r-1} (-1)^k e_k \sum_{j \geq 0} h_{j+i-k} z^{j+1} \right) u_{-i} =
\]

\[
= \frac{u_{-r}}{z^r} + \sum_{0 \leq i \leq r-1} \left( \frac{1}{z^i} - \sum_{i+1 \leq k \leq r-1} (-1)^k e_k \sum_{j \geq 0} h_{j+i-k} z^{j+1} \right) u_{-i} =
\]

\[
= \frac{u_{-r}}{z^r} + \sum_{0 \leq i \leq r-1} \left( \frac{1}{z^i} - \sum_{i+1 \leq k \leq r-1} (-1)^k e_k \sum_{j \geq 0} h_{j+i-k} z^{j+1} \right) u_{-i} =
\]

\[
= \frac{u_{-r}}{z^r} + \sum_{0 \leq i \leq r-1} \left( \frac{1}{z^i} - \sum_{i+1 \leq k \leq r-1} (-1)^k e_k \sum_{j \geq 0} h_{j+i-k} z^{j+1} \right) u_{-i} =
\]

which proves (30). If \( r = \infty \) one has

\[
\sum_{i \geq -r} u_i z^i = \sum_{i \in Z} u_i z^i = \sum_{i \geq 0} u_{-i} z^{-i} + \sum_{j \geq 0} u_j z^j
\]

and the same proof works in this case as well, up to expressing each \( u_j \) as a (infinite) linear combination of the \( u_{-i} \)'s.

\[\Box\]

6.3 Proposition. Let \( \lambda \in \mathcal{P}_k, k \in [0, r] \cap \mathbb{N}. \) Then

\[
(X(z) \wedge \Phi_{-1, \lambda}') \otimes_0 1_B_r = \frac{1}{E_r(z)} \cdot \exp \left( \frac{t}{z} \right) \wedge \Phi_{-1, \lambda}.
\]

(33)
Proof. By Lemma 6.2:
\[
(X(z) \land \Phi'_{r,1,\lambda}) \otimes_{\mathbb{Q}} 1_{B_r} = \frac{1}{E_r(z)} \left( u_0 + \frac{u_{\lambda-1}}{z} E_1(z) + \ldots + \frac{u_{-k}}{z^k} E_k(z) \right) \land \Phi'_{r,1,k}
\]
because \( u_{-k-j} \land \Phi'_{r,1,\lambda} = 0 \) for all \( j \geq 1 \). But:
\[
u_0 + \frac{u_{\lambda-1}}{z} E_1(z) + \ldots + \frac{u_{-k}}{z^k} E_k(z) = \sum_{j=0}^{k-1} \frac{1}{z^j} \left( u_{-j} + \sum_{i=1}^{k-j} (-1)^i e_i u_{-j-i} \right) = u_0 - e_1 u_{-1} + \ldots + (-1)^{k-1} e_{k-1} u_{-k} + \frac{1}{z} \left( u_{-1} - e_1 u_{-2} + \ldots + (-1)^{k-1} e_{k-1} u_{-k} \right) + \ldots + \frac{1}{z^k} \left( u_{-k} - e_1 u_{-k-1} + \ldots + (-1)^{k-1} e_{k-1} u_{-k} \right).
\]
Now, for each \( 0 \leq j \leq k \), because of (18):
\[
u_j + \sum_{i=1}^{k-j} (-1)^j e_i u_{-j-i} = \frac{t^j}{j!} + \sum_{p \geq 1} (-p)^j e_{k-j+p} u_{-k-p}
\]
and so
\[
u_j + \sum_{i=1}^{k-j} (-1)^j e_i u_{-j-i} \land \Phi'_{r,1,\lambda} = \left( \frac{t^j}{j!} + \sum_{p \geq 1} (-p)^j e_{k-j+p} u_{-k-p} \right) \land \Phi'_{r,1,\lambda} = \frac{t^j}{j!} \land \Phi'_{r,1,\lambda}.
\]
Notice now that \( \frac{t^{k+1}}{(k+1)!} \land \Phi'_{r,1,\lambda} = 0 \) because \( t^{k+1}/(k+1)! \) is a linear combination of \( u_{-k-j} \), \( j \geq 1 \). This prove that formula (30) can be put in the form (33), and the claim is proven. \( \blacksquare \)

6.4 For convenience we define \( G_r(z) : B_r \rightarrow B_r[z^{-1}, z] \) through the equality
\[
G_r(z) \Delta_\lambda(H_r) = E_r(z) (\Gamma_r(z) \Delta_\lambda(H_r)).
\]
Lemma 6.2 says that indeed \( G_r(z) \) takes values in the polynomial ring \( B_r[z^{-1}] \).

6.5 Theorem. Notation as in 5.5. For each partition \( \lambda \) of length \( k \in [0, r] \cap \mathbb{N} \):
\[
G_r(z) \Delta_\lambda(H_r) = \frac{1}{z} \Delta_\lambda_{r-1}(H_r) + \frac{1}{z^2} \Delta_\lambda_{r-2}(H_r) + \ldots + (-1)^{k-1} \frac{1}{z^k} \Delta_\lambda_{r-k}(H_r). \tag{34}
\]

Proof. If \( \lambda = (\lambda_1, \ldots, \lambda_k) \) is a partition of length \( k \leq r \), then:
\[
G_r(z) \Delta_\lambda(H_r) \Phi'_{r,0} = \exp \left( \frac{t}{z} \right) \land \Phi'_{r,1,\lambda} = 1 \land \Phi'_{r,1,\lambda} + \frac{t}{z} \land \Phi'_{r,1,\lambda} + \ldots + \frac{t^k}{z^k} \land \Phi'_{r,1,\lambda} \tag{35}
\]
Now, for each \( 0 \leq j \leq k \):
\[
\frac{t^j}{j!} \land \Phi'_{r,1,\lambda} = \left( u_{-j} - e_1 u_{-j-1} + \ldots + (-1)^{k-j} e_{k-j} u_{-k} \right) \land \Phi'_{r,1,\lambda} = \left( u_{-j} \land \Phi'_{r,1,\lambda} - e_1 u_{-j-1} \land \Phi'_{r,1,\lambda} + \ldots + (-1)^{k-j} e_{k-j} u_{-k} \land \Phi'_{r,1,\lambda} \right) = u_{-j} \land \Phi'_{r,1,\lambda} - (u_{-j} \land \Phi'_{r,1,\lambda} + u_{-j-1} \land \Phi'_{r,1,\lambda+1} + \ldots + (u_{-j-1} \land \Phi'_{r,1,\lambda+1} + u_{-j-2} \land \Phi'_{r,1,\lambda+2} + \ldots + (u_{-j-k+1} \land \Phi'_{r,1,\lambda+k-1} + u_{-k} \land \Phi'_{r,1,\lambda+k-j}) =
\]
so that only the term \((-1)^{k-j} u_{-k} \wedge \Phi_{1}^{c} \lambda_{+x-k-j} \) survives to cancellation. This last term can be written as follows:

\[
(-1)^{k-j} u_{-k} \wedge \Phi_{1}^{c} \lambda_{+x-k-j} = (-1)^{k-j} u_{-k} \wedge \sum_{(j_{1}, \ldots, j_{k}) \in m(j)} u_{\lambda_{1}+j_{1}} \wedge \ldots \wedge u_{-k+\lambda_{k}+j_{k}} \wedge \Phi_{- \lambda_{k}-1}^{c} = \\
= (-1)^{j} \sum_{(j_{1}, \ldots, j_{k}) \in m(j)} u_{\lambda_{1}+j_{1}} \wedge u_{-1+\lambda_{2}+j_{2}} \wedge \ldots \wedge u_{-k+\lambda_{k}+j_{k}} \wedge u_{-k} \wedge \Phi_{- \lambda_{k}-1}^{c} = \\
= (-1)^{j} \sum_{(j_{1}, \ldots, j_{k}) \in m(j)} \Delta_{j_{1}, \ldots, j_{k}}(H_{r}) \Phi_{0, \lambda} = (-1)^{j} \Delta_{\lambda, -r}(H_{r}) \Phi_{0}.
\]

Substitution into (35) gives (34).

Applying Theorem 6.5 to the case \(k = 1\) one obtains:

6.6 Corollary. For each \(r \geq 1\)

\[
\Gamma_{r}(z) h_{n} = \frac{1}{E_{r}(z)} \left( h_{n} - \frac{h_{n-1}}{z} \right). \tag{36}
\]

i.e.

\[
G_{r}(z) h_{n} = h_{n} - \frac{h_{n-1}}{z}.
\]

Let \(G_{r}(z) H_{r}\) be the sequence \((1, G_{r}(z) h_{1}, G_{r}(z) h_{2}, \ldots)\). Using Corollary 6.6 it is easily checked that

\[
\Delta_{\lambda}(G_{r}(z) H_{r}) = \Delta_{\lambda}(H_{r}) - \frac{1}{z} \Delta_{\lambda, -1}(H_{r}) + \ldots + (-1)^{r} \frac{1}{z^{r}} \Delta_{\lambda, -r}(H_{r})
\]

which so proves the first of our main results.

6.7 Theorem. The operator \(G_{r}(z)\) commutes with taking \(\Delta_{\lambda}:\)

\[
G_{r}(z) \Delta_{\lambda}(H_{r}) = \Delta_{\lambda}(G_{r}(z) H_{r}) \tag{37}
\]

and then:

\[
\Gamma_{r}(z) \Delta_{\lambda}(H_{r}) = \frac{1}{E_{r}(z)} \Delta_{\lambda}(G_{r}(z) H_{r}). \tag{38}
\]

6.8 Remark. The vector space \(V_{r} = \text{Span}_{0}(u_{i})_{i \in Z}\) is naturally a \(B_{r}\)-module via the multiplication

\[
P u_{j} := P \sum_{n \geq 0} h_{n+1} \frac{t^{n}}{n!} = \sum_{n \geq 0} P h_{n} \frac{t^{n}}{n!}, \quad (P \in B_{r}).
\]

One may so define \(\tilde{G}_{r}(z) : V_{r} \to V_{r}[z^{-1}]\) as

\[
\tilde{G}_{r}(z) u_{j} = \sum_{n \geq 0} G_{r}(z) h_{n} \frac{t^{n}}{n!} = u_{j} - \frac{1}{z} u_{j-1}
\]

Then (37) says that

\[
(G_{r}(z) \Delta_{\lambda}(H_{r})) \Phi_{0}^{c} = (\det \tilde{G}_{r}(z)) \cdot \Phi_{0, \lambda}^{c} = \\
= \tilde{G}_{r}(z) u_{\lambda_{1}} \wedge \tilde{G}_{r}(z) u_{-1+\lambda_{2}} \wedge \ldots \wedge \tilde{G}_{r}(z) u_{-r+1+\lambda_{r}} \wedge \Phi_{- \lambda_{r}}^{c}. \tag{39}
\]

6.9 Corollary. Let \(h_{i_{1}} \cdot \ldots \cdot h_{i_{s}}\) be an arbitrary product of terms of \(H_{r}\), with \(s \leq r\). Then

\[
G_{r}(z)(h_{i_{1}} \cdot \ldots \cdot h_{i_{s}}) = G_{r}(z) h_{i_{1}} \cdot \ldots \cdot G_{r}(z) h_{i_{s}}
\]

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Proof. Since \( s \leq r \), each monomial in the \( \Gamma \) is a \( \mathbb{Z} \)-linear combination of Schur polynomials associated to partitions of length at most \( r \). Suppose \( h_1 : \ldots : h_r = \sum_{\lambda \leq s} a_{\lambda} \Delta_{\lambda}(H_r) \). Then

\[
G_r(z)(h_1 \cdot \ldots \cdot h_r) = \sum_{|\lambda| \leq s} a_{\lambda} G_r(z) \Delta_{\lambda}(H_r) = \sum_{|\lambda| \leq s} a_{\lambda} \Delta_{\lambda}(G_r(z)H_r) = G_r(z)h_1 \cdots G_r(z)h_r.
\]

6.10 Example. Notice that if \( r < \infty \), \( G_r(z) \) is not a ring homomorphism. Consider e.g. \( \Gamma_1(z) : B_1 \to B_1[[z, z^{-1}]] \). Then

\[
\Gamma_1(z)h_2 = \frac{X_1(z) \land u_1 \land u_{-2} \land \ldots}{\Phi_0} = \frac{-u_1z \land u_{-1} \land u_{-2} \land \ldots}{\Phi_0} = \frac{h_1}{z} = \frac{1}{E_1(z)} \left( h_2 - \frac{h_1}{z} \right)
\]

Similarly

\[
\Gamma_1(z)h_1 = \frac{X_1(z) \land u_0 \land u_{-2} \land \ldots}{\Phi_0} = \frac{-z}{\Phi_0} = -\frac{z}{\Phi_1(z)} = \frac{h_1}{z} = \frac{1}{E_1(z)} \left( h_1 - \frac{1}{z} \right)
\]

Therefore

\[
G_1(z)h_2 = h_2 - \frac{h_1}{z} \quad \text{and} \quad G_1(z)h_1 = h_1 - \frac{1}{z}
\]

However in \( B_1 \) the relation \( h_2 = h_1^2 \) holds. Hence

\[
h_2 - \frac{h_1}{z} = G_1(z)(h_1^2) \neq G_1(z)h_1 \cdot G_1(z)h_1 = \left( h_1 - \frac{1}{z} \right)^2 = h_2 - 2\frac{h_1}{z} + \frac{1}{z^2}.\]

6.11 Corollary. If \( r = \infty \), then \( G_\infty(z) : B_\infty \to B_\infty[z^{-1}] \) is a ring homomorphism.

7 The Vertex Operator \( \Gamma^\vee_r(z) \)

Let \( G^\vee_r(z) : B_r \to B_r[[z^{-1}, z]] \) be the \( \mathbb{Q} \)-homomorphism

\[
\Delta_{\lambda}(H_r) \to G^\vee_r(z) \Delta_{\lambda}(H_r) := z^\left( \frac{\Gamma^\vee_r(z) \Delta_{\lambda}(H_r)}{E_r(z)} \right) = \left( z^\Gamma^\vee_r(z) \Delta_{\lambda}(H_r) \right) \cdot \sum_{n \geq 0} h_n z^n,
\]

where \( \Gamma^\vee_r(z) : B_r \to B_r[[z^{-1}, z]] \) is as in (29).

7.1 Lemma. For each \( n \geq 0 \)

\[
G^\vee_r(z)h_n = \sum_{i \geq 0}^n \frac{h_{n-i}}{z^i} = h_n + \frac{h_{n-1}}{z} + \ldots + \frac{h_1}{z^{n-1}} + \frac{1}{z^n}.
\]

Proof. By definition of \( G^\vee_r(z) \) one has

\[
E_r(z)(G^\vee_r(z)h_n) \Phi_0 = z(\Gamma_r(z)h_n) \Phi_0 = zX_r^\vee(z)h_n \Phi_1.
\]

For \( r < \infty \):

\[
zX_r^\vee(z)_j(h_n u_1 \land u_0 \land \ldots \land u_{-r+1} \land \Phi^f_{-r}) = zX_r^\vee(z)_j(u_{1+n} \land u_0 \land u_{-1} \ldots \land u_{-r+1} \land \Phi^f_{-r}) = z^{-n} \Phi^f_{-r} - u_{1+n} \land u_{-1} \land u_{-2} \land \ldots + z \cdot u_{1+n} \land u_0 \land u_{-2} \land \ldots + \ldots + \ldots + z^r u_{1+n} \land u_0 \land u_{-1} \land \ldots \land u_{-r-2} \land \Phi^f_{-r} = \ldots
\]
There are no further terms, because \( \Delta \chi(H_r) = 0 \) if \( \ell(\chi) > r \). An easy check shows that

\[
(-1)^J \Delta_{(n+1,J)}(H_r) = U_{j}(u_{n+1})
\]

so that

\[
zX^r(z)J_{n+1}\Phi_{\nu}^r = (z^{-n} - U_0(u_{n+1})z - U_1(u_{n+1})z^2 - \ldots - U_{r-1}(u_{n+1})z^{r-1}) \Phi_0^r
\]

(42)

On the other hand

\[
E_r(z) \left( \frac{1}{E_r(z)} - \sum_{p \geq n+1} h_p z^p \right) =
\]

\[
z^{-n} E_r(z) \left( \frac{1}{E_r(z)} - \frac{U_0(u_{n+1}) + U_1(u_{n+1})z + \ldots + U_{r-1}(u_{n+1})z^{r-1}}{E_r(z)} \right) =
\]

\[
z^{-n} - U_0(u_{n+1})z - U_1(u_{n+1})z^2 - \ldots - U_{r-1}(u_{n+1})z^{r-1} = \frac{zX^r(z)J_{n+1}\Phi_{\nu}^r}{\Phi_0^r}
\]

and this proves (41).

More generally:

**7.2 Lemma.** Let \((\lambda_1, \ldots, \lambda_r)\) be any partition of length at most \(r\). Then:

\[
z\Gamma^\nu_r(z) \Delta_{(\lambda_1, \ldots, \lambda_r)}(H_r) = \begin{vmatrix}
z^{-\lambda_1} & z^{-\lambda_2} & \ldots & z^{-\lambda_r} \\
h_{\lambda_1+1} & h_{\lambda_2} & \ldots & h_{\lambda_r+r+1} \\
\vdots & \vdots & \ddots & \vdots \\
h_{\lambda_1+r-1} & h_{\lambda_2+r-2} & \ldots & h_{\lambda_r}
\end{vmatrix} + \Delta_{(\lambda_1+1, \ldots, \lambda_r+1)}(H_r).
\]

(43)

where as usual one sets \(h_j = 0\) if \(j < 0\).

**Proof.** The proof of the equality is straightforward, as it merely consists in expanding the definition (29) of \(\Gamma^\nu_r(z)\).

\[
zX^\nu_r(z)\Delta \chi(H_r)\Phi^r_{1,0} = zX^\nu_r(z)J_{u_{1+1}}(H_r) = \begin{vmatrix}
z^{-\lambda_1} & z^{-\lambda_2} & \ldots & z^{-\lambda_r} \\
h_{\lambda_1+1} & h_{\lambda_2} & \ldots & h_{\lambda_r+r+1} \\
\vdots & \vdots & \ddots & \vdots \\
h_{\lambda_1+r-1} & h_{\lambda_2+r-2} & \ldots & h_{\lambda_r}
\end{vmatrix} + \Delta_{(\lambda_1+1, \ldots, \lambda_r+1)}(H_r)
\]

(44)

and the first summand of (44) is precisely the determinant occurring in (43).

**7.3 Theorem.** The operator \(G^\nu_r(z)\) commutes with taking Schur determinants, i.e.:

\[
G^\nu_r(z) \cdot \Delta \chi(H_r) = \Delta \chi(G^\nu_r(z)H_r).
\]

Therefore

\[
\Gamma^\nu_r(z)\Delta \chi(H_r) = \frac{1}{z}E_r(z) \cdot \Delta \chi(G^\nu_r(z)H_r).
\]
Proof. Again by definition of $G_r^\nu(z)$ one has
\[
G_r^\nu(z) \Delta_\lambda(H_r) = \frac{z \Gamma_r(z) \Delta_\lambda(z)}{E_r(z)} = z \Gamma_r(z) \Delta_\lambda(H_r) \sum_{n \geq 0} h_n z^n.
\]

Using Proposition 7.2:
\[
G_r^\nu(z) \Delta_\lambda(H_r) = z \Gamma_r(z) \Delta_{(\lambda_1, \ldots, \lambda_r)}(H_r) = \left( \begin{array}{ccc}
z^{-\lambda_1} & z^{1-\lambda_2} & z^{r-1-\lambda_r} \\
h_{\lambda_1+1} & h_{\lambda_2} & h_{\lambda_1+r-2} \\
\vdots & \vdots & \ddots \\
h_{\lambda_1+r-i} & h_{\lambda_2+r-i} & h_{\lambda_r}
\end{array} \right) + (-1)^r \Delta_{(\lambda_1+1, \ldots, \lambda_r+1)}(H_r) z^r \sum_{n \geq 0} h_n z^n. \tag{45}
\]

The key computational remark is: the coefficient of $z^n, n \in \mathbb{Z}$ in expression (45) is
\[
H(\lambda + n) + (-1)^r h_{n-r} \Delta_{(\lambda_1+1, \ldots, \lambda_r+1)}(H_r)
\]
where we set, for sake of brevity
\[
H(\lambda + n) := \begin{vmatrix}
h_{\lambda_1+i+n} & h_{\lambda_1+i+1+n} & \cdots & h_{\lambda_1+r+1+n} \\
h_{\lambda_1+1} & h_{\lambda_2} & \cdots & h_{\lambda_1+r-2} \\
\vdots & \vdots & \ddots & \vdots \\
h_{\lambda_1+r-i} & h_{\lambda_2+r-i} & \cdots & h_{\lambda_r}
\end{vmatrix} \tag{46}
\]
and $h_j = 0$ if $j < 0$. We claim that for $n > 0$
\[
H(\lambda + n) + (-1)^r h_{n-r} \Delta_{(\lambda_1+1, \ldots, \lambda_r+1)}(H_r) = 0.
\]

In fact if $1 \leq n \leq r-1$, one has $h_{n-r} \Delta_{(\lambda_1+1, \ldots, \lambda_r+1)} = 0$, since $h_{n-r} = 0$, while the first row of $H(\lambda + n)$ is equal to its $(n+1)$-th row and so vanishes by skew-symmetry. For $n = r$ one has
\[
H(\lambda + r) + (-1)^r \Delta_{(\lambda_1+1, \ldots, \lambda_r+1)}(H_r) = 0
\]
as an immediate check shows (substitute $n = r$ in (46)). For $1 \leq n-r \leq r-1$ one has:
\[
h_{\lambda_1+i+n} = \sum_{j=1}^{n-r} (-1)^{j-1} e_j h_{\lambda_1+i+1-j+n} + (-1)^{n-r+1} e_{n-r+1} h_{\lambda_1-i+r} + \ldots + (-1)^r e_r h_{\lambda_1+1+i+n-r}
\]
i.e.
\[
H(\lambda + n) = \begin{vmatrix}
h_{\lambda_1+1-j+n} & \cdots & h_{\lambda_1+r-i} \\
h_{\lambda_1+2-i} & \cdots & \vdots \\
\vdots & \ddots & \vdots \\
h_{\lambda_1+r-i} & \cdots & \vdots \\
\end{vmatrix} = \begin{vmatrix}
h_{\lambda_1+i+1-j+n} & \cdots & h_{\lambda_1+r-i} \\
h_{\lambda_1+2-i} & \cdots & \vdots \\
\vdots & \ddots & \vdots \\
h_{\lambda_1+r-i} & \cdots & \vdots \\
\end{vmatrix}
\tag{47}
\]
The second summand in (47) vanishes because linearity and skew symmetry of the determinant. Thus:
\[
H(\lambda + n) = e_1 H(\lambda - 1 + n) + \ldots + (-1)^{n-r} e_{n-r} H(\lambda - r + n).
\]
In particular

\[ H(\lambda + r + 1) + (-1)^r h_1 \Delta_{(\lambda_1+1, \ldots, \lambda_r+1)}(H_r) = e_1 H(\lambda + r) + (-1)^r h_1 \Delta_{(\lambda_1+1, \ldots, \lambda_r+1)}(H_r) = 0. \]

By induction, for all \(1 \leq n - r \leq r - 1\):

\[ H(\lambda + n) + (-1)^r h_{n-r} \Delta_{(\lambda_1+1, \ldots, \lambda_r+1)}(H_r) = \]

\[ = \sum_{j=1}^{n-r} (-1)^{j-1} e_j H(\lambda + n - j) - \sum_{j=1}^{n-r} (-1)^{j-1} e_j h_{n-r-j} \Delta_{(\lambda_1+1, \ldots, \lambda_r+1)}(H_r) = \]

\[ = \sum_{j=1}^{n-r} (-1)^j H(\lambda + n - j) - h_{n-r-j} \Delta_{(\lambda_1+1, \ldots, \lambda_r+1)}(H_r) = 0. \]

It follows that \( H(\lambda + n) + (-1)^r h_{n-r} \Delta_{(\lambda_1+1, \ldots, \lambda_r+1)}(H_r) = 0 \) for all \( r + 1 \leq n \leq 2r - 1 \). For \( n \geq 2r \) one uses

\[ H(\lambda + n) = e_1 H(\lambda + n - 1) - \ldots - (-1)^r e_r H(\lambda + n - r) \]

and induction, to prove that \( H(\lambda + n) - (-1)^r h_{n-r} \Delta_{(\lambda_1+1, \ldots, \lambda_r+1)}(H_r) = 0 \). Therefore \( G_r^\nu(z) \Delta_\lambda(H_r) \) involves no positive powers of \( z \). Let us look now for the coefficients of negative powers. For \( 0 \leq n \leq \lambda_1 \), the coefficient of \( z^{-n} \) is the determinant:

\[ H(\lambda - n) := \begin{vmatrix} h_{\lambda_1+1-i-n} & \cdots & \cdots & \cdots & \cdots \\ h_{\lambda_1+2-i} & \cdots & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots & \cdots \\ h_{\lambda_1+r-i} & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_{1 \leq i \leq r} \]

Therefore

\[ G_r^\nu(z) \Delta_\lambda(H_r) = \sum_{n=0}^{\lambda_1} \frac{1}{z^n} H(\lambda - n) = \sum_{n=0}^{\lambda_1} \frac{1}{z^n} \begin{vmatrix} h_{\lambda_1+1-i-n} & \cdots & \cdots & \cdots & \cdots \\ h_{\lambda_1+2-i} & \cdots & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots & \cdots \\ h_{\lambda_1+r-i} & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_{1 \leq i \leq r} = \sum_{n=0}^{\lambda_1} \frac{z^n}{h_{\lambda_1+2-i}} \begin{vmatrix} h_{\lambda_1+1-i-n} & \cdots & \cdots & \cdots & \cdots \\ h_{\lambda_1+2-i} & \cdots & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots & \cdots \\ h_{\lambda_1+r-i} & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_{1 \leq i \leq r} \]

(48)

Let \( R_1, R_2, \ldots, R_r \) be the rows of the matrix (48) and let

\[ G_r^\nu(z) R_j = (G_r^\nu(z) h_{\lambda_1+j-1}, \ldots, G_r^\nu(z) h_{\lambda_j}, \ldots, G_r^\nu(z) h_{\lambda_r+j-1}). \]

Then

\[ G_r^\nu(z) R_j = R_j + \frac{1}{z} G_r^\nu(z) R_{j-1}, \]

for all \( 2 \leq j \leq r \). Therefore, again by the skew-symmetry and multi-linearity of the determinant, one obtains:

\[ G_r^\nu(z) \Delta_\lambda(H_r) = \]

\[
\begin{vmatrix}
G_r^\nu(z) h_{\lambda_1} & G_r^\nu(z) h_{\lambda_2} & \cdots & G_r^\nu(z) h_{\lambda_r+1} \\
\h_{\lambda_1+1} & h_{\lambda_2} & \cdots & h_{\lambda_r+1} \\
\vdots & \vdots & \ddots & \vdots \\
\h_{\lambda_1+r-1} & \h_{\lambda_2+r-2} & \cdots & \h_{\lambda_r} \\
\end{vmatrix}
\]
Using the same argument, mutatis mutandis, as in Corollary 7.6.

Let \( (t) \) homomorphisms. Recall that

\[
\begin{array}{cccc}
G^{\nu}_{r}(z)h_{\lambda_{1}} & G^{\nu}_{r}(z)h_{\lambda_{2}} & \ldots & G^{\nu}_{r}(z)h_{\lambda_{r}+1} \\
G^{\nu}_{r}(z)h_{\lambda_{1}+1} & G^{\nu}_{r}(z)h_{\lambda_{2}} & \ldots & G^{\nu}_{r}(z)h_{\lambda_{r}+2} \\
\vdots & \vdots & \ddots & \vdots \\
G^{\nu}_{r}(z)h_{\lambda_{1}+r-1} & G^{\nu}_{r}(z)h_{\lambda_{2}+r-2} & \ldots & G^{\nu}_{r}(z)h_{r} \\
\end{array}
= \Delta_{\lambda}(G^{\nu}_{r}(z)H_{r})
\]

and the Theorem is proven. \( \blacksquare \)

7.4 Corollary. For each \( r \geq 0 \), by abuse of notation, let \( G_{r}(z), G_{r}^{\nu}(z) : B_{r}[z^{-1}] \rightarrow B_{r}[z^{-1}] \) be the \( \mathbb{Q}[z^{-1}] \) linear extension of the corresponding maps \( B_{r} \rightarrow B_{r}[z^{-1}] \). Then

\[
G_{r}(z) \circ G_{r}^{\nu}(z) = G_{r}^{\nu}(z) \circ G_{r}(z) = 1_{B_{r}[z^{-1}]}
\]
i.e. they are inverse of each other.

Proof. It suffices to evaluate each composition on \( \Delta_{\lambda}(H_{r}) \):

\[
G_{r}(z) \circ G_{r}^{\nu}(z) \Delta_{\lambda}(H_{r}) = G_{r}(z) \Delta_{\lambda}(G_{r}^{\nu}(z)H_{r}) = \Delta_{\lambda}(G_{r}(z)G_{r}^{\nu}(z)H_{r}) = \Delta_{\lambda}(G_{r}(z)).
\]

7.5 Remark. Analogously to Remark 6.8, if one defines

\[
\widetilde{G}^{\nu}_{r}(z)u_{j} = \sum_{n \geq 0} G_{r}(z)h_{n+j} \frac{t^{n}}{n!} = \sum_{i \geq 0} \frac{u_{j-i}}{z^{i}},
\]

then \( (G^{\nu}_{r}(z)\Delta_{\lambda}(H_{r})) \cdot \Phi \) = det(\( \widetilde{G}^{\nu}_{r}(z) \)) \cdot \Phi \)

\[
= \widetilde{G}^{\nu}_{r}(z)u_{\lambda_{1}} \wedge \widetilde{G}^{\nu}_{r}(z)u_{\lambda_{1} \ldots \lambda_{2}} \wedge \ldots \wedge \widetilde{G}^{\nu}_{r}(z)u_{r+1} \wedge \Phi.
\]

7.6 Corollary. Let \( h_{i_{1}}, \ldots, h_{i_{s}} \) be an arbitrary product of terms of \( H_{r} \), with \( s \leq r \). Then

\[
G_{r}^{\nu}(z)(h_{i_{1}} \cdot \ldots \cdot h_{i_{s}}) = G_{r}(z)h_{i_{1}} \cdot \ldots \cdot G_{r}(z)h_{i_{s}}
\]

Proof. Using the same argument, mutatis mutandis, as in Corollary 6.9. \( \blacksquare \)

7.7 Corollary. If \( r = \infty \), then \( G_{\infty}^{\nu}(z) : B_{\infty}[z^{-1}] \rightarrow B_{\infty}[z^{-1}] \) is a ring homomorphism. \( \blacksquare \)

8 The case \( r = \infty \)

8.1 We propose now an alternative proof for the expressions of \( \Gamma_{\infty}(z) \) and \( \Gamma^{\nu}_{\infty}(z) \), with respect to that shown e.g. in [10], using the fact \( G_{\infty}(z), G^{\nu}_{\infty}(z) : B_{\infty} \rightarrow B_{\infty}[z^{-1}] \) are ring homomorphisms. Recall that \( B_{\infty} = \mathbb{Q}[H_{\infty}] = \mathbb{Q}[X_{\infty}] \) and that the terms of the sequences \( H_{\infty} = (h_{1}, h_{2}, \ldots) \) and \( X_{\infty} = (x_{1}, x_{2}, \ldots) \) are algebraically independent in this case. Then relations (21) hold. Recall also that if \( D(z) \) is a formal power series whose coefficients are first order differential operators on a \( \mathbb{Q} \)-algebra \( A \), then

\[
\exp(D(z)) : A \rightarrow A[[z]]
\]
is a homomorphism of \( \mathbb{Q} \)-algebras, in the sense that \( \exp(D(z))(ab) = \exp(D(z))(a) \exp(D(z))(b) \).
8.2 By Corollary 6.6:
\[ G_\infty(z)h_n = h_n - \frac{h_{n-1}}{z} = \left(1 - \frac{1}{z} \frac{\partial}{\partial x_1}\right) h_n \]
where the last equality is because of (21). Using the well known identity:
\[ 1 - a = \exp \left( - \sum_{n \geq 0} \frac{a^n}{n} \right) \]
for \( a = \frac{1}{z} \frac{\partial}{\partial x_1} \), one obtains:
\[ G_\infty(z)h_n = \exp \left( - \sum_{i \geq 1} \frac{1}{iz^i} \frac{\partial}{\partial x_i} \right) h_n = \exp \left( - \sum_{n \geq 1} \frac{1}{iz^n} \frac{\partial}{\partial x_i} \right) h_n. \]
Since
\[ D \left( \frac{1}{z} \right) = \sum_{i \geq 1} \frac{1}{iz^i} \frac{\partial}{\partial x_i} \]
is a formal power series in the indeterminate \( 1/z \) and the coefficients are the first order differential operators \( \partial/\partial x_i \), it follows that \( \exp(-D(1/z)) : B_\infty \to B_\infty[z^{-1}] \) is a ring homomorphism such that \( G_\infty(z)h_n = \exp(-D(1/z))h_n \). Then \( G_\infty(z) = \exp(-D(1/z)) \), because \( (h_i)_{i \geq 1} \) generates \( B_\infty \) as a \( \mathbb{Q} \)-algebra. In conclusion:
\[ \Gamma(z) = \frac{1}{E_\infty(z)} G_\infty(z) = \exp \left( \sum_{i \geq 1} x_i t^i \right) \cdot \exp \left( - \sum_{n \geq 1} \frac{1}{nz^n} \frac{\partial}{\partial x_n} \right) \]
which is precisely expression [10, formula 5.25a] for \( m = -1 \), up to a factor used to keep track that \( X(z) \wedge \), which is defined on \( F_1^\infty \) for all \( i \in \mathbb{Z} \), is currently operating on \( F_1^\infty \).

The same kind of argument works for \( G'_\infty(z) \). One uses now the identity
\[ \frac{1}{1 - \frac{t}{z}} = 1 + \sum_{n \geq 1} \frac{t^n}{z^n} = \exp \left( \sum_{n \geq 1} \frac{t^n}{nz^n} \right) \]
Thus
\[ G'_\infty(z)h_n = \sum_{i \geq 0} h_{n-i} z^i = \left(1 + \sum_{i \geq 1} \frac{1}{iz^i} \frac{\partial}{\partial x_i} \right) h_n \]
Using (50):
\[ G'_\infty(z)h_n = \exp \left( \sum_{i \geq 1} \frac{1}{iz^i} \frac{\partial}{\partial x_i} \right) h_n = \exp \left( \sum_{i \geq 1} \frac{1}{iz^i} \frac{\partial}{\partial x_i} \right) h_n. \]
again by virtue of (21). Hence, for each \( n \geq 0 \), \( G'_\infty(z)h_n = \exp(D(1/z))h_n \), where \( \exp(D(1/z)) \) is as in (49). Then \( G'_\infty(z) = \exp(D(1/z)) \), because they are both algebra homomorphisms. In conclusion:
\[ \Gamma'_\infty(z) = \frac{E_\infty(z)}{z} \exp \left( \sum_{n \geq 1} \frac{1}{nz^n} \frac{\partial}{\partial x_n} \right) = \frac{1}{z} \exp\left( - \sum_{i \geq 1} x_i z^i \right) \exp \left( \sum_{n \geq 1} \frac{1}{nz^n} \frac{\partial}{\partial x_n} \right), \]
which is precisely expression [10, formula 5.25b] for \( m = 1 \), up to a factor used to keep track that \( X'(z) \wedge \), which is defined on \( F_1^\infty \), for all \( i \in \mathbb{Z} \), is currently operating on \( F_1^\infty \).
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