TOWARDS THE EQUIVALENCE OF THE ACC FOR $a$-LOG CANONICAL THRESHOLDS AND THE ACC FOR MINIMAL LOG DISCREPANCIES

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Abstract. In this paper we show that Shokurov’s conjectures on the ACC for $a$-lc thresholds and the ACC for minimal log discrepancies are equivalent for interval $[0, 1-t]$ and for every $t > 0$.

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1. Introduction

In this paper we work over the field of complex numbers $\mathbb{C}$.

Minimal log discrepancies are important invariants of singularities that play a fundamental role in higher dimensional birational geometry. They are not only invariants that characterize the singularities of varieties, but also behave nicely when running the minimal model program.

In [Sho04] Shokurov proved that for pairs of fixed dimension, the conjecture on the termination of flips follows from two conjectures on minimal log discrepancies (mlds for short): one is the ascending chain condition (ACC for short) conjecture for mlds (see Definition 2.9(1) below), the other is the lower-semicontinuity (LSC for short) conjecture for mlds (see [Amb99 Conjecture 2.4]).

However, it turns out that both the ACC conjecture and the LSC conjecture for mlds are very subtle problems. For instance, although the termination of flips in dimension 3 was established about 30 years ago (cf. [Mor88, Sho92]), the ACC conjecture for mlds in dimension 3 is still open for klt singularities. Indeed, it is only known for canonical pairs with coefficients contained in a finite set by using classification of terminal singularities.

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Moreover, the LSC conjecture for mlds is also only known up to dimension 3 (cf. [Mor85], [Nak16]).

In 2014, Hacon-McKernan-Xu has proved the conjecture on the ACC for lc thresholds (cf. [HMX14, Theorem 1.1]), an important conjecture that describes the structure of lc thresholds, which are very important invariants in birational geometry. On the other hand, since the lc threshold is a special case of the $a$-lc threshold, a more general birational invariant, it is natural to ask whether the ACC conjecture for $a$-lc thresholds holds for any $a$ (see Definition 2.9(2) below). This also turns out to be a very difficult conjecture, which remains open in dimension 3.

Birkar and Shokurov prove that the ACC conjecture for mlds implies the ACC conjecture for $a$-lc thresholds for any $a \geq 0$ (cf. [BS10]). It is then natural to ask the following:

**Question 1.1.** Is the conjecture on ACC for mlds equivalent to the ACC conjecture for $a$-lc thresholds?

In particular, as the ACC for lc thresholds is already proved (cf. [HMX14, Theorem 1.1]), it is our hope that it may be easier to prove results about the ACC for $a$-lc thresholds than prove results about the ACC for mlds. In a very recent paper, Kawakita shows that the ideal version of these two conjectures are equivalent for any fixed ambient variety with klt singularities (cf. [Kaw18, Theorem 4.6]) by using the method of generic limit. For arbitrary ambient varieties, however, we tend to use a different approach. Recently, Birkar (cf. [Bir16A]) proves the boundedness of complements for pairs of (relative) Fano type. Notice that for an isolated singularity, $(P \in X)$, $X$ is of Fano type over itself, hence Birkar’s result gives us a lot of information on the structure of $|\pi m K_X|$ for integers $m \geq 1$. In particular, we may construct monotonic local complements to analyze the behavior of mlds. Although we are unable to deduce the equivalence of these two conjectures in general, in this paper we give a positive answer to Question 1.1 for non-canonical singularities. More generally, we have the following:

**Theorem 1.2.** Let $d > 0$ be an integer and $t > 0$ a real number. Assume the ACC conjecture for $a$-lc thresholds holds for triples with coefficients in any finite set $\mathcal{I}_0 \subset [0, 1]$ and any $a \in [0, 1 - t]$. Then the ACC conjecture for mlds holds for the interval $[0, 1 - t]$. In other words, the accumulation points of $\text{MLD}(d, \mathcal{I})$ from below are not contained in $[0, 1 - t]$ for any DCC set $\mathcal{I} \subset [0, 1]$.

By similar arguments as in [BS10], we deduce the equivalence of the conjecture on ACC for mlds and the conjecture on ACC for $a$-lc thresholds for the interval $[0, 1 - t]$ for any $t > 0$.

**Theorem 1.3.** Let $d > 0$ be an integer and $t > 0$ a real number. The ACC conjecture for $a$-lc thresholds holds for every finite set $\mathcal{I} \subset [0, 1]$ and any $a \in [0, 1 - t]$ if and only if the ACC conjecture for mlds holds for the interval $[0, 1 - t]$. 
As an immediate corollary, we deduce the equivalence of these two conjectures for non-canonical singularities:

**Corollary 1.4.** Let \( d > 0 \) be an integer. The ACC conjecture for \( a \)-lct thresholds holds for every finite set \( I \subset [0, 1] \) and any \( a \in [0, 1) \) if and only if the ACC conjecture for mlds holds for the interval \([0, 1)\).

Since the total log discrepancy for any pair is \( \leq 1 \), we also deduce the following corollary:

**Corollary 1.5.** Let \( d > 0 \) be an integer and \( I \) a DCC set. Assume the ACC conjecture for \( a \)-lc thresholds for any \( a \in [0, 1) \). Then 1 is the only possible accumulation point of \( \{ \text{tmld}(X, B) \mid (X, B) \text{ is a pair, } B \in I \} \) where \( \text{tmld}(X, B) \) is the total minimal log discrepancy of \((X, B)\).

We give the readers some ideas of the proof of Theorem 1.2.

**Ideas of the proof of Theorem 1.2**

We start from the most simple case. First we start with pairs with \( \mathbb{Q} \)-coefficients. Assume that \((X, B)\) is a klt pair and \( x \in X \) is a closed point, such that \( \dim X \) is fixed and the coefficients of \( B \) belong to a finite set \( I \subset \mathbb{Q} \cap [0, 1] \). Let \( E \) be an exceptional divisor such that the \( \text{mld}(x, X, B) \) is attained at \( E \).

Suppose that there exists a \( \mathbb{Q} \)-divisor \( G \geq 0 \) on \( X \), such that

1. \((X, B + G)\) is lc near \( x \),
2. the coefficients of \( G \) belong to a finite set of rational numbers,
3. \( a(E, X, B + G) \) belongs to a finite set, and
4. \( \text{mld}(x, X, B + G) \) is also attained at \( E \).

Under the assumptions above, we may show that either \( a(E, X, B) = a(E, X, B + G) \), or \( a(E, X, B) \) accumulates to \( a(E, X, B + G) \), or there exists a real number \( a > 0 \) such that the \( a \)-lct \( (X, B; G) \) is attained at \( E \). In the first two cases, \( a(E, X, B) \) already belongs to an ACC set, and in the last case, according to the assumption that \( a \)-lct \( (X, B; G) \) satisfies the ACC, we deduce that \( a(E, X, B) \) satisfies the DCC after an elementary computation.

Thus, we may reduce the problem to “find \( G \) satisfying (1)-(4) as above”. It is almost immediate from the boundedness of complements (cf. [Bir16A] and [Bir16B]) to find \( G \) which satisfies (1), (2) and (3).

However, the key problem is to show that \( \text{mld}(x, X, B + G) \) is attained at \( E \). This is true in some cases, and we give an example for a baby case here:

**Example 1.6.** Assumptions as above. Suppose that there exists an extraction \( f : Y \to X \) of \( E \) such that \( K_Y + B_Y + E \) is lc, where \( B_Y \) is the strict transform of \( B \) on \( Y \) (this holds when \( E \) is a Kollár component). We show that the ACC for mlds holds in this case.

Since we want to prove the that \( \text{mld}(x, X, B) \) satisfies the ACC, we may suppose that \( \text{mld}(x, X, B) \geq \epsilon_0 \) for some \( \epsilon_0 > 0 \). According to [Bir16A],
Theorem 1.8], there exists $G_Y$ on $Y$ and an integer $n > 0$ such that $(Y/X \ni x, B_Y + E + G_Y)$ is a monotonic $n$-complement of $(Y/X \ni x, B_Y + E)$.

Let $G := f_*G_Y$, then $0 = a(E, X, B + G) = \text{mld}(x, X, B + G)$ and $\text{mld}(x, X, B)$ is also attained at $E$. Hence for any $a \in [0, \epsilon_0]$, $a\text{-lct}_x(X; B; G)$ is attained at $E$. By linearity of log discrepancies, we have

$$a = a(E, X, B + a\text{-lct}_x(X; B; G)G)$$

$$= a\text{-lct}_x(X; B; G)a(E, X, B + G) + (1 - a\text{-lct}_x(X; B; G))a(E, X, B)$$

$$= 0 + (1 - a\text{-lct}_x(X; B; G))a(E, X, B)$$

$$= (1 - a\text{-lct}_x(X; B; G))\text{mld}(x, X, B).$$

Since $a\text{-lct}_x(X; B; G)$ satisfies the ACC, we deduce that $\text{mld}(x, X, B)$ satisfies the ACC.

However, under the assumptions as in the example above, suppose that $K_Y + E$ is not lc, then for any $\mathbb{R}$-complement $(X \ni x, B + G)$ of $(X \ni x, B)$ such that $(X, B + G)$ is not klt at $x$, $a(E, X, B + G) > 0$. However, since $\text{mld}(x, X, B + G) = 0$, we deduce that $\text{mld}(x, X, B + G)$ is not attained at $E$. Therefore, we must modify (4) to some weaker assumptions.

Indeed, we only need to find a fixed number $0 < a < 1$, such that $a\text{-lct}_x(X; B; G)$ is attained at $E$. However, it is not even easy to show the existence of such $a$. The key problem is, when considering a sequence of pairs, the asymptotic structure of divisors with log discrepancies sufficiently close to the mlds may behave in a very subtle way. A detailed analysis of the structure of these divisors forms the main part of our proof (see Section 5).

Finally, we intend to improve our result, so that the statements also hold for irrational coefficients. We need to “approximate” pairs with irrational coefficients with pairs with rational coefficients. Fortunately for us, in a paper of Nakamura (cf. [Nak16]), an approximation theorem was proved (see Theorem 4.2 below). This theorem plays an important role in our proof.

Structure of the paper. In Section 3, we introduce some notation and tools which will be used in this paper. In Section 4, we state several approximation theorems and introduce “irrational monotonic $n$-complements”. This “monotonic complement” behaves very similar to the usual monotonic $n$-complement for pairs with rational coefficients in the rest of our proof. In Section 5, we give the proof of the main theorem.

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The author would like to dedicate this paper to his beloved grandpa who is fighting with cancer at his hometown.

**Major changes since Version 1.**

- The sketch of proof of Theorem 5.2 is moved from the introduction to Section 5.
- We add Corollary 1.5
- Example 1.6 is moved from Section 5 to the introduction.
- We adopt the notation of $b$-divisors, so that the notation in the proof of the main theorem is less confusing.
- Most of the results in this paper have been improved to not necessarily closed points.
- We clarify the notation of “near a point” and “at a point”.
- We adopt the notation of $(n, I_1, I_2)$-complements.
- Proof of Theorem 3.3 and Lemma 3.6 are given.
- Proof of Lemma 3.7 is re-written and simplified, and is moved from Section 5 to Section 3.
- We add Step 2 in the proof of Theorem 5.2 in order to deal with non-klt germs.
- We simplified the proof of Theorem 5.2 after Construction 5.4
- We add a proof of Claim 5.8.

## 2. Notation and conventions

We adopt the standard notation and definitions in [Sho92] and [KM98], and will freely use them.

**Definition 2.1 (Positivity definitions).** Let $X$ be a normal variety. $X$ is called $\mathbb{Q}$-factorial, if every $\mathbb{Q}$-divisor on $X$ is $\mathbb{Q}$-Cartier.

For any prime divisor $E$ and $\mathbb{R}$-divisor $D$, we define $\mu_E D$ to be the multiplicity of $E$ along $D$. For any irreducible $\mathbb{R}$-divisor $E \neq 0$ and $\mathbb{R}$-divisor $D$, suppose that $E = \alpha F$ for some prime divisor $F$ and real number $\alpha \neq 0$. We define $\mu_E D := \frac{1}{\alpha} \mu_F D$.

**Definition 2.2 ($b$-divisors).** Let $X$ be a normal variety. A $b$-$\mathbb{R}$ Cartier $b$-divisor ($b$-divisor for simplicity) over $X$ is the choice of a projective birational morphism $Y \to X$ from a normal variety and an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $M$ on $Y$ up to the following equivalence: another projective birational morphism $Y' \to X$ from a normal variety and an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $M'$ defines the same $b$-divisor if there is a common resolution $W \to Y$ and $W \to Y'$ on which the pullback of $M$ and $M'$ coincide. For and birational contraction $f : Y \to Z$, the center of $M$ on $Z$ is $f(M)$, and is denoted by $\text{center}_Z M$. If $\text{center}_Z M$ is an $\mathbb{R}$-divisor, we also use the notation $M_Z := \text{center}_Z M$. 
A $b$-divisor over $X$ is called prime if there is a choice of projective birational morphism $Y \to X$ and an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $M$ on $Y$ such that $M$ is a prime divisor.

**Definition 2.3** (Pairs). A pair $(X, B)$ is a normal variety $X$ and an effective $\mathbb{R}$-divisor $B$ on $X$ such that $K_X + B$ is $\mathbb{R}$-Cartier. A triple $(X, B; G)$ consists of a pair $(X, B)$ and an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $G$ on $X$.

Let $E$ be a prime $b$-divisor over $X$. $E$ is called exceptional over $X$ if the center $X_E$ is not a divisor.

Let $(X, B)$ be a pair and $D$ an $\mathbb{R}$-divisor on $X$. For any prime $b$-divisor $E$ over $X$, let $g : Y \to X$ be a log resolution of $(X, B)$ such that $E_Y := \text{center}_Y E$ is a divisor. Then there exist two uniquely determined $\mathbb{R}$-divisors $B_Y$ and $D_Y$, such that

$$K_Y + B_Y = g^*(K_X + B)$$

and

$$D_Y = g^*D.$$  

We define the log discrepancy of $E$ respect to $(X, B)$ to be $1 - \mu_{E, Y} B_Y$, and is denoted by $a(E, X, B)$. We define the multiplicity of $E$ along $D$ to be $\mu_{E, D_Y}$, and is denoted by $\text{mult}_E D$. For any real number $\alpha \neq 0$ and $b$-divisor $F = \alpha E$ over $X$, we define the multiplicity of $F$ along $D$ to be $\mu_{F, D_Y}$. Clearly, if $Y = X$, then $\mu_{F, D} = \text{mult}_F D$.

**Definition 2.4** (Singularities). Let $a \geq 0$ be a real number, $(X, B)$ a pair and $x \in X$ a (not necessarily closed) point.

$(X, B)$ is called a-log canonical (resp. a-kawamata log terminal) if for any prime $b$-divisor $E$ over $X$, $a(E, X, B) \geq a$ (resp. $a(E, X, B) > a$). For simplicity, a-log canonical and a-kawamata log terminal are usually called a-lc and a-klt respectively.

For any (not necessarily closed) point $x \in X$, $(X, B)$ is called a-lc (resp. a-klt) near $x$ if there exists an open subset $x \in U \subset X$ such that $(X|_U, B|_U)$ is a-lc (resp. a-klt). $(X, B)$ is called a-lc (resp. a-klt, lc, klt) at $x$ if for any prime $b$-divisor $E$ over $X$ such that $\text{center}_X E = \{\bar{x}\}$, $a(E, X, B) \geq a$ (resp. $a(E, X, B) > a$).

0-kawamata log terminal is usually called kawamata log terminal, or klt. 0-log canonical is usually called log canonical, or lc.

**Definition 2.5** (Minimal log discrepancies and a-lc thresholds). Let $(X, B)$ be a pair and $x \in X$ a (not necessarily closed) point. The minimal log discrepancy (mld for short) of $(X, B)$ is defined as

$$\text{mld}(X, B) := \inf \{0, a(E, X, B) | E \text{ is an exceptional prime } b\text{-divisor over } X\}.$$  

The minimal log discrepancy of $(X, B)$ at $x$ is defined as

$$\text{mld}(x, X, B) := \inf \{0, a(E, X, B) | E \text{ is a prime } b\text{-divisor over } X, \center_X E = \{\bar{x}\}\}.$$
The total minimal log discrepancy of $(X,B)$ is defined as
\[
\text{tmld}(X,B) := \inf\{a(E,X,B) | E \text{ is a prime } b\text{-divisor over } X\}.
\]

Notice that if $(X,B)$ is lc, these infimums are always minimums.

For any prime $b$-divisor $E$ that is exceptional over $X$, if $\text{mld}(X,B) = a(E,X,B)$, we say that $\text{mld}(X,B)$ is attained at $E$. For any prime $b$-divisor $E$ over $X$ such that $\text{center}_X E = \{x\}$, if $\text{mld}(x,X,B) = a(E,X,B)$, we say that $\text{mld}(x,X,B)$ is attained at $E$.

Let $a \geq 0$ be a real number, $(X,B;G)$ a triple and $x \in X$ be a (not necessarily closed) point such that $(X,B)$ is $a$-lc (resp. $a$-lc at $x$). The $a$-lc threshold (resp. the $a$-lc threshold at $x$) of the triple $(X,B;G)$, or the $a$-lc threshold (resp. the $a$-lc threshold at $x$) of the pair $(X,B)$ with respect to $G$, is defined as
\[
\sup\{c \geq 0 | (X,B + cG) \text{ is } a\text{-lc (resp. } a\text{-lc at } x\})\},
\]
and is denoted by $a\text{-lct}(X,B;G)$ (resp. $a\text{-lct}_x(X,B;G)$). For any prime $b$-divisor $E$ that is exceptional over $X$, we say that $a\text{-lct}(X,B;G)$ is attained at $E$ if
\[
a(E,X,B + a\text{-lct}(X,B;G)G) = a.
\]

For any prime $b$-divisor $E$ such that $\text{center}_X E = \{x\}$, we say that $a\text{-lct}_x(X,B;G)$ is attained at $E$ if
\[
a(E,X,B + a\text{-lct}_x(X,B;G)G) = a.
\]

0-lc threshold is usually called lc threshold.

The next example shows that it is possible that for any $0 < a < 1$, there does not exist any divisor that attains the $a$-lc threshold:

**Example 2.6** (Triple with no $b$-divisor attaining $a$-lc threshold). Let $X$ be a smooth surface and $H$ a general smooth curve on $X$. Then $(X,H)$ is plt, and for any closed point $x \in H \subset X$, $(X,H)$ is 1-lc at $x$. However, $(X,H)$ is not $a$-lc near $x$ for any $x \in H$. For any real number $0 < a < 1$, it is clear that
\[
a\text{-lct}_x(X,H;H) = a\text{-lct}(X,H;H) = 0.
\]
Thus for any $0 < a < 1$ and any closed point $x \in X$, neither is there a $b$-divisor that attains $a\text{-lct}_x(X,H;H)$, nor is there a $b$-divisor that attains $a\text{-lct}(X,H;H)$.

**Definition 2.7** (DCC and ACC sets). Let $\mathcal{I}$ be a set of real numbers. We say that $\mathcal{I}$ satisfies the descending chain condition (DCC) if any decreasing sequence $a_1 \geq a_2 \geq \cdots \geq a_k \geq \cdots$ in $\mathcal{I}$ stabilizes. We say that $\mathcal{I}$ satisfies the ascending chain condition (ACC) if any increasing sequence in $\mathcal{I}$ stabilizes. An accumulation point of $\mathcal{I}$ (from below (resp. above)) is a real number $s$ such that there exists a (strict increasing (resp. strict decreasing)) sequence $\{a_i\}_{i=1}^\infty \subset \mathcal{I}$ such that $s = \lim_{i \to \infty} a_i$.

Let $X$ be a normal variety and $B$ and $G$ two $\mathbb{R}$-divisors on $X$. The notion $B \in \mathcal{I}$ means that all the coefficients of $B$ belong to $\mathcal{I}$. We say “all
the coefficients of the pair \((X, B)\) (resp. the triple \((X, B; G)\)) belong to \(\mathcal{I}\)” if \(B \in \mathcal{I}\) (resp. \(B, G \in \mathcal{I}\)).

Let \(d > 0\) be an integer, \(a \geq 0\) a real number and \(\mathcal{I}, \mathcal{I}' \subset [0, \infty)\) two sets. We define

\[
\text{MLD}(d, \mathcal{I}) := \{\text{mld}(x, X, B)|(X, B)\text{ is a pair, } \dim X = d, B \in \mathcal{I}, x \in X \text{ is a (not necessarily closed) point}\}.
\]

and

\[
\text{a-LCT}(d, \mathcal{I}, \mathcal{I}') := \{\text{a-lct}_x(X, B; G)|(X, B; G)\text{ is a triple, } x \in X \text{ is a (not necessarily closed) point, } (X, B) \text{ is a-lc at } x, \dim X = d, B \in \mathcal{I}, G \in \mathcal{I}'\}.
\]

0-LCT\((d, \mathcal{I}, \mathcal{I}')\) is usually called LCT\((d, \mathcal{I}, \mathcal{I}')\).

**Definition 2.8 (Complements).** Let \(n > 0\) be an integer, \(X \to Z\) a contraction, \(B\) an effective \(\mathbb{R}\)-divisor on \(X\), and \(z \in Z\) a (not necessarily closed) point. An \(n\)-complement of \((X/Z \ni z, B)\) is a pair \((X/Z \ni z, B^+)\), such that over some neighborhood of \(z\), we have

1. \((X, B^+)\) is lc,
2. \(n(K_X + B^+) \sim 0\), and
3. \(B^+ \geq \lfloor B \rfloor + \frac{1}{n}((n + 1)\{B\})\).

We say that \((X/Z \ni z, B^+)\) is a monotonic \(n\)-complement of \((X/Z \ni z, B)\) if we additionally have \(B^+ \geq B\).

If \(Z = X\) and \((X/Z \ni z, B^+)\) is an \(n\)-complement (resp. monotonic \(n\)-complement) of \((X/Z \ni z, B)\), we may omit \(Z\) and say \((X \ni z, B^+)\) is an \(n\)-complement (resp. monotonic \(n\)-complement) of \((X \ni z, B)\), and in this case, we also say \((X, B^+)\) is a local \(n\)-complement (resp. monotonic local \(n\)-complement) of \((X, B)\) over \(z\).

**Definition 2.9 (Conjectures).** Our paper involves two conjectures and their special cases. These two conjectures are:

1. Let \(d > 0\) be an integer and \(\mathcal{I} \subset [0, 1]\) a DCC set. Then MLD\((d, \mathcal{I})\) satisfies the ACC.
2. Let \(d > 0\) be an integer, \(a \geq 0\) a real number and \(\mathcal{I} \subset [0, 1]\) and \(\mathcal{I}' \subset [0, \infty)\) two DCC sets. Then a-LCT\((d, \mathcal{I}, \mathcal{I}')\) satisfies the ACC.

Conjecture (1) is usually called ACC for minimal log discrepancies, or ACC for mlds for short. Conjecture (2) is usually called ACC for a-lc thresholds, or ACC for a-lcts for short.

**Definition 2.10 (Consensus).** For any integer \(d > 0\) and interval \(\mathcal{I}'' \subset \mathbb{R}\), and either for the conjecture on ACC for mlds or the conjecture on ACC for a-lc thresholds, we make the following consensus:

1. when we say “the conjecture in dimension \(d\)”, it means the dimension of the varieties we deal with is \(d\),
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(2) when we say “the conjecture for coefficient set $I$” it means that we only consider the ACC property of sets of the form $\text{MLD}(d, I)$ (or $a$-LCT$(d, I)$),

(3) when we say “the conjecture for finite coefficients”, it means the conjecture for any finite coefficient set,

(4) when we say “the conjecture for $\mathbb{Q}$-coefficients”, it means the conjecture for any coefficient set which belongs to $\mathbb{Q}$,

(5) when we say “the conjecture holds for the interval $I''$”, if we are dealing with the conjecture on ACC for mlcs, it means that for any proper $d$ and $I$, all the accumulation points of $\text{MLD}(d, I)$ from below do not belong to $I''$, and if we are dealing with the conjecture on ACC for $a$-lc thresholds, it means that the conjecture holds for every $a$ such that $a \in I''$.

3. Preliminaries

**Theorem 3.1** (Boundedness on number of components, [Kol92] Theorem 18.22). Let $(X, \sum b_i B_i)$ be an lc pair near a (not necessarily closed) point $x \in \cap B_i$. Assume that $K_X$ and $B_i$ are $\mathbb{Q}$-Cartier near $x$. Then $\sum b_i \leq \dim X$.

**Theorem 3.2** (ACC for lc thresholds, [HMX14 Theorem 1.1]). Let $d > 0$ be an integer and $I \subset [0, 1]$ and $I' \subset [0, \infty)$ two DCC sets. Then $\text{LCT}(d, I, I')$ satisfies the ACC.

**Theorem 3.3.** Let $d > 0$ be an integer, $a > 0$ be a real number. If for any DCC set $I \subset [0, 1]$, $a$ is not an accumulation point of $\text{MLD}(d, I)$ from below, then the conjecture on ACC for $a$-lc thresholds holds in dimension $d$.

**Proof.** It follows from the same lines as in [BS10 Proposition 2.1] and [BS10 Proposition 2.5]. For readers’ convenience, we give a full proof here. Suppose that there exist a sequence of pairs $(X_i, B_i)$ of dimension $d$, (not necessarily closed) points $x_i \in X_i$, and a strictly increasing sequence $t_i > 0$ of real numbers, such that for any $i$,

1. $B_i \in I$,
2. there exists an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $D_i$ on $X_i$, such that $D_i \in I$ and $t_i = a\text{-lct}_{x_i}(X_i, B_i; D_i)$.

Let $t_0 := \lim_{i \to \infty} t_i$. By Theorem 3.2 possibly passing to a subsequence, we may assume that $a > 0$ and $(X_i, B_i + t_0 D_i)$ is lc near $x_i$ for every $i$.

Let $\{\epsilon_i\}_{i=1}^\infty$ be a strictly decreasing sequence that converges to 0, such that $0 < \epsilon_i < 1$ for any $i$. Let $a_i := \text{mld}(x_i, X_i, B_i + t_0 D_i)$, $t_i' := t_i + \epsilon_i(t_0 - t_i)$ for any $i$. Then $(X_i, B_i + t_i' D_i)$ is klt near $x_i$. Since $t_i < t_i' < t_0$, possibly passing to a subsequence, we may assume that $t_i'$ is strictly increasing and hence all the coefficients of $B_i + t_i' D_i$ belong to a DCC set. By convexity of
mls, we have
\[ a > \text{mld}(x_i, X_i, B_i + t'_iD_i) \]
\[ = \text{mld}(x_i, X_i, \frac{t'_i - t_i}{t_0 - t_i}(B_i + t_0D_i) + \frac{t_0 - t'_i}{t_0 - t_i}(B_i + t_iD_i)) \]
\[ \geq \frac{t'_i - t_i}{t_0 - t_i} \text{mld}(x_i, X_i, B_i + t_0D_i) + \frac{t_0 - t'_i}{t_0 - t_i} \text{mld}(x_i, X_i, B_i + t_iD_i) \]
\[ = \frac{t'_i - t_i}{t_0 - t_i} t_0 - t'_i a = a - \frac{(t'_i - t_i)(a - a_i)}{t_0 - t_i} \]
\[ = a - \epsilon_i(a - a_i) \geq (1 - \epsilon_i)a. \]

Hence possibly passing to a subsequence, we may assume that \( \text{mld}(x_i, X_i, B_i + t'_iD_i) \) is strictly increasing and converges to \( a \), which contradicts to our assumptions. \( \square \)

**Theorem 3.4** (Precise inversion of adjunction). Let \((X, S + B)\) be a dlt pair where \( S \) is a prime divisor. Let \( x \in S \subset X \) be a (not necessarily closed) point, such that \( \text{mld}(x, X, S + B) < 1 \). Let \( K_S + B_S := (K_X + S + B)|_S \).

Then
\[ \text{mld}(x, X, S + B) = \text{mld}(x, S, B_S). \]

**Proof.** It is an immediate corollary of [BCHM10, Corollary 1.4.3]. See, for example, [Kol13, Theorem 7.10] for a detailed proof. \( \square \)

**Lemma 3.5.** Let \( d > 1 \) be an integer and \( t > 0 \) a real number. Assume the conjecture on ACC for \( a \)-lc thresholds in dimension \( d \) for finite coefficients and for the interval \([0, 1 - t]\). Then the conjecture on ACC for \( a \)-lc thresholds holds in dimension \( d - 1 \) for finite coefficients and for the interval \([0, 1 - t]\).

**Proof.** The lemma is immediate by noticing that for any triple \((X_i, B_i; G_i)\), any (not necessarily closed) point \( x_i \in X_i \), and any real number \( 0 \leq 1 \leq 1 - t \), \( a \text{-lct}_{x_i}(X_i, B_i; G_i) = a \text{-lct}_{x_i}(0)(X_i \times \mathbb{A}^1, B_i \times \mathbb{A}^1; G_i \times \mathbb{A}^1) \). \( \square \)

**Lemma 3.6.** Let \((X, B)\) be a pair and \( x \in X \) a (not necessarily closed) point. Suppose that \((X, B)\) is klt near \( x \). Then there are only finitely many prime \( b \)-divisors \( E \) over \( X \), such that \( a(E, X, B) \leq 1 \).

**Proof.** Let \( f : Y \to X \) be a log resolution of \((X, B)\) near \( x \), such that
\[ K_W + B_W := f^*(K_X + B). \]
There are only finitely many components of \( B_W \), and since \((X, B)\) is klt near \( x \), all the coefficients of \( B_W \) are \( < 1 \). Suppose that \( 1 - c \) is the maximum coefficient of \( B_W \). Let \( g : W' \to W \) be the blowing-up of the strata of \((W, B_W)\) for \( \lfloor \frac{1}{c} \rfloor \) times, such that
\[ K_{W'} + B_{W'} := g^*(K_W + B_W), \]
then any prime $b$-divisor over $W'$ has log discrepancy $> 1$ respect to $(X, B)$. Thus, all the prime $b$-divisors $E$ over $X$ such that $a(E, X, B) \leq 1$ have positive coefficient in $B_{W'}$, and in particular, their number is finite. \qed

**Lemma 3.7.** Let $(X, B)$ be a dlt pair and $n \geq 2$ an integer. Let $H_1, \ldots, H_n$ be $n$ different exceptional $b$-divisors over $X$, $\alpha_1, \ldots, \alpha_n > 0$ real numbers and $h : Z \to X$ a birational morphism, such that for any $1 \leq i \leq n$,

- $Z$ is $\mathbb{Q}$-factorial,
- $h$ is an extraction which exactly extracts $E_1, \ldots, E_n$,
- $H_i = \alpha_i E_i$ for some prime $b$-divisor $E_i$, and
- $a(E_i, X, B) < 1$.

Then there exists an integer $1 \leq i \leq n$, two birational morphisms $f_i : Y_i \to X$ and $g_i : Z_i \to Y_i$, such that:

1. $f_i$ is an extraction which exactly extracts $E_1, \ldots, E_{i-1}, E_{i+1}, \ldots, E_n$,
2. $g_i$ is the extraction which exactly extracts $E_i$,
3. $Y_i, Z_i$ are both $\mathbb{Q}$-factorial, and
4. $\text{mult}_{H_i} \sum_{j \neq i} H_j Y_i < 1$.

**Proof.** Since $(X, B)$ is dlt, possibly replacing $(X, B)$ with a $\mathbb{Q}$-factorialization, we may assume that $(X, B)$ is $\mathbb{Q}$-factorial. According to our assumptions and [Bir12 Theorem 1.8], we may run a $(\sum_{i=1}^n H_i, Z)$-MMP/$X$ which will terminate at $X$. For any step of this MMP, the only possible prime $b$-divisors that are contracted are $E_1, \ldots, E_n$. Suppose the first prime $b$-divisors contracted by this MMP is $E_i$. Let $Z \to Z_i$ be the first sequence of flips in this MMP, $g_i : Z_i \to Y_i$ the divisorial contraction of $E_i$, and $f_i : Y_i \to X$ the induced contraction. Then $f_i$ and $g_i$ satisfy (1) and (2), and $Y_i, Z_i$ are both $\mathbb{Q}$-factorial, hence we deduce (3). Moreover, let $\Sigma_i$ be the $((\sum_{j=1}^n H_{j,Z_i})$-extremal ray contracted by $g_i$, then we have

- $((\sum_{j=1}^n H_{j,Z_i}) \cdot \Sigma_i < 0$, as $\Sigma_i$ is a $(\sum_{j=1}^n H_{j,Z_i})$-extremal ray,
- $\text{mult}_{H_i} \sum_{j \neq i} H_{j,Y_i} \cdot \Sigma_i = 0$ according to the projection formula, and
- $H_{i,Z_i} \cdot \Sigma_i < 0$, as $-H_{i,Z_i}$ is $g_i$-nef,

and together we deduce (4). \qed

4. Complements

**Lemma 4.1.** Let $d > 0$ be an integer and $\mathcal{I} \subset [0, 1] \cap \mathbb{Q}$ a finite set. Then there exists an integer $n > 0$ which only depends on $d$ and $\mathcal{I}$ satisfying the following. Suppose $(X, B)$ is a pair of dimension $d$, $x \in X$ a (not necessarily closed) point, such that

- $(X, B)$ is lc,
- $B \in \mathcal{I}_0$, and
- $(X, \Delta)$ is a klt pair for some boundary $\Delta$,

then there is an effective $\mathbb{Q}$-divisor $G$ on $X$ such that $(X \ni x, B + G)$ is a monotonic local $n$-complement of $(X \ni x, B)$. In other words, we have

1. $(X, B + G)$ is lc near $x$, and
(2) \( n(K_X + B + G) \) is Cartier near \( x \).

Proof. By [Bir16A, Theorem 1.7], there exists an integer \( n > 0 \), such that for any \((X, B)\) as in the assumptions, there exists a monotonic local \( n\)-complement \((X \ni x, B^+)\) of \((X \ni x, B)\). We may just let \( G := B^+ - B \). \( \square \)

**Theorem 4.2** ([Nak16, Theorem 1.6]). Fix three positive integers \( d, c, m > 0 \). Let \( r_0 = 1, r_1, \ldots, r_c \) be positive real numbers that are linearly independent over \( \mathbb{Q} \) and suppose \( s_1, \ldots, s_m : \mathbb{R}^{c+1} \to \mathbb{R} \) are \( \mathbb{Q} \)-linear functions. Then, there exists a positive real number \( \epsilon > 0 \) which only depends on \( d \), \( r_1, \ldots, r_c \) and \( s_1, \ldots, s_m \) satisfying the following. For any \( \mathbb{Q} \)-Gorenstein normal variety \( X \) of dimension \( d \) and \( \mathbb{Q} \)-Cartier Weil divisors \( D_1, \ldots, D_m \geq 0 \) on \( X \), if

\[
(X, \sum_{i=1}^m s_i(r_0, \ldots, r_c)D_i)
\]
is lc, then

\[
(X, \sum_{i=1}^m s_i(r_0, \ldots, r_{c-1}, t)D_i)
\]
is lc for any \( t \) such that \( |t - r_c| \leq \epsilon \).

**Lemma 4.3.** Let \( d > 0 \) be an integer and \( \mathcal{I} \subset [0, 1] \) a finite set. Then there exist a finite set \( \mathcal{I}_1 \subset [0, 1] \) and a finite set \( \mathcal{I}_2 \subset [0, 1] \cap \mathbb{Q} \) which only depend on \( d \) and \( \mathcal{I} \), and a rational number \( u > 0 \) which only depend on \( \mathcal{I} \) satisfying the following.

Suppose \((X, B)\) is a \( \mathbb{Q} \)-factorial lc pair of dimension \( d \) such that \( B \in \mathcal{I} \). Then there exist \( a_i \in \mathcal{I}_1 \) and \( B_i \in \mathcal{I}_2 \), such that

1. \( \sum_{i=1}^u a_i = 1 \),
2. \( K_X + B = \sum_{i=1}^u a_i(K_X + B_i) \), and
3. \((X, B_i)\) is lc for any \( i \).

Proof. Since \( \mathcal{I} \) is a finite set of positive real numbers, \( \mathcal{I} \) is generated by finitely many \( \mathbb{Q} \)-linearly independent real numbers. Suppose that \( \mathcal{I} \) is generated by \( 1, r_1, \ldots, r_c \), such that \( 1, r_1, \ldots, r_c \) are \( \mathbb{Q} \)-linearly independent and \( r_1, \ldots, r_c \) only depend on \( \mathcal{I} \). We will additionally show that we may pick \( u = 2^c \).

We use induction on \( c \). When \( c = 0 \), we may take \( u := 1 = 2^c \), \( \mathcal{I}_1 := \{0, 1\} \) and \( \mathcal{I}_2 := \mathcal{I} \).

Assume the lemma holds for \( 1, 2, \ldots, c - 1 \). Since \( \mathcal{I} \) is a finite set, there exists an integer \( m > 0 \) which only depends on \( \mathcal{I} \), such that for any lc pair \((X, B)\) of dimension \( d \) such that \( B \in \mathcal{I} \), we may write \( B = \sum_{j=1}^m b_j B^j \) such that each \( B^j \) is reduced and \( b_j \in \mathcal{I} \) (we remark that we allow \( B^j = 0 \)).

We may write

\[
b_j = s_j(1, r_1, \ldots, r_c)
\]
for some \( \mathbb{Q} \)-linear functions \( s_1, \ldots, s_m \) which only depend on \( d \) and \( \mathcal{I} \). By Theorem 4.2 we may pick \( \epsilon > 0 \) and \( \delta > 0 \) which only depend on \( d \) and \( \mathcal{I} \),
such that \( r_c + \epsilon \) and \( r_c - \delta \) are both rational numbers, and

\[
(X, \sum_{j=1}^{m} s_j(1, r_1, \ldots, r_{c-1}, r_c + \epsilon)B^j), (X, \sum_{j=1}^{m} s_j(1, r_1, \ldots, r_{c-1}, r_c - \delta)B^j)
\]

are both lc.

Now

\[
K_X + B = \frac{\delta}{\epsilon + \delta}(K_X + \sum_{j=1}^{m} s_j(1, r_1, \ldots, r_{c-1}, r_c + \epsilon)B^j) + \frac{\epsilon}{\epsilon + \delta}(K_X + \sum_{j=1}^{m} s_j(1, r_1, \ldots, r_{c-1}, r_c - \delta)B^j).
\]

From our construction above, all the numbers

\[
s_j(1, r_1, \ldots, r_{c-1}, r_c + \epsilon) \text{ and } s_j(1, r_1, \ldots, r_{c-1}, r_c - \delta)
\]

belong to a finite set of positive real numbers \( \mathcal{I}' \) which only depends on \( d \) and \( \mathcal{I} \) such that \( \dim \mathcal{Q}\text{span}_\mathcal{Q}(\mathcal{I}') \leq c - 1 \). The lemma follows from the induction on \( c \).

The lemma above leads us to the following definition which is a generalization of monotonic \( n \)-complements.

**Definition 4.4.** Let \( \mathcal{I}_1, \mathcal{I}_2 \subset [0, 1] \) be two sets, \( (X, B) \) a pair, and \( X \to Z \) a contraction. An \((n, \mathcal{I}_1, \mathcal{I}_2)\)-complement of \((X/Z \ni z, B)\) is of the form \((X/Z \ni z, B_i)\) together with a decomposition

\[
K_X + B = \sum a_i(K_X + B_i^+)
\]
satisfying the following.

1. \( B_i^+ \geq B \),
2. \( a_i \in \mathcal{I}_1, \sum a_i = 1 \),
3. \( B_i^+ \in \mathcal{I}_2 \), and
4. there exists \( B_i \in \mathcal{I}_2 \), such that \((X, B_i)\) is lc, \( K_X + B_i \leq \sum a_i(K_X + B_i) \), \((X/Z \ni z, B_i^+)\) is a monotonic \( n \)-complement of \((X/Z \ni z, B_i)\).

The following lemma gives us the proof of existence of \((n, \mathcal{I}_1, \mathcal{I}_2)\)-complement in some cases. We refer the readers to [HLS19] for more general statements.

**Lemma 4.5.** Let \( d > 0 \) be an integer and \( \mathcal{I} \subset [0, 1] \) a finite set. Then there exists an integer \( n > 0 \), two finite sets \( \mathcal{I}_1 \subset [0, 1] \) and \( \mathcal{I}_2 \subset [0, 1] \cap \mathbb{Q} \) which only depend on \( d \) and \( \mathcal{I} \) satisfying the following. Let \((X, B)\) be a \( \mathbb{Q} \)-factorial pair of dimension \( d \) and \( x \in X \) a (not necessarily closed) point, such that \((X, B)\) is lc near \( x \) and \( B \in \mathcal{I} \). Then there exists an \((n, \mathcal{I}_1, \mathcal{I}_2)\)-complement \((X \ni x, B + G)\) of \((X \ni x, B)\).
Proof. By Lemma 4.3 there exist an integer $u > 0$ which only depend on $I$, a finite set $I_1 \subset [0, 1]$ and a finite set $I_2 \subset [0, 1] \cap \mathbb{Q}$ which only depend on $d$ and $I$, such that we may write

$$K_X + B = \sum_{i=1}^{u} a_i (K_X + B_i),$$

where $\sum a_i = 1$, and for any $i$, $a_i \in I_1$, $B_i \in I_2$, and $(X, B_i)$ is lc.

By Lemma 4.1 for any $(X, B)$ as above, there exists an integer $n > 0$ which only depends on $d$ and $I_2$, $\mathbb{Q}$-divisors $G_i$ on $X$ for each $i$, such that

1. $(X, B_i + G_i)$ is lc near $x$, and
2. $n(K_X + B_i + G_i)$ is Cartier near $x$.

We define $G := \sum_{i=1}^{u} a_i G_i$. Then from our construction, $(X, B + G)$ is lc near $x$, and we may write

$$K_X + B + G = \sum a_i (K_X + B_i + G_i)$$

where $(X \ni x, B_i + G_i)$ is an $n$-complement of $(X \ni x, B_i)$ near $x$ for any $i$. Thus $(X \ni x, B + G)$ an $(n, I_1, I_2)$-complement of $(X \ni x, B)$. □

Lemma 4.6. Let $d, n > 0$ be two integers, $I, I_1 \subset [0, 1]$ two finite sets, $I_2 \subset [0, 1] \cap \mathbb{Q}$ a finite set, and $M > 0$ a real number. Assume $(X, B)$ is a pair of dimension $d$, $x \in X$ is a (not necessarily closed) point, $G$ is an $\mathbb{R}$-divisor on $X$, and $E$ is a prime $b$-divisor over $X$, such that

1. $(X, B)$ is lc,
2. $B \in I$,
3. $\text{center}_XE = \{\bar{x}\}$,
4. $(X \ni x, B + G)$ is an $(n, I_1, I_2)$-complement of $(X \ni x, B)$, and
5. $a(E, X, B + G) < M$.

Then

$$a(E, X, B + G)$$

belongs to a finite set which only depends on $d, n, I, I_1, I_2$ and $M$. Moreover, suppose $a > 0$ is a real number such that the ACC for $a$-lc thresholds holds for $I$ and for the interval $[0, 1]$. If

6. $0 \leq a\text{-lct}_x(X, B; G) \leq 1$, and
7. $a\text{-lct}_x(X, B; G)$ is attained at $E$,

then $a(E, X, B)$ belongs to an ACC which only depends on $d, n, I, I_1$ and $I_2$.

Proof. Pick $X, B, G, E$ and $x \in X$ as in the assumptions. For simplicity, we define $a_0 := a(E, X, B)$ and $\alpha := a(E, X, B + G)$.

First we show that $\alpha$ belong to a finite set which only depends on $d, n, I, I_1, I_2$ and $M$. Let $f : Y \rightarrow X$ be a log resolution such that $E_Y := \text{center}_YE$ is a divisor on $Y$. We may write

$$K_Y + B_Y + (1 - a_0)E_Y + \Gamma_Y := f^*(K_X + B)$$
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and

$$K_Y + B_Y + G_Y + (1 - \alpha)E_Y + \Gamma_Y' := f^*(K_X + B + G)$$

where $B_Y, G_Y$ are the strict transforms of $B$ and $G$ on $Y$ respectively, and $\Gamma_Y$ and $\Gamma_Y'$ are uniquely determined $\mathbb{R}$-divisors which are exceptional over $X$. We have a decomposition

$$K_X + B + G = \sum c_i(K_X + B_i + G_i)$$

such that $(X \ni x, B_i + G_i)$ is a monotonic $n$-complement of itself, and $c_i \in \mathcal{I}_1$ for any $i$. We have

$$\alpha = \sum c_i a(E, X, B_i + G_i).$$

Since $a(E, X, B_i + G_i)$ belongs to the discrete set $\frac{1}{n} \mathbb{N}^+$ and $c_i \in \mathcal{I}_1$ for any $i$, $\alpha$ belongs to a discrete set. Since $\alpha < M$, $\alpha$ belongs to a finite set.

Now we show that additionally under the assumption (6) and (7), $a_0$ belongs to an ACC set. Since $a_\text{-lct}_x(X, B; G)$ is attained at $E$ and since $a_\text{-lct}_x(X, B; G) < 1$, by convexity of log discrepancies, we have that

$$a(E, X, B + \frac{a_0 - a}{a_0 - \alpha}G) = \frac{a_0 - a}{a_0 - \alpha}a(E, X, B + G) + \frac{a - \alpha}{a_0 - \alpha}a(E, X, B)$$

$$= \frac{a_0 - a}{a_0 - \alpha} + \frac{a - \alpha}{a_0 - \alpha}a_0 = a,$$

hence

$$a_\text{-lct}_x(X, B; G) = \frac{a_0 - a}{a_0 - \alpha} = 1 - \frac{a - \alpha}{a_0 - \alpha}.$$

Since $(X \ni x, B + G)$ is an $(n, \mathcal{I}_1, \mathcal{I}_2)$-complement of $(X \ni x, B)$, all the coefficients of $B + G$ belong to a finite set, and since all the coefficients of $B$ belong to a finite set, we have that all the coefficients of $G$ are belong to a finite set. Thus according to our assumption, $a_\text{-lct}_x(X, B; G)$ belongs to an ACC set. Since $a_0 > a > \alpha$, and since $\alpha$ belongs to a finite set, $a_0$ belongs to an ACC set, and the proof is finished. \hfill \square

**Remark 4.7.** Under condition (6) and (7) of Lemma 4.6, it is clear that $a(E, X, B + G) < a$, which implies condition (5) of Lemma 4.4 when $M = a$.

5. The Main Theorem

In this section we prove the main theorem. First we show that we only need to consider the conjecture on ACC for mlds for finite coefficients.

**Proposition 5.1** (Finite coefficients to DCC coefficients). Let $d > 0$ be an integer and $0 < t < 1$ a real number. Assume the conjecture on ACC for mlds in dimension $d$ for finite coefficients and for the interval $[0, 1 - t]$, and the conjecture on ACC for $\alpha$-lc thresholds in dimension $d$ for finite coefficients and for any $a \in [0, 1 - t]$.

Then the conjecture on ACC for mlds holds in dimension $d$ for the interval $[0, 1 - t]$. 


Proof. Suppose not, then there exists a DCC set $I \subset [0, 1]$, a sequence of pairs $(X_i, B_i)$ of dimension $d$, (not necessarily closed) points $x_i \in X_i$, a strict increasing sequence of real numbers $\{a_i\}$, such that for any $i$,

(1) $(X_i, B_i)$ is lc near $x_i$,  
(2) $B_i \in I$,  
(3) $\text{mld}(x_i, X_i, B_i) = a_i$, and  
(4) $\lim_{i \to \infty} a_i = \bar{a} < 1 - t$.

Possibly replacing $(X_i, B_i)$ with a dlt modification and replacing accordingly, we may assume that $(X_i, B_i)$ is Q-factorial. Suppose that $\text{mld}(x_i, X_i, B_i)$ is attained at $E_i$. Possibly replacing $B_i$, we may assume that all the irreducible components of $B_i$ pass through $x_i$. Since all the coefficients of $B_i$ belong to a DCC set, there exists a real number $\delta > 0$ which only depends on $I$, such that all the coefficients of $B_i$ are $\geq \delta$. Write

$$B_i = \sum_{j=1}^{u_i} b_{i,j} B_{i,j},$$

according to Theorem 3.1 $\sum b_{i,j} \leq d$, hence $u_i \leq \frac{d}{\delta}$ for every $i$. Thus possibly passing to a subsequence, we may assume that $u_i = u > 0$ is a constant. Possibly reordering the components of $B_i$, we may assume that $b_{i,j} \geq b_{i,j+1}$ for any $i$ and any $1 \leq j \leq u - 1$. Since $I$ is a DCC set, possibly passing to a subsequence, we may assume that for any $1 \leq j \leq u$, $\{b_{i,j}\}_{i=1}^{\infty}$ is an increasing sequence. Let

$$b'_j := \lim_{i \to \infty} b_{i,j}$$

and

$$B'_i := \sum_{j=1}^{u} b'_j B_{i,j}.$$

By Theorem 3.2 possibly passing to a subsequence, we may assume that $(X_i, B'_i)$ is lc for every $i$.

Let $a'_i := \text{mld}(x_i, X_i, B'_i)$. Possibly passing to a subsequence, we may assume that $a'_i$ has a unique accumulation point $\bar{a'}$. Since $B'_i \geq B_i$,

$$0 \leq a'_i = \text{mld}(x_i, X_i, B'_i) \leq \text{mld}(x_i, X_i, B_i) = a_i < 1 - t.$$

By ACC for mlds for finite coefficients for the interval $[0, 1 - t]$, possibly passing to a subsequence, we may assume that $a'_i$ is decreasing. Since $a_i$ is strictly increasing and converges to $\bar{a}$, we have $\bar{a'} < \bar{a}$.

Let $\epsilon := \frac{\bar{a} + \bar{a'}}{2}$. We have $0 < \epsilon < 1 - t$. Since $b'_j = \lim_{i \to \infty} b_{i,j}$, possibly passing to a subsequence, there exists a strictly increasing sequence of real numbers $\{\beta_i\}_{i=1}^{\infty}$ which converges to 1, such that

$$\beta_i B'_i \leq B_i \leq B'_i$$
Towards the equivalence of the ACC for $\alpha$-lcts and the ACC for mlds

for any $i$. Possibly passing to a subsequence, we may assume that $a_i > \epsilon$ and $a'_i < \epsilon$ for any $i$. Since $X_i$ is $\mathbb{Q}$-factorial, we have

$$\beta_i \leq \epsilon\text{-lct}(X_i, 0; B'_i) < 1.$$  

Possibly passing to a subsequence, we may assume that $\epsilon\text{-lct}(X_i, 0; B'_i)$ is a strict increasing sequence. But this contradicts to the ACC for $\epsilon$-lc thresholds for finite coefficients as $\epsilon \in [0, 1 - t]$. □

According to Proposition 5.1 to prove Theorem 1.2, we only need to show the following:

**Theorem 5.2.** Let $d > 0$ be an integer and $t > 0$ a real number. Assume the ACC conjecture for $\alpha$-lc thresholds for finite coefficients and for any $\alpha \in [0, 1 - t]$. Then the ACC conjecture for mlds for finite coefficients holds for the interval $[0, 1 - t]$. In other words, for any finite set $\mathcal{I} \subset [0, \infty)$, the accumulation points of MLD($d, \mathcal{I}$) from below are not contained in $[0, 1 - t]$.

We give a sketch of the proof first. In Step 1, we suppose the theorem does not hold, and construct a sequence of pairs $(X_i, B_i)$ and $x_i \in X_i$ such that MLD($x_i, X_i, B_i$) is strictly increasing to $\bar{a}$ for some real number $0 \leq a \leq 1 - t$.

In Step 2, we reduce to the case of klt germs. In Step 3, we reduce to the case when there is no exceptional divisor over $X_i$ with log discrepancy equals to $\bar{a}$. In Step 4, we reduce to the case when there is only one exceptional divisor over $X_i$ with log discrepancy $\leq \bar{a}$, which is exactly the divisor which attains MLD($x_i, X_i, B_i$). Step 5 constructs local $(n, I_1, I_2)$-complements (see Definition 4.4).

To simplify our following statements and for readers’ convenience, in Step 6 we make a construction (cf. Construction 5.4) of all the extractions and all the divisors we may need in the rest of the proof. We also give a table of notation. In Step 7, we reduce to a technical case, such that for any “bad exceptional divisor”, its extraction gives us a pair that is $\bar{a}$-lc. This will help us constructing $\bar{a}$-lc thresholds in the next step. In Step 8, we analyze the multiplicities of the divisor which attains the minimal log discrepancy with respect to the “bad exceptional divisors” by using the assumption on ACC for $\bar{a}$-lc thresholds. Finally, in Step 9 we reduce to the case when there does not exist any “bad exceptional divisor” and immediately get a contradiction to a claim proved in Step 6, and conclude the proof of the Theorem.

**Proof of Theorem 5.2.** We prove Theorem 5.2 by using contradiction.

**Step 1.** In this step we give some notation which will be repeatedly used in the rest of the proof. We also study some of their basic properties in this step.

Suppose the theorem does not hold. Then there exist a finite set of real numbers $\mathcal{I} \subset [0, 1]$, a sequence of pairs $(X_i, B_i)$ of dimension $d$, a sequence of (not necessarily closed) points $x_i \in X_i$, such that $B_i \in \mathcal{I},$
\( \{a_i := \mld(x_i, X_i, B_i)\}_{i=1}^{\infty} \) forms a strictly increasing sequence which converges to \( \bar{a} \) where \( \bar{a} \leq 1 - t < 1 \). Possibly replacing \((X_i, B_i)\) with its \( \mathbb{Q} \)-factorialization near \( x_i \), we may assume that \((X_i, B_i)\) is \( \mathbb{Q} \)-factorial near \( x_i \).

For every \( i > 0 \), suppose that \( \mld(x_i, X_i, B_i) \) is attained at \( E_i \) for some prime \( b \)-divisor \( E_i \) over \( X_i \). If center \( X_i E_i \) is a divisor, then \( 1 - \mld(x_i, X_i, B_i) \in I \cup \{0\} \) which is a DCC set, hence \( \mld(x_i, X_i, B_i) \) belongs to an ACC set. Thus, possibly passing to a subsequence, we may assume that \( E_i \) is exceptional over \( X_i \).

Step 2. In this step we reduce to the case when \((X_i, B_i)\) is klt near \( x_i \).

First we reduce to the case when \((X_i, B_i)\) is dlt near \( x_i \) for any \( i \). Since \( \mld(x_i, X_i, B_i) > 0 \), \((X_i, B_i)\) is klt at \( x_i \) for every \( i \). Let \( \rho_i : X'_i \to X_i \) be a dlt modification near \( x_i \), such that

\[
K_{X'_i} + B'_i + L_i = \rho_i^*(K_{X_i} + B_i)
\]

where \( B'_i \) is the strict transform of \( B_i \) and \( L_i \) is the reduced exceptional divisor of \( g_i \). Since \((X_i, B_i)\) is klt at \( x_i \) for every \( i \), \( E_i \) is exceptional over \( X'_i \), and we may let \( x'_i \) be the generic point of \( E_i \) on \( X'_i \) for every \( i \). Since

\[
\mld(x_i, X_i, B_i) \leq \mld(x'_i, X'_i, B'_i + L_i) \leq a(E_i, X'_i, B'_i + L_i)
\]

possibly replacing \( I \) with \( I \cup \{1\} \), \((X_i, B_i)\) with \((X'_i, B'_i + L_i)\) and \( x_i \) with \( x'_i \), we may assume that \((X'_i, B'_i)\) is dlt near \( x_i \).

Possibly passing to a subsequence, suppose that \((X_i, B_i)\) is not klt near \( x_i \) for any \( i \). Since \((X_i, B_i)\) is dlt near \( x_i \), we have \( |B_i| \neq 0 \), hence we may pick an irreducible component \( S_i \) of \( |B_i| \). According to Theorem 3.4

\[
\mld(x_i, X_i, B_i) = \mld(x_i, S_i, B_{S_i})
\]

where \( B_{S_i} \) is defined via the adjunction

\[
K_{S_i} + B_{S_i} := (K_{X_i} + B_i)|_{S_i}.
\]

Since all the coefficients of \( B_i \) belong to a finite set, all the coefficients of \( B_{S_i} \) belongs to a dcc set. Now Theorem 5.2 follows from Lemma 3.5, Proposition 5.1 and Theorem 5.2 in dimension \( d - 1 \).

Thus, in the rest of the proof, we may assume that \((X_i, B_i)\) is klt near \( x_i \) for any \( i \).

Step 3. For any \( i > 0 \), let \( \mathcal{A}_i = \mathcal{A}_i(x_i, X_i, B_i) \) be the set of prime \( b \)-divisors \( F_{i,j} \) over \( X_i \) such that \( a(F_{i,j}, X_i, B_i) = \bar{a} \) and center \( X_i F_{i,j} = \{\bar{x}_i\} \). In this step, we reduce to the case when \( \mathcal{A}_i = \emptyset \) for every \( i \).

By Step 2, \((X_i, B_i)\) is klt near \( x_i \). Since \( \bar{a} \leq 1 - t < 1 \), according to Lemma 3.7 for any \( i \), \( \mathcal{A}_i \) is a finite set. Let \( \phi_i : X'_i \to X_i \) be an extraction of all the prime \( b \)-divisors which belong to \( \mathcal{A}_i \), such that \( X'_i \) is \( \mathbb{Q} \)-factorial near the inverse image of \( x_i \). We have

\[
K_{X'_i} + B'_i + (1 - \bar{a})M_i = \phi_i^*(K_{X_i} + B_i)
\]
where $M_i$ is the reduced exceptional divisor of $\phi_i$ and $B'_i$ is the strict transform of $B_i$ on $X'_i$. Possibly replacing $\mathcal{I}, X_i, B_i, G_i$ and $x_i$ with $\mathcal{I} \cup \{1 - \bar{a}\}$, $X'_i, B'_i + (1 - \bar{a})M_i, \phi^*_iG_i$ and the generic point of center $X'_iE_i$ respectively, we may assume that $A_i = \emptyset$ for every $i$.

**Step 4.** For any $i > 0$, let $\mathcal{B}_i = \mathcal{B}_i(x_i, X_i, B_i)$ be the set of prime $b$-divisors $F_{i,j}$ such that $a(F_{i,j}, X_i, B_i) \leq \bar{a}$ and center $X_i F_{i,j} = \{x_i\}$. In this step, we reduce to the case when $\mathcal{B}_i = \{E_i\}$ for every $i$.

Since $\bar{a} \leq 1 - t < 1$, according to Lemma 3.6 and **Step 2**, for any $i$, $\mathcal{B}_i$ is a finite set, hence we may suppose that

$$\mathcal{B}_i = \{F_{i,1}, \ldots, F_{i,m_i}\}.$$  

For any $i$ and any $1 \leq j \leq m_i$, let

$$a_{F_{i,j}} := a(F_{i,j}, X_i, B_i)$$

and

$$H_{i,j} := (\bar{a} - a_{F_{i,j}})F_{i,j}.$$  

By **Step 3**, $a_{F_{i,j}} < \bar{a}$ for any $i, j$. By [BCHM10, Corollary 1.4.3] and Lemma 3.7 for any $i$, there exists $1 \leq j_0 \leq m_i$, an extraction $\psi_{i,j_0} : T_{i,j_0} \to X_i$ of $F_{i,1}, \ldots, F_{i,j_0-1}, F_{i,j_0+1}, \ldots, F_{i,m_i}$, and an extraction $t_{i,j_0} : Z_{i,j_0} \to T_{i,j_0}$ of $F_{i,j_0}$, such that $T_{i,j_0}$ and $Z_{i,j_0}$ are $\mathbb{Q}$-factorial near the inverse images of $x_i$,

$$K_{T_{i,j_0}} + B_{T_{i,j_0}} + \sum_{j \neq j_0} (1 - a_{F_{i,j}})F_{i,T_{i,j_0}} = \psi_{i,j_0}^*(K_{X_i} + B_i)$$

and

$$\sum_{j \neq j_0} H_{i,j}z_{i,j_0} + b_{i,j_0}H_{i,j_0}z_{i,j_0} = t_{i,j_0}^*(\sum_{j \neq j_0} H_{i,j}T_{i,j_0})$$

for some $b_{i,j_0} < 1$, where $B_{T_{i,j_0}}$ is the strict transform of $B_i$ on $T_{i,j_0}$.

Let $t_i$ be the generic point of center $T_{i,j_0} F_{i,j_0}$. From the equations above,

$$(T_{i,j_0}, B_{T_{i,j_0}} + (1 - \bar{a}) \sum_{j \neq j_0} F_{i,j,T_{i,j_0}})$$

is not $\bar{a}$-lc at $t_i$, all the coefficients of $B_{T_{i,j_0}} + (1 - \bar{a}) \sum_{j \neq j_0} F_{i,j,T_{i,j_0}}$ belong to $\mathcal{I} \cup \{1 - \bar{a}\}$, and $F_{i,j_0}$ is the only prime $b$-divisor that is exceptional over $T_{i,j_0}$ and whose log discrepancy with respect to $(T_{i,j_0}, B_{T_{i,j_0}} + (1 - \bar{a}) \sum_{j \neq j_0} F_{i,j,T_{i,j_0}})$ is $\leq \bar{a}$. Moreover, since

$$a_t < a_{F_{i,j}} \leq a(T_{i,j_0}, T_{i,j_0}, B_{T_{i,j_0}} + (1 - \bar{a}) \sum_{j \neq j_0} F_{i,j,T_{i,j_0}}) < \bar{a},$$

$F_{i,j_0}$ attains $\mld(t_i, T_{i,j_0}, B_{T_{i,j_0}} + (1 - \bar{a}) \sum_{j \neq j_0} F_{i,j,T_{i,j_0}})$, and possibly passing to a subsequence, we may assume that $\mld(t_i, T_{i,j_0}, B_{T_{i,j_0}} + (1 - \bar{a}) \sum_{j \neq j_0} F_{i,j,T_{i,j_0}})$ is strictly increasing and converges to $\bar{a}$.

Possibly replacing $\mathcal{I}, X_i, B_i, G_i$ and $x_i$ with $\mathcal{I} \cup \{1 - \bar{a}\}$, $T_{i,j_0}, B_{T_{i,j_0}} + (1 - \bar{a}) \sum_{j \neq j_0} F_{i,j,T_{i,j_0}}$, $\psi_{i,j_0}^*(K_{X_i} + B_i + G_i) - (K_{T_{i,j_0}} + B_{T_{i,j_0}} + \sum_{j \neq j_0}(1 - \bar{a})F_{i,T_{i,j_0}})$, $t_i$ and $F_{i,j_0}$ respectively, we may suppose that $\mathcal{B}_i = \{E_i\}$. 

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Claim 5.3. For any real number $1 - \alpha < a < \bar{a}$, possibly passing to a subsequence, $a \text{-lct}_{x_i}(X_i, B_i; G_i)$ is not attained at $E_i$.

Proof. Since $(X_i \ni x_i, B_i + G_i)$ is an $(n, \mathcal{I}_1, \mathcal{I}_2)$-complement of $(X_i \ni x_i, B_i)$ and since $a(E_i, X_i, B_i + G_i) \leq a(E_i, X_i, B_i) < \bar{a}$, $a(E_i, X_i, B_i + G_i)$ belongs to $\mathcal{J}$. In particular, $a(E_i, X_i, B_i + G_i) \leq \alpha$, hence possibly passing to a subsequence, $0 < a \text{-lct}_{x_i}(X_i, B_i; G_i) < 1$. The claim follows from Lemma 4.6. \hfill \Box

Step 6. According to Step 4, $E_i$ is the unique prime $b$-divisor that is exceptional over $X_i$ and has log discrepancy $\leq \bar{a}$. Since the rest of proof involves extractions of different divisors, to make the representation less complicated, in this step we make the following construction, which gives notation that we will use in the rest of the proof.

Construction 5.4. For any integer $i > 0$ and any finite set $\mathcal{I}'$, we let $\mathcal{D}_{i, \mathcal{I}'}$ be the set of triples $(U_i, B_{U_i}; G_{U_i})$ associated with (not necessarily closed) points $u_i \in U_i$, such that

- there exists a birational contraction $f_{U_i} : U_i \to X_i$, such that $K_{U_i} + B_{U_i} + G_{U_i} = f_{U_i}^*(K_{X_i} + B_i + G_i)$,
- $K_{U_i}$ is $\mathbb{Q}$-factorial near the inverse image of $x_i$,
- $B_{U_i} \in \mathcal{I}'$,
- $E_i$ is the unique prime $b$-divisor over $U_i$ whose center on $U_i$ is $\{\bar{u}_i\}$ and whose log discrepancy with respect to $(U_i, B_{U_i})$ is $\leq \bar{a}$, and
- $a_i \leq a_{U_i} := a(E_i, U_i, B_{U_i}) < \bar{a}$. 

Step 5. In this step we construct $(n, \mathcal{I}_1, \mathcal{I}_2)$-complements.

By Lemma 4.3, there exists an integer $n > 0$, a finite set $\mathcal{I}_1 \subset [0, 1]$ and a finite set $\mathcal{I}_2 \subset [0, 1] \cap \mathbb{Q}$ which only depend on $d$ and $\mathcal{I}$, such that for any $i > 0$, there exists an effective $\mathbb{R}$-divisor $G_i$ such that $(X_i \ni x_i, B_i + G_i)$ is an $(n, \mathcal{I}_1, \mathcal{I}_2)$-complement of $(X_i \ni x_i, B_i)$.

Let $f_i : Y_i \to X_i$ be the extraction of $E_i$ such that $Y_i$ is $\mathbb{Q}$-factorial near the inverse image of $x_i$.

Consider the set of non-negative real numbers

$$\mathcal{J} := \{ x = \sum_i c_i u_i | c_i \in \mathcal{I}_1, u_i \in \mathcal{I}_2 \} \cap [0, \bar{a}),$$

then $\mathcal{J}$ is a discrete, hence finite set. Let

$$\alpha := 1 - \max\{ j | j \in \mathcal{J} \}.$$ 

Then $\alpha > 1 - \bar{a}$, hence possibly passing to a subsequence, we may assume that there exists a real number $\delta > 0$ which only depends on $d$ and $\mathcal{I}$ such that $\alpha - (1 - a_i) > \delta$ for every $i$.

We finish this step by proving a claim.
For any real number \(0 < \epsilon < 1 - \bar{a}\) and any \((u_i, U_i, B_{U_i}, G_{U_i}) \in D_i,\mathcal{X}\), we define

\[
S^\epsilon_i,\mathcal{X}(u_i, U_i, B_{U_i}, G_{U_i}) := \{\Gamma^\epsilon_{i,j}(U_i) | \Gamma^\epsilon_{i,j}(U_i) \text{ is a prime } b\text{-divisor over } U_i,
\]

\[
\text{center}_{U_i,\Gamma^\epsilon_{i,j}(U_i)} = \{\bar{u}_i\},
\]

\[
\bar{a} < a(\Gamma^\epsilon_{i,j}(U_i), U_i, B_{U_i}) < \bar{a} + \epsilon, \text{ and}
\]

\[
a(\Gamma^\epsilon_{i,j}(U_i), U_i, B_{U_i} + G_{U_i}) < a(E_i, U_i, B_{U_i} + G_{U_i})\}.
\]

For simplicity, we usually use the notation \(S^\epsilon_i\) instead of \(S^\epsilon_i,\mathcal{X}(x_i, X_i, B_i, G_i)\), and \(\Gamma^\epsilon_{i,j}\) instead of \(\Gamma^\epsilon_{i,j}(X_i)\).

It is clear from the definition that for any \(i\), any \(\epsilon < \epsilon'\), \(S^\epsilon_i \subset S^{\epsilon'}_i\). By Lemma 3.6, \(S^\epsilon_i\) is a finite set for every \(i, \epsilon\). Let

\[
S^\epsilon := \bigcup_{i=1}^{\infty} S^\epsilon_i.
\]

For every \(i, j, \epsilon\), we define

\[
\gamma_{i,j}^\epsilon := a(\Gamma^\epsilon_{i,j}, X_i, B_i) - \bar{a},
\]

and

\[
n(i, j)^\epsilon := 1 - a(\Gamma^\epsilon_{i,j}, X_i, B_i + G_i).
\]

Since \((X_i \ni x_i, B_i + G_i)\) is an \((n, \mathcal{I}_1, \mathcal{I}_2)\)-complement of \((X_i \ni x_i, B_i)\), we deduce that \(n(i, j)^\epsilon\) belongs to a finite set of real numbers.

Moreover, for any \(i, j, \epsilon\), we let

\[
h^\epsilon_{i,j} : X^\epsilon_{i,j} \rightarrow X_i
\]

be the extraction of \(\Gamma^\epsilon_{i,j}\) such that \(X^\epsilon_{i,j}\) is \(\mathbb{Q}\)-factorial near the inverse image of \(x_i\),

\[
p^\epsilon_{i,j} : W^\epsilon_{i,j} \rightarrow X^\epsilon_{i,j}
\]

an extraction of all the prime \(b\)-divisors which belong to \(S^\epsilon_i\) except \(\Gamma^\epsilon_{i,j}\) satisfying the following:

- \(W^\epsilon_{i,j}\) is \(\mathbb{Q}\)-factorial near the inverse image of \(x_i\), and
- for any \(\epsilon \leq \epsilon'\), if \(\Gamma^\epsilon_{i,j} = \Gamma^{\epsilon'}_{i,j}\), then there exists a morphism

\[
W^\epsilon_{i,j} \rightarrow W^{\epsilon'}_{i,j}.
\]

Let

\[
g^\epsilon_{i,j} := p^\epsilon_{i,j} \circ h^\epsilon_{i,j},
\]

and

\[
f_i : Y_i \rightarrow X_i, f^\epsilon_{i,j} : Y^\epsilon_{i,j} \rightarrow X^\epsilon_{i,j}, \text{ and } q^\epsilon_{i,j} : V^\epsilon_{i,j} \rightarrow W^\epsilon_{i,j}
\]

the extractions of \(E_i\) from different varieties, such that \(Y_i, Y^\epsilon_{i,j}\) and \(V^\epsilon_{i,j}\) are \(\mathbb{Q}\)-factorial near the inverse images of \(x_i\).

For any \(i\), any set \(\mathcal{R}^\epsilon \subset S^\epsilon\) such that \(\mathcal{R}^\epsilon \neq \emptyset\), let \(\mathcal{R}_i^\epsilon := \mathcal{R}^\epsilon \cap S^\epsilon_i\). Then there exists an indices set \(\Lambda = (\Lambda(\mathcal{R}_i)\), a set of varieties \(\{X^\epsilon_{i,R}\}_{\lambda \in \Lambda}\) and a set of birational morphisms \(\{h^\epsilon_{i,R} : X^\epsilon_{i,R} \rightarrow X_i\}\), such that for any \(\lambda \in \Lambda\),
$X^\epsilon_{i,R}$ is $\mathbb{Q}$-factorial near the inverse image of $x_i$, $h^\epsilon_{i,R}$ is an extraction which extracts exactly prime $b$-divisors belonging to $R^\epsilon_i$. Let $f^\epsilon_{i,R} : Y^\epsilon_{i,R} \to X^\epsilon_{i,R}$ be the extraction of $E_i$ for any $i, R, \epsilon, \lambda$, such that $Y^\epsilon_{i,R}$ is $\mathbb{Q}$-factorial near the inverse image of $x_i$.

Finally, we define $C^\epsilon_{i,j} := \text{mult}_{E_i} \Gamma^\epsilon_{i,j,X^\epsilon_{i,j}}$, and $D^\epsilon_{i,j,k} := \text{mult}_{E_i} \Gamma^\epsilon_{i,j,W^\epsilon_{i,j}}$.

We conclude our construction above in the following diagrams. We also give notation for some strict transforms of $R$-divisors and center of $b$-divisors.

Space | Str. trans of $B_i$ | Str. trans of $G_i$ | Center of $E_i$ | Center of $\Gamma^\epsilon_{i,j}$
--- | --- | --- | --- | ---
$X_i$ | $B_i$ | $G_i$ | g.p. $x_i$ | g.p. $x_i$
$Y_i$ | $B_{Y_i}$ | $G_{Y_i}$ | $E_i$ | g.p. $y^\epsilon_{i,j}$
$X^\epsilon_{i,j}$ | $B_{X^\epsilon_{i,j}}$ | $G_{X^\epsilon_{i,j}}$ | g.p. $x^\epsilon_{i,j}$ | $\Gamma^\epsilon_{i,j,X^\epsilon_{i,j}}$
$Y^\epsilon_{i,j}$ | $B_{Y^\epsilon_{i,j}}$ | $G_{Y^\epsilon_{i,j}}$ | $E_i Y^\epsilon_{i,j}$ | $\Gamma^\epsilon_{i,j,Y^\epsilon_{i,j}}$
$W^\epsilon_{i,k}$ | $B_{W^\epsilon_{i,k}}$ | $G_{W^\epsilon_{i,k}}$ | g.p. $w^\epsilon_{i,k}$ | $\Gamma^\epsilon_{i,j,W^\epsilon_{i,k}}$
$V^\epsilon_{i,k}$ | $B_{V^\epsilon_{i,k}}$ | $G_{V^\epsilon_{i,k}}$ | $E_i V^\epsilon_{i,k}$ | $\Gamma^\epsilon_{i,j,V^\epsilon_{i,k}}$
$X^\epsilon_{i,R}$ | $B_{X^\epsilon_{i,R}}$ | $G_{X^\epsilon_{i,R}}$ | g.p. $x^\epsilon_{i,R}$ | $\Gamma^\epsilon_{i,j,X^\epsilon_{i,R}}$
$Y^\epsilon_{i,R}$ | $B_{Y^\epsilon_{i,R}}$ | $G_{Y^\epsilon_{i,R}}$ | $E_i Y^\epsilon_{i,R}$ | $\Gamma^\epsilon_{i,j,Y^\epsilon_{i,R}}$

Here g.p. means generic point.

We end this step by proving a claim.

**Claim 5.5.** For any $0 < \epsilon < 1 - t$, any $i$, any finite set $T'$ and any $(u_i, U_i, B_{U_i}, G_{U_i}) \in \mathcal{D}_{i,T'}$, $S^\epsilon_{i,T'}(u_i, U_i, B_{U_i}, G_{U_i}) \neq \emptyset$ except finitely many $i$.

**Proof of Claim 5.5.** Suppose not. Then there exists $\epsilon > 0$, such that possibly passing to a subsequence, $S^\epsilon_{i,T'}(u_i, U_i, B_{U_i}, G_{U_i}) = \emptyset$. Consider $(\bar{a} - \frac{\epsilon}{n + \delta})$-let$_{U_i}(U_i, B_{U_i}; G_{U_i})$, where $\delta > 0$ is defined in Step 1. Then for any prime $b$-divisor $F_i \neq E_i$ that is exceptional over $U_i$ and center$_{U_i} F_i = \{\bar{u}_i\}$, we have...
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\[ a(F_i, U_i, B_{U_i}) \geq \tilde{a} + \epsilon. \]

Since \((U_i, B_{U_i} + G_{U_i})\) is lc near \(u_i\), by linearity of log discrepancies, we have

\[ a(F_i, U_i, B_{U_i} + \frac{\epsilon}{\tilde{a} + \epsilon}G_{U_i}) = \frac{\epsilon}{\tilde{a} + \epsilon} a(F_i, U_i, B_{U_i} + G_{U_i}) + \frac{\tilde{a}}{\tilde{a} + \epsilon} a(F_i, U_i, B_{U_i}) \]

\[ \geq 0 + \frac{\tilde{a}}{\tilde{a} + \epsilon} (\tilde{a} + \epsilon) = \tilde{a} \]

and

\[ a(E_i, U_i, B_{U_i} + \frac{\epsilon}{\tilde{a} + \epsilon}G_{U_i}) = \frac{\epsilon}{\tilde{a} + \epsilon} a(E_i, U_i, B_{U_i} + G_{U_i}) + \frac{\tilde{a}}{\tilde{a} + \epsilon} a(E_i, U_i, B_{U_i}) \]

\[ \leq \frac{\epsilon}{\tilde{a} + \epsilon} (1 - \alpha) + \frac{\tilde{a}}{\tilde{a} + \epsilon} \tilde{a} \]

\[ = \tilde{a} - \frac{\epsilon}{\tilde{a} + \epsilon} (\alpha - (1 - \tilde{a})) < \tilde{a} - \frac{\epsilon}{\tilde{a} + \epsilon} \delta, \]

which implies that \((\tilde{a} - \frac{\epsilon}{\tilde{a} + \epsilon} \delta)\)-lct \(u_i(U_i, B_{U_i}; G_{U_i}) < \frac{\epsilon}{\tilde{a} + \epsilon} \) and \(E_i\) attains \((\tilde{a} - \frac{\epsilon}{\tilde{a} + \epsilon} \delta)\)-lct \(u_i(U_i, B_{U_i}; G_{U_i})\). This contradicts to Claim 5.3.

**Step 7.** In this step we reduce to the case when \((X^i_{i,j}, B^{X^i_{i,j}})\) is \(\tilde{a}\)-log canonical near \(x^i_{i,j}\) for any \(i, j\).

**Claim 5.6.** Let \(\epsilon_0 = \frac{1}{\tilde{a} + \epsilon}\). Then there exists a subset \(T^{\epsilon_0} \subset S^{\epsilon_0}\) and index \(\lambda_i \in \Lambda(T^{\epsilon_0}_i)\) for each \(i\) satisfying the following.

1. \((X^{\epsilon_0}_{i,T}, B^{\epsilon_0}_{i,T})\) is not \(\tilde{a}\)-lc near \(x^{\epsilon_0}_{i,T}\), and
2. for any \(\mathcal{P}^{\epsilon_0}_i \supseteq T^{\epsilon_0}_i\), any index \(\lambda'_i \in \Lambda(\mathcal{P}^{\epsilon_0}_i)\), \((X^{\epsilon_0}_{i,P_i}, B^{\epsilon_0}_{i,P_i})\) is \(\tilde{a}\)-lc near \(x^{\epsilon_0}_{i,P_i}\).

**Proof of Claim 5.6.** For any \(i\), we define

\[ Q^{\epsilon_0}_i := \{ R^{\epsilon_0}_i \subset S^{\epsilon_0}_i | R^{\epsilon_0}_i \neq \emptyset, \text{ and there exists } \lambda \in \Lambda(R^{\epsilon_0}_i), \]

such that \((X^{\epsilon_0}_{i,R}, B^{\epsilon_0}_{i,R})\) is not \(\tilde{a}\)-lc near \(x^{\epsilon_0}_{i,R}\}.\]

First we show that \(S^{\epsilon_0}_i \notin Q^{\epsilon_0}_i\) except finitely many \(i\). Suppose not, let \(\lambda_{i,0} \in \Lambda(S^{\epsilon_0}_i)\) be the index such that \((X^{\epsilon_0}_{i,S}, B^{\epsilon_0}_{i,S})\) is not \(\tilde{a}\)-lc near \(x^{\epsilon_0}_{i,S}\).

Then since

\[ K_{X^{\epsilon_0}_{i,S} + B^{\epsilon_0}_{i,S} + G_{i,S}^{\epsilon_0}_{i,S}} + \sum_j n(i, j) \Gamma^{\epsilon}_{i,j, x^{\epsilon_0}_{i,S} + G_{i,S}^{\epsilon_0}_{i,S}} = (h^{\epsilon}_{i,S})^*(K_{X_i + B_i + G_i}) \]

we have

\[ (x^{\epsilon_0}_{i,S}, X^{\epsilon_0}_{i,S}, B^{\epsilon_0}_{i,S}, G_{i,S}^{\epsilon_0}_{i,S}) + \sum_j n(i, j) \Gamma^{\epsilon}_{i,j, x^{\epsilon_0}_{i,S} + G_{i,S}^{\epsilon_0}_{i,S}} \in D_i. \]

However, from our construction,

\[ S^{\epsilon_0}_{i,T}(x^{\epsilon_0}_{i,S}, X^{\epsilon_0}_{i,S}, B^{\epsilon_0}_{i,S}, G_{i,S}^{\epsilon_0}_{i,S}) + \sum_j n(i, j) \Gamma^{\epsilon}_{i,j, x^{\epsilon_0}_{i,S} + G_{i,S}^{\epsilon_0}_{i,S}} = \emptyset, \]

which contradicts to Claim 5.3.
Now we may pick $T_i^{c_0} \in \mathcal{Q}_i^{c_0}$ be an element satisfying the following: for any $R_i^{c_0} \in \mathcal{Q}_i^{c_0},$
\[ |R_i^{c_0}| \leq |T_i^{c_0}|,\]
i.e. $T_i^{c_0}$ has maximum cardinality among elements of $\mathcal{Q}_i^{c_0}$. From the definition of $\mathcal{Q}_i^{c_0}$, there exists $\lambda_i \in \Lambda(T_i^{c_0})$ such that $(X_i^{c_0\lambda_i}, B_i^{c_0\lambda_i})$ is not $\bar{a}$-lc near $x_i^{c_0\lambda_i}$, hence (1). For any $P_i^{c_0} \supset T_i^{c_0}$, $P_i^{c_0}$ has a strictly larger cardinality than the cardinality of any element of $\mathcal{Q}_i^{c_0}$, hence it is not contained in $\mathcal{Q}_i^{c_0}$, hence (2), and the proof is concluded. \hfill $\square$

We may pick $T_i^{c_0}$ and $\lambda_i$ as in Claim 5.6. Possibly replacing $X_i, B_i, C_i$ and $x_i$ with $X_i^{c_0\lambda_i}, B_i^{c_0\lambda_i}, C_i^{c_0\lambda_i} + \sum n(i,j)\Gamma_{i,j}^{c_0\lambda_i}$ (where the sum is taken for every $\Gamma_{i,j}^{c_0\lambda_i}$ whose center on $X_i^{c_0\lambda_i}$ is divisorial) and $x_{i,j}^{c_0\lambda_i}$ respectively, we reduce to the case when $(X_i^{c_0}, B_i^{c_0})$ is $\bar{a}$-log canonical near $x_i^{c_0}$ for any $i, j$.

In particular, since $S^c \subset S^{c_0}$ for any $\epsilon \leq \epsilon_0$, $(X_{i,j}^\epsilon, B_{X_{i,j}}^\epsilon)$ is $\bar{a}$-log canonical near $x_{i,j}^\epsilon$ for any $i, j$ and $\epsilon \leq \epsilon_0$.

Step 8. According to Step 7, from now on we may assume that $(X_i^\epsilon, B_i^\epsilon)$ is $\bar{a}$-lc near $x_i^\epsilon$ for every $i, j$ and $\epsilon < \epsilon_0$. We analyze the behaviors of the multiplicities of $\Gamma_{i,j}^\epsilon$ along $E_{i,j}^\epsilon$, i.e. the numbers $C_{i,j}^\epsilon$ and $D_{i,j,k}^\epsilon$ defined in Step 6. We also define a constant $s_1 > 0$ in this step.

First we study $C_{i,j}^\epsilon$. We show the following claim:

Claim 5.7. \hspace{1cm} (1) $C_{i,j}^\epsilon \neq 0$ for any $i, j$ and $0 < \epsilon \leq \epsilon_0$,
(2) $\bar{a}$-lct$_{x_{i,j}^\epsilon}(X_{i,j}^\epsilon, B_{i,j}^\epsilon; \Gamma_{i,j}^\epsilon, X_{i,j}^\epsilon) = 1 - \bar{a} - \gamma_{i,j}^\epsilon - \frac{\bar{a} - a_i}{C_{i,j}^\epsilon}$,
and
(3) \hspace{1cm} $\{\gamma_{i,j}^\epsilon + \frac{\bar{a} - a_i}{C_{i,j}^\epsilon}\}_{i,j,0<\epsilon<\epsilon_0}$ satisfies the DCC.

Proof of Claim 5.7. If $C_{i,j}^\epsilon = 0$ for some $i, j, \epsilon$, then
\[
a(E_i, X_{i,j}^\epsilon, B_{i,j}^\epsilon) = a(E_i, X_{i,j}^\epsilon, B_{i,j}^\epsilon + (1 - a(\Gamma_{i,j}^\epsilon, X_{i,j}^\epsilon, B_{i,j}^\epsilon))\Gamma_{i,j}^\epsilon) \\
= a(E_i, X_i, B_i) = a_i < \bar{a},
\]
hence $(X_{i,j}^\epsilon, B_{i,j}^\epsilon)$ is not $\bar{a}$-lc near $x_{i,j}^\epsilon$, contradicts to our assumption as of Step 7, hence we prove (1).

To prove (2), consider $\bar{a}$-lct$_{x_{i,j}^\epsilon}(X_{i,j}^\epsilon, B_{i,j}^\epsilon; \Gamma_{i,j}^\epsilon, X_{i,j}^\epsilon)$.

Since $E_i$ is the only prime $b$-divisor over $X_{i,j}^\epsilon$ whose center on $X_{i,j}^\epsilon$ is $\{\overline{x_{i,j}}\}$ and whose log discrepancy with respect to $(X_i^\epsilon, B_i^\epsilon + (1 - a(\Gamma_{i,j}^\epsilon, X_{i,j}^\epsilon, B_{i,j}^\epsilon))\Gamma_{i,j}^\epsilon)$
is \( \bar{a} \), \( \bar{a} \)-lc threshold holds,

\[ \{1 - \bar{a} - \gamma_{i,j}^\epsilon - \frac{\bar{a} - a_i}{C_{i,j}^\epsilon} \}_{i,j,0<\epsilon \leq \epsilon_0} \]

satisfies the ACC, hence

\[ \{ \gamma_{i,j}^\epsilon + \frac{\bar{a} - a_i}{C_{i,j}^\epsilon} \}_{i,j,0<\epsilon \leq \epsilon_0} \]

satisfies the DCC.

According to Claim 5.7, we may define the constant \( s_1 \) in the following way. It is clear that

\[ \{ \gamma_{i,j}^\epsilon + \frac{\bar{a} - a_i}{C_{i,j}^\epsilon} \}_{i,j,0<\epsilon \leq \epsilon_0} \subset (0, \infty), \]

thus since \( \{ \gamma_{i,j}^\epsilon + \frac{\bar{a} - a_i}{C_{i,j}^\epsilon} \}_{i,j,0<\epsilon \leq \epsilon_0} \) satisfies the DCC, there exists a real number \( s_1 > 0 \) such that for any \( i, j \) and \( 0 < \epsilon \leq \epsilon_0 \),

\[ 0 < s_1 < \gamma_{i,j}^\epsilon + \frac{\bar{a} - a_i}{C_{i,j}^\epsilon}. \]

Next we show the following relationship between \( C_{i,j}^\epsilon \) and \( D_{i,j,k}^\epsilon \).

**Claim 5.8.** For any \( i, j, k \) and \( 0 < \epsilon < 1 - \bar{a} \), \( D_{i,j,k}^\epsilon \leq C_{i,j}^\epsilon \).

**Proof.** For any \( i, j, k \) and \( 0 < \epsilon < 1 - \bar{a} \), we may run a \((-\Gamma_{i,j,W_{i,k}^\epsilon})\)-MMP over \( X_i \). According to the uniqueness of lc model and [Bir12, Theorem 3.4], this MMP terminates at \( X_{i,j}^\epsilon \). We now have a \( \Gamma_{i,j,W_{i,k}^\epsilon}^\epsilon \)-positive birational map \( C_{i,j,k}^\epsilon : W_{i,k}^\epsilon \to X_{i,j}^\epsilon \). In particular, we have

\[ D_{i,j,k}^\epsilon = \text{mult}_{E_i} \Gamma_{i,j,W_{i,k}^\epsilon}^\epsilon \leq \text{mult}_{E_i} \Gamma_{i,j,X_{i,j}^\epsilon}^\epsilon = C_{i,j}^\epsilon. \]

**Step 9.** The proof of the theorem immediately follows from the next claim.
Claim 5.9. Let \( \epsilon_1 := \frac{s_1}{3(1-a)} \). Then

\[
(W_{i,k}^{e_1}, B_{W_{i,k}^{e_1}} + (1 - \tilde{a} - \epsilon_1) \sum_j \Gamma_{i,j,W_{i,k}^{e_1}}^{\epsilon_1})
\]

is not \( \tilde{a} \)-lc near \( w_{i,k}^{e_1} \).

Proof. According to our assumption, for any \( i, k \) and any \( 0 < \epsilon < \epsilon_0 \),

\[
(W_{i,k}^{\epsilon}, B_{W_{i,k}^{\epsilon}} + (1 - \tilde{a}) \sum_j \Gamma_{i,j,W_{i,k}^{\epsilon}}^{\epsilon})
\]
is lc near \( w_{i,k}^{\epsilon} \). By Theorem 3.1, there exists at most \( \frac{d}{1-a} \) different integers \( j \), such that \( w_{i,k}^{\epsilon} \in \Gamma_{i,j,W_{i,k}^{\epsilon}}^{\epsilon} \).

We have

\[
a(E_i, W_{i,k}^{e_1}, B_{W_{i,k}^{e_1}} + (1 - \tilde{a} - \epsilon_1) \sum_j \Gamma_{i,j,W_{i,k}^{e_1}}^{\epsilon_1}) \\
\leq a(E_i, W_{i,k}^{e_1}, B_{W_{i,k}^{e_1}} + (1 - \tilde{a} - \gamma_{i,j} - \epsilon_1) \sum_j \Gamma_{i,j,W_{i,k}^{e_1}}^{\epsilon_1}) \\
= a(E_i, W_{i,k}^{e_1}, B_{W_{i,k}^{e_1}} + (1 - \tilde{a} - \gamma_{i,j} - \frac{s_1}{3(1-a)}) \sum_j \Gamma_{i,j,W_{i,k}^{e_1}}^{\epsilon_1}) \\
\leq a(E_i, W_{i,k}^{e_1}, B_{W_{i,k}^{e_1}} + (1 - \tilde{a} - \gamma_{i,j} - \frac{1}{1-a} (\frac{2}{3}s_1 - \epsilon_1)) \sum_j \Gamma_{i,j,W_{i,k}^{e_1}}^{\epsilon_1}) \\
\leq a(E_i, W_{i,k}^{e_1}, B_{W_{i,k}^{e_1}} + \sum_j (1 - \tilde{a} - \gamma_{i,j} - \frac{1}{1-a} (\frac{2}{3}s_1 - \gamma_{i,j})) \Gamma_{i,j,W_{i,k}^{e_1}}^{\epsilon_1}) \\
= a(E_i, W_{i,k}^{e_1}, B_{W_{i,k}^{e_1}} + \sum_j (1 - \tilde{a} - \gamma_{i,j}) \Gamma_{i,j,W_{i,k}^{e_1}}^{\epsilon_1}) + \sum_j \frac{1}{1-a} (\frac{2}{3}s_1 - \gamma_{i,j}) \Gamma_{i,j,W_{i,k}^{e_1}}^{\epsilon_1} \\
= a_i + \frac{1}{1-a} \sum_j (\frac{2}{3}s_1 - \gamma_{i,j}) \Gamma_{i,j,W_{i,k}^{e_1}}^{\epsilon_1} = a_i + \frac{1}{1-a} \sum_{j:w_{i,k}^{e_1} \in \Gamma_{i,j,W_{i,k}^{e_1}}^{\epsilon_1}} (\frac{2}{3}s_1 - \gamma_{i,j}) \Gamma_{i,j,W_{i,k}^{e_1}}^{\epsilon_1} \\
\leq a_i + \frac{1}{1-a} \sum_{j:w_{i,k}^{e_1} \in \Gamma_{i,j,W_{i,k}^{e_1}}^{\epsilon_1}} (\frac{2}{3}s_1 - \gamma_{i,j}) \Gamma_{i,j,W_{i,k}^{e_1}}^{\epsilon_1} < a_i + \frac{1}{1-a} \sum_{j:w_{i,k}^{e_1} \in \Gamma_{i,j,W_{i,k}^{e_1}}^{\epsilon_1}} (s_1 - \gamma_{i,j}) \Gamma_{i,j,W_{i,k}^{e_1}}^{\epsilon_1} \\
< a_i + \frac{1}{1-a} \sum_{j:w_{i,k}^{e_1} \in \Gamma_{i,j,W_{i,k}^{e_1}}^{\epsilon_1}} (\tilde{a} - a_i) \Gamma_{i,j,W_{i,k}^{e_1}}^{\epsilon_1} \\
= a_i + \frac{1}{1-a} \sum_{j:w_{i,k}^{e_1} \in \Gamma_{i,j,W_{i,k}^{e_1}}^{\epsilon_1}} (\tilde{a} - a_i) \leq a_i + (\tilde{a} - a_i) = \tilde{a}.
\]

and the proof is finished. \( \square \)
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For any $i, k$, since

$$K_{W_{i,k}} + (B_{W_{i,k}} + (1 - \bar{a} - \epsilon_1)) \sum_j \Gamma_{i,j,W_{i,k}}^{\epsilon_1}$$

$$+ (G_{W_{i,k}} + \sum_j (n(i,j)^{\epsilon_1} - (1 - \bar{a} - \epsilon_1)) \Gamma_{i,j,W_{i,k}}^{\epsilon_1})$$

$$= K_{W_{i,k}} + B_{W_{i,k}} + G_{W_{i,k}} + \sum_j n(i,j)^{\epsilon_1} \Gamma_{i,j,W_{i,k}}^{\epsilon_1}$$

$$= (g_{i,k}^{\epsilon_1})^\epsilon (K_X + B_i + G_i),$$

we have

$$(w_{i,k}^{\epsilon_1}, W_{i,k}^{\epsilon_1}, B_{W_{i,k}}^{\epsilon_1} + (1 - \bar{a} - \epsilon_1)) \sum_j \Gamma_{i,j,W_{i,k}}^{\epsilon_1},$$

$$G_{W_{i,k}} + \sum_j n(i,j)^{\epsilon_1} - (1 - \bar{a} - \epsilon_1)) \Gamma_{i,j,W_{i,k}}^{\epsilon_1}) \in D_{\mathcal{I}, \{1 - \bar{a} - \epsilon_1\}}.$$  

According to Claim 5.9, we have

$$(w_{i,k}^{\epsilon_1}, W_{i,k}^{\epsilon_1}, B_{W_{i,k}}^{\epsilon_1} + (1 - \bar{a} - \epsilon_1)) \sum_j \Gamma_{i,j,W_{i,k}}^{\epsilon_1},$$

$$G_{W_{i,k}} + \sum_j n(i,j)^{\epsilon_1} - (1 - \bar{a} - \epsilon_1)) \Gamma_{i,j,W_{i,k}}^{\epsilon_1}) = \emptyset,$$

which contradicts to Claim 5.5 and the proof of Theorem 5.2 is concluded. $\Box$

**Proof of Theorem 1.2.** The theorem follows from Theorem 5.2 and Proposition 5.1. $\Box$

**Proof of Theorem 1.3.** The theorem follows from Theorem 1.2 and Theorem 3.3. $\Box$

**Proof of Corollary 1.4.** The corollary follows from Theorem 1.3 by taking every $t \in [0, 1)$. $\Box$

**Proof of Corollary 1.5.** Suppose $E$ is the prime $b$-divisor such that $a(E, X, B) = \text{tmld}(X, B)$. Since $\text{tmld}(X, B) \leq 1$, if $E$ is exceptional over $X$, the corollary follows from Corollary 1.4. Otherwise, $\text{tmld}(X, B) \in \{1 - a \mid a \in \mathcal{I}\}$ which is an ACC set. $\Box$

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