Of Moments and Matching: A Game-Theoretic Framework for Closing the Imitation Gap

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Abstract

We provide a unifying view of a large family of previous imitation learning algorithms through the lens of moment matching. At its core, our classification scheme is based on whether the learner attempts to match (1) reward or (2) action-value moments of the expert’s behavior, with each option leading to differing algorithmic approaches. By considering adversarially chosen divergences between learner and expert behavior, we are able to derive bounds on policy performance that apply for all algorithms in each of these classes, the first to our knowledge. We also introduce the notion of moment recoverability, implicit in many previous analyses of imitation learning, which allows us to cleanly delineate how well each algorithmic family is able to mitigate compounding errors. We derive three novel algorithm templates (AdvIL, AdRIL, and DaeQuIL) with strong guarantees, simple implementation, and competitive empirical performance.

1. Introduction

When formulated as a statistical learning problem, imitation learning is fundamentally concerned with finding policies that minimize some notion of divergence between learner behavior and that of an expert demonstrator. Existing work has explored various types of divergences including KL (Pomerleau 1989, Bojarski et al. 2016), Jensen-Shannon (Rhinehart et al. 2018), Reverse KL (Kostrikov et al. 2019), f (Ke et al. 2019), and Wasserstein (Dadashi et al. 2020).

At heart, though, we care about the performance of the learned policy under an objective function that is not known to the learner. As argued by (Abbeel and Ng 2004) and (Ziebart et al. 2008), this goal is most cleanly formulated as a problem of moment matching, or, equivalently, optimizing Integral Probability Metrics (Sun et al. 2019) (IPMs). This is because a learner that in expectation matches the expert on all the basis functions of a class that includes the expert’s objective function, or matches moments, must achieve the same performance and will thus be indistinguishable in terms of quality. Additionally, in sharp contrast to recently proposed approaches (Ke et al. 2019, Kostrikov et al. 2019, Jarrett et al. 2020, Rhinehart et al. 2018), moments, due to their simple forms as expectations of basis functions, can be effectively estimated via demonstrator samples and the uncertainty in these estimates can often be quantified to regularize the matching objective (Dudik et al. 2004). In short: these moment matching procedures are simple, effective, and provide the strongest policy performance guarantees we are aware of for imitation learning.

As illustrated in Fig. 1, there are three classes of moments a learner can focus on matching: (a) on-policy reward moments, (b) off-policy Q-value moments, and (c) on-policy Q-value moments, each of which have different requirements on the environment and on the expert. We abbreviate them as reward, off-Q, and on-Q moments, respectively.

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Our key insight is that reward moments have more discriminative power because they can pick up on differences in induced state visitation distributions rather than just action conditionals. Thus, reward moment matching is a harder problem with stronger guarantees than off-Q and on-Q moment matching.

Our work makes the following three contributions:

1. **We present a unifying framework for moment matching in imitation learning.** Our framework captures a wide range of prior approaches and allows us to construct, to our knowledge, the first formal lower bounds demonstrating that the choice between matching “reward”, “off-Q”, or “on-Q” moments is fundamental to the problem of imitation learning rather than an artifact of a particular algorithm or analysis.

2. **We clarify the dependence of imitation learning bounds on problem structure.** We introduce a joint property of an expert policy and moment class, moment recoverability, that helps us characterize the problems for which compounding errors are likely to occur, regardless of the kind of feedback the learner is exposed to.

3. **We provide three novel algorithms with strong performance guarantees.** We derive idealized algorithms that match each class of moments. We also provide practical instantiations of these ideas, AdVIL, AdRIL, and DAEQuIL. These algorithms have significantly different practical performance as well as theoretical guarantees in terms of compounding of errors over time steps of the problem.

**2. Related Work**

Imitation Learning. Imitation learning has been shown to be an effective method of solving a variety of problems, from getting cars to drive themselves (Pomerleau 1989) to achieving superhuman performance in games like Go (Silver et al. 2016; Sun et al. 2018), to sample-efficient learning of control policies for high DoF robots (Levine and Koltun 2013), to allowing human operators to effectively supervise and teach robot fleets (Swamy et al. 2020). The issue of compounding errors in imitation learning was first formalized by (Ross et al. 2011), with the authors proving that an interactive expert that can suggest actions in states generated via learner policy rollouts will be able to teach the learner to recover from mistakes.

Adversarial Imitation Learning. Starting with the seminal work of (Ho and Ermon 2016), numerous proposed approaches have framed imitation learning as a game between a learner’s policy and another network that attempts to discriminate between learner rollouts and expert demonstrations (Fu et al. 2018; Song et al. 2018). We build upon this work by elucidating the properties that result from the kind of feedback the learner is exposed to – whether they are able to see the consequences of their own actions via rollouts or if they are only able to propose actions in states from expert trajectories. Our proposed approaches also have stronger guarantees and less brittle performance than the popular GAIL (Ho and Ermon 2016).

**Mathematical Tools.** Our algorithmic approach combines two tools that have enjoyed success in imitation learning: functional gradients (Ratliff et al. 2009) and the Integral Probability Metric (Sun et al. 2019). We define two algorithms, AdRIL and AdVIL that are based on optimizing the value-directed IPM, with AdRIL having the discriminative player perform updates via functional gradient descent. The IPM is linear in the discriminative function, unlike other proposed metrics like the Donsker-Varadhan bound on KL divergence. Specifically, the Donsker-Varadhan bound includes an expectation of the exponentiated discriminative function, which makes estimation difficult with a few samples (McAllester and Stratos 2020). Our analysis makes repeated use of the **Performance Difference Lemma** (Kakade and Langford 2002; Bagnell et al. 2003) or PDL, which allows us to bound the suboptimality of the learner’s policy.

Our proposed algorithms bear some resemblance to previously proposed methods, with AdRIL resembling SQIL (Reddy et al. 2019) and AdVIL resembling ValueDICE (Kostrikov et al. 2019). We note that AdVIL, while cleanly derived from the PDL, can also be derived from an IPM by using a telescoping substitution similar to the ValueDICE derivation. Notably, because AdVIL is linear in the discriminator, it does not suffer from ValueDICE’s difficulty in estimating the expectation of an exponential. This difficulty might help explain why ValueDICE can underperform the behavioral cloning baseline on several benchmark tasks (Jarrett et al. 2020). Similarly, AdRIL can avoid the sharp degradation in policy performance that SQIL demonstrates (Barde et al. 2020). This is because SQIL hard-codes the discriminator while AdRIL adaptively updates the discriminator to account for changes in the policy’s trajectory distribution. DAEQuIL can be seen as the natural extension of DAgger (Ross et al. 2011) to the adversarial loss setting.

**3. Moment Matching Imitation Learning**

We begin by formalizing our setup and objective.

**3.1. Problem Definition**

Let $\Delta(\mathcal{X})$ denote the space of all probability distributions over a set $\mathcal{X}$. Consider an MDP parameterized by $(S, A, T, r, T_0)$, where $S$ is the state space, $A$ is the action space, $T : S \times A \rightarrow \Delta(S)$ is the transition operator, $r : S \times A \rightarrow [-1, 1]$ is a reward function, $T$ is the hori-
We assume all three function classes are closed under negatively.

Definition 2. A pair \( (\pi_E, F_Q) \) of an expert policy and set of expert Q-functions is said to be \( H \)-recoverable if \( s \in S \), \( \forall a \in A \), \( \forall f \in F_Q \), \( |f(s, a) - E_{a \sim \pi_E(s)}[f(s, a)]| < H \).

\( H \) is an upper bound on all possible advantages that could be obtained by the expert under a Q-function in \( F_Q \). Intuitively, \( H \) tells us how many time-steps it takes the expert to recover from an arbitrary mistake. We defer a more in-depth discussion of the implications of this concept to Section 4.5.

**Reward.** Matching reward moments entails minimizing the following expansion of the imitation gap:

\[
J(\pi_E) - J(\pi) = \mathbb{E}_{\tau \sim \pi} \sum_{t=1}^{T} r(s_t, a_t) - \mathbb{E}_{\tau \sim \pi} \sum_{t=1}^{T} r(s_t, a_t)
\]

\[
= \mathbb{E}_{\tau \sim \pi} \sum_{t=1}^{T} -r(s_t, a_t) - \mathbb{E}_{\tau \sim \pi} \sum_{t=1}^{T} -r(s_t, a_t)
\]

\[
\leq \sup_{f \in F_Q} \mathbb{E}_{\tau \sim \pi} \sum_{t=1}^{T} f(s_t, a_t) - \mathbb{E}_{\tau \sim \pi} \sum_{t=1}^{T} f(s_t, a_t)
\]

| Moment | Class | Env. | \( \pi_E \) Queries |
|--------|-------|------|-------------------|
| Reward | \( F_r : [-1, 1] \) | ✓ | ✗ |
| Off-Policy Q | \( F_Q : [-T, T] \) | ✗ | ✗ |
| On-Policy Q | \( F_Q^E : [-Q, Q] \) | ✓ | ✓ |

Table 1. An overview of the requirements for three classes of moment matching.

In the last step, we use the fact that \(-r(s, a) \in F_r\). Crucially, reward moment-matching demands on-policy rollouts \( \tau \sim \pi \) for the learner to calculate per-time step divergences.

Instead of matching moments of the reward function, we can consider matching moments of the action-value function. We can apply the Performance Difference Lemma (PDL) to expand the imitation gap \( J(\pi_E) \) into either on-policy or off-policy expressions.

**Off-Policy Q.** Starting from the PDL:

\[
J(\pi_E) - J(\pi) = \mathbb{E}_{\tau \sim \pi} \sum_{t=1}^{T} Q^r_t(s_t, a_t) - \mathbb{E}_{a \sim \pi_E(s_t)}[Q^r_t(s_t, a)]
\]

\[
\leq \sup_{f \in F_Q} \mathbb{E}_{\tau \sim \pi} \sum_{t=1}^{T} f(s_t, a_t) - \mathbb{E}_{a \sim \pi_E(s_t)}[f(s_t, a)]
\]

In the last step, we use the fact that \( Q^r_t(s, a) \in F_Q \) for all \( s \in \Pi \) and \( r \in F_r \). The above expression is off-policy – it only requires a collected dataset of expert trajectories to be evaluated and minimized. In general though, \( F_Q \) can be a far more complex class than \( F_r \) because it has to capture both the dynamics of the MDP and the choices of any policy.

**On-Policy Q.** Expanding in the reverse direction:

\[
J(\pi_E) - J(\pi) = -\mathbb{E}_{\tau \sim \pi} \sum_{t=1}^{T} Q^\pi_E(s_t, a_t) - \mathbb{E}_{a \sim \pi_E(s_t)}[Q^E_t(s_t, a)]
\]

\[
\leq \sup_{f \in F_Q^E} \mathbb{E}_{\tau \sim \pi} \sum_{t=1}^{T} f(s_t, a_t) - \mathbb{E}_{a \sim \pi_E(s_t)}[f(s_t, a)]
\]

In the last step, we use the fact that \( Q^\pi_E(s, a) \in F_Q^E \) for all \( r \in F_r \). In the realizable setting, \( \pi_E \in \Pi, F_Q^E \subseteq F_Q \). While \( F_Q^E \) is a smaller class, to actually evaluate this expression, we require an interactive expert that can tell us what action they would take in any state visited by the learner as well as on-policy samples from the learner’s current policy \( \tau \sim \pi \).

With this taxonomy in mind, we now turn our attention to deriving policy performance bounds.\(^2\)

\(^2\)See Sup \( \mathbb{E} \) for mixed moments and an alternative Q-moment scheme that can be extended to the IL from observation alone setting.
3.3. Moment Matching Games

A unifying perspective on the three moment matching variants can be achieved by viewing the learner as solving a game. More specifically, we consider variants of a two-player minimax game between a learner and a discriminator. The learner selects a policy \( \pi \in \Pi \), where \( \Pi \triangleq \{ \pi : \mathcal{S} \to \Delta(A) \} \). We assume \( \Pi \) is convex, compact and that \( \piE \in \Pi \). \(^3\) The discriminator (adversarially) selects a function \( f \in \mathcal{F} \), where \( \mathcal{F} \triangleq \{ f : \mathcal{S} \times \mathcal{A} \to \mathbb{R} \} \). We assume that \( \mathcal{F} \) is convex, compact, closed under negation, and finite dimensional. \(^4\) Depending on the class of moments being matched, we assume that \( \mathcal{F} \) is spanned by convex combinations of the elements of \( \mathcal{F}_{r} / 2, \mathcal{F}_{Q} / 2T \), or \( \mathcal{F}_{Q,B} / 2T \). Lastly, we set the learner as the minimization player and the discriminator as the maximization player.

**Definition 3.** The on-policy reward, off-policy Q, and on-policy Q payoff functions are, respectively:

\[
U_1(\pi, f) = \frac{1}{T} \left( \mathbb{E}_{\tau \sim \pi} \sum_{t=1}^{T} f(s_t, a_t) \right) - \mathbb{E}_{\tau \sim \piE} \left[ \sum_{t=1}^{T} f(s_t, a_t) \right]
\]

\[
U_2(\pi, f) = \frac{1}{T} \left( \mathbb{E}_{a \sim \piE} \sum_{t=1}^{T} f(s_t, a) \right) - \mathbb{E}_{a \sim \piE} \left[ \sum_{t=1}^{T} f(s_t, a) \right]
\]

\[
U_3(\pi, f) = \frac{1}{T} \left( \mathbb{E}_{a \sim \piE} \sum_{t=1}^{T} f(s_t, a_t) \right) - \mathbb{E}_{a \sim \piE} \left[ \sum_{t=1}^{T} f(s_t, a) \right]
\]

When optimizing over the policy class \( \Pi \) which contains \( \piE \), we have a minimax value of 0: for \( j \in \{1, 2, 3\} \),

\[
\min_{\pi \in \Pi} \max_{f \in \mathcal{F}} U_j(\pi, f) = 0
\]

Furthermore, for certain representations of the policy\(^5\) strong duality holds: for \( j \in \{1, 2, 3\} \),

\[
\min_{\pi \in \Pi} \max_{f \in \mathcal{F}} U_j(\pi, f) = \max_{f \in \mathcal{F}} \min_{\pi \in \Pi} U_j(\pi, f)
\]

We now study the properties that result from achieving an approximate equilibrium for each imitation game.

4. From Approximate Equilibria to Bounded Regret

A learner computing an equilibrium policy for any of the moment matching games will be imperfect due to many sources including restricted policy class, optimization error, or imperfect estimation of expert moments. More formally, in a game with payoff \( U_j \), a pair \( (\hat{\pi}, f) \) is a \( \delta\)-approximate equilibrium solution if the following holds:

\[
\sup_{f \in \mathcal{F}} U_j(f, \hat{\pi}) - \delta \leq U_j(f, \hat{\pi}) \leq \inf_{\pi \in \Pi} U_j(f, \hat{\pi}) + \frac{\delta}{2}
\]

We assume access to an algorithmic primitive capable of finding such strategies:

**Definition 4.** An imitation game \( \delta\)-oracle \( \Psi(\delta) \) takes payoff function \( U : \Pi \times \mathcal{F} \to [-k, k] \) and returns a \( (k\delta)\)-approximate equilibrium strategy for the policy player.

We now bound the imitation gap of solutions returned by such an oracle.

### 4.1. Example MDPs

For use in our analysis, we first introduce two MDPs, LOOP and CLIFF. As seen in Fig. 2 LOOP is an MDP where a learner can enter a state where it has seen no expert demonstrations \( (s_2) \) and make errors for the rest of the horizon. CLIFF is an MDP where a single mistake can result in the learner being stuck in an absorbing state.

\[
\text{Figure 2. Left: Borrowed from [Ross et al. 2011], the goal of LOOP is to spend time in } s_1. \text{ Right: a folklore MDP CLIFF, where the goal is to not "fall off the cliff" and end up in } s_2 \text{ evermore.}
\]

### 4.2. Reward Moment Performance Bounds

Let us first consider the reward moment-matching game.

**Lemma 1. Reward Upper Bound:** If \( \mathcal{F}_r \) spans \( \mathcal{F} \), then for all MDPs, \( \piE \), and \( \pi \leftarrow \Psi(\epsilon)(U_1) \), \( J(\piE) - J(\pi) \leq O(\epsilon T) \).
We contrast the preceding guarantees with those based on proof. The first line follows from the closure of all MDPs, unlike those presented in (Rajaraman et al. 2016) for all game $J$ and this would lead to an average cost of $\epsilon T$ for that timestep and $\epsilon$ on average. However, on rollouts, the learner would have an $\epsilon T$ chance of paying a cost of 1 for the rest of the horizon, leading to a lower bound of $J(\pi_\epsilon) - J(\pi) = \epsilon T^2 \geq \Omega(\epsilon T^2)$.

These bounds apply for all algorithms that can be written in the form of an off-$Q$ imitation game, including behavioral cloning (Pomerleau 1989) and ValueDICE (Kostrikov et al. 2019).

4.4. On-$Q$ Moment Performance Bounds

We now derive performance bounds for on-$Q$ algorithms with interactive experts.

**Lemma 5. On-$Q$ Upper Bound:** If $F_{Q_E}$ spans $F$, then for all MDPs with $H$-recoverable ($F_{Q_E}, \pi_E$), and $\pi \leftarrow \Psi_\epsilon(\pi(U_3), J(\pi_\epsilon) - J(\pi) \leq O(\epsilon HT)$.

**Proof.** Starting from the PDL:

$$J(\pi_\epsilon) - J(\pi) \leq \sup_{f \in F_{Q_E}} \sum_{t=1}^{T} \mathbb{E}_{s_t \sim \pi} f(s_t, a_t) - \mathbb{E}_{s_t \sim \pi} f(s_t, a_t)$$

$$\leq \sup_{f \in F_{Q_E}} \sum_{t=1}^{T} \mathbb{E}_{s_t \sim \pi} [2T f(s_t, a_t)] - \mathbb{E}_{s_t \sim \pi} [2T f(s_t, a_t)]$$

$$= 2T^2 \sup_{f \in F} U_3(\pi, f) \leq (H/2T)^2 T^2 = \epsilon HT$$

As before, the $T$ in the second to last line comes from the fact that $F_{Q_E}/2T \subseteq F$. The $H/2T$ factor comes from the scale of the payoff. Thus, a policy $\pi$ returned by $\Psi_\epsilon(\pi(U_3))$ must satisfy $J(\pi_\epsilon) - J(\pi) \leq O(\epsilon HT)$ – that is, it can do up to $O(\epsilon HT)$ worse than the expert.

**Lemma 6. On-$Q$ Lower Bound:** There exists an MDP, $\pi_E$, and $\pi \leftarrow \Psi_\epsilon(\pi(U_3))$ such that $J(\pi_\epsilon) - J(\pi) \geq \Omega(\epsilon T^2)$.

**Proof.** The proof of the reward lower bound holds verbatim because every policy, including the previously considered $\pi_\epsilon$, will be stuck in $s_x$ if it falls in.
4.5. Recoverability in Imitation Learning

The bounds we presented above beg the question of when on-Q moment matching has error properties similar to those of reward moment matching versus those of off-Q moment matching. Recoverability allows us to cleanly answer this question and others. We begin by providing more intuition for said concept.

Concretely, in Fig. 2, LOOP is 1-recoverable for the expert policy that always moves towards s1. CL1FF is not H-recoverable for any H < T if the expert never ends up in s2. A sufficient condition for H-recoverability is that the state occupancy distribution that results from taking an arbitrary action and then H − 1 actions according to πE is the same as that of taking H actions according to πE. We emphasize that recoverability is a property of the set of moments matched and the expert, not just of the expert, as has been previously considered [Pan et al. 2019].

**Bound Clarification.** Our previously derived upper bound for on-Q moment matching \( J(\pi_E) - J(\pi) \leq O(\kappa HT) \) tells us that for O(1)-recoverable MDPs, on-Q moment matching behaves like reward moment matching while for O(T)-recoverable MDPs, it instead behaves like off-Q moment matching and has an \( O(\kappa T^2) \) upper bound. Thus, O(1)-recoverability is in a certain sense necessary for achieving O(\( \epsilon T \)) error with on-Q moment matching.

Another perspective on recoverability is that it helps us delineate problems where compounding errors are hard to avoid for both on-Q and reward moment matching. Let \( I(s) = \sum_{a' \in A} |E_{a' \sim \pi_E(s)} [I_{\pi}(a)] - E_{a \sim \pi(s)} [I_{\pi}(a)] | \) be the classification error of a state \( s \). We prove the following lemma in Supplementary Material A.1.

**Lemma 7.** Let \( \kappa > 0 \). There exists a \( (\pi_E, \{r, -r\}) \) pair that for any \( H < T \) is not H-recoverable such that \( I(s) = \kappa \) on any state leads to \( J(\pi_E) - J(\pi) \geq \Omega(\kappa T^2) \).

Because some states might not appear on expert rollouts, evaluating \( I(s) \) can require an interactive expert. However, even with this strong form of feedback, a classification error of \( \kappa \) on a single state can lead to \( \Omega(\kappa T^2) \) imitation gap for O(T)-recoverable MDPs. This lemma also implies that in such an MDP, achieving \( O(\kappa T) \) imitation gap via on-policy moment matching would require the learner to have a classification error \( \propto \kappa / T \), or to make vanishingly rare errors as we increase the horizon. We note that this does not conflict with our previously stated bounds but reveals that achieving a moment-matching error of (time-independent) \( \epsilon \) might require achieving a classification error that scales inversely with time for O(T)-recoverable MDPs. Practically, this can be rather challenging. Thus, neither on-Q nor reward moment matching is a silver bullet for getting \( O(\epsilon T) \) error for O(T)-recoverable problems.

5. Finding Equilibria: Idealized Algorithms

We now provide reduction-based methods for computing (approximate) equilibria for these moment matching games which can be seen as blueprints for constructing our previously described oracle (Def. 5). We study, in particular, finite state problems and a complete policy class. We analyze an approach to equilibria finding where an outer player follows a no-regret strategy and the inner player follow a (modified) best response strategy, by which we can efficiently find policies with strong performance guarantees.

5.1. Preliminaries

An efficient no-regret algorithm over a class \( \mathcal{X} \) produces iterates \( x^1, \ldots, x^H \in \mathcal{X} \) that satisfy the following property for any sequence of loss functions \( l^1, \ldots, l^H \):

\[
\text{Regret}(H) = \sum_{t=1}^{H} l^t(x^t) - \min_{x \in \mathcal{X}} \sum_{t=1}^{H} l^t(x) \leq \beta_\mathcal{X}(H)
\]

where \( \beta_\mathcal{X}(H)/H \leq \epsilon \) holds for \( H \) that are \( O(\text{poly}(\frac{1}{\epsilon})) \).

5.2. Theoretical Guarantees

We are interested in obtaining a policy efficiently that is a near-equilibrium solution to the game. We consider two general strategies to do so:

**Primal.** We execute a no-regret algorithm on the policy representation, while a maximization oracle over the space \( \mathcal{F} \) computes the best response to those policies.

**Dual.** We execute a no-regret algorithm on the space \( \mathcal{F} \), while a minimization oracle over policies computes entropy regularized best response policies.

The asymmetry in the above is driven by the need to recover the equilibrium strategy for the policy player and the fact that a dual approach on the original, unregularized objective \( U \) will typical not converge to a single policy but rather shift rapidly as \( f \) changes.

By choosing the the policy representation to be a causally conditioned probability distribution over actions, \( P(A^T \mid S^T) = \prod_{t=1}^{T} P(A_t \mid S_{1:t}, A_{1:t-1}) \), we find each of the imitation games is bilinear in both policy and discriminator \( f \) and strongly dual [6]. Thus, we can efficiently compute a near-equilibrium assuming access to the optimization oracles in either primal or dual above:

\[\text{The average over iterations of the policies generated in an unregularized dual will also be near-equilibrium but can be in-convenient. Entropy regularization provides a convenient way to extract a single policy and meshes well with empirical practice.} \]

\[\text{Following Ziebart et al. 2010, we can represent the policy as an element of the causal conditioned polytope and regularize with the causal entropy } H(P(A^T \mid S^T)) \text{ to denote the causal Shannon entropy of a policy. An equivalent result can be proved for optimizing over occupancy measures (Ho and Ermon 2016).} \]
We can transform our IPM-based objective (2) into an expression only over \((s, a)\) pairs from expert data to fit into the off-\(Q\) moment matching framework by performing a series of substitutions. We refer interested readers to Supplementary Material B.1. We arrive at the following expression:

\[
\sup_{v \in \mathcal{F}} \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{E}_{a \sim \pi(s_t)} \left[ v(s_t, a_t) - v(s_{t-1}, a_{t-1}) \right] \right]
\]

Intuitively, by minimizing (3) over policies \(\pi \in \Pi\), one is driving down the extra cost the learner has over the expert.

We term this approach Adversarial Value-moment Imitation Learning (AdVIL) because if we view \(\pi\) as optimizing cumulative reward, \(-v\) could be viewed as a value function. We set the learning rate for \(f\) to be greater than that for \(\pi\), making AdVIL a primal algorithm.

Practically, AdVIL bears similarity to the Wasserstein GAN (WGAN) framework [Arjovsky et al. 2017], though we consider an IPM rather than the more restricted Wasserstein distances. However, the overlap is enough that techniques from the WGAN literature including gradient penalties on the discriminator [Gulrajani et al. 2017] and orthogonal regularization of the policy player [Brock et al. 2018] help.

6.2. AdRIL: Adversarial Reward-moment Imitation Learning

We now present our dual reward moment matching algorithm and refer interested readers to Supplementary Material B.2 for the full derivation. In brief, we solve for the discriminator in closed form via functional gradient descent in Reproducing Kernel Hilbert Space (RKHS) and have the policy player compute a best response via maximum entropy reinforcement learning. Let \(\mathbb{D}_E\) denote the aggregated dataset of policy rollouts. Assuming a constant number of training steps at each iteration and averaging functional gradients over \(k\) iterations of the algorithm, we get the cost function for the policy and round \(k\):

\[
\sum_{t=1}^{T} \frac{1}{\mathbb{D}_E} \sum_{\tau} K([s_t, a_t], \cdot) - \frac{1}{\mathbb{D}_E} \sum_{\tau} K([s_{t-1}, a_{t-1}], \cdot)
\]

For an indicator kernel and under the assumption we never see the same state twice, this is equivalent to maximizing a reward function that is 1 at each expert datapoint.
Our proposed methods (in orange) are able to match or out-perform the baselines across a variety of continuous control tasks. \( J(\pi) \) s are averaged across 10 evaluation episodes. Standard errors are across 5 runs of each algorithm.

\[ \frac{1}{t} \] along previous rollouts that do not perfectly match the expert, and 0 everywhere else. We term this approach Adversarial Reward-moment Imitation Learning (AdRIL).

We note that under these assumption, our objective resembles that of SQIL (Reddy et al. 2019). SQIL can be seen as a degenerate case of AdRIL that never updates the discriminator function. This oddity removes solution quality guarantees while introducing the need for early stopping (Arean and Neumann 2020).

### 6.3. DAeQuIL: DAGger-esque Qu-moment Imitation Learning

We present the natural extension of DAgger (Ross et al. 2011) to the space of moments: DAeQuIL (DAGger-esque Qu-moment Imitation Learning) in Algorithm 3. Like DAgger, one can view DAeQuIL as a primal algorithm that uses the Follow the Regularized Leader as the no-regret algorithm for the policy player. Per-round losses are adversarially chosen. It is a subtle point but as written, DAeQuIL is technically not solving the on-Q game directly because it optimizes over a history of learner state distributions instead of the current learner’s state distribution. However, it retains strong performance guarantees – see Sup. B.3 for more.

**Algorithm 3 DAeQuIL**

**Input:** Queryable expert \( \pi_E \), Policy class \( \Pi \), Discriminator class \( \mathcal{F} \), Performance threshold \( \delta \). Behavioral cloning loss \( \ell_{BC} : \Pi \to \mathbb{R} \), Strongly convex fn \( R : \Pi \to \mathbb{R} \)

**Output:** Trained policy \( \pi \)

Optimize: \( \pi \leftarrow \arg\min_{\pi' \in \Pi} \ell_{BC}(\pi') \).

Set \( L(\pi) = 2\delta, D = [], F = [], t = 1 \)

while \( L(\pi) > \delta \) do

Rollout \( \pi \) to generate \( \mathcal{D}_\pi \leftarrow \{(s, a), \ldots\} \), \( \text{Relabel } \mathcal{D}_\pi \text{ to } \mathcal{D}_E \leftarrow \{(s, a')| a' \sim \pi_E(s), \forall s \in \mathcal{D}_\pi \} \)

\( L(f) = \mathbb{E}_{(s, a) \sim \mathcal{D}_\pi}[f(s, a)] - \mathbb{E}_{(s, a) \sim \mathcal{D}_E}[f(s, a)] \)

Append: \( F \leftarrow F \cup \arg\max_{f \in \mathcal{F}} L(f) \).

Append: \( D \leftarrow D \cup \{(s, t) | \forall s \in \mathcal{D}_\pi \} \).

\( L(\pi) = \mathbb{E}_{(s, t) \in D}[\ell(f)(s, \pi(s))] + \ell_{BC}(\pi) + R(\pi) \)

Optimize: \( \pi \leftarrow \arg\min_{\pi' \in \Pi} L(\pi') \).

\( t \leftarrow t + 1 \)
end while

\( \frac{1}{t} \) for the policy player. Per-round losses are adversarially chosen though. It is a subtle point but as written, DAeQuIL is technically not solving the on-Q game directly because it optimizes over a history of learner state distributions instead of the current learner’s state distribution. However, it retains strong performance guarantees – see Sup. B.3 for more.

**7. Experiments**

**Figure 4. Left:** The expert demonstrates many feasible trajectories, causing a learner that attempts to just match the mean action to crash directly into the tree the expert was avoiding. **Center:** On-policy corrections do not help DAgger as it still tries to match the mean action and crashes into the first tree it encounters. **Right:** DAeQuIL, when run with moments that allow the learner to focus on swerving out of the way of trees, regardless of direction, is able to produce policies that successfully navigate through the forest.

We test our algorithms against several baselines on several higher-dimensional continuous control tasks from the PyBullet suite (Coumans and Bai 2016–2019). We measure the performance of off-Q algorithms as a function of the amount of data provided with a fixed maximum computational budget and of reward moment-matching algorithms as a function of the amount of environment interactions. We see from Fig. 3 that AdVIL can match the performance of ValueDICE and Behavioral Cloning across most tasks. AdRIL performs better than GAIL across all environments and does not exhibit the catastrophic collapse in performance SQLI does on the tested environments. On both environments, behavioral cloning is able to recover a strong policy with far less data than it takes Behavioral Cloning to, showcasing the potential benefit of the learner observing the consequences of its own actions. We refer readers to Sup. C for a description of our hyparameters and setup. Notably, AdVIL is able to converge reliably to a policy of the same quality as that found by ValueDICE with an order of magnitude less compute. As seen in Fig. 4, DAeQuIL is able to significantly out-perform DAgger on a toy UAV navigation task – see Sup. D for full information. We release our code at https://github.com/gkswamy98/pillbox.
8. Discussion

8.1. A Unifying View of Moment Matching IL

We present a cohesive perspective of moment matching in imitation learning in Table 3. We note that reward moment-matching dual algorithms have been a repeated success story in imitation learning but that there has been comparatively less work done in off-Q and on-Q dual algorithms.

| MOMENT | PRIMAL | DUAL |
|--------|--------|------|
| OFF-Q  | VDICE, AdVIL | ×    |
| REWARD | GAIL, MMP, LEARCH, MAXENT IOC, SQIL, AdRIL, A+N |     |
| ON-Q   | DAGGER, GPS iFAIL, DAEQUIL | ×    |

Table 3. An taxonomy of moment matching algorithms. Bold text indicates algorithms that are IPM-based.

8.2. The Hidden Cost of Reward Moment Matching

At first glance, the reward moment matching bound might seem to good to be true – reward matching algorithms don’t require a queryable expert like on-Q approaches yet their performance bound seems to be tighter. This better performance characteristic is a product of a potentially exponentially harder optimization problem for the learner. Consider the following tree-structured MDP with |A| actions at each step, each of which lead to a distinct state. Consider an expert that takes \(a_{|A|}\) at each timestep. Solving the reward matching problem requires the learner to simultaneously optimize over all T timesteps of the problem while considering the effect of past actions on future states. If we set \(T\) to be the class of deterministic policies, this is equivalent to optimizing over the set of all length \(T\) trajectories, of which there are \(O(|A|^T)\) in tree-structured problems. In contrast, for off-Q approaches, we attempt to match expert moments on a fixed expert state distribution. Similarly, we can optimize over a fixed history of past learner state distributions under a weak realizability assumption in the on-Q setting. In the preceding example, the Q-matching settings are like being handed a node at each level of the tree and being asked to choose between the \(|A|\) edges available, leading to a total of \(O(|A|^T)\) options to choose between. As we saw in Sec. 5, the policy player sometimes needs to compute a best response over the entire set of choices it has available, which means these search space sizes directly affect the per-iteration complexity of the moment-matching algorithms. Concisely, the price we pay for solving an easier optimization problem is looser policy performance bounds.

8.3. Takeaways

In this work, we tease apart the differences in requirements and performance guarantees that come from matching reward, on-Q, and off-Q adversarially chosen moments. Reward moment matching has strong guarantees but requires access to an accurate simulator or the real world. Off-Q moment matching can be done purely on collected data but incurs an \(O(\epsilon T^2)\) imitation gap.

We formalize a notion of recoverability that is both necessary and sufficient to understand recovering from errors in imitation learning. If a problem (due to expert or the MDP itself) is \(O(T)\)-recoverable, there exist problems where no algorithm can escape an \(O(T^2)\) compounding of errors; if it is \(O(1)\)-recoverable, we find on policy algorithms prevent compounding. Together, these constitute a cohesive picture of moment matching in imitation learning.

We derive idealized no-regret procedures and practical IPM-based algorithms that are conceptually elegant and correct for difficulties encountered by prior methods. While behavioral cloning equally weights action-conditional errors, AdVIL can prevent headaches with value moment-based weighting. AdRIL is simple to implement, does not require training a GAN, and enjoys strong performance both in theory and practice. DAEQuIL’s moment-based losses are able to help relieve hiccups from focusing on action-conditional that can stymie DAGGER.

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“As happens sometimes, a moment settled and hovered and remained for much more than a moment.” – John Steinbeck, Of Mice and Men.
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Supplementary Material

A. Proofs

A.1. Proof of Lemma \[7\]

Proof. Consider the following MDP:

![Diagram of MDP]

As illustrated above, \( c(s, a) = 1_{s=2} \) is the cost function for this MDP. Let the expert perfectly optimize this function by always taking \( a_1 \) in \( s_1 \). Thus, we are in the \( O(T) \)-recoverable setting. Then, for any \( \epsilon > 0 \), if the learner takes \( a_2 \) in \( s_1 \) with probability \( \epsilon \), \( J(\pi_E) - J(\pi) = \sum_{t=1}^T \epsilon (1 - \epsilon)^{t-1} (T - t) = \Omega(\epsilon T^2) \). There is only one action in \( s_2 \) so it is not possible to have a nonzero classification error in this state. □

A.2. Proof of Entropy Regularization Lemma

Lemma 8. Entropy Regularization Lemma: By optimizing \( U_j(\pi, f) - \alpha H(\pi) \) to a \( \delta \)-approximate equilibrium, one recovers at worst a \( Q_M \sqrt{\frac{25}{\alpha^2} + \alpha T(\ln |A| + \ln |S|)} \) equilibrium strategy for the policy player on the original game.

Proof. We optimize in the \( P(A^T | S^T) \) policy representation where strong duality holds and define the following:

\[
\pi^R = \arg\min_{\pi \in \Pi} (\max_{f \in F} U_j(\pi, f) - \alpha H(\pi))
\]

First, we derive a bound on the distance between \( \hat{\pi} \) and \( \pi^R \). We define \( M \) as follows:

\[
M(\pi) = \max_{f \in F} U_j(\pi, f) - \alpha H(\pi) + \alpha T(\ln |A| + \ln |S|)
\]

\( M \) is an \( \alpha \)-strongly convex function with respect to \( \| \cdot \|_1 \) because \( U \) is a max of linear functions, \(-H \) is 1-strongly convex, and the third term is a constant. This tells us that:

\[
M(\pi^R) - M(\hat{\pi}) \leq \nabla M(\pi^R)^T (\pi^R - \hat{\pi}) - \frac{\alpha}{2} ||\pi^R - \hat{\pi}||_1^2
\]

We note that because \( \pi^R \) minimizes \( M \), the first term on the RHS is negative, allowing us to simplify this expression to:

\[
\frac{\alpha}{2} ||\pi^R - \hat{\pi}||_1^2 \leq M(\hat{\pi}) - M(\pi^R)
\]

We now upper bound the RHS of this expression via the following series of substitutions:

\[
M(\hat{\pi}) = \max_{f \in F} U_j(\hat{\pi}, f) - \alpha H(\hat{\pi}) + \alpha T(\ln |A| + \ln |S|)
\]

\[
\leq U_j(\hat{\pi}, \hat{f}) - \alpha H(\hat{\pi}) + \alpha T(\ln |A| + \ln |S|) + \delta
\]

\[
\leq U_j(\pi^R, \hat{f}) - \alpha H(\pi^R) + \alpha T(\ln |A| + \ln |S|) + \delta
\]

\[
\leq \max_{f \in F} U_j(\pi^R, f) - \alpha H(\pi^R) + \alpha T(\ln |A| + \ln |S|) + \delta
\]

\[
= M(\pi^R) + \delta
\]

Rearranging terms to get the desired bound on strategy distance:

\[
M(\hat{\pi}) - M(\pi^R) \leq \delta
\]

\[
\Rightarrow ||\pi^R - \hat{\pi}||_1^2 \leq \frac{2\delta}{\alpha}
\]

\[
\Rightarrow ||\pi^R - \hat{\pi}||_1 \leq \sqrt{\frac{2\delta}{\alpha}}
\]

Next, we prove that \( \pi^R \) is an \( \alpha T(\ln |A| + \ln |S|) -\)-approximate equilibrium strategy for the original, unregularized game. We note that \( H(\pi) \in [0, T(\ln |A| + \ln |S|)] \) and then proceed as follows:

\[
\max_{f \in F} U_j(\pi^R, f) = M(\pi^R) + \alpha H(\pi^R) - \alpha T(\ln |A| + \ln |S|)
\]

\[
\leq M(\pi^R) + \alpha T(\ln |A| + \ln |S|)
\]

The last line comes from the fact that playing the optimal strategy in the original game on the regularized game could at worst lead to a payoff of \( \alpha T(\ln |A| + \ln |S|) \). Therefore, the value of the regularized game can at most be this quantity. Recalling that the value of the original game is 0 and rearranging terms, we get:

\[
\max_{f \in F} U_j(\pi^R, f) - \alpha T(\ln |A| + \ln |S|) \leq 0 = \max_{f \in F} \min_{\pi \in \Pi} U_j(\pi, f)
\]

Thus by definition, \( \pi^R \) must be half of an \( \alpha T(\ln |A| + \ln |S|) -\)-approximate equilibrium strategy pair.

Next, let \( Q_M \) denote the absolute difference between the minimum and maximum \( Q \)-value. For a fixed \( f \), the maximum amount the policy player could gain from switching to policies within an \( L_1 \) ball of radius \( r \) centered at the original

\[
Q_M = \max_{\pi \in \Pi} Q(\pi) - \min_{\pi \in \Pi} Q(\pi)
\]
policy is $rQ_M$ by the bilinearity of the game and Hölder’s inequality. Because the supremum over $k$-Lipschitz functions is known to be $k$-Lipschitz, this implies the same is true for the payoff against the best response $f$. To complete the proof, we can set $r = \sqrt{\frac{2\delta}{\alpha}}$ and combine this with the fact that $\pi^R$ achieves in the worst case a payoff of $\alpha T(\ln |A| + \ln |S|)$ to prove that $\hat{\pi}$ can at most achieve a payoff of $Q_M \sqrt{\frac{2\delta}{\alpha}} + \alpha T(\ln |A| + \ln |S|)$ on the original game, which establishes $\hat{\pi}$ as a $(Q_M \sqrt{\frac{2\delta}{\alpha}} + \alpha T(\ln |A| + \ln |S|))$-approximate equilibrium solution.

\[ \square \]

### A.3. Proof of Theorem [1]

We proceed in cases.

**Proof.** We first consider the **primal case**. Our goal is to compute a policy $\hat{\pi}$ such that:

$$\max_{f \in F} U_j(\hat{\pi}, f) \leq \delta$$

We prove that such a policy can be found efficiently by executing the following procedure for a polynomially large number of iterations:

1. For $t = 1 \ldots N$, do:
   2. No-regret algorithm computes $\pi^t$.
   3. Set $f^t$ to the best response to $\pi^t$.
   4. Return $\hat{\pi} = \pi^* = \arg\min_{\pi \in \Pi} U_j(\pi, f^t)$.

Recall that via our no-regret assumption we know that

$$\frac{1}{N} \sum_{t=1}^{N} U_j(\pi^t, f^t) - \frac{1}{N} \min_{\pi \in \Pi} \sum_{t=1}^{N} U_j(\pi, f^t) \leq \frac{\beta_\Pi(N)}{N} \leq \delta$$

for some $N$ that is poly($\frac{1}{\delta}$). We can rearrange terms and use the fact that $\pi^t \in \Pi$ to upper bound the average payoff:

$$\frac{1}{N} \sum_{t=1}^{N} U_j(\pi^t, f^t) \leq \delta + \frac{1}{N} \min_{\pi \in \Pi} \sum_{t=1}^{N} U_j(\pi, f^t) \leq \delta$$

Using the property that there must be at least one element in an average that is at most the value of the average:

$$\min_{t} U_j(\pi^t, f^t) \leq \frac{1}{N} \sum_{t=1}^{N} U_j(\pi^t, f^t) \leq \delta$$

To complete the proof, we recall that $f^t$ is chosen as the best response to $\pi^t$, giving us that:

$$\min_{t} \max_{f \in F} U_j(\pi^t, f) \leq \delta$$

In words, this means that by setting $\hat{\pi}$ to the policy with the lowest loss out of the $N$ computed, we are able to efficiently (within poly($N$) iterations) find a $\delta$-approximate equilibrium strategy for the policy player. Note that this result holds without assuming a finite $S$ and $A$ and does not require regularization of the policy. However, it requires us to have a no-regret algorithm over $\Pi$ which can be a challenge for the reward moment-matching game.

We now consider the **dual case**. As before, we wish to find a policy $\hat{\pi}$ such that:

$$\max_{f \in F} U_j(\hat{\pi}, f) \leq \delta$$

We run the following procedure on $U_j(\pi, f) - \alpha H(\pi)$:

1. For $t = 1 \ldots N$, do:
   2. No-regret algorithm computes $f^t$.
   3. Set $\pi^t$ to the best response to $f^t$.
   4. Return $\hat{\pi} = \arg\min_{\pi \in \Pi} U_j(\pi, f^t) - \alpha H(\pi)$.

By the classic result of [Freund and Schapire 1997], we know that the average of the $N$ iterates produced by the above procedure (which we denote $\hat{\pi}$ and $\hat{\pi}$) is a $\delta'$-approximate equilibrium strategy for some $N$ that is poly($\frac{1}{\delta'}$). Applying our Entropy Regularization Lemma, we can upper bound the payoff of $\pi$ on the original game:

$$\max_{f \in F} U_j(\pi, f) \leq Q_M \sqrt{\frac{2\delta'}{\alpha}} + \alpha T(\ln |A| + \ln |S|)$$

We now proceed similarly to our proof of the Entropy Regularization Lemma by first bounding the distance between $\pi$ and $\hat{\pi}$ and the appealing to the $Q_m$-Lipschitzness of $U_j$. Let $l(\pi) = U_j(\pi, f^t) - \alpha H(\pi)$. Then, while keeping the fact that $l$ is $\alpha$-strongly convex in mind:

$$l(\hat{\pi}) - l(\pi) \leq \nabla l(\hat{\pi})^T (\hat{\pi} - \pi) - \frac{\alpha}{2} ||\hat{\pi} - \pi||_1^2$$

$$\Rightarrow \frac{\alpha}{2} ||\hat{\pi} - \pi||_1^2 \leq l(\pi) - l(\hat{\pi}) + \nabla l(\hat{\pi})^T (\hat{\pi} - \pi)$$

$$\Rightarrow \frac{\alpha}{2} ||\pi - \hat{\pi}||_1^2 \leq l(\pi) - l(\hat{\pi})$$

$$\Rightarrow \frac{\alpha}{2} ||\pi - \hat{\pi}||_1^2 \leq \delta'$$

$$\Rightarrow ||\pi - \hat{\pi}||_1 \leq \sqrt{\frac{2\delta'}{\alpha}}$$

As before, the second to last step follows from the definition of a $\delta'$-approximate equilibrium. Now, by the bilinearity of the game, Hölder’s inequality, and the fact that supremum over $k$-Lipschitz functions is known to be $k$-Lipschitz, we can state that:

$$\max_{f \in F} U_j(\hat{\pi}, f) \leq 2Q_M \sqrt{\frac{2\delta'}{\alpha}} + \alpha T(\ln |A| + \ln |S|)$$
To ensure that the LHS of this expression is upper bounded by $\delta$, it is sufficient to set $\alpha = \frac{\delta}{2T(|A| + \ln(|S|))}$ and $\delta' = \frac{\delta^2}{32N}\alpha$. Plugging in these terms, we arrive at:

$$\sup_{f \in F} U_{\alpha}(\pi, f) \leq \frac{\delta}{2} + \frac{\delta}{2} \leq \delta$$

We note that in practice, $\alpha$ is rather sensitive hyperparameter of maximum entropy reinforcement learning algorithms (Haarnoja et al. 2018) and hope that the above expression might provide some rough guidance for how to set $\alpha$. To complete the proof, note that $N$ is poly($\frac{1}{\delta}$) and $\frac{\delta}{2} = \frac{64Q_2 \times T}{3}(|A| + \ln(|S|))$. Thus, $N$ is poly($\frac{1}{\delta}$, $T$, $\ln(|A|)$, $\ln(|S|)$).

\[ \square \]

\section*{B. Algorithm Derivations}

\subsection*{B.1. \textit{AdVIL} Derivation}

We begin by performing the following substitution: $f = v - B^\pi v$, where

$$B^\pi v = \sum_{s_t+1 \sim T(s_t, a_t)} \mathbb{E}_{a_t \sim \pi(s_t)} [v]$$

is the expected Bellman operator under the learner’s current policy. Our objective \[2\] then becomes:

$$\sup_{v \in F} \sum_{t=1}^{T} \mathbb{E}_{s_t \sim \tau} \left[ v(s_t, a_t) - B^\pi v(s_t, a_t) \right]$$

This expression telescopes over time, simplifying to:

$$\sup_{v \in F} \mathbb{E}_{s_0 \sim \pi} v(s_0, a_0) - \sum_{t=1}^{T} \mathbb{E}_{s_t \sim \tau} v(s_t, a_t) - B^\pi v(s_t, a_t)$$

We approximate $B^\pi v$ via a single-sample estimate from the respective expert trajectory, yielding the following off-policy expression:

$$\sup_{v \in F} \mathbb{E}_{s_0 \sim \pi} v(s_0, a_0) - \sum_{t=1}^{T} \mathbb{E}_{s_t \sim \tau} v(s_t, a_t) - \mathbb{E}_{a_t \sim \pi(s_{t+1})} [v(s_{t+1}, a)]$$

This resembles the form of the objective in ValueDICE (Kostrikov et al. 2019) but without requiring us to take the expectation of the exponentiated discriminator. We can further simplify this objective by noticing that trajectories generated by $\pi_E$ and $\pi$ have the same starting state distribution:

$$\sup_{v \in F} \mathbb{E}_{s_0 \sim \pi_E} \sum_{t=1}^{T} \mathbb{E}_{a_t \sim \pi(s_t)} [v(s_t, a_t)] - v(s_t, a_t)]$$

(4)

We also note that this \textit{AdVIL} objective can be derived straightforwardly via the Performance Difference Lemma.

\section*{B.2. \textit{AdRIL} Derivation}

Let $F$ be a RKHS be equipped with kernel $K : (S \times A) \times (S \times A) \to \mathbb{R}$. On iteration $k$ of the algorithm, consider a purely cosmetic variation of our IPM-based objective \[2\]:

$$\sup_{c \in F} \sum_{t=1}^{T} \mathbb{E}_{s_t \sim \pi} [c(s_t, a_t)] - \mathbb{E}_{s_t \sim \pi_E} [c(s_t, a_t)] = \sup_{L_k(c)}$$

We evaluate the first expectation by collecting on-policy rollouts into a dataset $D_k$ and the second by sampling from a fixed set of expert demonstrations $D_E$. Assume that $|D_k|$ is constant across iterations. Let $\mathcal{E}$ be the evaluation functional. Then, taking the functional gradient:

$$\nabla_{c} L_k(c) = \sum_{t=1}^{T} \frac{1}{|D_k|} \sum_{\tau \sim \pi} \mathbb{E}_{c} [\mathcal{E}(c; s_t, a_t)] - \frac{1}{|D_E|} \sum_{\tau \sim \pi_E} \mathbb{E}_{c} [\mathcal{E}(c; s_t, a_t)]$$

$$= \sum_{t=1}^{T} \frac{1}{|D_k|} \sum_{\tau \sim \pi} K([s_t, a_t], \cdot) - \frac{1}{|D_E|} \sum_{\tau \sim \pi_E} K([s_t, a_t], \cdot)$$

where $K$ could be an state-action indicator $1_{s,a}$ in discrete spaces and relaxed to a Gaussian in continuous spaces. Let $D_k = \bigcup_{k=0}^{K} D_k$ be the aggregation of all previous $D_k$. Averaging functional gradients over iterations of the algorithm (which, other than a scale factor that does not affect the optimal policy, is equivalent to having a constant learning rate of 1), we get the cost function our policy tries to minimize:

$$C(\pi_k) = \sum_{t=0}^{k} \nabla_{c} L_k(c)$$

(5)

$$= \sum_{t=1}^{T} \frac{1}{|D_k|} \sum_{\tau \sim \pi} K([s_t, a_t], \cdot) - \frac{1}{|D_E|} \sum_{\tau \sim \pi_E} K([s_t, a_t], \cdot)$$

\section*{B.3. \textit{DAeQuIL} Derivations}

Let $d_{\pi}$ denote the state-action visitation distribution of $\pi$. Then, \textit{DAeQuIL} can be seen as Follow The Regularized Leader on the following sequence of losses:

\begin{enumerate}
  \item $f_t = \arg \max_{f \in F} \mathbb{E}_{s_t, a_t \sim d_{\pi}} [f(s, a) - f(s, \pi_E(s))]$
  \item $l_t(\pi) = \mathbb{E}_{s_t \sim d_{\pi}} [f_t(s, \pi(s)) - f_t(s, \pi_E(s))]$
\end{enumerate}

Solving the on-$Q$ game proper would instead require $l_t'(\pi) = \mathbb{E}_{s_t \sim d_{\pi}} [f_t(s, \pi(s)) - f_t(s, \pi_E(s))]$ for the state distribution to depend on the policy that is passed to the loss. While this would allow our previous no-regret analysis to apply as written, we would need to re-sample trajectories after every gradient step, a burden we’d like to avoid.

Let us consider the no-regret guarantee we get from the \textit{DAeQuIL} losses:

$$\frac{1}{N} \sum_{t} l_t(\pi^t) - \frac{1}{N} \min_{\pi \in \Pi} \sum_{t} l_t(\pi) \leq \beta_t(N) \leq \delta$$
Notice that $l_\epsilon(\pi^t) = \max_{f \in \mathcal{F}} U_3(\pi^t, f)$, the exact quantity we'd like to bound. The tricky part comes from the second term in the regret – under realizability, $(\pi_E \in \Pi)$, this term is 0 and $\text{DAeQuIL}$ directly finds a $\delta$-approximate equilibrium for the on-$Q$ game. Otherwise, we require the following weak notion of realizability to maintain the on-$Q$ moment matching bounds: $\exists \pi' \in \Pi$ s.t.

$$\max_{d_s \in d_{\Pi}} \max_{f \in \mathcal{F}} [f(s, \pi'(a)) - f(s, \pi_E(a))] \leq O(\epsilon)$$

In words, this is saying that there exists a policy $\pi'$ that can match expert moments up to $\epsilon$ on any state visitation distribution generated by a policy in $\Pi$. If we instead solved the on-$Q$ game directly by using $l_\epsilon'(\pi)$, we would instead need the condition: $\exists \pi' \in \Pi$ s.t.

$$\max_{d_s \in d_{\Pi}} \max_{f \in \mathcal{F}} [f(s, \pi'(a)) - f(s, \pi_E(a))] \leq O(\epsilon)$$

This weaker condition is concomitant with a much more computationally expensive optimization procedure.

C. Experimental Setup

C.1. Expert

We use the Stable Baselines 3 (Raffin et al. 2019) implementation of PPO (Schulman et al. 2017) and SAC (Haarnoja et al. 2018) to train experts for each environment, mostly using the already tuned hyperparameters from (Raffin 2020). Specifically, we use the modifications in Tables 4 and 5 to the Stable Baselines Defaults.

| PARAMETER | VALUE |
|-----------|-------|
| Buffer Size | 300000 |
| Batch Size | 256 |
| $\gamma$ | 0.98 |
| $\tau$ | 0.02 |
| Training Freq. | 64 |
| Gradient Steps | 64 |
| Learning Rate | 7.3E-4 |
| Policy Architecture | 256 x 2 |
| State-Dependent Exploration | True |
| Training Timesteps | 1E6 |

Table 4. Expert hyperparameters for HalfCheetah Bullet Task.

C.2. Baselines

For all learning algorithms, we perform 5 runs and use a common architecture of 256 x 2 with ReLU activations. For each datapoint, we average the cumulative reward of 10 trajectories. For offline algorithms, we train on $\{5, 10, 15, 20, 25\}$ expert trajectories with a maximum of 500k iterations of the optimization procedure. For online algorithms, we train on a fixed number of trajectories (5 for HalfCheetah and 20 for Ant) for 500k environment steps. For GAIL (Ho and Ermon 2016) and behavioral cloning (Pomerleau 1989), we use the implementation produced by (Wang et al. 2020). We use the changes from the default values in Tables 6 and 7 for all tasks.

| PARAMETER | VALUE |
|-----------|-------|
| Entropy Weight | 0 |
| L2 Weight | 0 |
| Training Timesteps | 5e5 |

Table 7. Learner hyperparameters for Behavioral Cloning.

For SQIL (Reddy et al. 2019), we build a custom implementation on top of Stable Baselines with feedback from the authors. As seen in Table 9, we use the similar parameters for SAC as we did for training the expert.

We modify the open-sourced code for ValueDICE (Kostrikov et al. 2019) to be actually off-policy with feedback from the authors. The publicly available version of the ValueDICE code uses on-policy samples to compute a regularization term, even when it is turned off in the flags. We release our version.

Table 5. Expert hyperparameters for Ant Bullet Task.
In this section, we use bold text to highlight sensitive hyperparameters. Similarly to SQIL, AdRIL is built on top of the Stable Baselines implementation of SAC. AdVIL is written in pure PyTorch. We use the same network architecture choices as for the baselines. For AdRIL we use the hyperparameters in Table 10 across all experiments.

| PARAMETER     | VALUE           |
|---------------|-----------------|
| γ             | 0.98            |
| τ             | 0.02            |
| TRAINING FREQ. | 64              |
| GRADIENT STEPS | 64              |
| LEARNING RATE | LINEAR SCHEDULE OF 7.3e-4 |
| f UPDATE FREQ. | 1250            |

Table 10. Learner hyperparameters for AdRIL.

We note that AdRIL requires careful tuning of \( f \) UPDATE FREQ. for strong performance. To find the value specified, we ran trials with \{1250, 2500, 5000, 12500, 25000, 50000\} and selected the one that achieved the most stable updates. In practice, we would recommend evaluating a trained policy on a validation set to set this parameter. We also note because SAC is an off-policy algorithm, we are free to initialize the learner by adding all expert samples to the replay buffer at the start, as is done for SQIL.

We change one parameter between environments for AdRIL – for HalfCheetah, we perform standard sampling from the replay buffer while for Ant we sample an expert trajectory with \( p = \frac{1}{2} \) and a learner trajectory otherwise, similar to SQIL. We find that for certain environments, this modification can somewhat increase the stability of updates while for other environments it can significantly hamper learner performance. We recommend trying both options if possible but defaulting to standard sampling.

For AdVIL, we use the hyperparameters in Table 11 across all tasks. Empirically, small learning rates, large batch sizes, and regularization of both players are critical to stable convergence. We find that AdVIL converges significantly more quickly than ValueDICE, requiring only 50k steps for HalfCheetah and 100k Steps for Ant instead of 500k steps for both tasks. However, we also find that running AdVIL for longer than these prescribed amounts can lead to a collapse of policy performance. Fortunately, this can easily be caught by watching for sudden and large fluctuations in policy loss after a long period of steady decreases. One can perform this early-stopping check without access to the environment.

| PARAMETER                                      | VALUE           |
|-----------------------------------------------|-----------------|
| \( \eta \)                                    | 8e-6            |
| \( \eta_f \)                                  | 8e-4            |
| BATCH SIZE                                   | 1024            |
| \( f \) GRADIENT TARGET                      | 0.4             |
| \( f \) GRADIENT PENALTY WEIGHT              | 10              |
| \( \pi \) ORTHOGONAL REGULARIZATION          | 1e-4            |
| \( \pi \) MSE REGULARIZATION WEIGHT           | 0.2             |
| NORMALIZE STATES WITH EXPERT DATA             | TRUE            |
| NORMALIZE ACTIONS TO \([-1, 1]\)              | TRUE            |
| GRADIENT NORM CLIPPING \([-40, 40]\)         |                 |

Table 11. Learner hyperparameters for AdVIL.

We perform two experiments to tease out when one should apply DaeQuIL over DAgger. We first present results on a rocket-landing task from OpenAI Gym where behavioral cloning by itself is able to nearly solve the task, as has been
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previously noted (Spencer et al. 2021). To make the task more challenging, we truncate the last two dimensions of the state for the policy class, which corresponds to masking the location of the legs of the lander. We use two-layer neural networks with 64 hidden units as all our function classes, perform the optimization steps via ADAM with learning rate $3e^{-4}$, and sample 10 trajectories per update. Here, we see DAeQuIL do around as well as DAgger (Fig. 8), with both algorithms quickly learning a policy of quality equivalent to that of the expert. We list the full parameters of the algorithms in Tables 12 and 13. As in the previous section, **bold text** highlights sensitive hyperparameters.

Figure 8. As behavioral cloning alone is able to nearly match the expert, DAgger and DAeQuIL perform around the same.

| PARAMETER | VALUE |
|-----------|-------|
| BATCH SIZE | 32 |
| GRADIENT STEPS $\pi$ UPDATE | $3e3$ |
| GRADIENT STEPS $f$ UPDATE | $1e3$ |
| $f$ GRADIENT PENALTY TARGET | 0 |
| $f$ GRADIENT PENALTY WEIGHT | 5 |

Table 12. Learner hyperparameters for DAeQuIL on LunarLander-v2.

Figure 9. $J(\pi)$ is the longitudinal distance into the forest the learner is able to progress. All experiments are run on the forest layout shown in Fig. 4 and standard errors are computed across 10 trials.

We use the same function classes as the previous experiment but use a hidden size of 32 for the discriminator of DAeQuIL. We list the full set of parameters in Tables 14 and 15.

E. Additional Moment Types

E.1. A Fourth Moment Class: Mixed-Moment Value

We could instead plug in $Q$-moments to the reward moment payoff function $U_1$. Let $V$ and $V_E$ refer to the classes of policy and expert value functions. As before, we assume both of these classes are closed under negation and include the true value and expert value functions. For notational convenience, we assume both classes contain functions with
Table 14. Learner hyperparameters for DAbQuilt on Forest Navigation.

| Parameter           | Value |
|---------------------|-------|
| Batch Size          | 32    |
| Gradient Steps $\tau$ Update | 2e3   |
| $\ell_{BC}$ Scale   | 5e-2  |
| Gradient Steps $f$ Update | 1e3   |
| $f$ Gradient Penalty Target | 0     |
| $f$ Gradient Penalty Weight | 5     |

Table 15. Learner hyperparameters for DAgger on Forest Navigation.

| Parameter           | Value |
|---------------------|-------|
| Batch Size          | 32    |
| Gradient Steps $\tau$ Update | 5e3   |

type signatures $S \times A \rightarrow \mathbb{R}$, with the second argument being ignored. Starting from the PDL, we can expand as follows:

$$J(\pi_E) - J(\pi) = \sum_{t=1}^{T} \mathbb{E}_{\tau \sim \pi_E} [Q_t(\tau, a_t) - \mathbb{E}_{a \sim \pi(\tau)} [Q_t(\tau, a)]]$$

$$= \sum_{t=1}^{T} \mathbb{E}_{\tau \sim \pi_E} [Q_t^f(\tau, a_t) - \mathbb{E}_{a \sim \pi(\tau)} [Q_t^f(\tau, a)]] + \mathbb{E}_{\tau \sim \pi} [Q_t\tilde{Q}(\tau, a_t) - \mathbb{E}_{a \sim \pi(\tau)} [Q_t\tilde{Q}(\tau, a)]]$$

$$\leq \sup_{f \in \mathcal{F}_Q \cup \mathcal{F}_V} 2 \sum_{t=1}^{T} \mathbb{E}_{\tau \sim \pi} [f(\tau, a_t)] - \mathbb{E}_{\tau \sim \pi_E} [f(\tau, a_t)]$$

The last step follows from the fact that $\sup_{a \in A} f(a) + \sup_{b \in B} f(b) \leq \sup_{c \in A \cup B} 2f(c)$. An analogous bound for $\mathcal{F}_Q$ and $\mathcal{F}_V$ can be proved by expanding the PDL in the reverse direction. We can use these expansions to provide bounds related to the reward-moment bound:

**Lemma 9. Mixed Moment Value Upper Bound:** If $\mathcal{F}_Q / 2T$ and $\mathcal{F}_V / 2T$ spans $\mathcal{F}$ or $\mathcal{F}_Q / 2T$ and $\mathcal{F}_V / 2T$ do, then for all MDPs, $\pi_E$, and $\pi \leftarrow \Psi\{\epsilon\}(U_1)$, $J(\pi_E) - J(\pi) \leq O(\epsilon T^2)$.

**Proof.** We start by expanding the imitation gap:

$$J(\pi_E) - J(\pi) \leq \sup_{f \in \mathcal{F}_Q \cup \mathcal{F}_V} 2 \sum_{t=1}^{T} \mathbb{E}_{\tau \sim \pi} [f(\tau, a_t)] - \mathbb{E}_{\tau \sim \pi_E} [f(\tau, a_t)]$$

$$\leq \sup_{f \in \mathcal{F}_Q} \mathbb{E}_{\tau \sim \pi} \sum_{t=1}^{T} 2Tf(\tau, a_t) - \mathbb{E}_{\tau \sim \pi_E} \sum_{t=1}^{T} 2Tf(\tau, a_t)$$

$$= 4T^2 \sup_{f \in \mathcal{F}} U_1(\pi, f) \leq 4\epsilon T^2$$

The $T$ in the second to last line comes from the scaling down of either the $(\mathcal{F}_Q, \mathcal{F}_V)$ or the $(\mathcal{F}_Q, \mathcal{F}_V)$ pairs by $T$ to fit into the function class $\mathcal{F}$.

**Lemma 10. Mixed Moment Value Lower Bound:** There exists an MDP, $\pi_E$, and $\pi \leftarrow \Psi\{\epsilon\}(U_1)$ such that $J(\pi_E) - J(\pi) \geq \Omega(\epsilon T)$.

**Proof.** The proof of the reward lower bound holds verbatim.

These bounds show that solving this game, which might be more challenging than the reward-moment game, appears to offer no policy performance gains. However, in the imitation learning from observation alone setting, where one does not have access to action labels, reward-matching might be impossible, forcing one to use an approach similar to the above. This is because value functions are pure functions of state, not actions. [Sun et al. 2019] give an efficient algorithm for this setting.

### E.2. Combining Reward and Value Moments

For both the off-$Q$ and on-$Q$ setups, one can leverage the standard expansion of a $Q$-function into a sum of rewards to derive a flexible family of algorithms that allow one to include knowledge of both reward and $Q$ moments. Explicitly, for the off-$Q$ case:

$$J(\pi_E) - J(\pi) = \sup_{\max_{\ell \in \mathcal{E}}} 1 \mathbb{E}_{\tau \sim \pi_E} \left( \sum_{t=1}^{T} \sum_{\ell \in \mathcal{E}} \mathbb{E}_{\tau \sim \pi_E} [r(\tau, a_t) - r(\tau, a')] \right)$$

$$\leq \sup_{f \in \mathcal{F}_Q} \mathbb{E}_{\tau \sim \pi} \sum_{t=1}^{T} \sum_{\ell \in \mathcal{E}} f(\tau, a_t) - f(\tau, a')$$

Passing such a payoff to our oracle with $\mathcal{F}$ spanned by $\mathcal{F}_Q / 2T \times \mathcal{F}_V / 2T$ will recover the off-$Q$ bounds.

This expansion begs the question of when it is useful. One answer is a standard bias/variance trade-off with different values of $T'$, as has been explored in TD-Gammon [Tesauro 1995]. We can provide an alternative answer by considering the limiting case – when the $Q$ function is decomposed entirely into reward functions, the learner is required at timestep $t$ to match the sum of future reward moments. An efficient algorithm for such a problem can be derived as a natural extension of Policy Search by Dynamic Programming (PSPD) [Bagnell et al. 2003], where, starting from $t = T - 1$, the learner matches expert moments one timestep in the future, before moving one step backwards in
time along the expert’s trajectory. While this approach has the same performance characteristics as off-$Q$ algorithms, matching the class of reward moments might be simpler for some types of problems, like those with sparse rewards. However, it has the added complexity of producing a non-stationary policy.

We can perform an analogous expansion for the on-$Q$ case by utilizing the reverse direction of the PDL:

$$J(\pi_E) - J(\pi)$$

$$= \frac{1}{T}(\sum_{t=1}^{T} Q_{\pi_E}^{T}(s_t, a_t) - Q_{\pi}^{T}(s_t, a))$$

$$= \frac{1}{T}(\sum_{t=1}^{T} \sum_{t' = 1}^{T'} r(s_{t'}, a_{t'}) - r(s_{t'}, a))$$

$$+ Q_{T'}^{\pi_E}(s_{T'}, a_{T'}) - Q_{T'}^{\pi_E}(s_{T'}, a))$$

$$\leq \min_{\pi \in \Pi} \max_{f \in F_r} \max_{g \in F_{Q_E}} \frac{1}{T}(\sum_{t=1}^{T} \sum_{t'=t}^{T'} f(s_{t'}, a_{t'}) - f(s_{t'}, a))$$

$$+ g(s_{T'}, a_{T'}) - g(s_{T'}, a))$$  \hspace{1cm} (7)$$

Passing such a payoff to our oracle with $F$ spanned by $F_r/2 \times F_{Q_E}/2T$ would recover the on-$Q$ bounds. A backwards-in-time dynamic-programming procedure is not possible for this expansion because of the need to sample trajectories from the policy at previous timesteps.