THE OPTIMAL HARDY–LITTLEWOOD CONSTANTS FOR 2-HOMOGENEOUS POLYNOMIALS ON $\ell_p(\mathbb{R}^2)$ FOR $2 < p < 4$ ARE $2^{2/p}$

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Abstract. We show that the optimal constants for the Hardy–Littlewood inequalities for 2-homogeneous polynomials on $\ell_p(\mathbb{R}^2)$ are precisely $2^{2/p}$ for all $2 < p < 4$.

1. Introduction

The Hardy–Littlewood inequality for bilinear forms in $\ell_p$ spaces were proved in 1934 [16]; these inequalities and the classical Bohnenblust–Hille inequality [6] consist in optimal extensions of Littlewood’s $4/3$ inequality [17] (originally stated for $c_0$ spaces). In the last years the interest in this subject (which can be considered part of the theory of multiple summing and absolutely summing operators) was renewed with applications in several fields of Mathematics and even in Physics (see [3, [10, [18]) and several authors became interested in this field ([11, [4, [8, [12, [13, [14, [19, [20, [21, [22]). Very recently the subject became to be investigated via numerical and computational assistance due to several challenging problems which seemed quite difficult to be solved analytically (without computer assistance).

There is no doubt that computational assistance is important in this subject but some problems arise with this approach: for instance, the concrete estimates of the constants obtained with computational assistance are just approximations (even if we have thousands of decimal digits of confidence) of the exact values of the constants and closed (and elegant) formulas for the optimal constants are difficult (or even essentially impossible) to be achieved just with the help of computers.

As mentioned before, the search of optimal constants for these kind of inequalities has important applications but, as a matter of fact, even for 2-homogeneous polynomials the optimal constants are unknown. The same happens with the constants of the Hardy–Littlewood inequalities. In this note we obtain simple formulas for the optimal Hardy–Littlewood constants for 2-homogeneous polynomials on $\ell_p(\mathbb{R}^2)$ for all $2 < p \leq 4$. Up to now, the only known simple (explicit) formula for these constants is $2^{1/2}$ for $p = 4$, due to Araujo et al. [2], extending previous results of [9].

For $K$ be $\mathbb{R}$ or $\mathbb{C}$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we define $|\alpha| := \alpha_1 + \cdots + \alpha_n$. By $x^\alpha$ we shall mean the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for any $x = (x_1, \ldots, x_n) \in K^n$. The polynomial version of Littlewood’s $4/3$ theorem asserts that, given $n \geq 1$, there is a constant $B_{K,2}^{\text{pol}} \geq 1$ such that

$$
\left( \sum_{|\alpha| = 2} |a_\alpha|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq B_{K,2}^{\text{pol}} \|P\|
$$

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for all 2-homogeneous polynomials \( P : \ell_\infty^n \to \mathbb{K} \) given by
\[
P(x_1, \ldots, x_n) = \sum_{|\alpha|=2} a_\alpha x^\alpha,
\]
and all positive integers \( n \), where \( \| P \| = \sup_{z \in B_{\ell_\infty^n}} |P(z)| \). It is well-known that the exponent \( \frac{4}{3} \) is sharp.

The change of \( \ell_\infty^n \) by \( \ell_p^n \) gives us the polynomial Hardy–Littlewood inequality whose optimal exponents are \( \frac{4p}{3p-4} \) for \( 4 \leq p \leq \infty \) and \( \frac{p}{p-2} \) for \( 2 < p \leq 4 \). More precisely, given \( n \geq 1 \), there is a constant \( C_\text{pol}^{\mathbb{K},2,p} \geq 1 \) such that
\[
\left( \sum_{|\alpha|=2} |a_\alpha|^{\frac{4p}{3p-4}} \right)^{\frac{3p-4}{4p}} \leq C_\text{pol}^{\mathbb{K},2,p} \| P \|,
\]
for all 2-homogeneous polynomials on \( \ell_p^n \) with \( 4 \leq p \leq \infty \) given by \( P(x_1, \ldots, x_n) = \sum_{|\alpha|=m} a_\alpha x^\alpha \).

When \( 2 < p \leq 4 \) we have
\[
\left( \sum_{|\alpha|=2} |a_\alpha|^{\frac{p}{p-2}} \right)^{\frac{p-2}{p}} \leq C_\text{pol}^{\mathbb{K},2,p} \| P \|.
\]

The main result of this note shows that \( C_\text{pol}^{\mathbb{R},2,p} = 2^{2/p} \) when we are restricted to the case \( n = 2 \) and \( 2 < p \leq 4 \) (as mentioned before, the case \( p = 4 \) is already known). More precisely, our main result is the following:

**Theorem 1.1.** If \( 2 < p \leq 4 \), then the optimal Hardy–Littlewood constants for 2-homogeneous polynomials on \( \ell_p(\mathbb{R}^2) \) are \( 2^{2/p} \).

2. **Proof of the main result: part 1**

The following result due to B. Grecu \[15\] will be crucial for our goals:

**Theorem 2.1.** For \( p > 2 \), a 2-homogeneous norm one polynomial \( P \) is a extreme point of the unit ball of \( P(2,\ell_p^2) \) if, and only if,

(i) \( P(x, y) = ax^2 + cy^2 \), with \( ac > 0 \) and \( \| (a, c) \|_{p-2} = 1 \) or

(ii) \( P(x, y) = \pm \left( \frac{a^{p-2} + b^{p-2}}{a+b} \right) (x^2 - y^2) + 2ab \frac{a^{p-2} + b^{p-2}}{a+b} xy \), with \( a, b > 0 \) and \( \| (a, b) \|_p = 1 \).

We know that for all 2-homogeneous polynomials \( P : \ell_p^n \to \mathbb{K} \) given by
\[
P(x_1, \ldots, x_n) = \sum_{|\alpha|=2} a_\alpha x^\alpha,
\]
the formula
\[
|P|_q := \left( \sum_{|\alpha|=2} |a_\alpha|^q \right)^{\frac{1}{q}}
\]
defines a norm for all \( q \geq 1 \). Since \( \ell^n_p \) is finite-dimensional, it is obvious that \( \| \cdot \| \) and \( \| \cdot \|_q \) are equivalent; so there are constants \( C_{2,n,p,q} \) such that

\[
(1) \quad |P|_q \leq C_{2,n,p,q} \| P \|
\]

for all \( P \in \mathcal{P} \left( 2\ell^n_p \right) \). We shall investigate \( C_{2,n,p,q} \) in the particular case in which \( n = 2 \), and \( p > 2 \), and \( q \geq 1 \), i.e., we shall investigate \( C_{2,2,p,q} \). The following equality (for \( p > 2 \) and \( q \geq 1 \))

\[
(2) \quad C_{2,2,p,q} = \max_{a \in [0,1]} \left[ \frac{2a^p - 1}{a^2 + (1 - a)^2/p} \right]^{q \over p} + \left( 2a (1 - a)^{1 \over p} \right)^q \left( \frac{a_{p-2} - 1}{a^2 + (1 - a)^2/p} \right)^{q \over p}
\]

due to Araujo et al. (2), is also important for our goals. We present a proof of (2) for the sake of completeness.

From the Krein–Milman Theorem it is well-known that the optimal constants \( C_{2,2,p,q} \) shall be searched within extreme polynomials. So, using Theorem 2.1 we conclude that for \( 2 < p \) and \( q \geq 1 \) we have

\[
C_{2,2,p,q} = \max_{a \in [0,1]} \left[ \frac{a^q + \left( 1 - a_{p-2} \right)^{p-2}q}{q \over p} \right]^{1 \over q}
\]

Note that

\[
(3) \quad \max_{a \in [0,1]} \left[ \frac{a^q + \left( 1 - a_{p-2} \right)^{p-2}q}{q \over p} \right]^{1 \over q} \leq 2^{\frac{2}{p}}.
\]

In fact, since \( \| \cdot \|_q \leq \| \cdot \|_1 \), we have

\[
\left[ a^q + \left( 1 - a_{p-2} \right)^{p-2}q \right]^{1 \over q} \leq \left[ a + \left( 1 - a_{p-2} \right)^{p-2} \right]^{1 \over p}.
\]

On the other hand, fixing \( p > 2 \) and deriving the function \( f (a) = a + \left( 1 - a_{p-2} \right)^{p-2} \) we have

\[
1 + \frac{p-2}{p} \left( 1 - a_{p-2} \right) \frac{p}{p-2} \left( \frac{p}{p-2} a_{p-2} \right) = f' (a),
\]

Note that \( f' (a) \) is well-defined for all \( a \in (0,1) \), and to solve \( f' (a) = 0 \) is equivalent to solve

\[
\frac{2}{a_{p-2}^{p-2}} = \left( 1 - a_{p-2} \right)^{1 \over p},
\]

and the unique real value that verifies this equality is \( a_0 = 2^{\frac{2}{p}} \), and

\[
1 = f (0) = f (1) < f (a_0) = 2^{\frac{2}{p}}.
\]

To summarize, for each \( p \), \( f (a) \) attains its maximum in \( (0,1) \) at \( a_0 = 2^{\frac{2}{p}} \). We have

\[
\max_{a \in [0,1]} \left[ a^q + \left( 1 - a_{p-2} \right)^{p-2}q \right]^{1 \over q} \leq \max_{a \in [0,1]} \left[ a + \left( 1 - a_{p-2} \right)^{p-2} \right]^{1 \over p} = 2^{\frac{2}{p}},
\]
for all \( q \geq 1 \), and we obtain (3). On the other hand, estimating the function

\[
\left( 2 \left| \frac{2a^p - 1}{a^2 + (1 - a^p)^{2/p}} \right|^q + \left( 2a \left( 1 - a^p \right)^{\frac{1}{p}} \frac{a^{p-2} + (1 - a^p)^{\frac{p-2}{p}}}{a^2 + (1 - a^p)^{2/p}} \right)^q \right)^{\frac{1}{q}} = 2^p
\]

at the point \( a_1 = 2^\frac{1}{p} \) we obtain

\[
(4) \quad \left( 2 \left| \frac{2a_1^p - 1}{a_1^2 + (1 - a_1^p)^{2/p}} \right|^q + \left( 2a_1 \left( 1 - a_1^p \right)^{\frac{1}{p}} \frac{a_1^{p-2} + (1 - a_1^p)^{\frac{p-2}{p}}}{a_1^2 + (1 - a_1^p)^{2/p}} \right)^q \right)^{\frac{1}{q}} = 2^p
\]

and the proof of (2) is done.

### 3. Proof of the main theorem: part 2

It suffices to prove that if \( 2 < p \leq 4 \) and \( q \geq 2 \), then

\[
\max_{a \in [0,1]} \left[ \left( 2 \left| \frac{2a^p - 1}{a^2 + (1 - a^p)^{2/p}} \right|^q + \left( 2a \left( 1 - a^p \right)^{\frac{1}{p}} \frac{a^{p-2} + (1 - a^p)^{\frac{p-2}{p}}}{a^2 + (1 - a^p)^{2/p}} \right)^q \right)^{\frac{1}{q}} \right] = 2^\frac{2}{p}.
\]

We first prove the case \( 2 < p \leq 4 \) and \( q = 2 \). We have just seen in (4) that

\[
\left( 2 \left| \frac{2(2^{-1/p})^p - 1}{a^2 + (1 - (2^{-1/p})^p)^{2/p}} \right|^2 + \left( 2(2^{-1/p}) \left( 1 - (2^{-1/p})^p \right)^{\frac{1}{p}} \frac{(2^{-1/p})^{p-2} + (1 - (2^{-1/p})^p)^{\frac{p-2}{p}}}{(2^{-1/p})^{2} + (1 - (2^{-1/p})^p)^{2/p}} \right)^2 \right)^{\frac{1}{2}} = 2^2
\]

On the other hand, from

\[
C_{2,2,p,2} = \max_{a \in [0,1]} \left\{ \left( 2 \left| \frac{2a^p - 1}{a^2 + (1 - a^p)^{2/p}} \right|^2 + \left( 2a \left( 1 - a^p \right)^{\frac{1}{p}} \frac{a^{p-2} + (1 - a^p)^{\frac{p-2}{p}}}{a^2 + (1 - a^p)^{2/p}} \right)^2 \right)^{\frac{1}{2}} \right\},
\]

we will see that

\[
C_{2,2,p,2} = 2^2.
\]

In fact, defining for each \( 2 < p \leq 4 \) the function \( g : [0,1] \to \mathbb{R} \) given by

\[
g(a) = 2 \left| \frac{2a^p - 1}{a^2 + (1 - a^p)^{2/p}} \right|^2 + \left( 2a \left( 1 - a^p \right)^{\frac{1}{p}} \frac{a^{p-2} + (1 - a^p)^{\frac{p-2}{p}}}{a^2 + (1 - a^p)^{2/p}} \right)^2
\]

and estimating its derivative we obtain

\[
g'(a) = -8 \left( a^p \left( 1 - a^p \right)^{\frac{3}{p}} a^{p+1} - a^4 \right) \left( a^p \left( -a^p + 1 \right)^{\frac{2}{p}} \left( p - a^2p + 2a^p - 1 \right) - a^2 \left( a^p - 1 \right) \left( a^p p - 2a^p + 1 \right) \right)
\]

\[
a^3 \left( 1 - a^p \right)^{\frac{2}{p}} a^2 \right)^{\frac{3}{p}} \left( a^p - 1 \right) \left( 1 - a^p \right)^{\frac{2}{p}}.
\]
Now we observe that \( g'(a) \) is well-defined for all \( a \in (0, 1) \) and moreover has precisely one zero in the interval \((0, 1)\), and this zero is attained at \( 2^{-1/p} \). In fact, if \( a \neq 0, a \neq 1, a \neq 2^{-1/p} \), we have 
\[
g'(a) = 0 \text{ if, and only if, } a^p (-a^p + 1) \frac{2}{p} \left( p - a^p p + 2a^p - 1 \right) - a^2 (a^p - 1) (a^p p - 2a^p + 1) = 0
\]
and this never happens in \((0, 1)\), because in this interval
\[
a^p (-a^p + 1) \frac{2}{p} \left( p - a^p p + 2a^p - 1 \right) - a^2 (a^p - 1) (a^p p - 2a^p + 1) > 0.
\]
The reader can be convinced of (5) by observing that (5) is equivalent to prove that 
\[
a^p (-a^p + 1) \frac{2}{p} \left( p - a^p p + 2a^p - 1 \right) - a^2 (a^p - 1) (a^p p - 2a^p + 1) > 0
\]
and, this is equivalent to prove that 
\[
-a^p (-a^p + 1) \frac{2}{p} \left( p - a^p p + 2a^p - 1 \right) < a^2 (a^p p - 2a^p + 1).
\]
But this last inequality is true since the left-hand-side is always negative while the right-hand-side is always positive in \((0, 1)\), for \( p > 2 \). Thus we conclude that \( g'(a) \) has exactly one zero in the interval \((0, 1)\), and this zero is \( 2^{-1/p} \). Since 
\[
g(0) = 2,
g(1) = 2,
g\left(2^{-1/p}\right) = 2^\frac{4}{p},
\]
and since \( 2^\frac{4}{p} \geq 2 \) whenever \( 2 < p \leq 4 \), we finally conclude that 
\[
\max_{a \in [0, 1]} \left\{ \left( 2 \left| \frac{2a^p - 1}{a^2 + (1 - a^p)^{2/p}} \right|^q + \left( 2a (1 - a^p)^{\frac{1}{p}} \frac{a^{p-2} - (1 - a^p)^{\frac{p-2}{p}}}{a^2 + (1 - a^p)^{2/p}} \right)^2 \right) \right\} = 2^\frac{2}{p}.
\]
If \( q > 2 \), we have \( \ell_2 \subset \ell_q \) and 
\[
\|x\|_q \leq \|x\|_2,
\]
and thus 
\[
C_{2, 2, p, q} = \max_{a \in [0, 1]} \left\{ \left( 2 \left| \frac{2a^p - 1}{a^2 + (1 - a^p)^{2/p}} \right|^q + \left( 2a (1 - a^p)^{\frac{1}{p}} \frac{a^{p-2} - (1 - a^p)^{\frac{p-2}{p}}}{a^2 + (1 - a^p)^{2/p}} \right)^2 \right) \right\} = \frac{2^2}{p}.
\]
Now we just need to recall that
\[
\begin{pmatrix}
2 & \left| \frac{2(2^{-1/p})^p - 1}{a^2 + \left(1 - (2^{-1/p})^p\right)^{2/p}} \right|^q \\
2 & \left| \frac{2(2^{-1/p})^p - 1}{a^2 + \left(1 - (2^{-1/p})^p\right)^{2/p}} \right|^q \\
\end{pmatrix}
= 2^2 \frac{2}{p}
\]
to complete the proof.

4. Final comments

From the previous section we can conclude that if $2 < p \leq 4$ and $1 \leq q < 2$, then $C_{2,2,p,q} > 2^2$. The Theorem 1.1 seems somewhat surprising since Araújo et al. [2] mention in their Remark 4.4 that the “function cannot be optimized explicitly in general”. The possibility or not of finding a similar closed formula for the case $p > 4$ is still open and seems to be an interesting problem. We finish the paper by remarking that our main theorem recovers the computer assisted numerical table presented in ([2]), and also corrects the rounding errors since our estimates are exact. As a very particular case we conclude that the optimal Hardy–Littlewood constant for 2-homogeneous polynomials in $\ell_2(\mathbb{R}^2)$ is $\sqrt{2}$ (this result was obtained, numerically, as a lower bound in [9] and proved to be sharp in the aforementioned paper of Araújo et al. [2]).

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