PREScribing THE Q-CURVATURE ON THE SPHERE WITH CONICAL SINGULARITIES

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Abstract. In this paper we investigate the problem of prescribing the Q-curvature, on the sphere of any dimension with prescribed conical singularities. We also give the asymptotic behaviour of the solutions that we find and we prove their uniqueness in the negative curvature case. We focus mainly on the odd dimensional case, more specifically the three dimensional sphere.

1. Introduction and Main Results. In this paper we investigate the problem of prescribing the Q-curvature via a conformal change of the metric on the sphere $S^n$ with conical singularities. In the two dimensional case, the problem corresponds to removing a divisor from the manifold and then prescribing the Gaussian curvature. This was presented first in the work of Troyanov [33], along with an extension of the Gauss-Bonnet formula. There was an extensive work on this problem in the two dimensional case, one can check for instance [12], [13], [28] and [27]. The problem has different behaviours depending on the conical angles that we want to prescribe. In the recent work of Malchiodi and Carlotto [12], the authors use a variational approach based on the critical point at infinity argument as in [2] and they distinguish two main behaviours of the problem. The “sub-critical” case and the “supercritical” case.

Let us start by recalling the regular case (no conical singularities). The problem of prescribing the Gaussian curvature on a compact Riemann surface $\Sigma_g$, stated first in the work of Kahzdan and Warner [22], is equivalent to solving the following PDE:

$$-\Delta_{g_0} u + K_{g_0} = K_g e^{2u}$$

where $K_{g_0}$ is the Gaussian curvature of $g_0$ and $K_g$ is the Gaussian curvature corresponding to the metric $g = e^{2u} g_0$. There is an extensive literature on this problem for instance one could check [9], [11], [14] and [15] and the references therein. Now if one removes a divisor $D = \sum_{i=1}^{k} \alpha_i p_i$, then the right equation to be considered is

$$-\Delta_{g_0} u + K_{g_0} = K_g e^{2u} + 2\pi \sum_{i=1}^{k} \alpha_i \delta_{p_i}$$

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where \( \delta p_i \) is the Dirac mass at the point \( p_i \in \Sigma_{g_0} \).

The importance of this equation can be seen from the generalized Gauss-Bonnet formula (see [33]), that is

\[
\int_{\Sigma} K_g dv_g + 2\pi \sum_{i=1}^{k} \alpha_i = \chi(\Sigma).
\]

This last years there was a big interest from the mathematical community on higher dimensional conformal invariants and there geometric interpretations, especially after the introduction of the GJMS operators [20].

Those new conformal quantities were given the name of \( Q \)-curvatures. Let us for instance compare the two dimensional case with the 4-dimensional one. The equivalent of the \(-\Delta\) operator now, is a 4th order operator called the Paneitz operator [6] it has a rather complicated formula in a general manifold and for the sake of completion we present it here:

\[
P_{g,4} u = (\Delta_g)^2 u + \text{div}_g \left( \frac{2}{3} R_g g - 2 \text{Ric}_g \right) d\phi.
\]

The substitute of the Gaussian curvature in this case, is the \( Q \)-curvature, that reads as follows

\[
Q_g = -\frac{1}{12} \left( \Delta_g R_g + R_g^2 - 3|Ric_g|^2 \right).
\]

After a conformal change of the form \( g = e^{2u} g_0 \), the \( Q \)-curvature changes in the following way:

\[
P_{g_0,4} u + 2Q_{g_0} = Q_g e^{4u}.
\]

It is also important to mention that the \( Q \)-curvature has indeed a topological meaning similar to the Gaussian curvature in dimension 2. The reader can consult [26], [16], [7] and [10] for a better geometric interpretation. For instance, one has a Gauss-Bonnet-Chern Formula as follow

\[
\int_M Q_g + \frac{|W_g|^2}{8} dV_g = 4\pi^2 \chi(M),
\]

where \( W_g \) is the Weyl tensor. Hence if \( M \) is locally conformally flat, one has

\[
\int_M Q_g dV_g = 4\pi^2 \chi(M).
\]

It is also important to consider the case of 3-dimensional manifolds as a boundary of 4-manifolds to get the following Gauss-Bonnet formula

\[
\int_M Q_g dV_g + \int_{\partial M} \tilde{Q}_g d\sigma = 4\pi^2 \chi(M),
\]

if the boundary is totally geodesic and \( M \) locally conformally flat, where here \( \tilde{Q} \) is the 3-dimensional \( Q \)-curvature. It is always helpful to compare with the Gaussian curvature for surfaces with boundary, in that case \( \tilde{Q} \) is the substitute of the geodesic curvature.

So far the two equations that we considered have a differential operator in their principal part, and that is the case for all even dimensions. We want also to point out that the key step into prescribing the scalar curvature with conical singularity, is a singular version of the Moser-Trudinger inequality (see [28] and [17]).

The complication of the problem increases considerably in odd dimensions since the operator becomes a pseudo differential one (the principal part is of the form \((-\Delta_g)^\frac{3}{2}\)). Even though the regular problem was studied extensively, see [31], [35]
and the references therein, to the best of our knowledge, the singular one was not treated for dimensions greater than 2. Also we cannot use the approach developed in [12] and [13] in dimension 2, since we do not have a similar singular Moser-Trudinger inequality (a proof of such inequality would lead to a better understanding of the problem).

So in this paper, we will restrict our study to the spheres of any dimension. The operator on the sphere reads as follow

\[
P_n = \begin{cases} 
\prod_{k=0}^{\frac{n-2}{2}} (-\Delta_{g_0} + k(n-k-1)) & \text{if } n \text{ is even.} \\
\left(-\Delta_{g_0} + \frac{n-2}{2}\right)^{\frac{1}{2}} \prod_{k=0}^{\frac{n-3}{2}} (-\Delta_{g_0} + k(n-k-1)) & \text{if } n \text{ is odd.} 
\end{cases}
\]

This last definition is to be understood in the sense of spectral decomposition using spherical harmonics on the sphere. By stereographic projection, we transform it to a problem in \(\mathbb{R}^n\). Hence, the problem that we want to solve takes the form

\[
(-\Delta)^{\frac{n}{2}} u = K(x)e^{nu(x)} + \gamma_n \sum_{i=1}^{k} c_i \delta_{x_i^*}.
\]

Where \(x_i^*\) are given points in \(\mathbb{R}^n\) and \(\gamma_n\) a constant depending on \(n\). The operator \((-\Delta)^{\frac{n}{2}}\) is to be understood, if \(n\) is odd, in the sense of distribution as the composition \((-\Delta)^{\frac{n}{4}} \circ (-\Delta)^{\frac{n}{4}}\), where \((-\Delta)^{\frac{n}{4}}\) is defined on tempered distributions as in [32]. For a better understanding of this operator, we refer to the paper [21] and the references therein. A function \(H\) is called \(\frac{n}{2}\)-Harmonic if it satisfies \((-\Delta)^{\frac{n}{2}} H = 0\).

In order to transform the equation to a better one from an analytical point of view, we do a change of variable \(\tilde{u} = u + \sum_{1 \leq i \leq k} \gamma_n c_i G(x, x_i^*)\). The equation then transforms to

\[
(-\Delta)^{\frac{n}{2}} \tilde{u} = K \prod_{1 \leq i \leq k} |x - x_i^*|^{n c_i} e^{nu}.
\]

We will consider the following assumptions on the fixed sign function \(K\), after fixing an \(n\)-Harmonic function on \(\mathbb{R}^n\):

1. There exists a non-negative number \(q\) so that
   \[
   \int_{\mathbb{R}^n} \frac{|K(x)|}{\prod_{1 \leq i \leq k} |x - x_i^*|^{nc_i}} e^{nH(x)} |x|^q \, dx < \infty.
   \]
   And

2. We will call \(\alpha = \max_{1 \leq i \leq k} nc_i\) and assume that there exist \(\gamma > 0\) so that
   \[
   \int_{B_1(y)} |K(x)| e^{nH(x)} |x - y|^{-\gamma} \prod_{1 \leq i \leq k} |x - x_i^*|^{-nc_i} \, dx \to 0
   \]
as \(y \to \infty\).

Under those assumptions, we have the following
Theorem 1.1. Let $H$ be a $\frac{n}{2}$-Harmonic function on $\mathbb{R}^n$ and $K(x)$ a smooth signed function satisfying (A1) and (A2). Then there exists a one parameter family of distributional solution to problem (4), in the space $W^{n,p}_\text{loc}(\mathbb{R}^n)$ with $p \in [1, \frac{n}{2})$. Moreover, if $K \in C^{0,\delta}$ for some $\delta > 0$ we have that $u \in W^{n,p}_\text{loc}(\mathbb{R}^n) \cap C^{n,\alpha}(\mathbb{R}^n - \{x_1, \ldots, x_k\})$, for all $p \in [1, \frac{n}{2})$. This family depends on a parameter $\beta \in (\beta^*, 0)$ if $K \leq 0$ and $\beta \in [0, 2(n - \alpha))$ if $K \geq 0$.

In fact one have a good idea on the behaviour of the solutions that we get from the previous theorem

Corollary 1. The solutions obtained in the previous theorem have the following expansion at infinity :

$$u(x) = H(x) - \left(\frac{\beta}{n} + \frac{1}{\gamma_n} \sum_{i=1}^{k} c_i \right) \ln |x| + o(\ln(|x|)),$$

where $\beta$ varies as in the previous theorem. Moreover if $K < 0$ then this solution is unique.

Those kind of solutions where obtained in the even dimensional case by Chang and Chen [8] and in the odd dimensional case by Jin et al. [21] via a variational argument and when there is no conical singularities (that is all the $c_i = 0$). This result hence can be seen as an extension of theirs but also with a different and more general approach.

We even prove an existence result in the case of a worse singularity of the metric. More precisely a metric that has a singular set concentrated at a curve. Therefore this method seems to be effective in solving the prescribing $Q$-curvature problem in highly degenerate manifolds or stratified spaces as it was done for the Yamabe type problem in [1].

As we mentioned before, we will not use the PDE method developed in [12] and [13]. We will use here a probabilistic method based on a statistical mechanical approach. We will give the proof on a specific case, that is in dimension 3 and with just one Dirac mass prescribed. But there is no difference for the general case. This method was used in several contexts, the reader for instance can check the work [30], [23], [24] and [11]. Also a good reference for understanding the idea and the physical interpretation is the book [18]. In particular, we follow here the approach developed by S. Chanillo and M. Kiessling [11]. The sign assumption on $K$ is crucial for this approach to work, though we believe that this condition can be relaxed as the method was extended by Y. Wang [34] in her thesis to allow the change of sign of the curvature $K$.

2. Overview of the probabilistic method. We first introduce some probabilistic tools. For each $N \in \mathbb{N}$, we denote the probability measures on $\mathbb{R}^{3N}$ by $P(\mathbb{R}^{3N})$. For a probability measure $\varrho^{(N)} \in P(\mathbb{R}^{3N})$, we denote the associated Radon measure by $\varrho^{(N)}$ and we mean by this its action on functions, that is

$$\varrho^{(N)}(f) = \int_{\mathbb{R}^{3N}} f(y) \varrho(dy).$$

A measure $\mu^{(N)} \in P(\mathbb{R}^{3N})$ is called absolutely continuous w.r.t. a measure $\varrho^{(N)} \in P(\mathbb{R}^{3N})$, written $d\mu^{(N)} << d\varrho^{(N)}$, if there exists a positive $d\varrho^{(N)}$-integrable function $f(x_1, \ldots, x_N)$, called the density of $\mu^{(N)}$ w.r.t. $\varrho^{(N)}$, such that $d\mu^{(N)} = f(x_1, \ldots, x_N) d\varrho^{(N)}$. By $P^*(\mathbb{R}^{3N})$ we denote the exchangeable probabilities, i.e. the
subset of $P(\mathbb{R}^{3N})$ whose elements are permutation symmetric in $x_1, \ldots, x_N \in \mathbb{R}^3$. The $n^{th}$ marginal measure of $\varrho^{(N)} \in P^s(\mathbb{R}^{3N})$, $n < N$, is an element of $P^s(\mathbb{R}^{3n})$, given by integrating $\varrho^{(N)}$ with respect to $N-n$ variable. More precisely, given a measurable set $A \subset \mathbb{R}^{3n}$, then the $n^{th}$ marginal $\varrho^{(N)}_n(A)$ is given by

$$
\varrho^{(N)}_n(A) = \varrho^{(N)}(A \times \mathbb{R}^{3(N-n)}).
$$

We let $\Omega = (\mathbb{R}^3)^N$ the set of sequences with value in $\mathbb{R}^3$.

**Definition 2.1.** A sequence of probability measures $((\varrho_k)_{k \in \mathbb{N}}$ is called tight, if for every $\varepsilon > 0$, there exists a compact set $K$ such that $\varrho_k(K) > 1 - \varepsilon$, for every $k \in \mathbb{N}$.

Roughly speaking, this notion, is to say that there is no mass that escapes to infinity along the sequence. This guaranties somehow that the limiting measure is a probability measure, if we do have convergence. Indeed, we have the following theorem.

**Theorem 2.2.** Let $(\varrho_k)$ be a tight sequence of probability measures, then it is relatively compact. That is, we can extract a convergent subsequence, that weakly converge to a probability measure $P$.

The next result that we need here is about the convergence of the marginals. First let us define a compatibility concept. Let $P$ be a probability measure on $\Omega$. We define the projection $\pi_{m,k}$ on the space $\Omega$ by $\pi_{m,k}(w) = (w_{m+1}, \ldots, w_{m+k})$. Together with the projection maps $\phi_k$ and $\psi_k$, for $k \geq 2$, from $\mathbb{R}^{3k}$ to $\mathbb{R}^{3(k-1)}$, satisfying

$$
\psi_k \circ \pi_{m,k} = \pi_{m,k-1}
$$

and

$$
\phi_k \circ \pi_{m,k} = \pi_{m+1,k-1}.
$$

A sequence of probability measures $((\mu_{m,k})_{m \in \mathbb{N}}$ on $(\mathbb{R}^3)^k$ is said to satisfy the compatibility condition, if $\mu_{m,k-1} = \mu_{m,k}\psi_k^{-1}$ and $\mu_{m+1,k-1} = \mu_{m,k}\phi_k^{-1}$. This is to be understood as follows: for a measurable set $A \subset \mathbb{R}^{3(k-1)}$,

$$
\mu_{m,k-1}(A) = \mu_{m,k}(\psi_k^{-1}(A))
$$

and

$$
\mu_{m+1,k-1}(A) = \mu_{m,k}(\phi_k^{-1}(A)).
$$

**Theorem 2.3.** Let $(\mu_{m,k})_{m \in \mathbb{N}}$ be a sequence of probability measures on $\mathbb{R}^{3k}$, that satisfies the compatibility condition. Then there exists a unique probability measure $P$ on $\Omega$, such that $\mu_{m,k} = P\pi_{m,k}^{-1}$.

In other words, there exist a unique probability measure on $\Omega$ such that its marginals corresponds to the original sequence $\mu_{m,k}$. We refer to the book of P. Billingsley [5] (Appendix A.6).

To $\varrho \in P(\mathbb{R}^3)$ we assign the energy functional defined by

$$
E(\varrho) = \frac{1}{2} \int \varrho^2(\ln|x-y|) = \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \ln|x-y|\varrho(dx)\varrho(dy),
$$

whenever the integral on the right exists. We denote by $P_E(\mathbb{R}^3)$ the subset of $P(\mathbb{R}^3)$ for which $E(\varrho)$ exists. For $\mu \in P^s(\Omega)$ the mean energy of $\mu$ is defined by

$$
e(\mu) = \frac{1}{2} \mu^2(\ln|x-y|),
$$

whenever the integral on the right exists. Again the following proposition holds.
Proposition 1. The mean energy of \( \mu \), is well defined for those \( \mu \) whose decomposition measure \( \nu(d\varrho|\mu) \) is concentrated on \( P_\varepsilon(\mathbb{R}^3) \), and in that case given by

\[
e(\mu) = \int_{P_\varepsilon(\mathbb{R}^3)} \nu(d\varrho|\mu) \mathcal{E}(\varrho).
\] (7)

The idea of this approach originates from statistical mechanics. Indeed it consists of considering a number of \( N \) particles on the sphere (or on a given set \( \Sigma \)), with potential energy governed by a Hamiltonian of the form

\[
H^{(N)} = \frac{1}{2N} \sum_{i \neq j=1}^{N} V(x_i, x_j).
\]

Where \( V \) is the potential of interaction. In our case, \( V(x, y) = \ln |x - y| \). One can also add a kinetic energy term making the energy less singular in certain times. Those particles are assumed to be distributed on the surface according to the canonical ensemble

\[
\frac{1}{Z(N)} e^{-\beta H^{(N)}}.
\]

In fact, this does not correspond to the precise distribution, but rather its Laplace transform, to avoid having singular Dirac measures. This is well explained in the paper of M. Kiessling [25]. The last step is then to investigate the structure of a typical particle configurations as \( N \) grows to infinity. The reader can for instance check the paper of Messer and Spohn [30], where they use bounded interaction potential in a bounded set.

3. Proof of Theorem 1.1. We will give the proof in the case \( n = 3 \) where the operator is non-local. Up to minor changes the proof is the same for the other cases.

We will start first by prescribing one conical singularity with a coefficient \( c_1 \). Let \( K \geq 0 \) be an \( L^\infty \) function such that for some entire \( \frac{3}{2} \)-Harmonic function \( H \), we have

\[
\int_{B_1(x)} K(y)|x - y|^{-\gamma} |y - x^*|^{-\alpha} e^{3H(y)} dy \to 0,
\] (8)

for \( 0 < \gamma < 3 - \alpha \). And for some \( q > 0 \), we have

\[
\int_{\mathbb{R}^3} K(y)|y - x^*|^{-\alpha} e^{3H(y)} |y|^q dy < \infty
\] (9)

We set \( q^*(K, H) = \sup\{q; (9) \text{ holds}\} \).

3.1. First properties of the probability measures. First, we define the measure

\[
\tau(dx) = K(x) |x - x^*|^{-\alpha} e^{3H(x)} dx
\]

Clearly we have from (9) that

\[
M^{(1)} = \int_{\mathbb{R}^3} K(y)|y - x^*|^{-\alpha} e^{3H(y)} dy < \infty,
\]

hence, we consider the probability measure \( \mu^{(1)}(dx) = \frac{1}{M^{(1)}} \tau(dx) \).

We set now \( R^{(N)}(x_1, ..., x_N) = \prod_{1 \leq i < j \leq N} |x_i - x_j| \) and \( \beta^* = -2q^* \).

We introduce now the probability measure

\[
\mu^{(N)} = \frac{1}{M^{(N)(\beta)}} \left( R^{(N)} \right)^{\frac{\beta^*}{2}} \prod_{1 \leq i \leq N} \tau(dx_i)
\] (10)
For each \( g^{(N)}(dx_1...dx_N) \in P(\mathbb{R}^{3N}) \), its entropy with respect to the probability measure \( \mu(\cdot)(dx_1 \otimes ... \otimes dx_\beta) = \mu(\cdot)(dx_1...dx_N) \) is defined by

\[
S^{(\beta, N)}(g^{(N)}) = -\int_{\mathbb{R}^{3N}} \ln \left( \frac{d\mu^{(N)}}{d\mu(\cdot)(\otimes N)} \right) g^{(N)}(dx_1...dx_N)
\]  

(11)

if \( g^{(N)} \) is absolutely continuous w.r.t. \( d\tau \otimes N \), and provided the integral exists. In all other cases, \( S^{(\beta, N)}(g^{(N)}) = -\infty \). In particular, if \( \mu_n \) is the \( n \)-th marginal of a measure \( \mu \in P^p(\Omega) \), then the entropy of \( \mu_n \), \( n \in \{1, ..., N \} \), is given by \( S^{(n)}(\mu_n) \), with \( S^{(n)} \) defined as in (11) with \( g^{(n)} = \mu_n \). We also define \( S^{(0)}(\mu_0) = 0 \).

After defining the entropy function, we state some of its classical properties we can see for instance [25] or [34] for more details and proofs of the following. For each \( \mu \in P^p(\Omega) \), the sequence \( n \mapsto S^{(n)}(\mu_n) \) enjoys the following

**Proposition 2. Non-positivity**

For all \( n \),

\[
S^{(n)}(\mu_n) \leq 0.
\]

**Monotonic decrease**

If \( n < n_1 \), then

\[
S^{(n_1)}(\mu_{n_1}) \leq S^{(n)}(\mu_n).
\]

**Strong sub-additivity**

For \( n_1, n_2 \leq n \), and with \( S^{(n)}(\mu_{-m}) = 0 \) for \( m > 0 \),

\[
S^{(n)}(\mu_n) \leq S^{(n_1)}(\mu_{n_1}) + S^{(n_2)}(\mu_{n_2}) + S^{(n-n_1-n_2)}(\mu_{n_1-n_2}) - S^{(n_1+n_2-n)}(\mu_{n_1+n_2-n}).
\]

As a consequence of the sub-additivity of \( S^{(n)}(\mu_n) \), the limit

\[
s(\mu) = \lim_{n \to \infty} \frac{1}{n} S^{(n)}(\mu_n)
\]

exists whenever \( \inf_n n^{-1} S^{(n)}(\mu_n) > -\infty \); otherwise \( s(\mu) = -\infty \). The quantity \( s(\mu) \) is called the mean entropy of \( \mu \in P^p(\Omega) \). The mean entropy is an affine function, moreover one has the following representation.

**Proposition 3. The mean entropy of \( \mu \), is given by**

\[
s(\mu) = \int_{P(\mathbb{R}^3)} \nu(d\nu(\mu)) \mathcal{S}(1)(\nu).
\]

After stating those important properties, we can start investigating our problem. First we have the following integrability property.

**Proposition 4. For \( \beta \in (\beta^*, 6-2\alpha) \), the measure \( \mu^{(N)} \) satisfies \( d\mu^{(N)} < d\tau \otimes N \), moreover, the associated density belongs to \( L^p(\mathbb{R}^{3N}, d\tau \otimes N) \) for \( p \in [1, \frac{3}{2\beta}] \) if \( \beta < 0 \), \( p \in [1, \infty] \) if \( \beta = 0 \) and \( p \in [1, \frac{6-2\alpha}{\beta}] \) if \( \beta \in (0, 6-2\alpha) \).**

**Proof.** The proof is mainly based on the fact that in the case \( \beta < 0 \) we use the inequality

\[
|x_i - x_j| \leq (2 + |x_i|)(2 + |x_j|),
\]

so that we have

\[
M^{(N)}(\beta) \leq \left( \int_{\mathbb{R}^3} (1 + |x|)^{\frac{\beta}{2}} \tau(dx) \right)^N.
\]

Hence, if \( -\frac{\beta}{2} < q^* \), \( M^{(N)}(\beta) \) is finite.
The case of $\beta > 0$ follows from the fact that
\[
M^{(N)}(\beta) \leq \left( \sup_x \int_{\mathbb{R}^3} |x - y|^{-\beta} \tau(dy) \right)^{N-1} M^{(1)},
\]
but from assumption (8), $\int_{B_1(x)} |x - y|^{-\beta} \tau(dy)$ is continuous, bounded and converges to zero at infinity.

For higher integrability, we just iterate the previous estimates for $p\beta$ instead of $\beta$.

We set now the approximated variational problem by defining the functional $\mathcal{F}_\beta^{(N)}$ as follows
\[
\mathcal{F}_\beta^{(N)}(\rho^{(N)}) = \beta \rho^{(N)} \left( \ln \left( R^{(N)} \right) \right) - N \mathcal{S}^{(N)}(\rho^{(N)}),
\]
This functional is well defined on probability measures in $P(\mathbb{R}^{3N}) \cap \cup_{p \geq 1} L^p(\mathbb{R}^{3N}, d\mu^{(1) \otimes N})$ that are absolutely continuous with respect to $\tau^{\otimes N}$. We will denote their space by $X_N$.

**Lemma 3.1.** For $\beta \in (\beta^*, 6 - 2\alpha)$ the functional $\mathcal{F}_\beta^{(N)}$ has a unique minimum and it is achieved by the measure $\mu^{(N)}$. That is
\[
\inf_{\rho^{(N)} \in X_N} \mathcal{F}_\beta^{(N)}(\rho^{(N)}) = \mathcal{F}_\beta^{(N)}(\mu^{(N)}).
\]
Moreover,
\[
\mathcal{F}_\beta^{(N)}(\mu^{(N)}) = -N \ln \left( \frac{\rho^{(N)}}{\mu^{(N)}} \right)^{\frac{\beta}{N}}.
\]

**Proof.** Since $\ln(R^{(N)}) \in L^p(\mathbb{R}^{3N}, d\tau^{\otimes N})$ for all $p \in [1, \infty)$ by Lemma 3.4, $\mathcal{F}_\beta^{(N)}(\mu^{(N)})$ is well defined for $\beta \in (\beta^*, 6 - 2\alpha)$. An explicit computation gives exactly, equation (14).

Now,
\[
\mathcal{F}_\beta^{(N)}(\rho^{(N)}) = \beta \int_{\mathbb{R}^{3N}} \ln(R^{(N)}) \frac{d\rho^{(N)}}{d\mu^{(1) \otimes N}} d\mu^{(1) \otimes N}(dx_1, ..., dx_N) +
\]
\[+ N \int_{\mathbb{R}^{3N}} \ln \left( \frac{d\rho^{(N)}}{d\mu^{(1) \otimes N}} \right) \frac{d\rho^{(N)}}{d\mu^{(1) \otimes N}} d\mu^{(1) \otimes N}(dx_1, ..., dx_N)
\]

but
\[
\frac{d\rho^{(N)}}{d\mu^{(1) \otimes N}} = \frac{(M(1))^N}{M^{(N)}(\beta)} \left( R^{(N)} \right)^{-\frac{\beta}{N}} \frac{d\rho^{(N)}}{d\mu^{(N)}}.
\]

Hence
\[
\mathcal{F}_\beta^{(N)}(\rho^{(N)}) = N \int_{\mathbb{R}^{3N}} \ln \left( \frac{d\rho^{(N)}}{d\mu^{(N)}} \right) \rho^{(N)}(dx_1, ..., dx_N) + N \ln \left( \frac{(M(1))^N}{M^{(N)}(\beta)} \right)
\]
\[= N \int_{\mathbb{R}^{3N}} \ln \left( \frac{d\rho^{(N)}}{d\mu^{(N)}} \right) \rho^{(N)}(dx_1, ..., dx_N) + \mathcal{F}_\beta^{(N)}(\mu^{(N)}),
\]
and using the fact that $x \ln x \geq x - 1$, with equality iff $x = 1$, we find that
\[
\mathcal{F}_\beta^{(N)}(\rho^{(N)}) - \mathcal{F}_\beta^{(N)}(\mu^{(N)}) \geq 0,
\]
with equality holding if and only if $\rho^{(N)} = \mu^{(N)}$.

This proves Lemma 3.4 for $\beta \in (\beta^*, 6 - 2\alpha)$.  \(\square\)
The rest of this section will be dedicated to the study of the convergence of this sequence of minimization problems, as \( N \to \infty \).

3.2. **Bound from above.** We present in this part a lower bound for the limit of the rescaled sequence. This is done through several steps. First we start by proving the following result

**Lemma 3.2.** The function \( \beta \mapsto F(\beta) \) defined by

\[
F(\beta) \equiv \inf_{\varrho \in X_1} F_\beta(\varrho)
\]

is continuous for all \( \beta \in (\beta^*, 6 - 2\alpha) \).

**Proof.** By evaluating a trial product measure \( \varrho^{(N)} = \varrho^{\otimes N} \in P(\mathbb{R}^{3N}) \), with \( \varrho \in P(\mathbb{R}^3) \cap L^p(\mathbb{R}^3, d\tau) \) for some \( p > 1 \), we find

\[
\frac{1}{N^2} F_\beta^{(N)}(\mu^{(N)}) \leq \frac{1}{N^2} F_\beta^{(N)}(\varrho^{\otimes N}) = (1 - \frac{1}{N}) \beta \mathcal{E}(\varrho) - S^{(1)}(\varrho)
\]

for all \( N > 1 \). Now, using the fact that

\[
F_\beta^{(N)}(\mu^{(N)}) = N \ln \left( \frac{(M^{(1)})^N}{M^{(N)}(\beta)} \right)
\]

by the bounds that we got before for \( M^{(N)}(\beta) \) in the negative and positive case of \( \beta \), the left side in (15) is uniformly bounded below. Letting \( N \to \infty \) in (15) we obtain a lower bound for \( F_\beta(\varrho) \), uniformly over \( P(\mathbb{R}^3) \cap L^p(\mathbb{R}^3, d\tau) \), \( p > 1 \), for each \( \beta \in (\beta^*, 6 - 2\alpha) \). Thus,

\[
\beta \mathcal{E}(\varrho) - S^{(1)}(\varrho) \geq \limsup_{N \to \infty} \frac{1}{N^2} F_\beta^{(N)}(\mu^{(N)}) \geq \liminf_{N \to \infty} \frac{1}{N^2} F_\beta^{(N)}(\mu^{(N)}) \geq f_0(\beta) \quad (16)
\]

with

\[
f_0(\beta) = \begin{cases} -\ln \int_{\mathbb{R}^3} (2 + |x|)^{-\beta/2} \mu^{(1)}(dx) & \text{for } \beta \leq 0 \\ -\ln \sup_x \int_{\mathbb{R}^3} |x - y|^{-\beta/2} \mu^{(1)}(dy) & \text{for } \beta \geq 0 \end{cases}
\]

This proves that \( F_\beta \) is bounded below for \( \beta \in (\beta^*, 6 - 2\alpha) \).

Having a lower bound, continuity of \( F \) now follows from its definition. Indeed, assume that \( F \) is discontinuous at \( \beta_0 \in (\beta^*, 6 - \alpha) \). Without loss of generality, we can assume \( F(\beta_0^-) > F(\beta_0^+) \). Now let \( \beta = \beta_0 + \varepsilon \). Clearly, for each \( \varepsilon \) we can find a minimizing sequence \( \{ \varrho_k \}_{k \in \mathbb{N}} \) (depending on \( \varepsilon \)) such that \( F_{\beta_0}(\varrho_k) < F(\beta^+) + \delta \) if \( k > M(\delta) \). Pick a sufficiently small \( \delta \) and select a \( \varrho_\ast \in \{ \varrho_k \}_{k > M(\delta)} \). Insert this \( \varrho_\ast \) into \( F_{\beta_0-\varepsilon} \).

Using \( F_{\beta} = \beta \mathcal{E} - S^{(1)} \), one gets, for any \( \varepsilon \) and \( \delta \),

\[
F(\beta_0 - \varepsilon) \leq F_{\beta_0-\varepsilon}(\varrho_\ast) = F_{\beta_0+\varepsilon}(\varrho_\ast) - 2\varepsilon \mathcal{E}(\varrho_\ast) \leq F(\beta_0 + \varepsilon) + \delta - 2\varepsilon \mathcal{E}(\varrho_\ast)
\]

Letting \( \varepsilon \to 0 \) and \( \delta \to 0 \) we obtain \( F(\beta_0^-) \leq F(\beta_0^+) \), which is a contradiction. \( \square \)

Taking the infimum over \( \varrho \) in (15), letting \( N \to \infty \), and using Lemma 3.4, gives the proof of the following

**Proposition 5.** For all \( \beta \in (\beta^*, 6 - 2\alpha) \),

\[
\limsup_{N \to \infty} \frac{1}{N^2} F_\beta^{(N)}(\mu^{(N)}) \leq F(\beta)
\]
3.3. Bound from below.

**Proposition 6.** For all $\beta \in (\beta^*, 6 - 2\alpha)$,

$$\liminf_{N \to \infty} \frac{1}{N^2} \tau^{(N)}_{\beta} \left( \mu^{(N)} \right) \geq F(\beta).$$

To prove the last Proposition, we need to show that the sequence of the $n^{th}$ marginal measures $\mu^{(N)}_n$ is not leaking at $\infty$ as $N \to \infty$. When $\beta > 0$, we also need to show that the sequences of the densities $d\mu^{(N)}_n/d\tau^{\otimes n}$ of these marginal measures are uniformly in $L^p(\mathbb{R}^{2n}, d\tau^{\otimes n})$ for $N > N_0(\beta)$. However, since it gives a-priori regularity, we prove uniform $L^p$ bounds for all $\beta \in (\beta^*, 6 - 2\alpha)$.

We begin by deriving bounds on the expected value of $\ln R^{(N)}$ w.r.t. $\mu^{(N)}$ which, using permutation symmetry, can be written in terms of $\hat{\mu}_2^{(N)}$,

$$\hat{\mu}^{(N)}(\ln R^{(N)}) = N(N-1) \hat{\mu}_2^{(N)}(\ln |x - y|)$$

**Lemma 3.3.** For each $\beta \in (\beta^*, 6 - 2\alpha)$, there exist constants $C_1(\beta)$ and $C_2(\beta)$, independent of $N$, such that for all $N \geq 2$ we have the estimates

$$C_1(\beta) \geq \beta \hat{\mu}^{(1)\otimes 2}(\ln |x - y|) \geq \beta \hat{\mu}_2^{(N)}(\ln |x - y|) \geq C_2(\beta). \quad (17)$$

**Proof.** The first inequality in (17)

$$\hat{\mu}^{(1)\otimes 2}(\ln |x - y|) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \ln |x - y| \mu^{(1)}(dx)\mu^{(1)}(dy)$$

$$= \frac{1}{(M^{(1)})^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \ln |x - y| \tau(dx)\tau(dy)$$

using the fact that $\ln |x - y| \leq C |x - y|^{-\gamma}$ if $|x - y| \leq 1$ we have the required bound $C_1(\beta)$.

To obtain the second inequality, we study the functions $\beta \mapsto f_N(\beta)$, $N > 1$, given by

$$f_N(\beta) = -\frac{2}{N-1} \beta \hat{\mu}_{(N)}^{-\beta/N} \left( (R^{(N)})^{-\beta/N} \right),$$

for $\beta \in (\beta^*, 6 - \alpha)$.

Jensen’s inequality w.r.t. $\mu^{(1)\otimes N}$ applied for the previous formula, gives us

$$f_N(\beta) \leq \frac{2}{N-1} \frac{\beta}{N} \hat{\mu}_{(N)}^{(1)\otimes N} \left( \ln(R^{(N)}) \right),$$

Using symmetry, we have

$$f_N(\beta) \leq \beta \hat{\mu}_{(N)}^{(1)\otimes 2}(\ln |x - y|).$$

On the other hand, $N(N-1)f_N(\beta) = 2F^{(N)}_{\beta} (\mu^{(N)})$. Therefore, by Lemma 3.4 and the negativity of $S^{(N)}$, we have

$$f_N(\beta) = \beta \hat{\mu}_2^{(N)}(\ln |x - y|) - \frac{2}{N-1} S^{(N)}(\mu^{(N)}) \geq \beta \hat{\mu}_2^{(N)}(\ln |x - y|).$$

To prove the third estimate in (17), we note that for any $\beta \in (\beta^*, 6 - 2\alpha)$, there exists a small $\varepsilon > 0$ such that $(1 + \varepsilon)\beta \in (\beta^*, 6 - 2\alpha)$. Again, we use Jensen’s inequality w.r.t. $\mu^{(N)}$, 

|
\[ M^{(N)}((1 + \varepsilon)\beta) = \int_{\mathbb{R}^N} \left( R^{(N)} \right)^{\frac{-\beta}{N}} \left( R^{(N)} \right)^{\frac{-\beta}{N}} \tau(dx_1) \cdots \tau(dx_n) \]  
\[ = M^{(N)}(\beta) \hat{\mu}^{(N)} \left( \left( R^{(N)} \right)^{\frac{-\beta}{N}} \right) \]  
\[ \geq M^{(N)}(\beta) \exp \left( -\frac{1}{2} (N - 1)\varepsilon \beta \hat{\mu}_2^{(N)}(\ln |x - y|) \right). \]  

Now using the fact that  
\[ N(N - 1)f_N(\beta) = 2F^{(N)}(\mu^{(N)}) = 2N \ln \left( \frac{(M^{(1)})^N}{M^{(N)}(\beta)} \right), \]  
dividing (20) by \((M^{(1)})^N\), taking the logarithm and then multiplying by \(-2/(N - 1)\) gives  
\[ f_N((1 + \varepsilon)\beta) \leq f_N(\beta) + \varepsilon \beta \hat{\mu}_2^{(N)}(\ln |x - y|). \]  
Notice then that \(f_N(\beta)\) is bounded above and below independently of \(N\), \(N > 1\), for  
\[ 2F(\beta) \geq (1 - N^{-1})f_N(\beta) \geq 2f_0(\beta), \]  
\[ \beta \in (\beta^*, 6 - 2\alpha), \text{ and since } 1 - N^{-1} \to 1. \]  

The first inequality in (21) is Proposition 3.6, the second is (16). We now obtain, for \(N > 1\),  
\[ \beta \hat{\mu}_2^{(N)}(\ln |x - y|) \geq \frac{1}{\varepsilon} (f_N((1 + \varepsilon)\beta) - f_N(\beta)) \geq \frac{2}{(1 - N^{-1})\varepsilon} (f_0((1 + \varepsilon)\beta) - F(\beta)) \geq C_2(\beta) \]  
uniformly in \(N\), for all \(\beta \in (\beta^*, 6 - 2\alpha)\). \(\square\)

Next we need a uniform bound on the quantity \(\hat{\mu}^{(1)} \otimes \hat{\mu}_1^{(N)}(\ln |x - y|)\).

**Lemma 3.4.** Let \(\beta \in (\beta^*, 6 - 2\alpha)\) and \(N \geq 1\), then there exist a constant \(\tilde{C}(\beta)\) independent of \(N\), such that,  
\[ \beta \hat{\mu}^{(1)} \otimes \hat{\mu}_1^{(N)}(\ln |x - y|) \leq \tilde{C}(\beta). \]  

The proof of this lemma follows the same idea again as in [11], that needs to be adapted to our singular case. We will give it here for the sake of completion. The reader can check the book [19] for more details about the theory of functional integration.

**Proof.** For \(\beta = 0\) the statement is obvious.

For \(\beta \in (\beta^*, 0)\), we have  
\[ \beta \hat{\mu}^{(1)} \otimes \hat{\mu}_1^{(N)}(\ln |x - y|) \leq \hat{\mu}_1^{(N)} \left( \int_{B_1(z)} \beta \ln |x - y| \mu^{(1)}(dy) \right) \leq \hat{\mu}_1^{(N)}(\tilde{C}(\beta)) = \tilde{C}(\beta). \]
The second estimate follows from the fact that $\Psi : x \mapsto \int_{B_1(x)} \ln|x-y|\mu^{(1)}(dy) \in C^0(\mathbb{R}^3)$ because of the assumption (8). Now for $\beta \in (0, 6 - 2\alpha)$, we use (12) to estimate

$$\hat{\mu}^{(1)} \otimes \hat{\mu}_1^{(N)}(\ln|x-y|) \leq \hat{\mu}^{(1)}(\ln(2 + |x|)) + \hat{\mu}_1^{(N)}(\ln(2 + |y|)).$$

By (9),

$$\hat{\mu}^{(1)}(\ln(2 + |x|)) = C_1 < \infty.$$

Also from the same estimate, we have for $q \in (0, q^*)$,

$$\int_{\mathbb{R}^3} \exp(q \ln(2 + |y|)) \tau(dy) = C_2 < \infty.$$

To use the previous bound, we will make use of the convexity of the exponential function and use convex duality as follows

$$e^X + (Y \ln(Y) - Y) \geq XY.$$

So we start by writing all $\beta \in (0, 6 - 2\alpha)$,

$$\hat{\mu}_1^{(N)}(\ln(2 + |y|)) = \frac{M^{(N-1)}(\beta')}{M^{(N)}(\beta)} \int_{\mathbb{R}^{3(N-1)}} \frac{1}{M^{(N-1)}(\beta') |R^{(N-1)}|^{\frac{\beta'}{N-1}}} \times$$

$$\times \left( \int_{\mathbb{R}^3} \left( \prod_{1 \leq i < N} |x_i - y|^{\frac{\beta}{N-1}} \ln(2 + |y|) \right) \tau(dy) \right) \prod_{1 \leq i < N} \tau(dx_i),$$

where $\beta' = (1 - \frac{1}{N}) \beta$.

Now, by setting

$$Y = e^{q \ln(2 + |y|)}$$

and

$$X = \frac{1}{q} \int_{\mathbb{R}^3} \frac{1}{M^{(N-1)}(\beta') |R^{(N-1)}|^{\frac{\beta'}{N-1}}} \left( \prod_{1 \leq i < N} |x_i - y|^{\frac{\beta}{N-1}} \right) \prod_{1 \leq i < N} \tau(dx_i),$$

one has

$$\hat{\mu}_1^{(N)}(\ln(2 + |y|)) - \frac{M^{(N-1)}(\beta')}{M^{(N)}(\beta)} \int_{\mathbb{R}^3} e^{(q \ln(2 + |y|))} \tau(dy)$$

$$\leq -\frac{1}{q} \left( 1 + \ln q + \beta' \hat{\mu}_2^{(N)}(\ln|x-y|) \right)$$

$$\leq C^*(\beta).$$

In (22), $C^*(\beta)$ is independent of $N$, by Lemma 3.8. So to finish the proof, one needs to find a bound for $M^{(N-1)}(\beta')/M^{(N)}(\beta)$ from above uniformly in $N$, for each $\beta \in (0, 6 - 2\alpha)$.

To carry out this last step, we will regularize $M^{(N)}$ and prove an $N$-independent upper bound on the regularized ratio of $M$’s which is independent of the regularization parameter. We regularize $\ln|x-y|$ using the Lebesgue convergence formula as follow

$$-V_\epsilon(x, y) \equiv \left( \frac{4}{3} \pi \right)^2 \epsilon^{-6} \int_{B_\epsilon(x)} \int_{B_\epsilon(y)} \ln|\xi - \eta| d\xi d\eta.$$
As in [11], let \( \mathcal{H}_\varepsilon \) denote the Hilbert space obtained by completing the \( C^\infty(\mathbb{R}^3) \)
functions with vanishing integral, \( \int_{\mathbb{R}^3} f(x)dx = 0 \), with respect to the positive definite inner product

\[
\langle f, f \rangle_\varepsilon = N^{-1} \beta \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(x)V_\varepsilon(x, y)f(y)dxdy.
\]

If \( B^1 \equiv B_{1/\sqrt{\pi}}(0) \) denotes the ball of volume 1 centred at the origin, and \( \delta_y(x) \)
is the Dirac measure on \( \mathbb{R}^3 \) concentrated at \( y \), we note that

\[
\delta^\varepsilon_y(x) = \delta_y(x) - \chi_{B^1}(x), \in \mathcal{H}_\varepsilon.
\]

Accordingly,

\[
\delta^\varepsilon_{(N)}(x) \equiv \sum_{k=1}^N \delta^\varepsilon_{#k}(x) \in \mathcal{H}_\varepsilon,
\]
as well. We now define

\[
W_\varepsilon(x) \equiv \int_{B^1} V_\varepsilon(x, y)dy - \frac{1}{2} \int_{B^1} \int_{B^1} V_\varepsilon(x, y)dxdy
\]
and write

\[
\tau(dx) = e^{\beta W_\varepsilon(x)}\tilde{\tau}(dx).
\]

Note that, unless \( q^* > \beta \), \( \tilde{\tau} \) does not have finite mass, but we can write \( M^\varepsilon_{(N)}(\beta) \)
for \( M^{(N)}(\beta) \) with \( -\ln |x - y| \) replaced by \( V_\varepsilon(x, y) \). Recall that we can write

\[
M^{(N)}(\beta) = \int_{\mathbb{R}^{3N}} e^{-\frac{1}{\beta} \sum \ln|x_i - x_j|} \prod_{1 \leq l \leq N} \tau(dx_l).
\]

Therefore, we have

\[
M^\varepsilon_{(N)}(\beta) = \int_{\mathbb{R}^{3N}} e^{-\frac{1}{\beta} \sum_{i<j} V_\varepsilon(x_i, x_j)} \prod_{1 \leq l \leq N} e^{\beta W_\varepsilon(x_l)}\tilde{\tau}(dx_l).
\]

Using the fact that

\[
V_\varepsilon(x_i, x_i) = V_\varepsilon(0, 0),
\]
the previous formula reads as

\[
M^\varepsilon_{(N)}(\beta) = e^{-\frac{1}{2} \beta V_\varepsilon(0, 0)} \int_{\mathbb{R}^{3N}} e^{\frac{1}{2} \langle \delta^\varepsilon_{(N)}, \delta^\varepsilon_{(N)} \rangle} \prod_{1 \leq l \leq N} \tilde{\tau}(dx_l).
\]

We now use Gaussian functional integrals (see [19], Appendix two part I). Using Minlos’ theorem one has that \( N^{-1} \beta V_\varepsilon(x, y) \)
is the covariance operator of a Gaussian probability measure with mean zero, i.e., there exists a Gaussian average \( \text{Ave} \) on
a space of linear functionals \( \Phi \) on \( \mathcal{H}_\varepsilon \), with \( \text{Ave} (\Phi(\delta^\varepsilon_y)) = 0 \) and \( \text{Ave} (\Phi(\delta^\varepsilon_y)\Phi(\delta^\varepsilon_y)) = N^{-1} \beta V_\varepsilon(x, y) \). Hence,

\[
\text{Ave} \left( e^{\Phi(f)} \right) = e^{\frac{1}{2} \langle f, f \rangle_\varepsilon}
\]

with \( f = \delta^\varepsilon_{(N)} \), then integrating over \( \mathbb{R}^{3N} \) with respect to \( \tilde{\tau}^{\otimes N} \), we obtain

\[
M^\varepsilon_{(N)}(\beta) = e^{-\frac{1}{2} \beta V_\varepsilon(0, 0)} \text{Ave} \left( \left( \int_{\mathbb{R}^3} e^{\Phi(\delta^\varepsilon_y)\tilde{\tau}(dx)} \right)^N \right).
\]

(23)
Again, Jensen’s inequality applied to the right side of (23) gives, in terms of the $M_ε$’s,

$$M_ε^{(N)}(β) ≥ M_ε^{(N-1)}(β') \left( M_ε^{(N-1)}(β') \right)^{1/(N-1)}$$

for all $ε$. Hence, we can now let $ε → 0$ and then $N → ∞$ to obtain

$$\limsup_{N → ∞} \frac{M^{(N-1)}(β')}{M^{(N)}(β)} ≤ \limsup_{N → ∞} \left( M^{(N-1)}(β') \right)^{-1/(N-1)}$$

$$ ≤ \frac{1}{M(1)} \limsup_{N → ∞} e^{\frac{1}{(N-1)^2} F^{(N-1)}(μ(β))} ≤ \frac{1}{M(1)} e^{F(β)}.$$ 

And this finishes the proof. 

We now prepare for uniform $L^p$ bounds.

**Lemma 3.5.** For each $n ∈ N$, $β ∈ (β^*, 6 − 2α)$, there exist $N_n(β) ∈ N$ and $C(n, β) > 0$, such that for $N > N_n$ the density of $μ_n^{(N)}$ w.r.t. $τ^{⊗n}$ is bounded by

$$\frac{dμ_n^{(N)}}{dτ^{⊗n}}(x_1, ..., x_n) ≤ C(n, β) \left( R^{(n)}(x_1, ..., x_n) \right)^{-β/N}.$$ 

**Proof.** When $β = 0$, this is trivial. When $β ≠ 0$, we begin by writing

$$\frac{dμ_n^{(N)}}{dτ^{⊗n}}(x_1, ..., x_n) = \frac{1}{M^{(N)}(β)} G(x_1, ..., x_n) \left( R^{(n)}(x_1, ..., x_n) \right)^{-β/N},$$ 

where

$$G(x_1, ..., x_n) = \int_{\mathbb{R}^{(N-n)}} \prod_{1≤i≤n<j≤N} |x_i - x_j|^{-β/N} \prod_{n<k<j≤N} |x_k - x_j|^{-β/N} \tau(dx_j).$$

Given $β ∈ (β^*, 6 − 2α)$ and let $[.]$ denote integer part. We define

$$N_n(β) = \begin{cases} \left\lfloor \frac{2β - 3}{3β - β} \right\rfloor & \text{if } β ∈ (β^*, 0) \\ \left\lfloor \frac{2(6 - 2α) - β}{6 - 2α - β} \right\rfloor & \text{if } β ∈ (0, 6 − 2α) \end{cases}$$

Then, by Hölder’s inequality,

$$G(x_1, ..., x_n) ≤ \left( \int_{\mathbb{R}^{(N-n)}} \prod_{1≤i≤n<j≤N} |x_i - x_j|^{-β/2n} \tau(dx_j) \right)^{2n/N} \times$$

$$\left( \int_{\mathbb{R}^{(N-n)}} \prod_{n<k<j≤N} |x_i - x_j|^{-β/(N-2n)} \prod_{n<k<j≤N} \tau(dx_k) \right)^{1-2n/N}. \tag{25}$$

As for the first factor on the r.h.s. of (25), permutation symmetry gives

$$\int_{\mathbb{R}^{(N-n)}} \prod_{1≤i≤n<j≤N} |x_i - x_j|^{-β/2n} \tau(dx_j) = \left( \int_{\mathbb{R}^2} \prod_{1≤i≤n} |x_i - x|^{-β/2n} \tau(dx) \right)^{N-n}. \tag{26}$$

By the arithmetic-geometric mean inequality and permutation invariance, we have
Passing to the limit, we have
\[ N \Rightarrow \infty \]
which implies an \( \beta > 0 \) the ratio \( L \). Hence, the first term on the right-hand side of (25) is bounded by the \( 2 \)-th power of the right-hand side of (28), and this is done uniformly w.r.t. \( N \).

As for the second factor on the r.h.s. of (25), we split off the \((-2n/N)\)-th power. We set \( \alpha(N) = (N - n)/(N - 2n) \). Since \( N > N_n \), we have \( 1 < \alpha(N) < 4/3 \) if \( \beta > 0 \) and \( 1 < \alpha(N) < \beta/3 \) if \( \beta < 0 \). We also have \( \alpha(N) \to 1 \) as \( N \to \infty \).

Recall now that
\[
\mathcal{F}_\beta(\mu^{(N)}) = -N \ln \left( \frac{M^{(N)}(\beta)}{(M^{(1)})^N} \right)
\]
Thus
\[
\left( \int_{\mathbb{R}^3(N-n)} \prod_{n < i < j \leq N} |x_i - x_j|^{-\beta/(N-2n)} \prod_{n < k \leq N} \tau(dx_k) \right)^{-2n/N} = \left( M^{(N-n)}(\alpha(N)\beta) \right)^{-2n} \left( M^{(1)} e^{-\frac{1}{1-\alpha(N)\beta}} \mathcal{F}_\beta^{(N-n)} \right)^{-2n(1 - \frac{\beta}{N})}.
\]
Passing to the limit, we have
\[
\limsup_{N \to \infty} \left( \int_{\mathbb{R}^3(N-n)} \prod_{n < i < j \leq N} |x_i - x_j|^{-\beta/(N-2n)} \prod_{n < k \leq N} \tau(dx_k) \right)^{-2n/N} = \left( M^{(N-n)}(\alpha(N)\beta) \right)^{-2n} \leq \left( \frac{e^{F(\beta)}}{M^{(1)}} \right)^{2n}.
\]
which implies an \( N \)-independent bound on \( (M^{(N-n)}(\alpha(N)\beta))^{-2n/N} \).

Going back to the Hölder inequality of (25), we see that
\[
G(x_1, \ldots, x_n) \leq CM^{(N-n)}(\alpha(N)\beta).
\]
This already proves that for a given \( 1 < p < \infty \), there exists \( N \) big enough so that the density (24) is \( L^p(\mathbb{R}^{3n}, d\tau^{\otimes n}) \).

To prove that \( d\mu^{(N)}_n / dr^{\otimes n} \in L^p(\mathbb{R}^{3n} d\tau^{\otimes n}) \) uniformly in \( N \), it remains to estimate the ratio \( M^{(N-n)}(\alpha(N)\beta) / M^{(N)}(\beta) \) from above, independently of \( N \).

We first use the decomposition of \( R^{(N)} = R^{(N-n)} \times K_1 \times K_2 \), where
\[
K_1 = \prod_{1 \leq i \leq N-n \quad m < j \leq N} |x_i - x_j|,
\]
and
\[
K_2 = \prod_{N-n \leq i < j \leq N} |x_i - x_j|.
\]
Using this decomposition we have
\[
M^{(N)}(\beta) = \int_{\mathbb{R}^{2N}} \left( R^{(N-n)} \right) \left( R^{(N-n)} \right)^{-\frac{\alpha(N)\beta}{N}} \left( K_1K_2 \right)^{\frac{\beta}{N}} \tau(dx_1) \ldots \tau(dx_N) \\
= M^{(N-n)}(\alpha(N)\beta) \tilde{\mu}^{(N-n),\alpha} \\
\otimes \hat{\tau}^{\otimes n}(dx_{N-n+1} \ldots dx_N) \left( R^{(N-n)} \right)^{-\frac{\alpha(N)\beta}{N}} \left( K_1K_2 \right)^{\frac{\beta}{N}}
\]

Using Jensen’s inequality for \( \mu^{(N-n),\alpha} \) we get
\[
\frac{M^{(N)}(\beta)}{M^{(N-n)}(\alpha(N)\beta)} \leq e^{\left( n\beta\alpha(N)(1 - \frac{\alpha(N)\beta}{N}) \tilde{\mu}^{(N-n),\alpha}(\ln|x-y|) \right)} \times \\
\hat{\tau}^{\otimes n}(dx_{N-n+1} \ldots dx_N) \left( e^{-\frac{\beta}{N} \tilde{\mu}^{(N-n),\alpha}(\ln|K_1|+\ln|K_2|)} \right).
\]

Again using Jensen’s inequality for the probability measure \( \mu^{(1)}(dx_{N-n+1} \otimes \ldots \otimes \mu^{(1)}(dx_N) \) we find
\[
\hat{\tau}^{\otimes n}(dx_{N-n+1} \ldots dx_N) \left( e^{-\frac{\beta}{N} \tilde{\mu}^{(N-n),\alpha}(\ln|K_1|+\ln|K_2|)} \right) \\
\geq \left( M^{(1)} \right)^n e^{-\frac{n(n-1)}{2} \beta \tilde{\mu}^{(1)\otimes 2}(\ln|x-y|)} \times \\
\times e^{-n(1 - \frac{\beta}{N}) \tilde{\mu}^{(1)} \otimes \tilde{\mu}^{(N-n),\alpha}(\ln|x-y|)},
\]

where \( \tilde{\mu}^{(N-n),\alpha} \) stands for (10) with \( \alpha(N)\beta \) in place of \( \beta \). The first exponential factor on the r.h.s. is bounded above uniformly in \( N \) because \( \beta \tilde{\mu}^{(1)\otimes 2}(\ln|x-y|) \leq C_2(\beta) \) independently of \( N \), by the first inequality in Lemma 3.8, as for the second exponential factor, by re-identifying \( N \rightarrow N-n \) and \( \beta \rightarrow \alpha(N)\beta \), the \( N \)-independent upper bound in Lemma 3.9 gives \( \beta \tilde{\mu}^{(1)} \otimes \tilde{\mu}^{(N-n),\alpha}(\ln|x-y|) \leq \tilde{C}(\alpha(N)\beta) \). Since \( \alpha(N) \rightarrow 1 \) as \( N \rightarrow \infty \), the second exponential factor is bounded above uniformly in \( N \).

As for the third exponential factor, since \( \beta^* < \alpha(N)\beta < 6 - 2\alpha \), and since (17) holds for all \( \beta \in (\beta^*, 6 - 2\alpha) \), by Lemma 3.9, we have \( \beta \tilde{\mu}^{(N-n),\alpha}(\ln|x-y|) \geq C_1(\alpha(N)\beta) \). Again since \( \alpha(N) \rightarrow 1 \), we now see that also the third exponential factor is bounded above uniformly in \( N \). This proves Lemma 3.10.

So far we established that for each triple \( n \in \mathbb{N}, \beta \in (\beta^*, 6 - 2\alpha) \), \( p \in [1, \infty) \) there exists a \( \tilde{N}_n(\beta, p) \) \( ( > N_n(\beta) ) \) such that \( d\mu^{(N)}_n/d\tau^{\otimes n} \in L^p(\mathbb{R}^3, d\tau^{\otimes n}) \) uniformly in \( N \) when \( N > \tilde{N}_n(\beta, p) \). Hence, the sequence \( N \mapsto \mu^{(N)}_n \) is \( L^p(\mathbb{R}^3, d\tau^{\otimes n}) \)-weakly compact when \( N > \tilde{N}_n(\beta, p) \), for each \( p \in [1, \infty) \).

However, a weak \( L^p \) limit point of \( \mu^{(N)}_n \) need not be a probability measure because \( \mathbb{R}^3 \) is unbounded. To prove that indeed we converge to a probability measure we need the concept of tightness (see preliminaries).

**Lemma 3.6.** For each \( n \), the sequence \( \{\mu^{(N)}_n\}_{N \geq n} \) is tight.

**Proof.** First notice that \( \mu^{(N)}_n(dx_n) = \mu^{(N)}_m(dx_m \otimes \mathbb{R}^{m-n}) \), for every \( m > n \). Hence, tightness follows if proven for \( n = 1 \). We consider then the map \( h(x) = \int_{\mathbb{R}^3} \ln|x-y|\mu^{(1)}(dy) + C \). By construction of \( \mu^{(1)} \), \( h \) is continuous and diverges to infinity as
\[|x| \to \infty \text{ uniformly in } x. \] Thus we can choose \(C\) so that \(h > 0\). Also from this fact follows the existence of \(R(\epsilon) > 0\) for every choice of \(\epsilon << 1\) so that
\[
\inf_{x \notin B_{R(\epsilon)}} h(x) > \frac{1}{\epsilon} \hat{\mu}_1^{(N)}(h(x)),
\]
This is possible indeed from Lemma 3.8.

We set now \(\mathcal{I}_\Omega\), the indicator function of the set \(\Omega\). Then one has,
\[
\hat{\mu}_1^{(N)}(h(x)) \geq \hat{\mu}_1^{(N)}(h(x)\mathcal{I}_{\mathbb{R}^3 - B_{R(\epsilon)}})
\geq \frac{1}{\epsilon} \hat{\mu}_1^{(N)}(h(x))\hat{\mu}_1^{(N)}(\mathcal{I}_{B_{R(\epsilon)}})
= \frac{1}{\epsilon} \hat{\mu}_1^{(N)}(h(x)) \left(1 - \hat{\mu}_1^{(N)}(B_{R(\epsilon)})\right).
\]
From this, one gets \(\hat{\mu}_1^{(N)}(B_{R(\epsilon)}) \geq (1 - \epsilon)\), uniformly in \(N\), which gives the tightness.

Now in order for us to finish the proof of Proposition 3.7 we also need a lower bound on the mean entropy.

**Lemma 3.7.** For each \(\beta \in (\beta^*, 6 - 2\alpha)\), there exists a \(C(\beta)\), independent of \(N\), such that
\[
\frac{1}{N} S^{(N)}(\mu^{(N)}) \geq C(\beta)
\]

**Proof.** recall that
\[
\frac{1}{N} S^{(N)}(\mu^{(N)}) = \beta \frac{1}{N^2} \hat{\mu}^{(N)}(\ln(R^{(N)})) - \frac{1}{N^2} \mathcal{I}^{(N)}(\mu^{(N)})
\]
Now from the bounds that we got in (17) and Proposition 3.6, the result follows. \(\square\)

**Proof of Proposition.** Since the sequence of probability measures \(\{\mu_n^{(N)}|N = n, n+1,...\}\) is tight in \(P(\mathbb{R}^{3n})\) for all \(n\). Using Theorem 2.2, we can extract a subsequence \(k \mapsto N_k \in \mathbb{N}\), \(k \in \mathbb{N}\), such that for each \(n \in \mathbb{N}\), \(\mu_n^{(N(k))} \to \mu^\ell \in P(\mathbb{R}^{3n})\), as \(k \to \infty\). Since the marginals are consistent by Kolmogorov’s existence Theorem 2.3, the infinite family of marginals \(\{\mu_n^\ell\}_{n \in \mathbb{N}}\) now defines a unique \(\mu^\ell \in P^\ast(\Omega)\). Furthermore, for \(\beta^* < \beta < 6 - 2\alpha\), we have from Lemma 8.12 that, for any \(n\) and any \(p \in [1, \infty]\), the sequence \(\{\mu_n^{(N)}|N = n, n+1,...\}\) is bounded in \(L^p\) with a bound depending only on \(p\), \(\beta\) and \(n\). Therefore, as \(k \to \infty\), after at most selecting a sub-subsequence (also denoted by \(k \mapsto N_k \in \mathbb{N}\), \(k \in \mathbb{N}\)), we have that
\[
d\mu_n^{(N(k))}/dT^{\otimes n} \to d\mu^\ell/dT^{\otimes n},
\]
weakly in \(L^p(\mathbb{R}^{3n},dT^{\otimes n})\), for any \(p \in [1, \infty]\).

We first study the convergence of energy. By definition of the energy, we have
\[
\frac{1}{N_k} \hat{\mu}^{(N_k)}((\ln(R^{(N_k)}))) = (1 - \frac{1}{N_k}) \frac{1}{2} \hat{\mu}_2^{(N_k)}(\ln|x - y|)
\]
Since \(\ln|x - y| \in L^q(\mathbb{R}^6,dT^{\otimes 2})\), for \(\frac{1}{q} + \frac{1}{p} = 1\), by weak \(L^p(\mathbb{R}^6,dT^{\otimes 2})\) convergence of \(\mu_2^{(N_k)}\),
\[
\frac{1}{2} \hat{\mu}_2^{(N_k)}(\ln|x - y|) \to \frac{1}{2} \hat{\mu}_2^\ell(\ln|x - y|) = e(\mu^\ell)
\]
Using the fact that $1 - N_\ell(k)^{-1} \to 1$ as $k \to \infty$, we have
\[
\lim_{k \to \infty} \frac{1}{N_\ell(k)} \mathbb{S}^{(N_\ell(k))}\left(\mu^{(N_\ell(k))}\right) = e(\mu^\ell).
\] (32)

The second quantity that we need to deal with is the Entropy. We define $m = N_\ell(k) - \lceil [N_\ell(k)/n] \rceil n$. By sub-additivity and negativity of the entropy (see Section 2), we have, for any $n < N_\ell(k)$,
\[
\frac{1}{N_\ell(k)} \mathbb{S}^{(N_\ell(k))}\left(\mu^{(N_\ell(k))}\right)
\leq \frac{1}{N_\ell(k)} \left[\frac{N_\ell(k)}{n}\right] \mathbb{S}^{(n)}\left(\mu^{(N_\ell(k))}\right) + \frac{1}{N_\ell(k)} \mathbb{S}^{(n)}\left(\mu^{(N_\ell(k))}\right)
\leq \frac{1}{N_\ell(k)} \left[\frac{N_\ell(k)}{n}\right] \mathbb{S}^{(n)}\left(\mu^{(N_\ell(k))}\right).
\]

Clearly, $N_\ell(k)^{-1}[\lceil [N_\ell(k)/n] \rceil n] \to n^{-1}$. Moreover, for each $n$, weak upper semi-continuity of $\mathbb{S}^{(n)}$ gives us
\[
\limsup_{k \to \infty} \mathbb{S}^{(n)}\left(\mu^{(N_\ell(k))}\right) \leq \mathbb{S}^{(n)}\left(\mu^\ell\right).
\] (33)

Therefore, for all $n$,
\[
\limsup_{k \to \infty} \frac{1}{N_\ell(k)} \mathbb{S}^{(N_\ell(k))}\left(\mu^{(N_\ell(k))}\right) \leq \frac{1}{n} \mathbb{S}^{(n)}\left(\mu^\ell\right).
\] (34)

Recalling Propositions 3.1 and 3.2, we see that $s(\mu)$ exists. Hence, $n \to \infty$ in (33) gives
\[
\limsup_{k \to \infty} \frac{1}{N_\ell(k)} \mathbb{S}^{(N_\ell(k))}\left(\mu^{(N_\ell(k))}\right) \leq s(\mu^\ell)
\]
for each convergent subsequence $\mu^{(N_\ell(k))} \to \mu^\ell$.

Using the estimates (32) and (34), we find that for any $\beta \in (\beta^*, 6 - 2\alpha)$,
\[
\liminf_{k \to \infty} \frac{1}{N_\ell^2(k)} \mathcal{F}^{(N_\ell(k))}\left(\mu^{(N_\ell(k))}\right) \geq \beta e(\mu^\ell) - s(\mu^\ell).
\]

Therefore, we have
\[
\beta e(\mu^\ell) - s(\mu^\ell) = \int_{\mathbb{P}(\mathbb{R}^3)} \nu(d\varphi | \mu^\ell) \mathcal{F}_\beta(\varphi) \geq F(\beta).
\]

And this finishes the proof. \hfill \Box

3.4. Convergence. Now with those propositions proved, we can go on to the proof of the main tool of our approach

**Theorem 3.8.** The sequence of probability measures $N \mapsto \mu_n^{(N)}(dx_1...dx_n)$ is the union of weakly convergent sub-sequences, in the sense that there exist disjoint sequences $E_\ell = \{N_\ell(k)\}_{k \in \mathbb{N}}$, $E_\ell \cap E_{\ell'} = \emptyset$, for $\ell \neq \ell'$, such that for each $\ell$, the map $k \mapsto \mu_n^{(N_\ell(k))}(dx_1...dx_n)$ converges weakly in the sense of probability measures, with densities w.r.t. $d\tau^\otimes n$ converging weakly in $L^p(\mathbb{R}^{3n}, d\tau^\otimes n)$, for all $p \in [1, \infty)$.

Let $\mu^\ell_n$ denote the weak limit point of such a subsequence. Then there exists a unique $\mu^\ell \in \mathbb{P}(\Omega)$ (of which $\mu^\ell_n$ is the $n$-th marginal), and $\mu^\ell$ has its decomposition measure $\nu(d\varphi | \mu^\ell)$ concentrated on the subset of $\mathbb{P}(\mathbb{R}^3) \cap \bigcup_{p \geq 1} L^p(\mathbb{R}^3, d\tau)$, whose elements minimize the functional
\[
\mathcal{F}_\beta(\varphi) = \beta \mathcal{E}(\varphi) - \mathbb{S}^{(1)}(\varphi).
\] (35)
proof of Theorem. Combining Propositions 3.6 and 3.7, we conclude that
\[
\lim_{N \to \infty} \frac{1}{N^2} \mathcal{F}_\beta^{(N)}(\mu^{(N)}) = F(\beta).
\]
From the lower bound that we got in Proposition 3.6 and the upper bound in Proposition 3.7, we see that we have
\[
\int_{\mathbb{R}^2} \nu(dx) \mathcal{F}_\beta(\varrho) = F(\beta)
\tag{36}
\]
for every limit point \( \mu^\ell \) of \( \mu^{(N)} \). In the other hand, equation (36) implies that the decomposition measure \( \nu(dx) \) is concentrated on the minimizers of \( \mathcal{F}_\beta(\varrho) \). Indeed if this was not the case then by Lemma 3.5, we would have
\[
\int_{\mathbb{R}^2} \nu(dx) \mathcal{F}_\beta(\varrho) > F(\beta),
\]
which contradicts (36).

3.5. Proof of the main Theorem. The proof now is mainly a direct corollary form Theorem 3.13, in the previous subsection. We assume that we are under the hypothesis of the previous theorem. Then the functional \( F_\beta \) defined in (35), has a minimizer for all \( \beta \in (\beta^*, 6 - 2\alpha) \). The minimizers of (35) are of the form \( \varrho(dx) = \rho(x)dx \), with \( \rho \) satisfying the Euler-Lagrange equation
\[
\rho(x) = \frac{\tilde{K}(x) \exp \left(-\beta \int_{\mathbb{R}^3} \ln |x - y| \rho(y)dy + 3H(x)\right)}{\int_{\mathbb{R}^3} K(x) \exp \left(-\beta \int_{\mathbb{R}^3} \ln |x - y| \rho(y)dy + 3H(x)\right) dx}.
\tag{37}
\]

Where \( \tilde{K}(x) = \frac{K(x)}{|x - x^*|^n} \). Recall that \( K > 0 \), by hypothesis. If \( \beta \in (0, 6 - 2\alpha) \), we now identify \( K(x) \) with a positive \( Q \)-curvature function \( K \) and if \( \beta \in (\beta^*, 0) \) we identify \( -K \) with a negative \( Q \)-curvature \( K \). In either case, \( K \) satisfies the hypotheses (A1) and (A2).

We now pick a corresponding solution of (37), say \( \rho_{H,\beta} \), which exists by Theorem 3.13. With the help of this \( \rho_{H,\beta} \), we define, for all \( x \in \mathbb{R}^3 \), the function
\[
U_{H,\beta}(x) = H(x) - \frac{\beta}{3} \int_{\mathbb{R}^2} \ln |x - y| \rho_{H,\beta}(y)dy + U_0,
\]
the constant \( U_0 \) Is to be determined later. We also set
\[
\lambda = \int_{\mathbb{R}^3} \tilde{K}(x) \exp \left(-\beta \int_{\mathbb{R}^3} \ln |x - y| \rho(y)dy + 3H(x)\right) dx
\]
So that exp \(-3U_0 = \frac{\beta}{3} 2\pi^2 \lambda \).

Using Theorem 3.13, we have that \( \rho_{H,\beta} \in L^p(\mathbb{R}^3) \) for all \( p \in [1, \frac{3}{\alpha}) \). Now the regularity will depend on \( \alpha \). For instance, \( U \) is \( C^2 \) if \( \alpha < 1 \). In the general case, \( U \) belongs to \( W_{\text{loc}}^{3,p}(\mathbb{R}^3) \) for \( p < \frac{3}{\alpha} \). But away from the singularity \( x^* \) one has a better regularity. Indeed, if \( K \in C^{0,\delta} \), then \( U \in C^{3,\delta}(\mathbb{R}^3 - \{x^*\}) \).

Now, we consider the function \( u_{H,\beta} \), defined by
\[
u_{H,\beta}(x) = U_{H,\beta} - \frac{\alpha}{3} \ln |x - y|
\]
Then clearly it is a distributional solution of
\[
(-\Delta)u = K(x) e^{3u} + 2\pi^2 \alpha \delta_x.
\]
This concludes the proof of the existence and regularity.
And Clearly from the expression of the solution we have for $|x| \to \infty$,

$$u_{H,\beta} = H(x) - \left( \frac{\beta}{3} + \frac{\alpha}{6\pi^2} \right) \ln(|x|) + o(\ln |x|)$$

4. **Uniqueness.** We will show the uniqueness for the function $\rho$ obtained in equation (37) in the case $\beta < 0$. We will assume that we have two solutions $\rho_1$ and $\rho_2$ solving the equation. We set $\rho_{1,2} = \rho_1 - \rho_2$. First we have the following Lemma:

**Lemma 4.1.** Let $\mu$ be an absolutely continuous measure on $\mathbb{R}^n$ so that $\mu(\mathbb{R}^n) = 0$, then

$$-\hat{\mu} \ast \ln |x - y| \geq 0$$

with equality iff $\mu = 0$

**Proof.** This lemma was proved in [11], in the two dimensional setting, and that proof cannot be generalized to higher dimension since it relies heavily on the dimension. We will present here another proof that holds in all dimensions.

First, we assume that $\mu$ has a compactly supported density $g$. We define now the function

$$f(x) = -\int_{\mathbb{R}^n} \ln |x - y| d\mu(y) = -\ln \ast g.$$ 

This function $f$ solves the equation

$$(-\Delta)^{\frac{1}{2}} f = \mu_\varepsilon.$$ 

We approximate again the kernel $\ln |x|$ by a smooth $L^2$ kernel $\tilde{V}_\varepsilon$ and we call $f_\varepsilon$ the corresponding convolution with $g$. Notice, that

$$-\hat{\mu} \ast (\tilde{V}_\varepsilon(x, y)) = -\int_{\mathbb{R}^n} f_\varepsilon(x) g(x) dx$$

$$= -\int_{\mathbb{R}^n} f_\varepsilon(x) \tilde{g}(x) dx = -\int_{\mathbb{R}^n} \tilde{g}(x)^2 \tilde{V}_\varepsilon d\varepsilon \geq 0.$$ 

Hence the result follows by letting $\varepsilon \to 0$. It is clear that if $-\hat{\mu} \ast (\ln |x - y|) = 0$ then $f = 0$ hence $\mu = 0$. $\square$

As in the previous proof, we will use $f$ for $-\int_{\mathbb{R}^n} \ln |x - y| \rho_{1,2}(y) dy$. Let $\mathcal{P}(\rho)$ the functional defined by

$$\mathcal{P}(\rho) = \frac{\tilde{K}(x) \exp (-\beta \int_{\mathbb{R}^n} \ln |x - y| \rho(y) dy + 3H(x))}{\int_{\mathbb{R}^n} \tilde{K}(x) \exp (-\beta \int_{\mathbb{R}^n} \ln |x - y| \rho(y) dy + 3H(x)) dx}$$

So that $\rho_i = \mathcal{P}(\rho_i)$ for $i = 1, 2$. We define as in [11], the function $\rho_\lambda = \rho_1 + \lambda \rho_{1,2}$ and for a function $u$ we will denote its expectation with respect to the probability measure $\mathcal{P}(\rho_\lambda)$ by

$$< u > (\lambda) = \int_{\mathbb{R}^n} \mathcal{P}(\rho_\lambda) u(x) dx.$$ 

Therefore we have

$$-\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho_{1,2}(x) \ln |x - y| \rho_{1,2}(y) dxdy = -\int_{\mathbb{R}^n} \rho_{1,2}(x) (\mathcal{P}(\rho_1) - \mathcal{P}(\rho_1)) dx$$

$$= -\int_{\mathbb{R}^n} \rho_{1,2}(x) \left( \int_0^1 \frac{d}{d\lambda} \mathcal{P}(\rho_\lambda) d\lambda \right) dx$$

$$= \beta \int_0^1 \langle (f < f > (\lambda))^2 \rangle (\lambda) d\lambda$$
and since $\beta \leq 0$ we have that

$$-\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho_{1,2}(x) \ln |x - y| \rho_{1,2}(y) dx dy.$$  

Using the previous lemma now, we conclude that $\rho_1 = \rho_2$.

5. **Further singularity configurations.** For simplicity, here we will consider the case $n = 2$. We will consider a number of Dirac masses concentrating on the unit circle (see figure (1)). That is we want to study the limiting case singularity

$$\lim_{k \to \infty} \frac{\alpha}{2k} \sum_{j=0}^{k} \delta_{x_j}.$$  

![Distribution of Diracs](image)

Letting their number converge to infinity

![Figure 1. Limiting configuration](image)

If the points $x_j$ are chosen in a uniform way, then obviously

$$\lim_{k \to \infty} \frac{\alpha}{2k} \sum_{j=0}^{k} \delta_{x_j} = \alpha \nu_{S^1}$$

where $\nu_{S^1}$ is the Uniform probability measure on $S^1$. Hence first we want to solve

$$-\Delta R = \nu_{S^1}.$$  

For that we use the superposition principle to find that $R(x) = \frac{1}{2\pi} \ln (\frac{1}{1+|x|^2}) + L$, where $L$ is a bounded function. Hence dealing with the equation

$$-\Delta u = K(x)e^{2u} + 2\pi \alpha \nu_{S^1}$$
is equivalent to solving the equation

$$-\Delta u = \frac{K(x)}{1 + |x|^2} e^{2\alpha L} e^{2u}.$$ \tag{1}

But if we set \(\tilde{K} = \frac{K(x)}{(1 + |x|^2)^{p}} e^{2\alpha L} e^{2u}\) as our new Gaussian curvature, then one sees that the procedure that we did in the previous setting works perfectly, in fact it is even easier in this case since \(\tilde{K}\) is non-singular. Hence one have a solution of the form

$$u_{H,\beta} = H(x) + 2\pi \alpha R(x) + U_{H,\beta}$$

Where this time \(U_{H,\beta}\) is in \(W^{2,p}_{lo}(\mathbb{R}^2)\) for every \(p \in [1, \infty)\).

As mentioned before, this sheds light on a way of prescribing \(Q\)-curvature in stratified spaces (see (5)) as in [1]. This can be done by including the singularity as a measure in the equation and then solving for it to get a degenerate density in the measure.

\[\text{Figure 2. Stratified space with conical singularities.}\]

**Some remarks and Questions:** It is important to notice that in this work we prescribed the location and the weight of the singularity. A natural question then arises about the existence of an optimal configuration for the singularity. That is, if we leave the Dirac masses in the right hand side of the equation to move on the manifold, is there an optimal location for those Dirac masses. First, one needs to give a specific definition of the optimality, for instance one can consider the energy of the solution and consider the map

$$\begin{pmatrix} x_1^*, ..., x_k^* \end{pmatrix} \mapsto E(u_{H,\beta}(x_1^*, ..., x_k^*).$$

There are many choices for \(E\). One can for instance take \(E = F_{\beta}\), or consider the variational energy \(E\) of the PDE itself. This problem is tightly related to Smale’s seventh problem.

Another question would be to vary the weights \(c_i\). Indeed, if we denote \(u_{H,\beta}(f)\) the solution that we obtain by prescribing the Dirac masses with weights \(f(x_i^*)\). Then one could think about the best configuration by considering the map

$$f \mapsto E(u_{\beta,H}(f)).$$

We also believe that as in the regular case (at least in dimension 2) the solution is radially symmetric if \(K\) is radially symmetric and negative.

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