The Power of Global Knowledge on Self-stabilizing Population Protocols *

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Abstract

In the population protocol model, many problems cannot be solved in a self-stabilizing way. However, global knowledge, such as the number of nodes in a network, sometimes allow us to design a self-stabilizing protocol for such problems. In this paper, we investigate the effect of global knowledge on possibility of self-stabilizing population protocols in arbitrary graphs. Specifically, we clarify the solvability of the leader election problem, the ranking problem, the degree recognition problem, and the neighbor recognition problem by self-stabilizing population protocols with knowledge of the number of nodes and/or the number of edges in a network.

1 Introduction

We consider the population protocol (PP) model [2] in this paper. A network called population consists of a large number of finite-state automata, called agents. Agents make interactions (i.e., pairwise communication) with each other by which they update their states. The interactions are opportunistic, that is, they are unpredictable for the agents. Agents are strongly anonymous: they do not have identifiers and they cannot distinguish their neighbors with the same states. One example represented by this model is a flock of birds where each bird is equipped with a sensing device with a small transmission range. Two devices can communicate (i.e., interact) with each other only when the corresponding birds come sufficiently close to each other. Therefore, an agent cannot predict when it has its next interaction.

In the field of population protocols, many efforts have been devoted to devising protocols for a complete graph, that is, a population where every pair of agents interacts infinitely often. On the other hand, several works [2, 4, 5, 8, 9, 10, 15, 16, 19, 20] study the population represented by a general graph \( G = (V, E) \) where \( V \) is the set of agents and \( E \) specifies the set of interactable pairs. Each pair of agents \((u, v) \in E\) has interactions infinitely often, while each pair of agents \((u', v') \notin E\) never has an interaction.

Self-stabilization [11] is a fault-tolerant property that, even when any transient fault (e.g., memory crash) hits a network, it can autonomously recover from the fault. Formally, self-stabilization

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is defined as follows: (i) starting from an arbitrary configuration, a network eventually reaches a safe configuration (convergence), and (ii) once a network reaches a safe configuration, it keeps its specification forever (closure). Self-stabilization is of great importance in the PP model because self-stabilization tolerates any finite number of transient faults, and this is a necessary property in a network consisting of a huge number of cheap and unreliable nodes. (Such a network is the original motivation of the PP model.)

Consequently, it has been studied intensively to design self-stabilizing population protocols with oracles. Angluin et al. [4] gave self-stabilizing protocols for a variety of problems: the leader election in the rings whose size are not multiple of a given integer $k$ (in particular, the rings of odd size), the token circulation in rings with a pre-selected leader, the 2-hop coloring in degree-bounded graphs, the consistent global orientation in undirected rings, and the spanning-tree construction in regular graphs. The protocols for the first four problems use only a constant space of agent memory, while the protocol for the last problem requires $O(\log D)$ bits of agent memory, where $D$ is (a known upper bound on) the diameter of the graph. Chen and Chen [9] gave a constant-space and self-stabilizing protocol that solves the leader election in rings with arbitrary size.

On the negative side, Angluin et al. [4] proved that the self-stabilizing leader election (SS-LE) is impossible for arbitrary graphs. In particular, it immediately follows from their theorem that no protocol solves SS-LE in complete graphs with three different sizes, i.e., in all of $K_i$, $K_j$, and $K_k$ for any distinct integers $i, j, k \geq 2$, where $K_l$ is a complete graph with size $l$. Cai et al. [7] proved that no protocol solves SS-LE both in $K_l$ and in $K_{i+1}$ for any integer $i \geq 2$. In almost the same way, we can easily observe that no protocol solves SS-LE both in $K_i$ and $K_j$ for any distinct integers $i, j \geq 2$. (See a more detailed explanation in the second page of [21].) In other words, SS-LE is impossible unless the exact number of agents in the population is known to the agents. Because Cai et al. [7] also gave a protocol that solves SS-LE in $K_l$ for a given integer $l$, the knowledge of the exact number of agents is necessary and sufficient to solve SS-LE in a complete graph.

In addition to [4, 7, 9], many works have been devoted to SS-LE. This is because the leader election is one of the most fundamental and important problems in the PP model: several important protocols [2, 3, 4] require a pre-selected unique leader, especially, it is shown by Angluin et al. [4] that if we have a unique leader, all semi-linear predicates can be solved very quickly. However, we have strong impossibility as mentioned above: SS-LE can not be solved unless the knowledge of the exact number of agents is given to the agents. In the literature, there are three approaches to overcome this impossibility. One approach [6, 7] is to assume that every agent knows the exact number of agents. Cai et al. [7] took this approach for the first time. Their protocol uses $O(\log n)$ bits ($n$ states) of memory space per agent and converges within $O(n^3)$ steps in expectation in the complete graph of $n$ agents under the uniformly random scheduler, which selects a pair of agents to interact uniformly at random from all pairs at each step. Burman et al. [6] gave three faster SS-LE protocols than the protocol of Cai et al. [7], also for the complete graph of $n$ agents. These self-stabilizing protocols in [6, 7] solve not only the leader election problem but also the ranking problem, which requires ranking the $n$ agents by assigning them the different integers from $0, 1, \ldots, n - 1$. See Section 1.2 for the results of the other two approaches to overcome the impossibility, SS-LE protocols with oracles [5, 8, 12] and loosely-stabilizing protocols [14, 16, 17, 19, 20, 21].

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1In [4], $D$ is defined as the diameter of the graph, not a known upper bound on it. However, since we must take into account an arbitrary initial configuration, we require an upper bound on the diameter; Otherwise, the agents need the memory of unbounded size. Fortunately, the knowledge of the upper bound is not a strong assumption in this case: any upper bound which is polynomial in the true diameter is acceptable since the space complexity is $O(\log D)$ bits.
1.1 Our Contribution

As mentioned above, if we have knowledge of the exact number of agents, we can solve the self-stabilizing leader election in complete graphs, which we can never solve otherwise. In this paper, we investigate in detail how powerful global knowledge, such as the exact number of agents in the population, is to design self-stabilizing population protocols for arbitrary graphs. Specifically, we consider two kinds of global knowledge, the number of agents and the number of edges (i.e., interactable pairs) in the population, and clarify the relationships between the knowledge and the solvability of the following four problems:

- leader election (LE): Elect exactly one leader,
- ranking (RK): Assign the agents in the population $G = (V_G, E_G)$ distinct integers from 0 to $|V_G| - 1$,
- degree recognition (DR): Let each agent recognize its degree in the graph,
- neighbor recognition (NR): Let each agent recognize the set of its neighbors in the graph.

We denote $A_1 \preceq A_2$ if problem $A_2$ is not easier than $A_1$, that is, $A_1$ is reducible to $A_2$. We have $\text{LE} \preceq \text{RK}$ and $\text{DR} \preceq \text{NR}$. The first relationship holds because if the agents are labeled 0, 1, . . . , $|V_G| - 1$, LE is immediately solved by selecting the agent with label 0 as the unique leader. The second relationship is trivial.

To describe our contributions, we formally define the global knowledge that we consider. Define $\mathcal{G}_{n,m}$ as the set of all the simple, undirected, and connected graphs with $n$ nodes and $m$ edges. Let $\nu$ and $\mu$ be any sets of positive integers such that $\nu \subseteq \mathbb{N}_{\geq 2} = \{n \in \mathbb{N} \mid n \geq 2\}$ and $\mu \subseteq \mathbb{N}_{\geq 1} = \{m \in \mathbb{N} \mid m \geq 1\}$. Then, we define $\mathcal{G}_{\nu,\mu} = \bigcup_{n \in \nu, m \in \mu} \mathcal{G}_{n,m}$. For simplicity, we define $\mathcal{G}_{\nu,*} = \mathcal{G}_{\nu,\mathbb{N}_{\geq 1}}$ and $\mathcal{G}_{*,\mu} = \mathcal{G}_{\mathbb{N}_{\geq 2},\mu}$ for any $\nu \subseteq \mathbb{N}_{\geq 2}$ and $\mu \subseteq \mathbb{N}_{\geq 1}$. We consider that $\nu$ and $\mu$ are global knowledge on the population: $\nu$ is the set of the possible numbers of agents and $\mu$ is the set of the possible numbers of interactable pairs. In other words, when we are given $\nu$ and $\mu$, our protocol has to solve a problem only in the populations represented by the graphs in $\mathcal{G}_{\nu,\mu}$. We say that protocol $P$ solves problem $A$ in arbitrary graphs given knowledge $\nu$ and $\mu$ if $P$ solves $A$ in all graphs in $\mathcal{G}_{\nu,\mu}$.

In this paper, we investigate the solvability of LE, RK, DR, and NR for arbitrary graphs with the knowledge $\nu$ and $\mu$. Specifically, we prove the following propositions assuming that the agents are given knowledge $\nu$ and $\mu$:

1. When the agents know nothing about the number of interactable pairs, i.e., $\mu = \mathbb{N}_{\geq 1}$, there exists a self-stabilizing protocol that solves LE and RK in arbitrary graphs if and only if the agents know the exact number of agents i.e., $\mathcal{G}_{\nu,\mu} = \mathcal{G}_{n,*}$ for some $n \in \mathbb{N}_{\geq 2}$.

2. There exists a self-stabilizing protocol that solves NR ($\preceq$ DR) in arbitrary graphs if the agents know the exact number of agents and the exact number of interactable pairs i.e., $\mathcal{G}_{\nu,\mu} = \mathcal{G}_{n,m}$ holds for some $n \in \mathbb{N}_{\geq 2}$ and $m \in \mathbb{N}_{\geq 1}$.

3. The knowledge of the exact number of agents is not enough to design a self-stabilizing protocol that solves DR ($\preceq$ NR) in arbitrary graphs if the agents do not know the number of interactable pairs exactly. Specifically, no self-stabilizing protocol solves DR in all graphs in $\mathcal{G}_{\nu,\mu}$ if $\mathcal{G}_{n,m_1} \cup \mathcal{G}_{n,m_2} \subseteq \mathcal{G}_{\nu,\mu}$ holds for some $n \in \mathbb{N}_{\geq 2}$ and some distinct $m_1, m_2 \in \mathbb{N}_{\geq 1}$ such that $\mathcal{G}_{n,m_1} \neq \emptyset$ and $\mathcal{G}_{n,m_2} \neq \emptyset$.

In standard distributed computing models, generally, each node always has its local knowledge, e.g., its degree and the set of its neighbors. In the PP model, the agents do not have their local knowledge a priori, and many impossibility results (e.g., the impossibility of SS-LE in complete graphs [4, 7]) come from the lack of the local knowledge. Interestingly, the third proposition yields that, for self-stabilizing population protocols, obtaining some local knowledge (degree recognition
of each agent) is at least as difficult as obtaining the corresponding global knowledge (the number of interactable pairs). It is also worthwhile to mention that the PP model is empowered greatly if LE and NR are solved. After the agents recognize their neighbors correctly, the population can simulate one of the most standard distributed computing model, the message passing model, if each agent maintains a variable corresponding to a message buffer for each neighbor. Moreover, we have the unique leader in the population, by which we can easily break the symmetry of a graph and solve many important problems even in a self-stabilizing way. For example, we can construct a spanning tree rooted by the leader. This fact and the above propositions show how powerful this kind of global knowledge is when we design self-stabilizing population protocols.

1.2 Other Related Work

Several works use oracles, a kind of failure detectors, to solve SS-LE. Fischer and Jiang [12] took this approach for the first time. They introduced oracle \( \Omega \) that eventually tells all agents whether at least one leader exists or not and proposed two protocols that solve SS-LE for rings and complete graphs by using \( \Omega \). Beauquier et al. [5] presented an SS-LE protocol for arbitrary graphs that uses two copies of \( \Omega \); one is used to detect the existence of a leader and the other one is used to detect the existence of a special agent called a token. Canepa et al. [8] proposed two SS-LE protocols that use \( \Omega \) and require only 1 bit of each agent: one is a deterministic protocol for trees and the other is a randomized protocol for arbitrary graphs although the position of the leader is not static and moves among the agents forever.

To solve SS-LE without oracles or the knowledge of the exact number of agents, Sudo et al. [17] introduced the concept of loose-stabilization, which relaxes the closure requirement of self-stabilization, but keeps its advantage in practice. Specifically, loose-stabilization guarantees that, starting from any configuration, the population reaches a safe configuration within a relatively short time; after that, the specification of the problem (such as having a unique leader) must be sustained for a sufficiently long time, though not necessarily forever. In [17], a loosely-stabilizing leader election (LS-LE) protocol was given for the first time, which assumes that the population is a complete graph and every agent knows a common upper bound \( N \) of \( n \). Hence the number of agents in the population. This protocol is practically equivalent to an SS-LE protocol since it maintains the unique leader for an exponentially large number of steps in expectation (that is, practically forever) after reaching a safe configuration within \( O(nN \log N) \) steps in expectation. The assumption that we can use an upper bound \( N \) of \( n \) is practical because the protocol works correctly even if we make a large overestimation of \( n \), such as \( N = 10n \). Izumi [14] gave a method which reduces the number of steps for convergence to \( O(nN) \). Recently, Sudo et al. [21] gave a much faster loosely-stabilizing leader election protocol for complete graphs. Given parameter \( c > 0 \), it reaches a safe configuration within \( O(cn \log^3 N) \) steps and thereafter it keeps the unique leader for \( \Omega(cn^{10c+1}) \) steps, both in expectation. In [16] [19] [20], LS-LE protocols were presented for arbitrary graphs.

2 Preliminaries

A population is represented by a simple and connected graph \( G = (V_G, E_G) \), where \( V_G \) is the set of the agents and \( E_G \subseteq V_G \times V_G \) is the set of the interactable pairs of agents. If \((u, v) \in E_G\), two agents \( u \) and \( v \) can interact in the population \( G \), where \( u \) serves as the initiator and \( v \) serves as the responder of the interaction. We say that the population \( G \) is undirected if \((u, v) \in E_G \) yields \((v, u) \in E_G \) for any \( u, v \in V_G \). Throughout this paper, we consider only undirected populations. For each \( v \in V_G \), we define the set of the neighbors of agent \( v \) as \( N_G(v) = \{ u \in V_G \mid (v, u) \in E_G \} \).

A protocol \( P(Q, Y, T, \pi_{out}) \) consists of a finite set \( Q \) of states, a finite set \( Y \) of output symbols, a transition function \( T : Q \times Q \rightarrow Q \times Q \), and an output function \( \pi_{out} : Q \rightarrow Y \). When two agents interact, \( T \) determines their next states according to their current states. The output of an agent
is determined by $\pi_{\text{out}}$: the output of an agent in state $q$ is $\pi_{\text{out}}(q)$. As mentioned in Section 4, we assume that the agents can use knowledge $\nu$ and $\mu$. Therefore, the four parameters of protocol $P$, i.e., $Q$, $Y$, $T$, and $\pi_{\text{out}}$, may depend on $\nu$ and $\mu$. We sometimes write $P(\nu, \mu)$ explicitly to denote protocol $P$ with knowledge $\nu$ and $\mu$.

A configuration on population $G$ is a mapping $C : V_G \to Q$ that specifies the states of all the agents in $G$. We denote the set of all configurations of protocol $P$ on population $G$ by $C_{\text{all}}(P, G)$. We say that a configuration $C$ changes to $C'$ by an interaction $e = (u, v)$, denoted by $C \xrightarrow{P,G} C'$, if $(C'(u), C'(v)) = T(C(u), C(v))$ and $C'(w) = C(w)$ for all $w \in V \setminus \{u, v\}$. We also denote $C \xrightarrow{P,G} C'$ if $C \xrightarrow{P,G} C'$ holds for some $e \in E_G$.

An execution of protocol $P$ on population $G$ is an infinite sequence of configurations $\Xi = C_0, C_1, \ldots$ such that $C_i \xrightarrow{P,G} C_{i+1}$ for $i = 0, 1, \ldots$. We call $C_0$ the initial configuration of the execution $\Xi$. We have to assume some kind of fairness of an execution. Otherwise, for example, we cannot exclude an execution such that only one pair of agents have interactions in a row and no other pair has an interaction forever. Unlike most distributed computing models in the literature, the global fairness is usually assumed in the PP model\footnote{Since Angluin et al. introduced the global fairness in their seminal work \cite{Angluin1983}, which studied the PP model for the first time, almost all the work in the PP model has adopted the global fairness, including the uniformly random scheduler.}. We say that an execution $\Xi = C_0, C_1, \ldots$ of $P$ on population $G$ satisfies the global fairness (or $\Xi$ is globally fair) if for any configuration $C$ that appears infinitely often in $\Xi$, every configuration $C'$ such that $C \xrightarrow{P,G} C'$ also appears infinitely often in $\Xi$.

A problem is specified by a predicate on the outputs of the agents. We call this predicate the specification of the problem. We say that a configuration $C$ satisfies the specification of a problem if the outputs of the agents satisfy it in $C$. We consider the following four problems in this paper.

**Definition 1 (LE).** The specification of the leader election problem (LE) requires that exactly one agent outputs $L$ and all the other agents output $F$.

**Definition 2 (RK).** The specification of the ranking problem (RK) requires that in the population $G = (V_G, E_G)$, the set of the outputs of the agents in the population equals to $\{0, 1, \ldots, |V_G| - 1\}$.

**Definition 3 (DR).** The specification of the degree recognition problem (DR) requires that in the population $G = (V_G, E_G)$, every agent $v \in V_G$ outputs $|N_G(v)|$.

**Definition 4 (NR).** The specification of the neighbor recognition problem (NR) requires that in the population $G = (V_G, E_G)$, every agent $v \in V_G$ outputs a two-tuple $(c_v, S_v) \in \mathbb{Z} \times 2^Z$ such that, for all $v \in V_G$, we have $S_v = \{c_u \mid u \in N_G(v)\}$ and $|S_v| = |N_G(v)|$.

Note that the second condition in the definition of NR, i.e., $|S_v| = |N_G(v)|$, requires that the population is 2-hop colored, that is, every two distinct neighbors $u$ and $w$ of agent $v$ must have different colors $c_u$ and $c_w$.

Now, we define self-stabilizing protocols in Definitions 5 and 6, where we use the definitions given in Section 4 for knowledge $\nu$ and $\mu$ and the set $G_{\nu, \mu}$ of graphs. Note that Definition 5 is not enough if we consider dynamic problems such as the token circulation, where the specifications must be defined as predicates not on configurations but on executions. However, we consider only static problems in this paper, thus this definition is enough for our purpose.

**Definition 5 (Safe configuration).** Given a protocol $P$ and a population $G$, we say that a configuration $C \in C_{\text{all}}(P(\nu, \mu), G)$ is safe for problem $A$ if (i) $C$ satisfies the specification of problem $A$, and (ii) no agent changes its output in any execution of $P$ on $G$ starting from $C$.

**Definition 6 (Self-stabilizing protocol).** For any $\nu$ and $\mu$, we say that a protocol $P$ is a self-stabilizing protocol if solves problem $A$ in arbitrary graphs given knowledge $\nu$ and $\mu$ if every
globally-fair execution $\Xi = C_0, C_1, \ldots$ of $P(\nu, \mu)$ on any population $G$, which starts from any configuration $C_0 \in \mathcal{C}_{all}(P(\nu, \mu), G)$, reaches a safe configuration for $A$.

Finally, we define the uniformly random scheduler, which has been considered in most of the works [1] [2] [3] [13] [16] [17] [18] [19] [20] [21] in the PP model. Under this scheduler, exactly one ordered pair $(u, v) \in E_G$ is chosen to interact uniformly at random from all interactable pairs. We need this scheduler to evaluate time complexities of protocols because global fairness only guarantees that an execution makes progress eventually. Formally, the uniformly random scheduler is defined as a sequence of interactions $\Gamma = \Gamma_0, \Gamma_1, \ldots$, where each $\Gamma_i$ is a random variable such that $\Pr(\Gamma_t = (u, v)) = 1/|E_G|$ for any $t \geq 0$ and any $(u, v) \in E_G$. Given a population $G$, a protocol $P(\nu, \mu)$, and an initial configuration $C_0 \in \mathcal{C}_{all}(P(\nu, \mu), G)$, the execution under the uniformly random scheduler is defined as $\Xi_{P(\nu, \mu)}(G, C_0, \Gamma) = C_0, C_1, \ldots$ such that $C_t \xrightarrow{P(\nu, \mu)\cdot \Gamma_t} C_{t+1}$ for all $t \geq 0$. When we assume this scheduler, we evaluate time complexities of a population protocol, for example, the expected number of steps required to reach a safe configuration. Since the execution under the uniformly random scheduler is globally fair with probability 1, we have the following observation.

**Observation 1.** A protocol $P(\nu, \mu)$ is self-stabilizing for a problem $A$ if and only if $\Xi_{P(\nu, \mu)}(G, C_0, \Gamma)$ reaches a safe configuration for $A$ with probability 1 for any configuration $C_0 \in \mathcal{C}_{all}(P(\nu, \mu), G)$.

### 3 Random Walk in Population Protocols

In this paper, we gave two self-stabilizing protocols $P_{\text{rank}}$ and $P_{\text{neigh}}$. Both of them use $n$ tokens that make the random walk, where $n$ is the number of agents in the population. Specifically, all the agents in the population always has exactly one token, and two agents swap their tokens whenever they have an interaction. To analyze the expected number of steps until an execution of $P_{\text{rank}}$ or $P_{\text{neigh}}$ reaches a safe configuration, we give several lemmas about the movements of the tokens, Lemmas 1, 2, 3, 4, 5, and 6 in this section. Although Lemmas 1 and 4 were already proven by Sudo et al. [20], we also give proofs for them with the notations of this paper, to make this paper self-contained.

Fix a population $G = (V_G, E_G)$ and consider the execution $\Xi = C_0, C_1, \ldots$ of $P_{\text{rank}}$ or $P_{\text{neigh}}$ under the uniformly random scheduler starting from an arbitrary configuration $C_0$. Let $\Gamma = \Gamma_0, \Gamma_1, \cdots = (u_0, v_0), (u_1, v_1), \ldots$, that is, we denote the $i$-th interaction under the uniformly random scheduler $\Gamma$ by $(u_i, v_i)$. Formally, for each $w \in V_G$ we define token $t_w : \mathbb{N}_{\geq 0} \rightarrow V_G$ as follows:

- $t_w(0) = w$,
- $t_w(i) = \begin{cases} u_i & t_w(i-1) = v_{i-1} \\ v_i & t_w(i-1) = u_{i-1} \\ t_w(i-1) & \text{Otherwise} \end{cases}$ for each $i > 0$.

We say that token $t_v$ visits $u$ in the $i$-th step if $t_v(i) = u$. We also say that two tokens $t_u$ and $t_w$ meet in the $i$-th step if $\Gamma_i = (t_u(i), t_w(i))$ or $\Gamma_i = (t_u(i), t_w(i))$ holds. Throughout this section, we denote the number of agents, the number of interactable pairs, and the diameter of population $G$ by $n$, $m$, and $d$, respectively.

**Lemma 1 [20].** In execution $\Xi$, for any $u, v \in V_G$, token $t_u$ visits agent $v$ within $mn \cdot d(u, v)$ steps in expectation, where $d(u, v)$ is the distance between agent $u$ and $v$ in $G$.

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3 In this section, we do not care which protocol, $P_{\text{rank}}$ or $P_{\text{neigh}}$, we execute because we focus on only the movement of the tokens making the random walk.
Proof. We consider a Markov chain $X = \{X(t) \mid t = 0, 1, \ldots\}$, each $X(t) \in V_G$ represents the location of a token (i.e., the agents that the token stays on) in configuration $C_t$ (i.e., the t-th configuration in $\Xi$). For $t \geq 1$ and $x, y \in V_G$, the probability $\Pr(X(t) = y \mid X(t-1) = x)$ is independent of $t$, denoted by $P_X(x, y)$. Probability $P_X(x, y)$ is calculated as follows: $P_X(x, y) = 1/m$ if $(x, y) \in E_G$, $P_X(x, y) = 1 - \delta_x/m$ if $x = y$; otherwise, $P_X(x, y) = 0$, where $\delta_x = |N_G(v)|$. The symmetric structure of the chain, i.e., $P_X(y, x) = P_X(x, y)$ for all $x, y \in V$, gives $\sum_{x \in V} P_X(x, y) = 1$ for any $y$. Therefore, $\pi_X = (\pi_X(x_1), \pi_X(x_2), \ldots, \pi_X(x_n)) = (1/n, 1/n, \ldots, 1/n)$ is the unique stationary distribution on $X$ (i.e., $\pi_X P_X = \pi_X$), where $\{x_1, x_2, \ldots, x_n\}$. For $x, y \in V_G$, we define the hitting time $H_x(x, y)$ as the expected number of transition steps in the chain $X$ from state $x$ to $y$. We have $H_x(x, z) = 1/\pi(z) = n$ for any agent $z \in V$. We also have $H_x(x, z) = 1 + \sum_{w \in N_G(z)}(1/m) \cdot H_x(w, z)$. Therefore, $\sum_{w \in N_G(z)} H_x(w, z) = m(n-1)$. Thus, for any $(w, z) \in E_G$, we have $H_x(w, z) < mn$.

Let $w_0, w_1, \ldots, w_{d(u, v)}$ ($w_0 = u$ and $w_{d(u, v)} = v$) be the shortest path from $u$ to $v$ in $G$. Then, $H_x(u, v) = \sum_{i=0}^{d(u, v)-1} h_{w_i, w_{i+1}} < mn \cdot d(u, v)$, from which the lemma immediately follows.

\**Lemma 2.** In execution $\Xi$, for any $v \in V_G$, token $t_v$ visits all the agents in $V_G$ within $2mn^2$ steps in expectation.

Proof. Let $v_0, v_1, \ldots, v_{2n-2}$ ($v_0 = v_{2n-2} = v$ and $(v_i, v_{i+1}) \in E_G$ for all $i = 0, 1, \ldots, 2n-1$) be a tour on an arbitrary spanning tree of $G$. The lemma immediately follows from Lemma 1 because token $t_v$ moves from $v_i$ to $v_{i+1}$ within $mn$ steps in expectation for each $i = 0, 1, \ldots, 2n-1$.

\**Lemma 3.** In execution $\Xi$, for any $v \in V_G$, all the $n$ tokens visit agent $v$ within $O(mn^2 \log n)$ steps in expectation.

Proof. Let $u$ be any agent in $V_G$. It immediately follows from Lemma 1 that token $t_u$ visits agent $v$ within $mn$ steps in expectation. Therefore, by Markov’s inequality, $t_u$ visits $v$ within $2mn$ steps with probability at least $1/2$. Therefore, they meet within $2 \log n \cdot (2mn) = 4mn^2 \log n$ steps with probability at least $1 - 1/n^2$. By the union bound, all the $n$ tokens $(t_u)_{u \in V_G}$ visit $v$ within $4mn^2 \log n$ steps with probability $1 - O(1/n)$, from which the lemma immediately follows.

\**Lemma 4** ([20]). In execution $\Xi$, all the $n$ tokens meet each other within $O(mn^2 \log n)$ steps in expectation.

Proof. Let $S = V_G \times V_G \setminus \{(w, w) \mid w \in V_G\}$. Consider a Markov chain $Y = \{Y(t) \mid t = 0, 1, \ldots\}$, each $Y(t) \in S$ represents the locations of two token $t_u$ and $t_v$ in configuration $C_t$ (i.e., the t-th configuration in $\Xi$). For $t \geq 1$ and $x, y \in S$, the probability $\Pr(Y(t) = y \mid Y(t-1) = x)$ is independent of $t$, denoted by $P_Y(x, y)$. For $(a, b), (c, d) \in S$, we write $(a, b) \rightarrow (c, d)$ if (i) $(a, c) \in E_G \land b = d$, (ii) $a = c \land (b, d) \in E_G$, or (iii) $(a, b) \in E_G \land a = d \land b = c$. Intuitively, the first (resp. second) case represents that $t_u$ (resp. $t_v$) moves from $a$ to $c$ (resp. from $b$ to $d$). The third case represents that the agents $a$ and $b$ swaps the tokens $t_u$ and $t_v$. Probability $P_Y(x, y)$ is calculated as follows: $P_Y(x, y) = 1/m$ if $x \rightarrow y$, $P_Y(x, y) = 1 - |\{z \mid x \rightarrow z\}|/m$ if $x = y$, $P_Y(x, y) = 0$ otherwise. The symmetric structure of the chain, i.e., $P_Y(x, y) = P_Y(y, x)$ for all $x, y \in S$, gives $\sum_{x \in S} P_Y(x, y) = 1$ for any $y \in S$. Therefore, $\pi_Y = (\pi_Y(y_1), \pi_Y(y_2), \ldots, \pi_Y(y_{n(n-1)}) = \left(\frac{1}{n(n-1)}, \frac{1}{n(n-1)}, \ldots, \frac{1}{n(n-1)}\right)$ is the unique stationary distribution on $Y$ (i.e., $\pi_Y P_Y = \pi_Y$), where $S = \{y_1, y_2, \ldots, y_{n(n-1)}\}$. For $x, y \in S$, we define the hitting time $H_Y(x, y)$ as the expected number of transition steps in the chain $Y$ from state $x$ to $y$. We have $H_Y(x, z) = 1/\pi(z) = n(n-1)$ for any $z \in S$. We also have $H_Y(z, z) = 1 + \sum_{w \in S \setminus x \rightarrow z}(1/m) \cdot H_Y(w, z)$. Therefore, $\sum_{w \in S \setminus x \rightarrow z} H_Y(w, z) = mn(n-1)$. Thus, for any two distinct $x, y \in S$ such that $x \rightarrow y$, we have $H_Y(x, y) < mn^2$.

Let $u$ and $v$ be any two distinct agents and let $d(u, v)$ be the distance between $u$ and $v$ in $G$. Let $w_0, w_1, \ldots, w_l$ ($w_0 = u$, $w_l = v$, and $l = d(u, v)$) be an arbitrary shortest path from $u$ to $v$ in $G$. Clearly, the expected number of steps until two tokens $t_u$ and $t_v$ meet in execution...
any

on the proofs of [4, 7]. Since $(w_i, v) \rightarrow (w_{i+1}, v)$ holds for any $i = 0, 1, \ldots, l - 2$ and $(w_{l-1}, v) \rightarrow (v, w_{l-1})$ holds, we have $M_{u,v} < mn^2d(u,v)$. Therefore, by Markov’s inequality, any two distinct tokens meet within $2mn^2d$ steps with probability at least $1/2$. Therefore, they meet within $3 \log_2 n \cdot (2mn^2d) = 6mn^2d\log_2 n$ steps with probability at least $1 - 1/n^3$. By the union bound, all the $n$ tokens meet each other within $6mn^2d\log_2 n$ steps with probability $1 - O(1/n)$, from which the lemmas immediately follows.

**Lemma 5.** Let $k$ be any positive integer. There exists some agent $v \in V_G$ such that the expected number of steps until token $t_v$ moves $k$ times is $O(nk)$.

**Proof.** Let $s_u$ be the expected number of steps until token $t_u$ moves $k$ times. It suffices to show $\min_{u \in V_G} s_u = O(nk)$. To analyze $\min_{u \in V_G} s_u$, we introduce a Markov chain $Z = \{Z(t) \mid t = 0, 1, \ldots, l\}$, where each $Z(t) \in V_G$ represents the location of a token (i.e., the agents that the token stays on) after it moves $t$ times. For $t \geq 1$ and $x, y \in V_G$, the probability $Pr(Z(t) = y \mid Z(t-1) = x)$ is independent of $t$, denoted by $P_Z(x,y)$. Probability $P_Z(x,y)$ is calculated as follows: $P_Z(x,y) = 1/\delta_x$ if $(x,y) \in E_G$, $P_Z(x,y) = 0$ otherwise, where $\delta_x = |N_G(x)|$. Let $V_G = \{z_1, z_2, \ldots, z_n\}$. Then, $\pi_Z = (\pi_Z(z_i))_{i=1,2,\ldots,n}$ is a stationary distribution if $\pi_Z(z_i) = \delta_{z_i}/2m$ because $\pi_Z P_Z = \pi_Z$.

If a token visits agent $w \in V_G$, then it needs $m/\delta_w$ steps in expectation to leave $w$, i.e., move to another agent from $w$. Thus, we assign each agent $x \in V_G$ its weight $W(x) = m/\delta_x$; then, $s_u = \mathbb{E} \left[ \sum_{t=0}^{k-1} W(Z(t)) \mid Z(0) = u \right]$ holds for any $u \in V_G$. Assume that the initial state $Z_0$ is now set according to the stationary distribution, i.e., $Pr(Z(0) = z_i) = \pi_Z(z_i) = \delta_{z_i}/2m$ for any $i = 1, 2, \ldots, n$. Since $\pi_Z$ is a stationary distribution, we always have the same distribution thereafter, that is, we have $Pr(Z(t) = z_i) = \pi_Z(z_i)$ for any $t = 0, 1, \ldots$ and $i = 0, 1, \ldots, n$. Therefore, under this assumption, we have

$$\mathbb{E} \left[ \sum_{t=0}^{k-1} W(Z(t)) \right] = k \sum_{i=0}^{n} \pi_Z(z_i) W(z_i) = k \sum_{i=0}^{n} \delta_{z_i}/2m \cdot m = \frac{kn}{2}.$$ We also have $\mathbb{E} \left[ \sum_{t=0}^{k-1} W(Z(t)) \right] = \sum_{u \in V_G} \pi_Z(u) s_u$. Since $\sum_{u \in V_G} \pi_Z(u) = 1$, there must be at least one agent $u \in V_G$ such that $s_u \leq kn/2$. Thus, $\min_{u \in V_G} s_u = O(nk)$.

**Lemma 6.** Let $k$ be any positive integer. For any $v \in V_G$, the expected number of steps until token $t_v$ moves $k$ times is $O(nk + mn^2d)$.

**Proof.** By Lemma 5 there exists an agent $u \in V_G$ such that after visiting $u$, token $t_v$ moves $k$ times within $O(nk)$ steps in expectation. By Lemma 1 token $t_v$ visits $u$ within $mn^2d$ steps in expectation. In total, token $t_v$ moves $k$ times within $O(nk + mn^2d)$ steps in expectation.

### 4 Leader Election and Ranking

Our goal is to give a necessary and sufficient condition to solve RK and LE on knowledge $\nu$, provided that $\mu$ gives no information, i.e., $\mu = \mathbb{N}_{\geq 1}$. For a necessary condition, we have the following lemma.

**Lemma 7 (4, 7, 21).** Given knowledge $\nu$ and $\mu$, there exists no self-stabilizing protocol that solves LE in arbitrary graphs if $G_{n_1,*} \cup G_{n_2,*} \subseteq G_{n,\mu}$ for some two distinct $n_1, n_2 \in \mathbb{N}_{\geq 2}$.

**Proof.** The lemma immediately follows from the fact that there exists no self-stabilizing protocol that solves LE in complete graphs of two different sizes, i.e., both in $K_{n_1}$ and $K_{n_2}$ for any two integers $n_1 > n_2 \geq 2$. As mentioned in Section 4 Sudo et al. 21 gave how to prove this fact based on the proofs of 4 and 7.

This Markov chain $Z$ is not ergodic when $G$ is bipartite. However, this does not matter in this proof because we do not use the recurrent time $H_Z(z,z)$ unlike the proofs of Lemmas 1 and 4.
Algorithm 1 $P_{\text{rank}}(\nu, \mu)$

Assumption: $|\nu| = 1$. (Let $\nu = \{n\}$.)

Variables:
- $\text{id}_A, \text{id}_T \in \{0, 1, \ldots, n - 1\}$
- $\text{color}_A \in \{W, R, B\}$
- $\text{color}_T \in \{R, B\}$
- $\text{timer}_T \in \{0, 1, \ldots, U_T\}$

Output function $\pi_{\text{out}}$: $\text{id}_A$

Interaction between initiator $a_0$ and responder $a_1$:
1. $(a_0.\text{id}_T, a_0.\text{color}_T, a_0.\text{timer}_T) \leftrightarrow (a_1.\text{id}_T, a_1.\text{color}_T, a_1.\text{timer}_T)$  
   // Execute the random walk of two tokens
2. if $a_0.\text{id}_T = a_1.\text{id}_T$ then $a_1.\text{id}_T \leftarrow a_1.\text{id}_T + 1$ (mod $n$) endif
3. for all $i \in \{0, 1\}$ do $a_i.\text{timer}_T \leftarrow \max(0, a_i.\text{timer}_T - 1)$ endfor
4. for all $i \in \{0, 1\}$ such that $a_i.\text{id}_A = a_i.\text{id}_T$ do
5. if $a_i.\text{color}_A = W$ then $a_i.\text{color}_A \leftarrow a_i.\text{color}_T$ endif
6. if $a_i.\text{color}_A \neq a_i.\text{color}_T$ then
7. $a_i.\text{id}_A \leftarrow a_i.\text{id}_A + 1$ (mod $n$)
8. $a_i.\text{color}_A \leftarrow W$
9. else if $a_i.\text{timer}_T = 0$ then
10. $a_i.\text{timer}_T \leftarrow U_T$
11. if $a_i.\text{color}_A = R$ then $a_i.\text{color}_A \leftarrow a_i.\text{color}_T \leftarrow B$ endif
12. if $a_i.\text{color}_A = B$ then $a_i.\text{color}_A \leftarrow a_i.\text{color}_T \leftarrow R$ endif
13. end if
14. end for

To give a sufficient condition, we give a self-stabilizing protocol $P_{\text{rank}}$, which solves the ranking problem (RK) in arbitrary graphs given the knowledge of the exact number of agents in a population. Specifically, this protocol assumes that the given knowledge $\nu$ satisfies $|\nu| = 1$ while it does not care about the number of interactable pairs, that is, $P_{\text{rank}}(\nu, \mu)$ works even if $\mu$ does not give any knowledge (i.e., $\mu = \mathbb{N}_{\geq 1}$). Let $n$ be the integer such that $\nu = \{n\}$.

If we focus only on complete graphs, the following simple algorithm [7] is enough to solve self-stabilizing ranking with the exact knowledge $n$ of agents:

- Each agent $v$ has only one variable $v.\text{id} \in \{0, 1, \ldots, n - 1\}$, and
- Every time two agents with the same id meet, one of them (the initiator) increases its id by one modulo $n$.

Since this algorithm assumes complete graphs, every pair of agents in the population eventually has interactions. Therefore, as long as two agents have the same identifiers, they eventually meet and the collision of their identifiers is resolved. However, this algorithm does not work in arbitrary graphs, even if the exact number of agents is given. This is because some pair of agents may not be interactable in an arbitrary graph, then they cannot resolve the conflicts of their identifiers by meeting each other.

Protocol $P_{\text{rank}}$ detects the conflicts between any (possibly non-interactable) two agents by traversing $n$ tokens in a population where each agent always has exactly one token. This protocol is inspired by a self-stabilizing leader election protocol with oracles given by Beauquier et al. [5], where the agents traverse exactly one token in a population.

The pseudocode of $P_{\text{rank}}$ is shown in Algorithm 1. Our goal is to assign the agents the distinct labels $0, 1, \ldots, n - 1$. Each agent $v$ stores its label in a variable $v.\text{id}_A \in \{0, 1, \ldots, n - 1\}$ and
outputs it as it is. To detect and resolve the conflicts of the labels in arbitrary graphs, each agent maintains four other variables $\text{id}_A \in \{0, 1, \ldots, n - 1\}$, $\text{color}_A \in \{W, R, B\}$, $\text{color}_T \in \{R, B\}$, and $\text{timer}_T \in \{0, 1, \ldots, U_T\}$, where $U_T$ is a sufficiently large $\Omega(mn)$ value and $m$ is the number of interactable pairs in the population. We will explain later how to assign $U_T$ such a value. We say that $v$ has a token labeled $x$ if $v.\text{id}_T = x$. Each agent $v$ has one color, white ($W$), red ($R$), or blue ($B$), while $v$'s token has one color, red ($R$) or blue ($B$), maintained by variables $v.\text{id}_A$ and $v.\text{id}_T$, respectively.

The tokens always make the random walk: two agents swap their tokens whenever two agents interact (Line 1). If the two tokens have the same label, one of them increments its label modulo $n$ (Line 2). Since all tokens meet each other infinitely often by the random walk, they eventually have mutually distinct labels ($\text{id}_T$), after which they never change their labels. Thereafter, the conflicts of labels among the agents are resolved by using the tokens. Let $v$ increases its label by one modulo its color to white (Line 8). When $v$ meets $T_x$, changes its color to white (Line 7). The agent $v$, now labeled $x + 1 \pmod{n}$, changes its color to white (Line 8). When $v$ meets $T_{x+1} \pmod{n}$ the next time, it copies the color of the token to its color to synchronize a color with $T_{x+1} \pmod{n}$. Token $T_x$ changes its color periodically. Specifically, $T_x$ decreases its $\text{timer}_T$ whenever it moves unless $\text{timer}_T$ already reaches zero (Line 3). If token $T_x$ meets an agent labeled $x$, they have the same color, and the timer of the token is zero, then they change their color from blue to red or from red to blue (Lines 11–12). If there are two or more agents labeled $x$, this multiplicity is eventually detected because $T_x$ makes a random walk forever: $T_x$ eventually meets an agent labeled $x$ with a different color. By repeating this procedure, the population eventually reaches a configuration where all the agents have distinct labels and the agent labeled $x$ has the same color as that of $T_x$ for all $x = 0, 1, \ldots, n - 1$. No agent changes its label thereafter.

Note that this protocol works even if we do not use variable $\text{timer}_T$ and color $W$. We introduce them to make this protocol faster under the uniformly random scheduler. In the rest of this section, we prove the following theorem.

**Theorem 1.** Given knowledge $\nu$ and $\mu$, $P_{\text{rank}}(\nu, \mu)$ is a self-stabilizing protocol that solves RK in arbitrary graphs if $\nu = \{n\}$ for some integer $n$, regardless of $\mu$. Starting from any configuration $C_0$ on any population $G = (V_G, E_G) \in \mathcal{G}_{n,s}$, the execution of $P_{\text{rank}}(\nu, \mu)$ under the uniformly random scheduler (i.e., $\mathbb{E}_{\text{rank}(\nu, \mu)}(G, C_0, \Gamma)$) reaches a safe configuration within $O(mn^3d\log{n} + n^2U_T)$ steps in expectation, where $m = |E_G|/2$ and $d$ is the diameter of $G$. Each agent uses $O(\log{n})$ bits of memory space to execute $P_{\text{rank}}(\nu, \mu)$.

Recall that we require parameter $U_T$ to be a sufficiently large $\Omega(mn)$ value. If an upper bound $M$ of $m$ such that $M = \Theta(m)$ is obtained from knowledge $\mu$, we can substitute a sufficiently large $\Theta(mn)$ value for $U_T$. Then, $P_{\text{rank}}(\nu, \mu)$ converges in $O(mn^3d\log{n})$ steps in expectation. Even if such $M$ is not obtained from $\mu$, e.g., $\mu = \mathbb{N}_{\geq 1}$, we can substitute a sufficiently large $\Theta(n^3)$ value for $U_T$. Then, $P_{\text{rank}}(\nu, \mu)$ converges in $O(mn^3d\log{n} + n^5)$ steps in expectation.

In the rest of this section, we fix a population $G = (V_G, E_G) \in \mathcal{G}_{n,s}$, let $m = |E_G|/2$, and let $d$ be the diameter of $G$. To prove Theorem 1, we define three sets $S_{\text{token}}$, $S_{\text{sync}}$, and $S_{\text{rank}}$ of configurations in $C_{\text{all}}(P_{\text{rank}}(\nu, \mu), G)$ as follows.

- **$S_{\text{token}}$**: the set of all the configurations in $C_{\text{all}}(P_{\text{rank}}(\nu, \mu), G)$ where all tokens have distinct labels, i.e., $\forall u, v \in V_G : u.\text{id}_T \neq v.\text{id}_T$. In a configuration in $S_{\text{token}}$, there exists exactly one token labeled $x$ in the population for each $x \in \{0, 1, \ldots, n - 1\}$. We use notation $T_x$ to denote the unique token labeled by $x$ and to denote the agent on which this token stays.

- **$S_{\text{sync}}$**: the set of all the configurations in $S_{\text{token}}$ where proposition $Q_{\text{token}}(x) \overset{\text{def}}{=} V_G(x) \neq \emptyset \Rightarrow (\exists u \in V_G(x) : u.\text{color}_A = T_x.\text{color}_T \lor u.\text{color}_A = W)$ holds for any $x \in \{0, 1, \ldots, n - 1\}$,
where \( V_G(x) \) \( \text{def} \) \( \{v \in V \mid v.A = x\} \).

- \( \mathcal{S}_{\text{rank}} \): the set of all the configurations in \( \mathcal{S}_{\text{sync}} \) where all the agents in \( V_G \) have distinct labels, that is, \( \forall u, v \in V_G : u.A \neq v.A \).

We say that a set \( S \) of configurations is closed for protocol \( P \) if no execution of protocol \( P \) starting from any configuration in \( S \) reaches a configuration out of \( S \).

**Lemma 8.** The set \( \mathcal{S}_{\text{token}} \) is closed for \( P_{\text{rank}(\nu, \mu)} \).

*Proof.* A token changes its label only if it meets another token with the same label. Hence, no token changes its label in an execution starting from a configuration in \( \mathcal{S}_{\text{token}} \).

**Lemma 9.** Let \( x \in \{0, 1, \ldots, n - 1\} \). In an execution of \( P_{\text{rank}(\nu, \mu)} \) starting from a configuration in \( \mathcal{S}_{\text{token}} \), once \( Q_{\text{token}}(x) \) holds, it always holds thereafter.

*Proof.* This lemma holds because (i) an agent must be white just after it changes its label from \( x - 1 \) (mod \( n \)) to \( x \), (ii) a white agent labeled \( x \) changes its color only when token \( T_x \) visits it at an interaction, at which this white agent get the same color as that of \( T_x \), (iii) an agent labeled \( x \) with the same color as that of \( T_x \) changes its color only when token \( T_x \) visits it at an interaction, at which this agent and \( T_x \) get the same new color.

**Lemma 10.** The set \( \mathcal{S}_{\text{sync}} \) is closed for \( P_{\text{rank}(\nu, \mu)} \).

*Proof.* The lemma immediately follows from Lemma 9.

**Lemma 11.** Let \( x \in \{0, 1, \ldots, n - 1\} \). In an execution of \( P_{\text{rank}(\nu, \mu)} \) starting from a configuration in \( \mathcal{S}_{\text{sync}} \), once at least one agent is labeled \( x \), the number of agents labeled \( x \) never becomes zero thereafter.

*Proof.* This lemma holds in the same way as the proof of Lemma 9.

**Lemma 12.** The set \( \mathcal{S}_{\text{rank}} \) is closed for \( P_{\text{rank}(\nu, \mu)} \).

*Proof.* The lemma immediately follows from Lemmas 10 and 11.

The following lemma is useful to analyze the expected number of steps required to reach a configuration in \( \mathcal{S}_{\text{rank}} \) in an execution of \( P_{\text{rank}(\nu, \mu)} \).

**Lemma 13.** Consider the following game with \( n \) players \( p_0, p_1, \ldots, p_{n-1} \). Each player always has one state in \( \{0, 1, \ldots, n - 1\} \). At each step, an arbitrary pair of players is selected and they check their states each other. If they have the same state, one of them increases its state by one modulo \( n \). Otherwise, they do not change their states. Starting this game from any configuration (i.e., any combination of the states of all players), there is at least one state \( z \in \{0, 1, \ldots, n - 1\} \) such that no player changes its state from \( z - 1 \) (mod \( n \)) to \( z \). The set of such states is uniquely determined by a configuration from which the game starts.

*Proof.* Fix an initial configuration \( \psi_0 = (k_0, k_1, \ldots, k_{n-1}) \), where \( k_i \) represents the number of agents in state \( i \) in the configuration. In this proof, we make every addition and subtraction in modulo \( n \) and omit the notation “ (mod \( n \))”. It is trivial that for any \( x \in \{0, 1, \ldots, n - 1\} \), at least one player changes its state from \( x - 1 \) to \( x \) if and only if \( x \) satisfies the following predicate:

\[
\forall i \in \{1, 2, \ldots, n - 1\} : \sum_{j=1}^{i} k_{x-j} \leq i.
\]

Therefore, the set of states \( z \) such that no player changes its state from \( z - 1 \) to \( z \) is uniquely determined by the initial configuration \( \psi_0 \).
Hence, it suffices to show that there exists at least one state \( z \in \{0, 1, \ldots, n - 1\} \) such that no player changes its state from \( z - 1 \) to \( z \) in some execution of this game which starts from \( \psi_0 \). We say that a state \( x \in \{0, 1, \ldots, n - 1\} \) is filled if at least one player is in state \( x \). By definition of this game, once a state \( x \) is filled, \( x \) is always filled thereafter. Consider an arbitrary execution \( \Xi \) of this game which starts from \( \psi_0 \) and let \( z \) be the state that is filled for the last time in execution \( \Xi \). By definition, when \( z \) gets filled, all the \( n \) states are filled, which yields that all the \( n \) players have mutually distinct states at this time. Therefore, no player never changes its state from \( z - 1 \) to \( z \) in execution \( \Xi \).

**Lemma 14.** Starting from any configuration \( C_0 \in \mathcal{C}_{\text{all}}(P_{\text{rank}}(\nu, \mu), G) \), an execution of \( P_{\text{rank}}(\nu, \mu) \) under the uniformly random scheduler (i.e., \( \Xi_{P(\nu, \mu)}(G, C_0, \Gamma) \)) reaches a configuration in \( \mathcal{S}_{\text{token}} \) within \( O(mn^2 \log n) \) steps in expectation.

**Proof.** By Lemma 13 there exists an integer \( z \in \{0, 1, \ldots, n - 1\} \) such that no token changes its label from \( z - 1 \mod n \) to \( z \). Then, the number of tokens labeled \( z \) becomes exactly one before or when all the token meet each other. By Lemma 3 the number of tokens labeled \( z \) becomes exactly one within \( O(mn^2 \log n) \) steps in expectation. Therefore, no token changes its label from \( z \mod n \) to \( z + 1 \mod n \). Hence, the number of tokens labeled \( z + 1 \mod n \) becomes one in the next \( O(mn^2 \log n) \) steps in the same way. Repeating this procedure, all the tokens have distinct labels within \( O(mn^2 \log n) \) steps in expectation.

**Lemma 15.** Starting from any configuration \( C_0 \in \mathcal{S}_{\text{token}} \), an execution of \( P_{\text{rank}}(\nu, \mu) \) under the uniformly random scheduler (i.e., \( \Xi_{P(\nu, \mu)}(G, C_0, \Gamma) \)) reaches a configuration in \( \mathcal{S}_{\text{sync}} \) within \( O(mn^2) \) steps in expectation.

**Proof.** By Lemmas 8 and 9 it suffices to show that for each \( x \in \{0, 1, \ldots, n - 1\} \), \( Q_{\text{token}}(x) \) becomes true within \( O(mn^2) \) steps in expectation in an execution of \( P_{\text{rank}}(\nu, \mu) \) starting from \( C_0 \). We have \( Q_{\text{token}}(x) = \text{false} \) if and only if there exists at least one agent labeled \( x \) and all of them have colors different from that of \( T_x \) (i.e., the token labeled \( x \)). Even if \( Q_{\text{token}}(x) = \text{false} \) in \( C_0 \), \( Q_{\text{token}}(x) \) becomes true before or when \( T_x \) meets all of them. Hence, we obtain the lemma by Lemma 2.

**Lemma 16.** Assume that \( U_T \) is sufficiently large \( \Omega(mn) \) value. Starting from any configuration \( C_0 \in \mathcal{S}_{\text{sync}} \), an execution of \( P_{\text{rank}}(\nu, \mu) \) under the uniformly random scheduler (i.e., \( \Xi_{P(\nu, \mu)}(G, C_0, \Gamma) \)) reaches a configuration in \( \mathcal{S}_{\text{rank}} \) within \( O(mn^3 + n^2 U_T) \) steps in expectation.

**Proof.** By Lemmas 11 and 13 there exists an integer \( z \in \{0, 1, \ldots, n - 1\} \) such that no agent changes its label from \( z - 1 \mod n \) to \( z \). Therefore, at least one agent is labeled \( z \) in \( C_0 \). All of them get non-white color, i.e., blue or red, or gets a new label \( z + 1 \mod n \) before or when \( T_z \) meets all agents, which requires only \( O(mn^2) \) steps in expectation. (See Lemma 2). Without loss of generality, we assume that token \( T_z \) is red at this time. By Lemma 11 there is at least one red agent labeled \( z \). After that, the timer of \( T_z \) becomes zero within \( O(n U_T) \) steps in expectation, by Lemma 9. In the next \( O(mn^2) \) steps in expectation, \( T_z \) meets a red agent labeled \( z \), at which \( T_z \) and this agent changes their colors to blue, and \( T_z \) resets its timer to \( U_T \). It is well known that a token making the random walk visits all nodes of any undirected graph within \( O(mn) \) moves in expectation. Since a token decreases its timer only by one every time it moves, \( T_z \) meets all agents and makes each agent labeled \( z \) blue or pushes it to the next label (i.e., \( z + 1 \mod n \)) before its timer reaches zero again from \( U_T = \Omega(mn) \), with probability \( 1 - p \) for any small constant \( p \), by Markov’s inequality. This requires only \( O(mn^2) \) steps in expectation by Lemma 4. Similarly, (i) the timer of \( T_z \) becomes zero again in the next \( O(n U_T) \) steps, (ii) \( T_z \) meets a blue agent labeled \( z \), say \( v \), in the next \( O(mn^2) \) steps, at which \( T_z \) and \( v \) become red, and (iii) \( T_z \) meets all agents and pushes all agents labeled \( z \) except for \( v \) to the next label in the next \( O(mn^2) \) steps in expectation and with probability \( 1 - p \) for any small constant \( p \). Therefore, the number of agents labeled \( z \) becomes one within \( O(mn^2 + n U_T) \) steps in expectation. After that, no agent changes its label from \( z \) to \( z + 1 \mod n \). Hence, the number of agents labeled \( z + 1 \mod n \) becomes one
in the next \(O(mn^2 + nU_T)\) steps in expectation by the same reason. Repeating this procedure, all agents get mutually distinct labels (i.e., \(\text{id}_A\)) within \(O(mn^3 + n^2U_T)\) steps in expectation. \(\Box\)

**Proof of Theorem 4.** By Lemmas 13, 15, and 16, \(P_{\text{rank}}(\nu, \mu)(G, C_0, \Gamma)\) reaches a configuration in \(S_{\text{rank}}\) within \(O(mn^3d\log n + n^2U_T)\) steps in expectation. By Lemma 12, every configuration in \(S_{\text{rank}}\) is a safe configuration for the ranking problem. \(\Box\)

**Theorem 2.** Let \(\nu\) be any subset of \(\mathbb{N}_{\geq 2}\) and let \(\mu = \mathbb{N}_{\geq 1}\). Given knowledge \(\nu\) and \(\mu\) (\(= \mathbb{N}_{\geq 1}\)), there exists a self-stabilizing protocol that solves \(\text{LE}\) and \(\text{RK}\) in arbitrary graphs if and only if the agents know the exact number of agents i.e., \(G_{\nu, \mu} = G_{n, *}\) for some \(n \in \mathbb{N}_{\geq 2}\).

**Proof.** The theorem immediately follows from Lemma 7, Theorem 1, and the fact that \(\text{LE} \preceq \text{RK}\). \(\Box\)

## 5 Degree Recognition and Neighbor Recognition

Our goal is to prove the negative and positive propositions for \(\text{DR}\) and \(\text{NR}\) introduced in Section 1. First, we prove the negative proposition.

**Lemma 17.** Let \(\nu\) and \(\mu\) be any sets such that \(\nu \subseteq \mathbb{N}_{\geq 2}\) and \(\mu \subseteq \mathbb{N}_{\geq 1}\). There exists no self-stabilizing protocol that solves \(\text{DR}\) in all graphs in \(G_{\nu, \mu}\) if \(G_{n, m_1} \cup G_{n, m_2} \subseteq G_{\nu, \mu}\) holds for some \(n \in \mathbb{N}_{\geq 2}\) and some distinct \(n, m_1, m_2 \in \mathbb{N}_{\geq 1}\) such that \(G_{n, m_1} \neq \emptyset\) and \(G_{n, m_2} \neq \emptyset\).

**Proof.** Assume \(m_1 < m_2\) without loss of generality. By definition, there must exist two graphs \(G' = (V_{G'}, E_{G'}) \in G_{n, m_1}\) and \(G'' = (V_{G''}, E_{G''}) \in G_{n, m_2}\) such that \(V_{G'} = V_{G''}\) and \(E_{G'} \subseteq E_{G''}\). Then, there exists at least one agent \(v \in V_{G''}\) such that its degree differs in \(G'\) and \(G''\). Let \(\delta'\) and \(\delta''\) be the degrees of \(v\) in \(G'\) and \(G''\), respectively. Assume for contradiction that there is a self-stabilizing protocol \(P(\nu, \mu)\) that solves \(\text{DR}\) both in \(G'\) and \(G''\). By definition, there must be at least one safe configuration \(S\) of a protocol \(P(\nu, \mu)\) on \(G''\) for \(\text{DR}\). In every execution of \(P(\nu, \mu)\) starting from \(S\) on \(G''\), agent \(v\) must always output \(\delta''\) as its degree. However, \(P(\nu, \mu)\) is also self-stabilizing in \(G'\). Therefore, there must be a finite sequence of interactions \(\gamma_0, \gamma_1, \ldots, \gamma_t\) of \(G'\) that put configuration \(S\) to a configuration where \(v\) outputs \(\delta'\) as its degree. Since \(E_{G''} \subseteq E_{G'}\), \(\gamma_0, \gamma_1, \ldots, \gamma_t\) is also a sequence of interactions in \(G''\). This implies that this sequence changes the output of \(v\) from \(\delta''\) to \(\delta'\) starting from a safe configuration, a contradiction. \(\Box\)

To prove the positive proposition, we give a self-stabilizing protocol \(P_{\text{neigh}}\), which solves the neighbor recognition problem (NR) in arbitrary graphs given the knowledge of the exact number of agents and the exact number of interactable pairs, that is, given knowledge \(\nu\) and \(\mu\) such that \(|\nu| = |\mu| = 1\). In the rest of this section, let \(n\) and \(m\) be the integers such that \(\nu = \{n\}\) and \(\mu = \{m\}\).

The pseudocode of \(P_{\text{neigh}}\) is shown in Algorithm 2. Our goal is to let the agents recognize the set of their neighbors. Each agent \(v\) stores its label in a variable \(v.\text{id}_A \in \{0, 1, \ldots, n - 1\}\) and the set of the labels assigned to its neighbors in a variable \(\text{neighbors} \in 2^{\{0, 1, \ldots, n-1\}}\). Each agent \(v\) outputs \((v.\text{id}_A, v.\text{neighbors})\).

We use \(P_{\text{rank}}\) as a sub-algorithm to assign the agents the distinct labels \(0, 1, \ldots, n - 1\) and to let the \(n\) tokens make the random walk. Specifically, we first execute \(P_{\text{rank}}\) whenever two agents have an interaction (Line 1), substituting a sufficiently large \(\Theta(mn)\) value for \(U_T\). We do not update the variables used in \(P_{\text{rank}}\) in the other lines (Lines 2–17). Therefore, by Theorem 1, an execution of \(P_{\text{neigh}}\) starting from any configuration reaches a configuration in \(S_{\text{rank}}\) within \(O(mn^2d\log n)\) steps in expectation. To simplify explanation, we consider only an execution after reaching a configuration in \(S_{\text{rank}}\). Then, we can assume that the population always has exactly one agent labeled \(x\) and exactly one token labeled \(x\) for each \(x = \{0, 1, \ldots, n - 1\}\). We denote them by \(A_x\) and \(T_x\), respectively.
Algorithm 2 \(P_{\text{neigh}}(\nu, \mu)\)

Assumption: \(|\nu| = 1\) and \(|\mu| = 1\). (Let \(\nu = \{n\}\) and \(\mu = \{m\}\).)

Variables:
- \(\text{id}_A, \text{id}_T \in \{0, 1, \ldots, n-1\}\) // Updated only by \(P_{\text{rank}}\)
- \(\text{degree}_T \in \{0, 1, \ldots, n\}\)
- \(\text{sum} \in \{0, 1, \ldots, 2m+1\}\)
- \(\text{reset}_E \in \{0, 1, \ldots, U_E\}\)
- \(\text{timer}_P \in \{0, 1, \ldots, U_P\}\)
- \(\text{neighbors}, \text{counted} \in 2^{\{0, 1, \ldots, n-1\}}\)

Output function \(\pi_{\text{out}}(\text{id}_A, \text{neighbors})\)

Interaction between initiator \(a_0\) and responder \(a_1\):
1. Execute \(P_{\text{rank}}\) with substituting sufficiently large \(\Theta(mn)\) value for \(U_T\).
2. \(a_0.\text{degree}_T \leftarrow a_1.\text{degree}_T\)
   // Execute the random walk of two tokens with \(P_{\text{rank}}\)
3. \(a_0.\text{reset}_E \leftarrow a_1.\text{reset}_E \leftarrow \max(0, a_0.\text{reset}_E - 1, a_1.\text{reset}_E - 1)\)
4. if \(a_0.\text{reset}_E > 0\) then \(a_0.\text{neighbors} \leftarrow a_1.\text{neighbors} \leftarrow \emptyset\) endif
5. for all \(i \in \{0, 1\}\) do
6. \(a_i.\text{timer}_P \leftarrow \max(0, a_i.\text{timer}_P - 1)\)
7. if \(a_i.\text{timer}_P = 0\) then
8. \(\langle a_i.\text{sum}, a_i.\text{counted}, a_i.\text{timer}_P \rangle \leftarrow \langle 0, \emptyset, U_P \rangle\)
9. end if
10. \(a_i.\text{neighbors} \leftarrow a_i.\text{neighbors} \cup \{a_{1-i}.\text{id}_A\}\)
11. if \(a_i.\text{id}_A = a_i.\text{id}_T\) then \(a_i.\text{degree}_T \leftarrow |a_i.\text{neighbors}|\) endif
12. if \(a_i.\text{id}_T \notin a_i.\text{counted}\) then
13. \(a_i.\text{sum} \leftarrow \min(2m + 1, a_i.\text{sum} + a_i.\text{degree}_T)\)
14. \(a_i.\text{counted} \leftarrow a_i.\text{counted} \cup \{a_i.\text{id}_T\}\)
15. end if
16. if \(a_i.\text{sum} = 2m + 1\) then \(a_i.\text{reset}_E \leftarrow U_E\) endif
17. end for

The agents compute their \text{neighbors} in a simple way: every time two agents \(u\) and \(v\) have an interaction, \(u\) adds \(v.\text{id}_A\) to \(u.\text{neighbors}\) and \(v\) adds \(u.\text{id}_A\) to \(v.\text{neighbors}\) (Line 10). However, this simple way to compute \text{neighbors} is not enough to design a self-stabilizing protocol because we consider an arbitrary initial configuration. Specifically, in an initial configuration, \(v.\text{neighbors}\) may include \(u.\text{id}_A\) for some \(u \notin N_G(v)\). We call such \(u.\text{id}_A\) a fake label. To compute \(v.\text{neighbors}\) correctly, in addition to the above simple mechanism, it suffices to detect the existence of a fake label and reset the \text{neighbors} of all agents to the empty set if a fake label is detected.

Using the knowledge \(\mu = \{m\}\), we achieve the detection of fake labels with the following strategy. Each token \(T_x\) carries \(|A_x.\text{neighbors}|\) in a variable \(\text{degree}_T \in \{0, 1, \ldots, n\}\) (Line 2). Whenever \(T_x\) meets \(A_x\), the value of \(T_x.\text{degree}_T\) is updated by the current value of \(|A_x.\text{neighbors}|\) (Line 11). Each agent always tries to estimate \(\sum_{v \in V_G} |v.\text{neighbors}|\) using variables \(\text{sum} \in \{0, 1, \ldots, 2m+1\}\), \(\text{counted} \in 2^{\{0, 1, \ldots, n-1\}}\), and \(\text{timer}_P \in \{0, 1, \ldots, U_P\}\), where \(U_P\) is a sufficiently large \(\Theta(m \log n)\) value. It uses \(\text{timer}_P\) as a count-down timer to reset \(\text{sum}\) and \(\text{counted}\) periodically. Specifically, an agent \(v\) decreases \(v.\text{timer}_P\) by one every time it has an interaction and resets \(v.\text{sum}, v.\text{counted},\) and \(v.\text{timer}_P\) to 0, \(\emptyset\), and \(U_P\), respectively, when \(v.\text{timer}_P\) reaches zero (Lines 6-9). Whenever agent \(v\) meets \(T_x\) such that \(x \notin v.\text{counted}\), \(v\) executes \(v.\text{sum} \leftarrow \min(2m + 1, v.\text{sum} + T_x.\text{degree}_T)\) and adds \(x\) to \(v.\text{counted}\). (Lines 12-15) We expect
holds for all v. O interactable pairs in the next O in the next hence no agent emits the error signal, after which the error signal disappears from the population expectation for every x. ∑ reaches a configuration in S neighbors when it meets O(1). After that, no agent has a fake label in its neighbors, that is, v.neighbors ⊆ L(v) holds for all v ∈ V. G.

First, we show that execution Ξ = Ξ P G(C,G,Γ) reaches a configuration C′ in S rank within O(mn^3d log n) steps in expectation because U_T = Θ(mn). We assume C′ ∈ S rank because otherwise we need not discuss anything. Interactions happen between all interactable pairs within O(m log n) steps in expectation. Therefore, after reaching C′, Ξ reaches within O(m log n) steps in expectation a configuration C′′ where L(v) ⊆ v.neighbors for all v ∈ V. G or a configuration where u.reset > 0 for some u ∈ V. G. In the former case, ∑v∈V |v.neighbors| > 2m holds in C′′ since at least one agent has one or more fake labels in its neighbors. Thereafter, some agent v decreases its timer to zero and resets it to U_p in the next O(m U_p) = O(m^2 n d log n) steps in expectation. After that, v meets all tokens within O(m n d log n) steps in expectation (See Lemma 3). As a result, v.sum reaches 2m + 1 and v emits the error signal. To conclude, after Ξ reaches C′, some agent emits the error signal, i.e., it substitutes U_E for its reset. Since we set U_E to a sufficiently large Θ(n^2) value, the error signal is propagated to the whole population within O(mn) steps with probability 1 - O(1/n). (See Lemma 5 in [19].) Every time an agent receives the error signal, it resets its neighbors to the empty set. Therefore, Ξ reaches a configuration in S noFake within O(mn^3d log n) steps in expectation. After entering S noFake, Ξ reaches within O(m n d log n) steps in expectation a configuration where ∑x=0,1,...,n-1 T_x degree_x ≤ 2m holds; because every T_x meets A_x within O(mn) steps in expectation for every x ∈ {0,1,...,n-1}. Similarly, all agents reset their sum and counted in the next O(m U_p) = O(mn^3d log n) step in expectation. Thereafter, no agent sees sum = 2m + 1, hence no agent emits the error signal, after which the error signal disappears from the population in the next O(m U_p) = O(mn^2) steps in expectation. Therefore, interactions happen between all interactable pairs in the next O(m log n) steps in expectation, by which v.neighbors = L(v) holds for all v ∈ V. G. After that, no agent v changes v.neighbors, which yields that Ξ has reached a safe configuration.

6 Conclusion

In this paper, we clarified the solvability of the leader election problem, the ranking problem, the degree recognition problem, and the neighbor recognition problem by self-stabilizing population protocols with knowledge of the number of nodes and/or the number of edges in a network. The

\[ v.\text{sum} = \sum_{v \in V_G} |v.\text{neighbors}| \]

when it meets \(T_0, T_1, ..., T_{n-1}\). If \(v.\text{sum}\) reaches \(2m + 1\), agent \(v\) concludes that at least one agent has a fake label, i.e., \(v.\text{neighbors} \not\subseteq \{w.\text{id}_A \mid w \in N_G(u)\}\) for some \(u\).

When the existence of a fake label is detected, we reset the neighbors of all agents using a variable \(\text{reset}_E \in \{0, 1, ..., U_E\}\), where \(U_E\) is a sufficiently large \(\Theta(n^2)\) value. Specifically, when \(v.\text{sum} = 2m + 1\) holds, \(v\) emits the error signal by setting variable \(v.\text{reset}_E\) to \(U_E\) (Line 16). Thereafter, the error signal is propagated to the whole population via the larger value propagation: when two agents \(u\) and \(v\) meet, they substitute max(0, \(u.\text{reset}_E - 1\), \(v.\text{reset}_E - 1\)) for their \(\text{reset}_E\). (Line 3). Whenever an agent \(v\) receives the error signal, i.e., \(v.\text{reset}_E > 0\) holds, it resets its neighbors to the empty set (Line 4).

\textbf{Theorem 3}. Given knowledge \(\nu\) and \(\mu\), \(P_{\text{neigh}}(\nu, \mu)\) is a self-stabilizing protocol that solves NR in arbitrary graphs if \(\nu = \{n\}\) and \(\mu = \{m\}\) for some integers \(n\) and \(m\). Starting from any configuration \(C_0\) on any population \(G = (V_G, E_G) \in \mathcal{G}_{n,m}\), the execution of \(P_{\text{neigh}}(\nu, \mu)\) under the uniformly random scheduler (i.e., \(\mathbb{E}_{P_{\text{neigh}}(\nu, \mu)}(G, C_0, \Gamma)\)) reaches a safe configuration within \(O(mn^3d \log n)\) steps in expectation, where \(m = |E_G|/2\) and \(d\) is the diameter of \(G\). Each agent uses \(O(n)\) bits of memory space to execute \(P_{\text{neigh}}(\nu, \mu)\).

\textbf{Proof}. Define \(L_{\text{neigh}}(v) = \{u.\text{id}_A \mid u \in N_G(v)\}\) and define \(S_{\text{noFake}}\) as the set of all configurations in \(S_{\text{rank}}\) where no agent has a fake label in its neighbors, that is, \(v.\text{neighbors} \subseteq L_{\text{neigh}}(v)\) holds for all \(v \in V_G\).

First, we show that execution \(\Xi = \Xi_{P_{\text{neigh}}(\nu, \mu)}(G, C_0, \Gamma)\) reaches a configuration in \(S_{\text{noFake}}\) within \(O(mn^3d \log n)\) steps in expectation. By Theorem 1 \(\Xi\) reaches a configuration \(C'\) in \(S_{\text{rank}}\) within \(O(mn^3d \log n)\) steps in expectation because \(U_T = \Theta(mn)\). We assume \(C' \not\in S_{\text{noFake}}\) because otherwise we need not discuss anything. Interactions happen between all interactable pairs within \(O(m \log n)\) steps in expectation. Therefore, after reaching \(C'\), \(\Xi\) reaches within \(O(m \log n)\) steps in expectation a configuration \(C''\) where \(L_{\text{neigh}}(v) \subseteq v.\text{neighbors}\) for all \(v \in V_G\) or a configuration where \(u.\text{reset}_E > 0\) for some \(u \in V_G\). In the former case, \(\sum_{v \in V} |v.\text{neighbors}| > 2m\) holds in \(C''\) since at least one agent has one or more fake labels in its neighbors. Thereafter, some agent \(v\) decreases its timer to zero and resets it to \(U_p\) in the next \(O(m U_p) = O(m^2 n d \log n) \subseteq O(mn^3d \log n)\) steps in expectation. After that, \(v\) meets all tokens within \(O(m n d \log n)\) steps in expectation (See Lemma 3). As a result, \(v.\text{sum}\) reaches \(2m + 1\) and \(v\) emits the error signal. To conclude, after \(\Xi\) reaches \(C'\), some agent emits the error signal, i.e., it substitutes \(U_E\) for its \(\text{reset}_E\). Since we set \(U_E\) to a sufficiently large \(\Theta(n^2)\) value, the error signal is propagated to the whole population within \(O(mn)\) steps with probability \(1 - O(1/n)\). (See Lemma 5 in [19].) Every time an agent receives the error signal, it resets its neighbors to the empty set. Therefore, \(\Xi\) reaches a configuration in \(S_{\text{noFake}}\) within \(O(mn^3d \log n)\) steps in expectation.

After entering \(S_{\text{noFake}}\), \(\Xi\) reaches within \(O(m n d \log n)\) steps in expectation a configuration where \(\sum_{x=0,1,...,n-1} T_x \text{degree}_x \leq 2m\) holds; because every \(T_x\) meets \(A_x\) within \(O(mn)\) steps in expectation for every \(x \in \{0,1,...,n-1\}\). Similarly, all agents reset their sum and counted in the next \(O(m U_p) \subseteq O(mn^3d \log n)\) step in expectation. Thereafter, no agent sees sum = \(2m + 1\), hence no agent emits the error signal, after which the error signal disappears from the population in the next \(O(m U_p) = O(mn^2)\) steps in expectation. Therefore, interactions happen between all interactable pairs in the next \(O(m \log n)\) steps in expectation, by which \(v.\text{neighbors} = L_{\text{neigh}}(v)\) holds for all \(v \in V_G\). After that, no agent \(v\) changes \(v.\text{neighbors}\), which yields that \(\Xi\) has reached a safe configuration. \(\square\)
protocols we gave in this paper require exact knowledge on the number of agents and/or the number of interactable pairs. It is interesting and still open whether ambiguous knowledge such as “the number of interactable pairs is at most $M$” and “the number of agents is not a prime number” is useful to design self-stabilizing population protocols.

References

[1] D. Alistarh and R. Gelashvili. Polylogarithmic-time leader election in population protocols. In Proceedings of the 42nd International Colloquium on Automata, Languages, and Programming, pages 479–491, 2015.

[2] D. Angluin, J. Aspnes, Z. Diamadi, M. J. Fischer, and R. Peralta. Computation in networks of passively mobile finite-state sensors. Distributed Computing, 18(4):235–253, 2006.

[3] D. Angluin, J. Aspnes, and D. Eisenstat. Fast computation by population protocols with a leader. Distributed Computing, 21(3):183–199, 2008.

[4] D. Angluin, J. Aspnes, M. J. Fischer, and H. Jiang. Self-stabilizing population protocols. ACM Transactions on Autonomous and Adaptive Systems, 3(4):13, 2008.

[5] J. Beauquier, P. Blanchard, and J. Burman. Self-stabilizing leader election in population protocols over arbitrary communication graphs. In International Conference on Principles of Distributed Systems, pages 38–52, 2013.

[6] J. Burman, D. Doty, T. Nowak, E. E. Severson, and C. Xu. Efficient self-stabilizing leader election in population protocols. arXiv preprint arXiv:1907.06068, 2019.

[7] S. Cai, T. Izumi, and K. Wada. How to prove impossibility under global fairness: On space complexity of self-stabilizing leader election on a population protocol model. Theory of Computing Systems, 50(3):433–445, 2012.

[8] D. Canepa and M. G. Potop-Butucaru. Stabilizing leader election in population protocols. 2007. http://hal.inria.fr/inria-00166632.

[9] H.-P. Chen and H.-L. Chen. Self-stabilizing leader election. In Proceedings of the 2019 ACM Symposium on Principles of Distributed Computing, pages 53–59, 2019.

[10] G. Cordasco and L. Gargano. Space-optimal proportion consensus with population protocols. In International Symposium on Stabilization, Safety, and Security of Distributed Systems, pages 384–398, 2017.

[11] E. Dijkstra. Self-stabilizing systems in spite of distributed control. Communications of the ACM, 17(11):643–644, 1974.

[12] M. J. Fischer and H. Jiang. Self-stabilizing leader election in networks of finite-state anonymous agents. In International Conference on Principles of Distributed Systems, pages 395–409, 2006.

[13] L. Gąsieniec, G. Stachowiak, and P. Uznanski. Almost logarithmic-time space optimal leader election in population protocols. In The 31st ACM on Symposium on Parallelism in Algorithms and Architectures, pages 93–102. ACM, 2019.

[14] T. Izumi. On space and time complexity of loosely-stabilizing leader election. In International Colloquium on Structural Information and Communication Complexity, pages 299–312, 2015.
[15] G. B. Mertzios, S. E. Nikoletseas, C. L. Raptopoulos, and P. G. Spirakis. Determining majority in networks with local interactions and very small local memory. In International Colloquium on Automata, Languages, and Programming, pages 871–882, 2014.

[16] Y. Sudo, T. Masuzawa, A. K. Datta, and L. L. Larmore. The same speed timer in population protocols. In the 36th IEEE International Conference on Distributed Computing Systems, pages 252–261, 2016.

[17] Y. Sudo, J. Nakamura, Y. Yamauchi, F. Ooshita, H. Kakugawa, and T. Masuzawa. Loosely-stabilizing leader election in a population protocol model. Theoretical Computer Science, 444:100–112, 2012.

[18] Y. Sudo, F. Ooshita, T. Izumi, H. Kakugawa, and T. Masuzawa. Logarithmic expected-time leader election in population protocol model. In Proceedings of the 21st International Symposium on Stabilizing, Safety, and Security of Distributed Systems, pages 323–337, 2019.

[19] Y. Sudo, F. Ooshita, H. Kakugawa, and T. Masuzawa. Loosely-stabilizing leader election on arbitrary graphs in population protocols. In International Conference on Principles of Distributed Systems, pages 339–354, 2014.

[20] Y. Sudo, F. Ooshita, H. Kakugawa, and T. Masuzawa. Loosely stabilizing leader election on arbitrary graphs in population protocols without identifiers or random numbers. IEICE Transactions on Information and Systems, 103(3):489–499, 2020.

[21] Y. Sudo, F. Ooshita, H. Kakugawa, T. Masuzawa, A. K. Datta, and L. L. Larmore. Loosely-stabilizing leader election with polylogarithmic convergence time. Theoretical Computer Science, 806:617–631, 2020.