WHAT IS A TRUE SPECTRA OF A FINITE FOURIER TRANSFORM

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ABSTRACT. In this paper we deal with a finite abelian group $G$ and the abstract Fourier transform $\mathcal{F}: \mathbb{C}^G \to \mathbb{C}^{\hat{G}}$. Then, we consider $\tilde{j} \circ \mathcal{F}: \mathbb{C}^G \to \mathbb{C}^{\hat{G}}$ where $\tilde{j}: \mathbb{C}^{\hat{G}} \to \mathbb{C}^G$ is defined by the composition with a bijection $j: G \to \hat{G}$. ($\tilde{j}$ is a pullback of $j$.) In particular, we show that $(\tilde{j} \circ \mathcal{F})^2$ is a permutation if and only if $j$ is a group isomorphism. Then, we study how the spectra of $\tilde{j} \circ \mathcal{F}$ depends on the isomorphism $j$. 

1. INTRODUCTION

Let $G$ be a finite abelian group and $\hat{G}$ its dual. Abstractly, a Fourier transform is a linear operator $\mathcal{F}: \mathbb{C}^G \to \mathbb{C}^{\hat{G}}$. So, it is a linear map from one space to another and it is worthless to speak about its spectrum and period, etc. In order to do it, we have to identify $\mathbb{C}^G$ with $\mathbb{C}^{\hat{G}}$. To this end, we may take a bijection $j: G \to \hat{G}$ and consider $\tilde{j}: \mathbb{C}^{\hat{G}} \to \mathbb{C}^G$ defined as $(\tilde{j}f)(g) = f(j(g))$. A “concrete” Fourier transform is a composition $\tilde{j} \circ \mathcal{F}$. Our first result: $(\tilde{j} \circ \mathcal{F})^2$ is a permutation if and only if $j$ is an isomorphism. In this case $(\tilde{j} \circ \mathcal{F})^2$ is (a pullback of) $\hat{j}^{-1} \circ j$ which is an automorphism of $G$. Here, $j: G \to \hat{G}$ is the dual of $j$. This result distinguishes isomorphisms from other bijections by “Fourier transform’s point of view”. So, further we consider the case when $j$ is an isomorphism. But, which isomorphism $j$ should we take? From the pure group theoretic point of view, there are no natural choices for $j$. Of course, we may try to find $j$ with $\hat{j}^{-1} \circ j$ equal to a fixed automorphism. Such a choice is not always possible and, if possible, is not unique. For example, if $G$ is an additive group of $\mathbb{Z}/n\mathbb{Z}$, then $\hat{j}^{-1} \circ j$ is always the map $x \to -x$. Still, we show that, in this case, the spectra (the multiplicities of eigenvalues) of $j \circ \mathcal{F}$ does depend on $j$. If $G = (\mathbb{Z}/n\mathbb{Z})^m$, then there are more flexibility in the choice of $\hat{j}^{-1} \circ j$. Particularly, $\hat{j}^{-1} \circ j$ could be the identical transformation for even $m$. We consider several other examples and show how the spectra may be calculated in those cases.

We do not give a receipt for choosing $j$. We just point out that there are several choices and the properties of the Fourier transform do depend on these choices. In applied mathematics, for an additive group $\mathbb{Z}/n\mathbb{Z}$, which is a cyclic group $C_n$, up to an isomorphism, we choose $j(x) = w^x$, where $w$ is the $n$-th root of unity closest to 1. This corresponds to the choice of a generator of $C_n$. If $G = (\mathbb{Z}/n\mathbb{Z})^m$, we may choose $j$ to be the $m$-th power of the above defined $j$. Are those choices natural? Of course, the answer to this question depends on the problem. If $G$ comes with presentation as a direct product (which is not unique), then there is a naturally

2010 Mathematics Subject Classification. Primary 11T06; Secondary 13M10.

This paper is in final form and no version of it will be submitted for publication elsewhere.
defined presentation of $\hat{G}$ as a direct product. So, in this case, the choice of $j$ as a direct product might be natural. On the other hand, it is known that there are no natural isomorphism $G \to \hat{G}$ in the category of abelian groups. So, it looks like that for a pure group-theoretical problem (without some additional structure) there is no natural choice for an isomorphism $j$. So, it is interesting to study the dependence of $\tilde{j} \circ F$ on the isomorphism $j$.

2. Preliminaries

Let $G$ be a finite abelian group. Set $C^G = \{ f : G \to \mathbb{C} \}$, the $\mathbb{C}$-valued functions on $G$. This is a $\mathbb{C}$-vector space of functions. Every $f \in C^G$ can be expressed as a linear combination of the delta functions $\delta_g : G \to \{0, 1\}$ defined by

$$\delta_g(x) = \begin{cases} 1 & \text{if } x = g, \\ 0 & \text{if } x \neq g, \end{cases}$$

for every $g \in G$, as follows

$$f = \sum_{g \in G} f(g) \delta_g.$$  

Indeed, evaluate both sides at each $x \in G$ and we get the same value. The functions $\delta_g$ span $C^G$ and they are linearly independent: if $\sum_{g \in G} a_g \delta_g = 0$, then evaluating the sum at $x \in G$ shows $a_x = 0$. Thus, the functions $\delta_g$ are a basis of $C^G$, so $\dim C^G = |G|$.

A character of $G$ is a homomorphism $\chi : G \to S^1$. For a character $\chi$ on $G$, the conjugate character is the function $\overline{\chi} : G \to S^1$ given by $\overline{\chi}(g) := \bar{\chi}(g)$. Since for any complex number $z$ with $|z| = 1$, $\bar{z} = \frac{1}{z}$, we have that $\overline{\chi}(g) = \chi(g)^{-1} = \chi(g^{-1})$.

The dual group, or character group, of $G$ is the set of homomorphisms $G \to S^1$ with the group law of pointwise multiplication of functions: $(\chi \psi)(g) = \chi(g)\psi(g)$.

The dual group of $G$ is denoted by $\hat{G}$.

The following result is well known.

**Theorem 2.1.** If $G$ is a finite abelian group, then $G$ and $\hat{G}$ are isomorphic.

The next theorem is the first step leading to an expression for each $\delta_g$ as a linear combination of characters of $G$, which will lead to a Fourier series expansion of $f$ (see [2]).

**Theorem 2.2 (Orthogonality relations).** Let $G$ be a finite abelian group. Then

$$\sum_{g \in G} \chi(g) = \begin{cases} |G| & \text{if } \chi = \chi_0, \\ 0 & \text{if } \chi \neq \chi_0, \end{cases}$$

$$\sum_{\chi \in \hat{G}} \chi(g) = \begin{cases} |G| & \text{if } g = 1, \\ 0 & \text{if } g \neq 1. \end{cases}$$

The Fourier transform $F : C^G \to C^\hat{G}$ is the linear map defined as

$$F(f)(\chi) = \frac{1}{\sqrt{|G|}} \sum_{g \in G} f(g) \chi(g)$$

where $C^G$ and $C^\hat{G}$ are $\mathbb{C}$-vector spaces of dimension $|G| = |\hat{G}|$.

The process of recovering $f$ from its Fourier transform $F(f)$ is called Fourier inversion. The following theorem is a direct corollary of the Orthogonality relations.
**Theorem 2.3** (Fourier inversion). Let \( G \) be a finite abelian group. If \( f \in \mathbb{C}^G \), then
\[
f(x) = \frac{1}{\sqrt{|G|}} \sum_{\chi \in \hat{G}} \mathcal{F}(f)(\chi) \hat{\chi}(x)
\]
for all \( x \in G \).

Since \( \mathcal{F} \) is an isomorphism of two different vector spaces, we are not allowed to talk about \( \mathcal{F}^2 \), the spectra of \( \mathcal{F} \), etc.

On \( \mathbb{C}^G \) there is a natural unitary scalar product \( \langle \cdot, \cdot \rangle \) defined as follows
\[
\langle f_1, f_2 \rangle = \frac{1}{\sqrt{|G|}} \sum_{g \in G} \bar{f}_1(g)f_2(g).
\]

With this scalar product, \( \mathbb{C}^G \) and \( \hat{G} \) are unitary \( \mathbb{C} \)-vector spaces. The orthogonality relations imply that
\[
\langle \mathcal{F}(f_1), \mathcal{F}(f_2) \rangle = \langle f_1, f_2 \rangle
\]
for all \( f_1, f_2 \in \mathbb{C}^G \). So, \( \mathcal{F} \) is a unitary transform \( \mathbb{C}^G \to \hat{G} \).

3. **Main results of this paper**

Let us note, that \( \mathbb{C}^G \) forms an algebra under pointwise addition and multiplication:
\[
(f_1 + f_2)(g) = f_1(g) + f_2(g), \quad (f_1 f_2)(g) = f_1(g)f_2(g).
\]

So, \( \mathbb{C}^G \) contains multiplicative subgroup \( (S^1)^G \). Naturally, \( \hat{G} \subset (S^1)^G \subset \mathbb{C}^G \).

Let \( j : G \to \hat{G} \) be a function and \( \hat{j} : G \to \mathbb{C}^G \) be its dual function defined by \( \hat{j}(h)(x) = \overline{j(x)(h)} \). Notice that \( \hat{j}(G) \subseteq (S^1)^G \) and \( G \to (S^1)^G \) is always a homomorphism (even if \( j \) is not).

**Theorem 3.1.** \( \hat{j}(G) \subset \hat{G} \) if and only if \( j \) is a homomorphism.

**Proof.** Suppose that \( j \) is a homomorphism and let \( g \in G \). Then, for all \( x, y \in G \) we have
\[
\hat{j}(g)(xy) = \overline{j(xy)(g)} = \overline{j(x)j(y)(g)} = \overline{j(x)(g)j(y)(g)} = j(x)(g) \cdot j(y)(g)
\]
and hence \( \hat{j}(G) \subset \hat{G} \).

Conversely, suppose that \( \hat{j}(G) \subset \hat{G} \) and let \( x, y, h \in G \). Then,
\[
j(xy)(h) = j(xy)(h) = j(x)j(y)(h)
\]
\[
= \overline{j(x)(y)j(h)(y)} = \overline{j(x)(h)j(y)(h)} = (j(x)(h))(j(y)(h)) = \overline{(j(x)(h))(j(y)(h))},
\]
and therefore \( j(xy) = j(x)j(y) \) for all \( x, y \in G \), that is, \( j \) is a homomorphism. \( \square \)

**Corollary 3.2.** \( j \) is an isomorphism if and only if \( \hat{j}(G) = \hat{G} \).

**Proof.** Suppose that \( j \) is an isomorphism. By Theorem 3.1 we have \( \hat{j}(G) \subset \hat{G} \) and \( \hat{j} : G \to \hat{G} \). Moreover, \( j \) is a homomorphism. We will prove that \( \hat{j} \) is injective and thus it will be an isomorphism since \( |G| = |\hat{G}| \) and \( G \) is finite. Let \( h \in \ker \hat{j} \). Then, \( \hat{j}(h)(x) = 1 \) for all \( x \in G \). It follows that \( j(x)(h) = 1 \) for all \( x \in G \) and, therefore,
\[ \sum_{x \in G} \hat{j}(x)(h) = |G|. \] Since \( j \) is a bijection, \( j(x) \) runs over all \( \hat{G} \) as \( x \) runs over all \( G \). By the orthogonality relations we obtain that \( h \) is the identity of \( G \). Therefore, \( j \) is injective.

Suppose now that \( \hat{j}(G) = \hat{G} \). It follows that \( \hat{j} : G \to \hat{G} \) is an isomorphism and we deduce, applying the above arguments to \( \hat{j} \), that \( \hat{j}(G) = \hat{G} \) and \( \hat{j} \) is an isomorphism. We just have to check that \( \hat{j} \) is injective. 

Let \( j : G \to \hat{G} \) be a bijection. Then, \( j \) induces a unitary linear map \( \tilde{j} : \mathbb{C}^\hat{G} \to \mathbb{C}^G \) defined as
\[ \tilde{j}(\phi)(g) = \phi(j(g)). \]

Now, the composition \( \tilde{j} \circ F : \mathbb{C}^G \to \mathbb{C}^G \) is a unitary linear function. We will say that \( P : \mathbb{C}^G \to \mathbb{C}^G \) is a permutation if \( P(f) = f \circ p \) for all \( f \in \mathbb{C}^G \), where \( p : G \to G \) is a bijection.

**Theorem 3.3.** Let \( G \) be a finite abelian group, \( j : G \to \hat{G} \) a bijection and \( P = (\tilde{j} \circ F)^2 \). Then, \( P \) is a permutation if and only if \( j \) is an isomorphism. In this case \( p = \tilde{j}^{-1} \circ j \) is an isomorphism.

**Proof.** Suppose that \( P \) is a permutation. Then there is a bijection \( p : G \to G \) such that \( P(f) = f \circ p \) for all \( f \in \mathbb{C}^G \). It is enough to prove that \( j \) is a homomorphism since \( j \) is a bijection.

Recall that \( \delta_g : G \to \{0,1\} \) is the \( g \)-delta function \( (g \in G) \). Then, for every \( x \in G \) we have
\[ P(\delta_g)(x) = \frac{1}{\sqrt{|G|}} \sum_{h \in G} F(\delta_g)(j(h)) \cdot (j(x))(h) = \frac{1}{|G|} \sum_{h \in G} (j(h))(g) \cdot (j(x))(h). \]

Since \( j \) is a bijection there exists an inverse map \( j^{-1} : \hat{G} \to G \). Then, for every \( \chi \in \hat{G} \) we have
\[ P(\bar{j}(\chi))(j^{-1}(\chi)) = \frac{1}{|\hat{G}|} \sum_{h \in G} j(h)(g) \chi(h) = \frac{1}{|\hat{G}|} \sum_{h \in G} j(g)(h) \chi(h) = \frac{1}{\sqrt{|G|}} F(\bar{j}(\chi))(\chi). \]

On the other hand we have
\[ P(\delta_g)(j^{-1}(\chi)) = \delta_g(p(j^{-1}(\chi))) = \begin{cases} 1 & \text{if } g = p(j^{-1}(\chi)), \\ 0 & \text{if } g \neq p(j^{-1}(\chi)). \end{cases} \]

It follows that for every \( \chi \in \hat{G} \),
\[ F(\bar{j}(\chi))(\chi) = \begin{cases} \sqrt{|G|} & \text{if } g = p(j^{-1}(\chi)), \\ 0 & \text{if } g \neq p(j^{-1}(\chi)). \end{cases} \] (3.1)

Now, by the Fourier inversion formula and relation (3.1) we have that for every \( x \in G \)
\[ \bar{j}(g)(x) = \frac{1}{\sqrt{|G|}} \sum_{\chi \in \hat{G}} F(\bar{j}(\chi))(\chi) \chi(x) = \overline{\alpha(x)}, \]

where \( \alpha = j(p^{-1}(g)) \). Hence, \( \tilde{j} = j \circ p^{-1} \) and thus \( \hat{j}(G) \subset \hat{G} \). Now, by Theorem 3.1 it follows that \( j \) is a homomorphism as well as \( p = \tilde{j}^{-1} \circ j \).
Conversely, suppose that \( j \) is an isomorphism and let \( f \in C^G \). Then, for every \( g \in G \) we have
\[
P(f)(g) = (\hat{\cdot} \circ \mathcal{F})^2(f)(g) = (\hat{\cdot} \circ \mathcal{F})(\hat{\cdot}((\mathcal{F}(f))(g)) = \mathcal{F}((\hat{\cdot}(\mathcal{F}(f)))(g))
\]
\[
= \mathcal{F}((\hat{\cdot}(\mathcal{F}(f)))(g)) = \frac{1}{\sqrt{|G|}} \sum_{h \in G} \hat{j}(\mathcal{F}(f))(h) \cdot (j(g))(h)
\]
\[
= \frac{1}{\sqrt{|G|}} \sum_{h \in G} \mathcal{F}(f)(j(h)) \cdot (j(g))(h)
\]
\[
= \frac{1}{\sqrt{|G|}} \sum_{h \in G} \sqrt{|G|} \sum_{l \in G} f(l) \cdot (j(h))(l) \cdot (j(g))(h)
\]
\[
= \frac{1}{|G|} \sum_{l \in G} \sum_{h \in G} f(l) \cdot (j(h))(l) \cdot (j(g))(h)
\]
\[
= \frac{1}{|G|} \sum_{l \in G} \sum_{h \in G} f(l) \overline{j(l)(h)j(g)(h)}
\]
\[
= \frac{1}{|G|} \sum_{l \in G} f(l) \sum_{h \in G} \overline{j(l)j(g)(h)}.
\]

From the orthogonality relations we have that
\[
\sum_{h \in G} \overline{j(l)j(g)(h)} = \begin{cases} |G| & \text{if } \hat{j}(l) = j(g), \\ 0 & \text{if } \hat{j}(l) \neq j(g). \end{cases}
\]

Thus, for every \( g \in G \), we have \( P(f)(g) = f(l) \) where \( l = \hat{j}^{-1}(j(g)) \). This shows that \( P \) is a permutation and \( p = \hat{j}^{-1} \circ j \). \( \square \)

4. Examples

**Example 4.1.** Consider the additive group \((\mathbb{Z}/n\mathbb{Z})_{ad}\) of the ring \( \mathbb{Z}/n\mathbb{Z} \). So, starting from this point we will denote the group operation by \(+\). The multiplication will denote the ring multiplication. Consider the standard isomorphism \( s : (\mathbb{Z}/n\mathbb{Z})_{ad} \rightarrow (\mathbb{Z}/n\mathbb{Z})_{ad} \) defined by \( s(x)(y) = e_n(xy) \) where \( e_n(x) = e^{2\pi ix/n} \). Then any other isomorphism \( j : (\mathbb{Z}/n\mathbb{Z})_{ad} \rightarrow (\mathbb{Z}/n\mathbb{Z})_{ad} \) has the form \( j = \pi \circ h \) for some isomorphism \( h : (\mathbb{Z}/n\mathbb{Z})_{ad} \rightarrow (\mathbb{Z}/n\mathbb{Z})_{ad} \). The isomorphism \( h \) is defined by its value \( h(1) = l \) where \( l \) and \( n \) are relatively prime. So, for all \( x, y \in \mathbb{Z}/n\mathbb{Z} \)
\[
j(x)(y) = s(h(x))(y) = s(xl)(y) = e_n(xyl).
\]

Now, since \( j \) is an isomorphism, we have that \( P = (\hat{j} \circ \mathcal{F})^2 \) is a permutation with \( p = \hat{j}^{-1} \circ j \) by Theorem 3.3. Calculations shows that \( p(x) = -x \) for all such \( j \). This shows that \( P^2 = (\hat{j} \circ \mathcal{F})^4 \) is the identity on \( C^{\mathbb{Z}/n\mathbb{Z}} \). Thus, the spectrum of \( \hat{j} \circ \mathcal{F} \) is a subset of the set \( \{1, -1, i, -i\} \) of 4th roots of unity. In the next example we show that the multiplicities of eigenvalues do depend on \( j \), but not too much.

The following example is a particular case of Example 4.1.

**Example 4.2.** Consider the additive group \((\mathbb{Z}/p\mathbb{Z})_{ad} \) where \( p \) is an odd prime. To calculate the multiplicities of the eigenvalues in this case we use the multiplicative characters of \( \mathbb{Z}/p\mathbb{Z} \). The point is that \( \hat{j} \circ \mathcal{F} \) decomposes on \( 2 \times 2 \) and \( 1 \times 1 \) matrix blocks in the bases of multiplicative characters. It is well known but we do the
corresponding calculation here to point out the dependence on \( j \), and that we may calculate the possible spectra of \( \tilde{j} \circ F \) without calculating Gauss sums.

A multiplicative character is a function \( \psi : G \to \mathbb{C} \) such that \( \psi(xy) = \psi(x)\psi(y) \) for all \( x, y \in \mathbb{Z}/p\mathbb{Z} \) and \( \psi(0) = 0 \). Fix a generator \( g \) of a multiplicative group of \( \mathbb{Z}/p\mathbb{Z} \). Define \( \psi_0, \psi_1, \ldots, \psi_{p-2} : G \to \mathbb{C} \) by setting \( \psi_a(0) = 0, \psi_a(g^b) = e^{2\pi i ab/p-1} \). It is trivial to verify that \( \psi_a, a = 0, 1, \ldots, p-2 \) are multiplicative characters and account for all such. Setting

\[
\delta_0(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0, \end{cases}
\]

we have that \( \beta = (\delta_0, \alpha\psi_0, \alpha\psi_1, \ldots, \alpha\psi_{p-2}) \) is an orthonormal ordered basis of \( \mathbb{C}^G \), where \( \alpha = 1/\sqrt{p-1} \).

Let \( j : G \to G \) be an isomorphism. Then, from Example 4.1, we have that \( j := j_l \) for some \( l \in \{1, 2, \ldots, p-1\} \), where \( j_l(x)(y) = e_p(xy) \). It follows that for every \( f \in \mathbb{C}^G \) and for all \( x \in G \),

\[
\tilde{j}(F(f))(x) = F(f)(j_l(x)) = \frac{1}{\sqrt{p}} \sum_{y \in G} f(y)j_l(x)(y) = \frac{1}{\sqrt{p}} \sum_{y \in G} f(y)e_p(xy).
\]

In particular, we have that

\[
\tilde{j}(F(\delta_0))(x) = \frac{1}{\sqrt{p}} \delta_0(x) + \frac{1}{\sqrt{p}} \psi_0(x) = \frac{1}{\sqrt{p}} \delta_0(x) + \frac{\sqrt{p-1}}{\sqrt{p}} \alpha \psi_0(x)
\]

for all \( x \in G \).

Similarly, from the orthogonality relations we have that

\[
\tilde{j}(F(\psi_0))(x) = \begin{cases} -\frac{1}{\sqrt{p}} & \text{if } x \neq 0, \\ \frac{p-1}{\sqrt{p}} & \text{if } x = 0, \end{cases}
\]

\[
= \frac{p-1}{\sqrt{p}} \delta_0(x) - \frac{1}{\sqrt{p}} \psi_0(x)
\]

and so

\[
\tilde{j}(F(\alpha\psi_0))(x) = \frac{\sqrt{p-1}}{\sqrt{p}} \delta_0(x) - \frac{1}{\sqrt{p}} \alpha \psi_0(x).
\]

It follows, that the matrix of \( \tilde{j} \circ F \) relative to \( \beta \) begins in the “northwest” with the \( 2 \times 2 \) matrix block

\[
\begin{pmatrix}
\frac{1}{\sqrt{p}} & \frac{\sqrt{p-1}}{\sqrt{p}} \\
\frac{\sqrt{p-1}}{\sqrt{p}} & -\frac{1}{\sqrt{p}}
\end{pmatrix}.
\]

(4.1)

Now, let \( \psi \neq \psi_0 \) be a multiplicative character on \( G \). Then, for all \( x \in G \) we have that

\[
\tilde{j}(F(\psi))(x) = \frac{1}{\sqrt{p}} \sum_{y \in G} \psi(y)e_p(xy).
\]

If \( x = 0 \), then \( e_p(0yl) = 1 \) and \( \tilde{j}(F(\psi))(0) = \frac{1}{\sqrt{p}} \sum_{y \in G} \psi(y) = 0 \) where the last equality follows by the orthogonality relations.
If \( x \neq 0 \), then
\[
\hat{j}(\mathcal{F}(\psi))(x) = \frac{1}{\sqrt{p}} \sum_{y \in G} \psi(x^{-1}y) e_p(ly) = S_l(\psi) \cdot \overline{\psi}(x),
\]
where \( S_l(\psi) = \frac{1}{\sqrt{p}} \sum_{y \in G} \psi(y) e_p(ly) \).

If \( \psi \neq \overline{\psi} \), we have that \((\hat{j} \circ \mathcal{F})(\psi) = S_l(\psi) \cdot \overline{\psi} \) and \((\hat{j} \circ \mathcal{F})(\overline{\psi}) = S_l(\psi) \cdot \psi \).

Relative to the pair \((\psi, \overline{\psi})\), we get a \(2 \times 2\) matrix block of the form
\[
\begin{pmatrix}
0 & S_l(\psi) \\
S_l(\overline{\psi}) & 0
\end{pmatrix}.
\]

To determine the spectra of this matrix we must determine the product \( S_l(\overline{\psi}) \cdot S_l(\psi) \).

We have that
\[
S_l(\psi) \cdot S_l(\overline{\psi}) = \frac{1}{p} \sum_{x,y \in G} \psi(x) e_p(lx) \overline{\psi}(y) e_p(ly) = \frac{1}{p} \sum_{x,y \in G} e_p(l(x+y)) \psi(x) \overline{\psi}(y)
\]
\[
= \frac{1}{p} \sum_{a \in G} e_p(\lambda a) \cdot \sum_{y \in G} \psi(a-y) \overline{\psi}(y)
\]
\[
= \frac{1}{p} \sum_{y \in G} \psi(-y) \overline{\psi}(y) + \frac{1}{p} \sum_{a \in G} e_p(\lambda a) \sum_{y \in G} \psi(a-y) \overline{\psi}(y).
\]

Since \( \psi(0) = 0 \), for the first sum we have that
\[
\sum_{y \in G} \psi(-y) \overline{\psi}(y) = \sum_{y \in G^{\times}} \psi(-y) \overline{\psi}(y)^{-1} = \sum_{y \in G^{\times}} \psi(-1) = (p-1)\psi(-1).
\]

From the orthogonality relations, it follows that
\[
\sum_{y \in G} \psi(a-y) \overline{\psi}(y) = \sum_{y \in G^{\times}} \psi \left( \frac{a}{y} - 1 \right) = -\psi(-1)
\]
since for \( y \in G^{\times} \), \( x = \frac{a}{y} - 1 \) if and only if \( y = \frac{a}{x+1} \) for \( x \in G - \{1\} \).

Thus,
\[
S_l(\psi) \cdot S_l(\overline{\psi}) = \frac{p-1}{p} \psi(-1) - \frac{\psi(-1)}{p} \sum_{a \in G^{\times}} e_p(\lambda a) = \psi(-1) \left( \frac{p-1}{p} + \frac{1}{p} \right) = \psi(-1),
\]
since \( \sum_{a \in G^{\times}} e_p(\lambda a) = -1 \) by the orthogonality relations. It follows that \( S_l(\psi) \cdot S_l(\overline{\psi}) = \pm 1 \) because of \( 1 = \psi(1) = (\psi(-1))^2 \). Since \( \psi_k(g^r) = e^{2\pi irk/(p-1)} \) and \( g^{(p-1)/2} = -1 \), we have that \( \psi(-1) = -1 \) for \( \frac{p-1}{2} \) nontrivial multiplicative characters and \( \psi(-1) = 1 \) for \( \frac{p+1}{2} - 1 = \frac{p-3}{2} \) nontrivial multiplicative characters.

If \( \psi = \overline{\psi} \), then \( \psi(x) = \psi_{\frac{x}{p}} = (\frac{x}{p}) \) is the Legendre symbol of \( x \mod p \). In this case, \( S_l(\psi) = \frac{1}{\sqrt{p}} \sum_{y \in G} e_p(ly)(\frac{y}{p}) = \frac{1}{\sqrt{p}} \sum_{y \in G} e_p(ly^2) \) where the last equality follows by Theorem 4.17 of [3]. From the consideration above, we know that
\[
S_l(\psi_{\frac{x}{p}}) = \left( \frac{-1}{p} \right).
\]
Thus,
\[
S_l(\psi_{p-1}) = \begin{cases} 
\pm 1 & \text{if } p \equiv 1 \pmod{4}, \\
\pm i & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
\] (4.2)

So, we just need to choose the proper sign. Notice that \(S_l(\psi) = \overline{\psi(l)}S_1(\psi)\). It follows that any sign in Eq.4.2 is possible depending on whether \(l\) is a quadratic residue mod \(p\) or not. We know that all two possibilities happen, but we are not able to decide which one corresponds to which residue mod \(p\). The evaluation of the Gauss sum is less elementary than the calculations used here.

Finally, since \((\hat{j} \circ \mathcal{F})(\psi) = S_l(\psi) \cdot \psi\), relative to the pair \((\psi, \psi)\) we get a \(1 \times 1\) matrix of the form \((S_l(\psi))\).

It follows that the characteristic polynomial of \(\hat{j} \circ \mathcal{F}\) is
\[
(\lambda^2 - 1) \prod_{j=1}^{\frac{p-1}{2}} (\lambda^2 - \psi_j(-1))(\lambda \pm 1), \quad \text{if } p \equiv 1 \pmod{4}
\]
and
\[
(\lambda^2 - 1) \prod_{j=1}^{\frac{p-1}{2}} (\lambda^2 - \psi_j(-1))(\lambda \pm i), \quad \text{if } p \equiv 3 \pmod{4}
\]
where the choice of signs depend on \(l\) and, consequently, \(j\).

Let \(R_4 = \{1, -1, i, -i\}\) be a group of 4th roots of unity and \(\mathbb{Z}[R_4]\) its group algebra. We write an element \(a \in \mathbb{Z}[R_4]\) as \(a = a_1 + a_{-1}[1] + a_i[i] + a_{-i}[-i]\). Where we suppose that \([1] = 1\). The sum in \(\mathbb{Z}[R_4]\) is defined by
\[
(a_1 + a_{-1}[1] + a_i[i] + a_{-i}[-i]) + (b_1 + b_{-1}[1] + b_i[i] + b_{-i}[-i]) = (a_1 + b_1) + (a_{-1} + b_{-1})[1] + (a_i + b_i)[i] + (a_{-i} + b_{-i})[-i]
\]
and the multiplication is the prolongation of the multiplication in \(R_4\) \(\{[a] [\beta] = [a\beta]\}\) by linearity. We say that \(a \in \mathbb{Z}[R_4]\) represents the spectrum of an operator \(A\) \((A^4 = id)\) if \(a_\alpha\) is the multiplicity of the eigenvalue \(\alpha\). Let \(s = 1 + [-1] + [i] + [-i]\)

**Theorem 4.3.** Let \(p > 2\) be a prime.

If \(p \equiv 1 \pmod{4}\) then either \(\frac{p-1}{4}s + 1\) or \(\frac{p-1}{4}s + [-1]\) represents the spectrum of \(\hat{j} \circ \mathcal{F} : \mathbb{C}^{2p} \to \mathbb{C}^{2p}\). Both cases are possible depending on \(j\).

If \(p \equiv 3 \pmod{4}\) then either \(\frac{p+1}{4}s - [i]\) or \(\frac{p+1}{4}s - [-i]\) represents the spectrum of \(\hat{j} \circ \mathcal{F} : \mathbb{C}^{2p} \to \mathbb{C}^{2p}\). Both cases are possible depending on \(j\).

**Example 4.4.** Consider the abelian group \(G = (\mathbb{Z}/n\mathbb{Z})^m\). Then, the automorphism group of \(G\) is \(GL_m(\mathbb{Z}/n\mathbb{Z})\), the group of invertible \(m \times m\) matrices in \(\mathbb{Z}/n\mathbb{Z}\). Let \(\langle \cdot , \cdot \rangle : G \times G \to \mathbb{Z}/n\mathbb{Z}\) be the natural scalar product on \(G\):
\[
\langle h, g \rangle = h_1g_1 + h_2g_2 + \cdots + h_mg_m.
\]
For an isomorphism \(j : G \to \hat{G}\), there is \(M_j \in GL_m(\mathbb{Z}/n\mathbb{Z})\) such that
\[
j(g)(h) = e_n(\langle h, M_jg \rangle).
\]
We can check that \(M_j^t = -M_j^t\) where \(M_j^t\) denotes the transpose matrix of \(M_j\). It follows that, in this case, \(p(g) = -(M_j^t)^{-1}M_jg\).
Lemma 4.5. Let $M_{ij} = T^iT_jT$ for some $T \in GL_m(\mathbb{Z}/n\mathbb{Z})$. Then $\tilde{j} \circ F$ is unitary equivalent to $j \circ F$. The unitary equivalence comes from base change in $(\mathbb{Z}/n\mathbb{Z})^m$ by $T$.

If $M^t = M$, then $p(g) = -g$ for all $g \in G$ and, hence, $P^2 = id$. As in Example 4.1 it follows that the set of eigenvalues of $\tilde{j} \circ F$ is a subset of $R_4$. Let $n = p > 2$ be a prime. Then it is not hard to calculate the spectrum of $\tilde{j} \circ F$. Indeed, if $M \in GL_m(\mathbb{Z}/p\mathbb{Z})$ is symmetric ($M^t = M$), then there exists $T \in GL_m(\mathbb{Z}/p\mathbb{Z})$ such that

$$M = T^t \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 1 & l \end{pmatrix} T,$$

(see [1], Theorem 9.4). So, changing the base in $(\mathbb{Z}/p\mathbb{Z})^m$, we get that $\tilde{j} \circ F$ is isomorphic to a tensor product of $m$ Fourier transforms of $\mathbb{Z}/p\mathbb{Z}$, all but one with $l = 1$. We need the following lemma.

Lemma 4.6. Let $s = 1 + [-1] + [i] + [-i]$, $k \in \mathbb{Z}$ and $\alpha \in R_4$.

- Let $a, b \in \mathbb{Z}[R_4]$ represent the spectrum of $A$ and $B$, respectively. Then $ab$ represents the spectrum of $A \otimes B$.
- $s \cdot [\alpha] = s$.
- $s^m = 4^{m-1}s$.
- $(ks + [\alpha])^m = \frac{(4k+1)(m-1)}{4}s + [\alpha^m]$.
- $(ks - [\alpha])^m = \frac{(4k-1)(m-1)}{4}s + (-1)^m[\alpha^m]$.

This lemma with Theorem 4.3 imply the following corollary.

Corollary 4.7. Let $p > 2$ be a prime and the isomorphism $j : (\mathbb{Z}/p\mathbb{Z})^m \to (\mathbb{Z}/p\mathbb{Z})^m$ be symmetric (with $M_{ij}^t = M_{ij}$).

If $p \equiv 1 \mod 4$ then $\frac{p^m-1}{4}s + [\pm 1]$ represents the spectrum of $\tilde{j} \circ F$. Both cases are possible, depending on the choice of $j$.

If $p \equiv 3 \mod 4$ then $\frac{p^m-(1)^m}{4}s + (-1)^m[\pm(i)^m]$ represents the spectrum of $\tilde{j} \circ F$. Both cases are possible, depending on the choice of $j$.

In a similar way, if $M^t = -M$ then $p = id$ for all $g \in G$. In this case, the eigenvalues of $\tilde{j} \circ F$ are $\pm 1$.

In principle, using multiplicative characters we can calculate spectra of different Fourier transforms on $(\mathbb{Z}/p\mathbb{Z})^m$. As multiplicative characters with $\delta_0$ form a basis on $\mathbb{C}^{\mathbb{Z}/p\mathbb{Z}}$, the tensor products of these characters form a basis on $\mathbb{C}^{(\mathbb{Z}/p\mathbb{Z})^2}$. We show how to use it by an example. First of all, we remind that if $f_1, f_2 \in \mathbb{C}^{\mathbb{Z}/p\mathbb{Z}}$, then $f_1 \otimes f_2 \in \mathbb{C}^{(\mathbb{Z}/p\mathbb{Z})^2}$ is defined as $(f_1 \otimes f_2)(x_1, x_2) = f_1(x_1)f_2(x_2)$.

Example 4.8. Let $G = (\mathbb{Z}/p\mathbb{Z})^2$ and $j : G \to G$ be defined by

$$M_j = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Notice that $M_j$ is not symmetric. In this case we may find a transformation $T \in GL_2(\mathbb{Z}/p\mathbb{Z})$ such that

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = T \begin{pmatrix} 0 & 1 \\ k & 0 \end{pmatrix} T^t,$$
where \( k = \frac{1 - \sqrt{-1}}{2} \). Generally speaking, \( k \) is not necessarily in \( \mathbb{Z}/p\mathbb{Z} \) but may belong to its quadratic extension. Let \( p \equiv 1 \mod 3 \). Then \( k \in \mathbb{Z}/p\mathbb{Z} \). We only consider this case. The other case requires additional considerations. So, let \( p \equiv 1 \mod 3 \). Then by changing the base in \((\mathbb{Z}/p\mathbb{Z})^2\), we obtain

\[
\tilde{M}_j = \begin{pmatrix} 0 & 1 \\ k & 0 \end{pmatrix}.
\]

So, we have to study \( \tilde{j} \circ \mathcal{F} \) with \( j \) defined by \( \tilde{M}_j \). Let \( \psi_1, \psi_2 \) be nontrivial multiplicative characters on \( \mathbb{Z}/p\mathbb{Z} \). Then \((j \circ F)(\psi_1 \otimes \psi_2) = S(\psi_1)S(\psi_2)\tilde{\psi}_1(k)(\tilde{\psi}_2 \otimes \tilde{\psi}_1)\). So, for non-trivial multiplicative characters, we have the following two cases:

I: \( \psi_1 \neq \nu_2 \). Then the restriction of \( \tilde{j} \circ \mathcal{F} \) on subspace spanned by \( \psi_1 \otimes \nu_2, \nu_2 \otimes \nu_1 \) is

\[
\begin{pmatrix} 0 & \nu_1 \nu_2(k) \\ S(\nu_1)S(\nu_2)\nu_1(k) & 0 \end{pmatrix}.
\]

The corresponding eigenvalues are \( \lambda = \pm \sqrt{\nu_1(-k)}\nu_2(-k) \).

II: For nontrivial multiplicative characters \( \psi \), the vectors \( \nu \otimes \nu \) are eigenvectors with eigenvalues \( \lambda = \psi(-k) \).

We are left with the study of tensor products involving \( \delta_0 \) and trivial multiplicative character \( \psi_0 \). Let \( v_1 = \delta_0 + \frac{\sqrt{p-1}}{\sqrt{p-1}}\psi_0 \) and \( v_{-1} = \delta_0 - \frac{\sqrt{p+1}}{\sqrt{p-1}}\psi_0 \). Notice that \( v_\alpha \) is an \( \alpha \)-eigenvector of one-dimensional Fourier transform, see Eq.4.1.

III: Let \( \psi \) be a nontrivial multiplicative character. On subspaces spanned by \( \nu_\alpha \otimes \psi, \nu \otimes \nu_\alpha \), the \( \tilde{j} \circ \mathcal{F} \) acts as

\[
\begin{pmatrix} 0 & S(\psi) \nu_1(k) \\ S(\nu_1)\nu_1(k) & 0 \end{pmatrix}.
\]

The corresponding eigenvalues are \( \lambda = \pm \sqrt{\nu_1(-k)} \).

IV: The restriction of \( \tilde{j} \circ \mathcal{F} \) on 1-dimensional subspace spanned by \( \nu_\alpha \otimes \nu_\alpha \), is the identity operator, that is, \( \lambda = 1 \).

V: The restriction of \( \tilde{j} \circ \mathcal{F} \) on subspace spanned by \( \nu_1 \otimes v_{-1}, \nu_{-1} \otimes v_1 \) is

\[
\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.
\]

The corresponding eigenvalues are \( \lambda = \pm 1 \).

Notice that \((-k)^3 = 1\). It follows that there are

- \( \frac{p-4}{3} \) nontrivial multiplicative characters \( \psi \) on \( \mathbb{Z}/p\mathbb{Z} \) with \( \psi(-k) = 1 \);
- \( \frac{p+1}{3} \) nontrivial characters with \( \psi(-k) = e^{\frac{2\pi i}{3}} \);
- \( \frac{p-1}{3} \) nontrivial characters with \( \psi(-k) = e^{-\frac{2\pi i}{3}} \).

Based on the above considerations, we get the following theorem.

**Theorem 4.9.** Let \( p > 2 \) be a prime and \( j : (\mathbb{Z}/p\mathbb{Z})^2 \to (\mathbb{Z}/p\mathbb{Z})^2 \) be an isomorphism with

\[
M_j = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.
\]

Then, the eigenvalues of \( \tilde{j} \circ \mathcal{F} \) are the 6th roots of unity. If \( p \equiv 1 \mod 3 \) and \( p \geq 7 \), then the multiplicities of the eigenvalues are the following:
\[ \begin{array}{cc}
\lambda & \text{multiplicity} \\
+1 & \frac{1}{2} \left( \frac{(p-4)(p-7)}{9} \right) + \frac{(p-1)^2}{9} + p - 1 \\
-1 & \frac{1}{2} \left( \frac{(p-4)(p-7)}{9} \right) + \frac{(p-1)^2}{9} + 2p - 4 + 1 \\
e^{\frac{2\pi i}{3}} & \frac{3}{2} \left( \frac{(p-4)(p-1)}{9} \right) + 2p - 1 \\
e^{-\frac{2\pi i}{3}} & \frac{3}{2} \left( \frac{(p-4)(p-1)}{9} \right) + 2p - 1 \\
- e^{\frac{2\pi i}{3}} & \frac{3}{2} \left( \frac{(p-4)(p-1)}{9} \right) + p - 1 \\
- e^{-\frac{2\pi i}{3}} & \frac{3}{2} \left( \frac{(p-4)(p-1)}{9} \right) + p - 1 \\
\end{array} \]

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