Spin squeezed states are a class of entangled states of spins that have practical applications to precision measurements. In recent years, much progress has been made on the experimental squeezing of a large number of ultracold atoms. Spin squeezing may also be easily transferred to cases where the OA T Hamiltonian and the use of several coherent driving pulses. The scheme is robust against technical noise and can be readily implemented for spinor BECs or trapped ions with current technology.

Spin squeezed states [1] have attracted a lot of interest due to both its role in the fundamental study of many-particle entanglement and its practical application to precision measurements with Ramsey interferometers [2–6]. In recent years, much progress has been made on the experimental squeezing of a large number \(N\sim 10^3\) of ultracold atoms [7–11]. Many of these experiments follow the so-called one-axis twisting (OAT) scheme, which is known to reduce the noise/signal ratio from the classical case by a amount that scales as \(N^{-2/3}\) with the particle number \(N\) [1]. This reduction is not optimal yet and still above the so-called Heisenberg limit which scales as \(N^{-1}\). There have been several theoretical proposals to enhance the OAT [12, 13].

For example, one of the approaches [13] involves inducing a better squeezing Hamiltonian, the so-called two-axis twisting (TAT) Hamiltonian, with Raman assisted coupling for trapped spinor BECs. This is a hardware level engineering, requiring modification of a particular experimental setup and does not apply to other physical systems. Another approach [12] employs a digital quantum simulation technique to convert an OAT Hamiltonian to an effective TAT Hamiltonian by stroboscopically applying a large number of pulses. This software level solution is universal but sensitive to the accumulation of control errors. None of these proposals have been experimentally tested yet due to various difficulties.

Inspired by the idea of optimized quantum control, we propose an experimentally feasible scheme to greatly improve the performance of OAT, requiring only two or three additional coherent driving pulses to carry out collective spin rotations, which is a routine technique with the current technology. The scheme is shown to be robust to noise and imperfection in control pulses. Using this scheme, it is possible to generate more spin squeezing and detect a significantly larger entanglement depth for the many-particle atomic ensemble [2]. This new scheme enhances the OAT squeezing on the software level and therefore can be applied to any physical system that is endowed with these operations. The idea of optimized squeezing may also be easily transferred to cases where the interaction term deviates from the OAT Hamiltonian.

We consider the general scenario of one-axis twisting independent of the underlying physical system with the Hamiltonian \(H = \chi S_x^2 \left( S_z = \sum_{i=1}^{N} s_i \right)\). The system starts from a collective spin coherent state polarized along \(x\)-axis. As time goes on the initially homogenous spin fluctuation gets distorted and redistributed among different directions and the direction along which spin fluctuation gets suppressed gradually changes over time. The squeezing is measured by the parameter \(\xi^2\), defined as \(\xi^2 = N \langle S_x^2 \rangle / \langle |S_x| \rangle^2\), where \(\vec{n}\) is the direction along which spin fluctuation is minimized. The decreasing rate of \(\xi^2\) slows down with time, and after the optimal squeezing point, \(\xi^2\) increases again. Aside from the initial state, which is rotationally symmetric about \(x\)-axis, all the subsequent states breaks this symmetry and picks out a special direction, i.e. the direction along which fluctuation is minimized. It is well known that the two-axis twisting (TAT) Hamiltonian \(H_2 = \chi_2 \left( S_x^2 - S_y^2 \right)\) can produce better squeezing [1], which, after doing the Trotter decomposition with an infinitesimal time interval, could be seen as switching the squeezing axis back and forth very fast between two orthogonal directions [12]. To avoid the noise accumulation from a large number of switching pulses inherent in the Trotter expansion scheme, we take an alternative approach based on optimization of a few control pulses to maximize the squeezing of the final state. We consider an \(n\)-step squeezing protocol (where \(n\) is typically 2 or 3 for a practical scheme) defined as follows: at step \(j\) \((j = 1, 2, ..., n)\), we first apply an instantaneous collective spin rotation around \(x\)-axis, \(U(\alpha_j) = \exp(-i S_x \alpha_j)\), and then let the state evolve under the OAT Hamiltonian \(H = \chi S_x^2\) for a duration \(T_j\). Effectively, we squeeze the state along a different axis lying in the \(y-z\) plane in each step, so the effective evolution operator can be written as

\[
U(\theta_j, T_j) = \prod_{j=1}^{n} \exp(-i \chi S_x^2 T_j), \tag{1}
\]
where \( S_{\theta} = \cos \theta \hat{J}_z + \sin \theta \hat{J}_y \) and the factors are arranged from right to left with increase of \( j \). Since the initial state is assumed to be polarized along \( x \)-direction, which is symmetric around \( x \)-axis, \( \theta_j \) is irrelevant and can be chosen to be 0 (so no control pulse is needed for step 1). Therefore, for an \( n \)-step squeezing protocol, there are \((2n-1)\) tunable parameters: \( T_1 \) and \( \theta_i \) (excluding \( \theta_1 \)). The final squeezing parameter is thus a multi-variable function \( \xi^2(T_1, \theta_i) \). Our purpose is to find the best available squeezing \( \xi^2(T_1, \theta_i) \) with a minimum number \( n \) of the time steps.

In the case of \( n = 2 \) or 3, the landscape of \( \xi^2(T_1, \theta_i) \) in the parameter space is quite simple and well behaved. Take the \( n = 2 \) case as an example. For a typical value of \( T_1 \) smaller than the optimal OAT squeezing time, \(-log(\xi^2)\) as a function of \( \theta_2 \) and \( T_2 \) is shown in Fig.1. The optimal squeezing point marked by the cross lies way off the OAT trajectory, the horizontal line with \( \theta_2 = 0 \). For the \( n = 3 \) case, with \( \theta_2 \) and \( T_2 \) fixed near the optimal values of the \( n = 2 \) case, \(-log(\xi^2)\) as a function of \( \theta_3 \) and \( T_3 \) shows a similar landscape. These solutions already exceed that of the OAT scheme by a large margin. The results indicate that the optimization technique with \( n \) as small as 2 or 3 suffices to significantly improve over the OAT scheme.

Next, we investigate performance of the optimized squeezing scheme, focusing on the scaling of the squeezing \( \xi^2(T_1, \theta_i) \) as a function of the total particle number \( N \). For a given set of parameters, we can numerically calculate the evolution operator in Eq.1 by exactly diagonalizing the effective Hamiltonians \( S_{\theta_i} \) and then obtain the squeezing parameter \( \xi^2 \). We randomly sample from the parameter space for a large number of times, use these random samples as initial guesses to start unconstrained local optimization of the squeezing parameter, and pick the best one as our solution. Repeating this procedure for every system size \( N \) is extremely resource intensive especially when \( N \) gets as large as \( 10^5 \). Taking advantage of the fact that adding several more to \( 10^3 \) particles should not change the solution much, we can feed the previously found non-local optimal solution as an initial guess to the local optimizer of a larger system and obtain a near optimal solution quickly. In this way we managed to obtain (near) optimal solutions for systems all the way up to \( N = 10^5 \) particles, with only a cost of classical computing time on the order of tens of hours on a typical multi-core computer. As shown in Fig.2 with \( n = 2 \), the squeezing parameter \( \xi^2 \) gets reduced by a significant amount already compared with the OAT scheme, and with \( n = 3 \), \( \xi^2 \) decreases further. The scaling of \( \xi^2 \) with the number of particles shows a clear power law \( \xi^2 \sim 1/N^{3/2} \). A simple OAT scheme gives \( \beta = 2/3 \) and the TAT scheme gives \( \beta = 1 \) [1]. The Heisenberg limit of noise gives a bound \( \beta \leq 1 \) for the scaling, and this bound is saturated by the TAT scheme. Remarkably we observe that the optimized \( n = 2, 3 \) protocols can give \( \beta = 0.92 \) and 0.98, respectively, very close to the ultimate Heisenberg limit. Moreover, the \( n = 3 \) optimized scheme has a smaller multiplicative constant compared with the TAT scheme, so in the realistic range of particle number \( N \approx 10^6 \), it actually outperforms the TAT scheme. This shows that a moderate alternation of the OAT scheme through optimization can significantly increase the spin squeezing.

We have demonstrated a significant improvement over the conventional OAT by applying very few optimized control pulses. A cost of the proposed scheme is that it takes longer evolution time to achieve the optimal squeezing. A typical evolution of \( \xi^2 \) with time \( t \) is shown in Fig.3. We notice that in general the \((i+1)\)-th squeezing step takes longer time than the \(i\)-th step. Since the time cost in the first step is on the order of the optimal OAT duration, the overall duration of the new protocol is usually longer than that of the OAT scheme. An excessively long duration would be an obstacle in systems with short coherence time. The two relevant time scales here are the coherence time \( \tau \) and the inverse of interaction.
strength $1/\chi$. The time cost of the new scheme is around $0.01/\chi \sim 0.1/\chi$. If $\tau \gtrsim 0.1/\chi$ the new scheme can be implemented without compromise. On the other hand, if that is not the case, decoherence effect would play a role and our unconstrained optimization no longer yields the best result. However, we can work around this problem by performing an optimization with the total duration added as a cost function and get a compromised optimal pulse sequence. By tuning the weight of the cost function we could obtain a continuous series of compromised optimal solutions as shown in Fig. 4. These solutions of two-step and three-step schemes form two line segments, continuously connecting the optimal OAT squeezing protocol to that of the unconstrained optima, offering a trade off between the protocol duration and the squeezing magnitude. For each real experimental setup, one could correspondingly pick up the best point in accordance with the coherence time of the system. How much one can gain over the OAT scheme depends on how long the coherence time can reach.

Next we test noise resistance of the proposed scheme.

There are only 3(5) control parameters in the $n = 2(3)$ scheme, making the accumulation of control noise negligible. We have done numerical simulation of our scheme adding random pulse area/timing noise and confirmed the robustness of the squeezing parameter $\xi^2$ as shown in Fig. 5. This contrasts to the proposals [12, 14] requiring a large number of coherent rotation pulses where control errors accumulate and significantly degrade the performance. Thus our proposed scheme offers a useful alternative to the previous works. Another practical issue related to control noise is the uncertainty in number of particles in a real experiment. Our pulse scheme depends on the number of particles $N$ while in experiments such as ultracold gas we do not typically know the number $N$ exactly. Fortunately we notice that the control parameters vary slowly with $N$ and an uncertainty in $N$ is equivalent to a small extra noise in the control parameters, to which $\xi^2$ is not so sensitive as we have shown in Fig. 5.

Finally we discuss possible physical realizations of the scheme proposed here. The scheme only requires two ingredients, the nonlinear collective spin interaction $S_z^2$ and the ability to rotate the collective spin around an orthogonal axis, say $x$. Several experimental systems meet these requirements, e.g., trapped ions and spinor BECs. In trapped ion systems, depending on the ion species, one can use bichromatic lasers or two pairs of Raman laser beams (the Molmer-Sorensen scheme) to induce the $S_z^2$ or $S_x^2$ type of interaction. The strength of this interaction $\chi$ can reach kHz scale, giving $1/\chi \sim ms$. The coherence time usually exceeds $1/\chi$ and our scheme can apply without compromise. Collective spin rotation can be simply done by shining laser on all the ions driving the corresponding single-qubit $\sigma_z^{(\gamma)}$ or rotation. The rotation pulses have durations much shorter than $1/\chi$. While linear Paul traps [17] can now coherently control only about a dozen of ions, too few for the purpose of spin squeezing, planar Penning traps can manipulate more than 200.

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Figure 3: Evolution of the squeezing parameter $\xi^2$ with time, calculated with $N=2000$ spin-1/2 particles. The dash-dot line is for one-axis twisting (OAT), the dash line for the two-step optimized squeezing scheme, and the solid line for the three-step optimized squeezing.

Figure 4: Constrained optimization of $\xi^2$ with the total time duration as a cost function. We take $1/\chi$ as the time unit. Achievable squeezing $\xi^2$ as a function of the total duration is shown, together with one-axis twisting (OAT), calculated with $N = 2000$ spin-1/2 particles. OPT-2 (3) stands for optimized squeezing sequence with $n = 2(3)$ segments. Horizontal and vertical dashed lines are guides to the eye.

Figure 5: Optimized squeezing in the presence of control noise. We use the three-step optimization scheme as an example and assume all the five control parameters in this scheme have the same magnitude of relative errors as specified in this figure. The dash line is for the ideal case with no error in the control parameters, the solid line denotes the average of many random trajectories (about 50 random trials) and the shaded area marks the range of those trajectories. In the left panel, the shaded region is too small to be distinguished from the ideal case.
Figure 6: The entanglement depth achievable with different approaches for 200 spin-1/2 particles. The solid lines from top to bottom correspond respectively to the OAT scheme, the two-step optimized squeezing, the TAT, and the three-step optimized squeezing. The dashed lines from top to bottom correspond to the optimal squeezing for 50, 100, and 200 particles respectively. Lying below the curve of optimal squeezing for \( n \) particles is a certificate of genuine \( n \)-particle entanglement.

For the purpose of precision measurement, 200 ions may seem less impressive than \( 10^5 \) particles, but we show that using our scheme we can create genuine multiparticle entangled states with a significantly larger entanglement depth. The entanglement depth, defined in [5], is a way to measure how many particles within the whole sample have been prepared in a genuine entangled state. Our result is shown in Fig. 6. In this figure, a point lying below the optimal squeezing curve of \( n \) particles corresponds to a state that contains genuine \( n \)-particle entanglement. Our scheme produces states that lie below the OAT states in a large range of \( \langle S_z \rangle \) values, which means that experimentally one can achieve a significantly larger entanglement depth by this optimization technique.

Another class of physical system is a spinor Bose-Einstein condensate of atoms with two chosen internal states mimicking spin-1/2 particles [8, 9]. The desired \( S_z^2 \) interaction is induced by spin-dependent s-wave scattering as proposed in [4]. Coherent laser pulses illuminating the whole condensate can implement spin rotations similar to the trapped ion case. However, the strength of \( S_z \) interaction is much smaller compared with the trapped ion case, \( \chi = 0.3 \sim 0.5 \) Hz as reported in [8, 9]. The coherence time for the spinor BEC is also shorter. Hence we typically need to apply the compromised scheme, using the actual coherence time and interaction strength of the system as input parameters.

In summary, we have proposed a new method based on optimization to significantly enhance spin squeezing using the one axis twisting Hamiltonian. To achieve significant improvement in spin squeezing, we need to apply only one or two global rotation pulses at an appropriate evolution time and with optimized rotation angles. Using two pulses, the final squeezing is very close to the Heisenberg limit already. As we use a very small number of control pulses, the scheme is immune to accumulation of control errors and can be readily applied in experimental systems without significant modification of the setup.

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