Clique-Coloring of $K_{3,3}$-Minor Free Graphs

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Abstract

A clique-coloring of a given graph $G$ is a coloring of the vertices of $G$ such that no maximal clique of size at least two is monocolored. The clique-chromatic number of $G$ is the least number of colors for which $G$ admits a clique-coloring. It has been proved that every planar graph is 3-clique colorable and every claw-free planar graph, different from an odd cycle, is 2-clique colorable. In this paper, we generalize these results to $K_{3,3}$-minor free ($K_{3,3}$-subdivision free) graphs.

Keywords Clique-coloring · Clique chromatic number · $K_{3,3}$-Minor free graphs · Claw-free graphs

Mathematics Subject Classification 05C15 · 05C10

1 Introduction

Graphs considered in this paper are all simple and undirected. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices of $G$ is called the order of $G$. The set of vertices adjacent to a vertex $v$ is denoted by $N_G(v)$, and the size of $N_G(v)$ is called the degree of $v$ and is denoted by $d_G(v)$. A vertex with degree zero is called an isolated vertex. The maximum degree of $G$ is denoted by $\Delta(G)$. For a subset $S \subseteq V(G)$, the subgraph induced by $S$ is denoted by $G[S]$. An independent set is a set of vertices in graph that does not induce any edge and the size of maximum independent set in $G$ is written by $\alpha(G)$.

As usual, the complete bipartite graph with parts of cardinality $m$ and $n$ ($m, n \in \mathbb{N}$) is indicated by $K_{m,n}$. The graph $K_{1,3}$ is called a claw. The complete graph with $n$ vertices

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\{v_1, \ldots, v_n\} is denoted by $K_n$ or $[v_1, \ldots, v_n]$. The graph $\bar{G}$ is the complement of $G$ with the same vertex set as $G$, and $uv$ is an edge in $\bar{G}$ if and only if it is not an edge in $G$. The path and the cycle of order $n$ are denoted by $P_n$ and $C_n$, respectively. The length of a path and a cycle is the number of its edges. A path with end vertices $u$ and $v$ is denoted by $(u, v)$-path.

Edge $e$ is called an edge cut in connected graph $G$ if $G\setminus\{e\}$ is disconnected. A block in $G$ is a maximal 2-connected subgraph of $G$. A chord of a cycle $C$ is an edge not in $C$ whose end vertices lie in $C$. A hole is a chordless cycle of length greater than three. A hole is said to be odd if its length is odd; otherwise, it is said to be even. Given a graph $F$, a graph $G$ is called $F$-free if $G$ does not contain any induced subgraph isomorphic with $F$. A graph $G$ is a $(F_1, \ldots, F_k)$-free graph if it is $F_i$-free for all $i \in \{1, \ldots, k\}$. A graph $G$ is claw-free (resp. triangle-free) if it does not contain $K_{1,3}$ (resp. $K_3$) as an induced subgraph.

By a subdivision of an edge $e = uv$, we mean replacing the edge $e$ with a $(u, v)$-path. Any graph derived from graph $F$ by a sequence of subdivisions is called a subdivision of $F$ or an $F$-subdivision. The contraction of an edge $e$ with endpoints $u$ and $v$ is the replacement of $u$ and $v$ with a vertex such that edges incident to the new vertex are the edges that were incident with either $u$ or $v$ except $e$; the obtained graph is denoted by $G/\overline{e}$. Graph $F$ is called a minor of $G$ (or $F$-minor graph) if $F$ can be obtained from $G$ by a sequence of vertex and edge deletions and edge contractions. Given a graph $F$, graph $G$ is $F$-minor free if $F$ is not a minor of $G$. Obviously, any graph $G$ which contains an $F$-subdivision also has an $F$-minor. Thus an $F$-minor free graph is necessarily $F$-subdivision free, although in general the converse is not true. However, if $F$ is a graph of the maximum degree at most three, any graph which has an $F$-minor also contains an $F$-subdivision. Thus, a graph is $K_{3,3}$-minor free if and only if it is $K_{3,3}$-subdivision free. By the well-known Kuratowski’s theorem a graph is planar if and only if it is $K_5$-minor free and $K_{3,3}$-minor free. For further information on graph theory concepts and terminology we refer the reader to [17].

A vertex $k$-coloring of $G$ is a function $c : V(G) \rightarrow \{1, 2, \ldots, k\}$ such that for every two adjacent vertices $u$ and $v$, $c(u) \neq c(v)$. The minimum integer $k$ for which $G$ has a vertex $k$-coloring is called the chromatic number of $G$ and is denoted by $\chi(G)$. A hypergraph $\mathcal{H}$ is a pair $(V, \mathcal{E})$, where $V$ is the set of vertices of $\mathcal{H}$, and $\mathcal{E}$ is a family of non-empty subsets of $V$ called hyperedges of $\mathcal{H}$. A $k$-coloring of $\mathcal{H} = (V, \mathcal{E})$ is a mapping $c : V \rightarrow \{1, 2, \ldots, k\}$ such that for all $e \in \mathcal{E}$, where $|e| \geq 2$, there exist $u, v \in e$ with $c(u) \neq c(v)$. The chromatic number of $\mathcal{H}$, $\chi(\mathcal{H})$, is the smallest $k$ for which $\mathcal{H}$ has a $k$-coloring. Indeed, every graph is a hypergraph in which every hyperedge is of size two and a $k$-coloring of such hypergraph is a usual vertex $k$-coloring.

A clique of $G$ is a subset of mutually adjacent vertices of $V(G)$. A clique is said to be maximal if it is not properly contained in any other clique of $G$. We call a clique-hypergraph of $G$, the hypergraph $\mathcal{H}(G) = (V, \mathcal{E})$ with the same vertices as $G$ whose hyperedges are the maximal cliques of $G$ of cardinality at least two. A $k$-coloring of $\mathcal{H}(G)$ is also called a $k$-clique coloring of $G$, and the chromatic number of $\mathcal{H}(G)$ is called the clique-chromatic number of $G$, and is denoted by $\chi_c(G)$. In other words, a $k$-clique coloring of $G$ is a coloring of $V(G)$ such that no maximal clique in $G$ is monochromatic, and $\chi_c(G) = \chi(\mathcal{H}(G))$. A clique coloring of $\mathcal{H}(G)$ is strong if...
no triangle of $G$ is monochromatic. A graph $G$ is *hereditary $k$-clique colorable* if $G$ and all its induced subgraphs are $k$-clique colorable. The clique-hypergraph coloring problem was posed by Duffus et al. in [6]. To see more results on this concept, see [2,3,7,8,15].

Clearly, any vertex $k$-coloring of $G$ is a $k$-clique coloring, whence $\chi_c(G) \leq \chi(G)$. It is shown that in general, clique coloring can be a very different problem from usual vertex coloring and $\chi_c(G)$ could be much smaller than $\chi(G)$ [2]. On the other hand, if $G$ is triangle-free, then $\mathcal{H}(G) = G$, which implies $\chi_c(G) = \chi(G)$. Since the chromatic number of triangle-free graphs is known to be unbounded [10], we get that the same is true for the clique-chromatic number of triangle-free graphs. In addition, clique-chromatic number of claw-free graphs or even line graphs is not bounded. For instance for each constant $k$, there exists $N_k \in \mathbb{N}$ such that for each $n \geq N_k$, $\chi_c(L(K_n)) \geq k + 1$ that $L(K_n)$ is line graph of complete graph $K_n$ and is claw-free [2]. On the other hand, Défossez proved that a claw-free graph is hereditary 2-clique colorable if and only if it is odd-hole-free [5]. That is why recognizing the structure of graphs with bounded and unbounded clique-chromatic number could be an interesting problem.

For planar graphs, Mohar and Skrekovski in [9] proved the following theorem:

**Theorem 1.1** [9] *Every planar graph is strongly 3-clique colorable.*

Moreover, Shan et al. in [12] proved the following theorem:

**Theorem 1.2** [12] *Every claw-free planar graph, different from an odd cycle, is 2-clique colorable.*

Shan and Kang generalized the result of Theorem 1.1 to $K_5$-minor free graphs and the result of Theorem 1.2 to graphs which are claw-free and $K_5$-subdivision free [11] as follows:

**Theorem 1.3** [11] *Every $K_5$-minor free graph is strongly 3-clique colorable.*

**Theorem 1.4** [11] *Every graph which is claw-free and $K_5$-subdivision free, different from an odd cycle, is 2-clique colorable.*

In this paper, we generalize the result of Theorem 1.1 to $K_{3,3}$-minor free graphs and the result of Theorem 1.2 to claw-free and $K_{3,3}$-minor ($K_{3,3}$-subdivision) free graphs.

## 2 Preliminaries

In this section, we state the structure theorem of claw-free graphs that is proved by Chudnovsky and Seymour [4]. At first we need a number of definitions.

Two adjacent vertices $u$, $v$ of graph $G$ are called twins if they have the same neighbors in $G$, and if there are two such vertices, we say $G$ admits twins. For a vertex $v$ in $G$ and a set $X \subseteq V(G) \setminus \{v\}$, we say that $v$ is complete to $X$ or $X$-complete if $v$ is adjacent to every vertex in $X$; and that $v$ is anticomplete to $X$ or $X$-anticomplete if...
$v$ has no neighbor in $X$. For two disjoint subsets $A$ and $B$ of $V(G)$, we say that $A$ is complete, respectively, anticomplete, to $B$, if every vertex in $A$ is complete, respectively, anticomplete, to $B$. A vertex is called singular if the set of its non-neighbors induces a clique.

Let $G$ be a graph and $A, B$ be disjoint subsets of $V(G)$, the pair $(A, B)$ is called homogeneous pair in $G$, if for every vertex $v \in V(G) \setminus (A \cup B)$, $v$ is either $A$-complete or $A$-anticomplete and either $B$-complete or $B$-anticomplete. If one of the subsets $A$ or $B$, for instance $B$ is empty, then $A$ is called a homogeneous set.

Let $(A, B)$ be a homogeneous pair, such that $A, B$ are both cliques, and $A$ is neither complete nor anticomplete to $B$, and at least one of $A, B$ has at least two members. In these conditions the pair $(A, B)$ is called a $W$-join. A homogeneous pair $(A, B)$ is non-dominating if some vertex of $V(G) \setminus (A \cup B)$ has no neighbor in $A \cup B$, and it is coherent if the set of all $(A \cup B)$-complete vertices in $V(G) \setminus (A \cup B)$ is a clique.

Next, suppose that $V_1, V_2$ is a partition of $V(G)$ such that $V_1, V_2$ are non-empty and $V_1$ is anticomplete to $V_2$. The pair $(V_1, V_2)$ is called a 0-join in $G$.

Next, suppose that $V_1, V_2$ is a partition of $V(G)$, and for $i = 1, 2$ there is a subset $A_i \subseteq V_i$ such that:

1. $A_i$ is a clique, and $A_i, V_i \setminus A_i$ are both non-empty;
2. $A_1$ is complete to $A_2$;
3. $V_1 \setminus A_1$ is anticomplete to $V_2$, and $V_2 \setminus A_2$ is anticomplete to $V_1$.

In these conditions, the pair $(V_1, V_2)$ is a 1-join.

Now, suppose that $V_0, V_1, V_2$ is a partition of $V(G)$, and for $i = 1, 2$ there are subsets $A_i, B_i$ of $V_i$ satisfying the following properties:

1. $A_i, B_i$ are cliques, $A_i \cap B_i = \emptyset$, and $A_i, B_i$ and $V_i \setminus (A_i \cup B_i)$ are all non-empty;
2. $A_1$ is complete to $A_2$, and $B_1$ is complete to $B_2$, and there are no other edges between $V_1$ and $V_2$;
3. $V_0$ is a clique, and, for $i = 1, 2$, $V_0$ is complete to $A_i \cup B_i$ and anticomplete to $V_i \setminus (A_i \cup B_i)$.

The triple $(V_0, V_1, V_2)$ is called a generalized 2-join, and, if $V_0 = \emptyset$, the pair $(V_1, V_2)$ is called a 2-join.

The last decomposition is the following: Let $(V_1, V_2)$ be a partition of $V(G)$, such that for $i = 1, 2$, there are cliques $A_i, B_i, C_i \subseteq V_i$ with the following properties:

1. the sets $A_i, B_i, C_i$ are pairwise disjoint and have union $V_i$;
2. $V_1$ is complete to $V_2$ except that there are no edges between $A_1$ and $A_2$, between $B_1$ and $B_2$, and between $C_1$ and $C_2$; and
3. $V_1, V_2$ are both non-empty.

In these conditions it is said that $G$ is a hex-join of $V_1$ and $V_2$.

Now we define classes $F_0, \ldots, F_7$ as follows:

- $F_0$ is the class of all line graphs.
- The icosahedron is the unique planar graph with 12 vertices of all degree five. For $k = 0, 1, 2, 3$, $\text{icosa}(k)$ denotes the graph obtained from the icosahedron by deleting $k$ pairwise adjacent vertices. The class $F_1$ is the family of all graphs $G$ isomorphic to $\text{icosa}(0)$, $\text{icosa}(1)$, or $\text{icosa}(2)$.
Let $H$ be the graph with vertex set $\{v_1, \ldots, v_{13}\}$, with the following adjacency: $v_1v_2 \ldots v_6v_1$ is a hole in $G$ of length 6; $v_7$ is adjacent to $v_1, v_2$; $v_8$ is adjacent to $v_4, v_5$ and possibly to $v_7$; $v_9$ is adjacent to $v_6, v_1, v_2, v_3$; $v_{10}$ is adjacent to $v_5, v_4, v_5, v_6, v_9$; $v_{11}$ is adjacent to $v_3, v_4, v_6, v_1, v_9, v_{10}$; $v_{12}$ is adjacent to $v_3, v_5, v_6, v_9, v_{10}$; $v_{13}$ is adjacent to $v_1, v_2, v_4, v_5, v_7, v_8$ and no other pairs are adjacent. The class $F_2$ is the family of all graphs $G$ isomorphic to $H \setminus X$, where $X \subseteq \{v_{11}, v_{12}, v_{13}\}$.

Let $C$ be a circle, and $V(G)$ be a finite set of points of $C$. Take a set of subset of $C$ homeomorphic to interval $[0, 1]$ such that there are not three intervals covering $C$ and no two intervals share an end-point. Say that $u, v \in V(G)$ are adjacent in $G$ if the set of points $\{u, v\}$ of $C$ is a subset of one of the intervals. Such a graph is called circular interval graph. The class $F_3$ is the family of all circular interval graphs.

Let $H$ be the graph with seven vertices $h_0, \ldots, h_6$, in which $h_1, \ldots, h_6$ are pairwise adjacent and $h_0$ is adjacent to $h_1$. Let $H'$ be the graph obtained from the line graph $L(H)$ by adding one new vertex, adjacent precisely to the members of $V(L(H)) = E(H)$ that are not incident with $h_1$ in $H$. Then $H'$ is claw-free. Let $F_4$ be the class of all graphs isomorphic to induced subgraphs of $H'$. Note that the vertices of $H'$ corresponding to the members of $E(H)$ that are incident with $h_1$ in $H$ form a clique in $H'$. So the class $F_4$ is the family of graphs that is either a line graph or has a singular vertex.

Let $n \geq 0$. Let $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_n\}$, $C = \{c_1, \ldots, c_n\}$ be three cliques, pairwise disjoint. For $1 \leq i, j \leq n$, let $a_i, b_j$ be adjacent if and only if $i = j$, and let $c_i$ be adjacent to $a_j, b_j$ if and only if $i \neq j$. Let $d_1, d_2, d_3, d_4, d_5$ be five more vertices, where $d_1$ is $(A \cup B \cup C)$-complete; $d_2$ is complete to $A \cup B \cup \{d_1\}$; $d_3$ is complete to $A \cup \{d_2\}$; $d_4$ is complete to $B \cup \{d_2, d_3\}$; $d_5$ is adjacent to $d_3, d_4$; and there are no more edges. Let the graph just constructed be $H$. A graph $G \in F_5$ is (for some $n$) $G$ is isomorphic to $H \setminus X$ for some $X \subseteq A \cup B \cup C$. Note that vertex $d_1$ is adjacent to all the vertices but the triangle formed by $d_3, d_4$ and $d_5$, so it is a singular vertex in $G$.

Let $n \geq 0$. Let $A = \{a_0, \ldots, a_n\}$, $B = \{b_0, \ldots, b_n\}$, $C = \{c_1, \ldots, c_n\}$ be three cliques, pairwise disjoint. For $0 \leq i, j \leq n$, let $a_i, b_j$ be adjacent if and only if $i = j > 0$, and for $1 \leq i \leq n$ and $0 \leq j \leq n$ let $c_i$ be adjacent to $a_j, b_j$ if and only if $i \neq j \neq 0$. Let the graph just constructed be $H$. A graph $G \in F_6$ is (for some $n$) $G$ is isomorphic to $H \setminus X$ for some $X \subseteq (A \setminus \{a_0\}) \cup (B \setminus \{b_0\}) \cup C$.

A graph $G$ is prismatic, if for every triangle $T$ of $G$, every vertex of $G$ not in $T$ has a unique neighbor in $T$. A graph $G$ is antiprismatic if its complement is prismatic. The class $F_7$ is the family of all antiprismatic graphs.

The structure theorem in [4] is as follows:

**Theorem 2.1** [4] If $G$ is a claw-free graph, then either

- $G \in F_0 \cup \cdots \cup F_7$, or
- $G$ admits either twins, a non-dominating $W$-join, a 0-join, a 1-join, a generalized 2-join, or a hex-join.
3 $K_{3,3}$-Minor Free Graphs

In this section, we focus on the clique chromatic number of $K_{3,3}$-minor free graphs. In particular, we prove that every $K_{3,3}$-minor free graph is strongly 3-clique colorable. Moreover, it is 2-clique colorable if it is claw-free and different from an odd cycle.

For this purpose, first we need the Wagner characterization of $K_{3,3}$-minor free graphs [14]. Let $G_1$ and $G_2$ be graphs with disjoint vertex-sets. Also, let $k \geq 0$ be an integer, and for $i = 1, 2$, let $X_i \subseteq V(G_i)$ be a set of cardinality $k$ of pairwise adjacent vertices. For $i = 1, 2$, let $G'_i$ be obtained from $G_i$ by deleting a (possibly empty) set of edges with both ends in $X_i$. If $f : X_1 \rightarrow X_2$ is a bijection, and $G$ is the graph obtained from the union of $G'_1$ and $G'_2$ by identifying $x$ with $f(x)$ for all $x \in X_1$, then we say that $G$ is a $k$-sum of $G_1$ and $G_2$.

**Theorem 3.1** [13,14] A graph is $K_{3,3}$-minor free if and only if it can be obtained from planar graphs and complete graph $K_5$ by means of 0-, 1-, 2-sums.

In order to make the above characterization easier, we use the structural sequence for $K_{3,3}$-minor free graphs. In fact, graph $G$ is $K_{3,3}$-minor free if and only if there exists a sequence $T = T_1, T_2, \ldots, T_r$, in which for each $i$, $1 \leq i \leq r$, $T_i$ is either a planar graph or isomorphic with $K_5$, such that $G_1 = T_1$, and for each $i$, $2 \leq i \leq r$, $G_i$ is obtained from disjoint union of $G_{i-1}$ and $T_i$, or by gluing $T_i$ to $G_{i-1}$ on one vertex or one edge or two non-adjacent vertices and $G_r = G$. For a given $K_{3,3}$-minor free $G$, the sequence $T$ is called a Wagner sequence.

Also we need following lemma proposed in [9]:

**Lemma 3.2** [9] Let $G$ be a connected plane graph such that its outer cycle, $C$, is a triangle. If $\phi : V(C) \rightarrow \{1, 2, 3\}$ is a clique coloring of induced subgraph $C$, then $\phi$ can be extended to a strong 3-clique coloring of $G$.

In the following, we use the Wagner sequence to provide a strong 3-clique coloring for $K_{3,3}$-minor free graphs.

**Theorem 3.3** Every $K_{3,3}$-minor free graph is strongly 3-clique colorable.

**Proof** Let $G$ be a $K_{3,3}$-minor free graph. The assertion is trivial for $|V(G)| \leq 3$. So let $|V(G)| \geq 4$ and $T = T_1, T_2, \ldots, T_r$ be a Wagner sequence of $G$. We use induction on $r$. If $r = 1$, then $G = T_1$ is either $K_5$ or a planar graph. If $G$ is $K_5$, then the assertion is obvious, since by assigning color 1 to two vertices of $K_5$ and color 2 to two vertices of $K_5$ and color 3 to rest vertex, we have a strong 3-clique coloring of $K_5$. Also, if $G$ is a planar graph, then the assertion follows directly from Theorem 1.1.

Now let $r \geq 2$. By the induction hypothesis $G_{r-1}$ and $T_r$ have strong 3-clique coloring. If $G_r$ is 0-sum of $G_{r-1}$ and $T_r$, then there is nothing to say. Suppose that $G_r$ is obtained from $G_{r-1}$ and $T_r$ by gluing on vertex $\{v\}$. Thus, by a renaming of the colors, if it is necessary, we obtain a strong 3-clique coloring for $G_r$.

Next, we suppose that $G_r$ is obtained from $G_{r-1}$ and $T_r$ by gluing on edge $uv$ or two non-adjacent vertices $u$ and $v$. If $T_r$ is $K_5$, then we consider a strong 3-clique coloring of $G_{r-1}$, say $\phi$, and extend it to a strong 3-clique coloring of $G_r$ as follows: If $\phi(u) \neq \phi(v)$, then we assign three different colors $\{1, 2, 3\}$ to the other three vertices.
of $K_5$. If $\phi(u) = \phi(v)$, then we assign two different colors \{1, 2, 3\}$\setminus\{\phi(v)\}$ to the other three vertices of $K_5$. Obviously, the extended coloring is a strong 3-clique coloring of $G_r$.

Finally, let $T_r$ be a planar graph. We consider a strong 3-clique coloring of $G_{r-1}$, say $\phi$, and provide a strong 3-clique coloring of $G_r$ as follows: If $\phi(u) \neq \phi(v)$ and $e = uv$ is a maximal clique of $T_r$, then suppose that $\phi'$ is a strong 3-clique coloring of $T_r$. In this case, by a renaming the color of $\phi'(u)$ and $\phi'(v)$ in $T_r$, if it is necessary, we obtain a strong 3-clique coloring of $G_r$. If $e = uv$ is not a maximal clique in $T_r$, then there exists a triangle $T$ containing $e$ in $T_r$. Now we consider a planar embedding of $T_r$ in which $T$ is an outer face in it. Hence, by Lemma 3.2, it is enough to give a strong 3-clique coloring of outer cycle $T$ of plane graph $T_r$. That is obviously possible by coloring the third vertex of $T$ properly.

If $\phi(u) = \phi(v)$, then let $e = uv$ and $T_r = T_r \cdot e$. If there is no triangle consisting of $e = uv$ in $T_r$, then we consider a strong 3-clique coloring $\phi'$ of plane graph $T_r'$, such that $\phi'(u) = \phi'(v) = \phi(u) = \phi(v)$. Note that edge $e = uv$ is not maximal clique in $G_{r-1}$, so it is not maximal clique in $G_r$. Therefore, the coloring $\phi(x)$ for $x \in G_{r-1}$ and $\phi'(x)$ for $x \in T_r \cdot e$ is a strong 3-clique coloring for $G_r$. If $e = uv$ is in triangle $T$ in $T_r'$, then we consider a planar embedding of $T_r'$ in which $T$ is an outer face in it. By Lemma 3.2, it is enough to give a 3-clique coloring of outer cycle $T$ of plane graph $T_r'$. Thus, we give $\phi'(u = v) = \phi(u) = \phi(v)$ and assign two different colors \{1, 2, 3\}$\setminus\{\phi(v)\}$ to other two vertices of $T$; then we extend $\phi'$ to a strong 3-clique coloring of $T_r'$. This implies a strong 3-clique coloring of $T_r$ as desired, and again we obtain a strong 3-clique coloring of $G_r$.  

The rest of this section deals with the proof that, every claw-free and $K_{3,3}$-minor free graph $G$, different from an odd cycle of order greater than three, is 2-clique colorable. For this purpose, we need two following theorems:

**Theorem 3.4** [8] If $G \in F_1 \cup F_2 \cup F_3 \cup F_5 \cup F_6$ or $G$ admits a hex-join, different from an odd cycle of order greater than three, then $G$ is 2-clique colorable.

**Theorem 3.5** [8] Every connected claw-free graph $G$ with maximum degree at most seven, not an odd cycle of order greater than three, is 2-clique colorable.

From the proof of Theorem 3.5, we conclude the following corollary:

**Corollary 3.6** If $G$ is a connected $K_{3,3}$-minor free graph which admits either twins, or a non-dominating $W$-join, or a coherent $W$-join, or a 1-join, or a generalized 2-join, except an odd cycle of order greater than three, then $G$ is 2-clique colorable.

According to Theorem 3.4 and Corollary 3.6, it is sufficient to show that every $K_{3,3}$-minor free graph $G \in F_0 \cup F_4 \cup F_7$ except an odd cycle of order greater than three, is 2-clique colorable. First we show this result for class $F_0$ (the class of line graphs).

**Proposition 3.7** Every $K_{3,3}$-minor free graph in $F_0$, different from an odd cycle of order greater than three, is 2-clique colorable.
Proof Let $G$ be a $K_{3,3}$-minor free line graph. The assertion is trivial for $|V(G)| \leq 3$. Now, let $|V(G)| \geq 4$. Let $T = T_1, T_2, \ldots, T_r$ be a Wagner sequence of $G$. We use induction on $r$. If $r = 1$, then $G = T_1$ is either $K_5$ or a planar graph. If $G$ is $K_5$, then the assertion is obvious. If $G$ is a planar graph, then by Theorem 1.2, $G$ has a 2-clique coloring, since every line graph is claw-free.

Now let $r \geq 2$. By the induction hypothesis $G_{r-1}$ and $T_r$ have 2-clique coloring. If $G_r$ is 0-sum or 1-sum of $G_{r-1}$ and $T_r$, then the result is obvious. Now, we suppose that $G_r$ is 2-sum of $G_{r-1}$ and $T_r$ on edge $uv$. Note that if $uv$ is an edge cut, then $G$ can be considered as 1-sum of two graphs. So, later on we assume that $uv$ is not an edge cut. If $T_r$ is $K_5$ and $\phi$ is a 2-clique coloring of $G_{r-1}$, then we assign the colors $\phi(u)$ and $\phi(v)$ to vertices $u, v$ in $K_5$ and give two different colors $\{1, 2\}$ to the other three vertices of $K_5$.

If $T_r$ is a planar graph, then we have four possibilities:

(i) there exists 2-clique colorings $\phi$ and $\phi'$ of $G_{r-1}$ and $T_r$, such that $\phi(u) \neq \phi(v)$ and $\phi'(u) \neq \phi'(v)$;
(ii) there exists 2-clique colorings $\phi$ and $\phi'$ of $G_{r-1}$ and $T_r$, such that $\phi(u) = \phi(v)$ and $\phi'(u) = \phi'(v)$;
(iii) in every 2-clique colorings $\phi$ and $\phi'$ of $G_{r-1}$ and $T_r$, $\phi(u) \neq \phi(v)$ and $\phi'(u) = \phi'(v)$;
(iv) in every 2-clique colorings $\phi$ and $\phi'$ of $G_{r-1}$ and $T_r$, $\phi(u) = \phi(v)$ and $\phi'(u) \neq \phi'(v)$.

In the first two cases, only by a color renaming, if it is necessary, we obtain a 2-clique coloring for $G_r$. In the following, without loss of generality we consider the case (iii) and show that it is impossible:

The assumption (iii) concludes that vertex $u$ (and $v$) in $T_r$ belongs to a maximal clique $C_u$ (and $C_v$) such that in every 2-clique coloring of $T_r$, $C_u \setminus \{u\}$ (and $C_v \setminus \{v\}$) is monochromatic. Hence, $u \notin C_v$ and $v \notin C_u$. This implies that, $u$ has a non-neighbor vertex in $C_v$, say $v'$, also $v$ has a non-neighbor vertex in $C_u$, say $u'$. Moreover, assumption (iii) implies that $uv$ is a maximal clique in $G_{r-1}$. Thus, there exist vertex $u'' \in N_{G_{r-1}}(u)$ that $u'' \notin N_{G_{r-1}}(v)$ (or $v'' \in N_{G_{r-1}}(v)$ that $v'' \notin N_{G_{r-1}}(u)$). Hence, edge $uv$ among edges $uu'$ and $uu''$ (or $vv'$ and $vv''$) is a claw in $G_r$, that is a contradiction.

If in the operation 2-sum, the edge $uv$ is deleted, then by the following argument, we could change the coloring of vertices in $T_r$ such that $\phi'(u) \neq \phi'(v)$, that contradicts the assumption (iii). Note that since $uv$ is not an edge cut in $G_{r-1}$ and $T_r$, there are shortest $(u, v)$-paths $P : u_0 = uu_1 \ldots uu_s = v$ in $T_r/uv$ and $Q : v_0 = vv_1 \ldots vv_t = u$ in $G_{r-1}/uv$. Since $G_r$ is claw-free, vertices $u$ and $v$ in $T_r$ and $G_{r-1}$ belong to only one maximal clique. If $dt_r(u_i) = 2, i = 1, \ldots, s - 1$ and $dG_{r-1}(v_j) = 2, j = 1, \ldots, t - 1$, then by (iii), the length of $P$ is even and the length of $Q$ is odd. This implies $G_r$ is an odd cycle and contradicts our assumption. Thus, assume that $k \in \{0, 1, \ldots, s - 1\}$ is the smallest indices that $dt_r(u_k) \geq 3$ and $w \in N_{T_r}(u_k)$. Since $G_r$ is claw free, we must have $w \in N_{T_r}(u_{k+1})$. Let $C$ be a unique maximal clique consisting of $[u_k, u_{k+1}, w]$ (note that $N_{T_r}(u_k) \subseteq N_{T_r}(u_{k+1})$). If there exists a vertex in $C$ that its color is $\phi(u_k)$, then we swap the colors of vertices on $(u, u_k)$-path in $P$. Thus, we will obtain a 2-
clique coloring of $T_r$ such that $u$ and $v$ are assigned different colors. This contradicts the assumption (iii).

Now assume that the color of all vertices in $C$ is different from $\phi'(u_k)$. In this case, if there exists a vertex in $C$, say $w' \notin u_k$, such that $C$ is a unique maximal clique contains $w'$, then we assign $\phi'(u_k)$ to $w'$ and again swap the colors of vertices on $(u, u_k)$-path in $P$. Otherwise, every vertex in $C$ belongs to a maximal clique other than $C$. In this case, if there exists a vertex $w' \in C$, such that $w' \in C'$, where $C$ and $C'$ are maximal cliques in different blocks of $T_r$, then we swap the color of vertices in the component of $T_r / \{w'\}$ consisting of $C'$, assign $\phi'(u_k)$ to $w'$ and again swap the colors of vertices on $(u, u_k)$-path in $P$. Thus, we will obtain a 2-clique coloring of $T_r$ such that $u$ and $v$ are assigned different colors. This contradicts the assumption (iii).

The remaining case is that all vertices in $C$ belong to some other maximal cliques and all cliques are in one block in $T_r$. In this case, let $l$ be the smallest indices that there exists a path from $u_l$ to some vertices in $C / \{u_k, u_{k+1}\}$, whih we call $(w, u_l)$-path $P' : wv_1 \ldots v_m = u_l$. Note that if there is no such a path, then we can consider graph $G$ as a 2-sum of two graphs on edge $u_ku_{k+1}$, and we are done. If $m = 1$, then since $P$ is a shortest path, we have $l = k + 2$. Therefore, the induced subgraph on vertices $\{u_{k-1}, u_k, u_{k+1}, u_{k+2}, u_{k+3}, w\}$ is one of the nine forbidden structures in line graphs (see [16]) (note that if $k = 0$ or $k = s - 2$, then vertex $u_{k-1} = v_{l-1}$ or $u_{k+3} = v_1$). Hence, $m \geq 2$. Also, $w_{m-1}$ is adjacent to $u_{l+1}$, since $T_r$ is claw free. Now, by considering the first internal vertices in $P'$ and $(u_{k+1}, u_l)$-path in $P$ with degree greater than two, we do the similar above discussion in order to change the color of vertices $w$ or $u_{k+1}$ and subsequently change the color of $u$. Therefore, if we could not do that, then we conclude that pattern of colors in these paths are $a, b, a, b, \ldots$, where $a, b \in \{1, 2\}$. Now, we have $\phi'(w_{m-1}) = \phi'(u_{l+1}) \neq \phi'(u_l) \text{ or } \phi'(w_{m-1}) \neq \phi'(u_{l+1})$.

In the former case, we swap the color of vertices in path $w_{m-1}w_{m-2} \ldots w_1wu_k \ldots u$. In the latter case, we swap the color of vertices in path $u_mu_{l-1} \ldots u_{k+1}u_{k+2}u_{k+1} \ldots u$. Thus, in both cases, we obtain a 2-clique coloring for $T_r$ such that the vertices $u$ and $v$ receive different colors and this contradicts the assumption (iii). Therefore, the cases (iii) and (iv) are impossible and the proof is complete.

Now we show the 2-clique colorability of $K_{3,3}$-minor free graphs in class $F_4$. First, we need the following theorem:

Theorem 3.8 [2] For any graph $G \neq C_5$ with $\alpha(G) \geq 2$, we have $\chi_c(G) \leq \alpha(G)$.

Proposition 3.9 Every $K_{3,3}$-minor free graph in $F_4$ is 2-clique colorable.

Proof Let $G$ be a graph in $F_4$. Since a graph in $F_4$ is a line graph or has a singular vertex, by Proposition 3.7 it is sufficient to consider graphs in $F_4$ with singular vertex. So by the construction of graphs in $F_4$, we have $\alpha(G) \leq 3$. For case $\alpha(G) = 1$, the statement is obvious. If $\alpha(G) = 2$, then by Theorem 3.8, $G$ is 2-clique colorable; otherwise, $\alpha(G) = 3$. Let $x$ be a singular vertex and $S = \{r, s, t\}$ be a maximum independent set in $G$. Note that $x \notin S$, and since non-neighbor vertices of $x$ induce a clique, vertices $r, s$ are adjacent to $x$ and $t$ is not adjacent to $x$.

Now we propose a 2-clique coloring $\phi$ for $G$ as follows: let $\phi(x) = 1, \phi(t) = 2$ and assign color 1 to every non-neighbor vertex of $x$ except $t$. Now if $x$ and $t$ have
more than one common neighbor, then assign color 2 to one of them and color 1 to the other vertices; otherwise, assign color 1 to their common neighbor. Finally, assign color 2 to the other adjacent vertices to \( x \). It is easy to see that this assignment is a 2-clique coloring of \( G \).

Finally, we show the 2-clique colorability of \( K_{3,3} \)-minor free graphs in class \( F_7 \).

**Proposition 3.10** Every \( K_{3,3} \)-minor free graph in \( F_7 \) is 2-clique colorable.

**Proof** Let \( G \) be a graph in \( F_7 \). Since \( G \) is an antiprismatic, \( \bar{G} \) is prismatic. If \( \bar{G} \) has no triangle, then \( \alpha(G) = 2 \), and by Theorem 3.8, is 2-clique colorable. Now let \( T = [uvw] \) be a triangle in \( \bar{G} \), and \( S_1 = N_{\bar{G}}(v) \setminus \{u, w\} \), \( S_2 = N_{\bar{G}}(u) \setminus \{v, w\} \) and \( S_3 = N_{\bar{G}}(w) \setminus \{u, v\} \) be a partition of vertices \( V(G) \setminus \{v, u, w\} \).

Liang et al. in [8] prove that if

1. \( |S_i| = 0 \) for some \( i = 1, 2, 3 \), then \( G \) has a 2-clique coloring.
2. \( |S_i| = 1 \) for some \( i = 1, 2, 3 \), then \( G \) has a 2-clique coloring.
3. there is an edge \( xy \) in \( \bar{G} \) such that for \( i \neq j \in \{1, 2, 3\} \), \( x \) is an isolated vertex in \( \bar{G}[S_i] \) and \( y \) is an isolated vertex in \( \bar{G}[S_j] \), then there exists a 2-clique coloring of \( G \).
4. there exist \( i \neq j \in \{1, 2, 3\} \) such that \( S_i \cup S_j \) is an independent set in \( \bar{G} \), then \( G \) has a 2-clique coloring.

In the following for the remaining cases, we provide a 2-clique coloring for \( G \) or we show that \( G \) is \( K_{3,3} \)-minor that is a contradiction. Let \( S_1 = \{v_1, v_2\} \) and \( S_2 = \{u_1, u_2\} \) and \( S_3 = \{w_1, w_2\} \). There are \( i \neq j, i, j \in \{1, 2, 3\} \), say \( i = 1, j = 2 \), such that \( v_1 \) is adjacent to \( v_2 \) in \( \bar{G} \) and \( u_1 \) is adjacent to \( u_2 \) in \( \bar{G} \); otherwise by (iii) or (iv), we have \( \chi_c(G) \leq 2 \). Hence, we have triangles \([uu_1u_2]\) and \([vv_1v_2]\) in \( \bar{G} \). Since \( \bar{G} \) is a prismatic \( v_1, v_2, w_1, w_2 \) have a unique neighbor in \([uu_1u_2]\) and \( u_1, u_2, w_1, w_2 \) have a unique neighbor in \([vv_1v_2]\). Thus, \([u_1, u_2, v_1, v_2]\) induces a cycle in \( \bar{G} \) because, otherwise, for instance if \( u_1 \) and \( u_2 \) both are adjacent to \( v_1 \), then there exist two neighbors for \( u \) in triangle \([u_1u_2v_1]\). Without loss of generality, assume that \( u_1v_1 \) and \( u_2v_2 \) are edges in \( \bar{G} \). That means, \( u_1v_2 \) and \( u_2v_1 \) are edges in \( G \).

Now each two vertices \( w_1 \) and \( w_2 \) have unique neighbor in \([uu_1u_2]\) and \([vv_1v_2]\). If both vertices \( w_1 \) and \( w_2 \) are adjacent to \( u_1 \) (or \( u_2 \)) and \( v_1 \) (or \( v_2 \)) in \( \bar{G} \), then there exists two neighbors for \( w_2 \) in triangle \([v_1u_1w_1]\) (or \([v_2u_2w_1]\)) that contradicts \( \bar{G} \) is prismatic. If vertices \( w_1 \) and \( w_2 \) are both adjacent to \( u_1 \) (or \( u_2 \)) and \( v_2 \) (or \( v_1 \)) in \( \bar{G} \), then \( G \) has a \( K_{3,3} \)-minor, on vertices \([w, w_1, w_2; u, v, v_1]\) (or \([w, w_1, w_2; u, v, v_2]\)). Note that if \( w_1 \) is adjacent to \( w_2 \) in \( \bar{G} \), then we have triangle \([ww_1w_2]\) and since \( \bar{G} \) is prismatic, vertices \( w_1 \) and \( w_2 \) cannot be both adjacent to one vertex of \([v_1, v_2]\) or \([u_1, u_2]\). If \( w_1 \) is adjacent to \( u_1 \) (or \( u_2 \)) and \( v_1 \) (or \( v_2 \)) and \( w_2 \) is adjacent to \( u_2 \) (or \( u_1 \)) and \( v_2 \) (or \( v_1 \)) in \( \bar{G} \), then \( G \) has a \( K_{3,3} \)-minor, on vertices \([w, w_1, w_2; u, v, v_2]\) (or \([w, w_1, w_2; u, v, v_1]\)). Hence, all cases above contradict that \( G \) is \( K_{3,3} \)-minor free or \( \bar{G} \) is prismatic. Thus, it is enough to consider the two following remaining cases:

- \( w_1 \) is adjacent to \( u_1 \) and \( v_2 \), and \( w_2 \) is adjacent to \( u_2 \) and \( v_1 \) in \( \bar{G} \) (Fig. 1b shows graph \( G \)).
- \( w_1 \) is adjacent to \( u_2 \) and \( v_1 \), and \( w_2 \) is adjacent to \( u_1 \) and \( v_2 \) in \( \bar{G} \) (Fig. 1a shows graph \( G \)).
In both above cases $G$ is a claw free planar graph and by Theorem 1.2 is 2-clique colorable (in Fig. 1, and the dashed lines show the edges that may exist or not exist in $G$).

Finally, let $|S_i| \geq 3$ for some $i = 1, 2, 3$, say $|S_1| \geq 2$, $|S_2| \geq 2$ and $S_3 = \{w_1, w_2, w_3\}$. Since such graphs contain the graphs with $|S_i| \leq 2$, $i = 1, 2, 3$ as subgraph, we only need to consider graphs that contains one of the two graphs shown in Fig. 1. By case (iv) there are $i \neq j \in \{1, 2, 3\}$ such that $G[S_i]$ and $\overline{G}[S_j]$ both are not independent. Liang et al. in [8] show that $\overline{G}[S_i]$, $i \in \{1, 2, 3\}$, is not path and triangle. So we need to consider the case that $[uu_1u_2]$ and $[vv_1v_2]$ are triangles in $\overline{G}$, and $v_1w_3 \in E(\overline{G})$ or $v_2w_3 \in E(\overline{G})$. This implies $G$ has a $K_{3,3}$-minor, on vertex set $\{w, w_1, w_2; u, v, v_2\}$ or $\{w, w_1, w_2; u, v, v_1\}$, respectively. Note that, when $[uu_1u_2]$ and $[ww_1w_2]$ are triangles in $\overline{G}$, the proof is similar. Therefore, when $|S_i| \geq 3$ for some $i = 1, 2, 3$, $G$ is a $K_{3,3}$-minor, that is a contradiction. □

By Theorem 3.4, Corollary 3.6 and Propositions 3.7, 3.9, 3.10, the main result in this section is proved.

Theorem 3.11 If $G$ is claw-free and $K_{3,3}$-minor free graph except an odd cycle of order greater than three, then $G$ is 2-clique colorable.

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