Perspectives on Galactic Dynamics via General Relativity

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Abstract

Responses to questions, comments and criticism of our recent paper “General Relativity Resolves..” [1] are provided. It is emphasized that our model is entirely natural to describe the dynamics of an axially symmetric galaxy and that our solution, albeit idealized, contains the essence of the problem. The discontinuity of the metric derivative on the symmetry plane is necessarily interpreted as the effect of the mathematically idealized discontinuity of the gradient of the density and is shown to be naturally connected to the distributed volume density via the Gauss divergence theorem. We present arguments to the effect that for our approximate weak field model, we can choose the physically satisfactory mass distribution without an accompanying singular mass surface layer. To support this contention, we modify our solution slightly by removing the discontinuity with a region of continuous density gradient overlapping the \( z = 0 \) plane. The alternative of invoking a surface layer leads to the presence of a negative mass surface layer approaching the numerical value of the positive mass continuous region. This is in contradiction with the assumed stationarity of the model. We find that a test particle behaves normally as it approaches the \( z = 0 \) plane, the acceleration being towards the direction of this plane. This is in contradiction to the negative mass layer hypothesis as negative mass would repel the test particle. Thus, further support is added to the integrity of our original model.

Subject headings: galaxies: kinematics and dynamics-gravitation-relativity-dark matter
1 Introduction

Recently, [1] we presented a paper illustrating that Newtonian dynamics is inadequate to describe galactic dynamics. We showed that general relativity, the preferred theory of gravity, is required for the task even though the fields are weak and the motion is non-relativistic. This is because in such a gravitationally bound dynamical problem with an extended matter distribution, non-linearities cannot be neglected. We showed that general relativity allows the modeling of the essentially flat galactic rotation curves without exotic dark matter. In the process, we determined the mass density of the luminous threshold based upon data in the radial direction as given in [3]. In the short period since our presentation, we have received a large volume of correspondence with interesting questions, comments, suggestions and criticism (three such already posted [4], [5], [6]).

The essential thrust of [4] was the claim that our particular model contained a singular disk of mass in the symmetry plane of the galaxy and thereby, vitiated our solution as a proper model for a galaxy. Subsequently, various individuals have used this argument to claim that our work is flawed. More recently [5], it was suggested that the symmetry plane was the seat of exotic matter with one option being a negative mass sheet. Most recently, it was argued in [6] that the standard iterative perturbation scheme accounts for non-linearities and hence the galactic dynamics must be determined by Newtonian theory to lowest order. We will discuss these papers further in what follows.

In this paper, we respond to the criticisms. In the process, some new interesting

\footnote{It should be noted that in private communications, two colleagues independently alerted us to the same line of reasoning as in [4] prior to that posting.}
insights emerge. What must be emphasized is this: while it is certainly useful to have raised such criticisms, it is important to recognize that in the multitude of comments and correspondence that followed, no one to our knowledge has found valid reason to fault our central thesis, that general relativity, the preferred theory of gravity, exhibits essential non-linearities in sources of galactic scope. While in subsequent studies, one solution to the model may be preferred over another, the essential point is that a new route to astrophysical dynamics is opened up by the recognition that general relativity, long accepted as the key to cosmology, also comes into play with significance for the major building blocks within cosmology.

A frequent criticism of our work is the evidence that is presented for the existence of vast amounts of exotic dark matter in larger than galactic scales such as in the scale of clusters of galaxies. Two points must be made in this regard. Firstly, while some individuals have read into our work that we have claimed to have proved that such large exotic dark matter reservoirs do not exist, this is not the case. Thus far our work applies to the galactic scale. Secondly, we have pointed to the fact that it would be interesting to extend the general relativistic approach to the other relevant areas of astrophysics to determine whether or not exotic dark matter is truly required in those larger scale domains. For example, for the dynamics of clusters of galaxies, the virial theorem is used. This is based on Newtonian gravity theory. It would be of interest to introduce a general relativistic virial theorem for comparison. It is only after possible effects of general relativity are explored that we can be confident about the viability or non-viability of exotic dark matter in nature.

In Section 2, we review the essential structure and equations that were developed in
In Section 3, we reply to certain issues that had been raised and in Section 4, we consider the problem of matter distribution. We end with concluding remarks in Section 5.

2 Field Equations and Solution for Galactic Modeling

We had modeled a galaxy in terms of its essential characteristics as a uniformly rotating fluid without pressure and symmetric about its axis of rotation. Within the context of general relativity, we began with the general metric structure

\[
ds^2 = -e^{\nu-w}(u dz^2 + dr^2) - r^2 e^{-w} d\phi^2 + e^w (cdt - N d\phi)^2
\]

where \( u, \nu, w \) and \( N \) are functions of cylindrical polar coordinates \( r, z \). To the order required, we found from the field equations that \( u \) could be taken to be unity. As in previous studies \[12\] \[2\], we chose to work in a coordinate system co-moving with the matter having four-velocity

\[
U^i = \delta_0^i.
\]

Actually, with this choice, it follows that \( w = 0 \) as a result of the requirement

\[
g_{ik} U^i U^k = 1.
\]

As in \[2\], we performed a purely \emph{local} \((r, z \text{ held fixed})\) transformation \(^2\)

\[
\bar{\phi} = \phi + \omega(r, z) t
\]

\(^2\)The importance of the nature of this transformation will be discussed in reference to \[25\].
that locally diagonalizes the metric. For the weak fields under consideration, this gave
the approximate local angular velocity \( \omega \) and the tangential velocity \( V \) as

\[
\omega \approx \frac{N_c}{r^2},
\]

\[
V = \omega r.
\]

(5)

With \( w = 0 \), the Einstein field equations reduce to

\[
2r \nu_r + N_r^2 - N_z^2 = 0,
\]

\[
r \nu_z + N_r N_z = 0,
\]

\[
N_r^2 + N_z^2 + 2r^2 (\nu_{rr} + \nu_{zz}) = 0,
\]

\[
\frac{3}{4} r^{-2}(N_r^2 + N_z^2) + N_r^{-2} \left( N_{rr} + N_{zz} - \frac{N_r}{r} \right) - \frac{1}{2} (\nu_{rr} + \nu_{zz}) = \frac{8\pi G \rho c^2}{c^2}
\]

(6)

to order \( G^1 \) and are readily combined to yield

\[
N_{rr} + N_{zz} - \frac{N_r}{r} = 0
\]

(7)

\[
N r^{-2} \left( N_{rr} + N_{zz} - \frac{N_r}{r} \right) + \frac{N_r^2 + N_z^2}{r^2} = \frac{8\pi G \rho c^2}{c^2}
\]

(8)

for \( N \) and \( \rho \). When (7) applies \(^3\), the density is given by

\[
\frac{N_r^2 + N_z^2}{r^2} = \frac{8\pi G \rho c^2}{c^2}. \]

(9)

\( \nu \) is readily found from the remaining field equations.

We should note at this point, as some colleagues have indicated, that the full Einstein

\(^3\)While we would ordinarily set the first group of terms to 0 in (5) as a consequence of (7), we retain it here for the purposes of later discussion.
worth reiterating the fact that since we are not dealing with the exact Einstein equations but rather with equations and solutions only up to first order in $G$, it would be wrong to expect the degree of mathematical precision that is generally applied to exact solutions.

As in other works, we expressed (7) as

$$\nabla^2 \Phi = 0 \quad (10)$$

where

$$\Phi \equiv \int \frac{N}{r} dr. \quad (11)$$

It is to be emphasized that this $\Phi$ is not the potential of Newtonian theory. We have referred to $\Phi$ as the “generating potential” in [1]. With (5) and (11), we have the expression for the tangential velocity of the distribution in terms of the derivative of a harmonic function

$$V = c \frac{N}{r} \frac{\partial \Phi}{\partial r} = c \frac{\partial \Phi}{\partial r}. \quad (12)$$

By separating variables, we were able to express the solution as a superposition of lowest-order Bessel functions with exponential factors in the transverse dimension $z$ as

$$\Phi = \sum_n C_n e^{-k_n |z|} J_0(k_n r) \quad (13)$$

With (13) and (11), the tangential velocity is

$$V = -c \sum_n k_n C_n e^{-k_n |z|} J_1(k_n r). \quad (14)$$

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4 Since $\Phi$ contains a partial integral, we could replace the RHS of (10) by an arbitrary function of $z$.

5 We thank B.R. Steadman for bringing to our attention the paper by de Araujo and Wang [10] who consider as a solution, one term of the form in the series (13).
We chose appropriate parameters in (14) to model the observed galactic rotation curves and with (12) and (8), we deduced the corresponding galactic mass density distributions. These were intuitively satisfactory distributions, indicating mass primarily concentrated about a disk-like configuration with diminishing density with increasing radius, as observed for the luminous matter. Moreover, the integrated masses were less than the published values using Newtonian theory and greater than the MOND [9] values. Thus, the observed galactic dynamics was in accord with our calculations without the need for massive extended halos of exotic dark matter that had been predicted by previous studies using Newtonian gravity theory. We noted that with this form of solution, the absolute value of $z$ had to be used to provide a proper reflection of the distribution for negative $z$, the matter below the central plane. This results in a discontinuity in $N_z$ at $z = 0$ and as a consequence, in the strictest sense, the solution is restricted to $z$ values different from 0. The important point is this: the essential necessarily physical consequence of the use of $|z|$ is that the $z$ component of the gradient of $\rho$ is discontinuous at $z = 0$. Moreover, since the limits of the density are the same as the symmetry plane $z = 0$ is approached from above or below, we could usefully define the value of $\rho$ at $z = 0$ to have that limiting value. Thus, a globally continuous and finite density distribution source is established. $^6$ This is our choice to properly model the physics of galaxies as opposed to abstruse mathematical structures. We will discuss this further in what follows.

$^6$See, however, the issues surrounding the concept of a singular layer of mass at $z = 0$ in what follows.
3 Replies to Certain Issues Raised

There was a natural tendency for researchers to question the need for general relativistic analysis from the very outset. After all, the galactic gravitational fields are weak and the velocities of the stars are non-relativistic. In a variety of situations, it is indeed correct that weakness of field and slowness of velocity suffice to make Newtonian gravity theory an excellent approximation to physical phenomena, observed or predicted. In such situations, for example in planetary motion, there are only minute, albeit very interesting small changes to the orbits as a result of applying general relativity. Yet even in this, an essentially new phenomenon, the emission of gravitational waves, is predicted to occur within general relativity yet it is entirely absent in Newtonian gravity. Thus, it is helpful to keep an open mind with regard to forming conclusions as to what general relativity may yield in studies of dynamical sources.

In fact, even within the study of gravity waves, there are aspects of non-linearity that might not be expected upon first glance. Indeed, as Eddington had noted many years ago in the context of perturbative calculations for weak gravity waves generated by masses whose motions are driven by gravity itself (“gravitationally bound” or “free-fall” in the nomenclature of general relativity), non-linearities cannot be ignored. This was due to the fact that for gravitationally bound systems, the velocities are of the order $\sqrt{Gm/R}$ and hence non-linear terms that are ordinarily rejected in perturbation theory could be as large as the linear terms that are the usually sole terms that are retained. The novel aspect that we found is that the Eddington insight is also applicable for gravitationally bound stationary time-independent rotational systems (i.e. systems not
producing gravity waves) that exist in nature. We see this in (8): The source $8\pi G\rho/c^2$ is equated to quadratic terms in $N$ and from (5), we see that $N$ is of order $G^{1/2}$ as a consequence of the gravitational bound aspect. Thus the consistency of the procedure is established. This encapsulates the essential departure of our approach from that of Newtonian gravity.

It is correct that general relativistic effects are minute for the weak field gravitationally bound example of the solar system. But there, the dominant field is that of the sun and the planets are for most purposes properly treated as test particles in the solar field, guided by this field but not contributing to the global field. By contrast, in the galactic problem, the elements of matter are both guided by and essential contributors to the global field. By no means does this change the fact that the field is very weak but it does change the nature of the dynamics and the connection of field to source. Some have seen this effect as a peculiarity of our choice of co-moving coordinates. However, this is not the case: Other coordinates would have brought in an additional $\nabla^2 w$ into the equation for $\rho$, and $w$ which has a correspondence to the Newtonian potential, is of order $G^1$ as required. It is to be emphasized that the second non-linear term in $N$ in the $\rho$ equation (8) cannot be removed globally by any choice of coordinates for our stationary system and hence the non-linear aspect is essential. Linearized theory is simply inadequate to the task.

4 The Issue of Matter Distribution

An issue first raised privately to us by some colleagues and later in [4] [5] concerns the nature of the matter distribution. They have noted that given the existence of the
discontinuity of $N_z$ that we had pointed to in [1], a significant surface tensor $S^k_i$ can be constructed with a surface density component given by

$$(8\pi G/c^2)S^t_i = \frac{N[N_z]}{2r^2} - \frac{[\nu_z]}{2}$$

(15)

to order $G^1$. The notation $[.]$ denotes the jump over a discontinuity of the given function, here at $z = 0$. Using (6), this becomes

$$(8\pi G/c^2)S^t_i = \frac{N[N_z]}{2r^2} + \frac{N_z[N_z]}{2r}$$

(16)

It was claimed that this necessarily implied the existence of a singular physical surface of mass in the galactic plane above and beyond the continuous mass distribution that we had found, thus rendering our model unphysical.

Having received this challenge, we calculated the surface mass that was said to be present in the four galaxies that we had studied by integrating (16) over the surface without paying heed to the actual sign of the result. Suspicions were aroused from the
discovery that in each case gave a numerical value slightly less than the mass that we had derived from the volume integral of our continuous mass density distribution using , and . This pointed to a plausible explanation: in our case, with our choice of model, there is no physical mass layer present on the plane. The surface integral of this singular layer is merely a mathematical construct that indirectly describes most of the continuously distributed mass by means of the Gauss divergence theorem. To see this, consider the vector defined as

\[ \mathbf{F} \equiv A(r, z)e_r + B(r, z)e_z \]  

(17)

where

\[ (8\pi G/c^2)B \equiv \frac{NN_z}{2r^2} + \frac{N_rN_z}{2r} \]  

(18)
as a first option. We choose \( A(r, z) \) so that

\[ \int \nabla \cdot \mathbf{F}dV \equiv (8\pi G/c^2)M \]  

(19)

where \( M \) is the total mass. As a more transparent second option, we choose

\[ (8\pi G/c^2)B \equiv \frac{NN_z}{r^2} \]  

(20)

where we define

\[ \nabla \cdot \mathbf{F} \equiv (8\pi G/c^2)\rho \]  

(21)

From these definitions, we deduce the form of \( A(r, z) \) in order to produce the density as expressed through \( N \) in (9). We calculate the mass over the cylindrical volume defined

\footnote{It should be noted that the two terms in were found to contribute equally.} \footnote{\( e_r \) and \( e_z \) are unit vectors in the \( r \) and \( z \) directions.}
by $-\infty < z < \infty$, $0 < r < r_{\text{galaxy}}$. By the Gauss divergence theorem, the volume integral of $\rho$, via (21) is equal to the integral of the normal component of $\mathbf{F}$ over the bounding surfaces. However, the integration must be over a continuous domain and since the $e_z$ component is discontinuous over the $z = 0$ plane, the volume integral must be split into an upper and a lower half. The two new surface integrals together would constitute the jump integral of (16) in the first option if one were to be cavalier about the directions of unit outward normals, as we shall discuss in what follows. The surfaces above and below the galaxy give zero because of the exponential factors in $z$ and the final small contribution comes from the cylinder wall via the $A$ function.

In our solution, the actual physical distribution of mass is not in concentrated layers over bounding surfaces: the Gauss theorem gives the value of the distributed mass via equivalent purely mathematical surface constructs as we are familiar from elementary applications of this theorem. Physically, the density is well defined and continuous throughout, except on the $z = 0$ plane. In fact the limits as $z = 0$ is approached give the same finite values from above and below. While the field equations break down at $z = 0$, the density for a physically viable model is logically defined by this limit at $z = 0$. However, with the chosen form of solution, the density gradient in the $z$ direction is discontinuous on the $z = 0$ plane. This gradient undergoes a reversal for a galactic distribution with diminishing density in both directions away from the symmetry plane. It is most convenient to achieve this with an abrupt reversal as we have done. There is no indication that this choice alters the essential physics.

To achieve the reversal with significant smoothness requires fine-tuning as we demonstrate in what follows. Instead of exponential functions of the form that we chose in (13),
we now choose combinations of exponentials in the form of \( \cosh(\kappa_n z) \) for \(-z_0 < z < z_0\) and in the original form of exponentials for \( |z| > z_0 \). By the choice of the \( \cosh \kappa_n z \) functions to span the symmetry plane, we achieve smoothness over the interval that includes the \( z = 0 \) plane and it is accomplished symmetrically about this plane. Clearly, since the values of the \( \cosh \) function and the exponential functions at \( z = 0 \) are identical, the rotation curves now follow as before in \[1\]. However, this leads to the requirement that the functions \( N \) and \( N_z \) match at \( |z| = z_0 \). This was accomplished as is shown in Figure 2. While this result is not easily achieved, it should be kept in mind that such

\[9\] It is to be noted that we are not free to impose such functions to suit our convenience, for example of the form \( e^{-\kappa z^2} \) as suggested by some colleagues. While such functions would exhibit exquisite smoothness and symmetry about \( z = 0 \), they would not satisfy \[10\] in the separable form.

\[10\] Strictly speaking, the matching to the rotational velocity data could be seen to be effected at \( z = \epsilon \) in order to avoid the \( z = 0 \) value.

\[11\] The careful addition of more terms in the series with proper choices of \( \kappa_n \) and \( C_n \) values would
difficulties likely stem from our restricting the solution class to sequences of *separable* analytic functions in $z$ and $r$. Although these are most elegant to grasp and display, they invariably place restrictions on the forms of possible solutions. Had we resorted to direct numerical integration, the smoothing requirement issue about the $z = 0$ plane would not have arisen. Moreover, had we opted for non-separable harmonic functions as solution generators such as sequences of Weyl line mass potentials [11], this would also have averted the discontinuity issue. Interestingly, in its place thereby, we would have introduced a singularity at the origin. This would be pleasing to those who have suggested to us that it would be useful to model galaxies with black holes and/or naked singularities (see for example [13]) at the core, given the current suggested observational evidence.

It is important to focus upon and be guided by the essential physical situation and not become ensnared in mathematical minutiae. The physical situation is one of a galaxy of stars freely gravitating in approximately steady rotational motion. The model to describe it is a pressureless fluid with density symmetrically decreasing as we move vertically away from the $z = 0$ plane in both directions. By way of contrast, consider the simplest example where (16) comes into play in a directly *physical* context: a spherical shell with Minkowski metric within and with Schwarzschild metric with parameter $M \neq 0$ outside. The field equations are for vacuum both inside and outside yet the spacetime has mass as evidenced by the Schwarzschild mass parameter. This indicates that there is indeed a *necessarily physical* layer of mass on the dividing shell as there is the necessity to have the mass manifest itself and there is no other place for it to reside. There is improve the fit further at the join.
an essential physical discontinuity in this case in contrast to our rather benign galactic model density gradient discontinuity: mass in the shell in the former but not necessarily on the plane in the latter. The argument in [4] treats our benign discontinuity (which we have seen is a density gradient discontinuity) as a serious defect but we have seen that it has no necessarily essential effect on the physics. What is within our control is the decision as to whether we study a model with or without an actual singular layer of mass concentrated at \( z = 0 \). We have chosen the latter. There is no independent mass attributable to this discontinuity if we pursue this approach to develop a physically viable model. It was also argued that our spacetime has irregular behavior asymptotically. However, in [4], the harmonic coordinate conditions have been imposed in conjunction with the expansion that was used. By contrast, while we also deal with an expansion in powers of \( G^{1/2} \), we had already exhausted all of the available gauge freedom in choosing the particular metric form with co-moving coordinate conditions; there is no room left for additional gauge restrictions such as the harmonic conditions. Thus, the analysis in [4] does not apply to our situation. Moreover, we had noted in [1] that the Bessel functions fall off as \( 1/\sqrt{k^r} \). Indeed, the authors in [10] have nicely presented the evidence in favour of this type of solution having asymptotic flatness \(^{12}\) and noted some of the ambiguities associated with the issue.

In a similar vein to [4], [6] leads off with the well-known expression of the field equations in the harmonic gauge in Cartesian coordinates

\[
\partial_k \partial^k h^{ab} = \frac{16\pi G}{c^4} \tau^{ab} \tag{22}
\]

where \( \tau \) includes the energy-momentum tensor of the matter plus the non-linear terms

\(^{12}\)This includes the fact that the Kretschmann scalar approaches zero for such solutions.
in the Einstein equations. The standard description of the post-Newtonian perturbation scheme is invoked to conclude that the solution to the galactic problem must be the usual Newtonian one and that all corrections must be of higher order. However, what is unappreciated in the argument of [6] is that firstly we are not using this scheme, as we discussed above in conjunction with [4], and secondly, that for gravitationally bound systems, the metric components are of different orders in $G$. Thus, the structure of our solution does not proceed as in (22). In the latter, the lowest order base solution is the Newtonian solution whereas in the galactic problem, the lowest order equation for the density, (9), has non-linear terms in the metric in the form of the squares of the derivatives of $N$. Thus, our situation is unlike standard iterative perturbation scheme applications as envisaged in [6]. Hence there is no basis to draw the conclusions that are expressed therein.

It is particularly illuminating to compare our model with the Newtonian Mestel disk model [7] [8]. In the latter, Mestel uses a solution of (10) in the form of functions used in (13). However, in this Mestel case, $\Phi$ is the Newtonian potential and hence his solution in this context represents the physical condition of globally vanishing volume density $\rho$ by virtue of the Poisson equation of Newtonian gravity. There is mass present as

\footnote{Just as one would not logically choose Cartesian coordinates in the harmonic gauge to describe FRW cosmologies, one would not normally choose these for our stationary axially symmetric galactic problem. Our problem is greatly simplified with cylindrical polar coordinates co-moving with the matter. However, if one were to take the route as suggested in [6], the equations could be schematically expressed as

$$\nabla^2 h_{(1/2)} = 0, \quad \nabla^2 h_{(1)} = GT + h^2_{(1/2)}$$

where tensorial superscripts have been suppressed and the lower case numbers refer to orders in $G$. In this manner, we would have incorporated the non-linear structure of our system within the framework of the scheme suggested by [6]. The novel aspect is that the lowest order equation (of order $G^{1/2}$) has zero on the RHS and the second equation that would normally be the Newtonian Poisson equation, differs in that it has non-linear terms.}

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evidenced by the non-trivial potential that was selected. However, there is no place for it to reside except on the *singular physical layer* at \( z = 0 \). This is reminiscent of the situation with the Schwarzschild shell.

We contrast the Newtonian Mestel model with our general relativistic model. In both cases, solutions of the same type are chosen for (10). In the former, this implies vanishing volume mass density and hence an unambiguously physical surface mass layer. However, in the latter, a solution \( \Phi \) of (10) does not imply a vanishing volume density \( \rho \).

Our harmonic function plays a different role via general relativity from that of Mestel’s harmonic function in Newtonian gravity theory. It is useful to think of general relativity as having the effect of opening up the Mestel disk, spreading out the mass continuously and symmetrically about the \( z = 0 \) plane. By the choice of \(|z|\) functions, the gradient of this dispersal is necessarily discontinuous at the symmetry plane \( z = 0 \). However, the mass is dispersed: the surface discontinuity that is present in this case is not necessarily logically interpreted as an independent additional mass contribution. Indeed, let us examine this contribution more critically. Using the \( \mathbf{F} \) vector, we now evaluate the supposed mass that is harboured within the \( z = 0 \) plane as seen in Figure 1. We do so by integrating the divergence of \( \mathbf{F} \) over the volume from \( z = -\epsilon \) to \( z = +\epsilon \). As we see in the Appendix, the fact that the volume mass contributions that exclude the \( z = 0 \) plane are positive both for \( z > 0 \) and \( z < 0 \) individually, the contribution from the surface layer is actually *negative* 14 and integrations for the individual galaxies studied show the magnitudes to be close to those of the volume integrals 15. In these results, we see

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14In a private communication, W.B. Bonnor had informed us that he had deduced that the mass layer would be negative. We had come to this conclusion independently via the Gauss theorem. In fact Bonnor had conjectured that the singularity at the origin of his solution in [2] had negative mass.

15In fact, trial integrations for functions that are made to vanish at the \( r \) outer boundaries lead to
two things: firstly, we see that the surface layer of mass is intimately connected to the volume integral of the clearly physical continuous mass density. Secondly, we see that if we were to pursue this to its logical conclusion with a refined distribution of density that tapers off to zero for very large \( r \), we would be left with the patently false conclusion that the net galactic mass is zero.

We would argue that the correct interpretation is this: in the case of a physically simulated continuous density distribution having a gradient discontinuity as we have here, the \( S_i^k \) tensor reflects this discontinuity and it does not necessarily signal additional mass or stress or angular momentum.

Indeed this interpretation becomes more compelling when viewed as follows: suppose we were to begin in the more traditional physical manner, posit the energy-momentum distribution via the energy-momentum tensor and seek to solve the problem by solving for the corresponding metric. Let the \( T^{00} \) component (i.e. \( \rho \)) be given as

\[
(8\pi G/c^2)\rho = \frac{1}{r^2} \left\{ \left( 0.00126r J_1(0.0687r)e^{-0.0687|z|} + \cdots \right)_z \right\}^2 + \frac{1}{r^2} \left\{ \left( 0.00126r J_1(0.0687r)e^{-0.0687|z|} + \cdots \right)_r \right\}^2
\]

(23)

After differentiating and squaring, this leads to a well-defined, finite, positive and continuous function for all points including the \( z = 0 \) plane. Indeed we take our density to have this value for all \((r, z)\), even for \( z = 0 \). Since we are dealing with a pressureless fluid that is co-moving with the reference frame, all the remaining components of \( T^{ik} \) are zero. As a result, the equation for \( T^{02} \) yields \( \mathbb{I} \) everywhere, including the \( z = 0 \) plane. This perfect agreement between the galactic volume integrals and the absolute values of the surface integrals, as we would expect from the Gauss theorem.

\(^{16}\)For the purpose of illustration, we have used the numbers appropriate for the Milky Way.\[ \mathbb{I} \]
plane. From (8), this yields

\[
\frac{N_r^2 + N_z^2}{r^2} = \frac{8\pi G \rho}{c^2}
\]

as in (9) where the \( \rho \) is the posited one of (23). Clearly the solution is as before only through this route, we do not have an additional surface layer of matter. In effect, this is the approach that we followed in [1]. The difference is that in [1], we approached the problem from the opposite direction, but the intent was the physical model as developed here.

By contrast, we could have followed a variation of this approach as pursued, in effect, in [1] [5] [10]. We could have posited \( \rho \) as in (23) for \( z \neq 0 \) and having a singular layer at \( z = 0 \). In this case, we would have taken \( N_{zz} \) as a delta function for \( z = 0 \) and the component \( T^{02} \) would have had a delta function layer as well at \( z = 0 \). In this case, (8) would have been the operative equation for the density, now displaying the layer through the \( N_{zz} \) term now present. However, this leads, as we have seen in the Appendix, to the production of a negative density layer with almost the same numerical value as the positive mass continuous contribution.

As Bondi had noted in his writings, negative mass repels all masses, whether they be positive or negative. Thus, this enormous storage of negative mass on the plane could not maintain the envisaged stationary distribution and would blow itself apart. This underlines the problem associated with assuming that the \( S^k_l \) tensor will necessarily indicate a viable expression for physical mass in all situations. In our case, it leads to an untenable model yet at its source, the structure actually arises in physical terms from a rather benign density gradient discontinuity.

In this regard, it is of interest to consider the behaviour of a test particle having the
angular velocity and radial velocity of the bulk fluid but with a non-vanishing transverse velocity $dU^z/ds$ in the co-moving frame. The geodesic equation in the $z$ direction reduces to

$$\frac{dU^z}{ds} = \frac{N_r N_z (U^z)^2}{2r}$$  \hspace{1cm} (25)

We computed the complete $N$ series for NGC7331 for $r = 0.1$ to 30 and $z = 0.001$ to 1 for the right hand side of (25). All of the points gave a negative value as expected for the acceleration of a particle approaching a normal $z = 0$ boundary from above. However, if the $z = 0$ surface actually harboured a physical negative mass surface layer, indeed one of numerical value comparable to the positive mass of the normal galactic distribution, then one would have expected to witness a violent repulsion of the particle as the test particle approached the boundary. The absence of this occurrence adds further support to the integrity of our original model \[1\]. \[17\]

\[17\]This is assuming that a mechanism could be found to prevent the layer from exploding.

\[18\]In an interesting recent paper \[14\], the motion of a test particle for a different type of distribution was analyzed in the locally non-rotating frame produced via the transformation in \[14\]. In this case, the $z$ geodesic equation is dominated by $\Gamma^z_{00}$ for non-relativistic particles. In \[14\], this term indicated that $dU^z/ds$ was positive for $z > 0$ and hence implied an apparent repulsion of the test particle. However, the geodesic equation should apply to particles of the dust itself since they are geodesic. These particles are at rest in the original frame and have only tangential velocity in the locally non-rotating frame. The local transformation should not alter their having no $z$ velocity yet there is an apparent $z$ acceleration.

This is a contradiction which is resolved as follows: while we can use the local transformation to derive the local angular velocity (and hence tangential velocity) of the particles, it is not legitimate to take derivatives of the metric found after the local transformation has been applied in order to derive acceleration. The former usage simply reads off the required angular velocity to diagonalize the metric locally. No differentiation is required to do so. However, the latter usage would be legitimate only if the transformation would have been effected without constraints, in this case the constraint of holding $r$ and $z$ fixed. By holding these fixed to derive the new $ds^2$, we have metric tensor components that are correct as such only through a different transformation at each point. Hence there is an inherent discontinuity of transformation. In this manner, we recognize the derivative required to find $dU^z/ds$ as being illegitimately applied, thus resolving the contradiction.

This also brings into question the interpretation of the lack of a $1/r$ term in $g_{00}$ in the locally non-rotating frame for the same reason. The $1/r$ issue is a global one yet the transformation that brought the metric to the form that is being used is a purely local one. By contrast, no such problem arises
It is to be emphasized that the entire issue of singularities arose simply because of the wish to maintain the simplicity of solution with a sequence of exponential functions. The only irregularity that this necessarily entails is within the density gradient and not necessarily within the density itself. We have shown that even this irregularity can be averted by the introduction of a cosh series. However, this leads to the challenge of matching interior and exterior regions mathematically. It should be stressed that while this is mathematically challenging, the actual physical distinction is minimal: the density gradient discontinuity, a mathematical idealization, is not present physically. In actual physical terms, the density is necessarily rounded out to some extent, however abruptly. It is never infinitely sharp.

5 Summary and Concluding Remarks

In [1], we had modeled a galaxy as an axially symmetric pressureless stationary rotating fluid within the framework of general relativity to order \( G \). We had shown that the dynamics was driven by one linear and one non-linear equation as opposed to the linear equation of Newtonian gravity. A framework of solutions with separated variables was established as a sequence of Bessel functions. This enabled us to choose appropriate parameters to fit the flat galactic rotation curves without invoking massive halos of exotic dark matter as is required using Newtonian gravity. The masses were concentrated

\[ \text{in studying test particle motion relative to the dust co-moving frame with the particle following the motion of the dust cloud apart from a } z \text{ velocity component. The result is logical: for } U^r \text{ and } U^\phi \text{ being zero, acceleration occurs only for } U^z \text{ different from zero (otherwise it would be part of the dust cloud and hence stationary) and the acceleration is independent of the sign of the velocity. Moreover, the acceleration is negative for } z > 0 \text{ and positive for } z < 0. \text{ Thus, it is attracted} \]

\[ \text{to the central plane in both cases.} \]

We thank Professor W.B. Bonnor for bringing his paper to our attention.
primarily within the disk configuration. The mass values were found to be between those of Newtonian gravity and those predicted by the MOND model.

The separated variable approach led to exponential dependence in the transverse $z$ variable and reflection symmetry implied a discontinuity in density gradient at the symmetry plane. Critics had claimed that this necessitated an accompanying singular mass layer on this plane.

In this paper, we approached the problem in two ways: firstly, we took the normal route of specifying the source as a singularity-free density distribution with density gradient discontinuity and saw that this led to the solution that we had used in [1]. The key was the observation that the field equations only dealt with $z$ derivatives in a form that led to unique limits as one approached the symmetry plane from above or below. This enabled us to specify the values of the functions on the plane as the limit as the plane was approached, from above or below.

Secondly, we bridged the region of the symmetry plane where the density gradient was discontinuous by using a new solution there, one that was symmetric and smooth, in fact infinitely differentiable. We showed that this solution could be metric and metric derivative matched to the original satisfactory exponential fall-off exterior solution.

We also considered the consequences of not demanding a natural non-singular density distribution as was invoked in [1] [5] [10]. In this case, we found that the mass of the layer was necessarily connected to the mass of the continuum by the Gauss divergence theorem. In turn, this implied a negative mass in the layer that numerically approximated the positive mass of the continuum. Such an enormous negative mass would contradict the assumed stationarity of the model. Finally, we considered the motion of a test mass.
approaching the $z = 0$ plane, thoroughly searching the domain for any sign of repulsion as would be implied by the supposed vast store of negative mass in the $z = 0$ plane (assuming that the matter in the plane could somehow actually hold itself together). None was found in contradiction to the layer assumption.

In future work, it will be of interest to refine the process further with finer resolution of sequence solutions and with applications to more galaxies. It would also be of interest to explore the untapped area of non-separable solutions with their singular cores. These might be particularly useful should it be thoroughly established that galactic cores harbour physical singularities, be they clothed or naked.

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6 Appendix

From (17) and the Gauss theorem,

\[ \int \nabla \cdot \mathbf{F} \, dV = \int \mathbf{F} \cdot \mathbf{n} \, dS = \int \rho \, dV. \] (26)

Concentrating first on the upper cylinder, the surface integral contribution at \( z = +\infty \) gives zero because of the dependence on exponential functions. The contribution from the circular cylinder wall is small and would be totally negligible with more data points leading to an extension of the galactic model to very large \( r \). The key element to consider is the lower surface of the cylinder. The outward normal is \( \mathbf{n} = -\mathbf{e}_z \).

From (26), we have

\[ M_{\text{upper}} = \int B(r, \epsilon) \mathbf{e}_z \cdot (-\mathbf{e}_z) \, dS = -B(\epsilon)S \] (27)

where \( S \) is the lower surface area and \( z = +\epsilon \) on the lower face.

Similarly, for the lower cylinder, we have

\[ M_{\text{lower}} = +B(-\epsilon)S \] (28)

where \( z = -\epsilon \) on the upper face of the lower cylinder. By symmetry, \( M_{\text{upper}} = M_{\text{lower}} \). And since \( \rho \) is positive over the continuum, it follows that

\[ B(\epsilon) < 0, \quad B(-\epsilon) > 0, \quad B(-\epsilon) = -B(\epsilon) \] (29)

Now we perform the standard pill box calculation to deduce the mass of the surface layer when we do not impose the physical requirement that the density be taken as the limit value as in (23). The upper surface of the pill box (see Figure 1) is the same as the lower surface of the upper cylinder and this has unit outward normal \( +\mathbf{e}_z \). Hence the mass contribution is \( B(\epsilon)S \) which is negative. Similarly, the lower surface of the pill box gives a contribution \( -B(-\epsilon)S \) which, by (18) is also negative. Thus, the surface layer mass is deduced to be the negative of the continuum mass apart from a small contribution at the cylinder wall.