Quantum memories with zero-energy Majorana modes and experimental constraints

Matteo Ippoliti, Matteo Rizzi, Vittorio Giovannetti, and Leonardo Mazza

1 Department of Physics, Princeton University, Princeton, NJ 08544, USA
2 Scuola Normale Superiore, piazza dei Cavalieri 7, I-56126 Pisa, Italy
3 Università Mainz, Institut für Physik, Staudingerweg 7, D-55099 Mainz, Germany
4 NEST, Scuola Normale Superiore and Istituto Nanoscienze-CNR, piazza dei Cavalieri 7, I-56126 Pisa, Italy
5 Département de Physique, Ecole Normale Supérieure / PSL Research University, CNRS, 24 Rue Lhomond, 75005 Paris, France

In this work we address the problem of realizing a reliable quantum memory based on zero-energy Majorana modes in the presence of experimental constraints on the operations aimed at recovering the information. In particular, we characterize the best recovery operation acting only on the zero-energy Majorana modes and the memory fidelity that can be therewith achieved. In order to understand the effect of such restriction, we discuss two examples of noise models acting on the topological system and compare the amount of information that can be recovered by accessing either the whole system, or the zero-modes only, with particular attention to the scaling with the size of the system and the energy gap. We explicitly discuss the case of a thermal bosonic environment inducing a parity-preserving Markovian dynamics in which the memory fidelity achievable via a read-out of the zero modes decays exponentially in time, independent from system size. We argue, however, that even in the presence of said experimental limitations, the Hamiltonian gap is still beneficial to the storage of information.

PACS numbers: 03.65.Yz, 03.67.Pp

I. INTRODUCTION

Notwithstanding the impressive progress in the experimental control of genuine quantum systems [1–9], the reliable storage of quantum information over long times remains a problem of exceptional difficulty [10–12]. Among the ideas which have been put forward to solve this issue, proposals based on systems featuring topological order are among the most intriguing [13–15]. Central to these protocols is the encoding of one or more qubits in the degenerate ground states that appear in these models. Protection follows from the fact that such ground states are locally indistinguishable, and the information can be manipulated through some localized boundary modes, which may even feature non-Abelian statistics.

Due to their experimental relevance, topological superconductors featuring localized and unpaired Majorana modes have attracted considerable attention [16, 17]. Indeed, several reports accounting for the observation of experimental signatures of zero-energy Majorana modes have been recently published [18–22]. At the same time, many proposals have discussed the possibility of observing them in cold atomic gases [23–31]. The problem of assessing the robustness of quantum information protocols based upon these systems is now becoming of the highest relevance, and several works have already discussed many aspects of the problem [32–46].

In this work we consider the following scenario: the experimentalist is able to create and cool a topological superconductor, but can only manipulate the degrees of freedom related to the Majorana fermions which appear in the system. This assumption is motivated by realistic implementation proposals for Majorana qubits and quantum memories [47–50], and seems to be the mildest for discussing the near-future developments in the field. On one hand, experiments on many-body quantum systems have so far shown control over only a limited number of quantum degrees of freedom. On the other hand, if some quantum information has to be manipulated, it seems inevitable to assume some degree of control over the zero-energy Majorana modes; for the sake of simplicity, here we assume this control to be total. The upper bounds we derive apply a fortiori to more realistic situations of limited and imperfect control over the zero-modes. The question we address is the following: how can one realize a reliable quantum memory in the presence of such constraints?

Assuming that the noise model is local, we explicitly characterize the best recovery operation [36] to be applied after the action of noise and determine the associated fidelity, both in general and under the aforementioned constraints. In the latter case, which for clarity we dub decoding rather than recovery, we present an explicit recipe to be used in experiments in order to achieve the optimal fidelity. It is worth stressing that in this work we do not consider possible protection operations to be applied during the action of noise, but rather consider noise as given and optimize the read-out.

In order to understand the effect of the experimental limitations, we discuss examples where an appropriate recovery operation acting over the whole topological system can retrieve the encoded information with 100% fidelity even if some form of noise has affected the system. We show that a limited access to the Majorana modes of the system can compromise such possibility but we argue...
that in such cases information may still be protected by the presence of a significant energy gap in the topological Hamiltonian.

While the lack of protection of information in the constrained “decoding” case is in line with the general paradigm of quantum error correction, such behavior defies the expectation that parity preservation should automatically protect information by making the Majorana modes immune to errors in the first place, which would make quantum error correction unnecessary. Our study highlights the fact that a control over the whole many-body system is indeed usually necessary.

Our study comprises two other additional results which we want to highlight here. The first is about a topological system immersed in a thermal bosonic environment which induces a Markovian dynamics that does not change the parity of the number of fermions of the system. Notwithstanding this requirement, we find that the auto-correlation of a logical Pauli operator decays exponentially in time, with a time-scale given by the microscopic details of the system-environment coupling.

The latter result comprises some exact relations for the case of a topological qubit encoded into two fully-decoupled Kitaev chains which are perturbed by a noise that conserves the parity of the number of fermions in each chain. In particular, we characterize the maximal amount of information which can still be recovered from the system after the action of the noise, thus generalizing the results in Ref. [36].

The work is organized as follows. In Sec. II we present some introductory remarks about the encoding of a qubit in Majorana zero-modes [34, 44]. We consider a generic topological superconductor with $N$ fermionic modes, without any assumption about the setup geometry. We assume that the Hamiltonian is quadratic in the fermionic fields, and that the eigenmodes of the Hamiltonian include a group of four Majorana zero-modes $\{\hat{m}_i : i = 1, 2, 3, 4\}$ that satisfy the Clifford algebra $\{\hat{m}_i, \hat{m}_j\} = 2\delta_{i,j}$ and are localized very far away from one another, enough to make their overlap negligible for all practical purposes. The remaining fermionic operators are assumed to be arranged in gapped eigenmodes represented by $N - 2$ canonical annihilation (creation) Fermi operators $\{\hat{b}_i^{\dagger} : i = 1, \ldots, N - 2\}$ which satisfy $\{\hat{b}_i, \hat{b}_j\} = \delta_{i,j}$ and $\{\hat{b}_i, \hat{b}_j\} = 0$. Sometimes it will be convenient to use the related Majorana operators: $\hat{\gamma}_{i,1} = \hat{b}_i + \hat{b}_i^\dagger$ and $\hat{\gamma}_{i,2} = -i(\hat{b}_i - \hat{b}_i^\dagger)$. Analogously, the Majorana zero-modes can be algebraically combined to form non-local Dirac zero-modes $\hat{g}_0 \equiv \frac{1}{2}(\hat{m}_1 + i\hat{m}_2)$, $\hat{d}_0 \equiv \frac{1}{2}(\hat{m}_3 + i\hat{m}_4)$. At this level, the pairing choice is entirely arbitrary.

A qubit can be stored in a well defined parity sector of the zero-energy two-fermion system, e.g. the two-dimensional space spanned by the logical vectors $|0_L\rangle = |\Omega\rangle$, $|1_L\rangle = \hat{d}_0^\dagger|\Omega\rangle$. Here $|\Omega\rangle$ is the vacuum state of the fermionic system: $\hat{b}_i|\Omega\rangle = 0$ for $i = 1, \ldots, N-2$, $\hat{g}_0|\Omega\rangle = \hat{d}_0|\Omega\rangle = 0$. We remark that four Majorana modes are necessary to store a non-trivial qubit: with a single pair of Majorana modes (i.e. a single “bi-localized” Dirac mode) one can only encode a classical bit, as the fermionic parity superselection rule forbids coherent superpositions of the two degenerate ground states.

Denoting by $\Pi_G$ the projector on the ground-space of the system, $\Pi_G \equiv \prod \hat{b}_i^\dagger \hat{b}_i$, our prescription for the qubit encoding yields the following logical operators:

$$
\begin{align*}
\hat{\sigma}_0^L &= \frac{1}{2}(\mathbf{1} - \hat{m}_1\hat{m}_2\hat{m}_3\hat{m}_4)\Pi_G, \\
\hat{\sigma}_1^L &= \frac{i}{2}(\hat{m}_2\hat{m}_3 + \hat{m}_1\hat{m}_4)\Pi_G, \\
\hat{\sigma}_2^L &= -\frac{i}{2}(\hat{m}_1\hat{m}_3 - \hat{m}_2\hat{m}_4)\Pi_G, \\
\hat{\sigma}_3^L &= -\frac{i}{2}(\hat{m}_1\hat{m}_2 + \hat{m}_3\hat{m}_4)\Pi_G,
\end{align*}
$$

which play the role of effective Pauli operators for our encoded qubit. The $\hat{\sigma}_j^L$s can be rewritten in a more compact way by introducing the fermionic parity operator $\hat{P}_f^G$ for the ground-space,

$$
\hat{P}_f^G \equiv \frac{1}{2}(1 + \hat{P}_f^G)\Pi_G
$$

and letting $\hat{\Pi}_G^+ \equiv \frac{1}{2}(1 + \hat{P}_f^G)\Pi_G$ denote the projector onto the even parity sector of the ground space. Then, Eq. (1) re-writes as

$$
\begin{align*}
\hat{\sigma}_0^L &= \hat{\Pi}_G^+, \\
\hat{\sigma}_1^L &= \hat{\Pi}_G^+(i\hat{m}_2\hat{m}_3), \\
\hat{\sigma}_2^L &= \hat{\Pi}_G^+(-i\hat{m}_1\hat{m}_3), \\
\hat{\sigma}_3^L &= \hat{\Pi}_G^+(-i\hat{m}_1\hat{m}_2),
\end{align*}
$$

where $\hat{\Pi}_G^+$ commutes with the $\hat{m}_i\hat{m}_k$ monomials. This formalism makes the Pauli algebra more evident. It is worth stressing that the projector cannot generally be dropped in the qubit encoding operators $\{\hat{\sigma}_j^L\}$; indeed, an initial qubit encoding which goes as $\hat{\sigma}_1^L \rightarrow i\hat{m}_2\hat{m}_3$ et cetera physically corresponds to a fully mixed (i.e.,
infinite-temperature) state of the gapped modes, which also has mixed fermionic parity, and is a sub-optimal encoding choice.

From now on we assume that the experimentalist is able to create one such topological superconductor and to cool it to one of the two ground subspaces with fixed fermionic parity, e.g. $\Pi^0_G$. We further assume that via control over the Majorana modes $\hat{m}_i$, it is possible to initialize the system in a desired pure state $\hat{\rho}(0) = \frac{1}{2} (\hat{\sigma}^x_L + \vec{v} \cdot \vec{\sigma}^L)$, where $|\vec{v}| = 1$. Note that we can write $\hat{\rho}(0) = \hat{\rho}_0 \Pi_G$, where $\hat{\rho}_0$ is a function of the zero-modes $\{\hat{m}_i\}$ only.

As already mentioned, here and in the following we neglect the energy splitting which is associated to the overlap of the wavefunctions of the Majorana zero-modes [46]. This energy splitting has the consequence that the localized Majorana modes are not exactly eigenmodes of the system and the logical operators in Eq. (1) evolve in time. However, since the Majorana wavefunction decays exponentially, the energy splitting scales exponentially in the distance between the zero modes, and it can be made arbitrarily small by increasing the system size. The time-scale associated to the evolution of the logical operators can be made arbitrarily long and in this article we consider the physical situation in which it can be safely neglected.

### III. DYNAMICS UNDER NOISE AND DECODING OF INFORMATION

Once the qubit is initialized, the system is set free and noise and perturbations start acting on it. This drives the setup out of equilibrium by inducing a non-trivial and non-unitary dynamics, which may have disruptive effects on the encoded information. In order to describe these disturbances, we introduce the decoherence channel $\mathcal{D}_t$, which returns the state of the global system at time $t$: $\hat{\rho}(t) = \mathcal{D}_t(\hat{\rho}(0))$. The action of $\mathcal{D}_t$ includes the coherent evolution ruled by the topological Hamiltonian but does not entail any other additional passive protection mechanism. For each value of $t$, $\mathcal{D}_t$ is a quantum channel, namely a completely-positive trace-preserving (CPTP) map on the states of the system [53, 54]. This class of linear maps is known to describe the most general physical transformations of quantum states, so that our formalism is completely general. One additional physical requirement that we pose is that the dynamics must originate from a system-environment interaction that is local in space.

After a time $t$, the experimentalist attempts to “decode” the qubit, i.e. to recover the initially encoded information by reading only the four Majorana zero-modes. This limitation is physically relevant, since any protocol that allows the encoding in Eq. (1) implies the ability to control the zero-modes; on the other hand, a generic global recovery operation may be significantly harder (or technically impossible) to carry out.

Since the decoding operation can only involve the zero-modes, the full decoherence channel $\mathcal{D}_t$ contains much more information than what is necessary for our purposes. As we shall see below, we only need to know its restriction to the ground space, $\mathcal{D}_t$, which is obtained by tracing out the gapped modes $\{\hat{b}_i\}$:

$$\mathcal{D}_t(\hat{\rho}_0) = \text{tr}_{\{\hat{b}_i\}} [\mathcal{D}_t(\hat{\rho}(0))] = \text{tr}_{\{\hat{b}_i\}} \left[ \mathcal{D}_t(\hat{\rho}_0 \Pi_G) \right].$$

(4)

The trace over the gapped modes of an operator $\hat{O}$ can be computed as follows: (i) expand $\hat{O}$ over the Hilbert-Schmidt orthogonal basis given by the monomials $\hat{M}_{\alpha\beta}$ in all the Majorana modes of the system, defined as follows:

$$\hat{M}_{\alpha\beta} = \hat{m}_1^{\alpha_1} \cdots \hat{m}_4^{\alpha_4} \hat{\gamma}_{1,1}^{\beta_{1,1}} \hat{\gamma}_{1,2}^{\beta_{1,2}} \cdots \hat{\gamma}_{N-2,1}^{\beta_{N-2,1}} \hat{\gamma}_{N-2,2}^{\beta_{N-2,2}},$$

(5)

where the $\alpha_i$ and $\beta_{i,j}$ coefficients can take values 0 or 1; (ii) discard the monomials which contain at least one of the $\hat{\gamma}_{i,j}$; (iii) multiply all the remaining monomials by $2^{N-2}$, which is the trace of the identity operator on the space of states of the gapped $\{\hat{b}_i\}$ modes.

Let us now motivate the introduction of the $\mathcal{D}_t$ channel. A general qubit recovery operation is a channel which maps a state of the superconductor $\hat{\rho}$ to an abstract qubit state; it generally reads (see e.g. [36]):

$$\mathcal{R}(\hat{\rho}) = \frac{1}{2} \left( \sigma_0 + \sum_{i=1}^{3} \text{tr} \left[ \hat{\rho} \hat{H}_i \right] \sigma_i \right).$$

(6)

The qubit is defined in terms of the abstract Pauli matrices $\sigma_i$, $i = 1, 2, 3$ and of the identity $\sigma_0$, while the three coefficients which characterize the components of the Bloch vector are related to the measurement of three properly-selected (Hermitian) observables $\hat{H}_i$ (the request that $\mathcal{R}$ is a CPTP map imposes constraints on the $\{\hat{H}_i\}$) [36, 65]. The physical constraint that the experimentalist can only manipulate the zero-energy modes $\{\hat{m}_i\}$ translates mathematically into the fact that the $\{\hat{H}_i\}$ cannot be generic operators on the whole system; instead they must be polynomials in the zero-modes only.

One thus gets

$$\text{tr} \left[ \hat{H}_i \mathcal{D}_t(\hat{\rho}_0 \Pi_G) \right] = \text{tr}_{\{\hat{m}_i\}} \left[ \hat{H}_i \text{tr}_{\{\hat{b}_i\}} \left( \mathcal{D}_t(\hat{\rho}_0 \Pi_G) \right) \right]$$

$$\doteq \text{tr}_{\{\hat{m}_i\}} \left[ \hat{H}_i \mathcal{D}_t(\hat{\rho}_0) \right],$$

(7)

so that $\mathcal{D}_t(\hat{\rho}_0)$ becomes the relevant object to study under this constraint. The goal of this Section is to characterize the three observables $\hat{H}_i$ which define the optimal operation $\mathcal{R}$ in Eq. (6) within the class of decodings.

#### A. Constraints on the dynamics from locality and preservation of fermionic number parity

In the Hamiltonian setting, the existence of Majorana zero-modes is linked to the fact that perturbations (i)
are local and (ii) do not break the symmetry of the parity of the number of fermions \cite{55, 56}. It has often been suggested that similar constraints should also guarantee the robustness of information encoded into such modes \cite{57, 58}, although limitations have been pointed out \cite{46}. Leaving for later the discussion on the symmetry of the perturbation, let us here focus on the properties that a noise model should satisfy in order to be considered local.

In order to introduce the concept of locality in our discussion, we shall consider decoherence channels which satisfy a Lieb-Robinson bound (LRB) for all pairs of fermionic (i.e. odd-degree monomials in the fermionic fields and combinations thereof) operators with support in distant regions $A$, $B$,

$$\{D^*_t(\hat{O}_A), \hat{O}_B\} \simeq 0,$$  

and the clustering property

$$D^*_t(\hat{O}_A \hat{O}_B) \simeq D^*_t(\hat{O}_A) \cdot D^*_t(\hat{O}_B),$$

where $D^*_t$ is the adjoint channel which describes time-evolution in the Heisenberg picture. From now on, the approximation sign means that equality only holds up to LRB corrections, and thus in particular only for times $t \ll d_{AB}/v$, where $d_{AB}$ is the distance between regions $A$ and $B$ and $v$ is an effective group velocity. Intuitively, Eq. (8) and (9) imply that operators with distant, non-overlapping supports will take a long time to develop significant overlap and correlations.

The time interval within which the LRB corrections are negligible can be made arbitrarily long by increasing the size of the system and the distance between zero-modes $d_{AB}$. As a consequence, the approximation can be made arbitrarily accurate. The following results on the dynamics of the Majorana modes will be derived within this framework and thus are valid only up to a finite time. These points are quantitatively clarified in Appendix A.

Properties (8) and (9) can be used to prove that $\tilde{D}_t$ is approximately diagonal in the basis given by the monomials \{$M_\alpha \equiv \prod_{j=1}^4 \hat{m}_j^{\alpha_j}$, $\alpha \in \{0, 1\}^4$\}. In particular, we will show that:

$$\tilde{D}_t(\tilde{m}_i) \simeq \lambda_i(t) \cdot \tilde{m}_i; \quad i = 1, 2, 3, 4;$$

$$\tilde{D}_t(M_\alpha) \simeq \prod_{j=1}^4 \lambda_j^{\alpha_j}(t) \cdot M_\alpha; \quad \alpha \in \{0, 1\}^4,$$

up to LRB corrections, where $\lambda_j(t)$ are time-dependent scalar quantities. The result identifies the constraints on the dynamics of the ground states imposed by (i) the locality of the noise and (ii) the localization of the Majorana modes. As an interesting corollary, the average parity in the ground-space is simply estimated by

$$\langle \tilde{P}_f^G \rangle_t = tr[\tilde{P}_f^G \hat{\rho}(t)] = \prod_{i=1}^4 \lambda_i(t).$$

Moreover, the proof will return the proper definition of the adjoint channel:

$$\tilde{D}_t^*(\hat{O}) = tr[\hat{b}_4^\dagger \tilde{D}_t^* \hat{b}_4]$$

[notice the different position of the projector $\tilde{P}_f^0$ in definitions (4) and (12)]. The proof follows below; the reader uninterested in the technicalities can go directly to Sec. III.B.

Proof of Eqs. (10) and (11). In order to present the demonstration, we shall introduce the adjoint (or Heisenberg-picture) reduced channel $\tilde{D}_t^*$: if $\hat{A}$ and $\hat{B}$ are polynomials in the zero-modes, then

$$\tilde{D}_t^*(\tilde{m}_i) = tr[\hat{b}_4^\dagger \tilde{D}_t(\hat{A}) \tilde{b}_4],$$

$$\tilde{D}_t^*(\tilde{m}_i) \simeq \tilde{m}_i \cdot tr[\hat{b}_4^\dagger \tilde{D}_t^*(\tilde{b}_4)],$$

which implies the definition in Eq. (12).

Let us now prove Eq. (10a): the LRB (8) implies that $D_t^*(\tilde{m}_i) \simeq \tilde{m}_i \cdot \tilde{Q}_j(t) + \tilde{Q}_j(t)$, where $\tilde{Q}_j(t)$ and $\tilde{Q}_j(t)$ are polynomials in the gapped modes supported (approximately) in the region of radius $vt$ centered around $\tilde{m}_i$. Notice that, since $D_t$ is a physical evolution, $D_t^*(\tilde{m}_i)$ must be a fermionic operator, i.e. a linear combination of odd-degree monomials. This forces $\tilde{Q}_j(t)$ to be bosonic (i.e. even-degree) and $\tilde{Q}_j(t)$ to be fermionic. Therefore,

$$\tilde{D}_t^*(\tilde{m}_i) = tr[\hat{b}_4^\dagger \tilde{D}_t^*(\tilde{b}_4)] \simeq \tilde{m}_i \cdot tr[\hat{b}_4^\dagger \tilde{D}_t^*(\tilde{b}_4)] + \tilde{Q}_j(t),$$

where we have used that $tr[\hat{b}_4^\dagger \tilde{D}_t^*(\tilde{b}_4)] = 0$ because $\tilde{D}_t^*(\tilde{b}_4)$ is a fermionic operator, and thus is traceless.

By the same reasoning, applying the clustering property (9), one sees that $D_t^*(M_\alpha) = \Gamma_\alpha(t) M_\alpha$, where $M_\alpha$ are monomials in the zero-energy modes only [see Eq. (5)]. The eigenvalue $\Gamma_\alpha(t)$ is given by

$$\Gamma_\alpha(t) \equiv tr[\hat{b}_4^\dagger \tilde{D}_t^*(\tilde{b}_4)].$$

Let us recall that $\tilde{P}_f^G = \prod_i \hat{b}_i \hat{b}_i^\dagger$ is the ground-state projector: since the system (with the exception of the zero-modes) is gapped, correlations drop exponentially, and thus

$$\Gamma_\alpha(t) \simeq \sum \Gamma_\alpha(t).$$

Recalling that $tr[\hat{b}_4^\dagger \tilde{D}_t^*(\tilde{b}_4)] = \tilde{Q}_j(t)$, this yields $\Gamma_\alpha(t) = \langle \tilde{Q}_j(t) \rangle^{\alpha_1} \cdot \langle \tilde{Q}_j(t) \rangle^{\alpha_2}$ and proves that the reduced channel $\tilde{D}_t^*$ inherits the clustering property (9) from the full channel $D_t$: $\tilde{D}_t^*(\prod_i \tilde{m}_i) \simeq \prod_i \tilde{D}_t^*(\tilde{m}_i)$.

Finally, since the $\{M_\alpha\}$ monomials form a basis of the space of operators on the zero-modes and are Hilbert Schmidt orthogonal, we have that $D_t^*$ is (up to LRB corrections) diagonal in an orthogonal basis, and thus self-adjoint. Therefore all the conclusions we derived for $\tilde{D}_t^*$ apply equally to $\tilde{D}_t$. 
Using this result and the definition of the fermionic parity operator in Eq. (2), it is now easy to prove the corollary (11):

\[ \langle \tilde{P}_{f}^{G} \rangle_{t} = \text{tr}[\tilde{\rho}(t)\tilde{P}_{f}^{G}] = \text{tr}[\tilde{D}_{t}(\tilde{\rho}(0))\tilde{P}_{f}^{G}] = \text{tr}[\tilde{\rho}(0)\tilde{D}_{t}^{\dagger}(\tilde{P}_{f}^{G})] \]

\[ \simeq \prod_{i=1}^{4} \lambda_{i}(t)\text{tr}[\tilde{\rho}(0)\tilde{P}_{f}^{G}] = \prod_{i=1}^{4} \lambda_{i}(t) . \tag{16} \]

This concludes the proof. ■

B. Recovery fidelity and average parity

After a time \( t \) has elapsed, and noise and perturbations have acted on the system, the experimentalist attempts to recover the information previously encoded by applying a recovery operation (6) to the system. We define the fidelity of a recovery operation in the following way [36]:

\[ F[\mathcal{R}] = \int d\mu_{\phi} \langle \phi | \mathcal{R} \circ \mathcal{D}_{t} | \phi \rangle \langle \phi | \mathcal{R} | \phi \rangle \]

(17)

Here \( d\mu_{\phi} \) is the integration measure over the pure states of a qubit (i.e. the Bloch sphere). By optimizing over the whole set of recovery operations \( \mathcal{R} \), we define the optimal recovery fidelity, an operational measure of the total amount of information left in the system [36]:

\[ F_{\text{opt}} = \max_{\mathcal{R} \in \mathcal{R}} F[\mathcal{R}] . \tag{18} \]

In particular, given a decoherence channel, we are interested in the best recovery fidelity that can be attained under the constraint that \( \mathcal{R} \) only involves the manipulation of the Majorana zero-modes \( \{ \tilde{m}_{i} \} \). We thus limit the optimization to the set of “decoding operations” \( \mathcal{R} \) that act on the zero-modes only (see the previous discussion), and define

\[ \tilde{F}_{\text{opt}} = \max_{\mathcal{R} \in \mathcal{R}} F[\mathcal{R}] . \tag{19} \]

Note that because of the maximization procedure in (19), \( \tilde{F}_{\text{opt}} \) provides a measure of the information which can be retrieved by manipulating the Majorana zero-modes only. More information is generally available by accessing the whole system \( (F_{\text{opt}} \geq \tilde{F}_{\text{opt}}) \).

The use of \( F_{\text{opt}} \) as a figure of merit is an important difference between our discussion and previous work in which a specific physically-motivated class of recovery operations is considered (e.g. stabilizer codes, minimal weight perfect matching) [11, 12]. The optimal recovery fidelity represents a general upper limit to the performance of any possible recovery protocol, including all schemes previously considered in the literature. Thus, it might be used to test whether the known recovery operations are optimal or not, and it is ideally suited to discuss general limitations of quantum memory models. On the other hand, the recovery operation which achieves \( F[\mathcal{R}] = F_{\text{opt}} \) does not generically have an interpretation in terms of known error-correction schemes. The introduction of \( \tilde{F}_{\text{opt}} \) is exactly motivated by the problem of giving a related figure of merit to which an operation can be associated.

Let us consider a scenario in which the \( \lambda_{i}(t) \) from (14) are all equal to \( \lambda(t) \): in this simple symmetric case one can immediately compute the optimal decoding fidelity because \( \mathcal{D}_{t} \) is effectively a simple qubit channel (see Appendix B):

\[ \tilde{F}_{\text{opt}} \simeq \frac{1 + \lambda(t)^{2}}{2} , \tag{20} \]

which implies a relation between the decoding fidelity and the average parity in the ground-space: \( \tilde{F}_{\text{opt}} \simeq \frac{1}{2} (1 + \sqrt{\langle \tilde{P}_{f}^{G} \rangle_{t}}) \).

Moreover, at the end of this section we will show that this relation has a more general validity: if the \( \lambda_{j}(t) \) are distinct, the following is true:

\[ \tilde{F}_{\text{opt}} \simeq \frac{1}{2} (1 + \sqrt{\langle \tilde{P}_{f}^{G} \rangle_{t}}) , \tag{21} \]

with the approximate equality sign only holding when \( \lambda_{j}(t) = \lambda(t) \forall j \). To the best of our knowledge, this is the first mathematical relation linking the amount of information that can be stored in a system with zero-energy Majorana fermions and the parity of the fermionic number.

Note that the previous results for \( \tilde{F}_{\text{opt}} \) are true only up to LRB corrections. They are thus optimal only for times such that the effective light-cones originating from the Majorana zero-modes have not met yet. After such time, the value in Eq. (20) is not guaranteed to be achievable.

It is worth stressing that a decoherence channel \( \mathcal{D}_{t} \) that preserves the fermionic parity of the whole system \( \tilde{P}_{f} = \tilde{P}_{f}^{G} \cdot (-1)^{\sum_{i} \tilde{b}_{i}} \) does not necessarily preserve the ground-space parity \( \tilde{P}_{f}^{G} \). Even if no tunneling of individual fermions to or from the environment is allowed, transitions between the zero-modes and the gapped modes may take place. Since in general the conservation of the fermionic parity of the system \( \tilde{P}_{f} \) is conjectured to be a necessary condition for the preservation of the information, Eq. (21) points out that a limited control over the system might not be enough to benefit from it.

Nonetheless, since the Dirac modes \( \tilde{b}_{i} \) are gapped, one may argue that the transition of fermions from an excited mode \( \tilde{b}_{i} \) to e.g. \( \tilde{g}_{0} \), or vice-versa, should be strongly suppressed by the gap energy scale. In parallel, within an open system setting, as long as the noise strength or environment temperature are small compared to the gap scale, one may also envision some form of protection. In Sec. IV and V we will discuss examples where these mechanisms are highlighted.

On the contrary, if no parity conservation is enforced on the noise model, then \( \langle \tilde{P}_{f} \rangle_{t} \neq \langle \tilde{P}_{f} \rangle_{t=0} \) because e.g. of tunneling of fermions from the environment, and it is
difficult to envision protection at any level [32, 33, 35–40, 42, 43].

The proof of Eq. (21) follows below; the reader uninterested
in the technicalities can go directly to Sec. III C.

Proof of Eq. (21). For simplicity, within this proof we
use the notation \( \lambda_j \) instead of \( \lambda_j(t) \); the time-dependence
is implicitly assumed. We have, for any permutation
\( \{a, b, c, d\} \) of \( \{1, 2, 3, 4\} \), that

\[
\tilde{D}_t(i \hat{m}_a \hat{m}_b + i \hat{m}_c \hat{m}_d) \approx \lambda_a \lambda_b \hat{m}_a \hat{m}_b + \lambda_c \lambda_d \hat{m}_c \hat{m}_d . ~ (22)
\]

Now, assuming without loss of generality that the \( \lambda_j \) are
such that \( \lambda_1 \geq \cdots \geq \lambda_4 \), let us compute the recovery
fidelity for the operation \( \mathcal{R} \in \mathcal{R} \) defined in Eq. (6) with \( \hat{H} \) matrices given by:

\[
\begin{align*}
\hat{H}_1 &= \frac{i}{2}(\hat{m}_2 \hat{m}_3 + \hat{m}_1 \hat{m}_4), ~ (23a) \\
\hat{H}_2 &= -\frac{i}{2}(\hat{m}_1 \hat{m}_3 - \hat{m}_2 \hat{m}_4), ~ (23b) \\
\hat{H}_3 &= -\frac{i}{2}(\hat{m}_1 \hat{m}_2 + \hat{m}_3 \hat{m}_4). ~ (23c)
\end{align*}
\]

These matrices are related to the logical Pauli operators
introduced in Eq. (1) and describe an operation which simply attempts to decode the initially encoded information. The associated recovery fidelity is

\[
\begin{align*}
\tilde{F}[^R] &= \frac{1}{2} + \frac{1}{12} \sum_{i=1}^3 \text{tr}[\hat{H}_i \tilde{D}_t(\hat{\sigma}_i^\dagger)] \\
&\approx \frac{1}{2} \left[ 1 + \frac{\lambda_1 \lambda_2 + \lambda_3 \lambda_4 + \lambda_2 \lambda_3 + \lambda_1 \lambda_4 + \lambda_1 \lambda_3 + \lambda_2 \lambda_4}{6} \right]
\end{align*}
\]

The first line follows from the definitions (6) and (17), for
a proof see e.g. [36]; the second line comes from applying (22). The optimal fidelity will be at least as large as \( \tilde{F}[^R] \). Now, invoking the inequality between arithmetic and geometric mean, one has

\[
2\tilde{F}_{\text{opt}}[\tilde{D}_t] \geq 1 + (\lambda_1^2 \lambda_2^2 \lambda_3^2 \lambda_4^2)^{1/6} = 1 + (\lambda_1 \lambda_2 \lambda_3 \lambda_4)^{1/2}.
\]

Recalling Eq. (11), we get Eq. (21). Moreover, if \( \lambda_j = \lambda \forall j \), then the inequality is saturated and the choice of matrices in (23) is optimal. 

C. Best decoding operation

The formalism which we developed also allows us to
give a clear recipe for the best recovery operation to be performed by manipulating the Majorana zero-modes. Assuming for simplicity to be in the case \( \lambda_j(t) = \lambda(t) \) discussed in Eq. (20), its operative definition is the following:

\[
\mathcal{R}(\hat{\rho}) = \frac{1}{2} \left[ \hat{\sigma}_0 + \sum_{i=1}^3 \text{tr}[\hat{D}_t(\hat{\sigma}_i^\dagger)] \right], ~ (24)
\]

IV. KITAEV CHAIN AFTER A QUANTUM QUENCH

Let us now study some explicit examples of decoherence channels \( \mathcal{D}_t \), which in light of the previous discussion are chosen among those that preserve the parity of the number of fermions in the system. Our goals are (i) to compare the performance of recovery operations that act on the whole topological system to the performance of decodings of the zero modes, and (ii) to highlight the physical mechanisms which lead to protection of the information. As a first simple example, in this Section we study the case of a perturbation represented by a quantum quench. The second example, which concerns a bosonic Markovian dynamics, is discussed in Sec. V.
We consider a pair of Kitaev chains in the topological phase [17]. Each chain is described by a Hamiltonian $\hat{H}_K$ and hosts two Majorana zero-modes at its edges; these modes can be combined as described in Sec. II to store a single qubit. The two chains are assumed to be identical and hosts two Majorana zero-modes at its edges; these are the gapped modes of the system with energy $\Delta$. The inset highlights that increasing $\Delta$ the steady value $\lambda_{st}$ approaches 1 as $(\epsilon/\Delta)^2$.

The Hamiltonian of a single chain of length $L$ is:

$$\hat{H}_K = \Delta \sum_{i=1}^{L-1} \left( -a_i^\dagger a_i^\dagger + a_{i+1}^\dagger a_i^\dagger + \text{H.c.} \right), \quad \Delta > 0 \quad (25)$$

where the $\{\hat{a}_i^\dagger : i = 1, \ldots, L\}$ are canonical Fermi operators representing the annihilation (creation) of a fermion at the site $i$ of the chain. Hamiltonian (25) is the Kitaev model with the parameters tuned to the so-called “sweet point” [17]; this peculiar choice represents an idealized situation which allows for simple calculations but being in the middle of a topological phase captures all the general features of the dynamics. The Hamiltonian is diagonalized by a Bogolubov transformation, yielding $L-1$ eigenmodes with eigenenergy $\Delta$, denoted by $\{\hat{b}_i : i = 1, \ldots, L-1\}$ and two unpaired Majorana edge modes $\{\hat{m}_1, \hat{m}_2\}$:

$$\hat{H}_K = E_0 + \Delta \sum_{i=1}^{L-1} \hat{b}_i^\dagger \hat{b}_i = E_0 + \frac{\Delta}{2} \sum_{j=1}^{L-1} i \gamma_{j,1} \gamma_{j,2}. \quad (26)$$

The $\hat{b}_i$ are the gapped modes of the system with energy $\Delta$ and are defined by $\hat{b}_i = \frac{1}{2} (\hat{\gamma}_{i,1} + i \hat{\gamma}_{i,2})$ in terms of the original Majorana modes (see Fig. 1 for a sketch). $\Delta$ is the gap between the ground-space, spanned by $|\Omega\rangle$ and $\frac{1}{2} (\hat{m}_1 - i \hat{m}_2) |\Omega\rangle$, and the rest of the spectrum.

At time $t = 0$ the Hamiltonian is abruptly changed by adding a perturbation $\hat{H}_{1}$: the system is thus set out of equilibrium and starts evolving according to a non-trivial unitary dynamics. Mathematically, the $\mathcal{D}_t$ channel is defined as:

$$\mathcal{D}_t (\hat{\rho}(0)) = e^{-i(\hat{H}_K + \zeta \hat{H}_{1})t} \hat{\rho}(0) e^{i(\hat{H}_K + \zeta \hat{H}_{1})t}. \quad (27)$$

For earlier work on this model, see Refs. [34, 36].

We assume that $\hat{H}_K$ does not couple the two chains; it thus suffices to focus on only one of them. Such closed-system dynamics does not modify the parity of the number of fermions and moreover does not cause any information loss: the backward time evolution can in principle be used to restore the initial condition. In order to engineer such operation, however, it is necessary to control the whole system; if the experimentalist has limited access to the system, a non-trivial fraction of the information may not be recoverable. Here we characterize this type of information loss for a decoding operation acting only on the zero-energy Majorana modes.

As a simple example of perturbation, we consider:

$$\hat{H}_{1} = \epsilon \sum_{j=1}^{L} \hat{a}_j^\dagger \hat{a}_j; \quad \epsilon > 0. \quad (28)$$

In this case, it is well-known that as long as $|\zeta| < 2\Delta/\epsilon$ the system remains topological, with zero-energy boundary Majorana modes. We will restrict ourselves to such a case.

In order to evaluate the optimal fidelity $\bar{F}_{opt}$, we need to compute the $\lambda_j$ coefficients. Given the symmetry of the noise model, all the $\lambda_j$ coefficients are equal, therefore:

$$\lambda_j(t) \equiv \lambda(t) = \frac{1}{4} \text{tr} \left[ \mathcal{D}_t^*(\hat{m}_1) \hat{m}_1 \Pi_G \right], \quad (29)$$

where $\mathcal{D}_t^*(\hat{m}_1) = e^{i(\hat{H}_K + \zeta \hat{H}_1)t} \hat{m}_1 e^{-i(\hat{H}_K + \zeta \hat{H}_1)t}$. In Fig. 2 we plot the value of $\lambda(t)$ in Eq. (29), computed by exploiting the properties of quadratic time evolutions, for several values of $\Delta > \epsilon$. We consider finite system sizes, $L \sim 100$, and finite times such that correlations arising from one edge have not yet reached the other one. $\lambda(t)$ then displays a clear steady behavior $\lambda_{st}$, which we interpret as the value of the limit $\lim_{t \to \infty} \lim_{L \to \infty} \lambda(t)$. With the help of Eq. (20) it is possible to relate the best decoding fidelity $\bar{F}_{opt}$ to the value of $\lambda_{st}$. The fact that $\lambda_{st} < 1$ implies that the decoding is not perfect even in the limit $L \to \infty$. In presence of self-correction [52], the steady-state fidelity should scale as $1 - e^{-cL}$. Since in the regime considered here we find finite fidelity loss at infinite size, we can conclude that a simple decoding does not display self-correcting behavior. However, the figure highlights the protecting effect of the gap $\Delta$: $|\lambda_{st} - 1| \sim (\epsilon/\Delta)^2$.

Moving away from unitary dynamics, we can consider the case in which the parameter $\zeta$ is taken randomly from a distribution $g(\zeta)$ and the time evolution results from a classical average over such distribution:

$$\mathcal{D}_t^*(\hat{m}_1) = \int g(\zeta) e^{i(\hat{H}_K + \zeta \hat{H}_1)t} \hat{m}_1 e^{-i(\hat{H}_K + \zeta \hat{H}_1)t} d\zeta \quad (30)$$
In Fig. 3 we show the results, which are qualitatively analogous to those obtained without averaging. Note that the possibility of benefiting from full topological protection when acting on the whole system has been shown for this case in Ref. [30].

Summarizing, the examples discussed in this Section show that a decoding of the zero-modes only does not enjoy any topological protection. The system is however stabilized by the gap of the topological Hamiltonian, with a protection that scales as a power law in the gap amplitude.

V. KITAEV CHAIN IN THERMAL BOSONIC ENVIRONMENT

We now consider a Markovian decoherence channel which arises from a thermal bosonic environment (i.e., an environment that conserves the parity of the number of fermions in the system). We begin deriving the master equation from first principles (see Refs. [39, 65] for the case of fermionic environment). Subsequently, we characterize the noise it induces in the system and discuss the optimal recovery fidelity, both with and without the experimental limitation of acting on the zero-modes only.

A. Definition of the master equation

The system considered in this example is analogous to that discussed in Sec. IV, namely it consists of two Kitaev wires which are separated and completely decoupled. Each Kitaev wire is surrounded by its own environment, so that even the noise dynamics cannot induce any correlation between the two chains. Similarly to the previous example, we assume that also the Hamiltonian does not couple the wires, so that it suffices to focus on the dynamics of a single chain.

Let us first define the environment: we assume that every site \( i = 1, \ldots, L \) of the chain is coupled to a large number of spins, \( M \), via an interaction term which depends on whether the fermionic site is occupied. We thus introduce a collection of spin-1/2 \( \hat{\sigma}^{(m)}_i \) and the Hamiltonian \( \hat{H}_{\text{env}} = \sum_{i=1}^L \sum_{n=1}^M \omega_{i,n} \frac{1}{2} (I + \hat{Z}_{i,n}) \); the interaction with the chain is given by

\[
\hat{H}_{\text{int}} = -2\eta \sum_{i,n} (\hat{a}_i^\dagger \hat{a}_i - \frac{1}{2}) \hat{X}_{i,n} .
\]

While the coupling \( \eta \) is homogeneous, the energy splitting of the spin levels, \( \omega_{i,n} \), differs for each environment spin. The \( \{\omega_{i,n}\} \) energy gaps define an environment with density of states \( f(\omega) \) [59]. The situation is illustrated in Fig. 4.

In Appendix C we derive an effective master equation for this model in the weak-coupling limit \( \eta \to 0 \) and \( M \to \infty \), while \( M\eta^2 \) is kept constant with value \( g^2/\pi \). This, together with the assumption that there are no system-environment correlations in the initial state, ensures the Markovianity of the time evolution. For generality, the environment is assumed to be at inverse temperature \( \beta \). The associated Lindbladian defines the time evolution \( \partial_t \hat{\rho}(t) = \mathcal{L}[\hat{\rho}(t)] \) and the decoherence channel \( \mathcal{D}_t[\hat{\rho}(0)] = e^{t\mathcal{L}[\hat{\rho}(0)]} \). It consists of three parts:

\[
\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\Delta + \mathcal{L}_{2\Delta} ,
\]
Hamiltonian correction $\hat{H}_{LS}$ is given in Eq. (C19) and does not gap the zero-modes.

The term $\mathcal{L}_\Delta$ is given by dissipative processes that require the exchange of a unit $\Delta$ of energy from the system to the environment, or vice versa. These are:

$$\hat{L}_1^{(\Delta)} = \sqrt{\phi(\Delta)} \hat{m}_1 \hat{b}_1, \quad \hat{L}_1^{(-\Delta)} = \sqrt{\phi(-\Delta)} \hat{m}_1 \hat{b}_1^\dagger,$$

$$\hat{L}_2^{(\Delta)} = \sqrt{\phi(\Delta)} \hat{m}_2 \hat{b}_{L-1}, \quad \hat{L}_2^{(-\Delta)} = \sqrt{\phi(-\Delta)} \hat{m}_2 \hat{b}_{L-1}^\dagger.$$  

(35)

These operators describe the incoherent creation or annihilation of an excitation next to an edge, and involve explicitly the Majorana zero-modes.

Finally, the term $\mathcal{L}_{2\Delta}$ represents dissipative processes that require the exchange of two units of energy $\Delta$ between the system and the environment. These processes are represented by Lindblad operators

$$\hat{L}_1^{(2\Delta)} = \sqrt{\phi(2\Delta)} \hat{b}_{i-1} \hat{b}_i, \quad \hat{L}_1^{(-2\Delta)} = \sqrt{\phi(-2\Delta)} \hat{b}_{i-1} \hat{b}_i^\dagger.$$  

These describe the incoherent creation of a pair of excitations in neighboring empty bonds, or the annihilation of a pair of excitations in neighboring occupied bonds. Note that the $\hat{L}_1^{(0)}$ and the $\hat{L}_1^{(2\Delta)}$ do not depend on the zero-energy modes.

As a final consistency check, we can easily see that all the Lindblad operators we derived are quadratic in the Majorana modes (and thus bosonic). As we prove in Appendix D, this is equivalent to having a master equation that preserves fermionic number parity.

As anticipated earlier in the work, the gap of the Kitaev Hamiltonian can protect the information if it is large compared to temperature. If we set $\Delta \gg 1/\beta$, the processes that pump energy from the environment into the system are suppressed exponentially in $\beta$. Additionally, processes that drain energy from the system or that do not exchange energy act trivially because there are no excitations to move or annihilate in the ground space. In Sec. V C we consider a super-Ohmic density of states $f(\omega) \propto \omega^2$ and, resorting to numerical techniques, we discuss the temperature-dependence of the typical memory time-scale.

Before doing so, we analyze a model which allows for an interesting preliminary analysis obtained setting $\beta = 0$ and $f(2\Delta) \ll f(\Delta) \ll f(0)$. This latter assumption implies the relation $\phi(0) \gg \phi(-\Delta) \gg \phi(-2\Delta)$, which mimics the low-temperature regime, in which high-energy states become unavailable. However, setting $\beta = 0$, the asymmetry between $\phi(\Delta)$ and $\phi(-\Delta)$ characteristic of finite temperatures is not introduced, and thus no qualitative analogy should be in principle expected. We discuss this case because, thanks to several algebraic simplifications, it is possible to obtain analytical results. Similarly to the full thermal case, we obtain that in the limit $f(\Delta), f(2\Delta) \to 0$ the information is fully recoverable, as expected. In the following subsections we study the recoverability of the information when acting on the zero-modes or on the whole system for both cases.
B. Analytical model: $\beta = 0$ and $f(2\Delta) \ll f(\Delta) \ll f(0)$

1. Decoding fidelity

Let us discuss the properties of the channel $D_t$ related to the Markovian dynamics $e^{i(\tilde{L}_0 + \tilde{L}_\Delta + L_{2\Delta})}$ just introduced, for $\beta = 0$. In particular, we are interested in the behavior of the optimal decoding fidelity in the $L \gg 1$ limit. Starting from the formula in Eq. (20), we compute the $\lambda_1(t)$ coefficient:

$$\lambda_1(t) = \frac{1}{2} \text{tr}(\hat{m}_1 \tilde{D}_t(\hat{m}_1)) = \frac{1}{2} \text{tr}\left(\hat{m}_1 D_t[\hat{m}_1 \hat{\Pi}_G]\right) = \frac{1}{2} \text{tr}\left(D_t^*[\hat{m}_1] \hat{m}_1 \hat{\Pi}_G\right) = \frac{1}{2} \text{tr}\left(e^{tL^*}[\hat{m}_1] \hat{m}_1 \hat{\Pi}_G\right). \tag{36}$$

Since the channel considered here does not couple the two chains, we can focus on a single wire. In particular, Eq. (36) refers to a single chain and the normalizing factor of $2$ is such that $\frac{1}{2}\text{tr}\hat{\Pi}_G = 1$.

Since $\beta = 0$, we have $\phi(\omega) = \phi(-\omega)$, and thus $\tilde{L}^{(\omega)} \equiv \tilde{L}^{(-\omega)}$ for all Lindblad operators. Therefore, $L^*$ is equal to $L$, except for the sign of the Hamiltonian term (the only anti-Hermitian term in $L$). As $\hat{m}_1$ commutes with $\hat{H}$, $L^*$ acts on $\hat{m}_1$ just like $L$:

$$L^*[\hat{m}_1] = \hat{m}_1 L[\hat{1}] + L_\Delta[\hat{m}_1] + \hat{m}_1 L_{2\Delta}[\hat{1}] = -g^2 f(\Delta) \hat{m}_1, \tag{37}$$

Thus $e^{tL^*}[\hat{m}_1] = e^{-g^2 f(\Delta)t} \hat{m}_1$, and

$$\lambda_1(t) = \frac{1}{2} \text{tr}(e^{-g^2 f(\Delta)t} \hat{m}_1 \cdot \hat{m}_1 \hat{\Pi}_G) = e^{-g^2 f(\Delta)t}. \tag{38}$$

It is easy to see that the exact same argument applies to $\hat{m}_2$, and thus to both zero-modes of the other chain, so that all four $\lambda_i(t)$ coefficients are the same. The optimal decoding fidelity (20) is therefore:

$$\bar{F}_{\text{opt}}(t) = \frac{1 + e^{-2g^2 f(\Delta)t}}{2}. \tag{39}$$

A number of remarks are in order. First, let us stress that the formula for $\bar{F}_{\text{opt}}(t)$ in Eq. (39) does not depend on the size of the system, $L$, as it is derived assuming the $L \to \infty$ limit. This implies the absence of self-correcting behavior, which would give $\bar{F}_{\text{opt}}(t) \to 1$ as system size is increased.

Second, in the limit $f(\Delta) \to 0$, we observe perfect recoverability. Since $F_{\text{opt}} \geq \bar{F}_{\text{opt}}$, in this limit no error occurs in the system because, independently from the complicated dynamics which may arise in the bulk, the zero-energy modes are unperturbed. The corruption of information is thus due to the Lindbladian term $L_\Delta$.

2. Comparison with full recovery operations

Let us now compare the optimal decoding fidelity (39) with the optimal recovery fidelity achievable by acting on the whole system. In Appendix E we prove that in the presence of a decoherence channel which (i) does not couple the two Kitaev chains and (ii) preserves the fermionic parity of each chain, the optimal recovery fidelity is:

$$F_{\text{opt}}(t) = \frac{2}{3} + \frac{1}{12} \left(\|D_t(\hat{m}_1 \hat{\Pi}_G)\|_1\right)^2. \tag{40}$$

The formula is a simple generalization of the optimal recovery fidelity provided in Ref. [36] for the case of two Kitaev chains with noise acting on only one of them. Interestingly, the fact that the quantum channel $D_t$ acts in a product way on the two chains yields a significant advantage, as for the computation of $F_{\text{opt}}(t)$ one only needs to consider a single chain. From a numerical point of view, this is crucial for obtaining some meaningful size scaling in the cases in which the methods of fermionic linear optics cannot be applied (such as this one).

We numerically compute the optimal recovery fidelity with a Runge-Kutta (RK) integration of the differential equation $\partial_t \hat{A} = \hat{L}[\hat{A}]$. The exponential scaling of the dimension of the Hilbert space limits our computation to sizes of order $L = 11$ (length of a single chain). Results are reported in Fig. 6, where it is shown that the value of $F_{\text{opt}}(t)$ depends on the size.

In order to quantify how the memory properties of the system depend on the size $L$, we fix an arbitrary threshold value $F_{\text{thr}} < 1$ and define the memory time $t^*(L)$ as the first time at which $F_{\text{opt}}(t)$ crosses such value. In the inset in Fig. 6 we take $F_{\text{thr}} = 0.99$ and demonstrate that the scaling of $t^*(L)$ is linear in $L$. The increase of coherence time with system size represents a form of protection, even though $t^*(L)$ does not show the exponential scaling with $L$ which is considered a signature of self-correcting behavior.

![FIG. 6. (Color online) Optimal recovery fidelity for a system of two Kitaev wires, for several values of the wire length $L$. We set $\Delta = 1$ as the unit of energy (and inverse time), $g^2 f(0) = 4$, $f(\Delta) = 0.3 f(0)$, $f(2\Delta) = 0.3^2 f(0)$. The inset shows the linear scaling of coherence time $t^*$ with the chain length $L$. We define $t^*$ as the time at which $F_{\text{opt}}(t^*) = 0.99$.](image-url)
The obtained linear behavior $t^*(L) \sim L$ is rather surprising because at the single particle level the master equation induces a diffusive dynamics for the excitations, which yields a scaling $t^*(L) \sim L^2$. In order to understand the discrepancy between the expected diffusive behavior and the seemingly ballistic behavior we observe, we have studied the problem in some significant limits.

The quadratic (diffusive) scaling has been recovered by restricting the dynamics to the subspace of states with energy smaller than $2\Delta$ (and thus no more than two $b_i$ excitations in the chain) [not shown]. Additionally, setting the rate of creation and annihilation of excitation pairs to zero, $\phi(2\Delta) = \phi(-2\Delta) = 0$, the full many-body calculation shows a quadratic scaling $t^*(L) \sim L^2$ [not shown]. Both cases agree with the intuitive picture of fidelity being lost when an excitation created by $L_\Delta$ at one edge is diffused through the bulk by $L_0$ until it can annihilate with an excitation coming from the opposite edge.

This shows that the transition from diffusive to ballistic regime (i) is due to the possibility of creating pairs of excitations in the bulk, and (ii) it is a many-body problem which cannot be recast in terms of simple unbiased random walks. We leave the development of a simple microscopic model giving rise to this unexpected ballistic behavior for future work.

3. Comparison with other figures of merit

Before concluding the study of this setup, we consider two figures of merit which are defined independently from recovery operations and have already been discussed in the relevant literature.

The first figure of merit is related to the temporal resilience of the Majorana fermions, and was employed for instance in Ref. [43]:

$$\theta_1(t) = \text{tr}[i\hat{m}_1\hat{m}_2 \rho(t)]$$

Here, $\hat{\rho}(t) = e^{tC}(1 + i\hat{m}_1\hat{m}_2)\hat{\Pi}_G/2$. Recalling Eq. (7) we obtain:

$$\text{tr}[i\hat{m}_1\hat{m}_2 D_i(\hat{\rho}_0\hat{\Pi}_G)] = \text{tr}(\hat{m}_1) \left[i\hat{m}_1\hat{m}_2 D_i(i\hat{m}_1\hat{m}_2/2)\right].$$

We thus obtain:

$$\theta_1(t) = \lambda(t)^2 = e^{-2g^2f(\Delta)t}.$$

As a second figure of merit, we consider the autocorrelation time of the Majorana fermions, which was considered for example in Ref. [32]:

$$\theta_2(t) = \text{tr}[i\hat{m}_1\hat{m}_2 D_t^*(i\hat{m}_1\hat{m}_2)\hat{\rho}_0]$$

Here the trace is over two chains and the initial state is taken to be $|0_L\rangle$: $\hat{\rho}(0) = 1/2(\hat{\sigma}_0^+ + \hat{\sigma}_4^+) = 1/2(\hat{1} - i\hat{m}_1\hat{m}_2 - i\hat{m}_3\hat{m}_4 - \hat{m}_1\hat{m}_2\hat{m}_3\hat{m}_4)\hat{\Pi}_G^0\hat{\Pi}_G$. Chain $b$ can be traced out, yielding

$$\theta_2(t) = \frac{1}{2} \text{tr}_a[(1 - i\hat{m}_1\hat{m}_2) i\hat{m}_1\hat{m}_2 D_t^*(i\hat{m}_1\hat{m}_2)\hat{\Pi}_G^0].$$

Finally, the calculation of $D_t^*(\hat{m}_1\hat{m}_2)$ simply follows from Eq. (37):

$$\theta_2 = \frac{1}{2} \text{tr}_a[\hat{m}_1\hat{m}_2 e^{-2g^2f(\Delta)t}\hat{m}_1\hat{m}_2\hat{\Pi}_G] = e^{-2g^2f(\Delta)t}.$$

The similarity between these results and the constrained optimal fidelity $\tilde{F}_{\text{opt}}$ (39) underpins with information-theory techniques the methods which have been previously proposed. Additionally, this shows that even a bosonic environment can represent a danger for the quantum information encoded in Majorana zero-modes if control over the system is experimentally limited.

C. Environment at finite temperature

Let us now turn to the study of the temperature effects on the memory efficiency. As an example, we consider $f(\omega) \propto \omega^2 e^{-\omega/\Omega}$, which describes a super-Ohmic bath [59], where $\Omega$ is a high-energy cutoff necessary to make the integrals (C18) in the “Lamb shift” Hamiltonian (C19) convergent. For consistency, we work in the regime $\Omega \gg \Delta$ and $\Omega \gg 1/\beta$.

We then assume $\beta > 0$, so that there is an asymmetry between processes that pump energy into the system and those that drain it out of the system. Because of this asymmetry, $\hat{m}_1$ is no longer eigenmode of the Lindbladian.
\( \mathcal{L}^* \), and thus the analytical argument that we used to prove Eq. (39) breaks down. Thus, both fidelities \( F_{\text{opt}} \) and \( \bar{F}_{\text{opt}} \) must be computed numerically.

We thus consider a pair of chains of length \( L = 8 \) and we compute both kinds of optimal recovery fidelity at time \( t = 1/\Delta \). Given the chosen functional form for \( f(\omega) \), the relevant noise parameters obey \( f(0) = 0 \) and \( f(2\Delta) = 4e^{-\Delta/\Omega_f}f(\Delta) \). Note that in Refs [41, 44] a slightly different thermal environment is considered. We then vary independently \( \beta \) and the noise intensity \( g^2f(\Delta) \), while \( \Delta \) is kept constant, and plot the resulting fidelity loss \( 1 - F_{\text{opt}}(t = 1/\Delta) \).

We remark that the exact vanishing of \( f(0) \), while rather unrealistic and fine-tuned, is a useful limiting case to study: since it completely inhibits one of the diffusion mechanisms in the chains (the other being \( \mathcal{L}_{2\Delta} \)), it should improve the memory performance with respect to the \( f(0) \neq 0 \) case. Thus, any negative results obtained in this scenario will hold a fortiori for generic cases.

The results are represented in Fig. 7. As expected, a low environment temperature ensures the protection of information: as \( T \to 0 \) (i.e. \( \beta \to \infty \)), there is no energy in the environment nor in the initial state of the chain, so the system state is confined to the zero-energy sector, where no local interaction couples \( \hat{m}_1 \) and \( \hat{m}_2 \); thus information survives indefinitely. This ideal regime is approached exponentially in \( 1/T \), what is usually called an Arrhenius activation law [12]; however, we observe that the characteristic energy scale is different depending on whether one has full access to the system or is confined to the zero-modes. Once again, most of the information loss observed when acting on the edge modes only would actually be recoverable via a global operation, and only a tiny fraction is fatally lost to the environment. The ratio between the information lost by looking only at the edge modes and that really unrecoverable is expected to increase exponentially in \( \beta\Delta \).

Another significant result that emerges from Fig. 7 is that at a fixed time the fidelity loss scales only polynomially with the noise intensity \( g^2f(\Delta) \), while it is suppressed exponentially in \( \beta\Delta \). Note the difference with Eq. (39), where at fixed time the scaling with \( g^2f(\Delta) \) is exponential (no comparison with the temperature scaling is possible because there we set \( \beta = 0 \)). This means that increasing \( \Delta \), the gap of the Hamiltonian, leads to an exponential suppression of the thermal fluctuations, and to a polynomial increase in the noise \( (g^2f(\Delta) \sim \Delta^2) \): the result is an Arrhenius scaling typical of Hamiltonian protection.

As we already mentioned, in the finite-temperature case the analytical arguments we used for the \( \beta = 0 \) case break down, so we cannot obtain a simple generalization to (38) for \( \lambda(t) \). Nonetheless, it is possible to evaluate \( \lambda(t) \) perturbatively for short times. Going back to the definition (36) and expanding to first order in \( t \) yields

\[
\lambda(t) = 1 + \frac{1}{2} \text{tr} \left[ \mathcal{L}^*(\hat{m}_1)\hat{m}_1\hat{\Pi}_G \right] + O(t^2) ;
\]

so that

\[
\lambda(t) = 1 - g^2f(\Delta) \left[ 1 - \tanh \left( \frac{\beta\Delta}{2} \right) \right] + O(t^2). 
\]

Let us stress that this results returns the correct short-time expansion of (39) for \( \beta = 0 \). In the \( \beta\Delta \gg 1 \) limit, it yields:

\[
\lambda(t) \simeq 1 - 2g^2f(\Delta)e^{-\beta\Delta t}.
\]

Thus, as long as \( f(\Delta) \) is polynomial in \( \Delta \) [59], setting \( \Delta \gg k_BT \), i.e. increasing the Hamiltonian gap, leads to an Arrhenius-type protection of information in both the recovery and decoding scenarios, with different activation energies. This is a size-independent effect that shows, again, the absence of self-correction. This conclusion would only be reinforced by removing the constraint \( f(0) \neq 0 \) from the bath spectral density.

**VI. CONCLUSIONS**

In this article we have discussed the realization of a reliable quantum memory based on a topological superconductor hosting unpaired zero-energy Majorana modes. Motivated by the current technical limitations in the manipulation and control of such quantum fluids, we have addressed the problem of understanding whether a reliable storage of quantum information in such systems is possible with minimal assumptions regarding read-out capabilities. In particular, we have characterized the best recovery operation to be applied after the action of the noise when the control over the system is strictly limited to the zero-energy Majorana modes where the information is initially encoded. We have identified the fidelity of such operation, quantifying the information which remains in the initial encoding degrees of freedom, and provided experimental recipes for the optimal decoding.

Explicit calculations for two examples of noise models are also presented. The examples (a quantum quench in the Kitaev Hamiltonian and a bosonic thermal environment) are chosen because in those cases it is possible to prove that a recovery operation acting on the whole system is substantially more effective than a simple decoding of the Majorana zero-modes only. Thus their analysis best highlights the handicap introduced by the minimal experimental requirements.

While in both examples a decoding of the zero-modes does not enjoy any topological protection (i.e. finite fidelity loss persists even in the ideal infinite-size limit), we have highlighted that it benefits from the presence of the Hamiltonian energy gap, displaying Arrhenius-type behavior. This is true even when the environment has
a density of states which increases with the energy (we have explicitly considered a super-Ohmic behavior).

More generally, our results rely on the existence of a LRB in local unitary and open Markovian dynamics. Although in this article we have not entered the paradigm of non-Markovian dynamics, the employed formalism might still be used for such situations. It is rather intriguing to speculate that a LRB might exist also in those cases, as even non-Markovian dynamics should originate from a unitary dynamics in the larger system-environment setup: if such underlying unitary dynamics is local, it must obey the rule of light-cone propagation of correlations. Thus, we expect our findings to extend to more general scenarios.

Finally, in order to bridge our article to the previous literature, we have compared our results to figure of merits displayed in distant regions; which is the key argument used in the considered figures of merits display an exponential decay over time without size protection. This feature is not due to any fine-tuning in the figures of merit, and raises interesting questions about the role of parity in protecting the information stored in the system. Our study stresses that parity-conservation is not sufficient to protect information if the system cannot be properly manipulated. The precise conditions under which the encoding into Majorana fermions can protect information from a parity-preserving noise require further investigations.

ACKNOWLEDGMENTS

We gratefully acknowledge feedbacks and comments by M. Burrello and R. Fazio. We are personally indebted to C.-E. Bardyn for pointing out to us the diffusive nature of excitations under thermal noise. L. M. and M. R. acknowledge invaluable discussions with J. I. Cirac and M. D. Lukin in the early stages of the work. L.M. acknowledges financial support from Italian MIUR through FIRB project RBFR12NLNA. L. M. was supported by LabEX ENS-ICFP: ANR-10-LABX-0010/ANR-10-FIRB project RBFR12NLNA. L. M. was supported in the early stages of the work. L. M. and M. R. acknowledge invaluable discussions with J. I. Cirac and M. D. Lukin in the early stages of the work. L. M. and M. R. acknowledge invaluable discussions with J. I. Cirac and M. D. Lukin in the early stages of the work.

Appendix A: Local noise models

In this Appendix we discuss some properties of local Markovian noise models, which are used throughout the paper. We start from the original Lieb-Robinson bound (LRB) [60], an inequality which bounds the propagation of correlations in closed spin systems with local interactions:

\[ \|\hat{O}_A(t) \|_{op} \leq c V \|\hat{O}_A\|_{op} \|\hat{O}_B\|_{op} e^{-d_{AB} - vt}/\xi \],

(A1)
Here \(\hat{O}_A, \hat{O}_B\) are operators supported in distant regions \(A\) and \(B\), respectively; \(d_{AB}\) is the distance between such regions; \(V\) is the volume of the largest such region; \(c, v\) and \(\xi\) are model-dependent constants. This means that there is an effective light cone, with “speed of light” \(v\), that describes the propagation of correlations from a local region \(A\). Correlations are not strictly zero outside the light cone, but rather fall off on a distance scale \(\xi\).

Since Eq. (A1) deals with closed systems, the time-evolved operator (in the Heisenberg picture) is \(\hat{O}_A(t) = \hat{U}^\dagger(t) \hat{O}_A \hat{U}(t)\). However, it was shown by Poulin [62] that the same result holds if the unitary evolution is replaced by a Markovian channel (again in the Heisenberg picture) \(\hat{O}_A(t) = \mathcal{D}_t^f(\hat{O}_A)\), provided the Markovian dynamics is local. Locality in this context means that the Lindbladian \(\mathcal{L}\) can be decomposed as \(\sum_{X \subset \Lambda} \mathcal{L}_X\), where \(X\) are subregions of the system with diameter at most \(d\) and each \(\mathcal{L}_X\) has Lindblad operators supported in \(X\).

We observe that the same reasoning that leads to the LRB (A1) for spin systems can be mapped exactly to a fermionic setting [61], provided the Markovian dynamics is parity-preserving: by the result of Appendix D, parity preservation implies bosonic Lindblad operators, so that the commutation between distant operators (the key ingredient for the proof in the spin case) is recovered. One can thus prove that

\[ [\mathcal{D}_t^f(\hat{O}_A^f), \hat{O}_B] \approx 0 \], \quad \{\mathcal{D}_t^f(\hat{O}_A^f), \hat{O}_B^f\} \approx 0 \, , \] (A2)

where \(\hat{O}_A^f, \hat{O}_B^f\) are distant bosonic operators, \(\hat{O}_A^f, \hat{O}_B^f\) are distant fermionic operators, and the approximation symbol is meant as a shorthand for the right hand side of the inequality (A1), i.e. it holds up to LRB corrections.

Finally, one last useful property of local, Markovian, parity-preserving noise models that we use is the following:

\[ \mathcal{D}_t^f(\hat{O}_A \hat{O}_B) \approx \mathcal{D}_t^f(\hat{O}_A) \mathcal{D}_t^f(\hat{O}_B) \] , (A3)

where the approximation again means “up to LRB corrections”. We call (A3) a clustering property, as it implies the clustering of correlations for distant operators on uncorrelated (product) initial states:

\[ \langle \hat{O}_A \hat{O}_B \rangle_t \approx \langle \hat{O}_A \rangle_t \langle \hat{O}_B \rangle_t \] , (A4)

where \(\langle \hat{O} \rangle_t \doteq \text{tr}[\mathcal{D}_t(\hat{\rho}_A \hat{\rho}_B)] = \text{tr}[\mathcal{D}_t(\hat{\rho}) \hat{\rho}_A \hat{\rho}_B]\), for all states \(\hat{\rho}_A (\hat{\rho}_B)\) supported in region A (B). Property (A4) was proven by Poulin [62] for all local Markovian channels of spin systems; however, the same arguments are straightforwardly mapped to fermion systems, provided the dynamics is parity-preserving (i.e. all Lindblad operators are bosonic – see Appendix D), as that ensures the commutation between parts of the Lindbladian supported in distant regions, which is the key argument used in the proof. In [62], Eq. (A3) is actually proven as an intermediate step towards Eq. (A4).
Appendix B: Optimal single-qubit recovery operation

Here we prove a formula for the optimal recovery fidelity of a single qubit exposed to noise. It is known that a general single qubit noise can be expressed as follows [63]:

\[ \frac{1 + n \cdot \sigma}{2} \rightarrow \frac{1 + (\Lambda n + b) \cdot \sigma}{2}, \]  

where \( \Lambda \) is a matrix with singular values \( v_1, v_2, v_3 \in [0,1] \), namely: \( \Lambda = R \cdot D \cdot S \), with \( D = \text{diag}(v_1, v_2, v_3) \) and \( R, S \in O(3) \). The \( S \) rotation can be eliminated by a suitable unitary redefinition of the Pauli matrices: \( \sigma \rightarrow S \sigma, n \rightarrow Sn, b \rightarrow Sb \) preserves \( n \cdot \sigma \) and \( b \cdot \sigma \), while it yields

\[ (\Lambda n) \cdot \sigma = n^T S D R^T \sigma = n^T D R^T S^T \sigma. \]  

We can then simply call \( SR \triangleq R \) and effectively get rid of \( S \).

The optimal recovery operation is now the unitary gate that undoes the last rotation \( R \):

\[ \frac{1 + n \cdot \sigma}{2} \rightarrow \frac{1 + (R^T n) \cdot \sigma}{2}, \]  

which gives an average recovery fidelity

\[ F[\mathcal{R}] = \int d^2n \text{tr} \left( \frac{1 + n \cdot \sigma}{2} \right) \left( \frac{1 + (R^T n) \cdot \sigma}{2} \right) = \frac{1}{2} \int d^2n \left( 1 + n^T (R^T \Lambda n + R^T b) \right) = \frac{1}{2} \left[ 1 + \int d^2n \text{tr} (v) \right] = \frac{1}{2} \left( 1 + \frac{v_1 + v_2 + v_3}{3} \right). \]  

As discussed in Ref. [36], the fidelity of a recovery operation must obey the following inequality:

\[ F_{\text{opt}} \leq \frac{1}{2} + \frac{1}{12} \sum_{i=1}^3 \|D(\sigma_i)\|_1. \]  

Since by definition \( v_i = \|D(\sigma_i)\|_1/2 \), the recovery operation in Eq. (B3) saturates the upper bound and therefore is optimal. In the case discussed in the text, \( v_i = \lambda^2 \), as it can be explicitly verified:

\[ i \frac{\delta}{\delta t} \left( \hat{m}_2 \hat{m}_3 + \hat{m}_1 \hat{m}_4 \right) = \lambda^2 \frac{i}{2} (\hat{m}_2 \hat{m}_3 + \hat{m}_1 \hat{m}_4), \]  

(similar results hold for all qubit operators).

Appendix C: Microscopic derivation of a Markovian master equation for a Kitaev chain in bosonic environment

In this appendix we present a weak-coupling derivation of a Markovian master equation for the system and environment setup discussed in Section V. The derivation follows the general approach presented in [64] for spin systems. The discussion applies unchanged to our fermionic system, as all relevant operators are bosonic (i.e. even-degree monomials of Fermi fields and linear combinations thereof), and thus effectively behave like spins for our purposes. Similar results have been recently presented for master equations with Lindblad operators linear in the fermionic fields in Ref. [39] (see also Ref. [65]).

1. Generalities

We start from an uncorrelated system-environment state \( \hat{\rho}(0) \), and assume that the environment is in a thermal state at a generic inverse temperature \( \beta \), i.e. \( \hat{\rho}(0) = \hat{\rho}_\beta(0) \hat{\rho}_E(0) \) with \( \hat{\rho}_E(0) = \hat{\rho}_E \beta = \hat{\rho}_E \beta = e^{-\beta \hat{\mathcal{H}}_{\text{env}}}/\text{tr}[e^{-\beta \hat{\mathcal{H}}_{\text{env}}}] \). The evolution equation in the interaction picture is

\[ \frac{d}{dt} \hat{\rho}(t) = -i [\hat{H}_I(t), \hat{\rho}(t)] ; \]  

integrating it for \( \hat{\rho}(t) \) and plugging the result again into the right-hand side yields

\[ \frac{d}{dt} \hat{\rho}(t) = -i [\hat{H}_I(t), \hat{\rho}(t)] - \int_0^t ds [\hat{H}_I(t), [\hat{H}_I(s), \hat{\rho}(s)]] . \]  

The interaction Hamiltonian has the following structure: \( \hat{H}_I(0) = \sum_\alpha \hat{A}_\alpha \hat{B}_\alpha \), where the \( \{\hat{A}_\alpha\} \) are system operators and the \( \{\hat{B}_\alpha\} \) are environment operators, such that \( \hat{A}_\alpha \hat{B}_\alpha \) is bosonic \( \forall \alpha \) (i.e. for a given \( \alpha \), \( \hat{A}_\alpha \) and \( \hat{B}_\alpha \) are either both bosonic or both fermionic). This represents no loss of generality, since the fermionic parity superselection rule forces all observables to be bosonic operators, and \( \hat{H}_I \) is an observable. Since the Hamiltonian must be Hermitian, one can always choose \( \hat{B}_\alpha \) to be Hermitian, while \( \hat{A}_\alpha \) is either Hermitian (bosonic case) or anti-Hermitian (fermionic case), so that \( \hat{A}_\alpha \hat{B}_\alpha = (\hat{B}_\alpha)(\pm \hat{A}_\alpha) = \hat{A}_\alpha \hat{B}_\alpha \). We remark that both the \( \hat{A}_\alpha \) and the \( \hat{B}_\alpha \) can be taken traceless, up to a redefinition of the system and environment Hamiltonians. This implies that, when the environment is traced out in Eq. (C2), the first term vanishes. Then tracing out the environment and invoking the Born approximation to replace \( \hat{\rho}(s) \) by the instantaneous value \( \hat{\rho}(t) \) yields

\[ \frac{d}{dt} \hat{\rho}(t) = - \int_0^\infty ds \text{tr}_E [\hat{H}_I(t), [\hat{H}_I(t-s), \hat{\rho}(t)]] . \]  

We also removed the dependence on the initial time by integrating over the entire past history \( s \rightarrow \infty \); it is part of the assumptions of Markovianity that this does not change the result significantly. Introducing a Fourier decomposition of the \( \{\hat{A}_\alpha\} \) system operators such that
\[ \dot{\hat{A}}_\alpha(t) = \sum_\omega \hat{A}_\alpha(\omega) e^{-i\omega t} \] allows us to write
\[ \frac{d}{dt} \hat{\rho}(t) = -\sum_{\alpha,\omega, \alpha',\omega'} \int_0^\infty ds e^{-i\omega t} e^{-i\omega'(t-s)} \text{tr}_E \left[ \hat{A}_\alpha(\omega) \hat{B}_\alpha(t), [\hat{A}_{\alpha'}(\omega'), \hat{B}_{\alpha'}(t-s), \hat{\rho}(t)] \right]. \]

(C4)

At this point we invoke the\textit{ rotating wave approximation} to argue that all terms with \( \omega + \omega' \neq 0 \) cancel. Introducing the environment correlation function
\[ \Gamma_{\alpha\beta}(\omega) \equiv \int_0^\infty ds e^{i\omega s} \text{tr}_E \left[ \hat{B}_\alpha(t) \hat{B}_\beta(t-s) e^{-\beta \hat{H}_{\text{env}}} \right] / \text{tr}_E[ e^{-\beta \hat{H}_{\text{env}}} ] \]
(which does not depend on \( t \) because all involved Hamiltonians are time-independent), we have
\[ \frac{d}{dt} \hat{\rho} = -\sum_{\alpha,\beta,\omega} \Gamma_{\alpha\beta}(\omega) \left( \hat{A}_\alpha(\omega) \hat{\rho} \hat{A}_\beta^\dagger(\omega) - \hat{A}_\beta^\dagger(\omega) \hat{\rho} \hat{A}_\alpha(\omega) \right) - \hat{A}_\alpha(\omega) \hat{\rho} \hat{A}_\beta^\dagger(\omega) + \hat{A}_\beta^\dagger(\omega) \hat{\rho} \hat{A}_\alpha(\omega) \right). \]
(C6)

By finally introducing \( \gamma_{\alpha\beta}(\omega) \equiv \frac{1}{2} (\Gamma_{\alpha\beta}(\omega) + \Gamma_{\beta\alpha}(\omega) - \omega \) and \( S_{\alpha\beta}(\omega) \equiv \frac{1}{2} (\Gamma_{\alpha\beta}(\omega) - \Gamma_{\beta\alpha}(\omega)), \) we can recast (C6) in the form
\[ \frac{d}{dt} \hat{\rho} = -\sum_{\alpha,\beta,\omega} \gamma_{\alpha\beta}(\omega) \left( \hat{A}_\alpha(\omega) \hat{\rho} \hat{A}_\beta^\dagger(\omega) - \frac{1}{2} \left\{ \hat{A}_\beta^\dagger(\omega) \hat{A}_\alpha(\omega), \hat{\rho} \right\} \right) + i S_{\alpha\beta}(\omega) \left( \hat{A}_\alpha(\omega) \hat{A}_\beta^\dagger(\omega), \hat{\rho} \right) \]
(C7)

Here \( \sum_{\alpha,\beta,\omega} S_{\alpha\beta}(\omega) \hat{A}_\alpha(\omega) \hat{A}_\beta^\dagger(\omega) \equiv \hat{H}_{\text{LS}} \) is the \textit{Lamb shift} Hamiltonian, which renormalizes the system Hamiltonian. As for the other terms, they contribute to the dissipative part of the dynamics the Lindblad operators are obtained upon diagonalizing the Hermitian matrices \( \gamma_{\alpha\beta}(\omega) \):
\[ \hat{L}_{\alpha,\omega} = \sum_\beta \sqrt{\gamma_{\alpha\beta}(\omega)} \hat{A}_\beta(\omega), \]
(C8)

where for each \( \omega, c(\omega) \) is the unitary which diagonalizes \( \gamma(\omega) \) into \( d(\omega) : \gamma(\omega) = c(\omega)^d(\omega)c(\omega). \omega \) ranges over all the possible gaps in the system Hamiltonian, and the Fourier component \( \hat{A}_\alpha(\omega) \) is constructed as
\[ \hat{A}_\alpha(\omega) = \sum_{E_1, E_2, E_1-E_2=\omega} \hat{\Pi}(E_1) \hat{A}_\alpha \hat{\Pi}(E_2), \]
(C9)

where \( E_1, E_2 \) being energy levels for the system Hamiltonian and \( \Pi(E) \) being the projector on the \( E \)-energy eigenspace.

2. Derivation of the master equation studied in Sec. V

The decomposition of the interaction Hamiltonian (31) is as follows: \( \hat{H}_\text{int} = \sum_i \hat{A}_i \hat{B}_i \), with \( \hat{A}_i = 2 \hat{a}_i^\dagger \hat{a}_i - 1 \) and \( \hat{B}_i = -\eta \sum_n X_{i,n} \). The time evolution induced by the environment Hamiltonian \( \hat{H}_{\text{env}} = \sum_i \frac{1}{2} \omega_{i,n} \hat{Z}_{i,n} \) on the \( \hat{B}_i \) operators yields
\[ \hat{B}_i(\tau) = -\eta \sum_{n=1}^M e^{-i \frac{\omega_{i,n}}{2} \tau} \hat{X}_{i,n} e^{i \frac{\omega_{i,n}}{2} \tau} \approx \hat{X}_{i,n} + \sin(\omega_{i,n} \tau) \hat{Y}_{i,n} \]
(C10)

With this information, and by factoring the thermal state as a product of one-spin operators and using the identity \( e^{-\alpha \hat{Z}} = \cosh(\alpha) \hat{1} - \sinh(\alpha) \hat{Z} \), one can prove that
\[ \Gamma_{ij}(\tau) = \frac{\text{tr} \left[ \hat{B}_i(\tau) \hat{B}_j(0) e^{-\beta \hat{H}_{\text{env}}} \right]}{\text{tr}[ e^{-\beta \hat{H}_{\text{env}}} ]} = \eta^2 \delta_{ij} \sum_{n=1}^M [\cos(\omega_{i,n} \tau) - i \sin(\omega_{i,n} \tau) \tanh(\beta \omega_{i,n}/2)] . \]
(C11)

Hence, by Fourier-transforming and averaging over the environment spectra,
\[ \Gamma_{ij}(\omega) = M \eta^2 \delta_{ij} \int_0^\infty d\tau e^{i\omega \tau} \int dE f(E) \times \] \[ \times [\cos(\beta E) - i \sin(\beta E) \tanh(\beta E/2)] \]
(C12)

and finally, adding the conjugate transpose,
\[ \Gamma_{ij}(\omega) = -M \eta^2 \delta_{ij} \int_{-\infty}^\infty d\tau e^{i\omega \tau} \int dE f(E) \left[ \delta(\omega - E) \left( 1 + \tanh \left( \frac{\beta E}{2} \right) \right) + \delta(\omega + E) \left( 1 - \tanh \left( \frac{\beta E}{2} \right) \right) \right] ; \]
(C13)
as \( f(E) \) is a density of state supported in \( E > 0 \), only one of the two delta functions will contribute, depending on the sign of \( \omega \), therefore the result is

\[
\overline{\gamma}_{ij}(\omega) = g^2 \delta_{ij} \begin{cases} f(\omega) \frac{1 + \tanh(\beta |\omega|)}{2} & \text{if } \omega > 0 \\ f(0) \frac{1 - \tanh(\beta |\omega|)}{2} & \text{if } \omega = 0 \\ f(\omega) \frac{1 - \tanh(\beta |\omega|)}{2} & \text{if } \omega < 0 \end{cases}
\]  

(C14)

where \( g^2 \equiv \pi M \gamma^2 \) is the effective coupling constant, which is assumed to stay finite as \( M \to \infty \) and \( \eta \to 0 \).

This matrix is already diagonal in the \( \hat{S} \) that the \( \sqrt{f(0)} \), and for \( \omega \neq 0 \) \( \hat{L}_{i,\pm|\omega|} = g\sqrt{f(0)} \frac{1 + \tanh(\beta |\omega|)}{2} \hat{A}_i(\pm|\omega|) \). For notational ease, let us introduce the function \( \phi(\omega) \equiv \phi(\omega) \delta_{ij} \), i.e.

\[
\phi(\omega) \equiv g^2 f(\omega) \frac{1 + \tanh(\beta |\omega|)}{2} . \quad (C15)
\]

It is easy to determine the Fourier decomposition of the \( \hat{A}_i \) operators: one just needs to write them in terms of \( \hat{b}_i(\uparrow) \) operators, and count the energy quanta introduced into the system. Namely,

\[
\begin{align*}
\hat{A}_1 = i\hat{m}_1 \hat{\gamma}_{1,1} &= i\hat{m}_1 (\hat{b}_1 + \hat{b}_1^\dagger) , \\
\hat{A}_{1<i<L} &= i\hat{m}_{i-1} \hat{\gamma}_{1,2} = (\hat{b}_{i-1} - \hat{b}_{i-1}^\dagger)(\hat{b}_i + \hat{b}_i^\dagger) , \\
\hat{A}_L &= i\hat{m}_L \hat{\gamma}_{L-1,2} = i\hat{m}_L (\hat{b}_{L-1} - \hat{b}_{L-1}^\dagger) .
\end{align*}
\]  

(C16)

This leads to the following list of Lindblad operators:

\[
\begin{align*}
L_i^{(0)} &= \sqrt{2\phi(0)} \hat{b}_i \hat{b}_i^\dagger , \\
L_i^{(+\Delta)} &= \sqrt{\phi(\Delta)} i\hat{m}_1 \hat{b}_1 , \\
L_i^{(-\Delta)} &= \sqrt{\phi(-\Delta)} i\hat{m}_1 \hat{b}_1^\dagger , \\
L_i^{(+2\Delta)} &= \sqrt{\phi(2\Delta)} \hat{b}_{i-1} \hat{b}_i^\dagger , \\
L_i^{(-2\Delta)} &= \sqrt{\phi(-2\Delta)} \hat{b}_i^\dagger \hat{b}_{i-1} ,
\end{align*}
\]  

(C17)

where \( i \) ranges from 2 to \( L - 1 \).

As for the correction to the Hamiltonian, it can be seen by a computation analogous to the one leading to (C13) that the \( S \) term in Eq. (C7) is

\[
\overline{S}_{ij}(\omega) = g^2 \pi \delta_{ij} \mathcal{P} \int_0^\infty \frac{dE}{|\omega| - E} \frac{f(E) \omega + E \tanh(\beta E/2)}{|\omega| + E}
\]  

(C18)

with \( \mathcal{P} \) denoting the Cauchy principal part of the integral. Therefore, letting \( \overline{S}_{ij}(\omega) = S(\omega) \delta_{ij} \), we have, up to additive constants,

\[
\begin{align*}
\mathcal{H}_{LS} &= (S(\Delta) - S(-\Delta)) (\hat{b}_1 \hat{b}_1^\dagger + \hat{b}_{L-1} \hat{b}_{L-1}^\dagger) \\
&+ 2(S(2\Delta) - S(0)) \sum_{i=2}^{L-2} \hat{b}_i \hat{b}_i^\dagger \\
&+ (2S(0) - S(2\Delta) - S(-2\Delta)) \sum_{i=2}^{L-1} \hat{b}_{i-1} \hat{b}_i \hat{b}_i^\dagger \hat{b}_{i-1}^\dagger .
\end{align*}
\]  

(C19)

This renormalizes the energy gap by an amount that differs between the edge bonds and the interior bonds, and adds quartic interactions between nearest-neighbor bonds. The picture simplifies considerably when one takes \( \beta = 0 \) (i.e. \( T = \infty \)), since in that case \( S(\omega) + S(-\omega) = 0 \forall \omega \); thus no interactions emerge and one only has a renormalization of the Hamiltonian gaps:

\[
\mathcal{H}_{LS} = (S(2\Delta) - S(2\Delta)) (\hat{b}_1 \hat{b}_1^\dagger + \hat{b}_{L-1} \hat{b}_{L-1}^\dagger) \\
+ 2S(2\Delta) \mathcal{H}_K .
\]  

(C20)

**Appendix D: Parity-preserving noise models**

In this Appendix we prove a lemma that characterizes all parity-preserving Markovian noise models. Parity preservation (PP) for a generic channel \( D_i \) is stated as follows:

\[
\langle \hat{P}_t \rangle_t = \langle \hat{P}_0 \rangle_0 \quad \forall t ,
\]  

(D1)

which is equivalent to

\[
\partial_t \langle \hat{P}_t \rangle_t = \text{tr}[\hat{P}_t \partial_t \hat{\rho}] = \text{tr}[\hat{P}_t \mathcal{L}(\hat{\rho})] = \text{tr}[\hat{\rho} \mathcal{L}^*(\hat{P}_t)] = 0 .
\]  

(D2)

for any state \( \hat{\rho} \). We shall prove that a Markovian noise model is PP if and only if its associated Lindblad operators are all bosonic (BL).

(PP) \( \implies \) (BL): A state \( \hat{\rho} \) is a bosonic operator and as such it can be written as \( \hat{\rho} = \hat{P}_f \hat{\rho} \hat{P}_f \). Thus PP implies that \( \text{tr}[\hat{\rho} \mathcal{L}^*(\hat{P}_f) + \hat{P}_f \mathcal{L}^*(\hat{P}_f) \hat{P}_f] = 0 \) for any state \( \hat{\rho} \), i.e.

\[
\hat{P}_f \mathcal{L}^*(\hat{P}_f) + \mathcal{L}^*(\hat{P}_f) \hat{P}_f = 0 .
\]  

(D3)

Decomposing the Lindblad operators in their fermionic and bosonic parts, \( \hat{L}_i = \hat{L}_i^f + \hat{L}_i^b \), and exploiting the fact that \( \{\hat{L}_i^f, \hat{P}_f\} = 0 \) while \( \{\hat{L}_i^b, \hat{P}_f\} = 0 \), one gets

\[
\sum_i \left( (\hat{L}_i^b - \hat{L}_i^f)^\dagger (\hat{L}_i^b + \hat{L}_i^f) + (\hat{L}_i^b + \hat{L}_i^f)^\dagger (\hat{L}_i^b - \hat{L}_i^f) - \hat{L}_i^b \hat{L}_i^f - (\hat{L}_i^b - \hat{L}_i^f)^\dagger (\hat{L}_i^b - \hat{L}_i^f) \right) = 0 ,
\]  

(D4)

whose trace yields \( \sum_i \text{tr}((\hat{L}_i^b)^\dagger \hat{L}_i^b) = 0 \). Now, since \( \text{tr}(\hat{O}^\dagger \hat{O}) = \|\hat{O}\|_{HS}^2 \) is the squared Hilbert-Schmidt norm of operator \( \hat{O} \), \( \sum_i \|\hat{L}_i^b\|_{HS}^2 = 0 \) implies \( \hat{L}_i^b = 0 \forall i \), i.e. all Lindblad operators are purely bosonic (BL).
In this Appendix we prove a generalization of a result presented in Ref. [36]. We consider a system composed of two Kitaev wires as discussed in Sec. IV and V; the noise model $\mathcal{D}_t$ is parity-preserving and does not couple the two wires. Under these assumptions it is possible to characterize the best recovery operation and its fidelity.

Let us start from the general relation:

$$F(\mathcal{R}_t) \leq \frac{1}{2} + \frac{1}{12} \sum_{a=x,y,z} \|\mathcal{D}_t(\sigma^a_0)\|_1$$  

(E1)

The decoherence channel acts on two Kitaev chains, which are separated and decoupled; the noise does not couple them and conserves the parity of the fermionic number on each chain. Decoupling the chains has tremendous importance: for all times $t > 0$, the following holds:

$$\|\mathcal{D}_t(\sigma^L_0)\|_1 = \|\mathcal{D}_t(|0_L\rangle\langle 0_L|) - \mathcal{D}_t(|1_L\rangle\langle 1_L|)\|_1 = 2.$$  

(E2)

Proof: As the decoherence channel is parity preserving and does not couple the chains, $\mathcal{D}_t(|0_L\rangle\langle 0_L|)$ and $\mathcal{D}_t(|1_L\rangle\langle 1_L|)$ have an even (odd, respectively) number of particles in each chain. Therefore they are supported in orthogonal subspaces, and the trace norm is thus additive:

$$\|\mathcal{D}_t(\sigma^L_0)\|_1 = \|\mathcal{D}_t(|0_L\rangle\langle 0_L|)\|_1 + \|\mathcal{D}_t(|1_L\rangle\langle 1_L|)\|_1.$$  

(E3)

Now, $\mathcal{D}_t(|0_L\rangle\langle 0_L|)$ is a quantum state, and in particular a positive matrix; so its trace norm is simply its trace. As $\mathcal{D}_t$ is trace-preserving, one has $\|\mathcal{D}_t(|0_L\rangle\langle 0_L|)\|_1 = 1$. The same holds for $\mathcal{D}_t(|1_L\rangle\langle 1_L|)$, which concludes the proof of (E2).

Now let us determine the form of the optimal recovery operation. In the process, we shall also determine the value of $\|\mathcal{D}_t(\sigma^L_0)\|_1$ and $\|\mathcal{D}_t(\sigma^L_0)\|_1$.

In the notation introduced in (6), one needs to determine the optimal $\{\hat{H}_j\}$ matrices. A set of recovery matrices that saturates the upper bound (E1) is given by $\hat{H}_j = \text{sign}(\mathcal{D}_t(\sigma^L_0))$, where the sign of a Hermitian matrix is defined as $\text{sign}(A) = \sqrt{A^2} \cdot A^{-1}$. Unfortunately, this set is not guaranteed to induce a physical map, i.e. the upper bound (E1) might be unattainable. However, we shall argue, following [36], that in our case the matrices $\{\hat{H}_j\}$ obey the Pauli matrix algebra $\hat{H}_j \hat{H}_k = i\epsilon_{ijk} \hat{H}_l$, and this guarantees that the induced recovery map is physical.

For the operator $\sigma^L_0$, we have

$$\hat{H}_L = \text{sign}(\mathcal{D}_t(\sigma^L_0)) = \Pi_0^+ \Pi_0^- - \Pi_0^- \Pi_0^+,$$  

(E4)

where $\Pi_0^\pm$ is the projector on the ± fermionic number parity sector of the first chain, dubbed $\alpha$, and $\Pi_b^\pm$ is the same operator for the second chain, dubbed $b$. It is easy to verify that $\text{tr} [\hat{H}_L \mathcal{D}_t(\sigma^L_0)] = \|\mathcal{D}_t(\sigma^L_0)\|_1$.

Let us now consider the action of the decoherence channel on the operator $\sigma^L$, which is best analyzed through the operators $\hat{R}_a$ and $\hat{R}_b$:

$$\hat{R}_a = \mathcal{D}_t(|0_a\rangle\langle 1_a|), \quad \hat{R}_b = \mathcal{D}_t(|0_b\rangle\langle 1_b|).$$  

(E5)

where $|0_j\rangle$ and $|1_j\rangle$ denote states of the chain $j$ such that: $|0_L\rangle = |0_a\rangle \otimes |0_b\rangle$. In terms of these operators, one has

$$\mathcal{D}_t(\sigma^L_0) - \mathcal{D}_t(|1_L\rangle\langle 1_L|) = \hat{R}_a \hat{R}_b + \hat{R}_b^\dagger \hat{R}_a^\dagger.$$  

(E6)

Let us take the square of this operator:

$$(\mathcal{D}_t(\sigma^L_0))^2 = (\hat{R}_a \hat{R}_b + \hat{R}_b^\dagger \hat{R}_a^\dagger)^2 = -\hat{R}_a^2 \hat{R}_b^2 - \hat{R}_b^2 \hat{R}_a^2 + \hat{R}_a^\dagger \hat{R}_b^\dagger \hat{R}_a \hat{R}_b + \hat{R}_b^\dagger \hat{R}_a^\dagger \hat{R}_b \hat{R}_a,$$  

(E7)

This holds because $\hat{R}_a$ and $\hat{R}_b$ are fermionic operators with disjoint supports, and thus anti-commute. Since by definition $\hat{R}_a = \Pi_0^+ \hat{R}_a \Pi_0^-$, it squares to zero $\hat{R}_a^2 = \Pi_0^+ \hat{R}_a \Pi_0^- \Pi_0^+ \hat{R}_a \Pi_0^- = 0$; thus (E7) simplifies to

$$(\mathcal{D}_t(\sigma^L_0))^2 = \hat{R}_a \hat{R}_b + \hat{R}_b^\dagger \hat{R}_a^\dagger + \hat{R}_a^\dagger \hat{R}_b \hat{R}_a \hat{R}_b,$$  

(E8)

where we note that the first and second terms are positive Hermitian operators on the subspaces $\Pi_0^+ \Pi_0^-$ and $\Pi_0^- \Pi_0^+$ respectively. Taking the operator square root now yields

$$\|\mathcal{D}_t(\sigma^L_0)\|_1 = \sqrt{\hat{R}_a \hat{R}_b} \sqrt{\hat{R}_b \hat{R}_a} + \sqrt{\hat{R}_a^\dagger \hat{R}_b^\dagger} \sqrt{\hat{R}_b^\dagger \hat{R}_a^\dagger},$$  

(E9)

and the operator $\sqrt{\hat{R}_a \hat{R}_b}$ coincides the positive Hermitian operator $\hat{P}_a$ defined by the polar decomposition $\hat{R}_a = \hat{U}_a \hat{P}_a$ (the same holds for chain $b$). Finally, we can compute the trace norm via the definition $\|A\|_1 = \text{tr}[A]$:

$$\|\mathcal{D}_t(\sigma^L_0)\|_1 = 2 \text{tr}(\hat{P}_a) \text{tr}(\hat{P}_b),$$  

(E10)

Now we have that $\|\mathcal{D}_t(\sigma^L_0)\|_1 = \|\mathcal{D}_t(\sigma^L_0)\|_1$, as can be seen by plugging $\sigma^L = -i |0_0\rangle\langle 1_0| + h.c.$ instead of $\sigma^L = |0_0\rangle\langle 1_0| + h.c.$ into equation (E8). Therefore $\|\mathcal{D}_t(\sigma^L_0)\|_1 = \|\mathcal{D}_t(\sigma^L_0)\|_1$, and the optimal fidelity upper bound (E1) in our case where the noise has the same action on both chains reads

$$F_{\text{opt}} \leq \frac{2}{3} + \frac{1}{3} \left(\text{tr} \hat{P}\right)^2,$$  

(E11)

where $\hat{P}$ is meant as an operator on a single chain, and the trace goes over a single chain as well.
Let us now identify the optimal recovery operation, and check that the upper bound (E1) is indeed attainable. The recovery matrix for $\hat{\sigma}_{2}^{+}$ is

$$
\hat{H}_{1} = (\hat{R}_{a}\hat{R}_{b} + \hat{R}_{b}^{\dagger}\hat{R}_{a}^{\dagger}) \cdot \hat{R}_{a}\hat{R}_{b} + \hat{R}_{b}^{\dagger}\hat{R}_{a}^{\dagger} = \\
\hat{R}_{a}(\hat{R}_{b}^{\dagger}\hat{R}_{a}^{\dagger})^{-1/2} \hat{R}_{b}(\hat{R}_{b}^{\dagger}\hat{R}_{a}^{\dagger})^{-1/2} + \\
\hat{R}_{b}^{\dagger}(\hat{R}_{a}\hat{R}_{b})^{-1/2} \hat{R}_{a}(\hat{R}_{a}\hat{R}_{b})^{-1/2}; \quad (E12)
$$

now, since $\hat{R}_{a} = \hat{U}_{a}\hat{P}_{a}$, we have $\hat{R}_{a}(\hat{R}_{b}^{\dagger}\hat{R}_{a}^{\dagger})^{-1/2} = \hat{P}_{a}^{\dagger}\hat{U}_{a}\hat{P}_{a}$ and $\hat{R}_{b}^{\dagger}(\hat{R}_{a}\hat{R}_{b})^{-1/2} = \hat{P}_{b}^{\dagger}\hat{U}_{b}\hat{P}_{b}$ (and the same for chain b). Putting it all together, we get

$$
\hat{H}_{1}^{\dagger} = \hat{P}_{a}^{\dagger}\hat{U}_{a}\hat{P}_{a}^{\dagger}\hat{P}_{b}^{\dagger}\hat{U}_{b}\hat{P}_{b}^{\dagger} \hat{P}_{a}^{\dagger}\hat{U}_{a}\hat{P}_{a}^{\dagger}\hat{P}_{b}^{\dagger}\hat{U}_{b}\hat{P}_{b}^{\dagger} \cdot (E13)
$$

The same reasoning applied to $\hat{\sigma}_{2}^{-}$ yields

$$
\hat{H}_{2} = -i\hat{P}_{a}^{\dagger}\hat{U}_{a}\hat{P}_{a}^{\dagger}\hat{P}_{b}^{\dagger}\hat{U}_{b}\hat{P}_{b}^{\dagger} \cdot (E14)
$$

Both matrices are Hermitian and by construction they have operator norm equal to 1.

At this point, recalling the form of the third recovery matrix (E4), a simple computation exploiting the properties of unitary matrices and orthogonal projectors shows that the $\{\hat{H}_{i}\}$ matrices obey the Pauli algebra, and thus by following the discussion presented in [36] one can prove that the recovery map is CPTP, and the fidelity upper bound (E11) is attainable.

Finally, let us recast the optimal fidelity (E11) in a form that is more convenient for numerical computations. Let us consider a single chain (we drop the subscript $a$ for convenience). We have $\text{tr}[\hat{P}] = \|\hat{R}\|_{1}$; now, as $\hat{R}$ and $\hat{R}^{\dagger}$ act on orthogonal subspaces ($\hat{R} = \hat{P}^{\dagger}\hat{R}\hat{P}^{\dagger}$ and $\hat{R}^{\dagger} = \hat{P}^{\dagger}\hat{R}^{\dagger}\hat{P}^{\dagger}$), the trace norm is additive and thus $\|\hat{R}\|_{1} = \frac{1}{2}\|\hat{R}^{\dagger}\|_{1}$. The formula is convenient because:

$$
\hat{R} + \hat{R}^{\dagger} = D_{t}(|0\rangle\langle 1| + |1\rangle\langle 0|) = D_{t}((\hat{d}_{0} + \hat{d}_{0}^{\dagger})\hat{P}_{G}) \quad (E15)
$$

where $\hat{d}_{0}$ is the bi-localized Dirac zero mode and by definition $\hat{d}_{0} + \hat{d}_{0}^{\dagger} = \hat{n}_{1}$. Therefore, $\text{tr}[\hat{P}] = \frac{1}{2}\|D_{t}(\hat{n}_{1}\hat{P}_{G})\|_{1}$, and the optimal fidelity can be rewritten as

$$
F_{t}^{\text{opt}} = \frac{2}{3} + \frac{1}{12} \left(\|D_{t}(\hat{n}_{1}\hat{P}_{G})\|_{1}\right)^{2}, \quad (E16)
$$

which proves Eq. (40).

1. T. D. Ladd, D. Maryken, Y. Yamamoto, E. Abe and K. M. Itoh, Phys. Rev. B 71, 014401 (2005).
2. C. Langer, R. Ozeri, J. D. Jost, J. Chiaverini, B. DeMarco, A. Ben-Kish, R. B. Blakestad, J. Britton, D. B. Hume, W. M. Itano, D. Leibfried, R. Reichle, T. Rosenband, T. Schaetz, P. O. Schmidt and D. J. Wineland, Phys. Rev. Lett. 95, 060502 (2005).
3. M. V. Balabas, T. Karaulanov, M. P. Ledbetter and D. Budker, Phys. Rev. Lett. 105, 070801 (2010).
4. A. M. Tsyryshkin, S. Tojo, J. J. L. Morton, H. Riemann, N. V. Abrosimov, P. Becker, H.-J. Pohl, T. Schenkel, M. L. W. Thewalt, K. M. Itoh and S. A. Lyon, Nature Materials 11, 143 (2012).
5. P. C. Maurer, G. Kucsko, C. Latta, L. Jiang, N. Y. Yao, S. D. Bennett, F. Pastawski, D. Hunger, N. Chisholm, M. Markham, D. J. Twitchen, J. L. Cirac and M. D. Lukin, Science 336, 1283 (2012).
6. N. Bar-Gill, L. M. Pham, A. Jarmola, D. Budker and R.L. Walsworth, Nature Commun. 4 1743 (2013).
7. J. J. Pla, K. Y. Tan, J. P. Dehollain, W. H. Lim, J. J. L. Morton, F. A. Zwanenburg, D. N. Jamieson, A. Dzurak and A. Morelo, Nature and Science 496, 334 (2013).
8. K. Saeedi, S. Simmons, J. Z. Salvail, P. Dluhy, H. Riemann, N. V. Abrosimov, P. Becker, H.-J. Pohl, J. J. L. Morton and M. L. W. Thewalt, Science 342, 6160 (2013).
9. J. T. Muhonen, J. P. Dehollain, A. Laucht, F. E. Hudson, R. Kalra, T. Sekiguchi, K. M. Itoh, D. N. Jamieson, J. C. McCallum, A. S. Dzurak and A. Morelo, Nature Nanotechnology 9, 986 (2014).
10. T. D. Ladd, F. Jelezko, R. Lalanne, Y. Nakamura, C. Monroe and J. L. O’Brien, Nature 464, 45 (2010).
11. B. M. Terhal, Rev. Mod. Phys. 87, 807 (2015).
12. B. J. Brown, D. Loss, J. K. Pachos, C. N. Self and J. R. Woolton, Preprint at arXiv:1411.6643 (2014).
13. E. Dennis, A. Kitaev, A. Landahl and J. Preskill, J. Math. Phys. 43, 4452 (2002).
14. A. Kitaev, Annals Phys. 303, 2 (2003).
15. C. Nayak, S. H. Simon, A. Stern, M. Freedman and S. Das Sarma, Rev. Mod. Phys. 80, 1083 (2008).
16. N. Read and D. Green, Phys. Rev. B 61, 10267 (2000).
17. A. Yu. Kitaev, Phys.-Usp. 44, 131 (2001).
18. V. Mourik, K. Zuo, S. M. Frolov, S. R. Plissard, E. P. A. M. Bakkers, L. P. Kouwenhoven, Science 336 1003 (2012).
19. A. Das, Y. Ronen, Y. Most, Y. Oreg, M. Heiblum and H. Shtrikman, Nature Physics, 8, 887 (2012).
20. H. O. H. Churchill, V. Fatemi, K. Grove-Rasmussen, M. T. Deng, P. Caroff, H. Q. Xu and C. M. Marcus, Phys. Rev. B 87, 241401 (2013).
21. A. D. K. Finck, D. J. Van Harlingen, P. K. Mohseni, K. Jung and X. Li, Phys. Rev. Lett. 110, 126406 (2013).
22. S. Nadji-Perge, I. K. Drozdov, J. Li, H. Chen, S. Jeon, J. Seo, A. H. MacDonald, B. A. Bernevig and A. Yazdani, Science 346, 602 (2014).
23. L. Radzihovsky and V. Gurarie, Annals of Physics 322, 2 (2007).
24. M. Sato, Y. Takahashi and S. Fujimoto, Phys. Rev. Lett. 103, 020401 (2009).
25. J. D. Sau, R. Sensarma, S. Powell, I. B. Spielman and S. Das Sarma, Phys. Rev. B 83, 140510(R) (2011).
26. L. Jiang, T. Kitagawa, J. Alicea, A. R. Akhmerov, D. Pekker, G. Refael, J. I. Cirac, E. Demler, M. D. Lukin and P. Zoller, Phys. Rev. Lett. 106, 220402 (2011).
27. S. Diehl, E. Rico, M. A. Baranov and P. Zoller, Nature
[28] C. V. Kraus, S. Diehl, M. A. Baranov and P. Zoller, New J. Phys. 14 113036 (2012).
[29] S. Nascimbène, J. Phys. B: At. Mol. Opt. Phys. 46, 134005 (2013).
[30] C. V. Kraus, M. Dalmonte, M. A. Baranov, A. M. Läuchli and P. Zoller, Phys. Rev. Lett. 111, 173004 (2013).
[31] A. Bühler, N. Lang, C. V. Kraus, G. Möller, S. D. Huber and H. P. Büchler, Nature Comm. 5, 4504 (2014).
[32] G. Goldstein and C. Chamon, Phys. Rev. B 85, 205109 (2011).
[33] J. C. Budich, S. Walter and B. Trauzettel, Phys. Rev. B 85, 121405 (2012).
[34] S. Bravyi and R. Koenig, Commun. Math. Phys. 316, 641 (2012).
[35] D. Rainis and D. Loss, Phys. Rev. B 85, 174533 (2012).
[36] L. Mazza, M. Rizzi, M. D. Lukin and J. I. Cirac, Phys. Rev. B 88, 205142 (2013).
[37] S.-H. Ho, S.-P. Chao, C.-H. Chou and F.-L. Lin, New J. Phys. 16, 113062 (2014).
[38] H. T. Ng, Scientific Reports 5, 12530 (2015).
[39] E. T. Campbell, Preprint at arXiv:1502.05626 (2015).
[40] Y. Hu and M. A. Baranov, Phys. Rev. A 92, 053615 (2015).
[41] F. L. Pedrocchi and D. P. DiVincenzo, Phys. Rev. Lett. 115, 120402 (2015).
[42] F. L. Pedrocchi, N. E. Bonesteel and D. P. DiVincenzo, Phys. Rev. B 92, 115441 (2015).
[43] I. C. Fulga, B. van Heck, I. C. Fulga, M. Burrello, A. R. Akhmerov, and C. W. J. Beenakker, Phys. Rev. B 88, 155435 (2013).
[44] F. Hassler, A. R. Akhmerov and C. W. J. Beenakker, New J. Phys 12, 083039 (2010).
[45] T. Hyart, B. van Heck, I. C. Fulga, M. Burrello, A. R. Akhmerov, and C. W. J. Beenakker, Phys. Rev. B 88, 035121 (2013).
[46] M. A. Nielsen, I. L. Chuang, Quantum Computation and Quantum Information, Cambridge (2000).
[47] A. S. Holevo, Quantum Systems, Channels, Information (de Gruyter 2012).
[48] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82, 3045 (2010).
[49] X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. 83, 1057 (2011).
[50] A. R. Akhmerov, Phys. Rev. B 82, 020509 (2010).
[51] F. Hassler, preprint at arXiv:1404.0897 (2014).
[52] A. J. Leggett, S. Chakravarty, A. T. Dorsey, Matthew P. A. Fisher, Anupam Garg and W. Zwerger, Rev. Mod. Phys. 59, 1 (1987).
[53] E. Lieb, D. Robinson, Commun. Math. Phys. 28, 251 (1972).
[54] M. B. Hastings, Phys. Rev. Lett. 93, 126402 (2004).
[55] D. Poulin, Phys. Rev. Lett. 104, 190401 (2010).
[56] M. B. Ruskai, S. Szarek, E. Werner, Linear Algebra Appl. 347, 159 (2002).
[57] F. Petruccione, H. P. Breuer, The theory of open quantum systems (Oxford university press 2002).
[58] M. Ippoliti, Quantum recovery operations, Master thesis, University of Pisa (2014).