An interface formulation of the Laplace-Beltrami problem on piecewise smooth surfaces

Tristan Goodwill
Courant Institute, NYU
New York, NY 10012
tg1644@nyu.edu

Michael O’Neil
Courant Institute, NYU
New York, NY 10012
oneil@cims.nyu.edu

June 21, 2023

---

1Research supported in part by the Research Training Group in Modeling and Simulation funded by the National Science Foundation via grant RTG/DMS-1646339 and by the Office of Naval Research under awards #N00014-18-1-2307 and #N00014-21-1-2383.

2Research supported in part by the Simons Foundation/SFARI (560651, AB) and by the Office of Naval Research under awards #N00014-18-1-2307 and #N00014-21-1-2383.
Abstract

The Laplace-Beltrami problem on closed surfaces embedded in three dimensions arises in many areas of physics, including molecular dynamics (surface diffusion), electromagnetics (harmonic vector fields), and fluid dynamics (vesicle deformation). In particular, the Hodge decomposition of vector fields tangent to a surface can be computed by solving a sequence of Laplace-Beltrami problems. Such decompositions are very important in magnetostatic calculations and in various plasma and fluid flow problems. In this work we develop $L^2$-invertibility theory for the Laplace-Beltrami operator on piecewise smooth surfaces, extending earlier weak formulations and integral equation approaches on smooth surfaces. Furthermore, we reformulate the weak form of the problem as an interface problem with continuity conditions across edges of adjacent piecewise smooth panels of the surface. We then provide high-order numerical examples along surfaces of revolution to support our analysis, and discuss numerical extensions to general surfaces embedded in three dimensions.

Keywords: Laplace-Beltrami, harmonic vector field, Lipschitz, surface of rotation, Hodge decomposition.

Contents

1 Introduction 2

2 The Laplace-Beltrami problem 3
  2.1 Smooth surfaces .............................................. 3
  2.2 Lipschitz surfaces ......................................... 4

3 Interface conditions 7
  3.1 Interface problems in the plane .......................... 8
  3.2 Regularity of the pull-back of the Laplace-Beltrami problem ... 11
  3.3 Solution of the interface form around a single vertex ......... 17
  3.4 Interface form on a closed surface ........................ 21

4 A special case: The cone 23

5 Application: Surfaces of revolution with edges 25
  5.1 Separation of variables ...................................... 25
  5.2 A periodic ODE solver ....................................... 26
  5.3 A numerical solver .......................................... 28

6 Numerical examples 30
  6.1 Comparing Greens functions .................................. 30
  6.2 Laplace-Beltrami on a smooth surface ...................... 31
  6.3 Singular surface test ........................................ 33
  6.4 Harmonic vector field computation ......................... 34

7 Conclusions 37

A Error bounds for singular data 38
1 Introduction

The Laplace-Beltrami operator is a second-order elliptic differential operator defined along Riemannian manifolds in arbitrary dimensions. Practically, it can be thought of as the extension of the Laplace operator to curved surfaces [22, 40]. For the moment, let \( \Gamma \) be a smooth closed surface embedded in three dimensions, and let \( \nabla_\Gamma \cdot \) and \( \nabla_\Gamma \) be the intrinsic surface divergence and surface gradient operators along \( \Gamma \) (these are defined more carefully later on in the manuscript). Then the Laplace-Beltrami operator, also referred to as the surface Laplacian, is given as

\[
\Delta_\Gamma = \nabla_\Gamma \cdot \nabla_\Gamma.
\]  

The Laplace-Beltrami operator is particularly useful for problems in electromagnetics since it allows for the explicit construction of tangential vector fields along multiply-connected surfaces in terms of their gradient, divergence-free, and harmonic components. For example, any smooth tangential vector field \( \mathbf{F} \) along \( \Gamma \) can be written as

\[
\mathbf{F} = \nabla_\Gamma \alpha + \mathbf{n} \times \nabla_\Gamma \beta + \mathbf{H},
\]  

where \( \mathbf{n} \) is the unit normal to \( \Gamma \), the functions \( \alpha, \beta \) are smooth and scalar-valued along \( \Gamma \), and \( \mathbf{H} \) is a harmonic vector field tangent to \( \Gamma \), i.e. \( \nabla_\Gamma \cdot \mathbf{H} = 0 \) and \( \nabla_\Gamma \cdot (\mathbf{n} \times \mathbf{H}) = 0 \). See [17, 18, 40, 43] for more details regarding such decompositions and a proof of their uniqueness. Conversely, if the vector field \( \mathbf{F} \) above is known (but not its individual components), then, for example, its solenoidal (i.e. divergence-free) component involving \( \beta \) can be computed by taking \( \nabla_\Gamma \cdot \mathbf{n} \times \) of each side of (1.2), yielding

\[
\Delta_\Gamma \beta = -\nabla_\Gamma \cdot \mathbf{n} \times \mathbf{F}.
\]  

Solving the above PDE along \( \Gamma \) requires inverting the Laplace-Beltrami operator. Applications of this problem are myriad in electromagnetics, as mentioned, as well as plasma physics [38], vesicle deformation in biological flows [46, 49], surface diffusion [2, 20], and computational geometry [33] and computer vision [1].

Along smooth general surfaces, there have been several numerical methods proposed to solve the Laplace-Beltrami problem: finite element methods [2, 4, 10, 14, 15] (including eigenfunction computations [5]), the so-called virtual element method [3, 23], differencing methods [50], and integral equation methods [35, 43]. Of course, in specialized geometries, such as axisymmetric ones, separation of variables can be used to simplify the problem. In [19, 44], after separation of variables, an integral equation approach was used along the one-dimensional generating curve of the axisymmetric surface resulting in high-order convergence for the inversion of the surface Laplacian operator. Boundary-value problems for the Laplace-Beltrami problem were addressed in [34, 35] via a parametrix method and associated discretization via projection of the sphere.

However, while the related problem of electromagnetic scattering from non-smooth surfaces has been studied in some detail [8, 13], the Laplace-Beltrami problem on piecewise-smooth surfaces (or more generally Lipschitz surfaces) has not received as much attention. In order to extend various numerical methods in computational electromagnetics, namely integral equation methods based on generalized Debye source representations [11, 17–19], a thorough understanding of the Laplace-Beltrami problem on non-smooth surfaces is required (as well as high-order accurate methods for solving the problem numerically). This work aims to offer a mathematical discussion of the Laplace-Beltrami problem along piecewise smooth surfaces in the \( L^2 \) setting, which is compatible with many modern numerical methods (i.e. PDE vs. integral equation methods, Galerkin vs. Nyström discretizations). In particular, our main purpose for focusing on the \( L^2 \) theory of this problem is to
develop fast robust numerical solvers that can be incorporated into computational electromagnetics codes. These solvers often rely on iterative methods whose behavior is coupled with the spectrum of the operator, which depends strongly on the type of discretization used. A suitable $L^2$ embedding of the problem [6] is frequently the most straightforward way to obtain an accurate approximation to the spectrum of the continuous problem.

Previously, the regularity of solutions to the Laplace-Beltrami problem in the general Lipschitz setting was discussed in [24], and the regularity of solutions on polyhedral surfaces was discussed in [8]. Some special-case numerical methods on polyhedral surfaces were presented in [26, 51]. These methods are almost exclusively developed in the finite element setting whereby the use of polyhedral surfaces and linear finite elements reduce the continuous problem to a discrete system that can be studied in detail. Similar techniques have been used in the electromagnetics community for analyzing Hodge decompositions on polyhedra [7, 9].

The main result of our manuscript is Theorem 10, which proves a reformulation of the weak form of the Laplace-Beltrami problem \( \Delta \gamma u = f \) as an interface or transmission problem on the curved but piecewise smooth surface \( \Gamma \) with interface, or continuity, conditions along the edges of the smooth panels. This reformulation is akin to what are sometimes referred to as dielectric transmission problems or Laplace interface problems in the plane. In fact, our reformulation takes advantage of previous results for exactly this problem.

The outline of our approach and the paper is as follows: in Section 2 we recall some standard results regarding the weak Laplace-Beltrami problem on smooth and Lipschitz surfaces. In Section 3, we extend those results to an interface formulation on piecewise smooth surfaces. In Section 4, we extend these results to a flat cone and discuss how they might be extended to other surfaces with conic singularities. As an example of a case where a simple numerical scheme exists, in Section 5 we reformulate interface form of the Laplace-Beltrami problem as a sequence of decoupled ODEs along a surface of revolution, and introduce an integral equation reformulation of said ODEs. An associated high-order accurate numerical solver is then discussed. Section 6 contains various numerical experiments validating our formulation and demonstrating the accuracy of our solver. Finally, in Section 7 we offer some final remarks and point to outstanding problems related to the Laplace-Beltrami problem and future avenues of research.

## 2 The Laplace-Beltrami problem

In this section we lay out some well-known invertibility results for the Laplace-Beltrami problem on smooth surfaces, and then consolidate some existing results for Lipschitz domains. From the context, the assumptions on \( \Gamma \) should be clear (i.e. either smooth, general Lipschitz, or piecewise smooth).

### 2.1 Smooth surfaces

To begin with, let \( \Omega \subset \mathbb{R}^3 \) denote an open bounded domain whose boundary is given by \( \Gamma \). The boundary \( \Gamma \) can either be simply or multiply connected, and for now is assumed to be globally smooth. We shall start by recalling the definition of the Laplace-Beltrami operator and the Laplace-Beltrami problem on a smooth surface.

**Definition 1 (Laplace-Beltrami Operator).** If \( \Gamma \) is a smooth bounded surface, \( x = x(\theta, \varphi) \) is a local parameterization of \( \Gamma \) with respect to some set of variables \( \theta, \varphi \), the functions \( x_{\theta} \) and \( x_{\varphi} \) are the
partial derivatives of $x$, and $g$ is the associated metric tensor,

$$g(\theta, \varphi) = \begin{bmatrix} x_\theta \cdot x_\theta & x_\theta \cdot x_\varphi \\ x_\varphi \cdot x_\theta & x_\varphi \cdot x_\varphi \end{bmatrix},$$ (2.1)

then the Laplace-Beltrami operator $\Delta_\Gamma$ acting on a smooth function $f = f(\theta, \varphi)$ is defined by the formula

$$\Delta_\Gamma f = \frac{1}{\sqrt{\det g}} \left[ \partial_\theta \quad \partial_\varphi \right] \sqrt{\det g} g^{-1} \left[ \begin{array}{c} \partial_\theta \\ \partial_\varphi \end{array} \right] f$$

(2.2)

$$= \nabla_\Gamma \cdot \nabla_\Gamma f.$$

In the above definition, we have introduced the notation $\nabla_\Gamma \cdot$ and $\nabla_\Gamma$ to denote the surface divergence and surface gradient, respectively. The precise definitions in the case of interest, mainly along piecewise smooth surfaces, are given below in Section 2.2.

With the above definition of $\Delta_\Gamma$, the Laplace-Beltrami problem refers to solving the equation

$$\Delta_\Gamma u = f \quad \text{on } \Gamma$$ (2.3)

for the unknown function $u$. One must define the domains of the data $f$ and solution $u$ to this problem carefully in order to ensure that the problem is well-posed, as the Laplace-Beltrami operator is neither injective nor surjective. The operator has a well-known one-dimensional null space: the space of constant functions along $\Gamma$ [40]. Since the Laplace-Beltrami operator is self-adjoint (see below), the range of the operator is thus the space of functions with zero mean. A standard well-posed version of this problem is summarized in the following theorem, which is proved in [52]:

**Theorem 1.** Suppose that $\Gamma$ is a closed surface that is $C^k$ for some $k \geq 2$. Then, the Laplace-Beltrami problem (2.3) has a unique mean-zero solution for every mean-zero right hand side that is in $L^2(\Gamma)$. Furthermore, if the right hand side is in $H^s(\Gamma)$ for some $s \leq k - 2$, then the solution $u$ will be in $H^{s+2}(\Gamma)$.

We now move onto a practical discussion of the Laplace-Beltrami problem along Lipschitz surfaces, a topic which has received less attention in the literature.

### 2.2 Lipschitz surfaces

In this section, we shall recall the definitions required to clearly state the Laplace-Beltrami problem on a general Lipschitz surface $\Gamma$, as well as summarize the associated invertibility result presented in [24]. Equivalent definitions of the relevant functions spaces and operators are also discussed in [24] in greater detail. The first notion we outline is that of a Lipschitz surface. A similar definition to that which we give below is also contained in [39].

**Definition 2 (Lipschitz surface).** Let $\Gamma$ be a bounded surface embedded in $\mathbb{R}^3$. The surface $\Gamma$ is said to be a Lipschitz surface if there exists a finite open covering $\{O_j\}_{j=1}^N$ and an associated set of rigid rotations $\Sigma_j$ such that $\Sigma_k(\Gamma \cap O_j)$ is the graph of a Lipschitz function. That is to say, for each $j$ there exists an open domain $U_j \subset \mathbb{R}^2$ and a Lipschitz function $\varphi_j : U_j \rightarrow \mathbb{R}^3$ such that $\Sigma_j(\Gamma \cap O_j) = \{(x, y, \varphi_j(x, y)) \mid (x, y) \in U_j\}$.

In the above definition, a Lipschitz surface is written in terms of graphs of Lipschitz functions. It is also possible to describe a Lipschitz surface through local Lipschitz parameterizations. This description will make it simpler to define the relevant function spaces and differential operators. With this in mind, we define Sobolev spaces along $\Gamma$ as follows.
Definition 3 (Sobolev spaces). Let $\Gamma$ be a bounded Lipschitz surface with a finite open covering $\{O_j\}_{j=1}^N$. Also, let $\{x_j\}_{j=1}^N$ be a collection of local Lipschitz parameterizations, such that each $x_i$ maps an open neighborhood $U_i \subset \mathbb{R}^2$ onto $O_i$. Finally, let $\{\chi_j\}_{j=1}^N$ be a partition of unity on $\Gamma$ such that for each $j$, $\chi_j$ is supported on $O_j$. We then have the following definitions:

1. For all $0 \leq s \leq 1$ we define the Sobolev space along $\Gamma$ of order $s$ as
   \[ H^s(\Gamma) = \{ f \in L^2(\Gamma) | (\chi_j f) \circ x_j \in H^s(U_j) \text{ for all } j = 1, \ldots, N \}, \]
   with a norm given by
   \[ \|f\|_{H^s(\Gamma)} = \sum_{j=1}^N \| (\chi_j f) \circ x_j \|_{H^s(U_j)}. \]

2. For all $-1 \leq s < 0$, we define the Sobolev space of order $s$, $H^s(\Gamma)$, as the dual space of $H^{-s}(\Gamma)$.

3. For all $-1 \leq s \leq 1$, we define $H^s_{\text{mz}}(\Gamma)$ to be the subset of $H^s(\Gamma)$ with mean-zero. More explicitly, we set
   \[ H^s_{\text{mz}}(\Gamma) = \{ f \in H^s(\Gamma) | (f, 1)_s,\Gamma = 0 \}, \tag{2.4} \]
   where $(\cdot, \cdot)_s,\Gamma$ is the duality pairing between $H^s(\Gamma)$ and $H^{-s}(\Gamma)$.

4. We define the space of tangent vector fields $L^2_t(\Gamma)$ as the subset of three dimensional vector fields that are tangential to $\Gamma$. If $n$ is the unit normal for $\Gamma$, defined almost everywhere, then this can be written as
   \[ L^2_t(\Gamma) = \left\{ v \in (L^2(\Gamma))^3 \left| \int_{\Gamma} (v \cdot n)^2 = 0 \right. \right\}. \tag{2.5} \]

It should be noted that the above definitions are almost exactly those that are used when $\Gamma$ is smooth. The only difference is that the $x_i$’s are assumed to be Lipschitz instead of smooth, and we have restricted $s$ to $[-1, 1]$. These spaces thus coincide with the usual Sobolev spaces whenever $\Gamma$ happens to be smooth. We also note that for general Lipschitz surfaces, these Sobolev spaces only exist for $|s| \leq 1$; a local Lipschitz parameterization will only have fractional derivatives of order $s$ for $s \leq 1$. Lastly, we make the usual identification $H^0(\Gamma) = L^2(\Gamma)$ since $\|\cdot\|_{H^0(\Gamma)}$ is equivalent to $\|\cdot\|_{L^2(\Gamma)}$ [24].

With the above function spaces defined, we may now move onto defining the associated surface differential operators in the usual ways.

Definition 4. Let $\{x_j\}_{j=1}^N$ and $\{\chi_j\}_{j=1}^N$ be as in Definition 3 and let $g_j$ be the metric associated with $x_j$. Interpreting all partial derivatives in the distributional sense [21], we make the following definitions:

1. For $0 \leq s \leq 1$, the surface gradient $\nabla_{\Gamma}$ of a function $f \in H^s(\Gamma)$ is given by the formula
   \[ \nabla_{\Gamma} f = \sum_j \chi_j [x_{j,\theta} \ x_{j,\varphi} g_j^{-1} \partial_{\theta} \partial_{\varphi}] f, \tag{2.6} \]
   where $x_{j,\theta}$ and $x_{j,\varphi}$ are the weak partial derivatives of $x_j$ and $g_j$ is the associated metric tensor (see Definition 1). It is clear from this formula that $\nabla_{\Gamma} : H^1(\Gamma) \to L^2(\Gamma)$.
2. The surface divergence \( \nabla_T \cdot : \mathbf{L}^2_t(\Gamma) \rightarrow H^{-1}(\Gamma) \) is defined as the negative adjoint of the surface gradient map from \( H^1(\Gamma) \) to \( \mathbf{L}^2_t(\Gamma) \). This may be expressed through the duality product of \( H^{-1}(\Gamma) \) and \( H^1(\Gamma) \):

\[
(\nabla_T \cdot v, f)_{H^{-1}(\Gamma), H^1(\Gamma)} := - \int_{\Gamma} v \cdot \nabla_T f, \quad \forall f \in H^1(\Gamma). \tag{2.7}
\]

3. The Laplace-Beltrami operator \( \Delta_T : H^1(\Gamma) \rightarrow H^{-1}(\Gamma) \) is defined as the composition \( \Delta_T := \nabla_T \cdot \nabla_T \). The formula for applying \( \Delta_T \) to a function \( f \in H^1(\Gamma) \) is the same as for the smooth case (2.2), except that the derivatives must be interpreted in a weak sense.

From a practical point of view, the general definition of the surface divergence given above is difficult to implement numerically. This obstacle is most easily overcome by assuming a particular (piecewise) parameterization of \( f \) along \( \Gamma \) and using the following equivalent definition (see [40]):

if \( v \in \mathbf{L}^2_t(\Gamma) \) is written as \( v|_{\Gamma \cap O_j} = v_\theta x_{j,\theta} + v_\phi x_{j,\phi} \), then

\[
\nabla_T \cdot v|_{\Gamma \cap O_j} = \frac{1}{\sqrt{\det g}} \begin{bmatrix} \partial_\theta & \partial_\phi \end{bmatrix} \sqrt{\det g} \begin{bmatrix} v_\theta \\ v_\phi \end{bmatrix}. \tag{2.8}
\]

While the above formula still uses weak derivatives, we shall be able to reinterpret it in a stronger sense once we make further assumptions on \( \Gamma \).

We also note that the Laplace-Beltrami operator can only be defined on \( H^s(\Gamma) \) for \( s = 1 \). For any other value of \( s \) the domain of either the surface gradient or the surface divergence would not be well-defined on a general Lipschitz surface. We will therefore only consider right hand sides that are contained in \( H^{-1}(\Gamma) \) when attempting to solve the Laplace-Beltrami equation \( \Delta_T u = f \) on such surfaces. As in the smooth case, it will be necessary to assume that \( f \) is mean-zero, and we will also look for a mean-zero solution. The Laplace-Beltrami problem along a Lipschitz surface therefore becomes:

**Problem 1 (Laplace-Beltrami problem).** Let \( f \in H^{-1}_{\text{mz}}(\Gamma) \), the set of functions in \( H^{-1}(\Gamma) \) which are also mean zero. The Laplace-Beltrami problem is to find a \( u \in H^1_{\text{mz}}(\Gamma) \) such that \( \Delta_T u = f \).

The following theorem shows that the problem is well-posed. A proof can be found in [24].

**Theorem 2.** If \( \Gamma \) is a bounded Lipschitz surface without boundary, then the null-space of the Laplace-Beltrami operator is the set of constant functions and the range is \( H^{-1}_{\text{mz}}(\Gamma) \). The Laplace-Beltrami problem on \( \Gamma \) is thus well-posed.

In summary, we now have that the distributional form of our equation is well-posed on a general Lipschitz surface. If we also assume that \( f \in L^2_{\text{mz}}(\Gamma) := \{ h \in L^2(\Gamma) | \int_{\Gamma} h = 0 \} \), then we may reformulate this problem more explicitly in the following weak form.

Find \( u \in H^1_{\text{mz}}(\Gamma) \) such that

\[
\int_{\Gamma} \nabla_T u \cdot \nabla_T v = - \int_{\Gamma} f v \quad \text{for all } v \in H^1_{\text{mz}}(\Gamma). \tag{2.9}
\]

In the next section, we will convert the expression above into one with interface conditions along edges in piecewise smooth geometries.

6
3 Interface conditions

We shall now further restrict ourselves to the case that the surface \( \Gamma \) is a finite union of smooth and closed faces \( \Gamma_i \) and that \( f \in L^2_{m} (\Gamma) \). We will use this added smoothness of the geometry to turn the weak equation (2.9) into a strong interface form on each face augmented with matching conditions along the edges (but away from any corners). We start by defining the class of surfaces that we consider.

**Definition 5** (Piecewise smooth Lipschitz surface). Let \( \{ \Gamma_i \}_{i=1}^{N} \) be a collection of smooth surfaces such that for each \( \Gamma_i \), there exists a closed triangle \( T_i \subset \mathbb{R}^2 \) and a parameterization \( x_i : T_i \rightarrow \Gamma_i \) that is a smooth diffeomorphism. We suppose that each \( \Gamma_i \) is closed, so includes its boundary and that the interiors of the faces, denoted \( \{ \Gamma^o_i \} \), are pairwise disjoint. If the union \( \Gamma = \bigcup_{i=1}^{N} \Gamma_i \) is a Lipschitz surface, then the surface \( \Gamma \) is said to be a piecewise smooth Lipschitz surface with faces \( \{ \Gamma_i \}_{i=1}^{N} \).

**Remark 1.** The above definition may seem restrictive because it requires that every face is a curved triangle. This is not a true restriction as other piecewise smooth surfaces, such as cubes and hemispheres, may be reduced to this case by artificially splitting their faces.

By making this assumption, we can simplify our description of the surface. This in turn will simplify our proofs later on, particularly in Section 3.2. Similarly, we may ensure that each vertex of \( \Gamma \) is contained in at least three faces by splitting faces as necessary. This assumption will also simplify our proof of Theorem 7.

It is important to note that not every surface that is smooth almost everywhere can be made to satisfy Definition 5. For example, cones and other surfaces with conic singularities have unbounded mean curvature, and so are not-piecewise smooth. We will treat the special case of a flat cone in Section 4.

We now develop some notation for piecewise smooth Lipschitz surfaces. If it exists, the edge which is the intersection of the faces \( \Gamma_i \) and \( \Gamma_j \) will be denoted by \( e_{ij} \). We let \( n_i \) be the outward normal along \( \Gamma_i \), \( \tau_i \) be the tangent vector along the boundary \( \partial \Gamma_i \) to face \( \Gamma_i \), and \( b_i = \tau_i \times n_i \). We will refer to \( b_i \) as the binormal vector, and assume that the orientation of \( \tau_i \) was chosen so that \( b_i \) points away from \( \Gamma_i \). These definitions are demonstrated in Figure 1. We refer to the set of vertices of \( \Gamma \), i.e. the points where some \( \partial \Gamma_i \) is not smooth, as \( C \).

Using this notation, we compute the interface form of the equation on each face using integration by parts. For now, we will assume that \( u \) happens to be continuous and in \( C^\infty (\Gamma_i) \) for each \( i \), allowing straightforward integration by parts. After computing the strong interface form, we will go back and justify that the interface form has a solution that is also a weak solution. For this calculation, we also only consider test functions \( v \) in (2.9) that are continuous. Under these new assumptions, the...
weak form (2.9) becomes

$$\sum_i \left( \int_{\Omega_i} v \Delta u + \int_{\partial \Omega_i} v b_i \cdot \nabla u \right) = \sum_i \int_{\Omega_i} f v, \quad \text{for all } v \in C(\Gamma) \cap H^1(\Gamma).$$

(3.1)

Through the usual variational arguments [30], this equation tells us that $$\Delta u |_{\Gamma_i} = f |_{\Gamma_i}$$ almost everywhere for each $i$. By considering choices of $v$ that are supported near $e_{ij}$ for each $i$ and $j$, equation (3.1) also tell us that

$$\int_{e_{ij}} v \left( b_i \cdot \nabla u |_{\Gamma_i} + b_j \cdot \nabla u |_{\Gamma_j} \right) = 0 \quad \text{for all } v \in C(\Gamma) \cap H^1(\Gamma).$$

(3.2)

It is not hard to see that this implies that the one-sided binormal derivatives of $u$ agrees from both sides of an edge. We thus have that if the solution $u$ happens to be smooth enough that we can do the above integration by parts, then $u$ solves the following problem:

**Problem 2** (Interface form of the Laplace-Beltrami problem). *Let $\Gamma$ be a closed Lipschitz surface composed of smooth faces $\{\Gamma_i\}$ and suppose that $f \in L^2_{\text{mz}}(\Gamma)$. The interface form of the Laplace-Beltrami problem is to find $u \in H^1_{\text{mz}}(\Gamma)$ such that $\frac{\partial u}{\partial b_i} |_{\Gamma_i}$ exists in the trace sense defined below and such that

$$\Delta u |_{\Gamma_i} = f |_{\Gamma_i}, \quad \text{a.e. on } \Gamma_i,$$

$$\frac{\partial u}{\partial b_i} |_{\Gamma_i} = - \frac{\partial u}{\partial b_j} |_{\Gamma_j}, \quad \text{on } e_{ij},$$

(3.3)

where the edge conditions are interpreted in the trace sense.*

We note here that the interface form does not involve any corner conditions on $u$ beyond the requirement that the traces exist. This comes from the fact that integration by parts does not introduce any corner conditions on piecewise smooth domains.

Having identified the interface form of the Laplace-Beltrami equation, we will go back and prove that it is well posed and equivalent to the weak form whenever $f$ is in $L^2_{\text{mz}}(\Gamma)$. We do this in four stages. First, in Section 3.1, we recall standard results for elliptic interface problems in the plane. Then, in Section 3.2 we connect the Laplace-Beltrami problem on a surface patch to an interface problem in the plane and we show that there exists a covering by parameterizations such that the corresponding elliptic interface problem is uniformly elliptic. Subsequently, in Section 3.3 we check that the remaining requirements of the planar regularity theorems (Theorems 3 and 4) are satisfied so that we can therefore rigorously carry out the above integration by parts argument on a single surface patch (Theorem 9). Finally, in Section 3.4 we use these local results to prove that the weak solution of the Laplace-Beltrami problem satisfies the interface form on the whole surface (Theorem 10).

### 3.1 Interface problems in the plane

In order to prove the regularity of weak solutions of the Laplace-Beltrami problem, we shall leverage existing results for elliptic interface problems in the plane. To start with, with define such problems.

**Definition 6** (Elliptic interface problem in the plane). *Let the following hold:

1. The set $\Omega = \bigcup_{i=1}^N \Gamma_i \subset \mathbb{R}^2$ is a finite union of triangles such that every pair of triangles $\Gamma_i$ and $\Gamma_j$ either overlap along an entire edge, overlap only at a vertex shared by both triangles, or don't overlap.*
2. The vector field \( \mathbf{v}_i \) is the outward unit normal to the boundary of \( T_i \) for \( i = 1, \ldots, N \), defined almost everywhere on that boundary.

3. The differential operators \( L_i \) are elliptic on \( T_i \) for \( i = 1, \ldots, N \) and can be written as \( L_i = \text{trace}[g_i^{-1} \nabla^2] + \mathbf{h}_i \cdot \nabla \), where \( g_i^{-1} \) is a matrix-valued function and \( \mathbf{h}_i \) is a vector field defined on \( T_i \).

4. The function \( f \) is in \( L^2(\Omega) \).

5. The functions \( \alpha_i \) are positive functions on \( \partial T_i \).

6. The functions \( c_i \) are in \( L^2(\partial T_i \setminus \cup_{j \neq i} \partial T_j) \) for \( i = 1, \ldots, N \).

The elliptic interface problem is to find a function \( u \) such that

\[
L_i u |_{T_i} = f |_{T_i} \quad \text{a.e. on } T_i,
\]

\[
u |_{T_i} = u |_{T_j} \quad \text{a.e. on } T_i \cap T_j,
\]

\[
\alpha_i (g_i^{-1} \mathbf{v}_i) \cdot \nabla u |_{T_i} = -\alpha_j (g_j^{-1} \mathbf{v}_j) \cdot \nabla u |_{T_j} \quad \text{a.e. on } T_i \cap T_j,
\]

\[
u |_{T_i} = c_i \quad \text{a.e. on } \partial T_i \cap \partial \Omega,
\]

for all \( i \) and \( j \) such the corresponding sets are non-empty, that \( u \) is in \( H^1(T_i) \) for each \( i \), and that the trace of \( \nabla u \) can be taken in then sense discussed below Corollary 4.1.

The regularity of solutions to an elliptic interface problem will depend strongly on their behaviour near vertices. To analyze this, we look at homogeneous solutions of (3.4) in the region around a single interior or exterior vertex. These corner solutions are discussed in detail in [41]. Here we summarize by noting that they are of the form \( q^{s} \tau_s(\theta_s) \), where \( (r_s, \theta_s) \) are polar coordinated centered at the vertex \( s \). The class of allowable \( \lambda_s \) and \( \tau_s \) are defined below. For some higher order problems, there also exist corner solutions which behave as \( r_s^{d_s} (\ln r_s)^q \tau_s(\theta_s) \), where \( q \) is some positive integer, but those do not occur in our second-order problem (see Lemma 5 below), so we neglect them in this discussion.

**Definition 7.** Let \( s \in \mathbb{R}^2 \) be a vertex in an elliptic interface problem and let \( (r_s, \theta_s) \) be the polar coordinates centered at \( s \). The complex number \( \lambda_s \) and continuous, 2\( \pi \)-periodic, and piecewise smooth function \( \tau_s \) form an expansion pair for the vertex \( s \) in the elliptic interface problem if they satisfy the differential condition

\[
L_i r_s^{d_s} \tau_s(\theta_s) = o(r_s^{d_s-2}) \quad \text{as } r_s \to 0
\]

for all \( i \) such that \( T_i \) touches \( s \), the matching condition

\[
\alpha_i (g_i^{-1} \mathbf{v}_i) \cdot \nabla \left( r_s^{d_s} \tau_s \right) |_{T_i} = -\alpha_j (g_j^{-1} \mathbf{v}_j) \cdot \nabla \left( r_s^{d_s} \tau_s(\theta_s) \right) |_{T_j} + o(r_s^{d_s-1}) \quad \text{as } r_s \to 0
\]

for all \( i \) and \( j \) such that \( T_i \) and \( T_j \) share an edge touching \( s \), and the boundary condition \( \tau_s = 0 \) on \( \partial \Omega \). In the following, we refer to \( \lambda_s \) as an expansion power for the vertex \( s \) and to \( \tau_s \) as the angular function associated with \( \lambda_s \).

For the above expansion powers to be discrete, we must impose some more assumptions on the operators in the interface problem. We now define an alternative, more regular, interface problem.

**Definition 8 (Regular elliptic interface problem).** An elliptic interface problem is regular if the matrix valued functions \( g_i^{-1} \) are uniformly positive definite on their domains and the coefficient functions \( g_i^{-1}, h_i, \) and \( \alpha_i \) are smooth and bounded.
The following two theorems give the existence and regularity of the solution of an elliptic interface problem. These are Theorems 4.2 in [41] and 8.6 in [42], restricted to the case of second-order equations with Dirichlet boundary conditions on $\partial \Omega$.

**Theorem 3** (Existence, Nicaise and Sändig Theorem 4.2). Let $V$ be the set of corners of the domain $\Omega$ in Definition 6, i.e., points that are a vertex of some $T_i$. Also let $\{\lambda_{n,s}\}_n$ be the expansion powers of the vertex $s \in V$ for an elliptic interface problem. If all of the powers $\lambda_{n,s}$ are real and the right hand side $f$ is in $L^2(\Omega)$, then there exists a unique function $v \in H^1(T^0)$ that solves that elliptic interface problem in a sense described in Section 3 of [41] with $c_i = 0$ for each $i$.

**Theorem 4** (Regularity, Nicaise and Sändig Theorem 8.6). Suppose that the assumptions of Theorem 3 hold and $v$ is the identified solution. If $f$ is also in $H^k(T^0)$ for every $i$, where $k = 0, 1, 2$, and the neighborhood $V$ contains exactly one vertex of $\Omega$, then for any point in $V$ we may write the solution $v$ as

$$v = v_0 + \sum_{\lambda_{n,s} \leq (k+1), \lambda_{n,s} \neq 0} a_n r_s^{\lambda_{n,s}} \tau_{n,s}(\theta_s),$$

(3.7)

where the function $v_0$ is in $H^{k+2}(T^0 \cap V)$ for each $i$ such that $T_i$ touches $s$ and satisfies the interface conditions in (3.4) in the usual trace sense, where $a_n \in \mathbb{C}$ for each $n$, where $(r_s, \theta_s)$ are polar coordinates around the corner $s$, and where $\tau_{n,s}$ is the expansion function associated with $\lambda_{n,s}$.

**Remark 2.** Nicaise and Sändig Theorem 8.6 also includes the case where $k > 2$, but when looking for higher regularity, equation (3.21) must also include weak interface singularities. We thus restricted Theorem 4 to the case of moderate regularity ($k \leq 2$) in order to simplify our discussion.

The above theorem can be applied to give a global representation of the solution $v$. This extension will need the partition of unity given in the following lemma to ensure that each part of the decomposition satisfies the interface conditions.

**Lemma 1.** If $\{V_s\}_{s \in \gamma}$ is a collection of neighborhoods covering $\Omega$ such that each $V_s$ contains only the vertex $s$, then there exists a partition of unity $\{\xi_s\}$ such that each $\xi_s$ is supported in $V_s$, is smooth in each region $T_i$, and satisfies the interface conditions in (3.4).

**Proof.** Let $\{\eta_s\}_{s \in \gamma}$ be a smooth partition of unity covering $\Omega$ and such that $\eta_s$ is compactly supported in $V_s$. We shall modify this partition of unity to satisfy the interface conditions. We suppose that $t_{ij}$ is an arc-length parameter for the edge $e_{ij} = T_i \cap T_j$. For each value of $t_{ij}$, we define two line segments $C_{t_{ij}}^i$ and $C_{t_{ij}}^j$ that start on $e_{ij}$, are tangent to $g_i^{-1}v_i$ and $g_j^{-1}v_j$ respectively, and have length $\epsilon_{t_{ij}} > 0$ (see Figure 2). We set $\epsilon_{t_{ij}}$ such that $C_{t_{ij}}^i$ and $C_{t_{ij}}^j$ don’t include any other edge $e_{i'j'}$ or $\partial \Omega$. We also assume that $\epsilon_{t_{ij}}$ is chosen to vary smoothly with $t_{ij}$ and is less than the maximum of $e|g_i^{-1}\nabla_\eta_s|$ and $e|g_j^{-1}\nabla_\eta_s|$ for all $s$.

On each line segment, we define a piecewise smooth correction function $Q_{t_{ij}}$ as

$$Q_{t_{ij}}(u) = 1 + \begin{cases} \begin{array}{l} e \frac{g_i^{-1}v_i \cdot \nabla_\eta_s}{\epsilon_{t_{ij}}} \left| e \frac{1}{1/(u/\epsilon_{t_{ij}})^2} \right| \text{ if } -\epsilon_{t_{ij}} < u < 0 \\ e \frac{g_j^{-1}v_j \cdot \nabla_\eta_s}{\epsilon_{t_{ij}}} \left| e \frac{1}{1/(u/\epsilon_{t_{ij}})^2} \right| \text{ if } 0 \leq u < \epsilon_{t_{ij}} \end{array} \right.,$$

(3.8)

where $u$ is a parameter for the union of $C_{t_{ij}}^i$ and $C_{t_{ij}}^j$. With this definition, $Q_{t_{ij}}$ can be smoothly extended by one on either side of $e_{ij}$. The product rule gives that $g_i^{-1}v_i \cdot \nabla Q_{t_{ij},\eta_s}e_{ij} = 0$, so $Q_{t_{ij},\eta_s}e_{ij}$ satisfies the interface conditions on $e_{ij}$ (see Figure 2b).
We note that \( Q_{t_{ij},s} \) varies smoothly with \( t_{ij} \) because \( \eta_s \) and \( \epsilon_{t_{ij}} \) do. With these assumptions, we have that \( Q_{t_{ij},s} \eta_s \) is smooth on the interior of \( T_i \). The fact that \( \eta_s \) must be flat around either end of \( \epsilon_{t_{ij}} \) will then give that \( Q_{t_{ij},s} \eta_s \) is smooth up to \( \partial T_i \) for each \( i \), even though \( \epsilon_{t_{ij}} \) might vanish there. We also note that \( Q_{t_{ij},s} \) is positive because of the upper bound on \( \epsilon_{t_{ij}} \). We thus have that \( Q_{t_{ij},s} \eta_s \) has the same support as \( \eta_s \).

Including all such corrections, we define \( \tilde{\chi}_s = \bigcap_{ij} Q_{t_{ij},s} \eta_s \). By definition, \( \tilde{\chi}_s \) will be non-negative and satisfy the interface conditions in (3.4) and be smooth on \( T_i \) for each \( i \). We also know that \( \sum_{s \in \mathcal{V}} \tilde{\chi}_s \) is never zero in \( \Omega \) because \( \{ \eta_s \} \) cover \( \Omega \). The functions \( \tilde{\chi}_s = \tilde{\chi}_s / \sum_{s' \in \mathcal{V}} \tilde{\chi}_{s'} \) will thus form the desired partition of unity.

**Corollary 4.1.** Let \( \{ V_s \}_{s \in \mathcal{V}} \) and \( \{ \tilde{\chi}_s \} \) be as in Lemma 1. Then the solution of the elliptic interface problem \( v \) identified in Theorems 3 and 4 can be written as

\[
v = v_0 + \sum_{s \in \mathcal{V}} \tilde{\chi}_s \sum_{\lambda_n,s \leq (k+1)} a_{n,s} r^{\lambda_n,s} \tau_{n,s}(\theta_s),
\]

(3.9)

where the function \( v_0 \) is in \( H^{k+2}(\Gamma_i^o) \) for each \( i \) and satisfies the interface conditions.

In the above theorem, if \( \lambda_{n,s} < 1 \), for any \( n \), then the solution \( u \) will not be smooth enough for us to take the trace of \( \nabla u \) in the traditional sense. In this case, we compute the trace along an edge by evaluating the explicit function expansion terms at the edge and adding the result to the trace of \( v_0 \).

### 3.2 Regularity of the pull-back of the Laplace-Beltrami problem

In order to see the equivalence between the interface form of the Laplace-Beltrami problem (3.3) on \( \Gamma \) to problem (3.4), we must define the trace operator and a function space that is smooth enough so that we may take the trace of the derivative. To define this space, we note that on a piecewise smooth Lipschitz surface, the spaces \( H^s(\Gamma_i^o) \) may be defined for each \( i \) and for any \( s \in \mathbb{R} \) through an analogue of Definition 3 based on smooth parameterizations, even though \( H^s(\Gamma) \) may not exist. We thus define the space of piecewise \( H^s \) functions as follows:
**Definition 9.** For any $s \geq 0$, set
\[
\mathcal{H}^s(\Gamma) := \left\{ v \in L^2(\Gamma) : v|\Gamma_i^o \in H^s(\Gamma_i^o) \text{ for all } i \right\}. \tag{3.10}
\]

Having identified the appropriate function space, we now formally state the definition of the trace along an edge:

**Definition 10 (Trace).** Suppose $x_i$ is a smooth parameterization of $\Gamma_i$ with domain $T_i$. If $u \in H^s(\Gamma_i^o)$ for some $s > \frac{1}{2}$, then the trace of $u$ is defined as
\[
\text{tr}_{\partial T_i} u := \text{tr}_{\partial T_i}(u \circ x_i) \circ x_i^{-1}, \tag{3.11}
\]
where $\text{tr}_{\partial T}$ is the usual trace operator for the bounded Lipschitz domain $T$, which is defined in Theorem 18.1 of [37].

Furthermore, if $v \in \mathcal{H}^s(\Gamma)$ for some $s > \frac{1}{2}$, then the trace of $v$ along a surface edge is defined as
\[
\text{tr}_{\partial T_i} v = \text{tr}_{\partial T_i}\left(v|_{\Gamma_i^o}\right). \tag{3.12}
\]

By Theorem 18.1 of [37], we know that the range of the trace operator is $H^{s-\frac{1}{2}}(\partial \Gamma_i)$, which may be defined equivalently to $H^{\frac{1}{2}}(\Gamma)$ in Definition 3.

**Theorem 5.** The operator $\text{tr}_{\partial T_i}$ is independent of the local parameterization used in its definition.

**Proof.** We begin by showing that $C^\infty(\Gamma_i)$ is dense in $H^s(\Gamma_i^o)$ for all $s > 0$. We suppose that $x_i$ is a smooth parameterization of $\Gamma_i$ with domain $T_i$. By definition, $H^s(\Gamma_i^o) = \{ f \in L^2(\Gamma_i^o) : f \circ x_i \in H^s(T_i) \}$ and $C^\infty(\Gamma_i) = \{ f \in C(\Gamma_i) : f \circ x_i \in C^\infty(T_i) \}$. The standard density result then implies that $C^\infty(\Gamma_i)$ is dense in $H^s(\Gamma_i^o)$.

Next, we note that if $v \in C^\infty(\Gamma_i)$, then $\text{tr}_{\partial T_i} v$ is independent of the smooth local parameterization used in its definition. The density of $C^\infty(\Gamma_i)$ thus implies that the trace operator is independent of the local parameterization. \qed

We now state and prove the equivalence of the interface form of the Laplace-Beltrami problem with Dirichlet boundary conditions on a single surface patch to the above elliptic problem in the following theorem. We shall specifically consider surface patches consisting of several faces: $\Gamma = \bigcup_{i \in I} \Gamma_i$, where $I$ is a subset of $\{1, \ldots, N\}$.

**Theorem 6.** Let $\bar{\Gamma}$ be a patch of $\Gamma$ parameterized by a piecewise smooth parameterization $\bar{x}$ with domain $U$. Also suppose that $\Gamma_i = \bar{x}^{-1}(\bar{\Gamma} \cap \Gamma_i)$. Finally, let $f \in L^2(\bar{\Gamma})$ and let $c \in L^2(\partial \bar{\Gamma})$. A function $u$ solves the interface form of the Laplace-Beltrami problem on the $\Gamma$ with right hand side $f$ and Dirichlet boundary data $c$ if and only if $u \circ \bar{x}^{-1}$ solves the elliptic interface problem in the plane with right hand side $\bar{f} = f \circ \bar{x}^{-1}$, derivative matching coefficients $\alpha_i = (b_i^T g^{-1} b_i)^{-1/2}$, boundary data $c_i = (c \circ \bar{x}^{-1})|_{\partial \Gamma_i} \cup_{i \in I} \partial T_i$, and $L_i$ being the pull-back from $\Gamma_i$ of the Laplace-Beltrami operator by $\bar{x}$ (3.13).

**Proof.** We first check that the domain and operators in the plane satisfy the requirements of Definition 6. The domain of the piecewise smooth local parameterization $\bar{x}$ can be decomposed into subdomains $\Gamma_i$ corresponding to each face in $\bar{\Gamma}$. Each pair of these $\Gamma_i$’s either overlap along an edge or just at a vertex because the parameterization $\bar{x}$ is injective and $\bar{\Gamma}$ is a subsurface of a piecewise smooth Lipschitz surface.
Next, we may use the explicit forms of the surface gradient and divergence above, to see that the operators $L_i$ have the form

$$L_i u = \frac{1}{\sqrt{\det g_i}} \left[ \frac{\partial \theta}{\partial x} \frac{\partial \varphi}{\partial x} \right] \sqrt{\det g_i} g_i^{-1} \left[ \frac{\partial \theta}{\partial \varphi} \right] u$$

$$= \text{trace} \left[ g_i^{-1} \left[ \frac{\partial^2}{\partial \theta \partial \varphi} \frac{\partial \theta}{\partial \varphi} \frac{\partial \varphi}{\partial \varphi} \right] u \right] + \frac{1}{\sqrt{\det g_i}} \left( \left[ \frac{\partial \theta}{\partial \varphi} \right] \sqrt{\det g_i} g_i^{-1} \right) \left[ \frac{\partial \theta}{\partial \varphi} \right] u,$$

where $g_i$ is the surface metric on $T_i$ (2.1). Letting

$$h_i = \frac{1}{\sqrt{\det g_i}} \left( \left[ \frac{\partial \theta}{\partial \varphi} \right] \sqrt{\det g_i} g_i^{-1} \right),$$

using the definition of the metric, and the fact that $\tilde{x}$ is piecewise smooth, we have that $L_i$ is an elliptic operator on each $T_i$, though it may not be uniformly elliptic. The Lipschitz property of $\tilde{x}$ will also give that $\tilde{f}$, $\alpha_i$, and $c_i$ have the required integrability. We have thus identified the elliptic interface problem corresponding to the interface form Laplace-Beltrami problem on $\tilde{F}$.

We now check that (3.4) is equivalent to (3.3). The first equation follows directly from the definition of the pull-back of the Laplace-Beltrami operator. The second and fourth equations in (3.4) are equivalent to the trace conditions in (3.3) because of our definition of the trace. To see the equivalence of the derivative matching conditions, we let $t_i$ be the unit tangent to $\partial T_i$. The vector field $[\tilde{x}_\theta \ \tilde{x}_\varphi] t_i$ is then tangent to $\partial T_i$. We also write the binormal vector $b_i$ to $\partial T_i$ as $[\tilde{x}_\theta \ \tilde{x}_\varphi] w_i$. Combining these gives that

$$0 = \left[ \tilde{x}_\theta \ \tilde{x}_\varphi \right] t_i \cdot b_i = \left[ \tilde{x}_\theta \ \tilde{x}_\varphi \right] t_i \cdot \left[ \tilde{x}_\theta \ \tilde{x}_\varphi \right] w_i = t_i \cdot g_i w_i,$$

where $g_i$ is the surface metric (2.1). We thus have that $g_i w_i$ is perpendicular to $t_i$. This gives that $w_i = \alpha_i g_i^{-1} v_i$, where the value of $\alpha_i$ given above guarantees that $b_i$ is normalized. We may then use the definition of the surface gradient to compute

$$b_i \cdot \text{tr}_{\partial T_i} \nabla u = \alpha_i v_i \cdot \text{tr}_{\partial T_i} (g_i^{-1} g_i \nabla (u \circ \tilde{x}^{-1}))$$

$$= \alpha_i v_i \cdot \text{tr}_{\partial T_i} \nabla (u \circ \tilde{x}^{-1}).$$

Applying this identity to both sides of the edge $e_{ij}$ will give the equivalence of the derivative matching conditions. We have thus proved that $u \circ \tilde{x}^{-1}$ satisfies each part of (3.4) if and only if $u$ satisfies each part of (3.3). \hfill $\square$

With the above theorem, we can see how we might apply Theorems 3 and 4 to give results on the Laplace-Beltrami problem. It remains, however, to show that there are parameterizations such that the elliptic interface problem identified in Theorem 6 is regular. This will be simplest if we specifically consider $\tilde{F}$ to be the union of faces touching a surface vertex $S$: $\Gamma_S = \cup_{i \in T_i} \Gamma_i$. We thus wish to prove the following theorem.

**Theorem 7 (Suitable parameterization).** There exists a piecwise smooth parameterization $x_S$ of $\Gamma_S$ with domain $U_S$ such that the elliptic interface problems identified in Theorem 6 is regular.

To prove this theorem, we start by noting that the pull-back of the Laplace-Beltrami operator by a smooth parameterization of a single face is uniformly elliptic.
Lemma 2. Let $\Gamma_i$ be a closed face of a piecewise smooth Lipschitz surface $\Gamma$ with a smooth parameterization $x_i$ that maps the triangle $T_i$ to $\Gamma_i$. The pull-back $L_i$ of the Laplace-Beltrami operator under $x_i$,

$$L_i u := A_T(u \circ x_i^{-1}) \circ x_i,$$

(3.16)
is uniformly elliptic with smooth coefficients on $T_i$, the domain of $x_i$.

Proof. The coefficient in front of each derivative term in $(3.13)$ is in $C^\infty(T_i)$ because $x_i$ was assumed to be a smooth diffeomorphism on $T_i$ up to its boundary. To see that the pull-back is uniformly elliptic, we note that the partial derivatives of $x_i$ are bounded. The trace of $g$ is therefore bounded above. Since $g^{-1}$ is positive definite, this in turn ensures that the eigenvalues of $g^{-1}$ are bounded away from zero. The operator $L_i$ is thus uniformly elliptic on the closed triangle $T_i$. □

The parameterizations in Lemma 2 cannot be used to study the effect of surface edges on the solution of the Laplace-Beltrami problem because they do no cover the edges. Instead, we use them as building blocks to create the desired local parameterizations of the patches $\Gamma_S$: we shall combine the parameterizations of several faces into a single piecewise smooth parameterization by manipulating the domain of the parameterizations of the involved faces. The first such manipulation is contained in the following lemma.

Lemma 3. Let $\Gamma_i$ be a closed face of a piecewise smooth Lipschitz surface $\Gamma$. Also let $T_i \subset \mathbb{R}^2$ be any closed non-degenerate triangle. There exists a smooth parameterization $x_i$ of $T_i$ whose domain is $T_i$.

Proof. Let $\bar{\chi}$ be a smooth parameterization mapping some triangle $\bar{T} \subset \mathbb{R}^2$ to $\Gamma_i$, which must exist by definition. Let the vertices of $\bar{T}$ be $\bar{v}_1, \bar{v}_2,$ and $\bar{v}_3$. Let the vertices of $T_i$ be $v_1, v_2,$ and $v_3$. We define the affine transformation $\chi : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$\chi(v) = \begin{bmatrix} v_2 - v_1 & v_3 - v_1 \\ \bar{v}_2 - \bar{v}_1 & \bar{v}_3 - \bar{v}_1 \end{bmatrix}^{-1} (v - \bar{v}_1) + v_1.$$

It is clear that $\chi(\bar{T}) = T_i$, since if $a_1, a_2,$ and $a_3$ are three real numbers that add to 1, then

$$\chi(a_1 \bar{v}_1 + a_2 \bar{v}_2 + a_3 \bar{v}_3) = a_1 v_1 + a_2 v_2 + a_3 v_3.$$

The transformation $\chi$ is also a smooth diffeomorphism because the matrices in its definition are non-singular. The parameterization $x_i = \bar{x} \circ \chi^{-1}$ is therefore the desired one. □

In order to convert the interface form of the Laplace-Beltrami problem into a regular interface problem in the plane, we shall construct a piecewise smooth parameterization $x_S$ that maps its domain onto several adjacent faces. We construct $x_S$ piecewise, by using the above lemma to pick parameterizations of several faces such their domains are adjacent triangles in the plane. The above lemma is not sufficient, however, to ensure that $x_S$ is continuous: the parameterizations of two adjacent faces need not agree along the edge shared by their faces. To force the parameterizations to agree, we shall suppose that they preserve the arclength along that edge. We prove this is possible in the following lemma:

Lemma 4. Let $\Gamma_i$ be a face of a piecewise smooth surface $\Gamma$ and $e_{ij_1}$ and $e_{ij_2}$ be edges of $\Gamma_i$. There exists a parameterization $x_i$ of $\Gamma_i$ whose triangular domain $T$ has edges $e_1, e_2,$ and $e_3$, such that $e_1$ and $e_2$ are mapped onto $e_{ij_1}$ and $e_{ij_2}$ in an arclength-preserving manner.
Proof. Let \( \tilde{x} \) be a smooth parameterization mapping some triangle \( \tilde{T} \subset \mathbb{R}^2 \) to \( I_i \), let \( s_1 = |e_{ij_1}| \), and let \( s_2 = |e_{ij_2}| \). By Lemma 3, we may assume that

\[
\tilde{T} = \{(\theta, \varphi) \in \mathbb{R}^2 \mid 0 \leq \theta, \varphi \quad \text{and} \quad \theta/s_1 \leq 1 - \varphi/s_2\},
\]

and that \( \tilde{x} \) maps the lines \( \varphi = 0 \) and \( \theta = 0 \) onto \( e_{ij_1} \) and \( e_{ij_2} \), respectively. We now construct a reparameterization \( \Phi : \tilde{T} \to \tilde{T} \) such that this mapping is arc-length preserving for each \( \varphi \).

We begin by defining the family of curves \( r_\varphi(\theta) = \tilde{x}(\theta, \varphi) \), see Figure 3. We also let the arclength along the curve \( r_\varphi \) from \( r_\varphi(0) \) to \( r_\varphi(\theta) \) be denoted by

\[
s_\varphi(\theta) = \int_0^\theta |\partial_\theta \tilde{x}(\theta', \varphi)| \, du'.
\]

We define our reparameterization \( \Phi : \tilde{T} \to \mathbb{R}^2 \) by

\[
\Phi(\theta, \varphi) = \begin{pmatrix} s_\varphi^{-1}(\theta) \\ s_\varphi^{-1}(s_1 - \frac{s_1 \varphi}{s_2}) \end{pmatrix} \begin{pmatrix} s_1 - \frac{s_1 \varphi}{s_2} \\ \varphi \end{pmatrix},
\]

where \( s_\varphi^{-1} \) is the inverse of \( s_\varphi \). We see that \( x_i \) has the desired property that \( x_i'(\theta, 0) = \tilde{x}(\Phi(\theta, 0)) \) is an arclength parameterization of \( e_{ij_1} \) because

\[
|\partial_\theta x_i'(\theta, 0)| = |\partial_\theta [\tilde{x}(\Phi(\theta, 0))]| = \left| \partial_\theta \tilde{x} \big|_{\Phi(\theta, 0)} \right| \left| \frac{s_1}{s_0^{-1}(s_1)} \right| = 1.
\]

To see that \( x_i' \) is a smooth diffeomorphism mapping \( \tilde{T} \) to \( I_i \), it is enough to show that \( \Phi(\tilde{T}) = \tilde{T} \) and that \( \Phi \) is a smooth diffeomorphism from \( \tilde{T} \to \tilde{T} \). The fact that \( \Phi(\tilde{T}) = \tilde{T} \) follows from the monotonicity of \( s_\varphi^{-1}(\theta) \) with respect to \( \theta \). To see that \( \Phi \) is a smooth map from \( \tilde{T} \to \tilde{T} \), we note that
the inverse function theorem gives that
\[
\left( 1 - \frac{\varphi}{s_2} \right)^{-1} \left( s_2 - s_1 \frac{\varphi}{s_2} \right) \left( s_2 - \frac{\varphi}{s_2} \right) = \left( 1 - \frac{\varphi}{s_2} \right)^{-1} \left( s_1 - \frac{s_1 \varphi}{s_2} \right) |\partial_{\theta} \tilde{x}(\theta, \varphi)|^{-1} + O \left( \left( 1 - \frac{\varphi}{s_2} \right)^2 \right).
\]
Since the right side is a smooth function of \( \varphi \), even near \( \varphi = s_2 \), the reparameterization \( \Phi \) is smooth. A similar argument can be used to prove that \( \Phi^{-1} \) is also smooth so that \( \Phi \) is a smooth diffeomorphism.

We thus have that \( x'_i = \tilde{x} \circ \Phi \) is a smooth parameterization of \( I_i \) such that the edge \( \varphi = 0 \) of \( \tilde{T} \) is mapped to \( I_i \) in an arc-length preserving manner. Repeating the above argument with \( \theta \) and \( \varphi \) swapped will give a new parameterization \( x_i \) such that both the edges \( \theta = 0 \) and \( \varphi = 0 \) are mapped in an arc-length preserving manner.

We note that the affine transformation \( \chi \) in the proof of Lemma 3 preserves the arclength of any edge of \( T_i \) that gets mapped to an edge of \( \tilde{T} \) with the same length. Thus, if we compose the parameterization in the above proof with \( \chi \), we have a parameterization of \( I_i \) satisfying the requirements of the following theorem:

**Theorem 8.** Let \( I_i \) be a face of a piecewise smooth surface \( \Gamma \) with edges \( e_{ij}, e_{ij_1}, \) and \( e_{ij_2} \). Also let \( T \subseteq \mathbb{R}^2 \) be any triangle such that two of its sides, \( e_1 \) and \( e_2 \), have the same lengths \( e_{ij} \) and \( e_{ij_2} \). There exists a parameterization \( (T, x) \) of \( I_i \) such that \( e_1 \) and \( e_2 \) are mapped onto \( e_{ij} \) and \( e_{ij_2} \) in an arclength-preserving manner.

We now combine the parameterizations of all the faces that share a vertex \( S \) into a parameterization of the union of those faces, \( I_S \). This will give us the required piecewise smooth parameterization \( x_S \).

**Proof of Theorem 7.** For every vertex \( S \) of \( \Gamma \), we define \( \eta_S = \{ i_1, \ldots, i_{N_S} \} \) to be the set of faces that touch \( S \). We shall assume that \( \eta_S \) is sorted so that \( e_{ij_{j-1}} \) is non-empty for each \( j \), with \( j - 1 \) being interpreted mod \( N_S \). For each of these faces \( I_{ij} \), we define \( T_{ij} \) to be the triangle with vertices
\[
v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad v_2 = |e_{ij_{j-1}}| \begin{pmatrix} \cos \left( \frac{2\pi}{N_S} (j - 1) \right) \\ \sin \left( \frac{2\pi}{N_S} (j - 1) \right) \end{pmatrix}, \quad \text{and} \quad v_3 = |e_{ij_{j-1}}| \begin{pmatrix} \cos \left( \frac{2\pi}{N_S} j \right) \\ \sin \left( \frac{2\pi}{N_S} j \right) \end{pmatrix}.
\]
We use Theorem 8 to pick a parameterization \( (T_{ij}, x_{ij}) \) of \( I_{ij} \) such that the edges of \( T_{ij} \) touching the origin are mapped in an arc-length preserving manner to \( e_{ij_{j-1}} \) and \( e_{ij_{j+1}} \). The setup resulting from defining \( T_{ij} \) and \( x_{ij} \) for \( j = 1, \ldots, N_S \) is depicted in Figure 4.

We then define the neighborhood \( U_S \) to be the interior of \( \cup_{j=1}^{N_S} T_{ij} \) and define the map \( x_S : U_S \rightarrow \Gamma \) piecewise by \( x_S|_{T_{ij}} = x_{ij} \). We note that because the \( x_{ij} \)'s were chosen to preserve arc-length on the edges contained in \( U_S \), the parameterizations \( x_{ij} \) and \( x_{ij_{j-1}} \) will agree on the edge where their domains intersect. The map \( x_S \) will also be Lipschitz because each \( x_{ij} \) is smooth. We also know that since \( x_S|_{T_{ij}} \) is smooth for each \( j \), the pull-back of the Laplace-Beltrami operator will be uniformly elliptic on the pre-image of each face, by Lemma 2. The smoothness of \( x_S \) will also ensure that the functions \( g_i^{-1}, h_i \) and \( \alpha_i \) are smooth. The elliptic interface problem identified in Theorem 6 is thus regular when \( x_S \) is as defined above.

We have now proved that there exists a piecewise smooth parameterization such that the pull-back of the Laplace-Beltrami problem from a single patch is a regular elliptic interface form. In the next section, we identify the expansion pairs for each vertex in that interface problem. This will allows us to apply Theorems 3 and 4 to the interface form of the Laplace-Beltrami problem on a single patch.
3.3 Solution of the interface form around a single vertex

In the previous section, we identified the equivalence between the interface form of the Laplace-Beltrami problem and an interface problem in the plane. We shall now leverage results in the plane to prove that the weak solution of the Laplace-Beltrami problem solves the interface form. We do this as follows: First, we state the weak form of the Laplace-Beltrami problem on a single patch. Next we verify that the expansion powers for each vertex are real so that Theorems 3 and 4 can be used to prove the equivalence of the weak and interface forms on a single patch in Theorem 9.

We shall consider the Laplace-Beltrami problem on a single patch with homogeneous Dirichlet boundary conditions. We first define the functions space we work in, then give the problem statement.

Definition 11. Let $\tilde{\Gamma}$ be a Lipschitz surface with boundary. The Sobolev space $H^1_0(\tilde{\Gamma})$ is the set of $v$ in $H^1(\tilde{\Gamma})$ with $\text{tr}_{\partial\Gamma} v = 0$.

Problem 3 (Weak form on a single patch). Let $\tilde{\Gamma}$ be a Lipschitz surface with boundary and let $f \in L^2(\Gamma)$. A function $u \in H^1_0(\tilde{\Gamma})$ solves the the weak form of the Laplace-Beltrami problem on $\tilde{\Gamma}$ with homogeneous Dirichlet boundary conditions on $\partial\tilde{\Gamma}$ if

$$
- \int_{\tilde{\Gamma}} \nabla u \cdot \nabla v = \int_{\tilde{\Gamma}} f v, \quad \text{for all } v \in H^1_0(\tilde{\Gamma}).
$$

(3.18)

In order to apply Theorem 3 in a useful manner, we need the expansion pairs for each vertex of the surface $\Gamma$. As these only depend on the structure of the elliptic interface problem in an infinitesimal region of the vertex, and the surfaces we consider are piecewise smooth, it is enough to compute the expansion pairs for a corner formed by flat faces and straight edges. An example of such a geometry and the coordinate system that we consider is in Figure 5. We summarize the result in the following lemma.

Lemma 5. Suppose that $\Gamma_1, \ldots, \Gamma_n$ are a collection of two-dimensional infinite planar wedges embedded in $\mathbb{R}^3$. Let $\tilde{\Gamma}$ be the union of these faces and suppose that all faces meet at a single vertex, and that $\tilde{\Gamma}_i$ shares an edge with $\tilde{\Gamma}_{i+1}$ for $i = 1, \ldots, n - 1$. Lastly, let each face be parameterized by
the polar coordinates \((r_i, \theta_i)\) with \(\theta_i = 0\) on the edge \(e_{i(i-1)}\) and \(\theta_i = \gamma_i\) on \(e_{i(i+1)}\). The piecewise parameterization of \(\tilde{\Gamma}\) is then
\[
(r, \theta) = (r_i, \theta_i + \sum_{j=1}^{i-1} \gamma_j) \text{ on } \tilde{\Gamma}_i.
\] (3.19)

The sum \(\gamma := \sum_{i=1}^{n} \gamma_i\) is known as the conic angle of the vertex. If \(B_R\) is a ball of radius \(R\) centered on the vertex and \(u \in H^1(B_R \cap \tilde{\Gamma})\) is a solution of \(\Delta_{\tilde{\Gamma}} u = 0\), then one of the following holds.

1. The wedges \(\tilde{\Gamma}_i\) and \(\tilde{\Gamma}_n\) share an edge and for all \(r < R\) the solution \(u\) can be written as
\[
u(r, \theta) = \sum_{n=-\infty}^{\infty} a_n r^{\frac{2\pi}{\gamma} |n|} e^{i \frac{2\pi}{\gamma} n \theta}
\] (3.20)
for some set of coefficients \(a_n \in \mathbb{C}\). Furthermore, if \(k < 1 + 2\pi / \gamma\), then on each face \(\tilde{\Gamma}_i\) we have that \(u \in H^k(B_R \cap \tilde{\Gamma}_i)\).

2. The wedges \(\tilde{\Gamma}_i\) and \(\tilde{\Gamma}_n\) do not share an edge, we enforce that \(\text{tr}_{\partial \tilde{\Gamma}} u = 0\), and for all \(r < R\) the solution \(u\) can be written as
\[
u(r, \theta) = \sum_{n=1}^{\infty} a_n r^{\frac{2\pi}{\gamma} |n|} \sin \left( \frac{\pi}{\gamma} n \theta \right)
\] (3.21)
for some set of coefficients \(a_n \in \mathbb{C}\). Furthermore, if \(k < 1 + \pi / \gamma\), then on each face \(\tilde{\Gamma}_i\) we have that \(u \in H^k(B_R \cap \tilde{\Gamma}_i)\).

Before we prove this lemma, we make a few observations. First, we note that in the case where \(\tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_n\) are co-planar and \(\tilde{\Gamma}_1\) and \(\tilde{\Gamma}_n\) share an edge, \(\tilde{\Gamma}\) will be a plane in \(\mathbb{R}^3\). The conic angle \(\gamma\) will be \(2\pi\) and so (3.20) becomes the usual separation of variables solutions to Laplace’s equation in the plane and is therefore smooth, as we would expect. We also note that if a vertex \(S\) of \(\Gamma\) is artificial, in the sense that it was added in order to triangulate the surface, the conic angle at that vertex will be \(2\pi\). The solution will thus be smooth there, as we would expect. We now prove the lemma.

**Proof.** We begin with the case where \(\tilde{\Gamma}_1\) and \(\tilde{\Gamma}_n\) share an edge, so that vertex is an interior one. We shall use separation of variables and look for a solution of \(\Delta_{\tilde{\Gamma}} u_n = 0\) of the form \(u_n = c_n(r) \tau_n(\theta)\). In polar coordinates the Laplace-Beltrami operator will be
\[
\Delta_{\tilde{\Gamma}} = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}.
\] (3.22)
Plugging $u_n$ into this formula will give that $\tau_n(\theta) = e^{i \lambda_n \theta}$. Using the fact that $x(r, 0) = x(r, \gamma)$ will give that $\lambda_n = \frac{2 \pi}{\gamma} n$ for $n \in \mathbb{N}$. Some more calculus will give that

$$c_0(r) = a_0 + b_0 \log(r),$$
$$c_n(r) = a_n r^{\frac{2\pi n}{\gamma}} + b_n r^{-\frac{2\pi n}{\gamma}}, \quad \text{for } n \neq 0. \quad (3.23)$$

If we now impose that $u_n \in H^1(B_R \cap \tilde{\Gamma})$, then we find that

$$u_n = a_n r^{\frac{2\pi n}{\gamma}} e^{i \frac{2\pi n}{\gamma} \theta}. \quad (3.24)$$

It remains to show that we can write the solution $u$ as sum of the $u_n$’s.

We pick the Fourier coefficients $a_n$ so that $\sum u_n$ agrees with $u$ on $\partial B_R \cap \tilde{\Gamma}$. The uniqueness of weak solutions to the Laplace-Beltrami problem on $B_R \cap \tilde{\Gamma}$ with Dirichlet boundary conditions (Lemma 7 below) will then give that $u - \sum u_n \equiv 0$ on $B_R \cap \tilde{\Gamma}$. We have thus proved (3.20).

In order to see the higher regularity of $u$ on each face, we note that the most singular terms in the expansion occur when $n = \pm 1$. These terms will be contained in $H^k(B_R \cap \tilde{\Gamma}^0)$ if and only if

$$\left\| \nabla^k r^{\frac{2\pi n}{\gamma}} e^{\pm \frac{2\pi n}{\gamma} \theta} \right\|_{L^2(B_R \cap \tilde{\Gamma}^0)} \propto \left( \int_0^R r^{2(\frac{2\pi n}{\gamma} - k) + 1} dr \right)^{1/2} < \infty.$$

This occurs precisely when $k < 1 + \frac{2\pi}{\gamma}$.

The proof when $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_n$ do not share an edge is identical to the case above, except that the Fourier series are replaced by sine series because of the boundary conditions. \qed

**Remark 3.** We will use the above lemma to determine the singularities in the solution of the Laplace-Beltrami problem on surfaces with corners. The singularities in (3.20) were first derived in [8] for the special case of polyhedral surfaces, i.e. surfaces with only flat faces and straight edges, but we have presented Lemma 5 to make this section as complete as possible. In what follows, we will prove that the solution has these same singularities in the more general case where the surface faces are curved.

Now that we know the nature of the singularities of the solution of the interface form near interior and exterior corners of a surface, we may prove the following theorem.

**Lemma 6.** Let $\Gamma_S$ be the union of the faces of $\Gamma$ that touch the vertex $S$. Let $\Gamma_S$ be parameterized by the piecewise smooth map $x_S$ with domain $U_S$ (from Theorem 7), and let $f$ be a function in $H^k(\Gamma_S)$ with $k = 0, 1, 2$. Also let $L$ be the pull-back of the Laplace-Beltrami operator by $x_S$. If $\tilde{u}$ is the solution of the elliptic interface problem with homogeneous Dirichlet boundary conditions (3.4) on $U_S$ with right hand side $f \circ x_S \in H^k(U_S)$, then $u = \tilde{u} \circ x_S^{-1}$ is the unique solution of the interface forms of the Laplace-Beltrami problem $\Delta_{\Gamma_S} u = f$ with homogeneous Dirichlet boundary conditions on $\partial \Gamma_S$.

Further, there exists a neighborhood of each vertex $S'$ of $\Gamma_S$ we may write $u$ as

$$u = u_0 + \begin{cases} \sum_{|n| \leq \frac{\gamma}{2\pi} (k+3)} a_n S' \gamma_S |n| \frac{2\pi |n|}{\gamma_S} e^{i \frac{2\pi}{\gamma_S} n \theta_S}, & \text{if } S' = S, \\ \sum_{|n| \leq \frac{\gamma}{2\pi} (k+3)} a_n S' \gamma_S |n| \sin \left( \frac{\pi}{\gamma_S} n \theta_S \right), & \text{otherwise}, \end{cases} \quad (3.25)$$

where $u_0 \in H^{2+k}(\Gamma_S)$ and $\gamma_{S'}$ is the conic angle of the vertex $S'$, i.e. the sum of the corner angles of all of the faces in $\Gamma_S$ touching the vertex $S'$.
Proof. Let \( C_S \) be the set of corners of \( I \) that are included in \( I_S \) and let \( \tilde{S}' \) be the conic angle of the vertex \( S' \in C_S \) with respect to the surface \( I_S \). Since the radial distance in the plane and on the surface will be proportional in the limit as they get small, Lemma 5 tells us that the expansion powers at the corner \( S' \) are \( \{2\pi n/\tilde{S}'\}_{n \in \mathbb{Z}} \) if \( S' = S \) or \( \{\pi n/\tilde{S}'\}_{n \in \mathbb{Z}} \) if \( S' \neq S \). As these are all real and we have already verified that the planar elliptic interface problem is regular, we have thus verified that we may apply Theorem 3. The solution \( \tilde{u} \) of (3.4) thus exists and is unique. Theorem 6 then gives that \( \tilde{u} \circ x_S^{-1} \) is the unique solution of the interface form of the Laplace-Beltrami problem.

Theorem 4 will give the singularity expansions listed in (3.25). \( \square \)

We now prove that the solution of the interface form of the Laplace-Beltrami problem is the unique solution of the weak form.

**Lemma 7.** The weak solution of the Laplace-Beltrami problem on \( I_S \) with homogeneous Dirichlet boundary conditions on \( \partial I_S \) is unique.

**Proof.** By linearity, we need only check the case when \( f = 0 \). We recall that \( u \in H^1_0(I_S) \) is the weak solution if

\[
\int_{I_S} \nabla u \cdot \nabla v = 0 \quad \forall v \in H^1_0(I_S). \tag{3.26}
\]

Substituting \( u = v \) into the expression gives that \( \int_{I_S} |\nabla u|^2 = 0 \), so \( u \) must a constant. The boundary conditions then imply that \( u \equiv 0 \), and so the solution is unique. \( \square \)

**Theorem 9.** The solution of the interface form of the Laplace-Beltrami problem on \( I_S \) with homogeneous Dirichlet boundary conditions on \( \partial I_S \) exists and is the unique weak solution.

**Proof.** It only remains to show that the solution of the interface form identified in Lemma 6 \( u \) is a weak solution of the Laplace-Beltrami operator. To prove this, we let \( v \) be a test function in \( H^1_0(I_S) \) and let \( I_{i,e} = I_i \setminus \cup_{S' \in C_S} B_e(S') \). Since the expansion (3.25) implies that \( u \in \mathcal{H}^2(\cup_{i \in I_S} I_{i,e}) \), we have that

\[
- \int_{\cup_{i \in I_S} I_{i,e}} \nabla F' \cdot \nabla u = \sum_i \int_{I_{i,e}} v \nabla F' \cdot \nabla u + \int_{e_i \setminus \cup_{S' \in C_S}} \frac{\partial u}{\partial b_i} v + \sum_i \int \left( \frac{\partial u}{\partial b_i} + \frac{\partial u}{\partial b_j} \right) v - \sum_i \int_{\partial B_e(S')} v S' \cdot \nabla F' u. \tag{3.27}
\]

Since \( u \) solves the interface problem on \( I_S \), and \( v \in H^1_0(I_S) \), most of the boundary terms vanish and we are left with

\[
- \int_{\cup_{i \in I_S} I_{i,e}} \nabla F' \cdot \nabla u = \int_{\cup_{i \in I_S} I_{i,e}} v f + \sum_{S'} \int_{\partial B_e(S')} v S' \cdot \nabla F' u. \tag{3.28}
\]

We now show that the final term vanishes as \( e \to 0 \). First, suppose that \( v \) is bounded by a constant \( C \) and that \( S' = S \). If \( e \) is small, then (3.25) holds. The contribution of \( u_0 \) to this integral will vanish as \( e \to 0 \) since \( u \in \mathcal{H}^2(I_S) \). To see that that the singular terms vanish, we replace the metric in the surface gradient with its limit in the corner, denoted by \( \nabla I_S \), introducing an error that shrinks with \( e \):

\[
\left| \int_{\partial B_e(S)} v S' \cdot \nabla F' r_S^{2 \pi/5} e^{i \frac{2 \pi}{5} n \theta_S} \right| \leq C \int_{\partial B_e(S)} \left| r_S^{2 \pi/5} e^{i \frac{2 \pi}{5} n \theta_S} \right| + o (|\partial B_e(S)|). \tag{3.29}
\]
Evaluating the remaining derivative and integrating gives that
\[
\left| \int_{\partial B_\epsilon(S)} v \hat{\nabla} \cdot \nabla r S^\frac{2\pi}{\gamma S} |n| e^{i \frac{2\pi}{\gamma S} n \theta S} \right| \leq C \gamma_S \epsilon \left( \frac{2\pi}{\gamma S} |n| e^{i \frac{2\pi}{\gamma S} |n|} - 1 \right) + o(\gamma_S \epsilon). \tag{3.30}
\]
The right hand side above vanishes as \( \epsilon \to 0 \), and so
\[
\int_{\partial B_\epsilon(S)} v \hat{\nabla} \cdot \nabla r u \to 0.
\]
We may repeat the above argument for the other vertices of \( \Gamma_S \) to see that for any bounded \( v \in H^1_0(\Gamma_S) \),
\[
- \int_{\Gamma_S} \nabla v \cdot \nabla u = \int_{\Gamma_S} vf. \tag{3.31}
\]
Since bounded functions are dense in \( H^1_0(\Gamma_S) \), we have shown that \( u \) solves the weak form of the Laplace-Beltrami problem on \( \Gamma_S \). Finally, Lemma 7 thus gives that \( u \) is the unique weak solution. \( \Box \)

### 3.4 Interface form on a closed surface

We now extend the result for a single surface patch \( \Gamma_S \) to rigorously connect the weak and interface forms of the Laplace-Beltrami problem on the whole surface \( \Gamma \), which is the main result of this paper.

**Theorem 10** (Equivalence of the interface form). _If \( \Gamma \) is a piecewise smooth Lipschitz surface without boundary and \( f \in H^k(\Gamma) \) with \( k = 0, 1, \) or \( 2 \) is mean-zero, then the following hold:

1. The weak solution of the Laplace-Beltrami problem \( \Delta_{\Gamma} u = f \) solves the interface form of the Laplace-Beltrami problem (3.3).
2. There exists a smooth partition of unity over \( \Gamma, \{ \zeta_S \}_{S \in C} \), such that the support of \( \zeta_S \) is strictly contained in \( \Gamma_S \) and there exists a \( u_0 \in H^{2+k}(\Gamma) \) such that
\[
u = u_0 + \sum_{S \in C} \zeta_S \left( \sum_{|n| \leq \frac{2\pi}{\gamma S} (k+1)} a_{n,S} r S^\frac{2\pi}{\gamma S} |n| e^{i \frac{2\pi}{\gamma S} n \theta S} \right),
\]
    where \( \gamma_S \) is the conic angle for the corner \( S \) and \( (r_S, \theta_S) \) is a local polar coordinate system defined around the corner \( S \), as previously described in Lemma 5.

*Proof.* Let \( \{(U_S, x_S)\}_{S \in C} \) be the collection of the local parameterizations defined in the proof of Theorem 7. To construct the partition of unity, we define \( \tilde{x}_{x_S^1(S)}(S) \) as in the proof of Lemma 1 for each \( S \in C \). We then define \( \tilde{\zeta}_S \in C(\Gamma) \) to be \( \tilde{\zeta}_{x_S^{-1}(S)} \circ x_S^{-1} \) on \( \Gamma_S \) and zero on \( \Gamma \setminus \Gamma_S \). The functions \( \zeta_S = \tilde{\zeta}_S / \sum_{S \in C} \tilde{\zeta}_S \) are then smooth on each face, satisfy the interface conditions, and form the desired partition of unity.

Now that we have a suitable partition of unity, we define
\[
\tilde{f}_S := \Delta_{\Gamma}(\zeta_S u), \tag{3.33}
\]
for each \( S \in C \). In order to apply Theorem 9, we must verify that \( \tilde{f}_S \in L^2(\Gamma) \). To do this, we use the product rule to see that
\[
\tilde{f}_S = \nabla_{\Gamma} \cdot (u \nabla_{\Gamma} \zeta_S) + \nabla_{\Gamma} \cdot (\zeta_S \nabla_{\Gamma} u). \tag{3.34}
\]
For any \( v \in H^1(\Gamma) \), the first term is defined by
\[
(\nabla_\Gamma \cdot (u \nabla_\Gamma \xi_S), v) := -\int_\Gamma u \nabla_\Gamma \xi_S \cdot \nabla_\Gamma v = \sum_i \int_{I_i} \nabla_{I_i} \cdot (u \nabla_{I_i} (\xi_S)) v + \sum_j \int_{e_{ij}} b_i \cdot \nabla_{e_{ij}} (\xi_S) uv, \tag{3.35}
\]
where we have used integration by parts on each face. The edge integrals cancel because the trace of \( u \) and \( v \) along \( e_{ij} \) agrees from \( I_i \) and \( I_j \), since they are in \( H^1(\Gamma) \), and \( \xi_S \) satisfies the interface conditions by construction. Applying the product rule on each face gives that
\[
(\nabla_\Gamma \cdot (u \nabla_\Gamma \xi_S), v) = \sum_i \int_{I_i} (\nabla_{I_i} u \cdot \nabla_{I_i} \xi_S + u \Delta_{I_i} \xi_S) v. \tag{3.36}
\]

Since \( \xi_S \) is smooth on each \( I_i \) and \( u \in H^1(\Gamma) \), the function in brackets is in \( L^2(\Gamma) \). The density of \( H^1(\Gamma) \) in \( L^2(\Gamma) \) then gives that \( \nabla_\Gamma \cdot (u \nabla_\Gamma \xi_S) \in L^2(\Gamma) \).

For any \( v \in H^1(\Gamma) \), the second term in (3.34) is
\[
(\nabla_\Gamma \cdot (\xi_S \nabla_\Gamma u), v) := -\int_\Gamma \xi_S \nabla_\Gamma u \cdot \nabla_\Gamma v = -\int_\Gamma \nabla_\Gamma u \cdot (\xi_S \nabla_\Gamma v). \tag{3.37}
\]

Applying the product rule and using the fact that \( \xi_S \) is Lipschitz continuous gives that
\[
(\nabla_\Gamma \cdot (\xi_S \nabla_\Gamma u), v) = -\int_\Gamma \nabla_\Gamma u \cdot (\nabla_\Gamma (\xi_S v) - v \nabla_\Gamma \xi_S) = (\nabla_\Gamma \cdot (\nabla_\Gamma u), \xi_S v) + \int_\Gamma \nabla_\Gamma \xi_S \cdot \nabla_\Gamma uv. \tag{3.38}
\]
The density of \( H^1(\Gamma) \) in \( L^2(\Gamma) \) then gives that \( (\nabla_\Gamma \cdot (\xi_S \nabla_\Gamma u) \in L^2(\Gamma) \). Overall, we now have that \( \tilde{f}_S \in L^2(\Gamma) \).

We pause here to note that if \( \xi_S \) had not satisfied the interface conditions, then \( \tilde{f}_S \) would have involved a distribution supported on the surface edges and not have been in \( L^2(\Gamma) \).

Since \( \xi_S u \in H^1_0(I_S) \), the function \( \xi_S u \) is a solution of the weak form of the Laplace-Beltrami problem on \( I_S \) with homogeneous Dirichlet boundary conditions and right hand side \( \tilde{f}_S \). Since \( \tilde{f}_S \in L^2(\Gamma) \), Theorem 9 thus gives that \( \xi_S u \) satisfies the interface conditions on the interior of \( I_S \). Since \( \xi_S \) is zero in a neighborhood of \( \Gamma \setminus I_S \), we know that \( \xi_S u \) satisfies the interface conditions at all the other edges of \( \Gamma \). The linearity of our problem then tells us that \( u \) satisfies (3.3).

For the second part of the theorem, we note that \( \xi_S \) is flat near \( S \) and zero in a neighborhood every other vertex. The expansion of \( \xi_S u \) given by (3.25) therefore gives the expansion of \( u \) near \( S \). The global form of \( u \) in (3.32) can then be found using the argument in Corollary 4.1 with \{\xi_S\} as the partition of unity.

If \( f \in H^1(\Gamma) \), then we continue to show that the coefficients \( a_{n,s} \) can be chosen so that \( u_0 \) is in \( H^3(\Gamma) \). Since \( \xi_S \) must be flat near the vertices of \( \Gamma \), (3.33) implies that that \( \tilde{f}_S \in H^1(\Gamma) \). With this information, the above proof then implies that \( u_0 \) can be chosen to be in \( H^3(\Gamma) \). If \( f \in H^2(\Gamma) \), we may repeat this argument to see that \( u_0 \) can be chosen to be in \( H^4(\Gamma) \). Having exhausted the relevant cases, we have completed the proof.

Morrey’s inequality (see Theorem 12.55 in [37]) indicates that if one could prove a version of the above for \( f \in L^p(\Gamma) \) for some \( p > 2 \), then we could ask for \( u \) to be the solution of the following smoother problem:

**Problem 4** (Strong interface form of the Laplace-Beltrami problem). Let \( \Gamma \) be a surface composed of smooth faces \( I_i \) and \( f \) be a continuous mean-zero function on \( \Gamma \). The strong interface form of the Laplace-Beltrami problem is defined to be: find a \( u \in C^1(I_i) \cap H^2(\Gamma) \) for every \( i \), which satisfies (3.3) where the restrictions to surface edges are interpreted in the limit sense, rather than a trace sense.
We present the following conjecture about conditions for solvability of the strong interface form. The proof would follow from an $L^p$ version of Theorem 3 in the same way presented above and an application of Morrey’s inequality.

**Conjecture 1.** If $\Gamma$ is a piecewise smooth Lipschitz surface composed of faces $\Gamma_i$ with all conic angles less than or equal to $2\pi$ and $f$ is a function in $L^p(\Gamma)$ for some $p > 2$, then the solution of the weak form (2.9) is also a solution of the strong interface form of the Laplace-Beltrami problem.

The results of this section imply that numerical solvers for the Laplace-Beltrami problem on piecewise smooth surfaces can discretize the differential operator acting on functions in $L^2$ and obtain convergent results. We demonstrate this in Section 5 via a high-order numerical solver along surfaces of revolution. Before doing that however, we briefly discuss how the above result can be extended to cones.

4 A special case: The cone

Definition 5 excluded surfaces with cone-like singularities. This exclusion allowed us to simplify our arguments, but many applications involve surfaces with such singularities. In this section, we briefly address the Laplace-Beltrami problem on a simple cone of height $h$ with a base of radius one.

We begin by analytically solving the Laplace-Beltrami problem on this surface, showing that the expected singularities appear, and noting that the solution has the expected smoothness.

**Example 1.** Consider the case where $\Gamma$ is a cone of height $h$ with a base of radius one. We let $\Gamma_1$ be the curved part of the cone and let $\Gamma_2$ be the flat bottom of the cone. These pieces can be parameterized by

\[
\begin{align*}
  x_1 &= \left( \frac{h - z}{h} \cos(\varphi), \frac{h - z}{h} \sin(\varphi), z \right), & z \in [0, h], & \varphi \in [0, 2\pi), \\
  x_2 &= (r \cos(\varphi), r \sin(\varphi), 0), & r \in [0, 1], & \varphi \in [0, 2\pi),
\end{align*}
\]

respectively. Using these parameterizations $x_1$ and $x_2$, Mathematica [29] can analytically solve the Laplace-Beltrami problem with piecewise data $f_n$ given by:

\[
\begin{align*}
  f_n|_{\Gamma_1} &= (h - z)^\alpha e^{i n \varphi}, & \alpha > -1, & n \in \mathbb{N}, \\
  f_n|_{\Gamma_2} &= 0.
\end{align*}
\]

The analytic solution is given by

\[
\begin{align*}
  u_n|_{\Gamma_1} &= \left( c_1 (h - z)^{2+\alpha} + c_2 (h - z)^{\sqrt{1 + h^2}|n|} \right) e^{i n \varphi}, \\
  u_n|_{\Gamma_2} &= c_3 r^{|n|} e^{i n \varphi},
\end{align*}
\]

where $c_1$, $c_2$, and $c_3$ are constants that depend on $h$, $n$, and $\alpha$. We can clearly see that in this case the solution has two more integrable derivatives than $f_n$ and picks up terms whose higher order derivatives are singular near the corner. If we note that the conic angle for this cone is given by $\gamma = 2\pi/\sqrt{1 + h^2}$, then it becomes even clearer that the solution has the expected behavior near the top tip and the bottom edge.

Having seen that the solution of the Laplace-Beltrami problem has the expected behaviour on $\Gamma$, we now prove that Theorem 10 can be applied to this specific $\Gamma$. Since the cone is piecewise smooth away from its natural vertex $S = (0,0,h)$, it is enough to show that Lemma 6 holds on a patch containing the vertex $S$. The main requirement for this is included in the following lemma.
Lemma 8. There exists a parameterization $x_S$ of a patch $I_S$ around the vertex $S$ such that the pull-back of the Laplace-Beltrami operator has piecewise constant coefficients.

Proof. We begin by parameterizing the curved section of the cone $I_1$ as

$$\tilde{x}_1(x, y) = \begin{pmatrix} \sqrt{\frac{x^2+y^2}{1+h^2}} \cos \left( \sqrt{1+h^2} \tan^{-1} \left( \frac{y}{x} \right) \right) \\ \sqrt{\frac{x^2+y^2}{1+h^2}} \sin \left( \sqrt{1+h^2} \tan^{-1} \left( \frac{y}{x} \right) \right) \\ h \sqrt{\frac{x^2+y^2}{1+h^2}} \end{pmatrix}, \quad \text{for } (x, y) \in W \quad (4.4)$$

where the wedge $W$ is

$$W = \left\{ (x, y) \in \mathbb{R}^2 \left| \sqrt{x^2+y^2} \leq \sqrt{1+h^2} \text{ and } \tan^{-1} \left( \frac{y}{x} \right) \in \left[ 0, \frac{2\pi}{\sqrt{1+h^2}} \right] \right. \right\}. \quad (4.5)$$

Some calculus shows that under this parameterization, the pull-back of the Laplace-Beltrami operator is simply the Laplacian. To construct the patch $I_S$, we let $T_1$, $T_2$, and $T_3$ be the three triangular subsets of $W$ shown in Figure 6. For the purposes of this proof, we consider their images $\tilde{x}_i(T_i)$ to be faces of $I$. The patch around the vertex is then defined as $I_S := \cup_{i=1}^3 I_i$.

The construction of $x_S$ is then the same as in the proof of Theorem 7, except that the initial parameterizations already preserve arc-length along the overlapping edges and so Lemma 4 is not necessary. The required affine transformations make the pull-back of the Laplace-Beltrami operator a constant coefficient operator on each face, and so we have found the desired parameterizations. □

A similar argument to that in Lemma 5 could be applied on $W$ to find that the expansion powers are still $\frac{2\pi n}{2\pi}$. The rest of the proof of Lemma 6, and therefore Theorem 10, is the same. Our main result thus holds for this cone.

It is likely that our result also holds for Laplace-Beltrami problems along more generic surfaces with cone-like singularities. We therefore present the following conjecture.

Conjecture 2. Let $\Gamma$ be a Lipschitz surface than can be written as the union of faces $\{I_i\}_{i=1}^N$. Also suppose that the interiors of these faces are pairwise disjoint. Finally suppose that for each face $I_i$ there is a closed triangle $T_i$ and a bijective parameterization $x_i : T_i \rightarrow I_i$. If for each $i$, the parameterization $x_i$ is smooth away from the vertices of $T_i$ and can be written smoothly in polar coordinates around each verte, then the weak solution of the Laplace-Beltrami problem on $\Gamma$ satisfies the interface form whenever the right hand side is in $L^2(\Gamma)$. 

24
5 Application: Surfaces of revolution with edges

Having identified the correct form of the Laplace-Beltrami problem on a piecewise smooth Lipschitz surface $\Gamma$, we would like to develop numerical methods to solve it. In general, this will be challenging, as care would have to be taken to deal with the singularities in the solution near surface vertices. In this section, we shall restrict ourselves to the case that $\Gamma$ is a genus one surface of revolution (e.g. the surface in Figure 13), which cannot have vertices. In this case, we will able to use a simple numerical method to solve the interface form of the Laplace-Beltrami problem. Our method will be to use separation of variables to formulate the Laplace-Beltrami problem as a sequence of decoupled periodic ODEs, where the interface conditions become continuity conditions on the the solution of the ODEs. We then solve those ODEs using an integral equation approach which will automatically satisfy those continuity conditions.

5.1 Separation of variables

To begin with, denote by $(r, \theta, z)$ the usual cylindrical coordinate system in three dimensions. If $\Gamma$ is a genus one piecewise smooth surface of revolution about the $z$-axis, then let $\gamma$ denote its generating curve in the plane $\theta = 0$, which is assumed to be closed, piecewise smooth, and non-self-intersecting. The generating curve can be parameterized in terms of arclength, $\gamma : [0, L] \rightarrow \mathbb{R}^3$; let its cylindrical coordinate components (for $\theta = 0$) be parameterized as

$$\gamma(s) = (r(s), z(s)), \quad (5.1)$$

where $s \in [0, L]$ denotes arclength along the generating curve $\gamma$. As shown in the previous section, since the Laplace-Beltrami problem $\Delta_\Gamma u = f$ is uniquely solvable on the space of mean-zero square-integrable functions on a piecewise smooth surface, we can write the solution $u$ in a Fourier expansion in cylindrical coordinates as:

$$u(x) = u(r, \theta, z) = u(s, \theta) = \sum_{n=-\infty}^{\infty} u_n(s) e^{in\theta}, \quad (5.2)$$

and the right hand side $f$ as:

$$f(x) = \sum_{n=-\infty}^{\infty} f_n(s) e^{in\theta}. \quad (5.3)$$

Since a surface of revolution can only have edges at corners in the generating curve, the decomposition in (5.2) implies that the interface conditions become the requirement that $u_n$ and $u'_n$ are continuous at values of $s$ that correspond the surface edges.

Furthermore, we note that in the variables $s$ and $\theta$, the Laplace-Beltrami operator takes the form:

$$\Delta_\Gamma = \frac{\partial^2}{\partial s^2} + \frac{1}{r} \frac{\partial}{\partial s} \frac{\partial}{\partial s} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}. \quad (5.4)$$

Using the decompositions in (5.2) and (5.3), and the above form of the Laplace-Beltrami operator, we can transform the PDE into a sequence of decoupled periodic ODEs, one for each Fourier mode $n$. Specifically, the Fourier modes $u_n$ should be continuously differentiable, $L$-periodic, and such that

$$\frac{d^2 u_n}{ds^2} + \frac{1}{r} \frac{d}{ds} \frac{du_n}{ds} - \frac{n^2}{r^2} u_n = f_n, \quad \text{for } s \in [0, L]. \quad (5.5)$$
The solution \( u \) can then be easily synthesized via its Fourier ansatz. Note that the mean-zero condition on \( u \) for solvability of the Laplace-Beltrami problem reduces to a condition on only \( u_0 \) since every mode with \( n \neq 0 \) integrates to zero:

\[
\int_{\gamma} u = \int_0^{2\pi} \int_0^L \sum_n u_n(s) e^{i n \theta} r(s) \, ds \, d\theta = 2\pi \int_0^L u_0(s) r(s) \, ds. \quad (5.6)
\]

Enforcing this mean-zero condition on \( u_0 \) is discussed in the next section.

### 5.2 A periodic ODE solver

The separation of variables solution to the Laplace-Beltrami problem requires solving the sequence of periodic ODEs in (5.5) with the usual periodic boundary condition: continuity in the solution and its derivative \([12, 16, 48]\). In order to solve the ODE for each Fourier mode of the Laplace-Beltrami equation, we shall convert it into a second-kind integral equation. Doing this will allow us to easily use adaptive high-order quadrature methods to solve it accurately. This conversion is applicable to a broad class of periodic ODEs, so we shall present the procedure in a general framework. A similar approach for the Laplace-Beltrami problem on smooth surfaces of revolution was used in \([19]\) (using a global trapezoidal discretization scheme); a more general adaptive approach for two-point boundary value problems, coupled with a fast direct solver, was detailed in the widely known work of \([36]\). To this end, we shall consider a method to solve any ODE on \([-\frac{L}{4}, \frac{L}{4}]\) of the form:

\[
u'' + pu' + qu = f, \quad u(x + L) = u(x), \quad u'(x + L) = u'(x), \quad (5.7)
\]

where \( f, p, \) and \( q \) are known periodic functions in \( L'([0, L]) \) for some \( r > 1 \) and we are searching for a solution \( u \in C^1([0, L]) \). The boundary conditions above give that solution is \( L \)-periodic. If \( q = 0 \), then the solution can only be determined up to an additive constant; in this case, an additional constraint must be imposed to ensure well-posedness of the problem. Usually this constraint takes the form of a linear function of \( u \), such as

\[
\int_0^L u(x) w(x) \, dx = A. \quad (5.8)
\]

We address this special case where \( q = 0 \) later on in this section.

In order to convert the ODE (5.7) on \([0, L]\) into an integral equation on the same interval, first consider the kernel \( G_L \),

\[
G_L(x) = -\frac{1}{2L} \left( \text{mod}(x, L) - \frac{L}{2} \right)^2 + \frac{L}{24}. \quad (5.9)
\]

It is not hard to verify that if this kernel is convolved with a function \( f \) that is mean-zero on \([0, L]\), then the resulting function \( v = G_L \ast f \) solves the one-dimensional periodic Poisson equation \( v'' = f \). This \( v \) is in fact the unique mean-zero solution with \( \int_0^L v = 0 \), since \( G_L \) is also mean-zero on this interval. We next define the “single layer operator” \( S_L \) via the convolution

\[
S_L f(x) = \int_0^L G_L(x - t) f(t) \, dt. \quad (5.10)
\]

Since \( G_L \) is mean-zero on \([0, L]\), \( S_L f \) is also mean-zero on the same interval. We now represent the solution \( u \) to (5.7) as

\[
u = S_L \sigma + C, \quad (5.11)
\]
Young's inequality then tells us that gives that

\[ \int_0^L u = \int_0^L S_L \sigma + \int_0^L C \implies C = \frac{1}{L} \int_0^L u, \]  

(5.12)
i.e. \( C \) is the mean of the solution \( u \) on \([0, L]\). We have included the constant \( C \) in the representation in order to ensure that it is a complete representation; \( S_L \sigma \) will always be a mean-zero function on \([0, L]\), but the solution \( u \) may not be. This representation for \( u \) also ensures that the solution is automatically periodic since \( G_L \) is periodic.

With this representation, we have changed the problem of finding \( u \) into the problem of finding \( \sigma \) and \( C \). Inserting (5.11) in (5.7) yields a Fredholm second-kind integral equation for \( \sigma \) and \( C \):

\[ \sigma + pS'_L \sigma + qS_L \sigma + qC = f, \]

(5.13)
where

\[ S'_L \sigma(x) = \int_0^L G'_L(x - t) \sigma(t) \, dt. \]

(5.14)
The above integral equation is indeed invertible when \( q \neq 0 \) because the underlying ODE is invertible and we simply used a complete and unique representation for the solution \( u \). In order to see that our transformation has led to a well-conditioned equation, we note that (5.13) has the form of \((I + \mathcal{K})\sigma + qC = f\), where \( I \) is the identity operator and \( \mathcal{K} \) is a compact integral operator. The integral equation is therefore of Fredholm second-kind [32].

If \( q = 0 \), then we must explicitly enforce the additional integral condition \( \int_0^L u \, w = A \) from (5.8). We shall include this condition by simply adding it to (5.13), giving the integral equation

\[ \sigma + pS'_L \sigma + \int_0^L (S_L \sigma + C) \, w = f + A. \]

(5.15)
This method of adding a linear constraint to an integral equation is equivalent to adding a rank-one update to the original integral operator. It is not difficult to see that this update results in an invertible and well-conditioned equation provided the range of the update is not contained in the range of the original integral operator. See [47] for a discussion of this method in the matrix equation setting. In our case, this is equivalent to asking if there exists a mean-zero function on \([0, L]\) that is not in the null space of the adjoint, i.e. if there exists a mean-zero and periodic \( f \) such that \( f + S'_L [p f] \neq 0 \). Taking a derivative of this expressions gives the condition that \( f' + p f \neq 0 \). Plugging in \( f = \sin(\frac{2\pi}{L} x) \) or \( f = \cos(\frac{2\pi}{L} x) \) will yield at least one example.

Next, we check that the solution of (5.13) produces a \( u \) that is automatically in \( C^1([0, L]) \). To do this, we first note that for (5.13) to make sense, the solution \( \sigma \) must be in \( L^1([0, L]) \). Applying Young’s inequality then tells us that \( S_L \sigma \) and \( S'_L \sigma \) are uniformly bounded, and therefore (5.13) gives that \( \sigma \) is in \( L'([0, L]) \). Finally, we note that \( S_L \) maps \( L'([0, L]) \) to \( C^1([0, L]) \). This can easily be proven using Hölder’s inequality and the fact that \( G_L \) and \( G'_L \) are piecewise continuous and bounded. Knowing that \( \sigma \in L'([0, L]) \) then gives that \( S_L \sigma \), and thus the solution \( u \), are in \( C^1([0, L]) \). An analogous argument holds true in the \( q = 0 \) case as well.

We now have a suitable second-kind integral equation form of (5.7) and its solution will have the expected smoothness properties. In the Laplace-Beltrami case, we have that

\[ p = \frac{1}{r} \frac{dr}{ds}, \quad q = \frac{n^2}{r^2}, \]

(5.16)
and when \( n = 0 \) we enforce the additional constraint on \( u_0 \) of

\[ \int_0^L u_0(s) \, r(s) \, ds = 0, \]

(5.17)
which, as mentioned before, is equivalent to the mean-zero constraint \( \int_{\Gamma} u = 0 \) [19]. Since \( r \) is a piecewise smooth function bounded away from 0, \( p \) and \( q \) will be in \( L^r([0, L]) \) for any \( r \geq 1 \). Lastly, in light of the earlier discussion in the manuscript, the case of interest where \( f \in L^2[0, L] \) satisfies the earlier requirements.

**Remark 4.** The above integral equation formulation of (5.7) is not the only possible integral equation formulation. The above derivation could easily be repeated with other kernels (i.e. Green’s functions). One example can be found in [19], where they considered the function

\[
G_Y(x) = \frac{1}{2} e^{-|x|} - \frac{e^{-L}}{1-e^{-L}} \cosh(\tilde{x}), \quad \text{where } \tilde{x} = \mod \left( x - \frac{L}{2}, L \right) + \frac{L}{2},
\]

(5.18) instead of the function \( G_L \) used above. This function is the \( L \)-periodic Green’s function for the Yukawa problem \( v'' - v = f \). Since \( G_Y \) has the same smoothness properties as \( G_L \), the resulting integral equation will still be second-kind and give rise to a unique solution \( u \) that is periodic and in \( C^1([0, L]) \) whenever \( p, q, \) and \( f \) are in \( L^r([0, L]) \) for some \( r > 1 \).

We also note that \( G_Y \) has a minor advantage over \( G_L \) because it has a non-zero mean. Therefore, there is no need to ensure that \( \sigma \) is a priori mean-zero, and we would not need to include the constant \( C \) in the representation (but the mean-zero constraint (5.8) would still need to be enforced in the case that \( q = 0 \)). In practice, these advantages are small and we will see later that both kernels ultimately lead to very similar performance.

### 5.3 A numerical solver

In this section, we shall describe a numerical solver for the integral equations in (5.13). To summarize, our solver is based on a Nyström method for the equation using an adaptive discretization of the interval \([0, L]\) consisting of a piecewise 16th-order Gauss-Legendre quadrature. (A 16th-order quadrature was chosen so as to ensure a high-order discretization, of course our method extends to any other order discretization.) An overview of various methods for discretizing integral equations along curves in two dimensions is given in [25]. For problems in which the surface \( \Gamma \) is smooth (and therefore so is the generating curve \( \gamma \)), we partition the interval \([0, L]\) into uniformly sized panels. For piecewise smooth generating curves \( \gamma \), we dyadically refine the panels using knowledge of where the geometric singularities occur as a function of arclength.

Let \( \{P_i\}_{i=1}^M \) be a partition of the interval \([0, L]\), and \( \{x_{ij}, w_{ij}\}_{j=1}^{16} \), be the 16th-order Gauss-Legendre quadrature nodes and weights on panel \( i \). The integral equation in (5.13) is enforced at each of the nodes \( x_{ij} \):

\[
\sigma_{ij} + \sum_{k=1}^{M} \left( p(x_{ij}) \int_{P_k} G_L'(x_{ij} - t) \sigma(t) \, dt + q(x_{ij}) \int_{P_k} G_L(x_{ij} - t) \sigma(t) \, dt \right) + q(x_{ij}) C = f(x_{ij}),
\]

(5.19)

where \( \sigma_{ij} \) denotes the solution to the linear system which will be approximately equal to \( \sigma(x_{ij}) \). We will show how to enforce the a priori mean-zero condition on the \( \sigma_{ij} \)’s below. It remains now to replace the integrals above with discrete sums.

First, notice that the kernel \( G_L \) is piecewise smooth with a discontinuity in \( G_L' \) only at the origin. With this in mind, the integrals in (5.19) corresponding to \( k \neq i \) can be approximated using 16th-order Gauss-Legendre quadrature nodes, for example:

\[
\int_{P_k} G_L(x_{ij} - t) \sigma(t) \, dt \approx \sum_{\ell=1}^{16} w_{k\ell} G_L(x_{ij} - x_{k\ell}) \sigma_{k\ell}, \quad \text{for } k \neq i.
\]

(5.20)
For the “near field” integrals corresponding to when \( k = i \) in (5.19), standard Gauss-Legendre quadrature will fail to yield high-order convergence due to the irregularity in the kernel \( G_L \). With this in mind, we split the \( k = i \) integrals into two pieces precisely at the point of irregularity, \( x_{ij} \). On each of these new panels, the integrand is smooth and standard Gauss-Legendre quadrature can be used along with Lagrange interpolation on \( \sigma \) to obtain implied values at the extra quadrature support nodes.

To explain in more detail: if \( P_i = [a_i, b_i] \), then for each \( j \), we approximate this near field integral over \( P_i \) as

\[
\int_{P_i} G_L(x_{ij} - t) \sigma(t) \, dt = \int_{a_i}^{x_{ij}} G_L(x_{ij} - t) \sigma(t) \, dt + \int_{x_{ij}}^{b_i} G_L(x_{ij} - t) \sigma(t) \, dt \\
\approx \sum_{\ell = 1}^{16} u_{ij\ell} G_L(x_{ij} - s_{ij\ell}) \hat{\sigma}(s_{ij\ell}) + \sum_{\ell = 1}^{16} v_{ij\ell} G_L(x_{ij} - t_{ij\ell}) \hat{\sigma}(t_{ij\ell}),
\]

(5.21)

where \((s_{ij\ell}, u_{ij\ell})\) is the \( \ell \)th Gaussian quadrature node and weight pair on the interval \([a_i, x_{ij}]\), \((t_{ij\ell}, v_{ij\ell})\) is the \( \ell \)th Gaussian quadrature node and weight pair on the interval \([x_{ij}, b_i]\), and \( \hat{\sigma}(x) \) is the value obtained from the \( \sigma_i \)'s on \( P_i \) via Lagrange interpolation to the point \( x \in P_i \). The near field integrals in \( S' \) can be discretized similarly.

Finally, recall our representation for the solution \( u \): \( u = S_L \sigma + C \), where \( \int_0^L \sigma = 0 \). This assumption on \( \sigma \) must be explicitly enforced, and can easily be done so in one of two ways: (1) the condition can be discretized as

\[
\int_0^L \sigma(s) \, ds = 0 \approx \sum_{i=1}^{M} \sum_{j=1}^{16} w_{ij} \sigma_{ij}
\]

(5.22)

and appended to the system of equations yielding a square linear system of dimension \( 16M + 1 \) for the unknowns \( \sigma_{ij} \) and \( C \), or (2) the original representation can be replaced with one of the form

\[
u = S_L \left[ \sigma - \int_0^L \sigma \right] + \int_0^L \sigma.
\]

(5.23)

The above alternative representation ensures that the argument to \( S_L \) has mean-zero on \([0, L]\) and \( C \) is equated with the integral of \( \sigma \). This approach results in a square linear system of size \( 16M \) for only the unknowns \( \sigma_{ij} \). In our the subsequent numerical experiments, our solver implements the latter choice of changing the representation to equate the integral of \( \sigma \) with the integral of the solution \( u \).

Lastly, in the purely azimuthal component to the Laplace-Beltrami problem, i.e. the \( n = 0 \) mode, we must also enforce the mean-zero condition on the solution \( u_0 \) along the surface \( \Gamma \). A discretization of this condition in (5.8) can be obtained by an identical procedure as to that used in discretizing \( S_L \sigma \); the resulting equation can be put in the form

\[
\sum_{i=1}^{M} \sum_{j=1}^{16} c_{ij} \sigma_{ij} = 0
\]

(5.24)

and added to each row of the \( 16M \times 16M \) linear system.

**Remark 5.** The continuous Laplace-Beltrami equations are well-conditioned with respect to the \( L^2([0, L]) \) norm, \( || \cdot ||_{L^2([0, L])} \). In order to ensure that our discretized equations are similarly well-conditioned in \( \ell^2 \) (as an embedding of the continuous problem), we must use a discretization
method that that approximates \( \| \cdot \|_{L^2(0,L)} \) \cite{6}. This is especially important when we use local adaptive refinement, as that will cause the \( \ell^2 \) norm of the discretized functions to greatly diverge from their true \( L^2(0,L) \) norms. We address this issue by replacing \( \sigma_{ij} \) in the discretized linear system with \( \sqrt{w_{ij}} \sigma_{ij} \). An equivalent linear system is easily obtained by left and right diagonal preconditioners (as detailed in \cite{6}) with the effect that the \( \ell^2 \) norm of the discrete unknown approximates the true \( L^2 \) norm of the solution to the continuous problem:

\[
\| (\sqrt{w_{ij}} \sigma_{ij}) \| = \sum_{ij} w_{ij} \sigma_{ij}^2 \approx \int_{0}^{L} \sigma^2(s) \, ds. \quad (5.25)
\]

Furthermore, the resulting linear system has a spectrum which converges to the spectrum of the original continuous system, yielding increased performance when using iterative solvers such as GMRES.

6 Numerical examples

In this section, we give the results of some numerical examples demonstrating the ODE and Laplace-Beltrami solvers detailed above. In order to compute the right hand sides \( f_n \) in equation (5.5) we discretize the original function \( f \) using equispaced points in the azimuthal and compute \( f_n \) using the FFT (as in \cite{19}). In all of the tests, the linear systems were solved using GMRES to a relative tolerance of \( 10^{-14} \). The linear systems were of modest size, and could easily be applied in a matrix-free fashion. All codes were written in MATLAB and no attempt was made to accelerate the code beyond the use of the FFT.

All reported errors were estimated in the relative \( L^2 \) sense, e.g. the relative error in the solution \( u \) to an ODE was measured as

\[
\epsilon = \sqrt{\frac{\sum_{i} w_i (u_{\text{true}}(x_i) - u_i)^2}{\sum_{i} w_i u_{\text{true}}^2(x_i)}}, \quad (6.1)
\]

where we have used a single subscript \( i \) above to index all \( 16M \) values on the interval \([0, L]\) and \( u_{\text{true}} \) was either known a priori or estimated using a finer discretization or a different numerical solver. For Laplace-Beltrami problems on a surface \( \Sigma \), the formula above is modified to include the proper quadrature weight for a surface integral (obtained as a tensor product of piecewise Gauss-Legendre quadrature with the trapezoidal rule).

Lastly, in what follows, we will use \( N_s = 16M \) to denote the total number of discretization points on an interval \([0, L]\) and \( N_\theta \) to denote the total number of points used in the azimuthal direction for Fourier analysis/synthesis via the FFT.

6.1 Comparing Greens functions

In order to compare the Green’s function \( G_L \) discussed in Section 5.3 with the Green’s function \( G_Y \) used in \cite{19}, we test our ODE solver on a globally smooth problem. We set the coefficient functions in the ODE to

\[
p(x) = \sin(3x) - 2, \quad q(x) = 2 \sin(5x) - 3 \quad (6.2)
\]

and the right hand side to

\[
f(x) = \frac{d^2}{dx^2} e^{\sin(2x)}. \quad (6.3)
\]

These functions are smooth, periodic, and the resulting ODE is not trivialized by a simple change of coordinates. Since these functions are smooth, we use uniform panels to discretize the interval.
As the exact solution for this equation is not known, we compare the resulting solutions to the solution obtained with Chebfun, a well-known MATLAB package that uses global Chebyshev approximation to solve ODEs with spectral accuracy [45]. As a further test, we repeated this experiment with a Chebyshev-based discretization (as opposed to the Gauss-Legendre one described earlier). In this test, second-kind Clenshaw-Curtis quadratures were used to compute the required integrals.

Figure 7 depicts the convergence of the solution in this regime. We see that our method indeed converges and gives us an accurate solution to the problem with a moderate number of discretization points. We can also see that both choices of kernels result in comparable accuracy, and neither was consistently better. As such, we chose to use the Poisson kernel $G_L$ for the remainder of this paper. Furthermore, Figure 7 also makes it clear that using Gauss-Legendre points results in a more accurate solution than is achieved with the same number of Chebyshev points, but only slightly so and both solutions converge to near machine precision.

6.2 Laplace-Beltrami on a smooth surface

In order to validate our Laplace-Beltrami solver, we constructed a surface $\Gamma$ and right-hand side $f$ with a known solution. We did this by constructing a smooth surface and choosing the exact solution to be a constant plus the restriction of a smooth function $v$ defined in all of $\mathbb{R}^3$ (following the approach in [43]). We then generated $f$ through the following well-known analytic formula which applies in any neighborhood that a surface is smooth:

$$ f = \Delta_{\Gamma} (v|_{\Gamma}) = \Delta v - 2H \frac{\partial v}{\partial n} - \frac{\partial^2 v}{\partial n^2}. $$  

(6.4)

In the above formula, $H$ is the mean curvature of $\Gamma$ and $n$ is the normal to $\Gamma$. A derivation of this formula may be found in [40]. We evaluate $f$ by analytically computing the terms in (6.4) based on a global parameterization of the surface. In this example, $\Gamma$ is given by a circular torus with inner radius one and outer radius two, and we set $v$ to be the Newtonian potential centered at $x_0 = (0, 0.5, 0.5)$:

$$ v(x) = \frac{1}{|x - x_0|}. $$

The exact solution to this problem is given by $u = v|_{\Gamma} - \frac{1}{|\Gamma|} \int_{\Gamma} v$, where $|\Gamma|$ is the surface area of $\Gamma$. This solution and the corresponding right hand side are shown in Figure 8. We conducted a
(a) The solution of $\Delta_F u = f$.

(b) The right hand side of $\Delta_F u = f$.

Figure 8: The solution and right hand side of $\Delta_F u = f$, where $u$ was chosen to be the mean-zero restriction of the Newtonian potential to $\Gamma$. The Newtonian potential is centered at the black circledot.

Figure 9: The errors of the computed solution in our circular torus tests. $N_x$ is the number of points around the generating curve and $N_\theta$ is the number of points used in the $\theta$-direction.
convergence test by varying $N_\theta$ and $N_s$. We can see from the relative errors in Figure 9 that our method is capable of accurately solving this problem, and therefore the solver is working as expected.

### 6.3 Singular surface test

In this experiment, we tested our Laplace-Beltrami solver on a non-smooth surface with right-hand sides that are irregular at surface edges. We set the surface $\Gamma$ to be a square toroid: the surface resulting from revolving a unit square about the $z$-axis. Some care must be taken when discretizing this surface because of the geometric singularities. It is necessary to ensure that panel boundaries in our discretization of the generating curve coincided with the surface edges, as the coefficient functions in (5.4) are singular at the edges of $\Gamma$. The coarsest possible discretization thus had one panel per face. In order to resolve the singularity in the right-hand side, we dyadically refined this coarse discretization into the edge where $f$ is singular. We denote the width of the finest panel as $h_{\text{final}}$ and study how the errors depend on it. In the $\theta$-direction, our surface and right-hand sides were smooth, so we simply set $N_\theta = 10$.

In this example, we are not able to generate an exact solution and right-hand side through the same method as we used in the smooth surface case, as the restriction of a smooth function to our surface would not have been in $C^1(\Gamma)$. Instead, we generate them using Chebfun. Specifically, we specify that the true solution satisfies

$$\partial^2_s u(\theta, s) = \Theta(\theta) S(s),$$

where $\Theta$ and $S$ are specified functions. This allows us to use Chebfun’s anti-differentiation routine to compute $u$, $\partial_\theta u$, and $f = \Delta_F u$ in $\theta$-Fourier space. We note that this is different than our methodology in Section 6.1, where we instead specified the form of $f$ and used Chebfun to solve the ODE for $u$. We choose to use anti-differentiation here as it becomes trivial to ensure that $u$ is mean-zero on $\Gamma$. Furthermore, this scheme does not require us to solve a singular ODE in order to compute the reference solution.

Explicitly, we find each Fourier mode $n$ of $u$ and $f$ by performing the following calculations:

$$\partial_s u_n = \int \partial^2_s u_n - \frac{1}{4} \int_0^4 \left( \int \partial^2_s u_n \right),$$

and then for $n \neq 0$

$$u_n = \int \partial_s u_n - \frac{1}{4} \int_0^4 \left( \int \partial_s u_n \right),$$

and for $n = 0$

$$u_0 = \int \partial_s u_0 - \frac{1}{4} \int_0^4 w \left( \int \partial_s u_0 \right).$$

The right hand side is then computed (for all $n$) as

$$f_n = \partial^2_s u_n + \frac{\partial_s r}{r} \partial_s u_n + \frac{n^2}{r^2} u_n.$$

Above, we use $\int$ to denote anti-derivative.

We tested the solver with several choices of $S$ in (6.5). As a smooth test we set $S(s) = \cos(\pi s/2)$. We also tested several singular cases of the form $S(s) = |s-2|^\alpha$, where we set $\alpha = -\frac{1}{3}, -\frac{1}{2},$ and $-\frac{3}{4}$. Here $s = 2$ is the arclength parameter corresponding to the top inner edge, so $u$ is non-smooth at that edge (and the one directly across from it). For all of our tests, we set $\Theta(\theta) = \sin(3\theta)$. 33..
Cross sections of the resulting solutions are displayed in Figure 10. As expected, the solutions are continuously differentiable, with local extrema at $s = 2$. Further, as $\alpha$ approaches -1 and the minimum $r$ such that $f_n \in L^r([0, L])$ shrinks, the point of the extrema becomes sharper and the solution approaches the boundary of $C^1([0, L])$. This experiment is thus a good stress-test of our solver and will demonstrate how the solver behaves on problems that barely satisfy its basic requirements.

Figure 11a shows how the error in each test problem depended on the extent of the dyadic refinement. We see that the smooth case was resolved with a single panel on each face. From this, we can see that discontinuous coefficient functions did not prevent our method from computing accurate solutions with relatively few unknowns. In the singular tests, we can see that sufficient refinement was all that was necessary to achieve an accurate solution. Thus, the limited resolution of the singularity in the finest panel was the source of the dominant error in the solution. Furthermore, Figure 11b demonstrates that the error in fact decayed as $O(h^{1+\alpha}_{\text{final}})$, which was the order of accuracy in our evaluation of the left hand side of the integral equation. This fact is further discussed in Appendix A.

As a further verification, we performed a self convergence study that did not make use of Chebfun. To do this, we directly set $f(\theta, s) = \Theta(\theta)S(s)$, where $\Theta$ and $S$ were chosen to be the same as in the previous experiment. Figure 12a shows the relative difference between the solution on the finest grid and the solutions on other grids, for each of the choices of $S$. Most of the observations from the previous experiment apply here, except that in the case where $S(s) = \cos(\pi s/2)$. In this experiment, the corresponding solution was not smooth, so two panels on each face were necessary in order to accurately resolve it.

### 6.4 Harmonic vector field computation

The surfaces of revolution that we have considered so far are non-trivial and of genus one; it is well-known that they support a two-dimensional linear space of harmonic vector fields, i.e. square-integrable tangential vector fields $H \in L^2_r(\Gamma)$ such that $\nabla_{\Gamma} \cdot H = 0$ and $\nabla_{\Gamma} \cdot (n \times H) = 0$ in $H^{-1}(\Gamma)$. For all surfaces of revolution, an orthogonal basis for these harmonic vector fields is analytically given by

$$H_1 = \frac{1}{r} \hat{s}, \quad \text{and} \quad H_2 = -\frac{1}{r} \hat{\theta},$$

where $\hat{s}$ and $\hat{\theta}$ denote unit vectors along the generating curve and in the azimuthal direction, respectively [18, 19]. This fact may be easily verified by direct calculation. As a further test of our Laplace-Beltrami solver, we shall use it to construct a basis for the space of harmonic vector fields.
The relative $L^2$ errors of the square toroid tests. The dashed line indicates the scaling $O(h^{1+\alpha}_{\text{final}})$. Each marker indicates a trial.

Figure 11: The figures show how the relative error of the square toroid tests depended on the extent of the refinement.

The relative $L^2$ differences in the self-convergence study on the square toroid. The dashed line indicates the scaling $O(h^{1+\alpha}_{\text{final}})$. Each marker indicates a trial.

Figure 12: These figures show how the relative difference between the most refined solution and the other solutions depends on their refinement in the self-convergence study.
along the same surface $\gamma$ as in the previous experiment, a square toroid. The basis fields $H_1$ and $H_2$ for this surface are shown in Figure 13.

In order to compute a basis for the harmonic vector fields, we shall make use of the Hodge decomposition of a general vector field along $\gamma$, which was introduced in section 1. This decomposition splits a tangential vector field $F$ into a curl-free component $F_{cf}$ (i.e. one where $\nabla_\gamma \cdot (n \times F_{cf}) = 0$), a divergence-free component $F_{df}$, and a harmonic component $H$. The Hodge decomposition of a vector field $F \in H_1^1(\gamma)$ can be written explicitly as

$$F = \nabla_\gamma \alpha + n \times \nabla_\gamma \beta + H,$$  \hspace{1cm} (6.11)

where $\alpha$ and $\beta$ are mean-zero scalar functions defined on $\gamma$. See [28], for example, for a more detailed discussion of this representation in similar genus one geometries. Taking the surface divergence of (6.11), as well as the surface curl (6.11), shows that $\alpha$ and $\beta$ must satisfy

$$\Delta_\gamma \alpha = -\nabla_\gamma \cdot (n \times F), \quad \text{and} \quad \Delta_\gamma \beta = -\nabla_\gamma \cdot (n \times F).$$ \hspace{1cm} (6.12)

We note that because $F \in H_1^1$, the right hand sides $\nabla_\gamma \cdot F$ and $\nabla_\gamma \cdot (n \times F)$ will be in $L^2(\gamma)$ and mean-zero. The Laplace-Beltrami problems are thus well posed and there exist unique mean-zero $\alpha$ and $\beta$ satisfying the interface form with the smoothness described in Section 3.3.

With the above Hodge decomposition in mind, it becomes clear how to compute examples of Harmonic vector fields $H$: we can simply choose a tangential vector field $F$ and subtract off the components $\nabla_\gamma \alpha$ and $n \times \nabla_\gamma \beta$. In order to make this numerically feasible, we shall restrict our choice of $F$ to be smooth on each face of $\gamma$ and continuous across each edge of $\gamma$. We may then compute $\nabla_\gamma \cdot F$ and $\nabla_\gamma \cdot (n \times F)$ pseudo-spectrally by first Fourier decomposing $F$ as:

$$F = F^s \hat{s} + F^\theta \hat{\theta},$$

and then by applying the following formula mode-by-mode:

$$\nabla_\gamma \cdot F = \frac{dF^s}{ds} + \frac{1}{r} \frac{dr}{ds} F_s + \frac{1}{r} \frac{dF^\theta}{d\theta}.$$ \hspace{1cm} (6.14)

In order to compute $dF^s_n/ds$, we interpolate $F^s_n$ onto Chebyshev panels in arclength along the generating curve and use Chebyshev differentiation. Having computed the divergences (i.e. the right hand side to a Laplace-Beltrami problem), we use our method to solve the Laplace-Beltrami equation for $\alpha$ and $\beta$. Next, we compute the surface gradient of $\alpha$ and $\beta$ (again, mode-by-mode) through the formula

$$\nabla_\gamma u = \frac{du}{ds} \hat{s} + \frac{1}{r} \frac{du}{d\theta} \hat{\theta}.$$ \hspace{1cm} (6.15)

Note that differentiation with respect to $\theta$ is merely multiplication by $(in)$ in Fourier-space. Lastly, to compute $d\alpha/ds$ and $d\beta/ds$ we note that we already know $\alpha$ and $\beta$ as the solution of a system of integral equations with the representation, for example, $\alpha_n = S_L \sigma_n + C$. We can therefore use the formula $d\sigma_n/ds = S_L' \sigma_n$ to easily compute this quantity via quadrature on the integral representation. Once all terms are computed for each mode, we can synthesize the Fourier series and evaluate the harmonic component as

$$H = F - \nabla_\gamma \alpha - n \times \nabla_\gamma \beta.$$ \hspace{1cm} (6.16)
As a measure of accuracy, we then project $H$ onto the basis $H_1$ and $H_2$ in (6.10) and look at the $L^2$ norm of the remainder relative to the norm of $F$ to determine if the computed field lies in the space of harmonic vector fields. 

As a test field, we computed the harmonic component of $F = r \hat{s} + r^{-2} \hat{\theta}$. With two panels per face, the relative $L^2$ norm of the remainder was less than $10^{-14}$. We also validated our code by computing the harmonic components of the exact basis $\{H_1, H_2\}$. We found that the basis is within machine precision of being harmonic.

7 Conclusions

In this work, we reformulated the Laplace-Beltrami problem on piecewise smooth surfaces as a collection of smooth problems on each face combined with interface conditions at surface edges. To summarize, if the right hand side of a Laplace-Beltrami problem is in $L^2(\Gamma')$, then the solution in the usual weak sense will be well-behaved at the surface edges. Furthermore, we analytically computed an expansion of such solutions in corners having conic angle of $\gamma$; the leading order term is of the form $r^{2\frac{\gamma}{\pi}}$, where $r$ is the distance to the corner along the surface.

Furthermore, we used this reformulation to develop a numerical method that solves the Laplace-Beltrami problem on piecewise smooth surfaces of revolution. The numerical results support the theoretical results of the paper. This method converted each Fourier mode of the Laplace-Beltrami problem into a second-kind integral equation that automatically satisfied the interface conditions and could be accurately solved using standard numerical techniques for integral equations. The integral equation formulation for solving the associated one-dimensional periodic ODEs can be easily generalized to any second-order periodic ODE with coefficients and right hand side in $L^r([0, L])$ for some $r > 1$, even if they are non-smooth.

This Laplace-Beltrami solver, and the experiments used to verify it, demonstrated the ability to easily obtain high-order accuracy for the problem on piecewise smooth surfaces. However, this specific solver is limited to surfaces of revolution that are separated from their axis of rotation. In future work, we plan to develop a new integral equation based solver that can be applied to a more general classes of surfaces, for example piecewise smooth surfaces specified by a collection of charts with no symmetry assumptions at all. This will be based on a parametrix method [27], similar to the approach in [35]. Work in these directions is ongoing.
Acknowledgments

The authors would like to thank Charlie Epstein for several useful discussions.

A Error bounds for singular data

Here we discuss the error in our method for computing the left hand sides of (5.13) in the singular surface test. In this test, we chose the solution to satisfy $u''(s) = \Theta(\theta)|s - 2|^\alpha$. We will see that the error is $O(h_{\text{final}}^{1+\alpha})$, where $h_{\text{final}}$ denotes the width of the most refined panel. This will imply that the error in $\sigma$ has the same order.

In order to evaluate the left hand side of (5.13), we must evaluate integrals of the form $\int_4^0 K_\sigma n ds$ for various kernels $K$. This integral is challenging to compute because $\sigma_n = u''$, and therefore $\sigma_n(s) = C|s - 2|^\alpha$. We shall assume for this discussion that $K$ is smooth and bounded on each face. In reality, the kernel will only be piecewise smooth and we will the use panel splitting idea in Section 5.3 to accurately compute the integral. However, for the sake of clarity, we omit these details in this discussion.

To study the error, we consider the integral over an example face: $s \in (2, 4)$. We split the integral into two pieces, one over the finest panel $(2, 2 + h_{\text{final}})$ where $\sigma_n$ is singular, and one over the rest of the panels where $\sigma_n$ is smooth. On the finest panel, $K$ may be approximated as having its value at $s = 2$ since $h_{\text{final}}$ is sufficiently small. The integral thus becomes

$$\int_2^{2+h_{\text{final}}} K_\sigma n ds \approx h_{\text{final}} \int_0^1 K(2^+) C|h_{\text{final}} \tilde{s}|^\alpha d\tilde{s} = CK(2^+)^{h_{\text{final}}^{\alpha+1}} 1 + \alpha. \quad (A.1)$$

If we apply our quadrature rule to the integral we obtain

$$h_{\text{final}} \sum_{i=1}^{16} w_i CK(2^+)(h_{\text{final}} x_i)^\alpha = CK(2^+) h_{\text{final}}^{\alpha+1} \sum_{i=1}^{16} w_i x_i^\alpha, \quad (A.2)$$

where the $x_i$’s and $w_i$’s are the standard 16th-order Gauss-Legendre quadrature points and weights. The error on this panel thus approaches

$$|CK(2^+)|^{\alpha+1} \left| \frac{1}{1 + \alpha} - \sum_{i=1}^{16} w_i x_i^\alpha \right|. \quad (A.3)$$

Since the function $x^\alpha$ is not integrated exactly by Gauss-Legendre quadrature, the error on the finest panel will be $O(h_{\text{final}}^{1+\alpha})$.

On the panels where the function is smooth, we use the standard formula for the error resulting from applying $k$th-order Gauss-Legendre quadrature to integrate a function $f$ on the interval $(a, b)$, see §5.2 of [31]:

$$\left| \int_a^b f(x) dx - \sum_{j=1}^{k} w_j f(x_j) \right| = \frac{(b - a)^{2k+1}(k!)^4}{(2k + 1)(2k)13} f^{(2k)}(\xi), \quad \text{for some } \xi \in (a, b). \quad (A.4)$$

If we let $h_0, \ldots, h_N = h_{\text{final}}$ be the panel widths in our dyadic refinement, then by definition $h_i = h_0 2^{-i}$. Since $K$ is smooth on the interval $(2, 4)$, the dominant term in $(K_\sigma n)^{2n}$ will be
$CK(s)|s - 2|^{\alpha - 2k}$. On the panel of width $h_i$, this term may be bounded by $|C| \max |K|h_i^{\alpha - 2k}$, since that panel is a distance $h_i$ away from the singularity. The error on that panel is thus bounded by

$$|C| \max |K| \frac{h_i^{2k+1}(k!)^4}{(2k + 1)(2k)!^3} h_i^{\alpha - 2k}. \quad \text{(A.5)}$$

Summing our error over all of the smooth panels gives the bound

$$|C| \max |K| \frac{h_0^{1+\alpha}(k!)^4}{(2k + 1)(2k)!^3} \sum_{i=0}^{N} 2^{(1+\alpha)(-i)} \leq |C| \max |K| \frac{h_0^{1+\alpha}(k!)^4}{(2k + 1)(2k)!^3} \frac{1}{1 - 2^{1+\alpha}}. \quad \text{(A.6)}$$

Since we do not apply our scheme to the case where $\alpha$ is exponentially close to -1, and we are using $k = 16$, this error will be well below machine precision. The error from the final panel thus dominates the error in the smooth panels, and therefore the error in computing the left hand side of (5.13) will be $O(h_{\text{final}}^{1+\alpha})$. 

39
References

[1] S. Angenent, S. Haker, A. Tannenbaum, and R. Kikinis. On the Laplace-Beltrami Operator and Brain Surface Flattening. *IEEE Trans. Med. Imag.*, 18(8):700–711, 1999.

[2] E. Bänsch, P. Morin, and R. H. Nochetto. A finite element method for surface diffusion: The parametric case. *J. Comput. Phys.*, 203(1):321–343, 2005.

[3] L. Beirão da Veiga, F. Brezzi, L. D. Marini, and A. Russo. The hitchhiker’s guide to the virtual element method. *Math. Models and Meth. Appl. Sci.*, 24(08):1541–1573, 2014.

[4] A. Bonito, A. Demlow, and R. H. Nochetto. *Finite element methods for the Laplace–Beltrami operator*, volume 21. Elsevier B.V., 1st edition, 2020.

[5] A. Bonito, A. Demlow, and J. Owen. A Priori Error Estimates for Finite Element Approximations to Eigenvalues and Eigenfunctions of the Laplace–Beltrami Operator. *SIAM J. Num. Anal.*, 56(5):2963–2988, 2018.

[6] J. Bremer. On the Nyström discretization of integral equations on planar curves with corners. *Appl. Comput. Harm. Anal.*, 32(1):45–64, 2012.

[7] A. Buffa and P. Ciarlet. On traces for functional spaces related to Maxwell’s equations Part II: Hodge decompositions on the boundary of Lipschitz polyhedra and applications. *Math. Meth. Appl. Sci.*, 24(1):31–48, 2001.

[8] A. Buffa, M. Costabel, and C. Schwab. Boundary element methods for Maxwell’s equations on non-smooth domains. *Numerische Mathematik*, 92(4):679–710, 2002.

[9] A. Buffa, R. Hiptmair, T. von Petersdorff, and C. Schwab. Boundary Element Methods for Maxwell Transmission Problems in Lipschitz Domains. *Num. Math.*, 95:459–485, 2003.

[10] E. Burman, P. Hansbo, M. G. Larson, and A. Massing. A cut discontinuous Galerkin method for the Laplace-Beltrami operator. *IMA J. Num. Anal.*, 37(1):138–169, 2017.

[11] E. V. Chernokozhin and A. Boag. Method of generalized debye sources for the analysis of electromagnetic scattering by perfectly conducting bodies with piecewise smooth boundaries. *IEEE Transactions on Antennas and Propagation*, 61(4):2108–2115, 2013.

[12] E. A. Coddington and N. Levinson. *Theory of Ordinary Differential Equations*. Krieger, Malabar, FL, 1984.

[13] M. Costabel and M. Dauge. Singularities of Electromagnetic Fields in Polyhedral Domains. *Archive for Rational Mechanics and Analysis*, 151(3):221–276, 2000.

[14] A. Demlow and G. Dziuk. An adaptive finite element method for the Laplace–Beltrami operator on implicitly defined surfaces. *SIAM J. Num. Anal.*, 45(1):421–442, 2007.

[15] G. Dziuk. Finite elements for the Beltrami operator on arbitrary surfaces. In *Partial differential equations and calculus of variations*, pages 142–155. Springer, Berlin, Heidelberg, 1988.

[16] M. S. P. Eastham. *The spectral theory of periodic differential equations*. Scottish Academic Press, London, UK, 1973.
[17] C. L. Epstein and L. Greengard. Debye Sources and the Numerical Solution of the Time Harmonic Maxwell Equations. *Comm. Pure Appl. Math.*, 63:413–463, 2009.

[18] C. L. Epstein, L. Greengard, and M. O’Neil. Debye Sources and the Numerical Solution of the Time Harmonic Maxwell Equations II. *Comm. Pure Appl. Math.*, 66(5):753–789, 2013.

[19] C. L. Epstein, L. Greengard, and M. O’Neil. A high-order wideband direct solver for electromagnetic scattering from bodies of revolution. *J. Comput. Phys.*, 387:205–229, 2019.

[20] J. Escher, U. F. Mayer, and G. Simonett. The surface diffusion flow for immersed hypersurfaces. *SIAM J. Math. Anal.*, 29(6):1419–1433, 1998.

[21] G. B. Folland. *Introduction to Partial Differential Equations*. Princeton University Press, Princeton, NJ, 2nd edition, 1996.

[22] T. Frankel. *The Geometry of Physics*. Cambridge University Press, Cambridge, UK, 3rd edition, 2011.

[23] M. Frittelli and I. Sgura. Virtual element method for the Laplace-Beltrami equation on surfaces. *ESAIM: Mathematical Modelling and Numerical Analysis*, 52(3):965–993, 2018.

[24] F. Gesztesy, I. Mitrea, D. Mitrea, and M. Mitrea. On the nature of the Laplace-Beltrami operator on Lipschitz manifolds. *J. Math. Sci.*, 172(3):279–346, 2011.

[25] S. Hao, A. H. Barnett, P. G. Martinsson, and P. Young. High-order accurate methods for Nyström discretization of integral equations on smooth curves in the plane. *Adv. Comput. Math.*, 40:245–272, 2014.

[26] K. Hildebrandt and K. Polthier. On approximation of the Laplace–Beltrami operator and the Willmore energy of surfaces. *Computer Graphics Forum*, 30(5):1513–1520, 2011.

[27] L. Hörmander. *The Analysis of Linear Partial Differential Operators III*. Springer-Verlag, New York, NY, 1994.

[28] L. M. Imbert-Gérard and L. Greengard. Pseudo-Spectral Methods for the Laplace-Beltrami Equation and the Hodge Decomposition on Surfaces of Genus One. *Num. Meth. for Partial Diff. Eq.*, 33(3):941–955, 2017.

[29] W. R. Inc. Mathematica, Version 12.3.1. Champaign, IL, 2021.

[30] F. John. *Partial Differential Equations*. Springer-Verlag, New York, NY, fourth edition, 1982.

[31] D. Kahaner, C. Moler, and S. Nash. *Numerical methods and software*. Prentice-Hall, Inc., 1989.

[32] R. Kress and D. Colton. *Integral Equation Methods in Scattering Theory*. John Wiley and Sons, Inc., New York, 1983.

[33] J. Kromer and D. Bothe. Highly accurate numerical computation of implicitly defined volumes using the Laplace-Beltrami operator. *arXiv:1805.03136*, [physics.flu-dyn]:1–25, 2018.

[34] M. Kropinski, N. Nigam, and B. Quaife. Integral equation methods for the Yukawa-Beltrami equation on the sphere. *Adv. Comput. Math.*, 42(2):469–488, 2016.
[35] M. C. A. Kropinski and N. Nigam. Fast integral equation methods for the Laplace-Beltrami equation on the sphere. *Adv. Comput. Math.*, 40(2):577–596, 2014.

[36] J.-Y. Lee and L. Greengard. A Fast Adaptive Numerical Method for Stiff Two-Point Boundary Value Problems. *SIAM J. Sci. Comput.*, 18(2):403–429, 1997.

[37] G. Leoni. *A First Course in Sobolev Spaces: Second Edition*. American Mathematical Society, Providence, Rhode Island, second edition, 2017.

[38] D. Malhotra, A. Cerfon, L.-M. Imbert-Gérard, and M. O’Neil. Taylor States in Stellarators: A Fast High-order Boundary Integral Solver. *J. Comput. Phys.*, 397:108791, 2019.

[39] M. Mitrea. The method of layer potentials in electromagnetic scattering theory on nonsmooth domains. *Duke Mathematical Journal*, 77(1):111–133, 1995.

[40] J.-C. Nedéléc. *Acoustic and Electromagnetic Equations*. Springer-Verlag, New York, NY, 2001.

[41] S. Nicaise and A.-M. Sändig. General interface problems – I. *Math. Meth. in the Appl. Sci.*, 17(6):395–429, 1994.

[42] S. Nicaise and A.-M. Sändig. General interface problems – II. *Math. Meth. in the Appl. Sci.*, 17(6):431–450, 1994.

[43] M. O’Neil. Second-kind integral equations for the Laplace-Beltrami problem on surfaces in three dimensions. *Adv. Comput. Math.*, 44(5):1385–1409, 2018.

[44] M. O’Neil and A. J. Cerfon. An integral equation-based numerical solver for Taylor states in toroidal geometries. *J. Comput. Phys.*, 359:263–282, 2018.

[45] R. B. Platte and L. N. Trefethen. Chebfun: A new kind of numerical computing. In *Progress in Industrial Mathematics at ECMI 2008*, pages 69–87. Springer, 2010.

[46] A. Rahimian, I. Lashuk, S. Veerapaneni, A. Chandramowlishwaran, D. Malhotra, L. Moon, R. Sampath, A. Shringarpure, J. Vetter, R. Vuduc, et al. Petascale direct numerical simulation of blood flow on 200k cores and heterogeneous architectures. In *SC’10: Proceedings of the 2010 ACM/IEEE International Conference for High Performance Computing, Networking, Storage and Analysis*, pages 1–11. IEEE, 2010.

[47] J. Sifuentes, Z. Gimbutas, and L. Greengard. Randomized methods for rank-deficient linear systems. *Elec. Trans. Num. Anal.*, 44:177–188, 2015.

[48] L. N. Trefethen, A. Birkisson, and T. A. Driscoll. *Exploring ODEs*. SIAM, Philadelphia, PA, 2018.

[49] S. K. Veerapaneni, A. Rahimian, G. Biros, and D. Zorin. A fast algorithm for simulating vesicle flows in three dimensions. *J. Comput. Phys.*, 230(14):5610–5634, 2011.

[50] M. Wang, S. Leung, and H. Zhao. Modified Virtual Grid Difference for Discretizing the Laplace–Beltrami Operator on Point Clouds. *SIAM J. Sci. Comput.*, 40(1):A1–A21, 2018.

[51] M. Wardetzky. *Discrete Differential Operators on Polyhedral Surfaces - Convergence and Approximation*. PhD thesis, Freie Universität Berlin, 2007.
[52] F. W. Warner. *Foundations of Differentiable Manifolds and Lie Groups*. Springer, New York, NY, 2013.