Abstract. A simple hydrodynamic model of multiband superconductors describing Leggett interband collective excitations and their interactions with electromagnetic waves is applied to description of dynamics of distributed Josephson Junction between multiband superconductors. A modified Sin-Gordon equation is obtained and excitation of Leggett modes by moving Josephson vortices is investigated.

1. Introduction

It is well known that multifolded Fermi surface superconductors [1, 2] may feature the so-called Leggett modes [3], that essentially are collective excitations of phase difference of order parameters in different bands. Multiband superconductors, in accordance with the modern view, include the conventional ones such as Nb, and the recently discovered novel superconductors as MgB$_2$ and the FeAs based ones. Excitation of the Leggett modes in lumped Josephson junctions was earlier considered by some authors theoretically, see for example [4] and possibly observed experimentally [5]. Attempts have been made to generalize the theory for the case of distributed contacts [6], but the approach of tunnelling Hamiltonian used in this paper is seemed to be unsatisfactory for correct description of field distribution in the vicinity of Josephson Junction and, as a consequence, of Josephson Dynamics.

In this work we propose a hydrodynamic self-consistent model describing the dynamics of distributed Josephson junctions between multiband superconductors.

2. Hydrodynamic description of Multiband Superconductors

The dynamics of superconducting condensates within electrodes of Josephson junction can be described by hydrodynamical equations for superconducting velocities $v_j$ and concentrations $n_j$. These equations simply express general laws of number of particles and momentum conservation. We will limit yourself by the simplest case of sufficiently low temperatures so that the dynamics of quasiparticles will not be relevant. In such a case hydrodynamic equations look like

\[ \dot{N}_j + \text{div} N_j v_j = \frac{2}{\hbar} \sum_k J_{j,k} \sqrt{N_j N_k} \sin(\theta_j - \theta_k), \]

\[ \dot{v}_j + (v_j \nabla) v_j + \frac{\nabla p_j(N_j)}{m_j N_j} = -\frac{e}{m_j} \left\{ E + \frac{1}{e} [\mathbf{vB}] \right\} \]

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where $N_j$, $v_j$ are concentrations and velocities of superconducting electrons in $j$-th band, $J_{j,k}$ energies of interband couplings, $p_j$ are the pressure of electrons in $j$-th band for which we will use a simple expression for degenerate Fermi gas

$$p_j = \frac{(3\pi^2)^{2/3}}{5} \frac{\hbar^2}{m_j} N_j^{5/3}, \quad (3)$$

implying that corrections to compressibility due to superconductivity which are of order $\Delta/\varepsilon_F$ are very weak. It corresponds to well known Thomas-Fermi approximation. Calculating the derivatives $\frac{\partial p_j}{m_j \partial N_j}$ which are squared velocities of Bogoliubov-Anderson modes in corresponding bands $s_j^2$ we have well known relations $s_j^2 = v_{F,j}^2/3$ where Fermi velocities are defined by standard relations $m_j v_{F,j} = (3\pi^2 N_j)^{1/3} \hbar$.

Concentrations are related to module of order parameters $N_j = 2|\Psi_j|^2$ and velocities are expressed via their phase by standard gauge invariant relation

$$2m_j v_j = \hbar \nabla \theta_j + \frac{2e}{c} A, \quad (4)$$

where we explicitly took into account negative sign of electronic charge. Under Coulomb gauge condition $\text{div} A = 0$ the second term in RHS of Eq. 4 describes pure vortex motion while the first term is pure potential only if there are no Abrikosov vortices.

Expressing electromagnetic fields via vector and scalar potentials, using the vector identity $(\mathbf{v} \nabla) \mathbf{v} = \nabla v^2/2 - \mathbf{v} \times \nabla \times \mathbf{v}$ and substituting the expression 4 for velocity via phase into equations of motion 2 and integrating it we are able to find the Bernoulli-like equations

$$\hbar \dot{\theta} + m_j v_j^2 - 2e\varphi + 2\mu_j(N_j) - 2\mu_j(N_j^0) = 0, \quad (5)$$

where $\mu_j = m_j v_{F,j}^2/2$ are chemical potentials in corresponding bands, $N_j^0$ are equilibrium concentrations, which are, in such kind of description, phenomenological parameters which should be defined from spatially homogeneous many bands Bardeen-Cooper-Schriffer theory. These relations define an expression for concentration via phase and electromagnetic, vector and scalar, potentials. It can be called as extended London equation since it does not include derivatives of concentrations. It is naturally gauge invariant since phase and scalar potentials enters only in combination $\hbar \dot{\theta} - 2e\varphi$. With a good accuracy neglecting terms of order the relation 5 can be inverted and represented in the form

$$n = N_j - N_j^0 = \frac{\rho_j}{2} (2e\varphi - \hbar \dot{\theta} - m_j v_j^2), \quad (6)$$

where

$$\rho_j = \frac{N_j}{m_j s_j^2} = \left( \frac{\partial \mu_j}{\partial N_j} \right)^{-1} = \frac{p_{F,j}^2}{\pi^2 \hbar^3 v_{F,j}}$$

are the densities of states per unit volume per two spins. These quantities determine inertia of the phase difference oscillations of order parameters, $v_{F,j}, p_{F,j}$ are Fermi velocities and momenta. Taking into account quadratic terms give rise to somewhere more complicated expression

$$n_j = \frac{1}{2} \left( \frac{\partial \mu_j}{\partial N_j} \right)^{-1} \left[ 2e\varphi - \hbar \dot{\theta}_j - m_j v_j^2 - \frac{\partial^2 \mu_j}{2\partial N_j^2} \left( \frac{\partial \mu_j}{\partial n} \right)^2 (2e\varphi - \hbar \dot{\theta})^2 \right], \quad (7)$$

Let us notice here that that potential and vortex motions turns out to be connected via term $m_j v_j^2$. 


Electromagnetic fields obey Maxwell equations which being rewritten in terms of electromagnetic potentials in Coulomb gauge $\text{div} A = 0$ are

$$\Delta A - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} + \frac{4\pi}{c} j = \frac{1}{c} \frac{\partial \varphi}{\partial t}; \quad \Delta \varphi = 0 \quad (8)$$

Hydrodynamic equations 1 with state equation 3 and Maxwell equations form a closed system describing dynamics of multiband superconductor.

2.1. Linear approximation and Leggett modes

Now let us consider the theory of linear waves using hydrodynamic approach suggested. Introducing the perturbation of concentrations $n_j = N_j - N_j^0$ and linearizing LHS of continuity equation and Eq. 5 provided velocities and perturbation concentrations to be small we find expressions for concentrations via phases $2n_j = \rho_j (2e\varphi - \hbar \eta)$ and, then, assuming Coulomb gauge $\text{div} A = 0$, from continuity equations we find nonlinear wave equations for phases

$$\ddot{\theta}_j - 2e\hbar^{-1} \dot{\varphi} - s_j^2 \Delta \theta_j + \rho_j^{-1} \sum_k \beta \sin(\theta_j - \theta_k) = 0, \quad (9)$$

which describes potential collective excitations in superconductor. Electric potential here should be found from Poisson equation $\Delta \varphi = 2\pi e \sum_l \rho_l (2e\varphi - \hbar \theta_l)$. Here $n_j, v_j$ are perturbations of the electron densities and velocities in the j-band, the coefficient $\hbar \beta / 2 = 2J (\Psi_1^0 \Psi_2^0)$ defines the coupling between the phases of order parameters in different bands. We used the notations of Leggett [3], by which $J$ is the interband coupling energy,

$$\rho_j = \frac{N_j}{m_j s_j^2} = \left( \frac{\partial \mu}{\partial N} \right)^{-1} = \frac{p_{F,j}^2}{\pi^2 \hbar^3 v_{F,j}}$$

are the densities of states per unit volume per two spins, they determine inertia of the phase difference oscillations of order parameters so that the frequency of spatially homogeneous Leggett oscillations is $\omega_0^2 = 4(\rho_1^{-1} + \rho_2^{-1})/3$; $s_j^2 = v_{F,j}^2 / 3$ are velocities of the Bogolubov-Anderson modes in the respective bands; $(v, p)_{F,j}$ are, respectively, the Fermi velocities and momenta.

Let us consider a simple two-band case. It is possible two equilibrium homogeneous phase distributions depending on sign of interband interaction $\beta$. If $\beta > 0$ the stable configuration will be $\theta_1^0 = \theta_2^0$ but in the case of repulsive interband interaction $\beta < 0$ stable phase difference $\theta_1^0 = \theta_2^0 + \pi$. Linearizing the sine term in equations in the vicinities of corresponding steady state and excluding scalar potentials by using Poisson equation $-\Delta \varphi = -4\pi e \sum_l n_l$ we come to two equations for perturbations of electronic densities

$$\ddot{n}_j - s_j^2 \Delta n_j + \omega_j^2 \sum_l n_l + |\beta| \sum_k \left( \frac{n_j}{\rho_j} - \frac{n_k}{\rho_k} \right) = 0 \quad ,$$

where we introduce the notation $\omega_j$ for plasma frequencies in the j-th band. Plasma frequencies and densities of state are not independent since the values $q_j^2 = 4\pi e^2 \rho_j$ representing squared inverse screening length in the j-th band are related to plasma frequencies by $\omega_j = q_j s_j$. Searching for solution in the form $\sim \exp(-i\omega t + ik r)$ we find the dispersion relation for all potential waves, plasmon and Leggett ones

$$\prod_{j=1,2} \left( \omega^2 - s_j^2 \kappa^2 - \omega_j^2 \frac{\beta}{\rho_j} \right) = \left( \omega_1^2 - \frac{\beta}{\rho_2} \right) \left( \omega_2^2 - \frac{\beta}{\rho_1} \right).$$


This equation reproduce the dispersion of potential waves obtained in [3, 7] using different approach. Of course it does not take into account dissipation of potential waves due to Landau damping and due to pair breaking. In a such approach they should be taken into account phenomenologically. Dispersion of electromagnetic waves follow from equation for vector potential and is given by

\[ \omega^2 \varepsilon_t - c^2 k^2 = 0 \]

where \( \varepsilon_t = 1 - \omega_p^2 / \omega^2 \) and plasma frequency is defined as \( \omega_p^2 = \sum_j \omega_j^2 \). Typical dispersion of all modes, electromagnetic, plasma and Leggett ones are qualitatively shown on fig.1. In the next section we consider different mechanisms of Leggett mode excitation by Josephson vortices moving in distributed Josephson junction between multiband superconductors.

3. Modified Sine-Gordon Equation

In this section we consider dynamics of distributed Josephson Junction between multiband superconductors and will obtain a modified Sine-Gordon equation taking into account Leggett modes outside the junction, both localized and leaky ones. For simplicity we consider the symmetric case of Josephson junction between identical two band superconductor and limit yourself only by linear approximation. Josephson Junction represents a thin tunnelly transparent dielectric layer and from point of view of external problem it should be described as boundary conditions both for hydrodynamic and Maxwell equations. Let us begin with hydrodynamic boundary conditions. It can be obtained by trivial generalization of one-band boundary condition and can written in the form

\[ \frac{\partial \Psi_j}{\partial y} + \frac{2ie}{\hbar c} A_y \Psi_j \bigg|_a = \sum_k \delta_{j,k} \Psi_k \exp \left\{ \frac{-2ie}{\hbar c} \int_{-d/2}^{d/2} A_y dy \right\} \bigg|_d. \]
Two electrodes of Josephson junction we denote by indices $u$ for $y > d/2$ and $d$ for $y < -d/2$, $\delta_{j,k}$ are the coefficients giving tunneling rate from $k$-th band of the lower superconductor to $j$-th band of the upper one. We can rewrite it via modal parameters and phases of order parameters as

$$-\frac{e\hbar|\Psi_j^u|^2}{m_j} \left( \frac{\partial \theta_j}{\partial y} + \frac{2e}{\hbar c} A_y \right) = \frac{e\hbar}{m_j} \sum_k \delta_{j,k} |\Psi_j^u||\Psi_k^d| \sin \left( \theta_j^u - \theta_j^d + \frac{2e}{\hbar c} \int_{-d/2}^{d/2} A_y dy \right)$$

from there we just have the standard expression for Josephson current

$$j_{i,j} = e\hbar \sin \left( \theta_i^u - \theta_j^d + \frac{2e}{\hbar c} \int_{-d/2}^{d/2} A_y dy \right), \quad j_{j,k} = e\hbar \delta_{j,k} |\Psi_j^u||\Psi_k^d|, \quad (10)$$

where $j_{j,k}^c$ are critical current densities defining tunneling between different bands. This boundary condition defines tunnel current inside Josephson junction and it can be rewritten in the terms of velocities as

$$\gamma_{y,j}^u = \frac{\gamma_{j,k}^u \sin(\theta_j^d - \theta_k^u)}{m_j}, \quad \gamma_{y,k}^u = \delta_{j,k} \frac{\hbar |\Psi_j^u||\Psi_k^d|}{2m_j |\Psi_j^u||\Psi_k^d|}, \quad (11)$$

where we introduce the new coupling coefficients $\gamma_{j,k}$. To obtain electrodynamic boundary conditions we write down Maxwell equation for interior of the junction and take $y$ component of equation 8 for vector potential assuming $\varphi = 0$ what is possible due to fulfilment of equation $\Delta \varphi = 0$. Assuming a Josephson junction being sufficiently thin and searching for solution in the form, (common factor $\exp(-i\omega t + i\chi x)$ is omitted)

$$A_x = i\chi a_0 \frac{2\Lambda y}{(d+2\Lambda)} + i\chi f_0 \sinh \chi y, \quad A_y = a_0 \left( 1 + \frac{\chi^2 \Lambda y^2}{(d+2\Lambda)} \right) + \chi f_0 \cosh \chi y, \quad \varphi = \frac{i\omega}{c} f_0 \sinh \chi y,$$

where the terms with $f_0$ are gauge functions, for amplitude of $y$-component of vector potential $a_0$ we find an equation

$$c \left[ -\frac{\omega^2}{c^2} + \frac{\chi^2}{1 + 2\Lambda d} \right] a_0 + 4\pi j_y^t = 4\pi j_{y,ext} \quad (12)$$

where $\Lambda$ is an effective penetration depth related to impedance of Josephson interface $Z = E_x(d/2)/B_y(d/2)$ as $\Lambda = cZ/i\omega$. $j_t$ is a tunneling current, $j_{y,ext}$-external current flowing into the junction. Tunnel current and effective length To make this equation closed we should find from solution of external problem unknown quantities, effective length $\Lambda$ and tunnel current $j_{y,ext}$ To find external field we will use linearized in the vicinity of corresponding steady state the equation 9 for phases which is equivalent to equations 10 for concentrations. Let us write down the solutions of linearized equations for external with respect Josephson junction region

$$A_x = -i\frac{\kappa_t}{\chi} A_t e^{-\kappa_t \hat{y}} + i\chi f_1 e^{-\chi \hat{y}}, \quad A_y = A_t e^{-\kappa_t \hat{y}} - |\chi| f_1 e^{-\chi \hat{y}},$$

$$\varphi = \frac{i\omega}{c} \left( -A_t \sum_{i,t} \frac{b_i}{\kappa_t^2} n_{i,t} e^{-\kappa_t \hat{y}} + f_1 e^{-\chi \hat{y}} \right),$$

$$\delta \theta_j = \frac{2e}{\hbar c} \left\{ -\frac{A_t}{q_l^2} \sum_{i,t} \frac{b_i}{\kappa_t} e^{-\kappa_t \hat{y}} \left[ n_{j,i} + \frac{q_l^2}{\kappa_t^2} \sum_i n_{i,l} \right] + f_1 e^{-\chi \hat{y}} \right\},$$
\[ n_j = \frac{i \omega A_t}{e 4 \pi c} \sum_i b_{lj} n_{j,l} e^{-\kappa_{lj} \hat{y}}, \quad v_{yj} = \frac{e A_t}{m_j c} \left\{ -\frac{1}{q_j^2} \sum_i b_{lj} e^{-\kappa_{lj} \hat{y}} \left[ n_{j,i} + \frac{q_j^2}{k^2_l} \sum_i n_{i,l} \right] + e^{-\kappa_{lj} \hat{y}} \right\}. \quad (13) \]

Here we introduced phase perturbations as \( \theta_j = \theta_j^0 + \delta \theta_j, \ k_l = k_l(\omega), \ l = 1, 2 \) are solutions of dispersion equation for potential waves Eq.10, \( l = 1 \) denotes plasma wave, \( l = 2 \)-Leggett one, \( n_{j,l} \) -density perturbations eigenvectors what are found from 10, the indices \( j, l \) denote the band and sort of potential wave respectively, \( \kappa_l, \kappa_l \) transversal wave numbers of electromagnetic and potential waves defined as \( \kappa_l = \sqrt{\omega^2 - \varepsilon_l \omega^2 c^{-2}} \) and \( \kappa_l = \sqrt{\chi^2 - k_l^2 \omega} \), \( A_t \)-coefficient of excitation of electromagnetic wave, \( b_{lj} \)-relative, respect to \( A_t \) coefficients of potential mode excitation, the terms with \( f_1 \) is a gauge functions not affecting gauge conditions. Finding expressions for coefficients \( A_t, f_0, f_1, A_x \) via \( q_0 \) from electromagnetic boundary conditions expressing continuity of potentials and tangential fields we will be able to find jumps of phase differences \( \delta \theta_l - \delta \theta_l \delta \) and tunnel currents. Having this procedure carried out we come to the modified Sine-Gordon equation for phase variable \( \psi \) defined as \( 2e(d+2\lambda)B_x = h c \partial_x \psi \) where \( \lambda \) is a London penetration length.

\[ \frac{\hbar}{2 e \pi c} \frac{\chi^2}{d + 2 \epsilon} \left[ \frac{\epsilon \omega^2}{d + 2 \Lambda} - \frac{\epsilon \omega^2}{d + 2 \Lambda} \right] \psi + \sum_j \frac{m j^2 k \sin \left[ \left( \hat{H}_l + \hat{H}_k \right) \psi \right] = \hat{J}_{ext} \quad (14) \]

In such form this equation valid for positive interband coupling \( J > 0 \) when stable steady state is \( \theta_l^0 = \theta_l^0 \). For the case of negative \( J < 0 \) when stable unperturbed phase difference between bands is a \( \pi \) the critical currents \( j^c_j \) in Eq.14 should be replaced by \( (-1)^{j+k} j^c_{j,k} \). Expressions for effective length \( \Lambda \) and for operators \( \hat{H}_l \) defining phase jumps via \( \psi \) are nonlocal and in Fourier representation take the form

\[ \Lambda \approx \lambda - \chi^2 \chi^2 \sum_l \frac{f_l(\omega)}{\kappa_l}, \quad F_l(\omega) = \frac{b_j}{k^2_l} \sum_i n_{i,l}, \]

\[ \hat{H}_j \approx \left\{ \frac{1}{2} - \frac{\chi^2 \chi^2}{d + 2 \Lambda} \sum_l \frac{G_{j,l}(\omega)}{\kappa_l} \right\}, \quad G_{j,l}(\omega) = b_l \left[ \frac{1}{q_j^2} n_{j,l} + \frac{1}{k^2_l} \sum_i n_{i,l} \right], \]

where coefficients of potential wave excitation \( b_l \) are still left undefined. They should be found using hydrodynamic boundary condition. For sufficiently low transparency of Josephson junction a superconducting phase distribution in exterior can be found from approximate conditions \( v_{yj} = 0 \) what yields a simple equation for coefficients \( b_l \)

\[ \sum_l \frac{b_l \left[ n_{j,l} + \frac{q_j^2}{k^2_l} \sum_i n_{i,l} \right]}{q_j^2} = 0. \quad (15) \]

To accomplish all calculations we need to solve dispersion relation and find \( k_l(\omega) \), then find eigenvectors for electronic density perturbation \( n_{j,l}(\omega) \), and, in conclusion, to solve equations Eqs.15 for excitation coefficients. Then we have expression for functions \( F_l(\omega), G_{j,l}(\omega) \) To carry out such a procedure analytically we will use natural assumption that working frequency \( \omega \) and frequency of Leggett mode \( \omega_L^2 \approx \beta (\rho_1^{-1} + \rho_2^{-1}) \) is much smaller than plasma frequencies \( \omega_j \) and consider the case of strongly asymmetric bands so that \( \omega_1 \gg \omega_2, s_1 \gg s_2, \rho_1 \gg \rho_2 \). For such a case we have a simple formulas for transversal wave numbers for electromagnetic, plasma and Leggett modes

\[ \kappa_l \approx \lambda^{-1}, \kappa_1 \approx q_1, \kappa_2 \approx \sqrt{\frac{\omega_L^2 - \omega^2}{s_2^2} + \chi^2} \]
and explicit expressions for coefficients

\[ F_1 \approx \frac{\omega_j^2}{\omega_z^2}, \quad F_2 \approx -\frac{\omega_j^2 s_4^1}{\omega_z^2 s_2^2}, \quad \tilde{G}_{1,1} \approx 1, \quad \tilde{G}_{1,2} \approx \frac{q_0^2}{q_1^2}, \quad \tilde{G}_{2,1} \approx \frac{s_0^2}{s_2^2}, \quad \tilde{G}_{2,2} \approx -\frac{s_1^2}{s_2^2} \]  

(16)

and are able to write down explicitly a modified Sine-Gordon equation taking into account Leggett interband collective excitation. So far we assumed applicability of Fourier transform to phase distributions. But if we would understand \( \omega, \chi \) and their functions as operators \( \chi = -i\partial_x, \omega = i\partial_t \) we will be able to apply our approach to more general functions. It appear to be strongly nonlocal both in time and space due to presence of transverse wavenumber of Leggett mode \( \kappa_2(\omega, \chi) \) in denominators in expressions for effective length and for Josephson phase difference. Now let us apply this modified Sine-Gordon equation to some concrete problems

3.0.1. Swihart-Leggett modes in Josephson Junction. Here we consider distortion of dispersion of Josephson plasma mode sometimes called Swihart mode due of coupling to Leggett waves. Linearizing the term \( \sin \left( \hat{H}_i + \hat{H}_k \right) \psi \) we find dispersion of modified plasma waves

\[ \omega^2 - \Omega_j^2 = v_s^2 \chi^2 \left( \frac{2\varepsilon}{q_1d} + 1 + \frac{2\varepsilon \omega_j^2}{s_2q_1^2d\sqrt{\omega_j^2 - \omega^2 + s_2^2\chi^2}} \right), \]

where we introduced two effective frequencies \( \Omega_j^2 = (8\pi \varepsilon d)(\vec{h}\varepsilon)^{-1}(j_{1,1}^c \pm 2j_{2,2}^c + j_{2,2}^c), \quad \omega_j^2 = (8\pi \varepsilon d)(\vec{h}\varepsilon)(\pm j_{2,2}^c + j_{2,2}^c), \) the first one plays role of Josephson plasma frequency and Swihart velocity \( v_s^2 = c^2 d[\varepsilon(d + 2\lambda)]^{-1}. \) The first term in the parentis

3.0.2. Excitation of Swihart-Leggett modes by Josephson vortices. Modified Sine-Gordon equation as and the old one in high frequency limit has an approximate solution corresponding to dense vortex chain wave \( \psi^0 = \Omega t - \hbar x \) where \( \hbar, \Omega \) are dimensionless magnetic field and Josephson frequency \( h = 2e(hc)^{-1}(d + 2\lambda)B_z, \Omega = 2e\sigma d B_y h^{-1} \) where \( B_z, E_y \) are magnetic and electric field inside the junctions. For such a wave all nonlocal corrections in Josephson current disappear since \( \partial^2 \psi = \partial^2 \psi = 0 \) and the only small nonzero terms defined by contributions from imaginary part of dielectric permittivity resulted from quasiparticle tunneling \( \varepsilon = 4i\pi \sigma d^{-1} \) and external current in RHS of Eq.14 give unperturbed IV characteristic \( \Omega \sim \sigma^{-1} E_y \). In the equation for the next approximation the term \( \sum_{i,k} j_{i,k}^c \sin \psi^0 \) will be the source for perturbation \( \psi^1 \) for which we have an expression via complex amplitude \( \psi^1 = \text{Im}(\Psi^1 e^{ihx - i\Omega t}) \) where \( \Psi^1 \) is

\[ \Psi^1 = -\frac{8\pi \varepsilon d j^c}{\hbar \varepsilon} \frac{1}{\Omega^2 + i\gamma \Omega - v_s^2 \hbar^2} \left( \frac{2\omega_j^2}{s_2q_1^2d\sqrt{\omega_j^2 - \omega^2 + s_2^2\chi^2}} \right). \]

(17)

and we introduced \( j^c = \sum_{i,k} j_{i,k}^c \) and \( \gamma \sim \sigma/d \) as a dimensionless conductance of the junction. Dissipation of the Leggett mode can be taken into account by changing \( \Omega \to \Omega + iv \) under the square root in the denominator, where \( \nu \) is a damping rate of Leggett mode. Calculating the next approximation we find the correction for I-V curve \( j_c \psi^1 \cos \psi^0 = 0.5j_c \text{Im}(\Psi^1) \) and we can see that position and width of resonance peak on I-V curve will be changed due to Leggett modes excitation.
3.0.3. Excitation of Leggett modes by vortices due to Raman mechanism. Instead of linearized relation 6 let us use more precise equation 18 and neglect by small term \( \sim \frac{\partial^2 n_j}{\partial \theta^2} \left( \frac{\partial n_j}{\partial \theta} \right)^{-2} \). We come to the following relation between concentrations and phases of order parameters

\[
n = \frac{\rho_j}{2} \left[ 2e \phi - \hbar \dot{\theta} - m_j v_j^2 \right],
\]

what result in nonlinear equation

\[
\ddot{\theta}_j - 2e\hbar^{-1} \dot{\theta}_j - s_j^2 \Delta \theta_j + \rho_j^{-1} \sum_k \beta \sin(\theta_j - \theta_k) = - \frac{1}{4\hbar m_j} \frac{\partial}{\partial t} \left( \hbar \nabla \theta_j + \frac{2e}{c} A \right)^2,
\]

which differs from linear equation 9 by the term in RHS giving coupling of vortex and potential motions. If we consider the Josephson junction with external static magnetic field applied then this last term will bring to excitation of Leggett mode with Josephson frequency \( \Omega \) just as it was considered in previous section. Let us consider this additional with respect to considered process in more detail. Earlier considered mechanism was resulted from matching boundary conditions on Josephson junction interface and how it follows from Eq14 amplitude of Leggett mode is proportional to \( \chi^2 \). The new one can be more effective since excitation force \(-m_j \hbar^{-1} \dot{\theta} v_j^2\) can excite Leggett mode even for \( \chi = 0 \) due to imbalance of chemical potentials in different bands when this mode is pure neutral. Excitation of Leggett mode now can be estimated using perturbation theory. We can calculate the superconducting velocity staying in RHS of Eq.19 using the modified Sine-Gordon equation 14 derived above and expressions for fields found in linear approximation 13. Let us consider the case of strong magnetic field \( B = e^{-y/\lambda}(B^0 + B^1 \exp(\text{i}x - i\Omega_1 t)) \) and the term in RHS of Eq.19 is resulted from their combination. Solving linearized equation Eq.19 we find the amplitude of phase perturbation in exciting Leggett mode

\[
\approx \frac{i\Omega_j}{s_2 \Omega_L} e^{2\lambda^2 B^0 B^1} \left[ \frac{1}{m_1} - \frac{1}{m_2} \right] \frac{\lambda}{2 - iq_y \lambda},
\]

here \( q_y \) is a wavenumber of Leggett mode, and \( B^0, B^1 \) are defined by solution of Sine-Gordon equation Eq.14. Here we used an approximation of strong asymmetric band as we did earlier for derivation of relations 16. So the mechanism considered will result in resonance at \( \Omega_j \sim \Omega_L \) just as it was for pure linear mechanism. For low magnetic field the leading term for Leggett wave excitation due to such mechanism will be \( \sim (v^1)^2 e^{2\hbar x - 2i\Omega_1 t} \) and results in resonance peculiarity at \( 2\Omega_1 \sim \Omega_L \) on IV curve.

Thus we have developed a hydrodynamical model of multiband superconductors and derived a modified Sine-Gordon equation describing the dynamics of a distributed Josephson junction between multiband superconductors and studied the excitation of Leggett modes by dense chain of moving Josephson vortices. Though a hydrodynamical model is unable to describe kinetic effects resulting in decay of collective excitation such as Landau damping and Cooper pair breaking but qualitatively such effects can be taken into account in hydrodynamical scheme as some effective damping rates which should be calculated using kinetic equations [8].

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