TOEPLITZ OPERATORS ON WEIGHTED HARMONIC BERGMAN SPACES

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Abstract. In this paper, we study Toeplitz operators on the weighted harmonic Bergman spaces with nonnegative symbols, the weights we choose here are Muckenhoupt \( A_2 \) weights. Results obtained include characterizations of bounded Toeplitz operators, compact Toeplitz operators, invertible Toeplitz operators and Toeplitz operators in the Schatten classes.

1. Introduction

Let \( 1 \leq p < \infty \) and \( \omega \) be a nonnegative integrable function on the unit disk \( \mathbb{D} \). \( L^p(\omega) \) denotes the Banach space with norm

\[
\|f\|_{L^p(\omega)} := \left( \int_{\mathbb{D}} |f(z)|^p \omega(z) dA(z) \right)^{\frac{1}{p}}.
\]

The weighted harmonic (analytic) Bergman space \( L^p_h(\omega) \) (\( L^p_a(\omega) \)) is the subspace of \( L^p(\omega) \) which is consisting of harmonic (analytic) functions on \( \mathbb{D} \). The goal of this paper is to provide a framework to study operator properties (boundedness, compactness, Schatten classes and invertibility) of Toeplitz operators on \( L^p_h(\omega) \) with nonnegative symbols.

Weighted analytic function spaces and their Toeplitz operators have captured people’s attentions for a long time. It is now well known (\cite{24}) that several results on unweighted Bergman space can be extended to the standard weighted Bergman space \( L^p_\alpha(\omega_\alpha) \), where \( \omega_\alpha(z) = (1 + \alpha)(1 - |z|^2)^\alpha \) and \(-1 < \alpha < \infty\). In recent papers \cite{19} and \cite{20}, Peláez and Rättyä characterized the bounded and Schatten class Toeplitz operators (induced by a positive Borel measure) on a weighted Bergman space, here the weight is a radial function satisfying the doubling property

\[
\int_1^r \omega(s) ds \leq C \int_1^{r^2} \omega(s) ds.
\]

The first results of non-radial weighted Bergman space seems to be due to Luecking (\cite{13}) who investigated the structure of weighted Bergman space with Bézouté-Bonami weight. Based on Luecking’s representation and duality theorems in \cite{13}, Chacón (\cite{2}) and Constantin (\cite{5}, \cite{6}) studied the boundedness and compactness of Toeplitz operators on certain weighted Bergman spaces. In \cite{17}, Mitkovski and Wick established a reproducing kernel thesis for operators on Bergman type space, and their definitions include weighted versions of Bergman spaces on more complicated domains.

Key words and phrases. Toeplitz operator, weighted harmonic Bergman space, boundedness, compactness, invertibility.

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We will be primarily interested in weighted harmonic Bergman space \( L^2_h(\omega) \). Our choice of the weight \( \omega \) is motivated by the characterization of boundedness of \( P_h \) acting on \( L^2(\omega) \), where \( P_h \) is the unweighted harmonic Bergman (orthogonal) projection from \( L^2(dA) \) to \( L^2_h(dA) \). It is well known that \( L^2_h(dA) \) is a reproducing kernel Hilbert space and

\[
P_h f(z) = \int_{\mathbb{D}} f(\lambda) \left[ \frac{1}{(1 - \lambda z)^2} + \frac{1}{(1 - \overline{\lambda} z)^2} - 1 \right] dA(\lambda).
\]

Clearly, \( P_h \) is a Calderón-Zygmund operator on the homogeneous space \((\mathbb{D}, d, dA)\), where \( d \) is the Euclidean distance and \( dA \) is the Lebesgue area measure on \( \mathbb{D} \), normalized so that the measure of \( \mathbb{D} \) is 1. For the definitions of Calderón-Zygmund operator and homogeneous space, we refer to [1].

The most successful understanding of the (one) weight theory of Calderón-Zygmund operator was spurred by Muckenhoupt’s work in 1970s ([18]), which led to the introduction of the class of \( A_p \) weight and developments of weighted inequality. We will restrict our attentions to \( A_2 \) weight on \((\mathbb{D}, d, dA)\). Let \( 0 < \omega \in L^1(\mathbb{D}, dA) \), it is called a Muckenhoupt \( A_2 \) weight if

\[
[\omega]_{A_2} := \sup_{a \in \mathbb{D}, 0 < r < 1} \frac{|B(a, r)|_\omega |B(a, r)|_{\omega^{-1}}}{|B(a, r)|^2} < +\infty,
\]

where

\[
B(a, r) = \{ z \in \mathbb{D} : d(a, z) = |z - a| < r \},
\]

\[
|B(a, r)|_\omega = \int_{B(a, r)} \omega(z) dA(z)
\]

and \(| \cdot |\) is the normalized Lebesgue measure on \( \mathbb{D} \).

It follows form the remarkable \( A_2 \) theorem ([1], [9]) that \( P_h \) is bounded from \( L^2(\omega) \) to \( L^2_h(\omega) \) provided \( \omega \) is a Muckenhoupt \( A_2 \) weight. As mentioned above, we will focus on the weighted harmonic Bergman space \( L^2_h(\omega) \) with \( \omega \in A_2 \). Little is known about this nature function space. However, we will see in Section 2 that \( L^2_h(\omega) \) is a reproducing kernel Hilbert space with the reproducing kernel \( K_z^\omega(\lambda) \), i.e., \( f(z) = \langle f, K_z^\omega \rangle_{L^2(\omega)} \) for all \( f \) in \( L^2_h(\omega) \).

For a positive finite Borel measure \( \nu \) on \( \mathbb{D} \), we densely define the Toeplitz operator \( T_\nu \) on \( L^2_h(\omega) \) by

\[
T_\nu f(z) = \langle T_\nu f, K_z^\omega \rangle_{L^2(\omega)} = \int_{\mathbb{D}} f(\lambda) K_z^\omega(\lambda) d\nu(\lambda) \quad (z \in \mathbb{D}).
\]

For a bounded function \( \varphi \), using the integral representation for the projection operator (from \( L^2(\omega) \) to \( L^2_h(\omega) \)), we can express the Toeplitz operator \( T_\varphi \) (on \( L^2_h(\omega) \)) as the following:

\[
T_\varphi f(z) = \int_{\mathbb{D}} f(\lambda) K_z^\omega(\lambda) \varphi(\lambda) \omega(\lambda) dA(\lambda) \quad (z \in \mathbb{D}).
\]

Although we follow Luecking’s methods in [13] and [14] for the weighted Bergman spaces, some new difficulties arise in the study of the space \( L^2_h(\omega) \) and the corresponding operators. For instance, harmonic functions do not share many powerful tools with analytic functions. One can use the Cauchy formula to estimate the local values of analytic functions easily. However, because of the tedious remainder, the harmonic version Cauchy formula (known as Cauchy-Pompeiu formula) is not valid
The results illustrate that the invertibility of Toeplitz operators on Bergman space can imply a reverse Carleson inequality for \( L^p \) on the standard weighted harmonic Bergman space with nonnegative symbols on \( L^2 \), we obtain a relationship of the invertibility between Toeplitz operators with bounded \( L^p \) operators with nonnegative symbols (23) to the case of \( L^2 \) on this inequality, we generalize the result on the invertibility of Bergman Toeplitz Schatten classes on unweighted harmonic Bergman space \( L^2 \), we are pleased to mention here that Miao (15) have obtained characterizations for Toeplitz operators with nonnegative symbols to be bounded, compact and in \( L^p \) setting.

In Section 3, we characterize the boundedness, compactness and Schatten \( p \) class of Toeplitz operators \( T_\nu \) on \( L^2_\nu(\omega) \) by means of Berezin transform and Carleson measure. We are pleased to mention here that Miao (15) have obtained characterizations for the Toeplitz operators with nonnegative symbols to be bounded, compact and in Schatten classes on unweighted harmonic Bergman space \( L^2_h \).

Section 4 of this paper is devoted to studying the invertibility of Toeplitz operators on the standard weighted harmonic Bergman space \( L^2_\nu(\omega_a) \). A little surprising to us, the results illustrate that the invertibility of Toeplitz operators on Bergman space \( L^2_a \) can imply a reverse Carleson inequality for \( L^2_\nu(\omega_a) \), see Theorems 4.2 and 4.4. Based on this inequality, we generalize the result on the invertibility of Bergman Toeplitz operators with nonnegative symbols (23) to the case of \( L^2_\nu(\omega_a) \). As a consequence, we obtain a relationship of the invertibility between Toeplitz operators with bounded nonnegative symbols on \( L^2_\nu(\omega_a) \) and \( L^2_\nu(\omega_a) \), see Corollary 4.5.

In the final section, we establish a reverse Carleson type inequality for \( L^2_\nu(\omega) \) with \( \omega \in A_2 \). Indeed, we obtain a sufficient condition for \( \chi_G dA \) to be a reverse Carleson measure for \( L^2_\nu(\omega) \), where \( G \) is a measurable set in \( \mathbb{D} \), see Theorem 5.1 which extends Theorem 3.9 for the weighted (analytic) Bergman space in [13] to the harmonic setting.

Throughout the paper, positive constants will be explicitly denoted by \( C, C_0, C_1, \ldots \), which may depend on some fixed numbers and change at each occurrence.

2. The Space \( L^2_\nu(\omega) \) and its Representation

In this section, we present some elementary structures of \( L^2_\nu(\omega) \) with \( \omega \in A_2 \). To study the harmonic Bergman spaces, we need the following important properties of harmonic functions.

Lemma 2.1. (10) Suppose that \( f \) is a harmonic function on the disk \( \mathbb{D} \) and \( 0 < p < \infty \). There exists a positive constant \( C = C(p) \) such that for every ball \( B(a, r) = \{ z \in \mathbb{D} : |z - a| < r \} \) in \( \mathbb{D} \),

\[
|f(a)|^p \leq \frac{C}{|B(a, r)|} \int_{B(a, r)} |f(z)|^p dA(z).
\]

In particular, if \( p \geq 1 \), then the constant \( C = 1 \). Using this result one can get the following useful inequalities easily: given \( 0 < p < \infty \) and \( 0 < r < 1 \), there exist positive constants \( C_1 = C_1(p) \) and \( C_2 = C_2(p) \) such that

\[
|f(a)|^p \leq \frac{C_1}{(1 - r)^2} \cdot \frac{1}{|D(a, r)|} \int_{D(a, r)} |f(z)|^p dA(z) \quad (a \in \mathbb{D})
\]
and
\[ |\partial f(a)|^p \leq \frac{C_2}{(1-r)^{2+p}} \cdot \frac{1}{|D(a,r)|^{2+p}} \int_{D(a,r)} |f(z)|^p dA(z) \quad (a \in \mathbb{D}) \]
for all \( f \) harmonic on \( \mathbb{D} \), where \( \partial f = \frac{\partial f}{\partial z} \).

**Remark 1.** From the above inequalities, it is easy to show that point evaluations are bounded linear functionals on \( L^2_h(\omega) \) with \( \omega \in A_2 \). As a consequence, \( L^2_h(\omega) \) is a reproducing kernel Hilbert space.

**Remark 2.** It is not clear whether \( L^2_h(\omega) \) is complete for a general weight. However, if \( p \) is an analytic polynomial on \( \mathbb{D} \) and \( \omega(z) = |p(z)|^2 \), Douglas and Wang [7] showed that \( L^2_h(\omega) \) is complete, whose proof heavily depends on some particular properties of polynomials.

It is clear that \( L^2_h(\omega) \) coincides with its dual space with respect to the \( L^2(\omega) \) inner product. The next result illustrates that the dual space of \( L^2_h(\omega) \) can be identified with \( L^2_h(\omega^{-1}) \) via the unweighted inner product, which generalizes Luecking’s result for \( L^2_\omega(\omega) \) (see Theorem 2.1 in [13]) to the setting of \( L^2_h(\omega) \).

**Lemma 2.2.** Suppose that \( \omega \) is an \( A_2 \) weight. Then the dual space of \( L^2_h(\omega^{-1}) \) can be identified with \( L^2_h(\omega^{-1}) \). The pairing is given by
\[ \langle f, g \rangle = \int_{\mathbb{D}} f(z) \overline{g(z)} dA(z). \]
Consequently, there exists a bounded, bijective and linear operator \( \mathcal{F} : L^2_h(\omega^{-1}) \rightarrow (L^2_h(\omega))^* \) such that
\[ \mathcal{F}(f)(g) = \langle g, f \rangle_{L^2(\omega)} \]
for \( f \in L^2_h(\omega^{-1}) \) and \( g \in L^2_h(\omega) \).

**Proof.** Let \( \omega \) be an \( A_2 \) weight. Recall that orthogonal projection \( P_h : L^2(dA) \rightarrow L^2_h(dA) \) is a Calderón-Zygmund operator on \( (\mathbb{D}, d, dA) \). Then \( P_h \) is bounded on both \( L^2(\omega) \) and \( L^2(\omega^{-1}) \). Thus for each \( f \in L^2(\omega) \) and \( g \in L^2(\omega^{-1}) \), we have
\[ \langle P_h f, g \rangle = \langle f, P_h g \rangle. \]
Note that each \( f \in L^2_h(\omega) \) (or \( f \in L^2_h(\omega^{-1}) \)) has the following form:
\[ f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \bar{z}^n, \]
thus we obtain
\[
\begin{align*}
P_h f(z) &= \int_{\mathbb{D}} f(\lambda) R_z(\lambda) dA(\lambda) \\
&= \lim_{s \to 1^-} \int_{s\mathbb{D}} f(\lambda) \left[ \frac{1}{(1 - \bar{z}\lambda)^2} + \frac{1}{(1 - \lambda z)^2} - 1 \right] dA(\lambda) \\
&= f(z)
\end{align*}
\]
for \( z \in \mathbb{D} \). Then the rest of this proof are exactly the same as the one of Theorem 2.1 in [13], we omit the details. \( \square \)
Let \( a \in \mathbb{D} \) and \( 0 < r < 1 \). A pseudohyperbolic disk \( D(a, r) \) is defined by
\[
D(a, r) = \left\{ z \in \mathbb{D} : \rho(z, a) = \left| \frac{z - a}{1 - \overline{a}z} \right| < r \right\}.
\]

We will frequently use the following property of \( A_2 \) weights on pseudohyperbolic disks. For the sake of complete, we include a proof of this fact as follows.

**Lemma 2.3.** Let \( 0 < r \leq \frac{1}{4} \) and \( z \in \mathbb{D} \). If \( \xi \in D(z, r) \), then we have
\[
|D(z, r)|_\omega < 8[\omega]_{A_2}|D(\xi, r)|_\omega.
\]

**Proof.** Observe that \( D(z, r) \subset D(\xi, 2r) \). Now it suffices to show the following doubling inequality:
\[
|D(\xi, 2r)|_\omega < 8[\omega]_{A_2}|D(\xi, r)|_\omega \quad (\xi \in \mathbb{D}).
\]

Since \( \omega \) is an \( A_2 \) weight, we have
\[
\frac{|D(\xi, 2r)|_\omega |D(\xi, 2r)|_{\omega^{-1}}}{|D(\xi, 2r)|^2} \leq [\omega]_{A_2}.
\]

Recall that a pseudohyperbolic disk \( D(z, r) \) is a Euclidean disk with center and radius given by
\[
\mathcal{C} = \frac{1 - r^2}{1 - r^2|z|^2}z, \quad \mathcal{R} = \frac{1 - |z|^2}{1 - r^2|z|^2}r.
\]

Combining the above with Cauchy-Schwarz’s inequality gives
\[
|D(\xi, 2r)|_\omega \leq [\omega]_{A_2} \frac{|D(\xi, 2r)|^2}{|D(\xi, r)|_{\omega^{-1}}} \leq [\omega]_{A_2} \frac{|D(\xi, r)|^2}{|D(\xi, r)|_{\omega^{-1}}} \cdot \frac{|D(\xi, 2r)|^2}{|D(\xi, r)|^2} \leq 4[\omega]_{A_2}|D(\xi, r)|_\omega \cdot \left( \frac{1 - r^2|\xi|^2}{1 - 4r^2|\xi|^2} \right)^2
\]
\[
< 8[\omega]_{A_2}|D(\xi, r)|_\omega,
\]
where the last inequality is due to \( r \leq \frac{1}{4} \). This completes the proof of Lemma 2.3. \( \square \)

We now turn to the representation theory of the space \( L^2_h(\omega) \). These results and their proof strategies are motivated by Luecking’s works on weighted Bergman space \((13, 14)\).

Before studying the representation theory of \( L^2_h(\omega) \), we need to recall the concept of \( \epsilon \)-lattice in the unit disk. Let \( \epsilon \in (0, 1) \), a sequence \( \{a_n\}_{n=1}^\infty \) in the unit disk is called an \( \epsilon \)-lattice in the pseudo-hyperbolic metric if
\[
\bullet \quad \mathbb{D} = \bigcup_{n=1}^\infty D(a_n, \epsilon) \quad \text{and} \quad \inf_{n \neq m} \left| \frac{a_n - a_m}{1 - \overline{a_n}a_m} \right| \geq \frac{\epsilon}{2}.
\]

Now, we are ready to state the atomic decomposition for \( L^2_h(\omega) \).
Theorem 2.4. Let \( \omega \) be an \( A_2 \) weight. Then there is an \( \epsilon \)-lattice \( \{a_n\}_{n=1}^{\infty} \) such that for each \( f \in L^2_h(\omega) \) we have
\[
f(z) = \sum_{n=1}^{\infty} c_n (1 - |a_n|)^2 |D(a_n, \epsilon)|^{-1/2} R_{a_n}(z)
\]
for some sequence \( \{c_n\} \) in \( \ell^2(\mathbb{N}) \), where
\[
R_{\lambda}(z) = \frac{1}{(1 - z\lambda)^2} + \frac{1}{(1 - \lambda z)^2} - 1
\]
is the reproducing kernel for \( L^2_h \) at \( \lambda \in \mathbb{D} \).

Remarks. We have the following estimate of the module of \( R_{\lambda} \): there exists an \( r_0 \in (0, \frac{1}{4}] \) such that if \( 0 < r \leq r_0 \), then
\[
\frac{1}{2} (1 - |\lambda|)^2 \leq |R_{\lambda}(z)| \leq \frac{3}{(1 - |\lambda|)^2}
\]
for all \( z \in D(\lambda, r) \). For the proof of this fact, we refer to Lemma 2.2 in [4]. In what follows, we will use \( r_0 \) to denote the constant provided in this remarks.

To prove Theorem 2.4, we need to establish a harmonic version of Luecking’s theorems in [13] and [14].

Theorem 2.5. Let \( \omega \) be an \( A_2 \) weight. Then there exists an \( \epsilon \)-lattice \( \{a_n\}_{n=1}^{\infty} \) for some \( 0 < \epsilon < \frac{1}{16} \) such that
\[
\|f\|_{L^2_h(\omega)} \approx \sum_{n=1}^{\infty} |f(a_n)|^2 |D(a_n, \epsilon)|^{-1} \omega^{-1}
\]
for all \( f \in L^2_h(\omega) \). That is, there exist two positive constant \( C_1 \) and \( C_2 \) such that
\[
C_1 \|f\|_{L^2_h(\omega)}^2 \leq \sum_{n=1}^{\infty} |f(a_n)|^2 |D(a_n, \epsilon)|^{-1} \omega^{-1} \leq C_2 \|f\|_{L^2_h(\omega)}^2.
\]
for all \( f \) in \( L^2_h(\omega) \).

Once Theorem 2.5 is established, we can quickly present a proof of Theorem 2.4 as follows.

Proof of Theorem 2.4. Note that both \( \omega \) and \( \omega^{-1} \) are \( A_2 \) weights, it follows from Theorem 2.5 that we can choose \( \epsilon \in (0, \frac{1}{16}) \) and an \( \epsilon \)-lattice \( \{a_n\}_{n=1}^{\infty} \) such that
\[
\|g\|_{L^2_{h(\omega^{-1})}} \approx \sum_{n=1}^{\infty} |g(a_n)|^2 |D(a_n, \epsilon)|^{-1} \omega^{-1}.
\]
By Cauchy-Schwartz inequality and the definition of \( A_2 \) weight,
\[
|D(a_n, \epsilon)|^2 \leq |D(a_n, \epsilon)|_\omega \cdot |D(a_n, \epsilon)|_{\omega^{-1}} \leq [\omega]_{A_2} |D(a_n, \epsilon)|^2.
\]
Therefore,
\[
\|g\|_{L^2_{h(\omega^{-1})}} \approx \sum_{n=1}^{\infty} |g(a_n)|^2 (1 - |a_n|^2)^4 |D(a_n, \epsilon)|^{-1} \omega^{-1}.
\]
This implies that the linear operator \( \mathcal{L} : L^2_h(\omega^{-1}) \to \ell^2(\mathbb{Z}) \) defined by

\[
\mathcal{L}(g) := \left\{ g(a_n)(1 - |a_n|^2)^2|D(a_n, \epsilon)|^{-\frac{1}{2}} \right\}_{n=1}^{\infty}
\]

is bounded below and so its range is closed. It follows from the closed range theorem that \( \mathcal{L}^* \) is surjective.

From the proof of Lemma 2.2, we have

\[
g(a_n) = \langle g, R_{a_n} \rangle_{L^2(dA)} \quad (*)
\]

for each \( g \in L^2_h(\omega^{-1}) \) and every \( n \geq 1 \). Let \( \{c_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N}) \). Using (*) to obtain that

\[
\mathcal{L}^*(\{c_n\})(z) = \sum_{n=1}^{\infty} c_n R_{a_n}(z)(1 - |a_n|^2)^2|D(a_n, \epsilon)|^{-\frac{1}{2}},
\]

which gives the desired result. This completes the proof of Theorem 2.4. \( \square \)

We now turn to the proof of Theorem 2.5. Let \( 0 < \epsilon < \frac{1}{16} \) and \( \{a_n\}_{n=1}^{\infty} \subset \mathbb{D} \) be an \( \epsilon \)-lattice. Define a measure \( \mu = \mu_\epsilon \) on \( \mathbb{D} \) by

\[
\mu(z) = \sum_{n=1}^{\infty} \delta_{a_n}(z)|D(a_n, \frac{\epsilon}{4})|^{-\frac{1}{2}},
\]

where \( \delta_{a_n} \) is the Dirac measure concentrated at \( a_n \). Indeed, the conclusion of Theorem 2.4 tells us that \( \mu \) is a Carleson and reverse Carleson measure for \( L^2_h(\omega) \).

Proposition 2.6. Suppose that \( \nu \) is a positive Borel measure on \( \mathbb{D} \). If there exist an \( 0 < r \leq r_0 \) and a constant \( C > 0 \) independent of \( z \in \mathbb{D} \) such that

\[
\nu(D(z, r)) \leq C|D(z, r)|^{-\frac{1}{2}}
\]

for all \( z \in \mathbb{D} \), then \( \nu \) is a Carleson measure for \( L^p_h(\omega) \) \((0 < p < \infty)\), i.e., there is a positive constant \( C_p \) such that

\[
\int_{\mathbb{D}} |f(z)|^p d\nu(z) \leq C_p \|f\|_{L^p(\omega)}^p
\]

for \( f \in L^p_h(\omega) \). Consequently, \( \mu \) is a Carleson measure for \( L^2_h(\omega) \), i.e., there is an absolute constant \( C > 0 \) such that

\[
\int_{\mathbb{D}} |f(z)|^2 d\mu(z) \leq \sum_{n=1}^{\infty} |f(a_n)|^2|D(a_n, \frac{\epsilon}{4})|^{-\frac{1}{2}} \leq C \|f\|_{L^2(\omega)}^2
\]

for all \( f \) in \( L^2_h(\omega) \).

Proof. Fix an \( r \leq r_0 \). By Lemma 2.1 we obtain

\[
|f(z)|^2 \leq \frac{C}{|D(z, r)|} \int_{D(z, r)} |f(\xi)|^2 dA(\xi) \quad (z \in \mathbb{D}),
\]
where $0 < p < \infty$ and $C = C(p, r)$. Cauchy-Schwartz inequality and $A_2$ condition give us that
\[
|f(z)|^p \leq \frac{C^2}{|D(z, r)|^2} \left( \int_{D(z, r)} |f(\xi)|^p dA(\xi) \right)^2 \\
\leq \frac{C^2}{|D(z, r)|^2} \left( \int_{D(z, r)} |f(\xi)|^{p \omega(\xi)} dA(\xi) \right) \cdot \left( \int_{D(z, r)} \frac{1}{\omega(\xi)} dA(\xi) \right) \\
\leq C^2 [\omega]_{A_2}\frac{\int_{D(z, r)} |f(\xi)|^{p \omega(\xi)} dA(\xi)}{|D(z, r)|^2}.
\]
Integrating the above over the unit disk to obtain
\[
\int_{D} |f(z)|^p d\nu(z) \leq C \int_{D} |D(z, r)|^{-1} \int_{D(z, r)} |f(\xi)|^{p \omega(\xi)} dA(\xi) d\nu(z) \\
= C \int_{D} |D(z, r)|^{-1} \int_{D} |f(\xi)|^{p \omega(\xi)} \chi_{D(z, r)}(\xi) dA(\xi) d\nu(z) \\
= C \int_{D} \int_{D} |D(z, r)|^{-1} |f(\xi)|^{p \omega(\xi)} \chi_{D(\xi, r)}(z) dA(\xi) d\nu(z),
\]
here the constant $C$ depends only on $p$ and $r$. Note that $\xi \in D(z, r)$, we have by Lemma 2.3 that
\[
|D(z, r)|_{\omega} \geq C|D(\xi, r)|_{\omega}
\]
for some absolute constant $C$. Therefore,
\[
\int_{D} |f(z)|^p d\nu(z) \leq C \int_{D} |D(z, r)|^{-1} |f(\xi)|^{p \omega(\xi)} \chi_{D(\xi, r)}(z) dA(\xi) d\nu(z) \\
\leq C \int_{D} \int_{D} |D(\xi, r)|^{-1} |f(\xi)|^{p \omega(\xi)} \chi_{D(\xi, r)}(z) d\nu(z) dA(\xi).
\]
Now using our hypothesis on $\nu$ to get
\[
\int_{D} |f(z)|^p d\nu(z) \leq C \int_{D} |D(\xi, r)|^{-1} \nu(D(\xi, r)) |f(\xi)|^{p \omega(\xi)} dA(\xi) \\
\leq C_1 \int_{D} |f(\xi)|^{p \omega(\xi)} dA(\xi),
\]
the constant $C_1 > 0$ comes from the assumption and $C_1$ is independent of $f \in L^p_\nu(\omega)$.

For the second conclusion of this proposition, it sufficient for us to show the following inequality:
\[
\mu(D(a, \frac{1}{4})) \leq C \int_{D(a, \frac{1}{4})} \omega(z) dA(z) \quad (a \in \mathbb{D})
\]
for some absolute constant $C > 0$. Indeed, by the definition of $\mu$, we have
\[
\mu(D(a, \frac{1}{4})) = \sum_{\rho(a_n, a) < \frac{1}{4}} \int_{D(a_n, \frac{1}{4})} \omega dA = \sum_{\rho(a_n, a) < \frac{1}{4}} |D(a_n, \frac{\epsilon}{4})|_{\omega}.
\]
If $\rho(a, a_n) < \frac{1}{4}$, then for each $z \in D(a_n, \frac{1}{4})$ we get
\[
\rho(z, a) \leq \rho(z, a_n) + \rho(a_n, a) < \frac{\epsilon}{4} + \frac{1}{4} < \frac{1}{2}.
\]
Which shows that
\[ D(a_n, \epsilon) \subset D(a, \frac{1}{2}) \]
for every \( n \geq 1 \) provided \( \rho(a, a_n) < \frac{1}{4} \).

Since \( D(a_n, \epsilon) \cap D(a_m, \epsilon) = \emptyset \) for \( n \neq m \), we obtain
\[ \bigcup_{\rho(a_n, a) < \frac{1}{4}} D(a_n, \epsilon) \subset D(a, \frac{1}{2}) \]
and
\[ \mu(D(a, \frac{1}{2})) \leq |D(a, \frac{1}{2})|_\omega \leq C|D(a, \frac{1}{4})|_\omega \]
for every \( a \in \mathbb{D} \), the constant \( C > 0 \) (independent of \( a \)) comes from Lemma 2.3. This completes the proof of Proposition 2.6. □

In order to finish the proof of Theorem 2.5, we need to show that there is an \( \epsilon \in (0, \frac{1}{16}) \) such that \( \mu = \mu_\epsilon \) is a reverse \( L^2(\omega) \) Carleson measure. More precisely, we will prove the following proposition.

**Proposition 2.7.** There exists an \( \epsilon \in (0, \frac{1}{16}) \) such that
\[ \int_{\mathbb{D}} |f(z)|^2 d\mu(z) = \sum_{n=1}^{\infty} |f(a_n)|^2 |D(a_n, \epsilon)|_\omega \geq C\|f\|^2_{L^2(\omega)} \]
for all \( f \) in \( L^2(\omega) \), where \( C > 0 \) is an absolute constant.

The rest of this section is devoted to the proof of the above **reverse Carleson inequality**. To do so, we need to prove the following two lemmas related to harmonic functions, which extend the results in \([13]\) and \([14]\) for weighted Bergman spaces to the present situation.

**Lemma 2.8.** Let \( f \) be a harmonic function on \( \mathbb{D} \) and \( \epsilon \in (0, \frac{1}{16}) \). Then there exists a constant \( C_1 > 0 \) (independent of \( z, \epsilon \) and \( f \)) such that
\[ |f(z) - f(0)| \leq C_1 \epsilon \int_{D(0, \frac{1}{4})} |f(\lambda)| dA(\lambda) \]
when \( |z| < \epsilon \). As a consequence, there exists a constant \( C_2 > 0 \) (independent of \( z, \epsilon \) and \( f \)) such that
\[ |f(w) - f(\xi)|^2 \leq \frac{C_2 \epsilon^2}{|D(\xi, \frac{1}{4})|_\omega} \int_{D(\xi, \frac{1}{4})} |f(\lambda)|^2 \omega(\lambda) dA(\lambda) \]
when \( \xi \in D(w, \epsilon) \).

**Proof.** Observe that
\[ f(z) - f(0) = \int_0^1 \frac{\partial}{\partial t} [f(tz)] dt = \int_0^1 [\nabla f(tz) \cdot z] dt \]
for \( |z| < \epsilon < \frac{1}{16} \). Thus we have
\[ |f(z) - f(0)| \leq \sup_{|\xi| < \epsilon} |\nabla f(\xi)| \cdot |z|. \]
Recall that
\[ |\nabla f|^2 = 2(|\partial f|^2 + |\overline{\partial f}|^2) = 2(|\partial f|^2 + |\overline{\partial f}|^2) \leq \left[ \sqrt{2}(|\partial f| + |\overline{\partial f}|) \right]^2, \]
where \( \overline{\partial f} = \frac{\partial f}{\partial \overline{z}} \). By Lemma 2.1, there is an absolute constant \( C > 0 \) such that
\[ |\nabla f(\xi)| \leq \frac{2\sqrt{2}C}{(1 - \frac{1}{10})^3} \cdot \frac{1}{|D(\xi, \frac{1}{10})|^\frac{1}{2}} \int_{D(\xi, \frac{1}{10})} |f(\lambda)| dA(\lambda) \]
\[ = \frac{2\sqrt{2}C}{(1 - \frac{1}{10})^3} \cdot \left( \frac{1 - (\frac{1}{10})^2|\xi|^2}{\frac{1}{16}(1 - |\xi|^2)} \right) \int_{D(\xi, \frac{1}{10})} |f(\lambda)| dA(\lambda). \]
Note that \( |\xi| \leq \epsilon < \frac{1}{10} \) and if \( \lambda \in D(\xi, \frac{1}{10}) \), then
\[ |\lambda| < |\xi| + \frac{1}{16}|1 - \overline{\lambda}\xi| < \frac{1}{4}, \]
so we have \( D(\xi, \frac{1}{10}) \subset D(0, \frac{1}{4}) \) and
\[ |\nabla f(\xi)| \leq C_1 \int_{D(0, \frac{1}{4})} |f(\lambda)| dA(\lambda) \]
for all \( |\xi| \leq \epsilon \), where the constant \( C_1 \) is independent of \( z, \xi \) and \( \epsilon \). Therefore,
\[ |f(z) - f(0)| \leq \sup_{|\xi| \leq \epsilon} |\nabla f(\xi)| \cdot |z| \leq C_1 |z| \int_{D(0, \frac{1}{4})} |f(\lambda)| dA(\lambda) \]
for \( |z| < \epsilon \) with \( \epsilon \in (0, \frac{1}{16}) \), this proves the first part of the lemma.

Let \( \varphi_\xi \) be the Möbius map, then \( f \circ \varphi_\xi \) is harmonic on \( \mathbb{D} \). Changing of variable gives that
\[ |f(\varphi_\xi(z)) - f(\xi)| \leq \frac{C_3\epsilon}{|D(\xi, \frac{1}{4})|^\frac{1}{2}} \int_{D(\xi, \frac{1}{4})} |f(\lambda)| dA(\lambda) \]
for some absolute constant \( C_3 > 0 \). By Cauchy-Schwartz inequality we obtain
\[ |f(\varphi_\xi(z)) - f(\xi)|^2 \leq \frac{C_3^2\epsilon^2}{|D(\xi, \frac{1}{4})|^2} \left( \int_{D(\xi, \frac{1}{4})} |f(\lambda)|^2 \omega(\lambda) dA(\lambda) \right) \cdot \left( \int_{D(\xi, \frac{1}{4})} \frac{1}{\omega(\lambda)} dA(\lambda) \right) \]
\[ \leq \frac{C_3^2|\omega|_{A_2, \epsilon}^2}{|D(\xi, \frac{1}{4})|^\omega} \int_{D(\xi, \frac{1}{4})} |f(\lambda)|^2 \omega(\lambda) dA(\lambda). \]
Let \( w = \varphi_\xi(z) \), then \( |\varphi_\xi(w)| = |z| < \epsilon \), which gives that
\[ |f(w) - f(\xi)|^2 \leq \frac{C_2\epsilon^2}{|D(\xi, \frac{1}{4})|^\omega} \int_{D(\xi, \frac{1}{4})} |f(\lambda)|^2 \omega(\lambda) dA(\lambda) \]
if \( \xi \in D(w, \epsilon) \), as desired. \( \square \)

**Lemma 2.9.** Let \( f \) be a harmonic function and \( \epsilon \in (0, \frac{1}{10}) \). Let \( \mu \) be the measure defined in Section 2. Then there exists a constant \( C > 0 \) (independent of \( \epsilon \)) such that
\[ \int_{\mathbb{D}} \int_{\mathbb{D}} \chi_\epsilon(z, \xi) |D(\xi, \epsilon)|^{-1} |f(z) - f(\xi)|^2 \omega(z) dA(z) d\mu(\xi) \leq C \epsilon^2 \| f \|_{L^2(\omega)}^2 \]
for all \( f \in L^2_\alpha(\omega) \), where

\[
\chi_\epsilon(z, \xi) = \begin{cases} 
1, & \text{if } z \in D(\xi, \epsilon), \\
0, & \text{otherwise}.
\end{cases}
\]

**Proof.** By Lemma 2.8, we have

\[
\chi_\epsilon(z, \xi)\omega(z) \left| f(z) - f(\xi) \right|^2 \frac{\omega(z)}{|D(\xi, \epsilon)|_\omega} \leq \left( \frac{C\epsilon^2}{|D(\xi, \frac{1}{4})|_\omega} \int_{D(\xi, \frac{1}{4})} |f(\lambda)|^2 \omega(\lambda) dA(\lambda) \right) \frac{\chi_\epsilon(z, \xi)\omega(z)}{|D(\xi, \epsilon)|_\omega}.
\]

Integrating over \( z \in \mathbb{D} \) on both sides gives

\[
\int_{\mathbb{D}} \chi_\epsilon(z, \xi)\omega(z) \left| f(z) - f(\xi) \right|^2 \frac{\omega(z)}{|D(\xi, \epsilon)|_\omega} dA(z) \leq \left( \frac{C\epsilon^2}{|D(\xi, \frac{1}{4})|_\omega} \int_{D(\xi, \frac{1}{4})} |f(\lambda)|^2 \omega(\lambda) dA(\lambda) \right) \frac{\chi_\epsilon(z, \xi)\omega(z)}{|D(\xi, \epsilon)|_\omega}.
\]

Now integrate with respect to \( d\mu(\xi) \) to obtain

\[
\int_{\mathbb{D}} \int_{D(\xi, \frac{1}{4})} \chi_\epsilon(z, \xi)\omega(z) dA(z) d\mu(\xi) \leq \int_{\mathbb{D}} \frac{C\epsilon^2}{|D(\xi, \frac{1}{4})|_\omega} \int_{D(\xi, \frac{1}{4})} |f(\lambda)|^2 \omega(\lambda) dA(\lambda) d\mu(\xi).
\]

Using Fubini’s Theorem on the right side and noting \( \chi_{D(\xi, \frac{1}{4})}(\lambda) = \chi_{D(\lambda, \frac{1}{4})}(\xi) \), we have

\[
\int_{\mathbb{D}} \frac{C\epsilon^2}{|D(\xi, \frac{1}{4})|_\omega} \int_{D(\xi, \frac{1}{4})} |f(\lambda)|^2 \omega(\lambda) dA(\lambda) d\mu(\xi) = C\epsilon^2 \int_{\mathbb{D}} \left( \int_{\mathbb{D}} \frac{\chi_{D(\lambda, \frac{1}{4})}(\xi)}{|D(\xi, \frac{1}{4})|_\omega} d\mu(\xi) \right) |f(\lambda)|^2 \omega(\lambda) dA(\lambda)
\]

Since \( \lambda \in D(\xi, \frac{1}{4}) \), Lemma 2.3 tells us that there is an absolute constant \( C > 0 \) such that

\[
|D(\xi, \frac{1}{4})|_\omega \geq C |D(\lambda, \frac{1}{4})|_\omega.
\]

Thus we obtain

\[
\int_{\mathbb{D}} \frac{\chi_{D(\lambda, \frac{1}{4})}(\xi)}{|D(\xi, \frac{1}{4})|_\omega} d\mu(\xi) \leq C^{-1} \mu(D(\lambda, \frac{1}{4})) |\omega|_\omega.
\]

By Lemma 2.6, we have

\[
\mu(D(\lambda, \frac{1}{4})) \cdot |D(\lambda, \frac{1}{4})|_\omega^{-1} \leq C_1
\]

for some constant \( C_1 > 0 \) (independent of \( \epsilon \)), completing the proof.

**Proof of Proposition 2.7.** Recall that \( \mu \) satisfies the condition:

\[
\mu(D(a, \frac{1}{4})) \leq C |D(a, \frac{1}{4})|_\omega
\]

for all \( a \in \mathbb{D} \). Applying Lemma 2.9 to \( \epsilon \in (0, \frac{1}{10}) \) we have

\[
\left[ \int_{\mathbb{D}} \int_{D(\xi, \epsilon)} \chi_\epsilon(z, \xi)\omega(z) |f(z) - f(\xi)|^2 dA(z) d\mu(\xi) \right]^\frac{1}{2} \leq C\epsilon \|f\|_{L^2(\omega)}.
\]

The triangle inequality gives

\[
I - J \leq C\epsilon \|f\|_{L^2(\omega)},
\]
where
\[ I := \left[ \int_D \int_D \frac{\chi_\epsilon(z, \xi)}{|D(\xi, \epsilon)|_\omega} |f(z)|^2 \omega(z) dA(z) d\mu(\xi) \right]^\frac{1}{2} \]
and
\[ J := \left[ \int_D \int_D \frac{\chi_\epsilon(z, \xi)}{|D(\xi, \epsilon)|_\omega} |f(\xi)|^2 \omega(z) dA(z) d\mu(\xi) \right]^\frac{1}{2}. \]

For the first integral \( I \), we note that for each \( z \in D \),
\[ \int_D \frac{\chi_\epsilon(z, \xi)}{|D(\xi, \epsilon)|_\omega} d\mu(\xi) = \int_{D(z, \epsilon)} \frac{d\mu(\xi)}{|D(\xi, \epsilon)|_\omega} \geq C_1 \frac{\mu(D(z, \epsilon))}{|D(z, \epsilon)|_\omega}, \]
where the “\( \geq \)” follows from Lemma 2.3 and \( C_1 \) is an absolute constant.

Since \( D = \bigcup_{n=1}^\infty D(a_n, \epsilon) \), we can select a disk \( D(a_j, \epsilon) \) such that \( z \in D(a_j, \epsilon) \) for each \( z \in D \). Applying Lemma 2.3 to get
\[ \mu(D(z, \epsilon)) = \sum_{n=1}^\infty \delta_{a_n}(D(z, \epsilon)) \int_{D(a_n, 4\epsilon)} \omega dA \geq |D(a_j, \epsilon)|_\omega \geq C_2 |D(a_j, \epsilon)|_\omega \geq C_3 |D(z, \epsilon)|_\omega, \]
where \( C_2 \) and \( C_3 \) are absolute positive constants. Therefore, we have
\[ \int_D \frac{\chi_\epsilon(z, \xi)}{|D(\xi, \epsilon)|_\omega} d\mu(\xi) \geq \tilde{C} \]
for some absolute constant \( \tilde{C} > 0 \). These give us that
\[ I \geq \tilde{C} \|f\|_{L^2(\omega)}. \]

For the second integral, we observe that
\[ \int_D \chi_\epsilon(z, \xi) \omega(z) dA(z) = |D(\xi, \epsilon)|_\omega, \]
to get
\[ J = \left( \int_D |f(\xi)|^2 d\mu(\xi) \right)^\frac{1}{2}. \]

Thus we have
\[ \tilde{C} \|f\|_{L^2(\omega)} - \left( \int_D |f(\xi)|^2 d\mu(\xi) \right)^\frac{1}{2} \leq I - J \leq C_\epsilon \|f\|_{L^2(\omega)}. \]

Equivalently,
\[ (\tilde{C} - C_\epsilon) \|f\|_{L^2(\omega)} \leq \left( \int_D |f(\xi)|^2 d\mu(\xi) \right)^\frac{1}{2} \]
for each \( 0 < \epsilon < \frac{1}{16} \).

Since \( C, \tilde{C} \) are both independent of \( \epsilon \), we can choose
\[ 0 < \epsilon < \min \left\{ \frac{1}{16}, \frac{\tilde{C}}{2C}, r_0 \right\} \]
such that
\[ \|f\|_{L^2(\omega)}^2 \leq \frac{1}{(C - C\epsilon)^2} \int_{\mathbb{D}} |f(\xi)|^2 d\mu(\xi). \]
Recall the definition of $\mu$, we conclude that
\[ \|f\|_{L^2(\omega)}^2 \leq \frac{4}{C^2} \sum_{n=1}^{\infty} |f(a_n)|^2 |D(a_n, \frac{\epsilon}{4})|_\omega \]
\[ \leq \frac{4}{C^2} \sum_{n=1}^{\infty} |f(a_n)|^2 |D(a_n, \epsilon)|_\omega \]
This completes the proof.

The proof of Theorem 2.4 implies the following result immediately.

**Theorem 2.10.** Suppose that $\omega$ satisfies the $A_2$ condition. Then there is an $\epsilon$-lattice $\{a_n\}_{n=1}^{\infty} \subset \mathbb{D}$ such that any $f \in L^2_h(\omega)$ has the following form:
\[ f(z) = \sum_{n=1}^{\infty} c_n K_{a_n}^\omega(z) |D(a_n, \epsilon)|^{\frac{1}{2}} \]
for some sequence $\{c_n\}$ in $\ell^2(\mathbb{N})$, where $K_{a_n}^\omega$ is the reproducing kernel for $L^2_h(\omega)$.

**Proof.** We consider the linear map $\mathcal{S} : L^2_h(\omega) \rightarrow \ell^2(\mathbb{N})$:
\[ \mathcal{S} f = \left\{ f(a_n)|D(a_n, \epsilon)|^{\frac{1}{2}} \right\}_{n=1}^{\infty}. \]
Since $L^2_h(\omega)$ is a Hilbert space, we apply Propositions 2.6 and 2.7 to deduce that $\mathcal{S}^* : \ell^2(\mathbb{N}) \rightarrow L^2_h(\omega)$ is surjective and
\[ \left\langle \mathcal{S}^* \left( \{c_n\} \right), f \right\rangle_{L^2(\omega)} = \left\langle \sum_{n=1}^{\infty} c_n K_{a_n}^\omega(z) |D(a_n, \epsilon)|^{\frac{1}{2}}, f \right\rangle_{L^2(\omega)} \]
for $\{c_n\} \in \ell^2(\mathbb{N})$ and $f \in L^2_h(\omega)$. Therefore,
\[ \mathcal{S}^* \left( \{c_n\} \right)(z) = \sum_{n=1}^{\infty} c_n K_{a_n}^\omega(z) |D(a_n, \epsilon)|^{\frac{1}{2}}. \]
This completes the proof of this theorem.

**3. Boundedness and Compactness of $T_\nu$ on $L^2_h(\omega)$**

Recall Toeplitz operator $T_\nu$ initially defined on a dense subspace of $L^2_h(\omega)$ is given by
\[ T_\nu f(z) = \int_{\mathbb{D}} f(\lambda) \overline{K_z^\omega(\lambda)} d\nu(\lambda) \quad (z \in \mathbb{D}). \]
In this section, we will characterize the boundedness and compactness of $T_\nu$ on $L^2_h(\omega)$ via Berezin transform and Carleson measure for $L^2_h(\omega)$. Firstly, we define the Berezin transform $\tilde{\nu}$ of $\nu$ as follows:
\[ \tilde{\nu}(z) = \frac{1}{\|R_z\|_{L^2(\omega)}} \int_{\mathbb{D}} |R_z(\lambda)|^2 d\nu(\lambda), \]
where
\[ R_z(\lambda) = \frac{1}{(1 - \lambda z)^2} + \frac{1}{(1 - z\lambda)^2} - 1 \]
is the reproducing kernel for \( L^2_h \). The first main result of this section is Theorem 3.1.

**Theorem 3.1.** Let \( \nu \) be a positive finite Borel measure on \( D \) and \( \omega \) be an \( A_2 \) weight. The following conditions are equivalent:

1. \( T_\nu \) extends to a bounded linear operator on \( L^2_h(\omega) \);
2. \( \nu \) is a Carleson measure for \( L^2_h(\omega) \);
3. There exist an \( 0 < r \leq r_0 \) and a constant \( C > 0 \) independent of \( z \in D \) such that \( \nu(D(z, r)) \leq C|D(z, r)|_\omega \) for all \( z \in D \);
4. Berezin transform \( \tilde{\nu} \) is bounded.

To prove Theorem 3.1 we need the following useful lemma.

**Lemma 3.2.** Let \( \omega \in A_2 \). If \( 0 < r \leq r_0 \), then there is a constant \( C = C(r) \) such that
\[ \frac{|D(\lambda, r)|_\omega}{2(1 - |\lambda|)^4} \leq \|R_\lambda\|_{L^2(\omega)}^2 \leq C \frac{|D(\lambda, r)|_\omega}{(1 - |\lambda|)^4} \]
for all \( \lambda \in D \).

**Proof.** Let \( \lambda \in D \). By the remarks below Theorem 2.4, there exists an \( r_0 \in (0, \frac{1}{4}] \) such that if \( 0 < r \leq r_0 \), then
\[ \frac{1}{2} \frac{1}{(1 - |\lambda|)^2} \leq |R_\lambda(z)| \leq \frac{3}{(1 - |\lambda|)^2} \]
for all \( z \in D(\lambda, r) \). It follows that for each \( r \in (0, r_0] \) we have
\[ \|R_\lambda\|_{L^2(\omega)}^2 = \int_D |R_\lambda(z)|^2 \omega(z)dA(z) \]
\[ \geq \int_{D(\lambda, r)} |R_\lambda(z)|^2 \omega(z)dA(z) \]
\[ \geq \frac{1}{4} \int_{D(\lambda, r)} \frac{\omega(z)}{(1 - |\lambda|)^4}dA(z) \]
\[ = \frac{|D(\lambda, r)|_\omega}{4(1 - |\lambda|)^4} \]
To show the other inequality, observe that
\[ |z - \lambda| < r(1 - |\lambda|) < r|1 - z\lambda| \quad (z, \lambda \in D) \]
we have
\[ S(\lambda, r) := \{ z \in D : |z - \lambda| < r(1 - |\lambda|) \} \subset D(\lambda, r) \).
Thus, we have by Lemma 2.1 in [6] that
\[ C_1 \frac{|S(\lambda, r)|_\omega}{(1 - |\lambda|)^4} \leq \|K_\lambda\|_{L^2(\omega)}^2 \leq C_2 \frac{|S(\lambda, r)|_\omega}{(1 - |\lambda|)^4} \leq C_2 \frac{|D(\lambda, r)|_\omega}{(1 - |\lambda|)^4} \]
for some positive constants $C_1 = C_1(r)$ and $C_2 = C_2(r)$, here $K_\lambda(z) = \frac{1}{(1 - \lambda z)^2}$ is the reproducing kernel for $L^2_\omega$ at $\lambda$. Recall that 
\[ R_\lambda(z) = 2\Re(K_\lambda(z)) - 1, \]
we have 
\[ \|R_\lambda\|_{L^2(\omega)}^2 \leq \frac{4C|D(\lambda, r)|\omega}{(1 - |\lambda|)^4} + 2\|\omega\|_{L^1}. \]
Consequently, to complete the proof we need only to show that there is a constant $C_3$ depending only on $r$ such that 
\[ \|\omega\|_{L^1} \leq C_3 \frac{|D(\lambda, r)|\omega}{(1 - |\lambda|)^4} \]
for every $\lambda \in \mathbb{D}$. Indeed, we may assume that $\|\omega\|_{L^1} = 1$. Then it is easy to see that 
\[ C(1 - |\lambda|)^2 \leq |D(\lambda, r)| \]
for some constant $C = C(r)$. Thus we have 
\[
C(1 - |\lambda|)^2 \leq |D(\lambda, r)| = \int_{D(\lambda, r)} \omega(z)^{\frac{3}{2}}\omega(z)^{-\frac{1}{2}}dA(z)
\leq |D(\lambda, r)|^{\frac{3}{2}} \cdot |D(\lambda, r)|^{-\frac{1}{2}}
\leq |D(\lambda, r)|^{\frac{3}{2}} \cdot \|\omega^{-1}\|_{L^1}^{\frac{1}{2}}.
\]
This shows that 
\[ \frac{|D(\lambda, r)|\omega}{(1 - |\lambda|)^4} \geq C_3, \]
to complete the proof of Lemma 3.2.

Now we are ready to prove Theorem 3.1.

**Proof of Theorem 3.1** Our strategy is (4) \(\Rightarrow\) (3) \(\Rightarrow\) (2) \(\Rightarrow\) (1) \(\Rightarrow\) (4).

(4) \(\Rightarrow\) (3): Note that there exists an $r \in (0, r_0]$ such that 
\[ \frac{1}{2} \frac{(1 - |z|)^2}{(1 - |\lambda|)^2} \leq |R_z(\lambda)| \leq \frac{3}{(1 - |z|)^2} \]
for $\lambda \in D(z, r)$. From the definition of $\tilde{\nu}$, there is a constant $C > 0$ such that 
\[ \frac{1}{\|R_z\|_{L^2(\omega)}^2} \int_{D(z, r)} |R_z(\lambda)|^2 d\nu(\lambda) \leq \tilde{\nu}(z) \leq C \]
for all $z \in \mathbb{D}$. Combining these with Lemma 3.2 gives us that 
\[ \nu(D(z, r)) \leq C|D(z, r)|_\omega \quad (z \in \mathbb{D}) \]
for some positive constant $C = C(r)$.

(3) \(\Rightarrow\) (2): This was proved in Proposition 2.6.

(2) \(\Rightarrow\) (1): Assume that $\nu$ is a Carleson measure. By Condition (2) and Lemma 3.2, we obtain Condition (3). Now Proposition 2.6 implies that $\nu$ is also a Carleson.
measure for $L^1_h(\omega)$. Then for $f, g$ are harmonic on a neighborhood of $\overline{D}$ (by Theorem 2.4, these functions are dense in $L^2_h(\omega)$), we have

$$\left| \langle T_\nu f, g \rangle \right| \leq \int_D |f(z)g(z)|d\nu(z) \leq C\|fg\|_{L^1(\omega)} \leq C\|f\|_{L^2(\omega)}\|g\|_{L^2(\omega)};$$

which means that $T_\nu$ is bounded.

(1) $\Rightarrow$ (4): Suppose that $T_\nu$ is bounded on $L^2_h(\omega)$. We consider the partial sum $\sigma_N = \sum_{n=1}^{\infty} t_n K_{a_n}$, where $N \geq 1, \{t_n\}$ are complex numbers and $\{a_n\} \subset \mathbb{D}$. Direct calculation shows that

$$\|\sigma_N\|_{L^2(\nu)} \leq C\|\sigma_N\|_{L^2(\omega)}$$

for some constant $C > 0$. This implies that if $\lim_{N \to \infty} \|\sigma_N - g\|_{L^2(\omega)} = 0$ for some $g \in L^2_h(\omega)$, then

$$\lim_{N \to \infty} \langle f, \sigma_N \rangle_{L^2(\nu)} = \langle f, g \rangle_{L^2(\nu)}$$

for every $f \in L^2_h(\omega)$. Applying Theorem 2.10 to obtain

$$\langle T_\nu f, g \rangle_{L^2(\omega)} = \langle f, g \rangle_{L^2(\nu)}$$

for every $f, g \in L^2_h(\omega)$. In particular, we have

$$\langle T_\nu R_z, R_z \rangle_{L^2(\omega)} = \langle R_z, R_z \rangle_{L^2(\nu)} = \tilde{\nu}(z)\|R_z\|_{L^2(\omega)},$$

to get $\tilde{\nu}(z) \leq \|T_\nu\|$ for all $z \in \mathbb{D}$. The proof of Theorem 3.1 is complete now. \qed

From the above theorem, it is natural to characterize the compactness of $T_\nu$ via vanishing Carleson measure. In fact, we will characterize the compact Toeplitz operators with positive measures as the symbols via not only vanishing Carlson measure (for the $A_2$ weighted harmonic Bergman space) but also Berezin transform.

**Theorem 3.3.** Let $\nu$ be a positive finite Borel measure on $\mathbb{D}$ and $\omega \in A_2$. The following conditions are equivalent:

1. $T_\nu$ is compact on $L^2_h(\omega)$;
2. $\nu$ is a vanishing Carleson measure for $L^2_h(\omega)$, i.e.,

$$\lim_{|z| \to 1} \frac{\nu(D(z, r))}{|D(z, r)|_\omega} = 0$$

for some $r \in (0, 1)$;
3. The Berezin transform $\lim_{|z| \to 1^-} \tilde{\nu}(z) = 0$.

**Proof.** We will show that (2) $\Rightarrow$ (1) $\Rightarrow$ (3) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1): To prove (1), we need only to show that the inclusion operator $i : L^2_h(\omega) \to L^2(\nu)$ is compact, i.e.,

$$\lim_{n \to \infty} \int_D |f_n(z)|^2d\nu(z) = 0$$

whenever $\|f_n\|_{L^2(\omega)} \to 0 (n \to \infty)$, where $\{f_n\}_{n=1}^\infty$ is a bounded sequence in $L^2_h(\omega)$ which converges to zero uniformly on each compact subset of $\mathbb{D}$.
From the proof of Proposition 2.6 there exists a positive constant $C = C(r)$ such that
\[
\int_{\Omega} |f_n(z)|^2 d\nu(z) \leq C \int_{\Omega} |D(\xi, r)|^{-1} \nu(D(\xi, r)) |f_n(\xi)|^2 \omega(\xi) dA(\xi)
\]
\[
= C \left( \int_{|\xi| \leq s} + \int_{|\xi| > s} \right) \frac{\nu(D(\xi, r))}{|D(\xi, r)|^{-1}} |f_n(\xi)|^2 \omega(\xi) dA(\xi),
\]
where $s \in (0, 1)$. Under the assumption in (2), we can choose an $s_0 \in (0, 1)$ to make the second integral as small as we like; Fix $s_0$, it is easy to show the first integral converges to zero, since $f_n \to 0$ ($n \to \infty$) uniformly on compact subsets. This proves (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (3) : Observe that
\[
\int_{\Omega} \frac{R_z}{\|R_z\|_{L^2(\omega)}} \leq \left\| T_{\nu} \left( \frac{R_z}{\|R_z\|_{L^2(\omega)}} \right) \right\|_{L^2(\omega)}.
\]
So, it sufficient for us to show that $\frac{R_z}{\|R_z\|_{L^2(\omega)}}$ converges to zero weakly in $L^2(\omega)$ as $|z| \to 1^{-}$. Note that $\frac{R_z}{\|R_z\|_{L^2(\omega)}}$ is an unit vector in $L^2(\omega)$, we need only to show it converges to zero uniformly on compact subsets of $\Omega$ as $|z| \to 1^{-}$. Observe that Lemma 3.2 implies that there exists a positive constants $C = C(r_0)$ such that
\[
\left| \frac{R_z(\lambda)}{\|R_z\|_{L^2(\omega)}} \right|^2 \leq C |R_z(\lambda)|^2 \int_{D(z, r_0)} \omega^{-1} dA.
\]
It is clear that $\frac{R_z}{\|R_z\|_{L^2(\omega)}}$ converges to zero uniformly (as $|z| \to 1^{-}$) on each disk $|\lambda| \leq s < 1$, since $|D(z, r_0)| \to 0$ as $|z| \to 1^{-}$ and $\omega^{-1} \in L^1(dA)$.

(3) $\Rightarrow$ (2) : By the definition of $\nu$ and Lemma 3.2 there exists a constant $C = C(r_0) > 0$ such that
\[
\nu(z) = \frac{1}{\|R_z\|_{L^2(\omega)}} \int_{\Omega} |R_z(\lambda)|^2 d\nu(\lambda)
\]
\[
\geq C \frac{(1 - |z|)^4}{|D(z, r_0)|} \int_{D(z, r_0)} |R_z(\lambda)|^2 d\nu(\lambda)
\]
\[
\geq C \frac{\nu(D(z, r_0))}{4 |D(z, r_0)|}.
\]
Then the desired result follows, to complete the proof of Theorem 3.3.

In the rest of this section, we will consider the special class of compact Toeplitz operators to give a characterization of $\nu$ for $T_{\nu}$ to be in the Schatten class $S^p$ ($p \geq 1$). The following theorem is the third main result in Section 3.

**Theorem 3.4.** Let $\nu$ be a positive finite Borel measure on $\Omega$ and $\omega \in A_2$. Then for $1 \leq p < \infty$, $T_{\nu} \in S^p$ if and only if
\[
\sum_{n=1}^{\infty} \left( \frac{\nu(D(a_n, \epsilon))}{|D(a_n, \epsilon)|} \right)^p < +\infty,
\]
where $\{a_n\}_{n=1}^{\infty}$ is the $\epsilon$-lattice obtained by Theorem 2.4.
In order to prove the above result, we need one more lemma, which is a straightforward consequence of Lemmas \[2.1\] and \[3.2\].

**Lemma 3.5.** Let \( \omega \in A_2 \) and \( 0 < r \leq r_0 \). There exists a constant \( C = C(r) > 0 \) such that

\[
C^{-1} \leq K^\omega_z(z) \cdot |D(z, r)|_\omega \leq C
\]

for \( z \in \mathbb{D} \), where \( K^\omega_z \) is the reproducing kernel of \( L^2_h(\omega) \).

**Proof.** Note that \( K^\omega_\lambda(\lambda) = \|K^\omega_\lambda\|^2_{L^2(\omega)} \) for each \( \lambda \in \mathbb{D} \). Applying Lemma \[2.1\] to the function \( K^\omega_z(z) \), we get a constant \( C \) depends only on \( r \) such that

\[
|K^\omega_\lambda(\lambda)|^2 \leq \frac{C\|K^\omega_\lambda\|^2_{L^2(\omega)}}{|D(z, r)|_\omega} = \frac{CK^\omega_z(\lambda)}{|D(z, r)|_\omega}.
\]

Taking \( \lambda = z \) to get the inequality on the right hand side in Lemma \[3.5\].

For the reverse inequality, note that for each \( z \in \mathbb{D} \) we have

\[
\frac{1}{(1 - |z|^2)^2} - 1 \leq \frac{2}{(1 - |z|^2)^2},
\]

\[
= R_z(z) = \langle R_z, K^\omega_z \rangle_{L^2(\omega)} \leq \|R_z\|_{L^2(\omega)} \cdot \|K^\omega_z\|_{L^2(\omega)} \leq C \frac{|D(z, r)|_\omega^{\frac{3}{2}}}{(1 - |z|^2)^{\frac{1}{2}}} \cdot \sqrt{K^\omega_z(z)},
\]

where the constant \( C \) comes from Lemma \[3.5\]. This finishes the proof. \( \square \)

We are ready to prove Theorem \[3.4\]. The method of its proof is quite standard.

**Proof of Theorem 3.4.** Suppose that the series converges. We consider the \( \epsilon \)-lattice \( \{a_n\}_{n=1}^\infty \) given by Theorem \[2.4\], recall that \( \epsilon < r_0 \) (see the proof of Proposition \[2.7\]). For an arbitrary orthonormal basis \( \{e_n\}_{n=1}^\infty \) of \( L^2_h(\omega) \), we have

\[
\sum_{n=1}^\infty \langle T_\nu e_n, e_n \rangle = \sum_{n=1}^\infty \int_{\mathbb{D}} |e_n(z)|^2 d\nu(z) = \int_{\mathbb{D}} K^\omega_z(z) d\nu(z)
\]

\[
\leq \sum_{n=1}^\infty \int_{D(a_n, \epsilon)} K^\omega_z(z) d\nu(z)
\]

\[
\leq C \sum_{n=1}^\infty \int_{D(a_n, \epsilon)} |D(z, \epsilon)|_{\omega}^{-1} d\nu(z),
\]

the constant \( C \) comes from Lemma \[3.5\], which depends only on \( \epsilon \). Note that \( \rho(z, a_n) < \epsilon \) for every \( n \geq 1 \), by Lemma \[2.3\] and its proof we can choose a constant \( C_1 = C_1(\epsilon) \) such that

\[
\sum_{n=1}^\infty \langle T_\nu e_n, e_n \rangle \leq C_1 \sum_{n=1}^\infty \frac{\nu(D(a_n, \epsilon))}{|D(a_n, \epsilon)|_\omega}.
\]

This shows that \( T_\nu \in S^1 \) provided \( \sum_{n=1}^\infty \frac{\nu(D(a_n, \epsilon))}{|D(a_n, \epsilon)|_\omega} < +\infty \).

On the other hand, if \( \sup_{n \geq 1} \frac{\nu(D(a_n, \epsilon))}{|D(a_n, \epsilon)|_\omega} < +\infty \), then by the proof Theorem 7.4 in \[24\] (or the proof of (3) \( \Rightarrow \) (2) in Theorem \[3.1\]), we deduce that \( T_\nu \) is bounded on \( L^2_h(\omega) \),
i.e., $T_\nu \in \mathcal{S}^\infty$. Now applying the interpolation theorem for the Schatten classes (see Theorem 2.6 in [24] if needed), we obtain that $T_\nu \in \mathcal{S}^p$ for each $p \in (1, +\infty)$ if

$$\sum_{n=1}^\infty \left( \frac{\nu(D(a_n, \epsilon))}{|D(a_n, \epsilon)|_\omega} \right)^p < +\infty.$$ 

Conversely, we assume that $T_\nu \in \mathcal{S}^p$ for $1 \leq p < \infty$. We recall by Theorem 2.4 that each $f \in L^2_h(\omega)$ has the following form:

$$f(z) = \sum_{n=1}^\infty c_n (1 - |a_n|^2 R_{a_n}(z)|D(a_n, \epsilon)|^{-\frac{1}{2}}$$

$$:= \sum_{n=1}^\infty c_n h_n(z),$$

where $\{c_n\}_{n=1}^\infty \in \ell^2(\mathbb{N})$. Cauchy-Schwartz inequality shows that

$$\|f\|_{L^2(\omega)}^2 \leq \left( \sum_{n=1}^\infty |c_n|^2 \right) \cdot \left( \sum_{n=1}^\infty \frac{(1 - |a_n|^4)}{|D(a_n, \epsilon)|_\omega} \|R_{a_n}\|_{L^2(\omega)} \right).$$

Since $\epsilon < r_0$, we have by Lemma 3.2 that

$$\|f\|_{L^2(\omega)}^2 \leq C \sum_{n=1}^\infty |c_n|^2$$

where the constant $C > 0$ depends only on $\epsilon$.

Fix an orthonormal basis $\{e_n\}_{n=1}^\infty$ for $L^2_h(\omega)$ and define a linear operator $\mathcal{A}$ on $L^2_h(\omega)$ by

$$\mathcal{A} \left( \sum_{n=1}^\infty c_n e_n \right) = \sum_{n=1}^\infty c_n h_n.$$

Then $\mathcal{A}$ is a bounded surjective linear operator on $L^2_h(\omega)$. Thus the $\mathcal{A}^*$ is well-defined on $L^2_h(\omega)$ and $\mathcal{A}^* T_\nu \mathcal{A} \in \mathcal{S}^p$, so that

$$\sum_{n=1}^\infty \left| \langle \mathcal{A}^* T_\nu \mathcal{A} e_n, e_n \rangle_{L^2(\omega)} \right|^p < +\infty.$$ 

On the other hand,

$$\sum_{n=1}^\infty \left| \langle \mathcal{A}^* T_\nu \mathcal{A} e_n, e_n \rangle_{L^2(\omega)} \right|^p = \sum_{n=1}^\infty \left| \langle T_\nu \mathcal{A} e_n, \mathcal{A} e_n \rangle_{L^2(\omega)} \right|^p$$

$$= \sum_{n=1}^\infty \left| \langle T_\nu h_n, h_n \rangle_{L^2(\omega)} \right|^p \quad \text{(by the definition of $\mathcal{A}$)}$$

$$= \sum_{n=1}^\infty \left( \int_D |h_n(z)|^2 d\nu(z) \right)^p$$

$$\geq \sum_{n=1}^\infty \left( \int_{D(a_n, \epsilon)} |h_n(z)|^2 d\nu(z) \right)^p.$$
Recall
\[ |h_n(z)|^2 = \frac{(1 - |a_n|)^4 |R_{a_n}(z)|^2}{|D(a_n, \epsilon)|_{\omega}}, \]
we have by Lemma 3.2 that
\[ |h_n(z)|^2 \geq \frac{1}{4|D(a_n, \epsilon)|_{\omega}}, \]
if \( \rho(z, a_n) < \epsilon < r_0 \) for each \( n \geq 1 \). Therefore,
\[ \sum_{n=1}^{\infty} \left| \langle A^* T_{\nu} A, e_n \rangle_{L^2(\omega)} \right|^p \geq 4^{-p} \sum_{n=1}^{\infty} \left( \frac{\nu(D(a_n, \epsilon))}{|D(a_n, \epsilon)|_{\omega}} \right)^p. \]
This completes the proof of Theorem 3.4.

4. INVERTIBILITY OF TOEPLITZ OPERATORS ON \( L^2_h(\omega_{\alpha}) \)

A fundamental and interesting problem is to determine when a Toeplitz operator is invertible on the Hardy or Bergman space. In this section, we study the invertibility problem of Toeplitz operators on the standard weighted harmonic Bergman space \( L^2_h(\omega_{\alpha}) \) with \( \omega_{\alpha} = (1 + \alpha)(1 - |z|^2)^\alpha, \alpha > -1 \). Recall that the reproducing kernel for \( L^2_h(\omega_{\alpha}) \) is given by
\[ R_{\alpha}^\omega(z, \lambda) = K_{\alpha}^\omega(z) + \overline{K_{\alpha}^\omega(z)} - 1 \quad (z, \lambda \in \mathbb{D}), \]
where
\[ K_{\alpha}^\omega(z) = \frac{1}{(1 - \overline{z}\lambda)^2 + \alpha} \]
is the reproducing kernel for the weighted Bergman space \( L^2_\omega(\omega_{\alpha}) \), see [21] if needed.

For the (unweighted) Bergman space setting, the second author and Zheng provided a necessary and sufficient condition for the Toeplitz operators with nonnegative symbols to be invertible on \( L^2_\omega \) ([23]). The main tool used in [23] is Luecking’s result on the reverse Carleson measure for Bergman space ([11]), which also holds for the harmonic Bergman space. More precisely, Luecking established the following result.

**Lemma 4.1.** ([12]) Suppose that \( G \) is a measurable subset of \( \mathbb{D} \). Then the following are equivalent:
(i) There exists a \( \delta \in (0, 1) \) such that
\[ |G \cap K| \geq \delta |\mathbb{D} \cap K| \]
for every ball \( K \) whose center lies on \( \partial \mathbb{D} \);
(ii) \( \chi_G dA \) is a reverse Carleson measure for \( L^2_h(\omega_{\alpha}) \). That is, there is a constant \( C > 0 \) such that
\[ \int_{\mathbb{D}} |f(z)|^p \omega_{\alpha}(z) dA(z) \leq C \int_G |f(z)|^p \omega_{\alpha}(z) dA(z) \]
for all \( f \in L^2_h(\omega_{\alpha}) \).

Motivated by the ideas and techniques used in [23], we are able to characterize the invertibility of Toeplitz operator \( T_\varphi \ (\varphi \geq 0) \) on \( L^2_h(\omega_{\alpha}) \) in terms of the reverse Carleson measure and Berezin transform.
Theorem 4.2. Let $\varphi \geq 0$ be in $L^\infty(\omega_\alpha)$. The following conditions are equivalent:
(1) The Toeplitz operator $T_\varphi$ is invertible on $L^2_h(\omega_\alpha)$;
(2) The Berezin transform $\tilde{\varphi}$ is invertible in $L^\infty(\omega_\alpha)$, where
\[
\tilde{\varphi}(z) := \frac{1}{\|R^\alpha_z\|_{L^2(\omega_\alpha)}} \int_D \varphi(\lambda)|R_\alpha(z, \lambda)|^2 \omega_\alpha(\lambda) dA(\lambda)
\]
and
\[
\|R^\alpha_z\|_{L^2(\omega_\alpha)} = R^\alpha_z(z) = \frac{2}{(1 - |z|^2)^{2+\alpha}} - 1;
\]
(3) There exists $r > 0$ such that
\[
G := \{z \in \mathbb{D} : \varphi(z) > r\},
\]
$\chi_G dA$ is a reverse Carleson measure for $L^2_h(\omega_\alpha)$;
(4) There exists a constant $C > 0$ such that
\[
\int_D |f(z)|^2 \varphi(z) \omega_\alpha(z) dA(z) \geq C \int_D |f(z)|^2 \omega_\alpha(z) dA(z)
\]
for $f \in L^2_h(\omega_\alpha)$.

Before giving the proof the main theorem of this section, we need an other lemma, which was proved in [12], [16] and [22].

Lemma 4.3. Suppose that the ball $K$ has radius $0 < t < 1$ and center $u = (1, 0) \in \mathbb{R}^2$. Let $f$ be the harmonic function
\[
f(\lambda) = f_s(\lambda) := \sqrt{1 + \alpha R^\alpha_{z_0}(\lambda)(1 - |z_0|^2)^{\frac{2+\alpha}{2}}},
\]
where $z_0 = (1 - st)u$, $0 < s < 1$. Then for each $\epsilon > 0$, there exist $s = s(\epsilon)$ and a positive constant $C = C(\epsilon)$ (independent of $K$) such that
\[
\int_{B \setminus K} |f(\lambda)|^2 (1 - |\lambda|)^\alpha dA(\lambda) < \epsilon
\]
and
\[
\int_{G \cap K} |f(\lambda)|^2 (1 - |\lambda|)^\alpha dA(\lambda) \leq C \left( \frac{|G \setminus K|}{|D \setminus K|} \right)^\beta,
\]
where
\[
\beta = \begin{cases} 1, & \text{if } 0 \leq \alpha < \infty, \\ 1 - \frac{1}{\gamma}, & \text{if } -1 < \alpha < 0,
\end{cases}
\]
$\gamma$ is a number in $(1, \frac{1}{\alpha})$ if $-1 < \alpha < 0$.

Now we are ready to prove Theorem 4.2

Proof of Theorem 4.2. We will show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1). Without loss of generality, we may assume that $0 \leq \varphi \leq 1$.

(1) $\Rightarrow$ (2): This is trivial.
(2) $\Rightarrow$ (3): Suppose that $\tilde{\varphi}$ is bounded below by some positive constant $\delta$. By Lemma 4.1, it sufficient to show that there exists a $\delta' \in (0, 1)$ such that
\[
|G \cap K| \geq \delta'|D \cap K|
\]
for all balls $K$ whose centers lie on $\partial \mathbb{D}$.

Since $\omega_\alpha dA$ is a rotation invariant measure, it is no loss of generality to assume that $K$ has its center at the point $(1,0)$. It is also clear that we need only to prove the inequality for sufficient small radius $t$, say $t < 1$.

Now we consider the subset $G = \{ \lambda \in \mathbb{D} : \varphi(\lambda) > \frac{\delta}{4} \}$. For each $z \in \mathbb{D}$,

$$\delta \leq \tilde{\varphi}(z) = \frac{1}{\| R_z^\alpha \|_{L^2(\omega_\alpha)}} \int_{\mathbb{D}} \varphi(\lambda) |R_z^\alpha(\lambda)|^2 \omega_\alpha(\lambda) dA(\lambda)$$

$$= \frac{1}{\| R_z^\alpha \|_{L^2(\omega_\alpha)}} \left( \int_G + \int_{\mathbb{D}\setminus G} \right) \varphi(\lambda) |R_z^\alpha(\lambda)|^2 \omega_\alpha(\lambda) dA(\lambda)$$

$$\leq (1 - |z|^2)^{2+\alpha} \int_G \varphi(\lambda) |R_z^\alpha(\lambda)|^2 \omega_\alpha(\lambda) dA(\lambda) + \frac{\delta}{4}.$$

Let $L_z$ be the following integral:

$$L_z = \frac{1}{\| R_z^\alpha \|_{L^2(\omega_\alpha)}} \int_{G \cap K} \varphi(\lambda) |R_z^\alpha(\lambda)|^2 \omega_\alpha(\lambda) dA(\lambda).$$

Then for each $z \in \mathbb{D}$, we have

$$L_z = \frac{1}{\| R_z^\alpha \|_{L^2(\omega_\alpha)}} \left( \int_G - \int_{G \setminus K} \right) \varphi(\lambda) |R_z^\alpha(\lambda)|^2 \omega_\alpha(\lambda) dA(\lambda)$$

$$\geq \frac{1}{2} (1 - |z|^2)^{2+\alpha} \left( \int_G - \int_{G \setminus K} \right) \varphi(\lambda) |R_z^\alpha(\lambda)|^2 \omega_\alpha(\lambda) dA(\lambda)$$

$$\geq \frac{3\delta}{8} - \frac{1}{2} (1 - |z|^2)^{2+\alpha} \int_{G \setminus K} \varphi(\lambda) |R_z^\alpha(\lambda)|^2 \omega_\alpha(\lambda) dA(\lambda)$$

$$\geq \frac{3\delta}{8} - \frac{1}{2} (1 - |z|^2)^{2+\alpha} \int_{G \setminus K} |R_z^\alpha(\lambda)|^2 \omega_\alpha(\lambda) dA(\lambda) \quad \text{(using } 0 \leq \varphi \leq 1).$$

For the $\delta$ above, Lemma 4.3 guarantees that we can select $z_0 \in \mathbb{D}$ to define a function $f$ (as the one in Lemma 4.3) satisfies the following three inequalities:

$$L_{z_0} \geq \frac{3\delta}{8} - \frac{1}{2} (1 - |z_0|^2)^{2+\alpha} \int_{G \setminus K} |R_{z_0}^\alpha(\lambda)|^2 \omega_\alpha(\lambda) dA(\lambda)$$

$$= \frac{3\delta}{8} - \frac{1}{2} \int_{G \setminus K} |f(\lambda)|^2 (1 - |\lambda|^2)^\alpha dA(\lambda),$$

$$\int_{G \setminus K} |f(\lambda)|^2 (1 - |\lambda|^2)^\alpha dA(\lambda) < \frac{\delta}{4}$$

and

$$\int_{G \cap K} |f(\lambda)|^2 (1 - |\lambda|^2)^\alpha dA(\lambda) \leq C \left( \frac{|G \cap K|}{|\mathbb{D} \cap K|} \right)^{\beta},$$
where the constant $C$ depends only on $\delta$ and $\alpha$. Therefore,

$$
\frac{\delta}{4} \leq L_{z_0} \leq (1 - |z_0|^2)^{2+\alpha} \int_{|K \cap G|} |R_{z_0}^\alpha(\lambda)|^2 \omega_\alpha(\lambda) dA(\lambda)
$$

$$
= \int_{|G \cap K|} |f(\lambda)|^2 (1 - |\lambda|^2)^\alpha dA(\lambda)
$$

$$
\leq C \left( \frac{|G \cap K|}{|D \cap K|} \right)^\beta.
$$

Now we get (3) by Lemma 4.1.

(3) $\Rightarrow$ (4): Observe that

$$
\int_D |f(z)|^2 \varphi(z) \omega_\alpha(z) dA(z) \geq \int_G \varphi(z) |f(z)|^2 \omega_\alpha(z) dA(z)
$$

$$
> r \int_G |f(z)|^2 \omega_\alpha(z) dA(z)
$$

$$
\geq \frac{r}{C} \int_D |f(z)|^2 \omega_\alpha(z) dA(z),
$$

the last inequality follows from the definition of the reverse Carleson measure.

(4) $\Rightarrow$ (1): Use the same arguments as the proof of Corollary 3 in [11], we obtain that $\|I - T_\varphi\| < 1$, which implies that $T_\varphi$ is invertible on $L^2_\alpha(\omega_\alpha)$. This completes the whole proof of the theorem.

Let $T_\varphi$ denote the Toeplitz operator with symbol $\varphi$ on the weighted Bergman space $L^2_\alpha(\omega_\alpha)$. Combining the main result in [11] and the techniques used in the proof of Theorem 4.2, we can generalize Theorem 3.2 in [23] to the case of standard weighted Bergman space.

**Theorem 4.4.** Let $\varphi \geq 0$ be in $L^\infty(\omega_\alpha)$. Then the following are equivalent:

1. The Toeplitz operator $T_\varphi$ is invertible on $L^2_\alpha(\omega_\alpha)$;
2. The Berezin transform $\hat{\varphi}$ is invertible in $L^\infty(\omega_\alpha)$, where

$$
\hat{\varphi}(z) := \frac{1}{\|K_\varphi^\alpha\|_{L^2(\omega_\alpha)}} \int_D \varphi(\lambda) |K_\varphi^\alpha(\lambda)|^2 \omega_\alpha(\lambda) dA(\lambda);
$$

3. There exists $r > 0$ such that

$$
G := \{ z \in D : \varphi(z) > r \},
$$

$\chi_G dA$ is a reverse Carleson measure for $L^2_\alpha(\omega_\alpha)$;
4. There exists a $\delta \in (0, 1)$ such that

$$
|G \cap K| \geq \delta |D \cap K|
$$

for every ball $K$ whose center lies on $\partial D$;
5. There exists a constant $C > 0$ such that

$$
\int_D |f(z)|^2 \varphi(z) \omega_\alpha(z) dA(z) \geq C \int_D |f(z)|^2 \omega_\alpha(z) dA(z)
$$

for $f \in L^2_\alpha(\omega_\alpha)$. 


Proof. From the proof of Theorem 4.2, it is sufficient to show that one can replace the harmonic function $f$ defined in Lemma 4.3 by a suitable analytic function $g$. Indeed, we construct the desired function $g$ as follows. Suppose that $K$ has radius $0 < t < 1$ and center $u = (1, 0) \in \mathbb{R}^2$. Define

$$g(\lambda) = \sqrt{\alpha + 1} K^{\alpha}_{z_0}(\lambda)(1 - |z_0|^2)^{\frac{2\lambda}{1-\lambda}},$$

where $z_0 = (1-st)u$, $0 < s < 1$. Then it is not hard to check that the two inequalities in Lemma 4.3 both hold for $g$. Now the rest of the proof parallels exactly one given in Theorem 4.2. \hfill $\square$

Based on the theorem above, we can establish a relationship of the invertibility between Toeplitz operators with nonnegative symbols on $L^2_a(\omega_{\alpha})$ and $L^2_h(\omega_{\alpha})$.

**Corollary 4.5.** Let $\varphi$ be a nonnegative bounded measurable function on $\mathbb{D}$. The following four conditions are equivalent:

1. $\mathcal{T}_{\varphi}$ is invertible on $L^2_a(\omega_{\alpha})$;
2. $\mathcal{T}_{\varphi}$ is invertible on $L^2_h(\omega_{\alpha})$;
3. $\hat{\varphi}$ is invertible in $L^\infty(\omega_{\alpha})$;
4. $\tilde{\varphi}$ is invertible in $L^\infty(\omega_{\alpha})$.

To end this section, we study Toeplitz operators with bounded analytic symbols on $L^2_h(\omega_{\alpha})$. Unlike the Bergman space $L^2_a(\omega_{\alpha})$ setting, the spectrum of analytic Toeplitz operator $T_{\varphi}$ on $L^2_h(\omega_{\alpha})$ is not equal to the closure of $\varphi(\mathbb{D})$. Let $H^\infty$ denote space of analytic functions in $L^\infty(\omega_{\alpha})$, we have the following result.

**Theorem 4.6.** Suppose that $\varphi \in H^\infty$. If $\varphi$ is invertible, then the Toeplitz operator $T_{\varphi}$ is invertible on $L^2_h(\omega_{\alpha})$. However, the converse is not true.

Let $\varphi \in H^\infty$ be invertible. However, $T_{\varphi}T_{1/\varphi} = T_{1/\varphi}T_{\varphi}$ does not hold on the harmonic Bergman space in the general case (see Theorem 5 in [3]). For this reason, we need to find some relationships between $T_{\varphi}$ and $T_{\varphi}$ to study the invertibility problem. To do so, let us introduce some notations first.

For $\varphi \in L^\infty(\omega_{\alpha})$, we define

$$\varphi^*(z) = \varphi(\overline{z}) \quad \text{and} \quad \varphi^*(z) = \overline{\varphi(z)}.$$

Next, we define the unitary operator $W : L^2_h(\omega_{\alpha}) = zL^2_a(\omega_{\alpha}) \oplus L^2_a(\omega_{\alpha}) \to zL^2_a(\omega_{\alpha}) \oplus L^2_a(\omega_{\alpha})$ by

$$W = \begin{pmatrix} I & O \\ O & U \end{pmatrix},$$

where $U$ is a unitary operator on $L^2(\omega_{\alpha})$, defined by $Uf(z) = f(\overline{z})$.

It is clear that $W^*$ maps $zL^2_a(\omega_{\alpha}) \oplus L^2_a(\omega_{\alpha})$ to $L^2_h(\omega_{\alpha})$ and

$$W^* = \begin{pmatrix} I & O \\ O & U \end{pmatrix}.$$

For $f$ and $g$ in $L^2(\omega_{\alpha})$, let $f \otimes g$ be the rank one operator defined by

$$(f \otimes g)h = \langle h, g \rangle_{L^2(\omega_{\alpha})}f$$

for $h \in L^2(\omega_{\alpha})$. 

Let $P$ denote the orthogonal projection from $L^2(\omega_\alpha)$ to $L_a^2(\omega_\alpha)$, we define the Hankel operator with symbol $\varphi$ acting on $L_a^2(\omega_\alpha)$ by $\mathcal{H}_\varphi f = P(\varphi U f)$. Using the notations above, we obtain the following matrix representation of $WT_\varphi W^*$, which will be used to study the invertibility problem.

**Lemma 4.7.** Let $\varphi \in L^\infty(\omega_\alpha)$. On the space $zL^2_a(\omega_\alpha) \oplus L^2_a(\omega_\alpha)$ we have
\[
WT_\varphi W^* = \begin{pmatrix}
T_\varphi - (1 \otimes \overline{\varphi}) & \mathcal{H}_\varphi - (1 \otimes \varphi^*) \\
\mathcal{H}_{\varphi^*} & T_{\varphi^*}
\end{pmatrix}.
\]

**Proof.** The proof is exactly the same as the proof of Theorem 2.1 in [8], we omit the details here. \qed

We are now ready to prove Theorem 4.6.

**Proof of Theorem 4.6.** For $\varphi \in H^\infty$ and $f \in zL^2_a(\omega_\alpha)$ we have
\[
\mathcal{H}_\varphi f = P(\varphi^* U f) = P(\varphi(\overline{z}) f(\overline{z})) = 0
\]
and
\[
(1 \otimes \overline{\varphi}) f = \langle f, \overline{\varphi} \rangle_{L^2(\omega_\alpha)} 1 = 0,
\]
since $f(0) = 0$.

Consequently, the matrix representation of $WT_\varphi W^*$ with $\varphi \in H^\infty$ is given by
\[
WT_\varphi W^* = \begin{pmatrix}
T_\varphi & \mathcal{H}_\varphi - (1 \otimes \varphi^*) \\
O & T_{\varphi^*}
\end{pmatrix}.
\]

Note that $\varphi \in H^\infty$ is invertible implies that $\varphi^*$ is also invertible, so that $T_{\varphi^*}$ is invertible on $L^2_a(\omega_\alpha)$.

On the other hand, $T_\varphi$ is invertible on $zL^2_a(\omega_\alpha)$ follows from $\varphi$ is invertible. The above matrix representation tells us that $WT_\varphi W^*$ is invertible on $zL^2_a(\omega_\alpha) \oplus L^2_a(\omega_\alpha)$. Thus $T_{\varphi}$ is invertible on $L^2_a(\omega_\alpha)$.

To show there exists a function $\varphi$ in $H^\infty$ such that $\varphi$ is not invertible in $L^\infty(\omega_\alpha)$ but $T_{\varphi}$ is invertible on $L^2_a(\omega_\alpha)$, we consider the function $\varphi(z) = z$. It is easy to show that
\[
\text{Ker}(T_z) = \text{Ker}(T_z^*) = \text{Ker}(T_{\overline{z}}) = \{0\}.
\]
Observe that $T_z$ is Fredholm, we conclude that $T_z$ is invertible on $L^2_a(\omega_\alpha)$. This finishes the proof. \qed

5. A Reverse Carleson type Inequality for $L^2_h(\omega)$

In the preceding section, we study the invertibility problem of Toeplitz operators via the reverse Carleson measures for standard weighted harmonic Bergman spaces. In this section, we establish a sufficient condition for $\chi_G dA$ to be a reverse Carleson measure for the space $L^2_h(\omega)$, where $\omega \in A_2$ and $G$ is a measurable set in $\mathbb{D}$.

For $a \in \mathbb{D}$, $0 < r < 1$, recall that
\[
S(a, r) = \{ z \in \mathbb{D} : |z - a| < r(1 - |a|) \}.
\]
The main result in this section is Theorem 5.1, which is a harmonic version of Theorem 3.9 in [13].
**Theorem 5.1.** Suppose that $G \subset \mathbb{D}$ and $\omega$ satisfies $A_2$ condition. If there exist $\delta \in (0, 1)$ and $r \in (0, 1)$ such that for all $a \in \mathbb{D}$

$$|G \cap S(a, r)| \geq \delta|S(a, r)|,$$

then there exists a positive constant $C = C(r, \delta)$ such that

$$\int_{\mathbb{D}} |f(z)|^2 \omega(z) dA(z) \leq C \int_{G} |f(z)|^2 \omega(z) dA(z)$$

for all $f \in L^2_\omega(\omega)$.

To prove the above theorem, we will adopt some ideas and techniques in [13]. Firstly, we need to introduce a new (weight) function $\omega$ and discuss some properties of $\omega^*$. In the rest of this section, we use “$r$” and “$\delta$” to denote the numbers provided in Theorem 5.1.

Now we define a positive function $\omega^*$ on the open unit disk as follows:

$$\omega^*(z) = \omega^*_r(z) := \frac{|S(z, r)|_\omega}{|S(z, r)|}.$$

It is clear that $\omega^* \in L^1(dA)$, and so $\omega^*$ is a weight. Moreover, $\omega^*$ has the following important property.

**Lemma 5.2.** Let $z \in \mathbb{D}$, there exist constants $C_1$ and $C_2$ depending only on $r$ such that

$$C_1 \omega^*(a) \leq \omega^*(z) \leq C_2 \omega^*(a)$$

for all $a \in S(z, r)$. Consequently, we have

$$\int_{\mathbb{D}} \frac{\omega^*(a)}{|S(a, r)|} \chi_{S(a, r)}(z) dA(a) \leq C_3 \omega^*(z) \quad (z \in \mathbb{D}),$$

where $C_3 = C_3(r)$ is a constant.

**Proof.** By Lemma 2.2 in [6], there exists a positive constant $C$ depending only on $r$ such that

$$C^{-1} |S(a, r)|_\omega \leq |S(z, r)|_\omega \leq C |S(a, r)|_\omega.$$

Moreover, it is well known that $|S(z, r)|$ is equivalent to $|S(a, r)|$ (with constants independent of $a$ and $z$) if $a \in S(z, r)$. This gives the first conclusion of the lemma. Based on this result, we have

$$\int_{\mathbb{D}} \frac{\omega^*(a)}{|S(a, r)|} \chi_{S(a, r)}(z) dA(a) \leq C \int_{\mathbb{D}} \frac{\omega^*(z)}{|S(a, r)|} \chi_{S(a, r)}(z) dA(a) \quad \text{(using } z \in S(a, r))$$

$$\leq C \int_{\mathbb{D}} \frac{\omega^*(z)}{|S(a, r)|} \chi_{D(a, r)}(z) dA(a) \quad \text{(since } S(a, r) \subseteq D(a, r))$$

$$= C \int_{\mathbb{D}} \frac{\omega^*(z)}{|S(a, r)|} \chi_{D(z, r)}(a) dA(a)$$

$$= C \int_{D(z, r)} \frac{\omega^*(z)}{|S(a, r)|} dA(a)$$

$$\leq C_3 \omega^*(z),$$

where $C_3$ is a positive constant depending only on $r$, as desired. $\square$
Another property of $\omega^*$ is given by the following inequality, which will be used to estimate the integral of $|f|^2 \omega$ over the subset $G$.

**Lemma 5.3.** Let $\omega$ be an $A_2$ weight. Then there exists a constant $C > 0$ depending only on $r$ such that

$$\|f\|_{L^2(\omega^*)}^2 \leq C \|f\|_{L^2(\omega)}^2$$

for all $f \in L^2_0(\omega)$.

**Proof.** By definition, we have

$$\|f\|_{L^2(\omega^*)}^2 = \int_{D} |f(z)|^2 \omega^*(z) dA(z) = \int_{D} \omega(z) \left( \int_{D} |f(z)|^2 \frac{\chi_S(z,r)}{|S(z,r)|} dA(z) \right) dA(z),$$

We next deal with the bracketed expression. Observe

$$S\left(\xi, \frac{r}{2(1+r)}\right) \subset \{ z \in D : |z - \xi| < r(1 - |z|) \},$$

we obtain

$$\int_{D} |f(z)|^2 \frac{\chi_S(z,r)}{|S(z,r)|} dA(z) = \int_{\{ z \in D : |z - \xi| < r(1 - |z|) \}} |f(z)|^2 \frac{\chi_S(z,r)}{|S(z,r)|} dA(z) \geq C \int_{S(\xi, \frac{r}{2(1+r)})} |f(z)|^2 dA(z),$$

the last inequality follows from the subharmonicity of $|f|^2$ (see Lemma 2.1) and $C$ depends only on $r$. Thus we get

$$\|f\|_{L^2(\omega^*)}^2 \geq C \int_{D} |f(z)|^2 \omega(z) dA(z) = C \|f\|_{L^2(\omega)}^2,$$

which finishes the proof of Lemma 5.3. $\square$

In order to complete the proof the main theorem in this section, the following two key lemmas are also needed.

**Proposition 5.4.** Let $G$ be the subset which satisfies the assumption in Theorem 5.1. For $\eta \in (0, 1)$, we define a subset $F$ as the following:

$$F := \{ z \in D : \omega(z) \geq \eta \omega^*(z) \}.$$

Then one can choose $\eta$ (depending only on $\delta$ and $r$) sufficiently small such that

$$|F \cap S(a,r)| \geq (1 - \frac{\delta}{2})|S(a,r)|.$$
and
\[ |G \cap S(a, r) \cap F| \geq \frac{\delta}{2} |S(a, r)| \]
for all \( a \in \mathbb{D} \).

**Proof.** We first claim that for any \( \delta' \in (0, 1) \), there exists \( \eta' = \eta'(\delta') > 0 \) such that
\[ \left| \left\{ z \in S(a, r) : \omega(z) < \eta' \omega^*(a) \right\} \right| < \delta' |S(a, r)| \]
for all \( a \in \mathbb{D} \).

Indeed, for each \( \kappa \in (0, 1) \) and \( a \in \mathbb{D} \), we have
\[
\left| \left\{ z \in S(a, r) : \omega(z) < \kappa \omega^*(a) \right\} \right| \cdot \frac{1}{\kappa \omega^*(a)}
< \int_{\left\{ z \in S(a, r) : \omega(z) < \kappa \omega^*(a) \right\}} \frac{1}{\omega(z)} dA(z)
\leq |S(a, r)| \omega^{-1}
\leq [\omega]_{A_2} |S(a, r)|^2 \cdot |S(a, r)|^{-1}.
\]
So, we obtain
\[
\left| \left\{ z \in S(a, r) : \omega(z) < \kappa \omega^*(a) \right\} \right| < ([\omega]_{A_2} \kappa) |S(a, r)|
\]
for all \( a \in \mathbb{D} \) and \( \kappa \in (0, 1) \). By this inequality, we can choose any \( 0 < \eta' \leq [\omega]_{A_2}^{-1} \delta' \) to finish the proof of the claim.

Lemma 5.2 guarantees that there is a constant \( C = C(r) \) such that
\[ \{ z \in S(a, r) : \omega(z) < C \tau \omega^*(z) \} \subset \{ z \in S(a, r) : \omega(z) < \tau \omega^*(a) \} \quad (a \in \mathbb{D}) \]
for every \( \tau \in (0, 1) \). By the claim and its proof, there exists a \( \tau = \tau(\delta) < C^{-1} \) such that
\[
\left| \left\{ z \in S(a, r) : \omega(z) < \tau \omega^*(a) \right\} \right| < \frac{\delta}{2} |S(a, r)| \quad (a \in \mathbb{D}).
\]

Therefore, we can take \( \eta = \eta(\delta, r) = C \tau < 1 \) such that
\[
\left| \left\{ z \in S(a, r) : \omega(z) < \eta \omega^*(z) \right\} \right| < \frac{\delta}{2} |S(a, r)| \quad (a \in \mathbb{D}).
\]

Using this \( \eta \) to define the corresponding \( F \), so that
\[
|F \cap S(a, r)| = \left| \left\{ z \in S(a, r) : \omega(z) \geq \eta \omega^*(z) \right\} \right| \geq (1 - \frac{\delta}{2}) |S(a, r)|
\]
for all \( a \in \mathbb{D} \).

By assumption
\[
|G \cap S(a, r)| \geq \delta |S(a, r)|,
\]
we have
\[
\delta |S(a, r)| \leq |G \cap S(a, r)| \\
= \left| [G \cap S(a, r) \cap F] \cup [G \cap S(a, r) \cap (\mathbb{D} \setminus F)] \right| \\
\leq |G \cap S(a, r) \cap F| + |S(a, r) \cap (\mathbb{D} \setminus F)| \\
= |G \cap S(a, r) \cap F| + |S(a, r)| - |S(a, r) \cap F| \\
\leq |G \cap S(a, r) \cap F| + |S(a, r)| - (1 - \frac{\delta}{2})|S(a, r)|.
\]
This gives that
\[
|G \cap S(a, r) \cap F| \geq \frac{\delta}{2} |S(a, r)|
\]
for all \(a \in \mathbb{D}\), as desired. \(\square\)

**Lemma 5.5.** If \(G_0\) is a measurable subset of \(\mathbb{D}\) that satisfies
\[
|G_0 \cap S(a, r)| \geq \delta_0 |S(a, r)| \quad (a \in \mathbb{D})
\]
for some \(\delta_0 > 0\), then there exists constant \(C = C(\delta_0, r) > 0\) such that
\[
\int_{\mathbb{D}} |f(z)|^2 \omega^*(z) dA(z) \leq C \int_{G_0} |f(z)|^2 \omega^*(z) dA(z)
\]
for all \(f \in L^2_{\eta}(\omega^*)\).

The proof of the preceding lemma is some what long and it requires a number of technical lemmas. Let us assume this result for the moment and we will prove it at the end of this section. Now we give the proof of Theorem 5.1.

**Proof of Theorem 5.1** By Proposition 5.4 and Lemma 5.5, we have
\[
\int_{\mathbb{D}} |f(z)|^2 \omega^*(z) dA(z) \leq C_1 \int_{G \cap F} |f(z)|^2 \omega^*(z) dA(z) \leq C_1 \eta^{-1} \int_G |f|^2 \omega dA
\]
for all \(f \in L^2_{\eta}(\omega^*)\), where \(C_1\) is a constant depending only on \(r\) and \(\eta = \eta(\delta, r) < 1\) is chosen by Proposition 5.4.

From Lemma 5.3, it is clear that
\[
\int_{\mathbb{D}} |f(z)|^2 \omega(z) dA(z) \leq C_1 \eta^{-1} \int_G |f(z)|^2 \omega(z) dA(z)
\]
for all \(f \in L^2_{\eta}(\omega)\), which gives the desired inequality in Theorem 5.1. \(\square\)

Now we turn to the proof of Lemma 5.5. Before giving the proof, we need to introduce some notations and prove three technical lemmas.

Let \(0 < \theta < \frac{1}{2}\), we define the subset
\[
E_\theta(a) = E_\theta(f, a) := \left\{ z \in S(a, r) : |f(z)| > \theta |f(a)| \right\}
\]
and the following operator:

\[ B_\theta f(a) := \frac{1}{|E_\theta(a)|} \int_{E_\theta(a)} |f(z)|^2 dA(z) \quad (a \in \mathbb{D}). \]

It is clear that

\[ B_\theta f(a) \geq \frac{1}{|S(a,r)|} \int_{S(a,r)} |f(z)|^2 dA(z) \quad (a \in \mathbb{D}). \]

For \( \epsilon \in (0, 1) \), we consider the following two subsets, which are very useful to establish our main result. Define

\[ A = A_\epsilon := \left\{ a \in \mathbb{D} : |f(a)|^2 \leq \frac{\epsilon}{|S(a,r)|} \int_{S(a,r)} |f(z)|^2 dA(z) \right\} \]

and

\[ B = B_\epsilon := \left\{ a \in \mathbb{D} : |f(a)|^2 \leq \epsilon^2 B_\theta f(a) \right\}. \]

A useful estimation for the Lebesgue measure of \( \{ z \in S(a,r) : |f(z)| > \theta |f(a)| \} \) with \( f \) harmonic is the following inequality.

**Lemma 5.6.** Fix \( \epsilon \in (0,1) \). For any \( \delta' \in (0,1) \), there exists \( \theta \in (0, \frac{\delta}{2}) \) such that

\[ \left| \left\{ z \in S(a,r) : |f(z)| > \theta |f(a)| \right\} \right| > (1 - \frac{\delta'}{2})|S(a,r)| \]

for every \( f \) harmonic on \( \mathbb{D} \) and satisfies

\[ |f(a)|^2 > \frac{\epsilon^2}{|S(a,r)|} \int_{S(a,r)} |f(z)|^2 dA(z) \quad (a \in \mathbb{D}). \]

**Proof.** See the proof of Lemma 2 in [12]. \( \square \)

The next lemma provides an estimation for the integral of \( |f|^2 \omega^* \) over the set \( A \).

**Lemma 5.7.** Let \( \epsilon \in (0,1) \), then there exists a constant \( C \) (independent of \( \epsilon \)) such that

\[ \int_A |f(z)|^2 \omega^*(z) dA(z) \leq C \epsilon \int_{\mathbb{D}} |f(z)|^2 \omega^*(z) dA(z) \]

for all \( f \in L^2_h(\omega^*) \).

**Proof.** For \( a \in A \), we have

\[ |f(a)|^2 \leq \frac{\epsilon}{|S(a,r)|} \int_{\mathbb{D}} \chi_{S(a,r)}(z) |f(z)|^2 dA(z). \]

Multiplying the above inequality by \( \omega^*(a) \) and integrating over \( A \) to obtain

\[
\int_A |f(a)|^2 \omega^*(a) dA(a) \leq \int_A \frac{\epsilon}{|S(a,r)|} \int_{\mathbb{D}} \chi_{S(a,r)}(z) |f(z)|^2 dA(z) \omega^*(a) dA(a)
\]

\[ = \epsilon \int_{\mathbb{D}} |f(z)|^2 \left( \int_A \frac{\chi_{S(a,r)}(z)}{|S(a,r)|} \omega^*(a) dA(a) \right) dA(z). \]
By the second conclusion of Lemma 5.2, we have
\[ \int_A \frac{\chi_{S(a,r)}(z)}{|S(a,r)|} \omega^*(a) dA(a) \leq C \omega^*(z), \]
where \( C \) depends only on \( r \). This completes the proof. \( \square \)

The proof of Lemma 5.3 requires the following inequality, which can be proved by the preceding lemma.

**Lemma 5.8.** Let \( \epsilon \in (0,1) \), then there exists a constant \( C = C(r) \) such that
\[ \int_B |f(z)|^2 \omega^*(z) dA(z) \leq C \epsilon \int_D |f(z)|^2 \omega^*(z) dA(z) \]
for all \( f \in L^2_S(\omega^*) \).

**Proof.** Observe that
\[ \int_B |f(z)|^2 \omega^*(z) dA(z) = \int_{B \cap A} |f(z)|^2 \omega^*(z) dA(z) + \int_{B \setminus A} |f(z)|^2 \omega^*(z) dA(z) \]
\[ \leq \int_A |f(z)|^2 \omega^*(z) dA(z) + \int_{B \setminus A} |f(z)|^2 \omega^*(z) dA(z). \]

Based on Lemma 5.7, it sufficient for us to show the following inequality holds for some constant \( C = C(r) \):
\[ J := \int_{B \setminus A} |f(z)|^2 \omega^*(z) dA(z) \leq C \epsilon \int_D |f(z)|^2 \omega^*(z) dA(z). \]
Recall that for each \( a \in B \) we have
\[ |f(a)|^2 \leq \frac{\epsilon^2}{|E_\theta(a)|} \int_{E_\theta(a)} |f(z)|^2 dA(z). \]
From the above inequality we have
\[ J = \int_{B \setminus A} |f(a)|^2 \omega^*(a) dA(a) \]
\[ \leq \epsilon^2 \int_{B \setminus A} \left( \frac{1}{|E_\theta(a)|} \int_{E_\theta(a)} |f(z)|^2 dA(z) \right) \omega^*(a) dA(a) \]
\[ = \epsilon^2 \int_D \left( \int_{B \setminus A} \frac{\omega^*(a)}{|E_\theta(a)|} \chi_{E_\theta(a)}(z) dA(a) \right) |f(z)|^2 dA(z) \]
\[ \leq \epsilon^2 \int_D |f(z)|^2 \left( \int_{B \setminus A} \omega^*(a) \frac{\chi_{S(a,r)}(z)}{|E_\theta(a)|} dA(a) \right) dA(z). \]
The last \( \leq \) is due to \( E_\theta(a) \subset S(a,r) \).

To finish the proof, we need the following claim.

**Claim.** There is a positive constant \( C = C(r) \) such that
\[ |E_\theta(a)| \geq C \epsilon |S(a,r)| \quad \text{or} \quad |E_\theta(a)| \geq C |S(a,r)| \]
for each \( a \notin A \).
If the above claim is true, then we get
\[ \int_{B \setminus A} \omega^*(a) \frac{\chi_{S(a,r)}(z)}{|E_D(a)|} \, dA(a) \leq C^{-1} \epsilon \int_{B \setminus A} \omega^*(a) \frac{\chi_{S(a,r)}(z)}{|S(a,r)|} \, dA(a). \]

Use Lemma 5.2 again, we have
\[ \int_{B \setminus A} \omega^*(a) \frac{\chi_{S(a,r)}(z)}{|S(a,r)|} \, dA(a) \leq C_1 \omega^*(z), \]
the constant $C_1$ depends only on $r$. From the definition of $J$, we obtain
\[ J \leq C \epsilon \int_{\mathbb{D}} |f(z)|^2 \omega^*(z) \, dA(z) \]
for some positive constant $C = C(r)$.

Now we turn to prove the claim. For each $a \notin A$, we have
\[ |f(a)|^2 > \frac{\epsilon}{|S(a,r)|} \int_{S(a,r)} |f(z)|^2 \, dA(z) \]
\[ = \frac{\epsilon}{r^2 (1 - |a|^2)} \int_{S(a,r)} |f(z)|^2 \, dA(z). \]
Using the change of variables $z = a + r(1 - |a|)\lambda$ to get
\[ |f(a)|^2 > \epsilon \int_{\mathbb{D}} |f(a + r(1 - |a|)\lambda)|^2 \, dA(\lambda). \]
Let $g(\lambda) := f(a + r(1 - |a|)\lambda)$, then $g$ is also harmonic on $\mathbb{D}$ and
\[ |g(0)|^2 > \epsilon \int_{\mathbb{D}} |g(\lambda)|^2 \, dA(\lambda). \]

Applying Lemma 2.8 to the function $g$ to get a constant $C_0 = C_0(r)$ such that
\[ |g(z) - g(0)| \leq C_0 |z| \int_{D(0, \frac{r}{16})} |g(\lambda)| \, dA(\lambda) \leq C_0 |z| \int_{\mathbb{D}} |g| \, dA \]
whenever $|z| \leq \frac{r}{16}$. Cauchy-Schwartz inequality gives that
\[ |g(z) - g(0)| \leq C_0 |z| \left( \int_{\mathbb{D}} |g(\lambda)|^2 \, dA(\lambda) \right)^{\frac{1}{2}} \leq C_0 \epsilon^{-\frac{1}{2}} |g(0)| \cdot |z|. \]
provided $|z| \leq \frac{r}{16}$.

If
\[ |z| < \min \left\{ \frac{r}{16}, \frac{\epsilon^{\frac{1}{2}}}{2C_0} \right\}, \]
then
\[ |g(z)| \geq |g(0)| - |g(z) - g(0)| \geq \frac{|g(0)|}{2}. \]

Recall that $0 < \theta < \frac{1}{2}$, we have $|g(z)| > \theta |g(0)|$ for $|z| < \min \left\{ \frac{r}{16}, \frac{\epsilon^{\frac{1}{2}}}{2C_0} \right\}$. This means that
\[ B(0, \frac{r}{16}) \subset \left\{ z \in \mathbb{D} : |g(z)| > \theta |g(0)| \right\} \]
or
\[ B(0, \frac{\epsilon^2}{2C_0}) \subset \left\{ z \in \mathbb{D} : |g(z)| > \theta|g(0)| \right\}. \]

On the other hand, observe that
\[
|E_\theta(a)| = \int_{\{z \in S(a,r) : |f(z)| > \theta|f(a)|\}} dA(z)
\]
\[
= \int_{\{|\frac{z-a}{1-\overline{a}z}| < 1 : |f(z)| > \theta|f(a)|\}} dA(z)
\]
\[
= |S(a, r)| \int_{\{|\lambda| < 1 : |\lambda f(a) + 1 - |a|\lambda| > \theta f(a)\}} dA(\lambda)
\]
\[
= |S(a, r)| \int_{\{|\lambda| < 1 : |\lambda| > \theta|g(0)|\}} dA(\lambda)
\]
\[
= |S(a, r)| \cdot \left\{ \lambda \in \mathbb{D} : |\lambda| > \theta|g(0)| \right\}.
\]

Therefore, we obtain
\[
|E_\theta(a)| \geq \left( \frac{r}{16} \right)^2 |S(a, r)|
\]
or
\[
|E_\theta(a)| \geq \frac{\epsilon}{4C_0^2} |S(a, r)|.
\]

This gives the proof of the claim, and so we complete the proof of Lemma 5.8 \(\square\)

We are now ready to prove Lemma 5.5.

**Proof of Lemma 5.5.** Suppose that \(|G_0 \cap S(a, r)| \geq \delta_0|S(a, r)|\). From Lemmas 5.7 and 5.8 we can choose \(\epsilon\) small enough so that
\[
\int_{\mathbb{D}} |f(z)|^2 \omega^*(z) dA(z) < 2 \int_{\mathbb{D}
\setminus B} |f(z)|^2 \omega^*(z) dA(z).
\]

On the other hand, if \(a \notin B\), then
\[
|f(a)|^2 > \epsilon^2 B_\theta f(a) \geq \frac{\epsilon^2}{|S(a, r)|} \int_{S(a,r)} |f(z)|^2 dA(z).
\]

For the \(\delta_0\) above, we apply Lemma 5.6 to choose a \(\theta \in (0, \frac{1}{2})\) such that
\[
\left| \left\{ z \in S(a, r) : |f(z)| > \theta |f(a)| \right\} \right| > (1 - \frac{\delta_0}{2})|S(a, r)|.
\]

Since \(|G_0 \cap S(a, r)| \geq \delta_0|S(a, r)|\), we have
\[
\left| \left\{ z \in S(a, r) \cap G_0 : |f(z)| > \theta |f(a)| \right\} \right| \geq \frac{\delta_0}{2}|S(a, r)|,
\]
and so
\[
\frac{1}{|S(a, r)|} \int_{S(a,r) \cap G_0} |f(z)|^2 dA(z) > \frac{\theta^2 \delta_0}{2} |f(a)|^2 \quad (a \notin B).
\]
Multiplying the above inequality by $\omega^{*}(a)$ and integrating over $\mathbb{D}\setminus B$ to get
\[
\frac{\theta^2 \delta_0}{2} \int_{\mathbb{D}\setminus B} \omega^{*}(a)|f(a)|^2dA(a) < \int_{\mathbb{D}\setminus B} \frac{\omega^{*}(a)}{|S(a, r)|} \int_{S(a, r) \cap G_0} |f(z)|^2dA(z)dA(a)
\]
\[
= \int_{G_0} |f(z)|^2 \left( \int_{\mathbb{D}\setminus B} \frac{\omega^{*}(a)}{|S(a, r)|} \chi_{S(a, r)}(z)dA(a) \right)dA(z)
\]
\[
\leq C \int_{G_0} |f(z)|^2 \omega^{*}(z)dA(z),
\]
the last “$\leq$” follows from Lemma 5.2 and $C$ depends only on $r$.

Therefore,
\[
\int_{G_0} |f(z)|^2 \omega^{*}(z)dA(z) > \frac{\delta_0 \theta^2}{4C} \int_{\mathbb{D}} |f(z)|^2 \omega^{*}(z)dA(z)
\]
for all $f \in L^2_{\mu}(\omega^{*})$. This completes the proof of Lemma 5.5. \qed

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References

[1] T. Anderson, A. Vagharshakyan, A simple proof of the sharp weighted estimate for Calderon-Zygmund operators on homogeneous spaces, Journal of Geometric Analysis, 2014, 24(3): 1276-1297.

[2] R. Chacón, Toeplitz Operators on Weighted Bergman Spaces, Journal of Function Spaces and Applications, 2013, 2013: 1-5.

[3] B. Choe, Y. Lee, Commuting Toeplitz operators on the harmonic Bergman space, Michigan Mathematical Journal, 1999, 46(1): 163-174.

[4] S. Choi, Positive Toeplitz operators on pluriharmonic Bergman spaces, Journal of Mathematics of Kyoto University, 2007, 47(2): 247-267.

[5] O. Constantin, Discretizations of integral operators and atomic decompositions in vector-valued weighted Bergman spaces, Integral Equations and Operator Theory, 2007, 59(4): 523-554.

[6] O. Constantin, Carleson embeddings and some classes of operators on weighted Bergman spaces, Journal of Mathematical Analysis and Applications, 2010, 365(2): 668-682.

[7] R. Douglas, K. Wang, A harmonic analysis approach to essential normality of principal sub-modules, Journal of Functional Analysis, 2011, 261(11): 3155-3180.

[8] K. Guo, D. Zheng, Toeplitz algebra and Hankel algebra on the harmonic Bergman space, Journal of Mathematical Analysis and Applications, 2002, 276(1): 213-230.

[9] T. Hytönen, The sharp weighted bound for general Calderón-Zygmund operators, Annals of Mathematics, 2012, 175(3): 1473-1506.

[10] Ü. Kuran, Subharmonic Behaviour of $|h|^p$ ($p > 0$, h harmonic), Journal of the London Mathematical Society, 1974, 2(3): 529-538.

[11] D. Luecking, Inequalities on Bergman spaces, Illinois Journal of Mathematics, 1981, 25(1): 1-11.

[12] D. Luecking, Equivalent norms on $L^p$ spaces of harmonic functions, Monatshefte für Mathematik, 1983, 96(2): 133-141.

[13] D. Luecking, Representation and duality in weighted spaces of analytic functions, Indiana University Mathematics Journal, 1985, 34(2): 319-336.
[14] D. Luecking, Forward and reverse Carleson inequalities for functions in Bergman spaces and their derivatives, *American Journal of Mathematics*, 1985, 107(1): 85-111.

[15] J. Miao, Toeplitz operators on harmonic Bergman spaces, *Integral Equations and Operator Theory*, 1997, 27(4): 426-438.

[16] J. Miao, Reproducing kernels for harmonic Bergman spaces of the unit ball, *Monatshefte für Mathematik*, 1998, 125(1): 25-35.

[17] M. Mitkovski, B. Wick, A reproducing kernel thesis for operators on Bergman-type function spaces, *Journal of Functional Analysis*, 2014, 267(7): 2028-2055.

[18] B. Muckenhoupt, R. Wheeden, Weighted norm inequalities for fractional integrals, *Transactions of the American Mathematical Society*, 1974, 192: 261-274.

[19] J. Á. Peláez, J. Rättyä, Embedding theorems for Bergman spaces via harmonic analysis, *Mathematische Annalen*, 2015, 36(1-2): 205-239.

[20] J. Á. Peláez, J. Rättyä, Trace class criteria for Toeplitz and composition operators on small Bergman spaces, *Advances in Mathematics*, 2016, 293: 606-643.

[21] Y. Shu, X. Zhao, Positivity of Toeplitz operators on the harmonic Bergman spaces via Berezin transform, *Acta Mathematica Sinica (English Series)*, 2016, 32(2): 175-186.

[22] W. Sledd, A note on $L^p$ spaces of harmonic functions, *Monatshefte für Mathematik*, 1988, 106(1): 65-73.

[23] X. Zhao, D. Zheng, Invertibility of Toeplitz operators via Berezin transforms, to appear in *Journal of Operator Theory*.

[24] K. Zhu, Operator Theory in Function Spaces, 2nd ed, Mathematical Surveys and Monographs, 138, American Mathematical Society, 2007.

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