CONFORMAL BOUNDS FOR THE FIRST EIGENVALUE OF
THE $p$-LAPLACIAN

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Abstract. Let $M$ be a compact, connected, $m$-dimensional manifold without boundary and $p > 1$. For $1 < p \leq m$, we prove that the first eigenvalue $\lambda_{1,p}$ of the $p$-Laplacian is bounded on each conformal class of Riemannian metrics of volume one on $M$. For $p > m$, we show that any conformal class of Riemannian metrics on $M$ contains metrics of volume one with $\lambda_{1,p}$ arbitrarily large. As a consequence, we obtain that in two dimensions $\lambda_{1,p}$ is uniformly bounded on the space of Riemannian metrics of volume one if $1 < p \leq 2$, respectively unbounded if $p > 2$.

MSC: Primary 58C40; Secondary 53C21

Keywords: $p$-Laplacian, eigenvalue, conformal volume

1. Introduction

Let $M$ be a compact $m$-dimensional manifold. All through this paper we will assume that $M$ is connected and without boundary. The $p$-Laplacian ($p > 1$) associated to a Riemannian metric $g$ on $M$ is given by

$$\Delta_p u = \delta (|du|^p - 2 du),$$

where $\delta = -\text{div}_g$ is the adjoint of $d$ for the $L^2$-norm induced by $g$ on the space of differential forms. This operator can be viewed as an extension of the Laplace-Beltrami operator which corresponds to $p = 2$. The real numbers $\lambda$ for which the nonlinear partial differential equation

$$\Delta_p u = \lambda |u|^{p-2} u$$

has nontrivial solutions are the eigenvalues of $\Delta_p$, and the associated solutions are the eigenfunctions of $\Delta_p$. Zero is an eigenvalue of $\Delta_p$, the associated eigenfunctions being the constant functions. The set of the nonzero eigenvalues is a nonempty, unbounded subset of $\C$. The infimum $\lambda_{1,p}$ of this set is itself a positive eigenvalue, the first eigenvalue of $\Delta_p$, and has a Rayleigh type variational characterization $[15]$

$$\lambda_{1,p}(M, g) = \inf \left\{ \frac{\int_M |du|^p \nu_g}{\int_M |u|^p \nu_g} \mid u \in W^{1,p}(M) \setminus \{0\}, \int_M |u|^{p-2} u \nu_g = 0 \right\},$$

where $\nu_g$ denotes the Riemannian volume element associated to $g$.

The first eigenvalue of $\Delta_p$ can be viewed as a functional on the space of Riemannian metrics on $M$: $g \mapsto \lambda_{1,p}(M, g).$
Since $\lambda_{1,p}$ is not invariant under dilatations $(\lambda_{1,p}(M, cg) = c^{-\frac{p}{p-1}} \lambda_{1,p}(M, g))$, a normalization is needed when studying the uniform boundedness of this functional. It is common to restrict $\lambda_{1,p}$ to the set $\mathcal{M}(M)$ of Riemannian metrics of volume one on $M$. In the linear case $p = 2$ this problem has been extensively studied in various degrees of generality. The functional $\lambda_{1,2}$ was shown to be uniformly bounded on $\mathcal{M}(M)$ in two dimensions [7], [16], [8], and unbounded in three or more dimensions [13], [14], [12], [1], [2], [3]. However, $\lambda_{1,2}$ becomes uniformly bounded when restricted to any conformal class of Riemannian metrics in $\mathcal{M}(M)$ [4].

In the general case $p > 1$, the functional $\lambda_{1,p}$ is unbounded on $\mathcal{M}(M)$ in three or more dimensions [11]. In this paper we study the existence of uniform upper bounds for the restriction of $\lambda_{1,p}$ to conformal classes of Riemannian metrics in $\mathcal{M}(M)$:

- for $1 < p \leq m$ we extend the results from the linear case and obtain an explicit upper bound for $\lambda_{1,p}$ in terms of $p$, the dimension $m$ and the Li-Yau $n$-conformal volume.
- for $p > m$, we consider first the case of the unit sphere $S^m$ and we construct Riemannian metrics in $\mathcal{M}(S^m)$, conformal to the standard metric $\text{can}$ and with $\lambda_{1,p}$ arbitrarily large. We use then the result on spheres to show that any conformal class of Riemannian metrics on $M$ contains metrics of volume one with $\lambda_{1,p}$ arbitrarily large.

As a consequence, we obtain that in two dimensions, $\lambda_{1,p}$ is uniformly bounded on $\mathcal{M}(M)$ when $1 < p \leq 2$, and unbounded when $p > 2$.

2. The case $1 < p \leq m$ : Li-Yau type upper bounds

Let $g$ be a Riemannian metric on $M$ and denote by $[g] = \{fg \mid f \in C^\infty(M), f > 0\}$ the conformal class of $g$. Let $G(n) = \{\gamma \in \text{Diff}(S^n) \mid \gamma^*\text{can} \in [\text{can}]\}$ denote the group of conformal diffeomorphisms of $(S^n, \text{can})$.

For $n$ big enough, the Nash-Moser Theorem ensures (via the stereographic projection) that the set $I_n(M, [g]) = \{\phi : M \to S^n \mid \phi^*\text{can} \in [\text{can}]\}$ of conformal immersions from $(M, g)$ to $(S^n, \text{can})$ is nonempty. The $n$-conformal volume of $[g]$ is defined by [8]:

$$V^\circ_n(M, [g]) = \inf_{\gamma \in G(n)} \sup_{\phi \in I_n(M, [g])} \text{Vol}(M, (\gamma \circ \phi)^*\text{can}) ,$$

where $\text{Vol}(M, (\gamma \circ \phi)^*\text{can})$ denotes the volume of $M$ with respect to the induced metric $(\gamma \circ \phi)^*\text{can}$. By convention, $V^\circ_n(M, [g]) = \infty$ if $I_n(M, [g]) = \emptyset$.

**Theorem 2.1.** Let $M$ be an $m$-dimensional compact manifold and $1 < p \leq m$. For any metric $g \in \mathcal{M}(M)$ and any $n \in \mathbb{N}$ we have

$$\lambda_{1,p}(M, g) \leq m^{\frac{2}{p-1}} (n + 1)^\frac{2}{p-1} V^\circ_n(M, [g])^{\frac{p}{p-1}} .$$

**Remark 2.2.** In the linear case $p = 2$, this result was proved by Li and Yau [8] for surfaces and by El Soufi and Ilias [4] for higher dimensional manifolds.

**Remark 2.3.** Theorem 2.1 gives an explicit upper bound for $\lambda_{1,p}$, $1 < p \leq m$, in the case of some particular manifolds: the sphere $S^m$, the real projective space $\mathbb{R}P^m$, the complex projective space $\mathbb{C}P^d$, the equilateral torus $T^d_{eq}$, the generalized Clifford torus $S^r\left(\sqrt{r/r+q}\right) \times S^q(\sqrt{q/r+q})$, endowed with their canonical metrics.
For these manifolds we have [3]: $V_n^+ (M, \{can\}) = Vol(M, \frac{\lambda_{n+2} - \lambda_n}{m} \cdot can)$ for $n + 1$ greater or equal to the multiplicity of $\lambda_{1,2}$.

Using the relationships between the conformal volume and the genus of a compact surface [5] we obtain:

**Corollary 2.4.** Suppose $m = 2$ and $1 < p \leq 2$. Then for any metric $g \in \mathcal{M}(M)$

$$\lambda_{1,p} (M, g) \leq k_p \left[ \frac{\text{genus}(M) + 3}{2} \right]^\frac{p}{2},$$

where $[\cdot]$ denotes the integer part, $k_p = 3[\frac{p}{2} - 1](8\pi)^\frac{p}{2}$ if $M$ is orientable and $k_p = 5[\frac{p}{2} - 1](24\pi)^\frac{p}{2}$ if not.

**Remark 2.5.** In the case $p = 2$ and $M = S^2$, this result is the well known Hersch inequality [7]. For higher genus surfaces, the upper bound of $\lambda_{1,2}$ in terms of the genus was obtained by El-Soufi and Ilias [5] by improving a previous result of Yang and Yau [16].

In order to prove Theorem 2.1 we need two Lemmas:

**Lemma 2.6.** Let $\phi : (M, g) \rightarrow (S^n, \text{can})$ be a smooth map whose level sets are of measure zero in $(M, g)$. Then for any $p > 1$ there exists $\gamma \in G(n)$ such that

$$\int_M (|\gamma \circ \phi|_i)^p (\gamma \circ \phi)_i \nu_g = 0, \quad 1 \leq i \leq n + 1.$$

**Proof of Lemma 2.6** Let $a \in S^n$ and denote by $\pi_a$ the stereographic projection of pole $a$. Let $t \in (0, 1]$ and $H_{\frac{t}{1-t}} = e^{\frac{1-t}{t}} \cdot Id_{\mathbb{R}^n}$ (i.e. $H_{\frac{t}{1-t}}$ is the linear dilatation of $\mathbb{R}^n$ of factor $e^{\frac{1-t}{t}}$). Let $\gamma_t^a \in G(n)$, $\gamma_t^a(x) = \begin{cases} \pi_a^{-1} \circ H_{\frac{t}{1-t}} \circ \pi_a(x) & \text{if } x \in S^n \setminus \{a\} \\ a & \text{if } x = a \end{cases}$ and consider the continuous map

$$F : [0, 1] \times S^n \rightarrow \mathbb{R}^{n+1}$$

$$F(t, a) = \frac{1}{Vol(M, g)} \left( \int_M (|\gamma_t^a \circ \phi|_i)^p (\gamma_t^a \circ \phi)_i \nu_g, \ldots, \int_M (|\gamma_t^a \circ \phi|_{n+1})^p (\gamma_t^a \circ \phi)_{n+1} \nu_g \right).$$

For any $x \in M \setminus \{\phi^{-1}(-a)\}$ we have $\lim_{t \rightarrow 0+} \gamma_t^a \circ \phi(x) = a$. Since $\phi^{-1}(-a)$ is of measure zero in $M$, we can extend $F$ into a continuous function on $[0, 1] \times S^n$ by setting

$$F(0, a) = (|a_1|^{p-2} a_1, \ldots, |a_{n+1}|^{p-2} a_{n+1}).$$

The map $a \rightarrow F(0, a)$ is odd on $S^n$, and since $\gamma_1^n = I_{S^n}$, the map $a \rightarrow F(1, a)$ is constant. Assume $\|F(t, a)\| \neq 0$ for any $(t, a) \in [0, 1] \times S^n$. Then the map

$$G : [0, 1] \times S^n \rightarrow S^n$$

$$G(t, a) = \frac{F(t, a)}{\|F(t, a)\|}$$

gives a homotopy between the odd map $a \rightarrow G(0, a)$ and the constant map $a \rightarrow G(1, a)$, and this is impossible. Hence there exists $(t, a) \in [0, 1] \times S^n$ such that $\|F(t, a)\| = 0$, i.e. $\int_M (|\gamma_t^a \circ \phi|_i)^p (\gamma_t^a \circ \phi)_i \nu_g = 0, \quad 1 \leq i \leq n + 1. \quad \Box$
Lemma 2.7. Suppose \( g \in \mathcal{M}(M) \) and let \( \phi : (M, g) \to (S^n, can) \) be a smooth map whose level sets are of measure zero in \((M, g)\). Then there exists \( \gamma \in G(n) \) such that
\[
\lambda_{1,p}(M, g) \leq (n + 1)^{\frac{p}{\beta} - 1} \int_M |d(\gamma \circ \phi)|^p \nu_g,
\]
where \( |d(\gamma \circ \phi)| \) denotes the Hilbert-Schmidt norm of \( d(\gamma \circ \phi) \).

**Proof of Lemma 2.7** Lemma 2.6 implies there exists \( \gamma \in G(n) \) such that \( \psi = \gamma \circ \phi : M \to S^n \) verifies \( \int_M |\psi_i|^p - 2 \psi_i \nu_g = 0, \ 1 \leq i \leq n + 1 \). The variational characterization for \( \lambda_{1,p}(M, g) \) implies that
\[
\lambda_{1,p}(M, g) \leq \frac{\int_M |d\psi_i|^p \nu_g}{\int_M |\psi_i|^p \nu_g},
\]
\( \lambda_{1,p}(M, g) \leq \frac{\int_M \sum_{i=1}^{n+1} |d\psi_i|^p \nu_g}{\int_M \sum_{i=1}^{n+1} |\psi_i|^p \nu_g} \).

- **Case 1:** \( p \geq 2 \). It is straightforward that
\[
(2.2) \quad \sum_{i=1}^{n+1} |d\psi_i|^p = \sum_{i=1}^{n+1} (|d\psi_i|^2) \frac{p}{2} \leq \left( \sum_{i=1}^{n+1} |\psi_i|^2 \right)^{\frac{p}{2}} = |d\psi|^p.
\]

On the other hand
\[
(2.3) \quad \sum_{i=1}^{n+1} |\psi_i|^p \geq (n + 1)^{1 - \frac{p}{2}} \left( \sum_{i=1}^{n+1} |\psi_i|^2 \right)^{\frac{p}{2}} = (n + 1)^{1 - \frac{p}{2}},
\]
where we have used the fact that \( x \to x^{\frac{p}{2}} \) is convex and that \( \sum_{i=1}^{n+1} |\psi_i|^2 = 1 \).
Replacing \( 2.2 \) and \( 2.3 \) in \( 2.1 \) we obtain
\[
\lambda_{1,p}(M, g) \leq (n + 1)^{\frac{p}{2} - 1} \int_M |d\psi|^p \nu_g.
\]

- **Case 2:** \( 1 < p < 2 \). Since \( |\psi_i| \leq 1 \) we have \( |\psi_i|^2 \leq |\psi_i|^p \) and
\[
(2.4) \quad 1 = Vol(M, g) = \int_M \sum_{i=1}^{n+1} |\psi_i|^2 \nu_g \leq \int_M \sum_{i=1}^{n+1} |\psi_i|^p \nu_g.
\]

On the other hand
\[
(2.5) \quad \sum_{i=1}^{n+1} |d\psi_i|^p = \sum_{i=1}^{n+1} (|d\psi_i|^2) \frac{p}{2} \leq (n + 1)^{1 - \frac{p}{2}} \left( \sum_{i=1}^{n+1} |d\psi_i|^2 \right)^{\frac{p}{2}} = (n + 1)^{1 - \frac{p}{2}} |d\psi|^p,
\]
where the inequality follows from the concavity of \( x \to x^{\frac{p}{2}} \). Replacing \( 2.4 \) and \( 2.5 \) in \( 2.1 \) we obtain
\[
\lambda_{1,p}(M, g) \leq (n + 1)^{1 - \frac{p}{2}} \int_M |d\psi|^p \nu_g.
\]

**Proof of Theorem 2.1** Let \( \phi : (M, g) \to (S^n, can) \) be a conformal immersion. From Lemma 2.7 we have that there exists \( \gamma \in G(n) \) such that
\[
\lambda_{1,p}(M, g) \leq (n + 1)^{\frac{p}{\beta} - 1} \int_M |d(\gamma \circ \phi)|^p \nu_g.
\]
Since $g \in \mathcal{M}(M)$, Hölder’s inequality implies
\[ \int_M |d(\gamma \circ \phi)|^p \nu_g \leq \left( \int_M |d(\gamma \circ \phi)|^m \nu_g \right)^{\frac{p}{m}}. \]
On the other hand since $\gamma \circ \phi : (M, g) \to (S^n, \text{can})$ is a conformal immersion, $(\gamma \circ \phi)^* \text{can} = \frac{|d(\gamma \circ \phi)|}{m}^2 g$ and we have
\[ \int_M |d(\gamma \circ \phi)|^m \nu_g = m^\frac{p}{m} \text{Vol}(M, (\gamma \circ \phi)^* \text{can}) \leq m^\frac{p}{m} \sup_{\gamma \in G(n)} \text{Vol}(M, (\gamma \circ \phi)^* \text{can}). \]
Combining the inequalities above we obtain:
\[ \lambda_{1,p}(M, g) \leq m^\frac{p}{m} (n + 1)^{\frac{p}{m} - 1} \left( \sup_{\gamma \in G(n)} \text{Vol}(M, (\gamma \circ \phi)^* \text{can}) \right)^{\frac{p}{m}}. \]
Taking the infimum over all $\phi \in I_n(M, [g])$ we obtain the desired inequality. □

**Proof of Corollary 2.4.** In the case of surfaces, the $n$-conformal volume is bounded above by a constant depending only on the genus of the surface [5]. If $M$ is orientable we have
\[ V_c^n(M, [g]) \leq 4\pi \left( \text{genus}(M) + 3 \right) \]
for $n \geq 2$.

If $M$ is non orientable,
\[ V_c^n(M, [g]) \leq 12\pi \left( \text{genus}(M) + 3 \right) \]
for $n \geq 4$.

Theorem 2.1 implies now the desired result with $k_p = 3^{\frac{n}{2} - 1}(8\pi)^{\frac{n}{2}}$ when $M$ is orientable and $k_p = 5^{\frac{n}{2} - 1}(24\pi)^{\frac{n}{2}}$ when $M$ is non orientable. □

3. **The case $p > m$**

For the sake of self-containedness we include here the variational characterizations for the first eigenvalues for the Dirichlet and the Neumann problems for $\Delta_p$. Let $\Omega$ be a domain in $M$ and consider the Dirichlet problem:
\[ \begin{cases} 
\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases} \]
The infimum $\lambda_{1,p}^D(\Omega, g)$ of the set of eigenvalues for this problem is itself a positive eigenvalue with the variational characterization
\[ \lambda_{1,p}^D(\Omega, g) = \inf \left\{ \frac{\int_{\Omega} |du|^p \nu_g}{\int_{\Omega} |u|^p \nu_g} \mid u \in W^{1,p}_0(\Omega) \setminus \{0\} \right\}. \]
Consider now the Neumann problem on $\Omega$:
\[ \begin{cases} 
\Delta_p f = |f|^{p-2} f & \text{in } \Omega \\
\frac{df}{\eta} = 0 & \text{on } \partial \Omega,
\end{cases} \]
where $\eta$ denotes the exterior unit normal vector field to $\partial \Omega$. Here too, the infimum $\lambda_{1,p}^N(\Omega, g)$ of the set of nonzero eigenvalues is a positive eigenvalue with the variational characterization
\[ \lambda_{1,p}^N(\Omega, g) := \inf \left\{ \frac{\int_{\Omega} |df|^p \nu_g}{\int_{\Omega} |f|^p \nu_g} \mid f \in W^{1,p}(\Omega, g) \setminus \{0\}, \int_{\Omega} |f|^{p-2} f \nu_g = 0 \right\}. \]
We consider first the case of \((S^m, [\text{can}])\):

**Theorem 3.1.** For any \(p > m\), \(S^m\) carries Riemannian metrics of volume one, conformal to the standard metric \(\text{can}\), with \(\lambda_{1,p}\) arbitrarily large.

**Proof of Theorem 3.1.** Let \(r \in [0, \pi]\), denote the geodesic distance on \((S^m, \text{can})\) w.r.t. a point \(x_0 \in S^m\). Let \(\varepsilon > 0\) and define a radial function \(f_\varepsilon : S^m \to \mathbb{R}\) by

\[
(3.1) \quad f_\varepsilon(r) = \varepsilon (r^p/(\varepsilon^m - 1)) \cdot \chi_{[0, \pi]}(\varepsilon(r)) + \chi_{(\pi, \pi + \varepsilon)}(r).
\]

Let \(\lambda_{1,p}(\varepsilon) = \inf \left\{ R_\varepsilon(u) := \frac{\int_{S^m} |du|^p f_\varepsilon^{m-p} \nu_{\text{can}}}{\int_{S^m} |u|^p f_\varepsilon^p \nu_{\text{can}}} \mid u \in W^{1,p}(S^m) \setminus \{0\} \right\}, \)

where \(V_{\text{can}} = \{ \int_{S^m} |u|^p f_\varepsilon^{m-p} \nu_{\text{can}} = 0 \} \). We will show first that

\[
(3.2) \quad \limsup_{\varepsilon \to 0} \lambda_{1,p}(\varepsilon) \cdot \varepsilon^m = \infty.
\]

Classical density arguments imply that there exists \(u_\varepsilon \in W^{1,p}(S^m) \setminus \{0\}\) with \(\int_{S^m} |u_\varepsilon|^{p-2} u_\varepsilon f_\varepsilon^p \nu_{\text{can}} = 0\) such that \(\lambda_{1,p}(\varepsilon) = R_\varepsilon(u_\varepsilon)\). Let \(u^:\varepsilon \in \mathcal{M} \to \mathcal{R}\) be a radial function defined by

\[
(3.3) \quad \check{u}_\varepsilon^p(r) = \frac{1}{V} \int_{S^m} |u_\varepsilon(r, \cdot)|^p \nu_{\text{can}}
\]

where \(V = \text{Vol}(S^m, \text{can})\). Differentiating w.r.t. \(r\) we obtain

\[
\check{u}_\varepsilon^{p-1} \check{u}_\varepsilon' = \frac{p}{V} \int_{S^m} |u_\varepsilon|^{p-2} u_\varepsilon \frac{\partial u_\varepsilon}{\partial r} \nu_{\text{can}}.
\]

By Hölder’s inequality we obtain

\[
\check{u}_\varepsilon^{p-1} |\check{u}_\varepsilon'| \leq \frac{1}{V} \int_{S^m} |u_\varepsilon|^{p-1} \left| \frac{\partial u_\varepsilon}{\partial r} \right| \nu_{\text{can}} \leq \frac{1}{V} \left( \int_{S^m} |u_\varepsilon|^p \nu_{\text{can}} \right)^{\frac{p-1}{p}} \left( \int_{S^m} \left| \frac{\partial u_\varepsilon}{\partial r} \right|^p \nu_{\text{can}} \right)^{\frac{1}{p}}.
\]

It follows that

\[
|\check{u}_\varepsilon'| \leq \frac{1}{V} \int_{S^m} \left| \frac{\partial u_\varepsilon}{\partial r} \right|^p \nu_{\text{can}} \leq \frac{1}{V} \int_{S^m} |du_\varepsilon|^p \nu_{\text{can}}.
\]

On the other hand

\[
\int_{S^m} |\check{u}_\varepsilon|^p f_\varepsilon^{m-p} \nu_{\text{can}} = V \cdot \int_0^\pi |\check{u}_\varepsilon|^p f_\varepsilon^m \sin r^{m-1} dr
\]

\[
= \int_0^\pi \left[ \int_{S^m} |u_\varepsilon|^p \nu_{\text{can}} \right] f_\varepsilon^m \sin r^{m-1} dr
\]

\[
= \int_{S^m} |u_\varepsilon|^p f_\varepsilon^m \nu_{\text{can}},
\]

where the second equality follows from \((3.3)\). Similarly \((3.4)\) implies

\[
\int_{S^m} |\check{u}_\varepsilon'|^p f_\varepsilon^{m-p} \nu_{\text{can}} \leq \int_{S^m} |du_\varepsilon|^p f_\varepsilon^{m-p} \nu_{\text{can}}.
\]
In particular, we obtain that \( \bar{u}_\varepsilon \in W^{1,p}(S^m) \) and

\[
\lambda_{1,p}(\varepsilon) = R_\varepsilon(u_\varepsilon) \geq \frac{\int_{S^m_m} |\bar{u}_\varepsilon'|^p \bar{f}_\varepsilon^{m-p} \nu_{\text{can}}}{\int_{S^m_m} |\bar{u}_\varepsilon|^p f_\varepsilon^{m-n} \nu_{\text{can}}} \geq \min \left\{ \frac{\int_{S^m_m} |\bar{u}_\varepsilon'|^p \bar{f}_\varepsilon^{m-p} \nu_{\text{can}}}{\int_{S^m_m} |\bar{u}_\varepsilon|^p f_\varepsilon^{m-n} \nu_{\text{can}}} \right\},
\]

where \( S^m_+ \), \( S^m_- \) denote the hemispheres centered at \( x_0 \), respectively \(-x_0\). Without loss of generality we may assume that

\[
(3.5) \quad \lambda_{1,p}(\varepsilon) \geq \frac{\int_{S^m_+} |\bar{u}_\varepsilon'|^p \bar{f}_\varepsilon^{m-p} \nu_{\text{can}}}{\int_{S^m_+} |\bar{u}_\varepsilon|^p f_\varepsilon^{m-n} \nu_{\text{can}}},
\]

Let \( w_\varepsilon \in W^{1,p}(S^m_+) \), \( w_\varepsilon = \begin{cases} \bar{u}_\varepsilon & \text{on } [0, \frac{\varepsilon}{2} - \varepsilon] \\ \bar{u}_\varepsilon(\frac{\varepsilon}{2} - \varepsilon) & \text{on } (\frac{\varepsilon}{2} - \varepsilon, \frac{\varepsilon}{2}) \end{cases} \) and \( v_\varepsilon = \bar{u}_\varepsilon - w_\varepsilon \). Then \( v_\varepsilon = 0 \) on \([0, \frac{\varepsilon}{2} - \varepsilon]\) and \( w_\varepsilon' = 0 \) on \((\frac{\varepsilon}{2} - \varepsilon, \frac{\varepsilon}{2})\). Since \( u_\varepsilon' \) and \( w_\varepsilon' \) have disjoint supports, we have \( |\bar{u}_\varepsilon'|^p = |v_\varepsilon'|^p + |w_\varepsilon'|^p \). On the other hand \( |\bar{u}_\varepsilon|^p = |v_\varepsilon + w_\varepsilon|^p \leq 2^{p-1}(|v_\varepsilon|^p + |w_\varepsilon|^p) \). Then (3.5) and (3.1) imply

\[
\lambda_{1,p}(\varepsilon) \geq 2^{1-p} \frac{\int_{S^m_+} (|v_\varepsilon'|^p + |w_\varepsilon'|^p) \bar{f}_\varepsilon^{m-p} \nu_{\text{can}}}{\int_{S^m_+} (|v_\varepsilon|^p + |w_\varepsilon|^p) f_\varepsilon^{m-n} \nu_{\text{can}}} = 2^{1-p} \frac{\int_{S^m_+} |v_\varepsilon'|^p \nu_{\text{can}} + \varepsilon^{-\frac{2}{m}} \int_{S^m_+} |w_\varepsilon'|^p \nu_{\text{can}}}{\int_{S^m_+} |v_\varepsilon'|^p \nu_{\text{can}} + \int_{S^m_+} |w_\varepsilon|^p f_\varepsilon^{m-n} \nu_{\text{can}}}
\]

Quite to multiply \( \bar{u}_\varepsilon \) by a constant we may assume \( \int_{S^m_+} |v_\varepsilon|^p \nu_{\text{can}} + \int_{S^m_+} |w_\varepsilon|^p f_\varepsilon^{m-n} \nu_{\text{can}} = 1 \) and the inequality above becomes

\[
(3.6) \quad \lambda_{1,p}(\varepsilon) \geq 2^{1-p} \int_{S^m_+} |u_\varepsilon'|^p \nu_{\text{can}} + \varepsilon^{-\frac{2}{m}} \int_{S^m_+} |w_\varepsilon'|^p \nu_{\text{can}}.
\]

- **Case 1**: \( \limsup_{\varepsilon \to 0} \int_{S^m_+} |w_\varepsilon'|^p \nu_{\text{can}} > 0 \). Inequality (3.6) implies that \( \lambda_{1,p}(\varepsilon) \varepsilon^{\frac{2}{m}} \geq 2^{1-p} \varepsilon^{-\frac{2}{m}} \int_{S^m_+} |w_\varepsilon'|^p \nu_{\text{can}} \), and therefore (3.2) is verified.
- **Case 2**: \( \lim_{\varepsilon \to 0} \int_{S^m_+} |w_\varepsilon'|^p \nu_{\text{can}} = 0 \). Then we may find a sequence \( \varepsilon_n \to 0 \) such that \( w_{\varepsilon_n} \to c \) strongly in \( L^p(M) \), where \( c \) is a constant. In particular since \( p > m \), \( \{f_\varepsilon_n\} \) is uniformly bounded and we have \( \lim_{n \to \infty} \int_{S^m_+} f_\varepsilon_n \nu_{\text{can}} = 0 \). It follows that \( \lim_{n \to \infty} \int_{S^m_+} |w_{\varepsilon_n}|^p f_\varepsilon_n \nu_{\text{can}} = \lim_{n \to \infty} \int_{S^m_+} (|w_{\varepsilon_n}| - |c|^p) f_\varepsilon_n \nu_{\text{can}} + |c|^p \lim_{n \to \infty} \int_{S^m_+} f_\varepsilon_n \nu_{\text{can}} = 0 \). Hence for \( \varepsilon_n \)...
small enough, \( \int_{S^m} \left| v_{\varepsilon_n} \right|^p \nu_{can} = 1 - \int_{S^m} \left| w_{\varepsilon_n} \right|^p \int \frac{\nu_{can}}{2} \) and (3.6) implies (3.7)

\[
\lambda_{1,p}(\varepsilon_n) \geq 2^{1 - \frac{p}{2}} \int_{S^m} |v_{\varepsilon_n}'|^p \nu_{can} \geq 2^{1 - \frac{p}{2}} \int_{S^m} |v_{\varepsilon_n}'|^p \nu_{can} = 2^{1 - \frac{p}{2}} \frac{\int_{S^m} \left| v_{\varepsilon_n}' \right|^p \sin m^{-1} dr}{\int_{S^m} \left| v_{\varepsilon_n} \right|^p \sin m^{-1} dr} \geq 2^{1 - \frac{p}{2}} \sin (\frac{\pi}{2} - \varepsilon_n) \left| v_{\varepsilon_n}' \right|^p \frac{dr}{\int_{S^m} \left| v_{\varepsilon_n} \right|^p \sin m^{-1} dr}.
\]

Let \( \bar{v}_{\varepsilon_n} \in W_0^1, p(-\varepsilon_n, \varepsilon_n) \) be an even function such that \( \bar{v}_{\varepsilon_n}(s) = v_{\varepsilon_n}(s + \frac{\pi}{2} - \varepsilon_n) \) for \( 0 \leq s \leq \varepsilon_n \). We have then (3.8)

\[
\int_{\bar{S}^m} |v_{\varepsilon_n}'|^p \bar{\nu}_{\varepsilon_n} \frac{dr}{\int_{S^m} \left| v_{\varepsilon_n} \right|^p \sin m^{-1} dr} = \int_{\bar{S}^m} \left| v_{\varepsilon_n}' \right|^p \bar{\nu}_{\varepsilon_n} \frac{dr}{\int_{S^m} \left| v_{\varepsilon_n} \right|^p \sin m^{-1} dr} \geq \lambda_{1,p}(-\varepsilon_n, \varepsilon_n) = \varepsilon_n^{-p} \lambda_{1,p}(-1, 1).
\]

Inequalities (3.7), (3.8) imply \( \lambda_{1,p}(\varepsilon_n) \geq \varepsilon_n^{-p} \lambda_{1,p}(-1, 1) \) and (3.2) is verified again.

Fix now \( \varepsilon > 0 \) and let \( \tilde{f}_\varepsilon \in C^\infty(S^m) \), radial with respect to \( x_0 \) and such that \( \tilde{f}_\varepsilon \leq f_\varepsilon \), \( \tilde{f}_\varepsilon(r) = f_\varepsilon(r) = 1 \) on \( \left[ \frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon \right] \) and \( \tilde{f}_\varepsilon(\pi - r) = \tilde{f}(r) \). Then

\[
Vol(S^m, \tilde{f}_\varepsilon can) = \int_{S^m} \tilde{f}_\varepsilon^m \nu_{can} = \int_{S^m-1} \int_{\frac{\pi}{2} - \varepsilon}^{\frac{\pi}{2} + \varepsilon} \sin m^{-1} dr \nu_{can} > V \int_{\frac{\pi}{2} - \varepsilon}^{\frac{\pi}{2} + \varepsilon} \sin m^{-1} dr > \varepsilon V \sin (\frac{\pi}{2} - \varepsilon)^{m-1}, \quad \text{where} \quad V = Vol(S^{m-1}, can).
\]

We will compare now \( \lambda_{1,p}(S^m, \tilde{f}_\varepsilon can) \) and \( \lambda_{1,p}(\varepsilon) \). Let \( \tilde{u}_\varepsilon \) be an eigenfunction for \( \lambda_{1,p}(S^m, \tilde{f}_\varepsilon can) \) and denote by \( \tilde{u}_\varepsilon^+ \), \( \tilde{u}_\varepsilon^- \) the positive, respectively, the negative part of \( \tilde{u}_\varepsilon \). Then [\#]

\[
\lambda_{1,p}(S^m, \tilde{f}_\varepsilon can) = \int_{S^m} |d\tilde{u}_\varepsilon|^p \tilde{f}_\varepsilon^m \nu_{can} = \int_{S^m} |d\tilde{u}_\varepsilon|^p \tilde{f}_\varepsilon^m \nu_{can} = \int_{S^m} |\tilde{u}_\varepsilon|^p \tilde{f}_\varepsilon^m \nu_{can}.
\]

Let \( t \in \mathbb{R} \) and \( \tilde{u}_{\varepsilon,t} = t\tilde{u}_\varepsilon^+ + \tilde{u}_\varepsilon^- \). Then there is \( t_0 \) such that \( \int_{S^m} |\tilde{u}_{\varepsilon,t_0}|^p \tilde{f}_\varepsilon \nu_{can} = 0 \) and the equation above implies (3.10)

\[
\lambda_{1,p}(S^m, \tilde{f}_\varepsilon can) = \int_{S^m} |d\tilde{u}_{\varepsilon,t_0}|^p \tilde{f}_\varepsilon^m \nu_{can} = \int_{S^m} |d\tilde{u}_{\varepsilon,t_0}|^p \tilde{f}_\varepsilon^m \nu_{can} \geq \lambda_{1,p}(\varepsilon),
\]

where the first inequality follows from the fact that \( \tilde{f}_\varepsilon \leq f_\varepsilon \) and the second from the variational characterization for \( \lambda_{1,p}(\varepsilon) \). Inequalities (3.9), (3.10) and (3.2) yield

\[
\limsup_{\varepsilon \to 0} \lambda_{1,p}(S^m, \tilde{f}_\varepsilon can) Vol(S^m, \tilde{f}_\varepsilon can) = V \tilde{f}_\varepsilon \cdot \limsup_{\varepsilon \to 0} \lambda_{1,p}(\varepsilon) \cdot \tilde{f}_\varepsilon = \infty.
\]

Finally, let \( h_\varepsilon = Vol(S^m, \tilde{f}_\varepsilon can) \frac{\varepsilon}{\tilde{f}_\varepsilon} \frac{1}{\tilde{f}_\varepsilon} \). We have then

\[
Vol(S^m, h_\varepsilon can) = 1 \quad \text{and} \quad \limsup_{\varepsilon \to 0} \lambda_{1,p}(S^m, h_\varepsilon can) = \infty.
\]

\[\square\]
We will extend the construction from \((S^m, [can])\) to \((M, [g])\) by means of the first eigenvalue for the Neumann problem for \(\Delta_p\) on a domain \(\Omega\) in \(M\).

**Theorem 3.2.** Let \((M, g)\) be a compact Riemannian manifold of dimension \(m\). Then for any \(p > m\), \([g]\) contains Riemannian metrics of volume one with \(\lambda_{1,p}\) arbitrarily large.

**Proof of Theorem 3.2.** Let \(r\) denote the geodesic distance on \((S^m, can)\) w.r.t. a point \(x_0\). Let \(f \in C^\infty(S^m)\) be a function radial w.r.t. \(x_0\), such that \(f(r) = f(\pi - r)\) and \(\text{Vol}(S^m, can) = 1\). As before, let \(S^m_+\) denote the hemisphere centered at \(x_0\). Let \(v\) be an eigenfunction for \(\lambda_1^N(S^m_+, f can)\) and let \(w \in W^1, p(S^m)\), \(w(r) = \begin{cases} v(r) & \text{if } 0 \leq r \leq \frac{\pi}{2} \\ v(\pi - r) & \text{if } \frac{\pi}{2} < r \leq \pi \end{cases}\). Then \(\int_{S^m} |w|^p - 2w f \nu_{can} = 2 \int_{S^m} |v|^p - 2v f \nu_{can} = 0\) and the variational characterization for \(\lambda_1^N(S^m, f can)\) implies

\[
\lambda_1(S^m, f can) \leq \frac{\int_{S^m} |dw|^p f \frac{m - p}{2} \nu_{can}}{\int_{S^m} |w|^p f \frac{m - p}{2} \nu_{can}} = \frac{\int_{S^m} |dv|^p f \frac{m - p}{2} \nu_{can}}{\int_{S^m} |v|^p f \frac{m - p}{2} \nu_{can}} = \lambda_1^N(S^m_+, f can)
\]

Let \(\Omega\) be a domain in \(M\) such that there exists a diffeomorphism \(\Phi : \Omega \to S^m\). We may assume \(\Omega\) is included in the open region of a local chart of \(M\). In this chart we have \(\nu_{g} = \sqrt{\text{det}(g_{ij})} dx^1 \wedge dx^2 \wedge \ldots \wedge dx^m\) and \(\nu_{\Phi^* can} = \sqrt{\text{det}((\Phi^* can)_{ij})} dx^1 \wedge dx^2 \wedge \ldots \wedge dx^m\). There exist positive constants \(c_1, c_2\) such that

\[
c_1 \sqrt{\text{det}(g_{ij})} \leq \sqrt{\text{det}((\Phi^* can)_{ij})} \leq c_2 \sqrt{\text{det}(g_{ij})}\text{ on } \Omega.
\]

We will compare now \(\lambda_1^N(S^m_+, f can)\) and \(\lambda_1^N(\Omega, (f \circ \Phi)\Phi^* can)\). Note first that since \(\Phi\) is an isometry between \((\Omega, (f \circ \Phi)\Phi^* can)\) and \((S^m_+, f can)\) we have

\[
\lambda_1^N(S^m_+, f can) = \lambda_1^N(\Omega, (f \circ \Phi)\Phi^* can)
\]

Let \(u\) be an eigenfunction for \(\lambda_1^N(\Omega, (f \circ \Phi)\Phi^* can)\) and denote by \(u^+, u^-\) the positive, respectively, the negative part of \(u\). Then there is \(s \in \mathbb{R}\) such that the function \(u_s = su^+ + u^-\) verifies \(\int_{\Omega} |u_s|^p - 2u_s f \Phi \Phi^* \nu_{\Phi^* can} = 0\). Furthermore

\[
\lambda_1^N(\Omega, (f \circ \Phi)\Phi^* can) \geq \frac{c_1}{c_2} \lambda_1^N(S^m, f can),
\]

where the first inequality follows from \((3.12)\) and the second from the variational characterization of \(\lambda_1^N(\Omega, (f \circ \Phi)\Phi^* can)\). From \((3.11), (3.13)\) and \((3.14)\) we obtain

\[
\lambda_1^N(\Omega, (f \circ \Phi)\Phi^* can) \geq \frac{c_1}{c_2} \lambda_1^N(S^m, f can).
\]

Let now \(\delta > 0\); there is an extension \(\widetilde{f \circ \Phi}\) of \(f \circ \Phi\) on the entire manifold \(M\) such that the metric \(\tilde{g} = \tilde{f} \circ \Phi g\) verifies \([10]\): \(\lambda_1(M, \tilde{g}) > \lambda_1^N(\Omega, (f \circ \Phi)\Phi^* can) - \delta\). Inequality \((3.15)\) implies

\[
\lambda_1(M, \tilde{g}) > \frac{c_1}{c_2} \lambda_1^N(S^m, f can) - \delta
\]
On the other hand

\begin{equation}
\text{Vol}(M, \tilde{g}) > \text{Vol}(\Omega, (f \circ \Phi)g) \geq \frac{1}{c_2} \text{Vol}(\Omega, (f \circ \Phi)\Phi^*can)
\end{equation}

\begin{equation}
= \frac{1}{c_2} \text{Vol}(S^m_+, fcan) = \frac{1}{2c_2} \text{Vol}(S^m, fcan) = \frac{1}{2c_2}.
\end{equation}

Let \( K > 0 \); from the proof of Theorem 3.1 we may assume that \( f \) is chosen such that 

\[ \lambda_{1,p}(S^m, fcan) > 2^{m+1}c_1^{-1}c_2^{m+1} K. \]

For \( \delta \) small enough such that \((2c_2)^{-\frac{m}{2}} \delta < K\), inequalities (3.16) and (3.17) imply 

\[ \lambda_{1,p}(M, \tilde{g}) \text{Vol}(M, \tilde{g})^{\frac{m}{2}} \geq \left( \frac{c_1}{c_2} \lambda_{1,p}(S^m, fcan) - \delta \right)(2c_2)^{-\frac{m}{2}} > K. \]

Finally, let \( h = \text{Vol}(M, \tilde{g})^{-\frac{m}{2}} \tilde{g} \). Then \( h \in [g] \), \( \text{Vol}(M, h) = 1 \) and \( \lambda_{1,p}(M, h) > K \).

\[ \square \]

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