THE ALGEBRA OF CLOSED FORMS IN A DISK IS KOSZUL

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Abstract. We prove that the algebra of closed differential forms in an (algebraic, formal, or analytic) disk with logarithmic singularities along several coordinate hyperplanes is (both nontopologically and topologically) Koszul. The connection with variations of mixed Hodge–Tate structures is discussed in the introduction.

Introduction

In this paper we consider the algebras of closed differential forms in a disk, regular outside of several chosen coordinate hyperplanes and having at most logarithmic singularities along these hyperplanes, with respect to the operation of product of differential forms. Such algebras occur in connection with mixed Hodge–Tate sheaves on smooth algebraic varieties [2]. More precisely, the above algebras of closed forms in a disk play a role in the local description of such sheaves in the neighborhood of a point that may belong either to the original variety, or to a normal crossing divisor lying at infinity in its smooth compactification.

Let $D$ be a complex analytic disk and $V$ be the complement to several coordinate hyperplanes in $D$. According to [2], the real mixed Hodge–Tate sheaves on $V$ with admissible singularities in $D \setminus V$ can be described in terms of an associative, super-commutative, positively internally graded DG-algebra $\mathbb{R}\mathcal{HT}(V,D)$. The cohomology of the DG-algebra $\mathbb{R}\mathcal{HT}(V,D)$ lie in the union of two half-lines: the diagonal where the internal grading is equal to the cohomological one and the axis where the cohomological grading is equal to one. The diagonal part of the cohomology is isomorphic to the algebra of closed forms in $V$ with logarithmic singularities along $D \setminus V$, while the part lying in the latter axis (which is responsible for the mixed Hodge structures over a point) has a one-dimensional component in every positive internal degree.

The commutative Hopf algebra describing the category of mixed Hodge–Tate sheaves on $(V,D)$ in the Tannakian formalism is the algebra of zero cohomology of the reduced bar-construction of the DG-algebra $\mathbb{R}\mathcal{HT}(V,D)$. It follows from the Koszul property of the algebra of closed forms, proven in this paper, that this bar-construction has no cohomology in the cohomological gradings different from zero. To deduce this, it suffices to consider the spectral sequence converging from the cohomology of the bar-construction of the cohomology algebra of $\mathbb{R}\mathcal{HT}(V,D)$ to the cohomology of the bar-construction of $\mathbb{R}\mathcal{HT}(V,D)$ itself, and use the well-known description of the Ext algebra of the connected direct sum of augmented algebras [5, 5].
Proposition 1.1 of Chapter 3. The commutative Hopf algebra of zero cohomology is the cofree product of the Hopf algebra quadratic dual to the algebra of closed differential forms and the cofree Hopf algebra with homogeneous cogenerators indexed by the positive integers.

Various Koszul properties of the algebras of motivic cohomology (Milnor K-theory, Galois cohomology, etc.) play an important role in the theory of motives (see [3, 4] and the other present author’s papers on the subject; the arguments above can be viewed as a new illustration to this general observation). However, the present status of these properties is mostly that of conjectures rather than theorems. The algebras of closed differential forms in a disk, which are considered in this paper, provide an interesting family of algebras of motivic significance whose Koszulity can be readily established. That will be demonstrated below.

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1. Module Koszulity

Let $D$ be a disk with the coordinates $z_1, \ldots, z_u$. With few exceptions, it will not matter for us which particular geometric category is presumed. So $D$ can be the algebraic affine space over a field of characteristic zero, the formal disk over such a field, a complex analytic disk, or a smooth real disk. One can also take $D$ to be the spectrum of the algebra of polynomials or formal power series with divided powers over a field of prime characteristic.

Let $0 \leq v \leq u$. For any $1 \leq s \leq v$, let $L_s$ denote the coordinate hyperplane $\{z_s = 0\} \subset D$. Denote by $\Omega$ the de Rham DG-algebra of regular differential forms in $D \setminus \bigcup_{s=1}^v L_s$ with logarithmic singularities along $L_s$. Let $Z \subset \Omega$ denote the subalgebra of closed forms, i. e., the kernel of the de Rham differential $d: \Omega \rightarrow \Omega$. Let $H = H(\Omega, d)$ be the cohomology algebra of $\Omega$.

Denote by $A$ the exterior algebra generated by the closed 1-forms $dz_s/z_s, \ s \leq v$, and $dz_r, \ r > v$. It is only important for us to have control over the homological properties of the $A$-modules $\Omega$ and $H$. In particular, one can replace the disk with any space endowed with an étale map to the disk and satisfying an appropriate version of the Poincaré lemma. In the above examples, $\Omega$ is the free $A$-module generated by $\Omega^0$, and $H$ is the exterior algebra generated by $dz_s/z_s$ with the obvious $A$-module structure in which $dz_s/z_s \in A$ act freely and $dz_r \in A$ act trivially in $H$.

The notion of a Koszul algebra, introduced by S. Priddy [6] in the context of locally finite-dimensional algebras with respect to an additional grading and studied mostly for algebras with a finite-dimensional space of generators [5], can be easily generalized to the completely infinite-dimensional case [3]. The same applies [4] to the notion of a Koszul module introduced by A. Beilinson, V. Ginzburg, and W. Soergel [1].
The main difference with the locally finite-dimensional case is that in the infinite-dimensional situation the quadratic duality connects algebras with coalgebras and modules with comodules.

Let us recall these definitions. A nonnegatively graded algebra $A$ over a field $k$ is called Koszul if $A_0 = k$ and $\text{Tor}^A_{ij}(k, k) = 0$ for $i \neq j$. For a Koszul algebra $A$, a nonnegatively graded left $A$-module $M$ is called Koszul if $\text{Tor}^A_{ij}(k, M) = 0$ for $i \neq j$. Here the first index $i$ denotes the homological grading of the Tor, while the second index $j$ denotes the internal grading (see the references above). Note that for any nonnegatively graded algebra $A$ and module $M$ the condition $A_0 = k$ implies the vanishing of $\text{Tor}^A_{ij}(k, k)$ and $\text{Tor}^A_{ij}(k, M)$ for $i > j$. We will use the lower and upper indices interchangeably for denoting our internal gradings; no sign change is presumed when passing from the upper to the lower indices and back.

**Lemma.** Let $M$ be a nonnegatively graded left module over a Koszul algebra $A$. Then the graded vector space $\text{Tor}^A_i(k, M)$ is concentrated in the gradings $i$ and $i + 1$ for all $i$ if and only if $M_+ = M_1 \oplus M_2 \oplus \cdots$ is a Koszul left $A$-module in the grading shifted by 1 (i.e., so that the component $M_1$ be put in degree 0).

**Proof.** The assertion follows from the long exact sequence $\cdots \to \text{Tor}^A_{i+1}(k, M_0) \to \text{Tor}^A_i(k, M_+) \to \text{Tor}^A_i(k, M) \to \text{Tor}^A_i(k, M_0) \to \cdots$, since the graded vector space $\text{Tor}^A_i(k, M_0)$ is concentrated in degree $i$ (due to Koszulity of the algebra $A$). \hfill \Box

**Theorem.** Let $(\Omega, d)$ be a nonnegatively graded DG-algebra over a field $k$ with the differential of degree 1; set $Z = \ker d$ and $H = H(\Omega, d)$. Let $A$ be a Koszul algebra and $f: A \to Z$ be a morphism of graded algebras. Assume that $\Omega$ and $H$ are Koszul left $A$-modules in the module structures induced by $f$. Then $Z_+ = Z^1 \oplus Z^2 \oplus \cdots$ is a Koszul left $A$-module in the grading shifted by 1.

**Proof.** For a graded $A$-module $M$ and $j \in \mathbb{Z}$, denote by $M(j)$ the graded $A$-module with the components $M(j)^m = M^{m-j}$ and the action of $A$ defined by the rule $a \cdot x(j) = (-1)^m(a \cdot x)(j)$ for $x \in M$ and $a \in A^n$. Consider the complex of graded $A$-modules

$$\cdots \to \Omega(3) \to \Omega(2) \to \Omega(1) \to Z.$$  

We are interested in the two hyperhomology spectral sequences with the same limit that are obtained by applying the derived functor $\text{Tor}^A_{\cdot, -}(k, -)$ to this complex $C$. Specifically, we have $E_{pq}^2 = E_{pq}^1 \implies \text{Tor}^A_{p+q}(k, C)$, where $E_{pq}^2 = \text{Tor}^A_p(k, H(q))$ and $E_{pq}^1 = \text{Tor}^A_q(k, \Omega(p))$ for $p > 0$, while $E_{0, q}^1 = \text{Tor}^A_q(k, Z)$.

By the assumption, the term $E_{pq}^2$ is concentrated in the internal degree $p+q$, hence the limit term $\text{Tor}^A_{\cdot, -}(k, C)$ is concentrated in the internal degree $i$. Furthermore, the term $E_{pq}^1$ is concentrated in the internal degree $p + q$ for all $p > 0$. Now any components of the term $E_{0, i}^1$ of the internal degree different from $i$ have to be killed by the differentials $d^r: E_{r, i-1}^r \to E_{r, i}^r$, hence the term $E_{0, i}^1$ is concentrated in the internal degrees $i$ and $i-1$. It remains to use Lemma. \hfill \Box

For a nonnegatively graded $k$-algebra $B$, set $B' = k \oplus B_1 \oplus B_2 \oplus \cdots$. According to Lemma and [4, Theorem 6.1], if $A \to B'$ is a homomorphism of graded algebras, the
algebra $A$ is Koszul, and $B'$ is a Koszul left $A$-module in the grading shifted by 1, then the algebra $B'$ is Koszul. (For another proof of the same result, see Theorem in Section 3 below.) Thus in the assumptions of Theorem all the three graded algebras $\Omega'$, $H'$, and $Z'$ are Koszul.

**Corollary.** In any of the geometric categories listed above, the algebra $Z$ of closed differential forms in the disk $D$ with logarithmic singularities along the several coordinate hyperplanes $L_s$ is Koszul.

**Proof.** It is clear from the discussion at the beginning of the section that the algebra $\Omega$ of logarithmic differential forms in $D$ with respect to $\{L_s\}$ and its algebra of de Rham cohomology $H$ are Koszul modules over the exterior algebra $A$ generated by $dz_s/z_s$ and $dz_r$. Besides, $Z^0 = k$. Thus the assertion follows from Theorem above and Theorem 6.1 from [4]. □

2. Remarks on Topological Koszulity

Koszulity of a graded algebra $Z$ is an exactness property of the bar-complex of $Z$, whose components are direct sums of tensor products of the grading components of $Z$. However, tensor products of the grading components of an algebra $Z$ are not always a natural object to consider. It is perfectly natural to consider such tensor products when $Z$ is the algebra of functions or forms on an algebraic variety, but perhaps not so when $Z$ is the algebra of forms on a formal, complex analytic, or smooth real disk. In the latter cases, it might be better to consider completed tensor products. Notice in particular that it is the algebra of closed forms in a complex analytic disk that appears in the problem of local description of mixed Hodge–Tate sheaves (see Introduction to the present paper and the preprint [2]).

Specifically, one may wish to define the completed tensor product $Z_{n_1} \hat{\otimes} \cdots \hat{\otimes} Z_{n_m}$ as the space of closed forms in $D^n$ of degree $n_t$ with respect to the $t$-th group of variables, with logarithmic singularities along the hyperplanes $D^t \times L_s \times D^{m-t-1}$. Then one may construct the completed bar-complex out of such completed tensor products and ask oneself whether it is exact outside of the diagonal. Moreover, one may wish to consider the diagonal homology of this complex as the completed version of the coalgebra Koszul dual to $Z$. The component of degree $n$ of this completed coalgebra is the space of all closed forms in $D^n$ of degree 1 with respect to each group of variables with vanishing pull-backs under the maps $\text{Id}_D^{t-1} \times \Delta_D \times \text{Id}_D^{n-t-1} : D^{n-1} \to D^n$, where $\Delta_D : D \to D^2$ is the diagonal map.

From the point of view of a homological algebraist, topological algebra as such is a treacherous ground which is better avoided whenever possible. So we propose a simple linear algebra formalism for describing topological Koszulity in the above sense, independent on any notion of a topological tensor product.

With any associative algebra $Z$ with unit over a field $k$, one can associate the family of vector spaces $Z_n = Z^{\otimes n}, \ n \geq 0$, endowed with linear maps induced by the multiplication and unit in $Z$. The structure one obtains in this way is that of
a simplicial $k$-vector space $Z_\bullet$ endowed with a morphism $k \to Z_\bullet$ into it from the constant simplicial vector space $k$.

When $Z$ is a graded algebra, one obtains a much richer structure. A \textit{(quasi-associative graded) quasi-algebra} over a field $k$ is a family of vector spaces $Z_{n_1, \ldots, n_m}$, where $m \geq 0$ and $n_t \in \mathbb{Z}$, endowed with the \textit{quasi-multiplication} maps

$$Z_{n_1, \ldots, n_m} \longrightarrow Z_{n_1, \ldots, n_t-1, n_t+n_t+1, n_t+2, \ldots, n_m}$$

and the \textit{quasi-unit} maps

$$Z_{n_1, \ldots, n_m} \longrightarrow Z_{n_1, \ldots, n_t, 0, n_t+1, \ldots, n_m}$$

satisfying the conventional properties of the multiplication and unit maps between the tensor products $Z_{n_1, \ldots, n_m} = Z_{n_1} \otimes_k \cdots \otimes_k Z_{n_m}$ of the grading components of an associative algebra with unit. Specifically, the compositions of maps

$$Z_{n_1, \ldots, n_m} \longrightarrow Z_{n_1, \ldots, n_s+n_s+1, n_t, \ldots, n_m} \longrightarrow Z_{n_1, \ldots, n_s+n_s+1, \ldots, n_t+n_t+1, \ldots, n_m}$$

and

$$Z_{n_1, \ldots, n_m} \longrightarrow Z_{n_1, \ldots, n_t, n_t+n_t+1, \ldots, n_m}$$

should coincide; the compositions of maps

$$Z_{n_1, \ldots, n_m} \longrightarrow Z_{n_1, \ldots, n_t-1, n_t-1+n_t+1, \ldots, n_m} \longrightarrow Z_{n_1, \ldots, n_t-1+n_t+1, \ldots, n_m}$$

and

$$Z_{n_1, \ldots, n_m} \longrightarrow Z_{n_1, \ldots, n_t-1+n_t+1, \ldots, n_m}$$

should coincide. The quasi-unit maps must commute with the quasi-multiplication maps; the compositions

$$Z_{n_1, \ldots, n_m} \longrightarrow Z_{n_1, \ldots, n_t, 0, n_t+1, \ldots, n_m} \longrightarrow Z_{n_1, \ldots, n_t+0, n_t+1, \ldots, n_m} = Z_{n_1, \ldots, n_m}$$

and

$$Z_{n_1, \ldots, n_m} \longrightarrow Z_{n_1, \ldots, n_t, 0, n_t+1, \ldots, n_m} \longrightarrow Z_{n_1, \ldots, n_t, 0+0, n_t+1, \ldots, n_m} = Z_{n_1, \ldots, n_m}$$

should be the identity maps. The component $Z_m$ with $m = 0$ indices has to be identified with $k$.

A quasi-algebra is said to be \textit{nonnegative} if $Z_{n_1, \ldots, n_m} = 0$ whenever $n_t < 0$ for some $1 \leq t \leq m$. A nonnegative quasi-algebra is said to be \textit{positive} if all its quasi-unit maps are isomorphisms. A positive quasi-algebra is determined by its components $Z_{n_1, \ldots, n_m}$ with positive indices $n_m > 0$ and the quasi-multiplication maps between them subject to the quasi-associativity equations (i.e., those of the above equations which depend on the quasi-multiplication maps only).

Let $Z$ be a positively graded associative algebra over $k$, i.e., $Z$ is nonnegatively graded in the obvious sense and $Z_0 = k$. Set $Z_+ = Z/k$. Then one can associate with $Z$ its reduced bar-complex $B$ of the form

$$k \longleftarrow Z_+ \longleftarrow Z_+ \otimes_k Z_+ \longleftarrow Z_+ \otimes_k Z_+ \otimes_k Z_+ \longleftarrow \cdots,$$

consider its grading component $B_n$ of degree $n$, and take the tensor product $B_{n_1, \ldots, n_m} = B_{n_1} \otimes_k \cdots \otimes_k B_{n_m}$ of several such complexes. The components of the complex $B_{n_1, \ldots, n_m}$ are direct sums of tensor products of the grading components of
the algebra $Z$. Replacing all such tensor products with the components $Z_{n_1', \ldots, n'_m}$ of an arbitrary positive quasi-algebra, one defines the complex $B_{n_1, \ldots, n_m}$ as an additive functor on the abelian category of positive quasi-algebras. A positive quasi-algebra $Z$ is called Koszul if all the complexes $B_{n_1, \ldots, n_m}$ have no homology except at the homological degree $n_1 + \cdots + n_m$.

Inverting all the arrows in the above definitions, one defines (quasi-coassociative graded) quasi-coalgebras and, in particular, positive quasi-coalgebras and Koszul quasi-coalgebras. In particular, the quasi-comultiplications are the maps

$$C_{n_1, \ldots, n_{t-1}, n_t + n_{t+1}, n_{t+2}, \ldots, n_m} \longrightarrow C_{n_1, \ldots, n_m},$$

the quasi-counits are the maps

$$C_{n_1, \ldots, n_t, 0, n_{t+1}, \ldots, n_m} \longrightarrow C_{n_1, \ldots, n_m},$$

and Koszulity of positive quasi-coalgebras is defined in terms of the complexes emulating tensor products of the grading components of reduced cobar-complexes of positively graded coalgebras.

The additive categories of Koszul quasi-algebras and Koszul quasi-coalgebras are equivalent. The equivalence functor assigns to a Koszul quasi-algebra $Z$ the Koszul quasi-coalgebra $C$ with the components $C_{n_1, \ldots, n_m} = H_{n_1 + \cdots + n_m} (B_{n_1, \ldots, n_m})$. The natural surjections of complexes

$$B_{n_1, \ldots, n_{t-1}, n_t + n_{t+1}, n_{t+2}, \ldots, n_m} \longrightarrow B_{n_1, \ldots, n_m}$$

induce the quasi-comultiplications in $C$. The inverse functor is defined in the similar way in terms of the cobar-complexes of quasi-coalgebras.

In particular, all the quasi-multiplication maps between components with non-negative indices in a Koszul quasi-algebra are surjective. This is the quasi-algebra analogue of the condition that the graded algebra $Z$ be generated by $Z_1$. There is also an analogue of the quadraticity condition, and there are the dual conditions for Koszul quasi-coalgebras. Due to these conditions, the dual Koszul quasi-algebra $Z$ and quasi-coalgebra $C$ are uniquely determined by the vector spaces $Z_{1, \ldots, 1} \simeq C_{1, \ldots, 1}$ and the exact sequences

$$0 \longrightarrow C_{1, \ldots, 1, 2, 1, \ldots, 1} \longrightarrow C_{1, \ldots, 1} \simeq Z_{1, \ldots, 1} \longrightarrow Z_{1, \ldots, 1, 2, 1, \ldots, 1} \longrightarrow 0.$$

For the corresponding quasi-coalgebra $C$ and quasi-algebra $Z$ to be Koszul, the collection of $n - 1$ subspaces $C_{1, \ldots, 1, 2, 1, \ldots, 1}$ in the vector space $C_{1, \ldots, 1}$ ($n$ units) has to be distributive for all $n$ (cf. [3, Subsection 2.2]).

A quasi-algebra with external multiplications is a quasi-algebra $Z$ endowed with linear maps

$$Z_{n_1, \ldots, n_t} \otimes_k Z_{n_{t+1}, \ldots, n_m} \longrightarrow Z_{n_1, \ldots, n_m}$$

compatible with the identification $Z_\emptyset = k$ and commuting with the quasi-multiplication and quasi-unit maps. Quasi-coalgebras with external multiplications are defined in the similar way. This time, arrows are not inverted, i.e., external multiplications in a quasi-coalgebra have the form

$$C_{n_1, \ldots, n_t} \otimes_k C_{n_{t+1}, \ldots, n_m} \longrightarrow C_{n_1, \ldots, n_m}.$$
The quasi-algebra or quasi-coalgebra corresponding to a graded algebra or coalgebra is naturally endowed with external multiplications; and a quasi-(co)algebra with external multiplications comes from a (uniquely defined) graded (co)algebra if and only if all its external multiplication maps are isomorphisms.

The equivalence between the categories of Koszul quasi-algebras and Koszul quasi-coalgebras transforms quasi-algebras with external multiplications to quasi-coalgebras with external multiplications and back. In order to see this, it suffices to construct natural external multiplications

$$B_{n_1, \ldots, n_l} \otimes_k B_{l_{t+1}, \ldots, n_m} \longrightarrow B_{n_1, \ldots, n_m}$$
onumber

on the bar-complexes of a quasi-algebra with external multiplications, and similar external multiplications on the cobar-complexes of a quasi-coalgebra with external multiplications.

### 3. Module Koszulity for Quasi-Algebras

Let $A$ be a Koszul algebra and $\mathcal{Z}$ be a positive quasi-algebra over a field $k$. Assume that $A$ acts in $\mathcal{Z}$ from the left in the following sense: for all $l_1, \ldots, l_t, n_0, \ldots, n_q \geq 0$, $p, q \geq 0$ there are linear maps

$$A_n \otimes_k Z_{l_{p}, t, \ldots, l_{t}, n_0, \ldots, n_q} \longrightarrow Z_{l_{p}, t, \ldots, l_{t}, n_0 + n_1, \ldots, n_q},$$

making $\bigoplus_{n_0=0}^{\infty} Z_{l_{p}, t, \ldots, l_{t}, n_0, \ldots, n_q}$ a graded $A$-module for any fixed $l_1, \ldots, l_t, n_1, \ldots, n_q$ and commuting with the quasi-multiplication maps

$$Z_{l_{p}, t, \ldots, l_{t}, n_0, \ldots, n_q} \longrightarrow Z_{l_{p}, t, \ldots, l_{t}, n_0 + n_1, \ldots, n_q}$$

for all $0 \leq t < q$.

For any positively graded associative algebra $Z$ over $k$, consider the tensor product $E_{l_{p}, t, \ldots, l_{t}, n_1, \ldots, n_q} = Z_{l_{p}, t, \ldots, l_{t}, n_1, \ldots, n_q} \otimes_k B_{t} \otimes_k B_{n_1} \otimes_k \cdots \otimes_k B_{n_q}$ of the grading components of the algebra $Z$ and its reduced bar-complex $B$ (with the trivial coefficients). The terms of the complex $E_{l_{p}, t, \ldots, l_{t}, n_1, \ldots, n_q}$ are direct sums of tensor products of grading components of the algebra $Z$; replacing all such tensor products with the components $Z_{n_0', \ldots, n_q'}$ of a positive quasi-algebra $\mathcal{Z}$, we obtain the complex $\mathcal{E}_{l_{p}, t, \ldots, l_{t}, n_1, \ldots, n_q}$. The action of $A$ in the quasi-algebra $\mathcal{Z}$ induces its action

$$A_n \otimes_k \mathcal{E}_{l_{p}, t, \ldots, l_{t}, n_1, \ldots, n_q} \longrightarrow \mathcal{E}_{l_{p}, t, \ldots, l_{t}, n_0 + l_0, n_1, \ldots, n_q}$$
onumber

on the complexes $\mathcal{E}_{l_{p}, t, \ldots, l_{t}, n_1, \ldots, n_q}$, hence also on their cohomology.

**Theorem.** Suppose that the graded $A$-module

$$\bigoplus_{n_0=1}^{\infty} H_{n_0 + \cdots + n_q} (\mathcal{E}_{l_{p}, t, \ldots, l_{t}, n_1, \ldots, n_q})$$

is Koszul in the grading shifted by 1 for any fixed $l_1, \ldots, l_t, n_1, \ldots, n_q \geq 1$, $p, q \geq 0$. Then the quasi-algebra $\mathcal{Z}$ is Koszul.

**Proof.** Let us show that the homology of the complex $\mathcal{E}_{l_{p}, t, \ldots, l_{t}, n_0, \ldots, n_q}$ are concentrated in the degree $n_0 + \cdots + n_q$ for all $l_1, n_0, \ldots, n_q \geq 1$, $p, q \geq 0$. We will argue by induction in $n_0 + \cdots + n_q$. 


Imagine that we have a morphism of positively graded algebras $A \to Z$ and consider the bicomplex $F = \bigoplus_{l',l'' \geq 0} A_{l'}^l \otimes_k Z \otimes_k Z_{l''}^{l''}$ with one differential induced by the differential of the reduced bar-complex of $A$ with coefficients in the left $A$-module $Z$ and the other one induced by the differential of the reduced bar-complex of $Z$ with coefficients in the right $Z$-module $Z$. Consider the component $F_{n_0}$ of the complex $F$ of the (total) internal degree $n_0$ and take its tensor product $Z_{l_1} \otimes_k \cdots \otimes_k Z_{l_n} \otimes_k F_{n_0} \otimes_k B_{n_1} \otimes_k \cdots \otimes_k B_{n_q}$ with the components of the algebra $Z$ and its reduced bar-complex $B$ (with the trivial coefficients). Denote the complex so obtained by $F_{l_1;\cdots;l_n;\cdots;l_q}$. The components of this complex are direct sums of tensor products of the grading components of the algebras $A$ and $Z$. Moving the tensor products of components of the algebra $A$ to the left in these tensor products and replacing all the tensor products of components of an algebra $Z$ with the components $Z_{n_1;\cdots;n_q}$ of a positive quasi-algebra, we obtain the complex $F_{l_1;\cdots;l_n;\cdots;l_q}$ depending as an additive functor on the quasi-algebra $Z$ with a left action of $A$.

Using the quasi-algebra version of the contracting homotopy of the bar-complex of $Z$ with coefficients in $Z$, which is induced by the isomorphisms $Z_{l_1;\cdots;l_n;\cdots;l_q} \to Z_{l_1;\cdots;l_n;\cdots;l_q}$, one can see that the complex $F_{l_1;\cdots;l_n;\cdots;l_q}$ is quasi-isomorphic to the tensor product of the component of internal degree $n_0$ of the reduced bar-complex of $A$ (with the trivial coefficients) and the complex $E_{l_1;\cdots;l_n;\cdots;l_q}$. So it follows from the induction hypothesis that the homology of $F_{l_1;\cdots;l_n;\cdots;l_q}$ is concentrated in the homological degree $n_0 + \cdots + n_q$. On the other hand, there is a natural surjective morphism of complexes $F_{l_1;\cdots;l_n;\cdots;l_q} \to E_{l_1;\cdots;l_n;\cdots;l_q}$. We will show that the homology of the kernel of this map is concentrated in the homological degrees $n_0 + \cdots + n_q$ and $n_0 + \cdots + n_q - 1$; so it will follow that the homology of $E_{l_1;\cdots;l_n;\cdots;l_q}$ is concentrated in the degree $n_0 + \cdots + n_q$.

Indeed, let $F'_{l_1;\cdots;l_n;\cdots;l_q}$ be the subcomplex of $F_{l_1;\cdots;l_n;\cdots;l_q}$ consisting of all the components in which all the components of the tensor product $Z_{l_1;\cdots;l_n;\cdots;l_q}$ are positive (i.e., $n_0' \geq 1$). Then the quotient complex of $F_{l_1;\cdots;l_n;\cdots;l_q}$ by $F'_{l_1;\cdots;l_n;\cdots;l_q}$ is isomorphic to the direct sum of the tensor products of the components of internal grading $n$ of the reduced bar-complex of $A$ (with the trivial coefficients) and the complex $E_{l_1;\cdots;l_n;\cdots;l_q}$. It follows from the induction assumption that the kernel of the map

$$F_{l_1;\cdots;l_n;\cdots;l_q} / F'_{l_1;\cdots;l_n;\cdots;l_q} \to E_{l_1;\cdots;l_n;\cdots;l_q}$$

has homology in the homological grading $n_0 + \cdots + n_q$ only.

On the other hand, the complex $F'_{l_1;\cdots;l_n;\cdots;l_q}$ has a finite increasing filtration whose indices are the sums $n_0' + \cdots + n_q'$ of indices of the tensor factors $Z_{l_1;\cdots;l_n;\cdots;l_q}$. The associated quotients of this filtration are the components of internal grading $n$ of the bar-complexes of $A$ with coefficients in the complexes of graded $A$-modules $\bigoplus_{l_0'=1}^\infty E_{l_1;\cdots;l_n;\cdots;l_q}$. It follows from the induction assumption that they are quasi-isomorphic to the components of internal grading $n$ of the bar-complexes of $A$ with coefficients in the graded modules

$$\bigoplus_{l_0=1}^\infty H_{n_0+\cdots+n_q-n}(E_{l_1;\cdots;l_n;\cdots;l_q}).$$
placed in the homological grading $n_0 + \cdots + n_q - n$. By the module Koszulity assumption of Theorem, their homology are concentrated in the homological degree $n_0 + \cdots + n_q - 1$. (Cf. [4, proof of Theorem 6.1].)

Let us finally return to the case when $\mathcal{Z}_{n_1, \ldots, n_m} = Z_{n_1} \hat{\otimes} \cdots \hat{\otimes} Z_{n_m}$ is the quasi-algebra of closed forms in a formal, complex analytic, or smooth real disk, as defined in Section 2. One defines the quasi-multiplication in $\mathcal{Z}$ in terms of the diagonal embedding $\Delta_D: D \rightarrow D \times D$. The vector space $H_{n_1 + \cdots + n_q}(\mathcal{E}_{l_0, \ldots, l_0; n_1, \ldots, n_q})$ is described as the space of closed forms in $D^{p+1+n_1+\cdots+n_q}$ of the degrees $l_1, \ldots, l_0$ with respect to the first $p+1$ groups of variables and 1 with respect to the last $n_1 + \cdots + n_q$ groups, regular outside of the hyperplanes $D^t \times L_s \times D^{p+1+n_1+\cdots+n_q-t}$ and having at most logarithmic singularities along these hyperplanes, with vanishing inverse images with respect to the maps

$$\text{Id}_D^{p+n_1+\cdots+n_t-s+1+n_1+\cdots+n_q} \times \Delta_D \times \text{Id}_D^{n_1-s-1+n_1+\cdots+n_q}: D^{p+n_1+\cdots+n_q} \rightarrow D^{p+1+n_1+\cdots+n_q}$$

for all $1 \leq s \leq n_t - 1$.

To show that such spaces of closed forms satisfy the module Koszulity assumption of Theorem above, it suffices to consider them as spaces of closed forms of the degree $l_0$ in the disk $D$ with coordinates from the $(p+1)$-th group of variables, depending on the variables from all the other groups as parameters. One has to include such parameters into the assertion of Theorem from Section 1 and use the Poincaré lemma with parameters (provable by the usual integration procedure, which preserves the conditions imposed on the dependence of our forms from parameters).

Hence $\mathcal{Z}$ is Koszul, and its quadratic dual quasi-coalgebra $\mathcal{C}$ described in Section 2 is Koszul, too. Besides, $\mathcal{Z}$ and $\mathcal{C}$ are obviously a quasi-algebra and a quasi-coalgebra with external multiplications, and their external multiplication structures correspond to each other under the Koszul duality.

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