We prove that the set of right 4-Engel elements of a group $G$ is a subgroup for locally nilpotent groups $G$ without elements of orders 2, 3 or 5; and in this case the normal closure $\langle x \rangle^G$ is nilpotent of class at most 7 for each right 4-Engel elements $x$ of $G$.

Keywords: Right 4-Engel elements of a group; 4-Engel groups.

Mathematics Subject Classification 2000: 20D45

1. Introduction and Results

Let $G$ be any group and $n$ a nonnegative integer. For any two elements $a$ and $b$ of $G$, we define inductively $[a, n] b$ the $n$-Engel commutator of the pair $(a, b)$, as follows:

$[a, 0] b := a, \quad [a, 1] b := a^{-1} b^{-1} a b$ and $[a, n+1] b = [[a, n] b, b]$ for all $n > 0$.

An element $x$ of $G$ is called right $n$-Engel if $[x, n] g = 1$ for all $g \in G$. We denote by $R_n(G)$ the set of all right $n$-Engel elements of $G$. A group $G$ is called $n$-Engel if $G = R_n(G)$. It is clear that $R_1(G) = Z(G)$ is the center of $G$ and Kappe [5] proved $R_2(G)$ is a characteristic subgroup of $G$. Macdonald [6] has shown that the inverse or square of a right 3-Engel element need not be right 3-Engel. Nickel [8] generalized Macdonald’s result to all $n \geq 3$. Although Macdonald’s example shows that $R_3(G)$ is not in general a subgroup of $G$, Heineken [4] has already shown that if $A$ is the subset of a group $G$ consisting of all elements $a$ such that $a^{±1} \in R_3(G)$, then $A$ is a subgroup if either $G$ has no element of order 2 or $A$ consists only of elements having finite odd order. Newell [7] proved that the normal closure of every right 3-Engel element is nilpotent of class at most 3. In Sec. 2, we prove that if $G$ is a 2′-group, then $R_3(G)$ is a subgroup of $G$. Nickel’s example shows that the set
of right 4-Engel elements is not a subgroup in general (see also the first example in Sec. 4 of [1]). In Sec. 3, we prove that if \( G \) is a locally nilpotent \( \{2, 3, 5\}' \)-group, then \( R_4(G) \) is a subgroup of \( G \).

Traustason [11] proved that any locally nilpotent 4-Engel group \( H \) is Fitting of degree at most 4. This means that the normal closure of every element of \( H \) is nilpotent of class at most 4. More precisely he proved that if \( H \) has no element of order 2 or 5, then \( H \) has Fitting degree at most 3. Now by a result of Havas and Vaughan-Lee [3], one knows any 4-Engel group is locally nilpotent and so Traustason’s result is true for all 4-Engel groups. In Sec. 3, by another result of Traustason [12] we show that the normal closure of every right 4-Engel element in a locally nilpotent \( \{2, 3, 5\}' \)-group, is nilpotent of class at most 7.

Throughout the paper, we have frequently used \texttt{nq} package of Nickel [9] which is implemented in \texttt{GAP} [10]. All given timings were obtained on an Intel Pentium 4-1.70GHz processor with 512 MB running Red Hat Enterprise Linux 5.

2. Right 3-Engel Elements

Throughout, for any positive integer \( k \) and any group \( H \), \( \gamma_k(H) \) denotes the \( k \)-th term of the lower central series of \( H \). The main result of this section implies that \( R_3(G) \) is a subgroup of \( G \) whenever \( G \) is a \( 2' \)-group. Newell [7] proved that

**Theorem 2.1.** Let \( G = \langle a, b, c \rangle \) be a group such that \( a, b \in R_3(G) \). Then

1. \( \langle a, c \rangle \) is nilpotent of class at most 5 and \( \gamma_5(\langle a, c \rangle) \) has exponent 2.
2. \( G \) is nilpotent of class at most 6.
3. \( \gamma_5(G)/\gamma_6(G) \) has exponent 10. Furthermore \( [a, c, b, c, c]^2 \in \gamma_6(G) \).
4. \( \gamma_6(G) \) has exponent 2.

**Theorem 2.2.** Let \( G \) be a group such that \( \gamma_5(G) \) has no element of order 2. Then \( R_3(G) \) is a subgroup of \( G \).

**Proof.** Let \( a, b \in R_3(G) \) and let \( c \) be an arbitrary element of \( G \). Thus

1. \( [a, c, c, c] = 1 \).
2. \( [b, c, c, c] = 1 \).

Since by our assumption \( \gamma_5(G) \) has no element of order 2, it follows from Theorem 2.1 parts (1), (3), and (4), respectively that

3. the subgroup \( \langle a, c \rangle \) is nilpotent of class at most 4.
4. \( [a, c, b, c, c] = 1 \).
5. the subgroup \( \langle a, b, c \rangle \) is nilpotent of class at most 5.
To prove $R_3(G)$ is a subgroup, we have to show that both $a^{-1}$ and $ab$ belong to $R_3(G)$. We first prove that $a^{-1} \in R_3(G)$. It easily follows from (1) and (3) that:

$$[a^{-1},c,c,c] = [a,c,c,c]^{-1} = 1.$$ 

Therefore $a^{-1} \in R_3(G)$.

We now show that $ab \in R_3(G)$.

$$[ab,c,c,c] = [[a,c,c,b][b,c],c,c]$$

$$= [[a,c,b][b,c][b,c],c,c]$$

$$= [[a,c,b][b,c][b,c],c,c]$$

$$= [[a,c,b][b,c],c,c]$$

$$= [a,c,b,c]$$

by (4)

This completes the proof.

Now we give a proof of Theorem 2.2 by using $nq$ package of Nickel [9] which is implemented in GAP [10]. Note that the knowledge of Theorem 2.1 is crucial in the following proof. The package $nq$ has the capability of computing the largest nilpotent quotient (if it exists) of a finitely generated group with finitely many identical relations and finitely many relations. For example, if we want to construct the largest nilpotent quotient of a group $G$ as follows

$$\langle x_1, \ldots, x_n | r_1(x_1, \ldots, x_n) = \cdots = r_m(x_1, \ldots, x_n) = 1, w(x_1, \ldots, x_n, y_1, \ldots, y_k) = 1 \rangle,$$

where $r_1, \ldots, r_m$ are relations on $x_1, \ldots, x_n$ and $w(x_1, \ldots, x_n, y_1, \ldots, y_k) = 1$ is an identical relation in the group $\langle x_1, \ldots, x_n \rangle$, one may apply the following code to use the package $nq$ in GAP:

```cpp
LoadPackage("$nq$"); # nq package of Werner Nickel #
F:=FreeGroup(n+k);
L:=F/[r1(F.1,...,F.n),... ,rm(F.1,...,F.n),w(F.1,...,F.n,F.(n+1),...,F.(n+k))];
H:=NilpotentQuotient(L,[F.(n+1),...,F.(n+k)]);
```

Note that we need to construct the free group of rank $n+k$ because as well as the $n$ generators for $G$ we also have an identical relation with $k$ free variables.

Note that the function $\text{NilpotentQuotient}(L)$ attempts to compute the largest nilpotent quotient of $L$ and it will terminate only if $L$ has a largest nilpotent quotient.

**Second Proof of Theorem 2.2.** By Theorem 2.1, we know that $(x, y, z)$ is nilpotent if $x, y \in R_3(G)$ and $z \in G$. We now construct the largest nilpotent group $H = \langle a, b, c \rangle$ such that $a, b \in R_3(H)$ and $c \in H$, by $nq$ package.
LoadPackage("nq");
F:=FreeGroup(4); a1:=F.1; b1:=F.2; c1:=F.3; x:=F.4;
L:=F/\LeftNormedComm([a1,x,x,x]),\LeftNormedComm([b1,x,x,x])];
H:=NilpotentQuotient(L,[x]);
a:=H.1; b:=H.2; c:=H.3; d:=\LeftNormedComm([a^{-1},c,c,c]);
e:=\LeftNormedComm([a*b,c,c,c]); Order(d); Order(e);
C:=LowerCentralSeries(H); d in C[5]; e in C[5];

Then if we consider the elements $d = [a^{-1},c,c,c]$ and $e = [ab,c,c,c]$ of $H$, we can see by above command in GAP that $d$ and $e$ are elements of $\gamma_5(H)$ and have orders 2 and 4, respectively. So, in the group $G$, we have $d = e = 1$. This completes the proof. □

Note that, the second proof of Theorem 2.2 also shows the necessity of assuming that $\gamma_5(G)$ has no element of order 2.

3. Right 4-Engel Elements

Our main result in this section is to prove the following.

**Theorem 3.1.** Let $G$ be a $\{2,3,5\}'$-group such that $\langle a, b, x \rangle$ is nilpotent for all $a, b \in R_4(G)$ and any $x \in G$. Then $R_4(G)$ is a subgroup of $G$.

**Proof.** Consider the “freest” group, denoted by $U$, generated by two elements $u, v$ with $u$ a right 4-Engel element. We mean this by the group $U$ given by the presentation

$$\langle u, v \mid [u,4,x] = 1 \text{ for all words } x \in F_2 \rangle,$$

where $F_2$ is the free group generated by $u$ and $v$. We do not know whether $U$ is nilpotent or not. Using the nq package shows that the group $U$ has a largest nilpotent quotient $M$ with class 8. By the following code, the group $M$ generated by a right 4-Engel element $a$ and an arbitrary element $c$ is constructed. We then see that the element $[a^{-1},c,c,c]$ of $M$ is of order $375 = 3 \times 5^3$. Therefore, the inverse of a right 4-Engel element of $G$ is again a right 4-Engel element. The following code in GAP gives a proof of the latter claim. The computation was completed in about 248s.

```gap
F:=FreeGroup(3); a1:=F.1; b1:=F.2; x:=F.3;
U:=F/\LeftNormedComm([a1,x,x,x,x]);
M:=NilpotentQuotient(U,[x]);
a:=M.1; c:=M.2;
h:=\LeftNormedComm([a^{-1},c,c,c,c]);
Order(h);
```

We now show that the product of every two right 4-Engel elements in $G$ is a right 4-Engel element. Let $a, b \in R_4(G)$ and $c \in G$. Then we claim that $H = \langle a, b, c \rangle$ is nilpotent of class at most 7. (*)
By induction on the nilpotency class of $H$, we may assume that $H$ is nilpotent of class at most 8. Now we construct the largest nilpotent group $K = \langle a_1, b_1, c_1 \rangle$ of class 8 such that $a_1, b_1 \in R_4(K)$.

F:=FreeGroup(4); A:=F.1; B:=F.2; C:=F.3; x:=F.4;
W:=F/[LeftNormedComm([A,x,x,x,x]),LeftNormedComm([B,x,x,x,x])];
K:=NilpotentQuotient(W,[x],8);
LowerCentralSeries(K);

The computation took about 22.7h. We see that $\gamma_8(K)$ has exponent 60. Therefore, as $H$ is a $(2,3,5)'$-group, we have $\gamma_8(H) = 1$ and this completes the proof of our claim ($\ast$).

Therefore, we have proved that any nilpotent group without elements of orders 2, 3, or 5 which is generated by three elements two of which are right 4-Engel, is nilpotent of class at most 7.

Now we construct, by the $\text{nq}$ package, the largest nilpotent group $S$ of class 7 generated by two right 4-Engel elements $s, t$ and an arbitrary element $g$. Then one can find by GAP that the order of $[s, g, g, g]$ in $S$ is 300. Since $H$ is a quotient of $S$, we have that $[ab, c, c, c]$ is of order dividing 300 and so it is trivial, since $H$ is a $(2,3,5)'$-group. This completes the proof.  

**Corollary 3.2.** Let $G$ be a $(2,3,5)'$-group such that $\langle a, b, x \rangle$ is nilpotent for all $a, b \in R_4(G)$ and for any $x \in G$. Then $R_4(G)$ is a nilpotent group of class at most 7. In particular, the normal closure of every right 4-Engel element of group $G$ is nilpotent of class at most 7.

**Proof.** By Theorem 3.1, $R_4(G)$ is a subgroup of $G$ and so it is a 4-Engel group. In [12], it is shown that every locally nilpotent 4-Engel $(2,3,5)'$-group is nilpotent of class at most 7. Therefore, $R_4(G)$ is nilpotent of class at most 7. Since $R_4(G)$ is a normal set, the second part follows easily.

Therefore, to prove that the normal closure of any right 4-Engel element of a $(2,3,5)'$-group $G$ is nilpotent, it is enough to show that $\langle a, b, x \rangle$ is nilpotent for all $a, b \in R_4(G)$ and for any $x \in G$.

**Corollary 3.3.** In any $(2,3,5)'$-group, the normal closure of any right 4-Engel element is nilpotent if and only if every 3-generator subgroup in which two of the generators can be chosen to be right 4-Engel, is nilpotent.

**Proof.** By Corollary 3.2, it is enough to show that a $(2,3,5)'$-group $H = \langle a, b, x \rangle$ is nilpotent whenever $a, b \in R_4(H)$, $x \in H$ and both $\langle a \rangle^H$ and $\langle b \rangle^H$ are nilpotent. Consider the subgroup $K = \langle a \rangle^H \langle b \rangle^H$ which is nilpotent by Fitting’s theorem. Now we prove that $K$ is finitely generated. We have $K = \langle a, b \rangle^{(x)}$ and since $a$ and $b$ are both right 4-Engel, it is well-known that  

$$\langle a \rangle = \langle a, a^x, a^{x^2}, a^{x^3} \rangle$$

and

$$\langle b \rangle = \langle b, b^x, b^{x^2}, b^{x^3} \rangle,$$
and so 
\[ K = (a, a^x, a^{x^2}, a^{x^3}, b, b^x, b^{x^2}, b^{x^3}). \]

It follows that \( H \) satisfies maximal condition on its subgroups as it is (finitely generated nilpotent)-by-cyclic. Now by a famous result of Baer [2] we have that \( a \) and \( b \) lie in the \((m + 1)\)th term \( \zeta_m(H) \) of the upper central series of \( H \) for some positive integer \( m \). Hence \( H/\zeta_m(H) \) is cyclic and so \( H \) is nilpotent. This completes the proof.

We conclude this section with the following interesting information on the group \( M \) in the proof of Theorem 3.1. In fact, for the largest nilpotent group \( M = \langle a, b \rangle \) relative to \( a \in R_4(M) \), we have that \( M/T \) is isomorphic to the largest (nilpotent) 2-generated 4-Engel group \( E(2, 4) \), where \( T \) is the torsion subgroup of \( M \) which is a \( \{2, 3, 5\} \)-group. Therefore, in a nilpotent \( \{2, 3, 5\}' \)-group, a right 4-Engel element with an arbitrary element generate a 4-Engel group. This can be seen by comparing the presentations of \( M/T \) and \( E(2, 4) \) as follows. One can obtain two finitely presented groups \( G_1 \) and \( G_2 \) isomorphic to \( M/T \) and \( E(2, 4) \), respectively by GAP:

\[
\begin{align*}
\text{MoverT:=FactorGroup(M,TorsionSubgroup(M));} \\
E24:=NilpotentEngelQuotient(FreeGroup(2),4); \\
isol1:=IsomorphismFpGroup(MoverT);isol2:=IsomorphismFpGroup(E24); \\
G1:=Image(isol1);G2:=Image(isol2);
\end{align*}
\]

Next, we find the relators of the groups \( G_1 \) and \( G_2 \) which are two sets of relators on 13 generators by the following command in GAP.

\[
\begin{align*}
r1:=\text{RelatorsOfFpGroup}(G1);r2:=\text{RelatorsOfFpGroup}(G2);
\end{align*}
\]

Now, save these two sets of relators by LogTo command of GAP in a file and go to the file to delete the terms as

\[
<\text{identity } \ldots> 
\]

in the sets \( r1 \) and \( r2 \). Now call these two modified sets \( R1 \) and \( R2 \). We show that \( R1=R2 \) as two sets of elements of the free group \( f \) on 13 generators \( f1,f2,\ldots,f13 \).

\[
\begin{align*}
f:=\text{FreeGroup}(13); \\
f1:=f.1;f2:=f.2;f3:=f.3;f4:=f.4;f5:=f.5;f6:=f.6; \\
f7:=f.7;f8:=f.8;f9:=f.9;f10:=f.10;f11:=f.11;f12:=f.12;f13:=f.13;
\end{align*}
\]

Now by Read function, load the file in GAP and type the simple command \( R1=R2 \). This gives us true which shows \( G_1 \) and \( G_2 \) are two finitely presented groups with the same relators and generators and so they are isomorphic. We do not know if there is a guarantee that if someone else does as we did, then he/she finds the same relators for \( \text{Fp} \) groups \( G_1 \) and \( G_2 \), as we have found. Also we remark that using function IsomorphismGroups to test if \( G_1 \cong G_2 \), did not give us a result in less than 10h and we do not know whether this function can give us a result or not.
We summarize the above discussion as following.

**Theorem 3.4.** Let $G$ be a nilpotent group generated by two elements, one of which is a right 4-Engel element. If $G$ has no element of order 2, 3, or 5, then $G$ is a 4-Engel group of class at most 6.

**Acknowledgments**

The authors are grateful to the referee for his/her careful reading and insightful comments. The research of the first author is financially supported by the Center of Excellence for Mathematics, University of Isfahan.

**References**

[1] A. Abdollahi and H. Khosravi, On the right and left 4-Engel elements, to appear in Comm. Algebra 38(3) (2010) 933–943.
[2] R. Baer, Engelsche elemente Noetherscher Gruppen, Math. Ann. 133 (1957) 256–270.
[3] G. Havas and M. R. Vaughan-Lee, 4-Engel groups are locally nilpotent, Int. J. Algebra Comput. 15(4) (2005) 649–682.
[4] H. Heineken, Engelsche Elemente der Länge drei, Illinois J. Math. 5 (1961) 681–707.
[5] W. P. Kappe, Die A-Norm einer Gruppe, Illinois J. Math. 5 (1961) 187–197.
[6] I. D. Macdonald, Some examples in the theory of groups, in Mathematical Essays Dedicated to A. J. Macintyre (Ohio University Press, Athens, Ohio, 1970), pp. 263–269.
[7] M. L. Newell, On right-Engel elements of length three, Proc. Roy. Irish. Acad. Sect. A. 96(1) (1996) 17–24.
[8] W. Nickel, Some groups with right Engel elements, Groups St. Andrews 1997 in Bath, II, London Math. Soc. Lecture Note Ser., Vol. 261 (Cambridge Univ. Press, Cambridge, 1999), pp. 571–578.
[9] W. Nickel, NQ, 1998, A refereed GAP 4 package, see [10].
[10] The GAP Group, GAP — Groups, Algorithms, and Programming, Version 4.4.12 (2008), (http://www.gap-system.org).
[11] G. Traustason, Locally nilpotent 4-Engel groups are fitting groups, J. Algebra 270(1) (2003) 7–27.
[12] G. Traustason, On 4-Engel groups, J. Algebra 178(2) (1995) 414–429.