THE INTEGRAL MONODROMY OF THE CYCLE TYPE SINGULARITIES

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Abstract. The middle homology of the Milnor fiber of a quasi-homogeneous polynomial with an isolated singularity is a \( \mathbb{Z} \)-lattice and comes equipped with an automorphism of finite order, the integral monodromy. Orlik (1972) made a precise conjecture, which would determine this monodromy in terms of the weights of the polynomial. Here we prove this conjecture for the cycle type singularities. A paper of Cooper (1982) with the same aim contained two mistakes. Still it is very useful. We build on it and correct the mistakes. We give additional algebraic and combinatorial results.

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1. INTRODUCTION AND MAIN RESULT

The main objects of the paper are a quasihomogeneous singularity, the monodromy on its Milnor lattice, Orlik’s conjecture for this monodromy, and our proof in the case of a cycle type singularity. In order to make precise statements, we start with some algebraic definitions.

Definition 1.1. (a) Start with a product \( p \in \mathbb{Z}[t] \) of cyclotomic polynomials which has only simple zeros. The Orlik block \( \text{Or}(p) \) is a pair

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\((H, h)\) where \(H\) is a \(\mathbb{Z}\)-lattice of rank \(\deg(p)\) and \(h : H \to H\) is an automorphism of finite order with characteristic polynomial \(p\) such that an element \(a_0 \in H\) with
\[
H = \bigoplus_{j=0}^{\deg(p)-1} \mathbb{Z} \cdot h^j(a_0).
\]exists. Such an element is called a generating element. The Orlik block \(\text{Or}(p)\) is up to isomorphism uniquely determined by \(p\), which justifies the notion \(\text{Or}(p)\).

(b) Consider a pair \((H, h)\) where \(H\) is a \(\mathbb{Z}\)-lattice and \(h : H \to H\) is an automorphism of finite order. It admits a decomposition into Orlik blocks if it is isomorphic to a direct sum of Orlik blocks.

(c) Consider a pair \((H, h)\) as in (b). Then the characteristic polynomial \(p_{H,h}\) of \(h\) is a product of cyclotomic polynomials. It has a unique decomposition
\[
p_{H,h} = \prod_{l=1}^{t} p_{l}
\]with \(p_{l} | p_{l-1} | \cdots | p_{2} | p_{1}\) and \(p_{l} \neq 1\) and all \(p_{l}\) unitary and such that \(p_{1}\) has only simple zeros. The pair \((H, h)\) admits a standard decomposition into Orlik blocks, if an isomorphism \((H, h) \cong \bigoplus_{l=1}^{t} \text{Or}(p_{l})\) exists.

A polynomial \(f \in \mathbb{C}[x_1, \ldots, x_n]\) is called quasihomogeneous if for some weight system \((w_1, \ldots, w_n)\) with \(w_i \in (0, 1) \cap \mathbb{Q}\) each monomial in \(f\) has weighted degree 1. It is called an isolated quasihomogeneous singularity if it is quasihomogeneous and the functions \(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\) vanish simultaneously only at \(0 \in \mathbb{C}^n\). Then the Milnor lattice \(H_{\text{Mil}} := H^{(\text{red})}_{n-1}(f^{(1)}(1), \mathbb{Z})\) (here \(H^{(\text{red})}_{n-1}\) means the reduced homology in the case \(n = 1\) and the usual homology in the cases \(n \geq 2\)) is a \(\mathbb{Z}\)-lattice of some finite rank \(\mu \in \mathbb{N}\), which is called the Milnor number \([\text{Mi68}]\). It comes equipped with an automorphism \(h_{\text{Mil}}\) of finite order, the monodromy.

Orlik conjectured the following.

**Conjecture 1.2.** (Orlik’s conjecture [Or72, Conjecture 3.1]) For any isolated quasihomogeneous singularity, the pair \((H_{\text{Mil}}, h_{\text{Mil}})\) admits a standard decomposition into Orlik blocks.

Here we will prove this conjecture for the cycle type singularities. A cycle type singularity is a polynomial \(f \in \mathbb{C}[x_1, \ldots, x_n]\) in \(n \geq 2\) variables of the following shape,
\[
f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_1 \quad \text{where } a_1, \ldots, a_n \in \mathbb{N}, \quad (1.2)
\]for even \(n\) neither \(a_j = 1\) for all even \(j\) nor \(a_j = 1\) for all odd \(j\). \(1.3\)
It is an isolated quasihomogeneous singularity. Define \( d := \prod_{j=1}^{n} a_j - (-1)^n \). Then (see e.g. Lemma 4.1 in [HZ19])

\[
\mu = \prod_{j=1}^{n} a_j = d + (-1)^n,
\]

and there are natural numbers \( v_1, \ldots, v_n \) (which are given in (5.1)) such that \( (w_1, \ldots, w_n) = \left( \frac{v_1}{d}, \ldots, \frac{v_n}{d} \right) \) is the unique weight system for which \( f \) is quasihomogeneous of weighted degree 1. They satisfy also \( \gcd(v_1, d) = \ldots = \gcd(v_n, d) \). Define \( b := d / \gcd(v_1, d) \in \mathbb{N} \). An easy calculation which builds on the formula in [MO70] for the characteristic polynomial in terms of \( (w_1, \ldots, w_n) \) (see e.g. Lemma 4.1 in [HZ19]) shows that the characteristic polynomial of \( h_{\text{Mil}} \) on \( H_{\text{Mil}} \) is

\[
p_{H_{\text{Mil}}, h_{\text{Mil}}} = (tb - 1)^{\gcd(d,v_1)} \cdot (t - 1)^{(-1)^n}.
\]

Therefore Orlik's conjecture says here the following.

**Theorem 1.3.** For a cycle type singularity as above,

\[
(H_{\text{Mil}}, h_{\text{Mil}}) \cong \begin{cases} 
\gcd(d,v_1) \text{Or}(tb - 1) \oplus \text{Or} \left( \frac{tb - 1}{t - 1} \right) & \text{if } n \text{ is odd}, \\
\gcd(d,v_1) \text{Or}(tb - 1) \oplus \text{Or}(t - 1) & \text{if } n \text{ is even}.
\end{cases}
\]

Our proof in section 5 builds on Cooper’s work [Co82]. By [Mi68], the set \( F_0 := f^{-1}(\mathbb{R}_{\geq 0}) \cap S^{2n-1} \subset S^{2n-1} \subset \mathbb{C}^n \) is diffeomorphic to the Milnor fiber \( f^{-1}(1) \). Cooper studied for the cycle type singularities a beautiful subset \( G \subset F_0 \), which is a probably a deformation retract of the Milnor fiber (Cooper’s Lemma 3 is slightly weaker). He considered certain cells from which this set \( G \) is built up and which allow to filter \( G \) by a sequence of subsets \( G = G_n \supset G_{n-1} \supset \ldots \supset G_1 \supset G_0 = \emptyset \). Finally, he studied a spectral sequence which comes from this filtration.

Cooper claimed to have proved Orlik’s conjecture for the cycle type singularities. But his paper contains two serious mistakes. The second one leads in the case of even \( n \) to the wrong claim

\[
(H_{\text{Mil}}, h_{\text{Mil}}) \cong (\gcd(d,v_1) - 1) \text{Or}(tb - 1) \oplus \text{Or} \left( \frac{tb - 1}{t - 1} \right) \oplus 2\text{Or}(t - 1).
\]

The right hand side is a decomposition into Orlik blocks, but not a standard decomposition.

We will discuss the relation of this paper to Cooper’s work and the two mistakes in the Remarks 5.1. His work is the basis for the sections 3 and 4 below. Section 2 gives new algebraic results. Section 3 gives the set \( G \), its cells and an inductive construction of cycles of which only the beginning is in [Co82]. Section 4 makes good use of the spectral sequence which Cooper considered and determines \( H_{n-1}(G, \mathbb{Z}) \). A
combination of the results of the sections 2, 3 and 4 and a discussion of the monodromy proves Theorem 1.3 in section 5.

2. Algebraic results

The following two lemmata 2.3 and 2.4 are elementary. They will be used in the proof of Orlik’s conjecture for cycle type singularities with an even number of variables. They say something about the $\mathbb{Z}$-lattices $H^{(d,c)}$ with automorphisms $h^{(d,c)}$ of finite order, which are defined in Definition 2.1.

**Definition 2.1.** Let $d \in \mathbb{N}$ and $c \in \mathbb{Z}$. Define the pair $\text{Lo}^{(d,c)} = (H^{(d,c)}, h^{(d,c)})$ as follows. $H^{(d,c)} = \mathbb{Z} \cdot \gamma \oplus \bigoplus_{j=1}^{d-1} \mathbb{Z} \cdot \delta_j$ is a $\mathbb{Z}$-lattice of rank $d$. Define additionally $\delta_d := c \cdot \gamma - \sum_{j=1}^{d-1} \delta_j \in H^{(d,c)}$. Then $h^{(d,c)} : H \to H$ is the automorphism of finite order $d$ which is defined by

$$h^{(d,c)} : \gamma \mapsto \gamma, \quad \delta_d \mapsto \delta_1, \quad \delta_j \mapsto \delta_{j+1} \text{ for } j \in \{1, \ldots, d-1\}. \quad (2.1)$$

**Remark 2.2.** The characteristic polynomial of $h^{(d,c)}$ is $t^d - 1$. If $c \neq 0$, then $\sum_{j=1}^{d} \mathbb{Z} \cdot \delta_j = \bigoplus_{j=1}^{d} \mathbb{Z} \cdot \delta_j$ is an $h^{(d,c)}$-invariant sublattice of index $|c|$ in $H^{(d,c)}$, and $(\bigoplus_{j=1}^{d} \mathbb{Z} \cdot \delta_j, h^{(d,c)}) \cong \text{Or}(t^d - 1)$. Therefore $\text{Lo}(d, 1) \cong \text{Or}(t^d - 1)$. If $c = 0$ then the summands $\mathbb{Z} \cdot \gamma$ and $\bigoplus_{j=1}^{d-1} \mathbb{Z} \cdot \delta_j$ of $H^{(d,c)}$ are $h^{(d,c)}$-invariant with $(\mathbb{Z} \cdot \gamma, h^{(d,c)}) \cong \text{Or}(t-1)$ and $(\bigoplus_{j=1}^{d-1} \mathbb{Z} \cdot \delta_j, h^{(d,c)}) \cong \text{Or}(\frac{t^d - 1}{t-1})$.

**Lemma 2.3.** Let $d, v \in \mathbb{N}$, $c \in \mathbb{Z}$ and $b := d / \gcd(d, v) \in \mathbb{N}$. Then

$$(H^{(d,c)}, (h^{(d,c)})^v) \cong (\gcd(d, v) - 1) \cdot \text{Or}(t^b - 1) \oplus \text{Lo}^{(b,c)}. \quad (2.2)$$

**Proof:** Write $h := (h^{(d,c)})^v$. The elements $\delta_1, \ldots, \delta_d$ can be renumbered to elements $\tilde{\delta}_1, \ldots, \tilde{\delta}_d$ (i.e. $\{\delta_1, \ldots, \delta_d\} = \{\tilde{\delta}_1, \ldots, \tilde{\delta}_d\}$) such that these form $\gcd(d, v)$ many cycles of length $b$ with respect to $h$:

$$h : \tilde{\delta}_{ab+1} \mapsto \tilde{\delta}_{ab+2} \mapsto \ldots \mapsto \tilde{\delta}_{ab+b} \mapsto \tilde{\delta}_{ab+1}$$

for $a \in \{0,1,\ldots,\gcd(d, v) - 1\}$.

Define

$$\beta_j := \sum_{a=0}^{\gcd(d, v) - 1} \tilde{\delta}_{ab+j} \quad \text{for } j \in \{1, \ldots, b\}.$$
Then these also form a cycle of length \( b \) with respect to \( h \), and their sum is \( c \gamma \),

\[
\begin{align*}
h : \beta_1 &\mapsto \beta_2 \mapsto \ldots \mapsto \beta_b \mapsto \beta_1, \\
\sum_{j=1}^{b} \beta_j &\equiv \sum_{j=1}^{d} \bar{\delta}_j = \sum_{j=1}^{d} \delta_j = c \gamma.
\end{align*}
\]

We obtain

\[
(H^{(d,c)}, h) \cong \bigoplus_{a=0}^{\gcd(d,v)-2} \left( \bigoplus_{j=1}^{b} \mathbb{Z} \cdot \tilde{\delta}_{ab+j}, h \right) \oplus \left( \mathbb{Z} \cdot \gamma + \sum_{j=1}^{b} \mathbb{Z} \cdot \beta_j, h \right)
\]

\[
\cong \left( \gcd(d,v) - 1 \right) \text{Or}(t^b - 1) \oplus \text{Lo}^{(b,c)}.
\]

\[
\text{Lemma 2.4. Let } d \in \mathbb{N} \text{ and } c, \tilde{c} \in \mathbb{Z}. \text{ The following three conditions are equivalent.}
\]

\( (i) \) \( \text{Lo}^{(d,c)} \cong \text{Lo}^{(d,\tilde{c})} \).

\( (ii) \) \( \text{Lo}^{(d,c)} \oplus \text{Or}(t-1) \cong \text{Lo}^{(d,\tilde{c})} \oplus \text{Or}(t-1) \).

\( (iii) \) \( \gcd(d, c) = \gcd(d, \tilde{c}) \).

And

\[
\text{Lo}^{(d,c)} \oplus \text{Or}(t-1) \cong \text{Or}(t^d - 1) \oplus \text{Or}(t-1) \quad (2.3)
\]

\[
\Longleftrightarrow \gcd(d, c) = 1.
\]

\textbf{Proof:} Keep the notations \( \delta_1, \ldots, \delta_d, \gamma \) of Definition 2.1 for the elements of \( H^{(d,c)} \). And extend them by \( \delta_j := \delta_{j_0} \) if \( j \in \mathbb{Z} \setminus \{1, \ldots, d\} \), \( j_0 \in \{1, \ldots, d\} \) and \( d \mid (j - j_0) \).

\( (iii) \Rightarrow (i) \): We start with \( \text{Lo}^{(d,c)} \) for some \( c \in \mathbb{N} \) with \( c \mid d \). We will present a construction which leads to certain \( \tilde{c} \in \mathbb{Z} \) with \( \text{Lo}^{(d,c)} \cong \text{Lo}^{(d,\tilde{c})} \). Then we will show that these are all \( \tilde{c} \in \mathbb{Z} \) with \( \gcd(d, \tilde{c}) = c \).

Choose \( a \in \mathbb{N} \) with \( \gcd(a, d) = 1 \), and choose \( b \in \mathbb{Z} \). Define

\[
\tilde{\gamma} := \gamma, \quad \tilde{\delta}_j := b \gamma + \sum_{i=0}^{a-1} \delta_{j+i} \quad \text{for } j \in \mathbb{Z},
\]

so that \( \tilde{\delta}_{j_1} = \tilde{\delta}_{j_2} \) if \( d \mid (j_1 - j_2) \). Of course, \( h^{(d,c)} \) acts by

\[
h^{(d,c)} : \tilde{\gamma} \mapsto \tilde{\gamma}, \quad \tilde{\delta}_j \mapsto \tilde{\delta}_{j+1},
\]

and we have

\[
\sum_{j=1}^{d} \tilde{\delta}_j = (bd + ac) \cdot \tilde{\gamma}.
\]
Furthermore, the condition $\gcd(a, d) = 1$ implies
\[
\mathbb{Z} \cdot \tilde{\gamma} + \sum_{j=1}^{d} \mathbb{Z} \cdot \tilde{\delta}_j = H^{d,c}(\mathbb{Z} \cdot \gamma + \sum_{j=1}^{d} \mathbb{Z} \cdot \delta_j).
\] (2.6)

To see this, choose $b_1, b_2 \in \mathbb{N}$ with $ab_1 - db_2 = 1$. Then for $j \in \mathbb{Z}$
\[
-(b_1b + b_2c)\tilde{\gamma} + \sum_{k=0}^{b_1-1} \tilde{\delta}_{j+ak} = -b_1b\tilde{\gamma} - \sum_{k=1}^{db_2} \delta_{j+k} + b_1b\tilde{\gamma} + \sum_{k=0}^{db_2} \delta_{j+k} = \delta_j,
\]
which shows (2.6). Together, (2.4), (2.5) and (2.6) give
\[
\text{Lo}^{(d,c)} \cong \text{Lo}^{(d, bd + ac)}.
\] (2.7)

Now choose any $\tilde{c} \in \mathbb{Z}$ with $\gcd(d, \tilde{c}) = c$. It remains to see that there exist $a \in \mathbb{N}$ and $b \in \mathbb{Z}$ with $\gcd(a, d) = 1$ and $\tilde{c} = bd + ac$.

For any integer $m \in \mathbb{Z} - \{0\}$, write $m = \frac{m}{|m|} \cdot \prod_p$ prime number $p^{v_p(m)}$, where $v_p(m) \in \mathbb{Z}_{\geq 0}$. We choose
\[
\tilde{b} := \prod_{p \text{ prime number with } v_p(d) = v_p(\tilde{c}) > 0} p,
\]
\[
b := -\tilde{b} - |\tilde{c}| \in \mathbb{Z}_{< 0},
\]
\[
a := \frac{\tilde{c}}{c} - b \cdot \frac{d}{c} = \frac{\tilde{c}}{c} + \frac{\tilde{d}}{c} + \frac{|\tilde{c}|}{c} \in \mathbb{N}.
\]

Then $\tilde{c} = bd + ac$. For a prime number $p$ with $v_p(d) > v_p(\tilde{c})$, $v_p(\tilde{c}) = 0$ and $v_p(c) > 0$ and $v_p(a) = 0$. For a prime number $p$ with $v_p(d) = v_p(\tilde{c}) > 0$ and $v_p(\tilde{c}) = 0$ and $v_p(b) > 0$ and $v_p(a) = 0$. For a prime number $p$ with $0 < v_p(d) < v_p(\tilde{c})$, $v_p(\tilde{c}) > 0$ and $v_p(b) = 0$ and $v_p(a) = 0$. Therefore $\gcd(a, d) = 1$.

(i)⇒(ii): This is trivial.

(ii)⇒(iii): We proved already (iii)⇒(i) and (i)⇒(ii). Therefore, in (ii) we can suppose $c = \gcd(d, c)$ and $\tilde{c} = \gcd(d, \tilde{c})$. Then we have to show $c = \tilde{c}$.

Write the elements in Definition 2.1 for $H^{d,c}$ with a tilde, so as $\tilde{\delta}_1, \ldots, \tilde{\delta}_d, \tilde{\gamma}$. Write generators of $\text{Or}(t - 1)$ on the left hand side respectively right hand side of (ii) as $\beta$ respectively $\tilde{\beta}$. The automorphisms of the left hand side respectively right hand side of (ii) which extend $h^{d,c}$ respectively $h^{d,\tilde{c}}$ by id on $\text{Or}(t - 1)$, are called $h$ respectively $\tilde{h}$.

Let
\[
g : \text{Lo}^{(d,c)} \oplus \text{Or}(t - 1) \rightarrow \text{Lo}^{(d,\tilde{c})} \oplus \text{Or}(t - 1)
\]
be an isomorphism. Then
\[ \tilde{h} \circ g = g \circ h. \] (2.8)
This and \( \ker(h - \text{id}) = \mathbb{Z} \cdot \gamma + \mathbb{Z} \cdot \beta \) and \( \ker(\tilde{h} - \text{id}) = \mathbb{Z} \cdot \tilde{\gamma} + \mathbb{Z} \cdot \tilde{\beta} \) imply \( g(\gamma) = b_1 \tilde{\gamma} + b_2 \tilde{\beta}, \quad g(\beta) = b_3 \tilde{\gamma} + b_4 \tilde{\beta} \) with \( \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in GL(2, \mathbb{Z}) \).

Because of (2.8) and the action of \( h \) on \( \delta_1, ..., \delta_d \) and \( \tilde{h} \) on \( \tilde{\gamma}, \tilde{\beta}, \tilde{\delta}_1, ..., \tilde{\delta}_d \), numbers \( a_1, ..., a_{d+2} \in \mathbb{Z} \) with
\[ g(\delta_j) = a_{d+1} \tilde{\gamma} + a_{d+2} \tilde{\beta} + \sum_{i=1}^{d} a_i \tilde{\delta}_{i+j-1} \quad \text{for} \quad j \in \{1, ..., d\} \]
exist. Write \( a := \sum_{i=1}^{d} a_i \). Then
\[ c(b_1 \tilde{\gamma} + b_2 \tilde{\beta}) = g(c\gamma) = g(\sum_{j=1}^{d} \delta_j) \]
\[ = da_{d+1} \tilde{\gamma} + da_{d+2} \tilde{\beta} + \left( \sum_{i=1}^{d} a_i \right) \cdot \sum_{j=1}^{d} \tilde{\delta}_j \]
so \( cb_1 = da_{d+1} + a\tilde{c}, \quad cb_2 = da_{d+2} \).
\[ \pm 1 = b_1 b_4 - b_2 b_3 \] implies \( \gcd(b_1, b_2) = 1 \). Therefore \( c \) is in the ideal generated by \( d \) and \( \tilde{c} \), which is the ideal generated by \( \tilde{c} \), because \( \tilde{c}|d \).
Thus \( c|\tilde{c} \). Because of the symmetry of the situation also \( c|\tilde{c} \). We obtain \( c = \tilde{c} \). This shows (ii)⇒(iii).

Now the equivalence of (i), (ii) and (iii) is proved. It rests to see (2.3). It follows with the isomorphy \( \text{Lo}^{(d,1)} \cong \text{Or}(t^d - 1) \) in Remark 2.2 and the equivalence of (i), (ii) and (iii). \( \square \)

3. Cells, chains and cycles

Throughout this section we fix a cycle type singularity \( f(x_1, ..., x_n) \) as in (1.2) with \( n \geq 2 \) and \( a_1, ..., a_n \in \mathbb{N} \) with (1.3). By [Mi68], the Milnor fiber \( f^{-1}(1) \subset \mathbb{C}^n \) and the set \( \overline{F_0} := f^{-1}(\mathbb{R}_{\geq 0}) \cap S^{2n-1} \subset S^{2n-1} \subset \mathbb{C}^n \) are diffeomorphic. Cooper [Co82] considers the subset \( G \subset \overline{F_0} \) which is defined as follows,
\[ G = \{ z \in S^{2n-1} | z_j^{a_j} z_{j+1} \geq 0 \ \forall \ j \in \{1, ..., n - 1\}, \ z_n^{a_n} z_1 \geq 0 \}. \] (3.1)
He conjectures that \( G \) is a deformation retract of \( \overline{F_0} \). He proves a slightly weaker deformation lemma (stated at the end of 3. in [Co82]).
which implies especially that the inclusion map \( i_g : G \hookrightarrow F_0 \) induces epimorphisms in homology. For him and for us, this property suffices.

Cooper builds \( G \) up from certain cells. We will need these cells, and also refinements of them. For this, quite some notations are needed. They are given now.

**Notations 3.1.** \( N := \{1, ..., n\} \), so that \( \mathbb{R}^N = \mathbb{R}^n \). The map

\[
\mathcal{E} : \mathbb{R}^n \to T^n := (S^1)^n \subset \mathbb{C}^n, \quad (r_1, ..., r_n) \mapsto (e^{2\pi i r_1}, ..., e^{2\pi i r_n}),
\]

induces an isomorphism \( \mathcal{E}_T : \mathbb{R}^n / \mathbb{Z}^n \to T^n \). Also the projection \( \text{pr}_T : \mathbb{R}^n \to \mathbb{R}^n / \mathbb{Z}^n \) will be useful. Then \( \mathcal{E} = \mathcal{E}_T \circ \text{pr}_T \). The following binary operation \( \odot \) is not standard, but it will also be useful,

\[
\odot : \mathbb{R}_{\geq 0}^n \times (S^1)^n \to \mathbb{C}^n, \quad (a_1, ..., a_n) \odot (b_1, ..., b_n) := (a_1b_1, ..., a_nb_n).
\]

Given a finite tuple \((v_1, ..., v_k) \in (\mathbb{R}^n)^k\) of vectors in \( \mathbb{R}^n \), we consider the subset of \( \mathbb{R}^n \)

\[
C(v_1, ..., v_k) := \{ \sum_{i=1}^k t_i v_i \mid t_1, ..., t_k \in [0, 1] \}.
\]

If \((v_1, ..., v_k)\) is a tuple of linearly independent vectors (which will be the case almost always), \(C(v_1, ..., v_k)\) is a hypercube of dimension \( k \). Then it inherits an orientation from the ordered tuple \((v_1, ..., v_k)\). And then its boundary is

\[
\partial C(v_1, ..., v_k) = \sum_{i=1}^k (-1)^{i-1} C(v_1, ..., \widehat{v_i}, ..., v_k) \quad (3.2)
\]

\[
- \sum_{i=1}^k (-1)^{i-1} (v_i + C(v_1, ..., \widehat{v_i}, ..., v_k)),
\]

where \( \widehat{v_i} \) means that \( v_i \) is erased in the tuple. The following observation will be useful, because we will consider the images of hypercubes under \( \text{pr}_T \) in \( \mathbb{R}^n / \mathbb{Z}^n \). If \( v_1 \in \mathbb{Z}^n \), then

\[
\text{pr}_T(C(v_1, ..., v_k)) = \text{pr}_T(C(v_1, v_2 + \lambda_2 v_1, ..., v_k + \lambda_k v_1)) \quad (3.3)
\]

for any \( \lambda_2, ..., \lambda_k \in \mathbb{R} \),

and similarly for \( v_j \in \mathbb{Z}^n \) instead of \( v_1 \in \mathbb{Z}^n \).
The exponents $a_1, \ldots, a_n \in \mathbb{N}$ in the monomials of the cycle type singularity are the source of useful integers and vectors of integers:

Recall $\mu = a_1 \cdot \ldots \cdot a_n$, $d := \mu - (-1)^n$.

For $k \in N$:

- $a_k := (-1)^{k-1}a_1 \cdot \ldots \cdot a_{k-1} \in \mathbb{Z}$, so $a_1 = 1$.
- $a_{n+1} := (-1)^n a_1 \cdot \ldots \cdot a_n = (-1)^n \mu = (-1)^n d + 1$.

For $k \in N$:

- $b_k := (a_1, \ldots, a_{k-1}, 0, \ldots, 0) \in \mathbb{Z}^n$, so $b_1 = (0, \ldots, 0)$.
- $b_{n+1} := (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$.

For $k \in N$:

- $c_k := (a_{n+1} a_1, \ldots, a_{n+1} a_{k-1}, a_k, a_{k+1}, \ldots, a_n)$
  - $= a_{n+1} \cdot b_k + (0, \ldots, 0, a_k, \ldots, a_n)$
  - $= (-1)^n d \cdot b_k + c_1 \in \mathbb{Z}^n$, especially $c_1 = b_{n+1}$.

Also the following vectors in $\mathbb{Q}^n$ will be used,

for $j \in \{1, \ldots, d\}$:

\[ p_j := \frac{j}{d} \cdot c_1 \in \mathbb{Q}^n. \]

In fact, their images $\text{pr}_T(p_j)$ in $\mathbb{R}^n / \mathbb{Z}^n$ will be the only 0-chains in $\mathbb{R}^n / \mathbb{Z}^n$ which we will need. The reason is the equality

\[ \text{pr}_T(p_{j+1}) = \text{pr}_T(p_j + \frac{1}{d} c_j) \quad \text{for any } k \in N. \quad (3.4) \]

For any subset $A \subset N$ with $A \neq \emptyset$, define

\[
\begin{align*}
\mathbb{R}^A &:= \{ r = (r_1, \ldots, r_n) \in \mathbb{R}^n \mid r_j = 0 \text{ for } j \notin A \} \subset \mathbb{R}^n, \\
\mathbb{Z}^A &\subset \mathbb{Z}^n \text{ and } \mathbb{R}^A_{\geq 0} \subset \mathbb{R}^n_{\geq 0} \text{ and } \mathbb{C}^A \subset \mathbb{C}^n \text{ analogously,} \\
\Delta_A &:= \mathbb{R}^A_{\geq 0} \cap S^{2n-1} \subset S^{2n-1}, \\
\text{pr}_A &\colon \mathbb{R}^n \to \mathbb{R}^A \text{ the projection.}
\end{align*}
\]

$\Delta_A$ is a deformation of a simplex of dimension $|A| - 1$. We call it a deformed simplex. Its boundary consists of the deformed simplices $\Delta_B$ with $B \subsetneq A$.

We want to consider $N$ and its subsets $A$ as cyclic: 1 follows $n$. In order to write this down, we denote by $(k)_{\text{mod} n} \in N$ for $k \in \mathbb{Z}$ the number with $n | (k - (k)_{\text{mod} n})$.

For $A \subsetneq N$ with $A \neq \emptyset$, the blocks are the maximal sequences $k, (k+1)_{\text{mod} n}, \ldots, (k+b_{A})_{\text{mod} n}$ within $A$. Their number is called $b(A)$. And the block beginnings are the first numbers in the blocks. Explicitly, they are the numbers in the set

\[\{ k_1^A, \ldots, k_{b(A)}^A \} = \{ k \in A \mid (k-1)_{\text{mod} n} \notin A \},\]

with $1 \leq k_1^A < \ldots < k_{b(A)}^A \leq n$. 


The *gaps* of $A$ are the blocks of $N - A$. Also their number is $b(A)$. A set $A \subseteq N$ is *thick* if $b(A) = n - |A|$. Equivalent is that each gap of $A$ consists of a single number. For thick $A \subset N$ define the sign

$$\text{sign}(A) := \text{sign} \begin{pmatrix} 1 & \ldots & |A| & |A| + 1 & \ldots & n \\ \alpha_1 & \ldots & \alpha_{|A|} & k_1^A & \ldots & k_{b(A)}^A \end{pmatrix} \in \{\pm 1\}$$

where

$$N - \{k_1^A, \ldots, k_{b(A)}^A\} = \{\alpha_1, \ldots, \alpha_{|A|}\} \text{ with } \alpha_1 < \ldots < \alpha_{|A|}.$$

A set $A \not\subseteq N$ with $A \neq \emptyset$ is *almost thick* if $b(A) = n - |A| - 1$. Equivalent is that one gap consists of two numbers and each other gap consists of a single number. For an almost thick set $B$ with gap

$$\{k_0, (k_0 + 1)_{\text{mod } n}\} \text{ of two elements, denote}$$

$$B^{(1)} := B \cup \{k_0\}, \quad B^{(2)} := B \cup \{(k_0 + 1)_{\text{mod } n}\}.$$ 

$B^{(1)}$ and $B^{(2)}$ are the unique thick sets with $B \subset B^{(i)}$ and $|B^{(i)}| = |B| + 1$. They satisfy $b(B) = b(B^{(i)})$.

For $A \not\subseteq N$ with $A \neq \emptyset$, we will define a subtorus $T_A \subset T^n$ below. For this we define for each block beginning $k_j^A$ in $A$ a vector of integers,

$$d_j^A := \frac{a_{k_j^A}^{-1}(b_{k_j^A+1} - b_{k_j^A})}{a_{b(A)}^{-1}} \text{ for } j \in \{1, \ldots, b(B) - 1\}$$

$$= (0, \ldots, 0, 1, (-a_{k_j^A}), \ldots, (-a_{k_j^A})(-a_{k_j^A+1} - 2), 0, \ldots, 0) \in \mathbb{Z}^n,$$

$$d_{b(A)}^A := \frac{a_{b(A)}^{-1}(a_{b(A)+1}b_{k_1^A} + c_1 - b_{k_{b(A)}^A})}{a_{b(A)-1}}$$

$$= (\ldots, a_{b(A)+1}a_{b(A)}^{-1}a_{k_{b(A)}^A} - 1, 0, \ldots, 0, 1, (-a_{k_{b(A)}^A}), \ldots, a_{a_{b(A)}^{-1}a_{b(A)}}, \ldots) \in \mathbb{Z}^n.$$ 

Then the subtorus $T_A \subset T^n$ is the set

$$T_A := \{c(d_1^A, \ldots, d_{b(A)}^A)\} \subset T^n.$$

It is a torus of dimension $b(A)$. Finally, observe that the $b(A)$-dimensional hypercube $p_j + d^{-1}C(a_{k_1^A}, \ldots, a_{k_{b(A)}^A})$ maps by $c$ to a subset of $T_A$,

$$c \left( p_j + d^{-1}C(a_{k_1^A}, \ldots, a_{k_{b(A)}^A}) \right) \subset T_A, \quad (3.5)$$

because the vectors $p_j$ and $d^{-1}C_{k_1^A}$ are linear combinations of the vectors $d_1^A, \ldots, d_{b(A)}^A$. And observe that many of the $2^{b(A)}$ vertices of the image in $T_A$ of this hypercube coincide and that the vertices form the set

$$\{c(p_j), c(p_{(j+1)_{\text{mod } d}}), \ldots, c(p_{(j+b(A))_{\text{mod } d}})\}, \text{ because of } (3.4).$$

Some of these notations are due to Cooper [Co82], namely the deformed simplices $\Delta_A$, the tori $T_A$, the *block beginnings*, their number $b(A)$ and the *thick* sets $A$. The hypercubes, the integers $a_k$ and the
vectors $b_k, c_k, d_k^A$, and $p_j$ are new. The observations in the following lemma are all due to Cooper [Co82].

**Lemma 3.2.** [Co82] We stick to the cycle type singularity $f$ above and all induced data in the Notations 3.1.

\[
G = \bigcup_{j=1}^{d} \Delta_N \circ \{e(p_j)\} \cup \bigcup_{A \subseteq N, A \neq \emptyset} \Delta_A \circ T_A \tag{3.6}
\]

\[
= \bigcup_{j=1}^{d} \Delta_N \circ \{e(p_j)\} \cup \bigcup_{A \subseteq N \text{ thick}} \Delta_A \circ T_A.
\]

For $A \subseteq N$ with $A \neq \emptyset$,

\[G \cap C^A = \Delta_A \circ T_A. \tag{3.7}\]

The sets $\Delta_N \circ \{e(p_j)\}$ and $\Delta_A \circ T_A$ are called cells of $G$. The natural map $\text{Int}(\Delta_A) \times T_A \to \text{Int}(\Delta_A) \circ T_A$ is a diffeomorphism. The natural map $\Delta_N \to \Delta_N \circ \{e(p_j)\}$ is a diffeomorphism.

\[\dim_R(\Delta_N \circ \{e(p_j)\}) = n - 1, \tag{3.8}\]

\[\dim_R(\Delta_A \circ T_A) = |A| - 1 + b(A) \begin{cases} = n - 1 & \text{if } A \text{ is thick,} \\ < n - 1 & \text{if } A \text{ is not thick.} \end{cases}\]

If $A, B \subseteq N$ with $A \neq B$, then $\text{Int}(\Delta_A) \circ T_A \cap \text{Int}(\Delta_B) \circ T_B = \emptyset$. If $B \subseteq A \neq N$ with $b(B) = b(A)$ then $\Delta_B \circ T_B \subset \Delta_A \circ T_A$.

$A \subset N$ thick implies $|A| \in \{\lfloor \frac{n+1}{2} \rfloor, \lfloor \frac{n+1}{2} \rfloor + 1, \ldots, n - 1\}$. In the case $n$ even, there are only two thick sets $A$ with $|A| = \frac{n}{2}$, the set $A_{od} := \{1, 3, \ldots, n - 1\}$ and the set $A_{ev} := \{2, 4, \ldots, n\}$. The cells $\Delta_{A_{od}} \circ T_{A_{od}}$ and $\Delta_{A_{ev}} \circ T_{A_{ev}}$ are the unit spheres in $\mathbb{C}^{A_{od}}$ respectively $\mathbb{C}^{A_{ev}}$.

$A \subset N$ almost thick implies $|A| \in \{\lfloor \frac{n}{2} \rfloor, \ldots, n - 2\}$.

The proof is easy. We will not give details. If $z_j, z_{j+1} \in \mathbb{C}^*$ with $z_j^a z_{j+1} > 0$ then $\arg(z_{j+1}) \equiv (-a_j) \arg(z_j) \mod 2\pi$. This observation is crucial.

Now we will build up chains, starting with the cells $\Delta_N \circ \{e(p_j)\}$, and ending with cycles which represent elements of $H_{n-1}(G, \mathbb{Z})$. In section 4 we will show that these cycles (and the cells $\Delta_{A_{od}} \circ T_{A_{od}}$ and $\Delta_{A_{ev}} \circ T_{A_{ev}}$ in the case $n$ even) generate $H_{n-1}(G, \mathbb{Z})$. Of course, it is necessary to refine the cells of $G$ to a simplicial chain complex. We will not describe this precisely. Except from the cells $\Delta_N \circ \{e(p_j)\}$, we will...
work only with the following chains (compare (3.5)),
\[
C(A, j) := \Delta_A \circ \varepsilon \left( p_j + d^{-1} C(\xi_{k_1}^A, \ldots, \xi_{k_{\ell,(A)}}^A) \right) \subset \Delta_A \otimes T_A \quad (3.9)
\]
for \( A \subseteq N \) thick, \( j \in \{1, \ldots, d\} \),
\[
C(B, A, j) := \Delta_B \circ \varepsilon \left( p_j + d^{-1} C(\xi_{k_1}^A, \ldots, \xi_{k_{\ell,(A)}}^A) \right) \subset \Delta_B \otimes T_A \quad (3.10)
\]
for \( B \subseteq A \subseteq N \) with \( A \) thick, \( |B| = |A| - 1 \geq 1 \), \( j \in \{1, \ldots, d\} \).

Here our notation is not precise in two ways. (1) If the projection boundary into two parts is well defined. For \( \partial \) is not injective, the chain \( C \) the vectors \( c \) into account. (2) The chain obtains an orientation from the order of \( C \).

We decompose the boundary of a chain \( C(A, j) \) into two parts, \( \partial C(A, j) = \partial_1 C(A, j) + \partial_2 C(A, j) \) with
\[
\partial_1 C(A, j) = \partial \Delta_A \circ \varepsilon \left( p_j + d^{-1} C(\xi_{k_1}^A, \ldots, \xi_{k_{\ell,(A)}}^A) \right), \quad (3.11)
\]
\[
\partial_2 C(A, j) = (-1)^{|A|-1} \Delta_A \circ \partial \varepsilon \left( p_j + d^{-1} C(\xi_{k_1}^A, \ldots, \xi_{k_{\ell,(A)}}^A) \right) = (-1)^{|A|-1} \Delta_A \circ \varepsilon \left( p_j + d^{-1} \partial C(\xi_{k_1}^A, \ldots, \xi_{k_{\ell,(A)}}^A) \right). \quad (3.12)
\]

The definitions of the following chains \( R_j^{(k)} \) and \( X_j^{(k)} \) are crucial. Here \( j \in \{1, \ldots, d\} \) and \( k \in \{0, 1, \ldots, \left[\frac{n+1}{2}\right]\} \).

\[
R_j^{(k)} := \sum_{A \text{ thick}, |A|=n-k} \text{sign}(A) \cdot C(A, j) \quad \text{for } k \in \{1, 2, \ldots, \left[\frac{n}{2}\right]\}, (3.13)
\]

\[
R_j^{(\frac{n+1}{2})} := 0 \quad \text{if } n \text{ is odd},
\]

\[
R_j^{(0)} := \Delta_N \circ \{ \varepsilon(p_j) \}. \]

For \( R_j^{(k)} \) with \( k \geq 1 \), the decomposition \( \partial R_j^{(k)} = \partial_1 R_j^{(k)} + \partial_2 R_j^{(k)} \) of its boundary into two parts is well defined. For \( k = 0 \) we define \( \partial_2 R_j^{(0)} := 0 \) and \( \partial_1 R_j^{(0)} := \partial R_j^{(0)} \). The definition of the next chains is inductive. Again \( j \in \{1, \ldots, d\} \).

\[
X_j^{(0)} := \Delta_N \circ \{ \varepsilon(p_j) \} = R_j^{(0)}, \quad (3.14)
\]

\[
X_j^{(k)} := X_j^{(k-1)} - X_j^{(k-1)} + R_j^{(k)} \quad \text{for } k \in \{1, \ldots, \left[\frac{n+1}{2}\right]\}. \quad (3.15)
\]
Theorem 3.3. Again, \( j \in \{1, \ldots, d\} \) and \( k \in \{0, 1, \ldots, \left\lceil \frac{n+1}{2} \right\rceil\} \).

\[
\partial X_j^{(k)} = \partial_1 R_j^{(k)}, \tag{3.16}
\]

and especially
\[
\partial X_j^{\left\lceil \frac{n+1}{2} \right\rceil} = 0. \tag{3.17}
\]

So, the chains \( X_j^{\left\lceil \frac{n+1}{2} \right\rceil} \) are cycles and induce homology classes in \( H_{n-1}(G, \mathbb{Z}) \). (Theorem 4.6 will tell more.) If \( n \) is odd then
\[
\sum_{j=1}^d X_j^{\left\lceil \frac{n+1}{2} \right\rceil} = 0. \tag{3.18}
\]

If \( n \) is even then
\[
\sum_{j=1}^d X_j^{\left\lceil \frac{n}{2} \right\rceil} = (-1)^{\frac{n}{2}+1} \cdot a_1 a_3 \ldots a_{n-1} \cdot \left( \Delta_{A_{od}} \odot T_{A_{od}} + (-1)^{\frac{n}{2}} a_1 a_3 \ldots a_{n-1} \Delta_{A_{ev}} \odot T_{A_{ev}} \right). \tag{3.19}
\]

**Proof:** (3.16) for \( k = 0 \) is \( \partial X_j^{(0)} = \partial_1 R_j^{(0)} \). It is true by definition of \( \partial_1 R_j^{(0)} \). By induction, we obtain for \( k \geq 0 \)
\[
\partial X_j^{(k+1)} = \partial X_j^{(k)} - \partial X_j^{(k+1)} + \partial R_j^{(k+1)}
= \left( \partial_1 R_j^{(k)} - \partial_1 R_j^{(k+1)} + \partial_2 R_j^{(k+1)} \right) + \partial_1 R_j^{(k+1)}.
\]

Therefore we have to show for \( k \geq 0 \)
\[
\partial_1 R_j^{(k)} - \partial_1 R_j^{(k+1)} + \partial_2 R_j^{(k+1)} = 0. \tag{3.20}
\]

In fact, \( R_j^{(k+1)} \) was chosen so that (3.20) holds. In order to show (3.20), first we study \( \partial_1 R_j^{(k)} \).

For \( A = \{k_1, \ldots, k_{|A|}\} \subset N \) with \( k_1 < \ldots < k_{|A|} \) if \( n \notin A \) and with \( k_1 = n \) & \( k_2 < \ldots < k_{|A|} \) if \( n \in A \) and for \( B = \{k_1, \ldots, \hat{k}_j, \ldots, k_{|A|}\} \subset A \) define
\[
\text{sign}(B, A) := (-1)^{j-1}. \tag{3.21}
\]

Then
\[
\partial_1 R_j^{(k)} = \sum_{A \text{ thick}, |A| = n-k} \sum_{B \subset A, |B| = |A|-1} \text{sign}(B, A) \cdot \text{sign}(A) \cdot C(B, A, j)
\text{ for } k \geq 1, \tag{3.22}
\]
\[
\partial_1 R_j^{(0)} = \sum_{B : |B| = n-1} \text{sign}(B, N) \cdot \Delta_B \odot \{e(p_j)\}.
\]
First we consider the cases \( k \geq 1 \). A part of the following arguments will be valid also for the case \( k = 0 \) and will give this case. In general, the subsets \( B \) of thick sets \( A \) with \( |B| = |A| - 1 = n - k - 1 \) are of three different types,

| type I   | type II   | type III  |
|----------|-----------|-----------|
| \( b(B) = b(A) + 1 \) | \( b(B) = b(A) \) | \( b(B) = b(A) - 1 \) |
| \( \Rightarrow B \) thick | \( \Rightarrow B \) almost thick | \( \) |

First, consider a set \( B \) of type III. Then only one thick set \( A \) with \( |A| = n - k \) and \( A \supset B \) exists. It contains a block \( \{k_j^A\} \) which consists of a single number \( k_j^A \), and \( B = A - \{k_j^A\} \). The gap of \( B \) which contains \( k_j^A \) consists of \( k_j^A \) and the gaps of \( A \) left and right of \( k_j^A \), so this gap of the set \( B \) has 3 elements. Therefore the set \( B \) is neither thick nor almost thick. All other gaps of \( B \) consist of a single number. Observe

\[
\text{pr}_B(\overline{\mathcal{L}}_{k_j^A}) = \text{pr}_B(\overline{\mathcal{L}}_{k_{j+1}^A}), \quad \text{thus}
\]

\[
\text{pr}_B(d^{-1}C(\overline{\mathcal{L}}_{k_1^A}, ..., \overline{\mathcal{L}}_{k_{n(A)}^A})) = \text{pr}_B(d^{-1}C(\overline{\mathcal{L}}_{k_1^B}, ..., \overline{\mathcal{L}}_{k_{b(B)}^B})), \quad \text{thus}
\]

\[
\mathcal{C}(B, A, j) = \mathcal{C}(B, j) \subset \Delta_B \ominus T_B, \\
\dim \mathcal{C}(B, A, j) = |B| - 1 + b(B) = |A| - 2 + b(A) - 1 = \dim \mathcal{C}(A, j) - 2 = \dim R_j^{(k)} - 2.
\]

Therefore this part \( \mathcal{C}(B, A, j) \) of the boundary \( \partial_1 R_j^{(k)} \) has too small dimension and can be ignored.

Next, consider a set \( B \) of type II. It is almost thick. It has one gap which consists of two numbers \( k_0 \) and \( (k_0 + 1) \text{ mod } n \). All other gaps consist of a single number. The only two thick sets \( A \) with \( A \supset B \) and \( |A| = n - k \) are \( B^{(1)} := B \cup \{k_0\} \) and \( B^{(2)} := B \cup \{(k_0 + 1) \text{ mod } n\} \). All possibilities for \( k_0 \) except one are easy to treat, namely the cases \( k_0 \in \{1, ..., n\} - \{n - 1\} \). In all these cases

\[
\text{pr}_B(\overline{\mathcal{L}}_{k_i^{B^{(1)}}}) = \text{pr}_B(\overline{\mathcal{L}}_{k_i^{B^{(2)}}}) = \text{pr}_B(\overline{\mathcal{L}}_{k_j^B}) \text{ for any } i \in \{1, ..., b(B)\},
\]

thus \( \mathcal{C}(B, B^{(1)}, j) = \mathcal{C}(B, B^{(2)}, j) = \mathcal{C}(B, j) \).

One checks also

\[
\text{sign}(B, B^{(1)}) = \text{sign}(B, B^{(2)}), \quad \text{sign}(B^{(1)}) = -\text{sign}(B^{(2)}). \quad (3.23)
\]

These observations show that these contributions to \( \partial_1 R_j^{(k)} \) cancel.

The only difficult possibility for \( k_0 \) is the case \( k_0 = n - 1 \). Then

\[
k_j^{B^{(1)}} = 1 < k_j^{B^{(1)}} < ... < k_{b(B)}^{B^{(1)}}, \quad k_j^{B} = k_j^{B}, \quad k_j^{B^{(2)}} < ... < k_{b(B)-1}^{B^{(2)}} < k_b^{B^{(2)}} = n,
\]

\[
\text{pr}_B(\overline{\mathcal{L}}_{k_j^{B^{(1)}}}) = \text{pr}_B(\overline{\mathcal{L}}_{k_j^{B^{(2)}}}) = \text{pr}_B(\overline{\mathcal{L}}_{k_j^B})\text{ for any } i \in \{1, ..., b(B)\},
\]

thus \( \mathcal{C}(B, B^{(1)}, j) = \mathcal{C}(B, B^{(2)}, j) = \mathcal{C}(B, j) \).
\[ \zeta_{kB^{(2)}} = \zeta_0 = (-1)^n d \cdot \bar{b} + \zeta_1, \]
\[ \text{pr}_B(\zeta_{kB^{(2)}}) = ((-1)^n d + 1) \text{pr}_B(\zeta_0), \]
\[ 1 = \text{sign}(B, B^{(2)}) = (-1)^{|B|} \text{sign}(B, B^{(1)}), \]
\[ \text{sign}(B^{(2)}) = (-1)^{n-1} \text{sign}(B^{(1)}), \]
\[ \text{sign}(B, B^{(2)}) \text{sign}(B^{(2)}) = (-1)^{|B|} \text{sign}(B, B^{(1)}) \text{sign}(B^{(1)}). \]

This together with (3.3), formulas for \( \zeta_k \) and \( \zeta_j^B \) and especially \( k_1^B = 1 \) and \( \bar{b}_1 = (0, ..., 0) \) shows

\[
\begin{align*}
\text{pr}_T \text{pr}_B(p_j + d^{-1} C(\zeta_{k_1^B}, ..., \zeta_{k_B^{(2)}})) \\
+ (-1)^{|B|} \text{pr}_T \text{pr}_B(p_j + d^{-1} C(\zeta_{k_1^B}, ..., \zeta_{k_B^{(1)}})) \\
= \text{pr}_T \text{pr}_B(p_j + d^{-1} C(\zeta_{k_1^B}, ..., \zeta_{k_B^{(1)}}), (-1)^n d \zeta_1)) \\
= \text{pr}_T \text{pr}_B(p_j + C(d^{-1} \zeta_{k_1^B}, ..., d^{-1} \zeta_{k_B^{(1)}}, (-1)^n \zeta_1)) \\
\equiv \text{pr}_T \text{pr}_B(p_j + C(d^{-1} \zeta_{k_1^B}, ..., d^{-1} \zeta_{k_B^{(1)}}, (-1)^n \zeta_1)) \\
= (-1)^{|B|} a_1 a_2 \cdots a_B \cdot \text{pr}_T \text{pr}_B(C(d_1^B, d_2^B, ..., d_B^B)). 
\end{align*}
\]

This implies

\[
\begin{align*}
C(B, B^{(2)}, j) + (-1)^{|B|} C(B, B^{(1)}, j) \\
= (-1)^{|B|} a_1 a_2 \cdots a_B \cdot \Delta_B \otimes T_B. 
\end{align*}
\]

This is up to the sign \( \text{sign}(B, B^{(2)}) \cdot \text{sign}(B^{(2)}) \) the contribution of \( B \) to \( \partial_1 R_j^{(k)} \). Especially, it is independent of \( j \). Therefore a set \( B \) of type II makes the same contribution to \( \partial_1 R_j^{(k)} \) and to \( \partial_1 R_{j+1}^{(k)} \). Thus its contribution to the difference \( \partial_1 R_j^{(k)} - \partial_1 R_{j+1}^{(k)} \) is zero.

Finally, consider a set \( B \) of type I. It is thick. Its gaps are the sets \( \{(k_1^B - 1) \mod n\}, \ldots, \{(k_B^{(1)} - 1) \mod n\} \). Exactly \( b(B) \) thick sets \( A \) with \( A \supset B \) and \( |A| = n - k \) exist. They are the sets \( A^{(i)} = B \cup \{(k_i^B - 1) \mod n\} \) for \( i \in \{1, ..., b(B)\} \).
The set of block beginnings of $A^{(i)}$ is the set \( \{k_1^B, ..., k_j^B, ..., k_{b(B)}^B\} \).

Therefore the contribution of $B$ to the boundary $\partial_1 R_j^{(k)}$ is

\[
\sum_{i=1}^{b(B)} \text{sign}(B, A^{(i)}) \cdot \text{sign}(A^{(i)}) \cdot \Delta_B \odot \mathcal{E} \left( \frac{p_j}{B} + d^{-1}C(\xi_{k_1}^B, ..., \xi_{k_j}^B, ..., \xi_{k_{b(B)}}^B) \right),
\]

and the contribution of $B$ to $\partial_1 R_j^{(k+1)}$ looks analogously, with $j$ replaced by $j + 1$. The following calculation of signs will be useful,

\[
\begin{align*}
\text{sign}(B) &= (-1)^{(i-1)+(n-k^B)+(b(B)-i)} \cdot \text{sign}(A^{(i)}) \\
&= (-1)^{k^B+|B|-1} \cdot \text{sign}(A^{(i)}), \\
\text{sign}(B, A^{(i)}) &= (-1)^{(k^B-i)-(i-1)} = (-1)^{k^B-i}, \\
\text{sign}(B, A^{(i)}) \cdot \text{sign}(A^{(i)}) &= \text{sign}(B) \cdot (-1)^{|B|+i-1}.
\end{align*}
\]

On the other side, the boundary $\partial_2 R_j^{(k+1)}$ has only contributions from sets $B$ of type I. The contribution of one such set $B$ is as follows. Here we use \( (3.12), (3.2) \) and \( (3.4) \).

contribution of $B$ to $\partial_2 R_j^{(k+1)}$

\[
\begin{align*}
&\sum_{i=1}^{b(B)} \text{sign}(B) (-1)^{|B|-1} \cdot \Delta_B \odot \mathcal{E} \left( \frac{p_j}{B} + d^{-1}\partial C(\xi_{k_1}^B, ..., \xi_{k_j}^B, ..., \xi_{k_{b(B)}}^B) \right) \\
&\sum_{i=1}^{b(B)} \text{sign}(B) (-1)^{|B|-1} \cdot \left( \sum_{i=1}^{b(B)} (-1)^{i-1} \Delta_B \odot \mathcal{E} \left( \frac{p_j}{B} + d^{-1}C(\xi_{k_1}^B, ..., \xi_{k_j}^B, ..., \xi_{k_{b(B)}}^B) \right) \right) \\
&\sum_{i=1}^{b(B)} (-1)^{i-1} \Delta_B \odot \mathcal{E} \left( \frac{p_j}{B} + d^{-1}C(\xi_{k_1}^B, ..., \xi_{k_j}^B, ..., \xi_{k_{b(B)}}^B) \right) \\
&\sum_{i=1}^{b(B)} \text{sign}(B) (-1)^{|B|+i} \Delta_B \odot \mathcal{E} \left( \frac{p_j}{B} + d^{-1}C(\xi_{k_1}^B, ..., \xi_{k_j}^B, ..., \xi_{k_{b(B)}}^B) \right) \\
&- \sum_{i=1}^{b(B)} \text{sign}(B) (-1)^{|B|+i} \Delta_B \odot \mathcal{E} \left( \frac{p_{j+1}}{B} + d^{-1}C(\xi_{k_1}^B, ..., \xi_{k_j}^B, ..., \xi_{k_{b(B)}}^B) \right) \\
&= -(\text{contribution of } B \text{ to } \partial_1 R_j^{(k)}) \\
&+ (\text{contribution of } B \text{ to } \partial_1 R_j^{(k+1)}).
\end{align*}
\]

Therefore the contribution of $B$ to $\partial_1 R_j^{(k)} - \partial_1 R_{j+1}^{(k)} + \partial_2 R_j^{(k+1)}$ is zero.
So, no set $B \subset N$ with $|B| = n - k - 1$ gives a contribution to this sum. Therefore this sum is zero. (3.20) is proved for $k \geq 1$.

Now we will show (3.20) for $k = 0$. $\partial_1 R_j^{(0)}$, $\partial_1 R_{j+1}^{(0)}$ and $\partial_2 R_j^{(1)}$ are sums over sets $B \subset N$ with $|B| = n - 1$. These sets are the sets $B^{(i)} := N - \{i\}$ for $i \in N$. The set $B^{(i)}$ is thick with $b(B^{(i)}) = 1$. It can be treated as the sets of type I above. The same calculations as above for sets of type I show that the contribution of $B^{(i)}$ to the sum $\partial_1 R_j^{(0)} - \partial_1 R_{j+1}^{(0)} + \partial_2 R_j^{(1)}$ is zero. Therefore this sum is zero. Therefore (3.20) holds also for $k = 0$, so it holds for all $k \geq 0$.

This implies (3.16) for $k \geq 1$. It holds for $k = 0$ anyway.

It remains to prove (3.17), (3.18) and (3.19). (3.17) is because of (3.16) equivalent to $\partial_1 R^{(n+1)/2}_j = 0$. In the case $n$ odd this is trivial as then $R^{(n+1)/2}_j = 0$ by definition. In the case $n$ even, we have to show $\partial_1 R^{(n+1)/2}_j = 0$. By Lemma 3.2 there are only two thick sets $A$ with $|A| = \frac{n}{2}$, the sets $A_{od} = \{1, 3, \ldots, n-1\}$ and $A_{ev} = \{2, 4, \ldots, n\}$. In the discussion above of $\partial_1 R_j^{(k)}$ for $k \geq 1$, we distinguished 3 different types of sets $B$ with $|B| = |A| - 1 = n - k - 1$ and $B \subset A$. In the cases $A \in \{A_{od}, A_{ev}\}$, we have only sets $B$ of type III. Above we stated that the part $C(B, A, j)$ of $\partial_1 R_j^{(k)}$ (here for $k = \frac{n}{2}$) has too small dimension and can be ignored. Therefore $\partial_1 R^{(n+1)/2}_j = 0$. We proved (3.17).

(3.18) holds for odd $n$ and is an immediate consequence of (3.15) and $R^{(n+1)/2}_j = 0$.

(3.19) holds for even $n$. It requires two calculations which are similar to the one which led to (3.25). The details are as follows. (3.15) and (3.13) show

$$\sum_{j=1}^{d} X^{(n+1)/2}_j = \sum_{j=1}^{d} R^{(n+1)/2}_j$$

$$= \sum_{j=1}^{d} \text{sign}(A_{od}) \cdot C(A_{od}, j) + \sum_{j=1}^{d} \text{sign}(A_{ev}) \cdot C(A_{ev}, j)$$

$$= \sum_{j=1}^{d} \text{sign}(A_{od}) \cdot \Delta_{A_{od}} \odot \xi \left( p_j + d^{-1}C(\zeta_1, \zeta_3, \ldots, \zeta_{n-1}) \right)$$

$$+ \sum_{j=1}^{d} \text{sign}(A_{ev}) \cdot \Delta_{A_{ev}} \odot \xi \left( p_j + d^{-1}C(\zeta_2, \zeta_4, \ldots, \zeta_n) \right).$$
Here
\[
\text{sign}(A_{\text{od}}) = (-1)^{\frac{n(n+1)}{2}},
\]
\[
\text{sign}(A_{\text{ev}}) = (-1)^{\frac{n(n-1)}{2}} = (-1)^{\frac{n}{2}} \cdot \text{sign}(A_{\text{od}}).
\]
Because of \( p_{j+1} = p_j + d^{-1}c_1 \) and (3.3) (and \((-1)^n = 1\) as \(n\) is even),
we have
\[
\sum_{j=1}^{d} \text{pr}_T(p_j + d^{-1}C(c_1, c_3, \ldots, c_{n-1})) = \text{pr}_T(p_1 + C(c_1, d^{-1}c_3, \ldots, d^{-1}c_{n-1})).
\]
\[
\overset{(3.3)}{=} \text{pr}_T(p_1 + C(c_1, d^{-1}c_3 - d^{-1}c_1, \ldots, d^{-1}c_{n-1} - d^{-1}c_1))
\]
\[
= \text{pr}_T(p_1 + C(c_1, (-1)^n b_3, \ldots, (-1)^n b_{n-1}))
\]
\[
= (-1)^{n/2-1} \text{pr}_T(C(b_3, \ldots, b_{n-1}, c_1))
\]
If we identify \( \text{pr}_{A_{\text{od}}}(b_j) \) with a column vector in \( M_{n/2 \times 1}(\mathbb{Z}) \) then the tuple \((\text{pr}_{A_{\text{od}}}(b_3), \ldots, \text{pr}_{A_{\text{od}}}(b_{n-1}), \text{pr}_{A_{\text{od}}}(c_1))\) is an upper triangular \( \frac{n}{2} \times \frac{n}{2} \)-matrix, and its determinant is \(a_1 a_3 \ldots a_{n-1}.\) Therefore
\[
\sum_{j=1}^{d} \Delta_{A_{\text{od}}} \odot e(p_j + d^{-1}C(c_1, c_3, \ldots, c_{n-1})) = (-1)^{n/2-1} a_1 a_3 \ldots a_{n-1} \cdot \Delta_{A_{\text{od}}} \odot T_{A_{\text{od}}}.
\]
Observe
\[
\text{pr}_{A_{\text{ev}}}(c_{2l}) = \text{pr}_{A_{\text{ev}}}(c_{2l-1}).
\]
Therefore
\[
\sum_{j=1}^{d} \text{pr}_{A_{\text{ev}}} \text{pr}_T(p_j + d^{-1}C(c_1, c_3, \ldots, c_{n})) = \sum_{j=1}^{d} \text{pr}_{A_{\text{ev}}} \text{pr}_T(p_j + d^{-1}C(c_1, c_3, \ldots, c_{n-1}))
\]
\[
= (-1)^{n/2-1} \text{pr}_{A_{\text{ev}}} \text{pr}_T(C(b_3, \ldots, b_{n-1}, c_1)).
\]
If we identify \( \text{pr}_{A_{\text{ev}}}(b_j) \) with a column vector in \( M_{n/2 \times 1}(\mathbb{Z}) \) then the tuple \((\text{pr}_{A_{\text{ev}}}(b_3), \ldots, \text{pr}_{A_{\text{ev}}}(b_{n-1}), \text{pr}_{A_{\text{ev}}}(c_1))\) is an upper triangular \( \frac{n}{2} \times \frac{n}{2} \)-matrix, and its determinant is
\[
a_2 a_4 \ldots a_n = a_1 a_3 \ldots a_{n-1} \cdot a_1 a_3 \ldots a_{n-1}.
\]
Therefore
\[
\sum_{j=1}^{d} \Delta_{A_{ev}} \otimes \Delta_{\left(p_{j} + d^{-1} C(c_{2}, c_{4}, \ldots, c_{n})\right)}
= (-1)^{n/2 - 1} \frac{a_{1}a_{3}...a_{n-1}}{a_{1}a_{3}...a_{n-1}} \cdot \Delta_{A_{ev}} \otimes T_{A_{ev}}.
\]
(3.19) follows. \hfill \Box

4. A SPECTRAL SEQUENCE, FOLLOWING AND CORRECTING

Cooper

Throughout this section we fix a cycle type singularity \( f(x_1, \ldots, x_n) \) as in (1.2) with \( n \geq 2 \) and \( a_1, \ldots, a_n \in \mathbb{N} \) with (1.3), as in section 3. Cooper [Co82] considered a filtration of the set \( G \subset \mathbb{F}_{0} = f^{-1}(\mathbb{R}_{\geq 0}) \cap S^{2n-1} \subset \mathbb{S}^{2n-1} \subset \mathbb{C}^{n} \), which was defined in (3.1), by a sequence of subsets,

\[ G = G_{n} \supset G_{n-1} \supset \ldots \supset G_{1} \supset G_{0} = \emptyset. \]

He studied a spectral sequence \( (E_{r,s}^{s,t}, d_{r,s}^{s,t})_{r \geq 1, s \geq 1, s+t \geq 0} \) which is associated to this filtration. It allows to determine \( H_{n-1}(G, \mathbb{Z}) \). Though he made two serious mistakes. Here we will partly follow his line of thoughts, but correct the mistakes. Remark 5.1 will explain the differences to [Co82].

Cooper does not give a reference for the construction of the spectral sequence. We found a manuscript of Hatcher [Ha04] very useful and will cite from it.

First we will define the subspaces \( G_{s} \subset G \) and make some elementary observations. Then we will introduce the spectral sequence, following Hatcher. Then we will study it in detail (following partly Cooper) in Lemma 4.3 and Theorem 4.5. Theorem 4.6 will give the conclusion for \( H_{n-1}(G, \mathbb{Z}) \). It complments Theorem 3.3.

**Definition 4.1.** [Co82] Recall the definition (3.1) of the set \( G \subset \mathbb{C}^{n} \). Define \( G_{n} := G \), \( G_{0} := \emptyset \), and define for \( s \in \{1, \ldots, n-1\} \)

\[
G_{s} := \{ z \in G \mid \text{at most } s \text{ of the coordinates } z_{1}, \ldots, z_{n} \text{ are } \neq 0 \}
= \bigcup_{A \subset \mathbb{N} : |A| \leq s} \Delta_{A} \otimes T_{A}.
\] (4.1)
Here the second equality follows from Lemma 3.2. Lemma 3.2 gives also the differences $G_s - G_{s-1}$,

$$G_n - G_{n-1} = \bigcup_{j=1}^{d} \text{Int}(\Delta_N) \odot \{e(p_j)\},$$

$$G_s - G_{s-1} = \bigcup_{A \subset N : |A| = s} \text{Int}(\Delta_A) \odot T_A \text{ for } s \in \{1, \ldots, n-1\}.$$ (4.3)

In this paper, all considered homology groups will have coefficients in $\mathbb{Z}$. Especially, we consider for $(s, t) \in N \times \mathbb{Z}$ the homology groups of the spaces $G_s$ and of the pairs $(G_s, G_{s-1})$,

$$A^1_{s,t} := H_{s+t}(G_s) := H_{s+t}(G_s, \mathbb{Z}),$$

$$E^1_{s,t} := H_{s+t}(G_s, G_{s-1}) := H_{s+t}(G_s, G_{s-1}, \mathbb{Z}).$$ (4.4) (4.5)

We extend this definition by

$$A^1_{s,t} = E^1_{s,t} = 0 \text{ for } s \leq 0,$$

$$A^1_{s,t} = H_{s+t}(G), \ E^1_{s,t} = 0 \text{ for } s \geq n + 1.$$ (4.6)

Of course

$$A^1_{s,t} = E^1_{s,t} = 0 \text{ for } s + t < 0 \text{ or } s + t \geq n,$$ (4.7)

as $\dim \mathbb{R} G_s \leq n-1$. Cooper [Co82] observed that Lemma 3.2 (especially that the map $\text{Int}(\Delta_A) \times T_A \rightarrow \text{Int}(\Delta_A) \odot T_A$ is a diffeomorphism) and (4.3) imply

$$H_*(G_n, G_{n-1}) = \bigoplus_{j=1}^{d} H_{n-1}(\Delta_N \odot \{e(p_j)\}, \partial \Delta_N \odot \{e(p_j)\}),$$

$$H_*(G_s, G_{s-1}) = \bigoplus_{A \subset N : |A| = s} H_*(\Delta_A \odot T_A, \partial \Delta_A \odot T_A)$$

for $s \leq n - 1.$
And here

\[ H_{n-1}(\Delta_N \cap \{c(p_j)\}, \partial \Delta_N \cap \{c(p_j)\}) \cong \mathbb{Z} \]

with generator the class \([X^0_j]\),  \hspace{1cm} (4.9)

\[ H_s(\Delta_A \cap T_A, \partial \Delta_A \cap T_A) \cong H_{|A|-1}(\Delta_A, \partial \Delta_A) \otimes H_s(T_A), \]

\hspace{1cm} (4.10)

\[ H_{|A|-1}(\Delta_A, \partial \Delta_A) \cong \mathbb{Z} \text{ with generator the class } [\Delta_A], \]

\hspace{1cm} (4.11)

\[ H_s(T_A) \cong \bigotimes_{j=1}^{b(A)} H_s(S^1), \quad H_s(S^1) = H_0(S^1) \oplus H_1(S^1), \]

\hspace{1cm} (4.12)

\[ H_0(S^1) \cong \mathbb{Z} \text{ and } H_1(S^1) \cong \mathbb{Z} \]

with generators \([p]\) and \([S^1]\) for any \(p \in S^1\).  \hspace{1cm} (4.13)

By (4.7) and (4.9) the group

\[ E^1_{n-1} = H_{n-1}(G_n, G_{n-1}) = H_s(G_n, G_{n-1}) = \bigoplus_{j=1}^{d} \mathbb{Z} \cdot [X^0_j] \]

(4.14)

is a \(\mathbb{Z}\)-lattice of rank \(d\) with generators the classes \([X^0_1], \ldots, [X^0_d]\). All the groups \(E^1_{s,t}\) are \(\mathbb{Z}\)-lattices (= finitely generated free \(\mathbb{Z}\)-modules).

The long exact homology sequence for the pair \((G_s, G_{s-1})\) for \(s \in N\) reads as follows,

\[
\cdots A^1_{n-1,t+1} \xrightarrow{j^1} A^1_{s,t} \xrightarrow{j^1} E^1_{s,t} \xrightarrow{k^1} A^1_{s-1,t} \xrightarrow{j^1} A^1_{s,t-1} \xrightarrow{k^1} E^1_{s,t-1} \xrightarrow{k^1} A^1_{s-1,t-1} \cdots
\]

Here \(j^1\) is induced by the embedding \(G_{s-1} \hookrightarrow G_s\); \(j^1\) is the natural map from absolute homology to relative homology, and \(k^1\) is the boundary map. Together, these exact homology sequences can be put into a large diagram which Hatcher [Ha04] calls a *staircase diagram*, because the long exact sequences look like staircases in this diagram:

\[
\begin{align*}
A^1_{s-1,t+1} & \xrightarrow{j^1} E^1_{s-1,t+1} \xrightarrow{k^1} A^1_{s-2,t+1} \xrightarrow{j^1} E^1_{s-2,t+1} \xrightarrow{k^1} A^1_{s-3,t+1} \\
A^1_{s,t} & \xrightarrow{j^1} E^1_{s,t} \xrightarrow{k^1} A^1_{s-1,t} \xrightarrow{j^1} E^1_{s-1,t} \xrightarrow{k^1} A^1_{s-2,t} \\
A^1_{s+1,t-1} & \xrightarrow{j^1} E^1_{s+1,t-1} \xrightarrow{k^1} A^1_{s+1,t-1} \xrightarrow{j^1} E^1_{s+1,t-1} \xrightarrow{k^1} A^1_{s+1,t-1}
\end{align*}
\]

The map

\[ d^1_{s,t} := d^1 := j^1 \circ k^1 : E^1_{s,t} \to E^1_{s-1,t} \]

is the cellular boundary map [Ha04]. The system \((E^1_{s,t}, d^1_{s,t})_{s,t \in \mathbb{Z}}\) of spaces and maps is the first page of a spectral sequence which is associated to the filtration \(G = G_n \supseteq G_{n-1} \supseteq \ldots \supseteq G_1 \supseteq G_0 = \emptyset\). The other
(ii) in it. The indices in general properties. It follows from Lemma 5.1 and Proposition 5.2 in \(\text{[Ha04]}\). Proposition 5.2 in \(\text{[Ha04]}\) applies with the conditions (i) and (ii) in it. The indices in \(A_{s,t}^r\) and \(E_{s,t}^1\) here are chosen differently (more standard) from those in \(\text{[Ha04]}\).

**Theorem 4.2.** \(\text{[Ha04]}\), Lemma 5.1 and Proposition 5.2

(a) (Properties) Fix \(r \geq 2\). The tuple \((A^r_{s,t}, E^r_{s,t}, i^r, j^r, k^r, d^r)\) will be constructed from the tuple \((A^{r-1}_{s,t}, E^{r-1}_{s,t}, i^{r-1}, j^{r-1}, k^{r-1}, d^{r-1})\) in part (c). It has the following properties:

\[
\begin{align*}
i^r &: A^r_{s,t} \to A^r_{s+1,t-1}, \\
\quad & \text{(4.16)} \\
\quad j^r &: A^r_{s,t} \to E^r_{s-r+1,t+r-1}, \\
\quad & \text{(4.17)} \\
k^r &: E^r_{s,t} \to A^r_{s-1,t}, \\
\quad & \text{(4.18)} \\
d^r &: E^r_{s,t} \to E^r_{s-r,t+r-1}, \\
\quad & \text{d}^r \circ d^r = 0. \quad \text{(4.19)} \\
\end{align*}
\]

The restriction of \(i^r\) to \(A^r_{s,t}\) is also called \(i^r_{s,t}\), and similarly for \(j^r, k^r\) and \(d^r\). There are long exact sequences,

\[
\cdots A^r_{s+r-2,t-r+2} \overset{i^r}{\to} A^r_{s+r-1,t-r+1} \overset{j^r}{\to} E^r_{s,t} \overset{k^r}{\to} A^r_{s-1,t} \overset{i^r}{\to} A^r_{s,t-1} \overset{j^r}{\to} E^r_{s-r+1,t+r-2} \overset{k^r}{\to} A^r_{s-r,t+r-2} \cdots
\]

They can be put together into the following staircase diagram (the positions of the arrows for \(j^r\) are not precise).

![Staircase Diagram](image)

(b) (Spectral sequence) The part \((E^r_{s,t}, d^r_{s,t})_{r \geq 1, s,t \in \mathbb{Z}}\) of the tuples above for all \(r \geq 1\) is a spectral sequence. It converges to \(H_s(G)\). This imprecise statement means the following. For each \((s,t) \in \mathbb{Z}^2\), a bound \(r(s,t) \in \mathbb{N}\) exists such that the space \(E^\infty_{s,t} := E^{r(s,t)}_{s,t}\) coincides with \(E^r_{s,t}\) for any \(r \geq r(s,t)\), and

\[
E^\infty_{s,t} \cong \frac{F^s_{s+t}}{F^{s-1}_{s+t}}, \quad \text{where} \\
F^s_{s+t} := \text{Im}(A^1_{s,t} \to A^1_{n,s+t-n}) = \text{Im}(H_{s+t}(G_s) \to H_{s+t}(G)). \quad \text{(4.20)}
\]

\(\text{[Ha04]}\).
(c) (Construction) Fix \( r \geq 2 \). Suppose that the tuple \((A_{s,t}^{-1}, E_{s,t}^{-1}, r^{-1}, j^{-1}, k^{-1}, d^{-1})\) has been constructed. Then especially \( d^{-1} \circ d^{-1} = 0 \). Therefore the quotient

\[ E_{s,t}^r := \ker(d_{s,t}^{-1}) / \operatorname{Im}(d_{s+r-1,t-r+2}^{-1}) \]  

is well-defined. It is a subquotient of \( E_{s,t}^{-1} \). The space

\[ A_{s,t}^r := i^{-1}(A_{s-1,t+1}^{-1}) \subset A_{s,t}^{-1} \]  

is well-defined, anyway. It is a subspace of \( A_{s,t}^{-1} \). The map \( i^r \) is the restriction of \( i^{-1} \) to \( A_{s,t}^r \). The map \( j^r \) on \( A_{s,t}^r = i^{-1}(A_{s-1,t+1}^{-1}) \) is defined on \( i^{-1}(a) \) for \( a \in A_{s-1,t+1}^{-1} \) by \( j^r(i^{-1}(a)) := [j^{-1}(a)] \). A priori, this is in \( E_{s-r+1,t+r-1}^r \) but it turns out to be in \( E_{s-r+1,t+r-1}^r \). The map \( k^r \) on \( E_{s,t}^r \) is defined on \( [a] \) for \( a \in \ker(d_{s,t}^{-1}) \subset E_{s,t}^{-1} \) by \( k^r([a]) := k^{-1}(a) \) which is a priori in \( A_{s-1,t}^{-1} \). But it turns out to be in \( A_{s-1,t}^r \) and to be well-defined.

Theorem 4.2 gives basic and general properties, which are not specific to the geometry of \( G \) and its subspaces \( G_s \) (except for our definition of \( F_{s,t} \) where we used \( A_{n,n+t-n}^1 = H_{s+t}(G) \)). Now we will use the geometry of \( G \) and its subspaces \( G_s \) to make more specific statements. We are interested especially in \( E_{s,n-1-s}^\infty \) as that is the subquotient \( F_{n-1}^s / F_{n-1}^{s-1} \) of \( H_{n-1}(G) \) by part (b). Lemma 4.3 gives first elementary observations. Algebraic statements in Lemma 4.4 make a part of Theorem 4.5 more transparent. Theorem 4.5 states the main results on the spectral sequence. Theorem 4.6 gives the conclusion for \( H_{n-1}(G) \). It complements Theorem 3.3.

**Lemma 4.3.** (a) For \( r \geq 2 \)

\[ E_{s,n-1-s}^r = \ker(d_{s,n-1-s}^{-1}) \subset E_{s,n-1-s}^{-1} \subset \ldots \subset E_{s,n-1-s}^1. \]  

(b) For \( r \geq 1 \) and for \( s < [\frac{n+1}{2}] \)

\[ E_{s,n-1-s}^r = E_{s,n-1-s}^\infty = 0 \]  

and thus \( F_{n-1}^s = 0 \).

(c) Consider even \( n \). For \( r \geq 1 \)

\[ F_{n-1}^{n/2} = \operatorname{Im}((H_{n-1}(G_{\frac{n}{2}}) \to H_{n-1}(G)) \cong E_{\frac{n}{2}, \frac{n}{2}-1}^\infty \cong E_{\frac{n}{2}, \frac{n}{2}-1}^r = E_{\frac{n}{2}, \frac{n}{2}-1}^r = H_{n-1}(G_{\frac{n}{2}}, G_{\frac{n}{2}-1}) \cong \mathbb{Z}^2. \]  

The generators are the classes in \( H_{n-1}(G) \) respectively in \( H_{n-1}(G_{\frac{n}{2}}, G_{\frac{n}{2}-1}) \) of the spheres \( \Delta_{A_{od}} \circ T_{A_{od}} \subset \mathbb{C}^{A_{od}} \) and \( \Delta_{A_{ev}} \circ T_{A_{ev}} \subset \mathbb{C}^{A_{ev}} \) (see Lemma 3.2).
Proof: (a) Because of (4.6) $E_{s+1}^{-r} = 0$ for $s + t \geq n$. Therefore $d_{s+n-1-s-r+2}^{-r} = 0$, and by (4.21) $E_{s,n-1-s}^{-r} = \ker(d_{s,n-1-s}^{-r})$.

(b) For $s \leq n-1$, the set $E_{s,n-1-s} = H_{n-1}(G,s-1)$ has by (4.8), (4.10) and (4.12) only contributions from sets $A \subset N$ with $b(A) = n - |A|$, so from thick sets $A$. By Lemma 3.2, if $s < \lfloor \frac{n+1}{2} \rfloor$, there are no thick sets $A$ with $|A| = s$. Therefore then $E_{s,n-1-s}^{-r} = 0$ and $E_{s,n-1-s}^{-r} = 0$ for $r \geq 1$ and $E_{s,n-1-s}^{-r} = 0$. This and (4.20) imply inductively $F_{s,n-1}^{-r} = 0$ for $s < \lfloor \frac{n+1}{2} \rfloor$.

(c) The first equality is the definition of $E_{n-1}^{-r}$. The isomorphism $F_{n-1}^{-r} \cong E_{n-1}^{-r}$ follows with (4.20) and $F_{n-1}^{-r} = 0$ for $s < \frac{n}{2}$.

The equality $E_{n-1}^{-r} = H_{n-1}(G \frac{s}{2}, G \frac{s}{2} - 1)$ is the definition of $E_{n-1}^{-r}$. By (4.8), (4.10) and (4.12) it has a contribution isomorphic to $\mathbb{Z}$ with generator $[\Delta_A \circ T_A]$ for any thick set $A$ with $|A| = \frac{s}{2}$. By Lemma 3.2 these are only the sets $A_{od}$ and $A_{ev}$. Therefore $E_{n-1}^{-r} \cong \mathbb{Z}^2$. As the spheres $\Delta_{A_{od}} \circ T_{A_{od}}$ and $\Delta_{A_{ev}} \circ T_{A_{ev}}$ have no boundary, the boundary map

$$k_{n-1}^{1} : E_{n-1}^{-r} \to A_{n-1}^{-1} = H_{n-2}(G \frac{n}{2} - 1)$$

and also the induced maps $E_{n-1}^{-r}$ and the maps $d_{n-1}^{-r}$ are zero maps. Therefore for $r \geq 1$

$$E_{n-1}^{-r} = E_{n-1}^{r} \cong E_{n-1}^{-r}.$$

Part (c) is proved. \[ \square \]

The sole purpose of the following elementary algebraic lemma is to make Theorem 4.5 (d) more transparent.

**Lemma 4.4.** Recall from (4.13) that the group $E_{n-1} = H_{n-1}(G,G_{n-1})$ is a $\mathbb{Z}$-lattice of rank $d$ with generators the classes $[X_{1}^{(0)}], ..., [X_{d}^{(0)}]$. Define for $r \in \{1, \ldots, \lfloor \frac{n+3}{2} \rfloor \}$ the sublattice

$$E_{n-1}^{-r} := \sum_{j=1}^{d} \mathbb{Z} \cdot [X_{j}^{(r)}] \subset E_{n-1}^{-r}, \quad (4.26)$$

which is generated by the classes $[X_{1}^{(r-1)}], ..., [X_{d}^{(r-1)}]$ in $H_{n-1}(G,G_{n-1})$. Then $E_{n-1}^{-r} = E_{n-1}^{-r}$. For $r \geq 2$

$$[X_{j}^{(r-1)}] = [X_{j}^{(r-2)}] - [X_{j+1}^{(r-2)}] \mod d \quad \text{for } j \in \{1, \ldots, d\}, \quad (4.27)$$

$$\sum_{j=1}^{d} [X_{j}^{(r-1)}] = 0, \quad (4.28)$$
and $\tilde{E}_{n,-1}^r \subset E_{n,-1}^r$ is a sublattice of rank $d - 1$ which is generated by any $d - 1$ of the $d$ elements $[X_1^{(r-1)}], ..., [X_d^{(r-1)}]$. The lattice $\tilde{E}_{n,-1}^2$ is a primitive sublattice of $E_{n,-1}^1$. For $r \geq 3$, $\tilde{E}_{n,-1}^r$ has index $d$ in $\tilde{E}_{n,-1}^{r-1}$.

Proof: (4.27) follows from the definition of $X_j^{(r-1)}$ in (3.15) and from $R_j^{(r-1)} \subset G_{n,-1}$. (4.28) is an immediate consequence of (4.27). $\tilde{E}_{n,-1}^r$ is obviously a primitive sublattice of $E_{n,-1}^1$ of rank $d - 1$. For $r \geq 3$, $\tilde{E}_{n,-1}^r$ has index $d$ in $\tilde{E}_{n,-1}^{r-1}$ because

$$\tilde{E}_{n,-1}^r = \left\{ \sum_{j=1}^d z_j \cdot [X_j^{(r-2)}] \mid z_j \in \mathbb{Z}, \sum_{j=1}^d z_j \in d\mathbb{Z} \right\}$$

(4.29)

by (4.27) and (4.28) for $[X_j^{(r-2)}]$. □

Theorem 4.5. (a) [Co82 18.] For $s \in \{[n+2], [n+4], ..., n-1\}$

$$d_{s,n-1-s}^1 : E_{s,n-1-s}^1 \to E_{s-1,n-1-s}^1$$

is injective, (4.30)

$$E_{s,n-1-s}^r = E_{s,n-1-s}^\infty$$

for $r \geq 2$. (4.31)

(b) For $t \in \{3, ..., [n+3]\}$ and $3 \leq r \leq t$

$$E_{n-t+1,t-3}^{r-1} \subset \frac{E_{n-t+1,t-3}^1}{d^1(E_{n-t+2,t-3}^1)}.$$ (4.32)

(c) For $s < [n+2]$ and $r \geq 1$

$$E_{s-1,n-1-s}^r = 0.$$ (4.33)

(d) Recall the definition of $\tilde{E}_{n,-1}^r$ in Lemma 4.4

$$E_{n,-1}^r = \tilde{E}_{n,-1}^r \text{ for } r \in \{1, ..., \left\lfloor \frac{n+3}{2} \right\rfloor\},$$

$$E_{n,-1}^\infty = \tilde{E}_{n,-1}^{(n+3)/2} \text{ for } r \geq \left\lfloor \frac{n+3}{2} \right\rfloor.$$ (4.34)

Proof: (a) (4.31) is an immediate consequence of (4.30), because of the definition of $E_{s,n-1-s}^r$ in (4.21).

The injectivity of $d_{s,n-1-s}^1$ was proved in [Co82 18]. Our proof differs in a way which allows to apply it also to the proof of (4.34) in part (d). First we consider some useful sets and maps. Fix $s \in \{[n+2], [n+4], ..., n-1\}$.

$$\mathcal{A}_1(s) := \{ A \subset N \mid A \text{ thick, } |A| = s \},$$

$$\mathcal{A}_2(s) := \{(A, j) \mid A \in \mathcal{A}_1(s), j \in \{1, ..., b(A)\} \}$$

with $k_j^A, (k_j^A + 1) \mod n \in A$,

$$\text{pr}_1 : \mathcal{A}_2(s) \to \mathcal{A}_1(s), (A, j) \mapsto A,$$

the canonical projection.
For the given $s$, each $A \in A_1(s)$ contains a block which consists of $\geq 2$ elements. Therefore $\text{pr}_1 : A_2(s) \to A_1(s)$ is surjective.

$$\begin{align*}
B(s) & := \{ B \subset N \mid B \text{ almost thick, } |B| = s - 1 \}, \\
\beta_1 & : B(s) \to N, \beta_1(B) := k_0 \text{ if } \{ k_0, (k_0 + 1)_{\text{mod } n} \} \subset N - B, \\
\beta_2 & : B(s) \to N, \beta_2(B) := (\beta_1(B) + 1)_{\text{mod } n}, \\
\alpha_1 & : A_2(s) \to B(s), (A, j) \mapsto A - \{(k_{(j+1)_{\text{mod } d}} - \text{2}_{\text{mod } n}\}\}, \\
\alpha_2 & : A_2(s) \to B(s), (A, j) \mapsto A - \{k_j^A\}.
\end{align*}$$

$\beta_1(B)$ and $\beta_2(B)$ are the first and last element of the unique block of $N - B$ with two elements. $k_j^A$ is the beginning of a block of $A$ with $\geq 2$ elements, and $\{(k_{(j+1)_{\text{mod } d}} - \text{2}_{\text{mod } n}\}\}$ is the last element of this block. $\alpha_1$ and $\alpha_2$ are bijections, and (see the notations $3.1$ for $B(1)$ and $B(2)$)

$$\begin{align*}
B(1) &= B \cup \{ \beta_1(B) \} = \text{pr}_1(\alpha_1^{-1}(B)), \\
B(2) &= B \cup \{ \beta_2(B) \} = \text{pr}_1(\alpha_2^{-1}(B)).
\end{align*}$$

Recall what (4.8), (4.10) and (4.12) say about $E_{s,n-1-s}^1$ and $E_{s-1,n-1-s}^1$. Both are $\mathbb{Z}$-lattices. The generators of $E_{s,n-1-s}^1$ are simply the classes $[\Delta_A \odot T_A], A \in A_1(s)$,

$$E_{s,n-1-s}^1 = H_{n-1}(G_s, G_{s-1}) = \bigoplus_{A \in A_1(s)} H_{n-1}((\Delta_A \odot T_A), \partial \Delta_A \odot T_A) = \bigoplus_{A \in A_1(s)} \mathbb{Z} \cdot [\Delta_A \odot T_A],$$

$E_{s-1,n-1-s}^1$ splits into two parts, one from almost thick sets, the other from thick sets,

$$\begin{align*}
E_{s-1,n-1-s}^1 &= H_{n-2}(G_{s-1}, G_{s-2}) \\
&= (H_{n-2}(G_{s-1}, G_{s-2}))_B \oplus (H_{n-2}(G_{s-1}, G_{s-2}))_A \text{ with} \\
(H_{n-2}(G_{s-1}, G_{s-2}))_B &:= \bigoplus_{B \in B(s)} \mathbb{Z} \cdot [\Delta_B \odot T_B], \\
(H_{n-2}(G_{s-1}, G_{s-2}))_A &:= \bigoplus_{A \in A_1(s-1)} H_{s-1}(\Delta_A, \partial \Delta_A) \otimes H_{b(A)-1}(T_A) \\
&= \bigoplus_{A \in A_1(s-1), i=1} \bigoplus_{b(A)} \mathbb{Z} \cdot [\Delta_A \odot T_A \cup \{(k_i^{A-1})_{\text{mod } n}\}]\).
\end{align*}$$

We will work mainly with the part from almost thick sets. The projection to this part is called $\text{pr}_B$,

$$\text{pr}_B : E_{s-1,n-1-s}^1 = H_{n-2}(G_{s-1}, G_{s-2}) \to (H_{n-2}(G_{s-1}, G_{s-2}))_B.$$
We will prove that \( \text{pr}_B \circ d_{s,n-1-s}^1 \) is injective. This implies that \( d_{s,n-1-s}^1 \) is injective. Recall \( d_{s,n-1-s}^1 = j_{s-1,n-1-s}^1 \circ k_{s,n-1-s}^1 \), and \( k_{s,n-1-s}^1 \) is a boundary map. Therefore for \( A \in \mathcal{A}_1(s) \)

\[
\begin{align*}
d_{s,n-1-s}^1([\Delta_A \circ T_A]) &= [\partial \Delta_A \circ T_A] \\
&= \sum_{B \subset A : B \in B(s) \cup \mathcal{A}_1(s-1)} \text{sign}(B, A) \cdot [\Delta_B \circ T_A],
\end{align*}
\]

(4.38)

We care only about the terms for \( B \subset A \) with \( B \in B(s) \). These sets \( B \) split into two types,

\[
\{ B \subset A \mid B \in B(s) \} = \{ B \subset A \mid A = B^{(1)} \} \cup \{ B \subset A \mid A = B^{(2)} \}.
\]

(4.39)

We claim to have on the level of chains

\[
\Delta_B \circ T_A = \begin{cases} 
\Delta_B \circ T_B & \text{if } A = B^{(1)}, \\
(-a_{\beta_2(B)}) \cdot \Delta_B \circ T_B & \text{if } A = B^{(2)}, \beta_2(B) \neq n, \\
(-1)^{(B)-1}( -a_n) \Delta_B \circ T_B & \text{if } A = B^{(2)}, \beta_2(B) = n.
\end{cases}
\]

(4.40)

This follows from comparison of \( T_A \) with \( T_B \). If \( B = \alpha_1(A, j) \) then \( \beta_1(B) = (k_{(j+1) \mod n} - 2) \mod n \) is the last element of the \( j \)-th block of \( A \). If \( B = \alpha_2(A, j) \) then \( \beta_2(B) = k_j^A \) is the beginning of the \( j \)-th block of \( A \). The generating vectors of \( T_A \) and \( T_B \) from these blocks of \( A \) and \( B \) differ as follows,

\[
\begin{align*}
\text{pr}_B(d_j^A) &= \text{pr}_B(d_j^B) & \text{if } B = \alpha_1(A, j), \\
\text{pr}_B(d_j^A) &= (-a_{\beta_2(B)}) \cdot \text{pr}_B(d_j^B) & \text{if } B = \alpha_2(A, j),
\end{align*}
\]

(4.41)

where \( \tilde{j} = j \) if \( \beta_2(B) \neq n \) and where in the case \( \beta_2(B) = n \)

\[
\begin{align*}
j &= b(B) = b(A), \quad \tilde{j} = 1, \\
\text{pr}_B(d_i^A) &= \text{pr}_B(d_{i+1}^B) & \text{for } i \in \{1, \ldots, b(B) - 1\}.
\end{align*}
\]

This shows (4.40). Together (4.38), (4.39) and (4.40) give

\[
\begin{align*}
\text{pr}_B \left( d_{s,n-1-s}^1([\Delta_A \circ T_A]) \right) &= \sum_{B \in \alpha_1(\text{pr}_1^{-1}(A))} \text{sign}(B, A) \cdot [\Delta_B \circ T_B] \\
&+ \sum_{B \in \alpha_2(\text{pr}_1^{-1}(A)), \beta_2(B) \neq n} \text{sign}(B, A) \cdot (-a_{\beta_2(B)}) \cdot [\Delta_B \circ T_B],
\end{align*}
\]

(4.42)

\[
\begin{align*}
&+ \sum_{B \in \alpha_2(\text{pr}_1^{-1}(A)), \beta_2(B) = n} (-1)^{(B)-1} \text{sign}(B, A) \cdot (-a_n) \cdot [\Delta_B \circ T_B].
\end{align*}
\]
Recall from (3.23) and (3.24)

\[
\text{sign}(B, B^{(2)}) = \text{sign}(B, B^{(1)}) \quad \text{if} \quad \beta_2(B) \neq n,
\]

\[
1 = \text{sign}(B, B^{(2)}) = (-1)^{|B|} \text{sign}(B, B^{(1)}) \quad \text{if} \quad \beta_2(B) = n,
\]

and

\[
(-1)^{\beta(B) - 1 + |B|} = (-1)^n.
\]

In view of these signs and (4.42), an arbitrary linear combination

\[
\sum_{A \in A_1(s)} z_A \cdot [\Delta_A \odot T_A], \quad z_A \in \mathbb{Z},
\]

is mapped by \(\text{pr}_B \circ d_{s,n}^{1} \) to

\[
\text{pr}_B(d_{s,n-1-s}^{1} \left( \sum_{A \in A_1(s)} z_A \cdot [\Delta_A \odot T_A] \right))
\]

\[= \sum_{B \in B(s), \beta_2(B) \neq n} \text{sign}(B, B^{(1)})(z_B^{(1)} - a_{\beta_2(B)} z_B^{(2)})[\Delta_B \odot T_B]
\]

\[+ \sum_{B \in B(s), \beta_2(B) = n} \text{sign}(B, B^{(1)})(z_B^{(1)} - (-1)^n a_n z_B^{(2)})[\Delta_B \odot T_B].
\]

The following Claim 1 will be useful here and in the proof of (4.34) in part (d).

**Claim 1:** Fix any \(B \in B(s)\). A sequence \((B_i)_{i \in \mathbb{N}}\) of elements \(B_i \in B(s)\) with \(B = B_n\) and with the following property exists,

\[
B^{(1)}_{(i+1) \mod n} = B^{(2)}_i \quad \text{for} \quad i \in \mathbb{N}.
\]

Additionally,

the map \( N \rightarrow N, \quad i \mapsto \beta_2(B_i), \) is a bijection.

**Proof of Claim 1:** We will describe such a sequence from a point of view which will make its existence clear. Suppose we have such a sequence \((B_i)_{i \in \mathbb{N}}\). Compare the set of gaps of \(B^{(2)}_{(i-1) \mod n} = B^{(1)}_i\) with the set of gaps of \(B^{(2)}_i = B^{(1)}_{(i+1) \mod n}\). These sets almost coincide. Only the gap \{\beta_2(B_i)\} of \(B^{(1)}_i\) is shifted one position to the left to the gap \{\beta_1(B_i)\} of \(B^{(2)}_i\). Starting from the set of gaps of \(B^{(2)}_n = B^{(1)}_1\), one shifts in \(n\) steps all gaps to the left, so that the final position of each gap is the original position of the gap left of it. One has to take care that the gaps stay always apart from one another. It is clear that starting from the set of gaps of \(B^{(2)}_n\), one can find such \(n\) steps. Also (4.45) is clear. This finishes the proof of Claim 1. \(\square\)

Consider now a linear combination \(\sum_{A \in A_1(s)} z_A \cdot [\Delta_A \odot T_A], \quad z_A \in \mathbb{Z},\) which is mapped by \(\text{pr}_B \circ d_{s,n-1-s}^{1}\) to 0. Choose any \(A \in A_1(s)\) and
choose \( B \in \mathcal{B}(s) \) with \( B^{(2)} = A \). Claim 1 provides a sequence \((B_i)_{i \in \mathbb{N}}\) with \( B_n = B, \ B_n^{(2)} = A, \ (4.44) \) and \((4.45)\). Then \((4.43)\) gives the relations

\[
z^{(2)}_{B^{(i-1)} \bmod n} = z^{(1)}_{B_i} = \begin{cases} a\beta_2(B_i)z^{(2)}_{B_i} & \text{if } \beta_2(B_i) \neq n, \\ (-1)^n a_n z^{(2)}_{B_i} & \text{if } \beta_2(B_i) = n. \end{cases} \tag{4.46}
\]

Together they give because of \((4.45)\) especially

\[
z_A = a_{n+1} \cdot z_A, \quad \text{so } (-1)^n d \cdot z_A = 0, \quad \text{so } z_A = 0.
\]

Therefore \( \text{pr}_B \circ d^1_{s,n-1-s} \) is injective, and thus also \( d^1_{s,n-1-s} \) is injective. This finishes the proof of part (a).

(b) The definition \((4.21)\) of \( E_{n-t+1,t-3}^{r-1} \) for \( r \geq 3 \) gives

\[
E_{n-t+1,t-3}^{r-1} \subset \frac{E_{n-t+1,t-3}^{r-2}}{d^{r-2}(E_{n-t+r-1,t-r}^{r-2})}.
\]

If \( r = 3 \), this is \((4.32)\). If \( r \geq 4 \), \((4.31)\) gives \( E_{n-t+r-1,t-r}^{r-2} = 0 \), so \( E_{n-t+1,t-3}^{r-1} \subset E_{n-t+1,t-3}^{r-2} \). Induction gives \((4.32)\).

(c) \( E_{s-1,n-1-s}^1 = H_{n-2}(G_{s-1}, G_{s-2}) \) has by \((4.8)\), \((4.10)\) and \((4.12)\) only contributions from sets \( A \subset N \) with \( |A| = s - 1 \) and \( b(A) \in \{n - |A|, n - 1 - |A|\} \). These are thick or almost thick sets. By Lemma 3.2 for \( s < \lceil \frac{n+2}{2} \rceil \), there are no thick or almost thick sets with \( |A| = s - 1 \). Therefore then \( E_{s-1,n-1-s}^1 = 0 \) and \( E_{s-1,n-1-s}^r = 0 \) for \( r \geq 1 \).

(d) By \((4.23)\), for \( r \geq 2 \)

\[
E_{n-1}^r = \ker(d_{n-1}^{r-1} : E_{n-1}^{r-1} \rightarrow E_{n-r+1,r-3}^{r-1}) \subset E_{n-1}^{r-1} \subset E_{n-1}^1. \tag{4.47}
\]

For \( r \geq \lceil \frac{n+3}{2} \rceil + 1 \), \( E_{n-r+1,r-3}^r = 0 \) by part (c). Therefore then \( E_{n-1}^r = E_{n-1}^{r-1} \). Inductively we obtain

\[
E_{n-1}^\infty = E_{n-1}^r = E_{n-1}^{\lceil \frac{n+3}{2} \rceil} \quad \text{for } r \geq \lceil \frac{n+3}{2} \rceil.
\]
It remains to prove (4.34). We will prove it by induction in $r$. By definition $E^1_{n-1} = E^1_{n-1}$. First we treat the special case $r = 2$:

$$E^2_{n-1} = \ker(d^1 : E^1_{n-1} \to E^1_{n-1-1}),$$

$$d^1([X_j^{(0)}]) = \left(\sum_{B \subseteq N : |B| = n-1} \text{sign}(B, N) \cdot [\Delta_B \circ \{\xi(p_j)\}]\right) \cdot \sum_{j=1}^d \sum_{z_j = 0}^d z_j \cdot [X_j^{(0)}] \in \bigoplus_{i=1}^n \mathbb{Z} \cdot [\Delta_{N-\{i\}} \circ \{\xi(p_i)\}] \cong \mathbb{Z}^n.$$

The last equality follows from (4.8) and (4.10). Obviously $d^1([X_j^{(0)}])$ is independent of $j$ and is $\neq 0$. Therefore

$$E^2_{n-1} = \ker(d^1 : E^1_{n-1} \to E^1_{n-1-1})$$

$$= \{ \sum_{j=1}^d z_j \cdot [X_j^{(0)}] \mid z_j \in \mathbb{Z}, \sum_{j=1}^d z_j = 0 \} = \tilde{E}^2_{n-1}.$$

Now we suppose $r \in \{3, \ldots, \left[\frac{n+3}{2}\right]\}$ and (induction hypothesis) $E^r_{n-1} = \tilde{E}^r_{n-1}$. This induction hypothesis and (4.26) give

$$E^r_{n-1} = \tilde{E}^r_{n-1} = \bigoplus_{j=1}^{d-1} \mathbb{Z} \cdot [X_j^{(r-2)}] = \sum_{j=1}^d \mathbb{Z} \cdot [X_j^{(r-2)}] \subset E^1_{n-1-1}. \quad (4.48)$$

We have to control $d^r_{n-1}([X_j^{(r-2)}])$. Recall four points:

(i) $d^r_{n-1} = j^r_{n-1-1} \circ k^r_{n-1-1}$.

(ii) $k^r_{n-1-1}$ is a boundary map with image in

$$A^r_{n-1-1} = (i^1)^{r-2}(A^1_{n-r+1-1, -3}) = \text{Im}(H_{n-2}(G_{n-r+1}) \to H_{n-2}(G_{n-1})).$$

(iii) $j^r_{n-1-1}$ maps this space to $E^r_{n-r+1, r-3}$, which satisfies because of (4.32) (for $r = t$)

$$E^r_{n-r+1, r-3} \subset \frac{E^1_{n-r+1, r-3}}{d^1(E^r_{n-r+2, r-3})} = \frac{H_{n-2}(G_{n-r+1}, G_{n-r})}{d^1(E^1_{n-r+2, r-3})}.$$ 

(iv) (3.10) in Theorem 3.3 gives $\partial X_j^{(r-2)} = \partial_1 R_j^{(r-2)}$, and this is a chain in $G_{n-r+1}$, which fits.
These four points imply
\[ d_{r-1}^{-1}([X_j^{(r-2)}]) = [\partial_1 R_j^{(r-2)}] \in \frac{H_{n-2}(G_{n-r+1}, G_{n-r})}{d^1(E_{n-r+2,r-3}^{1})}. \] (4.49)

We will show the following claim.

**Claim 2:** The class \([\partial_1 R_j^{(r-2)}]\) in the quotient in (4.49) of the chain \(\partial_1 R_j^{(r-2)}\) is independent of \(j\), and for \(m \in \mathbb{Z}\)
\[ m \cdot [\partial_1 R_j^{(r-2)}] = 0 \iff d|m. \] (4.50)

Claim 2 implies
\[ E_{n-1}^r = \ker(d_{n-1}^{-1} : E_{n-1}^{r-1} \to E_{n-r+1,r-3}^{r-1}) \]
\[ = \left\{ \sum_{j=1}^{d} z_j \cdot [X_j^{(r-2)}] \mid z_j \in \mathbb{Z}, \sum_{j=1}^{d} z_j \in d\mathbb{Z} \right\} \overset{(4.29)}{=} \tilde{E}_{n-1}^r. \]

It rests to prove Claim 2.

The chain \(\partial_1 R_j^{(r-2)}\) was already studied in the proof of Theorem 3.3. Formula (3.22) showed contributions from pairs \((B, A)\) of sets \(B \subset A \subset N\) with \(|B| = |A| - 1, |A| = n - r + 2\) and \(A\) thick. A priori \(B\) could by of one of three types I, II and III. But a set \(B\) of type III gives nothing. A set \(B\) of type II is almost thick and gives either nothing or the contribution in (3.25) (up to a sign) which is independent of \(j\). A set \(B\) of type I is thick and gives the contribution in (3.26), which we call now \(\text{Cont}(B, \partial_1 R_j^{(r-2)})\).

First we will show that \(\text{Cont}(B, \partial_1 R_j^{(r-2)})\) is a relative cycle, so that it gives a class
\[ [\text{Cont}(B, \partial_1 R_j^{(r-2)})] \in H_{n-2}(G_{n-r+1}, G_{n-r}). \] (4.51)

Then we will show that this class is independent of \(j\).

\(\partial_1 \left(\text{Cont}(B, \partial_1 R_j^{(r-2)})\right)\) contains \(\partial \Delta_B\), so it is in \(G_{n-r}\), and we can ignore it. For the signs in (3.26), recall (3.27). Then
\[ \partial_2 \left(\text{Cont}(B, \partial_1 R_j^{(r-2)})\right) \]
\[ = \sum_{i=1}^{b(B)} \text{sign}(B)(-1)^{|B|+i-1} \cdot \Delta_B \odot \left( p_j + d^{-1}\partial C(\xi_i^B, ..., \xi_{i-1}^B, ..., \xi_{b(B)} B) \right) \].

This is equal to 0, because for each pair \((i, l) \in \{1, ..., b(B)\}\) with \(i < l\), the term \(p_j + d^{-1}\partial C(\xi_i^B, ..., \xi_{i-1}^B, ..., \xi_{b(B)} B)\) and the term
\[ p_{j+1} + d^{-1}C(\beta_1^B, \ldots, \beta_i^B, \ldots, \beta_{n-B}^B) \text{ turn up twice and with different signs. Therefore } \text{Cont}(B, \partial_1 R_j^{(r-2)}) \text{ is a relative cycle. In the proof of Theorem 3.3, we found}
\]

\[
\begin{align*}
- & \text{Cont}(B, \partial_1 R_j^{(r-2)}) + \text{Cont}(B, \partial_1 R_{j+1}^{(r-2)}) \\
= & \text{Cont}(B, \partial_2 R_j^{(r-1)}) \\
= & \partial_2(\text{Cont}(B, R_j^{(r-1)})) \\
= & \partial(\text{Cont}(B, R_j^{(r-1)})) - \partial_1(\text{Cont}(B, R_j^{(r-1)})).
\end{align*}
\]

In the last difference, the first term is a boundary, and the second term is in \( G_{n-r} \). Therefore the class in \( H_{n-2}(G_{n-r+1}, G_{n-r}) \) of this difference is 0. Therefore the class in (4.51) is independent of \( j \).

We see that all contributions to the class \([\partial_1 R_j^{(r-2)}] \in H_{n-2}(G_{n-r+1}, G_{n-r})\) are independent of \( j \), so this class is independent of \( j \). This implies the first statement in Claim 2.

By (4.28) \( 0 = \sum_{j=1}^{d}[X_j^{(r-2)}] \in E_{n-1}^n \). Thus

\[
0 = \sum_{j=1}^{d} d^{-1}_{n-1}([X_j^{(r-2)}]) = d \cdot [\partial_1 R_j^{(r-2)}]
\]

in the quotient in (4.49). This is \( \Leftarrow \) of (4.50).

It rests to prove \( \Rightarrow \) of (4.50). Define \( s := n - r + 2 \). Then \( r \in \{3, \ldots, \left[ \frac{n+3}{2} \right] \} \) implies \( s \in \{ \left[ \frac{n+2}{2} \right], \ldots, n-1 \} \). So part (a) and its proof apply. The map \( \text{pr}_B \circ d^1_{n-r+2, r-3} \) is injective, and formula (4.43) gives its image in \( H_{n-2}(G_{n-r+1}, G_{n-r}) \). For \( \Rightarrow \) of (4.50), it is sufficient to show for \( m \in \mathbb{Z} \)

\[
m \cdot \text{pr}_B([\partial_1 R_j^{(r-2)}]) \in \text{pr}_B(\text{Im}(d^1_{n-r+2, r-3})) \Rightarrow d|m, \quad (4.52)
\]

here \([\partial_1 R_j^{(r-2)}]\) denotes the class in \( H_{n-2}(G_{n-r+1}, G_{n-r}) \) (not the class in the quotient in (4.49)).

For \( \text{pr}_B([\partial_1 R_j^{(r-2)}]) \), we need only the contributions of sets \( B \in \mathcal{B}(n-r+2) \). By the proof of Theorem 3.3, the contribution of such a set \( B \) is 0 if \( \beta_2(B) \neq n \), and it is given by (3.25) if \( \beta_2(B) = n \). We obtain

\[
\text{pr}_B([\partial_1 R_j^{(r-2)}]) = \sum_{B \in \mathcal{B}(n-r+2): \beta_2(B) = n} (\pm 1)a_1 a_2^B \ldots a_{k_B^B} \cdot [\Delta_B \circ T_B].
\]

Suppose that the left hand side of (4.52) holds for some \( m \in \mathbb{Z} \). Write \( \lambda_B := m \cdot (\pm 1) a_1 a_2^B \ldots a_{k_B^B} \). Then we have a linear combination
\[ \sum_{A \in A_1(n-r+2)} z_A \cdot [\Delta_A \odot T_A] \quad \text{with} \]
\[ m \cdot \text{pr}_B([\partial_1 P^{(r-2)}]) = \sum_{B \in B(n-r+2): \beta_2(B) = n} \lambda_B \cdot [\Delta_B \odot T_B] \tag{4.53} \]
\[ \quad = \text{pr}_B(d_{n-r+2,r-3}^1 \left( \sum_{A \in A_1(n-r+2)} z_A \cdot [\Delta_A \odot T_A] \right)). \]

Choose a set \( B \in B(n-r+2) \) with \( \beta_2(B) = n \). Claim 1 provides a sequence \( (B_i)_{i \in \mathbb{N}} \) of elements \( B_i \in B(n-r+2) \) with \( B = B_n, \) (4.44) and (4.45). Then (4.53) and (4.43) give the first line of (4.46), so
\[ z_{B(2)} = \left( \prod_{i=1}^{n-1} a_i \right) \cdot z_{B(1)}. \]

By (4.43), the coefficient \( \lambda_B \) is
\[ \lambda_B = \text{sign}(B, B^{(1)}) \cdot \left( z_{B(1)} - (-1)^n a_n z_{B(2)} \right) \tag{4.54} \]
\[ = \text{sign}(B, B^{(1)}) \cdot z_{B(1)} \cdot (1 - a_{n+1}) \]
\[ = \text{sign}(B, B^{(1)}) \cdot z_{B(1)} \cdot (-1)^{n+1} d. \]

Therefore \( \lambda_B \in d\mathbb{Z} \). Observe \( \gcd(d, a_1 a_{k_2} \ldots a_{k_B}) = 1. \) Therefore, \( d|m. \) (4.52) is proved. This finishes the proof of part (d).

**Theorem 4.6.** The homology group \( H_{n-1}(G, \mathbb{Z}) = H_{n-1}(G) \) is a \( \mathbb{Z} \)-lattice of rank \( \mu = (-1)^n a_{n+1} + d + (-1)^n. \) A \( \mathbb{Z} \)-basis of it is
\[ [X^{(\frac{n}{2})}_1, \ldots, X^{(\frac{n}{2})}_d], [\Delta_{\text{od}} \odot T_{\text{od}}], [\Delta_{\text{ev}} \odot T_{\text{ev}}], \quad \text{if } n \text{ is even,} \]
\[ [X^{(\frac{n}{2}+1)}_1, \ldots, X^{(\frac{n}{2}+1)}_d], \quad \text{if } n \text{ is odd.} \tag{4.55} \]

Together with \( [X^{(n+1)}_d]\), these elements satisfy the relation (3.18) if \( n \) is odd and the relation (3.19) if \( n \) is even.

**Proof:** By Lemma 3.2, in the case \( n \) even, \( [\Delta_{\text{od}} \odot T_{\text{od}}] \) and \( [\Delta_{\text{ev}} \odot T_{\text{ev}}] \) are homology classes in \( H_{n-1}(G) \). By Theorem 3.3 \( [X^{(\frac{n}{2}+1)}_1], \ldots, [X^{(\frac{n}{2}+1)}_d] \) are homology classes in \( H_{n-1}(G) \), the relation (3.18) holds if \( n \) is odd, and the relation (3.19) holds if \( n \) is even. We have to show that the elements in (4.55) form a \( \mathbb{Z} \)-basis of \( H_{n-1}(G) \).

First we consider even \( n \). By Lemma 4.3 (b) and (c) and Theorem 4.5 (a), \( E_{s,n-1-s} = 0 \) for all \( s \) except \( s = n \) and \( s = \frac{n}{2} \). Therefore \( F^{n}_{n-1} = 0 \) for all \( s = \frac{n}{2} \), and by Lemma 4.3 (c)
\[ F^{\frac{n}{2}}_{n-1} = F^{\frac{n}{2}+1}_{n-1} = \ldots = F^{n-1}_{n-1} = \text{Im}(H_{n-1}(G_{\frac{n}{2}}) \to H_{n-1}(G)) \]
\[ \cong E^{\infty}_{\frac{n}{2},\frac{n}{2}-1} = E^{\frac{n}{2}}_{\frac{n}{2},\frac{n}{2}-1} = H_{n-1}(G_{\frac{n}{2}}, G_{\frac{n}{2}-1}) \cong \mathbb{Z}^2 \tag{4.56} \]
with generators the classes in \(H_{n-1}(G)\) respectively in \(H_{n-1}(G_{\frac{1}{2}}, G_{\frac{3}{2}})\)
of the spheres \(\Delta_{A_{od}} \odot T_{A_{od}} \subset C^{A_{od}}\) and \(\Delta_{A_{ev}} \odot T_{A_{ev}} \subset C^{A_{ev}}\). By Theorem 4.5 (d) and Lemma 4.4

\[
E_{n-1}^\infty = \tilde{E}_{n-1}^{\frac{n+2}{4}} = \bigoplus_{j=1}^{d-1} \mathbb{Z} \cdot [X_j^{\left(\frac{n}{2}\right)}] \subset E_{n-1}^1 = H_{n-1}(G, G_{n-1}), \quad (4.57)
\]

\(E_{n-1}^\infty\) is a \(\mathbb{Z}\)-lattice of rank \(d - 1\), and

\[
E_{n-1}^\infty \cong \frac{F_{n-1}^n}{F_{n-1}^2} = \frac{H_{n-1}(G)}{\text{Im}(H_{n-1}(G_{\frac{1}{2}}) \to H_{n-1}(G))}. \quad (4.58)
\]

Therefore \(H_{n-1}(G)\) is a \(\mathbb{Z}\)-lattice of rank \(d + 1\) with \(\mathbb{Z}\)-basis as in (4.55), and \(F_{n-1}^2\) is a primitive sublattice of rank 2 with generators the classes of the spheres \(\Delta_{A_{od}} \odot T_{A_{od}}\) and \(\Delta_{A_{ev}} \odot T_{A_{ev}}\).

Now we consider odd \(n\). By Lemma 4.3 (b) and Theorem 4.5 (a), \(E_{s,n-1-s}^\infty = 0\) for all \(s \neq n\). Therefore \(F_{n-1}^s = 0\) for all \(s \neq n\) and

\[
H_{n-1}(G) = F_{n-1}^n \cong E_{n-1}^\infty. \quad (4.59)
\]

By Theorem 4.5 (d) and Lemma 4.4

\[
E_{n-1}^\infty = \tilde{E}_{n-1}^{\frac{n+3}{4}} = \bigoplus_{j=1}^{d-1} \mathbb{Z} \cdot [X_j^{\left(\frac{n+3}{4}\right)}] \subset E_{n-1}^1 = H_{n-1}(G, G_{n-1}), \quad (4.60)
\]

and \(E_{n-1}^\infty\) is a \(\mathbb{Z}\)-lattice of rank \(d - 1\). Therefore \(H_{n-1}(G)\) is a \(\mathbb{Z}\)-lattice of rank \(d - 1\) with \(\mathbb{Z}\)-basis as in (4.55). \(\square\)

Remarks 4.7. In the proof of part (d) of Theorem 4.5, we considered the contribution \(\text{Cont}(B, \partial_1 R_j^{(r-2)})\) of a set \(B \in A_1(n - r + 1)\) (\(B\) is thick with \(|B| = n - r + 1\)) to the chain \(\partial_1 R_j^{(r-2)}\). We showed that it is a relative cycle in \(H_{n-2}(G_{n-r+1}, G_{n-r})\), and we saw that it is independent of \(j\). We did not make precise which cycle it is, as we did not need this. Now we will say which cycle it is, but leave the proof to the reader:

\[
\text{Cont}(B, \partial_1 R_j^{(r-2)}) = \varepsilon_1 \cdot [\Delta_B \odot \mathcal{e}(d^{-1}C(\mathcal{e}_{2 B} - \mathcal{e}_{1 B}, \ldots, \mathcal{e}_{k B} - \mathcal{e}_{k B_{(k B)}}))]
\]

\[
= \varepsilon_2 \cdot [\Delta_B \odot \mathcal{e}(C(\mathcal{e}_{2 B} - \mathcal{e}_{1 B}, \ldots, \mathcal{e}_{k B} - \mathcal{e}_{k B_{(k B)}}))].
\]

\[
= \varepsilon_2 \cdot \mathcal{e}_{1 B} \mathcal{e}_{2 B} \ldots \mathcal{e}_{k B_{(k B)}} \cdot [\Delta_B \odot \mathcal{e}(C(d_{1 B}, \ldots, d_{k B_{(k B)}}))],
\]

with \(\varepsilon_1 = \text{sign}(B)(-1)^{|B|}, \varepsilon_2 = \varepsilon_1 \cdot (-1)^{n(B)-1}\).
5. Proof of the main result

Throughout this section we fix a cycle type singularity \( f(x_1, \ldots, x_n) \) as in \([1.2]\) with \( n \geq 2 \) and \( a_1, \ldots, a_n \in \mathbb{N} \) with \([1.3]\), as in the sections \([3]\) and \([4]\). We want to prove the main result of this paper, Theorem \([1.3]\).

The deformation lemma in \([Co82, 3.]\) implies that the inclusion map \( i_g : G \hookrightarrow F_0 \) induces an epimorphism \( (i_g)_* : H_{n-1}(G, \mathbb{Z}) \to H_{n-1}(F_0, \mathbb{Z}) \). Both groups are \( \mathbb{Z} \)-lattices of rank \( \mu \). This holds by Theorem \([4.6]\) for \( H_{n-1}(G, \mathbb{Z}) \) and by \([Mi68]\) for \( H_{n-1}(F_0, \mathbb{Z}) \). Therefore the epimorphism is an isomorphism \( (i_g)_* : H_{n-1}(G, \mathbb{Z}) \to H_{n-1}(F_0, \mathbb{Z}) \).

It rests to determine the monodromy on \( H_{n-1}(G, \mathbb{Z}) \). We call this monodromy \( h_{\text{mon}} \). First we give the weights \((w_1, \ldots, w_n)\) of the quasi-homogeneous polynomial \( f \) (see e.g. Lemma \([4.1]\) in \([HZ19]\)):

\[
\begin{align*}
  w_j &= \frac{v_j}{d} \in \mathbb{Q} \cap (0, 1) \quad \text{for } j \in N \quad \text{with} \\
  d &= \prod_{j=1}^{n} a_j - (-1)^n = \mu - (-1)^n, \\
  v_j &= \sum_{l=1}^{n} (-1)^l \left( \prod_{k=j+l} a_{(k) \mod n} \right) \\
  &\quad \text{with } (5.1) \\
  a_j v_j + v_{(j+1) \mod n} &= d.
\end{align*}
\]

One sees easily \( \gcd(v_1, d) = \ldots = \gcd(v_n, d) \). We defined already \( b := d/\gcd(v_1, d) \). In the case of a quasihomogeneous singularity, one can give explicitly a diffeomorphism \( \Phi_{\text{mon}} : \mathbb{C}^n \to \mathbb{C}^n \) of the Milnor fibration which induces the monodromy on the homology of the Milnor fiber and on \( H_{n-1}(G, \mathbb{Z}) \). It looks as follows.

\[
\Phi_{\text{mon}} : \mathbb{C}^n \to \mathbb{C}^n, \quad (z_1, \ldots, z_n) \mapsto (e^{2\pi i w_1} z_1, \ldots, e^{2\pi i w_n} z_n).
\]

It maps the point \( e(p_j) \) to the point \( e(p_{(j+v_1) \mod d}) \), because of \( a_j v_j + v_{(j+1) \mod n} = d \). Therefore it maps the chain \( R_j^{(k)} \) to the chain \( R_{(j+v_1) \mod d}^{(k)} \) and the chain \( X_j^{(k)} \) to the chain \( X_{(j+v_1) \mod d}^{(k)} \). In the case of even \( n \), it maps the spheres \( \Delta_{A_{\text{od}}} \cap T_{A_{\text{od}}} \subset \mathbb{C}^{A_{\text{od}}} \) and \( \Delta_{A_{\text{ev}}} \cap T_{A_{\text{ev}}} \subset \mathbb{C}^{A_{\text{ev}}} \) to themselves.

Now we can apply the results of section \([2]\) In order to make the relation to these results transparent, we introduce the following notations,

\[
\delta_j := [X_j^{([n+1]/2)}] \in H_{n-1}(G, \mathbb{Z}) \quad \text{for } j \in \{1, \ldots, d\}.
\]
for odd $n$ or even $n$, and
\[
\gamma := \left[\Delta_{\text{od}} \odot T_{\text{od}}\right] + (-1)^{\frac{n}{2}} a_1 a_3 \ldots a_{n-1} \cdot \left[\Delta_{\text{ev}} \odot T_{\text{ev}}\right] \in H_{n-1}(G, \mathbb{Z}),
\]
\[
\beta := \left[\Delta_{\text{ev}} \odot T_{\text{ev}}\right] \in H_{n-1}(G, \mathbb{Z}),
\]
\[
c := (-1)^{\frac{n}{2}+2(\frac{n-1}{2})} a_1 a_3 \ldots a_{n-1} \in \mathbb{Z},
\]
for even $n$. Then
\[
H_{n-1}(G, \mathbb{Z}) = \bigoplus_{j=1}^{d-1} \mathbb{Z} \cdot \delta_j \oplus \left\{ \begin{array}{ll}
0 & \text{if } n \text{ is odd}, \\
\mathbb{Z} \cdot \gamma \oplus \mathbb{Z} \cdot \beta & \text{if } n \text{ is even},
\end{array} \right.
\]
\[
h_{\text{mon}}(\delta_j) = \delta_{(j+1) \mod d},
\]
\[
h_{\text{mon}}(\gamma) = \gamma, \quad h_{\text{mon}}(\beta) = \beta, \quad \text{if } n \text{ is even},
\]
\[
\sum_{j=1}^{d} \delta_j = \left\{ \begin{array}{ll}
0 & \text{if } n \text{ is odd (because of (3.18))}, \\
c \cdot \gamma & \text{if } n \text{ is even (because of (3.19))}.
\end{array} \right.
\]

The proof of Lemma 2.3 can be adapted to show for odd $n$
\[
(H_{n-1}(G, \mathbb{Z})) \cong (\gcd(v_1, d) - 1) \text{Or}(t^b - 1) \oplus \text{Or} \left(\frac{t^b - 1}{t - 1}\right).
\]

For even $n$, we see with Definition 2.1 Lemma 2.3 and Lemma 2.4
\[
(H_{n-1}(G, \mathbb{Z}), h_{\text{mon}}) \cong (H^{(d,c)}, (h^{(d,c)} v_1) \oplus \text{Or}(t - 1))
\]
\[
\text{Lemma 2.3} \cong (\gcd(v_1, d) \text{Or}(t^b - 1) \oplus \text{Lo}^{(b,c)} \oplus \text{Or}(t - 1))
\]
\[
\text{Lemma 2.4} \cong \gcd(v_1, d) \text{Or}(t^b - 1) \oplus \text{Or}(t - 1),
\]
the last isomorphism uses $\gcd(b, c) = 1$ and (2.3). This finishes the proof of Theorem 1.3.

Remarks 5.1. Cooper’s paper [Co82] studied the integral monodromy of the cycle type singularities. It is split into 22 sections. The sections 1 and 2 are an introduction and a discussion of a degenerate case. The sections 3 to 14 are devoted to the deformation lemma in section 3, which yields that $(i_q)_*$ is surjective.

Section 15 introduces the deformed simplices $\Delta_A$, the tori $T_A$, the chains $\Delta_A \odot T_A$, the blocks of sets $A \subset \mathbb{N}$, and it gives most of the statements of Lemma 3.2.

Section 16 states formulas which are close to our formula (4.40).

At the end of section 17, a formula which is close to our formula (4.43) is derived. Section 17 also defines the subsets $G_s$ and thick sets $A$ and states a part of the formulas (4.7)–(4.13) for the relative homology groups.
Section 18 first introduces the spectral sequence \((E^r_{s,t}, d^r_{s,t})\) of the filtration \(G = G_n \supset G_{n-1} \supset ... \supset G_1 \supset G_0 = \emptyset\). But it gives neither a reference, nor the properties in Theorem 4.2. Then it states and proves the injectivity of the maps \(d^1_{s,n-1-s}\) in our Theorem 4.5 (a). Though it does not use our Claim 1, but uses a specific sequence of \(n \cdot (n - s)\) elements \(B_i \in \mathcal{B}(s)\). Claim 1 is more efficient and can be used also for the proof of Theorem 4.5 (d).

The lemma in Section 19 is crucial, but not correct. It claims correctly \([X^{(1)}_j] \in E^2_{n-1}\), but wrongly \([X^{(2)}_j] \in E^\infty_{n-1}\). By Theorem 4.5 (d), this is true only if \(n \in \{3,4\}\). The wrong argument is in the proof of the lemma at the end of section 19: A part of the boundary is missed. This makes Cooper believe that \(X^{(2)}_j\) is always a cycle. He constructs and sees only the beginning for \(k \in \{0,1,2\}\) of the sequence \((X^{(k)}_j)_{k \in \{1,\ldots, \lfloor \frac{n+1}{2} \rfloor\}}\) of chains.

The lemma in section 20 has some similarity with Claim 2 (in the proof of Theorem 4.5 (d)). But it is too special.

The lemma in section 21 concludes wrongly that the classes \([X^{(2)}_j]_{j=1,\ldots,d-1}\) form a \(\mathbb{Z}\)-basis of \(E^\infty_{n-1}\). This is true only for \(n \in \{3,4\}\). It builds on the lemma in section 19. The wrong statements on the elements and generators of \(E^\infty_{n-1}\) in the sections 19 and 21 form the first serious mistake.

The final section 22 makes the second serious mistake, in the case of even \(n\). It states correctly that for even \(n\) \(E^\infty_{s,n-1-s}\) is 0 only if \(s \in \{\frac{n}{2}, n\}\). And it derives correctly \((E^\infty_{\frac{n}{2},n-1}, h_{mon}) \cong 2 \text{Or}(t-1)\) and

\[
(E^\infty_{n-1}, h_{mon}) \cong (\gcd(v_1,d) - 1)\text{Or}(t^b - 1) \oplus \text{Or}(\frac{t^b - 1}{t - 1}).
\]

But then it supposes wrongly that the quotients \(E^\infty_{\frac{n}{2},n-1}\) and \(E^\infty_{n-1}\) of the filtration \(F^*_n\) of \(H_{n-1}(G)\) lift to a splitting of \(H_{n-1}(G)\) which is invariant by the monodromy. This would give a non-standard decomposition of \((H_{n-1}(G), h_{mon})\) into Orlik blocks (and this is what Cooper claims to have for even \(n\)).

But Cooper’s paper contains most of the ingredients which we need for the proof of Orlik’s conjecture. We corrected the two mistakes, we were more careful with the signs and proofs, we introduced more notations, we had the full sequence of chains \((X^{(k)}_j)_{k \in \{1,\ldots, \lfloor \frac{n+1}{2} \rfloor\}}\) and we had the algebraic statements in our section 2. But Cooper’s paper [Co82] was an indispensable basis on which we could build.

**Remarks 5.2.** (i) Orlik and Randell study in [OR77] mainly the integral monodromy of the chain type singularities. But section 3 says...
also something about the cycle type singularities. In (3.4.1) they make a conjecture on the integral monodromy of the cycle type singularities which would imply Orlik’s conjecture. Their ansatz for a proof is very different from [Co82]. They point themselves to a gap in this ansatz.

(ii) [HM20] combines the result in [OR77] for the chain type singularities with an algebraic result. Together they imply that Orlik’s conjecture holds for the chain type singularities. And [HM20] gives a relative result, which says that the Thom-Sebastiani sum of any two quasihomogeneous singularities satisfies Orlik’s conjecture if the two singularities satisfy Orlik’s conjecture. As Orlik’s conjecture is valid for the chain type singularities by [HM20] and for the cycle types singularities by this paper, it is valid also for all their (iterated) Thom-Sebastiani sums.

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