Lossy SU(1,1) interferometers in the single-photon-pair regime

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Abstract
The success of quantum technologies is intimately connected to the possibility of using them in real-world applications. This requires the system to be comprehensively modeled including various relevant experimental parameters. To this aim, in this paper, we study the performance of lossy SU(1,1) interferometers in the single-photon pair regime, posing particular attention to the different amount of information contained in the measurement of single counts and of coincidences at the output of the interferometer. To this aim, we derive the classical Fisher information (FI) of both single and coincidence events, and study it as a function of the internal and external losses of the system. Our analysis shows that, in the absence of external losses, the FI of the coincidence events is always higher or equal than the one of single events. On the other hand, in the presence of external losses, the FI of the singles can increase above the one of the coincidences. Moreover, our analysis shows that coincidence measurement can be exploited to partially mitigate the effect of internal losses in the absence of external losses. Finally, comparing SU(1,1) and classical SU(2) interferometers, we find that the former can outperform the classical systems when the internal losses are above 50%.

1. Introduction
Precise determination of physical quantities, such as dispersion and absorption, is often achieved via accurate phase measurement in an interferometric setup. In the never-ending quest to improve the maximum phase resolution achievable, the new class of SU(1,1) interferometers was introduced in 1986 by Yurke et al [1].

Traditional interferometers are usually characterised by the presence of two passive, linear elements: one of the elements splits the input of the interferometer into two separate modes, while the other recombines the two modes. In an SU(1,1) interferometer these passive, linear elements are replaced with nonlinear parametric processes [2]. In this way, the state generated after the first element is an entangled quantum state, whose evolution in the interferometer is then read out by the second element. In the context of optical interferometry, the nonlinear elements usually employed are nonlinear mediums performing parametric down conversion (PDC), such as lithium niobate or potassium titanyl phosphate.

This class of interferometers has been widely studied in the context of quantum metrology [2–8], particularly in the high-gain regime, i.e. when a high mean photon number, ⟨n⟩ ≫ 1, is generated inside the interferometer. In this operation regime, the achievable precision of SU(1,1) interferometers can scale as ∼1/⟨n⟩, the so-called Heisenberg scaling. Hence, these systems can outperform linear MZI with classical input light, whose scaling is limited to the standard quantum limit (SQL) ∼1/√⟨n⟩. These interferometers have been theoretically investigated extensively, in particular within the context of Gaussian-state interferometry [9], both in the ideal, lossless case [10] and in the presence of losses [8, 11–13]. It was found
that they can be particularly robust against detection losses, while losses within the interferometer can have a detrimental impact and even prevent reaching the Heisenberg scaling [11]. Improvement beyond the SQL has been shown experimentally, demonstrating the validity of the theoretical studies so far [14, 15]. Moreover, new variants of Yurke’s SU(1,1) interferometer have been proposed and analysed. Some examples are the schemes employing parity detection [4, 7] or homodyne detection [6], as well as those considering not only vacuum as the input of the interferometer, but also coherent states and displaced squeezed vacuum [7, 16]. All this work is currently stimulating an interesting debate over what is, ultimately, the actual resource that allows better-than-SQL resolution in an SU(1,1) interferometer [17, 18].

On the other side of the spectrum, these systems have also attracted much attention in the low-gain regime, where \( \langle n \rangle \ll 1 \). In these conditions, SU(1,1) interferometers do not show Heisenberg scaling [9, 19]. However, these systems, and variations thereof, have been utilised to investigate fundamental quantum mechanical concepts, such as state superposition, the quantum Zeno effect, and induced coherence without interaction [20–25]. Moreover, the correlations of the states generated within an SU(1,1) interferometer have enabled microscopy and spectroscopy with undetected photons [26–32], as well as the generation of tailored biphoton entangled states [33–37]. Realisation of integrated SU(1,1) interferometers for metrological application has also been explored recently, both theoretically and experimentally [38, 39], showing promising results. Finally, it has recently been demonstrated that it is even possible to achieve phase sensitivity beyond the one provided by SU(1,1) interferometers by cascading several nonlinear elements, in analogy with classical multi-source interference [40].

Some of the works mentioned so far investigated effect of losses onto fringe visibility at the output of the interferometer [20, 24, 41]. However, despite the great interest in this system, little work has been done so far to analyse the phase sensitivity of low-gain, low-gain SU(1,1) interferometers and their relationship with the SQL, especially when considering non-Gaussian detection schemes. The most relevant study in this area so far is the work by Michael et al [42], whose analysis focuses mainly on the performance of SU(1,1) interferometer with and without a seeded input. Understanding under which conditions these systems can improve upon the SQL will pave the way to quantum metrology in photon-starved scenarios and enable real-world applications such as quantum-enhanced spectroscopy and microscopy in photosensitive materials.

In this work we study the properties of lossy SU(1,1) interferometers in the low-gain regime. We analyse the dependence of the classical Fisher information (FI) on the losses present in the system and identify the operation conditions under which quantum SU(1,1) interferometers outperform their respective classical SU(2) counterparts. Importantly, we identify realistic experimental parameters to illustrate these regimes of operations, highlighting the practical advantage offered by SU(1,1) interferometers.

This paper is structured in two main parts. In section 2, SU(1,1) interferometers in the low-gain regime are discussed. Analytic expressions for the FI of different detection schemes are derived and examined. In section 3, we investigate in which experimental conditions the SU(1,1) interferometer can provide an advantage, when compared to classical MZI. The results of this comparison show that lossy, low-gain SU(1,1) interferometers can beat the SQL if the internal average transmission is above 50%.

### 2. SU(1,1) in the low-gain regime, click detection, and classical Fisher information

The system under investigation is sketched in figure 1. The interferometer consists of two nonlinear cascaded stages with gains \( g_1 \) and \( g_2 \), respectively, which generate photon pairs in two different modes, A and B, via a nonlinear process, e.g. via type-II PDC. The phase \( \phi \) between the two stages is set by a phase-shifting element, or sample, located in mode A. After the second stage, the generated photons are detected at detectors \( D_A \) and \( D_B \). Internal losses \( \alpha_{A/B} \) inside the interferometer are modelled with beam splitters placed in the path of modes A and B, which transmit mode A/B with probability \( T_{A/B} = 1 - \alpha_{A/B} \). Without loss of generality, internal losses in mode A are assumed to occur after the phase shifter. Note that placing the internal losses before or after the phase element imparts only a constant phase shift to the interference pattern. Imperfect detection efficiencies in the output modes—which also include the external losses after the interferometer—are similarly modelled with a pair of beam splitters with transmission \( T_{A/B} \) for mode A/B.

Throughout most of this work, the interferometer is operated in the low-gain regime, i.e. with \( g_{1,2} \ll 1 \). This means that the mean number of photon pairs generated in each stage is \( \langle n \rangle = \sinh^2 g \approx g^2 \ll 1 \). Moreover, we consider the detectors \( D_A \) and \( D_B \) to be binary detectors, which cannot discriminate whether more than one photon have impinged on them. Note that these types of detectors are also routinely used and readily available, making this restriction meaningful in terms of applicability. This allows us to derive simple equations describing the system in the single-photon-pair regime. To understand the limits of validity of the model, we compare the analytic results with numerical simulations, where we include gains up to \( g \sim 1 \) to understand the impact of multiphoton components. Finally, to consider realistic sensing scenarios, \( g_1 \) is fixed, i.e. the number of photons that, on average, interacts with the sample is constant. This allows us to...
investigate how much information is present in the photons that interacted with the sample by varying the gain $g_2$ of the second stage.

Under the assumptions of $g_{1/2} \ll 1$ and click detectors, the click probability for a single detector event $p_{A/B}$ (here onward referred to as singles) and for a coincidences event $p_{CC}$ can be approximated as (more details are shown in the Suppl. Mat.)

$$
p_{A} = \eta_A \left[ T_A g_1^2 + g_2^2 + 2\sqrt{T_A T_B} g_1 g_2 \cos(\phi) \right],
\tag{1a}
$$

$$
p_{B} = \eta_B \left[ T_B g_1^2 + g_2^2 + 2\sqrt{T_A T_B} g_1 g_2 \cos(\phi) \right],
\tag{1b}
$$

$$
p_{CC} = \eta_A \eta_B \left[ T_A T_B g_1^2 + g_2^2 + 2\sqrt{T_A T_B} g_1 g_2 \cos(\phi) \right].
\tag{1c}
$$

A close inspection of equation (1) reveals that the click probabilities are determined by the interplay of three different terms. The first two terms, proportional to $g_1^2$ and $g_2^2$, describe the probability of measuring a photon (or a photon pair, in the case of the coincidences) generated in either the first or the second stage, respectively. They represent incoherent terms that are always present at the output of the interferometer. The third term, proportional to $\sqrt{T_A T_B} g_1 g_2 \cos(\phi)$, describes the interference between the probability of measuring a pair generated in either the first or in the second stage; therefore, this last term is present only when coherence between the two stages is ensured. Interestingly, this third term is the one that allows phase sensing in this interferometer since it is the only one that depends on the internal phase shift $\phi$.

The ratio between the coherent and the incoherent contributions to the click probabilities is related to how well the two stages interfere and thus can provide interesting information about the interferometer properties and sensing capabilities. This ratio is the so-called visibility $V$ of the interference pattern, which is formally defined as $V = (p_{\text{max}} - p_{\text{min}}) / (p_{\text{max}} + p_{\text{min}})$.

From equation (1), the visibilities of the singles ($V_{A/B}$) and the coincidences ($V_{CC}$) are given by

$$
V_A = \frac{2\sqrt{T_A T_B} g_1 g_2}{T_A g_1^2 + g_2^2},
\tag{2a}
$$

$$
V_B = \frac{2\sqrt{T_A T_B} g_1 g_2}{T_B g_1^2 + g_2^2},
\tag{2b}
$$

$$
V_{CC} = \frac{2\sqrt{T_A T_B} g_1 g_2}{T_A T_B g_1^2 + g_2^2}.
\tag{2c}
$$

One can analyse the behaviour of the visibilities in equation (2) by calculating the global maxima of these functions. It can be found that the maxima of $V_A$, $V_B$ and $V_{CC}$ are found for $g_1^2 = T_A g_1^2$, $g_2^2 = T_B g_2^2$ and $g_1^2 = T_A T_B$, respectively. Inserting these values in equation (2) reveals that the visibilities of the singles are always limited by the internal transmission of the interferometer, i.e. $V_A \leq \sqrt{T_B}$ and $V_B \leq \sqrt{T_A}$. This means that it is impossible to achieve perfect interference for the singles in a lossy SU(1,1) interferometer. On the other hand, regardless of the detection efficiency and internal losses, the visibility of the coincidences is always maximized ($V_{CC} = 1$) at $g_1^2 = g_2^2 = T_A T_B$. Given the importance of this operating point, we refer to the condition $g_1^2 T_A T_B = g_2^2$ as loss-balanced gains and an interferometer operated at such a point shall be called a loss-balanced interferometer.

Let us dwell for a moment on the meaning of the condition just derived: in a loss-balanced interferometer, one can always set the internal phase $\phi$ such that the total number of coincidences exiting the

![Figure 1. Schematic for the SU(1,1) interferometer with losses and non-ideal detection efficiency.](image-url)
interferometer is zero. In this case, detection losses do not matter and the visibility of the coincidences is thus 100%. Such an unexpected behaviour arises thanks to the coherence properties of the SU(1,1) interferometer and suggests that photon-number correlations between the modes of the interferometer can contain more information than a single arm.

A more accurate quantification of the information contained in the singles and the coincidence measurement is provided by the classical FI. For a given state and measurement scheme, one can calculate the classical FI of the measurement, given the probability \( p_i \) of measuring \( i \) photons, as

\[
FI = \sum_i (\partial_\phi \log p_i)^2 p_i. \tag{3}
\]

Since we are considering a scenario where multiphoton components are negligible, one can only measure either a click or no click, corresponding perfectly to the presence of a single photon or no photons, respectively. This means that equation (3) can be simplified considering only the two complementary probabilities where the detector clicks (\( p \)) or does not click (\( 1 - p \)). This simplifies the expression of the FI to

\[
FI = \frac{(\partial_\phi p)^2}{p(1 - p)} \approx \frac{(\partial_\phi p)^2}{p}, \tag{4}
\]

where the last approximation is valid when \( p \to 0 \), as is true in the case considered here.

From equations (1) and (4), one can calculate the maximum classical FI for singles (\( FI_{\text{max}} \)) and coincidence measurements (\( FI_{\text{max}}^{\text{CC}} \)) in a lossy SU(1,1) interferometer,

\[
FI_{A}^{\text{max}} = 2\eta_A \left[ T_A g_1^2 + g_2^2 - \sqrt{(T_A g_1^2 + g_2^2)^2 - 4T_A T_B g_1^2 g_2^2} \right], \tag{5a}
\]

\[
FI_{B}^{\text{max}} = 2\eta_B \left[ T_B g_1^2 + g_2^2 - \sqrt{(T_B g_1^2 + g_2^2)^2 - 4T_A T_B g_1^2 g_2^2} \right], \tag{5b}
\]

\[
FI_{\text{CC}}^{\text{max}} = 2\eta_A\eta_B \left[ T_A T_B g_1^2 + g_2^2 - \sqrt{(T_A T_B g_1^2 + g_2^2)^2 - 4T_A T_B g_1^2 g_2^2} \right], \tag{5c}
\]

\[
= \begin{cases} 
4\eta_A\eta_B g_1^2 & \text{if } \frac{g_1}{g_2} < \sqrt{\frac{1}{T_A T_B}}, \\
4\eta_A\eta_B T_A T_B g_1^2 & \text{if } \frac{g_1}{g_2} \geq \sqrt{\frac{1}{T_A T_B}}.
\end{cases}
\]

The first two expressions describe the information of modes A and B locally while the latter FI can be thought of as the cross-information between the two modes.

In the following subsections, we analyse these equations to discuss the properties of low-gain lossy SU(1,1) interferometers.

### 2.1. Perfect detection efficiency

The expressions in equation (5) involve quite a complex interplay between the gains of the two stages, the internal transmissions, and the detection efficiencies. To isolate the impact of the internal transmissions on the FI, we begin by considering the case of ideal detection, i.e. \( \eta_A = \eta_B = 1 \).

The behaviour of the FI for different values of internal transmission \( T_A \) and \( T_B \) is shown by the solid lines in figure 2, where the different colours—red, green and blue—correspond to high, medium and low average internal transmission, respectively. The darker lines correspond to the highest FI of the singles, i.e. max\(\{FI_{A}^{\text{max}}, FI_{B}^{\text{max}}\}\), while the lighter lines correspond to \( FI_{\text{CC}}^{\text{max}} \). The analysis of the figure reveals two interesting features.

The first feature of interest is that the FI of the coincidences is always higher than the FI of the singles, i.e. \( FI_{\text{CC}}^{\text{max}} \geq FI_{A/B}^{\text{max}} \). Indeed, with some lengthy calculations, it can be proved that this observation is general and independent of the gains and the internal transmissions. This finding is quite remarkable: the coincidence measurement contains more information than either of the two modes, as was suggested by the analysis of the visibilities. This is somewhat unexpected since coincidences are affected by the losses in both arms while the singles only by those in their arm. In other words, the useful resources in an SU(1,1) interferometer are the number of pairs generated in the first stage. This finding, along with the observation that one can always obtain perfect visibility for the coincidences, highlights the fact that the information in our SU(1,1) interferometer is not encoded in the photons that pass through the sample in mode A, but in the photon pairs generated in the first stage. In fact, if we destroy all photon pair correlations generated in the first stage by blocking mode B in the interferometer—i.e. by setting \( T_B = 0 \) in equation (1)—there will still be photons...
Figure 2. FI of an SU(1,1) interferometer with perfect detection efficiency, gain of the first stage $g_1 = 0.05$ and internal transmissions of $T_A$ and $T_B$ (see the inset in the figure), as a function of the gain $g_2$ of the second stage. The solid lines correspond to the analytic theory of equation (5) while the dashed lines correspond to the numerical simulations (more details in the main text). Lighter lines correspond to the FI of the coincidence measurement, while darker lines correspond to the highest FI of the single measurements. The filled markers correspond to the loss-balanced interferometer, i.e. when $g_2^2 = g_1^2 T_A T_B$, where $V_{CC} = 1$. The vertical solid black line corresponds to the case where the second stage is pumped with the same gain as the first stage. The deviations of the simulations from the analytic theory for $g_2 > 0.3$ are due to the non-negligible contributions of multiphoton effects.

passing through the sample, but we will not be able to observe any interference pattern at the output of the interferometer.

To further clarify why the coincidences always have the highest FI, let us consider in greater detail the different detection events possible. For the system under consideration, there are only two possible cases: we can measure either a coincidence event or an exclusive single-photon event i.e. a single click in either one of the two detectors, without a coincidence. If we measure a coincidence event, we do not gain any information on which stage generated the pair. On the other hand, if we measure an exclusive single-photon event, we know that the photon must have been generated in the first stage because internal losses can only affect the pairs generated in the first stage. Knowing which stage generated the photon pair breaks the coherence of the system [23] and thus destroys the interference between the two stages—often referred to as which-crystal interference [43, 44]. This is analogous to what happens in Young’s double slit experiment, where knowledge of which path the photon has taken erases the interference pattern on the screen. Therefore, all the events where one knows exactly where the photon pair was generated cannot exhibit any interference pattern, and thus they carry no phase information. Since this occurs only in the single arms, this leads to the condition $FI_{CC}^{\max} \leq FI_{A/B}^{\max}$.

The second interesting feature that can be observed in figure 2 is that the FI of both singles and coincidences increases proportionally to $g_2^2$ until it reaches a saturation level which depends on the internal transmission of the system and on the number of resources used to probe the sample, $g_1^2$. Even more interestingly, $FI_{CC}^{\max}$ saturates abruptly to the value of $4g_1^2 T_A T_B$ when the coincidences reach maximum visibility for loss-balanced gains, as shown by the round markers in the plot.

To understand the reason behind the observed behaviour of $FI_{CC}^{\max}$, we begin by noting that the interference pattern of the coincidence clicks in equation (1c) arises due to the interference of two probabilities, one with amplitude $g_1 \sqrt{T_A T_B}$ and one with amplitude $g_2$. These correspond to the probability of detecting a photon pair that was generated in the first or in the second stage, respectively. Since the
amplitude of the first probability is fixed to \( g_1 \sqrt{T_A T_B} \). Three main operating regimes for the interferometer can be identified, depending on the magnitude of \( g_2 \).

If \( g_2 < g_1 \sqrt{T_A T_B} \), then the probability of measuring a pair generated in the second stage is too small compared to the one of the first stage. This results in a poor interference pattern \( V_{CC} < 1 \), meaning that it is not possible to read out all the information available inside the interferometer. As the gain of the second stage is increased, the two probabilities start to become comparable in magnitude, i.e. a detection event has similar likelihood of coming from a pair generated in the first or in the second stage. This is why, in this regime, \( F_{max}^{CC} \) depends on \( g_2^2 \); we can vary the amount of information retrieved by changing the probability of generating photon pairs in the second stage.

As the interferometer reaches loss-balanced gains \( (g_2 = g_1 \sqrt{T_A T_B}) \), the two probabilities have the same amplitude, so perfect visibility \( V_{CC} = 1 \) is reached. This means that one can extract all the information available inside the interferometer. Thus, in a loss-balanced interferometer, maximum \( F_{max}^{CC} \) is reached. From equation (5c), one can notice that, in this case, \( F_{max}^{CC} \) is proportional to \( g_1^2 T_A T_B \). The reason is that the total number of pairs that carry phase information is limited to \( g_1^2 T_A T_B \), which is the number of pairs that is generated in the first stage and survives the internal losses.

Further increasing the gain \( g_2 \) does not allow one to retrieve more information since this is limited by the pairs generated in the first stage. This is the reason why the coincidences saturate at a value that is proportional to \( g_1^2 T_A T_B \). The only effect of increasing \( g_2 \) is to reduce the visibility \( V_{CC} \) as the extra events generated in the second stage constitute only a noise floor.

Differently from the coincidences, in figure 2 one can see that the \( FI \) of the singles saturates later than \( F_{max}^{CC} \) and with a flatter slope. The reason is that, as we have discussed above, the exclusive single-photon events carry no information about the internal phase of the interferometer and thus pollute the interference signal. In fact, since there are no exclusive-single-photon events in the case of perfect internal transmission, the behaviour of the singles is identical to the one of the coincidences, i.e. \( F_{max}^A = F_{max}^B = F_{max}^{CC} \), as can be derived from equation (5).

The analytic description discussed so far is based on the approximation that multiphoton components are not present, such that click detectors are sufficient to accurately reconstruct the state generated inside the interferometer. However, it is not yet clear until when this approximation holds true. Moreover, it is interesting to see what happens to the \( FI \) obtained with binary detectors, when the state generated by the interferometer has non-negligible higher order components. For these reasons, we perform numerical simulations using the Python package QuTiP \(^3\). The simulations are performed in the photon-number basis with a Hilbert space size of 10 and consider \( g_1 = 0.05 \) and \( g_2 \in [10^{-3}, 1] \). For an ideal, lossless system, this corresponds to generating at most a mean photon pair number \( \langle n_{\text{pair}} \rangle \in [0.01, 1.57] \) after the second stage. Finally, three different sets of transmissions \( (T_A, T_B) \) were randomly generated, corresponding to low, moderate and high internal losses. The values of the transmissions are shown in figure 2.

The results of the simulation are shown by the dashed lines in figure 2. From the figure, one can see that our analytic theory becomes inaccurate when \( g_2 \approx 0.2 \), regardless of the values of the internal transmissions. In these conditions, the mean photon pair number generated at the output of the interferometer is mainly determined by the second stage, which is not influenced by the internal losses, and lies in the range \( \langle n_{\text{pair}} \rangle \sim 0.045 - 0.063 \) — depending on the internal transmissions of the system. For such high mean photon pair numbers, the multiphoton events already constitute \( \sim 5\% \) of the total number of single-photon events, and therefore they cannot be neglected, which explains the deviations from the analytic theory. These results are not specific to the values of gains and transmission chosen for the simulations but stem from the observation that multiphoton components become non-negligible when \( \langle n_{\text{pair}} \rangle \) reaches values around 0.05.

### 2.2. Imperfect detection

So far we have seen that, in the case of perfect detection, it is always better to measure coincidences between the output modes, since their \( FI \) is always the highest. When detection losses are considered, the situation changes significantly. In fact, detection losses affect the coincidences much more than the singles, as can be seen from the click probabilities in equation (1): the singles depends only on \( \eta_A \) or \( \eta_B \), while the coincidences depend on their product \( \eta_A \eta_B \). This observation implies that, in the presence of external losses, the singles can achieve the absolute maximum \( FI \) in an \( SU(1,1) \) interferometer. This bound can be calculated by finding the maximum of equation (5) which yields:

\[
F_{SU(1,1)}^{max} = 4\eta_{max} g_1^2 T_A T_B.
\]

One can notice that this value is limited by the maximum detection efficiency \( \eta_{max} = \max\{\eta_A, \eta_B\} \).

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\(^3\) The simulation scripts are available upon request.
Figure 3. Visualization of the conditions in equation (7), representing the experimental conditions where it is advantageous to measure the singles in mode A instead of coincidences ($F_{\text{max}}^{\text{A}} / F_{\text{max}}^{\text{CC}} \geq F_{\text{max}}^{\text{B}}$), depending on the internal transmission $T_B$ and the detection efficiency $\eta_B$ of the other mode. The text in the different colored regions reports the conditions on the gains $g_1$ and $g_2$ where the given inequality holds true. The values of $\alpha$ and $\beta$ are reported in equation (8). Due to symmetry arguments, the same results hold true if we consider $F_{\text{max}}^{\text{B}}$. For a better understanding of the relation between the FI of the singles and the one of the coincidences in the case of imperfect detection, we compare in the following $F_{\text{max}}^{\text{CC}}$ and $F_{\text{max}}^{\text{A}}$ — note that the choice of mode A is arbitrary as mode A and mode B are interchangeable due to the vacuum seeding in both modes. With straightforward but rather lengthy calculations, it can be shown that the $F_{\text{max}}^{\text{A}}$ is greater than $F_{\text{max}}^{\text{CC}}$ in the following scenarios:

\[ F_{\text{max}}^{\text{A}} \geq F_{\text{max}}^{\text{CC}} \iff \begin{cases} 
\frac{g_2^2}{g_1^2} / g_1^2 \geq \beta & \text{if } T_B < \eta_B, \\
\frac{g_2^2}{g_1^2} \leq \alpha & \text{if } \eta_B \leq T_B \leq \frac{\eta_B}{1 - \eta_B + \eta_B^2}, \\
\text{Always} & \text{if } T_B > \frac{\eta_B}{1 - \eta_B + \eta_B^2}. 
\end{cases} \tag{7} \]

with

\[ \alpha = \frac{T_A(T_B - \eta_B)}{\eta_B(1 - \eta_B)} \quad \text{and} \quad \beta = \eta_B T_A \frac{1 - \eta_B T_B}{1 - \eta_B}. \tag{8} \]

An intuitive representation of the conditions in equation (7) is represented in figure 3. Note that, from symmetry arguments, analogous conditions can be found for $F_{\text{max}}^{\text{B}} \geq F_{\text{max}}^{\text{CC}}$.

The main conclusion of equation (7) is that, in the presence of detection losses, the FI of the singles eventually becomes greater than $F_{\text{max}}^{\text{CC}}$ if the gain $g_2$ is sufficiently high. Moreover, in the presence of low internal losses and relatively high detection losses, measurement of the singles always outperforms the coincidence measurements. This result is quite interesting, given that we have seen that the phase information is encoded in the photon pairs and not in the singles. What happens is that, thanks to the photon-number correlations of states generated in the nonlinear vacuum-seeded PDC stages, the singles at the output of the interferometer still retain the information carried by the interference of the photon pairs, which can be extracted even in the case of low detection efficiencies.

Similar to the case of ideal detection, we have simulated the effect of imperfect detection for a number of systems to investigate when our analytic theory starts to become invalid. For simplicity, the simulations
Figure 4. FI in a lossy SU(1,1) interferometer with imperfect detection. The gain of the first stage is set to $g_1 = 0.05$, and the internal transmissions are $T_A = 20\%$ and $T_B = 22\%$. The solid colour lines correspond to the analytic theory while the dashed colour lines correspond to the simulations (more details in the text). The darker lines correspond to the FI of the singles while the lighter lines correspond to the FI of the coincidences. For $g_2 > 0.01$, multiphoton components cause a deviation from the constant FI predicted by the analytic theory, especially for low detection efficiency.

considered an interferometer with $T_A = 20\%$ and $T_B = 22\%$, corresponding to case (3) in figure 2, and with varying but equal detection efficiencies in both arms, i.e. $\eta_A = \eta_B = \eta_{\text{detection}}$.

The results of the simulations are displayed in figure 4. In agreement with our analytic theory, one can see that, when the detection efficiency is lower than the internal transmission of the system ($\eta = 10\%$), the FI of the singles is always above the FI of the coincidences. As the detection efficiency increases, the FI of the coincidences becomes higher than the one of the single, for $g_2/g_1 < \beta$.

Finally, the figure shows that the deviation between the analytic theory and the numerical simulations now depends on the detection efficiency $\eta$. This is somewhat expected since now the photon statistics of the state that exits the interferometer can be affected by the presence of detection losses. Interestingly, for low detection efficiencies, we observe higher FI for both singles and coincidences than predicted by the analytic theory, in contrast with what was observed in the cases of high detection efficiency. We attribute this to the fact that having multiphoton components now help to overcome the high detection losses.

2.3. Optimization of a low-gain, lossy SU(1,1) interferometer

The results presented so far provide different strategies for maximizing the FI measured using a low-gain, lossy SU(1,1) interferometer, assuming the gain $g_1$ is constrained. We can consider two main scenarios: in the first one, we are allowed to freely choose the gain $g_2$ of the second stage; in the second one, we are restricted to a maximum level $g_2$, e.g. due to laser power limitations.

In the first case, the results of the last section show that one can simply set the interferometer such that $g_2 \gg g_1$, as we have demonstrated that, for this condition, the FI of the singles is as good as the FI of the coincidences—or even better, in the presence of detection losses.

In the second case, a more refined analysis of the interferometer is necessary. In particular, it is necessary to quantify both the internal losses $T_{A/B}$ and the detection losses $\eta_{A/B}$ of the system. The theory exposed so far provides insight in how to derive these parameters with simple measurements. The first step is to characterise the detection losses of the setup. This can be done simply by pumping only the second stage of the interferometer and measuring the Klyshko efficiencies [45], which correspond to the detection losses. Then, the first stage is pumped with gain $g_1$ while the gain $g_2$ is scanned and the visibilities $V_A$, $V_B$ and $V_{CC}$
are monitored. In this way, it is possible to use equation (2) to evaluate the values of the internal transmissions $T_A$ and $T_B$. Finally, now that both the internal and the detection losses are known, one can use equation (7) to determine the optimal $g_2$. As a general rule of thumb, if the internal transmission is higher than the detection efficiency, it is advised to measure the singles; if, on the other hand, the internal transmission is lower than the detection efficiency, then it is better to measure the coincidences when $g_2^A < g_2^B$, and the singles when $g_2^A > g_2^B$.

Interestingly, there is a second way to determine the internal transmissions of the system. At loss-balanced gains ($g_2^A = g_2^B T_A T_B$), one can use equations (2a) and (2b) to easily calculate the transmissions of modes $A$ and $B$ inside the interferometer as

\[ T_A = \frac{V_B}{2 - V_B}, \quad (9a) \]
\[ T_B = \frac{V_A}{2 - V_A}. \quad (9b) \]

This result is quite striking: it is possible to use a loss-balanced SU(1,1) interferometer to estimate the losses inside the interferometer by simply measuring the visibility of the singles.

3. Comparison between SU(2) and SU(1,1) interferometers

In the previous section, we have discussed the properties of lossy, low-gain SU(1,1) interferometers and how internal and external losses impact on the FI of the measured singles and coincidences. However, it is not obvious yet when it is advantageous to implement these interferometers instead of classical MZIs. In other words, we have not discussed in which experimental conditions SU(1,1) interferometers can beat the SQL. To answer this question, in this last part of our work, we investigate in which context it is beneficial to set up an SU(1,1) interferometer, as opposed to a classical MZI—also known as SU(2) interferometer. Please note that, SU(1,1) interferometers have the ability of coherently combining two fields with different polarizations and wavelengths, while SU(2) interferometers can only coherently combine the fields having same polarization and frequency.

A fair comparison between the two interferometers can be performed by analysing their FIs under the assumption that they are using the same amount of resources. This is easier said than done since the choice of which are the resources to be counted is not unique. Possible choices in the SU(1,1) interferometer are the total number of photons generated at the output of the interferometer, the number of photon pairs generated in the first stage, the number of photons passing through the sample, or the number of photons in the pump field that is used to generate the photon pairs.

In the following, we decide to consider as resources the number of photons passing through the sample. This choice has both a practical and a theoretical motivation. From an experimental perspective, we are quite often interested in keeping track of how many photons interact with the sample under test, in particular when it can be damaged. From a theoretical perspective, this choice is motivated by the fact that, in an SU(1,1) interferometer, the FI is ultimately limited by the number of photon pairs generated by the first stage (see equation (6)), which also corresponds to the number of photons that pass through the sample. For this reason, we compare the FI of classical SU(2) and SU(1,1) interferometers assuming that the same number $\langle n \rangle = g_1^2 \ll 1$ of photons is used to probe the phase element. In the case of an SU(1,1) interferometer, this corresponds to setting the gain of the first stage to $g_1^2 = \langle n \rangle$. In the case of a classical SU(2) interferometer, this corresponds to injecting the MZI with a coherent state $|\alpha\rangle$ having $|\alpha|^2 = 2\langle n \rangle$. Note that here we consider $g_1$ to be fixed, since we are not interested here in the scaling properties of the two systems as a function of the mean photon number $\langle n \rangle$ inside the interferometer as both systems scale with $\sqrt{\langle n \rangle}$ for $\langle n \rangle \ll 1$ [9, 19].

To compare SU(1,1) and SU(2) interferometers, we need to describe the SU(2) interferometer in the same framework that was used in the previous sections. The system under consideration is the MZI sketched in figure 5. It consists of a 50:50 input beam splitter, an output beam splitter with variable reflectivity $R$. The internal transmissions and detection efficiencies are modelled similarly to the SU(1,1) interferometers, with beam splitters with transmissions $T_{A/B}$ and $\eta_{A/B}$, respectively. We consider the state $|\alpha, 0\rangle$ as input, where a coherent state with amplitude $|\alpha|^2 = 2\langle n \rangle = 2g_1^2 \ll 1$ is injected in one input arm and vacuum is present at the other. For the system described, the probability of measuring singles in mode $A/B$ is given by

\[ P_{A/B} = \eta_{A/B} g_1^2 \left[ (1 - R) T_{A/B} + RT_{B/A} - 2\sqrt{(1 - R)RT_A T_B} \cos(\phi) \right]. \quad (10) \]

We do not consider here the probability of coincidence events between the output ports of the MZI because it tends to zero, since we are injecting a state with $\langle n \rangle \ll 1$. 


Following the same strategy as in section 2, it is possible to calculate the maximum $FI$ for the SU(2) interferometer. If $\eta_{\text{max}} = \max \{\eta_A, \eta_B\}$ is the highest detection efficiency of the setup, the maximum $FI$ of the system under study is given by

$$FI_{\text{SU}(2)} = 2\eta_{\text{max}}|\alpha|^2 \frac{T_A T_B}{T_A + T_B}$$  \hspace{1cm} (11a)

$$= 4\eta_{\text{max}} \frac{g^2}{T_A + T_B}.$$  \hspace{1cm} (11b)

One can use the upper bounds of equations (6) and (11) to quickly compare different SU(1,1) and SU(2) systems. In fact, these equations can be used to answer two interesting questions: for the classical system, how many more photons need to interact with the sample to achieve the same $FI$ of an identical SU(1,1) interferometer? And, for the same number of resources, how high should the detection efficiency of a classical system be, in order to achieve the same performance of an identical SU(1,1) interferometer? To answer the first question, we consider equations (6) and (11a) to be equal and assume that internal transmissions and detection efficiencies are the same. In this way, one derives the mean photon number impinging on the phase element in an SU(2) interferometer as $T_A + T_B$ times higher than the mean photon pairs generated by the first stage in an SU(1,1) interferometer. Similarly, to answer the second question, we consider equations (6) and (11b) to be equal and assume that the internal transmissions of the two systems are the same. Solving for the detection efficiency, one can derive that an SU(2) interferometer needs a detection efficiency that is $T_A + T_B$ times higher than the one of an SU(1,1) interferometer in order to achieve the same $FI$. For the same interferometers, if the mean photon number sampling the phase element is the same, the SU(2) interferometer needs a detection efficiency 20% higher than an SU(1,1) interferometer.

The result of this analysis shows that the sum of the internal transmission of an SU(1,1) interferometer is the figure of merit to compare it to a classical SU(2) interferometer. This means that SU(1,1) interferometers characterized by low losses and high-quality detectors can gain up to a factor of 2 over the SQL.

As we have seen from the examples above, it appears that one of the main metrics of comparison between SU(1,1) and SU(2) interferometers is the sum of the internal transmissions $T_A + T_B$. Indeed, using equations (6) and (11) to compare the $FI$ performance of SU(1,1) and the SU(2) interferometers with the
same internal transmissions $T_A/B$ and the same maximum detection efficiency $\eta_{\text{max}}$ in the asymptotic limit $g^2_A \gg g^2_1$, we discover that the SU(1,1) interferometer can outperform the SU(2) systems when

$$F_{SU(1,1)}^{\text{max}} > F_{SU(2)}^{\text{max}} \iff T_A + T_B > 1. \quad (12)$$

We call this condition asymptotic conditional advantage because it was derived in the asymptotic limit and because it is conditioned to both systems having the same internal losses.

One can also perform a stricter comparison and consider a lossy SU(1,1) and a lossless SU(2) interferometer. In this case, the SU(1,1) system can still outperform the classical system when

$$F_{SU(1,1)}^{\text{max}} > F_{\text{ideal SU(2)}}^{\text{max}} \iff 2\eta_{\text{max}}T_AT_B > 1. \quad (13)$$

We refer to this condition as asymptotic unconditional advantage because we have imposed no constraints on the classical system.

The conditions (12) and (13) are very handy rules of thumb to assess whether the SU(1,1) interferometers can, in general, beat classical systems. For example, assuming equal transmissions $T_A = T_B = T$ and perfect detection, one can see that asymptotic conditional advantage can be achieved with internal transmission $T > 50\%$ while asymptotic unconditional advantage can be achieved with $T > 71\%$.

The asymptotic conditions are useful to determine whether the SU(1,1) interferometers can, in principle, beat the SQL. However, they do not provide direct information of what are the experimental conditions that allow one to actually beat the SQL. The answer to this question can be obtained with a direct comparison of equations (5) and (11). In this way, we calculate the minimum gain ratio $g^2_A/g^2_1$ necessary to achieve the conditional/unconditional advantage, when considering both the internal and the external transmissions of the interferometers. The results of this comparison are quite involved and are reported in table 1.

As a final comment, it is clear that also other systems can beat the SQL in the regime of $\langle n \rangle < 1$ considered here, e.g. NOON state interferometry [46]. However, the analysis presented here highlights that SU(1,1) interferometers can perform well even in the presence of relatively high internal losses, which typically constitutes a problem for other quantum-enhanced systems.

4. Conclusions

In this paper, we derived and discussed the classical FI of lossy SU(1,1) interferometers when states with $\langle n \rangle < 1$ are generated inside the setup and when utilizing click detectors. We have shown that, in the case of lossless detection, coincidence measurement always performs better than measuring a single arm and that the classical FI of the coincidence measurement saturates to the highest possible value when the interferometer is loss-balanced. Interestingly, when detection losses are considered, we found that measuring the single events can yield higher FI than coincidences.

Finally, we discussed the conditions where lossy SU(1,1) interferometers outperform their classical SU(2) counterparts, in the limits of low photon numbers inside the setup. We discovered that the main figure of merit of an SU(1,1) interferometer is the sum of the internal transmissions $T_A + T_B$. When this parameter is above 1, the SU(1,1) interferometer can be more efficient than a similar SU(2) interferometer.

The results presented in this paper are of fundamental importance for the understanding of SU(1,1) interferometers, in particular in the regime of low mean photon numbers. Our findings show the power of photon-number correlations and how they can help overcome the presence of internal losses inside the interferometer.

These findings are of further significance for the implementation of SU(1,1) interferometers for the investigation of lossy and/or photosensitive samples and thus will likely impact the future development of these interferometers for practical applications.
Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors. All data that support the findings of this study are included within the article (and any supplementary files).

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