PRIMITIVE, PROPER POWER, AND SEIFERT CURVES IN THE BOUNDARY OF A GENUS TWO HANDLEBODY

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Abstract. A simple closed curve $\alpha$ in the boundary of a genus two handlebody $H$ is primitive if adding a 2-handle to $H$ along $\alpha$ yields a solid torus. If adding a 2-handle to $H$ along $\alpha$ yields a Seifert-fibered space and not a solid torus, the curve is called Seifert. If $\alpha$ is disjoint from an essential separating disk in $H$, does not bound a disk in $H$, and is not primitive in $H$, then it is said to be proper power.

As one of the background papers of the classification project of hyperbolic primitive/Seifert knots in $S^3$ whose complete list is given in [BK20], this paper classifies in terms of R-R diagrams primitive, proper power, and Seifert curves. In other words, we provide up to equivalence all possible R-R diagrams of such curves. Furthermore, we further classify all possible R-R diagrams of proper power curves with respect to an arbitrary complete set of cutting disks of a genus two handlebody.

1. Introduction and main results

In this paper, we provide the classifications of three types of simple closed curves lying in the boundary of a genus two handlebody: primitive, proper power, and Seifert curves. These classifications will be used in the classification project of hyperbolic primitive/Seifert knots in $S^3$ whose complete list is given in [BK20].

Primitive/Seifert (or simply P/SF) knots, which were introduced in [D03], are a natural generalization of primitive/primitive (or simply P/P) knots defined by Berge in [B90] or an available version [B18]. Both P/P knots and P/SF knots are represented by simple closed curves lying in a genus two Heegaard surface of $S^3$ bounding two handlebodies such that 2-handle additions to the handlebodies along the curves are solid tori for P/P knots and one 2-handle addition is a solid torus and the other is a Seifert-fibered space and not a solid torus for P/SF knots. One component of the intersection of a regular neighborhood of a knot and the Heegaard surface defines a so-called surface-slope.

Berge constructed twelve families of P/P knots which are referred to as the Berge knots. The Berge knots admit lens space Dehn surgeries at surface-slopes. The Berge conjecture, which is still unsolved, says that if a knot in $S^3$ admits a lens space Dehn surgery, then it is a Berge knot and the surgery is the corresponding surface-slope surgery. Therefore the conjecture implies the complete classification of knots admitting lens space Dehn surgeries. Toward the Berge conjecture, it is proved in [B08] or independently in [G13] that all P/P knots are the Berge knots. This implies that the Berge knots are the complete list of P/P knots.

Meanwhile, P/SF knot are also of interest, because P/SF knots admit Seifert-fibered Dehn surgeries at surface-surface slopes and knots with Dehn surgeries yielding Seifert-fibered spaces are not well understood. The classification project...
of hyperbolic primitive/Seifert knots in $S^3$ has been carried out for years and has recently been completed. The complete list of hyperbolic primitive/Seifert knots in $S^3$ is given in [BK20] where the surface-slope of the exceptional surgery on each P/SF knot that yields a Seifert-fibered space and the indexes of each exceptional fiber in the resulting Seifert-fibered space are also provided.

Now we describe the results of this paper, which is the classifications of primitive, proper power, and Seifert curves in the boundary of a genus two handlebody. The definitions of such curves are as follows.

**Definition 1.1.** Let $H$ be a genus two handlebody, $\alpha$ an essential simple closed curve in $\partial H$, and $H[\alpha]$ the 3-manifold obtained by adding a 2-handle to $H$ along $\alpha$.

1. $\alpha$ is said to be **primitive** if $H[\alpha]$ is a solid torus.
2. $\alpha$ is said to be **proper power** if $\alpha$ is disjoint from an essential separating disk in $H$, does not bound a disk in $H$, and is not primitive in $H$.
3. $\alpha$ is said to be **Seifert** if $H[\alpha]$ is a Seifert-fibered space and not a solid torus.

There are subtypes of Seifert curves in $H$. Since $H$ is a genus two handlebody, that $\alpha$ is Seifert in $H$ implies that $H[\alpha]$ is an orientable Seifert-fibered space over $D^2$ with two exceptional fibers, or an orientable Seifert-fibered space over the Möbius band with at most one exceptional fiber. Therefore, we further divide Seifert curves into two subtypes. If $H[\alpha]$ is Seifert-fibered over $D^2$, we say that $\alpha$ is **Seifert-d**. If $H[\alpha]$ is Seifert-fibered over the Möbius band, we say that $\alpha$ is **Seifert-m**.

The following theorems present the classifications of primitive, proper power, and Seifert curves. They are described in terms of R-R diagrams. For the definition and properties of R-R diagrams, see [K20].

**Theorem 1.2.** Suppose $\alpha$ is a simple closed curve in the boundary of a genus two handlebody $H$.

1. If $\alpha$ is a primitive curve, then $\alpha$ has an R-R diagram of the form shown in Figure 1.
2. If $\alpha$ is a Seifert-d curve, then $\alpha$ has an R-R diagram of the form shown in Figure 2 with $n, s > 1, a, b > 0$, and $\gcd(a, b) = 1$.
3. If $\alpha$ is a Seifert-m curve, then $\alpha$ has an R-R diagram of the form shown in Figure 3 with $s > 1$.
4. If $\alpha$ is a proper power curve, then $\alpha$ has an R-R diagram of the form shown in Figure 4 with $s > 1$.

Regarding Seifert curves, if $\alpha$ has an R-R diagram of the form shown in Figure 2a (b, respectively), then $H[\alpha]$ is a Seifert-fibered space over $D^2$ with two exceptional fibers of indexes $n$ and $s$ ($n(a+b)+b$ and $s$, respectively). If $\alpha$ has an R-R diagram of the form shown in Figure 3 then $H[\alpha]$ is a Seifert-fibered space over the Möbius band with one exceptional fiber of index $s$.

We can further classify proper power curves in the following theorem. From now on, to distinguish proper power curves from primitive or Seifert curves, we use the letter $\beta$ instead of $\alpha$ to represent proper power curves. We will see such a situation in Sections 4 and 5.
Figure 1. If $\alpha$ is a primitive curve in the boundary of a genus two handlebody $H$, then $\alpha$ has an R-R diagram with the form of this figure.

Figure 2. If $\alpha$ is a Seifert-d curve in the boundary of a genus two handlebody $H$, then $\alpha$ has an R-R diagram with the form of one of these figures with $n, s > 1$, $a, b > 0$, and $\gcd(a, b) = 1$. If $\alpha$ has an R-R diagram of the form shown in Figure 2a (2b, respectively), then $H[\alpha]$ is a Seifert-fibered space over $D^2$ with two exceptional fibers of indexes $n$ and $s$ ($n(a+b)+b$ and $s$, respectively).

Figure 3. If $\alpha$ is a Seifert-m curve in the boundary of a genus two handlebody $H$, then $\alpha$ has an R-R diagram with the form of this figure with $s > 1$ in which $\alpha = AB^*A^{-1}B^*$ in $\pi_1(H)$ and $H[\alpha]$ is a Seifert-fibered space over the Möbius band with one exceptional fiber of index $s$. 
Theorem 1.3 (Further classification of proper power curves). Suppose \( H \) is a genus two handlebody with a complete set of cutting disks \( \{D_A, D_B\} \) with \( \pi_1(H) = F(A, B) \), where the generators \( A \) and \( B \) are dual to \( D_A \) and \( D_B \) respectively. If \( \beta \) is a proper power curve in \( H \), then \( \beta \) has one of the following R-R diagrams with respect to the complete set of cutting disks \( \{D_A, D_B\} \) up to the homeomorphisms of \( H \) inducing the automorphisms exchanging \( A \) and \( B \), and replacing \( A^{-1} \) by \( A \) or \( B^{-1} \) by \( B \):

1. Type I: \( \beta \) has an R-R diagram with a 0-connection in at least one of the handles.
2. Type II: \( \beta \) has an R-R diagram of the form shown in Figure 4 with \( s > 1 \).
3. Type III: \( \beta \) has an R-R diagram of the form shown in Figure 5 with \( a, b > 0 \) and \( s > 0 \).
4. Type IV: \( \beta \) has an R-R diagram of the form shown in Figure 6 with \( a, b, c > 0 \).
5. Type V: \( \beta \) has an R-R diagram of the form shown in Figure 7 with \( a, b, c, d > 0 \).

The classifications of such curves play very important role in the classification of all hyperbolic primitive/Seifert(or simply P/SF) knots in \( S^3 \) whose complete list is given in [BK20]. The classifications of primitive and Seifert curves are the first step...
Figure 6. Type IV of proper power curves $\beta$: $[\beta] = (AB)^{a+b+c}$, where $a, b, c > 0$.

Figure 7. Type V of proper power curves $\beta$, where $a, b, c, d > 0$.

in the classification of hyperbolic P/SF knots in $S^3$. Additionally if $\alpha$ is Seifert in $H$ such that $H[\alpha]$ embeds in $S^3$, then $H[\alpha]$ is homeomorphic to the exterior of a torus knot. Therefore the classification of Seifert curves in $H$ naturally carries that of all simple closed curves $\alpha$ such that $H[\alpha]$ is homeomorphic to the exterior of a torus knot.

The classification of proper power curves has various applications. First it is used to determine if P/P(primitive/primitive) or P/SF knots are hyperbolic or not. It turns out that if P/P or P/SF knots are not hyperbolic, then there exists a proper power curve in some circumstances. Another application is that when a curve $\alpha$ is Seifert in $H$, i.e., $H[\alpha]$ is an orientable Seifert-fibered space, a proper power curve disjoint from $\alpha$ becomes a regular fiber of the Seifert-fibered space $H[\alpha]$ and can be used to compute the indexes of exceptional fibers. Also in order to classify some type of primitive/Seifert knots, called knots in Once-Punctured Tori(or simply OPT), properties of proper power curves are used.

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The present paper, which provides some of the background materials necessary to carry out the project, is originated from the joint work with John Berge for the project. I should like to express my gratitude to John Berge for his support and collaboration. I would also like to thank Cameron Gordon and John Luecke for their support while I stayed in the University of Texas at Austin.

2. R-R diagrams of primitive curves

In this section, we classify R-R diagrams of primitive curves in the boundary of a genus two handlebody. Let $\alpha$ be a primitive curve in a genus two handlebody $H$. In other words, $H[\alpha]$ is a solid torus. In order to obtain R-R diagrams of $\alpha$ we use the following lemma which shows equivalent conditions of primitivity. The proof may be found in [W36], [Z70], or [G87].

**Lemma 2.1.** The following are equivalent:

1. $\alpha$ is primitive in $H$, i.e., $H[\alpha]$ is a solid torus;
2. $\alpha$ belongs to a basis for the free group $\pi_1(H)$;
3. $\alpha$ is transverse to a properly embedded disk in $H$.

**Theorem 2.2.** If $\alpha$ is a nonseparating simple closed curve in the boundary of a genus two handlebody $H$ such that $\alpha$ is primitive on $H$, then $\alpha$ has an R-R diagram with the form of Figure 1.

**Proof.** By Lemma 2.1 there exists a cutting disk $D_A$ in $H$ such that $\alpha$ meets $\partial D_A$ transversely in a single point. Consider the regular neighborhood $N$ of $\partial D_A \cup \alpha$ in $\partial H$. Then $N$ is a once-punctured torus which contains $\partial D_A$, and whose boundary $\partial N$ bounds a separating disk of $H$. Let $N' = \partial H - N$. Then it follows by cutting $H$ open along $D_A$ that there exists a unique cutting disk $D_B$ of $H$ up to isotopy whose boundary $\partial D_B$ lies on $N'$.

This partition $\{N, N'\}$ of $\partial H$ with $\partial D_A \subset N$ and $\partial D_B \subset N'$ gives rise to an R-R diagram of $\alpha$ in which $N$ and $N'$ correspond to the $A$-handle and $B$-handle respectively. Since $\alpha$ lies in $N$ and intersects $\partial D_A$ in a single point, $\alpha$ has an R-R diagram of the form shown in Figure 1. This completes the proof. □

3. R-R diagrams of a proper power curve and more classifications

In this section, we classify proper power curves. Let $\beta$ be a proper power curve in the boundary of a genus two handlebody $H$. The following lemma is an easy consequence of Lemma 2.1.

**Lemma 3.1.** The following are equivalent:

1. $\beta$ is a proper power curve in $H$, i.e., $\beta$ is disjoint from an essential separating disk, does not bound a disk, and is not primitive in $H$;
2. $\beta$ is conjugate to $w^n$, $n > 1$, of $\pi_1(H)$, where $w$ is a free generator of $\pi_1(H)$;
3. There exists a complete set of cutting disks $\{D_A, D_B\}$ of $H$ such that $\beta$ is disjoint to, say, $D_A$ and is transverse to $D_B$ $n > 1$ times.

The main results of this section are the following.

**Theorem 3.2.** Suppose $\beta$ is a simple closed curve in the boundary of a genus two handlebody $H$. If $\beta$ is a proper power curve in $H$, then $\beta$ has an R-R diagram of the form shown in Figure 3.
Theorem 3.3. Suppose \( H \) is a genus two handlebody with a complete set of cutting disks \( \{D_A, D_B\} \) with \( \pi_1(H) = F(A, B) \), where the generators \( A \) and \( B \) are dual to \( D_A \) and \( D_B \) respectively. If \( \beta \) is a proper power curve in \( H \), then \( \beta \) has one of the following R-R diagrams with respect to the complete set of cutting disks \( \{D_A, D_B\} \) up to the homeomorphisms of \( H \) inducing the automorphisms exchanging \( A \) and \( B \), and replacing \( A^{-1} \) by \( A \) or \( B^{-1} \) by \( B \):

1. Type I: \( \beta \) has an R-R diagram with a 0-connection in at least one of the handles.
2. Type II: \( \beta \) has an R-R diagram of the form shown in Figure 4 with \( s > 1 \).
3. Type III: \( \beta \) has an R-R diagram of the form shown in Figure 5 with \( a, b > 0 \) and \( s > 0 \).
4. Type IV: \( \beta \) has an R-R diagram of the form shown in Figure 6 with \( a, b, c > 0 \).
5. Type V: \( \beta \) has an R-R diagram of the form shown in Figure 7 with \( a, b, c, d > 0 \).

The proof of Theorem 3.2 follows immediately from the definition of a proper power curve. In order to prove Theorem 3.3, we need the following minor generalization of a result of Cohen, Metzler, and Zimmerman [CMZ81] which allows one to determine easily if a given cyclically reduced word in a free group of rank two is a primitive or a proper power of a primitive.

Theorem 3.4. [CMZ81] Suppose a cyclic conjugate of

\[ W = A^{n_1}B^{m_1} \ldots A^{n_l}B^{m_l} \]

is a member of a basis of \( F(A, B) \) or a proper power of a member of a basis of \( F(A, B) \), where \( l \geq 1 \) and each indicated exponent is nonzero. Then, after perhaps replacing \( A \) by \( A^{-1} \) or \( B \) by \( B^{-1} \), there exists \( e > 0 \) such that:

\[ n_1 = \cdots = n_l = 1, \quad \text{and} \quad \{m_1, \ldots, m_l\} \subseteq \{e, e + 1\}, \]

or

\[ \{n_1, \ldots, n_l\} \subseteq \{e, e + 1\}, \quad \text{and} \quad m_1 = \cdots = m_l = 1. \]

The proof of Theorem 3.3 In order to find all possible R-R diagrams of a proper power curve \( \beta \), we consider the following cases separately.

1. \( \beta \) has a 0-connection in at least one of the handles in its R-R diagram.
2. \( \beta \) has no 0-connections in either handle in its R-R diagram.

The case where \( \beta \) has a 0-connection in one handle gives restriction to other curves in \( \partial H \) disjoint from \( \beta \). In other words, if \( \gamma \) is a simple closed curve in \( \partial H \) disjoint from \( \beta \), then \( \gamma \) must have only bands of connections labeled by 0 or 1 in that handle. Therefore we put this case into one type of proper power curves, which gives Type I in Theorem 3.2.

Now we assume that \( \beta \) has no 0-connections in either handle.

Suppose \( \beta \) has only one generator in \( \pi_1(H) = F(A, B) \). Then up to replacement of \( A \) with \( A^{-1} \), \( B \) with \( B^{-1} \), or exchange of \( A \) and \( B \), we may assume that \( [\beta] = B^s \) for some \( s > 1 \). Since \( \beta \) has no 0-connections, this implies that \( \beta \) has no connections in the \( A \)-handle and only one connection in the \( B \)-handle. Thus this case gives rise to Type II of a proper power curve in Theorem 3.2 with an R-R diagram of the form shown in Figure 4.
Now suppose that $\beta$ has the two generators $A$ and $B$ in $\pi_1(H)$. By Theorem 3.4 we may assume that $\beta = AB^{m_1} \cdots AB^{m_l}$, where $\{m_1, \ldots, m_l\} \subseteq \{s, s + \epsilon\}$ with $\epsilon = \pm 1$ and $\min\{s, s + \epsilon\} > 0$. This implies that every connection in the $A$-handle is labeled by 1. There are two cases to consider:

1. $\beta$ has only one band of 1-connections in the $A$-handle,
2. $\beta$ has two bands of 1-connections in the $A$-handle.

**Case (1):** Assume that $\beta$ has only one band of 1-connections in the $A$-handle. Suppose $\{m_1, \ldots, m_l\} \subseteq \{s, s + \epsilon\}$. Without loss of generality, we may assume that $\{m_1, \ldots, m_l\} = \{s\}$. If $s > 1$, then there must be only one band of $s$-connections in the $B$-handle, in which case $\beta$ is a primitive curve with $[\beta] = AB^s$. Therefore $s = 1$ and there must be two bands of 1-connections in the $B$-handle. Then with $A$ and $B$ exchanged, this case belongs to Type III of a proper power curve in Theorem 3.2 with an R-R diagram of the form shown in Figure 8.

Suppose $\{m_1, \ldots, m_l\} = \{s, s + \epsilon\}$. Then $\beta$ has at least two bands of connections in the $B$-handle. If $\beta$ has two bands of connections in the $B$-handle, $\beta$ must have an R-R diagram of the form shown in Figure 9 and by Lemma 3.5, $\beta$ is not a proper power curve. If $\beta$ has three bands of connections in the $B$-handle, then these three bands of connections should be labeled by 1, 2 and 1 respectively, and $\beta$ has the
Figure 9. An R-R diagram of \( \beta \) where \( s > 0, \epsilon = \pm 1 \), and \( a, b > 0 \).

R-R diagram of the form shown in Figure 11. By Lemma 3.6, \( \beta \) is not a proper power curve.

Case (2): \( \beta \) has two bands of 1-connections in the \( A \)-handle.

Suppose \( \{m_1, \ldots, m_l\} \subsetneq \{s, s + \epsilon\} \). Without loss of generality, we may assume that \( \{m_1, \ldots, m_p\} = \{s\} \). Then if \( \beta \) has only one band of connections in the \( B \)-handle, then the band must be labeled by \( s \), in which case Type III of a proper power curve in Theorem 3.2 arises with an R-R diagram of the form shown in Figure 5. If \( \beta \) has two bands of connections in the \( B \)-handle, then \( s \) must be 1 and this case yields Type IV of a proper power curve in Theorem 3.2 with an R-R diagram of the form shown in Figure 6.

Suppose \( \{m_1, \ldots, m_l\} = \{s, s + \epsilon\} \). If there are only two bands of connections in the \( B \)-handle, then \( \beta \) must have R-R diagram of the form shown in Figure 12 and by Lemma 3.7, \( \beta \) is not a proper power curve.

If there are three bands of connections in the \( B \)-handle, then \( \beta \) has two types of R-R diagrams as shown in Figure 8. By Lemma 3.8, the R-R diagram of \( \beta \) in Figure 8a cannot be a proper power curve. Therefore we have the R-R diagram of \( \beta \) in Figure 8b, which gives Type V of a proper power curve in Theorem 3.2.

Thus we complete the proof of Theorem 3.2.

Lemma 3.5. Suppose a simple closed curve \( \beta \) has an R-R diagram with the form shown in Figure 9 where \( s > 0, \epsilon = \pm 1 \) with \( \min\{s, s + \epsilon\} > 0 \), and \( a, b > 0 \). Then \( \beta \) is a primitive curve.

Proof. Note that since \( \beta \) is a simple closed curve, \( \gcd(a, b) = 1 \). Let \( b = \rho a + \eta \), where \( \rho \geq 0 \) and \( 0 \leq \eta < a \) (if \( \eta = 0 \), then \( \rho > 0 \) and \( a = 1 \)). Now we record the curve \( \beta \) algebraically by starting the \( a \) parallel arcs, i.e., the band of width \( a \), entering into the \( (s + \epsilon) \)-connection in the \( B \)-handle. It follows that \( \beta \) is the product of two subwords \( AB^{s+\epsilon}(AB^s)^\rho \) and \( AB^{s+\epsilon}(AB^s)^\rho+1 \) with \( \rho \geq 0 \) and \( |AB^{s+\epsilon}(AB^s)^\rho| = a - \eta \) and \( |AB^{s+\epsilon}(AB^s)^\rho+1| = \eta \). Here, for example, \( |AB^{s+\epsilon}(AB^s)^\rho| \) denotes the total number of appearances of \( AB^{s+\epsilon}(AB^s)^\rho \) in the word of \( \beta \) in \( \pi_1(H) = F(A, B) \).

There is a change of cutting disks of the handlebody \( H \), which induces an automorphism of \( \pi_1(H) \) that takes \( A \leftrightarrow AB^{-s} \) and leaves \( B \) fixed. Then by this change of cutting disks, \( AB^{s+\epsilon}(AB^s)^\rho \) and \( AB^{s+\epsilon}(AB^s)^\rho+1 \) are sent to \( A^{\rho+1}B^\epsilon \) and \( A^{\rho+2}B^\epsilon \) respectively. Therefore the resulting Heegaard diagram of \( \beta \) realizes
Figure 10. The R-R diagram after change of cutting disks of the handlebody $H$ inducing an automorphism of $\pi_1(H)$ that takes $A \mapsto AB^{-\eta}$ and leaves $B$ fixed.

Figure 11. An R-R diagram of $\beta$ where $a, b, c > 0$.

a new R-R diagram of the form in Figure 10 where the positions of the $A$ and $B$-handles are switched. The new R-R diagram has the same form as that in Figure 9 with less number of arcs. One can continue inductively until one of the labels of parallel arcs is 0, in which case $[\beta] = AB^j$ for some $j > 0$ up to replacement of $A$ with $A^{-1}$, $B$ with $B^{-1}$, or exchange of $A$ and $B$. Such a curve $\beta$ is a primitive curve.

Lemma 3.6. Suppose a simple closed curve $\beta$ has an R-R diagram with the form shown in Figure 11 where $a, b, c > 0$. Then $\beta$ is not a proper power curve.

Proof. First, note that $\gcd(a + b, b + c) = 1$ in order for $\beta$ to be a simple closed curve. Since $AB$ and $AB^2$ appear in $[\beta]$, by considering $\{AB, AB^2\}$ as a generating set of $F(A, B)$ and by Theorem 3.4 one of the two must appear with exponent 1. However, from the R-R diagram one can see that if one reads $\beta$ from the right-hand edge of the band of width $a$ entering the 1-connection in the $B$-handle, then $AB$ appears twice consecutively. Therefore $AB^2$ must have only exponent 1.

If $b = 1$, then $AB^2$ appears only once and thus $\beta$ is a primitive curve. Suppose $b > 1$. Consider following the band of width $b$ around the R-R diagram. Since $\beta$ is
Figure 12. An R-R diagram of $\beta$ where $s > 0$, $\epsilon = \pm 1$, and $a, b, c > 0$.

Figure 13. The hybrid diagram of $\beta$ and change of cutting disks of $H$.

Lemma 3.7. Suppose a simple closed curve $\beta$ has an R-R diagram with the form shown in Figure 12 where $s > 0$, $\epsilon = \pm 1$ with $\min\{s, s + \epsilon\} > 0$, and $a, b, c > 0$. Then $\beta$ is not a proper power curve.

Proof. We use the argument of hybrid diagrams which are introduced in [K20]. Consider the corresponding hybrid diagram as shown in Figure 13a. Then we drag the vertex $A^-$ together with the edges meeting the vertex $A^-$ over the $s$-connection in the $B$-handle. This corresponds to the change of the cutting disks of $H$ inducing an automorphism of $\pi_1(H)$ which takes $A \mapsto AB^{-s}$ and leaves $B$ fixed. The resulting hybrid diagram is shown in Figure 13b.
Transforming the hybrid diagram in Figure 13b back into an R-R diagram, there are three possible cases to consider as follows:

1. There is only one band of connections in the $A$-handle;
2. There are only two bands of connections in the $A$-handle;
3. There are three bands of connections in the $A$-handle.

Note from Figure 13b that since $b + c > 0$, all of the labels of bands of connections in the $A$-handle are greater than 0 and at least one of the bands of connections has label greater than 1.

1. Suppose that there is only one band of connections in the $A$-handle. Then $a$ must be 1 and $\beta = B^c A^{b+c+1}$, which is primitive.
2. Suppose that there are only two bands of connections in the $A$-handle. Let $p$ and $q$ be the labels of the two bands of connections. Since $b + c > 0$ and thus at least one of the bands of connections has label greater than 1, $p \neq q$. Theorem 3.4 forces $|p - q|$ to be 1. However, by Lemma 3.5 such a curve $\beta$ is a primitive curve.
3. Suppose that there are three bands of connections in the $A$-handle. Let $p$, $q$, and $r$ be the labels of the three bands of connections with $q = p + r$. In order for $\beta$ to be a proper power, $(p, q, r) = (1, 2, 1)$. However, by Lemma 3.6 $\beta$ is not a proper power curve.

Lemma 3.8. Suppose a simple closed curve $\beta$ has an R-R diagram with the form shown in Figure 14 where $a, b, c, d > 0$. Then $\beta$ is not a proper power curve.

Proof. Consider the corresponding hybrid diagram as shown in Figure 15a.

Then we drag the vertex $A^-$ together with the edges meeting the vertex $A^-$ over the 2-connection in the $B$-handle. This corresponds to the change of the cutting disks of $H$ inducing an automorphism of $\pi_1(H)$ which takes $A \mapsto AB^{-2}$ and leaves $B$ fixed. The resulting hybrid diagram is shown in Figure 15b.

Transforming the hybrid diagram in Figure 15b back into an R-R diagram, there are three possible cases to consider as follows:

1. There is only one band of connections in the $A$-handle;
2. There are only two bands of connections in the $A$-handle;
3. There are three bands of connections in the $A$-handle.
Figure 15. The hybrid diagram of $\beta$ and the change of cutting disks of $H$.

Note from Figure 15b that since $b > 0$, all of the labels of bands of connections in the $A$-handle are greater than 0 and at least one of the bands of connections has label greater than 1.

(1) Suppose that there is only one band of connections in the $A$-handle. Let $p$ be the label of the band of connections. Since there are the $b$ edges connecting $A^+$ and $A^-$, $p > 1$. On the other hand, in Figure 14, consider chasing back and forth the outermost arc in the band of width $a$ entering the 1-connection in the $A$-handle. This represents $ABAB\cdots$, which is carried into $AB^{-1}AB^{-1}\cdots$ by the automorphism $A \mapsto AB^{-2}$. This implies that $\beta$ has a 1-connection in the $A$-handle, a contradiction.

(2) Suppose that there are only two bands of connections in the $A$-handle. Let $p$ and $q$ be the labels of the two bands of connections. By the similar argument in the proof of Lemma 3.7, $|p - q| = 1$. However, by Lemma 3.7 such a curve $\beta$ cannot be a proper power.

(3) Suppose that there are three bands of connections in the $A$-handle. Let $p, q,$ and $r$ be the labels of the three bands of connections with $q = p + r$. In order for $\beta$ to be a proper power, $(p, q, r) = (1, 2, 1)$. Then by switching the $A$- and $B$-handles, and the signs of the labels, the R-R diagram of $\beta$ has the same form as in Figure 13 with less number of the arcs connecting the $A$- and $B$-handles. So if we can continue to perform the change of cutting disks of $H$, then since the number of the edges is strictly decreasing under the change of cutting disks, this case must eventually belong to the cases (1) and (2).

4. R-R diagrams of Seifert-d curves

In this section, we classify the R-R diagrams of a simple closed curve which is Seifert-d in a genus two handlebody. If $\alpha$ is a Seifert-d curve in a genus two handlebody $H$, then by its definition $H[\alpha]$ is a Seifert-fibered space over $D^2$ with two exceptional fibers. In order to compute the type of exceptional fibers in $H[\alpha]$, we need the following notation and lemma.

Notation. If $U = (a, b)$ is an element of $\mathbb{Z} \oplus \mathbb{Z}$, let $U^\perp$ denote the element $(-b, a)$ of $\mathbb{Z} \oplus \mathbb{Z}$, and let ‘$\cdot$’ denote the usual inner product or dot product of vectors.

Lemma 4.1. Let $U = (a, b)$, $V = (c, d)$ and $W = (e, f)$ be three elements of $\mathbb{Z} \oplus \mathbb{Z}$ such that $ad - bc = \pm 1$. If $W$ is expressed as a linear combination of $U$ and $V$, say $W = xU + yV$, then $y = \pm (U^\perp \cdot W)$.
Figure 16. If \( \alpha \) is a nonseparating simple closed curve in the boundary of a genus two handlebody \( H \) such that \( H[\alpha] \) is Seifert-fibered over \( D^2 \) with two exceptional fibers, then \( \alpha \) has an R-R diagram with the form of one of these figures with \( n, s > 1, a, b > 1, \) and \( \gcd(a, b) = 1. \) The converse also holds. (See Figure 17.)

Proof. We take an inner product by \( U^\perp \) on both sides of \( W = xU + yV. \) Then since \( U^\perp \circ U = 0 \) and \( U^\perp \circ V = \pm 1, \) the result follows.

The main result of this section is the following theorem.

**Theorem 4.2.** If \( \alpha \) is a nonseparating simple closed curve in the boundary of a genus two handlebody \( H \) such that \( H[\alpha] \) is Seifert-fibered over \( D^2 \) with two exceptional fibers, then \( \alpha \) has an R-R diagram with the form of Figure 16a with \( n, s > 1, \) or Figure 16b with \( n, s > 1, a, b > 0, \) and \( \gcd(a, b) = 1. \)

Conversely, if \( \alpha \) has an R-R diagram with the form of Figure 16a with \( n, s > 1, \) or Figure 16b with \( n, s > 1, a, b > 0, \) and \( \gcd(a, b) = 1, \) then \( H[\alpha] \) is Seifert-fibered over \( D^2 \) with two exceptional fibers of indexes \( n \) and \( s \) in Figure 16a, or indexes \( n(a + b) + b \) and \( s \) in Figure 16b.

In addition, a curve \( \beta \) shown in Figure 17 which is an augmentation of Figure 16 is a regular fiber of \( H[\alpha]. \)

Proof. We start by showing that if \( H[\alpha] \) is Seifert-fibered over \( D^2 \) with two exceptional fibers, then \( \alpha \) has an R-R diagram of the claimed form.

The key idea is that \( H[\alpha], \) which is defined to be the manifold obtained by adding a 2-handle to \( H \) along \( \alpha, \) induces a genus two Heegaard decomposition of a Seifert-fibered space over \( D^2 \) with two exceptional fibers. However Heegaard decompositions of a Seifert-fibered space over \( D^2 \) are well understood. For instance, Theorem 4.4 of Boileau, Rost and Zieschang, completely describes the genus two Heegaard diagrams of a Seifert-fibered space over \( D^2 \) with two exceptional fibers. Using this result, Theorem 4.5 shows how the Heegaard diagrams described in Theorem 4.4 translate into R-R diagrams. And then Lemma 4.6 adds a finishing detail to the proof of this direction by showing that it is always possible to assume that \( n > 1 \) in Figure 16.

With the proof of one direction of Theorem 4.2 finished, it remains to show that if \( \alpha \) has an R-R diagram with the form of Figure 16a with \( n, s > 1, \) or Figure 16b with \( n, s > 1, a, b > 0, \) and \( \gcd(a, b) = 1, \) then \( H[\alpha] \) is Seifert-fibered over \( D^2 \) with two exceptional fibers.
To see that this is the case, consider Figure 17 in which each of the R-R diagrams of Figure 16 has been augmented with a simple closed curve $\beta$ disjoint from $\alpha$. Then, in each diagram of Figure 17, two parallel copies of $\beta$ bound an essential separating annulus $A$ in $H$. (Figure 19 illustrates the situation when $s = 2$, and $D_A$ and $D_B$ are cutting disks of $H$ underlying the A-handle and B-handle of the R-R diagram of $\alpha$.)

Cutting $H$ apart along $A$ yields a genus two handlebody $W$ and a solid torus $V$. Note that $\alpha$ lies in $\partial W$ as a primitive curve in $W$ implying that $W[\alpha]$ is a solid torus, because any component of $D_B \cap W$ is a cutting disk $D_C$ of $W$ such that an R-R diagram of $\alpha$ with respect to $\{D_A, D_C\}$ of $W$ has the form in Figure 16 with $s$ replaced by 1, in which case $\alpha$ in Figure 16 intersects $D_C$ only once and thus is primitive, and $\alpha$ in Figure 19 is primitive by Lemma 3.5. It follows that $H[\alpha]$ is obtained by gluing the two solid tori $W[\alpha]$ and $V$ together along $A$. So $H[\alpha]$ is Seifert-fibered over $D^2$ with $\beta$ as a regular fiber and the cores of $W[\alpha]$ and $V$ as exceptional fibers.

The last step is to compute the indexes of the two exceptional fibers of $H[\alpha]$. It is clear that the annulus $A$ wraps around the solid torus $V$ $s$ times longitudinally, so the core of $V$ is an exceptional fiber of index $s > 1$. For the other index, it follows by computing $\pi_1((W[\alpha])[\beta])(= \pi_1((W[\beta])[\alpha]) = \mathbb{Z}_n$ that if $\alpha$ has an R-R diagram with the form of Figure 16b, then the core of $W[\alpha]$ is an exceptional fiber of index $n$.

Finally, suppose the R-R diagram of $\alpha$ has the form of Figure 16. In this case, Lemma 4.1 can be used to compute the index of the second exceptional fiber formed by the core of $W[\alpha]$.

Abelianizing $\pi_1(W)$, we have: $[\alpha] = (n(a + b) + b, a + b)$ and $[\beta] = (0, 1)$. By Lemma 4.1

$$[\beta] = \pm((\alpha^{-1} \circ [\beta]) = \pm((-a + b, n(a + b) + b) \circ (0, 1)) = \pm(n(a + b) + b)$$

in $H_1(W[\alpha])$. Thus the regular fiber $\beta$ wraps around the solid torus $W[\alpha]$ longitudinally $n(a + b) + b > 1$ times, so the core of $W[\alpha]$ is an exceptional fiber of index $n(a + b) + b$.

\[\square\]

**Remark 4.3.** (1) If $\alpha$ has an R-R diagram of the form shown in Figure 16, resp., $\alpha$ is said to be a Seifert-d curve of rectangular(non-rectangular, resp.) form.

(2) The regular fiber $\beta$ is a proper power curve representing $B^s$ in $\pi_1(H) = F(A, B)$, where $A$ and $B$ are dual to the cutting disks $D_A$ and $D_B$ underlying the A-handle and B-handle respectively of the R-R diagram. In addition, since the R-R diagram of Figure 16 is symmetric, the argument of the fifth paragraph in the proof above shows that that the proper power curve representing $A^n$ in the R-R diagram of Figure 16 is also a regular fiber of $H[\alpha]$.

Turning to the description of Heegaard decompositions of orientable Seifert-fibered spaces over $D^2$ with two exceptional fibers, let $S(\nu/p, \omega/q)$ denote an orientable Seifert-fibered space over the disk $D^2$ which has two exceptional fibers of types $\nu/p$ and $\omega/q$ with $0 < \nu < p$ and $0 < \omega < q$.

Also let $W_{m,n}(x, y)$ be the unique primitive word up to conjugacy in the free group $F(x, y)$ which has $(m, n)$ as its abelianization. Then, if $v$ and $w$ are words in
Figure 17. In this figure, each of the R-R diagrams of Figure 16 has been augmented with a simple closed curve $\beta$ disjoint from $\alpha$. Then, in each case, two parallel copies of $\beta$ bound an essential separating annulus $A$ in $H$. Cutting $H$ apart along $A$ yields a genus two handlebody $W$ and solid torus $V$. Then $\alpha$ lies in $\partial W$ and $W[\alpha]$ is a solid torus. It follows that $H[\alpha]$ is obtained by gluing two solid tori together along $A$. So $H[\alpha]$ is Seifert-fibered over $D^2$ with $\beta$ as regular fiber and the cores of $W[\alpha]$ and $V$ as exceptional fibers. As for the indexes of the exceptional fibers: If the R-R diagram of $\alpha$ has the form of Figure 16a, the indexes are $n$ and $s$ respectively. If the R-R diagram of $\alpha$ has the form of Figure 16b, the indexes are $n(a+b)+b$ and $s$ respectively.

Figure 18. If $H$ is a handlebody of genus two with cutting disks $D_A$ and $D_B$ and $\beta$ is a nonseparating simple closed curve in $\partial H$ such that $|\beta \cap \partial D_A| = 0$, and $|\beta \cap \partial D_B| = s > 1$, then two parallel copies of $\beta$ bound an essential separating annulus $A$ in $H$. This figure illustrates $D_A$, $D_B$, and $A$ in the special case $s = 2$.

Then the following theorem of Boileau, Rost and Zieschang (Theorem 5.4 in [BRZ88]) completely describes the genus two Heegaard diagrams of $S(\nu/p, \omega/q)$. For the notations in Theorem 4.4 see Sections 2, 4, and 5 in [BRZ88].

**Theorem 4.4.** [BRZ88] The manifold $S(\nu/p, \omega/q)$ admits three genus two Heegaard decompositions $HD_0$, $HD_S$, and $HD_T$, represented by the following Heegaard
Suppose $\nu$, $\omega$, $p$, and $q$ are positive integers such that $0 < \nu < p$, $0 < \omega < q$, $\gcd(\nu, p) = \gcd(\omega, q) = 1$, and $H$ is a genus two handlebody. Then the manifold $H[\alpha]$, obtained by adding a 2-handle to $\partial H$ along a simple closed curve $\alpha$ in $\partial H$ that has an R-R diagram with the form of this figure, is a Seifert-fibered space over $D^2$ with exceptional fibers of types $\nu/p$ and $\omega/q$.

**Theorem 4.5.** The R-R diagrams in Figures 19, 20, and 21 correspond to the Heegaard diagrams $HD_0$, $HD_S$, and $HD_T$ of Theorem 4.4 respectively.

**Proof.** First, observe that the curves $\alpha$ in Figures 19, 20, and 21 represent $s^p t^{-q}$, $W_{p, \nu}(u^{-1}, t^q)$, and $W_{q, \omega}(v^{-1}, s^p)$ respectively in $\pi_1(H)$.

Next, consider the diagram of Figure 19 and let $C_A$ and $C_B$ be cores of the A-handle and B-handle of $H$. Then $C_A$ and $C_B$ are the exceptional fibers of the Seifert-fibration of $H[\alpha]$. Let $N(C_A)$ and $N(C_B)$ be closed regular neighborhoods of $C_A$ and $C_B$ respectively in $H$, and let $M_A$ and $M_B$ be meridional disks of $N(C_A)$ and $N(C_B)$ respectively.

Observe that the pair of dotted curves $\beta$, $\gamma_s$ on the A-handle of Figure 19 can be considered to lie on $\partial N(C_A)$, while the pair of dotted curves $\beta'$, $\gamma_t$ on the B-handle of Figure 19 can be considered to lie on $\partial N(C_B)$. Also observe that as indicated in Remark 4.3, the curves $\beta$ and $\beta'$ represent regular fibers of the Seifert-fibration of $H[\alpha]$. Then $\partial M_A = (\beta' \gamma_s^p)_{s^p}^{\pm 1}$ in $\pi_1(\partial N(C_A))$, while $\partial M_B = (\beta' \gamma_t^q)_{s^p}^{\pm 1}$ in $\pi_1(\partial N(C_B))$. So $C_A$ and $C_B$ are exceptional fibers of types $\nu/p$ and $\omega/q$ in the Seifert-fibration of $H[\alpha]$.

This leaves the diagrams of Figures 20 and 21. Since these diagrams are similar, we will only consider Figure 20 in detail. To start, note that the configuration of
Figure 20. Suppose $\nu$, $\omega$, $p$, and $q$ are positive integers such that $1 < \nu < p$, $0 < \omega < q$, and $\gcd(\nu, p) = \gcd(\omega, q) = 1$. In addition, suppose $a$, $b$, and $n$ are positive integers such that $a + b = \nu$, $n\nu + a = p$, and $H$ is a genus two handlebody. Then the manifold $H[\alpha]$, obtained by adding a 2-handle to $\partial H$ along a simple closed curve $\alpha$ in $\partial H$ that has an R-R diagram with the form of this figure, is a Seifert-fibered space over $D^2$ with exceptional fibers of types $\nu/p$ and $\omega/q$.

Figure 21. Suppose $\nu$, $\omega$, $p$, and $q$ are positive integers such that $0 < \nu < p$, $1 < \omega < q$, and $\gcd(\nu, p) = \gcd(\omega, q) = 1$. In addition, suppose $a$, $b$, and $n$ are positive integers such that $a + b = \omega$, $n\omega + a = q$, and $H$ is a genus two handlebody. Then the manifold $H[\alpha]$, obtained by adding a 2-handle to $\partial H$ along a simple closed curve $\alpha$ in $\partial H$ that has an R-R diagram with the form of this figure, is a Seifert-fibered space over $D^2$ with exceptional fibers of types $\nu/p$ and $\omega/q$.

The curves $\beta$ and $\gamma_t$ on the B-handle of Figure 20 is identical to that of $\beta'$ and $\gamma_t$ on the B-handle of Figure 19. Since $\beta$ is again a regular fiber in the Seifert-fibration of $H[\alpha]$ when $\alpha$ has an R-R diagram on $\partial H$ with the form of Figure 20, the core of the B-handle $C_B$ of $H$ is again an exceptional fiber of type $\omega/q$. The other exceptional fiber that exists in $H[\alpha]$ when $\alpha$ has an R-R diagram on $\partial H$ with the form of Figure 20 arises in a slightly different way.
Suppose $\alpha$ has an R-R diagram on $\partial H$ with the form of Figure 20. Let $W$ be the genus two handlebody obtained when $H$ is cut open along the essential separating annulus in $H$ bounded by two parallel copies of the regular fiber $\beta$ in Figure 20. Then $\alpha$ lies on $\partial W$, and $\alpha$ has an R-R diagram on $\partial W$ with the form of this figure. Then $W[\alpha]$ is a solid torus, and the curves $\beta$ and $\gamma_u$ are a basis for $\partial W[\alpha]$. By using Lemma 4.1 to compute the images of $\beta$ and $\gamma_u$ in $H_1(W[\alpha])$, it is possible to see that the core of $W[\alpha]$ is an exceptional fiber of type $\nu/p$ in the Seifert-fibration of $H[\alpha]$. As in Figure 18, two parallel copies of the regular fiber $\beta$ in Figure 20 bound an essential separating annulus $A$ in $H$. Cutting $H$ open along $A$ cuts $H$ into a genus two handlebody $W$ and a solid torus $V$ which has $C_B$ as its core. The curve $\alpha$ lies on $\partial W$, and the R-R diagram of $\alpha$ on $\partial W$ appears in Figure 22. Since $\alpha$ is primitive in $W$, $W[\alpha]$ is a solid torus $V'$. Let $C_{V'}$ be the core of $V'$. Then $C_{V'}$ is the second exceptional fiber of the Seifert-fibration of $H[\alpha]$.

Let $M$ be the meridional disk of $V'$, and note that the curves $\beta$ and $\gamma_u$ of Figure 22 lie on $\partial W$ and $\beta$ and $\gamma_u$ form a basis for $\pi_1(\partial W[\alpha])$. The next step is to obtain an expression for $\partial M$ in $\pi_1(\partial W[\alpha])$ in terms of the basis $\beta, \gamma_u$ of $\pi_1(\partial W[\alpha])$.

We can do this by using Lemma 4.1. Abelianizing $\pi_1(W)$, we have: $[\alpha] = (-n(a+b) - a, a + b) = (-p, \nu)$, $[\gamma_u] = (-1, 0)$, and $[\beta] = (0, 1)$. By Lemma 4.1

$$[\gamma_u] = \delta([\alpha] \perp [\gamma_u]) = \delta((\nu, p) \circ (-1, 0)) = -\delta \nu,$$

and

$$[\beta] = \delta([\alpha] \perp [\beta]) = \delta((\nu, p) \circ (0, 1)) = \delta p$$

in $H_1(W[\alpha])$, where $\delta = \pm 1$. It follows that $\partial M = (\beta \, \gamma_u) \pm 1$ in $\pi_1(\partial W[R])$. So $C_{V'}$ is an exceptional fiber of type $\nu/p$ in the Seifert-fibration of $H[\alpha]$ when $\alpha$ has an R-R diagram with the form of Figure 20.

Similarly, one sees that if $\alpha$ has an R-R diagram with the form of Figure 21, then $H[\alpha]$ is also Seifert-fibered over $D^2$ with two exceptional fibers of types $\nu/p$ and $\omega/q$. $\square$

It will be convenient to be able to assume that $n > 1$ in Figure 16b. The following lemma shows this can always be done.
Figure 23. Suppose \( \alpha \) has an R-R diagram on \( \partial H \) with the form of Figure [21]. Let \( W \) be the genus two handlebody obtained when \( H \) is cut open along the essential separating annulus in \( H \) bounded by two parallel copies of the regular fiber \( \beta \) in Figure [21]. Then \( \alpha \) lies on \( \partial W \), and \( \alpha \) has an R-R diagram on \( \partial W \) with the form of this figure. Then \( W[\alpha] \) is a solid torus, and the curves \( \beta \) and \( \gamma \) are a basis for \( \partial W[\alpha] \). By using Lemma 4.1 to compute the images of \( \beta \) and \( \gamma \) in \( H_1(W[\alpha]) \), it is possible to see that the core of \( W[\alpha] \) is an exceptional fiber of type \( \omega/q \) in the Seifert-fibration of \( H[\alpha] \).

Lemma 4.6. Suppose \( S(\nu/p, \omega/q) \) has an R-R diagram with the form of Figure 20 or 21 with \( n = 1 \). Then \( S(\nu/p, \omega/q) \) also has another R-R diagram with the form of Figure 20 or 21 in which \( n > 1 \).

Proof. Suppose a simple closed curve \( \alpha \) in the boundary of a genus two handlebody \( H \) has an R-R diagram with the form of Figure 20 with \( n = 1 \). For the form of Figure 21, the similar argument can apply. Note that since \( n = 1 \), \( p = 2a + b \) and \( \nu = a + b \). It is easy to see that the underlying Heegaard diagram of \( \alpha \) on \( \partial H \) does not have minimal complexity. Thus in order to have minimal complexity, we can perform a change of cutting disks of \( H \), i.e., replace the cutting disk \( D_B \) of \( H \) which underlies the \( B \)-handle with a new cutting disk \( D'_B \) by bandsumming \( D_B \) with the cutting disk \( D_A \) in the \( A \)-handle along the arc of \( \alpha \).

Specifically, for the weights \( a \) and \( b \) in the R-R diagram, since \( \gcd(a, b) = 1 \), we can let \( b = \rho a + r \), where \( \rho \geq 0 \) and \( 0 \leq r < a \). However we may assume \( r > 0 \), otherwise \( a = 1 \) and thus \( p = 2 + b \) and \( \nu = 1 + b \), which implies that \( \nu \equiv -1 \mod p \) and it follows from Theorem [4.4] that this Heegaard decomposition is homeomorphic to \( HD_0 \).

Now we record \( \alpha \) by starting the \( a \) parallel arcs entering into the \( -2 \)-connection in the \( A \)-handle. It follows from the R-R diagram that \( \alpha \) is the product of two subwords \( A^{-2}B^\rho(A^{-1}B)^\rho \) and \( A^{-2}B^\rho(A^{-1}B)^\rho+1 \) with \( |A^{-2}B^\rho(A^{-1}B)^\rho| = a - r \) and \( |A^{-2}B^\rho(A^{-1}B)^\rho+1| = r \). We perform a change of cutting disks of the handlebody \( H \) which induces an automorphism of \( \pi_1(H) \) that takes \( A^{-1} \mapsto A^{-1}B^{-q} \). Then by this change of cutting disks, \( A^{-2}B^\rho(A^{-1}B)^\rho \) and \( A^{-2}B^\rho(A^{-1}B)^\rho+1 \) are sent to \( A^{-1}B^{-q}(A^{-1})^\rho+1 \) and \( A^{-1}B^{-q}(A^{-1})^\rho+2 \). Therefore the resulting Heegaard diagram of \( \alpha \) has minimal complexity and realizes a new R-R diagram of the form in Figure 24, which is the same form as in Figure 20 with \( n > 1 \). \(\square\)
Figure 24. A new R-R diagram of $\alpha$ obtained by performing a change of cutting disks of the handlebody $H$ which induces an automorphism of $\pi_1(H)$ that takes $A^{-1} \mapsto A^{-1}B^{-q}$.

Remark 4.7. The following observations are relevant for Lemma 4.6.

1. The change of cutting disks of $H$ inducing the automorphism $A^{-1} \mapsto A^{-1}B^{-q}$ in $\pi_1(H)$ corresponds to the change of the cutting disk $D_B$ to a new cutting disk $D'_B$ which is obtained by bandsumming $D_B$ with the cutting disk $D_A$ along the arcs of $\alpha q$ times.

2. The diagram of Figure 24 corresponds to the Heegaard decomposition of $S((p-\nu)/p, -\omega/q)$, which is homeomorphic to $S(\nu/p, \omega/q)$ as desired.

5. R-R diagrams of Seifert-M curves

If $\alpha$ is a Seifert-M curve on a genus two handlebody $H$, then by its definition $H[\alpha]$ is a Seifert-fibered space over the Möbius band with at most one exceptional fiber. The main result of this section is the following theorem.

Theorem 5.1. If $\alpha$ is a nonseparating simple closed curve in the boundary of a genus two handlebody $H$ such that $H[\alpha]$ is a Seifert-fibered space $M$ over the Möbius band, then $\alpha$ has an R-R diagram of the form shown in Figure 25, and $\alpha$ represents $AB^sA^{-1}B^s$ in $\pi_1(H)$. There is no loss in taking $s > 0$, and then $M$ has an exceptional fiber if and only if $s > 1$, in which case, $s$ equals the index of the exceptional fiber of $M$.

Conversely, if $\alpha$ has an R-R diagram of the form shown in Figure 25, then $H[\alpha]$ is Seifert-fibered over the Möbius band with one exceptional fiber of index $s$ provided that $s > 1$. If $s = 1$ in Figure 25, then $H[\alpha]$ is Seifert-fibered space over the Möbius band with no exceptional fibers.

In addition, a curve $\beta$ shown in Figure 26 which is an augmentation of Figure 25 is a regular fiber of $H[\alpha]$.

Proof. Suppose $H[\alpha]$ is homeomorphic to a Seifert-fibered space $M$ over the Möbius band. Then $M$ contains an essential nonseparating annulus which is vertical in the Seifert fibration of $M$. (Such an annulus can be easily obtained by starting with a nonseparating arc in the Möbius band—taking care to choose an arc which misses any exceptional fiber of $M$—and then saturating that arc in the Seifert-fibration of $M$.)
Figure 25. If attaching a 2-handle to a genus two handlebody $H$ along a simple closed curve $\alpha$ yields a Seifert-fibered space over the Möbius band with one exceptional fiber of index $s > 1$, then $\alpha$ has an R-R diagram with the form of this figure in which $\alpha = AB^s A^{-1} B^s$ in $\pi_1(H)$.

Since $M$ is Seifert-fibered, and not a solid torus, $H[\alpha]$ is $\partial$-irreducible. This implies $\alpha$ intersects every cutting disk of $H$. Now a theorem of Eudave-Muñoz applies. It is shown in [EM94] that if $H[\alpha]$ contains an essential nonseparating annulus, then there exists an essential nonseparating annulus $A$ in $H$, with $\partial A$ and $\alpha$ disjoint, such that $A$ is essential in $H[\alpha]$. (Note that in [EM94] the definition of an essential annulus which is properly embedded in a 3-manifold $M$ is that it is incompressible and not $\partial$-parallel.) Furthermore, it follows from Lemmas 1.10, 1.11 and the argument following them in [H07] that $A$ is vertical in the Seifert-fibration of $H[\alpha]$.

This suggests looking for all possible R-R diagrams of $\alpha$ by starting with an R-R diagram $D$ of the boundary components of a nonseparating annulus $A$ in $\partial H$. Then any R-R diagram of $\alpha$ must be obtained by adding $\alpha$ to $D$ so that $\alpha$ is disjoint from the curves $\beta$ and $\hat{\beta}$ of $\partial A$, and $\alpha$ intersects every cutting disk of $H$.

Lemma 5.3 carries out the first step of this scheme by showing that if $A$ is a nonseparating essential annulus in a genus two handlebody $H$, and $\beta$ and $\hat{\beta}$ are the components of $\partial A$ in $\partial H$, then $\beta$ and $\hat{\beta}$ have an R-R diagram $D$ of the form shown in Figure 28.

Next, Lemma 5.5 shows that if $D$ is an R-R diagram with the form of Figure 28 and a simple closed curve $\alpha$ is added to $D$ so that $\alpha$ is disjoint from $\beta$ and $\hat{\beta}$ in $D$, and $\alpha$ intersects every cutting disk of $H$, then the resulting R-R diagram must have the form shown in Figure 31.

Lemmas 5.6 and 5.7 finish the argument by showing that if $H[\alpha]$ is Seifert-fibered over the Möbius band, then $(a, b) = (0, 1)$ in Figure 31 so Figure 31 reduces to Figure 25. In addition, these lemmas show that $H[\alpha]$ has an exceptional fiber if and only if $s > 1$, and when $s > 1$, $s$ equals the index of the exceptional fiber of $H[\alpha]$.

Now we prove the second statement of the theorem. First, suppose $\alpha$ and $\beta$ have an R-R diagram of the form shown in Figure 26 with $s$ replaced by 1, which is an augmentation of the R-R diagram of $\alpha$ in Figure 25. We will show that $H[\alpha]$ is a
Figure 26. Suppose \( s \) and \( \omega \) are positive integers such that \( 0 < 2\omega \leq s \), \( \gcd(s, \omega) = 1 \), and \( H \) is a genus two handlebody. Then the manifold \( H[\alpha] \), obtained by adding a 2-handle to \( \partial H \) along a simple closed curve \( \alpha \) in \( \partial H \) that has an R-R diagram with the form of this figure, is a Seifert-fibered space over the Möbius band with one exceptional fiber of type \( \omega/s \), whose regular fiber is the curve \( \beta \).

Seifert-fibered space over the Möbius band with no exceptional fibers whose regular fiber is represented by the curve \( \beta \). Consider a properly embedded nonseparating annulus \( A \) in \( H \) whose boundary consists of the curves \( \beta \) and \( \hat{\beta} \), where \( \hat{\beta} \) is a curve illustrated in Figure 28. Figure 27a shows the genus two handlebody \( H \), the simple closed curve \( \alpha \) on \( \partial H \), and the annulus \( A \) with \( \partial A = \beta \cup \hat{\beta} \) oriented, which realize the R-R diagram of \( \alpha \) and \( \beta \) in Figure 26 with \( s = 1 \) and the R-R diagram of \( \hat{\beta} \) in Figure 28. Let \( H/A \) be the manifold obtained by cutting \( H \) along \( A \). Let \( A_1 \), \( \beta_1 \), \( \hat{\beta}_1 \) and \( \beta_2 \), \( \hat{\beta}_2 \) be the copies of \( A \), \( \beta \), and \( \hat{\beta} \) in \( H/A \). Then it is easy to see that \( H/A \) is a genus two handlebody such that \( A_1 \) and \( A_2 \) together with \( \partial A_1 = \beta_1 \cup \hat{\beta}_1 \) and \( \partial A_2 = \beta_2 \cup \hat{\beta}_2 \) lie in \( \partial (H/A) \) as shown in Figure 27b.

Since \( \alpha \) is disjoint from \( A \), \( H[\alpha] \) is obtained from \( (H/A)[\alpha] \) by gluing the two copies \( A_1 \) and \( A_2 \) such that the orientations of their boundaries \( \beta_1 \cup \hat{\beta}_1 \) and \( \beta_2 \cup \hat{\beta}_2 \) match. However, we can observe from Figure 27 that \( \alpha \) is primitive in the genus two handlebody \( H/A \) and thus \( (H/A)[\alpha] \) is a solid torus. Thus gluing \( A_1 \) and \( A_2 \) in the boundary of the solid torus \( (H/A)[\alpha] \) yields Seifert-fibered over the Möbius band with no exceptional fibers such that \( \beta \) is a regular fiber. Therefore \( H[\alpha] \) is a Seifert-fibered space over the Möbius band with no exceptional fibers whose regular fiber is represented by the curve \( \beta \).

Now we suppose that \( \alpha \) and \( \beta \) have an R-R diagram of the form shown in Figure 26 with \( s > 1 \). Similarly as in the proof of Theorem 4.2, the two parallel copies of \( \beta \) bound an essential separating annulus \( A' \) in \( H \) as shown in Figure 18 which cuts \( H \) apart into a genus two handlebody \( W \) and a solid torus \( V \). Note that \( \alpha \) lies in the boundary of \( W \). To complete the proof, it suffices to show that \( W[\alpha] \) is a Seifert-fibered space over the Möbius band with no exceptional fibers and the curve \( \beta \) is a regular fiber of \( W[\alpha] \).

A component of \( D_B \cap W \) is a cutting disk \( D_C \) of \( W \) such that \( \alpha \) and \( \beta \) intersect \( D_C \) transversely once. This implies that the R-R diagram of \( \alpha \) and \( \beta \) with respect
Figure 27. The genus two handlebody $H$, the simple closed curve $\alpha$ on $\partial H$, and the annulus $A$ with $\partial A = \beta \cup \hat{\beta}$ oriented, which realize the R-R diagram of $\alpha$ in Figure 25 with $s = 1$ and the R-R diagram of $\beta$ and $\hat{\beta}$ in Figure 28 in a), and the manifold $H/A$ obtained by cutting $H$ along $A$ in b), which is a genus two handlebody.

Remark 5.2. In Theorem 5.1, if $s = 1$, then $H[\alpha]$ is a Seifert-fibered space over the Möbius band with no exceptional fibers and the curve $\beta$ is a regular fiber of $W[\alpha]$, as desired. □

(2) Figure 26 shows an R-R diagram of $\alpha$ such that $H[\alpha]$ is a Seifert-fibered space over the Möbius band with one exceptional fiber of type $\omega/s$, whose regular fiber is the curve $\beta$.

Lemma 5.3. Suppose $A$ is an essential nonseparating annulus properly embedded in a genus two handlebody $H$. Let $\beta$ and $\hat{\beta}$ be the two curves in $\partial H$ that form $\partial A$. Then the pair $\beta$, $\hat{\beta}$ have an R-R diagram that appears in Figure 28.

Proof. Given $A$ and its boundary components $\beta$ and $\hat{\beta}$, we claim the following.
Figure 28. If \( A \) is a nonseparating essential annulus in a genus two handlebody \( H \) with \( \partial A = \beta \cup \hat{\beta} \), then there exists \( s > 0 \) such that \( \beta \) and \( \hat{\beta} \) have an R-R diagram with the form of this figure.

Figure 29. Gluing two copies \( D' \) and \( D'' \) of \( D \) to the disk \( D^* \) of \( A \) along \( \sigma' \) and \( \sigma'' \) yields a disk \( D_C \).

Claim 5.4. There exists a complete set of cutting disks \( \{D_A, D_B\} \) of \( H \) such that one of the cutting disks, say \( D_A \), is disjoint from \( A \) and \( A \cap D_B \) consists of a set of \( s > 0 \) essential spanning arcs in \( A \).

Proof. First, we show that there exists a cutting disk of \( H \) disjoint from \( A \). Let \( D_A \) be a cutting disk of \( H \) which intersects \( A \) minimally.

Suppose \( D_A \cap A \neq \emptyset \). Then we may assume that \( D_A \) intersects \( A \) essentially and \( D_A \cap A \) consists of properly embedded disjoint arcs and disjoint circles. However by the incompressibility of \( A \), irreducibility of \( H \), and the minimality condition rule out circle intersections. Suppose \( \gamma \) is an outermost arc of \( D_A \cap A \) which cuts a disk \( D \) of \( A \). Then \( \gamma \) also cuts \( D_A \) into two subdisks \( D_1 \) and \( D_2 \) of \( D_A \). Consider two disks \( D \cup_\gamma D_1 \) and \( D \cup_\gamma D_2 \), which are obtained by gluing \( D \) and \( D_1 \), and \( D \) and \( D_2 \) respectively along \( \gamma \). Then since \( D_A \) is a cutting disk which means that it is nonseparating, at least one of \( D \cup_\gamma D_1 \) and \( D \cup_\gamma D_2 \) is nonseparting and thus is a cutting disk. But this cutting disk intersects \( A \) less than \( D_A \). This is a contradiction to the minimality.

Suppose \( \sigma \) is a spanning arc of \( D_A \cap A \) in \( A \) which is outermost in \( D_A \). Let \( D' \) and \( D'' \) be two copies of \( D \), and let \( \sigma' = D' \cap A \) and \( \sigma'' = D'' \cap A \) as shown in Figure 29. Also let \( D^* = A - N(\sigma) \) such that \( \sigma' \) and \( \sigma'' \) are included in \( \partial D^* \). Gluing the two copies \( D' \) and \( D'' \) to the disk \( D^* \) along \( \sigma' \) and \( \sigma'' \) yields a disk \( D_C \). Since \( A \) is nonseparating, from the construction \( D_C \) is nonseparating (and thus is a cutting disk) and also does not intersect \( A \), a contradiction. Therefore there exists a cutting disk \( D_A \) of \( H \) disjoint from \( A \).
Figure 30. This figure shows that a simple closed curve \( \alpha \) on \( \partial H \), which is disjoint from the curves \( \beta \) and \( \hat{\beta} \) forming \( \partial A \), can not contain both a 0-connection and a 1-connection on the A-handle of this diagram. (Otherwise \( \alpha \) is forced to spiral endlessly.)

Now let \( D_B \) be a cutting disk of \( H \) chosen so that \( \{D_A, D_B\} \) is a complete set of cutting disks of \( H \) and \( D_B \) intersects \( A \) minimally. By applying the same argument above, we can show that \( D_B \cap A \) consists of spanning arcs of \( A \). Note that \( |D_B \cap A| > 0 \), otherwise \( A \) would embed properly in a 3-ball and thus not be essential. This completes the proof of the claim.

By the claim, there exists a complete set of cutting disks \( \{D_A, D_B\} \) of \( H \) such that \( D_A \) is disjoint from \( A \) and \( A \cap D_B \) consists of a set of \( s > 0 \) essential spanning arcs in \( A \). Now consider the solid torus \( V \) obtained by cutting \( H \) open along \( D_A \). Then \( \partial V \) contains two disks \( D^+_A \) and \( D^-_A \), which are copies of \( D_A \). The simple closed curves \( \beta \) and \( \hat{\beta} \) also lie in \( \partial V \) and cut \( \partial V \) into two annuli, say \( A^+ \) and \( A^- \). Since \( \beta \) and \( \hat{\beta} \) are not isotopic in \( \partial H \), \( D^+_A \) lies in the interior of one of these annuli, and \( D^-_A \) lies in the interior of the other. It is easy to see that there exists an arc \( \tau \) in \( \partial V \) connecting \( D^+_A \) and \( D^-_A \) which \( |\tau \cap (\beta \cup \hat{\beta})| = 1 \). Furthermore since a regular neighborhood of \( D^+_A \cup D^-_A \cup \tau \) is a disk in \( \partial V \), we can isotope \( D^+_A \cup D^-_A \cup \tau \) keeping \( |\tau \cap (\beta \cup \hat{\beta})| = 1 \) so that \( D^+_A \cup D^-_A \cup \tau \) is disjoint from the meridional disk \( D_B \) of \( V \). Since \( |\tau \cap (\beta \cup \hat{\beta})| = 1 \), now we may assume that \( |\tau \cap \hat{\beta}| = 1 \).

Next, let \( T_A \) be a once-punctured torus in \( \partial H \) which is a regular neighborhood in \( \partial H \) of \( \tau \cup \partial D_A \), with \( T_A \) chosen so that \( T_A \) is disjoint from \( \partial D_B \), and so that \( T_A \) intersects \( \beta \cup \hat{\beta} \) minimally. Then, if \( D \) is the R-R diagram of \( \beta \) and \( \hat{\beta} \), whose A-handle corresponds to \( T_A \), then \( \hat{\beta} \) crosses the A-handle of \( D \) in a single connection without intersecting \( D_A \), while \( \beta \) lies completely in the B-handle of \( D \). So \( D \) has the form of the diagram in Figure 28.

Lemma 5.5. If \( D \) is an R-R diagram with the form of Figure 28 and a simple closed curve \( \alpha \) is added to \( D \) so that \( \alpha \) is disjoint from \( \beta \) and \( \hat{\beta} \) in \( D \), and \( \alpha \) intersects every cutting disk of \( H \), then the resulting R-R diagram must have the form shown in Figure 31.
Figure 31. If a 2-handle is added to the genus two handlebody $H$ of Figure 28 along a simple closed curve $\alpha$, disjoint from the curves $\beta$ and $\hat{\beta}$ forming $\partial A$, the annulus $A$ of Figure 28 is essential in $H[\alpha]$, and $\alpha$ intersects every cutting disk of $H$, then there are nonnegative integers $a$ and $b$ such that $\alpha$, $\beta$ and $\hat{\beta}$ have an R-R diagram with the form of this figure.

Proof. Since $\alpha$ must intersect every cutting disk of $H$, it must traverse both the A-handle and the B-handle of $D$. In particular, $\alpha$ must have 1-connections on the A-handle of $D$ and s-connections on the B-handle of $D$. Now Figure 30 shows that $\alpha$ can’t have both 1-connections and 0-connections on the A-handle of $D$, otherwise $\alpha$ is forced to spiral endlessly. So $\alpha$ has only 1-connections on the A-handle of $D$. Now it is not hard to see that there must exist nonnegative integers $a$ and $b$ such that the diagram of $\alpha$, $\beta$, and $\hat{\beta}$ has the form of Figure 31.

$\square$

Lemma 5.6. Suppose $\alpha$ has an R-R diagram with the form of Figure 31. $H[\alpha]$ is Seifert-fibered over the Möbius band, and $s > 1$ in Figure 31. Then $(a, b) = (0, 1)$, Figure 31 reduces to Figure 25 and $\alpha = AB^sA^{-1}B^s$ in $\pi_1(H)$. Furthermore, the core of the B-handle of $H$ is an exceptional fiber of index $s$ in the Seifert-fibration of $H[\alpha]$.

Proof. Since the annulus $A$ is vertical in the Seifert-fibration of $H[\alpha]$, its boundary components $\beta$ and $\hat{\beta}$ are regular fibers. Then as in Figure 18 two parallel copies of $\beta$ bound an essential separating annulus $A'$ in $H$ such that $A'$ is saturated and vertical in the Seifert-fibration of $H[\alpha]$, and $A'$ cuts $H$ into a solid torus $V$ and a genus two handlebody $W$, with $\alpha$ lying on $\partial W$.

Let $\lambda$ be the core of $V$. (Note that $\lambda$ is also a core of the B-handle of $H$.) Then, since $s > 1$, $\lambda$ is an exceptional fiber of index $s$ in the Seifert-fibration of $H[\alpha]$. And, since the Seifert-fibration of $H[\alpha]$ can have at most one exceptional fiber, $\lambda$ is the only exceptional fiber in $H[\alpha]$. It follows that the manifold $W[\alpha]$ is Seifert-fibered over the Möbius band with no exceptional fibers. Then, using well-known formulas for presentations of Seifert-fibered spaces, one gets $\pi_1(W[\alpha]) = \langle x, y | x^2y^2 \rangle$.

By a result of Zieschang in [Z77], $\pi_1(W[\alpha])$ has only one Nielsen equivalence class of generators. It follows that if $(x, y | R)$ is a one-relator presentation of $\pi_1(W[\alpha])$, then there is an automorphism of the free group $F(x, y)$ which carries $R$ onto a cyclic conjugate of $x^2y^2$ or its inverse.
Now it is not hard to see that one obtains a one-relator presentation of $\pi_1(W[\alpha])$ from the one-relator presentation $\langle A, B \mid \alpha \rangle$ of $\pi_1(H[\alpha])$ by setting $s = 1$ in Figure 31. But if $s = 1$ in Figure 31 then the Heegaard diagram underlying the R-R diagram in Figure 31 has a graph with the form of Figure 32. This graph shows $\alpha$ has minimal length under automorphisms of $F(A, B)$. It follows that if $\alpha$ is a cyclic conjugate of $A^2B^2$ or its inverse in $F(A, B)$, then $a + b = 1$. So $(a, b) = (0, 1)$. If $(a, b) = (1, 0)$, then $\alpha = ABA^{-1}B^{-1}$, which is a commutator, $\alpha$ separates $\partial H$, and $\alpha$ is not an automorph of $(A^2B^2)^{\pm 1}$ in $F(A, B)$. The only remaining possibility is $(a, b) = (0, 1)$. In this case, Figure 31 reduces to Figure 25, and $\alpha = AB^sA^{-1}B^s$ in $\pi_1(H)$, as desired. \[\Box\]

**Lemma 5.7.** Suppose $\alpha$ has an R-R diagram with the form of Figure 31. $H[\alpha]$ is Seifert-fibered over the Möbius band, and $s = 1$ in Figure 31. Then $(a, b) = (0, 1)$, Figure 31 reduces to Figure 25, and $\alpha = ABA^{-1}B$ in $\pi_1(H)$. Furthermore, the Seifert-fibration of $H[\alpha]$ has no exceptional fibers.

**Proof.** As in Lemma 5.6, the annulus $A$ is vertical in the Seifert-fibration of $H[\alpha]$, and its boundary components $\beta$ and $\tilde{\beta}$ are regular fibers. Then two parallel copies of $\beta$ bound a separating annulus $A'$ in $H$ such that $A'$ is saturated and vertical in the Seifert-fibration of $H[\alpha]$, and $A'$ cuts $H$ into a solid torus $V$ and a genus two handlebody $W$, with $\alpha$ lying on $\partial W$. Let $\lambda$ be the core of $V$. (Note that like the $s > 1$ case of Lemma 5.6, $\lambda$ is also a core of the B-handle of $H$, but unlike the $s > 1$ case, here $A'$ is parallel into $\partial H$.) Then, since $s = 1$, $\lambda$ is a regular fiber in the Seifert-fibration of $H[\alpha]$. By performing Dehn surgery on $\lambda$, one can change $H$ into another genus two handlebody $H'$ such that $H'[\alpha]$ is Seifert-fibered over the Möbius band with $\lambda$ as an exceptional fiber of index $s' > 1$. And then, since $H'[\alpha]$ is Seifert-fibered over the Möbius band, $\lambda$ must be its only exceptional fiber. This implies that the Seifert-fibration of $H[\alpha]$ had no exceptional fibers.

To finish, notice that the R-R diagram of $\alpha$ on $\partial H'$ is obtained from the diagram of $\alpha$ on $\partial H$ by replacing $s$ in Figure 31 with $s'$. Now the argument used in Lemma 5.6 applies and shows that, as before, $(a, b) = (0, 1)$ and the diagram of Figure 31 reduces to that of Figure 25. \[\Box\]

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