ON STABLE RANK OF $H^\infty$ ON COVERINGS OF FINITE BORDERED RIEMANN SURFACES

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ABSTRACT. We prove that the Bass stable rank of the algebra of bounded holomorphic functions on an unbranched covering of a finite bordered Riemann surface is equal to one.

1. Formulation of Main Results

Let $S'$ be a (not necessarily connected) unbranched covering of a finite bordered Riemann surface $S$. In this paper we continue the study initiated in [Br2] of the algebra $H^\infty(S')$ of bounded holomorphic functions on $S'$. (We write $H^\infty := H^\infty(\mathbb{D})$, where $\mathbb{D} \subset \mathbb{C}$ is the open unit disk.) It was shown in our previous work that algebras $H^\infty(S')$ and $H^\infty$ share many common properties (e.g., they are Hermite, their maximal ideal spaces are two-dimensional with vanishing second Čech cohomology groups, etc., see [Br2]–[Br4] for the corresponding results). The purpose of this paper is to prove that these algebras have also the same Bass stable rank. The latter notion is defined as follows.

Let $A$ be an associative ring with unit. For a natural number $n$ let $U_n(A)$ denote the set of unimodular elements of $A^n$, i.e.,

$$U_n(A) = \{(a_1, \ldots, a_n) \in A^n : Aa_1 + \cdots + Aa_n = A\}.$$ 

An element $(a_1, \ldots, a_n) \in U_n(A)$ is called reducible if there exist $c_1, \ldots, c_{n-1} \in A$ such that $(a_1 + c_1a_n, \ldots, a_{n-1} + c_{n-1}a_n) \in U_{n-1}(A)$. The stable rank $sr(A)$ is the least $n$ such that every element of $U_{n+1}(A)$ is reducible. The concept of the stable rank introduced by Bass [B] plays an important role in some stabilization problems of algebraic $K$-theory. Following Vaserstein [V2] we call a ring of stable rank 1 a $B$-ring. (We refer to this paper for some examples and properties of $B$-rings.)

In [T] Treil proved the following result.

**Theorem A.** Let $f, g \in H^\infty$, $\|f\|_{H^\infty} \leq 1$, $\|g\|_{H^\infty} \leq 1$ and

$$\inf_{z \in \mathbb{D}} (|f(z)| + |g(z)|) =: \delta > 0.$$ 

Then there exists a function $G \in H^\infty$ such that the function $\Phi = f + gG$ is invertible in $H^\infty$, and moreover $\|G\|_{H^\infty} \leq C$, $\|\Phi^{-1}\|_{H^\infty} \leq C$, where the constant $C$ depends only on $\delta$.

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By the Carleson corona theorem condition (1.1) is satisfied if and only if \((f, g) \in U_2(H^\infty)\). Hence, Treil’s theorem implies that \(H^\infty\) is a \(B\)-ring.

Theorem A was used by Tolokonnikov [To] to prove that algebras \(H^\infty(U)\) are \(B\)-rings for finitely connected domains and some Behrens domains \(U\). Until now no other classes of Riemann surfaces \(U\) for which \(H^\infty(U)\) are \(B\)-rings were known. In the present paper, we prove the following extension of Theorem A.

**Theorem 1.1.** Let \(S'\) be an unbranched covering of a finite bordered Riemann surface \(S\). Let \(f, g \in H^\infty(S')\), \(\|f\|_{H^\infty(S')} \leq 1\), \(\|g\|_{H^\infty(S')} \leq 1\) and

\[
\inf_{z \in S'}(|f(z)| + |g(z)|) =: \delta > 0.
\]

Then there exists a function \(G \in H^\infty(S')\) such that the function \(\Phi = f + gG\) is invertible in \(H^\infty(S')\), and moreover \(\max\{\|G\|_{H^\infty(S')}, \|\Phi^{-1}\|_{H^\infty(S')}\} \leq C\), where the constant \(C\) depends only on \(\delta\) and \(S\).

By the corona theorem for \(H^\infty(S')\) (see [Br2, Cor. 1.6]) condition (1.2) is satisfied if and only if \((f, g) \in U_2(H^\infty(S'))\). Hence, Theorem 1.1 implies

**Theorem 1.2.** \(H^\infty(S')\) is a \(B\)-ring.

**Remark 1.3.** It is known that every \(B\)-ring is Hermite (see, e.g., [V2, Thm. 2.7]), i.e., any finitely generated stably free right module over the ring is free (equivalently, any rectangular left-invertible matrix over the ring can be extended to an invertible matrix). Let \(J \subset H^\infty(S')\) be a closed ideal and \(H^\infty_J := \{c + f : c \in \mathbb{C}, f \in J\}\) be the unital closed subalgebra generated by \(J\). Then Corollary 1.2 implies that \(H^\infty\) is a \(B\)-ring (see, e.g., [V1, Thm. 4]); hence, it is Hermite. This gives a generalization of [Br3, Thm. 1.1] proved by a different method.

Let \(M_n(H^\infty(S'))\) be the algebra of \(n \times n\) matrices with entries in \(H^\infty(S')\) regarded as the subspace of bounded linear operators on \((H^\infty(S'))^n\) equipped with the operator norm. We use Theorem 1.1 to describe the structure of the group \(SL_n(H^\infty(S')) \subset M_n(H^\infty(S'))\) of matrices with determinant 1.

Recall that a matrix in \(SL_n(H^\infty(S'))\) is elementary if it differs from the identity matrix by at most one non-diagonal entry.

**Theorem 1.4.** Every matrix in \(SL_n(H^\infty(S'))\) of norm \(\leq M\) is a product of at most \((n - 1)(3n^2 + 1)\) elementary matrices whose norms are bounded from above by a constant depending only on \(M\), \(n\) and \(S\).

This result is new even for matrices with entries in \(H^\infty\).

The proof of Theorem 1.1 is based on Theorem A and some results of the author presented in [Br4] and [Br5] along with some topological results. In the next section we collect some results required for the proof of Theorem 1.1. The proof is given in Section 4.
2. Auxiliary Results

2.1. Let \( \mathfrak{M}(A) \) denote the maximal ideal space of a commutative complex unital Banach algebra \( A \), i.e., the set of nonzero homomorphisms \( A \to \mathbb{C} \) equipped with the Gelfand topology. In this part we present some facts about the maximal ideal space \( \mathfrak{M}(H^\infty(S')) \), where \( r : S' \to S \) is a (not necessarily connected but second-countable) unbranched covering of a bordered Riemann \( S \), see \[BF3\] Sect. 2, \[BF4\] Sect. 4 for details.

Recall that \( H^\infty(S') \) separates points of \( S' \) and the map \( \iota : S' \to \mathfrak{M}(H^\infty(S')) \) sending \( x \in S' \) to the evaluation functional \( \delta_x \in (H^\infty(S'))^* \) at \( x \) embeds \( S' \) into \( \mathfrak{M}(H^\infty(S')) \) as an open dense subset (the corona theorem for \( H^\infty(S') \)).

The covering \( r : S' \to S \) can be viewed as a fiber bundle over \( S \) with a discrete (at most countable) fiber \( F \). Let \( E(S, \beta F) \) be the space obtained from \( S' \) by taking the Stone-
Čech compactifications of fibres under \( r \). It is a normal Hausdorff space and \( r \) extends to a continuous map \( r_E : E(S, \beta F) \to S \) such that \( (E(S, \beta F), S, r_E, \beta F) \) is a fibre bundle on \( S \) with fibre \( \beta F \) and \( S' \) is an open dense subbundle of \( E(S, \beta F) \). Each \( f \in H^\infty(S') \) admits an extension \( \hat{f} \in C(E(S, \beta F)) \) and the algebra formed by such extensions separates points of \( E(S, \beta F) \). Thus \( \iota \) extends to a continuous injection \( \hat{i} : E(S, \beta F) \to \mathfrak{M}(H^\infty(S')) \), \( (\hat{i}(\xi))(f) := \hat{f}(\xi) \).

In what follows, we identify \( E(S, \beta F) \) with its image under \( \hat{i} \). Also, for \( K \subset S \) we set \( K' := r^{-1}(K) \), \( K_E := r_E^{-1}(K) \) and for a subset \( U \) of a topological space we denote by \( \hat{U} \), \( \check{U} \) and \( \partial U \) its interior, closure and boundary.

It is well known that \( S \) can be regarded as a domain in a compact Riemann surface \( R \) such that \( R \setminus \hat{S} \) is the finite disjoint union of open disks with analytic boundaries. Let \( A(S) \subset H^\infty(S) \) be the subalgebra of functions continuous up to the boundary. We denote by \( \hat{r} : \mathfrak{M}(H^\infty(S')) \to \hat{S} \) the continuous surjective map induced by the transpose of the homomorphism \( A(S) \to H^\infty(S') \), \( f \mapsto f \circ r \). Then \( E(S, \beta F) \) coincides with the open set \( \hat{r}^{-1}(S) \) and \( \hat{r}|_{E(S, \beta F)} = r_E \).

Let \( U \subset R \) be open such that \( V := U \cap \hat{S} \neq \emptyset \). Then \( \hat{r}^{-1}(V) \) is an open subset of \( \mathfrak{M}(H^\infty(S')) \) and due to the corona theorem \( \hat{V} := \hat{r}^{-1}(V) \), \( \check{V} := U \cap S \), is an open dense subset of \( \hat{r}^{-1}(V) \).

**Proposition 2.1.** Each \( f \in H^\infty(\check{V}') \) admits an extension \( \hat{f} \in C(\hat{r}^{-1}(V)) \).

**Proof.** We reduce the statement to some known results proved earlier by the author.

We have to extend \( f \) continuously to each point \( \xi \in \hat{r}^{-1}(V) \). The set \( \hat{r}^{-1}(V) \) is the disjoint union of the open set \( \check{V}_E = \hat{r}^{-1}(\check{V}) \) and the set \( \hat{r}^{-1}(V \cap \partial S) \). So we consider two cases.

1. \( \xi \in \hat{r}^{-1}(\check{V}) \).

Let \( O \subset \check{V} \) be an open simply connected neighbourhood of \( \hat{r}(\xi) \). By the definition of the bundle \( E(S, \beta F) \), the set \( O_E = r_E^{-1}(O) \) is homeomorphic to \( O \times \beta F \) and this homeomorphism maps \( O' = r^{-1}(O) \) biholomorphically onto \( O \times F \). Then Lemma 3.1 of
Let $\hat{\xi}$ be the Arens-Royden theorem implies that $f|_{O'} \in H^\infty(O')$ admits an extension $\hat{f} \in C(O_E)$ as required (because $O_E$ is an open neighbourhood of $\xi$).

According to [Br5, Lm. 5.3] the homomorphism of the Čech cohomology groups $H^\infty(C(S'), C^*)$ is holomorphically convex (with respect to the algebra $H^\infty(S')$) if for every $\xi \not\in K$ there is $f \in H^\infty(S')$ such that

$$\max_K |\hat{f}| < |\hat{f}(\xi)|;$$

here $\hat{f} \in C(M(H^\infty(S')))$ is the Gelfand transform of $f$.

A holomorphically convex subset $Z \subset M(H^\infty(S'))$ is called a hull if there is a proper ideal $I \subset H^\infty(S')$ such that

$$Z = \{ \xi \in M(H^\infty(S')) : \hat{f}(\xi) = 0 \quad \forall f \in I \}.$$ 

The algebra $H^\infty(S')$ is a $B$-ring if and only if for every hull $Z \subset M(H^\infty(S'))$ the map $C(M(H^\infty(S'))), C^*) \rightarrow C(Z, C^*)$, $C^* := C \setminus \{0\}$, induced by restriction to $Z$ is onto, see [CS].

In the next two lemmas, $S = \mathbb{D}$ and $S' = S \times \mathbb{N}$ (the countable disjoint union of open unit disks).

**Lemma 2.3.** If $K \subset M(H^\infty(S'))$ is holomorphically convex, then for every $g \in C(K, C^*)$, there exists $\tilde{g} \in C(M(H^\infty(S'))), C^*)$ such that $\tilde{g}|_K = g$.

**Proof.** According to [Br5, Lm. 5.3] the homomorphism of the Čech cohomology groups $H^1(M(H^\infty(S')), \mathbb{Z}) \rightarrow H^1(K, \mathbb{Z})$ induced by the restriction map to $K$ is surjective. In turn, by the Arens-Royden theorem $H^1(K, \mathbb{Z})$ and $H^1(M(H^\infty(S')), \mathbb{Z})$ are connected components of topological groups $C(K, C^*)$ and $C(M(H^\infty(S'))), C^*)$, respectively. Hence, for each $g \in C(K, C^*)$, there is $g_1 \in C(M(H^\infty(S'))), C^*)$ such that $g \cdot (g_1^{-1})|_K = e^h$ for some $h \in C(K)$.

Let $\tilde{h} \in C(M(H^\infty(S')))$ be an extension of $h$ (existing by the Titze-Urysohn theorem). Then $\tilde{g} = g_1 e^{\tilde{h}}$ is the required extension of $g$. \qed
Lemma 2.4. Suppose $K \subset \mathcal{M}(H^\infty(S'))$ is holomorphically convex and $Z \subset \mathcal{M}(H^\infty(S'))$ is a hull. Then $K \cup Z \subset \mathcal{M}(H^\infty(S'))$ is holomorphically convex.

Proof. Let $\xi \not\in K \cup Z$. By the hypothesis, there exist $f, g \in H^\infty(S')$ such that $\hat{f}(\xi) = \hat{g}(\xi) = 1$ and

$$\max_{K} |f| =: c < 1, \quad g|_{Z} = 0.$$  

Let $M := \max_{K} |g|$. We choose $n \in \mathbb{N}$ such that $c^n M < 1$. Then for $h := f^n g \in H^\infty(S')$ we have

$$\max_{K} |\hat{h}| \leq c^n M < 1 = |\hat{h}(\xi)|$$

This shows that the set $K \cup Z$ is holomorphically convex. \hfill \Box

2.3. For the proof of Theorem 1.1 we require the following topological result.

Lemma 2.5. There is a finite cover $(U_j)_{j=1}^m$ of $\bar{S}$ by compact subsets homeomorphic to $\mathbb{D}$ such that each $U_i$ is contained in an open simply connected set $V_i \subset R$ with simply connected intersection $V_i \cap S$, each $U_i$ intersects with at most two other sets of the family and each non-void $U_i \cap U_j$ is homeomorphic to $I := [0,1]$.

Proof. Since $\bar{S}$ is triangulable, we may regard it as a two dimensional polyhedral manifold. It follows from the Whitehead theorem [W Thm. (3.5)] that there are a (finite) one-dimensional polyhedron $L \subset \bar{S}$ with sets of edges $E_L$ and vertices $V_L$ and a piecewise linear strong deformation retraction $F: \bar{S} \times I \rightarrow \bar{S}$ of $\bar{S}$ onto $L$ such that

(a) $F^{-1}(x,1) \subset \bar{S}$ is a connected polyhedron homeomorphic to a star tree with internal vertex $x$ of degree 2 if either $x \in \partial e$ for some $e \in E_L$ or $x \in V_L$ is of degree $\leq 2$ and of degree $> 2$ if $x \in V_L$ is of degree $> 2$, and this homeomorphism maps $F^{-1}(x,1) \cap \partial S$ onto the set of external points of the tree.

(b) If $e \in E_L$, then $F^{-1}(e,1) \cap \partial S$ is the disjoint union of two sets homeomorphic to $I$.

Let $E_L := \{e_1, \ldots, e_m\}$. We define

$$U_i := F^{-1}(e_i,1), \quad 1 \leq i \leq m.$$  

Then every $U_i$ is a polyhedral submanifold of $\bar{S}$ homeomorphic to $\mathbb{D}$ with the boundary formed by some arcs in $\partial S$ along with some subsets of $F^{-1}(v_{ij},1), j = 1, 2$, homeomorphic to $I$; here $v_{i1}, v_{i2} \in V_L$ are endpoints of $e_i$. Clearly every non-void intersection $U_i \cap U_j \subset F^{-1}(e_i \cap e_j,1)$ is homeomorphic to $I$. Moreover, it is readily seen that each $U_i$ is contained in an open simply connected subset $V_i \subset R$ with simply connected intersection $V_i \cap S$ because $\bar{S}$ is the strong deformation retract of some of its open neighbourhoods in $R$ (see, e.g., [W Thm. (3.3)]). \hfill \Box

3. Proof of Theorem 1.2

We retain notation of Lemma 2.5. We set

$$\partial U_i^\circ = \overline{\partial U_i \setminus \partial S} \quad \text{and} \quad W_i := V_i \cap S, \quad 1 \leq i \leq m.$$  

Then $\partial U_i^\circ$ consists of two connected components homeomorphic to $I$ and $W_i$ is an open simply connected subset of $S$. By the definition, $\partial U_i^\circ \subset \overline{W_i}$. 

Let $A(W_i) \subset H^\infty(W_i)$ be the subalgebra of functions continuous up to the boundary. We denote by $\hat{r}_i : \mathcal{M}(H^\infty(W'_i)) \to \overline{W}_i$ the continuous surjective map induced by the transpose of the homomorphism $A(W_i) \to H^\infty(W'_i)$, $f \mapsto f \circ r_i$.

Let $K$ be either $\partial U^o_i$ or its connected component. We set

$$\overline{K} := \hat{r}_i^{-1}(K).$$

**Lemma 3.1.** The set $\overline{K} \subset \mathcal{M}(H^\infty(W'_i))$ is holomorphically convex.

**Proof.** By our construction the open set $V_i \setminus K$ is connected. By the Riemann mapping theorem there is a biholomorphic map $\psi_i$ of $V_i$ onto $\mathbb{D}$. Then $\mathbb{D} \setminus \psi_i(K)$ is a connected open subset of $\mathbb{D}$. This implies that the compact set $\psi_i(K) \subset \mathbb{C}$ is polynomially convex. Hence, $K \subset V_i$ is holomorphically convex with respect to the algebra $H^\infty(V_i)$ and so it is holomorphically convex in $\overline{W}_i$ with respect to the algebra $A(W_i)$. Since $\hat{r}_i$ is a surjection onto $\overline{W}_i$ and $K \subset \mathcal{M}(H^\infty(W'_i))$ is the preimage of $K$, it is holomorphically convex. \hfill $\Box$

Due to Remark 2.2 the transpose of the restriction homomorphism $H^\infty(S') \to H^\infty(W'_i)$ induces a continuous map $s_i : \mathcal{M}(H^\infty(W'_i)) \to \mathcal{M}(H^\infty(S'))$ with image $\hat{r}_i^{-1}(W'_i)$ one-to-one on $s_i^{-1}(\hat{r}_i^{-1}(V_i \cap \bar{S}))$.

Let $Z \subset \mathcal{M}(H^\infty(S'))$ be a hull and $g \in C(Z, C^*)$. To prove the theorem we have to extend $g$ to a function $\tilde{g} \in C(\mathcal{M}(H^\infty(S')), C^*)$, see Section 2.2 above.

Clearly $Z_i := s_i^{-1}(Z)$ is a hull for the algebra $H^\infty(W'_i)$ and $s_i^*g \in C(Z_i, C^*)$. Since $W'_i := r_i^{-1}(V_i)$ is biholomorphic to $\mathbb{D} \times F$, the Treil theorem implies that there is $g_i \in C(\mathcal{M}(H^\infty(W'_i)), C^*)$ which extends $s_i^*g (= g \circ s_i)$. Hence $\tilde{g}_i := g_i \circ s_i^{-1} \in C(\hat{r}_i^{-1}(V_i \cap \bar{S}), C^*)$ extends $g|_{Z \cap \hat{r}_i^{-1}(U_i)}$ (because $U_i \subset V_i \cap \bar{S}$). If $Z \cap \hat{r}_i^{-1}(U_i) = \emptyset$, we define $\tilde{g}_i = 1$.

Next, we order the sets of the cover $(U_i)_{i=1}^m$ as follows. Choose some $U_{i_1} \subset \{U_1, \ldots, U_m\}$. If $U_{i_p}$ is already chosen, we choose $U_{i_{p+1}}$ so that

$$U_{i_{p+1}} \cap (\cup_{j=1}^p U_{i_j}) \neq \emptyset.$$

This is always possible because $\bar{S}$ is a connected set. We extend $g$ by induction on the indices of the order.

For $j = 1$ we set $\tilde{g} = \tilde{g}_{i_1}$ on $\hat{r}_i^{-1}(U_{i_1})$. Suppose that $\tilde{g}$ is already defined on $\cup_{j=1}^p \hat{r}_i^{-1}(U_{i_j})$. Let us define it on $\cup_{j=1}^{p+1} \hat{r}_i^{-1}(U_{i_j})$. To this end let

$$g_{p,p+1} := \tilde{g}_{i_{p+1}}^{-1} \quad \text{on} \quad \hat{r}_i^{-1}(\cup_{j=1}^p U_{i_j}) \cap \hat{r}_i^{-1}(U_{i_{p+1}}).$$

By the definition, the above intersection, say $X$, is either the preimage of $\partial U^o_{i_{p+1}}$ or its connected component under $\hat{r}$. Due to Lemma 3.1 the set $s_{i_{p+1}}^{-1}(X) \cup Z_{i_{p+1}}$ is holomorphically convex with respect to $H^\infty(W_{i_{p+1}})$. Moreover, $s_{i_{p+1}}^*g_{p,p+1} \in C(s_{i_{p+1}}^{-1}(X), C^*)$ and equals 1 on $s_{i_{p+1}}^{-1}(X) \cap Z_{i_{p+1}}$. Hence, it can be extended to a function in $C(s_{i_{p+1}}^{-1}(X) \cup Z_{i_{p+1}}, C^*)$ attaining value 1 on $Z_{i_{p+1}}$. Due to Lemma 2.3 the extended function can further be extended to a function from $C(s_{i_{p+1}}^{-1}(\hat{r}_i^{-1}(U_{i_{p+1}})), C^*)$. Composing this extension with $s_{i_{p+1}}^{-1}$ we obtain an extension $\tilde{g}_{p,p+1}$ of $g_{p,p+1}$ equal to 1 on $Z \cap \hat{r}_i^{-1}(U_{i_{p+1}})$. Let us define

$$\tilde{g}|_{\hat{r}_i^{-1}(U_{i_{p+1}})} := \tilde{g}_{p+1} \tilde{g}_{p,p+1}.$$
Then \( \tilde{g}|_{\hat{r}^{-1}(U_{i+p+1})} \) extends \( g|_{Z \cap \hat{r}^{-1}(U_{i+p+1})} \) and

\[
\tilde{g}|_{\hat{r}^{-1}(U_{i+p+1})} \cdot \tilde{g}^{-1}|_{\hat{r}^{-1}(U_{i+p+1})} = \tilde{g}_{p+1} \tilde{g}_{p} \tilde{g}^{-1} = 1 \quad \text{on} \quad \hat{r}^{-1}(\bigcup_{j=1}^{p} U_{ij}) \cap \hat{r}^{-1}(U_{i+p+1}),
\]

i.e., \( \tilde{g}|_{\hat{r}^{-1}(U_{i+p+1})} \) is the required extension of \( \tilde{g}|_{\hat{r}^{-1}(U_{i+p+1})} \) to \( \hat{r}^{-1}(U_{ij}) \). This completes the proof of the induction step and hence of the theorem.

4. Proofs of Theorems 1.1 and 1.4

**Proof of Theorem 1.1.** Without loss of generality we may assume that \( S' \) is a connected unbranched covering of \( S \). Let \( f, g \in H^\infty(S') \), \( \|f\|_{H^\infty(S')} \leq 1 \), \( \|g\|_{H^\infty(S')} \leq 1 \) and

\[
(4.1) \quad \sup_{z \in S'} (|f(z)| + |g(z)|) =: \delta > 0.
\]

Due to Theorem 1.2 there exists a function \( G \in H^\infty(S') \) such that the function \( f + gG \) is invertible in \( H^\infty(S') \). By \( \mathcal{G}_{f,g,\delta,S'} \) we denote the class of such functions \( G \). We have to prove that

\[
(4.2) \quad C = C(\delta, S) := \sup_{f,g \in \mathcal{G}_{f,g,\delta,S'}} \inf_{G \in \mathcal{G}_{f,g,\delta,S'}} \max \{ \|G\|_{H^\infty(S')}, \|f + gG\|_{H^\infty(S')}^{-1} \}
\]

is finite. (Here the first supremum is taken over all functions \( f, g \) satisfying the above hypotheses and all connected unbranched coverings \( S' \) of \( S \).)

Let \( \{S'_i\}_{i \in \mathbb{N}} \) and \( \{f_i\}_{i \in \mathbb{N}}, \{g_i\}_{i \in \mathbb{N}}, f_i, g_i \in H^\infty(S'_i) \), be sequences satisfying assumptions of the theorem such that

\[
(4.3) \quad C = \lim_{i \to \infty} \inf_{G \in \mathcal{G}_{f_i,g_i,\delta,S'_i}} \max \{ \|G\|_{H^\infty(S'_i)}, \|(f_i + g_iG)^{-1}\|_{H^\infty(S')} \}.
\]

The disjoint union \( S' := \sqcup_{i \in \mathbb{N}} S'_i \) is clearly an unbranched covering of \( S \) and functions \( f, g \in H^\infty(S') \) defined by the formulas

\[
\begin{align*}
f|_{S'_i} := f_i, & \quad g|_{S'_i} := g_i, \quad i \in \mathbb{N},
\end{align*}
\]

are of norms \( \leq 1 \) and satisfy condition \((4.1)\) on \( S' \). Then due to Theorem 1.2 there exists a function \( G \in H^\infty(S') \) such that the function \( f + gG \) is invertible in \( H^\infty(S') \). We set

\[
G_i := G|_{S'_i}, \quad i \in \mathbb{N}.
\]

Then due to \((4.3)\)

\[
C \leq \sup_{i \in \mathbb{N}} \max \{ \|G_i\|_{H^\infty(S'_i)}, \|(f_i + g_iG_i)^{-1}\|_{H^\infty(S')} \} = \max \{ \|G\|_{H^\infty(S')}, \|(f + gG)^{-1}\|_{H^\infty(S')} \}.
\]

This completes the proof of the theorem.

**Proof of Theorem 1.4.** According to [Br4 Thm. 1.3 (b)] the covering dimension of the maximal ideal space \( \mathfrak{M}(H^\infty(S')) \) is 2. In turn, due to the Browder theorem [Br4 Thm. 6.11] the second homotopy group \( \pi_2(SL_n(\mathbb{C})) = 0 \) (here \( SL_n(\mathbb{C}) \) is the group of \( n \times n \) complex matrices with determinant 1). These two facts and the Hu theorem [Hu (11.4)] imply that the homotopy classes of the continuous mappings \( f : \mathfrak{M}(H^\infty(S')) \to SL_n(\mathbb{C}) \) are in a one-to-one correspondence with the elements of the first Čech cohomology group.
This group is trivial because the space $SL_n(\mathbb{C})$ is simply connected. Hence, each $f \in C(\mathbb{R}(H^\infty(S'))$, $SL_n(\mathbb{C})$) is homotopic to a constant map with value $1_n$ (the unit of $SL_n(\mathbb{C})$), i.e., the space $C(\mathbb{R}(H^\infty(S'))$, $SL_n(\mathbb{C})$) is connected. Next, by the Arens theorem [A] the Gelfand transform induces a bijection between the sets of connected components of the spaces $SL_n(H^\infty(S'))$ and $C(\mathbb{R}(H^\infty(S'))$, $SL_n(\mathbb{C})$). Therefore the group $SL_n(H^\infty(S'))$ is connected as well. In particular, each matrix in $SL_n(H^\infty(S'))$ can be presented as a finite product of elementary matrices. (This is a well-known fact; it can be deduced, e.g., from [BRS, Thm. 2.1].) Then since $H^\infty(S')$ is a $B$-ring (by Theorem [12]), Lemma 9 and Remark 10 of [DV] imply that each matrix $F \in SL_n(H^\infty(S'))$ can be presented as a product of at most $(n-1)(\frac{2n}{2}+1)$ elementary matrices. Let us show that if

$$\|F\|_{M_n(H^\infty(S'))} \leq M,$$

these matrices would be chosen so that their norms were bounded from above by a constant depending only on $M$, $n$ and $S$.

As before we may assume that $S'$ is connected. Let $\mathcal{F}_{M,S',n}$ be the class of matrices $F \in SL_n(H^\infty(S'))$ satisfying (4.4). For every $F \in \mathcal{F}_{M,S',n}$ by $\Pi_{F,M,S',n}$ we denote the set of all possible products of $F$ by at most $(n-1)(\frac{2n}{2}+1)$ elementary matrices. By the above arguments the set $\Pi_{F,M,S',n}$ is non-void. For each $\pi \in \Pi_{F,M,S',n}$ by $\|\pi\|$ we denote maximum of norms of elementary matrices in $\pi$. We have to prove that

$$C = C(S, M, n) := \sup_{S', F \in \mathcal{F}_{M,S',n}} \inf_{\pi \in \Pi_{F,M,S',n}} \|\pi\| < \infty;$$

here $S'$ runs over all connected unbranched coverings of $S$.

Let $S'_i$ and $F_i \in \mathcal{F}_{M,S'_i,n}$, $i \in \mathbb{N}$, be such that

$$C = \lim_{i \to \infty} \inf_{\pi \in \Pi_{F_i,M,S'_i,n}} \|\pi\|.$$

It is clear that the disjoint union $S' := \sqcup_{i \in \mathbb{N}} S'_i$ is an unbranched covering of $S$ and the matrix $F \in H^\infty(S')$ defined by the formula

$$F|_{S'_i} := F_i, \quad i \in \mathbb{N},$$

belongs to the class $\mathcal{F}_{M,S',n}$. Then there is $\pi \in \Pi_{F,M,S',n}$. By $\pi_i$ we denote the product obtained by restriction of elementary matrices in $\pi$ to $S'_i$. Then each $\pi_i \in \Pi_{F_i,M,S'_i,n}$ and so due to (4.6)

$$C \leq \sup_{i \in \mathbb{N}} \|\pi_i\| = \|\pi\| < \infty.$$

This completes the proof of the theorem. \hfill \Box

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