NEW INFINITE HIERARCHIES OF POLYNOMIAL IDENTITIES RELATED TO THE CAPPARELLI PARTITION THEOREMS

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Abstract. We prove a new polynomial refinement of the Capparelli's identities. Using a special case of Bailey's lemma we prove many infinite families of sum-product identities that root from our finite analogues of Capparelli’s identities. We also discuss the \( q \rightarrow 1/q \) duality transformation of the base identities and some related partition theoretic relations.

1. Introduction

Let \( a \) and \( q \) be variables and define the \( q \)-Pochhammer symbol \( (a; q)_n := (1 - a)(1 - aq) \ldots (1 - aq^{n-1}) \) for any non-negative integer \( n \). For \( |q| < 1 \), we define \( (a; q)_\infty := \lim_{n \to \infty} (a; q)_n \). For \( a \) some finite number of variables we define the shorthand notation \( (a_1, a_2, \ldots, a_k; q)_n := (a_1; q)(a_2; q) \ldots (a_k; q)_n \). Finally note that \( 1/(q; q)_n = 0 \) for all negative \( n \).

A partition of a positive integer \( n \) is a non-increasing sequence of natural numbers whose sum is \( n \). For example, the partitions of 4 are \((4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1)\). The sum of all the parts of a partition \( \pi \) is called the size of \( \pi \) and it is denoted with \(|\pi|\). The total number of parts of a partition \( \pi \) is denoted with \(#(\pi)\). We denote the set of all the partitions by \( P \) and the set of all the partitions into distinct parts by \( D \). Then, it is widely known that the generating function for the number of partitions into ordinary and distinct parts are given (both as a sum representation and as a product representation) as follows

\[
\sum_{\pi \in P} q^{\left|\pi\right|} = \sum_{n \geq 0} q^n (q; q)_n = \frac{1}{(q; q)_\infty}, \quad \text{and} \quad \sum_{\pi \in D} q^{\left|\pi\right|} = \sum_{n \geq 0} \frac{q^{n(n+1)/2}}{(q; q)_n} = (-q; q)_\infty.
\]

Notice that both generating functions’ \( q \)-series starts with the constant term 1. This constant represents the conventional partition of 0. We consider the empty sequence with 0 parts to be the only partition of 0.

Let \( C_m(n) \) be the number of partitions of \( n \) into distinct parts where no part is congruent to \( \pm m \) modulo 6. Define \( D_m(n) \) to be the number of partitions of \( n \) into parts, not equal to \( m \), where the minimal difference between consecutive parts is 2. In fact, the difference between consecutive parts is greater than or equal to 4 unless consecutive parts are \( 3k \) and \( 3k + 3 \) (yielding a difference of 3), or \( 3k - 1 \) and \( 3k + 1 \) (yielding a difference of 2) for some \( k \in \mathbb{Z}_{>0} \).

In his thesis \[11\], S. Capparelli stated two (then) conjectural identities, which we will present as the following theorem.

Theorem 1.1. For any non-negative integer \( n \) and \( m \in \{1, 2\} \),

\[
C_m(n) = D_m(n).
\]

The \( m = 1 \) case was first proven by G. E. Andrews \[5\] shortly after its debut. Two years later Lie theoretic proofs for both cases of the conjecture were supplied by Tamba and Xie \[24\] and by Capparelli \[10\]. The Lie theoretic proofs were followed by Alladi, Andrews, and Gordon by a refinement of these identities, where they introduced restrictions on the number of occurrences of parts belonging to certain congruence classes \[2\]. In recent years some

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new refinements of Capparelli’s identities were discovered by Dousse and Dousse joint with Lovejoy [12, 13]. We invite interested readers to check these resources.

After a long wait, finally in 2018, the analytic counterparts of Theorem 1.1 were independently found by Kanade–Russell [15] and Kurşungöz [18].

Theorem 1.2.

\[
\sum_{m,n \geq 0} \frac{q^{2m^2+6mn+6n^2}}{(q; q)_m(q^3; q^3)_n} = (-q^2, -q^4; q^6)_{\infty}(-q^3; q^3)_{\infty},
\]

\[
\sum_{m,n \geq 0} \frac{q^{2m^2+6mn+6n^2+3m+6n+1}}{(q; q)_m(q^3; q^3)_n} = (-q, -q^5; q^6)_{\infty}(-q^3; q^3)_{\infty}.
\]

Theorem 1.2 relied on the original proofs [10, 5] of the Capparelli identities. In [7], we proved polynomial identities that directly imply Theorem 1.2 and, hence, the Capparelli identities.

Theorem 1.3. For any \(L \in \mathbb{Z}_{\geq 0}\), we have

\[
(1.2) \quad \sum_{m,n \geq 0} q^{2m^2+6mn+6n^2} \left[3(L - 2n - m) \atop m\right]_q \left[2(L - 2n - m) + n \atop n\right]_q = \sum_{j=-\infty}^{\infty} q^{3j^2 + 2j} \left( \frac{L, 2j + 1}{2j + 1} ; q^3 \right)_2,
\]

where

\[
[m + n \atop m]_q := \begin{cases} \frac{(q^2)^{m + n}}{(q; q)_m(q; q)_n} & \text{for } m, n \geq 0, \\ 0 & \text{otherwise}, \end{cases}
\]

are the \(q\)-binomial coefficients, and

\[
\left( \frac{L, b}{a} ; q \right)_2 := \sum_{j=0}^{L} q^{j+b} \left[ L \atop j \right]_q \left[ L - j \atop j + a \right]_q
\]

are the \(q\)-trinomial coefficients defined by Andrews and Baxter [4].

Later, in [8], we proved different finite versions of Capparelli’s theorems.
Theorem 1.4.

\begin{equation}
\sum_{m,n \geq 0} \frac{q^{2m^2+6mn+6n^2} (q^3; q^3)_M}{(q; q)_m (q^3; q^3)_n (q^3; q^3)_{M-2n-m}} = \sum_{j=-M}^{M} q^{3j^2+j} \frac{2M}{M-j} q^3,
\end{equation}

\begin{equation}
\sum_{m,n \geq 0} \frac{q^{2m^2+6mn+6n^2+3m+3n} (q^3; q^3)_M}{(q; q)_m (q^3; q^3)_n (q^3; q^3)_{M-2n-m}} + q \sum_{m,n \geq 0} \frac{q^{2m^2+6mn+6n^2+3m+6n} (q^3; q^3)_M}{(q; q)_m (q^3; q^3)_n (q^3; q^3)_{M-2n-m}} = \sum_{j=-M-1}^{M} q^{3j^2+2j} \frac{2M+1}{M-j} q^3,
\end{equation}

\begin{equation}
\sum_{m,n \geq 0} \frac{q^{2m^2+6mn+6n^2-2m-3n} (q^3; q^3)_M}{(q; q)_m (q^3; q^3)_n (q^3; q^3)_{M-2n-m}} (1 + q^{3M}) = \sum_{j=-M}^{M} q^{3j^2-2j(1+q^3)} \frac{2M}{M-j} q^3.
\end{equation}

As \( M \to \infty \), the identities (1.3) and (1.4), with the help of Jacobi triple product identity (see (2.5)), prove Theorem 1.2. The third identity (1.5) gives a new relation that equates a double sum to the sum of the products that appear in Theorem 1.2.

Furthermore, in a follow up work [9], we found a doubly bounded identity that unifies both (1.2) and (1.3).

\textbf{Theorem 1.5.} For \( L \) and \( M \) non-negative integers, we have

\begin{equation}
\sum_{i+m \equiv 0 (\mod 2)} q^{\frac{a^2+2a}{2}} \left[ \frac{L+M-i}{L}; q^a \right] \left[ \frac{3(L-i)}{m}; q^a \right] \left[ \frac{2(L-i)+i-m}{2}; q^{3a} \right] = \sum_{j=-\infty}^{\infty} q^{3j^2+j} S\left( \frac{L}{2j}, \frac{M}{3}; q^3 \right),
\end{equation}

where

\[ S\left( \frac{L}{a}, \frac{M}{b}; q^3 \right) := \sum_{n \geq 0} q^{n(n+a)} \left[ \frac{M+L-a-2n}{M}; q^a \right] \left[ \frac{M-a+b}{n}; q^a \right] \left[ \frac{M+a-b}{a}; q^a \right].\]

a refinement of \( q \)-trinomial coefficients first defined by Wararna [25].

As \( M \to \infty \), (1.6) reduces to (1.2) and as \( L \to \infty \), (1.6) reduces to (1.3).

On top of it all, this unifying identity and an analogue of Bailey’s Lemma helped us identify two infinite hierarchies of polynomial identities that extends (1.6). One of such infinite hierarchies is the following.

\textbf{Theorem 1.6.} Let \( \nu \) be a positive integer, and let \( N_k = n_k + n_{k+1} + \cdots + n_{\nu}, \) for \( k = 1, 2, \ldots, \nu. \) Then,

\[ \sum_{i,m,n_1,n_2,\ldots,n_\nu \geq 0, \atop i+m \equiv N_1+N_2+\cdots+N_\nu (\mod 2)} q^{\frac{m^2+3(N_1^2+N_2^2+\cdots+N_\nu^2)}{2}} \left[ \frac{L+M-i}{L}; q^3 \right] \left[ \frac{L-N_1}{i}; q^3 \right] \left[ \frac{3n_\nu}{m}; q^3 \right] \times \left[ \frac{2n_\nu+i-m-N_1-N_2-\cdots-N_\nu}{2n_\nu}; q^3 \right] \prod_{j=1}^{\nu-1} \left[ \frac{i-\sum_{l=1}^{j} N_l + n_j}{n_j}; q^3 \right] = \sum_{j=-\infty}^{\infty} q^{3(j+\nu)^2+2j} S\left( \frac{L}{(\nu+2)j}, \frac{M}{(\nu+1)j}; q^3 \right).\]

Tending \( L \to \infty \) and \( M \to \infty \) in Theorem 1.6 yields the following infinite hierarchy of sum-product identities [9, Theorem 15].
Theorem 1.7. Let \( \nu \) be a positive integer, and let \( N_k = n_k + n_{k+1} + \cdots + n_\nu \), for \( k = 1, 2, \ldots, \nu \). Then,

\[
\sum_{i,m,n_1,n_2,\ldots,n_\nu \geq 0, \atop i+m \equiv N_1+N_2+\cdots+N_\nu \pmod{2}} q^{m^2+3(n_1^2+n_2^2+\cdots+n_\nu^2)} \frac{(q^2;q^3)_\nu}{(q^4;q^3)_\nu} \left[ \frac{3n_\nu}{m} \right] \frac{[2n_\nu+i-N_1-N_2-\cdots-N_\nu-m]}{[2n_\nu]} \prod_{j=1}^{\nu-1} \left[ \frac{i - \sum_{k=1}^{j} N_k + n_j}{n_j} \right]_{q^3} = \frac{(q^{6(\nu+2)}, q^{3(\nu+2)} - 1, q^{6(\nu+2)} \infty)}{(q^4;q^3)_\nu}.
\]

In this paper we prove a new set of polynomial identities that imply Capparelli’s theorems.

Theorem 1.8. Let \( L \in \mathbb{Z}_{\geq 0} \), then

\[
(1.7) \sum_{m,n \geq 0} \frac{q^{2m^2+6mn+6n^2} (q;q)_L}{(q;q)_{L-3n-2m}(q;q)_m(q^3;q^3)_n} = \sum_{j=-L}^{L} \left( \frac{j+1}{3} \right) q^{j^2} \left[ \frac{2L}{L-j} \right]_q,
\]

\[
(1.8) \sum_{m,n \geq 0} \frac{q^{2m^2+6mn+6n^2+m+3m} (q;q)_L}{(q;q)_{L-3n-2m}(q;q)_m(q^3;q^3)_n} + q \sum_{m,n \geq 0} \frac{q^{2m^2+6mn+6n^2+3m+6n} (q;q)_L}{(q;q)_{L-3n-2m-1}(q;q)_m(q^3;q^3)_n} = \sum_{j=-L}^{L} \left( \frac{j+1}{3} \right) q^{j(j+1)} \left[ \frac{2L}{L-j} \right]_q,
\]

where \((\cdot)\) is the Jacobi symbol.

The theorem above is analogous to Theorem 1.4.

Then, by appeal to a special case of Bailey’s lemma, we prove various infinite polynomial identities which tend to infinite hierarchies of sum-product identities asymptotically. One of such sum-product hierarchies is below.

Theorem 1.9. Let \( f \in \mathbb{N}, s \in \{0, 1, 2, \ldots, f\} \) and \( N_i := n_i + n_{i+1} + \cdots + n_{f} \) with \( i = 1, 2, \ldots, f \), where \( N_{f+1} := 0 \), then

\[
(1.9) \sum_{m,n,n_1,n_2,\ldots,n_f \geq 0} \frac{q^{2m^2+6mn+6n^2+N_1^2+N_2^2+\cdots+N_f^2+N_{f+1}^2+\cdots+N_{f-s+1}^2+\cdots+N_{f-1}^2+\cdots+N_1^2} (q;q)_{N_f}}{(q;q)_m(q^3;q^3)_n(q;q)_n_f-3n-2m(q;q)_n_1(q;q)_n_2 \cdots (q;q)_{n_{f-1}}(q;q)_{2f})} = \frac{(q^{f+1-s}, q^{5f+5+s}, q^{6f+1}\infty, q^{6f+1}\infty)}{(q^4;q^3)\infty}.
\]

The organization of this paper is as follows. In Section 2, we give useful formulas that will aid in the proofs. Section 3 has the proof of Theorem 1.8 and other similar formulas that will lead to the discovery of infinite hierarchies. These infinite hierarchies and the proof of Theorem 1.9 are given in Section 4. In Section 5, we look at the dual identities to the ones in Theorem 1.8 and some related partition theoretic consequences. The last Section has some short concluding remarks.

2. SOME USEFUL FORMULAS

For completeness, in this section we would also like to present some essential ingredients of our proofs. We start with two well known limits of the \( q \)-binomial coefficients. For any \( j \in \mathbb{Z}_{\geq 0} \) and \( a = 0 \) or \( 1 \),

\[
\lim_{L \to \infty} \left[ \frac{L}{j} \right]_q = \frac{1}{(q;q)_m},
\]

\[
\lim_{L \to \infty} \left[ \frac{2L + a}{L - j} \right]_q = \frac{1}{(q;q)_\infty},
\]

and for \( n, m \in \mathbb{Z}_{\geq 0} \)

\[
(2.1) \left[ \frac{n + m}{m} \right]_q = q^{-mn} \left[ \frac{n + m}{m} \right]_{q-1}.
\]
We would also like to recall the \( q \)-binomial recurrences \([14, I.45, p.353]\):

\[
\binom{m+n}{m}_q = \binom{m+n-1}{m}_q + q^n \binom{m+n-1}{m-1}_q.
\]

It is easy to verify that

\[
(1 - q^j) \binom{L}{j}_q = (1 - q^{L-j}) \binom{L-1}{j-1}_q.
\]

The \( q \)-binomial theorem \([14, II.3, p.354]\) states that

\[
\sum_{n \geq 0} \binom{a; q}{n} z^n = \binom{az; q}{\infty} \binom{z; q}{\infty}.
\]

For \( z \neq 0 \), the Jacobi triple product identity \([14, (1.6.1), p.15]\) is

\[
\sum_{j=-\infty}^{\infty} q^{j^2} z^j = (-qz, -q/z, q^2)_{\infty}.
\]

The quintuple product identity \([14, ex.5.6, p.147]\) for \( z \neq 0 \) is given as

\[
\sum_{j=-\infty}^{\infty} (-1)^j q^{(3j-1)/2} z^{3j} (1 + q^j) = (q, -z, -q/z; q)_{\infty} (qz^2, q/z^2, q^2)_{\infty}.
\]

We now present a special case of Bailey’s Lemma \([3, 21]\) that will be instrumental to some of our proofs.

**Theorem 2.1.** For \( a = 0, 1 \), if

\[
F_a(L, q) = \sum_{n \geq 0} \frac{(a q)_n}{(q q)_n} z^n = \frac{(az q)_\infty}{(z q)_\infty}.
\]

Observe that the right-hand side of the second equation in the Theorem 2.1 is of the same form as the right-hand side of the first equation. Thus, we may iterate Theorem 2.1 as often as we desire by updating \( \alpha_j(q) \)'s in each step. This procedure gives rise to an infinite hierarchy of polynomial identities.

**3. New Polynomial Identities**

We prove Theorem 1.8 by showing that both sides of (1.7) and (1.8) satisfy the recurrences

\[
a_L = (1 + q - q^{2L-1}) a_{L-1} - q (1 - q^{2L-2}) (1 - q^{2L-3}) a_{L-2},
\]

and

\[
b_L = (1 + q - q^{2L}) b_{L-1} - q (1 - q^{2L-1}) (1 - q^{2L-2}) b_{L-2},
\]

respectively, and by checking the appropriate initial conditions.

**Proof.** The \( q \)-Zeilberger algorithm \([19]\) implemented in Riese’s \texttt{qZeil} package \([20]\) (for single-fold sums), Sister Celine’s algorithm implemented in Riese’s \texttt{qMultiSum} package \([22]\), Creative Telescoping implemented in the Mathematica package \texttt{HolonomicFunctions} of Koutschan \([17]\), and Schneider’s \texttt{Sigma} package \([23]\)) are all sufficient tools to
find and prove recurrences satisfied by the expressions in (1.7) and (1.8). However, the outcome recurrences heavily depend on the representation of these functions. For that reason, we first rewrite the right-hand side expressions first

\[
\sum_{j=-L}^{L} \left( \frac{j+1}{3} \right) q^{j} \left[ \frac{2L}{L+j} \right] = \sum_{k=-\infty}^{\infty} \left( q^{(3k)^{2}} \left[ \frac{2L}{L+3k} \right] - q^{(3k+1)^{2}} \left[ \frac{2L}{L+3k+1} \right] \right),
\]

\[
= \sum_{j=-\infty}^{\infty} q^{(3j)^{2}} \left[ \frac{2L}{L+3j} \right] - q^{(3j+1)^{2}} \left[ \frac{2L}{L+3j+1} \right] - \delta_{3j}(L-2)q^{L},
\]

\[
= \sum_{j=-\infty}^{\infty} q^{(3(j+1))^{2}} \left[ \frac{2L}{L+(j+1)} \right] - q^{(3(j+1)+1)^{2}} \left[ \frac{2L}{L+(j+1)+1} \right] - \delta_{3(j+1)}(L-2)q^{L}. \tag{3.6}
\]

where \(\delta_{a|b} = 1\) if \(a \mid b\), and 0 otherwise.

If one chooses to calculate the recurrence from the representations (3.3) and (3.5), the orders of the recurrences found/proven would be 4 and 6, respectively.

In certain cases, we find and prove better recurrences. When we start with the alternative representations (3.4) we get the recurrence (3.1).

With the alternative representation (3.6) of the right-hand side of (1.8), which is analogous to (3.3), we can only find a 4th order recurrence (3.7) using the automated proof techniques:

\[
b_{L} = -(1 + q^{2})(1 + q - q^{2L-2})b_{L-1} + q((1 + q^{2})(1 + q + q^{2}) - 2q^{2L-3}(1 + q)(1 + q^{2}) - q^{4L-7}(1 + q^{2} + q^{4}))b_{L-2}
\]

\[- q^{3}(1 + q^{2})(1 - q^{2L-4})(1 - q^{2L-5})(1 + q - q^{2L-3})b_{L-3} + q^{6}(1 - q^{2L-4})(1 - q^{2L-5})(1 - q^{2L-6})(1 - q^{2L-7})b_{L-4}.
\]

However, we can do better than that. Experimentally, using the \texttt{qFunctions} package [1] of the second author and Abinger, we can guess that the right-hand side of (1.8) satisfies the 2nd order recurrence (3.2). Moreover, again by using \texttt{qFunctions} package [1] we can prove that the greatest common divisor of (3.7) and (3.2) is (3.2). To demonstrate that (3.2) is a factor of (3.7), let

\[
r_{L} := b_{L} - (1 + q - q^{2L})b_{L-1} + q(1 - q^{2L-1})(1 - q^{2L-2})b_{L-2}. \tag{3.8}
\]

Then it is easy to observe that

\[r_{L} - q^{2}(1 + q - q^{2L-4})r_{L-1} + q^{5}(1 - q^{2L-4})(1 - q^{2L-7})r_{L-2} = 0\]

is equivalent to (3.7).

By checking the initial conditions, this is enough to prove that the right-hand sides of (1.8) satisfy the guessed recurrence (3.2). This is done for \(b_{n}\) as follows: We know that (3.7) and 4 initial values fully determine the sequence \(b_{n}\). We also know that (3.2) divide this sequence and only require 2 initial conditions to determine a sequence fully. We seed the first two initial conditions of (3.7) to (3.2) and check that the next 2 sequence terms we get from (3.2) are the same as the next 2 terms for (3.2). This way we prove that not only that the guessed recurrence (3.2) divides the proven recurrence (3.7) that (1.8) satisfies, but the recurrence (3.2) also determines the same sequence fully with the 2 initial conditions of (3.7).

Now, using Zeilberger’s Creative Telescoping algorithm [19] (implemented in Koutschan’s \texttt{HolonomicFunctions} package [17]) we find the recurrences satisfied by the left-hand sides of (1.7) and (1.8). The said implementation directly proves that, just like the right-hand side, the left-hand side of (1.7) satisfies (3.1). Therefore, since both sides of (1.7) satisfy the same 2nd order recurrence, we finish the proof of that identity by checking the two initial conditions at \(L = 0\) and 1.
For finding the recurrence of the left-hand side of (1.8), first we define

\[ S_{1,L} := \sum_{m,n \geq 0} \frac{q^{2m^2 + 6mn + 6n^2 + m + 3n}(q; q)_L}{(q; q)_{L-3n-2m}(q; q)_m(q^3; q^3)_n} \quad \text{and} \quad S_{2,L} := \sum_{m,n \geq 0} \frac{q^{2m^2 + 6mn + 6n^2 + 3n + 6n+1}(q; q)_L}{(q; q)_{L-3n-2m-1}(q; q)_m(q^3; q^3)_n}. \]

Then, we get that these sums satisfy the recurrences

\[ S_{1,L} = (1 + q + q^3 - q^L - q^{L+2})S_{1,L-1} - q(1 - q^{L-1})(1 + q^2 + q^3 - q^{L+1} - q^{2L-1} - q^{2L-2})S_{1,L-2} + q^3(1 - q^{L-1})(1 - q^{L-2})(1 - q^{2L-3})(1 - q^{2L-4})S_{1,L-3} \]

and

\[ (1 - q^{L-1})S_{2,L} = -(1 - q^L)(1 + q + q^2 - q^{L-1} - q^L)S_{2,L-1} + q(1 - q^L)(1 + q^2)(1 + q + q^2 - q^{L-1} - q^{2L-1} - q^{2L-2})S_{2,L-2} - q^3(1 - q^L)(1 - q^{L-1})(1 - q^{L-2})(1 - q^{2L-3})(1 - q^{2L-4})S_{2,L-3}. \]

These recurrences are similar but unlike the situation we encountered on the right-hand sides of (3.3) and (3.5), they are not identical.

We can find a recurrence that is satisfied by both sums on the left-hand side of (1.8) by using the closure properties of holonomic (recurrent) sequences. This is merely a specialized substitution of one of the recurrences (3.9) and (3.10) into the other one. This is implemented in the \texttt{GeneratingFunctions} package of Kauers and Koutschan [16]. This way we prove that the whole left-hand side satisfies the 5th order recurrence

\[ c_L = (1 + q + q^2 + q^3 + q^4 - q^{2L-1} - q^L - q^{L+2})c_{L-1} - q((1 + q^2)(1 + q^2 + q^3 + q^4) - q^{L-1}(1 + q)(1 + q^2)^2 \]

\[ -q^{2L-2}(1 + q)(2 + q^2 + q^3 + q^4 - q^{L-1}(1 + q^2 + q^3 + q^4) - q^{3L-5}(1 + q^2 + q^{2L-5})c_{L-1} \]

\[ + q^3(1 - q^{L-2})((1 + q^2)(1 + q + q^2 + q^3 + q^4) - q^2(1 + q^2 - q^{2L-4}(1 + q)(1 + 2q + 2q^2 + q^3 + q^4) - q^{3L-7}(1 + q^2 + q^{2L-7} + 2q + q^2 - q^{5L-7})c_{L-3} \]

\[ - q^6(1 - q^{L-2})(1 - q^{2L-6})((1 + q + q^2 + q^3 + q^4) - q^{L-3}(1 + q^2 + q^3 - q^{2L-5}(1 + q + q^2 + q^{3L-6})c_{L-4} \]

\[ + q^{10}(1 - q^{L-2})(1 - q^{L-4})(1 - q^{2L-5})(1 - q^{2L-6})(1 - q^{2L-7})(1 - q^{2L-8})c_{L-5}. \]

Comparing this with (3.2), we see that the greatest common divisor of these two recurrences is (3.2). One can easily verify this by checking that

\[ r_L = q^2(1 + q + q^2 - q^{L-2} + q^L + q^{2L-2} + q^{2L-3})r_{L-1} + q^5(1 - q^{L-2})(1 + q^2 - q^{L-3} - q^{2L-4} - q^{2L-5} + q^{3L-6} - q^{3L-7})r_{L-2} + q^9(1 - q^{L-2})(1 - q^{L-4})(1 - q^{2L-5})(1 - q^{2L-6})r_{L-3} = 0 \]

is equivalent to (3.11), where \( r_L \) is defined as in (3.8).

Analogous to moving from the 4th degree recurrence (3.7) to the shorter recurrence (3.2), we prove that the left-hand side of (1.8) satisfies this much shorter recurrence. This proves that the both the left- and right-hand sides of (1.8) satisfies (3.2) and by only checking the two initial conditions \( L = 0 \) and \( 1 \) we finish the proof of (1.8).

Furthermore, we can prove a simple transformation formula for the right-hand side sum of (1.8).
Theorem 3.1. Let $L \in \mathbb{Z}_{>0}$, then

\[
\sum_{j=-L}^{L} \left( \frac{j+1}{3} \right) q^{j(j+1)} \left[ \frac{2L}{L-j} \right] = \sum_{j=-L}^{L+1} \left( \frac{j+1}{3} \right) q^{j(j+1)} \left[ \frac{2L+1}{L-j} \right].
\]

Proof. We start by applying (2.2) to the right-hand side of (3.12):

\[
\sum_{j=-L}^{L+1} \left( \frac{j+1}{3} \right) q^{j(j+1)} \left[ \frac{2L+1}{L-j} \right] = \sum_{j=-L}^{L} \left( \frac{j+1}{3} \right) q^{j(j+1)} \left[ \frac{2L}{L-j} \right] + \sum_{j=-L}^{L+1} \left( \frac{j+1}{3} \right) q^{j(j+1)+(L+j+1)} \left[ \frac{2L}{L-j-1} \right].
\]

Therefore, it is enough to prove that the rightmost series above vanishes. With the change of variables $j \mapsto j-1$, we get

\[
\sum_{j=-L-1}^{L+1} \left( \frac{j+1}{3} \right) q^{j(j+1)+(L+j+1)} \left[ \frac{2L}{L-j-1} \right] = qL \sum_{j=-L}^{L} \left( \frac{j+1}{3} \right) q^{j^2} \left[ \frac{2L}{L-j} \right].
\]

We point out that

\[
\left( \frac{j}{3} \right) = - \left( \frac{j}{3} \right), \text{ and } \left[ \frac{2L}{L-j} \right] = \left[ \frac{2L}{L+j} \right].
\]

Hence, by changing $j \mapsto -j$, we can now clearly see that

\[
\sum_{j=-L}^{L} \left( \frac{j}{3} \right) q^{j^2} \left[ \frac{2L}{L-j} \right] = - \sum_{j=-L}^{L} \left( \frac{j}{3} \right) q^{j^2} \left[ \frac{2L}{L-j} \right] = 0.
\]

\[
\square
\]

Now, we move onto the proof of a series transformation involving $q$-Binomial coefficients.

Theorem 3.2. Let $L \in \mathbb{Z}_{>0}$ and $k = 1, 2, \ldots$, then

\[
\sum_{j=-L}^{L} \left( \frac{j+1}{3} \right) q^{kj(j-1)} \left[ \frac{2L}{L-j} \right] = q^L \sum_{j=-L}^{L} \left( \frac{j+1}{3} \right) q^{kj^2-(k-1)j} \left[ \frac{2L}{L-j} \right].
\]

Proof. We take the difference of the two sides of (3.13) and show that it reduces to zero. Making use of (2.3), we have

\[
\sum_{j=-L}^{L} \left( \frac{j+1}{3} \right) q^{kj(j-1)} (1 - q^{L+j}) \left[ \frac{2L}{L+j} \right] = (1 - q^{2L}) \sum_{j=-L}^{L} \left( \frac{j+1}{3} \right) q^{kj(j-1)} \left[ \frac{2L-1}{L+j-1} \right] - q^{2L} \sum_{j=-L}^{L} \left( \frac{j+1}{3} \right) q^{kj(j-1)} \left[ \frac{2L-1}{L-j} \right]
\]

\[
= (1 - q^{2L}) \sum_{j=-L}^{L} \left( \frac{j+1}{3} \right) q^{kj(j-1)} \left[ \frac{2L-1}{L-3j} \right] - q^{2L} \sum_{j=-L}^{L} \left( \frac{j+1}{3} \right) q^{kj(j-1)} \left[ \frac{2L-1}{L-3j-1} \right]
\]

Replacing $j \mapsto -j$ in the last term in the braces in the above term, we can notice that the terms in the braces vanish. Therefore the difference of the sides of (3.13) vanishes for each non-negative $L$ and positive integers $k$.

\[
\square
\]

Multiplying (1.7) by $q^L$ and using (3.13) with $k = 1$ on the right-hand side of the resulting equation we arrive at Corollary 3.3.

Corollary 3.3. Let $L \in \mathbb{Z}_{>0}$, then

\[
\sum_{m,n \geq 0} \frac{q^{L+2m^2+6mn+6n^2}(q; q)_L}{(q; q)L-3L-2m(q; q)_m(q^3; q^3)_n} = \sum_{j=-L}^{L} \left( \frac{j+1}{3} \right) q^{j(j-1)} \left[ \frac{2L}{L-j} \right].
\]
4. NEW INFINITE HIERARCHIES

Applying Theorem 2.1 with \( a = 0 \) and \( q \rightarrow q^3 \), \( f \) times in iterative fashion to (1.3), we derive:

**Theorem 4.1.** Let \( L \in \mathbb{Z}_{\geq 0}, f \in \mathbb{N} \) and \( N_i := n_i + n_{i+1} + \cdots + n_f \) with \( i = 1, 2, \ldots, f \), then

\[
\sum_{m,n,n_1,n_2,\ldots,n_f \geq 0} q^{2m^2+6mn+6n^2+3(N_1^2+N_2^2+\cdots+N_f^2)} [q^3; q^3]_n (q^3; q^3)_n (q^3; q^3)_n \cdots (q^3; q^3)_n \prod_{j=1}^{f+1} \left[ \frac{2L-j}{L-j} \right]_{q^3}.
\]

And in the limit \( L \rightarrow \infty \), with the help of (2.5), (4.1) yields

**Theorem 4.2.** Let \( f \in \mathbb{N} \) and \( N_i := n_i + n_{i+1} + \cdots + n_f \) with \( i = 1, 2, \ldots, f \), then

\[
\sum_{m,n,n_1,n_2,\ldots,n_f \geq 0} q^{2m^2+6mn+6n^2+3(N_1^2+N_2^2+\cdots+N_f^2)} [q^3; q^3]_n (q^3; q^3)_n \cdots (q^3; q^3)_n \prod_{j=1}^{f+1} \left[ \frac{2L-j}{L-j} \right]_{q^3}.
\]

It is clear that the products in Theorems 4.2 and 1.7 are identical when \( 2f+2 = (\nu+1)(\nu+2) \). This observation implies the following transformation.

**Corollary 4.3.** For \( f = \nu(\nu+3)/2 \), where \( \nu \) is a positive integer, \( N_j := n_j + \cdots + n_\nu \) for \( j = 1, 2, \ldots, \nu \) and \( N_i := n_i + n_{i+1} + \cdots + n_f \) with \( i = 1, 2, \ldots, f \), we have

\[
\sum_{i,m,n,n_1,n_2,\ldots,n_\nu \geq 0, \ i+m \equiv N_1^2+N_2^2+\cdots+N_\nu^2 \ (\text{mod} \ 2)} q^{m^2+3(2+N_1^2+N_2^2+\cdots+N_\nu^2)} [q^3; q^3]_n (q^3; q^3)_n \cdots (q^3; q^3)_n \prod_{j=1}^{f+1} \left[ \frac{2L-j}{L-j} \right]_{q^3}.
\]

Applying Theorem 2.1 with \( a = 1 \) and \( q \rightarrow q^3 \), \( f \) times in iterative fashion to (1.4), we derive:

**Theorem 4.4.** Let \( L \in \mathbb{Z}_{\geq 0}, f \in \mathbb{N} \) and \( N_i := n_i + n_{i+1} + \cdots + n_f \) with \( i = 1, 2, \ldots, f \), then

\[
\sum_{m,n,n_1,n_2,\ldots,n_f \geq 0} q^{2m^2+6mn+6n^2+3(N_1^2+N_2^2+\cdots+N_f^2) \prod_{j=1}^{f+1} \left[ \frac{2L-j}{L-j} \right]_{q^3}}.
\]

Notice that as \( L \rightarrow \infty \), with the help of (2.5), (4.2) yields

**Theorem 4.5.** Let \( f \in \mathbb{N} \) and \( N_i := n_i + n_{i+1} + \cdots + n_f \) with \( i = 1, 2, \ldots, f \), then

\[
\sum_{m,n,n_1,n_2,\ldots,n_f \geq 0} q^{2m^2+6mn+6n^2+3(N_1^2+N_2^2+\cdots+N_f^2) \prod_{j=1}^{f+1} \left[ \frac{2L-j}{L-j} \right]_{q^3}}.
\]
We can apply Theorem 2.1 with \( a = 0 \) and \( q \rightarrow q^3 \) iteratively to derive the infinite family that roots from (1.5).

**Theorem 4.6.** Let \( L \in \mathbb{Z}_{\geq 0}, f \in \mathbb{N} \) and \( N_i := n_i + n_{i+1} + \cdots + n_f \) with \( i = 1, 2, \ldots, f \), then

\[
q^{2m^2+6mn+6n^2-2m-3n+3(N_i^2+N_i^2+N_i^2+\cdots+N_f^2)}(q^3;q^3)_m(q^3;q^3)_n(q^3)_{n_f}(1+q^{3n_f}) = \sum_{m,n,n_1,n_2,\ldots,n_f \geq 0} \frac{q^{2m^2+6mn+6n^2-2m-3n+3(N_i^2+N_i^2+N_i^2+\cdots+N_f^2)}(q^3;q^3)_m(q^3;q^3)_n(q^3)_{n_f}(1+q^{3n_f})}{(q;q)_m(q^3;q^3)_n(q^3)_{n_f}(1+q^{3n_f})}
\]

We can apply the Jacobi triple product identity (2.5) to (4.3) twice after tending \( L \to \infty \), and this yields:

**Theorem 4.7.** Let \( f \in \mathbb{N} \) and \( N_i := n_i + n_{i+1} + \cdots + n_f \) with \( i = 1, 2, \ldots, f \), then

\[
\sum_{m,n,n_1,n_2,\ldots,n_f \geq 0} \frac{q^{2m^2+6mn+6n^2-2m-3n+3(N_i^2+N_i^2+N_i^2+\cdots+N_f^2)}(q^3;q^3)_m(q^3;q^3)_n(q^3)_{n_f}(1+q^{3n_f})}{(q;q)_m(q^3;q^3)_n(q^3)_{n_f}(1+q^{3n_f})} = \frac{(q^6f+1;q^6f+1)_\infty}{(q^3;q^3)_\infty} \left( -q^{3f+1}, -q^{3f+5}, -q^{6f+1}_\infty + (-q^{3f+2}, -q^{3f+4}, -q^{6f+1}_\infty) \right)
\]

Applying Theorem 2.1 with \( a = 0 \), \( f \) times in iterative fashion to (1.7) we derive:

**Theorem 4.8.** Let \( L \in \mathbb{Z}_{\geq 0}, f \in \mathbb{N} \) and \( N_i := n_i + n_{i+1} + \cdots + n_f \) with \( i = 1, 2, \ldots, f \), then

\[
\sum_{m,n,n_1,n_2,\ldots,n_f \geq 0} \frac{q^{2m^2+6mn+6n^2+3N_i^2+3N_i^2+\cdots+3N_f^2}q^{2L}(q;q)_m(q^3q^3)_n(q^3)_{n_f}(q)_{2n_f}}{(q;q)_m(q^3q^3)_n(q^3)_{n_f}(q)_{2n_f}} = \sum_{j=-L}^L \left( \frac{j+1}{3} \right) q^{(j+1)/2} \left[ \frac{2L}{L-j} \right]_q
\]

As \( L \to \infty \), with the aid of quintuple product identity (2.6), we get

**Theorem 4.9.** Let \( f \in \mathbb{N} \) and \( N_i := n_i + n_{i+1} + \cdots + n_f \) with \( i = 1, 2, \ldots, f \), then

\[
\sum_{m,n,n_1,n_2,\ldots,n_f \geq 0} \frac{q^{2m^2+6mn+6n^2+3N_i^2+3N_i^2+\cdots+3N_f^2}q^{2L}(q;q)_m(q^3q^3)_n(q^3)_{n_f}(q)_{2n_f}}{(q;q)_m(q^3q^3)_n(q^3)_{n_f}(q)_{2n_f}} = \frac{1}{(q;q)_\infty} \sum_{j=-\infty}^{\infty} \left( \frac{j+1}{3} \right) q^{(j+1)/2} = \frac{(q^{f+1};q^{f+1})_\infty(q^2f+1);q^{2f+1})_\infty(-q^{2f+1}, -q^{4f+1}, -q^{6f+1})_\infty.
\]

Note that the summants of (4.4) is not necessarily made out of terms with non-negative \( q \)-series coefficients. In fact, some \( q \)-series coefficients can be negative depending on the choice of \( L \), due to the \( (q;q)_{2L} \) term in the numerator in the summants. As \( L \to \infty \), these sign changes disappear. The summants of (4.5) are all manifestly positive.

Now we move onto the new infinite hierarchy related to the (1.8). Applying Theorem 2.1 with \( a = 0 \), \( f \) times in iterative fashion to (1.8) we derive:
Let \( L \in \mathbb{Z}_{\geq 0}, f \in \mathbb{N} \) and \( N_i := n_i + n_{i+1} + \cdots + n_f \) with \( i = 1, 2, \ldots, f \), then

\[
\sum_{m,n,n_1,n_2,\ldots,n_f \geq 0} \frac{q^{2m^2+6mn+6n^2+m+3n+N_1^2+N_2^2+\cdots+N_f^2}(q;q)_{2L}(q;q)_{n_f}}{(q;q)_m(q^3;q^3)_n(q;q)_{n_f-3n-2m}(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_f-1}(q;q)_{2n_f}}
\]

(4.6)

As \( L \to \infty \), with the help of quintuple product identity (2.6), we get

Theorem 4.11. Let \( L \in \mathbb{Z}_{\geq 0}, f \in \mathbb{N} \) and \( N_i := n_i + n_{i+1} + \cdots + n_f \) with \( i = 1, 2, \ldots, f \), then

\[
\sum_{m,n,n_1,n_2,\ldots,n_f \geq 0} \frac{q^{2m^2+6mn+6n^2+m+3n+N_1^2+N_2^2+\cdots+N_f^2}(q;q)_{2L}(q;q)_{n_f}}{(q;q)_m(q^3;q^3)_n(q;q)_{n_f-3n-2m}(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_f-1}(q;q)_{2n_f}}
\]

(4.6)

\[
+ q \sum_{m,n,n_1,n_2,\ldots,n_f \geq 0} \frac{q^{2m^2+6mn+6n^2+m+3n+N_1^2+N_2^2+\cdots+N_f^2}(q;q)_{2L}(q;q)_{n_f}}{(q;q)_m(q^3;q^3)_n(q;q)_{n_f-3n-2m}(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_f-1}(q;q)_{2n_f}} = \frac{(q^{f+2}, q^{5f+4}, q^{6f+6})_{\infty}(q^{4f+2}, q^{8f+10}, q^{12f+12})_{\infty}}{(q;q)_{\infty}}.
\]

In particular, for \( f = 1 \) we have

Corollary 4.12.

\[
\sum_{m,n,n_1 \geq 0} \frac{q^{2m^2+6mn+6n^2+m+3n+N_1^2}(q;q)_{n_1}}{(q;q)_m(q^3;q^3)_n(q;q)_{n_1-3n-2m}(q;q)_{2n_1}} + q \sum_{m,n,n_1 \geq 0} \frac{q^{2m^2+6mn+6n^2+m+3n+N_1^2}(q;q)_{n_1}}{(q;q)_m(q^3;q^3)_n(q;q)_{n_1-3n-2m}(q;q)_{2n_1}} = \frac{(q^3;q^3)_{\infty}}{(q;q)_{\infty}}.
\]

Alternatively, instead of applying Theorem 2.1 to (1.8) as is, one can replace the right-hand side of (1.8) using (3.12) and then apply Theorem 2.1 with \( a = 1 \) in an iterative fashion. This yields an analogue of Theorem 4.10.

Theorem 4.13. Let \( L \in \mathbb{Z}_{\geq 0}, f \in \mathbb{N} \) and \( N_i := n_i + n_{i+1} + \cdots + n_f \) with \( i = 1, 2, \ldots, f \), then

\[
\sum_{m,n,n_1,n_2,\ldots,n_f \geq 0} \frac{q^{2m^2+6mn+6n^2+m+3n+N_1^2+N_2^2+\cdots+N_f^2}(q;q)_{2L+1}(q;q)_{n_f}}{(q;q)_m(q^3;q^3)_n(q;q)_{n_f-3n-2m}(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_f-1}(q;q)_{2n_f+1}}
\]

(4.7)

\[
+ q \sum_{m,n,n_1,n_2,\ldots,n_f \geq 0} \frac{q^{2m^2+6mn+6n^2+m+3n+N_1^2+N_2^2+\cdots+N_f^2}(q;q)_{2L+1}(q;q)_{n_f}}{(q;q)_m(q^3;q^3)_n(q;q)_{n_f-3n-2m}(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_f-1}(q;q)_{2n_f+1}} = \sum_{j=-L}^{L} \left( \frac{j+1}{3} \right) q^{(f+1)(j^2+j)} \frac{[2L+1]}{L - j}.
\]

Letting \( L \to \infty \) in (4.7) yields the analogue of Theorem 4.11 with the help of the quintuple product identity (2.6).
Theorem 4.14. Let \( L \in \mathbb{Z}_{\geq 0} \), \( f \in \mathbb{N} \) and \( N_i := n_i + n_{i+1} + \cdots + n_f \) with \( i = 1, 2, \ldots, f \), then
\[
\sum_{m,n_1,n_2 \ldots, n_f \geq 0} \frac{q^{2m^2 + 6mn + n^2 + m + 3n + N_i^2 + N_{1+1}^2 + N_i + N_{1+2} + \cdots + N_f(q; q)_{n_f}}}{(q; q)_m(q; q)^3 n(q; q)_{n_j} - 3n - 2m(q; q)_n_1(q; q)_n_2 \cdots (q; q)_{n_{j-1}}(q; q)_{2n_{j+1}} + q}
\]
\[
+ q \sum_{m,n_1,n_2 \ldots, n_f \geq 0} \frac{q^{2m^2 + 6mn + n^2 + m + 3n + N_i^2 + N_{1+1}^2 + N_i + N_{1+2} + \cdots + N_f(q; q)_{n_f}}}{(q; q)_m(q; q)^3 n(q; q)_{n_f} - 3n - 2m(q; q)_n_1(q; q)_n_2 \cdots (q; q)_{n_{j-1}}(q; q)_{2n_{j+1}}}
\]
\[
= \frac{(q^{2(f+1)}; q^{2(f+1)})_{\infty}}{(q; q)^{20(f+1)}; q^{12(f+1)}},
\]

Theorem 4.15. Let \( f \in \mathbb{N} \) and \( N_i := n_i + n_{i+1} + \cdots + n_f \), with \( i, s = 1, 2, \ldots, f \), where \( N_{f+1} := 0 \), then
\[
\sum_{m,n_1,n_2 \ldots, n_f \geq 0} \frac{q^{2m^2 + 6mn + n^2 + N_i^2 + N_{1+1}^2 + N_i + N_{1+2} + \cdots + N_{f+1}(q; q)_L(q; q)_{n_f}}}{(q; q)_m(q; q)^3 n(q; q)_{n_f} - 3n - 2m(q; q)_n_1(q; q)_n_2 \cdots (q; q)_{n_{j-1}}(q; q)_{2n_{j+1}}}
\]
\[
= \sum_{j=-L}^{L} \left( \frac{j+1}{3} \right) q^{(f+1)^2 - sj} \left[ \frac{2L}{L-j} \right]_q.
\]

Proof. We first apply Theorem 2.1 with \( a = 0 \) to (3.14). This yields
\[
\sum_{m,n_1,n_2 \ldots, n_f \geq 0} \frac{q^{2m^2 + 6mn + n^2 + n_1 + n_2 + n_3 + n_4 + n_5 + \cdots + n_f(q; q)_L(q; q)_{2n_f}}}{(q; q)_m(q; q)^3 n(q; q)_n - 3n - 2m(q; q)_n_1(q; q)_n_2 \cdots (q; q)_{n_{j-1}}(q; q)_{2n_{j+1}}}
\]
\[
= \sum_{j=-L}^{L} \left( \frac{j+1}{3} \right) q^{2j - sj} \left[ \frac{2L}{L-j} \right]_q.
\]

Next, we multiply both sides of (4.9) with \( q^L \) and apply (3.13) to the right-hand side of (4.9) with \( k = 2 \) to get
\[
\sum_{m,n_1,n_2 \ldots, n_f \geq 0} \frac{q^{L+2m^2 + 6mn + n^2 + n_1 + n_2 + n_3 + n_4 + n_5 + \cdots + n_f(q; q)_L(q; q)_{2n_f}}}{(q; q)_m(q; q)^3 n(q; q)_n - 3n - 2m(q; q)_n_1(q; q)_n_2 \cdots (q; q)_{n_{j-1}}(q; q)_{2n_{j+1}}}
\]
\[
= \sum_{j=-L}^{L} \left( \frac{j+1}{3} \right) q^{2j - 2j} \left[ \frac{2L}{L-j} \right]_q.
\]

The outcome equation (4.10) is in a form suitable for the application of Theorem 2.1 with \( a = 0 \). We can once again apply Bailey’s lemma with \( a = 0 \), directly follow this step with multiplying both sides with \( q^L \), and rewriting the right-hand side of the outcome expression with (3.13) with \( k = 3 \) as in the previous case. This yields the next step in this succession:
\[
\sum_{m,n_1,n_2 \ldots, n_s \geq 0} \frac{q^{L+2m^2 + 6mn + n^2 + (n_1 + n_2 + n_3 + n_4 + n_5 + \cdots + n_f(q; q)_L(q; q)_{2n_f}}}{(q; q)_m(q; q)^3 n(q; q)_n - 3n - 2m(q; q)_n_1(q; q)_n_2 \cdots (q; q)_{n_{j-1}}(q; q)_{2n_{j+1}}}
\]
\[
= \sum_{j=-L}^{L} \left( \frac{j+1}{3} \right) q^{3j - 3j} \left[ \frac{2L}{L-j} \right]_q.
\]

Proceeding in this fashion, for any \( s = 1, 2, \ldots \), we arrive at
\[
\sum_{m,n_1,n_2 \ldots, n_s \geq 0} \frac{q^{L+2m^2 + 6mn + n^2 + (n_1 + n_2 + n_3 + n_4 + n_5 + \cdots + n_s(q; q)_L(q; q)_{2n_s}}}{(q; q)_m(q; q)^3 n(q; q)_n - 3n - 2m(q; q)_n_1(q; q)_n_2 \cdots (q; q)_{n_{j-1}}(q; q)_{2n_{j+1}}}
\]
\[
= \sum_{j=-L}^{L} \left( \frac{j+1}{3} \right) q^{(s+1)j - sj} \left[ \frac{2L}{L-j} \right]_q,
\]
where \( N_i := n_i + \cdots + n_s \) for \( i = 1, 2, \ldots, s \). Finally, applying Theorem 2.1 with \( a = 0 \) directly to (4.11), without going through the multiplication with the factor \( q^L \), \( f - s \) times (for any \( f \geq s \)) yields (4.8).

We would like to remark that Theorem 4.8 can be seen as the \( s = 0 \) case of Theorem 4.15. Also note that if one apply Theorem 2.1 with \( a = 0 \) iteratively to (3.14), then one gets the \( s = 1 \) case Theorem 4.15.

Tending \( L \rightarrow \infty \) in (4.8) and using (2.6) we get Theorem 1.9 for \( s \in \mathbb{N} \). The \( s = 0 \) case of Theorem 1.9 is Theorem 4.9 and it is already proven. □
5. Dual Identities

Replacing $q \mapsto 1/q$ in (1.7) and (1.8), using (2.1) followed by multiplying both sides with $q^{L^2}$ and $q^{L^2+L}$, respectively, yields the following theorem.

**Theorem 5.1.**

\[
\sum_{n,m \geq 0 \atop L \equiv n + 2m \pmod{3}} (-1)^m q^{m(m-1)/2 + Ln(q; q)_L} = \sum_{j=-\infty}^{\infty} \left[ \frac{2L}{L-j_q} \right]^{j+1/3},
\]

where $(\cdot)$ is the Jacobi symbol.

The asymptotic behaviors of (5.1) and (5.2) as $L \to \infty$ should also be considered. It is easier to understand the asymptotic behavior of these identities through (5.2), so we will only be focusing on that.

All the summands of the left-hand side series of (5.2) with any non-zero $n$-values vanishes as $L \to \infty$ for $|q| < 1$. This reduces the double sums on the left-hand sides of (5.2) to single sums for the limit discussions.

Letting $L \mapsto 3L$, we see that the left-hand side summation conditions ' $L \equiv n + 2m \pmod{3}$' and ' $L \equiv n + 2m + 1 \pmod{3}$' imply that ' $n \equiv 0 \pmod{3}$' and ' $n \equiv 1 \pmod{3}$', respectively. On the right-hand side of (5.2), we do the simple $k = 3L - j$ substitution before taking any limits. We then see that $L \to \infty$ implies

\[
\sum_{m \geq 0 \atop m \equiv 0 \pmod{3}} (-1)^m q^{m(m+1)/2} = \sum_{k \geq 0} \left( 1 - \frac{k}{3} \right) \frac{q^k}{(q; q)_k}.
\]

The identity (5.3) can be simplified by writing the left-hand side sums explicitly and then combining the terms. For the right-hand side we can use the simple observation about the Jacobi symbols that $(1-k/3) = -(k+2)/3$. Then we get,

**Corollary 5.2.**

\[
\sum_{m \geq 0} (-1)^{m+1} q^{3m(3m+1)/2} = \sum_{k \geq 0} \left( \frac{k+2}{3} \right) \frac{q^k}{(q; q)_k}.
\]

Similar to Corollary 5.2, we can take the limit of (5.2) after $L \mapsto 3L + 2$ and $L \mapsto 3L + 1$. These considerations imply the two identities of the following corollary, respectively.

**Corollary 5.3.**

\[
\sum_{m \geq 0} (-1)^m q^{3(m+1)(3m+2)/2} = \sum_{k \geq 0} \left( \frac{k}{3} \right) \frac{q^k}{(q; q)_k},
\]

\[
\sum_{m \geq 0} (-1)^m q^{3m(3m-1)/2} = \sum_{k \geq 0} \left( \frac{k+1}{3} \right) \frac{q^k}{(q; q)_k}.
\]
We can combine and rewrite the equations (5.4)-(5.6) as
\[
\sum_{k \geq 0} \left( \frac{k + b}{3} \right) \frac{q^k}{(q; q)_k} = \frac{(q; q)_\infty}{(q^3; q^3)_\infty} \sum_{m \geq 0} (-1)^{m+1} \left( \frac{m - b}{3} \right) \frac{q^{m(m+1)/2}}{(q; q)_m},
\]
where \( b = 0, 1, 2 \). This identity can also be proven directly by appeal to the \( q \)-binomial theorem (2.4).

Corollaries 5.2 and 5.3 can be interpreted as weighted partition theorems. For example, (5.5) has the following partition theoretic interpretation.

**Theorem 5.4.** Let
\[
P_1 := \{ \pi \in D : \#(\pi) \not\equiv 0 \pmod{3} \}, \quad P_2 := \{ \pi = (\lambda_1, \ldots) \in \mathcal{P} : 3 \nmid \lambda_1 \lambda_2 \ldots \lambda_{\#(\pi)} \}, \quad P_3 := \{ \pi \in \mathcal{P} : \#(\pi) \not\equiv 0 \pmod{3} \},
\]
then
\[
(5.7) \quad \sum_{\pi \in P_1} (-1)^{\mu(\pi)} q^{\#(\pi)} = \sum_{(\pi_1, \pi_2) \in P_2 \times P_3} (-1)^{\sigma(\pi_2)} q^{\#(\pi_1) + \#(\pi_2)},
\]
where \( \sigma(\pi) \) is 1 if \( \#(\pi) \equiv 2 \pmod{3} \) and 0 otherwise, and \( \mu(\pi) := \#(\pi) + \sigma(\pi) + 1 \).

For example, for partitions of 3, the left-hand side of (5.7) counts the partitions (3) and (2,1). Both of these partitions are counted with positive weight and grants the total count of 2. On the right-hand side we consider the following pair of partitions with a total size of 3 and their respective weights \( \omega \):
\[
\begin{array}{ccc}
(1,1) & (2,1) & (1,2) \\
1 & -1 & 1 \\
\end{array}
\]
The total of all these weights is also 2.

To prove Theorem 5.4, we rewrite (5.5) as
\[
\sum_{m \equiv 2 \pmod{3}} (-1)^m \frac{q^{m(m+1)/2}}{(q; q)_m} = \frac{1}{(q, q^2; q^3)_\infty} \sum_{k \geq 0} \frac{k^2}{3} \frac{q^k}{(q; q)_k}.
\]
In light of (1.1), the left-hand side sums are the generating functions for distinct partitions with number of parts \( \equiv 2 \pmod{3} \) and \( \equiv 1 \pmod{3} \) parts, respectively, where \(-1\) is raised to the number of parts with one extra negative sign for partitions with number of parts congruent to 1 modulo 3. On the right-hand side, it is clear that the \( (q, q^2; q^3)_\infty^{-1} \) is the generating function for partitions where no part is divisible by 3. The last sum on the far right is the generating function for partitions with number of parts \( \equiv 0 \pmod{3} \) where partitions with 2 modulo 3 number of parts are weighted with a \(-1\) coming from the Jacobi symbol.

Similarly, (5.4) and (5.6) can be interpreted as the two following weighted partition theorems, respectively.

**Theorem 5.5.** Let
\[
P_1^* := \{ \pi \in D : \#(\pi) \not\equiv 2 \pmod{3} \}, \quad \text{and} \quad P_3^* := \{ \pi \in \mathcal{P} : \#(\pi) \not\equiv 1 \pmod{3} \},
\]
then
\[
\sum_{\pi \in P_1^*} (-1)^{\mu^*(\pi)} q^{\#(\pi)} = \sum_{(\pi_1, \pi_2) \in P_2 \times P_3^*} (-1)^{\sigma^*(\pi_2)} q^{\#(\pi_1) + \#(\pi_2)},
\]
where \( \sigma^*(\pi) \) is 1 if \( \#(\pi) \equiv 0 \pmod{3} \) and 0 otherwise, and \( \mu^*(\pi) := \#(\pi) + \sigma^*(\pi) \).
Theorem 5.6. Let
\[ P_1' := \{ \pi \in D : \#(\pi) \neq 1 \pmod{3} \} \quad \text{and} \quad P_3' := \{ \pi \in P : \#(\pi) \neq 2 \pmod{3} \}, \]
then
\[ \sum_{\pi \in P_1'} (-1)^{\mu^*(\pi)} q^{\pi} = \sum_{(\pi_1, \pi_2) \in P_2 \times P_3'} (-1)^{\sigma^*(\pi_2)} q^{\pi_1 \cdot \pi_2}, \]
where \( \sigma^*(\pi) \) is 1 if \( \#(\pi) \equiv 0 \pmod{3} \) and 0 otherwise, and \( \mu^*(\pi) := \#(\pi) + \sigma^*(\pi) \).

6. OUTLOOK

Although the study here is systematic and seemingly complete, there are still some interesting leads to be explored and missing cases to be found. For example, we could not find a result analogous to (1.5) for Theorem 1.8. Also the simple extension of (1.3) analogous to (1.9) does not seem to exist.

However, the missing applications of Bailey’s lemma to Theorem 5.1 is not an oversight. Instead it is a deliberate choice that we make. Applying Theorem 2.1 to Theorem 5.1 yields the same polynomial right-hand sides the application of Bailey’s lemma to Theorem 1.8 with messier polynomials on the left-hand side.

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REFERENCES

[1] J. Ablinger and A. K. Uncu, qFunctions - A Mathematica package for q-series and partition theory applications, Journal of Symbolic Computation 107, (2021), 145-166.
[2] K. Alladi, G. E. Andrews, and B. Gordon, Refinements and Generalizations of Capparelli’s Conjecture on Partitions, Journal of Algebra 174 (1995), no. 2, 636-658.
[3] G. E. Andrews, Multiple series Rogers–Ramanujan type identities, Pacific. J. Math.114, 267-283 (1984).
[4] G. E. Andrews, and R. J. Baxter, Lattice gas generalization of the hard hexagon model. III. q-Trinomial coefficients, J. Statist. Phys. 47 (1987), no: 3-4, 297-330.
[5] G. E. Andrews, Schur’s theorem. Capparelli’s conjecture and q-trinomial coefficients, Contemporary Mathematics 166 (1994), 141–154.
[6] G. E. Andrews, The theory of partitions, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1998. Reprint of the 1997 original. MR1634067 (99c:11126)
[7] A. Berkovich and A. K. Uncu, Polynomial identities implying Capparelli’s partition theorems, J. Number Theory 201 (2019), 77-107.
[8] A. Berkovich and A. K. Uncu, Elementary polynomial identities involving q-trinomial coefficients, Ann. Comb. 23 (2019), no. 3-4, 549-560.
[9] A. Berkovich and A. K. Uncu, Refined q-Trinomial Coefficients and Two Infinite Hierarchies of q-Series Identities, Algorithmic Combinatorics: Enumerative Combinatorics, Special Functions and Computer Algebra. Texts & Monographs in Symbolic Computation (A Series of the Research Institute for Symbolic Computation, Johannes Kepler University, Linz, Austria). Springer, Cham. https://doi.org/10.1007/978-3-030-44559-1_4
[10] S. Capparelli, A combinatorial proof of a partition identity related to the level 3 representation of twisted affine Lie algebra, Communications in Algebra 23 (1995), no. 8, 2959-2969.
[11] S. Capparelli, Vertex operator relations for affine algebras and combinatorial identities, Ph.D Thesis Rutgers University (1988).
[12] J. Dousse, On partition identities of Capparelli and Primc, Adv. Math. 370 (2020), 107245.
[13] J. Dousse, and J. Lovejoy, Generalizations of Capparelli’s identity, Bull. Lond. Math. Soc. 51, Issue 2 (2019), pp. 193-206.
[14] G. Gasper and M. Rahman, Basic hypergeometric series, Cambridge University Press, 2004.
[15] S. Kanade and M. Russell, Staircases to analytic sum-sides for many new integer partition identities of Rogers–Ramanujan type, Electron. J. Combin. 26, no. 1 (2019), Paper 1.1.
[16] M. Kauers and C. Koutschan, A Mathematica package for q-holonomic sequences and power series, The Ramanujan Journal, 19 (2), 137-150, Springer, 2009, ISSN 1382-4090.
[17] C. Koutschan, Advanced Applications of the Holonomic Systems Approach, RISC, Johannes Kepler University, Linz. PhD Thesis. September 2009.
[18] K. Kurşungöz, Andrews–Gordon type series for Capparelli’s and Göllnitz–Gordon identities, J.Combin.Theory Ser.A 165 (2019), 117-138.
[19] M. Petkovšek, H. S. Wilf, and D. Zeilberger, A = B (With a foreword by Donald E. Knuth. With a separately available computer disk). A K Peters, Ltd., Wellesley, MA, 1996. xii+212 pp. ISBN: 1-56881-063-6

[20] P. Paule and A. Riese, A Mathematica q-Analogue of Zeilberger’s Algorithm Based on an Algebraically Motivated Approach to q-Hypergeometric Telescoping, in Special Functions, q-Series and Related Topics, Fields Inst. Commun., Vol. 14, pp. 179-210, 1997.

[21] P. Paule, Zwei neue Transformationen als elementare Anwendungen der q-Vandermonde Formel, Ph.D. Thesis (1982), University of Vienna.

[22] A. Riese, qMultiSum - A Package for Proving q-Hypergeometric Multiple Summation Identities, Journal of Symbolic Computation 35 (2003), 349-376.

[23] C. Schneider, Symbolic Summation Assists Combinatorics, Sem. Lothar. Combin. 56, (2007), pp.1-36. Article B56b.

[24] M. Tamba, C. F. Xie Level three standard modules for $A_2$ and combinatorial identities, J. Pure Appl. Algebra 105 (1995), no. 1, 53–92.

[25] S. O. Warnaar, The generalized Borwein conjecture. II. Refined q-trinomial coefficients, Discrete Math. 272 (2003), no. 2-3, 215-258.

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