On integers \( n \) for which \( \sigma(2n + 1) \geq \sigma(2n) \)

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Abstract

We show that the natural density of positive integers \( n \) for which \( \sigma(2n + 1) \geq \sigma(2n) \) is between 0.053 and 0.055.

1 Introduction

Let \( \sigma(n) \) denote the sum of divisors function. While its average value is well-behaved (see, e.g. [6, §18.3]), the local behavior of \( \sigma(n) \) is, as with many interesting arithmetical functions, erratic. Consider, for example, a result from Erdős, Györy, and Papp [3] (see also [12, p. 89]) that says that the chain of inequalities

\[
\sigma(n + m_1) > \sigma(n + m_2) > \sigma(n + m_3) > \sigma(n + m_4)
\]

holds for infinitely many \( n \), where the \( m_i \) are any permutations of the numbers 1, 2, 3, 4.

We consider here the problem of counting those \( n \) such that \( \sigma(2n + 1) \geq \sigma(2n) \). When \( 2n + 1 \) is prime the left side is \( 2n + 2 \) whereas the right side is at least \( 2n + 1 + n + 2 = 3n + 3 \). This shows that the inequality is false infinitely often. Empirically, it appears to be false very frequently. Let \( B \) be the set of natural numbers \( n \) such that \( \sigma(2n + 1) \geq \sigma(2n) \) and let \( B(x) \) be the number of those \( n \) in \( B \) with \( n \leq x \). From Table 1 one may be tempted to conjecture that \( B(x)/x \sim 0.0546 \ldots \).

Laub [9] posed the question of estimating the size of \( B(x)/x \). Mattics [11] answered this, and records a remark of Hildebrand that \( \lim_{x \to \infty} B(x)/x \) exists. We will call this limit

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Table 1: Proportion of integers $n \leq x$ with $\sigma(2n + 1) \geq \sigma(2n)$

| $x$   | Proportion |
|-------|------------|
| $10^3$ | 0.06       |
| $10^4$ | 0.0551     |
| $10^5$ | 0.549      |
| $10^6$ | 0.054603   |
| $10^7$ | 0.0546879  |
| $10^8$ | 0.0546537  |
| $10^9$ | 0.05465173 |

the natural density of $B$, denoted $d_B$. Although Mattics was not able to calculate this density, he was able to establish the existence of constants $\lambda$ and $\mu$ with $0 < \lambda < \mu < 1$ such that $\lambda x < B(x) < \mu x$ for $x$ sufficiently large. Specifically, he showed that one could take $\lambda = 1/3000$ and $\mu = 25/28$.

We refine Mattics’ result and prove the following.

**Theorem 1.** Let $B = \{n \geq 1 : \sigma(2n + 1) \geq \sigma(2n)\}$ and let $B(x) = |\{n \in B : n \leq x\}|$. Then $d_B$ exists and we have

$$0.0539171 \leq d_B \leq 0.0549445. \quad (1)$$

The precision in (1) is not as high as in the analogous problem concerning abundant numbers, that is, those numbers $n$ such that $\sigma(n)/n \geq 2$. Let $d_A$ be the natural density of abundant numbers. We have that $0.247617 < d_A < 0.247648$, due to the first author [7, 8]. We shall draw on methods used in [1, 11] to establish Theorem 1.

In §2 we prove that the density of $B$ exists. In §3 we set up the tools to bound $d_B$ and in §4 we complete the proof of Theorem 1.

## 2 Existence of $d_B$

Let $h(n) = \sigma(n)/n$. It will be convenient to work with the set

$$C := \{n : h(2n + 1) \geq h(2n)\}.$$

We will prove that the sets $B$ and $C$ have equal densities. First observe that

$$\frac{h(2n + 1)}{h(2n)} = \frac{\sigma(2n + 1)}{\sigma(2n)} \cdot \frac{2n}{2n + 1},$$

so $C \subseteq B$. By [13], $C$ has a density, so it remains to prove that the set

$$B - C = \left\{ n : 0 \leq \sigma(2n + 1) - \sigma(2n) < \frac{\sigma(2n)}{2n} \right\}$$
has density zero. On the one hand, Grönwall’s theorem [5] states that
\[
\limsup_{n \to \infty} \frac{\sigma(n)/n}{\log \log n} = e^\gamma,
\]
where \( \gamma \) is the Euler–Mascheroni constant. Hence, for \( n \in B - C \) we have that
\[
\sigma(2n + 1) - \sigma(2n) = O(\log \log n).
\]
On the other hand, Lemma 2.1 of [10] gives that on a set \( S \) of asymptotic density 1, \( p \mid \sigma(n) \) for every prime \( p \leq \log \log n / \log \log \log n \). Writing \( K(n) \) for the product of the primes satisfying this inequality, the prime number theorem yields
\[
K(n) = \log(n)^{(1+o(1))}/\log \log \log n.
\]
Thus, for almost all \( n \), \( K(2n) \mid \sigma(2n + 1) - \sigma(2n) \). It follows that in set \( B - C \), either \( \sigma(2n + 1) = \sigma(2n) \) or
\[
\log(n)^{(1+o(1))}/\log \log \log n = K(2n) \leq \sigma(2n + 1) - \sigma(2n) = O(\log \log n),
\]
a contradiction for sufficiently large \( n \). In the case of equality, we invoke the result in [2] or [4] that the set of \( n \) satisfying the equality has density zero. This establishes that the set \( B \) has a density and that \( d_B = d_C \).

3 Preparatory results

In this section, we partition the set \( C \) into subsets and bound the densities of these subsets.

3.1 Smooth partitions

Let \( y \geq 2 \). We say a number \( n \) is \( y \)-smooth if its largest prime divisor \( p \) has \( p \leq y \), and write \( S(y) \) for the set of \( y \)-smooth numbers. Let \( Y(n) \) be the largest \( y \)-smooth divisor of \( n \). We define
\[
S(a, b) := \{ n \in \mathbb{N} : Y(2n + 1) = a, Y(2n) = b \}.
\]
Note that the sets \( S(a, b) \), \( a, b \in S(y) \) partition \( \mathbb{N} \), and that \( S(a, b) = \varnothing \) unless \( b \) is even and \( \gcd(a, b) = 1 \). We partition \( C \) via \( C(a, b) := C \cap S(a, b) \).

We will express bounds of \( d_C(a, b) \) in terms of \( d_S(a, b) \). To see that \( S(a, b) \) has a natural density and to determine the value of the density, we will show that \( S(a, b) \) is a finite union of arithmetic progressions. Denote the set of totatives modulo \( N \) by
\[
\Phi(N) := \{ t \in \mathbb{N} : 1 \leq t \leq N, \gcd(t, N) = 1 \}.
\]
We define \( P = P(y) \) as the product of primes \( p \), \( p \leq y \). For any \( n \in \mathbb{N} \) we have \( \gcd(n/Y(n), P) = 1 \), so we may partition \( S(a, b) \) by
\[
S(a, b; t_1, t_2) := \{ n \in S(a, b) : (2n + 1)/a \equiv t_1 \mod P, 2n/b \equiv t_2 \mod P \},
\]
for \( t_1, t_2 \in \Phi(P) \). We will show that these sets are either empty or are arithmetic progressions.

For \( n \in S(a,b; t_1, t_2) \), the condition \( n \in S(a,b) \) implies \( 2n + 1 = ax, 2n = by \) for some \( x, y \in \mathbb{Z} \). We thus study the linear Diophantine equation

\[
xax - by = 1. \tag{2}
\]

Writing the congruence conditions as

\[
x = t_1 + x'p, \quad y = t_2 + y'p, \quad x', y' \in \mathbb{Z},
\]

the equation in (2) becomes

\[
apx' - bpy' = 1 - at_1 + bt_2. \tag{3}
\]

This equation has solutions if and only if \( P \mid 1 - at_1 + bt_2 \). In this case, write \( P\ell = 1 - at_1 + bt_2 \).

Then (3) simplifies to

\[
ax' - by' = \ell,
\]

which has the general solution \( x' = x_0\ell + kb, y' = y_0\ell + ka, k \in \mathbb{Z} \), where \( x = x_0, y = y_0 \) is a particular solution for (2). We conclude that \( n \in S(a,b; t_1, t_2) \) has the form

\[
2n + 1 = a(t_1 + P\ell)x_0 + abPk,
\]

\[
2n = b(t_2 + P\ell)y_0 + abPk,
\]

and any choice of \( k \) such that \( n \in \mathbb{N} \) puts \( n \) in \( S(a,b; t_1, t_2) \). Thus \( S(a,b; t_1, t_2) \) is an arithmetic progression when nonempty and

\[
dS(a,b; t_1, t_2) = \begin{cases} 
0 & P \mid 1 - at_1 + bt_2, \\
\frac{2}{ab} & P \mid 1 - at_1 + bt_2.
\end{cases}
\]

To determine \( dS(a,b) \), we must count the number of ordered pairs \( (t_1, t_2) \) satisfying \( P \mid 1 - at_1 + bt_2 \). We check for valid pairs modulo each prime \( p \mid P \). If \( p \mid a \), then \( p \mid 1 - at_1 + bt_2 \) if and only if \( t_2 \equiv a^{-1} \mod p \), so \( t_1 \) is free and \( t_2 \) is completely determined modulo \( p \). Thus, there are \( p - 1 \) ordered pairs modulo \( p \). Similarly, if \( p \mid b \), \( p \mid 1 - at_1 + bt_2 \) if and only if \( t_1 \equiv a^{-1} \mod p \), so again there are \( p - 1 \) ordered pairs modulo \( p \). Finally, if \( p \nmid ab \), \( p \mid 1 - at_1 + bt_2 \) if and only if \( t_2 \equiv b^{-1}(at_1 - 1) \mod p \). For each \( t_1 \in \Phi(P) \), we fail to get a valid \( t_2 \in \Phi(P) \) only if \( t_1 \equiv a^{-1} \mod p \). Thus, there are \( p - 2 \) valid ordered pairs modulo \( p \). We conclude by the Chinese remainder theorem that

\[
\#\{(t_1, t_2) \in \Phi(P)^2 : P \mid 1 - at_1 + bt_2 \} = \prod_{p\mid ab}(p - 1) \prod_{p\mid P, p\nmid ab} (p - 2),
\]

so that

\[
dS(a,b) = \frac{2}{ab} \prod_{p\mid ab} \left(1 - \frac{1}{p}\right) \prod_{p\mid P, p\nmid ab} \left(1 - \frac{2}{p}\right).
\]

4
3.2 Moments of $h(2n+1)$ and $h(2n)$

To bound $dC(a,b)$ we will also need bounds on the following moments of $h(2n+1)$ and $h(2n)$ over $n \in S(a,b)$

$$\sum_{n \leq x, n \in S(a,b)} h^r(2n+1) \quad \text{and} \quad \sum_{n \leq x, n \in S(a,b)} h^r(2n).$$

To this end, we prove a higher-moments analogue of the lemma in \cite{11} using ideas in \cite{1}.

Lemma 1. Let

$g(n) := \left(\frac{\sigma(n)}{n}\right)^r$, \quad $\rho(p^\alpha) := g(p^\alpha) - g(p^\alpha - 1)$,

and

$$\Lambda_k(r) := \prod_{p \nmid k} \left(1 + \frac{\rho(p)}{p} + \frac{\rho(p^2)}{p^2} + \cdots \right).$$

If $h$ and $k$ are given coprime positive integers, $r \geq 1$ and $x \geq 2$, then

$$\sum_{\substack{n \leq x, \ n \equiv h \mod k}} \left(\frac{\sigma(n)}{n}\right)^r = x^r \frac{\Lambda_k(r)}{k} + O((\log k)^r).$$

(Note that, although we are borrowing the notation of Deléglise, our meaning for $k$ differs from his.)

Proof. We generalize the lemma in \cite{11} which proves the case $r = 1$. Fix a real number $r \geq 1$. By Möbius inversion, we express $g(n)$ as the divisor sum

$$g(n) = \sum_{d|n} \rho(d),$$

where

$$\rho(n) = \sum_{d|n} \mu \left(\frac{n}{d}\right) g(d).$$

Since $g$ is multiplicative, so is $\rho$, and on prime powers $p^\alpha$ we have

$$\rho(p^\alpha) = g(p^\alpha) - g(p^\alpha - 1).$$

Note that $\rho$ is always positive.

If $\chi$ is a character modulo $k$, we have

$$\sum_{n \leq x} \chi(n) g(n) = \sum_{n \leq x} \chi(n) \sum_{d|n} \rho(d)$$

$$= \sum_{d \leq x} \chi(d) \rho(d) \sum_{m \leq x/d} \chi(m).$$
If $\chi$ is non-principal, we have

$$
\sum_{n \leq x} \chi(n) g(n) = O \left( \sum_{d \leq x} \rho(d) \right).
$$

If $\chi$ is the principal character, and letting a dash on a summation denote sums restricted to integers relatively prime to $k$, we have

$$
\sum_{n \leq x} \chi(n) g(n) = \sum_{d \leq x} \rho(d) \left( \frac{\varphi(k)}{k} \frac{x}{d} + O(1) \right)
= \frac{\varphi(k)}{k} x \sum_{d \leq x} \frac{\rho(d)}{d} + O \left( \sum_{d \leq x} \rho(d) \right)
= \frac{\varphi(k)}{k} x \sum_{d=1}^{\infty} \frac{\rho(d)}{d} + O \left( \sum_{d>x} \frac{\rho(d)}{d} + \sum_{d \leq x} \rho(d) \right)
= \frac{\varphi(k)}{k} x \Lambda_k(r) + O \left( \sum_{d>x} \frac{\rho(d)}{d} + \sum_{d \leq x} \rho(d) \right),
$$

where

$$
\Lambda_k(r) := \sum_{d=1}^{\infty} \frac{\rho(d)}{d}.
$$

Again by multiplicativity of $\rho$, we have

$$
\Lambda_k(r) = \prod_{p \mid k} \left( 1 + \frac{\rho(p)}{p} + \frac{\rho(p^2)}{p^2} + \ldots \right).
$$

Multiplying by $\chi(h)$ and summing over the characters $\chi$ modulo $k$, we obtain

$$
\sum_{\frac{n}{n \equiv h \mod k}} g(n) = \frac{1}{\varphi(k)} \sum_{\chi} \chi(h) \sum_{n \leq x} \chi(n) g(n)
= x \frac{\Lambda_k(r)}{k} + O \left( \sum_{d>x} \frac{\rho(d)}{d} + \sum_{d \leq x} \rho(d) \right).
$$

It remains to estimate the error. Since $\sum_d \frac{\rho(d)}{d}$ is a convergent series, its tail is $o(1)$. We now estimate

$$
\sum_{d \leq x} \rho(d).
$$
We have
\[ \sum_{d \leq x} \rho(d) \leq \prod_{p \leq x} \left( 1 + \rho(p) + \rho(p^2) + \cdots \right) \]
\[ = \prod_{p \leq x} \lim_{a \to \infty} g(p^a) \]
\[ = \prod_{p \leq x} \left( 1 + \frac{1}{p-1} \right)^r \]
\[ = \exp \log \left( \prod_{p \leq x} \left( 1 + \frac{1}{p-1} \right)^r \right) \]
\[ = \exp \left( r \sum_{p \leq x} \log \left( 1 + \frac{1}{p-1} \right) \right) \]
\[ \leq \exp \left( r \sum_{p \leq x} \frac{1}{p-1} \right), \]
where we have used the bound \( \log(1 + x) \leq x \) for \( x > 0 \). Since
\[ \frac{1}{p-1} = \frac{1}{p} + O \left( \frac{1}{p^2} \right), \]
and
\[ \sum_{p \leq x} \frac{1}{p} = \log \log x + O(1), \quad \sum_{p \leq x} \frac{1}{p^2} = O(1), \]
we conclude that
\[ \sum_{d \leq x} \rho(d) = O \left( (\log x)^r \right). \]

By Lemma \[ \square \] and our characterization of the set \( S(a, b; t_1, t_2) \) as an arithmetic progression when \( P | 1 - at_1 + bt_2 \), we conclude that for such pairs \( (t_1, t_2) \) we have
\[ \sum_{n \leq x \atop n \in S(a, b; t_1, t_2)} h^r(2n + 1) = h^r(a) \sum_{m \leq (2x+1)/a \atop m \equiv (t_1 + P) x_0 \, \text{mod} \, bP} h^r(m) \]
\[ = h^r(a) \Lambda_P(r) \frac{2}{abP} x + O \left( \log^r x \right). \]

Summing over all pairs \( (t_1, t_2) \), we have
\[ \sum_{n \leq x \atop n \in S(a, b)} h^r(2n + 1) \sim h^r(a) \Lambda_P(r) \mathcal{D} S(a, b)x, \quad x \to \infty. \]
In the case

\[ \sum_{n \leq x} h^r(2n) \sim h^r(b) \Lambda_P(r) d S(a, b)x, \quad x \to \infty. \]

Likewise,

\[ \sum_{n \in S(a, b)} h^r(2n) \sim h^r(b) \Lambda_P(r) d S(a, b)x, \quad x \to \infty. \]

### 4 Bounds on d B

We can now place bounds on d C, and thus on d B, by bounding d C(a, b). We call 0 and d S(a, b) trivial bounds for d C(a, b). For a nontrivial upper bound, we observe that

\[ \sum_{n \in S(a, b)} h^r(2n + 1) = \sum_{n \in S(a, b)} h^r(2n + 1) + \sum_{n \in S(a, b)} h^r(2n + 1) \]

\[ \geq \sum_{n \in S(a, b)} h^r(2n) + \sum_{n \in S(a, b)} h^r(2n + 1) \]

\[ \geq h^r(b) |C(a, b) \cap [1, x]| + h^r(a) (|S(a, b) \cap [1, x]| - |C(a, b) \cap [1, x]|). \]

Dividing by x and taking \( x \to \infty \) we have

\[ h^r(a) \Lambda_P(r) d S(a, b) \geq h^r(b) d C(a, b) + h^r(a) d S(a, b) - h^r(a) d C(a, b). \]

In the case \( h(b) > h(a) \), we arrive at the upper bound

\[ d C(a, b) \leq \frac{h^r(a) (\Lambda_P(r) - 1)}{h^r(b) - h^r(a)} d S(a, b). \]

For this upper bound to be nontrivial, we require \( \Lambda_P(r)^{1/r} < h(b)/h(a) \). Note that since \( \Lambda_P(r) > 1 \) for all \( r \geq 1 \), this condition implies \( h(b) > h(a) \).

For a nontrivial lower bound, we proceed similarly:

\[ \sum_{n \in S(a, b)} h^r(2n) = \sum_{n \in S(a, b)} h^r(2n) + \sum_{n \in S(a, b)} h^r(2n) \]

\[ \geq \sum_{n \in S(a, b)} h^r(2n) + \sum_{n \in S(a, b)} h^r(2n + 1) \]

\[ \geq h^r(b) |C(a, b) \cap [1, x]| + h^r(a) (|S(a, b) \cap [1, x]| - |C(a, b) \cap [1, x]|). \]

Thus, asymptotically we have

\[ h^r(b) \Lambda_P(r) d S(a, b) \geq h^r(b) d C(a, b) + h^r(a) d S(a, b) - h^r(a) d C(a, b). \]
In the case \( h(b) - h(a) < 0 \), we have

\[
d C(a, b) \geq \frac{h^r(a) - h^r(b)\Lambda_P(r)}{h^r(a) - h^r(b)} d S(a, b).
\]

This bound is nontrivial when \( h(a)/h(b) > \Lambda_P(r)^{1/r} \), and this condition implies \( h(b) < h(a) \).

For upper bounds \( \Lambda^+_P(r) \) for \( \Lambda_P(r) \) we use the work of Delégise [1] when \( r > 1 \), where we have taken 65536 to be the maximum prime bound:

\[
\Lambda^+_P(r) = \prod_{\substack{p \text{ prime} \ y < p < 65536}} \left(1 + \frac{(1+1/p)^r - 1}{p} + \frac{r}{(p^4 - p^2)(1 - \frac{1}{p})^{r-1}} \right) \exp(1.6623114 \times 10^{-6} r).
\]

When \( r = 1 \) we use

\[
\Lambda_P(1) = \Lambda^+_P(1) = \zeta(2) \prod_{\substack{p | r \ p \text{ prime}}} \left(1 - \frac{1}{p^2}\right).
\]

To summarize, we use the following bounds for \( d C(a, b) \):

\[
\begin{align*}
d C(a, b) &\geq d C^-(a, b) = \begin{cases} 
\frac{h^r(a) - h^r(b)\Lambda_P(r)}{h^r(a) - h^r(b)} d S(a, b) & \text{for } h(a)/h(b) > \Lambda^+_P(r)^{1/r}, \\
0 & \text{for } h(a)/h(b) \leq \Lambda^+_P(r)^{1/r},
\end{cases} \\
d C(a, b) &\leq d C^+(a, b) = \begin{cases} 
\frac{h^r(a)(\Lambda^+_P(r) - 1)}{h^r(b) - h(a)} d S(a, b) & \text{for } h(b)/h(a) > \Lambda^+_P(r)^{1/r}, \\
d S(a, b) & \text{for } h(b)/h(a) \leq \Lambda^+_P(r)^{1/r}.
\end{cases}
\end{align*}
\]

Then

\[
\sum_{a, b \in S(y)} d C^-(a, b) \leq \sum_{a, b \in S(y)} d C \leq \sum_{a, b \in S(y)} d C^+(a, b).
\]

In practice, we fix the parameters \( y, z, \) and \( r_{\text{max}} \), then recursively run through odd \( a \in S(y) \cap [1, z] \). For each \( a \) we recursively run through even \( b \in S(y) \cap [1, z/a] \). For a given pair \( (a, b) \), we calculate \( C^\pm(a, b) \) for \( 1 \leq r \leq \min(r_1, r_{\text{max}}) \) where \( r_1 \) is the value of \( r \) that produces a locally optimum bound. For example, in the case of \( d C^+(a, b) \), we calculate bounds consecutively from \( r = 1 \) until the values stop decreasing or we reach \( r = r_{\text{max}} \), then keep the minimum value found.

By experimentation, we find that different values of the parameters \( y \) and \( z \) optimize the upper and lower bounds over a comparable time period. For the lower bound, the choice \( y = 353, z = 10^{13}, r_{\text{max}} = 2000 \) yielded the value 0.0539171 in 34.4 hours. For the upper bound, the choice \( y = 157, z = 10^{16}, r_{\text{max}} = 2000 \) yielded the value 0.0549446 in 25.1 hours. Both of these calculations were done on a Dell XPS 13 9370 laptop. This proves Theorem [1]
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