Lévy-Khintchine type representation of Dirichlet generators and Semi-Dirichlet forms

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Abstract

Let $U$ be an open set of $\mathbb{R}^n$, $m$ a positive Radon measure on $U$ such that $\text{supp}[m] = U$, and $(P_t)_{t>0}$ a strongly continuous contraction sub-Markovian semigroup on $L^2(U; m)$. We investigate the structure of $(P_t)_{t>0}$.

(i) Denote respectively by $(A, D(A))$ and $(\hat{A}, D(\hat{A}))$ the generator and the co-generator of $(P_t)_{t>0}$. Under the assumption that $C_0^\infty(U) \subset D(A) \cap D(\hat{A})$, we give an explicit Lévy-Khintchine type representation of $A$ on $C_0^\infty(U)$.

(ii) If $(P_t)_{t>0}$ is an analytic semigroup and hence is associated with a semi-Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$, we give an explicit characterization of $\mathcal{E}$ on $C_0^\infty(U)$ under the assumption that $C_0^\infty(U) \subset D(\mathcal{E})$.

We also present a LeJan type transformation rule for the diffusion part of regular semi-Dirichlet forms on general state spaces.

Keywords: Dirichlet generator; Semi-Dirichlet form; Markov process; Lévy-Khintchine type representation; LeJan type transformation rule; Beurling-Deny formula

1 Introduction and main results

Let $(X_t)_{t \geq 0}$ be a Lévy process on $\mathbb{R}^n$. By the celebrated Lévy-Khintchine formula, we know that the infinitesimal generator $A$ of $(X_t)_{t \geq 0}$ is characterized by (cf. 29
Theorem 31.5])

\[ Au(y) = \frac{1}{2} \sum_{i,j=1}^{n} Q_{ij} \frac{\partial^2 u}{\partial y_i \partial y_j}(y) + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial y_i}(y) \]

+ \int_{\mathbb{R}^n} \left( u(y + x) - u(y) - \sum_{i=1}^{n} x_i \frac{\partial u}{\partial y_i}(y) I_{\{|x| \leq 1\}}(x) \right) \nu(dx), \quad u \in C_0^\infty(\mathbb{R}^n), \quad (1.1) \]

where \( Q = (Q_{ij})_{1 \leq i,j \leq n} \) is a symmetric nonnegative-definite \( n \times n \) matrix, \( (b_1, \ldots, b_n) \in \mathbb{R}^n \), and \( \nu \) is a Lévy measure satisfying \( \nu(\{0\}) = 0 \) and \( \int_{\mathbb{R}^n} (1 \wedge |x|^2)\nu(dx) < \infty \). Hereafter, \(| \cdot |\) denotes the Euclidean metric of \( \mathbb{R}^n \), \( C(\mathbb{R}^n) \) denotes the set of all continuous functions on \( \mathbb{R}^n \), and \( C_0^\infty(\mathbb{R}^n) \) denotes the set of all infinitely differentiable functions on \( \mathbb{R}^n \) with compact supports.

The decomposition of type (1.1) also holds for Feller processes on \( \mathbb{R}^n \). In [8], Courrège proved that if \( A \) is a linear operator from \( C_0^\infty(\mathbb{R}^n) \) to \( C(\mathbb{R}^n) \) satisfying the positive maximum principle, then \( A \) is decomposed as

\[ Au(y) = \frac{1}{2} \sum_{i,j=1}^{n} q_{ij}(y) \frac{\partial^2 u}{\partial y_i \partial y_j}(y) + \sum_{i=1}^{n} l_i(y) \frac{\partial u}{\partial y_i}(y) + \gamma(y)u(y) \]

+ \int_{\mathbb{R}^n} \left( u(y + x) - u(y)w(x) - \sum_{i=1}^{n} x_i \frac{\partial u}{\partial y_i}(y)w(x) \right) \mu(y, dx), \]

where \( \sum_{i,j=1}^{n} q_{ij}(y)\xi_i\xi_j \geq 0 \) for all \( y \in \mathbb{R}^n \) and \( (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \), the function \( y \to \sum_{i,j=1}^{n} q_{ij}(y)\xi_i\xi_j \) is upper semicontinuous, \( l_i \in C(\mathbb{R}^n) \), \( 1 \leq i \leq n \), \( \gamma \in C(\mathbb{R}^n) \) with \( \gamma \leq 0 \), \( \mu \) is a kernel on \( \mathbb{R}^n \times \mathcal{B}(\mathbb{R}^n) \), and \( w \in C_0^\infty(\mathbb{R}^n) \) with \( 0 \leq w \leq 1 \) and \( w = 1 \) on \( \{ x \in \mathbb{R}^n : |x| \leq 1 \} \) (cf. [21] §4.5).

Suppose now that \( (X_t)_{t \geq 0} \) is a general right continuous Markov process on \( \mathbb{R}^n \), or more generally, on an open set \( U \) of \( \mathbb{R}^n \). In this paper, we are interested in describing the analytic structure of \( (X_t)_{t \geq 0} \). Denote by \( (P_t)_{t \geq 0} \) the transition semigroup of \( (X_t)_{t \geq 0} \). Suppose that there is a positive Radon measure \( m \) on \( U \) such that \( (P_t)_{t \geq 0} \) acts as a strongly continuous contraction semigroup on \( L^2(U;m) \). Note that this condition is fulfilled if, for example, \( m \) is an excessive measure of \( (X_t)_{t \geq 0} \). Denote by \( (A, D(A)) \) the \( L^2 \)-generator of \( (P_t)_{t \geq 0} \). Then, \( (A, D(A)) \) is a Dirichlet operator, i.e., \( (Au, (u - 1) \vee 0) \leq 0 \) for all \( u \in D(A) \) (cf. [24] Proposition I.4.3). Hereafter \( (\cdot, \cdot) \) denotes the inner product of \( L^2(U;m) \).

Denote by \( (\hat{A}, D(\hat{A})) \) the co-generator of \( (P_t)_{t \geq 0} \). Note that generally \( (\hat{A}, D(\hat{A})) \) may not be a Dirichlet operator (see [24] Remark 2.2(ii)] for an example). We assume that \( C_0^\infty(U) \subset D(\hat{A}) \cap D(A) \) and consider the following bilinear form

\[ \mathcal{E}(u, v) := (-Au, v) \quad \text{for } u, v \in C_0^\infty(U). \quad (1.2) \]

Here we would like to remind the reader that a generator on an \( L^2 \)-space is a Dirichlet operator if and only if its associated semigroup is sub-Markovian. It does not imply that its associated bilinear form is a (pre-) semi-Dirichlet form since
the sector condition might not be satisfied. Denote by \((G_\beta)_{\beta > 0}\) and \((\hat{G}_\beta)_{\beta > 0}\) the resolvent and co-resolvent of \((P_t)_{t > 0}\), respectively. Similar to [17 §2] (cf. also [12 §3.2]) and noting that the sector condition is not used therein, we can prove the following lemma by virtue of the fact that \(E(u,v) = \lim_{\beta \to \infty} \beta (u - \beta G_\beta u, v)\) for \(u, v \in C^\infty_0(U)\).

**Lemma 1.1.** (i) For \(\beta > 0\), there exist unique positive Radon measures \(\sigma_\beta\) and \(\hat{\sigma}_\beta\) on \(U \times U\) satisfying

\[
(\beta G_\beta u, v) = \int_{U \times U} u(x)v(y)\sigma_\beta(dx,dy) \tag{1.3}
\]

and

\[
(\beta \hat{G}_\beta u, v) = \int_{U \times U} u(x)v(y)\hat{\sigma}_\beta(dx,dy)
\]

for \(u, v \in L^2(U;m)\).

(ii) There exist a unique positive Radon measure \(J\) on \(U \times U\) off the diagonal \(d\) and a unique positive Radon measure \(K\) on \(U\) such that for \(v \in C^\infty_0(U)\) and \(u \in \{g \in C^\infty_0(U) : g\) is constant on a neighbourhood of \(\text{supp}[v]\}\),

\[
E(u,v) = \int_{U \times U \setminus d} 2(u(y) - u(x))v(y)J(dx,dy) + \int_{U} u(x)v(x)K(dx). \tag{1.4}
\]

Hereafter \(\text{supp}[u]\) denotes the support of \(u\). \(J\) and \(K\) are called the jumping and killing measures, respectively.

(iii) \((\beta/2)\sigma_\beta \to J\) and \((\beta/2)\hat{\sigma}_\beta \to \hat{J}\) vaguely on \(U \times U \setminus d\) as \(\beta \to \infty\), where \(\hat{J}(dx,dy) := J(dy,dx)\).

For \(\delta > 0\), we define

\[
U^\delta := \{x \in U : \inf_{y \in \partial U} |x - y| > \delta\}.
\]

Hereafter, for \(B \subset \mathbb{R}^n\), we denote by \(\partial B\) its boundary in \(\mathbb{R}^n\).

Now we can state the first main result of this paper.

**Theorem 1.2.** Let \(U\) be an open set of \(\mathbb{R}^n\) and \(m\) a positive Radon measure on \(U\) such that \(\text{supp}[m] = U\). Suppose that \((A, D(A))\) is a generator on \(L^2(U;m)\) such that \(A\) is a Dirichlet operator and \(C^\infty_0(U) \subset D(A) \cap D(\hat{A})\). Let \(\delta > 0\) be a constant such that \(U^\delta \neq \emptyset\). Then, we have the decomposition:

\[
(-Au, v) = \frac{1}{2} \sum_{i,j=1}^{n} \int_{U} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \nu_{ij}(dx) + \sum_{i=1}^{n} \int_{U^\delta} \frac{\partial u}{\partial x_i}(x)v(x)\nu_i^\delta(dx)
\]

\[
+ \int_{U \times U \setminus d} \sum_{i=1}^{n} (y_i - x_i) \left(\frac{\partial u}{\partial y_i}(y)v(y) - \frac{\partial u}{\partial x_i}(x)v(x)\right) I_{\{|x-y| \leq \delta\}}(x,y)J(dx,dy)
\]
+ \int_{U \times U \setminus d} 2 \left( u(y) - u(x) - \sum_{i=1}^{n} (y_i - x_i) \frac{\partial u}{\partial y_i}(y) I_{|x-y| \leq \delta}(x,y) \right) v(y) J(dx \, dy) \\
+ \int_{U} u(x)v(x)K(dx), \quad \forall u, v \in C^\infty_0(U^\delta), \tag{1.5}

where J and K are the jumping and killing measures, respectively, \{\nu_{ij}\}_{i,j=1}^{n} are signed Radon measures on U such that for any compact set K \subset U, \nu_{ij}(K) = \nu_{ji}(K) and \sum_{i,j=1}^{n} \xi_i \xi_j \nu_{ij}(K) \geq 0 \text{ for all } (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n, \text{ and } \{\nu_t^\delta\}_{t=1}^{n} are signed Radon measures on U^\delta.

On the one hand, Theorem 1.2 is a result of analysis, which characterizes a large class of Dirichlet generators on \mathbb{R}^n; on the other hand, it generalizes the classical result of Courrège from the Feller process setting to the right continuous Markov process setting. The representation (1.5) improves our understanding of Markov processes and has many potential applications. For example, it sheds light on the long-standing open problem, “when does a Markov process satisfy Hunt’s hypothesis (H)?” (cf. [2, 3, 10, 14, 16, 20, 28, 31] and the references therein). For a dual diffusion on an open set of \mathbb{R}^n, (1.5) indicates the strong connection between Hunt’s hypothesis (H) and the condition that the diffusion is locally associated with a semi-Dirichlet form. Here we would like to point out that Theorem 1.2 does not assume the sector condition although its proof is motivated by the theory of Dirichlet forms, and that the assumption \( C^\infty_0(U) \subset D(A) \cap D(\hat{A}) \) is reasonable for many applications, for example, when the martingale problem of Markov processes is studied (cf. [9, Chapter 4]).

If the diffusion part of \((X_t)_{t \geq 0}\) corresponds to a differential operator with very singular coefficients, then it is not suitable to assume that \( C^\infty_0(U) \subset D(A) \cap D(\hat{A}) \) any more. In this case, we will adopt the framework of semi-Dirichlet forms to investigate the analytic structure of \((X_t)_{t \geq 0}\). Suppose that \((A, D(A))\) satisfies the sector condition, i.e., there exists a positive constant \( \kappa \) such that

\[ |(1 - A)u, v| \leq \kappa |(1 - A)u, u|^{1/2} |(1 - A)v, v|^{1/2}, \quad \forall u, v \in D(A). \tag{1.6} \]

Note that \((A, D(A))\) satisfies the sector condition (1.6) if and only if \((P_t)_{t \geq 0}\) is an analytic semigroup (cf. [25, Corollary I.2.21]). Denote by \((\mathcal{E}, D(\mathcal{E}))\) the semi-Dirichlet form obtained by completing \( D(A) \) w.r.t. the \((1 - A)u, u)^{1/2}\)-norm. Assume that \( C^\infty_0(U) \subset D(\mathcal{E}) \). Then, one finds that Lemma 1.1 also holds for \((\mathcal{E}, D(\mathcal{E}))\). We make the following assumption.

**Assumption 1.3.** Let \( O \) be a relatively compact open set of \( U \). Suppose that \( \{f_n\}_{n=1}^{\infty} \subset C^\infty_0(O) \) and \( f \in C^\infty_0(O) \) satisfying \( f_n \) and all of its partial derivatives converge uniformly to \( f \) and its corresponding partial derivatives as \( n \to \infty \). Then, \( \mathcal{E}(f, g) = \lim_{n \to \infty} \mathcal{E}(f_n, g) \) and \( \mathcal{E}(g, f) = \lim_{n \to \infty} \mathcal{E}(g, f_n) \) for any \( g \in C^\infty_0(U) \).

We will obtain the following Lévy-Khintchine type representation of semi-Dirichlet forms, which generalizes the classical Beurling-Deny formula of symmetric Dirichlet forms on open sets of \( \mathbb{R}^n \) (cf. [12, Theorem 3.2.3]).
Theorem 1.4. Let $U$ be an open set of $\mathbb{R}^n$ and $m$ a positive Radon measure on $U$ such that $\text{supp}[m] = U$. Suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a semi-Dirichlet form on $L^2(U; m)$ such that $C^\infty_0(U) \subset D(\mathcal{E})$ and Assumption 1.3 holds. Let $\delta > 0$ be a constant such that $U^\delta \neq \emptyset$. Then, we have the decomposition:

$$
\mathcal{E}(u, v) = \frac{1}{2} \sum_{i,j=1}^{n} \int_U \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \nu_{ij}(dx) + \sum_{i=1}^{n} \left\langle \Psi_i^\delta, \frac{\partial u}{\partial x_i} v \right\rangle 
+ \int_{U \times \mathbb{R}^d} \sum_{i=1}^{n} (y_i - x_i) \left( \frac{\partial u}{\partial y_i}(y) v(y) - \frac{\partial u}{\partial x_i}(x) v(x) \right) I_{|x-y| \leq \delta}(x,y) J(dx dy)
+ \int_{U \times \mathbb{R}^d} 2 \left( u(y) - u(x) - \sum_{i=1}^{n} (y_i - x_i) \frac{\partial u}{\partial y_i}(y) I_{|x-y| \leq \delta}(x,y) \right) v(y) J(dx dy)
+ \int_{U} u(x)v(x)K(dx), \quad \forall u,v \in C^\infty_0(U^\delta),
$$

where $J$ and $K$ are the jumping and killing measures, respectively, $\{\nu_{ij}\}_{i,j=1}^{n}$ are signed Radon measures on $U$ such that for any compact set $K \subset U$, $\nu_{ij}(K) = \nu_{ji}(K)$ and $\sum_{i,j=1}^{n} \xi_i \xi_j \nu_{ij}(K) \geq 0$ for all $$(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$$, and $\{\Psi_i^\delta\}_{i=1}^{n}$ are generalized functions on $U^\delta$.

We will prove Theorems 1.2 and 1.4 in Section 2. If Assumption 1.3 is replaced by the assumption that $(\mathcal{E}, D(\mathcal{E}))$ is locally controlled by Dirichlet forms, then we can obtain a clearer characterization of the generalized functions $\{\Psi_i^\delta\}_{i=1}^{n}$ given in Theorem 1.4; see Corollary 2.9 below.

In Section 3, we will apply some ideas of Section 2 to investigate the structure of general regular semi-Dirichlet forms. Recently, there is new interest in further developing the theory of semi-Dirichlet forms. For example, semi-Dirichlet forms are used to construct and study jump-type Hunt processes ([13, 30]), the stochastic calculus of nearly-symmetric Markov processes has been generalized to the semi-Dirichlet form setting ([23, 27, 32]). However, the structure of semi-Dirichlet forms is still not completely known until now.

Let us first recall some known results on the structures of Dirichlet forms and semi-Dirichlet forms. For notation and terminology used in the paper, we refer to [12, 25]. Suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a regular symmetric Dirichlet form on $L^2(E; m)$, where $E$ is a locally compact separable metric space and $m$ is a positive Radon measure on $E$ with $\text{supp}[m] = E$. Recall that “regular” implies

(i) $C_0(E) \cap D(\mathcal{E})$ is dense in $D(\mathcal{E})$ w.r.t. the $\tilde{\mathcal{E}}_1^{1/2}$-norm.
(ii) $C_0(E) \cap D(\mathcal{E})$ is dense in $C_0(E)$ w.r.t. the uniform norm $\| \cdot \|_\infty$.

The Beurling-Deny formula tells us that $$(\mathcal{E}, D(\mathcal{E}))$$ can be expressed for $u,v \in C_0(E) \cap D(\mathcal{E})$ as

$$
\mathcal{E}(u, v) = \mathcal{E}(u, v) + \int_{E \times E \setminus d} (u(x) - u(y))(v(x) - v(y))J(dx dy)
$$
Here $\mathcal{E}^c(u, v)$ is a symmetric bilinear form with domain $D(\mathcal{E}^c) = C_0 (E) \cap D(\mathcal{E})$ and satisfies the strong local property:

$$\mathcal{E}^c (u, v) = 0 \text{ for } u \in D(\mathcal{E}^c) \text{ and } v \in I(u),$$

where

$$I(u) := \{ g \in D(\mathcal{E}^c) : g \text{ is constant on a neighbourhood of supp}[u] \}.$$ 

$J$ is a symmetric positive Radon measure on $E \times E \setminus d$ and $K$ is a positive Radon measure on $E$. Such $\mathcal{E}^c$, $J$ and $K$ are uniquely determined by $\mathcal{E}$.

Furthermore, the structure of $\mathcal{E}^c$ is characterized by the mutual energy measures. Let $u, v \in C_0 (E) \cap D(\mathcal{E})$. Then, there exists a unique signed Radon measure $\mu^c_{<u,v>}$ on $E$ such that

$$\int_E f d\mu^c_{<u,v>} = \mathcal{E}^c (uf, v) + \mathcal{E}^c (vf, u) - \mathcal{E}^c (uv, f), \quad f \in C_0 (E) \cap D(\mathcal{E}).$$

We have $\mathcal{E}^c (u, v) = \frac{1}{2} \mu^c_{<u,v>} (E)$ and $\mu^c_{<u,v>}$ obeys LeJan’s transformation rule:

$$d\mu^c_{\Phi(u_1, \ldots, u_m), v} = \sum_{i=1}^m \Phi_i (u_1, \ldots, u_m) d\mu^c_{<u_i, v>},$$

for any $\Phi \in C^1 (\mathbb{R}^m)$ with $\Phi (0) = 0$ and $u_1, \ldots, u_m, v \in C_0 (E) \cap D(\mathcal{E})$.

Proofs of the above structure results on symmetric Dirichlet forms can be found in [12, §3.2]. When non-symmetric Dirichlet forms, or more generally, semi-Dirichlet forms are considered, things become complicated. Through introducing the SPV integrable condition, [17] has generalized (1.7) to the semi-Dirichlet forms setting. Suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a regular semi-Dirichlet form on $L^2 (E; m)$. Then, there exist a unique positive Radon measure $J$ on $E \times E \setminus d$ and a unique positive Radon measure $K$ on $E$ such that for $v \in C_0 (E) \cap D(\mathcal{E})$ and $u \in I(v),$

$$\mathcal{E} (u, v) = \int_{E \times E \setminus d} 2 (u(y) - u(x)) v(y) J(dx dy) + \int_E u(x) v(x) K(dx).$$

Define $\mathcal{A}(v) := \{ f \in C_0 (E) \cap D(\mathcal{E}) : (f(y) - f(x)) v(y) \text{ is SPV integrable w.r.t. } J \}$.

Then, for $v \in C_0 (E) \cap D(\mathcal{E})$ and $u \in \mathcal{A}(v)$, we have the unique decomposition:

$$\mathcal{E} (u, v) = \mathcal{E}^c (u, v) + SPV \int_{E \times E \setminus d} 2 (u(y) - u(x)) v(y) J(dx dy)$$

$$+ \int_E u(x) v(x) K(dx), \quad (1.8)$$

where $\mathcal{E}^c (u, v)$ satisfies the left strong local property in the sense that $I(v) \subset \mathcal{A}(v)$ and $\mathcal{E}^c (u, v) = 0$ whenever $v \in C_0 (E) \cap D(\mathcal{E})$ and $u \in I(v)$. In general, the SPV
integrable condition cannot be dropped for the decomposition (1.8) to hold (see [19] for an example).

[18] [19] investigate the structure of non-symmetric Dirichlet forms and characterize their diffusion parts. Suppose that \((\mathcal{E}, D(\mathcal{E}))\) is a regular (non-symmetric) Dirichlet form. Since the dual form \((\hat{\mathcal{E}}, D(\mathcal{E})))\) also satisfies the semi-Dirichlet property, we have the decomposition:

\[
\hat{\mathcal{E}}(u, v) = \hat{\mathcal{E}}^c(u, v) + SPV \int_{E \times E \setminus d} 2(u(y) - u(x))v(y)\hat{J}(dxdy)
+ \int_E u(x)v(x)\hat{K}(dx) \tag{1.9}
\]

for \(v \in C_0(E) \cap D(\mathcal{E})\) and \(u \in \hat{A}(v) := \{ f \in C_0(E) \cap D(\mathcal{E}) : (f(y) - f(x))v(y)\) is SPV integrable w.r.t. \(\hat{J}\}.\) Note that \(\hat{J}(dxdy) = J(dydx)\) and it can be shown that \(\hat{A}(v) = A(v)\) for Dirichlet forms (cf. [19]). Let \(u, v \in C_0(E) \cap D(\mathcal{E})\) satisfying \((u(y) - u(x))v(y)\) is SPV integrable w.r.t. \(J.\) By [18] and (1.9), we get

\[
\hat{\mathcal{E}}(u, v) := \frac{1}{2}(\mathcal{E}(u, v) - \mathcal{E}(v, u))
= \frac{1}{2}(\mathcal{E}^c(u, v) - \hat{\mathcal{E}}^c(u, v)) + SPV \int_{E \times E \setminus d} 2(u(y) - u(x))v(y)\frac{J - \hat{J}}{2}(dxdy)
+ \int_E u(x)v(x)\frac{K - \hat{K}}{2}(dx).
\]

Define

\[
\hat{\mathcal{E}}^c(u, v) := \frac{1}{2}(\mathcal{E}^c(u, v) - \hat{\mathcal{E}}^c(u, v))
\]

and refer it as the co-symmetric diffusion part. Then, the diffusion part \(\mathcal{E}^c\) is uniquely decomposed into the symmetric part and the co-symmetric part as follows:

\[
\mathcal{E}^c(u, v) = \hat{\mathcal{E}}^c(u, v) + \hat{\mathcal{E}}^c(u, v),
\]

where \(\hat{\mathcal{E}}\) denotes the symmetric part of \(\mathcal{E}\) and \((\hat{\mathcal{E}}, D(\mathcal{E}))\) is a regular symmetric Dirichlet form.

Since \(\hat{\mathcal{E}}^c\) obeys LeJan’s transformation rule, to understand the structure of \(\mathcal{E},\) we need only concentrate on \(\hat{\mathcal{E}}^c.\) In [19], a LeJan type transformation rule is derived for \(\hat{\mathcal{E}}^c\) under the SPV integrable condition. This result has been used to study Markov processes associated with non-symmetric Dirichlet forms. For example, it plays a crucial role in investigating the strong continuity of generalized Feynman-Kac semigroups for nearly-symmetric Markov processes (see [26]).

In Section 3 of this paper, we will generalize the LeJan type transformation rule of [19] to the semi-Dirichlet forms setting, see Theorems 3.2 and 3.3 below. Note that if \((\mathcal{E}, D(\mathcal{E}))\) is only a semi-Dirichlet form, its dual form \((\hat{\mathcal{E}}, D(\mathcal{E}))\) generally does not satisfy the semi-Dirichlet property. So we do not have the decomposition (1.9). In particular, the existence of the dual killing measure \(\hat{K}\) is not ensured.
Also, the symmetric part $\tilde{E}$ of $E$ is only a symmetric positivity preserving form but not a symmetric Dirichlet form, which causes extra difficulty in characterizing the structure of $E$.

We hope the Lévy-Khintchine type representation and the LeJan type transformation rule obtained in this paper can help us better understand semi-Dirichlet forms and further their applications. We will apply these results in a forthcoming work to consider the strong continuity of generalized Feynman-Kac semigroups for Markov processes associated with semi-Dirichlet forms. We refer the interested reader to [1, 4, 5, 11, 15, 26] and the references therein for the topic of perturbation of Markov processes and Dirichlet forms. Finally, we would like to point out that by quasi-homeomorphisms (cf. [6, 17, 22]) many results obtained in this paper can be extended to quasi-regular semi-Dirichlet forms.

2 Lévy-Khintchine type representation of Dirichlet generators and semi-Dirichlet forms on open sets of $\mathbb{R}^n$

Throughout this section, we let $U$ be an open set of $\mathbb{R}^n$ which is equipped with the subspace topology of $\mathbb{R}^n$ and $m$ a positive Radon measure on $U$ such that $\text{supp}[m] = U$. We will give a Lévy-Khintchine type representation for Dirichlet generators and semi-Dirichlet forms on $U$. All the results of this section, except for those given in §2.3, apply to both of the following two cases.

Case 1: $(A, D(A))$ is a Dirichlet operator on $L^2(U; m)$ and is the generator of a strongly continuous contraction semigroup on $L^2(U; m)$. We assume that $C_0^\infty(U) \subset D(A) \cap D(\hat{A})$ and define the bilinear form $E$ as in (1.2).

Case 2: $(E, D(E))$ is a semi-Dirichlet form on $L^2(U; m)$ such that $C_0^\infty(U) \subset D(E)$ and Assumption 1.3 holds.

Let $J$ be the jumping measure given in Lemma 1.1. We choose a sequence of relatively compact open sets $\Omega_l \uparrow U$ and a sequence of numbers $\varsigma_l \downarrow 0$ such that the set $\Gamma_l := \{(x, y) \in \Omega_l \times \Omega_l : |x - y| \geq \varsigma_l\}$ is a continuous set w.r.t. $J$ for every $l \in \mathbb{N}$. Hereafter when we say that a set $B$ is a relatively compact set of an open set $V$ of $\mathbb{R}^n$, we mean that $B \subset V$ and $B$ is relatively compact w.r.t. the subspace topology of $V$ inherited from $\mathbb{R}^n$. Denote $\Lambda_l := \{(x, y) \in \Omega_l \times \Omega_l : |x - y| < \varsigma_l\}$. Define $\hat{E}(u, v) := E(v, u)$ for $u, v \in C_0^\infty(U)$.

2.1 Decomposition of $\mathcal{E}$

Lemma 2.1. Let $u, v \in C_0^\infty(U)$ and $F$ be a compact set of $U$. Then
(i) \[
\int_{U \times F \setminus d} (u(y) - u(x))^2 J(\,dxdy) < \infty.
\]

(ii) \[
\int_{F \times F \setminus d} |x - y|^2 J(\,dxdy) < \infty.
\]

(iii) For \(\varepsilon > 0\),
\[
\int_{(U \times U) \cap \{|x - y| > \varepsilon\}} |(u(y) - u(x))v(y)| J(\,dxdy) < \infty.
\]

Proof. (i) We choose a \(w \in C_0^\infty(U)\) satisfying \(w \geq 0\) and \(w|_F \equiv 1\). By (1.3) and the sub-Markovian property of \((G_\beta)_{\beta > 0}\), we get
\[
\int_{U \times F \setminus d} (u(y) - u(x))^2 J(\,dxdy) \leq \int_{U \times U \setminus d} (u(y) - u(x))^2 w(y) J(\,dxdy)
= \lim_{l \to \infty} \int_{\Gamma_l} (u(y) - u(x))^2 w(y) J(\,dxdy)
= \lim_{l \to \infty} \lim_{\beta \to \infty} \frac{\beta}{2} \int_{\Gamma_l} (u(y) - u(x))^2 w(y) \sigma(\,dxdy)
\leq \lim_{\beta \to \infty} \frac{\beta}{2} \int_{U \times U} (u(y) - u(x))^2 w(y) \sigma(\,dxdy)
= \lim_{\beta \to \infty} \frac{\beta}{2} ((\beta G_\beta \Gamma_U, u^2 w) - 2(\beta G_\beta u, uw) + (\beta G_\beta u^2, w))
\leq \lim_{\beta \to \infty} \left\{ \beta (u - \beta G_\beta u, uw) - \frac{\beta}{2} (u^2 - \beta G_\beta u^2, w) \right\}
= \mathcal{E}(u, uw) - \frac{1}{2} \mathcal{E}(u^2, w)
< \infty.
\]

(ii) We choose a \(w' \in C_0^\infty(U)\) satisfying \(w'|_F \equiv 1\). For \(1 \leq i \leq n\), we define \(u_i(x) = x_i \cdot w'(x)\) for \(x = (x_1, \ldots, x_n) \in U\). Then, \(u_i \in C_0^\infty(U)\) satisfying \(u_i(x) = x_i\) for \(x \in F\). By (i), we get
\[
\int_{F \times F \setminus d} |x - y|^2 J(\,dxdy) = \sum_{i=1}^n \int_{F \times F \setminus d} (x_i - y_i)^2 J(\,dxdy)
= \sum_{i=1}^n \int_{F \times F \setminus d} (u_i(x) - u_i(y))^2 J(\,dxdy)
< \infty.
\]
(iii) By (i), we get
\[
\int_{(U \times U) \cap \{|x-y|>\varepsilon\}} |(u(y) - u(x))v(y)| J(\text{d}x \text{d}y)
\]
\[
= \int_{(U \times \text{supp}[v]) \cap \{|x-y|>\varepsilon\}} |(u(y) - u(x))v(y)| J(\text{d}x \text{d}y)
\]
\[
= \int_{(U \times \text{supp}[v]) \cap \{|x-y|>\varepsilon\}} \left| (u(y) - u(x))(v(y) - v(x)) + (u(y) - u(x))v(x) \right| J(\text{d}x \text{d}y)
\]
\[
\leq \int_{U \times \text{supp}[v]} |(u(y) - u(x))(v(y) - v(x))| J(\text{d}x \text{d}y)
\]
\[
+ \int_{(\text{supp}[v] \times \text{supp}[v]) \cap \{|x-y|>\varepsilon\}} |(u(y) - u(x))v(x)| J(\text{d}x \text{d}y)
\]
\[
\leq \left( \int_{U \times \text{supp}[v]} (u(y) - u(x))^2 J(\text{d}x \text{d}y) \right)^{1/2} \left( \int_{U \times \text{supp}[v]} (v(y) - v(x))^2 J(\text{d}x \text{d}y) \right)^{1/2}
\]
\[
+ 2\|u\|_\infty \|v\|_\infty J((\text{supp}[v] \times \text{supp}[v]) \cap \{|x-y|>\varepsilon\})
\]
\[
< \infty.
\]

\[\square\]

Let \( \delta > 0 \) be a constant such that \( U^\delta \neq \emptyset \). Suppose that \( u, v \in C^\infty_0(U^\delta) \). Let \( \chi \in C^\infty_0(U) \) satisfying \( \chi = 1 \) on a neighbourhood of \( \text{supp}[u] \cup \text{supp}[v] \). By Taylor’s theorem and Lemma \[2.1\](ii), one finds that \( (u(y) - u(x)) - \sum_{i=1}^n (y_i - x_i) \frac{\partial u}{\partial y_i}(y) I_{\{|x-y|\leq \delta\}}(x,y) \) is integrable w.r.t. both \( J \) and \( \hat{J} \). Hereafter, we define
\[
F^\delta_v := \left\{ x \in U : \inf_{y \in \text{supp}[v]} |x-y| \leq \delta \right\}.
\]

\( F^\delta_v \) is a compact set of \( U \). By Lemma \[2.1\](i) and (ii), for \( 1 \leq i \leq n \), we have
\[
\int_{U \times U \setminus d} \left| (y_i - x_i) \frac{\partial u}{\partial y_i}(y) I_{\{|x-y|\leq \delta\}}(x,y) v(y)(1 - \chi(x)) \right| (J(\text{d}x \text{d}y) + \hat{J}(\text{d}x \text{d}y))
\]
\[
\leq 2 \left\| \frac{\partial u}{\partial y_i} \cdot v \right\|_\infty \left( \int_{F^\delta_v \times F^\delta_v \setminus d} |x-y|^2 J(\text{d}x \text{d}y) \right)^{1/2} \left( \int_{F^\delta_v \times F^\delta_v \setminus d} (\chi(y) - \chi(x))^2 J(\text{d}x \text{d}y) \right)^{1/2}
\]
\[
< \infty.
\]

Hence \( \sum_{i=1}^n (y_i - x_i) \frac{\partial u}{\partial y_i}(y) I_{\{|x-y|\leq \delta\}}(x,y) v(y)(1 - \chi(x)) \) is integrable w.r.t. both \( J \) and \( \hat{J} \). Therefore,
\[
\left( (u(y) - u(x)) \chi(x) - \sum_{i=1}^n (y_i - x_i) \frac{\partial u}{\partial y_i}(y) I_{\{|x-y|\leq \delta\}}(x,y) \right) v(y)
\]
is integrable w.r.t. both \( J \) and \( \hat{J} \).
We assume temporarily that $\mathcal{J}(\{(x, y) \in U \times U : |x - y| = \delta\}) = 0$. Then, we obtain by the vague convergence of $(\beta/2)\sigma_\delta$ to $\mathcal{J}$ that

$$
\mathcal{E}(u, v) = \lim_{\beta \to \infty} \beta(u - \beta G_\beta u, v)
$$

$$= \lim_{\beta \to \infty} \beta \left\{ \int_{U \times U} (u(y) - u(x))v(y)\chi(x)\sigma_\delta(dxdy)
+ \int_U \chi(x)u(x)v(x)m(dx) - \int_{U \times U} \chi(x)u(y)v(y)\sigma_\delta(dxdy) \right\}
$$

$$= \lim_{\beta \to \infty} \beta \left( \int_{U \times U} (u(y) - u(x))v(y)\chi(x)\sigma_\delta(dxdy) + \mathcal{E}(\chi, uv) \right)
$$

$$= \lim_{\beta \to \infty} \lim_{\beta \to \infty} \beta \left\{ \int_{\Lambda_i} (u(y) - u(x))v(y)\sigma_\delta(dxdy)
+ \int_{\Gamma_1} \left( (u(y) - u(x))\chi(x) - \sum_{i=1}^n (y_i - x_i) \frac{\partial u}{\partial y_i}(y)I_{|x-y|\leq \delta}(x, y) \right) v(y)\sigma_\delta(dxdy)
+ \int_{\Gamma_1} \sum_{i=1}^n (y_i - x_i) \frac{\partial u}{\partial y_i}(y)I_{|x-y|\leq \delta}(x, y) v(y)\sigma_\delta(dxdy) \right\} + \mathcal{E}(\chi, uv)
$$

$$= \lim_{\beta \to \infty} \lim_{\beta \to \infty} \beta \left\{ \int_{\Lambda_i} (u(y) - u(x))v(y)\sigma_\delta(dxdy)
+ \int_{\Gamma_1} \sum_{i=1}^n (y_i - x_i) \frac{\partial u}{\partial y_i}(y)I_{|x-y|\leq \delta}(x, y) v(y)\sigma_\delta(dxdy) \right\}
+ \int_{U \times U \setminus \delta} 2 \left( (u(y) - u(x))\chi(x) - \sum_{i=1}^n (y_i - x_i) \frac{\partial u}{\partial y_i}(y)I_{|x-y|\leq \delta}(x, y) \right) v(y)\mathcal{J}(dxdy)
+ \mathcal{E}(\chi, uv). \quad (2.2)
$$

Similarly, we get

$$
\tilde{\mathcal{E}}(u, v) = \lim_{\beta \to \infty} \lim_{\beta \to \infty} \beta \left\{ \int_{\Lambda_i} (u(y) - u(x))v(y)\tilde{\sigma}_\delta(dxdy)
+ \int_{\Gamma_1} \sum_{i=1}^n (y_i - x_i) \frac{\partial u}{\partial y_i}(y)I_{|x-y|\leq \delta}(x, y) v(y)\tilde{\sigma}_\delta(dxdy) \right\}
+ \int_{U \times U \setminus \delta} 2 \left( (u(y) - u(x))\chi(x) - \sum_{i=1}^n (y_i - x_i) \frac{\partial u}{\partial y_i}(y)I_{|x-y|\leq \delta}(x, y) \right) v(y)\tilde{\mathcal{J}}(dxdy)
+ \tilde{\mathcal{E}}(\chi, uv). \quad (2.3)
$$

By (2.2) and (2.3), we can introduce the following definition.

**Definition 2.2.** Let $\{\delta_n\}_{n=1}^\infty$ be a sequence of constants satisfying $\delta = \lim_{n \to \infty} \delta_n$, $\delta_n \geq \delta$ and $\mathcal{J}(\{(x, y) \in U \times U : |x - y| = \delta_n\}) = 0$ for each $n \in \mathbb{N}$. For
\(u, v \in C_0^\infty(U^\delta)\), we define

\[
\mathcal{E}^{c,\delta}(u, v) := \lim_{n \to \infty} \lim_{l \to \infty} \lim_{\beta \to \infty} \left\{ \int_{\Lambda_l} (u(y) - u(x))v(y)\sigma_\beta(dx\,dy) \right. \\
+ \int_{\Gamma_l} \sum_{i=1}^{n} (y_i - x_i) \frac{\partial u}{\partial y_i}(y)I_{\{|x-y| \leq \delta_n\}}(x, y)v(y)\sigma_\beta(dx\,dy) \left. \right\} \quad (2.4)
\]

and

\[
\hat{\mathcal{E}}^{c,\delta}(u, v) := \lim_{n \to \infty} \lim_{l \to \infty} \lim_{\beta \to \infty} \left\{ \int_{\Lambda_l} (u(y) - u(x))v(y)\hat{\sigma}_\beta(dx\,dy) \right. \\
+ \int_{\Gamma_l} \sum_{i=1}^{n} (y_i - x_i) \frac{\partial u}{\partial y_i}(y)I_{\{|x-y| \leq \delta_n\}}(x, y)v(y)\hat{\sigma}_\beta(dx\,dy) \left. \right\}. \quad (2.5)
\]

By (2.2), (2.3) and the fact that \(J\) is a positive Radon measure \(J\) on \(U \times U \setminus d\), one finds that the definitions of \(\mathcal{E}^{c,\delta}\) and \(\hat{\mathcal{E}}^{c,\delta}\) are independent of the selections of \(\{\Omega_l\}\) and \(\{\delta_n\}\). Both \(\mathcal{E}^{c,\delta}(u, v)\) and \(\hat{\mathcal{E}}^{c,\delta}(u, v)\) satisfy the left strong local property in the sense that \(\mathcal{E}^{c,\delta}(u, v) = \hat{\mathcal{E}}^{c,\delta}(u, v) = 0\) whenever \(u\) is constant on a neighbourhood of \(\text{supp}[v]\).

**Theorem 2.3.** Suppose \(u, v \in C_0^\infty(U^\delta)\).

(i) We have the decomposition

\[
\mathcal{E}(u, v) = \mathcal{E}^{c,\delta}(u, v) \\
+ \int_{U \times U \setminus d} 2 \left( (u(y) - u(x)) - \sum_{i=1}^{n} (y_i - x_i) \frac{\partial u}{\partial y_i}(y)I_{\{|x-y| \leq \delta_n\}}(x, y) \right) v(y)J(dx\,dy) \\
+ \int_u u(x)v(x)K(dx).
\] \quad (2.6)

(ii) Let \(\chi \in C_0^\infty(U)\) satisfying \(\chi = 1\) on a neighbourhood of \(\text{supp}[u] \cup \text{supp}[v]\). Then, we have

\[
\mathcal{E}(u, v) = \mathcal{E}^{c,\delta}(u, v) \\
+ \int_{U \times U \setminus d} 2 \left( (u(y) - u(x))\chi(x) - \sum_{i=1}^{n} (y_i - x_i) \frac{\partial u}{\partial y_i}(y)I_{\{|x-y| \leq \delta_n\}}(x, y) \right) v(y)J(dx\,dy) \\
+ \mathcal{E}(\chi, uv)
\] \quad (2.7)

and

\[
\hat{\mathcal{E}}(u, v) = \hat{\mathcal{E}}^{c,\delta}(u, v) \\
+ \int_{U \times U \setminus d} 2 \left( (u(y) - u(x))\chi(x) - \sum_{i=1}^{n} (y_i - x_i) \frac{\partial u}{\partial y_i}(y)I_{\{|x-y| \leq \delta_n\}}(x, y) \right) v(y)\hat{J}(dx\,dy) \\
+ \hat{\mathcal{E}}(\chi, uv).
\] \quad (2.8)
Proof. (ii) is a direct consequence of (2.2)–(2.5). We only prove (i). By (1.4), we have

\[ \mathcal{E}(\chi, uv) = \int_{U \times U \setminus d} 2(1 - \chi(x))u(y)v(y)J(dxy) + \int_U u(x)v(x)K(dx). \] (2.9)

Here the integrability of \((1 - \chi(x))u(y)v(y)\) w.r.t. \(J\) is also ensured by Lemma 2.1(iii). Then, we obtain (2.6) by (2.7) and (2.9). \(\Box\)

By (2.6), to understand the structure of \(\mathcal{E}\), we may concentrate on the left strong local part \(\mathcal{E}^{c,\delta}\).

Suppose that \(u, f \in C^\infty_0(U^\delta)\). By (2.4), we get

\[ 2\mathcal{E}^{c,\delta}(u, uf) - \mathcal{E}^{c,\delta}(u^2, f) = \lim_{l \to \infty} \lim_{\beta \to \infty} \beta \int_{\Lambda_l} (u(y) - u(x))^2 f(y)\sigma_{\beta}(dxdy). \] (2.10)

Since \(\delta\) is arbitrary, \(\lim_{l \to \infty} \lim_{\beta \to \infty} \beta \int_{\Lambda_l} (\varphi(y) - \varphi(x))^2 g(y)\sigma_{\beta}(dxdy)\) exists for any \(\varphi, g \in C^\infty_0(U)\).

Let \(\varphi \in C^\infty_0(U)\). For \(r \in \mathbb{N}\), we choose a \(w \in C^\infty_0(U)\) satisfying \(w \geq 0\) and \(w|_{\Omega_r} \equiv 1\). For \(g \in C^\infty_0(\Omega_r)\), we obtain by the sub-Markovian property of \((G_\beta)_{\beta \geq 0}\) that

\[
\left| \lim_{l \to \infty} \lim_{\beta \to \infty} \beta \int_{\Lambda_l} (\varphi(y) - \varphi(x))^2 g(y)\sigma_{\beta}(dxdy) \right| \\
\leq \|g\|_\infty \lim_{\beta \to \infty} \beta \int_{U \times U} (\varphi(y) - \varphi(x))^2 w(y)\sigma_{\beta}(dxdy) \\
\leq \|g\|_\infty \lim_{\beta \to \infty} \left\{ 2\beta(\varphi - \beta G_\beta \varphi, \varphi w) - \beta(\varphi^2 - \beta G_\beta \varphi^2, w) \right\} \\
= (2\mathcal{E}(\varphi, \varphi w) - \mathcal{E}(\varphi^2, w))\|g\|_\infty.
\]

Then, there exists a unique Radon measure \(\mu^{r, c}_{<\varphi>}\) on \(\Omega_r\) such that

\[
\int_{\Omega_r} gd\mu^{r, c}_{<\varphi>} = \lim_{l \to \infty} \lim_{\beta \to \infty} \beta \int_{\Lambda_l} (\varphi(y) - \varphi(x))^2 g(y)\sigma_{\beta}(dxdy), \quad \forall g \in C^\infty_0(\Omega_r).
\]

It is easy to see that \(\{\mu^{r, c}_{<\varphi>}\}\) is a consistent sequence of Radon measures. Therefore, we can well define the measure \(\mu^c_{<\varphi>}\) by \(\mu^c_{<\varphi>} = \mu^{r, c}_{<\varphi>}\) on \(\Omega_r\), which satisfies

\[
\int g d\mu^c_{<\varphi>} = \lim_{l \to \infty} \lim_{\beta \to \infty} \beta \int_{\Lambda_l} (\varphi(y) - \varphi(x))^2 g(y)\sigma_{\beta}(dxdy), \quad \forall g \in C^\infty_0(U).
\]

For \(\varphi, \phi \in C^\infty_0(U)\), we define

\[
\mu^c_{<\varphi, \phi>} := \frac{1}{2}(\mu^c_{<\varphi + \phi>} - \mu^c_{<\varphi + \phi>} - \mu^c_{<\phi>}).
\]
Then, for any \( g \in C_{0}^{\infty}(U) \), we have

\[
\int_{U} gd\mu_{c,\phi} = \lim_{l \to \infty} \lim_{\beta \to \infty} \beta \int_{\Lambda_{l}} (\varphi(y) - \varphi(x)) (\phi(y) - \phi(x)) g(y) \sigma_{\beta}(dxy). \tag{2.11}
\]

Suppose now that \( u, v, f \in C_{0}^{\infty}(U^{\delta}) \). We obtain by (2.10) and (2.11) that

\[
\int_{U} f d\mu_{c,<u,v>} = \mathcal{E}^{c,\delta}(u,v) + \mathcal{E}^{c,\delta}(v,u) - \mathcal{E}^{c,\delta}(uv,f). \tag{2.12}
\]

Hence, for any \( h \in C_{0}^{\infty}(U^{\delta}) \) satisfying \( h|_{\text{supp}[u] \cup \text{supp}[v]} \equiv 1 \), we have

\[
\mathcal{E}^{c,\delta}(u,v) + \mathcal{E}^{c,\delta}(v,u) = \int_{U} h d\mu_{c,<u,v>} + \mathcal{E}^{c,\delta}(uv,h). \tag{2.13}
\]

For \( u, v \in C_{0}^{\infty}(U^{\delta}) \), we define a linear functional \( L^{\delta}(u, v) \) on \( C_{0}^{\infty}(U^{\delta}) \) by

\[
<L^{\delta}(u, v), f > := \frac{1}{2} (\mathcal{E}^{c,\delta}(u,vf) - \mathcal{E}^{c,\delta}(u,vf)), \quad f \in C_{0}^{\infty}(U^{\delta}). \tag{2.14}
\]

Then, for any \( h \in C_{0}^{\infty}(U^{\delta}) \) satisfying \( h|_{\text{supp}[v]} \equiv 1 \), we have

\[
\mathcal{E}^{c,\delta}(u,v) - \mathcal{E}^{c,\delta}(u,v) = 2 < L^{\delta}(u, v), h >. \tag{2.15}
\]

Let \( \chi \in C_{0}^{\infty}(U^{\delta}) \) satisfying \( \chi = 1 \) on a neighbourhood of \( \text{supp}[u] \cup \text{supp}[v] \). Then, we obtain by (2.6), (2.8) and (2.15) that

\[
\mathcal{E}^{c,\delta}(u,v) - \mathcal{E}^{c,\delta}(u,v) = \mathcal{E}^{c,\delta}(u,v) - \mathcal{E}(\chi, uv) + \int_{U} u(x)v(x)K(dx)
\]

\[
+ \int_{U \times U \setminus d} 2 \left( v(y) - v(x) - \sum_{i=1}^{n} (y_{i} - x_{i}) \frac{\partial v}{\partial y_{i}}(y) I_{\{x-y|\leq \delta\}} (x,y) \right) u(y) J(dx dy)
\]

\[
= \mathcal{E}^{c,\delta}(u,v) - \mathcal{E}^{c,\delta}(u,v) - \mathcal{E}(\chi, uv) + \int_{U} u(x)v(x)K(dx)
\]

\[
- \int_{U \times U \setminus d} 2 \left( (u(y) - u(x)) \chi(x) - \sum_{i=1}^{n} (y_{i} - x_{i}) \frac{\partial u}{\partial y_{i}}(y) I_{\{x-y|\leq \delta\}} (x,y) \right) v(y) J(dx dy)
\]

\[
+ \int_{U \times U \setminus d} 2 \left( v(y) - v(x) - \sum_{i=1}^{n} (y_{i} - x_{i}) \frac{\partial v}{\partial y_{i}}(y) I_{\{x-y|\leq \delta\}} (x,y) \right) u(y) J(dx dy)
\]

\[
= 2 < L^{\delta}(u, v), \chi > - \mathcal{E}(uv, \chi) + \int_{U} u(x)v(x)K(dx)
\]

\[
- \int_{U \times U \setminus d} 2 \left( (u(y) - u(x)) \chi(x) - \sum_{i=1}^{n} (y_{i} - x_{i}) \frac{\partial u}{\partial y_{i}}(y) I_{\{x-y|\leq \delta\}} (x,y) \right) v(y) J(dx dy)
\]
Theorem 2.4. Suppose \( \text{supp}[\mu] \neq \emptyset \).

Theorem 2.5. (i) For \( u, v, w \in C^\infty_0(U^\delta) \) satisfying \( \chi = 1 \) on a neighbourhood of \( \text{supp}[u] \cup \text{supp}[v] \), then

\[
\mathcal{E}^{\epsilon, \delta}(u, v) = \frac{1}{2} \int_U \chi d\mu_{<u,v>} + <L^\delta(u, v), \chi>
\]

\[
+ \int_{U \setminus U^\delta} \sum_{i=1}^{n} (y_i - x_i) \left( \frac{\partial u}{\partial y_i}(y)v(y) - \frac{\partial u}{\partial x_i}(x)v(x) \right) I_{\{|x-y| \leq \delta\}}(x, y)J(\text{d}xdy). \tag{2.17}
\]

2.2 Transformation rules for the symmetric and co-symmetric diffusion parts

In this subsection, we will derive transformation rules for the sign Radon measure \( \mu_{<\cdot, \cdot>} \) and the lineal functional \( L^\delta(\cdot, \cdot) \) introduced in §2.1.

Theorem 2.5. (i) For \( u, v, w \in C^\infty_0(U^\delta) \),

\[
d\mu_{<uw,w>} = ud\mu_{<v,w>} + vd\mu_{<u,w>}. 
\]

(ii) For \( u, v, w, f \in C^\infty_0(U^\delta) \),

\[
<L^\delta(u, vw), f >= <L^\delta(u, v), wf >.
\]

(iii) For \( u, v, w, f \in C^\infty_0(U^\delta) \),

\[
<L^\delta(uw, w), f >= <L^\delta(u, w), vf > + <L^\delta(v, w), uf >.
\]
Proof. We assume without loss of generality that \( u, v, f \in C_0^\infty(U^\delta) \).

(i) We need only show that \( d\mu^c_{<u^2,v>} = 2ud\mu^c_{<u,v>} \). We choose a \( \chi \in C_0^\infty(U) \) satisfying \( \chi = 1 \) on a neighbourhood of \( \text{supp}[u] \cup \text{supp}[v] \). Let \( g \in C_0^\infty(U^\delta) \). Then, by (2.11) and Lemma 2.1(i), we get

\[
\int_{U^\delta} gd\mu^c_{<u^2,v>} - 2 \int_{U^\delta} gud\mu^c_{<u,v>}
\]

\[
= - \lim_{l \to \infty} \lim_{\beta \to \infty} \int_{\Lambda_l} (u(y) - u(x))^2(v(y) - v(x))g(y)\sigma_\beta(dx dy)
\]

\[
= - \lim_{l \to \infty} \lim_{\beta \to \infty} \int_{\Lambda_l} (u(y) - u(x))^2(v(y) - v(x))g(y)\chi(x)\sigma_\beta(dx dy)
\]

\[
- \lim_{l \to \infty} \lim_{\beta \to \infty} \int_{\Lambda_l} u^2(y)v(y)g(y)(1 - \chi(x))\sigma_\beta(dx dy)
\]

\[
= - \lim_{l \to \infty} \int_{\Lambda_l} 2(u(y) - u(x))^2(v(y) - v(x))g(y)\chi(x)J(dx dy)
\]

\[
= 0.
\]

(ii) is obvious by (2.14).

(iii) We need only show that \( < L^\delta(u^2, v), f >= 2 < L^\delta(u, v), uf > \). By (2.4), (2.5) and (2.14), we get

\[
< L^\delta(u, v), f >
\]

\[
= \lim_{n \to \infty} \lim_{l \to \infty} \lim_{\beta \to \infty} \frac{\beta}{2} \left\{ \int_{\Lambda_l} (u(y) - u(x))(v(y)f(y) + v(x)f(x))\sigma_\beta(dx dy)
\]

\[
+ \int_{\Gamma_i} \sum_{i=1}^{n} (y_i - x_i) \frac{\partial u}{\partial y_i}(y)I_{\{x-y \leq \delta_n\}}(x, y)v(y)f(y)\sigma_\beta(dx dy)
\]

\[
- \int_{\Gamma_i} \sum_{i=1}^{n} (y_i - x_i) \frac{\partial u}{\partial y_i}(y)I_{\{x-y \leq \delta_n\}}(x, y)v(y)f(y)\sigma_\beta(dx dy)
\}
\]

We choose a \( \chi \in C_0^\infty(U) \) satisfying \( \chi = 1 \) on a neighbourhood of \( \text{supp}[u] \cup \text{supp}[v] \). Then, by Lemma 2.1(i), we get

\[
< L^\delta(u^2, v), f > - 2 < L^\delta(u, v), uf >
\]

\[
= - \lim_{l \to \infty} \lim_{\beta \to \infty} \frac{\beta}{2} \int_{\Lambda_l} (u(y) - u(x))^2(v(y)f(y) - v(x)f(x))\sigma_\beta(dx dy)
\]

\[
= - \lim_{l \to \infty} \lim_{\beta \to \infty} \frac{\beta}{2} \int_{\Lambda_l} (u(y) - u(x))^2(v(y)f(y) - v(x)f(x))\chi(x)\chi(y)\sigma_\beta(dx dy)
\]

\[
- \lim_{l \to \infty} \lim_{\beta \to \infty} \frac{\beta}{2} \int_{\Lambda_l} u^2(y)v(y)f(y)(1 - \chi(x))\sigma_\beta(dx dy)
\]

\[
+ \lim_{l \to \infty} \lim_{\beta \to \infty} \frac{\beta}{2} \int_{\Lambda_l} u^2(x)v(x)f(x)(1 - \chi(y))\sigma_\beta(dx dy)
\]
\[
\begin{align*}
&= - \lim_{l \to \infty} \int_{\Lambda_l} (u(y) - u(x))^2(v(y)f(y) - v(x)f(x))\chi(x)\chi(y)J(dx dy) \\
&= 0.
\end{align*}
\]

Let \( w \in C_0^\infty(U) \) and \( V \) be a relatively compact open set of \( U \). If \( w = k \) (constant) on \( V \), then \( \mu_{<w>} = 0 \) on \( V \). In fact, taking an \( f \in C_0^\infty(V) \), we obtain by Theorem 2.5(i) that
\[
kd\mu_{<f,w>} = d\mu_{<f,w>} = f d\mu_{<w>} + w d\mu_{<f,w>},
\]
which implies that \( f d\mu_{<w>} = 0 \) on \( V \). Since \( f \in C_0^\infty(V) \) is arbitrary, \( \mu_{<w>} = 0 \) on \( V \). For \( u, v \in C^\infty(U) \), we choose a sequence of functions \( \{u_l, v_l\} \subset C_0^\infty(U) \) such that \( u = u_l \) and \( v = v_l \) on \( \Omega_l \). Therefore, we can well define the measure \( \mu_{<u,v>} \) by \( \mu_{<u,v>} = \mu_{<u_l,v_l>} \) on \( \Omega_l \). The definition of \( \mu_{<u,v>} \) is independent of the selections of \( \{\Omega_l\} \) and \( \{u_l, v_l\} \).

For \( u, v \in C^\infty(U^\delta) \), we choose a sequence of relatively compact open sets \( V_l \uparrow U^\delta \) and a sequence of functions \( \{u_l, v_l\} \subset C_0^\infty(U^\delta) \) such that \( u = u_l \) and \( v = v_l \) on \( V_l \). By (2.14) and the left strong local property of \( \mathcal{E}^c,\delta \) and \( \mathcal{E}^c,\delta \), we can well define the linear functional \( L^\delta(u, v) \) by \( \langle L^\delta(u, v), f \rangle = \lim_{l \to \infty} \langle L^\delta(u_l, v_l), f \rangle \) for \( f \in C_0^\infty(U^\delta) \). The definition of \( L^\delta(u, v) \) is independent of the selections of \( \{V_l\} \) and \( \{u_l, v_l\} \).

**Theorem 2.6.** Let \( \Phi \in C^\infty(\mathbb{R}^m) \).

(i) For \( u_1, \ldots, u_m, v, w \in C^\infty(U) \),
\[
d\mu_{<\Phi(u_1, \ldots, u_m), v>} = \sum_{i=1}^m \Phi_x(u_1, \ldots, u_m) d\mu_{<u_i, v>}.
\]

(ii) For \( u_1, \ldots, u_m, v, w \in C^\infty(U^\delta) \) and \( f \in C_0^\infty(U^\delta) \),
\[
\langle L^\delta(\Phi(u_1, \ldots, u_m), vw), f \rangle = \sum_{i=1}^m \langle L^\delta(u_i, v), \Phi_x(u_1, \ldots, u_m) w f \rangle.
\]

**Proof.** Since the constant function belongs to \( C^\infty(U) \), to prove the theorem, we may assume without loss of generality that \( \Phi \in C^\infty(\mathbb{R}^m) \) with \( \Phi(0) = 0 \) and \( u_1, \ldots, u_m, v, w, f \in C_0^\infty(U^\delta) \). To simplify notation, we denote \( u = (u_1, \ldots, u_m) \).

Let \( C \) be the family of all \( \Phi \) satisfying (i) and (ii). By Theorem 2.5 we know that if \( \Psi, \Gamma \in C \), then \( \Psi \Gamma \in C \). Since \( C \) contains the coordinate functions, it contain all polynomials vanishing at the origin.

Let \( V \) be a finite cube containing the range of \( u \). Then, there exists a sequence \( \{\Phi^{(k)}\} \) of polynomials vanishing at the origin such that \( \Phi^{(k)} \) and all of its partial derivatives converge uniformly to \( \Phi \) and its corresponding partial derivatives on \( V \) (cf. [7] II §4)).

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\[ (i) \text{ Let } g \in C_0^\infty(U^\delta). \text{ We choose a } \phi \in C_0^\infty(U) \text{ satisfying } \phi = 1 \text{ on } F^\delta_{u_1} \cup \cdots \cup F^\delta_{u_m} \cup F^\delta_v \cup F^\delta_g \text{ (see (2.11) for the definition of } F^\delta). \text{ Then, we obtain by (2.7), (2.12), the assumption that } C_0^\infty(U) \subset D(A) \cap D(\hat{A}) \text{ or Assumption 1.3, Taylor's theorem and Lemma 2.1(ii), the finiteness of } J \text{ on } (\text{supp}[\phi] \times \text{supp}[\phi]) \cap \{|x-y| > \delta\}, \text{ and the dominated convergence theorem that} \]

\[ \int_{U^\delta} g d\mu^c_{\phi(x),v} > \int_{U^\delta} \frac{\partial \Phi(x)}{\partial y_i}(y) I_{\{x-y \leq \delta\}}(x, y) (v(y))(y) \phi(x) J(dxdy) \]

\[ = \lim_{k \to \infty} \{E^{c,\delta}(\Phi(x), v) + E^{c,\delta}(v, \Phi(x)) - E^{c,\delta}(\Phi(x)v, g) \}
\]

\[ = \lim_{k \to \infty} \int_{U^\delta} g d\mu^c_{\phi(x),v} > \int_{U^\delta} \frac{\partial \Phi(x)}{\partial y_i}(y) I_{\{x-y \leq \delta\}}(x, y) (v(y))(y) \phi(x) J(dxdy) \]

\[ = \lim_{k \to \infty} \int_{U^\delta} g d\mu^c_{\phi(x),v} > \int_{U^\delta} \frac{\partial \Phi(x)}{\partial y_i}(y) I_{\{x-y \leq \delta\}}(x, y) (v(y))(y) \phi(x) J(dxdy) \]

\[ = \lim_{k \to \infty} \int_{U^\delta} g d\mu^c_{\phi(x),v} > \int_{U^\delta} \frac{\partial \Phi(x)}{\partial y_i}(y) I_{\{x-y \leq \delta\}}(x, y) (v(y))(y) \phi(x) J(dxdy) \]

\[ = \lim_{k \to \infty} \int_{U^\delta} g d\mu^c_{\phi(x),v} > \int_{U^\delta} \frac{\partial \Phi(x)}{\partial y_i}(y) I_{\{x-y \leq \delta\}}(x, y) (v(y))(y) \phi(x) J(dxdy) \]

\[ = \lim_{k \to \infty} \int_{U^\delta} g d\mu^c_{\phi(x),v} > \int_{U^\delta} \frac{\partial \Phi(x)}{\partial y_i}(y) I_{\{x-y \leq \delta\}}(x, y) (v(y))(y) \phi(x) J(dxdy) \]

\[ = \lim_{k \to \infty} \sum_{i=1}^{m} \int_{U^\delta} g \Phi_{x_i}(u) d\mu^c_{\phi(x),v} > \sum_{i=1}^{m} \int_{U^\delta} g \Phi_{x_i}(u) d\mu^c_{\phi(x),v} > \]
Taylor’s theorem and Lemma 2.1(ii), the finiteness of $J$ on $(\text{supp}[\phi] \times \text{supp}[\phi]) \cap \{|x - y| > \delta\}$, and the dominated convergence theorem, we get

$$< L^\delta(\Phi(u), v), f >$$

$$= \frac{1}{2} (\mathcal{E}^c.\delta(\Phi(u), vf) - \hat{\mathcal{E}}^c.\delta(\Phi(u), vf))$$

$$= \frac{1}{2} \left[ \mathcal{E}(\Phi(u), v) f - \mathcal{E}(\phi, \Phi(u)v) f - \hat{\mathcal{E}}(\Phi(u), v) f + \hat{\mathcal{E}}(\phi, \Phi(u)v) f \right]$$

$$- \int_{U \times U \setminus d} \left( \Phi(u)(y) - \Phi(u)(x) - \sum_{i=1}^{n} (y_i - x_i) \frac{\partial \Phi(u)}{\partial y_i}(y) I_{\{|x-y| \leq \delta\}}(x, y) \right) (vf)(y) \phi(x) J(dx dy)$$

$$+ \int_{U \times U \setminus d} \left( \Phi(u)(y) - \Phi(u)(x) - \sum_{i=1}^{n} (y_i - x_i) \frac{\partial \Phi(u)}{\partial y_i}(y) I_{\{|x-y| \leq \delta\}}(x, y) \right) (vf)(y) \phi(x) \hat{J}(dx dy)$$

$$= \lim_{k \to \infty} \left\{ \frac{1}{2} \left[ \mathcal{E}(\Phi^{(k)}(u), v) f - \mathcal{E}(\phi, \Phi^{(k)}(u)v) f - \hat{\mathcal{E}}(\Phi^{(k)}(u), v) f + \hat{\mathcal{E}}(\phi, \Phi^{(k)}(u)v) f \right] \right\}$$

$$- \int_{U \times U \setminus d} \left( \Phi^{(k)}(u)(y) - \Phi^{(k)}(u)(x) - \sum_{i=1}^{n} (y_i - x_i) \frac{\partial \Phi^{(k)}(u)}{\partial y_i}(y) I_{\{|x-y| \leq \delta\}}(x, y) \right) (vf)(y) \phi(x) J(dx dy)$$

$$+ \int_{U \times U \setminus d} \left( \Phi^{(k)}(u)(y) - \Phi^{(k)}(u)(x) - \sum_{i=1}^{n} (y_i - x_i) \frac{\partial \Phi^{(k)}(u)}{\partial y_i}(y) I_{\{|x-y| \leq \delta\}}(x, y) \right) (vf)(y) \phi(x) \hat{J}(dx dy)$$

$$= \lim_{k \to \infty} \frac{1}{2} (\mathcal{E}^c.\delta(\Phi^{(k)}(u), vf) - \hat{\mathcal{E}}^c.\delta(\Phi^{(k)}(u), vf))$$

$$= \lim_{k \to \infty} < L^\delta(\Phi^{(k)}(u), v), f >$$

$$= \lim_{k \to \infty} \sum_{i=1}^{m} < L^\delta(u_i, v), \Phi^{(k)}(u) f >$$

$$= \lim_{k \to \infty} \sum_{i=1}^{m} \frac{1}{2} (\mathcal{E}^c.\delta(u_i, v\Phi^{(k)}(u) f) - \hat{\mathcal{E}}^c.\delta(u_i, v\Phi^{(k)}(u) f))$$

$$= \lim_{k \to \infty} \sum_{i=1}^{m} \left\{ \frac{1}{2} \left[ \mathcal{E}(u_i, v\Phi^{(k)}(u) f) - \mathcal{E}(\phi, u_i v\Phi^{(k)}(u) f) - \hat{\mathcal{E}}(u_i, v\Phi^{(k)}(u) f) + \hat{\mathcal{E}}(\phi, u_i v\Phi^{(k)}(u) f) \right] \right\}$$

$$- \int_{U \times U \setminus d} (u_i(y) - u_i(x) - \sum_{j=1}^{n} (y_j - x_j) \frac{\partial u_i}{\partial y_j}(y) I_{\{|x-y| \leq \delta\}}(x, y) (v\Phi^{(k)}(u) f)(y) \phi(x) J(dx dy)$$

$$+ \int_{U \times U \setminus d} (u_i(y) - u_i(x))$$
\[ -\sum_{j=1}^{n} (y_j - x_j) \frac{\partial u_i}{\partial y_j}(y) I_{|x-y|\leq \delta}(x, y)(v\Phi_{x_i}(u)f)(y)\phi(x) \hat{J}(dxdy) \}\]

\[ = \sum_{i=1}^{m} \left\{ \frac{1}{2} [\mathcal{E}(u_i, v\Phi_{x_i}(u)f) - \mathcal{E}(\phi, u_i v\Phi_{x_i}(u)f) - \hat{\mathcal{E}}(u_i, v\Phi_{x_i}(u)f) + \hat{\mathcal{E}}(\phi, u_i v\Phi_{x_i}(u)f)] \right. \]

\[ - \int_{U \times U \setminus \delta} (u_i(y) - u_i(x)) \]

\[ - \sum_{j=1}^{n} (y_j - x_j) \frac{\partial u_i}{\partial y_j}(y) I_{|x-y|\leq \delta}(x, y)(v\Phi_{x_i}(u)f)(y)\phi(x) \hat{J}(dxdy) \]

\[ + \int_{U \times U \setminus \delta} (u_i(y) - u_i(x)) \]

\[ - \sum_{j=1}^{n} (y_j - x_j) \frac{\partial u_i}{\partial y_j}(y) I_{|x-y|\leq \delta}(x, y)(v\Phi_{x_i}(u)f)(y)\phi(x) \hat{J}(dxdy) \}\]

\[ = \sum_{i=1}^{m} \frac{1}{2} (\mathcal{E}^{c, \delta}(u_i, v\Phi_{x_i}(u)f) - \hat{\mathcal{E}}^{c, \delta}(u_i, v\Phi_{x_i}(u)f)) \]

\[ = \sum_{i=1}^{m} \langle L^\delta(u, v), \Phi_{x_i}(u)f \rangle . \]

Therefore, the proof is complete by noting Theorem 2.5(ii).

### 2.3 Proofs of Theorems 1.2 and 1.4

**Proofs of Theorems 1.2 and 1.4.** We first characterize the first two terms of (2.17). Suppose that \( u, v \in C_0^\infty(U^\delta) \) and \( \chi \in C_0^\infty(U^\delta) \) satisfying \( \chi = 1 \) on a neighbourhood of \( \text{supp}[u] \cup \text{supp}[v] \). Denote by \( x_i, 1 \leq i \leq n \), the coordinate functions of \( \mathbb{R}^n \). For \( 1 \leq i, j \leq n \), we define \( \nu_{ij} := \mu_{<x_i,x_j>}^c \), which is a Radon measure on \( U \). Then, by Theorem 2.6(i), we get

\[ \int_U \chi d\mu_{<u,v>}^c = \sum_{i,j=1}^{n} \int_U \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \nu_{ij}(dx). \]  

(2.18)

For \( 1 \leq i \leq n \), we define the linear functional \( \Psi_i^\delta \) on \( C_0^\infty(U^\delta) \) by

\[ \langle \Psi_i^\delta, f \rangle = \langle L^\delta(x_i, 1), f \rangle, \quad f \in C_0^\infty(U^\delta). \]  

(2.19)

Then, by Theorem 2.6(ii) and (2.19), we get

\[ \langle L^\delta(u, v), \chi \rangle = \sum_{i=1}^{n} \left\langle \Psi_i^\delta, \frac{\partial u}{\partial x_i} v \right\rangle . \]

(2.20)

We now show that each \( \Psi_i^\delta \) is a generalized function on \( U^\delta \). Let \( O \) be an arbitrary relatively compact open set of \( U^\delta \). Suppose that \( \{f_n\} \) is a sequence of functions...
in $C_0^\infty(O)$ such that $f_n$ and all of its partial derivatives converge uniformly to some $f \in C_0^\infty(O)$ and its corresponding partial derivatives as $n \to \infty$. We fix a $\xi_i \in C_0^\infty(U^\delta)$ satisfying $\xi_i = x_i$ on $O$ and choose a $\psi \in C_0^\infty(U)$ satisfying $\psi = 1$ on $F^\delta_i \cup \{ x \in U : \inf_{y \in O} |x - y| \leq \delta \}$. For $g \in C_0^\infty(O)$, by (2.21), (2.17), (2.18) and (2.20), we get

\[
< \Psi_i^\delta, g > = < L^\delta(x_i, 1), g > = \frac{1}{2} \langle \mathcal{E}_{c,\delta}(\xi_i, g) - \mathcal{E}_{c,\delta}^{\hat{\xi}}(\xi_i, g) \rangle = \frac{1}{2} \langle \mathcal{E}(\xi_i, g) - \mathcal{E}(\psi, \xi_i g) - \mathcal{E}(\xi_i, g) + \mathcal{E}(\psi, \xi_i g) \rangle - \int_{U \times U^\delta} (\xi_i(y) - \xi_i(x) - (y_i - x_i)I_{\{|x - y| \leq \delta\}}(x, y)) g(y)\psi(x)J(dx dy) + \int_{U \times U^\delta} (\xi_i(y) - \xi_i(x) - (y_i - x_i)I_{\{|x - y| \leq \delta\}}(x, y)) g(y)\psi(x)\hat{J}(dx dy). \tag{2.21}
\]

Then, we obtain by (2.21), the assumption that $C_0^\infty(U) \subset D(A) \cap D(\hat{A})$ or Assumption 1.3, Taylor’s theorem and Lemma 2.1(ii), the finiteness of $J$ on $(\text{supp}[\psi] \times \text{supp}[^{\hat{\psi}}]) \cap \{|x - y| > \delta\}$, and the dominated convergence theorem that $< \Psi_i^\delta, f >= \lim_{n \to \infty} < \Psi_i^\delta, f_n >$. Therefore, the proof of Theorem 1.4 is complete by (2.6), (2.17), (2.18) and (2.20).

To complete the proof of Theorem 1.2, we need only show that there exist signed Radon measures $\{\nu_i^\delta\}_{i=1}^n$ on $U^\delta$ such that for each $1 \leq i \leq n$,

\[
< \Psi_i^\delta, g > = \int_{U^\delta} g(x)\nu_i^\delta(dx), \quad \forall g \in C_0^\infty(U^\delta).
\]

In fact, let $O$ be an arbitrary relatively compact open set of $U^\delta$. Then, by (2.21), the assumption that $C_0^\infty(U) \subset D(A) \cap D(\hat{A})$ and Lemma 2.1(ii) and (iii), one finds that there exists a unique signed Radon measure $\nu_i^O$ on $O$ such that

\[
< \Psi_i^\delta, g > = \int_{O} g(x)\nu_i^O(dx), \quad \forall g \in C_0^\infty(O).
\]

Therefore, we can well define $\nu_i^\delta = \nu_i^O$ for each $O$. The proof is complete.

From now on till the end of this section, we suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a semi-Dirichlet form on $L^2(U; m)$ satisfying $C_0^\infty(U) \subset D(\mathcal{E})$.

**Remark 2.7.** Assumption 1.3 is implied by the following assumption.

**Assumption 2.8.** There exist a sequence of Dirichlet forms $(\mathcal{Q}_i^l, D(\mathcal{Q}_i^l))$ on $L^2(\Omega_l; m)$ and a sequence of positive constants $C_i$ such that $C_0^\infty(\Omega_l) \subset D(\mathcal{Q}_i^l)$ and

\[
\mathcal{E}_1(g, g) \leq C_i \mathcal{Q}_i^l(g, g), \quad \forall g \in C_0^\infty(\Omega_l).
\]
In fact, suppose Assumption 2.8 holds and \( O \) is a relatively compact open set of \( U^\delta \). Then, there exist an open set \( O_0 \) of \( U^\delta \) satisfying \( \overline{O} \subset O_0 \) and a regular symmetric Dirichlet form \( (\mathcal{Q}, D(\mathcal{Q})) \) on \( L^2(O_0; m) \) such that \( C_0^\infty(O_0) \subset D(\mathcal{Q}) \) and
\[
\mathcal{E}_1(g, g) \leq C \mathcal{Q}_1(g, g), \quad \forall g \in C_0^\infty(O_0),
\]
for some positive constant \( C \). We consider the classical Beurling-Deny formula for \( (\mathcal{Q}, D(\mathcal{Q})) \) (cf. [12, Theorem 3.2.3]):
\[
\mathcal{Q}(u, v) = \frac{1}{2} \sum_{i,j=1}^n \int_{O_0} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \nu^Q_i (dx) + \int_{O_0 \times O_0 \setminus d} (u(x) - u(y))(v(x) - v(y)) J^\mathcal{Q}(dxdy) + \int_{O_0} u(x)v(x) K^\mathcal{Q}(dx),
\]
where \( u, v \in C_0^\infty(O_0) \) and we use the superscript “\( \mathcal{Q} \)” to emphasize that the corresponding Radon measures are for \( (\mathcal{Q}, D(\mathcal{Q})) \). Note that for any compact set \( K \) and open set \( O_1 \) with \( K \subset O_1 \subset O_0 \) (cf. [12, (1.2.4)]),
\[
\int_{K \times K \setminus d} |x - y|^2 J^\mathcal{Q}(dxdy) < \infty, \quad J^\mathcal{Q}(K, O_0 - O_1) < \infty.
\]
Suppose \( \{f_n\}_{n=1}^\infty \subset C_0^\infty(O) \) and \( f \in C_0^\infty(O) \) satisfying \( f_n \) and all of its partial derivatives converge uniformly to \( f \) and its corresponding partial derivatives as \( n \to \infty \). By (2.23), (2.24) and the dominated convergence theorem, we find that \( f_n \) converges to \( f \) w.r.t. the \( \tilde{\mathcal{Q}}^{1/2} \)-norm as \( n \to \infty \). Therefore, we obtain by (2.22) that \( \lim_{n \to \infty} \mathcal{E}_1(f_n - f, f_n - f) = 0 \).

**Corollary 2.9.** Assume the setting of Theorem 1.4 but with Assumption 2.8 replaced by Assumption 2.8. Then, we have the decomposition given in Theorem 1.4. Moreover, for any relatively compact open set \( O \) of \( U^\delta \), there exist signed Radon measures \( \{\mu^Q_i\}_{i=1}^n \) and \( \{\mu^Q_{ij}\}_{i,j=1}^n \) on \( O \) such that for \( 1 \leq i \leq n, \)
\[
< \Psi^\delta_i, g > = \int_{O} g(x) \mu^Q_i (dx) + \sum_{j=1}^n \int_{O} \frac{\partial g}{\partial x_j} (x) \mu^Q_{ij} (dx), \quad \forall g \in C_0^\infty(O).
\]

**Proof.** Let \( O \) be a relatively compact open set of \( U^\delta \). By Assumption 2.8 there exist an open set \( O_0 \) of \( U^\delta \) satisfying \( \overline{O} \subset O_0 \) and a regular symmetric Dirichlet form \( (\mathcal{Q}, D(\mathcal{Q})) \) on \( L^2(O_0; m) \) such that \( C_0^\infty(O_0) \subset D(\mathcal{Q}) \) and (2.22) holds.

By (2.21), (2.22), the sector condition and Lemma 2.1 (ii) and (iii), to prove the corollary, we need only show that for any \( u \in D(\mathcal{Q}) \) there exist signed Radon measures \( \mu^u \) and \( \{\mu^u_{ij}\}_{i,j=1}^n \) on \( O \) such that
\[
\mathcal{Q}(u, v) = \int_{O} v(x) \mu^u (dx) + \sum_{j=1}^n \int_{O} \frac{\partial v}{\partial x_j} (x) \mu^u_{ij} (dx), \quad \forall v \in C_0^\infty(O).
\]
By [12, Theorems 3.2.2 and 5.3.1], we get
\[
\mathcal{Q}(u, v) = \frac{1}{2} \sum_{j=1}^{n} \int_{O_0} \frac{\partial u}{\partial x_j} \mu_{\xi_j, \xi_j} \, (dx) \\
+ \int_{O_0 \times O_0 \setminus d} (\tilde{u}(x) - \tilde{u}(y))(v(x) - v(y)) \mathcal{Q}(dx dy) + \int_{O_0} u(x)v(x) \mathcal{K}(dx),
\]
where \( \xi_j \in C_0^\infty((U_\delta)) \) satisfying \( \xi_j = x_j \) on \( O \) for \( 1 \leq j \leq n \) as in \[2.21\], \( \mu^c \) denotes the local part of the energy measure of \( (\mathcal{Q}, D(\mathcal{Q})) \), \( \tilde{u} \) denotes a quasi-continuous version of \( u \). Therefore, the proof is complete by the mean value theorem, \[2.24\] and the Riesz representation theorem.

### 3 LeJan type transformation rule for the diffusion part of regular semi-Dirichlet forms

In this section, we will apply some ideas of Section 2 to investigate the structure of general regular semi-Dirichlet forms. Throughout this section, we let \( E \) be a locally compact separable metric space, \( m \) a positive Radon measure on \( E \) with \( \text{supp}[m] = E \), and \( (\mathcal{E}, D(\mathcal{E})) \) a regular semi-Dirichlet form on \( L^2(E; m) \).

Following Section 1, we use \( J \) and \( K \) to denote respectively the jumping and killing measures of \( (\mathcal{E}, D(\mathcal{E})) \). By [17, Corollary 2.2], there exists a unique positive Radon measure \( \sigma_{\beta} \) on \( E \times E \) satisfying
\[
(\beta G_{\beta} u, v) = \int_{E \times E} u(x)v(y)\sigma_{\beta}(dxdy) \quad \text{for } u, v \in L^2(E; m).
\]
Hereafter \( (\cdot, \cdot) \) denotes the inner product of \( L^2(E; m) \) and \( (G_{\beta})_{\beta \geq 0} \) denotes the resolvent of \( (\mathcal{E}, D(\mathcal{E})) \). We have \( (\beta/2)\sigma_{\beta} \to J \) vaguely on \( E \times E \setminus d \) as \( \beta \to \infty \) (cf. the proof of [17, Theorem 2.6]). Define \( \hat{J}(dxdy) := J(dydx) \), \( \hat{\sigma}_{\beta}(dxdy) := \sigma_{\beta}(dydx) \), and denote by \( (\hat{G}_{\beta})_{\beta \geq 0} \) the co-resolvent of \( (\mathcal{E}, D(\mathcal{E})) \). Then, we have
\[
(\beta \hat{G}_{\beta} u, v) = \int_{E \times E} u(x)v(y)\hat{\sigma}_{\beta}(dxdy) \quad \text{for } u, v \in L^2(E; m)
\]
and \( (\beta/2)\hat{\sigma}_{\beta} \to \hat{J} \) vaguely on \( E \times E \setminus d \) as \( \beta \to \infty \).

Let \( \rho \) be the metric on \( E \). We choose a sequence of relatively compact open sets \( \Omega_l \uparrow E \) and a sequence of numbers \( q_l \downarrow 0 \) such that the set \( \Gamma_l = \{(x, y) \in \Omega_l \times \Omega_l : \rho(x, y) \geq q_l \} \) is a continuous set w.r.t. \( J \) for every \( l \in \mathbb{N} \). Denote \( \Lambda_l = \{(x, y) \in \Omega_l \times \Omega_l : \rho(x, y) < q_l \} \).

We make the following assumption.

**Assumption 3.1.** For \( f, g \in C_0(E) \cap D(\mathcal{E}) \), we have \( fg \in C_0(E) \cap D(\mathcal{E}) \) and \( (f(y) - f(x))g(y) \) is integrable w.r.t. \( J \).
Suppose \( u, v \in C_0(E) \cap D(\mathcal{E}) \). Let \( \chi \in C_0(E) \cap D(\mathcal{E}) \) satisfying \( \chi = 1 \) on a neighbourhood of \( \text{supp}[u] \cup \text{supp}[v] \). Then, by Assumption 3.1, we get

\[
\hat{E}(u, v) = \lim_{\beta \to \infty} \beta(u - \beta \hat{G}_\beta u, v)
= \lim_{\beta \to \infty} \beta \left\{ \int_{E \times E} (u(y) - u(x))v(y)\chi(x)\hat{\sigma}_\beta(dx \, dy)
+ \int_{E} \chi(x)u(x)v(x)m(dx) - \int_{E \times E} \chi(x)u(y)v(y)\hat{\sigma}_\beta(dx \, dy) \right\}
= \lim_{\beta \to \infty} \beta \int_{E \times E} (u(y) - u(x))v(y)\chi(x)\hat{\sigma}_\beta(dx \, dy)
+ \int_{E \times E \setminus \Lambda} 2(u(y) - u(x))v(y)\chi(x)\hat{J}(dx \, dy) + \hat{E}(\chi, uv). \tag{3.1}
\]

Hence we can well define

\[
\hat{E}^c(u, v) := \lim_{l \to \infty} \lim_{\beta \to \infty} \beta \int_{\Lambda_l} (u(y) - u(x))v(y)\hat{\sigma}_\beta(dx \, dy). \tag{3.2}
\]

\( \hat{E}^c(u, v) \) satisfies the left strong local property in the sense that \( \hat{E}^c(u, v) = 0 \) whenever \( u \) is constant on a neighbourhood of \( \text{supp}[v] \). By (3.1) and (3.2), we obtain the decomposition

\[
\hat{E}(u, v) = \hat{E}^c(u, v) + \int_{E \times E \setminus \Lambda} 2(u(y) - u(x))v(y)\chi(x)\hat{J}(dx \, dy) + \hat{E}(\chi, uv). \tag{3.3}
\]

Similar to (3.1), we can show that

\[
E(u, v) = \lim_{l \to \infty} \lim_{\beta \to \infty} \beta \int_{\Lambda_l} (u(y) - u(x))v(y)\sigma_\beta(dx \, dy)
+ \int_{E \times E \setminus \Lambda} 2(u(y) - u(x))v(y)\chi(x)J(dx \, dy) + E(\chi, uv). \tag{3.4}
\]

By (1.8) and (3.4), we get

\[
E^c(u, v) = \lim_{l \to \infty} \lim_{\beta \to \infty} \beta \int_{\Lambda_l} (u(y) - u(x))v(y)\sigma_\beta(dx \, dy)
+ \int_{E \times E \setminus \Lambda} 2(u(y) - u(x))v(y)\chi(x)J(dx \, dy)
+ \int_{E \times E \setminus \Lambda} 2(\chi(y) - \chi(x))(uv)(y)J(dx \, dy) + \int_{E} u(x)v(x)K(dx)
- \int_{E \times E \setminus \Lambda} 2(u(y) - u(x))v(y)J(dx \, dy) - \int_{E} u(x)v(x)K(dx)
= \lim_{l \to \infty} \lim_{\beta \to \infty} \beta \int_{\Lambda_l} (u(y) - u(x))v(y)\sigma_\beta(dx \, dy). \tag{3.5}
\]

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For $r \in \mathbb{N}$, we choose a $w \in C_0(E) \cap D(E)$ satisfying $w \geq 0$ and $w|_{\Omega_r} \equiv 1$. For $f \in C_0(\Omega_r) \cap D(E)$, we obtain by (3.5) and the sub-Markovian property of $(G_\beta)_{\beta > 0}$ that

$$
|2 \mathcal{E}^c(u, uf) - \mathcal{E}^c(u^2, f)|
= \left| \lim_{\beta \to \infty} \lim_{\beta \to \infty} \beta \int_{\Lambda_1} (u(y) - u(x))^2 f(y)\sigma_\beta(dxdy) \right|
\leq \|f\|_\infty \lim_{\beta \to \infty} \beta \int_{E \times E} (u(y) - u(x))^2 w(y)\sigma_\beta(dxdy)
\leq \|f\|_\infty \lim_{\beta \to \infty} \{2\beta(u - \beta G_\beta u, uw) - \beta(u^2 - \beta G_\beta u^2, w)\}
= (2\mathcal{E}(u, uw) - \mathcal{E}(u^2, w))\|f\|_\infty.
$$

Then, there exists a unique Radon measure $\mu_{<u>}^{E^c}$ on $\Omega_r$ such that

$$
\int_{\Omega_r} f d\mu_{<u>}^{E^c} = 2 \mathcal{E}^c(u, uf) - \mathcal{E}^c(u^2, f), \quad \forall f \in C_0(\Omega_r) \cap D(E).
$$

It is easy to see that $\{\mu_{<u>}^{E^c}\}$ is a consistent sequence of Radon measures. Therefore, we can well define the measure $\mu_{<u>}^E$ by $\mu_{<u>}^E = \mu_{<u>}^{E^c}$ on $\Omega_r$, which satisfies

$$
\int_{E} f d\mu_{<u>}^E = 2 \mathcal{E}^c(u, uf) - \mathcal{E}^c(u^2, f), \quad \forall f \in C_0(E) \cap D(E).
$$

We define

$$
\mu_{<u,v>}^E := \frac{1}{2}(\mu_{<u+v>}^E - \mu_{<u>}^E - \mu_{<v>}^E).
$$

Then

$$
\int_{E} f d\mu_{<u,v>}^E = \mathcal{E}^c(u, vf) + \mathcal{E}^c(v, uf) - \mathcal{E}^c(uv, f), \quad f \in C_0(E) \cap D(E).
$$

Hence, for any $h \in C_0(E) \cap D(E)$ satisfying $h|_{\text{supp}[u] \cup \text{supp}[v]} \equiv 1$, we have

$$
\mathcal{E}^c(u, v) + \mathcal{E}^c(v, u) = \int_{E} h d\mu_{<u,v>}^E + \mathcal{E}^c(uv, h). \quad (3.6)
$$

We define a linear functional $L(u, v)$ on $C_0(E) \cap D(E)$ by

$$
< L(u, v), f > := \frac{1}{2}(\mathcal{E}^c(u, vf) - \mathcal{E}^c(u, vf)), \quad f \in C_0(E) \cap D(E).
$$

Then, for any $h \in C_0(E) \cap D(E)$ satisfying $h|_{\text{supp}[u]} \equiv 1$, we have

$$
\mathcal{E}^c(u, v) - \mathcal{E}^c(u, v) = 2 < L(u, v), h >. \quad (3.7)
$$
By (1.8), (3.3) and (3.7), we get
\[ E^c(u, v) - E^c(v, u) \]
\[ = E^c(u, v) - E(v, u) + \int_E u(x)v(x)K(dx) + \int_{E \times E \setminus d} 2(v(y) - v(x))u(y)J(dxdy) \]
\[ = E^c(u, v) - \hat{E}^c(u, v) - \hat{E}(\chi, uv) + \int_E u(x)v(x)K(dx) \]
\[ - \int_{E \times E \setminus d} 2(u(y) - u(x))v(y)\chi(x)\hat{J}(dxdy) + \int_{E \times E \setminus d} 2(v(y) - v(x))u(y)J(dxdy) \]
\[ = 2 < L(u, v), \chi > - E(uv, \chi) + \int_U u(x)v(x)K(dx) \]
\[ - \int_{E \times E \setminus d} 2(u(y) - u(x))v(y)\chi(x)\hat{J}(dxdy) + \int_{E \times E \setminus d} 2(v(y) - v(x))u(y)J(dxdy) \]
\[ = 2 < L(u, v), \chi > - E^c(uv, \chi) \]
\[ - \int_{E \times E \setminus d} 2((uv)(y) - (uv)(x))\chi(y)J(dxdy) \]
\[ - \int_{E \times E \setminus d} 2(u(x) - u(y))v(x)\chi(y)J(dxdy) + \int_{E \times E \setminus d} 2(v(y) - v(x))u(y)J(dxdy) \]
\[ = 2 < L(u, v), \chi > - E^c(uv, \chi). \] (3.8)

By (3.6) and (3.8), we obtain the following expression of the diffusion part \( E^c \).

**Theorem 3.2.** Suppose Assumption 3.1 holds. Let \( u, v \in C_0(E) \cap D(E) \) and \( \chi \in C_0(E) \cap D(E) \) satisfying \( \chi = 1 \) on a neighbourhood of \( \text{supp}[u] \cup \text{supp}[v] \). Then
\[ E^c(u, v) = \frac{1}{2} \int_E \chi d\mu^c_{\leq u,v} + < L(u, v), \chi >. \]

Similar to Theorem 2.5, we can derive the following transformation rules for \( \mu^c_{\leq u,v} \) and \( L(\cdot, \cdot) \).

**Theorem 3.3.** Let \( u, v, w, f \in C_0(E) \cap D(E) \). Then
\( (i) d\mu^c_{\leq u,w} = ud\mu^c_{\leq u,w} + vd\mu^c_{\leq u,w}. \)
\( (ii) < L(u, vw), f > = < L(u, v), w f >. \)
\( (iii) < L(uw, w), f > = < L(u, w), v f > + < L(v, w), u f >. \)

We use \( \mathcal{F}_{loc} \) to denote the set of all functions \( u \) such that for any relatively compact open set \( V \) there exists a \( w \in C_0(E) \cap D(E) \) such that \( u = w \) on \( V \). Then, by an argument similar to that given after the proof of Theorem 2.5, we can extend \( \mu^c_{\leq u,v} \) and \( L(u, v) \) to \( u, v \in \mathcal{F}_{loc} \). The transformation rules given in Theorem 3.3 still hold with \( C_0(E) \cap D(E) \) replaced by \( \mathcal{F}_{loc} \).

Now we make the following assumption.
**Assumption 3.4.** There exist a sequence of Dirichlet forms $(\mathcal{Q}_l, D(\mathcal{Q}_l))$ on $L^2(\Omega_l; m)$ and a sequence of positive constants $C_l$ such that $C_0(\Omega_l) \cap D(\mathcal{E}) = C_0(\Omega_l) \cap D(\mathcal{Q}_l)$ and

$$\mathcal{E}_1(g, g) \leq C_l \mathcal{Q}_1^1(g, g), \forall g \in C_0(\Omega_l) \cap D(\mathcal{E}).$$

**Theorem 3.5.** Suppose Assumption 3.4 holds and $J$ is a finite measure on $E \times E \setminus d$. Let $\Phi \in C^2(\mathbb{R}^m)$, $u_1, \ldots, u_m, v, w \in \mathcal{F}_{loc}$ and $f \in C_0(E) \cap D(\mathcal{E})$. Then

(i) $d\mu_{\Phi(u_1, \ldots, u_m), v} = \sum_{i=1}^m \Phi_{x_i}(u_1, \ldots, u_m) d\mu_{u_i, v}.$

(ii) $\langle L(\Phi(u_1, \ldots, u_m), vw), f \rangle = \sum_{i=1}^m \langle L(u_i, v), \Phi_{x_i}(u_1, \ldots, u_m)wf \rangle.$

The proof of Theorem 3.5 is similar and simpler than that of Theorem 2.6. We omit the details here. We only point out that [12, (3.2.27)] and Assumption 3.4 ensure the convergence of $\Phi^{(k)}(u)$ (resp. $\Phi_{x_i}^{(k)}(u) - \Phi_{x_i}^{(k)}(0)$) to $\Phi(u)$ (resp. $\Phi_{x_i}(u) - \Phi_{x_i}(0)$) w.r.t. the $\mathcal{Q}_1^{1/2}$-norm and hence the $\mathcal{E}_1^{1/2}$-norm, and the finiteness of $J$ ensures that the dominated convergence theorem can be applied directly. Theorems 1.4 and 3.5 will be applied in a forthcoming work to obtain the strong continuity of generalized Feynman-Kac semigroups for Markov processes associated with semi-Dirichlet forms.

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