Distribution of singular values of random band matrices; Marchenko-Pastur law and more

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Abstract

We consider the limiting spectral distribution of matrices of the form \( \frac{1}{2b_n + 1} (R + X)(R + X)^* \), where \( X \) is an \( n \times n \) band matrix of bandwidth \( b_n \) and \( R \) is a non random band matrix of bandwidth \( b_n \). We show that the Stieltjes transform of ESD of such matrices converges to the Stieltjes transform of a non-random measure. And the limiting Stieltjes transform satisfies an integral equation. For \( R = 0 \), the integral equation yields the Stieltjes transform of the Marchenko-Pastur law.

Keywords: Marchenko-Pastur law, Fixed noise with random band matrices, Norm of random band matrices

1 Introduction

Random matrices play a crucial role in several scientific research including Nuclear Physics, Signal Processing, Numerical linear algebra etc. In 1950s’, Wigner studied Random Band Matrices (RBM) in the context of Nuclear Physics [26]. Tridiagonal RBM can be used to approximate random Schrödinger operator. RBM can also be used to model a particle system where interactions are stronger for nearby particles. Casati et al. studied RBM in the context of quantum chaos [5]. A study of RBM in the framework of supersymmetric approach can be found in [9]. Properties of RBM with strongly fluctuating diagonal entries and sparse RBM were studied by Fyodorov, Mirlin, and co-authors [10], [8]. In addition, RBM appear in the studies of conductance fluctuations of quasi-one dimensional disordered systems [6], the kicked quantum rotator [18], systems of interacting particles in a random potential [19], [14].

In this paper, we consider random band matrices of the form \( \frac{1}{2b_n + 1} (R + X)(R + X)^* \), where \( X \) is an \( n \times n \) band matrix of bandwidth \( b_n \) with iid entries and \( R \) is a nonrandom band matrix. We study the limiting empirical distribution of the eigenvalues of such matrices.

Let \( M_n \) be an \( n \times n \) matrix. Let \( \lambda_1, \ldots, \lambda_n \) be the eigenvalues of \( M_n \) and

\[
\mu_n(x, y) := \frac{1}{n} \# \{ \lambda_i, 1 \leq i \leq n : \Re(\lambda_i) \leq x, \Im(\lambda_i) \leq y \}
\]

be the empirical spectral distribution (ESD) of \( M \). Ginibre [11] showed that if \( M_n = \frac{1}{\sqrt{n}} X_n \), where \( x_{ij} \), the entries of \( X_n \), are iid complex normal variables, then the joint density of \( \lambda_1, \ldots, \lambda_n \) is given by

\[
f(\lambda_1, \ldots, \lambda_n) = c_n \prod_{i<j} |\lambda_i - \lambda_j|^2 \prod_{i=1}^n e^{-n|\lambda_i|^2},
\]

where \( c_n \) is the normalizing constant. Using this, Mehta [17] showed that \( \mu_n \) converges to the uniform distribution on the unit disk. Later on Girko [12] and Bai [3] proved the result under more relaxed assumptions,

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namely under the assumption that \( \mathbb{E}[X_{ij}]^6 < \infty \). Proving the result only under second moment assumption was open until Tao and Vu [22, 23].

Following the method used by Girko, and Bai, the real part of the Stieltjes transform \( m_n(z) := \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\lambda_i - z} \) can be written as

\[
m_{nr}(z) := \Re(m_n(z)) = \frac{1}{n} \sum_{i=1}^{n} \Re(\lambda_i - z) = -\frac{1}{2} \frac{\partial}{\partial \Re(z)} \int_{0}^{\infty} \log x \nu_n(dx, z),
\]

where \( \nu_n(\cdot, z) \) is the ESD of \( \left( \frac{1}{\sqrt{n}} X_n - z I \right) \left( \frac{1}{\sqrt{n}} X_n - z I \right)^* \), and \( z \in \mathbb{C}^+ := \{ z \in \mathbb{C} : \Re(z) > 0 \} \). And secondly the characteristic function of \( \frac{1}{\sqrt{n}} X_n \) satisfies [12, section 1]

\[
\int \int e^{i(ux + vy)} \mu_n(dx, dy) = \frac{u^2 + v^2}{i4\pi u} \int \int \frac{\partial}{\partial s} \left[ \int_{0}^{\infty} (\log x) \nu_n(dx, z) \right] e^{i(us + vt)} \ dt ds,
\]

for any \( uv \neq 0 \), and where \( z = s + it \).

So, finding the limiting behaviour of \( \nu_n(\cdot, z) \) is an essential ingredient in finding the limiting behaviour of \( \mu_n(\cdot, \cdot) \). However, as described in [23], a good estimate of the smallest singular value of random matrix was open until Tao and Vu [22, 23]. In this article, we will focus on finding the limiting behaviour of \( \nu_n(\cdot, z) \) for RBM so that it can be used for finding the limiting behaviour of \( \mu_n(\cdot, \cdot) \) for RBM.

We consider the limiting ESD of matrices of the form \( \frac{1}{2b_n+1}(R+X)(R+X)^* \), where \( X \) is an \( n \times n \) band matrix of bandwidth \( b_n \) and \( R \) is a non RBM. Silverstein, Bai, and Dozier considered the ESD of \( \frac{1}{n}(R+X)(R+X)^* \) type of matrices where \( X \) was \( m \times n \) rectangular matrix with iid entries, \( R \) was a matrix independent of \( X \), and the ratio \( \frac{m}{n} \rightarrow c \in (0, \infty) \) [20, 21, 24]. Having the same bandwidth for \( R \) and \( X \) simplifies the calculation. But we do not think that we need the same bandwidth. Thanks to the referees for pointing this out.

This paper is organized in the following way; in the section 2 we formulate the band matrix model and state the main results. In section 3 we give the main idea of the proof. In section 7 we prove two concentration results which are the main ingredients of the proof. And in the section 8 we provide some tools and the proofs for interested readers.

## 2 Main Results

**Definition 2.1** (Periodic band matrix). An \( n \times n \) matrix \( M = (m_{ij})_{n \times n} \) is called a periodic band matrix of bandwidth \( b_n \) if \( m_{ij} = 0 \) whenever \( b_n < |i - j| < n - b_n \).

\( M \) is called a non-periodic band matrix of bandwidth \( b_n \) if \( m_{ij} = 0 \) whenever \( b_n < |i - j| \).

Notice that in case of a periodic band matrix, the maximum number of non-zero elements in each row is \( 2b_n + 1 \). On the other hand, in case of a non-periodic band matrix, the number of non-zero elements in a row depends on the index of the row. For example, in the first row there are at most \( b_n + 1 \) non-zero elements, and in the \( (b_n + 1) \)th row there are at most \( 2b_n + 1 \) many non-zero elements. In general, the \( i \)th row of a non-periodic band matrix has at most \( b_n + 1 \leq i \leq b_n + 1 \) \( 1 \leq i < n - b_n \) many non-zero elements. In any case, the maximum number of non-zero elements is \( O(b_n) \). In this context, let us define two types of index sets.

Let \( M = (m_{ij})_{n \times n} \) be a RBM (periodic or non-periodic), then we define

\[
I_j = \{ 1 \leq k \leq n : m_{jk} \text{ are not identically zero} \},
I_k' = \{ 1 \leq j \leq n : m_{jk} \text{ are not identically zero} \}.
\]
Notice that in case of periodic band matrices, $|I_j| = 2b_n + 1$. Now we proceed to our main results.

Let $X = (x_{ij})_{n \times n}$ be an $n \times n$ periodic band matrix of bandwidth $b_n$, where $b_n \to \infty$ as $n \to \infty$. Let $R$ be a sequence of $n \times n$ deterministic periodic band matrices of bandwidth $b_n$. Let us denote the ESD of $M$ by $\mu_M$. We define $$c_n = 2b_n + 1$$ for convenience in writing. Assume that

(a) $\mu_{\frac{R}{c_n}} \rightarrow H$, for some non random probability distribution $H$,

(b) $H$ is compactly supported,

(c) $\{x_{jk} : k \in I_j, 1 \leq j \leq n\}$ is an iid set of random variables,

(d) $E[x_{11}] = 0, E[|x_{11}|^2] = 1$.

Define

$$Y = \frac{1}{\sqrt{c_n}}(R + \sigma X), \text{ where } \sigma > 0 \text{ is fixed}. \quad (3)$$

For notational convenience, we assume that the band matrix is periodic. However, the following results can easily be extended to the case when the band matrix is not periodic. We will give the outline of the proof in the section.

Let $M$ be an $n \times n$ matrix. For convenience, let us introduce the following notation

$$\{\lambda_i(M) : 1 \leq i \leq n\} = \text{eigenvalues of } M, \quad m_j := (m_{1j}, m_{2j}, \ldots, m_{nj})^T$$

It is easy to see that $MM^* = \sum_{j=1}^n m_j m_j^*.$

**Definition 2.2** (Poincaré inequality). Let $X$ be a $\mathbb{R}^d$ valued random variable with probability measure $\mu$. The random variable $X$ is said to satisfy the Poincaré inequality with constant $\kappa > 0$, if for all continuously differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$,

$$\text{Var}(f(X)) \leq \frac{1}{\kappa} E(|\nabla f(X)|^2).$$

It can be shown that if $\mu$ satisfies the Poincaré inequality with constant $\kappa$, then $\mu \otimes \mu$ also satisfies the Poincaré inequality with the same constant $\kappa$ [13] Theorem 2.5]. It can also be shown that if $\mu$ satisfies Poincaré inequality and $f : \mathbb{R}^d \to \mathbb{R}$ is a continuously differentiable function then

$$P_\mu(|f - E_\mu(f)| > t) \leq 2K \exp \left(-\frac{\sqrt{\kappa}}{\sqrt{2||\nabla f||_\infty}} t\right), \quad (4)$$

where $K = -\sum_{i>0} 2^i \log(1 - 2^{-2i-1}),$ and $\nabla f$ denotes the gradient of the function $f.$ A proof of the above fact can be found in [1] Lemma 4.4.3].

For example, the Gaussian distribution satisfies the Poincaré inequality.

**Theorem 2.3.** Let $Y$ be defined in (3). In addition to the assumptions made in (2), assume that

(i) $\frac{(\log n)^2}{c_n} \to 0,$

(ii) Both $\Re(x_{ij})$ and $\Im(x_{ij})$ satisfy Poincaré inequality with constant $m$.

Then there exists a non-random probability measure $\mu$ such that $E|m_n(z) - m(z)| \to 0$ uniformly for all $z \in \{z : \Im(z) > \eta\}$ for any fixed $\eta > 0$, where $m_n(z) = \frac{1}{n} \sum_{i=1}^n (\lambda_i(YY^*) - z)^{-1}$ is the Stieltjes transform of
ESD of $YY^*$, and $m(z) = \int_{\mathbb{R}} \frac{d\mu(x)}{x - z}$. In particular, the expected ESD of $YY^*$ converges weakly as a measure. In addition, $m(z)$ satisfies

$$m(z) = \int_{\mathbb{R}} \frac{dH(t)}{1 + \sigma^2m(z)} - (1 + \sigma^2m(z))z \quad \text{for any } z \in \mathbb{C}^+.$$  

In particular, the above result is true for standard Gaussian random variables. The Poincaré inequality in the Theorem 2.3 simplifies the proof a lot. A similar result can also be obtained without Poincaré. However in that case, we prove the Theorem under the assumption that the bandwidth grows sufficiently faster. The Theorem is formulated below.

**Theorem 2.4.** Let $Y$ be defined in (3). In addition to the assumptions made in (2), assume that

(i) $\frac{n}{c_n} \rightarrow 0$,

(ii) $\mathbb{E}[|x_{11}|^{4p}] < \infty$, for some $p \in \mathbb{N}$.

Then there exists a non-random probability measure $\mu$ such that $\mathbb{E}[m_n(z) - m(z)]^{2p} \rightarrow 0$ uniformly for all $z \in \{z : \Im(z) > \eta\}$ for any fixed $\eta > 0$, and the Stieltjes transform of $\mu$ satisfies (5).

Moreover, if $c_n = n^\alpha$, where $\alpha > 0$, then the $m_n(z)$ in Theorem 2.3 converges almost surely to $m(z)$. And the same is true for Theorem 2.4 when $c_n = n^\beta$ where $\beta = \frac{1}{2} + \frac{1}{4p}$. We will prove it at the end of the sections 3 and 4 respectively.

Notice that if we take $R = 0$ and $\sigma = 1$, then $H$ is supported only at the real number 0. In that case (5), becomes

$$m(z)(1 + m(z))z + 1 = 0,$$

which is the same quadratic equation satisfied by the Stieltjes transform of Marchenko-Pastur law.

Proof of the Theorem 2.4 contains the main idea of the proof of both of the Theorems. Main structure of the proof is similar to the method described in (7). However in case of band matrices, we need to proof a generalised version of the Lemma 3.1 in (7), which is proven in the Propositions 7.1 and 7.3. In addition, Lemma 7.2 gives a large deviation estimate of the norm of a RBM.

Also, the assumption that $H$ is compactly supported can be weakened by truncating the singular values of $R$ at a threshold of $\log(c_n)$ and have the same result as the Theorems 2.3 and 2.4. But, in that case we need the bandwidth $c_n$ to grow a little faster, $\log(c_n)$ times faster than the existing rate of divergence. We will prove it in the section 3.

### 3 Proof of Theorem 2.4

Let us define the empirical Stieltjes transform of $YY^*$ as $m_n = \frac{1}{n} \sum_{i=1}^{n} (\lambda_i(YY^*) - z)^{-1}$. It is clear from the context that $m_n$ depends on $z$. So we omit it hereafter to avoid unnecessary cluttering. We introduce the following notations which will be used in the proof of the Theorems.

$$A = \frac{RR^*}{c_n(1 + \sigma^2m_n)} - \sigma^2z m_n I$$

$$B = A - zI$$

$$C = YY^* - zI$$

$$C_j = C - y_j y_j^*$$

$$m_n^{(j)} = \frac{1}{n} \sum_{i=1}^{n} [\lambda_i(YY^* - y_j y_j^*) - zI]^{-1} = \frac{1}{n} \sum_{i=1}^{n} (\lambda_i(C_j))^{-1}$$

$$A_j = \frac{RR^*}{c_n(1 + \sigma^2m_n^{(j)})} - \sigma^2z m_n^{(j)} I$$

$$B_j = A_j - zI.$$
Since \( YY^* = \sum_{j=1}^{n} y_j y_j^* \), we observe that \( m_n^{(j)} \), \( A_j, B_j, C_j \) are independent of \( y_j \). This fact is crucial in our proofs, in particular, in the proof of Proposition 7.1.

**Remark 3.1.** We notice that the eigenvalues of \( A - zI \) are given by \( \lambda_i/(1 + \sigma^2 m) - (1 + \sigma^2 m) z \), where \( \lambda_i \)s are eigenvalue of \( \frac{1}{c_n} RR^* \). Therefore \( \int_k \frac{1}{(1 + \sigma^2 m) - (1 + \sigma^2 m) z} dH(t) \) can be thought of as \( \frac{1}{n} \text{tr}(A - zI)^{-1} \) for large \( n \). So heuristically, proving the Theorem is equivalent to showing that \( \frac{1}{n} \text{tr}(A - zI)^{-1} - m_n \rightarrow 0 \) as \( n \rightarrow \infty \).

Using the definition (6) and Lemma 8.2, we obtain

\[
B^{-1} - C^{-1} = B^{-1}(YY^* - A)C^{-1} = \frac{1}{c_n} B^{-1} \left[ RR^* + \sigma RX^* + \sigma X R^* + \sigma^2 XX^* - \frac{1}{1 + \sigma^2 m} RR^* + c_n \sigma^2 z m_n \right] C^{-1} = \frac{1}{c_n} \sum_{j=1}^{n} B^{-1} \left[ \frac{\sigma x_j C^{-1} r_j}{1 + \sigma^2 m} + \sigma x_j r_j^* + \sigma r_j x_j + \sigma^2 x_j x_j^* - \frac{c_n}{n} 1 + y_j^* C_j^{-1} y_j \sigma^2 \right] C^{-1}.
\]

Taking the trace, dividing by \( n \), and using (4), we have

\[
\frac{1}{n} \text{tr} B^{-1} - m_n = \frac{1}{n} \sum_{j=1}^{n} \left[ \frac{\sigma x_j C^{-1} r_j}{1 + \sigma^2 m} + \frac{1}{c_n} \sigma x_j C^{-1} B^{-1} r_j + \frac{1}{c_n} \sigma C^-1 B^{-1} x_j \right] + \frac{1}{n} \sum_{j=1}^{n} [T_{1,j} + T_{2,j} + T_{3,j} + T_{4,j} + T_{5,j}].
\]
For convenience of writing $T_{i,j}$, let us introduce some notations
\[
\rho_j = \frac{1}{c_n}r_j^*C_j^{-1}r_j, \quad \omega_j = \frac{1}{c_n}\sigma^2x_j^*C_j^{-1}x_j,
\]
\[
\beta_j = \frac{1}{c_n}\sigma r_j^*C_j^{-1}x_j, \quad \gamma_j = \frac{1}{c_n}\sigma x_j^*C_j^{-1}r_j,
\]
\[
\hat{\rho}_j = \frac{1}{c_n}r_j^*C_j^{-1}B^{-1}r_j, \quad \hat{\omega}_j = \frac{1}{c_n}\sigma^2x_j^*C_j^{-1}B^{-1}x_j,
\]
\[
\hat{\beta}_j = \frac{1}{c_n}\sigma r_j^*C_j^{-1}B^{-1}x_j, \quad \hat{\gamma}_j = \frac{1}{c_n}\sigma x_j^*C_j^{-1}B^{-1}r_j,
\]
\[
\alpha_j = 1 + \frac{1}{c_n}(r_j + \sigma x_j)^*C_j^{-1}(r_j + \sigma x_j) = 1 + \rho_j + \beta_j + \gamma_j + \omega_j.
\]

Using Lemma 5.2 for $C = C_j + y_jy_j^* = C_j + \frac{1}{c_n}(r_j + \sigma x_j)(r_j + \sigma x_j)^*$ and the above notations, we can compute
\[
T_{1,j} = \frac{1}{c_n}\frac{\sigma^2m_n}{1 + \sigma^2m_n} \left[ r_j^*C_j^{-1}B^{-1}r_j - \frac{1}{\alpha_j}r_j^*C_j^{-1}y_jy_j^*C_j^{-1}B^{-1}r_j \right]
= \frac{1}{c_n}\frac{\sigma^2m_n}{\alpha_j + \sigma^2m_n} \left[ \alpha_j r_j^*C_j^{-1}B^{-1}r_j - \frac{1}{c_n}r_j^*C_j^{-1}(r_jr_j^* + \sigma r_jx_j^* + \sigma x_jr_j^* + \sigma^2x_jx_j^*)C_j^{-1}B^{-1}r_j \right]
= \frac{1}{\alpha_j + \sigma^2m_n} \left[ (1 + \gamma_j + \omega_j)\hat{\rho}_j - (\rho_j + \beta_j)\hat{\gamma}_j \right].
\]

Similarly,
\[
T_{2,j} = \frac{1}{\alpha_j}[(1 + \rho_j + \beta_j)\hat{\gamma}_j - (\gamma_j + \omega_j)\hat{\beta}_j],
\]
\[
T_{3,j} = \frac{1}{\alpha_j}[(1 + \gamma_j + \omega_j)\hat{\beta}_j - (\rho_j + \beta_j)\hat{\omega}_j],
\]
\[
T_{4,j} = \frac{1}{\alpha_j}[(1 + \rho_j + \beta_j)\hat{\omega}_j - (\gamma_j + \omega_j)\hat{\beta}_j],
\]
and,
\[
T_{5,j} = -\frac{1}{\alpha_j n}\sigma^2trC_j^{-1}B^{-1}.
\]

Using the equations 4 and 8 and the above expressions, we can write
\[
\frac{1}{n}trB^{-1} - m_n = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\alpha_j} \left[ \frac{1}{1 + \sigma^2m_n}(\sigma^2m_n - \gamma_j - \omega_j)\hat{\rho}_j \right. \\
+ \left. \frac{1}{1 + \sigma^2m_n}(1 + \rho_j + \beta_j + \sigma^2m_n)\hat{\gamma}_j + \hat{\beta}_j + \hat{\omega}_j - \frac{1}{n}\sigma^2trC_j^{-1}B^{-1} \right].
\]

We would like to show that the above quantity converges to zero as $n \to \infty$. Now, we start listing up some basic observations.

Since $x_{ij}$ are iid and $E[|x_{ij}|^2] = 1$, by the strong law of large numbers,
\[
\frac{1}{nc_n}trXX^* = \frac{1}{nc_n} \sum_{i,j} |x_{ij}|^2 \frac{1}{n} \to 1.
\]
So, $\mu_{XX}$ is almost surely tight. Using the condition \(2\) (a) and Lemma \(8.1\) we conclude that $\mu_{YY'}$ is almost surely tight. Therefore,

$$\delta := \inf_n \frac{1}{n} \int \frac{1}{|\lambda - z|^2} d\mu_{YY'}(\lambda) > 0.$$  

As a result, for any $z \in \mathbb{C}^+$, we have

$$\Im(zm_n) = \int \frac{\lambda \Im(z)}{|\lambda - z|^2} d\mu_{MM'}(\lambda) \geq 0,$$

$$\Im(m_n) = \int \frac{\Im(z)}{|\lambda - z|^2} d\mu_{MM'}(\lambda) \geq \Im(z) \delta > 0. \quad (11)$$

Let $z \in \mathbb{C}^+$, where $\Im(z)$ stands for the imaginary part of $z$. For any Hermitian matrix $M$, $\|(M - zI)^{-1}\| \leq \frac{1}{\Im(z)}$. Therefore

$$\|C^{-1}\| \leq \frac{1}{\Im(z)}, \quad \|C_j^{-1}\| \leq \frac{1}{\Im(z)}.$$  

We also have a similar bound for $B^{-1}$. If $\lambda$ is an eigenvalue of $\frac{1}{c_n}RR^*$, then $\lambda(B) := \frac{1}{1 + \frac{\sigma^2 mM_n}{\|z\|^2}} \lambda - (1 + \sigma^2 m_n)z$ is the corresponding eigenvalue of $B$. So

$$|\lambda(B)| \geq \|\Im(B)\| \geq \frac{\sigma^2 \Im(m_n)}{1 + \sigma^2 m_n} \lambda + \sigma^2 \Im(zm_n) + \Im(z) \geq \Im(z),$$

where the last inequality follows from \((11)\).

We can do the similar calculations for $B_j$. As a result we have

$$\|B^{-1}\| \leq \frac{1}{\Im(z)}, \quad \|B_j^{-1}\| \leq \frac{1}{\Im(z)}.$$  

Secondly, we would like to estimate the effect of rank one perturbation on $C$ and $B$. More precisely, we would like to estimate $C^{-1} - C_j^{-1}$ and $B^{-1} - B_j^{-1}$. Using the Lemma \(5.3\) we have

$$|\text{tr}(C^{-1} - C_j^{-1})| \leq \frac{1}{\Im(z)},$$

$$|m_n - m_n^{(j)}| = \frac{1}{n} |\text{tr}(C^{-1} - C_j^{-1})| \leq \frac{1}{n|\Im(z)|}. \quad (14)$$

Using the estimates \((11)\) for $z \in \mathbb{C}^+$, we have

$$|1 + \sigma^2 m_n| = \frac{|z + \sigma^2 zm_n|}{|z|} \geq \frac{1}{|z|} |\Im(z) + \sigma^2 \Im(zm_n)| \geq \frac{\Im(z)}{|z|}.$$  

Similarly, we also have $|1 + \sigma^2 m_n^{(j)}| \geq \frac{\Im(z)}{|z|}$ for $z \in \mathbb{C}^+$.

Therefore, using the estimates \((13), (14)\) and the estimate of $\|RR^*\|$ from subsection \(5.1\) we have

$$\|B^{-1} - B_j^{-1}\| = \|B^{-1}(B_j - B)B_j^{-1}\| \leq \frac{1}{|\Im(z)|^2} \|B_j - B\| \leq |m_n - m_n^{(j)}| \frac{\sigma^2}{|\Im(z)|^2} \left\| \frac{1}{c_n(1 + \sigma^2 m_n)(1 + \sigma^2 m_n^{(j)})} RR^* + zI \right\| \leq \frac{K \sigma^2}{n}. \quad (15)$$

Here and in the following estimates, $K > 0$ is a constant that depends only on $p, \Im(z)$, and the moments of $x_{ij}$.

Now, we start estimating several components of the equation \((10)\).
3.1 Estimates of $\hat{\rho}_j$ and $\rho_j$

According to our assumptions we have $\mu_{\frac{1}{c_n}RR^*} \to H$, where $H$ is compactly supported. Therefore, there exists $K > 0$ such that

$$\|r_j\|^2 = \|r_j r_j^*\| \leq \|RR^*\| \leq Kc_n. \tag{16}$$

Using the estimates (12) and (13), we have

$$|\hat{\rho}_j| \leq Kc_n, \quad |\rho_j| \leq Kc_n,$$

where $K > 0$ is a constant which depends only on the imaginary part of $z$.

3.2 Estimates of $\gamma_j, \beta_j, \hat{\gamma}_j$ and $\hat{\beta}_j$

Using Proposition 7.1 and equations (12), (13), (16), we have

$$\mathbb{E}[|\gamma_j|^4] = \frac{1}{c_n^4} \mathbb{E}\left[|x_j^* (C_j^{-1})_j r_j r_j^* (C_j^{-1}*)_j x_j|^{2p}\right] \leq \frac{K}{c_n^{4p}} \mathbb{E}\left[|x_j^* C_j^{-1} J^{-1} r_j r_j^* (C_j^{-1}*)_j x_j|^{2p}\right] + \frac{K}{c_n^{2p} n^{2p}} \mathbb{E}\left[|r_j^* C_j^{-1} 4 J^{-1} x_j|^{2p}\right] \leq \frac{K n^p}{c_n^{4p}} \|r_j r_j^*\|^{2p} + \frac{K}{c_n^{2p} n^{2p}} \mathbb{E}[|3(z)|^{4p}] \leq \frac{K n^p}{c_n^{2p}}.$$

Similarly, we can show that

$$\mathbb{E}[|\beta_j|^4] \leq \frac{K n^p}{c_n^{2p}}.$$

Notice that there are $c_n$ many non-trivial elements in the vector $x_j$ and $\mathbb{E}[|x_{11}|^2] = 1$. Therefore $\mathbb{E}[|x_j|^2] = c_n$. Similarly,

$$\mathbb{E}[|x_j|^2] \leq Kc_n^p.$$

To estimate $\hat{\gamma}_j$, we are going to use Proposition 7.1 and equations (12), (13), (16), (15).

$$\mathbb{E}[|\hat{\gamma}_j|^4] = \frac{1}{c_n^4} \mathbb{E}\left[|x_j^* C_j^{-1} B^{-1} r_j|^{4p}\right] \leq \frac{K}{c_n^{4p}} \mathbb{E}[|x_j^* C_j^{-1} B^{-1} r_j|^{4p}] + \frac{K}{c_n^{2p} n^{2p}} \mathbb{E}[|x_j^* C_j^{-1} B^{-1} r_j|^{4p}] \leq \frac{K}{c_n^{4p}} \mathbb{E}\left[|x_j^* C_j^{-1} B^{-1} r_j r_j^* B_j^{-1} 4 C_j^{-1} x_j|^{2p}\right] + \frac{K}{c_n^{2p} n^{2p}} \mathbb{E}\left[|4 C_j^{-1} B_j^{-1} r_j r_j^* B_j^{-1} 4 C_j^{-1} x_j|^{2p}\right] \leq \frac{K n^p}{c_n^{4p}} + \frac{K}{n^{4p}} \leq \frac{K n^p}{c_n^{2p}}.$$

Similarly,

$$\mathbb{E}[|\hat{\beta}_j|^4] \leq \frac{K n^p}{c_n^{2p}}.$$
3.3 Estimates of \( \omega_j \) and \( \hat{\omega}_j \)

Using the Proposition \( \text{7.3} \), Lemma \( \text{8.3} \) and the estimates \( \{12, 13, 14, 15\} \), we can write

\[
\frac{1}{\sigma^4} \mathbb{E} \left| \omega_j - \frac{\sigma^2}{n} \text{tr} C^{-1} B^{-1} \right|^{2p} = \frac{1}{\sigma^4} \mathbb{E} \left| \frac{1}{c_n} \sigma^2 x_j^* C_j^{-1} B_j^{-1} x_j - \frac{\sigma^2}{n} \text{tr} C^{-1} B^{-1} \right|^{2p} \\
\leq \frac{K}{c_n^p} \mathbb{E} \left| x_j^* C_j^{-1} (B_j^{-1} - B_j) x_j \right|^{2p} + \frac{K}{c_n^p} \mathbb{E} \left| x_j^* C_j^{-1} B_j^{-1} - B_j^{-1} \right|^{2p} \\
+ \frac{K}{n^{2p}} \mathbb{E} \left| \text{tr} (C^{-1} - C_j^{-1}) \right|^{2p} + \frac{K}{n^{2p}} \mathbb{E} \left| \text{tr} C_j^{-1} (B_j^{-1} - B_j) \right|^{2p} \\
\leq \frac{K}{c_n^p n^{2p}} \mathbb{E} \left\| x_j \right\|^{2p} + \frac{K n^p}{c_n^p} + \frac{K n^p}{n^{2p}} \leq \frac{K n^p}{c_n^p}.
\]

Similarly, it can be shown that

\[
\frac{1}{\sigma^4} \mathbb{E} \left| \omega_j - \frac{\sigma^2}{m_n} \right|^{2p} = \frac{1}{\sigma^4} \mathbb{E} \left| \omega_j - \frac{\sigma^2}{n \text{tr} C^{-1}} \right|^{2p} \leq \frac{K n^p}{c_n^p}.
\]

This completes the estimates of the main components of \( 10 \). Finally, we notice that if \( z \in \mathbb{C}^+ \), then \( \Im(zy_j^* (C_j - zI)^{-1} y_j) \geq 0 \). As a result, we have \( |\alpha_j| \geq |\Im(z)| \).

Plugging in all the above estimates into \( 10 \), we obtain

\[
\mathbb{E} \left| \frac{1}{n} \text{tr} B^{-1} - m_n \right|^{2p} \leq \frac{K}{n} \sum_{j=1}^n \frac{K n^p}{c_n^p} \leq \frac{K n^p}{c_n^p} \rightarrow 0.
\]

Since \( |m_n| \leq \frac{1}{\Im(z)} \), there exists a subsequence \( \{m_{n_k}\}_k \) such that \( \{m_{n_k}\}_k \) converges. Uniqueness of the solution of \( 4 \) can be proved in the exact same way as described in \( 2 \) Section 4]. Also following the same exact procedure as described in \( 2 \) End of section 3], it can be proved that

\[
\frac{1}{n} \text{tr} B_n^{-1} \rightarrow \int_{1+\sigma^2 m(z)} dH(t) \quad \text{a.s.}
\]

We skip the details here. This completes the proof of the Theorem \( 2.4 \).

From the above estimate, we also see that if \( c_n = n^\beta \), where \( \beta > \frac{1}{4} + \frac{1}{2p} \), then \( \sum_{n=1}^\infty \frac{n^p}{c_n^p} < \infty \). Therefore by Borel-Cantelli Lemma, we can conclude that \( \frac{1}{n} \text{tr} B_n^{-1} - m_n \rightarrow 0 \) almost surely.

4 Proof of Theorem \( 2.3 \)

Proof of this Theorem is exactly same as the proof of Theorem \( 2.3 \). We notice that we obtained the bound \( O \left( \frac{n^p}{c_n^p} \right) \) using the Proposition \( 14 \). So while estimating the bounds of several components of equation \( 14 \), instead of using the Proposition \( 7.1 \) we will use the Proposition \( 7.3 \). And by doing so we can obtain that \( \mathbb{E} \left| \frac{1}{n} \text{tr} B^{-1} - m_n \right|^2 = O(1/c_n) \). Which will conclude the Theorem \( 2.3 \).

To prove the almost sure convergence, we can truncate all the entries of the matrix \( X \) at \( 6 \sqrt{2} \log n \). Let us denote that truncated matrix as \( \tilde{X} \). Since \( x_{ij} \)'s satisfy the Poincaré inequality, from \( 14 \) we have

\[
\mathbb{P} \left( |x_{ij}| > t \right) \leq 2K \exp \left( -\frac{K}{2} t \right).
\]

Therefore,

\[
\mathbb{P} \left( X \neq \tilde{X} \right) \leq 2Kn^2 \exp (-6 \log n) \leq \frac{K}{n^4}.
\]
Now using the second part of Proposition 7.3 and following the same method as described in section 3 we have

$$\mathbb{E}\left[\frac{1}{n}\text{tr}B^{-1} - m_n\right]^{2l}\mathbb{1}_{\{|X|^2\leq 1\}} \leq K\frac{(\log n)^{2l}}{c_n^l}.$$  

Since \(\frac{1}{n}\text{tr}B^{-1}, |m_n| \leq |\mathbb{E}|^{-1}\), we have

$$\mathbb{E}\left[\frac{1}{n}\text{tr}B^{-1} - m_n\right]^{2l} \leq K\frac{(\log n)^{2l}}{c_n^l} + \frac{K}{|\mathbb{E}|^{2l/2n^4}}.$$  

If \(c_n = n^\alpha, \alpha > 0\), then taking \(l\) large enough and using the Borel-Cantelli Lemma we may conclude the almost sure convergence.

5 Truncation of \(R\)

In several estimates, it was convenient when we had bounded \(r_{ij}\). However, we can achieve the same results as described in the Theorems 2.4 and Theorem 2.2 by truncating the Singular values of \(R\). Below, we have described the truncation method by following the same procedure as described in [7].

Let \(R = USV^*\) be the singular value decomposition of \(R\), where \(S = \text{diag}[s_1, \ldots, s_n]\) are the singular values of \(R\) and \(U, V\) are orthonormal matrices. Let us construct a diagonal matrix \(S_\alpha\) as \(S_\alpha = \text{diag}[s_1 \ldots 1(s_1 \leq \alpha), \ldots, s_n 1(s_n \leq \alpha)]\), and consider the matrices \(R_\alpha = US_\alpha V, Y_\alpha = \text{diag}(R_\alpha + \sigma X)\). Then by Lemma 8.5 we have

$$\|\mu_{YY^*} - \mu_{Y_\alpha Y^*_\alpha}\| \leq \frac{2}{n} \text{rank} \left(\frac{R}{\sqrt{c_n}^\alpha - \sqrt{c_n}^\alpha}\right)$$  

$$= \frac{2}{n} \sum_{i=1}^n 1(s_i > \alpha)$$  

$$= 2H(\alpha^2, \infty).$$

If we take \(\alpha^2 \to \infty\) for example \(\alpha = \log(c_n)\) then \(\|\mu_{YY^*} - \mu_{Y_\alpha Y^*_\alpha}\| \to 0\). So without loss of generality we can assume that \(\|r_j\| \leq \|RR^*\| \leq c_n \log(c_n)\). In that case, we have

$$\|r_j\|^2 = \|r_j r_j^*\| \leq \|RR^*\| \leq c_n \log(c_n).$$

So, using the estimates (12) and (13) we have

$$|\hat{r}_j| \leq Kc_n \log(c_n), \quad |r_j| \leq Kc_n \log(c_n),$$

where \(K > 0\) is a constant which depends only on the imaginary part of \(z\). Similarly, all the places in the proof of Theorem 2.4 we can replace the estimates \(|r_j r_j^*| \leq Kc_n\) by the estimates \(|r_j r_j^*| \leq Kc_n \log(c_n)\).

6 Extension of the results to non-periodic band matrices

The result can easily be extended to non-periodic band matrices. We observe that for the purpose of our proof, the main difference between a periodic and a non-periodic band matrix is the number of elements in certain rows. In the case of a periodic band matrix, the number of non-trivial elements in any row is \(|J_j| = 2b_n + 1 = c_n\), which is fixed for any \(1 \leq j \leq n\). Therefore, in the definition (7) we divide by \(c_n\). For a non periodic band matrix \(|J_j| = b_n + i 1(i \leq b_n + 1) + (b_n + 1) 1(b_n + 1 < i \leq n-b_n)(n + 1 - i) 1(i \geq n - b_n) = O(b_n)\). Once in the definition (7) and in the Proposition 7.3 Proposition 7.3 if we replace \(c_n\) by \(|J_j|\), everything works out as before.
7 Two concentration results

In this section we list two main concentration results which are used in the proofs of the Theorems 2.3, 2.4

Proposition 7.1. Let $M$ be one of $C_j^{-1}, C_j^{-1}B_j^{-1},$ and $N$ be one of $C_j^{-1}r_jr_j^*C_j^{-1*},$ or $C_j^{-1}B_j^{-1}r_jr_j^*B^{-1}C_1^{-1*}.$ Let $x_j$ be the $j$th column of $X$ as defined in Theorem 2.4. Let us also assume that $E|x_{11}|^4 < \infty.$ Then for any $l \in \mathbb{N},$

$$E\left|x_j^*Mx_j - \frac{c_n}{n}trM\right|^{2l} \leq Kn^l$$
$$E\left|x_j^*Nx_j - \frac{c_n}{n}trN\right|^{2l} \leq Kn^l\|r_jr_j^*\|^{2l},$$

where $K > 0$ is a constant that depends on $l,$ $\Im(z),$ and the moments of $x_j,$ but not on $n.$

Proof. From the estimates (12) and (13) we know that $\|C_j^{-1}\| \leq 1/|\Im(z)|$ and $\|B_j^{-1}\| \leq 1/|\Im(z)|.$ So for convenience of writing the proof, let us assume that $\|M\| \leq 1$ and $\|N\| \leq \|r_jr_j^*\|.$ Also without loss of generality, we can assume that $j = 1,$ and recall the definition of $I_j$ from (1). We can write $M = P + iQ,$ where $P$ and $Q$ are the real and imaginary parts of $M$ respectively. Then we can write

$$E\left|x_j^*Mx_j - \frac{c_n}{n}trM\right|^{2l} \leq 2^{2l-1}E\left|x_j^*Px_1 - \frac{c_n}{n}trP\right|^{2l} + 2^{2l-1}E\left|x_j^*Qx_1 - \frac{c_n}{n}trQ\right|^{2l}.$$ 

We can write the first part as

$$\left|x_j^*Px_1 - \frac{c_n}{n}trP\right|^{2l} = \left|x_j^*Px_1 - \sum_{k \in I_1} P_{kk} + \sum_{k \in I_1} P_{kk} - \frac{c_n}{n}trP\right|^{2l} \leq 3^{2l-1}E\left[\sum_{k \in I_1} (|x_{1k}|^2 - 1)P_{kk}\right]^{2l} + 3^{2l-1}E\left[\sum_{i\neq j \in I_1} P_{ij}x_{1i}x_{1j}\right]^{2l} + 3^{2l-1}\left|\sum_{k \in I_1} P_{kk} - \frac{c_n}{n}trP\right|^{2l} =: 3^{2l-1}(S_1 + S_2 + S_3).$$

Following the same procedure as in (21), we can estimate the first part. Note that $\|P^m\| \leq \|P\|^m \leq \|M\|^m \leq 1$ for any $m \in \mathbb{N}.$ In the expansion of $\left[\sum_{k \in I_1} (|x_{1k}|^2 - 1)P_{kk}\right]^{2l},$ the maximum contribution (in terms of $c_n$) will come from the terms like

$$\sum_{k_1, \ldots, k_l \in I_1} (|x_{1k_1}|^2 - 1)^2 \cdots (|x_{1k_l}|^2 - 1)^2 (P_{i_1,i_1} \cdots P_{i_l,i_l})^2,$$

when all $i_1, \ldots, i_l$ are distinct. Note that $(P_{i_1,i_1} \cdots P_{i_l,i_l})^2 \leq 1.$ Consequently, expectation of the above term is bounded by $Kc_n^l,$ where $K$ depends only on the fourth moment of $x_{1j}.$ Therefore

$$S_1 = E\left[\sum_{k \in I_1} (|x_{1k}|^2 - 1)P_{kk}\right]^{2l} \leq Kc_n^l,$$

where $K$ depends only on $l$ and the moments of $x_{1j}.$

Since $C_1^{-1}, C_1^{-1}B_1^{-1}, C_1^{-1}r_i r_i^* C_1^{-1*}$ or $C_1^{-1}B_1^{-1}r_i r_i^* B^{-1} C_1^{-1*}$ are independent of $x_{1j},$ for the second sum we have

$$\sum_{i_1 \neq j_1, \ldots, i_l \neq j_l, i_1, j_1, \ldots, i_l, j_l \in I_1} E[P_{i_1,j_1} \cdots P_{i_l,j_l}] E[|x_{1i_1}x_{1j_1}|^2 \cdots |x_{1i_l}x_{1j_l}|^2].$$
The expectation will be zero if a term appears only once and the maximum contribution (in terms of \( c_n \)) will come from the case when each of \( x_{ij} \) and \( x_{ji} \) appears only twice. In that case, the contribution is

\[
\sum_{i \neq j \atop i,j \in I_1} P_{ij}^2 \cdots \sum_{i \neq j \atop i,j \in I_1} P_{ij}^2 \leq c_n^l,
\]

where the last inequality follows from the fact that \( \sum_{i,j \in I_1} P_{ij}^2 = \text{tr}(LPL^T L^T L^T) \leq c_n \), where \( L_{c_n \times n} \) is the projection matrix onto the co-ordinates indexed by \( I_1 \). As a result, we have

\[
S_2 = \mathbb{E} \left[ \sum_{i \neq j \atop i,j \in I_1} P_{ij} x_{ij} \right]^{2l} \leq K c_n^l,
\]

where \( K \) depends only on \( l \) and the moments of \( x_{ij} \).

To estimate the \( S_3 \), we can write it as

\[
S_3 = \left| \sum_{k \in I_1} P_{kk} - \frac{c_n}{n} \text{tr} P \right|^{2l} = 2^{2l-1} \left| \sum_{k \in I_1} P_{kk} - \mathbb{E} \sum_{k \in I_1} P_{kk} \right|^{2l} + 2^{2l-1} \left| \mathbb{E} \sum_{k \in I_1} P_{kk} - \frac{c_n}{n} \text{tr} P \right|^{2l}.
\]

Since \( |P_{kk} - \mathbb{E}[P_{kk}]| \leq \| (C_1^{-1})_{kk} - \mathbb{E}[(C_1^{-1})_{kk}] \| \), from Lemma 7.2 we have an exponential tail bound on \( \sum_{k \in I_1} P_{kk} - \mathbb{E} \sum_{k \in I_1} P_{kk} \). As a result,

\[
\mathbb{E} \left| \sum_{k \in I_1} P_{kk} - \mathbb{E} \sum_{k \in I_1} P_{kk} \right|^{2l} \leq K n^l,
\]

where \( K \) depends only on \( l \). Hence we have

\[
S_3 \leq K(n^l + c_n^l).
\]

Combining all the above estimates, we get

\[
\mathbb{E} \left| x_1^* P x_1 - \frac{c_n}{n} \text{tr} P \right|^{2l} \leq K n^l.
\]

Repeating the above computation, we can do the same estimate \( \mathbb{E} \left| x_1^* Q x_1 - \frac{n}{n} \text{tr} Q \right|^{2l} \leq K n^l \). This completes the proof.

\[\Box\]

**Lemma 7.2 (Norm of a random band matrix).** Let \( X \) and \( Y \) be defined in \( \mathbb{E} \), \( x_{ij} \) satisfy the Poincaré inequality with constant \( m \), and \( c_n > (\log n)^2 \). Then \( \mathbb{E} \|XX^*\| \leq K c_n^2 \) for some universal constant \( K \) which may depend on the Poincaré constant \( m \). In particular, if the limiting ESD of \( \frac{1}{c_n} RR^* \) i.e., \( H \) is compactly supported then \( \mathbb{E} \|YY^*\| \leq K c_n \).
Proof. We will follow the method described in [25, 16, 24] and the references therein. The analysis becomes somewhat easier if we assume that all non-zero entries of $X$ are standard Gaussian random variables. However, it contains the main idea of the analysis.

**Case I** ($x_{jk}$ are standard Gaussian random variables): Using the Markov’s inequality, we have

$$P\left(\frac{1}{c_n} \|XX^*\| > t\right) \leq e^{-t} \mathbb{E} \left[ \exp\left(\frac{1}{c_n} \|XX^*\|\right)\right] \leq e^{-t} \mathbb{E} \left[ \text{tr} \exp\left(\frac{1}{c_n} XX^*\right)\right],$$

To estimate the right hand side, we will use the Lieb’s Theorem. Let $H$ be any $n \times n$ fixed Hermitian matrix. From Lieb’s Theorem ([15], Theorem 6), we know that the function $f(A) = \text{tr} \exp(H + \log A)$ is a concave function on the convex cone of $n \times n$ positive definite Hermitian matrices.

Let us write $\frac{1}{c_n} XX^* = \sum_{k=1}^{n} x_k x_k^*$, where $x_k$ is the $k$th column vector of $X/\sqrt{c_n}$. Then using Lieb’s Theorem and Jensen’s inequality, we have

$$\mathbb{E} \left[ \text{tr} \exp\left(\frac{1}{c_n} XX^*\right)^{|x_1, \ldots, x_{n-1}|} \right] = \mathbb{E} \left[ \text{tr} \exp\left(\frac{1}{c_n} \sum_{k=1}^{n-1} x_k x_k^* + \log \exp\left(\frac{1}{c_n} x_n x_n^*\right)\right)^{|x_1, \ldots, x_{n-1}|}\right]$$

$$\leq \text{tr} \exp\left(\frac{1}{c_n} \sum_{k=1}^{n-1} x_k x_k^* + \log \mathbb{E} \exp\left(\frac{1}{c_n} x_n x_n^*\right)\right).$$

Proceeding in this way, we obtain

$$\mathbb{E} \left[ \text{tr} \exp\left(\frac{1}{c_n} XX^*\right)\right] \leq \text{tr} \exp\left[ \sum_{k=1}^{n} \log \mathbb{E} \exp\left(\frac{1}{c_n} x_k x_k^*\right)\right].$$

Therefore

$$P\left(\frac{1}{c_n} \|XX^*\| > t\right) \leq e^{-t} \text{tr} \exp\left[ \sum_{k=1}^{n} \log \mathbb{E} \exp\left(\frac{1}{c_n} x_k x_k^*\right)\right].$$

(18)

It is easy to see that

$$\exp\left(\frac{1}{c_n} x_k x_k^*\right) = I + \left(\sum_{l=1}^{\infty} \frac{1}{l! c_n} \|x_k\|^{2(l-1)}\right) x_k x_k^*$$

$$= I + \frac{\|x_k\|^2/c_n - 1}{\|x_k\|^2} x_k x_k^*$$

$$\leq I + \frac{\|x_k\|^2/c_n}{\|x_k\|^2} x_k x_k^*,$$

where $A \preceq B$ denotes that $(B - A)$ is positive semi-definite. Since $\{x_{jk}\}_{1 \leq j \leq n, \ j \neq l}$ are independent standard Gaussian random variables, we have

$$\mathbb{E} \left[ e^{\|x_k\|^2/c_n} x_j x_l \right] = 0, \text{ if } j \neq l$$

$$\mathbb{E} \left[ e^{\|x_k\|^2/c_n} |x_{jk}|^2\right] = \left(1 - \frac{1}{c_n}\right)^{-(c_n+1)}.$$

As a result,

$$\text{tr} \exp\left[ \sum_{k=1}^{n} \log \mathbb{E} \exp\left(\frac{1}{c_n} x_k x_k^*\right)\right] \leq n \left(1 + \frac{e}{c_n}\right)^{c_n}.$$
Substituting this estimate in \((18)\), we have
\[
\mathbb{P}\left( \frac{1}{c_n} \|XX^*\| > t + \log n \right) \leq e^{ne^{-t+t+\log n}} = e^te^{-t}.
\]
(19)

As a result,
\[
\frac{1}{c_n} \mathbb{E}[\|XX^*\|] = \int_0^\infty \mathbb{P}\left( \frac{1}{c_n} \|XX^*\| > u \right) du \leq \int_0^{\log n} du + \int_0^\infty \mathbb{P}\left( \frac{1}{c_n} \|XX^*\| > t + \log n \right) dt \leq \log n + e^t \leq Kc_n.
\]

This completes the proof.

**Case II** \((x_{jk}\ v s\ tisfy\ tje\ Poincaré\ inequlity):** First of all, let us write the random matrix \(X\) as \(X = X_1 + iX_2\), where \(X_1\) and \(X_2\) are the real and imaginary parts of \(X\) respectively. Since \(\|X\| \leq \|X_1\| + \|X_2\|\), it is enough to estimate \(\|X_1\|\) and \(\|X_2\|\) separately. In other words, without loss of generality, we can assume that \(x_{ij}\) are real valued random variables.

Let us construct the matrix
\[
\tilde{X} = \begin{bmatrix} O & X \\ X & O \end{bmatrix}.
\]

It is easy to see that \(\|	ilde{X}\| = \|X\|\). Therefore it is enough to bound \(\mathbb{E}[\|	ilde{X}\|^2]\).

We can write \(\tilde{X}\) as
\[
\tilde{X} = \sum_{i=1}^n \sum_{j \in I} x_{ij} (E_{i,n+j} + E_{n+j,i}),
\]
where \(E_{ij}\) is a \(2n \times 2n\) matrix with all 0 entries except 1 at the \((i,j)\)th position. Proceeding in the same way as case I, we may write
\[
\mathbb{P}\left( \frac{1}{\sqrt{c_n}} \|	ilde{X}\| > t \right) \leq e^{-t} \exp \left[ \sum_{i=1}^n \sum_{j \in I} \log \mathbb{E} \exp \left( \frac{1}{\sqrt{c_n}} x_{ij} (E_{i,n+j} + E_{n+j,i}) \right) \right].
\]
(20)

Let us consider the \(2 \times 2\) matrix \(H = \begin{bmatrix} 0 & \gamma \\ \gamma & 0 \end{bmatrix}\), where \(\gamma\) is a real valued random variable. By the spectral calculus, we have
\[
\log \mathbb{E}[\exp(H)] = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \log \mathbb{E}e^{-\gamma} & 0 \\ 0 & \log \mathbb{E}e^{-\gamma} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \log[\mathbb{E}e^{\gamma}/\mathbb{E}e^{-\gamma}] & \log[\mathbb{E}e^{\gamma}/\mathbb{E}e^{-\gamma}] \\ \log[\mathbb{E}e^{\gamma}/\mathbb{E}e^{-\gamma}] & \log[\mathbb{E}e^{\gamma}/\mathbb{E}e^{-\gamma}] \end{bmatrix}.
\]

Since \(x_{ij}\)s are iid, let us assume that all \(x_{ij}\) have the same probability distribution as a real valued random variable \(\gamma\). Then proceeding as above, we can see that
\[
\log \mathbb{E} \exp \left( \frac{1}{\sqrt{c_n}} x_{ij} (E_{i,n+j} + E_{n+j,i}) \right) = \frac{1}{2} \log[\mathbb{E}e^{\gamma}/\sqrt{c_n}/\mathbb{E}e^{-\gamma}/\sqrt{c_n}] (E_{ii} + E_{n+j,n+j}) + \frac{1}{2} \log[\mathbb{E}e^{\gamma}/\sqrt{c_n}/\mathbb{E}e^{-\gamma}/\sqrt{c_n}] (E_{i,n+j} + E_{n+j,i}).
\]
Therefore,
\[
\sum_{i=1}^{n} \sum_{j \in I_i} \log \mathbb{E} \exp \left( \frac{1}{\sqrt{c_n}} x_{ij} (E_{i,n+j} + E_{n,j,i}) \right) = \frac{c_n}{2} \log \left[ \mathbb{E} e^{\gamma/\sqrt{c_n}} e^{-\gamma/\sqrt{c_n}} \right] \, I
\]
\[
+ \frac{1}{2} \log \left[ \mathbb{E} e^{\gamma/\sqrt{c_n}} e^{-\gamma/\sqrt{c_n}} \right] \sum_{i=1}^{n} \sum_{j \in I_i} (E_{i,j+n} + E_{j+n,i}).
\]

From Golden–Thompson inequality, if \( A \) and \( B \) are two \( d \times d \) real symmetric matrices then \( \text{tr} e^{A+B} \leq \text{tr}(e^A e^B) \).

In our case, let us take
\[
A = \frac{c_n}{2} \log \left[ \mathbb{E} e^{\gamma/\sqrt{c_n}} e^{-\gamma/\sqrt{c_n}} \right] I
\]
\[
B = \frac{1}{2} \log \left[ \mathbb{E} e^{\gamma/\sqrt{c_n}} e^{-\gamma/\sqrt{c_n}} \right] \sum_{i=1}^{n} \sum_{j \in I_i} (E_{i,j+n} + E_{j+n,i}).
\]

Then
\[
e^A = \left[ \mathbb{E} e^{\gamma/\sqrt{c_n}} e^{-\gamma/\sqrt{c_n}} \right]^{c_n/2} I.
\]

\[
\text{tr} \exp \left[ \sum_{i=1}^{n} \sum_{j \in I_i} \log \mathbb{E} \exp \left( \frac{1}{\sqrt{c_n}} x_{ij} (E_{i,n+j} + E_{n,j,i}) \right) \right]
\]
\[
\leq \text{tr} \left( \left[ \mathbb{E} e^{\gamma/\sqrt{c_n}} e^{-\gamma/\sqrt{c_n}} \right]^{nc_n/2} e^{B} \right)
\]
\[
\leq \left\{ \left[ \mathbb{E} e^{\gamma/\sqrt{c_n}} e^{-\gamma/\sqrt{c_n}} \right]^{nc_n/2} \right\} \text{ne} \|B\|.
\]

It is easy to see that \( \left\| \sum_{i=1}^{n} \sum_{j \in I_i} (E_{i,j+n} + E_{j+n,i}) \right\| \leq c_n \). Combining all the estimates and plugging them in (20), we obtain
\[
\mathbb{P} \left( \frac{1}{\sqrt{c_n}} \|\tilde{X}\| > t \right) \leq ne^{-t} \left[ \mathbb{E} e^{\gamma/\sqrt{c_n}} e^{-\gamma/\sqrt{c_n}} \right]^{c_n/2} \left[ \mathbb{E} e^{\gamma/\sqrt{c_n}} e^{-\gamma/\sqrt{c_n}} \right]^{c_n/2}
\]
\[
= ne^{-t} \left\{ \mathbb{E} e^{\gamma/\sqrt{c_n}} \right\}^{c_n}.
\]

From the concentration estimate (11), we have that \( \mathbb{P}(|\gamma| > t) \leq \exp \{ -t \sqrt{\kappa} / \sqrt{2} \} \)
\[
\mathbb{E}[e^{\gamma/\sqrt{c_n}}] = \int_{0}^{\infty} \mathbb{P} \left( \frac{\gamma}{\sqrt{c_n}} > \log t \right) \, dt
\]
\[
\leq \int_{0}^{1} \mathbb{P}(\gamma > \sqrt{c_n} \log t) \, dt + \int_{1}^{\infty} \mathbb{P}(\gamma > \sqrt{c_n} \log t) \, dt
\]
\[
\leq 1 + \int_{1}^{\infty} t^{-\sqrt{c_n} \log t} \, dt
\]
\[
= 1 + \left( \frac{\sqrt{c_n}}{2} - 1 \right)^{-1}.
\]

As a result,
\[
\mathbb{P} \left( \frac{1}{\sqrt{c_n}} \|\tilde{X}\| > t \right) \leq ne^{-t} e^{\sqrt{2c_n}/\sqrt{\kappa}}.
\]
Therefore,
\[ \frac{1}{c_n} \mathbb{E} \| \hat{X} \|^2 \leq (\log n + \sqrt{2c_n}/\sqrt{n})^2 \leq K c_n. \]

**Proposition 7.3.** Let \( M \) be one of \( C_j^{-1}, C_j^{-1}B_j^{-1}, C_j^{-1}r_j^*C_j^{-1*} \) or \( C_j^{-1}B_j^{-1}r_j^*B^{-1*}C_j^{-1*} \), and \( x_j \) be the \( j \)th column of \( X \). In addition, let us also assume that the random variables \( x_{ij} \) satisfy the Poincaré inequality with constant \( m \), and \( c_n > (\log n)^2 \). Then we have
\[ \mathbb{E} \left| x_{ij}^* M x_j - \frac{c_n}{n} trM \right|^2 \leq K c_n, \]
where \( K > 0 \) is a constant depends on \( \Im(z) \), \( \sigma \), and the Poincaré constant \( m \). Moreover, if the entries of the matrix \( X \) are bounded by \( 6 \sqrt{\frac{2}{\pi}} \log n \), then
\[ \mathbb{E} \left| x_{ij}^* M x_j - \frac{c_n}{n} trM \right|^{2l} \leq K c_n (\log n)^{2l}, \]
where \( K > 0 \) depends on \( l, \Im(z), \sigma \), and the Poincaré constant \( \kappa \).

**Proof.** Let us first prove this for \( M = C_j^{-1} = (YY^* - y_jy_j^* - zI)^{-1} \). Since \( x_{ij} \) satisfy the Poincaré inequality, they have exponential tails and consequently they have all moments. As a result, we can repeat the same proof of Proposition 7.2. However, notice that in Proposition 7.2 we are getting the order \( n^l \) instead of \( c_n^l \) solely because of the estimate (17). So, it boils down to obtain an estimate of \( O(c_n) \) for (17) when \( x_{ij} \) satisfy Poincaré inequality.

Since \( x_{ij} \) satisfy the Poincaré inequality we can write
\[ \text{Var} \left( \sum_{p \in I_j} M_{pp} \right) \leq \frac{1}{\kappa} \sum_{s,t} \mathbb{E} \left| \sum_{p \in I_j} \frac{\partial M_{pp}}{\partial x_{st}} \right|^2 + \frac{1}{\kappa} \sum_{s,t} \mathbb{E} \left| \sum_{p \in I_j} \frac{\partial M_{pp}}{\partial x_{st}} \right|^2, \]
where \( \kappa > 0 \) is the constant of Poincaré inequality. Let \( m_{kl} := \sum_{i \neq j} y_{ki}y_{li} = \frac{1}{c_n} \sum_{i \neq j} (r_{ki} + \sigma x_{ki})(\bar{r}_{li} + \sigma \bar{x}_{li}) \) be the \( kl \)th entry of \( YY^* - y_jy_j^* \). It is very easy to compute, and done in the literature in past, that
\[ \frac{\partial M_{pp}}{\partial m_{kl}} = -\frac{1}{1 + \delta_{kl}} \left[ M_{pk}M_{lp} + M_{pl}M_{kp} \right] = -\frac{2}{1 + \delta_{kl}} M_{kp}M_{pl}. \]

Now, it is easy to see that
\[ \frac{\partial m_{kl}}{\partial x_{st}} = \frac{\sigma}{c_n} \sum_{i \neq j} \delta_{ks}\delta_{it}(r_{li} + \sigma x_{li}) = \frac{\sigma}{c_n} \delta_{ks}(r_{lt} + \sigma x_{lt}) 1_{\{t \neq j \}}. \]

Consequently,
\[ \sum_{p \in I_j} \frac{\partial M_{pp}}{\partial x_{st}} = -\frac{\sigma}{c_n} \sum_{p \in I_j} \frac{2\delta_{ks}}{1 + \delta_{kl}} M_{kp}M_{pl}(r_{lt} + \sigma x_{lt}) 1_{\{t \neq j \}} = -\frac{\sigma}{c_n} \sum_{p \in I_j} \sum_{l} \frac{2}{1 + \delta_{st}} M_{sp}M_{pl}(r_{lt} + \sigma x_{lt}) 1_{\{t \neq j \}} = -\frac{\sigma}{c_n} \sum_{l} (\hat{M}_j)_{st}(r_{lt} + \sigma x_{lt}) 1_{\{t \neq j \}} = -\frac{\sigma}{\sqrt{c_n}} (\hat{M}_jY^j)_{st}, \]

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where \((M_j)_{st} = \frac{1}{1 + \tau_{st}} \sum_{p \in I_j} M_{sp} M_{pt}\), and \(Y_j\) is the matrix \(Y\) with \(j\)th column replaced by zeros.

Therefore, rank\((\tilde{M}_j)\) ≤ \(c_n\). As a result,

\[
\sum_{s, t} \mathbb{E} \left( \sum_{p \in I_j} \left| \frac{\partial M_{pp}}{\partial x_{st}} \right|^{2} \right) \leq \frac{\sigma^2}{c_n} \mathbb{E} \tr(\tilde{M}_j Y_j Y_j^*) \leq \frac{\sigma^2}{|\Im(z)|^4} \mathbb{E}\|Y_j Y_j^*\|, \tag{21}
\]

where in the last inequality we have used the fact that \(\|\tilde{M}_j\| \leq 1/|\Im(z)|\). Consequently, using the Lemma \ref{lemma:bound} we have

\[
\sum_{s, t} \mathbb{E} \left( \sum_{p \in I_j} \left| \frac{\partial M_{pp}}{\partial x_{st}} \right|^{2} \right) \leq Kc_n.
\]

Repeating the above calculations for \(\sum_{s, t} \mathbb{E} \left( \sum_{p \in I_j} \left| \frac{\partial M_{pp}}{\partial x_{st}} \right|^{2} \right)\), we can obtain the same bounds. Hence the result follows for \(M = C_j^{-1}\).

Since \(\|B_j^{-1}\| \leq 1/|\Im(z)|\) and \(\|r_j r_j^*\| \leq Kc_n\), the result follows for \(C_j^{-1} B_j^{-1}, C_j^{-1} r_j r_j^* C_j^{-1}\), \(C_j^{-1} B_j^{-1} r_j r_j^* B^{-1} C_j^{-1}\) too.

To prove the second part, we invoke the equation \ref{eq:bound}.

\[
P\left( \left| \sum_{k \in I_j} M_{kk} - \mathbb{E} \sum_{k \in I_j} M_{kk} \right| > t \right) \leq 2 K \exp\left( -\sqrt{\frac{\kappa}{2 \|\nabla \sum_{k \in I_j} M_{kk}\|_{\infty}} t} \right).
\]

From the equation \ref{eq:bound}, we have

\[
\left\| \nabla \sum_{k \in I_j} M_{kk} \right\|_{2}^{2} \leq \frac{2 \sigma^2}{|\Im(z)|^4} \|Y_j Y_j^*\|.
\]

Since all the entries of \(X\) are bounded by \(6 \sqrt{2 \kappa \log n}\), we have \(\|XX^*\| \leq Kc_n^2 (\log n)^2\). And we know that \(\|RR^*\| \leq Kc_n\) for large \(n\). Therefore \(\|YY^*\| \leq Kc_n (\log n)^2\). We can get the same bound for \(\|Y_j Y_j^*\|\). As a result,

\[
P\left( \left| \sum_{k \in I_j} M_{kk} - \mathbb{E} \sum_{k \in I_j} M_{kk} \right| > t \right) \leq 2 K \exp\left( -\sqrt{\frac{\kappa}{K' \sqrt{2c_n \log n}}} t \right).
\]

Which implies that

\[
\left\| \sum_{k \in I_j} M_{kk} - \mathbb{E} \sum_{k \in I_j} M_{kk} \right\|_{2}^{2l} \leq Kc_n^l (\log n)^{2l}.
\]

Plugging this in \ref{eq:bound}, and following the same procedure as in Proposition \ref{prop:bound} we have the result.

Observe that the second result of this Proposition is somewhat stronger than the first result, as it leads to the almost sure convergence (see section \ref{sec:asymp}) and it does not need the help of Lemma \ref{lemma:bound}. However the method used in Lemma \ref{lemma:bound} is interesting by itself. So we keep it.
8 Appendix

In this section we list the results which were used in the section

Lemma 8.1 (Lemma 2.3, [21]). Let $P, Q$ be two rectangular matrices of the same size. Then for any $x, y \geq 0$,

$$
\mu((P + Q)^*(P + Q)^{-1})(x, \infty) \leq \mu_{P^T}(x, \infty) + \mu_{Q^T}(y, \infty).
$$

Lemma 8.2 (Sherman-Morrison formula). Let $P_{n \times n}$ and $(P + vv^*)$ be invertible matrices, where $v \in \mathbb{C}^n$. Then we have

$$(P + vv^*)^{-1} = P^{-1} - \frac{P^{-1}v v^* P^{-1}}{1 + v^* P^{-1}v}.$$

In particular,

$$v^*(P + vv^*)^{-1} = \frac{v^* P^{-1}}{1 + v^* P^{-1}v}.$$

Lemma 8.3 (Lemma 2.6, [21]). Let $P, Q$ be $n \times n$ matrices such that $Q$ is Hermitian. Then for any $r \in \mathbb{C}^n$ and $z = E + i\eta \in \mathbb{C}^+$ we have

$$|\text{tr}((Q - zI)^{-1} - (Q + rr^* - zI)^{-1} P)| \leq \frac{\text{rank}(Q)}{1 + r^* (Q - zI)^{-1} r} |E| \leq \frac{\|P\|}{\eta}.$$

Lemma 8.4 ([2], Lemma 1). Let $\{X_n\}_n$ be a sequence of random variables such that $|X_n| \leq K_n$ almost surely, and $\mathbb{E}[X_{i_1} X_{i_2} \ldots X_{i_k}] = 0$ for all $k \in \mathbb{N}$, $i_1 < i_2 < \cdots < i_k$. Then for every $\lambda \in \mathbb{R}$ we have

$$\mathbb{E}\left[ \exp \left\{ \lambda \sum_{i=1}^{n} X_i \right\} \right] \leq \exp \left\{ \frac{\lambda^2}{2} \sum_{i=1}^{n} K_i^2 \right\}.$$

In particular, for any $t > 0$ we have

$$\mathbb{P}\left( \left| \sum_{i=1}^{n} X_i \right| > t \right) \leq 2 \exp \left\{ -\frac{t^2}{2 \sum_{i=1}^{n} K_i^2} \right\}.$$

Lemma 8.5. Let $P, Q$ be two $n \times n$ matrices, then

$$\|\mu_{P^T} - \mu_{Q^T}\| \leq \frac{2}{n} \text{rank}(P - Q),$$

where $\| \cdot \|$ denotes the total variation norm between probability measures.

Proof. By Cauchy’s interlacing property,

$$\|\mu_{P^T} - \mu_{Q^T}\| \leq \frac{1}{n} \text{rank}(P P^T - QQ^T)$$

$$\leq \frac{1}{n} \text{rank}((P - Q) P^T) + \frac{1}{n} \text{rank}(Q (P - Q)^T)$$

$$\leq \frac{2}{n} \text{rank}(P - Q).$$

Lemma 8.6 ([3], Lemma C.3). Let $P$ and $Q$ be $n \times n$ Hermitian matrices, and $I \subset \{1, 2, \ldots, n\}$, then

$$\left| \sum_{k \in I} (P - zI)^{-1} - \sum_{k \in I} (Q - zI)^{-1} \right| \leq \frac{2}{\Im(z)} \text{rank}(P - Q).$$
Lemma 8.7. Let $C_j$ and $B_j$ be defined in $\S 8$, $r_j$ be the $j$th column of $R$, and $I_j \subset \{1,2,\ldots,n\}$ be same as $\S 1$, and $z \in \mathbb{C}^+$. Then

\[
\begin{align*}
P \left( \sum_{k \in I_j} (C_j^{-1})_{kk} - \mathbb{E} \sum_{k \in I_j} (C_j^{-1})_{kk} \right) > t \right) & \leq 2 \exp \left\{ -\frac{3(z)^2 l^2}{32n} \right\} \\
P \left( \sum_{k \in I_j} (\mathbb{E} C_j^{-1} B_j^{-1})_{kk} \right) > t \right) & \leq 2 \exp \left\{ -\frac{3(z)^2 l^2}{32n} \right\} \\
P \left( \sum_{k \in I_j} (C_j^{-1} B_j^{-1} r_j^* C_j^{-1*})_{kk} - \mathbb{E} \sum_{k \in I_j} (C_j^{-1} B_j^{-1} r_j^* C_j^{-1*})_{kk} \right) > t \right) & \leq 2 \exp \left\{ -\frac{3(z)^2 l^2}{32n} \right\} \\
P \left( \sum_{k \in I_j} (C_j^{-1} B_j^{-1} r_j^* B_j^{-1*} C_j^{-1*})_{kk} - \mathbb{E} \sum_{k \in I_j} (C_j^{-1} B_j^{-1} r_j^* B_j^{-1*} C_j^{-1*})_{kk} \right) > t \right) & \leq 2 \exp \left\{ -\frac{3(z)^2 l^2}{32n} \right\}.
\end{align*}
\]

Proof. Let $\mathcal{F}_l = \sigma\{y_1,\ldots,y_l\}$ be the $\sigma$-algebra generated by the column vectors $y_1,\ldots,y_l$. Then, we can write

\[
\begin{align*}
\sum_{k \in I_j} (C_j^{-1})_{kk} - \mathbb{E} \sum_{k \in I_j} (C_j^{-1})_{kk} & = \sum_{i=1}^n \left[ \mathbb{E} \left\{ \sum_{k \in I_j} (C_j^{-1})_{kk} \middle| \mathcal{F}_l \right\} - \mathbb{E} \left\{ \sum_{k \in I_j} (C_j^{-1})_{kk} \middle| \mathcal{F}_{l-1} \right\} \right] \\
\end{align*}
\]

Notice that for any two matrices $P,Q$, we have $\text{rank}(PP^* - QQ^*) \leq 2\text{rank}(P - Q)$ (from Lemma 8.5). Therefore, using the Lemma 8.6 and Lemma 8.4 we can conclude the result. The remaining three equations can also be proved in the same way.

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