On the Asymptotic Distribution of Variance Weighted
KS Statistics

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Abstract

This paper derives the asymptotic distribution of variance weighted Kolmogorov-Smirnov statistics for conditional moment inequality models for the case of a one dimensional covariate. The asymptotic distribution depends on the data generating process only through the variance of a single random variable, leading to critical values that can be calculated analytically. By arguments in Armstrong (2011b), the resulting tests achieve the best minimax rate for local alternatives out of available approaches in a broad class of settings.

1 Introduction

This paper derives the asymptotic distribution of variance weighted Kolmogorov-Smirnov (KS) statistics for conditional moment inequality models for the case of a one dimensional conditioning variable. The arguments can be extended to higher dimensional conditioning variables as well. The asymptotic distribution is extreme value, with a scaling that can be easily estimated. Thus, critical values for this test are easy to compute and do not require resampling or simulation from complicated random processes. By arguments in Armstrong (2011b), these tests are, in a broad class of models, the only tests available that achieve the best minimax rate for power against local alternatives out of available tests. Loosely

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speaking, the test will be optimal in this sense generically when the conditioning variable is continuously distributed and the model may not be point identified. In cases where point identification is suspected, other approaches may be desirable, although the results in Armstrong (2011b) show that the tests in the present paper will still have good power properties.

The tests proposed in this paper are most closely related to those studied by Armstrong (2011b) and Chetverikov (2012). Armstrong (2011b) uses a similar class of statistics, but proposes confidence regions that satisfy the stronger criterion of containing the entire identified set. That paper derives critical values that are conservative even for this stronger criterion. While the critical values proposed in that paper would be valid in this context, they rely on conservative bounds for the null distribution of the test statistic. The asymptotic distribution result in the present paper allows for less conservative critical values, leading to more powerful tests. Armstrong (2011b) shows that, even with these conservative critical values, the confidence regions proposed in that paper achieve the best minimax rate of convergence in the Hausdorff metric to the identified set out of available methods in a broad class of models. These arguments will give analogous local power results for the tests proposed in the present paper. Chetverikov (2012), which was written at around the same time as the present paper, proposes critical values for the tests treated in the present paper using simulation from an approximating random process. That paper uses a novel argument that proves validity of simulation methods without deriving an asymptotic distribution or even showing that one exists. The asymptotic distribution result of the present paper shows that such arguments are not necessary, at least in the one dimensional case, and provides critical values that can be computed easily without simulation. Asymptotic distribution results are also important for assessing the asymptotic behavior of the critical values for use in power calculations. On the other hand, the methods of Chetverikov (2012) apply to higher dimensional conditioning variables. While the results of the present paper could be extended to this case, the approach of Chetverikov (2012) allows computation of valid critical values even without this extension. It also seems likely that, while being more difficult to compute, the simulation based critical values of Chetverikov (2012) enjoy higher order accuracy properties similar to resampling methods in other settings, and that the methods of that paper would be useful in showing this.

Other approaches to inference on conditional moment inequalities include Andrews and Shi (2009), Kim (2008), Khan and Tamer (2009), Chernozhukov, Lee, and Rosen (2009), Lee, Song, and Whang (2011), Ponomareva (2010), Menzel (2008) and Armstrong (2011a). See Armstrong (2011b).
for a discussion of some of these approaches, including power results that show that the approach in the present paper is optimal among these approaches in a certain sense in a broad class of models in the set identified case. The literature on the related problem of inference with finitely many unconditional moment inequalities is also recent, but more developed. Articles include Andrews, Berry, and Jia (2004), Andrews and Jia (2008), Andrews and Guggenberger (2009), Andrews and Soares (2010), Chernozhukov, Hong, and Tamer (2007), Romano and Shaikh (2010), Romano and Shaikh (2008), Bugni (2010), Beresteanu and Molinari (2008), Moon and Schorfheide (2009), Imbens and Manski (2004) and Stoye (2009).

2 Setup

We wish to test the null hypothesis

\[ E(Y_i | X_i) \geq 0 \ a.s. \] (1)

using iid observations \((X_1, Y_1), \ldots, (X_n, Y_n)\). Typically, \(Y_i = m(W_i, \theta)\) for some parametric function \(m\) of data \(W_i\) that may include \(X_i\). See Armstrong (2011b) for several examples of such models along with references to the literature. The moment inequality defines the identified set \(\Theta_0\) of parameter values that satisfy this inequality, and it is of interest to design tests that are powerful against alternative hypotheses within the model defined by parameter values \(\theta\) such that \(m(W_i, \theta)\) does not satisfy the moment restriction (1). Arguments in Armstrong (2011b) show that the tests in the present paper are most powerful among existing approaches against such alternatives in a local minimax sense in a broad class of models used in empirical economics.

This paper considers inference using the test statistic

\[ T_n = \inf_{\{s,t|\sigma(s,t) > 0\}} \frac{E_nY_iI(s < X < s + t)}{\hat{\sigma}(s,t)} \] (2)

The results in the following section derive the asymptotic distribution of \(T_n\). The result gives easily computable critical values for this test statistic.
3 Asymptotic Distribution

The following theorem gives the asymptotic distribution of $T_n$. The result is stated for a single moment inequality (that is, $Y_i$ is one dimensional). This result can be used immediately for the case of higher dimensional $Y_i$ using Bonferroni bounds, which will be conservative, but will not result in a first order loss in power. Alternatively, the results could also be extended to this case to get less conservative critical values.

Theorem 1. Suppose that $E(Y_i|X_i) \geq 0$ a.s., the data are iid and

(i) $X_i$ is one dimensional, and, for some $\underline{x} < \overline{x}$, $E(Y_i|X_i) = 0$ only on $[\underline{x}, \overline{x}]$.

(ii) $X_i$ has a density bounded from above away from infinity and from below away from zero.

(iii) $\sigma^2_X(x) \equiv \text{var}(Y_i|X_i = x)$ is bounded away from zero and infinity.

(iv) $Y_i$ is bounded by a nonrandom constant with probability one.

(v) $\sigma_n n^{1/2}/(\log n)^{5/2} \to \infty$.

Define $c_n = E(\sigma^2_X(x)I(\underline{x} \leq X_i \leq \overline{x}))/\sigma_n$ and $\hat{c}_n = E(\sigma^2_Y(x)I(\underline{x} \leq X_i \leq \overline{x}))/\sigma_n$. Then

$$\liminf_n P \left( \sqrt{n}T_n \leq (2 \log c_n)^{1/2} + \frac{3}{2} \log \log c_n - \log(2\pi^{1/2}) + r \right) \geq \exp(\exp(r)),$$

and the same statement holds with $c_n$ replaced with $\hat{c}_n$.

It is interesting to note that the critical value for $T_n$ is, to a first order approximation, given by $(2 \log(1/\sigma_n^2))/n^{1/2}$. This is the same as the first order approximation to the critical value if the infimum were taken only over $\hat{\sigma}(s,t) = \sigma_n$, which would essentially be a kernel estimator with bandwidth proportional to $\sigma_n$. Thus, up to a first order power comparison, there is asymptotically no loss in power in considering larger bandwidths. If $\sigma_n = Kn^{-\delta}$ for some $0 < \delta < 1/2$, the critical value will be approximately equal to $(n/(4\delta \log n))^{1/2}$, so, loosely, the critical value is proportional to $\delta^{-1/2}$ times a scaling that does not depend on $\sigma_n$.

The asymptotic distribution derived in Theorem 1 will be attained when the conditional moment inequality binds on $[\underline{x}, \overline{x}]$. If the conditional moment inequality is binding for some, but not all, values of $x$ in $[\underline{x}, \overline{x}]$, the theorem gives an upper bound for the limiting distribution, resulting in a potentially conservative critical value. In practice, one can compute
the critical values setting \( [x, \overline{x}] \) equal to the entire support of \( X_i \), or by using a first stage estimate of this set. Perhaps surprisingly, Theorem 1 shows that using the entire support of \( X_i \) does not lead to an asymptotic increase in the critical value up to a first order comparison relative to using a smaller, nondegenerate set.

4 Conclusion

This paper derives the asymptotic distribution of variance weighted KS statistics for conditional moment inequality models. The results apply to a one dimensional conditioning variable, but could be extended to higher dimensional conditioning variables. The result gives easily computable critical values for these test statistics. The tests give the best available minimax rate for local alternatives in a broad class of models used in empirical economics.

Appendix: Proof of Theorem 1

We derive the asymptotic distribution of (2) when the infimum is taken over a fixed set \( \mathcal{X} \) and the inequality binds on this set. This paper treats the case where \( X_i \) is one dimensional and continuously distributed and \( \mathcal{X} = [x, \overline{x}] \), although the latter assumption can be relaxed without changing the argument in any essential way. We also assume that \( X_i \) has a density bounded from above away from infinity and from below away from zero, although this can be relaxed by transforming \( X_i \).

First, we derive the asymptotic distribution of

\[
\tilde{T}_n \equiv \left\{ \inf_{\{s,t|\tilde{\sigma}(s,t) \geq \overline{\sigma},(s,s+t) \in \mathcal{X}\}} \frac{E_n \tilde{Y}_i I(s < X < s + t)}{\tilde{\sigma}(s,t)} \right\}
\]

where \( \tilde{Y}_i = Y_i - E(Y_i|X_i) \) and \( \tilde{\sigma}^2(s,t) = var(\tilde{Y}_i I(s < X_i < s + t)) \). Then, we show that

\[
\inf_{\tilde{\sigma}(s,t) \geq \overline{\sigma}} \tilde{\sigma}(s,t) - \tilde{\sigma}(s,t)
\]

is bounded from below away from zero when scaled by an appropriate rate, so that replacing \( \tilde{\sigma}(s,t) \) by \( \tilde{\sigma}(s,t) \) will not increase the statistic too much.
Define
\[ \tilde{X}_i = g(X_i) \equiv \int_{\underline{x}}^{X_i} \sigma_Y^2(x) f_X(x) \, dx. \]

Then
\[ E[\tilde{Y}_i^2(0 \leq \tilde{X}_i \leq t)] = E[\tilde{Y}_i^2(0 \leq \tilde{X}_i \leq g^{-1}(t))] = \int_{\underline{x}}^{g^{-1}(t)} \sigma_Y^2(x) f_X(x) \, dx \]
\[ = g(g^{-1})(t) = t. \]

Since monotone transformations of \( X_i \) do not change the statistic, we can work with \( \tilde{X}_i \).

The corresponding value of \( \tilde{\sigma}^2(s, t) \) will be
\[ E(\tilde{Y}_i^2(s < \tilde{X}_i < s + t)) = E(\tilde{Y}_i^2(0 < \tilde{X}_i < s + t)) - E(\tilde{Y}_i^2(0 < \tilde{X}_i < s)) = s + t - s = t. \]

This transformation takes \( \underline{x} \) to 0 and \( \overline{x} \) to
\[ \int_{\underline{x}}^{\overline{x}} \sigma_Y^2(x) f_X(x) \, dx = E(\tilde{Y}_i^2(\underline{x} \leq X_i \leq \overline{x})) \equiv \tilde{x}_u. \]

The following lemma gives an approximation to the process by a Brownian motion using a Skorohod embedding.

**Lemma 1.** Let \( A_n \) be any sequence of sets of \((s, t)\) such that \( t \geq \overline{\sigma}_n \) for all \( n \) and \((s, t) \in A_n\) and let \( a_n \) be an arbitrary sequence going to infinity. There exists a sequence of Brownian motions \( B_n(t) \) such that
\[ \inf_{(s, t) \in A_n} \frac{nE_n\tilde{Y}_i(s < \tilde{X}_i < s + t)}{\sqrt{t}} \geq \inf_{(s, t) \in B_n} \frac{|B_n(s + t) - B_n(s)|}{\sqrt{t}} \cdot (1 + O_P((\log n)^{1/4}/(n^{1/4} \sigma_n^{1/2}))) \]

where \( B_n \) is the set of \((s', t')\) such that \(|s - s'|\) and \(|t' - t|/t\) are both less than \( a_n \cdot (\log n)^{1/2}/(n^{1/2} \sigma_n)\) for some \((s, t) \in A_n\).

**Proof.** Let \( \tilde{X}_{i:n} \) be the \( i \)th least value of \( \tilde{X}_i \), and let \( \tilde{Y}_{i:n} \) be the corresponding value of \( \tilde{Y}_i \). By a Skorohod embedding conditional on the \( X_i \)s, there is a Brownian motion \( B_n(s) \) and
stopping times $\tau_{1,n}, \ldots, \tau_{n,n}$ such that

$$
\frac{1}{\sqrt{n}} S_{n,k} \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^{k} \tilde{Y}_{i:n} = \mathbb{B}_n(\tau_{k,n}/n).
$$

We have

$$
\frac{\sqrt{n} E_n \tilde{Y}_i I(s < \tilde{X}_i < s + t)}{\sqrt{t}} = \frac{1}{\sqrt{n}} \frac{S_{n,k(s+t)} - S_{n,k(s)}}{\sqrt{\tau_{k(s+t),n}/n - \tau_{k(s),n}/n}} \left( \frac{\tau_{k(s+t),n}/n - \tau_{k(s),n}/n}{t} \right)^{1/2}
$$

where $k(s) = \min\{i|\tilde{X}_{i,n} > s\}$.

The second term is the square root of

$$
\frac{\tau_{k(s+t)} - \tau_{k(s)}}{nt} = \frac{1}{nt} \sum_{i=k(s)+1}^{k(s+t)} (\tau_{i,n} - \tau_{i-1,n}). \tag{4}
$$

We have

$$
E(\tau_{i,n} - \tau_{i-1,n}|\tilde{X}_1, \ldots, \tilde{X}_n) = \sigma_Y^2(\tilde{X}_{i:n,n})
$$

so that the expectation of (4) given $\tilde{X}_1, \ldots, \tilde{X}_n$ is

$$
\frac{1}{nt} \sum_{i=k(s)+1}^{k(s+t)} \sigma_Y^2(\tilde{X}_{i:n,n}) = \frac{1}{t} E_n \sigma_Y^2(\tilde{X}_i) I(s < X_i < s + t) = \frac{E_n \sigma_Y^2(\tilde{X}_i) I(s < X_i < s + t)}{E \sigma_Y^2(\tilde{X}_i) I(s < X_i < s + t)}. \tag{5}
$$

This converges to 1 at a $\sqrt{n\sigma_Y^2/\log n}$ by Theorem 37 in Pollard (1984). We can bound the difference between (4) and (5) using Bernstein’s inequality conditional on the $X_i$s. We have, for any $k$ and $k'$,

$$
P\left( \left| \frac{1}{nt} \sum_{i=k}^{k'} (\tau_{i,n} - \tau_{i-1,n}) - E(\tau_{i,n} - \tau_{i-1,n}|\tilde{X}_1, \ldots, \tilde{X}_n) \right| > \varepsilon \right) \leq 2 \exp\left( -\frac{1}{2} \frac{n(t\varepsilon)^2}{K_1 + Mt\varepsilon/3} \right)
$$

where $K_1$ is a bound for the variance of the stopping times and $M$ is a bound for their
support. For some constant $K_2$ and $t \epsilon$ bounded from above, this is less than

$$2 \exp(-n(t \epsilon)^2/K_2) \leq 2 \exp(-n(\sigma_n \epsilon)^2/K_2).$$

Setting $\epsilon = K_3 \sqrt{(\log n)/(n \sigma_n^2)}$ for $K_3$ large enough, this gives a bound of $2 \exp(-K_3^2 \log n/K_2)$. The probability of the supremum of the difference between (4) and (5) being greater than $\epsilon$ is bounded by $n^2$ times this quantity $((k(s), k(s + t))$ can take on no more than $n^2$ values), and this can be made to go to zero with $n$ by making $K_3$ large enough.

This argument shows that the following event holds with probability approaching one for any sequence $a_n \to \infty$. For any $(s, t) \in A_n$, there are $(s', t')$ with $|s - s'|$ and $|t' - t|/t$ less than $a_n \cdot (\log n)^{1/2}/(n^{1/2} \sigma_n)$ such that

$$\sqrt{n}E_n \tilde{Y}_i I(s < \tilde{X}_i < s + t) = [\mathbb{B}_n(s + t) - \mathbb{B}_n(s)]/\sqrt{t} \cdot (1 + O_p((\log n)^{1/4}/(n^{1/4} \sigma^2/n))).$$

This completes the proof.

We apply this lemma with

$$A_n = \{(s, t) | (s, s + t) \in [0, \tilde{x}_n], t \geq \sigma_n^2\}$$

to get a lower bound of

$$\inf_{0 \leq s \leq s + t \leq \tilde{x}_u + b_n, t \geq \sigma_n^2/(1 - b_n)} [\mathbb{B}_n(s + t) - \mathbb{B}_n(s)]/\sqrt{t} \cdot (1 + b_n^{1/2})$$

with probability approaching one for the term in (3) where $b_n = a_n \cdot (\log n)^{1/2}/(n^{1/2} \sigma_n)$. By the invariance properties of the Brownian motion, this has the same distribution as

$$\inf_{0 \leq s \leq s + t \leq (\tilde{x}_u + b_n)/[\sigma_n^2(1 - b_n)], t \geq 1} [\mathbb{B}(s + t) - \mathbb{B}(s)]/\sqrt{t} \cdot (1 + b_n^{1/2})$$

for a Brownian motion $\mathbb{B}$. By arguments in Kabluchko (2011), we have, letting $c'_n = (\tilde{x}_u + b_n)/[\sigma_n^2(1 - b_n)]$

$$P \left( \inf_{0 \leq s \leq s + t \leq (\tilde{x}_u + b_n)/[\sigma_n^2(1 - b_n)], t \geq 1} [\mathbb{B}(s + t) - \mathbb{B}(s)]/\sqrt{t} \leq (2 \log c'_n)^{1/2} + \frac{3}{2} \log \log c'_n - \log(2\pi^{1/2} + r) \right) \to \exp(-\exp(-r)).$$
This will hold with $c'_n$ replaced by $c_n = \tilde{x}_u/\tilde{\sigma}_n^2$ as long as $c_n/c'_n \to 1$, which will hold as long as $b_n \to 0$. This gives
\[
P \left( \sqrt{n}I_n/(1 + b_n^{1/2}) \leq (2 \log c_n)^{1/2} + \frac{3}{2} \log \log c_n - \log(2\pi^{1/2}/r) \right) \nonumber \to \infty \exp(- \exp(-r)).
\]

This will hold with $b_n$ replaced by 0 if
\[
b_n^{1/2} \log c_n = a_n^{1/2} \cdot (\log n)^{1/4}/(n^{1/4} \tilde{\sigma}_n^{1/2}) \log(\tilde{x}_u/\tilde{\sigma}_n) \to 0
\]
which will hold for $a_n$ increasing slowly enough as long as
\[
\frac{\sigma_n n^{1/2}}{(\log n)^{5/2}} \to \infty.
\]

To get the same extreme value limit with $\sigma(s, t)$ replaced by $\hat{\sigma}(s, t)$, it suffices to show that
\[
\frac{\hat{\sigma}(s, t) - \bar{\sigma}(s, t)}{\bar{\sigma}(s, t)} \to 1
\]
converges to 1 at a $(\log n)$ rate uniformly in $\tilde{\sigma}(s, t)$ such that $\tilde{\sigma}(s, t) \geq \tilde{\sigma}_n^2/2$. To show that replacing $\sigma(s, t)$ with its estimate gives a limit that is asymptotically no greater than an extreme value random variable, it suffices to show that this quantity is bounded from below by $1 - o_p(\log n)$ term, and converges to 1 at any rate (the latter condition is needed for the sets on which the infimum is taken to converge quickly enough). We have
\[
E\hat{\sigma}^2(s, t) = \text{var}(Y_i I(s < X < s + t))
= E[\text{var}(Y_i|X_i)I(s < X < s + t)] + \text{var}(E(Y_i|X_i)I(s < X < s + t))
= \hat{\sigma}^2(s, t) + \text{var}(E(Y_i|X_i)I(s < X < s + t)).
\]
By Theorem 37 in Pollard (1984), $(\hat{\sigma}^2(s, t) - E\hat{\sigma}^2(s, t))/\hat{\sigma}^2(s, t)$ converges to 1 at a $(n\tilde{\sigma}_n^2/\log n)^{1/2}$ rate uniformly in $\hat{\sigma}(s, t) \geq \sigma_n/2$, which translates into at least a $(n\tilde{\sigma}_n^2/\log n)^{1/4}$ rate for
\[
\frac{\hat{\sigma}(s, t) - \sqrt{E\hat{\sigma}^2(s, t)}}{\hat{\sigma}(s, t)} \geq \frac{\hat{\sigma}(s, t) - \bar{\sigma}(s, t)}{\bar{\sigma}(s, t)},
\]
which will be fast enough as long as (6) holds. For (7) to converge to 1, it now suffices to have

$$\frac{\text{var}(E(Y_i|X_i)I(s < X < s + t))}{\sigma^2(s, t)} \leq C \text{var}(E(Y_i|X_i)I(s < X < s + t))/t \to 0.$$  

Consistency of $\hat{\sigma}(s, t)$ also implies that $\hat{c}_n/c_n$ converges in probability to one, which allows $c_n$ to be replaced with $\hat{c}_n$.

References

ANDREWS, D. W., S. BERRY, AND P. JIA (2004): “Confidence regions for parameters in discrete games with multiple equilibria, with an application to discount chain store location,” .

ANDREWS, D. W., AND P. GUGGENBERGER (2009): “Validity of Subsampling and ?plug-in Asymptotic? Inference for Parameters Defined by Moment Inequalities,” Econometric Theory, 25(03), 669–709.

ANDREWS, D. W., AND X. SHI (2009): “Inference Based on Conditional Moment Inequalities,” Unpublished Manuscript, Yale University, New Haven, CT.

ANDREWS, D. W. K., AND P. JIA (2008): “Inference for Parameters Defined by Moment Inequalities: A Recommended Moment Selection Procedure,” SSRN eLibrary.

ANDREWS, D. W. K., AND G. SOARES (2010): “Inference for Parameters Defined by Moment Inequalities Using Generalized Moment Selection,” Econometrica, 78(1), 119–157.

ARMSTRONG, T. (2011a): “Asymptotically Exact Inference in Conditional Moment Inequality Models,” Unpublished Manuscript.

——— (2011b): “Weighted KS Statistics for Inference on Conditional Moment Inequalities,” Unpublished Manuscript.

BERESTEANU, A., AND F. MOLINARI (2008): “Asymptotic Properties for a Class of Partially Identified Models,” Econometrica, 76(4), 763–814.

BUGNI, F. A. (2010): “Bootstrap Inference in Partially Identified Models Defined by Moment Inequalities: Coverage of the Identified Set,” Econometrica, 78(2), 735–753.
Chernozhukov, V., H. Hong, and E. Tamer (2007): “Estimation and Confidence Regions for Parameter Sets in Econometric Models,” *Econometrica*, 75(5), 1243–1284.

Chernozhukov, V., S. Lee, and A. M. Rosen (2009): “Intersection bounds: estimation and inference,” *Arxiv preprint arXiv:0907.3503*.

Chetverikov, D. (2012): “Adaptive Test of Conditional Moment Inequalities,”

Imbens, G. W., and C. F. Manski (2004): “Confidence Intervals for Partially Identified Parameters,” *Econometrica*, 72(6), 1845–1857.

Kabluchko, Z. (2011): “Extremes of the standardized Gaussian noise,” *Stochastic Processes and their Applications*, 121(3), 515–533.

Khan, S., and E. Tamer (2009): “Inference on endogenously censored regression models using conditional moment inequalities,” *Journal of Econometrics*, 152(2), 104–119.

Kim, K. i. (2008): “Set estimation and inference with models characterized by conditional moment inequalities,”

Lee, S., K. Song, and Y. Whang (2011): “Testing functional inequalities,”

Menzel, K. (2008): “Estimation and Inference with Many Moment Inequalities,” *Preprint, Massachusetts Institute of Technology*.

Moon, H. R., and F. Schorfheide (2009): “Bayesian and Frequentist Inference in Partially Identified Models,” *National Bureau of Economic Research Working Paper Series*, No. 14882.

Pollard, D. (1984): *Convergence of stochastic processes*. David Pollard.

Ponomareva, M. (2010): “Inference in Models Defined by Conditional Moment Inequalities with Continuous Covariates,”

Romano, J. P., and A. M. Shaikh (2008): “Inference for identifiable parameters in partially identified econometric models,” *Journal of Statistical Planning and Inference*, 138(9), 2786–2807.

Romano, J. P., and A. M. Shaikh (2010): “Inference for the Identified Set in Partially Identified Econometric Models,” *Econometrica*, 78(1), 169–211.
STOYE, J. (2009): “More on Confidence Intervals for Partially Identified Parameters,”
Econometrica, 77(4), 1299–1315.