Radiative Corrections to the $2E1$ Decay Rate of the $2s$-State in Hydrogen-Like Atoms

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Abstract

Radiative corrections to the $2E1$ decay width of the $2s_{1/2}$-state in the low-$Z$ hydrogen-like system are examined within logarithmic approximation. The correction is found to be $2.025(1) \alpha(Z\alpha)^2 / \pi \log(Z\alpha)^2$ in units of the non-relativistic rate.
A leading radiative correction to the $E1$ one-photon decay width of the hydrogenic levels was analyzed recently by the authors in Refs. [1, 2]. The correction has order of $\alpha(Z\alpha)^2 \log(Z\alpha)$ in relative units. This work is devoted to the leading radiative correction to the $2E1$ two-photon decay of the metastable $2s_{1/2}$ state in hydrogen-like atoms, which has the same relative order of the magnitude.

The rate of the $2s$ metastable state decay in the hydrogen atom was first examined in Ref. [3] and the investigations were developed in a number of works (see Refs. [4, 5, 6, 7] etc). The decay of the level in the atom with one Schrödinger electron and the infinite nuclear mass was determined with a high precision in Ref. [5]. Calculations on the atom with one Dirac electron were performed in Ref. [6]. The nuclear mass corrections were carried out in Ref. [7], where the results of Ref. [8] for the $E1$ decays were adjusted for the two-photon $2E1$ transition. However, any radiative corrections have not yet been discussed previously. In this work a calculation is made of the leading radiative corrections of the relative order of $\alpha(Z\alpha)^2$ to the $2E1$ two-photon decay rates in the low-$Z$ hydrogen-like systems within the logarithmic approximation.

The evaluation of the $\alpha(Z\alpha)^2 \log(Z\alpha)$ correction can be essentially simplified by using the Yennie gauge [9] for virtual photons, in which their propagator in the momentum space has the form

$$D_{\mu\nu}^Y(k) = \frac{1}{k^2} \left( g_{\mu\nu} + 2\frac{k_{\mu}k_{\nu}}{k^2} \right).$$

In this gauge it is enough to take into account only the logarithmic corrections to the energy and the wave functions. Both
of them can be described with using an effective local delta-like potential\footnote{Relativistic units in which $\hbar = c = 1$ and $\alpha = e^2$ are used and $Z$ is the nuclear charge in units of the proton one.}

$$V(\mathbf{r}) = \frac{4}{3} \frac{\alpha(Z\alpha)}{m^2} \log \frac{1}{(Z\alpha)^2} \delta(\mathbf{r}),$$

(1)

leading to zero result for all, but $s$ states (see for details \cite{10,11}).

As it is well known (see e.g. Ref. \cite{11}) the two-photons decay rate of the $2s$-state in the non-relativistic approximation is of the form

$$\Gamma^{(0)}_{2\gamma}(2s_{1/2}) = C \int_0^1 \, dy \, y^3 (1-y)^3 \left| \sum_{q \neq 2} d_{1q} d_{2q} f_q(y) \right|^2,$$

(2)

where

$$C = \frac{4 \alpha^2 (E_{2s} - E_{1s})^5}{3 \pi} \frac{1}{(Z\alpha m)^4},$$

(3)

$$d_{n'n} = \langle n's||Z\alpha mr||np \rangle,$$

(4)

and the sum has to run over all intermediate $p$-states. For the discrete $p$-states the function in eq. (2) is equal to

$$f_n(y) = \frac{1}{1 - 4/n^2 + 3y} + \frac{1}{4 - 4/n^2 - 3y}.$$

(5)

The correction can be written as a sum

$$\Delta \Gamma^{Rad}_{2\gamma}(2s) = \frac{8}{3} \frac{\alpha(Z\alpha)^2}{\pi} \log \frac{1}{(Z\alpha)^2} (R^c + R^f + R^d) \Gamma^{(0)}_{2\gamma}(2s),$$

(6)
where the contributions labeled by the superscripts $C$, $f$ and $d$ associate with the corrections to the constant $C$ (eq. (3)), the function $f_q(y)$ (eq. (5)) and the reduced dipole matrix elements of eq. (4). The results for two first terms presented in Table are easy to compute in the same techniques as the leading contribution of eq. (2).

The calculation of the third term can be carried out with using analytic expressions for the radiative corrections to dipole matrix elements in the Coulomb field. For the discrete $p$-state the expression

$$
\delta d_{n'n'} = \frac{\Delta E_{1s}}{(Z\alpha)^2m} \frac{(-1)^{n'+1}}{24} \sqrt{\frac{n^3(n^2-1)}{n'^3}} \times
$$

$$
x^4 \frac{n'}{\Gamma(n')} \sum_{s=0}^{n'} y^{n'-s} \frac{(-1)^s}{s!} \left[ \frac{\Gamma(n'+1)}{\Gamma(n'+1-s)} \right]^2 \frac{n'-s}{n'} \Gamma(n'-s+4) \times
$$

$$
\left\{ \left[ \psi(n'-s+4) - \psi(n') + 2\psi(n'+1) - 2\psi(n'+1-s) + \log y \right.ight.
$$

$$
- \frac{n'-s}{n+n'} + \frac{4n}{n(n+n')} + \frac{s}{n(n'-s)} - \frac{3}{2n'} \right\} 2F_1(n'-s+4, 2-n, 4, x)
$$

$$
+ 2F_{1:a}(n'-s+4, 2-n, 4, x) + \frac{nx}{n'(n+n')} 2F_1'(n'-s+4, 2-n, 4, x) \right\},
$$

where

$$
x = \frac{2n'}{n+n'},
$$

(7)
\[ y = \frac{2n}{n + n'} , \]

\[ 2F_{1:a}(a_0, b, c, x) = \left[ \frac{\partial}{\partial a} 2F_1(a, b, c, x) \right]_{a=a_0} , \]

\[ 2F_1'(a, b, c, x) = \frac{\partial}{\partial x} 2F_1(a, b, c, x) , \]

and

\[ \psi(z) = \Gamma'(z)/\Gamma(z) \]

were obtained earlier in Ref. [2].

One can perform the analytic continuation of the corrections \( \delta d_{1n} \) and \( \delta d_{2n} \) as functions on \( n \) to the continuous \( p \)-states. The sum done over all intermediate \( p \)-states of the discrete and continuous parts of the spectrum is evaluated numerically and the result is summarized in Table.

Our final result of the correction to the decay rate of the \( 2s_{1/2} \rightarrow 1s_{1/2} \) transition is found to be

\[ \frac{\Delta \Gamma_{2\gamma}^{Rad}(2s_{1/2})}{\Gamma_{2\gamma}^{(0)}(2s_{1/2})} = -2.025(1) \frac{\alpha(Z\alpha)^2}{\pi} \log \frac{1}{(Z\alpha)^2} . \quad (8) \]

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References

[1] S. G. Karshenboim, ZhETF \textbf{107} (1995) 1061 /in Russian/; JETP \textbf{80} (1995) 593.

[2] V. G. Ivanov and S. G. Karshenboim, ZhETF \textbf{109} (1996) 1219 /in Russian/; JETP \textbf{82} (1996) 656; Phys. Lett. \textbf{A210} (1996) 313.

[3] G. Breit and E. Teller, Astrophys. J. \textbf{91} (1940) 215.

[4] L. Spitzer and J. L. Greenstein, Astrophys. J. \textbf{114} (1951) 407; J. Shapiro and G. Breit, Phys. Rev. \textbf{113} (1959) 179; B. A. Zon and N. L. Manakov, Pis’ma ZhETF \textbf{7} (1968) 70 /in Russian/; JETP Lett. \textbf{7} (1968) 52.

[5] S. Klarsfeld, Phys. Lett. \textbf{30A} (1969) 382.

[6] S. P. Goldman and G. W. F. Drake, Phys. Rev. \textbf{A24} (1981) 183; F. A. Parpia and W. R. Johnson, Phys. Rev. \textbf{A26} (1982) 1142.

[7] R. Bacher, Z. Phys. \textbf{315} (1984) 1351.

[8] Z. Fried and A. D. Martin, Nuovo Cim. \textbf{29} (1963) 574.

[9] H. M. Fried and D. R. Yennie, Phys. Rev. \textbf{112} (1958) 1391.

[10] S. G. Karshenboim, ZhETF \textbf{106} (1994) 414 /in Russian/; JETP \textbf{79} (1994) 230; Yad. Fiz. \textbf{58} (1995) 901 /in Russian/; Phys. At. Nucl. \textbf{58} (1995) 835.
[11] V. B. Berestetskii, E. M. Lifshitz and L. P. Pitaevskii. 
Quantum Electrodynamics. Pergamon Press, Oxford (1982).
| $R^C$  | $R^f$    | $R^d$    | $R^{tot}$ |
|-------|---------|---------|---------|
| -5.8333 | 2.2115(1) | 2.8625(2) | -0.7593(2) |

**Caption to Table.**

Radiative corrections to the $2s_{1/2}$ decay rate