Characterization of convex $\mu$-compact sets

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Abstract

The class of $\mu$-compact sets can be considered as a natural extension of the class of compact metrizable subsets of locally convex spaces, to which the particular results well known for compact sets can be generalized. This class contains all compact sets as well as many noncompact sets widely used in applications. In this paper we give a characterization of a convex $\mu$-compact set in terms of properties of functions defined on this set. Namely, we prove that the class of convex $\mu$-compact sets can be characterized by continuity of the operation of convex closure of a function (= the double Fenchel transform) with respect to monotonic pointwise converging sequences of continuous bounded and of lower semicontinuous lower bounded functions.

The properties of compact sets in the context of convex analysis have been studied by many authors (see [1, 2, 3] and the references therein). It is natural to ask about possible generalizations of the results proved for compact convex sets to noncompact sets. In [4] one such generalization concerning the particular class of sets called $\mu$-compact sets is considered. In [4, 5] it is shown that for this class of sets, which includes all compact convex sets as well as some noncompact sets widely used in applications, many results of the Choquet theory [1] and of the Vesterstrom-O’Brien theory [2, 3] can be proved. In this paper we give a characterization of a convex $\mu$-compact set in terms of properties of functions defined on this set.

In what follows $\mathcal{A}$ is a bounded convex complete separable metrizable subset of some locally convex space. Let $C(\mathcal{A})$ be the set of all continuous bounded functions on the set $\mathcal{A}$ and $M(\mathcal{A})$ be the set of all Borel probability measures on the set $\mathcal{A}$ endowed with the weak convergence topology [6, Chapter II, §6]. Let $\text{co}f$ and $\overline{\text{co}}f$ be the convex hull and the

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1This means that the topology on the set $\mathcal{A}$ is defined by a countable subset of the family of seminorms, generating the topology of the entire locally convex space, and this set is separable and complete in the metric generated by this subset of seminorms.
convex closure of a function $f$, which are defined respectively as the maximal convex and the maximal convex closed (that is, lower semicontinuous) functions majorized by $f$.\footnote{The convex closure of a function is also called the lower (convex) envelope of this function.}

With an arbitrary measure $\mu \in M(\mathcal{A})$ we associate its barycenter (average) $b(\mu) \in \mathcal{A}$, which is defined by the Pettis integral (see \cite{8,9})

$$b(\mu) = \int_{\mathcal{A}} x \mu(dx). \quad (1)$$

For arbitrary $x \in \mathcal{A}$ let $M_x(\mathcal{A})$ be a convex closed subset of the set $M(\mathcal{A})$ consisting of such measures $\mu$ that $b(\mu) = x$.

The barycenter map

$$M(\mathcal{A}) \ni \mu \mapsto b(\mu) \in \mathcal{A} \quad (2)$$

is continuous (this can be shown easily by applying Prokhorov’s theorem \cite[Ch.II, Th.6.7]{6}). Hence the image of any compact subset of $M(\mathcal{A})$ under this map is a compact subset of $\mathcal{A}$. The $\mu$-compact sets are defined in \cite{4,5} by the converse requirement.

**Definition.** A set $\mathcal{A}$ is called $\mu$-compact if the preimage of any compact subset of $\mathcal{A}$ under barycenter map (2) is a compact subset of $M(\mathcal{A})$.

Any compact set is $\mu$-compact, since compactness of $\mathcal{A}$ implies compactness of $M(\mathcal{A})$ \cite{6}. The $\mu$-compactness property is studied in detail in \cite{5}, where simple criteria of this property have been established. By using these criteria $\mu$-compactness of the following noncompact sets has been proved:

- the positive parts of the unit balls of the Banach space $\ell_1$ and of the Banach space $\mathfrak{T}(\mathcal{H})$ of trace class operators in a separable Hilbert space $\mathcal{H}$;

- the set of positive Borel measures on an arbitrary complete separable metric space with the total variation $\leq 1$ endowed with the weak convergence topology;

- the positive parts of the unit balls of the Banach spaces of linear bounded operators in $\ell_1$ and in $\mathfrak{T}(\mathcal{H})$ endowed with the strong operator topology.

In particular, this implies $\mu$-compactness of the set of all Borel probability measures on an arbitrary complete separable metric space endowed with the weak convergence topology, of the set of quantum states and of the set of quantum operations endowed with the strong operator topology \cite{10}.

It is essential to note that the $\mu$-compactness property of a convex set is not purely topological but reflects the special relation between the topology and the convex structure of this set \cite{5}.

The following theorem shows that the class of convex $\mu$-compact sets can be characterized by continuity of the operation of convex closure (coinciding with the double Fenchel transform) with respect to monotonic pointwise converging sequences of functions.
Theorem. The following properties are equivalent:

(i) the set \( \mathcal{A} \) is \( \mu \)-compact;

(ii) for an arbitrary increasing sequence \( \{f_n\} \subset C(\mathcal{A}) \), converging pointwise to a function \( f_0 \in C(\mathcal{A}) \), the sequence \( \{\overline{\mu} f_n\} \) converges pointwise to the function \( \overline{\mu} f_0 \);

(iii) for an arbitrary increasing sequence \( \{f_n\} \) of lower semicontinuous lower bounded functions on \( \mathcal{A} \), converging pointwise to a function \( f_0 \), the sequence \( \{\overline{\mu} f_n\} \) converges pointwise to the function \( \overline{\mu} f_0 \).

If these equivalent properties hold then for an arbitrary decreasing sequence \( \{f_n\} \) of lower semicontinuous bounded functions on the set \( \mathcal{A} \), converging pointwise to a lower semicontinuous bounded function \( f_0 \), the sequence \( \{\overline{\mu} f_n\} \) converges pointwise to the function \( \overline{\mu} f_0 \).

Remark 1. The functions \( \overline{\mu} f_n \) in (ii) are not necessarily continuous (only lower semicontinuous). By the generalized Vesterstrom-O’Brian theorem (Theorem 1 in [5]) these functions are continuous provided the set \( \mathcal{A} \) is \( \mu \)-compact and stable (the last property means openness of the convex mixing map \( \mathcal{A} \times \mathcal{A} \ni (x, y) \mapsto \frac{1}{2}(x + y) \in \mathcal{A} \) [11]).

Proof. It is sufficient to show that (i) \( \Rightarrow \) (iii) and (ii) \( \Rightarrow \) (i).

(i) \( \Rightarrow \) (iii) Let \( \{f_n\} \) be an increasing sequence of lower semicontinuous lower bounded functions on the set \( \mathcal{A} \), converging pointwise to a function \( f_0 \). We may assume that this sequence consists of nonnegative functions. It suffices to show that the assumption on existence of such \( x_0 \in \mathcal{A} \) that

\[ \overline{\mu} f_n(x_0) \leq \overline{\mu} f_0(x_0) - \Delta, \quad \Delta > 0, \quad \forall n, \]

where ”\( \leq +\infty - \Delta \)” means ”\( \leq \Delta \)”, leads to a contradiction.

By Proposition 6 in [5] we have

\[ \overline{\mu} f_n(x_0) = \inf_{\mu \in M_0(\mathcal{A})} \mu(f_n), \quad n = 0, 1, 2, \ldots, \] \( \mu(f) = \int_{\mathcal{A}} f(y) \mu(dy), \) \( \overline{\mu} f_0(x_0) = \mu_n(f_n). \) \( \overline{\mu} f_0(x_0) = \mu_n(f_n). \)

and this infimum is attained at a particular measure \( \mu_n \in M_{x_0}(\mathcal{A}) \), t.i. \( \overline{\mu} f_n(x_0) = \mu_n(f_n). \)

For definiteness suppose \( \overline{\mu} f_0(x_0) < +\infty \) (the case \( \overline{\mu} f_0(x_0) = +\infty \) is considered similarly). By Fenchel’s theorem (see [7]) there exists a continuous affine function \( \alpha \) on \( \mathcal{A} \) such that

\[ \alpha(x) \leq f_0(x), \quad \forall x \in \mathcal{A}, \] \( \overline{\mu} f_0(x_0) = \alpha(x_0) + \frac{1}{2} \Delta. \)

Since the function \( \alpha \) is affine, we have

\[ \mu_n(\alpha) - \mu_n(f_n) = \alpha(x_0) - \overline{\mu} f_n(x_0) \]

\[ = [\alpha(x_0) - \overline{\mu} f_0(x_0)] + [\overline{\mu} f_0(x_0) - \overline{\mu} f_n(x_0)] \geq -\frac{1}{2} \Delta + \Delta = \frac{1}{2} \Delta. \]
The assumed \( \mu \)-compactness of the set \( \mathcal{A} \) implies relative compactness of the sequence \( \{ \mu_n \} \). By Prokhorov's theorem this sequence is \( \text{tight} \), which means that for any \( \varepsilon > 0 \) there exists such compact set \( \mathcal{K}_\varepsilon \subset \mathcal{A} \) that \( \mu(\mathcal{A} \setminus \mathcal{K}_\varepsilon) < \varepsilon \) \([6]\). Let

\[
M = \sup_{x \in \mathcal{A}} |\alpha(x)| \quad \text{and} \quad \varepsilon_0 = \frac{1}{4M}\Delta.
\]

By using \([5]\) we obtain

\[
\int_{\mathcal{K}_{\varepsilon_0}} (\alpha(x) - f_n(x))\mu_n(dx) \geq \frac{1}{2}\Delta - \int_{\mathcal{A} \setminus \mathcal{K}_{\varepsilon_0}} (\alpha(x) - f_n(x))\mu_n(dx) \geq \frac{1}{4}\Delta.
\]

Hence the set \( \mathcal{C}_n = \{ x \in \mathcal{K}_{\varepsilon_0} | \alpha(x) \geq f_n(x) + \frac{1}{4}\Delta \} \) is not empty for all \( n \).

\[\text{(ii) } \Rightarrow \text{(i)} \quad \text{Suppose the set } \mathcal{A} \text{ is not } \mu \text{-compact. Then there exist a sequence } \{ \mu_k \} \subset M(\mathcal{A}), \text{ which is not relatively compact and such that the sequence } \{ x_k = b(\mu_k) \} \text{ converges. By the below Lemma 1 one can consider that this sequence consists of finitely supported measures. By Prokhorov's theorem the sequence } \{ \mu_k \} \text{ is not tight. The below Lemma 2 (with the remark after it) guarantees existence of such } \varepsilon > 0 \text{ and } \delta > 0 \text{ that for any compact set } \mathcal{K} \subset \mathcal{A} \text{ and any natural } N \text{ there is such } k > N \text{ that } \mu_k(U_\delta(\mathcal{K})) < 1 - \varepsilon, \text{ where } U_\delta(\mathcal{K}) \text{ is the closed } \delta \text{-vicinity of the set } \mathcal{K} \text{ (as a subset of the metric space } \mathcal{A}) \text{. Let } \{ \mathcal{K}_n \} \text{ be an increasing sequence of compact subsets of } \mathcal{A} \text{ such that } \bigcup_{n \in \mathbb{N}} U_{\delta/2}(\mathcal{K}_n) = \mathcal{A}. \text{ Denote by } d(\cdot, \cdot) \text{ the metric in } \mathcal{A}. \text{ For each natural } n \text{ consider the continuous bounded function}
\]

\[
f_n(x) = 1 - 2\delta^{-1}\inf_{y \in U_{\delta/2}(\mathcal{K}_n)} d(x, y)
\]

on the set \( \mathcal{A} \) such that \( f_n(x) = 1, \) if \( x \in U_{\delta/2}(\mathcal{K}_n), \) and \( f_n(x) < 0, \) if \( x \in \mathcal{A} \setminus U_{\delta}(\mathcal{K}_n). \) It is clear that \( f_0(x) = \lim_{n \to +\infty} f_n(x) \equiv 1 \) and hence \( \overline{\text{co}}f_0(x) \equiv 1. \) Let \( x_0 \) be a limit of the sequence \( \{ x_k \}. \) To obtain a contradiction it suffices to show that

\[
\overline{\text{co}}f_n(x_0) < 1 - \varepsilon \quad \forall n \in \mathbb{N}.
\]

By the above-stated property of the sequence \( \{ \mu_k \} \) for each \( n \) and any natural \( N \) there exists such \( k > N \) that \( \mu_k(U_{\delta}(\mathcal{K}_n)) < 1 - \varepsilon \) and hence

\[
\overline{\text{co}}f_n(x_k) \leq \text{co}f_n(x_k) \leq \int_{\mathcal{A}} f_n(x)\mu_k(dx) < 1 - \varepsilon,
\]

since \( \mu_k \) is a measure with finite support. This inequality and lower semicontinuity of the function \( \overline{\text{co}}f_n \) imply \([7]\).

By using representation \([3]\) and the monotonic convergence theorem, it is easy to prove the last assertion of the theorem. \( \square \)

In the above proof the following assertions (well known in the measure theory) are used.

**Lemma 1.** For an arbitrary sequence \( \{ \mu_k \} \subset M(\mathcal{A}), \) which is not relatively compact, there exists a sequence \( \{ \tilde{\mu}_k \} \subset M(\mathcal{A}) \) of finitely supported measures, which is not relatively

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4
compact as well, such that $b(\tilde{\mu}_k) = b(\mu_k)$ for all $k$.

Since the set $M(A)$ can be considered as a complete separable metric space [6, Chapter II], the above lemma is easily proved by using density of the set of finitely supported measures in $M_x(A)$ for each $x \in A$ [4, Lemma 1].

**Lemma 2.** A subset $M_0 \subseteq M(A)$ is tight if and only if for any $\varepsilon > 0$ and $\delta > 0$ there exists a compact subset $K(\varepsilon, \delta) \subseteq A$ such that

$$\mu(\bigcup_{n \in \mathbb{N}} U_{\delta}(K_n)) \geq 1 - \varepsilon$$

for all $\mu \in M_0$, where $U_{\delta}(K(\varepsilon, \delta))$ is the closed $\delta$-vicinity of the set $K(\varepsilon, \delta)$.

Since any finite subset of $M(A)$ is tight, the words ”for all $\mu \in M_0”$ in the above criterion may be replaced by ”for all $\mu \in M_0 \setminus M'$, where $M'$ is a finite subset of $M_0”$.

**Proof.** It is easy to see that tightness of the set $M_0$ implies validity of the condition in the lemma. Suppose this condition holds. For arbitrary $\varepsilon > 0$ and each $n \in \mathbb{N}$ let $K_n = K(\varepsilon 2^{-n}, \varepsilon 2^{-n})$. Then for the compact set $K = \bigcap_{n \in \mathbb{N}} U_{\varepsilon 2^{-n}}(K_n)$ we have

$$\mu(A \setminus K) \leq \sum_{n=1}^{+\infty} \mu(A \setminus U_{\varepsilon 2^{-n}}(K_n)) < \sum_{n=1}^{+\infty} \varepsilon 2^{-n} < \varepsilon$$

for all $\mu \in M_0$, which means that the set $M_0$ is tight. □

It is well known that an arbitrary lower semicontinuous lower bounded function on a metric space can be represented as a pointwise limit of some increasing sequence of continuous bounded functions [9]. By using the above Theorem and the generalized Vesterstrom-O’Brien theorem mentioned in Remark 1 this observation can be strengthened as follows.

**Corollary.** An arbitrary lower semicontinuous lower bounded convex (correspondingly, concave) function on a stable $\mu$-compact set $A$ can be represented as a pointwise limit of some increasing sequence of convex (correspondingly, concave) continuous bounded functions.

**Remark 2.** In [5] the weaker version of the $\mu$-compactness property of a set $A$ defined by the requirement of compactness of the set $M_x(A)$ for each $x \in A$ is considered. This property called pointwise $\mu$-compactness was used to show that even slight relaxing of the $\mu$-compactness condition in the generalized Vesterstrom-O’Brien theorem leads to breaking its validity. The class of pointwise $\mu$-compact sets is wider than the class of $\mu$-compact sets, in particular, it contains the simplex $\{\{x_i\}_{i=1}^{+\infty} \mid x_i \geq 0, \forall i, \sum_{i=1}^{+\infty} x_i \leq 1\} \subset \ell_p$ for any $p \geq 1$, which is $\mu$-compact only for $p = 1$ [5, Proposition 13]. For an arbitrary convex pointwise $\mu$-compact set the assertions of the Krein-Milman theorem and of the Choquet theorem are valid [5, Proposition 5], but representation (3) for the convex closure of a lower semicontinuous lower bounded function $f$ does not hold in general [5, Example 1].
Similar to a convex $\mu$-compact set a convex pointwise $\mu$-compact set can be characterized in terms of properties of functions defined on this set. Namely, one can show that the following properties are equivalent:

(i) the set $\mathcal{A}$ is pointwise $\mu$-compact;

(ii) for an arbitrary increasing sequence $\{f_n\} \subset C(\mathcal{A})$, converging pointwise to a function $f_0 \in C(\mathcal{A})$, the sequence $\{\text{co}f_n\}$ converges pointwise to the function $\text{co}f_0$.

Since

$$\text{co}f_n(x) = \inf_{\mu \in M_f^f(\mathcal{A})} \int_{\mathcal{A}} f_n(y) \mu(dy), \quad x \in \mathcal{A}, \quad n = 0, 1, 2, \ldots,$$

where $M_f^f(\mathcal{A})$ is a subset of $M_\mathcal{A}$ consisting of finitely supported measures, the implication (i) $\Rightarrow$ (ii) in the above assertion is proved by noting that pointwise $\mu$-compactness of $\mathcal{A}$ and Prokhorov’s theorem implies tightness of $M_f^f(\mathcal{A})$ and by using Dini’s lemma. The implication (ii) $\Rightarrow$ (i) can be established by using the proof of the implication (ii) $\Rightarrow$ (i) in the Theorem with $x_k = x_0$ for all $k$.

The above characterization of a convex pointwise $\mu$-compact set, the Theorem and Corollary 2 in [5] show that

$$\{\text{pointwise } \mu\text{-compactness of } \mathcal{A}\} \land \{\text{co}f = \text{co}f \forall f \in C(\mathcal{A})\} = \{\mu\text{-compactness of } \mathcal{A}\}.$$

The implication (i) $\Rightarrow$ (iii) in the Theorem can be used in study of entropic characteristics of quantum states [12, §6.2].

Bibliography

[1] E.Alfsen, ”Compact convex sets and boundary integrals”, Springer Verlag, 1971.

[2] J.Vesterstrom, ”On open maps, compact convex sets and operator algebras”, J. London Math. Soc. V.6, N.2, P.289-297, 1973.

[3] R.O’Brien, ”On the openness of the barycentre map”, Math. Ann., V.223, N.3, P.207-212, 1976.

[4] M.E.Shirokov, ”On the strong CE-property of convex sets”, Mathematical Notes, V.82, N.3, P.395-409, 2007.

[5] V.Yu.Protasov, M.E.Shirokov, ”Generalized compactness in linear spaces and its applications”, Sbornik:Mathematics V.200, N.5, P.697-722, 2009; [arXiv:1002.3610] [math-ph].

[6] K.Parthasarathy, ”Probability measures on metric spaces”, Academic Press, New York and London, 1967.
[7] A.D. Joffe, W.M. Tikhomirov, "Theory of extremum problems", AP, NY, 1979.

[8] N.N. Vahania, V.I. Tarieladze, "Covariant operators of probability measures in locally convex spaces", Theory of Probability and its Applications, V.23, N.1, P.1-23, 1978.

[9] C.D. Aliprantis, K.C. Border, "Infinite dimensional analysis", Springer Verlag, 2006.

[10] A.S. Holevo, "Statistical structure of quantum theory", Springer Verlag, 2001.

[11] S. Papadopoulou, "On the geometry of stable compact convex sets", Math. Ann. V.229, P.193-200, 1977.

[12] M.E. Shirokov, "On properties of the space of quantum states and their application to construction of entanglement monotones", Izvestiya: Mathematics, V.74, N.4, 2010; arXiv:0804.1515 [math-ph].