$L^\infty$ NORMS OF HUSIMI DISTRIBUTIONS OF EIGENFUNCTIONS

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ABSTRACT. We give two term pointwise Weyl laws for analytic continuations of eigenfunctions $\varphi_j^C(\zeta)$ of a real analytic Riemannian manifold $(M, g)$ without boundary to a Grauert tube $M_\tau$. The Weyl laws are asymptotic formulae for Weyl sums $\sum_{j: \lambda_j \leq \lambda} e^{-2\tau \lambda_j} |\varphi_j^C(\zeta)|^2$ with $\zeta \in \partial M_\tau$. The summands, when $L^2$ normalized, are special types of ‘microlocal lifts’ or Husimi distributions $\frac{|\varphi_j^C(\zeta)|^2}{\|\varphi_j\|^2_{L^2(\partial M_\tau)}}$, whose weak limits are the microlocal defect measures studied in quantum chaos. Rather than weak* limits, we study the asymptotics of their sup norms. The asymptotics depend on whether or not $\zeta$ is a periodic point of the geodesic flow, and on whether the periodic orbit is of elliptic type or not. The two-term Weyl law is analogous to the two-term Weyl asymptotics of Y. Safarov in the real domain. The remainder estimate gives universal growth bounds on $|\varphi_j^C(\zeta)|^2$ for $\zeta \in \partial M_\tau$, which are shown to be sharp (they are attained by analytic continuations of Gaussian beams).

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This article is concerned with analytic continuations $\varphi_j^C(\zeta)$ of eigenfunctions of the Laplacian $\Delta$ of a real analytic Riemannian manifold $(M, g)$ of dimension $m$ without boundary to a Grauert tube $M_\tau \subset M_C$ in the complexification of $M$; see §2.1 for definitions and background.

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In the ‘real domain’ $M$, we denote by

$$\Delta_g \varphi_j = \lambda_j^2 \varphi_j, \quad \langle \varphi_j, \varphi_k \rangle = \delta_{jk}$$

an orthonormal basis of eigenfunctions with $\lambda_0 = 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$. The classical pointwise Weyl law of Avakumovic, Levitan and Hörmander in the real domain asserts that,

$$N(\lambda, x) := \sum_{j: \lambda_j \leq \lambda} |\varphi_j(x)|^2 = C_m \lambda^m + R(\lambda, x). \quad (1)$$

where $C_m$ is a dimensional constant and where the remainder satisfies,

$$R(\lambda, x) = O(\lambda^{m-1}), \quad \text{uniformly in } x.$$

An important application of the pointwise Weyl law is to bound sup-norms of eigenfunctions, since $\varphi_j^2(\lambda, x)$ is bounded above by the jump in the remainder at an eigenvalue, i.e. there exists a constant $C_g > 0$ depending only on the metric $g$ so that,

$$\|\varphi_j\|_{L^\infty} \leq \sup_x |R(\lambda_j + 0, x) - R(\lambda_j - 0, x)| \leq C_g \lambda_j^{\frac{m-1}{2}}. \quad (2)$$

In [Saf, SV, SoZ, SoZ16] it is shown that the size of the remainder and the sup norm bounds depend on the structure of the set $\mathcal{L}_x$ of geodesic loops based at $x$. For a real analytic surface, it is shown in [SoZ16] that the sup norm bound (2) is only achieved if there exists a point $p \in M$ so that all geodesics through $p$ are closed. The author has conjectured that, in all dimensions, the sup norm bound is achieved only when there exists a point $p \in M$ so that all geodesics through $p$ are closed.

The purpose of this article is to formulate and prove a phase version of (1) and (2) in terms of analytic continuations $\varphi_j^C$ of the eigenfunctions $\varphi_j$ to Grauert tubes $M_\tau$ in the complexification $M_C$ of $M$. As reviewed in §2.1, the metric $g$ induces a Grauert tube radius function $\sqrt{\rho}$, essentially (half) the distance between $\zeta$ and $\bar{\zeta}$ in a natural metric on $M_\tau$. For $\tau$ sufficiently small, $\partial M_\tau = \{ \zeta \in M_C : \sqrt{\rho}(\zeta) = \tau \}$ is equivalent (under the complexified exponential map (11)) to the cosphere bundle $S^*_\tau M$ of radius $\tau$. The purpose of this article is to prove a pointwise ‘phase-space’ Weyl law (1) and sharp universal sup norm bounds (2) for the Husimi distributions,

$$U^\tau_j(\zeta) := \frac{|\varphi_j^C(\zeta)|^2}{\|\varphi_j\|^2_{L^2(\partial M_\tau)}} \quad (3)$$

Husimi distributions are special constructions of ‘Wigner distributions’ or ‘microlocal lifts’ of eigenfunctions (see §0.7 for other microlocal lifts); their weak* limits are the well-known microlocal defect measures or quantum limits studied in quantum chaos. They are probability distributions on $\partial M_\tau \simeq S^*_\tau M$, and are viewed as giving the probability density of finding a quantum particle at the phase space point $\zeta \in \partial M_\tau$. What makes the construction in terms of analytic continuations of eigenfunctions attractive is that these microlocal lifts are positive, relatively concrete and can be studied using complex analytic methods (see [ZJDG] and its references for applications to nodal sets).

The motivation to study sup-norms of Husimi functions (3) is similar to that in the physical space $M$: namely, to determine the maximal degree of concentration at a point

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1We use the convention where Laplacian $\Delta$ is minus the usual sum of squares, hence is a positive operator, because we will take many square roots.
in phase space of the probability density of the quantum particle. The main results of this article give universal upper bounds on the sup-norms of (3) and show that, when a sequence of Husimi distribution attains the maximal sup norm bounds, there must exist an elliptic closed geodesic along which it attains the bounds see Theorem 0.7). This is a novel kind of ‘scarring’. A natural question under investigation is the relation of this kind of scarring to that of weak* limits of the Husimi distributions.

The first result gives the universal upper bound (the lower bound is not important for this article).

**Theorem 0.1.** Suppose \((M, g)\) is real analytic, with \(\dim M = m\), and let \(\zeta \in \partial M_r\). Then, for any \(C > 0\), there exists \(\mu, A, a > 0\) (independent of \(\lambda\)) so that, for \(\frac{C}{\sqrt{r}} \leq \sqrt{p}(\zeta) \leq \tau\),

\[
a\lambda_j^{-\mu} e^{\tau \lambda_j} \leq \sup_{\zeta \in M_r} |\varphi_j^C(\zeta)| \leq A\lambda_j^{-\frac{m-1}{2}} e^{\tau \lambda_j}. \quad (4)
\]

Moreover, the square roots of the Husimi distributions (3) satisfy the bounds,

\[
a\lambda_j^{-\mu} \lambda_j^{-\frac{m-1}{2}} \leq \sup_{\zeta \in M_r} \frac{|\varphi_j^C(\zeta)|}{||\varphi_j||_{L^2(\partial M_r)}} \leq A\lambda_j^{-\frac{m}{2}} \quad (5)
\]

The upper bound of (4) is sharp and is attained by complexified highest weight spherical harmonics on \(S^m\) and by general Gaussian beams (Section 12). The bounds of Theorem 0.1 substantially improve the estimates

\[
\sup_{\zeta \in M_r} |\varphi_j^C(\zeta)| \leq C\tau^{m+1} e^{\tau \lambda_j}.
\]

in \([\text{Bou}, \text{GLS}]\) and their improvements in \([\text{ZPSH1}, \text{Corollary 3}]\) (see also \([\text{L18}]\)).

The upper bounds on the Husimi distributions follow from the bounds on \(\varphi_j^C(\zeta)\) and from the following \(L^2\) norm asymptotics:

**Lemma 0.2.** Under the assumptions of Theorem 0.1, then there exists a universal postive constant \(C(m, \tau) > 0\) so that,

\[
||\varphi_j^C||_{L^2(\partial M_r)}^2 = C(m, \tau) e^{2\tau \lambda_j} \lambda_j^{-\frac{m-1}{2}} (1 + O(\lambda_j^{-1})).
\]

See (34) for the explicit calculation for plane waves \(e_k\) on a flat torus.

The upper bound (5) on the Husimi distribution is also sharp and is also obtained by Gaussian beams. In the case of the standard basis \(Y^m_l\) on standard spheres, it is straightforward to relate the sup norms of \(e^{-2\tau \lambda_j} |\varphi_j^C(\zeta)|^2\) and of the Husimi distributions of \(Y^m_l\), and to show that Husimi distributions of highest weight spherical harmonics of \(S^m\) attain the upper bound (5) (see Section 11.1). It is also attained by complex coherent states (see Definition 4 for their definition), but they are not quite of the form (3).

As in the real domain, a motivating problem is to characterize the Riemannian metrics \(g\) possessing sequences of eigenfunctions whose Husimi distributions attain the maximal sup-norm bounds, then to characterize the points \(\zeta\) at which the sup norm is attained, and to characterize the associated sequence \(\{\varphi_jk\}\) of eigenfunctions. One may conjecture that the
sup norm bound is only attained by complexified Gaussian beams. Theorem 0.7 at least shows that the elliptic geodesic must exist on such manifolds.

Of course, the $L^\infty$ norms of the Husimi distributions (3) are just one type of norm to study. The most significant norms to study in phase space $M_\tau$ (i.e. $\mathcal{B}_\tau^*M$) are not necessarily the same ones as in configuration space $M$. Most studies of Husimi distributions concern the weak * limits of the sequence (3), which are invariant probability measures under the geodesic flow. The relation between $L^\infty$ norms along subsequences and its weak * limits is currently under exploration. Another interesting norm is the microlocal Kakeya-Nikodym norm,

$$||\varphi_j||^2_{MKN} := \sup_{\gamma \in \Pi} \int_{\lambda_j^{1/2}(\gamma)} U^\tau_j dV_\tau,$$

where $\lambda_j^{1/2}(\gamma)$ is the tube in $\partial M_\tau$ around the phase space geodesic arc $\gamma$, i.e. the orbit of a point $\zeta \in \partial M_\tau$ under the geodesic flow (transported to $\partial M_\tau$). As Theorem 0.7 shows, maximal sup norm growth only occurs at points along elliptic closed geodesics, and such Husimi distributions also seem to saturate the microlocal Kakeya-Nikodym norm. A related microlocal Kakeya-Nikodym norm was also defined and studied in [BlS17, page 515]. The definitions are apriori quite different; the exact relations between the two microlocal Kakeya-Nikodym norms is also under investigation.

0.1. Background and precise statement of results. To state the next results, we need to introduce some further notation and background regarding Grauert tubes and their relation to $T^*M$. More systematic expositions of the relevant background can be found in [GS1, LS1, GLS, ZPSH1, L18].

A real analytic Riemannian manifold $M$ always possesses a complexification $M_\mathbb{C}$ into which it embeds as a totally real submanifold. A real analytic metric $g$ induces a unique plurisubharmonic exhaustion function $\sqrt{\rho}$ known as the Grauert tube function. It is related to the square $r^2(x,y)$ of the Riemannian distance function on $M \times M$ by

$$\sqrt{\rho}(\zeta) = \frac{1}{2i} \sqrt{r^2_\mathbb{C}(\zeta,\bar{\zeta})}$$

(6)

where $r^2_\mathbb{C}$ is the holomorphic extension of $r^2(x,y)$ to a small neighborhood of the anti-diagonal $(\zeta,\bar{\zeta})$ in $M_\mathbb{C} \times M_\mathbb{C}$. The open Grauert tube of radius $\tau$ is defined by

$$M_\tau = \{ \zeta \in M_\mathbb{C}, \sqrt{\rho}(\zeta) < \tau \}.$$

There is a maximal radius $\tau_{\text{max}} \in (0, \infty]$ such that $M_\tau$ is an embedded tube for $\tau < \tau_{\text{max}}$, and all eigenfunctions $\varphi_\lambda$ admit analytic continuations to $M_\tau$ and are smooth up to the boundary. For further background on Grauert tubes, we refer to Section 2.

As in the real domain (see (2)), sup norms of normalized complexified eigenfunctions are bounded by the associated jump of the remainder term in the pointwise Weyl law at the eigenvalue. In the complex domain, there are several choices of the relevant Weyl law. One way to study the average growth of modulus squares of analytic continuations of eigenfunctions is to is complexify the spectral function,

$$\Pi^\mathbb{C}_{I_\lambda}(\zeta,\bar{\zeta}) := \sum_{j: \lambda_j \in I_\lambda} |\varphi^\mathbb{C}_j(\zeta)|^2,$$

(7)
of $\Delta$ for an interval $I_\lambda = [a(\lambda), b(\lambda)]$, and restrict it to the totally real diagonal of $M_C \times \overline{M_C}$. In [ZPSH1] it is proved that the kernels (7) grow exponentially at the rate $e^{2\lambda_j \sqrt{\rho(\zeta)}}$. To obtain polynomial growth, we introduce the ‘tempered’ spectral projections

$$P_{[0,\lambda]}^\tau(\zeta, \bar{\zeta}) = \sum_{j: \lambda_j \leq \lambda} e^{-2\tau \lambda_j |\varphi_j^c(\zeta)|^2}, \quad (\sqrt{\rho}(\zeta) \leq \tau). \tag{8}$$

More generally, as in (7), we could study $P_{I_\lambda}^\tau(\zeta, \bar{\zeta})$, where $I_\lambda$ could be a short interval $[\lambda, \lambda + 1]$ of frequencies or a long window $[0, \lambda]$. We let $I_\lambda = [\lambda - 1, \lambda + 1]$ when $(M, g)$ is a Zoll manifold, with the intervals centered so that $I_\lambda$ contains exactly one full cluster. But for the main results, we only consider $I_\lambda = [0, \lambda]$ and focus on the special case (8). The tempered kernels $P_{I_\lambda}^\tau(\zeta, \bar{\zeta})$ are in some ways analogous to the semi-classical Szegő kernels $\Pi_{\mu}(x_1, x_2)$ of positive line bundles over Kähler manifolds (see Section 2.9). Henceforth, we generally assume that $\sqrt{\rho}(\zeta) = \tau$ when we study (8), since the sums (8) are exponentially decaying if $\sqrt{\rho}(\zeta) < \tau$.

Our pointwise Weyl law is a two-term asymptotic expansion for (8) in terms of a certain function $Q_\zeta(\lambda)$ (Theorem 0.4). The $Q_\zeta$ function is the phase space analogue of a function introduced by Yu. Safarov [Saf, SV] in the real domain. Before stating the result, we digress to define $Q_\zeta$.

**Remark 0.3.** Rather than use (8), one may wish to work with sums of Husimi distributions,

$$\tilde{P}_{[0,\lambda]}^\tau(\zeta, \bar{\zeta}) = \sum_{j: \lambda_j \leq \lambda} \frac{|\varphi_j^c(\zeta)|^2}{||\varphi_j^c||_{L^2(\partial M_\tau)}}, \quad (\sqrt{\rho}(\zeta) = \tau),$$

But analytic continuations $\varphi_j^c$ of an orthonormal basis of eigenfunctions are rarely orthogonal on $\partial M_\tau$, unless $M$ has a large symmetry forcing the orthogonality. In general, we are unable to analyze sums of Husimi distributions directly; instead, we first use (8) and then derive results for Husimi distributions by using Lemma 0.2.

0.2. **The $Q_\zeta(\lambda)$ function.** To define $Q_\zeta$ we first need to introduce the *osculating Bargmann-Fock space* $\mathcal{H}_\zeta^2$ at $\zeta \in \partial M_\tau$ (Definition 4.1). This is the Bargmann-Fock space constructed from the complexification $H_{\zeta}^{1,0} \oplus H_{\zeta}^{0,1}$ of the CR (Cauchy-Riemann) subspace $H_{\zeta}(\partial M_\tau)$ in the complexified tangent space $T_\zeta \partial M_\tau \otimes \mathbb{C}$ at $\zeta$ (see (70) and Section 2.4 for background on CR structures and on Bargmann-Fock spaces and Section 3 for more details). Thus, $\mathcal{H}_\zeta^2$ is the space of entire holomorphic functions on $H_{\zeta}^{1,0}$ (with respect to the complex structure $J_\zeta$ of $M_\tau$ at $\zeta$) which are square integrable with respect to the *ground state* $\Omega_{\partial M_\tau}$ (defined in (55)).

The (real) CR subspace $H_{\zeta} \subset T_\zeta(\partial M_\tau)$ is tangent to a symplectic transversal to the geodesic flow. In this article, we use a distinguished symplectic transversal that we term a ‘Phong-Stein leaf’ (Section 2.6). Given a periodic point $\zeta$ of $g^\tau_t$ of period $n$, we obtain a complexified linear symplectic Poincaré map (62),

$$D_\zeta g^{nT(\zeta)} : H_{\zeta}(\partial M_\tau) \to H_{\zeta}(\partial M_\tau), \tag{9}$$
onumber

on the CR subspace. In a symplectic basis of $H_{\zeta}(\partial M_\tau)$,

$$D_\zeta g^{nT(\zeta)} = \begin{pmatrix} A_n(\zeta) & B_n(\zeta) \\ C_n(\zeta) & D_n(\zeta) \end{pmatrix} \in Sp(n, \mathbb{R}), \tag{10}$$

where
where as usual $Sp(n, \mathbb{R})$ denotes the symplectic group. For simplicity of notation, we often write

$$S^n_\zeta := D_\zeta g^{nT(x)}_\tau.$$  

The symplectic matrix (10) will arise often and is discussed in Section 3.2 (see (54)).

Since $(M,g)$ is real analytic, its exponential map $\exp_x t\xi$ admits an analytic continuation in $t$ and the imaginary time exponential map

$$E : B^{*\varepsilon}_\zeta M \to M_C, \quad E(x,\xi) = \exp_x i\xi$$  

is, for small enough $\varepsilon$, a diffeomorphism from the ball bundle $B^{*\varepsilon}_\zeta M$ of radius $\varepsilon$ in $T^{*}M$ to the Grauert tube $M_\varepsilon$ in $M_C$. As reviewed in §2.1 (see [GS1, LS1, ZPSH1, ZJDG] for more details), $E$ conjugates the homogeneous geodesic flow $G^t$ on $B^{*\varepsilon}_\zeta M$ to the Hamiltonian flow of the Grauert tube function $\sqrt{\rho}$ with respect to the Kähler form $\omega = i\partial\bar{\partial}\rho$. We denote by

$$g^t_\tau = E \circ G^t \circ E^{-1}|_{\partial M_\varepsilon}$$  

the transfer of the geodesic flow of $S^{*\varepsilon}_\tau M$ to $\partial M_\tau$. We say that $\zeta$ a periodic point if it is a periodic point of $g^t_\tau$ and denote the set of periodic points by,

$$\zeta \in \mathcal{P} \iff \zeta \text{ is aperiodic point for } g^t_\tau.$$  

We denote its primitive period by $T(\zeta)$, Thus,

$$T(\zeta) = \inf\{t > 0 : g^t_\tau(\zeta) = \zeta\}.$$  

The set of periodic points of period $T$ is the set of fixed points of $g^T_\tau$. As usual, we say that the fixed point set $F$ of a map $T$ is clean if $F$ is a manifold and $T_x F = \text{Fix}(D_x T)$.

Next, we define the metaplectic representation $W_{J_\zeta}$ of the derivative $Dg^\tau_\zeta$ on the osculating Bargmann-Fock space. In the model case of $\mathbb{R}^{2m}$ with complex structure $J$, a symplectic linear map $S \in Sp(m,\mathbb{R})$ can be quantized, $S \to W_J(S)$ by the metaplectic representation as a unitary operator on Bargmann-Fock space (reviewed in §3.2; see also Sections 3 and 4). Identifying (9) with a symplectic map (10) on the model space, (9) may be quantized as a unitary operator on the osculating Bargmann-Fock space at $\zeta$,

$$W_{J_\zeta} (Dg^{nT(\zeta)}_\zeta) : \mathcal{H}^2_\zeta \to \mathcal{H}^2_\zeta.$$  

The space $\mathcal{H}^2_\zeta$ has a distinguished ground state $\Omega_{J_\zeta}$, a Gaussian associated to the complex structure $J_\zeta$ (see Section 3.5 and (55) for background). We denote by

$$G_n(\zeta) := \langle W_{J_\zeta} (Dg^{nT(\zeta)}_\zeta) \Omega_{J_\zeta}, \Omega_{J_\zeta} \rangle$$  

the matrix element of (15) relative to the ground state $\Omega_{J_\zeta}$. As reviewed in Sections 3.2 and 3 (see in particular, Lemma 3.3),

$$G_n(\zeta) = 2^{n/2}(\det (A_n(\zeta) + D_n(\zeta) + i(B_n(\zeta) - C_n(\zeta)))^{-\frac{1}{2}} = \det P_{J_\zeta} S^n_\zeta P_{J_\zeta},$$  

where $P_{J_\zeta} S^n_\zeta P_{J_\zeta}$ is the holomorphic block of a unitary conjugate of (10). Thus, the ‘quantum invariant’ (16) equals the ‘classical invariant’ (17). We then define the function $Q_\zeta(\lambda)$ by:
**Definition 1.** Recalling the set (13),

\[
Q_\zeta(\lambda) = \begin{cases} 
0, & \zeta \notin \mathcal{P} \\
\sum_{n=1}^{\infty} \frac{\sin \lambda n T(\zeta)}{n T(\zeta)} G_n(\zeta), & \zeta \in \mathcal{P}.
\end{cases}
\]  

(18)

Note that since (17) is purely classical, (18) gives a formula for \(Q_\zeta(\lambda)\) defined purely in terms of classical quantities. On the other hand, (16) gives a ‘quantum formula’. By (16),

\[
Q_\zeta(\lambda) = \frac{1}{2\pi} \left( \sum_{n=1}^{\infty} \frac{e^{i\lambda n T(\zeta)}}{n T(\zeta)} \langle W_J(S_\zeta)^n \Omega_{J_\zeta}, \Omega_{J_\zeta} \rangle - \sum_{n=1}^{\infty} \frac{e^{-i\lambda n T(\zeta)}}{n T(\zeta)} \langle W_J(S_\zeta)^{-n} \Omega_{J_\zeta}, \Omega_{J_\zeta} \rangle \right).
\]  

(19)

The classical formula (17) seems simpler than the quantum formula, since it comes down to the diagonalization of \(S_\zeta \in \text{Sp}(n, \mathbb{R})\). This classical formula does not seem to have an analogue in the real domain, hence does not have an analogue in \([\text{Saf, SV}]\) (it does, of course, have an analogue in \([\text{ZZ18}]\)).

### 0.3. Statement of the two term pointwise Weyl asymptotics

The main result on pointwise Weyl asymptotics encompasses three scenarios: (i) where \(\zeta\) is not a periodic point; (ii) where \(\zeta\) is a periodic point, and where \(Q_\zeta(\lambda)\) is uniformly continuous, and \(P_{[0, \lambda]}(\zeta, \zeta)\) admits asymptotics with a well-defined ‘middle term’; (iii) where \(\zeta\) is a periodic point, and \(Q_\zeta\) has jumps.

**Definition 2.** Let \(\zeta \in \mathcal{P}\). We define the jump-set of \(Q_\zeta\) by,

\[\mathcal{J}(\zeta) := \{\lambda \in \mathbb{R}_+: [Q_\zeta(\lambda)] := Q_\zeta(\lambda + 0) - Q_\zeta(\lambda - 0) > 0\}.\]

In Section 10, we study the possible jumps of \(Q_\zeta(\lambda)\). As mentioned above, the two equations (16), resp. (17) indicate that there is a ‘quantum dynamical ’ definition of \(\mathcal{J}(\zeta)\), resp. a ‘classical mechanical’ definition. Of course, they must agree.

In the following, we enumerate the jump points as the sequence \(\{\nu_k\}_{k=1}^{\infty}\).

**Theorem 0.4.** Suppose \((M, g)\) is real analytic, and \(\zeta \in M_T\). Then, for fixed \(\tau > 0\),

1. When \(Q_\zeta(\lambda)\) is uniformly continuous in \(\lambda\), then for \(\sqrt{\rho(\zeta)} \geq C_\tau\),

\[P_{[0, \lambda]}(\zeta, \bar{\zeta}) = \lambda \left( \frac{\lambda}{\sqrt{\rho}} \right)^{\frac{m-1}{2}} \left( 1 + O(\lambda^{-1}) \right);\]

where the remainders are uniform in \(\zeta\). Moreover, \(Q_\zeta(\lambda) = 0\) if \(\zeta \notin \mathcal{P}\).

2. If \(\sqrt{\rho(\zeta)} = \tau\), if \(\zeta \in \mathcal{P}\) and the fixed point set of \(g_T^\tau(\zeta)\) is clean, and if \(Q_\zeta(\lambda)\) has jumps at the points \(\mathcal{J}(\zeta) = \{\nu_k\}_{k=1}^{\infty}\), then there exists a sequence \(\varepsilon_k = O(\nu_k^{-\frac{1}{2}})\) and eigenvalues \(\{\lambda_{jk}\}_{k=1}^{\infty}\) such that \(\lambda_{jk} = \nu_k + O(\varepsilon_k)\) and a positive constant \(C > 0\), such that

\[P_{[\nu_k - \varepsilon_k, \nu_k + \varepsilon_k]}(\zeta, \bar{\zeta}) \geq C \nu_k^{\frac{m-1}{2}}.\]
If \( \sqrt{\rho}(\zeta) = \tau \), if \( \zeta \in \mathcal{P} \), then for all \( \lambda \in \mathbb{R}_+ \), and functions \( o(\lambda) \) tending monotonically to 0 as \( \lambda \to \infty \),

\[
\lambda \left( \frac{\lambda}{\zeta} \right)^{\frac{m-1}{2}} (1 + Q_\zeta(\lambda - o(\lambda))\lambda^{-1} + o(\lambda^{-1})) \\
\leq P_{[0,\lambda]}(\zeta, \bar{\zeta}) \\
\leq \lambda \left( \frac{\lambda}{\zeta} \right)^{\frac{m-1}{2}} (1 + Q_\zeta(\lambda + o(\lambda))\lambda^{-1} + o(\lambda^{-1})).
\]

In (3) we do not have asymptotics in the middle term when \( Q_\zeta(\lambda) \) has a non-empty jumpset (Definition 2). The necessity of the inequalities (20) is due to the fact that the jump-set of \( Q_\zeta(\lambda) \) is not necessarily contained in the jumpset of \( P_{[0,\lambda]} \), namely the set \( \{\lambda_j\} \). The exact relation is explained below Corollary 0.9. Examples illustrating the scenarios are given in Section 1. The Zoll case illustrates the need for the somewhat imprecise inequalities in (3). As discussed in Section 1 and in Section 11.2 in the Zoll case, there will exist a cluster of eigenvalues in \( \lambda_j \in [\nu_k - \varepsilon_k, \nu_k + \varepsilon_k] \) of cardinality comparable to \( \lambda^{m-1} \), for which each Husimi distribution has non-extremal sup norm but for which the sum has the lower bound (b). However, in the Zoll case cases there is a more precise result than Theorem 0.4 (see Theorem 1.2) and the unwieldy inequalities in (3) are only necessary if one sums over intervals \([0, \lambda]\) which contain incomplete portions of the eigenvalue clusters reviewed in Section 11.2. One obtains much better asymptotics for the Weyl sums \( P_{I_\lambda} \) over intervals \( I_\lambda \) containing exactly one cluster.

The next result (Theorem 0.7) relates the continuity properties of \( Q_\zeta \) to the dynamical properties of the geodesic through \( \zeta \). To prepare for it we study we now give further information on \( Q_\zeta \).

0.4. Further properties of \( Q_\zeta \). We denote by \( \{x\}_{2\pi} = x + 2\pi\mathbb{Z} \in [0, 2\pi) \) the residue of \( x \) modulo \( 2\pi \), which we identify with the function \( x \) on \([0, 2\pi]\), extended periodically of period \( 2\pi \) to \( \mathbb{R} \). Equivalently, \( \{x\}_{2\pi} = 2\pi \left\{ \frac{x}{2\pi} + \frac{1}{2} \right\} - \pi \) where \( \{x\} \) is the fractional part. Its Fourier series is given by \( \{x - \pi\}_{2\pi} = \sum_{n \neq 0} \frac{\sin nx}{n} = 2\sum_{n=1}^{\infty} \sin nx \). \(^2\)

In the following Proposition, we recall that eigenvalues of a symplectic matrix occur in quadruples \( \lambda, \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1} \); if \( \lambda \in \mathbb{R} \) or \( |\lambda| = 1 \), the eigenvalues come just in pairs. We say that \( S \) is semi-simple if it is diagonalizable over \( \mathbb{C} \), that \( S \) is elliptic if its eigenvalues all have modulus 1, i.e. \( S \in U(m) \), and that it is hyperbolic if it is positive symmetric symplectic and none of its eigenvalues have modulus 1.

**Proposition 0.5.** Assume that \( \zeta \in \mathcal{P} \) and that \( S_\zeta \) is semi-simple. Then,

1. \( Q_\zeta(\lambda) \) is uniformly continuous if and only if \( S_\zeta \) is not elliptic. Hence, \( \mathcal{J}(\zeta) = \emptyset \) unless \( S_\zeta \) is elliptic.

2. If \( S_\zeta \) is non-degenerate elliptic, and if \( \det P_3S_\zeta P_3 = e^{i\alpha_0} \), then \( Q_\zeta(\lambda) = \{\alpha_0 + \lambda T(\zeta) - \pi\}_{2\pi} \), and

\[
\mathcal{J}(\zeta) = \{\lambda : \alpha_0 + \lambda T(\zeta) = \pi + 2\pi\mathbb{Z}\}.
\]

\(^2\)In [SV], \( \{x\}_{2\pi} = x + 2\pi\mathbb{Z} \in [-\pi, \pi) \).
0.4.1. *Metaplectic quantum approach.* Although it is more complicated, it is also natural and interesting to use the quantum formula (16) as in [Saf, SV], and to compare the results with the classical approach. To some extent, we adapt the notation and arguments of [Saf, SV] on the quantum mechanical approach to the jump behavior of \( Q_\zeta(\lambda) \) to our setting. However, even the spectral measure formula (23) does not seem to be mentioned in [Saf, SV], so our approach is not the same. The results in the real domain and complex domain are to some degree analogous, but there are significant differences. In the real domain, for analytic metrics, rather than the metaplectic unitary quantization \( W_{\zeta}(D_\zeta g^{nT(\zeta)}) \) of the Poincaré map of a closed geodesic, one has a nonlinear first return map on directions of loops at a point \( p \in M \), whose quantization is an operator on half-densities on \( S^*_p M \). In the complex domain, we have only the periodic orbit of \( \zeta \), its linear Poincaré map, and the metaplectic quantization of the Poincaré map. The classical mechanical approach does not have a simple analogue in the real domain.

Although the exponential map \( \exp : \mathfrak{sp}(n, \mathbb{R}) \to Sp(n, \mathbb{R}) \) from the symplectic Lie algebra to the symplectic group is not surjective, we will assume with a rather small loss of generality that \( S = e^{iH} \) where \( H \in \mathfrak{sp}(n, \mathbb{R}) \). The exponent \( iH \) is not unique, and in particular is only defined up to the addition by \( 2\pi \mathbb{Z} \) times the identity. However, all of the choices of logarithms will given the same results. With an abuse of notation we denote the inverse exponential map by \( \arg : Sp(n, \mathbb{R}) \to \mathfrak{sp}(n, \mathbb{R}) \) on the image of \( \exp \), so that \( e^{i\arg(S)} = S \). Thus, as in [SV, Proposition 1.8.12], \( \arg(W_\zeta(S)) \) is a self-adjoint operator so that \( W_\zeta(S) = \exp i \arg(W_\zeta(S)) \).

In the quantum mechanical approach, we use the spectral decomposition of the quadratic Hamiltonian \( \arg(W_\zeta(S_\zeta)) \). We denote the possible pure point eigenvalues/eigenfunctions of \( \arg(W_\zeta(S_\zeta)) \) by \( \{s_\ell\} \), resp. \( \{v_\ell\}_{\ell=1}^\infty \). Then,

\[
W_{\zeta}(S_\zeta)v_\ell = e^{is_\ell}v_\ell. \tag{21}
\]

Of course, \( W_{\zeta}(S_\zeta) \) often has continuous spectrum as well (or, only has continuous spectrum). For \( \zeta \) such that there exist \( s_\ell \) satisfying (21), we define

\[
\Lambda_{\ell,j} = \frac{2\pi j + s_\ell}{T(\zeta)}. \tag{22}
\]

We observe that \( \langle W_{\zeta}(S_\zeta)^n\Omega_{\zeta,j}, \Omega_{\zeta,j} \rangle \) is, by definition, the \( n \)th moment of the spectral measure \( d\mu_\zeta \) on \( S^1 = \{z : |z| = 1\} \) of the unitary operator \( W_{\zeta}(S_\zeta) \) with respect to the ground state \( \Omega_{\zeta,j} \), i.e.

\[
\langle W_{\zeta}(S_\zeta)^n\Omega_{\zeta,j}, \Omega_{\zeta,j} \rangle = \int_{S^1} e^{in\theta}d\mu_\zeta. \tag{23}
\]

The utility of (23) depends on the extent to which the properties of the spectral measure \( d\mu_\zeta \) can be determined. We denote by \( \pi_{\Omega_\zeta} := \Omega_{\zeta,j} \otimes \Omega_{\zeta,j}^* \) the orthogonal projection in the osculating Bargmann-Fock space \( \mathcal{H}_\zeta^2 \) onto the ground state.

**Proposition 0.6.** *With the above notation and conventions,*

1. *In terms of the spectral measure (23),

\[
Q_\zeta(\lambda) = \frac{1}{T(\zeta)} \int_0^{2\pi} \{\theta + \lambda T(\zeta) - \pi\}2\pi d\mu_\zeta.
\]

2. *\( Q_\zeta \) is uniformly continuous if and only if \( d\mu_\zeta \) is an absolutely continuous measure.*
(3) The atoms of \( d\mu_\zeta(\theta) \) occur at the eigenvalues \( e^{i\alpha} \) of \( W_{J_\ell}(S_\zeta) \), and \( \mu_\zeta(\{e^{i\alpha}\}) = |\pi\Omega_\zeta v_\ell|^2 \).

(4) \( Q_\zeta \) has jumps at the points (22) with \( \pi\Omega_\zeta v_\ell \neq 0 \).

See [SV, Theorem 1.8.17] for the corresponding statement in the real domain.

In Section 12, \( Q_\zeta \) is calculated in the case where \( \zeta \) generates a non-degenerate elliptic closed geodesic whose Poincaré map has eigenvalues \( e^{i\alpha_j} \) with frequencies \( (\alpha_1, \ldots, \alpha_{m-1}) \) independent, together with \( \pi \), over \( \mathbb{Q} \). In this case, \( W_{J_\zeta}(S_\zeta) = \exp i\arg(W_{J_\zeta}(S_\zeta)) \) is the unitary operator generated by a Harmonic oscillator Hamiltonian. The spectrum of \( \arg(W_{J_\zeta}(S_\zeta)) \) is of the form \( \{s_\ell\} = \{\sum_{j=1}^{m-1} \alpha_j(k_j + \frac{1}{2})\}_{k\in\mathbb{N}^{m-1}} \) in the notation above. In this case, \( \Omega_\zeta \) is itself the ground state eigenfunction corresponding to \( k = 0 \) and all other eigenfunctions are orthogonal to it. Hence, in Proposition 0.6 there is a single \( \ell \), \( |\pi\Omega_\zeta v_\ell|^2 = 1 \) and \( s_\ell = \frac{1}{2} \sum_j \alpha_j \). See also Proposition 10.1.1 for a general result in the elliptic case.

0.5. Sup-norms of Husimi distributions and dynamics. We have now assembled enough background to state the main result: Combining Theorem 0.4 with Proposition 0.5 and Proposition 0.6 gives the following.

**Theorem 0.7.** Among real analytic Riemannian manifolds \( (M, g) \) for which \( D_\zeta g_\tau^T \) is semi-simple for all periodic points \( \zeta \in \partial M \), the universal sup norm upper bound bound of Theorem 0.1(2) is attained by \( (M, g, \zeta, \{\varphi_{j,k}\}) \) only if:

1. \( \zeta \) is a periodic orbit point of the geodesic flow \( g_\tau^T \) of some period \( T(\zeta) > 0 \),

2. \( S_\zeta := D_\zeta g_\tau^{T(\zeta)} \) is an elliptic semi-simple symplectic matrix, i.e. the orbit of \( \zeta \) is an elliptic closed geodesic.

3. With \( \Lambda_{\ell,k} \) as in (22), there exist \( \varepsilon_{\ell,k} \to 0 \), and a subsequence \( j_{\ell,k} \) so that \( |\lambda_{j_{\ell,k}} - \Lambda_{\ell,k}| < \varepsilon_{\ell,k} \), and

\[
\sum_{j: \Lambda_{\ell,k} - \varepsilon_{\ell,k} \leq \lambda_j \leq \Lambda_{\ell,k+1} - \varepsilon_{\ell,k}} e^{-2\tau\lambda_j} |\varphi_j^\zeta(\zeta)|^2 = 2\pi \Lambda_{\ell,k}^{m-1} |\pi\Omega_\zeta v_\ell|^2_{L^2} + o(\Lambda_{\ell,k}^{m-1}).
\]

Under these conditions, there exist eigenvalues of \( \sqrt{\Delta} \) lying in shrinking neighborhoods of jump points in \( J(\zeta) \).

The assumption of semi-simplicity in Theorem 0.7 is to simplify the discussion of the symplectic normal forms of symplectic matrices. In the Jordan normal form decomposition, the semi-simple and nilpotent parts need not be symplectic in general. Hence, we assume for simplicity that all Poincaré type maps are semi-simple. This is an open-dense condition on symplectic matrices [Gutt]. A simple example where it is not satisfied is the flat torus, but it is easy to see (and proved in Section 1.2) that the universal sup norm bound is not attained in this case either. There are more general examples of manifolds without conjugate points which are not covered by Theorem 0.7, and which doubtless do not attain the universal upper bound, but we omit these for the sake of brevity.

It follows from Theorem 0.7 that if a sequence of Husimi distributions \( U_\tau^\zeta \) (3) attains the maximal \( L^\infty \) bound at \( \zeta \), then it obtains the bound on the whole closed geodesic (Hamiltonian orbit) \( \gamma_\zeta \) with initial data \( \zeta \).
Remark 0.8. An interesting question is whether Theorem 0.4 (2) and Theorem 0.7(2) may be improved in some situations so that they imply an extremal lower bound on the sup-norm of a Husimi distribution $\lambda_j \in [\nu_k - \varepsilon_k, \nu_k + \varepsilon_k]$ when $(M, g)$ possesses an elliptic closed geodesic. In the general elliptic case, (b) only reflects the existence of a Gaussian beam quasi-mode, not an actual eigenfunction with of Gaussian beam type. Such eigenfunctions exist on convex surfaces of revolution (see Section 12).

The sup norm estimate of Theorem 0.1 is obtained from the jump of (8) at an eigenvalue. In the next Corollary, we equate the jumps on the two sides of (1) of Theorem 0.4 and implies the result of Theorem 0.1 and the main input into the results of Theorem 0.7.

Corollary 0.9. For any $(M, g)$, for fixed $\tau$ and for $\zeta \in \partial M_\tau$,
\[
\sum_{j : \lambda_j = \lambda} e^{-2r\lambda_j} |\varphi_j^C(\zeta)|^2 = P_{[0, \lambda_j + 0]}^\tau(\zeta, \bar{\zeta}) - P_{[0, \lambda_j - 0]}^\tau(\zeta, \bar{\zeta}).
\]
If $Q_\zeta(\lambda)$ is uniformly continuous in $\lambda$, i.e. has no jumps, then
\[
e^{-2r\lambda_j} |\varphi_j^C(\zeta)|^2 = o_r(\lambda_j^{m-1}).
\]
Hence, if $Q_\zeta$ is uniformly continuous in $\lambda$ for all $\zeta$, then
\[
\sup_{\zeta \in \partial M_\tau} e^{-2r\lambda_j} |\varphi_j^C(\zeta)|^2 = o_r(\lambda_j^{m-1}).
\]
The analogous results for (3) follow from Lemma 0.2.

We observe that jumps in spectral functions arise in two ways in Corollary 0.9:

• (i) Jumps $P_{[0, \lambda_j + 0]}^\tau(\zeta, \bar{\zeta}) - P_{[0, \lambda_j - 0]}^\tau(\zeta, \bar{\zeta})$ in the Weyl function at eigenvalues of $\sqrt{\Delta}$, which are non-zero as long as $\varphi_j^C(\zeta) \neq 0$ for some $j$ with $\lambda_j = \lambda$;

• (ii) Jumps $Q_\zeta(\nu_k + 0) - Q_\zeta(\nu_k - 0)$ in $Q_\zeta$ function at its jump discontinuities $\nu_k$.

If $Q_\zeta(\lambda)$ has jump discontinuities at points $\nu_k$ with $Q_\zeta(\nu_k + 0) - Q_\zeta(\nu_k - 0) \geq C_1 > 0$, then there exists a sequence $\varepsilon_k \to 0$ such that
\[
P_{[0, \nu_k + \varepsilon_k]}^\tau(\zeta, \bar{\zeta}) - P_{[0, \nu_k - \varepsilon_k]}^\tau(\zeta, \bar{\zeta}) \geq C_2 \nu_k^{m-1}.
\]
When the fixed point sets are non-degenerate (and necessarily elliptic in the jump case), one may take $\varepsilon_k = \frac{1}{k}$. This implies that there exist eigenvalues $\lambda_{jk}$ such that $|\lambda_{jk} - \nu_k| < \nu_k^{1/2}$. There are several ways that such eigenvalues can arise. First, $(M, g)$ might be a Zoll manifold, all of whose geodesics are closed. In that case, the spectrum of $\sqrt{\Delta}$ occurs in clusters of width $k^{-1}$ around an arithmetic progression $\{\nu_k\}_{k=1}^\infty$; see Theorem 1.2 and Section 11.2 for precise statements. In this case, $\varepsilon_k = O(k^{-1})$ is the minimal possible size for the lower bound above, because the clusters centered at the points $\nu_k$ have widths $O(k^{-1})$ (see [DG]). In intervals $\lambda \in I_k$ outside the union of the clusters, there are no jumps in $Q_\zeta(\lambda)$ and $P_{[0, \lambda_j]}^\tau(\zeta, \bar{\zeta})$ is constant. A second scenario is illustrated by a generic convex surface of revolution $(S^2, g)$. In this case, the spectrum of $\sqrt{\Delta}$ is evenly distributed in intervals $[\lambda, \lambda + 1]$, so there is no clustering of its eigenvalues; hence, jump behavior is not a spectral invariant but is due to the existence of special eigenfunctions (Gaussian beams) centered along elliptic closed geodesics $\gamma$ whose Husimi measures attain maximal size. If $\gamma$ is the orbit of $\zeta$, then $Q_\zeta(\lambda)$ exhibits
jumps along a dynamically defined arithmetic progression \( \{ \nu_k \} \) (see Section 0.4), and there exist eigenvalues \( \{ \lambda_j \} \) close to \( \nu_k \) at which \( P_{\nu_k}^* \frac{r}{\eta}((\zeta, \bar{\zeta})) \) has the jumps above. These associated eigenvalues can be constructed by the elliptic quasi-mode construction [Ral82, BB91] and from this construction one can see that \( |\lambda_j - \nu_k| \leq k^{-\frac{1}{2}} \).

### 0.6. Outline of the proofs.

The proof of Theorem of 0.4 is based on Fourier Tauberian arguments relating tempered spectral projection measures

\[
d_{\lambda}P_{\nu_k}^*((\zeta, \bar{\zeta})) = \sum_j \delta(\lambda - \lambda_j)e^{-2\tau\lambda_j}|\varphi_j^C(\zeta)|^2, \quad (\tau = \sqrt{\rho}(\zeta))
\]  

(24)

to their Fourier transforms. Note that (24) is a temperate distribution on \( \mathbb{R} \) for each \( \zeta \) satisfying \( \sqrt{\rho}(\zeta) \leq \tau \).

We study analytic continuations of eigenfunctions, as in [ZJDG], using the Poisson kernel,

\[
P^\tau(\zeta, y) := \sum_j e^{-\tau\lambda_j}\varphi_j^C(\zeta)\varphi_j(y),
\]

(25)

which has the property,

\[
P^\tau(\varphi_j(\zeta) = e^{-\tau\lambda_j}\varphi_j^C(\zeta).
\]

To obtain the Weyl asymptotics for (8) we study the singularities in \( t \) of the Fourier transform of (24),

\[
U_C(t + 2it, \zeta, \bar{\zeta}) := F_{\lambda \rightarrow t}d_{\lambda}P_{\nu_k}^*((\zeta, \bar{\zeta})) = \sum_j e^{(-2\tau + it)\lambda_j}|\varphi_j^C(\zeta)|^2,
\]

(26)

whose properties may be deduced from those of (25). Here, the wave kernel \( U(t, x, y) \) is the kernel of \( e^{it\sqrt{\Delta}} \) and (26) is the Poisson wave kernel obtained by analytically continuing the wave kernel in time and in space. The asymptotics are obtained by constructing a parametrix for (26) as a Fourier integral Toeplitz operator (or dynamical Toeplitz operator) in Proposition 7.1. We used such a construction in [ZJDG] in studying analytic continuation of eigenfunctions.

**Proposition 0.10.** For each \( \zeta, \) (26) is a homogeneous Lagrangian (Fourier integral) distribution with complex phase in \( t \) which is singular at \( t = 0 \) and at the periods \( t = nT(\zeta) \) of \( \zeta \) if it is a periodic point (Definition 2.5). The principal symbol of \( t \rightarrow U_C(t + 2it, \zeta, \bar{\zeta}) \) at a period \( nT(\zeta) \) is given by \( G_n(\zeta) = \langle W_{J_\zeta}(Dg_{nT(\zeta)}^\nu(\zeta), \Omega_{J_\zeta}, \Omega_{J_\zeta}) \rangle \) (16).

To prove the Proposition, we construct the wave group in the complex domain (as in [ZJDG]) as a dynamical Toeplitz operator of the form,

\[
V^t_r := \Pi_rg^t_r\sigma_{t, \tau}\Pi_r,
\]

(27)

where \( \Pi_r \) is the Szegö kernel of \( \partial M_r, g^t_r \) is translation by the geodesic flow (12) and \( \sigma_{t, \tau} \) is a certain symbol, designed to make (27) a unitary group. In Proposition 7.1 it is shown (roughly speaking) that

\[
U_C(t + 2it, \zeta, \bar{\zeta}) = V^t_r(\zeta, \bar{\zeta}).
\]

(28)

To determine the singularities in Proposition 0.10, we convolve with a suitable test function \( \chi \) and determine the leading term as \( \lambda \rightarrow \infty \) of the smoothed temperate sums,

\[
\chi \ast dP_{\nu_k}^*((\zeta, \bar{\zeta})) = \int_\mathbb{R} \hat{\chi}(t)e^{i\lambda t}U_C(t + 2it, \zeta, \bar{\zeta})dt.
\]

(29)
The asymptotics of (29) are determined by substituting this expression into (27), using the Boutet de Monvel-Sjöstrand parametrix for $\Pi$, and then employing the stationary phase method in the complex domain in §9, to obtain,

**Theorem 0.11.** Let $m = \dim M$. Let $\zeta \in \partial M_\tau$ be a periodic point of $g^t_\tau$. Then, for $\chi \in \mathcal{S}(\mathbb{R})$ with $\hat{\chi} \in C^\infty_c(\mathbb{R})$, there exist positive universal dimensional constants $C_m, C'_m$ so that $\chi * dP^\tau_{[0,\lambda]}(\zeta, \bar{\zeta})$ admits a complete asymptotic expansions as $\lambda \to \infty$, satisfying

$$
\chi * dP^\tau_{[0,\lambda]}(\zeta, \bar{\zeta}) = \begin{cases} 
C_m \lambda^{\frac{m-1}{2}} + \mathcal{O}(\lambda^{\frac{m-3}{2}}), & \zeta \notin \mathcal{P}, \\
C_m \lambda^{\frac{m-1}{2}} + C'_m \lambda^{\frac{m-3}{2}} \Re \sum_{n=1}^\infty \hat{\chi}(nT(\zeta)) e^{-i\lambda nT(\zeta)} \mathcal{G}_n(\zeta) + \mathcal{O}(\lambda^{\frac{m-3}{2}}), & \zeta \in \mathcal{P}.
\end{cases}
$$

The main difficulty in the proof lies in interpreting the Hessian determinants in the stationary phase expansion explicitly in geometric terms, which is necessary in understanding the convergence of the $Q_\zeta(\lambda)$ function. As is proved in Lemma 3.3, the principal term on the right side of Theorem 0.11 is the Gaussian integral,

$$
\langle W_{\zeta} (S_\zeta) (\Omega_{\zeta}), \Omega_{\zeta} \rangle = \int_{\mathbb{C}^{m-1}} e^{-\frac{1}{\lambda}(|u|_{L(\zeta)}^2 + |S_\zeta(u)|_{L(\zeta)}^2)} du.
$$

where $S_\zeta = D_\zeta g^T_\tau$ and where $L_\zeta$ is the Levi metric at $\zeta$. In principle, the principal symbol could be calculated using the Boutet de Monvel-Guillemin symbol calculus for Toeplitz operators [BoGu], whose purpose is to explicitly evaluate Hessian determinants in the stationary phase formulae in a metaplectic way. But their calculus involves a somewhat abstract comparison to a Grushin system of harmonic oscillators on $\mathbb{R}^n$. Instead, we use a simpler and more natural approach in the complex setting of combining the Boutet de Monvel-Sjöstrand parametrix and the ‘osculating Bargmann-Fock’ representation. A novelty is that we use a geometric construction of Phong-Stein [PhSt1, PhSt2] of a certain foliation to construct representations of the relevant oscillatory integrals (see below). It is the codimension one foliation $\Im \psi(\zeta, w) = 0$ of $\partial M_\tau$ associated to the phase $\psi(x, y)$ of the Boutet-de-Monvel-Sjöstrand parametrix. Expressing oscillatory integrals as integrals over the leaves allows one to compute the principal term of the asymptotics in a geometrically transparent fashion. The Phong-Stein leaves are introduced in Section 2.6 and their application to the computation is given in Section 8.3.

With no additional effort we prove the analogue of Theorem 0.11 for a ‘purely dynamical’ operator kernel in the Grauert tube setting, namely the spectral projections kernel

$$
\Pi_{\chi, \tau}(\lambda) := \chi(\Pi_\tau \sqrt{\rho} \Pi_\tau - \lambda) = \Pi_\tau \int_{\mathbb{R}} \hat{\chi}(t) e^{-it\lambda} e^{it\Pi_\tau D_\sqrt{\rho} \Pi_\tau} dt,
$$

(30)
of the Toeplitz differential operator

$$
\Pi_\tau D_\sqrt{\rho} \Pi_\tau : H^2(\partial M_\tau) \to H^2(\partial M_\tau)
$$

(31)
on $L^2(\partial M_\tau)$ where $D_\sqrt{\rho}$ is $\frac{1}{2} \Xi_\sqrt{\rho}$, where $\Xi_\sqrt{\rho}$ is the Hamilton vector field of the Grauert tube function $\sqrt{\rho}$ acting as a differential operator. Here, $\chi \in \mathcal{S}(\mathbb{R})$ (Schwartz space) with $\hat{\chi} \in C^\infty_c(\mathbb{R})$; the extra $\Pi_\tau$ is needed to define the unitary group

$$
\mathcal{W}_\tau(t) := \Pi_\tau e^{it\Pi_\tau D_\sqrt{\rho} \Pi_\tau} : H^2(\partial M_\tau) \to H^2(\partial M_\tau)
$$

(32)
We note that (32) is very similar to (27) and that (30) is very similar to (29). The proof of Theorem 0.11 applies with no essential change to (30), and Theorem 0.4 is also valid for (31). We introduce (32) because it is the natural Koopman dynamics in the Grauert tube setting, defined entirely in terms of the complex structure and the geodesic flow. Its spectral theory seems of independent interest. We denote the Schwartz kernel of $\Pi_{\chi,\tau}(\lambda)$ on the diagonal by $\Pi_{\chi,\tau}(\lambda, \zeta, \bar{\zeta})$.

**Theorem 0.12.** Let $m = \dim M$. Let $\zeta \in \partial M$ be a periodic point of $g_t^l$. Then, for $\chi \in \mathcal{S}(\mathbb{R})$ with $\hat{\chi} \in C_\infty^0(\mathbb{R})$, there exist positive universal dimensional constants $C_m, C_m'$ so that $\chi * dP_{[0,\lambda]}^\tau(\zeta, \bar{\zeta})$ admits a complete asymptotic expansions as $\lambda \to \infty$, satisfying

$$
\Pi_{\chi,\tau}(\lambda, \zeta, \bar{\zeta}) = \begin{cases} 
C_m \lambda^{m-1} + O(\lambda^{m-3}), & \zeta \notin \mathcal{P}, \\
C_m \lambda^{m-1} + C_m' \lambda^{m-1} \text{Re} \sum_{n=1}^{\infty} \hat{\chi}(nT(\zeta))e^{-i\lambda nT(\zeta)}G_n(\zeta) + O(\lambda^{m-3}), & \zeta \in \mathcal{P}.
\end{cases}
$$

The only significant difference between Theorem 0.11 and Theorem 0.12 lies in the power of $\lambda$, reflecting that the two operators have a different power of $|\zeta|$ in their principal symbols. The origin of this difference lies in the operator $A_\tau$ in Lemma 7.2, which does not arise in (30) and which lowers the order of $P_{[0,\lambda]}^\tau$ relative to (30). Since $P_{[0,\lambda]}^\tau$ is the main focus of this article, we will carry out the analysis in more detail for this kernel and then explain the very simple modifications necessary to deal with (30). A comparison of the two types of operator kernels in Theorem 0.11, and a comparison with Szegö's kernel asymptotics on line bundles, is given in Section 2.9.

To complete the proof of Theorem 0.4, we use the Fourier Tauberian method of Safarov [Saf, SV] (see §13.2). The singularities of $U_C(t + 2i\tau, \zeta, \bar{\zeta})$ at periods $t \neq 0$ of the periodic orbit through $\zeta$ are all of the same degree (strength) as the singularity at $t = 0$. That is why one needs to sum over periods. As discussed above, they give rise to an oscillating second term or possibly a discontinuous middle term depending on the continuity of the function $Q_\zeta(\lambda)$ (18).

### 0.7. Related problems and results.

Q-functions in the real domain were introduced by Safarov (see [Saf, SV]) to obtain two-term Weyl laws. The pointwise asymptotics are sharper in the complex domain than in the real domain, and resemble the two term pointwise quasi-Weyl asymptotics of Safarov et al [Saf, SV]. An interesting aspect of Theorem 0.4 is that the formula for $Q_\zeta$ is valid even if the closed geodesic through $\zeta$ is degenerate as a closed geodesic in the real domain. For instance, it is valid on a sphere, Zoll manifold or flat torus (Section 11.2). The analogous two-term formula in the real domain involves an integration over the set $L_x \subset S^*_x M$ of loop directions. In the real domain, $L_x$ might be dense in $S^*_x M$ and additionally might fail to have the kind of cleanliness or transversality properties that are required for application of the stationary phase method. In contrast, in the complex domain there is a single critical point when $\zeta$ is a periodic point and the stationary phase method is always applicable.

As mentioned above, it follows from Theorem 0.7 that if a sequence of Husimi distributions $U_\zeta(3)$ attains the maximal $L^\infty$ bound at $\zeta$, then it obtains the bound on the whole closed geodesic (Hamiltonian orbit) $\gamma_\zeta$ with initial data $\zeta$. It would be interesting to relate this result to C. Sogge's result [Sog11, Proposition 3.1] that if $\gamma$ is not an arc of a periodic
geodesic, then the configuration space Kakeya-Nikodym norms are not achieved. We expect that Sogge’s result has an analogue for phase space Kakeya-Nikodym norms. Eigenfunctions cannot decay faster than a Gaussian in the transverse directions to \( \gamma \) in configuration space \( M \). The corresponding statement for the Husimi distributions is more complicated, since analytic continuation adds directions in which the complexified eigenfunctions may grow exponentially. But one expects that the \( L^\infty \) bound over the \( \lambda^{-\frac{1}{2}} \)-tube around \( \gamma_\zeta \) to saturate microlocal Kakeya-Nikodym norms. The relation between the norms is currently under investigation. See [ZCBMS, Section 3.3] and the Problems in [ZCBMS, Problem 10.3].

Another comparison in terms of techniques and results in the real domain is to compare the complexification techniques of this article to the use of Gaussian beam decompositions in the work of Canzani-Galkowski (see [CG19, CG19b]) and the use of defect measures in [G19]. As mentioned above, it would be interesting to study the relation between \( L^\infty \) norms of Husimi distributions (3) and the microlocal Kakeya-Nikodym norms defined above. It would also be interesting to compare these norms to the ones in [BlS17] and explore the relations to \( L^p \) norms on \( M \).

The Husimi distributions (3) and pointwise Weyl asymptotics for \( P_T^T(\zeta, \bar{\zeta}) \) can be pushed-forward to \( M \) by integrating over the fibers of the natural projections \( \pi : \partial M \rightarrow M \) (essentially the cosphere bundle of radius \( \tau \)). This is straightforward but postponed to a latter occasion. Recalling that Wigner distributions and other ‘microlocal lifts’ of eigenfunctions push forward to the squares of the eigenfunctions, it would be interesting to determine the pushforwards of the Husimi distributions. The interesting aspect is that only \( \zeta \in \pi^{-1}(x) \) which lie on closed geodesics contribute sub-principal terms to the asymptotics, whereas sup-norm bounds on eigenfunctions reflect the measure of all closed geodesic loops (see [SoZ16]).

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1. Examples illustrating the continuity properties of \( Q_\zeta(\lambda) \)

Let us give simple examples where \( Q_\zeta(\lambda) \) and others where it has jumps.

**Proposition 1.1.** If \( \zeta \) generates a hyperbolic closed geodesic, i.e. if \( D_\zeta g^{nT(\zeta)} (9) \) is a real hyperbolic symplectic map, then \( Q_\zeta(\lambda) \) is uniformly continuous in \( \lambda \).

Indeed, in this case \( P = (S^T S)^{\frac{1}{2}} \) is diagonalizable with real positive eigenvalues that come in pairs \( e^{\mu_j}, e^{-\mu_j} \). Since \( D_\zeta g^{nT(\zeta)} = (D_\zeta g^{T(\zeta)})^n \), with eigenvalues \( e^{\pm n\mu_j} \), it is not hard to prove that the series (18) convergences. Hence, \( Q_\zeta(\lambda) \) is continuous in \( \lambda \) for manifolds with only hyperbolic geodesics, such as negatively curved manifolds.

As is easily proved by elementary calculations, \( Q_\zeta(\lambda) \) is also uniformly continuous on flat tori (Section 1.2). It is natural to conjecture that \( Q_\zeta(\lambda) \) is continuous at all periodic points on a general \((M, g)\) without conjugate points.

As a further example with conjugate points, all geodesics through an umbilic point \( u \) of a tri-axial ellipsoid \( \mathcal{E} \subset \mathbb{R}^3 \) are geodesic loops at \( u \) of period \( 2\pi \), but only one direction \( \zeta \) at \( u \) (up to time reversal) gives a smoothly closed hyperbolic geodesic. If \( \zeta \in S^n_u \mathcal{E} \), then \( Q_\zeta = 0 \) unless \( \zeta \) is the closed geodesic direction. Since it is hyperbolic, there are no jumps in \( Q_\zeta \) for such \( \zeta \). The ellipsoid does contain elliptic closed geodesics (namely, the elliptical
slices by coordinate hyperplanes \( x_1 = 0 \), or \( x_3 = 0 \), so it is not clear that the universal sup norm bounds are not attained by a sequence of eigenfunctions of \( \mathcal{E} \). The results of [SoZ16] disqualify all points of \( \mathcal{E} \) for attainment of the universal sup-norm bound (2) in the real domain.

1.0.1. Examples where \( Q_\zeta \) has jumps. In contrast to Proposition 1.1, \( Q_\zeta(\lambda) \) has jump discontinuities of a rigid kind in the case of the standard sphere (Section 11.1) or Zoll manifolds (Section 11.2). The existence of possible jumps explains why one must write the two-term asymptotics as the inequalities (20) rather than as an asymptotic equality as in the continuous case. An alternative which is available in pure Zoll cases is to prove asymptotics for \( P^\tau_{I_\lambda} \) where the intervals are adapted to the eigenvalue clusters of the Zoll manifold (Theorem 1.2). We now illustrate the jump with an elementary calculation on the circle \( S^1 \). In Section 11.2 we study Zoll examples.

In the real domain, the asymptotics of the spectral projection kernels are constant since \( S^1 \) acts by isometries. There is a single closed geodesic, of period \( 2\pi \), and

\[
\Pi_{[0,\lambda]}(x, x) = \sum_{k \in \mathbb{Z}, |k| \leq \lambda} 1 = 2(\lambda - \{\lambda\}) + 1,
\]

where, as above, \( \{\lambda\} \) is the fractional part of \( \lambda \); it is periodic with jump discontinuities at \( \lambda \in \mathbb{N} \).

Now consider the long interval damped spectral projections (8). We fix \( \zeta \in S^1_\mathbb{C} = \mathbb{C}/\mathbb{Z} \) and assume that \( \text{Im} \zeta \) is greater than zero. Then,

\[
P^\tau_{[0,\lambda]}(\zeta, \bar{\zeta}) = \sum_{k: |k| \leq \lambda} e^{-2\tau|k|} e^{2\text{Im} \zeta}
= 1 + \sum_{k: 0 < k \leq \lambda} 1 + \sum_{k < 0: |k| \leq \lambda} e^{-4\tau|k|}
= \sum_{0 < k \leq \lambda} 1 + C(\tau) + o(1), \quad C(\tau) = 1 + \sum_{k=1}^\infty e^{-4\tau|k|}
= \lambda - \{\lambda\} + C(\tau) + o(1),
\]

We note that there is a single geodesic of period \( T = 2\pi \), and \( W_1(S) = I \) has a single eigenvalue with \( s_\ell = 0 \). It has a single eigenfunction \( \Omega_{J_\lambda} \), which is not orthogonal to the ground state. Proposition 0.6 asserts that \( Q_\zeta(\lambda) = \frac{1}{2}\{\lambda T - \pi\}_{2\pi} = \{\lambda - \frac{1}{2}\}_{2\pi} \). One may compare this simple asymptotic with Theorem 0.4(3), when \( m = 1 \).

The discontinuity of the subprincipal term is reflected by the nature of the singularities at \( t \neq 0 \) of the “complex wave group” (see (26) below),

\[
U_C(t + 2i\tau, \zeta, \bar{\zeta}) = \sum_{k \in \mathbb{Z}} e^{(it-2\tau)|k|} e^{2(k\text{Im} \zeta)} = \sum_{k \geq 0} e^{itk} + R(t, \zeta, \bar{\zeta}), \quad (33)
\]

where \( R(t) \) is analytic in \( t \). By the Poisson summation formula, the singularities correspond to closed geodesics of \( S^1 \).

1.1. Zoll manifolds. One higher dimensional generalization of a circle is a Zoll manifold, all of whose geodesics are closed. As reviewed in §13, the spectrum of \( \sqrt{\Delta} \) on a Zoll manifold consists of a union of small eigenvalue clusters surrounding a certain arithmetic progression \( \{(k + \beta)\}_{k=0}^\infty \), where \( \beta \) is the common Morse index of the periodic orbits. If the interval \( I_k \) is chosen to cover the kth cluster and not to intersect any other cluster, then the cluster
projection $\Pi_{I_\rho}$ is a Riesz projector (i.e. a contour integral of the resolvent) and its analytic continuation $P^r_{I_\rho}$ has a complete asymptotic expansion.

**Theorem 1.2.** Suppose that $(M, g)$ is a real analytic Zoll metric. Then, if $\sqrt{\rho}(\zeta) \geq \frac{\zeta}{\xi}$, there exist geometric coefficients $R_j(\zeta)$ such that $P^r_{I_\rho}(\zeta, \bar{\zeta})$ admits a complete asymptotic expansion as $k \to \infty$ of the form,

$$P^r_{I_\rho}(\zeta, \bar{\zeta}) = \left( \frac{(k+\frac{\xi}{\sqrt{\rho}(\zeta)})^{\frac{m+1}{2}}}{(1 + \sum_{j=1}^{\infty} R_j(\zeta)(k + \frac{\xi}{\sqrt{\rho}(\zeta)}))^{-j}} \right)$$

Here, we have avoided the inequalities in Theorem 0.4 (3) by choosing to locate $\lambda$ at the center of each eigenvalue cluster. More precisely, to obtain the Weyl sum we simply sum the above result in $k$. (The notation $P^r_{I_\lambda}$ is discussed below (8).)

Although Zoll manifolds are not a primary focus in this article, they are the Riemannian analogues of the unit circle bundles associated to positive line bundles over Kähler manifolds, and as in the Kähler case we obtain complete asymptotic expansions if the spectral intervals $I_\lambda$ contain precisely one eigenvalue cluster (see Section 11.2). A brief review of spectral asymptotics in the real domain, and their dependence on the periodicity of the geodesic flow, is given in §13.

As mentioned in Remark 0.8, $Q_\zeta$ will also have jumps for convex surfaces of revolution if $\zeta$ is closed geodesic between the poles. This is an example where the spectrum is uniformly distributed modulo 1, i.e. does not come in clusters.

### 1.2. Parabolic closed geodesics and flat tori.

As a simple example to illustrate the terminology, the setting and the results, we work out the details for product eigenfunctions on a flat torus $M = \mathbb{R}^m/2\pi\mathbb{Z}^m$. The real eigenfunctions are the exponentials $e_k(x) = e^{i(x,k)}$ with $k \in \mathbb{Z}^m$. Since $M_C = \mathbb{C}^m/2\pi\mathbb{Z}^m = \mathbb{R}^m/2\pi\mathbb{Z}^m \times \mathbb{R}^m$ we see that $\sqrt{\rho}(\zeta) = |\xi|$, and for $\zeta = x + i\xi \in \partial M_\tau$, their complexifications are the complex exponentials $e_k^C(x + i\xi,k) = e^{i(x + i\xi,k)}$, and clearly

$$|e^{i(x + i\xi,k)}|^2 = e^{-2i(x,k)}.$$

Hence,

$$P^r_{[0,\lambda]}(\zeta, \bar{\zeta}) = \sum_{k \in \mathbb{Z}^m : |k| \leq \lambda} e^{-2\xi||k||} e^{-2\xi(x,k)}, \quad (\zeta = x + i\xi).$$

The point $\zeta \in \partial M_\tau$ is a periodic point if and only if $\operatorname{Im} \zeta = \frac{k}{|k|} \tau$ for some lattice point $k \in \mathbb{Z}^m$. Then $T(\zeta) = |k|$, and

$$Q_\zeta(\lambda) \sim \sum_{n=1}^{\infty} \sin(n\lambda |k|) \left( \frac{\lambda}{|nk| + 2i\tau} \right)^{\frac{m+1}{2}}.$$

For $m \geq 2$, the function $Q_\zeta(\lambda)$ is a bounded uniformly continuous function. If $\sqrt{\zeta} = \tau$,

$$P^r_{[0,\lambda]}(\zeta, \bar{\zeta}) \sim C_m \lambda^{\frac{m+1}{2}} \left( 1 + Q_\zeta(\lambda) \lambda^{-1} + o(\lambda^{-1}) \right).$$

To check Lemma 0.2, we note that the $L^2$ norm-square of $e_k^C(x + i\xi)$ over the Grauert tube $\partial M_\tau$, or equivalently over the co-sphere bundle $S^*_\tau \mathbb{R}^m/\mathbb{Z}^m$ is, by the usual steepest descent calculation of asymptotics of Bessel functions,
\[ \| e_k^C(x + i\xi) \|^2_{L^2(\partial M_\tau)} = \int_{S^{m-1}} e^{-2\langle \xi, k \rangle} d\mu_\tau(\xi) \]
\[ = \tau^{m-1} \int_{S^{m-1}} e^{-2|k|\tau \langle \xi, k \rangle} d\mu(\xi) \]
\[ = \tau^{m-1} \int_{S^{m-1}} e^{-2|k|\tau \langle \xi, e \rangle} d\mu(\xi) \]
\[ \sim C_m e^{2\tau |k|} \tau^{m-1} (\det(2|k|\tau I_{m-1})^{1/2} = C_m |k|^{-\frac{m-1}{2}} e^{2\tau |k|} \tau^{\frac{m-1}{2}}, \]

The Husimi functions,
\[ \frac{|e^i(x + i\xi, k)|^2}{\|e_k^C(x + i\xi)\|^2_{L^2(\partial M_\tau)}} = C_m \tau^{-\frac{m-1}{2}} |k|^{\frac{m-1}{2}} e^{-2\langle \xi, k \rangle} e^{2\tau |k|}, \]
attain their maximum when \( \xi = -\tau \frac{\bar{e}}{|k|} \) and at that point take the value \( C_m \tau^{-\frac{m-1}{2}} |k|^{\frac{m-1}{2}} \).

\[ \text{2. Grauert tubes: Geometry and Analysis} \]

In this section, we review geometry and analysis on Grauert tubes. The relevant geometry and analysis have already been presented in the prior articles [ZPSH1] and [ZJDG]. To avoid duplication, we only briefly introduce the basic objects and results and refer to these articles for further background.

2.1. Grauert tubes, the Hamilton flow of \( \sqrt{\rho} \) and the complex geodesic flow.

In this section, we briefly review the basic objects regarding Grauert tubes and establish notation. There is considerable overlap with the exposition in [ZPSH1] and we refer there for many details.

The Grauert tube function (6) induces a distinguished 1-form,
\[ \alpha = \frac{1}{i} \partial \rho |_{\partial M_\tau} \]
\[ \text{on } \partial M_\tau. \]

It induces the Kähler form
\[ \omega = d\alpha = i \partial \bar{\partial} \rho \]
\[ \text{with } \rho \text{ as Kähler potential, and also the volume form} \]
\[ d\mu_\xi := \alpha \wedge \omega^{m-1}, \quad (m = \text{dim } M). \]

For generic analytic Riemannian metrics \( g \), there is a finite maximal radius \( \tau_{\max} \) of the Grauert tubes, which is finite for all but a few real analytic Riemannian metrics. The eigenfunctions are known to extend holomorphically to the maximal open Grauert tube but do not extend further. For instance, the maximal Grauert tube radius for hyperbolic space of constant curvature \(-1\) is \( \frac{\pi}{2} \). As a result, the Grauert tube Weyl laws are only valid for \( \tau < \tau_{\max} \) and should blow up when \( \tau = \tau_{\max} \). We refer to [LS1, Szo] and to §2.1 for background.
2.2. **The diastasis.** As with any real analytic Kähler potential, we may consider \( \rho(z) \) as a function of \((z, \bar{z})\) and analytically extend it to a function \( \rho(z, w) \) on the complexification \( M_\tau \times \overline{M_\tau} \) of \( M \), as a function holomorphic in \( z \) and \( \bar{w} \). Thus, the analytic extension of \( \rho(z) \) has the form \( \rho(z, w) = f(z, \bar{w}) \) where \( f \) is holomorphic in both variables and satisfies \( f(z, w) = f(w, z) \), \( \overline{f(z, w)} = f(\bar{z}, \bar{w}) \).

The defining function of the Grauert tube \( M_\tau \) is \( \sqrt{\rho - 2\tau} \), or equivalently \( (\rho(z, z) - 4\tau^2) \). Thus, \( \rho < 4\epsilon^2 \) in \( M_\tau \), \( \rho \neq 0 \) when \( \rho = 4\epsilon^2 \), and the Levi matrix \( \left( \partial^2 \rho / \partial z_j \partial z_k \right) \) is positive Hermitian non-degenerate. Indeed, \( i\partial\bar{\partial}\rho = \omega_\rho \) is a Kähler form on \( M_\tau \).

Following the notational conventions of [BoSj, Proposition 1.1],

\[
\psi_\tau(\zeta, w) = \frac{1}{i}(\rho(\zeta, w) - 4\tau^2), \quad (\zeta, w \in \partial M_\tau).
\]

Thus, \( \psi_\tau(z, w) \) is holomorphic in \( z \), anti-holomorphic in \( w \) and satisfies

\[
\psi(z, w) = -\overline{\psi(w, z)}.
\]

The imaginary part of (38) is minus the Calabi diastasis function,

\[
D(z, w) = - (\rho(z, w) + \rho(w, z) - \rho(z, z) - \rho(w, w)).
\]

Near the diagonal, (38) admits the Taylor expansion,

\[
\frac{1}{i} (\psi(x, y) + \psi(y, x) - \psi(x, x) - \psi(y, y)) = L_\rho(x - y) + O(|x - y|^3).
\]

where \( L_\rho \) is the Levi form (or Kähler form), and so as a function on \( \Omega \times \Omega \),

\[
\text{Im} \psi(z, w) \geq C (d(z, \partial \Omega) + d(w, \partial \Omega) + |z - w|^2) + O(|z - w|^3).
\]

**Remark 2.1.** The notational convention of [BoSj] of putting \( i^{-1} \) in front of \( \rho - 4\tau^2 \) disagrees with that of Phong-Stein [PhSt1] (page 96), who use the notation \( \psi \) rather than \( i\psi \) for the complex phase.

2.3. **Kähler normal coordinates.** Since \((M_\tau, i\partial\bar{\partial}\rho)\) is a Kähler manifold, we may define Kähler normal coordinates around any point \( p \in M_\tau \). Local holomorphic coordinates \((z_1, \ldots, z_m)\) in a neighborhood \( U \) of \( p \) are called Kähler normal coordinates centered at \( \zeta \in \partial M_\tau \) with \( z(\rho) = 0 \) if the Kähler potential \( \rho \) takes the form,

\[
\rho(z) = 4\tau^2 + |z|^2 + \sum_{jK} a_{jK} z^j \bar{z}^K, \quad \text{with } |J| \geq 2, |K| \geq 2,
\]

so that the Kähler form is locally given by,

\[
\omega = \omega_0 + \sum_{ijkt} R_{ijkt} z_i \bar{z}_j dz_k \wedge d\bar{z}_t + \cdots, \quad \omega_0 = \sum_j dz_j \wedge d\bar{z}_j.
\]

where \( R_{ijkt} \) is the curvature.

**Lemma 2.2.** In Kähler normal coordinates centered at \( \zeta \in \partial M_\tau \), so that \( z = 0 \) denotes the point \( \zeta \) the diastasis has the form,

\[
D(\zeta, z) = |z|^2 + O(|z|^3).
\]

**Proof.** Set \( w = \zeta \) in (39) with \( \rho(\zeta, \zeta) = 4\tau^2 \), to get \( \rho(z, w) = 4\tau^2 + z\bar{w} + O^3 \). Then,

\[
D(z, w) = -(z\bar{w} + w\bar{z} - |z|^2 - |w|^2 + O^3) = |z - w|^2 + O^3.
\]

\[\square\]
2.4. **CR geometry of \( \partial M_\tau \).** Let us denote by \( J \) the adapted complex structure on \( M_\tau \) arising from the complexification of \( M \).

As a real hypersurface of the complex manifold \( M_\tau, \partial M_\tau \) has a \( CR \) structure, i.e. a real \( J \)-invariant horizontal symplectic hyperplane bundle defined by

\[
H = \ker \alpha \subset T\partial M_\tau, \quad J : H \rightarrow H, \quad H = JT\partial M_\tau \cap T\partial M_\tau. \tag{42}
\]

Then

\[
T\partial M_\tau = H \oplus \mathbb{R}T \tag{43}
\]

where \( T \) is the characteristic vector field satisfying

\[
\alpha(T) = 1, \quad d\alpha(T,\cdot) = \omega_\rho(T,\cdot) = 0 \quad \text{on} \quad \partial M_\epsilon. \tag{36}
\]

\( T = \Xi_{\sqrt{\rho}} \) is the just the Hamilton vector field of \( \sqrt{\rho} \) on \( M_\tau \) with respect to \( \omega \) \((36)\). After complexifying each horizontal space \( H_\zeta \otimes \mathbb{C} = H_{1,0}^\zeta \oplus H_{0,1}^\zeta \), we have the decomposition

\[
T_\zeta\partial M_\tau = H_{1,0}^\zeta \oplus H_{0,1}^\zeta \oplus \mathbb{C}T, \tag{44}
\]

We define the boundary Cauchy-Riemann operator \( \bar{\partial}_b \) operator by

\[
\bar{\partial}_b = df|_{H_{1,0}^\zeta}. \tag{45}
\]

2.5. **Geodesic and Hamiltonian flows.** We denote by \( \Xi_H \) the Hamiltonian vector field of a Hamiltonian \( H \) and its flow by \( \exp t\Xi_H \). Given the symplectic form \((36)\), we define the Hamiltonian flow

\[
g^t := \exp t\Xi_{\sqrt{\rho}}, \quad \text{on} \quad M_\tau. \tag{45}
\]

We denote the restriction of the Hamilton flow on the right side to the energy surface \( \partial M_\tau \) by,

\[
g^t_\tau : \partial M_\tau \rightarrow \partial M_\tau. \tag{46}
\]

We denote by \( G^t = \exp t\Xi_{|\xi|} \) the homogeneous Hamiltonian flow on \( T^*M \setminus \{0\} \). We also define the exponential map \( \exp_x : T^*M \rightarrow M \) of \( g \); as usual in geometry, the exponential map is defined by the Hamiltonian flow of \( |\xi|^2 \) rather than \( |\zeta| \).

The analytic continuation of the exponential map to imaginary time defines a diffeomorphism \((11)\) satisfying \( E^* \sqrt{\rho} = |\xi| \) \([GS1, LS1]\). It follows that \( E^* \) conjugates the geodesic flow on \( B^*M \) to the Hamiltonian flow \((45)\), i.e.

\[
E(G^t(x,\xi)) = g^t(\exp_x i\xi). \tag{46}
\]

We often restrict \((11)\) to the unit cosphere bundle \( S^*_\tau M \) of radius \( \tau \) and we then \((11)\)- \((46)\) become

\[
E_\tau : S^*_\tau \rightarrow \partial M_\tau, \quad g^t_\tau := E_\tau G^t E^{-1}_\tau. \tag{47}
\]

2.6. **Phong-Stein leaves as symplectic transversals.** In dealing with the geodesic flow on \( \partial M_\tau \) it is very useful to introduce symplectic transversals to the flow, and to define the time coordinate by the flow-time to a transversal. A very nice set of transversals (or, leaves) were introduced in \([PhSt1]\) (see also \([PhSt2, Example 4]\) on a general strictly pseudo-convex domain \( D \). In the Grauert tube setting, they are defined by,

\[
\mathcal{M}_\zeta = \{ w \in \partial M_\tau : \text{Re } \psi(\zeta, w) = 0 \},
\]

where \( \psi \) is defined in \((38)\). The Phong-Sturm leaves \( \mathcal{M}_\zeta \) are not complex submanifolds of \( M_\tau \) in general, but at the point \( z \) they are tangent to \( H_{\zeta}^{*(1,0)} M_\tau \). We summarize the result in \([PhSt1]\):
Lemma 2.3. \( \mathcal{T}_\zeta \mathcal{M}(\zeta) = H^1_{\zeta, 0} \), hence \( \mathcal{M}_\zeta \) is transversal to \( T = \Xi_{\sqrt{2}} \) at \( \zeta \), hence is a local symplectic to the flow \( g^t_r \) (by (44)).

2.7. Heisenberg coordinates. We will be linearizing phases of integrals by taking Taylor expansions in special coordinate systems, generalizing the Kähler normal coordinates in the line bundle setting and the Heisenberg coordinates in [ShZ02] on boundaries of unit co-disc bundles. Since \( \mathcal{M}_\zeta \) is not a complex hypersurface in \( M_\tau \) it is a fortiori not a Kähler manifold and it does not make sense to introduce Kähler normal coordinates on it. However, there exist Kähler normal coordinates on \( M_\tau \) (see Section 2.3) and one may define ‘Heisenberg coordinates’ as follows:

Definition 2.4. We define ‘slice-orbit’ coordinates on \( \partial M_\tau \) near \( \zeta \) as the (locally defined) inverse of the slice-orbit parametrization,

\[
(w', t) \in \mathcal{M}_\zeta \times \mathbb{R} \to g^t_r(w').
\]

By slice-orbit Kähler normal coordinates we mean slice orbit coordinates with \( w' \) defined by the restriction of Kähler normal coordinates centered at \( \zeta \) on \( \mathcal{M}_\zeta \).

2.8. Return times and return maps. The map (48) is well defined for all \( t \) but is not one to one, and has many local inverses. The inverses will be important below and are best described in terms of the return times and (non-linear) local Poincaré first return map for the transversal near \( \zeta \). For points \( z \in \mathcal{M}_\zeta \) near \( \zeta \), the orbit \( g^t_r(z) \) will return to \( \mathcal{M}_\zeta \) at some minimal time \( T(z) \) near \( T(\zeta) \).

Definition 2.5. Denote by

\[
\Phi_\zeta(z) = g^{T(z)}(z) : \mathcal{M}_\zeta \to \mathcal{M}_\zeta
\]

first return map to \( \mathcal{M}_\zeta \). We further define the \( n \)th return time \( T_n \) so that \( T_n(\zeta) = nT(\zeta) \). The \( n \)th return map \( \Phi_\zeta^n(z) \) is defined to equal \( g^{nT(z)}(z) \).

When \( \zeta \) is a periodic point, we obviously have

\[
(s, \zeta) \simeq (s + nT(\zeta), \zeta)
\]

in the sense that the two sides get taken to the same point under the slice-orbit parametrization.

The \( n \)th return map is only well-defined for \( z \) sufficiently close to \( \zeta \), but this is sufficient for the proof of the pointwise Weyl laws. We then consider \( z \in \mathcal{M}_\zeta \) near \( \zeta \) and use slice orbit coordinates for the \( n \)th return \( \Phi_\zeta^n(z) \in \mathcal{M}_\zeta \). For \( t \) near \( T_n(\zeta) \) and \( w \) sufficiently close to \( \zeta \), we have the following equivalence relation on slice-orbit coordinates:

\[
(t, z) \simeq (nT(\zeta) + (t - nT(\zeta)), \Phi_\zeta^n(z))
\]

2.9. Comparison to the line bundle setting. In this section, we compare the geometry of the Grauert tube setting to that of the line bundle setting. Grauert tubes are the Riemannian analogue of (co-) disc bundles \( D^* \subset L^* \) of positive Hermitian line bundles \( L \to M \) over Kähler manifolds. The line bundle setting is that of a positive Hermitian holomorphic line bundle \( (L, h) \to (M, \omega) \) over a Kähler manifold, where \( i\partial\bar{\partial}\log h = \omega \).

Let \( L^* \) be the dual line bundle, let \( h^* : L^* \to \mathbb{R} \) be the dual Hermitian metric and let \( X_h = \{ h^* = 1 \} \) be the boundary of the unit co-disk bundle. It is a strictly pseudo-convex CR hypersurface in the complex manifold \( L^* \) and is the analogue of \( \partial M_\tau \). The Reeb vector...
field $\frac{\partial}{\partial \theta}$ is the analogue of the Hamilton vector field $\Xi_H$ on $\partial M$, but has two significantly simpler properties: First, it generates an $S^1$ action ($S^1 = \mathbb{R}/\mathbb{Z}$), which in the Riemannian case is only true on a Zoll manifold; second, and more important, it generates a holomorphic action which complexifies to a $\mathbb{C}^*$ action. In the Riemannian setting, the geodesic flow is never holomorphic, and (except for Zoll manifolds), the orbits of the geodesic flow almost never form a fiber bundle over a quotient space and there is no analogue of $M$.

There are also significant differences in the behavior of Szegö kernels and the linearization of other relevant kernels to osculating Heisenberg spaces. We make substantial use of the Phong-Stein foliation and their analysis of adapted coordinates in [PhSt1, PhSt2] in the linearization procedure. Moreover, this article is fundamentally about analytic continuations of eigenfunctions from $M$ to the Grauert tube, while in [ZZ18] the results pertain to holomorphic sections of powers of $L$ that have no underlying real structure. But we are able to refer to [ZZ18] for many of the details on local Bargmann-Fock-Heisenberg approximations and linearizations of Fourier integral Toeplitz operators. We refer to Section 5.4 for further comparisons to the line bundle case. The pointwise Weyl law in the line bundle setting was proved by the author and P. Zhou in [ZZ18], and there are many overlaps in this article and [ZZ18].

3. Linear and Heisenberg models

In this section, we review the linear Bargmann-Fock models. As in [ZZ18] and elsewhere, we often reduce calculations in the nonlinear setting to osculating Bargmann-Fock and Heisenberg models on the tangent spaces. Much of the exposition below repeats that in [ZZ18]; proofs are generally omitted unless they are short, and the reader is referred to [ZZ18] for further details.

3.1. Heisenberg group. The space $\mathbb{C}^m \times S^1$ can be identified with the reduced Heisenberg group $H^m_{\text{red}}$, where the group multiplication is given by

$$(z, \theta) \circ (z', \theta') = (z + z', \theta + \theta' + \text{Im}(zz')).$$

We repeat some background from [ZZ18].

The generators of the Heisenberg group action are contact vector fields on the Heisenberg group generated by a linear Hamiltonian function $H : \mathbb{C}^m \to \mathbb{R}$. For any $\beta \in \mathbb{C}^m$, we define a linear Hamiltonian function on $\mathbb{C}^m$ by $H(z) = z\beta + \beta \bar{z}$. The Hamiltonian vector field on $\mathbb{C}^m$ is $\Xi_H = -i\beta \partial_z + i\bar{\beta} \partial_{\bar{z}}$, and it lifts to a contact vector field on $H^m_{\text{red}}$,

$$\hat{\Xi}_H = -i\beta \partial_z + i\bar{\beta} \partial_{\bar{z}} - \frac{1}{2}(z\bar{\beta} + \beta \bar{z}) \partial_\theta,$$

with respect to the contact form $\alpha = d\theta + \frac{i}{2} \sum_j (z_j d\bar{z}_j - \bar{z}_j dz_j)$, and generates the flow,

$$\hat{g}_t(z, \theta) = (-i\beta t, 0) \circ (z, \theta) = (z - i\beta t, \theta - t \text{Re}(\beta \bar{z})).$$

3.2. Symplectic Linear Algebra. We refer to [deG, Gutt, Gutt2, Gutt3] for background on symplectic linear algebra and symplectic normal forms.

Let $(V, \sigma)$ be a real symplectic vector space of dimension $2m$ and let $J$ be a compatible complex structure on $V$. There exists a symplectic basis in which $V \simeq \mathbb{R}^{2m}$, $\sigma$ takes the standard form $\omega = 2 \sum_{j=1}^m dx_j \wedge dy_j$ and $J$ has the standard form, $J_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. We
generally identify $(V, \sigma)$ with the standard real symplectic vector space $\mathbb{R}^{2m}$, $\omega = 2 \sum_{j=1}^{m} dx_j \wedge dy_j$.

We denote the symplectic group of $(V, \sigma)$ by $\text{Sp}(m, \mathbb{R})$, and its Lie algebra by $\text{sp}(m, \mathbb{R})$. The group $\text{Sp}(m, \mathbb{R})$ consists of linear transformation $S : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$, such that $S^* \omega = \omega$, and as in [F89] it may be expressed in block form as,

$$
\begin{bmatrix}
x'
y'
\end{bmatrix} = S
\begin{bmatrix}
x
y
\end{bmatrix} =
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}.
$$

(50)

The symplectic Lie algebra consists of skew symplectic matrices; it may be identified with the Poisson algebra of quadratic Hamiltonians.

The maximal compact subgroup of $\text{Sp}(m, \mathbb{R})$ is the unitary group $K = U(m) := \text{Sp}(m, \mathbb{R}) \cap \text{O}(2n, \mathbb{R})$ of $(\mathbb{R}^{2m}, J)$. It is the group of orthogonal matrices $U$ on $\mathbb{R}^{2m}$ satisfying $UJ = JU$.

One has

$$
U = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}, \quad AB^t = B^tA, \quad AA^t + BB^t = I, \quad U^{-1} = \begin{pmatrix} A^t & B^t \\ -B^t & A^t \end{pmatrix} = U^t.
$$

A symplectic matrix admits a polar decomposition $S = UP_S$ where $P_S = (S^*S)^{\frac{1}{2}}$ is a non-negative symplectic matrix and where $U \in U(m)$; see [F89, Proposition 4.3].

If $S \in \text{Sp}(m, \mathbb{R})$, then its transpose $S^t = JS^{-1}J^{-1}$ also lies in $\text{Sp}(m, \mathbb{R})$ and $SJ = J(S^t)^{-1}$. $S \in \text{Sp}(m, \mathbb{R})$ is a symmetric symplectic matrix if it satisfies $S^t = S$, and then $SJ = JS^{-1}$. We say that $S \in \text{Sp}(m, \mathbb{R})$ is a normal symplectic matrix if $[S, S^t] = 0$. In the polar decomposition $S = U\hat{P}_S$ of a normal $S \in \text{Sp}(m, \mathbb{R})$, one has $P_S U = U\hat{P}_S$. A symplectic matrix $S$ is symmetric positive definite if and only if $S = e^X$ with $X \in \text{sp}(m)$ and $X = X^t$. See for instance [F89, Proposition 4.7].

3.3. **Semi-simple symplectic matrices.** Let $S \in \text{Sp}(m, \mathbb{R})$. An element $S \in \text{Sp}(m, \mathbb{R})$ is called semi-simple if $\mathbb{R}^{2m} \otimes \mathbb{C} = \mathbb{C}^{2m}$ is the direct sum of the eigenspaces $E_\lambda$ of $S$. The eigenvalues of $S$ arise in quadruples $[\lambda] = \{ \lambda, \lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1} \}$.

A semi-simple matrix $M$ is one that may be diagonalized over $\mathbb{C}$, i.e. if $M \otimes \mathbb{C}$ is conjugate in $\text{GL}(2m, \mathbb{C})$ to a diagonal matrix. In general, it need not be conjugate in $\text{Sp}(m, \mathbb{R})$ to a diagonal matrix. If $M \in \text{Sp}(m, \mathbb{R})$ is diagonalizable over $\mathbb{R}$, then $\mathbb{R}^{2m}$ admits a symplectic basis consisting of eigenvectors of $M$, hence is symplectically diagonalizable; equivalently there exists $U \in U(m)$ so that $U^tSU = \Lambda$ is the diagonal matrix

$$
\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_m; \lambda_1^{-1}, \ldots, \lambda_m^{-1}).
$$

Indeed if $e_1, \ldots, e_n$ are orthonormal eigenvectors of $S$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$ then since $SJ = JS^{-1},$

$$
SJ e_k = JS^{-1} e_k = \frac{1}{\lambda_k} Je_k.
$$

Hence $\pm Je_1, \ldots, \pm Je_n$ are orthonormal eigenvectors. The basis $\{e_j, Je_k\}$ is symplectic.

Symplectic matrices such as $U \in U(m)$ with complex eigenvalues are not conjugate to their diagonalizations in $\text{Sp}(m, \mathbb{R})$. According to [Gutt, Gutt2, Gutt3], the conjugacy classes of $A \in \text{Sp}(m, \mathbb{R})$ are determined by three types of data: (i) the eigenvalues of $A$, (ii) $\dim(\ker(A-$

\[3\text{It is sometimes denoted Sp}(2m, \mathbb{R})\]
λ) for \( r \geq 1 \) for one eigenvalue in each \([\lambda]\); and for \( \lambda = \pm 1 \), the rank and signature data of an associated quadratic form. To avoid excessive technicalities, we only consider semi-simple symplectic matrices.

Let
\[
W_{[\lambda]} = E_\lambda \oplus E_{\lambda^{-1}} \oplus E_{\bar{\lambda}} \oplus E_{\bar{\lambda}^{-1}}.
\]
Then \( W_{[\lambda]} = V_{[\lambda]} \otimes \mathbb{C} \) where \( V_{[\lambda]} \) is a real symplectic subspace, and one has symplectic orthogonal decompositions,
\[
\mathbb{R}^{2m} = \bigoplus_{j=1}^{k} V_{[\lambda_j]}, \quad \mathbb{C}^{2m} = \bigoplus_{j=1}^{k} W_{[\lambda_j]}, \tag{51}
\]
where \( k \) is the number of distinct 4-tuples of eigenvalues. Note that eigenvalues \( \pm 1 \) are special, since then \( E_\lambda = E_{\lambda^{-1}} \), and the 4-tuple collapses to a singleton.

- We say that \( S \in \text{Sp}(m, \mathbb{R}) \) is a positive definite symmetric symplectic matrix (or, a real hyperbolic symplectic matrix) if it is conjugate in \( \text{Gl}(2m, \mathbb{R}) \) to the diagonal matrix \( \Lambda \) with \( \lambda_j > 0 \). In this case, there exists \( U \in U(m) \) so that \( S = U^t \Lambda U \).

- \( S \) is complex hyperbolic if it is semi-simple and its decomposition (51) contains eigenvalue quadruples \( \lambda \in \mathbb{C} \) with \( \text{Re} \lambda \neq 0 \).

- \( S \) is elliptic if all \( \lambda \) in (51) satisfy \( |\lambda| = 1, \lambda \neq \pm 1 \).

- \( S \) is degenerate elliptic if all \( \lambda \) in (51) satisfy \( |\lambda| = 1 \), and there exists \( \lambda \) with \( \lambda = \pm 1 \).

3.4. **Complex structures.** Let \( V \) be a real symplectic vector space and let \( V_\mathbb{C} = V \otimes \mathbb{C} \) be its complexification. Given a complex structure \( J \) on \( V \), let \( H^{1,0}_J \) resp. \( H^{0,1}_J \), denote the \( \pm i \) eigenspaces of \( J \) in \( V \otimes \mathbb{C} \). The projections onto these subspaces are denoted by
\[
P_J = \frac{1}{2}(I - iJ) : V \otimes \mathbb{C} \to H^{1,0}_J, \quad \bar{P}_J = \frac{1}{2}(I + iJ) : V \otimes \mathbb{C} \to H^{0,1}_J. \tag{52}
\]
In complex coordinates \( z_i = x_i + iy_i \), we have then
\[
\begin{bmatrix} z' \\ \bar{z}' \end{bmatrix} = \begin{bmatrix} P & Q \\ \bar{Q} & \bar{P} \end{bmatrix} \begin{bmatrix} z \\ \bar{z} \end{bmatrix} =: M \begin{bmatrix} z \\ \bar{z} \end{bmatrix},
\]
where (in the block notation of (50)) \( M = \mathcal{W}^{-1} S \mathcal{W} \), i.e.
\[
M := \begin{bmatrix} P & Q \\ \bar{Q} & \bar{P} \end{bmatrix} = \mathcal{W}^{-1} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mathcal{W}, \quad \mathcal{W} = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -iI & iI \end{bmatrix}. \tag{53}
\]
This conjugate group of \( M \) is denoted by \( \text{Sp}_c(m, \mathbb{R}) \subset \text{GL}(2m, \mathbb{C}) \) in [F89], and one has the identities [F89, Prop. 4.17],
\[
\begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}^{-1} = \begin{pmatrix} P^* & -Q^t \\ -\bar{Q}^* & \bar{P}^t \end{pmatrix}.
\]
By (52), the upper left block \( P \) is given in terms of the real blocks (50) of \( S \) by,
\[
P_J S P_J = P = \frac{1}{2}(A + D + i(C - B)). \tag{54}
\]
In the following, we assume \( D \) is the identity. The orthogonal projection onto \( L \) is the 'vacuum state' and \( S \) is a normal symplectic matrix, i.e. that \( S \) commutes with \( S^* \).

**Lemma 3.1.** Let \( S \) be a normal symplectic matrix, and let \( S = U \hat{P}_S \) be the polar decomposition of \( S \). Then, \( \det P_JSP_J = \det P_JUP_J \cdot \det P_J \hat{P}_SP_J \), with \( |\det P_JUP_J| = 1 \).

**Proof.** For \( U \in U(m) \), \( UJ = JU \), so by (52) \( U \hat{P}_J = P_JU \).

Since \( S \) is normal, \( P_JU = UP_J \), and since \( P_J \) is a projection, \( P_JSP_J = P_JU \hat{P}_SP_J = (P_JUP_J)(P_J \hat{P}_SP_J) \). Hence, it suffices to prove that \( \det P_JUP_J = 1 \). But if we use (53) to conjugate \( U \in O(2m) \cap Sp(m, \mathbb{R}) \) to \( Sp_c(m, \mathbb{R}) \) it is a block diagonal matrix with \( Q = 0 \), \( P = \hat{U} \) unitary on \( \mathbb{C}^m \). It follows that \( P_JUP_J = \hat{U} \) and \( \det P_JUP_J \in S^1 \).

\[ \square \]

### 3.5. Heisenberg and metaplectic representations.

The metaplectic representation is a representation of the double cover of \( Sp(m, \mathbb{R}) \simeq Sp_c(m, \mathbb{R}) \) on Bargamann-Fock space. The Bargmann-Fock space of a symplectic vector space \((V, \sigma)\) with compatible complex structure \( J \in J \) is the Hilbert space,

\[ \mathcal{H}_J = \{ fe^{-\frac{1}{2} \sigma(v,Jv)} \in L^2(V, dL); f \text{ is entire } J \text{- holomorphic} \}. \]

Here,

\[ \Omega_J(v) := e^{-\frac{1}{2} \sigma(v,Jv)} \]

is the 'vacuum state' and \( dL \) is normalized Lebesgue measure (normalized so that square of the symplectic Fourier transform is the identity). The orthogonal projection onto \( \mathcal{H}_J \) is denoted by \( P_J \) in [D] but we denote it by \( \Pi_J \) in this article. Its Schwartz kernel relative to \( dL(w) \) is denoted by \( \Pi_J(z, w) \).

The Heisenberg group acts on Bargmann-Fock space by phase space translations,

\[ W(a) : \mathcal{H}_J \to \mathcal{H}_J \]

for \( a \in V \), defined by

\[ (W(a)\psi)(v) = e^{i\sigma(a,v)}\psi(v - a). \]

The (double cover) \( Mp(m, \mathbb{R}) \) of \( Sp(m, \mathbb{R}) \) acts on the Bargmann-Fock space \( \mathcal{H} \) by unitary integral operators. Following [D], we denote by \( W_J(S) \) the unitary operator associated to \( S \in Mp(m, \mathbb{R}) \) but for simplicity we view \( S \) as an element of \( Sp(m, \mathbb{R}) \). Let \( U_S \) be the unitary translation operator on \( L^2(\mathbb{R}^{2n}, dL) \) defined by \( U_S F(x, \xi) := F(S^{-1}(x, \xi)) \). The metaplectic representation of \( S \) on \( \mathcal{H}_J \) is given by (\([D], (5.5) \) and (6.3 b))

\[ W_J(S) = \eta_{J,S} \Pi_J U_S \Pi_J, \]

where

\[ \eta_{J,S} = 2^{-n} \det(I - iJ) + S(I + iJ)^{1/2}. \]

Given \( M \) as in (53), the Schwartz kernel of \( W_J(M) \) is given by

\[ \mathcal{K}_M(z, w) = (\frac{1}{2\pi})^m |\det P|^{-1/2} \exp \left\{ \frac{1}{2} (zQP^{-1}z + \bar{w}P^{-1}z - \bar{w}P^{-1}Q\bar{w}) \right\}. \]

\[ \text{The notation } \hat{P}_S \text{ for a positive matrix should not be confused with the block } P = P_JSP_J. \]
where the ambiguity of the sign the square root \((\det P)^{-1/2}\) is determined by the lift to the double cover. The following Proposition ties together the dynamical Toeplitz formula \((56)\) for \(W_J(S)\) with the kernel formula \((58)\).

**Proposition 3.2.** Let \(M\) be a linear symplectic map \((53)\). Then, the Schwartz kernel \((58)\) of its metaplectic quantization may be expressed in terms of the Szegő projector \(\Pi_J\) onto \(\mathcal{H}_J\) by,

\[
K_M(z, w) = (\det P^*)^{1/2} \int_{\mathbb{C}^m} \Pi(z, Mu) \hat{\Pi}(u, w) du
\]

We only use this formula as motivation for the dynamical Toeplitz quantization of geodesic flows on Grauert tubes. Proposition 3.2 is extensively discussed in [D, Z97, ZZ18], and we refer there for further discussion.

### 3.6. Matrix elements and determinants.

In this section, we review determinant formulae from [D] which relate the determinant \(\det P\) in \((58)\) and in Proposition 3.2 with \(\eta_{J,S}\) in \((57)\) and \((56)\). The same determinant arises in the principal symbol in Proposition 0.10 and in Theorem 0.11, and is the origin of \(G_n(\zeta)\) \((16) - (17)\) in the \(Q_\zeta(\lambda)\) function in Definition 1 Theorem 0.4.

The symplectic form \(\omega\) induces a notion of determinant of a linear transformation \(T\) by \(\det_\omega T := \frac{P^\omega(T)}{\omega^n}\). A choice of symplectic basis identifies \((V, \omega)\) with \((\mathbb{R}^{2n}, \omega_0)\) (the standard symplectic form), and then \(\det T\) is the standard determinant. Given \(S \in \text{Sp}(2n, \mathbb{R})\), the polar decomposition of \(S\) has the form \(S = PQ\) where \(P = (S^*S)^{1/2}\) is the polar part and \(Q = P^{-1}S\) is the orthogonal part.

Given \(J \in \mathcal{J}\) and \(S \in \text{Sp}(V, \omega)\) we define (see [D] \((6.1)\) and \((6.3a)\)),

\[
\beta_{J,SJS^{-1}} = 2^{-n/2} [\det(SJ + JS)]^{1/4}
\]

The determinants \((57)\) and \((59)\) are related by \(|\eta_{J,S}| = \beta_{J,SJS^{-1}}\). In fact (see [D], above \((6.3a)\), and \((B6)\))

\[
|2^{-n} \det(I - iJ) + S(I + iJ)^{1/2}| = [\det(SJ + JS)]^{1/2} = 2^n \beta_{J,SJS^{-1}}^2.
\]

We further record the identities,

\[
\det(SJ + JS) = \det(I + J^{-1}S^{-1}JS) = \det(I + S^*S).
\]

If we express \(S\) as a block matrix \((50)\), then (cf. [D], p. 1388,

\[
(\eta_{J,S})^{-1} = \det((I + iJ) + S(I - iJ)) = 2^n \det(A + D + i(B - C)).
\]

This is the origin of the determinant \((17)\).

The following explains the relations between the determinants \((17)\) and the matrix elements in the ground state in \((16)\).

**Lemma 3.3.** ([D] p. 1388 (Above Appendix C); see \((57)\)) Let \((V, \omega)\) be a real symplectic vector space, and let \(\det = \det_\omega\). Let \(S \in \text{Sp}(V, \omega)\) be as in \((50)\), let \(W_J(S)\) be as in \((56)\), and let \(\Omega_J\) be as in \((55)\). Then,

\[
\langle \Omega_J, W_J(S)\Omega_J \rangle = 2^{n/2} \det(A + D + i(B - C))^{1/2}.
\]
Although the proof is well-known (and is given in [ZZ18]) we include it here for the reader’s convenience.

**Proof.** The following identities are proved on p. 1388 of [D]

\[
\langle \Omega_J, W_J(S)\Omega_J \rangle = \eta_{JS}(\Omega_J, \Omega_{JS^{-1}})
\]

\[
= \eta_{JS}\beta_{JS^{-1}}^{-2} = (\eta_{JS})^{-1}
\]

\[
= 2^n(\det(I + iJ + S(I - iJ)))^{-\frac{1}{2}}
\]

\[
= 2^{n/2}(\det(A + D + iB - iC)^{-\frac{1}{2}})
\]

\[\Box\]

## 4. Osculating Bargmann-Fock-Heisenberg space and Heisenberg coordinates

We now tie together the linear theory of §3 with the nonlinear CR setting of Grauert tubes §2.4 by defining the osculating Bargmann-Fock space at a point \( \zeta \in \partial M_\tau \). As mentioned in the introduction and in §2.1, the maximal radius \( \tau_{\text{max}} \) of the Grauert tubes is generally finite, and so the data defining the osculating Bargmann-Fock space should become singular at this radius. In particular, the identification map \( E_\tau(47) \) become and therefore \( g_t^\prime \) become singular.

We recall from §2.4 that \((H, \omega|_H)\) is a real \( J \)-invariant symplectic vector space (42) of dimension \( 2m - 2 \) \((m = \dim \mathbb{R} M)\). \( T \) is Hamilton vector field of \( \sqrt{\rho} \) on \( M_\tau \) with respect to \( \omega_\rho \) (36). Since \( Dg_t \) preserves \( T \) and \( \alpha \) it also preserves \( H \) and the splitting (43).

Thus \( Dg_t \) induces a linear symplectic Poincaré map

\[
Dg_t : H_z \rightarrow H_{g_t^\prime z}.
\]

The complexification of \( H \) is invariant under the complex structure \( J \) of \( M_\tau \) and we have a splitting

\[
H \otimes \mathbb{C} = H_1^1 \oplus H_0^0
\]

into the \( \pm i \) eigenspaces of \( J \). Thus we have the complex decomposition,

\[
T_C \partial M_\varepsilon = H_1^1 \oplus H_0^1 \oplus \mathbb{C} T.
\]

Extending by scalars, we also have

\[
(D_\zeta g_t) : H_1^1 \oplus H_0^1 \rightarrow H_1^1 \oplus H_0^1
\]

\( Dg_t \) never commutes with \( J \) in the Riemannian setting, and the extended \( Dg_t \) never preserves \( H_1^1 \). In the following, we use the notation and terminology of Section 3.5.

**Definition 4.1.** Given a point \( \zeta \in M \), we define the osculating Bargmann-Fock space at \( \zeta \) to be the Bargmann-Fock space of \( (H_\zeta, J_\zeta, \omega_\zeta) \) and denote it by \( \mathcal{H}_{J_\zeta, \omega_\zeta} \).

If \( \zeta \) is a periodic point for \( g_t \), let \( \gamma = \bigcup_{0 \leq s \leq t} g_s^\prime \zeta \) be the corresponding closed geodesic and we may apply the metaplectic representation to define \( W_{J_\zeta}(D_\zeta g_t^\prime) \) as a unitary operator on \( (H_\zeta, J_\zeta, \omega_\zeta) \). There is a square root ambiguity which can be resolved as in [D] but for our purposes it is irrelevant.
4.1. Determinants. We now apply the results of Section 3.4 and Lemma 3.3 to $D_\zeta g^{nT}_\tau$. Relative to a symplectic basis $\{e_j, Je_k\}$ of $H_\zeta(\partial M_\tau)$ in which $J$ assumes the standard form $J_0$, the matrix of $D_\zeta g^{nT(\zeta)}_\tau$ has the form (50),

$$D_\zeta g^{nT}_\tau := S^n := \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \in Sp(m, \mathbb{R}).$$

If we conjugate to the complexification $T_zM \otimes \mathbb{C}$ by the natural map $W$ defined in (53), then (17) conjugates to

$$\begin{pmatrix} P_n & Q_n \\ Q_n & \bar{P}_n \end{pmatrix} \in Sp_c(m).$$

Then, by (54)

$$P_n = (A_n + D_n + i(-B_n + C_n)) = P_J S^n P_J : H^1_J \rightarrow H^1_J.$$

We then obtain a formula for the leading order symbol in Proposition 0.10 from Lemma 3.3 and (63).

**Lemma 4.2.** Let $P_n$ be as in (54) and $G_n(\zeta)$ as in (16). Then (as stated in (17)),

$$G_n(\zeta) := \langle W_{J_\zeta} (D_\zeta g^{nT(\zeta)}) \Omega_{J_\zeta}, \Omega_{J_\zeta} \rangle = (\det P_n)^{-\frac{1}{2}}.$$

5. Dynamical Toeplitz operators and the spectral projections $\Pi_{\tau, \chi}(\lambda)$

To prove Theorem 0.11, we construct a parametrix for the smoothed spectral projectors (29) - (30) as ‘dynamical Toeplitz operators’ of a type deployed in [ZJDG] (and elsewhere). To prepare for the parametrix construction, we briefly review the definition and properties of the Szegő kernel for a Grauert tube in section 5.1. We then review dynamical Toeplitz operators in Sections 5.2 - 5.3. The spectral projections (30) are analogous to the Fourier (or, spectral) decompositions of the Szegő projector in the line bundle setting. We include a brief comparison between the Fourier components in the line bundle and the Grauert tube setting in Section 5.4, since the analogy has guided the definition of (30). The proof of Theorem 0.11 requires further preparations in the following two sections, and is given in Section 8.

We use the following notation: for any manifold $X$, let $\Psi^s(X)$ denote the class of pseudo-differential operators of order $s$ on $X$.

5.1. Szegő kernel. The semi-classical asymptotics of Theorem 0.4 in the complex domain is based on the microlocal construction of the Szegő kernel $\Pi_\tau$ of $\partial M_\tau$ and of the wave group (27). The leading order term of the asymptotics is tantamount to calculating the principal symbol of the wave group (28) at singular times $t$. But the Szegő kernel and the wave group are Fourier integral operators with complex phase and the symbol calculus for such operators is rather complicated and not well developed. We therefore calculate the asymptotics directly from the Boutet-de-Monvel-Sjöstrand parametrix for $\Pi_\tau$ without using either symbol calculus. We also quote some results of the symplectic spinor symbol calculus of [BoGu] to identify the principal symbol of the wave group (27), essentially because the calculation was already done in [ZJDG]. When it comes to calculating the asymptotics of (28)
we find that it is simpler to work by hand with the parametrix for $\Pi_\tau$ and the Phong-Stein foliation.

We denote by $\mathcal{O}^{s+\frac{m-1}{4}}(\partial M_\tau)$ the Sobolev spaces of CR holomorphic functions on the boundaries of the strictly pseudo-convex domains $M_\tau$, i.e.

$$\mathcal{O}^{s+\frac{m-1}{4}}(\partial M_\tau) = W^{s+\frac{m-1}{4}}(\partial M_\tau) \cap \mathcal{O}(\partial M_\tau),$$

where $W_s$ is the $s$th Sobolev space and where $\mathcal{O}(\partial M_\tau)$ is the space of boundary values of holomorphic functions. The inner product on $\mathcal{O}^0(\partial M_\tau) =: H^2(\partial M_\tau)$ is with respect to the Liouville measure or contact volume form $(37)$.

The study of norms of complexified eigenfunctions is intimately related to the study of the Szegő kernels $\Pi_\tau$ of $M_\tau$, namely the orthogonal projections

$$\Pi_\tau : L^2(\partial M_\tau, d\mu_\tau) \rightarrow H^2(\partial M_\tau, d\mu_\tau)$$

onto the Hardy space of boundary values of holomorphic functions in $M_\tau$ which belong to $L^2(\partial M_\tau, d\mu_\tau)$. The Szegő projector $\Pi_\tau$ is a complex Fourier integral operator with a positive complex canonical relation. The real points of its canonical relation form the graph $\Delta_{\Sigma}$ of the identity map on the symplectic one $\Sigma_\tau \subset T^*\partial M_\tau$ defined by the spray $\Sigma_\tau = \{(\zeta, rd^c\sqrt{\rho}(\zeta) : r \in \mathbb{R}_+ \} \subset T^*(\partial M_\tau)$.

There exists a symplectic equivalence $\iota_\tau : T^*M - 0 \rightarrow \Sigma_\tau$, $\iota_\tau(x, \xi) = (E(x, \tau \frac{\xi}{|\xi|}), |\xi|d^c\sqrt{\rho}_E(x, \tau \frac{\xi}{|\xi|})$.

The well-known parametrix construction of Boutet-de Monvel-Sjöstrand [BoSj, Theorem 1.5] for Szegő kernels of strictly pseudo-convex domains applies to the Grauert tube setting, and we have

$$\Pi_\tau(\zeta, \zeta') \sim \int_0^\infty e^{i\sigma\psi_\tau(\zeta, \zeta')} s(\zeta, \zeta', \sigma) d\sigma,$$

where the phase $\psi_\tau$ is defined in $(38)$.

Also, the symbol $s \in S^n(\partial M_\tau \times \partial M_\tau \times \mathbb{R}_+)$ is of the classical type and of order $m$,

$$s(\zeta, \zeta', \sigma) \sim \sum_{k=0}^\infty \sigma^{m-k} s_k(\zeta, \zeta').$$

### 5.2. Dynamical Toeplitz operators.

Let $\Pi_\tau$ be as in $(65)$, and let $g^t_\tau$ be as in $(46)$ (see also $(9)$). The time evolution of $\Pi_\tau$ under the flow $g^t_\tau$ is defined by

$$\Pi^t_\tau = g^{-t}_\tau \Pi_\tau g^t_\tau.$$
the analysis generalizes to Grauert tubes (see [ZPSH1, ZJDG]). We summarize the results in this section. Under $Dg_t^\tau$, the Lagrangian $\Lambda$ goes to a new Lagrangian $\Lambda_t$ and $\sigma_{\Pi^t}$ is a rank one projector onto a ground state $e_{\Lambda_t}$ depending on $t$. As in [Z97, ZPSH1, ZJDG], we define the symbol,

$$\sigma^0_{\tau,t} = \langle e_{\Lambda_t}, e_{\Lambda_t} \rangle^{-1}$$

(66)

to be the (inverse of) the inner product of the ground states with respect to $\Lambda$ and $\Lambda_t$. In the linear model, this inner product is calculated in Lemma 3.3 and the formula for the linear quantization (56) implies that

$$\Pi^t_\tau \sigma_{\tau,t} (g^{-1}_\tau)^* \Pi^t_\tau$$

is unitary in the Bargmann-Fock setting. In the nonlinear setting, it is unitary modulo compact operators (i.e. a Toeplitz operator of order $-1$). To see this, we observe that the composite symbol is

$$\sigma(\Pi^t_\tau \Pi^t_\tau) = |\langle e_{\Lambda_t}, e_{\Lambda_t} \rangle|^2 \sigma_{\Pi^t_\tau}.$$ 

(67)

In the linear case, it is the matrix element given in Lemma 3.3 and (56).

The change of $\Pi^t_\tau$ under $g^t_\tau$ reflects the change in complex structure. Let $J$ be the complex structure of $M^\tau$. Under $g^t_\tau$ it is moved to a different complex structure

$$J^t_t := g^t_\tau J := Dg^t_\tau JDg^{-t}_\tau.$$ 

The Szegö projector with respect to this deformed complex structure is $\Pi^t_\tau$ above.

5.3. $W^\tau_t$ as a dynamical Toeplitz operator.

**Proposition 5.1.** The unitary group $W^\tau_t$ of (32) is a one-parameter group of unitary dynamical Toeplitz operators.

**Proof.** We deploy the ingenious argument of [BoGu, Lemma 12.2]. According to this Lemma (but in the notation and setting of this article) given any first order Toeplitz operator $\Pi^t_\tau P \Pi^t_\tau$ with $P \in \Psi^1(\partial M^\tau)$, there exists a first order pseudo-differential operator $Q$ on $\partial M^\tau$ such that $[Q, \Pi^t_\tau] = 0$ and such that $\Pi^t_\tau P \Pi^t_\tau = \Pi^t_\tau Q \Pi^t_\tau$. Let $Q_{\sqrt{\sigma}}$ be this operator $Q$ in the case where $P = D_{\Xi_{\sqrt{\sigma}}}$. Then,

$$W^\tau_t = \Pi^t_\tau e^{itQ_{\sqrt{\sigma}}}.$$ 

Indeed, both sides solve the evolution equation,

$$\begin{align*}
\frac{d}{dt} W(t) &= Q_{\sqrt{\sigma}} W(t) = \Pi D_{\Xi_{\sqrt{\sigma}}} \Pi W(t), \\
W(0) &= \Pi^t_\tau.
\end{align*}$$

Now, $e^{itQ_{\sqrt{\sigma}}}$ is a unitary group of Fourier integral operators on $L^2(\partial M^\tau)$ by standard Fourier integral operator theory (see e.g. [Ho, Volume IV]). By the composition theorem for the composition of the Fourier integral operator $e^{itQ_{\sqrt{\sigma}}}$ with the Szegö projection $\Pi^t_\tau$ of [BoGu] (or, alternatively, for Fourier integral operators with positive complex phases of [MeSj]), $W^\tau_t$ is a Toeplitz Fourier integral operator adapted to the Hamilton flow of $\Xi_{\sqrt{\sigma}}$ on $\Sigma_\tau$ (see [BoGu, Appendix] for adapted Fourier integral Toeplitz operators).

---

$\Psi^k(X)$ denotes the space of $k$th order poly-homogeneous pseudo-differential operators on a manifold $X$. See [Ho] for background.
On the other hand, for any zeroth order pseudo-differential operator \( \sigma(x, D) \) on \( L^2(\partial M_\tau) \), the dynamical Toeplitz operator \( \Pi_\tau \sigma(t, x, D)(g^\tau_t)^* \Pi_\tau \) is also a Fourier integral Toeplitz operator or, equivalently, a Fourier integral operator with complex phase that commutes with \( \Pi_\tau \). Therefore, \( \mathcal{W}_\tau(t) \) and \( \Pi_\tau \sigma(x, D)(g^\tau_t)^* \Pi_\tau \) are both Fourier integral Toeplitz operators with the same canonical relation, i.e. adapted to the geodesic flow on \( \Sigma_\tau \). We now choose \( \sigma(t, x, D) \) so that the principal symbols of \( \Pi_\tau \sigma(t, x, D)(g^\tau_t)^* \Pi_\tau \) and of \( \mathcal{W}_\tau(t) \) coincide, i.e so that \( \sigma(t, x, \xi)|_{\Sigma_\tau} \) equals (66). By induction on the order of the symbol, one can improve \( \sigma(t, x, D) \) so that its complete symbol (restricted to \( \Sigma_\tau \)) agrees with that of \( \mathcal{W}_\tau(t) \). Then, \( \Pi_\tau \sigma(t, x, D)(g^\tau_t)^* \Pi_\tau - \mathcal{W}_\tau(t) \) is a smoothing operator, and the Proposition follows.

\[\square\]

**Remark 5.2.** The operator \( Q \) may be thought of as Wick normal-ordering \( P \). It would be interesting to construct \( Q \) explicitly when \( P = D_{\Xi_\tau, \pi} \).

**Proposition 5.3.** There exists a poly-homogeneous pseudo-differential operator \( \hat{\sigma}_\tau (w, D_{\Xi_\tau, \pi}) \) on \( \partial M_\tau \) with complete symbol of the classical form

\[
\sigma_\tau (w, r) \sim \sum_{j=0}^{\infty} \sigma_{t, \tau, j}(w)r^{-j}
\]

and with \( \sigma_{t, \tau, 0} = \sigma_{t, \tau}^0 \), so that for \( \zeta \in \partial M_\tau \), modulo smoothing Toeplitz operators,

\[
\mathcal{W}_\tau(t, \zeta, \zeta) \simeq \int_{\partial M_\tau} \Pi_\tau(\zeta, w)\hat{\sigma}_\tau(\zeta, w, \zeta)d\mu_\tau(w).
\]

The symbol \( \sigma_\tau (w, r) \) is a zeroth order polyhomogeneous function on \( \Sigma_\tau \), i.e. a classical symbol of order zero.

### 5.4. Comparison of dynamical Toeplitz operators in the Grauert and line bundle settings.

In this section, we extend the comparisons in Section 2.9 between the CR geometry in the Grauert and line bundle settings to the spectral theory of dynamical Toeplitz operators. The Szegö projector in the line bundle setting is the orthogonal projection \( \hat{\Pi} : L^2(X_h) \rightarrow H^2(X_h) \) where \( H^2 \) is the Hardy space of \( L^2 \) CR holomorphic functions. Under the \( S^1 \) action on \( L^2 \) it has the Fourier decomposition \( \hat{\Pi} = \sum_{N=0}^{\infty} \hat{\Pi}_N \), and there is a canonical lift, \( s \rightarrow \hat{s} \) from \( H^0(M, L^N) \rightarrow \text{Range}(\hat{\Pi}_N) \), from holomorphic sections of \( L^N \) to equivariant functions on \( X_h \), which conjugates \( \hat{\Pi}_N \) with the standard Bergman-Szegö kernels \( \Pi_{\pi, N} \) on \( H^0(M, L^N) \).

It follows that the Fourier decomposition of \( \hat{H} \) is the same as the spectral decomposition of \( D_\theta := \frac{\partial}{\partial \tilde{\theta}} \) on \( H^2(X_h) \).

The analogue of this spectral decomposition in the Grauert tube setting is that of the Toeplitz operator \( \Pi_\tau D_{\sqrt{\tau}} \Pi_\tau \), where \( D_{\sqrt{\tau}} = \frac{1}{\sqrt{\tau}} \partial \sqrt{\tau} \) is the differential operator induced by the Hamiltonian vector field of \( \sqrt{p} \). One might also consider \( \Pi_\tau (g^\tau_t)^* \Pi_\tau \), the compression to \( H^2(\partial M_\tau) \) of the pullback (or, composition) operator with the Hamilton flow, but as mentioned above, \( [D_{\sqrt{p}}, \Pi_\tau] \neq 0 \) and unitary group generated by \( \Pi_\tau D_{\sqrt{p}} \Pi_\tau \) on \( H^2(\partial M_\tau) \) is not quite the same as the 1-parameter family \( \Pi_\tau (g^\tau_t)^* \Pi_\tau \), which is not unitary and not a group.

\( \Pi_\tau D_{\sqrt{p}} \Pi_\tau \) is an elliptic Toeplitz operator with a discrete spectrum, which is very close to that of \( \sqrt{-\Delta} \) in the sense that, after the identifications discussed in [ZJDG], the two
operators have the same principal symbol. The analogue of the Fourier decomposition of \( \hat{\Pi} \) in the line bundle setting is, then, the spectral decomposition of \( \Pi_r D_{\sqrt{\rho}} \Pi_r \) on \( H^2(\partial M_r) \). In the line bundle case, the spectrum lies in \( \mathbb{Z}_+ \) and the eigenvalues have large multiplicities. In the Grauert analogue, one may expect that the spectrum of \( \Pi_r D_{\sqrt{\rho}} \Pi_r \) is quite irregular and, generically, the eigenvalues have multiplicity; we will not prove this here, but it is a simple Toeplitz analogue of well-known theorems in the Riemannian setting (the Helton clustering theorem and the Uhlenbeck generic simplicity of the spectrum of \( \Delta_g \); see [BoGu] for background).

In the setting of line bundles \( L \to M \), the semi-classical Szegő kernels \( \Pi_{hk}(z, w) \) are Fourier components,

\[
\Pi_{hk}(z, w) = \frac{1}{2\pi} \int_0^{2\pi} \Pi_h(x, r\theta y) e^{-ik\theta} d\theta
\]

of the Szegő projector \( \Pi_h(x, y) : L^2(\partial D^*_h) \to H^2(\partial D^*_h) \) onto boundary values of holomorphic functions in the strictly pseudo-convex domain \( D^*_h \subset L^* \) in the dual line bundle \( L^* \) where \( D^*_h \) is the dual unit disk bundle \( \{ \lambda \in L^* : h^*(\lambda) < 1 \} \). One obtains their asymptotic expansions as \( k \to \infty \) by applying using a Boutet de Monvel - Sjöstrand parametrix for \( \Pi_h \) and by applying a complex stationary phase argument. In the setting of Grauert tubes one also has a Boutet de Monvel - Sjöstrand parametrix for \( \Pi_r \) and can try to adapt the argument to obtain analogous asymptotics for \( \Pi_{h,\tau}(\lambda) \) (30). The direct analogue would apply to the integral,

\[
S_{\lambda,\chi,\tau}(x, y) := \int_{\mathbb{R}} \hat{\chi}(t)e^{-i\lambda t}\Pi_r(x, \hat{g}^t y) dt,
\]

for some \( \hat{\chi} \in C^\infty_0(\mathbb{R}) \). This is not quite the right analogue, however, because unlike the \( S^1 \) action, \( \hat{g}^t \) does not act holomorphically, hence composition with \( \hat{g}^t \) does not commute with \( \Pi_r \), and therefore \( S_{\lambda,\chi,\tau}(x, y) \) fails to be CR holomorphic in the \( y \) variable. Indeed, \( S_{\lambda,\chi,\tau} \) is the Schwartz kernel of the operator \( \int_{\mathbb{R}} \hat{\chi}(t)\Pi_r \circ \hat{g}^t dt \). Using the Boutet de Monvel - Sjöstrand parametrix for \( \Pi_r \), this one obtains

\[
S_{\lambda,\chi,\tau}(x, y) = \int_{\mathbb{R}} \int_0^\infty \hat{\chi}(t)e^{-i\lambda t}e^{\theta|\psi(x, \hat{g}^t y)|} S(x, \hat{g}^t y, \theta) dt d\theta,
\]

where \( S(x, y, \lambda) \) is a semi-classical symbol of order \( m \). By stationary phase, one finds that if \( \text{supp} \hat{\chi} \) is close to 0, then the only critical point occurs at \( \theta = 1, t = 0 \) and

\[
S_{\lambda,\chi,\tau}(x, y) \simeq \lambda^m e^{\lambda\psi(x, y)} \tilde{S}(x, y, \lambda),
\]

where \( \tilde{S}(x, y, \lambda) \) is classical symbol of order zero.

For our problem, \( \hat{g}^t \) is not holomorphic and it is necessary to work with the more complicated operator (30). The pointwise values on the anti-diagonal of \( \int_{\mathbb{R}} \Pi_r \hat{\chi}(t)\Pi_r \circ \hat{g}^t ds \) are quite different from those of (68), as the next result shows.

Theorem 0.11 in the Riemannian Grauert setting is somewhat analogous to [ZZ18, Theorem 0.9] in the line bundle setting (see Section 0.7). However, there are significant differences in the two settings and the analogy only goes so far. The most obvious difference is that, in the line bundle setting, there are two Hamiltonians: the generator \( \frac{\partial}{\partial s} \) of rotations in the fibers of the line bundle \( L \to M \) and an independent Toeplitz Hamiltonian \( \hat{H}_k \) whose spectrum is the main object of study. In the Riemannian setting, there is just one operator, \( \Pi_r D_{\sqrt{\rho}} \Pi_r \) (31), or alternatively (and essentially equivalently) \( \sqrt{\Delta} \). As mentioned above, \( \Pi_r D_{\sqrt{\rho}} \Pi_r \) is
the analogue of $\frac{\partial}{\partial \theta}$, but is spectrum is the main object of study in the Grauert tube setting, and it simultaneously plays the role of $\frac{\partial}{\partial \theta}$ and of $\widetilde{H}_k$.

6. Analytic continuation of the Poisson kernel

For the remainder of the article, we analyze the Laplacian and associated operators. In the next two sections, we build up enough background to show that $U(t + i\tau, \zeta, \bar{\zeta})$ (33) is also a dynamical Toeplitz operator of the same type as $W_\tau(t, \zeta, \bar{\zeta})$ in Section (5.3). Much of this statement is proved in [ZPSH1, ZJDG], using the analytic continuation of the Poisson-wave kernel, and we review that material in this section. We state the result in the language of adapted Fourier integral operators of the Appendix of [BoGu], where only the real points of canonical relations are considered. We use a slight extension of the notion of adapted Fourier integral operators, in which the homogeneous symplectic map may be a symplectic embedding rather than a symplectic isomorphism. All of the composition results of [BoGu] extend readily to this case. For the definitions of Hermite Fourier integral operators, and operators “adapted” to the graph of the Hamiltonian flow of $\sqrt{\rho}$ on the symplectic cone $\Sigma_\tau$ we refer to the Appendix of [BoGu].

The wave group of $(M, g)$ is the unitary group $U(t) = e^{it\sqrt{\Delta}}$. Its kernel $U(t, x, y)$ solves the ‘half-wave equation’,

$$\left( \frac{1}{i} \frac{\partial}{\partial t} - \sqrt{\Delta} \right) U(t, x, y) = 0, \quad U(0, x, y) = \delta_y(x).$$

It is well known [Ho, DG] that $U(t, x, y)$ is the Schwartz kernel of a Fourier integral operator, $U(t, x, y) \in I^{-1/4}(\mathbb{R} \times M \times M, \Gamma)$ with underlying canonical relation

$$\Gamma = \{(t, \tau, x, \xi, y, \eta) : \tau + |\xi| = 0, G^t(x, \xi) = (y, \eta)\} \subset T^*\mathbb{R} \times T^*M \times T^*M.$$

6.1. Poisson wave kernel. The Poisson-wave kernel is the analytic continuation $U(t + i\tau, x, y)$ of the wave kernel with respect to time, $t \rightarrow t + i\tau \in \mathbb{R} \times \mathbb{R}_+$. For $t = 0$ we obtain the Poisson semi-group $U(i\tau) = e^{-\tau\sqrt{\Delta}}$ on $L^2(M)$. For general $t + i\tau$ we define the Poisson-wave kernel in the real domain by the eigenfunction expansion for $\tau > 0$,

$$U(i\tau, x, y) = \sum_j e^{(t+i\tau)\lambda_j} \varphi_j(x) \varphi_j(y).$$

As discussed in [ZPSH1], this kernel is globally real analytic on $M \times M$ for any $\tau > 0$. The Poisson-wave kernel $U(t + i\tau, x, y)$ admits an analytic continuation $U_\mathbb{C}(t + i\tau, \zeta, y)$ in the first variable to $M_{\tau} \times M$. When the real time $t = 0$, the operator kernel $U_\mathbb{C}(i\tau, \zeta, y)$ $P^\tau$ defines the operator

$$P^\tau := \Pi_\tau \circ U_\mathbb{C}(i\tau) : L^2(M) \rightarrow H^2(\partial M_\tau)$$

with Schwartz kernel (25). The Szegö kernel is not needed here, since $U_\mathbb{C}(i\tau, \zeta, y)$ is holomorphic in $\zeta$, but is put in to emphasize that point. We also define the adjoint operator $P^{\tau^*} : H^2(\partial M_\tau) \rightarrow L^2(M)$ which has the Schwartz kernel

$$P^{\tau^*}(y, \zeta) = \sum_j e^{-\tau\lambda_j} \overline{\varphi_j(\zeta)} \varphi_j(y), \quad y \in M, \zeta \in \partial M_\tau.$$ (69)
The following result was stated by Boutet de Monvel (and given a detailed proof in three recent articles [ZPSH1, L18].

**Theorem 6.1.** For sufficiently small $\tau$, $P^\tau := \Pi_{\tau} \circ U_\mathcal{C}(i\tau) : L^2(M) \to H^2(\partial M_\tau)$ is a Fourier integral operator with complex phase in the sense of [MeSi] of order $-\frac{m-1}{4}$ adapted to the canonical relation

$$\Gamma = \{(y, \eta, \iota_\tau(y, \eta)) \} \subset T^*M \times \Sigma_\tau.$$  

Moreover, for any $s$,

$$P^\tau = \Pi_{\tau} \circ U_\mathcal{C}(i\tau) : W^s(M) \to \mathcal{O}^{s+\frac{m-1}{4}}(\partial M_\tau)$$

is a continuous isomorphism.

Theorem 6.1 readily extends to $U_\mathcal{C}(t + i\tau)$. Referring to (65),

**Proposition 6.2.** $P^\tau \circ U_\mathcal{C}(t) : C_c(\mathbb{R} \times M) \to H^2(\partial M_\tau)$ is a Fourier integral operator with complex phase of order $-\frac{m-1}{4}$ adapted to the canonical relation

$$(t, E, \chi_{\tau,t}(y, \eta), y, \eta) : E + |\eta| = 0 \subset T^*M \times \Sigma_\tau \times T^*M,$$

where $\chi_{\tau,t}$ is the symplectic isomorphism

$$\chi_{\tau,t}(y, \eta) = \iota_{\tau}(G^t(y, \eta), y, \eta) : T^*M - 0 \to \Sigma_\tau.$$  

Equivalently, $P^\tau \circ U(t)$ is a Fourier integral operator of Hermite type of order $-\frac{m-1}{4}$ associated to the canonical relation

$$\Gamma_\tau = \{(t, E), (\iota_{\tau}(G^t(y, \eta), y, \eta)) \} \subset \Sigma_\tau \times T^*M.$$  

**Proof.** This follows from Theorem 6.1 and from the fact proved in [BoGu], Theorems 3.4 and 7.5, that the compositon of a Fourier integral operator and a Fourier integral operator of Hermite type is also a Fourier integral operator of Hermite type, with a certain addition law for the orders and a composition law for the symbols. \qed

### 6.2. Singular support of $U_\mathcal{C}(t + 2i\tau, \zeta, \bar{\zeta})$

In this section, we extend the discussion of the analytic continuation of the Poisson kernel to the Poisson wave kernel on the anti-diagonal,

$$U(t + 2i\tau, \zeta, \bar{\zeta}) \in \mathcal{D}'(\mathbb{R} \times \Delta_{M_\tau \times M_\tau}).$$

The main result determines the singularities of (33) for fixed $\zeta$ as a distribution in $t$. It shows that $U_\mathcal{C}(t + 2i\tau, \zeta, \bar{\zeta})$ is singular in $t$ only if $\zeta$ corresponds to a point $(x, \xi) \in S^*M$ for which the geodesic $G^t(x, \xi)$ is periodic and then the singular times are multiples of the lengths of the corresponding closed geodesic. This should be compared with the well-known fact (see e.g. [SV, SoZ]) that in the real domain, $U(t, x, y)$ is singular at the lengths of all geodesic segments from $x$ to $y$. The same result will be proved below by a parametrix construction, but it is possible to prove this statement just using the results of the previous section. The parametrix construction is valuable in computing the leading coefficient, which is not easy to obtain from the abstract approach.

To analyze the singularities, we use the calculus of Hermite Fourier integral operators adapted to symplectic maps in the framework of [BoGu].

**Proposition 6.3.** For fixed $\zeta \in \partial M_\tau$, the singular support of the distribution $t \in \mathbb{R} \to U_\mathcal{C}(t + 2i\tau, \zeta, \bar{\zeta})$ consists of times $T$ such that $g^T_\tau(\zeta) = \zeta$. If no $T \neq 0$ exists, the singular support is $\{0\}$.  

Proof. The first step is to express (33) as a composition of Fourier integral operators.

**Lemma 6.4.** We have,

\[ P^{\tau}U(t)P^{\tau*}(\zeta, \bar{\zeta}) = \sum_{j,k} e^{(-2\tau + it)\lambda_j} \int_M \varphi_j^C(\zeta)\varphi_j(y)\overline{\varphi_{\lambda_k}(\zeta)}\varphi_k(y) dV(y) \]

\[ = U_C(t + 2i\tau, \zeta, \bar{\zeta}) \]

**Proof.** The identity follows directly from the eigenfunction expansions (25) and (69) and orthonormality of \( \varphi_j(y) \) in the real domain:

\[ U_C(t + 2i\tau, \zeta, \bar{\zeta}) = P^{\tau}U(t)P^{\tau*}(\zeta, \bar{\zeta}) \]

\[ = \sum_{j,k} e^{(-2\tau + it)\lambda_j} \int_M \varphi_j^C(\zeta)\varphi_j(y)\overline{\varphi_{\lambda_k}(\zeta)}\varphi_k(y) dV(y) \]

□

By Lemma 6.4 we can calculate the wave front set of \( U(t + 2i\tau, \zeta, \bar{\zeta}) \) by composing wave front sets in the real domain of the adapted Hermite Fourier integral operators \( P^{\tau}U(t) \) and \( P^{\tau*} \). Proposition 6.2 implies that, for \( \sqrt{\rho}(\zeta) = \tau \), and \( t \in \mathbb{R} \), the singular support of the distribution \( t \to U_C(t + 2i\tau, \zeta, \bar{\zeta}) \) is the set

\[ \text{SingSupp}(t \to U_C(t + 2i\tau, \zeta, \bar{\zeta})) = \{ t : \exists (y, \eta) \in T^*M : |\eta| = \tau, \exp_y(i\eta) = \zeta, \exp_y(t + i\tau)(-\eta) = \bar{\zeta} \}. \]

To complete the proof of Proposition 6.3, we observe that \( \exp_y i\eta = \zeta \) implies \( \exp_y(-i\eta) = \bar{\zeta} \). Since \( G^{\tau}(y, -\eta) \) must be tangent to \( \Sigma_\tau \), the terminal momentum must be \( d^c\sqrt{\rho} \). It follows that

\[ (y, \eta) = G^{\tau}(\bar{\zeta}, d^c\sqrt{\rho}). \]

If \( t \) lies in the singular support, then \( \exp_y(t + i\tau)(-\eta) = \bar{\zeta} \) and since the terminal momentum must again be tangent to \( \Sigma \) we have

\[ (y, \eta) = G^{-t - \tau}(\bar{\zeta}, d^c\sqrt{\rho}) \]

hence \( G^t(y, \eta) = (y, \eta) \).

\[ \square \]

7. The wave group in the complex domain as a dynamical Toeplitz operator

In this section, we prove the identity (28). Consequently, (33) is a dynamical Toeplitz operator of the same type as \( W_\tau(t, \zeta, \bar{\zeta}) \) in Section (5.3). We use symbol calculus of Toeplitz Fourier integral operators to calculate the symbol of (33), which is apparently more complicated than \( W_\tau(t) \), and prove (28). In effect, the main result is proved in [ZJDG, Proposition 44.] and we review the relevant background. In Section 7.2, we introduce the kernel \( K_\tau \) and prove Lemma 0.2.

As above, let \( \frac{1}{i}D_{\Xi, \sqrt{\rho}} \) denote the self-adjoint directional derivative in the direction of \( \Xi, \sqrt{\rho} \). The directional derivative \( D_{\Xi, \sqrt{\rho}} \) is elliptic on the kernel of \( \partial_\rho \), i.e. its symbol is nowhere vanishing on \( \Sigma_\tau \setminus \{0\} \). Hence \( \Pi_\tau D_{\Xi, \sqrt{\rho}} \Pi_\tau \) is an elliptic Toeplitz operator. The symbol \( \sigma_{\tau}(w, r) \)
is a polyhomogeneous function on $\Sigma_\tau$. Also as above, for any manifold $X$, let $\Psi^s(X)$ denote the class of pseudo-differential operators of order $s$ on $X$.

The next Proposition is [ZJDG, Proposition 44.] and is analogous to Proposition 5.3 for $W_\tau(t)$ and Proposition 8.1.

**Proposition 7.1.** There exists a poly-homogeneous pseudo-differential operator $\hat{\sigma}_{t\tau}(w, D_{\Sigma, \pi})$ on $\partial M_\tau$ with complete symbol of the classical form

$$\sigma_{t\tau}(w, r) \sim \sum_{j=0}^{\infty} \sigma_{t,\tau,j}(w) r^{-\frac{m-1}{2} - j}$$

on $\Sigma_\tau$, and with $\sigma_{t,\tau,0} = \sigma_{t,\tau,0}^0$, so that for $\zeta \in \partial M_\tau$, modulo smoothing Toeplitz operators,

$$U_\zeta(t + 2i\tau, \zeta, \bar{\zeta}) \sim \int_{\partial M_\tau} \Pi(\zeta, w) \hat{\sigma}_{t,\tau}(\zeta) \Pi(\eta^t(w, \zeta)) d\mu_\tau(w). \quad (71)$$

The main point of the proof is to show that $U_\zeta(t + i\tau, \zeta, \bar{\zeta})$ may be constructed as the dynamical Toeplitz operator $V_\tau^t$ of (27). The proof consists of a sequence of Lemmas from [ZPSH1, ZJDG].

**Proof.** The following Lemma is [ZPSH1, Lemma 8.2] (see also [ZJDG, Section 3.1]).

**Lemma 7.2.** Let $A_\tau = (P^{**}P^\tau)^{-\frac{1}{2}}$. Then,

- (i) $A_\tau \in \Psi^{-m-1}(M)$, with principal symbol $|\xi|^{\frac{m+1}{2}}$.

- (ii) $U_\zeta(i\tau)^*U_\zeta(i\tau) \in \Psi^{-\frac{m-1}{2}}(M)$ with principal symbol $|\xi|_g^{-\frac{m-1}{2}}$.

We note that $P^{**}P^\tau : L^2(M) \to L^2(M)$. It is proved in [ZPSH1, Lemma 8.2] (see also [ZJDG]) that $P^{**}P^\tau \in \Psi^{-\frac{m-1}{2}}(M)$ with principal symbol $|\xi|^{-\frac{m-1}{2}}$, proving (i). Statement (ii) follows from Theorem 6.1.

To prove (28), we introduce a slightly modified version of $P^*U(t)P^{**}$ from [ZJDG].

**Definition 3.** As above, let $A_\tau = (P^{**}P^\tau)^{-\frac{1}{2}}$ and define

$$\tilde{V}_\tau^t := P^* A_\tau U(t) A_\tau P^{**} : H^2(\partial M_\tau) \to H^2(\partial M_\tau).$$

As the notation suggests, $\tilde{V}_\tau^t$ can be constructed in the form $V_\tau^t$ of (27). The first step is the following Lemma, which is proved in [ZJDG, Proposition 4.4].

**Lemma 7.3.** $\tilde{V}_\tau^t$ is a unitary Fourier integral operator with positive complex phase of Hermite type on $H^2(\partial M_\tau) \subset L^2(\partial M_\tau)$ adapted to the graph of the Hamiltonian flow of $\sqrt{\rho}$ on $\Sigma_\tau$.

We briefly indicate the proof.

**Proof.** By Proposition 4.3 of [ZJDG], $\tilde{V}_\tau^t$ is a unitary group with eigenfunctions

$$\tilde{V}_\tau^t P^* A_\tau \varphi_j = e^{it\lambda_j} P^* A_\tau \varphi_j.$$ 

Just like $V_\tau^t$, $\tilde{V}_\tau^t$ is a composition of Fourier integral operators with complex phase, and all are associated to canonical graphs and equivalence relations. Moreover all are operators of
Hermite type. It follows that the composition is transversal, so that \( \tilde{V}_\tau^t \) is also a Fourier integral operator with complex phase and of Hermite type. It follows that \( \tilde{V}_\tau^t \) is adapted to the graph of \( EG^t E^{-1} = \exp t \Xi \sqrt{r} \) on \( \Sigma_\tau \) (see (11)).

\[ \square \]

The next Lemma is [ZJDG, Proposition 4.5]. It shows that \( \tilde{V}_\tau^t(\zeta, \bar{\zeta}) \) can be constructed as a unitary group of Toeplitz dynamical operators \( V_\tau^t \) (27).

**Lemma 7.4.** There exists a polyhomogeneous pseudo-differential operator \( \sigma_{\tau_\tau} \) on \( \partial M_\tau \) so that

\[ \tilde{V}_\tau^t = \Pi_\tau \sqrt{\sigma_{t,\tau}(g^{-1}_t)^*} \sqrt{\sigma_{t,\tau}} \Pi_\tau. \]

Thus, \( \tilde{V}_\tau^t \) is equivalent to (27).

We note that \( \tilde{V}_\tau^t \) only differs from (27) in the definition of its symbol. One can interchange the order of \( (g^{-1}_t)^* \sqrt{\sigma_{t,\tau}} \) by translating the symbol. Since Lemma 7.4 is proved in [ZJDG], we only briefly sketch the proof for the sake of completeness.

**Proof.** Each side of each formula is an elliptic Toeplitz Hermite Fourier integral operator adapted to the graph of \( g_t \) on \( \Sigma_\tau \). In the case of \( \tilde{V}_\tau^t \) this follows directly from the definitions and by the composition theorem for such operators in [BoGu]. In the case of \( \Pi_\tau g_t \sigma_{\tau,\tau} \Pi_\tau \) it follows similarly from the fact that \( \Pi_\tau \) is a Toeplitz operator and from the simple composition with pullback by \( g_t \).

By Proposition 7.3, \( \tilde{V}_\tau^t \) is unitary. Hence its principal symbol is unitary. We also have by the composition calculus of Toeplitz symplectic spinor symbols (see (67)) that

\[ \sigma_{\Pi_\tau} \circ \sigma_{g_{-1}^{-1} g_t^*} \circ \sigma_{\Pi_\tau} = \sigma_{\Pi_\tau} \leftrightarrow |\sigma_{\tau,\tau}|^2 \sigma_{\Pi_\tau} \circ \sigma_{g_{-1}^{-1} g_t^*} \circ \sigma_{\Pi_\tau} = \sigma_{\Pi_\tau}. \]

Then

\[ \sigma_{\Pi_\tau} \circ \sigma_{g_{-1}^{-1} g_t^*} \circ \sigma_{\Pi_\tau} = |\langle e_{\Lambda_t}, e_{\Lambda} \rangle|^2. \]

It follows that

\[ |\sigma_{\tau,\tau}^0|^2 = |\langle e_{\Lambda_t}, e_{\Lambda} \rangle|^{-2}. \]

Thus the principal symbol can only differ from \( \sigma_{\tau,\tau}^0 \) (66) by a multiplicative factor of modulus one. We can choose the factor to make the principal coincide with the principal symbol of the linearization on the osculating Bargmann-Fock space in (58) and Proposition 3.2. The symbol then equals (17).

Using the composition calculus, one can define the lower order terms recursively so that \( V^t_\tau \) and \( \tilde{V}_\tau^t \) have the same complete symbol, i.e. differ by a smoothing Toeplitz operator.

\[ \square \]

The final Lemma compares \( \tilde{V}_\tau^t(\zeta, \bar{\zeta}) \) and \( U_\tau(t + 2i\tau, \zeta, \bar{\zeta}) \).

**Lemma 7.5.** ([ZJDG] Proposition 4.4) \( \tilde{V}_\tau^t \) is a Fourier integral operator with complex phase of Hermite type on \( H^2(\partial M_\tau) \subset L^2(\partial M_\tau) \) adapted to the graph of the Hamiltonian flow of \( \sqrt{r} \) on \( \Sigma_\tau \). \( \tilde{V}_\tau^t(\zeta, \bar{\zeta}) \) has the same canonical relation as \( U_\tau(t + 2i\tau, \zeta, \bar{\zeta}) \) and the same principal symbol multiplied by \( |\xi|^{m-1/2}. \)

Indeed, \( \tilde{V}_\tau^t \) only differs from \( U(t + 2i\tau, \zeta, \bar{\zeta}) \) by the insertion of two \( A_t \) factors, and by Lemma 7.2 this only changes the principal symbol by the factor \( |\xi|^{m-1/2}. \)

Combining Lemma 7.3 and Lemma 7.5 proves (28) and (71).
7.1. **Completion of the proof of Proposition 7.1.** By (29), we see that $\chi \ast dP^r_{[0,\lambda]}$ is the Fourier transform of $U_\mathcal{C}(t + 2i\tau, \zeta, \bar{\zeta})$, weighted by $\hat{\chi}$. By Lemma 7.5, one has a Toeplitz parametrix for $U_\mathcal{C}(t + 2i\tau, \zeta, \bar{\zeta})$ as in Lemma 7.4, hence in the form (71) but with the symbol of $\hat{V}_r^2$ multiplied by $|\xi|^{-\frac{m-1}{2}}$, accounting for the order of the amplitude.

This completes the proof of Proposition 7.1.

7.2. **Proof of Lemma 0.2.** In this section, we prove Lemma 0.2. That is, we prove

$$||\varphi_j^C||^2_{L^2(\partial M_r)} \simeq e^{2\tau\lambda_j \lambda_j^{-\frac{m-1}{2}}} (1 + O(\lambda_j^{-1})).$$

**Proof.** We have,

$$||\varphi_j^C||^2_{L^2(\partial M_r)} = e^{2\tau\lambda_j} ||P^r \varphi_j||^2_{L^2(\partial M_r)} = e^{2\tau\lambda_j} \langle P^r P^r \varphi_j, \varphi_j \rangle_{L^2(M)}. \qed$$

By Lemma 7.2 (see also [ZJDG, Proposition 3.6]) $A_r$ is an elliptic self-adjoint pseudo-differential operator of order $\frac{m-1}{2}$ with principal symbol $|\xi|^{\frac{m-1}{2}}$. That is, $(P^r P^r)$ is a pseudo-differential operator of order $-\frac{m-1}{2}$ with principal symbol $|\xi|^{-\frac{m-1}{2}}$. Hence,

$$(P^r P^r) = \Delta^{-\frac{m-1}{4}} + R, \quad R \in \Psi^{-\frac{m-3}{2}}.$$ It follows that

$$\langle P^r P^r \varphi_j, \varphi_j \rangle_{L^2(M)} \lambda_j^{-\frac{m-1}{2}} (1 + O(\lambda_j^{-1})).$$

Hence,

$$||\varphi_j^C||^2_{L^2(\partial M_r)} = e^{2\tau\lambda_j \lambda_j^{-\frac{m-1}{2}}} (1 + O(\lambda_j^{-1})). \qed$$

8. **Proof of Theorems 0.11 - 0.12**

We now use the Boutet de Monvel-Sjöstrand parametrix for $\Pi_r$ to construct an oscillatory integral parametrix with complex phase for $U_\mathcal{C}(t + 2i\tau, \zeta, \bar{\zeta})$ of (71). The expression in Corollary 5.3 can be put in an explicit form as an oscillatory integral with complex phase.

8.1. **An oscillatory integral parametrix for $\chi \ast dP^r_{[0,\lambda]}$.** In this section, we prove,

**Proposition 8.1.** Define the phase

$$\Phi(t, \zeta, w, \sigma_1, \sigma_2) = -t + \sigma_1 \psi(\zeta, w) + \sigma_2 \psi(g_1(t)w, \zeta). \quad (72)$$

Let $\chi \in \mathcal{S}(\mathbb{R})$ be an even function with $\hat{\chi} \in C_0^\infty$. Then, There exists a semi-classical amplitude $A_\lambda(\zeta, \bar{\zeta}, \sigma_1, \sigma_2, t, w)$ of order $-\frac{m-1}{2}$ such that

$$\chi \ast dP^r_{[0,\lambda]}(\zeta, \bar{\zeta}) = \sum_j e^{(-2\tau\lambda_j)\chi(\lambda_j - \lambda)||\varphi_j^C(\zeta)||^2}, \quad (73)$$

$$= \lambda^{2m} \int_{\mathbb{R}} \int_0^\infty \int_{\partial M_r} \hat{\chi}(t) e^{i\lambda \Phi(t, \zeta, \bar{\zeta}, \sigma_1, \sigma_2)} A_\lambda(\zeta, \bar{\zeta}, \sigma_1, \sigma_2, t, w) d\sigma_1 d\sigma_2 dw dt.$$ The same type of parametrix exists for $\Pi_{\chi, r}(\lambda, \zeta, \bar{\zeta})$ but with an amplitude of order 0.
\( L^\infty \text{ norms of Husimi distributions of eigenfunctions } \)

\textbf{Proof.} We deploy the Boutet de Monvel-Sjöstrand parametrix for \( \Pi_t \) in Section 5.1 to pass from \((71)\) to \((73)\). We combine \((24)\) and \((26)\) and \((71)\) with Proposition 7.1 to obtain,

\[
\chi \ast dP_{\omega,\lambda}^\tau(\zeta, \tilde{\zeta}) = \int_\mathbb{R} \hat{\chi}(t)e^{-i\lambda t}\hat{\Pi}_t \hat{\sigma}_{t,\tau} \hat{g}_t^\tau \Pi_t(\zeta, \tilde{\zeta})dt
\]

\[
= \int_\mathbb{R} \int_0^\infty \int_{\partial M_t} \hat{\chi}(t)e^{-i\lambda t}e^{i\sigma_1 t}e^{i\sigma_2 t}g_t^\tau w, \zeta\big) \sigma_{t,\tau}(w, \sigma_1)\big) s(g_t^\tau w, \zeta, \sigma_2) = \sigma_{t,\tau}(w, \sigma_1)\big) s(g_t^\tau w, \zeta, \sigma_2) d\sigma_1 d\sigma_2 dwdt.
\]

Thus, the phase is \( \Phi(t, \zeta, w, \sigma_1, \sigma_2) \) as given in \((72)\). Here, \( \sigma_{t,\tau}(w, \sigma_1)\) is a polyhomogeneous function determined by the complete symbol of \( \hat{\sigma}_{t,\tau} \) in Proposition 7.1 and with the same principal term \( \sigma_{t,\tau}^0 \), and \( s \) in \( |\xi|^{-\frac{m+1}{2}} \) times the amplitude of the Szegö kernel. The order of each factor \( s \) of the symbol of the Szegö kernel is \( m - 1 \). Changing variables \( \sigma_j \rightarrow \lambda \sigma_j \) we obtain an oscillatory integral with large parameter \( \lambda \), with an amplitude of order \( \lambda^{2+2(m-1)-\frac{m-1}{2}} \), and with the complex phase \((72)\). The factors of \( A_t \) account for the factor \( \lambda^{-\frac{m-1}{2}} \).

The only change in the proof for \( \Pi_{\chi,t}(\lambda, \zeta, \tilde{\zeta}) \) is in the order of the amplitude, which now is 0.

\( \square \)

8.2. \textbf{Proof of Theorem 0.11 for } \( W_t(t, \zeta, \tilde{\zeta}) \). We now apply a complex stationary phase method to prove Theorem 0.11.

8.2.1. \textit{Critical set of the complex phase}. The critical set of \((72)\) with respect to the integration variables is given by the equations:

\[
\begin{align*}
\psi(\zeta, w) &= \psi(g_t^\tau w, \zeta) = 0; \\
\sigma_2 d_t \psi(g_t^\tau w, \zeta) &= 1, \\
d_w(\sigma_1(\psi(\zeta, w) + \sigma_2 \psi(g_t^\tau w, \zeta))) &= 0 \\
\end{align*}
\]

As in Proposition 6.3, we have:

\textbf{Lemma 8.2.} Given \( \zeta \), the critical set of \((72)\) is empty unless \( \psi_t^\tau(\zeta) = \zeta \). It then consists of

\[
\{(t, \sigma_1, \sigma_2, w) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \partial M_t : w = \zeta, \sigma_1 = \sigma_2 = 1, t = nT(\zeta)\}.
\]

It follows that for fixed \( \zeta \), the values of \( t \) for which one has critical points in the support of \( \hat{\psi} \) are \( t = 0 \) and \( t \) in the period set \( \mathcal{P}(\zeta) \), i.e. the set of \( t \) so that \( \psi_t^\tau(\zeta) = \zeta \). Thus \( t = nT(\zeta) \) for some \( n \), where \( T(\zeta) \) is the primitive period.

Indeed, by \((41)\) the first equation holds if and only if

\[
w = \zeta, \quad g_t^\tau \omega = \zeta \in \partial M_t \implies g_t^\tau \omega = \zeta.
\]

We restrict the second equation to \( w = \zeta \) and get \( \sigma_2 d_t \psi(g_t^\tau \omega, \zeta) = 1 \). Since the period set of \( \zeta \) is discrete the left side equals \( \sigma_2 d_t \rho(g_t^\tau \zeta, \zeta)|_{t = L(\zeta)} \) where \( L \) is a value of \( t \) so that \( g_t^\tau \zeta = \zeta \), and then \( \sigma_2 \partial \rho(\zeta) \cdot d_t g_t^\tau \zeta = \sigma_2 \alpha(\frac{d_t g_t^\tau \zeta}{\alpha}) = \sigma_2 = 1 \). Here we use that \( g_t^\tau \) is a contact flow for the contact form \( \alpha \).

We then consider the third equation. We may set \( t = nT(\zeta) \) and get \( \sigma_1 d_w \psi(\zeta, w) + \sigma_2 d_w \psi(w, \zeta) = 0 \). But \( d_w \psi(\zeta, w)|_{w = \zeta} = \alpha \) by \((35)\). If follows that \( \sigma_2 = \sigma_1 = 1 \).
Corollary 8.3. The critical value of phase (72) on critical set of Lemma 8.2 is given by
\[ \Phi(t, w, \sigma_1, \sigma_2; \zeta) = nT(\zeta), \]
where \( T(\zeta) \) is the primitive period of \( \zeta \) and \( n \) is the iterate number.

8.3. Localization to a Phong-Stein leaf. In slice-orbit coordinates (Definition 2.4) the phase (72) takes the form
\[ \Psi(t, \zeta, (s, z), \sigma_1, \sigma_2) = -t + \sigma_1 \psi(\zeta, (s, z)) + \sigma_2 \psi(g_s^t(s, z), \zeta) \]
Fix \( \zeta \) and \( \varepsilon > 0 \) and consider the time \( \varepsilon \) flow-out of the leaf in both positive and negative time:
\[ \mathcal{M}_\zeta(\varepsilon) = \bigcup_{|s| \leq \varepsilon} g^s \mathcal{M}_\zeta. \]

Let \( \theta(s) \) be a smooth cutoff in \( |t| \) supported in \( [-\varepsilon, \varepsilon] \) which equals one for \( |s| \leq \frac{\varepsilon}{2} \). As above we parametrize a neighborhood by the slice-orbit coordinates \( (z, s) \in \mathcal{M}_\zeta \times [-\varepsilon, \varepsilon] \rightarrow g^s z \) (48). Denote the volume density on \( \partial \mathcal{M}_\tau \) in slice-orbit coordinates by \( J(w, s) \).

Lemma 8.4. With the same notation as in Proposition 8.1. Fix \( \zeta \). Then, modulo a rapidly decaying error in \( \lambda \),
\[ \chi \ast dP^\tau_{[0, \lambda]}(\zeta, \bar{\zeta}) \simeq A_\lambda(\zeta, \bar{\zeta}, \sigma_1, \sigma_2, t, g^s z) J(w, s)dzdsdt, \]
where the Jacobian factor \( J(w, s) \) is the volume density on \( \mathcal{M}_\zeta \times [-\varepsilon, \varepsilon] \) and where \( A_\lambda \) has order \( \lambda^{-\frac{m-1}{2}} \). Similarly for \( \Pi_{\chi, \tau}(\lambda, \zeta, \bar{\zeta}) \) with the changes in \( A_\lambda \) mentioned in Proposition 8.1.

Proof. We have merely localized the integral over \( \partial \mathcal{M}_\tau \) to \( \mathcal{M}_\zeta \times [-\varepsilon, \varepsilon] \). This is possible, by the standard Lemma of Stationary Phase, i.e. the use of integration by parts to show that the integral is negligible on the complement of any neighborhood of the critical point. It applies since \( \mathcal{M}_\zeta(\varepsilon) \) covers a neighborhood of the stationary phase point. By the Lemma of stationary phase the remaining part of the integral is rapidly decaying.

In the next Lemma we apply stationary phase in the variables \( (\sigma_1, \sigma_2, s, t) \) to reduce the integral to a Phong-Stein leaf. This reduction is reminiscent of the steepest descent method for an oscillatory integral with complex phase, where the contour is deformed to one on which the imaginary part of the phase \( \text{Im} i \Phi \) equals zero. We do not deform contours but use the stationary phase method to obtain the reduction.

Next we evaluate the phase in Lemma 8.3 more explicitly. We retain the notation of that Lemma.

Lemma 8.5. Fix \( \zeta \in \mathcal{P} \) and let \( T_n(\zeta) \) be the return time to \( \mathcal{M}_\zeta \) of Definition 2.5, and let \( D \) be the diastasis (39). Then there exists a zeroth order amplitude \( B_\lambda(\cdot, z) \) supported in an
arbitrarily small neighborhood of \( z = \zeta \), so that, modulo rapidly decreasing functions of \( \lambda \),

\[
\chi \ast dP_{[0,\lambda]}^r(\zeta, \tilde{\zeta}) \simeq \lambda^{2m-2-m-1 \over 2} \sum_{n=0}^{\infty} \hat{\chi}(T_n(\zeta)) e^{-i\lambda T_n(\zeta)} \int_{\mathcal{M}_\zeta} e^{i(\lambda T_n(\zeta) + D(g^{T_n(\zeta)}_{\zeta}, z, \zeta))} dz.
\]

A similar formula holds for \( \Pi_{\chi,r}(\lambda) \) (30) but without the factor of \( \lambda^{-m-1 \over 2} \).

**Proof.** At a critical point of the phase, \( \psi(g_t^z, \zeta, \zeta) = 0 \) and so \( \text{Im} \psi(g_t^z, \zeta, \zeta) = 0 \) and also \( \psi(\zeta, z) = 0 \) so that \( w \in \mathcal{M}_\zeta \) and also \( g_t^z, \zeta \in \mathcal{M}_\zeta \). This forces \( \zeta = z, g_t^z = \zeta \), and again we see that \( t = T_n(\zeta) \) for some \( n \in \mathbb{Z} \). We then introduce Phong-Stein coordinates using a local inverse to (48) for \( t \) near \( T_n(\zeta) \) in the sense of the equivalence relation (49), and consider critical points in \( t, s, \sigma_1, \sigma_2 \). If we take \( \partial_s \) and consider only the imaginary part of the critical point equation, we get \( \sigma_1 = \sigma_2 \). If we take \( \partial_t \) and consider only the imaginary part we get \( \sigma_2 = 1 \). If we take \( D_{\sigma_1} \) and consider only the imaginary part we get \( s = 0 \); for \( D_{\sigma_2} \) we get \( t - T_n(\zeta) = -s = 0 \). Thus, the only critical point occurs at \( (s, t, \sigma_1, \sigma_2) = (0, T_n(\zeta), 1, 1) \).

The Hessian of the phase in \( (\sigma_1, \sigma_2, s, t) \) at this critical point is

\[
\begin{pmatrix}
\sigma_1 & \sigma_2 & s & t \\
\sigma_1 & 0 & 0 & -4i\tau \\
\sigma_2 & 0 & 0 & 4i\tau & 4i\tau \\
s & -4i\tau & 4i\tau & * & * \\
t & 0 & 4i\tau & * & * \\
\end{pmatrix}
\]

Here, \( T = T_n(\zeta) = nT(\zeta) \) for some \( n \). It is not necessary to calculate the lower right block. Note that by (35),

\[
\begin{align*}
\partial_s \psi(\zeta, (s, z))|_{s=0, z=\zeta} &= \partial_s \psi(\zeta, (s, \zeta))|_{s=0} = -4i\alpha_\zeta(\Xi, \sqrt{\rho}) = -4i\tau \\
\partial_s \psi(g_t^{s+z}, \zeta)|_{s=0, z=\zeta, t=T_n(\zeta)} &= \partial_s \psi(g_t^{T_n(\zeta)+s}, \zeta)|_{s=0} = 4i\tau.
\end{align*}
\]

The \( \partial_t \) derivative is the same as the \( \partial_s \) derivative in the last line.

Since the determinant is non-singular, we may apply stationary phase in the variables \( (\sigma_1, \sigma_2, s, t) \). Since the only stationary phase points are \( \sigma_1 = \sigma_2 = 1, s = 0, t = T_n(\zeta) \), the integral localizes to \( \mathcal{M}_\zeta \) and has the phase \( \Psi|_{\sigma_1=\sigma_2=1, s=0, t=T_n(\zeta)} \). Applying stationary phase in \( (\sigma_1, \sigma_2, s, t) \) concludes the proof of the Lemma. Since Hessian is non-degenerate, the integration produces a factor of \( \lambda^{-2} \). We then get,

\[
\chi \ast dP_{[0,\lambda]}^r(\zeta, \tilde{\zeta}) \simeq \lambda^{2m-2-m-1 \over 2} \sum_{n=0}^{\infty} \hat{\chi}(T_n(\zeta)) \int_{\mathcal{M}_\zeta} e^{i\lambda \tilde{\Psi}(T_n(\zeta), \zeta, \tilde{\zeta}, z, 1, 1)} B_\lambda(\zeta, \tilde{\zeta}, 1, 1, T_n(z), z) J(z, 0) dz,
\]

where \( B_\lambda(\cdot, z) \) is a 0th order amplitude supported in an arbitrarily small neighborhood of \( z = \zeta \). The phase \( \tilde{\Psi} \) and amplitude \( B_\lambda \) are obtained from the amplitude \( A_\lambda \) and phase \( \Psi \).
by the standard stationary phase method. The factor \( \lambda^{-\frac{m-1}{2}} \) in \( A_\lambda \) is moved outside the integral. \( \hat{\Psi} = \Psi|_{\sigma_1=\sigma_2=1,s=0,t=T_n(z)} \) is the critical value of \( \Psi \).

Since \( \mathrm{Im} \psi(\zeta, z) = \mathrm{Im} \psi(g_\tau^{T_n(\zeta)} z, \zeta) = 0 \) on \( \mathcal{M}_\zeta \), the value of the phase on the critical set is

\[
\Phi(t, \zeta, z, 1, 1) = -T_n(\zeta) + \Re \psi(\zeta, z) + \Re \psi(g_\tau^{T_n(\zeta)} z, \zeta) = -T_n(\zeta) + D(\zeta, z) + D(g_\tau^{T_n(\zeta)} z, \zeta),
\]

where \( D(z, w) \) is the Calabi diastasis \( (39) \).

The same form is valid for \( \Pi_{\chi, \tau}(\lambda, \zeta, \bar{\zeta}) \) with a different amplitude and without the factor \( \lambda^{-\frac{m-1}{2}} \). This completes the proof of the Lemma.

\[
\square
\]

8.4. **Proof of Theorem 0.11.** To complete the proof of Theorem 0.11, we need to calculate the integral over \( \mathcal{M}_\zeta \) asymptotically using the method of complex stationary phase (steepest descent). We next deploy the estimate on the phase from \( (41) \).

**Lemma 8.6.** In Kähler normal (or, Heisenberg normal) coordinates centered at \( \zeta \), there exists \( \varepsilon > 0 \) and there exists a positive constant \( C_\varepsilon > 0 \) so that for \( z \in B_\varepsilon(\zeta) \), we have

\[
D(\zeta, z) + D(g_\tau^{T_n(\zeta)} z, \zeta) \geq C_\varepsilon |z|^2 + |g_\tau^{T_n(\zeta)}(z) - \zeta|^2.
\]

**Proof.** We use Lemma 2.2. In the estimate \( (41) \) it suffices to choose \( \varepsilon \) so that the \( O(|z-w|^3) \) term of \( (41) \) is smaller than the term \( C|z-w|^2 \). That is, if the implicit constant in \( O(|z-w|^3) \) is \( D|z-w|^3 \) then we choose \( \varepsilon \) so that \( C \geq D\varepsilon \).

\[
\square
\]

To complete the proof of Theorem 0.11, we study the \( n \)th term in the sum in Corollary 8.5. The integral of concern is,

\[
S_{\lambda,n}(\zeta) := \int_{\mathcal{M}_\zeta} e^{N(D(\zeta, z) + D(g_\tau^{T_n(\zeta)} z, \zeta))} B_\lambda(\zeta, \bar{\zeta}, 1, 1, T_n(\zeta), z) J(z; \zeta) dz.
\]

As noted in Corollary 8.5, the integral of that Lemma may be cut off to the ball \( B_{\varepsilon(0)} \subset \mathbb{C}^{m-1} \) of Lemma 8.6 around \( \zeta \) (= 0 in the Heisenberg coordinates) without changing the asymptotics, since the phase has no critical points in this case. For simplicity of notation, we retain the notation \( z \in \mathbb{C}^{m-1} \) for the Heisenberg coordinates and the previous notation for the disastasis and the geodesic flow, without explicitly putting in the conjugation to Heisenberg coordinates. Thus,

\[
S_{\lambda,n}(\zeta) = \int_{B_{\varepsilon(0)}} e^{\Psi_n(z; \zeta)} A(\lambda, z; \zeta) dz.
\]

(74)

where \( A(\lambda, z; \zeta) = B_\lambda(\zeta, \bar{\zeta}, 1, 1, T_n(\zeta), z) J(z, 0) \) is supported in the ball of radius \( \varepsilon \) around 0, and where

\[
\Psi_n(z; \zeta) := (D(\zeta, z) + D(g_\tau^{T_n(\zeta)} z, \zeta)).
\]
8.4.1. Proof by steepest descents. The phase $\Psi$ is positive and real, so we may apply the method of (real) steepest descent on $\mathbb{C}^{m-1}$.

Proof. The steepest descent point is the minimum of the phase. We note that the phase vanishes at $z = \zeta$, and since the phase is positive this is a global minimum of the phase. By 8.6, $D(z, \zeta) \neq 0$ for $z \neq \zeta$, at least when $w \in B_r(\zeta)$, and note also that $g_r^{T_r(\zeta)} z \neq \zeta$ if $z \neq \zeta$. Hence, the phase does not vanish at any other $z \in B_r(\zeta)$.

By [HoI, Theorem 7.1], (74) admits a complete asymptotic expansion of the type,

$$S_{\lambda,n}(\zeta) \simeq \lambda^{-(m-1)} \frac{1}{\det \text{Hess}_\zeta \Psi} \sum_k \lambda^{-k} L_k(A e^{i\lambda R_3}).$$

We thus need to compute $\det \text{Hess}_\zeta \Psi$. For this purpose, we give the Taylor expansion to order 2 of the phase in a Heisenberg coordinate chart $(z, t, \rho) = (z, \text{Re} z, \rho(w)) \in \partial M_r \times \mathbb{R}_+$ centered at $\zeta = 0$, of Section 2.7. We fix $\rho = \tau^2$. To compute the Taylor expansion in the local coordinates, we write

$$z = \zeta + \frac{1}{\sqrt{\lambda}} u.$$

Here, $\zeta = 0$ in the coordinates, but (by abuse of notation) we leave it in to remember that the coordinates are centered at $\zeta$. The factor $\frac{1}{\sqrt{\lambda}}$ is natural in the Kähler or Heisenberg scaling. It is not necessary to put in this factor but it helps to keep track of the order of the terms.

Let $L(\zeta)$ denote the Levi form on $T_\zeta M_\zeta$. We now prove the following,

**Lemma 8.7.** Let $D_\zeta g^{T_n(\zeta)} = : S : T_\zeta M_\zeta \to T_\zeta M_\zeta \simeq \mathbb{C}^{d-1}$. With the above notation,

$$\lambda(D(\zeta, \zeta + \frac{u}{\sqrt{\lambda}}) + D(g^{T_n(\zeta)}(\zeta + \frac{u}{\sqrt{\lambda}}), \zeta)) = |u|^2_{L(\zeta)} + |S(u)|^2_{L(\zeta)} + O((\lambda^{-1}|u|)^3).$$

More generally, by polarization,

$$\lambda(D(\zeta + \frac{u}{\sqrt{\lambda}}, \zeta + \frac{v}{\sqrt{\lambda}}) + D(g^{T_n(\zeta)}(\zeta + \frac{u}{\sqrt{\lambda}}), \zeta)) = \langle u, v \rangle_{L(\zeta)} + \langle S u, S v \rangle_{L(\zeta)} + O((\lambda^{-1}|u, v|)^3).$$

**Proof.** By Lemma 2.2 and by (40),

$$D(x, y) = L_\rho(x - y) + O(|x - y|^3) = \sum_{j,k=1}^{m-1} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(\zeta)(x_j - y_j)(\bar{x}_k - \bar{y}_k) + O^3$$

so if $x = \zeta, y = \zeta + \frac{1}{\sqrt{\lambda}} u$,

$$D(\zeta, \zeta + \frac{1}{\sqrt{\lambda}} u) = \frac{1}{\lambda} \sum_{j,k=1}^{m-1} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(\zeta) u_j u_k + R_3(\frac{u}{\sqrt{\lambda}}),$$

where $R_3$ is the third order Taylor remainder satisfying,

$$R_3(\frac{u}{\sqrt{\lambda}}) = O((|u|/\lambda)^3).$$

Further,

$$g^{T_n(\zeta)}(\zeta + \frac{1}{\sqrt{\lambda}} u) = \zeta + \frac{1}{\sqrt{\lambda}} S u + O(\frac{|u|}{\lambda}),$$
so
\[ D_\zeta(g^T_\zeta)(\zeta + \frac{1}{\sqrt{\lambda}}u), \zeta) = D(\zeta + \frac{1}{\sqrt{\lambda}}Su + O(\frac{|u|}{\lambda}), \zeta) \]
\[ = \sum_{j,k=1}^{m-1} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(\zeta) (Su)_j (Su)_k + R'_3(\frac{u}{\sqrt{\lambda}}), \]
where \( R'_3 \) is the third order Taylor remainder satisfying,
\[ R'_3(\frac{u}{\sqrt{\lambda}}) = O((\frac{|u|}{\lambda})^3). \]

Since \( |z|^2_{h(\zeta)} = \omega_\zeta (J_\zeta(z), z) \), it follows from Lemma 8.7 and the fact that \( S \) is symplectic that
\[ \text{Hess}\Psi_\zeta(0) = \omega_\zeta (J_\zeta, \cdot) + \omega_\zeta (SJ_\zeta S^{-1}, \cdot), \]
so that, by (57) and (60), and then by (61) and Lemma 3.3,
\[ \det \text{Hess}\Psi_\zeta(0) = \det [\omega_\zeta (J_\zeta, \cdot) + \omega_\zeta (SJ_\zeta S^{-1}, \cdot)] \]
\[ = \langle W_{J_\zeta} (Dg^T_\zeta) (\Omega_{J_\zeta}), \Omega_{J_\zeta} \rangle. \]
It follows that
\[ S_{\lambda,n}(\zeta) \simeq \lambda^{-(m-1)} e^{i\lambda nT(\zeta)} \left( \langle W_{J_\zeta} (Dg^T_\zeta) (\Omega_{J_\zeta}), \Omega_{J_\zeta} \rangle \right). \]
Combining with Lemma 8.5, and using (19) and (17) (see also (16)), we get
\[ \chi * dP^\tau_{[0,\lambda]}(\zeta, \tilde{\zeta}) = \lambda^{\frac{m-1}{2}} \sum_n \hat{\chi}(nT(\zeta)) e^{i\lambda nT(\zeta)} G_n(\zeta) + O(\lambda^{\frac{m-3}{2}}) \]
\[ = C_m \lambda^{\frac{m-1}{2}} + C'_m \lambda^{\frac{m-1}{2}} \Re \sum_{n=1}^{\infty} \hat{\chi}(nT(\zeta)) e^{-i\lambda nT(\zeta)} G_n(\zeta), \]
and obtain the result stated in Theorem 0.11. Indeed, there is a complete asymptotic expansion with the given principal terms. Here, \( C_m \) denotes a dimensional constant. The proof for \( \Pi_{\lambda,\psi} \) is essentially the same, but without the factor of \( \lambda^{\frac{m-1}{2}} \) throughout.

**Remark 8.8.** The power of \( \lambda \) results from \( \lambda^{2m-2-\frac{m-1}{2}} \lambda^{-(m-1)} = \lambda^{\frac{m-1}{2}}. \)

In the case of \( \Pi_{\chi,\tau}(\lambda) \) (30), the factor \( \lambda^{\frac{m-1}{2}} \) does not arise, and the order is \( \lambda^{m-1}. \)
As mentioned above, \( \chi * dP^\tau_{[0,\lambda]}(\zeta, \tilde{\zeta}) \) (29) and \( \Pi_{\chi,\tau}(\zeta, \tilde{\zeta}) \) (30) are both dynamical Toeplitz operators, with the same canonical relation. They only differ in their amplitudes.

**9. Tauberian arguments: Completion of proof of Theorem 0.4**

To complete the proof of Theorem 0.4, we apply the Tauberian argument of [SV] (pages 225-6). See also Theorem 13.2 (cf. [SV], Appendix B (Theorem B.4.1)); the statement and proof are reviewed in §13.2.
We let \( N_{2,\tau,\zeta}(\lambda) = P^\tau_{[0,\lambda]}(\zeta, \tilde{\zeta}), \) and also define
\[ N_{1,\tau,\zeta}(\lambda) = 1_{[0,\infty)} \left( C'_m \lambda^{\frac{m-1}{2}} + Q_\zeta(\lambda) \lambda^{\frac{m-1}{2}} \right), \]
where $1_{[0,\infty)}$ is the indicator function. Both $N_j = N_{j;\tau,\zeta}(\lambda)$ are monotone non-decreasing functions of polynomial growth which vanish for $\lambda \leq 0$ and both satisfy the estimate of the Tauberian theorem 13.1. It follows from [SV] (p. 198 and p. 225) that if the support of $\hat{\chi}$ contains only \{0\} among the critical points of (73) and $\hat{\chi} \equiv 1$ in some smaller neighborhood of 0, then

$$N_{2;\tau,\zeta} \ast \chi(\lambda) = N_{1;\tau,\zeta}(\lambda) \ast \psi(\lambda) + O\left(\frac{\lambda}{\sqrt{\rho(\zeta)}}\right)^{m-1/2-1}.$$ 

Moreover, Theorem 0.11 shows that if $\hat{\gamma} \in C_0^\infty$ has support in $(0, \infty)$,

$$\gamma \ast dN_{2;\tau,\zeta}(\lambda) = \gamma \ast dN_{1;\tau,\zeta}(\lambda)(\lambda) + O\left(\frac{\lambda}{\sqrt{\rho(\zeta)}}\right)^{m-1/2-1}.$$ 

By Theorem 13.2 (see Theorem B.4 of [SV]),

$$N_{1;\tau,\zeta}(\lambda - o(1)) - o\left(\frac{\lambda}{\sqrt{\rho(\zeta)}}\right)^{m-1/2-1} \leq N_{2,\tau,\zeta}(\lambda) \leq N_{1;\tau,\zeta}(\lambda + o(1)) + o\left(\frac{\lambda}{\sqrt{\rho(\zeta)}}\right)^{m-1/2-1}.$$ 

Here, $o(1)$ is a positive monotone function which tends to zero as $\lambda \to \infty$. This proves (20). When $Q_\zeta$ is uniformly continuous, we may simplify $Q_\zeta(\lambda - o(\lambda))$ to $Q_\zeta(\lambda)$ and absorb the remainder into the error term.

10. Jump behavior: Proof of Proposition 0.5 and Proposition 0.6

In this section, we study the continuity or jumps of $Q_\zeta(\lambda)$ and prove 0.5 and Proposition 0.6.

10.1. Classical dynamical approach: Proof of Proposition 0.5. The natural approach to studying the right side of (64) is to put $S$ into a normal form. It is tempting to put $S$ into standard additive (resp. multiplicative) Jordan normal form as a sum (resp. product) of a semi-simple matrix and a nilpotent (resp. unipotent) matrix, but these matrices need not be symplectic in general and we cannot quantize the components by the metaplectic representation, and the symplectic Jordan normal forms are rather lengthy and complicated (see [Gutt]). Even when the matrix is put into normal form one must still extract its holomorphic part $P = P_jSP_j$. The map $S \to P_jSP_j$ does not behave well with respect to multiplicative normal forms, and its determinant $detP_jSP_j$ does not behave well with respect to additive normal forms. For that reason, we study only the open dense set of semi-simple symplectic matrices (see [Gutt] for the proof of density).

We refer to Section 3.2 for background on the symplectic Linear algebra. The article [MU00] contains a list of all possible symplectic normal forms of matrices in $Sp(2, \mathbb{R})$. Since semi-simple symplectic matrices are direct sums of symplectic matrices in $Sp(1, \mathbb{R})$, $Sp(2, \mathbb{R})$, the list in [MU00] contains the building blocks (under symplectic direct sum) of all normal forms relevant to this article.

The proof of Proposition 0.5 consists of a series of Lemmas dealing with the cases of (i) elliptic symplectic matrices; (ii) positive definite symmetric symplectic matrices (hyperbolic blocks), and (iii) semi-simple normal symplectic matrices with complex eigenvalues (sometimes
called loxodromic blocks). Since loxodromic blocks are not so familiar, we recall their
definition: If one of $\pm \alpha \pm i\beta$ is an eigenvalue of $A$ for some $\alpha, \beta > 0$, then there exists
$S \in Sp(2, \mathbb{R})$ so that
$$s^{-1} A s = \begin{pmatrix} A_4 & 0 \\ 0 & D_4 \end{pmatrix}, \text{ where } A_4 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \text{ } D_4 = \begin{pmatrix} -\alpha & \beta \\ -\beta & -\alpha \end{pmatrix}. \quad (75)$$
The associated symplectic linear map acts by complex dilation, i.e. a mixture of rotation
and real dilation.

10.1.1. **Elliptic symplectic matrices.** The following Lemma proves one direction of Proposition
0.5.

**Lemmas 10.1.** If $\zeta$ is elliptic, and if $\det P_J SP_J = e^{is_0}$, then
$$Q_\zeta(\lambda) = \sum_{n \neq 0} e^{\lambda n T(\zeta)} e^{in s_0} = \{s_0 + \lambda T(\zeta) - \pi\}_{2\pi}. \quad (76)$$

**Proof.** If $\zeta$ is elliptic, and if the polar part is the identity, i.e. $\zeta \in U(m)$. Then by Lemma
3.1, $\det P_J \zeta P_J = e^{is_0}$ for some $s_0 \in [0, 2\pi]$. It follows from (19) that (76) holds.

Next we consider non-elliptic semi-simple $S_\zeta$. Thus, we assume that $S_\zeta$ is diagonalizable
over $\mathbb{C}$ but that it has some eigenvalues which are not of modulus 1.

10.1.2. **Positive symmetric symplectic matrices.** In this section, we assume that $S$ is a
symmetric symplectic matrix, and, slightly more, that all of its eigenvalues are positive. As
discussed in Section 3.2, if $S$ is symmetric, there exists $U \in U(m)$ conjugating $S$ to its
diagonal form. Proposition 1.1 is an immediate consequence of the following Proposition,
adapted from [ZZ18] in the line bundle setting.

**Proposition 10.2.** If $S$ is positive definite symmetric symplectic, and if the spectrum of $S$
is $\{e^{\lambda_j}, e^{-\lambda_j}\}_{j=1}^{n}$ with $\lambda_j \geq 0$ then
$$\det P_J SP_J |_{T_{0,0}^{1,0} \mathbb{R}^{2n}} = \prod_{j=1}^{n} \cosh \lambda_j.$$ Consequently, $Q_\zeta(\lambda)$ is uniformly continuous.

**Proof.** The proof is through a series of Lemmas from [ZZ18]; since the proofs are short, we
repeat them here.

**Lemmas 10.3.** If $S$ is positive definite symplectic, then
$$P_J SP_J = \frac{1}{2} P_J (S + S^{-1}) = \frac{1}{2} (S + S^{-1}) P_J$$

**Proof.** If $S$ is positive definite symmetric, then $SJ = JS^{-1}$. Hence,
$$P_J SP_J = \frac{1}{4} (I - iJ) S (I - iJ) = \frac{1}{4} [S - iJS - iJS - JSJ]$$
$$= \frac{1}{4} [S + S^{-1}] - \frac{i}{4} [S + S^{-1}] = \frac{1}{4} ((S + S^{-1}) - iJ(S + S^{-1})) = \frac{1}{2} P_J (S + S^{-1}).$$
Also, $J(S + S^{-1}) = JS + SJ = (S^{-1} + S)J$, so that $P_J (S + S^{-1}) = (S + S^{-1}) P_J$. \qed


Lemma 10.4. Let $S$ be positive definite symmetric symplectic and $e_j$ be eigenvectors of $S$ for eigenvalues $\lambda_1, \ldots, \lambda_n$. Consider the basis $P_J e_k$ of $H_J^{1,0}$. Then

$$[P_J S P_J] P_J e_k = \cosh(\lambda_j) P_J e_k,$$

and $[P_J S P_J]^{-1} = P_J [S + S^{-1}]^{-1} P_J$.

Proof. Follows from the previous Lemma and the fact that $(S + S^{-1})$ commutes with $P_J$:

$$[P_J S P_J] P_J e_k = \frac{1}{2} P_J (S + S^{-1}) e_k = \frac{1}{2} (e^{\lambda_j} + e^{-\lambda_j}) P_J e_k = \cosh(\lambda_j) P_J e_k.$$

The determinant formula of Proposition 10.2 follows from the fact that the eigenvalues of $P_J S P_J$ are $\cosh \lambda_j$ by Lemma 10.4. When the closed geodesic through $\zeta \in \partial M_r$ is positive definite symplectic (or, real hyperbolic for short), then the determinant formula obviously implies that the Fourier series (18) for $Q_\zeta(\lambda)$ converges absolutely to uniformly continuous function, proving Proposition 1.1.

10.1.3. Semi-simple symplectic matrices with complex eigenvalues. We recall from Section 3.2 that if $S \in \text{Sp}(m, \mathbb{R})$ is a normal symplectic matrix, its polar decomposition $S = U \hat{P}_S$ satisfies $\hat{P}_S U = U \hat{P}_S$, with $\hat{P} = (S^* S)^{\frac{1}{2}}$. Since $U$ is unitary, $P_J U \hat{P} P_J = (P_J U P_J)(P_J \hat{P} P_J)$ and $\det P_J S P_J = \det(P_J U P_J) \det(P_J \hat{P} P_J)$. Proposition 10.2 applies to $\det(P_J \hat{P} P_J)$, and obviously $|\det P_J S^n P_J| \leq |\det(P_J \hat{P}^n P_J)| \leq \cosh(n \lambda_j)$. Hence, $Q_\zeta(\lambda)$ is an absolutely and uniformly convergent Fourier series.

10.1.4. Completion of the proof. If $S_\zeta$ is semi-simple symplectic, it is a direct sum of the three cases above and the coefficients (64) are products of those in the three cases. Only one block needs to be non-elliptic for the series to converge absolutely and uniformly.

This completes the proof of Proposition 0.5.

10.2. Quantum approach: Proof of Proposition 0.6. In this section, we use the real Schrödinger representation to prove Proposition 0.6.

The first statement (1) follows immediately from the definition of (23) and the fact that

$$\frac{1}{2i T(\zeta)} \int_0^{2\pi} \sum_{n \neq 0} e^{i \lambda n T(\zeta)} n e^{i n \theta} d\theta = \frac{1}{T(\zeta)} \int_0^{2\pi} \{\theta + \lambda T(\zeta) - \pi\} d\mu_\zeta.$$

The second statement follows since $\{\theta + \lambda T(\zeta) - \pi\} d\mu_\zeta$ is continuous in $\lambda$ on $[0, 2\pi]$, so if $d\mu_\zeta$ is absolutely continuous it is uniformly continuous in $\lambda$.

On the other hand, if $d\mu_\zeta$ has an atom at $e^{i \theta_0}$ then, $Q_\zeta(\lambda)$ has the jump of $\mu_\zeta(e^{i \theta_0})\{\theta_0 + \lambda T(\zeta) - \pi\}$. Hence, (2) is proved.

The atoms of the spectral measure of any unitary operator $W$ on a Hilbert space occur at its eigenvalues. Hence, (3) is true. Moreover, by definition of the spectral measure with respect to a normalized eigenvector $\Omega$, the $\mu_\zeta(s_0)$ equals $|\langle v_0, \Omega \rangle|^2$, proving (4).

This completes the proof of Proposition 0.6.
Remark 10.5. The argument in [SV] is to use the spectral theorem to write,

\[ Q_\zeta(\lambda) = \frac{1}{T(\zeta)} \sum_\ell \{s_\ell + \lambda T(\zeta) - \pi\}_2 \|\pi_{\Omega_\zeta} v_\ell\|^2, \]

10.2.1. Spectral theory of metaplectic operators. The above proof is rather abstract. To apply it to the metaplectic operators, we need to determine when they have eigenvalues (i.e. \( L^2 \) eigenfunctions) to produce atoms in the spectral measures, and moreover we need to determine the projections \( \pi_{\Omega_\zeta} v_\ell \). Although quadratic Hamiltonians and their propagators are very classical, the only reference we are aware of regarding their spectral decomposition is in [MU96, MU00]. In [MU00, Proposition 3.1] the Hörmander classification of symplectic normal forms of quadratic Hamiltonians on \( T^*\mathbb{R}^2 \) is recalled, and in [MU00, Proposition 3.2] the corresponding Schrödinger operators are listed. In addition to harmonic oscillators such as \( -\Delta + \|x\|^2 \), there are magnetic Schrödinger operators with potential such as \( (iD_{x_1} - bx_2)^2 + (iD_{x_2} + bx_1)^2 + \langle Kx, x\rangle \) where \( K \) is a real symmetric matrix; here \( D_{x_j} = \frac{1}{i} \frac{\partial}{\partial x_j} \). More generally, a magnetic Schrödinger operator with potential has the form, \( \sum_{j=1}^n (iD_{x_j} - (Bx)_j)^2 + \langle Kx, x\rangle \) where \( B \) is a real skew-symmetric \( n \times n \) matrix.

In [MU00, Proposition 3.3, Theorem 3.5, Theorem 4.7], the nature of the spectrum is determined for four types of quadratic Hamiltonians on \( \mathbb{R}^2 \). Of these, only the harmonic oscillator has eigenvalues. The others have absolutely continuous spectrum. These results imply that for \( \dim M = 2, 3 \), \( Q_\zeta(\lambda) \) is uniformly continuous in all cases except for elliptic closed geodesics. Thus, Proposition 0.6 is proved when \( \dim M = 2, 3 \). The nature of the spectrum in the special case of magnetic Schrödinger operators in higher dimension is studied in [MU00].

The cases of \( T^*\mathbb{R}, T^*\mathbb{R}^2 \) are fundamental by the normal form theorems above, since by (51) in the semi-simple case every quadratic Hamiltonian is a symplectic direct sum of model quadratic Hamiltonians on \( T^*\mathbb{R}^2 \) or \( T^*\mathbb{R} \). Clearly, it would be very laborious to determine the nature of the spectrum by this method for every possible type of symplectic linear transformation, or every possible quadratic in the symplectic classification. Hence we restrict again to normal symplectic transformations. We now present some simple proofs of Proposition 0.6 using this decomposition in the semi-simple case.

The case of general harmonic oscillators can be reduced to the one-dimensional case, as the next Lemma shows.

Lemma 10.6. Let \( S = e^B \) be positive definite symplectic, where \( B \in \mathfrak{sp}(m, \mathbb{R}) \) and \( B^* = B \). Then, the Weyl quantization \( W(S) \) of \( S \) (56) (with the standard \( J \)) has an \( L^2 \) eigenvector \( v \) if and only if \( v \) is an \( L^2 \) eigenvector of \( W(B) \) if and only if the Weyl quantizations \( W(B_j) \) of the diagonal blocks \( B_j \) of \( B \) have definite Weyl symbols.

Proof. As reviewed above, \( B \) is unitarily conjugate in \( Sp(m, \mathbb{R}) \) to a diagonal matrix. Its Weyl quantization \( W(B) \) is then a sum of squares of vector fields \( B_j \), and the symbol is a quadratic form in \( x, \xi \) which is a sum of squares \( c_j x_j^2 + d_j \xi_j^2 \). If the coefficients \( c_j, d_j \) are all positive, then \( |\sigma_B(x, \xi)| \to \infty \) as \( (x, \xi) \to \infty \), and \( W(B) \) has discrete spectrum. If any coefficient is negative, then it has continuous spectrum. The generalized eigenfunctions are tensor products \( v_1 \otimes v_2 \otimes \cdots \otimes v_m \) in the tensor decomposition \( L^2(\mathbb{R}^m) = \bigotimes_{j=1}^m L^2(\mathbb{R}, B_j) \), where the jth component \( v_j \in L^2(\mathbb{R}) \). In order that \( v \) be an \( L^2 \) eigenfunction it is necessary and sufficient that \( v_j \) be an \( L^2 \) eigenfunction of \( B_j \) for all \( j \).
For one-dimensional symmetric quadratic Schrödinger operators, it is known that the spectrum is discrete in the definite case and continuous in the indefinite case.

More generally, we have

**Proposition 10.7.** Suppose that \( \zeta \in \partial M \) is a periodic point such that \( D_\zeta g^T(\zeta) \) is a normal symplectic matrix that lies in the image of the exponential map and whose polar part \( (S^* S)^{1/2} \) has at least one positive eigenvalue, then \( Q_\zeta(\lambda) \) is uniformly continuous as long as \( \zeta \) is not an elliptic closed geodesic.

**Proof.** Its eigenvalues come in 4-tuples \( \lambda, \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1} \), though it may happen that \( \lambda = \bar{\lambda} \) or \( \lambda = \lambda^{-1} \). Using (51), the generator of \( W(S) \) is a symplectic direct sum of the 2 and 4 dimensional cases studied in [MU00]. In order that the generator have eigenvalues, it is necessary that every factor has an eigenvalue. But only harmonic oscillators (or their opposites) in dimension 2 have eigenvalues.

We may also prove the statement using polar decomposition. Given \( S \in Sp(m, \mathbb{R}) \), with \([S^*, S] = 0 \), we get \( W(S) = W(U)W(P) \), with \( W(U), W(P) \) unitary, and \([W(U), W(P)] = 0 \); here \( W = W_{J_0} \) (the standard complex structure). Hence, \( L^2 \) eigenfunctions are sums of joint eigenfunctions of \( W(U), W(P) \), i.e. \( W(S) \) has an eigenvalue \( e^{i\theta} \) if and only if there exists \( v \in L^2(\mathbb{R}^m) \) such that \( W(U)v = e^{i\theta}v, W(P)v = e^{i\tau}v \) with \( e^{i\theta} e^{i\tau} = e^{is} \). If \( U \) is unitary, then \( U = e^{iH} \) where \( H \in sp(n, \mathbb{R}) \) and \( H^* = B \). \( U \) has an \( L^2 \) eigenvector \( v \) if and only if \( v \) is an \( L^2 \) eigenvector of \( H \) if and only if the diagonal blocks \( H_j \) of \( B \) are definite.

It follows that the spectrum of \( W_{J_\zeta}(S_\zeta) \) has no eigenvalues unless \( \zeta \) is elliptic when \( S_\zeta \) is non-degenerate and semi-simple. This can also be read off [MU00, Proposition 3.3]. The loxodromic case is,

\[
P_{A_4} = -\alpha(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}) + \frac{\beta}{2\pi i}(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}),
\]
in the notation of that article (see (75) for the classical matrices), and it is proved there to have absolutely continuous spectrum. The only operators with discrete spectrum in [MU00, Proposition 3.3] is the harmonic oscillator.

**Remark 10.8.** There are further cases in [MU00, Theorem 3.3, Theorem 4.2] which are either degenerate or not semi-simple, and which can have dense pure point spectrum or eigenvalues of infinite multiplicity. We are not considering them here, for the sake of brevity, but the same methods apply to them.

### 11. Spheres and Zoll manifolds

In this section and the next Section 12, we exhibit extremals for the sup norm in the complex domain in the case of spheres, and then prove Theorem 1.2. In particular, the results show that the upper bound of Theorem 0.1 is sharp.

#### 11.1. Spheres

We now prove that the sup norm bounds are sharp by showing that they are obtained for analytic continuations of highest weight spherical harmonics. The eigenspaces \( \mathcal{H}_N \) of the Laplacian on the standard sphere \( S^m \) are the same as the spaces of
spherical harmonics of degree \( N \), i.e. restrictions of homogeneous harmonic polynomials of degree \( N \) to the surface of \( S^m \). We assume a basic familiarity with spherical harmonics in what follows, and refer to [SoZ] for background and references. Many calculations with Poisson transforms and complexified spectral projections on spheres can be found in [L80, G84]; the complexification of \( S^m \) is the homogeneous cone \( z \cdot z = 0 \) rather than the actual complexification \( z \cdot z = 1 \). In particular, in [G84, Section 6], the \( L^2 \) norms in the real and complex domains are compared.

Since the \( \Delta \) commutes with the \( SO(m + 1) \) action on \( S^m \), the Poisson transform \( P^\tau \) conjugates the \( SO(m + 1) \) action on \( S^m \) and on \( \partial S^m \). In particular, the operator \( A \) of Definition 3 is a function of \( \Delta \), hence is a scalar on each \( \mathcal{H}_N \).

In the real domain, as stated in (2), an \( L^2 \) normalized eigenfunction of a compact Riemannian manifold \( (M, g) \) has sup-norm at most \( C \lambda^{\frac{m-1}{2}} \), and by Theorem 0.1, an \( L^2 \) normalized Husimi distribution (3) on \( \partial M \) has sup-norm at most \( C \lambda^{\frac{m-1}{2}} \) where \( m = \dim M \). The real sup norm bound is attained by zonal spherics on \( S^m \). The Husimi sup norm bound is attained by analytic continuations of highest weight spherical harmonics. Since the explicit formula become complicated for \( m > 2 \), we illustrate the results only on \( S^2 \).

11.1.1. **Highest weight spherical harmonics on \( S^2 \).** Highest weight spherical harmonics on \( S^2 \) of degree \( N \) are denoted by \( Y_N^m \) and are “Gaussian beams” along the equator; see Section 12 for general Gaussian beams. In this section, we consider the \( L^2 \) norm and sup norm of the analytic continuation of \( Y_N^m \) to \( S^2_e \). The \( L^2 \) norm comparison may also be found in [L80, (5.9)] and [G84, Section 6].

We recall that in dimension 2, the normalized spherical harmonics are defined by

\[
Y_N^m(\theta, \varphi) = \sqrt{\frac{(2N+1)(N-m)!}{(N+m)!}} P_N^m(\cos \varphi) e^{im\theta},
\]

where

\[
P_N^m(\cos \varphi) = \frac{1}{2\pi} \int_0^{2\pi} (i \sin \varphi \cos \theta + \cos \varphi) e^{-im\theta} d\theta
\]
is an associated Legendre polynomial.

Up to a constant normalizing factor, the highest weight spherical harmonic \( Y_N^m \) is the restriction of the homogeneous harmonic polynomial \( (x_1 + ix_2)^N \) to \( S^2 \). It is independent of \( x_3 \) and is a holomorphic function of \( x_1 + ix_2 \). We claim that \( ||(x + iy)^N||_{L^2(S^2)} \sim N^{-1/4} \). Indeed we compute it using Gaussian integrals:

\[
\int_{\mathbb{R}^3}(x^2 + y^2)^N e^{-(x^2+y^2+z^2)} dxdydz
\]

\[
= \| (x + iy)^N \|^2_{L^2(S^2)} \int_0^\infty r^{2N} e^{-r^2} r^2 dr,
\]

\[
\Rightarrow \| (x + iy)^N \|^2_{L^2(S^2)} = \frac{\Gamma(N+1)}{\Gamma(N+\frac{1}{2})} \sim C_0 N^{-1/2}.
\]

Therefore the normalized highest weight spherical harmonics or Gaussian beams are \( Y_N^m \sim C_0 N^{1/4}(x + iy)^N \). It achieves its \( L^\infty \) norm at \( (1, 0, 0) \) where it has size \( N^{1/4} \).

The analytic continuation of \( Y_N^m \) to \( S^2_e = \{ (z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^2 + z_2^2 + z_3^2 = 1 \} \) is given in the usual holomorphic coordinates on \( \mathbb{C}^3 \) by \( C_0 N^{1/4}(z_1 + iz_2)^N \). The calculation of its \( L^2 \) norms in the real and complex domains are compared.
norm on $\partial \mathbb{S}^2_\tau$ is lengthy, so we opt for a simpler approach using Fermi normal coordinates. The calculation is valid in all dimensions.

**Lemma 11.1.** Highest weight spherical harmonics on the sphere $\mathbb{S}^m$ achieve the maximal sup norm bound of Theorem 0.1 and Corollary 0.9.

**Proof.** Gaussian beams (highest weight spherical harmonics) may be constructed in Fermi normal coordinates $(s, y)$ along a closed geodesic $\gamma$ in the form $N^{(m-1)/4} e^{iN\tau} e^{-Ny^2/2}$; see Section 12 for a detailed construction. Here, $s$ is arc-length along $\gamma$ and $y$ are exponential normal coordinates on the normal bundle. The factor $N^{(m-1)/4}$ is due to the $L^2$ normalization since the integral of $e^{-Ny^2}$ over $\mathbb{R}^{m-1}$ equals $c_m N^{-(m-1)/2}$ up to a dimensional constant $c_m$. The sup norm is achieved along the complexified equator.

We then complexify $s \to s + i\sigma$, $y \to y + i\eta \in \partial \mathbb{S}^m$, to get

$$Y_N^N(s + i\sigma, y + i\eta) = N^{(m-1)/4} e^{iN(s+i\sigma)} e^{-N(y+i\eta)^2},$$

where the $\sqrt{\rho}(s + i\sigma, y \to y + i\eta) = \tau$. A point of this kind is $y = \eta = 0$ and $\sigma = -\tau$, since the equator is isometrically imbedded in $\mathbb{S}^m$ and the tube function of the complexified geodesic equals the restriction of the tube function of $\mathbb{S}^m_\tau$. At this point,

$$|Y_N^N(s + i\sigma, 0)| = N^{(m-1)/4} = N^{(m-1)/4}e^{N\tau},$$

and we see that the sup norm bound of Theorem 0.1 is attained. $\square$

The analytic continuation could also be calculated by analytically continuing the oscillatory integral formula, given by

$$P_N^N(\cos \varphi) = 2\pi \int_0^{2\pi} (i \sin \varphi \cos \theta + \cos \varphi)^N e^{-iN\theta} d\theta$$

where $\varphi$ is now complex, with complex phase $\log(i \sin \varphi \cos \theta + \cos \varphi) - i\theta$. Recall that $x = \sin \varphi \cos \theta$, $y = \sin \varphi \sin \theta$. Hence,

$$Y_N^N(\theta, \varphi) = C_N(\cos \theta \sin \varphi + i \sin \theta \sin \varphi)^N = C_N e^{iN\theta}(\sin \varphi)^N.$$

This formula can be used to calculate that $C_N = \sqrt{N+1}$, but we omit this classical calculation. Then we analytically continue to get

$$(Y_N^N)^\mathbb{C}(\theta + ip_\theta, \varphi + ip_\varphi) = \sqrt{N + 1/2} e^{iN(\theta + ip_\theta)}(\cos(\varphi + ip_\varphi))^N.$$

On the set $p_\theta = -\tau, \varphi = \pi/2, p_\varphi = 0$ we find that

$$|(Y_N^N)^\mathbb{C}(\theta + ip_\theta, \varphi + ip_\varphi)| \simeq N^{1/4} e^{N\tau}.$$

11.1.2. Coherent states in the complex domain. Next we use the relation between coherent states and orthogonal projections to calculate the $L^2$ norm and $L^\infty$ norm of coherent states on spheres of general dimensions.

In the real domain, the spectral projections $\Pi_N : L^2(\mathbb{S}^m) \to \mathcal{H}_N$ onto the space of spherical harmonics of degree $N$ commute with the action of $SO(m + 1)$. Let $Y_N^m$ denote the joint eigenfunctions $Y_N^m$ of $\Delta$ and of the maximal torus of $SO(m + 1)$. They are orthogonal for different joint eigenvalues of $\Delta$ and of the maximal torus of $SO(m + 1)$, and the kernel

$$\Pi_N(x, y) = \sum_{\vec{m}} Y_N^m(x) \overline{Y_N^m(y)}$$
of $\Pi_N$ satisfies

$$\Pi_N(gx, gy) = \Pi_N(x, y), \quad g \in SO(m + 1).$$

Here $dS$ is the standard surface measure. Hence $\Pi_N(x, x)$ is a constant independent of $x$. For each $y$, $\Pi_N(\cdot, y)$ is spherical harmonic of degree $N$ with $L^2$ norm squared,

$$||\Pi_N(\cdot, y)||^2_{L^2} = \int_{S^m} \Pi_N(x, y)\Pi_N(y, x)dS(x) = \Pi_N(y, y).$$

Its integral is $\dim \mathcal{H}_N$, hence, $\Pi_N(y, y) = \frac{1}{\sqrt{\text{Vol}(S^m)}} \dim \mathcal{H}_N$. Hence the normalized projection

$$Y^0_N(x) = \frac{\Pi_N(x, y_0)\sqrt{\text{Vol}(S^m)}}{\sqrt{\dim \mathcal{H}_N}}$$

kernel with ‘peak’ at $y_0$ achieves the maximum possible sup norm of $\sqrt{\dim \mathcal{H}_N}$. Moreover, since $\Pi_N$ is the orthogonal projection, a standard argument using the reproducing property and the Schwartz inequality shows that $Y^0_N(y_0)$ is maximal among all $L^2$-normalized spherical harmonics of degree $N$. We note that if $y_0$ is the fixed point of the $S^1$ action (or, in general dimensions, the maximal torus action), then $Y^0_N(x_0) = 0$ for $m \neq 0$ and the identity above is obvious.

The zonal spherical harmonic also admits the Legendre representation,

$$Y^0_N(\theta, \varphi) = \sqrt{(2N + 1)P^N_0(\cos \varphi)},$$

where

$$P^0_N(\cos \varphi) = \frac{1}{2\pi} \int_0^{2\pi} (i \sin \varphi \cos \theta + \cos \varphi)^N d\theta.$$

If we analytically continue $\varphi$ to $S^2$, we obtain an oscillatory integral with complex phase $\log(i \sin \varphi \cos \theta + \cos \varphi)$. It has a critical point when, and only when, $\sin \theta = 0$. The stationary phase expansion brings in the additional factor of $N^{1/2}$, explaining why the complexified zonal harmonic is not an extremal. Note that the analytic continuation of (78) to $M_\tau$ is,

$$(Y^0_N)^C(z) = \frac{\Pi_N^C(z, y_0)\sqrt{\text{Vol}(S^m)}}{\sqrt{\dim \mathcal{H}_N}}, \quad (78)$$

However we now eschew oscillatory integrals to work with projection kernels in order to identify the extremals. Denote by $\mathcal{H}^C_N$ the holomorphic extensions of the spherical harmonics of degree $N$. For each $\tau$ the restrictions of the harmonics to $\partial S^m_\tau$ is a space $\mathcal{H}^C_N(\tau)$ of CR holomorphic functions, and it is easy to see that the joint eigenfunctions $Y^m_N$ of $\Delta$ and of the maximal torus of $SO(m + 1)$ are orthogonal for different joint eigenvalues. We denote by

$$\Pi_N^C(z, w) = \sum_m (Y^m_N(z))^C(Y^m_N)^C(w)$$

(79)

the analytic extension of $\Pi_N$. We denote by $\Pi_N^C(z, w)$ the restriction of the kernel to $z, w \in \partial S^m_\tau$. Using the natural complex conjugation on $S^m_\tau$ we also consider the kernel $\Pi_N^C(z, \bar{w})$, which is holomorphic in $z$ and anti-holomorphic in $w$.

**Definition 4.** Given $\tau > 0$ and $w \in \partial M_\tau$, we define the ‘coherent state’ centered at $w$ by,

$$\Phi^w_N(z) = \frac{\Pi_N^C(z, w)}{||\Pi_N^C(\cdot, w)||_{L^2(\partial M_\tau)}}, \quad z, w \in \partial S^m_\tau$$

(80)
where
\[ ||\Pi_N(z, w)||^2_{L^2(\partial M_\tau)} = \sum_{\tilde{m}} ||Y^{\tilde{m}}_N||^2_{L^2(\partial M_\tau)}. \]

For each \( w \), the coherent state (80) is an element of \( \mathcal{H}_N^C \), but is a scalar multiple of (79) and is not the analytic continuation of (77).

**Proposition 11.2.** Norm-squares of coherent states of \( \mathbb{S}^m \) attain the asymptotically maximal sup norm of (5).

**Proof.** \( \Pi_N(z, w) \) is the orthogonal sum of \( Y^{\tilde{m}}_N \otimes \overline{Y^{\tilde{m}}_N} \), and is not normalized to be an orthogonal projection, so we cannot immediately apply the argument in the real domain to find the value on the diagonal, nor can we immediately conclude that either defines an extremal for the sup norm (when properly normalized). But we observe that, by orthogonality of the terms,

\[ \Pi_N Y^{\tilde{m}}_N = ||Y^{\tilde{m}}_N||^2_{L^2(\partial M_\tau)} Y^{\tilde{m}}_N. \]

We introduce the orthonormal basis
\[ \tilde{Y}^{\tilde{m}}_N = \frac{(Y^{\tilde{m}}_N)^C}{|||Y^{\tilde{m}}_N||^2_{L^2(\partial M_\tau)}}. \]

The orthogonal projection on by \( \mathcal{H}_N^C(\varepsilon) \) is then
\[ \tilde{\Pi}_N(z, w) = \sum \tilde{Y}^{\tilde{m}}_N(z) \overline{Y^{\tilde{m}}_N(w)}, \]

and, as for any reproducing kernel,
\[ ||\tilde{\Pi}_N(\cdot, w)||^2_{L^2(\partial M_\tau)} = \tilde{\Pi}(w, w) = \frac{\dim \mathcal{H}_N}{\text{Vol}(\partial M_\tau)}. \]

For fixed \( w \) this \( z \to \tilde{\Pi}_N(z, w) \) defines an element of \( \mathcal{H}_N^C(\varepsilon) \), and we define a variant of the coherent state centered at \( w \) by,
\[ \Phi^w_N(z) := \frac{\tilde{\Pi}_N(z, w)}{||\tilde{\Pi}_N(\cdot, w)||_{L^2(\partial M_\tau)}} = \sqrt{\text{Vol}(\partial M_\tau)} \frac{\tilde{\Pi}_N(z, w)}{\sqrt{\dim \mathcal{H}_N}}. \]

**Lemma 11.3.** For any \( w \in \partial M_\tau \), (81) achieves the sup norm bound (5) of Theorem 0.1

**Proof.** As in the real case, (81) is the evaluation functional on \( \mathcal{H}_N^C(\tau) \). By the standard argument, the reproducing kernel achieves the extremal \( L^2 \)-normalized element of \( \mathcal{H}_N^C(\tau) \) for the sup norm. Namely, for any \( s_N \in \mathcal{H}_N^C(\tau) \),
\[ |s_N(z)| = |\langle s_N, \Phi^w_N \rangle| \leq ||s_N||_{L^2} ||\Phi^w_N||, \]

with equality if \( s_N = \Phi^w_N \).

Both of the kernels \( \Pi_N(z, w) \), resp. \( \tilde{\Pi}_N(z, w) \), are invariant under the diagonal action of \( SO(m+1) \). Indeed, by analytic continuation from the real domain, also have \( \Pi_N(gz, gw) = \Pi_N(z, w) \) for all \( z, w \). The group \( SO(m+1) \) acts transitively on \( S^m \mathbb{S}^m \) and hence on \( \partial S^m \) (for any \( \tau > 0 \)). It is also a holomorphic action on \( \mathbb{S}^m \). It follows that \( \Pi_N(\zeta, \zeta) \) is constant as \( \zeta \) varies. Since the orthogonal projection commutes with \( SO(m+1) \), we also have \( \tilde{\Pi}_N(gz, gw) = \tilde{\Pi}_N(z, w) \). This implies that its \( L^2 \) norm equals \( \frac{\dim \mathcal{H}_N}{\sqrt{\text{Vol}(\partial M_\tau)}} \). \( \square \)
To complete the proof we use Lemma 0.2 to compare (80) and (81), or equivalently the kernels \( \Pi_N^C(z, w) \) and \( \tilde{\Pi}_N(z, w) \). By this Lemma, \( \| (Y^m_N)^C \|_{L^2(\partial M_\tau)}^2 \approx e^{2\tau \lambda - \frac{m+1}{2}} (1 + O(\lambda^{-1})) \), uniformly in \( \bar{m} \). Hence,

\[
||\Pi_N^C(\bar{m}, w)||_{L^2(\partial M_\tau)}^2 = \sum_{\bar{m}} ||(Y^m_N)^C||_{L^2(\partial M_\tau)}^2 \approx e^{2\tau \lambda - \frac{m+1}{2}} \frac{\dim H_N}{\text{Vol}(\partial M_\tau)} (1 + O(\lambda^{-1})).
\]

Also, using Lemma 0.2 term by term,

\[
\tilde{\Pi}_N(z, w) = e^{-2\tau \lambda - \frac{m+1}{2}} (1 + o(1)) \Pi_N(z, w).
\]

The constant is canceled when we divide each side by its \( L^2 \) norm, By the definitions (80) and (81), it follows that, modulo terms of one lower order in \( N \),

\[
\Phi^w_N(z) \simeq \tilde{\Phi}^w_N(z),
\]

completing the proof of the Proposition.

\[ \square \]

Remark 11.4. Note that the analytic continuation (78) of (77), with \( z \in \partial M_\tau, y_0 \in M \), is analytically continued only in the first variable. Although (79) and (80) are complex analogues of (77), there does not exist a fixed point of the torus action in \( \partial M_\tau \), so (80) does not have a single term (as (77) does). This explains why Proposition 11.2 does not assert that (78) is an extremal.

11.2. Zoll case: Proof of Theorem 1.2. We normalize the metric so that the geodesic flow is periodic of period \( 2\pi \) and we assume that \( 2\pi \) is the minimal period of periodic orbits. We then center the intervals \( I_k \) at the points \( k + \beta \frac{\tau}{4} \) where \( \beta \) is the common Morse index of the \( 2\pi \)-periodic geodesics.

Proof. The proof is similar to the real off-diagonal asymptotics in [ZZoll], and we only sketch it here. The key point is that \( I_k \) are the Fourier coefficients of the \( 2\pi \) periodic unitary group \( U(t) = e^{it(A + \frac{\beta}{4})} \) in the sense that

\[
\Pi_{I_k} = \frac{1}{2\pi} \int_0^{2\pi} e^{-i(k + \frac{\beta}{4})t} U(t) dt.
\]

Here, \( A = k \) in the kth cluster of eigenvalues. It follows that

\[
P_{I_k}^T(\zeta, \bar{\zeta}) = \frac{1}{2\pi} \int_0^{2\pi} e^{-i(k + \frac{\beta}{4})t} \Pi_{\tau t}^\sigma \sigma_{\tau t}^g \Pi_\tau (\zeta, \bar{\zeta}) dt.
\]

We then proceed through the steps of Theorem 0.11 but using (82) instead of the oscillatory integral in Lemma 8.4. The calculations are of the same type with \( T(\zeta) = 2\pi \) and with \( Dg_\tau^T = I \) for all \( \zeta \) in the Zoll case.

The principal new feature is that one does not need to use a Tauberian theorem to obtain the asymptotics for \( P_{I_k}^T(\zeta, \bar{\zeta}) \), but only to use the fact that

\[
\sum_{k=1}^{\infty} e^{i(k + \frac{\beta}{4})t} P_{I_k}^T(\zeta, \bar{\zeta})
\]
is a Fourier series with only positive terms. One can then obtain complete asymptotic expansions of the Fourier coefficients by matching terms of Hardy series. We refer to Proposition 13.10 of [BoGu] for the details.

The result is a complete asymptotic expansion of the type stated in Theorem 1.2.

\[\square\]

**Remark 11.5.** To obtain an ‘integrated’ expansion on \([0, \lambda]\) we would form the sums \(\sum_{k=1}^{N} P^\tau_k(\zeta, \bar{\zeta})\) and substitute the asymptotic expansion for each term. The rather complicated inequalities of Theorem 0.4 (3) are only necessary for choices of \(\lambda\) which do not contain the full cluster of eigenvalues below the endpoint \(\lambda\).

### 12. Extremals: Gaussian beams associated to non-degenerate elliptic closed geodesics

Since Gaussian beams along elliptic closed geodesics are the extremals for sup-norms in Grauert tubes, we provide some background on the construction of Gaussian beams associated to an elliptic closed geodesic \(\gamma\). We follow the presentation in [Z97b] and [BB91, Section 9] (see also [Ral82]).

We denote by \(J_\gamma^\perp \otimes \mathbb{C}\) the space of complex normal Jacobi fields along \(\gamma\), a symplectic vector space of (complex) dimension \(2n\) (\(n = \dim M - 1\)) with respect to the Wronskian

\[
\omega(X, Y) = g(X, \frac{D}{ds}Y) - g(D\frac{ds}{ds}X, Y).
\]

The linear Poincare map \(P_\gamma\) is defined to be the linear symplectic map on \(J_\gamma^\perp \otimes \mathbb{C}\) defined by

\[P_\gamma Y(t) = Y(t + L_\gamma)\]

The closed geodesic \(\gamma\) is called non-degenerate elliptic if the eigenvalues of \(P_\gamma\) are of the form \(\{e^{\pm i\alpha_j}, j = 1, ..., n\}\) where the exponents \(\{\alpha_1, ..., \alpha_n\}\), together with \(\pi\), are independent over \(\mathbb{Q}\). The associated normalized eigenvectors will be denoted \(\{Y_j, \bar{Y}_j, j = 1, ..., n\}\),

\[
P_\gamma Y_j = e^{i\alpha_j}Y_j \quad P_\gamma \bar{Y}_j = e^{-i\alpha_j}\bar{Y}_j \quad \omega(Y_j, \bar{Y}_k) = \delta_{jk}. \tag{83}
\]

Let \((s, y)\) denote Fermi normal coordinates in a tubular neighborhood of \(\gamma\). Let \(L\) denote the length of \(\gamma\). Roughly speaking, Gaussian beams \(\Phi_{kq}(s, y)\) along \(\gamma\) are oscillatory sums with positive complex phases. They have a real oscillatory factor \(e^{iks}\) and a transverse Hermite factor \(D_q(y)\), which is the \(q\)th Hermite function in the normal direction to \(\gamma\), with \(q \in \mathbb{N}^n\). The special case \(q = 0\) is the ground state Gaussian beam, and the higher \(q\) are Gaussians times higher Hermite polynomials. In general, Gaussian beams are only approximate eigenfunctions (quasi-modes) but in special cases such as surfaces of revolution (and many other \((M, g)\) with completely integrable geodesic flow), they are exact eigenfunctions. Given \((k, q)\) the effective ‘Planck constant’ for the sequence with fixed \(q\) and \(k \to \infty\) is,

\[
r_{kq} := \frac{1}{L}(2\pi k + \sum_{j=1}^{n}(q_j + \frac{1}{2})\alpha_j).
\]

The associated sequence of eigenvalues of \(\sqrt{\Delta}\) has the expansion,

\[
\lambda_{kq} \equiv r_{kq} + \frac{p_1(q)}{r_{kq}} + \frac{p_2(q)}{r_{kq}^2} + ..., \tag{88}
\]

where \(p_j(q)\) are polynomials whose parity and degrees are described in [BB91, Section 9].
We now introduce the precise Hermite functions in the Gaussian beam. Relative to a parallel normal frame \( e(s) := (e_1(s), ..., e_n(s)) \) along \( \gamma \) the Jacobi eigenfields have the form, 
\[
Y_j(s) = \sum_{k=1}^{n} y_{jk}(s)e_k(s).
\]
We denote by, 
\[
\Gamma(s) := \frac{dY(s)}{ds}Y(s)^{-1}.
\]
\( \Gamma(s) \) satisfies a matrix Riccati equation, 
\[
\dot{\Gamma} + \Gamma^2 + K = 0,
\]
where \( K \) is the curvature matrix \( R(\dot{\gamma}, Y_j)\dot{\gamma} \), and is a complex symmetric \( n \times n \) matrix with positive definite imaginary part [BB91, Page 229]. In fact, by [BB91, 9.3.11], 
\[
\text{Im} \Gamma(s) = \frac{1}{2}(Y(s)Y^*(s))^{-1},
\]
where as usual \( Y(s)^* \) is the adjoint of \( Y(s) \). We will use the equations [BB91, (9.2.22)],
\[
\begin{cases}
Y(s)^*\dot{Y}(s) - \dot{Y}(s)^*Y(s) = iI, \\
Y(s)^t\dot{Y}(s) - \dot{Y}(s)^tY(s) = 0
\end{cases},
\]
where \( Y^t \) is the transpose of \( Y \). We multiply the second equation on the left by \( (Y(s)^t)^{-1} \) and on the right by \( Y(s)^{-1} \) to get 
\[
\dot{Y}(s)Y(s)^{-1} - (Y(s)^t)^{-1}\dot{Y}(s) = 0, \text{ or } Y^{-1*}\dot{Y}(s)^* - \dot{Y}(s)^*(Y(s)^t)^{-1*} = 0
\]

The transverse ground state Gaussian is defined in Fermi normal coordinates by,
\[
U_0(s, y) = (\det Y(s))^{-1/2}e^{\frac{i}{2}Y(s)y}.
\]
Although they are of secondary interest here, the higher Hermite functions have the form, 
\[
U_q = \Lambda_1^q \cdots \Lambda_n^q U_0,
\]
where \( \Lambda_j \) are certain creation operators associated to the Jacobi data [BB91, Section 9].

The Gaussian beams can now be defined by the formal series,
\[
\Phi_{kq}(s, \sqrt{r_{kq}}, y) = e^{ir_{kq}s} \sum_{j=0}^{\infty} r_{kq}^{-j/2} U_j^q(s, \sqrt{r_{kq}y}, r_{kq}^{-1})
\]
with \( U_0^q = U_q \) (see [BB92]). The functions \( U_j^q \) are found by solving transport equations. As is usual in the theory of quasi-modes, the infinite series represents a formal asymptotic expansion, and means that if one truncates the series at \( j = N \), then the resulting finite series solves the Laplace equation up to a remainder of order \( r_{kq}^{-N} \). We are mainly interested in the case \( q = 0 \), in which case the Gaussian beam is given by,
\[
\Phi_{k0}(s, \sqrt{r_{k0}y}) = e^{ir_{k0}s} \sum_{j=0}^{\infty} r_{k0}^{-j/2} U_j^0(s, \sqrt{r_{k0}y}, r_{k0}^{-1}).
\]
Remark 12.1. The calculations here are much simpler on spheres than for general Gaussian beams in Section 12; due to the constant curvature on spheres, the fact that all geodesics are closed, the matrix $\Gamma(s)$ (84) is simply $iI$ for a closed geodesic of $S^m$.

We say that an eigenfunction is a Gaussian beam when it admits such an asymptotic expansion. The Gaussian beam is exponentially concentrated in a tubular neighborhood of radius $\frac{1}{\sqrt{r_{k_0}}}$. around $\gamma$. Changing variables to $u = \sqrt{r_{k_0}}y$, one may approximate its $L^2$ norm-square by,

$$\int_0^L \int_{|y| \leq \frac{1}{\sqrt{r_{k_0}}}} |U_0(s, \sqrt{r_{k_0}}y)|^2 dsdy \simeq C_0(r_{k_0})^{-\frac{(\text{dim } M - 1)}{2}}.$$  

We are only interested in orders of magnitude and omit further details. It follows that the $L^2$ normalized Gaussian beam has the form,

$$C_0 \frac{k}{2} (\frac{\text{dim } M - 1}{2}) \Phi_{k_0}(s, \sqrt{r_{k_0}}y),$$

where $C_0 > 0$ is a positive constant. Thus, the sup-norm in the real domain of the Gaussian beam is asymptotically $C_0 \frac{k}{2} (\frac{\text{dim } M - 1}{2})$.

The linear Poincaré map $D_\zeta g^L_\tau$ in this case is given by (83) or in the Grauert tube notation by,

$$S_\zeta = \begin{pmatrix} \text{Im } \dot{Y}(L)^* & \text{Im } Y(L)^* \\ \text{Re } \dot{Y}(L)^* & \text{Re } Y(L)^* \end{pmatrix}.$$  

By (83), it is diagonalizable over $\mathbb{C}$ as a block-diagonal matrix

$$S_\zeta \simeq \bigoplus_{j=1}^n \begin{pmatrix} e^{i\alpha_j} & 0 \\ 0 & e^{-i\alpha_j} \end{pmatrix},$$

where $\simeq$ denotes unitary equivalence in $GL(n, \mathbb{C})$; the right side is of course not in $\text{Sp}(n, \mathbb{R})$. The metaplectic quantization of $S_\zeta$ is the exponential of a Harmonic oscillator Hamiltonian $\hat{H}_{\vec{\alpha}}$ with frequencies $\alpha_j$, i.e.

$$W_J(S_\zeta) = \exp i\hat{H}_{\vec{\alpha}}, \quad \hat{H}_{\vec{\alpha}} = \sum_{j=1}^n D_{x_j}^2 + \alpha_j x_j^2,$$

with eigenvalues $\lambda_{\vec{k}} = \sum_{j=1}^n \alpha_j (k_j + \frac{1}{2})$, $\vec{k} \in \mathbb{N}^n$. Thus, $W_J(S_\zeta)$ has eigenvalues $\exp(i \sum_{j=1}^n \alpha_j (k_j + \frac{1}{2}))$. In this model, the ground state $\Omega_\zeta$ is the standard Gaussian, which is the eigenfunction of eigenvalue $\lambda_{\vec{0}}$. Hence, $W_J(S_\zeta)\Omega_\zeta = e^{i\frac{1}{2}|\vec{\alpha}|}\Omega_\zeta$ with $\vec{\alpha} = \sum_j \alpha_j$, and therefore, $\langle W_J(S_\zeta^\ell)\Omega_\zeta, \Omega_\zeta \rangle = e^{i\frac{1}{2}\ell|\vec{\alpha}|}$. By (19),

$$Q_\zeta(\lambda) = \frac{1}{2i} \sum_{n \neq 0} \frac{e^{in(\frac{1}{2}L + \frac{1}{4}|\vec{\alpha}|)}}{n} = \{\lambda L + \frac{1}{2}|\vec{\alpha}| - \pi\}_{2\pi}.$$  

It is not straightforward to calculate the $L^2$ norm and sup norm of the analytic continuation of the Gaussian beam to $\partial M_\tau$. The analytic continuation is given in analytic Fermi normal
coordinates \((s + i\sigma, y + i\eta)\) by,

\[
\Phi_{k_0}^C(s + i\sigma, \sqrt{r_{k_0}}(y + i\eta)) = k^{(\dim M - 1)/2} e^{i r_{k_0}(s + i\sigma)} \sum_{j=0}^{\infty} r_{k_0}^{-j/2} U_{0,j}^C(s + i\sigma, \sqrt{r_{k_0}}(y + i\eta)),
\]

with leading order term,

\[
U_{0,j}^C(s + i\sigma, \sqrt{r_{k_0}}(y + i\eta)) = (\det Y(s + i\sigma))^{-1/2} e^{i r_{k_0}(s + i\sigma)} e^{\frac{j}{2} r_{k_0}(\Gamma(s + i\sigma)(y + i\eta)),(y + i\eta))}.
\]

Here, \(\sqrt{\rho}(s + i\sigma, y + i\eta) = \tau\). Upon analytic continuation, it is not clear that the Gaussian beam need should be concentrated in the complexification of the real tube around \(\gamma\), i.e. in a phase space tube around the phase space orbit \(\gamma\), since the damping Hermite factor in the real domain can grow exponentially outside the tube once it is analytically continued. When \(\sigma = -\tau, y = \eta = 0\) it is evident that it attains the maximal value \(k^{(\dim M - 1)/2} e^{k\tau}\). By Lemma 0.2, the \(L^2\) norm is asymptotically \(k^{-\frac{(\dim M - 1)}{2}} e^{k\tau}\), so the sup norm of the Husimi distribution is of order \(k^{\frac{(\dim M - 1)}{2}}\).

12.1. Geometric interpretation. We briefly explain the symplectic geometry underlying the extremals for sup-norms in both the real and complex domain.

In the real domain, the extremal eigenfunctions for sup-norms are zonal spherical harmonics of each degree \(N\) (i.e. eigenfunctions invariant under rotations around the third axis). The proof is that all other eigenfunctions vanish at the fixed points (poles) of these rotations, hence the universal pointwise Weyl asymptotics (1) can only hold at a pole \(x\) if the zonal harmonics attain the maximal sup norm bound at \(x\). The symplectic geometry underlying this sup norm behavior is that zonal spherics harmonics \(Y^N_0\) (indeed, the entire basis of joint eigenfunctions \(Y^N_m\) of \(\Delta\) and of rotation around the third axis) are semi-classical Lagrangian distributions associated to the meridian Lagrangian \(\Lambda_0 \subset S^*S^2\) of unit co-vectors tangent to the family of meridian geodesics between the poles. The extremal sup norm is attained at the poles and reflects the ‘blow-down’ singularity of the Lagrange projection \(\pi : \Lambda_0 \to S^2\) over the poles.

The vanishing of modes \(Y^N_m\) with \(m \neq 0\) at the poles has no analogue for Husimi distributions in the Grauert tube (i.e. phase space) setting, because there are no fixed points in \(\partial M_\tau\) for the lift of the rotation group. Indeed, analytic continuations of zonal spherical harmonics do not attain maximal sup-norm growth in the complex domain. Rather, the extremals are Gaussian beams (highest weight spherical harmonics), which are extremals for low \(L^p\) norms, but not high \(L^p\) norms, in the real domain. As mentioned above, coherent states (Definition 4) also attain the maximum, but are not Husimi distributions of eigenfunctions.

From the symplectic geometric viewpoint, the explanation requires background in theory of Toeplitz operators and their associated symplectic cones in [BoGu]. Briefly, the Hardy space \(H^2(\partial M_\tau)\) of boundary values of holomorphic functions in \(M_\tau\) is a Hilbert space associated to the symplectic cone \(\Sigma_\tau \subset T^*\partial M_\tau\) spanned by the action form \(\alpha_\tau\). That is, \(\pi : \Sigma_\tau \to \partial M_\tau\) is an \(\mathbb{R}_+\) bundle whose fiber over \(\zeta\) consists of \(\mathbb{R}_+ \alpha_\zeta\). As reviewed in Section 2, the metric \(g\) induces an identification of \(\Sigma_\tau \simeq S^*_\tau M\) (covaectors of length \(\tau\)). Hence, the Lagrangian submanifold \(\Lambda_0 \subset S^*_\tau M\) may be identified with a Lagrangian submanifold of \(\partial M_\tau\) and of \(\Sigma_\tau\). Obviously, the natural projection \(\pi : \Lambda_0 \to \partial M_\tau\) is an embedding rather than a Lagrangian projection. Consequently, there is no ‘singularity’ to cause sup norm blowup. On the other hand, the symplectic geometry associated to the highest weight spherical harmonics \(Y^N_N\) (or
any Gaussian beam) is the closed geodesic along which it concentrates, lifted by its unit
tangent vectors to $\partial M$. This geodesic is a singular leaf of the foliation of $\partial M$ by orbits of
the Hamiltonian torus $\mathbb{R}^2/\mathbb{Z}^2$ action generated by the geodesic flow together with rotations.
This singularity does cause extremal behavior in the associated modes.

13. Appendix

13.1. Integrated Weyl laws in the real domain. The geodesic flow $G^t$ of $(M,g)$ of a
real analytic Riemannian manifold is of one of the following two types:

(1) aperiodic: The Liouville measure of the closed orbits of $G^t$, i.e. the set of vectors
lying on closed geodesics, is zero; or
(2) periodic = Zoll: $G^T = \text{id}$ for some $T > 0$; henceforth $T$ denotes the minimal period.
The common Morse index of the $T$-periodic geodesics will be denoted by $\beta$.

In the real domain, the two-term Weyl laws counting eigenvalues of $\sqrt{\Delta}$ are very different
in these two cases.

(1) Let $I_\lambda = [0, \lambda]$ and let $N(\lambda) = \int_M \Pi_\lambda(x,x)dV(x)$. In the aperiodic case, the
Duistermaat-Guillemin-Ivrii two term Weyl law states

$$N(\lambda) = \# \{ j : \lambda_j \leq \lambda \} = c_m \text{Vol}(M,g) \lambda^m + o(\lambda^{m-1})$$

where $m = \text{dim } M$ and where $c_m$ is a universal constant.

(2) In the periodic case, the spectrum of $\sqrt{\Delta}$ is a union of eigenvalue clusters $C_N$ of the
form

$$C_N = \{ (\frac{2\pi}{T})(N + \frac{\beta}{4}) + \mu_{Ni}, i = 1 \ldots d_N \}$$

with $\mu_{Ni} = 0(N^{-1})$. The number $d_N$ of eigenvalues in $C_N$ is a polynomial of degree
$m - 1$.

In the aperiodic case, we can choose the center of the spectral interval $I_\lambda$ arbitrarily. In
the Zoll case we center it along the arithmetic progression $\{(\frac{2\pi}{T})(N + \frac{\beta}{4})\}$. We refer to
\[Ho, SV, ZZoll\] for background and further discussion.

13.2. Tauberian Theorems. We record here the statements of the Tauberian theorems
that we use in the article. Our main reference is [SV], Appendix B and we follow their
notation.

We denote by $F_+$ the class of real-valued, monotone nondecreasing functions $N(\lambda)$ of
polynomial growth supported on $\mathbb{R}_+$. The following Tauberian theorem uses only the
singularity at $t = 0$ of $\hat{dN}$ to obtain a one term asymptotic of $N(\lambda)$ as $\lambda \rightarrow \infty$:

**Theorem 13.1.** Let $N \in F_+$ and let $\psi \in \mathcal{S}(\mathbb{R})$ satisfy the conditions: $\psi$ is even, $\psi(\lambda) > 0$
for all $\lambda \in \mathbb{R}$, $\hat{\psi} \in C_0^\infty$, and $\hat{\psi}(0) = 1$. Then,

$$\psi \ast dN(\lambda) \leq A\lambda^\nu \implies |N(\lambda) - N \ast \psi(\lambda)| \leq C A\lambda^\nu,$$

where $C$ is independent of $A, \lambda$.

To obtain a two-term asymptotic formula, one needs to take into account the other
singularities of $d\hat{N}$. We let $\psi$ be as above, and also introduce a second test function $\gamma \in \mathcal{S}$
with $\hat{\gamma} \in C_0^\infty$ and with the supp $\hat{\gamma} \subset (0, \infty)$.

**Theorem 13.2.** Let $N_1, N_2 \in F_+$ and assume:
Then,

\[ N_1(\lambda - o(1)) - o(\lambda^\nu) \leq N_2(\lambda) \leq N_1(\lambda + o(1)) + o(\lambda^\nu). \]

This Tauberian theorem is useful when the non-zero singularities of \( \hat{d}N_2 \) are as strong as the singularity at \( t = 0 \) and \( N_2 \) does not have two term polynomial asymptotics.

13.3. Notational Appendix. In this section, we list the main notations.

13.3.1. Notation for Husimi distributions and Weyl sums.

(1) Husimi distributions: (3)

\[ \frac{|\varphi_{C_j}^\zeta(\zeta)|^2}{||\varphi_\lambda||_{L^2(\partial M_r)}^2}. \]

(2) (7) Analytic continuations of spectral projections with eigenvalues in the interval \( I_\lambda \):

\[ \Pi_{I_\lambda}^\zeta(\zeta, \bar{\zeta}) := \sum_{j: \lambda_j \in I_\lambda} |\varphi_{C_j}^\zeta(\zeta)|^2. \]

\( I_\lambda \) could be a short interval \([\lambda, \lambda + 1]\) of frequencies or a long window \([0, \lambda]\).

(3) (8) ‘Tempered’ spectral projections

\[ P_{I_\lambda}^\tau(\zeta, \bar{\zeta}) = \sum_{j: \lambda_j \in I_\lambda} e^{-2\tau \lambda_j} |\varphi_{C_j}^\zeta(\zeta)|^2 \tau. \]

(4) Renormalized spectral projections:

\[ \tilde{P}_{[0,\lambda]}^\tau(\zeta, \bar{\zeta}) = \sum_{j: \lambda_j \leq \lambda} \frac{|\varphi_{C_j}^\zeta(\zeta)|^2}{||\varphi_{C_j}^\zeta||_{L^2(\partial M_r)}^2}, \quad (\sqrt{\rho}(\zeta) = \tau), \]

adapted to the Husimi distributions (3).

(5) Dual Poisson-wave group: (26)

\[ U_C(t + 2i\tau, \zeta, \bar{\zeta}) = \sum_{j} e^{(-2\tau + it)\lambda_j} |\varphi_{C_j}^\zeta(\zeta)|^2. \]

(6) Poisson kernel (25):

\[ P^\tau(\zeta, y) = \sum_{j} e^{-\tau \lambda_j} \varphi_{C_j}^\zeta(\zeta) \varphi_j(y). \]
13.3.2. Notation for Grauert tubes, CR geometry, geodesic flow.

1. $J_\zeta$: complex structure at $T_\zeta M_\tau$.

2. Complexified CR subspace and type decomposition:
   \[ H^{1,0}_\zeta \oplus H^{0,1}_\zeta. \]

3. Geodesic flow in the Grauert tube setting (45):
   \[ g^t = \exp t\Xi_\sqrt{\rho}, \quad G^t = \exp t\Xi_\rho. \]
   Its restriction to $\partial M_\tau$ (46)
   \[ g^t_\tau : \partial M_\tau \rightarrow \partial M_\tau. \]

4. $\mathcal{P}$: periodic orbits of $g^t_\tau$. $T(\zeta)$ (14): period of the periodic point $\zeta \in \mathcal{P}$.

5. Linearization (Poincaré map) of $g^t_\zeta$ at $\zeta \in \mathcal{P}$ (9)
   \[ Dg^{nT(\zeta)}_\zeta : H^{1,0}_\zeta \oplus H^{0,1}_\zeta \rightarrow H^{1,0}_\zeta \oplus H^{0,1}_\zeta. \]

6. Osculating Bargmann-Fock space (Definition 4.1): $\mathcal{H}^2_\zeta$ at $\zeta \in \partial M_\tau$.

7. Vacuum (ground) state in $\mathcal{H}^2_\zeta$: $\Omega_{J_\zeta}$.

8. Metaplectic representation $W_{J_\zeta} (Dg^{nT(\zeta)}_\zeta)$ of $(Dg^{nT(\zeta)}_\zeta)$.

9. (16)
   \[ \mathcal{G}_n(\zeta) := \langle W_{J_\zeta} (Dg^{nT(\zeta)}_\zeta) \Omega_{J_\zeta} \rangle. \]

13.4. Notation for linear symplectic algebra.

1. Projection to $H^{1,0}_J$ (52):
   \[ P_J = \frac{1}{2}(I - iJ) : V \otimes \mathbb{C} \rightarrow H^{1,0}_J, \quad P_J = \frac{1}{2}(I + iJ) : V \otimes \mathbb{C} \rightarrow H^{0,1}_J. \]

2. Holomorphic block of a linear symplectic map (54):
   \[ P_J S P_J = P = \frac{1}{2}(A + D + i(C - B)). \]

3. Ground state of Bargmann-Fock space (55):
   \[ \Omega_J(v) := e^{-\frac{1}{2}\sigma(v,Jv)}. \]

4. The Bargmann-Fock space of a symplectic vector space $(V, \sigma)$ with compatible complex structure $J \in \mathcal{J}$ (Section 3.5):
   \[ \mathcal{H}_J = \{ f e^{-\frac{1}{2}\sigma(v,Jv)} \in L^2(V, dL) \mid f \text{ is entire } J\text{-holomorphic} \}. \]

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