ON THE INFRARED LIMIT OF THE
CHERN-SIMONS-PROCA THEORY

Antti J. Niemi * and V. V. Sreedhar †

Department of Theoretical Physics, Uppsala University
P.O. Box 803, S-75108, Uppsala, Sweden

and

* Research Institute for Theoretical Physics, Helsinki University,
Siltavuorenpenger 20 C, FIN-00170 Helsinki, Finland

We investigate a modification of the 2+1 dimensional abelian Chern-Simons theory, obtained by adding a Proca mass term to the gauge field. We are particularly interested in the infrared limit, which can be described by two a priori different ”topological” quantum mechanical models. We apply methods of equivariant cohomology and the ensuing supersymmetry to analyze the partition functions of these quantum mechanical models. In particular, we find that a previously discussed phase-space reductive limiting procedure which relates these two models can be seen as a direct consequence of our supersymmetry.

E-mail: niemi@rhea.teorfys.uu.se sreedhar@rhea.teorfys.uu.se

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The abelian and nonabelian versions of three-dimensional Chern-Simons theory have found several applications both in Physics and Mathematics [1]. Here we discuss a simple, gauge invariant modification of the abelian Chern-Simons theory, obtained by adding a Proca mass term [2] to the gauge field. This modification introduces a new dimensionful parameter which is a relevant perturbation of the Chern-Simons action in the infrared (or long-distance) limit. Thus it could have even physically relevant consequences, for example in the quantum Hall effect or high-temperature superconductivity where it could e.g. parametrize finite size effects in an experimental set-up.

In the present Letter we shall apply loop space equivariant cohomology and the ensuing supersymmetry [3] to investigate the infrared limit of the abelian Chern-Simons-Proca theory. In particular, we find that this limit can be described by two a priori different "topological" quantum mechanical models that have been discussed previously in [4]. Furthermore, we conclude that the phase-space reductive limiting procedure which was used in [4] to relate these two quantum mechanical models to each other, is a direct consequence of a supersymmetry which emerges when the corresponding path integrals are formulated in terms of loop space equivariant cohomology as described in [3].

The abelian Chern-Simons gauge theory is defined by the action

\[ S_{CS} = - \int \frac{B}{2} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda = \int \frac{B}{2} (A_1 \dot{A}_2 - A_2 \dot{A}_1) - B \cdot A_0 (\partial_1 A_2 - \partial_2 A_1) \] (1)

where \( B \) is a parameter. Using a Hamiltonian interpretation, we can view \( A_1 \) and \( A_2 \) as canonically conjugated variables with the Poisson bracket

\[ \{ A_i(x), A_j(y) \} = - \frac{1}{B} \delta_{ij} \delta(x - y) \] (2)

Alternatively, we can use the standard definition

\[ E_i = \frac{\delta L}{\delta \dot{A}_i} = - \frac{B}{2} \epsilon_{ij} A_j \] (3)

of canonically conjugated momenta to conclude that (2) is the Dirac bracket that follows from the second class constraints

\[ E_i + \frac{B}{2} \epsilon_{ij} A_j \approx 0 \] (4)
In both interpretations, the time component $A_0$ is a Lagrange multiplier for the first class constraint
\[ F_{12} = \partial_1 A_2 - \partial_2 A_1 \approx 0 \] (5)
which defines the abelian gauge transformation that leaves (1) invariant. We can fix the gauge e.g. by using the Coulomb gauge condition,
\[ \partial_i A_i = \partial_1 A_1 + \partial_2 A_2 \approx 0 \] (6)

With the bracket (2) we then have
\[ \{ F_{12}(x), \partial_i A_i(y) \} = \frac{1}{B} \Box_x \delta(x-y) \] (7)
and as explained e.g. in [5] we may interpret the Coulomb-gauge Chern-Simons theory as a second class constrained system, with constraints
\[ \phi_1 = F_{12} \approx 0 \] (8)
\[ \phi_2 = \partial_i A_i \approx 0 \] (9)
and
\[ \{ \phi_i(x), \phi_j(y) \} = \frac{1}{B} \epsilon_{ij} \Box_x \delta(x-y) \] (10)
as the second-class constraint algebra. The determinant of (10) is nonsingular when evaluated over the non-zeromodes of the two dimensional Laplacian, and the path integral
\[ Z = \int [dA_1][dA_2] \delta(\phi_i) \text{det}'[\{ \phi_i(x), \phi_j(y) \}] \exp\left\{ i \int \frac{B}{2} (A_1 \dot{A}_2 - A_2 \dot{A}_1) \right\} \] (11)
describes the quantum Chern-Simons theory in the Coulomb gauge. We recall [5] that a change in the gauge condition (6) corresponds to a change of variables in this path integral.

Notice that in the action in (11) we do not have any Hamiltonian, only a kinetic term appears. Consistent with general properties of constrained systems, we can consider modifications where we add a Hamiltonian $H$ which is consistent with the symmetries of the theory, i.e. the brackets $\{ \phi_i, H \}$ vanish on the constraint surface (8)-(9)
\[ \{ \phi_i, H \} \approx 0 \] (12)
An example of such a Hamiltonian is the Coulomb gauge version of the Proca mass term \([2]\)
\[
H_P = \int \frac{m}{2} A_i^2 = \int \frac{m}{2} (A_1^2 + A_2^2)
\]  
(13)
This gives rise to the following modification of the Chern-Simons action in (11),
\[
S_{CS} \rightarrow S_P = \int \frac{B}{2} (A_1 \dot{A}_2 - A_2 \dot{A}_1) - \frac{m}{2} (A_1^2 + A_2^2)
\]  
(14)
Indeed, since
\[
\{ \phi_1, H_P \} = \{ F_{12}, H_P \} = -\frac{m}{B} \partial_i A_i = -\frac{m}{B} \phi_2 \approx 0
\]  
(15)
and
\[
\{ \phi_2, H_P \} = \{ \partial_i A_i, H_P \} = \frac{m}{B} F_{12} = \frac{m}{B} \phi_1 \approx 0
\]  
(16)
this action is gauge invariant in the Coulomb gauge. This is fully consistent with the standard result \([2]\) that the Proca mass term is a gauge invariant perturbation of the Maxwell action.

Instead of (14) we could also consider the following modification of the Chern-Simons action,
\[
S_{CS} \rightarrow -\int \frac{B}{2} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + \frac{m}{2} A_\mu^2
\]  
(17)
which is manifestly Lorentz-invariant, and gauge invariant in the Lorentz gauge
\[
\partial_\mu A_\mu = 0
\]  
(18)
Here we have the standard Proca mass term, and we refer e.g. to \([2]\) for a discussion of its properties in the context of standard Maxwell theory. Here we shall restrict ourselves to the Proca mass term in the Coulomb gauge Chern-Simons theory, since in condensed matter applications this is quite sufficient.

The gauge field \(A_\mu\) has dimensions \([A] \sim \frac{1}{L}\) in terms of a length scale \(L\). As a consequence the parameter \(m\) in (14) has the proper dimensions of a mass. If we set \(m = \frac{c}{L}\) where \(c\) is a dimensionless constant we conclude that in the double scaling limit \(c \to \infty\), \(L \to \infty\) with \(m\) constant, the Proca mass term in (14) grows linearly with the length scale \(L\). Consequently it is a relevant perturbation of the Chern-Simons action in the long distance, infrared limit. In particular, we can
expect that the Proca mass term could have physically relevant consequences e.g. in condensed matter applications of the Chern-Simons theory such as quantum Hall effect and high temperature superconductivity.

Notice in particular, that the infrared behavior of the Proca mass term is in contrast with the infrared behavior of the conventional Maxwell term $\frac{1}{\alpha} F_{ij} F_{ij}$ which is an irrelevant perturbation of the Chern-Simons action because in three dimensions it goes as $\frac{1}{\ell}$ in the long distance limit.

In order to investigate the infrared properties of the Coulomb gauge Chern-Simons-Proca theory (14), (8)-(9) we expand the gauge field $A_i$ in a Fourier series and separate out the part which plays an important role at long distances. This is the term which is constant in space, and is denoted in the following by $q_i$,

$$A_i(x) = q_i(t) + \tilde{A}_i(x)$$

In the infrared limit the Chern-Simons-Proca action then reduces to

$$S_P \rightarrow \int \left( \frac{B}{2} \epsilon_{ij} q_i \dot{q}_j - \frac{m}{2} q_i q_i \right)$$

We observe that this is precisely the topological Chern-Simons quantum mechanical model considered in [4].

From (20) we identify the Hamiltonian

$$H = \frac{m}{2} q_i q_i$$

while the first term in (20) yields the bracket

$$\{q_i, q_j\} = -\frac{1}{B} \epsilon_{ij}$$

which we can again view either as a Poisson bracket in the phase space with coordinates $q_i$ or alternatively as a Dirac bracket in a larger phase space with canonical momenta

$$p_i \equiv \frac{\partial L_P}{\partial \dot{q}_i} = -\frac{B}{2} \epsilon_{ij} q_j$$

and second class constraints

$$p_i + \frac{B}{2} \epsilon_{ij} q_j \approx 0$$
In particular, if we use (24) to eliminate one of the coordinates in (20), say \( q_2 \), we conclude that (24) describes a harmonic oscillator with frequency \( \omega = \frac{2mB}{\hbar} \), so that the corresponding quantum mechanical partition function is

\[
Z = \exp\{-\beta H_P\} = \frac{1}{2} \cdot \frac{1}{\sinh \beta \frac{2mB}{\hbar}} \tag{25}
\]

As explained in [4], the spectrum of (20) can also be described by a topologically massive quantum mechanical model, using a phase-space reductive limiting procedure. In the remaining of this Letter we shall show how this result also follows directly from the loop space equivariant cohomology and the ensuing supersymmetry developed in [3]: We first construct a supersymmetric version of (20) that contains the topologically massive quantum mechanical model of [4] in its bosonic sector. The relation in [4] between the spectra of the two theories is then a simple consequence of this supersymmetry.

In order to derive a supersymmetric version of (20), we first consider a generic coordinate system \( z^a \) on a \( 2n \) dimensional phase space \( \Gamma \). In these coordinates the Poisson (Dirac) brackets are given by

\[
\{z^a, z^b\} = \theta^{ab}(z) \tag{26}
\]

which determines the components of the symplectic two form \( \theta \) on \( \Gamma \),

\[
\theta = \frac{1}{2} \theta_{ab} \, dz^a \wedge dz^b \tag{27}
\]

In the present case we may for example identify \( q^i \) with \( z^a \) which implies that

\[
\theta_{ab} = B \epsilon_{ab} \tag{28}
\]

In these variables the canonical phase space path integral for the Chern-Simons-Proca quantum mechanics (20) is

\[
Z = \int [dz^a] \sqrt{\det |B \epsilon_{ab}|} \exp\{i \int dt \left( \frac{B}{2} \epsilon_{ab} z^a \dot{z}^b - \frac{m}{2} z^a z^a \right)\} \tag{29}
\]

\[\text{Notice that if we substitute (24) in (20), we need to re-scale both } p \text{ and } q \text{ by a factor of } \sqrt{2} \text{ in order to properly normalize their commutator. This explains the discrepancy of a factor 2 between our result and that in [4].}\]
and if we introduce anticommuting variables $c^a$ we can rewrite this as

$$Z = \int [dz^a][dc^a] \exp i(S_B + S_F)$$

where

$$S_B = \int dt \left( \frac{B}{2} \epsilon_{ab} z^a z^b - \frac{m}{2} z^a z^a \right)$$

and

$$S_F = \int dt \frac{B}{2} c^a \epsilon_{abc} c^b$$

Following [3] we interpret (30) as a loop space integral in a loop space $L\Gamma$, parametrized by the time evolution $z^a \to z^a(t)$ with periodic boundary conditions $z^a(t_i) = z^a(t_f)$. In this loop space we define exterior derivative by lifting the exterior derivative on the phase space $\Gamma$,

$$d = \int dt \frac{\delta}{\delta z^a(t)} d(z^a(t)) = dz^a \frac{\delta}{\delta z^a}$$

The $dz^a(t)$ obtained by lifting the basis of one forms $dz^a$ on $\Gamma$ then constitutes a basis of one forms on the loop space $L\Gamma$, and we can identify $dz^a$ with the anticommuting variables $c^a(t)$.

We define a loop space Hamiltonian vector field $\mathcal{X}^a_P$, determined by the bosonic part (31) of the action through the equation

$$\frac{\delta S_B}{\delta z^a(t)} = \Theta_{ab}(z; t, t') \mathcal{X}^b_P(t')$$

where $\Theta_{ab}$ are the components of a loop space symplectic two form $\Theta$, obtained by lifting the components of the symplectic two form $\theta$ to $L\Gamma$ through the relation

$$\Theta_{ab}(z; t, t') = \theta_{ab}(z) \delta(t - t')$$

In the present case this implies

$$\Theta_{ab}(t, t') = B \epsilon_{ab} \delta(t - t')$$

so that the Hamiltonian vector field is

$$\mathcal{X}^a_P = \frac{\dot{z}^a}{2} + \frac{m}{B} \epsilon_{ab} z^b$$
Notice in particular, that the zeroes of (37) yield the classical solutions of the Chern-Simons-Proca theory.

Let \( i_P \) denote loop space contraction along the Hamiltonian vector field \( \mathcal{X}_P \),

\[
i_P = \mathcal{X}_P^a i_a
\]

where the \( i_a(t) \) form a basis of loop space contractions which is dual to \( c^a(t) \),

\[
i_a(t) c^b(t') = \delta^b_a(t - t')
\]

We define a loop space equivariant exterior derivative by

\[
d_P = d + i_P
\]

It gives the following loop space supersymmetry transformation on the variables \( z^a \) and \( c^a \),

\[
d_P z^a = c^a
\]

\[
d_P c^a = \mathcal{X}_P^a
\]

More precisely, if \( L\Lambda \) is the DeRham complex on the loop space \( L\Gamma \), \( d_P \) maps
the subspace of even forms in \( L\Lambda \) onto the subspace of odd forms and vice versa. Furthermore,

\[
d_P^2 = di_P + i_Pd = \mathcal{L}_P
\]

is the loop space Lie derivative along the Hamiltonian vector field \( \mathcal{X}_P \), and if we
consider the invariant subspace

\[
L\Lambda_{inv} = \{ \xi \in L\Lambda \mid \mathcal{L}_P \xi = 0 \}
\]

in this subspace \( d_P \) is nilpotent and acts like an exterior derivative. In particular,
we find that the action \( S_B + S_F \) is invariant under the supersymmetry transformation (41)-(42)

\[
d_P(S_B + S_F) = 0
\]

and consequently determines an element in the invariant subspace (44).

If \( g_{ab} \) is a metric on \( L\Gamma \) which is Lie derived by \( \mathcal{X}_P \), the function

\[
\psi = g_{ab} z^a c^b
\]
is also Lie derived by $X_P$

$$\mathcal{L}_P g = 0 \implies \mathcal{L}_P \psi = 0 \quad (47)$$

In the present case, we can simply choose the flat Euclidean metric $g_{ab} = \delta_{ab}$. As explained in [3], if we now add to the action $S_B + S_F$ a $d_P$-exact piece of the form

$$\lambda \cdot d_P \psi = \lambda \delta_{ab} [\dot{c}^a \dot{c}^b + \frac{1}{2} \dot{z}^a \dot{z}^b + \frac{m}{B} \epsilon_{bc} \dot{z}^a z^c] \quad (48)$$

where the parameter $\lambda$ has dimensions $[\lambda] \sim L$, the corresponding path integral is independent of $\lambda$ and coincides with the original path integral (29).

Explicitly, this gives for our infrared limit of the Chern-Simons-Proca action (we now specialize to $z^a \rightarrow q^i$ and $c^a \rightarrow c^i$)

$$S_P \rightarrow \int \lambda \delta_{ab} \dot{q}^i \dot{q}^j + \frac{B'}{2} \epsilon_{ij} \dot{q}^i \dot{q}^j - \frac{m}{2} \delta_{ij} \dot{q}^i \dot{q}^j + \lambda \delta_{ij} \dot{c}^i \dot{c}^j + \frac{B'}{2} \epsilon_{ij} \dot{c}^i \dot{c}^j \quad (49)$$

where

$$\frac{B'}{2} = \left( \frac{B}{2} - \frac{\lambda m}{B} \right) \quad (50)$$

From the functional form of (49) we conclude in the usual manner that we can introduce momenta $p_i, \pi_i$ which are canonically conjugate to $q^i, c^i$,

$$p_i \equiv \frac{\partial L}{\partial \dot{q}^i} = \lambda \dot{q}^i - \frac{B'}{2} \epsilon_{ij} \dot{q}^j \quad (51)$$

$$\pi_i \equiv \frac{\partial L}{\partial \dot{c}^i} = \lambda c^i \quad (52)$$

In this way we conclude that the Hamiltonian of (49) is

$$H = H_B + H_F = \frac{1}{2\lambda} \left[ p_i + \frac{B'}{2} \epsilon_{ij} q^j \right] \left[ p_i + \frac{B'}{2} \epsilon_{ik} q^k \right] + \frac{m}{2} \dot{q}^i \dot{q}^i - \frac{B'}{2} \epsilon_{ij} \dot{c}^i \dot{c}^j \quad (53)$$

which is the desired supersymmetric extension of the Chern-Simons-Proca Hamiltonian (21). In particular, we observe that the bosonic part

$$H_B = \frac{1}{2\lambda} \left[ p_i + \frac{B'}{2} \epsilon_{aij} q^j \right] \left[ p_i + \frac{B'}{2} \epsilon_{ik} q^k \right] + \frac{m}{2} \dot{q}^i \dot{q}^i \quad (54)$$

of (53) coincides with the topologically massive Hamiltonian discussed in [4], except for the shift proportional to $\lambda$ displayed in equation (50). This shift, innocuous as
it is in so far as the qualitative features of the spectrum are concerned, will play an important role when we proceed to reproduce the results in [4]. Indeed, according to general arguments [3] which are based on the supersymmetry (41)-(42), we expect that the partition function of (53) will be independent of λ, and coincides with the partition function (25) of the original Chern-Simons-Proca quantum mechanical model (20). This reproduces the results obtained in [4] using a phase-space reductive limiting procedure.

We shall now proceed to explicitly verify that the partition functions of (20) and (49) indeed coincide, independently of λ. For this we introduce the following variables

\[ p_\pm = \left( \frac{\omega_\pm}{2\lambda \Omega} \right)^{1/2} p_1 \pm \left( \frac{\lambda \Omega \omega_\pm}{2} \right)^{1/2} q^2 \]

\[ q_\pm = \left( \frac{\lambda \Omega}{2 \omega_\pm} \right)^{1/2} q^1 \mp \left( \frac{1}{2 \lambda \Omega \omega_\pm} \right)^{1/2} p_2 \]

with the nonvanishing commutators

\[ [q_+, p_+] = [q_-, p_-] = i \]

where

\[ \Omega = \sqrt{\left( \frac{B}{2\lambda} \right)^4 + \frac{m^2}{B^2}} \]

\[ \omega_\pm = \Omega \pm \frac{B}{2\lambda} \pm \frac{m}{B} \]

In terms of these variables the Hamiltonian \( H_B \) splits into two decoupled one dimensional oscillators with frequencies \( \omega_\pm \). In terms of creation and annihilation operators

\[ a^*_\pm = \frac{1}{2\omega_\pm} (p_\pm + i \omega_\pm q_\pm) \]

\[ a_{\pm} = \frac{1}{2\omega_\pm} (p_\pm - i \omega_\pm q_\pm) \]

such that

\[ [a_+, a^*_+] = 1 \]

\[ [a_-, a^*_+] = 1 \]
with all other commutators being zero, we get for $H_B$

$$H_B = \omega_+ (a^*_+ a_+ + \frac{1}{2}) + \omega_- (a^*_- a_- + \frac{1}{2})$$  \hfill (64)

We now proceed to discuss the fermionic part $H_F$: Recalling the definition of the canonical momentum conjugate to $c^i$ from equation (52) and imposing the anticommutation relations

$$\{ c^i, \pi_j \} = i \delta^i_j$$  \hfill (65)

we get the Dirac brackets

$$\{ c^i, c^j \} = i \frac{\delta^{ij}}{\lambda}$$  \hfill (66)

Consequently we can realize the $c^i$ in terms of Pauli matrices $\sigma^1$ and $\sigma^2$ through the relations

$$c^1 = \sqrt{\frac{i}{2\lambda}} \sigma^1$$  \hfill (67)

$$c^2 = \sqrt{\frac{i}{2\lambda}} \sigma^2$$  \hfill (68)

and we find that we can rewrite $H_F$ as

$$H_F = \frac{B}{2\lambda} \sigma^3$$  \hfill (69)

and combining (64), (69) we get for the Hamiltonian (53),

$$H = \omega_+ (a^*_+ a_+ + \frac{1}{2}) + \omega_- (a^*_- a_- + \frac{1}{2}) + \frac{B}{2\lambda} \sigma^3$$  \hfill (70)

Let

$$| N_+, N_-, N_F \rangle \quad N_+ = 0, 1, 2, \cdots \quad N_F = 0, 1$$  \hfill (71)

denote the simultaneous eigenstates of the number operators for our two bosonic and one fermionic oscillators. The spectrum of the Hamiltonian (70) is then given by

$$E(N_+, N_-, N_F) = \left[ \Omega - (-1)^{N_F} \frac{B}{2\lambda} \right] + \omega_+ N_+ + \omega_- N_-$$  \hfill (72)

and we can directly evaluate the partition function

$$Z \equiv Tr \{ (-1)^{N_F} e^{-\beta H} \} = \sum_{N_+, N_-, N_F} (-1)^{N_F} \exp \{-\beta E(N_+, N_-, N_F)\}$$  \hfill (73)
We find
\[ Z = \exp\{-\beta \left( \frac{B}{\lambda} \right)\} - \exp\{-\beta \left( -\frac{B}{\lambda} \right)\} \cdot \frac{\exp\{-\frac{1}{2}\beta \omega_+\}}{1 - \exp\{-\beta \omega_+\}} \cdot \frac{\exp\{-\frac{1}{2}\beta \omega_-\}}{1 - \exp\{-\beta \omega_-\}} \] (74)

and using (59) we get after a little algebra
\[ Z = \frac{1}{2} \cdot \frac{1}{\sinh \beta \frac{m}{B}} \] (75)

which coincides with (25), as expected. In particular, consistent with our general arguments all \( \lambda \)-dependence has disappeared in (75). As a consequence we have reproduced the results of [4] directly, using supersymmetry which is based on loop space equivariant cohomology. Notice in particular, that the infinite subtraction that is required in [4] for the zero point spectra of the two theories to coincide, is here taken care of by the supersymmetry.

In conclusion, we have investigated a simple gauge invariant modification of the three dimensional abelian Chern-Simons theory, obtained by adding the Proca mass term. This modification of the Chern-Simons theory survives in the long-distance limit, and consequently determines a relevant infrared perturbation of the original theory that could have experimental consequences \textit{e.g.} in quantum Hall effect and high-temperature superconductivity. Furthermore, we have found that in this limit our model reproduces the topological quantum mechanical models investigated in [4]. In particular, we have derived the results in [4] directly, using supersymmetry which is determined by loop space equivariant cohomology. Since this supersymmetry emerges from the symplectic structure of the theory, we expect that similar techniques could yield interesting results also in the three dimensional context of our model.

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