AN EXTENSION OF A SUPERCONGRUENCE OF LONG AND RAMAKRISHNA

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Abstract. We prove two supercongruences for specific truncated hypergeometric series. These include an uniparametric extension of a supercongruence that was recently established by Long and Ramakrishna. Our proofs involve special instances of various hypergeometric identities including Whipple’s transformation and the Karlsson–Minton summation.

1. Introduction

Let \((a)_n = a(a+1)\cdots(a+n-1)\) denote the Pochhammer symbol. For complex numbers \(a_0, a_1, \ldots, a_r\) and \(b_1, \ldots, b_r\), the (generalized) hypergeometric series \(\binom{r+1}{F_r} \) is defined as

\[
\binom{a_0, a_1, \ldots, a_r}{b_1, \ldots, b_r; z} = \sum_{k=0}^{\infty} \frac{(a_0)_k(a_1)_k\cdots(a_r)_k z^k}{k!(b_1)_k\cdots(b_r)_k}.
\]

Summation and transformation formulas for generalized hypergeometric series play an important part in the investigation of supercongruences. See, for instance, [7, 11, 12, 15–17, 19]. In particular, Long and Ramakrishna [15, Theorems 3 and 2] proved the following two supercongruences:

\[
\sum_{k=0}^{p-1} \frac{(1/7)^3}{k!^3} \equiv \begin{cases} 
-\Gamma_p \left( \frac{1}{4} \right)^4 \pmod{p^3}, & \text{if } p \equiv 1 \pmod{4}, \\
-\frac{p^2}{16} \Gamma_p \left( \frac{1}{4} \right)^4 \pmod{p^3}, & \text{if } p \equiv 3 \pmod{4}, 
\end{cases}
\] (1.1)

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and
\[ \sum_{k=0}^{p-1} (6k + 1) \frac{(1/3)_k^6}{k!^6} \equiv \begin{cases} 
- p \Gamma_p \left( \frac{1}{3} \right)^9 \pmod{p^6}, & \text{if } p \equiv 1 \pmod{6}, \\
- \frac{10}{27} p^4 \Gamma_p \left( \frac{1}{3} \right)^9 \pmod{p^6}, & \text{if } p \equiv 5 \pmod{6},
\end{cases} \] (1.2)

where \( \Gamma_p(x) \) is the \( p \)-adic Gamma function. The restriction of the supercongruence (1.1) modulo \( p^2 \) was earlier established by Van Hamme [18, Equation (H.2)]. The supercongruence (1.2) is even stronger than a conjecture made by Van Hamme [18, Equation (D.2)] who asserted the corresponding supercongruence modulo \( p^4 \) for \( p \equiv 1 \pmod{6} \). Long and Ramakrishna also mentioned that (1.2) does not hold modulo \( p^7 \) in general.

The first purpose of this paper is to prove the following supercongruence. Note that the \( r = \pm 1 \) cases partially confirm the \( d = 5 \) and \( q \to 1 \) case of [8, Conjectures 1 and 2].

**Theorem 1.** Let \( r \leq 1 \) be an odd integer coprime with 5. Let \( p \) be a prime such that \( p \equiv -\frac{r}{2} \pmod{5} \) and \( p \geq \frac{5 - r}{2} \). Then
\[ \sum_{k=0}^{p-1} (10k + r) \frac{(\frac{r}{3})^5_k}{k!^5} \equiv 0 \pmod{p^4}, \] (1.3)

Recently, the second author [13] established the following supercongruence related to (1.2):

\[ \sum_{k=0}^{p-1} (6k - 1) \frac{(-1/3)_k^6}{k!^6} \equiv \begin{cases} 
140p^4 \Gamma_p \left( \frac{2}{3} \right)^9 \pmod{p^5}, & \text{if } p \equiv 1 \pmod{6}, \\
378p \Gamma_p \left( \frac{2}{3} \right)^9 \pmod{p^5}, & \text{if } p \equiv 5 \pmod{6},
\end{cases} \] (1.4)

where \( p \) is a prime.

The second purpose of this paper is to give the following common generalization of the second supercongruence in (1.2), restricted to modulo \( p^5 \), and the first supercongruence in (1.4).

**Theorem 2.** Let \( r \leq 1 \) be an integer coprime with 3. Let \( p \) be a prime such that \( p \equiv -r \pmod{3} \) and \( p \geq 3 - r \). Then
\[ \sum_{k=0}^{p-1} (6k + r) \frac{(\frac{r}{3})^6_k}{k!^6} \equiv (-1)^{r+1} \frac{80rp^4}{81} \cdot \frac{\Gamma_p(1 + \frac{r}{3})^2}{\Gamma_p(1 + \frac{2r}{3})^3 \Gamma_p(1 - \frac{r}{3})^4} \times \sum_{k=0}^{1-r} (r - 1)_k \left( \frac{r}{3} \right)_k^3 \pmod{p^5}. \] (1.5)

Letting \( r = 1 \) and \( r = -1 \) in (1.5) and using (1.9) and (1.11), we arrive at the \( p \equiv 5 \pmod{6} \) case of (1.2) modulo \( p^5 \) and the \( p \equiv 1 \pmod{6} \) case of (1.4), respectively.
Our proof of Theorem 1 will require Whipple’s well-poised \( \genfrac{[}{]}{0pt}{}{7}{6} \) transformation formula (see \[2, p. 28\]):

\[
\begin{align*}
\genfrac{[}{]}{0pt}{}{7}{6}
\begin{bmatrix}
  a, & 1 + \frac{1}{2} a, & b, & c, & d, & e, & -n \\
  \frac{1}{2} a, & 1 + a - b, & 1 + a - c, & 1 + a - d, & 1 + a - e, & 1 + a + n
\end{bmatrix}
\end{align*}
\]

\[
= \frac{(a + 1)n(a - d - e + 1)n}{(a - d + 1)n(a - e + 1)n} \genfrac{[}{]}{0pt}{}{4}{3}
\begin{bmatrix}
  1 + a - b - c, & d, & e, & -n \\
  d + e - a - n, & 1 + a - b, & 1 + a - c
\end{bmatrix}
\quad (1.6)
\]

where \( n \) is a non-negative integer, and Karlsson–Minton’s summation formula (see, for example, \[4, Equation (1.9.2)\]):

\[
\begin{align*}
\genfrac{[}{]}{0pt}{}{r + 1}{r}
\begin{bmatrix}
  -n, & b_1 + m_1, & \ldots, & b_r + m_r \\
  b_1, & \ldots, & b_r
\end{bmatrix}
\end{align*}
= 0,
\quad (1.7)
\]

where \( n, m_1, \ldots, m_r \) are non-negative integers and \( n > m_1 + \cdots + m_r \). Our proof of Theorem 2 relies on a \( \genfrac{[}{]}{0pt}{}{7}{6} \) transformation formula slightly different from Whipple’s \( \genfrac{[}{]}{0pt}{}{7}{6} \) transformation formula (1.6), obtained as a result from combining (1.6) with a \( \genfrac{[}{]}{0pt}{}{4}{3} \) transformation formula. The transformation was already utilized by the second author to prove \( (1.4) \).

Furthermore, in order to prove Theorem 2 we require some properties of the \( p \)-adic Gamma function, collected in the following two lemmas.

**Lemma 1.** \[3, Section 11.6\] Suppose \( p \) is an odd prime and \( x \in \mathbb{Z}_p \). Then

\[
\begin{align*}
\Gamma_p(0) &= 1, \quad \Gamma_p(1) = -1, \\
\Gamma_p(x)\Gamma_p(1 - x) &= (-1)^{v_p(x)}, \\
\Gamma_p(x) &\equiv \Gamma_p(y) \pmod{p} \text{ for } x \equiv y \pmod{p},
\end{align*}
\]

\[
\Gamma_p(x + 1) = \begin{cases} 
-1 & \text{if } v_p(x) = 0, \\
-1 & \text{if } v_p(x) > 0,
\end{cases}
\quad (1.10)
\]

where \( a_p(x) \in \{1, 2, \ldots, p\} \) with \( x \equiv a_p(x) \pmod{p} \) and \( v_p(\cdot) \) denotes the \( p \)-order.

**Lemma 2.** \[15, Lemma 17, (4)\] Let \( p \) be an odd prime. If \( a \in \mathbb{Z}_p, n \in \mathbb{N} \) such that none of \( a, a + 1, \ldots, a + n - 1 \) are in \( p\mathbb{Z}_p \), then

\[
(a)_n = (-1)^n \frac{\Gamma_p(a + n)}{\Gamma_p(a)}.
\]

**Lemma 2.** \[15, Lemma 17, (4)\] Let \( p \) be an odd prime. If \( a \in \mathbb{Z}_p, n \in \mathbb{N} \) such that none of \( a, a + 1, \ldots, a + n - 1 \) are in \( p\mathbb{Z}_p \), then

\[
(a)_n = (-1)^n \frac{\Gamma_p(a + n)}{\Gamma_p(a)}.
\]

In the following Sections 2 and 3, we give proofs of Theorems 1 and 2 respectively. The final Section 4 is devoted to a discussion and includes two conjectures.
2. Proof of Theorem

Motivated by the work of McCarthy and Osburn [10] and Mortenson [17], we take the following choice of parameters in (1.6). Let \( a = \frac{5}{3}, b = \frac{r+5}{10}, c = \frac{r+3p}{5}, d = \frac{r+3p}{5}, e = \frac{r-3p}{5}, \)
and \( n = \frac{3p-r}{5}, \) where \( i^2 = -1. \) Then we conclude that

\[
\begin{align*}
\sum_{k=0}^{3p-r} \frac{(1 + \frac{r}{10})_k \left(\frac{r}{5}\right)_k^5}{(\frac{r}{10})_k (1)_k^5} &= \frac{1}{r} \sum_{k=0}^{3p-r} (10k + r) \left(\frac{r}{5}\right)_k^5 k!^5 \\
&\equiv \frac{1}{r} \sum_{k=0}^{p-1} (10k + r) \left(\frac{r}{5}\right)_k^5 (\mod p^4),
\end{align*}
\]

where we have used the fact that \( \left(\frac{r}{5}\right)_k \equiv 0 \pmod{p} \) for \( 3p-r \leq k \leq p-1 \) (the condition \( p \geq \frac{5-r}{2} \) in the theorem is to guarantee \( 3p-r \leq p-1 \)). Since \( 3p-r \geq \frac{2p+r}{5} \), we have

\[
\frac{(1 + \frac{r}{5})_{3p-r} \left(1 - \frac{r}{5}\right)_{3p-r}}{(1 - \frac{3p}{5})_{3p-r} \left(1 + \frac{3p}{5}\right)_{3p-r}} = \frac{(1 + \frac{r}{5})_{3p-r} \left(1 - \frac{r}{5}\right)_{3p-r}}{(1 + \frac{9p^2}{25})_{3p-r}} \equiv 0 \pmod{p^2}.
\]

Finally, by the congruences

\[
(a + bp)_k (a - bp)_k (a + bip)_k (a - bip)_k \equiv (a)_k^4 \pmod{p^4}.
\]
for any $p$-adic integer $b$, we obtain

$$\begin{align*}
4F3 \left[ \frac{5-r-6p}{10}, \frac{r+3p}{5}, \frac{r-3p}{5}, \frac{r-3p}{5}; \frac{1}{10} \right] & \equiv 4F3 \left[ \frac{5-r-6p}{10}, \frac{r}{5}, \frac{r}{5}, \frac{r-3p}{5}; \frac{1}{10} \right] \\
& \equiv 4F3 \left[ \frac{5-r-6p}{10}, \frac{r+p}{5}, \frac{r-p}{5}, \frac{r-3p}{5}; \frac{1}{10} \right] \\
& \equiv 4F3 \left[ \frac{5-r-6p}{10}, \frac{r+p}{5}, \frac{r-p}{5}, \frac{r-3p}{5}; \frac{1}{10} \right] \\
& = 0 \pmod{p^2},
\end{align*}$$

where we have utilized Karlsson–Minton’s summation (1.7) with $n = \frac{3p-r}{b}, b_1 = \frac{2r-3p}{5}, b_2 = \frac{r+5}{10}, b_3 = \frac{5-3p}{5}, m_1 = \frac{1-r}{2}, m_2 = \frac{2p+r-5}{10}$, and $m_3 = \frac{2p+r-5}{5}$ in the last step.

### 3. Proof of Theorem 2

We can verify (1.5) for $r = 1$ and $p = 2$ by hand. In what follows, we assume that $p$ is an odd prime. Recall the following transformation formula [13, Equation (4.2)]:

$$\begin{align*}
7F6 \left[ t, 1 + \frac{t}{3}, \frac{-n}{3}, \frac{-n}{3}, t-a, t-b, t-c, 1-t-m+n+a+b+c; 1 \right] \\
& = \frac{(1+t)n(a+b+2-m-t)n(a+c+2-m-t)n(b+c+2-m-t)n}{(1+a)n(1+b)n(1+c)n(a+b+c+1-m-2t)n} \\
& \times \frac{(a+b+1-m-t)(a+c+1-m-t)(b+c+1-m-t)}{(a+b+n+1-m-t)(a+c+n+1-m-t)(b+c+n+1-m-t)} \\
& \times 4F3 \left[ \frac{-m}{3}, \frac{-n}{3}, a+b+c+1-m-2t, a+b+c+1+n-m-t; \frac{1}{3} \right]. \quad (3.1)
\end{align*}$$

Let $\zeta$ be a fifth primitive root of unity. Setting $m = 1-r$, $t = \frac{r}{3}, n = \frac{2p-r}{3}, a = \frac{2p\zeta}{3}, b = \frac{2p\zeta^2}{3}$ and $c = \frac{2p\zeta^3}{3}$ in (3.1) and using $1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 = 0$, the left-hand side of (3.1) becomes

$$\begin{align*}
7F6 \left[ \frac{1+\frac{r}{6}}{3}, \frac{r}{3}, \frac{r-2p}{3}, \frac{r-2p\zeta}{3}, \frac{r-2p\zeta^2}{3}, \frac{r-2p\zeta^3}{3}, \frac{r-2p\zeta^4}{3}; 1 \right] \\
& \equiv \frac{1}{r} \sum_{k=0}^{2r-1} (6k + r) \left( \frac{r}{3} \right)_k^{16} \pmod{p^5},
\end{align*}$$
where we have used the facts that none of the denominators in \( \tau F_6 \) contain a multiple of \( p \) (the condition \( p \geq 3 - r \) in the theorem is to guarantee \( \frac{2p-r}{3} \leq p-1 \)) and

\[
(u + vp)_k (u + vp\zeta)_k (u + vp\zeta^2)_k (u + vp\zeta^3)_k (u + vp\zeta^4)_k \equiv (u)_k^5 \pmod{p^5}.
\]

Furthermore, for \( \frac{2p-r}{3} < k \leq p-1 \) we have \( \left( \frac{r}{3} \right)_k \equiv 0 \pmod{p} \). Thus,

\[
\tau F_6 \left[ 1 + \frac{r}{3}, \frac{r}{3}, \frac{r-2p}{3}, \frac{r-2p\zeta}{3}, \frac{r-2p\zeta^2}{3}, \frac{r-2p\zeta^3}{3}, \frac{r-2p\zeta^4}{3}; 1 \right] \\
\equiv \frac{1}{r} \sum_{k=0}^{p-1} (6k+r) \left( \frac{r}{3} \right)_k^6 \pmod{p^5}.
\] (3.2)

On the other hand, we determine the terminating hypergeometric series on the right-hand side of (3.1) modulo \( p \):

\[
\frac{(a+b+1-m-t)(a+c+1-m-t)(b+c+1-m-t)}{(a+b+n+1-m-t)(a+c+n+1-m-t)(b+c+n+1-m-t)} \\
\times \, _4F_3 \left[ -m, -n, a+b+c+1-m-2t, a+b+c+1+n-m-t \right] \\
\left. \begin{array}{l} \\
\quad \begin{array}{l}
\quad a+b+1-m-t, \quad a+c+1-m-t, \quad b+c+1-m-t
\end{array} \\
end{array} \right] ; 1
\equiv 8 \sum_{k=0}^{1-r} \frac{(r-1)_k \left( \frac{r}{3} \right)_k^3}{(1)_k \left( \frac{2r}{3} \right)_k^3} \pmod{p}.
\] (3.3)

Moreover,

\[
\frac{(1+t)_n(a+b+2-m-t)_n(a+c+2-m-t)_n(b+c+2-m-t)_n}{(1+a)_n(1+b)_n(1+c)_n(a+b+c+1-m-2t)_n} \\
= \frac{(1+\frac{r}{3})_{\frac{2p-r}{3}} \left( 1 + \frac{2r+2p(\zeta^2+\zeta^3)}{3} \right)_{\frac{2p-r}{3}} \left( 1 + \frac{2r+2p(\zeta^2+\zeta^3)}{3} \right)_{\frac{2p-r}{3}} \left( 1 + \frac{2r+2p(\zeta^2+\zeta^3)}{3} \right)_{\frac{2p-r}{3}}}{(-1)_{\frac{2p-r}{3}} \left( 1 + \frac{2p\zeta^2}{3} \right)_{\frac{2p-r}{3}} \left( 1 + \frac{2p\zeta^2}{3} \right)_{\frac{2p-r}{3}} \left( 1 + \frac{2p\zeta^2}{3} \right)_{\frac{2p-r}{3}} \left( 1 + \frac{2p\zeta^2}{3} \right)_{\frac{2p-r}{3}}}. \] (3.4)

Note that

\[
(1+\frac{r}{3})_{\frac{2p-r}{3}} = \frac{2p}{3} \left( 1 + \frac{r}{3} \right)_{\frac{2p-r-3}{3}},
\] (3.5)
and
\[
\left(1 + \frac{2r + 2p(\zeta + \zeta^2)}{3}\right)^{2p-r} \left(1 + \frac{2r + 2p(\zeta + \zeta^3)}{3}\right)^{2p-r} \left(1 + \frac{2r + 2p(\zeta^2 + \zeta^3)}{3}\right)^{2p-r} = \frac{5p^3}{27} \left(1 + \frac{2r + 2p(\zeta + \zeta^2)}{3}\right)^{\frac{p-2r-3}{3}} \left(1 + \frac{2r + 2p(\zeta + \zeta^3)}{3}\right)^{\frac{p-2r-3}{3}} \times \left(1 + \frac{2r + 2p(\zeta^2 + \zeta^3)}{3}\right)^{\frac{p+r}{3}} \left(3 + p(2\zeta + 2\zeta^2 + 1)\right)^{\frac{p+r}{3}} \times \left(3 + p(2\zeta^2 + 2\zeta^3 + 1)\right)^{\frac{p+r}{3}}.
\]

Combining (3.5) and (3.6), we arrive at
\[
(1 + \frac{r}{3})^{2p-r} \left(1 + \frac{2r}{3}\right)^{\frac{p-r-3}{3}} \left(1 + \frac{2r}{3}\right)^{\frac{p-2r-3}{3}} \left(1 + \frac{2r}{3}\right)^{\frac{p+r}{3}} (1 + \frac{r}{3})^{\frac{p+r}{3}} \equiv \frac{(-1)^{\frac{p-r}{3}} 10p^4}{81} \cdot \frac{1 + \frac{r}{3}}{1^\frac{p-r}{3}} \left(1 + \frac{2r}{3}\right)^{\frac{p-r-3}{3}} \left(1 + \frac{2r}{3}\right)^{\frac{p-2r-3}{3}} \left(1 + \frac{2r}{3}\right)^{\frac{p+r}{3}} (\text{mod } p^5).
\]

It follows from (3.2)–(3.4) and (3.7) that
\[
\sum_{k=0}^{p-1} (6k + r) \left(\frac{r}{3}\right)_k^6 \equiv \frac{(-1)^{\frac{p-r}{3}} 80rp^4}{81} \cdot \frac{1 + \frac{r}{3}}{1^\frac{p-r}{3}} \left(1 + \frac{2r}{3}\right)^{\frac{p-r-3}{3}} \left(1 + \frac{2r}{3}\right)^{\frac{p-2r-3}{3}} \left(1 + \frac{2r}{3}\right)^{\frac{p+r}{3}} \times \sum_{k=0}^{1-r} (r - 1)_k \left(\frac{2r}{3}\right)_k^3 \left(\text{mod } p^5\right).
\]

By Lemmas 11 and 24 we have
\[
\frac{(-1)^{\frac{p-r}{3}} \Gamma_p \left(\frac{2r}{3}\right) \Gamma_p \left(\frac{r}{3}\right) \Gamma_p \left(1 + \frac{r}{3}\right)^3 \Gamma_p \Gamma_p \left(1 + \frac{p+r}{3}\right)^4}{\Gamma_p \left(1 + \frac{r}{3}\right) \Gamma_p \left(1 + \frac{2r}{3}\right)^3 \Gamma_p \left(1 + \frac{2p-r}{3}\right)^4} \equiv \frac{(-1)^{\frac{2p-r+r}{3}} \Gamma_p \left(\frac{2p}{3}\right) \Gamma_p \left(\frac{p}{3}\right) \Gamma_p \left(1 + \frac{p+r}{3}\right)^3 \Gamma_p \Gamma_p \left(1 + \frac{2p-r}{3}\right)^4}{\Gamma_p \left(1 + \frac{r}{3}\right) \Gamma_p \left(1 + \frac{2r}{3}\right)^3 \Gamma_p \left(1 + \frac{2p-r}{3}\right)^4}.
\]
\[
(1.10) \quad \equiv \frac{(-1)^{2p-r} \Gamma_p(0)^4 \Gamma_p(1 + \frac{r}{3})^2 \Gamma_p(1)}{\Gamma_p(1 + \frac{2r}{3})^3 \Gamma_p(1 - \frac{r}{3})^4} \quad (\text{mod } p)
\]

\[
(1.8) \quad \equiv \frac{(-1)^{2p-r} \Gamma_p(1 + \frac{r}{3})^2}{\Gamma_p(1 + \frac{2r}{3})^3 \Gamma_p(1 - \frac{r}{3})^4}.
\]

The proof of (1.5) then follows from (3.8) and (3.9).

4. Discussion

We know that many supercongruences have nice \(q\)-analogues (see [5, 6, 8–10, 14, 20]). For example, we have the following conjectural \(q\)-analogue of (1.3): for the same \(p\) and \(r\) as in Theorem 1,

\[
P - 1 \sum_{k=0}^{p-1} 6k + r \cdot \frac{(\frac{r}{3})^6}{k!^6} \equiv 0 \quad (\text{mod } [p]^4),
\]

(4.1)

where \([n] = 1 + q + \cdots + q^{n-1}\) is the \(q\)-integer and \((a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})\) denotes the \(q\)-shifted factorial.

Although there are \(q\)-analogues of Whipple’s well-poised \(7F_6\) transformation and of Karlsson–Minton’s summation (see [4, Appendix (II.27) and (III.18)]), we are unable to give a proof of (4.1). This is because we only know a \(q\)-analogue of (2.3) (see [9, Lemma 1]) but do not know any \(q\)-analogues of (2.2). Besides, we do not know how to prove (4.1) by using the method of ‘creative microscoping’ devised in [10] either.

While in Theorem 2 we were able to provide a common generalization of the second supercongruence in (1.2) (restricted to modulo \(p^5\)) and the first supercongruence in (1.4), it appears to be rather difficult to extend Theorem 1 to a higher supercongruence involving the \(p\)-adic Gamma function in the spirit of Theorem 2 even in the special cases \(r = 1\) or \(r = -1\).

We end our paper with two further conjectures for future research. Conjecture 1 concerns a stronger version of Theorem 2 and includes the second supercongruence in (1.2) as a special case. Conjecture 2 concerns a common generalization of the first supercongruence in (1.2) and the second supercongruence in (1.4).

Conjecture 1. The supercongruence (1.5) holds modulo \(p^6\) for any prime \(p > 3\).

Conjecture 2. Let \(r \leq 1\) be an integer coprime with 3. Let \(p \geq 7\) be a prime such that \(p \equiv r \pmod{3}\) and \(p \geq 3 - 2r\). Then

\[
P - 1 \sum_{k=0}^{p-1} (6k + r) \cdot \frac{(\frac{r}{3})^6}{k!^6} \equiv \frac{(-1)^r 8rp}{3} \cdot \frac{\Gamma_p(1 + \frac{r}{3})^2}{\Gamma_p(1 + 2r/3)^3 \Gamma_p(1 - r/3)^4} \times \sum_{k=0}^{1-r} \frac{(r-1)_k (\frac{r}{3}_3)_k}{(1)_k (\frac{2r}{3})_k^3} \quad (\text{mod } p^6).
\]

(4.2)
We remark that by using (3.1) and the same method as in the proof of Theorem 2, we can only show that (4.2) holds modulo \( p^2 \). A new technique is needed to prove Conjecture 2.

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