Understanding the Fierz Ambiguity of Partially Bosonized Theories

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A useful tool in non perturbative studies of fermionic theories is partial bosonization. However, partial bosonization is often connected to an ambiguity due to Fierz rearrangement in the original theory. We discuss two different approximations for the calculation of the effective action $\Gamma$ with respect to a spurious dependence on the choice of Fierz transformation: Mean field theory and the truncated flow of an exact renormalization group equation for the effective average action.

1 Introduction

Mean field theory is a widely used method in many body statistical physics and quantum field theory, in particular if the ground state is characterized by condensates and spontaneous symmetry breaking. For example, mean field solutions of the Nambu-Jona-Lasinio (NJL) model or extensions of it are one of the main theoretical tools in nuclear physics see e.g. [1], [2], [3]. The recent discussion of color superconductivity at high but realistic baryon density is mainly based on this method [4], [5], [6], [7], [8]. An example from statistical physics is a mean field description of antiferromagnetic and superconducting condensates in the Hubbard model. Quite generally, mean field theory (MFT) seems to be well suited for systems with multifermion interactions and bosonic condensates.

Unfortunately, in these systems MFT has a basic ambiguity which is connected with the possibility to perform Fierz transformations (FT) for the underlying local multifermion interaction. This becomes apparent already in the simplest NJL-type model (for only one fermion species) with a chirally invariant pointlike four fermion interaction:

$$\Gamma_k = \int d^4x \left[ \bar{\psi}i\partial\psi + \frac{1}{2} \lambda_{\sigma,k} [(\bar{\psi}\gamma^5\psi)^2] + \frac{1}{2} \lambda_{V,k} [(\bar{\psi}\gamma^\mu\gamma^5\psi)^2] + \frac{1}{2} \lambda_{A,k} [(\bar{\psi}\gamma^\mu\gamma^5\psi)^2] \right].$$

(1)

Due to the Fierz identity

$$[(\bar{\psi}\gamma^\mu\psi)^2 - (\bar{\psi}\gamma^\mu\gamma^5\psi)^2] + 2 [(\bar{\psi}\gamma^5\psi)^2 - (\bar{\psi}\gamma^5\psi)^2] = 0$$

(2)

only two of the quartic couplings are independent and we write

$$\lambda_{\sigma} = \bar{\lambda}_{\sigma} - 2\gamma \bar{\lambda}_V, \quad \lambda_V = (1 - \gamma) \bar{\lambda}_V, \quad \lambda_A = \gamma \bar{\lambda}_V.$$ 

(3)

The parameter $\gamma$ is redundant since it multiplies just the vanishing expression (2). No physical quantity can depend on $\gamma$ in a full computation of the functional integral for partition function and expectation values of field operators.
The index $k$ for the effective action denotes a cutoff scale. We will see in sect. 2 that the MFT results do strongly depend on $\gamma$, limiting their quantitative reliability. For example the critical coupling for the onset of a non vanishing condensate $\sigma \sim \langle \bar{\psi} (1 - \gamma^5) \psi \rangle$ depends strongly on $\gamma$ for fixed physical couplings $\bar{\lambda}_\sigma$ and $\bar{\lambda}_V$. MFT is tightly connected to the method of partial bosonization. Indeed, MFT can be thought of as simply performing the fermionic functional integral of the partially bosonized model introduced below. In order to make progress one has to find a method where MFT appears as some type of first step in a more systematic expansion. As a test of such a method one may investigate if the results become independent of $\gamma$ as it should be. In this talk (for a more detailed discussion see [10]) we want to discuss such a method based on the exact renormalization group (RG) equation for the effective average action [11]. Perturbation theory is an alternative approach to the model (1) (see Fig. 1). Results of perturbation theory are unambiguous. However, perturbation theory is limited to small coupling. Therefore we cannot observe the interesting phenomena of spontaneous symmetry breaking (SSB). Doing RG-improvement it is possible to observe the onset of SSB [12]. Nevertheless, in the simple model of (1) it is impossible to proceed to the SSB phase. To do this we would need to enlarge the truncation to include eight and higher fermion interactions which is quite difficult. Partial bosonization seems to be the ideal remedy to this difficulty [13]. Using this technique the model (1) can be rewritten as an equivalent Yukawa type model with scalars $\phi$, vectors $V^\mu$ and axial vectors $A^\mu$ representing the corresponding fermion bilinears:

$$L_B = \int d^4x \left\{ \mu_\sigma^2 \phi^* \phi + \frac{1}{2} \mu_V^2 (V^\mu)^2 + \frac{1}{2} \mu_A^2 (A^\mu)^2 \right\} + h_{\sigma} \left[ \bar{\psi} \left( \frac{1 + \gamma^5}{2} \right) \phi \psi - \bar{\psi} \left( \frac{1 - \gamma^5}{2} \right) \phi^* \psi \right] + h_V \bar{\psi} \gamma^\mu V^\mu \psi + h_A \bar{\psi} A^\mu A^\mu \psi$$

Taking

$$\mu_\sigma^2 = \frac{h_{\sigma}^2}{2 \lambda_\sigma}, \quad \mu_V^2 = -\frac{h_V^2}{\lambda_V}, \quad \mu_A^2 = -\frac{h_A^2}{\lambda_A}$$

at some cutoff scale $k = \Lambda$ this model is equivalent to the NJL-type model (2). Indeed, partial bosonization is nothing more than the introduction of a factor of unity into the functional integral.

Spontaneous symmetry breaking can now be dealt with by computing the effective potential for $\phi$ and looking for a minimum at $\phi \neq 0$. For example, a term $\sim \phi^4$ stands for an eight quark interaction. Unfortunately, partial bosonization brings back the "Fierz ambiguity" of MFT.

## 2 Critical couplings from MFT

For a mean field calculation we treat the fermionic fluctuations in a homogeneous background of a $\phi \sim \bar{\psi} (1 - \gamma^5) \psi$ field. Partial bosonization introduces just such a composite field and we use the
action (4). MFT means that we perform the functional integral in a homogeneous background $\phi$-field. For the correction to the mass of the $\phi$-field the corresponding Feynman diagram is given in Fig. 1(a). Since we only want to determine the critical couplings we are satisfied to calculate the mass term $\sim \phi^\star \phi$ and look when it turns negative. Therefore we do not need to consider the other background fields such as $V^\mu \sim \bar{\psi} \gamma^\mu \psi$ and $A^\mu \sim \bar{\psi} \gamma^\mu \gamma^5 \psi$. So, for these fields we set $V^\mu = A^\mu = 0$. Including the fluctuations from $k = \Lambda$ to $k = 0$ we find:

$$\Gamma = \Gamma_\Lambda + \Delta \Gamma_{\text{MFT}} = \left( \mu^2_{\sigma,\Lambda} - \frac{1}{8\pi^2} h^2_{\sigma,\Lambda} \Lambda^2 \right) \phi^\star \phi + \text{const} + O \left( (\phi^\star \phi)^2 \right),$$

where we have expanded in powers of the $\phi$-field to better see the mass term.

The mass term turns negative at the critical coupling

$$\lambda_{\text{crit}} = \frac{h^2_{\sigma,\Lambda}}{2\mu^2_{\sigma,\Lambda}} = \frac{4\pi^2}{\Lambda^2}, \quad \bar{\lambda}_{\text{crit}} = \frac{4\pi^2}{\Lambda^2} + 2\gamma \bar{\lambda}_{V,\Lambda}$$

where we used Eqs. (5) and (3) to express the result in terms of the underlying fermionic model. The result depends on the unphysical parameter $\gamma$ and is therefore ambiguous.

### 3 Invariant Bosonic Flow

In the context of the RG-equation for the effective average action the way to specify an approximation is to choose an ansatz for the effective average action. Using (4) as an ansatz it is possible to calculate the flow equations for the (now $k$-dependent) couplings and mass terms. This calculation includes not only the mass shift Fig. 1(a) but also the vertex corrections 1(b). However the results (e.g. the critical coupling) still depend strongly on $\gamma$ and therefore we do not reproduce perturbation theory.

At this point we note a discrepancy between the claim that bosonization is an exact identity and the fact that we do not reproduce perturbation theory. Furthermore we get a dependence on the unphysical Fierz parameter $\gamma$ (actually this is an expression of the fact that we do not reproduce perturbation theory which is invariant).

Of course, the exact equivalence between the bosonized and the non bosonized model is ensured only if we calculate the complete flow. When we do approximations it might be violated. That is what has happened in our model. Where did this happen? The bosonization procedure cancelled all four fermion interactions at the bosonization scale $\Lambda$. However, during the flow to $k < \Lambda$ new four fermion interactions are generated by box diagrams with internal bosonic lines Figs. 1(c) and 1(d). These are not included in the truncation to (1). Nevertheless, all diagrams of Figs. 1(a)-1(d) are of the same order $\sim h^4$. So it seems inconsistent to neglect the two box diagrams and therefore the generated four fermion interactions. However, including a four fermion interaction into the truncation seems not to be a sensible thing to do since we bosonized to get rid of these complicated multi fermion interactions. Luckily, a method to absorb four fermion interactions into the bosonic flow has been developed in [18]. Applying this method to the model (4) we can absorb the interactions generated by the diagrams Figs. 1(c) and 1(d). This completes the flow in the sense that now all diagrams at this order are taken into account. We are now able to reproduce one loop perturbation theory in the bosonic model. A thorough analysis shows that the RG calculation for this truncation is now indeed equivalent to the RG calculation for the purely fermionic model specified by (1). Since the latter is invariant under FT’s for the initial action the former also exhibits this feature.

### 4 Conclusions

We have shown that MFT leads to ambiguous results depending on the choice of FT for the underlying fermionic theory. An RG calculation which includes also the vertex correction for
the Yukawa coupling does not improve on this point. The reason for this problem is that we have neglected the four fermion interactions generated during the flow even though they are of the same order in the coupling. Including these interactions we are able to reproduce (RG-improved) one-loop perturbation theory. Moreover we find invariance of the result under FT’s.

Having established a way to produce FT invariant results at the lowest level it seems now possible to look toward more complicated (and more useful) truncations e.g. including kinetic terms for the bosons. It might be difficult to achieve complete independence of the results on the FT but one can hope at least for a weaker dependence.

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