ORDERS OF $\pi$-BASES

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Abstract. We extend the scope of B. Shapirovskii’s results [6] on the order of $\pi$-bases in compact spaces and answer some questions of V. Tkachuk in [7].

Introduction

The notion of $\pi$-base is an essential tool for studying the internal structure of a topological space as well as its external properties (embeddings, functions and the like); this was established, primarily, in the work of Boris Shapirovskii in the 1970s ([5] and [6] containing major discoveries). In this paper we attempt to show the full natural scope of his ideas regarding the order of $\pi$-bases.

In Section 1, we decipher and refine the method of induction used by Shapirovskii in Section 3 of [6]. We develop a purely set-theoretic technique which will be applied to generalize Shapirovskii’s results and answer some questions of Tkachuk; this technique provides a formalism interesting in itself and apt to have other applications.

In Section 2, using the results of Section 1, we describe a canonical form for $\pi$-bases in regular spaces and prove that canonical $\pi$-bases always exist. In our Lemma 2.4 we give a characterization of free sequences and with its help we derive a series of new results, starting with the central Theorem 2.6. By carefully numbering the points of our argument, we have tried to achieve “sufficiency with precision,” and, hopefully, to avoid accidental and irrelevant conditions in our statements. This also gave us reliable guidance as to where exactly to look for counterexamples when the sufficiency of a weaker condition was in question.

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Section 3 deals with the natural question as to whether or not the assumptions in our theorems could be further relaxed. We give some examples to the contrary which also solves three problems of V. Tkachuk from [7].

The idea for this paper originated from the observation that our Lemma 2.4 could be used, in place of final compactness (that is, small $L(X)$), even in the original Shapirovskii argument, made for compact spaces.

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We have used [1] and [2] as general references for definitions and notation. $\mathcal{ON}$ is the class of ordinal numbers. Additions and multiplications are ordinal operations. We have written $[\gamma, \delta]$, or $\delta \setminus \gamma$, for $\{\alpha : \gamma \leq \alpha < \delta\}$. We have denoted by $T_X$ the family of all non-empty open subsets of a topological space $X$.

1. Canonical $\kappa$-functions

Definition 1.1. For an infinite cardinal $\kappa$, a canonical $\kappa$-function is a class function

$$\phi = \phi_\kappa : \mathcal{ON} \longrightarrow [\mathcal{ON} \times \kappa]^{<\omega}$$

satisfying the following two conditions:

1. For every ordinal $\alpha$, $\phi(\alpha) \subseteq \alpha \times \kappa$.
2. For every ordinal $\delta$ of the form $\delta = \kappa \cdot \epsilon$ there is $\gamma(\delta) < \delta$ such that $[\gamma(\delta), \delta] \times \kappa]^{<\omega} \subseteq \phi^\sim \delta$.

Definition 1.2. A $\tau$-strong canonical $(\kappa, \lambda)$-function is a function

$$\psi : \lambda \longrightarrow [\lambda \times \kappa]^\tau$$

satisfying the following two conditions:

1. $\forall \alpha \in \text{dom}(\psi)$ $\psi(\alpha) \subseteq \alpha \times \kappa$.
2. For every ordinal $\delta \leq \lambda$ with $\text{cf}(\delta) = \kappa^+$ there is $\gamma(\delta) < \delta$ such that $[\gamma(\delta), \delta] \times \kappa]^{<\tau} \subseteq \psi^\sim \delta$.

Definition and Lemma 1.3. Let $\kappa$ be an infinite cardinal. Define a class-function $\sigma = \sigma_\kappa : \mathcal{ON} \longrightarrow \mathcal{ON}$ by the following rule:

- $\sigma(0) = 0$,
- $\sigma(1) = \kappa$,
- $\sigma(\alpha + 1) = \sigma(\alpha) + |\sigma(\alpha)|$, for $\alpha > 0$,
- $\sigma(\beta) = \sup\{\sigma(\alpha) : \alpha < \beta\}$, for $\beta$ limit.
Then every ordinal \( \delta \) has the following unique \( \sigma_\kappa \)-normal form:

\[
\delta = \sigma(\alpha_0) + \sigma(\alpha_1) + \cdots + \sigma(\alpha_{n-1}) + \Delta,
\]

where \( n \in \omega, |\sigma(\alpha_0)| > |\sigma(\alpha_1)| > |\sigma(\alpha_2)| > \cdots > |\sigma(\alpha_{n-2})| > |\sigma(\alpha_{n-1})|, \alpha_{n-1} > 0, \) and \( \Delta < \kappa \).

**Proof.** To visualize, we partition \( ON \) into intervals \([0], [1, \kappa^+], \ldots, [\mu, \mu^+], \ldots\) This is the finest partition of \( ON \) into intervals that are closed under \( \sigma \). Then we choose descending \( \alpha_i \) from different intervals, excluding the first.

Existence. Since \( \sigma \) is increasing continuous, and \( \sigma(1) = \kappa \), if \( \delta \geq \kappa \), then \( \exists \alpha_0 > 0 \) such that \( \sigma(\alpha_0) \leq \delta < \sigma(\alpha_0 + 1) \). Similarly, if type(\( \delta \setminus \sigma(\alpha_0) \)) \( \geq \kappa \), then \( \exists \alpha_1 > 0 \) such that \( \sigma(\alpha_1) \leq \text{type}(\delta \setminus \sigma(\alpha_0)) < \sigma(\alpha_1 + 1) \). Eventually, we’ll get to \( \alpha_{n-1} > 0 \) (if any, otherwise set \( n = 0 \)) such that \( \sigma(\alpha_{n-1}) \leq \text{type}(\delta \setminus (\sigma(\alpha_0) + \sigma(\alpha_1) + \cdots + \sigma(\alpha_{n-2})) < \sigma(\alpha_{n-1} + 1) \), but now type(\( \delta \setminus (\sigma(\alpha_0) + \sigma(\alpha_1) + \cdots + \sigma(\alpha_{n-1})) \) \( < \kappa \). Put

\( \Delta = \text{type}(\delta \setminus (\sigma(\alpha_0) + \sigma(\alpha_1) + \cdots + \sigma(\alpha_{n-1})) \).

Uniqueness is now easily proved by induction on the length of the normal form. It follows that the lexicographic ordering of the \( \sigma \)-normal forms (that is of the ordinal sequences \(< \alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \Delta >\) ) coincides with the natural order of their values in \( ON \), but we will not need this explicitly. \( \square \)

**Definition 1.4.** Define a total pressing-down (save for \( \gamma(0) = 0 \)) class-function \( \gamma = \gamma_\kappa : ON \rightarrow ON \) as follows:

For every ordinal \( \delta \) with the \( \sigma_\kappa \)-normal form \( \delta = \sigma(\alpha_0) + \sigma(\alpha_1) + \cdots + \sigma(\alpha_{n-2}) + \sigma(\alpha_{n-1}) + \Delta, \) set

\( \gamma(\delta) = 0, \) if \( n = 0, \)

\( \gamma(\delta) = \sigma(\alpha_0) + \sigma(\alpha_1) + \cdots + \sigma(\alpha_{n-2}), \) otherwise.

**Theorem 1.5.** For every infinite cardinal \( \kappa \) there is a canonical \( \kappa \)-function \( \phi = \phi_\kappa \).

**Proof.** 1) For every ordinal \( \delta \) with \( \Delta = 0 \) in its normal form, let \( \delta' = \gamma(\delta) + \sigma(\alpha_{n-1} + 1) \) (for \( \delta > 0 \) this is also \( \delta' = \delta + |\sigma(\alpha_{n-1})| \)). Then fix a function \( f_\delta : [\delta, \delta') \rightarrow [\gamma(\delta), \delta'] \times \kappa \) such that

(1) \( f_\delta \) is onto, and

(2) \( \forall \xi \in [\delta, \delta') f_\delta(\xi) \subseteq \xi \times \kappa \).

This is very easy to arrange, because, for every \( \alpha, |\sigma(\alpha + 1)| = |\sigma(\alpha), \sigma(\alpha + 1)| \) \( \geq \kappa \). We may start with an arbitrary surjection mapping \( |\sigma(\alpha_{n-1} + 1)| \) -many times to every member of the range.

2) Now consider ordinals \( \delta \) of the form \( \delta = \kappa \cdot \epsilon \). These are the same as just considered ordinals with \( \Delta = 0 \) in their normal form.
Suppose that we have finitely many functions \( h_0, \ldots, h_{n-1} \) such that
\[
(\forall i < n) \ h_i : [\delta, \delta + \kappa) \to ([\delta + \kappa] \times \kappa)^{<\omega} \quad \text{and} \quad (\forall \xi \in \text{dom}(h_i)) \ h_i(\xi) \subseteq \xi \times \kappa.
\]
Then denote by \( H = H[h_0, \ldots, h_{n-1}] \), and fix, a function with the same domain and co-domain such that
\[
(\forall i) (\forall \xi \in [\delta, \delta + \kappa]) \ (\exists \eta \geq \xi) \ H(\eta) = h_i(\xi) \quad \text{and so} \quad H(\eta) = h_i(\xi) \subseteq \xi \times \kappa \subseteq \eta \times \kappa.
\]
In other words, \( H \) is a combination of \( h_0, \ldots, h_{n-1} \) mapping onto the union of their ranges.

3) Finally, define \( \phi = \phi_\kappa \) on the ordinal intervals of the form \([\delta, \delta + \kappa)\) with \( \delta = \kappa \cdot \epsilon \), simultaneously for all such \( \delta \), by the following explicit rule. Find the normal form \( \kappa \cdot \epsilon = \delta = \sigma(\alpha_0) + \sigma(\alpha_1) + \cdots + \sigma(\alpha_{n-1}) + 0 \). Then set \( \phi \upharpoonright [\delta, \delta + \kappa) = H[h_0, \ldots, h_{n-1}] \), where \( h_i = f_\sigma(\alpha_0) + \sigma(\alpha_1) + \cdots + \sigma(\alpha_i) \upharpoonright [\delta, \delta + \kappa) \).

4) We are left to check that the function \( \phi \) just defined satisfies the Definition 1.1. It is transparent that the first condition is satisfied, and the second is in the following assertion.

**Claim 1.6.** Suppose that, for every \( \delta \) with \( \Delta = 0 \) (and \( n \geq 0 \)) in its normal form,
\[
\text{ran}(f_\delta) \subseteq \text{ran}(\phi \upharpoonright [\delta, \delta')).
\]
Then, for every such \( \delta \) with \( n \geq 1 \),
\[
[[\gamma(\delta), \delta) \times \kappa)^{<\omega} \subseteq \text{ran}(\phi \upharpoonright [\gamma(\delta), \delta)).
\]

**Proof.** This is straightforward by induction on \( n \), and then by a sub-induction on \( \alpha_{n-1} \) in the normal form for \( \delta \).

The case \( \alpha_{n-1} = \beta + 1 \) is explicit, and for \( \alpha_{n-1} \) a limit ordinal use
\[
[[\gamma(\delta), \gamma(\delta) + \sigma(\alpha_{n-1})] \times \kappa)^{<\omega} = \bigcup_{\beta < \alpha_{n-1}} [[\gamma(\delta), \gamma(\delta) + \sigma(\beta)] \times \kappa)^{<\omega}.
\]
The equation is true, because \( \beta < \alpha_{n-1} : \gamma(\gamma(\delta) + \sigma(\beta)) = \gamma(\delta) \) is cofinal in \( \alpha_{n-1} \).

**Theorem 1.7.** If \( (\kappa^+)^\kappa = \kappa^+ \) and, for every cardinal \( \mu \) with \( \kappa^+ \leq \mu < \lambda \), we have \( \mu^\kappa = \mu \), then there is a \( \kappa \)-strong \( (\kappa, \lambda) \)-function. Under CH, there is an \( \omega \)-strong \( (\omega, \mathbb{R}_\omega) \)-function.

**Proof.** This time consider \( \sigma_{\kappa^+} \)-normal forms for ordinals \( \delta \in \mathcal{ON} \) and the regressive function \( \gamma_{\kappa^+} \). Otherwise do - *mutatis mutandis* - as in the proof of Theorem 1.5. It will be still possible to define \( f_\delta : [\delta, \delta') \to [[\gamma(\delta), \delta') \times \kappa]^\kappa \), because, for every \( \delta \) of the form \( \delta = \kappa^+ \cdot \epsilon \), we have \( ||[\delta, \delta')]^\kappa = ||[\delta, \delta')] \geq \kappa^+ \).

2. **Shapirovskii \( \pi \)-bases in regular spaces**

Recall ([1]) that \( \mathcal{R} \subseteq \mathcal{T}_X \) is a \( \pi \)-base for the topology \( \mathcal{T}_X \) of \( X \) iff \( \forall U \in \mathcal{T}_X \ \exists R \in \mathcal{R} \ \text{with} \ R \subseteq U \). A family \( \mathcal{R} \subseteq \mathcal{T}_X \) is a local \( \pi \)-base for
\[ p \in X \text{ iff } \forall U \in T_X \text{ with } p \in U \exists R \in \mathcal{R} \text{ with } R \subseteq U. \] The \( \pi \)-character of a point \( p \) in \( X \) is the cardinal \( \pi_X(p, X) = \min \{ |\mathcal{R}| : \mathcal{R} \subseteq T_X \text{ is a local } \pi \)-base for \( p \} \) and the \( \pi \)-character of \( X \) is \( \pi_X(X) = \sup \{ \pi_X(p, X) : p \in X \} \).

For \( P = \{ p_\alpha : \alpha < \mu \} \) and \( \delta < \mu \), let’s write \( P_\delta = \{ p_\alpha : \alpha < \delta \} \) and \( P^\delta = \{ p_\alpha : \delta \leq \alpha < \mu \} \).

\( P = \{ p_\alpha : \alpha < \mu \} \subseteq X \) is called \textbf{left-separated}, if \( \overline{P_\delta} \cap P^\delta = \emptyset \), for every \( \delta < \mu \) (this means that all initial segments of \( P \) are relatively closed in \( P \)). It is well-known and easy to see that every space \( X \) has a dense subspace left-separated in the order-type \( d(X) \).

\textbf{Definition 2.1.} Suppose \( X \) is a regular topological space with \( \pi_X(X) = \kappa \). Suppose the density of \( X \), \( d(X) = \lambda \), is an infinite cardinal and \( P = \{ p_\alpha : \alpha < \lambda \} \), is a left-separated dense subspace, left-separated as written. Then \( S = \{ S_{\alpha,i} : < \alpha, i > \in \lambda \times \kappa \} \subseteq T_X \), or \( S \) together with \( P \), is called a \textbf{Shapirovskii} \( \pi \)-base for \( X \) if the following conditions are satisfied:

\begin{enumerate}
  \item \( \{ S_{\alpha,i} : i < \kappa \} \) a local \( \pi \)-base for \( p_\alpha \) in \( X \);
  \item \( \overline{P_\alpha} \cap \{ S_{\beta,i} : \alpha \leq \beta, i < \kappa \} = \emptyset \).
  \item \( (\forall \delta = \kappa \cdot \epsilon)(\forall A \in [\gamma(\delta), \delta) \times \kappa]^{<\omega}) \) if \( \bigcap_{\alpha \in A} \overline{S_\alpha} \neq \emptyset \), then \( \bigcap_{\alpha \in A} \overline{S_\alpha} \cap \bigcup \{ P_\alpha : \alpha < \delta \} \neq \emptyset \), and therefore \( \bigcap_{\alpha \in A} \overline{S_\alpha} \cap \overline{P_\delta} \neq \emptyset \).
\end{enumerate}

We will also say that \( S \) as above is a \( \kappa \)-\textbf{strong Shapirovskii} \( \pi \)-base, if the condition (c) is replaced by the following:

\( (c^+) \) \( (\forall \delta = \kappa^+ \cdot \epsilon)(\forall A \in [\gamma(\delta), \delta) \times \kappa]^\omega \) if \( \bigcap_{\alpha \in A} \overline{S_\alpha} \neq \emptyset \), then \( \bigcap_{\alpha \in A} \overline{S_\alpha} \cap \overline{P_\delta} \neq \emptyset \).

\textbf{Theorem 2.2.} Every regular space has a Shapirovskii \( \pi \)-base.

\textit{Proof.} Let \( Q = \{ q_\alpha : \alpha < \lambda \} \) be a left-separated dense subspace of \( X \). Define \( P = \{ p_\alpha : \alpha < \lambda \} \) and \( S = \{ S_{\alpha,i} : < \alpha, i > \in \lambda \times \kappa \} \) by induction on \( \delta < \lambda \).

Suppose at the stage \( \delta \geq 0 \) we have \( P_\delta = \{ p_\alpha : \alpha < \delta \} \) and \( S = \{ S_{\alpha,i} : < \alpha, i > \in \delta \times \kappa \} \).

1) If \( \bigcap \{ \overline{S_\alpha} : a \in \phi_\delta(\delta) \} \cap \overline{P_\delta} = \emptyset \) and \( \bigcap \{ \overline{S_\alpha} : a \in \phi_\delta(\delta) \} \neq \emptyset \), pick \( p_\delta \) in \( \bigcap \{ \overline{S_\alpha} : a \in \phi_\delta(\delta) \} \).

Otherwise, put \( p_\delta = q_\xi \), where \( \xi \) is the least index of a member of \( Q \setminus \overline{P_\delta} \).

2) Next, \( p_\delta \) being thus defined, pick a \( \pi \)-base \( \mathcal{B} \) for \( p_\delta \) of size \( |\mathcal{B}| \leq \kappa \) and with \( \overline{P_\delta} \cap \mathcal{B} = \emptyset \) for each member \( \mathcal{B} \) of \( \mathcal{B} \), and index it as \( \{ S_{\delta,i} : i \in \kappa \} \). End of induction. \( \square \)
Theorem 2.3. Under the cardinal assumptions of Theorem 1.7, every regular space with \( \pi(X) \leq \kappa \) and \( d(X) \leq \lambda \) has a \( \kappa \)-strong Shapirovskii \( \pi \)-base.

Recall ([1]) that \( P = \{ p_\alpha : \alpha < \mu \} \subseteq X \) is a free sequence in the space \( X \), if \( \bar{P}_\delta \cap \bar{P}_\delta = \emptyset \), for every \( \delta < \mu \). Let \( \mathcal{F}(X) = \sup \{|P| : P \) is a free sequence in \( X \} \).

The following characterization of free sequences parallels Shapirovskii’s characterization of discrete sets in [4]. It says that small \( \mathcal{F}(X) \) can be viewed as a compactness-like reflection property of the space \( X \), and this is precisely what we will need in the sequel.

Lemma 2.4. Let \( X \) be any topological space and \( \kappa \) any infinite cardinal. Then (a) \( \mathcal{F}(X) \leq \kappa \) if and only if (b) for every \( Y \subseteq X \), every family \( U \subseteq T_X \) such that \( (\forall A \in [Y]^{\leq \kappa}) \ (\exists U \in U) \ A \subseteq U \) has a subfamily \( V \subseteq U \) of size \( |V| \leq \kappa \) covering \( Y \).

Proof. Sufficiency. Suppose \( Y \) and \( U \) are as in (b), but (b) fails, so there is no \( V \in [U]^{\leq \kappa} \) covering \( Y \). We will pick up a free sequence \( P = \{ p_\alpha : \alpha < \kappa^+ \} \) by induction on \( \delta < \kappa^+ \). Suppose that at the stage \( \delta \geq 0 \) we have \( P_\delta = \{ p_\alpha : \alpha < \delta \} \) and \( \{ U_\alpha : \alpha < \delta \} \subseteq U \).

Then pick \( U_\delta \in U \) with \( \bar{P}_\delta \subseteq U_\delta \) and \( p_\delta \in Y \setminus \bigcup_{\alpha \leq \delta} U_\alpha \).

We claim that \( P \) is a free sequence. Indeed, for \( \delta < \kappa^+ \), \( \bar{P}_\delta \subseteq U_\delta \) and \( P_\delta \cap U_\delta = \emptyset \), whereupon \( \bar{P}_\delta \cap \bar{P}_\delta = \emptyset \).

Necessity. Now assume that (a) fails, and there is a free sequence \( P = \{ p_\alpha : \alpha < \kappa^+ \} \). Let \( U = \{ X \setminus \bar{P}_\delta : \delta < \kappa^+ \} \). Because \( \kappa^+ \) is a regular cardinal, \( (\forall A \in [P]^{\leq \kappa}) \ (\exists \delta < \kappa^+) \ A \subseteq \bar{P}_\delta \subseteq X \setminus \bar{P}_\delta \). Therefore, \( U \) is as in (b) with \( Y = P \). Let \( V \) be a subfamily of \( U \) of size \( |V| \leq \kappa \). Then, again by regularity of \( \kappa^+ \), \( (\exists \delta < \kappa^+) \) such that \( \bigcup V \subseteq \bigcup_{\delta \leq \delta} (X \setminus \bar{P}_\delta) = X \setminus \bar{P}_\delta \). Therefore \( \bigcup V \) is disjoint from \( P_\delta \), and thus does not cover \( P \).

Recall (see [0]) that a space \( X \) is initially \( \kappa \)-compact, if every cover of cardinality at most \( \kappa \) has a finite subcover, and \( X \) is \( (\kappa, \kappa^+] \)-compact, if every cover of \( X \) of cardinality \( \kappa^+ \) has a subcover of cardinality \( \kappa \).

The fact that \( t(X) = \mathcal{F}(X) \) in compact spaces is well known, but we will need these weaker covering properties of \( X \) as a factor in the relationship between \( t(X) \) and \( \mathcal{F}(X) \). The following fact is folklore.

Fact 2.5. Suppose that \( X \) is a regular space. Then

\[ ^1 \text{For an important different (external, or “algebraic”) approach to free sequences see [9] and [10].} \]
(a) $X$ is initially $\kappa$-compact $+$ $\mathcal{F}(X) \leq \kappa \implies t(X) \leq \kappa$.
(b) $X$ is $(\kappa, \kappa^+]$-compact $+$ $t(X) \leq \kappa \implies \mathcal{F}(X) \leq \kappa$.

Proof. (a) 1) Let $Y \subseteq X$ with $Y = \bigcup \{A : A \in [Y]^{\leq \kappa}\}$ and observe that $Y$ is initially $\kappa$-compact.
   
   2) It is sufficient to show that $Y$ is closed. So fix $p \notin Y$. We will find a neighbourhood of $p$ disjoint from $Y$.
   
   3) For every $A \in [Y]^{\leq \kappa}$ find $U_A$, a neighbourhood of $p$ with $\overline{U_A} \cap A = \emptyset$.
   
   4) $\mathcal{U} = \{X \setminus \overline{U_A} : A \in [Y]^{\leq \kappa}\}$ is a cover of $Y$ as in (b) of Lemma 2.4.
   
   5) Therefore, there is a $\mathcal{V} \subseteq \mathcal{U}$, $|\mathcal{V}| \leq \kappa$, $\mathcal{V}$ covers $Y$, and since $Y$ is initially $\kappa$-compact, a finite $\mathcal{W} \subseteq \mathcal{V} \subseteq \mathcal{U}$ which also covers $Y$.
   
   6) But then $\bigcap \{U_A : X \setminus \overline{U_A} \in \mathcal{W}\}$ is a neighbourhood of $p$ disjoint from $Y$, as wanted.

(b) Suppose $P = \{p_\alpha : \alpha < \kappa^+\} \subseteq X$. By tightness, $\overline{P} = \bigcup \{\overline{P_\alpha} : \alpha < \kappa^+\}$. If $P$ were a free sequence in $X$, then $\mathcal{U} = \{X \setminus \overline{P_\delta} : \delta < \kappa^+\}$ would be an increasing $\kappa^+$-cover of $X$, contradicting $(\kappa, \kappa^+]$-compactness of $X$. $\blacksquare$

For a family $\mathcal{R}$ of subsets of $X$ and a point $p \in X$, the order of $p$ in $\mathcal{R}$ is the cardinal $\text{ord}(p, \mathcal{R}) = |\{R \in \mathcal{R} : p \in R\}|$. The order of $\mathcal{R}$ is $\text{ord}(\mathcal{R}) = \sup\{\text{ord}(p, \mathcal{R}) : (p \in X)\}$. Finally, a family $\mathcal{R}$ is is point-$\kappa$ if $\text{ord}(\mathcal{R}) \leq \kappa$, i.e. if every point belongs to at most $\kappa$ members of $\mathcal{R}$.

**Theorem 2.6.** Suppose $X$ is a regular initially $\kappa$-compact space with $\pi\chi(X) = \kappa$ and no free sequences of length $\kappa^+$ (that is $\mathcal{F}(X) \leq \kappa$). Then any Shapirovskii $\pi$-base is point-$\kappa$.

Proof. Let $\mathcal{S}$ be a Shapirovskii $\pi$-base for $X$, as displayed in the definition, and suppose that $\mathcal{R} \subseteq \mathcal{S}$, $|\mathcal{R}| = \kappa^+$. We have to show that $\bigcap \mathcal{R} = \emptyset$.

1) For some $I \in [\lambda \times \kappa]^{\kappa^+}$, $\mathcal{R} = \{S_{\alpha,i} : \langle \alpha, i > \in I\}$, and so $|\pi_0^{-1}I| = \kappa^+$, where $\pi_0$ denotes the projection from the square to the first coordinate, $\pi_0(< a, b >) = a$.

2) Pick $\delta \in \lambda$, the least ordinal such that $|(\pi_0^{-1}I) \cap \delta| = \kappa^+$. Then $\text{cf}(\delta) = \kappa^+$. Let $J = I \bigcap (\langle \gamma(\delta), \delta \rangle \times \kappa)$. Then $|\pi_0^{-1}J| = \kappa^+$, because $\gamma(\delta) < \delta$.

Let $\mathcal{Q} = \{S_{\alpha,i} : \langle \alpha, i > \in J\} \subseteq \mathcal{R}$.

3) By Fact 2.5 (a), $t(X) \leq \kappa$.

4) Since $t(X) \leq \kappa < \kappa^+ = \text{cf}(\delta)$, $\overline{P_\delta} = \bigcup \{\overline{P_\alpha} : \alpha < \delta\}$.

5) Since, by the choice of $\delta$, $\pi_0^{-1}J$ is cofinal in $\delta$, $\bigcap \{\overline{Q} : Q \in \mathcal{Q}\} \bigcap \overline{P_\delta} = \emptyset$. This uses 4) above and the property (b) of Definition 2.1.
6) Therefore, \( U = \{X \setminus \overline{Q} : Q \in \mathcal{Q}\} \) is an open cover of \( \overline{\mathcal{P}_\delta} \), and
\[
(\forall A \in [\overline{\mathcal{P}_\delta}]^{\leq \kappa}) (\exists Q \in \mathcal{Q}) \text{ with } A \subseteq X \setminus \overline{Q}.
\]

7) Since \( \mathcal{F}(X) \leq \kappa \), Lemma 2.4 applies (with \( Y = \overline{\mathcal{P}_\delta} \)), and \( \exists V \subseteq U \), \( |V| \leq \kappa \), such that \( V \) is also a cover of \( \overline{\mathcal{P}_\delta} \).

8) Since \( \overline{\mathcal{P}_\delta} \) is initially \( \kappa \)-compact, there is a finite \( W \subseteq V \subseteq U \) which is a cover of \( \overline{\mathcal{P}_\delta} \). Say, \( W = \{X \setminus \overline{S_a} : a \in A\} \), for some finite \( A \subseteq J \), and so we have \( (\cap_{a \in A} \overline{S_a}) \cap \overline{\mathcal{P}_\delta} = \emptyset \).

9) Since \( \delta \) with \( \text{cf}(\delta) = \kappa^+ \) is a fortiori of the form \( \delta = \kappa \cdot \epsilon \) for some \( \epsilon \), this implies (by the property (c) in the Definition 2.1) that \( \cap_{a \in A} \overline{S_a} = \emptyset \), and therefore \( \cap_{a \in A} S_a = \emptyset \). But \( \{S_a : a \in A\} \subseteq \mathcal{Q} \subseteq \mathcal{R} \), and so \( \cap \mathcal{R} = \emptyset \). \( \square \)

The numbering of points in this proof, and in some of our other proofs, hopefully, helps to see exactly what ideas are involved (and seem needed, but maybe are not), at each point of the argument.

**Corollary 2.7.** Every regular countably compact space with countable \( \pi \)-character and no uncountable free sequences has a point-countable \( \pi \)-base.

This corollary gives a partial positive answer to Problem 4.4 of [7].

The core of our argument, in fact, also proves the following variations.

**Corollary 2.8.** Every regular initially \( t(X)^+ \)-compact space with \( \kappa = \max\{\pi\chi(X), t(X)\} \) has a point-\( \kappa \) \( \pi \)-base.

**Corollary 2.9.** Every first-countable initially \( \omega_1 \)-compact regular space has a point-countable \( \pi \)-base.

**Corollary 2.10.** Let \( \kappa = \max\{\pi\chi(X), t(X)\} \). If \( d(X) \leq \kappa^+ \), then \( X \) has a point-\( \kappa \) \( \pi \)-base.

This is, in essence, Tkachuk’s theorem 3.2 of [7].

**Corollary 2.11.** Suppose \( X \) is a regular space which is initially \( \mathcal{F}(X) \)-compact. Let \( \kappa = \max\{\mathcal{F}(X), \pi\chi(X)\} \). Then \( X \) has a point-\( \kappa \) \( \pi \)-base. In fact, any Shapirovaskii \( \pi \)-base is point-\( \kappa \).

**Corollary 2.12.** Every regular countably compact space with no uncountable free sequences has a point-\( \pi\chi(X) \) \( \pi \)-base.

In the presence of a nice cardinal arithmetic, the covering restrictions can be altogether omitted, but only when the density of the space is not too large.
**Theorem 2.13.** Suppose that $\kappa$ and $\lambda$ are cardinals such that $(\kappa^+)^\kappa = \kappa^+$ and, for every $\mu$ with $\kappa^+ \leq \mu < \lambda$, $\mu^\kappa = \mu$. Then every regular space $X$ with $\pi \chi(X) \leq \kappa$, $\mathcal{F}(X) \leq \kappa$ and $d(X) \leq \lambda$ has a point-$\kappa$ $\pi$-base.

**Corollary 2.14.** Under CH, every regular space with $\mathcal{F}(X) = \omega$ and $d(X) \leq \aleph_\omega$ has a point-$\pi \chi(X)$ $\pi$-base.

**Corollary 2.15.** Under CH, every regular first-countable space with $d(X) \leq \aleph_\omega$ and no uncountable free sequences has a point-countable $\pi$-base.

### 3. Counterexamples to weaker assumptions

The main tool for proving that a space does not have a point-$\kappa$ $\pi$-base is Shapirovskii’s Theorem 3.2 in [6] which we will need in the following weak form:

\[(\ast) \quad \text{"If } \max\{\kappa^+, s(X)\} < d(X), \text{ then } X \text{ does not have a point-}\kappa \text{ $\pi$-base."} \]

This criterion is used in every example below.

**Example 3.1.** There is, in ZFC, a first-countable zero-dimensional left-separated space $X$ such that $d(X) = |X| \geq (\beth_\omega)^+, hL(X) = \beth_\omega$, and hence $s(X) = \beth_\omega$.

By $(\ast)$, $X$ cannot have a point-countable $\pi$-base. This is one of the celebrated generalized $L$-spaces of Stevo Todorčević (cf. Theorem 16 in [8]). This example gives a negative answer to Problem 4.1 of [7].

**Example 3.2.** There is, in ZFC, a zero-dimensional first-countable space left-separated in the order-type $\beth$ with no discrete subspace of size $\beth$. It has a point-countable $\pi$-base if and only if $\beth = \omega_1$.

This is another $L$-space of Stevo Todorčević from the same paper. In the case of $\beth = \omega_1$, whatever the value of $\mathcal{F}(X)$ is, the space has a point-countable $\pi$-base by 2.10.

**Example 3.3.** Consistently, relative to the existence of a supercompact cardinal, there is a first-countable hereditarily Lindelöf (hence with $\mathcal{F}(Y) = s(Y) = \omega$) space $Y$ left-separated in the order-type $\omega_2 = 2^\omega$, without a point-countable $\pi$-base.

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\[2\text{If we define } m(X) = \min\{\sup\{(\text{ord}(p, \mathcal{R}))^+: p \in X\} : \mathcal{R} \text{ is a $\pi$-base for } X \}, \text{ it states that } d(X) \leq m(X) \cdot s(X).\]
There is, in $ZFC$, a zero-dimensional space $X$ left-separated in the order-type $d(X) = |X| = 2^\omega_1$ with $\chi(X) = \omega_1$ and $s(X) = hL(X) = \beth_\omega$. This is still another generalized $L$-space of Todorcevic from the same paper. By ($\star$), it does not have a point-$\omega_1$ $\pi$-base.

By a result of Magidor (see [3], Corollary 3), $V| = 2^{\beth_\omega} = (\beth_\omega)^{++}$ is consistent, relative to the existence of a supercompact cardinal. Force with $Fn(\omega, \beth_\omega)$ from $V$ as a ground model. This will preserve $(\beth_\omega)^+$ (in the form of $(\aleph_1)^{V[G]}$) and all cardinals above it, while collapsing all cardinals below it to $\aleph_0$. By a routine computation (counting names and using the generic collapsing function), $2^{\aleph_0} = \aleph_2$ in $V[G]$. We claim that the space $X$ from the ground model $V$ will possess in the generic extension $V[G]$ all the properties of $Y$ stated above. As usual, the topology of $Y$ is understood to be generated in $V[G]$ as a base.

All topological base properties (i.e. those which can be formulated in terms of a base and are invariant under choosing a base) of $X$ will be inherited by $Y$. These include regularity, left-separated structure, and “$p$ is a complete accumulation point of $A$,” provided $A$ is in $V$ (even if the cardinal $|A|^V$ is collapsed). The only property that needs an argument is the hereditary Lindelöfness of $Y$. It is sufficient to check that a set of size $\aleph_1$ in the extension contains a point of complete accumulation of itself. Now, a set $A$ of cardinality $\aleph_1$ in $V[G]$ has a name $\dot{A}$ in $V$ indexed by the ordinals in $(\beth_\omega)^+$. Since our forcing poset has size $\beth_\omega$, there is a single condition in $G$ which evaluates in $V$ $(\beth_\omega)^+$-many points of $\dot{A}$, say a set $B$. Now a complete accumulation point $b \in B$ of $B$ in $X$ (which exists by $hL(X) < |B|$ in $V$), as we remarked, is the same for $B \subseteq A$ in $V[G]$.

This space doesn’t have a point-countable $\pi$-base for the cardinal arithmetic reason alone (since it has $\chi(Y) = \mathcal{F}(Y) = \aleph_0$ and $|Y| = \aleph_2 \leq \aleph_\omega$). This shows that the cardinal assumption in the Corollary 2.15 (and a fortiori in Theorem 2.13) is necessary. This example also gives a negative answer to Problems 4.3 and 4.6 of [7].

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