THE FIXED POINT PROPERTY FOR \((c)\)-MAPPINGS AND UNBOUNDED SETS

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Abstract. We prove that a closed convex subset \(C\) of a real Hilbert space \(X\) has the fixed point property for \((c)\)-mappings if and only if \(C\) is bounded. Some convergence results about the iterations are obtained.

Keywords. Banach space; \((c)\)-mapping; unbounded closed convex subset; uniformly convex Banach space; fixed point; Picard sequence.

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1. INTRODUCTION

Let \(X\) be a real Banach space with norm \(\|\cdot\|\) and let \(C\) be a nonempty subset of \(X\). A mapping \(T : C \rightarrow C\) is said to be nonexpansive if \(\|Tx - Ty\| \leq \|x - y\|\) for all \(x, y \in C\). \(T\) is said to be a \((c)\)-mapping if there exist \(a, c \in [0, 1], c > 0\) and \(a + 2c = 1\) such that
\[
\|Tx - Ty\| \leq a\|x - y\| + c(\|Tx - y\| + \|Ty - x\|).
\]
for all \(x, y \in C\).

A closed convex subset \(C\) of \(X\) is said to have the fixed point property for nonexpansive mappings (in short, FPP) if every nonexpansive mapping \(T : C \rightarrow C\) has at least a fixed point in \(C\) (see \([7, 9, 10, 11, 12, 13, 19]\)).

A closed convex subset \(C\) of \(X\) is said to have the fixed point property for \((c)\)-mappings (in short, \((c)\)-FPP) if every \((c)\)-mapping \(T : C \rightarrow C\) has at least a fixed point in \(C\) (see \([20, 22]\)).

It is an open problem whether these two fixed point properties hold simultaneously. The answer is affirmative if \(C\) is a bounded set of uniformly convex Banach space (see \([10, 12, 13]\)). However, the situation seems to be unknown when \(C\) is unbounded. The contributions related to this subject are very few. In 1980, W. Ray (see \([16]\)) proved that the boundedness of \(C\) characterizes FPP in Hilbert spaces. Ray’s result was simplified by R. Sine \([21]\) who observed that the metric projection in Hilbert space is nonexpansive. After that, T. Benavides (see \([5]\)) established the same result in the Banach space \(c_0\). In \([22]\), M. A. Smyth investigated the existence of fixed points for \((c)\)-mappings defined on weakly compact convex subsets which does not have necessarily normal structure and he wondered about assumptions.

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on $C$ to be a weakly compact convex subset having FPP, to possess $(c)$-FPP. Recall that in the setting of Banach space $L^1([0,1])$, the weakly compact convex subset

$$C = \{ f \in L^1([0,1]) : 0 \leq f \leq 2, a.e., \int_0^1 f(t) dt = 1 \}$$

fails to have FPP (see [1]) but we do not know if $C$ has $(c)$-FPP.

In this paper, by the insights in the contributions of W. Takahashi et al in [25], we prove the variant of Ray’s result for $(c)$-mappings. Some convergence of iterations associated to $(c)$-mappings are studied. Finally, we conclude this work by asking some interesting questions.

2. MAIN RESULTS

First of all, let us define the concept of firmly nonexpansive mappings.

**Definition 2.1.** Let $C$ be a nonempty subset of a Banach space $X$. A mapping $T : C \rightarrow C$ is said to be $\lambda$-firmly nonexpansive ($\lambda \in (0, 1)$) if

$$\|Tx - Ty\| \leq \|(1 - \lambda)(x - y) + \lambda(Tx - Ty)\|$$

for all $x, y \in C$. $T$ is said to be firmly nonexpansive if $T$ is $\lambda$-firmly nonexpansive for all $\lambda \in (0, 1)$.

**Remark 2.2.** It is obvious that a $\lambda$-firmly nonexpansive mapping is nonexpansive while the converse is in general not true (it suffices to take $T : X \rightarrow X$ defined by $Tx = -x$).

For more details on $\lambda$-firmly nonexpansive mappings, we quote [2, 9, 10, 17, 23]).

The first result in this section is the following proposition.

**Proposition 2.3.** Let $C$ be a nonempty subset of a Banach space $X$. Then every $\lambda$-firmly nonexpansive mapping is a $(c)$-mapping.

**Proof.** Let $x, y \in C$. Since $T$ is a $\lambda$-firmly nonexpansive mapping, we have

$$\|Tx - Ty\| \leq \|\lambda(Tx - Ty) + (1 - \lambda)(x - y)\|$$

$$= \|(1 - \lambda)x + \lambda Tx - y + \lambda y - \lambda Ty\|$$

$$= \|(1 - \lambda)[(1 - \lambda)x + \lambda Tx - y] + \lambda[(1 - \lambda)x + \lambda Tx - Ty]\|$$

$$\leq (1 - \lambda)\|[(1 - \lambda)x + \lambda Tx] - y\| + \lambda\|[(1 - \lambda)x + \lambda Tx - Ty]\|$$

$$\leq (1 - \lambda)\|[(1 - \lambda)x + \lambda Tx] - [(1 - \lambda)y + \lambda y]\|$$

$$+ \lambda\|[(1 - \lambda)x + \lambda Tx] - [(1 - \lambda)Ty + \lambda Ty]\|$$

$$\leq (1 - \lambda)[(1 - \lambda)]\|x - y\| + \lambda\|Tx - Ty\|$$

$$+ \lambda[(1 - \lambda)]\|x - Ty\| + \lambda\|Tx - Ty\|$$

$$= (1 - \lambda)^2\|x - y\| + (1 - \lambda)\|Tx - y\| + (1 - \lambda)\|x - Ty\|$$

$$+ \lambda^2\|Tx - Ty\|.$$
\[(1 - \lambda^2) ||T x - T y|| \leq (1 - \lambda) ||x - y|| + \lambda (1 - \lambda) ||T x - y|| + \lambda (1 - \lambda) ||x - T y||.\]

Therefore

\[||T x - T y|| \leq \frac{1 - \lambda}{1 + \lambda} ||x - y|| + \frac{\lambda}{1 + \lambda} (||T x - y|| + ||x - T y||).\]  \hspace{1cm} (2.2)

which means that \( T \) is a \((c)\)-mapping.

The following example shows that the class of \((c)\)-mappings is wider than that of firmly nonexpansive mappings.

**Example 2.4.** (see [24]) Let \((X, ||.||) = (\mathbb{R}, |.|)\) and \(C = [0, 3]\). Define \(T : [0, 3] \rightarrow [0, 3]\) by

\[T x = \begin{cases} 
0 & \text{if } x \in [0,3], \\
1 & \text{if } x = 3.
\end{cases}\]

A simple calculation shows that \(T\) is a \((c)\)-mapping for \(c = \frac{1}{2}\) and \(c = \frac{1}{3}\). However, \(T\) is not firmly nonexpansive, since \(T\) is not nonexpansive.

The following theorem was established by W. Takahashi et al in [25].

**Theorem 2.5.** Let \(H\) be a Hilbert space and let \(C\) be a nonempty closed convex subset of \(H\). Then the following conditions are equivalent.

(i) Every firmly nonexpansive mapping \(T : C \rightarrow C\) has a fixed point in \(C\).

(ii) \(C\) is bounded.

The main theorem of this paper is the following.

**Theorem 2.6.** Let \(H\) be a Hilbert space and let \(C\) be a nonempty closed convex subset of \(H\). Then the following conditions are equivalent.

(i) Every \((c)\)-mapping \(T : C \rightarrow C\) has a fixed point in \(C\).

(ii) \(C\) is bounded.

**Proof.** \((ii) \implies (i)\) Since \(C\) is bounded then \(C\) is a weakly compact convex subset of \(H\) (which has a normal structure). So the result is an immediate consequence of Theorem 2 in [6] (see also [13]).

\((i) \implies (ii)\) Assume that \(C\) is unbounded. By using Theorem 2.5, there exists a free fixed point firmly nonexpansive mapping \(T : C \rightarrow C\). But following Proposition 2.3, \(T\) is a \((c)\)-mapping which contradicts \((i)\). Hence \(C\) must be bounded.

**Corollary 2.7.** Let \(C\) be a nonempty closed convex subset of a Hilbert space \(H\) and let \(T : C \rightarrow C\) be a mapping satisfying
9\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2 \tag{2.3}

If \( C \) is bounded then \( T \) has a (unique) fixed point in \( C \).

**Proof.** For the uniqueness, assume that \( T \) has two distincts fixed points \( z_1, z_2 \in C \) such that \( z_1 \neq z_2 \). Then

\[ 9\|z_1 - z_2\|^2 \leq \|z_1 - z_2\|^2 + \|z_1 - z_2\|^2 + \|z_1 - z_2\|^2 = 3\|z_1 - z_2\|^2. \]

which is a contradiction.

Now, if \( T \) satisfies (4), then

\[ \|Tx - Ty\|^2 \leq \frac{1}{9}(\|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2). \]

Consequently,

\[ \|Tx - Ty\| \leq \frac{1}{3}(\|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2)^{\frac{1}{2}}. \]

By using the inequality

\[ \sqrt{x^2 + y^2 + z^2} \leq x + y + z \text{ for all } x, y, z \geq 0, \]

we get

\[ \|Tx - Ty\| \leq \frac{1}{3}(\|x - y\| + \|Tx - y\| + \|Ty - x\|), \tag{2.4} \]

which proves that \( T \) is a \((c)\)-mapping. Now, the result is an immediate consequence of the implication \((u) \implies (i)\) of Theorem 2.6.

**Definition 2.8.** Let \( X \) be a Banach space. The modulus \( \delta \) of convexity of \( X \) is defined by

\[ \delta(\varepsilon) = \inf\left\{ 1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon \right\} \]

for every \( 0 \leq \varepsilon \leq 2 \). A Banach space \( X \) is said to be uniformly convex if \( \delta(\varepsilon) > 0 \) for all \( \varepsilon \in (0, 2] \).

For a mapping \( T : C \rightarrow C \), we define the orbit \( O(x_0) \) of \( x_0 \in C \) by \( O(x_0) = \{ T^n x_0 \}_{n \geq 0} \) \( (T^0 x_0 = x_0) \).

**Theorem 2.9.** Let \( X \) be a uniformly convex Banach space and let \( C \) be a closed convex subset of \( X \). Assume that \( T : C \rightarrow C \) is a \((c)\)-mapping satisfying

\( (i) \) There exists \( x_0 \in C \) such that \( O(x_0) \) is bounded.

Then \( T \) has a fixed point in \( C \).

If moreover, \( C = -C \) and \( T \) is an odd mapping satisfying

\( (u) \) For all integer \( i \geq 1 \) and all \( x, y \in C \), the sequence \( (\|T^{n+i} x - T^ny\|) \) is decreasing.

Then the Picard sequence \( (T^n(x_0))_n \) converges in norm to a fixed point of \( T \).
Proof. (i) Since $X$ is uniformly convex, then the asymptotic center $A(C, (T^n(x_0)))$, associated to the Picard sequence $(T^n(x_0))$, is a singleton (see assertions (a) and (c) of Theorem 5.2 in [10]). On the other hand, Lemma 2 in [3] shows that

$$\lim_{n \to +\infty} \|Ty_n - y_n\| = 0.$$  

(2.5)

where $y_n = T^n(x_0)$. A simple argument implies that there exists $z_0 \in C$ such that $Tz_0 = z_0$ which proves the first claim.

(ii) Now, since $C$ is a nonempty convex subset with $C = -C$, we have

$$0 = \frac{x + (-x)}{2} \in C.$$  

Next, since $T$ is a $(c)$-mapping then for all $x \in C$, we have

$$\|Tx - T0\| \leq a\|x - 0\| + c(\|Tx - 0\| + \|T0 - x\|).$$

But $T$ is odd, so $T0 = 0$. Therefore

$$\|Tx\| \leq a\|x\| + c\|Tx\| + c\|x\|.$$  

This leads to

$$\|Tx\| \leq \left(\frac{a + c}{1 - c}\right)\|x\|.$$

$$= \|x\|.$$  

(2.6)

Thus, by induction, we deduce that the sequence $(\|T^n x\|)_n$ is decreasing in $[0, +\infty]$, and

$$\lim_{n \to +\infty} \|T^n x\| = \gamma \geq 0.$$  

(2.7)

Furthermore, since $T$ is odd, we have $T^n(-x) = -T^n x$ for all integer $n \geq 1$. So, by replacing $y$ by $-x$ in (ii), we observe that the sequence $(\|T^{n+i}x - T^n(-x)\| = \|T^{n+i}x + T^n(-x)\|)_n$ is non-increasing for all fixed integer $i \geq 1$.

Afterwards, by the triangle inequality, we infer that

$$\|T^{n+i}x_0 - T^n x_0\| \leq \sum_{k=1}^{i} \|T^{n+k}x_0 - T^{n+k-1}x_0\|.$$  

(2.8)

It follows, from Lemma 2 in [3] that for all fixed integer $i$, we have

$$\lim_{n \to +\infty} \|T^{n+i}x_0 - T^n x_0\| = 0.$$  

(2.9)
The rest of the proof is similar to that given in Theorem 1.1 in [4].

The next example shows that the hypothesis of uniform convexity is important in Theorem 2.9.

**Example 2.10.** Let \( X = C([0, 1]) \) equipped with the sup norm and let

\[
C = \{ f \in X : f(0) = 0 \}
\]

and let \( T : C \rightarrow C \) defined by \( Tf(t) = tf(t) \).

Clearly, \( C \) is a closed convex subset of \( X \) with \( C = -C \). In addition, \( T \) is an odd \( (c) \)-mapping (see Example in [3]). The formula

\[
T^n f(t) = t^n f(t) \quad (n \geq 1) \quad (2.10)
\]

shows that the orbit \( O(f) \) of any \( f \in C \) is bounded. On the other hand, it is obvious that \( T \) is also nonexpansive then for any fixed integer \( i \geq 1 \), the sequence \( (\|T^n f_i - T^n f_2\|)_n \) is decreasing and 0 is the unique fixed point of \( T \) in \( C \). But if we take \( f_0(t) = \sin(t\pi/2) \) then \( f_0 \in C \) and \( T^n f_0(t) = t^n \sin(t\pi/2) \) does not converge to 0 in \( X \) since \( \|T^n f_0 - 0\| = 1 \not\rightarrow 0 \).

**Definition 2.11.** A mapping \( T : C \rightarrow C \) is called Chatterjea mapping if \( T \) is a \( (c) \)-mapping with \( c = \frac{1}{2} \).

In the sequel, we will denote by \( R(I - T) \) the range of the mapping \( I - T : C \rightarrow X \).

Now, we are in a position to state our next result

**Theorem 2.12.** Let \( C \) be a closed convex subset of a Banach space \( X \) and let \( T : C \rightarrow C \) be a \( (c) \)-mapping. Then

I) If \( X \) is uniformly convex then

\[
0 \in R(I - T) \iff O(x_0) \text{ is bounded for some } x_0.
\]

II) If \( T \) is a Chatterjea mapping satisfying the following assumptions:

\( \mathcal{H}_1 \) For all \( x \in C \) and all integer \( k \geq 2 \), the sequence \( (\|T^{n+k} x - T^nx\|)_n \) is decreasing.

\( \mathcal{H}_2 \) There exists an integer \( k_0 \geq 1 \) such that \( T^n \) is uniformly lipschitzian for all \( n \geq k_0 \).

a) Then we have

\[
0 \notin \overline{R(I - T)} \iff \lim_{n \rightarrow +\infty} \frac{\|T^n x\|}{n} = \alpha > 0 \text{ for all } x \in C.
\]

b) If \( X \) is uniformly convex. Then

\[
0 \in \overline{R(I - T)} \text{ and } 0 \notin R(I - T) \iff \lim_{n \rightarrow +\infty} \|T^n x\| = \infty \text{ and } \lim_{n \rightarrow +\infty} \frac{\|T^n x\|}{n} = 0
\]

for all \( x \in C \).
**Proof.** The proof of I) can be obtained by combining Lemma 2 in [3] and the equivalence between assertions (a) and (c) of Theorem 5.2 in [10].

Now, we will prove II)

II) a) $\implies$ Assume that

$$\lim_{n \to +\infty} \frac{\|T^nx_0\|}{n} = 0 \text{ for some } x_0 \in C.$$  

From Corollary 2.8 in [8], we have

$$\lim_{n \to +\infty} \|T^{n+1}x_0 - T^nx_0\| = 0.$$  

But

$$\lim_{n \to +\infty} \|T(T^nx_0) - T^nx_0\| = \lim_{n \to +\infty} \|(I - T)(T^nx_0)\| = 0$$

and

$$(I - T)(T^nx_0) \in R(I - T),$$

so

$$0 \in R(I - T).$$

II) a) $\impliedby$ If $0 \in R(I - T)$. Then

$$0 = \inf\{\|y\| : y \in R(I - T)\},$$

this implies the existence of a sequence $(x_k)_k$ in $C$ such that

$$\lim_{k \to +\infty} \|Tx_k - x_k\| = 0.$$  

On the other hand, since

$$T^nx_k = x_k + \sum_{s=1}^{n} (T - I)T^{s-1}x_k,$$  \hspace{1cm} (2.11)

and $T^n$ is uniformly lipschitzian for $n \geq k_0$, then

$$\|T^n x\| \leq \|T^nx_0\| + M\|x - x_0\| (M > 1),$$  \hspace{1cm} (2.12)

But from (2.11), we have

$$\|T^nx_k\| \leq \|x_k\| + \sum_{s=1}^{n} \|(T - I)T^{s-1}x_k\|,$$

$$\leq \|x_k\| + n\|Tx_k - x_k\|,$$  \hspace{1cm} (2.14)
it follows that

\[ \|T^nx\| \leq \|T^nx_k\| + M\|x_k - x\|, \]
\[ \leq \|x_k\| + n\|T x_k - x_k\| + M\|x - x_k\|, \]  
(2.15)

By dividing by \(n\), we get

\[ \frac{\|T^nx\|}{n} \leq \frac{\|x_k\|}{n} + \frac{M\|x_k - x\|}{n} + \frac{\|T x_k - x_k\|}{n}. \]
(2.17)

For a fixed integer \(k \geq 1\), letting \(n \to +\infty\), we infer that

\[ \limsup_n \frac{\|T^nx\|}{n} \leq \|T x_k - x_k\|. \]
(2.18)

Now, by letting \(k \to +\infty\), we obtain that

\[ \lim_n \frac{\|T^nx\|}{n} = 0. \]
(2.19)

II) \(b)\) Can be deduced immediately from I) and \(a)\) of II).

To illustrate Theorem 2.11, we give the following examples

**Example 2.13.** Let \((X, \|\cdot\|) = (\mathbb{R}, |\cdot|)\) and let \(T : \mathbb{R} \to \mathbb{R}\) defined by \(Tx = x + a\) \((a \neq 0)\). It is easy to see that \(T\) is a free fixed point \((c)\)-mapping. Obviously, we have

\[ \lim_n |T^nx| = \lim_n |x + na| = \infty. \]
(2.20)

which illustrates the assertion I) of Theorem 2.12. This example illustrates also assertion \(a)\) of II) since in this case, we have \(R(I-T) = R(I-T) = \{-a\}\).

**Example 2.14.** Let \(X = C([0, 1])\) equipped with the sup norm and

\[ C = \{f \in X : f(0) = 0 \leq f(t) \leq f(1) = 1\} \]

and let \(T : C \to C\) be defined as in Example 2.10. Then \(T\) is a free fixed point \((c)\)-mapping. But since

\[ \lim_n \|T^{n+1}x - T^nx\| = 0 \text{ (by Lemma 2 in [3])}, \]

we infer that \(0 \in R(T-I)\). Next, the fact that all orbits are bounded in this case, we get

\[ \lim_n \frac{\|T^nx\|}{n} = 0. \]

This contradicts \(b)\) of II). Indeed, \(C([0, 1])\) is not uniformly convex.
3. Some questions

We conclude this work by the following interesting questions

*Question 1:* Does Banach space $L^1([0,1])$ have $(c)$-FPP?

*Question 2:* Does Benavides’s result in $c_0$ hold for $(c)$-mappings?

*Question 3:* Can we extend Benavides’s result to orthogonally convex spaces? (see [22] for the definition)

*Question 4:* Let $X$ be a Banach space and let $T : X \rightarrow X$ be a $(c)$-mapping. Is it true that $\overline{R(I-T)}$ is a convex subset of $X$.

Recall that when $T$ is nonexpansive, the convexity of $\overline{R(I-T)}$ was proved by A. Pazy in the case of Hilbert spaces (see [15]) and the result was generalized by S. Reich to the setting of uniformly convex Banach spaces (see [18]).

*Question 5:* Let $C$ be a nonempty closed convex subset of a Hilbert space $X$ and let $\mathcal{S}$ be a representation of a semigroup $S$ of $(c)$-mappings on $C$. Suppose that $\{T_c : s \in S\}$ is relatively weakly compact for some $c \in C$. Does $\mathcal{F}(\mathcal{S}) \neq \emptyset$? (Here $\mathcal{F}(\mathcal{S})$ is the set of common fixed points of $\mathcal{S}$).

Competing interests

The authors declare that they have no competing interests.

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