Quantum Hall Effect on the Flag Manifold $F_2$

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Abstract

The Landau problem on the flag manifold $F_2 = SU(3)/U(1) \times U(1)$ is analyzed from an algebraic point of view. The involved magnetic background is induced by two $U(1)$ abelian connections. In quantizing the theory, we show that the wavefunctions, of a non-relativistic particle living on $F_2$, are the $SU(3)$ Wigner $D$-functions satisfying two constraints. Using the $F_2$ algebraic and geometrical structures, we derive the Landau Hamiltonian as well as its energy levels. The Lowest Landau level (LLL) wavefunctions coincide with the coherent states for the mixed $SU(3)$ representations. We discuss the quantum Hall effect for a filling factor $\nu = 1$, where the obtained particle density is constant and finite for a strong magnetic field. In this limit, we also show that the system behaves like an incompressible fluid. We study the semi-classical properties of the system confined in LLL. These will be used to discuss the edge excitations and construct the corresponding Wess-Zumino-Witten action.
1 Introduction

Two-dimensional quantum Hall effect (QHE) [1] remains among the successful phenomena in condensed matter physics. In fact, this subject continues nowadays to be investigated in different manifolds [2, 3, 4, 5, 6] and various contexts [7]. The first attempt towards a high dimensional generalization of QHE was formulated by Hu and Zhang [2] on $S^4$. Their main motivation is based on the fact that QHE on $S^4$ could give a way to formulate a quantum theory of gravitation. More precisely, the edge excitations for the quantum Hall droplet could lead to higher spin massless fields, in particular the graviton. Subsequently, many interesting studies have been done on different higher dimensional manifolds [7]. Among them, Karabali and Nair [3] who have employed a method based on the group theory approach to deal with QHE and related issues on the complex projective spaces $CP^k$ as well as the fuzzy spaces [6].

The noncompact counterpart of $CP^k$, say the Bergman ball $B^k$, was considered recently both analytically [8] and algebraically [9]. Using the group theory approach and considering a system of particles living on $B^k$ in the presence of a $U(1)$ background magnetic field, we have investigated QHE. This was based on the fact that $B^k$ can be viewed as the coset space $SU(k,1)/U(k)$. This was used to get wavefunctions as the Wigner $D$-functions submitted to a set of suitable constraints and to map the corresponding Hamiltonian in terms of the $SU(k,1)$ right generators. This latter coincides with the generalized Maass Laplacian in the complex coordinates. The Landau levels on $B^k$ are obtained by using the correspondence between the two manifolds $CP^k$ and $B^k$. In the lowest Landau levels (LLL), the obtained wavefunctions were nothing but the $SU(k,1)$ coherent states. Restricting to LLL, we have derived a generalized effective Wess-Zumino-Witten action that describes the quantum Hall droplet of radius proportional to $\sqrt{M}$, with $M$ is the number of particles in LLL. In order to obtain the boundary excitation action, we have defined the star product and the density of states. Also we have introduced the perturbation potential responsible of the degeneracy lifting in terms of the magnetic translations of $SU(k,1)$. Finally, we have discussed the nature of the edge excitations and illustrated this discussion by giving the disc as example. Based on the previous results related to QHE on $CP^k$ and $B^k$, it is natural to consider the Landau problem on other spaces as for instance the flag manifold $F_k = SU(k+1)/U(1)^k$ and discuss QHE.

The flag manifolds [10] have appeared in physics in different contexts as target manifolds for sigma model or in a geometric formulation of the harmonic superspace. These special homogeneous spaces have interesting geometric properties, which are relevant to discuss different issues. Indeed, they are Kähler manifolds and therefore possess a symplectic form, which is relevant to discuss QHE. This suggests to consider the Landau problem on the coset space $SU(k+1)/U(1)^k$ and discuss its basic features. In the present paper, we restrict ourselves to the particular case $k=2$. The case $k=1$ corresponding to two-sphere $F_1 = CP^1$ was considered previously in many works, for instance see [3].

More precisely, we consider a system of particles living on the flag manifold $F_2$. Taking advantage of the fact that the space $F_2$ can be seen as the coset space $SU(3)/U(1) \times U(1)$, we analyze the quantum mechanics of the present system. Due to the geometrical nature of the considered manifold, we show that the particles are submitted to the action of two magnetic backgrounds. In quantizing the theory on $F_2$, we obtain the wavefunctions as the $SU(3)$ Wigner $D$-functions satisfying two constraints. To derive the corresponding Hamiltonian $H$, we consider the right $SU(3)$ generators. By establishing
the relations between the right generators and the covariant derivatives, we obtain the second order differential form of $H$. Using the $SU(3)$ representation theory, we derive the Landau energy levels indexed by four integer quantum numbers. Restricting to LLL, we find a ground state completely different from that of the same system on $\mathbb{CP}^3$ or $\mathbb{R}^6$. We analyze QHE by building the generalized Laughlin states and evaluating the particle density. The incompressibility of these states is also considered. On the other hand, we analyze the semi-classical properties of the system confined in LLL. These will be used to discuss the edge excitations and construct the Wess-Zumino-Witten action.

The present paper is organized as follows. In section 2, we review some mathematical tools related to the flag manifold $F_2$ needed for our task. In particular we review the parametrization of $F_2$, mixed unitary representations and the Perelomov coherent states of the group $SU(3)$. In section 3, by quantizing the dynamics of a system of particles on $F_2$, we express the wavefunctions as the Wigner $D$-functions satisfying two constraints. The geometrical origin of the magnetic background will be discussed in section 4. Also we show that the magnetic field is a superposition of two abelian background species. Moreover, we construct the Hamiltonian as second order differential in terms of the $F_2$ local coordinates. In section 5, using the $SU(3)$ representation theory, we give the energy levels and wavefunctions. We construct the Laughlin states for the fractional QHE at $\nu = \frac{1}{m}$, with $m$ odd integer. We evaluate the particle density as well as two-point correlation function. In fact, we show the incompressibility of Hall system for large magnetic field strength. In section 6, we analyze the semi-classical properties of a large collection of particles confined in LLL for $n_1$ and $n_2$ large. In particular, we derive the density distribution, the symbol associated to a product of two operators acting on LLL (the star product) and give the excitation potential inducing a degeneracy lifting. These will be used to discuss the edge excitations of a quantum Hall droplet in the Flag manifold and constructing their Wess-Zumino-Witten action in section 7. We conclude and give some discussions as well as perspectives in the last section.

2 Flag manifold $F_2$

We begin by introducing the flag manifold $F_2$ and related matters. In fact, to discuss the quantum mechanics of a particle living on $F_2$, we need to consider the parametrization of the present manifold. Note that, $F_2$ is a compact Kähler manifold and homogeneous but nonsymetric parametrized by three local complex coordinates $u_\alpha$, with $\alpha = 1, 2, 3$. Algebraically, $F_2$ can be realized as the coset space

$$F_2 = SU(3)/U(1) \times U(1).$$

This realization is interesting in sense that it will allow us to use the group theory approach needed for our task. The flag manifold is equipped with the hermitian Riemannian metric

$$ds^2 = g_{\alpha\beta}du^\alpha d\bar{u}^\beta.$$  \hspace{1cm} (2)

The corresponding Kähler form is

$$\omega = ig_{\alpha\beta}du^\alpha d\bar{u}^\beta.$$  \hspace{1cm} (3)

Since that $\omega$ is closed, i.e. $d\omega = 0$, the components of the magnetic field expressed in terms of the frame fields defined by the metric are constants. This is interesting because it will be used to discuss
QHE on the flag manifold. The metric elements $g_{\alpha \bar{\beta}}$, which form a positive definite matrix, can be defined by

$$g_{\alpha \bar{\beta}} = \frac{\partial}{\partial u^\alpha} \frac{\partial}{\partial \bar{u}^\beta} K$$

where $K = K(u, \bar{u})$ is the Kähler potential, such as

$$K(u, \bar{u}) = \ln [\Delta_1(u, \bar{u}) \Delta_2(u, \bar{u})], \quad u = (u_1, u_2, u_3).$$

The functions $\Delta_1$ and $\Delta_2$ are given by

$$\Delta_1(u, \bar{u}) = 1 + |u_1|^2 + |u_3|^2, \quad \Delta_2(u, \bar{u}) = 1 + |u_2|^2 + |u_3 - u_1 u_2|^2.$$  

It is clear that $\omega$ is related to $K(u, \bar{u})$ by

$$\omega = i\partial \bar{\partial} K.$$  

This suggests that, one can decompose $\omega$ into two components

$$\omega = \omega_1 + \omega_2$$

where $\omega_1$ and $\omega_2$ read as

$$\omega_j = i\partial \bar{\partial} \ln \Delta_j(u, \bar{u}), \quad j = 1, 2.$$  

With the coset space realization (1), an element of the manifold $F_2$ can be written as lower triangular matrix in terms of the local coordinates. This is

$$u = \begin{pmatrix} 1 & 0 & 0 \\ u_1 & 1 & 0 \\ u_3 & u_2 & 1 \end{pmatrix}.$$  

Note that, the elements of the group $SU(3)$ are represented by $3 \times 3$ unitary matrices with determinants equal one. Moreover, they are generated by traceless Hermitian matrices, which are linearly independent generators $t_a, a = 1, 2, \cdots, 8$. These can be mapped in terms of the Gell-Mann matrices $\lambda_a$, such as

$$t_a = \frac{\lambda_a}{2}.$$  

They verify the normalization conditions

$$\text{Tr}(t_a t_b) = \frac{1}{2} \delta_{ab}.$$  

In terms of the matrices $t_a$, the Weyl generators, which are the raising and lowering operators, can be realized as

$$t_{\pm 1} = t_1 \pm it_2, \quad t_{\pm 2} = t_4 \pm it_5, \quad t_{\pm 3} = t_6 \pm it_7.$$  

The Cartan subalgebra corresponding to $SU(3)$ is generated by the elements

$$h_1 = \frac{1}{2} \text{diag}(1, -1, 0), \quad h_2 = \frac{1}{2} \text{diag}(0, 1, -1).$$  

From (10), it is clear that $F_2$ can be also written as another coset space. This is

$$F_2 = SL(3, \mathbb{C})/B_+.$$  

where $B_+$ is the Borel subgroup of the upper triangular matrices with determinants equal to one. This is the so-called Iwasawa decomposition [10]. Comparing (15) with the definition (1), one can see that there is an isomorphism:

$$SU(3)/U(1) \times U(1) \cong SL(3, \mathbb{C})/B_+.\,
$$

The mapping $SU(3)/U(1) \times U(1) \to SL(3, \mathbb{C})/B_+$ is a generalization of the stereographic projection in the $SU(2)$ case.

Note that, $u$ given by (10) is not necessarily, in general, an unitary matrix. To obtain the corresponding unitary matrix $v \in SU(3)$, we firstly consider $u$ as element of $SL(3, \mathbb{C})$. It can be expressed in terms of the column vectors

$$u = (c_1, c_2, c_3) \in SL(3, \mathbb{C}) = SU(3)^c$$

given by

$$c_1 = (1 \ u_1 \ u_3)^t, \quad c_2 = (0 \ 1 \ u_2)^t, \quad c_3 = (0 \ 0 \ 1)^t$$

where $t$ stands for matrix transposition. Secondly, by applying the Gramm-Schmidt orthogonalization process, we obtain, from $(c_1, c_2, c_3)$, a set of mutually orthogonal vectors $(e_1, e_2, e_3)$. They are

$$e_1 = c_1, \quad e_2 = c_2 - \frac{(c_2, e_1)}{(e_1, e_1)} e_1, \quad e_3 = c_3 - \frac{(c_3, e_2)}{(e_2, e_2)} e_2 - \frac{(c_3, e_1)}{(e_1, e_1)} e_1$$

where the inner product is defined as usually

$$(c_i, c_j) = c_i^t c_j.$$

Defining the normalized vectors by

$$v_i := e_i/\sqrt{(e_i, e_i)}$$

we get another element in $SU(3)$ mapped in terms of the local coordinates $u_\alpha$, namely

$$v = (v_1, v_2, v_3) \in SU(3)$$

which verifies $\det v = 1$ and $v^\dagger v = 1$. Explicitly, $v$ can be written as

$$v = \begin{pmatrix} \frac{1}{\sqrt{\Delta_1}} & -\frac{\bar{a}_1 u_2 \bar{u}_3}{\sqrt{\Delta_1 \Delta_2}} & -\frac{\bar{a}_1 \bar{u}_2 u_3}{\sqrt{\Delta_2}} \\ \frac{u_1}{\sqrt{\Delta_1}} & \frac{1+|u_3|^2-u_1 u_2 \bar{u}_3}{\sqrt{\Delta_1 \Delta_2}} & \frac{\bar{a}_2 \bar{u}_3}{\sqrt{\Delta_2}} \\ \frac{u_3}{\sqrt{\Delta_1}} & \frac{u_2 u_1 |u_3|^2-\bar{u}_3}{\sqrt{\Delta_1 \Delta_2}} & \frac{1}{\sqrt{\Delta_2}} \end{pmatrix}.$$

This form is convenient to calculate the Maurer-Cartan one-form and then generate the magnetic background indispensable to discuss the Landau problem as well as QHE on the flag manifold.

At this level, it is interesting to note that there is a one-to-one correspondence between the coset representative $u \in SU(3)/U(1) \times U(1)$ and the coherent state representation. Our interest in the $SU(3)$ coherent states is mainly motivated by the fact that they are exactly the LLL wavefunctions of the quantum system living on the manifold $\mathbf{F}_2$, as we will see later. The unitary irreducible representations (UIR) of $SU(3)$, denoted by $J \equiv (p, q)$, are finite dimensional and labeled by two positive integers $p$ and $q$. The dimension of the corresponding Hilbert space $\mathcal{H}_{(p,q)}$ is

$$\dim \mathcal{H}_{(p,q)} = \frac{1}{2} (p+1)(q+1)(p+q+2).$$
The orthonormal basis of \( \mathcal{H}_{(p,q)} \) writes as
\[
|\psi\rangle_{j_1,j_2,\ldots,j_p}^{k_1,k_2,\ldots,k_q} \equiv |\psi\rangle_{q_1,q_2,q_3}^{p_1,p_2,p_3}, \quad j,k = 1,2,3
\] (25)
where the sets of non-negative integers \((p_1,p_2,p_3)\) and \((q_1,q_2,q_3)\) satisfy two constraints
\[
p_1 + p_2 + p_3 = p, \quad q_1 + q_2 + q_3 = q.
\] (26)
It is well-known that \( J \) can be realized via a tensor \( O \) with \( p \) indices belonging to UIR \((1,0)\) and \( q \) indices to UIR \((0,1)\), which has \((p+1)(q+1)(p+q+2)/2\) complex components \( O_{j_1,j_2,\ldots,j_p}^{k_1,k_2,\ldots,k_q} \). It is completely symmetric separately in the upper and lower scripts and traceless, i.e. contraction of any upper index with any lower one gives zero. The explicit correspondence between the tensor components and the basis vectors (25) can be found in [11]. In \( \mathcal{H}_{(p,q)} \), the highest weight vector
\[
|\lambda\rangle \equiv |(p,q)\rangle = |\psi\rangle_{p,0,0}^{0,0,q}
\] (27)
verifies the condition
\[
t_{+i}|\lambda\rangle = 0, \quad i = 1,2,3.
\] (28)
Also it is a common eigenvector of the Cartan subalgebra generators of \( SU(3) \)
\[
h_1|\lambda\rangle = \frac{1}{2} p|\lambda\rangle, \quad h_2|\lambda\rangle = \frac{1}{2} q|\lambda\rangle.
\] (29)
As we will show next, the LLL wavefunctions of the quantum particle on \( F_2 \) coincide with the \( SU(3) \) coherent states. For this, we shall sketch some important facts about the definition and construction of the coherent states. To begin, we choose the highest vector \(|\lambda\rangle\) as a reference state and denote by \( T \) a stationary subgroup. It is defined as a subgroup of \( SU(3) \) leaving \(|\lambda\rangle\) invariant up to a phase factor, namely
\[
h|\lambda\rangle = |\lambda\rangle e^{i\psi(h)}, \quad h \in T.
\] (30)
Note that, the isotropy subgroup \( T \) includes the Cartan subgroup \( U(1) \times U(1) \). As any element \( g \in SU(3) \) can be uniquely decomposed into \( g = \phi h \), one can have
\[
g|\lambda\rangle = \phi|\lambda\rangle e^{i\psi(h)}.
\] (31)
Thus, the coherent states can be defined by
\[
|\phi,\lambda\rangle = \phi|\lambda\rangle
\] (32)
and therefore they are functions of the coset space \( SU(3)/T \). The maximal stability group \( T \) is \( U(2) \) for the completely symmetric representation \((p,0)\) or its adjoint \((0,q)\). In such case, \( SU(3)/T \) is the complex projective space \( \mathbb{CP}^2 \). For a generic representation of type \((p \neq 0, q \neq 0)\), \( T = U(1) \times U(1) \) and thus the coset space is the flag manifold \( F_2 \), which is of interest in the present analysis.

The coset representative element \( \phi \) can be identified with the unitary element \( v \) (23). It can be written also as
\[
v = \begin{pmatrix}
1 & 0 & 0 \\
u_1 & 1 & 0 \\
u_3 & u_2 & 1
\end{pmatrix}
\begin{pmatrix}
1/\sqrt{\Delta_1} & 0 & 0 \\
0 & \sqrt{\Delta_1/\Delta_2} & 0 \\
0 & 0 & \sqrt{\Delta_2}
\end{pmatrix}
\begin{pmatrix}
1 & \bar{w}_1 & \bar{w}_3 \\
0 & 1 & \bar{w}_2 \\
0 & 0 & 1
\end{pmatrix}
\] (33)
where the functions $w_i, i = 1, 2, 3$, are given by
\begin{align*}
w_1 &= -\frac{1}{\sqrt{\Delta_2}} (u_1 + \bar{u}_2 u_3), \\
w_2 &= \frac{1}{\sqrt{\Delta_1}} \left[ \bar{u}_1 u_3 - u_2 (1 + |u_1|^2) \right], \\
w_3 &= -\sqrt{\frac{\Delta_1}{\Delta_2}} (u_3 - u_1 u_2). \quad (34)
\end{align*}

Furthermore, in the defining representation, one can verify that $v$ takes another form. This is

$$v = \exp \left( \sum_{i=1}^3 \tau_i^- t_{-i} \right) \exp \left[ -(\ln \Delta_1) h_1 - (\ln \Delta_2) h_2 \right] \exp \left( \sum_{i=1}^3 \tau_i^+ t_{+i} \right) \quad (35)$$

which is more appropriate in constructing the required coherent states. The parameters $\tau_i^-$ and $\tau_i^+$ read as

$$\begin{align*}
\tau_1^- &= u_1, & \tau_2^- &= u_2, & \tau_3^- &= u_3 - \frac{1}{2} u_1 u_2, \\
\tau_1^+ &= \bar{w}_1, & \tau_2^+ &= \bar{w}_2, & \tau_3^+ &= \bar{w}_3 - \frac{1}{2} \bar{w}_1 \bar{w}_2. \quad (36)
\end{align*}$$

From (32), we can write the coherent states as follows

$$|u_1, u_2, u_3, \lambda \rangle := v(u_1, u_2, u_3) |\lambda \rangle. \quad (37)$$

To completely determine the required states $|u_1, u_2, u_3, \lambda \rangle$, we use the highest weight conditions (28) and (29). Thus, we show that

$$|u_1, u_2, u_3, \lambda \rangle = N(u, \bar{u}) \exp \left( \sum_{i=1}^3 \tau_i^- t_{-i} \right) |\lambda \rangle \quad (38)$$

where the normalization constant $N(u, \bar{u})$ is

$$N(u, \bar{u}) = \Delta_1^{-\frac{p}{2}} \Delta_2^{-\frac{q}{2}}. \quad (39)$$

Note that, the explicit expression of the coherent states (38) has been derived in [12]. This derivation is based on the Schwinger realization of the mixed representation $(p, q)$ and the bosonic construction of the vector basis (25).

### 3 Quantization of the flag manifold

We discuss now the quantization of a particle living on the flag manifold $F_2$. As it will be shown, the particle is submitted to the action of two abelian magnetic backgrounds $U(1)$. The wavefunctions of the present system can be obtained as functions on $SU(3)$ with specific transformation properties under the $U(1) \times U(1)$ subgroup. In other words, the quantum description of a "free" particle on $F_2$ can be performed by reducing the free motion on the group manifold $SU(3)$. This reduction can be established by imposing some suitable constraints on the $SU(3)$ wavefunctions.

The classical dynamics of a free particle on $SU(3)$ is described by the Lagrangian

$$L = \frac{1}{2} \text{Tr} \left( g^{-1} \dot{g} \right)^2 \quad (40)$$
where dot stands for time derivative. Quantum mechanically, the Hilbert space $\mathcal{H}_{(p,q)}$ is given by the square integrable functions on the group manifold $SU(3)$, i.e. $\mathcal{H} = L^2 (SU(3))$. The wavefunctions on $SU(3)$ can be expanded as

$$f(g) = \sum f_{n_l,n_r}^J \mathcal{D}_{n_l,n_r}^J (g)$$

(41)

where $\mathcal{D}_{n_l,n_r}^J (g)$ are the Wigner $\mathcal{D}$-functions, such as

$$\mathcal{D}_{n_l,n_r}^J (g) = \langle J, n_l | g | J, n_r \rangle$$

(42)

with $g \in SU(3)$, $J \equiv (p,q)$ and

$$n_l \equiv (p_1, p_2, p_3, q_1, q_2, q_3)_l, \quad n_r \equiv (p_1, p_2, p_3, q_1, q_2, q_3)_r$$

(43)

are two sets of quantum numbers specifying the right $R_a$ and left $L_a$ actions. The vectors $|J, n_r\rangle$ and $|J, n_l\rangle$, which are nothing but the ones defined by (25), generate, respectively, the basis of $SU(3)_R$ and $SU(3)_L$ unitary irreducible representation $J$. The right $R_a$ and left $L_a$ actions are defined by

$$R_a g = g t_a, \quad L_a g = t_a g, \quad a = 1, 2, \cdots, 8.$$  

(44)

The Wigner $\mathcal{D}$-functions (42) are orthogonal and form a basis of $\mathcal{H}_{(p,q)}$. The states of a $SU(3)$ representation $J$ correspond to a tensor of the form $O$ introduced previously. Under the action of $(g_l, g_r) \in SU(3) \times SU(3)$, (42) transforms as

$$\mathcal{D}_{n_l,n_r}^J (g) \rightarrow \sum_{p,q} \mathcal{D}_{n_l,n_r}^{J'} (g_l) \mathcal{D}_{p,q}^J (g) \mathcal{D}_{p,q}^{J'} (g_r).$$

(45)

This relation shows that the quantum numbers $n_r$ and $n_l$ transform, respectively, in the representation $J$ and the complex conjugate representation $\bar{J}$. Thus, $\mathcal{H}_{(p,q)}$ decomposes into the sum of irreducible representations, namely

$$\mathcal{H} \cong \bigoplus J V_J \otimes \bar{V}_J$$

(46)

where $V_J$ and $\bar{V}_J$ are, respectively, the vector spaces in which the representation $J$ and $\bar{J}$ are acting. The sum is over all inequivalent unitary irreducible representations of $SU(3)$. The basis of $\mathcal{H}_{(p,q)}$, introduced in the previous section, coincides with that associated to the space $V_J$. Then, we can set the following identification:

$$\mathcal{D}_{(p,q)}^{(p,q)} (g) \rightarrow \langle (p,q) n_l \rangle \otimes |(p,q) n_r\rangle.$$  

(47)

The quantum dynamics on $\mathcal{F}_2$ can be described by reducing the free motion, or imposing constraints, on the group manifold $SU(3)$. In this sense, following the standard procedure of quantization on the coset spaces, the classical motion on the flag manifold is described by the Lagrangian [14]

$$L_{\mathcal{F}_2} = \frac{1}{2} \text{Tr} \left( g^{-1} \dot{g} \big|_{\mathcal{F}_2} \right)^2 - \text{Tr} \left[ t \left( \dot{g} g^{-1} \right)^T \right]$$

(48)

where the symbols $|_T$ and $|_{\mathcal{F}_2}$ stand for the projection to the isotropy subgroup $T = U(1) \times U(1)$ and $\mathcal{F}_2$ in $G = SU(3)$, respectively. The first term in the above Lagrangian is invariant under $g \rightarrow gh$ for an element $h \in U(1) \times U(1)$ and thus depends only on $SU(3)/U(1) \times U(1)$ coordinates. The effect of reducing the motion from $SU(3)$ to the coset space $\mathcal{F}_2$ are contained in the second term of the Lagrangian and given in terms of $l$. This latter can be written as linear combination of the Cartan
generators $h_1$ and $h_2$. It follows that to get a quantized theory on the flag space $F_2$, we should quantize the following action [14]

$$S = i \int dt \ Tr \left( l g^{-1} g \right)$$

where $l$ is a combination of the Cartan generators, such as

$$l = n_1 h_1 + n_2 h_2.$$

For the $U(1) \times U(1)$ transformations of the form $g \rightarrow gh$ with

$$h = \exp(i \varphi_1 h_1 + i \varphi_2 h_2)$$

the action $S$ changes by a boundary term \( \left( \frac{1}{2} n_1 \Delta \varphi_1 + \frac{1}{2} n_2 \Delta \varphi_2 \right) \). Thus, the equations of motion are not affected by this gauge transformation and the classical theory is defined on the coset space $SU(3)/U(1) \times U(1) = F_2$. The canonical momenta associated to the direction parametrized by the angles $\varphi_1$ and $\varphi_2$, respectively, are given by $\frac{1}{2} n_1$ and $\frac{1}{2} n_2$. In this case, the physical states, denoted by $\psi(g)$, in the quantum theory should satisfy two constraints. These are

$$R_3 \psi(g) \equiv \psi(gh_1) = \frac{1}{2} n_1 \psi(g), \quad \frac{1}{2} (\sqrt{3} R_8 - R_3) \psi(g) \equiv \psi(gh_2) = \frac{1}{2} n_2 \psi(g).$$

There is another easy way to see the latter conditions. Indeed, under the transformation $g \rightarrow gh$, the variation of the action is given by

$$\Delta S = -\frac{1}{2} (n_1 \Delta \varphi_1 + n_2 \Delta \varphi_2)$$

and the state $\psi(g)$ transforms as

$$\psi(gh) = \psi(g) \exp \left[ -i \left( \frac{1}{2} n_1 \varphi_1 + \frac{1}{2} n_2 \varphi_2 \right) \right].$$

Using the conditions (52), one can show that the right generators satisfy the commutation relations

$$[R_{-1}, R_{+1}] = -n_1, \quad [R_{-2}, R_{+2}] = -n_1 - n_2, \quad [R_{-3}, R_{+3}] = -n_2$$

when they act on the states $\psi(g)$. The right generators, or covariant derivatives, play the role of the creation and annihilation operators for the harmonic oscillators. Thus, the groundstate should be annihilated by $R_{+i}$, namely

$$R_{+i} \psi(g) = 0.$$

This is the so-called polarization condition in the geometric quantization and implies that the groundstate satisfying (56) is holomorphic. Physically, it describes the LLL condition.

The wavefunctions of a quantum theory on the Flag manifold $F_2$ are the Wigner $D$-functions verifying (52). As we will see later, the polarization condition (56) will lead to the LLL analysis of the present system. Note that, the constraints (52) and (56) are exactly the defining relations for a highest weight state (28-29). Thus, the groundstate wavefunctions coincide with the $SU(3)$ coherent states for the mixed representations.
4 Induced magnetic background

It is well-known that the magnetic field is an important ingredient one should define in order to formulate QHE in any space. Thus, it is natural to ask about this physical quantity in the present analysis. More precisely, how to generate a magnetic background on the flag manifold $F_2$. This issue will be treated by considering the geometric features of $F_2$. The $SU(3)$ parametrization, introduced in the first section, will provide us with the Maurer-Cartan one-form and the $U(1)$ connections for $SU(3)/U(1) \times U(1)$. To perform this, we identify $g \in SU(3)$ with the element $v \in SU(3)/T$ given by (23). It follows that a basis of invariant one-forms is given by

$$g^{-1} dg = -i \epsilon^\alpha t_+ - i \epsilon^\alpha t_- - i \theta^j h_j, \quad \alpha = 1, 2, \quad j = 1, 2.$$  \hfill (57)

The elements $e^\alpha \equiv e^\alpha_\beta du_\beta$, with summation over repeated indices, are

$$e^1 = \frac{-i}{\Delta_1 \sqrt{\Delta_2}} \left\{ [1 + u_3(\bar{u}_3 - \bar{u}_1 \bar{u}_2)] du_1 + [\bar{u}_2 - u_1(\bar{u}_3 - \bar{u}_1 \bar{u}_2)] du_3 \right\},$$

$$e^2 = \frac{i}{\sqrt{\Delta_1 \Delta_2}} (u_2 du_1 - du_3),$$

$$e^3 = \frac{i}{\Delta_2 \sqrt{\Delta_1}} \left\{ -u_2(\bar{u}_1 + u_2 \bar{u}_3) du_1 - \Delta_1 du_2 + (\bar{u}_1 + u_2 \bar{u}_3) du_3 \right\}. \hfill (58)$$

The $U(1)$-connections $\theta^j$ are defined by

$$\theta^j = i du^\alpha \frac{\partial}{\partial u_\alpha} \ln \Delta_j + c.c., \quad j = 1, 2. \hfill (59)$$

They can be also written as

$$\theta^j = i \theta^j_\alpha du^\alpha + c.c., \quad \theta^j_\alpha = \frac{\partial}{\partial u_\alpha} \ln \Delta_j \hfill (60)$$

reflecting that $\theta^j$ are related to the Kähler potential (5). Actually, we have two abelian connections $\theta^1$ and $\theta^2$. They correspond to the vector potentials generating the magnetic background field, under which the quantum particle is constrained to move in the six-dimensional manifold $F_2$. To make contact with previous works on QHE in higher dimensions, the present situation should be compared with the $\mathbb{CP}^3$ analysis [3] where the particle is submitted only to one $U(1)$ magnetic field. Note that, the symplectic two-form (7) can be derived from the Maurer-Cartan one-form. Indeed, we have

$$\omega = -\text{Tr} \left[ 2(h_1 + h_2) g^{-1} dg \wedge g^{-1} dg \right]. \hfill (61)$$

This implies

$$\omega = e^1 \wedge e^1 + 2e^2 \wedge e^2 + e^3 \wedge e^3 \hfill (62)$$

which agrees with the $\omega$ form given by (7).

Let us denote the elements of the inverse of the $3 \times 3$ matrix $e = (e^1, e^2, e^3)$ as $(e^{-1})^\beta_\alpha$. They are

$$e^{-1} = i \begin{pmatrix} \frac{\Delta_1}{\sqrt{\Delta_2}} & -\frac{\Delta_1}{\Delta_2} (u_2 - u_1(\bar{u}_3 - \bar{u}_1 \bar{u}_2)) & 0 \\ 0 & \sqrt{\frac{\Delta_2}{\Delta_1}} (\bar{u}_1 + u_2 \bar{u}_3) & \frac{\Delta_2}{\sqrt{\Delta_1}} \\ \frac{\Delta_2}{\sqrt{\Delta_2}} u_2 & \sqrt{\frac{\Delta_1}{\Delta_2}} (1 + u_3(\bar{u}_3 - \bar{u}_1 \bar{u}_2)) & 0 \end{pmatrix}. \hfill (63)$$
To derive the Hamiltonian describing the system under consideration, we should define the $U(1) \times U(1)$ gauge covariant differentials on $F_2$. In this order, from the Maurer-Cartan one-form, we have
\[ g^{-1} \frac{\partial g}{\partial u^\beta} = -ie^{\beta}_\alpha t^{+\alpha} - i\theta^\beta_h. \] (64)

Using this relation, one can show that the right generators, $R^+g = gt^+\alpha$, defined by
\[ R_{\pm 1} = R_1 \pm iR_2, \quad R_{\pm 2} = R_4 \pm iR_5, \quad R_{\pm 3} = R_6 \pm iR_7 \] (65)
can be written as
\[ R^+ = \frac{i(e^{-1})^\beta_\alpha}{a^\beta_\alpha} \left[ \frac{\partial}{\partial u^\beta} - \frac{1}{2}(n_1 \theta_{\beta 1}^1 + n_2 \theta_{\beta 2}^2) \right] \] (66)
where we have used the constraints (52). They can be mapped in terms of the gauge field as
\[ R^+ = \frac{i(e^{-1})^\beta_\alpha}{a^\beta_\alpha} \left[ \frac{\partial}{\partial u^\beta} - ia_\beta \right] \] (67)
with $a_\beta$ given by
\[ a_\beta = -\frac{i}{2}(n_1 \theta_{\beta 1}^1 + n_2 \theta_{\beta 2}^2). \] (68)

Similarly, one can show that the following relation holds
\[ R^- = -\bar{R}^+. \] (69)

The gauge potential can be written as
\[ a = a_\beta du^\beta + a^\beta du^\beta = -\frac{i}{2}(n_1 \theta^1 + n_2 \theta^2). \] (70)

Therefore the corresponding electromagnetic field is
\[ F = da = -\frac{i}{2} (n_1 d\theta^1 + n_2 d\theta^2) \] (71)
where $n_1$ and $n_2$ are integers in agreement with the Dirac quantization. It is obvious that $F$ is also as a superposition of two abelian parts $F_1$ and $F_2$.

At this stage, we have the necessary ingredients to write down the required Hamiltonian. It can be mapped, in terms of the $SU(3)$ right generators, as [13]
\[ H = -\frac{1}{4m} \sum_{\alpha = 1}^{3} (R^+R^- - R^-R^+). \] (72)

By introducing $D_\alpha$ and $\bar{D}_{\alpha'}$
\[ D_\alpha = \frac{\partial}{\partial u_\alpha} - \frac{\partial}{\partial \bar{u}_\alpha} \ln \left( \Delta_{1}^{n_1} \Delta_{2}^{n_2} \right), \quad \bar{D}_{\alpha'} = \frac{\partial}{\partial \bar{u}_\alpha} - \frac{\partial}{\partial u_\alpha} \ln \left( \bar{\Delta}_{1}^{n_1} \bar{\Delta}_{2}^{n_2} \right) \] (73)
the operator $H$ takes the form
\[ H = -\frac{1}{4m} \sum_{\beta}(e^{-1})^\beta_\alpha(\bar{e}^{-1})^\alpha_{\beta'} (D_\alpha D_{\alpha'} + \bar{D}_{\alpha'} D_\alpha). \] (74)

The forms (72) and (74) show that there is a bridge between the algebraic analysis and the spectral theory. The Hamiltonian is written in terms of the local coordinates and thus one may analytically determine the spectrum of a particle living on the flag manifold $F_2$. But next, we use the $SU(3)$ representation theory to get the corresponding spectrum.
5 Spectrum and lowest Landau levels

At this point, it is clear that to derive the spectrum of the present system, the $U(1)$ gauge fields, or "monopoles" labeled by two integers $n_1$ and $n_2$, will play a crucial role. Note that, $n_1$ and $n_2$ are related to the third component of isospin and the hypercharge of a $SU(3)$ irreducible representation. To analyze the Landau problem on $\mathbf{F}_2$, we adopt an approach similar to that developed in [3] by studying the Landau spectrum for a quantum particle living on the complex projective spaces $\mathbf{CP}^k$.

As we have noticed above, the $SU(3)$ mixed representation $(p, q)$ can be realized via the irreducible tensor $O_q^p \equiv O_{k_1\ldots k_q}^{j_1\ldots j_p}$, $(j, k = 1, 2, 3)$. It transforms under $A \in SU(3)$ according to the rule

$$O_{k_1\ldots k_q}^{j_1\ldots j_p} = A_{i_1}^{j_1} \cdots A_{i_p}^{j_p} \overline{A}_{l_1}^{k_1} \cdots \overline{A}_{l_q}^{k_q} O_{l_1\ldots l_q}^{i_1\ldots i_p}. \quad (75)$$

In the presence of two abelian magnetic fields, it is convenient to label the irreducible representation $SU(3)_R$ by $(p, q)$ satisfying the relations (26) and corresponding to the irreducible tensor $O_{q_1 q_2 q_3}^{p_1 p_2 p_3}$. The wavefunctions rewrite as

$$\psi(g) = D_{n_1 n_2}^{(p_1+p_2+p_3,q_1+q_2+q_3)}(g) = \langle (p, q), n_l | g | (p, q), n_r \rangle. \quad (76)$$

Combining the rule transformations (75) where

$$A = \exp(+i\varphi_1 h_1 + i\varphi_2 h_2) = \text{diag}(e^{+\frac{1}{2}\varphi_1}, e^{-\frac{1}{2}(\varphi_1-\varphi_2)}, e^{-\frac{1}{2}\varphi_2}) \quad (77)$$

and using the constraints (52), we obtain two conditions on the integer right quantum numbers $(p_1, p_2, p_3, q_1, q_2, q_3)$. These are

$$n_1 = (p_1 - q_1) - (p_2 - q_2), \quad n_2 = (p_2 - q_2) - (p_3 - q_3). \quad (78)$$

The states verifying (52) are now labeled by four integers. The corresponding energy levels can be derived from of the Hamiltonian (72) as

$$E = \frac{1}{2m} \left[ C_2(p, q) - R_3^2 - R_8^2 \right] \quad (79)$$

where the quadratic Casimir $C_2(p, q)$ of the $(p, q)$ representation is given by

$$C_2(p, q) = \frac{1}{2} \left[ p(p+3) + q(q+3) + pq \right]. \quad (80)$$

Using (52) together with the constraints (78), one can write $E$ as

$$E(q_1, q_2, p_2, p_3) = \frac{1}{6m} \left[ 3C_2(n_1 + 2p_2 + p_3 + q_1 - q_2, q_2 + q_1 + 2q_2 + p_3 - p_2) - (n_1^2 + n_1 n_2 + n_2^2) \right]. \quad (81)$$

This show that actually the Landau levels are specified by four quantum numbers. In particular, the lowest energy eigenstates, for $n_1$ and $n_2$ fixed, correspond to $q_1 = q_2 = p_2 = p_3 = 0$. This is

$$E_0 = \frac{1}{2m} (n_1 + n_2) \quad (82)$$

with the degeneracy

$$d_0 = \frac{1}{2} (n_1 + 1)(n_2 + 1)(n_1 + n_2 + 2) \quad (83)$$
This is exactly the dimension of the \( (p = n_1, q = n_2) \) representation or more precisely

\[
p_1 = n_1, \quad p_2 = p_3 = 0, \quad q_1 = q_2 = 0, \quad q_3 = n_2.
\]

The last constraints arise from the polarization (or lowest Landau) condition (56). Therefore, from (42), the wavefunctions describing a free charged particle living on \( \mathbf{F}_2 \) in LLL are given by

\[
\psi_{\text{LLL}} = \langle (n_1, n_2)(s_1, s_2, s_3)(t_1, t_2, t_3) | g | \lambda \rangle
\]

where \( s_j, r_j \) \((j = 1, 2, 3)\) stand for the left quantum numbers of the states, which encode the degeneracy of LLL and satisfy the relation

\[
s_1 + s_2 + s_3 = n_1, \quad t_1 + t_2 + t_3 = n_2.
\]

In (85), \( |\lambda\rangle \) is the highest weight vector for the \( (n_1, n_2) \) unitary irreducible representation. As far as the flag manifold is concerned, one can identify the group element \( g \) with \( v \) given by (32). Consequently, the action of \( g \) on the state \( |\lambda\rangle \) gives the \( SU(3) \) coherent states discussed in section 2. Thus, the LLL wavefunctions coincide with the \( SU(3) \) coherent states associated to the mixed \((n_1, n_2)\) representation. They are given by [12]

\[
\Psi_{\text{LLL}}(u_1, u_2, u_3) = \left[ \frac{n_1!n_2!}{s_1!s_2!s_3!t_1!t_2!t_3!} \right]^{\frac{1}{2}} \Delta_1^{-\frac{n_1}{2}} \Delta_2^{-\frac{n_2}{2}} u_1^{s_1} u_2^{s_2} u_3^{s_3} (u_3 - u_1 u_2)^{t_1} u_2^{t_2}.
\]

It is interesting to note that the LLL wavefunctions are in correspondence with the zero modes of the Dirac operators on the flag manifold [13]. We recall that the LLL wavefunctions for complex projective space \( \mathbb{CP}^k \) [3, 5, 6] and Bergman ball \( \mathbb{B}^k \) [9] are, respectively, given by the coherent states of the groups \( SU(k + 1) \) and \( SU(k, 1) \) in the symmetric representations. Usually, the Perelomov coherent states for \( SU(3) \) mixed representation are

\[
|u_1, u_2, u_3\rangle = \sum \psi_{\text{LLL}}(u_1, u_2, u_3) |(n_1, n_2)(s_1, s_2, s_3)(t_1, t_2, t_3)\rangle
\]

where the sum runs over the quantum numbers labeling the LLL wavefunctions. They constitute an over-complete basis

\[
\int d\mu |u_1, u_2, u_3\rangle \langle u_1, u_2, u_3| = I
\]

where \( I \) is the identity operator and the measure

\[
d\mu = \frac{(n_1 + 1)(n_2 + 1)(n_1 + n_2 + 1)}{\pi^3 \Delta_1 \Delta_2} \prod_{i=1}^3 d^2 u_i
\]

is simply obtained from the \( SU(3) \) Haar measure by integrating over the angles \( \varphi_1 \) and \( \varphi_2 \), see (51), associated to the isotropy group \( U(1) \times U(1) \). The coherent states are not orthogonal and the overlapping given by

\[
\langle u'_1, u'_2, u'_3 | u_1, u_2, u_3 \rangle = \left[ \frac{1 + u'_1 u_1 + u'_3 u_3}{\sqrt{\Delta_1 \Delta'_1}} \right]^{n_1} \left[ \frac{1 + u'_2 u_2 + (u'_3 - u'_1 u'_2)(u_3 - u_1 u_2)}{\sqrt{\Delta_2 \Delta'_2}} \right]^{n_2}
\]

will be useful to deal with the incompressibility of a collection of \( N \) particles living on \( \mathbf{F}_2 \).
The \( N \)-body wavefunctions can be obtained as the Slater determinant

\[
\Psi^{(1)}_N = \epsilon^{i_1 \cdots i_N} \Psi_{i_1} \Psi_{i_2} \cdots \Psi_{i_N}
\]

where each \( \Psi_{i_j} \) has the form given by (87) and \( \epsilon^{i_1 \cdots i_N} \) is the fully antisymmetric tensor. This is the first Laughlin state corresponding to the filling factor \( \nu = 1 \). Other similar Laughlin states can be obtained as

\[
\Psi^{(m)}_N = \{ \epsilon^{i_1 \cdots i_N} \Psi_{i_1} \Psi_{i_2} \cdots \Psi_{i_N} \}^m
\]

where \( m \) is an odd integer value.

The definition of the filling factor

\[
\nu = \frac{N}{N_\phi}
\]

where \( N_\phi \) is the quantized flux and also represents the degrees of the Landau level degeneracy, tells us that the particle density is relevant in QHE and it should be kept constant by varying the magnetic field. In the first Laughlin state, i.e. \( \nu = 1 \), the density is given by

\[
\rho_0 = \frac{(n_1 + 1)(n_2 + 1)(n_1 + n_2 + 2)}{64\pi^3R^6}
\]

where we have introduced the radius \( R \) of \( F_2 \), such as

\[
Ru_\alpha = x_\alpha + ix_{\alpha+3}
\]

and considered the volume of the flag space as [13]

\[
\text{vol}(F_2) = 32\pi^3R^6.
\]

The thermodynamic limit corresponds to the situation in which the radius \( R \) and the number of available LLL states are large (\( R \to \infty, n_1, n_2 \sim n \to \infty \)). To determine the particle density in this limit, one may use the Dirac quantization for the flag manifold

\[
F_1 = \mathbb{C}P^1 = \frac{SU(2)}{U(1)}
\]

where the total magnetic field \( B \) is submitted to the constraint

\[
n = 2BR^2.
\]

From the above tools, the density can be approximated as

\[
\rho_0 \sim \left( \frac{B}{2\pi} \right)^3
\]

which is constant and has a finite value. This is exactly the particle density on the flat geometry \( \mathbb{R}^6 \) and therefore corresponds to the fully occupied state \( \nu = 1 \). It is also interesting to note that the obtained density coincides with that derived in the \( \mathbb{C}P^3 \) space [5] as expected since, in the limit of large radius, the geometry of both spaces is flat.

In QHE, the quantized plateaus come from the realization of an incompressible liquid. This property is important since it is related to the energy. It means that by applying an infinitesimal
pressure to an incompressible system the volume remains unchanged [15]. This condition can be checked for our system by calculating the two-point correlation function. This can be derived by integrating the density $\Psi_N^{(1)}\Psi_N^{(1)}$ over all particles except two. As result, we obtain

$$I(12) \sim 1 - |\langle u_{11}, u_{21}, u_{31}|u_{12}, u_{22}, u_{32}\rangle|^2$$  \hspace{1cm} (101)

in terms of the kernel of two states localized at the position $s (u_{1s}, u_{2s}, u_{3s})$, with $s = 1, 2$. Using (86) together with (87), it is easy to see that, in the limit $n_1, n_2 \rightarrow \infty$, the correlation function $I(12)$ goes like

$$I(12) \sim 1 - \exp \left( -n_1(|u_{11} - u_{12}|^2 + |u_{31} - u_{32}|^2) - n_2(|u_{21} - u_{22}|^2 + |u_{31} - u_{21} - u_{32} + u_{12}u_{22}|^2) \right).$$  \hspace{1cm} (102)

This result shows that for a large magnetic field ($n_1, n_2 \sim n = 2BR^2$), the $F_2$ quantum Hall system $\nu = 1$ is incompressible and the probability to find two particles at the same position vanishes, as it is usual in the flat geometry.

6 Semi-classical analysis on the lowest Landau levels

Recall that LLL of particles living on $F_2$ are described by the $SU(3)$ coherent states (88). This provides us with a simple way to establish a correspondence between operators and classical functions on the phase space of the present system for large magnetic fields. In this section, we investigate the semi-classical properties of a large collection of particles confined in LLL for $n_1$ and $n_2$ large. In particular, we derive the density distribution, the symbol associated to a product of two operators acting on LLL (the star product) and give the excitation potential inducing a degeneracy lifting. This will be useful in driving the edge excitations of a quantum Hall droplet in the Flag manifold.

6.1 Density matrix and Hall droplet

To investigate the classical behavior of a collection of particles in LLL, we first derive the mean value of the density matrix corresponding to an abelian droplet configuration for large magnetic fields. Since the coherent states (87) (LLL eigenfunctions) are labeled by four quantum occupation numbers, one may fill the LLL states with a large number of particles $M = N_1 + N_2 + M_1 + M_2$ such that the density operator is

$$\rho_0 = \sum_{s_1=0}^{N_1} \sum_{s_2=0}^{N_2} \sum_{t_1=0}^{M_1} \sum_{t_2=0}^{M_2} |s_1, s_2, t_1, t_2\rangle\langle s_1, s_2, t_1, t_2|$$  \hspace{1cm} (103)

where the states read as

$$|s_1, s_2, t_1, t_2\rangle \equiv |(n_1, n_2)(s_1, n_1 - (s_1 + s_2), s_2), (t_1, t_2, n_2 - (t_1 + t_2))\rangle.$$  \hspace{1cm} (104)

The mean value of the density matrix is defined by

$$\rho_0(\bar{u}, u) = \langle u|\rho_0|u\rangle$$  \hspace{1cm} (105)

with $u$ stands for the variables ($u_1, u_2, u_3$) labeling the $SU(3)$ coherent states. $\rho_0(\bar{u}, u)$ is the symbol associated with the density operator. As we are concerned with the situation when $n_1$ and $n_2$ are
large, we analyze the spacial shape of $\rho_0(\bar{u}, u)$. Thus, using (87-88), one obtains

$$\rho_0(\bar{u}, u) = \Delta_1^{n_1} \Delta_2^{n_2} \sum_{s_1=0}^{N_1} \sum_{s_2=0}^{N_2} \sum_{t_1=0}^{M_1} \sum_{t_2=0}^{M_2} n_1! n_2! \frac{|u_1|^{2s_1} |u_3|^{2s_2}}{s_1! s_2! t_1! t_2!} \frac{|u_3 - u_1 u_2|^{2t_1} |u_2|^{2t_2}}{(n_1 - (s_1 + s_2))! (n_2 - (t_1 + t_2))!}. \quad (106)$$

For $n_1$ and $n_2$ large, we get

$$\Delta_1^{n_1} \Delta_2^{n_2} = \exp(-n_1(|u_1|^2 + |u_3|^2)) \exp(-n_2(|u_3 - u_1 u_2|^2 + |u_2|^2)). \quad (107)$$

Furthermore, one can verify the relation

$$\sum_{s_1=0}^{N_1} \sum_{s_2=0}^{N_2} \frac{n_1!}{s_1! s_2!} \frac{|u_1|^{2s_1} |u_3|^{2s_2}}{(n_1 - (s_1 + s_2))!} = \sum_{s=0}^{N_1+N_2} \frac{(n_1(|u_1|^2 + |u_3|^2))^s}{s!}. \quad (108)$$
as well as

$$\sum_{t_1=0}^{M_1} \sum_{t_2=0}^{M_2} \frac{n_2!}{t_1! t_2!} \frac{|u_3 - u_1 u_2|^{2t_1} |u_2|^{2t_2}}{(n_2 - (t_1 + t_2))!} = \sum_{t=0}^{M_1+M_2} \frac{(n_2(|u_3 - u_1 u_2|^2 + |u_2|^2))^t}{t!}. \quad (109)$$

It follows that the term involving the sum in the expression of $\rho_0$ behaves like

$$\sum_{s=0}^{M} \frac{(n_1(|u_1|^2 + |u_3|^2) + n_2(|u_3 - u_1 u_2|^2 + |u_2|^2))^s}{s!}. \quad (110)$$

Combining (107) and (110), the density can be approximated by

$$\rho_0(\bar{u}, u) \simeq \Theta(M - (n_1(|u_1|^2 + |u_3|^2) + n_2(|u_3 - u_1 u_2|^2 + |u_2|^2))) \quad (111)$$

for a large number $M$ of particles. Clearly, $\rho_0(\bar{u}, u)$ is a step function for $n_1, n_2 \to \infty$ and $M \to \infty$ ($\frac{M}{n_1}, \frac{M}{n_2}$ fixed). Note that, a large magnetic field corresponds to a large radius $R$, see (96) and (99), one can identify $u_3 - u_1 u_2$ with $u_3$. Then, introducing the rescaled variables

$$z_1 = \sqrt{\frac{n_1}{n}} u_1, \quad z_2 = \sqrt{\frac{n_2}{n}} u_2, \quad z_3 = \sqrt{\frac{n_1 + n_2}{n}} u_3 \quad (112)$$

the density function takes the simple form

$$\rho_0(\bar{z}, z) \simeq \Theta(M - n\bar{z} \cdot z) \quad (113)$$

where dot stands for the usual scalar product and $n$ is related to the total magnetic field defined by (96). Clearly, (113) corresponds to a droplet configuration with boundary defined by $n\bar{z} \cdot z = M$ and its radius is proportional to $\sqrt{M}$. The derivative of this density tends to a $\delta$-function. This property play a crucial role in deriving the edge excitations, see next.

### 6.2 Star product and Moyal bracket

An important tool to write the action describing the edge excitations of a quantum Hall droplet in $\mathbf{F}_2$ is the star product. In fact for $n_1$ and $n_2$ large the mean value of the product of two operators leads to the Moyal star product. To show this, to every operator $A$ acting on LLL, we associate the function

$$\mathcal{A}(\bar{u}, u) = \langle u|A|u \rangle = \langle u_1, u_2, u_3|A|u_1, u_2, u_3 \rangle. \quad (114)$$
An associative star product of two functions \( \mathcal{A}(\bar{u}, u) \) and \( \mathcal{B}(\bar{u}, u) \) is defined by

\[
\mathcal{A}(\bar{u}, u) \star \mathcal{B}(\bar{u}, u) = \langle u|AB|u \rangle = \int d\mu(\bar{u}', u') \langle u|A|u' \rangle \langle u'|B|u \rangle
\]  

(115)

where the measure \( d\mu(\bar{u}, u) \) is given by (90). To calculate (115), we exploit the analytical properties of coherent states defined above. Indeed, using (87-88), one can see that the function

\[
\mathcal{A}(\bar{u}', u) = \frac{\langle u'|A|u \rangle}{\langle u'|u \rangle}
\]

(116)

satisfies the holomorphic and anti-holomorphic conditions:

\[
\frac{\partial}{\partial \bar{u}_i} \mathcal{A}(\bar{u}', u) = 0, \quad \frac{\partial}{\partial u_i} \mathcal{A}(\bar{u}', u) = 0, \quad i = 1, 2, 3, \quad u \neq u'.
\]

(117)

Consequently, the action of the translation operator on \( \mathcal{A}(\bar{u}', u) \) gives

\[
\exp \left( u' \frac{\partial}{\partial u} \right) \mathcal{A}(\bar{u}', u) = \mathcal{A}(\bar{u}', u + u').
\]

(118)

This gives \( \mathcal{A}(\bar{u}, u') \) in terms of the function \( \mathcal{A}(\bar{u}, u) \), namely

\[
\exp \left( -u \cdot \frac{\partial}{\partial u} \right) \mathcal{A}(\bar{u}, u) = \exp \left( (u' - u) \cdot \frac{\partial}{\partial u} \right) \mathcal{A}(\bar{u}, u) = \mathcal{A}(\bar{u}, u').
\]

(119)

Similarly, one obtains

\[
\exp \left( -\bar{u} \cdot \frac{\partial}{\partial \bar{u}'} \right) \mathcal{A}(\bar{u}, u) = \exp \left( (\bar{u}' - \bar{u}) \cdot \frac{\partial}{\partial \bar{u}} \right) \mathcal{A}(\bar{u}, u) = \mathcal{A}(\bar{u}', u).
\]

(120)

Equivalently, (119-120) can also be cast in the following forms

\[
\exp \left( (u' - u) \cdot \frac{\partial}{\partial u} \right) \mathcal{A}(\bar{u}, u) = \mathcal{A}(\bar{u}', u), \quad \exp \left( (\bar{u}' - \bar{u}) \cdot \frac{\partial}{\partial \bar{u}} \right) \mathcal{A}(\bar{u}, u) = \mathcal{A}(\bar{u}', u).
\]

(121)

Combining all we write the star product as

\[
\mathcal{A}(\bar{u}, u) \star \mathcal{B}(\bar{u}, u) = \int d\mu(\bar{u}', u') \exp \left( (u' - u) \cdot \frac{\partial}{\partial u} \right) \mathcal{A}(\bar{u}, u) |\langle u|u' \rangle|^2 \exp \left( (\bar{u}' - \bar{u}) \cdot \frac{\partial}{\partial \bar{u}} \right) \mathcal{B}(\bar{u}, u)
\]

(122)

where the overlapping of coherent states is given by (91). For large magnetic field, it can be expressed as

\[
|\langle u|u' \rangle| = 1 \text{ if and only if } u = u', \quad |\langle u|u' \rangle| < 1 \text{ and } |\langle u|u' \rangle| \rightarrow 0 \text{ for } n_1 \text{ and } n_2 \text{ large.}
\]

This provides us with a simple way to calculate the star product between two functions. Indeed, one can see from (123) that the quantity \( |\langle u|u' \rangle| \) gives contribution only in the domain near to point \( u' \simeq u \). It follows that the sum (122) can be evaluated by decomposing the integral near this point and integrating over \( \eta = u' - u \). Thus, we get

\[
\mathcal{A}(\bar{u}, u) \star \mathcal{B}(\bar{u}, u) = \int \frac{d\eta.d\bar{\eta}}{\pi^3} \exp \left( \eta \cdot \frac{\partial}{\partial u} \right) \mathcal{A}(\bar{u}, u) \exp \left( -s(\eta, \bar{\eta}) \right) \exp \left( \bar{\eta} \cdot \frac{\partial}{\partial \bar{u}} \right) \mathcal{B}(\bar{u}, u).
\]

(124)

where

\[
s(\eta, \bar{\eta}) = n_1(|\eta_1|^2 + |\eta_3|^2) + n_2(|\eta_3|^2 + |\eta_2|^2).
\]

(125)
Finally, by a direct calculation, one verifies that the star product between two functions is

$$A(\bar{u}, u) \star B(\bar{u}, u) = AB - \left( \frac{1}{n_1} \frac{\partial A}{\partial u_1} \frac{\partial B}{\partial \bar{u}_1} + \frac{1}{n_2} \frac{\partial A}{\partial u_2} \frac{\partial B}{\partial \bar{u}_2} + \frac{1}{n_1 + n_2} \frac{\partial A}{\partial u_3} \frac{\partial B}{\partial \bar{u}_3} \right) + O \left( \frac{1}{n^2} \right).$$  \hspace{1cm} (126)$$

Then, the symbol or function associated with the commutator of two operators $A$ and $B$

$$\langle u | [A, B] | u \rangle = \{A(\bar{u}, u), B(\bar{u}, u)\}_s$$  \hspace{1cm} (127)$$
is given in terms of the Moyal bracket

$$\{A(\bar{u}, u), B(\bar{u}, u)\}_s = A(\bar{u}, u) \star B(\bar{u}, u) - B(\bar{u}, u) \star A(\bar{u}, u).$$  \hspace{1cm} (128)$$

This will be helpful in building the WZW action describing the edge excitations.

### 6.3 Excitation potential

Note that LLL is degenerate and the degeneracy is given by (82). To generate excitations, we consider the Hamiltonian

$$H_0 = E_0 + V$$  \hspace{1cm} (129)$$

where $E_0$ is the LLL energy (82) and $V$ is the excitation potential defined by

$$V|s_1, s_2, t_1, t_2\rangle = \omega(s_1 + s_2 + t_1 + t_2)|s_1, s_2, t_1, t_2\rangle.$$  \hspace{1cm} (130)$$

The perturbation $V$ induces a lifting of the LLL degeneracy. Using (88), one can show that the symbol $\mathcal{V}(\bar{u}, u)$ associated to $V$ is

$$\langle u|V|u\rangle = \mathcal{V}(\bar{u}, u) = \omega(n_1(|u_1|^2 + |u_3|^2) + n_2(|u_3 - u_1 u_2|^2 + |u_2|^2)).$$  \hspace{1cm} (131)$$

It can also be written as

$$\mathcal{V} = n\omega \bar{z} \cdot z$$  \hspace{1cm} (132)$$

which is just the classical harmonic oscillator potential.

### 7 Edge excitations and WZW action

The quantum droplet under consideration is specified by the density matrix $\rho_0$. The excitations of this configuration can be described by an unitary time evolution operator $U$ which gives information concerning the dynamics of the excitations around $\rho_0$. The excited states will be characterized by a density operator:

$$\rho = U \rho_0 U^\dagger.$$  \hspace{1cm} (133)$$

In this section, we derive the effective action for excitations living on the edge of this quantum droplet. The derivation is based on semi-classical analysis presented in the previous section. As mentioned above, the dynamical information, related to degrees of freedom of the edge states, is contained in the unitary operator $U$. The corresponding action is [17]

$$S = \int dt \ Tr \left( \rho_0 U^\dagger (i\partial_t - H_0) U \right).$$  \hspace{1cm} (134)$$
It is compatible with the Liouville evolution equation for the density matrix
\[
\frac{i}{\hbar} \frac{\partial \rho}{\partial t} = [H_0, \rho] .
\] (135)

To write down an effective action describing the edge excitations, we evaluate the quantities occurring in (134) as classical functions on the basis of the semi-classical analysis performed above. Note that, the strategy adopted here is similar to those developed in references [4, 5, 9]. Indeed, we start by calculating the term \( i \int dt \, \text{Tr}(\rho_0 U^\dagger \partial_t U) \) with \( U = e^{i\Phi} \) and \( \Phi^\dagger = \Phi \). A direct calculation gives
\[
dU = \sum_{k=1}^{\infty} \frac{(i)^k}{k!} \sum_{p=0}^{k-1} \Phi^p d\Phi \Phi^{k-1-p} .
\] (136)

It leads
\[
U^\dagger dU = i \int_0^1 d\alpha \, e^{-i\alpha \Phi} d\Phi e^{i\alpha \Phi} .
\] (137)

Thus, we have
\[
e^{-i\Phi} \partial_t e^{+i\Phi} = i \int_0^1 d\alpha e^{-i\alpha \Phi} \partial_t \Phi e^{i\alpha \Phi} .
\] (138)

Using Baker-Campbell-Hausdorff formula, one can show
\[
i \int dt \, \text{Tr}(\rho_0 U^\dagger \partial_t U) = \int dt \sum_{k=0}^{\infty} \frac{-(i)^k}{(k+1)!} \text{Tr}([\Phi, \cdots [\Phi, \rho_0] \cdots ] \partial_t \Phi) .
\] (139)

Due to the coherent states completeness, the trace of any operator \( A \) is
\[
\text{Tr} A = \int d\mu(u, u) \langle u | A | u \rangle .
\] (140)

It follows that
\[
i \int dt \, \text{Tr}(\rho_0 U^\dagger \partial_t U) = \int d\mu dt \sum_{k=0}^{\infty} \frac{-(i)^k}{(k+1)!} \{ \Phi, \cdots [\Phi, \rho_0] \cdots \} \star \partial_t \Phi
\] (141)

where the star product and the Moyal bracket are those defined before. It is important to stress that \( \rho_0 \) and \( \Phi \) are now classical functions. It is easy to obtain
\[
i \int dt \, \text{Tr}(\rho_0 U^\dagger \partial_t U) \simeq -\frac{i}{2} \int d\mu dt \{ \Phi, \rho_0 \} \star \partial_t \Phi
\] (142)

here we have dropped terms containing the total time derivative as well as those of higher orders. We show that the Moyal bracket reads as
\[
\{ \Phi, \rho_0 \} \star = \frac{i}{n} (\mathcal{L}\Phi) \frac{\partial \rho_0}{\partial (\bar{z} \cdot z)}
\] (143)

where the first order differential operator is given by
\[
\mathcal{L} = i \left( z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}} \right).
\] (144)

in terms of the rescaled variables \( z_i \). Since the derivative of the density (113) is a \( \delta \)-function with support on the boundary \( \partial \mathcal{D} \) of the droplet \( \mathcal{D} \) defined by \( n \bar{z} \cdot z = M \), we get
\[
i \int dt \, \text{Tr}(\rho_0 U^\dagger \partial_t U) \simeq -\frac{1}{2} \int d\mu dt \delta(M - n \bar{z} \cdot z) (\mathcal{L}\Phi)(\partial_t \Phi) = -\frac{1}{2} \int_{\partial \mathcal{D} \times \mathbb{R}^+} dt \, (\mathcal{L}\Phi)(\partial_t \Phi).
\] (145)
Now we come to the valuation of the second term in the action (134). By a straightforward calculation, we obtain

$$\text{Tr}(\rho_0 U^\dagger VU) \simeq \text{Tr}(\rho_0 V) + i \text{Tr}([\rho_0, V]\Phi) + \frac{1}{2} \text{Tr}([\rho_0, \Phi][V, \Phi]).$$  \hspace{1cm} (146)

The first term in r.h.s of (146) ($\Phi$-independent) does not contain any information about the edge excitations of the Hall droplet. Thus, it can be ignored. The second term in r.h.s of (146) rewrites as

$$i \text{Tr}([\rho_0, V]\Phi) \approx i \int d\mu \{\rho_0, V\} \Phi$$  \hspace{1cm} (147)

in terms of the Moyal bracket. It is easy to see that the star product in (147) is zero and we have

$$i \text{Tr}([\rho_0, V]\Phi) \rightarrow 0.$$  \hspace{1cm} (148)

The last term in r.h.s of (146) gives

$$\frac{1}{2} \text{Tr}([\rho_0, \Phi][V, \Phi]) \approx -\frac{1}{2n} \int d\mu dt (\mathcal{L}\Phi) \frac{\partial \rho_0}{\partial (\bar{z} \cdot z)} (\mathcal{L}\Phi) \frac{\partial V}{\partial (\bar{z} \cdot z)}.$$  \hspace{1cm} (149)

Using (113) and (132), we find

$$\int dt \text{Tr}(\rho_0 U^\dagger HU) = \frac{\omega}{2} \int d\mu dt \delta(M - n\bar{z} \cdot z)(\mathcal{L}\Phi)^2.$$  \hspace{1cm} (150)

Note that we have eliminated the term containing the groundstate energy $E_0$ which does not contribute to the edge dynamics. Finally, combining all together to get

$$S \approx -\frac{1}{2} \int_{\partial D \times \mathbb{R}^+} d\mu dt \delta(M - n\bar{z} \cdot z)(\mathcal{L}\Phi) ((\partial_t \Phi) + \omega(\mathcal{L}\Phi)).$$  \hspace{1cm} (151)

This action involves only the time derivative of $\Phi$ and the tangential derivatives $\mathcal{L}\Phi$. It generalizes the chiral abelian WZW theory describing a bosonized theory of a system of a large number of fermions in two-dimensions [17]. It is interesting that the obtained WZW action is similar to one describing the edge excitations for Hall droplets in the six-dimensional complex projective $\mathbb{C}P^3$ [4] and in the Bergman Ball $B^3$ [9].

8 Conclusion and discussions

We have analyzed, through this paper, some aspects of the quantum Hall effect at the filling factor $\nu = 1$ on the flag manifold $\mathbb{F}_2$. More precisely, we have algebraically investigated the eigenvalue problem of a collection of $N$ non-interacting particles living on $\mathbb{F}_2$. We have shown, in quantizing the theory, that the wavefunctions write as the Wigner $D$-functions. They satisfy two constraints which are in correspondence with the $U(1)$ abelian gauge fields. Obtaining the energy levels is an easy task thanks to the $SU(3)$ representation theory. Also, we have derived the analytical expression of the Landau Hamiltonian describing the dynamics of a non-relativistic particle living on $\mathbb{F}_2$. We have clearly established that the lowest Landau level wavefunctions coincide with $SU(3)$ coherent states expressed in terms of the $\mathbb{F}_2$ coordinates. We have constructed the Laughlin states describing the fractional quantum Hall effect at $\nu = \frac{1}{m}$, with $m$ odd integer. For the state $\nu = 1$, we have shown for
large magnetic field that the particle density is finite as well as constant and the system behaves like an incompressible fluid.

On the other hand, we have analyzed the semi-classical properties of a large collection of particles confined in LLL for \( n_1 \) and \( n_2 \) large. In particular, we have derived the density distribution, the symbol associated to a product of two operators acting on LLL (the star product) and given the excitation potential inducing a degeneracy lifting. This is used to discuss the edge excitations of a quantum Hall droplet in the Flag manifold and constructing their Wess-Zumino-Witten action.

It is obvious, from previous analysis, that one can obtain the Landau spectrum of a system living on \( \mathbb{CP}^2 \). This can be performed by reducing the mixed representation \( (p, q) \) to the completely symmetric one \( (p, 0) \) or its adjoint \( (0, q) \). In this way, one recovers the results of Karabali and Nair [3]. Furthermore, because they are six dimensional, one can compare \( F_2 \) and \( \mathbb{CP}^3 \) analysis [3]. In the \( \mathbb{CP}^3 \) case, QHE can be approached following two different ways. The first one corresponds to the situation in which only one \( U(1) \) abelian gauge is involved. In the second situation, the particle evolves in the \( SU(2) \) magnetic background. Another interesting comparison concerns the particle density for large magnetic field. It is remarkable that, in both spaces \( F_2 \) and \( \mathbb{CP}^3 \), we find the same value which coincides with one obtained for particles living on \( \mathbb{R}^6 \).

To close this discussion, as examples of quantum systems submitted to two magnetic fields, we quote the composite fermions and multi-layers or a set of electrons and holes together. Composite fermions are a new kind of particles which appear in condensed matter physics to provide an explanation of the behavior of electrons moving in a strong magnetic field \( B \) [16]. Electrons possessing \( 2l\Phi_0, \ l = 1, 2, \cdots \), flux quanta (vortices) can be thought of being composite fermions. One of the most important features of them is that they feel effectively the magnetic field

\[
B^* = B - 2l\Phi_0\rho
\]  

(152)

where \( \rho \) is the electron density. This magnetic field can be seen as a superposition of two abelian parts.

Of course, the prolongations of the present work are numerous and some interesting questions still open. The first point concerns the analytical derivation of Landau spectrum (energies and wave-functions) by solving the Schrödinger equation for the Hamiltonian (74). The second question is related to a possible generalization of the present study to other higher dimensional flag manifolds, i.e. \( k \geq 3 \). Finally, one may ask about the topological excitations on the flag manifold generalizing those constructed by Haldane [18] on two-sphere \( S^2 \).

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References

[1] For instance see R.E. Prange and S.M. Girvin (editors), ”The Quantum Hall Effect” (Springer, New York 1990).

[2] S.C. Zhang and J.P. Hu, Science 294 (2001) 823, cond-mat/0110572; J.P. Hu and S.C. Zhang, cond-mat/0112432; S.C. Zhang, Quantum Hall effect in higher dimensions, (Talk given at the Conference on Higher Dimensional Quantum Hall Effect, Chern-Simons Theory and Non-Commutative Geometry in Condensed Matter Physics and Field Theory, 1-4/03/2005 AS-ICTP Trieste); J.P Hu and S.C. Zhang, Phys. Rev. B66 (2002) 125301; B.A. Bernevig, J.P. Hu, N. Thombas and S.C. Zhang, Phys. Rev. Lett. 91 (2003) 236803.

[3] D. Karabali and V.P. Nair, Nucl. Phys. B641 (2002) 533-546, hep-th/0203264; D. Karabali, Quantum Hall droplets on $\text{CP}^k$ and edge effective actions, (Talk given at the Conference on Higher Dimensional Quantum Hall Effect, Chern-Simons Theory and Non-Commutative Geometry in Condensed Matter Physics and Field Theory, 1-4/03/2005 AS-ICTP Trieste).

[4] D. Karabali and V.P. Nair, Nucl. Phys. B679 (2004) 427, hep-th/0307281; ibid B697 (2004) 513, hep-th/0403111; D. Karabali, hep-th/0605006; V.P. Nair, hep-th/0605007; Dimitra Karabali and V.P. Nair, hep-th/0606161.

[5] V.P. Nair and S. Randjbar-Daemi, Nucl. Phys. B679 (2004) 447, hep-th/0309212.

[6] D. Karabali, V.P. Nair and S. Randjbar-Daemi, Fuzzy spaces, the M(atrix) model and the quantum Hall effect, hep-th/0407007.

[7] M. Fabinger, JHEP 0205 (2002) 037, hep-th/0201016; B.A. Bernevig, J.P. Hu, N. Thombas and S.C. Zhang, Phys. Rev. Lett. 91 (2003) 236803; G. Sparling, cond-mat/0211679; Y.X. Chen, hep-th/0210059, hep-th/0209182; Y.X. Chen and B.Y. Hou, Nucl. Phys. B638 (2002) 220; H. Elvang and J. Polchinski, hep-th/0209104; Y.D. Chong and R.B. Laughlin, Ann. Phys. 308 (2003) 237; S. Bellucci, P.Y. Casteill and A. Nersessian, Phys. Lett. B574 (2003) 21; G. Meng, J. Phys. A36 (2003) 9415; A.P. Polychronakos, Nucl. Phys. B705 (2005) 457, hep-th/0408194; K. Hasebe and Y. Kimura, Phys. Lett. B602 (2004) 255; A.P. Polychronakos, Nucl. Phys. B711 (2005) 505, hep-th/0411065; G. Landi, hep-th/0504092; L. Mardoyan and A. Nersessian, Phys. Rev. B72 (2005) 233303, hep-th/0508062; R. Ahl Laamara, L.B. Drissi and El Hassan Saidi, hep-th/0604001; ibid hep-th/0605209.

[8] A. Jellal, Nucl. Phys. B725 (2005) 554, hep-th/0505095.

[9] M. Daoud and A. Jellal, Nucl. Phys. B764 (2006) 109, hep-th/0605289; Int. J. Geo. Meth. Mod. Phys. 4 (2007) 0407, hep-th/0605290.

[10] R.F. Picken, J. Math. Phys. 31 (1990) 616.

[11] S. Chaturvedi and N. Mukunda, J. Math. Phys. 43 (2002) 5262.

[12] M. Mathur and D. Sen, J. Math. Phys. 42 (2001) 4181.
[13] B.P. Dolan, *JHEP* **0305** (2003) 018.

[14] D. McMullen and I. Tsutsui, *Annals of Phys.* **237** (1995) 269, hep-th/9308027.

[15] Z.F. Ezawa, ”Quantum Hall Effects: Field Theoretical Approach and Related Topics” (World Scientific, Singapore 2000).

[16] J.V. Jain, *Phys. Rev. Lett.* **63** (1989) 199; *Phys. Rev. B** **41** (1990) 7653, *Adv. Phys.* **41** (1992) 105, O. Heinonen (editor), Composite Fermions: A Unified View of Quantum Hall Regime, (World Scientific, 1998).

[17] B. Sakita, *Phys. Lett.* **B315** (1993) 124; ibid **B387** (1996) 118.

[18] F.D. Haldane, *Phys. Rev. Lett.* **51** (1983) 605.