The Superparticle on the Surface $S_2$

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Abstract

A superparticle action which is globally supersymmetric in the target space is proposed. The supersymmetry is the supersymmetric extension of the rotation group $O(3)$.

1 Introduction

The superparticle action [1,2] was designed to be invariant under a supersymmetry transformation in the target space. This supersymmetry was taken to be the supersymmetric extension of the Poincaré group.

It is possible to make supersymmetric extensions of other groups; in particular groups of transformations on spaces of constant curvature [3]. In this paper we devise an action for a superparticle which has a target space invariance that is a supersymmetric extension of the rotation group $O(3)$, associated with the sphere $S_2$ defined by

$$ (x^1)^2 + (x^2)^2 + (x^3)^2 = r^2. \quad (1) $$

2 The Superparticle

The rotation group $O(3)$ leaves the quadratic form of eq. (1) invariant. The generators of this group, $J^a$ have the algebra defined by

$$ [J^a, J^b] = i\epsilon^{abc} J^c. \quad (2) $$

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One can extend this group [3,4] by introducing Fermionic spinor generators $Q_i, Q_i^\dagger$ plus an additional Bosonic generator $Z$ so that one has the algebras defined by the (anti-)commutators
\[
\{Q_i, Q_j^\dagger\} = Z \delta_{ij} \mp 2\tau^a_{ij} J^a \tag{3a}
\]
\[[J^a, Q_i] = -\frac{1}{2}(\tau^a Q)_i \tag{3b}
\]
\[[Z, Q_i] = \mp Q_i \tag{3c}
\]
in addition to eq. (2). The Pauli matrices $\tau^a$ satisfy
\[
\tau^a \tau^b = \delta^{ab} + i \varepsilon^{abc} \tau^c \tag{4a}
\]
\[
\tau^a_{ij} \tau^a_{kl} = 2\delta_{ik} \delta_{kj} - \delta_{ij} \delta_{kl} \tag{4b}
\]
\[
\tau^a_{ij} \delta_{kl} + \tau^a_{kl} \delta_{ij} = \tau^a_{ik} \delta_{lj} + \tau^a_{lj} \delta_{ik} \tag{4c}
\]
\[
\varepsilon^{abc} \tau_{ij} \tau^c_{kl} = i \left( \tau^a_{ij} \delta_{kj} - \tau^a_{kj} \delta_{ij} \right) \tag{4d}
\]
\[
\tau^a_{ij} \tau^b_{kl} = \frac{1}{2} \left[ \delta^{ab} \delta_{ik} \delta_{kj} + i \varepsilon^{abc} \left( \tau^a_{ik} \delta_{kj} - \delta_{ik} \tau^a_{kj} \right) + \tau^a_{ik} \tau^b_{kj} + \tau^b_{ik} \tau^a_{kj} - \delta^{ab} \delta_{ik} \delta_{kj} \right] \tag{4e}
\]
(There is an additional extension of the $O(3)$ algebra whose (anti-)commutators are
\[
\left\{Q_i^\dagger, Q_j^\dagger\right\} = \tau^a_{ij} J^a \quad \left\{Q_i^\dagger, Q_j^\dagger\right\} = \tau^a_{ij} Z^a \tag{5a, b}
\]
\[[J^a, Q_i] = -\frac{1}{2} \tau^a_{ij} Q_j \quad [Z^a, Q_i] = \frac{1}{2} Q^a_{ij} \tau^a_{ji} \tag{5c, d}
\]
\[[J^a, J^b] = i \varepsilon^{abc} J^c \quad [Z^a, Z^b] = -i \varepsilon^{abc} J^c \tag{5e, f}
\]
\[[J^a, Z^b] = i \varepsilon^{abc} Z^c \tag{5g}
\]
where $\tilde{Q} = Q^T \tau^2$ and $Z^a$ is a Bosonic vector.)

We now introduce Bosonic vector coordinates $x^a(\tau)$, a Bosonic scalar coordinate $\beta(\tau)$, and Fermionic spinor coordinates $\theta_i(\tau)$ for a superparticle moving along a trajectory parameterized by $\tau$ on the sphere $S_2$ defined by eq. (1). Next, we define
\[
Y^a = \dot{\beta} x^a - \beta \dot{x}^a + \varepsilon^{abc} x^b \ddot{x}^c + i \left( \dot{\theta}^\dagger \tau^a \theta - \theta^\dagger \tau^a \dot{\theta} \right) \tag{6}
\]
and note that $\delta Y^a = 0$ under the supersymmetry transformation
\[
\delta \beta = -\epsilon^\dagger \theta - \theta^\dagger \epsilon \tag{7a}
\]
\[
\delta \theta = (-i \tau \cdot x - \beta) \epsilon \tag{7b}
\]
\[
\delta \dot{\theta}^\dagger = \epsilon^\dagger (i \tau \cdot x - \beta) \tag{7c}
\]
\[
\delta x^a = i (\epsilon^\dagger \tau^a \theta - \theta^\dagger \tau^a \epsilon) \tag{7d}
\]
where $\epsilon$ is a constant Fermionic Dirac Spinor. In addition, the quantity
\[ R^2 = x^a x^a - 2\theta\theta + \beta^2 \] (8)
is also invariant under the transformation of eq. (7). For an ordinary massless particle moving on a sphere $S_2$ we would have the Lagrangian
\[ L_0 = \frac{1}{2e} (e^{abc} x^b x^c)^2 + \lambda (x^a x^a - r^2) \] (9)
where $\lambda(\tau)$ is a Lagrange multiplier and $e(\tau)$ is an einbein field. This we now generalize to
\[ L = \frac{1}{2e} Y^a Y^a + \lambda (x^a x^a - 2\theta\theta + \beta^2 - R^2). \] (10)
This Lagrangian is obviously invariant under the global supersymmetry transformation of eq. (7).

The generator of the transformation of eq. (7) is $\epsilon^\dagger Q + Q^\dagger \epsilon$ where
\[ Q = -\beta \frac{\partial}{\partial \beta} + (i\tau \cdot x - \beta) \frac{\partial}{\partial \theta^\dagger} + i\tau \cdot \nabla \theta \] (11a)
\[ Q^\dagger = -\theta \frac{\partial}{\partial \beta} + \frac{\partial}{\partial \theta^\dagger} (i\tau \cdot x + \beta) - i\theta \tau \cdot \nabla. \] (11b)
These generators appear also in refs. [3,4]. By using eq. (4), it can be verified that if
\[ \delta_i A = (\epsilon_i^\dagger Q + Q_i^\dagger \epsilon) A \] (12)
then the Jacobi identity
\[ ([\delta_1, [\delta_2, \delta_3]] + [\delta_2, [\delta_3, \delta_1]] + [\delta_3, [\delta_1, \delta_2]]) A = 0 \] (13)
is satisfied.

The canonical momenta conjugate to $(x^a, \beta, \lambda, e, \theta^\dagger, \theta)$ respectively are given by
\[ p^a = \frac{1}{e} (-\beta Y^a + e^{abc} Y^b x^c) \] (14a)
\[ p_\beta = \frac{1}{e} x^a Y^a \] (14b)
\[ p_\lambda = 0 \] (14c)
\[ p_e = 0 \] (14d)
\[ \pi = \frac{i}{e} (\tau \cdot Y \theta) \] (14e)
\[ \pi^\dagger = \frac{i}{e} (\theta^\dagger \tau \cdot Y). \] (14f)
(We use the left hand derivative for Fermionic variables.) From eqs. (14a,b) it follows immediately that we have the primary constraint

\[ \Pi = x^a p^a + \beta p_\beta = 0. \]  

(15)

Furthermore, from eq. (14a) we obtain

\[ Y^a = \frac{e}{x^2 + \beta^2} \left( -\beta \delta^{ab} + e^{apb} x^b - \frac{1}{\beta} x^a x^b \right) p^b \]  

(16)

and so eqs. (14d,e) result in two more primary constraints

\[ \chi = \pi - i \Xi \theta = 0 \]  

(17a)

\[ \chi^\dagger = \pi^\dagger - i \theta^\dagger \Xi = 0 \]  

(17b)

where

\[ \Xi = \frac{1}{x^2 + \beta^2} \left( -\beta p^a + e^{abc} x^b p^c - \frac{1}{\beta} x^a x^p \right) \tau^a. \]

Eq. (17) is immediately seen to provide a pair of primary second class constraints as \[ \{ \chi, \chi^\dagger \} = 2i \frac{\Lambda}{\Lambda + \beta} \left( p \cdot \tau^\dagger \theta - \theta^\dagger p \cdot \tau \theta \right) + 4i \frac{\Lambda}{\Lambda + \beta} \left( \Xi \theta^\dagger \theta - \theta^\dagger \Xi \theta \right) + 2i \Xi \]  

(18)

The canonical Hamiltonian for our system is given by

\[ H_c = \dot{\pi} = \dot{\pi}^T \pi^T + \dot{x}^a p^a + \dot{\beta} p_\beta - L 
= \frac{1}{2e} Y^a Y^a - \lambda \left( x^2 - 2\theta^\dagger \theta + \beta^2 - R^2 \right) \]  

(19)

which by eq. (16) becomes

\[ = \frac{1}{2} \frac{e}{x^2 + \beta^2} \left( p^2 + p_\beta^2 \right) - \lambda \left( x^2 - 2\theta^\dagger \theta + \beta^2 - R^2 \right) \]  

(20)

\[ \text{For} \ \text{Poisson Brackets (PB) involving Bosonic variables } B_i \ \text{and Fermionic variables } F_i, \ \text{depending on Bosonic canonical pairs } (q, p) \ \text{and Fermionic canonical pairs } (\psi, \pi), \ \text{we use the conventions} \]

\[ \{ B_1, B_2 \} = (B_{1,q} B_{2,p} - B_{2,q} B_{1,p}) + (B_{1,\psi} B_{2,\pi} - B_{2,\psi} B_{1,\pi}) \]

\[ \{ B, F \} = - \{ F, B \} = (B_{q,F} p - F_{q,B} p) + (B_{\psi,F} \pi + F_{\psi,B} \pi) \]

\[ \{ F_1, F_2 \} = (F_{1,q} F_{2,p} + F_{2,q} F_{1,p}) - (F_{1,\psi} F_{2,\pi} + F_{2,\psi} F_{1,\pi}) \].
Eqs. (14c,d; 15; 17a,b) are all primary; eqs. (14c,d) imply the secondary constraints

\[ \Sigma_1 = x^2 - 2\theta^\dagger\theta + \beta^2 - R^2 = 0 \quad (21a) \]

and

\[ \Sigma_2 = \frac{p^2 + p^2_\beta}{x^2 + \beta^2} = 0. \quad (21b) \]

The PB \{\Pi, \Sigma_2\} weakly vanishes but the PB \{\Pi, \Sigma_1\} does not. We note that by eq. (18), all Fermionic constraints are second class; since there are no first class Fermionic constraints, there is no local Fermionic symmetry that would be the analogue of the \(\kappa\)-symmetry discussed in ref. [5].

**Conclusions**

We have introduced a Lagrangian that is invariant under a global supersymmetry transformation that is an extension of the rotation group \(O(3)\).

We note that the stereographic projection

\[ x^a = \frac{2\eta^a r^2}{r^2 + \eta^2} \quad (a = 1, 2) \quad (22a) \]

\[ x^3 = r \left( \frac{r^2 - \eta^2}{r^2 + \eta^2} \right) \quad (22b) \]

can be used to map coordinates on the surface of a sphere \(S_2\) defined by eq. (1) onto the Euclidean plane defined by \((\eta^1, \eta^2)\). If instead of eq. (1), we have the constraint of eq. (21a), we would replace \(r^2\) in eq. (22) with \(R^2 + 2\theta^\dagger\theta - \beta^2\).

It would be of interest to find superparticle Lagrangians that are invariant under supersymmetric extensions of other symmetry groups associated with spaces of constant curvature, such as \(AdS\) and \(dS\) spaces. Quantization of such models would be non-trivial.

We note that there recently has been some interest in exploring supersymmetry in curved spaces; see for example ref. [9].

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