Logarithmic corrections to the FRW brane cosmology from 5d Schwarzschild-deSitter black hole

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ABSTRACT

Thermodynamics of 5d SdS black hole is considered. Thermal fluctuations define the (sub-dominant) logarithmic corrections to black hole entropy and then to Cardy-Verlinde formula and to FRW brane cosmology. We demonstrate that logarithmic terms (which play the role of effective cosmological constant) change the behavior of 4d spherical brane in dS, SdS or Nariai bulk. In particularly, bounce Universe occurs or 4d dS brane expands to its maximum and then shrinks. The entropy bounds are also modified by next-to-leading terms. Out of braneworld context the logarithmic terms may suggest slight modification of standard FRW cosmology.

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1 Introduction

The deSitter space always attracts much attention in cosmology and gravity. This is caused by several reasons. First of all, according to the theory of inflationary Universe the very early Universe eventually has passed deSitter (dS) phase. Second, recent astrophysical data indicate that modern Universe is (or will be in future) also in deSitter phase. Third, dS is very attractive from the theoretical point of view due to its highly symmetric nature (like flat space). This is also the reason why dS space was frequently considered as candidate for ground state in quantum gravity.

According to recent studies the dS quantum gravity should be quite unusual theory in many respects [1]. In connection with braneworld scenario [2] it is expected that there occurs dS/CFT correspondence [3, 4]. In one of its versions, dS/CFT correspondence indicates that properties of 5d classical dS space are related with those of dual CFT living on the four-dimensional boundary (which may be also dS). Despite the fact that explicit examples of such consistent dual CFTs are not constructed yet, one can still get a lot of information from dS/CFT correspondence. In particular, starting from five-dimensional Schwarzschild-deSitter (SdS) black hole (which should be relevant to the description of 4d dual CFTs at non-zero temperature) one can easily get the Friedmann-Robertson-Walker (FRW) brane cosmology. The corresponding FRW brane equation may be often written in so-called Cardy-Verlinde (CV) form [5].

There was much activity recently (see [6, 7]) in the study of FRW brane cosmology in CV form when bulk space is dS or SdS space and in the corresponding investigations of thermodynamical properties of dS black holes.

In the present paper we consider 5d SdS black hole and calculate the corresponding thermodynamical quantities. Taking into account thermal fluctuations defines the logarithmic corrections to both cosmological and black hole entropies. As a result the CV formula and FRW brane cosmology receive the (sub-dominant) logarithmic corrections, in the way similar to FRW brane cosmology in AdS black hole bulk [8]. It is interesting that such sub-dominant terms slightly change the entropy bounds appearing in CV formulation.

The paper is organized as follows. In the next section we find entropy, free energy and thermodynamical energy for SdS black hole which is the space with two horizons and for Nariai black hole (when both horizons coincide). Using the logarithmic corrections to the entropy the corrected CV formula is established. The corresponding FRW brane equation with logarithmic terms is found. Section three is devoted to the qualitative study of FRW brane cosmology where next-to-leading (logarithmic) terms play the role of small effective cosmological constant. It is explicitly demonstrated that 4d spherical brane behaves in a different way when log-terms are present. In particular, bounce Universe may occur or dS brane reaches its maximum and then shrinks. Its behavior depends also from the choice of bulk: dS, SdS or Nariai space. In section four we consider standard 4d FRW cosmology and show that even in this case, due to log-corrections to four-dimensional cosmological entropy the FRW equa-
tion and CV formula may get modified. The correction terms may be interpreted as dust. Some summary and outlook are given in the last section. General derivation of logarithmic corrections to entropy (due to thermal fluctuations) is presented in the Appendix A. Penrose diagram of SdS space is drawn in Appendix B.

2 Logarithmic Corrections to Cardy-Verlinde formula and FRW brane cosmology in SdS bulk

Let us consider the thermodynamics of SdS black hole in five dimensions. The SdS black hole is a constant curvature solution of the Einstein equation, which follows from the action

\[
S = \int d^5x \sqrt{-\hat{G}} \left\{ \frac{1}{16\pi G_5} \hat{R} + \Lambda \right\} .
\]

(1)

Here \(\hat{R}\) is the scalar curvature, \(\Lambda\) is the (positive) cosmological constant and \(G_5\) denotes the five–dimensional Newton constant.

The metric of the five–dimensional SdS black hole is given by

\[
ds_5^2 = \hat{G}_{\mu\nu} dx^\mu dx^\nu = -e^{2\rho} dt^2 + e^{-2\rho} da^2 + a^2 \sum_{ij} g_{ij} dx^i dx^j,
\]

\[e^{2\rho} = \frac{1}{a^2} \left( -\mu + \frac{k}{2} a^2 - \frac{a^4}{l^2} \right),\]

(2)

where \(l\) represents the curvature radius of the SdS bulk space and is related with the cosmological constant \(\Lambda = \frac{12}{l^2}\). The three-dimensional metric, \(g_{ij}\), is the metric of an Einstein space with Ricci tensor given by \(r_{ij} = kg_{ij}\), where the constant \(k = \{-2, 0, +2\}\) in our notations. The metric of the hypersurface with constant \(a\) is then negatively–curved, spatially flat, or positively–curved depending on the sign of \(k\). For SdS background, only \(k = 2\) is the static solution, but when \(k = -2, 0\), the coordinate \(a\) plays a role of the (second) time coordinate and the solutions are time-evolving and cosmological ones with a big-bang singularity at \(a = 0\).

The mass of the black hole is parametrized by a constant \(\mu\), and \(\mu\) can be expressed in terms of the horizon radius \(a_H\):

\[
\mu = a_H^2 \left( -\frac{a_H^2}{l^2} + \frac{k}{2} \right) .
\]

(3)

The horizon radius \(a_H\) is the solution of the equation \(\exp[2\rho(a_H)] = 0\), which corresponds to (3)

\[
a_H^2 = \frac{k l^2}{4} \pm \frac{1}{2} \sqrt{\frac{k^2 l^4}{4} - 4\mu l^2}
\]

(4)
Note that, when \( k = 2 \) SdS black hole has two horizons \( a_H \), that corresponds to the upper and lower signs in Eq. (1) (the cosmological and black hole horizons, respectively). Hereafter we denote black hole horizon by \( a_{BH} \) and cosmological one by \( a_{CH} \). When \( k = 0, -2 \), there is no horizon since the right-hand side in (1) becomes imaginary or negative for positive \( \mu \). Then in the following we consider mainly \( k = 2 \) case.

One can define two Hawking temperatures corresponding to the two horizons:

\[
T_H = \left| \frac{1}{4\pi} \frac{d e^{2\phi}}{d a} \right|_{a=a_{BH},a_{CH}} = \left\{ \begin{array}{ll}
\frac{1}{2a_{BH}} - \frac{a_{BH}}{2a_{CH}} & \text{for the black hole horizon} \\
\frac{a_{CH}}{2a_{BH}} + \frac{a_{BH}}{2a_{CH}} & \text{for the cosmological horizon}
\end{array} \right.
\]  

(5)

The Cardy-Verlinde (CV) formula [5] (see also [9]) is derived from the thermodynamical properties of the five–dimensional black hole. So let us summarize the calculation of the thermodynamical quantities like the free energy \( F \), the entropy \( S \), and the energy \( E \) by following the method in [10]. After Wick-rotating the time variable \( t \to i\tau \), the free energy \( F \) can be obtained from the action Eq.(1) as

\[
F = -TS
\]

The classical solutions for \( R \) and \( \Lambda \) are given by

\[
R = \frac{20}{l^2} \quad \text{and} \quad \Lambda = -\frac{12}{16\pi G_5 l^2}
\]

Then the classical action (1) takes the form

\[
S = \frac{8}{16\pi G_5 l^2} \int d^5x \sqrt{-G},
\]

\[
= \frac{W_3}{T} \frac{8}{16\pi G_5 l^2} \int_{a_H}^{\infty} da \ a^3,
\]

(6)

where \( W_3 \) is the volume of the unit three–sphere and \( \tau \) has a period of \( \frac{1}{T} \). The expression for \( S \) contains the divergence coming from large \( a \). In order to subtract the divergence, we regularize \( S \) by cutting off the integral at a large radius \( a_{\text{max}} \) and subtracting the solution with \( \mu = 0 \) in the same way as in [10]:

\[
S = \frac{W_3}{T} \frac{8}{16\pi G_5 l^2} \left\{ \int_{a_H}^{a_{\text{max}}} da \ a^3 - e^{\rho(a_{\text{max}}) - \rho(a_{\text{max}}; \mu=0)} \int_0^{a_{\text{max}}} da \ a^3 \right\}.
\]

(7)

The factor \( e^{\rho(a_{\text{max}}) - \rho(a_{\text{max}}; \mu=0)} \) is chosen so that the proper length of the circle which corresponds to the period \( \frac{1}{T} \) in the Euclidean time at \( a = a_{\text{max}} \) coincides with each other in the two solutions. Taking \( a_{\text{max}} \to \infty \), one finds

\[
F = \left\{ \begin{array}{ll}
-\frac{8W_3}{16\pi G_5 l^2} \left( \frac{\tau^2 \mu}{8} - \frac{a_{BH}^4}{4} \right) & \text{for the black hole horizon} \ (a_H = a_{BH}) \\
\frac{8W_3}{16\pi G_5 l^2} \left( \frac{\tau^2 \mu}{8} - \frac{a_{CH}^4}{4} \right) & \text{for the cosmological horizon} \ (a_H = a_{CH})
\end{array} \right.
\]

(8)

One can rewrite \( F \) by using Eq.(3) as

\[
F = \left\{ \begin{array}{ll}
\frac{W_3 a_{BH}^4}{16\pi G_5} \left( \frac{\tau^2 \mu}{l^2} + 1 \right) & \text{for black hole horizon} \ (a_H = a_{BH}) \\
-\frac{W_3 a_{CH}^4}{16\pi G_5} \left( \frac{\tau^2 \mu}{l^2} + 1 \right) & \text{for cosmological horizon} \ (a_H = a_{CH})
\end{array} \right.
\]

(9)
The entropy $S$ and the thermodynamical energy $E$ are

\[
S = -\frac{dF}{dT_H} = -\frac{dF}{da_H} \frac{da_H}{dT_H} = \frac{W_3 a_H^3}{4G_5} \text{ for both of black hole and cosmological horizons (}a_H = a_{BH}, a_{CH})
\]

\[
E = F + T_H S = \pm \frac{3W_3 \mu}{16\pi G_5} + \text{ for black hole horizon and } - \text{ for cosmological one}.
\]  

Note that there can be two definitions of the temperature, the entropy and the energy, associated with two horizons. From Appendix B we find that the future black hole horizon and the future cosmological horizon are causally separated from each other. Then it is clear that any particle, which exists between the black hole horizon and the cosmological horizon, always may pass through only one of two future horizons. The particle which crosses the black hole horizon observes the temperature and the entropy associated with the black hole horizon but the other particle which crosses the cosmological horizon observes the thermodynamical quantities associated with the cosmological horizon.

In this paper, we are interested primarily in the corrections to the entropy (10) that arise due to thermal fluctuations. The leading–order correction has been found for a generic thermodynamic system [11]. The entropy is calculated in terms of a grand canonical ensemble, where the corresponding density of states, $\rho$, is determined by performing an inverse Laplace transformation of the partition function$^4$. The integral that arises in this procedure is then evaluated in an appropriate saddle–point approximation. The correction to the entropy follows by assuming that the scale, $\epsilon$, defined such that $S \equiv \ln(\epsilon \rho)$, varies in direct proportion to the temperature, since this latter parameter is the only parameter that provides a physical measure of scale in the canonical ensemble. The final result is then of the form [12]:

\[
S = S_0 - \frac{1}{2} \ln C_v + \ldots ,
\]

where $C_v$ is the specific heat of the system evaluated at constant volume and $S_0$ represents the uncorrected entropy. The derivation of (11) is given in Appendix A.

In the case of the SdS black hole, the entropy is given by Eq. (10). The specific heat of the black hole is determined in terms of this entropy:

\[
C_v \equiv \frac{dE}{dT_H} = \frac{32a_H^2 - l^2}{2a_H^2 + l^2} S_0 .
\] 

The above expression (12) is valid for both of the black hole and cosmological cases. For consistency, the condition $a_H^2 > l^2/2$ should be satisfied to ensure that the specific

$^4$The reader is referred to Refs. [11, 12] for details and related discussion in Refs. [13].
heat is positive. In the limit $a_H^2 \gg l^2/2$, $C_v \approx 3S_0$, and this implies that

$$S = S_0 - \frac{1}{2} \ln S_0 + \cdots .$$

(13)

Using the form of the logarithmic correction to the entropy, it is now possible to derive the corresponding corrections to CV formula. We begin by recalling that the four–dimensional energy, which can be derived from the FRW equation of motion for a brane propagating in an SdS bulk is given by

$$E_4 = \pm \frac{3W_3 l \mu}{16\pi G_5 a}$$

(14)

(+ corresponds to the black hole horizon and − to the cosmological one) and is related to the five–dimensional energy of the bulk black hole such that $E_4 = (l/a)E$. This implies that the temperature $T$, associated with the brane should differ from the Hawking temperature by a similar factor:

$$T = \frac{l}{a} T_H = \left\{ \begin{array}{ll}
-\frac{a_{BH}}{\pi a} & \text{for the black hole horizon} \\
\frac{l}{2\pi a a_{BH}} & \text{for the cosmological horizon}
\end{array} \right.$$

(15)

In determining the corrections to the entropy, a crucial physical quantity is the Casimir energy $E_C$, defined in terms of the four–dimensional energy $E_4$, pressure $p$, volume $W = a^3W_3$, temperature $T$ and entropy $S$:

$$E_C = 3(E_4 + pW - TS).$$

(16)

This quantity vanishes in case that the energy and entropy are purely extensive, but in general, this condition does not hold. For the present discussion, the total entropy is assumed to be of the form, where the uncorrected entropy $S_0$, corresponds to that associated with the black hole in Eq. (10) (due to the dS/CFT correspondence). It then follows by employing (14) and (15) that the Casimir energy can be expressed in terms of the uncorrected entropy:

$$E_C = \pm \left( \frac{3a_H^2 W_3}{8\pi G_5 a} + \frac{3}{2} T \ln S_0 \right),$$

(17)

where the direct dependence on the pressure has been eliminated by assuming the relation $p = E_4/(3W_3)$, which tells the 4d matter is the conformal one.

Moreover, in the limit where the logarithmic correction in Eq. (17) is small, it can be shown, after substitution of Eqs. (10), (14) and (17), that the four–dimensional and Casimir energies are related to the uncorrected entropy by

$$\frac{4\pi a}{3\sqrt{2}} \left[ E_C \left( E_4 - \frac{1}{2} E_C \right) \right] \sim S_0 + \frac{\pi a l}{2a_H^3} T \left( \frac{a_H^4}{l^2} + a_H^2 \right) \ln S_0.$$
In the limit where the correction is small, the coefficient of the logarithmic term on the right-hand side of Eq. (18) can be expressed in terms of the four-dimensional and Casimir energies through the relationship:

\[ \frac{\pi al}{2a_H^3} T \left( \frac{a_H^4}{l^2} + a_H^2 \right) = \frac{(4E_4 - E_C)(E_4 - E_C)}{2(2E_4 - E_C)E_C}, \]  

(19)

where we have substituted Eq. (15) for the temperature and have also employed the relation

\[ \frac{E_4 - \frac{1}{2}E_C}{E_C} = -\frac{a_H^2}{2l^2}. \]  

(20)

We may conclude, therefore, that in the limit where the logarithmic corrections are sub–dominant, Eq. (18) can be rewritten to express the entropy in terms of the four–dimensional and Casimir energies (corrected Cardy-Verlinde formula derived in Refs. [8, 15] for AdS black hole):

\[ S_0 \approx \frac{4\pi a}{3\sqrt{2}} \sqrt{E_C \left( E_4 - \frac{1}{2}E_C \right)} \]  

\[ -\frac{(4E_4 - E_C)(E_4 - E_C)}{2(2E_4 - E_C)E_C} \ln \left( \frac{4\pi a}{3\sqrt{2}} \sqrt{E_C \left( E_4 - \frac{1}{2}E_C \right)} \right) \]  

(21)

and, consequently, the total entropy Eq. (13) to first order in the logarithmic term, is given by

\[ S \approx \frac{4\pi a}{3\sqrt{2}} \sqrt{E_C \left( E_4 - \frac{1}{2}E_C \right)} \]  

\[ -\frac{E_4 (4E_4 - 3E_C)}{2(2E_4 - E_C)E_C} \ln \left( \frac{4\pi a}{3\sqrt{2}} \sqrt{E_C \left( E_4 - \frac{1}{2}E_C \right)} \right). \]  

(22)

It then follows that the logarithmic corrections to CV formula are given by the second term in the right hand side of Eq. (22) and the magnitude of this correction can be deduced by taking the logarithm of the original CV formula. As we saw in above discussion these corrections are caused by thermal fluctuations of the dS black hole.

If we consider the brane universe in the SdS black hole background, the four–dimensional FRW equation, which describes the motion of the brane universe, also receives corrections as a direct consequence of the logarithmic correction arising in Eq. (22). In general, the Hubble parameter \( H \) is related with the four–dimensional (Hubble) entropy (which is identified with bulk black hole entropy, see corresponding proof for AdS bulk in Ref. [14] and for dS bulk in [7]. For a brief review in the context of the brane world, see Appendix C)

\[ H^2 = \left( \frac{2G_4}{W} \right)^2 S^2, \]  

(23)
Here the effective four-dimensional Newton constant $G_4$ is related to the five-dimensional Newton constant $G_5$ by $G_4 = 2G_5/l$. The formula (23) is correct in both cases: when the brane crosses black hole horizon ($a = a_{BH}$) and when the brane crosses the cosmological one ($a = a_{CH}$). As we will see in the next section, we extend the equation corresponding to (23) to the case for $a \neq a_{BH}, a_{CH}$. The extended equation (36) coincides with (23) in both cases $a = a_{BH}$ and $a = a_{CH}$. By substituting Eq. (22) into Eq. (23), it can be shown by employing Eqs. (3), (10), (15), (17), (21) and (22) that the four-dimensional FRW equation is

$$H^2 = \left(\frac{2G_4}{W}\right)^2 \left[\left(\frac{4\pi a}{3\sqrt{2}}\right)^2 |E_C\left(E_4 - \frac{1}{2}E_C\right)| - \frac{4\pi a}{3\sqrt{2}} \left(2E_4 - E_C\right) E_C\right]$$

$$\times \sqrt{E_C\left(E_4 - \frac{1}{2}E_C\right)} \ln\left(\frac{4\pi a}{3\sqrt{2}} \left|E_C\left(E_4 - \frac{1}{2}E_C\right)\right|\right)$$

$$= \frac{1}{a_H^2} - \frac{8\pi G_4}{3} \rho - \frac{2G_4}{Wl} \ln S_0. \quad (24)$$

Here the logarithmic corrections have been included up to first-order in the logarithmic term, the effective energy density is defined by $\rho = |E_4|/W$ and $W = a_H^3 W_3$ parametrizes the spatial volume of the world-volume of the brane\(^5\). In the limit where the scale factor $a$ of the brane coincides with the horizon radius $a_H$ of the black hole, the first and second terms on the right-hand-side of Eq. (24) are identical to the FRW equation for the space-like brane in SdS background whose signs of the terms are the opposite to SAdS background [7, 15]. It is interesting that even if we did not assume the space-like brane, the Hubble equation (24) agrees with the case for space-like brane which is the brane for the Wick-rotated version of standard FRW equation.

Then the logarithmic corrections for the FRW equation are given by the third term on the right-hand-side in terms of the uncorrected entropy (10) of the black hole.

In the usual four-dimensional cosmology, the (first) FRW equation is given by

$$H^2 = \frac{8\pi G}{3} \rho - \frac{1}{a^2} ,$$

$$\rho = \rho_m + \frac{\Lambda}{8\pi G} . \quad (25)$$

Here $\Lambda$ is a cosmological constant and $\rho_m$ corresponds to the energy density of the matter. Typically in case that the matter is radiation, $\rho$ is proportional to $1/a^4$. Then

\(^5\)Since, at least, the brane receives the thermal radiation from the black hole, the thermal correction should change the dynamics of the brane from the leading order or zero temperature behavior. Since the detailed mechanism is not clear, here we naively assume that Eq. (23) is valid even if we include the thermal logarithmic correction.
Eq. (24) tells that, if we neglect the logarithmic correction, the obtained energy density corresponds to the radiation and the cosmological constant should vanish. On the other hand, by comparing (24) and (25), the logarithmic correction can be regarded as a small effective cosmological constant by identifying

\[ \Lambda_{\ln} = -\frac{6G_4}{W_l} \ln S_0 , \]

although it depends on the size of the universe since \( W \propto a^3 \). As \( \rho \) behaves as \( \frac{1}{a^4} \), the \( \Lambda_{\ln} \propto \frac{1}{a^3} \) varies slowly with the expansion of the universe when \( a \) is small. For large \( a \), where the approximation used here might not be valid, \( \Lambda_{\ln} \) becomes dominant if compared with the radiation.

Particularly, we consider the Nariai black hole which is the most simple case. In this case, the second term of (24) is zero, namely \( a_H^2 = \frac{l^2}{2} \). Then black hole horizon coincides with cosmological horizon.

The Hubble equation (24) for this case looks

\[ H^2 = \frac{1}{l^2} - \frac{16G_4}{2^{3/2}W_3^4 l^4} \ln S_0 , \]

\[ S_0 = \frac{W_3^2 3^{3/2} l^2}{16G_4} . \]

In the Nariai limit where \( \mu = \frac{l^2}{4} \), the expression (5) for the Hawking temperature seems to vanish but this might not be true since \( e^{2\rho} = 0 \) in the region between the black hole and cosmological horizons, which tells that the coordinates \( t \) and \( a \) are degenerate or ill-defined in the region. Since the SdS solution is not asymptotically flat, there is an ambiguity to rescale the time coordinate by a constant factor. We now introduce the following new coordinates \( \tilde{a} \) and \( \tilde{t} \):

\[ a^2 = \frac{l^2}{2} + \frac{\tilde{a}}{2} \sqrt{l^4 - 4\mu l^2} , \quad t = \frac{\tilde{t}}{\sqrt{l^4 - 4\mu l^2}} , \]

Then there are horizons at \( \tilde{a} = \pm 1 \) and the metric has the following form:

\[ ds^2 = -\frac{(1 - \tilde{a}^2) d\tilde{t}^2}{4l^2 \left( \frac{l^2}{2} + \frac{\tilde{a}}{2} \sqrt{l^4 - 4\mu l^2} \right)} + \frac{l^2 d\tilde{a}^2}{4(1 - \tilde{a}^2)} \]

\[ + \left( \frac{l^2}{2} + \frac{\tilde{a}}{2} \sqrt{l^4 - 4\mu l^2} \right) \sum_{ij} g_{ij} dx^i dx^j , \]

and the Hawking temperature \( T_H \) is also rescaled as

\[ T_H \to \tilde{T}_H \equiv \frac{T_H}{\sqrt{l^4 - 4\mu l^2}} = \frac{1}{2\pi l^2 a_H} . \]
Then in the Nariai limit, the metric has the following form:

\[ ds^2 = -\frac{(1 - \tilde{a}^2)}{2l^4}dt^2 + \frac{l^2d\tilde{a}^2}{4(1 - \tilde{a}^2)} + \frac{l^2}{2} \sum_{ij}^{3} g_{ij}dx^idx^j, \]

(32)

and the rescaled Hawking temperature is finite

\[ T_H = \frac{1}{\pi l^3 \sqrt{2}}. \]

(33)

If we further rewrite the coordinate \( \tilde{a} \) as \( \tilde{a} = -\cos \theta \), we obtain

\[ ds^2 = -\frac{\sin^2 \theta}{2l^4}dt^2 + \frac{l^2d\theta^2}{4} + \frac{l^2}{2} \sum_{ij}^{3} g_{ij}dx^i dx^j, \]

(34)

which might be a standard form of the metric in the Nariai space.

3 Qualitative Dynamics of the Brane Cosmology

In this section, we investigate the asymptotic behavior of the FRW brane cosmology when the logarithmic corrections to the CV formula are included. Formally, the FRW equation (24) holds precisely at the instant when the brane crosses black hole and cosmological horizons. Here we extend the analysis to consider an arbitrary scale factor \( a \) where the world–volume of the brane is given by the line–element

\[ ds_4^2 = d\tau^2 + a^2(\tau)g_{ij}dx^i dx^j. \]

Thus, around each horizon we assume the FRW equation as follows:

\[ H^2 = \frac{1}{a^2} - \frac{8\pi G_4}{3} \rho - \frac{2G_4}{Wl} \ln S_0, \]

(35)

where \( W = a^3W_3 \) and \( S_0 = W_3a^3/(4G_5) \). This equation differs by several signs from the corresponding FRW brane equation with log-corrections obtained in Ref. [8] where bulk is AdS black hole. On the black hole and cosmological horizon Eq. (35) agrees with Eq. (24). We also extend the result of the previous section to the general \( k \):

\[ H^2 = \frac{k}{2a^2} - \frac{8\pi G_4}{3} \rho - \frac{2G_4}{Wl} \ln S_0, \]

(36)

Eq. (36) can be rewritten in such a way that it represents the conservation of energy of a point particle moving in a one–dimensional effective potential, \( V(a) \):

\[ \left( \frac{da}{d\tau} \right)^2 = \frac{k}{2} - V(a) \]

(37)

\[ V(a) = \frac{8\pi G_4}{3} a^2 \rho + \frac{2G_4 a^2}{Wl} \ln S_0 \]

\[ = \frac{\mu}{a^2} + \frac{2G_4}{W_3a} \ln \left( \frac{W_3a^3}{2lG_4} \right), \]

(38)
where, in this interpretation, the variable \( a \) represents the position of the particle. Since \( \rho \propto a^{-4} \), the first term in the effective potential (38) redshifts as \( a^{-2} \) as the brane moves away from the black hole horizon. This term is often referred to as the ‘dark radiation’ term.

To proceed in the analogy with [8], let us briefly recall the behavior of the standard FRW cosmology, whose effective potential includes only the first term on the right-hand side of Eq. (38). The behavior of this potential is illustrated in Figure 1. The brane exists in the regions where the line \( k/2 \geq V(a) \) (so that \( H^2 > 0 \)). Then, we only have the case of \( k = 2 \) which is the spherical brane. The spherical brane starts from \( a = \infty \) and reaches its minimum size at \( a = a_{\text{min}} = \sqrt{\mu} \) and then it re-expands.

From Eq. (14) the energy density \( \rho = E_4/W \) looks like \( \frac{3\mu}{8\pi G_4 a^4} \), then the first term in Eq. (38) is rewritten as \( \mu/a^2 \):

\[
V(a) \equiv \frac{\mu}{a^2} + \frac{2G_4}{W_3 l a} \ln S_0 .
\]

For the Nariai black hole, the mass \( \mu \) takes the particular value \( \mu = \frac{k^2 l^2}{16} \) which is the largest mass for the SdS black hole, since the inside of square root in Eq. (4) must be positive. Then the behavior of the potential for Nariai BH is bigger than in SdS case as illustrated by thin line in Figure 2 so that its minimum size \( a_{\text{min}} \) is bigger than the minimum size of SdS.

Next, one considers the behavior of the effective potential with logarithmic corrections. From Eq. (39), there are several cases which depend on the parameters \( G_4, W_3, l, \mu \). If the coefficient \( \frac{2G_4}{W_3 l a} \) of second term in Eq. (39) is equal to or less than \( \sqrt{\mu} \), the behavior of the potential is not so changed from Figure 1. But when the
coefficient $\frac{2G_4}{W_3 l}$ of the second term in Eq. (39) is large compared with $\sqrt{\mu}$, the behavior of the potential changes from thin line to thick line as illustrated in Figure 2.

Figure 2: The effective potential for the FRW Universe in SdS bulk when logarithmic corrections are included. There are several cases which depend on the parameters $G_4, W_3, l$.

For Nariai black hole, the ratio of $\sqrt{\mu}$ and the coefficient of the second term in Eq. (39) is smaller than that of SdS case since $\mu$ is always bigger than the mass of SdS black hole. Then the behavior of the effective potential is similar to Figure 2 but smaller than that of SdS case.

As an explicit example one can take five-dimensional deSitter background instead of SdS background. The dS metric is given by

$$ds_5^2 = -e^{2\rho}dt^2 + e^{-2\rho}da^2 + a^2 \sum g_{ij}dx^idx^j,$$

$$e^{2\rho} = 1 - \frac{a^2}{l^2},$$

which is the massless case $\mu = 0, k = 2$ in Eq. (40). Then, horizon radius looks like $a_H = l$. From Eq. (24), the FRW equation for dS case also takes simple form as

$$H^2 = \frac{1}{a_H^2} - \frac{2G_4}{Wl} \ln S_0,$$

where $S_0 = \frac{W_3 a^3}{4G_4 \mu}$.

One can extend the FRW equation to general $a$ and $k$:

$$H^2 = \frac{k}{2a^2} - \frac{2G_4}{Wl} \ln S_0.$$
Here $S_0 = \frac{W_0^3}{4G_5}$, $W = a^3W_3$, again. This equation defines the effective potential $V(a)$ as

$$V(a) = \frac{2G_4}{aW_3l} \ln S_0.$$  \hfill (43)

The behavior for the effective potential for dS bulk is illustrated in Figure 3. When $k = 0, -2$, the brane starts from $a = 0$ and reaches its maximal size $a_{\text{max}}$ and then it re-collapses. Note that the behavior of the effective potential with logarithmic correction for FRW universe in deSitter bulk differs from the one in SdS bulk (Figure 2). If the maximum of the effective potential $V_{\text{max}}$ is larger then 1, there are two solutions for $k = 2$ case. In one case, the brane started at $a = 0$ reaches its maximum and shrinks. In another case, the brane started at $a = \infty$ shrinks and reaches its minimum and reexpands, which is the bounce universe case.

When there are no logarithmic corrections, Eq.(42) has a simple form:

$$\dot{a}^2 = k^2,$$  \hfill (44)

Then when $k = -2$, there is no solution, when $k = 0$, the brane is static ($\dot{a} = 0$). When $k = 2$, the solution is given by

$$a = |\tau|.$$  \hfill (45)

The solution has a singularity at $\tau = 0$. The second, logarithmic correction term in (42) can be neglected compared with the first term for large $a$. Then the behavior of the brane with $k = 2$ when $a$ is large is given by (45) even if we include the logarithmic correction. When $a$ is small, the logarithmic correction becomes dominant. Then (42) can be approximated as

$$\dot{a}^2 \sim -\frac{2G_4}{aW_3l} \ln \frac{W_3a^3}{4G_5},$$  \hfill (46)

Then $a$ behaves as $a \sim |\tau|^{\frac{2}{3}}$ (by approximating $\ln a$ to be constant, compared with $\frac{1}{a}$).

Thus, we demonstrated that the evolution of spherical brane which could correspond to our observable early Universe depends explicitly from the choice of bulk (dS, SdS or Nariai space) and from the inclusion (or not) of log-corrections.

4 Logarithmic corrections for four-dimensional FRW cosmology

In this section, we forget for the moment about the braneworld and discuss the role of logarithmic corrections to usual 4d cosmology and to 4d CV formula (see [5] and [16] for CV formula in 4d dS space). One starts from the Einstein gravity with
Figure 3: The behavior of the effective potential for FRW Universe in deSitter bulk with logarithmic corrections. There are two types of behavior for spherical brane $k = 2$. For the case of thick line, the brane starts from $a = 0$ and reaches its maximal size $a = a_{\text{max}}$ and then it re-collapses, or the brane starts from $a = \infty$ and reaches its minimum size at $a = a_{\text{min}}$ then it re-expands. For the case of thin line, the brane starts from $a = 0$ and expands to infinity.

Positive cosmological constant $\Lambda_4 > 0$. Then the standard FRW equation has the following form:

$$H^2 = \frac{8\pi G_4}{3} \rho_m - \frac{1}{a^2} + \frac{1}{l^2}.$$  \hspace{1cm} (47)

Here $\rho_m$ is the energy density of the matter and the length parameter $l$ is given by $\Lambda_4 = \frac{3}{l^4}$. We also consider only $k = 2$ case. If the matter energy can be neglected as $\rho_m \ll \frac{3}{8\pi G_4 l^2}$, the spacetime becomes deSitter space (in the static coordinates)

$$ds^2 = -e^{2\rho} dt^2 + e^{-2\rho} da^2 + a^2 d\Omega_2^2, \hspace{0.5cm} e^{2\rho} \equiv 1 - \frac{a^2}{l^2}.$$  \hspace{1cm} (48)

Here $d\Omega_2^2$ is the metric of the two-sphere. Then the cosmological horizon is given by $a = a_H \equiv l$ and the Hawking temperature $T_H$ is defined as

$$T_H = \left. \frac{1}{4\pi} \left| \frac{de^{2\rho}}{da} \right| \right|_{a=l} = \frac{1}{2\pi l}.$$  \hspace{1cm} (49)

Then the entropy is found to be

$$S_0 = \frac{\pi l^2}{G_4}.$$  \hspace{1cm} (50)

The expression for the Casimir energy $E_C$

$$E_C = 3 \left( E + pV - TS \right),$$  \hspace{1cm} (51)
suggests that the logarithmic correction to the entropy

\[ S_0 \rightarrow S_0 - \frac{1}{2} \ln S_0 \]  

may shift the energy by

\[ \delta E = -\frac{T}{2} \ln S_0 = -\frac{1}{4\pi l} \ln \frac{\pi l^2}{G_4}. \]  

This suggests the following modification of FRW equation (51)

\[ H^2 = \frac{8\pi G_4 \frac{\delta E}{V} - \frac{1}{a^2} + \frac{1}{l^2}}{3} = -\frac{G_4}{3\pi^2 l a^3} \ln \frac{\pi l^2}{G_4} - \frac{1}{a^2} + \frac{1}{l^2}. \]  

Here \( V = 2\pi^2 a^3 \) is the volume of the three-sphere. The correction to the energy effectively shifts the cosmological constant and might be dominant for small \( a \). When \( \frac{\pi l^2}{G_4} > 1 \), the effective cosmological constant decreases due to the correction and when \( \frac{\pi l^2}{G_4} > 0 \), it increases. The \( a^{-3} \) behavior of \( \frac{\delta E}{V} \) tells the correction part of the energy behaves as the (effective) dust where the pressure vanish \( p = 0 \).

If we assume \( E = \delta E \) (in the absence of matter), by using (51) one obtains an expression for the Casimir energy:

\[ E_C = -\frac{24\pi^2 l}{G_4}. \]  

The expression (55) is not changed by the logarithmic correction. Then, using (50) one gets

\[ S_0 = \frac{l}{24\pi} |E_C|, \]  

or using (52), we may obtain

\[ S \equiv S_0 - \frac{1}{2} \ln S_0 = \frac{l}{24\pi} |E_C| - \frac{1}{2} \ln \left( \frac{l}{24\pi} |E_C| \right), \]  

which corresponds to the Cardy-Verlinde formula in the situation under consideration. Even if the logarithmic correction is not included, the formula is rather different from the usual one:

\[ S = \frac{l}{24\pi} |E_C|. \]  

We may compare the expression (57) with the Cardy-Verlinde formula with the logarithmic correction (22), which has been obtained in the braneworld context. By putting \( E_4 = 0 \) in (22), the logarithmic correction vanishes and we obtain

\[ S \approx \frac{2\pi a}{3} |E_C|, \]  

which is similar to (58), rather than (57), and identical with (58) if we put \( a = \frac{l}{16\pi} \).

Thus, we found that logarithmic correction to the entropy may lead to inducing of small effective cosmological constant in FRW equation. Eventually, this may have some cosmological applications.
5 Discussion

In the present paper we discussed the role of logarithmic corrections which appear in SdS black hole entropy to FRW brane cosmology. The relation between black hole entropy and Hubble parameter is controlled by dS/CFT correspondence. These, next-to-leading corrections in FRW equation may be interpreted as small effective cosmological constant which qualitatively changes the evolution of spherical brane. The examples of the spherical brane evolution are presented without (or with) logarithmic terms and for different bulk: dS, SdS or Nariai space. Eventually, if our brane FRW Universe is embedded into SdS bulk (or AdS black hole [8]), these next-to-leading terms may be very important in cosmology as we explicitly demonstrated.

Let us now comment their role in the entropy bounds estimations. If we define the four-dimensional Hubble, Bekenstein-Hawking and Bekenstein entropies by [5]

\[ S_H \equiv \frac{H W}{2G_4}, \quad S_{BH} \equiv \frac{W}{4G_4 a_H}, \quad S_B \equiv \frac{2\pi a}{3} \rho W, \quad (60) \]

we can rewrite (24) as

\[ S_H^2 = (S_{BH} - S_B)^2 - S_B^2 - S_C \ln S_C. \quad (61) \]

Here \( S_C \) is defined, in a similar way to \( S_{BH} \), by

\[ S_C \equiv \frac{W}{2G_5 l} = \frac{W}{4G_5}, \quad (62) \]

which may give a lower limit of \( S_{BH} \) since \( a_H < l \) and we have \( a_H = l \) for cosmological horizon in pure deSitter space \( (\mu = 0) \). Eq.(62) tells that \( S_C \) is the entropy of the 5d black hole, whose horizon area is equal to the space-like volume of the brane. We also note that if we conjecture the redefined Bekenstein-Hawking entropy as

\[ S_{BH} \rightarrow \hat{S}_{BH} = S_{BH} + \frac{S_C \ln S_C}{2(S_{BH} - S_B)}; \quad (63) \]

Eq.(24) or (61) can be rewritten as

\[ S_H^2 = (\hat{S}_{BH} - S_B)^2 - S_B^2. \quad (64) \]

This looks as standard CV formula which defines the entropy bounds in expanding Universe but the interpretation of Bekenstein-Hawking entropy is changed.

Finally, one can note that similar considerations are applied in the study of FRW or anisotropic brane cosmology with logarithmic corrections for another types of dS bulk black holes.
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Appendix

A Logarithmic corrections to the entropy

In this section, we review briefly the calculation of the log correction to the entropy. First, let us recall the expression for the partition function in the grand canonical ensemble by

$$Z(\beta) = \int e^{-\beta E} \rho(E) dE.$$  \hspace{1cm} (65)

Here $\rho(E)$ is the density of states, $\beta$ is the inverse temperature, $\beta = \frac{1}{T}$, and we set $k_B = 1$, so that the temperature has the dimension of energy. From the classical thermodynamical relation between free energy $F$, energy $E$ and entropy $S$:

$$F = E - TS, \quad F = -T \ln Z,$$ \hspace{1cm} (66)

the partition function Eq.(65) can be written as follows:

$$e^{-\beta F} = \int dE \frac{e^{-\beta E + S(E)}}{L},$$ \hspace{1cm} (67)

where $\rho(E) = Le^{S(E)}$. The parameter $L$ has the dimension of the length and can be determined by the details of the system under consideration.

The function $-\beta E + S(E)$ can be expanded around the energy of thermal equilibrium point $E_0$ as

$$-\beta E + S(E) = -\beta E_0 + S(E_0) + \frac{1}{2} \beta^2 B(E_0)(E - E_0)^2 + O((E - E_0)^3).$$ \hspace{1cm} (68)

Here the coefficient $B(E_0)$ is the dimensionless constant related with $E_0$. Of course, $\frac{1}{B(E_0)}$ is nothing but the square of the standard deviation. Then Eq.(67) can be calculated by Gaussian integral up to second order on $(E - E_0)^2$ as

$$e^{-\beta F} = \sqrt{\frac{\pi L^2}{\beta^2 B(E_0)}} e^{-\beta E_0 + S(E_0)},$$

$$= e^{-\beta E_0 + S(E_0) + \frac{1}{2} \ln \frac{\pi L^2}{\beta^2 B(E_0)}},$$ \hspace{1cm} (69)
which leads to the relation
\[- \beta F = - \beta E_0 + S(E_0) + \frac{1}{2} \ln \frac{C(E_0)L^2}{\beta^2}\]  
(70)

where \(C(E_0) = \frac{\pi}{B(E_0)}\). Since the specific heat is given by
\[\frac{\partial \langle E \rangle}{\partial T} \bigg|_V = \beta^2 \left( \langle E^2 \rangle - \langle E \rangle^2 \right) = \frac{1}{B(E_0)}\]  
(71)

\(\frac{C(E)}{\pi}\) can be regarded as the specific heat. If we assume the second and third terms in the right hand side of Eq.(70) as uncorrected entropy \(S_0\),
\[S_0 = S(E_0) + \frac{1}{2} \ln \frac{C(E_0)L^2}{\beta^2},\]  
(72)

the similar equation to (13) is obtained:
\[S(E_0) = S_0 - \frac{1}{2} \ln \frac{C(E_0)L^2}{\beta^2}.\]  
(73)

Therefore one can realize that the entropy \(S(E_0)\) always has the logarithmic correction from the classical thermodynamical considerations.

If one can choose \(L = \beta\), Eq.(11) is reproduced but this might not be justified since \(L\) might be determined independently from the temperature. Then there could be extra \(\beta\) or temperature dependence inside the logarithmic function. In the case that we are considering in this paper, the temperature \(T\) is determined by the radius of the horizon, then \(T\) might not depend on the radius \(a\) of the brane universe or \(T\) might scale as \(l/a\) as in Eq.(15). Even in the latter case, only the power of \(a\) inside the logarithmic term changes from \(a^3\) to \(a\), which changes the coefficient in front of the logarithmic term but the qualitative structure is be changed.

For Schwarzschild-deSitter spacetime, there are two kinds of temperatures corresponding to two horizons, black hole horizon and cosmological one. The system is not thermodynamically stable. However, the system should be adiabatic since one can define the temperature in the vicinity of each horizon. Furthermore, future black hole and cosmological horizons are separated from each other as we will see in the next Appendix. Then we may discuss the thermodynamics near the horizon.

**B Penrose diagram for Schwarzschild-deSitter black hole.**

The Penrose diagram for Schwarzschild-deSitter black hole is given in Figure. We can find the future black hole horizon is causally separated from the cosmological one. Then any particle in a region between the black hole and cosmological horizons will cross one and only one of the future horizons. Then such a particle observes the energy (the entropy, etc.) associated with the horizon that the particle crosses.
C A brief review of the Cardy-Verlinde Formula in the context of the brane world

In this appendix, we briefly explain how the Cardy-Verlinde formula can be understood in the context of the brane world in the SAdS bulk. For the SdS bulk case, see [7]. Here we do not include the logarithmic corrections.

We start with the Minkowski signature action $S$ which is the sum of the Einstein-Hilbert action $S_{EH}$ with the cosmological term, the Gibbons-Hawking surface term $S_{GH}$, and the surface counter term $S_1$:

$$
S = S_{EH} + S_{GH} + 2S_1 , \quad (74)
$$

$$
S_{EH} = \frac{1}{16\pi G} \int d^5x \sqrt{-g(5)} \left( R(5) + \frac{12}{l^2} \right) , \quad (75)
$$

$$
S_{GH} = \frac{1}{8\pi G} \int d^4x \sqrt{-g(4)} \nabla_{\mu} n^\mu , \quad (76)
$$

$$
S_1 = -\frac{6}{16\pi G l} \int d^4x \sqrt{-g(4)} . \quad (77)
$$

Here the quantities in the 5 dimensional bulk spacetime are specified by the suffices (5) and those in the boundary 4 dimensional spacetime are specified by (4). In (76), $n^\mu$ is the unit vector normal to the boundary. The Gibbons-Hawking term $S_{GH}$ is necessary in order to make the variational method well-defined when there is boundary in the spacetime. In (77), the coefficient of $S_1$ is determined from AdS/CFT. The factor 2 in front of $S_1$ is coming from that we have two bulk regions which are connected with...
each other by the brane. Then on the brane, we have the following equation:

\[ 0 = A_{,z} - \frac{1}{l}. \] (78)

This equation is derived from the condition that the variation of the action on the brane, or the boundary of the bulk spacetime, vanishes under the variation over \( A \). The first term proportional to \( A_{,z} \) expresses the bulk gravity force acting on the brane and the term proportional to \( \frac{1}{l} \) comes from the brane tension. In (78), one uses the form of the metric as

\[ ds^2 = dz^2 + e^{2A(z, \tau)} \tilde{g}_{\mu\nu} dx^\mu dx^\nu, \quad \tilde{g}_{\mu\nu} dx^\mu dx^\nu \equiv l^2 \left(-d\tau^2 + d\Omega_3^2\right). \] (79)

Here \( d\Omega_3^2 \) corresponds to the metric of 3 dimensional unit sphere.

As a bulk space, we consider 5d AdS-Schwarzschild black hole spacetime, whose metric is given by,

\[ ds_{\text{AdS-S}}^2 = \frac{1}{h(a)} da^2 - h(a) dt^2 + a^2 d\Omega_3^2, \quad h(a) = \frac{a^2}{l^2} + 1 - \frac{16\pi GM}{3V_3a^2}. \] (80)

Here \( V_3 \) is the volume of the unit 3 sphere. If one chooses new coordinates \((z, \tau)\) by

\[ \frac{e^{2A}}{h(a)} A_{,z} - h(a) t_{,z} = 1, \quad \frac{e^{2A}}{h(a)} A_{,\tau} - h(a) t_{,\tau} = 0 \]
\[ \frac{e^{2A}}{h(a)} A_{,\tau} - h(a) t_{,\tau} = -e^{2A}. \] (81)

the metric takes the warped form (79). Here \( a = le^A \). In general we might be unable to rewrite globally the metric in (80) in the form of (79). Nevertheless, it can be done in the neighbourhood of the brane, what is necessary here. Further choosing a coordinate \( \tilde{t} \) by \( d\tilde{t} = le^A d\tau \), the metric on the brane takes FRW form:

\[ e^{2A} \tilde{g}_{\mu\nu} dx^\mu dx^\nu = -d\tilde{t}^2 + l^2 e^{2A} d\Omega_3^2. \] (82)

By solving Eqs.(81), we have

\[ H^2 = A_{,z} - he^{-2A} = A_{,z} - \frac{1}{l^2} - \frac{1}{a^2} + \frac{16\pi GM}{3V_3a^4}. \] (83)

Here the Hubble constant \( H \) is introduced: \( H = \frac{dA}{dt} \). By using (78), we find

\[ H^2 = -\frac{1}{a^2} + \frac{16\pi GM}{3V_3a^4}. \] (84)

Further by differentiating Eq.(84) with respect to \( \tilde{t} \), we obtain

\[ H_{,\tilde{t}} = \frac{1}{a^2} - \frac{32\pi GM}{3V_3a^4}. \] (85)
One can rewrite the above equations (84) and (85) in the form of the standard FRW equations:

\[ H^2 = \frac{-1}{a^2} + \frac{8\pi G_4\rho}{3}, \quad \rho = \frac{Ml}{V_3a^4}, \]  
\[ H_{\tilde{t}} = \frac{1}{a^2} - 4\pi G_4(\rho + p), \quad \rho + p = \frac{4Ml}{3V_3a^4}. \]  

(86)
(87)

Here 4d Newton constant \( G_4 \) is given by

\[ G_4 = \frac{2G}{l}. \]  

(88)

For SAdS, The entropy and the thermodynamical energy have the following form:

\[ S = \frac{V_3\pi r_H^3}{4\pi G}, \]  
\[ E = M. \]  

(89)
(90)

Here \( r_H \) is the radius of the BH horizon. By comparing (90) and (86), we find

\[ E = \frac{l}{a}E_4, \quad E_4 = \rho V_3a^3. \]  

(91)

Note that when \( a \) is large, the metric (80) has the following form:

\[ ds_{\text{AdS-S}}^2 \rightarrow \frac{a^2}{l^2} \left( -dt^2 + l^2 \sum_{i,j} g_{ij}dx^i dx^j \right), \]  

(92)

which tells that the time \( \tilde{t} \) on the brane is equal to the AdS time \( t \) times the factor \( \frac{a}{l} \):

\[ t_{\text{CFT}} = \frac{a}{l}t. \]  

(93)

Therefore Eq. (91) expresses that the energy on the brane is related with the energy \( E \) in AdS by a factor \( \frac{l}{a} \).

By using (86) and (87), we find

\[ 0 = -\rho + 3p, \]  

(94)

which tells that the trace of the energy-stress tensor coming from the matter on the brane vanishes:

\[ T_{\text{matter}}^{\mu \nu} = 0. \]  

(95)

Therefore the matter on the brane can be regarded as the radiation, i.e., the massless fields. In other words, field theory on the brane should be conformal one.
In [5], it was shown that the FRW equation in $d$-dimensions can be regarded as a $d$-dimensional analogue of the Cardy formula of 2d conformal field theory (CFT):

$$\tilde{S} = 2\pi \sqrt{\frac{c}{6} \left( L_0 - \frac{k}{d-2} \frac{c}{24} \right)}.$$  \hspace{1cm} (96)

In the present case for $d = 4$ case, identifying

$$\frac{2\pi E_4 a}{3} \Rightarrow 2\pi L_0 ,$$
$$\frac{V}{8\pi Ga} \Rightarrow \frac{c}{24} ,$$
$$\frac{HV}{2G} \Rightarrow \tilde{S} ,$$

the FRW-like equation (86) has the form (96).

The total entropy of the universe could be conserved in the expansion. Then one can evaluate holographic (Hubble) entropy $\tilde{S}$ in (97) when the brane crosses the horizon $r = r_H$. When $r = r_H$, Eq. (86) tells that

$$H = \pm \frac{1}{l} .$$  \hspace{1cm} (98)

Here the plus sign corresponds to the expanding brane universe and the minus one to the contracting universe. Taking the expanding case and using (97), we find

$$\tilde{S} = \frac{r_H^3 V_3}{4G} ,$$  \hspace{1cm} (99)

which is nothing but the black hole entropy in (89). Then the expression (23) can be naturally understood from the context of the brane world.
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