Security and complexity of the McEliece cryptosystem based on quasi-cyclic low-density parity-check codes

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Abstract: In the context of public key cryptography, the McEliece cryptosystem represents a very smart solution based on the hardness of the decoding problem, which is believed to be able to resist the advent of quantum computers. Despite this, the original McEliece cryptosystem based on Goppa codes, has encountered limited interest in practical applications, partly because of some constraints imposed by this very special class of codes. The authors have recently introduced a variant of the McEliece cryptosystem including low-density parity-check codes, that are state-of-the-art codes, now used in many telecommunication standards and applications. In this study, the authors discuss the possible use of a bit-flipping decoder in this context, which gives a significant advantage in terms of complexity. The authors also provide theoretical arguments and practical tools for estimating the trade-off between security and complexity, in such a way to give a simple procedure for the system design.

1 Introduction

In recent years, a renewed interest has been devoted to the McEliece cryptosystem [1], which is one of the most attractive options for ‘post-quantum’ public key cryptography. It exploits error correcting codes to obtain both the private and the public key. Its security relies on the difficulty of decoding a linear block code without any known structure. More precisely, two kinds of attacks can be mounted against this system. The former aims at retrieving the private key from the public key, whereas the latter tries to recover the cleartext from the ciphertext, without the knowledge of the private key. The first kind of attack can be avoided through a suitable choice of the codes to be used in the system and of their parameters. The second kind of attack basically consists in decrypting the intercepted ciphertext without knowing the private key. This can be achieved by using information set decoding algorithms on the public code. Algorithms of this kind have been investigated since a long time [1, 2]. Other approaches exploit improvements of the probabilistic algorithm first proposed by Stern [3]. These are presented in [4] and, more recently, in [5–9].

The original version of the McEliece cryptosystem, based on binary Goppa codes with irreducible generator polynomials, is faster than the widespread Rivest, Shamir and Adleman (RSA) cryptosystem. However, it has two major drawbacks: large keys and low transmission rate, the latter being coincident with the code rate. The McEliece cryptosystem uses generator matrices and encodes the messages into codewords of the public code. Niederreiter [10] proposed a code-based cryptosystem using the parity-check matrix and generalised Reed–Solomon (GRS) codes. This proposal was broken by Sidelnikov and Shestakov [11]; however, it still works with Goppa codes, as shown in [12]. The main advantage of Niederreiter’s variant, which encodes the messages into syndrome vectors, is to achieve a significant reduction in the number of operations for encryption, although this is paid with a moderate increase in the number of operations for decryption.

The most effective way to overcome the drawbacks of the McEliece cryptosystem is to replace Goppa codes with other families of codes, yielding a more compact representation of their characteristic matrices, and permitting to increase the code rate. Unfortunately, although several families of codes with such characteristics exist, it is very difficult to replace Goppa codes with other codes without incurring into serious security flaws, as occurred, for example, with Gabidulin codes [13] and GRS subcodes [14].

Among the most recent proposals, quasi-cyclic (QC) [15], quasi-dyadic (QD) [16] and QC low-density parity-check (QC-LDPC) codes [17] have been considered for possible inclusion in the McEliece cryptosystem and also in symmetric key secure channel coding schemes [18]. However, the solutions [15, 16] have been successfully attacked in [19, 20]. An updated variant of the QC solution has been recently proposed in [21], and it should be more secure; however, the complexity of the attack in [20] for the binary QD case is still open, and work is in progress on such issue. The attack procedure described in [20, 22] exploits an algebraic approach, based on a system of bi-homogeneous polynomial equations, which holds for the whole class of alternant codes. Hence, such attack concerns all cryptosystems using codes in this family.

LDPC codes are state-of-art error correcting codes, first introduced by Gallager in the 1960s [23], and more recently rediscovered [24–26]. Although random-based LDPC codes are able to approach the channel capacity [27],
structured LDPC codes have the advantage of an easier implementation of the encoding and decoding operations, and benefit from reduced storage requirements [28]. QC-LDPC codes are one of the most important examples of structured LDPC codes, and they have also been proved to achieve very good performance [29]. The existence of efficient iterative decoding algorithms for LDPC codes is the distinguishing feature of this class of codes. The rationale of these algorithms is an iterated updating and exchange of messages along a bipartite graph, also known as Tanner graph, which represents the code parity-check matrix. Very good decoding performance is achieved as long as the code Tanner graph is free of short cycles, that is, closed loops starting and ending at one node. More details on LDPC decoding will be given in Section 3.

Concerning the use of LDPC codes in the McEliece cryptosystem, they were initially thought to be unable to give significant advantages, because of the fact that the sparse nature of their matrices cannot be exploited for reducing the key size [30]. Furthermore, adopting very large codes was found to be necessary for avoiding that the intrinsic code sparsity is exploited by an attack to the dual code. However, it has also been shown that, by replacing the permutation matrix used for obtaining the public key with a more general transformation matrix, the code sparsity can be hidden and the attack to the dual code avoided [32]. Unfortunately, the proposal in [32] still used only sparse transformations, which exposed it to a total break attack [33]. Subsequently, however, we have presented a simple modification that allows to avoid such flaw, so obtaining a QC-LDPC code-based cryptosystem that is immune to any known attack [34]. This version of the cryptosystem is able to reduce the key size with respect to the original version and also to use higher code rates, which is in line with the most recent proposals concerning McEliece variants. Moreover, the size of its public keys increases linearly with the code dimension; so, it scales favourably when larger keys are needed for facing the increasing computing power.

In this paper, we elaborate on our last proposal; first by describing bit-flipping (BF) decoding [35] for the considered QC-LDPC codes, which yields a significant reduction in the decoding complexity, at the cost of a moderate loss in terms of error correction. The performance of BF decoding can be easily predicted through theoretical arguments, and this helps dimensioning the system, without the need of long numerical simulations. We also consider the most effective attack procedures known up to now and estimate analytically their work factor (WF). This way, we provide tools that allow to easily find the best set of system parameters aiming at optimising the trade-off between security and complexity.

The paper is organised as follows: in Section 2, we describe the proposed version of QC-LDPC code-based cryptosystem; in Section 3, we describe the encryption and decryption algorithms and evaluate their complexity; in Section 4, we assess the security level of the system; finally, Section 5 concludes the paper.

## 2 McEliece cryptosystem based on QC-LDPC codes

The main functions of the McEliece cryptosystem based on QC-LDPC codes are shown in Fig. 1: QC-LDPC codes with length \( n = n_0 p \), dimension \( k = k_0 p \) and redundancy \( r = p \) are adopted, where \( n_0 \) is a small integer (e.g. \( n_0 = 3 \), 4), \( k_0 = n_0 - 1 \) and \( p \) is a large integer (on the order of some thousands). For fixed values of the parameters, the private key is formed by the sparse parity-check matrix \( H \) of one of these codes, randomly chosen, having the following form

\[
H = \begin{bmatrix}
H_0 & H_1 & \ldots & H_{n_0-1}
\end{bmatrix}
\]  

(1)

that is, a row of \( n_0 \) circulant blocks \( H_i \), each with row (column) weight \( d_v \). Without loss of generality, we can suppose that \( H_{n_0-1} \) is non-singular; so, a systematic generator matrix for the code is \( G = [I \mathbf{P}] \), where \( I \) represents the \( k \times k \) identity matrix and

\[
P = \begin{bmatrix}
\begin{pmatrix}
H_{n_0-1}^{-1} & H_0
\end{pmatrix}^T \\
\begin{pmatrix}
H_{n_0-2}^{-1} & H_{n_0-1}
\end{pmatrix}^T \\
\vdots \\
\begin{pmatrix}
H_1^{-1} & H_{n_0-2}
\end{pmatrix}^T
\end{bmatrix}
\]  

(2)

where superscript \( T \) denotes transposition. Concerning the computation of \( H_i^{-1} \), we observe that the inverse of a circulant matrix can be computed through techniques that are significantly more efficient than naive inversion [36].

Let us denote by \( h_i, i = 0 \ldots n_0 - 1 \), the vector containing the positions of symbols 1 in the first row of the matrix \( H_i \), \( i = 0 \ldots n_0 - 1 \). It is easy to show that, if all the \( h_i \) vectors have disjoint sets of differences modulo \( p \), the matrix \( H \) is

![Fig. 1 McEliece cryptosystem based on QC-LDPC codes](image-url)
free of length-4 cycles in its associated Tanner graph. The secret code can be easily constructed by randomly selecting \(n_0\) vectors \(h\), with such property. This permits us to obtain large families of codes with identical parameters [32]. Under the LDPC decoding viewpoint, most of the codes in a family have the same properties; so, they show comparable error correction performance when ‘belief propagation’ [37] decoding algorithms are adopted.

In the QC-LDPC code-based cryptosystem, Bob chooses a secret QC-LDPC code by generating its parity-check matrix, \(H\), and chooses two more secret matrices: a \(k \times k\) non-singular scrambling matrix \(S\) and an \(n \times n\) non-singular transformation matrix \(Q\) with row/column weight \(m\). Then, he obtains a systematic generator matrix \(G\) for the secret code, in the form \(G = [IP]\), and produces his public key as

\[
G' = S^{-1} G Q^{-1}
\]  

The public key is a dense matrix, but, since we adopt QC-LDPC codes, the knowledge of one row of each circulant block is sufficient to describe it. We note that, differently from the original McEliece cryptosystem, the public code is not permutation equivalent to the private code. In fact, the permutation matrix used in the original system [1] has been replaced by \(Q\), that is a sparse \(n \times n\) matrix, with row and column weight \(m > 1\). This way, the LDPC matrix of the secret code (\(H\)) is mapped into a new parity-check matrix valid for the public code

\[
H' = HQ^T
\]

and, through a suitable choice of \(m\), the density of \(H'\) can be made high enough to avoid attacks to the dual code.

Alice fetches \(G'\) from the public directory, divides her message into \(k\)-bit words, and applies the encryption map as follows

\[
x = uG' + e
\]

where \(x\) is the ciphertext corresponding to the cleartext \(u\) and \(e\) is a random vector of \(t\) intentional errors. After receiving \(x\), Bob inverts the transformation as follows

\[
x' = xQ = uS^{-1} G + eQ
\]

thus obtaining a codeword of the secret LDPC code affected by the error vector \(eQ\), with weight \(\leq t = \lceil tm\rceil\). Bob must be able to correct all the errors through LDPC decoding and to obtain \(u' = uS^{-1}\). Finally, he can recover \(u\) from \(u'\), through multiplication by \(S\).

We note from (6) that the introduction of the matrix \(Q\) causes an error propagation effect (at most by a factor \(m\)) within each received frame. This is compensated by the high error correction capability of the QC-LDPC code, that must be able to correct up to \(t\) errors. Suitable QC-LDPC codes can be designed for such purpose. However, we must also note that, in contrast to the McEliece cryptosystem based on Goppa codes, which corrects all errors of a certain prescribed weight, the decoding radius of LDPC codes is usually unknown. Hence, there is a small probability that Bob fails to recover the secret message. To prevent such event, different procedures can be implemented. First, Bob can make a careful selection of the private code, rather than just picking up the first code randomly generated. In fact, the number of codes that can be obtained through random-based approaches, like random difference families [32], is impressively high. Second, when the cryptosystem is used for data transmissions, an automatic repeat request protocol can allow Alice to know whether Bob is able to correct all the errors she has randomly introduced or not. Indeed, Bob is able to detect uncorrected frames through the parity check performed by the LDPC decoder, and, consequently, he can request retransmission. In this case, a new random vector is generated by Alice, and the procedure is repeated until a correctable error pattern is obtained. In principle, this exposes the system to message-resend attacks, but simple modifications of the cryptosystem are known which prevent these attacks without significant drawbacks [38, 39]. As will be observed in the next section, using these conversions is also advantageous from the key size standpoint. Obviously, this additional effort increases the latency, but the problem is not serious if the number of errors is properly chosen and controlled.

### 3 Encryption, decryption and their complexity

#### 3.1 Key size and transmission rate

In the QC-LDPC code-based cryptosystem, because of the special form (1) of the matrix \(H\), the code rate is \((n_0 - 1)/n_0\). In the following, we will focus on two values of \(n_0\), namely: \(n_0 = 3, 4\), which give transmission rates equal to 2/3 and 3/4, respectively.

Concerning the key size, we observe that in the considered system, the public key is a binary matrix formed by \(k_0 \times n_0 = (n_0 - 1) \times n_0\) circulant blocks, each with size \(p \times p\). Since each circulant block is completely described by a single row (or column), that is, \(p\) bits, the public key size is \(k_0n_0p = (n_0 - 1)n_0p\) bits.

This size can be further reduced if we consider that a suitable conversion is needed to make the McEliece cryptosystem secure against some classical attacks, like partial message knowledge and message-resend attacks [39]. Attacks of this kind can be avoided by using some CCA2-secure variants of the McEliece cryptosystem, which have in common the idea of scrambling the input messages. When these variants are used, the public matrix can be put in systematic form [5], so the memory needed to store it becomes \(k_0(n_0 - k_0)p = (n_0 - 1)p\) bits.

The values of the key size (expressed in bytes), estimated considering the use of a CCA2-secure variant, are reported in Table 1, for \(n_0 = 3, 4\) and for a set of values of \(p\) we will consider throughout the paper. All choices of the system parameters we have considered give smaller key size and

| Table 1  Public key size expressed in bytes |
|-------------------------------------------|
| \(p\), bits   | 4096 | 5120 | 6144 | 7168 | 8192 | 9216 | 10240 | 11264 | 12288 | 13312 | 14336 | 15360 | 16384 |
| \(n_0 = 3\) | 1024 | 1280 | 1536 | 1792 | 2048 | 2304 | 2560 | 2816 | 3072 | 3328 | 3584 | 3840 | 4096 |
| \(n_0 = 4\) | 1536 | 1920 | 2304 | 2688 | 3072 | 3456 | 3840 | 4224 | 4608 | 4992 | 5376 | 5760 | 6144 |
higher transmission rate than those of the original McEliece cryptosystem [1] and its Niederreiter version [10]. Considering a CCA2-secure conversion, both have a key length of 32750 bytes, and rates 0.51 and 0.57, respectively.

### 3.2 Multiplication by circulant matrices

A fundamental point for reducing complexity in the considered cryptosystem is to adopt efficient algorithms for performing multiplication of a circulant matrix by a vector.

Since circulant matrices are also Toeplitz matrices, an effective algorithm for fast computation of vector–matrix products is the Winograd convolution [40]. The Winograd algorithm is a generalisation of the Karatsuba–Ofman algorithm, that has been reviewed even recently, in the perspective to allow fast very large scale integrations (VLSI) implementations [41]. If we consider a \( p \times p \) Toeplitz matrix \( T \), with even \( p \), we can decompose it as follows

\[
T = \begin{bmatrix} T_0 & T_1 \\ T_2 & T_0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \times \begin{bmatrix} T_1 - T_0 & 0 & 0 \\ 0 & T_2 - T_0 & 0 \\ 0 & 0 & T_0 \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}
\]

where \( I \) and \( 0 \) are the \( p/2 \times p/2 \) identity and null matrices, respectively, and \( T_0, T_1, T_2 \) are \( p/2 \times p/2 \) Toeplitz matrices, as well as \( T_1 - T_0 \) and \( T_2 - T_0 \). It follows that the multiplication of a vector \( V = [V_0 V_1] \) by the matrix \( T \) can be split into three phases:

- **Evaluation phase**: multiplication of \( V \) by the first matrix translates into the addition of two \( p/2 \times p \) vectors \( (V_0 \text{ and } V_1) \); so, its cost, in terms of binary operations, is \( p/2 \).
- **Multiplication phase**: the vector resulting from the evaluation phase must be multiplied by the second matrix. This translates into three vector–matrix products by \( p/2 \times p/2 \) Toeplitz matrices. If \( p/2 \) is even, the three multiplications can be computed in a recursive way, by splitting each of them into four \( p/4 \times p/4 \) blocks. If \( p/2 \) is odd (or sufficiently small to make splitting no more advantageous), the vector–matrix multiplication can be performed in the traditional way and its complexity is about \( (p/2)^2/2 \).
- **Interpolation phase**: the result of the multiplication phase must be multiplied by the third matrix. This requires two additions of \( p/2 \times p \)-bit vectors, that is, further \( p \) binary operations.

The matrix \( G \) used in the QC-LDPC code-based cryptosystem is formed by \( k_0 \times n_0 \) circulant blocks with size \( p \times p \). When a vector is multiplied by such matrix, we can split the vector into \( k_0 \times p \)-bit subvectors and consider \( k_0 p \times n_0 \) vector–matrix multiplications. However, we must take into account that the evaluation phase on the \( k_0 \times p \)-bit subvectors must be performed only once, and that further \( (k_0 - 1)n_0 p \) binary operations are needed for recombining the result of multiplication by each column of circulants.

### 3.3 Encryption operations and complexity

Encryption is performed by calculating the product \( uG \) and then adding the intentional error vector \( e \). So, the encryption complexity can be estimated by considering the cost of a vector–matrix multiplication through the Winograd convolution and adding \( n \) binary operations for summing the intentional error vector.

Table 2 reports the values of the encryption complexity, expressed in terms of the number of binary operations needed for each encrypted bit, as a function of the circulant matrix size \( p \), for \( n_0 = 3, 4 \). The use of the Winograd convolution is particularly efficient when \( p \) is a power of 2, since, in such cases, recursion can be exploited to the utmost.

The values reported in Table 2 refer to the case of a non-systematic \( G \), that is, a generator matrix formed by \((n_0 - 1) \times n_0 \) generic circulant matrices. Actually, when a CCA2-secure variant of the system is used, \( G \) can be put in systematic form, and, in this case, only \( n_0 - 1 \) vector–matrix multiplications are needed, for the non-systematic part. However, to implement a CCA2-secure variant, some suitable scrambling operation must be performed on the message, before multiplication by \( G \). In this case, the complexity depends on the chosen variant, and becomes more involved to estimate. Since message scrambling followed by systematic coding is approximately equivalent to non-systematic coding, we prefer to consider the latter, which allows for a straightforward complexity estimation.

### 3.4 Decryption operations and complexity

Bob must perform the following three operations for decrypting the received message:

1. calculate the product \( xQ \);
2. decode the secret LDPC code;
3. calculate the product \( uS \).

Matrices \( Q \) and \( S \) are formed, respectively, by \( n_0 \times n_0 \) and \( k_0 \times k_0 \) circulant blocks. These two matrices have different density: the matrix \( S \) is dense (with row/column weight about \( k_2 \)), and the matrix \( Q \) is sparse (with row/column weight \( m \ll n_0 \)). So, it is advantageous to use the traditional multiplication (requiring \( nm \) binary operations) for calculating the product \( xQ \). On the contrary, the complexity of step 3 can be reduced by resorting to the Winograd convolution for efficient multiplication of a vector by a circulant matrix. Concerning step 2, Bob must exploit the secret LDPC matrix to implement a suitable decoding algorithm for trying to correct all intentional errors (that are \( \leq t = t/m \)). LDPC decoding is usually accomplished through iterative decoding algorithms, which work on the code Tanner graph, and implement the belief propagation.

### Table 2: Binary operations needed for each encrypted bit

| \( p \), bits | 4096 | 5120 | 6144 | 7168 | 8192 | 9216 | 10 240 | 11 264 | 12 288 | 13 312 | 14 336 | 15 360 | 16 384 |
|-------------|-----|-----|-----|-----|-----|-----|-------|-------|-------|-------|-------|-------|-------|
| \( n_0 = 3 \) | 726 | 823 | 919 | 1005 | 1092 | 1178 | 1236 | 1351 | 1380 | 1524 | 1510 | 1697 | 1639 |
| \( n_0 = 4 \) | 956 | 1081 | 1206 | 1321 | 1437 | 1552 | 1624 | 1783 | 1811 | 2013 | 1984 | 2244 | 2157 |
principle to provide very good error correction capability. Among them: the sum–product algorithm (SPA) [42] and the BF algorithm [35]. The SPA exploits real-valued messages and ensures the best performance on channels with soft information, although with some dependence on finite precision issues [43]. When soft information from the channel is not available, as it occurs in our case, it may be advantageous to use the BF algorithm, which works on binary messages and requires very low complexity, although its performance is not as good as that of the SPA.

The principle of the BF algorithm was devised in Gallager’s seminal work for LDPC codes with a tree representation [35]. Given an LDPC parity-check matrix with column weight \( d_c \), the variable nodes of its Tanner graph are initially filled in with the received codeword bits. During an iteration, every check node \( c_i \) sends each neighbouring variable node \( v_j \) the binary sum of all its neighbouring variable nodes other than \( v_j \). So, each variable node receives \( d_c \) parity-check sums. In order to send a message back to each neighbouring check node \( c_i \), the node \( v_j \) counts the number of unsatisfied parity-check sums from check nodes other than \( c_i \). Let us denote by \( b \leq d_c - 1 \) a suitably chosen integer; if the number of unsatisfied parity-check sums counted by \( v_j \) is greater than or equal to \( b \), then \( v_j \) flips its value and sends it to \( c_i \); otherwise, it sends its initial value unchanged to \( c_i \). At the next iteration, the check sums are updated with such new values, until all of them are satisfied or a maximum number of iterations is reached.

A relevant issue concerns the choice of \( b \). Two algorithms, commonly named A and B, were originally proposed by Gallager [35]: in Algorithm A, the value is fixed to \( b = d_c - 1 \), whereas in Algorithm B it can vary between \( \lceil d_c/2 \rceil \) and \( d_c - 1 \) during decoding (\( \lceil \cdot \rceil \) is the ceiling function). Algorithm A is simpler to implement, but Algorithm B ensures better performance.

We have already observed that, differently from algebraic hard-decision codes, the decoding radius of LDPC codes is generally unknown. So, numerical simulations are usually exploited for estimating the performance, but such approach is time demanding and impractical for the purpose of dimensioning the QC-LDPC code-based cryptosystem. In the following, we show how to estimate the performance of the BF algorithm, when applied in the considered scenario, through theoretical arguments that are very similar to those developed in [44].

Let us suppose that Bob, after having received the ciphertext, performs decoding through Algorithm A. At each iteration of the algorithm, we denote by \( p_c \) the probability that a bit is not in error and a generic parity-check equation evaluates it correctly. Instead, \( p_s \) is the probability that a bit is not in error and a parity-check equation evaluates it incorrectly. Similarly, \( p_u \) are the probabilities that a bit is in error and a parity-check equation evaluates it correctly and incorrectly, respectively.

In the considered context, by using simple combinatorial arguments, it is possible to verify that the following expressions hold

\[
p_c(q_t) = \sum_{j=0}^{\min(d_c-1,q_t)} \left( \frac{d_c - 1}{j} \right) \left( \frac{n - d_c}{q_t - j} \right) \cdot \left( \frac{n}{q_t - 1} \right)
\]

\[
p_s(q_t) = \sum_{j=0}^{\min(d_c-1,q_t)} \left( \frac{d_c - 1}{j} \right) \left( \frac{n - d_c}{q_t - j} \right) \cdot \left( \frac{n}{q_t} \right)
\]

\[
p_u(q_t) = \sum_{j=0}^{\min(d_c-1,q_t)} \left( \frac{d_c - 1}{j} \right) \left( \frac{n - d_c}{q_t - 1 - j} \right) \cdot \left( \frac{n}{q_t} \right)
\]

where \( d_c = n d_t \) is the row weight of the matrix \( H \) and \( q_t \) is the average number of residual errors after the \( t \)th iteration. It must be \( q_0 \leq t = t_m \); we fix \( q_0 = t = t_m \) in order to obtain worst-case estimates (maximum error propagation).

Let us suppose that, after the \( t \)th iteration, the estimate of a bit is in error. Based on (8), we can calculate the probability that, during the subsequent iteration, the message originating from its corresponding variable node is correct; this can be expressed as

\[
f^b(q_t) = \sum_{j=b}^{d_c-1} \left( \frac{d_c - 1}{j} \right) \left[ p_c(q_t) \right]^j \left[ p_s(q_t) \right]^{d_c-1-j}
\]

Similarly, the probability of incorrectly evaluating, in a single iteration of the algorithm, a bit that is not in error can be expressed as

\[
g^b(q_t) = \sum_{j=b}^{d_c-1} \left( \frac{d_c - 1}{j} \right) \left[ p_s(q_t) \right]^j \left[ p_u(q_t) \right]^{d_c-1-j}
\]

Under the ideal assumption of a cycle-free Tanner graph (which implies to consider an infinite-length code), the average number of residual bit errors at the \( t \)th iteration, \( q_t \), results in

\[
q_t = t - t f^b(q_{t-1}) + (n - t) g^b(q_{t-1})
\]

Based on this recursive procedure, we can calculate a waterfall threshold by finding the maximum value \( t = t_{th} \) such that \( \lim_{t \to \infty} (q_t) = 0 \).

| \( \rho \) | 4096 | 5120 | 6144 | 7168 | 8192 | 9216 | 10 240 | 11 264 | 12 288 | 13 312 | 14 336 | 15 360 | 16 384 |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| \( d_c = 13 \) | 190 | 237 | 285 | 333 | 380 | 428 | 476 | 523 | 571 | 619 | 666 | 714 | 762 |
| \( d_c = 15 \) | 182 | 240 | 288 | 336 | 384 | 432 | 479 | 527 | 575 | 622 | 670 | 718 | 766 |
| \( d_c = 17 \) | 174 | 232 | 280 | 328 | 376 | 424 | 471 | 519 | 567 | 615 | 662 | 710 | 758 |

**Table 3** Threshold values for BF decoding with fixed (optimal) \( \rho \)
Actually, different values of $t_{th}$ can be found by different choices of $b$. So, rather than resorting only to Algorithm A (in which $b = d_v - 1$ is fixed), we can also optimise the choice of $b$ by looking for the minimum $t_{th}$ for each $b \in \{d_v/2, \ldots, d_v - 1\}$. This way, variants of Algorithm A with better choices of $b$ can be obtained. For each set of code parameters, we will refer to the optimal choice of $b$ in the following.

Table 3 reports the threshold values, so obtained, for several values of the circulant block size $p$, code rates $2/3$ ($n_0 = 5$) and $3/4$ ($n_0 = 4$), and two values of column weight: $d_v = 13, 15$.

In more realistic scenarios, with finite code lengths and closed loops in the Tanner graphs, and also adopting a finite number of decoding iterations, there is no guarantee that the error rate is arbitrarily small for $t \leq t_{th}$. In this sense, the values in Table 3 should be seen as an optimistic assumption. However, we can observe that the performance achievable by BF with fixed $b$ can be improved in a number of ways.

One of these improvements has been mentioned above, and consists in using Algorithm B (i.e. variable $b$). On the other hand, more recently, the original Gallager algorithms have been made more efficient through further and more elaborated variants [45, 46]. Such improved versions reduce the gap in performance with respect to the SPA, which is able to reach extremely small error rates for values of $t$ even above the BF threshold $t_{th}$ [17]. So, taking into account these aspects, we can consider the BF threshold values as reliable approximations of the decoding radius of the considered QC-LDPC codes.

As concerns complexity, we can estimate the number of binary operations needed for each iteration of the algorithm over the code Tanner graph. During an iteration, each check node receives $d_v$ binary values and EX-ORS them, for a total of $d_v - 1$ binary sums. The result is then EX-OREd again with the message coming from each variable node before sending it back to the same node, thus requiring further $d_v$ binary sums. So, the total number of operations at check nodes is $r(2d_v - 1)$. Similarly, each variable node receives $d_v$ check sum values and counts the number of them that are unsatisifed; this requires $d_v$ operations. After that, for each neighbouring check node, any variable node updates the number of unsatisfied check sums by excluding the message received from that node and compares the result with the threshold $b$; this requires further $2d_v$ operations. So, the total number of operations at variable nodes is $n(3d_v)$. In conclusion, the cost of one iteration of BF can be estimated as

$$C_{BF}^{(i)} = r(2d_v - 1) + n(3d_v) = 5nd_v - r \quad (12)$$

Based on (12), and considering the computational effort required for calculating the $xQ$ and $u^*S$ products, we can estimate the total cost, in terms of binary operations, for each decrypted bit. The values obtained are reported in Table 4, where $m = 7$ has been assumed and a BF algorithm with ten average iterations has been considered.

By using the same parameters, and considering $v = 6$ quantisation bits for the decoder messages, we have estimated the decryption complexity with SPA decoding [17]; the results are reported in Table 5. To decode by using the SPA guarantees the best error correction performance at the threshold value $t = t_{th}$. However, in comparison with Table 4, the adoption of BF decoding gives a significant advantage over the SPA in terms of decryption complexity.

### 4 Security level

Attacks can be divided into two classes:

- structural attacks, aimed at recovering the secret code;
- decoding attacks, aimed at decrying the transmitted ciphertext.

#### 4.1 Structural attacks

Structural attacks against the McEliece cryptosystem aim at recovering the secret code from the public one; thus, they are strongly influenced by the family of codes used.

The original proposal of using binary Goppa codes has still never suffered a structural attack. Recently, a new class of distinguishers has been proposed for high rate McEliece cryptosystems [47]. They allow to distinguish the generator matrix of a Goppa code from a randomly picked binary matrix, under the condition that the code rate is very high (i.e. close to 1). Although this is an important result, it does not...

### Table 4 Binary operations needed for each decrypted bit by using BF decoding

| $p$, bits | 4096 | 5120 | 6144 | 7168 | 8192 | 9216 | 10 240 | 11 264 | 12 288 | 13 312 | 14 336 | 15 360 | 16 384 |
|-----------|------|------|------|------|------|------|--------|--------|--------|--------|--------|--------|--------|
| $n_0 = 3$ |      |      |      |      |      |      |        |        |        |        |        |        |        |
| $d_v = 13$ | 1476 | 1544 | 1611 | 1668 | 1726 | 1784 | 1827   | 1899   | 1928   | 2014   | 2014   | 2130   | 2101   |
| $d_v = 15$ | 1626 | 1694 | 1761 | 1818 | 1876 | 1934 | 1977   | 2049   | 2078   | 2164   | 2164   | 2280   | 2251   |
| $n_0 = 4$ |      |      |      |      |      |      |        |        |        |        |        |        |        |
| $d_v = 13$ | 1598 | 1694 | 1790 | 1877 | 1963 | 2050 | 2107   | 2223   | 2252   | 2396   | 2381   | 2569   | 2511   |
| $d_v = 15$ | 1731 | 1828 | 1924 | 2010 | 2097 | 2183 | 2241   | 2356   | 2385   | 2529   | 2515   | 2702   | 2644   |

### Table 5 Binary operations needed for each decrypted bit by using SPA decoding

| $p$, bits | 4096 | 5120 | 6144 | 7168 | 8192 | 9216 | 10 240 | 11 264 | 12 288 | 13 312 | 14 336 | 15 360 | 16 384 |
|-----------|------|------|------|------|------|------|--------|--------|--------|--------|--------|--------|--------|
| $n_0 = 3$ |      |      |      |      |      |      |        |        |        |        |        |        |        |
| $d_v = 13$ | 9791 | 9859 | 9926 | 9983 | 10041| 10099| 10142  | 10214  | 10243  | 10329  | 10329  | 10445  | 10416  |
| $d_v = 15$ | 11261| 11329| 11396| 11453| 11511| 11569| 11612  | 11684  | 11713  | 11799  | 11799  | 11915  | 11886  |
| $n_0 = 4$ |      |      |      |      |      |      |        |        |        |        |        |        |        |
| $d_v = 13$ | 9068 | 9164 | 9260 | 9347 | 9433 | 9520 | 9577   | 9693   | 9722   | 9866   | 9851   | 10039  | 9981   |
| $d_v = 15$ | 10375| 10471| 10567| 10653| 10740| 10826| 10884  | 10999  | 11028  | 11172  | 11158  | 11345  | 11288  |

*IET Inf. Secur., 2013, Vol. 7, Iss. 3, pp. 212–220
doi:10.1049/iet-ifs.2012.0127 © The Institution of Engineering and Technology 2013
not concern most Goppa-based instances of the McEliece cryptosystem, in which code rates below 0.8 are used.

With the aim of reducing the size of the public key, it has been attempted many times to replace binary Goppa codes with other families of codes. In order to meet the target, the codes must have a structured nature, which allows achieving compact matrix representations, but, on the other hand, may be exploited by suitably devised structural attacks.

Two of the most interesting proposals of this type are those considering QC and QD codes [15, 16], which are able to achieve strong reductions in the key size with respect to classical Goppa codes. The codes used in these systems are still algebraic codes which fall in the class of alternate codes. A first attack against these variants has been presented in [19], and exploits linear redundancies in subfield subcodes of GRS codes. An even more effective attack procedure against them has been proposed in [20, 22], and exploits a system of bi-homogeneous polynomial equations which hold for alternate codes. The structured nature of the QC and QD codes used in [15] and [16] results in a set of highly structured algebraic equations, which allow to mount an efficient key-recovery attack.

The system we consider is based on codes which are basically designed at random, apart from the need to avoid short cycles in their Tanner graphs. So, they do not have any algebraic structure, which prevents structural attacks of this kind.

The proposed system is also immune against the new class of distinguishers proposed in [47]. In fact, the transformation proposed there cannot be applied to the QC-LDPC codes of the type we consider, because of the lack of algebraic structure. Although the existence of a distinguisher cannot be considered as a proof of weakness, the non-existence is a further argument in favour of the robustness of the QC-LDPC code-based cryptosystem.

The most dangerous structural attacks against the considered cryptosystem come from the existence of a sparse matrix representation for the private code. In fact, if one tries to exploit the sparse nature of LDPC codes for reducing the public key size, density reduction attacks can be mounted, which are able to recover the private parity-check matrix [30, 31, 48]. Even though the sparse representation is hidden for the public code, the latter is still permutation equivalent to the private code, it may be recovered by an attacker through a search for low-weight codewords in the dual of the public code [31]. So, when using LDPC codes, the public code must not be permutation equivalent to the private one, as instead occurs in the original McEliece cryptosystem. We have shown in [17, 34], that this can be achieved by using the transformation matrix \( Q \) in the place of a permutation \( P \) for computing the public key.

As already shown in Section 2, the matrix \( Q \) must be sparse in order to allow correcting all intentional errors. If also the matrix \( S \) is chosen to be sparse [32], a structural attack still exists [33], which is able to recover a sparse representation for the secret code. However, it suffices to choose a dense matrix \( S \), as in the original McEliece cryptosystem, to avoid such attack [34].

### 4.2 Decoding attacks

Owing to the low weight \( t' \) of the intentional error vector, decoding attacks against the considered system are often more dangerous than structural attacks, and provide the smallest WF. These attacks aim at solving the decoding problem, that is, obtaining the error vector \( e \) used for encrypting a ciphertext. A way for finding \( e \) is to search for the minimum weight codewords of an extended code, generated by

\[
G' = \begin{bmatrix} G \end{bmatrix}_x
\]

(13)

The WF of such attacks can be determined by referring to the Stern’s algorithm [3]. More precisely, we have used an updated version of this algorithm [6], that results in minimum WF for the class of codes considered here. It must be said that several advances have recently appeared in the literature for improving the running time of the best decoding algorithms for binary random codes (see [7, 9], for example). These papers, however, often aim at evaluating the performance of information set decoding in asymptotic conditions, that is, for codes with infinite length, whereas we prefer to rely on actual operation counts, which are not reflected in these recent works. Another recent advance in this direction is represented by ‘ball collision decoding’ [8], which is able to achieve important WF reductions asymptotically. However, for finite code lengths and security levels even above those of interest here, such improvement is negligible [8].

On the other hand, we must observe that, in the QC-LDPC code-based cryptosystem, a further speedup is obtained by considering that, because of the QC property of the codes, each block-wise cyclically shifted version of the ciphertext \( x \) is still a valid ciphertext. So, the eavesdropper can continue extending \( G' \) by adding block-wise shifted versions of \( x \), and can search for one among as many shifted versions of the error vector. So, in order to estimate the minimum WF, we have considered the optimum number of shifted ciphertexts that can be used by an attacker in the generator matrix of the extended code.

For each QC-LDPC code, we have calculated the maximum number of intentional errors \( t' = [t/m] \) by considering \( m = 7 \) and the estimated error correction capability \( t \) reported in Table 3. The values obtained are reported in Table 6.

The minimum WF values, obtained in such conditions, are shown in Table 7. For \( m_0 = 3 \), the WF of the attack to the dual code, also based on the improved version of Stern’s algorithm, is about \( 2^{161} \), when \( d_s = 13 \), and \( 2^{184} \), when

### Table 6  Number of intentional errors \( t' \) introduced by Alice

| \( p \) | \( n \) | \( d_s \) | \( d_f \) | \( d_r \) |
|---|---|---|---|---|
| 4096 | 5120 | 6144 | 7168 | 8192 | 9216 |
| 10 240 | 11 264 | 12 288 | 13 312 | 14 336 | 15 360 |
| 16 384 | 2184 |
| \( n_0 = 3 \) | 13 | 13 | 15 | 15 |
| \( n_0 = 4 \) | 27 | 27 | 26 | 26 |
d_v = 15. So, we have reported the former of such values in Table 7 for those cases in which the decoding attack WF would be higher. The same has been done for n_0 = 4, for which the WF of the attack to the dual code is about 2^{154} and 2^{176} for d_v = 13 and 15, respectively.

In order to give an example of system design, we can consider the parameters of the Goppa code suggested in [5] for achieving 80-bit security (i.e. WF = 2^{1632}, that are: n = 1632, k = 1269 and t = 33. Under a suitable CCA2-secure conversion, we give a key size of 57 581 bytes for both the McEliece cryptosystem and the Niederreiter version. The encryption and decryption complexities, estimated through the formulas in [49, p. 27], result in 817 and 2472 operations per bit, respectively, for the McEliece cryptosystem and 48 and 7890 operations per bit for the Niederreiter version. The transmission rate is 0.78 for the McEliece cryptosystem and 0.63 for the Niederreiter version.

A similar security level can be reached by the QC-LDPC code-based cryptosystem with n_0 = 4, p = 6144 and d_v = 13. In this case, as reported in Table 1, the public key size is 2304 bytes, that is, 25 times smaller than in the Goppa code-based McEliece and Niederreiter cryptosystems. The transmission rate is 0.75, similar to that of the Goppa code-based McEliece cryptosystem and higher than in the Niederreiter version. The encryption and decryption complexities, as reported in Tables 2 and 4, result in 1206 and 1790 operations per bit, respectively. So, the complexity increases in the encryption stage, but, by exploiting the BF algorithm, the decryption complexity is reduced.

So, we can conclude that, for achieving the same security level, the QC-LDPC code-based cryptosystem can adopt smaller keys and comparable or higher transmission rates with respect to the classical Goppa code-based McEliece and Niederreiter cryptosystems. Moreover, this does not come at the expense of a significantly increased complexity.

## 5 Conclusions

We have deepened the analysis of a variant of the McEliece cryptosystem using QC-LDPC codes in the place of Goppa codes. Such modification is aimed at overcoming the main drawbacks of the original system, while still allowing to reach a satisfactory security level.

We have proposed to adopt BF algorithms for decoding the QC-LDPC codes, in such a way as to achieve a rather good performance and strongly reducing the decoding complexity with respect to the SPA. The adoption of BF decoding has also allowed to develop simple analytical tools for estimating the error correction capability of the considered codes, thus simplifying the system design by avoiding the need for long numerical simulations. Together with the methods we have described to evaluate complexity, these tools provide the system designer with a fast procedure for optimising the choice of the cryptosystem parameters.

## 6 Acknowledgment

The authors thank Rafael Misoczki for having suggested improvements in Table 3, and Ludovic Perret for fruitful discussion on the complexity of the attacks to the cryptosystems based on QC codes and QC-LDPC codes. They are also grateful to the editor and the anonymous reviewers, who provided valuable comments and suggestions which significantly helped to improve the quality of this paper. This work was supported in part by the MIUR project ‘ESCAPEAD’ (Grant RBFR105NLCL) under the ‘FIRB – Futuro in Ricerca 2010’ funding program.

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