Hermite-Hadamard type integral inequalities for geometric-arithmetically \((s, m)\) convex functions

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Abstract

In this paper, we introduce a definition of geometric-arithmetically \((s, m)\) convex function and give some new inequalities of Hermite-Hadamard type for the geometric-arithmetically \((s, m)\) convex function. Finally, we discuss applications of these inequalities to special means.

Keywords: Integral inequality, Hermite-Hadamard type integral inequality, geometric-arithmetically \((s, m)\) convex function, Hölder inequality.

2020 MSC: 26D15, 26A51.

1. Introduction

The following definition is well known in the literature.

Definition 1.1 ([13]). Let \(f(x) : I \subseteq \mathbb{R} \rightarrow \mathbb{R}\) be a continuous function. The function \(f(x)\) is said to be convex on \(I\) if

\[ f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) \]

holds for all \(x, y \in I\) and \(t \in [0, 1]\).

For such a kind of convex function on \(I\) with \(a, b \in I\) and \(a < b\), we have the double inequality

\[ f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}. \]

The convex function can be generalized and the corresponding the Hermite-Hadamard’s integral inequality has been refined and generalized by many mathematicians.

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doi: 10.22436/jnsa.015.04.01

Received: 2022-01-27 Revised: 2022-02-28 Accepted: 2022-03-10
Definition 1.2 ([9]). A function \( f : I \subseteq \mathbb{R}_0 = [0, \infty) \to \mathbb{R}_0 \) is said to be geometric-arithmetically-convex if the inequality
\[
 f(xy) \leq \lambda f(x) + (1 - \lambda) f(y)
\]
holds for all \( x, y \in I \) and \( \lambda \in [0, 1) \).

Theorem 1.3 ([20]). Let \( f : I \subseteq \mathbb{R}_0 = [0, \infty) \to \mathbb{R}_0 \) be a differentiable function on \( I \), \( a, b \in I \) with \( a < b \), and \( f' \in L([a, b]) \). If \( |f'(x)|^q \) is geometric-arithmetically-convex on \( [a, b] \) for \( q \geq 1 \), then
\[
 \left| \frac{bf(b) - af(a)}{n} - \int_a^b f(x)dx \right| \leq \frac{[(b - a)A(a, b)]^{1 - \frac{1}{q}}}{2\pi} \times \{ \left| L(a^2, b^2) - a^2 \right| |f'(a)|^q + \left| L(a^2, b^2) b^2 - L(a^2, b^2) \right| |f'(b)|^q \}^{\frac{1}{q}},
\]
where \( A(x, y) \) and \( L(x, y) \) denote arithmetic and logarithmic mean, respectively, which may be defined in (4.1).

Definition 1.4 ([15]). Let \( f : I \subseteq \mathbb{R}_0 = [0, \infty) \to \mathbb{R}_0 \) and \( s \in (0, 1] \). If
\[
 f(x^t y^{1-t}) \leq t^s f(x) + (1 - t)^s f(y)
\]
holds for all \( x, y \in I \) and \( t \in [0, 1] \), then \( f(x) \) is said to be geometric-arithmetically \( s \)-convex function or simply speaking, an \( s \)-GA-convex function.

Theorem 1.5 ([12]). Let \( f : I \subseteq \mathbb{R}_0 = [0, \infty) \to \mathbb{R} \) be a differentiable function and \( f' \in L([a, b]) \) for \( 0 < a < b < \infty \). If \( |f'|^p \) is an \( s \)-GA-convex function on \( [0, b] \), \( s \in (0, 1] \) and \( p \geq 1 \), then
\[
 \left| \frac{b^n f(b) - a^n f(a)}{n} - \int_a^b x^{n-1} f(x)dx \right| \leq \frac{\ln b - \ln a}{n} \left[ L(a^{n+1}, b^{n+1}) \right]^{1 - \frac{1}{p}} \left[ G(s, n + 1) |f'(b)|^p + H(s, n + 1) |f'(a)|^p \right]^{\frac{1}{p}},
\]
where \( G(s, l), L(x, y), H(s, l) \) are given in (3.1).

Definition 1.6 ([8]). For some \( (s, m) \in [-1, 1] \times (0, 1) \), a function \( f : (0, b] \to \mathbb{R} \) is called to be extended \( (s, m) \)-GA-convex on \( (0, b] \) if
\[
 f \left( x^{\lambda} y^{m(1-\lambda)} \right) \leq \lambda^s f(x) + m(1 - \lambda)^s f(y)
\]
holds for all \( x, y \in (0, b] \) and \( \lambda \in (0, 1) \).

Definition 1.7 ([18]). For some \( (s, m) \in [-1, 1] \times (0, 1) \) and \( \epsilon \geq 0 \), a function \( f : (0, b] \to \mathbb{R} \) is called to be extended \( (s, m) \)-\( e \)-GA-convex on \( (0, b] \) if
\[
 f \left( x^{\lambda} y^{m(1-\lambda)} \right) \leq \lambda^s f(x) + m(1 - \lambda)^s f(y) + \epsilon.
\]

Theorem 1.8 ([18]). Let \( (s, m) \in [-1, 1] \times (0, 1] \) and \( \lambda \in (0, 1) \) and let \( f : (0, b^*) \to \mathbb{R} \) be a differentiable mapping on \( (0, b^*) \), where \( a, b \in (0, b^*), a < b, b^{\frac{1}{m}} < b^*, \) and \( f' \in L_1([a, b]) \). If \( |f'|^q \) is extended \( (s, m) \)-\( e \)-GA-convex on \( (0, \max \{ b^{\frac{1}{m}}, b \}) \) for \( q \geq 1 \), then
\[
 \left| f(a^{\lambda} b^{1-\lambda}) - \frac{1}{\ln b - \ln a} \int_a^b f(x)dx \right| \leq \frac{\ln b - \ln a}{6^q} \{ F^{-\frac{1}{q}}(a, b; \lambda) (6G(a, b; \lambda, s)) |f'(a)|^q + 6mH(a, b; \lambda, s) |f'(b^{\frac{1}{m}})|^q + \epsilon \lambda^2 |2\lambda a + (3 - 2\lambda)b|^{\frac{1}{q}} + F^{-\frac{1}{q}}(b, a; 1 - \lambda, s) (6H(b, a, 1 - \lambda, s) |f'(a)|^q) + 6mG(b, a; 1 - \lambda, s) |f'(b^{\frac{1}{m}})|^q + \epsilon (1 - \lambda)^2 ((1 + 2\lambda)a + 2(1 - \lambda)b)^{\frac{1}{q}} \},
\]
where
\[
F(a, b; \lambda, s) = \int_0^{\lambda} (1-t)^a b^{1-t} dt, \quad F(a, b; 1-\lambda, s) = \int_0^{1-\lambda} (1-t)^a b^{1-t} dt, \\
G(x, y; \lambda, s) = \int_0^{\lambda} t[x+(1-t)y]t^s dt, \quad H(x, y; \lambda, s) = \int_0^{\lambda} t[x+(1-t)y](1-t)^s dt.
\]

Now we introduce the definition of geometric-arithmetically \((s, m)\) convex function.

**Definition 1.9.** Let \(f : I \subseteq \mathbb{R}_0 \to \mathbb{R}_0 = [0, +\infty)\) and \(\lambda \in [0,1]\). A function \(f(x)\) is said to be geometric-arithmetically \((s, m)\) convex on \(I\) if
\[
f\left(x^\lambda y^{m(1-\lambda)}\right) \leq \lambda^s f(x) + m(1-\lambda)^s f(y) \tag{1.1}
\]
holds for \(x, y \in I\) and \(s, m \in (0,1]\).

In recent years, a number of mathematicians researched Hermite-Hadamard type inequalities for some kinds of convex functions, for example, \([2–7, 10, 11, 14–17, 19, 21, 22]\). In this paper, we will establish some integral inequalities of Hermite-Hadamard type related to \((s, m)\)-GA-convex functions and then apply these inequalities to special means.

### 2. Two lemmas

To establish the inequalities for geometric-arithmetically \((s, m)\) convex functions, we recite the following lemmas.

**Lemma 2.1** ([8]). Let \(f : I \subseteq \mathbb{R}_0 = (0, \infty) \to \mathbb{R}\) be differentiable on \(I\) and \(a, b \in I\) with \(a < b\). If \(f' \in L([a, b])\), then
\[
\frac{b^n f(b) - a^n f(a)}{n} - \int_a^b t^{n-1} f(t) dt = \frac{\ln b - \ln a}{n} a^{(n+1)(1-t)} b^{(n+1)t} f'(a^{1-t} b^t) dt.
\]

**Lemma 2.2** ([1]). Let \(f : I \subseteq \mathbb{R} \to \mathbb{R}\) be differentiable on \(I\) and \(a, b \in I\) with \(a < b\). If \(f' \in L([a, b])\), then
\[
\frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du = \frac{(x-a)^2}{b-a} \int_0^1 (t-1)f'(tx+(1-t)a) dt + \frac{b-x}{b-a} \int_0^1 (1-t)f'(tx+(1-t)b) dt
\]
for \(x \in [a, b]\).

### 3. Main results

We now set off to establish some integral inequalities of Hermite-Hadamard type for geometric-arithmetically \((s, m)\) convex functions.

**Theorem 3.1.** Suppose \(f : I \subseteq \mathbb{R}_0 = [0, \infty) \to \mathbb{R}\) be differentiable on \(I\), \(a, b \in I\) with \(1 < a < b\), \(f' \in L([a, b])\) and \(|f'|\) be decreasing on \([a, b]\). If \(|f'|^p\) is geometric-arithmetically \((s, m)\) convex function on \([a, b]\) for \((s, m) \in (0,1]^2\) and \(p \geq 1\), then
\[
\left|\frac{b^n f(b) - a^n f(a)}{n} - \int_a^b t^{n-1} f(t) dt\right| \leq \frac{\ln b - \ln a}{n} a^{\lambda n+1} b^{\lambda n+1})^{1-\frac{1}{p}} \times (G(s, n+1) |f'(b)|^p + mH(s, n+1) |f'(a)|^p)^{\frac{1}{p}},
\]
where
\[
G(s, l) = \int_0^1 t^s a^{(1-t)t} b^{l-t} dt, \quad L(x, y) = \frac{y-x}{\ln y - \ln x}, \quad H(s, l) = \int_0^1 (1-t)^s a^{(1-t)t} b^{l-t} dt, \tag{3.1}
\]
for all \(x > 0, y > 0, l > 0, \) with \(x \neq y.\)

**Proof.** Since \(|f'|^p\) is an \((s, m)\)-GA-convex function on \([a, b]\) and \(|f'|\) is decreasing on \([a, b],\) from Lemma 2.1 and Hölder inequality, we get

\[
\left| \frac{b^nf(b) - a^nf(a)}{n} - \int_a^b x^{n-1}f(x)dx \right| \leq \frac{\ln b - \ln a}{n} \left( \frac{1}{s+1} \right)^{\frac{1}{p}} \left[ L \left( a^{\frac{(n+1)p}{s+1}}, b^{\frac{(n+1)p}{s+1}} \right) \right]^{1-\frac{1}{p}} \times \left[ |f'(b)|^p + m|f'(a)|^p \right]^{\frac{1}{p}}.
\]

**Theorem 3.2.** Suppose \(f : I \subseteq \mathbb{R}_0 = [0, \infty) \rightarrow \mathbb{R}\) be differentiable on \(I, a, b \in I\) with \(1 < a < b, f' \in L([a, b])\) and \(|f'|\) be decreasing on \([a, b].\) If \(|f'|^p\) is geometric-arithmetically \((s, m)\) convex function on \([a, b]\) for \((s, m) \in (0, 1)^2\) and \(p > 1,\) then

\[
\left| \frac{b^nf(b) - a^nf(a)}{n} - \int_a^b x^{n-1}f(x)dx \right| \leq \frac{\ln b - \ln a}{n} \left( \frac{1}{s+1} \right)^{\frac{1}{p}} \left[ L \left( a^{\frac{(n+1)p}{s+1}}, b^{\frac{(n+1)p}{s+1}} \right) \right]^{1-\frac{1}{p}} \times \left[ |f'(b)|^p + m|f'(a)|^p \right]^{\frac{1}{p}}.
\]

**Proof.** Since \(|f'|^p\) is an \((s, m)\)-GA-convex function on \([a, b]\) and \(|f'|\) is decreasing on \([a, b],\) from Lemma 2.1 and Hölder inequality, it follows that

\[
\left| \frac{b^nf(b) - a^nf(a)}{n} - \int_a^b x^{n-1}f(x)dx \right| \leq \frac{\ln b - \ln a}{n} \left( \frac{1}{s+1} \right)^{\frac{1}{p}} \left[ L \left( a^{\frac{(n+1)p}{s+1}}, b^{\frac{(n+1)p}{s+1}} \right) \right]^{1-\frac{1}{p}} \times \left[ |f'(b)|^p + m|f'(a)|^p \right]^{\frac{1}{p}}.
\]

**Theorem 3.3.** Suppose \(f : I \subseteq \mathbb{R}_0 = [0, \infty) \rightarrow \mathbb{R}\) be differentiable on \(I, a, b \in I\) with \(1 < a < b, f' \in L([a, b])\) and \(|f'|\) be decreasing on \([a, b].\) If \(|f'|^p\) is a geometric-arithmetically \((s, m)\) convex function on \([a, b]\) for \((s, m) \in (0, 1)^2\) and \(p > 1,\) then

\[
\left| \frac{b^nf(b) - a^nf(a)}{n} - \int_a^b x^{n-1}f(x)dx \right| \leq \frac{\ln b - \ln a}{n} \left[ G(s, (n+1)p) |f'(b)|^p + mH(s, (n+1)p) |f'(a)|^p \right]^{\frac{1}{p}}.
\]
Proof. Since $|f'|^p$ is an $(s, m)$-GA-convex function on $[a, b]$ and $|f'|$ is decreasing on $[a, b]$, from Lemma 2.1 and Hölder inequality, we have

$$\left| \frac{b^n f(b) - a^n f(a)}{n} - \int_a^b x^{n-1} f(x) \, dx \right| \leq \frac{\ln b - \ln a}{n} \left[ \int_0^1 a^{(n+1)(1-t)} b^{(n+1)t} \left| f'(a^{m(1-t)} b^t) \right| \, dt \right]^{1-\frac{1}{p}}$$

$$= \frac{\ln b - \ln a}{n} \left[ G(s, (n+1)p) |f'(b)|^p + mH(s, (n+1)p) |f'(a)|^p \right]^{\frac{1}{p}}.$$

Theorem 3.4. Suppose $f : I \subseteq \mathbb{R}_0 = [0, \infty) \to \mathbb{R}$ be differentiable on $I$, $a, b \in I$ with $a < b$, $f' \in L([a, b])$ and $|f'|$ be decreasing on $[a, b]$. If $|f'|^p$ is a geometric-arithmetically $(s, m)$ convex function on $[a, b]$ for $(s, m) \in (0, 1]^2$ and $p > 1$, $p > q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\left| \frac{b^n f(b) - a^n f(a)}{n} - \int_a^b x^{n-1} f(x) \, dx \right| \leq \frac{\ln b - \ln a}{n} \left[ L \left( a^{\frac{(n+1)(p-q)}{p-1}}, b^{\frac{(n+1)(p-q)}{p-1}} \right) \right]^{1-\frac{1}{p}}$$

$$\times \left[ G(s, (n+1)q) |f'(b)|^p + mH(s, (n+1)q) |f'(a)|^p \right]^{\frac{1}{p}}.$$

Proof. Since $|f'|^p$ is an $(s, m)$-GA-convex function on $[a, b]$ and $|f'|$ is decreasing on $[a, b]$, by Lemma 2.1 and Hölder inequality, we have

$$\left| \frac{b^n f(b) - a^n f(a)}{n} - \int_a^b x^{n-1} f(x) \, dx \right| \leq \frac{\ln b - \ln a}{n} \left[ \int_0^1 a^{(n+1)(1-t)} b^{(n+1)t} \left| f'(a^{m(1-t)} b^t) \right| \, dt \right]^{1-\frac{1}{p}}$$

$$= \frac{\ln b - \ln a}{n} \left[ L \left( a^{\frac{(n+1)(p-q)}{p-1}}, b^{\frac{(n+1)(p-q)}{p-1}} \right) \right]^{1-\frac{1}{p}} \left[ G(s, (n+1)q) |f'(b)|^p + mH(s, (n+1)q) |f'(a)|^p \right]^{\frac{1}{p}}.$$

Theorem 3.5. Suppose $f : I \subseteq \mathbb{R}_0 = [0, \infty) \to \mathbb{R}$ be differentiable on $I$, $a, b \in I$ with $a < b$, $f' \in L([a, b])$ and $|f'|$ be decreasing on $[a, b]$. If $|f'|^p$ is a geometric-arithmetically $(s, m)$ convex function on $[a, b]$ for $(s, m) \in (0, 1]^2$ and $p > 1$, then

$$\left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) \, du \right|$$

$$\leq \frac{(x-a)^2}{b-a} \left[ \frac{(p+s+1)B(s+1, p+1) |f'(x)|^p + m|f'(a)|^p}{p+s+1} \right]^{1/p}$$

$$+ \frac{(b-x)^2}{b-a} \left[ \frac{(p+s+1)B(s+1, p+1) |f'(x)|^p + m|f'(b)|^p}{p+s+1} \right]^{1/p},$$
Proof.  

\[
B(r, s) = \int_{0}^{1} t^{r-1}(1 - t)^{s-1} dt, \tag{3.2}
\]

for \( r > 0 \) and \( s > 0 \) is the noted Beta function.

\[
\frac{(b - x)f(b) + (x - a)f(a)}{b - a} - \frac{1}{b - a} \int_{a}^{b} f(u) du
\]

\[
\leq \frac{(x - a)^2}{b - a} \int_{0}^{1} (1 - t)|f'(tx + m(1 - t)a)|dt + \frac{(b - x)^2}{a - b} \int_{0}^{1} (1 - t)|f'(tx + m(1 - t)b)|dt
\]

\[
\leq \frac{(x - a)^2}{b - a} \left[ \int_{0}^{1} (1 - t)^{p-1} |f'(tx + m(1 - t)a)|^p dt \right]^{1/p}
\]

\[
+ \frac{(b - x)^2}{b - a} \left[ \int_{0}^{1} (1 - t)^{p-1} |f'(tx + m(1 - t)b)|^p dt \right]^{1/p}
\]

\[
\leq \frac{(x - a)^2}{b - a} \left[ \int_{0}^{1} (1 - t)^{p} \left( t^s |f'(x)|^p + m(1 - t)^s |f'(a)|^p \right) dt \right]^{1/p}
\]

\[
+ \frac{(b - x)^2}{b - a} \left[ \int_{0}^{1} (1 - t)^{p} \left( t^s |f'(x)|^p + m(1 - t)^s |f'(b)|^p \right) dt \right]^{1/p}
\]

\[
= \frac{(x - a)^2}{b - a} \left[ \frac{(p + s + 1)B(s + 1, p + 1)|f'(x)|^p + m|f'(a)|^p}{p + s + 1} \right]^{1/p}
\]

\[
+ \frac{(b - x)^2}{b - a} \left[ \frac{(p + s + 1)B(s + 1, p + 1)|f'(x)|^p + m|f'(b)|^p}{p + s + 1} \right]^{1/p}.
\]

Thus, the theorem is proved. \( \square \)

**Corollary 3.6.**

1. If \( p = 1 \), we have

\[
\frac{(b - x)f(b) + (x - a)f(a)}{b - a} - \frac{1}{b - a} \int_{a}^{b} f(u) du
\]

\[
\leq \frac{(x - a)^2}{b - a} \left[ \frac{|f'(x)| + m(s + 1)|f'(a)|}{(s + 1)(s + 2)} \right] + \frac{(b - x)^2}{b - a} \left[ \frac{|f'(x)| + m(s + 1)|f'(b)|}{(s + 1)(s + 2)} \right].
\]

2. If \( x = \frac{a + b}{2} \), we have

\[
\frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_{a}^{b} f(u) du
\]

\[
\leq \frac{b - a}{4} \left\{ \left[ \frac{(p + s + 1)B(s + 1, p + 1)|f'(\frac{a + b}{2})|^p + m|f'(a)|^p}{p + s + 1} \right]^{1/p}
\]

\[
+ \left[ \frac{(p + s + 1)B(s + 1, p + 1)|f'(\frac{a + b}{2})|^p + m|f'(b)|^p}{p + s + 1} \right]^{1/p} \right\}.
\]
3. If \( p = 1 \) and \( x = \frac{a + b}{2} \), we have
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq \frac{b-a}{4(s+1)(s+2)} \left[ 2 \left| f' \left( \frac{a+b}{2} \right) \right| + m(s+1) \left( |f'(a)| + |f'(b)| \right) \right].
\]

**Theorem 3.7.** Let \( f : I \subseteq \mathbb{R}_0 = [0, \infty) \to \mathbb{R} \) be differentiable on \( I \), \( a, b \in I \) with \( a < b \), \( f' \in L([a, b]) \) and let \( |f'| \) be decreasing on \([a, b]\). If \( |f'|^p \) is geometric-arithmetically \((s, m)\) convex on \([a, b]\) for \((s, m) \in (0, 1]^2\) and \( p > 1 \), then
\[
\left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq \left( \frac{p-1}{2p-1} \right)^{(p-1)/p} \left\{ \frac{(x-a)^2}{b-a} \left[ \frac{m|f'(a)|^p + |f'(x)|^p}{s+1} \right]^{1/p} + \frac{(b-x)^2}{b-a} \left[ \frac{|f'(x)|^p + m|f'(b)|^p}{s+1} \right]^{1/p} \right\}
\]
for \( x \in [a, b], \ t \in [0, 1]. \)

**Proof.** Since \( |f'| \) is decreasing on \([a, b]\), by Lemma 2.2 and Hölder inequality, we have
\[
\left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq \left\{ \frac{(x-a)^2}{b-a} \int_0^1 (1-t)|f'(tx + m(1-t)a)| \, dt + \frac{(b-x)^2}{b-a} \int_0^1 (1-t)|f'(tx + m(1-t)b)| \, dt \right\}^{1/(p-1)}
\]
\[
\leq \left\{ \frac{(x-a)^2}{b-a} \left[ \int_0^1 (1-t)^{p/(p-1)} \, dt \right]^{(p-1)/p} \left[ \int_0^1 |f'(x^1 a^{m(1-t)})|^p \, dt \right] \right\}^{1/p}
\]
\[
+ \left\{ \frac{(b-x)^2}{b-a} \left[ \int_0^1 (1-t)^{p/(p-1)} \, dt \right]^{(p-1)/p} \left[ \int_0^1 |f'(x^1 b^{m(1-t)})|^p \, dt \right] \right\}^{1/p},
\]
where
\[
\int_0^1 (1-t)^{p/(p-1)} \, dt = \frac{p-1}{2p-1}.
\]
Making use of the \((s, m)\)-geometric-arithmetic convexity of \( |f'(x)|^p \) on \([a, b]\) again, we get
\[
\int_0^1 |f'(x^1 a^{m(1-t)})|^p \, dt \leq \left[ \int_0^1 (t^s |f'(x)|^p + m(1-t)^s |f'(a)|^p) \, dt \right] \frac{|f'(x)|^p + m|f'(a)|^p}{s+1}
\]
and
\[
\int_0^1 |f'(x^1 b^{m(1-t)})|^p \, dt \leq \left[ \int_0^1 (t^s |f'(x)|^p + m(1-t)^s |f'(b)|^p) \, dt \right] \frac{|f'(x)|^p + m|f'(b)|^p}{s+1}.
\]
Therefore, the inequality
\[
\left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq \left( \frac{p-1}{2p-1} \right)^{(p-1)/p} \left\{ \frac{(x-a)^2}{b-a} \left[ \frac{m|f'(a)|^p + |f'(x)|^p}{s+1} \right]^{1/p} + \frac{(b-x)^2}{b-a} \left[ \frac{|f'(x)|^p + m|f'(b)|^p}{s+1} \right]^{1/p} \right\}
\]
is derived. \(\square\)
Corollary 3.8. Under the conditions of Theorem 3.7, if \( x = \frac{a+b}{2} \), we have
\[
\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(u) \, du \leq \frac{b-a}{4} \left( \frac{p-1}{2p-1} \right)^{(p-1)/p} \left\{ \frac{m|f'(a)|^p + |f'(\frac{a+b}{2})|^p}{s+1} \right\}^{1/p} + \left\{ \frac{|f'(\frac{a+b}{2})|^p + m|f'(b)|^p}{s+1} \right\}^{1/p}.
\]

Theorem 3.9. Let \( f : I \subseteq \mathbb{R}_0 = [0, \infty) \rightarrow \mathbb{R} \) be differentiable on \( I \), \( a, b \in I \) with \( a < b \), \( f' \in L([a, b]) \) and let \( |f'| \) be decreasing on \([a, b]\). If \( |f'|^p \) is a geometric-arithmetically \((s, m)\) convex on \([a, b]\) for \((s, m) \in (0, 1]^2\) and \( p \geq 1 \), then
\[
\left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq \left( \frac{1}{2} \right)^{(p-1)/p} \left\{ \left( \frac{x-a}{b-a} \right)^2 \frac{\int_0^1 (1-t)|f'(tx + m(1-t)a)| \, dt}{b-a} + \frac{\int_0^1 (1-t)|f'(tx + m(1-t)b)| \, dt}{b-a} \right\}^{1/p} + \left( \frac{b-x}{b-a} \right)^2 \left[ \frac{|f'(x)|^p + m(s+1)|f'(b)|^p}{s+1} \right]^{1/p}.
\]

Proof. Since \(|f'|\) is decreasing on \([a, b]\) and \(|f'|^p\) is geometric-arithmetically \((s, m)\) convex on \([a, b]\), by Lemma 2.2 and Hölder inequality, we have
\[
\left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq \left( \frac{x-a}{b-a} \right)^2 \left[ \frac{\int_0^1 (1-t)|f'(tx + m(1-t)a)| \, dt}{b-a} + \frac{\int_0^1 (1-t)|f'(tx + m(1-t)b)| \, dt}{b-a} \right]^{(p-1)/p} + \left( \frac{b-x}{b-a} \right)^2 \left[ \frac{|f'(x)|^p + m(s+1)|f'(b)|^p}{s+1} \right]^{1/p}.
\]

Corollary 3.10. Under the conditions of Theorem 3.9, if \( x = \frac{a+b}{2} \), we have
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq \left( \frac{1}{2} \right)^{(p-1)/p} \frac{b-a}{4} \left\{ \left[ \frac{|f'(\frac{a+b}{2})|^p + m(s+1)|f'(a)|^p}{s+1} \right]^{1/p} + \left[ \frac{|f'(\frac{a+b}{2})|^p + m(s+1)|f'(b)|^p}{s+1} \right]^{1/p} \right\}.
\]
4. Application to special means

For positive numbers $b > a > 0$, define

$$\Lambda(a, b) = \frac{a + b}{2}, \quad H(a, b) = \frac{2ab}{a + b}, \quad L(a, b) = \frac{b - a}{\ln b - \ln a},$$

and

$$L_r(a, b) = \begin{cases} \left[ \frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)} \right]^{1/r}, & r \neq 0, -1, \\ L(a, b), & r = -1, \\ \frac{1}{r} \left( \frac{b}{a} \right)^{1/(b-a)}, & r = 0. \end{cases}$$

These quantities are respectively called the arithmetic, harmonic, logarithmic, generalized logarithmic means of two positive numbers $a$ and $b$.

Now let $f(x) = x^r$ for $x > 0$, $r \in \mathbb{R}$ with $r \neq 0$, and $(s, m) \in (0, 1)^2$. Then

$$|f'(x^\lambda y^{m(1-\lambda)})|^p = |r|^p |x^\lambda y^{m(1-\lambda)}|^{p(r-1)} \leq |r|^p \left[ \lambda^p x^{p(r-1)} + m(1-\lambda) y^{p(r-1)} \right]$$

for $\lambda \in [0, 1]$, $x, y > 0$ and $p \geq 1$. We can see a function $|f'|^p$ is said to be geometric-arithmetically $(s, m)$ convex on $I$. Applying this function to Corollaries 3.6, 3.8, and 3.10 derives the following inequalities for means.

**Theorem 4.1.** Let $B(r, s)$ be defined by (3.2) and let $b > a > 0$, $r \in (-\infty, 0) \cup (0, 1)$, $p \geq 1$ and $0 < s \leq 1, 0 < m \leq 1$.

1. If $r \neq -1$ and $x = \frac{a + b}{2}$, we have

$$|\Lambda(a^r, b^r) - L_r(a, b)| \leq \frac{|r|(b-a)}{4(p+s+1)^{1/p}} \left\{ \left[ (p+s+1)B(s+1, p+1)A^{p(r-1)}(a, b) + ma^{p(r-1)} \right]^{1/p} \\
+ \left[ (p+s+1)B(s+1, p+1)A^{p(r-1)}(a, b) + mb^{p(r-1)} \right]^{1/p} \right\}.$$

2. If $r = -1$ and $x = \frac{a + b}{2}$, we have

$$\left| \frac{1}{H(a, b)} - \frac{1}{L(a, b)} \right| \leq \frac{(b-a)}{4(p+s+1)^{1/p}} \left\{ \left[ \frac{(p+s+1)B(s+1, p+1)}{A^{2p}(a, b)} + \frac{m}{a^{2p}} \right]^{1/p} \\
+ \left[ \frac{(p+s+1)B(s+1, p+1)}{A^{2p}(a, b)} + \frac{m}{b^{2p}} \right]^{1/p} \right\}.$$

**Proof.** According to Corollary 3.6, we have

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq \frac{b-a}{4} \left\{ \left[ (p+s+1)B(s+1, p+1)\left| f'\left( \frac{a+b}{2} \right) \right|^p + m|f'(a)|^p \right]^{1/p} \\
+ \left[ (p+s+1)B(s+1, p+1)\left| f'\left( \frac{a+b}{2} \right) \right|^p + m|f'(b)|^p \right]^{1/p} \right\}.$$
1. If \( r \neq -1 \), then \( f(x) = x^r \),

\[
|A(a^r, b^r) - L_f^r(a, b)| = \frac{|a^r + b^r - b^{r+1} - a^{r+1}|}{(r+1)(b-a)} = \frac{f(a) + f(b) - \frac{1}{b-a} \int_a^b f(u) \, du}{2}
\]

\[
\leq \frac{b-a}{4} \left\{ \left[ (p+s+1)B(s+1, p+1) \left| f' \left( \frac{a+b}{2} \right) \right|^p + m|f'(a)|^p \right]^{1/p} \frac{p}{p+s+1} + \left[ (p+s+1)B(s+1, p+1) \left| f' \left( \frac{a+b}{2} \right) \right|^p + m|f'(b)|^p \right]^{1/p} \right\}
\]

\[
\leq \frac{|r|(b-a)}{4(p+s+1)^{1/p}} \left\{ \left[ (p+s+1)B(s+1, p+1) f' \left( \frac{a+b}{2} \right) \right|^p + m|f'(a)|^p \right\}^{1/p} \frac{p}{p+s+1} + \left[ (p+s+1)B(s+1, p+1) f' \left( \frac{a+b}{2} \right) \right|^p + m|f'(b)|^p \right\}^{1/p}
\]

2. If \( r = -1 \), then \( f(x) = \frac{1}{x} \),

\[
\left| \frac{1}{H(a, b)} - \frac{1}{L(a, b)} \right| = \frac{|a + b\ ln b - \ln a|}{2ab} - \frac{1}{b-a} \int_a^b f(u) \, du
\]

\[
\leq \frac{b-a}{4} \left\{ \left[ (p+s+1)B(s+1, p+1) \left| f' \left( \frac{a+b}{2} \right) \right|^p + m|f'(a)|^p \right]^{1/p} \frac{p}{p+s+1} + \left[ (p+s+1)B(s+1, p+1) \left| f' \left( \frac{a+b}{2} \right) \right|^p + m|f'(b)|^p \right]^{1/p} \right\}
\]

\[
\leq \frac{b-a}{4(p+s+1)^{1/p}} \left\{ \left[ (p+s+1)B(s+1, p+1) \left| f' \left( \frac{a+b}{2} \right) \right|^p + m|f'(a)|^p \right]^{1/p} \frac{p}{p+s+1} + \left[ (p+s+1)B(s+1, p+1) \left| f' \left( \frac{a+b}{2} \right) \right|^p + m|f'(b)|^p \right]^{1/p} \right\}
\]

\[
\leq \frac{b-a}{4(p+s+1)^{1/p}} \left\{ \left[ (p+s+1)B(s+1, p+1) \left| f' \left( \frac{a+b}{2} \right) \right|^p + m|f'(a)|^p \right]^{1/p} \frac{p}{p+s+1} + \left[ (p+s+1)B(s+1, p+1) \left| f' \left( \frac{a+b}{2} \right) \right|^p + m|f'(b)|^p \right]^{1/p} \right\}
\]

\[
\leq \frac{(b-a)}{4(p+s+1)^{1/p}} \left\{ \left[ (p+s+1)B(s+1, p+1) \left| f' \left( \frac{a+b}{2} \right) \right|^p + m|f'(a)|^p \right]^{1/p} \frac{p}{p+s+1} + \left[ (p+s+1)B(s+1, p+1) \left| f' \left( \frac{a+b}{2} \right) \right|^p + m|f'(b)|^p \right]^{1/p} \right\}
\]
Theorem 4.2. Let $b > a > 0$, $r \in (-\infty, 0) \cup (0, 1)$, $p > 1$ and $0 < s \leq 1$, $0 < m \leq 1$.

1. If $r \neq -1$ and $x = \frac{a+b}{2}$, we have

\[
|A(a^r, b^r) - L_r(a, b)| \leq \frac{|r|(b-a)}{4} \left( \frac{p-1}{2p-1} \right)^{(p-1)/p} \frac{1}{(s+1)^{1/p}} \\
\times \left\{ \left[ m a^{p(r-1)} + A^p(r-1)(a, b) \right]^{1/p} + \left[ A^p(r-1)(a, b) + m b^{p(r-1)} \right]^{1/p} \right\}.
\]

2. If $r = -1$ and $x = \frac{a+b}{2}$, we have

\[
|\frac{1}{H(a, b)} - \frac{1}{L(a, b)}| \leq \left( \frac{p-1}{2p-1} \right)^{(p-1)/p} \\
\times \frac{b-a}{4(s+1)^{1/p}} \left\{ \left[ \frac{m f'(a)|p + |f'(\frac{a+b}{2})|p}{s+1} \right]^{1/p} + \left[ \frac{|f'(\frac{a+b}{2})|p + m|f'(b)|p}{s+1} \right]^{1/p} \right\}.
\]

Proof. According to Corollary 3.8, we get

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq \frac{b-a}{4} \left( \frac{p-1}{2p-1} \right)^{(p-1)/p} \\
\times \left\{ \left[ \frac{m f'(a)|p + |f'(\frac{a+b}{2})|p}{s+1} \right]^{1/p} + \left[ \frac{|f'(\frac{a+b}{2})|p + m|f'(b)|p}{s+1} \right]^{1/p} \right\}.
\]

1. If $r \neq -1$, then $f(x) = x^r$,

\[
|A(a^r, b^r) - L_r(a, b)| = \left| \frac{a^r + b^r}{2} - \frac{b^{r+1} - a^{r+1}}{r+1} \frac{b-a}{r+1} \right| = \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq \frac{b-a}{4} \left( \frac{p-1}{2p-1} \right)^{(p-1)/p} \\
\times \left\{ \left[ \frac{m f'(a)|p + |f'(\frac{a+b}{2})|p}{s+1} \right]^{1/p} + \left[ \frac{|f'(\frac{a+b}{2})|p + m|f'(b)|p}{s+1} \right]^{1/p} \right\}.
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2. If $r = -1$, then $f(x) = \frac{1}{x}$,

\[
\left| \frac{1}{H(a, b)} - \frac{1}{L(a, b)} \right| = \left| \frac{a + b}{2ab} - \ln b - \ln a \right| - \frac{1}{b - a} \left( f(a) + f(b) \right) - \frac{1}{b - a} \int_a^b f(u) \, du \leq \frac{b - a}{4} \left( \frac{p - 1}{2p - 1} \right)^{(p - 1)/p} \times \left\{ \left[ \frac{m|f'(a)|^p + |f'(\frac{a+b}{2})|^p}{s + 1} \right]^{1/p} + \left[ \frac{|f'(\frac{a+b}{2})|^p + m|f'(b)|^p}{s + 1} \right]^{1/p} \right\} \leq \left( \frac{p - 1}{2p - 1} \right)^{(p - 1)/p} \times \left\{ \left[ \frac{2^p}{(a + b)^{2p}} + \frac{m}{a^{2p}} \right]^{1/p} + \left[ \frac{2^p}{(a + b)^{2p}} + \frac{m}{b^{2p}} \right]^{1/p} \right\}.
\]

\[\lim_{r \to -1} A(a^r, b^r) = \frac{1}{2} \left( \frac{b - a}{4[(s + 1)(s + 2)]^{1/p}} \right) \left\{ \left[ A^p(r^{-1})(a, b) + m(s + 1)a^{p(r - 1)} \right]^{1/p} + \left[ A^p(r^{-1})(a, b) + m(s + 1)b^{p(r - 1)} \right]^{1/p} \right\}.\]

**Theorem 4.3.** Let $b > a > 0$, $r \in (-\infty, 0] \cup [0, 1)$, $p \geq 1$ and $0 < s \leq 1$, $0 < m \leq 1$. If $r \neq -1$ and $x = \frac{a + b}{2}$, we have

\[|A(a^r, b^r) - L_r^*(a, b)| \leq \left( \frac{1}{2} \right)^{(p - 1)/p} \frac{(b - a)}{4[(s + 1)(s + 2)]^{1/p}} \left\{ \left[ A^p(r^{-1})(a, b) + m(s + 1)a^{p(r - 1)} \right]^{1/p} + \left[ A^p(r^{-1})(a, b) + m(s + 1)b^{p(r - 1)} \right]^{1/p} \right\}.
\]

**Proof.** According to Corollary 3.10, we have

\[\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(u) \, du \right| \leq \left( \frac{1}{2} \right)^{(p - 1)/p} \frac{b - a}{4} \left\{ \left[ |f'(\frac{a+b}{2})|^p + m(s + 1)|f'(a)|^p \right]^{1/p} \right\} \leq \left( \frac{1}{2} \right)^{(p - 1)/p} \frac{b - a}{4} \left\{ \left[ |f'(\frac{a+b}{2})|^p + m(s + 1)|f'(a)|^p \right]^{1/p} \right\}.
\]

1. If $r \neq -1$, then $f(x) = x^r$,

\[|A(a^r, b^r) - L_r^*(a, b)| = \left| \frac{a^r + b^r}{2} - \frac{b^{r+1} - a^{r+1}}{(r + 1)(b - a)} \right| = \left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(u) \, du \right| \leq \left( \frac{1}{2} \right)^{(p - 1)/p} \frac{b - a}{4} \left\{ \left[ |f'(\frac{a+b}{2})|^p + m(s + 1)|f'(a)|^p \right]^{1/p} \right\}.
\]
2. If \( r = -1 \), then \( f(x) = \frac{1}{x} \),

\[
\left| \frac{1}{H(a, b)} - \frac{1}{L(a, b)} \right| = \frac{a + b}{2ab} - \frac{\ln b - \ln a}{b - a} = \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(u) \, du
\]

\[
\leq \left( \frac{1}{2} \right)^{(p-1)/p} \frac{b - a}{4} \left\{ \left[ \frac{f'(\frac{a+b}{2})^p + m(s+1)|f'(b)|^p}{(s+1)(s+2)} \right] + \left[ \frac{2^p m(s+1)}{(a+b)^{2p}} + \frac{m(s+1)}{a^{2p}} \right] + \left[ \frac{2^p m(s+1)}{(a+b)^{2p}} + \frac{m(s+1)}{b^{2p}} \right] \right\}
\]

\[
\leq \left( \frac{1}{2} \right)^{(p-1)/p} \frac{b - a}{4[(s+1)(s+2)]^{1/p}} \times \left\{ \left[ \frac{1}{A^{2p}(a, b)} + \frac{m(s+1)}{a^{2p}} \right] + \left[ \frac{1}{A^{2p}(a, b)} + \frac{m(s+1)}{b^{2p}} \right] \right\}.
\]

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