Brans-Dicke theory with the cosmological constant from $M_4 \times Z_2$ geometry

Kunihiko Uehara

Department of Physics, Tezukayama University
Tezukayama 631-8501, Japan
and
Abdus Salam International Centre for Theoretical Physics
P. O. Box 568, 34100 Trieste, Italy

Abstract

The theory on $M_4 \times Z_2$ geometry is applied to the Einstein gravity to yield the Brans-Dicke theory on $M_4$ geometry. The geometrical meaning and the relation between the curvatures and the torsions are clarified. The cosmological constant is also introduced into the pure Einstein action on $M_4 \times Z_2$ in order to determine the explicit form of the cosmological term in the Brans-Dicke theory on $M_4$ geometry.

PACS numbers: 02.40.-k, 04.50.+h, 46.10.+z, 98.80.-k

*Permanent e-mail address: uehara@tezukayama-u.ac.jp
The noncommutative geometry (NCG) of Connes\cite{1} has been successful in producing geometrical interpretations of many grand unification models. In these circumstances we have been giving geometrical meanings to them using the gauge theories\cite{2} on \( M_4 \times \mathbb{Z}_N \) geometry without recourse to NCG, where the \( \mathbb{Z}_N \) is the supplemented extra discrete space. These approaches appear to be geometrically much simpler and clearer than NCG.

Recently we have shown\cite{3} that the pure Einstein action on \( M_4 \times \mathbb{Z}_2 \) geometry exactly leads to the Brans-Dicke theory\cite{4} in four dimensional space-time \( M_4 \), where we have used the equivalence assumption\cite{5} previously developed by three collaborators. We have clarified the geometrical meaning of Riemann curvature tensors which are classified to three kinds on this \( M_4 \times \mathbb{Z}_2 \) manifold. Just after that our collaborators have written a paper\cite{6} faithful to the fundamental concept of covariant differences. But one may still ask why the new isometric condition in Ref.\cite{6} would change the form of the curvature given by connections in Ref.\cite{3}, because the concept of the curvature given by the parallel transport has nothing to do with that of the distance which is introduced by the metric and the isometric condition above is given by the metric.

It is the main purpose of this note to clarify the interrogative point above together with the definition of the curvature. And we also introduce the cosmological constant into the pure Einstein action on \( M_4 \times \mathbb{Z}_2 \) geometry in order to determine the explicit form of the Brans-Dicke theory with the cosmological term on \( M_4 \) geometry in the Jordan frame, where the non-minimal coupling exists. Hereafter we mainly follow the notations in Ref.\cite{6}.

To begin with we introduce the tangent space \( T(x, g) \) whose origin is \( O(x, g) \) at a point \((x^\mu, g)\) on \( M_4 \times \mathbb{Z}_2 \), where \( x^\mu \in M_4 \) and \( g = \{e(\text{unit element}), r\} \in \mathbb{Z}_2 \). We consider a mapping of the origin \( O(x + \Delta x, g) \) onto \( T(x, g) \) from \( T(x + \Delta x, g) \) which is another tangent space at a point \((x^\mu + \Delta x^\mu, g)\) neighboring to \((x^\mu, g)\). The mapped point is denoted by a notation \( U(x, x + \Delta x, g)O(x + \Delta x, g) \). A covariant difference between \( U(x, x + \Delta x, g)O(x + \Delta x, g) \) and \( O(x, g) \) defines a vector \( e_\mu(x, g) \) on \( T(x, g) \)

\[
\triangle_x O(x, g) = U(x, x + \Delta x, g)O(x + \Delta x, g) - O(x, g) = e_\mu(x, g)\Delta x^\mu.
\]

(1)

Another vector \( e_r(x, g) \) can be got by the covariant difference between the mapped point \( V(x, g, g + r)O(x, g + r) \) and \( O(x, g) \) on \( T(x, g) \) in the same way above as follows:

\[
\triangle_r O(x, g) = V(x, g, g + r)O(x, g + r) - O(x, g) = e_r(x, g)\Delta^r z(g),
\]

(2)
where \( z(g) \) is the coordinate corresponding to \( g \) and
\[
\Delta^r z(g) = z(g + r) - z(g)
\]  \hspace{1cm} (3)

is proportional to the limiting process parameter\[5\] which tends to zero where we assume the physics on two \( M_4 \) sheets should be equivalent to each other. A set of vectors
\[
e_N(x, g) = \{ e_\mu(x, g), e_r(x, g) \mid g = (e, r) \in \mathbb{Z}_2 \}
\]  \hspace{1cm} (4)

forms a basis on \( T(x, g) \). The parallel transport of the basis \( e_N(x+\Delta x, g) \) from \( T(x+\Delta x, g) \) onto \( T(x, g) \), which is given by a rotation \( H^M_N(x, x+\Delta x, g) \) of \( e_M(x, g) \), generates the parallel transported basis \( e^H_N(x+\Delta x, g) \) as
\[
e^H_N(x+\Delta x, g) = e_M(x, g) H^M_N(x, x+\Delta x, g).
\]  \hspace{1cm} (5)

The affine connection \( \tilde{\Gamma}^M_{N\mu}(x, g) \) is defined by
\[
H^M_N(x, x+\Delta x, g) = \delta^M_N + \tilde{\Gamma}^M_{N\mu}(x, g) \Delta x^\mu + O(\Delta x^2).
\]  \hspace{1cm} (6)

The covariant difference of \( e_N(x, g) \) can be defined on \( M_4 \) by use of two equations above, (5) and (6), as follows:
\[
\Delta_x e_N(x, g) \equiv e^H_N(x+\Delta x, g) - e_N(x, g) = e_M(x, g) \tilde{\Gamma}^M_{N\mu}(x, g) \Delta x^\mu + O(\Delta x^2).
\]  \hspace{1cm} (7)

In the same way the covariant difference of \( e_N \) along \( \mathbb{Z}_2 \) is defined by
\[
\Delta_r e_N(x, g) \equiv e^H_N(x, g+r) - e_N(x, g) = e_M(x, g) \tilde{\Gamma}^M_{N\mu}(x, g) \Delta^r z(g).
\]  \hspace{1cm} (8)

The reason why the covariant difference in Eq. (8) is not associated with a term of \( O(\Delta^r z^2) \) is that the rotation matrix along \( \mathbb{Z}_2 \) has the form \( H^M_N(x, g, g+r) = \delta^M_N + \tilde{\Gamma}^M_{N\mu}(x, g) \Delta^r z(g) \) due to the discreteness of \( \mathbb{Z}_2 \).

There are three kinds of torsion tensors on \( M_4 \times \mathbb{Z}_2 \) geometry corresponding to the three path differences depicted in the Figure. They are defined by
\[
[\Delta_{1x}, \Delta_{2x}] \ O(x, g) = e_M(x, g) \tilde{T}^M_{\mu\nu}(x, g) \Delta_1 x^\mu \Delta_2 x^\nu,
\]  \hspace{1cm} (9a)
\[
[\Delta x, \Delta_r] \ O(x, g) = e_M(x, g) \tilde{T}^M_{\mu r}(x, g) \Delta x^\mu \Delta^r z(g),
\]  \hspace{1cm} (9b)
\[
[\Delta_r \Delta_r + 2 \Delta_r] \ O(x, g) = e_M(x, g) \tilde{T}^M_{r r}(x, g) \Delta^r z(g) \Delta^r z(g).
\]  \hspace{1cm} (9c)
Figure. Path differences where parallel transports are compared for (a) the first, (b) the second and (c) the third kind.

The torsion of the first kind $\hat{T}^M_{\mu\nu}$ is the conventional one, which is derived by comparing with two mappings of the origin $O(x + \Delta_1x + \Delta_2x, g)$ from $T(x + \Delta_1x + \Delta_2x, g)$ onto $T(x, g)$ along two paths $C_1$ and $C_2$ depicted in Fig.(a). They are given by

$$C_1 = U(x, x + \Delta_1x, g)U(x + \Delta_1x, x + \Delta_1x + \Delta_2x, g)O(x + \Delta_1x + \Delta_2x, g),$$  \hfill (10)  

$$C_2 = U(x, x + \Delta_2x, g)U(x + \Delta_2x, x + \Delta_1x + \Delta_2x, g)O(x + \Delta_1x + \Delta_2x, g).$$  \hfill (11)  

The straightforward calculation of the difference between $C_1$ and $C_2$ gives the torsion of the first kind as follows:

$$C_1 - C_2 = (\Delta_1x\Delta_2x - \Delta_2x\Delta_1x)O(x, g),$$  \hfill (12)  

which is just Eq.(10a). The torsion of the second kind $\hat{T}^M_{\mu r}$ is obtained in the same way above by considering two mappings of the origin $O(x + \Delta x, g + r)$ from $T(x + \Delta x, g + r)$ onto $T(x, g)$ along two paths $C_3$ and $C_4$ depicted in Fig.(b). They are given by

$$C_3 = U(x, x + \Delta x, g)V(x + \Delta x, g, g + r)O(x + \Delta x, g + r),$$  \hfill (13)  

$$C_4 = V(x, g, g + r)U(x, x + \Delta x, g + r)O(x + \Delta x, g + r).$$  \hfill (14)  

The difference between $C_3$ and $C_4$ gives the torsion of the second kind by the similar calculation above as

$$C_3 - C_4 = (\Delta x\Delta_r - \Delta_r\Delta_x)O(x, g),$$  \hfill (15)  

which is identical to Eq.(13). Finally the torsion of the third kind $\hat{T}^M_{rr}$ is derived considering the mappings of the origin $O(x, g)$ from $T(x, g)$ onto $T(x, g + r)$ and again onto the same
The torsion after some calculations \[\text{as} \ V(x, g + r) V(x, g + r, g) O(x, g) - O(x, g) = (\Delta r \Delta r + 2 \Delta r) O(x, g). \quad (16)\]

By using the relation \(\Delta^r z(g + r) = -\Delta^r z(g)\) and the Eqs. (1), (2), (7) and (8), one will find the following three kinds of torsions in terms of connections:

\[
\begin{align*}
\hat{T}^M_{\mu \nu}(x, g) &= \hat{\Gamma}^M_{\mu \nu}(x, g) - \hat{\Gamma}^M_{\mu \nu}(x, g), \quad (17a) \\
\hat{T}^M_{\mu r}(x, g) &= \hat{\Gamma}^M_{\mu r}(x, g) - \hat{\Gamma}^M_{\mu r}(x, g), \quad (17b) \\
\hat{T}^M_{rr}(x, g) &= -\hat{\Gamma}^M_{rr}(x, g). \quad (17c)
\end{align*}
\]

The torsions of the first and second kind vanish when the affine connection \(\hat{\Gamma}^L_{MN}(x, g)\) is symmetric with respect to \(M\) and \(N\), namely,

\[
\hat{R}^L_{MN}(x, g) = \hat{R}^L_{NM}(x, g), \quad (18)
\]

nevertheless the torsion of the third kind \(\hat{T}^M_{rr}\) generally remains finite.

There are also three kinds of curvature tensors corresponding to three kinds of path differences depicted in the figure. They are defined by

\[
\begin{align*}
[\Delta_1 x, \Delta_2 x] e_N &= e_M \hat{R}^M_{N \mu \nu} \Delta_1 x^\mu \Delta_2 x^\nu + e_M \hat{\Gamma}^M_{N \mu \nu} \Delta_1 x^\mu \Delta_2 x^\nu, \quad (19a) \\
[\Delta x, \Delta r] e_N &= e_M \hat{R}^M_{N \mu r} \Delta x^\mu \Delta^r z + e_M \hat{\Gamma}^M_{N \mu r} \Delta x^\mu \Delta^r z, \quad (19b) \\
[\Delta r, \Delta r + 2 \Delta r] e_N &= e_M \hat{R}^M_{N r r} \Delta^r z \Delta^r z + e_M \hat{\Gamma}^M_{N r r} \Delta^r z \Delta^r z. \quad (19c)
\end{align*}
\]

As far as we constrain ourselves in the theory where \(\hat{R}^L_{MN}\) is symmetric with respect to \(M\) and \(N\), Eqs. (19a) and (19b) reproduce the same definitions in Ref. [3]. But even in this case the curvature of the third kind in Eq. (19c) appears different due to the existence of the torsion term as follows together with curvatures of the first and second kind:

\[
\begin{align*}
\hat{R}^M_{N \mu \nu} &= \partial_\mu \hat{R}^M_{N \nu \mu} - \partial_\nu \hat{R}^M_{N \mu \nu} + \hat{\Gamma}^M_{L \mu \nu} \hat{R}^L_{N \mu} - \hat{\Gamma}^M_{L \nu \mu} \hat{R}^L_{N \nu}, \quad (20a) \\
\hat{R}^M_{N \mu r} &= \partial_\mu \hat{R}^M_{N \nu r} - \partial_\nu \hat{R}^M_{N \mu r} + \hat{\Gamma}^M_{L \mu r} \hat{R}^L_{N r} - \hat{\Gamma}^M_{L \nu r} \hat{R}^L_{N \nu}, \quad (20b) \\
\hat{R}^M_{N r r} &= -\partial_r \hat{R}^M_{N r r} - \hat{\Gamma}^M_{L r r} \hat{R}^L_{N r} + \hat{\Gamma}^M_{N L r} \hat{R}^L_{r r}. \quad (20c)
\end{align*}
\]

In order to evaluate these curvatures in the concept of distance, the metric must be introduced onto the manifold.
The manifold $M_4 \times Z_2$ can be regarded as the five dimensional Kaluza-Klein space except the fifth dimension is replaced by two points $z(e)$ and $z(r)$. The line element $\Delta s$ of this space is assumed here as

$$\Delta s^2 = g_{\mu\nu}(x) \Delta x^\mu \Delta x^\nu + \lambda^2(x) \Delta z^2 = G_{MN}(x) \Delta x^M \Delta x^N,$$

(21)

where

$$\Delta x^N = (\Delta x^\mu, \Delta x^r) \equiv \Delta z = z(r) - z(e)$$

(22)

and $G_{MN}(x)$ is the five dimensional metric on $M_4 \times Z_2$, which is defined by the inner product of basis vectors

$$G_{MN}(x) = e_M(x, g) \cdot e_N(x, g).$$

(23)

Let the manifold $M_4 \times Z_2$ be isometric, namely, any inner product of vectors is invariant under the parallel transport as

$$G_{MN}(x + \Delta x) = e_M(x + \Delta x, g) \cdot e_N(x + \Delta x, g) = e_M^H(x + \Delta x, g) \cdot e_N^H(x + \Delta x, g).$$

(24)

Substituting Eqs. (7) and (8) into (24), the relations between the metric and the connections can be obtained for $M_4$ and $Z_2$, respectively

$$\partial_\lambda G_{MN} = \hat{\Gamma}_M^{\lambda MN},$$

(25)

$$\partial_r G_{MN} = \hat{\Gamma}_{MrN} + \hat{\Gamma}_{NrM} + \hat{\Gamma}_{KrM} \Delta r z,$$

(26)

where $\hat{\Gamma}_{MLN} \equiv G_{MK} \hat{\Gamma}_{LN}^{K}$. The limit $\Delta r z \to 0$ in Eqs. (25) and (26) yields

$$\partial_\lambda G_{MN} = \Gamma_{M\lambda N} + \Gamma_{N\lambda M},$$

(27)

$$\partial_r G_{MN} = \Gamma_{MrN} + \Gamma_{NrM},$$

(28)

where notations without a hat mean that these quantities are independent of $g$. One can obtain the well-known expression of $\Gamma_{LMN}$ in terms of $G_{MN}$ using Eqs. (18), (27) and (28)

$$\Gamma_{LMN} = \frac{1}{2} (\partial_M G_{LN} + \partial_N G_{LM} - \partial_L G_{MN}).$$

(29)

Since the metric $G_{MN}$ is given by (21), one immediately get followings:

$$\Gamma_{\lambda\mu\nu} = \frac{1}{2} (\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}),$$

(30)

$$\Gamma_{r\mu\nu} = \Gamma^\mu_{\mu r} = \Gamma^\mu_{\mu r} = 0,$$

(31)

$$\Gamma_{rr\mu} = \Gamma_{r\mu r} = -\Gamma^r_{\mu r} = \lambda \partial_\mu \lambda,$$

(32)

$$\Gamma_{rrr} = 0.$$  

(33)
On the other hand, in the same limit $\Delta^r z \to 0$ after differentiating the Eq. (26) with respect to $z(g)$ together with the fact $\partial_z \Delta^r z(g) = -2$, one can get

$$ (\partial_r \hat{\Gamma}_{MN} + \partial_r \hat{\Gamma}_{NM})_0 = 2 \Gamma_{K}^{K} \Gamma_{rN}^{r}, \quad (34) $$

where the quantity with a subscript 0 also means that quantity is independent of $g$. By using Eqs. (31), (32) and (34) with $M = \mu$ and $N = \nu$, one can see

$$ (\partial_r \hat{\Gamma}_{\mu\nu} + \partial_r \hat{\Gamma}_{\nu\mu})_0 = 2 \Gamma_{K\mu}^{K} \Gamma_{r\nu}^{r} = 2 \Gamma_{r\nu}^{r} \Gamma_{r\mu}^{r} \quad (35) $$

and by Eq. (28)

$$ \partial_v G_{\mu\nu} = \hat{\Gamma}_{\mu\nu} + \hat{\Gamma}_{\nu\mu} = 0, \quad (36) $$

which with Eq. (18) reads

$$ \hat{\Gamma}_{\mu\nu} = -\hat{\Gamma}_{\nu\mu} = -\hat{\Gamma}_{r\mu} = \hat{\Gamma}_{\nu\mu} \quad (37) $$

From this equation (37) one can see that the first or the second term in the left-hand side of (35) is equal to each other, hence

$$ (\partial_r \hat{\Gamma}_{\mu\nu})_0 = \Gamma_{r\mu}^{r} \Gamma_{r\nu}^{r} = -(\partial_r \hat{\Gamma}_{r\mu})_0. \quad (38) $$

Another important equation to evaluate the curvatures comes from Eq. (34) with $M = N = r$

$$ (\partial_r \hat{\Gamma}_{r\nu})_0 = \Gamma_{K\nu}^{K} \Gamma_{r\nu}^{r} = \Gamma_{r\nu}^{r} \Gamma_{r\nu}^{r}. \quad (39) $$

In the limit $\Delta^r z \to 0$, three curvature tensors (20a)-(20c) can be rewritten using Eqs. (27) and (28) as follows:

$$ R_{MN\mu\nu} = \partial_\mu \Gamma_{MN\nu} - \partial_\nu \Gamma_{MN\mu} - \Gamma_{LM\mu} \Gamma_{N\nu}^{L} + \Gamma_{LM\nu} \Gamma_{N\mu}^{L}, \quad (40a) $$

$$ R_{MN\mu\nu} = \partial_\mu \Gamma_{MNr} - (\partial_r \hat{\Gamma}_{MN})_0 - \Gamma_{LMr} \Gamma_{N\nu}^{L} + \Gamma_{LMr} \Gamma_{N\mu}^{L}, \quad (40b) $$

$$ R_{MNrr} = -(\partial_r \hat{\Gamma}_{MNr})_0 + \Gamma_{LMr} \Gamma_{N\nu}^{L} + \Gamma_{LMr} \Gamma_{N\mu}^{L}. \quad (40c) $$

The equation (10a) with $M = \rho$ and $N = \nu$ reads

$$ R_{\rho\sigma\mu\nu} = \partial_\mu \Gamma_{\rho\sigma\nu} - \partial_\nu \Gamma_{\rho\sigma\mu} - \Gamma_{L\rho\mu} \Gamma_{\sigma\nu}^{L} + \Gamma_{L\rho\nu} \Gamma_{\sigma\mu}^{L} $$

$$ = \partial_\mu \Gamma_{\rho\sigma\nu} - \partial_\nu \Gamma_{\rho\sigma\mu} - \Gamma_{\lambda\rho\mu} \Gamma_{\sigma\nu}^{\lambda} + \Gamma_{\lambda\rho\nu} \Gamma_{\sigma\mu}^{\lambda} = R_{\rho\sigma\mu\nu}. \quad (41) $$

which gives the conventional scalar curvature on $M_4$

$$ R = g^{\mu\nu} R_{\mu\rho\nu}^{\rho}. \quad (42) $$
The equation (40b) together with (31), (32) and (38) gives the relevant component $R_{\nu r\mu r}$ as

\[
R_{\nu r\mu r} = \partial_\mu \Gamma_{\nu rr} - (\partial_r \hat{\Gamma}_{\nu \mu r})_0 - \Gamma_{L\nu \mu} \Gamma^L_{rr} + \Gamma_{L \nu r} \Gamma^L_{r\mu} \\
= \partial_\mu \Gamma_{\nu rr} - \Gamma_{rr \nu} \Gamma^r_{\mu r} - \Gamma_{r \nu \mu} \Gamma^r_{rr} + \Gamma_{r \nu r} \Gamma^r_{r \mu} \\
= -\nabla_\mu (\lambda \partial_\nu \lambda),
\]

(43)

where $\nabla_\mu$ is the covariant derivative in $M_4$, hence, one get

\[R^\rho_{r \rho r} = -\nabla^\rho (\lambda \partial_\rho \lambda).
\]

(44)

In the same way one can get

\[
R_{r\nu\mu\nu} = (\partial_r \hat{\Gamma}_{r \nu \mu})_0 - \partial_\mu \Gamma_{r \nu r} - \Gamma_{L r \nu} \Gamma^L_{\nu \mu} + \Gamma_{L \nu r} \Gamma^L_{r \mu} \\
= -\Gamma_{r \nu \nu} \Gamma^r_{\mu r} - \partial_\mu \Gamma_{r \nu r} - \Gamma_{r r \nu} \Gamma^r_{\nu \mu} + \Gamma_{r r \nu} \Gamma^r_{r \mu} \\
= -\nabla_\mu (\lambda \partial_\nu \lambda),
\]

(45)

so that

\[R^\rho_{r \rho r} = G^{\rho r} R_{r \rho r} = -\lambda^{-2} \nabla_\nu (\lambda \partial_\mu \lambda).
\]

(46)

The equation (40d) with $M = N = r$ together with (32) and (33) gives

\[
R_{r r r r} = -(\partial_r \hat{\Gamma}_{r r r})_0 + \Gamma_{L r r} \Gamma^L_{r r} + \Gamma_{L L r} \Gamma^L_{L r} \\
= -(\partial_r \hat{\Gamma}_{r r r})_0 + \Gamma_{\rho r r} \Gamma^\rho_{r r} + \Gamma_{r \rho \rho} \Gamma^r_{r r} \\
= -\lambda^2 \partial_\rho \lambda \partial^\rho \lambda,
\]

(47)

hence

\[R^r_{r r r} = G^{r r} R_{r r r r} = -\partial_\rho \lambda \partial^\rho \lambda,
\]

(48)

which is identical to that in the previous work[3] as is expected. The Ricci curvature on $M_4 \times Z_2$ is therefore given by

\[R_{MN} \equiv R^L_{MLN} = R^\rho_{M \rho N} + R^r_{MrN},
\]

(49)

hence

\[
R_{\mu \nu} = R_{\mu \nu} + R^r_{\mu \nu r} \\
= R_{\mu \nu} - \lambda^{-2} \nabla_\nu (\lambda \partial_\mu \lambda),
\]

(50)

\[
R_{r r} = R^\rho_{r r \rho} + R^r_{r r r} \\
= -\nabla_\rho (\lambda \partial^\rho \lambda) - \partial_\rho \lambda \partial^\rho \lambda,
\]

(51)
where $R_{\mu\nu} \equiv R^\rho_{\mu\rho\nu}$ is the conventional four dimensional Ricci curvature. Finally, therefore, one get the scalar curvature on $M_4 \times \mathbb{Z}_2$

$$R \equiv G^{MN} R_{MN}$$
$$= g^{\mu\nu} R_{\mu\nu} + G^{rr} R_{rr}$$
$$= R - 2\lambda^{-2} \nabla_\rho (\lambda \partial^\rho \lambda) - \lambda^{-2} \partial_\rho \lambda \partial^{\rho} \lambda.$$  \hfill (52)

The action of the gravity can be now obtained by using the fact $\det(G_{MN}) = \det(g_{\mu\nu}) \lambda^2$

$$I = \int_{M_4} \int_{\mathbb{Z}_2} \sqrt{-\det(G_{MN})} \cdot R$$
$$= \int_{M_4} \sqrt{-\det(g_{\mu\nu})} \lambda [R - 2\lambda^{-2} \nabla_\rho (\lambda \partial^\rho \lambda) - \lambda^{-2} \partial_\rho \lambda \partial^{\rho} \lambda]$$
$$= \int_{M_4} \sqrt{-\det(g_{\mu\nu})} [\lambda R - 3 \lambda \partial_\mu \lambda \partial^{\mu} \lambda],$$  \hfill (53)

which is just the Brans-Dicke theory with Dicke constant $\omega = 3$.

Inspired by the result above, we also introduce the cosmological constant into the Einstein gravity on $M_4 \times \mathbb{Z}_2$ geometry in order to determine the explicit form of the cosmological term in the Brans-Dicke theory on $M_4$ geometry. The calculation itself is as same as the pure Einstein case, and one obtains the action of the Brans-Dicke theory with the cosmological term $\Lambda$ in the Jordan frame as

$$I = \int_{M_4} \int_{\mathbb{Z}_2} \sqrt{-\det(G_{MN})} (R - 2\Lambda)$$
$$= \int_{M_4} \sqrt{-\det(g_{\mu\nu})} [\lambda (R - 2\Lambda) - 3 \lambda \partial_\mu \lambda \partial^{\mu} \lambda],$$  \hfill (54)

to which we got the exact solutions\textsuperscript{7} with an undetermined Dicke constant for the spatially flat cosmology some years ago. The choice of this form was done at that time by another reason, that is, such a cosmological term simply leads to the mass term of the scalar field, although the general form of the scalar-tensor gravitation theory\textsuperscript{8} including the cosmological term was already proposed to attempt the general theory.

On the basis of the equivalence assumption and also of the isometric conditions (25) and (26), we have derived the Brans-Dicke theory from the manifold $M_4 \times \mathbb{Z}_2$. We have also clarified the geometrical relation between torsions and curvature in this space. In the work\textsuperscript{2} the contribution from the torsion to the curvature in Eq.(20c) has not been taken into
account. The existence of the torsion term has actually played an essential role to reproduce the result given by the concept of the parallel transport. As for the Dicke constant, one may get a greater value of \( \omega \) if the structure of the discrete space has more degrees of freedom. Finally we have determined the explicit form of the cosmological term in the Brans-Dicke theory on \( M_4 \) geometry within the framework of the theory.

Acknowledgements
The author would like to express his gratitude for the hospitality at the Abdus Salam International Centre for Theoretical Physics. This work is supported partly by the research abroad grant of Tezukayama university and partly by the promotion and mutual aid corporation for private schools of Japan.

References
[1] A. Connes, The Interface of Math. & Particle Phys., ed. D. Quillen, G. B. Segal and S.T. Tsou (Clarendon Press, Oxford, 1990).
A. Connes and J. Lott, Nucl. Phys.(Proc. Suppl.) B18, 29 (1990).
See also A. Connes, Noncommutative Geometry (Academic Press, Inc., 1994).

[2] B. Chen, T. Saito, H.B. Teng, K. Uehara and K. Wu, Prog. Theor. Phys., 95, 1173 (1996).
T. Saito and K. Uehara, Phys. Rev. D56, 2390 (1997).

[3] A. Kokado, G. Konisi, T. Saito and K. Uehara, Prog. Theor. Phys. 96, 1291 (1996).

[4] C. Brans and R.H. Dicke, Phys. Rev. 124, 925 (1961).
R.H. Dicke, Phys. Rev. 125, 2163 (1962).

[5] G. Konisi, T. Saito and K. Wu, Prog. Theor. Phys. 93, 621 (1995).

[6] A. Kokado, G. Konisi, T. Saito and Y. Tada, “Scalar-Tensor Theory of Gravity on \( M_4 \times Z_2 \) Geometry”, hep-th/9705192.

[7] K. Uehara and C.W. Kim, Phys. Rev. D26, 2575 (1982).

[8] R.V. Wagoner, Phys. Rev. D1, 3209 (1970).