Proof of a generalized Geroch conjecture for the hyperbolic Ernst equation

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Abstract

We enunciate and prove here a generalization of Geroch’s famous conjecture concerning analytic solutions of the elliptic Ernst equation. Our generalization is stated for solutions of the hyperbolic Ernst equation that are not necessarily analytic, although it can be formulated also for solutions of the elliptic Ernst equation that are nowhere axis-accessible.

1 A generalized Geroch conjecture

In terms of Weyl canonical coordinates \((z, \rho)\), the Ernst equation of general relativity can be expressed in the form

\[
(\text{Re } \mathcal{E}) \left\{ \frac{\partial^2 \mathcal{E}}{\partial z^2} \pm \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \mathcal{E}}{\partial \rho} \right) \right\} = \left( \frac{\partial \mathcal{E}}{\partial z} \right)^2 \pm \left( \frac{\partial \mathcal{E}}{\partial \rho} \right)^2,
\]

where the upper signs correspond to the elliptic equation associated with stationary axisymmetric (spinning body) gravitational fields and the lower signs correspond to the hyperbolic equation associated with colliding gravitational plane wave pairs and cylindrical gravitational waves. In 1972 R. Geroch asserted a conjecture concerning the solution manifold of the elliptic Ernst equation that was eventually proved by the present authors, who used their own homogeneous Hilbert problem version of the Kinnersley–Chitre realization of the Geroch group.

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1In the latter case, one of the Weyl coordinates has the character of a time coordinate. In practice a notation more appropriate for the physical problem being treated would be in order.

2R. Geroch, J. Math. Phys. 13, 394-404 (1972).

3I. Hauser and F. J. Ernst, A new proof of an old conjecture, in Gravitation and Geometry, Eds. Rindler and Trautman, Bibliopolis, Naples (1987).
In 1986, at the suggestion of S. Chandrasekhar, we turned our attention from stationary axisymmetric fields to colliding gravitational plane wave pairs. While the Kinnersley–Chitre transformations could still be used to generate scores of exact analytic solutions of the hyperbolic Ernst equation, we were aware of the fact that there might exist a significantly larger group, for, whereas any $C^3$ solution of the axis-accessible elliptic Ernst equation can be shown to be automatically an analytic solution, a solution of the hyperbolic Ernst equation can be even $C^\infty$ without being analytic. Clearly, one should not expect a non-analytic solution of the hyperbolic Ernst equation to be related to Minkowski space by a K–C transformation, for these transformations preserve analyticity.

A. Linear systems for the Ernst equation

Any discussion of the Geroch group or its extensions requires a knowledge of at least one linear system

\[ dF(x, \tau) = \Gamma(x, \tau)F(x, \tau) \]  

(1A.1) for the Ernst equation. Here $x$ is shorthand for the nonignorable spacetime coordinates (e.g., $z$ and $\rho$), $\tau$ is a spacetime-independent complex-valued parameter, and the $2 \times 2$ matrix $\Gamma(x, \tau)$ satisfies the integrability condition

\[ d\Gamma(x, \tau) - \Gamma(x, \tau)\Gamma(x, \tau) = 0 \]  

(1A.2)

if and only if the Ernst equation is satisfied. The symbol $\Gamma(x, \tau)$ was chosen because of the resemblance of the last equation to a zero-curvature condition for a connection 1-form.

If there exists one such $\Gamma(x, \tau)$ for the Ernst equation, then there are infinitely many, for if

\[ \Gamma'(x, \tau) := p(x, \tau)\Gamma(x, \tau)p(x, \tau)^{-1} + dp(x, \tau)p(x, \tau)^{-1}, \]  

(1A.3)

where $p(x, \tau)$ is an invertible matrix, then

\[ d\Gamma'(x, \tau) - \Gamma'(x, \tau)\Gamma'(x, \tau) = p(x, \tau)\{d\Gamma(x, \tau) - \Gamma(x, \tau)\Gamma(x, \tau)\} p(x, \tau)^{-1}. \]  

(1A.4)

This transformation is nothing but a gauge transformation, the analog of the effect that a mere change of basis has upon a connection 1-form. Under such a gauge transformation, the matrix $F(x, \tau)$ transforms into the matrix

\[ F'(x, \tau) := p(x, \tau)F(x, \tau). \]  

(1A.5)

While, in one sense, the various possible representations of the linear system may be regarded as equivalent, in another sense they may be quite different, with the matrices $F(x, \tau)$ and $F'(x, \tau)$ possibly having very different domains in the space $R^2 \times C$, as well as different continuity and/or differentiability properties. Often one representation is more useful for one part of the analysis, while another representation is more useful for another part.

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4Even the elliptic equation admits a larger group if solutions are considered that are everywhere axis-inaccessible.

5Such linear systems have been found by many authors, including China, Harrison, Kinnersley and Chitre, Maison, Neugebauer and Papanicolaou. A more complicated type of linear system was found by Belinski and Zakharov.
Different formalisms may also differ with respect to the number of columns that the matrix \( F \) has. Here we shall follow an approach that we described long ago that effectively sidesteps the question of number of columns by introducing an auxiliary \( 2 \times 2 \) matrix potential \( \mathcal{F}(x, \tau) \) such that

\[
F(x, \tau) = \mathcal{F}(x, \tau)F(x_0, \tau),
\]

\[
d\mathcal{F}(x, \tau) = \Gamma(x, \tau)\mathcal{F}(x, \tau)
\]

and

\[
\mathcal{F}(x_0, \tau) = I,
\]

where \( I \) is a unit matrix, and \( x_0 \) is a selected spacetime point within the domain of \( \mathcal{E}(x) \).

Clearly, under a gauge transformation (1A.5), \( \mathcal{F}(x, \tau) \) transforms into

\[
\mathcal{F}'(x, \tau) := p(x, \tau)\mathcal{F}(x, \tau)p(x_0, \tau)^{-1}.
\]

One of the simplest formulations of the linear system is that of G. Neugebauer in which \( \Gamma(x, \tau) = \Gamma_N(x, \tau) \), where

\[
\Gamma_N(x, \tau) := \left( \frac{\tau - z \pm \rho^*}{\tau - z \mp \rho^*} \right)^{1/2} \begin{pmatrix}
0 & \frac{d\mathcal{E}(x)}{df(x)} \\
\frac{d^*\mathcal{E}(x)}{2f(x)} & 0
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
\frac{d\mathcal{E}(x)}{2f(x)} & 0 \\
0 & \frac{d^*\mathcal{E}(x)}{2f(x)}
\end{pmatrix},
\]

where \( \ast \) is a 2-dimensional duality operator such that

\[
\ast d\rho = \pm dz, \quad \ast dz = -d\rho,
\]

the upper signs applying in the stationary axisymmetric (elliptic) case, and the lower signs applying in the gravitational wave (hyperbolic) case. Here \( \Gamma(x, \tau) \) is expressed directly in terms of the Ernst potential \( \mathcal{E}(x) \) and its complex conjugate, with \( f(x) := \text{Re} \mathcal{E}(x) \). Using these notations, the Ernst equation (1.1) can be expressed as

\[
(\text{Re} \mathcal{E})d(\rho \ast d\mathcal{E}) = \rho d\mathcal{E} \ast d\mathcal{E}.
\]

A slightly different linear system that is due to the authors and is more suited to our purpose employs \( \Gamma = \Gamma_{HE} \), where

\[
\Gamma_{HE}(x, \tau) := \left( \frac{\tau - z \pm \rho^*}{\tau - z \mp \rho^*} \right)^{1/2} \begin{pmatrix}
0 & \frac{Idf(x) \mp Jd\chi(x)}{2f(x)} \\
\frac{d^*\mathcal{E}(x)}{2f(x)} & 0
\end{pmatrix} \sigma_3 \mp J \frac{d\chi(x)}{2f(x)}
\]

and

\[
\chi := \text{Im} \mathcal{E}, \quad J := \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}, \quad \sigma_3 := \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}.
\]

The 1-form \( \Gamma_{HE} \) can be obtained from \( \Gamma_N \) by the gauge transformation (1A.3) corresponding to \( p = p_{N \rightarrow HE} \), where

\[
p_{N \rightarrow HE}(x, \tau) = \frac{1}{2\sqrt{|f(x)|}} \begin{pmatrix}
1 \pm i & -1 \mp i \\
1 \mp i & 1 \pm i
\end{pmatrix}.
\]

\(^6\)G. Neugebauer, Bäcklund transformations of axially symmetric stationary gravitational fields, Phys. Lett. A 12, L67 (1979).
On the other hand, the Kinnersley–Chitre formulation of the linear system corresponds to the choice \( \Gamma(x, \tau) = \Gamma_{KC}(x, \tau) \), where

\[
\Gamma_{KC}(x, \tau) := \frac{1}{2} \Lambda(x, \tau)^{-1} dH(x)\Omega, \quad \Omega := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},
\]

(1A.16)

with

\[
\Lambda(x, \tau) := \tau - (z \pm \rho^*)
\]

(1A.17)

and \( H(x) \) a 2 \times 2 matrix generalization of the Ernst potential \( E(x) \) that can be introduced in the following manner.

It is well-known that any vacuum spacetime possessing two commuting Killing vector fields can be described in terms of a 2 \times 2 real symmetric matrix \( h(x) \) (a 2 \times 2 block of the metric tensor) that depends exclusively on the nonignorable coordinates, and that this matrix satisfies the equation

\[
d[\rho \star dh^{-1}] = d[\rho \star h^{-1} dh] = 0,
\]

(1A.18)

where

\[
\rho := \sqrt{|\det h|}.
\]

Equation (1A.18) can be used to justify the introduction of a complex \( H \)-potential that satisfies the equations

\[
\rho d(\text{Im } H) = i h \Omega \star dh \quad \text{and} \quad \text{Re } H = -h,
\]

(1A.20)

or, equivalently,

\[
2(z \pm \rho^*) dH = (H + H^\dagger) \Omega dH,
\]

(1A.21)

where

\[
H - H^T = 2 z \Omega \quad \text{and} \quad \text{Re } H = -h.
\]

(1A.22)

Then it is not difficult to establish that \( \Gamma(x, \tau) \) as given by Eq. (1A.16) satisfies the zero-curvature condition (1A.2) if and only if \( E := H_{22} \) satisfies the Ernst equation.

The reader can verify that the K–C connection (1A.16) is related to the H–E connection (1A.13) by

\[
\Gamma_{KC}(x, \tau) := p(x, \tau) \Gamma_{HE}(x, \tau)p(x, \tau)^{-1} + dp(x, \tau)p(x, \tau)^{-1},
\]

(1A.23)

and \( F_{KC}(x, \tau) \) is related to \( F_{HE}(x, \tau) \) by

\[
F_{KC}(x, \tau) = p(x, \tau) F_{HE}(x, \tau)p(x_0, \tau)^{-1},
\]

(1A.24)

where

\[
p(x, \tau) = \frac{1}{\sqrt{|h_{22}(x)|}} \begin{pmatrix} 1 & \mp h_{12}(x) \\ 0 & |h_{22}(x)| \end{pmatrix} P^M(x, \tau),
\]

(1A.25)

\[
P^M(x, \tau) := \begin{pmatrix} 1 & \pm i(\tau - z) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \mu(x, \tau)^{-1} \end{pmatrix} \frac{1}{\sqrt{2}} (\mp \sigma_3 - \sigma_2)
\]

(1A.26)

\footnote{W. Kinnersley and D. M. Chitre, \textit{Symmetries of the stationary Einstein-Maxwell field equations, III}, J. Math. Phys. 19, 1926–1931 (1978).}
and

$$\mu(x, \tau) := \sqrt{(\tau - z)^2 + \rho^2}, \quad \lim_{\tau \to \infty} \frac{\mu(x, \tau)}{\tau} := 1.$$  \hspace{1cm} (1A.27)

Note that, for fixed $x$, $\mu(x, \tau)$ is a holomorphic function of $\tau$ throughout a cut complex plane. It has branch points of index 1/2 at the zeroes of $\mu(x, \tau)$, which are at the end points of the branch cut, and a simple pole at $\tau = \infty$.

We shall assume that this brief review of the three formulations of the linear system for the Ernst equation and the relationships among these formulations will suffice. In the rest of this paper we shall suppress the subscript $KC$ on $\Gamma_{KC}(x, \tau)$ and $F_{KC}(x, \tau)$ as we proceed to discuss how a group $K$ such as the Geroch group can be described in terms of its action upon the potentials $F_{KC}(x, \tau)$ associated with the spacetimes in question.

**B. The set $S_F$ of Kinnersley–Chitre $F$-potentials**

In order to discuss in a meaningful way the action of the group $K$ upon the potentials $F(x, \tau)$, we must first identify the set $S_F$ of $F$-potentials being considered, and this requires, in particular, the specification of the domain of $F(x, \tau)$. This can best be done by first specifying the domain of $H(x)$ [and $E(x)$], and then choosing the gauge of $F(x, \tau)$ so as to minimize its singularities in the complex $\tau$-plane. Throughout the rest of this paper we shall be concerned exclusively with the hyperbolic Ernst equation, where we find it convenient to introduce null coordinates $r := z - \rho$ and $s := z + \rho$ and to adopt the $E$-potential domain (see Fig. 1)

$$D := \text{dom } E := \{(r, s) : r_1 < r < r_2, s_2 < s < s_1, r < s\}. \hspace{1cm} (1B.1)$$

It is to be understood that $r_1$ may be $-\infty$ and/or $s_1$ may be $+\infty$. Moreover, we restrict attention to domains $D$ such that $r_1 < s_2$ and $r_2 < s_1$; i.e., $\rho > 0$ at both the lower left vertex $(r_1, s_2)$ and the upper right vertex $(r_2, s_1)$, while $\rho$ may be greater than, less than or equal to zero at the lower right vertex $x_2 := (r_2, s_2)$. Finally, we select one point $x_0 := (r_0, s_0) \in D$ such that the null line segments $\{(r, s_0) : r_1 < r < r_2\}$ and $\{(r_0, s) : s_2 < s < s_1\}$ lie entirely within $D$; and at this point we assign the Minkowski space value $E(x_0) = -1$ to the complex $E$-potential.\footnote{We have also considered more general domains and a more general choice for $x_0$, but to include discussion of these extensions here would unnecessarily complicate our exposition.} It is our intention to solve an initial value problem in which $E(x)$ is determined throughout $D$ from its values specified on the two null line segments through the point $x_0$.

For a given choice of the triple $(x_0, x_1, x_2)$, we shall define

$$S_E := \{\text{the set of all complex-valued functions } E \text{ such that }$$

$\text{dom } E = D$, the derivatives $E_r(x)$, $E_s(x)$ and $E_{rs}(x)$ exist and are continuous at all $x \in D$, $f := \text{Re } E > 0$ and $E$ satisfies Eq. (1A.12) throughout $D$, and $E(x_0) = -1$. $\hspace{1cm} (1B.2)$

The metric components $h_{ab}$ corresponding to each given $E \in S_E$ are defined by $h_{22} := -f$, $\omega := \rho f^{-2} * d\chi$ such that $\omega(x_0) := 0$, $h_{12} := \omega h_{22}$ and $h_{11} := [(h_{12})^2 + \rho^2]/h_{22}$.
Figure 1: An $\mathcal{E}$-potential domain $D$ for which $s_2 < r_2$ is illustrated. The null line segments through $\mathbf{x}_0$ are represented by the vertical and horizontal dashed lines.
Naturally, we shall let \( \text{dom } H = D \) and assign the value
\[
H(x_0) = H^M(x_0),
\]
where \( H^M \) is the Minkowski space \( H \)-potential with values
\[
H^M(x) = -\begin{pmatrix}
\rho^2 & 0 \\
2\i z & 1
\end{pmatrix}.
\]

For a given choice of the triple \((x_0, x_1, x_2)\), we shall define
\[
\mathcal{S}_H := \text{the set of all complex-valued } 2 \times 2 \text{ matrix functions } H \text{ with } \text{dom } H := D \text{ such that there exists } E \in \mathcal{S}_E \text{ for which } \text{Re } H = -h, d(\text{Im } H) \text{ exists and satisfies } \\
\rho d(\text{Im } H) = i h \Omega \ast dh \text{ and the gauge condition } (1B.3) \text{ holds.}
\]

Let \( \mathcal{I}^{(3)}(x) \) denote the open interval with end points \( r, r_0 \) and \( \mathcal{I}^{(4)}(x) \) denote the open interval with end points \( s, s_0 \), and let \( \mathcal{I}^{(3)}(x) \) and \( \mathcal{I}^{(4)}(x) \) denote, respectively, the closures of these two intervals. Furthermore, let
\[
\mathcal{I}(x) := \mathcal{I}^{(3)}(x) \cup \mathcal{I}^{(4)}(x), \text{ and } \\
\mathcal{I}(x) := \bar{\mathcal{I}}^{(3)}(x) \cup \bar{\mathcal{I}}^{(4)}(x).
\]

Note that \( \bar{\mathcal{I}}^{(3)}(x) \) is empty if \( r = r_0 \) and \( \bar{\mathcal{I}}^{(4)}(x) \) is empty if \( s = s_0 \). When neither \( r = r_0 \) nor \( s = s_0 \), the set \( \mathcal{I}(x) \) comprises two disjoint closed sets (for \( x_0 \) chosen as indicated earlier). The gauge of the \( F \)-potential can be chosen so that
\[
\text{dom } F := \{(x, \tau) : x \in D, \tau \in C - \bar{\mathcal{I}}(x)\}.
\]

For a given choice of the triple \((x_0, x_1, x_2)\), we shall define
\[
\mathcal{S}_F := \text{the set of all complex-valued } 2 \times 2 \text{ matrix functions } F \text{ with domain } (1B.8) \text{ such that there exists } H \in \mathcal{S}_H \text{ such that, for all } x \in D \text{ and } \tau \in [C - \bar{\mathcal{I}}(x)] - \{r_0, s_0\}, \\
dF(x, \tau) \text{ exists and Eq. (1A.7) holds, subject to the condition } (1A.8) \text{, and, for each } (r, s) \in D, F((r, s), \tau) \text{ are continuous functions of } \tau \text{ at } \\
\tau = s_0 \text{ and at } \tau = r_0, \text{ respectively.}
\]

Remember that at \( x = x_0 \), \( F(x, \tau) \) reduces to the \( 2 \times 2 \) unit matrix.

With these definitions one can establish the properties enumerated in the following theorem, the proof of which is (except for conventions and notations and the choice of the domain \( D \)) essentially the same as that given in two earlier papers on the IVP (initial value problem) for colliding gravitational plane wave pairs by the present authors. The complex-valued functions \( \mathcal{E}^{(3)} \) and \( \mathcal{E}^{(4)} \) with respective domains \( \mathcal{I}^{(3)} := \{r : r_1 < r < r_2\} \) and \( \mathcal{I}^{(4)} := \{s : s_2 < s < s_1\} \) serve as initial value data for the \( \mathcal{E} \)-potential on the null line segments through the point \( x_0 \).

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9I. Hauser and F. J. Ernst, *Initial value problem for colliding gravitational plane waves-III/IV*, J. Math. Phys. 31, 871–881 (1990), 32, 198–209 (1991). In these papers we used ‘\( P \)' in place of ‘\( F \)’. 7
THEOREM 1 (Initial Value Problem)

(i) For each $H \in S_H$, the corresponding $F \in S_F$ exists and is unique; and, for each $x \in D$, $F(x, \tau)$ is a holomorphic function of $\tau$ throughout $C - \bar{T}(x)$ and, in at least one neighborhood of $\tau = \infty$,

$$F(x, \tau) = I + (2\tau)^{-1} [H(x) - H(x_0)] \Omega + O(\tau^{-2}).$$  \hspace{1cm} (1B.10)

(ii) For each $F \in S_F$, there is only one $H \in S_H$ for which $dF(x, \tau) = \Gamma(x, \tau)F(x, \tau)$.

(iii) With the understanding that

$$\text{dom } \nu := \{(x, \tau) : x \in D \text{ and } \tau \in C - \bar{T}(x)\}$$  \hspace{1cm} (1B.11)

and that $\nu(x, \infty) = 1$, we have

$$\det F(x, \tau) = \nu(x, \tau) := \frac{\mu(x_0, \tau)}{\mu(x, \tau)} = \left(\frac{\tau - r_0}{\tau - r}\right)^{1/2} \left(\frac{\tau - s_0}{\tau - s}\right)^{1/2}. \hspace{1cm} (1B.12)$$

(iv) The member of $S_F$ that corresponds to $E^M$ is given by

$$F^M(x, \tau) = \begin{pmatrix} 1 & -i(\tau - z) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \nu(x, \tau) \end{pmatrix} \begin{pmatrix} 1 & i(\tau - z_0) \\ 0 & 1 \end{pmatrix}. \hspace{1cm} (1B.13)$$

(v) For each $E \in S_E$, there is exactly one $H \in S_H$ such that $E = H^{22}$.

(vi) If, for each $i \in \{3, 4\}$, $E^{(i)}$ is $C^{n_i}$ ($n_i \geq 1$), then, for all $0 \leq k < n_3$ and $0 \leq m \leq n_4$, the partial derivatives $\partial^{k+m} H(x)/\partial r^k \partial s^m$ exist and are continuous throughout $D$. If, for each $i \in \{3, 4\}$, $E^{(i)}$ is analytic, then $H$ is analytic.

(vii) For each choice of complex valued functions $E^{(3)}$ and $E^{(4)}$ for which (for $i \in \{3, 4\}$) $\text{dom } E^{(i)} = \mathcal{I}^{(i)}, E^{(i)}$ is $C^1$, $f^{(i)} := \text{Re } E^{(i)} < 0$ throughout $\mathcal{I}^{(i)}$, and $E^{(3)}(r_0) = -1 = E^{(4)}(s_0)$, there exists exactly one $E \in S_E$ such that

$$E^{(3)}(r) = E(r, s_0) \text{ and } E^{(4)}(s) = E(r_0, s)$$

for all $r \in \mathcal{I}^{(3)}$ and $s \in \mathcal{I}^{(4)}$, respectively.

C. Homogeneous Hilbert problem

The HHP that we developed for effecting K–C transformations\textsuperscript{10}\hspace{1cm}(adapted to the hyperbolic case) involved a closed contour in the complex $\tau$-plane surrounding the arcs that comprise $\bar{T}(x)$. This was fine as long as we were dealing with the analytic case, but now we must instead formulate an HHP on those arcs themselves, and this will involve the limiting values of $F(x, \tau)$ as $\tau$ approaches points on those arcs. What we discovered concerning these

\textsuperscript{10}I. Hauser and F. J. Ernst, A homogeneous Hilbert problem for the Kinnersley–Chitre transformations, J. Math. Phys. 21, 1126-1140 (1980).
limiting values is contained in the following theorems, the proofs of which are based upon a classic method of reducing the solving of a total differential equation to the solving of a pair of ordinary linear differential equations along characteristic lines in $D$. The Picard method of successive approximations and certain well known theorems of infinite sequences of functions are used to demonstrate existence, continuity and differentiability properties of the solution.\footnote{For each $\sigma \in R^1$ and for fixed $x \in D$, the limits of $F(x, \sigma \pm \zeta)$ as $\zeta \to 0 (\text{Im} \; \zeta > 0)$ exist. Moreover, $F(x, \tau^*) = F(x, \tau^*)$ and $\text{det} \; F(x, \tau) = 1$. For these reasons, we found it convenient to use the H–E representation of the linear system in developing this proof, translating the results into corresponding results for the K–C representation.}

**THEOREM 2 (Limits of $F$)**

(i) For each $x \in D$ and and $\sigma \in I(x)$ the limits $F^\pm(x, \sigma) := \lim_{\zeta \to 0} F(x, \sigma \pm \zeta) (\text{Im} \; \zeta > 0)$ exist.

(ii) Further, let $\alpha$ and $\beta$ be points of $I(x)$ such that $\alpha \in \{r_0, s_0\}$ and $\beta \in \{r, s\}$, and let $\tau \in C - I(x)$. Then the following limits all exist and are equal as indicated:

$$
\begin{align*}
\lim_{\sigma \to \alpha} F^\pm(x, \sigma) &= \lim_{\tau \to \alpha} F(x, \tau), \\
\lim_{\sigma \to \beta} [F^\pm(x, \sigma)]^{-1} &= \lim_{\tau \to \beta} [F(x, \tau)]^{-1}.
\end{align*}
$$

(1C.1) (1C.2)

We shall employ $\Box$ as a generic superscript that stands for $n, n+, \infty$ or ‘an’ (analytic). The symbols $C^n$ and $C^\infty$ are self explanatory. We shall say that $f$ is $C^{n+}$ if its $n$th derivative $D^n f$ exists throughout $\text{dom} \; f$ and $D^n f$ obeys a H"older condition of arbitrary index on each closed subinterval of $\text{dom} \; f$\footnote{The index may be different for different closed subintervals of $\text{dom} \; f$.}

If $f$ is a real- or complex-valued function, the domain of which is a union of disjoint intervals of $R^1$, and $[a, b]$ is a given closed subinterval of $\text{dom} \; f$, then $f$ is said to obey a H"older condition of index $0 < \gamma \leq 1$ on $[a, b]$; i.e., to be $H(\gamma)$ on $[a, b]$, if there exists $M(a, b, \gamma) > 0$ such that $|f(x') - f(x)| \leq M(a, b, \gamma)|x' - x|^\gamma$ for all $x, x' \in [a, b]$. The same terminology is used if $f(x)$ is a matrix with real or complex elements, and $|f(x)|$ is its norm.

**Dfn. of the groups $K^\Box$ and $K$**

In order to describe our extensions $K^\Box$ of the Geroch group, we shall introduce groups $K^\Box$ of $2 \times 2$ matrix pairs; namely, the multiplicative groups of all ordered pairs $v = (v(3), v(4))$ of $2 \times 2$ matrix functions such that, for both $i = 3$ and $i = 4$,

$$
\text{dom} \; v(i) = I(i), \quad \text{det} v(i) = 1, \quad v(i) \text{ is } C^\Box
$$

(1C.3)

and the condition

$$
v(i)(\sigma)\hat{A}^M(x_0, \sigma)v(i)(\sigma) = A^M(x_0, \sigma) \text{ for all } \sigma \in I(i)
$$

(1C.4)

holds, where

$$
A^M(x_0, \sigma) := (\sigma - z_0)\Omega + \Omega h^M(x_0)\Omega, \quad h^M(x_0) := \begin{pmatrix} \rho^2 & 0 \\ 0 & 1 \end{pmatrix}.
$$

(1C.5)
Moreover, the symbol $K$ will denote the multiplicative group of all ordered pairs $v = (v^{(3)}, v^{(4)})$ of $2 \times 2$ matrix functions such that, for both $i = 3$ and $i = 4$,

$$
dom v^{(i)} = \mathcal{I}^{(i)}, \quad \det v^{(i)} = 1, \quad v^{(i)} \text{ is } H(1/2) \text{ on each closed subinterval of } \mathcal{I}^{(i)} \quad (1C.6)
$$

and the condition (1C.4) holds.

End of Dfn.

**Dfn. of the HHP corresponding to $(v, F_0)$**

For each $v \in K^{\square}$ and $F_0 \in S_F$, the **HHP corresponding to $(v, F_0)$** will mean the set of all functions $F$ [which are not presumed to be members of $S_F$] such that $dom F = \{(x, \tau) : x \in D, \tau \in C - \bar{\mathcal{I}}(x)\}$ and such that, for each $x \in D$, the functions $F(x)$ whose domains are $\mathcal{I}^{(i)}(x)$ and whose values are $F(x, \tau)$ is a solution of the HHP corresponding to $(v, F_0, x)$, i.e., a member of the set of all $2 \times 2$ matrix functions $F(x)$ such that

1. $F(x)$ is holomorphic throughout $dom F(x) := C - \mathcal{I}(x)$,
2. $F(x, \infty) = I$,
3. $F^\pm(x)$ exist, and
   $$
   Y^{(i)}(x, \sigma) := F^{+}(x, \sigma)v^{(i)}(\sigma)[F^{+}_0(x, \sigma)]^{-1} = F^{-}(x, \sigma)v^{(i)}(\sigma)[F^{-}_0(x, \sigma)]^{-1}
   $$
   for each $i \in \{3, 4\}$ and $\sigma \in \mathcal{I}^{(i)}(x)$,
4. $F(x)$ is bounded at $x_0$ and $\nu(x)^{-1}F(x)$ is bounded at $x$, and the function $Y(x)$ whose domain is $\mathcal{I}(x)$ and whose values are given by $Y(x, \sigma) := Y^{(i)}(x, \sigma)$ for each $\sigma \in \mathcal{I}^{(i)}(x)$ is bounded at $x_0$ and at $x$.

The members of the HHP corresponding to $(v, F_0)$ will be called its **solutions**.

End of Dfn.

Notes:

- $F^+(x)$ and $F^-(x)$ denote the functions that have the common domain $\mathcal{I}(x)$ and the values ($\text{Im } \zeta > 0$)
  $$
  F^\pm(x, \sigma) := \lim_{\zeta \to 0} F(x, \sigma \pm \zeta). \quad (1C.8)
  $$
  It is understood that $F^+(x)$ and $F^-(x)$ exist if and only if the above limits exist for every $\sigma \in \mathcal{I}(x)$. $\nu^+(x)$ and $\nu^-(x)$ are similarly defined.

- $\nu(x)$ denotes the function whose domain is $C - \mathcal{I}(x)$ and whose values $\nu(x, \tau)$ are defined in Eq. (1B.12).
• It is to be understood that $F(x)$, with domain $C - \bar{I}(x)$, is bounded at $x_0$ if there exists a neighborhood $\text{nbd}(x_0)$ of the set $\{r_0, s_0\}$ in the space $C$ such that

$$\{F(x, \tau) : \tau \in \text{nbd}(x_0) - \bar{I}(x)\}$$

is bounded. Likewise, $F(x)$ is said to be bounded at $x$ if there exists a neighborhood $\text{nbd}(x)$ of the set $\{r, s\}$ in the space $C$ such that

$$\{F(x, \tau) : \tau \in \text{nbd}(x) - \bar{I}(x)\}$$

is bounded.

• We say that $Y(x)$, with domain $I(x)$, is bounded at $x_0$ if there exists a neighborhood $\text{nbd}(x_0)$ of the set $\{r_0, s_0\}$ in the space $R^1$ such that

$$\{Y(x, \sigma) : \sigma \in \text{nbd}(x_0) \cap I(x)\}$$

is bounded. Likewise, $Y(x)$ is bounded at $x$ if there exists a neighborhood $\text{nbd}(x)$ of the set $\{r, s\}$ in the space $R^1$ such that

$$\{Y(x, \sigma) : \sigma \in \text{nbd}(x) \cap I(x)\}$$

is bounded.

**THEOREM 3 (Properties of HHP solution)**

Suppose that $v \in K^\Omega$, $F_0 \in S_F$ and $x \in D$ exist such that a solution $F(x)$ of the HHP corresponding to $(v, F_0, x)$ exists. Then

(i) $F^+(x)$, $F^-(x)$ and $Y(x)$ are continuous throughout $I(x)$,

(ii) $F^\pm(x)$ are bounded at $x_0$, and $[\nu^\pm(x)]^{-1}F^\pm(x)$ are bounded at $x$,

(iii) $\det F(x) = \nu(x)$, $\det Y(x) = 1$,

(iv) the solution $F(x)$ is unique, and

(v) the solution of the HHP corresponding to $(v, F_0, x_0)$ is given by

$$F(x_0, \tau) = I$$

for all $\tau \in C$.

**Proofs:**

(i) The statement that $F^+(x)$ and $F^-(x)$ are continuous is a direct consequence of a theorem by P. Painlevé which is stated and proved by N. I. Muskhelishvili. The continuity of $Y(x)$ then follows from its definition by Eq. (1C.7), the fact that $\nu^{(i)}$ is continuous and the fact that $F_{0+}^+(x)$ and $F_{0-}^-(x)$ are continuous. End of proof.

---

13N. I. Muskhelishvili, *Singular Integral Equations*, Ch. 2, Sec. 14, pp. 33-34 (Dover, 1992).
(ii) From Eq. (1C.7),
\[ F^\pm(x) = Y^{(i)}(x)F_0^\pm(x)[v^{(i)}]^{-1} \]  
(1C.14)
for each \( i \in \{3, 4\} \). The function \( Y(x) \) is bounded at \( x \) and at \( x_0 \) according to condition (4) in the definition of the HHP, and \( v^{(i)} \) and its inverse are continuous throughout \( \mathcal{I}^{(i)} \). Finally, \( F_0^\pm(x) \) is bounded at \( x_0 \) and \( [v^{\pm}(x)]^{-1}F_0^\pm(x) \) is bounded at \( x \), so, from Eq. (1C.14), \( F^\pm(x) \) is bounded at \( x_0 \), and \( [v(x)]^{-1}F^\pm(x) \) is bounded at \( x \). *End of proof.*

(iii) Conditions (1), (2), (3) and (4) of the definition of the HHP imply that
\[ Z_1(x) := \det \mathcal{F}(x)/\nu(x) \text{ is holomorphic throughout } C - \bar{\mathcal{I}}(x), \]  
(1C.15)
\[ Z_1(x, \infty) = 1, \]  
(1C.16)
the limits \( Z^\pm_1(x) \) exist and
\[ \det Y(x, \sigma) = Z^+_1(x, \sigma) = Z^-_1(x, \sigma) \text{ for all } \sigma \in \mathcal{I}(x), \]  
(1C.17)
\[ \nu(x)Z_1(x) \text{ is bounded at } x_0 \text{ and } \]  
\[ \nu(x)^{-1}Z_1(x) \text{ is bounded at } x, \]  
(1C.18)
and
\[ \det Y(x) = Z^\pm_1(x) \text{ is bounded at } x \text{ and at } x_0. \]  
(1C.19)
From the above statements (1C.15) and (1C.17) together with the theorem of Riemann\(^\text{14}\) on analytic continuation across an arc, \( Z_1(x) \) has a holomorphic extension to the domain \( C - \{r, s, r_0, s_0\} \); and, from the statements (1C.18) and (1C.19), together with the theorem of Riemann\(^\text{14}\) on isolated singularities of holomorphic functions, \( Z_1(x) \) has a further holomorphic extension \( Z^{ex}_1(x) \) to \( C \). Finally, the theorem of Liouville\(^\text{16}\) on entire functions that do not have an essential singularity at \( \tau = \infty \), together with Eq. (1C.16), then yields
\[ Z^{ex}_1(x, \tau) = 1 \text{ for all } C. \]  
(1C.20)
Thus, we have shown that \( \det \mathcal{F}(x) = \nu(x) \), whereupon Eq. (1C.17) yields \( \det Y(x) = 1 \). *End of proof.*

(iv) Suppose that \( \mathcal{F}'(x) \) is also a solution of the HHP corresponding to \((v, F_0, x)\). Since \( \det \mathcal{F}(x) = \nu(x) \), \( \mathcal{F}(x) \) is invertible. Conditions (1), (2), (3) and (4) in the definition of the HHP imply that
\[ Z_2(x) := \mathcal{F}'(x)\mathcal{F}(x)^{-1} \text{ is holomorphic throughout } C - \bar{\mathcal{I}}(x), \]  
(1C.21)
\[ Z_2(x, \infty) = I, \]  
(1C.22)
the limits \( Z^\pm_2(x) \) exist and
\[ Y'(x)Y(x)^{-1} = Z^+_2(x) = Z^-_2(x) \text{ throughout } \mathcal{I}(x), \]  
(1C.23)
\[^{14}\text{See Sec. 24, Ch. 1, of A Course of Higher Mathematics, Vol. III, Part Two, by V. I. Smirnov (Addison-Wesley, 1964).}\]
\[^{15}\text{See Sec. 133 of Theory of Functions of a Complex Variable, Vol. 1, by C. Caratheodory, 2nd English edition (Chelsea Publishing Company, 1983).}\]
\[^{16}\text{See Secs. 167-168 of the text by Caratheodory cited above.}\]
\[ Z_2(x) \text{ is bounded at } x \text{ and at } x_0, \quad (1C.24) \]

and
\[ Y'(x)Y(x)^{-1} = Z_2^\pm(x) \text{ is bounded at } x \text{ and at } x_0. \quad (1C.25) \]

The same kind of reasoning that was used in the proof of part (iii) of the theorem nets
\[ Z(x) = I. \text{ So } F'(x) = F(x). \]

\textit{End of proof.}

(v) When \( x = x_0 \), \( I(x) \) and its closure \( \bar{I}(x) \) are empty. So, condition (1) of the HHP definition implies that \( F(x_0) \) is holomorphic throughout \( C \), whereupon condition (2) tells us that \( F(x_0) \) has the value \( I \) throughout \( C \). \([F^\pm(x)] \) are empty sets when \( x = x_0 \); and conditions (3) and (4) hold trivially when \( x = x_0 \).

\textit{End of proof.}

D. The generalized Geroch conjecture

At this point we shall conjecture that for each \( \Box \), where \( \Box \) may be \( n \) or \( n+ \), where \( n \geq 3 \), \( \infty \) or ‘an’ (analytic), the following theorems hold:

- There exists a subset \( S_\Box^F \) of \( S_\Box \) such that, for each \( F_0 \in S_\Box^F \) and each \( \nu \in K_\Box \), there exists exactly one solution \( F \in S_\Box^F \) of the HHP corresponding to \((\nu, F_0)\), enabling us to define a mapping
  \[ [\nu] : S_\Box^F \to S_\Box^F \]

  such that, for each \( F_0 \in S_\Box^F \),
  \[ [\nu](F_0) = F \]

  is that unique solution of the HHP corresponding to \((\nu, F_0)\). We then define our extension \( K_\Box \) of the K–C group by
  \[ K_\Box := \{[\nu] \in K_\Box \} \]

  \[ (1D.3) \]

- The mapping \([\nu]\) is the identity map on \( S_\Box^F \) iff \( \nu \in Z^{(3)} \times Z^{(4)} \), where

  \[ Z^{(i)} := \{\delta^{(i)}, -\delta^{(i)}\} \]

  \[ (1D.4) \]

  and

  \[ \delta^{(i)}(\sigma) = I \text{ for all } \sigma \in I^{(i)}. \]

  \[ (1D.5) \]

- The set \( K_\Box \) is a group of permutations of \( S_\Box^F \) such that the mapping \( \nu \to [\nu] \) is a homomorphism of \( K_\Box \) onto \( K_\Box \); and the mapping \( \{\nu \nu' : \nu \in Z^{(3)} \times Z^{(4)}\} \to [\nu] \) is an isomorphism [viz, the isomorphism of \( K_\Box / (Z^{(3)} \times Z^{(4)}) \) onto \( K_\Box \)].

- The group \( K_\Box \) is transitive [i.e., for each \( F_0, F \in S_\Box^F \) there exists at least one element of \( K_\Box \) that transforms \( F_0 \) into \( F \)].

It will later be seen when we come to Thm. 35 that to prove the first part of the above generalized Geroch conjecture it is sufficient to prove that, for each \( \nu \in S_\Box^F \) with \( \Box = n \), \( n + (n \geq 3) \), \( \infty \) or ‘an’, the solution \( F \) of the HHP corresponding to \((\nu, F^M)\) exists, and \( F \in S_\Box^F \). For this reason, we shall now focus on the HHP corresponding to \((\nu, F^M)\).
We shall begin with a study of an Alekseev-type singular integral equation and a Fredholm integral equation of the second kind that are, under suitable circumstances, equivalent to the HHP corresponding to $(\nu, \mathcal{F}^M)$. Ultimately we shall have to return to the identification of the sets $\mathcal{S}^\nu_{\pm}$ for $\square = n, n+, \infty$ and ‘an’ (analytic), which will require us to introduce the concept of generalized Abel transforms of the initial data functions $\mathcal{E}^{(3)}$ and $\mathcal{E}^{(4)}$.

2 An Alekseev-type singular integral equation that is equivalent to the HHP corresponding to $(\nu, \mathcal{F}^M)$ when $\nu \in K^{1+}$

Using an ingenious argument G. A. Alekseev\(^{17}\) derived a singular integral equation, supposing that $\mathcal{F}(\tau)$ was analytic in a neighborhood of $\{r, s\}$ except for branch points of index $1/2$ at $\tau = r$ and $\tau = s$. We shall now show that the same type integral equation arises in connection with solutions of our new HHP that need not be analytic.

A. A preliminary theorem

Henceforth, whenever there is no danger of ambiguity, the arguments ‘x’ and ‘x\(_0\)’ will be suppressed. For example, ‘$\mathcal{F}(\tau)$’ and ‘$\mathcal{F}^{\pm}(\sigma)$’ will generally be used as abbreviations for ‘$\mathcal{F}(x, \tau)$’ and ‘$\mathcal{F}^{\pm}(x, \sigma)$’, respectively; and ‘$\nu(\tau)$’, ‘$\nu^{\pm}(\sigma)$’ and ‘$\bar{I}(x)$’ will generally stand for ‘$\nu(x, \tau)$’, ‘$\nu^{\pm}(x, \sigma)$’ and ‘$\bar{I}(x)$’, respectively.

THEOREM 4 (Alekseev preliminaries)

(i) Suppose that the solution $\mathcal{F}(x)$ of the HHP corresponding to $(\nu, \mathcal{F}_0, x)$ exists. Then, for each $\tau \in C - \bar{I}(x)$,

$$[\nu^{\pm}(\sigma)]^{-1} \frac{\mathcal{F}^{\pm}(\sigma') + \mathcal{F}^{-}(\sigma')}{\sigma' - \tau}$$

is summable over $\sigma' \in \bar{I}(x)$, with assigned orientation in the direction of increasing $\sigma'$,

$$(2A.1)$$

and

$$[\nu(\tau)]^{-1} \mathcal{F}(\tau) = I + \frac{1}{2\pi i} \int_{\bar{I}} d\sigma' [\nu^{\pm}(\sigma')]^{-1} \frac{\mathcal{F}^{\pm}(\sigma') + \mathcal{F}^{-}(\sigma')}{\sigma' - \tau},$$

$$(2A.2)$$

where the meaning we attribute to the symbol $\int_{\bar{I}}$ should be obvious.

(ii) Moreover, for each $\sigma \in \mathcal{I}(x)$,

$$[\nu^{\pm}(\sigma')]^{-1} \frac{\mathcal{F}^{\pm}(\sigma') + \mathcal{F}^{-}(\sigma')}{\sigma' - \sigma}$$

is summable over $\sigma' \in \bar{I}(x)$ in the principal value (PV) sense,

$$(2A.3)$$

\(^{17}\)G. A. Alekseev, *The method of the inverse scattering problem and singular integral equation for interacting massless fields*, Dokl. Akad. Nauk SSSR 283, 577–582 (1985) [Sov. Phys. Dokl. (USA) 30, 565 (1985)], *Exact solutions in the general theory of relativity*, Trudy Matem. Inst. Steklova 176, 215–262 (1987).
and

\[
\frac{1}{2} [\nu^+(\sigma)]^{-1} \{ F^+(\sigma) - F^-(\sigma) \} = I + \frac{1}{2\pi i} \int_{\bar{I}} d\sigma' [\nu^+(\sigma')]^{-1} \frac{F^+(\sigma') + F^-(\sigma')}{\sigma' - \sigma}. \tag{2A.4}
\]

**Proofs:**

(i) From Thms. 3(i) and (ii), the function of \( \sigma' \) given by \( \nu \pm \sigma' \) is continuous throughout \( \bar{I} \) and is bounded at \( x \), while \( F^\pm(\sigma')(\sigma' - \tau)^{-1} \) is bounded at \( x_0 \). Moreover, it is clear that \( \nu \pm \sigma' \) and \( \nu \pm (\sigma') \) are summable on \( \bar{I} \), and \( \nu^-(\sigma') = -\nu^+(\sigma') \) throughout \( \bar{I} \). Statement (2A.1) can now be obtained by employing the well-known theorem\(^{18}\) that the product of any complex-valued function which is summable on \([a,b] \subset \mathbb{R}^1\) by a function which is continuous and bounded on \([a,b]-\)any given finite set) is also summable on \([a,b] \).

To obtain the conclusion (2A.2), one employs Cauchy’s integral formula and the HHP condition \( F(\infty) = I \) to infer that

\[
\nu^{-1}(\tau) = I - \frac{1}{2\pi i} \int_{\Lambda} d\tau' \frac{[\nu(\tau')]^{-1} F(\tau')}{\tau' - \tau}, \tag{2A.5}
\]

where \( \Lambda \) is a closed positively oriented contour enclosing \( \bar{I} \) but not the point \( \tau \), which we may assume to be rectangular. This equation can be expressed in the form

\[
\nu^{-1}(\tau) = I - \frac{1}{2\pi i} \int_{\Lambda^+} d\tau' \frac{[\nu^+(\tau')]^{-1} F^+(\tau')}{\tau' - \tau} - \frac{1}{2\pi i} \int_{\Lambda^-} d\tau' \frac{[\nu^-(\tau')]^{-1} F^-(\tau')}{\tau' - \tau}, \tag{2A.6}
\]

where \( \Lambda^\pm := \Lambda \cap \bar{C}^\pm \) denote the parts of the contour \( \Lambda \) that lie respectively in the upper and lower half planes, \( \bar{C}^\pm \).

To evaluate each of the integrals, one applies a well known generalization\(^{19}\) of Cauchy’s integral theorem which asserts that the integral of a function about a simple piecewise smooth contour \( K \) is zero if the given function is holomorphic throughout \( K_{int} \) and is continuous throughout \( K \cup K_{int} \). In the case of the first integral, we select the contour as in Fig. 2. The other integral is evaluated in a similar way, using a contour in \( \bar{C}^- \).

![Figure 2](image)

Here \( a^i \) and \( b^i \) are the left and right endpoints, respectively, of the arc \( \bar{I}^{(i)} \). The radius of each semicircular arc is \( \alpha \) and each of the vertical segments of the closed contours has length \( \sqrt{2} \alpha \). One ultimately takes the limit as \( \alpha \to 0 \).

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\(18\)See *Integration*, by Edward J. McShane (Princeton University Press, 1944).

\(19\)See Remark 2 in Sec. 2, Ch. II, of *Analytic Functions* by M. A. Evgrafov (Dover Publications, 1978).
From a well known theorem on Lebesgue integrals,

\[
\int_{a^i+\alpha}^{b^i} d\sigma' \left[ \nu^\pm(\sigma') \right]^{-1} \mathcal{F}^\pm(\sigma') \sigma' - \tau \to \int_{a^i}^{b^i} d\sigma' \left[ \nu^\pm(\sigma') \right]^{-1} \mathcal{F}^\pm(\sigma') \sigma' - \tau \quad \text{as} \quad \alpha \to 0.
\] 

(2A.7)

Upon applying the above statement (2A.7) and the easily proved statement that the integral on each semicircular arc \( \to 0 \) as \( \alpha \to 0 \), and using the fact that \( \nu^- (\sigma') = -\nu^+ (\sigma') \) for all \( \sigma' \in \mathcal{I}(x) \), one obtains the conclusion (2A.2). End of proof.

(ii) To obtain statement (2A.3) and Eq. (2A.4) when \( \sigma \in \mathcal{I}(3) \), we again employ the Cauchy integral formula and the generalized Cauchy integral theorem, this time using (for the integral over \( \Lambda^+ \)) the positively oriented closed contours depicted in Fig. 3. The case \( \sigma \in \mathcal{I}(4) \) is treated similarly.

![Figure 3:](image)

Here the radius of the semicircular arc about \( \sigma \) is \( \beta \) and each of the vertical segments of the left closed contour has length \( \sqrt{2} \beta \). The radius of each of the other semicircular arcs is \( \alpha \), and each of the vertical segments of the right closed contour has length \( \sqrt{2} \alpha \). One ultimately takes the limit as \( \alpha \to 0 \) followed by the limit as \( \beta \to 0 \). It is clear that the integral on the semicircular arc with center \( \sigma \) has the limit \( \frac{1}{2} \nu^+(\sigma) \mathcal{F}^+(\sigma) \) as \( \beta \to 0 \).

End of proof.

B. Derivation of an Alekseev-type singular integral equation

Proceeding from equations (2A.2) and (2A.4), one can construct a singular integral equation of the Alekseev type and, if \( v \in K^{1+} \), a Fredholm equation of the second kind.

We begin by observing that Eq. (2B.13) implies that, for each \( \sigma \in \mathcal{I}(x) \cup \{r_0, s_0\} \),

\[
\frac{1}{2} \left\{ \mathcal{F}^{M+}(\sigma) + \mathcal{F}^{M-}(\sigma) \right\} = \begin{pmatrix} 1 & 0 & -i(\sigma-z) \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & i(\sigma-z_0) \\ 0 & 1 \end{pmatrix} , \quad (2B.1)
\]

and

\[
\frac{1}{2} [\nu^+(\sigma)]^{-1} \left\{ \mathcal{F}^{M+}(\sigma) - \mathcal{F}^{M-}(\sigma) \right\} = \begin{pmatrix} 1 & -i(\sigma-z) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & i(\sigma-z_0) \\ 0 & 1 \end{pmatrix} . \quad (2B.2)
\]

If \( \mathcal{F} \) is a solution of the HHP corresponding to \( (v, \mathcal{F}^M) \), Eq. (2B.7) tells us that, for any \( \sigma \in \mathcal{I}(x) \),

\[
\mathcal{F}^\pm(\sigma) v^{(i)}(\sigma) = Y^{(i)}(\sigma) \mathcal{F}^{M\pm}(\sigma) , \quad (2B.3)
\]

\[^{20}\text{See Cor. 27.7 in Ref. [18].}\]
and, therefore,

\[
\frac{1}{2} \{ \mathcal{F}^+(\sigma) + \mathcal{F}^-(\sigma) \} v^{(i)}(\sigma) =
Y^{(i)}(\sigma) \begin{pmatrix} 1 & -i(\sigma - z) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & i(\sigma - z_0) \\ 0 & 1 \end{pmatrix},
\]

(2B.4)

and

\[
\frac{1}{2} [\nu(\sigma)]^{-1} \{ \mathcal{F}^+(\sigma) - \mathcal{F}^-(\sigma) \} v^{(i)}(\sigma) =
Y^{(i)}(\sigma) \begin{pmatrix} 1 & -i(\sigma - z) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & i(\sigma - z_0) \\ 0 & 1 \end{pmatrix}.
\]

(2B.5)

This motivates the introduction of two new \(2 \times 2\) matrices.

**Dfn. of functions** \(W^{(i)}(x)\) **and** \(Y^{(i)}(x)\)

For each \(v \in K\), we let \(W^{(i)}(x)\) denote the function whose domain is \(I^{(i)}\) and whose value for each \(\sigma \in I^{(i)}\) is

\[
W^{(i)}(x, \sigma) := W^{(i)}(x)(\sigma) := v^{(i)}(\sigma) \begin{pmatrix} 1 & -i(\sigma - z_0) \\ 0 & 1 \end{pmatrix},
\]

(2B.6)

and, for each solution \(\mathcal{F}(x)\) of the HHP corresponding to \((v, \mathcal{F}^M, x)\), we let \(Y^{(i)}(x)\) denote the function whose domain is \(I^{(i)}(x)\) and whose value for each \(\sigma \in I^{(i)}(x)\) is

\[
Y^{(i)}(x, \sigma) := Y^{(i)}(x)(\sigma) := Y^{(i)}(x, \sigma) \begin{pmatrix} 1 & -i(\sigma - z) \\ 0 & 1 \end{pmatrix}.
\]

(2B.7)

End of Dfn.

In terms of these matrices we may write [suppressing ‘\(x\)’]

\[
\mathcal{F}^\pm(\sigma) W^{(i)}(\sigma) = Y^{(i)}(\sigma) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \nu^\pm(\sigma)
\]

(2B.8)

as well as

\[
\frac{1}{2} \{ \mathcal{F}^+(\sigma) + \mathcal{F}^-(\sigma) \} W^{(i)}(\sigma) = Y^{(i)}(\sigma) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\]

(2B.9)

and

\[
\frac{1}{2} [\nu(\sigma)]^{-1} \{ \mathcal{F}^+(\sigma) - \mathcal{F}^-(\sigma) \} W^{(i)}(\sigma) = Y^{(i)}(\sigma) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

(2B.10)

**Dfns. of** \(W(x)\), \(Y(x)\), \(W_a(x)\) **and** \(Y_a(x)\)
Let \( W(x) \) and \( \mathcal{Y}(x) \) denote the functions with domain \( I(x) \) and values

\[
W(x, \sigma) := W(x)(\sigma) := W^{(i)}(x, \sigma) \quad \text{and} \quad \mathcal{Y}(x, \sigma) := \mathcal{Y}(x)(\sigma) := \mathcal{Y}^{(i)}(x, \sigma)
\]

for each \( i \in \{3, 4\} \) and \( \sigma \in I^{(i)}(x) \).

Moreover, let

\[
W_a(x, \sigma) := a^{th} \text{ column of } W(x, \sigma) \quad \text{and} \quad \mathcal{Y}_a(x, \sigma) := a^{th} \text{ column of } \mathcal{Y}(x, \sigma), \quad \text{where } a \in \{1, 2\}.
\]

THEOREM 5 (Alekseev-type equation)

For each \( v \in K, x \in D \), solution \( F(x) \) of the HHP corresponding to \( (v, F^M, x) \), \( \tau \in C - \bar{I}(x) \) and \( \sigma \in \bar{I}(x) \), the following statement holds:

\[
[\nu^{+}(\sigma')]^{-1}\mathcal{Y}_1(\sigma')W_2^T(\sigma')(\sigma' - \tau)^{-1} \text{ is summable over } \sigma' \in \bar{I}(x),
\]

\[
\nu(\tau)^{-1}F(\tau) = I + \frac{1}{\pi i} \int_{\bar{I}} d\sigma'[\nu^{+}(\sigma')]^{-1}\mathcal{Y}_1(\sigma')\frac{W_2^T(\sigma')J}{\sigma' - \tau},
\]

\[
[\nu^{+}(\sigma')]^{-1}\mathcal{Y}_1(\sigma')W_2^T(\sigma')(\sigma' - \sigma)^{-1} \text{ is summable over } \sigma' \in \bar{I}(x) \text{ in the PV sense},
\]

\[
\mathcal{Y}_2(\sigma) = W_2(\sigma) - \frac{1}{\pi i} \int_{\bar{I}} d\sigma'[\nu^{+}(\sigma')]^{-1}\mathcal{Y}_1(\sigma')\frac{W_2^T(\sigma')JW_2(\sigma)}{\sigma' - \sigma},
\]

and

\[
0 = W_1(\sigma) + \frac{1}{\pi i} \int_{\bar{I}} d\sigma'[\nu^{+}(\sigma')]^{-1}\mathcal{Y}_1(\sigma')\frac{W_2^T(\sigma')JW_1(\sigma)}{\sigma' - \sigma}.
\]

Here we have employed the symbol \( J := -i\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \).

Proof: The statements \( (2B.13) \) to \( (2B.15) \) are obtained by using Eqs. \( (2B.9) \) and \( (2B.10) \) together with the relation

\[
W(\sigma)^{-1} = -JW^T(\sigma)J
\]

to replace \( F^{+}(\sigma') - F^{-}(\sigma') \) and \( [\nu^{+}(\sigma')]^{-1}[F^{+}(\sigma') + F^{-}(\sigma)] \) in statements \( (2A.1) \) to \( (2A.3) \) of Thm. \( 4 \). The same replacements are to be made in the integrands on the right side of Eq. \( (2A.4) \) in Thm. \( 4 \). Equation \( (2B.16) \) is obtained by multiplying both sides of Eq. \( (2A.4) \) by \( W_2(\sigma) \) and replacing the product on the left side with the second column of \( (2B.10) \) multiplied by \( [\nu^{+}(\sigma')]^{-1} \). Equation \( (2B.17) \) is obtained by multiplying both sides of Eq. \( (2A.4) \) by \( W_1(\sigma) \) and replacing the product on the left side with the first column of \( (2B.10) \).

End of proof.

Equation \( (2B.17) \) has the form of the singular integral equation which Alekseev obtained in the analytic case.

\[21\text{We shall frequently suppress } x.\]
C. Extension of the function $\mathcal{Y}(x)$ from $\mathcal{I}(x)$ to $\tilde{\mathcal{I}}(x)$

Since $C^TJC = 0$ (the zero matrix) for any $2 \times 1$ matrix $C$, Eq. (2B.16) is expressible in the following form for each $i \in \{3, 4\}$:

$$
\mathcal{Y}^{(i)}_2(\sigma) = W^{(i)}_2(\sigma)
- \frac{1}{\pi i} \int_{a^i}^{b^i} d\sigma'[\nu^+(\sigma')]^{-1} \mathcal{Y}^{(i)}_1(\sigma')W^{(i)}_2(\sigma')^T J \left[ \frac{W^{(i)}_2(\sigma) - W^{(i)}_2(\sigma')}{\sigma' - \sigma} \right]
- \frac{1}{\pi i} \int_{a^7-i}^{b^7-i} d\sigma'[\nu^+(\sigma')]^{-1} \mathcal{Y}^{(7-i)}_1(\sigma')W^{(7-i)}_2(\sigma')^T J \left[ \frac{W^{(i)}_2(\sigma) - W^{(7-i)}_2(\sigma')}{\sigma' - \sigma} \right],
$$

(2C.1)

for all $\sigma \in \mathcal{I}^{(i)}(x)$, where recall that $a^i := \inf \{x^i, x^0_i\}$ and $b^i := \sup \{x^i, x^0_i\}$. Without indicating the parallel proof, we simply remark that one can also show that

$$
\mathcal{Y}^{(i)}_1(\sigma) = W^{(i)}_1(\sigma)
+ \frac{1}{\pi i} \int_{a^i}^{b^i} d\sigma' \nu^+(\sigma') \mathcal{Y}^{(i)}_2(\sigma')W^{(i)}_1(\sigma')^T J \left[ \frac{W^{(i)}_1(\sigma) - W^{(i)}_1(\sigma')}{\sigma' - \sigma} \right]
+ \frac{1}{\pi i} \int_{a^{7-i}}^{b^{7-i}} d\sigma' \nu^+(\sigma') \mathcal{Y}^{(7-i)}_2(\sigma')W^{(7-i)}_1(\sigma')^T J \left[ \frac{W^{(i)}_1(\sigma) - W^{(7-i)}_2(\sigma')}{\sigma' - \sigma} \right],
$$

(2C.2)

Now, from Thms. (i) and (ii), Eq. (2B.3) and Eq. (2B.10),

$$
\nu^+(\sigma') \mathcal{Y}^{(i)}_2(\sigma')W^{(i)}_1(\sigma')^T J
\text{ and } [\nu^+(\sigma')]^{-1} \mathcal{Y}^{(i)}_1(\sigma')W^{(i)}_2(\sigma')^T J
$$

are summable over $\sigma' \in \tilde{\mathcal{I}}(x)$.

From the definition of $W^{(i)}$ by Eq. (2B.3) and the definition of $K^\Box$, the following statement holds for each $x \in D$ and $i \in \{3, 4\}$:

If $v \in K^1$, then $W^{(i)}$ is $C^1$ throughout $\mathcal{I}^{(i)}$,

$[W^{(i)}(\sigma') - W^{(i)}(\sigma)](\sigma' - \sigma)^{-1}$ is a continuous function of $(\sigma', \sigma)$ throughout $\mathcal{I}^{(i)} \times \mathcal{I}^{(i)}$, and

$W^{(i)}(\sigma)(\sigma' - \sigma)^{-1}$ is a $C^1$ function of $(\sigma', \sigma)$ throughout $\tilde{\mathcal{I}}^{(7-i)}(x) \times \tilde{\mathcal{I}}^{(i)}(x^{7-i})$,

(2C.4)

where

$$
\tilde{\mathcal{I}}^{(3)}(s) := \{\sigma \in \mathcal{I}^{(3)} : \sigma < s\}, \text{ and } \tilde{\mathcal{I}}^{(4)}(r) := \{\sigma \in \mathcal{I}^{(4)} : r < \sigma\}.
$$

(2C.5)

Note that (See Fig. 4)

$$
\mathcal{I}^{(i)}(x) \subset \tilde{\mathcal{I}}^{(i)}(x^{7-i}) \subset \mathcal{I}^{(i)}.
$$

(2C.6)

From the above statements (2C.3) and (2C.4), and from the theorem that asserts the summability over a finite interval of the product of a summable function by a continuous function, the extension of $\mathcal{Y}^{(i)}(x)$ that we shall define below exists. Note that $\Box$ is $n \geq 1$, $n+$ (with $n \geq 1$), $\infty$ or ‘an’.
Figure 4: Illustrating the relation \( I^{(3)}(x) \subset \bar{I}^{(3)}(x^4) \subset I^{(3)} \). In this example, \( \bar{I}^{(4)}(x^3) = I^{(4)} \).
Dfn. of an extension of $\mathcal{Y}^{(i)}(x)$ when $v \in K^\Box$

For each $v \in K^\Box$, $x \in D$, solution $\mathcal{F}(x)$ of the HHP corresponding to $(v, F^M, x)$ and $i \in \{3, 4\}$, let $\mathcal{Y}^{(i)}(x)$ denote the function whose extended domain is $\mathcal{I}(x^{7-i})$ and whose value for each $\sigma \in \mathcal{I}^{(i)}(x^{7-i})$ is given by [suppressing 'x']

$$
\mathcal{Y}_1^{(i)}(\sigma) := \text{right side of Eq. (2C.2)}, \quad (2C.7) \\
\mathcal{Y}_2^{(i)}(\sigma) := \text{right side of Eq. (2C.1)}. \quad (2C.8)
$$

End of Dfn.

**LEMMA 6 (Continuity and differentiability of $W^{(i)}$)**

(i) If $v \in K^\Box$, then $W^{(i)}$ is $C^\Box$ throughout its domain $\mathcal{I}^{(i)}$, and the function whose domain is $\mathcal{I}^{(7-i)}(x) \times \mathcal{I}^{(i)}(x^{7-i})$ and whose values for each $(\sigma', \sigma)$ in this domain is $W^{(i)}(\sigma)(\sigma' - \sigma)^{-1}$ is also $C^\Box$.

(ii) If $v \in K^\Box$, then the function of $(\sigma', \sigma)$ whose domain is $\mathcal{I}^{(i)} \times \mathcal{I}^{(i)}$ and whose value for each $(\sigma', \sigma)$ in this domain is $[W^{(i)}(\sigma) - W^{(i)}(\sigma')]/(\sigma' - \sigma)$ is $C^{n-1}$ if $\Box$ is $n \geq 1$, is $C^{(n-1)+}$ if $\Box$ is $n^+$ ($n \geq 1$), is $C^\infty$ if $\Box$ is $\infty$, and is $C^\infty$ if $\Box$ is 'an'.

**Proofs:**

(i) The conclusion follows by using the definition of $W^{(i)}$ by Eq. (2B.6) together with the definition of $K^\Box$. End of proof.

(ii) The conclusions when $\Box$ is $n$, $\infty$ or 'an' are well known. As regards the case when $\Box$ is $n^+$ ($n \geq 1$), one can construct a simple proof (which we shall not reproduce here) using the relation

$$
\frac{W^i(\sigma) - W^i(\sigma')}{\sigma - \sigma'} = \int_0^1 dt (DW^i)(t\sigma + (1-t)\sigma'), \quad (2C.9)
$$

where $D^pW^i$ ($1 \leq p \leq n$) denotes the function whose domain is $\mathcal{I}^{(i)}$ and whose value for each $\sigma \in \mathcal{I}^{(i)}$ is

$$(D^pW^i)(\sigma) := \frac{\partial^pW^i(\sigma)}{\partial \sigma^p}; \quad (2C.10)$$

and $DW^i := D^1W^i$. End of proof.

We shall leave the proof of the following basic lemma to the reader.

**LEMMA 7 (Integral of product)**

Suppose $[a, b] \subset R^1$, $S$ is a connected open subset of $R^m$ ($m \geq 1$), $f$ is a real-valued function defined almost everywhere on and summable over $[a, b]$, and $g$ is a real-valued function whose domain is $[a, b] \times S$ and which is continuous. Let $\sigma := (\sigma^1, \ldots, \sigma^m)$ denote any member of $S$, and let $F$ denote the function whose domain is $S$ and whose value at each $\sigma \in S$ is

$$
F(\sigma) := \int_a^b d\sigma' f(\sigma')g(\sigma', \sigma). \quad (2C.11)
$$

Then the following statements hold:
LEMMA 8 (Generalization of Lem. 7)

All of the conclusions of the preceding lemma remain valid when the only alteration in the premises is to replace the statement that \( f \) and \( g \) are real valued by the statement that they are complex valued or are finite matrices (such that the product \( fg \) exists) with complex-valued elements.

Proof: Use the definition

\[
\int_a^b d\sigma' h(\sigma') := \int_a^b d\sigma' \text{Re} h(\sigma') + i \int_a^b d\sigma' \text{Im} h(\sigma')
\]

for any complex-valued function \( h \) whose real and imaginary parts are summable over \([a, b]\). The rest is obvious. End of proof.

THEOREM 9 (Continuity and differentiability of extended \( \mathcal{Y}^{(i)}(x) \))

For each \( v \in K^i \), \( x \in D \), solution \( F(x) \) of the HHP corresponding to \( (v, \mathcal{F}^M, x) \) and \( i \in \{3, 4\} \), \( \mathcal{Y}^{(i)}(x) \) [see Eqs. (2C.7) and (2C.8)] is \( C^{n-1} \) if \( \square \) is \( n \), is \( C^{(n-1)+} \) if \( \square \) is \( n+ \), is \( C^{\infty} \) if \( \square \) is \( \infty \) and is \( C^{an} \) if \( \square \) is ‘an’.

Proof: Apply Lemmas 3, 4 and 5 to the definitions (2C.7) and (2C.8) of \( \mathcal{Y}_1^{(i)}(x) \) and \( \mathcal{Y}_2^{(i)}(x) \). It is then easily shown that the second term on the right side of each of the Eqs. (2C.2) and (2C.1) [with \( \sigma \in \mathcal{I}^{(i)}(x^{7-i}) \)] is \( C^{n-1} \) if \( \square \) is \( n \), is \( C^{(n-1)+} \) if \( \square \) is \( n+ \), is \( C^{\infty} \) if \( \square \) is \( \infty \) and is \( C^{an} \) if \( \square \) is ‘an’. The first and third terms on the right sides of each of the Eqs. (2C.2) and (2C.1) are, on the other hand, both \( C^i \) even when \( \square \) is \( n \) or is \( n+ \). However, a \( C^n \) function is also a \( C^{n-1} \) function; and a \( C^{n+} \) function is also a \( C^{(n-1)+} \) function. End of proof.

Dfns. of \( \mathcal{Y}^{(i)} \), \( \mathcal{Y} \) and the partial derivatives of \( \mathcal{Y} \)
Henceforth, \( \mathcal{Y}(i) \) \((i \in \{3, 4\})\) will denote the function whose domain is
\[
\text{dom} \ \mathcal{Y}(i) := \{(x, \sigma) : x \in D, \sigma \in \tilde{I}(x^{7-i})\}
\] (2C.14)
and whose values are given by
\[
\mathcal{Y}(i)(x, \sigma) := \mathcal{Y}(i)(x)(\sigma),
\] (2C.15)
where \( \mathcal{Y}(i)(x) \) is the extension of the original \( \mathcal{Y}(i)(x) \) that is defined by Eqs. (2C.4) and (2C.8). We shall let \( \mathcal{Y} \) denote the function whose domain is
\[
\text{dom} \ \mathcal{Y} := \{(x, \sigma) : x \in D, \sigma \in \tilde{I}(x)\}
\]
and whose values are given by
\[
\mathcal{Y}(x, \sigma) := \mathcal{Y}(i)(x, \sigma) \text{ whenever } \sigma \in \tilde{I}(i)(x).
\]
[Thus, \( \mathcal{Y}(x, \sigma) = \mathcal{Y}(x)(\sigma) \).] Also, for each \( x \in D, i \in \{3, 4\} \) and \( \sigma \in \tilde{I}(i)(x) \), we shall let
\[
\frac{\partial^{l+m+n} \mathcal{Y}(x, \sigma)}{\partial r^l \partial s^m \partial \sigma^n} := \frac{\partial^{l+m+n} \mathcal{Y}(i)(x, \sigma)}{\partial r^l \partial s^m \partial \sigma^n},
\]
if the above partial derivative of \( \mathcal{Y}(i) \) exists.

End of Dfn.

The domain of \( \mathcal{Y}(i) \), as defined above, is an open subset of \( R^3 \); and (though the domain of \( \mathcal{Y} \) is not an open subset of \( R^3 \)) the partial derivatives of \( \mathcal{Y} \) are defined in terms of partial derivatives of \( \mathcal{Y}(i) \) and, therefore, employ sequences of points in \( R^3 \) which may converge to a given point along any direction in \( R^3 \). This has formal advantages when one employs the derivatives of \( \mathcal{Y} \) at the boundary of its domain.

**COROLLARY 10 (The extension \( \mathcal{Y}(x) \) when \( v \in K^\square \))**

(i) Suppose \( v \in K^1, x \in D \) and the solution \( \mathcal{F}(x) \) of the HHP corresponding to \( (v, \mathcal{F}^M, x) \) exists. Then \( \mathcal{Y}(x) \) has a unique continuous extension to \( \tilde{I}(x) \).

(ii) If \( v \in K^\square \), then the extension \( \mathcal{Y}(x) \) is \( C^{n-1} \) if \( \square \) is \( n \), is \( C^{(n-1)+} \) if \( \square \) is \( n+ \), is \( C^\infty \) if \( \square \) is \( \infty \) and is \( C^\square \) if \( \square \) is ‘an’.

**Proof:** Statement (ii) of this corollary follows from Thm. [4]. The uniqueness follows, of course, from the fact that a function defined and continuous on an open subset of \( R^3 \) has no more than one continuous extension to the closure of that subset.  

End of proof.
D. Equivalence of the HHP to an Alekseev-type equation when $v \in K^{1+}$

THEOREM 11 (HHP-Alekseev equivalence theorem)
Suppose $v \in K^{1+}$ and $x \in D$, and suppose that $F(x)$ and $Y_1(x)$ are $2 \times 2$ and $2 \times 1$ matrix functions, respectively, such that

\begin{align*}
\text{dom } F(x) = C - \mathcal{I}(x), \quad \text{dom } Y_1(x) = \mathcal{I}(x) \text{ and } Y_1(x) \text{ is } C^{0+}. \tag{2D.1}
\end{align*}

Then the following two statements are equivalent to one another:

(i) The function $F(x)$ is a solution of the HHP corresponding to $(v, F^M, x)$, and $Y_1(x)$ is the function whose restriction to $\mathcal{I}(x)$ is defined in terms of $F^+(x) + F^-(x)$ by Eq. (2B.1) [where $x$ is suppressed] and whose existence and uniqueness [for the given $F(x)$] is asserted by Cor. 10 when $\Box$ is $1+$.

(ii) The restriction of $Y_1(x)$ to $\mathcal{I}(x)$ is a solution of the singular integral equation (2B.17), and $F(x)$ is defined in terms of $Y_1(x)$ by Eq. (2B.14) [where $x$ is suppressed].

Proof: That (i) implies (ii) has already been proved. [See Thm. 5 and Cor. 10.] The proof that (ii) implies (i) will be given in four parts:

(1) Assume that statement (ii) is true. From the definition of $F(x)$ by Eq. (2B.14),

\begin{align*}
F(x) \text{ is holomorphic; } \tag{2D.2}
\end{align*}

and, from two theorems of Plemelj\footnote{See Secs. 16 and 17 of Ch. II of Ref. 13 (pp. 37-43).}

\begin{align*}
F^+(x) \text{ and } F^-(x) \text{ exist } \tag{2D.3}
\end{align*}

and, since $\nu^-(\sigma) = -\nu^+(\sigma)$ for all $\sigma \in \mathcal{I}(x),

\begin{align*}
\frac{1}{2} \left[ F^+(\sigma) + F^-(\sigma) \right] = -Y_1(\sigma) W_2^T(\sigma) J \tag{2D.4}
\end{align*}

and

\begin{align*}
\frac{1}{2} \left[ \nu^+(\sigma)^{-1} \left[ F^+(\sigma) - F^-(\sigma) \right] = I - \frac{1}{\pi i} \int_{\mathcal{I}} \text{d} \sigma' \nu^+(\sigma')^{-1} Y_1(\sigma') \frac{W_2^T(\sigma') J}{\sigma' - \sigma} \right] \tag{2D.5}
\end{align*}

for all $\sigma \in \mathcal{I}(x)$. Upon multiplying Eqs. (2D.4) and (2D.3) through by $W_2(\sigma)$ on the right, one obtains, for all $\sigma \in \mathcal{I}(x),

\begin{align*}
\frac{1}{2} \left[ F^+(\sigma) + F^-(\sigma) \right] W_2(\sigma) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2D.6}
\end{align*}

and

\begin{align*}
\frac{1}{2} \left[ \nu^+(\sigma)^{-1} \left[ F^+(\sigma) - F^-(\sigma) \right] W_2(\sigma) = Y_2(\sigma), \tag{2D.7}
\end{align*}
where \( \mathcal{Y}_2(x) \) has the domain \( \tilde{I}(x) \) and the values
\[
\mathcal{Y}_2(\sigma) := W_2(\sigma)
\]
\[
- \frac{1}{\pi I} \int_{\mathcal{I}} d\sigma' [\nu^+(\sigma')]^{-1} \mathcal{Y}_1(\sigma') \frac{W_2^T(\sigma')J[W_2(\sigma) - W_2(\sigma')]}{\sigma' - \sigma}
\]
for all \( \sigma \in \tilde{I}(x) \).

\text{(2D.8)}

From Lemmas 6(ii), 7(iv) and 8, \( \mathcal{Y}_2(x) \) is \( C_{0+} \).

\text{(2D.9)}

Upon multiplying Eqs. (2D.4) and (2D.5) through by \( W_1(\sigma) \) on the right, upon using the fact that \( \det W(\sigma) = 1 \) is equivalent to the equation
\[
W_2^T(\sigma)JW_1(\sigma) = -(1),
\]
\text{(2D.10)}

and, upon using Eq. (2B.17), one obtains, for all \( \sigma \in \mathcal{I}(x) \),
\[
\frac{1}{2} \left[ \mathcal{F}^+(\sigma) + \mathcal{F}^-(\sigma) \right] W_1(\sigma) = \mathcal{Y}_1(\sigma)
\]
\text{(2D.11)}

and
\[
\frac{1}{2} \left[ \mathcal{F}^+(\sigma) - \mathcal{F}^-(\sigma) \right] W_1(\sigma) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]
\text{(2D.12)}

(2) We next note that the four equations (2D.6), (2D.7), (2D.11) and (2D.12) are collectively equivalent to the single equation
\[
\mathcal{F}^\pm(\sigma)W(\sigma) = \mathcal{Y}(\sigma) \begin{pmatrix} 1 & 0 \\ 0 & \nu^\pm(\sigma) \end{pmatrix}
\]
for all \( \sigma \in \mathcal{I}(x) \),
\text{(2D.13)}

where \( \mathcal{Y}(\sigma) \) is defined to be the \( 2 \times 2 \) matrix whose first and second columns are \( \mathcal{Y}_1(\sigma) \) and \( \mathcal{Y}_2(\sigma) \), respectively. From the definition of \( W(\sigma) \) by Eqs. (2B.6) and (2B.11), and from the expression for \( \mathcal{F}^M(\tau) \) that is given by Eq. (1B.13), Eq. (2D.13) is equivalent to the statement
\[
\mathcal{F}^+(\sigma)v^{(i)}(\sigma)[\mathcal{F}^{M+}(\sigma)]^{-1} = \mathcal{F}^-(\sigma)v^{(i)}(\sigma)[\mathcal{F}^{M-}(\sigma)]^{-1} = Y(\sigma) \text{ for all } \sigma \in \mathcal{I}^{(i)}(x),
\]
\text{(2D.14)}

where
\[
Y(\sigma) := \mathcal{Y}(\sigma) \begin{pmatrix} 1 & i(\sigma - z) \\ 0 & 1 \end{pmatrix} \text{ for all } \sigma \in \tilde{I}(x).
\]
\text{(2D.15)}

From the above Eq. (2D.15) and from statements (2D.1) and (2D.9),

the function \( Y(x) \) whose domain is \( \tilde{I}(x) \) and whose value for each \( \sigma \in \tilde{I}(x) \) is \( Y(x)(\sigma) := Y(x, \sigma) \) is \( C_{0+} \) and is, therefore, continuous.

\text{(2D.16)}

So,
\[
Y(x) \text{ is bounded at } x \text{ and at } x_0.
\]
\text{(2D.17)}
We now return to the definition of $F(x)$ in terms of $Y_1(x)$ by Eq. (2B.14). From Lemma 3(i) when $\square$ is 1, and from statement (2D.1) concerning $Y_1(x)$ being $C^{0+}$ on its domain $\bar{I}(x)$, note that the factors in the numerator of the integrand in Eq. (2B.14) have the following properties:

$$Y_1(\sigma'|W_2(\sigma')^TJ$$

is defined for all $\sigma' \in \bar{I}(x)$ and obeys a Hölder condition on $\bar{I}(x)$; (2D.18)

and

$$[\nu(\sigma')]^{-1}$$

is $H(1/2)$ on each closed subinterval of $I(x)$ and converges to zero as $\sigma' \to r$ and as $\sigma' \to s$. (2D.19)

Also, recall that

$$\nu(\tau)$$

is that branch of $(\tau - r_0)^{1/2}(\tau - s_0)^{1/2}(\tau - r)^{-1/2}(\tau - s)^{-1/2}$ which has the cut $I(x)$ and the value 1 at $\tau = \infty$. (2D.20)

Several theorems on Cauchy integrals near the end points of the lines of integration are given in Ref. 13, Sec. 29, Ch. 4. In particular, by applying Muskelishvili’s Eq. (29.4) to our Eq. (2B.14), one obtains the following conclusion from the above statements (2D.18) and (2D.19):

$$\nu(\tau)^{-1}F(\tau)$$

converges as $\tau \to r$ and as $\tau \to s$. (2D.21)

Moreover, by applying Muskelishvili’s Eqs. (29.5) and (29.6) to our Eq. (2B.14), one obtains the following conclusion from the above statements (2D.18) to (2D.20):

$$F(\tau)$$

converges as $\tau \to r_0$ and as $\tau \to s_0$. (2D.22)

(4) From the above statements (2D.2), (2D.3), (2D.14), (2D.17), (2D.21) and (2D.22), all of the defining conditions for a solution of the HHP corresponding to $(v, F^M, x)$ are satisfied by $F(x)$ as defined in terms of $Y_1(x)$ by Eq. (2B.14).

End of proof.

We already know from Thm. 3(iv) that there is not more than one solution of the HHP corresponding to $(v, F^M, x)$.

**COROLLARY 12 (Uniqueness of $Y_1(x)$)**

For each $v \in K^{1+}$ and $x \in D$, there is not more than one $2 \times 1$ matrix function $Y_1(x)$ such that

$$\text{dom } Y_1(x) = \bar{I}(x),$$

$Y_1(x)$ is $C^{0+}$

and $Y_1(x, \sigma) := Y_1(x)(\sigma)$ satisfies the singular integral equation (2B.17) for all $\sigma \in I(x)$. (2D.23)
Proof: Suppose that \( Y_1(x) \) and \( Y_1'(x) \) are 2 \( \times \) 1 matrix functions, both of which have domain \( \bar{I}(x) \), are \( C^{0+} \) and satisfy Eq. (2B.17) for all \( \sigma \in I(x) \) and for the same given \( \mathbf{v} \in K^{1+} \). Let \( F(x) \) and \( F'(x) \) be the 2 \( \times \) 2 matrix functions with domain \( C - \bar{I}(x) \) that are defined in terms of \( Y_1(x) \) and \( Y_1'(x) \), respectively, by Eq. (2B.14). Then, from the preceding Thm. 11, \( F(x) \) and \( F'(x) \) are both solutions of the HHP corresponding to (\( \mathbf{v}, F^{M}, x \)); and, therefore, from Thm. 3(iv),

\[
F(x) = F'(x); 
\tag{2D.25}
\]

and, from Eq. (2D.14) in the proof of Thm. 11 and from statements (2D.23) and (2D.24),

\[
Y_1(x) = Y_1'(x). 
\tag{2D.26}
\]

End of proof.

3 A Fredholm integral equation of the second kind that is equivalent to the Alekseev-type singular integral equation when \( \mathbf{v} \in K^{2+} \)

If \( \mathbf{v} \in K^{1+} \) and the particular solution \( Y_1(x) \) of Eq. (2B.17) that has a \( C^{0+} \) extension to \( \bar{I}(x) \) exists, then it can be shown that \( Y_1(x) \) is also a solution of a Fredholm integral equation of the second kind.

A. Derivation of Fredholm equation from Alekseev-type equation

We shall employ a variant of the Poincaré-Bertrand commutator theorem. Suppose that \( L \) is a smooth oriented line or contour in \( C - \{\infty\} \) and \( \phi \) is a complex-valued function whose domain is \( L \times L \) and which obeys a Hölder condition on \( L \times L \). Then the conventional Poincaré-Bertrand theorem asserts

\[
\left[ \frac{1}{\pi i} \int_L d\tau'', \frac{1}{\pi i} \int_L d\tau' \right] \frac{\phi(\tau', \tau'')}{(\tau'' - \tau)(\tau' - \tau'')} = \phi(\tau, \tau) \quad \text{for all } \tau \in L
\]

minus its end points, \( \tag{3A.1} \)

where the above bracketed expression is the commutator of the path integral operators. We are, of course, concerned here only with the case \( L = \bar{I}(x) \); and our variant asserts that, for any function \( \phi \) which is \( C^{0+} \) on \( \bar{I}(x)^2 \),

\[
\left[ \frac{1}{\pi i} \int_{\bar{I}} d\sigma'' \nu^+(\sigma'')^{-1}, \frac{1}{\pi i} \int_{\bar{I}} d\sigma' \nu^+(\sigma')^{-1} \right] \frac{\phi(\sigma', \sigma'')}{(\sigma'' - \sigma)(\sigma' - \sigma'')} = \phi(\sigma, \sigma) \quad \text{for all } \sigma \in \bar{I}(x), 
\tag{3A.2}
\]

or, alternatively,

\[
\left[ \frac{1}{\pi i} \int_{\bar{I}} d\sigma'' \nu^+(\sigma'')^{-1}, \frac{1}{\pi i} \int_{\bar{I}} d\sigma' \nu^+(\sigma') \right] \frac{\phi(\sigma', \sigma'')}{(\sigma'' - \sigma)(\sigma' - \sigma'')} = \phi(\sigma, \sigma) \quad \text{for all } \sigma \in \bar{I}(x), \tag{3A.3}
\]

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We shall not supply the proof here, as an elegant and thorough proof of the Poincaré-Bertrand theorem (3A.1) is given by Sec. 23 of Muskhelishvili’s treatise, and what we have done is to construct proofs of (3A.2) and (3A.3) that parallel his proof step by step.

We shall now apply Eq. (3A.2) to the Alekseev-type equation (2B.17), which we express in the form

$$\frac{1}{\pi i} \int_{\tilde{I}} d\sigma' \nu^{+}(\sigma')^{-1} \frac{\mathcal{Y}_1(\sigma') d_{21}(\sigma', \sigma'')}{\sigma' - \sigma''} = -W_1(\sigma'') \text{ for all } \sigma'' \in \mathcal{I}(x),$$

(3A.4)

where, for all \( \sigma \in \tilde{I}(x) \) and \( \sigma' \in \tilde{I}(x) \),

$$d_{21}(\sigma', \sigma) := W_{22}(\sigma') W_{11}(\sigma) - W_{12}(\sigma') W_{21}(\sigma).$$

(3A.5)

We suppose that, for a given \( v \in K^{1+} \) and \( x \in D \), a solution \( \mathcal{Y}_1(x) \) of the Alekseev-type equation (2B.17) exists and is \( C^{0+} \) on \( \tilde{I}(x) \). Then the product \( \mathcal{Y}_1(x) d_{21} \) is \( C^{0+} \) on \( \tilde{I}(x) \).

Also, \( \det W(\sigma) = d_{21}(\sigma, \sigma) = 1 \). Therefore, upon multiplying both sides of Eq. (3A.4) by \((\sigma'' - \sigma)^{-1}\) and then applying the PV integral operator

$$\frac{1}{\pi i} \int_{\tilde{I}} d\sigma'' \nu^{+}(\sigma''),$$

Eq. (3A.2) gives us

$$\mathcal{Y}_1(\sigma) - \frac{1}{\pi i} \int_{\tilde{I}} d\sigma' \nu^{+}(\sigma')^{-1} \mathcal{Y}_1(\sigma') K_{21}(\sigma', \sigma) = U_1(\sigma),$$

(3A.6)

where, for each \( \sigma \in \tilde{I}(x) - \{r, s\} \),

$$U_1(\sigma) := -\frac{1}{\pi i} \int_{\tilde{I}} d\sigma' \nu^{+}(\sigma') \frac{W_1(\sigma')}{\sigma' - \sigma};$$

(3A.7)

and, for each \((\sigma', \sigma) \in \tilde{I}(x) \times [\tilde{I}(x) - \{r, s\}]\),

$$K_{21}(\sigma', \sigma) := -\frac{1}{\pi i} \int_{\tilde{I}} d\sigma'' \nu^{+}(\sigma'') \frac{d_{21}(\sigma', \sigma'')}{(\sigma'' - \sigma)(\sigma' - \sigma'')}.$$ 

(3A.8)

So far, we have only established that Eq. (3A.6) holds for all \( \sigma \in \mathcal{I}(x) \). However, using the expressions

$$U_1(\sigma) = W_1(\sigma) - \frac{1}{\pi i} \int_{\tilde{I}} d\sigma' \nu^{+}(\sigma') \frac{W_1(\sigma') - W_1(\sigma)}{\sigma' - \sigma},$$

(3A.9)

$$K_{21}(\sigma', \sigma) = k_{21}(\sigma', \sigma) - \frac{1}{\pi i} \int_{\tilde{I}} d\sigma'' \nu^{+}(\sigma'') \left[ \frac{k_{21}(\sigma', \sigma'') - k_{21}(\sigma', \sigma)}{\sigma'' - \sigma} \right],$$

(3A.10)

where

$$k_{21}(\sigma', \sigma) := \frac{d_{21}(\sigma', \sigma) - 1}{\sigma' - \sigma},$$

(3A.11)

it is not difficult to prove the following lemma.
LEMMA 13 (Properties of $U_1(x)$ and $K_{21}(x)$)

For each $x \in D$ and $v \in K^\square$, $U_1(x)$ is $C^{n-1}$ if $\Box$ is $n$ and $n \geq 2$, is $C^{(n-1)^+}$ if $\Box$ is $n^+$ and $n \geq 2$, are $C^\infty$ if $\Box$ is $\infty$ and is $C^\infty$ if $\Box$ is $\infty$; and $K_{21}(x)$ is $C^{n-2}$ if $\Box$ is $n$ and $n \geq 2$, is $C^{(n-2)^+}$ if $\Box$ is $n^+$ and $n \geq 2$, is $C^\infty$ if $\Box$ is $\infty$, and is $C^\infty$ if $\Box$ is $\infty$.

If $v \in K^1$, then $U_1(x)$ is $C^0^+$ and $K_{21}(x)$ is also $C^0^+$ {but, as we recall, its domain is only $I(x) \times [I(x) - \{r, s\}]$}.

From this it follows that

$$U_1(x) \text{ is continuous on } I(x) \tag{3A.12}$$

and

$$K_{21}(x) \text{ is continuous on } dom \ K_{21}(x). \tag{3A.13}$$

Moreover, Lem. [4] remains valid if $S$ is a closed or a semi-closed subinterval of $R^1$. Therefore, from (3A.13) and Lem. [7(i) with a closed or a semi-closed $S \subset R^1$, the integral in Eq. (3A.6) is a continuous function of $\sigma$ throughout $I(x)$ if $v \in K^2$, and throughout $I(x) - \{r, s\}$ if $v \notin K^2$; and it then follows from the fact that

$$\mathcal{J}_1(x) \text{ is continuous on } I(x) \tag{3A.14}$$

and from (3A.12) that Eq. (3A.4) holds for all $\sigma \in I(x)$ if $v \in K^2$, and for all $\sigma \in I(x) - \{r, s\}$ if $v \notin K^2$. Thus, we have the following theorem.

THEOREM 14 (Fredholm equation)

Suppose that, for a given $v \in K^1$ and $x \in D$, a solution $\mathcal{J}_1(x)$ of the Alekseev-type equation (2B.17) exists and is $C^0^+$ on $I(x)$. Then the Fredholm equation (3A.6) holds for all $\sigma \in I(x)$ if $v \in K^2$ and for all $\sigma \in I(x) - \{r, s\}$ if $v \notin K^2$.

B. Equivalence of Alekseev-type equation and Fredholm equation when $v \in K^2$

The Fredholm equation (3A.6) generally has a singular kernel and is generally not equivalent to the Alekseev-type equation (2B.17). In this section we shall restrict our attention to the case $v \in K^2$.

THEOREM 15 (Alekseev-Fredholm equivalence theorem)

Suppose $v \in K^2$, $x \in D$ and $\mathcal{J}_1(x)$ is a $2 \times 1$ column matrix function with domain $I(x)$. Then $U_1(x)$ is $C^{1^+}$ and $K_{21}(x)$ is $C^{0^+}$. Also, the following two statements are equivalent to one another:

(i) $\mathcal{J}_1(x)$ is $C^{0^+}$ and is the solution of Eq. (2B.17) for all $\sigma \in I(x)$.

(ii) $\mathcal{J}_1(x)$ is summable over $I(x)$ and is a solution of Eq. (3A.6) for all $\sigma \in I(x)$.

Proof: From Lem. [13], $U_1(x)$ is $C^{1^+}$ and $K_{21}(x)$ is $C^{0^+}$; and Thm. [14] already asserts that statement (i) implies statement (ii). It remains only to prove that statement (ii) implies statement (i).
Grant statement (ii). Since \( U_1(x) \) is \( C^{1+} \) and \( K_{21}(x) \) is \( C^{0+} \) on \( \bar{I}(x) \) and since \( Y_1(x) \) is summable over \( \bar{I}(x) \), Eq. (3A.6) and Lem. 7(iv) yield
\[
Y_1(x) \text{ is } C^{0+} \text{ on } \bar{I}(x).
\] (3B.1)

Next, using the Poincaré–Beltrami variant, one deduces the following equivalent of the Fredholm equation (3A.6):
\[
Y_1(\sigma) + \frac{1}{\pi i} \int_{\bar{I}} d\sigma' \nu^+(\sigma') \frac{\psi(\sigma') + W_1(\sigma')}{\sigma' - \sigma} = 0,
\] (3B.2)
where
\[
\psi(\sigma) := \frac{1}{\pi i} \int_{\bar{I}} d\sigma' \nu^+(\sigma')^{-1} Y_1(\sigma') k_{21}(\sigma', \sigma).
\] (3B.3)
From Lem. 7(iv) and (3B.1), \( \psi \) is \( C^{0+} \) on \( \bar{I}(x) \). (3B.4)

Next, after replacing ‘\( \sigma \)’ by ‘\( \sigma'' \)’ in Eq. (3B.2) and then applying the operator
\[
\frac{1}{\pi i} \int_{\bar{I}} d\sigma'' \nu^+(\sigma'')^{-1} \frac{1}{\sigma'' - \sigma},
\]
one finds that
\[
\frac{1}{\pi i} \int_{\bar{I}} d\sigma'' \nu^+(\sigma'')^{-1} \frac{Y_1(\sigma'')}{\sigma'' - \sigma} + \psi(\sigma) + W_1(\sigma) = 0,
\] (3B.5)
from which equation one can derive the Alekseev-type equation (2B.17). \( \Box \)

Let us summarize the results given by Thm. 11 and Thm. 15 when \( v \in K^{2+} \).

**THEOREM 16 (Summary)**
Suppose \( v \in K^{2+} \), \( x \in D \), and \( F(x) \) and \( Y_1(x) \) are \( 2 \times 2 \) and \( 2 \times 1 \) matrix functions, respectively, such that
\[
\text{dom } F(x) = C - \bar{I}(x) \text{ and dom } Y_1(x) = \bar{I}(x).
\] (3B.6)

Then the following three statements are equivalent to one another:

\( (i) \) The function \( F(x) \) is the solution of the HHP corresponding to \( (v, F^M, x) \), and \( Y_1(x) \) is the function whose restriction to \( \bar{I}(x) \) is defined by Eq. (2B.3) and whose extension to \( \bar{I}(x) \) is then defined by Eqs. (2C.1) and (2C.2). \( \Box \)

\( (ii) \) The function \( Y_1(x) \) is \( C^{0+} \) and its restriction to \( \bar{I}(x) \) is a solution of the Alekseev-type equation (2B.17); and \( F(x) \) is defined in terms of \( Y_1(x) \) by Eq. (2B.14).

\( (iii) \) The function \( Y_1(x) \) is summable over \( \bar{I}(x) \) and is a solution of the Fredholm equation (3A.6) for all \( \sigma \in \bar{I}(x) \).

**Proof:** Directly from Thm. 11 and Thm. 15. \( \Box \)

**COROLLARY 17 (Uniqueness of solutions)**
When \( v \in K^{2+} \), each of the solutions defined in (i), (ii) and (iii) of the preceding theorem is unique if it exists.

**Proof:** This follows from the preceding theorem and the uniqueness theorem [Thm. 3(iv)] for the HHP. \( \Box \)
4 Existence and properties of the HHP solution $\mathcal{F}$ when $\mathbf{v} \in K^{2+}$

A. Homogeneous equations, theorems, etc.

By considering a homogeneous version of the Fredholm equation (3A.6), we found it possible to employ the Fredholm alternative theorem to establish the existence of the solution of the HHP corresponding to $(\mathbf{v}, \mathcal{F}^M)$ when $\mathbf{v} \in K^{2+}$.

Dfn. of HHP$_0$

The HHP that is defined as in Sec. I except that the condition (2) is replaced by the condition

$$\mathcal{F}(\mathbf{x}, \infty) = 0 \text{ (HHP}_0 \text{ condition)}$$

(4A.1)

will be called the HHP$_0$ corresponding to $(\mathbf{v}, \mathcal{F}_0, \mathbf{x})$.

End of Dfn.

Clearly, the $2 \times 2$ matrix function $\mathcal{F}(\mathbf{x})$ with the domain $C - \mathcal{I}(\mathbf{x})$ and the value $\mathcal{F}(\mathbf{x}, \tau) = 0$ for all $\tau$ in this domain is a solution of the HHP$_0$ corresponding to $(\mathbf{v}, \mathcal{F}_0, \mathbf{v})$. It will be called the zero solution.

Dfn. of equation number with attached subscript ‘0’

To each linear integral equation that occurs in these notes from Thm. 4 to Thm. 16, inclusive, and that has a term that is an integral whose integrand involves ‘$\mathcal{F}$’, ‘$\mathcal{F}^\pm$’, ‘$\mathcal{Y}$’ or ‘$\mathcal{Y}^{(i)}$’ (or one of their columns), there corresponds a homogeneous integral equation that will be designated by the symbol that results when the subscript ‘0’ is attached to the equation number for the inhomogeneous integral equation.

End of Dfn.

Dfn. of theorem label (etc.) with attached subscript ‘0’

When a new valid assertion results from subjecting a labelled assertion to the following substitutions, that new valid assertion will bear the same label with an attached subscript ‘0’.

1. ‘HHP’ $\rightarrow$ ‘HHP$_0$’
2. $\mathcal{F}(\mathbf{x}, \infty) = I$ $\rightarrow$ ‘$\mathcal{F}(\mathbf{x}, \infty) = 0$’ in condition (2) of the HHP
3. each integral equation $\rightarrow$ the corresponding homogeneous integral equation
4. each equation number for an integral equation $\rightarrow$ the same equation number with an attached subscript ‘0’.

End of Dfn.
B. Only a zero solution of homogeneous equation

For our immediate purpose, we shall need the following explicit version of Thm. 16:

**THEOREM 18 (Theorem 16)***

Suppose $v \in K^{2+}$, $x \in D$, and $F(x)$ and $Y_1(x)$ are $2 \times 2$ and $2 \times 1$ matrix functions, respectively, such that

$$
\text{dom } F(x) = C - \mathcal{I}(x) \text{ and } \text{dom } Y_1(x) = \mathcal{I}(x).
$$

Then the following three statements are equivalent to one another:

(i) The function $F(x)$ is a solution of the HHP corresponding to $(v, F^M, x)$; and $Y_1(x)$ is the continuous function whose restriction to $\mathcal{I}(x)$ is defined in terms of $F^\pm(x)$ by Eq. (2B.9), and whose existence and uniqueness are asserted by Cor. 10.

(ii) The function $Y_1(x)$ is $C^{0+}$ and its restriction to $\mathcal{I}(x)$ is a solution of Eq. (2B.17); and $F(x)$ is defined in terms of $Y_1(x)$ by Eq. (2B.14).

(iii) The function $Y_1(x)$ is summable over $\mathcal{I}(x)$ and is a solution of the homogeneous Fredholm integral equation (3A.6) for all $\sigma \in \mathcal{I}(x)$.

**Proof:** This theorem summarizes Thms. 11 and 15 for the case $v \in K^{2+}$. End of proof.

**THEOREM 19 (Only a zero solution of HHP)***

For each $v \in K$, $F_0 \in S_F$ and $x \in D$, the only solution of the HHP corresponding to $(v, F_0, x)$ is its zero solution.

**Proof:** The proof will be given in four parts:

1. From the hypothesis $F_0 \in S_F$,

$$
[F_0(x, \tau^*)]^\dagger A_0(x, \tau) F_0(x, \tau) = A_0(x_0, \tau) \text{ for all } \tau \in C - \mathcal{I}(x),
$$

where

$$
A_0(x, \tau) := (\tau - z)\Omega + \Omega h_0(x)\Omega
$$

and $h_0(x)$ is computed from $E_0 \in S_E$ in the usual way.

Since

$$
h_0(x_0) := h^M(x_0) = \begin{pmatrix} \rho_0^2 & 0 \\ 0 & 1 \end{pmatrix}
$$

in our gauge,

$$
A_0(x_0, \tau) = A^M(x_0, \tau).
$$

Equation (4B.2) is clearly expressible in the alternative form

$$
F_0(x, \tau) \left[A^M(x_0, \tau)\right]^{-1} [F_0(x, \tau^*)]^\dagger = [A_0(x, \tau)]^{-1}
$$

for all $\tau \in C - \mathcal{I}(x)$.

---

23 To prove Eq. (4B.3), one first shows that Eq. (1A.21) is equivalent to $A_0 \Gamma_0 = \frac{1}{2} \Omega dH_0\Omega$ and then uses (1A.7) to show that the differential of the left side of Eq. (4B.2) vanishes. The rest is obvious.
since \([F_0(x, \tau)]^{-1}\) exists for all \(\tau \in C - \tilde{I}(x)\), and
\[
[A_0(x, \tau)]^{-1} = \frac{B_0(x, \tau)}{\rho^2 - (\tau - z)^2},
\]  
(4B.7)
where
\[
B_0(x, \tau) := h_0(x) - (\tau - z)\Omega,
\]  
(4B.8)
exists for all \(\tau \in C - \{r, s\}\).

(2) Next, condition (3) in the definition of the HHP (and the HHP_0) that is given in Sec. I asserts that \(F^\pm(x)\) exist, and Eq. (1C.7) is expressible in the form
\[
F^\pm(x, \sigma) = Y^{(i)}(\sigma)F^\pm_0(x, \sigma)[v^{(i)}(\sigma)]^{-1} \text{ for each } i \in \{3, 4\} \text{ and } \sigma \in I(x).
\]  
(4B.9)
From the definition of the group \(K\),
\[
[v^{(i)}(\sigma)]^{-1}[A^M(x_0, \sigma)]^{-1}[v^{(i)}(\sigma)]^{-1} = A^M(x_0, \sigma)^{-1} \text{ for all } \sigma \in I(i) - \{r, s\}.
\]  
(4B.10)
Therefore, from Eqs. (4B.9), (4B.10) and (4B.6),
\[
F^\pm(x, \sigma)[A^M(x_0, \sigma)]^{-1}[F^\mp(x, \sigma)]^\dagger = Y(x, \sigma)[A_0(x, \sigma)]^{-1}Y(x, \sigma)^\dagger
\]  
for all \(\sigma \in I(x)\);  
(4B.11)
or, equivalently, with the aid of Eqs. (4B.7), (4B.8) and (4B.4),
\[
\left[\frac{\rho^2 - (\sigma - z)^2}{\rho^2_0 - (\sigma - z_0)^2}\right]F^\pm(x, \sigma)B^M(\sigma)[F^\mp(x, \sigma)]^\dagger =
Y(x, \sigma)B_0(x, \sigma)Y(x, \sigma)^\dagger \text{ for all } \sigma \in I(x),
\]  
(4B.12)
where
\[
B^M(\tau) := \begin{pmatrix} \rho^2_0 & -i(\tau - z_0) \\ i(\tau - z_0) & 1 \end{pmatrix}.
\]  
(4B.13)

(3) Next, let \(Z(x)\) denote the function with the (tentative) domain \(C - \tilde{I}(x)\) and the values
\[
Z(x, \tau) := Z(x)(\tau)
\]
\[
:= \nu(x, \tau)^{-1}F(x, \tau)B^M(\tau)[\nu(x, \tau^*)^{-1}F(x, \tau^*)]^\dagger
\]  
for all \(\tau \in C - \tilde{I}(x)\),
(4B.14)
where note that
\[
\nu(x, \tau)^{-2} = \frac{(\tau - r)(\tau - s)}{(\tau - r_0)(\tau - s_0)} = \frac{(\tau - z)^2 - \rho^2}{(\tau - z_0)^2 - \rho^2_0}.
\]  
(4B.15)
We again appeal to the trilogy of elementary theorems due to Riemann and Liouville.\textsuperscript{24}
Using these, we shall define an extension of \(Z(x)\), and we shall let \(Z(x)\) denote this extension as well.

\textsuperscript{24}See Refs. 14, 15 and 14.
From condition (1) in the definition of the HHP (and the HHP₀), and from Eqs. (4B.14), (4B.13) and (1A.1),

\[ Z(x, \tau) \text{ is a holomorphic function of } \tau \text{ throughout } C - \mathcal{I}(x), \quad (4B.16) \]

and

\[ Z(x, \infty) = 0. \quad (4B.17) \]

Let \( \text{Im} \ \zeta > 0 \)

\[ Z^\pm(x, \sigma) := \lim_{\zeta \to 0} Z(x, \sigma \pm \zeta) \text{ for all } \sigma \in \mathcal{I}(x), \quad (4B.18) \]

which exist according to condition (3) in the definition of the HHP (and the HHP₀). Then, from Eqs. (4B.14), (4B.15) and (4B.12),

\[ Z^+(x, \sigma) = Z^-(x, \sigma) = Y(x, \sigma) B_0(x, \sigma) Y(x, \sigma)^\dagger \text{ for all } \sigma \in \mathcal{I}(x). \quad (4B.19) \]

The above equation permits us to define a single valued extension of \( Z(x) \) to the domain \( C - \{r, s, r_0, s_0\} \) by letting

\[ Z(x, \sigma) := Z^\pm(x, \sigma) = Y(x, \sigma) B_0(x, \sigma) Y(x, \sigma)^\dagger \text{ for all } \sigma \in \mathcal{I}(x), \quad (4B.20) \]

whereupon, from (4B.16), (4B.20) and the theorem on analytic continuation across an arc,

\[ Z(x, \tau) \text{ is a holomorphic function of } \tau \text{ throughout } C - \{r, s, r_0, s_0\}. \quad (4B.21) \]

We next apply condition (4) in the definition of the HHP (and HHP₀). Since, according to condition (4), \( \nu(x)^{-1} \mathcal{F}(x) \) and \( Y(x) \) are both bounded at \( x \), Eqs. (4B.14) and (4B.20) yield

There exists a positive real number \( M_1(x) \) such that

\[ ||Z(x, \tau)|| < M_1(x) \text{ as } \tau \to r \text{ and as } \tau \to s \text{ through any sequence of points in } C - \{r, s, r_0, s_0\}. \quad (4B.22) \]

Since \( \mathcal{F}(x) \) and \( Y(x) \) are both bounded at \( x_0 \), Eqs. (4B.14), (4B.13) and (4B.20) yield

There exists a positive real number \( M_2(x) \) such that

\[ ||(\tau - r_0)(\tau - s_0)Z(x, \tau)|| < M_2(x) \text{ as } \tau \to r_0 \text{ and as } \tau \to s_0 \text{ through any sequence of points in } C - \{r, s, r_0, s_0\}. \quad (4B.23) \]

However, since \( Y(x) \) is bounded at \( x_0 \), Eq. (4B.20) yields

There exists a positive real number \( M_3(x) \) such that

\[ ||Z(x, \sigma)|| < M_3(x) \text{ as } \sigma \to r_0 \text{ and as } \sigma \to s_0 \text{ through any sequence of points in } \mathcal{I}(x). \quad (4B.24) \]

The theorem on isolated singularities, together with statements (4B.21) to (4B.24), now informs us that

\[ Z(x) \text{ has a holomorphic extension [which we also denote by } Z(x)\text{]} \text{ to } C, \quad (4B.25) \]

whereupon Eq. (4B.17) and the (generalized) theorem of Liouville yield

\[ Z(x, \tau) = 0 \text{ for all } \tau \in C. \quad (4B.26) \]
(4) Putting (4B.14) and (4B.26) together, one obtains

\[ F(x, \sigma)B^M(\sigma)F(x, \sigma)^\dagger = 0 \text{ for all } \sigma \in C - \bar{I}(x). \] (4B.27)

Note from Eq. (4B.13), \( B^M(\sigma) \) is hermitian,

\[ \text{tr } B^M(\sigma) = 1 + \rho_0^2 \text{ and } \det B^M(\sigma) = (s_0 - \sigma)(\sigma - r_0). \] (4B.28)

Recall that \( |r, r_0| < |s, s_0| \) for any type A triple \((x_0, x_1, x_2)\); and it is clear that

\[ B^M(\sigma) \text{ is hermitian and positive definite for all } |r, r_0| < \sigma < |s, s_0|. \] (4B.29)

Therefore, Eq. (4B.27) implies

\[ F(x, \sigma) = 0 \text{ for all } \sigma \text{ such that } |r, r_0| < \sigma < |s, s_0|. \] (4B.30)

However, \( F(x, \tau) \) is a holomorphic function of \( \tau \) throughout \( C - \bar{I}(x) \), and this domain contains the open interval between \( |r, r_0| \) and \( |s, s_0| \). So,

\[ F(x, \tau) = 0 \text{ for all } \tau \in C - \bar{I}(x). \] (4B.31)

End of proof.

**THEOREM 20 (Only a zero solution of (3A.6)_0)**

The only solution of the homogeneous Fredholm integral equation of the second kind Eq. (3A.6)_0 is its zero solution.

**Proof:** Let \( \mathcal{Y}_1(x) \), with domain \( \bar{I}(x) \), denote a solution of Eq. (3A.6)_0; and let \( F(x) \), with domain \( C - \bar{I}(x) \), be defined in terms of \( \mathcal{Y}_1(x) \) by Eq. (2B.14). Using Thm. 18, one obtains

\[ F(x) \text{ is a solution of the HHP}_0 \text{ corresponding to } (v, F^M, x), \] (4B.32)

whereupon Thm. 19 delivers

\[ F(x, \tau) = 0 \text{ for all } \tau \in C - \bar{I}(x). \] (4B.33)

It follows that

\[ F^\pm(x, \sigma) = 0 \text{ for all } \sigma \in I(x), \] (4B.34)

whereupon, from Thm. 19(i), Eq. (2B.9) and the continuity of \( \mathcal{Y}_1(x) \),

\[ \mathcal{Y}_1(x, \sigma) = 0 \text{ for all } \sigma \in \bar{I}(x). \] (4B.35)

End of proof.
C. Existence and uniqueness of HHP solution

At this point, we note that Eq. (3A.6) is a regular Fredholm equation in disguise when \( v \in K^{2+} \). In integrals such as those in Thm. 5, it is sometimes useful to introduce a new variable of integration for the purpose of getting rid of the singularities of the integrands at \( \sigma' \in \{r, s, r_0, s_0\} \). This is especially important when one has to consider derivatives of the integrals with respect to \( r \) and \( s \).

**Dfns. of \( \Theta, \theta(x) \) and \( \sigma(x) \)**

Let \( \Theta \) denote that union of arcs

\[
\Theta := \left[0, \frac{\pi}{2}\right] + \left[\frac{\pi}{2}, 3\pi/2\right]
\]

whose assigned orientations are in the direction of increasing \( \theta \in [0, \pi/2] \) and \( \theta \in [\pi, 3\pi/2] \). For each \( x \in D \), let

\[
\theta(x) : \bar{I}(x) \to \Theta
\]

be a mapping such that

\[
\theta(x)(\sigma) := \theta(x, \sigma),
\]

where

\[
0 \leq \theta(x, \sigma) \leq \frac{\pi}{2} \quad \text{and} \quad \cos[2\theta(x, \sigma)] := \frac{2\sigma - (r_0 + r)}{r_0 - r} \quad \text{when} \quad \sigma \in \bar{I}^{(3)}(x)
\]

and

\[
\pi \leq \theta(x, \sigma) \leq \frac{3\pi}{2} \quad \text{and} \quad \cos[2\theta(x, \sigma)] := \frac{2\sigma - (s_0 + s)}{s_0 - s} \quad \text{when} \quad \sigma \in \bar{I}^{(4)}(x).
\]

Also let

\[
\sigma(x) : \Theta \to \bar{I}(x)
\]

be a mapping such that

\[
\sigma(x)(\theta) := \sigma(x, \theta),
\]

where

\[
\sigma(x, \theta) := r_0 \cos^2 \theta + r \sin^2 \theta \quad \text{when} \quad \theta \in \left[0, \frac{\pi}{2}\right]
\]

and

\[
\sigma(x, \theta) := s_0 \cos^2 \theta + s \sin^2 \theta \quad \text{when} \quad \theta \in \left[\pi, \frac{3\pi}{2}\right].
\]

End of Dfn.
The mapping $\theta(x)$ is monotonic and is a continuous bijection (one-to-one and onto) of $I(x)$ onto $\Theta$, and $\sigma(x)$ is its inverse mapping. Moreover, $\sigma(x)$ is analytic [which means that it has an analytic extension to an open subset of $R^1$]. Note, in particular, that

$$\sqrt{\frac{s - \sigma(x, \theta')}{s_0 - \sigma(x, \theta')}}$$

is a real positive-valued analytic function of $(x, \theta')$ on $D \times \left[0, \frac{\pi}{2}\right]$ (4C.10)

and

$$\sqrt{\frac{\sigma(x, \theta') - r}{\sigma(x, \theta') - r_0}}$$

is a real positive-valued analytic function of $(x, \theta')$ on $D \times \left[\pi, \frac{3\pi}{2}\right]$, (4C.11)

since the left and right cuts are assumed not to overlap.

The following equation is equivalent to Eq. (3A.6) and has a $C_{0+}$ kernel and a $C_{1+}$ inhomogeneous term:

$$y_1(x, \theta) = \frac{1}{2\pi} \int_{\Theta} d\theta' y_1(x, \theta') \kappa_{21}(x, \theta', \theta)$$

$$= u_1(x, \theta) \text{ for all } \theta \in \Theta := [0, \pi/2] \cup [\pi, 3\pi/2], \text{ (4C.12)}$$

where

$$y_1(x, \theta) := Y_1(x, \sigma(x, \theta)), \text{ (4C.13)}$$

$$u_1(x, \theta) := U_1(x, \sigma(x, \theta)), \text{ (4C.14)}$$

$$\kappa_{21}(x, \theta', \theta) := q(x, \theta') K_{21}(x, \sigma(x, \theta'), \sigma(x, \theta)) \text{ (4C.15)}$$

and

$$q(x, \theta) := \begin{cases} 
(r_0 - r) \cos^2 \theta \sqrt{\frac{s - \sigma(x, \theta)}{s_0 - \sigma(x, \theta)}} & \text{when } \theta \in [0, \pi/2], \\
(s_0 - s) \cos^2 \theta \sqrt{\frac{\sigma(x, \theta) - r}{\sigma(x, \theta) - r_0}} & \text{when } \theta \in [\pi, 3\pi/2].
\end{cases} \text{ (4C.16)}$$

Equations (3A.9) and (3A.10) are expressible in the following forms, in which $x$ and $x_0$ are no longer suppressed:

$$U_1(x, \sigma) = W_1(\sigma) - \frac{2}{\pi} \int_{\Theta} d\theta' p(x, \theta') \frac{W_1(\sigma(x, \theta')) - W_1(\sigma)}{\sigma(x, \theta') - \sigma}, \text{ (4C.17)}$$

$$K_{21}(x, \sigma', \sigma) = k_{21}(\sigma', \sigma) - \frac{2}{\pi} \int_{\Theta} d\theta'' p(x, \theta'') \frac{k_{21}(\sigma', \sigma''(x, \theta'')) - k_{21}(\sigma', \sigma)}{\sigma(x, \theta'') - \sigma}, \text{ (4C.18)}$$

where

$$p(x, \theta) := \begin{cases} 
(r_0 - r) \sin^2 \theta \sqrt{\frac{s_0 - \sigma(x, \theta)}{s - \sigma(x, \theta)}} & \text{when } \theta \in [0, \pi/2], \\
(s_0 - s) \sin^2 \theta \sqrt{\frac{\sigma(x, \theta) - r_0}{\sigma(x, \theta) - r}} & \text{when } \theta \in [\pi, 3\pi/2].
\end{cases} \text{ (4C.19)}$$
THEOREM 21 (Fredholm determinant not zero)
The Fredholm determinant corresponding to the kernel \( \kappa_{21}(x) \) is not zero. Therefore, there exists exactly one solution of Eq. (4.12) for each given \( v \in K^2 + \) and \( x \in D \); or, equivalently, there exists exactly one solution of Eq. (3A.8) for each given \( v \in K^2 + \) and \( x \in D \).

Proof: This follows from Thm. 20 and the Fredholm alternative. End of proof.

Thus, in summation, we have the following theorem:

THEOREM 22 (Existence and uniqueness of HHP solution)

(i) If \( v \in K^2 + \), then the HHP\(_0\) corresponding to \( (v, F^M, x) \) is equivalent to the homogeneous Fredholm equation of the second kind that is obtained from Eq. (3A.4) by deleting the term \( U_1(\sigma) \), provided that the term \( W_1(\sigma) \) is also deleted from the expression (2B.16) for \( Y_2(\sigma) \).

(ii) For any given \( x \in D \), and \( v \in K \), the HHP\(_0\) corresponding to \( (v, F^M, x) \) has the unique solution \( F(x, \tau) = 0 \) for all \( \tau \in C - I(x) \).

(iii) Therefore, if \( v \in K^2 + \), the only solution of the homogeneous Fredholm equation is the zero solution. Hence from the Fredholm alternative theorem, the inhomogeneous Fredholm equation (3A.8) has exactly one solution. We conclude that there exists one and only one solution of the HHP corresponding to \( (v, F^M) \) when \( v \in K^2 + \).

Proof: Directly from Thms. 18, 19, 20 and 21. End of proof.

D. The 2×2 matrix \( H(x) \) associated with each solution of the HHP corresponding to \( (v, F_0, x) \) when \( v \in K \)

THEOREM 23 (Properties of \( H(x) \) and \( h(x) \))
For each \( v \in K, F_0 \in S_F, x \in D \) and solution \( F(x) \) of the HHP corresponding to \( (v, F_0, x) \), there exists exactly one 2×2 matrix \( H(x) \) such that

\[
F(x, \tau) = I + (2\tau)^{-1} \left[ H(x) - H^M(x_0) \right] \Omega + O(\tau^{-2})
\]
in at least one neighborhood of \( \tau = \infty \). \hspace{1cm} (4D.1)

Moreover,

\[
H(x_0) = H^M(x_0), \hspace{1cm} (4D.2)
\]
\[
H(x) - H(x)^T = 2z\Omega, \hspace{1cm} (4D.3)
\]
\[
h(x) := -\text{Re} \ H(x) \text{ is symmetric}, \hspace{1cm} (4D.4)
\]

and

\[
h(x_0) = \begin{pmatrix}
\rho_0^2 & 0 \\
0 & 1 
\end{pmatrix}. \hspace{1cm} (4D.5)
\]
Proof: From conditions (1) and (2) in the definition of the HHP, there exists exactly one $2 \times 2$ matrix $B(x)$ such that

$$F(x, \tau) = I + (2\tau)^{-1}B(x) + O(\tau^{-2})$$

in at least one neighborhood of $\tau = \infty$.

Let

$$H(x) := H^M(x_0) + B(x)\Omega,$$

whereupon statement (4D.1) follows. From Thm. 3(v) [Eq. (1C.13)], $B(x_0) = 0$, whereupon Eq. (4D.2) follows.

Next, from Thm. 3(iii),

$$\det F(x, \tau) = \nu(x, \tau) = 1 + (2\tau)^{-1}(r + s - r_0 - s_0) + O(\tau^{-2})$$

$$= 1 + \tau^{-1}(z - z_0) + O(\tau^{-2})$$

in at least one neighborhood of $\tau = \infty$. \hfill (4D.6)

Moreover, from Eq. (1B.4),

$$H^M(x_0) - [H^M(x_0)]^T = 2z_0\Omega. \hfill (4D.7)$$

For any $2 \times 2$ matrix $M$, $M\Omega M^T = \Omega \det M$. In particular,

$$F(x, \tau)\Omega F(x, \tau)^T = \Omega \nu(x, \tau). \hfill (4D.8)$$

The next step is to consider Eq. (4D.8) in at least one neighborhood of $\tau = \infty$ for which the expansions given by Eqs. (4D.1) and (4D.6) hold. The reader can then easily deduce Eq. (4D.3) by using Eq. (4D.7) and the relations $Q_T = -\Omega$ and $\Omega^2 = I$.

The statement (4D.4) follows from Eq. (4D.3) and the relation $\Omega^* = -\Omega$. Equation (4D.5) is derived from Eqs. (4D.2) and (1B.4).

End of proof.

THEOREM 24 (Quadratic relation)

For each $v \in K$, $F_0 \in S_F$, $x_0 \in D$ and solution $F(x)$ of the HHP corresponding to $(v, F_0, x)$, let $h(x)$ be defined as in the preceding theorem, and let

$$A(x, \tau) = (\tau - z)\Omega + \Omega h(x)\Omega. \hfill (4D.9)$$

Then

$$F^\dagger(x, \tau)A(x, \tau)F(x, \tau) = A(x_0, \tau) \text{ for all } \tau \in [C - \tilde{I}(x)] - \{\infty\}, \hfill (4D.10)$$

where

$$F^\dagger(x, \tau) := [F(x, \tau^*)]^\dagger \text{ for all } \tau \in C - \tilde{I}(x). \hfill (4D.11)$$
Proof: Note that parts (1) and (2) in the proof of Thm. 19 remain valid here. For the sake
of convenience, we repeat below Eq. (4B.12) from part (2) of that proof.

\[
\nu(x, \sigma) - 2 F^{\pm}(x, \sigma) B^M(\sigma)[F^{\pm}(x, \sigma)]^\dagger = Y(x, \sigma) B_0(x, \sigma) Y(x, \sigma)^\dagger
\]

for all \( \sigma \in \mathcal{I}(x) \), (4D.12)

where

\[
B^M(\tau) := \begin{pmatrix} \rho_0^2 & -i(\tau - z_0) \\ i(\tau - z_0) & 1 \end{pmatrix}, \quad (4D.13)
\]

\[
B_0(x, \tau) := h_0(x) - (\tau - z)\Omega
\]

\[
= [\rho^2 - (\tau - z)^2] A_0(x, \tau)^{-1}, \quad (4D.14)
\]

\[
\nu(x, \tau)^{-2} = \frac{(\tau - \tau_0)(\tau - s_0)}{(\tau - s)(\tau - z)} \frac{(\tau - s_0)(\tau - s_0)}{(\tau - z_0)^2 - \rho_0^2}.
\]

Next, let \( Z(x) \) denote the function with the (tentative) domain \([C - \mathcal{I}(x)] - \{\infty\}\) and the values

\[
Z(x, \tau) := \nu(x, \tau)^{-1} F(x, \tau) B^M(\tau) \left[\nu(x, \tau)^{-1} F(x, \tau^*)\right]^\dagger. \quad (4D.16)
\]

From conditions (1) and (2) in the definition of the HHP, and from Eqs. (4D.13) and (4D.16),

\[
Z(x, \tau) \text{ is a holomorphic function of } \tau
\]

throughout \([C - \mathcal{I}(x)] - \{\infty\}\)

and has a simple pole at \( \tau = \infty \). (4D.17)

Note that Eq. (4D.5) enables us to express (4D.13) in the form

\[
B^M(\tau) = h(x_0) - (\tau - z_0)\Omega. \quad (4D.18)
\]

Also, note that Eqs. (4D.3) and (4D.4) imply that

\[
H(x) + H(x)^\dagger = -2h(x) + 2\zeta\Omega \quad (4D.19)
\]

and that Eq. (4D.15) yields

\[
\nu(x, \tau)^{-2} = 1 + 2\tau^{-1}(z_0 - \zeta) + O(\tau^{-2})
\]

in at least one neighborhood of \( \tau = \infty \). (4D.20)

Upon using the relation \( \nu(x, \tau^*)^\dagger = \nu(x, \tau) \) and upon inserting (4D.1), (4D.18) and (4D.20)

into the right side of Eq. (4D.16), one obtains the following result with the aid of Eqs. (4D.2)

and (4D.13):

\[
Z(x, \tau) = -(\tau - z)\Omega + h(x) + O(\tau^{-1})
\]

in at least one neighborhood of \( \tau = \infty \). (4D.21)
We again appeal to the trilogy of elementary theorems due to Riemann and Liouville. We let $Z^\pm(x, \sigma)$ be defined for all $\sigma \in \mathcal{I}(x)$ by Eq. \((4B.18)\), whereupon Eqs. \((4D.12)\) and \((4D.10)\) yield
\[
Z^+(x, \sigma) = Z^-(x, \sigma) = Y(x, \sigma)B_0(x, \sigma)Y(x, \sigma)^\dagger \text{ for all } \sigma \in \mathcal{I}(x).
\]
(4D.22)

The above equation permits us to define a single valued extension of $Z(x)$ to the domain $C - \{r, s, r_0, s_0, \infty\}$ by letting
\[
Z(x, \sigma) := Z^\pm(x, \sigma) = Y(x, \sigma)B_0(x, \sigma)Y(x, \sigma)^\dagger \text{ for all } \sigma \in \mathcal{I}(x),
\]
(4D.23)

whereupon \((4D.17)\), \((4D.23)\) and the theorem on analytic continuation across an arc tell us that
\[
Z(x, \tau) \text{ is a holomorphic function of } \tau \text{ throughout } C - \{r, s, r_0, s_0, \infty\} \text{ and has a simple pole at } \tau = \infty.
\]
(4D.24)

We next use condition (4) in the definition of the HHP, and we obtain the statements \((4B.22)\), \((4B.23)\) and \((4B.24)\) exactly as we did in the proof of Thm. 19. The theorem on isolated singularities, together with the statements \((4D.24)\), \((4B.22)\), \((4B.23)\) and \((4B.24)\) now inform us that
\[
Z(x) \text{ has a holomorphic extension [which we also denote by } Z(x) \text{] to } C - \{\infty\} \text{ and has a simple pole at } \tau = \infty,
\]
(4D.25)

whereupon Eq. \((4D.21)\) and the theorem on entire functions that do not have an essential singularity at $\tau = \infty$ yield
\[
Z(x, \tau) = -(\tau - z)\Omega + h(x) \text{ for all } \tau \in C - \{\infty\}.
\]
(4D.26)

We are now close to completing our proof. From Thm. 3(iii), Eqs. \((4D.10)\), \((4D.13)\) and \((4D.13)\),
\[
\det Z(x, \tau) = \rho^2 - (\tau - z)^2.
\]
(4D.27)

Therefore, from Eqs. \((4D.1)\) and \((4D.26)\), the matrix $-(\tau - z)\Omega + h(x)$ is invertible when
\[
\tau \notin \{r, s, \infty\},
\]
and
\[
[-(\tau - z)\Omega + h(x)]^{-1} = \frac{\mathcal{A}(x, \tau)}{\rho^2 - (\tau - z)^2}
\]
(4D.28)

[Above, we have used the fact that $M^{-1} = \Omega M^T\Omega/ \det M$ for any invertible $2 \times 2$ matrix $M$.]

One then obtains from Eqs. \((4D.10)\), \((4D.18)\), \((4D.26)\) and \((4D.28)\),
\[
\mathcal{F}(x, \tau)[\mathcal{A}(x_0, \tau)]^{-1}\mathcal{F}^\dagger(x, \tau) = \mathcal{A}(x, \tau)^{-1}
\]
for all $\tau \in [C - \bar{\mathcal{I}}(x)] - \{\infty\}$,

whereupon the conclusion \((4D.10)\) follows.

End of proof.

\[\text{See Refs. 14, 15 and 16.}\]
THEOREM 25 (More properties of $h(x)$)

Grant the same premises as in the preceding two theorems, and let $h(x)$ be defined as before. Then

$$ \det h(x) = \rho^2 $$

(4D.29)

and

$$ h(x) \text{ is positive definite} $$

(4D.30)

as well as real and symmetric.

Proof: Since $h(x)$ is symmetric

$$ \det [h(x) - (\tau - z)\Omega] = \det [h(x) - (\tau - z)^2]. $$

Therefore, Eq. (4D.26) and (4D.27) imply that $\det h(x) = \rho^2$.

From Eqs. (4D.16), (4D.26) and (4D.15),

$$ Z(x, \sigma) = \frac{(\sigma - r)(s - \sigma)}{(\sigma - r_0)(s_0 - \sigma)} F(x, \sigma) B^M(\sigma) F(x, \sigma)^\dagger $$

$$ = - (\sigma - z)\Omega + h(x) \text{ for all } |r, r_0| < \sigma < |s, s_0|. $$

Equation (4D.31) provides us with

$$ \det B^M(\sigma) = (s_0 - \sigma)(\sigma - r_0) \text{ and } \text{tr } B^M(\sigma) = 1 + \rho_0^2. $$

(4D.32)

Therefore,

$$ \frac{(\sigma - r)(s - \sigma)}{(\sigma - r_0)(s_0 - \sigma)} B^M(\sigma) $$

is a positive definite hermitian matrix when $|r, r_0| < \sigma < |s, s_0|$. Therefore, the left side of Eq. (4D.31) is a positive definite hermitian matrix when $|r, r_0| < \sigma < |s, s_0|$ and must, therefore, have a real positive trace when $|r, r_0| < \sigma < |s, s_0|$. So,

$$ \text{tr } [- (\sigma - z)\Omega + h(x)] = \text{tr } h(x) > 0; $$

(4D.33)

and, since the determinant of $h(x)$ is also positive, $h(x)$ is positive definite. End of proof.

We caution the reader that the HHP solution $F$ whose existence has been proved in this section when $v \in K^{2+}$ is not necessarily a member of $S_F$; and $H$ as defined by Eq. (4D.1) is not necessarily a member of $S_H$. However, as we shall prove in Sec. 5, $F \in S_F$ and $H \in S_H$ when $v \in K^3$. To prepare for this proof, we shall now investigate the differentiability of $F$ and $H$ when $v \in K^3$.

5 Derivatives of $F$ and $H$ when $v \in K^3$

A. Fredholm equation solution $y_1$ corresponding to $v \in K^3$

We again refer the reader to the mappings $\theta(x) : \tilde{I}(x) \to \Theta$ and $\sigma(x) : \Theta \to \tilde{I}(x)$, for we shall first be discussing the solution $y_1$ of the Fredholm equation (4C.12) with kernel $\kappa_{21}$ and
inhomogeneous term $u_1$ rather than the solution $Y_1$ of the Fredholm equation (3A.4) with kernel $K_{21}$ and inhomogeneous term $U_1$.

When $v \in K^{2+}$, the solution $y_1$ need not be differentiable. However, when $v \in K^3$, the kernel $K_{21}(x, \sigma', \sigma)$ and the inhomogeneous term $U_1(x, \sigma)$ in the Fredholm equation (3A.6) are $C^1$ and $C^2$ functions of $(x, \sigma', \sigma)$ and $(x, \sigma)$, respectively; and the result is a differentiable $y_1$ as we shall see in Thm. 28. The following lemma is required for the proof of Thm. 28.

**LEMMA 26 (Differentiability properties of $u_1$ and $\kappa_{21}$ when $v \in K^3$)**

When $v \in K^3$, $u_1$ is $C^2$ and $\kappa_{21}$ is $C^1$. Moreover, $\partial^2 \kappa_{21}(x, \theta', \theta)/\partial r \partial s$ exists and is a continuous function of $(x, \theta', \theta)$ throughout $D \times \Theta^2$ [whereupon, from a theorem of the calculus, $\partial^2 \kappa_{21}/\partial s \partial r$ also exists and is equal to $\partial^2 \kappa_{21}/\partial r \partial s$].

**Proof:** The proof will be given in three parts:

1. From Eqs. (4C.8) and (4C.9), $\sigma(x, \theta)$ is a real analytic function of $\theta$ throughout $D \times \Theta$,

\[
\sigma(x, \theta) \in |r, r_0| \text{ when } \theta \in [0, \pi/2],
\]

\[
\sigma(x, \theta) \in |s, s_0| \text{ when } \theta \in [\pi, 3\pi/2],
\]

Therefore,

\[
W(\sigma(x, \theta)) \text{ is a } C^3 \text{ function of } (x, \theta) \text{ throughout } D \times \Theta
\]

and

\[
\lambda_{21}(\sigma(x, \theta'), \sigma(x, \theta''), \sigma(x, \theta)) \text{ is a } C^1 \text{ function of } (x, \theta', \theta'', \theta) \text{ throughout } D \times \Theta^3.
\]

2. To prove that

\[
\frac{\partial^2 \lambda_{21}(\sigma(x, \theta'), \sigma(x, \theta''), \sigma(x, \theta))}{\partial r \partial s}
\]

exists and is a continuous function of

$(x, \theta', \theta'', \theta)$ throughout $D \times \Theta^3$, (5A.5)

we consider three distinct cases, (a), (b) and (c):

(a)

\[
(\theta'', \theta) \in [0, \pi/2] \times [\pi, 3\pi/2] \text{ or } (\theta'', \theta) \in [\pi, 3\pi/2] \times [0, \pi/2].
\]

(b)

\[
(\theta'', \theta) \in [0, \pi/2]^2 \text{ and } \theta' \in [\pi, 3\pi/2], \text{ or }
(\theta'', \theta) \in [\pi, 3\pi/2]^2 \text{ and } \theta' \in [0, \pi/2].
\]

(c)

\[
(\theta', \theta'') \in [0, \pi/2]^3 \text{ or } (\theta', \theta'') \in [\pi, 3\pi/2]^3.
\]

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In case (a) and case (b), it is easily seen that the denominator \( \sigma(x, \theta''') - \sigma(x, \theta) \) is different from zero, and hence
\[
\lambda_{21} (\sigma(x, \theta'), \sigma(x, \theta''), \sigma(x, \theta)) \text{ is a } C^2 \text{ function of } (x, \theta', \theta'', \theta),
\] (5A.9)
from which the desired conclusion follows.

In case (3) we employ
\[
\frac{\partial \sigma(x, \theta)}{\partial s} = 0 \text{ when } \theta \in [0, \pi/2],
\] (5A.10)
and
\[
\frac{\partial \sigma(x, \theta)}{\partial r} = 0 \text{ when } \theta \in [\pi, 3\pi/2],
\] (5A.11)
to show that the mixed second derivative of \( \lambda_{21} \) exists and equals zero.

(3) From Eqs. (4C.14) to (4C.18),
\[
u_1(x, \theta) = W_1(\sigma(x, \theta)) - \frac{2}{\pi} \int_\Theta d\theta' p(x, \theta') L_1(\sigma(x, \theta'), \sigma(x, \theta))
\]
for all \((x, \theta) \in D \times \Theta\) (5A.12)
and
\[
\kappa_{21}(x, \theta', \theta) = q(x, \theta') \left[ k_{21}(\sigma(x, \theta'), \sigma(x, \theta)) - \frac{2}{\pi} \int_\Theta d\theta'' p(x, \theta'') \lambda_{21}(\sigma(x, \theta'), \sigma(x, \theta)) \right]
\]
for all \((x, \theta', \theta) \in D \times \Theta^2\), (5A.13)
where \(p(x, \theta)\) is defined by Eq. (4C.19), and \(q(x, \theta)\) is defined by Eq. (4C.16). From statements (4C.10) and (4C.11), \(p(x, \theta)\) and \(q(x, \theta)\) are real analytic functions of \((x, \theta)\) throughout \(D \times \Theta\). (5A.14)

From statements (5A.3), (5A.4), (5A.5) and (5A.14), it is clear that the functions \(u_1\) and \(\kappa_{21}\) whose values are given by Eqs. (5A.12) and (5A.13), respectively, satisfy the conclusions of our lemma.

**End of proof.**

**Dfn. of a function that is** \(C^{N_1, \ldots, N_L}\) **on** \(X \subset R^L\)

Suppose that \(X\) is an open subset of \(R^L\) or a closed or semi-closed subinterval of \(R^L\), \(x = (x^1, \ldots, x^L)\) denotes any point in \(X\), \(T\) is a topological space, \(t\) denotes any point in \(T\), and \(N_1, \ldots, N_L\) are \(L\) non-negative integers. Suppose, furthermore, that \(F: (X \times T) \to C\) and that, for each \(L\)-tuple of integers \((n_1, \ldots, n_L)\) such that \(0 \leq n_k \leq N_k\) for all \(k = 1, \ldots, L\),
\[
\partial_{1-n_L} F(x, t) := \left( \frac{\partial}{\partial x^1} \right)^{n_1} \cdots \left( \frac{\partial}{\partial x^k} \right)^{n_k} \cdots \left( \frac{\partial}{\partial x^L} \right)^{n_L} F(x, t)
\] (5A.15)
exists and is a continuous function of \((x,t)\) throughout \(X \times T\). [It is understood that \((\partial/\partial x^k)^0 = 1\).] Then, we shall say that \(F\) is \(C^{N_1,\ldots,N_L}\) on \(X\).

Also, if \(F : X \to C\) and \(\partial^{n_1\ldots n_L} F(x)\) exists and is a continuous function of \(x\) throughout \(X\) for each choice of \((n_1,\ldots,n_L)\) that satisfies \(0 \leq n_k \leq N_k\) for all \(1 \leq k \leq L\), then we shall say that \(F\) is \(C^{N_1,\ldots,N_L}\) on \(X\).

End of Dfn.

Note: If \(F\) is \(C^{N_1,\ldots,N_L}\) on \(X\), then a theorem of the calculus tells us that, for each \((n_1,\ldots,n_L)\) satisfying \(0 \leq n_k \leq N_k\) for all \(1 \leq k \leq L\), the existence and value of \(\partial^{n_1\ldots n_L} F\) are unchanged when the operator factors \(\partial/\partial x^k\) are subject to any permutation.

The following lemma is applicable to a broad class of Fredholm integral equations and is clearly capable of further generalization in several directions. A \(2 \times 2\) matrix version of the lemma for the case \(L = 2\) was covered in a paper by the authors on the initial value problem for colliding gravitational plane wave pairs.\(^{26}\) As regards the current notes, the lemma will play a key role in the proof of Thm. 28.

**Lemma 27 (Fredholm minor \(M\) and determinant \(\Delta\))**

Let \(X\), \(x\) and \(N_k\) \((k = 1,\ldots,L)\) be assigned the same meanings as in the preceding definition; and let \(Y\) denote a compact, oriented, \(m\)-dimensional differentiable manifold, \(y\) denote any point in \(Y\), and \(dy\) denote a volume element at point \(y\) (the value of a distinguished non-zero \(m\)-form at \(y\)). Suppose that \(K : X \times (Y \times Y) \to C\) and \(K\) is \(C^{N_1,\ldots,N_L}\) on \(X\). Let us regard \(K\) as an \(L\)-parameter family of Fredholm kernels that is employed in Fredholm integral equations of the form

\[
f(x, y) - \int_Y dy f(x, y') K(x, y', y) = g(x, y) \text{ for all } (x, y) \in X \times Y,
\]

(5A.16)

where \(X\) is the parameter space. Then, the corresponding Fredholm minor \(M\) and Fredholm determinant \(\Delta\) are \(C^{N_1,\ldots,N_L}\) on \(X\).

**Proof:** The Fredholm construction of \(M\) and \(\Delta\) are given by

\[
M(x, y', y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} M^{(n)}(x, y', y),
\]

(5A.17)

\[
\Delta(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Delta^{(n)}(x),
\]

(5A.18)

where

\[
M^{(0)}(x, y', y) := K(x, y', y),
\]

(5A.19)

\[
M^{(n)}(x, y', y) := \int_Y dy_1 \cdots \int_Y dy_n D^{(n+1)} \left( x \begin{array}{c} y \\ y' \\ y_1 \cdots y_n \end{array} \right)
\]

for all \(n > 0\),

(5A.20)

\[
\Delta^{(0)}(x) := 1,
\]

(5A.21)

\[
\Delta^{(n+1)}(x) := \int_Y dy M^{(n)}(x, y, y) \text{ for all } n \geq 0,
\]

(5A.22)

\[^{26}\text{I. Hauser and F. J. Ernst, J. Math. Phys. 32, 198 (1991), Sec. V.}\]
and
\[ D^{(n)} \left( \begin{array}{c|c} y_1 \cdots y_n \\ y'_1 \cdots y'_n \end{array} \right) := \text{the determinant of that } n \times n \text{ matrix whose element in the } k \text{th row and } l \text{th column is } K(x, y'_k, y_l). \] (5A.23)

In particular,
\[ D^{(0)} \left( \begin{array}{c|c} y' \end{array} \right) := K(x, y', y). \] (5A.24)

For each bounded and closed subspace \( U \) of \( X \), let
\[ ||K_u|| := \sup \{ |\partial_{n_1 \cdots n_L}^{n_1 \cdots n_L} M^n(x, y', y)| : (x, y', y) \in U \times Y^2, \text{ and } 0 \leq n_k \leq N_k \text{ for all } k = 1, \ldots, L \}. \] (5A.25)

Also let
\[ V := \int_Y dy. \] (5A.26)

Then, from Eqs. (5A.19) and (5A.20), and from a generalization of Hadamard’s inequality that was formulated and proved by the authors in the aforementioned paper on the initial value problem for colliding gravitational plane wave pairs [see Thm. 7 in that paper],
\[ |\partial_{1 \cdots L}^{n_1 \cdots n_L} M^n(x, y', y)| \leq V^n||K_U||^{n+1}(n+1)^{N_1+\ldots+N_L+(n+1)/2} \]
for all \((x, y', y) \in U \times Y^2\) and all \((n_1, \ldots, n_L)\) such that \(0 \leq n_k \leq N_k\) for each \( k = 1, \ldots, L \). (5A.27)

It follows that, for each positive integer \( N \),
\[ \sum_{n=0}^{N} \frac{1}{n!} |\partial_{1 \cdots L}^{n_1 \cdots n_L} M^{(n)}(x, y', y)| \leq \sum_{n=0}^{N} \frac{V^n||K_U||^{n+1}}{n!}(n+1)^{N_1+\ldots+N_L+(n+1)/2} \]
for all \((x, y', y) \in U \times Y^2\) and all choices (the usual) of \((n_1, \ldots, n_i)\). (5A.28)

The application of the ratio test to the series on the right side of the above inequality (5A.28) is straightforward and demonstrates that this series converges as \( N \to \infty \). Hence, from the comparison test, the series on the left side of (5A.28) converges for all \((x, y', y) \in U \times Y^2\) and all choices of \((n_1, \ldots, n_L)\). The theorems in the calculus of the continuity and term-by-term differentiability of a uniformly convergent infinite series of functions then supply us with the following conclusions:

For all choices of \((n_1, \ldots, n_L)\) for which \(0 \leq n_k \leq N_k(1 \leq k \leq L)\), \(\partial_{1 \cdots n_L}^{n_1 \cdots n_L} M(x, y', y)\) exists and is a continuous function of \((x, y', y)\) throughout \( X \times Y^2 \);

\[ \text{See Sec. 2, Ch. IV, of Differential and Integral Calculus by R. Courant (Interscience Publishers, Inc., 1936).} \]
and
\[ \partial_{1\cdots L}^{n_1\cdots n_L} M(x, y', y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left[ \partial_{1\cdots L}^{n_1\cdots n_L} M^{(n)}(x, y', y) \right], \]
the infinite series converges absolutely and converges uniformly on each compact subspace of \( X \times Y^2 \).

Hence, \( M \) is \( C^{N_1, \ldots, N_L} \) on \( X \). The proof that \( \Delta \) is also \( C^{N_1, \ldots, N_L} \) on \( X \) is left for the reader. \textit{End of proof.}

The following theorem concerns the solution \( y_1(x, \theta) \) of the Fredholm equation (4C.12) for all \((x, \theta) \in D \times \Theta\).

**THEOREM 28 (Differentiability properties of \( y_1 \) when \( v \in K^3 \))**

If \( v \in K^3 \), then \( y_1 \) is \( C^{1,1} \) on \( D \); i.e., \( \partial y_1(x, \theta)/\partial r \), \( \partial y_1(x, \theta)/\partial s \) and \( \partial^2 y_1(x, \theta)/\partial r \partial s \) exist and are continuous functions of \((x, \theta)\) throughout \( D \times \Theta \).

Proof: Consider the inhomogenous Fredholm equation of the second kind (4C.12). According to Thm. 21, the Fredholm determinant for Eq. (4C.12) is not zero for all choices of \( x \in D \). Therefore, a unique solution of the Fredholm equation exists and is given by

\[ y_1(x, \theta) = u_1(x, \theta) + \frac{2}{\pi} \int_{\Theta} d\theta' u_1(x, \theta') R(x, \theta', \theta) \]  

for all \((x, \theta) \in D \times \Theta\), where the resolvent kernel \( R(x, \theta', \theta) \) is the following ratio of the Fredholm minor and determinant:

\[ R(x, \theta', \theta) = \frac{M(x, \theta', \theta)}{\Delta(x)}. \]

From Lem. 26, \( \kappa_{21} \) is \( C^1 \). Moreover, \( \partial^2 \kappa_{21}(x, \theta', \theta)/\partial r \partial s \) exists and is a continuous function of \((x, \theta', \theta)\) throughout \( D \times \Theta^2 \). Therefore,

\[ \kappa_{21} \text{ is } C^{1,1} \text{ on } D. \]  

The preceding Lem. 27 is now applied to the present case, for which

\[ X = D, \quad Y = \Theta, \quad L = 2, \quad m = 1, \quad dy = 2d\theta/\pi. \]

Thereupon, one obtains

\[ R \text{ is } C^{1,1} \text{ on } D. \]  

Lemma 26 also tells us that (amongst other things)

\[ u_1 \text{ is } C^{1,1} \text{ on } D. \]

Therefore, from Eq. (5A.31), statements (5A.35) and (5A.36), and the theorems of the calculus on the continuity and differentiability of an integral with respect to parameters,

\[ y_1 \text{ is } C^{1,1} \text{ on } D. \]
Note that, in terms of standard notation and terminology, \( \lambda = 1 \) for our particular Fredholm equation; and the statement that \( \Delta(x) \neq 0 \) is equivalent to the statement that 1 is not a characteristic value (eigenvalue) of our kernel.

**B. Concerning the partial derivatives of \( \mathcal{Y}, \mathcal{F}, H \) and \( \mathcal{F}^\pm \) when \( v \in K^3 \)**

Dfn. of \( L^{(i)}(\sigma', \sigma) \) for each \( x \in D \) and \( i \in \{3, 4\} \)

For each \( \sigma' \in \mathcal{I}(x) \) and \( \sigma \in \mathcal{I}^{(i)} \), let

\[
L^{(i)}(\sigma', \sigma) := \frac{W(\sigma') - W^{(i)}(\sigma)}{\sigma' - \sigma}.
\]

End of Dfn.

Employing the transformation defined by Eqs. (4C.1) to (4C.9), the definition of \( p(x, \theta) \) by Eq. (4C.10), the definition of \( q(x, \theta) \) by Eq. (4C.16) and the definition of \( L^{(i)}(\sigma', \sigma) \) that we just gave, one finds that Eqs. (2C.2), (2C.1) and (2B.14) are expressible in the forms (in which ‘\( x \)’ is no longer suppressed)

\[
\mathcal{Y}^{(i)}_1(x, \sigma) = W^{(i)}_1(\sigma) - \frac{2}{\pi} \int_{\Theta} d\theta' p(x, \theta') y_2(x, \theta') W^{T}_1(\sigma(x, \theta')) J L^{(i)}_1(\sigma(x, \theta'), \sigma)
\]

for all \( x \in D \) and \( \sigma \in \bar{\mathcal{I}}^{(i)}(x^{7-i}) \)

[after the extension defined by (2C.7)], \hspace{1cm} (5B.1)

\[
\mathcal{Y}^{(i)}_2(x, \sigma) = W^{(i)}_2(\sigma) + \frac{2}{\pi} \int_{\Theta} d\theta' q(x, \theta') y_1(x, \theta') W^{T}_2(\sigma(x, \theta')) J L^{(i)}_2(\sigma(x, \theta'), \sigma)
\]

for all \( x \in D \) and \( \sigma \in \bar{\mathcal{I}}^{(i)}(x^{7-i}) \)

[after the extension defined by (2C.7)], \hspace{1cm} (5B.2)

and

\[
\nu(x, \tau^{-1}) \mathcal{F}(x, \tau) = I - \frac{2}{\pi} \int_{\Theta} d\theta' q(x, \theta') y_1(x, \theta') W_2(\sigma(x, \theta')) J \frac{W_2(\sigma(x, \theta'))}{\sigma(x, \theta') - \tau}
\]

for all \( x \in D \) and \( \tau \in C - \bar{\mathcal{I}}(x) \).

(5B.3)

Furthermore, from Eqs. (4D.1), (4D.6) and (5B.3),

\[
H(x) = H^M(x_0) + 2(z - z_0) \Omega - \frac{4i}{\pi} \int_{\Theta} d\theta' q(x, \theta') y_1(x, \theta') W_2^T(\sigma(x, \theta'))
\]

for all \( x \in D \).

(5B.4)

When proving the following theorem, one should bear in mind that \( \sigma(x, \theta) \), \( p(x, \theta) \) and \( q(x, \theta) \) are analytic functions of \( (x, \theta) \) throughout \( D \times \Theta \).
THEOREM 29 (Differentiability properties of $\mathcal{Y}^{(i)}$, $\mathcal{F}$ and $H$ when $\nu \in K^3$)

If $\nu \in K^3$, then

$$
\begin{align*}
&\frac{\partial \mathcal{Y}^{(i)}(x, \sigma)}{\partial r}, \frac{\partial \mathcal{Y}^{(i)}(x, \sigma)}{\partial s}, \frac{\partial^2 \mathcal{Y}^{(i)}(x, \sigma)}{\partial r \partial s}, \\
&\frac{\partial^2 \mathcal{Y}^{(i)}(x, \sigma)}{\partial r^2}, \frac{\partial^2 \mathcal{Y}^{(i)}(x, \sigma)}{\partial r \partial \sigma}, \text{ and } \frac{\partial^2 \mathcal{Y}^{(i)}(x, \sigma)}{\partial s \partial \sigma}
\end{align*}
$$

exist and are continuous functions of $(x, \sigma)$ throughout

$$\{(x, \sigma) : x \in D, \sigma \in \mathcal{I}^{(i)}(x^{7-i})\}.$$ (5B.5)

Also, upon letting $\hat{\mathcal{F}}$ denote the restriction of $\mathcal{F}$ to

$$\text{dom } \hat{\mathcal{F}} := \{(x, \tau) : x \in D, \tau \in C - \mathcal{I}(x) - \{r, s, r_0, s_0\}\},$$

one has

$$\frac{\partial \hat{\mathcal{F}}(x, \tau)}{\partial r}, \frac{\partial \hat{\mathcal{F}}(x, \tau)}{\partial s}, \text{ and } \frac{\partial^2 \hat{\mathcal{F}}(x, \tau)}{\partial r \partial s}$$

exist and are continuous functions of $(x, \tau)$ throughout dom $\hat{\mathcal{F}}$; and, for each $x \in D$, these partial derivatives are holomorphic functions of $\tau$ throughout $C - \mathcal{I}(x) - \{r, s, r_0, s_0\}$. (5B.6)

Furthermore,

$$H \text{ is } C^{1,1} \text{ on } D.$$ (5B.7)

Proof: From Thm. 28, statement (5A.3) and the fact that $L^{(i)}$ is $C^2$, one concludes from Eq. (5B.2) that

$$\begin{align*}
&\frac{\partial \mathcal{Y}^{(i)}_1(x, \sigma)}{\partial r}, \frac{\partial \mathcal{Y}^{(i)}_1(x, \sigma)}{\partial s}, \\
&\frac{\partial^2 \mathcal{Y}^{(i)}_2(x, \sigma)}{\partial r \partial s}, \frac{\partial^2 \mathcal{Y}^{(i)}_2(x, \sigma)}{\partial r \partial \sigma}, \text{ and } \frac{\partial^2 \mathcal{Y}^{(i)}_2(x, \sigma)}{\partial s \partial \sigma}
\end{align*}$$

exist and are continuous functions of $(x, \sigma)$ throughout $\{(x, \sigma) : x \in D, \sigma \in \mathcal{I}^{(i)}(x^{7-i})\}$. Then, from Eq. (5B.1), one obtains like conclusions for $\mathcal{Y}^{(i)}_1(x, \sigma)$, whereupon the statement (5B.5) follows.

Statements (5B.6) and (5B.7) follow from Thm. 28, statement (5A.3), the known differentiability and holomorphy properties of $\nu(x, \tau)^{-1}$ on dom $\hat{\mathcal{F}}$, and the theorem on the holomorphy of functions given by Cauchy-type integrals. 

End of proof.

LEMMA 30 (d($\nu(x, \tau)^{-1}\hat{\mathcal{F}}(x, \tau)$))

If $\nu \in K^3$, then the first partial derivatives of

$$\frac{\nu^+(x, \sigma')^{-1}\mathcal{Y}_1(x, \sigma')W_2^T(\sigma')J_{\sigma' - \tau}}{\sigma' - \tau}$$

with respect to $r$ and with respect to $s$ are summable over $\sigma' \in \mathcal{I}(x)$; and

$$d\left[\nu(x, \tau)^{-1}\hat{\mathcal{F}}(x, \tau)\right] = -\frac{1}{\pi i} \int_{\bar{\mathcal{I}}} d\sigma' d\left[\nu^+(x, \sigma')^{-1}\mathcal{Y}_1(x, \sigma')W_2^T(\sigma')J_{\sigma' - \tau}\right]$$

for all $(x, \tau) \in \text{dom } \hat{\mathcal{F}}$. (5B.9)

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Proof: We shall tacitly employ statements (5B.5) and (5B.6) of Thm. 29 in some steps of this proof. We shall supply the proof only for $\partial[\nu(x, \tau)^{-1}\mathcal{F}(x, \tau)]/\partial r$ and leave the proof for the partial derivative with respect to $s$ for the reader. The summability over $\mathcal{I}(x)$ of the partial derivative with respect to $r$ of (5B.8) is seen from the facts that

$\nu^+(x, \sigma')^{-1} = M^+(\sigma' - r)M^+(\sigma' - s) \left[M^+(\sigma' - r_0)M^+(\sigma' - s_0)\right]^{-1}$  \hspace{1cm} (5B.10)

and

$\frac{\partial \nu^+(x, \sigma')^{-1}}{\partial r} = -\frac{1}{2}M^+(\sigma' - s) \left[M^+(\sigma' - r)M^+(\sigma' - r_0)M^+(\sigma' - s_0)\right]^{-1}$, (5B.11)

where

$M^+(\sigma) := \begin{cases} \sqrt{\sigma} & \text{if } \sigma \geq 0, \\ i\sqrt{\sigma} & \text{if } \sigma \leq 0, \end{cases}$

are both summable over $\mathcal{I}(x)$, and a summable function times a continuous function over a bounded interval is summable.

In the proofs of this lemma and the next lemma, we shall employ the shorthand notations

$f(x, \sigma') := \gamma_1(x, \sigma')W^T_2(\sigma')J, \\ g(x, \sigma') := \nu(x, \tau)^{-1}\mathcal{F}(x, \tau), \hspace{1cm} (5B.12)$

whereupon Eq. (2B.14) becomes

$g(x, \tau) = I - \frac{1}{\pi i} \int_{\mathcal{I}} d\sigma' \nu^+(x, \sigma')^{-1} \frac{f(x, \sigma')}{\sigma' - \tau}$

$= I - g_1(x, \tau) - f(x, r)g_2(x, \tau)$,  \hspace{1cm} (5B.13)

where

$g_1(x, \tau) := \frac{1}{\pi i} \int_{\mathcal{I}} d\sigma' \nu^+(x, \sigma')^{-1} \frac{f(x, \sigma') - f(x, r)}{\sigma' - \tau}$, \hspace{1cm} (5B.14)

and

$g_2(x, \tau) := \frac{1}{\pi i} \int_{\mathcal{I}} d\sigma' \nu^+(x, \sigma')^{-1} \frac{\sigma'}{\sigma' - \tau}$. \hspace{1cm} (5B.15)

We shall first deal with the term $f(x, r)g_2(x, \tau)$. It is easy to show that

$g_2(x, \tau) = \nu(x, \tau)^{-1} - 1$.  \hspace{1cm} (5B.16)

Therefore, for all $(x, \tau) \in \text{dom } \mathcal{F}$,

$\frac{\partial g_2(x, \tau)}{\partial r} = -\frac{1}{2(\tau - r)}\nu(x, \tau)^{-1}$. \hspace{1cm} (5B.17)

Also, note that

$\frac{1}{\pi i} \int_{\mathcal{I}} d\sigma' \frac{\partial \nu^+(x, \sigma')^{-1}/\partial r}{\sigma' - \tau} = -\frac{1}{2\pi i} \int_{\mathcal{I}} d\sigma' \frac{\nu^+(x, \sigma')^{-1}}{(\sigma' - r)(\sigma' - \tau)}$

$= -\frac{\nu(x, \tau)^{-1}}{2(\tau - r)}$. \hspace{1cm} (5B.18)
That takes care of the term \( f(x, r)g_2(x, \tau) \).

So, from Eqs. (5B.13), (5B.17) and (5B.18),
\[
\frac{\partial}{\partial r} [f(x, r)g_2(x, \tau)] = \frac{1}{\pi i} \int_\mathcal{I} d\sigma' \frac{\partial [\nu^+(x, \sigma')^{-1} f(x, r)] / \partial r}{\sigma' - \tau}
\]
for all \((x, \tau) \in \text{dom } \mathcal{F}\).  

(5B.19)

That takes care of the term \( f(x, r)g_2(x, \tau) \).

We shall next deal with the term \( g_1(x, \tau) \). From statement (5B.3) in Thm. 29 and from Eq. (5B.12), one can see that
\[
\frac{\partial}{\partial r} \left\{ M^+(\sigma' - r)M^+(\sigma' - s) [f(x, \sigma') - f(x, r)] \right\}
\]
exists and is a continuous function of \((x, \sigma')\) throughout \(\{(x, \sigma') : x \in D, \sigma' \in \mathcal{I}(x)\}\). [We leave details for the reader.] No loss of generality will be incurred if we tentatively introduce a closed and bounded convex neighborhood \(\mathcal{N}\) of the point \(x_0\) in the space \(D\), whereupon it is seen that
\[
\{(x, \sigma') : x \in \mathcal{N}, \sigma' \in \mathcal{I}(x)\}
\]
is a bounded closed subspace of \(R^3\); and, therefore,
\[
M(\mathcal{N}) := \sup \left\{ \left\| \frac{\partial}{\partial r} \left\{ M^+(\sigma' - r)M^+(\sigma' - s) [f(x, \sigma') - f(x, r)] \right\} \right\| : x \in \mathcal{N}, \sigma' \in \mathcal{I}(x) \right\}
\]
is finite; and the integrand in the expression for \(g_1(x, \tau)\) that is given by Eq. (5B.14) satisfies
\[
\left\| \frac{\partial}{\partial r} \left[ \nu^+(x, \sigma')^{-1} f(x, \sigma') - f(x, r) \right] \right\| \leq \left[ \sqrt{\|\sigma' - r_0\|} \|\sigma' - s_0\| \|\sigma' - \tau\| \right]^{-1} M(\mathcal{N}).
\]

(5B.22)

Since the right side of the above inequality is summable over \(\mathcal{I}(x)\) and is independent of \(x\), a well-known theorem\(^{29}\) on differentiation of a Lebesgue integral with respect to a parameter tells us that \(\partial g_1(x, \tau)/\partial r\) exists (which, it happens, we already know) and is given by
\[
\frac{\partial g_1(x, \tau)}{\partial r} = \frac{1}{\pi i} \int_\mathcal{I} d\sigma' \frac{\partial}{\partial r} \left\{ \nu^+(x, \sigma')^{-1} [f(x, \sigma') - f(x, r)] \right\}
\]
for all \(x \in \mathcal{N}\) and \(\tau \in C - \mathcal{I}(x) - \{r, s, r_0, s_0\}\), where we have used the fact that the contribution to \(\partial g_1(x, \tau)/\partial r\) due to differentiation of the integral with respect to the endpoint \(r \in \{a^3, b^3\}\) of the integration interval \(\mathcal{I}^{(3)}(x)\) vanishes, because the integrand in Eq. (5B.14) vanishes when \(\sigma' = r\).

However, since \(\mathcal{N}\) can always be chosen so that it covers any given point in \(D\), Eq. (5B.23) holds for all \((x, \tau) \in \text{dom } \mathcal{F}\); and upon combining (5B.23), (5B.19) and (5B.13), one obtains
\[
\frac{\partial g(x, \tau)}{\partial r} = \frac{1}{\pi i} \int_\mathcal{I} d\sigma' \frac{\partial [\nu^+(x, \sigma')^{-1} f(x, \sigma')] / \partial r}{\sigma' - \tau}
\]
for all \((x, \tau) \in \text{dom } \mathcal{F}\),

(5B.24)

\(^{29}\)See Ref. 18, Sec. 39.
which is the coefficient of $dr$ in Eq. (2B.9).

Before we give the next lemma, note that application of the Plemelj relations to Eq. (2B.14) yields

$$\frac{1}{2}[\mathcal{F}^+(x, \sigma) + \mathcal{F}^-(x, \sigma)] = -\mathcal{V}_1(x, \sigma)W_2^T(\sigma)\mathcal{J}$$

for all $x \in D$ and $\sigma \in \mathcal{I}(x)$, \hspace{1cm} (5B.25)

and

$$\frac{1}{2}\nu^+(x, \sigma)^{-1}[\mathcal{F}^+(x, \sigma) - \mathcal{F}^-(x, \sigma)] = I - \frac{1}{\pi i} \int_{\mathcal{I}} d\sigma' \nu^+(x, \sigma')^{-1}\mathcal{V}_1(x, \sigma')W_2^T(\sigma')\mathcal{J}$$

for all $x \in D$ and $\sigma \in \mathcal{I}(x)$. \hspace{1cm} (5B.26)

**LEMMA 31** (Differentiability properties of $\mathcal{F}^\pm$ when $v \in K^3$)

As in the preceding lemma, suppose that $v \in K^3$ and $\mathcal{F}$ is the solution of the HHP corresponding to $(v, \mathcal{F}^M)$. Then the following three statements hold:

(i) The partial derivatives $\partial \mathcal{F}^\pm(x, \sigma)/\partial r$, $\partial \mathcal{F}^\pm(x, \sigma)/\partial s$ and $\partial^2 \mathcal{F}^\pm(x, \sigma)/\partial r \partial s$ exist and are continuous functions of $(x, \sigma)$ throughout $\{(x, \sigma) : x \in D, \sigma \in \mathcal{I}(x)\}$.

(ii) The 1-form

$$d[\nu^+(x, \sigma')^{-1}\mathcal{V}_1(x, \sigma')W_2^T(\sigma')\mathcal{J}]$$

is, for each $x \in D$ and $\sigma \in \mathcal{I}(x)$, summable over $\mathcal{I}(x)$ in the PV sense.

(iii) For all $x \in D$ and $\sigma \in \mathcal{I}(x)$,

$$d \left\{ \frac{1}{2}\nu^+(x, \sigma)^{-1}[\mathcal{F}^+(x, \sigma) - \mathcal{F}^-(x, \sigma)] \right\} =$$

$$-\frac{1}{\pi i} \int_{\mathcal{I}} d\sigma' \frac{d[\nu^+(x, \sigma')^{-1}\mathcal{V}_1(x, \sigma')W_2^T(\sigma')\mathcal{J}]}{\sigma' - \sigma}. \hspace{1cm} (5B.27)$$

**Proofs:**

(i) This follows from statement (5B.3), Eq. (5B.25) and Eq. (2B.10). \hspace{1cm} End of proof.

The proofs of parts (ii) and (iii) will be supplied only for the coefficients of $dr$ in Eqs. (5B.25) and (5B.26). The proofs for the coefficients of $ds$ are left to the reader.

(ii) As functions of $\sigma'$, $W_2^T(\sigma')$ is $C^3$, $\mathcal{V}_1(x, \sigma')$ is $C^2$ and $\partial \mathcal{V}_1(x, \sigma')/\partial r$ is $C^1$ on $\mathcal{I}(x)$; and $\nu^+(x, \sigma')^{-1}$ and $\partial \nu^+(x, \sigma')^{-1}/\partial r$ are summable over $\mathcal{I}(x)$. Therefore, for a sufficiently small $\epsilon > 0$,

$$\frac{\partial}{\partial \sigma} \left[ \nu^+(x, \sigma')^{-1}\mathcal{V}_1(x, \sigma') \right] W_2^T(\sigma')\mathcal{J}$$

is summable over $\mathcal{I}(x) - |\sigma - \epsilon, \sigma + \epsilon|$. Moreover, since the numerator of $\frac{\partial}{\partial \sigma} \mathcal{V}_1$ is a $C^1$ function of $\sigma'$, it is well known that $\frac{\partial}{\partial \sigma} \mathcal{V}_1(x, \sigma')$ is summable over $[\sigma - \epsilon, \sigma + \epsilon]$ in the PV sense.

Therefore, (5B.27) is summable over $\mathcal{I}(x)$ in the PV sense. \hspace{1cm} End of proof.

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(iii) In terms of the shorthand notations \((5B.12)\), Eq. \((5B.26)\) is expressible in the form
\[
\frac{1}{2}[g^+(x, \sigma) + g^-(x, \sigma)] = I - \frac{1}{\pi i} \int_{I} d\sigma' \nu^+(x, \sigma')^{-1} f(x, \sigma') \overline{\sigma' - \sigma},
\]  
where Thm. \(29\) furnishes the following properties of \(f(x, \sigma')\):

\[
\partial f(x, \sigma')/\partial r, \quad \partial f(x, \sigma')/\partial s, \quad \partial^2 f(x, \sigma')/\partial r \partial s, \quad \partial^2 f(x, \sigma')/\partial r \partial \sigma', \quad \text{and} \quad \partial^2 f(x, \sigma')/\partial s \partial \sigma'
\]
exist and are continuous functions of \((x, \sigma')\) throughout \(\{(x, \sigma') : x \in D, \sigma' \in \bar{I}(x)\}\).

Let us introduce the additional shorthand notations
\[
f_0(x, \sigma', \sigma) := \frac{f(x, \sigma') - f(x, \sigma)}{\sigma' - \sigma}, \quad (5B.32)
f_1(x, \sigma', \sigma) := f_0(x, \sigma', \sigma) - f_0(x, r, \sigma), \quad (5B.33)
g_1(x, \sigma) := \frac{1}{\pi i} \int_{I} d\sigma' \nu^+(x, \sigma')^{-1} f_1(x, \sigma', \sigma), \quad (5B.34)
g_2(x, \sigma) := \frac{1}{\pi i} \int_{I} d\sigma' \nu^+(x, \sigma')^{-1} \overline{\sigma' - \sigma} \overline{\sigma - \sigma'}^{-1}, \quad (5B.35)
\]
and
\[
g_3(x, \sigma) := \frac{1}{\pi i} \int_{I} d\sigma' \nu^+(x, \sigma')^{-1}. \quad (5B.36)
\]

Then Eq. \((5B.30)\) is expressible in the form
\[
\frac{1}{2}[g^+(x, \sigma) + g^-(x, \sigma)] = I - g_1(x, \sigma) - f(x, \sigma) g_2(x, \sigma) - f_0(x, r, \sigma) g_3(x, \sigma). \quad (5B.37)
\]

Let us first consider the above terms that contain \(g_2\) and \(g_3\). A well-known formula yields
\[
g_2(x, \sigma) = -1, \quad (5B.38)
\]

while the usual contour integration technique yields
\[
g_3(x, \sigma) = \frac{1}{2}(r + s - r_0 - s_0). \quad (5B.39)
\]

Therefore, by using
\[
\frac{1}{\pi i} \int_{I} d\sigma'' \nu^+(\sigma'')^{-1}(\sigma'' - \sigma)^{-1}(\sigma' - \sigma'')^{-1} = 0 \quad \text{for all} \ \sigma \in \bar{I}(x) \setminus \{r_0, s_0\}. \quad (5B.40)
\]

and the fact that
\[
\partial \nu^+(x, \sigma')^{-1}/\partial r = -\frac{1}{2}(\sigma' - r)^{-1} \nu^+(x, \sigma')^{-1},
\]

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the reader can prove that
\[ \frac{\partial g_2(x, \sigma)}{\partial r} = -\frac{1}{\pi i} \int_{\bar{I}} d\sigma' \frac{\partial \nu^+(x, \sigma')^{-1}/\partial r}{\sigma' - \sigma}, \] (5B.41)
and
\[ \frac{\partial g_3(x, \sigma)}{\partial r} = -\frac{1}{\pi i} \int_{\bar{I}} d\sigma' \frac{\partial \nu^+(x, \sigma')^{-1}/\partial r}{\sigma' - \sigma}, \] (5B.42)
whereupon
\[ \frac{\partial}{\partial r} \left[ f(x, \sigma)g_2(x, \sigma) \right] = -\frac{1}{\pi i} \int_{\bar{I}} d\sigma' \frac{\partial [f(x, \sigma)\nu^+(x, \sigma')^{-1}]/\partial r}{\sigma' - \sigma} \] (5B.43)
and
\[ \frac{\partial}{\partial r} \left[ f(x, \sigma)g_3(x, \sigma) \right] = -\frac{1}{\pi i} \int_{\bar{I}} d\sigma' \frac{\partial \nu^{-1}[f_0(x, r, \sigma)\nu^+(x, \sigma')^{-1}]}{\partial r}. \] (5B.44)

That completes the analysis of the terms in Eq. (5B.37) that contain \( g_2 \) and \( g_3 \).

We next consider \( g_1 \). From (5B.31) to (5B.33), one sees that

\[ \frac{\partial}{\partial r} \left[ f_1(x, \sigma') \nu^+(x, \sigma')^{-1} \right] \]

exist and are continuous functions of \( (x, \sigma') \) throughout \( \{(x, \sigma') : x \in D, \sigma' \in \bar{I}(x)\} \).

Therefore, as regards the integrand in the definition (5B.34) of \( g_1(x, \sigma) \), one readily deduces (by an argument similar to the one used in the proof of the preceding lemma) that, corresponding to each closed and bounded neighborhood \( \mathcal{N} \) of the point \( x_0 \) in the space \( D \), and each \( \sigma \in \mathcal{I}(x) \), there exists a positive real number \( M(\mathcal{N}, \sigma) \) such that

\[ \left| \left| \frac{\partial}{\partial r} \left[ \nu^+(x, \sigma')^{-1}f_1(x, \sigma', \sigma) \right] \right| \right| \leq \frac{M(\mathcal{N}, \sigma)}{\sqrt{|\sigma' - r_0| |\sigma' - s_0|}} \]

for all \( x \in \mathcal{N} \) and \( \sigma' \in \bar{I}(x) - \{r_0, s_0\} \). (5B.46)

The remainder of the proof employs the same theorem on differentiation of a Lebesgue integral with respect to a parameter that was used in the proof of the preceding lemma. The result is

\[ \frac{\partial g_1(x, \sigma)}{\partial r} = \frac{1}{\pi i} \int_{\bar{I}} d\sigma' \frac{\partial}{\partial r} \left[ \nu^+(x, \sigma')^{-1}f_1(x, \sigma', \sigma) \right]. \] (5B.47)

Upon combining the results given by Eqs. (5B.43), (5B.44) and (5B.47), one obtains with the aid of Eqs. (5B.30), (5B.32) to (5B.34), and Eq. (5B.37),

\[ \frac{1}{\partial r} \left[ g^+(x, \sigma) + g^-(x, \sigma) \right] = -\frac{1}{\pi i} \int_{\bar{I}} d\sigma' \frac{\partial [\nu^+(x, \sigma')^{-1}f(x, \sigma')]/\partial r}{\sigma' - \sigma} \]

for all \( x \in D \) and \( \sigma \in \mathcal{I}(x) \). (5B.48)

\[ \text{End of proof.} \]

The point of the preceding two lemmas is the following crucial theorem.
THEOREM 32 (Limits of $dF$ when $v \in K^3$)
Suppose $v \in K^3$ and $F$ is the solution of the HHP corresponding to $(v,F^M)$. Then, the following three statements hold:

(i) For each $x \in D$ and $\sigma \in \mathcal{I}(x)$, $d\hat{F}(x,\sigma \pm \zeta)$ converges as $\zeta \to 0$ ($\Im \zeta > 0$) and

$$
\lim_{\zeta \to 0} d\hat{F}(x,\sigma \pm \zeta) = dF^\pm(x,\sigma).
$$

Note: The existences of $d\hat{F}(x,\tau)$ and $dF^\pm(x,\sigma)$ are guaranteed by Thm. 29 [statement (5B.6)] and by Lem. 31(i), respectively.

(ii) $F(x,\tau)$ converges as $\tau \to r_0$ and as $\tau \to s_0$ [$\tau \in C - \bar{\mathcal{I}}(x)$]; and $\nu(x,\tau)^{-1}F(x,\tau)$ converges as $\tau \to r$ and as $\tau \to s$.

(iii) For each $i \in \{3,4\}$,

$$
(\tau - x_i) \frac{\partial \hat{F}(x,\tau)}{\partial x_i}
$$

converges as $\tau \to r_0$ and as $\tau \to s_0$, while

$$
\nu(x,\tau)^{-1}(\tau - x_i) \frac{\partial \hat{F}(x,\tau)}{\partial x_i}
$$

converges as $\tau \to r$ and as $\tau \to s$.

Proofs:

(i) We shall prove statement (i) for the coefficient of $dr$ in $d\hat{F}(x,\tau)$ and leave the proof for the coefficient of $ds$ to the reader.

Employ the shorthand notation

$$
f(x,\sigma') := \mathcal{Y}_1(x,\sigma')W^T_2(\sigma')J
$$

in the integrand of Eq. (2B.14), which then becomes

$$
\nu(x,\tau)^{-1}F(x,\tau) = I - \frac{1}{\pi i} \int_{\mathcal{I}} d\sigma' \nu(x,\sigma')^{-1} \frac{f(x,\sigma')}{\sigma' - \tau}
$$

for all $(x,\tau) \in \text{dom } F$,

whereupon, from Eq. (5B.9) in Lem. 31 and from Eq. (5B.18),

$$
-\frac{\nu(x,\tau)^{-1}F(x,\tau) + \nu(x,\tau)^{-1}\frac{\partial F(x,\tau)}{\partial r}}{2(\tau - r)} = -\Phi(x,\tau) + \frac{\nu(x,\tau)^{-1}}{2(\tau - r)} f(x,\tau)
$$

for all $(x,\tau) \in \text{dom } \hat{F}$,
where

\[
\Phi(x, \tau) := \frac{1}{\pi i} \int_{\mathcal{I}} d\sigma' \nu^+ (x, \sigma')^{-1} \frac{\phi(x, \sigma')}{\sigma' - \tau},
\]

and

\[
\phi(x, \sigma') := \frac{\partial f(x, \sigma')}{\partial r} - \frac{f(x, \sigma') - f(x, r)}{2(\sigma' - r)}. \tag{5B.56}
\]

From Eq. (5B.56) and the properties of \(f(x, \sigma')\) given by statement (5B.31),

\[
\frac{\partial \phi(x, \sigma')}{\partial \sigma'} \text{ exists and is a continuous function of } (x, \sigma') \text{ throughout } \{(x, \sigma') : x \in D, \sigma' \in \mathcal{I}(x)\}. \tag{5B.57}
\]

Therefore, \(\nu(x, \sigma')^{-1} \phi(x, \sigma')\) obeys a Hölder condition of index 1 on each closed subinterval of \(\mathcal{I}(x)\); and it follows from the theorem in Sec. 16 in Muskhelishvili’s treatise\(^{30}\) that (5B.55) satisfies

\[
\Phi(\pm x, \sigma) := \lim_{\zeta \to 0} \Phi(x, \sigma \pm \zeta) \text{ exists for all } \sigma \in \mathcal{I}(x). \tag{5B.58}
\]

Moreover, from the Plemelj relations [Eq. (17.2) in Sec. 17 of Muskhelishvili’s treatise],

\[
\Phi^\pm (x, \sigma) = \pm \nu^+ (x, \sigma)^{-1} \phi(x, \sigma) + \frac{1}{\pi i} \int_{\mathcal{I}} d\sigma' \nu^+ (x, \sigma')^{-1} \frac{\phi(x, \sigma')}{\sigma' - \sigma}. \tag{5B.59}
\]

[The existence of the above PV integral is demonstrated in Sec. 12 of Muskhelishvili’s treatise.] From Eq. (5B.54), condition (3) in the definition of the HHP [the one about the existence of \(F^\pm (x)\)] and statement (5B.58),

\[
\lim_{\zeta \to 0} \frac{\partial \hat{F}(x, \sigma \pm \zeta)}{\partial r} \text{ exists for each } x \in D \text{ and } \sigma \in \mathcal{I}(x); \tag{5B.60}
\]

and, with the aid of Eqs. (5B.25), (6B.52) and (5B.59),

\[
\lim_{\zeta \to 0} \frac{1}{2} \left[ \frac{\partial \hat{F}(x, \sigma + \zeta)}{\partial r} + \frac{\partial \hat{F}(x, \sigma - \zeta)}{\partial r} \right] = -\frac{\partial f(x, \sigma)}{\partial r} \tag{5B.61}
\]

and

\[
\lim_{\zeta \to 0} \frac{1}{2} \frac{\partial}{\partial r} \left[ \nu(x, \sigma + \zeta)^{-1} \hat{F}(x, \sigma + \zeta) + \nu(x, \sigma - \zeta)^{-1} \hat{F}(x, \sigma - \zeta) \right] = -\frac{1}{\pi i} \int_{\mathcal{I}} d\sigma' \nu^+ (x, \sigma')^{-1} \frac{\phi(x, \sigma')}{\sigma' - \sigma}. \tag{5B.62}
\]

However, from Eq. (5B.40),

\[
\frac{1}{\pi i} \int_{\mathcal{I}} d\sigma' \nu^+ (x, \sigma')^{-1} \frac{f(x, r)}{(\sigma' - r)(\sigma' - \sigma)} = 0.
\]

\(^{30}\)See footnote 13.
Therefore, from Eq. (5B.56), Eq. (5B.62) becomes
\[
\lim_{\zeta \to 0} \frac{1}{2} \frac{\partial}{\partial r} \left[ \nu(x, \sigma + \zeta)^{-1} \hat{F}(x, \sigma + \zeta) + \nu(x, \sigma - \zeta)^{-1} \hat{F}(x, \sigma - \zeta) \right] = -\frac{1}{\pi i} \int d\sigma' \frac{\partial [\nu^+(x, \sigma')^{-1} f(x, \sigma')]}{\partial r} / \sigma' - \sigma . \tag{5B.63}
\]

Next, from Eqs. (5B.25) and (5B.52),
\[
\frac{1}{2} \left[ \frac{\partial F^+(x, \sigma)}{\partial r} + \frac{\partial F^-(x, \sigma)}{\partial r} \right] = -\frac{\partial f(x, \sigma)}{\partial \sigma} ; \tag{5B.64}
\]
and, from Eq. (5B.28) in Lem. 31,
\[
\frac{1}{2} \frac{\partial}{\partial r} \{ \nu^+(x, \sigma)^{-1} [F^+(x, \sigma) - F^-(x, \sigma)] \} = -\frac{1}{\pi i} \int d\sigma' \frac{\partial [\nu^+(x, \sigma')^{-1} f(x, \sigma')]}{\partial r} / \sigma' - \sigma . \tag{5B.65}
\]

A comparison of the above Eqs. (5B.64) and (5B.65) with Eqs. (5B.64) and (5B.63), together with the fact that
\[
\lim_{\zeta \to 0} \frac{\partial \nu(x, \sigma \pm \zeta)^{-1}}{\partial r} = \frac{\partial \nu^\pm(x, \sigma)^{-1}}{\partial r} ,
\]
now yields
\[
\lim_{\zeta \to 0} \frac{\partial \hat{F}(x, \sigma \pm \zeta)}{\partial r} = \frac{\partial F^\pm(x, \sigma)}{\partial r} . \tag{5B.66}
\]
Statements (5B.60) and (5B.66) complete the proof of part (i) of our theorem for \( \partial F(x, \tau)/\partial r \).

End of proof.

(ii) Since
\[
\nu^+(x, \sigma)^{-1} = \frac{M^+(\sigma - r)M^+(\sigma - s)}{M^+(\sigma - r_0)M^+(\sigma - s_0)} , \tag{5B.67}
\]
one has
\[
\nu^+(x, \sigma)^{-1} f(x, \sigma) = 0 \text{ when } \sigma = r \text{ and when } \sigma = s . \tag{5B.68}
\]
Therefore, from statement 1º in Sec. 29 of Muskhelishvili's treatise, and from our Eq. (5B.53),
\[
\nu(x, \tau)^{-1} F(x, \tau) \text{ converges as } \tau \to r \text{ and as } \tau \to s \text{ [} \tau \in C - \bar{I}(x) \text{]} . \tag{5B.69}
\]
Furthermore, from Eqs. (5B.15), (5B.16) and (5B.53),
\[
F(x, \tau) = \nu(x, \tau) I + [\nu(x, \tau) - 1] f(x, r_0) - \frac{\nu(x, \tau)}{\pi i} \int d\sigma' \nu^+(x, \sigma')^{-1} \left[ \frac{f(x, \sigma') - f(x, r_0)}{\sigma' - r} \right] . \tag{5B.70}
\]
From statement (5B.31), \( \partial f(x, \sigma')/\partial \sigma' \) exists and is a continuous function of \( \sigma' \) throughout \( \mathcal{I}(x) \). Therefore, as one can see from Eq. (5B.67),

\[
\nu^+(x, \sigma)^{-1} [f(x, \sigma') - f(x, r_0)] = 0 \quad \text{when} \quad \sigma = r_0; \quad (5B.71)
\]

and it then follows from Eq. (5B.70) and the same statement 1° in Sec. 29 of Muskhelishvili that was used before that

\[
\mathcal{F}(x, \tau) \text{ converges [to } -f(x, r_0)\text{]} \quad \text{as} \quad \tau \to r_0. \quad (5B.72)
\]

Similarly, one proves that

\[
\mathcal{F}(x, \tau) \text{ converges [to } -f(x, s_0)\text{]} \quad \text{as} \quad \tau \to s_0. \quad (5B.73)
\]

Statements (5B.69), (5B.72) and (5B.73) together constitute part (ii) of our theorem.

End of proof.

(iii) We shall prove this part of our theorem for \( i = 3 \), and the proof for \( i = 4 \) is left to the reader.

We start with the definition (5B.55) of \( \Phi(x, \tau) \). The proof that we have just given for part (ii) of this theorem is also applicable to \( \Phi(x, \tau) \). Specifically, the proof of part (ii) remains valid if one makes all of the following substitutions in its wording and equations:

- \( f(x, \sigma') \rightarrow \phi(x, \sigma') \),
- Eq. (5B.53) \( \rightarrow \text{Eq. (5B.53)} \),
- \( \nu(x_0, x, \tau) \mathcal{F}(x, \tau) \rightarrow \Phi(x, \tau) \),
- statement (5B.31) \( \rightarrow \text{condition (5B.57)} \).

Therefore, the conclusion of part (ii) of our theorem remains valid if one makes the substitution \( '\nu(x, \tau)^{-1} \mathcal{F}(x, \tau)' \rightarrow '\Phi(x, \tau)' \). So, for all \((x, \tau) \in \text{dom } \mathcal{F}\),

\[
\Phi(x, \tau) \text{ converges as } \tau \to r \text{ and as } \tau \to s, \text{ and } \\
\nu(x, \tau) \Phi(x, \tau) \text{ converges as } \tau \to r_0 \text{ and as } \tau \to s_0. \quad (5B.74)
\]

When the above statement (5B.74) is applied to Eq. (5B.54), one obtains the statement in part (iii) of our theorem for the case \( i = 3 \). End of proof.

Note: The meanings that we assigned above to \('f(x, \sigma')', \(\phi(x, \sigma')\)' and \('\Phi(x, \tau)'\) will not be used in the remainder of these notes. They were temporary devices for the purpose of abbreviating the proofs of the preceding theorem and two lemmas.

6 Proof of the generalized Geroch conjecture
A. Generalized Abel transforms of the initial data and the identification of the sets \( S_f^\square \) and \( S_e^\square \)

In Sec. 1A, we introduced a linear system \( \mathcal{F}_{HE} \) for the Ernst equation that is related to \( \mathcal{F} = \mathcal{F}_{KC} \) by Eqs. (1A.24) to (1A.26). It will now be useful to introduce one more linear system \( \tilde{\mathcal{F}}_{HE} \) such that

\[
\tilde{\mathcal{F}}_{HE}(x, \tau) := P^M(x_0, \tau)\mathcal{F}_{HE}(x, \tau)P^M(x_0, \tau)^{-1},
\]

(6A.1)

whereupon Eq. (1A.24) and the fact that

\[
\mathcal{F}^M(x, \tau) = P^M(x, \tau)P^M(x_0, \tau)^{-1}
\]

(6A.2)

yields

\[
\mathcal{F}(x, \tau) = A(x)\mathcal{F}^M(x, \tau)\mathcal{F}_{HE}(x, \tau),
\]

(6A.3)

where

\[
A := \frac{1}{\sqrt{h_{22}}} \begin{pmatrix} 1 & h_{12} \\ 0 & h_{22} \end{pmatrix}.
\]

(6A.4)

Note that

\[
h = Ah^M A^T, \quad h^M = \begin{pmatrix} \rho^2 & 0 \\ 0 & 1 \end{pmatrix}.
\]

(6A.5)

Therefore, from Eqs. (4B.2) to (4B.5), and the fact that

\[
A^T \Omega A = (\det A)\Omega = \Omega,
\]

\[
\begin{bmatrix} \tilde{\mathcal{F}}_{HE}(x, \tau^*) \end{bmatrix}^\dagger A^M(x_0, \tau)\tilde{\mathcal{F}}_{HE}(x, \tau) = A^M(x_0, \tau).
\]

(6A.6)

Obviously, \( d\tilde{\mathcal{F}}_{HE} = \tilde{\Gamma}_{HE} \tilde{\mathcal{F}}_{HE} \), where

\[
\tilde{\Gamma}_{HE}(x, \tau) = P^M(x_0, \tau)\Gamma_{HE}(x, \tau)P^M(x_0, \tau)^{-1},
\]

(6A.7)

and \( \Gamma_{HE} \) is given by Eq. (1A.13). Note that \( \tilde{\Gamma}_{HE} \) can be obtained by making the following substitutions in \( \Gamma_{HE} \):

\[
J \rightarrow \tilde{J}(\tau) := P^M(x_0, \tau)JP^M(x_0, \tau)^{-1} = \begin{pmatrix} -i & 2(\tau - z_0) \\ 0 & i \end{pmatrix},
\]

(6A.8)

\[
N(\tau) \rightarrow \tilde{N}(\tau) := P^M(x_0, \tau)N(\tau)P^M(x_0, \tau)^{-1} = \begin{pmatrix} -\tau + z_0 & -i\rho_0^2 \\ -i & \tau - z_0 \end{pmatrix},
\]

(6A.9)

where

\[
J := i\sigma_2 \text{ and } N(\tau) := \mu(x_0, \tau)\sigma_3.
\]

(6A.10)

The properties of \( \tilde{\mathcal{F}}_{HE} \) can be deduced from those of \( \mathcal{F}_{HE} \). For example, consider the generalized Abel transforms\(^{31}\) (our term)

\[
\alpha^{(3)}(\sigma) := \mathcal{F}_{HE}^+((\sigma, s_0), \sigma)^{-1} \text{ for } \sigma \in \mathcal{I}^{(3)}
\]

and

\[
\alpha^{(4)}(\sigma) := \mathcal{F}_{HE}^+((\tau_0, \sigma), \sigma)^{-1} \text{ for } \sigma \in \mathcal{I}^{(4)}
\]

(6A.11)

\(^{31}\)In the Weyl case the \( \alpha^{(i)} \) are easily expressed in terms of Abel transforms.
of the initial data functions

\[ E^{(3)}(r) = E(r, s_0) \text{ for } r \in I^{(3)} \text{ and } \]
\[ E^{(4)}(s) = E(r_0, s) \text{ for } s \in I^{(4)}. \]  

(6A.12)

Analysis\(^3\) yields

\[ \alpha^{(i)} = I\alpha^{(i)}_0 + J\alpha^{(i)}_1 + \mathcal{N}^{(i)}(s_0)[I\alpha^{(i)}_2 + J\alpha^{(i)}_3], \]  

(6A.13)

where \( \mathcal{N}^{(i)}(s) = \mu^+(x_0, s_0)s_3, \)

\[ \alpha^{(i)}_k : I^{(i)} \rightarrow \mathbb{R}^1 (k = 0, 1, 2, 3), \]  

(6A.14)

\[ \alpha^{(i)}_k \text{ is } H(1/2) \text{ on each closed subinterval of } I^{(i)}, \]  

(6A.15)

\[ \alpha^{(i)}_k \text{ is } C^{n-1} \text{ if } E^{(i)} \text{ is } C^n \text{ and } \alpha^{(i)}_k \text{ is analytic if } E^{(i)} \text{ is analytic}, \]  

(6A.16)

and

\[ \det \alpha^{(i)} = [\alpha^{(i)}_0]^2 + [\alpha^{(i)}_1]^2 + (\sigma - r_0)(\sigma - s_0) \left\{ [\alpha^{(i)}_2]^2 + [\alpha^{(i)}_3]^2 \right\} = 1. \]  

(6A.17)

Instead of \( \alpha^{(3)} \) and \( \alpha^{(4)} \), we shall be employing

\[ V^{(3)}(\sigma) := \tilde{F}^{+}_{HE}(\sigma, s_0, \sigma)^{-1} \text{ for } \sigma \in I^{(3)} \text{ and } \]
\[ V^{(4)}(\sigma) := \tilde{F}^{+}_{HE}(\sigma, s_0, \sigma)^{-1} \text{ for } \sigma \in I^{(4)}, \]  

(6A.18)

whose pertinent properties are easily deduced from those of \( \alpha^{(3)} \) and \( \alpha^{(4)} \) by using Eq. (6A.1). For example,

\[ V^{(i)} = I\alpha^{(i)}_0 + J(\sigma)\alpha^{(i)}_1 + \tilde{N}(\sigma)[I\alpha^{(i)}_2 + J(\sigma)\alpha^{(i)}_3]. \]  

(6A.19)

Furthermore, with the aid of Eq. (6A.6) and the definitions of \( K \) and \( K^\uparrow \) by Eqs. (1C.3) to (1C.6), one readily deduces from Eqs. (6A.14) to (6A.17) that

\[ V \in K \text{ where } V := (V^{(3)}, V^{(4)}), \]  

(6A.20)

and

\[ V \in K^{n-1} \text{ if } E^{(3)} \text{ and } E^{(4)} \text{ are } C^n, \]
\[ V \in K^\infty \text{ if } E^{(3)} \text{ and } E^{(4)} \text{ are } C^\infty \text{ and } \]
\[ V \in K^{an} \text{ if } E^{(3)} \text{ and } E^{(4)} \text{ are } C^{an}. \]  

(6A.21)

Defining

\[ S_V := \text{the set of all ordered pairs } V = (V^{(3)}, V^{(4)}), \]
where \( V^{(i)} \) is a \( 2 \times 2 \) matrix function with the domain \( I^{(i)} \) and there exists \( F \in S_F \)

such that Eqs. (6A.22) hold,

\[ ^{32}\text{For details, see our Magnum Opus (gr-qc/9903104).} \]
\[ B(\mathcal{I}^{(i)}) := \text{the multiplicative group of all } \]

\[ \exp \left( \tilde{j} \varphi^{(i)} \right) = I \cos \varphi^{(i)} + \tilde{j} \sin \varphi^{(i)} \]  

(6A.23)

such that \( \varphi^{(i)} \) is any real-valued function

that has the domain \( \mathcal{I}^{(i)} \) and is \( H(1/2) \)
on every closed subinterval of \( \mathcal{I}^{(i)} \),

it will turn out to be possible to identify the sets \( \mathcal{S}_F^\Box \) involved in the generalized Geroch conjecture in terms of the more fundamental sets

\[ \mathcal{S}_V^\Box := \{ V \in \mathcal{S}_V : \text{there exists } w \in B(\mathcal{I}^{(3)}) \times B(\mathcal{I}^{(4)}) \text{ for which } Vw \in k^\Box \} \]  

(6A.24)

where

\[ k^\Box = k \cap K^\Box, \quad k := \{ Vw : V \in \mathcal{S}_V, w \in B(\mathcal{I}^{(3)}) \times B(\mathcal{I}^{(4)}) \} \]  

(6A.25)

and, for any members \( v = (v^{(3)}, v^{(4)}) \) and \( v' = (v'^{(3)}, v'^{(4)}) \) of \( K \),

\[ vv' := (v^{(3)}v'^{(3)}, v^{(4)}v'^{(4)}) \].

Specifically, we let

\[ \mathcal{S}_F^\Box := \text{the set of all } F \in \mathcal{S}_F \text{ for which } V \in \mathcal{S}_V^\Box. \]  

(6A.26)

Having defined \( \mathcal{S}_F^\Box \), we can easily identify the remaining important sets. Thus,

\[ \mathcal{S}_E^\Box := \text{the set of all } E \in \mathcal{S}_E \text{ for which } F \in \mathcal{S}_F^\Box, \]  

(6A.27)

with a like definition of \( \mathcal{S}_H^\Box \).

We leave the proof of the following theorem, which actually motivated how we formulated our HHP corresponding to \((v, F_0)\), to the reader:

**THEOREM 33 (Motivation)**

For all \( v \in K \) and for all \( \mathcal{S}_F \) members \( F \) and \( F_0 \) whose corresponding \( \mathcal{S}_V \) members are \( V \) and \( V_0 \), respectively, the following statements (i) and (ii) are equivalent to one another:

(i) There exists \( w \in B(\mathcal{I}^{(3)}) \times B(\mathcal{I}^{(4)}) \) such that

\[ v = VwV_{0}^{-1}. \]  

(6A.28a)

(ii) For each \( x \in D, i \in \{3, 4\} \) and \( \sigma \in \mathcal{I}^{(i)}(x) \),

\[ F^+(x, \sigma)v^{(i)}(\sigma)[F_0^+(x, \sigma)]^{-1} = F^-(x, \sigma)v^{(i)}(\sigma)[F_0^-(x, \sigma)]^{-1}. \]  

(6A.28b)

Moreover, if \( \mathcal{E}^{(i)} \) and \( \mathcal{E}_0^{(i)} \) are \( C^{n_i} \) (resp. analytic) and \( v^{(i)} \) is \( C^{n_i-1} \) (resp. analytic), then the function of \( \sigma \) that equals each side of Eq. \( (6A.28b) \) has a \( C^{n_i-1} \) (resp. analytic) extension \( Y^{(i)}(x) \) to the interval

\[ \text{dom } Y^{(i)}(x) = \hat{\mathcal{I}}^{(i)}(x^{7-i}) \]  

(6A.28c)

and, if \( v \in K^\Box \) and \( F_0 \in \mathcal{S}_F^\Box \), then \( V \in \mathcal{S}_V^\Box \) and \( F \in \mathcal{S}_F^\Box \).
THEOREM 34 (Relation of $F_0$ and $V_0$)
For each $F_0 \in S_{\mathcal{F}}$ whose corresponding member of $S_V$ is $V_0$, and for each $w \in B(\mathcal{I}^{(3)}) \times B(\mathcal{I}^{(4)})$, $F_0$ is a solution of the HHP corresponding to $(V_0 w, F^M)$.

Proof: For each $x \in D$ and $i \in \{3, 4\}$,

(1) Thm. 34(i) states that $F_0(x)$ is holomorphic throughout its domain $C - I(x)$,

(2) Thm. 34 states that $F^\pm(x)$ exist and, from Thm. 33 and the fact that $V^M = (I, I)$,

$$Y_0^{(i)}(x, \sigma) := F_0^+(x, \sigma)\nu^{(i)}(\sigma)[F^M+(x, \sigma)]^{-1}$$

$$= F_0^-(x, \sigma)\nu^{(i)}(\sigma)[F^M-(x, \sigma)]^{-1}$$

for all $\sigma \in I^{(i)}(x)$; (6A.29a)

and Thm. 34 and Thm. 34(iii) imply that $F_0(x)$ is bounded at $x_0$ and $\nu(\sigma)^{-1}F_0(x)$ is bounded at $x$, while the function $Y_0(x)$ whose domain is $I(x)$ and whose values are given by $Y_0^{(i)}(x, \sigma)$ at each $\sigma \in I^{(i)}(x)$ satisfies the condition

$$Y_0(x)$$

is bounded at $x$ and at $x_0$. (6A.29b)

Thus, $F_0$ is a solution of the HHP corresponding to $(Vw, F^M)$.

End of proof.

THEOREM 35 (Reduction theorem)
For each $x \in D$ and $2 \times 2$ matrix function $F(x)$ with the domain $C - I(x)$, for each $v \in K$ and $F_0 \in S_{\mathcal{F}}$ whose corresponding member of $S_V$ is $V_0$, and for each $w \in B(\mathcal{I}^{(3)}) \times B(\mathcal{I}^{(4)})$, the following two statements are equivalent to one another:

(1) The function $F(x)$ is a solution of the HHP corresponding to $(v, F_0, x)$.

(2) The function $F(x)$ is a solution of the HHP corresponding to $(vV_0 w, F^M, x)$.

Proof: Suppose that statement (i) is true. Then $F(x)$ satisfies all four conditions (1) through (4) in the definition of the HHP corresponding to $(v, F_0, x)$. In particular, from conditions (3) and (4),

$$Y^{(i)}(x, \sigma) := \nu^{(i)}(\sigma)[F^M+(x, \sigma)]^{-1}$$

for all $i \in \{3, 4\}$ and $\sigma \in I^{(i)}(x)$; (6A.30a)

and

$$Y(x)$$

is bounded at $x$ and at $x_0$. (6A.30b)

So, from the preceding Thm. 34 and Eqs. (6A.29a) and (6A.30a),

$$X^{(i)}(x, \sigma) := \nu^{(i)}(\sigma)[F^M+(x, \sigma)]^{-1}$$

for all $i \in \{3, 4\}$ and $\sigma \in I^{(i)}(x)$, (6A.30c)
where 
\[ u := v(V_0w) \] 
(6A.30d)

(which is a member of \( K \), since \( \mathcal{S}_V \subset K \) and \( B \subset K \)); and, furthermore,

\[ X(x) = Y(x)Y_0(x) \] 
(6A.30e)

and, from (6A.29f), (6A.30b) and (6A.30d),

\[ X(x) \text{ is bounded at } x \text{ and } x_0. \] 
(6A.30f)

Therefore, we have proved that statement (ii) is true if statement (i) is true.

Next, suppose statement (ii) is true. Then \( \mathcal{F}(x) \) satisfies all four conditions in the definition of the HHP corresponding to \((u, \mathcal{F}^M, x)\), where \( u \) is defined by Eq. (6A.30d). In particular, from conditions (3) and (4), Eq. (6A.30c) and the statement (6A.30f) hold. Since \( \det V_0^{(i)} = \det w^{(i)} = 1 \) and since \( \det \mathcal{F}_0(x) = \det \mathcal{F}^M(x) = \nu(x) \) [Thm. 1(iii)], Eq. (6A.29a) yields \( \det Y^{(i)}(x) = 1 \). Therefore, both sides of Eq. (6A.29a) are invertible, and

\[ [Y_0^{(i)}(x, \sigma)]^{-1} = [\mathcal{F}^{M+}(x, \sigma)[V_0^{(i)}(\sigma)w^{(i)}(\sigma)]^{-1}[\mathcal{F}_0^{+}(x, \sigma)]^{-1} = [\mathcal{F}^{M-}(x, \sigma)[V_0^{(i)}(\sigma)w^{(i)}(\sigma)]^{-1}[\mathcal{F}_0^{-}(x, \sigma)]^{-1} \]

for all \( i \in \{3, 4\} \) and \( \sigma \in \mathcal{I}^{(i)}(x) \); 
(6A.30g)

and, from (6A.29b),

\[ Y_0(x)^{-1} \text{ is bounded at } x \text{ and at } x_0. \] 
(6A.30h)

So, by multiplying both sides of Eq. (6A.30c) by the corresponding sides of Eq. (6A.30g), and then using (6A.30c), (6A.30d) and (6A.30f), we establish that \( \mathcal{F} \) is a solution of the HHP corresponding to \((v, \mathcal{F}_0, x)\).

End of proof.

B. The HHP solution \( \mathcal{F} \) is a member of \( \mathcal{S}_\square \) when \( v \in K^\square \) and \( \square \) is \( n \geq 3, n + (n \geq 3), \infty \) or ‘an’

THEOREM 36 \((\partial \mathcal{F}/\partial x^i = \Gamma_i\mathcal{F})\)

When \( v \in K^3 \), \( \mathcal{F} \) is the solution of the HHP corresponding to \((v, \mathcal{F}^M)\) and \( H \) is the function defined by Eq. (4D.1) in Thm. 23, then [from Thm. 23] \( d\mathcal{F}(x, \tau) \) and \( dH(x) \) exist; and, for each \( i \in \{3, 4\} \),

\[ \frac{\partial \mathcal{F}(x, \tau)}{\partial x^i} = \Gamma_i(x, \tau)\mathcal{F}(x, \tau) \text{ for all } (x, \tau) \in \text{dom } \mathcal{F}, \]
(6B.1)

where

\[ \Gamma_i(x, \tau) := \frac{1}{2(\tau - x^i)} \frac{\partial H(x)}{\partial x^i}. \]
(6B.2)

Proof: From Thm. 1(ii), \( \mathcal{F}(x, \tau)^{-1} \) exists for all \( (x, \tau) \in \text{dom } \mathcal{F} \); and, for the continuous extension of \( Y \) that is defined by Cor. 10 (also, see the beginning of Sec. 4F) and Eq. (2B.7), \( Y(x, \sigma)^{-1} \) exists for all \( x \in D \) and \( \sigma \in \mathcal{I}(x) \). From Thm. 23, \( d\mathcal{F}(x, \tau), dY(x, \sigma) \) and \( dH(x) \) exist and are continuous functions of \( (x, \tau) \), \( (x, \sigma) \) and \( x \) throughout \( \text{dom } \mathcal{F} \),
dom $Y := \{(x, \sigma) : x \in D, \sigma \in \mathcal{I}(x)\}$ and $D$, respectively; and, for each $x \in D$, $d\hat{F}(x, \tau)$ is a holomorphic function of $\tau$ throughout $C - \mathcal{I}(x) - \{r, s, r_0, s_0\}$. It then follows, with the aid of conditions (1) through (3) in the definition of the HHP, Eq. (4D.1) in Thm. 23, and Thm. 32 (ii) that, for each $x \in D$,

$$Z_i(x, \tau) := (\tau - x^i) \frac{\partial \hat{F}(x, \tau)}{\partial x^i} \hat{F}(x, \tau)^{-1} \text{ exists and is a holomorphic function of } \tau \text{ throughout } C - \mathcal{I}(x) - \{r, s, r_0, s_0\}$$

(6B.3)

$$Z_i(x, \tau) = \frac{1}{2} \frac{\partial H(x)}{\partial x^i} \Omega + O(\tau^{-1}) \text{ in at least one neighborhood of } \tau = \infty,$$

(6B.4)

$$Z_i^+(x, \sigma) \text{ exists for each } \sigma \in \mathcal{I}(x)$$

(6B.5)

and

$$Z_i^+(x, \sigma) = Z_i^-(x, \sigma) = (\sigma - x^i) \frac{\partial Y(x, \sigma)}{\partial x^i} Y(x, \sigma)^{-1} + Y(x, \sigma) \frac{1}{2} \frac{\partial H^M(x)}{\partial x^i} \Omega Y(x, \sigma)^{-1} \text{ for all } \sigma \in \mathcal{I}(x),$$

(6B.6)

where we have used the fact that the defining equation in condition (3) for the HHP corresponding to $(\mathbf{v}, \mathcal{F}^M, \mathbf{x})$ is expressible in the form

$$\mathcal{F}^\pm(x, \sigma) = Y^{(j)}(x, \sigma) \mathcal{F}^M(x, \sigma)[\nu^{(j)}(\sigma)]^{-1} \text{ for all } \sigma \in \mathcal{I}^{(j)}(x);$$

(6B.7)

and we have used the fact that, since $\mathcal{F}^M \in \mathcal{S}_\mathcal{F}$,

$$\frac{\partial \hat{F}^M(x, \tau)}{\partial x^j} = \Gamma^M_i(x, \tau) \hat{F}^M(x, \tau) \text{ for all } \tau \in C - \mathcal{I}(x) - \{r, s, r_0, s_0\}. \text{ (6B.8)}$$

We next define a continuous extension of $Z_i(x)$ [which we also denote by $Z_i(x)$] to the domain $C - \{r, s, r_0, s_0\}$ by letting

$$Z_i(x, \sigma) := Z_i^\pm(x, \sigma). \text{ (6B.9)}$$

Then, from the statement (6B.3) and the theorem of Riemann that we have already used in a different context,

$$Z_i(x, \tau) \text{ is a holomorphic function of } \tau \text{ throughout } C - \{r, s, r_0, s_0\}. \text{ (6B.10)}$$

However, from Eq. (6B.3) and Thms. 32 (ii) and (iii),

$$\nu(x, \tau) Z_i(x, \tau) \text{ converges as } \tau \rightarrow r_0 \text{ and as } \tau \rightarrow s_0,$$

and $\nu(x, \tau)^{-1} Z_i(x, \tau)$ converges as $\tau \rightarrow r$ and as $\tau \rightarrow s$. \text{ (6B.11)}

Also, from Eq. (6B.6) and the continuity on $\mathcal{I}(x)$ of $dY(x, \sigma)$ and $Y(x, \sigma)^{-1} = \Omega Y(x, \sigma)^T \Omega$,

$$Z_i(x, \sigma) \text{ converges as } \sigma \rightarrow r_0, \sigma \rightarrow s_0, \sigma \rightarrow r, \sigma \rightarrow s.$$ \text{ (6B.12)}

Combining (6B.10), (6B.11) and (6B.12), one obtains, by reasoning that should now be familiar to us, $Z_i(x, \tau) = Z_i(x, \infty)$, whereupon the conclusion of our theorem follows from Eqs. (6B.3) and (6B.4).

End of proof.
COROLLARY 37 ($d\hat{F} = \Gamma \hat{F}$)
For each $(x, \tau) \in \text{dom} \ \hat{F}$,
\[
d\hat{F}(x, \tau) = \Gamma(x, \tau) \hat{F}(x, \tau),
\]
where
\[
\Gamma(x, \tau) := \frac{1}{2}(\tau - z + \rho^\star)^{-1}dH(x)\Omega = \sum_i dx^i \Gamma_i(x, \tau).
\]

Proof: Obvious. End of proof.

THEOREM 38 ($A\Gamma = \frac{1}{2}\Omega dH\Omega$)
Suppose $v \in K^3$ and $F$ is the solution of the HHP corresponding to $(v, F^M)$. Then
\[
A\Gamma = \frac{1}{2}\Omega dH\Omega,
\]
where $H, A$ and $\Gamma$ are defined by Eqs. (4D.1), (4D.9) and (6B.14), respectively.

Proof: The proof will be given in three parts:

(1) For each $H' \in S_H$, note that
\[
\text{Re } H' = -h'
\]
and that the defining differential equation for $\text{Im } H'$ in terms of $\text{Re } H'$ is expressible in the form
\[
h'\Omega d(\text{Re } H') = \rho^\star (\text{iIm } H').
\]
Recall that $h'$ is symmetric and $\det h' = \rho^2$. So,
\[
(h'\Omega)^2 = \rho^2 I.
\]
From Eq. (6B.18), Eq. (6B.17) is equivalent to the equation
\[
h'\Omega d(\text{iIm } H') = \rho^\star d(\text{Re } H');
\]
and, therefore, Eq. (6B.17) is equivalent to the equation
\[
h'\Omega dH' = \rho^\star dH'.
\]
Furthermore, the above Eq. (6B.20) yields
\[
A\Gamma' = [(\tau - z)\Omega + \Omega h'\Omega] \frac{1}{2}(\tau - z + \rho^\star)^{-1}dH'\Omega = \frac{1}{2}(\tau - z + \rho^\star)^{-1}[(\tau - z)\Omega dH'\Omega + \rho^\star \Omega dH'\Omega].
\]
So, Eq. (6B.20) implies
\[
A\Gamma' = \frac{1}{2}\Omega dH'\Omega \text{ for each } H' \in S_H.
\]

The reader should have no difficulty in proving that, conversely, Eq. (6B.21) implies (6B.20). We shall use the above result later in our proof.
(2) We now obtain a second result that we shall need for the proof. From Eq. (4D.10) in Thm. 24,
\[ [\mathcal{F}^\pm(x, \sigma)]^\dagger \mathcal{A}(x, \sigma) \mathcal{F}^\pm(x, \sigma) = \mathcal{A}(x_0, \sigma) \quad \text{for all } \sigma \in \mathcal{I}(x). \]  
(6B.22)

Now, recall that \( \mathcal{A}(x_0, \sigma) = \mathcal{A}^M(x_0, \sigma) \) in our gauge. Also, recall that
\[ [\mathcal{F}'(x, \tau^*)]^\dagger \mathcal{A}'(x, \tau) \mathcal{F}'(x, \tau) = \mathcal{A}^M(x_0, \sigma) \quad \text{for all } \mathcal{F}' \in \mathcal{S}_F. \]  
(6B.23)

Therefore, we obtain the following result by using Eqs. (6B.7) [condition (3) in the definition of the HHP corresponding to \( (v, F^M, x) \)], (6B.22), (1C.4) and (6B.23) [for \( \mathcal{F}' = F^M \)]:
\[ Y^\dagger(x, \sigma) \mathcal{A}(x, \sigma) Y(x, \sigma) = \mathcal{A}^M(x, \sigma) \quad \text{for all } \sigma \in \mathcal{I}(x). \]

However, recall that \( Y(x, \sigma) \) is now a continuous function of \( \sigma \) throughout \( \overline{\mathcal{I}}(x) \). Therefore,
\[ Y^\dagger(x, \sigma) \mathcal{A}(x, \sigma) Y(x, \sigma) = \mathcal{A}^M(x, \sigma) \quad \text{for all } \sigma \in \overline{\mathcal{I}}(x). \]  
(6B.24)

We shall use this result below.

(3) From the definition of \( \mathcal{A} \) and \( \Gamma \), each component of \( \mathcal{A}(x, \tau) \Gamma(x, \tau) \) is a holomorphic function of \( \tau \) throughout \( C - \{r, s\} \) and has no essential singularity at \( \tau = r \) and at \( \tau = s \). In fact, if there are any singularities at these points, they are simple poles. That much is obvious.

From Eqs. (6B.13), (6B.7) and (6B.8),
\[ \mathcal{A}(x, \sigma) \Gamma(x, \sigma) = \mathcal{A}(x, \sigma) [d \mathcal{F}^\pm(x, \sigma)][\mathcal{F}^\pm(x, \sigma)]^{-1} = \mathcal{A}(x, \sigma) [dY(x, \sigma) + Y(x, \sigma) \Gamma^M(x, \sigma)] [Y(x, \sigma)]^{-1} \]
for all \( \sigma \in \mathcal{I}(x) \).  
(6B.25)

The above equation becomes, after using Eq. (6B.24),
\[ \mathcal{A}(x, \sigma) \Gamma(x, \sigma) = \left\{ \mathcal{A}(x, \sigma) dY(x, \sigma) + [Y(x, \sigma)^\dagger]^{-1} \mathcal{A}^M(x, \sigma) \Gamma^M(x, \sigma) \right\} [Y(x, \sigma)]^{-1}, \]
which becomes, after using Eq. (6B.21) with \( H' = H^M \),
\[ \mathcal{A}(x, \sigma) \Gamma(x, \sigma) = \left\{ \mathcal{A}(x, \sigma) dY(x, \sigma) + [Y(x, \sigma)^\dagger]^{-1} \frac{1}{2} \Omega dH^M(x) \Omega \right\} [Y(x, \sigma)]^{-1}. \]  
(6B.26)

From Thm. 29 and the fact that \( \det Y(x, \sigma) = 1 \), the right side of the above equation is a continuous function of \( \sigma \) throughout \( \overline{\mathcal{I}}(x) \). Therefore, \( \mathcal{A}(x, \tau) \Gamma(x, \tau) \) is extendable to a holomorphic function of \( \tau \) throughout \( C \); and it follows that
\[ \mathcal{A}(x, \tau) \Gamma(x, \tau) = [\mathcal{A}(x, \tau) \Gamma(x, \tau)]_{\tau=\infty} = \frac{1}{2} \Omega dH(x) \Omega. \]

End of proof.

**COROLLARY 39** \( (h\Omega dH = \rho \ast dH) \)

Suppose \( H \) is defined as in the preceding theorem. Then
\[ h\Omega dH = \rho \ast dH. \]  
(6B.27)
Proof: Multiply both sides of Eq. (6B.14) through by \( 2\Omega(\tau - z + \rho^*) \) on the left, and by \( \Omega \) on the right; and then set \( \tau = z \).

End of proof.

THEOREM 40 (HHP solution \( \mathcal{F} \in S_\pi^\square \))

(i) For each \( \mathbf{v} \in K^\square \), where \( \square \) is \( n \geq 3 \), \( n+ \ (n \geq 3) \), \( \infty \) or \('an', \) and, for each \( \mathcal{F}_0 \in S_\pi^\square \), there exists exactly one solution \( \mathcal{F} \) of the HHP corresponding to \((\mathbf{v}, \mathcal{F}_0)\).

(ii) Let \( H \) be defined in terms of \( \mathcal{F} \) by Eq. (4D.1), and let \( \mathcal{E} := H_{22} \). Then \( \mathcal{E} \in S_\mathcal{E} \), and \( H \) is identical with the unique member of \( S_H \) that is constructed from \( \mathcal{E} \) in the usual way.

(iii) Furthermore, \( \mathcal{F} \) is identical with the member of \( S_\mathcal{F} \) that is defined in terms of \( H \) in Sec. \[ \] [and whose existence and uniqueness for a given \( H \in S_H \) is asserted in Thm. \[ \]].

(iv) Finally, let \( \mathcal{F}_{HE} \) be defined in terms of \( \mathcal{F}_{KC} = \mathcal{F} \) by Eq. (1A.24), and let \( \mathbf{V} \) denote the member of \( S_\mathcal{V} \) that is defined in terms of \( \mathcal{F}_{HE} \) by Eq. (6A.18). Then \( \mathbf{V} \in S_\mathcal{V} \) and, therefore (by definition), \( \mathcal{E} \in S_\mathcal{E} \), \( H \in S_H^\square \) and \( \mathcal{F} \in S_\mathcal{F}^\square \).

Proofs:

(i) Let \( \mathbf{V}_0 \) denote the member of \( S_\mathbf{V} \) that corresponds to \( \mathcal{F}_0 \). Since \( \mathcal{F}_0 \in S_\mathcal{F}^\square \), there exists (by definition of \( S_\mathcal{F}^\square \)) \( \mathbf{w} \in B(\mathcal{F}_3) \times B(\mathcal{F}_4) \) such that

\[
\mathbf{V}_0 \mathbf{w} \in k^\square \subset K^\square; \tag{6B.28}
\]

and, since \( K^\square \) is a group,

\[
\mathbf{v} \mathbf{V}_0 \mathbf{w} \in K^\square. \tag{6B.29}
\]

From Thm. \[ \], there exists exactly one solution \( \mathcal{F} \) of the HHP corresponding to \((\mathbf{v} \mathbf{V}_0 \mathbf{w}, \mathcal{F}^M)\); and, from Thm. \[ \], it then follows that \( \mathcal{F} \) is also a solution of the HHP corresponding to \((\mathbf{v}, \mathcal{F}_0)\). Finally, from Thm. \[ \] (iv), there is no other solution of the HHP corresponding to \((\mathbf{v}, \mathcal{F}_0)\).

End of proof.

(ii) From the premises of this theorem, \( \mathbf{v} \in K^3 \). Therefore, from statement (5B.7) in Thm. \[ \],

\[
dH \text{ exists and is continuous} \tag{6B.30}
\]

throughout \( D \); and since

\[
(d^2 H)(\mathbf{x}) = drds \left[ \frac{\partial^2 H(\mathbf{x})}{\partial r \partial s} - \frac{\partial^2 H(\mathbf{x})}{\partial s \partial r} \right] \tag{6B.31}
\]

and

\[
(d \times dH)(\mathbf{x}) = -drds \left[ \frac{\partial^2 H(\mathbf{x})}{\partial r \partial s} + \frac{\partial^2 H(\mathbf{x})}{\partial s \partial r} \right], \tag{6B.32}
\]

it is true that

\[
d^2 H \text{ exists and vanishes} \tag{6B.33}
\]
and
\[ d \star dH \text{ exists and is continuous} \] (6B.34)
throughout \( D \). Also, Eq. (6B.27) in Cor. 39 asserts that
\[ \rho \star dH = h \Omega dH, \] (6B.35)
where we recall from Eq. (1D.2) in Thm. 23 that
\[ h := -\text{Re } H = h^T \] (6B.36)
and, from Thm. 25,
\[ \det h = \rho^2 \text{ and } f := \text{Re } \mathcal{E} = -g_{22} < 0, \] (6B.37)
where \( g_{ab} \) denotes the element of \( h \) in the \( a \)th row and \( b \)th column. Since \( \star \star = 1 \), Eq. (6B.35) is equivalent to the equation
\[ \rho dH = h \Omega \star dH \] (6B.38)
from which we obtain
\[ \rho dH^\dagger \Omega dH = dH^\dagger \Omega h \Omega \star dH. \] (6B.39)
Upon taking the hermitian conjugates of the terms in the above equation, and upon noting that \( \Omega^\dagger = \Omega \), \( h^\dagger = h \),
\[ (\omega \eta)^T = -\eta^T \omega^T \text{ and } \omega \star \eta = -(\star \omega) \eta \text{ for any } n \times n \text{ matrix 1-forms}, \] (6B.40)
one obtains
\[ -\rho dH^\dagger \Omega dH = dH^\dagger \Omega h \Omega \star dH. \] (6B.41)
From Eqs. (6B.41) and (6B.33),
\[ dH^\dagger \Omega dH = 0. \] (6B.42)
[The above result (6B.42) was first obtained by the authors in a paper\(^{33}\) which introduced an abstract geometric definition of the Kinnersley potential \( H \) and which derived other properties of \( H \) that we shall not need in these notes.]
We next consider the \((2, 2)\) matrix elements of Eqs. (6B.33) and (6B.42). With the aid of Eqs. (6B.36) and (6B.37), one obtains
\[ \rho \star d\mathcal{E} = i(g_{12} d\mathcal{E} + f dH_{12}) \] (6B.43)
and
\[ dH^*_{12} d\mathcal{E} - d\mathcal{E}^* dH_{12} = 0. \] (6B.44)
\(^{33}\)I. Hauser and F. J. Ernst, J. Math. Phys. \textbf{21}, 1116-1140 (1980). See Eq. (37).
From Eq. (6B.43),
\[
fd(\rho \star dE) - \rho dE \star dE = if(dg_{12}dE + dfdH_{12} - dE dH_{12})
\]
\[
= if \left[ -\frac{1}{2}(dh_{12} + dH_{12})dE + \frac{1}{2}(dE + \mathcal{E}^*)dH_{12} - dE dH_{12} \right]
\]
\[
= \frac{if}{2}(-dH_{12}dE + dE^*dH_{12}).
\]
Therefore, from Eq. (6B.44),
\[
fd(\rho \star dE) - \rho dE \star dE = 0.
\] (6B.45)

Furthermore, from Eqs. (1B.3), (4D.2) and (6B.37),
\[
\mathcal{E}(x_0) = -1 \quad \text{and} \quad \text{Re} \mathcal{E} < 0.
\] (6B.46)

Therefore,
\[
\mathcal{E} \in \mathcal{S}_\mathcal{E},
\] (6B.47)
since \( \mathcal{E} \) satisfies the Ernst equation (6B.45) and the requisite gauge conditions (6B.46)

Next, let
\[
\chi := \text{Im} \mathcal{E} \quad \text{and} \quad \omega := g_{12}/g_{22}.
\] (6B.48)

Then, by taking the imaginary parts of the terms in Eq. (6B.43), one deduces
\[
d\omega = \rho f^{-2} \star d\chi.
\] (6B.49)

Furthermore, Eqs. (6B.36) and (6B.37) enable us to express \( h \) in the form
\[
h = A \begin{pmatrix} \rho^2 & 0 \\ 0 & 1 \end{pmatrix} A^T,
\] (6B.50)

where
\[
A := \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{-f} & 0 \\ 0 & \sqrt{-f} \end{pmatrix}.
\] (6B.51)

Finally, the imaginary parts of the terms in Eq. (6B.38) give us
\[
\rho d(\text{Im} H) = -h J \star dh.
\] (6B.52)

A comparison of Eqs. (4D.2), (4D.4) and (6B.49) to (6B.52) with the definition of \( \mathcal{S}_H \) that is given in Sec. 4 demonstrates that \( H \) is precisely that member of \( \mathcal{S}_H \) that is computed from \( \mathcal{E} \) in the usual way. \textit{End of proof.}

(iii) From statement (5B.6) in Thm. 29,
\[
d\hat{\mathcal{F}}(x, \tau) \text{ exists for all } x \in D \text{ and } \tau \in C - \mathcal{I}(x) - \{r, s, r_0, s_0\};
\] (6B.53)

and, from Cor. 34,
\[
d\hat{\mathcal{F}}(x, \tau) = \Gamma(x, \tau) \hat{\mathcal{F}}(x, \tau) \text{ for all } x \in D \text{ and } \tau \in C - \mathcal{I}(x) - \{r, s, r_0, s_0\}.
\] (6B.54)
Furthermore, from Thm. 3(v),
\[ F(x_0, \tau) = I \text{ for all } \tau \in C. \] (6B.55)

Finally, consider that \( \bar{I}(x) = \bar{I}^{(3)}(x) \) when \( s = s_0 \), and \( \bar{I}(x) = \bar{I}^{(4)}(x) \) when \( r = r_0 \); and, from condition (1) in the definition of the HHP, \( F(x, \tau) \) is a holomorphic function of \( \tau \) throughout \( C - \bar{I}(x) \). Therefore,
\[ F((r, s_0), \tau) \text{ and } F((r_0, s), \tau) \]
are continuous functions of \( \tau \) at \( \tau = s_0 \) and at \( \tau = r_0 \), respectively. (6B.56)

From the above statements (6B.53) to (6B.56) and from the definition of \( S_F \) in Sec. 1, it follows that \( F \in S_F \).

End of proof.

(iv) From condition (3) in the definition of the HHP and from Thm. 33, there exists \( w' \in B(\mathcal{I}^{(3)}) \times B(\mathcal{I}^{(4)}) \) such that
\[ v = Vw'V_0^{-1}. \] (6B.57)

Therefore,
\[ V = vV_0(w')^{-1}. \] (6B.58)

However, from the proof of part (i) this theorem [see Eq. (6B.29)], there then exists \( w \in B(\mathcal{I}^{(3)}) \times B(\mathcal{I}^{(4)}) \) such that
\[ V(w'w) = vV_0w \in K^{\Box}. \] (6B.59)

Therefore, since \( w'w \in B(\mathcal{I}^{(3)}) \times B(\mathcal{I}^{(4)}) \), it follows from the definition of \( S_V^{\Box} \) given by Eq. (6A.24) that \( V \in S_V^{\Box} \). Hence, by definition, \( E \in S_E^{\Box} \), \( H \in S_H^{\Box} \) and \( F \in S_F^{\Box} \).

End of proof.

COROLLARY 41 (\( k^{\Box} = K^{\Box} \))
 Suppose that \( \Box \) is \( n \geq 3 \), \( n+ \) (\( n \geq 3 \)), \( \infty \) or ‘an’. Then
\[ k^{\Box} := \{ Vw : V \in S_V, w \in B(\mathcal{I}^{(3)}) \times B(\mathcal{I}^{(4)}), Vw \in K^{\Box} \} = K^{\Box}. \] (6B.60)

Proof: From its definition
\[ k^{\Box} \subset K^{\Box}. \] (6B.61)

Now, suppose \( v \in K^{\Box} \). Since
\[ V^M = (\delta^{(3)}, \delta^{(4)}), \] (6B.62)
where
\[ \delta^{(i)}(\sigma) := I \text{ for all } \sigma \in \mathcal{I}^{(i)}, \] (6B.63)
we know that
\[ F^M \in S_F^{2n} \subset S_F^{\Box}. \] (6B.64)
Therefore, from the preceding theorem, there exists $\mathcal{F} \in S_{\mathcal{F}}$ such that $\mathcal{F}$ is the solution of the HHP corresponding to $(v, \mathcal{F}^M)$; and, if $\textbf{V}$ denotes the member of $S_{\textbf{V}}$ that corresponds to $\mathcal{F}$, Eq. (6B.57) in the proof of the preceding theorem informs us that $w' \in B(I(3)) \times B(I(4))$ exists such that $v = Vw'$. So $v \in k^\Box$.

We have thus proved that
\[ K^\Box \subset k^\Box, \]
whereupon (6B.61) and (6B.62) furnish us with the conclusion $k^\Box = K^\Box$. \textit{End of proof.}

C. The generalized Geroch group $K^\Box$

Dfn. of $Z^{(i)}$

Let $Z^{(i)}$ denote the subgroup of $K(x_0, I^{(i)})$ that is given by
\[ Z^{(i)} := \{ \delta^{(i)}, -\delta^{(i)} \}, \]
where $\delta^{(i)}$ is defined by Eq. (6B.63). \textit{End of Dfn.}

\textbf{THEOREM 42 (Center of $K$)}

The center of $K(x_0, I^{(i)})$ is $Z^{(i)}$. Hence the center of $K$ is $Z^{(3)} \times Z^{(4)}$.

\textit{Proof}: Left for the reader. Hint: See the proof of Lem. 43(i). \textit{End of proof.}

Dfn. of $[v]$ for each $v \in K^3$

For each $v \in K^3$, let $[v]$ denote the function such that
\[ \text{dom} \ [v] := S^3_{\mathcal{F}} \]
and, for each $\mathcal{F}_0 \in S^3_{\mathcal{F}}$,
\[ [v](\mathcal{F}_0) := \text{the solution of the HHP corresponding to } (v, \mathcal{F}_0). \]
Note that the existence of $[v]$ is guaranteed by Thm. 33 and Thm. 22(iii). \textit{End of Dfn.}

Dfn. of $K^\Box(x_0, I^{(3)}, I^{(4)})$ when $\Box$ is $n \geq 3$, $n+ (n \geq 3)$, $\infty$ or ‘an’

Let
\[ K^\Box := \{ [v] : v \in K^\Box \}. \]
\textit{End of Dfn.}
The following lemma concerns arbitrary members \(v\) and \(v'\) of \(K\), and arbitrary members \(F_0\) and \(F\) of \(S_F\). Therefore, the lemma could have been given as a theorem in Sec. 1. However, we have saved it for now, because the lemma is directly applicable in the proof of the next theorem.

**LEMMA 43 (Properties of the next theorem)**

(i) Suppose that \(v \in K\), \(F_0 \in S_F\) and \(F \in S_F\). Then \(F\) is the solution of the HHP corresponding to \((v, F_0)\) if and only if \(V^{-1}vV_0 \in B(I^{(3)}) \times B(I^{(4)})\), where \(V_0\) and \(V\) are the members of \(S_V\) corresponding to \(F_0\) and \(F\), respectively.

In particular, the solution of the HHP corresponding to \((v, F_0)\) is \(F_0\) if and only if \(V_0^{-1}vV_0 \in B(I^{(3)}) \times B(I^{(4)})\); and the solution of the HHP corresponding to \((v, F^M)\) is \(F^M\) if and only if \(v \in B(I^{(3)}) \times B(I^{(4)})\).

(ii) In addition to the premises of part (i) of this lemma, suppose that \(v' \in K\). Thereupon, if \(F\) is the solution of the HHP corresponding to \((v, F_0)\), and \(F'\) is the solution of the HHP corresponding to \((v', F)\), then \(F'\) is the solution of the HHP corresponding to \((v', F_0)\).

If \(F\) is the solution of the HHP corresponding to \((v, F_0)\), then \(F_0\) is the solution of the HHP corresponding to \((v^{-1}, F)\).

Proofs:

(i) This theorem follows from Thm. 33 and the properties of members of \(S_F\) that are given in Thm. 43 [specifically, the properties \(F(x, \infty) = I\) and (iv)] and Thm. 32. The reader can easily fill in the details of the proof.

(ii) This follows from the obvious facts that the equations

\[ Y^{(i)}(x, \sigma) := F^{+(x, \sigma)}v^{(i)}(\sigma)[F^{+(x, \sigma)}]^{-1} \]

\[ = F^{-(x, \sigma)}v^{(i)}(\sigma)[F^{-(x, \sigma)}]^{-1} \]

and

\[ Y''^{(i)}(x, \sigma) := F'^{+(x, \sigma)}v'^{(i)}(\sigma)[F'^{+(x, \sigma)}]^{-1} \]

\[ = F'^{-(x, \sigma)}v'^{(i)}(\sigma)[F'^{-(x, \sigma)}]^{-1} \]

imply

\[ Y^{(i)}(x, \sigma)Y^{(i)}(x, \sigma) = F'^{+(x, \sigma)}v'^{(i)}(\sigma)v^{(i)}(\sigma)[F'^{+(x, \sigma)}]^{-1} \]

\[ = F'^{-(x, \sigma)}v'^{(i)}(\sigma)v^{(i)}(\sigma)[F'^{-(x, \sigma)}]^{-1} \]

and

\[ [Y^{(i)}(x, \sigma)]^{-1} = F^{+(x, \sigma)}v^{(i)}(\sigma)[F^{+(x, \sigma)}]^{-1} \]

\[ = F^{-(x, \sigma)}v^{(i)}(\sigma)[F^{-(x, \sigma)}]^{-1} \]

for all \(i \in \{3, 4\}\) and \(\sigma \in I^{(i)}\).
Finally, we prove the following generalized Geroch conjecture:

**THEOREM 44**

(i) The mapping \([v]\) is the identity map on \(S_F\) iff \(v \in Z^3 \times Z^4\).

(ii) The set \(K^\Box\) is a group of permutations of \(S_F^\Box\) such that the mapping \(v \rightarrow [v]\) is a homomorphism of \(K^\Box\) onto \(K^\Box\); and the mapping \(\{vw : w \in Z^3 \times Z^4\}\) \(\rightarrow [v]\) is an isomorphism [viz, the isomorphism of \(K^\Box/(Z^3 \times Z^4)\) onto \(K^\Box\)].

(iii) The group \(K^\Box\) is transitive [i.e., for each \(F_0, F \in S_F^\Box\) there exists at least one element of \(K^\Box\) that transforms \(F_0\) into \(F\)].

**Proofs:**

(i) The statement that \([v]\) is the identity mapping on \(S_F\) means that each \(F_0 \in S_F\) is the solution of the HHP corresponding to \((v, F_0)\); and, from Lem. 43(i), this is equivalent to the following statement:

For each \(V_0 \in S_V\), \(V_0^{-1}vV_0 \in B(I^3) \times B(I^4)\).  
\((6C.5)\)

Since \(K^\Box = K^\Box\) (Cor. 11), each \(v' \in K^\Box\) is also a member of \(K^\Box\), and this means that there exist \(V' \in S_V\) and \(w' \in B(I^3) \times B(I^4)\) such that \(v' = V'w'\). Therefore, from statement \((6C.5)\),

For each \(v' \in K^\Box\), there exists \(w' \in B(I^3) \times B(I^4)\) such that \(w'(v')^{-1}vV'(w')^{-1} \in B(I^3) \times B(I^4)\).

So, since \(B(I^3) \times B(I^4)\) is a group,

For each \(v' \in K^\Box\), \((v')^{-1}vv' \in B(I^3) \times B(I^4)\).  
\((6C.6)\)

In particular, since \(V^M \in K^\Box\) [see Eqs. \((6B.62)\) to \((6B.64)\) and \((V^M)^{-1}vv^M = v\),

\[v \in B(I^3) \times B(I^4).\]  
\((6C.7)\)

Therefore, there exist

\[\alpha_0^{(i)} : I(i) \rightarrow R^1 \text{ and } \alpha_1^{(i)} : I(i) \rightarrow R^1\]  
\((6C.8)\)

such that

\[v^{(i)} = I\alpha_0^{(i)} + J\alpha_1^{(i)};\]  
\((6C.9)\)

and, since \(v \in K^\Box\) and \(\det v^{(i)} = 1\),

\[\alpha_0^{(i)} \text{ and } \alpha_1^{(i)} \text{ are } C^\Box\]  
\((6C.10)\)

and

\[(\alpha_0)^2 + (\alpha_1)^2 = 1.\]  
\((6C.11)\)
Also [see Eq. (6A.9)], the function whose domain is $\mathcal{I}^{(i)}$ and whose values are given by

$$u^{(i)}(\sigma) = \exp \tilde{N}(\sigma)$$

is a member of $K^{\alpha_n}(\mathcal{I}^{(i)}) \subset K^\square(\mathcal{I}^{(i)})$, where $K^\square = K^\square(\mathcal{I}^{(3)}) \times K^\square(\mathcal{I}^{(4)})$. Upon letting $v' = (u^{(3)}, u^{(4)})$ in Eq. (6C.6), and upon using Eq. (6C.9), one obtains

$$I\alpha_0^{(i)} + J\alpha_1^{(i)}u^{(i)}2 \in B(\mathcal{I}^{(i)});$$

and this is true if and only if

$$\alpha_1^{(i)}(\sigma) \sinh[2\tilde{N}(\sigma)] = 0 \text{ for all } \sigma \in \mathcal{I}^{(i)}. \tag{6C.12}$$

However, $\alpha_1^{(i)}$ is continuous. Therefore, the condition (6C.12) can hold if and only if $\alpha_1^{(i)}$ is identically zero, whereupon (6C.9) and (6C.11) yield $v^{(i)} = \pm\delta^{(i)}$. Hence $v \in Z^{(3)} \times Z^{(4)}$ is a necessary and sufficient condition for $[v] = \text{the identity map on } S^\square_{\mathcal{F}}$.

(ii) Suppose $V \in K^\square$ and suppose $F \in S^\square_{\mathcal{F}}$ and the corresponding member of $S^\square_{\mathcal{F}}$ is $V$. From the definition of $S^\square_{\mathcal{F}}$ and Cor. 11, there exists $w \in B(\mathcal{I}^{(3)}) \times B(\mathcal{I}^{(4)})$ such that

$$Vw \in k^\square = K^\square.$$ 

Therefore, since $v \in K^\square$ and $K^\square$ is a group,

$$v^{-1}Vw \in K^\square = k^\square.$$ 

Therefore, from the definition of $k^\square$, there exist $V_0 \in S_{\mathcal{F}}$ and $w' \in B(\mathcal{I}^{(3)}) \times B(\mathcal{I}^{(4)})$ such that

$$v^{-1}Vw = V_0w'.$$

So, since $B(\mathcal{I}^{(3)}) \times B(\mathcal{I}^{(4)})$ is a group,

$$V^{-1}vV_0 \in B(\mathcal{I}^{(3)}) \times B(\mathcal{I}^{(4)}).$$

It then follows from Lem. 43(i) that $F$ is the solution of the HHP corresponding to $(v, F_0)$, where $F_0$ is the member of $S^\square_{\mathcal{F}}$ that corresponds to $V_0$.

We have thus shown that every member $F$ of $S^\square_{\mathcal{F}}$ is in the range of $[v]$; i.e.,

$$[v] \text{ is a mapping of } S^\square_{\mathcal{F}} \text{ onto } S^\square_{\mathcal{F}}. \tag{6C.13}$$

Next, suppose $F_0$ and $F'_0$ are members of $S^\square_{\mathcal{F}}$ such that

$$F := [v](F_0) = [v](F'_0).$$

Then, $F$ is the solution of the HHP’s corresponding to $(v, F_0)$ and to $(v, F'_0)$, whereupon Lem. 43(ii) informs us that $F'_0$ is the solution of the HHP corresponding to $(v^{-1}v, F_0)$. Hence, $F'_0 = F_0$. 

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We have thus shown that $[v]$ is one-to-one. Upon combining this result with (6C.13), we obtain

$$\text{For each } v \in K^\square, [v] \text{ is a permutation of } S^\square_F \quad \{ \text{i.e., } [v] \text{ is a one-to-one mapping of } S^\square_F \text{ onto } S^\square_F \}. \quad (6C.14)$$

Furthermore, the reader can easily show from Lem. 43(ii) that, if

$$[v'] \circ [v] \coloneqq \text{the composition of the mappings } [v'] \text{ and } [v], \quad (6C.15)$$

then

$$[v'] \circ [v] = [v'v]. \quad (6C.16)$$

Lemma 43(ii) also yields

$$[v]^{-1} = [v^{-1}]. \quad (6C.17)$$

Therefore, since $K^\square$ is a group, $K^\square$ is a group with respect to composition of mappings. The remainder of the proof is straightforward and is left to the reader. \textit{End of proof.}

(iii) Let $F_0$ and $F$ be any members of $S^\square_F$ such that the corresponding members of $S^\square_V$ are $V_0$ and $V$, respectively. By definition of $S^\square_V$, there exist members $w_0$ and $w$ of the group $B(I^{(i)}) \times B(I^{(4)})$ such that

$$V_0w_0 \text{ and } Vw \text{ are members of } k^\square = K^\square.$$

Then, from Lem. 43(i), $F$ is the solution of the HHP corresponding to $(v, F_0)$, where

$$v \coloneqq Vw(w_0)^{-1}V_0^{-1},$$

and is clearly a member of $K^\square$. So, for each $F_0 \in S^\square_F$ and $F \in S^\square_F$, there exists $[v] \in K^\square$ such that $F = [v](F_0)$; and that is what is meant by the statement that $K^\square$ is transitive. \textit{End of proof.}

As a final note, the K–C subgroup of $K^3$ is

$$\{[v] \in K^{an} : v^{(3)} \text{ and } v^{(4)} \text{ have equal analytic extensions to the domain } ]r_1, s_1[ \}. \quad (6C.18)$$

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