DECOMPOSITION THEOREMS FOR ASYMPTOTIC
PROPERTY C AND PROPERTY A

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Abstract. We combine aspects of the notions of finite decomposition complexity and asymptotic property C into a notion that we call finite APC-decomposition complexity. Any space with finite decomposition complexity has finite APC-decomposition complexity and any space with asymptotic property C has finite APC-decomposition complexity. Moreover, finite APC-decomposition complexity implies property A for metric spaces. We also show that finite APC-decomposition complexity is preserved by direct products of groups and spaces, amalgamated products of groups, and group extensions, among other constructions.

1. Introduction

Dranishnikov introduced asymptotic property C (APC) for metric spaces as a large-scale analog of topological property C [11]. APC is a weaker condition than finite asymptotic dimension [13], but it is strong enough that discrete metric spaces with asymptotic property C have Yu’s property A [20].

Later, Guentner, Tessera, and Yu introduced another large-scale property of metric spaces called finite decomposition complexity (FDC) [15, 16]. FDC is again a weaker condition that finite asymptotic dimension, but is still sufficiently strong to imply Yu’s property A.

Both APC and FDC have received a lot of attention recently, e.g. [1, 2, 3, 6, 10, 12, 17, 19]. At the end of [6], we briefly described a plan to apply the decomposition technique in that paper to define a concept we proposed to call ω-APC. The current paper is the realization of that plan. Similar ideas of applying FDC-like decompositions to coarse properties (including APC) recently appeared in a paper of Dydak, [12]. Some of our corollaries can also be deduced as special cases of Dydak’s theorems; however, unlike Dydak’s approach, we introduce a notion of decomposition length and provide upper bound estimates on this length in our permanence results.

In particular, we combine the decomposition ideas from FDC with Dranishnikov’s APC into a concept we call finite APC-decomposition complexity. More precisely, we study the permanence of asymptotic property C and of property A with respect to the decomposition notion given in the following definition.

Definition 1.1. Let \( \mathcal{X} \) and \( \mathcal{Y} \) be families of metric spaces. Let \( R \in \mathbb{R}^N \). We say that \( \mathcal{X} \) is uniformly \( R \)-decomposable over \( \mathcal{Y} \) if there exists an integer \( k \) such that for each \( X \in \mathcal{X} \) there exists a sequence \( \mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_k \) of families of subsets of \( \mathcal{Y} \) such that each \( \mathcal{U}_i \) is \( R_i \)-disjoint and \( \bigcup \mathcal{U}_i \) covers \( X \). We denote this by \( \mathcal{X} \xrightarrow{R} \mathcal{Y} \).
Using this notion we can rephrase the definition of asymptotic property C in the following way.

**Definition 1.2.** We say that a family $\mathcal{X}$ of metric spaces has **uniform asymptotic property C** if for each $R \in \mathbb{R}^\mathbb{N}$ there exists a family $\mathcal{Y}_R$ of uniformly bounded metric spaces such that $\mathcal{X} \xrightarrow{R} \mathcal{Y}_R$. We say that a metric space $X$ has **asymptotic property C** if the family $\{X\}$ has uniform asymptotic property C.

The following is a decomposition theorem for uniform asymptotic property C.

**Theorem 1.3.** Let $\mathcal{Y}$ be a family of metric spaces with uniform asymptotic property C. If a family $\mathcal{X}$ of metric spaces admits a uniform $R$-decomposition over $\mathcal{Y}$ for each sequence $R$, then $\mathcal{X}$ has uniform asymptotic property C.

Note the order of the quantifiers in the assumption: $\exists \forall R \mathcal{Y} \xrightarrow{R} \mathcal{Y}$. The family $\mathcal{Y}$ does not depend on the sequence $R$. In [6] a special case of Theorem 1.3 was proven for $\mathcal{Y}$ equal to the family of $k$-dimensional subsets of a single space $X$, under the name Decomposition Lemma. As mentioned above, this decomposition lemma was used to prove that the free product $X \ast Y$ of two discrete metric spaces $X$ and $Y$ with asymptotic property C has asymptotic property C.

We define uniform property A for a family of spaces in the following way.

**Definition 1.4.** We say that a map $\xi: X \to \ell^1(X)$ has $\varepsilon$-variation if for each $k \in \mathbb{N}$ and each $x_1, x_2 \in X$ with $d(x_1, x_2) \leq k$ we have $\|\xi_{x_1} - \xi_{x_2}\|_1 \leq k\varepsilon$.

**Definition 1.5.** Let $\mathcal{X}$ be a family of metric spaces. We say that $\mathcal{X}$ has **uniform property A** if for each $\varepsilon > 0$ there exists $S > 0$ such that for each $X \in \mathcal{X}$ there exists a map $\xi: X \to \ell^1(X)$ such that

1. $\|\xi_x\|_1 = 1$ for all $x \in X$,
2. $\xi$ has $\varepsilon$-variation,
3. $\text{supp} \xi_x \subset \overline{B}(x, S)$ for all $x \in X$.

The following is a decomposition theorem for uniform property A.

**Theorem 1.6.** For each sequence $R$ let $\mathcal{Y}_R$ be a family of metric spaces with uniform property A. If a family $\mathcal{X}$ of metric spaces admits a uniform $R$-decomposition over $\mathcal{Y}$ for each sequence $R$, then $\mathcal{X}$ has uniform property A.

Note the order of quantifiers in the assumption: $\forall R \exists \mathcal{Y}_R: \mathcal{X} \xrightarrow{R} \mathcal{Y}_R$. The family $\mathcal{Y}_R$ may vary with a change of the sequence $R$.

Following the ideas of [16] we use the notion of uniform $R$-decomposition to define complexity classes $\mathcal{C}_\alpha$ for each ordinal $\alpha$.

**Definition 1.7.** Let $\mathcal{C}_0$ be the class of all uniformly bounded families of metric spaces. For each ordinal $\alpha > 0$ let

$$\mathcal{C}_\alpha = \{X: \forall R, \exists \beta < \alpha, \exists \mathcal{Y} \in \mathcal{C}_\beta, X \xrightarrow{R} \mathcal{Y}\}.$$  

We let

$$\mathcal{C} = \bigcup_\alpha \mathcal{C}_\alpha \text{ and } \mathcal{APC} = \bigcup \mathcal{C}.$$  

Given a property of metric spaces one would like to understand its so-called permanence properties; i.e., the extent to which the property is preserved by forming unions, products, etc. While FDC enjoys very strong permanence properties [14],
permanence properties for APC are more elusive. Indeed, only recently, some 16 years after APC first appeared, was it shown that APC is preserved by direct products \( [6, 8] \) and free products \([6]\).

In \([12]\), Dydak considered decomposition complexity with respect to several coarse properties, including asymptotic property \( C \). There it was shown that this class is closed under finite unions and some natural types of infinite unions. It was also shown that spaces in a collection in \( \mathcal{C}_\alpha \) have property \( A \) and satisfy so-called limit permanence. Our definition in terms of ordinals provides more control on the decompositions and so leads to statements involving upper bounds on the depth \( \alpha \) of the decomposition complexity.

Note that if \( X \) has FDC, then \( \{ X \} \in \mathcal{C}_\alpha \) for some countable ordinal \( \alpha \). We say that \( \{ X \} \) has finite APC-decomposition complexity if it belongs to \( \mathcal{Apc} \). Theorem 1.6 implies that if \( X \in \mathcal{Apc} \), then \( X \) has uniform property \( A \).

We show that the class \( \mathcal{Apc} \) is closed under many group operations. These results are summarized in the theorem below.

**Theorem 1.8.** Let \( H \) and \( K \) be countable groups with proper left-invariant metrics. If \( H, K \in \mathcal{Apc} \) then

1. \( H \times K \in \mathcal{Apc} \);
2. \( H \ast_C K \in \mathcal{Apc} \), where \( C \) is some common subgroup;
3. \( G \in \mathcal{Apc} \) where \( 1 \to K \to G \to H \to 1 \) is exact; and
4. \( H \wr K \in \mathcal{Apc} \).

We observe that in \([6]\), it was shown that products as in item (1) or item (2) with \( C \) equal to the trivial subgroup were stable with respect to property \( \mathcal{C}_1 \). The status of the corresponding permanence questions for \( \mathcal{C}_1 \) remains open for items (3) and (4) and also for item (2) with non-trivial \( C \).

We leave it as an open question whether Theorem 1.3 is still true if we change the order of quantifiers to the order used in Theorem 1.6. If this were true, it would show that if \( X \in \mathcal{C}_\alpha \) for some ordinal \( \alpha \), then \( X \) has uniform asymptotic property \( C \). This would show that finite decomposition complexity implies asymptotic property \( C \).

2. Preliminaries

2.1. Metric families. In this paper we are concerned with applying properties of coarse geometry to metric spaces. Often, these properties will need to be applied in some uniform way to a family of spaces. It will therefore be convenient to define coarse geometric notions for families of metric spaces.

We begin by describing the terms uniformly expansive and effectively proper as they apply to maps \( F \) between families of metric spaces. These definitions appear in \([16]\).

**Definition 2.1.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be families of metric spaces. A map \( F : \mathcal{X} \to \mathcal{Y} \) is a collection of maps \( f : X_f \to Y_f \) with \( X_f \in \mathcal{X} \) and \( Y_f \in \mathcal{Y} \). We say that \( F \) is uniformly expansive if there is some non-decreasing \( \rho_2 : \mathbb{R}_+ \to \mathbb{R}_+ \) such that for every \( f \in F \), and for every pair of points \( x, x' \in X_f \),

\[
d(f(x), f(x')) \leq \rho_2(d(x, x')).
\]

We say that \( F : \mathcal{X} \to \mathcal{Y} \) is effectively proper if there is some proper non-decreasing \( \rho_1 : \mathbb{R}_+ \to \mathbb{R}_+ \) such that for every \( f \in F \) and for each pair of
points $x, x' \in X_f$, 
$$
\rho_1(d(x, x')) \leq d(f(x), f(x')).
$$
We call $F$ a \textbf{coarse embedding} if it is uniformly expansive and effectively proper. We say that the metric spaces $X$ and $Y$ are coarsely equivalent if there is a coarse embedding $F : \{X\} \to \{Y\}$ and a positive number $C$ so that if $y \in Y$ then there is some $x \in X$ such that $d_Y(f(x), y) < C$.

\textbf{Definition 2.2.} Let $X$ and $Y$ be families of metric spaces. We write $X \prec Y$ to mean that for every $X \in \mathcal{X}$ there is some $Y \in \mathcal{Y}$ such that $X \subseteq Y$.

\textbf{Definition 2.3.} A family of metric spaces $\mathcal{B}$ will be said to be \textbf{uniformly bounded} if there is some positive real number $D$ so that $\sup\{\text{diam}(B) : B \in \mathcal{B}\} < D$. We also describe such a $\mathcal{B}$ as a \textbf{uniformly bounded family}. When speaking of such families, it has become customary in the literature to refer to this property as simply \textbf{bounded} or to refer to such a family as a \textbf{bounded family}. Let $r$ be a real number. We say that the family $\mathcal{U}$ of metric subspaces of $X$ is \textbf{$r$-disjoint} if $d(x, y) > r$ whenever $x \in U$, $y \in U'$ and $U \neq U'$ are elements of $\mathcal{U}$.

2.2. Groups as metric spaces. Coarse geometry can often be fruitfully applied to discrete groups. We recall that a metric on a set is said to be \textbf{proper} if closed balls are compact. A metric on a group is called \textbf{left-invariant} if the action of the group on itself by left multiplication is an isometry. Finitely generated groups carry a unique (up to coarse equivalence) left-invariant proper metric called the \textbf{word metric}, which is given by fixing a finite symmetric generating set $S$ and taking the distance $d_S(g, h)$ between the group elements $g$ and $h$ to be the length of the shortest $S$-word that presents the element $g^{-1}h$. Here, we adopt the convention that the neutral element is represented by the empty word.

For a countable (discrete) group $G$ that is not finitely generated, Dranishnikov and Smith showed that up to coarse equivalence $G$ carries a unique proper left-invariant metric $\mathcal{C}_α$ of infinite type [9]. This metric is given by taking a (countably) infinite symmetric generating set $S$ and computing a weighted word metric in which the weights associated to the elements of the infinite generating set $S$ are a proper function, i.e., for any $N \in \mathbb{N}$ and the set of $s \in S$ with weight at most $N$ is finite.

Thus, if we restrict our attention to proper left-invariant metrics, then the coarse geometric properties of any countable group are group properties. Whenever we consider such countable groups, we will always assume them to have proper left-invariant metrics.

3. A Decomposition theorem for $\mathcal{C}_α$

The goal of this section is to prove a decomposition theorem for $\mathcal{C}_α$. To begin, we show that $\mathcal{C}_α$ is a coarse invariant.

\textbf{Theorem 3.1} (Coarse Invariance). The property $\mathcal{C}_α$ is a coarse invariant. More precisely, if $F : \mathcal{X} \to \mathcal{Y}$ is a coarse embedding and $\mathcal{Y} \in \mathcal{C}_α$, then $\mathcal{X} \in \mathcal{C}_α$.

\textbf{Proof.} Let $ρ_1$ and $ρ_2$ denote the control functions for the collection of maps $F$. Suppose first that $\mathcal{Y} \in \mathcal{C}_0$; i.e., $\mathcal{Y}$ is a (uniformly) bounded family – say $B > 0$ is a uniform bound on the diameters of elements of $\mathcal{Y}$. Given $X \in \mathcal{X}$ there is some $Y \in \mathcal{Y}$ and an $f_X \in F$ so that $f_X^{-1}(Y) = X$. Since $f_X$ is effectively proper, if $x, x' \in X$, then $ρ_1(d(x, x')) \leq d(f_X(x), f_X(x')) \leq B$. Since $ρ_1$ is proper, there is some $B' > 0$ so that $ρ_1(t) \leq B$ implies $t \leq B'$. Thus, $d(x, x') \leq B'$. Since $B'$ only
depends on the uniform $B$ and the uniform $\rho_1$ (and is independent of $Y$), we see that $X \in \mathcal{C}_0$.

Now, if $\alpha > 0$ is an ordinal number, we suppose that some $R \in \mathbb{R}^N$ is given and consider the sequence $S$ given by $S_i = \rho_2(R_i)$. Using $S$ we find some $\beta < \alpha$ and some $Z \in \mathcal{C}_\beta$ such that $Y \xrightarrow{S} Z$. Now, if $X \in \mathcal{X}$, we find $Y \in \mathcal{Y}$ and $f_X \in F$ so that $f_X^{-1}(Y) = X$. Take $k$ and $\mathcal{V}_1, \ldots, \mathcal{V}_k$ in $Z$ such that each $\mathcal{V}_j$ is $S_j$-disjoint and the union of the $\mathcal{V}_j$ covers $Y$. Then, put $U_j = f_X^{-1}(\mathcal{V}_j)$, i.e. $U_j = \{f_X^{-1}(V) : V \in \mathcal{V}_j \}$. Clearly, the union of the $U_j$ covers $X$ and if $v \in V$ and $v' \in V'$ with $V \neq V'$ in $\mathcal{V}_j$, then $\rho(R_j) = S_j < d(f(v), f(v')) \leq \rho_2(d(v, v'))$. Since $\rho_2$ is non-decreasing, we see that each $\mathcal{V}_j$ is $R_j$-disjoint as required. Since $\beta < \alpha$ we have $f_X^{-1}(Z) \in \mathcal{C}_\beta$. We are done. 

\[ \Box \]

**Corollary 3.2.** [12 Corollary 10.2] The property of having finite APC-decomposition complexity is a coarse invariant.

Next, we record three simple facts in a lemma; these will be needed in the proof of the decomposition theorem for uniform asymptotic property $C$.

**Lemma 3.3.** Let $\mathcal{X}$ and $\mathcal{Y}$ be families of metric spaces. Let $\alpha$ and $\beta$ be ordinal numbers.

1. If $X \in \mathcal{C}_\beta$ and $\beta < \alpha$, then $X \in \mathcal{C}_\alpha$.
2. If $X, Y \in \mathcal{C}_\alpha$, then $X \cup Y \in \mathcal{C}_\alpha$.
3. If $\mathcal{X} \prec \mathcal{Y}$ and $\mathcal{Y} \in \mathcal{C}_\beta$, then $\mathcal{X} \in \mathcal{C}_\alpha$.

**Proof.** Statement (1) follows from the definition. Statement (2) is an easy consequence of the definition. Statement (3) follows from coarse invariance. 

**Decomposition Theorem.** Let $\alpha > 0$ be an ordinal number. If $\mathcal{Y} \in \mathcal{C}_\alpha$, and for each $R \in \mathbb{R}^N$ we have $\mathcal{X} \xrightarrow{R} \mathcal{Y}$, then $\mathcal{X} \in \mathcal{C}_\alpha$.

**Proof.** For each $R \in \mathbb{R}^N$ let $\tilde{R} \in \mathbb{R}^{N^2}$ be the rearrangement shown in the table below (any fixed rearrangement would work).

| : | : | : | : | : | : | : | : | : | : | : | : | : | : | : | : | : | : | : | : | : | : | : | : |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tilde{R}_{4,1}$ | $\tilde{R}_{4,2}$ | $\tilde{R}_{4,3}$ | $\tilde{R}_{4,4}$ | $\ldots$ | $\tilde{R}_{10}$ | $\tilde{R}_{14}$ | $\tilde{R}_{19}$ | $\tilde{R}_{25}$ | $\ldots$ |
| $\tilde{R}_{3,1}$ | $\tilde{R}_{3,2}$ | $\tilde{R}_{3,3}$ | $\tilde{R}_{3,4}$ | $\ldots$ | $\tilde{R}_6$ | $\tilde{R}_9$ | $\tilde{R}_{13}$ | $\tilde{R}_{18}$ | $\ldots$ |
| $\tilde{R}_{2,1}$ | $\tilde{R}_{2,2}$ | $\tilde{R}_{2,3}$ | $\tilde{R}_{2,4}$ | $\ldots$ | $\tilde{R}_3$ | $\tilde{R}_5$ | $\tilde{R}_8$ | $\tilde{R}_{12}$ | $\ldots$ |
| $\tilde{R}_{1,1}$ | $\tilde{R}_{1,2}$ | $\tilde{R}_{1,3}$ | $\tilde{R}_{1,4}$ | $\ldots$ | $\tilde{R}_1$ | $\tilde{R}_2$ | $\tilde{R}_4$ | $\tilde{R}_7$ | $\ldots$ |

We will prove that for each $R \in \mathbb{R}^N$ there exists a $\beta < \alpha$, a $Z \in \mathcal{C}_\beta$, and a number $m$ such that for each $X \in \mathcal{X}$ there exist at most $m$-many collections $\mathcal{U}_{i,j}$ such that $\mathcal{U}_{i,j} \subset Z$, $U_{i,j}$ is $\tilde{R}_{i,j}$-disjoint, and $\bigcup \mathcal{U}_{i,j}$ covers $X$. This means that $\mathcal{X} \xrightarrow{R} Z$ and implies that $\mathcal{X} \in \mathcal{C}_\alpha$.

Let some $R \in \mathbb{R}^N$ be given and form $\tilde{R}$. For each pair of natural numbers $(i, j)$, put $S^j_i = \tilde{R}_{i,j}$. Then, for each fixed $j$, we consider the sequence $S^j = (S^j_i)_{i \geq 1}$ that corresponds to the $j$-th column of $\tilde{R}$. Without loss of generality, we may assume $S^j$ to be increasing for each $j$. 

By the assumption that $\mathcal{Y} \in \mathcal{C}_\alpha$, for each $j$ there is an ordinal $\beta_j < \alpha$, a family $Z_j \in \mathcal{C}_{\beta_j}$ and an integer $k_j \geq 1$ such that $\mathcal{Y}$ uniformly $S_j$-decomposes of $Z_j$ into $k_j$-many families satisfying the conditions of Definition 3.1.1.

Next, for each $j$, put $P_j = S_{k_j}^j$ and consider the sequence $P = (P_j)_{j \geq 1}$. By the assumptions, there is a $P$-decomposition of $\mathcal{X}$ over $\mathcal{Y}$.

Fix $X \in \mathcal{X}$ and take the number $k$ from the $P$-decomposition and find families $U_1, U_2, \ldots, U_k$ in $\mathcal{Y}$ such that $U_j$ is $P_j$-disjoint and so that $\bigcup_j U_j$ covers $X$.

Write the elements of $U_j$ as $\cap_j = \{U_j^i\}$. Since we have $U_j \xrightarrow{S_j} Z_j \in \mathcal{C}_{\beta_j}$ with $k_j$ families, for each $\ell$ we have a sequence of families $V_{1,\ell}^j, V_{2,\ell}^j, \ldots, V_{k_j,\ell}^j$ such that $V_{i,\ell}^j \subset Z_j$, $V_{i,\ell}^j$ is $R_{i,j}$-disjoint and $\bigcup_{i} V_{i,\ell}^j$ covers $\bigcup_{i} U_j$.

Put $U_{i,j} = \bigcup_{\ell} V_{i,\ell}^j$. Let $\beta$ be the largest among $\{\beta_1, \beta_2, \ldots, \beta_k\}$ and observe that $\beta < \alpha$. We see that from Lemma 3.3 $U_{i,j} \in \mathcal{C}_\beta$ for each $j = 1, 2, \ldots, k$, and $i = 1, 2, \ldots, k_j$.

Next, $\bigcup_{i,j} U_{i,j}$ covers $X$ since $\bigcup_{i,j} U_{i,j}$ covers $\bigcup_{i} U_j$ and $\bigcup_{j} U_j$ covers $X$.

Finally, we show that each $U_{i,j}$ is $\tilde{R}_{i,j}$-disjoint. We know that each family $V_{i,\ell}^j$ is $\tilde{R}_{i,j}$-disjoint. Since $U_i$ is $\tilde{R}_{k_i,j}$-disjoint, we know that when $\ell_1 \neq \ell_2$, then $U_{i,\ell_1}^j$ and $U_{i,\ell_2}^j$ are at least $\tilde{R}_{k_i,j}$ apart. Because $\tilde{R}_{k_i,j} \geq \tilde{R}_{i,j}$ whenever $i \in \{1, 2, \ldots, k_j\}$, we conclude that $U_{i,j}$ is $\tilde{R}_{i,j}$-disjoint for $j = 1, 2, \ldots, k$ and $i = 1, 2, \ldots, k_j$.

To complete the proof, we unravel the indexing $\tilde{R}_{i,j}$ back to the original $R_i$ and observe that we have at most $m := k \cdot \max\{k_1, \ldots, k_k\}$ families, where we fill in any gaps in the sequence with empty families as necessary. Note that the value of $m$ depends only on $R$ and not on the space $X$.

\[ \Box \]

**Lemma 3.4.** A family $\mathcal{X}$ of metric spaces has uniform asymptotic property C if and only if $\mathcal{X} \in \mathcal{C}_1$.

**Proof.** This is a simple consequence of the definitions.

The family $\mathcal{X}$ is said to have uniform asymptotic property C if for every sequence $R \in \mathbb{R}^N$ there is a family $\mathcal{Y}_R$ of metric spaces that is uniformly bounded with the property that $\mathcal{X} \not\xrightarrow{R} \mathcal{Y}$. But, $\mathcal{Y}_R$ is uniformly bounded if and only if $\mathcal{Y}_R \in \mathcal{C}_0$. Finally, we observe that $\mathcal{X} \in \mathcal{C}_1$ if and only if for every $R$ there is some ordinal $\alpha < 1$ and some $\mathcal{Y} \in \mathcal{C}_\alpha$ so that $\mathcal{X} \not\xrightarrow{R} \mathcal{Y}$.

\[ \Box \]

**Proof of Theorem 3.3.** Apply the Decomposition Theorem for $\mathcal{Y}$ and then apply Lemma 3.4.

\[ \Box \]

4. Permanence Theorems for $\mathfrak{ApC}$

4.1. Permanence results for spaces. In Guentner’s survey [14], he describes four so-called “primitive permanence results” for spaces: coarse invariance, union permanence, fibering permanence, and limit permanence. We establish these results before stating the “derived permanence results” that follow from the primitive results.

We have already seen that $\mathcal{C}_\alpha$ is a coarse invariant in Theorem 3.1.

**Lemma 4.1.** Let $\mathcal{X}$ be a family of metric spaces. Suppose that there is some $\alpha$ so that for every $R \in \mathbb{R}^N$ there is some $\mathcal{Y} \in \mathcal{C}_\alpha$ so that whenever $X \in \mathcal{X}$, there exists a
A decomposition of $X$ into subsets $A_X \cup W_X$ such that $W_X \in \mathcal{V}$ and $\{A_X\}_{X \in X} \overset{R}{\rightarrow} \mathcal{V}$. Then $\mathcal{X} \overset{R}{\rightarrow} \mathcal{Y}$.

**Proof.** Let $R \in \mathbb{R}^N$ be given and take $X \in \mathcal{X}$. Take the uniform $k$ from the decomposition of $\{A_X\}$ over $\mathcal{V}$. Then, we can cover $A_X$ by $k$ families $U_1, \ldots, U_k$ of sets from $\mathcal{V}$ such that $U_j$ is $R_j$-disjoint. Then, we append to this list the single-element family $\{W_X\}$. This family is trivially $R_{k+1}$ disjoint for any value of $R_{k+1}$.

Thus, $\mathcal{X} \overset{R}{\rightarrow} \mathcal{Y}$ with at most $k + 1$ families. \qed

As a corollary, we obtain Statement (2) from Lemma 4.3.

**Corollary 4.2.** [12 Proposition 6.5] If $X$ and $Y$ have finite APC-decomposition complexity, then $X \cup Y$ has finite APC-decomposition complexity.

The proof of Lemma 4.1 shows that if $\mathcal{X}$ admits an “excisive” $\mathcal{C}_\alpha$ decomposition then $\mathcal{X} \in \mathcal{C}_{\alpha+1}$. We apply the decomposition theorem to improve this upper bound to the expected $\mathcal{C}_\alpha$.

**Theorem 4.3** (Union Permanence). Let $\mathcal{X}$ be a family of metric spaces. Suppose that there is some $\alpha$ so that for every $R \in \mathbb{R}^N$ there is some $\mathcal{Y} \in \mathcal{C}_\alpha$ so that whenever $X \in \mathcal{X}$, there exists a decomposition of $X$ into subsets $A_X \cup W_X$ such that $W_X \in \mathcal{V}$ and $\{A_X\}_{X \in X} \overset{R}{\rightarrow} \mathcal{V}$. Then $\mathcal{X} \in \mathcal{C}_\alpha$. \qed

Before proving Fibering Permanence, we consider the case of direct products. We consider this separately to compare with the product theorem in [6].

**Definition 4.4.** For any collections $\mathcal{X}, \mathcal{Y}$ we let

$$\mathcal{X} \otimes \mathcal{Y} = \{X \times Y : X \in \mathcal{X}, Y \in \mathcal{Y}\}.$$  

We give spaces in $\mathcal{X} \otimes \mathcal{Y}$ the product metric:

$$d_{X \times Y}(x \times y, x' \times y') = \sqrt{d_X(x, x')^2 + d_Y(y, y')^2}.$$  

**Product Permanence.** Let $\mathcal{X} \in \mathcal{C}_\alpha$ and $\mathcal{Y} \in \mathcal{C}_\beta$. Then, $\mathcal{X} \otimes \mathcal{Y} \in \mathcal{C}_{\alpha+\beta}$.

**Proof.** We prove this by transfinite induction on $\alpha + \beta$. Note that this is obvious if $\alpha = \beta = 0$ since the $\otimes$-product of two bounded families is bounded.

Since $\mathcal{Y} \in \mathcal{C}_\beta$, for any $R \in \mathbb{R}^N$, there is an integer $k$ and some $\gamma < \beta$ so that $\mathcal{Y}$ admits an $R$-uniform decomposition over some family $Z \in \mathcal{C}_\gamma$.

Let $X \times Y \in \mathcal{X} \otimes \mathcal{Y}$. Then, take $k$ from the $R$-uniform decomposition of $\mathcal{Y}$ and find $U_1, U_2, \ldots, U_k$ in $Z$ such that each $U_j$ is $R_j$-disjoint and $\bigcup U_j$ covers $Y$. Then, the families $\{X\} \otimes U_j$ ($j = 1, 2, \ldots, k$) are each $R_j$-disjoint and their union covers $X \times Y$.

Now, since $\gamma < \beta$ we have $\alpha + \gamma < \alpha + \beta$; by the inductive assumption we are done. \qed

This bound is not sharp, as we now show.

**Theorem 4.5.** Let $\mathcal{X}$ and $\mathcal{Y}$ be families in $\mathcal{C}_1$. Then, $\mathcal{X} \otimes \mathcal{Y}$ is in $\mathcal{C}_1$.

**Proof.** Let $R \in \mathbb{R}^N$ be given. We must find $k$ so that whenever $X \times Y \in \mathcal{X} \otimes \mathcal{Y}$, there are families $U_1, U_2, \ldots, U_k$ in $\mathcal{C}_0$ such that $U_j$ is $R_j$-disjoint and $\bigcup U_j$ covers $X \times Y$. We use the same technique as [6 Theorem 3.1].
First, we rearrange the given sequence $R$ as in the proof of our Decomposition Theorem. Next, we fix $j$ and consider the column $R_{i,j}$ for $i = 1, 2, 3, \ldots$. Applying property $\mathcal{C}_1$ to each column produces a sequence of integers $k_j = k_j(\{R_{i,j}\}_i)$ corresponding to the each sequence $R_{1,j}, R_{2,j}, R_{3,j}, \ldots$. Next, we apply the $\mathcal{C}_1$ condition to the sequence $R_{1,k_1}, R_{2,k_2}, R_{3,k_3}, \ldots$ to obtain some integer $l$. Set $k = l \cdot \max\{k_1, k_2, \ldots, k_l\}$.

Now, for any $X \times Y \in \mathcal{X} \otimes \mathcal{Y}$, the construction in the proof of [6, Theorem 3.1] provides at most $k$ uniformly bounded subsets $W_j$ of $X \times Y$ such that $W_j$ is $R_j$ disjoint, and such that $\cup W_j$ covers $X \times Y$.

**Lemma 4.6.** Let $f : X \to Y$ be a uniformly expansive map of metric spaces. Let $\mathcal{Y} \subset 2^Y$. Then the following are equivalent:

1. There is some ordinal $\alpha$ so that if $Y \in \mathcal{C}_\alpha$, then $f^{-1}(Y) \in \mathcal{C}_\alpha$.
2. There is some ordinal $\alpha$ so that for any ordinal $\beta$ such that $Y \in \mathcal{C}_\beta$, it follows that $f^{-1}(Y) \in \mathcal{C}_{\alpha+\beta}$.

**Proof.** Obviously (2) is stronger than (1), so we need only prove (1) implies (2). We proceed by transfinite induction.

Take $\alpha$ as in (1). Suppose that $\beta$ is an ordinal with $Y \in \mathcal{C}_\beta$. Let $R \in \mathbb{R}^3$ be given and consider the sequence $S$ defined by $S_i = \rho(R_i)$, where $\rho$ is the control function corresponding to the uniformly expansive map $f$. Using the sequence $S$, we find a $k$ so that $Y \prec Z$, and $Z \in \mathcal{C}_\gamma$, with $\gamma < \beta$. By the induction hypothesis, $f^{-1}(Z) \in \mathcal{C}_{\alpha+\gamma}$. To finish the proof, it suffices to show that $f^{-1}(Y)$ admits a uniform $R$-decomposition over $f^{-1}(Z)$ with the same $k$ as above. Then, since $\alpha + \gamma < \alpha + \beta$, this will show that $f^{-1}(Y) \in \mathcal{C}_{\alpha+\beta}$, as desired.

To this end, let $W \in f^{-1}(Y)$; i.e., $W = f^{-1}(A)$ for some $A \in \mathcal{Y}$. Take $k$ as above and find families $U_1, U_2, \ldots, U_k$ of elements from $Z$ such that $U_j$ is $S_j$-disjoint and so that the union $\bigcup U_j$ covers $A$. Then, put $\mathcal{V}_i = \{f^{-1}(U) : U \in U_j\}$. It is clear that $\bigcup \mathcal{V}_j$ covers $W$ and that the $\mathcal{V}_j$ are in $f^{-1}(Z)$. Finally, if $j$ is fixed and if $a \in V$, and $a' \in V'$ with $V \neq V'$ in $\mathcal{V}_j$, then we see that $\rho(R_j) = S_j < d(f(a), f(a')) \leq \rho(d(a, a'))$. Since $\rho$ is non-decreasing, we see that each $\mathcal{V}_j$ is $R_j$-disjoint as required.

**Remark 4.7.** We observe that Lemma 4.6 applies equally well in the case that $X$ and $Y$ are so-called $\infty$-pseudo-metric spaces; i.e., spaces with a metric that is allowed to assume the value infinity. For such spaces, we extend the notion of uniformly expansive to maps $\rho : [0, +\infty] \to [0, +\infty]$ in the natural way.

**Fibering Permanence.** Let $F : \mathcal{X} \to \mathcal{Y}$ be a uniformly expansive map of metric families $\mathcal{X}$ and $\mathcal{Y}$. Suppose that there is some $\alpha$ so that for any bounded family $\mathcal{B} \prec \mathcal{Y}$, the set $F^{-1}(\mathcal{B}) \in \mathcal{C}_\alpha$. Then, if there is some $\beta$ such that $\mathcal{Y} \in \mathcal{C}_\beta$, then $\mathcal{X} \in \mathcal{C}_{\alpha+\beta}$.

**Proof.** Put $X = \sqcup \mathcal{X}$ and $Y = \sqcup \mathcal{Y}$. Then, condition (1) of Lemma 4.6 is satisfied by $\mathcal{B}$, so condition (2) holds, which is what we needed to show.

**Corollary 4.8.** Suppose $f : X \to Y$ is a uniformly expansive map of metric spaces, $\{Y\} \in \mathcal{C}_\beta$ for some $\beta$ and suppose that $f^{-1}(\mathcal{B}) \in \mathcal{C}_\alpha$ for every bounded family $\mathcal{B}$ of subsets of $Y$. Then, $\{X\} \in \mathcal{C}_{\alpha+\beta}$.

**Limit Permanence.** Let $\mathcal{X} = \{X_a\}_{a \in J}$ be a collection of metric spaces indexed by some indexing set $J$. Suppose that for every real number $r > 0$, there is an
expression $X_a = \bigcup_i X_i^a$ as an $r$-disjoint union such that for each $a$, the family $\{X_i^a\}$ belongs to $\mathcal{C}_\alpha$. Then $X$ belongs to $\mathcal{C}_{\alpha+1}$.

Proof. Let $R \in \mathbb{R}^N$ be given. Let $X_a \in \mathcal{X}$. Write $X_a = \bigcup_i X_i^a$ as an $R_1$-disjoint union over the family $\{X_i^a\}$ with $\{X_i^a\}_i \in \mathcal{C}_\alpha$.

This is useful in that it shows that $\mathcal{A}_{\text{pc}}$ satisfies limit permanence.

Corollary 4.9. [12, Corollary 11.3] Finite APC-decomposition complexity satisfies limit permanence.

4.2. Permanence results for groups. Having proven the primitive permanence results, we turn our attention to derived results. Most of these are interesting in the context of groups. Throughout this section, all groups are assumed to be countable discrete groups in left-invariant proper metrics.

Theorem 4.10. Let

$$1 \to K \to G \to H \to 1$$

be a short exact sequence of groups. If $K$ and $H$ have finite APC-decomposition complexity, then $G$ has finite APC-decomposition complexity.

Proof. As stated, this follows from the fact that any coarse property that satisfies subspace, finite union, and fibering permanence is closed under group extensions [14, Corollary 7.5].

We can apply our Fibering Permanence to prove the stronger result that if $\{H\} \in \mathcal{C}_\alpha$ and $\{K\} \in \mathcal{C}_\beta$, then $\{G\} \in \mathcal{C}_{\alpha+\beta}$.

To see this, we fix a proper left-invariant metric on $H$ arising from a weighting of a generating set $T$. Let the surjective homomorphism from $G$ to $H$ be denoted by $\varphi$ so that $K = \ker \varphi$. For each $t_i \in T$, take some $s_i \in G$ such that $\varphi(s_i) = t_i$. Give each $s_i$ the same weight as was assigned to $t_i$ and adjoin elements to the collection $\{s_i\}$ to form a generating set $S$ for $G$. Take a weighting function on $S$ that extends the values on the $s_i$ and consider $G$ in this left-invariant proper metric. Note, that this metric is unique up to coarse equivalence. In these metrics $\phi$ is 1-Lipschitz and so it is uniformly expansive. Next, take the metric on $K$ that it inherits as a subgroup of $G$. If $B \subset H$ is any bounded subset, then $\varphi^{-1}(B)$ is contained in a neighborhood of $K$ (see, for example [3, Theorem 3]) and so $\{\varphi^{-1}(B)\}$ has $\mathcal{C}_\beta$ because $\{K\}$ does. By Fibering Permanence we are done.

Theorem 4.11. Let $G$ be a countable discrete group acting (without inversion) on a tree in such a way that the vertex stabilizers have finite APC-decomposition complexity. Then, $G$ has finite APC-decomposition complexity.

Proof. This follows from [14, Theorem 7.6] as a consequence of subspace permanence, coarse invariance, union permanence, and fibering permanence.

The next result follows from the Bass-Serre theory, (see [7] or [18]).

Corollary 4.12. Let $A$ and $B$ be countable (discrete) groups in proper left-invariant metrics. If $A$ and $B$ are in $\mathcal{A}_{\text{pc}}$, then the amalgamated free product $A \ast_C B$ and the HNN extension $A \ast_C$ are in $\mathcal{A}_{\text{pc}}$.

Theorem 4.13. Let $G$ and $H$ be countable (discrete) groups in proper left-invariant metrics. If $G$ and $H$ are in $\mathcal{A}_{\text{pc}}$ then $H \wr G \in \mathcal{A}_{\text{pc}}$. 
5. Finite APC Complexity and Property A

In this section we show that spaces with finite APC-decomposition complexity have property A; we also show a decomposition theorem for uniform property A.

The notion of $\varepsilon$-variation was defined in the introduction. The following lemma is trivial.

**Lemma 5.1.** If $\xi_i$ has $\varepsilon_i$-variation, then $\sum_i \xi_i$ has $\sum_i \varepsilon_i$-variation.

**Definition 5.2.** We say that a map $\xi: X \to \ell^1(X)$ is normed if $\|\xi_x\|_1 = 1$ for each $x \in X$. We describe such a $\xi$ as locally supported if there is some $S > 0$ so that $\text{supp}(\xi_x) \subset B(x,S)$ for all $x \in X$.

**Lemma 5.3.** If $\xi$ has $\varepsilon$-variation and for each $x \in X$ we have $\|\xi_x\|_1 \geq 1$, then $\bar{\xi}$ defined by the formula $\bar{\xi}_x = \xi_x/\|\xi_x\|_1$ is normed and has $2\varepsilon$-variation.

**Proof.** For any maps nonzero maps $u$ and $v$ in $\ell^1(X)$, we have

$$\left\| \frac{u}{\|u\|_1} - \frac{v}{\|v\|_1} \right\|_1 \leq \frac{1}{\|u\|_1} \|u - v\|_1 + \left\| \frac{\|v\|_1 v - \|u\|_1 u}{\|u\|_1 \|v\|_1} \right\|_1 \leq \frac{2}{\|u\|_1} \|u - v\|_1.$$

The result easily follows. □

**Lemma 5.4.** Let $U \subset X$. Let $\xi: U \to \ell^1(U)$ be a normed, locally supported map with $\varepsilon$-variation. Let $R \in \mathbb{N}$. Let $N(U,R)$ denote the open $R$-neighborhood of $U$ in $X$. There exists a locally supported map $\bar{\xi}: X \to \ell^1(X)$ such that

1. $\|\bar{\xi}_x\|_1 = 0$ for $x \in X$ such that $d(x,U) \geq R$.
2. $\bar{\xi}_x = \xi_x$ for $x \in U$.
3. $\xi$ has $(2R + 1)\varepsilon + \frac{1}{R}$-variation.

**Proof.** Define $\alpha: X \to [0,1]$ by the formula

$$\alpha(x) = \begin{cases} 1 & d(x,X \setminus N(U,R))/R \quad x \in U \\ d(x,X \setminus N(U,R))/R & x \not\in U. \end{cases}$$

The map $\alpha$ is $\frac{1}{R}$-Lipschitz.

For $x \in X$ let $u(x)$ be $x$ if $x \in U$; any point in $U$ such that $d(u(x),x) \leq R$ if $x \in N(U,R)$; any point in $U$ otherwise. We let

$$\bar{\xi}_x = \alpha(x) \cdot \xi_{u(x)}$$

Let $x_1, x_2 \in X$ with $d(x_1,x_2) \leq k$. We have

$$\|\bar{\xi}_{x_1} - \bar{\xi}_{x_2}\|_1 = \|\alpha(x_1) \cdot \xi_{u(x_1)} - \alpha(x_2) \cdot \xi_{u(x_2)}\|_1$$

We apply the inequality

$$\|au - bv\|_1 \leq a\|u - v\|_1 + |a - b|\|v\|_1$$

to get

$$\|\alpha(x_1) \cdot \xi_{u(x_1)} - \alpha(x_2) \cdot \xi_{u(x_2)}\|_1 \leq \alpha(x_1)\|\xi_{u(x_1)} - \xi_{u(x_2)}\|_1 + |\alpha(x_1) - \alpha(x_2)|\|\xi_{u(x_2)}\|_1.$$
If \( x_1 \in X \setminus N(U, R) \), then \( \alpha(x_1) = 0 \) and
\[
\| \xi_{x_1} - \xi_{x_2} \|_1 \leq |\alpha(x_1) - \alpha(x_2)| \leq \frac{k}{R}
\]
since \( \alpha \) is \( \frac{1}{R} \)-Lipschitz.
If \( x_2 \in X \setminus N(U, R) \), then the situation is analogous.
If \( x_1, x_2 \in N(U, R) \), then
\[
d(u(x_1), u(x_2)) \leq R + d(x_1, x_2) + R \leq 2R + k.
\]
Since \( \xi \) has \( \varepsilon \)-variation, we have
\[
\| \xi_{x_1} - \xi_{x_2} \|_1 \leq (2R + k)\varepsilon + \frac{k}{R} \leq k \left( (2R + 1)\varepsilon + \frac{1}{R} \right)
\]
Combining both cases we have
\[
\| \xi_{x_1} - \xi_{x_2} \|_1 \leq k \left( (2R + 1)\varepsilon + \frac{1}{R} \right)
\]
for each \( x_1, x_2 \in X \) with \( d(x_1, x_2) \leq k \). Hence \( \xi \) has \((2R + 1)\varepsilon + \frac{1}{R}\)-variation. If \( \xi \) is \( S \)-locally supported, then it follows from the construction that \( \xi \) is \((R + S)\)-locally supported.

\textbf{Lemma 5.5.} Let \( X \) be a metric space. Let \( \mathcal{U}_i \) be a finite sequence of \( R_i \)-disjoint families of subsets of \( X \) such that \( \bigcup \mathcal{U}_i \) covers \( X \). For \( U \in \mathcal{U}_i \), let \( \xi^U : U \to \ell^1(U) \) be a normed, locally supported map with \( \varepsilon_i \)-variation. Then there exists a normed, locally supported map \( \xi : X \to \ell^1(X) \) with \( E \)-variation, where
\[
E = 2 \left( \sum (2R_i + 1)\varepsilon_i + \frac{1}{R_i} \right).
\]

\textbf{Proof.} This follows immediately from Lemmas 5.1, 5.3, and 5.4. \( \square \)

\textbf{Corollary 5.6.} Let \( X \) be a metric space. Let \( \varepsilon = \frac{1}{R} > 0 \). Let \( R_i = 2^{i+1}N \) and \( \varepsilon_i = \frac{1}{2^{i+2}N} \). Let \( \mathcal{U}_i \) be a finite sequence of \( R_i \)-disjoint families of subsets of \( X \) such that \( \bigcup \mathcal{U}_i \) covers \( X \). If for each \( U \in \mathcal{U}_i \) there exists a locally supported normed map \( \xi^U : U \to \ell^1(U) \) with \( \varepsilon_i \)-variation, then there exists a locally supported normed map \( \xi : X \to \ell^1(X) \) with \( \varepsilon \)-variation.

\textbf{Definition 5.7.} Let \( X \) be a bounded geometry metric space. We say that \( X \) has property A if for each \( R, \varepsilon > 0 \) there exists a map \( \xi : X \to \ell^1(X) \) such that
\begin{enumerate}
\item \( \| \xi_x \|_1 = 1 \) for all \( x \in X \),
\item if \( x_1, x_2 \in X \) and \( d(x_1, x_2) \leq R \), then \( \| \xi_{x_1} - \xi_{x_2} \|_1 \leq \varepsilon \),
\item there exists \( S > 0 \) such that \( \text{supp} \xi_x \subset B(x, S) \) for all \( x \in X \).
\end{enumerate}

We are now in a position to prove our decomposition theorem for uniform property A from the introduction.

\textbf{Proof of Theorem 1.6.} Suppose that \( X \) has the property that for every \( R \in \mathbb{R}^N \) there is a family \( \mathcal{Y}_R \) of metric spaces with uniform property A such that \( X \) admits a uniform \( R \)-decomposition over \( \mathcal{Y}_R \).

Let \( \varepsilon > 0 \) be given; we may assume \( \varepsilon \) to be of the form \( \frac{1}{n} \) for some \( n \in \mathbb{N} \). Then, as in Corollary 5.6, take the sequence \( R \) given by \( R_i = 2^{i+1} \frac{1}{n} \) and put \( \varepsilon_i = \frac{1}{2^{i+2}n} \).

By assumption, we can find a family \( \mathcal{Y}_R \) with uniform property A with the property that there is some \( k \) so that for any \( X \in \mathcal{X} \) there are families \( \mathcal{U}_1, \ldots, \mathcal{U}_k \) of subsets...
from \( Y_R \) whose union covers \( X \). Use the uniform assumption with \( \varepsilon_i \) as above to find \( S_i \) and maps \( \xi_i^U : U \to \ell^1(U) \) realizing the uniform property A condition. Then, with \( S = \max \{ S_i \} \), we have a map \( \xi : X \to \ell^1(X) \) that is \((R_k + S)\)-locally supported, is normed and is of \( \varepsilon \)-variation.

\[ \square \]

We can use this result to conclude that spaces with finite APC-decomposition complexity have property A. We remark that this was also shown in [12, Proposition 11.1] using different techniques.

**Theorem 5.8.** If \( X \) is a metric space with finite APC-decomposition complexity, then \( X \) has property A.

**Proof.** It is clear that any bounded family has uniform property A. Suppose therefore that \( \{ X \} \in C_\alpha \) for some \( \alpha > 0 \). Then, for any \( R \), \( X \) admits an \( R \)-decomposition over some family \( Y \in C_\beta \) with \( \beta < \alpha \). By the inductive assumption, \( Y \) has uniform property A. By Theorem 1.6, \( \{ X \} \) has uniform property A. We conclude that \( X \) has property A as desired. \[ \square \]

### 6. Open Questions

We end with several open questions regarding APC and finite APC-decomposition complexity.

The first question was stated in the introduction:

**Question 6.1.** Does finite decomposition complexity imply asymptotic property C?

Several of the results in this paper involve an increase in the depth of the APC-decomposition complexity, \( \alpha \). We could ask several questions when \( \alpha = 1 \), which is the case of (uniform) APC.

**Question 6.2.** Let \( H \) and \( K \) be countable groups with proper left-invariant metrics and APC.

1. Does \( H \ast_C K \) have APC, where \( C \) is some common subgroup?
2. Does \( G \) have APC where \( 1 \to K \to G \to H \to 1 \) is exact?
3. Do the groups \( \bigoplus H \) or \( H \wr K \) have APC?

More generally, we can ask whether the complexity level necessarily increases in the product or fibering theorems, e.g.:

**Question 6.3.** Let \( X \) and \( Y \) be in \( C_\alpha \) with \( \alpha > 1 \). Does it follow that \( X \otimes Y \in C_\alpha \)?

When we compare \( Apc \) to FDC, we see that \( X \) has FDC if and only if \( X \) is in \( D_\alpha \) for a countable ordinal \( \alpha \), see [16, Theorem 2.2.2]. One could formulate this question about whether \( C_\alpha \) makes sense for uncountable ordinals \( \alpha \).

**Question 6.4.** Is it true that \( X \in Apc \) is equivalent to \( X \in C_\alpha \) for a countable ordinal \( \alpha \)?

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