Quantization of massive scalar fields
over axis symmetric space-time backgrounds

Owen Pavel Fernández Piedra
Departamento de Física y Química, Universidad de Cienfuegos, Cuba

Alejandro Cabo Montes de Oca
Grupo de Física Teórica, ICIMAF, Cuba.

(Dated: March 24, 2022)

Abstract

The renormalized mean value of the quantum Lagrangian and the Energy-Momentum tensor for scalar fields coupled to an arbitrary gravitational field configuration are analytically evaluated in the Schwinger-DeWitt approximation, up to second order in the inverse mass value. The cylindrical symmetry situation is considered. The results furnish the starting point for investigating iterative solutions of the back-reaction problem related with the quantization of cylindrical scalar field configurations. Due to the homogeneity of the equations of motion of the Klein-Gordon field, the general results are also valid for performing the quantization over either vanishing or non-vanishing mean field configurations. As an application, compact analytical expressions are derived here for the quantum mean Lagrangian and Energy-Momentum tensor in the particular background given by the Black-String space-time.
I. INTRODUCTION

Semiclassical gravity considers the quantum dynamics of fields in a gravitational background, which at this level of description is considered as a classical external field. That is, all fields are considered as quantum ones, with the only exception of the external gravitational field, that remains satisfying the classical Einstein field equations, associated with sources given by the vacuum expectation values of the stress energy tensor of the matter fields \[ T_{\mu\nu} \]. In that situation, it is necessary to have adequate mathematical methods to obtain explicit analytical expressions for the renormalized stress tensor \( <T_{\mu\nu}>_{\text{ren}} \), the quantity that enters as a source in the semiclassical Einstein equations \[ 4, 5, 6, 7, 8, 9, 21 \]. This stress tensor and the expectation value \( <\varphi^2>_{\text{ren}} \) of a quantum field \( \varphi \) are the main objects to calculate from quantum field theory in curved spacetime.

Having the components for \( <T_{\mu\nu}>_{\text{ren}} \), the backreaction of the quantized fields in the spacetime geometry of black holes can in principle be determined, unless the (unknown) effects of quantum gravity become important. For the calculation of \( <T_{\mu\nu}>_{\text{ren}} \) we need, in principle, the exact knowledge of the functional dependence of this tensor over all the possible metrics. For this reason it is improbable to have an exact analytical formula for this object. Except for very special spacetimes, on which quantum matter fields propagates, and for boundary conditions with a high degree of symmetry \[ 10, 11, 12, 13, 14 \], it is not possible to obtain exact expressions for this quantity. That mathematical difficulty has led to the development of approximate methods to build the effective action, starting from which the energy momentum tensor can be calculated by functional differentiation with respect to the metric. One of the developed techniques, the so called Schwinger-De Witt expansion, is based on a series development expansion of the effective action in inverse powers of the field mass. It is well-known that this method can be used to investigate effects like the vacuum polarization of massive fields in curved backgrounds, whenever the Compton's wavelength of the field is less than the characteristic radius of curvature \[ 5, 6, 7, 8, 21 \]. Also numerical computations of \( <T_{\mu\nu}>_{\text{ren}} \) and \( <\varphi^2>_{\text{ren}} \) have been performed by a number of authors \[ 4, 15, 16, 17, 18, 19, 20 \].

In General Relativity there exists a four parameter family of black hole solutions called the generalized Kerr-Newmann family. The solutions belonging to this family are characterized by the four parameters: mass \( M \), angular momentum \( J \), charge \( Q \) and the Cosmological Constant \( \Lambda \) \[ 2 \]. These are axis-symmetric solutions that show different asymptotic behavior...
depending on the sign of the cosmological constant. There are two important cases of axial symmetry. One is the spherical symmetry that have been studied in great detail since the birth of General Relativity. The other one is the cylindrical symmetry. As it has been shown by Lemos in Ref. [2], in the case of negative cosmological constant, there exists a Black hole solution showing cylindrical symmetry: the so called Black String. Charged rotating black string solutions has many similarities with the Kerr-Newman black hole, apart from spacetime being asymptotically anti-de Sitter in the radial direction (and not asymptotically flat). The existence of black strings suggests that they could be the final state of the collapse of matter having cylindrical symmetry.

The problems of determining $\langle \varphi^2 \rangle_{\text{ren}}$ and investigate the renormalized stress tensor components for conformally coupled massless scalar fields in black String backgrounds were studied by DeBenedictis in [22, 23]. Also the results obtained for $\langle T^\nu_{\mu} \rangle_{\text{ren}}$ were used for the calculation of gravitational backreaction of the quantum field. It was found that the perturbations initially strengthen the singularity, an effect similar to the case of spherical symmetry without a cosmological constant, indicating that the behaviour of quantum effects may be universal and not depend on the geometry of the spacetime nor the presence of non-zero cosmological constant.

In this paper we address the problem of evaluating the components of the renormalized vacuum expectation values of the Stress-Energy Tensor for a massive scalar field in a background space-time having cylindrical symmetry. The general results are applied to explicitly evaluate closed expression for those quantities in the special background formed by a neutral and non-rotating cylindrical Black String. In Section II first we build the effective action and the Stress-Energy tensor taking into account terms up to the second order in the inverse mass of the scalar field. Section III is devoted to review the metric tensor which solves the Einstein-Maxwell in the considered cylindric symmetry situation. Finally, employing the explicit form of the Black-String metric, close expressions for the renormalized components of the Energy-Momentum tensor are derived in Section IV. These results can be used to study the vacuum polarization and the back-reaction of the quantum scalar field in the gravitational background. Final comments and possible future extensions of the work are given in the section on Conclusions.

In the following we use for the Riemann tensor, its contractions, and the covariant derivatives the sign conventions of Misner, Thorne and Wheeler [24]. Our units are such that
\[ h = c = G = 1. \]

II. RENORMALIZED EFFECTIVE ACTION

This paper first consider the derivation of the Schwinger-De Witt approximation for the renormalized Lagrangian and Stress-Energy tensor of massive scalar field subject to an arbitrary background space-time. Consider a single massive scalar field \( \phi(x) \) interacting with gravity with non minimal coupling constant \( \xi \) in four dimensions. The action for the system is:

\[ S = S_{\text{gravity}} + S_{\text{matter}} \quad (1) \]

with \( S_{\text{gravity}} \) the Einstein-Hilbert action for the gravitational background field and \( S_{\text{matter}} \) that of the scalar field:

\[ S_{\text{gravity}} = \frac{1}{2} \int d^4x \sqrt{-g} \left( R - \Lambda \right) \quad (2) \]

and:

\[ S_{\text{matter}} = -\frac{1}{2} \int d^4x \sqrt{-g} \left[ g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi + (m^2 + \xi R) \phi^2 \right] \quad (3) \]

where \( m \) is the mass of the field. The action (1) leads to the usual Einstein equations for the gravitational field and the Klein-Gordon one for the scalar field:

\[ R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R + \Lambda g_{\mu \nu} = 8\pi T_{\mu \nu} \quad (4) \]

\[ (\Box - \xi g - m^2) \phi = 0, \quad (5) \]

In the above equations \( \Box = g^{\mu \nu} \nabla_\mu \nabla_\nu \) is the covariant D'Alembertian and \( T_{\mu \nu} \) the stress tensor of the classical scalar field. Note that the values of the scalar fields in the classical approximation can be non zero. However, in the particular case to be considered here in more detail, the Black String, the scalar field can be considered as vanishing. The homogeneity of the scalar field equation, implies that the Black String solution after fixing the classical scalar field as vanishing, is also a solution of the classical Einstein equations including scalar fields.

For a general background geometry, the most direct approach to evaluate the renormalized Lagrangian is to use the first non-vanishing term of the renormalized effective action calculated using the Schwinger-DeWitt approximation [5]. The advantage of this approach lies in the purely geometric nature of the approximation that reflects its local nature. The
effective action of the quantized massive scalar field differs from the analogous actions constructed for fields of higher spins only by numerical coefficients, and one can generalize the presented results to fields of other spins. However, it is important to stress that the method is restricted to cases in which we avoid the presence of strong or rapidly varying gravitational fields. Moreover, nonphysical divergences that appears in the massless limit obstacle its application in that case. The construction of the first-order renormalized stress-energy tensor should be carried out in an analytically continued Euclidean space-time. The analytic continuation to the physical space is performed at the last stage of the calculations.

Using DeWitt’s effective action approach and applying Schwinger’s regularization prescription \[5, 8\] one gets the renormalized effective action for the quantized scalar field satisfying equation (1) as

\[ W_{\text{ren}} = \int d^4x \sqrt{-g} \mathcal{L}_{\text{ren}} \]  (6)

where the renormalized effective Lagrangian reads:

\[ \mathcal{L}_{\text{ren}} = \frac{1}{2(4\pi)^2} \sum_{k=3}^{\infty} \frac{\text{str} a_k(x, x)}{k(k-1)(k-2)m^{2(n-2)}} \]  (7)

and:

\[ \text{str} F = (-1^i) F_{i}^{A} = \int d^4x (-1^4) F_{A}^{A}(x) \]  (8)

is the functional supertrace \[8\]. The coefficients \([a_k] = a_k(x, x')\), whose coincidence limit appears under the supertrace operation in (7) are the Hadamard-Minakshisundaram-DeWitt-Seeley coefficients (HMDS), whose complexity rapidly increases with \(k\). As usual, the first three coefficients of the DeWitt-Schwinger expansion, \(a_0, a_1,\) and \(a_2\), contribute to the divergent part of the action and can be absorbed in the classical gravitational action by renormalization of the bare gravitational and cosmological constants. Various authors have calculated some of the HDSM coefficients in exact form up to \(n \geq 4\). DeWitt \[7\] have calculated the coefficient \([a_2]\), which is proportional to the trace anomaly of the renormalized Stress-Energy tensor of the quantized, massless, and conformally invariant fields. The coincidence limit of the coefficient \(a_3\) has been obtained by Gilkey \[9\], whereas the coefficient \([a_4]\) has been calculated by Avramidi \[8\].

Restricting ourselves here to the terms proportional to \(m^{-2}\), using integration by parts and the elementary properties of the Riemann tensor, we obtain for the renormalized effective lagrangian,

\[ \mathcal{L}_{\text{ren}} = \mathcal{L}_{\text{ren}}^{\text{conformal}} + \tilde{\mathcal{L}}_{\text{ren}}, \]  (9)
where the conformal part of the effective lagrangian is given by
\[
\mathcal{L}_{\text{conformal}}^{\text{ren}} = \frac{1}{192\pi^2m^2} \left[ \Theta R\Box R + \frac{1}{140} R_{\mu\nu} \Box R^{\mu\nu} \right. \\
- \frac{8}{945} R_{\mu}^\nu R_{\gamma}^\rho R_{\mu}^\gamma + \frac{2}{315} R_{\mu\nu} R_{\gamma\epsilon} R_{\mu}^\gamma R_{\epsilon}^\nu + \frac{1}{1260} R_{\mu\nu} R_{\mu}^\gamma R_{\sigma\gamma\epsilon} R^{\nu\sigma\epsilon} \\
\left. + \frac{17}{7560} R_{\sigma\epsilon} R_{\mu\nu} \sigma^\tau R_{\sigma\tau\gamma\epsilon} - \frac{1}{270} R_{\mu}^\gamma R_{\mu\nu} R_{\nu\sigma} R_{\gamma\epsilon} \right],
\tag{10}
\]
and the mass dependent contribution takes the form
\[
\mathcal{L}_{\text{ren}} = \frac{1}{192\pi^2m^2} \left[ \frac{1}{30} \eta (R R_{\mu\nu} R^{\mu\nu} - R R_{\mu\nu\gamma\epsilon} R^{\mu\nu\gamma\epsilon}) \right. \\
\left. + \frac{1}{2} \eta^2 R\Box R - \eta^3 R^3 \right].
\tag{11}
\]
where we use \( \Theta = \frac{1}{252} - \frac{1}{30} \xi \) and \( \eta = \xi - \frac{1}{6} \).

By standard functional differentiation of the effective action with respect to the metric, the renormalized Stress-Energy tensor is obtained according to the known formula:
\[
\langle T_{\mu\nu}\rangle_{\text{ren}} = \frac{2}{\sqrt{-g}} \frac{\delta W_{\text{ren}}}{\delta g^{\mu\nu}}
\tag{12}
\]
The result can be written in a general form as
\[
\langle T_{\mu}^{\ \mu}\rangle_{\text{ren}} = C_{\mu}^{\ \nu} + D_{\mu}^{\ \nu},
\tag{13}
\]
where the \( C_{\mu}^{\ \nu} \) and \( D_{\mu}^{\ \nu} \) tensors take the somewhat cumbersome forms
\[
C_{\mu}^{\ \nu} = \frac{1}{96\pi^2m^2} \left[ \Theta (\nabla_\mu R \nabla^\nu R + \nabla^\nu \nabla_\mu (\Box R) + \nabla_\mu \nabla^\nu (\Box R) - 2 \Box R \delta_{\mu}^{\ \nu} \right. \\
- \frac{1}{2} \delta_{\mu}^{\ \nu} \nabla_\gamma R \nabla^\gamma R - 2 \Box R \nabla^\nu \nabla_\mu R) + \frac{1}{140} \left[ \nabla_\mu R_{\gamma\lambda} \nabla^\nu R_{\lambda}^{\gamma\lambda} - \nabla^\nu R_{\gamma\lambda} \nabla^\lambda R_{\mu}^{\gamma\lambda} \right. \\
- \nabla_\mu R_{\gamma\lambda} \nabla^\lambda R_{\nu}^{\gamma\lambda} + \nabla_\gamma R_{\gamma\lambda} \nabla^\nu R_{\mu}^{\lambda\nu} + \nabla_\gamma \nabla^\nu (\Box R_{\mu\nu}) - 2 \Box R_{\mu}^{\ \nu} \\
+ \nabla_\gamma \nabla_\nu (\Box R_{\mu\nu}) - \frac{1}{2} \nabla_\rho R_{\gamma\lambda} \nabla^\rho R_{\nu}^{\gamma\lambda} \delta_{\mu}^{\ \nu} - \nabla_\gamma \nabla^\lambda (\Box R_{\gamma\lambda}) \delta_{\mu}^{\ \nu} + \nabla_\lambda \nabla^\nu R_{\gamma\lambda} R_{\mu}^{\gamma\lambda} \\
+ \nabla_\lambda \nabla_\nu R_{\gamma}^{\gamma\lambda} R_{\gamma\lambda} R_{\mu}^{\gamma\lambda} - \nabla_\gamma \nabla^\nu R_{\gamma\lambda} R_{\mu}^{\gamma\lambda} - \left( \nabla_\lambda \nabla_\sigma R_{\gamma}^{\lambda\sigma} R_{\gamma\lambda} R_{\mu}^{\gamma\lambda} - \frac{1}{2} \nabla_\nu \nabla_\gamma R_{\lambda}^{\gamma\lambda} \right) R_{\mu}^{\ \gamma} \\
+ \nabla_\mu R_{\gamma\lambda} R_{\gamma\lambda} R_{\mu}^{\gamma\lambda} - \Box R_{\gamma\lambda} R_{\mu}^{\gamma\lambda} \left. \right] - \frac{8}{945} \left[ \frac{3}{2} \nabla_\nu R_{\gamma\lambda} \nabla^\lambda R_{\mu}^{\gamma\lambda} \right.
\tag{10}
\]
\[
\left. + \frac{3}{2} \nabla_\gamma R_{\gamma\lambda} \nabla_\mu R_{\lambda}^{\mu\nu} - \frac{3}{2} \nabla_\gamma R_{\gamma\lambda} \nabla_\epsilon R_{\lambda}^{\gamma\epsilon} R_{\mu}^{\gamma\nu} - \frac{3}{2} \nabla_\epsilon R_{\gamma\lambda} \nabla_\lambda R_{\gamma\epsilon} R_{\mu}^{\gamma\nu} R_{\lambda}^{\epsilon\nu} + \frac{3}{2} \nabla_\lambda \nabla^\nu R_{\gamma\lambda} R_{\mu}^{\gamma\lambda} \\
+ \frac{3}{2} \nabla_\lambda \nabla_\mu R_{\gamma}^{\gamma\lambda} R_{\gamma\lambda} R_{\mu}^{\gamma\lambda} - \frac{3}{2} \nabla_\lambda \nabla_\epsilon R_{\gamma}^{\gamma\lambda} R_{\epsilon}^{\nu} R_{\mu}^{\gamma} + \frac{3}{2} \nabla_\lambda \nabla^\nu R_{\gamma}^{\gamma\lambda} R_{\mu}^{\gamma} - \frac{3}{2} \nabla_\nu \nabla_\gamma R_{\lambda}^{\gamma\lambda} \sigma_{\nu} \\
\left. + \frac{1}{2} \nabla_\nu \nabla_\gamma R_{\lambda}^{\gamma\lambda} \right] - R_{\lambda}^{\gamma\lambda} R_{\gamma}^{\lambda\nu} R_{\gamma\lambda} R_{\mu}^{\gamma\lambda} - \frac{3}{2} \Box R_{\gamma\lambda} R_{\mu}^{\gamma\lambda} - \frac{3}{2} \nabla_\nu \nabla_\gamma R_{\lambda}^{\gamma\lambda} R_{\epsilon}^{\nu} \delta_{\mu}^{\ \nu} + R_{\gamma\lambda} R_{\epsilon}^{\gamma\lambda} R_{\mu}^{\gamma} \delta_{\mu}^{\ \nu}
\tag{10}
\]
\[
+ \frac{3}{2} \nabla_\lambda \nabla_\mu R_{\gamma\lambda} R_{\gamma\lambda} R_{\mu}^{\gamma\lambda} - \frac{3}{2} \Box R_{\gamma\lambda} R_{\mu}^{\gamma\lambda} - \frac{3}{2} \nabla_\nu \nabla_\gamma R_{\lambda}^{\gamma\lambda} R_{\mu}^{\gamma} \delta_{\mu}^{\ \nu} + \frac{3}{2} \nabla_\lambda \nabla^\nu R_{\gamma\lambda} R_{\mu}^{\gamma\lambda} \delta_{\mu}^{\ \nu} + R_{\gamma\lambda} R_{\epsilon}^{\gamma\lambda} R_{\mu}^{\gamma} \delta_{\mu}^{\ \nu}
\tag{10}
\]
\[-3R_{\gamma\lambda\mu} \gamma R^{\lambda\nu} + \frac{3}{2} \nabla_\mu R_{\gamma\lambda} \nabla^\lambda R^{\gamma\nu} - 3\nabla_\gamma R_{\gamma\mu} \nabla^\gamma R^{\gamma\nu} + \frac{3}{2} \nabla_{\gamma} R_{\gamma\lambda} \nabla_{\nu} R_{\mu} \gamma \]

\[+ \frac{2}{315} \left( \nabla_\gamma R_{\gamma\mu} \nabla_\lambda R_{\lambda} \nu + \nabla_\lambda R_{\gamma\nu} \nabla_\gamma R_{\mu} \lambda - 2\nabla_\gamma R_{\gamma\lambda} \nabla^\lambda R_{\mu} \nu - \nabla_{\nu} R_{\gamma\lambda} \nabla_{\nu} R_{\mu} \gamma \right) + \nabla_\nu R_{\gamma\lambda} \nabla_\mu R_{\gamma\lambda} R_{\gamma\mu} R_{\gamma\nu} + \nabla_\nu R_{\gamma\lambda} \nabla_\nu R_{\gamma\lambda} R_{\gamma\mu} R_{\gamma\lambda} R_{\gamma\nu} + \frac{1}{2} \nabla_\gamma R_{\gamma\lambda} \nabla_{\nu} R_{\mu} \gamma \]

\[= \left( \begin{array}{l}
\frac{2}{315} (3\nabla_\gamma R_{\gamma\mu} \nabla_\lambda R_{\lambda} \nu + \nabla_\lambda R_{\gamma\nu} \nabla_\gamma R_{\mu} \lambda - 2\nabla_\gamma R_{\gamma\lambda} \nabla^\lambda R_{\mu} \nu - \nabla_{\nu} R_{\gamma\lambda} \nabla_{\nu} R_{\mu} \gamma) \\
\nabla_\nu R_{\gamma\lambda} \nabla_\mu R_{\gamma\lambda} R_{\gamma\mu} R_{\gamma\nu} + \nabla_\nu R_{\gamma\lambda} \nabla_\nu R_{\gamma\lambda} R_{\gamma\mu} R_{\gamma\lambda} R_{\gamma\nu} + \frac{1}{2} \nabla_\gamma R_{\gamma\lambda} \nabla_{\nu} R_{\mu} \gamma
\end{array} \right) \]

[7]
and:

\[
D_{\mu}^{\nu} = \frac{1}{96\pi^2m^2} \left[ \frac{1}{30} \eta \left( \nabla^{\nu} R \nabla^{\gamma} R_{\gamma\mu} + \nabla_{\mu} R \nabla^{\gamma} R_{\gamma}^{\nu} + 2 \nabla^{\nu} R_{\gamma\lambda} \nabla_{\mu} R_{\gamma\lambda}^{\gamma} - \Box R R_{\mu}^{\nu} \right) + \nabla_{\gamma} R \nabla^{\nu} R_{\mu}^{\gamma} + R \nabla^{\gamma} \nabla^{\nu} R_{\gamma\mu} + R \nabla^{\gamma} \nabla_{\mu} R_{\gamma}^{\gamma} \\
- R \nabla_{\lambda} \nabla^{\nu} R_{\mu}^{\lambda} - \frac{1}{2} R \nabla^{\nu} \nabla_{\mu} R + R R_{\mu}^\lambda R_{\mu}^\nu - R R_{\mu}^\lambda R_{\gamma}^\lambda - 2 \nabla_{\gamma} R \nabla_{\lambda} R_{\gamma}^\nu \right) \nabla^{\nu} \nabla_{\mu} R_{\gamma}^{\lambda} + \nabla_{\mu} \nabla^{\nu} R_{\gamma\lambda} R_{\gamma\lambda}^{\gamma} + \nabla_{\lambda} \nabla_{\nu} R_{\gamma\lambda} R_{\gamma\lambda}^{\lambda} \delta_{\nu}^{\gamma} - 2 \Box R_{\gamma\lambda} R_{\gamma\lambda}^{\gamma} \delta_{\nu}^{\gamma} \\
+ \frac{1}{2} R R_{\gamma\lambda} R_{\gamma\lambda}^{\gamma} \delta_{\nu}^{\gamma} + \nabla_{\gamma} \nabla_{\nu} R_{\mu}^{\gamma} - 2 R R_{\gamma}^{\gamma} R_{\mu}^{\gamma} + \nabla_{\mu} \nabla_{\gamma} R R_{\mu}^{\gamma} - R R_{\gamma\lambda} R_{\gamma\lambda}^{\gamma} R_{\mu}^{\gamma} \\
+ 4 \nabla_{\gamma} R \nabla_{\lambda} R_{\lambda\mu}^{\gamma} + 2 \nabla_{\gamma} R_{\gamma\lambda\sigma\gamma} \nabla^{\gamma} R_{\lambda\sigma\gamma}^{\lambda\sigma} - \nabla_{\mu} \nabla^{\nu} R_{\gamma\lambda\sigma\gamma} R_{\gamma\lambda\sigma\gamma}^{\gamma} + 2 R \nabla^{\gamma} \nabla_{\lambda} R_{\gamma\lambda}^{\gamma} \\
+ 2 \Box R_{\gamma\lambda\sigma\gamma} R_{\gamma\lambda\sigma\gamma}^{\gamma} \delta_{\nu}^{\gamma} - \frac{1}{2} R R_{\gamma\lambda\sigma\gamma} R_{\gamma\lambda\sigma\gamma}^{\gamma} \delta_{\nu}^{\gamma} + R_{\mu}^{\nu} R_{\gamma\lambda\sigma\gamma} R_{\gamma\lambda\sigma\gamma}^{\gamma} + 2 R R_{\gamma\lambda\sigma\gamma} R_{\gamma\lambda\sigma\gamma}^{\gamma} \\
+ 2 \nabla_{\gamma} \nabla_{\lambda} R R_{\gamma\lambda}^{\gamma} R_{\gamma\lambda}^{\gamma} - 2 \nabla_{\mu} R_{\gamma\lambda} \nabla^{\gamma} R_{\gamma\lambda}^{\gamma} \delta_{\nu}^{\gamma} - R \nabla^{\gamma} \nabla_{\lambda} R_{\gamma\lambda} \delta_{\nu}^{\gamma} \\
+ \frac{1}{2} R_{\gamma\lambda} \nabla^{\gamma} R_{\gamma\lambda}^{\gamma} \nabla_{\mu} \nabla^{\nu} \nabla_{\mu} \nabla^{\nu} (\Box R) + \nabla_{\mu} \nabla^{\nu} (\Box R) - \frac{1}{2} \delta_{\nu}^{\gamma} \nabla_{\gamma} R \nabla^{\gamma} R \\
- 2 \Box R_{\mu}^{\nu} - 2 \Box R \nabla^{\nu} \nabla_{\mu} R - \eta(6 \nabla_{\mu} R \nabla^{\nu} R + 6 R \nabla^{\nu} \nabla_{\mu} R + \frac{1}{2} R^3 \delta_{\nu}^{\gamma} \\
- 6 R \Box R_{\mu}^{\nu} - 3 R^2 R_{\mu}^{\nu} - 6 \nabla_{\gamma} R \nabla^{\gamma} R \delta_{\nu}^{\gamma}) \right],
\]

III. THE BLACK STRING METRIC

Let us review in this section the particular metric associated to the Black String solutions. Consider the Einstein-Hilbert action in four dimensions with a cosmological term in the presence of an electromagnetic field. The total action will be

\[
S = S_{\text{gravity}} + S_{\text{em}},
\]

where \( S_{\text{gravity}} \) is given by (2) and:

\[
S_{\text{em}} = -\frac{1}{2} \int d^4x \sqrt{-g} F_{\mu \nu} F^{\mu \nu}.
\]

The Maxwell tensor is

\[
F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,
\]

\( A_\mu \) being the vector potential. We will consider in this work solution of the Einstein-Maxwell system admitting a commutative two dimensional Lie group \( G_2 \) of isometries. These are
space-times with cylindrical symmetry. The group $G_2$ generates two dimensional spaces with three possible topologies. The first is $R \times S^1$, the standard cylindrically symmetric model, with orbits diffeomorphic either to cylinders or to $R$ (i.e., $G_2 = R \times U(1)$), the second one is $S_1 \times S_1$ the flat torus $T_2$ model ($G_2 = U(1) \times U(1)$), and finally the third possible topology is $R^2$. However, we will focus only on the first case. Then a cylindrical coordinate system $(x^0, x^1, x^2, x^3) = (t, \rho, \phi, z)$ with $-\infty < t < \infty$, $0 \leq \rho < \infty$, $-\infty < z < \infty$, $0 \leq \phi < \infty$ will be chosen. An stationary cylindrically symmetric spacetime that solves the Einstein-Maxwell equations from (1) is (see Ref. [2]):

$$ds^2 = -(\alpha^2 \rho^2 - \frac{2(M + \Omega)}{\alpha \rho}) dt^2 + \frac{1}{\alpha^2 \rho^2} d\rho^2 + \rho^2 d\phi^2 + \alpha^2 \rho^2 dz^2$$

(17)

where $M$, $Q$, and $J$ are the mass, charge, and angular momentum per unit length of the string respectively. $\Omega$ is given by

$$\Omega = \sqrt{M^2 - \frac{8J^2 \alpha^2}{9}}.$$  

(18)

The constant $\alpha$ is defined as follows:

$$\alpha^2 = -\frac{1}{3} \Lambda,$$  

(19)

where $\Lambda$ is a negative Cosmological Constant, giving to the spacetime its asymptotically anti-De Sitter behavior. In this paper we are only concerned with space-times showing both charge and angular momentum equal to zero, thus yielding the following form to relation (17):

$$ds^2 = -(\alpha^2 \rho^2 - \frac{4M}{\alpha \rho}) dt^2 + \frac{1}{(\alpha^2 \rho^2 - \frac{4M}{\alpha \rho})} d\rho^2 + \rho^2 d\phi^2 + \alpha^2 \rho^2 dz^2.$$  

(20)

As immediately can be seen from (20), the considered metric behaves as the one corresponding to the anti-De Sitter space-time in the limit $\rho \to \infty$, and therefore is not globally hyperbolic. This solution has an event horizon located at $\rho_H = \frac{3M}{\alpha}$ and the apparent singular behavior at this horizon is a coordinate effect and not a true one. The only true singularity is a polynomial one at the origin, as can it be seen after calculating the Kretschmann scalar. It results in

$$K = R_{\alpha \beta \xi \gamma} R^{\alpha \beta \xi \gamma} = 24 \alpha^4 \left(1 + \frac{M^2}{\alpha^6 \rho^6}\right).$$  

(21)
IV. RENORMALIZED STRESS-ENERGY TENSOR FOR SCALAR FIELDS IN A BLACK STRING BACKGROUND

In the space-time of a static Black String metric given by (20) simple results were obtained for the renormalized Stress Tensor of massive scalar field showing an arbitrary coupling to the background gravitational field. After a direct calculation, for the conformal part of the stress tensor we evaluated in this work the result:

\[ C_{tt} = \frac{1}{2520m^2\pi^2\alpha^3\rho^9} \left( 11\alpha^9\rho^9 - 201\alpha^3M^2\rho^3 + 1252M^3 \right), \]  

(22)

\[ C_{zz} = C_{\phi\phi} = \frac{1}{2520m^2\pi^2\alpha^3\rho^9} \left( 11\alpha^9\rho^9 - 183\alpha^3M^2\rho^3 + 1468M^3 \right), \]  

(23)

\[ C_{\rho\rho} = \frac{1}{2520m^2\pi^2\alpha^3\rho^9} \left( 11\alpha^9\rho^9 + 189\alpha^3M^2\rho^3 - 308M^3 \right). \]  

(24)

The above components of the Stress Tensor do not depend in any way of the coupling constant \( \xi \) because of the constant value of the Ricci scalar in this space-time: \( R = -12\alpha^2 = 4\Lambda \). The coupling parameter arises only in the term proportional to the DÁlembertian of the Ricci scalar that in our case is identically zero. The results for the components of the \( D_\mu^\nu \) tensor are

\[ D_{tt} = \eta \left[ \frac{1}{80\alpha^3\rho^9\pi^2m^2} \left( 112\alpha^3\rho^3M^2 - 704M^3 + \alpha^9\rho^9 \right) \right] - \frac{9\alpha^6}{2\pi^2m^2\eta^3}, \]  

(25)

\[ D_{zz} = D_{\phi\phi} = \eta \left[ \frac{1}{80\alpha^3\rho^9\pi^2m^2} \left( 112\alpha^3\rho^3M^2 - 896M^3 + \alpha^9\rho^9 \right) \right] - \frac{9\alpha^6}{2\pi^2m^2\eta^3}, \]  

(26)

\[ D_{\rho\rho} = \eta \left[ \frac{1}{80\alpha^3\rho^9\pi^2m^2} \left( -112\alpha^3\rho^3M^2 + 192M^3 + \alpha^9\rho^9 \right) \right] - \frac{9\alpha^6}{2\pi^2m^2\eta^3}. \]  

(27)

It is interesting to evaluate the above components of the stress tensor at the event horizon of the black string. We obtain the following results:

\[ T_{tt}|_{\text{horizon}} = -\frac{3\alpha^6}{2\pi^2m^2} \left( \frac{1}{40} + 3\eta^2 \right) + \frac{\alpha^6}{140\pi^2m^2} \]  

(28)

\[ T_{zz}|_{\text{horizon}} = T_{\phi\phi}|_{\text{horizon}} = -\frac{3\alpha^6}{2\pi^2m^2} \left( \frac{1}{20} + 3\eta^2 \right) + \frac{\alpha^6}{112\pi^2m^2} \]  

(29)

\[ T_{\rho\rho}|_{\text{horizon}} = -\frac{3\alpha^6}{2\pi^2m^2} \left( \frac{1}{40} + 3\eta^2 \right) + \frac{\alpha^6}{140\pi^2m^2}. \]  

(30)
In the general case, all the components of the renormalized stress energy tensor of the quantized scalar field will be positive at the horizon for the values of the coupling constant satisfying the relation:

\[ 3\eta^3 + \frac{1}{40}\eta < \frac{1}{210} \quad (31) \]

There are some particular cases in which the above relation is always satisfied. The simplest case of the conformal coupling \( \xi = \frac{1}{6} \) and the minimal one are two important examples. Also for the case \( \xi < \frac{1}{6} \) the spacetime components of the quantized scalar field at the horizon of the black string are always positive quantities. If we define the energy density as usual:

\[ \varepsilon = -T_{tt} \quad (32) \]

then we can conclude that for the particular cases mentioned above (satisfying condition 32) the weak energy condition is violated. However, violations of the weak energy condition for quantum matter are common, as in the case of Casimir Effect, and it is unknown how relevant the classical energy conditions are for this cases. Also, they are in fact required for a self consistent picture of the Hawking evaporation effect.

V. CONCLUDING REMARKS

The quantization of a massive scalar field with arbitrary coupling to a gravitational background corresponding to cylindrical symmetry was considered. The renormalized quantum mean values of the Lagrangian and the corresponding components of the Energy-Momentum tensor are explicitly evaluated for arbitrary axis symmetric space-time metrics configurations up to the second order in the inverse mass of the scalar field. It is found that, as is usual for the case of quantum fields in curved backgrounds, exist regions in which the weak energy condition is violated. In the case of the Black String considered in this work, we proof that the violation occur at the horizon of the spacetime for values of the coupling constant satisfying the relation (32), that include as particular cases the most interesting of minimal and conformal coupling. Also for values of the coupling constant \( \xi < \frac{1}{6} \) the violation occur. This work furnish the starting point for of an iterative solution of the quantum back-reaction problems of the quantum dynamics of the matter fields on the space-time metric. Due to the homogeneity of the scalar field equation of motion, the results turn to be also valid for the quantization over non-vanishing scalar mean field configurations. The general
formula are here employed to explicitly evaluate compact expressions for the mean value of the quantum Lagrangian and Energy Momentum tensor for the scalar field quantized over the metric associated to the Black String solution. As mentioned above, the results are expected to be employed to investigate the back-reaction of the quantum scalar field, on the Black String metric. For this purpose, the Einstein equations for the metric should be solved after including in them the calculated Energy-Momentum tensor for the Black String solution. After that, the new value of the metric can be employed to determine the modified Energy-Momentum tensor by substituting in (13). Then, these new back-reacted components can be employed to solve the Einstein equations with them included again, and so on... iteratively. We expect to implement this program in coming works.

**Acknowledgments**

One of the authors (O.P.F.P) greatly acknowledges M. Chacón Toledo and E. R. Bezerra de Mello for helpful discussions. This work was supported by the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

[1] N.D. Birrel and P. C. Davies, *Quantum Fields in Curved Space*, (Cambridge University Press, Cambridge, 1982).

[2] J. P. S. Lemos and V. T. Zanchin, *Phys. Rev. D* 54, 3840 (1996).

[3] S. A. Fulling, *Aspects of Quantum Field Theory in Curved Space Time*, (Cambridge University Press, Cambridge, 1985).

[4] P. R. Anderson, W. A. Hiscock and D. A. Samuel, *Phys. Rev. D* 51, 4337 (1995).

[5] V. P. Frolov and A. I. Zelnikov, *Phys. Lett. 115B*, 372 (1982), V. P. Frolov and A. I. Zelnikov, *Phys. Lett. 123B*, 197 (1983), V. P. Frolov and A. I. Zelnikov, *Phys. Rev. D* 29, 1057 (1984).

[6] A. O. Barvinsky and G. A. Vilkovisky, *Phys. Rept.* 119, 1 (1985).

[7] B. S. DeWitt, *Phys. Rept* 53, 1615 (1984).

[8] I. G. Avramidi, *Nucl. Phys. B* 355, 712 (1991), I. G. Avramidi, PhD Thesis, hep-th/9510140.

[9] P. B. Gilkey *J. Diff. Geom.* 10, 601 (1975).

[10] J. S. Dowker and R. Critchley, *Phys. Rev. D* 13, 3224 (1976).
[11] L. S. Brown and J. P. Cassidy, *Phys. Rev. D* **15**, 2810 (1977).

[12] T. S. Bunch and P. C. W. Davies, *Proc. R. Soc. London A* **360**, 117 (1978); T. S. Bunch *J. Phys. A* **12**, 517 (1979).

[13] B. Allen and A. Folacci, *Phys. Rev. D* **35**, 3771 (1987); A. Folacci, *J. Math. Phys* **32**, 2813 (1991).

[14] K. Kirsten and J. Garriga, *Phys. Rev. D* **48**, 567 (1993).

[15] K. H. Howard and P. Candelas, *Phys. Rev. Lett.* **53**, 403 (1984).

[16] P. Candelas, *Phys. Rev. D* **21**, 2185 (1980).

[17] M. S. Fawcet, *Commun. Math. Phys.* **89**, 103 (1983).

[18] B. P. Jensen and A. C. Ottewill, *Phys. Rev. D* **39**, 1130 (1989); B. P. Jensen, J. G. McLaughlin and A. C. Ottewill, *Phys. Rev. D* **45**, 3002 (1992).

[19] P. R. Anderson, W. A. Hiscock, and D. J. Loranz *Phys. Rev. Lett.* **74**, 4365 (1995).

[20] E. R. Bezerra de Mello, V. B. Bezerra, and N. R. Khusnutdinov *Phys. Rev. D* **60**, 063506 (1999).

[21] J. Matyjasek, *Phys. Rev. D* **61**, 124019 (2000).

[22] A. DeBenedictis, *Gen. Rel. Grav.* **31**, 1549 (1999).

[23] A. DeBenedictis, *Class. Quant. Grav.* **16**, 1955 (1999).

[24] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation,* (Freeman, San Francisco, 1973).