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RANK GRADIENT IN CO-FINAL TOWERS OF
CERTAIN KLEINIAN GROUPS

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RANK GRADIENT IN CO-FINAL TOWERS OF CERTAIN KLEINIAN GROUPS

by

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Dedicated to my wife Ivy, and my sons, Murilo and Alvaro.
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This dissertation provides the first known examples of finite co-volume Kleinian groups which have co-final towers of finite index subgroups with positive rank gradient. We prove that if the fundamental group of an orientable finite volume hyperbolic 3-manifold has finite index in the reflection group of a right-angled ideal polyhedron in $\mathbb{H}^3$ then it has a co-final tower of finite sheeted covers with positive rank gradient. The manifolds we provide are also known to have co-final towers of covers with zero rank gradient. We also prove that the reflection groups of compact right-angled hyperbolic polyhedra satisfying mild conditions have co-final towers of finite sheeted covers with positive rank gradient.
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Chapter 1

Introduction

Let $G$ be a finitely generated group. The rank of $G$ is the minimal cardinality of a generating set, and is denoted by $\text{rk}(G)$. If $G_j$ is a finite index subgroup of $G$, the Reidemeister-Schreier process ([LS]) gives an upper bound on the rank of $G_j$.

$$\text{rk}(G_j) - 1 \leq [G : G_j](\text{rk}(G) - 1)$$

Given a finitely generated group $G$ and a collection $\{G_j\}$ of finite index subgroups, the rank gradient of the pair $(G, \{G_j\})$ is defined by (see [La1])

$$\text{rgr}(G, \{G_j\}) = \lim_{j \to \infty} \frac{\text{rk}(G_j) - 1}{[G : G_j]}$$

We say that the collection of finite index subgroups $\{G_j\}$ is co-final if $\bigcap_j G_j = \{1\}$, and we call it a tower if $G_{j+1} < G_j$. Note that if $\{G_j\}$ is a tower then the sequence $\left\{ \frac{\text{rk}(G_j) - 1}{[G : G_j]} \right\}$ is non-increasing and therefore converges.

An important line of research in low dimensional topology is the study of the behavior of the topology, geometry and algebra of the finite sheeted covers of a finite volume hyperbolic 3-manifold $M$. One problem of particular interest is the behavior of the rank of the fundamental groups of the finite
sheeted covers of such manifolds. In some particular cases it is easy to determine the rank gradient for families of finite covers, but in general this is a very hard problem. For instance

**Question 1.** *Does there exist an orientable finite volume hyperbolic 3-manifold* $M$ *with a co-final family of covers* $\{M_j\}$ *such that* $\text{rgr}(\pi_1(M_1), \pi_1(M_j)) > 0$?

This is the main focus of this thesis. In Chapters 2 and 3 we provide what seems to be the first examples of orientable finite volume hyperbolic 3-manifolds which have co-final towers of finite sheeted covers with positive rank gradient. The manifolds we provide are those whose fundamental group have finite index in the group of reflections of certain hyperbolic right-angled polyhedra. In Chapter 4 we relate our results to other outstanding problems in 3-manifold topology.

### 1.1 Preliminary material

#### 1.1.1 Examples for rank gradient

In some particular cases it is easy to determine rank gradient, for example:

**Example 1.** When $G$ is a free group, the rank gradient of any pair $(G, \{G_j\})$ is positive. This follows since the Reidemeister-Schreier process produces an equality for free groups.

**Example 2.** The same is true if $G$ is the fundamental group of a closed surface $S$ with $\chi(S) < 0$. Let $S$ be a surface with genus $g$ such that $\chi(S) = 2 - 2g < 0$. Note that $\text{rk}(\pi_1(S)) = 2g$. Let $S' \rightarrow S$ a covering of degree $d$ and assume
\( S' \) has genus \( g' \) so that \( \text{rk}(\pi_1(S')) = 2g' \). From \( \chi(S') = d \cdot \chi(S) \) we get
\[ 2g' = d(2g) - 2d + 2. \]
Since \( 2g > 2 \) we see that the number of generators of the fundamental groups of such surfaces grows linearly with their genera.

**Example 3.** If \( \phi : G \twoheadrightarrow F_2 \), where \( F_2 \) is the free group on two generators then, using example 1, one can find a families of subgroups with positive rank gradient. These families are given by the finite index subgroups \( G_n < G \) such that \( \phi|_{G_n} : G_n \twoheadrightarrow F_n \), where \( F_n \) is the free group on \( n \) letters. We remark that these families are not co-final, as \( \ker(\phi) \) is a subgroup of each \( G_n \).

**Example 4.** A group \( G \) is called virtually abelian if it has a finite index abelian subgroup. Let \( H < G \) be a finite index abelian subgroup of rank \( h \). If \( \{H_i\} \) is a tower of finite index subgroups of \( H \), then \( \text{rk}(H_i) \leq h \). It is then easy to see that the pair \( (G, \{H_i\}) \) has zero rank gradient.

**Example 5.** A 3-manifold \( M \) is called virtually fibered if it has a fibered finite sheeted cover \( M' \). This means that
\[ M' \cong \frac{S \times [0, 1]}{(x, 0) \sim (\phi(x), 1)} \]
where \( S \) is the fiber and \( \phi : S \longrightarrow S \) is an orientation preserving homeomorphism. We see that \( \pi_1(M') \) is a HNN-extension of \( \pi_1(S) \) and thus \( \text{rk}(\pi_1(M')) \leq \text{rk}(\pi_1(S)) + 1 \). Choosing \( \{M'_i \longrightarrow M'\} \) to be a tower of covers dual to the fiber \( S \), i.e., surface bundles associated to \( \phi^p \), we see that \( \text{rk}(\pi_1(M')) \leq \text{rk}(\pi_1(S)) + 1 \) as well. Therefore the pair \( (\pi_1(M), \{\pi_1(M_i)\}) \) has zero rank gradient.

**Example 6.** For each \( k \in \mathbb{Z} \), consider the reduction map \( \text{SL}(n, \mathbb{Z}) \longrightarrow \text{SL}(n, \mathbb{Z}/k\mathbb{Z}) \). The kernel of this map is a *congruence subgroup* of \( \text{SL}(n, \mathbb{Z}) \).
When $n > 2$ then $\text{SL}(n, \mathbb{Z})$, has zero rank gradient with respect to towers of congruence subgroups (J. Tits [Ti]).

1.1.2 Hyperbolic geometry

We review some basic facts and terminology from hyperbolic geometry. For more details about hyperbolic space and its isometries see [Ra].

*Hyperbolic space*

The hyperbolic space is defined by

$$\mathbb{H}^3 = \{ (x, y, z) \in \mathbb{R}^3 | z > 0 \}$$

with metric

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2}$$

We will also often use the *Poincaré conformal ball model* for $\mathbb{H}^3$. This is useful for visualizing hyperbolic polyhedra. It is defined as

$$\mathbb{B}^3 = \{ (x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 < 1 \}$$

with metric

$$ds^2 = 4\frac{dw^2}{(1 - w^2)^2}$$

where $w^2 = x^2 + y^2 + z^2$ and $dw^2 = dx^2 + dy^2 + dz^2$.

With this metric, the geodesic planes in $\mathbb{H}^3$ correspond to vertical Euclidean planes and hemispheres in $\{ z > 0 \}$ perpendicular to the plane $\{ z = 0 \}$.
Geodesic lines correspond to the intersection of two geodesic planes, i.e., vertical lines and semi-circles perpendicular to \( \{ z = 0 \} \).

In the conformal ball model \( \mathbb{B}^3 \), planes correspond to the intersection of \( \mathbb{B}^3 \) with spheres and planes in \( \mathbb{R}^3 \) perpendicular to \( \partial \mathbb{B}^3 \). Geodesic lines are obtained as the intersection of geodesic planes. Therefore they correspond to circular arcs and lines perpendicular to \( \partial \mathbb{B}^3 \).

The group of orientation preserving isometries of \( \mathbb{H}^3 \) can be identified to \( \text{PSL}_2(\mathbb{C}) \). The elements \( \gamma \in \text{PSL}_2(\mathbb{C}) \) are classified as \textit{elliptic}, \textit{parabolic} or \textit{hyperbolic} according to the traces of their lifts to \( \text{SL}_2(\mathbb{C}) \). Denote a lift of \( \gamma \) to \( \text{SL}_2(\mathbb{C}) \) by \( \gamma' \): \( \gamma \) is elliptic if \( |\text{tr}(\gamma')| < 2 \), parabolic if \( |\text{tr}(\gamma')| = 2 \) and hyperbolic if \( |\text{tr}(\gamma')| > 2 \).

1.1.3 Discrete groups and fundamental domains

Here we discuss some basic facts about discrete groups of isometries of \( \mathbb{H}^3 \). We remark that all the theorems below hold for hyperbolic, Euclidean or spherical spaces of any dimension. For a more detailed treatment of discrete groups refer to [Ra].

**Definition 1.** A \textit{discrete group} is a topological group \( \Gamma \) whose points are open.

**Definition 2.** A group \( G \) acts \textit{discontinuously} on \( \mathbb{H}^3 \) if and only if \( G \) acts on \( \mathbb{H}^3 \) and for each compact subset \( K \) of \( \mathbb{H}^3 \), the set \( K \cap gK \) is nonempty for only finitely many \( g \) in \( G \).

The main point of these definitions is that in our context they are equivalent (see [Ra]):
Theorem 1.1.1. A group $\Gamma$ of isometries of $\mathbb{H}^3$ is discrete if and only if $\Gamma$ acts discontinuously on $\mathbb{H}^3$.

A subset $R$ of $\mathbb{H}^3$ is called a fundamental region for a group $\Gamma$ of isometries of $\mathbb{H}^3$ if

1. the set $R$ is open;
2. the members of $\{gR|g \in \Gamma\}$ are mutually disjoint; and
3. $\mathbb{H}^3 = \bigcup\{g\bar{R}|g \in \Gamma\}$.

When $\Gamma$ is a discrete group of isometries of $\mathbb{H}^3$, a convex fundamental polyhedron for $\Gamma$ is a convex polyhedron $P$ in $\mathbb{H}^3$ whose interior is a fundamental domain for $\Gamma$. $P$ is called exact if for each side $S$ of $P$ there is an element $g$ of $\Gamma$ such that $S = P \cap gP$. It is known that every discrete group $\Gamma$ has an exact convex fundamental polyhedron. The element $g$ is called a side-pairing.

The main result we need concerning convex fundamental polyhedra for discrete groups is (see [Ra])

Theorem 1.1.2. Let $S$ be a side of an exact convex polyhedron for a discrete group $\Gamma$ of isometries of $\mathbb{H}^3$. Then there is a unique element $g \in \Gamma$ such that $S = P \cap gP$. Moreover, $g^{-1}S$ is also a side of $P$ and $\Gamma$ is generated by the set

$$\Phi = \{g \in \Gamma|P \cap gP, \text{ is a side of } P\}$$
1.1.4 Reflection groups

We now discuss reflection groups of convex polyhedra in $\mathbb{H}^3$. Again, all the results below hold for hyperbolic, Euclidean or spherical spaces of any dimension. For a more details on reflections groups please refer to [Ra].

Let $S$ be a side of an $n$-dimensional convex polyhedron $P$ in $\mathbb{H}^3$. The reflection of $\mathbb{H}^3$ in the side $S$ is the reflection of $\mathbb{H}^3$ in the hyperplane $\langle S \rangle$ spanned by $S$. The group $G$ generated by reflections of $\mathbb{H}^3$ in the sides of $P$ is called reflection group of $P$.

**Theorem 1.1.3.** Let $G$ be the reflection group of a convex polyhedron $P$ in $\mathbb{H}^3$ of finite volume. Then

$$\mathbb{H}^3 = \{gP | g \in G\}$$

Let $P$ be an exact convex fundamental polyhedron for a discrete group $\Gamma$ of isometries of $\mathbb{H}^3$. Then for each side $S$ of $P$, there is a unique element $g$ such that $S = P \cap gP$. We say $\Gamma$ is a discrete reflection group with respect to $P$ when $g$ is the reflection in the hyperplane $\langle S \rangle$.

Our main interest is in discrete reflection groups. These are very common, as shown on the theorem below (see [Ra]).

**Theorem 1.1.4.** Let $P$ be a finite sided convex polyhedron in $\mathbb{H}^3$ of finite volume all of whose dihedral angles are submultiples of $\pi$. Then the group $\Gamma$ generated by reflections in the sides of $P$ is a discrete reflection group with respect to the polyhedron $P$. 
1.1.5 Combinatorial description of hyperbolic right-angled polyhedra

In this section we provided a combinatorial description of right-angled hyperbolic polyhedra. This is given by Andreev’s Theorem and is one of the main tools in the proof of our main results.

An abstract polyhedron $P_1$ is a cell complex on $S^2$ which can be realized by a convex Euclidean polyhedron. A labeling of $P_1$ is a map

$$\Theta : \text{Edges}(P_1) \longrightarrow (0, \pi/2]$$

The pair $(P_1, \Theta)$ is a labeled abstract polyhedron. A labeled abstract polyhedron is said to be realizable as a hyperbolic polyhedron if there exists a hyperbolic polyhedron $P_1$ such that there is a label preserving graph isomorphism between the 1-skeleton of $P_1$ with edges labeled by dihedral angles and the 1-skeleton of $P_1$ with edges labeled by $\Theta$.

By a right-angled polyhedron we mean a polyhedron whose all of its dihedral angles are $\pi/2$. Let $P_1$ be a totally geodesic right-angled polyhedron in $\mathbb{H}^3$ (that is, faces of $P_1$ are contained in hyperplanes). We call a vertex of $P_1$ ideal if it lies in the boundary at infinity $S^2_\infty$, where we here we consider the ball model for $\mathbb{H}^3$.

We consider the 1-skeleton of $P_1$ as a graph $\Gamma_1 \subset S^2$ with labels $\theta_e = \pi/2$. Let $\Gamma_1^*$ be its dual graph, i.e., vertices of $\Gamma_1^*$ correspond to faces of $P_1$ and two vertices are joined by an edge if their corresponding faces in $P_1$ share a common edge. A $k$-circuit is a simple closed curve composed of $k$ edges in $\Gamma_1^*$. A prismatic $k$-circuit is a $k$-circuit $\gamma$ so that no two edges of $\Gamma_1$ which
correspond to edges traversed by $\gamma$ share a vertex. Andreev’s theorem for right-angled polyhedra in $\mathbb{H}^3$ ([An], see also [At]) can be stated as:

**Theorem 1.1.5** (Andreev). Let $\mathcal{P}_1$ be an abstract polyhedron. Then $\mathcal{P}_1$ is realizable as a hyperbolic right-angled polyhedron $P_1$ if and only if

1. $P_1$ has at least 6 faces;
2. Vertices have valence 3 or 4;
3. For any triple of faces of $P_1$, $(f_i, f_j, f_k)$, such that $f_i \cap f_j$ and $f_j \cap f_k$ are edges of $P_1$ with distinct endpoints, $f_i \cap f_k = \emptyset$;
4. There are no prismatic 4-circuits.

Furthermore, each vertex of valence 3 in $P_1$ corresponds to a finite vertex in $\mathcal{P}_1$, each vertex of valence 4 in $P_1$ corresponds to an ideal vertex in $\mathcal{P}_1$, and the realization is unique up to isometry.

### 1.1.6 Kleinian groups, hyperbolic manifolds and orbifolds

By a *Kleinian group* $\Gamma$ we mean a discrete subgroup of $PSL_2(\mathbb{C})$. This is equivalent to saying that the action of $\Gamma$ in $\mathbb{H}^3$ is properly discontinuous (see section 1.1.3 for definition).

When $\Gamma$ is torsion free (i.e., has no non-trivial elements of finite order) the quotient $M_\Gamma = \mathbb{H}^3/\Gamma$ is a hyperbolic 3-manifold with metric induced from $\mathbb{H}^3$. When $\Gamma$ has torsion elements we call the quotient, $O_\Gamma = \mathbb{H}^3/\Gamma$, a hyperbolic orbifold.
When $M_\Gamma$ has finite volume, then it is the interior of a compact manifold with (possibly empty) toroidal boundary. Each of these toroidal components correspond to a cusp of $M_\Gamma$, where a cusp is topologically of the form $T^2 \times [0, \infty)$. These are obtained as the quotient of a set of the form $B = \{(x, y, z)|z > 1\}$ by a subgroup of $\Gamma$ isomorphic to $\mathbb{Z} \times \mathbb{Z}$, consisting of parabolic elements.

In the case that $\Gamma$ contains elliptic elements, the quotient of their fixed point set is called singular locus of the orbifold $O_\Gamma$. When $\Gamma$ is finitely generated, Selberg’s lemma (see [Ra], page 331) implies that every finite volume orbifold $O_\Gamma$ has a finite sheeted covering $M \rightarrow O_\Gamma$, where $M$ is a manifold.

We will say a Kleinian group $\Gamma$ has finite co-volume if the corresponding manifold or orbifold has finite volume.
Chapter 2

Main theorem

The contents of this chapter are mostly those that appear in [Gi].

If $M_1$ is an orientable finite volume hyperbolic 3-manifold, we call the family of covers $\{M_j \rightarrow M_1\}$ co-final (resp. a tower) if $\{\pi_1(M_j)\}$ is co-final (resp. a tower). By rank gradient of the the pair $(M_1, \{M_j\})$, rgr$(M_1, \{M_j\})$, we mean the rank gradient of $(\pi_1(M_1), \{\pi_1(M_j)\})$.

Our main result is:

**Theorem 2.0.6.** Let $M_1$ be an orientable finite volume hyperbolic 3-manifold whose fundamental group has finite index in the reflection group of a right-angled ideal polyhedron $P_1$ in $\mathbb{H}^3$. Then there exists a co-final tower of finite sheeted covers $\{M_j \rightarrow M\}$ for which rgr$(M_1, \{M_j\}) > 0$.

The main idea of the proof of Theorem 2.0.6 is as follows: given $P_1$ as in the theorem, construct a collection of polyhedra $\{P_j\}$ whose reflection groups have finite index $2^{j-1}$ in the reflection group of $P_1$. If one is given an orientable hyperbolic 3-manifold $M_1$ whose fundamental group has finite index in the reflection group of $P_1$ then $M_1$ has at least as many cusps as the number of vertices of $P_1$. We may find manifold covers $M_j \rightarrow M_1$ so that $M_j$ is a
2j−1-sheeted covering and has at least as many cusps as the number of ideal vertices of \( P_j \). We then show that the \( P_j \) can be chosen so that the number of its vertices is of the same magnitude as \( 2^j \).

This chapter will be organized as follows: In section 2.1 we use the characterization of right-angled ideal polyhedra given by Andreev’s theorem ([An]) to show how the construction of the family \( \{P_j\} \) will be done. In section 2.2 we prove Theorem 2.0.6. In section 2.3 we prove all the technical results we need to estimate \( \text{rk}(\pi_1(M_j)) \). In section 2.4 we show how to construct \( \{P_j\} \) so that the family \( \{M_j\} \) is co-final. The idea for this appears in [Ag] (Theorem 2.2) and we include a proof here for completeness.

2.1 Construction of the family \( \{P_j\} \)

Andreev’s theorem implies that, in the present setting, the 1-skeleton of \( P_1 \) is a 4-valent graph. The faces can therefore be checkerboard colored. Reflecting \( P_1 \) along a face \( f_1 \) gives a polyhedron \( P_2 \) which is also right-angled, ideal and totally geodesic with checkerboard colored faces (see figure below). We construct a sequence of polyhedra \( P_1, P_2, \ldots, P_j, \ldots \) recursively, whereby \( P_{j+1} \) is obtained from \( P_j \) by reflection along a face \( f_j \). The faces of \( P_{j+1} \) are colored accordingly with the coloring of the faces of \( P_j \).

The notation for the remainder of this work is as follows: the number of vertices in the face \( f_j \) is denoted by \( S_{f_j} \) and \( \phi_{f_j} \) denotes the reflection along \( f_j \). \( B_j \) and \( W_j \) represent the maximal number of ideal vertices on a black or white face of the polyhedron \( P_j \), respectively. \( V_j \) denotes the total number of
vertices on $P_j$.

Throughout, the construction of the polyhedra $P_j$ will be done in an alternating fashion with respect to the color of the faces: $P_{2j}$ is obtained from $P_{2j-1}$ by reflection along a black face and $P_{2j+1}$ is obtained from $P_{2j}$ by reflection along a white face.

### 2.2 The Proof (construction of $\{M_j\}$)

Our construction of the family $\{M_j\}$ was inspired by the proof of Theorem 2.2 of Agol’s paper [Ag]. The proof that this family can be made co-final is given in section 2.4 (following [Ag]).

**Proof of Theorem 2.0.6.** Consider the family of polyhedra $\{P_j\}$ obtained from $P_1$ as described above. Denote by $G_j$ the reflection group of $P_j$ and observe that $G_{j+1}$ is a subgroup of $G_j$ of index 2. $G_1$ acts on $\mathbb{H}^3$ with fundamental domain $P_1$. The orbifold $\mathbb{H}^3/G_1$ is non-orientable, and may be viewed as $P_1$ with its faces mirrored. The singular locus is the 2-skeleton of $P_1$. Each ideal
vertex of $P_1$ corresponds to a cusp of $\mathbb{H}^3/G_1$.

Let $M_1$ be an orientable cusped hyperbolic 3-manifold such that $\pi_1(M_1)$ has finite index in $G_1$. Let $M_j \to M_1$ be the cover of $M_1$ whose fundamental group is $\pi_1(M_j) = \pi_1(M_1) \cap G_j$. Since $[G_j : G_{j+1}] = 2$, we must have $[\pi_1(M_j) : \pi_1(M_{j+1})] \leq 2$. Also note that since $\text{vol}(P_j) = 2^{j-1}\text{vol}(P_1)$, for all but finitely many $j$ (at most $[G_1 : \pi_1(M_1)]$) we must have $[\pi_1(M_j) : \pi_1(M_{j+1})] = 2$. We may thus assume that $[\pi_1(M_j) : \pi_1(M_{j+1})] = 2$. By mirroring the faces of $P_j$, it may be regarded as a non-orientable finite volume orbifold (as described before). This implies that $M_j \to P_j$ is an orientable finite sheeted cover for $j = 1, 2, \ldots$.

Note that $[\pi_1(M_1) : \pi_1(M_j)] = 2^{j-1}$. Thus to show that the family $\{M_j \to M_1\}$ has positive rank gradient we will establish that $\text{rk}(\pi_1(M_j))$ grows with the same magnitude as $2^j$.

By “half lives half dies” lemma (see [Ha], Theorem 3.5), an easy lower bound on the rank of the fundamental group of an orientable finite volume hyperbolic 3-manifold is the number of its cusps. Since the cusps of $P_j$ correspond to its ideal vertices and the number of cusps does not go down under finite sheeted covers, it must be that $M_j$ has at least as many cusps as the number of ideal vertices of $P_j$.

Recall that $B_j$ and $W_j$ are the maximal number of ideal vertices on a black or white face of the polyhedron $P_j$, respectively, and $V_j$ is the total number of vertices on $P_j$. The claims below (proved in section 2.3) give us the estimates we need for $V_j$ in terms of $V_1$, $B_1$ and $W_1$. 

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Claim 1. \( V_1 \geq B_1 + W_1 - 1 \)

Claim 2. For any \( j \geq 6 \),

\[ V_j \geq 2^{j-1}V_1 - 2^{j-1}(B_1 + W_1) + 2^{j-1} + 2^{j-2} \]

Given these, we argue as follows:

\[
gr(M_1, \{M_j\}) = \lim_{j \to \infty} \frac{\text{rk}(\pi_1(M_j)) - 1}{[\pi_1(M_1) : \pi_1(M_j)]} \geq
\]

\[
\lim_{j \to \infty} \frac{V_j - 1}{2^{j-1}} \geq \lim_{j \to \infty} \frac{2^{j-1}V_1 - 2^{j-1}(B_1 + W_1) + 2^{j-1} + 2^{j-2} - 1}{2^{j-1}} \geq
\]

\[
\lim_{j \to \infty} \frac{2^{j-1}(B_1 + W_1 - 1) - 2^{j-1}(B_1 + W_1) + 2^{j-1} + 2^{j-2} - 1}{2^{j-1}} \geq
\]

\[
\lim_{j \to \infty} \frac{2^{j-2} - 1}{2^{j-1}} = \frac{1}{2}
\]

which proves the theorem. \( \Box \)

2.3 Lower bounds on the number of ideal vertices of \( P_j \)

We now proceed to prove Claims 1 and 2. This requires several preliminary results.

Lemma 2.3.1. Let \( P_{j+1} \) be obtained from \( P_j \) by reflection along a face \( f_j \).

Then \( V_{j+1} = 2V_j - S_{f_j} \).
Proof. Here we abuse notation and write \( v \in f_j \) if \( v \) is an ideal vertex of the face \( f_j \) and write \( v \not\in f_j \) otherwise. Note that if \( v \not\in f_j \), then \( v \) yields two vertices on \( P_{j+1} \), namely, \( v \) and \( \phi_{f_j}(v) \). If \( v \in f_j \), then it yields a single vertex (\( v \) itself).

If \( v \not\in f_j \), then, by the observation above, \( v \) yields two ideal vertices on \( P_{j+1} \). Since a total of \( S_{f_j} \) ideal vertices lie in \( f_j \) and \( V_j - S_{f_j} \) do not, it must be that that

\[
V_{j+1} = 2(V_j - S_{f_j}) + S_{f_j} = 2V_j - S_{f_j}
\]

Recall also that the construction of the family of polyhedra \( \{ P_j \} \) is made in an alternating fashion with respect to the color of the faces: \( P_{2j} \) is obtained from \( P_{2j-1} \) by reflection along a black face and \( P_{2j+1} \) is obtained from \( P_{2j} \) by reflection along a white face.

**Corollary 2.3.2.** For \( j \geq 1 \)

1. \( V_{2j} \geq 2V_{2j-1} - B_{2j-1} \)
2. \( V_{2j+1} \geq 2V_{2j} - W_{2j} \)

Proof. \( P_{2j} \) is obtained from \( P_{2j-1} \) by reflection along a black face \( f_{2j-1} \), thus \( S_{f_{2j-1}} \leq B_{2j-1} \). By the lemma, \( V_{2j} = 2V_{2j-1} - S_{f_{2j-1}} \) and therefore \( V_{2j} \geq 2V_{2j-1} - B_{2j-1} \). The second inequality is similar.

With the notation established above we now find lower bounds for the \( V_j \) in terms of \( V_1, B_1 \) and \( W_1 \). First we need to find upper bounds for \( B_j \) and
$W_j$ in terms of $B_1$ and $W_1$. To do this in a way that will fit our purposes we establish two properties of the family $\{P_j\}$. As before, denote by $\phi_{f_j}$ the reflection along the face $f_j$.

**Lemma 2.3.3.** (1) If $P_j$ is reflected along a white (resp. black) face $f_j$, all black faces $f_*$ (resp. white faces $f_*$) adjacent to $f_j$ yield new black faces $\tilde{f}_*$ (resp. white faces $\tilde{f}_*$) on $P_{j+1}$. The number $S_{f_*}$ (resp. $S_{\tilde{f}_*}$) of ideal vertices on $\tilde{f}_*$ (resp. $\tilde{f}_*$) is $2S_{f_*} - 2$ (resp. $2S_{\tilde{f}_*} - 2$).

(2) A face $f_*$ not adjacent to $f_j$ yield two new faces, $f_*$ itself and $\phi_f(f_*)$, both with $S_{f_*}$ vertices.

**Proof.** For the first property, reflecting $f_*$ along $f_j$ gives a face $\phi_{f_j}(f_*)$ in $P_{j+1}$ adjacent to $f_*$. The dihedral angle between $f_*$ and $\phi_f(f_*)$ is $\pi$. Thus, on $P_{j+1}$, they correspond to a single face denoted by $\tilde{f}_*$. The number of ideal vertices on $\tilde{f}_*$ is exactly $2S_{f_*} - 2$. The second property should be clear. See figure 1 for an illustration of these properties.

As an immediate consequence we have

**Corollary 2.3.4.**

(1) \[
\begin{cases}
B_{2j} = B_{2j-1} \\
W_{2j} \leq 2W_{2j-1} - 2
\end{cases}
\]

(2) \[
\begin{cases}
B_{2j+1} \leq 2B_{2j} - 2 \\
W_{2j+1} = W_{2j}
\end{cases}
\]

We are now in position to estimate the values $B_j$ and $W_j$ in terms of $B_1$ and $W_1$. 17
Theorem 2.3.5. With the notation as before we have

\[(1) \quad W_{2j+1} = W_{2j} \leq 2^j W_1 - \sum_{l=1}^{j} 2^l \]

\[(2) \quad B_{2j+2} = B_{2j+1} \leq 2^j B_1 - \sum_{l=1}^{j} 2^l \]

Proof. We proceed by induction. By corollary 2.3.4 these statements are true for \( j = 1 \). Suppose it is also true for \( j \leq n \). We now want to estimate \( B_{2n+3} = B_{2n+4} \) and \( W_{2n+2} = W_{2n+3} \). The hypothesis is that

\[W_{2j+1} = W_{2j} \leq 2^n W_1 - \sum_{l=1}^{n} 2^l \]

\[B_{2n+2} = B_{2n+1} \leq 2^n B_1 - \sum_{l=1}^{n} 2^l \]

\( P_{2n+2} \) is obtained from \( P_{2n+1} \) by reflection along a black face, denoted by \( f \). White faces on \( P_{2n+1} \) adjacent to \( f \) yield new white faces on \( P_{2n+2} \) with at most \( 2W_{2n+1} - 2 \) vertices, by Corollary 2.3.4. But

\[2W_{2n+1} - 2 \leq 2[2^n W_1 - \sum_{l=1}^{n} 2^l] - 2 = 2^{(n+1)} W_1 - \sum_{l=1}^{n+1} 2^l \]

which gives the desired result for \( W_{2n+2} \) and \( W_{2n+3} \). Finally, \( P_{2n+3} \) is obtained from \( P_{2n+2} \) by a reflection along a white face, again denoted by \( f \). Since black faces of \( P_{2n+2} \) have at most \( B_{2n+2} (= B_{2n+1}) \) vertices, black faces of \( P_{2n+3} \) will have at most \( 2B_{2n+1} - 2 \) vertices, again by corollary 2.3.4. But

\[2B_{2n+1} - 2 \leq 2[2^n B_1 - \sum_{l=1}^{n} 2^l] - 2 = 2^{(n+1)} B_1 - \sum_{l=1}^{n+1} 2^l \]

vertices. This establishes the result for \( B_{2n+3} \) and \( B_{2n+4} \). \( \square \)
Theorem 2.3.6. With the notation as before, and for \( j \geq 3 \),

\[
(1) \quad V_{2j} \geq 2^{2j-1}V_1 - B_1 \sum_{l=j-1}^{2j-2} 2^l - W_1 \sum_{l=j}^{2j-1} 2^l + \sum_{l=j+2}^{2j-1} 2^l + 2^j + 2
\]

\[
(2) \quad V_{2j+1} \geq 2^{2j}V_1 - B_1 \sum_{l=j}^{2j-1} 2^l - W_1 \sum_{l=j+2}^{2j-1} 2^l + \sum_{l=j+2}^{2j} 2^l + 2
\]

Proof. Lower bounds estimates for \( V_1, \ldots, V_7 \) are found recursively. \( V_1, V_2, V_3, V_4 \) and \( V_5 \) do not fit these formulas but \( V_6 \) and \( V_7 \) do. The statement is then true for \( j = 3 \). We now proceed by induction, using the previous proposition and corollary 2.3.2. Suppose it is true for \( j \leq n, n \geq 3 \). We want to show this implies true for \( j = n + 1 \). By corollary 2.3.2, \( V_{2n+2} \geq 2V_{2n+1} - B_{2n+1} \). The hypothesis is that

\[
V_{2n+1} \geq 2^{2n}V_1 - B_1 \sum_{l=n}^{2n-1} 2^l - W_1 \sum_{l=n}^{2n-1} 2^l + \sum_{l=n+2}^{2n} 2^l + 2
\]

We also know that

\[
B_{2n+1} \leq 2^n B_1 - \sum_{l=1}^{n} 2^l
\]

Thus

\[
V_{2n+2} \geq 2V_{2n+1} - B_{2n+1} \geq
\]

\[
2[2^{2n}V_1 - B_1 \sum_{l=n}^{2n-1} 2^l - W_1 \sum_{l=n}^{2n-1} 2^l + \sum_{l=n+2}^{2n} 2^l + 2] - [2^n B_1 - \sum_{l=1}^{n} 2^l] =
\]

\[
2^{2n+1}V_1 - B_1 \sum_{l=n}^{2n-1} 2^{l+1} - W_1 \sum_{l=n}^{2n-1} 2^{l+1} + \sum_{l=n+2}^{2n} 2^{l+1} + 2^2 + \sum_{l=1}^{n} 2^l =
\]
\[2^{2n+1}V_1 - B_1 \sum_{l=n}^{2n} 2^l - W_1 \sum_{l=n+1}^{2n} 2^l + \sum_{l=n+3}^{2n+1} 2^l + 2 + 2 \]

which establishes (1) for \(2(n + 1) = 2n + 2\).

We use the exact same idea to and the estimate for \(V_{2n+2}\) to establish (2) for \(2(n + 1) + 1 = 2n + 3\). \qed

**Corollary 2.3.7.** For any \(j \geq 6\),

\[V_j \geq 2^{j-1}V_1 - 2^{j-1}(B_1 + W_1) + 2^{j-1} + 2^{j-2}\]

Hence Claim 2 in the proof of Theorem 2.0.6 is proved. We now prove

**Claim 1.** \(V_1 \geq B_1 + W_1 - 1\)

*Proof.* Let \(f_b\) and \(f_w\) be black and white faces of \(P_1\) with maximal number of vertices, i.e., \(S_{f_b} = B_1\) and \(S_{f_w} = W_1\).

**Case 1:** The faces \(f_b\) and \(f_w\) are not adjacent

Here we get \(V_1 \geq B_1 + W_1\) and the claim follows.

**Case 2:** The faces \(f_b\) and \(f_w\) are adjacent.

Since \(f_b\) and \(f_w\) share exactly 2 vertices we see that \(V_1 \geq B_1 + W_1 - 2\).

Suppose we have equality. Then every vertex of \(P_1\) must be a vertex of either \(f_b\) or \(f_w\). Recall that we can visualize the 1-skeleton of \(P_1\) as lying in \(S^2\). Label the vertices of \(P_1\) by \(\{v_1, ..., v_k\}\). The assumption is that all these vertices lie in the boundary of the disk \(D = (f_b \cup f_w) \subset S^2\). By Andreev’s theorem (refer
back to 1.1.5), $P_1$ has at least 6 faces, every face is at least 3-sided and all vertices are 4-valent. Denoting by $F_1$ and $E_1$ the number of faces and edges of $P_1$ respectively we have the relation $V_1 - E_1 + F_1 = 2$. Since vertices are 4-valent we also have $E_1 = 2V_1$. From these relations and $F_1 \geq 6$, we get $V_1 \geq 4$. At two of the vertices, say $v_1$ and $v_2$, three of the emanating edges lie in $D$ and one does not. Denote the ones that do not lie in $D$ by $e_1$ and $e_2$, respectively. At all other $v_i$ we have two edges that lie in $D$ and two that do not. Denote the latter by $e_i, e'_i$. We have a total of $2(k - 2) + 2 = 2k - 2$ edges not in $D$. The problem we have now is combinatorial:

**Proposition 2.3.8.** Consider the disk $D' = S^2 - D$ and the points $v_1, \ldots, v_k \in \partial D'$, $k \geq 4$. Then it is not possible to subdivide $D'$ by $2k - 2$ edges in a way that exactly one edge emanates from both $v_1$ and $v_2$ and exactly two edges emanate from $v_3, \ldots, v_k$ in such a way that no pair of edges intersect and every face on the subdivision of $D'$ is at least 3-sided (here we also consider sides coming from the boundary).

This completes the proof of the claim.

**Proof of Proposition.** Orient the boundary of $D'$ counterclockwise. Starting at $v_1$, draw the edge $e_1$ emanating from it. The other endpoint of $e_1$ is some vertex $v_{i_1}$. Consider the vertices contained in the segment $[v_1, v_{i_1}] \subset \partial D'$ in the given orientation. If there are no vertices at all, then we must have a 2-sided face, which is not possible. Therefore, by relabeling, we may assume $v_2$ is the first vertex between $v_1$ and $v_{i_1}$. Observe that the edges emanating from $v_2$ are trapped between the edge $e_1$ and $\partial D'$. Draw an edge $e_2$ emanating from
v_2 \text{ with the second endpoint } v_{i_2}. \text{ It must be that } v_{i_2} \text{ also lies in } [v_1, v_{i_1}], \text{ or else we find a pair of intersecting edges. As above, there must be a vertex in the segment } [v_2, v_{i_2}]. \text{ By repeating the above argument eventually we find a 2-sided face, which is not possible. Therefore it must be that } V_1 > B_1 + W_1 - 2. \ 

\section{2.4 Co-finalness}

In this section we provide a way of choosing the black or white faces on the polyhedra \( P_j \) along which it is reflected in such a way that the resulting family \( \{M_j\} \) of manifolds is cofinal. The main result of this section, Theorem 2.4.1, appears as part of the proof of Theorem 2.2 of [Ag]. We include a proof here for completeness. To better describe this construction we need to change notation slightly by adding another index.

Start with \( P_1 \) and relabel it \( P_{11} \). Reflect along a black face \( f_{11} \) obtaining \( P_{12} \). Let \( \phi_{f_{11}} \) represent such reflection. Observe that if \( f \) is adjacent to \( f_{11} \), then \( f \cup \phi_{f_{11}}(f) \) corresponds to a single face on \( P_{12} \). We call \( f \) and \( \phi_{f_{11}}(f) \) subfaces of \( f \cup \phi_{f_{11}}(f) \). Next reflect \( P_{12} \) along a white face \( f_{12} \), which is also a face of \( P_{11} \) or contains a face of \( P_{11} \) as a subface, obtaining \( P_{13} \). We construct a subcollection \( P_{11}, ..., P_{1k_1} \) of polyhedra such that

(i) If \( P_{1j} \) is obtained from \( P_{1(j-1)} \) by reflection along a white (black) face then \( P_{1(j+1)} \) is obtained from \( P_{1j} \) by reflection along a black (white) face.

(ii) Whenever possible, the face \( f_{1j} \) must be a face of \( P_{11} \) or contain a face of \( P_{11} \) as a subface.
(iii) No faces of $P_{11}$ are subfaces of $P_{1k_1}$.

Now set $P_{1k_1} := P_{21}$. Suppose $P_{n1}$ has been constructed. Construct the subcollection of polyhedra $P_{n1}, ..., P_{nk_n}$ such that

(i) The reflections were performed in a alternating fashion with respect to the color of the faces;

(ii) Whenever possible, the face $f_{nj}$ must be a face of $P_{n1}$ or contain a face of $P_{n1}$ as a subface.

(iii) No faces of $P_{n1}$ are subfaces of $P_{nk_n}$.

Now set $P_{nk_n} := P_{(n+1)1}$. Inductively we obtain a collection of polyhedra $P_{11}, P_{12}, ..., P_{1k_1} := P_{21}, ..., P_{2k_2} := P_{31}, ..., P_{nk_n} := P_{(n+1)1}, ...$

satisfying (i), (ii) and (iii) above.

Let $G_{ij}$ be the reflection group of $P_{ij}$ and let $M_{ij}$ be the cover of $M_{11}$ whose fundamental group is $\pi_1(M_{ij}) = \pi_1(M_{11}) \cap G_{ij}$. Co-finalness of the family $\{M_{ij} \rightarrow M_{11}\}$ is an immediate consequence of

**Theorem 2.4.1.** Let $G_{ij}$ be as above. Then $\cap_{ij} G_{ij} = \{1\}$.

In order to prove this theorem we consider the base point for the fundamental group of each $P_{ij}$ (viewed as orbifolds with their faces mirrored) to be the barycenter $x_0$ of $P_{11}$.
Figure 2.2: Construction of the family \( \{P_{ij}\} \)

**Proof of Theorem.** Set \( R_{ij} = \inf_{\gamma} \{ \ell(\gamma) \} \), where \( \gamma \) is an arc with endpoints in faces (possibly edges) of \( P_{ij} \) going through \( x_0 \). Note that, by construction, 
\[
\lim_{i \to \infty} R_{ij} = \infty.
\]
For a non-trivial element \( g \in G_{11} \) set \( R_g = \inf_{[\alpha]=g} \{ \ell(\alpha) \} \), where \( \alpha \) is a loop in \( P_{11} \) based at \( x_0 \) and \( [\alpha] \) represents its homotopy class. Let \( \alpha_g \) be a loop in \( P_{11} \) based at \( x_0 \) such that \( [\alpha_g] = g \) and \( \ell(\alpha_g) \leq R_g + 1 \).

We claim that for sufficiently large \( i \) one cannot have \( g \in G_{ij} \). In fact,
if $\alpha_{ij}$ is any loop in $P_{ij}$ based at $x_0$, then this loop bounces off faces of $P_{ij}$, yielding an arc $\gamma_{ij}$ through $x_0$. Therefore $\ell(\alpha_{ij}) \geq \ell(\gamma_{ij}) \geq R_{ij}$. Since covering maps preserve length of curves, this implies that if $i$ is large enough no such $\alpha_{ij}$ maps to $\alpha_g$. Thus it is not possible to find a loop representative for $g$ in $P_{ij}$. □

2.5 Remarks

2.5.1 A related theorem

An important result, closely related to our work is the following

**Theorem 2.5.1.** Let $M$ be a virtually fibered oriented hyperbolic 3-manifold of finite volume. Then there exists a co-final tower of regular finite sheeted covers $\{M_j' \to M\}$ such that $rgr(M, \{M_j'\}) = 0$.

For the proof of this theorem we need the notion of **residual finiteness**.

**Definition 3.** A group $G$ is residually finite if the intersection of all its subgroups of finite index is trivial.

It is known that the fundamental group of a 3-manifold is residually finite (see [He]).

The following lemma will also be used.

**Lemma 2.5.2.** Let $G$ be a finitely generated group, $\{G_j\}$ and $\{H_j\}$ be two collections of finite index subgroups such that $H_j < G_j$ and $[G_j : H_j] < \infty$. Then $rgr(G, \{H_j\}) \leq rgr(G, \{G_j\})$.  

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Proof. Just note that,
\[
    \text{rgr}(G, \{H_j\}) = \lim_{j \to \infty} \frac{\text{rk}(H_j) - 1}{[G : H_j]} \leq \lim_{j \to \infty} \frac{[G : H_j](\text{rk}(G_j) - 1)}{[G : H_j]}
\]
\[
    = \lim_{j \to \infty} \frac{[G_j : H_j](\text{rk}(G_j) - 1)}{[G : G_j][G_j : H_j]} = \lim_{j \to \infty} \frac{\text{rk}(G_j) - 1}{[G : G_j]} = \text{rgr}(G, \{G_j\})
\]

Proof of theorem. If \( M \) is virtually fibered then it is possible to find a tower of finite sheeted covers \( \{\Gamma_j\} \) with \( \text{rgr}(\pi_1(M), \{\Gamma_j\}) = 0 \) (refer back to section 1.1.1). Consider the core of \( \Gamma_j \) in \( \pi_1(M) \) (i.e., \( \text{core}(\Gamma_j) = \cap_{g \in \pi_1(M)} g\Gamma_j g^{-1} \)). Since \( \Gamma_j \) is a finite index subgroup, there are only finitely many of its conjugacy classes in \( \pi_1(M) \) and thus \( [\Gamma_j : \text{core}(\Gamma_j)] < \infty \). The above lemma implies that the tower of normal subgroups \( \{\text{core}(\Gamma_j)\} \) also has zero rank gradient. This tower may not be co-final. Using residual finiteness we get a co-final tower \( \{\tilde{\Gamma}_j\} \) of finite index subgroups of \( \pi_1(M) \). Another application of the above lemma will give us the desired co-final tower with zero rank gradient. It is given by the covers \( \{M_j' \to M\} \) corresponding to the subgroups \( \{\tilde{\Gamma}_j \cap \text{core}(\Gamma_j)\} \).

Remark 1. By the work of Agol ([Ag]), if \( M_1 \) is as in Theorem 2.0.6 then it virtually fibers. The above theorem shows that these manifolds also have towers with zero rank gradient. We discuss this further in Chapter 4.

2.5.2 Examples with large rank gradient

It is also easy to give examples of families \( \{M_j \to M_1\} \) with arbitrarily large rank gradient. Using the methods above it suffices to provide examples of polyhedra \( P_1 \) for which the difference \( V_1 - (B_1 + W_1) \) is arbitrarily large.
Below we illustrate some cases in which this happens: consider the right-angled ideal polyhedron $P_0$ pictured below, viewed as lying in $S^2$.

\[ \text{Figure 2.3: Polyhedron } P_0 \]

Note that, by Andreev's theorem, this polyhedron can be realized as a totally geodesic right-angled ideal polyhedron in $\mathbb{H}^3$. Reflecting $P_0$ along the white face containing the point at infinity of $S^2$ will give us a polyhedron $P_1$. Since $P_1$ is obtained from two copies of $P_0$ by gluing together the white faces containing the point at infinity, we have a maximum of 6 ideal vertices per white face of $P_1$ and a maximum of 4 per black faces. Obviously this construction can be made so that $P_1$ has arbitrarily many ideal vertices. Thus, given any $C > 0$ we may find $P_1$ such that for the family $\{M_j \to M_1\}$ as above

\[
\lim_{j \to \infty} \frac{\text{rk}(\pi_1(M_j)) - 1}{[\pi_1(M_1) : \pi_1(M_j)]} \geq \lim_{j \to \infty} \frac{2^{j-1}(V_1 - (B_1 + W_1)) - 1}{2^{j-1}} > C
\]

2.5.3 Number of vertices versus volume

There is a strong relationship between the volume of a (ideal) right-angled polyhedron and its number of vertices. Let $O$ denote the right-angled ideal
octahedron. Atkinson ([At]) proved that if $P$ is a right-angled ideal polyhedron with $V$ ideal vertices then

$$ (V - 2) \cdot \frac{\text{vol}(O)}{4} \leq \text{vol}(P) \leq (V - 4) \frac{\text{vol}(O)}{2} $$

A key ingredient in the proof of theorem 2.0.6 was the fact that the number of ideal vertices of the polyhedra in the family $\{P_j\}$ grows with magnitude $2^j$. One may at first suspect that the growth we need follows directly from Atkinson’s inequality. Note, however, this is not the case.
In this chapter we provide a result similar to that of Theorem 2.0.6 but for orbifolds arising as compact hyperbolic right-angled polyhedra. Recall that all vertices in these polyhedra are 3-valent. Therefore such a polyhedron can have its faces 4-colored. Here we denote these colors by black (B), white (W), red (R) and yellow (Y). For a collection \( \{P_j\} \) of polyhedra, denote by \( B_j, W_j, R_j, W_j \) the maximum number of vertices in a black, white, red or yellow face of \( P_j \) respectively. Again, let \( G_j \) denote the reflection group of \( P_j \). Let \( r_j \) denote the number of faces of \( P_j \). It is not hard to see that \( \text{rk}(G_j) = r_j \). In fact, \( H_1(G_j, \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^{r_j} \). This gives \( \text{rk}(G_j) \geq r_j \). Now observe that \( G_j \) is generated by reflections along faces of \( P_j \). Thus we also have \( \text{rk}(G_j) \leq r_j \).

For a compact hyperbolic right-angled polyhedron \( P \), let \( v, e \) and \( f \) denote its number of vertices, edges and faces respectively. By Andreev’s theorem (refer back to Theorem 1.1.5), all vertices are 3 valent. Since the boundary of this polyhedron is topologically a sphere, an Euler characteristic argument gives us

\[
f = \frac{1}{2}v + 2
\]
and therefore, in order to estimate the rank of the reflection group of such polyhedra, one only needs to estimate their number of vertices.

### 3.1 A theorem for compact polyhedra

**Theorem 3.1.1.** Let $P_1$ be a compact hyperbolic right-angled polyhedra, $G_1$ its reflection group and $V_1$ the number of vertices in $P_1$. If $2V_1 \geq 2(B_1 + W_1 + R_1 + Y_1)$ then there exists a co-final tower of finite index subgroups $\{G_j\}$ of $G_1$ such that $\text{rgr}(G_1\{G_j\}) > 0$.

An example of a such polyhedron is obtained from a compact hyperbolic dodecahedron $D$ (see Figure 3.1 below). $D$ has 20 vertices and all the faces have 5 vertices. Therefore $D$ does not satisfy the hypothesis of the theorem. However, we can obtain a polyhedron $P_1$ from $D$ by performing certain reflections.

![Hyperbolic right-angled dodecahedron](image)

**Figure 3.1:** Hyperbolic right-angled dodecahedron.
Another way to visualize $D$ is as an abstract polyhedron on $S^2$, which we identify with the extended plane.

Figure 3.2: Dodecahedron visualized in the extended plane.

Let $f_1$ be the central face of $D$ (as seen in figure 3.2). Reflect $D$ along $f_1$ obtaining a polyhedron $P_0$.

Figure 3.3: Left: dodecahedron $D$. Right: reflect along central face obtaining $P_0$.

$P_0$ has 30 vertices and all faces have either 5 or 6 vertices (this follows
from Lemmas 3.2.1 and 3.2.2 below). Thus this polyhedron does not satisfy
the hypothesis of the theorem either.

Finally reflect $P_0$ along the outer face (corresponding to the unbounded
region of the plane) obtaining $P_1$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{polyhedra.png}
\caption{Left: polyhedron $P_0$. Right: reflect along outer face obtaining $P_1$.}
\end{figure}

$P_1$ has 50 vertices and again all the faces have either 5 or 6 vertices.
Therefore, no matter how one colors this polyhedron, the coloring will satisfy
the hypothesis of the theorem.

The proof of this theorem is similar to the proof of Theorem 2.0.6. The
groups $G_j$ arise as reflection groups of polyhedra $P_j$ obtained from $P_1$ by the
same type of construction as in the case of polyhedron with all vertices ideal.
Here we perform reflections in an alternating fashion, but now with respect
to the colors black, white, red and yellow, in this order. Recall that in the
proof of Theorem 2.0.6 a key point was Claim 2. The corresponding result for
compact polyhedra is
Lemma 3.1.2. For $n \geq 1$ we have

$$V_{4n+1} \geq 2^{4n} V_1 - 2^{4n+1}(B_1 + W_1 + R_1 + Y_1) + 2^{4n+2}$$

We omit the technical results needed for this lemma for now and proceed to the proof of Theorem 3.1.1:

Proof of Theorem 3.1.1. From the observation above, it suffices to show that the number of vertices the polyhedra in the $\{P_{4j+1}\}$ grows linearly with the degree $[G_1 : G_{4j+1}] = 2^{4n}$. We have

$$\lim_{j \to \infty} \frac{V_{4j+1}}{2^{4j}} \geq \lim_{j \to \infty} \frac{2^{4n} V_1 - 2^{4n+1}(B_1 + W_1 + R_1 + Y_1) + 2^{4n+2}}{2^{4j}} \geq$$

If $2V_1 \geq 2(B_1 + W_1 + R_1 + Y_1)$ then

$$\lim_{j \to \infty} \frac{2^{4n} V_1 - 2^{4n+1}(B_1 + W_1 + R_1 + Y_1) + 2^{4n+2}}{2^{4j}} \geq$$

$$\lim_{j \to \infty} \frac{2^{4j+2}}{2^{4j}} = 4$$

which proves that this family has positive rank gradient.

The proof that this family can be made cofinal is the same as the proof of Theorem 5.

Remark 2. Without the requirement $V_1 \geq 2(B_1 + W_1 + R_1 + Y_1)$ the construction in the proof of Theorem 3.1.1 does not work for general compact right-angled polyhedra. Consider for instance the right-angled Euclidean cube in $\mathbb{R}^3$. No matter how one performs reflections, in each step on the construction of $\{P_j\}$ we have a polyhedron (a parallelepiped) with exactly 6 faces and 8 vertices.
3.2 Lower bounds on the number of vertices

The key ingredients in the proof of Lemma 3.1.2 are the lemmas below. Recall that we perform reflections in a alternating fashion with respect to the colors black, white, red and yellow.

Using the fact that we perform reflections in a alternating fashion with respect to the colors black, white, red and yellow, in this order, we obtain

**Lemma 3.2.1.**

\[
\begin{align*}
V_{4j+1} &\geq 2V_{4j} - 2Y_{4j} \\
V_{4j+2} &\geq 2V_{4j+1} - 2B_{4j+1} \\
V_{4j+3} &\geq 2V_{4j+2} - 2W_{4j+2} \\
V_{4(j+1)} &\geq 2V_{4j+3} - 2R_{4j+3}
\end{align*}
\]

**Proof.** Given the polyhedron \( P_k \), note that when we perform reflection along a face \( f_k \), each vertex not in \( f_k \) generates two new vertices in \( P_{k+1} \). Vertices in \( f_k \) do not yield any new vertices in \( P_{k+1} \). Depending on the color of the face the reflection is performed, we have at most \( B_k, W_k, R_k \) or \( Y_k \) vertices in such a face. The inequalities follow easily from these observations.

Since we build the family of polyhedra in an alternating fashion with respect to the 4 colors, we also obtain

**Lemma 3.2.2.** For \( j \geq 1 \) we have

\[
\begin{align*}
B_{4j+1} &\leq 2^{3j} B_1 - (2^{3j} + \cdots + 2^2 + 2) \\
W_{4j+1} &\leq 2^{3j} W_1 - (2^{3j} + \cdots + 2^2 + 2) \\
R_{4j+1} &\leq 2^{3j} R_1 - (2^{3j} + \cdots + 2^2 + 2) \\
Y_{4j+1} &= Y_{4j} \leq 2^{3j} Y_1 - (2^{3j} + \cdots + 2^2 + 2)
\end{align*}
\]
\begin{align}
\begin{cases}
B_{4j+2} = B_{4j+1} & \leq 2^{3j} B_1 - (2^{3j} + \cdots + 2^2 + 2) \\
W_{4j+2} & \leq 2^{3j+1} W_1 - (2^{3j+1} + \cdots + 2^2 + 2) \\
R_{4j+2} & \leq 2^{3j+1} R_1 - (2^{3j+1} + \cdots + 2^2 + 2) \\
Y_{4j+2} & \leq 2^{3j+1} Y_1 - (2^{3j+1} + \cdots + 2^2 + 2)
\end{cases}
\tag{2}
\end{align}

\begin{align}
\begin{cases}
B_{4j+3} & \leq 2^{3j+1} B_1 - (2^{3j+1} + \cdots + 2^2 + 2) \\
W_{4j+3} = W_{4j+2} & \leq 2^{3j+1} W_1 - (2^{3j+1} + \cdots + 2^2 + 2) \\
R_{4j+3} & \leq 2^{3j+2} R_1 - (2^{3j+2} + \cdots + 2^2 + 2) \\
Y_{4j+3} & \leq 2^{3j+2} Y_1 - (2^{3j+2} + \cdots + 2^2 + 2)
\end{cases}
\tag{3}
\end{align}

\begin{align}
\begin{cases}
B_{4(j+1)} & \leq 2^{3j+2} B_1 - (2^{3j+2} + \cdots + 2^2 + 2) \\
W_{4(j+1)} & \leq 2^{3j+2} W_1 - (2^{3j+2} + \cdots + 2^2 + 2) \\
R_{4(j+1)} = R_{4j+3} & \leq 2^{3j+2} R_1 - (2^{3j+2} + \cdots + 2^2 + 2) \\
Y_{4(j+1)} & \leq 2^{3j+2} Y_1 - (2^{3j+2} + \cdots + 2^2 + 2)
\end{cases}
\tag{4}
\end{align}

Proof. The arguments here are very similar to those of Corollary 2.3.4 and Theorem 2.3.5. The only difference is that, starting with \( P_1 \), we perform reflections in an alternating fashion with respect to the colors black, white, red and yellow, in this order. By construction, one can easily verify all the inequalities above for \( j = 1 \).

Suppose they hold for \( j = k \). The last set of inequalities for \( j = k \) is
\begin{align}
\begin{cases}
B_{4(k+1)} & \leq 2^{3k+2} B_1 - (2^{3k+2} + \cdots + 2^2 + 2) \\
W_{4(k+1)} & \leq 2^{3k+2} W_1 - (2^{3k+2} + \cdots + 2^2 + 2) \\
R_{4(k+1)} = R_{4j+3} & \leq 2^{3k+2} R_1 - (2^{3k+2} + \cdots + 2^2 + 2) \\
Y_{4(k+1)} & \leq 2^{3k+2} Y_1 - (2^{3k+2} + \cdots + 2^2 + 2)
\end{cases}
\end{align}

We now reflect \( P_{4(k+1)} \) along a yellow face, obtaining \( P_{4(k+1)+1} \). We have
\begin{align}
\begin{cases}
B_{4(k+1)+1} & \leq 2B_{4(k+1)} - 2 \leq 2[2^{3k+2} B_1 - (2^{3k+2} + \cdots + 2^2 + 2)] \\
W_{4(k+1)+1} & \leq 2W_{4(k+1)} - 2 \leq 2[2^{3k+2} W_1 - (2^{3k+2} + \cdots + 2^2 + 2)] \\
R_{4(k+1)+1} & \leq 2R_{4(k+1)} - 2 \leq 2[2^{3k+2} R_1 - (2^{3k+2} + \cdots + 2^2 + 2)] \\
Y_{4(k+1)+1} = Y_{4(k+1)} & \leq 2^{3k+3} Y_1 - (2^{3k+3} + \cdots + 2^2 + 2)
\end{cases}
\end{align}
which gives the first inequality of the lemma for $j = k + 1$. The other three inequalities are obtained similarly by reflecting along black, white and yellow faces.

Lemma 3.1.2 follows directly from the following inequality:

**Lemma 3.2.3.** Set $S = B_1 + W_1 + R_1 + Y_1$. Then, for $n \geq 1$, we have

$$V_{4n+1} \geq 2^{4n}V_1 - S \sum_{j=0}^{n-1} 2^{4n-j} + 2^{4n+2}$$

**Proof.** One easily finds

$$V_5 \geq 2^4V_1 - 2^4S + 2^6$$

Therefore the statement is true for $n = 1$. Suppose now it is true for $n = k$. We wish to use induction and Lemmas 3.2.2 and 3.2.1 to prove it is true for $n = k + 1$.

Induction hypothesis gives

$$V_{4k+1} \geq 2^{4k}V_1 - S \sum_{j=0}^{k-1} 2^{4k-j} + 2^{4k+2}$$

and Lemmas 3.2.2 and 3.2.1 give

$$V_{4k+2} \geq 2V_{4k+1} - 2B_{4k+1}$$

$$\geq 2[2^{4k}V_1 - S \sum_{j=0}^{k-1} 2^{4k-j} + 2^{4k+2}] - 2[2^{3k}B_1 - \sum_{j=1}^{3k} 2^j]$$

$$= 2^{4k+1}V_1 - S \sum_{j=0}^{k-1} 2^{4k-j+1} + 2^{4k+3} - 2^{3k+1}B_1 + \sum_{j=1}^{3k} 2^{j+1}$$
The above estimate and the lemmas give

\[ V_{4k+3} \geq 2V_{4k+2} - 2W_{4k+2} \]

\[ \geq 2[2^{4k+1}V_1 - S \sum_{j=0}^{k-1} 2^{4k-j+1} + 2^{4k+3} - 2^{3k+1}B_1 + \sum_{j=1}^{3k} 2^{j+1}] - 2[2^{3k+1}W_1 - \sum_{j=1}^{3k+1} 2^j] \]

\[ = 2^{4k+2}V_1 - S \sum_{j=0}^{k-1} 2^{4k-j+2} + 2^{4k+4} - 2^{3k+2}(B_1 + W_1) + \sum_{j=1}^{3k} 2^{j+2} + \sum_{j=1}^{3k+1} 2^{j+1} \]

\[ \geq 2^{4k+2}V_1 - S \sum_{j=0}^{k-1} 2^{4k-j+2} + 2^{4k+4} - 2^{3k+2}(B_1 + W_1) \]

Again, the estimates above and the lemmas give

\[ V_{4(k+1)} = V_{4k+4} \geq 2V_{4k+3} - 2R_{4k+3} \]

\[ V_{4k+1} \geq 2[2^{4k+2}V_1 - S \sum_{j=0}^{k-1} 2^{4k-j+2} + 2^{4k+4} - 2^{3k+2}(B_1 + W_1)] - 2[2^{3k+2}R_1 - \sum_{j=1}^{3k+2} 2^j] \]

\[ \geq 2^{4k+3}V_1 - S \sum_{j=0}^{k-1} 2^{4k-j+3} + 2^{4k+5} - 2^{3k+3}(B_1 + W_1 + R_1) + \sum_{j=1}^{3k+2} 2^{j+1} \]

\[ \geq 2^{4k+3}V_1 - S \sum_{j=0}^{k-1} 2^{4k-j+3} + 2^{4k+5} - 2^{3k+3}(B_1 + W_1 + R_1) \]

Finally

\[ V_{4(k+1)+1} \geq 2V_{4(k+1)} - 2Y_{4(k+1)} \]

\[ \geq 2[2^{4k+3}V_1 - S \sum_{j=0}^{k-1} 2^{4k-j+3} + 2^{4k+5} - 2^{3k+3}(B_1 + W_1 + R_1)] - 2[2^{3(k+1)}Y_1 - \sum_{j=1}^{3(k+1)} 2^j] \]

\[ \geq 2^{4k+4}V_1 - S \sum_{j=0}^{k-1} 2^{4k-j+4} + 2^{4k+6} - 2^{3(k+1)+1}(B_1 + W_1 + R_1 + Y_1) + \sum_{j=1}^{3(k+1)} 2^{j+1} \]

\[ \geq 2^{4(k+1)}V_1 - S \sum_{j=0}^{k-1} 2^{4(k+1)-j} + 2^{4(k+1)+2} - 2^{3(k+1)+1}S \]
which is the desired inequality for $n = k + 1$.

3.3 Remarks

3.3.1 Number of vertices versus volume

Atkinson ([At]) estimates the volume of a hyperbolic compact right-angled polyhedra in terms of its number of vertices. Let $O$ denote the right-angled ideal octahedron, $T$ denote the ideal tetrahedron, $P$ a hyperbolic compact right-angled polyhedron and $V$ the number of vertices in $P$. Then

$$(V - 8)\frac{\text{vol}(O)}{32} \leq \text{vol}(P) \leq (V - 10)\frac{5 \cdot \text{vol}(T)}{8}$$

Again one sees that our estimates for the growth of the number of vertices in the family $\{P_j\}$ do not follow from Atkinson’s estimate.

3.3.2 Further generalizations

It should be clear that a similar result as that of theorems 2.0.6 and 3.1.1 exists for general hyperbolic right-angled polyhedra (those with both types of vertices). The faces of such polyhedra can be 4-colored. Informally, we may regard compact polyhedra as the worst case scenario one may have and ideal polyhedron as the best. Note that, by reflecting a polyhedron $P_j$ along a face $f_j$, all ideal vertices of $f_j$ will be vertices of $P_{j+1}$ and all non-ideal vertices disappear. Estimates for the number of vertices in a family $\{P_j\}$ of such polyhedra should therefore lie between the estimates for the families of ideal and compact polyhedra.
Chapter 4

Final Remarks

4.1 Heegaard genus and Heegaard gradient

Here we summarize what we need on Heegaard splittings. For details see [Sc].

A handlebody is a 3-manifold with boundary constructed as follows: begin with the 3-ball $B^3$ and in its boundary pick out two disjoint 2-disks $D_0$ and $D_1$. Using these disks, attach to $B^3$ a handle, that is, a copy of $D^2 \times I$, by identifying $D^2 \times \{i\}$ with $D_i$, $i = 0, 1$. One can continue to add more handles in a similar way. The result of adding $g$ handles is a handlebody of genus $g$. Note that $g$ is precisely the genus of the boundary surface.

Every closed 3-manifold $M$ admits a decomposition into two handlebodies, i.e., there exist handlebodies $H_1, H_2$ such that $M$ is obtained by attaching $H_1$ to $H_2$ by a homeomorphism of their boundaries (this implies that $H_1$ and $H_2$ have the same genus). Write $M = H_1 \cup_S H_2$, where $S$ is the surface $\partial H_i$.

There is an analogous decomposition for 3-manifolds with boundary. A compression body is a connected 3-manifold obtained from a closed surface (not necessarily connected), denoted by $\partial_- H$. Consider $\partial_- H \times I$. Attach 1-handles to $\partial_- H \times \{1\}$ in such a way that we obtain a connected manifold.
The resulting manifold is called a compression body. Let \( H \) be a compression body. Its boundary is subdivided as \( \partial_- H \) and the remaining is denoted \( \partial_+ H \). The genus of \( H \) is the genus of \( \partial_+ H \).

Every 3-manifold with boundary \( M \) can be decomposed into two compression bodies, i.e., there exists compression bodies \( H_1, H_2 \) such that \( M \) is obtained by attaching \( H_1 \) to \( H_2 \) by a homeomorphism of their boundary \( \partial_+ H_i \). It is interesting to notice that, given any partition of \( \partial M \) as the disjoint union of two sets of connected components \( \partial M_1 \) and \( \partial M_2 \), we find a decomposition of \( M \) into compression bodies \( H_1, H_2 \) such that \( \partial_- H_i = \partial M_i \).

It is conventional to consider a handlebody as a compression body in which \( \partial_- H = \emptyset \).

This type of decomposition of a 3-manifold is called a Heegaard decomposition. The Heegaard genus of \( M \), \( \text{Hg}(M) \), is defined as \( \min_{S} \{ g(S) \} \), where \( S = \partial_+ H \) and \( H \) is a compression body (or handlebody) in a decomposition of \( M \). The surfaces \( S \) is called a Heegaard surfaces for \( M \). The disks \( D \) in the 1-handles \( D \times I \) of a compression body \( H \) are called meridian disks. A collection of meridian disks is called complete if each of its complementary components is either a 3-ball or \( \partial_- H \times I \).

If we restrict ourselves to Heegaard decompositions in which one of the compression bodies is a handlebody then we can obtain upper bounds for \( \text{rk}(\pi_1(M)) \) in terms of \( \text{Hg}(M) \). To see this consider a minimal genus Heegaard decomposition for \( M \), say, \( M = H_1 \cup_S H_2 \), where \( H_1 \) is a handlebody and \( g(S) = s \). It is obvious that \( \pi_1(H_1) \) is a free group on \( s \) letters. The attaching
map of \( \partial_+ H_1 \) and \( \partial_+ H_2 \) provides us with a presentation for \( \pi_1(M) \) as follows: consider a complete disk system \( \{D_1, ..., D_k\} \) for \( H_2 \). Each curve \( \partial D_j \), when viewed as (a conjugacy class) in \( \pi_1(H_1) \) is a relator for the fundamental group. \( \pi_1(M) \) has a presentation given by

\[
\langle x_1, ..., x_r \mid r_1, ..., r_k \rangle
\]

where the \( x_j \) are the generators for \( \pi_1(H_1) \) and the \( r_i \) are the relators described above.

Figure 4.1: (a) Manifolds without boundary; (b) Manifolds with boundary

In 1960’s Waldhausen ([Wa]) asked whether or not \( \text{rk}(\pi_1(M)) = \text{Hg}(M) \).

In the early 1980’s Boileau-Zieschang ([BZ]) provided the first examples where strict inequality holds. These examples were Seifert fibered spaces. Other examples of closed 3-manifolds where strict inequality holds were given by
Schultens-Weidmann ([SW]). This problem remained opened until very recently for hyperbolic manifolds. Tao Li has recently announced counter-examples for closed hyperbolic 3-manifolds ([Li]). However, the exact relationship between rank of fundamental groups and Heegaard genus (of finite volume hyperbolic 3-manifolds) still remains unknown. For instance

**Question 2.** Is there a universal constant \( C \) such that if \( M \) is a finite volume hyperbolic 3-manifold then \( \text{Hg}(M) \leq C \cdot \text{rk}(\pi_1(M)) \)?

Another concept due to Lackenby is that of *Heegaard gradient* ([La2]). Given a orientable 3-manifold \( M \) and a family \( \{M_j\} \) of finite sheeted covers, we define the Heegaard grandient of \( \{M_j \rightarrow M\} \) by

\[
\text{Hgr}(M, \{M_j\}) = \lim_{j \to \infty} \frac{-\chi(S_j)}{d_j}
\]

where \( d_j \) is the degree of the cover \( M_j \rightarrow M \) and \( S_j \) is a minimal genus Heegaard surface for \( M_j \). By the above discussion on Heegaard genus we have that if \( \text{rgr}(M, \{M_j\}) > 0 \), then \( \text{Hgr}(M, \{M_j\}) > 0 \).

In [LLR] Long–Lubotzky–Reid prove that every orientable finite volume hyperbolic 3-manifold \( M \) has a co-final family of regular finite sheeted covers \( \{M_j\} \) for which the Heegaard gradient is positive.

An important open problem for finite volume hyperbolic 3-manifolds associated to the work of Long–Lubotzky–Reid is

**Question 3** (Rank vs. Heegaard gradient). Let \( M \) be a finite volume hyperbolic 3-manifold and \( \{M_i \rightarrow M\} \) a family of finite sheeted covers. Is it true that \( \text{rgr}(M, \{M_i\}) > 0 \) if and only if \( \text{Hgr}(M, \{M_i\}) > 0 \)?
These were also motivation for our work. Our results provide examples for which this question has positive answer.

4.2 Relation to the cost of group actions

Our work also relates to the work of Abért and Nikolov ([AN]), and in particular to a question about cost of group actions ([Ga]). For a more general treatment refer to [AN] and [Fa].

**Question 4.** Let $G$ be a finitely generated group and $\{G_j\}$ be a co-final tower of finite index normal subgroups of $G$. Does $\operatorname{rgf}(G, \{G_j\})$ depend on the tower $\{G_j\}$?

Given the result of Theorem 2.0.6 one may naturally ask

**Question 5.** Is it possible, in the setting of Theorem 2.0.6, to obtain a co-final tower of regular covers $\{M_j \to M_1\}$ with positive rank gradient?

A positive answer to this would be very relevant, as it implies that Question 4 has a positive answer. However, the tower constructed in Theorem 2.0.6 cannot consist of normal subgroups. To see this we argue as follows: as a particular case of the main theorem in [Ma] we have

**Theorem 4.2.1.** Let $P$ be a finite volume right-angled polyhedron in $\mathbb{H}^3$ and let $G$ be its reflection group. Then $\operatorname{injrad}(G) < \cosh^{-1}(7) = 2.6639...$, where $\operatorname{injrad}(G)$ denotes half of the shortest translation length among hyperbolic elements of $G$. 

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Therefore we can find a sequence \( \{ \gamma_j \} \) of hyperbolic elements, \( \gamma_j \in G_j \), whose translation lengths are bounded above by 2.634. Since there exist at most finitely many conjugacy classes of hyperbolic elements of bounded translation length in \( G_1 \), it must be that an infinite subsequence \( \{ \gamma_{j_k} \} \) lie in the same conjugacy class in \( G_1 \). Let \( \gamma \) be a representative of this class and \( g_{j_k} \in G_1 \) be such that \( \gamma_{j_k} = g_{j_k} \gamma g_{j_k}^{-1} \). If the tower \( \{ G_j \} \) consists of normal subgroups, then \( \gamma \in G_{j_k} \) for all \( j_k \), contradicting the fact that \( \{ G_{j_k} \} \) is co-final.

These covers are actually far from being normal: Lück Approximation Theorem ([Lu1]) implies these covers do not even satisfy a weaker condition (called Farber). See [Fa] for details.

Question 5 is relevant also because of the following result (see [AN]):

**Theorem** (Abért-Nikolov). *If Question 4 has a negative answer then both the Rank vs. Heegaard gradient problem (see Question 3) and Question 2 above have a negative answers.*

### 4.3 More on right-angled polyhedra

As remarked before, the proof of Theorem 3.1.1 does not apply to an Euclidean cube. Questions that arises naturally are:

**Question 6.** *Can the estimates in Lemma 3.1.2 be improved so that Theorem 3.1.1 is true for every compact hyperbolic polyhedra?*

**Question 7.** *Can one find estimates for general right-angled hyperbolic polyhedra (those possibly having both types of vertices) so that results similar to those of Theorem 3.1.1 are valid for every such polyhedra?*
Let $M_1$ be an orientable finite volume hyperbolic 3-manifold such that $\pi_1(M_1)$ has finite index in the reflection group of a hyperbolic right-angled polyhedron $P_1$.

**Question 8.** Does $M_1$ have a co-final tower of finite sheeted covers $\{M_j \to M_1\}$ such that $rgr(M_1, \{M_j\}) > 0$?

We also expect a much broader generalization of the results in this thesis.

**Conjecture.** Let $M_1$ be an orientable finite volume hyperbolic 3-manifold that decomposes into right-angled polyhedra. Then $M_1$ has a co-final tower of finite sheeted towers $\{M_j \to M_1\}$ such that $rgr(M_1, \{M_j\}) > 0$. 
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