Auto-stabilized Electron

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Abstract
We include effects of self-gravitation in the self-interaction of single electrons with the electromagnetic field. When the effect of gravitation is included there is an inevitable cut-off of the k-vector - the upper limit is finite. The inward pressure of the self-gravitating field balances the outward pressure of self-interaction. Both pressures are generated by self-interactions of the electron with two fields - the vacuum electromagnetic field and the self-induced gravitational field. Specifically we demonstrate that gravitational effects must be included to stabilize the electron. We use the Einstein equation to perform an exact calculation of the bare mass and electron radius. We find a close-form solution. We find the electron radius \( r_e = 9.2\sqrt{\alpha/4\pi\sqrt{\hbar G/c^3}} = 9.2\sqrt{\alpha/4\pi l_P} \approx 10^{-36} m \). \( l_P \) is the Planck length, which is educed from first principles. We find that the electromagnetic and gravitational fields merge at \( (8/3)\sqrt{\alpha/4\pi m_P} = 10^{17} GeV \) in terms of the Planck mass \( m_P \). Renormalisation is accomplished by requiring continuity of the interior and exterior metrics at \( r_e \).

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I. INTRODUCTION

The paradox of electron stability has been recognized since its discovery \[1\]. Instability is inevitable in any system with charge distributed over an extended volume. There have been several conjectures to explain stability (i) a shell of non-electromagnetic origin to contain the field (Poincare’ stress), (ii) using radiation reaction to compensate the outward pressure (Abraham-Lorentz equation), (iii) dimensional regularization etc.

Poincare’s suggestion was to trap the charge in a rigid shell subjecting it to a stress \[2\]. This was found to be untenable.

When radiation reaction is included as in the Abraham-Lorentz equation \[3\], one gets runaway solutions instead of stability.

Yet another suggestion was to treat the space dimension as a free parameter \( d \) (dimensional regularization) \[4\], then use its variation to solve a divergent
integral. Choosing $d=4$ unfortunately leads to a pole, which then must be compensated, raising new issues.

It was recognized that renormalisation was required to compute the self-energy of the electron. In this scheme the measured mass of electrons $m$ was used to terminate the divergence.

With the advent of quantum electrodynamics new schemes were proposed, but even these were partially successful. Although the calculations are well-established, the result is inevitably divergent (albeit logarithmically at best). Various work-arounds have been proposed, among them a modification of electrodynamics at short distances, terminating the integral at the Compton or Planck length etc. We will discuss some of these later.

These approaches have one common motivation: that infinities are a mathematical anomaly which must be side-stepped, removed, truncated or at worst ignored. Needless to say these approaches suffer from deficiencies. Infinities remain because they are intrinsic to the theory.

We claim instead that the appearance of infinities is the consequence of ignoring basic physical phenomena. The infinities are real. In the sections below we describe what these are and calculate the resulting corrected mass from first principles.

A. Computation algorithm

Referring to a free electron, Feynman’s final result is

$$\Delta m = \frac{4\pi e^2}{2mi} \int_{-\infty}^{\infty} \frac{\tilde{u}(2m + 2k)u}{k^2 - 2\vec{p} \cdot \vec{k}} \frac{d^4k}{(2\pi)^4 k^2} \frac{1}{k^2} \tag{1}$$

for the mass correction $\Delta m$ \cite{5} where $m$ is the experimental mass and $e$ the charge of the electron; $k$ is the four-momentum of photons and $p$ is the four-momentum of the electron.

The integral is of the form $d^4k/k^4$ which is intrinsically divergent. Feynman gets around the divergence by modifying the photon kernel to $(1/k^2) c(k^2)$ from $(1/k^2)$. The function $c(k^2)$ is chosen so that $c(0) = 1$ and $c(k^2) \to 0$ as $k^2 \to \infty$. This conjectured change in the photon kernel is a consequence of altering the laws of electrodynamics at short distances. With this alteration the integral is still divergent, albeit logarithmically; a free parameter is introduced. As a consequence the high frequency components of the Fourier expansion or equivalently, the short-range contributions are modified. No reason exists to justify this alteration, since there is no evidence that the laws of electrodynamics are inadequate at short distances. The puzzle remains unsolved.

The diagram Fig. 1 represents the electron emitting and absorbing a single photon

1. Near field

A way out of this dilemma is to re-interpret the one-loop correction and relate it to fields in the vicinity of a classical radiation source. Surrounding


any source radiating at a wavelength \( \lambda \) there is a near zone which extends to \( r = \lambda/2\pi \). In terms of the wave-vector \( \kappa \) the condition that makes this so is
\[
\kappa \cdot r = 1 .
\]
The extent of the near zone scales as the inverse of the wave-vector. Within the near zone the fields are Coulomb-like. The fields exert a radially outward pressure. Outside the near zone, fields acquire a transverse component; they propagate.

Energy is stored within the near zone. It is this energy that shows up as an excess mass in Eq.(1). Energy is continuously exchanged between fields and source in the near zone; pictorially represented as the emission and absorption of single photons Fig.(1). Higher order diagrams correspond to quadrupole, octupole and higher order fields. Appropriately, fields in this zone may be treated as that in the diagram above.

B. Gravitation

We can see how gravity comes into the picture. If we squeeze an electron into a sphere of radius equal to a Compton wavelength (\( \lambda_C \)) the energy density can be estimated. Setting \( r = \lambda_C \)

\[
\frac{e^2}{4\pi\epsilon_0 \left(4\pi/3\right) r^3} = \frac{e^2}{4\pi\epsilon_0} \frac{3}{4\pi} \frac{1}{(\lambda_C)^3} = \frac{e^2}{4\pi\epsilon_0} \frac{3}{4\pi} \left(\frac{mc}{\hbar}\right)^4 = 10^{18} \text{ J/m}^3
\]

which is \( \approx 10^{13} \) atmospheres. By comparison the pressure at the center of the Sun is \( 10^{11} \) atmospheres. This shows that such high energy densities are not just the purview of astrophysical sources but are common among elementary particles.

Clearly under these conditions energy densities are high enough to alter the metric in the vicinity of the electron. At these densities virtual excitations follow geodesics of curved metrics rather than flat space. Virtual excitations loop back to the source along geodesics of the distorted metric. For example in Fig.(1) the emission and absorption of virtual photons occurs in a curved metric. General relativistic effects not only cannot be ignored; they become essential part of the dynamics of the electron. Gravitation is part of the electron.

It is evident that theories that rely on flat space geometry are inadequate; the engendered divergences are evidence that in such theories a major reservoir
of energy is being ignored. The enormous outward forces cannot be balanced in flat space; curved space-time must be included.

There have been attempts to include gravity in QED phenomena. An early example is Isham, Salam and Strathdee, [6] and others.

1. Gravitating electron

In this paper we insert gravitation first by integrating Eq.(1) up to an upper limit for $k$. The upper limit is an unknown for now. We set the momentum

$$k = \hbar \kappa = \frac{2\pi}{\lambda}$$

where $\kappa$ is the wave number. The corresponding near zone radius for $\lambda$ is $r = \lambda / 2\pi$.

Performing the integral Eq.(1) we get for the mass correction (see Appendix)

$$\Delta m \equiv \mu(\eta) = \frac{\alpha m}{2\pi} \left[ -\frac{\eta}{2} \sqrt{1 + \frac{1}{\eta}} + \eta + \ln \left\{ \sqrt{\eta} \left( \eta \sqrt{1 + \frac{1}{\eta}} + 1 \right) \right\} \right]$$

(4)

in terms of a dimensionless variable

$$\eta \equiv \hbar / 2mcr = \lambda C / 4\pi r$$

(5)

We have redefined $\Delta m \equiv \mu(\eta)$.

There is an energy density associated with the self-field within the near zone. The energy density, or the stress tensor, alters the metric within the near zone. The net result is an inward pressure. Analogous to the Sun where the radiation pressure (or Fermi pressure in the case of white dwarfs or neutron stars) is balanced by the inward gravitational pressure.

It is this pressure that compensates the outward pressure from the electron field that stabilises the electron.

Increasing values of the $k$– vector raise the stress tensor which in turn increases the inward gravitation induced pressure. The electron is auto-stabilized.

Equating the two competing pressures yields an upper limit for $k$. We will calculate this limit.

We use the Einstein equation to calculate the resulting Einstein tensor. We start with a line element of the form

$$ds^2 = -e^{2\Phi} c^2 dt^2 + e^{2\Lambda} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

(6)

Fields within the near zone are treated as a perfect fluid. The elements of the stress tensor are

$$T^\hat{0}\hat{0} = \rho, T^\hat{r}\hat{r} = T^\hat{\theta}\hat{\theta} = T^\hat{\phi}\hat{\phi} = P$$

(7)

in the fluid’s orthonormal rest-frame basis vectors. Imposing momentum conservation and spherical symmetry the relevant Einstein equations are

$$G^\hat{0}\hat{0} = -\frac{1}{r^2} \frac{d}{dr} \left[ r \left( 1 - e^{-2\Lambda} \right) \right] = 8\pi T^\hat{0}\hat{0} = 8\pi \rho$$

(8)
\[ G^{\hat{1}\hat{1}} = -\frac{1}{r^2} \frac{d}{dr} \left( 1 - e^{-2\Lambda} \right) + \frac{2}{r} e^{-2\Lambda} \frac{d\Phi}{dr} = 8\pi T^{\hat{1}\hat{1}} = 8\pi P \]  

We define a new metric coefficient \( \mu (r) \) (same as in Eq. (4) as 

\[ g_{11} = e^{2\Lambda} \equiv \frac{1}{1 - \frac{2\mu}{r}} \]  

\( \mu (r) \) is the corrected mass inside the radius \( r \). The time-time component of Eq. (8) takes the form 

\[ \frac{d\mu}{dr} = 4\pi r^2 \rho \]  

whereas radial-radial component of Eq. (9) takes the form 

\[ \frac{d\Phi}{dr} = \frac{\mu + 4\pi r^3 P}{r (r - 2\mu)} \]  

The proper density is 

\[ \rho = \frac{1}{4\pi r^2} \frac{1}{\sqrt{|g_{11}|}} \frac{d\mu}{dr} \]  

Since the volume element 

\[ dV = \sqrt{|g_{11}|} r^2 \sin \theta \ d\theta \ d\phi \ dr \]  

These equations when combined with the condition for hydrostatic equilibrium lead to the Tolman-Oppenheimer-Volkov equation 

\[ -\frac{dP (r)}{dr} = (\rho (r) + P (r)) \left( \frac{\mu (r) + 4\pi r^3 P (r)}{r^2 \left( 1 - \frac{2\mu (r)}{r} \right)} \right) \]  

This first order differential equation can be solved once the relation between pressure \( P (r) \) and density \( \rho (r) \) is known. The negative slope guarantees the decrease of \( P (r) \) until for some value of \( r \), \( P (r) = 0 \). We seek this value of \( r \). 

**C. Equation of state** 

We can derive an equation of state - relate \( P (r) \) with \( \rho (r) \). We use the work equation 

\[ dE = -P \ dV; \quad \frac{dE}{dV} = -P \]  

Also \( E = \rho V \)

\[ \frac{dE}{dV} = V \frac{dho}{dV} + \rho \frac{dV}{dV} = \frac{d\rho}{dV} V + \rho = -P \]  

\[ \frac{d\rho}{dV} = \frac{d\rho}{dr} \frac{dr}{dV} = \frac{d\rho}{dr} \left( \frac{3}{4\pi} \right)^{1/3} V^{-2/3} \]  

\[ \frac{dE}{dV} = \frac{d\rho}{dr} \left( \frac{3}{4\pi} \right)^{1/3} V^{-2/3} V + \rho = -P \]
The near zone fields have an equation of state which is

\[ P = -\rho - r \frac{d\rho}{dr} \tag{21} \]

Or in terms of \( \mu (r) \) the equation of state can be written as

\[ P = \frac{1}{4\pi r^2} \frac{1}{\sqrt{|g_{11}|}} \frac{d\mu}{dr} - \frac{1}{4\pi r} \frac{1}{\sqrt{|g_{11}|}} \frac{d^2\mu}{dr^2} \tag{22} \]

Eq. (15) can be re-written in terms of the variable \( \eta \).

\[ \frac{dP}{d\eta} = \frac{dP}{dr} \frac{dr}{d\eta} \]

We calculate \( P(\eta) \) by integrating \( \frac{dP(\eta)}{d\eta} \)

\[ P(\eta) = \int \frac{dP(\eta)}{d\eta} d\eta \tag{23} \]

and find the root of the integral \( \eta_e \) for which \( P(\eta_e) = 0 \); this is the value of \( \eta \) and thus the lower limit \( r \) or the upper limit \( k \) we are looking for.

D. Results

In order to simplify the computation we will use an approximation where \( \eta \gg 1 \) to find the root of Eq. (15). This allows us to simplify the terms \( \mu(\eta) \), \( \rho(\eta) \) and \( P(\eta) \) (which has \( \frac{\partial \rho}{\partial \eta} \)).

For example for \( \eta \gg 1 \), \( \rho(\eta) \to \frac{n^4}{2} \frac{\partial \rho}{\partial \eta} \to 2\eta^3 \), \( \mu(\eta) \to \frac{\eta}{2} \). For \( \eta \) we get a fourth order algebraic equation with roots, of which one is positive, one negative and two complex. We take the positive root and get

\[ \eta_e = \frac{8}{15} \sqrt{2} \left( \frac{c^2}{2mG} \frac{2\pi}{\alpha} \frac{\hbar}{2mc} \right)^{1/2} = \frac{8}{15} \frac{1}{m} \sqrt{\frac{\pi}{\alpha}} \sqrt{\frac{\hbar c}{G}} \tag{24} \]

where \( P(\eta) = 0 \).

The corresponding electron radius is

\[ r_e = 7.5 \sqrt{\frac{\alpha}{4\pi}} \sqrt{\frac{\hbar G}{c^3}} \tag{25} \]

The electron radius is greater than the Schwarzschild radius \( 2m \). We note that the interior line element develops a singularity in \( g_{11} \) when

\[ 1 - \frac{\alpha mG}{\pi} \frac{2mc \eta^2}{\hbar} \frac{\eta}{2} = 0 \tag{26} \]

That is when

\[ r_0 = \sqrt{\frac{\alpha \hbar G}{4\pi c^3}} \tag{27} \]
also \( r_0 > 2m \). And so the electron radius

\[
 r_e = 7.5r_0 = 2.95 \times 10^{-36}m
\]  

(28)

We observe that \( r_e > r_0 \) - the pressure falls to zero outside the singularity.

Simplifying the equation the radius is

\[
 r_e = 7.5 \sqrt{\frac{G}{c^4}}
\]  

(29)

The radius is independent of the electron mass \( m \) and \( \hbar \) and is entirely in terms of fundamental constants \( e \), \( G \) and \( c \). Since it is a radius independent of the electron mass we re-write the radius as a universal radius \( r_* \) in terms of the Planck length as

\[
 r_* = 7.5 \sqrt{\frac{\alpha}{4\pi}} l_P
\]  

(30)

We note that what is called the Planck length \( l_P \) appears naturally.

If instead we integrate Eq.(23) numerically we find

\[
 r_e = 9.21 \sqrt{\frac{\alpha \hbar G}{4\pi}} = 5.5 \times 10^{-36}m
\]  

(31)

or

\[
 r_e = 9.21 \sqrt{\frac{\alpha}{4\pi}} l_P
\]  

(32)

which is close to Eq.(30).

One may ask if general relativity is valid at such short lengths. The existence of black holes and the Big Bang offer proof that there is no known lower limit to lengths in general relativity.

In terms of energy

\[
 r_* = 10^{20}GeV
\]  

(33)

If we substitute \( \eta_e \) from Eq.(24) into Eq.(4) we get a mass independent of the electron mass. We call it the universal mass \( \mu_* \)

\[
 \mu_* = 8 \sqrt{\frac{\alpha \hbar c G}{4\pi}} = 8 \sqrt{\frac{\alpha}{4\pi}} m_P
\]  

(34)

in terms of the Planck mass \( m_P \). The Planck mass also appears naturally.

Simplifying the result

\[
 \mu_* = \frac{8}{3} \sqrt{\frac{\alpha \hbar c}{4\pi G}}
\]  

(35)

dependent on \( e \) and \( G \) alone and independent of \( \hbar \) and \( c \). Numerically

\[
 \mu_* \approx 10^{17} GeV
\]  

(36)

This result is remarkable for several reasons. Since \( \mu_* \) is enormously larger than the physical mass \( m \) it cannot be the corrected mass. Furthermore \( \mu_* \) is
independent of mass; it is solely in terms of the fundamental constants \(e\) and \(G\). Since it depends on \(e^2\) it is also independent of the sign of the charge. The inescapable conclusion is that the result is a general result applicable to all charged particles irrespective of the mass and sign of the charge.

Although our goal was to derive the corrected electron mass we have found instead a mass that applies to all charged particles. A possible interpretation is that \(\mu_*\) is the universal bare mass.

These observations also apply to the radius \(r_e\) since it too is independent of mass, and is in terms of the fundamental constants \(G, c\) and \(e\).

The value of \(\mu_*\) is close to the GUT energy \((10^{16} \text{ GeV})\) where it is conjectured that all forces except gravity merge. It would appear that all forces, including gravity, merge at \(10^{17} \text{ GeV}\).

Since the unified field energy is independent of \(\hbar\) one may also conclude that the unified field is not quantized but is a continuum. This is a serendipitous result.

The bare mass, is now exact. The integral converges to an exact value. Physical laws remain unaltered. The momentum upper limit is

\[
k^\text{max} = \frac{\hbar}{r_e}
\]  

We reiterate that this is a self-regulating mechanism since if \(k\) exceeds \(k^\text{max}\) it engenders a proportional reaction from the metric such that the system reverts to a state of equilibrium. The equilibrium is stable.

At \(r = 10^{-36} \text{ m}\) where the two competing pressures are equal; the pressures are

\[
10^{70} \text{ m}^{-2}
\]  

in geometric units, or \(10^{114} \text{ N/m}^2\) in standard units. Evidently the electron surface is highly stressed: the pressure is \(\approx 10^{109}\) atmospheres on the surface. By comparison the pressure inside neutron stars is merely \(\approx 10^{30}\) atmospheres.

At this value the outward pressure due to the energy density of self-interaction equals the inward pressure of the curved metric.

Pictorially, the distorted metric in the vicinity of the electron looks like this photo-representation Fig. (2):

The inward pressure is a consequence of the distorted metric.

E. Renormalization

We are left to compute \(g_{00}\) from Eq. (12)

\[
g_{00} = e^{2\Phi}
\]

where

\[
\frac{d\Phi_i}{dr} = \frac{\mu + 4\pi r^3 P}{r^2(1 - 2\mu/r)}
\]  

(39)
Figure 2: Pictorial representation of metric alteration due to self-gravity of self-energy of electron. The dark disk is a stand-in for the electron. The photograph is that of a wire tip immersed in a thin layer of water in a clear glass baking dish. A graph paper is glued to the underside of the dish. Surface tension distorts the water surface.

We calculate the interior metric element $\Phi_i$ by integrating Eq. (39). As expected $\Phi_i = 0$ on the surface defined by the radius $r_e$. Integration also yields a constant $C$. We choose the constant $C$ to match the interior and exterior solutions at the surface. This is how we renormalize the mass. At the surface we require

$$\Phi_e = \Phi_i$$  \hspace{1cm} (40)

The exterior metric is the Schwarzschild solution for an electron of mass $m$. So for $r > r_e$

$$g_{00} = e^{2\Phi_e} = \left(1 - \frac{2m}{r}\right)$$

and for $r \leq r_e$

$$g_{00} = \exp\left(1 - \frac{2m}{r_e}\right) \exp 2\Phi_i$$  \hspace{1cm} (41)

where $\Phi_i$ is obtained from Eq. (39). The same algorithm is used to ensure the continuity of the $g_{11}$ term (Eq. (10)).

Although we have renormalized the metric for electrons, we note that for other particles such as protons or neutrons the same algorithm can be used with the appropriate masses instead.
Although there have been previous attempts to introduce gravitation to remove infinities in the self-energy correction to the electron mass [6] we believe this work demonstrates the existence of an auto-generated curved metric in the vicinity of an electron as well as an explicit computation of the upper limit on the momentum vector. The self-energy integral is shown to be finite; infinities in the integral have been removed. The mass correction has an exact value, free of infinities.

We have demonstrated that gravitation is essential to stabilize the electron. Competing pressures from the electromagnetic and gravitational fields auto-stabilize the electron. We have identified the Poincare shell.

Furthermore instead of using dimensional analysis to derive the Planck length and mass we show that both are a consequence of merging quantum electrodynamics with gravitation. We calculate from first principles the radius of the electron as well as the Planck length and Planck mass. In deriving the electron radius and bare mass we identify the mechanism (inward pressure of the induced metric) that stabilises the electron. The mass correction is close to the GUT scale - an unexpected result. In demanding a smooth transition of the metric across the electron surface we uncover a rigorous mechanism for renormalisation.

The solution is exact; since it is independent of mass it is valid for strong fields as well.

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I. Appendix

Derivation of Eq. (1): start by using the identity: we follow Feynman [5].

\[
\frac{1}{ab} = \int_0^1 \frac{dx}{[ax + b(1 - x)]^2}
\]  

(42)

Define two quantities

\[
k^2 - 2p \cdot k = a; k^2 \equiv b
\]  

(43)
Then
\[
\frac{1}{(k^2 - 2p \cdot k) k^2} = \int_0^1 \frac{dx}{[(k^2 - 2p \cdot k) x + k^2 (1 - x)]^2} = \int_0^1 \frac{dx}{[k^2 - 2p \cdot k x]^2} \tag{44}
\]
So Eq. (1) becomes
\[
\mu = \frac{4 \pi e^2}{2m} \int_{-\infty}^{\infty} \int_0^1 \tilde{u}(2m + 2\tilde{k}) u \, d^4k \, dx \tag{45}
\]
\[
d^4k = d\omega \, d^3\tilde{K} \, k^2 - 2p \cdot k x = \omega^2 - (\tilde{K}^2 + 2p \cdot k x) \tag{46}
\]
Let \(xp \equiv p\) then
\[
\mu = \frac{4 \pi e^2}{2m} \frac{1}{p} \int_{-\infty}^{\infty} \int_0^1 \tilde{u}(2m + 2\tilde{k}) u \, d\omega \, d^3\tilde{K} \, dx \tag{47}
\]
Do the \(\omega\) integral first using residues and for \(\varepsilon \ll L + \tilde{K}^2\):
\[
\int_{-\infty}^{\infty} \frac{d\omega}{\left[\omega^2 - i\varepsilon + (\tilde{K}^2 + 2p \cdot k)\right]^2} = \frac{-2\pi i}{2\sqrt{\tilde{K}^2 + 2p \cdot k}} \equiv \frac{-2\pi i}{2\sqrt{\tilde{K}^2 + L}} \tag{48}
\]
Since the only contribution to the integral comes from the pole we choose the limits as \(\pm \infty\) for \(\omega\). Take derivatives with respect to \(L\) on both sides:
\[
\int_{-\infty}^{\infty} \frac{d\omega}{\left[\omega^2 - i\varepsilon + (\tilde{K}^2 + 2p \cdot k)\right]^2} = \frac{\pi i}{2 \left(\tilde{K}^2 + 2p \cdot k\right)^{3/2}} \tag{49}
\]
\[
\int_{0}^{\infty} \frac{d\omega}{\left[\omega^2 - i\varepsilon + (\tilde{K}^2 + 2p \cdot k)\right]^2} = \frac{\pi i}{4 \left(\tilde{K}^2 + 2p \cdot k\right)^{3/2}} \tag{50}
\]
The remaining integral is
\[
\frac{\pi i}{4} \int_{0}^{k} \frac{4 \pi^2 K^2}{(K^2 + L)^{3/2}} \, dK = i \pi^3 \left[\frac{-K}{\sqrt{K^2 + L}} + \ln \left(\frac{K + \sqrt{K^2 + L}}{\sqrt{K^2 + L}}\right)\right]_0^K \tag{51}
\]
\[
= i \pi^3 \left[\frac{-K}{\sqrt{K^2 + L}} + \ln \left(\frac{K + \sqrt{K^2 + L}}{\sqrt{L}}\right)\right] \tag{52}
\]
The last integral is over \(x\). Insert \(x\) back into the integral
\[
\int_0^1 \left[\frac{-K}{\sqrt{K^2 + 2xp \cdot k}} + \ln \left(\frac{K + \sqrt{K^2 + 2xp \cdot k}}{\sqrt{2xp \cdot k}}\right)\right] dx \tag{53}
\]
The two integrals are

$$\int_0^1 \left[ \frac{-K}{\sqrt{K^2 + 2xp \cdot k}} \right] dx = \left[ \frac{-2K\sqrt{K^2 + 2xp \cdot k}}{2p \cdot k} \right]_0^1 = \frac{-K\sqrt{K^2 + 2mck}}{mck} + \frac{1}{mc}$$

(54)

$$\int_0^1 \ln \left( \frac{K + \sqrt{K^2 + 2xp \cdot k}}{\sqrt{2xp \cdot k}} \right) dx = \left[ \frac{k\sqrt{K^2 + 2mck}}{2mck} + \ln \left( \frac{\sqrt{K^2 + 2mck} + k}{\sqrt{2mck}} \right) \right] - \frac{k^2}{2mck}$$

(55)

Substituting for $k = \hbar\kappa$ Eq.(3) and $\eta \equiv \hbar/2mc = \lambda C/4\pi r$ the mass correction is:

$$\Delta m \equiv \mu (r) = \frac{\alpha m}{2\pi} \left[ -\eta \sqrt{1 + \frac{1}{\eta}} + \eta + \ln \left\{ \sqrt{\eta} \left( \sqrt{1 + \frac{1}{\eta}} + 1 \right) \right\} \right]$$

For a stationary electron $p = mc$.

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