A remark on the $\mathcal{M}_H(G)$-conjecture and Akashi series

Meng Fai Lim*

Abstract

In this article, we give a criterion for the dual Selmer group of an elliptic curve which has either good ordinary reduction or multiplicative reduction at every prime above $p$ to satisfy the $\mathcal{M}_H(G)$-conjecture. As a by-product of our calculations, we are able to define the Akashi series of the dual Selmer groups assuming the conjectures of Mazur and Schneider. Previously, the Akashi series are defined under the stronger assumption that the dual Selmer group satisfies the $\mathcal{M}_H(G)$-conjecture. We then establish a criterion for the vanishing of the dual Selmer groups using the Akashi series. We will apply this criterion to prove some results on the characteristic elements of the dual Selmer groups. Our methods in this paper are inspired by the work of Coates-Schneider-Sujatha and can be extended to the Greenberg Selmer groups attached to other ordinary representations, for instance, those coming from an $p$-ordinary modular form.

Keywords and Phrases: Selmer groups, $\mathcal{M}_H(G)$-conjecture, admissible $p$-adic Lie extensions, Akashi series, characteristic element.

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1 Introduction

Throughout the paper, $p$ will always denote an odd prime. Let $G$ be a compact $p$-adic Lie group with a closed normal subgroup $H$ such that $G/H \cong \mathbb{Z}_p$. We denote $\mathcal{M}_H(G)$ to be the category of a special class of $\mathbb{Z}_p[G]$-torsion modules, namely those finitely generated left $\mathbb{Z}_p[G]$-modules $M$ such that $M/M(p)$ is finitely generated over $\mathbb{Z}_p[H]$; here $M(p)$ is the $\mathbb{Z}_p[G]$-submodule of $M$ consisting of elements of $M$ which are annihilated by some power of $p$. In this paper, we give a criterion for the dual Selmer group of an elliptic curve to be in $\mathcal{M}_H(G)$. Such a criterion is already known in certain cases when $G$ is the Galois group of a $p$-adic Lie extension of dimension 2 and when $G$ is the Galois group of the $p$-adic Lie extension obtained by adjoining all the $p$-division points of the elliptic curve. The criterion we obtained here can therefore be viewed as a generalization of those. We remark that the knowledge of such dual Selmer groups lying in $\mathcal{M}_H(G)$ is crucial in the formulation of the main conjectures of non-commutative Iwasawa theory, as one requires this property in order to be able to define suitable characteristic elements for these modules in certain relative $K_0$-groups (see [CFKSV]).

It will follow from our calculations that the Selmer group has better properties than a general module in $\mathcal{M}_H(G)$ under certain assumptions (see Proposition 5.1). This in turn allows us to define the “Akashi series” of the dual Selmer group. Previously, these are defined under the assumption that the dual Selmer group lies in $\mathcal{M}_H(G)$ (see [CFKSV] Section 3 and [Z2] Section 2). We then establish a criterion for the vanishing of the dual Selmer group of an elliptic curve with good ordinary reduction at all primes above

*Department of Mathematics, University of Toronto, 40 St. George St., Toronto, Ontario, Canada M5S 2E4
We say that the module choice of $G$ is a torsion $M$. It is well known that $\mathbb{Z}_p[G]$ is an Auslander regular ring (cf. [V1] Theorems 3.26). Furthermore, the ring $\mathbb{Z}_p[G]$ has no zero divisors (cf. [Neu]), and therefore, admits a skew field $K(G)$ which is flat over $\mathbb{Z}_p[G]$ (see [GW] Chapters 6 and 10 or [Lam] Chapter 4, §9 and §10). If $M$ is a finitely generated $\mathbb{Z}_p[G]$-module, we define the $\mathbb{Z}_p[G]$-rank of $M$ to be

$$\text{rank}_{\mathbb{Z}_p[G]} M = \dim_{K(G)} K(G) \otimes_{\mathbb{Z}_p[G]} M.$$ 

We say that the module $M$ is a torsion $\mathbb{Z}_p[G]$-module if $\text{rank}_{\mathbb{Z}_p[G]} M = 0$. Now suppose that $N$ is a $\mathbb{F}_p[G]$-module. We then define its $\mathbb{F}_p[G]$-rank by

$$\text{rank}_{\mathbb{F}_p[G]} N = \frac{\text{rank}_{\mathbb{Z}_p[G]} N}{[G : G_0]},$$

where $G_0$ is an open normal uniform pro-$p$ subgroup of $G$. This is integral and independent of the choice of $G_0$ (see [Ho] Proposition 1.6)). Similarly, we will say that $N$ is a torsion $\mathbb{F}_p[G]$-module if $\text{rank}_{\mathbb{F}_p[G]} N = 0$.

For a general compact $p$-adic Lie group $G$ and a finitely generated $\mathbb{Z}_p[G]$-module $M$, we say that $M$ is a torsion $\mathbb{Z}_p[G]$-module if there exists an open normal uniform pro-$p$ subgroup $G_0$ of $G$ such that $M$ is a torsion $\mathbb{Z}_p[G_0]$-module in the above sense. We will also make use of a well-known equivalent definition for $M$ to be torsion $\mathbb{Z}_p[G]$-module, namely: $\text{Hom}_{\mathbb{Z}_p[G]}(M, \mathbb{Z}_p[G]) = 0$. The notion of a torsion $\mathbb{F}_p[G]$-module for a general compact $p$-adic Lie group $G$ is extended in a similar fashion.

For a given finitely generated $\mathbb{Z}_p[G]$-module $M$, we denote $M(p)$ to be the $\mathbb{Z}_p[G]$-submodule of $M$ consisting of elements of $M$ which are annihilated by some power of $p$. Since the ring $\mathbb{Z}_p[G]$ is Noetherian, the module $M(p)$ is finitely generated over $\mathbb{Z}_p[G]$. Therefore, one can find an integer $r \geq 0$ such that $p^r$ annihilates $M(p)$. Following [Ho] Formula (33)], we define

$$\mu_G(M) = \sum_{i \geq 0} \text{rank}_{\mathbb{F}_p[G]} (p^i M(p)/p^{i+1}).$$

(For another alternative, but equivalent, definition, see [V1] Definition 3.32.) By the above discussion, the sum on the right is a finite one. Also, it is clear from the definition that $\mu_G(M) = \mu_G(M(p))$. Finally, it is not difficult to see that this definition coincides with the classical notion of the $\mu$-invariant for $\Gamma$-modules when $G = \Gamma$. We now record certain properties of this invariant.

**Lemma 2.1.** Let $G$ be a compact pro-$p$ $p$-adic Lie group without $p$-torsion. Then we have the following statements.
(a) For every finitely generated $\mathbb{Z}_p[[G]]$-module $M$, one has
\[ \mu_G(M) = \sum_{i \geq 0} (-1)^i \text{ord}_p(H_i(G, M(p))). \]

(b) Suppose that $G$ has a closed normal subgroup $H$ such that $G/H \cong \mathbb{Z}_p$. If $M$ is a $\mathbb{Z}_p[[G]]$-module which is finitely generated over $\mathbb{Z}_p[[H]]$, then one has $\mu_G(M) = 0$.

(c) Suppose that we are given a short exact sequence of finitely generated $\mathbb{Z}_p[[G]]$ modules
\[ 0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0. \]

(i) One has $\mu_G(M) \leq \mu_G(M') + \mu_G(M'')$. Moreover, if $M$, and hence also $M'$ and $M''$, is $\mathbb{Z}_p[G]$-torsion, the inequality is an equality.

(ii) If $\mu_G(M'') = 0$, then one has $\mu_G(M') = \mu_G(M)$.

(iii) If $M'$ is finitely generated over $\mathbb{Z}_p[[H]]$, then one has $\mu_G(M) = \mu_G(M'')$.

(d) Suppose that we are given an exact sequence of finitely generated $\mathbb{Z}_p[[G]]$-modules
\[ A \longrightarrow B \longrightarrow C \longrightarrow D \]

such that $A$ is finitely generated over $\mathbb{Z}_p[[H]]$ and $\mu_G(D) = 0$. Then one has the equality $\mu_G(B) = \mu_G(C)$.

Proof. (a), (b) and (c)(i) are proven in [Ho, Corollary 1.7], [Ho, Lemma 2.7] and [Ho, Proposition 1.8] respectively. The remaining statements can be deduced from the previous statements without too much difficulties.

We make a note here mentioning that the conclusion of (c)(iii) is false in general if one replaces the assumption “$M'$ is finitely generated over $\mathbb{Z}_p[[H]]$” by “$\mu_G(M') = 0$”. An example will be to take $M' = M = \mathbb{Z}_p[[G]]$ and $M'' = \mathbb{Z}_p[[G]]/\mathbb{Z}_p$, and consider the canonical exact sequence
\[ 0 \longrightarrow \mathbb{Z}_p[[G]] \longrightarrow \mathbb{Z}_p[[G]]/\mathbb{Z}_p \longrightarrow 0. \]

Clearly, we have $\mu_G(M') = 0$ but $\mu_G(M) \neq \mu_G(M'')$.

We end the section with the following useful relative formula for the $\mu$-invariant.

Lemma 2.2. Let $G$ be a compact pro-$p$ $p$-adic Lie group without $p$-torsion and let $N$ be a closed normal subgroup of $G$ such that $G/N$ has no $p$-torsion. Then for every finitely generated torsion $\mathbb{Z}_p[[G]]$-module $M$, we have
\[ \mu_G(M) = \sum_{i \geq 0} (-1)^i \mu_{G/N}(H_i(N, M(p))). \]
Proof.
\[
\mu_G(M) = \sum_{i \geq 0} (-1)^i \text{ord}_p(H_i(G, M(p))) \\
= \sum_{i,j \geq 0} (-1)^{i+j} \text{ord}_p(H_i(G/N, H_j(N, M(p)))) \\
= \sum_{j \geq 0} (-1)^j \mu_{G/N}(H_j(N, M(p))),
\]
where the first and fourth equality follow from Lemma 2.1(a) (and noting that \(H_j(N, M(p))\) is \(p\)-torsion), the third is obvious and the second is a consequence of the following bounded spectral sequence

\[
H_i(G/N, H_j(N, M(p))) = \Rightarrow H_{i+j}(G, M(p)).
\]

3 Selmer groups

In this section, we recall the definition of the \((p\)-primary\) Selmer group of an elliptic curve. As before, \(p\) denote an odd prime. Let \(F\) be a number field, and let \(E\) be an elliptic curve defined over \(F\). Let \(v\) be a prime of \(F\). For every finite extension \(L\) of \(F\), we define

\[
J_v(E/L) = \bigoplus_{w|v} H^1(L_w, E)(p),
\]
where \(w\) runs over the (finite) set of primes of \(L\) above \(v\). If \(\mathcal{L}\) is an infinite extension of \(F\), we define

\[
J_v(E/\mathcal{L}) = \lim\limits_{\rightarrow} J_v(E/L),
\]
where the direct limit is taken over all finite extensions \(L\) of \(F\) contained in \(\mathcal{L}\). For any algebraic (possibly infinite) extension \(\mathcal{L}\) of \(F\), the Selmer group of \(E\) over \(\mathcal{L}\) is defined to be

\[
S(E/\mathcal{L}) = \ker \left( H^1(\mathcal{L}, E_{p\infty}) \longrightarrow \bigoplus_v J_v(E/\mathcal{L}) \right),
\]
where \(v\) runs through all the primes of \(F\).

We say that \(F_{\infty}\) is an admissible \(p\)-adic Lie extension of \(F\) if (i) \(\text{Gal}(F_{\infty}/F)\) is a compact \(p\)-adic Lie group, (ii) \(F_{\infty}\) contains the cyclotomic \(\mathbb{Z}_p\)-extension \(F_{\text{cyc}}\) of \(F\) and (iii) \(F_{\infty}\) is unramified outside a finite set of primes of \(F\). Write \(G = \text{Gal}(F_{\infty}/F)\), \(H = \text{Gal}(F_{\infty}/F_{\text{cyc}})\) and \(\Gamma = \text{Gal}(F_{\text{cyc}}/F)\). Let \(S\) be a finite set of primes of \(F\) which contains the primes above \(p\), the infinite primes, the primes at which \(E\) has bad reduction and the primes that are ramified in \(F_{\infty}/F\). Denote \(F_S\) to be the maximal algebraic extension of \(F\) unramified outside \(S\). For each algebraic (possibly infinite) extension \(\mathcal{L}\) of \(F\) contained in \(F_S\), we write \(G_{\mathcal{L}} = \text{Gal}(F_{\mathcal{L}}/F)\). The following alternative equivalent description of the Selmer group of \(E\)

\[
S(E/\mathcal{L}) = \ker \left( H^1(G_{\mathcal{L}}, E_{p\infty}) \xrightarrow{\lambda_{\mathcal{L}}(F_{\infty})} \bigoplus_{v \in S} J_v(E/\mathcal{L}) \right)
\]
is well-known. We will denote $X(E/L)$ to be the Pontryagin dual of $S(E/L)$.

**From now on, we will assume that for every prime $v$ of $F$ above $p$, our elliptic curve $E$ has either good ordinary reduction or multiplicative reduction at $v$.** The following conjecture is well-known.

**Conjecture.** $X(E/F^\text{cyc})$ is a torsion $\mathbb{Z}_p[\Gamma]$-module.

The conjecture was first stated in [Maz] for elliptic curves that have good ordinary reduction at all primes of $F$ above $p$. The general form we have here was stated in [Sch] (see also [HO, OcV2]). At present, the best result in support of the conjecture is due to Kato [K], who has proven it when $F$ is abelian over $\mathbb{Q}$ and $E$ is an elliptic curve defined over $\mathbb{Q}$ with good ordinary reduction at $p$.

One has a natural generalization of the above conjecture to admissible $p$-adic extensions, namely that $X(E/F_\infty)$ should be $\mathbb{Z}_p[G]$-torsion for every admissible $p$-adic Lie extension $F_\infty/F$. In this direction, this has been studied in [HV] when $G$ has dimension 2, and later in [HO] when $G$ is a solvable uniform pro-$p$ group. However, such a generalization of the conjecture does not allow one to define a suitable characteristic element which is required in the formulation of the main conjectures of non-commutative Iwasawa theory. To overcome this difficulty, the following stronger conjecture, when $G$ has dimension $> 1$, was introduced in [CFKSV] for elliptic curve having good ordinary reduction at all primes above $p$.

In the case when the elliptic curve has multiplicative reduction, this was introduced in [Lee] and was an important condition required in order to apply Iwasawa-theoretical methods to the study of root numbers (see also [CFKS] for a similar study in the case when the elliptic curve has good ordinary reduction at all primes above $p$).

**$\mathfrak{M}_H(G)$-Conjecture.** For every admissible $p$-adic Lie extension $F_\infty$ of $F$, $X(E/F_\infty)/X(E/F_\infty)(p)$ is a finitely generated $\mathbb{Z}_p[H]$-module.

The best evidence in support of the $\mathfrak{M}_H(G)$-conjecture is in the “$\mu = 0$ situation”, i.e., the $\mathfrak{M}_H(G)$-conjecture holds for $X(E/F_\infty)$ if $X(E/F^\text{cyc})$ is finitely generated over $\mathbb{Z}_p$ and $F_\infty$ is a pro-$p$ extension of $F$ (see [CFKSV] Proposition 5.6 or [CS3, Theorem 2.1]). In this article, we will investigate the above conjecture from a general point of view, that is namely the “$\mu \neq 0$ situation”. Our first goal is to prove the following criterion for the conjecture to hold. From now on, we write $X_f(E/F_\infty) = X(E/F_\infty)/X(E/F_\infty)(p)$.

**Theorem 3.1.** Let $p$ be an odd prime. Assume that (i) $E$ has either good ordinary reduction or multiplicative reduction at every prime of $F$ above $p$, (ii) $X(E/F^\text{cyc})$ is $\mathbb{Z}_p[\Gamma]$-torsion, (iii) $G$ is pro-$p$ without $p$-torsion, (iv) $H^2(G_S(F_\infty), E_{p^\infty}) = 0$ and (v) $\lambda_S(F_\infty)$ is surjective. Then $X_f(E/F_\infty)$ is a finitely generated $\mathbb{Z}_p[H]$-module if and only if

$$\mu_G(X(E/F_\infty)) = \mu_H(X(E/F^\text{cyc}))$$

and $H_i(H, X_f(E/F_\infty))$ is finitely generated over $\mathbb{Z}_p$ for every odd $i \leq \dim H - 1$.

For the remainder of the section, we will compare our theorem with existing known results, deferring the proof to Section 4. Before we begin the comparison, we first discuss the relationship between the torsionness of the dual Selmer group and the conditions: $H^2(G_S(F_\infty), E_{p^\infty}) = 0$ and surjectivity of $\lambda_S(F_\infty)$. We begin with the following proposition.
Proposition 3.2. Let $F_\infty/F$ be an admissible $p$-adic Lie extension. If $H^2(G_S(F_\infty), E_{p^{\infty}}) = 0$ and $\lambda_S(F_\infty)$ is surjective, then $X(E/F_\infty)$ is a torsion $\mathbb{Z}_p[G]$-module.

Proof. This follows from a standard rank calculation noting the formulas in [OcV2 Theorem 4.1] and [HV Proposition 7.4].

The next proposition is a partial converse to the preceding one. Although we make use of the proposition mainly under case (i), we feel it is important to note down a few other cases. We mention that the proof of the proposition under case (ii) is a standard well-known argument which we have included for the reader’s convenience.

Proposition 3.3. Let $F_\infty/F$ be an admissible $p$-adic Lie extension. Suppose that at least one of the following statements holds.

(i) $E_{p^{\infty}}$ is not rational over $F_\infty$.

(ii) For every $v \in S$, the decomposition group of $G$ at $v$ has dimension $\geq 2$.

(iii) $E$ has no additive reduction, $G$ is pro-$p$ and has no $p$-torsion, and the set $S$ consists precisely of the primes above $p$, the infinite primes, the primes at which $E$ has bad reduction and the primes that are ramified in $F_\infty/F$.

Then $X(E/F_\infty)$ is a torsion $\mathbb{Z}_p[G]$-module if and only if $H^2(G_S(F_\infty), E_{p^{\infty}}) = 0$ and $\lambda_S(F_\infty)$ is surjective.

Proof. By Proposition 3.2 it suffices to show that if $X(E/F_\infty)$ is a torsion $\mathbb{Z}_p[G]$-module, then we have $H^2(G_S(F_\infty), E_{p^{\infty}}) = 0$ and $\lambda_S(F_\infty)$ is surjective. We first consider the case that $E_{p^{\infty}}$ is not rational over $F_\infty$. Then by [Z1] Proposition 10], we have that $E(F_\infty)_{p^{\infty}}$ is finite. The vanishing of $H^2(G_S(F_\infty), E_{p^{\infty}})$ and the surjectivity of $\lambda_S(F_\infty)$ then follow from an application of [HV Theorem 7.2].

Now we will prove the above assertion under the assumption of (ii). Since $X(E/F_\infty)$ is a torsion $\mathbb{Z}_p[G]$-module by hypothesis, we have that the dual fine Selmer group (see [CS2 Section 3] for definition) is also a torsion $\mathbb{Z}_p[G]$-module. By [CS2 Lemma 3.1], this is equivalent to $H^2(G_S(F_\infty), E_{p^{\infty}}) = 0$. Now the Cassels-Poitou-Tate sequence (for example, see [CS1 1.7]) gives an exact sequence

$$0 \rightarrow S(E/F_\infty) \rightarrow H^1(G_S(F_\infty), E_{p^{\infty}}) \xrightarrow{\lambda_S(F_\infty)} \bigoplus_{v \in S} J_v(F_\infty) \rightarrow \left( \hat{S}(E/F_\infty) \right)^\vee \rightarrow H^2(G_S(F_\infty), E_{p^{\infty}}) \rightarrow 0,$$

where $\hat{S}(E/F_\infty)$ is defined as the kernel of the map

$$\lim_L H^1(G_S(L), T_p E) \rightarrow \lim_{w \mid S} T_p H^1(L_w, E).$$

Since we have already shown that $H^2(G_S(F_\infty), E_{p^{\infty}}) = 0$, it follows from the Cassels-Poitou-Tate exact sequence that $\text{coker } \lambda_S(F_\infty) = \hat{S}(E/F_\infty)^\vee$. One may now apply the formulas in [OcV2 Theorem 4.1] and [HV Proposition 7.4] to conclude that

$$\text{rank}_{\mathbb{Z}_p[G]} H^1(G_S(F_\infty), E_{p^{\infty}})^\vee = [F : Q] = \text{rank}_{\mathbb{Z}_p[G]} \left( \bigoplus_{v \in S} J_v(F_\infty) \right)^\vee.$$
Therefore, it follows that \(X(E/F_\infty)\) is a torsion \(\mathbb{Z}_p[\Gamma]\)-module and only if \(\hat{S}(E/F_\infty)\) is a torsion \(\mathbb{Z}_p[\Gamma]\)-module. On the other hand, the Poitou-Tate sequence gives an exact sequence

\[
H^2(G_S(F_\infty), E_{p^\infty})^\vee \longrightarrow \lim_{\leftarrow L} H^1(G_S(L), T_p E) \xrightarrow{\phi} \lim_{\leftarrow L} \bigoplus_{w|S} H^1(L_w, T_p E).
\]

Since \(H^2(G_S(F_\infty), E_{p^\infty}) = 0\), we have that \(\phi\) is injective. In particular, \(\hat{S}(E/F_\infty)\) is contained in \(\lim_{\leftarrow L} \bigoplus_{w|S} H^1(L_w, T_p E)\). Now by virtue of the assumption in (ii), we may apply [OCV2] Proposition 4.5 to conclude that \(\lim_{\leftarrow L} \bigoplus_{w|S} H^1(L_w, T_p E)\), and hence \(\hat{S}(E/F_\infty)\), is \(\mathbb{Z}_p[\Gamma]\)-torsionfree. On other hand, we have shown above that \(\hat{S}(E/F_\infty)\) is a torsion \(\mathbb{Z}_p[\Gamma]\)-module, and therefore, we must have \(\hat{S}(E/F_\infty) = 0\). This gives the surjectivity of \(\lambda_S(F_\infty)\), as required.

Suppose that we are in the situation of (iii). Now if \(E_{p^\infty}\) is not rational over \(F_\infty\), then we are already done by (i). Therefore, we may assume that \(F(E_{p^\infty}) \subseteq F_\infty\). It then follows from either [C] Lemma 2.8 or [CH] Lemma 5.1 that the field \(F(E_{p^\infty})\), and hence \(F_S\), has the property that for each \(v\) which is either above \(p\) or is a bad reduction prime for \(E\) (noting that \(E\) has no additive reduction), the dimension of \(G\) at \(v\) is \(\geq 2\). Now let \(v\) be a prime of \(F\) that is ramified in \(F_\infty\) and does not divide \(p\). Since \(v\) does not divide \(p\), it is unramified in \(F^{\text{cycl}}/F\). Therefore, every prime \(w\) of \(F^{\text{cycl}}\) above \(v\) must ramify in \(F_\infty/F^{\text{cycl}}\).

Since \(G\) has no \(p\)-torsion by one of the hypotheses in (iii), the inertia group of \(w\) in \(F_\infty/F^{\text{cycl}}\) must be infinite and hence of dimension \(\geq 1\). Adding this to the decomposition component coming from \(F^{\text{cycl}}/F\) (since \(v\) does not divide \(p\)), it follows that the decomposition group of \(v\) in \(F_\infty/F\) has dimension \(\geq 2\). Thus, we are now in the situation of (ii), and so we are done.

We now note the following corollary and give some remarks on it.

**Corollary 3.4.** Let \(F_\infty/F\) be an admissible \(p\)-adic Lie extension. If \(X(E/L^{\text{cycl}})\) is a torsion \(\mathbb{Z}_p[\text{Gal}(L^{\text{cycl}}/L)]\)-module for every finite extension \(L\) of \(F\) contained in \(F_\infty\), then we have that \(H^2(G_S(F_\infty), E_{p^\infty}) = 0\) and that \(\lambda_S(F_\infty)\) is surjective. In particular, \(X(E/F_\infty)\) is a torsion \(\mathbb{Z}_p[\Gamma]\)-module.

**Proof.** Since \(E_{p^\infty}\) is not rational over \(L^{\text{cycl}}\), it follows from Proposition 3.3 that \(H^2(G_S(L^{\text{cycl}}), E_{p^\infty}) = 0\) and \(\lambda_S(L^{\text{cycl}})\) is surjective. Note that \(H^2(G_S(F_\infty), E_{p^\infty}) = \lim_{\leftarrow L} H^2(G_S(L^{\text{cycl}}), E_{p^\infty})\) and \(\lambda_S(F_\infty) = \lim_{\leftarrow L} \lambda_S(L^{\text{cycl}})\), where \(L\) runs through all finite extensions of \(F\) contained in \(F_\infty\). Therefore, we have that \(H^2(G_S(L^{\text{cycl}}), E_{p^\infty}) = 0\) and \(\lambda_S(L^{\text{cycl}})\) is surjective. The final assertion of the corollary then follows from an application of Proposition 3.2. \(\square\)

It has been a long asked question (see [CFKSV] [CS3]) on whether one can prove the \(\mathcal{M}_H(G)\)-conjecture under the hypothesis that \(X(E/L^{\text{cycl}})\) is a torsion \(\mathbb{Z}_p[\text{Gal}(L^{\text{cycl}}/L)]\)-module for every finite extension \(L\) of \(F\) contained in \(F_\infty\). Although we can do no better than Theorem 3.1 and Corollary 3.4 in so far as showing the \(\mathcal{M}_H(G)\)-conjecture, we are able to show that our dual Selmer group exhibits “\(\mathcal{M}_H(G)\)-like properties” (see Proposition 5.1). We finally remark that there is a related result of Hachimori-Ochiai in the direction of Corollary 3.4. Namely, their result [HO] Theorem 2.3 states that if \(X(E/F^{\text{cycl}})\) is a torsion \(\mathbb{Z}_p[\Gamma]\)-module and \(G\) is a uniform solvable pro-\(p\) group, then \(X(E/F_\infty)\) is a torsion \(\mathbb{Z}_p[\Gamma]\)-module.
We now describe how our Theorem 3.1 compares with existing results.

Let $H$ be pro-$p$ of dimension 1 with no $p$-torsion. Such a group $H$ is necessarily solvable. Then by [HO, Theorem 2.3] (see also [HV, Theorem 2.8]), the $\mathbb{Z}_p[\Gamma]$-torsionness of $X(E/F^\infty)$ implies the $\mathbb{Z}_p[G]$-torsionness of $X(E/F_\infty)$. Therefore, if $E$ has either good ordinary reduction or multiplicative reduction at every prime of $F$ above $p$, $X(E/F^\infty)$ is $\mathbb{Z}_p[\Gamma]$-torsion and $G$ is pro-$p$ of dimension $\leq 2$ with no $p$-torsion, then we have that $X_f(E/F_\infty)$ is a finitely generated $\mathbb{Z}_p[H]$-module if and only if

$$\mu_G(X(E/F_\infty)) = \mu_T(X(E/F^\infty)).$$

This recovers [CS3, Corollary 3.2].

We now consider the case that $H$ is of dimension either 2 or 3, and has no $p$-torsion. Then under the same hypotheses as Theorem 3.1 $X_f(E/F_\infty)$ is a finitely generated $\mathbb{Z}_p[H]$-module if and only if $H_1(H, X_f(E/F_\infty))$ is finite and

$$\mu_G(X(E/F_\infty)) = \mu_T(X(E/F^\infty)).$$

In this form, this criterion has been observed in [CFKSV, Lemmas 5.3 and 5.4] when $F_\infty$ is the field generated by all the $p$-power division points of an elliptic curve without complex multiplication.

We conclude this section with two auxiliary results on the $\mu$-invariant of the Selmer group. The first gives an inequality under an extra condition of $F_\infty$.

**Proposition 3.5.** Let $p$ be an odd prime. Assume that (i) $E$ has either good ordinary reduction or multiplicative reduction at every prime of $F$ above $p$, (ii) that $X(E/F^\infty)$ is $\mathbb{Z}_p[\Gamma]$-torsion and (iii) that there is a finite family of closed normal subgroups $H_i$ ($0 \leq i \leq r$) of $G$ such that $1 = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_r = H$ and $H_i/H_{i-1} \cong \mathbb{Z}_p$ for $1 \leq i \leq r$. Then we have that $H^2(G, E_{p^\infty}) = 0$ and $\lambda_S(F_\infty)$ is surjective. Furthermore, we have

$$\mu_G(X(E/F_\infty)) \leq \mu_T(X(E/F^\infty)).$$

**Proof.** This follows by applying Proposition 4.8 iteratively.

The next result gives a necessary condition for $X_f(E/F_\infty)$ to be finitely generated over $\mathbb{Z}_p[H]$.

**Proposition 3.6.** Retain the assumptions of Theorem 3.1. Suppose that $X_f(E/F_\infty)$ is a finitely generated $\mathbb{Z}_p[H]$-module. Then for every finite extension $L$ of $F$ contained in $F_\infty$, we have

$$\mu_{\Gamma_L}(X(E/L^{cyb})) = [L : F]\mu_{\Gamma_p}(X(E/F^\infty)),$$

where $\Gamma_L = \text{Gal}(L^{cyb}/L)$.

**Proof.** Write $G_L = \text{Gal}(F_\infty/L)$. Then we have

$$\mu_{\Gamma_L}(X(E/L^{cyb})) = \mu_{G_L}(X(E/F_\infty)) = [L : F]\mu_G(X(E/F_\infty)) = [L : F]\mu_T(X(E/F^\infty)).$$

In the case when $G = \mathbb{Z}_p^2$, the conclusion of the preceding proposition turns out to be a sufficient condition for $X_f(E/F_\infty)$ to be finitely generated over $\mathbb{Z}_p[H]$ (cf. [CS3, Theorem 3.8]). In view of this, a natural question will be whether the conclusion of the preceding proposition is a sufficient condition for $X_f(E/F_\infty)$ to be a finitely generated $\mathbb{Z}_p[H]$-module for a general $G$. We do not have an answer to this at this point of writing.
4 Proof of Theorem 3.1

In this section, we will prove a result relating the quantities \( \mu_G(X(E/F_\infty)) \) and \( \mu_\Gamma(X(E/F^{\text{cyc}})) \). This relationship will allow us to establish the required criterion in Theorem 3.1. As before, \( p \) will denote an odd prime number and \( E \) will denote an elliptic curve defined over \( F \) with either good ordinary reduction or multiplicative reduction at every prime of \( F \) above \( p \). We let \( F_\infty \) denote a fixed admissible \( p \)-adic Lie extension of \( F \). Let \( S \) be a finite set of primes of \( F \) which contains the primes above \( p \), the infinite primes, the primes at which \( E \) has bad reduction and the primes that are ramified in \( F_\infty/F \). We continue to write \( G = \text{Gal}(F_\infty/F) \), \( H = \text{Gal}(F_\infty/F^{\text{cyc}}) \) and \( \Gamma = \text{Gal}(F^{\text{cyc}}/F) \). We split the section into three subsections, where we will study the \( H \)-cohomology of \( J_v(E/F_\infty) \) and \( S(E/F_\infty) \) in Subsection 4.1 and Subsection 4.2 respectively. The calculations done in these two subsections will be used in Subsection 4.3 to give a relationship between the respective \( \mu \)-invariants of \( X(E/F_\infty) \) and \( X(E/F^{\text{cyc}}) \). We then use this relationship to give a proof of Theorem 3.1. Again, we like to remind the reader that our line of attack is deeply inspired by the approach in [CSS, Section 2].

4.1 Local cohomology calculations

Before calculating the \( H \)-cohomology of the local terms, we give another description of \( J_v(\mathcal{L}) \), where \( \mathcal{L} \) is an algebraic extension of \( F^{\text{cyc}} \). By [CG] P. 150, for each prime \( v \) of \( F \) above \( p \), we have a short exact sequence

\[
0 \rightarrow C_v \rightarrow E_{v,\infty} \rightarrow D_v \rightarrow 0
\]

of discrete \( \text{Gal}(\bar{F}_v/F_v) \)-modules which is characterized by the fact that \( C_v \) is divisible and that \( D_v \) is the maximal quotient of \( E_{v,\infty} \) by a divisible subgroup such that the inertia group acts on \( D_v \) via a finite quotient. Since our elliptic curve \( E \) has either good ordinary reduction or multiplicative reduction at each prime of \( F \) above \( p \), we have that \( C_v \) and \( D_v \) are divisible abelian groups of corank one. If fact, these groups can be explicitly described when \( E \) has either good ordinary reduction or split multiplicative reduction at \( v \). For instance, if \( E \) has good ordinary reduction at \( v \), we may take \( D_v \) to be \( \bar{E}_{v,\infty} \), where \( \bar{E}_v \) is the reduction of \( E \) mod \( v \), and take \( C_v \) to be the kernel of the natural surjection \( E_{v,\infty} \rightarrow \bar{E}_{v,\infty} \). If \( E \) has split multiplicative reduction at \( v \), we have \( C_v = \mu_{p^\infty} \) and \( D_v = \mathbb{Q}_p/\mathbb{Z}_p \) (cf. [Gr] P. 69-70)).

Let \( \mathcal{L} \) be an algebraic (possibly infinite) extension of \( F^{\text{cyc}} \). For each non-archimedean prime \( w \) of \( \mathcal{L} \), define \( \mathcal{L}_w \) to be the union of the completions at \( w \) of the finite extensions of \( F \) contained in \( \mathcal{L} \). We can now state the following lemma.

Lemma 4.1. Let \( \mathcal{L} \) be an algebraic extension of \( F^{\text{cyc}} \) which is unramified outside a set of finite primes of \( F \). Then we have an isomorphism

\[
J_v(\mathcal{L}) \cong \begin{cases} 
\lim \bigoplus_{\mathcal{L}' \mid \mathcal{L}_w} H^1(\mathcal{L}_w, D_v), & \text{if } v \text{ divides } p \\
\lim \bigoplus_{\mathcal{L}' \mid \mathcal{L}_w} H^1(\mathcal{L}_w, E_{p,\infty}), & \text{if } v \text{ does not divide } p
\end{cases}
\]

where the direct limit is taken over all finite extensions \( \mathcal{L}' \) of \( F^{\text{cyc}} \) contained in \( \mathcal{L} \).
Proof. It suffices to prove the lemma in the case when \(\mathcal{L}\) is a finite extension of \(F^\mathrm{cyc}\). Now by definition, we have
\[
J_v(\mathcal{L}) = \bigoplus_{w|v} H^1(\mathcal{L}_w, E)(p).
\]

By the local Kummer theory of elliptic curves, we have an exact sequence
\[
0 \to E(\mathcal{L}_w) \otimes \mathbb{Q}_p/\mathbb{Z}_p^\kappa \mathcal{L}_w \to H^1(\mathcal{L}_w, E_{p^\infty}) \to H^1(\mathcal{L}_w, E)(p) \to 0.
\]

Now if \(v\) does not divide \(p\), then \(E(\mathcal{L}_w) \otimes \mathbb{Q}_p/\mathbb{Z}_p = 0\). The required isomorphism is then immediate in this case. Now suppose that \(v\) divides \(p\). Since the profinite degree of \(\mathcal{L}_w\) over \(F_v\) is divisible by \(p^\infty\), it follows that \(\text{Gal}(\bar{F}_v/\mathcal{L}_w)\) has \(p\)-cohomological dimension \(\leq 1\) (cf. [NSW, Theorem 7.1.8(i)]). Therefore, we obtain an exact sequence
\[
H^1(\mathcal{L}_w, C_v) \xrightarrow{\varphi_{\mathcal{L}_w}} H^1(\mathcal{L}_w, E_{p^\infty}) \to H^1(\mathcal{L}_w, D_v) \to 0.
\]

Now it follows from [CG, Proposition 4.3] and [Gr, P. 69-70] that \(\text{im} \kappa_{\mathcal{L}_w} = \text{im} \varphi_{\mathcal{L}_w}\). Hence we have \(H^1(\mathcal{L}_w, E)(p) \cong H^1(\mathcal{L}_w, D_v)\), and this gives the required conclusion for the case when \(v\) divides \(p\).

Lemma 4.2. \(H^i(H, \bigoplus_{v \in S} J_v(F_{\infty}))\) is cofinitely generated over \(\mathbb{Z}_p\) for every \(i \geq 1\). Moreover, if \(H\) has no \(p\)-torsion, then we have
\[
H^i(H, \bigoplus_{v \in S} J_v(F_{\infty})) = 0 \text{ for } i \geq d - 1,
\]
where \(d\) is the dimension of \(H\).

Proof. We denote \(A_v\) to be \(E_{p^\infty}\) or \(D_v\) according as \(v\) does not or does divide \(p\). By the Shapiro lemma, we have
\[
H^1(H, \bigoplus_{v \in S} J_v(F_{\infty})) \cong \bigoplus_{w} H^1(H_w, H^1(F_{\infty,w}, A_v)),
\]
where \(w\) runs through the (finite) set of primes of \(F_{\infty}\) above \(S\) and \(H_w\) is the decomposition group in \(H\) of some fixed prime of \(F_{\infty}\) lying above \(w\). It then suffices to show that \(H^1(H_w, H^1(F_{\infty,w}, E_{p^\infty}))(p)\) is cofinitely generated over \(\mathbb{Z}_p\) for every \(i \geq 1\). For any extension \(\mathcal{L}\) of \(F_{\infty}^\mathrm{cyc}\), the profinite degree of \(\mathcal{L}\) over \(F_v\), is divisible by \(p^\infty\), and therefore, it follows that \(\text{Gal}(\bar{F}_v/\mathcal{L})\) has \(p\)-cohomological dimension \(\leq 1\) (cf. [NSW, Theorem 7.1.8(i)]). Hence the spectral sequence
\[
H^i(H_w, H^j(F_{\infty,w}, A_v)) \Rightarrow H^{i+j}(F_{\infty}^\mathrm{cyc}, A_v)
\]
degenerates to yield
\[
H^i(H_w, H^1(F_{\infty,w}, A_v)) \cong H^{i+2}(H_w, A_v(F_{\infty,w})) \text{ for } i \geq 1.
\]

Since \(H_w\) is a \(p\)-adic Lie group and \(A_v(F_{\infty,w})\) is cofinitely generated over \(\mathbb{Z}_p\), the latter cohomology group, and hence \(H^i(H_w, H^1(F_{\infty,w}, A_v))\), is cofinitely generated over \(\mathbb{Z}_p\). This proves the first assertion. The second assertion is also an immediate consequence from the above isomorphism.

We need another lemma on the cokernel of the restriction map of the local cohomology groups.
Lemma 4.3. The cokernel of the restriction map $\bigoplus_{v \in S} J_v(F_{cyc}) \xrightarrow{\gamma} \left( \bigoplus_{v \in S} J_v(F_{\infty}) \right)^H$ is cofinitely generated over $\mathbb{Z}_p$.

Proof. As before, we denote $A_v$ to be $E_{p\infty}$ or $D_v$ according as $v$ does not or does divide $p$. By an application of the Shapiro lemma and the Hochschild-Serre spectral sequence, we have

$$\text{coker } \gamma \cong \bigoplus_w H^2(H_w, A_v(F_{\infty},w)),$$

where $w$ runs through the (finite) set of primes of $F_{cyc}$ above $S$ and $H_w$ is the decomposition group in $H$ of some fixed prime of $F_{\infty}$ lying above $w$. Clearly, each $H^2(H_w, A_v(F_{\infty},w))$ is cofinitely generated over $\mathbb{Z}_p$ and since the sum is a finite one, the lemma follows.

4.2 Global cohomology calculations

In this subsection, we will study the $H$-cohomology of the Selmer group. We first record a lemma on the $H$-cohomology of $H^1(G_S(F_{\infty}), E_{p\infty})$.

Lemma 4.4. If $H^2(G_S(F_{\infty}), E_{p\infty}) = H^2(G_S(F_{cyc}), E_{p\infty}) = 0$, then $H^i(H, H^1(G_S(F_{\infty}), E_{p\infty}))$ is cofinitely generated over $\mathbb{Z}_p$ for every $i \geq 1$. Moreover, if $H$ has no $p$-torsion, then we have

$$H^i(H, H^1(G_S(F_{\infty}), E_{p\infty})) = 0 \text{ for } i \geq d - 1,$$

where $d$ is the dimension of $H$.

Proof. Under the given assumptions of the lemma, the spectral sequence

$$H^i(H, H^j(G_S(F_{\infty}), E_{p\infty})) \Rightarrow H^{i+j}(G_S(F_{cyc}), E_{p\infty})$$

degenerates to yield

$$H^i(H, H^1(G_S(F_{\infty}), E_{p\infty})) \cong H^{i+2}(H, E_{p\infty}(F_{\infty})) \text{ for } i \geq 1,$$

and the latter group is easily seen to be cofinitely generated over $\mathbb{Z}_p$. The second assertion is also immediate.

Proposition 4.5. Assume that hypotheses (i), (ii), (iv) and (v) of Theorem 3.1 hold. Then $H^i(H, S(E/F_{\infty}))$ is cofinitely generated over $\mathbb{Z}_p$ for every $i \geq 1$. Moreover, if hypothesis (iii) of Theorem 3.1 holds, we then have

$$H^i(H, S(E/F_{\infty})) = 0 \text{ for } i \geq \dim H.$$

Proof. Consider the following commutative diagram

\[
\begin{array}{cccccccc}
0 & \to & S(E/F_{cyc}) & \to & H^1(G_S(F_{cyc}), E_{p\infty}) & \to & \bigoplus_{v \in S} J_v(F_{cyc}) & \to & 0 \\
& & \downarrow & & \downarrow & & \gamma & & \\
0 & \to & S(E/F_{\infty})^H & \to & H^1(G_S(F_{\infty}), E_{p\infty})^H & \to & \bigoplus_{v \in S} J_v(F_{\infty})^H & \to & H^1(H, S(E/F_{\infty})) & \to & \cdots
\end{array}
\]
with exact rows, where the vertical maps are given by restriction. Note that the top and bottom rows are exact by Proposition 3.3. To simplify notation, we write $W_\infty = H^1(G_S(F_\infty), E_{p^\infty})$ and $J_\infty = \bigoplus_{v \in S} J_v(F_\infty)$. By a diagram chasing argument, we have a long exact sequence

$$\text{coker} \gamma \to H^1(H, S(E/F_\infty)) \to H^1(H, W_\infty) \to H^1(H, J_\infty) \to \cdots$$

$$\cdots \to H^{i-1}(H, J_\infty) \to H^i(H, S(E/F_\infty)) \to H^i(H, W_\infty) \to H^i(H, J_\infty) \to \cdots$$

It then follows from Lemmas 4.2, 4.3 and 4.4, and the above long exact sequence that $H^i(H, S(E/F_\infty))$ is cofinitely generated over $\mathbb{Z}_p$ for every $i \geq 1$. The second assertion follows from Lemmas 4.2 and 4.4. □

We record an immediate corollary.

**Corollary 4.6.** Suppose that all the hypotheses in Theorem 3.7 hold. Denoting $d$ to be the dimension of $H$, we have

$$H_i(H, X(E/F_\infty)(p)) = 0 \text{ for } i \geq d, \text{ and}$$

$$H_i(H, X_f(E/F_\infty)) = 0 \text{ for } i \geq d.$$  

**Proof.** The vanishing is clear for $i \geq d + 1$. Now by Proposition 4.5, we have $H_d(H, X(E/F_\infty)) = 0$. Since $H_d(H, -)$ is left exact, and both $X(E/F_\infty)(p)$ and $X_f(E/F_\infty) = p^r X(E/F_\infty)$ for some integer $r \geq 0$ are submodules of $X(E/F_\infty)$, we have the vanishing for $i = d$ too. □

### 4.3 Relation between $\mu_G(X(E/F_\infty))$ and $\mu_\Gamma(X(E/F^{\text{cyc}}))$

We can now give the required relation (compare with [CSS Proposition 2.13]).

**Proposition 4.7.** Let $p$ be an odd prime. Assume that (i) $E$ has either good ordinary reduction or multiplicative reduction at every prime of $F$ above $p$, (ii) $X(E/F^{\text{cyc}})$ is $\mathbb{Z}_p[\Gamma]$-torsion, (iii) $G$ has no $p$-torsion, (iv) $H^2(G_S(F_\infty), E_{p^\infty}) = 0$ and (v) $\lambda_S(F_\infty)$ is surjective. Then we have that $H_i(H, X_f(E/F_\infty))$ is a finitely generated $\mathbb{Z}_p[\Gamma]$-torsion module for all $i \geq 0$ and

$$\mu_G(X(E/F_\infty)) = \mu_\Gamma(X(E/F^{\text{cyc}})) + \sum_{i=0}^{d-1} (-1)^{i+1} \mu_\Gamma(H_i(H, X_f(E/F_\infty))),$$

where $d$ is the dimension of $H$.

**Proof.** By Lemma 2.2, we have

$$\mu_G(X(E/F_\infty)) = \sum_{i \geq 0} (-1)^i \mu_\Gamma(H_i(H, X(E/F_\infty)(p))).$$

Write $X = X(E/F_\infty)$ and $X_f = X_f(E/F_\infty)$. Taking $H$-homology of the following short exact sequence

$$0 \to X(p) \to X \to X_f \to 0,$$

we obtain a long exact sequence

$$\cdots \to H_{i+1}(H, X) \to H_{i+1}(H, X_f) \to H_i(H, X_f) \to H_i(H, X(p)) \to H_i(H, X) \to \cdots$$
Combining these with Corollary 4.6, we obtain the required equality.

Clearly, \( H_i(H, X(p)) \) is a finitely generated \( \mathbb{Z}_p[\Gamma] \)-torsion module. Combining this with Proposition 4.7 and the above long exact sequence, we have that \( H_i(H, X_f) \) is a finitely generated \( \mathbb{Z}_p[\Gamma] \)-torsion module for \( i \geq 1 \). By a standard argument (for instance, see [CS3, Lemma 2.4]), the natural map \( H_0(H, X(E/F_\infty)) \to X(E/F_{cyc}) \) has kernel and cokernel which are finitely generated over \( \mathbb{Z}_p \). Therefore, this implies that \( H_0(H, X(E/F_\infty)) \) is a \( \mathbb{Z}_p[\Gamma] \)-torsion module and

\[
\mu_\Gamma(H_0(H, X(E/F_\infty))) = \mu_\Gamma(X(E/F_{cyc})).
\]

Applying Lemma 2.1(b), (c) and (d) to the long exact sequence, we obtain

\[
\mu_\Gamma(H_i(H, X(p))) = \mu_\Gamma(H_{i+1}(H, X_f)) \text{ for } i \geq 1,
\]

and

\[
\mu_\Gamma(H_0(H, X(p))) = \mu_\Gamma(H_0(H, X)) + \mu_\Gamma(H_0(H, X_f)) - \mu_\Gamma(H_1(H, X_f)).
\]

Combining these with Corollary 4.6, we obtain the required equality.

We can now prove Theorem 3.1.

**Proof of Theorem 3.1** To see that the “if” direction holds, one observes that it follows from Lemma 2.1(b) and the equality in Proposition 4.7 that \( \mu_\Gamma(H_0(H, X_f(E/F_\infty))) = 0 \). Since \( H_0(H, X_f(E/F_\infty)) \) is a finitely generated \( \mathbb{Z}_p[\Gamma] \)-module, this in turns implies that \( H_0(H, X_f(E/F_\infty)) \) is finitely generated over \( \mathbb{Z}_p \). Since \( G \), and hence \( H \), is pro-\( p \), we may apply Nakayama Lemma to conclude that \( X_f(E/F_\infty) \) is finitely generated over \( \mathbb{Z}_p \). To see that the “only if” direction holds, we see that if \( X_f(E/F_\infty) \) is finitely generated over \( \mathbb{Z}_p[H] \), then \( H_i(H, X_f(E/F_\infty)) \) is finitely generated over \( \mathbb{Z}_p \) for all \( i \geq 0 \). By Lemma 2.1(b), these modules have trivial \( \mu_\Gamma \)-invariant. Putting these into the equality in Proposition 4.7, we obtain \( \mu_G(X(E/F_\infty)) = \mu_\Gamma(X(E/F_{cyc})) \).

We mention that one can also prove an analogue of Proposition 4.7 replacing \( F_{cyc} \) by an intermediate admissible subextension \( F'_\infty \) of \( F_\infty \). We will only state the following special form of such a statement whose omitted proof is similar to those in this section.

**Proposition 4.8.** Let \( p \) be an odd prime. Let \( F_\infty \) and \( F'_\infty \) be two admissible \( p \)-adic extensions of \( F \) with \( F'_\infty \subseteq F_\infty \) and \( N := \text{Gal}(F'_\infty/F_\infty) \cong \mathbb{Z}_p \). Assume that (i) \( E \) has either good ordinary reduction or multiplicative reduction at every prime of \( F \) above \( p \), (ii) \( X(E/F_{cyc}) \) is \( \mathbb{Z}_p[\Gamma] \)-torsion, (iii) \( G \) and \( G/N \) have no \( p \)-torsion, (iv) \( H^2(G_S(F'_\infty), E_{p^\infty}) = 0 \) and (v) \( \lambda_S(F'_\infty) \) is surjective. Then we have that \( H^2(G_S(F_\infty), E_{p^\infty}) = 0 \) and \( \lambda_S(F_\infty) \) is surjective. Furthermore, we have

\[
\mu_G(X(E/F_\infty)) = \mu_{G/N}(X(E/F'_\infty)) - \mu_{G/N}(X_f(E/F_\infty)N).
\]

## 5 Akashi Series of Selmer Groups

Firstly, we record the following proposition.

**Proposition 5.1.** Assume that (i) \( E \) has either good ordinary reduction or multiplicative reduction at every prime of \( F \) above \( p \), (ii) \( X(E/F_{cyc}) \) is \( \mathbb{Z}_p[\Gamma] \)-torsion, (iii) \( G \) has no \( p \)-torsion, (iv) \( H^2(G_S(F_\infty), E_{p^\infty}) = 0 \) and (v) \( \lambda_S(F_\infty) \) is surjective. Then the following statements hold.
(a) $H_0(H, X(E/F_\infty))$ is $\mathbb{Z}_p[\Gamma]$-torsion and its $\mu_1$-invariant is precisely the quantity $\mu_1(X(E/F^{\text{cyc}}))$.

(b) $H_i(H, X(E/F_\infty))$ is finitely generated over $\mathbb{Z}_p$ for every $i \geq 1$.

(c) $H_i(H, X(E/F_\infty)) = 0$ for $i \geq \dim H$.

Proof. (a) is shown in the proof of Proposition 4.4. (b) and (c) are restatements of Proposition 4.5.

Remark 5.2. By the preceding proposition, we see that the Selmer groups satisfy much stronger properties than a general module $M$ in $\mathcal{M}_H(G)$ as in [CFKSV] Lemma 3.1. Even so we are still not able to show that $X(E/F_\infty)$ lies in $\mathcal{M}_H(G)$ in general.

Now, assuming that all the hypotheses in Proposition 5.1 are valid, we set $f_i$ to be the characteristic power series of the module $H_i(H, X(E/F_\infty))$. Then the Akashi series of $X(E/F_\infty)$ is given by

$$\prod_{i=0}^{d-1} f_i^{(-1)^i},$$

where $d$ is the dimension of $H$. Note that it follows from Proposition 5.1(b) that $p$ does not divide $f_i$ for every $i \geq 1$.

Combining results of Kato and Hachimori-Ochiai, we can find examples where one can construct Akashi series of Selmer groups unconditionally as follows: Assume for now that our elliptic curve $E$ is defined over $\mathbb{Q}$ and has good ordinary reduction at $p$. For a given finite abelian extension $F$ of $\mathbb{Q}$, Kato’s result [K] Theorem 12.4 tells us that $X(E/F^{\text{cyc}})$ is a torsion $\mathbb{Z}_p[\Gamma]$-module. If $F_\infty$ is a solvable admissible $p$-adic extension of $F$, it then follows from [HO] Theorem 2.3 that $X(E/F_\infty)$ is a torsion $\mathbb{Z}_p[\Gamma]$-module. If our elliptic curve satisfies one of the conditions in Proposition 5.1 (for instance, $E$ has no complex multiplication or no additive reduction), then hypotheses (iv) and (v) of Proposition 5.1 are satisfied. Therefore, we can define Akashi series for $X(E/F_\infty)$ via the above discussion.

We now mention how one can prove the results [Z2] (1.1), Theorem 1.3 and Theorem 6.2] by replacing the condition “$X(E/F_\infty) \in \mathcal{M}_H(G)$” with the (weaker) three conditions: $X(E/F^{\text{cyc}})$ is $\mathbb{Z}_p[\Gamma]$-torsion, $H^2(G_S(F_\infty), E_{p^\infty}) = 0$ and $\lambda_S(F_\infty)$ is surjective. Going through the arguments in [Z2], one sees that the condition “$X(E/F_\infty) \in \mathcal{M}_H(G)$” is used in the following ways.

1. To define the Akashi series for $X(E/F_\infty)$.
2. To deduce that $X(E/F^{\text{cyc}})$ is $\mathbb{Z}_p[\Gamma]$-torsion which is required in [Z2] Proposition 5.1, Proposition 5.5.

3. To yield $H^2(G_S(F^{\text{cyc}}), E_{p^\infty}) = 0$ which is required in [Z2] Lemma 5.2.
4. To yield $H^2(G_S(F_\infty), E_{p^\infty}) = 0$ which is required in [Z2] Lemma 5.2.
5. To deduce that $\lambda_S(F_\infty)$ is surjective which is required in [Z2] Formula (5.3), Lemma 5.7.

As seen above, the replaced conditions suffice for us to define the Akashi series for $X(E/F_\infty)$, and the necessary deductions in (2)-(5) for the argument in [Z2] are consequences of the replaced conditions by Propositions 5.2 and 5.5.

We will now establish the following criterion result for the vanishing of the Selmer group of an elliptic curve with good ordinary reduction at all primes above $p$. Recall that a finitely generated $\mathbb{Z}_p[G]$-module $M$ is said to be pseudo-null if $\text{Ext}_{\mathbb{Z}_p[G]}^i(M, \mathbb{Z}_p[G]) = 0$ for $i = 0, 1$.  


Theorem 5.3. Assume that (i) $E$ has good ordinary reduction at every prime of $F$ above $p$, (ii) $X(E/F^\infty)$ is $\mathbb{Z}_p[\Gamma]$-torsion, (iii) $G$ is pro-$p$ with no $p$-torsion, (iv) $H^2(G_S(F_\infty), E_{p^\infty}) = 0$ and (v) $\lambda_S(F_\infty)$ is surjective. Furthermore, suppose that at least one of the following statements holds.

1. $X(E/F_\infty)$ has no nonzero pseudo-null $\mathbb{Z}_p[G]$-submodule.

2. The number field $F$ is not totally real.

Then $X(E/F_\infty) = 0$ if and only if its Akashi series is a unit in $\mathbb{Z}_p[\Gamma]$. In particular, if $X(E/F_\infty) \neq 0$, then its Akashi series is not in $\mathbb{Z}_p[\Gamma] \times$.

We mention that condition (1) is satisfied in many cases (see [HO, Theorem 3.2], [HV, Theorem 2.6], [KT, Theorem 6.5] and [OcV1, Theorem 5.1]). The above theorem under condition (1) is probably known (for instance, see [DD, Proposition A.9] or [HV, Proposition 4.12] for the case of a false Tate extension), but nevertheless, we have included it for completeness and also because the author could not find a reference for it in the general case.

The proof of Theorem 5.3 will follow from a series of lemmas. Before stating and proving these lemmas, we introduce certain preliminary notation. Let $G$ be a compact pro-$p$ $p$-adic group without $p$-torsion, and let $H$ be a closed normal subgroup of $G$ with $\Gamma = G/H \cong \mathbb{Z}_p$. For a given finitely generated $\mathbb{Z}_p[\Gamma]$-module $M$, we denote $\text{Ak}(M)$ to be the Akashi series of $M$ which is given by the alternating product of the characteristic polynomials of its $H$-homology groups. Of course, the Akashi series is only well-defined (up to a unit in $\mathbb{Z}_p[\Gamma]$) if all the $H$-homology groups of $M$ are $\mathbb{Z}_p[\Gamma]$-torsion. We can now state the following.

Proposition 5.4. Let $G$ be a compact pro-$p$ $p$-adic group without $p$-torsion, and let $H$ be a closed normal subgroup of $G$ with $G/H \cong \mathbb{Z}_p$. Let $M$ be a finitely generated $\mathbb{Z}_p[\Gamma]$-module which satisfies the following properties:

(i) $H_i(H, M)$ is a finitely generated $\mathbb{Z}_p[\Gamma]$-torsion module for every even $i$.

(ii) $H_i(H, M)$ is finitely generated over $\mathbb{Z}_p$ for every odd $i$.

(iii) $\text{Ak}(M)$ lies in $\mathbb{Z}_p[\Gamma]^\times$.

Then $M$ is a finitely generated torsion $\mathbb{Z}_p[H]$-module. In particular, $M$ is a finitely generated pseudo-null $\mathbb{Z}_p[\Gamma]$-module.

We record the following corollary which will establish Theorem 5.3 under condition (1).

Corollary 5.5. Retain the notation and assumptions of the preceding lemma. Suppose further that $M$ has no nonzero pseudo-null $\mathbb{Z}_p[G]$-submodule. Then $\text{Ak}(M)$ is a unit in $\mathbb{Z}_p[\Gamma]$ if and only if $M = 0$.

Proof. One direction of the corollary is obvious. Conversely, suppose that $\text{Ak}(M)$ is a unit in $\mathbb{Z}_p[\Gamma]$. Then Proposition 5.4 tells us that $M$ is a finitely generated pseudo-null $\mathbb{Z}_p[G]$-module. Since $M$ has no nonzero pseudo-null $\mathbb{Z}_p[G]$-submodule, we must have $M = 0$. 

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Remark 5.6. Corollary 5.5 is a refinement of an observation made in [CFKSV, P. 182]. In the case when $G$ is abelian, the converse of Proposition 5.4 does hold (cf. [CSS, Lemma 4.4]). When $G$ is not abelian, the converse is false in general (see [CSS, Examples 3 and 4]), although it is known to hold in certain special cases (see [CSS, Lemma 4.5] and [Z2, Proposition 2.3]). For another variant of Proposition 5.4, we refer the reader to Proposition 5.9 below.

From now on, we will identify $\mathbb{Z}_p[\Gamma] \cong \mathbb{Z}_p[T]$ under a choice of a generator of $\Gamma$. A polynomial $T^n + c_{n-1}T^{n-1} + \cdots + c_0$ in $\mathbb{Z}_p[T]$ is said to be a Weierstrass polynomial if $p$ divides $c_i$ for every $0 \leq i \leq n - 1$. We record the following lemma.

Lemma 5.7. Let $f$ and $g$ be two Weierstrass polynomials, and let $a$ and $b$ be two non-negative integers. Then $p^a f$ and $p^b g$ generate the same ideal in $\mathbb{Z}_p[T]$ if and only if if $a = b$ and $(f) = (g)$. Furthermore, we have $\deg f = \deg g$.

Proof. Since $p^a f$ and $p^b g$ generate the same ideal in $\mathbb{Z}_p[T]$, we have a pseudo-isomorphism

$$\mathbb{Z}_p[T]/(p^a) \oplus \mathbb{Z}_p[T]/(f) \sim \mathbb{Z}_p[T]/(p^b) \oplus \mathbb{Z}_p[T]/(g).$$

This in turns implies that $a = b$ and $(f) = (g)$. Then we have

$$\deg f = \text{rank}_{\mathbb{Z}_p} \mathbb{Z}_p[T]/(f) = \text{rank}_{\mathbb{Z}_p} \mathbb{Z}_p[T]/(g) = \deg g,$$

thus proving the lemma.

We can now give the proof of Lemma 5.4.

Proof of Lemma 5.4. For each $i$, choose a Weierstrass polynomial $f_i$ such that $p^a f_i$ is a characteristic element of $H_i(H, M)$, where $a_i$ is the $\mu_1$-invariant of $H_i(H, M)$. Write

$$a = \sum_{i \text{ even}} a_i, \quad f = \prod_{i \text{ even}} f_i \quad \text{and} \quad g = \prod_{i \text{ odd}} f_i.$$

Then we have $Ak(M) = p^a f / g$. Now suppose that $Ak(M)$ is a unit. It then follows from Lemma 5.7 that $a = 0$ and $\deg f = \deg g$. In particular, we have $a_0 = 0$ which in turn implies that $H_0(H, M)$ is a finitely generated $\mathbb{Z}_p$-module. Since $G$ (and hence $H$) is pro-$p$, we may apply Nakayama Lemma to conclude that $M$ is finitely generated over $\mathbb{Z}_p[H]$. Therefore, it follows that $H_1(H, M)$ is finitely generated over $\mathbb{Z}_p$ and that $\deg f_i = \text{rank}_{\mathbb{Z}_p} H_i(H, M)$. Then we have

$$\text{rank}_{\mathbb{Z}_p[H]} M = \sum_{i \geq 0} (-1)^i \text{rank}_{\mathbb{Z}_p} H_i(H, M) = \sum_{i \geq 0} (-1)^i \deg f_i = \deg f - \deg g = 0,$$

where the first equality follows from Howson’s formula (cf. [Ho, Theorem 1.1]). Therefore, we have established the first assertion of Proposition 5.4. The second assertion follows from the first by a well-known result of Venjakob (see [V2, Example 2.3 and Proposition 5.4]).

It remains to show the validity of Theorem 5.3 under condition (2), and this will follow from combining Lemma 5.4 with the following lemma. We note that this lemma does not assume any additional condition on the structure of $X(E/F_\infty)$ (i.e., torsioness or $\mathfrak{M}_H(G)$) other than being nonzero.
Lemma 5.8. Assume that $E$ has good ordinary reduction at every prime of $F$ above $p$, and assume that $F$ is not totally real. If $X(E/F_{\infty}) \neq 0$, then $X(E/F_{\infty})$ is not a finitely generated torsion $\mathbb{Z}_p[H]$-module.

Proof. If $X(E/F_{\infty})$ is not finitely generated over $\mathbb{Z}_p[H]$, then we are done. Therefore, we may assume that $X(E/F_{\infty})$ is finitely generated over $\mathbb{Z}_p[H]$. In particular, this implies that $X(E/L^{\text{cycl}})$ is finitely generated over $\mathbb{Z}_p$ for every finite extension $L$ of $F$ contained in $F_{\infty}$. Note that $X(E/F_{\infty}) = \lim_{L} X(E/L^{\text{cycl}})$, where $L$ runs through all finite extensions of $F$ contained in $F_{\infty}$. Since $X(E/F_{\infty}) \neq 0$, it follows that $X(E/L^{\text{cycl}}) \neq 0$ for some $L$. Since $F$ is not totally real, so is $L$. Therefore, we may apply $[\text{Mat}]$ Proposition 7.5 to conclude that $X(E/L^{\text{cycl}})$ has positive $\mathbb{Z}_p$-rank. By an application of Hachimori’s formula (see $[\text{HSh}]$ Theorem 5.4 or Proposition 7.1), this in turn implies that $X(E/F_{\infty})$ has positive $\mathbb{Z}_p[H]$-rank. Hence we have established the lemma.

The conclusion of Theorem 5.3 follows immediately from Proposition 5.4, Corollary 5.5 and Lemma 5.8. We now proceed to establish an analogous statement of Theorem 5.3 for the Artin twists of the Selmer group under the assumption that the Selmer group satisfies the $\mathfrak{M}_H(G)$-conjecture. By abuse of notation, we will denote $\mathfrak{M}_H(G)$ to be the category of all finitely generated $\mathbb{Z}_p[G]$-modules $M$ such that $M/M(p)$ is finitely generated over $\mathbb{Z}_p[H]$. As a start, we record the following analog of Proposition 5.4.

Proposition 5.9. Let $G$ be a compact pro-$p$ $p$-adic group without $p$-torsion, and let $H$ be a closed normal subgroup of $G$ with $G/H \cong \mathbb{Z}_p$. Let $M$ be a finitely generated $\mathbb{Z}_p[G]$-module which lies in $\mathfrak{M}_H(G)$. If $\text{Ak}(M) \in \mathbb{Z}_p[G]^{\times}$, then $M$ is a finitely generated pseudo-null $\mathbb{Z}_p[G]$-module.

Furthermore, if one assumes that $M$ has no nontrivial pseudo-null $\mathbb{Z}_p[G]$-submodule, then we have $\text{Ak}(M) \in \mathbb{Z}_p[G]^{\times}$ if and only if $M = 0$.

Proof. It suffices to show the first assertion. Write $M_f = M/M(p)$. Then $\text{Ak}(M) = \text{Ak}(M(p))\text{Ak}(M_f)$. Note that $\text{Ak}(M(p)) = p^{\mu_G(M)}$ by $[\text{Ho}]$ Corollary 1.7. Since $M_f$ is finitely generated over $\mathbb{Z}_p[H]$ by assumption, it follows that $\text{Ak}(M_f) = g/h$ for some Weierstrass polynomials $g$ and $h$. Hence we have $\text{Ak}(M) = p^{\mu_G(M)}g/h$. Since $\text{Ak}(M) \in \mathbb{Z}_p[G]^{\times}$, it follows from Lemma 5.7 that $\mu_G(M(p)) = \mu_G(M) = 0$ and $\text{Ak}(M_f) = g/h \in \mathbb{Z}_p[G]^{\times}$. By $[\text{VI}]$ Remark 3.33, the equality $\mu_G(M(p)) = 0$ in turn implies that $M(p)$ is a pseudo-null $\mathbb{Z}_p[G]$-module. On the other hand, since $\text{Ak}(M_f) \in \mathbb{Z}_p[G]^{\times}$ and $M_f$ is a finitely generated $\mathbb{Z}_p[H]$-module, we may apply Proposition 5.4 to conclude that $M_f$ is a pseudo-null $\mathbb{Z}_p[G]$-module. Hence $M$ is a pseudo-null $\mathbb{Z}_p[G]$-module, since it is an extension of the pseudo-null $\mathbb{Z}_p[G]$-modules $M(p)$ and $M_f$.

Suppose that we are given an Artin representation $\rho : G \rightarrow GL_{d_{\rho}}(\mathcal{O}_p)$, where $\mathcal{O} = \mathcal{O}_p$ is the ring of integers of some finite extension of $\mathbb{Q}_p$. Denote $W_{\rho}$ to be a free $\mathcal{O}$-module of rank $d_{\rho}$ realizing $\rho$. If $M$ is a $\mathbb{Z}_p[G]$-module, we define $\text{tw}_{\rho}(M)$ to be the $\mathcal{O}_{\rho}$-module $W_{\rho} \otimes_{\mathbb{Z}_p} M$ with $G$ acting diagonally.

For an admissible $p$-adic extension $F_{\infty}$ with $G = \text{Gal}(F_{\infty}/F)$, we denote the twisted Selmer group by $S(\text{tw}_{\rho}(E)/F_{\infty})$ which is obtained by taking $A = E_{p_{\infty}} \otimes_{\mathbb{Z}_p} W_{\rho}$ and $A_v = C_v \otimes_{\mathbb{Z}_p} W_{\rho}$ in the definition of the Greenberg Selmer group in Section 7 (see also $[\text{CFKS}]$ P. 47-48). We denote $X(\text{tw}_{\rho}(E)/F_{\infty})$ to be the Pontryagin dual of $S(\text{tw}_{\rho}(E)/F_{\infty})$. By $[\text{CFKS}]$ Lemma 3.4, we have

$$\text{tw}_{\rho}X(E/F_{\infty}) = X(\text{tw}_{\rho}(E)/F_{\infty}),$$

where $\rho^*$ is the contragredient of $\rho$. 17
Lemma 5.10. Suppose that $E$ has good ordinary reduction at every prime of $F$ above $p$ and suppose that $F$ is not totally real. If $X(E/F_\infty) \neq 0$, then $X_{tw}(E/F_\infty)$ is not a finitely generated torsion $O[H]$-module.

Proof. We will prove the lemma by contradiction. Suppose that $X_{tw}(E/F_\infty)$ is a finitely generated torsion $O[H]$-module. Let $L$ be a finite extension of $F$ contained in $F_\infty$ such that $\text{Gal}(F_\infty/L)$ is contained in $\ker \rho$. Write $H_L = \text{Gal}(F_\infty/L^{\text{cyc}})$. Since $H_L$ is a subgroup of $H$ with finite index and $O$ is a finite free $\mathbb{Z}_p$-algebra, we have that $O[H]$ is a finite free $\mathbb{Z}_p[H_L]$-algebra and

$$O[H] \otimes_{\mathbb{Z}_p[H_L]} \text{Hom}_{\mathbb{Z}_p[H_L]}(X_{tw}(E/F_\infty), \mathbb{Z}_p[H])$$

$$\cong \text{Hom}_{O[H]}(O[H] \otimes_{\mathbb{Z}_p[H_L]} X_{tw}(E/F_\infty), O[H])$$

$$\cong \text{Hom}_{O[H]}(X_{tw}(E/F_\infty), O[H])^{\text{H} : H_L \otimes O[Z_p]}$$

$$= 0.$$ 

This in turn implies that $\text{Hom}_{\mathbb{Z}_p[H_L]}(X_{tw}(E/F_\infty), \mathbb{Z}_p[H_L]) = 0$. On the other hand, we have

$$0 = \text{Hom}_{\mathbb{Z}_p[H_L]}(X_{tw}(E/F_\infty), \mathbb{Z}_p[H_L])$$

$$= \text{Hom}_{\mathbb{Z}_p[H_L]}(X(E/F_\infty) \otimes_{\mathbb{Z}_p} W_\rho, \mathbb{Z}_p[H_L])$$

$$\cong \text{Hom}_{\mathbb{Z}_p[H_L]}(W_\rho, \text{Hom}_{\mathbb{Z}_p[H_L]}(X(E/F_\infty), \mathbb{Z}_p[H_L])).$$

(The last isomorphism is the adjointness isomorphism which makes sense here, since $H_L$ acts trivially on $W_\rho$ by our choice of $L$.) Since $W_\rho$ is a free $\mathbb{Z}_p$-module, it follows from the above that

$$\text{Hom}_{\mathbb{Z}_p[H_L]}(X(E/F_\infty), \mathbb{Z}_p[H_L]) = 0.$$ 

This in turns implies that $X(E/F_\infty)$ is a torsion $\mathbb{Z}_p[H_L]$-module, contradicting Lemma 5.8. Thus, we have proven our lemma. \hfill \qed

We can now prove the analogue of Theorem 5.3 for the Artin twists of the Selmer group. This result may also be viewed as a generalization of [DD, Corollary A.14].

Theorem 5.11. Assume that (i) $E$ has good ordinary reduction at every prime of $F$ above $p$, (ii) the number field $F$ is not totally real, (iii) $G$ is pro-$p$ and has no $p$-torsion and (iv) $X_f(E/F_\infty)$ is finitely generated over $\mathbb{Z}_p[H]$. Then the following statements are equivalent.

(a) $X(E/F_\infty) = 0$.

(b) $\text{Ak}(X_{tw}(E/F_\infty))$ is a unit in $O_\rho[\Gamma]$ for some Artin representation $\rho$ of $G$.

(c) $\text{Ak}(X_{tw}(E/F_\infty))$ is a unit in $O_\rho[\Gamma]$ for every Artin representation $\rho$ of $G$.

Proof. Clearly, one has the implications (a) $\Rightarrow$ (c) $\Rightarrow$ (b). Now suppose that (b) holds. For convenience, we will write $O = O_\rho$. Since $X_f(E/F_\infty)$ is finitely generated over $\mathbb{Z}_p[H]$, it follows from [CFKSV, Lemma 3.2] that $X_f(tw_\rho(E/F_\infty))$ is a finitely generated $O[H]$-module. Therefore, we may apply an $O$-analogue of Theorem 3.1 to conclude that

$$\mu_G(X_{tw}(E/F_\infty)) = \mu_G(X_{tw}(E/F^{\text{cyc}})).$$

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We then apply an $O$-analogue of Proposition 5.9 to conclude that $X(\text{tw}_p(E)/F_\infty)$ is a finitely generated pseudo-null $O[G]$-module. In particularly, it follows from the proof of Proposition 5.9 that one has $\mu_G(X(\text{tw}_p(E)/F_\infty)) = 0$. Combining this with the above, we obtain $\mu_{\Gamma}(X(\text{tw}_p(E)/F^{\text{cyc}})) = 0$. By the structure theory of finitely generated $O[\Gamma]$-modules, this in turn implies that $X(\text{tw}_p(E)/F^{\text{cyc}})$ is finitely generated over $O$. It then follows from an application of Nakayama Lemma that $X(\text{tw}_p(E)/F_\infty)$ is finitely generated over $O[H]$. Since $X(\text{tw}_p(E)/F_\infty)$ is also a pseudo-null $O[G]$-module, we can apply a well-known result of Venjakob (see [V2, Example 2.3 and Proposition 5.4]) to conclude that $X(\text{tw}_p(E)/F_\infty)$ is a finitely generated torsion $O[\Gamma]$-module. By Lemma 5.10 this in turn implies that $X(E/F_\infty) = 0$. 

We end the section considering an analogue of Proposition 5.1 for the Galois group of the maximal abelian extension of $F_\infty$ unramified outside $S$. The statement may have been known among the experts but does not seem to be written down anywhere.

**Proposition 5.12.** Assume that $F_\infty$ is an admissible extension of $F$ such that $G = \text{Gal}(F_\infty/F)$ has no $p$-torsion. Let $S$ be a finite set of primes of $F$ containing those above $p$, the infinite primes and the primes that ramify in $F_\infty/F$. Write $X_S(F_\infty) = G_S(F_\infty)^{ab}(p)$. Then the following statements hold:

(a) $H_0(H, X_S(F_\infty))$ has $\mathbb{Z}_p[\Gamma]$-rank $r_2(F)$ and its $\mu_\Gamma$-invariant is precisely the quantity $\mu_{\Gamma}(X_S(F^{\text{cyc}}))$.

(b) $H_i(H, X_S(F_\infty))$ is finitely generated over $\mathbb{Z}_p$ for $i \geq 1$.

(c) $H_i(H, X_S(F_\infty)) = 0$ for $i \geq \dim G - 1$.

**Proof.** It is a fundamental knowledge that $H^2(G_S(F^{\text{cyc}}), \mathbb{Q}_p/\mathbb{Z}_p) = 0$ and $H^2(G_S(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p) = 0$. Therefore, the spectral sequence

$$H^i(H, H^j(G_S(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p)) \implies H^{i+j}(G_S(F^{\text{cyc}}), \mathbb{Q}_p/\mathbb{Z}_p)$$

degenerates to give an exact sequence

$$0 \rightarrow H^1(H, \mathbb{Q}_p/\mathbb{Z}_p) \rightarrow H^1(G_S(F^{\text{cyc}}), \mathbb{Q}_p/\mathbb{Z}_p) \rightarrow H^1(G_S(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p)^H \rightarrow H^2(H, \mathbb{Q}_p/\mathbb{Z}_p)$$

and an isomorphism

$$H^i(H, H^1(G_S(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p)) \cong H^{i+2}(H, \mathbb{Q}_p/\mathbb{Z}_p) \text{ for each } i \geq 1.$$

All the statements in the proposition now follow immediately from the facts that $H^1(H, \mathbb{Q}_p/\mathbb{Z}_p)$ is cofinitely generated over $\mathbb{Z}_p$ and that $H^1(G_S(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p)^H \cong X_S(F_\infty)$.

### 6 On the integrality of the Akashi series

In this section, we discuss certain integrality properties of the Akashi series of Selmer groups. We will also establish [CFKSV] Conjecture 4.8 (Case 4) for the characteristic elements attached to certain classes of modules in $\mathfrak{M}_H(G)$.

As before, we say that $F_\infty$ is an admissible $p$-adic Lie extension of $F$ if (i) $\text{Gal}(F_\infty/F)$ is a compact $p$-adic Lie group, (ii) $F_\infty$ contains the cyclotomic $\mathbb{Z}_p$-extension $F^{\text{cyc}}$ of $F$ and (iii) $F_\infty$ is unramified.
outside a finite set of primes of $F$. We continue to write $G = \text{Gal}(F_\infty / F)$, $H = \text{Gal}(F_\infty / F^{\text{cyc}})$ and $\Gamma = \text{Gal}(F^{\text{cyc}} / F)$.

Let $\rho : G \longrightarrow GL_{d_\rho}(\mathcal{O}_p)$ be an Artin representation, where $\mathcal{O} = \mathcal{O}_p$ is the ring of integers of some finite extension of $\mathbb{Q}_p$. As in the previous section, the twisted Selmer group $S(tw_\rho(E)/F_\infty)$ is defined by taking $A = E_{p\infty} \otimes \mathbb{Z}_p W_\rho$ and $A_\psi = C_\psi \otimes \mathbb{Z}_p W_\rho$ in the definition of the Greenberg Selmer group in Section 7 (see also [CFKS, P. 47-48]). We denote by $\lambda_\rho(F_\infty)$ the localization map and by $X(tw_\rho(E)/F_\infty)$ the Pontryagin dual of $S(tw_\rho(E)/F_\infty)$. We can now state the following proposition which slightly refines [CFKSV] Lemma 3.9 for $X(E/F_\infty)$.

**Proposition 6.1.** Assume that (i) $E$ has good ordinary reduction at every prime of $F$ above $p$, (ii) $G = \text{Gal}(F_\infty / F)$ is pro-$p$ and has no $p$-torsion, (iii) $X_f(E/F_\infty)$ is finitely generated over $\mathbb{Z}_p[H]$ and (iv) $H_i(H', X(E/F_\infty))$ is finite for every open normal subgroup $H'$ of $H$ and $i \geq 1$.

Then $\text{Ak}(tw_\rho(X(E/F_\infty))) \in \mathcal{O}[\Gamma][\mathcal{O}[\Gamma]]$ for every Artin representation $\rho$ of $G$.

**Proof.** Since $X_f(E/F_\infty)$ is finitely generated over $\mathbb{Z}_p[H]$, it follows from [CFKSV] Lemma 3.2 and [CFKS] Lemma 3.4 that $X_f(tw_\rho(E)/F_\infty)$ is a finitely generated $\mathcal{O}[H]$-module. By an application of [CS3, Proposition 2.5], this in turn implies that $X(tw_\rho(E)/L^{\text{cyc}})$ is a finitely generated torsion $\mathcal{O}[\text{Gal}(L^{\text{cyc}}/L)]$-module for every finite extension $L$ of $F$ contained in $F_\infty$.

We claim that $(E[p^{\infty} ] \otimes \mathbb{Z}_p W_\rho)(L^{\text{cyc}})$ is finite. Suppose for now that this claim holds. Then by a similar argument to that of Corollary 4.4, we have that $H^2(G_S(L^{\text{cyc}}), E[p^{\infty} ] \otimes \mathbb{Z}_p W_\rho) = 0$ and the localization map $\lambda_\rho(L^{\text{cyc}})$ is surjective for all $L$ (and in particular for $F$), and that $H^2(G_S(F_\infty), E[p^{\infty} ] \otimes \mathbb{Z}_p W_\rho) = 0$ and the localization map $\lambda_\rho(F_\infty)$ is surjective. Therefore, by an entirely parallel argument to that of Proposition 5.1, we conclude that $H_i(H, tw_\rho(X(E/F_\infty)))$ is finitely generated over $\mathcal{O}$ for every $i \geq 1$. On the other hand, by virtue of the assumptions (iii) and (iv), it follows from the proof of [CFKSV] Lemma 3.9 that, for every $i \geq 1$, $H_i(H, tw_\rho(X(E/F_\infty)))$ is annihilated by some power of $p$. Hence we conclude that $H_i(H, tw_\rho(X(E/F_\infty)))$ is finite for every $i \geq 1$. Therefore, the Akashi series of $tw_\rho(X(E/F_\infty))$ is precisely the $\mathbb{Z}_p[[\Gamma]]$-characteristic power series of $H_0(H, tw_\rho(X(E/F_\infty)))$, and so, necessarily lies in $\mathcal{O}[[\Gamma]] / \mathcal{O}[[\Gamma]]^\times$.

It remains to verify our claim. Since $\rho$ is an Artin representation, we may find some finite extension $L_0$ of $L$ contained in $F_\infty$ such that $\rho$ factors though $\text{Gal}(L_0/L)$. Then we have

$$
(E_{p^{\infty}} \otimes \mathbb{Z}_p W_\rho)(L_0^{\text{cyc}}) = E_{p^{\infty}}(L_0^{\text{cyc}}) \otimes \mathbb{Z}_p,
$$

where the latter is finite by [W] Theorem 4.3. Thus, it follows that $(E_{p^{\infty}} \otimes \mathbb{Z}_p W_\rho)(L^{\text{cyc}})$ is also finite. \[ \square \]

To prove the next set of results, we need to introduce some further notion and notation. Let

$$
\Sigma = \{ s \in \mathbb{Z}_p[G] \mid \mathbb{Z}_p[G] / \mathbb{Z}_p[G] s \text{ is a finitely generated } \mathbb{Z}_p[H]-\text{module}\}.
$$

By [CFKSV] Theorem 2.4, $\Sigma$ is a left and right Ore set consisting of non-zero divisors in $\mathbb{Z}_p[G]$. Set $\Sigma^* = \bigcup_{n \geq 0} p^n \Sigma$. It follows from [CFKSV] Proposition 2.3 that a finitely generated $\mathbb{Z}_p[G]$-module $M$ is annihilated by $\Sigma^*$ if and only if $M/M(p)$ is finitely generated over $\mathbb{Z}_p[H]$. Therefore, the category $\mathfrak{M}_H(G)$ can also be thought of as the category of all finitely generated $\mathbb{Z}_p[G]$-modules which are $\Sigma^*$-torsion.

By the discussion in [CFKSV] Section 3, we have the following exact sequence

$$
K_1(\mathbb{Z}_p[G]) \longrightarrow K_1(\mathbb{Z}_p[G]_{\Sigma^*}) \overset{\partial_{\Sigma^*}}{\longrightarrow} K_0(\mathfrak{M}_H(G)) \longrightarrow 0
$$

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of $K$-groups. For each $M$ in $\mathcal{M}_H(G)$, we define a characteristic element for $M$ to be any element $\xi_M$ in $K_1(\mathbb{Z}_p[G][\Sigma^\infty])$ with the property that

$$\partial_G(\xi_M) = [M].$$

Let $\rho: G \to GL_d(O_\rho)$ denote a continuous group representation (not necessarily an Artin representation), where $O = O_\rho$ is the ring of integers of some finite extension of $Q_p$. For $g \in G$, we write $\bar{g}$ for its image in $\Gamma = G/H$. We define a continuous group homomorphism

$$G \to M_d(O) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Gamma], \quad g \mapsto \rho(g) \otimes \bar{g}.$$  

By [CFKSV] Lemma 3.3, this in turn induces a map

$$\Phi': K_1(\mathbb{Z}_p[G][\Sigma^\infty]) \to Q_O(\Gamma),$$

where $Q_O(\Gamma)$ is the field of fraction of $O[\Gamma]$. Let $\varphi: O[\Gamma] \to O$ be the augmentation map and denote its kernel by $p$. One can extend $\varphi$ to a map $\varphi: O[\Gamma]_p \to K$, where $K$ is the field of fraction of $O$. Let $\xi$ be an arbitrary element in $K_1(\mathbb{Z}_p[G][\Sigma^\infty])$. If $\Phi'(\xi) \in O[\Gamma]_p$, we define $\xi(\rho)$ to be $\varphi(\Phi'(\xi))$. If $\Phi'(\xi) \notin O[\Gamma]_p$, we set $\xi(\rho)$ to be $\infty$.

We will write $\alpha$ for the natural map

$$\mathbb{Z}_p[G][\Sigma^\infty] \to K_1(\mathbb{Z}_p[G][\Sigma^\infty]).$$

We can now state our results which will prove [CFKSV] Conjecture 4.8 Case 4 for the characteristic elements attached to certain modules.

**Proposition 6.2.** Let $M$ be an object in $\mathcal{M}_H(G)$. Assume that $M$ has no nonzero pseudo-null $\mathbb{Z}_p[G]$-submodule. Let $\xi_M$ be a characteristic element of $M$. Then the following statements are equivalent.

(a) $\xi_M \in \alpha(\mathbb{Z}_p[G]^\times)$, where $\alpha$ is the map $\mathbb{Z}_p[G][\Sigma^\infty] \to K_1(\mathbb{Z}_p[G][\Sigma^\infty]).$

(b) $\xi_M(\rho)$ is finite and lies in $O_\rho$ for every continuous group representation $\rho$ of $G$.

(c) $\Phi'(\xi_M) \in O_\rho[\Gamma]^\times$ for every continuous group representation $\rho$ of $G$.

(d) $\Phi'(\xi_M) \in O_\rho[\Gamma]^\times$ for every Artin representation $\rho$ of $G$.

**Proof.** By [CFKSV] Lemma 4.9, we have the implications (a)$\Rightarrow$(b)$\Leftrightarrow$(c)$\Rightarrow$(d). It remains to show (d)$\Rightarrow$(a). Taking $\rho: G \to \mathbb{Z}_p^\times$ to be the trivial representation, one has

$$\Phi'(\xi_M) = Ak(M) \mod \mathbb{Z}_p[\Gamma]^\times$$

for this particular $\rho$ by [CFKSV] Lemma 3.7. Applying (d), we have that $Ak(M) \in \mathbb{Z}_p[\Gamma]^\times$. By Corollary 5.3, this in turn implies that $\partial_G(\xi_M) = 0$. It then follows from the above exact sequence of $K$-groups that there exists an element in $K_1(\mathbb{Z}_p[G])$ which maps to $\xi_M$. On the other hand, it is well-known that $\mathbb{Z}_p[G]^\times$ maps onto $K_1(\mathbb{Z}_p[G])$, thus yielding (a).

In particular, if $X(E/F_\infty)$ lies in $\mathcal{M}_H(G)$ and has no nonzero pseudo-null $\mathbb{Z}_p[G]$-module, the above proposition applies to the characteristic elements of $X(E/F_\infty)$. In the case when we do not know whether $X(E/F_\infty)$ has no nonzero pseudo-null $\mathbb{Z}_p[G]$-module, we have the following.

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Proposition 6.3. Assume that (i) $E$ has good ordinary reduction at every prime of $F$ above $p$, (ii) the number field $F$ is not totally real, (iii) $G$ is pro-$p$ and has no $p$-torsion and (iv) $X(E/F_\infty) \in \mathfrak{M}_H(G)$. Let $\xi_E$ be a characteristic element of $X(E/F_\infty)$. Then the following statements are equivalent.

(a) $\xi_E \in \alpha(\mathbb{Z}_p[G]^{\times})$, where $\alpha$ is the map $\mathbb{Z}_p[G]^{\times} \rightarrow K_1(\mathbb{Z}_p[G]^{\times})$.

(b) $\xi_E(\rho)$ is finite and lies in $\mathcal{O}_\rho$, for every continuous group representation $\rho$ of $G$.

(c) $\Phi_p'(\xi_E) \in \mathcal{O}_p[\Gamma]^{\times}$ for every continuous group representation $\rho$ of $G$.

(d) $\Phi_p'(\xi_E) \in \mathcal{O}_p[\Gamma]^{\times}$ for every Artin representation $\rho$ of $G$.

Proof. The proof is similar to that of the preceding proposition, where one makes use of Theorem 5.11 in place of Corollary 5.5.

We end the section mentioning that under the assumptions of Proposition 6.1, one has $\Phi_p'(\xi_E) \in \mathcal{O}_p[\Gamma]$ which in turn implies that assertion (d) of [CFKSV Conjecture 4.8 Case 2] is satisfied. However, the author is not able to prove assertions (a), (b) and (c) of [CFKSV Conjecture 4.8 Case 2] (or even case 1 of the said conjecture) for $\xi_E$.

7 Complement: Other Ordinary Representations

As seen in the previous argument, most of the material in this article can be applied to other situations.

Indeed, this is the case, and we will briefly describe here a general context that one may apply the argument to.

Denote $\mathcal{O}$ to be the ring of integers of some finite extension $K$ of $\mathbb{Q}_p$. Fix a local parameter $\pi$ for $\mathcal{O}$. The material in Section 2 can be extended over for modules over $\mathcal{O}[G]$. One simply replaces any occurrences of “$p$” by “$\pi$”, “$\mathbb{Z}_p$” by “$\mathcal{O}$” and “$\mathbb{F}_p$” by “$\mathcal{O}/\mathcal{O}$”.

Let $A$ be a cofinitely generated $\mathcal{O}$-module with a continuous, $\mathcal{O}$-linear $\text{Gal}(\bar{F}/F)$-action which is unramified outside a finite set of primes of $F$. Assume further that $A^\vee$ has $\mathcal{O}$-rank $2d$, and for each $v$ above $p$, there is a $\mathcal{O}$-submodule $A_v$ of $A$ invariant under the action of $\text{Gal}(\bar{F}_v/F_v)$ and such that $(A_v)^\vee$ has $\mathcal{O}$-rank $d$. Write $D_v = A/A_v$. Note that $(D_v)^\vee$ has $\mathcal{O}$-rank $d$.

For an algebraic (possibly infinite) extension $\mathcal{L}$ of $F^{\text{cyc}}$, we define the Greenberg Selmer group of $A$ over $\mathcal{L}$ by

$$S(A/\mathcal{L}) = \ker \left( H^1(\mathcal{L}, A) \rightarrow \prod_{w|p} H^1(\mathcal{L}, A) \times \prod_{w|p} H^1(\mathcal{L}, D_w) \right),$$

where we set $D_w = D_v$ whenever $w$ divides $v$. Now for every finite extension $\mathcal{L}$ of $F^{\text{cyc}}$, we define $J_v(A/\mathcal{L})$ to be

$$\bigoplus_{w|v} H^1(\mathcal{L}_w, A) \text{ or } \bigoplus_{w|v} H^1(\mathcal{L}_w, D_w)$$

according as $v$ does not or does divide $p$. In the case that $\mathcal{L}$ is an infinite extension of $F^{\text{cyc}}$, we will define

$$J_v(A/\mathcal{L}) = \lim_{\mathcal{L}'} J_v(A/\mathcal{L}'),$$

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where the direct limit is taken over all finite extensions $L'$ of $\mathbb{F}^{cyc}$ contained in $L$. Let $S$ be a finite set of primes of $F$ which contains the primes above $p$, the ramified primes of $A$ and the infinite primes. It then follows from a standard argument (see [CS3 Corollary 3.2]) that for every extension $L$ of $\mathbb{F}^{cyc}$ contained in $F_S$, one has the following exact sequence

$$0 \longrightarrow S(A/L) \longrightarrow H^1(G_S(L), A) \xrightarrow{\lambda_S(L)} \bigoplus_{v \in S} J_v(A/L).$$

We will write $X(A/L) = S(A/L)^\vee$. Then one can prove results analogous to Theorem 5.1 and Proposition 5.7 for $X(A/L)$. Note that we may have to replace the condition “$X(A/\mathbb{F}^{cyc})$ is $\mathcal{O}[\mathcal{G}]$-torsion” by the conditions “$H^2(G_S(\mathbb{F}^{cyc}), A) = 0$” and “$\lambda_S(\mathbb{F}^{cyc})$ is surjective”. This is because we do not have an analog statement as in Proposition 3.3(i) for a general $A$. The point is that we may not have the finiteness of $A(\mathbb{F}^{cyc})$. However, for many interesting representations which we shall see below, this does hold.

Typical examples of the above (besides the case of an elliptic curve and its Artin twist as considered in the main body of the paper) are abelian varieties $A$ with good ordinary reduction at all primes above $p$. We note that the finiteness of $A(\mathbb{F}^{cyc})$ follows from [W Theorem 4.3]. The nonexistence of nontrivial pseudo-null submodules for the dual Selmer groups was established in certain cases [Oc], although they contain less situation of the $p$-adic extensions than the case of an elliptic curve. Despite this, if one assumes that the base field $F$ is not totally real, one can still make use of the result of Matsuno [Mat, Proposition 7.5] and Proposition 6.16 to obtain similar results to those in Theorems 5.3 and 5.11. Similarly, one can obtain analogous results to those in Section 6.

We now state the following analogue of Hachimori’s formula [HSh, Theorem 5.4] (see also [Bh Theorem 16], [CH Corollary 6.10], [Ho Theorem 2.8] and [HV Theorem 3.1]) for a general Galois representation, and we give a proof of this statement for completeness.

**Proposition 7.1.** Assume that (i) $X(A/\mathbb{F}^{cyc})$ is finitely generated over $\mathcal{O}$, (ii) $A(\mathbb{F}^{cyc})$ is finite, and (iii) $F_\infty$ is an $S$-admissible $p$-adic extension of $F$ with $G = \text{Gal}(F_\infty/F)$ being a pro-$p$ group with no $p$-torsion. Then $H^2(G_S(\mathbb{F}^{cyc}), A) = 0$, $\lambda_S(\mathbb{F}^{cyc})$ is surjective and we have the following formula

$$\text{rank}_{\mathcal{O}[H]} X(A/\mathbb{F}^{cyc}) = \text{rank}_{\mathcal{O}} X(A/\mathbb{F}^{cyc}) + \sum_w \left( \text{rank}_{\mathcal{O}} B_w(\mathbb{F}^{cyc}) - \text{rank}_{\mathcal{O}[H_w]} B_w(\mathbb{F}_w) \right),$$

where $w$ runs through the primes of $\mathbb{F}^{cyc}$ above $S$, $H_w$ is the decomposition group of $H$ at some prime of $F_\infty$ above $w$, and $B_w$ denotes $A$ or $D_w$ according as $w$ does not or does divide $p$. In particular, we have

$$\text{rank}_{\mathcal{O}[H]} X(A/\mathbb{F}^{cyc}) \geq \text{rank}_{\mathcal{O}} X(A/\mathbb{F}^{cyc}).$$

**Proof.** Since $X(A/\mathbb{F}^{cyc})$ is finitely generated over $\mathcal{O}$ and $G$ is pro-$p$ with no $p$-torsion, it follows that $X(A/\mathbb{F}^{cyc})$ is finitely generated over $\mathcal{O}[H]$. In particular, we have that $H^2(G_S(\mathbb{F}_\infty), A) = 0$ and $\lambda_S(\mathbb{F}_\infty)$ is surjective. Then we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S(A/\mathbb{F}^{cyc}) & \longrightarrow & H^1(G_S(\mathbb{F}^{cyc}), A) & \longrightarrow & \bigoplus_{v \in S} J_v(A/\mathbb{F}^{cyc}) & \longrightarrow & 0 \\
\alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & & & \\
0 & \longrightarrow & S(A/\mathbb{F}_\infty)^H & \longrightarrow & H^1(G_S(\mathbb{F}_\infty), A)^H & \longrightarrow & \bigoplus_{v \in S} J_v(A/\mathbb{F}_\infty)^H & \longrightarrow & H^1(H, S(A/\mathbb{F}_\infty)^H) & \longrightarrow & \cdots
\end{array}$$

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with exact rows. To simplify notation, we write $W_\infty = H^1(G_S(F_\infty), A)$ and $J_\infty = \bigoplus_{v \in S} J_v(A/F_\infty)$, which in turns yield a long exact sequence

$$0 \to \ker \alpha \to \ker \beta \to \ker \gamma \to \coker \alpha \to \coker \beta$$

$$\to \coker \gamma \to H^1(H, S(A/F_\infty)) \to H^1(H, W_\infty) \to H^1(H, J_\infty) \to \cdots$$

$$\cdots \to H^{i-1}(H, J_\infty) \to H^i(H, S(A/F_\infty)) \to H^i(H, W_\infty) \to H^i(H, J_\infty) \to \cdots$$

of cofinitely generated $O$-modules. By assumption (i), the following exact sequence

$$0 \to \ker \alpha \to S(A/F_{\text{cyc}}) \to S(A/F_\infty)^H \to \coker \alpha \to 0$$

is an exact sequence of cofinitely generated $O$-modules. Comparing the $O$-ranks of the terms in both exact sequences, we obtain

$$\sum_{i \geq 0} (-1)^i \text{rank}_O H_i(H, X(A/F_\infty)) = \text{rank}_O X(A/F_{\text{cyc}}) + \sum_{j \geq 1} (-1)^j H_j(H, A(F_\infty))$$

$$+ \sum_w \sum_{j \geq 1} (-1)^{j+1} H_j(H_w, B_w(F_\infty)).$$

The required formula then follows from an application of a formula of Howson [Ho, Theorem 1.1] on each of the alternating sum. \qed

Another interesting example one may consider is the Galois representation coming from a primitive Hecke eigenform of weight $k > 2$ for $GL_2/\mathbb{Q}$, which is ordinary at $p$. Here the finiteness of $A(F_{\text{cyc}})$ is established in [Su, Proof of Lemma 2.2]. Also, as in the case of an elliptic curve, by combining results of Kato and Hachimori-Ochiai, we can find examples where one can construct Akashi series for these Selmer groups unconditionally. The nonexistence of nontrivial pseudo-null submodules for the dual Selmer groups is established in certain cases [Su, Theorem 3.1]. As in the case of abelian varieties, one can therefore obtain results for the Akashi series of these Selmer groups similar to Theorem 5.3.

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References

[Bh] A. Bhave, Analogue of Kida’s formula for certain strongly admissible extensions, J. Number Theory 122 (2007), no. 1, 100–120.

[C] J. Coates, Fragments of the $GL_2$ Iwasawa theory of elliptic curves without complex multiplication. Arithmetic theory of elliptic curves (Cetraro, 1997), 1–50, Lecture Notes in Math., 1716, Springer, Berlin, 1999.

[CFKS] J. Coates, T. Fukaya, K. Kato and R. Sujatha Root numbers, Selmer groups and noncommutative Iwasawa theory, J. Algebraic Geom. 19 (2010), no. 1, 19–97.

[CFKSV] J. Coates, T. Fukaya, K. Kato, R. Sujatha and O. Venjakob, The $GL_2$ main conjecture for elliptic curves without complex multiplication, Publ. Math. IHES 101 (2005), 163–208.

[CG] J. Coates and R. Greenberg, Kummer theory for abelian varieties over local fields, Invent. Math. 124 (1996), 129–174.
[CH] J. Coates and S. Howson, Euler characteristics and elliptic curves II, *J. Math. Soc. Japan* **53** (2001), no. 1, 175–235.

[CSS] J. Coates, P. Schneider and R. Sujatha, Links between cyclotomic and $GL_2$ Iwasawa theory. Kazuya Kato’s fiftieth birthday. *Doc. Math.* 2003, Extra Vol., 187–215 (electronic).

[CS1] J. Coates and R. Sujatha, *Galois Cohomology of Elliptic Curves*, Tata Institute of Fundamental Research Lectures on Mathematics, **88**, Published by Narosa Publishing House, New Delhi; for the Tata Institute of Fundamental Research, Mumbai, 2000.

[CS2] ———, Fine Selmer groups of elliptic curves over $p$-adic Lie extensions, *Math. Ann.* **331** (2005), no. 4, 809–839.

[CS3] ———, On the $\mathfrak{M}_H(G)$-conjecture. *Non-abelian fundamental groups and Iwasawa theory*, 132–161, London Math. Soc. Lecture Note Ser., **393**, Cambridge Univ. Press, Cambridge, 2012.

[DD] T. Dokchitser and V. Dokchitser, Computations in non-commutative Iwasawa theory. With an appendix by J. Coates and R. Sujatha. *Proc. Lond. Math. Soc. (3)* **94** (2007), no. 1, 211–272.

[GW] K. R. Goodearl and R. B. Warfield, *An introduction to non-commutative Noetherian rings*, London Math. Soc. Stud. Texts **61**, Cambridge University Press, 2004.

[Gr] R. Greenberg, Iwasawa theory for elliptic curves. *Arithmetic theory of elliptic curves (Cetraro, 1997)*, 51–144, Lecture Notes in Math., **1716**, Springer, Berlin, 1999.

[HO] Y. Hachimori and T. Ochiai, Notes on non-commutative Iwasawa theory, *Asian J. Math.* **14** (2010), no. 1, 11–17.

[HSh] Y. Hachimori and R. Sharifi, On the failure of pseudo-nullity of Iwasawa modules, *J. Algebraic Geom.* **14** (2005), no. 3, 567–591.

[HV] Y. Hachimori and O. Venjakob, Completely faithful Selmer groups over Kummer extensions. Kazuya Kato’s fiftieth birthday. *Doc. Math.* 2003, Extra Vol., 443–478 (electronic).

[Ho] S. Howson, Euler characteristic as invariants of Iwasawa modules, *Proc. London Math. Soc. (3)* **85** (2002), no. 3, 634–658.

[K] K. Kato, $p$-adic Hodge theory and values of zeta functions of modular forms, in: *Cohomologies $p$-adiques et applications arithmétiques. III.*, Astérisque **295**, 2004, ix, 117–290.

[KT] Y. Kubo and Y. Taguchi, A generalization of a theorem of Imai and its applications to Iwasawa theory, *Math. Z.* **275** (2013), no. 3-4, 1181–1195.

[Lam] T. Y. Lam, *Lectures on Modules and Rings*, Grad. Texts in Math. **189**, Springer-Verlag, New York, 1999.

[Lee] C-Y Lee, Non-commutative Iwasawa theory of elliptic curves at primes of multiplicative reduction, *Math. Proc. Cambridge Philos. Soc.* **154** (2013), no. 2, 303–324.

[Mat] K. Matsuno, Finite $\Lambda$-submodules of Selmer groups of abelian varieties over cyclotomic $\mathbb{Z}_p$-extensions, *J. Number Theory* **99** (2003), no. 2, 415–443.

[Maz] B. Mazur, Rational points of abelian varieties with values in towers of number fields, *Invent. Math.* **18** (1972), 183–206.

[NSW] J. Neukirch, A. Schmidt and K. Wingberg, *Cohomology of Number Fields*, 2nd Ed., Grundlehren Math. Wiss. **323**, Springer-Verlag, Berlin, 2008.

[Neu] A. Neumann, Completed group algebras without zero divisors, *Arch. Math.** 51**(1988), no. 6, 496–499.

[Oc] Y. Ochi, A note on Selmer groups of abelian varieties over the trivializing extensions, *Proc. Amer. Math. Soc.* **134** (2006), no. 1, 31–37.

[OcV1] Y. Ochi and O. Venjakob, On the structure of Selmer groups over $p$-adic Lie extensions, *J. Algebraic Geom.* **11** (2002), no. 3, 547–580.

[OcV2] ———, On the ranks of Iwasawa modules over $p$-adic Lie extensions, *Math. Proc. Cambridge Philos. Soc.* **135** (2003), no. 1, 25–43.

[Sch] P. Schneider, $p$-adic height pairings II, *Invent. Math.* **79** (1985), no. 2, 329–374.
[Su] R. Sujatha, Iwasawa theory and modular forms, Pure Appl. Math. Q. 2 (2006), no. 2, 519–538.

[V1] O. Venjakob, On the structure theory of the Iwasawa algebra of a $p$-adic Lie group, J. Eur. Math. Soc. 4 (2002), no. 3, 271–311.

[V2] ______________, A noncommutative Weierstrass preparation theorem and applications to Iwasawa theory. With an appendix by Denis Vogel. J. reine angew. Math. 559 (2003), 153–191.

[W] K. Wingberg, On the rational points of abelian varieties over $\mathbb{Z}_p$-extensions of number fields, Math. Ann. 279 (1987), no. 1, 9–24.

[Z1] S. L. Zerbes, Selmer groups over $p$-adic Lie extensions I, J. London Math. Soc. (2) 70 (2004), no. 3, 586–608.

[Z2] ______________, Akashi series of Selmer groups, Math. Proc. Cambridge Philos. Soc. 151 (2011), no. 2, 229–243.