Determination of functional parameters in boundary conditions of linear hyperbolic systems by optimal control methods

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Abstract. An inverse problem of a functional parameter determination is considered for a hybrid system of population dynamics. Dynamics of several interacting populations is described by a system of first-order linear hyperbolic equations. The boundary conditions for this system are determined from the initial problem for ordinary differential equations. The problem is interpreted as an optimal control problem. A non-classic optimality condition of variational maximum principle type is proved. The original problem in distributed parameter systems is reduced to the optimal control problem for ordinary differential equations.

1. Introduction
When modeling population dynamics in some cases it is important to consider not only a total population size but also other parameters (age, weight, size, etc.). In this case, continuous mathematical models are described by partial differential equations. The question of setting parameters is relevant for a correct description of ecological processes. If the parameters are constants, statistical methods (usually the least squares method) are generally used to identify them. The main problem is that parameters are often functional in terms of spatial coordinates and time. In the paper, a problem of restoring one of the parameters of a hybrid system of differential equations describing population dynamics is interpreted as an optimal control problem. Dynamics of age-structured populations is described by linear hyperbolic equations. An unknown functional parameter determines a fertility pattern. A non-classic optimality condition of a variational maximum principle type is obtained. This condition makes it possible to reduce the optimal control by the hybrid system of differential equations to the optimal control problem by a system of ordinary differential equations.

2. Problem statement
Consider one of the classic models [1, 2] of age-structured population dynamics

$$\frac{\partial x}{\partial t} + \frac{\partial x}{\partial s} = B(t, s)x + b(t, s), \quad (t, s) \in P.$$  

Here $t$ represents time, $s$ denotes age, $t \in T = [0, t_1]$, $s \in S = [0, s_1]$, $P = T \times S$, $s_1$ is the maximum age. Scalar functions $x_i(t, s)$ describe age densities of corresponding populations,
\( i = 1, 2, ..., n \). So, \( x(t, s) \) is a \( n \)-dimensional vector function. Diagonal elements of a matrix \( B(t, s) \) are mortality rates. Other matrix elements describe the interaction of populations. "Predator-prey" interactions in particular can be described by this model. Let \( b_{ij} \) be elements of the matrix \( B \). If \( b_{ij} = 0, i \neq j \), then populations \( i \) and \( j \) do not interact.

For simplicity, the initial values are assumed to be fixed, that is

\[
x(0, s) = x^0(s), \quad s \in S.
\]  

(2)

The boundary conditions are determined from the controlled system of ordinary differential equations

\[
\frac{\partial x(t, 0)}{\partial t} = C(u(t), t)x(t, s_0) + g(u(t), t),
\]  

(3)

where \( u = (u_1, u_2, ..., u_m) \) is a \( m \)-dimensional functional parameter, \( C(u, t) \) is a \( n \times n \) matrix function, \( g(u, t) \) is a \( n \)-dimensional vector function.

The initial conditions for (3) are agreed with (2):

\[
x(0, 0) = x^0(0).
\]  

(4)

Consider an inverse problem for (1)-(4). Let we know population densities \( \bar{x}_i(s), i = 1, 2, ..., n \) at the final time moment \( t_1 \). Denote \( \bar{x} = (\bar{x}_1, \bar{x}_2, ..., \bar{x}_n) \). The goal is in determination of a functional parameter \( u(t) \) under restrictions

\[
u(t) \in U \subset E^m.
\]  

(5)

3. Interpretation of the inverse problem as an optimal control problem

We will consider the inverse problem as an optimal control problem. The functional parameter \( u(t) \) will be considered as a \( m \)-dimensional control function. The problem is in minimization of a quadratic cost functional

\[
J(u) = \int \|(x(t_1, s) - \bar{x}(s))\|^2 ds \to \min.
\]  

(6)

Here \( \| . \| \) is a designation of the Euclidean norm in \( n \)-dimensional space.

The problem (1)-(6) is considered under the following assumptions.

1) Functions \( \bar{x}(s), x^0(s) \) are continuous with respect to \( s \).
2) Functions \( C(u, t), g(u, t) \) are continuous with respect to their arguments.
3) The set of admissible controls is assumed to consist of bounded and measurable vector functions \( u(t) \) defined on \( T \) and satisfying almost everywhere in this interval a constraint of the inclusion type (5).

Note that the boundary conditions \( x(t, 0) \), which are determined by equations (3), are absolutely continuous functions in \( T \). The above assumptions are not sufficient for the existence of a classical (i.e., continuously differentiable) solution of the problem (1)-(4). Based on the equivalent integral representation of (1)-(2), the existence and uniqueness of the generalized solution is established in [3, 4]. This is a continuous in \( P \) vector function \( x(t, s) \) all components of which are continuously differentiable along the characteristics of (1).

4. Variational maximum principle

The optimal control problem (1)-(6) is the "almost" linear-quadratic problem with a convex cost functional. In spite of this fact, the Pontryagin maximum principle is not a sufficient optimality condition. This is explained by the dependence of the matrix of coefficients in (3) on controls.
The system (3) is a bilinear system. Usually such problems are solved by general optimal control methods which had been developed for general nonlinear problems.

We applied for this problem an approach based on non-standard increment formulas of the cost functional (6). Seems to be the first time this approach has been used by V.A. Srochko [5, 6] for optimal control problems in bilinear ordinary differential equations. However in ordinary differential equations this approach leads to discontinuous with respect to state functions differential equations.

For our problem this approach allows to prove the following optimality condition of a variational maximum principle type.

**Theorem.** Let $u(t)$ be an arbitrary admissible control in (1)-(6). The admissible function $u^*(t)$ be an optimal control in the problem if and only if $u^*$ is optimal in a special optimal control problem by ordinary differential equations

$$
I(v) = \int_T \langle q(t,u) + Q(t,u)(y(t,v) - x(0,t,v)),
(C(v(t),t) - C(u(t),t))y(t,v) + g(v(t),t) - g(u(t),t)\rangle dt \rightarrow \min,
$$

$$
\dot{y} = C(v(t),t)y + b(v(t),t), \quad t \in T, 
$$

$$
y(0) = x^0(0).$$

Here $\langle ., . \rangle$ is a designation of a scalar product in $n$-dimensional Euclidean space, vector function $q(t,u)$ and matrix function $Q(t,u)$ are determined from the corresponding boundary value problems for the adjoint systems.

We describe the solution scheme for the original optimal control problem based on the result obtained.

1. An arbitrary admissible control $u(t)$ is specified, and the corresponding solution $x(t,s)$ of the problem (1)–(4) for the original hyperbolic system is found. The adjoint problems are solved.
2. The auxiliary problem of the minimization of the quadratic performance index (7) for the system of ordinary differential equations, which are linear in the state variables, is solved. The matrix of coefficients in the cost functional can be transformed to a symmetric form by means of the standard symmetrization operation. Let $v^*(t)$ be an optimal control for the problem (7).
3. It follows from the increment formula that this control is also a solution of the original problem.

Thus, in order to solve the problem (1)–(6) it is required to integrate three times the system of hyperbolic equations (to find $x(t,u)$, to calculate a solution of an adjoint hyperbolic system and to find an optimal state function $x^*(t,s)$). Iterative methods in general nonlinear problems require integrations of hyperbolic systems during each iteration.

The result makes it possible to reduce the problem of optimal control (1) - (6) to the problem of optimal control of a bilinear system of ordinary differential equations with quadratic cost functional. All the effective tools developed previously for this class of problems can be used to solve the reduced problem [7, 8, 9].

Non-local improvement techniques for this purpose have been applied to the problems of functional parameter identification in models describing the dynamics of several invertebrate species in Lake Baikal.
5. Conclusion
A specific feature of the optimal control problems to which the original problem is reduced is as follows: all systems of ordinary differential equations are linear in the state variables, and the matrices of the coefficients of the phase variables depend on the control. The last circumstance makes it possible to solve the auxiliary problems, which are optimal control problems for systems of ordinary differential equations, by applying efficient approaches based on nonclassical exact formulas of the performance index increment [5, 6, 8]. The important feature of these approaches is that they permit integration of systems discontinuous in the phase or adjoint variables. The nonuniqueness of the solutions makes it possible, in a number of cases, to improve nonoptimal controls satisfying the maximum principle. In the case of the quadratic performance index, the methods are improved considerably through the appearance of additional quadratic "regularizing" terms.

The optimal control problems of form (1)–(6) appear to arise principally when studying transfer-related processes (for example, the processes of contamination, radiation, etc.) and those associated with the development of systems for which the age structure is important (basic economical funds, etc.) In these problems, the boundary conditions of a distributed system are generated by a controlled system of ordinary differential equations (e.g., an electromagnetic oscillator in the radiation intensity transfer problem).

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