Integrating Interval Constraints into Logic Programming

M.H. van Emden
Department of Computer Science
University of Victoria
Research report DCS-133-IR

Abstract

The clp scheme uses Horn clauses and SLD resolution to generate multiple constraint satisfaction problems (csp's). The possible csp's include rational trees (giving Prolog) and numerical algorithms for solving linear equations and linear programs (giving clp(R)). In this paper we develop a form of CSP for interval constraints. In this way one obtains a logic semantics for the efficient floating-point hardware that is available on most computers.

The need for the method arises because in the practice of scheduling and engineering design it is not enough to solve a single csp. Ideally one should be able to consider thousands of csp's and efficiently solve them or show them to be unsolvable. This is what clp/ncsp, the new subscheme of CLP described in this paper is designed to do.

1 Introduction

Floating-point arithmetic is marvelously cheap, and it works most of the time. Many textbooks on numerical analysis contain examples of how spectacularly, or insidiously, it can go wrong when it does not work. It would seem that in a mature computing technology there is only place for reliable techniques. Yet floating-point arithmetic is not to be lightly dismissed: it is one of the main beneficiaries of the enormous increase in processor performance of the last few decades. In combination with the insatiable demand for more computationally intensive mathematical modeling, this gives every motivation to use interval arithmetic [21] as a way to safely use the dangerous technology that is floating-point arithmetic.

Interval arithmetic ensures that, in spite of the errors inherent in floating-point arithmetic, a computation can be interpreted as a proof that the real-valued result is contained in the (interval) result of the computation. However, the correctness provided by interval arithmetic is limited to the evaluation of a single expression; it does not extend to the algorithm in which such evaluations take place. To ensure correctness of the way an algorithm combines expression evaluations one could of course
use verification methods for imperative programs, such as Floyd’s assertions. In this paper we consider the alternative of replacing the algorithms by logic programs, thus allowing programs to be read as specifications.

Logic programming is more than just an alternative to Floyd’s assertions. The logic framework suggests a relational form for interval computation. Such a relational form is provided by interval constraints, an improvement to interval arithmetic itself. Incorporating interval constraints into logic programming has the added advantage that the result goes beyond the constraint processing paradigm by yielding programs that generate multiple constraint satisfaction problems in addition to solving them. In scheduling and in engineering design it is typically the case that one has an entire search space of such problems. CLP/NCSP, the integrated system described in this paper, generates such search spaces. Solving is not only used for obtaining results, but also for pruning the search spaces by inducing early failure.

In Section 4 we start at the logic end with a review of the CLP scheme. We use Clark’s method for the semantics of logic programming schemes. As this method uses a mild form of algebraic logic, it needs some introduction; this happens in Section 3. In Section 5 we start at the opposite end with a suitably modified version of the main features of the Constraint Processing framework (CSP). To bring together the two established constraint approaches of the literature we develop in Section 6 what we call here the dc subscheme of the CLP scheme. The integration of interval constraints (reviewed in Section 7) into logic programming is described in Section 8.

2 Related work

The pioneering work in constraint logic programming is实施 [15], implemented as CHIP [13]. Prolog has sld resolution as sole inference rule; prolog added Forward Checking, Look-Ahead, and Partial Look-Ahead as additional inference rules, to be applied to goals, depending how they are declared.

CHIP was restricted to finite domains. ICCHP [20] proposed extending CHIP to include floating-point intervals as domains for real-valued variables. Descendants of CHIP such as the Eclipse system (see [4] for a recent description), implemented floating-point intervals.

The earliest design for integrating interval arithmetic into Prolog is Cleary’s [10], which served as basis for BNR-Prolog [8,17]. Cleary’s proposal of a “logical arithmetic” for Prolog described an implementation, but not a logical semantics. His paper and [12] are the first to describe relational, rather than functional, interval arithmetic. It remains to be seen whether the mathematical model given by Older and Vellino [22] can be connected to logic. BNR Prolog and Prolog IV [11] are mentioned here because of their connection with Prolog, but not because of connection with logic programming.

The CLP scheme [17] gives a logical semantics that combines pure Prolog with constraint solving. This scheme supersedes CHIP and its descendants as it is both simpler and more general.

The CLP scheme served as the basis for the CLP(R) system [19].
uses the scheme to generate answers to numerical problems in the form of “active constraints”. In the derivation of these, floating-point arithmetic is used without due precaution, so that the validity of answers is lost through rounding errors.

In [23] it was shown that the CLP scheme is general enough to accommodate both interval and finite-domain constraints. It does this by introducing “value constraints” without suggesting any way of interfacing these with intervals. This is done in this paper by means by means of CLP/DC, the DC subscheme of the CLP scheme. In this way we obtain CLP/NCSP, the first logic programming language (as distinct from extension of Prolog) with real variables in which only the precision, but not the validity, of answers is affected by rounding errors.

3 Logic Preliminaries

3.1 Relations

Relations play a central role in the integration of interval constraints into logic programming: both constraints and the meanings of logic predicates are relations. Here we do not attempt to define relations as generally as possible: we only strive for adequacy for the purpose of this paper. For a more drastic generalization of the usual notion of relation, see [24].

As usually defined, a relation is a subset of a Cartesian product \( S_1 \times \cdots \times S_k \). That is, it consists of tuples \( (a_1, \ldots, a_k) \) with \( a_i \in S_i \) for \( i = 1, \ldots, k \). Such tuples are indexed by the integers \( 1, \ldots, k \). In the following we will need such relations, as for example the ternary relation \( \text{sum} = \{ (x, y, z) \in \mathbb{R}^3 | x + y = z \} \), indexed by the set \( \{1, 2, 3\} \).

But we will also need relations consisting of tuples indexed by variables instead of integers. For example, the constraint written as \( \text{sum}(x_2, x_2, x_1) \) is intended to be a relation distinct for the sum relation just mentioned. As another example, \( \text{sum}(x_2, x_2, x_1) \land \text{sum}(x_3, x_4, x_1) \) is intended to be a relation. If so, which set of tuples? How indexed?

In this section we introduce the suitable type of relation; in Section 3.4 we define how they arise as meaning of the constraint expressions just shown.

**Definition 1** Given a set \( X = \{x_1, \ldots, x_N\} \) of variables, a relation \( \rho \subset S_1 \times \cdots \times S_n \) consisting of tuples indexed by \( \{1, \ldots, n\} \) and a sequence \( \langle v_1, \ldots, v_n \rangle \) of variables (not necessarily distinct), the relation \( \rho \) on \( v \) is the set of tuples \( \tau \) indexed by the set of the \( k \leq n \) variables in \( v \) such that \( \tau(v_i) = t_i \) for a tuple \( t \in \rho \), for all \( i = 1, \ldots, n \).

**Example 1** Let \( \rho \) be the ternary sum relation over the set \( N \) of natural numbers, \( X = \{x_1, \ldots, x_{100}\} \), and \( v = (x_2, x_2, x_1) \). Then we have as example of a tuple \( \tau \) in the relation \( \rho \) on \( v \):

\[
\begin{align*}
\tau(v_1) &= \tau(x_2) = t_1 \\
\tau(v_2) &= \tau(x_2) = t_2 \\
\tau(v_3) &= \tau(x_1) = t_3
\end{align*}
\]
Table 1: On the left, tabular form of the relation \( \text{sum} \) on \( \langle x_2, x_2, x_1, \rangle \) where
\[
\text{sum} = \{ (x, y, z) \in \mathbb{R}^3 \mid x + y = z \}
\]. On the right, tabular form of \( \rho_1 \Join \rho_2 \) from Example 2.

For such a \( \tau \) to exist, only tuples \( t \in \rho \) qualify where the first two elements are equal to each other. \( \tau \) consists of tuples indexed by the set \( \{ x_1, x_2 \} \).

The tabular form of \( \tau \) is as shown in Table 1.

| \( \tau \) | \( x_1 \) | \( x_2 \) | \( \rho_1 \Join \rho_2 \) | \( x_1 \) | \( x_2 \) | \( x_3 \) |
|-----|-----|-----|----------------|-----|-----|-----|
| 0   | 0   | 0   | 0              | 0   | 0   | 0   |
| 2   | 1   | 2   | 1              | 0   | 2   |
| 4   | 2   | 2   | 1              | 1   | 1   |
| 6   | 3   | 2   | 1              | 2   | 0   |
| 8   | 4   | 4   | 2              | 0   | 4   |
| 10  | 5   | 4   | 2              | 1   | 3   |
| \ldots | \ldots | \ldots | \ldots | \ldots | \ldots | \ldots |

Example 2 Let \( \rho \) be the ternary sum relation over the set \( \mathbb{N} \) of natural numbers, \( \chi = \{ x_1, \ldots, x_{100} \} \), \( v_1 = \langle x_2, x_2, x_1 \rangle \), and \( v_2 = \langle x_3, x_4, x_1 \rangle \). Let \( \rho_1 \) be \( \rho \) on \( v_1 \) and \( \rho_2 \) be \( \rho \) on \( v_2 \). Then \( \rho_1 \Join \rho_2 \) is a relation of which some tuples are shown in Table 1.

3.2 Language

The vocabulary of logic is formalized as a signature \( \Sigma = \langle P, F, V \rangle \), a tuple of disjoint, countably infinite, sets of predicates, functors, and variables. \( P \) is partitioned according to whether it may occur in a constraint or in a program. Thus we have “constraint predicates” and “program predicates”. The constraint predicates include the nullary \( \text{true} \) and \( \text{false} \) and the binary \( = \).

A term is a variable or an expression of the form \( f(t_0, \ldots, t_{k-1}) \), where \( f \in F \) and \( t_0, \ldots, t_{k-1} \) are terms. If \( k = 0 \), then the term is a constant.

An atom (or atomic formula) is an expression of the form \( p(t_0, \ldots, t_{k-1}) \), where \( p \in P \) is a predicate and \( t_0, \ldots, t_{k-1} \) are terms. If \( p \) is a program (constraint) predicate, then an atom with \( p \) as predicate is a program (constraint) atom.

A goal statement is a conjunction of program atoms or constraint atoms. A constraint is a conjunction of constraint atoms.
3.3 Interpretations
Interpretations depend on a language’s signature. They are formalized as \( \Sigma \)-structures \( \mathcal{I} = (D, P, F) \) where

- \( D \) is a non-empty set called the domain of the interpretation.
- \( P \) is a function mapping every \( k \)-ary predicate in \( P \) to a subset of \( D^k \). \( P \) maps true to \( \text{true} = \{\} \), false to \( \text{false} = \{\} \), and = to \( \{ (a, a) \mid a \in D \} \).
- \( F \) is a function mapping every functor \( f \) in \( F \) to a function mapping each \( k \)-ary functor in \( F \) to a \( k \)-adic function in \( D^k \rightarrow D \).

3.4 Denotations
An interpretation \( \langle D, P, F \rangle \) determines a function \( M \) mapping variable-free terms to their denotations, as follows:

- \( M(t) = F(t) \in D \) if \( t \) is a constant
- \( M(f(t_0, \ldots, t_{k-1})) = (F(f))(M(t_0), \ldots, M(t_{k-1})) \) for \( k > 0 \)
- A ground atom \( p(t_0, \ldots, t_{k-1}) \) is true in an interpretation iff \( \langle M(t_0), \ldots, M(t_{k-1}) \rangle \in P(p) \)

We give denotations of non-atomic formulas later, via relations.

We now consider denotations of terms and atoms that contain variables. Let \( \mathcal{A} \) be an assignment, which is a function in \( V \rightarrow D \), assigning an individual in \( D \) to every variable. (In other words, \( \mathcal{A} \) is a tuple of elements of \( D \) indexed by \( V \)). As denotations of formulas with free variables depend on \( \mathcal{A} \), we write \( M_\mathcal{A} \).

- \( M_\mathcal{A}(t) = \mathcal{A}(t) \) if \( t \in V \)
- \( M_\mathcal{A}(f(t_0, \ldots, t_{k-1})) = (F(f))(M_\mathcal{A}(t_0), \ldots, M_\mathcal{A}(t_{k-1})) \) for \( k > 0 \).
- An atom \( p(t_0, \ldots, t_{k-1}) \) is true in an interpretation iff \( \langle M_\mathcal{A}(t_0), \ldots, M_\mathcal{A}(t_{k-1}) \rangle \in P(p) \)

We give denotations of non-atomic formulas later, via relations.

The existential closure \( \exists x_0, \ldots, x_{n-1} \) of a set \( C \) of atoms is true in an interpretation iff there is an assignment \( \mathcal{A} \) such that \( M_\mathcal{A}(A) = \text{true} \) for every atom \( A \in C \).

**Definition 3** Let \( \mathcal{X} \subset V \) be the set of the free variables in formula \( C \). \( R(C) \), the relation denoted by \( C \), given the interpretation determining \( M_\mathcal{A} \), is defined as

\[
R(C) = \{ \mathcal{A} \downarrow \mathcal{X}_C \mid \mathcal{A} \text{ is an assignment and } M_\mathcal{A}(C) = \text{true} \}.
\]

By \( \mathcal{A} \downarrow \mathcal{X}_C \) we mean the function \( \mathcal{A} : V \rightarrow D \) restricted to arguments in \( \mathcal{X}_C \subset V \).

Thus \( R(C) \) consists of tuples indexed by variables. \( R \) allows us to translate between algebraic expressions in terms of relations and formulas of logic. This is useful because the results in constraint satisfaction
problems are expressed in terms of relations, whereas the constraint logic programming scheme is expressed in terms of first-order predicate logic.

We have of course $R(\text{true}) = M(\text{true}) = \text{true}$; also $R(\text{false}) = M(\text{false}) = \text{false}$. More interestingly, we may have $R(C_1 \land C_2) = R(C_1) \cap R(C_2)$ and $R(C_1 \lor C_2) = R(C_1) \cup R(C_2)$. But these hold only when $C_1$ and $C_2$ have the same set of variables. As this is not always the case, we also need to define $R(Z, C)$, where $Z$ is a set of variables containing $X_C$, the set of the free variables of $C$:

**Definition 4**

$$R(Z, C) = \{ A \downarrow Z \mid A \text{ is an assignment and } M_A(C) = \text{true} \}.$$  

Definitions (3) and (4) were suggested by a similar device first brought to our attention by [9]. The version here is modified to allow translations of a wider class of formulas. Their advantage is that of simplicity compared to other systems of algebraic logic such as [14].

$$R(X_{C_2}, C_1 \land C_2) = R(C_1) \cap R(C_2)$$

$$R(X_{C_1} \cup X_{C_2}, C_1 \land C_2)$$

$$R(X_{C_1} \cup X_{C_2}, C_1 \lor C_2) = R(X_{C_1} \cup X_{C_2}, C_1) \cup R(X_{C_1} \cup X_{C_2}, C_2).$$

To be able to interface the CLP scheme, expressed in terms of predicate logic formulas with CSPs, expressed in terms of relations, we will use the following lemma.

**Lemma 1** $R(X_{C_1} \cup X_{C_2}, C_1 \land C_2) = R(C_1) \cap R(C_2)$.

### 3.5 Logical implication

In the usual formulation of first-order predicate logic we find the notation $T \models S$ for the sentence $S$ being logically implied by sentence $T$, where “sentence” means closed formula. The meaning of the implication is that $S$ is true in all models of $T$. The denotations just defined allow logical implication to be generalized to apply to formulas that have free variables [9]:

**Definition 5** Let $S$ and $T$ be formulas and let $Z$ be the set of variables occurring in them. Then we write $T \models S$ to mean that in all interpretations $R(Z, S) \subset R(Z, T)$. Likewise, $T \models R(Z, S) \subset R(Z, T)$ means that $R(Z, S) \subset R(Z, T)$ holds in all models of $T$.

### 4 Review of the CLP scheme

The clp scheme is based on the observation that in logic programming the Herbrand base can be replaced by any of many other semantic domains. Hence the scheme has as parameter a tuple $\langle \Sigma, I, L, T \rangle$, where $\Sigma$ is a signature, $I$ is a $\Sigma$-structure, $L$ is a class of $\Sigma$-formulas, and $T$ is a first-order $\Sigma$-theory. These components play the following roles. $\Sigma$ determines the relations and functions that can occur in constraints. $I$ is the structure
over which computations are performed. \( \mathcal{L} \) is the class of constraints that can be expressed. Finally, \( \mathcal{T} \) axiomatizes properties of \( \mathcal{L} \).

Derivations in the clp scheme are defined by means of transitions between states. A state is defined as a tuple \( \langle G, A, P \rangle \) where the goal statement \( G \) is a set of atoms and constraints and \( A \) and \( P \) are sets of constraint. Together \( A \) and \( P \) form the constraint store. The constraints in \( A \) are called the active constraints; those in \( P \) the passive constraints.

The query \( Q \) corresponds to the initial state \( \langle Q, \emptyset, \emptyset \rangle \). A successful derivation is one that ends in a state of the form \( \langle \emptyset, A, P \rangle \).

The role of \( A \) and \( P \) in this formula is to describe the answer to the query \( Q \). \( A \land P \) is clp’s generalization of Prolog’s answer substitution. It describes an answer, if consistent. Such an answer may not be useful, as \( P \) may still represent a difficult computational problem. All that the derivation has done is to reduce the program atoms to constraint atoms, directly or indirectly via program atoms. Derivations also transfer as much as possible the computational burden of the passive constraints \( P \) to the easily solvable active constraints \( A \).

### 4.1 Operational semantics

A derivation is a sequence of states such that each next state is obtained from the previous one by a transition. There are four transitions, \( \rightarrow_r \), \( \rightarrow_c \), \( \rightarrow_i \), and \( \rightarrow_s \):

1. **The resolution transition** \( \rightarrow_r \): \( \langle G \cup \{a\}, A, P \rangle \rightarrow_r \langle G \cup B, A, P \cup \{s_1 = t_1, \ldots, s_n = t_n\} \rangle \) if \( a \) is the atom selected out of \( G \cup \{a\} \) by the computation rule, \( h \leftarrow B \) is a rule of \( \mathcal{P} \), renamed to new variables, and \( h = p(t_1, \ldots, t_n) \) and \( a = p(s_1, \ldots, s_n) \).

2. **The constraint transfer transition** \( \rightarrow_c \): \( \langle G \cup \{c\}, A, P \rangle \rightarrow_c \langle G, A, P \cup \{c\} \rangle \) if constraint \( c \) is selected by the computation rule.

3. **The constraint store management transition** \( \rightarrow_i \): \( \langle G, A, P \rangle \rightarrow_i \langle G, A', P' \rangle \) if \( \langle A', P' \rangle = \text{infer}(A, P) \).

4. **The consistency test transition** \( \rightarrow_s \): \( \langle G, A, P \rangle \rightarrow_s \langle G, A, P \rangle \) if \( A \) is consistent; \( \langle G, A, P \rangle \rightarrow_s \text{fail} \) otherwise.

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1. \( \mathcal{L} \) is a structure consisting of a set \( \mathcal{D} \) of values (the carrier of the structure) together with relations and functions over \( \mathcal{D} \) as specified by the signature \( \Sigma \). For example, there is a complete ordered field that has \( \mathcal{R} \), the set of real numbers, as carrier.

2. We will often regard \( A \) and \( P \) as formulas. Then they are the conjunctions of the atoms they contain.
4.2 Logic semantics

For the logic semantics of the clp scheme we follow [9].

Theorem 1 (soundness) Whenever we have a successful derivation from query \( Q \) resulting in \( P \) and \( A \) as passive and active constraints we have \( P, T \vdash R(\exists (P \land A)) \subset R(Q) \), where the quantification is over the free variables in \( P \land A \) that do not occur free in \( Q \). Note Definition 5 for “\( \vdash \)”.

Theorem 2 (completeness) Let \( Q \) be a query with variables \( X_Q \). If \( P, T \vdash R(X_Q, \Gamma) \subset R(Q) \) for a constraint atom \( \Gamma \), then there are \( k \) successful derivations from \( Q \) with answer constraints \( \Gamma_1, \ldots, \Gamma_k \) such that \( T \vdash R(X_Q, \Gamma) \subset R(X_Q, \Gamma_1) \cup \cdots \cup R(X_Q, \Gamma_k) \).

For credits see [9].

5 Constraint Satisfaction Problems

Constraint Satisfaction Problems (csp) can be defined as a framework to cover a variety of specific situations, each exploiting an algorithmic opportunity. For example, the csp framework can be instantiated to graph-colouring problems exploiting an efficient algorithm for the all-different constraint based on matching in bipartite graphs. It can also be instantiated to the solution of arithmetical constraints over real-valued variables using efficient algorithms and hardware for floating-point intervals. It is for this latter instantiation that we are interested in csp. But before describing it, first the general framework.

5.1 CSPs according to Apt

K. Apt was early in recognizing [2] that csp can be defined rigorously, yet in such a way as to be widely applicable. The following definition is distilled from [2, 5], and uses his notation.

Definition 6 A csp \( \langle X, D, C \rangle \) consists of a sequence \( X = \langle x_1, \ldots, x_n \rangle \) of variables, a sequence \( D = \langle D_1, \ldots, D_n \rangle \) of sets called domains, and a set \( C = \{ c_1, \ldots, c_k \} \) of constraints. Each constraint is a constraint on a subsequence of \( X \). An \( n \)-tuple \( (d_1, \ldots, d_n) \in D_1 \times \cdots \times D_n \) is a solution to \( \langle X, D, C \rangle \) iff for every \( c \in C \) on a sequence of variables \( \langle x_{i_1}, \ldots, x_{i_m} \rangle \) from \( X \) we have \( \langle d_{i_1}, \ldots, d_{i_m} \rangle \in c \).

In this definition \( X \) is probably intended to consist of \( n \) different variables. Once that condition is assumed, \( X \) need not be a sequence, but can be a set without further qualification.

Example 3 To see Apt’s definition at work, consider the following example. \( X = \langle x_1, x_2, x_3, x_4 \rangle \), \( D = \langle N, N, N, N \rangle \), and \( C = \{ c_1, c_2 \} \). Constraints \( c_1 \) and \( c_2 \) are on \( \langle x_2, x_2, x_1 \rangle \) and \( \langle x_3, x_4, x_1 \rangle \), respectively. To determine some of the solutions we construct Table 3.

The table for \( c_1 \) is constructed according to the rule \( x_2 + x_2 = x_1 \); for \( c_2 \) the rule is \( x_3 + x_4 = x_1 \). In Definition 5 a constraint remains a black box: there is no opportunity to specify a rule according to which the tuples
are constructed. This omission can be a disadvantage, as is seen in the important type of discrete CSP that can be viewed as a graph-colouring problem. In practical applications such CSPs have a small domain, consisting of the “colours”. At the same time they have a large number of variables and a large number of constraints, both numbers running in the thousands. Yet all these constraints have an important property in common: they derive from the “all different” constraint that requires that no two of their arguments have the same value.

The remedy for this problem was prepared by Definition 1, which is used in our alternative Definition 7 for CSP. If the definition of CSP included a language for expressing constraints, then these expressions would clarify the connection between \( c_1 \) and \( c_2 \). For example, \( \text{sum}(x_2, x_2, x_1) \) would be a good expression for \( c_1 \) and \( \text{sum}(x_3, x_4, x_1) \) for \( c_2 \).

Predicate logic is a potential candidate for a formal constraint language. To realize this potential we modify Apt’s definition to obtain the definition given in the following section. To be able to interface the solving algorithm for CSPs with the CLP/DC scheme, we modify the algorithm also. In the section after that we define how predicate logic can be used as the constraint language.

### 5.2 A modified definition of CSPs

**Definition 7** A Constraint Satisfaction Problem (CSP) consists of a finite set \( X = \{x_1, \ldots, x_n\} \) of variables, a finite set \( C = \{c_1, \ldots, c_m\} \) of constraints, each of which is a relation over a sequence of elements of \( X \) in the sense of Definition 1. With each variable \( x_i \) is associated a universe \( D_i \), which is the set of values that \( x_i \) can assume. A solution of a CSP is an assignment to each variable \( x_i \) of an element of \( D_i \) such that each constraint in \( C \) is satisfied.
Apparently, the solution set of a CSP with set \( \mathcal{X} \) of variables is a relation on \( \mathcal{X} \) in the sense of Definition 1. A compact characterization of the solution set can be given as follows.

**Lemma 2** The solution set equals \( c_1 \times \cdots \times c_m \) where \( \cdots \) is as in Definition 3.

For certain CSPs it is practical to enumerate the solutions. In other cases the solution set, though finite, is too large to be enumerated. And it may be the case that the solution set is uncountable; moreover its individual solution tuples may consist of reals that are not computer-representable.

Thus it is often necessary to approximate the solution set. A convenient form is that of a Cartesian product \( D_1 \times \cdots \times D_n \) that is contained in \( D_1 \times \cdots \times D_n \). Such an approximation has the property that \( x_i \notin D_i \) for any \( i \in \{1, \ldots, n\} \) ensures that \( \langle x_1, \ldots, x_n \rangle \) is not a solution. Making \( D_1, \ldots, D_n \) as small as possible gives us as much information about the solution set as is possible for approximations of this form.

\( D_i \) is called the **domain** for \( x_i \) for \( i \in \{1, \ldots, n\} \). We need to ensure that the subsets \( D_i \) of \( D_i \) are computer-representable. This may not be a restriction when \( D_i \) is finite and small. It is when \( D_i = \mathbb{R} \). In general we require that the subsets of \( D_i \) that are allowed as \( D_i \) include \( D_i \) itself and are closed under intersection. We call such subsets a **domain system**.

**Lemma 3** Given sets \( D_1, \ldots, D_n \), each with a domain system and \( S \subset D_1 \times \cdots \times D_n \). There is a unique least Cartesian product of domain system elements containing \( S \).

**Definition 8** Given sets \( D_1, \ldots, D_n \), each with a domain system and \( S \subset D_1 \times \cdots \times D_n \). The least Cartesian product of domain system elements containing \( S \), which exists according to Lemma 3, is denoted \( \sqcap S \).

With each constraint there is associated a **domain reduction operation** (DRO), which is intended to reduce the domains of one or more variables occurring in the constraint.

**Definition 9** Given a CSP and a relation \( \rho \subset D_1 \times \cdots \times D_{ik} \). Let constraint \( c \) be a relation \( \rho \) on \( \langle x_1, \ldots, x_{ik} \rangle \). A **domain reduction operation** (DRO) \( c \) is a function that maps Cartesian products \( D_1 \times \cdots \times D_{ik} \subset D_1 \times \cdots \times D_{ik} \) to Cartesian products of the same type. The map of the function is given by \( D_1 \times \cdots \times D_{ik} \mapsto D'_1 \times \cdots \times D'_{ik} \) where \( D'_1 \times \cdots \times D'_{ik} \) satisfies

\[
\Box((D_1 \times \cdots \times D_{ik}) \cap \rho) \subset D'_1 \times \cdots \times D'_{ik} \subset D_1 \times \cdots \times D_{ik}.
\]

If the left inclusion is equality, then we call the DRO a **strong one**.

This operation was introduced by [7] under the name “narrowing”. The intended application had intervals for the domains, hence the name.

Note that domains are reduced only by removing non-solutions. As one can see, DROS are contracting: if they do not succeed in removing anything, they leave the domains unchanged. Strong DROS are idempotent: multiple successive applications of the same DRO have the same effect as a single application.

Success of the constraint satisfaction method of solving problems depends on finding efficiently executable strong DROS.
5.2.1 Constraint Propagation

Definition 10 A computation state of a CSP is $D_1 \times \cdots \times D_n$ where $D_i \subseteq D_i$ is a domain and is associated with $x_i$, for $i = 1, \ldots, n$.

A computation of a CSP is a sequence of computation states in which each (after the initial one) is obtained from the previous one by applying the DRO of one of the constraints.

The limit of a computation is the intersection of its states.

A fair computation of a CSP is a computation in which each of the constraints is represented by its DRO infinitely many times.

Fair computations have infinite length. However, no change occurs from a certain point onward (domain systems have a finite number of sets). By the idempotence of strong DROs, this is detectable by algorithms that generate fair computations, so that they can terminate accordingly.

Theorem 3 \[3\] The limit of a fair computation of a CSP is equal to the intersection of the initial state of the computation with the greatest fixpoint common to all DROs.

For a given CSP the intersection of the states of any fair computation only depends on the initial state. It is therefore independent of the computation itself. Apparently the CSP maps the set of Cartesian products to itself. It is a contracting, idempotent mapping.

Lemma 4 Let $D$ be the initial state of a fair computation of a CSP. Then the limit of the fair computation contains the intersection of $D$ with the solution set.

Definition 11 The transition from the initial state of a computation to the limit of that computation is called constraint propagation.

The reason for the name is that the effect of a DRO application on a domain may cause a subsequent DRO applications to reduce other domains.

5.2.2 Enumeration

Constraint propagation only goes part way toward solving a CSP: it results in a single Cartesian product containing all solutions. In general this single Cartesian product needs to be split up to give more information about any solutions that might be contained in it. This is what enumeration does.

Before a more precise definition, let us sketch the solving process by means of the CSP arising from a graph-colouring problem. In case constraint propagation yields an empty domain in the computation state, the solving process is over: absence of solutions has been proved. Suppose the resulting computation state does not have an empty domain. We only know that any solutions that may exist are elements of the Cartesian product of the domains. If all domains are singletons, then the corresponding tuple is a solution. If not, one enumerates a domain with more than one element (say, the smallest such). In turn, for each element in that domain, one assumes it as the value of the variable concerned and leaves the other domains unchanged. To the smaller CSP thus obtained, one applies
constraint propagation. This may, in turn, require enumeration; and so on.

To make the idea applicable to the case where there are infinite domains, we split a domain instead of enumerating it. Then it works as above if the domains are countable.

To split an uncountable domain, then we need the property that the domain system is finite. Splitting is restricted to producing results that belong to the domain system. This implies that only a finite number of splits are possible. In case of an uncountable domain it is not in general possible to identify solutions.

Enumeration yields tuples consisting of domains that are as small as the domain system allows that together contain all solutions, if any exist. And of course solving the CSP results in eliminating almost all of the Cartesian product of the initial domains as not containing any solutions.

**Enumeration algorithm**

To enumerate computation state \( S \):

- If a domain is empty, then halt.
- If one of the domains is a singleton,
  - then substitute the element as value of the corresponding variable
  - and construct the computation state \( S' \) with that variable eliminated.
  - Enumerate \( S' \).
- else split a domain \( d \) into domain system elements \( d_0 \) and \( d_1 \).
  - Construct computation states \( S_i \) by replacing \( d \) in \( S \) by \( d_i \), for \( i = 0, 1 \).
  - Enumerate \( S_0 \); Enumerate \( S_1 \).

Often too many enumeration results are generated. Sometimes the domain system comes with a suitable notion of adjacency so that adjacent enumeration results can be consolidated into a single one. Such a consolidation may trigger further consolidations.

### 6 The Domain Constraint subscheme of the CLP Scheme

The CLP scheme is open-ended: it is basically a scheme for using Horn-clause rules to generate a multitude of constraint-satisfaction problems. The parameters of the scheme allow a great variety of useful algorithms and of data-types for these to act on. A first step in reducing the vast variety of options is \( \text{clp}/\text{dc} \), the domain-constraint subscheme of the CLP scheme. We define \( \text{clp}/\text{dc} \) by visiting first the parameters \( \langle \Sigma, I, L, T \rangle \), and then the transitions of the CLP scheme.

#### 6.1 The parameters

\( \Sigma \): Some domains are such that individual elements may not be representable in a computer, if only because there are infinitely many of them. Satisfactory results can still be obtained by designating a finite set of subsets of the domain that are computer-representable. To accommodate
these the signature $\Sigma$ includes a unary \textit{representability predicate} for each of these subsets.

$I$: the domain component $D$ of the $\Sigma$-structure $I$ has to admit a domain system: a finite set of subsets of $D$ that includes $D$ and is closed under intersection.

$L$: the language of constraints consists of conjunctions of atomic formulas.

$T$: the theory giving the semantics of the constraints links unary representability predicates to representable subsets of $D$. This is done in part by clauses describing the effects of the DROs. In Definition 6 let the constraint $c$ be $r(x_{i_1}, \ldots, x_{i_k})$. Then the clauses describing the DRO of $c$ are

\[ d_j'(x_{i_j}) \leftarrow d_1(x_{i_1}), \ldots, d_k(x_{i_k}), r(x_{i_1}, \ldots, x_{i_k}) \quad (1) \]

for $j = 1, \ldots, k$. Further details depend on the instance concerned of CLP/DC. The idea of expressing the action of a DRO in the form of an implication like the one above in a theory.

6.2 The transitions

The $\rightarrow_r$ and $\rightarrow_c$ transitions: These only serve to transform goal atoms into constraint atoms, and are needed unchanged in the CLP/DC subscheme.

The $\rightarrow_i$ transition: In the CLP scheme this transition is intended to accommodate any inference that transfers the burden of constraint from the passive constraints $P$ to the efficiently solvable active constraints $A$. In the CLP/DC subscheme such inference is restricted to those forms that leave $P$ unchanged: the information contained in them is only used to strengthen the active constraints $A$. Moreover, $A$ is restricted to the form $\{d_1(x_1), \ldots, d_n(x_n)\}$ where each variable in the passive constraint $P$ occurs exactly once and where $\{d_1, \ldots, d_n\}$ are unary representability predicates.

As $P$ is the unchanging conjunction of the constraints, we refer to it as $C$ in the CLP/DC subscheme. As $A = \{d_1(X_1), \ldots, d_n(X_n)\}$ only states of each of the variables that it belongs to a certain domain we refer to it as $D$ in the CLP/DC subscheme. As a result of these renamings we have a close relationship between CSP and CLP/DC: $C$ and $D$ in CSP and in CLP/DC are counterparts of each other. As a result of these restrictions and renamings, the constraint store management transition becomes $\langle G, D, C \rangle \rightarrow_i \langle G, D', C \rangle$ if $\langle D', C \rangle = \text{infer}(D, C)$.

The \textit{infer} operation is performed by setting up a CSP with an initial state and determining the limit of the fair computations from the initial state. This limit is then the $D'$ in $\langle D', C \rangle = \text{infer}(D, C)$.

The CSP that implements \textit{infer} in this way has the following components:

1. The variables are those that occur in the passive constraint $C$.
2. The universes over which the variables range are equal to each other and to $D$. 

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3. If a constraint atom \( c_j \) of CLP/DC is \( r(x_{i_1}, \ldots, x_{i_{k_j}}) \), then the corresponding constraint of the CSP is \( \rho \), the meaning of \( r \), on \( \langle x_{i_1}, \ldots, x_{i_{k_j}} \rangle \), with “on” as in Definition 1.

4. If the active constraint is \( \{ d_1(x_1), \ldots, d_n(x_n) \} \), then the initial computation state in the CSP is \( D_1 \times \cdots \times D_n \) with \( D_i = \{ x \in D \mid d_i(x) \} \), for \( i = 1, \ldots, n \).

In the CSP thus obtained a fair computation is constructed with limit \( D'_1 \times \cdots \times D'_n \). These domains are then used to determine the active constraint \( D' = \{ d'_1(x_1), \ldots, d'_n(x_n) \} \), where the \( d_i \) are obtained from \( D'_i = \{ x \in D \mid d'_i(x) \} \), for \( i = 1, \ldots, n \).

In this way CSP computations can be used in CLP/DC.

The \( \rightarrow_s \) transition: In the CLP scheme this transition checks as best as it can whether \( P \land A \) is consistent. In the CLP/DC subscheme no attempt is made to check \( C \) for consistency. It does this only for \( D = d_1(x_1) \land \cdots \land d_n(x) \) and this is simply a check whether any of the \( d_i \) is the predicate for the empty subset of \( D \).

**Lemma 5** The existence of a successful derivation implies that \( C, T \models R(C) \subset R(Q) \).

**Proof 1** By theorem 1 we have \( C, T \models R(C \land D) \subset R(Q) \) and we have \( C, T \models R(C) \subset R(D) \).

### 7 Interval constraints

We have used in Section 4 the CLP scheme as starting point. To establish the direction in which to proceed, we identified in Section 5 a desirable point outside of logic programming: the CSP paradigm. Here are to be found useful algorithms for computational tasks of interest. These range from the “most discrete” such as graph colouring to the “most continuous” such as solving non-linear equalities and inequalities over the reals.

After thus establishing a line along which to travel, we went back in Section 6 to establish a subscheme of the CLP scheme, that of the domain constraints, to emulate within logic the main features of the CSP paradigm.

It is now time to declare our main interest: real valued variables rather than discrete ones. It so happens that there is a real-variable specialization of the CSP paradigm, interval constraints, and it will be useful to take an excursion from logic again and review this next.

We are interested in CSPs with the following characteristics. The variables range over the reals; that is, all universes \( D_1, \ldots, D_n \) are equal to the set \( R \) of reals. The domain system is that of the floating-point intervals. The constraints include the binary \( \leq \) and the ternary \( \text{sum and prod} \). The reason is that these have strong DROS that are efficiently computable. Strong DROS are also available for \( =, \max, \text{abs} \), and for rational powers. For the constraints corresponding to the transcendental functions DROS are available that are idempotent, but not strong. The definition of “strong” requires them to be the least floating-point box containing the intersection of the relation with the argument box. That the DRO is not
strong has to do with the difficulty of bounding these function values between adjacent floating-point numbers. But DROs closely approximating this ideal are used in some systems [10].

Let us consider an example of a DRO for use with real-valued variables constrained by the relation

$$\text{sum} = \{ (x, y, z) \in \mathbb{R} \mid x + y = z \}.$$ 

Suppose the domains for $$x$$, $$y$$, and $$z$$ are $$[0, 2]$$, $$[0, 2]$$, and $$[3, 5]$$. Clearly, neither $$x$$ nor $$y$$ can be close to 0, nor can $$z$$ be close to 5. Accordingly, when this DRO is applied, these intervals are reduced to $$[1, 2]$$, $$[1, 2]$$, and $$[3, 4]$$.

The numbers 1 and 4 arise by computing $$3 - 2$$ and $$2 + 2$$. Here no rounding errors were made. This is exceptional. Let us now consider the case in which the initial intervals are scaled down by a factor of ten to $$[0.0, 0.2]$$, $$[0.0, 0.2]$$, and $$[0.3, 0.5]$$.

Now the corresponding operations $$0.3 - 0.2$$ and $$0.2 + 0.2$$ do incur rounding errors. $$0.3 - 0.2$$ is evaluated to a floating-point number we shall name $$0.1^-$$. Similarly, $$0.2 + 0.2$$ is evaluated to $$0.4^+$$, so that the DRO gives the intervals $$[0.1^-, 0.2^+]$$, $$[0.1^-, 0.2^+]$$, and $$[0.3^-, 0.4^+$$] for $$x$$, $$y$$, and $$z$$, respectively. Here $$0.1^-$$ may equal $$0.1^-$$ or $$0.1^-$$.

The decimal equivalents of the binary floating-point numbers computed here are so lengthy that users are neither willing to write nor to read them, so that further containment precautions are called for on (decimal) input and output.

In this way single arithmetic operations find their counterpart in interval constraints. To give an idea of how the arbitrarily complex arithmetic expressions in nonlinear equalities and inequalities are translated to interval constraints consider the equation $$1/x + 1/y = 1/z$$ relating the resistance $$z$$ of two resistors in parallel with resistances $$x$$ and $$y$$. Constraint processing is not directly applicable when, as we assume here, we only have DROs for $$\text{sum}$$ and $$\text{inv}$$. We therefore convert the equation to the equivalent form

$$\exists u, v, w \in \mathbb{R}. \text{inv}(x, u) \land \text{inv}(y, v) \land \text{inv}(z, w) \land \text{sum}(u, v, w).$$

Accordingly, the equation is translated to a CSP with $$\mathcal{X} = \langle x, y, z, u, v, w \rangle$$ and $$\mathcal{C} = \{ \text{inv}(x, u), \text{inv}(y, v), \text{inv}(z, w), \text{sum}(u, v, w) \}$$.

In numerical CSPs we can conclude, according to Theorem [11] that the solution set is empty when the limit of the computation is empty. However, a nonempty limit can still coexist with an empty solution set.

It is possible to develop DROs for complex expressions such as $$1/x + 1/y = 1/z$$ [12]. It is useful to know that this paper is antedated by the technical report version of [7].

8 CLP/NCSP: the CLP/DC subscheme with a numerical CSP

In Section [8] we described how the open-ended CLP scheme is narrowed down to the domain-constraints subscheme CLP/DC. In this section we
take a step further in this direction to obtain a subscheme suitable for
numerical computation. We do this by following the specification in Sec-

8.1 The hierarchy of theories

Σ: The signature contains the language elements needed for the usual
type of the real numbers: constants including 0 and 1; the unary func-
tion symbol −; the binary function symbols +, −, ⋅, and ÷; the binary
predicates ≤ and ≥. To these we add:

• A unary representability predicate \( d_{a,b} \) for every floating-point inter-
val in a given floating-point number system. For the IEEE-standard
double-length floating-point numbers this means in the orde
r of \( 2^{127} \) unary predicates. Not a mathematically elegant signature, but a
finite one.
• Ternary predicates \( \text{sum} \) and \( \text{prod} \).

\[ I \]: the domain component \( D \) of the \( \Sigma \)-structure \( I \) is the set \( \mathbb{R} \) of
real numbers. The domain system consists of the floating-poi
nt intervals, which are sets of reals. The floating-point intervals include \( \mathbb{R} \) itself
and are closed under intersection, so include the empty inter
val.

\[ T \]: to the axioms of the usual theory of the reals we add:

\[
\forall x. [d_{-\infty,b}(x) \leftrightarrow x \leq b] \text{ for every floating-point number } b
\]
\[
\forall x. [d_{a,b}(x) \leftrightarrow a \leq x, x \leq b] \text{ for every pair of flpt numbers such that } a \leq b
\]
\[
\forall x. [d_{a,\infty}(x) \leftrightarrow a \leq x] \text{ for every floating-point number } a
\]
\[
\forall x, y, z. [\text{sum}(x, y, z) \leftrightarrow x + y = z]
\]

We refer to the resulting theory as \( T_1 \). The only difference with the usual
axiomatization of the reals is that meanings are established for the newly
introduced predicates.

The effect of the dro of a constraint is described in clauses as in
equation (1) in Section 6. For each of the atomic constraints in the passive
constraint \( C \) this causes clauses to be added to \( T_1 \). We call the resulting
theory \( T_2 \).

**Theorem 4** Let \( C \) be the passive constraint, let \( D \) be the initial active
constraint, and let \( D'_1, \ldots, D'_m \) be the active constraints corresponding to
the results of a CSP enumeration starting with initial constraint corre
sponding to \( D \) and constraints corresponding to \( C \). Then \( T_2 \models R(C \land D) \subset
[R(D'_1) \cup \cdots \cup R(D'_m)] \).

It would be more convincing if we could assert that \( T \models R(C \land D) \subset
[R(D'_1) \cup \cdots \cup R(D'_m)] \), as \( T \) is the usual theory of the reals, without
computer-related artifacts. This is not possible, as \( R(C) \) and \( R(D) \) con
tain constraint predicates and these do not occur in \( T \). However, all
axioms that are in \( T_2 \) and not in \( T \) are logical consequences of \( T \).

**Proof 2** Every application of a dro corresponds to an inference with one
of the rules in \( T_2 \) of the form of Equation (1).
It is now time to look at examples of what we can do with the tools developed so far. The first two examples concern a polynomial in a single real variable and represent it by a term \( p \) in the variable \( x \). In these examples the problem is stated in a single constraint, so only uses a part of the CLP paradigm. The third example is a toy design problem. Here the CLP paradigm is fully exercised: multiple derivations are generated, each of which is potentially a significant numerical CSP.

### 8.2 Semantics of solving numerical inequalities

Consider the problem of determining where the given polynomial is non-positive. This corresponds to the constraint \( p \leq 0 \). In \( \mathcal{T}_2 \) we can translate \( p \leq 0 \) to a set \( C \) of constraints. For example, if \( p \) is \( x \ast (x - 2) \) we have in \( \mathcal{T}_2 \)

\[
\forall x [x \ast (x - 2) = 0 \leftrightarrow \exists v, w. \ sum(v, 2, x) \land prod(x, v, w) \land w \leq 0]
\]

so that we have the constraint \( \Gamma \) equal to \( \{ \sum(v, 2, x) \land \prod(x, v, w) \land w \leq 0 \} \). A highly complex \( p \) will give rise to a \( C \) with many atoms and many variables.

Soundness (Theorem 4) implies that the active constraints in the answer constraint for this problem \( \text{clp}/\text{ncsp} \) contains all intervals in which \( p \) is zero or negative. Completeness implies that whenever we have for a constraint \( \Gamma \) that

\[
\mathcal{T}_2 \models R(\Gamma) \subset R(p \leq 0)
\]

there are \( m > 0 \) derivations ending in answer constraints \( \Gamma_1, \ldots, \Gamma_m \) such that \( \mathcal{T}_2 \models R(\Gamma) \subset R(\Gamma_1) \cup \cdots \cup R(\Gamma_m) \). We cannot replace Equation (2) by \( \mathcal{T}_2 \models R(p \leq 0) \neq \emptyset \). This would be reducible to the problem of deciding equality between two reals, a problem shown to be unsolvable [1].

### 8.3 Semantics of equation solving

A well-known numerical problem that can present computational difficulties is the one of determining \( R(p = 0) \).

Theorem 4 shows that the active constraints in the answer constraint for this problem \( \text{clp}/\text{ncsp} \) contain all zeroes of the polynomial. It also shows that in case of finite failure the polynomial has no zeroes. The possibility remains that finite failure does not occur, yet there are no zeroes. This is unavoidable. The problem of deciding whether \( \mathcal{T}_2 \models R(p = 0) = \emptyset \) reduces again to the problem of deciding equality between two reals. The best we can hope for is attained here: showing emptiness or finding small intervals in which all solutions, if any, are contained.

Completeness (Theorem 2) has nothing to say about this problem: it is rare for a polynomial \( p \) to make \( \mathcal{T}_2 \models R(\Gamma) \subset R(p = 0) \) true for non-empty \( R(\Gamma) \). With respect to \( \mathcal{T}_2 \) the set \( R(p = 0) \) is a finite set of reals, and it is rare for these to be a floating-point number. For most \( p \), the least \( R(\Gamma) \) containing containing any root of it is an interval of positive width.
Conventional numerical computation produces single floating-point numbers that are intended to be near a solution, and mostly are. Sometimes they are not, and one cannot tell from the program’s output. Interval arithmetic and numerical CSPs improve on this by returning intervals that contain the solutions, if any, and by failing to return any intervals in which it is certain that no solutions exist. CLP/NCSP improves on this by giving a logic semantics, of which Theorem 4 is an example. However, interval arithmetic and interval constraints are limited in that they only solve a single CSP. A more important advantage of CLP/NCSP is that, in addition to solving CSPs, it automates the generation of the multiple CSPs that are often required in scheduling and in engineering design. We close by giving an example of this mode of operation.

8.4 A toy example in CLP/NCSP

Consider an electrical network in which resistors are connected to each other. The network as a whole has a certain resistance. We have available twelve resistors; three each of 100, 150, 250, and 500 ohms. From this inventory we are to build a network that has a specified resistance so that it can function as part in a larger apparatus. Fortunately there is a certain latitude: the resistance of the resulting network has to lie between 115 and 120 ohms. *The structure of the network is not given.* This is a design problem in addition to being a computational problem.

Even with the dozen components given in this problem there is a large number of ways in which they can be connected. We can nest parallel networks inside a series network, or the other way around, to several levels deep. Evaluation of each such combination requires a non-negligible amount of computation involving real-valued variables. The search space is sizable, hence the importance of constraint propagation to eliminate most of it.

Let us imagine for CLP/NCSP a Prolog-like syntax. Please do not be misled by the type writer-like font into believing in an implementation: none exists. The figures given in the example are for illustration only and are chosen to be merely plausible.

According to CLP bodies of clauses contain both constraint atoms and program atoms. We separate them with a semicolon: the constraints, if any, come first. Instead of writing \( d_{a,b}(x) \) for the domain constraints, we write for ease of typing \( <a\mid x\mid b> \) in the style of Dirac’s bra and ket notation. When \( a \) is infinite, we write \(-\text{inf}\); this is a single mnemonic identifier, denoting that particular floating-point value. Similarly for \( b \) and \( \text{inf} \) or \(+\text{inf}\). We omit constraints like \(-\text{inf}\mid x\mid +\text{inf}\), which do not constrain their argument.

The predicate \( \text{netw}(A, N, B, R, PL) \) asserts that network represented by \( N \) connects terminals \( A \) and \( B \), has resistance \( R \), and has parts list \( PL \). The term \( N \) can be \( \text{at}(X) \) for an atomic network, which is in this case a single resistor; it can be \( \text{ser}(N1,N2) \), for two networks in series, or \( \text{par}(N1,N2) \), for two networks in parallel.

1: \( \text{netw}(A, \text{at}(R), B, R, (r150:1).\,\text{nil}) \)
   \( :- <149.9\mid R\mid 150.1>;. \)
Similarly for 100, 250, and 500 ohms.

2: netw(A, ser(N1, N2), C, R, PL)
   :- sum(R1, R2, R);
   netw(A, N1, B, R1, PL1), netw(B, N2, C, R2, PL2),
   merge(PL1, PL2, PL).

3: netw(A, par(N1, N2), B, R, PL)
   :- inv(R, RR), inv(R1, RR1), inv(R2, RR2), sum(RR1, RR2, RR);
   netw(A, N1, B, R1, PL1), netw(A, N2, B, R2, PL2),
   merge(PL1, PL2, PL).

Clause 1 says that the network can be atomic, consisting of a single resistor with resistance $R$, represented by the term \texttt{at(R)}. Its parts list is a list consisting of a single item \texttt{r150:1}, being a resistor of nominal value 150 ohms in quantity 1. The condition of clause 1 states that the actual resistance $R$, a real variable, belongs to the interval $[149.9, 150.1]$, which expresses the tolerance. There are similar clauses to represent the other sizes of resistor that are available.

Clause 2 says the network can be \texttt{ser(N1, N2)}, the series composition of two networks $N1$ and $N2$ of unspecified structure, with resistance $R$, that satisfies the constraint for the resistance of a serial composition of networks: $\text{sum}(R1, R2, R)$, which means that $R_1 + R_2 = R$.

In clause 3 the constraint means $1/R_1 + 1/R_2 = 1/R$, which is the constraint for resistances $R_1$ and $R_2$ in parallel giving resistance $R$.

The predicate \texttt{merge} is left as a black box. Suffice it to know that the goal \texttt{merge(PL1, PL2, PL)} merges parts lists $PL1$ and $PL2$, which satisfy the inventory restrictions, into parts list $PL$ unless the latter does not satisfy the inventory restrictions, in which case the goal fails.

The query
\begin{verbatim}
:- <149.9|R150|150.1>, ..., 
   netw(a, par(at(R150), ser(at(R500), par(at(R100), at(R250)))), b, R, PL).
\end{verbatim}

succeeds without search to an answer that could include something like $<117.1|R|119.3>$. The program looks like it has been written with such queries in mind. However, as explained below, it also succeeds, though with some search, to answer
\begin{verbatim}
:- <115.0|R|120.0>; netw(A, N, B, R, PL).
\end{verbatim}

with

$N = \text{par}(\text{at}(R150), \text{ser}(\text{at}(R500), \text{par}(\text{at}(R100), \text{at}(R250))))$  

and

$<117.1|R|119.3>, <149.9|R150|150.1>, ..., $

In response to the latter query \texttt{CLP/NCSP} has \textit{synthesized} a suitable network, thereby solving the design problem. It traversed a search space consisting of multiple \texttt{csp}s that was generated by \texttt{CLP} derivations. Many of these derivations were cut short by failing \texttt{csp}s.

As in the first two examples, the soundness of Theorem\[\text{[1]}\] guarantees for this problem that all networks that are found have a resistance contained in the required interval. We noticed that in the case of polynomial roots completeness has no interesting consequence. This was true because the problem had the form of a single constraint with equality. In the design of
a resistor network there is a goal statement with a program atom. It gives rise to multiple derivations. Completeness (theorem 2) implies that in case a solution exists, derivation are generated to cover the given interval for the network’s resistance: all solutions are found.

9 Concluding remarks

CLP/NCSP only incorporates numerical CSPs into the CLP scheme. Other types of CSP such as those dealing with finite domains, can be incorporated in the same way. In fact, the CLP scheme is not restricted to including special-purpose computation into the logic framework: it has remedies for those difficulties that prevented Prolog from being a logic programming language.

Pure Prolog held out the promise of a programming language with logical-implication semantics. The impracticality of the occurs check in unification and of symbolic implementation of numerical computation caused standard Prolog to compromise semantics. In this paper we described a method for including the power of hardware floating-point arithmetic without semantical compromise. We should not lose sight of the fact that the CLP scheme also has a remedy for the other blemish of standard Prolog: compromised unification. As Clark [9] showed, the Herbrand Equality Theory, which requires the occurs check, is only one possible unification theory for the CLP scheme. It can be replaced by Colmerauer’s Rational Tree Equality Theory, so that we have the prospect of a fully practical programming language with logic-implication semantics.

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