The diameter game

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Maker/Breaker games

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Maker/Breaker games

A **positional game** is one in which players **Maker**
Maker/Breaker games

A positional game is one in which players Maker and Breaker
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A **positional game** is one in which players **Maker** and **Breaker** occupy vertices of a hypergraph.
Maker/Breaker games

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**Maker**’s goal is to occupy each vertex in some hyperedge.

**Breaker**’s goal is to prevent this; i.e., occupy at least one vertex from **each** hyperedge. There is no draw.
(1:1) diameter 2 game

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**GRAPH GAMES:**
**MAKER** and **BREAKER** in turns occupy edges of $K_n$. 
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**GRAPH GAMES:**
**Maker** and **Breaker** in turns occupy edges of $K_n$.

- **Maker** gets to choose one edge not already taken.
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Most properties we want are monotone properties ($\mathcal{P}$).
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**THE PROPERTY $\mathcal{P}$ GAME:**
- **Maker** wins if his graph has property $\mathcal{P}$. 
(1:1) diameter 2 game

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**THE DIAMETER 2 GAME:**
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\textbf{GRAPH GAMES:} Maker and Breaker in turns occupy edges of $K_n$.

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Most properties we want are monotone properties ($\mathcal{P}$).

\textbf{THE DIAMETER 2 GAME:}
- Maker wins if his graph is a diameter two graph.
- Breaker wins if he can prevent this.

\textit{Does anyone have a winning strategy?}
Winning the game

If \( n \geq 4 \), it doesn’t matter who goes first!
Winning the game

If $n \geq 4$, it doesn’t matter who goes first!  
**BREAKER** always has a winning strategy!
Winning the game

If \( n \geq 4 \), it doesn’t matter who goes first! **Breaker** always has a winning strategy!

Even if **Maker** goes first,
Winning the game

If \( n \geq 4 \), it doesn’t matter who goes first! \textbf{Breaker} always has a winning strategy!

Even if \textbf{Maker} goes first, \textbf{Breaker} can choose an edge \( \{x, y\} \) not incident to it.
Winning the game

If \( n \geq 4 \), it doesn’t matter who goes first! \textcolor{red}{\textsc{Breaker}} always has a winning strategy!

Even if \textcolor{blue}{\textsc{maker}} goes first, \textcolor{red}{\textsc{breaker}} can choose an edge \( \{x, y\} \) not incident to it.

Then \textcolor{red}{\textsc{breaker}} plays a mimic strategy, keeping \( x \) and \( y \) apart.
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If $n \geq 4$, it doesn’t matter who goes first! \textbf{Breaker} always has a winning strategy!

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If $n \geq 4$, it doesn’t matter who goes first! Breaker always has a winning strategy!

Even if Maker goes first, Breaker can choose an edge $\{x, y\}$ not incident to it.

Then Breaker plays a mimic strategy, keeping $x$ and $y$ apart.
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If $n \geq 4$, it doesn’t matter who goes first! **Breaker** always has a winning strategy!

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Winning the game

If \( n \geq 4 \), it doesn’t matter who goes first! \( \text{BREAKER} \) always has a winning strategy!

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Then **BREAKER** plays a mimic strategy, keeping $x$ and $y$ apart.

**BREAKER** wins!
Winning the game

If $n \geq 4$, it doesn’t matter who goes first! \textsc{Breaker} always has a winning strategy!

Even if \textsc{Maker} goes first, \textsc{Breaker} can choose an edge $\{x, y\}$ not incident to it.

Then \textsc{Breaker} plays a mimic strategy, keeping $x$ and $y$ apart.

\textsc{Breaker} wins! There is no way for \textsc{Maker} to create a path of length at most 2 between $x$ and $y$. 
(a:b) games

We often generalize games, so that \textbf{Maker} gets to make \(a\) moves in each turn and \textbf{Breaker} gets to make \(b\) moves in each turn.
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- **Erdős-Selfridge-Beck**: **Breaker** wins the hypergraph game on $\mathcal{H}$ if

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\sum_{A \in E(\mathcal{H})} (1 + a)^{-|A|/b} < \frac{1}{b + 1}
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- **Frieze-Krivelevich-Pikhurko-Szabó**: Maker can create a pseudo-random graph if $a = b = 1$. 
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- **Łuczak-Bednarska**: If \( a = 1 \) and \( b = n - s \), then Maker can guarantee the size of the largest component is \( s + O(\sqrt{n}) \).

- **Frieze-Krivelevich-Pikhurko-Szabó**: Maker can create a pseudo-random graph* if \( a = b = 1 \).

  * Degrees approximately \( n/2 \), codegrees approximately \( n/4 \).
Probabilistic intuition

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For every monotone property $\mathcal{P}$, there is a threshold probability $p_0$ such that the random graph $G(n, p)$
- fails to have property $\mathcal{P}$ with probability $\rightarrow 1$. 
Probabilistic intuition

These graph games seem to confirm the \textit{probabilistic intuition}:

For every monotone property $\mathcal{P}$, there is a threshold probability $p_0$ such that the random graph $G(n, p)$

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For every monotone property \( \mathcal{P} \), there is a threshold probability \( p_0 \) such that the random graph \( G(n, p) \)

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Recall $p_0 = \Theta(\ln n/n)$ for both connectivity and Hamiltonicity.
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Let $p = \frac{a}{a+b}$, the proportion of edges belonging to *Maker* when the game is over. In the connectivity and Hamiltonicity games, *Maker* wins if $p \gg p_0$. 

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These graph games seem to confirm the *probabilistic intuition*:

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Recall $p_0 = \Theta(\ln n/n)$ for both connectivity and Hamiltonicity.

Let $p = \frac{a}{a+b}$, the proportion of edges belonging to Maker when the game is over. In the connectivity and Hamiltonicity games, Maker wins if $p \gg p_0$.

In fact, the graph games all seem to follow probabilistic intuition.
Until now!

We know that $G(n, p)$ has diameter 2 if $p \gg \ln n / \sqrt{n}$. 
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But that says that $G(n, 1/2)$ has diameter 2 and Maker should win the (1:1)-game!
Until now!

We know that $G(n, p)$ has diameter 2 (whp) if $p \gg \ln n / \sqrt{n}$.

But that says that $G(n, 1/2)$ has diameter 2 and MAKER should win the (1:1)-game!

More strongly, Frieze, et al. said that MAKER can create a pseudo-random graph in the (1:1)-game. However, BREAKER can ensure that at least one pair is of distance at least 3.

Probabilistic Intuition Fails!
(a:b) diameter 2 game

What if $a > b$?
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**Lemma.** If $a \leq n/(4 \ln n)$ and $n$ is large enough
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**Lemma.** If $a \leq n/(4 \ln n)$ and $n$ is large enough then \textsc{Maker} wins the (a:b)-minimum-degree $\geq k$ game on $K_n$ if
(a:b) diameter 2 game

What if $a > b$?

**Lemma.** If $a \leq n/(4 \ln n)$ and $n$ is large enough then Maker wins the $(a:b)$-minimum-degree $\geq k$ game on $K_n$ if

$$k \leq \frac{a}{a + b} n - \frac{6ab}{(a + b)^{3/2}} \sqrt{n \ln n}.$$
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**Lemma.** If \( a \leq n/(4 \ln n) \) and \( n \) is large enough then Maker wins the \((a:b)\)-minimum-degree \( \geq k \) game on \( K_n \) if

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I.e., if \( b < a < \frac{n}{4 \ln n} \), then Maker can ensure that his graph has minimum-degree at least \((n - 1)/2\), if \( n \) is large enough.
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Consider the \((2:2)\)-diameter 2 game. This is called *accelerating the game*. 
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Consider the \((2:2)\)-diameter 2 game.
This is called *accelerating the game*.

The mimic strategy doesn’t work, here.
Solution to (a:b) diameter 2 game

**THEOREM.**
Solution to (a:b) diameter 2 game

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**MAKER** wins the \(2 : c \left(\frac{n}{\ln^3 n}\right)^{1/5}\)-diameter 2 game and
Solution to (a:b) diameter 2 game

**THEOREM.**

**-maker** wins the \(\left(2 : c \left(\frac{n}{\ln^3 n}\right)^{1/5}\right)\)-diameter 2 game and

**breaker** wins the \(\left(2 : 3 \sqrt{\frac{n}{\ln n}}\right)\)-diameter 2 game,
Solution to (a:b) diameter 2 game

**Theorem.**

**Maker** wins the $\left(2 : c \left(\frac{n}{\ln^3 n}\right)^{1/5}\right)$-diameter 2 game and **Breaker** wins the $\left(2 : 3 \sqrt{n/\ln n}\right)$-diameter 2 game, for some $c$, if $n$ is big enough.
Solution to \((a:b)\) diameter 2 game

**Theorem.**

**Maker** wins the \(\left(2 : c \left(\frac{n}{\ln^3 n}\right)^{1/5}\right)\)-diameter 2 game and

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for some \(c\), if \(n\) is big enough.

The upper bound is close to probabilistic intuition.
Other results

**THEOREM.**
Let $d \geq 3$ and $a > 1$. 
Other results

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Let \( d \geq 3 \) and \( a > 1 \).

**MAKER** wins the
- \((1 : c_1 n^{1 - \lfloor d/2 \rfloor^{-1}} / \ln n)\)-diameter \( d \) game.
Other results

**THEOREM.**
Let $d \geq 3$ and $a > 1$.

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- $\left(1 : c_1n^{1-\lceil [d/2]^{-1} \rceil} / \ln n \right)$-diameter $d$ game.

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- $(a : c_3 n^{1-d^{-1}})$-diameter $d$ game,

for absolute constants $c_1, c_2, c_3 > 0$, if $n$ is big enough.
Diameter 3

For $d = 3$,

- **Maker** wins the $(1 : O(n^{1/2}/\ln n))$-diameter 3 game and
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For probabilistic intuition, note that $G(n, p)$:
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For probabilistic intuition, note that $G(n, p)$:

- has diameter 3 if $p \gg \frac{\ln n}{n^{2/3}}$
- fails to have diameter 3 if $p \ll \frac{\ln n}{n^{2/3}}$. 
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This means the probabilistic intuition would give $\frac{a}{a+b}$ should be about $n^{-2/3}$. So, diameter 3 violates probabilistic intuition for $a = 1$. 
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But *acceleration*; i.e., increasing $a$, seems not to contradict probabilistic intuition.
Proof ideas

To show, for example, that Maker wins the \((2, n^{1/5})\)-diameter two game,
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To show, for example, that $\text{MAKER}$ wins the $(2, n^{1/5})$-diameter two game, we divide $\text{MAKER}$’s strategy into two phases, playing alternating subgames in each phase.

For example: The subgames of Phase I:
Proof ideas

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For example: The subgames of Phase I:

- **Degree Game.**
Proof ideas

To show, for example, that \textsc{Maker} wins the \((2, n^{1/5})\)-diameter two game, we divide \textsc{Maker}'s strategy into two phases, playing alternating subgames in each phase.

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- \textbf{Degree Game}. \textsc{Maker} can ensure that the minimum degree of his graph is \(\frac{an}{a+b} - O\left(\sqrt{n \ln n}\right)\).
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For example: The subgames of Phase I:

- **Degree Game.** \textsc{Maker} can ensure that the minimum degree of his graph is \( \frac{an}{a+b} - O\left(\sqrt{n \ln n}\right) \).

- **Expansion Game.**
Proof ideas

To show, for example, that Maker wins the \((2, n^{1/5})\)-diameter two game, we divide Maker’s strategy into two phases, playing alternating subgames in each phase.

For example: The subgames of Phase I:

- **Degree Game.** Maker can ensure that the minimum degree of his graph is \(\frac{an}{a+b} - O\left(\sqrt{n \ln n}\right)\).

- **Expansion Game.** Maker can ensure that second neighborhoods are large.
Proof ideas

To show, for example, that \textsc{Maker} wins the \((2, n^{1/5})\)-diameter two game, we divide \textsc{Maker}'s strategy into two phases, playing alternating subgames in each phase.

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- **Ratio Game.**
Proof ideas

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- **Connecting High Vertices.**
Proof ideas

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- **Ratio Game.** If Breaker occupies many edges incident to a vertex, so does Maker.

- **Connecting High Vertices.** Vertices with many incident edges chosen will connect to each other.
Proof structure for degree game

For the **DEGREE GAME**, we use a potential function argument, choosing the weights carefully.
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There is a function $T_i$, computed after the $i^{th}$ step, with the following properties:
Proof structure for degree game

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There is a function $T_i$, computed after the $i^{th}$ step, with the following properties:

(a). If Breaker wins in the $i^{th}$ step, then $T_i \geq 1$, 
Proof structure for degree game

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There is a function $T_i$, computed after the $i^{th}$ step, with the following properties:

(a). If **BREAKER** wins in the $i^{th}$ step, then $T_i \geq 1$,

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(c). $T_0 < 1$. 
Proof structure for degree game

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(a). If **Breaker** wins in the $i^{th}$ step, then $T_i \geq 1$,
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(c). $T_0 < 1$.

Clearly, the existence of such a $\{T_i\}$ gives that **Breaker** cannot win the degree game.
Thanks

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