A NON-LOCAL POREOUS MEDIA EQUATION

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ABSTRACT. In this manuscript we consider a non-local porous medium equation with non-local diffusion effects given by a fractional heat operator
\[
\begin{align*}
\partial_t u &= \text{div} (u \nabla p), \\
\partial_t p &= -(-\Delta)^s p + u^2,
\end{align*}
\]
in three space dimensions for $3/4 \leq s \leq 1$. Global in time existence of weak solutions is shown by employing a time semi-discretization of the equations, an energy inequality and a uniform estimate for the first moment of the solution to the discretized problem.

1. Introduction

In this manuscript we study existence of weak solutions to a porous medium equation with non-local diffusion effects:
\[
\begin{align*}
\partial_t u &= \text{div} (u \nabla p), \\
\partial_t p &= -(-\Delta)^s p + u^2.
\end{align*}
\]
Here $u(x, t) \geq 0$ denotes the density function and $p(x, t) \geq 0$ the pressure. We analyze the problem when $x \in \mathbb{R}^3$ and $3/4 \leq s \leq 1$. The model describes the time evolution of a density function $u$ that evolves under the continuity equation
\[
\partial_t u = \text{div} (uv),
\]
where the velocity is conservative, $v = \nabla p$, and $p$ is related to $u^2$ by the inverse of the fractional heat operator $\partial_t + (-\Delta)^s$.

Problem (1) is the parabolic-parabolic version of a parabolic-elliptic problem recently studied in [2]. In [2] the authors studied existence of sign-changing weak solutions to
\[
\partial_t u = \text{div} (|u|^\alpha - 1 (|u|^m - 2u)),
\]
which is the parabolic-elliptic version of the system (1) considered in this current manuscript.

Notice that for $m = 3$ and $\alpha = 2 - 2s$ equation (2) reads as
\[
\partial_t u = \text{div} (u \nabla p), \quad p = (-\Delta)^{-s} u^2, \quad 0 < s \leq 1,
\]
which is the parabolic-elliptic version of the system (1) considered in this current manuscript. The introduction of $\partial_t p$ in the equation for the pressure makes our system quite different from [2]. For example, techniques such as maximum principle and Stroock-Varopoulos inequality do not work in the current parabolic-parabolic setting. We overcome these major shortcomings with the introduction of ad-hoc regularization terms, together with suitable compact embeddings and moment estimates (see later for a more detailed explanation).

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A linear parabolic-elliptic version of (1)

\[ \partial_t u = \text{div} (u \nabla p), \quad p = (-\Delta)^{s} u, \quad 0 < s \leq 1, \]

was studied by the first author and collaborators in a series of papers: existence of weak solutions for (4) as proven in \cite{6, 4, 14} and Hölder regularity in \cite{5}. The case \( s = 1 \) also appeared in \cite{1} as a model for superconductivity.

Systems (4) and (1) are also reminiscent to a well-studied macroscopic model proposed for phase segregation in particle systems with long range interaction:

\[ \begin{cases} 
\partial_t u = \Delta u + \text{div} (\sigma(u) \nabla p), \\
p = K * u.
\end{cases} \]

Here \( \sigma(u) := u(1 - u) \) denotes the mobility of the system and \( K \) a bounded, symmetric and compactly supported kernel. This model was proposed in \cite{7} and appears as the hydrodynamic (or mean field) limit of a microscopic model undergoing phase segregation with particles interacting under a short-range and long-range Kac potential. System (5) can also be reformulated as the gradient flow associated to the free energy functional

\[ H[u, p] = \int u \log u + (1 - u) \log(1 - u) + pu \, dx. \]

Several variants of (5) have been considered in the literature and we refer to \cite{13, 7, 9, 8} and references therein for more detailed discussions on this topic. In fact, any system that, at certain temperatures, exhibits coexistence of different densities (for example fluid and vapor or fluid and solid) has equilibrium configurations that segregate into different regions; the surface of these regions are minimizers of a free energy functional. Given a perturbation of such equilibrium state, the relaxation of the density function \( u(x, t) \) can be described in general by nonlinear integro-differential equations.

We also mention \cite{12} for the study of a deterministic particle method for heat and Fokker-Planck equations of porous media type where the non-locality appears in the coefficients.

The condition that the pressure satisfies a parabolic equation introduces non-trivial complications in the analysis of (1). The non-local structure prevents the equation from satisfying a comparison principle. Moreover maximum principle does not give useful insights, since at any point of maximum for \( u \) we only know that \( \partial_t u \leq u \Delta p \). We also remark that the techniques for (4) do not apply here, as they rely on the fact that \( u \) and \( p \) are linked by an elliptic operator. We overcome the difficulties related to the lack of comparison and maximum with the introduction of several regularizations. Stampacchia truncation arguments yield non-negativity of the solutions and a modified Minty’s trick will be used to identify the limit for \( u^2 \).

The main result of this manuscript is summarized in the following theorem:

**Theorem 1.** Let \( u_{in}, p_{in} : \mathbb{R}^3 \to (0, +\infty) \) be functions such that \( u_{in}, p_{in} \in L^1(\mathbb{R}^3) \), \( \int_{\mathbb{R}^3} u_{in}^2 + |\nabla p_{in}|^2 \, dx < +\infty \) and \( \int_{\mathbb{R}^3} u_{in} \gamma(x) \, dx < +\infty \), with \( \gamma(x) := \sqrt{1 + |x|^2} \). Let \( q > 3/s \). There exist functions \( u, p : \mathbb{R}^3 \times [0, \infty) \to [0, +\infty) \) such that for every \( T > 0 \)

\[ u \in L^\infty(0, T, L^1 \cap L^2(\mathbb{R}^3)), \quad p \in L^\infty(0, T, H^1 \cap L^1(\mathbb{R}^3)), \]

\[ p \in L^2(0, T, H^{s+1}(\mathbb{R}^3)), \quad \sup_{[0, T]} \int_{\mathbb{R}^3} u \gamma \, dx < +\infty, \]
\[ \partial_t u \in L^2(0, T, (W^{1,q}(\mathbb{R}^3))'), \quad \partial_t p \in L^2(0, T, (L^2 \cap L^4(\mathbb{R}^3))'), \]

which satisfy the following weak formulation to (1):

\[
\begin{align*}
\int_0^T \langle \partial_t u, \phi \rangle dt + \int_0^T \int_{\mathbb{R}^3} u \nabla p \cdot \nabla \phi dx dt = 0 & \quad \forall \phi \in L^2(0, T; W^{1,q}(\mathbb{R}^3)), \\
\int_0^T \langle \partial_t p, \phi \rangle dt + \int_0^T \int_{\mathbb{R}^3} ((-\Delta)^s p - u^2) \psi dx dt = 0 & \quad \forall \psi \in L^2(0, T; L^2 \cap L^4(\mathbb{R}^3)), \\
\lim_{t \to 0} u(t) = u_{in} & \quad \text{in } W^{1,q}(\mathbb{R}^3)', \quad \lim_{t \to 0} p(t) = p_{in} & \quad \text{in } (L^2 \cap L^4(\mathbb{R}^3)').
\end{align*}
\]

The starting point about our analysis is the observation that

\[ H[u, p] := \int_{\mathbb{R}^d} \left( u^2 + \frac{1}{2} |\nabla p|^2 \right) dx \]

is a Lyapunov functional for (1) and satisfies the bound

\[
\int_{\mathbb{R}^d} \left( u^2 + \frac{1}{2} |\nabla p|^2 \right) dx + \int_0^T \int_{\mathbb{R}^d} (-\Delta)^s |\nabla p|^2 dx dt = \int_{\mathbb{R}^d} \left( u_{in}^2 + \frac{1}{2} |\nabla p_{in}|^2 \right) dx.
\]

Indeed, formal computations show that

\[
\frac{d}{dt} \int_{\mathbb{R}^d} u^2 dx = \langle \text{div}(u \nabla p), 2u \rangle = -2 \int_{\mathbb{R}^d} u \nabla u \cdot \nabla p dx = - \int_{\mathbb{R}^d} \nabla u^2 \cdot \nabla p dx.
\]

Testing the equation for \( p \) against \( \Delta p \) we obtain

\[
\begin{align*}
\int_{\mathbb{R}^d} \nabla u^2 \cdot \nabla p dx &= \frac{d}{dt} \int_{\mathbb{R}^d} \frac{|\nabla p|^2}{2} dx + \int_{\mathbb{R}^d} \nabla p \cdot (-\Delta)^s \nabla p dx \\
&= \frac{d}{dt} \int_{\mathbb{R}^d} \frac{|\nabla p|^2}{2} dx + \int_{\mathbb{R}^d} |(-\Delta)^{s/2} \nabla p|^2 dx,
\end{align*}
\]

which leads to

\[
\frac{d}{dt} H[u, p] + \int_{\mathbb{R}^d} (-\Delta)^{s/2} \nabla p|^2 dx = 0 \quad t > 0.
\]

The proof of Theorem 1 is based on a time semi-discretization of (1). The discretization and regularization of the problem yield existence of a solution to the semi-discretized problem that is uniform in terms of the time-step. This is achieved via the mass conservation and regularization of the problem yield existence of a solution to the semi-discretized problem.

In the following, we will allow existence of a solution to the semi-discretized problem. This, together with strong convergence results for \( p \), will allow for strong convergence of \( u \).

Therefore, a by-product of our analysis is the following lemma:
Lemma 1. Let $K$ be the fundamental solution to the fractional heat operator $\partial_t + (-\Delta)^s$. For any regular enough function $w$ we have
\[
A[w, w] := \int_0^T \int_0^T \int_{\mathbb{R}^6} K(x-y, |t-\tau|)w(x,t)w(y,\tau) \, dx \, dy \, d\tau \, dt \geq 0,
\]
and $A[w, w] = 0$ if and only if $w = 0$.

This last inequality is the starting point (work in progress in [3]) for the analysis of (1) when the pressure satisfies the equation
\[
\partial_t p = -(-\Delta)^s p + u.
\]

The rest of the paper is organized as follows: in Section 2 we show two preliminary technical lemmas. The proof of the main theorem is in Section 3. Lemma 1 is shown at the end of Section 3.4.1

2. SOME TECHNICAL RESULTS

We first prove two lemmas that will be used later.

Lemma 2. Let $g : [0, \infty) \to [0, \infty)$ be a continuous, nondecreasing function such that $\lim_{r \to \infty} g(r) = \infty$. For $\kappa \in [1, 2]$ define the functional space $V_{g, \kappa}$ as
\[
V_{g, \kappa} := H^1(\mathbb{R}^3) \cap L^\kappa(\mathbb{R}^3), g(|x|)dx = \left\{ f \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} f(x)^\kappa g(|x|)dx < \infty \right\}.
\]

Then $V_{g, \kappa}$ is compactly embedded in $L^q(\mathbb{R}^3)$ for any $\kappa \leq q < 6$.

Proof. Let $\{f_n\}$ be a uniformly bounded sequence in $V_{g, \kappa}$. We first notice that there exists a subsequence, still denoted with $f_n$ such that
\[
f_n \rightharpoonup f \text{ weakly in } V_{g, \kappa}.
\]

Denote with $B_R$ the ball of center $x = 0$ and radius $R$. Since $H^1(B_R)$ is compactly embedded in $L^q(B_R)$ for any $1 \leq q < 6$, there exists a subsequence, still denoted with $f_n$ such that $f_n \to f$ strongly in $L^q(B_R)$, for any $1 \leq q < 6$. Thanks to a Cantor diagonal argument, the subsequence $f_n$ can be chosen to be independent of $R$.

Next we show that $\{f_n\}$ is a Cauchy sequence in $L^\kappa(\mathbb{R}^3)$: for $n, m$ big enough
\[
\int_{\mathbb{R}^3} |f_n - f_m|^{\kappa} \, dx = \int_{B_R} |f_n - f_m|^{\kappa} \, dx + \int_{B_R^c} |f_n - f_m|^{\kappa} \, dx \\
\leq \varepsilon + \frac{1}{g(R)} \int_{B_R^c} g(|x|)|f_n - f_m|^{\kappa} \, dx \leq \varepsilon
\]
by choosing $R$ big enough. Therefore $\|f_n - f\|_{L^\kappa(\mathbb{R}^3)} \to 0$.

Strong convergence in any other $L^q(\mathbb{R}^3)$ for any $\kappa < q < 6$ follows from Gagliardo-Nirenberg-Sobolev inequality:
\[
\|f_n - f\|_{L^q(\mathbb{R}^3)} \leq \|\nabla(f_n - f)\|_{L^2(\mathbb{R}^3)}^{\theta} \|f_n - f\|_{L^\kappa(\mathbb{R}^3)}^{1-\theta}.
\]

Lemma 3. Define $\eta(x) = (1 + |x|^2)^{-\alpha/2}$ with $\alpha > 4$ and for every $R \geq 1$ we set $\eta_R(x) = \eta(x/R)$. For $s > 1/2$ we have
\[
|(-\Delta)^s \eta(x/R)| \leq \frac{C}{R}.
\]
Proof. First we observe that
\[ \exists C > 0 : \ |\nabla \eta_r| + |\Delta \eta_r| \leq CR^{-1} \quad \text{in } \mathbb{R}^3, \quad \forall R \geq 1. \]

Then we write
\[
(-\Delta)^s \eta_r(x/R) = c(s) P.V. \int_{B_1} \frac{\eta_r(x+y) - \eta_r(x-y) - 2\eta_r(x)}{|y|^{3+2s}} dy
\]
\[+ c(s) P.V. \int_{B_1^c} \frac{\eta_r(x+y) + \eta_r(x-y) - 2\eta_r(x)}{|y|^{3+2s}} dy \leq c(s) \|D^2 \eta_r\|_{L^\infty} \int_{B_1} \frac{1}{|y|^{1+2s}} dy + c(s) \|\nabla \eta_r\|_{L^\infty} \int_{B_1^c} \frac{1}{|y|^{2+2s}} dy \leq \frac{C}{R}. \]

\[\square\]

3. Proof of the main theorem

For \( \delta > 0 \) (small enough), define the spaces
\[ X := L^2 \cap L^{6-\delta}(\mathbb{R}^3), \quad Y := \left\{ g \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} g^2 \gamma dx < \infty \right\}, \]
where \( \gamma(x) := \sqrt{1 + |x|^2}. \)

For every measurable function \( g : \mathbb{R}^3 \to \mathbb{R} \cup \{\pm \infty\} \) we denote by \( g_+ := \max\{g, 0\} \) and \( g_- := \min\{g, 0\} \) its positive and negative part, respectively.

For given constants \( q, \tau, \varepsilon > 0 \), functions \( u^* \in Y \) and \( p^* \in H^{2s}(\mathbb{R}^3) \) such that \( u^*, p^* \geq 0 \) a.e. in \( \mathbb{R}^3 \), consider the discrete problem
\[
\int_{\mathbb{R}^3} \frac{u - u^*}{\tau} \phi + u \nabla p \cdot \nabla \phi dx + \gamma \nabla u \cdot \nabla \phi + \varepsilon u \phi \gamma dx = 0 \quad \forall \phi \in Y, \tag{6}
\]
\[
\int_{\mathbb{R}^3} \frac{p - p^*}{\tau} + (-\Delta)^s p - q \Delta p - u^2 = 0. \tag{7}
\]

The proof of Theorem 3 is divided into several steps: we first show existence and uniqueness of solution to \((6), \ (7)\) by Leray-Schauder fixed point theorem. After we perform the limit \( \varepsilon \to 0 \) and then \( \tau \to 0 \). The last limit is \( q \to 0 \); it is the most complicated one since we need compactness for \( u \) without relying on the term \( q \int_{\mathbb{R}^3} \nabla u \cdot \nabla \phi dx \). Here is where the restriction for \( s \) comes into play and where Lemma 3 is used.

3.1. Existence and uniqueness for \((6)-(7)\). For given constants \( q, \tau, \varepsilon > 0, \sigma \in [0, 1], \)
functions \( z \in X, u^* \in Y \) and \( p^* \in H^{2s}(\mathbb{R}^3) \) such that \( u^*, p^* \geq 0 \) a.e. in \( \mathbb{R}^3 \), consider the linear problem
\[
\int_{\mathbb{R}^3} (\tau^{-1} (u - u^*)) \phi + \sigma z^+ \nabla p \cdot \nabla \phi dx + \gamma \int_{\mathbb{R}^3} \nabla u \cdot \nabla \phi dx + \varepsilon \int_{\mathbb{R}^3} u \phi \gamma dx = 0 \quad \forall \phi \in Y, \tag{8}
\]
\[
\int_{\mathbb{R}^3} (\tau^{-1} (p - p^*)) \psi + (-\Delta)^{s/2} p \cdot (-\Delta)^{s/2} \psi + \gamma \nabla p \cdot \nabla \psi - z^2 \psi) dx = 0 \quad \forall \psi \in H^1(\mathbb{R}^3). \tag{9}
\]

We first solve \((9)\). Choosing \( \delta \leq 2 \) we have that \( z^2 \in L^2(\mathbb{R}^3) \). Therefore Lax-Milgram lemma yields existence of a unique solution \( p \in H^1(\mathbb{R}^3) \). Moreover standard elliptic regularity results imply that \( p \in H^2(\mathbb{R}^3) \) and consequently \( \nabla p \in L^2 \cap L^6(\mathbb{R}^3) \).
We now search for uniform bounds with respect to $\sigma$. Since $z \in L^4(\mathbb{R}^3)$ and $\nabla p \in L^4(\mathbb{R}^3)$, the linear mapping
\[
\phi \in Y \mapsto \int_{\mathbb{R}^3} (\tau^{-1}u^\ast \phi - \sigma z_x \nabla p \cdot \nabla \phi) dx \in \mathbb{R}
\]
is continuous for $\delta > 0$ small enough. Once more, Lax-Milgram lemma yields the existence of a unique solution $u \in Y$ to (8).

We can now define the mapping
\[
F : (z, \sigma) \in X \times [0, 1] \mapsto u \in X,
\]
where $(u, p) \in Y \times H^2(\mathbb{R}^3)$ is the unique solution to (8), (9). Clearly $F(\cdot, 0)$ is a constant mapping. Moreover $F$ is continuous and also compact due to the compact embedding $Y \hookrightarrow X$, see Lemma [2]

Next we show that any fixed point $u \in Y$ of $F(\cdot, \sigma)$ is nonnegative and uniformly bounded in $\sigma$. Via Stampacchia truncation argument (i.e. choosing $\phi = u_-$ and $\psi = p_-$ as test functions) it follows that $u, p \geq 0$ a.e. in $\mathbb{R}^3$. The nonnegativity of $u$ and the $H^2(\mathbb{R}^3)$-regularity of $p$ allow for the formulation
\[
\begin{align*}
(10) & \quad \int_{\mathbb{R}^3} (\tau^{-1}(u - u^\ast) + \sigma u \nabla p \cdot \nabla \phi) dx + \varrho \int_{\mathbb{R}^3} \nabla u \cdot \nabla \phi dx + \varepsilon \int_{\mathbb{R}^3} u \phi \gamma dx = 0 \quad \forall \phi \in Y, \\
(11) & \quad \tau^{-1}(p - p^\ast) + (-\Delta)^s p - \varrho \Delta p - u^2 = 0 \quad \text{in } \mathbb{R}^3.
\end{align*}
\]

We now search for uniform bounds with respect to $\sigma$: choosing $\phi = u$ in (10) leads to
\[
\int_{\mathbb{R}^3} (\tau^{-1}(u - u^\ast) u + \varrho \int_{\mathbb{R}^3} |\nabla u|^2 dx + \varepsilon \int_{\mathbb{R}^3} u^2 dx) = -\sigma \int_{\mathbb{R}^3} u \nabla p \cdot \nabla u dx = \frac{\sigma}{2} \int_{\mathbb{R}^3} u^2 \Delta p dx.
\]

On the other hand, multiplying (11) by $\Delta p \in L^2(\mathbb{R}^3)$ and integrating in $\mathbb{R}^3$ yields
\[
\begin{align*}
\int_{\mathbb{R}^3} u^2 \Delta p dx &= \int_{\mathbb{R}^3} (\tau^{-1}(p - p^\ast) + (-\Delta)^s p - \varrho \Delta p) \Delta p dx \\
&= -\tau^{-1} \int_{\mathbb{R}^3} (\nabla p - \nabla p^\ast) \cdot \nabla p dx - \int_{\mathbb{R}^3} |(-\Delta)^{s/2} \nabla p|^2 dx - \varrho \int_{\mathbb{R}^3} (\Delta p)^2 dx.
\end{align*}
\]

Given that
\[
(u - u^\ast)u \geq u^2/2 - (u^\ast)^2/2, \quad (\nabla p - \nabla p^\ast) \cdot \nabla p \geq |\nabla p|^2/2 - |\nabla p^\ast|^2/2,
\]
we deduce
\[
\begin{align*}
\frac{1}{\tau} \int_{\mathbb{R}^3} \left( \frac{u^2}{2} + \frac{\sigma}{4} |\nabla p|^2 \right) dx + \varrho \int_{\mathbb{R}^3} |\nabla u|^2 dx + \varepsilon \int_{\mathbb{R}^3} u^2 dx \\
&\quad + \frac{\sigma}{2} \int_{\mathbb{R}^3} |(-\Delta)^{s/2} \nabla p|^2 dx + \varrho \frac{\sigma}{2} \int_{\mathbb{R}^3} (\Delta p)^2 dx \\
&\leq \frac{1}{\tau} \int_{\mathbb{R}^3} \left( \frac{(u^\ast)^2}{2} + \frac{\sigma}{4} |\nabla p^\ast|^2 \right) dx.
\end{align*}
\]

The above estimate yields a bound for $u$ in $H^1(\mathbb{R}^3)$ which is uniform in $\sigma$. Together with the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^2 \cap L^{6-\delta}(\mathbb{R}^3) = X$ we have that $u$ belongs to $X$, with $\|u\|_X$ bounded uniformly with respect to $\sigma$. Leray-Schauder fixed point theorem yields the existence of a fixed point $u \in Y$ for $F(\cdot, 1)$, i.e. a solution $(u, p) \in Y \times H^2(\mathbb{R}^3)$ to
\[
\begin{align*}
(13) & \quad \int_{\mathbb{R}^3} \frac{u - u^\ast}{\tau} \phi dx + \int_{\mathbb{R}^3} u \nabla p \cdot \nabla \phi dx + \varrho \int_{\mathbb{R}^3} \nabla u \cdot \nabla \phi dx + \varepsilon \int_{\mathbb{R}^3} u \phi \gamma dx = 0 \quad \forall \phi \in Y,
\end{align*}
\]
such that \( u, p \geq 0 \) a.e. in \( \mathbb{R}^3 \) and (12) holds for \( \sigma = 1 \):

\[
\frac{p - p^s}{\tau} + (-\Delta)^s p - \varrho \Delta p - u^2 = 0 \quad \text{in } \mathbb{R}^3,
\]

Choosing \( \varepsilon \), \( \tau \), \( R \), \( \varsigma \) satisfying (15), (16) provide us with uniform (w.r.t. \( \varepsilon, \tau, R, \varsigma \)) bounds provided by (15) are enough to pass to the limit except for the second term of (13) and the last one of (14). For these we need the existence of a subsequence \( u^{(e)} \) strongly convergent in some \( L^p \) space.

3.2. The limit \( \varepsilon \to 0 \). The next step is to take the \( \lim_{\varepsilon \to 0} \) in (13)-(15). Note that all the bounds provided by (15) are enough to pass to the limit except for the second term of (13) and the last one of (14). For these we need the existence of a subsequence \( u^{(e)} \) strongly convergent in some \( L^p \) space.

Since (15) already shows uniform bound for the \( H^1 \)-norm of \( u \), one possible way to show compactness is using Lemma 2 with \( \kappa = 1 \), meaning to show a bound for \( \int_{\mathbb{R}^3} u^{(e)} \) in \( \mathbb{R}^3 \). To this aim we first assume that \( u^* \) satisfies \( \int_{\mathbb{R}^3} u^* \gamma dx < \infty \). We define \( \eta(x) = (1 + |x|^2)^{-\alpha/2} \) with \( \alpha > 4 \) and for every \( R \geq 1 \) we set \( \eta_R(x) = \eta(x/R) \). It follows that \( \gamma \eta_R \in L^1 \cap L^\infty(\mathbb{R}^3) \). Moreover, it is easy to see that a constant \( C > 0 \) exists such that \( |\nabla \eta| + |D^2 \eta| \leq C \eta \) in \( \mathbb{R}^3 \); a fortiori \( |\nabla \eta_R| + |D^2 \eta_R| \leq C \eta_R \) in \( \mathbb{R}^3 \) with the same \( R \)-independent constant \( C \). From this property it is straightforward to show that

\[
\exists C > 0 : \quad \frac{|\nabla (\eta_R \gamma)| + |D^2 (\eta_R \gamma)|}{\eta_R \gamma} \leq C \quad \text{in } \mathbb{R}^3, \quad \forall R \geq 1.
\]

Therefore one is allowed to choose \( \phi = \eta_R \gamma \) in (13) and deduce

\[
\tau^{-1} \int_{\mathbb{R}^3} (\varepsilon \nabla (\eta_R \gamma) + (u^{(e)} - u^*) \nabla (\eta_R \gamma) \cdot \nabla (\eta_R \gamma) + \varrho u^{(e)} \Delta (\eta_R \gamma) - \varepsilon u^{(e)} (\eta_R \gamma)^2) dx.
\]

Since \( |\nabla (\eta_R \gamma)| \leq C \) in \( \mathbb{R}^3 \) for \( R \geq 1 \), the uniform bounds for \( u \) and \( \nabla p \) in \( L^2(\mathbb{R}^3) \) yield a uniform (w.r.t. \( \varepsilon, \tau, R, \varrho \)) bound for \( u \nabla p \cdot \nabla (\eta_R \gamma) \) in \( L^1(\mathbb{R}^3) \). Moreover, the fact that \( |D^2 (\eta_R \gamma)| \leq C \eta_R \gamma \) with \( C \) independent of \( R \) implies that

\[
\varrho \int_{\mathbb{R}^3} u^{(e)} \Delta (\eta_R \gamma) dx \leq C \varrho \int_{\mathbb{R}^3} u^{(e)} (\eta_R \gamma) dx.
\]

We deduce

\[
(1 - C \tau \varrho) \int_{\mathbb{R}^3} u^{(e)} \eta_R \gamma dx \leq \int_{\mathbb{R}^3} u^* \eta_R \gamma dx + C' \tau \leq \int_{\mathbb{R}^3} u^* \gamma dx + C' \tau.
\]

Choosing \( \tau < C^{-1/2} \) we can take the limit \( R \to \infty \) in the above inequality and conclude (by Beppo Levi monotone convergence theorem)

\[
(1 - C \tau \varrho) \int_{\mathbb{R}^3} u^{(e)} \gamma dx \leq \int_{\mathbb{R}^3} u^* \gamma dx + C' \tau.
\]

The estimates obtained in (15), (16) provide us with \( \varepsilon \)-uniform bounds for \( u^{(e)} \) in the space

\[
Z \equiv H^1(\mathbb{R}^3) \cap L^1(\mathbb{R}^3, \gamma dx),
\]

which embeds compactly in \( L^1 \cap L^{6-\delta}(\mathbb{R}^3) \) for every \( \delta \in (0, 5] \). Therefore \( u^{(e)} \) is (up to subsequences) relatively strongly compact in \( L^1 \cap L^{6-\delta}(\mathbb{R}^3) \) as well as relatively weakly compact.
in $H^1(\mathbb{R}^3)$. Then there exists a subsequence (still denoted with $u^{(\varepsilon)}$) such that
\begin{equation}
\tag{17}
 u^{(\varepsilon)} \rightarrow u \text{ strongly in } L^1 \cap L^{6-\delta}(\mathbb{R}^3), \quad u^{(\varepsilon)} \rightarrow u \text{ a.e.}
\end{equation}

Therefore
\begin{equation}
\tag{18}
(u^{(\varepsilon)})^2 \rightarrow u^2 \text{ strongly in } L^2(\mathbb{R}^3), \quad u^{(\varepsilon)} \rightarrow u \text{ strongly in } L^{3/2}(\mathbb{R}^3).
\end{equation}

Going back to the limit in (14) and (13) we have that as $\varepsilon \to 0$
\begin{equation}
\int_{\mathbb{R}^3} (u^{(\varepsilon)})^2 \phi \, dx \rightarrow \int_{\mathbb{R}^3} u^2 \phi \, dx, \quad \forall \phi \in L^2(\mathbb{R}^3),
\end{equation}
\begin{equation}
\int_{\mathbb{R}^3} u^{(\varepsilon)} \nabla p^{(\varepsilon)} \cdot \nabla \phi \, dx \rightarrow \int_{\mathbb{R}^3} u \nabla p \cdot \nabla \phi \, dx, \quad \forall \phi \in H^1(\mathbb{R}^3),
\end{equation}

where we used (18) for the first limit, and (18) together with $\nabla p^{(\varepsilon)} \rightharpoonup \nabla p$ in $L^{6/(3-2s)}(\mathbb{R}^3)$ to obtain the second limit (remember that $p^{(\varepsilon)}$ is relatively weakly compact in $H^{1+s}(\mathbb{R}^3)$). Summarizing, taking the limit $\varepsilon \to 0$ in (13), (14) and subsequently employing a standard density argument we get
\begin{equation}
\tag{19}
\int_{\mathbb{R}^3} (\tau^{-1}(u - u^*)\phi + u \nabla p \cdot \nabla \phi) \, dx + \theta \int_{\mathbb{R}^3} \nabla u \cdot \nabla \phi \, dx = 0 \quad \forall \phi \in H^1(\mathbb{R}^3),
\end{equation}
\begin{equation}
\tag{20}
\tau^{-1}(p - p^*) + (-\Delta)^s p - \theta \Delta p - u^2 = 0 \quad \text{in } \mathbb{R}^3.
\end{equation}

Moreover $u, p \geq 0$ a.e. in $\mathbb{R}^3$ and
\begin{equation}
\tag{21}
\frac{1}{\tau} \int_{\mathbb{R}^3} \left( \frac{u^2}{2} + \frac{1}{4} |\nabla p|^2 \right) \, dx + \theta \int_{\mathbb{R}^3} |\nabla u|^2 \, dx
+ \frac{1}{2} \int_{\mathbb{R}^3} |(-\Delta)^{s/2} \nabla p|^2 \, dx + \frac{\theta}{2} \int_{\mathbb{R}^3} (\Delta p)^2 \, dx \leq \frac{1}{\tau} \int_{\mathbb{R}^3} \left( \frac{(u^*)^2}{2} + \frac{1}{4} |\nabla p^*|^2 \right) \, dx.
\end{equation}

Fatou’s lemma and almost everywhere convergence (17) imply that
\begin{equation}
(1 - C\tau \theta) \int_{\mathbb{R}^3} u^* \gamma \, dx \leq \int_{\mathbb{R}^3} u^* \gamma \, dx + C' \tau.
\end{equation}

We show now conservation of mass for $u$ and $p$. Since
\begin{equation}
\exists C > 0 : \ |\nabla \eta_R| + |\Delta \eta_R| \leq CR^{-1} \quad \text{in } \mathbb{R}^3, \quad \forall R \geq 1,
\end{equation}

by choosing $\phi = \eta_R$ in (19) and integrating by parts we get
\begin{equation}
\tau^{-1} \left|\int_{\mathbb{R}^3} (u - u^*) \eta_R \, dx \right| = \left|\int_{\mathbb{R}^3} (-u \nabla p \cdot \nabla \eta_R + \theta u \Delta \eta_R) \, dx \right|
\leq CR^{-1}(\|u\|_{L^2} \|\nabla p\|_{L^2} + \|u\|_{L^1}).
\end{equation}

The right-hand side of the above inequality is finite thanks to (16). The limit $R \to \infty$ in the above inequality (by monotone convergence) leads to
\begin{equation}
\int_{\mathbb{R}^3} (u - u^*) \, dx = 0.
\end{equation}

In the same way, multiplying (21) by $\eta_R$, integrating in $\mathbb{R}^3$ and integrating by parts leads to
\begin{equation}
\tau^{-1} \int_{\mathbb{R}^3} (p - p^*) \eta_R \, dx = \int_{\mathbb{R}^3} (u^2 \eta_R + \theta p \Delta \eta_R - p(-\Delta)^s \eta_R) \, dx.
\end{equation}
Since there exists a constant $C > 0$ such that $|(-\Delta)^s \eta_R| \leq CR^{-1}$ in $\mathbb{R}^3$ for all $R \geq 1$, see Lemma 3, the bound for the mass of $p$ follows

$$
\int_{\mathbb{R}^3} p \, dx = \int_{\mathbb{R}^3} p^* \, dx + \tau \int_{\mathbb{R}^3} u^2 \, dx.
$$

At this point we have proved the existence of sequences $(u_k)_{k \in \mathbb{N}} \subset H^1(\mathbb{R}^3)$, $(p_k)_{k \in \mathbb{N}} \subset H^2(\mathbb{R}^3)$ such that $u_0 = u_{in}$, $p_0 = p_{in}$, and for $k \geq 1$ $u_k, p_k \geq 0$ a.e. in $\mathbb{R}^3$,

(23) \quad \int_{\mathbb{R}^3} (\tau^{-1}(u_k - u_{k-1})\phi + u_k \nabla p_k \cdot \nabla \phi) \, dx + \theta \int_{\mathbb{R}^3} \nabla u_k \cdot \nabla \phi \, dx = 0 \quad \forall \phi \in H^1(\mathbb{R}^3),

(24) \quad \tau^{-1}(p_k - p_{k-1}) + (-\Delta)^s p_k - \theta \Delta p_k - u_k^2 = 0 \quad \text{in } \mathbb{R}^3,

with the estimates

(25) \quad \frac{1}{\tau} \int_{\mathbb{R}^3} \left( \frac{u_k^2}{2} + \frac{1}{4} |\nabla p_k|^2 \right) \, dx + \theta \int_{\mathbb{R}^3} |\nabla u_k|^2 \, dx

+ \frac{1}{2} \int_{\mathbb{R}^3} |(-\Delta)^{s/2} \nabla p_k|^2 \, dx + \frac{\theta}{2} \int_{\mathbb{R}^3} (\Delta p_k)^2 \, dx \leq \frac{1}{\tau} \int_{\mathbb{R}^3} \left( \frac{(u_{k-1})^2}{2} + \frac{1}{4} |\nabla p_{k-1}|^2 \right) \, dx,

(26) \quad (1 - C\tau \theta) \int_{\mathbb{R}^3} u_k \gamma \, dx \leq \int_{\mathbb{R}^3} u_{k-1} \gamma \, dx + C'\tau,

(27) \quad \int_{\mathbb{R}^3} u_k \, dx = \int_{\mathbb{R}^3} u_{k-1} \, dx, \quad \int_{\mathbb{R}^3} p_k \, dx = \int_{\mathbb{R}^3} p_{k-1} \, dx + \tau \int_{\mathbb{R}^3} u_k^2 \, dx.

Choose $T > 0$ arbitrary. Define $N = T/\tau$, $u^{(\tau)}(t) = u_0 \chi_{(0)}(t) + \sum_{k=1}^N u_k \chi_{((k-1)\tau, k\tau]}(t)$, $p^{(\tau)}(t) = p_0 \chi_{(0)}(t) + \sum_{k=1}^N p_k \chi_{((k-1)\tau, k\tau]}(t)$. Moreover define the backward finite difference w.r.t. time $D_\tau$ as

$$D_\tau f(t) \equiv \tau^{-1}(f(t) - f(t - \tau)), \quad t \in [\tau, T].$$

We can rewrite (23)–(26) with the new notation:

(28) \quad \int_0^T \int_{\mathbb{R}^3} ((D_\tau u^{(\tau)} \phi + u^{(\tau)} \nabla p^{(\tau)} \cdot \nabla \phi) \, dx \, dt + \theta \int_0^T \int_{\mathbb{R}^3} \nabla u^{(\tau)} \cdot \nabla \phi \, dx \, dt = 0

\forall \phi \in L^2(0, T; H^1(\mathbb{R}^3)),

(29) \quad \int_0^T \int_{\mathbb{R}^3} ((D_\tau p^{(\tau)} \psi + ((-\Delta)^{s/2} p^{(\tau)}))((-\Delta)^{s/2} \psi) + \theta \nabla p^{(\tau)} \cdot \nabla \psi - (u^{(\tau)})^2 \psi \, dx = 0

\forall \psi \in L^2(0, T; H^1(\mathbb{R}^3)),

(30) \quad \int_{\mathbb{R}^3} \left( \frac{(u^{(\tau)})^2}{2} + \frac{1}{4} |\nabla p^{(\tau)}|^2 \right) \, dx + \theta \int_0^t \int_{\mathbb{R}^3} |\nabla u^{(\tau)}|^2 \, dx \, dt' + \frac{\theta}{2} \int_0^t \int_{\mathbb{R}^3} (\Delta p^{(\tau)})^2 \, dx \, dt'

+ \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} |(-\Delta)^{s/2} \nabla p^{(\tau)}|^2 \, dx \, dt' \leq \int_{\mathbb{R}^3} \left( \frac{(u_{in})^2}{2} + \frac{1}{4} |\nabla p_{in}|^2 \right) \, dx,

(31) \quad \int_{\mathbb{R}^3} u^{(\tau)}(t) \gamma \, dx \leq (1 - C\tau \theta)^{-t/\tau} \int_{\mathbb{R}^3} u_{in} \gamma \, dx + C'(1 - C\tau \theta)^{-1},

(32) \quad \int_{\mathbb{R}^3} u^{(\tau)}(t) \, dx = \int_{\mathbb{R}^3} u_{in} \, dx, \quad \int_{\mathbb{R}^3} p^{(\tau)}(t) \, dx \leq \int_{\mathbb{R}^3} p_{in} \, dx + Ct \quad t \in [0, T],

where all the constants in (32) and (31) only depend on the entropy at initial time.
3.3. The limit $\tau \to 0$. We first estimate the time derivative of the density function:

$$\left| \int_0^T \int_{\mathbb{R}^3} (D_\tau u^{(\tau)}) \phi \, dx \, dt \right| \leq \int_0^T \int_{\mathbb{R}^3} u^{(\tau)} \nabla \rho^{(\tau)} \cdot \nabla \phi \, dx \, dt + \rho \int_0^T \int_{\mathbb{R}^3} \nabla u^{(\tau)} \cdot \nabla \phi \, dx \, dt$$

$$\leq \| \nabla \rho^{(\tau)} \|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \| u^{(\tau)} \|_{L^2(0,T;L^6(\mathbb{R}^3))} \| \nabla \phi \|_{L^2(0,T;L^3(\mathbb{R}^3))}$$

$$+ \rho \| u^{(\tau)} \|_{L^2(0,T;H^1(\mathbb{R}^3))} \| \phi \|_{L^2(0,T;H^1(\mathbb{R}^3))}$$

$$\leq C(T) \| \phi \|_{L^2(0,T;W^{1,3} \cap H^1(\mathbb{R}^3))},$$

using (30) and the Sobolev embedding $L^2(0,T;H^1(\mathbb{R}^3)) \hookrightarrow L^2(0,T;L^6(\mathbb{R}^3))$. This yields

$$\| D_\tau u^{(\tau)} \|_{L^2(0,T;W^{1,3} \cap H^1(\mathbb{R}^3)')} \leq C(T).$$

The embedding $H^1(\mathbb{R}^3) \cap L^1(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$ is compact thanks to Lemma 2. Therefore Aubin-Lions Lemma applied to $u^{(\tau)}$ with the functional spaces

$$L^2(0,T,H^1(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)) \hookrightarrow L^2(0,T,L^2(\mathbb{R}^3))$$

$$\hookrightarrow L^2(0,T,(W^{1,3} \cap H^1(\mathbb{R}^3))'),$$

yields the existence of a subsequence of $u^{(\tau)}$ (which we denote again with $u^{(\tau)}$) such that

$$u^{(\tau)} \to u \quad \text{strongly in} \quad L^2(0,T;L^2(\mathbb{R}^3)),$$

and

$$u^{(\tau)} \to u \quad \text{a.e. in} \quad \mathbb{R}^3 \times [0,T].$$

Since $(u^{(\tau)})^2 \in L^\infty(0,T,L^1(\mathbb{R}^3)) \cap L^1(0,T,L^3(\mathbb{R}^3))$ interpolation yields

$$\| (u^{(\tau)})^2 \|_{L^2(0,T,L^{3/2}(\mathbb{R}^3))} \leq \| (u^{(\tau)})^2 \|_{L^\infty(0,T,L^1(\mathbb{R}^3))} \| (u^{(\tau)})^2 \|_{L^1(0,T,L^3(\mathbb{R}^3))},$$

and consequently

$$u^{(\tau)} \to u^2 \quad \text{in} \quad L^2(0,T,L^{3/2}(\mathbb{R}^3)),$$

thanks to (34). Hence as $\tau \to 0$:

$$\int_0^T \int_{\mathbb{R}^3} (u^{(\tau)})^2 \psi \, dx \, dt \to \int_0^T \int_{\mathbb{R}^3} u^2 \psi \, dx \, dt, \quad \text{for all} \ \psi \in L^2(0,T,H^1(\mathbb{R}^3)).$$

From the uniform bounds for $u^{(\tau)}$ in $L^\infty(0,T,L^1(\mathbb{R}^3))$ in (31) and the strong convergence of $u^{(\tau)}$ in $L^2(0,T;L^2(\mathbb{R}^3))$ it follows $u^{(\tau)}$ is a Cauchy sequence in the $L^2(0,T,L^1(\mathbb{R}^3))$-norm. In fact, for $R$ big enough we have

$$\int_0^T \left( \int_{\mathbb{R}^3} |u^{(\tau)} - u^{(\tau_m)}|^2 \, dx \right) \, dt \leq \int_0^T \left( \int_{B_R} |u^{(\tau)} - u^{(\tau_m)}| \, dx + \frac{1}{R} \int_{B_R} |u^{(\tau)} - u^{(\tau_m)}| \gamma(x) \, dx \right) \, dt$$

$$\leq C(R^3) \int_0^T \int_{B_R} |u^{(\tau)} - u^{(\tau_m)}|^2 \, dx \, dt + \frac{1}{R^2} \int_0^T \left( \int_{B_R} |u^{(\tau)} - u^{(\tau_m)}| \gamma(x) \, dx \right)^2 \, dt \leq \varepsilon,$$

for $k, m \geq N$ with $N$ big enough. Therefore

$$u^{(\tau)} \to u \quad \text{strongly in} \quad L^2(0,T;L^1(\mathbb{R}^3)),$$

and via Gagliardo-Nirenberg inequality

$$u^{(\tau)} \to u \quad \text{strongly in} \quad L^2(0,T;L^1 \cap L^{6-\epsilon}(\mathbb{R}^3)) \quad \forall \epsilon \in (0,5].$$
Moreover directly from (30)
\[ u^{(r)} \rightharpoonup u \quad \text{weakly in } L^2(0, T; H^1(\mathbb{R}^3)), \]
\[ u^{(r)} \rightharpoonup u \quad \text{weakly}^* \text{ in } L^\infty(0, T; L^2(\mathbb{R}^3)). \]

From (38), (37) it follows
\[ D_{\tau} u^{(r)} \rightharpoonup \partial_t u \quad \text{weakly in } L^2(0, T, (W^{1,3} \cap H^1(\mathbb{R}^3))'). \]

Since \( p^{(r)} \) is uniformly bounded in \( L^\infty(0, T, L^1(\mathbb{R}^3)) \) and \( \nabla p^{(r)} \) is uniformly bounded in \( L^\infty(0, T, L^2(\mathbb{R}^3)) \), Gagliardo-Nirenberg and the entropy inequality (30) yield
\[ \| p^{(r)} \|_{L^\infty(0, T, H^1(\mathbb{R}^3))} + \| p^{(r)} \|_{L^2(0, T; H^{s+1}(\mathbb{R}^3))} \leq C, \]
where \( C \) only depends on the initial data. Hence there exists a subsequence of \( p^{(r)} \) (which we denote again with \( p^{(r)} \)) such that
\[ p^{(r)} \rightharpoonup p \quad \text{weakly in } L^2(0, T; H^{s+1}(\mathbb{R}^3)), \]
\[ p^{(r)} \rightarrow^* p \quad \text{weakly}^* \text{ in } L^\infty(0, T, H^1(\mathbb{R}^3)). \]

In particular
\[ \| p \|_{L^\infty(0, T, H^1(\mathbb{R}^3))} + \| p \|_{L^2(0, T, H^{s+1}(\mathbb{R}^3))} \leq C. \]

We now look at the term
\[ \int_0^T \int_{\mathbb{R}^3} (u^{(r)} \nabla p^{(r)} - u \nabla p) \cdot \nabla \phi \, dx \, dt, \]
with \( \phi \in L^2(0, T, W^{1,q} \cap H^1(\mathbb{R}^3)) \), \( q > 3 \), and rewrite it as
\[ \int_0^T \int_{\mathbb{R}^3} (u^{(r)} \nabla p^{(r)} - u \nabla p) \cdot \nabla \phi \, dx \, dt = \int_0^T \int_{\mathbb{R}^3} (u^{(r)} - u) \nabla p^{(r)} \cdot \nabla \phi \, dx \, dt \]
\[ + \int_0^T \int_{\mathbb{R}^3} u (\nabla p^{(r)} - \nabla p) \cdot \nabla \phi \, dx \, dt = I_1 + I_2. \]

Since
\[ |I_1| \leq \| u^{(r)} - u \|_{L^2(0, T, L^{6-\epsilon}(\mathbb{R}^3))} \| \nabla \phi \|_{L^2(0, T, L^6(\mathbb{R}^3))} \| \nabla p^{(r)} \|_{L^\infty(0, T, L^2(\mathbb{R}^3))}, \quad q > 3, \]
we conclude that \( |I_1| \rightarrow 0 \) as \( \tau \rightarrow 0 \) thanks to (37). Moreover also \( |I_2| \rightarrow 0 \) as \( \tau \rightarrow 0 \) since \( u \nabla \phi \in L^2(0, T, L^{4/3}(\mathbb{R}^3)) \), with a bound independent on \( \tau \), and \( \nabla p^{(r)} \rightarrow p \) in \( L^2(0, T, L^2 \cap L^6(\mathbb{R}^3)) \).

Let us look at the discrete time derivatives of the pressure function. Thanks to (35) we have
\[ \int_0^T \int_{\mathbb{R}^3} (u^{(r)})^2 \psi \, dx \, dt \leq \| (u^{(r)})^2 \|_{L^2(0, T, L^{3/2}(\mathbb{R}^3))} \| \psi \|_{L^2(0, T, L^3(\mathbb{R}^3))} \]
\[ \leq C(T, q) \| \psi \|_{L^2(0, T, H^1(\mathbb{R}^3))}, \]
which implies
\[ \left| \int_0^T \int_{\mathbb{R}^3} (D_{\tau} p^{(r)}) \psi \, dx \, dt \right| \leq C(T, q) \| \psi \|_{L^2(0, T, H^1(\mathbb{R}^3))}. \]

Since \( p^{(r)} \) is bounded in \( L^\infty(0, T, H^1(\mathbb{R}^3)) \) and \( D_{\tau} p^{(r)} \) is bounded in \( L^2(0, T, H^{-1}(\mathbb{R}^3)) \), we can invoke Aubin-Lions lemma to conclude that \( p^{(r)} \rightarrow p \) strongly in \( L^1(0, T, L^1(\Omega)) \), for
every \( \Omega \subset \mathbb{R}^3 \) bounded, and (up to a subsequence) \( p^{(\tau)} \to p \) a.e. in \( \Omega \). A Cantor’s diagonal argument yields

\[
(41) \quad p^{(\tau)} \to p \text{ strongly in } L^1(0,T,L^1(\mathbb{R}^3)), \quad p^{(\tau)} \to p \text{ a.e. in } \mathbb{R}^3.
\]

Finally, (40) and (41) imply

\[
D_\tau p^{(\tau)} \rightharpoonup \partial_t p \quad \text{weakly in } L^2(0,T,H^{-1}(\mathbb{R}^3)).
\]

This last limit should be understood as \( D_\tau p^{(\tau)} \) converges to a function that equals \( \partial_t p \) in the sense of distribution.

At this point we can take the limit \( \tau \to 0 \) in (28) and (29), which yields

\[
(42) \quad \int_0^T \langle \partial_t u, \phi \rangle dt + \int_0^T \int_{\mathbb{R}^3} u \nabla p \cdot \nabla \phi dx \, dt
\]

\[
+ \frac{q}{2} \int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 dx \, dt = 0 \quad \forall \phi \in L^2(0,T; W^{1,q} \cap H^1(\mathbb{R}^3)), \quad q > 3,
\]

\[
(43) \quad \int_0^T \langle \partial_t p, \psi \rangle dt + \int_0^T \int_{\mathbb{R}^3} (-\Delta)^s p - u^2 \psi dx \, dt
\]

\[
+ \frac{q}{2} \int_0^T \int_{\mathbb{R}^3} \nabla p \cdot \nabla \psi dx \, dt = 0 \quad \forall \psi \in L^2(0,T; H^1(\mathbb{R}^3)).
\]

Using Fatou’s lemma and the lower weak semicontinuity of the \( L^p \) norm we deduce from (30) the following entropy inequality

\[
(44) \quad \int_{\mathbb{R}^3} \left( \frac{u^2}{2} + \frac{1}{4} |\nabla p|^2 \right) dx + \frac{q}{2} \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 dx \, dt' + \frac{q}{2} \int_0^t \int_{\mathbb{R}^3} (\Delta p)^2 dx \, dt'
\]

\[
+ \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} |(-\Delta)^s \nabla p|^2 dx \, dt' \leq \int_{\mathbb{R}^3} \left( \frac{(u_{in})^2}{2} + \frac{1}{4} |\nabla p_{in}|^2 \right) dx.
\]

We point out that \( \log[(1 - C\tau \theta)^{-t/\tau}] = -\frac{t}{\tau} \log(1 - C\tau \theta) \to C\theta t \) as \( \tau \to 0 \), therefore \( (1 - C\tau \theta)^{-t/\tau} \to e^{Ct\theta} \) as \( \tau \to 0 \). Then Fatou’s lemma applied to (31), thanks to the a.e. convergence in (32), yields

\[
(45) \quad \int_{\mathbb{R}^3} u \gamma dx \leq e^{Ct\theta} \int_{\mathbb{R}^3} u_{in} \gamma dx + C't.
\]

We now show conservation of mass for \( u \) and \( L^\infty(0,T,L^1(\mathbb{R}^3)) \)-norm for \( p \). Since \( u^{(\tau)} \to u \) a.e. in \( \mathbb{R}^3 \times [0,T] \) and \( \|u^{(\tau)}\|_{L^\infty(0,T,L^2(\mathbb{R}^3))} \leq C \), Vitali’s convergence theorem ensures that \( u^{(\tau)} \to u \) strongly in \( L^1(B_R) \) for a.e. \( t > 0 \). This, together with (31), implies that \( u^{(\tau)} \) strongly converges in \( L^1(\mathbb{R}^3) \) for a.e. \( t > 0 \). In fact for \( R \geq \frac{C}{\varepsilon} \) one has

\[
\int_{\mathbb{R}^3} |u^{(\tau_k)} - u^{(\tau_m)}| \, dx = \int_{B_R} |u^{(\tau_k)} - u^{(\tau_m)}| \, dx + \frac{1}{R} \int_{B_R^c} |u^{(\tau_k)} - u^{(\tau_m)}| \, \gamma(x) \, dx \leq \varepsilon
\]

for \( k \) and \( m \) big enough. Therefore \( \int_{\mathbb{R}^3} u^{(\tau)} \, dx \to \int_{\mathbb{R}^3} u \, dx \) and thanks to (32)

\[
(46) \quad \int_{\mathbb{R}^3} u(t) \, dx = \int_{\mathbb{R}^3} u_{in} \, dx.
\]
Furthermore, thanks to the a.e. convergence of \( p^{(\tau)} \) in (42) and get

\[
\int_{\mathbb{R}^3} p(t) dx \leq \int_{\mathbb{R}^3} p_{in} dx + C t, \quad t \in [0, T].
\]

3.4. The limit \( \rho \to 0 \). Since \( u \in L^2(0, T; H^1(\mathbb{R}^3)) \), it is an admissible test function in (43):

\[
\int_0^T \int_{\mathbb{R}^3} u^3 dx dt = \int_0^T \left( \int_{\mathbb{R}^3} (\partial_t u, \rho u) + \int_{\mathbb{R}^3} (u(-\Delta)^s p + \rho \nabla p \nabla u) dx \right) dt
\]

\[
= \int_{\mathbb{R}^3} (u(T)p(T) - u_0p_0) dx + \int_0^T \int_{\mathbb{R}^3} (-\partial_t u, p) + u(-\Delta)^s p + \rho \nabla p \nabla u dx dt.
\]

Furthermore, thanks to (44) and (47) imply that

\[
\int_{\mathbb{R}^3} u^3 dx dt = \int_{\mathbb{R}^3} (u(T)p(T) - u_0p_0) dx + \int_0^T \int_{\mathbb{R}^3} (u|\nabla p|^2 + u(-\Delta)^s p + 2\rho \nabla p \cdot \nabla u) dx dt.
\]

We estimate the first and the third term as

\[
\int_{\mathbb{R}^3} u(T)p(T) dx \leq \| u \|_{L^\infty(0, T; L^2(\mathbb{R}^3))} \| p \|_{L^\infty(0, T; L^2(\mathbb{R}^3))},
\]

\[
\int_0^T \int_{\mathbb{R}^3} u|\nabla p|^2 dx dt \leq \int_0^T \left( \int_{\mathbb{R}^3} u^{\frac{2}{3}} dx \right)^{\frac{3}{2}} \left( \int_{\mathbb{R}^3} |\nabla p|^{\frac{6}{1-2s}} dx \right)^{\frac{1}{3-2s}} dt.
\]

Since \( s \geq \frac{3}{4} \) then \( \frac{3}{2s} \leq 2 \), the bounds for \( u \) in \( L^\infty(0, T, L^2(\mathbb{R}^3)) \) and \( \nabla p \) in \( L^2(0, T, L^{6/(3-2s)}(\mathbb{R}^3)) \) thanks to (44) and (47) imply that

\[
\int_0^T \int_{\mathbb{R}^3} u|\nabla p|^2 dx dt \leq C(T).
\]

Using again (44) we obtain an uniform bound for \( \int_0^T \int_{\mathbb{R}^3} u(-\Delta)^s p dx dt \) since

\[
\int_0^T \int_{\mathbb{R}^3} u(-\Delta)^s p dx dt \leq \int_0^T \int_{\mathbb{R}^3} u^2 + \|(-\Delta)^s p\|^2 dx dt
\]

\[
\leq C(T)\| u \|^2_{L^\infty(0, T; L^2(\mathbb{R}^3))} + \| p \|^2_{L^2(0, T; H^{2s}(\mathbb{R}^3))}
\]

\[
\leq C(T)\| u \|^2_{L^\infty(0, T; L^2(\mathbb{R}^3))} + \| p \|^2_{L^2(0, T; H^{s+1}(\mathbb{R}^3))}
\]

\[
\leq C(T, u_{in}, p_{in}).
\]

Combining all the previous estimates we conclude that

\[
\| u^{(\rho)} \|_{L^3(\mathbb{R}^3 \times (0, T))} \leq C_T, \quad u^{(\rho)} \rightharpoonup u \quad \text{in} \quad L^3(0, T, L^3(\mathbb{R}^3)).
\]

Moreover, thanks to (44) and (48) we have that

\[
\| (u^{(\rho)})^2 \|_{L^{3/2}(0, T; L^{3/2}(\mathbb{R}^3)) \cap L^\infty(0, T; L^1(\mathbb{R}^3))} \leq C;
\]
and by interpolation
\[ \|(u^{(\ell)})^2\|_{L^2(0,T;L^{4/3}(\mathbb{R}^3))} \leq C. \]
Hence there exists a subsequence \( u^{(\ell)} \) such that
\[ (u^{(\ell)})^2 \rightharpoonup v \quad \text{weakly in} \quad L^{3/2}(0,T;L^{3/2}(\mathbb{R}^3)) \cap L^2(0,T;L^{4/3}(\mathbb{R}^3)), \]
\[ (u^{(\ell)})^2 \rightharpoonup^* v \quad \text{weakly}^* \text{ in} \quad L^\infty(0,T;\mathcal{M}(\mathbb{R}^3)), \]
where \( \mathcal{M}(\mathbb{R}^3) \equiv (C_c(\mathbb{R}^3))' \) is the space of Radon measures. By the weak lower semicontinuity of the norm we also deduce
\[ \|v\|_{L^2(0,T;L^{4/3}(\mathbb{R}^3))} \leq C. \]
We now look at (43) and estimate the time derivative of \( p^{(\ell)} \) using (44):
\[
\left| \int_0^T \langle \partial_t p^{(\ell)}, \psi \rangle dt \right| \leq \rho \|\nabla p^{(\ell)}\|_{L^\infty(0,T,L^2(\mathbb{R}^3))} \|\nabla \psi\|_{L^2(0,T,L^2(\mathbb{R}^3))} \\
+ \|(u^{(\ell)})^2\|_{L^2(0,T;L^{4/3}(\mathbb{R}^3))} \|\psi\|_{L^2(0,T,L^4(\mathbb{R}^3))} \\
+ \|(-\Delta)^{s/2} p^{(\ell)}\|_{L^2(0,T,H^1(\mathbb{R}^3))} \|(-\Delta)^{s/2} \psi\|_{L^2(0,T,H^{-1}(\mathbb{R}^3))} \\
\leq C(T) \left( \|\psi\|_{L^2(0,T;H^1(\mathbb{R}^3))} + \|\psi\|_{L^2(0,T;L^4(\mathbb{R}^3))} + \|\psi\|_{L^2(0,T;H^{s-1}(\mathbb{R}^3))} \right) \\
\leq C(T) \|\psi\|_{L^2(0,T;H^1(\mathbb{R}^3))}. 
\]
Hence
\[ \|\partial_t p^{(\ell)}\|_{L^2(0,T;H^{-1}(\mathbb{R}^3))} \leq C, \]
and therefore
\[ \partial_t p^{(\ell)} \rightharpoonup \partial_t p \quad \text{weakly in} \quad L^2(0,T;H^{-1}(\mathbb{R}^3)). \]
We can now pass to the limit \( \ell \to 0 \) in (43) and obtain
\[ \langle \partial_t p, \psi \rangle + \int_0^T \int_{\mathbb{R}^3} ((-\Delta)^{s/2} p \cdot (-\Delta)^{s/2} \psi - v \psi) dx dt = 0, \]
for every \( \psi \in L^2(0,T;H^1(\mathbb{R}^3)) \).

3.4.1. The limit \( \ell \to 0 \) in (42). We first seek for strong convergence for \( \nabla p^{(\ell)} \). From (44) we have that \( \|p^{(\ell)}\|_{L^2(0,T,H^{s+1}(\mathbb{R}^3))} \leq C \); therefore in any bounded set \( \Omega \subset \mathbb{R}^3 \) we can apply Aubin-Lions lemma with the functional spaces
\[ L^2(0,T,H^{s+1}(\Omega)) \hookrightarrow L^2(0,T,H^1(\Omega)) \hookrightarrow L^2(0,T;H^{-1}(\mathbb{R}^3)) \]
and thanks to (51) we obtain that
\[ p^{(\ell)} \to p \quad \text{strongly in} \quad L^2(0,T,H^1(\Omega)), \]
and, repeating a similar argument as for (44),
\[ p^{(\ell)} \to p \quad \text{a.e. in} \quad \mathbb{R}^3. \]
The next goal is to show strong convergence for \( u^{(\ell)} \). From this convergence one will be able to identify \( v = u^2 \) and to pass to the limit in (42). Preliminarily, we point out that
\[ \|p^{(\ell)}\|_{L^8(0,T,L^8(\mathbb{R}^3))} \leq C(T). \]
This is a simple consequence of (44), interpolating \( p \in L^2(0, T, H^{1+s}(\mathbb{R}^3)) \hookrightarrow L^2(0, T, L^\infty(\mathbb{R}^3)) \) with \( p \in L^\infty(0, T, H^1(\mathbb{R}^3)) \hookrightarrow L^\infty(0, T, L^6(\mathbb{R}^3)) \).

The next step is to prove that

\[
\lim_{\theta \to 0} \int_0^T \int_{\mathbb{R}^d} ((u^{(\theta)})^2 - v)(p^{(\theta)} - p)dxdt = 0.
\]

To show this, let us choose an arbitrary \( R > 0 \), denote the ball with radius \( R \) and center 0 with \( B_R \) and its complementary with \( B_R^c \), and write

\[
\int_0^T \int_{\mathbb{R}^d} ((u^{(\theta)})^2 - v)(p^{(\theta)} - p)dxdt = \int_0^T \int_{B_R} ((u^{(\theta)})^2 - v)(p^{(\theta)} - p)dxdt
\]

\[
- \int_0^T \int_{B_R^c} v(p^{(\theta)} - p)dxdt + \int_0^T \int_{B_R^c} (u^{(\theta)})^2(p^{(\theta)} - p)dxdt.
\]

Since \( p^{(\theta)} \) is bounded in \( L^\infty(0, T, H^1(\mathbb{R}^3)) \) and \( \partial_t p^{(\theta)} \) is bounded in \( L^2(0, T, H^{-1}(\mathbb{R}^3)) \), we can invoke Aubin-Lions lemma to conclude that \( p^{(\theta)} \to p \) strongly in \( L^3(0, T, L^3(\Omega)) \), for every \( \Omega \subset \mathbb{R}^3 \) bounded.

Consequently, using (49), as \( \theta \to 0 \) we have that

\[
\int_0^T \int_{B_R} (u^{(\theta)})^2 p^{(\theta)} dxdt \to \int_0^T \int_{B_R} v p dxdt.
\]

With similar arguments we conclude

\[
\lim_{\theta \to 0} \int_0^T \int_{B_R^c} v(p^{(\theta)} - p)dxdt = 0.
\]

Let’s look now at the two integrals in the complement of the ball. Using once more the fact that \( p^{(\theta)} \in L^\infty(0, T, H^1 \cap L^1(\mathbb{R}^3)) \), we know that \( p^{(\theta)} \to p \) and \( L^q(0, T, L^q(\mathbb{R}^3)) \) for any \( 1 < q \leq 6 \) which yields

\[
\lim_{\theta \to 0} \int_0^T \int_{B_R^c} v(p^{(\theta)} - p)dxdt = 0.
\]

Finally, the last integral can be estimated as follows:

\[
\left| \int_0^T \int_{B_R^c} (u^{(\theta)})^2(p^{(\theta)} - p)dxdt \right| \leq \int_0^T \int_{B_R^c} [(1 + |x|)u^{(\theta)}]^\theta (1 + |x|)^{-\theta} (u^{(\theta)})^{2-\theta}(p^{(\theta)} + p)dxdt
\]

\[
\leq \|(1 + |x|)^\theta (u^{(\theta)})^\theta\|_{L^\infty(0,T;L^7(B_R^c))} \|p^{(\theta)} + p\|_{L^8(B_R^c \times (0,T))} \|(1 + |x|)^{-\theta}(u^{(\theta)})^{2-\theta}\|_{L^8(0,T;L^{7/(7-8\theta)}(B_R^c))},
\]

with \( 0 < \theta < 7/8 \). It holds

\[
\|(1 + |x|)^\theta (u^{(\theta)})^\theta\|_{L^\infty(0,T;L^7(B_R^c))} \leq C_T^\theta \leq \max\{1, C_T\}
\]

where \( C_T \) is as in (45). Choosing \( \theta = 5/16 \) implies that \( \frac{8(2-\theta)}{7-8\theta} = 3 \) and \( \frac{8}{7}(2-\theta) = 27/14 < 3 \), so we can write

\[
\|(1 + |x|)^{-\theta}(u^{(\theta)})^{2-\theta}\|_{L^8(0,T;L^{7/(7-8\theta)}(B_R^c))}
\]

\[
\leq (1 + R)^{-\theta}\|u^{(\theta)}\|^{2-\theta}_{L^8(0,T;L^{8/(2-\theta)}(B_R^c))}
\]

\[
\leq C(T)(1 + R)^{-\theta}\|u^{(\theta)}\|^{2-\theta}_{L^3(0,T;L^3(B_R^c))}.
\]
We wish to write the above-written integral in a more symmetrical form. It holds that is,

\[ \lim \inf_{\rho \to 0} \int_0^T \int_{\mathbb{R}^3} ((u^{(\rho)})^2 - v)(p^{(\rho)} - p) dx dt \leq C_T (1 + R)^{-\theta}. \]

As a consequence

\[ -C_T (1 + R)^{-\theta} \leq \lim \inf_{\rho \to 0} \int_0^T \int_{\mathbb{R}^3} ((u^{(\rho)})^2 - v)(p^{(\rho)} - p) dx dt \]

\[ \leq \lim \sup_{\rho \to 0} \int_0^T \int_{\mathbb{R}^3} ((u^{(\rho)})^2 - v)(p^{(\rho)} - p) dx dt \leq C_T (1 + R)^{-\theta}. \]

Since \((1 + R)^{-\theta} \to 0\) as \(R \to \infty\) while \(\int_0^T \int_{\mathbb{R}^3} ((u^{(\rho)})^2 - v)(p^{(\rho)} - p) dx dt\) does not depend on \(R\), we conclude that (55) holds.

Semigroup theory allows us to write \(p_n\) as

\[ p_n(t) = K(t) * p_{in} + \int_0^t K(t - \tau) * (u^{(\rho)})^2(\tau) d\tau, \quad t > 0, \]

with \(K(x, t)\) being the fundamental solution to the fractional heat equation

\[ g_t + (-\Delta)^{s} g = 0, \quad t > 0 \in \mathbb{R}^3. \]

Then (55) can be rewritten as

\[ \lim_{\rho \to 0} \int_0^T \int_{\mathbb{R}^3} ((u^{(\rho)})^2 - v) \int_0^t K(t - \tau) * ((u^{(\rho)})^2 - v) d\tau dx dt = 0, \]

that is,

\[ \lim_{\rho \to 0} \int_0^T \int_0^t \int_{\mathbb{R}^6} K(x - y, t - \tau)((u^{(\rho)})^2 - v)(x, t)((u^{(\rho)})^2 - v)(y, \tau) dxdyd\tau dt = 0. \]

We wish to write the above-written integral in a more symmetric form. It holds

\[ \int_0^T \int_0^t \int_{\mathbb{R}^6} K(x - y, t - \tau)((u^{(\rho)})^2 - v)(x, t)((u^{(\rho)})^2 - v)(y, \tau) dxdyd\tau dt \]

\[ = \int_0^T \int_0^T \int_{\mathbb{R}^6} K(x - y, t - \tau)((u^{(\rho)})^2 - v)(x, t)((u^{(\rho)})^2 - v)(y, \tau) dxdydtd\tau \]

\[ = \int_0^T \int_0^T \int_{\mathbb{R}^6} K(x - y, \tau - t)((u^{(\rho)})^2 - v)(y, \tau)((u^{(\rho)})^2 - v)(x, t) dxdyd\tau dt \]

\[ = \frac{1}{2} \int_0^T \int_0^T \int_{\mathbb{R}^6} K(x - y, |t - \tau|)((u^{(\rho)})^2 - v)(x, t)((u^{(\rho)})^2 - v)(y, \tau) dxdyd\tau dt. \]

Therefore, if we define the bilinear form

\[ \mathcal{A}[w_1, w_2] = \frac{1}{2} \int_0^T \int_0^T \int_{\mathbb{R}^6} K(x - y, |t - \tau|)w_1(x, t)w_2(y, \tau) dx dy d\tau dt \]

for all \(w_1, w_2 \in X \equiv L^\infty(0, T; L^1(\mathbb{R}^3)) \cap L^{3/2}(\mathbb{R}^3 \times (0, T))\) we proved that

\[ \lim_{\rho \to 0} \mathcal{A}[(u^{(\rho)})^2 - v, (u^{(\rho)})^2 - v] = 0. \]

Note that thanks to (49) the form \(\mathcal{A}\) is well-defined in \(X\).
We wish to prove that $A$ is positive, namely,
$$\forall w \in X, \ w \neq 0 \implies A(w, w) > 0.$$ 
Define $\hat{w}(k, t) := \int_{\mathbb{R}^3} w(x, t)e^{-2\pi i k \cdot x} dx$ be the transform of $w$. By Plancherel’s theorem we have
$$A[w, w] = \frac{1}{2} \int_0^T \int_0^T \int_{\mathbb{R}^d} e^{-|k|^2 t} |\hat{w}(k, t)|^2 dkdt$$
since $\hat{K}(k, t) = e^{-|k|^2 t}$. Let us now prolong $\hat{w}(k, t)$ to an even, $2T$—periodic function of $t$ defined in $\mathbb{R}^d \times \mathbb{R}$. It follows
$$A[w, w] = \frac{1}{4} \int_{-T}^T \int_{-T}^T \int_{\mathbb{R}^3} e^{-|k|^2 |t-\tau|} \hat{w}(k, t) \hat{w}(k, \tau) dkdt d\tau.$$ 
Now we find the Fourier series of $\hat{w}(k, t)$ with respect to the time variable:
$$\hat{w}(k, t) = \sum_{m=0}^{+\infty} \omega_m(k) e^{im\pi t/T}, \quad \omega_m(k) := \frac{1}{2T} \int_{-T}^T e^{-im\pi s/T} \hat{w}(k, s) ds.$$ 
Define $f_m(k) := \int_{-T}^T e^{-im\pi s/T} e^{-|k|^2 s} dy$. We obtain
$$A[w, w] = \frac{1}{4} \int_{-T}^T \int_{-T}^T \int_{\mathbb{R}^d} e^{-|k|^2 |t-\tau|} \hat{w}(k, t) \hat{w}(k, \tau) dkdt d\tau$$
$$= \frac{1}{4} \int_{\mathbb{R}^3} \sum_{n=0}^{+\infty} \omega_n(k) e^{in\pi \tau/T} \left( \sum_{m=0}^{+\infty} \omega_m(k) \int_{-T}^T e^{-im\pi \tau/T} e^{-|k|^2 |t-\tau|} dt \right) d\tau dk$$
$$= \frac{1}{4} \int_{\mathbb{R}^3} \sum_{n=0}^{+\infty} \omega_n(k) e^{in\pi \tau/T} \left( \sum_{m=0}^{+\infty} \omega_m(k) e^{-im\pi \tau/T} \int_{-T}^T e^{-i\pi s/T} e^{-|k|^2 |s|/T} ds \right) d\tau dk$$
$$= \frac{1}{4} \int_{\mathbb{R}^3} \sum_{n=0}^{+\infty} \omega_n(k) \left( \sum_{m=0}^{+\infty} \omega_m(k) f_m(k) \int_{-T}^T e^{i\pi s/T} e^{-i\pi \tau/T} d\tau \right) dk$$
$$= \frac{2T}{4} \sum_{m=0}^{+\infty} \int_{\mathbb{R}^3} f_m(k) |\omega_m(k)|^2 dk,$$
by orthogonality. Let us compute $f_m(k)$:
$$f_m(k) = \int_0^T e^{-im\pi t/T} e^{-|k|^2 t} dt + \int_{-T}^0 e^{-im\pi t/T} e^{-|k|^2 t} dt$$
$$= \frac{e^{-im\pi (T-|k|^2)T} - 1}{-im\pi T + |k|^2} + 1 - e^{-(-im\pi T+|k|^2)T}$$
$$= \frac{1}{i\pi m/T + |k|^2} + \frac{1}{-i\pi m/T + |k|^2} - e^{-|k|^2 T} \left( \frac{e^{-im\pi m/T + |k|^2} + e^{i\pi m}}{i\pi m/T + |k|^2} \right)$$
$$= (1 + (-1)^{m-1} e^{-|k|^2 T}) \left( \frac{1}{i\pi m/T + |k|^2} + \frac{1}{-i\pi m/T + |k|^2} \right)\]
\[ = \frac{2|k|^s(1 + (-1)^{m-1}e^{-|k|^s T})}{|k|^{2s} + \pi^2m^2/T^2}. \]

As a consequence,
\[ f_m(k) \geq \frac{2|k|^s(1 - e^{-|k|^s T})}{|k|^{2s} + \pi^2m^2/T^2} > 0 \quad k \neq 0, \quad m \geq 0. \]

We deduce that \( A[w, w] \geq 0 \) and \( A[w, w] = 0 \) if and only if \( \omega_m \equiv 0 \) in \( \mathbb{R}^3 \), \( m \geq 0 \) (being the Fourier transform of an \( L^1 \) function, \( \omega_m \) is continuous), which means that \( w \equiv 0 \) a.e. in \( \mathbb{R}^3 \times (0, T) \). This concludes the proof of Lemma \( \mathbb{I} \).

Consequently \( A[w_1, w_2] \) defines a scalar product on (a subset of) \( X \) and its corresponding norm is \( \|w\|_A \equiv \sqrt{A[w, w]} \). We have already proved that \( \lim_{n \to \infty} \|(u^{(\ell)})^2 - v\|_A = 0 \). In particular, \( (u^{(\ell)})^2 \) is a Cauchy sequence w.r.t. \( \| \cdot \|_A \).

We next show that it is also a Cauchy sequence w.r.t. the sup norm. Since \( u^{(\ell)} \) is a Cauchy sequence w.r.t. \( \| \cdot \|_A \), we can find a strictly increasing sequence \( (n_k)_{k \in \mathbb{N}} \subset \mathbb{N} \) of indexes such that
\[
\|(u^{(\ell)})^2 - (u^{(n_{k+1})})^2\|_A < 2^{-k}, \quad n, m \geq n_k, \quad k \geq 1.
\]
In particular, being \( n_{k+1} > n_k \) for \( k \geq 1 \),
\[
\|(u^{(\ell)})^2 - (u^{(n_{k+1})})^2\|_A < 2^{-k}, \quad k \geq 1,
\]
and
\[
\sum_{k=1}^{\infty} \|(u^{(n_{k})})^2 - (u^{(n_{k+1})})^2\|_A < 1.
\]

Let us define
\[
w_\ell(x, t) \equiv \sum_{k=1}^{\ell} [(u^{(n_k)})^2(x, t) - (u^{(n_{k+1})})^2(x, t)], \quad \ell \geq 1,
\]
\[
w(x, t) \equiv \sum_{k=1}^{\infty} [(u^{(n_k)})^2(x, t) - (u^{(n_{k+1})})^2(x, t)].
\]
It follows
\[
\|w_\ell\|_A \leq \sum_{k=1}^{\ell} \|(u^{(n_k)})^2 - (u^{(n_{k+1})})^2\|_A \leq \sum_{k=1}^{\infty} \|(u^{(n_k)})^2 - (u^{(n_{k+1})})^2\|_A < 1.
\]
However,
\[
\|w_\ell\|_A^2 = A[w_\ell, w_\ell] = \frac{1}{2} \int_0^T \int_0^T \int_{\mathbb{R}^{2d}} K(x - y, |t - s|)w_\ell(x, t)w_\ell(y, s) \, dx \, dy \, ds \, dt.
\]
Since \( w_\ell \) is an increasing sequence of nonnegative measurable functions, by Beppo Levi monotone convergence theorem it follows that
\[
\|w\|_A = \lim_{n \to \infty} \|w_\ell\|_A \leq 1.
\]
In particular,
\[
\int_0^T \int_0^T \int_{\mathbb{R}^{2d}} K(x - y, |t - s|)w(x, t)w(y, s) \, dx \, dy \, ds \, dt < \infty.
\]
This means that \( w < \infty \) a.e. \( \mathbb{R}^3 \times (0, T) \). In fact, if \( w = \infty \) on \( \Omega \subset \mathbb{R}^3 \times (0, T) \) with \( |\Omega| > 0 \), then
\[
\int_0^T \int_0^T \int_{\mathbb{R}^6} K(x - y, |t - s|)w(x, t)w(y, s) \, dx \, dy \, ds \, dt \\
\geq \int_{\Omega} \int_{\Omega} K(x - y, |t - s|)w(x, t)w(y, s) \, dx \, dt \, dy \, ds = \infty.
\]

We have thus proved that
\[
\sum_{k=1}^{\infty} |(u(^{\varepsilon}n_k))^2(x, t) - (u(^{\varepsilon}n_{k+1}))^2(x, t)| = w(x, t) < \infty \quad \text{a.e. } x \in \mathbb{R}^3, \ 0 < t < T.
\]
Therefore, for a.e. \( x \in \mathbb{R}^3, \ 0 < t < T, \) and for every \( \varepsilon > 0 \), there exists \( \ell_\varepsilon(x, t) \in \mathbb{N} \) such that
\[
\sum_{k=\ell}^{\infty} |(u(^{\varepsilon}n_k))^2(x, t) - (u(^{\varepsilon}n_{k+1}))^2(x, t)| < \varepsilon \quad \forall \ell \geq \ell_\varepsilon(x, t).
\]
In particular,
\[
|u(^{\varepsilon}n_r))^2(x, t) - (u(^{\varepsilon}n_k))^2(x, t)| \leq \sum_{k=\ell}^{r} |u(^{\varepsilon}n_k))^2(x, t) - (u(^{\varepsilon}n_{k+1}))^2(x, t)| < \varepsilon \quad r > \ell \geq \ell_\varepsilon(x, t).
\]
This means that, for a.e. \( x \in \mathbb{R}^3, \ 0 < t < T, \) the sequence \((u(^{\varepsilon}))^2\) is a Cauchy sequence, and therefore converging. We conclude that
\[
(\epsilon)^2 \rightarrow u^2, \quad \text{a.e. in } \mathbb{R}^3 \rightarrow (0, T).
\]
Since \( u(\epsilon) \in L^\infty(0, T, L^1(\mathbb{R}^3)) \) by (44) and (45), and \( u(\epsilon) \in L^3(0, T, L^3(\mathbb{R}^3)) \) by (48), Vitali's convergence theorem yields
\[
\begin{align*}
\epsilon & \rightarrow u \quad \text{strongly in } L^\infty(0, T; L^r(\Omega)), \quad q \geq 1, \ 1 \leq r < 2, \\
\epsilon & \rightarrow u \quad \text{strongly in } L^{r+1}(\Omega \times (0, T)).
\end{align*}
\]
We can identify the limit \( v \) of \((u(\epsilon))^2\) with \( v = u^2\).

We now look for estimates for \( \partial_t \epsilon \). Inequality (44) yields
\[
\left| \int_0^T \int_{\mathbb{R}^3} \partial_t (\epsilon)^2 \phi \, dx \right| \leq \| (\epsilon)^2 \|_{L^\infty(0, T; L^2(\mathbb{R}^3))} \| \nabla \phi \|_{L^2(0, T, L^{3/s}(\mathbb{R}^3))} \| \nabla (\epsilon)^2 \|_{L^2(0, T, L^{6/(3-2s)})},
\]
which implies
\[
\left| \int_0^T \partial_t (\epsilon) \phi \, dt \right| \leq C(T) \| \phi \|_{L^2(0, T, W^{1,3/s}(\mathbb{R}^3))},
\]
and, as \( \epsilon \rightarrow 0, \)
\[
\partial_t (\epsilon) \rightarrow \partial_t u \quad \text{weakly in } L^2(0, T, (W^{1,3/s} \cap H^1(\mathbb{R}^3))').
\]
Let now \( \phi \in L^2(0, T, W^{1,q} \cap H^1(\mathbb{R}^3)) \) for \( q > 3/s \). We have
\[
\int_0^T \int_{\mathbb{R}^3} (\epsilon) \nabla (\epsilon) \cdot \nabla \phi \, dx \, dt = \int_0^T \int_{\mathbb{R}^3} (\epsilon - u) \nabla (\epsilon) \cdot \nabla \phi \, dx \, dt.
\]
we have that \( \lim \)
\[ I_1 + I_2. \]

Since for any \( r < 2 \) there exists \( q > 3 \) such that
\[ |I_1| \leq \|u^{(r)} - u\|_{L^\infty(0,T,T,L^r(\mathbb{R}^3))} \|\nabla \phi\|_{L^2(0,T,L^p(L^2(\mathbb{R}^3)))} \|\nabla \rho\|_{L^q(0,T,L^6/((3-2s)\mathbb{R}^3))}, \]
we conclude that \( |I_1| \to 0 \) as \( \tau \to 0 \) thanks to (58). Moreover also \( |I_2| \to 0 \) as \( q \to 0 \) since \( u\nabla \phi \in L^2(0,T,L^{4/3}(\mathbb{R}^3)) \) and \( \nabla p^{(q)} \to p \) in \( L^2(0,T,L^2 \cap L^6(\mathbb{R}^3)) \).

Furthermore
\[
\rho \int_0^T \int_{\mathbb{R}^3} \nabla u^{(q)} \cdot \nabla \phi \, dx \, dt \to 0,
\]
since \( \rho \|\nabla u^{(q)}\|_{L^2(0,T,T,L^2(\mathbb{R}^3))} \leq C. \) Summarizing up, when \( q \to 0 \) in (12), (13) we get
\[
\langle \partial_t u, \phi \rangle + \int_0^T \int_{\mathbb{R}^3} u\nabla p \cdot \nabla \phi \, dx \, dt = 0 \quad \forall \phi \in L^2(0,T; W^{1,q} \cap H^1(\mathbb{R}^3)), \quad q > 3/s,
\]
\[
\langle \partial_t p, \phi \rangle + \int_0^T \int_{\mathbb{R}^3} ((-\Delta)^{s/2} p(-\Delta)^{s/2} \psi - v\psi) \, dx \, dt = 0 \quad \forall \psi \in L^2(0,T; H^1(\mathbb{R}^3)),
\]
with the estimates
\[
\int_{\mathbb{R}^3} \left( u^2 + \frac{1}{2} |\nabla p|^2 \right) \, dx + \int_0^t \int_{\mathbb{R}^3} |(-\Delta)^{s/2} \nabla p|^2 \, dx \, dt' \leq \int_{\mathbb{R}^3} \left( (u_{in})^2 + \frac{1}{2} |\nabla p_{in}|^2 \right) \, dx.
\]

Taking the \( \liminf_{q \to 0} \) in both sides of (17), Fatou’s Lemma and (53) lead to
\[
\int_{\mathbb{R}^3} p(t) \, dx \leq \int_{\mathbb{R}^3} p_{in} \, dx + Ct, \quad t \in [0,T],
\]
while the uniform bounds for \( p \) and the weak semicontinuity of the \( L^p \) norm implies
\[
\|p\|_{L^2(0,T,H^{s+1}(\mathbb{R}^3))} + \|p\|_{L^\infty(0,T,H^1(\mathbb{R}^3))} \leq C.
\]

Finally, bounds (50), (59) and a density argument allow us to state that (60) holds for every \( \phi \in L^2(0,T; W^{1,q}(\mathbb{R}^3)) \), \( q > 3/s \), while (61) holds for every \( \psi \in L^2(0,T; L^2(\mathbb{R}^3)) \).

Mimicking the same argument as in (46) one can show that
\[
\int_{\mathbb{R}^3} u(t) \, dx = \int_{\mathbb{R}^3} u_{in} \, dx.
\]

Finally, since
\[
u \in H^1(0,T,(W^{1,q}(\mathbb{R}^3)))', \quad p \in H^1(0,T,(L^4 \cap L^2(\mathbb{R}^3)))'
\]
we have that \( \lim_{t \to 0} u(t) = u_{in} \) in \( W^{1,q}(\mathbb{R}^3) \)', \( p(t) = p_{in} \) in \( (L^4 \cap L^2(\mathbb{R}^3))' \).

This concludes the proof of Theorem 1.

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