On the Numerical Solution of Non-linear First Order Ordinary Differential Equation Systems
Applications to a Flight Mechanics Problem

Fabio Silva Botelho

Abstract In this article, firstly we develop a method of solution for a type of difference equations, applicable to solve approximately a class of first order ordinary differential equation systems.

In a second step, we apply the results obtained to solve a non-linear two point boundary value problem relating a flight mechanics model. We highlight the algorithm obtained seems to be robust and of easy computational implementation.

Keywords Ordinary differential equations · Two point boundary value problem · Flight mechanics

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1 Introduction

In this article we develop an algorithm to solve a class of first order non-linear ordinary differential equations.

We start by presenting a general procedure for solving the linearized equations, and in a second step, we apply it to solve a problem in flight mechanics in a Newton’s method context. In fact, a sequence of linear problems is solved intending to obtain a solution for the original non-linear problem. We emphasize the method here proposed has a performance considerably better than those so far known, particularly concerning the computation time for a standard commercial PC, which for the present one corresponds to a few seconds, even for a non-linear problem with about 40,000 degrees of freedom.
At this point we present a remark on the references.

Remark 1 We highlight that a similar problem is addressed in \cite{2} for a nuclear physics model. The main difference is that now our results are more general and applicable to a much larger class of problems. Specifically in the present work, we apply them to a flight mechanics model found in \cite{4}.

For the numerical results we have used finite differences. Details about finite differences schemes may be found in \cite{3}.

Finally, details on the Sobolev spaces in which the original problem is established may be found in \cite{1}.

2 The main results

Consider the following system of difference equations in \{(u_k)_n\}, given by

\[
(u_k)_{n+1} = \sum_{j=1}^{4} (a_{kj})_n (u_j)_n + (g_k)_n, \quad \forall k \in \{1, 2, 3, 4\}, n \in \{0, \ldots, N - 1\},
\]  

(1)

where

\[
(u_k)_n, (a_{kj})_n, (g_k)_n \in \mathbb{R}, \quad \forall j, k \in \{1, 2, 3, 4\}, n \in \{0, \ldots, N - 1\}.
\]

Assume the following boundary conditions are intended to be satisfied:

\[
\begin{aligned}
(u_1)_0 &= h_0, \\
(u_3)_0 &= V_0 \\
(u_4)_0 &= x_0 \\
(u_1)_N &= h_f.
\end{aligned}
\]  

(2)

For \(n = 0\) and \(k = 1\) we obtain

\[
(u_1)_1 = \sum_{j=1}^{4} (a_{1j})_0 (u_j)_0 + (g_1)_0,
\]

that is

\[
(u_2)_0 = \frac{(u_1)_1 - (g_1)_0 - (a_{11})_0 (u_1)_0 - (a_{13})_0 (u_3)_0 - (a_{14})_0 (u_4)_0}{(a_{12})_0},
\]

so that we write

\[
(u_2)_0 = m_2[0](u_1)_1 + \tilde{z}_2[0],
\]  

(3)

where,

\[
m_2[0] = \frac{1}{(a_{12})_0},
\]

and

\[
\tilde{z}_2[0] = \frac{-(g_1)_0 - (a_{11})_0 (u_1)_0 - (a_{13})_0 (u_3)_0 - (a_{14})_0 (u_4)_0}{(a_{12})_0}.
\]
Replacing (3) into
\[(u_2)_1 = \sum_{j=1}^{4} (a_{2j})_0 (u_j)_0 + (g_2)_0,\]
we get
\[(u_2)_1 = m_2[1](u_1)_1 + z_2[1],\]
where
\[m_2[1] = (a_{22})_0 m_2[0],\]
and,
\[z_2[1] = (a_{21})_0 (u_1)_0 + (a_{22})_0 \tilde{z}_2[0] + (a_{23})_0 (u_3)_0 + (a_{24})_0 (u_4)_0 + (g_2)_0.\]

Also, replacing (3) into
\[(u_3)_1 = \sum_{j=1}^{4} (a_{3j})_0 (u_j)_0 + (g_3)_0,\]
we obtain
\[(u_3)_1 = m_3[1](u_1)_1 + z_3[1],\]
where
\[m_3[1] = (a_{32})_0 m_2[0],\]
and,
\[z_3[1] = (a_{31})_0 (u_1)_0 + (a_{32})_0 \tilde{z}_2[0] + (a_{33})_0 (u_3)_0 + (a_{34})_0 (u_4)_0 + (g_3)_0.\]

Finally, replacing (3) into
\[(u_4)_1 = \sum_{j=1}^{4} (a_{4j})_0 (u_j)_0 + (g_4)_0,\]
we may obtain
\[(u_4)_1 = m_4[1](u_1)_1 + z_4[1],\]
where
\[m_4[1] = (a_{42})_0 m_2[0],\]
and,
\[z_4[1] = (a_{41})_0 (u_1)_0 + (a_{42})_0 \tilde{z}_2[0] + (a_{43})_0 (u_3)_0 + (a_{44})_0 (u_4)_0 + (g_4)_0.\]

Reasoning inductively, having for \(k \in \{2, 3, 4\},\)
\[(u_k)_{n-1} = m_k[n-1](u_1)_{n-1} + z_k[n-1], \quad (4)\]
for \(n \geq 2,\) replacing these last equations into (1) for \(k = 1,\) we may obtain
\[(u_1)_{n-1} = \tilde{m}_1[n-1](u_1)_n + \tilde{z}_1[n], \quad (5)\]
where
\[ \tilde{m}_1[n-1] = \{(a_{11})_{n-1} + (a_{12})_{n-1}m_2[n-1] + (a_{13})_{n-1}m_3[n-1] + (a_{14})_{n-1}m_4[n-1]\}^{-1}, \] (6)

and
\[ \tilde{z}_1[n-1] = -\tilde{m}_1[n-1]\{(a_{12})_{n-1}z_2[n-1] + (a_{13})_{n-1}z_3[n-1] + (a_{14})_{n-1}z_4[n-1]\}. \] (7)

Finally, replacing (5) into (4), we may obtain
\[ (u_k)_{n-1} = \tilde{m}_k[n-1](u_1)_n + \tilde{z}_k[n-1], \] (8)

\[ \forall k \in \{2, 3, 4\}, \]

where
\[ \tilde{m}_k[n-1] = m_k[n-1]\tilde{m}_1[n-1], \]
\[ \tilde{z}_k[n-1] = m_k[n-1]\tilde{z}_1[n-1] + z_k[n-1]. \]

Replacing (5) and (8) into the system (1), we get
\[ (u_k)_n = m_k[n](u_1)_n + z_k[n], \]

where,
\[ m_k[n] = (a_{k1})_n\tilde{m}_1[n-1] + (a_{k2})_n\tilde{m}_2[n-1] + (a_{k3})_n\tilde{m}_3[n-1] + (a_{k4})_n\tilde{m}_4[n-1], \]

and,
\[ z_k[n] = \sum_{j=1}^{4}(a_{kj})_n\tilde{z}_j[n-1] + (g_k)_n. \]

Summarizing, we have obtained,
\[ (u_1)_{n-1} = \tilde{m}_1[n-1](u_1)_n + \tilde{z}_1[n-1], \]
\[ (u_k)_n = m_k[n](u_1)_n + z_k[n], \quad \forall k \in \{2, 3, 4\} \]

\[ \forall n \in \{1, ..., N\}. \]

Therefore, having \((u_1)_N = h_f\), we may obtain
\[ (u_k)_N = m_k[N](u_1)_N + z_k[N], \quad \forall k \in \{2, 3, 4\} \]

and
\[ (u_1)_{N-1} = \tilde{m}_{N-1}(u_1)_N + \tilde{z}_1[N-1]. \]

Having, \((u_1)_{N-1}\), we may obtain
\[ (u_k)_{N-1} = m_k[N-1](u_1)_{N-1} + z_k[N-1], \quad \forall k \in \{2, 3, 4\} \]

and
\[ (u_1)_{N-2} = \tilde{m}_{N-2}[(u_1)_{N-1} + \tilde{z}_1[N-2]], \]

and so on, up to finding \((u_k)_1, \forall k \in \{1, 2, 3, 4\}, \)

and finally,
\[ (u_2)_0 = m_2[0](u_1)_1 + \tilde{z}_2[0]. \]

At this point, the problem is solved.
3 Numerical Results

We present numerical results for the following system of equations, which models the in plane climbing motion of an airplane (please, see more details in [4]).

\[
\begin{align*}
\dot{h} &= V \sin \gamma, \\
\dot{\gamma} &= \frac{1}{m_f} (T \sin(e_3) + L) - g \cos \gamma, \\
\dot{V} &= \frac{1}{m_f} (T \cos(e_3) - D) - g \sin \gamma, \\
\dot{x} &= V \cos \gamma,
\end{align*}
\]  

(9)

with the boundary conditions,

\[
\begin{align*}
h(0) &= h_0, \\
V(0) &= V_0 \\
x(0) &= x_0 \\
h(t_f) &= h_f,
\end{align*}
\]  

(10)

where \( t_f = 40 \) s, \( h \) is the airplane altitude, \( V \) is its speed, \( \gamma \) is the angle between its velocity and the horizontal axis, and finally \( x \) denotes the horizontal coordinate position.

For numerical purposes, we assume:

\[
m_f = 120,000(2.2) \text{ lb}, S_f = 260(3.2)^2 \text{ ft}^2, a = 12^\circ, e_3 = 0.19, R_e = 2.09 \times 10^7 \text{ ft}, g = 32 \text{ ft}^2/\text{s}^2, \rho_0 = 0.00239 \text{ slug/ft}^3, B_0 = 26,600 \text{ ft}, \rho = \rho_0 e^{-h/B_0},
\]

\[
C_L = -0.0005225a^2 + 0.003506a + 0.1577, \\
C_D = 0.0001432a^2 + 0.0058a + 0.2204,
\]

\[
L = \frac{1}{2} \rho V^2 C_L S_f, \\
D = \frac{1}{2} \rho V^2 C_D S_f,
\]

and where units refer to the British system and,

\[
T = D + m_f g \sin \gamma,
\]

which refers to a slightly negatively accelerated motion.

To simplify the analysis, we redefine the variables as below indicated:

\[
\begin{align*}
h &= u_1, \\
\gamma &= u_2 \\
V &= u_3 \\
x &= u_4.
\end{align*}
\]  

(11)

Thus, denoting \( u = (u_1, u_2, u_3, u_4) \in U = W^{1,2}([0, t_f]; \mathbb{R}^4) \), the system above indicated may be expressed by
Finally, we obtain the following approximate system in finite differences, that is,

\[
\begin{aligned}
(u_1)_{n+1} &= (u_1)_n + d f_1(\bar{u}_n) + \frac{d}{d \bar{u}_n} \left( \sum_{j=1}^{d} \frac{\partial f_1(\bar{u}_n)}{\partial u_j} (u_j)_n - (\bar{u}_j)_n \right) \\
(u_2)_{n+1} &= (u_2)_n + d f_2(\bar{u}_n) + \frac{d}{d \bar{u}_n} \left( \sum_{j=1}^{d} \frac{\partial f_2(\bar{u}_n)}{\partial u_j} (u_j)_n - (\bar{u}_j)_n \right) \\
(u_3)_{n+1} &= (u_3)_n + d f_3(\bar{u}_n) + \frac{d}{d \bar{u}_n} \left( \sum_{j=1}^{d} \frac{\partial f_3(\bar{u}_n)}{\partial u_j} (u_j)_n - (\bar{u}_j)_n \right) \\
(u_4)_{n+1} &= (u_4)_n + d f_4(\bar{u}_n) + \frac{d}{d \bar{u}_n} \left( \sum_{j=1}^{d} \frac{\partial f_4(\bar{u}_n)}{\partial u_j} (u_j)_n - (\bar{u}_j)_n \right) 
\end{aligned}
\]  

\hspace{1cm} (15)

where, 

\[
\begin{align*}
 f_1(u) &= u_3 \sin(u_2), \\
 f_2(u) &= \frac{1}{m_f} (T(u) \sin(e_3) + L(u)) - \frac{d}{d u_3} \cos(u_2), \\
 f_3(u) &= \frac{1}{m_f} (T(u) \cos(e_3) - D(u)) - g \sin(u_2), \\
 f_4(u) &= u_3 \cos(u_2). 
\end{align*}
\]  

\hspace{1cm} (13)

At this point we shall write the system indicated in (12) in finite differences, that is,

\[
\begin{aligned}
(u_1)_{n+1} &= (u_1)_n + f_1(\bar{u}_n)d \\
(u_2)_{n+1} &= (u_2)_n + f_2(\bar{u}_n)d \\
(u_3)_{n+1} &= (u_3)_n + f_3(\bar{u}_n)d \\
(u_4)_{n+1} &= (u_4)_n + f_4(\bar{u}_n)d, 
\end{aligned}
\]  

\hspace{1cm} (14)

here \(d = 40/N\), we \(N\) refers to the number of nodes concerning the discretization in \(t\) (in our numerical example \(N = 10000\)).

Intending to apply the Newton’s method we linearize the system indicated in (14) about an initial guess 
\[
\bar{u} = (\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4).
\]

We obtain the following approximate system 

\[
\begin{aligned}
(u_1)_{n+1} &= (u_1)_n + \sum_{j=1}^{d} \frac{\partial f_1(\bar{u}_n)}{\partial u_j} (u_j)_n - (\bar{u}_j)_n \\
(u_2)_{n+1} &= (u_2)_n + \sum_{j=1}^{d} \frac{\partial f_2(\bar{u}_n)}{\partial u_j} (u_j)_n - (\bar{u}_j)_n \\
(u_3)_{n+1} &= (u_3)_n + \sum_{j=1}^{d} \frac{\partial f_3(\bar{u}_n)}{\partial u_j} (u_j)_n - (\bar{u}_j)_n \\
(u_4)_{n+1} &= (u_4)_n + \sum_{j=1}^{d} \frac{\partial f_4(\bar{u}_n)}{\partial u_j} (u_j)_n - (\bar{u}_j)_n. 
\end{aligned}
\]  

\hspace{1cm} (15)

Observe that such a system is in the form, 

\[
(u_k)_{n+1} = (u_k)_n + \sum_{j=1}^{d} (a_{kj})_n (u_j)_n + (g_k)_n,
\]
On the Numerical Solution of ODEs

Fig. 1 The solution $h$ (in ft) for $t_f = 40s$.

where

$$(a_{kj})_n = \frac{\partial f_k(\tilde{u}_n)}{\partial u_j} d, \text{ for } j \neq k$$

and,

$$(a_{jj})_n = 1 + \frac{\partial f_j(\tilde{u}_n)}{\partial u_j} d, \text{ for } j = k;$$

and

$$(g_k)_n = f_k(\tilde{u}_n) d - \sum_{j=1}^{4} \frac{\partial f_k(\tilde{u}_n)}{\partial u_j} (\tilde{u}_j)_n d.$$ We solve this last system for the following boundary conditions:

$$\left\{
\begin{array}{l}
h(0) = 0 \text{ ft}, \\
V(0) = 960 \text{ ft/s}, \\
x(0) = 0 \text{ ft}, \\
h(t_f) = 35000 \text{ ft}.
\end{array}
\right.$$ (16)

We have obtained \{u_n\}. In a Newton’s method context, the next step is to replace \(\tilde{u}_n\) by \{u_n\} and thus to repeat the process up to the satisfaction of an appropriate convergence criterion.

We have obtained the following solutions for $h, \gamma, V$ and $x$. Please see figures 1, 2, 3 and 4 respectively.

4 Conclusion

In this article, we have developed a method for solving a class of first order ordinary differential equations.

The results are applied to a flight mechanics problem which models the in-plane climbing of an airplane. It is worth mentioning the algorithm obtained is of easy implementation and very efficient from a computational point of view.
Finally, we would highlight the numerical results obtained are perfectly consistent with the physical problem context. In future works we intend to apply the method to solve relating optimal control problems.

References

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Fig. 4 The solution $x$ (in ft) for $t_f = 40s$. 