COMPLETENESS IN QUASI-PSEUDOMETRIC SPACES

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Abstract. The aim of this paper is to discuss the relations between various notions of sequential completeness and the corresponding notions of completeness by nets or by filters in the setting of quasi-metric spaces. We propose a new definition of right $K$-Cauchy net in a quasi-metric space for which the corresponding completeness is equivalent to the sequential completeness. In this way we complete some results of R. A. Stoltenberg, Proc. London Math. Soc. 17 (1967), 226–240, and V. Gregori and J. Ferrer, Proc. Lond. Math. Soc., III Ser., 49 (1984), 36.

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1. Introduction

For a mapping $d : X \times X \to \mathbb{R}$ on a set $X$ consider the following conditions:

- $(M1)$ $d(x, y) \geq 0$ and $d(x, x) = 0$;
- $(M2)$ $d(x, y) = d(y, x)$;
- $(M3)$ $d(x, z) \leq d(x, y) + d(y, z)$;
- $(M4)$ $d(x, y) = 0 \Rightarrow x = y$;
- $(M4')$ $d(x, y) = d(y, x) = 0 \Rightarrow x = y$,

for all $x, y, z \in X$.

The mapping $d$ is called a pseudometric if it satisfies $(M1)$, $(M2)$ and $(M3)$ and a metric if it further satisfies $(M4)$.

The open and closed balls in a pseudometric space $(X, d)$ are defined by

$$B_d(x, r) = \{ y \in X : d(x, y) < r \} \quad \text{and} \quad B_d[x, r] = \{ y \in X : d(x, y) \leq r \},$$

respectively.

A filter on a set $X$ is a nonempty family $\mathcal{F}$ of nonempty subsets of $X$ satisfying the conditions

- $(F1)$ $F \subseteq G$ and $F \in \mathcal{F}$ $\Rightarrow$ $G \in \mathcal{F}$;
- $(F2)$ $F \cap G \in \mathcal{F}$ for all $F, G \in \mathcal{F}$.

It is obvious that $(F2)$ implies

- $(F2')$ $F_1, \ldots, F_n \in \mathcal{F}$ $\Rightarrow$ $F_1 \cap \ldots \cap F_n \in \mathcal{F}$.

for all $n \in \mathbb{N}$ and $F_1, \ldots, F_n \in \mathcal{F}$.

A base of a filter $\mathcal{F}$ is a subset $\mathcal{B}$ of $\mathcal{F}$ such that every $F \in \mathcal{F}$ contains a $B \in \mathcal{B}$.

A nonempty family $\mathcal{B}$ of nonempty subsets of $X$ such that

- $(BF1)$ $\forall B_1, B_2 \in \mathcal{B}, \exists B \in \mathcal{B}, B \subset B_1 \cap B_2$.
generates a filter $\mathcal{F}(\mathcal{B})$ given by

$$\mathcal{F}(\mathcal{B}) = \{ U \subseteq X : \exists B \in \mathcal{B}, \ B \subseteq U \}.$$  

A family $\mathcal{B}$ satisfying (BF1) is called a filter base.

A uniformity on a set $X$ is a filter $\mathcal{U}$ on $X \times X$ such that

1. $\Delta(X) \subseteq U$, $\forall U \in \mathcal{U}$;
2. $\forall U \in \mathcal{U}$, $\exists V \in \mathcal{U}$, such that $V \circ V \subseteq U$,
3. $\forall U \in \mathcal{U}$, $U^{-1} \in \mathcal{U}$.

where

$$\Delta(X) = \{(x,x) : x \in X\} \text{ denotes the diagonal of } X,$$

$$M \circ N = \{(x,z) \in X \times X : \exists y \in X, (x,y) \in M \text{ and } (y,z) \in N\}, \text{ and }$$

$$M^{-1} = \{(y,x) : (x,y) \in M\},$$

for any $M, N \subseteq X \times X$.

The sets in $\mathcal{U}$ are called entourages. A base for a uniformity $\mathcal{U}$ is a base of the filter $\mathcal{U}$.

The composition $V \circ V$ is denoted sometimes simply by $V^2$. Since every entourage contains the diagonal $\Delta(X)$, the inclusion $V^2 \subseteq U$ implies $V \subseteq U$.

For $U \in \mathcal{U}$, $x \in X$ and $Z \subseteq X$ put

$$U(x) = \{ y \in X : (x,y) \in U \} \text{ and } U[Z] = \bigcup \{ U(z) : z \in Z \}.$$  

A uniformity $\mathcal{U}$ generates a topology $\tau(\mathcal{U})$ on $X$ for which the family of sets

$$\{ U(x) : U \in \mathcal{U} \}$$

is a base of neighborhoods of any point $x \in X$.

A base for a uniformity $\mathcal{U}$ is any base of the filter $\mathcal{U}$. The following characterization of bases can be found in Kelley [6].

**Proposition 1.1.** A nonempty family $\mathcal{B}$ of subsets of a set $X \times X$ is a base of a uniformity $\mathcal{U}$ if and only if

1. $\Delta(X) \subseteq B$ for any $B \in \mathcal{B}$;
2. $\forall B \in \mathcal{B}$, $\exists C \in \mathcal{B}$, $C^2 \subseteq B$;
3. $\forall B_1, B_2 \in \mathcal{B}$, $\exists C \in \mathcal{B}$, $C \subseteq B_1 \cap B_2$;
4. $\forall B \in \mathcal{B}$, $\exists C \in \mathcal{B}$, $C \subseteq B^{-1}$.

The corresponding uniformity is given by

$$\mathcal{U} = \{ U \subseteq X \times X : \exists B \in \mathcal{B}, \ B \subseteq U \}.$$  

A subbase of a uniformity $\mathcal{U}$ is a family $\mathcal{B} \subseteq \mathcal{U}$ such that any $U \in \mathcal{U}$ contains the intersection of a finite family of sets in $\mathcal{B}$.

**Remark 1.2.** It can be shown (see, e.g., Kelley [6]) that a nonempty family $\mathcal{B}$ of subsets of a set $X \times X$ is a subbase of a uniformity $\mathcal{U}$ if and only if it satisfies the conditions (B1), (B2) and (B4) from Proposition 1.1.

In this case the corresponding uniformity is given by

$$\mathcal{U} = \{ U \subseteq X \times X : \exists n \in \mathbb{N}, \exists B_1, \ldots, B_n \in \mathcal{B}, B_1 \cap \cdots \cap B_n \subseteq U \}.$$
Let \((X, d)\) be a pseudometric space. Then the pseudometric \(d\) generates a topology \(\tau_d\) for which

\[ B_d(x, r), \ r > 0, \]

is a base of neighborhoods for every \(x \in X\).

The pseudometric \(d\) generates also a uniform structure \(U_d\) on \(X\) having as basis of entourages the sets

\[ U_\varepsilon = \{(x, y) \in X \times X : d(x, y) < \varepsilon\}, \ \varepsilon > 0 \]

Since

\[ U_\varepsilon(x) = B_d(x, \varepsilon), \ x \in X, \ \varepsilon > 0, \]

it follows that the topology \(\tau(U_d)\) agrees with the topology \(\tau_d\) generated by the pseudometric \(d\).

A sequence \((x_n)\) in \(X\) is called Cauchy (or fundamental) if for every \(\varepsilon > 0\) there exists \(n_\varepsilon \in \mathbb{N}\) such that

\[ d(x_n, x_m) < \varepsilon \text{ for all } m, n \in \mathbb{N} \text{ with } m, n \geq n_\varepsilon, \]

a condition written also as

\[ \lim_{m,n \to \infty} d(x_m, x_n) = 0. \]

A sequence \((x_n)\) in a uniform space \((X, U)\) is called \(U\)-Cauchy (or simply Cauchy) if for every \(U \in \mathcal{U}\) there exists \(n_0 \in \mathbb{N}\) such that

\[ (x_m, x_n) \in U \text{ for all } m, n \in \mathbb{N} \text{ with } m, n \geq n_\varepsilon. \]

It is obvious that in the case of a pseudometric space \((X, d)\) a sequence is Cauchy with respect to the pseudometric \(d\) if and only if it is Cauchy with respect to the uniformity \(U_d\).

The Cauchyness of nets in pseudometric or in uniform spaces is defined by analogy with that of sequences.

A filter \(F\) in a uniform space \((X, \mathcal{U})\) is called \(\mathcal{U}\)-Cauchy (or simply Cauchy) if for every \(U \in \mathcal{U}\) there exists \(F_0 \in \mathcal{F}\) such that

\[ F \times F \subseteq U. \]

**Definition 1.3.** A pseudometric space \((X, d)\) is called complete if every Cauchy sequence in \(X\) converges. A uniform space \((X, \mathcal{U})\) is called sequentially complete if every \(\mathcal{U}\)-Cauchy sequence in \(X\) converges and complete if every \(\mathcal{U}\)-Cauchy net in \(X\) converges (or, equivalently, if every \(\mathcal{U}\)-Cauchy filter in \(X\) converges).

**Remark 1.4.** We can define the completeness of a subset \(Y\) of \(X\) by the condition that every Cauchy sequence in \(Y\) converges to some element of \(Y\). A closed subset of a pseudometric space is complete and a complete subset of a metric space is closed. A complete subset of a pseudometric space need not be closed.

The following result holds in the metric case.

**Theorem 1.5.** For a pseudometric space \((X, d)\) the following conditions are equivalent.

1. The metric space \(X\) is complete.
2. Every Cauchy net in \(X\) is convergent.
3. Every Cauchy filter in \((X, \mathcal{U}_d)\) is convergent.

An important result in metric spaces is Cantor characterization of completeness.
Theorem 1.6 (Cantor theorem). A pseudometric space \((X, d)\) is complete if and only if every descending sequence of nonempty closed subsets of \(X\) with diameters tending to zero has nonempty intersection. This means that for any family \(F_n, n \in \mathbb{N}\), of nonempty closed subsets of \(X\)

\[ F_1 \supseteq F_2 \supseteq \ldots \text{ and } \lim_{n \to \infty} \text{diam}(F_n) = 0 \Rightarrow \bigcap_{n=1}^{\infty} F_n \neq \emptyset. \]

If \(d\) is a metric then this intersection contains exactly one point.

The diameter of a subset \(Y\) of a pseudometric space \((X, d)\) is defined by

\[(1.1) \quad \text{diam}(Y) = \sup \{d(x, y) : x, y \in Y\}.\]

2. Quasi-pseudometric and quasi-uniform spaces

2.1. Quasi-pseudometric spaces. Dropping the symmetry condition \((M2)\) in the definition of a metric one obtains the notion of quasi-pseudometric, that is, a quasi-pseudometric on an arbitrary set \(X\) is a mapping \(d : X \times X \to \mathbb{R}\) satisfying the conditions \((M1)\) and \((M3)\). If \(d\) satisfies further \((M4')\) then it called a quasi-metric. The pair \((X, d)\) is called a quasi-pseudometric space, respectively a quasi-metric space.1

The conjugate of the quasi-pseudometric \(d\) is the quasi-pseudometric \(\bar{d}(x, y) = d(y, x), x, y \in X\). The mapping \(d^+(x, y) = \max\{d(x, y), d(y, x)\}, x, y \in X\), is a pseudometric on \(X\) which is a metric if and only if \(d\) is a quasi-metric.

If \((X, d)\) is a quasi-pseudometric space, then for \(x \in X\) and \(r > 0\) we define the balls in \(X\) by the formulae

\[ B_d(x, r) = \{y \in X : d(x, y) < r\} \text{ - the open ball, and} \]
\[ B_d[x, r] = \{y \in X : d(x, y) \leq r\} \text{ - the closed ball.} \]

The topology \(\tau_d\) (or \(\tau(d)\)) of a quasi-pseudometric space \((X, d)\) can be defined through the family \(V_d(x)\) of neighborhoods of an arbitrary point \(x \in X\):

\[ V \in V_d(x) \iff \exists r > 0 \text{ such that } B_d(x, r) \subseteq V \]
\[ \iff \exists r' > 0 \text{ such that } B_d[x, r'] \subseteq V. \]

The topological notions corresponding to \(d\) will be prefixed by \(d\)- (e.g. \(d\)-closure, \(d\)-open, etc).

The convergence of a sequence \((x_n)\) to \(x\) with respect to \(\tau_d\), called \(d\)-convergence and denoted by \(x_n \xrightarrow{d} x\), can be characterized in the following way

\[(2.1) \quad x_n \xrightarrow{d} x \iff d(x, x_n) \to 0.\]

Also

\[(2.2) \quad x_n \xrightarrow{d} x \iff d(x, x_n) \to 0 \iff d(x_n, x) \to 0.\]

As a space equipped with two topologies, \(\tau_d\) and \(\tau_{\bar{d}}\), a quasi-pseudometric space can be viewed as a bitopological space in the sense of Kelly.2

Asymmetric normed spaces

Let \(X\) be a real vector space. A mapping \(p : X \to \mathbb{R}\) is called an asymmetric seminorm on \(X\) if

\[ p(x) \geq 0 \quad \text{for all } x \in X \quad \text{and} \quad p(\lambda x) = |\lambda| p(x) \quad \text{for all } x \in X, \lambda \in \mathbb{R}. \]

1In [4] the term “quasi-semimetric” is used instead of “quasi-pseudometric”
(AN1) \[ p(x) \geq 0; \]
(AN2) \[ p(\alpha x) = \alpha p(x); \]
(AN3) \[ p(x + y) \leq p(x) + p(y), \]
for all \( x, y \in X \) and \( \alpha \geq 0 \).

If, further,

(AN4) \[ p(x) = p(-x) = 0 \Rightarrow x = 0, \]
for all \( x \in X \), then \( p \) is called an asymmetric norm.

To an asymmetric seminorm \( p \) one associates a quasi-pseudometric \( d_p \) given by

\[ d_p(x, y) = p(y - x), \quad x, y \in X, \]

which is a quasi-metric if \( p \) is an asymmetric norm. All the topological and metric notions in an asymmetric normed space are understood as those corresponding to this quasi-pseudometric \( d_p \) (see [4]).

The following topological properties are true for quasi-pseudometric spaces.

**Proposition 2.1** (see [4]). If \((X, d)\) is a quasi-pseudometric space, then the following hold.

1. The ball \( B_d(x, r) \) is \( d \)-open and the ball \( B_d[x, r] \) is \( d \)-closed. The ball \( B_d[x, r] \) need not be \( d \)-closed.
2. The topology \( d \) is \( T_0 \) if and only if \( d \) is a quasi-metric.
   The topology \( d \) is \( T_1 \) if and only if \( d(x, y) > 0 \) for all \( x \neq y \) in \( X \).
3. For every fixed \( x \in X \), the mapping \( d(x, \cdot) : X \to (\mathbb{R}, |\cdot|) \) is \( d \)-usc and \( d \)-lsc.
   For every fixed \( y \in X \), the mapping \( d(\cdot, y) : X \to (\mathbb{R}, |\cdot|) \) is \( d \)-lsc and \( d \)-usc.

**Remark 2.2.** It is known that the topology \( \tau_d \) of a pseudometric space \((X, d)\) is Hausdorff (or \( T_2 \)) if and only if \( d \) is a metric if and only if any sequence in \( X \) has at most one limit.

The characterization of Hausdorff property of quasi-pseudometric spaces can also be given in terms of uniqueness of the limits, as in the metric case: the topology of a quasi-pseudometric space \((X, d)\) is Hausdorff if and only if every sequence in \( X \) has at most one \( d \)-limit if and only if every sequence in \( X \) has at most one \( \bar{d} \)-limit (see [17]).

In the case of an asymmetric seminormed space there exists a characterization in terms of the asymmetric seminorm (see [4], Proposition 1.1.40).

Recall that a topological space \((X, \tau)\) is called:

- \( T_0 \) if, for every pair of distinct points in \( X \), at least one of them has a neighborhood not containing the other;
- \( T_1 \) if, for every pair of distinct points in \( X \), each of them has a neighborhood not containing the other;
- \( T_2 \) (or Hausdorff) if every two distinct points in \( X \) admit disjoint neighborhoods;
- regular if, for every point \( x \in X \) and closed set \( A \) not containing \( x \), there exist the disjoint open sets \( U, V \) such that \( x \in U \) and \( A \subseteq V \).

### 2.2. Quasi-uniform spaces

Again, the notion of quasi-uniform space is obtained by dropping the symmetry condition \((U3)\) from the definition of a uniform space, that is, a quasi-uniformity on a set \( X \) is a filter \( \mathcal{U} \) in \( X \times X \) satisfying the conditions \((U1)\) and \((U2)\). The sets in \( \mathcal{U} \) are called entourages and the pair \((X, \mathcal{U})\) is called a quasi-uniform space, as in the case of uniform spaces.
As uniformities, a quasi-uniformity \( \mathcal{U} \) generates a topology \( \tau(\mathcal{U}) \) on \( X \) in a similar way: the sets
\[
\{ U(x) : U \in \mathcal{U} \}
\]
form a base of neighborhoods of any point \( x \in X \).

The topology \( \tau(\mathcal{U}) \) is \( T_0 \) if and only if \( \bigcap \mathcal{U} \) is a partial order on \( X \), and \( T_1 \) if and only if \( \bigcap \mathcal{U} = \Delta(X) \).

The family of sets
\[
(2.3) \quad \mathcal{U}^{-1} = \{ U^{-1} : U \in \mathcal{U} \}
\]
is another quasi-uniformity on \( X \) called the quasi-uniformity conjugate to \( \mathcal{U} \). Also \( \mathcal{U} \cup \mathcal{U}^{-1} \) is a subbase of a uniformity \( \mathcal{U}^s \) on \( X \), called the associated uniformity to the quasi-uniformity \( \mathcal{U} \).

A basis for \( \mathcal{U}^s \) is formed by the sets
\[
\{ U \cap U^{-1} : U \in \mathcal{U} \}.
\]

If \( (X, d) \) is a quasi-pseudometric space, then
\[
\mathcal{U}_\varepsilon = \{ (x, y) \in X \times X : d(x, y) < \varepsilon \}, \quad \varepsilon > 0,
\]
is a basis for a quasi-uniformity \( \mathcal{U}_d \) on \( X \). The family
\[
\mathcal{U}_\varepsilon^- = \{ (x, y) \in X \times X : d(x, y) \leq \varepsilon \}, \quad \varepsilon > 0,
\]
generates the same quasi-uniformity. Since \( U_\varepsilon(x) = B_d(x, \varepsilon) \) and \( U_\varepsilon^-(x) = B_d[x, \varepsilon] \), it follows that the topologies generated by the quasi-pseudometric \( d \) and by the quasi-uniformity \( \mathcal{U}_d \) agree, i.e., \( \tau_d = \tau(\mathcal{U}_d) \).

In this case
\[
\mathcal{U}_d^{-1} = \mathcal{U}_d \quad \text{and} \quad \mathcal{U}_d^s = \mathcal{U}_d^s.
\]

3. **Cauchy sequences and sequential completeness in quasi-pseudometric and quasi-uniform spaces**

In contrast to the case of metric or uniform spaces, completeness, total boundedness and compactness look very different in quasi-metric and quasi-uniform spaces, due to the lack of symmetry of the distance. The present paper is concerned only with completeness. There are several notions of completeness in quasi-metric and quasi-uniform spaces, all agreeing with the usual notion of completeness in the case of metric or uniform spaces, each of them having its advantages and weaknesses.

We introduce now some of these notions following [10] (see also [4]).

**Definition 3.1.** A sequence \( (x_n) \) in \( (X, d) \) is called
- **left** (right) \( d \)-Cauchy if for every \( \varepsilon > 0 \) there exist \( x \in X \) and \( n_0 \in \mathbb{N} \) such that
  \[
d(x, x_n) < \varepsilon \quad \text{(respectively} \quad d(x_n, x) < \varepsilon)\]
  for all \( n \geq n_0 \);
- \( d^s \)-Cauchy if it is a Cauchy sequence is the pseudometric space \( (X, d^s) \), that is for every \( \varepsilon > 0 \) there exists \( n_0 \in \mathbb{N} \) such that
  \[
d^s(x_n, x_k) < \varepsilon \quad \text{for all} \quad n, k \geq n_0,
\]
or, equivalently,
  \[
d(x_n, x_k) < \varepsilon \quad \text{for all} \quad n, k \geq n_0;
\]
- **left** (right) \( K \)-Cauchy if for every \( \varepsilon > 0 \) there exists \( n_0 \in \mathbb{N} \) such that
  \[
d(x_k, x_n) < \varepsilon \quad \text{(respectively} \quad d(x_n, x_k) < \varepsilon)
\]
  for all \( n, k \in \mathbb{N} \) with \( n_0 \leq k \leq n \);
• weakly left (right) $K$-Cauchy if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that
$$d(x_{n_0}, x_n) < \varepsilon \quad \text{(respectively } d(x_n, x_{n_0}) < \varepsilon\text{)},$$
for all $n \geq n_0$.

Sometimes, to emphasize the quasi-pseudometric $d$, we shall say that a sequence is left $d$-$K$-Cauchy, etc.

It seems that $K$ in the definition of a left $K$-Cauchy sequence comes from Kelly [7] who considered first this notion.

Some remarks are in order.

Remark 3.2 ([10]). Let $(X, d)$ be a quasi-pseudometric space.

1. These notions are related in the following way:
$$d^*\text{-Cauchy } \Rightarrow \text{ left } K\text{-Cauchy } \Rightarrow \text{ weakly left } K\text{-Cauchy } \Rightarrow \text{ left } d\text{-Cauchy}.$$  
The same implications hold for the corresponding right notions. No one of the above implications is reversible.

2. A sequence is left Cauchy (in some sense) with respect to $d$ if and only if it is right Cauchy (in the same sense) with respect to $\bar{d}$.

3. A sequence is $d^*$-Cauchy if and only if it is both left and right $d$-$K$-Cauchy.

4. A $d$-convergent sequence is left $d$-Cauchy and a $\bar{d}$-convergent sequence is right $d$-Cauchy.

For the other notions, a convergent sequence need not be Cauchy.

5. If each convergent sequence in a regular quasi-metric space $(X, d)$ admits a left $K$-Cauchy subsequence, then $X$ is metrizable ([9]).

We also mention the following simple properties of Cauchy sequences.

Proposition 3.3 ([2] [11]). Let $(x_n)$ be a left or right $K$-Cauchy sequence in a quasi-pseudometric space $(X, d)$.

1. If $(x_n)$ has a subsequence which is $d$-convergent to $x$, then $(x_n)$ is $d$-convergent to $x$.

2. If $(x_n)$ has a subsequence which is $\bar{d}$-convergent to $x$, then $(x_n)$ is $\bar{d}$-convergent to $x$.

3. If $(x_n)$ has a subsequence which is $d^*$-convergent to $x$, then $(x_n)$ is $d^*$-convergent to $x$.

To each of these notions of Cauchy sequence corresponds two notions of sequential completeness, by asking that the corresponding Cauchy sequence be $d$-convergent or $d^*$-convergent. Due to the equivalence $d$-left Cauchy $\iff$ $d$-right Cauchy one obtains nothing new by asking that a $d$-left Cauchy sequence is $d$-convergent. For instance, the $d$-convergence of any left $d$-$K$-Cauchy sequence is equivalent to the right $K$-completeness of the space $(X, \bar{d})$.

Definition 3.4 ([10]). A quasi-pseudometric space $(X, d)$ is called:

• sequentially $d$-complete if every $d^*$-Cauchy sequence is $d$-convergent;

• sequentially left $d$-complete if every left $d$-Cauchy sequence is $d$-convergent;

• sequentially weakly left (right) $K$-complete if every weakly left (right) $K$-Cauchy sequence is $d$-convergent;

• sequentially left (right) $K$-complete if every left (right) $K$-Cauchy sequence is $d$-convergent;

• sequentially left (right) Smyth complete if every left (right) $K$-Cauchy sequence is $d^*$-convergent;

• bicomplete if the associated pseudometric space $(X, d^*)$ is complete, i.e., every $d^*$-Cauchy sequence is $d^*$-convergent. A bicomplete asymmetric normed space $(X, p)$ is called a biBanach space.
As we noticed (see Remark 3.2.4), each \( d \)-convergent sequence is left \( d \)-Cauchy, but for each of the other notions there are examples of \( d \)-convergent sequences that are not Cauchy, which is a major inconvenience. Another one is that a complete (in some sense) subspace of a quasi-metric space need not be closed.

The implications between these completeness notions are obtained by reversing the implications between the corresponding notions of Cauchy sequence from Remark 3.2.1.

**Remark 3.5.** (a) These notions of completeness are related in the following way:

- sequentially \( d \)-complete \( \Rightarrow \) sequentially weakly left \( K \)-complete \( \Rightarrow \) sequentially left \( K \)-complete \( \Rightarrow \) sequentially left \( d \)-complete.

The same implications hold for the corresponding notions of right completeness.

(b) sequentially left or right Smyth completeness implies bicompleteness.

No one of the above implications is reversible (see [10]), excepting that between weakly left and left \( K \)-sequential completeness, as it was surprisingly shown by Romaguera [11].

**Proposition 3.6** ([12], Proposition 1). A quasi-pseudometric space is sequentially weakly left \( K \)-complete if and only if it is sequentially left \( K \)-complete.

A series \( \sum_n x_n \) in an asymmetric seminormed space \((X, p)\) is called convergent if there exists \( x \in X \) such that \( x = \lim_{n \to \infty} \sum_{k=1}^{n} x_k \) (i.e., \( \lim_{n \to \infty} p(\sum_{k=1}^{n} x_k - x) = 0 \)). The series \( \sum_n x_n \) is called absolutely convergent if \( \sum_{n=1}^{\infty} p(x_n) < \infty \). It is well-known that a normed space is complete if and only if every absolutely convergent series is convergent. A similar result holds in the asymmetric case too.

**Proposition 3.7.** Let \((X, d)\) be a quasi-pseudometric space.

1. If a sequence \((x_n)\) in \( X \) satisfies \( \sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty \) \( (\sum_{n=1}^{\infty} d(x_{n+1}, x_n) < \infty) \), then it is left (right) \( d \)-\( K \)-Cauchy.

2. The quasi-pseudometric space \((X, d)\) is sequentially left (right) \( d \)-\( K \)-complete if and only if every sequence \((x_n)\) in \( X \) satisfying \( \sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty \) (resp. \( \sum_{n=1}^{\infty} d(x_{n+1}, x_n) < \infty \)) is \( d \)-convergent.

3. An asymmetric seminormed space \((X, p)\) is sequentially left \( K \)-complete if and only if every absolutely convergent series is convergent.

**Cantor type results**

Concerning Cantor-type characterizations of completeness in terms of descending sequences of closed sets (the analog of Theorem 1.6) we mention the following result. The diameter of a subset \( A \) of a quasi-pseudometric space \((X, d)\) is defined by

\[
\text{diam}(A) = \sup\{d(x, y) : x, y \in A\}.
\]

It is clear that, as defined, the diameter is, in fact, the diameter with respect to the associated pseudometric \( d^s \). Recall that a quasi-pseudometric space is called sequentially \( d \)-complete if every \( d^s \)-Cauchy sequence is \( d \)-convergent (see Definition 3.2).

**Theorem 3.8** ([10], Theorem 10). A quasi-pseudometric space \((X, d)\) is sequentially \( d \)-complete if and only if each decreasing sequence \( F_1 \supseteq F_2 \ldots \) of nonempty closed sets with \( \text{diam}(F_n) \to 0 \) as \( n \to \infty \) has nonempty intersection, which is a singleton if \( d \) is a quasi-metric.

The following characterization of right \( K \)-completeness was obtained in [3], using a different terminology.
Proposition 3.9. A quasi-pseudometric space \((X, d)\) is sequentially right \(K\)-complete if and only if any decreasing sequence of closed \(d\)-balls
\[
B_d[x_1, r_1] \supseteq B_d[x_2, r_2] \supseteq \ldots \quad \text{with} \quad \lim_{n \to \infty} r_n = 0,
\]
has nonempty intersection.

If the topology \(d\) is Hausdorff, then \(\bigcap_{n=1}^{\infty} B_d[x_n, r_n]\) contains exactly one element.

4. Completeness by nets and filters

The Cauchy properties of a net \((x_i : i \in I)\) in a quasi-pseudometric space \((X, d)\) are defined by analogy with that of sequences, by replacing in Definition 3.1 the natural numbers with the elements of the directed set \(I\).

The situation is good for left Smyth completeness (see Definition 3.4).

Proposition 4.1 ([13], Prop. 1). For a quasi-metric space \((X, d)\) the following are equivalent.

1. Every left \(d-K\)-Cauchy sequence is \(d^s\)-convergent.
2. Every left \(d-K\)-Cauchy net is \(d^s\)-convergent.

A quasi-uniform space \((X, \mathcal{U})\) is called bicomplete if \((X, \mathcal{U}^s)\) is a complete uniform space. This notion is useful and easy to handle, because one can appeal to well known results from the theory of uniform spaces, but it is not appropriate for the study of the specific properties of quasi-uniform spaces, so one introduces adequate definitions, by analogy with quasi-pseudometric spaces.

Definition 4.2. Let \((X, \mathcal{U})\) be a quasi-uniform space.

A filter \(\mathcal{F}\) on \((X, \mathcal{U})\) is called:

- left (right) \(\mathcal{U}\)-Cauchy if for every \(U \in \mathcal{U}\) there exists \(x \in X\) such that \(U(x) \in \mathcal{F}\) (respectively \(U^{-1}(x) \in \mathcal{F}\));
- left (right) \(\mathcal{U}\)-\(K\)-Cauchy if for every \(U \in \mathcal{U}\) there exists \(F \in \mathcal{F}\) such that \(U(x) \in \mathcal{F}\) (resp. \(U^{-1}(x) \in \mathcal{F}\)) for all \(x \in F\).

A net \((x_i : i \in I)\) in \((X, \mathcal{U})\) is called:

- left \(\mathcal{U}\)-Cauchy (right \(\mathcal{U}\)-Cauchy) if for every \(U \in \mathcal{U}\) there exists \(x \in X\) and \(i_0 \in I\) such that \((x, x_i) \in U\) (respectively \((x_i, x) \in U\)) for all \(i \geq i_0\);
- left \(\mathcal{U}\)-\(K\)-Cauchy (right \(\mathcal{U}\)-\(K\)-Cauchy) if
\[
\forall U \in \mathcal{U}, \exists i_0 \in I, \forall i, j \in I, i_0 \leq i \leq j \Rightarrow (x_i, x_j) \in U \quad (\text{resp.} \quad (x_j, x_i) \in U)
\]

The notions of left and right \(\mathcal{U}\)-\(K\)-Cauchy filter were defined by Romaguera in [12].

Observe that
\[
(x_j, x_i) \in U \iff (x_i, x_j) \in U^{-1},
\]
so that a filter is right \(\mathcal{U}\)-\(K\)-Cauchy if and only if it is left \(\mathcal{U}^{-1}\)-\(K\)-Cauchy. A similar remark applies to \(\mathcal{U}\)-nets.

Definition 4.3. A quasi-uniform space \((X, \mathcal{U})\) is called:

- left \(\mathcal{U}\)-complete by filters (left \(K\)-complete by filters) if every left \(\mathcal{U}\)-Cauchy (respectively, left \(\mathcal{U}\)-\(K\)-Cauchy) filter in \(X\) is \(\tau(\mathcal{U})\)-convergent;
- left \(\mathcal{U}\)-complete by nets (left \(\mathcal{U}\)-\(K\)-complete by nets) if every left \(\mathcal{U}\)-Cauchy (respectively, left \(\mathcal{U}\)-\(K\)-Cauchy) net in \(X\) is \(\tau(\mathcal{U})\)-convergent;
- Smyth left \(\mathcal{U}\)-\(K\)-complete by nets if every left \(K\)-Cauchy net in \(X\) is \(\mathcal{U}^s\)-convergent.

The notions of right completeness are defined similarly, by asking the \(\tau(\mathcal{U})\)-convergence of the corresponding right Cauchy filter (or net) with respect to the topology \(\tau(\mathcal{U})\) (or with respect to \(\tau(\mathcal{U}^s)\) in the case of Smyth completeness).
As we have mentioned in Introduction, in pseudometric spaces the sequential completeness is equivalent to the completeness defined in terms of filters, or of nets. Romaguera [12] proved a similar result for the left $K$-completeness in quasi-pseudometric spaces.

**Remark 4.4.** In the case of a quasi-pseudometric space the considered notions take the following form.

A filter $\mathcal{F}$ in a quasi-pseudometric space $(X, d)$ is called *left $K$-Cauchy* if it is left $\mathcal{U}_d$-$K$-Cauchy. This is equivalent to the fact that for every $\varepsilon > 0$ there exists $F_\varepsilon \in \mathcal{F}$ such that

$$\forall x \in F_\varepsilon, \quad B_d(x, \varepsilon) \in \mathcal{F}.$$  

Also a net $(x_i : i \in I)$ is called *left $K$-Cauchy* if it is left $\mathcal{U}_d$-$K$-Cauchy or, equivalently, for every $\varepsilon > 0$ there exists $i_0 \in I$ such that

$$\forall i, j \in I, \quad i_0 \leq i \leq j \Rightarrow d(x_i, x_j) < \varepsilon.$$  

**Proposition 4.5** ([12]). For a quasi-pseudometric space $(X, d)$ the following are equivalent.

1. The space $(X, d)$ is sequentially left $K$-complete.
2. Every left $K$-Cauchy filter in $X$ is $d$-convergent.
3. Every left $K$-Cauchy net in $X$ is $d$-convergent.

In the case of left $\mathcal{U}_d$-completeness this equivalence does not hold in general.

**Proposition 4.6** (Künzi [8]). A Hausdorff quasi-metric space $(X, d)$ is sequentially left $d$-complete if and only if the associated quasi-uniform space $(X, \mathcal{U}_d)$ is left $\mathcal{U}_d$-complete by filters.

**4.1. Right $K$-completeness in quasi-pseudometric spaces.** It is strange that for the right completeness the things look worse than for the left completeness.

As remarked Stoltenberg [15] Example 2.4] a result similar to Proposition 4.5 does not hold for right $K$-completeness: there exists a sequentially right $K$-complete $T_1$ quasi-metric space which is not right $K$-complete by nets. Actually, Stoltenberg [15] proved that the equivalence holds for a more general definition of a right $K$-Cauchy net, see Proposition 4.15.

An analog of Proposition 4.5 for right $K$-completeness can be obtained only under some supplementary hypotheses on the quasi-pseudometric space $X$.

A quasi-pseudometric space $(X, d)$ is called $R_1$ if for all $x, y \in X$, $d-cl\{x\} \neq d-cl\{y\}$ implies the existence of two disjoint $d$-open sets $U, V$ such that $x \in U$ and $y \in V$.

**Proposition 4.7** ([1]). Let $(X, d)$ be a quasi-pseudometric space. The following are true.

1. If $X$ is right $K$-complete by filters, then every right $K$-Cauchy net in $X$ is convergent. In particular, every right $K$-complete by filters quasi-pseudometric space is sequentially right $K$-complete.
2. If the quasi-pseudometric space $(X, d)$ is $R_1$ then $X$ is right $K$-complete by filters if and only if it is sequentially right $K$-complete.

**Stoltenberg’s example**

As we have mentioned, Stoltenberg [15] Example 2.4] gave an example of a sequentially right $K$-complete $T_1$ quasi-metric space which is not right $K$-complete by nets, which we shall present now.

Denote by $A$ the family of all countable subsets of the interval $[0, \frac{1}{3}]$. For $A \in A$ let

$$A_1^A = A, \quad A_{k+1}^A = A \cup \left\{ \frac{1}{2^k}, \frac{3}{2^k}, \ldots, \frac{2^k - 1}{2^k} \right\}, \quad k \in \mathbb{N},$$

$$A_\infty^A = A \cup \left\{ \frac{2^k - 1}{2^k} : k \in \mathbb{N} \right\} = \bigcup \left\{ A_k^A : k \in \mathbb{N} \right\}.$$
Put $S = \{X^A_k : A \in \mathcal{A}, k \in \mathbb{N} \cup \{\infty\}\}$ and define $d : S \times S \to [0, \infty)$ by
\[
d(X^A_k, X^B_j) = \begin{cases} 
0 & \text{if } A = B \text{ and } k = j, \ A, B \in \mathcal{A}, k, j \in \mathbb{N} \cup \{\infty\} \\
2^{-j} & \text{if } X^B_j \not\subseteq X^A_k, \ A, B \in \mathcal{A}, k \in \mathbb{N} \cup \{\infty\}, j \in \mathbb{N}, \\
1 & \text{otherwise}.
\end{cases}
\]

**Proposition 4.8.** $(S, d)$ is a sequentially right $K$-complete $T_1$ quasi-metric space which is not right $K$-complete by nets.

**Proof.** The proof that $d$ is a $T_1$ quasi-metric on $S$ is straightforward.

I. $(S, d)$ is sequentially right $K$-complete.

Let $(X_n)_{n \in \mathbb{N}}$ be a right $K$-Cauchy sequence in $S$. Then there exists $n_0 \in \mathbb{N}$ such that
\[
d(X_m, X_n) < 1 \quad \text{for all} \quad m, n \in \mathbb{N} \text{ with } n_0 \leq n \leq m.
\]

For $i \in N_0 = \mathbb{N} \cup \{0\}$ let
\[
X_{n_0+i} = X^A_{k_i} \quad \text{where } A_i \in \mathcal{A} \text{ and } k_i \in \mathbb{N} \cup \{\infty\}.
\]

Since
\[
d\left(X^{A_{i+1}}_{k_{i+1}}, X^{A_i}_{k_i}\right) < 1,
\]
it follows $k_i \in \mathbb{N}$ for all $i \in N_0$. For $0 < \varepsilon < 1$ there exists $i_0 \in \mathbb{N}$ such that
\[
d(X_{n_0+i}, X_{n_0+i+1}) < \varepsilon \quad \text{for all } i \geq i_0,
\]
which means that
\[
2^{-k_i} = d\left(X^{A_{i+1}}_{k_{i+1}}, X^{A_i}_{k_i}\right) < \varepsilon \quad \text{for all } i \geq i_0.
\]
This shows that $\lim_{i \to \infty} k_i = \infty$.

Let $A = \bigcup \{A_i : i \in N_0\}$ and $X = X^A_{\infty}$. Then $X^A_{k_i} \not\subseteq X^A_{\infty}$, so that
\[
d(X, X_{n_0+i}) = d\left(X^A_{\infty}, X^{A_i}_{k_i}\right) = 2^{-k_i} \to 0 \quad \text{as } i \to \infty,
\]
which shows that the sequence $(X_n)$ is $d$-convergent to $X$.

II. The quasi-metric space $(S, d)$ is not right $K$-complete by nets.

Let $S_0 = \{X^A_k : A \in \mathcal{A}, k \in \mathbb{N}\}$ ordered by
\[
X \leq Y \iff X \subseteq Y, \text{ for } X, Y \in S_0.
\]

We have
\[
X^A_i \leq X^B_j \iff \begin{cases} 
A \subseteq B \text{ and } \\
i \leq j
\end{cases}
\]
for $X^A_i, X^B_j \in S_0$, $(S_0, \leq)$ is directed and the mapping $\phi : S_0 \to S$ defined by $\phi(X) = X$, $X \in S_0$, is a net in $S$.

Let us show first that the net $\phi$ is right $K$-Cauchy. For $\varepsilon > 0$ choose $k \in \mathbb{N}$ such that $2^{-k} < \varepsilon$. For some $C \in \mathcal{A}$, $X^C_k$ belongs to $S_0$ and
\[
d(X^A_i, X^B_j) = 2^{-i} \leq 2^{-k} < \varepsilon
\]
for all $X^A_j, X^B_i \in S_0$ with $X^C_k \subseteq X^B_j \subseteq X^A_i$, $X^A_j \neq X^B_i$,
showing that the net $\phi$ is right $K$-Cauchy.
Let \( X = X^C_k \) be an arbitrary element in \( S \). We show that for every \( X^A_i \in S_0 \) there exists \( X^B_j \in S_0 \) such that \( d(X, X^B_j) = 1 \), which will imply that the net \( \phi \) is not \( d \)-convergent to \( X \).

Since \( C \) is a countable set, there exists \( x_0 \in [0, \frac{1}{3}] \setminus C \). For an arbitrary \( X^A_i \in S_0 \) let \( B = A \cup \{x_0\} \). Then \( X^B_j \in S_0 \), \( X^A_i \leq X^B_j \) and \( X^C_k \nsubseteq X^B_j \), so that, by the definition of the metric \( d \), \( d(X^C_k, X^B_j) = 1 \).

\[ \square \]

Stoltenberg-Cauchy nets

Stoltenberg [15] also considered a more general definition of a right \( K \)-Cauchy net as a net \((x_i : i \in I)\) satisfying the condition: for every \( \varepsilon > 0 \) there exists \( i_\varepsilon \in I \) such that
\[ (4.4) \quad d(x_i, x_j) < \varepsilon \quad \text{for all} \quad i, j \geq i_\varepsilon \quad \text{with} \quad i \nleq j. \]

Let us call such a net \textit{Stoltenberg-Cauchy} and \textit{Stoltenberg completeness} the completeness with respect to Stoltenberg-Cauchy nets.

It follows that, for this definition,
\[ d(x_j, x_i) < \varepsilon \quad \text{and} \quad d(x_j, x_i) < \varepsilon \quad \text{for all} \quad i, j \geq i_\varepsilon \quad \text{with} \quad i \nleq j, \]
where \( i \nleq j \) means that \( i, j \) are incomparable (that is, no one of the relations \( i \leq j \) or \( j \leq i \) holds).

Gregori-Ferrer-Cauchy nets

Later, Gregori and Ferrer [5] found a gap in the proof of Theorem 2.5 from [15] and provided a counterexample to it, based on Example 2.4 of Stoltenberg (see Proposition 4.8).

\textbf{Example 4.9} ([5]). Let \( A, (S, d) \) be as in the preamble to Proposition 4.8 and \( I = \mathbb{N} \cup \{a, b\} \), where the set \( \mathbb{N} \) is considered with the usual order and \( a, b \) are two distinct elements not belonging to \( \mathbb{N} \) with
\[ k \leq a, \quad k \leq b, \quad \text{for all} \quad k \in \mathbb{N}, \]
\[ a \leq a, \quad b \leq b \quad a \leq b, \quad b \leq a. \]

Consider two sets \( A, B \in A \) with \( A \subsetneq B \) and let \( \phi : I \to S \) be given by
\[ \phi(k) = X^A_k, \quad k \in \mathbb{N}, \quad \phi(a) = X^A_\infty, \quad \phi(b) = X^B_\infty. \]

Then the net \( \phi \) is right Cauchy in the sense of (4.4) but not convergent in \((S, d)\).

Indeed, for \( 0 < \varepsilon < 1 \) let \( k_0 \in \mathbb{N} \) be such that \( 2^{-k_0} < \varepsilon \).

Since
\[ i \leq a, \quad i \leq b, \quad i \geq k_0, \quad \forall i \in \mathbb{N}, \quad k_0 \leq a \leq b, \quad k_0 \leq b \leq a, \]
it follows that the condition \( i \nleq j \) can hold for some \( i, j \in I, \quad i, j \geq k_0 \), in the following cases:

(a) \( i, j \in \mathbb{N}, \quad i, j \geq k_0, \quad j < i; \)
(b) \( i = a, \quad j \in \mathbb{N}, \quad j \geq k_0 \)
(c) \( i = b, \quad j \in \mathbb{N}, \quad j \geq k_0 \)

In the case (a), \( X^A_j \subsetneq X^A_i \) and
\[ d(\phi(i), \phi(j)) = d(X^A_i, X^A_j) = 2^{-j} \leq 2^{-k_0} < \varepsilon. \]

In the case (b), \( X^A_j \subsetneq X^A_\infty \) and again
\[ d(\phi(a), \phi(j)) = d(X^A_\infty, X^A_j) = 2^{-j} \leq 2^{-k_0} < \varepsilon. \]

The case (c) is similar to (b).

To show that \( \phi \) is not convergent let \( X \in S \setminus \{X^B_\infty\} \). Then \( b \geq i \) for any \( i \in I \) and

\[ d(X, \phi(b)) = d(X, X^B_\infty) = 1, \]

so that \( \phi \) does not converge to \( X \). If \( X = X^B_\infty \), then \( a \geq i \) for any \( i \in I \) and

\[ d(X^B_\infty, \phi(a)) = d(X^B_\infty, X^A_\infty) = 1. \]

Gregori and Ferrer [5] proposed a new definition of a right K-Cauchy net, for which the equivalence to sequential completeness holds.

**Definition 4.10.** A net \((x_i : i \in I)\) in a quasi-metric space \((X, d)\) is called GF-Cauchy if one of the following conditions holds:

1. For every maximal element \( j \in I \) the net \((x_i)\) converges to \( x_j \);
2. \( I \) has no maximal elements and the net \((x_i)\) converges;
3. \( I \) has no maximal elements and the net \((x_i)\) satisfies the condition (4.4).

**Maximal elements and net convergence**

For a better understanding of this definition we shall analyze the relations between maximal elements in a preordered set and the convergence of nets. Recall that in the definition of a directed set \((I, \preceq)\) the relation \( \preceq \) is supposed to be only a preorder, i.e. reflexive and transitive and not necessarily antireflexive (see [6]). Notice that some authors suppose that in the definition of a directed set \( \preceq \) is a partial order (see, e.g., [16]). For a discussion of this matter see [14], §7.12, p. 160]

Let \((I, \preceq)\) be a preordered set. An element \( j \in I \) is called:

- **strictly maximal** if there is no \( i \in I \setminus \{j\} \) with \( j \preceq i \), or, equivalently,

\[ j \preceq i \Rightarrow i = j, \quad \text{for every } i \in I; \]

- **maximal** if

\[ j \preceq i \Rightarrow i \preceq j, \quad \text{for every } i \in I. \]

**Remark 4.11.** Let \((I, \preceq)\) be a preordered set.

1. A strictly maximal element is maximal, and if \( \preceq \) is an order, then these notions are equivalent.

Suppose now that the set \( I \) is further directed. Then the following hold.

2. Every maximal element \( j \) of \( I \) is a maximum for \( I \), i.e. \( i \preceq j \) for all \( i \in I \).
3. If \( j \) is a maximal element and \( j' \in I \) satisfies \( j \preceq j' \), then \( j' \) is also a maximal element.
4. (Uniqueness of the strictly maximal element) If \( j \) is a strictly maximal element, then \( j' = j \) for any maximal element \( j' \) of \( I \).

**Proof.** 1. These assertions are obvious.

2. Indeed, suppose that \( j \in I \) satisfies (4.6). Then, for arbitrary \( i \in I \), there exists \( i' \in I \) with \( i' \geq j, i \). But \( j \preceq i' \) implies \( i' \preceq j \) and so \( i \preceq i' \preceq j \). (We use the notation \( i \geq j, k \) for \( i \geq j \land i \geq k \).)

3. Let \( i \in I \) be such that \( j' \preceq i \). Then \( j \preceq i \) and, by the maximality of \( j \), \( i \preceq j \preceq j' \).

4. If \( j \) is strictly maximal and \( j' \) is a maximal element of \( I \), then, by 2, \( j' \preceq j \) so that, by \( (4.6) \) applied to \( j' \), \( j \preceq j' \) and so, by (4.5) applied to \( j, j' = j \).

We present now some remarks on maximal elements and net convergence.
Remark 4.12. Let \((X, d)\) be a quasi-metric space, \((I, \preceq)\) a directed sets and \((x_i : i \in I)\) a net in \(X\).

1. If \((I, \preceq)\) has a strictly maximal element \(j\), then the net \((x_i)\) is convergent to \(x_j\).
2. (a) If the net \((x_i)\) converges to \(x \in X\), then \(d(x, x_j) = 0\) for every maximal element \(j\) of \(I\).

If the topology \(\tau_d\) is \(T_1\) then, further, \(x_j = x\).

(b) If the net \((x_i)\) converges to \(x_j\) and to \(x_{j'}\), where \(j, j'\) are maximal elements of \(I\), then \(x_j = x_{j'}\).

(c) If \(I\) has maximal elements and, for some \(x \in X\), \(x = x\) for every maximal element \(j\), then the net \((x_i)\) converges to \(x\).

Proof. 1. For an arbitrary \(\varepsilon > 0\) take \(i_\varepsilon = j\). Then \(i \geq j\) implies \(i = j\), so that

\[
d(x_j, x_i) = d(x_j, x_j) = 0 < \varepsilon.
\]

2. (a) For every \(\varepsilon > 0\) there exists \(i_\varepsilon \in I\) such that \(d(x, x_i) < \varepsilon\) for all \(i \geq i_\varepsilon\). By Remark 4.11.2, \(j \geq i_\varepsilon\) for every maximal \(j\), so that \(d(x, x_j) < \varepsilon\) for all \(\varepsilon > 0\), implying \(d(x, x_j) = 0\).

If the topology \(\tau_d\) is \(T_1\), then, by Proposition 2.1.2, \(x_j = x\).

(b) By (a), \(d(x_j, x_{j'}) = 0\) and \(d(x_{j'}, x_j) = 0\), so that \(x_j = x_{j'}\).

(c) Let \(x \in X\) be such that \(x_j = x\) for every maximal element \(j\) of \(I\) and let \(j\) be a fixed maximal element of \(I\). For any \(\varepsilon > 0\) put \(i_\varepsilon = j\). Then, by Remark 4.11.3, any \(i \in I\) such that \(i \geq j\) is also a maximal element of \(I\), so that \(x_i = x\) and \(d(x, x_i) = 0 < \varepsilon\).

Let us say that a quasi-metric space \((X, d)\) is GF-complete if every GF-Cauchy net (i.e. satisfying the conditions (a), (b), (c) from Definition 4.10) is convergent. Remark that, with this definition, condition (b) becomes tautological and so superfluous, so it suffices to ask that every net satisfying (a) and (c) be convergent.

By Remarks 4.11.1 and 4.12.1, (a) always holds if \(\preceq\) is an order, so that, in this case, a net satisfying condition (c) is a GF-Cauchy net and so GF-completeness agrees with that given by Stoltenberg.

Strongly Stoltenberg-Cauchy nets

In order to avoid the shortcomings of the preorder relation, as, for instance, those put in evidence by Example 4.19, we propose the following definition.

Definition 4.13. A net \((x_i : i \in I)\) in a quasi-metric space \((X, d)\) is called strongly Stoltenberg-Cauchy if for every \(\varepsilon > 0\) there exists \(i_\varepsilon \in I\) such that, for all \(i, j \geq i_\varepsilon\),

\[
(j \leq i \lor i \sim j) \Rightarrow d(x_i, x_j) < \varepsilon.
\] (4.7)

We present now some remarks on the relations of this notion with other notions of Cauchy net (Stoltenberg and GF), as well as the relations between the corresponding completeness notions. It is obvious that in the case of a sequence \((x_k)_{k \in \mathbb{N}}\) each of these three notions agrees with the right \(K\)-Cauchyness of \((x_k)\).

Remark 4.14. Let \((x_i : i \in I)\) be a net in a quasi-metric space \((X, d)\).

1. (a) We have

\[
i \not\sim j \Rightarrow (j \leq i \lor i \sim j),
\] (4.8)

for all \(i, j \in I\). If \(\preceq\) is an order, then the reverse implication also holds for all \(i, j \in I\) with \(i \neq j\).

(b) If the net \((x_i : i \in I)\) satisfies (4.7), then it satisfies (4.4), i.e. every strong Stoltenberg-Cauchy net is Stoltenberg-Cauchy. If \(\preceq\) is an order, then these notions are equivalent.

Hence, net completeness with respect to (4.4) (i.e. Stoltenberg completeness) implies net completeness with respect to (4.7).
2. Suppose that the net \((x_i : i \in I)\) satisfies (4.7).
   (a) If \(j, j'\) are maximal elements of \(I\), then \(x_j = x_{j'}\). Hence, if \(I\) has maximal elements, then there exists \(x \in X\) such that \(x_j = x\) for every maximal element \(j\) of \(I\), and the net \((x_i)\) converges to \(x\).
   (b) Consequently, the net \((x_i)\) also satisfies the conditions (a) and (c) from Definition 4.10 so that, GF-completeness implies completeness with respect to (4.7).

Proof. 1.(a) Let \(i, j \in I\) with \(i \neq j\). Since \(j \leq i\) if \(i, j\) are comparable, the implication (4.8) holds. If \(\leq\) is an order and \(i \neq j\), then \(j \leq i \Rightarrow i \not\sim j \Rightarrow i \neq j\).

(b) Since it suffices to ask that (4.4) and (4.7) hold only for distinct \(i, j \geq i_\varepsilon\), the equivalence of these notions in the case when \(\leq\) is an order follows.

Suppose that the net \((x_i)\) satisfies (4.7). For \(\varepsilon > 0\) choose \(i_\varepsilon \in I\) according to (4.7) and let \(i, j \geq i_\varepsilon\) with \(i \not\sim j\). Taking into account (4.8) it follows \(d(x_i, x_j) < \varepsilon\), i.e. \((x_i)\) satisfies (4.4).

Suppose now that every net satisfying (4.4) converges and let \((x_i)\) be a net in \(X\) satisfying (4.7). Then it satisfies (4.4) so it converges.

2.(a) Let \(j, j'\) be maximal elements of \(I\). For \(\varepsilon > 0\) choose \(i_\varepsilon \in I\) according to (4.7). By Remark 4.11 2, \(j, j' \geq i_\varepsilon\), \(j \leq j'\), \(j' \leq j\), so that \(d(x_{j'}, x_j) < \varepsilon\) and \(d(x_j, x_{j'}) < \varepsilon\). Since these inequalities hold for every \(\varepsilon > 0\), it follows \(d(x_{j'}, x_j) = 0 = d(x_j, x_{j'})\) and so \(x_j = x_{j'}\). The convergence of the net \((x_i)\) follows from Remark 4.11 2.(c).

(b) The assertions on GF-Cauchy nets follow from (a).

We show now that completeness by nets with respect to (4.7) is equivalent to sequential right K-completeness.

Proposition 4.15 ([15], Theorem 2.5). A \(T_1\) quasi-metric space \((X, d)\) is sequentially right K-complete if and only if every net in \(X\) satisfying (4.7) is \(d\)-convergent.

Proof. We have only to prove that the sequential right K-completeness implies that every net in \(X\) satisfying (4.7) is \(d\)-convergent.

Let \((x_i : i \in I)\) be a net in \(X\) satisfying (4.7). Let \(i_k \geq i_{k-1}\), \(k \geq 2\), be such that (4.7) holds for \(\varepsilon = 1/2^k\), \(k \in \mathbb{N}\).

This is possible. Indeed, take \(i_1\) such that (4.7) holds for \(\varepsilon = 1/2\). If \(i_2\) is such that (4.7) holds for \(\varepsilon = 1/2^2\), then pick \(i_2 \in I\) such that \(i_2 \geq i_1, i_2\). Continuing by induction one obtains the desired sequence \((i_k)_{k \in \mathbb{N}}\).

We distinguish two cases.

Case I. \(\exists j_0 \in I\), \(\exists k_0 \in \mathbb{N}\), \(\forall k \geq k_0\), \(i_k \leq j_0\).

Let \(i \geq j_0\). Then for every \(k\), \(i_k \leq j_0 \leq i\) implies \(d(x_i, x_{j_0}) < 2^{-k}\) so that \(d(x_i, x_{j_0}) = 0\). Since the quasi-metric space \((X, d)\) is \(T_1\), it follows \(x_i = x_{j_0}\) for all \(i \geq j_0\) (see Proposition 2.1), so that the net \((x_i : i \in I)\) is \(d\)-convergent to \(x_{j_0}\).

Case II. \(\forall j \in I\), \(\forall k \in \mathbb{N}\), \(\exists k' \geq k\), \(i_{k'} \not\sim j\).

The inequalities \(d(x_{i_{k+1}}, x_{i_k}) < 2^{-k}\), \(k \in \mathbb{N}\), imply that the sequence \((x_{i_k})_{k \in \mathbb{N}}\) is right K-Cauchy (see Proposition 3.7), so it is \(d\)-convergent to some \(x \in X\).

For \(\varepsilon > 0\) choose \(k_0 \in \mathbb{N}\) such that \(2^{-k_0} < \varepsilon\) and \(d(x, x_{i_k}) < \varepsilon\) for all \(k \geq k_0\).

Let \(i \in I\), \(i \geq i_{k_0}\). By hypothesis, there exists \(k \geq k_0\) such that \(i_k \not\sim i\), implying \(i \leq i_k \vee i_k \sim i\).

Since \(i_{k_0} \leq i_k, i\), by the choice of \(i_{k_0}\), \(d(x_{i_k}, x_i) < 2^{-k_0} < \varepsilon\) in both of these cases. But then

\[d(x, x_i) \leq d(x, x_{i_k}) + d(x_{i_k}, x_i) < 2\varepsilon,\]

proving the convergence of the net \((x_i)\) to \(x\). 
\[\square\]
The proof of Proposition 4.15 in the case of GF-completeness

As the result in [5] is given without proof, we shall supply one. We shall show that:

A $T_1$ quasi-metric space $(X,d)$ is right $K$-sequentially complete if and only if every net satisfying the conditions (a) and (c) from Definition 4.10 is convergent.

Obviously, a proof is needed only for in the case (c).

Suppose that the directed set $(I,\leq)$ has no maximal elements and let $(x_i : i \in I)$ be a net in a quasi-metric space $(X,d)$ satisfying (4.4).

The proof follows the ideas of the proof of Proposition 4.15 with some further details. Let $i_k \leq i_{k+1}$, $k \in \mathbb{N}$, be a sequence of indices in $I$ such that $d(x_i, x_j) < 2^{-k}$ for all $i, j \geq i_k$ with $i \neq j$. We show that we can further suppose that $i_{k+1} \neq i_k$.

Indeed, the fact that $I$ has no maximal elements implies that for every $i \in I$ there exists $i' \in I$ such that

\begin{equation}
    i \leq i' \quad \text{and} \quad i' \neq i.
\end{equation}

Let $i'_1 \in I$ be such that (4.4) holds for $\varepsilon = 2^{-1}$. Take $i_1$ such that $i'_1 \leq i_1$ and $i_1 \neq i'_1$. Let $i'_2 \leq i_2$ be such that (4.4) holds for $\varepsilon = 2^{-2}$ and let $i_2 \in I$ satisfying $i'_2 < i_2$ and $i_2 \neq i'_2$. Then $i_1 \leq i_2$ and $i_2 \neq i_1$, because $i_2 \leq i_1 \leq i'_2$ would contradict the choice of $i_2$.

By induction one obtains a sequence $(i_k)$ in $I$ satisfying $i_k \leq i_{k+1}$ and $i_{k+1} \neq i_k$ such that (4.4) is satisfied with $\varepsilon = 2^{-k}$ for every $i_k$.

We shall again consider two cases.

**Case I.** $\exists j_0 \in I, \exists k_0 \in \mathbb{N}, \forall k \geq k_0, i_k \leq j_0$.

Let $i \geq j_0$. By (4.4) there exists $i' \in I$ such that $i \leq i'$ and $i' \neq i$, implying $d(x_{i'}, x_i) < 2^{-k}$ for all $k \geq k_0$, that is $d(x_{i'}, x_i) = 0$, so that, by $T_1$, $x_{i'} = x_i$.

We also have $i' \neq j_0$ because $i' \leq j_0$ would imply $i' \leq i$, in contradiction to the choice of $i'$.

But then, $d(x_{i'}, x_{j_0}) < 2^{-k}$ for all $k \geq k_0$, so that, as above, $d(x_{i'}, x_i) = 0$ and $x_{i'} = x_{j_0}$.

Consequently, $x_i = x_{j_0}$ for every $i \geq j_0$, proving the convergence of the net $(x_i)$ to $x_{j_0}$.

**Case II.** $\forall j \in I, \forall k \in \mathbb{N}, \exists k' \geq k, i_{k'} \neq j$.

The condition $d(x_{i_k}, x_{i_{k+1}}) < 2^{-k}$, $k \in \mathbb{N}$, implies that the sequence $(x_{i_k})_{k \in \mathbb{N}}$ is right $K$-Cauchy, so that there exists $x \in X$ with $d(x, x_{i_k}) \to 0$ as $k \to \infty$.

For $\varepsilon > 0$ let $k_0 \in \mathbb{N}$ be such that $2^{-k_0} < \varepsilon$ and $d(x, x_{i_k}) < \varepsilon$ for all $k \geq k_0$.

Let $i \geq i_{k_0}$. By II, for $j = i$ and $k = k_0$, there exists $k \geq k_0$ such that $i_k \neq i$. The conditions $k \geq k_0$, $i_k \leq i$, $i_k \leq i_k$ and $i_k \neq i$ imply

\[ d(x, x_{i_k}) < \varepsilon \quad \text{and} \quad d(x_{i_k}, x_i) < \varepsilon, \]

so that

\[ d(x, x_i) \leq d(x, x_{i_k}) + d(x_{i_k}, x_i) < 2\varepsilon, \]

for all $i \geq i_{k_0}$.

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