The property of phase velocity in the cloak shell

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Abstract

According to classical EM field theory, the dispersion relation for anisotropic media in the 2-D cylinder cloak have been calculated in the KDB system (consisting of $\vec{k}$ vector and the DB plane). Due to the special and variable Electromagnetic (EM) parameters, theoretical results indicate that the phase velocities of EM waves are non-uniform in the cloak shell. The numerical simulations show that the phase velocity is reduced, and less than that of light in vacuum in the front and rear of cloak shell. While in the upper and lower region of cloak shell, the phase velocities are lager than that of other regions, and exceed that of light in vacuum.

\textbf{Keywords:} electromagnetic fields, phase velocity, energy flow, superluminal

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1 Introduction

Recently, artificially structured metamaterials have attracted worldwide attention due to their unprecedented flexibility in manipulating electromagnetic (EM) waves and producing new functionalities\cite{1, 2, 3, 4, 5}. One of the novel features is that metamaterials can produce negative permeability and permittivity simultaneously. Due to the negative refraction, there are lots of unusual properties and potential applications, such as the superlens, the energy localization, the super waveguide, and so on\cite{1, 6, 7}. Besides negative refraction and cloaks, more wave manipulation strategies using metamaterials have been drawn out, such as EM field concentrators, rotators, direction changing, illusion optics, and so forth\cite{8, 9, 10, 11, 12, 13, 14, 15, 16}.

We all know that the novel properties of metamaterials in the electromagnetic fields are due to the flexible, variable and anisotropic electromagnetic parameters, and the special materials can be designed and fabricated by man-made periodic and aperiodic structure units. In these peculiar materials, there exist many physical laws which cannot be found in nature materials. One of the unusual peculiarities is propagation properties of EM waves in the metamaterials, such as phase velocity, energy flow velocity, which has been few reported in the literatures. When a plane waves incident on a cloak (Fig. 1a), these fields can be compressed in the shell as required or made to avoid objects to be found, and flow around them like a fluid, returning undisturbed to their original trajectories. In the cloak shell, the EM fields in the inner regions are required to follow a more curved and longer trajectory than that in the outer regions, and all the EM waves in the cloak shell are required the same phase as that in the free space.

In this paper, we have focused on the propagation characteristics of the electromagnetic waves in the general anisotropic medium, and investigated the distribution of phase velocities in the cloak shell. Firstly, the dispersion relation for general anisotropic medium has been deduced in the KD-B system\cite{17} based on classical EM fields theory. Secondly, the phase velocity of in the general
Figure 1: (a) A 2-D cylindrical cloak model along z-direction with inner radius $R_1$ and outer radius $R_2$, which divided space into three region: surrounding media(region 1), cloak shell(region 2), shelter object(region 3). (b) The KDB system established in the rectangular coordinate system(0-xyz), consists of $\vec{k}$ vector($\vec{e}_3$) and the DB plane($\vec{e}_1$ and $\vec{e}_2$).

Anisotropic medium has been derived and analyzed theoretically. Thirdly, the distribution of the phase velocity in cloak shell has been investigated in numerical methods. Metamaterials as man-made materials have some novel properties for variable EM parameters, and make up for lack of nature materials. There are many unknown fundamental processes in physics when EM waves transmit through the metamaterials. In order to comprehend and design the novel devices, it is inevitable to know how the distributions of phase velocity and energy flow velocity of EM waves are in the metamaterials. The investigation of transmitting properties of EM waves in metamaterials not only have extensive applications of the metamaterials in the military, communication and antenna, but also enrich the classical theory of electromagnetic field.

2 Theoretical Analysis

For simplicity, we restrict the problem to 2-D cylindrical cloak in which EM fields are excluded from an infinite circular cylinder with radius $R_1$, as shown in Fig.1. The cloaking region is consisted
of a concentric cylindrical shell with inside radius $R_1$ and outside radius $R_2$, which is filled with the following radius-dependent, anisotropic relative permittivity and permeability (in rectangular coordinates):

$$\bar{\varepsilon}_r = \bar{\mu}_r = \begin{pmatrix} e_{xx} & e_{xy} & 0 \\ e_{yx} & e_{yy} & 0 \\ 0 & 0 & e_{zz} \end{pmatrix}$$

(1)

where $e_{xx} = \frac{x}{\sqrt{R_1^2 - x^2}} - \frac{(2R_1r - R_2^2)x^2}{r^3(R_1 - R_2)}$, $e_{xy} = e_{yx} = -\frac{(2R_1r - R_2^2)y}{r^3(R_1 - R_2)}$, $e_{yy} = \frac{r}{\sqrt{R_1^2 - r^2}} - \frac{(2R_1r - R_2^2)y^2}{r^3(R_1 - R_2)}$ and $e_{zz} = \frac{R_2}{(R_2 - R_1)}^2 \frac{r - R_1}{r}$. The cloak shell separate space into three region: free space(region 1), cloak shell(region 2), sheltered region(region 3).

In order to investigate the EM properties in the anisotropic media, a new coordinated system called KDB system has been established in the rectangular coordinate system, which is consisted of $\vec{k}$ vector and the DB plane(Fig.1b). The KDB system has unit vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$, and $\theta$ is angles between $\vec{e}_3$ and $\vec{z}$, $\phi$ is angles between $\vec{e}_2-\vec{e}_3$ plane and $\vec{x}$. The components of field vectors and EM parameters in the rectangular coordinate system can be transform into a new forms in the KDB system using transition matrix $T$:

$$T = \begin{pmatrix} \sin\phi & -\cos\phi & 0 \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix}$$

(2)

Without loss of generality, a transverse-electric (TE) plane waves along $z-$polarization are incident from the left side of cloak, and the electric field can be written as:$E_{1z} = E_0 e^{i(k_x y)}$, where the time-dependent factor $e^{i\omega t}$ is assumed and omitted in notations. In this case, $\phi = \frac{1}{2}\pi$, the transition
matrix $T$ can be simplified as follows:

$$T = \begin{vmatrix}
1 & 0 & 0 \\
0 & \cos\theta & -\sin\theta \\
0 & -\sin\theta & \cos\theta
\end{vmatrix}$$  \hfill (3)

The relation between a vector $\vec{A}$ in the rectangular coordinate system and the same vector $\vec{A}_k$ in the KDB system is governed by $\vec{A}_k = T\vec{A}$, where subscript $k$ represents a vector in the KDB system. Therefore, the relations of the field vectors in two different systems can be expressed as:

$$\vec{E}_k = \vec{\kappa}_k \vec{D}_k$$  \hfill (4)

$$\vec{H}_k = \vec{\nu}_k \vec{D}_k$$  \hfill (5)

where $\epsilon_0$ and $\mu_0$ are the permittivity and permeability in a vacuum, respectively. $\vec{\kappa}_k$ and $\vec{\nu}_k$ are quantities related to relative permittivity and permeability of anisotropic medium in the KDB system, which are defined as: $\vec{\kappa}_k = T\vec{\kappa}T^{-1}$, $\vec{\nu}_k = T\vec{\nu}T^{-1}$, where $T^{-1}$ is the inverse of $T$. In our system, $\vec{\kappa}_k$ and $\vec{\nu}_k$ can be written as:

$$\vec{\kappa}_r = \vec{\nu}_r = \begin{vmatrix}
e_{kxx} & e_{kxy}\cos\theta & e_{kxy}\sin\theta \\
e_{kxy}\cos\theta & e_{kyy}\cos^2\theta + e_{kzz}\sin^2\theta & 0 \\
e_{kxy}\sin\theta & (e_{kyy} - e_{kzz})\sin\theta\cos\theta & e_{kzz}\cos^2\theta + e_{kyy}\sin^2\theta
\end{vmatrix}$$  \hfill (6)

in which $e_{kxx} = e_{xx}/m$, $e_{kxy} = e_{kyx} = -e_{xy}/m$, $e_{kyy} = e_{yy}/m$, $e_{kzz} = 1/e_{zz}$, and $m = e_{xx}e_{yy} - e_{xy}e_{yx}$.

Within the frame of KDB system, the Maxwell equations for the TE plane waves take the same form as those in the rectangular coordinate system. Due to wave vector $\vec{k}$ in the $\hat{e}_3$ direction, it is easy to have $\vec{B}_{k\hat{e}_3} = \vec{D}_{k\hat{e}_3} = 0$ according to $\vec{k} \cdot \vec{B}_k = 0$ and $\vec{k} \cdot \vec{D}_k = 0$. Based on $\vec{k} \times \vec{E}_k = \omega \vec{B}_k$ and $\vec{k} \times \vec{H}_k = -\omega \vec{D}_k$, the EM fields components of $\hat{e}_1$ and $\hat{e}_2$ direction can be rearranged and written
in the matrix form:

\[
\begin{bmatrix}
\kappa_{11} & \kappa_{12} \\
\kappa_{21} & \kappa_{22}
\end{bmatrix}
\begin{bmatrix}
D_{k1} \\
D_{k2}
\end{bmatrix}
= \frac{\epsilon_0}{\omega}
\begin{bmatrix}
0 & -k \\
k & 0
\end{bmatrix}
\begin{bmatrix}
B_{k1} \\
B_{k2}
\end{bmatrix}
\]

(7)

\[
\begin{bmatrix}
\nu_{11} & \nu_{12} \\
\nu_{21} & \nu_{22}
\end{bmatrix}
\begin{bmatrix}
B_{k1} \\
B_{k2}
\end{bmatrix}
= \frac{\mu_0}{\omega}
\begin{bmatrix}
0 & k \\
-k & 0
\end{bmatrix}
\begin{bmatrix}
D_{k1} \\
D_{k2}
\end{bmatrix}
\]

(8)

where \( \omega \) is angular frequency. Eliminating \( B_k \) or \( D_k \) in the equation (7) and (8), and defining \( c_0 = 1/\sqrt{\epsilon_0 \mu_0} \) (the speed of light in a vacuum), the dispersion relation for anisotropic media can be obtained:

\[
\omega^4 - c_0^2(a_1 k_z^2 + b_1 k_s^2) \omega^2 + c_0^4(c_1 k_z^4 + d_1 k_s^4 + e_1 k_z^2 k_s^2) = 0
\]

(9)

where \( k_z = k \cos \theta \) and \( k_s = k \sin \theta \) are wave numbers in \( z \)-direction and transverse direction, respectively, and \( a_1 = 2(e_{kxx}^2 - e_{kxy}^2) - 2e_{kzz}^2, b_1 = e_{kxx} e_{kzz}, c_1 = 2(e_{kxx}^2 - e_{kzy}^2), d_1 = (e_{kxx} e_{kzz})^2, \) and \( e_1 = 2(e_{kxx}^2 - e_{kzy}^2)(e_{kxx} e_{kzz}). \) For given angular frequency \( \omega \), there are four solutions for \( k \) in the equation (9), which are corresponding to the type 1 wave and the type 2 wave of forward propagation and backward propagation in the general anisotropic media. If the \( e_{xy} = e_{yx} = 0 \) in the formula 1, the general anisotropic media is reduces to biaxial anisotropy medium. The equation (9) can be simplified as follows:

\[
[\omega^2 - c_0^2 \left( \frac{1}{e_{xx}} k_z^2 + \frac{1}{e_{zz}} k_s^2 \right)] [\omega^2 - c_0^2 \left( \frac{1}{e_{xx}} k_z^2 + \frac{1}{e_{zz}} k_s^2 \right)] = 0
\]

(10)

This equation can be decomposed into the two familiar dispersion relations of TE mode and TM mode waves.

Applying phase velocity \( u = \omega/k \) and formula (9), the phase speed in the anisotropic media can be obtained:

\[
v^4 - \left( \frac{c_0}{k} \right)^2(a_1 k_z^2 + b_1 k_s^2) v^2 + \left( \frac{c_0}{k} \right)^4(c_1 k_z^4 + d_1 k_s^4 + e_1 k_z^2 k_s^2) = 0
\]

(11)
The equation (11) shows the different anisotropic parameters decide different phase velocities in the medium. According to formula (1), the EM parameters in the cloak are the functions of the positions, then the phase velocities in the cloak shell vary with positions.

3 Simulation Results

In order to calculate the phase velocity in cloak and around medium, full-wave simulations have been carried out. We choose a square calculating region with $-2m \leq x \leq 2m$ and $-2m \leq y \leq 2m$,.
and a 2-D cylindrical cloak with an inner radius \( R_1 = 0.8 \text{m} \) and outer radius \( R_2 = 1.1 \text{m} \) is located at the center of square region. A plane wave is incident from left boundary of calculating region with frequency \( f = 1 \text{GHz} \).

If a plane wave is incident from free space into a cloak, as we know, the impedance of metamaterials in the cloak shell, which parameters are acquired by method of transformation optics, match with surrounding materials (free space) in essence. Thus, for the TE-polarized incident wave \( (E_{1z} = E_0 e^{ik_0y}) \), the \( z \)-components of the electromagnetic field in the region 1 (free space: isotropic media) and region 2 (cloak shell: anisotropic media) in the cylindrical coordinates can be written in the following forms:

\[
E_{1z} = \sum_{n=1}^{\infty} i^{-n} E_0 J_n(k_0 r) e^{in\theta} \quad (12)
\]

\[
E_{2z} = \sum_{n=1}^{\infty} i^{-n} E_0 J_n(mk_0(r - R_1)) e^{in\theta} \quad (13)
\]

where \( k_0 \) is wave number in the free space, \( J_n \) is \( n \)-order bessel function, and \( m = \frac{R_2}{R_2 - R_1} \).

Fig. 2a shows the distribution of electric field \( E_z \) of TE mode EM field, where we choose \( E_0 = 1 \).

When the plane wave passes through the region of cloak, the wave is compressed in the cloak shell, and diffracted smoothly the cloak region. Because the impedance of metamaterial in the cloak region is matched with around media, there is no reflected wave from cloak shell. Therefore, the shape of plane wave can be maintained when the plane wave transmit through the cloak.

In the cloak shell, the equiphase surface of the plane wave is distorted by the shelved object in the center of circular cylinder, but the EM wave is well guided and diffracted smoothly through the cloak shell by the special metamaterial.

In free space, as is well-known, the phase velocity is uniform, and equal to the speed of light in vacuum \( (3 \times 10^8 \text{m/s}) \). While in the cloak shell, the equiphase surface is not a plane, that is, the
phase velocities in the same equiphase surface are not equivalence and non-uniform distribution. According to formula (11), the distribution of phase velocities in the $x-y$ plane are shown in the Fig.(2b). In the front and the rear of cloak shell, the EM waves are compressed in the cloak shell, and the equiphase surfaces lag behind those in the free space. At the same time, in the two regions, the phase velocity is reduced, and less than that of the velocity of light in vacuum. Fig.3a shows that the distribution of phase velocity along $y-$direction at $x = 1m$. Between $y = -0.5m$ and $y = 0.5m$, the phase velocity is only $0.6 - 0.7$ times speed of light in vacuum. We also can see the lower the phase velocity is, the closer to outer boundary the positions are in the front and the rear of cloak shell( Fig.(2b)). However, in the middle region(the upper and the lower of circle cylinder along direction of propagation) of cloak shell, the phase velocities are larger than that of other region, and exceed that of light in vacuum(Fig.2b). In the region, the phase velocity is sped up to make up the velocity loss in the front and rear of cloak shell. Fig.3b shows that the distribution of phase velocity along $x-$direction at $y = 1m$. Between $x = -0.5m$ and $x = 0.5m$, the phase velocity reaches $1.5 - 3.5$ times velocity of light in vacuum. Also, the phase velocity is not non-uniform distributions, and the phase velocity increase sharply near the inner boundary of cloak shell( Fig.(2b)).

Although the superluminal phenomena is rare in the normal nature medium, the special phenomena can be appeared in the anomalous dispersion medium. Based on Einstein’s special theory of relativity, the speed of a moving object can not exceed that of light in vacuum. Because the speed of moving object in the Einstein’s special theory of relativity is related to the speed of energy flow, the superluminal phenomena of phase velocity in the the anomalous dispersion medium is not inconsistent with it. In the isotropic nondispersive media, the phase velocity of EM waves can be simplified as $v = \frac{1}{\sqrt{\mu \epsilon}}$, in which $\epsilon$ and $\mu$ are the EM parameters of medium. Because EM parameters in the free space is less than that of other media, the velocity of the phase in the general media is less than the velocity of light in vacuum. Metmaterial as a special man-made materials, has variable
EM parameters (Formula (1)), which can be designed and fabricated by structure units arranged periodically or aperiodically. The special variable EM parameters can be larger or less than the values of nature materials, even equal to zero or less than zero. According to classical theories on electromagnetic field, the less EM parameters are, the larger of the phase velocity and energy flow of EM waves are. Thus, when the EM parameters in the metamaterials are less than that of nature materials, the faster-than-light EM waves in the metamaterials are common phenomena.

4 Conclusion

In this paper, the dispersion relation and the phase velocity for anisotropic media in the 2-D cylinder cloak have been calculated in the $KDB$ system according to classical EM field theory. The theoretical analysis and simulations show that there is superluminal phenomena in the cloak shell consisting of metamaterials. In the upper and the lower region of cloak shell, the phase velocities are larger than that of other region, and exceed that of light in vacuum. While in the front and the rear of cloak shell, the phase velocity is reduced and less than that of light in vacuum. The investigation of transmitting properties of EM waves in metamaterials not only have extensive applications in the military, communication and antenna, but also enrich the classical theory of electromagnetic field.

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