INTERSECTION PROPERTIES OF TYPICAL COMPACT SETS

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ABSTRACT. We prove that a typical compact set does not contain any similar copy of a given pattern. We also prove that a typical compact set of $[0, 1]^d$ $(d \geq 2)$ intersects any $(d - 1)$-dimensional plane in at most $d$ points. We study the “hitting probabilities” of compact sets in the sense of Baire category. In the end we study the arithmetic properties of typical compact sets in $[0, 1]$ and the “hitting probabilities” of continuous functions.

1. Introduction

A subset of a metric space $X$ is of first category if it is a countable union of nowhere dense sets (i.e. whose closure in $X$ has empty interior); otherwise it is called of second category. We say that a typical element $x \in X$ has property $P$, if the complement of

$$\{x \in X : x \text{ satisfies } P\}$$

is of first category. For the basic properties and various applications of Baire Category, we refer to [13, 17]. Let $\mathcal{K} = \mathcal{K}([0, 1]^d)$ be all the compact subsets of unite cube $[0, 1]^d$. We endow $\mathcal{K}$ with Hausdorff metric. Recall that the Hausdorff distance of two compact sets $E$ and $F$ of $\mathcal{K}$ is defined by

$$d_H(E, F) = \inf\{\varepsilon > 0 : E \subset F^{\varepsilon} \text{ and } F \subset E^{\varepsilon}\},$$

where $E^{\varepsilon} = \{x \in \mathbb{R}^d : \text{dist}(x, E) < \varepsilon\}$.

Davies, Mastrand and Taylor [5] constructed a compact set $A \subset [0, 1]$ with Hausdorff dimension zero containing a similar copy of any finite set. Chen and Rossi [4] showed that a typical compact set is locally rich which means that we can “see” all the compact sets when we zooming in at any point of this compact set. Feng and Wu [7] proved that a typical compact set has Hausdorff dimension zero. It is natural to ask
that does a typical compact set containing a similar copy of any finite set. We have the following negative answer.

**Theorem 1.1.** A typical compact set does not contain a similar copy of a given set $P$ with three distinct points.

Note that the Lebesgue density theorem implies that any set of $\mathbb{R}^d$ with positive Lebesgue measure contains a similar copy of any finite set. However, Keleti [8, 9] constructed an 1-dimensional compact set that does not contain the non-trivial 3-term arithmetic progressions. Recently, Shmerkin [15] constructed an 1-dimensional Salem set without 3-term arithmetic progressions also. For more backgrounds and further results we refer to [1, 3, 10, 16]. For the basic properties of Hausdorff dimension we refer to [6, 12].

It is not hard to see that if the complement of $A \subset \mathbb{R}^d$ is of first category, then $A$ contains a similar copy of any countable set. This follows by the fact that for any countable set $\{t_i \in \mathbb{R}^d : i \in \mathbb{N}\}$, the intersection $\bigcap_{i \in \mathbb{N}} (A + t_i)$ is not empty. Note that $A$ is not a compact set. However, there exists a second category set $E$ in the plane such that any line intersects $E$ in at most two points, see [13, Theorem 15.5]. For a typical compact set of $\mathcal{K}$ we have the following result.

**Theorem 1.2.** A typical compact set of $\mathcal{K}([0,1]^d)(d \geq 2)$ intersects any $(d-1)$-dimensional plane in at most $d$ points.

Let $A \subset [0,1]^d$ and $\mathcal{K}_A = \{E \in \mathcal{K} : E \cap A \neq \emptyset\}$. It is reasonable to think that if $A$ is a “small” set in $[0,1]^d$ then $\mathcal{K}_A$ will be a “small” set in $\mathcal{K}$ also.

**Theorem 1.3.** A set $A \subset [0,1]^d$ is nowhere dense in $[0,1]^d$ if and only if $\mathcal{K}_A$ is nowhere dense in $\mathcal{K}$.

If $A \subset [0,1]^d$ is of first category in $[0,1]^d$, then $A = \bigcup_{i \in \mathbb{N}} A_i$ where each $A_i$ is nowhere dense in $[0,1]^d$. Observe that

$$\mathcal{K}_A = \bigcup_{i \in \mathbb{N}} \mathcal{K}_{A_i}.$$  

Theorem 1.3 claims that $\mathcal{K}_{A_i}$ is nowhere dense in $\mathcal{K}$ for each $i \in \mathbb{N}$, and hence $\mathcal{K}_A$ is of first category in $\mathcal{K}$. It follows that a typical compact set of $\mathcal{K}$ does not intersects $A$. Šalát [14] proved that the set of *normal numbers* is of first category. It is also known that the complementary set of *Liouville numbers* is of first category, see [13, Chapter 2]. Thus we obtain that a typical compact set of $\mathcal{K}([0,1])$ is a subset of non-normal Liouville numbers. We collect these facts as the following corollary.
Corollary 1.4. (a) If $A \subset [0, 1]^d$ is of first category in $[0, 1]^d$ then $\mathcal{K}_A$ is of first category in $\mathcal{K}$.

(b) A typical compact set of $\mathcal{K}([0, 1])$ is a subset of non-normal Liouville numbers.

We do not know that whether $\mathcal{K}_A$ is of first category implies that $A$ is of first category.

In the following, we study the size of sets formulated under finite steps arithmetic operations of a typical set $A$ of $\mathcal{K}([0, 1])$. Let $S^m(A)$ be the $m$-th sum set of $A$, and $\tilde{P}(A)$ be a set formed under the rule of the polynomial $P$. We show these definitions in Section 4. Under these notations we have the following result.

Theorem 1.5. For a typical compact set $A$ of $\mathcal{K}([0, 1])$, we have that $\dim H S^m(A) = 0$ for any $m \in \mathbb{N}$ and $\dim H \tilde{P}(A) = 0$ for any polynomial $P$.

The paper is organized as follows. Theorems 1.1 and 1.2 are proved in section 2. Theorem 1.3 is proved in section 3. Theorem 1.5 is proved in Section 4. In the end we study the “hitting probabilities” of continuous function.

2. Proofs of Theorems 1.1 and 1.2

Let $A, B \subset \mathbb{R}^d$. If there exist $\lambda > 0$ and an isometric map $\varphi$ on $\mathbb{R}^d$ such that $\varphi(\lambda A) = B$, then we say that $A$ is similar to $B$ and denote this by $A \sim B$. If there is a subset $A' \subset A$ such that $A' \sim B$, then we say that $A$ contains a similar copy of $B$. For each $n \in \mathbb{N}$, let $\mathcal{D} = \bigcup_{n \in \mathbb{N}} \mathcal{D}_n$ where $\mathcal{D}_n$ is the family of $2^n$-adic closed subcubes of $[0, 1]^d$, i.e.

$$\mathcal{D}_n = \left\{ \prod_{k=1}^d [i_k 2^{-n}, (i_k + 1)2^{-n}] : 0 \leq i_k \leq 2^n - 1 \right\}.$$  

For a set $A \subset \mathbb{R}^d$, denote by $\partial A$ the boundary of $A$, denote by $|A|$ the diameter of $A$. For two points $x, y \in \mathbb{R}^d$, denote by $|x - y|$ the Euclidean metric. Denote by $\mathcal{L}^d$ the $d$-dimensional Lebesgue measure. Let $\{x_1, x_2, x_3\} \subset \mathbb{R}^d$ be three distinct points. Define

$$R(x_1, x_2, x_3) = \left\{ \frac{|x_i - x_k|}{|x_j - x_k|} : i \neq j, j \neq k, i \neq k \right\}.$$  

Lemma 2.1. Let $P = \{p_1, p_2, p_3\}$ be three distinct points of $\mathbb{R}^d$ and $a, b \in \mathbb{R}^d$, $a \neq b$. Then $\mathcal{L}^d(P') = 0$, where

$$P' = \{ x \in \mathbb{R}^d : P \sim \{a, b, x\} \}.$$
Proof. If $P \sim \{a, b, x\}$, then $R(p_1, p_2, p_3) = R(a, b, x)$. It follows that there are at most $N = N(d)$ balls $\{B_i\}_{i=1}^N$ such that $P' \subset \bigcup_{i=1}^N \partial B_i$, and hence $L^d(P') = 0$. \hfill \Box

Lemma 2.2. Let $P = \{p_1, p_2, p_3\}$ be three distinct points of $[0, 1]^d$. Then for any $n \in \mathbb{N}$, there exists $\Gamma_n = \{x_Q : Q \in \mathcal{D}_n\}$ with $x_Q \in Q$ such that any three distinct points of $\Gamma_n$ is not similar to $P$. Moreover there exists $\varepsilon = \varepsilon_n$ such that the following two conditions hold.

\begin{enumerate}
\item [(C1)] $B(x_Q, \varepsilon) \subset Q$ for each $Q \in \mathcal{D}_n$.
\item [(C2)] For any $\{a_1, a_2, a_3\} \subset \bigcup_{Q \in \mathcal{D}_n} B(x_Q, \varepsilon)$ which is similar to $P$, there exists $Q \in \mathcal{D}_n$ such that $\{a_1, a_2, a_3\} \subset B(x_Q, \varepsilon)$.
\end{enumerate}

Proof. Let $\mathcal{D}_n = \{(Q_i : 1 \leq i \leq 2^n)\}$. Assume that we have chosen $m$ points $K_m = \{x_i : 1 \leq i \leq m\}$ with $x_i \in Q_i$, $1 \leq i \leq m$ such that any three distinct points of $K_m$ is not similar to $P$. For any two points $x_i, x_j$ of $K_m$, by Lemma 2.1 we obtain that the set

$$\{x \in \mathbb{R}^d : P \sim \{x_i, x_j, x\}\}$$

has Lebesgue measure zero. Note that there are at most $m(m-1)/2$ pairs of $(i, j)$. It follows that there exists an interior point $x_{m+1}$ of $Q_{m+1}$ such that any three points of $\{x_i : 1 \leq i \leq m+1\}$ is not similar to $P$. We use the same way to find points $x_{m+2}, \ldots$. In the end, we obtain a point $x_{2dn}$ from $Q_{2dn}$. Let $\Gamma_n$ be the collection of chosen points.

Since each $x_i$ is an interior point of $Q_i$, $1 \leq i \leq 2^n$, there is $\varepsilon' > 0$ such that the condition $C_1$ holds. Observe that for any three distinct points $\{x_{i_1}, x_{i_2}, x_{i_3}\} \subset \Gamma_n$, there is a positive constant $\varepsilon_{i_1, i_2, i_3}$ such that any three points $\{a_1, a_2, a_3\}$ with $a_k \in B(x_{i_k}, \varepsilon_{i_1, i_2, i_3})$, $k = 1, 2, 3$ is not similar to $P$. Let $\varepsilon''$ be the minimal value over all the possible $\varepsilon_{i_1, i_2, i_3}$, and $\varepsilon = \min\{\varepsilon', \varepsilon''\}$. Thus we complete the proof. \hfill \Box

Definition 2.3. Recall that the points $\{x_1, x_2, \ldots, x_m\}$ are called affinely independent if the vectors $x_2 - x_1, \ldots, x_m - x_1$ are linearly independent. Let $A \subset \mathbb{R}^d, d \geq 2$. We say $A$ is affinely independent if any $k(3 \leq k \leq d + 1)$ distinct points of $A$ is affinely independent.

Let $A \subset \mathbb{R}^d, d \geq 2$ be a set that intersects any $(d-1)$-dimensional plane in at most $d$ points. Observe that this is equivalent to say that any $(d + 1)$ points $\{x_1, \ldots, x_{d+1}\} \subset A$ is affinely independent.

Lemma 2.4. For each $\mathcal{D}_n$, $n \in \mathbb{N}$, there exists $\Gamma_n = \{x_Q : Q \in \mathcal{D}_n\}$ with $x_Q \in Q$ such that $\Gamma_n$ is affinely independent. Moreover there exists $\varepsilon = \varepsilon_n$ such that the following two conditions hold.

\begin{enumerate}
\item [(C1)] $B(x_Q, \varepsilon) \subset Q$ for each $Q \in \mathcal{D}_n$.
\end{enumerate}
\((C_2)\) For any \(\{a_1, \ldots, a_{d+1}\} \subset \bigcup_{Q \in \mathcal{D}_n} B(x_Q, \varepsilon)\) which is not affinely independent, there exists \(Q \in \mathcal{D}_n\) and \(\{a_i, a_j\} \subset \Gamma_n\) such that \(\{a_i, a_j\} \subset B(x_Q, \varepsilon)\).

Proof. Let \(\mathcal{D}_n = \{Q_i : 1 \leq i \leq 2^n\}\). Assume that we have chosen \(m\) points \(K_m = \{x_i : 1 \leq i \leq m\}\) with \(x_i \in Q_i, 1 \leq i \leq m\) such that \(K_m\) is affinely independent. Observe that for any \(d\) points \(x_{ik}, 1 \leq k \leq d\) of \(K_m\), the set
\[
\{x \in \mathbb{R}^d : \{x\} \cup \{x_{ik} : 1 \leq k \leq d\} \text{ is not affinely independent}\}
\]
has Lebesgue measure zero. Note that there are at most finite elements of \((i_1, i_2, \ldots, i_d)\). It follows that there exists an interior point \(x_{m+1}\) of \(Q_{m+1}\) such that \(\{x_i : 1 \leq i \leq m + 1\}\) is affinely independent. We use the same way to find points \(x_{m+2}, \ldots\). In the end, we obtain a point \(x_{2dn}\) from \(Q_{2dn}\). Let \(\Gamma_n\) be the collection of chosen points.

Since each \(x_i\) is an interior point of \(Q_i, 1 \leq i \leq 2^n\), there is \(\varepsilon' > 0\) such that the condition \(C_1\) holds. Observe that for any \(d + 1\) distinct points \(\{x_{i_1}, \ldots, x_{i_{d+1}}\} \subset \Gamma_n\), there is a positive constant \(\varepsilon_{i_1, \ldots, i_{d+1}}\) such that any \(\{a_1, \ldots, a_{d+1}\}\) with
\[
\varepsilon_{i_k} = B(x_{i_k}, \varepsilon_{i_1, \ldots, i_{d+1}}), 1 \leq k \leq d + 1
\]
is not affinely independent. Let \(\varepsilon''\) be the minimal value over all the possible \(\varepsilon_{i_1, \ldots, i_{d+1}}\) and \(\varepsilon = \min\{\varepsilon', \varepsilon''\}\). □

In fact we can also choose the sets \(\Gamma_n\) of Lemma 2.2 and Lemma 2.4 in a probability way. For each \(Q \in \mathcal{D}_n\), we randomly choose a point \(x_Q \in Q\) under the law of uniform distribution. The choices are independent for different cubes of \(\mathcal{D}_n\). Denote by \(\Gamma^\omega_n\) the random chosen points. It is not hard to show that with probability one \(\Gamma^\omega_n\) has the same properties as \(\Gamma_n\). We show the outline for this argument.

Proposition 2.5. With probability one \(\Gamma^\omega_n\) has the same properties as \(\Gamma_n\) in Lemma 2.2 and Lemma 2.4.

Proof. Let \(\mathcal{D}_n = \{Q_1, Q_2, \ldots, Q_{2^n}\}\). Lemma 2.1 implies that conditional on \(x_1 \in Q_1, x_2 \in Q_2\), the probability of the event \(P \sim \{x_1, x_2, x_3\}\) is zero. Therefore we have that \(\mathbb{P}(P \sim \{x_1, x_2, x_3\}) = 0\). It follows that
\[
\mathbb{P}(\text{exists } \{x_i, x_j, x_k\} \subset \Gamma^\omega_n \text{ such that } \{x_i, x_j, x_k\} \sim P) \leq \sum_{i,j,k} \mathbb{P}(\{x_i, x_j, x_k\} \sim P) = 0.
\]

Thus we obtain that with probability one any three points of \(\Gamma^\omega_n\) is not similar to \(P\).
Observe that
\[ P(\{x_1, x_2, x_3\} \text{ is affinely independent} ) = 1. \]
Let \( A_k \) be the event
\[ \{x_1, x_2, \cdots, x_k\} \text{ is affinely independent}, \ 3 \leq k \leq d + 1. \]
Then it is not hard to see that \( P(A_{k+1} | A_k) = 1 \), and \( P(A_{k+1}^c | A_k) = 0 \).
Thus we have
\[ P(A_{d+1}) = P(A_{d+1} | A_d) P(A_d) + P(A_{d+1} | A_d^c) P(A_d^c) \]
\[ = P(A_d) = \cdots = P(A_3) = 1. \]
It follows that \( P(A_{d+1}^c) = 0 \), and thus
\[ \mathbb{P}(\Gamma_n^c \text{ is not affinely independent} ) \leq \sum_{i_1, \cdots, i_{d+1}} P(\{x_{i_1}, \cdots, x_{i_{d+1}}\} \text{ is not affinely independent} ) = 0. \]
Since the boundary of cube has Lebesgue measure zero, we obtain that with probability one one \( x_i \) is an interior point of \( Q_i, 1 \leq i \leq 2^n. \)

Proof of Theorem 1.1. Let \( P = \{p_1, p_2, p_3\} \subset [0, 1]^d \). For each \( n \in \mathbb{N} \), let \( \Gamma_n \) be the set in Lemma 2.2 and \( P_n \) be the power set of \( \Gamma_n \). Recall that the power set of a set \( X \) is the collection of all the subset of \( X \).
Let
\[ \mathcal{G} = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \bigcup_{\gamma \in P_n} U_{d_H}(\gamma, \varepsilon_n), \]
where \( U_{d_H}(\gamma, \varepsilon_n) \) is an open set of \( (\mathcal{K}, d_H) \) with center \( \gamma \) and radius \( \varepsilon_n \).
Note that \( \{\gamma : \gamma \in P_n, n \in \mathbb{N}\} \) is a countable dense subset in \( \mathcal{K} \). Thus
\[ \bigcup_{n=k}^{\infty} \bigcup_{\gamma \in P_n} U_{d_H}(\gamma, \varepsilon_n) \]
is a dense open set in \( \mathcal{K} \). It follows that the complementary set of \( \mathcal{G} \) is of first category. In the following we intend to show that any element of \( \mathcal{G} \) does not contain a similar copy of \( \{p_1, p_2, p_3\} \).

Let \( E \in \mathcal{G} \), then there exist \( n_k \nearrow \infty \) and \( \gamma_{n_k} \in P_{n_k} \) such that \( E \in \bigcap_{k=1}^{\infty} U_{d_H}(\gamma_{n_k}, \varepsilon_{n_k}). \) Suppose that there is \( \{x_1, x_2, x_3\} \subset E \) which is similar to \( F \). By the condition \( C_2 \) of Lemma 2.2 there is \( Q \in D_{n_k} \) such that
\[ \{x_1, x_2, x_3\} \subset B(x_Q, \varepsilon_{n_k}), \]
and hence
\[ |\{x_1, x_2, x_3\}| \leq 2\varepsilon_{n_k} \leq \sqrt{d}2^{-n_k} \rightarrow 0. \]
This is a contradiction. Thus we complete the proof. \( \square \)
Proof of Theorem 1.2. For each \( n \in \mathbb{N} \), let \( \Gamma_n \) be the set in Lemma 2.4 and \( P_n \) be the power set of \( \Gamma_n \). Let
\[
G = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \bigcup_{\gamma \in P_n} U_{d_H}(\gamma, \varepsilon_n).
\]
Then the complementary set of \( G \) is of first category.

Let \( E \in G \), then there exist \( n_k \uparrow \infty \) and \( \gamma_{n_k} \in P_{n_k} \) such that \( E \in \bigcap_{k=1}^{\infty} U_{d_H}(\gamma_{n_k}, \varepsilon_{n_k}) \). Suppose that \( E \) is not affinely independent. Thus there exists \( \{a_1, \ldots, a_{d+1}\} \subset E \) such that \( \{a_1, \ldots, a_{d+1}\} \) is not affinely independent. For each \( n_k \), there exists \( \gamma \in \Gamma_{n_k} \) such that \( \{a_1, \ldots, a_{d+1}\} \subset \bigcup_{x \in \gamma} B(x, \varepsilon_{n_k}) \).

By the condition \( C_2 \) of Lemma 2.4, we obtain that there exists two distinct points \( a_i, a_j \) with \( |a_i, a_j| \leq 2\varepsilon_{n_k} \). Note that we may choose \( \varepsilon_{n_k} \) such that \( \varepsilon_{n_k} \searrow 0 \). It follows that there exist two points of \( \{a_1, \ldots, a_{d+1}\} \) with distance zero which is a contradiction. \( \square \)

Remark 2.6. Let \( E \subset \mathbb{R}^d, d \geq 2 \). We say \( E \) contains the angle \( \theta \) if there are three points \( \{x, y, z\} \subset E \) such that the angle between the vectors \( y - x \) and \( z - x \) is \( \theta \), and write \( \angle \theta \in E \). For some results on this topic and further references we refer to [3, 16].

Let \( \theta \in [0, \pi) \) and \( a, b \in \mathbb{R}^d, d \geq 2 \). Then by some elementary geometric arguments, we have
\[
\mathcal{L}^d(\{x \in \mathbb{R}^d : \angle \theta \in \{a, b, x\}\}) = 0.
\]
It follows that for each \( n \in \mathbb{N} \), there is \( \Gamma_n = \{x_Q : Q \in D_n\} \) such that \( \Gamma_n \) does not contain the angle \( \theta \). Applying the similar argument in the proofs of Theorems 1.1 and 1.2, we obtain that a typical compact set of \( \mathcal{K}(\{0, 1\}^d), d \geq 2 \) does not contain the angle \( \theta \). We omit the details here.

3. Proof of Theorem 1.3

Proof of Theorem 1.3. Suppose \( A \) is a nowhere dense subset of \( [0, 1]^d \). Let \( E \in \mathcal{K}, \varepsilon = 2^{-n} \sqrt{d} \). Assume first that \( E \cap A \neq \emptyset \). We define
\[
\mathcal{E}_n = \{Q \in D_n : Q \cap E \neq \emptyset\} = \mathcal{E}'_n \cup \mathcal{E}''_n
\]
where
\[
\mathcal{E}'_n = \{Q \in \mathcal{E}_n : Q \cap A = \emptyset\}, \quad \mathcal{E}''_n = \mathcal{E}_n \setminus \mathcal{E}'_n.
\]
For every \( Q \in \mathcal{E}'_n \), let \( c_Q \) be the center point of \( Q \). For every \( Q \in \mathcal{E}''_n \), since \( A \) is nowhere dense, there exists \( x_Q \in Q, r_Q > 0 \) such that
\[
U(x_Q, r_Q) \subset Q \text{ and } U(x_Q, r_Q) \cap A \neq \emptyset.
\]
Let $F$ be the collection of points $c_{Q}$ for $Q \in \mathcal{E}_{n}'$ and $x_{Q}$ for $Q \in \mathcal{E}_{n}''$. Then $F \in U_{dH}(E, 2^{-n}\sqrt{d})$. Let

$$
\varepsilon' = \min\{r_{Q} : Q \in \mathcal{E}_{n}''\}.
$$

Then $U_{dH}(F, \varepsilon') \cap \mathcal{K}_{A} = \emptyset$.

For the case $E \cap A = \emptyset$ we have that $\mathcal{E}_{n}'' = \emptyset$. Let $F$ be the collection of points $c_{Q}$ for $Q \in \mathcal{E}_{n}'$. Then

$$
F \in U_{dH}(E, 2^{-n}\sqrt{d}) \text{ and } U_{dH}(F, 2^{-n-1}) \cap \mathcal{K}_{A} = \emptyset.
$$

By the arbitrary choice of $E \in \mathcal{K}$ and $\varepsilon = 2^{-n}\sqrt{d}$, we obtain that $\mathcal{K}_{A}$ is nowhere dense in $\mathcal{K}$.

Now we assume that there is an open ball $U \subset \overline{A}$ where $\overline{A}$ is the closure of $A$. Let $\mathcal{K}(U)$ be all the compact subsets of $U$, then $\mathcal{K}(U)$ is an open set in $\mathcal{K}$. Observe that $\mathcal{K}(U) \subset \overline{\mathcal{K}_{A}}$. Thus we obtain that if $\mathcal{K}_{A}$ is nowhere dense in $\mathcal{K}$ then $A$ is nowhere dense in $[0, 1]^{d}$. \hfill \square

4. Proof of Theorem 1.5

For $A, B \subset \mathbb{R}$ we define their sum set

$$
A + B = \{a + b : a \in A, b \in B\}.
$$

Let $\lambda \in \mathbb{R}$ and $\lambda A = \{\lambda \times a : a \in A\}$. For $m \in \mathbb{N}$, define

$$
S^{m}(A) := \left\{ \sum_{i=1}^{m} x_{i} : x_{i} \in A, 1 \leq i \leq m \right\},
$$

$$
A^{m} = \{(x_{1}, \ldots, x_{m}) : x_{i} \in A, 1 \leq i \leq m\},
$$

and

$$
T^{m}(A) := \left\{ x_{1} \times \cdots \times x_{m} : x_{i} \in A, 1 \leq i \leq m \right\}.
$$

Let $P(x) = \sum_{k=0}^{n} a_{k}x^{k}, a_{k} \in \mathbb{R}$ be a polynomial. For a set $A \subset \mathbb{R}$, let

$$
\tilde{P}(A) := \sum_{k=0}^{n} a_{k}T^{k}(A).
$$

Note that $\tilde{P}(A)$ is the sum set of $\{a_{k}T^{k}(A)\}_{k=0}^{n}$. The Hausdorff dimension of $E$ is defined as

$$
\dim_{H} E = \inf\{s \geq 0 : \mathcal{H}^{s}(E) = 0\},
$$

where $\mathcal{H}^{s}(E) = \lim_{\delta \to 0} \mathcal{H}_{\delta}^{s}(E)$, and

$$
\mathcal{H}_{\delta}^{s}(E) = \inf \left\{ \sum_{i=1}^{\infty} |U_{i}|^{s} : E \subset \bigcup_{i \in \mathbb{N}} U_{i}, |U_{i}| \leq \delta, i \in \mathbb{N} \right\}.
$$
For each \( n \in \mathbb{N} \), let \( \varepsilon_n = 2^{-n^2} \),
\[
\mathcal{D}'_n = \left\{ \frac{k}{2^n} : 0 \leq k \leq 2^n \right\},
\]
and \( P_n \) be the power set of \( \mathcal{D}'_n \). Define
\[
\mathcal{G} = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \bigcup_{\gamma \in P_n} U_{d_H}(\gamma, \varepsilon_n).
\]
Applying the same argument as in the proof of Theorem 1.1, we have that the complement of \( \mathcal{G} \) is of first category in \( \mathcal{K} \).

**Lemma 4.1.** Let \( A \in \mathcal{G} \) then \( \dim H A^m = 0 \) for any \( m \in \mathbb{N} \).

**Proof.** For any \( k \in \mathbb{N} \) there exist \( n \geq k \) and 
\[
\gamma = \{x_1, \cdots, x_N\} \in P_n
\]
such that \( A \in U_{d_H}(\gamma, \varepsilon_n) \). It follows that 
\[
A \subset \bigcup_{i=1}^{N} B_i
\]
where \( B_i := B(x_i, \varepsilon_n), 1 \leq i \leq N. \) Let \( m \in \mathbb{N} \), then 
\[
A^m \subset \bigcup_{i_1, \cdots, i_m \in \mathcal{I}^m} B_{i_1} \times \cdots \times B_{i_m}
\]
where \( \mathcal{I} = \{1, 2, \cdots, N\} \). Note that
\[
|B_{i_1} \times \cdots \times B_{i_m}| \leq \sqrt{m} \varepsilon_n \text{ for any } i_1, \cdots, i_m \in \mathcal{I}^m.
\]
Since \( N \leq 2^n + 1 \) and \( \varepsilon_n = 2^{-n^2} \), for any \( s > 0 \) we have 
\[
\mathcal{H}^s_{\varepsilon_n \sqrt{m}}(A^m) \leq N^m (\varepsilon_n \sqrt{m})^s \leq 2^{n+1}2^{-n^2s} \sqrt{m}^s.
\]
It follows that \( \mathcal{H}^s(A^m) = 0 \). By the arbitrary choice of \( s > 0 \) we have that \( \dim H A^m = 0 \). Thus we complete the proof. \( \square \)

**Proof of Theorem 1.5.** It is clear that \( S^m(A) = \sqrt{m} \pi_e(A^m) \) where \( \pi_e(A^m) \) is the orthogonal projection of \( A^m \) on to the line with direction \( e = \sqrt{m}^{-1}(1, 1, \cdots, 1) \). Thus \( \dim H S^m(A) \leq \dim H A^m \). Therefore by Lemma 4.1 we obtain \( \dim H S^m(A) = 0 \).

Suppose that 
\[
P(x) = \sum_{k=0}^{n} a_k x^k, a_k \in \mathbb{R}, a_n \neq 0.
\]
Does not lose general we may assume $a_0 = 0$. Note that

$$\tilde{P}(A) = \left\{ \sum_{k=1}^{n} a_k x_{k,1} \cdots x_{k,k} : 1 \leq i \leq k, x_{k,i} \in A \right\}.$$ 

Define a new function

$$\varphi : [0, 1]^{\frac{(n+1)n}{2}} \rightarrow \mathbb{R}$$

by

$$\varphi(x_{1,1}, x_{2,1}, x_{2,2}, \cdots, x_{mn}) = \sum_{k=1}^{n} a_k x_{k,1} \cdots x_{k,k}.$$ 

By the mean value theorem we have that $\varphi$ is a Lipschitz map on $[0, 1]^{\frac{(n+1)n}{2}}$. Observe that

$$\tilde{P}(A) = \varphi(A^{\frac{(n+1)n}{2}}).$$

Thus by Lemma 1.1 and the fact that Lipschitz map will not increase the Hausdorff dimension, we obtain that $\dim_H \tilde{P}(A) = 0$. $\square$

**Remark 4.2.** Let $A \subset \mathbb{R}^d$. Then we can also consider the sets $S^m(A), A^m$. By applying the same arguments in Lemma 1.1 and in the proof of Theorem 1.5 we have that for a typical compact set $A \in \mathcal{K}([0,1]^d)$,

$$\dim_H A^m = 0 \text{ and } \dim_H S^m(A) = 0 \text{ for any } m \in \mathbb{N}. $$

We omit the details here.

Denote

$$e^A := \sum_{n=0}^{\infty} \frac{T^n(A)}{n!}.$$ 

We consider $e^A$ as the limit point of $S_m$ in the space $(\mathcal{K}(\mathbb{R}), d_H)$ where

$$S_m := \sum_{n=0}^{m} \frac{T^n(A)}{n!}$$

is a sum set of $\{ \frac{T^n(A)}{n!} \}_{n=0}^{m}$. Note that $\{S_m\}_{m \in \mathbb{N}}$ is a Cauchy sequence in the complete metric space $(\mathcal{K}(\mathbb{R}), d_H)$. Thus the set $e^A$ is well defined.

**Question 4.3.** Is it true that a typical $A \in \mathcal{K}$ has $\dim_H e^A = 0$?
5. Typical continuous functions

Let $\mathcal{C} = \mathcal{C}([0, 1])$ be all the continuous functions on $[0, 1]$. The distance of continuous functions $f, g \in \mathcal{C}$ is defined by

$$d(f, g) = \max\{|f(x) - g(x)| : x \in [0, 1]\}.$$ 

Let $A \subset [0, 1] \times \mathbb{R}$. Define

$$\mathcal{C}_A = \{f \in \mathcal{C} : G(f) \cap A \neq \emptyset\}$$

where $G(f)$ is the graph of function $f$. Let $x \in [0, 1]$ and $V_x = \{x\} \times \mathbb{R}$. In the following we only consider the case $A \subset V_x$. For this special case, we have the following similar result to Theorem 1.3.

**Proposition 5.1.** A subset $A \subset V_x, x \in [0, 1]$ is nowhere dense in $V_x$ if and only if $\mathcal{C}_A$ is nowhere dense in $\mathcal{C}$.

**Proof.** Suppose $A$ is nowhere dense in $V_x$. Let $U_C(f, \varepsilon)$ be an open ball in $\mathcal{C}$ with center $f$ and radius $\varepsilon$. Then by the nowhere dense of $A$ there exist $g \in \mathcal{C}, \varepsilon' > 0$ such that

$$U(g(x), \varepsilon') \cap A = \emptyset, \text{ and } U(g(x), \varepsilon') \subset U(f(x), \varepsilon).$$

Here $U(f(x), \varepsilon)$ is an open ball in $V_x$ with center $f(x)$ and radius $\varepsilon$. Note that $U_C(g, \varepsilon') \cap \mathcal{C}_A = \emptyset$. By the arbitrary choice of $f \in \mathcal{C}$ and $\varepsilon$ we obtain that $\mathcal{C}_A$ is nowhere dense.

By applying the same argument as in the proof of Theorem 1.3 we obtain that if $\mathcal{C}_A$ is nowhere dense then $A$ is nowhere dense. \qed

Applying the same argument as in the introduction, we obtain that if $A \subset V_x, x \in [0, 1]$ is of first category in $V_x$ then $\mathcal{C}_A$ is of first category in $\mathcal{C}$. Again we do not know that if the converse claim is also true. Let $z \in [0, 1] \times \mathbb{R}$ then Proposition 5.1 claims that $\mathcal{C}_z$ is nowhere dense in $\mathcal{C}$. Since the rational points in plane is countable, we obtain that the graph of a typical continuous function of $\mathcal{C}$ does not contain any rational points in plane.

Maga [11] proved that for any distinct points $\{x, y, z\} \subset \mathbb{R}^2$, there exists a compact set $E \subset \mathbb{R}^2$ with $\dim_H E = 2$ and $E$ does not contain a similar copy of $\{x, y, z\}$. Motivated by this result and Theorem 1.1 we ask the following question.

**Question 5.2.** Does the graph of a continuous function with Hausdorff dimension larger than one contains three points which are the vertices of an equilateral triangle?

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