Quantization and Intrinsic Dynamics

Mikhail Karasev*

Applied Math. Department
Moscow Institute of Electronics and Mathematics
Moscow 109028, Russia
karasev@miem.edu.ru

Abstract

A dynamical scheme of quantization of symplectic manifolds is described. It is based on intrinsic Schrödinger and Heisenberg type nonlinear evolutionary equations with multidimensional time running over the manifold. This is the restricted version of the original article to be published in “Asymptotic Methods for Wave and Quantum Problems” (M. Karasev, ed.), Advances in Math. Sci., AMS.

Key words: quantization, symplectic geometry, semiclassical approximation.

2000 Math. Subject Classification: 81S10, 81S30, 53D55.

1 Introduction

The paper deals with constructing irreducible quantum manifolds, that is, quantizing symplectic manifolds [1]–[14].

Reviewing results of previous works [15, 16], we investigate an intrinsic evolutionary differential equation for the integral kernel of a quantum associative product of functions over the symplectic manifold. The “time” variable in this evolutionary equation is multidimensional and runs over the

*This research was supported in part by the Russian Foundation for Basic Research, Grant 02-01-00952.
same symplectic manifold. We describe the solution of this equation in the
semiclassical approximation, as well as in the sense of weak asymptotics, by
purely geometric means.

It is an open question how to solve this equation exactly (and so to
construct exact quantization) for a general symplectic manifold. However, in
some nontrivial classes of examples, this occurs to be possible.

The quantization which we deal with is formal in the sense that we do
not consider analytic conditions on function spaces, Hilbert norms, etc. Nev-
ertheless, this approach is not completely formal, it is adapted to the strict
quantization scheme \[17\] and directly related to operator representations.
This is a program for the future: exact operator realization of the this ap-
proach over general symplectic manifolds. The important case, where the
operator representation is possible, are Kählerian manifolds, in particular,
symplectic leaves in some Poisson manifolds with partial complex structure.
The equations for the $\ast$-product kernel in this case were used in \[18, 19\].
About examples for the non-Kählerian situation see, for instance, \[20, 21, 22\].

Besides the algebraic and analytic machinery, which we explain below,
there is an important geometric framework around such a dynamical scheme
of quantization. In the present paper we would like to pay attention to the
fact that the old Ether idea, or more precisely, the idea of intrinsic dynamics
on the phase space level, is mathematically fruitful and, moreover, automat-
ically arises from the quantization problem. Hopefully, on this way, one can
realize the early Weyl’s objectives and his intuitive program of exploring the
continuum “relationship between a part and the whole” (see \[23\]).

2 Basic wave equations

Let us look first at the stationary Schrödinger equation for wave functions $\psi$
and for the energy levels $\lambda$:

$$
\left(\hat{p}^2 + V(q)\right)\psi = \lambda\psi, 
$$

(2.1)

where $\hat{q} = q$, $\hat{p} = -i\hbar \partial / \partial q$. Solving this equation is difficult, and the proper-
ties of the wave functions are complicated. Say, in the semiclassical approx-
imation we have

$$
\psi \sim c_\hbar e^{\hbar S(\rho + O(\hbar))}, \quad c_\hbar = \text{const}.
$$

(2.2)
But, this expression is very far from the asymptotics of the wave function in the presence of focal or turning points.

On the other hand, the weak limit $\psi^0$ of the function $\psi$ as $\hbar \to 0$ is very simple. Indeed, if $\lambda^0$ denotes the limit of $\lambda$, then Eq. (2.1) at $\hbar = 0$ becomes

$$V(q)\psi^0(q) = \lambda^0 \psi^0(q).$$

The solutions are obvious:

$$\psi^0(q) = \delta(q^0), \quad \lambda^0 = V(q^0), \quad (2.3)$$

where $q^0$ is arbitrary. So, the quantum wave functions in the weak limit $\hbar = 0$ are just $\delta$-functions characterizing the singular distribution of the probability amplitude of a classical particle to be localized at a given point.

Now let us look at the algebraic picture behind Eq. (2.1), namely, at the Heisenberg commutation relations

$$[\hat{q}, \hat{p}] = i\hbar \mathbf{I}. \quad (2.4)$$

These relations introduce a noncommutative product $*$ into the function algebra over the $q, p$-phase space as follows:

$$\hat{f} \overset{\text{def}}{=} f(\hat{q}, \hat{p}), \quad \hat{f} \hat{g} = \hat{f} * \hat{g}. \quad (2.5)$$

Here we use the Weyl-symmetrized functions in the noncommuting operators $\hat{q}, \hat{p}$ so that, if $f(q, p) = \exp\{i(\alpha q + \beta p)\}$, then $\hat{f} = \exp\{i(\alpha \hat{q} + \beta \hat{p})\}$.

If $\hbar = 0$, then the algebra (2.4) becomes commutative and the product $f * g$ becomes the usual commutative product $fg$ of functions. One can represent it via the integral kernel as

$$(fg)(z) = \int \int \delta_x(z) \delta_y(z) f(x) g(y) \, dm(x) dm(y). \quad (2.6)$$

Here the arguments $x, y, z$ run over the $q, p$-phase space and $\delta$ denotes the Dirac delta function with respect to a certain measure $dm$ on the phase space.

If $\hbar \neq 0$, then the noncommutative product $f * g$ in (2.5) can be represented in integral form as well:

$$(f * g)(z) = \frac{1}{(2\pi\hbar)^{2n}} \int \int K_{x,y}(z) f(x) g(y) \, dm(x) dm(y), \quad (2.7)$$
where $2n$ is the dimension of the phase space. The integral kernel here is the noncommutative product of $\delta$-functions:

$$\frac{1}{(2\pi\hbar)^{2n}} K_{x,y} = \delta_x \ast \delta_y.$$  

From (2.6) we see that in the weak limit as $\hbar \to 0$ the integral kernel is equal to the commutative product of delta functions:

$$\lim_{\hbar \to 0} \frac{1}{(2\pi\hbar)^{2n}} K_{x,y} = \delta_x \delta_y.$$  

This is the same picture as for the wave function $\psi$ in (2.1) and its weak limit $\psi^0$ (2.3). Looking at this analogy, we conclude that a natural interpretation of the kernel of the $\ast$-product could be a “wave function” of something.

Of what?

In (2.4), (2.5), and (2.7), there are no particles, no $a$ priori Hamiltonians. Nevertheless, one can introduce a Hamiltonian of the product $\ast$ itself by mimicking the Schrödinger evolutionary equation.

Let us first note that the operators $\hat{f}$ in (2.5) depend on $f$ linearly and so can be written as integrals

$$\hat{f} = \frac{1}{(2\pi\hbar)^n} \int f S \, dm$$  

over the phase space. Here $S = \{S_x\}$ is a family of operators parametrized by points $x$ of the phase space. In the case of the flat phase space $\mathbb{R}^{2n}$ one could choose the Liouville measure, $dm(x) = dx = dqdp$; then the explicit formula for the integral kernel of the operators $S_x$ in the Hilbert space $L^2(\mathbb{R}^n)$ is

$$S_x \sim \delta \left( q - \frac{q' + q''}{2} \right) \exp \left\{ \frac{i}{\hbar} p(q' - q'') \right\}, \quad x = (q,p).$$

An important fact about the representation (2.8) is that it is easily inverted [25]:

$$f = \text{tr}(\hat{f}S).$$  

(2.8a)

The integral kernel $K_{x,y}$ in (2.7) is related to $S_x$ by the formula

$$\hat{K}_{x,y} = S_x S_y.$$  

(2.9)
It is not surprising that each $S_x$ is self-adjoint (if one wants to associate self-adjoint operators $\hat{f}$ with real symbols $f$ by (2.8)), but it is remarkable that all $S_x$ are almost unitary; namely, $S_x$ differs from a unitary only by a constant multiplier
\[ \mu = 2^n. \]
Thus we have
\[ S^* = S, \quad S^2 = \mu^2 \cdot I. \quad (2.10) \]

The first consequence from this almost unitarity is the representation
\[ S = \mu(I - 2P), \quad (2.11) \]
where $P = \{P_x\}$ is a family of orthogonal projections.

Second, if one looks at the parameter $x$ of the almost unitary family $S = \{S_x\}$ as at a “time” variable, then one can derive the generator of $S$ with respect to this multidimensional “time”:
\[ i\hbar \partial_x S = \hat{H}_x \cdot S. \quad (2.12) \]
Here the differential $\partial = \partial/\partial x$ is taken with respect to the variable $x$ running over the phase space. The operator $\hat{H}_x$ is the self-adjoint generator given by
\[ \hat{H}_x = 2i\hbar(\partial P \cdot P - P \cdot \partial P). \quad (2.13) \]

Since $S$ and $\hat{H}_x$ are self-adjoint, we conclude from (2.12) that
\[ \hat{H}_x S = -S \hat{H}_x \quad (2.14) \]
and so the equation (2.12) can be transformed as
\[ i\hbar \partial_x S = \frac{1}{2}[\hat{H}_x, S]. \quad (2.12a) \]
This dynamical equation looks like the Heisenberg evolution equation, or like a Lax type equation for the “$L - A$ pair.”

By means of (2.8a) and (2.9), we return to the integral kernel $K_{x,y}$ and obtain from (2.12):
\[ i\hbar \partial_x K_{x,y} = \mathcal{H}_x^h \ast K_{x,y}. \quad (2.15) \]
The Cauchy data are
\[ K_{y,y} = \mu^2, \quad \forall y. \] (2.16)

The constant \( \mu^2 \) is determined by the condition that the unity function \( 1 \) is the unity element for product (2.7), i.e., the function
\[ \frac{1}{(2\pi \hbar)^{2n}} \int K_{x,y}(z) \, dm(y) \sim \delta_x(z) \] (2.17)
in arguments \( x, z \) serves as the kernel of the unity operator in the quantum function algebra, that is, it equals \( \delta_x(z) \) for the Euclidean phase space.

Also from the self-adjointness (2.10), we have
\[ K_{x,y}(z) = K_{y,x}(z), \quad \text{or} \quad \overline{f \ast g} = \overline{g \ast f}. \] (2.18)

In addition to these simple conditions, there is also the fundamental cyclicity condition, which follows from the trace representation \( K_{x,y}(z) = \text{tr}(S_x S_y S_z) \), namely,
\[ K_{x,y}(z) = K_{z,x}(y), \quad \text{or} \quad \int f \ast g \, dm = \int f g \, dm. \] (2.19)

Actually, this is the condition for the choice of the measure \( dm \).

Note that, after integrating (2.15) with the function \( 1(y) f(z) \) and using the property (2.19), one obtains the identities
\[ i\hbar df(x) = (f \ast \mathcal{H}_x^\hbar)(x) = -\mathcal{H}_x^\hbar \ast f)(x), \quad \forall f. \] (2.20)

These identities determine the “germ” of the \( \ast \)-product near a given point \( x \) of the phase space. Global information about the \( \ast \)-product is contained in the wave equation (2.15), which can be resolved using the operator multiplicative exponential or the \( \ast \)-exponential\footnote{Such exponentials were introduced in early works on deformation quantization \cite{5, 26} and were intensively exploited, for instance, see \cite{27} and the references therein.} as follows:
\[ \hat{K}_{x,y} = \mu^2 \cdot \text{Exp} \left\{ -\frac{2i}{\hbar} \int_{y}^{x} \hat{\mathcal{H}}^{\hbar} \right\}, \]
or
\[ f \ast g = \frac{\mu^2}{(2\pi \hbar)^{2n}} \int \int f(x) \text{Exp*} \left\{ -\frac{2i}{\hbar} \int_{y}^{x} \mathcal{H}^{\hbar} \right\} g(y) \, dm(x) \, dm(y), \] (2.21)
This formula manifests the dynamical character of quantization. We see a sum of parallel-transported amplitudes multiplied by values of functions $g$ and $f$ at the initial and final “time” points. The role of “time” is played by points of the phase space.

We stress that (2.15), as well as (2.12), are Schrödinger type dynamical equations behind any particles. We see no particles but, nevertheless, something is “moving” and controlled by the Hamiltonian $\mathcal{H}^h$. We call $\mathcal{H}^h$ the *Ether Hamiltonian*.

Of course, in the case of the standard phase space $\mathbb{R}^{2n}$, the kernel $K$ is well known independently of the calculations performed above, namely,

$$K_{x,y}(z) = \exp\left\{\frac{i}{\hbar}\Phi_{x,y}(z)\right\}, \quad (2.22)$$

where $\Phi_{x,y}(z)$ is the symplectic area of the triangle with mid-points $x, y, z$ (see [28, 29] and [3, 30] for general flat symplectic spaces).

The Ether Hamiltonian $\mathcal{H}^h$ is easily derived in the case $\mathbb{R}^{2n}$ from Eq. (2.12) or from (2.15), (2.16):

$$\mathcal{H}^h_x(z) = \frac{i\hbar}{\mu^2} \frac{\partial_x K_{x,y}(z)}{y=z} = 2\omega(z-x) \cdot dx. \quad (2.23)$$

Here $\omega$ is the constant matrix of the symplectic form on $\mathbb{R}^{2n}$ in the Cartesian coordinates. In the Darboux coordinates $\omega = J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

Even for the simplest integral kernel (2.22) the differential equation (2.15) seems to be new. Other attempts to find some differential equations for the $\ast$-product kernel were initiated in [31, 32], but they are certainly different from our approach.

Note that the operator $\hat{\mathcal{H}}^h$ in (2.12) is defined via (2.8), and so Eqs. (2.12), (2.12a) are actually nonlinear:

$$i\hbar \partial_x S_x = \int \mathcal{H}^h_x(y) S_y S_x dy, \quad (2.24)$$

or

$$i\hbar \partial_x S_x = \frac{1}{2} \int \mathcal{H}^h_x(y) [S_x, S_y] dy. \quad (2.24a)$$
Similarly, the product $\ast$ in the evolutionary equation (2.15) for the “wave function” $K$ is determined by the same function $K$ via (2.7), and so this equation is actually nonlinear. Explicitly, it looks as

$$i\hbar\partial_x K_{x,y}(z) = \frac{1}{(2\pi\hbar)^{2n}} \iint \mathcal{H}^\hbar_x(z')K_{z',y'}(z')K_{x,y'}(y') \, dm(z')dm(y'). \quad (2.25)$$

One can “resolve” this nonlinearity by using the representation of the $\ast$-product via operators $L$ of the left regular representation (see general definitions, e.g., in [16]). In the case of the phase space $\mathbb{R}^{2n}$ it is well known that

$$f \ast g = f(L)g, \quad (2.26)$$

the operators where $L = (L_q, L_p)$ are Weyl-symmetrized and given precisely by

$$L = x + \frac{i\hbar}{2} J \frac{\partial}{\partial x}, \quad x = (q, p) \in \mathbb{R}^{2n}, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$ 

So Eq. (2.15) reads as a linear equation:

$$i\hbar\partial_x K_{x,y} = \mathcal{H}^\hbar_x(L)K_{x,y}, \quad (2.27)$$

where $\mathcal{H}^\hbar$ is the Ether Hamiltonian.

The derivation of the wave equations for the $\ast$-product kernel demonstrated above is universal and could be produced over curvilinear phase spaces. All formulas, except (2.22) and (2.23), are general. Equation (2.15) can be considered as a fundamental nonlinear equation determined by the Ether Hamiltonian. This equation generates the $\ast$-product kernel $K$ starting from the Cauchy data (2.16). And Eq. (2.27) can also be well established by means of an a priori computation of the left regular representation (2.26) of the $\ast$-product algebra\(^\text{2}\). But first of all, one needs to know what the Ether Hamiltonian $\mathcal{H}^\hbar$ is in general.

\(^2\)The derivation (2.7)–(2.27) in the case of the phase space $\mathbb{R}^{2n}$ with symplectic form $\omega = dp \wedge dq + \frac{1}{2}F(q) \, dp \wedge dq$ containing an additional “magnetic” summand was given in the author’s talk in Manitoba University, Winnipeg (July 2000). The observation was that Eq. (2.27) for the $\ast$-product kernel is immediately generalized to such “magnetic” case and the operators $L$ are explicitly known in that case [33]. For the properties of the kernel $K_{x,y}$ and the family $S_x$ and for their relations with symplectic transformations and connections in the “magnetic” case, see [34].
3 Zero curvature equation

In the case of a general phase space, i.e., of a curvilinear symplectic manifold, the Ether Hamiltonian $\mathcal{H}_x^\hbar$ is a 1-form on this space (with respect to the “time” variable) whose values are functions on the same space (with respect to the “space” variable):

$$\mathcal{H}_x^\hbar(z) = \sum_{j=1}^{2n} \mathcal{H}_x^\hbar(z)_j \, dx^j.$$  \hfil (3.1)

Equations (2.12), (2.15), or (2.27) are actually systems of equations, and so their generators $\hat{\mathcal{H}}_j^\hbar (j = 1, \ldots, 2n)$ should satisfy a compatibility condition. It can be written in the form

$$i\hbar \partial \hat{\mathcal{H}}^\hbar = \frac{1}{2} [\hat{\mathcal{H}}^\hbar \wedge \hat{\mathcal{H}}^\hbar],$$

or

$$\partial \mathcal{H}^\hbar + \frac{i}{2\hbar} \left[ \mathcal{H}^\hbar \wedge \mathcal{H}^\hbar \right]_* = 0.$$  \hfil (3.2)

Here the differential $\partial$ is taken only with respect to the “time” variable and the brackets $[f, g]_* = f \ast g - g \ast f$ are taken only with respect to the “space” variable.

Condition (3.2) is the zero curvature equation for the connection over the phase space determined by the Ether Hamiltonian on the bundle whose fibers are quantum function algebras over the same phase space.

In the classical limit $\hbar = 0$, Eq. (3.2) reads

$$\partial \mathcal{H} + \frac{1}{2} \{\mathcal{H} \wedge \mathcal{H}\} = 0.$$  \hfil (3.3)

Here we omit the index $\hbar$ in the Ether Hamiltonian when considering the limit $\hbar = 0$. The Poisson brackets in (3.3) correspond to the symplectic form

$$\omega(z) = \frac{1}{2} \omega_{jk}(z) \, dz^k \wedge dz^j.$$  \hfil (3.4)

\footnote{In \cite{35}, where the construction of the work \cite{9} was analyzed in the classical limit $\hbar = 0$, a zero curvature equation was obtained, which in a sense is analogous to (3.3). An essential distinction from our equation (3.3) is that, instead of the phase space Poisson brackets $\{\cdot, \cdot\}$, the commutator of formal vector fields along tangent fibers is used in \cite{35}. On the quantum level, the so-called “Weyl bundle” used in \cite{7, 8} consists of algebras of polynomials along tangent fibers whose algebraic structure is just the Groenewold–Moyal product, in contrast to (3.2), where the $\ast$-product is taken over the whole phase manifold.}
Note that the symplectic case under study is a particular case of the general Poisson situation. In the general Poisson picture, the “wave” equations (2.12), (2.15) and the zero curvature equations (3.2), (3.3) were used in [15, 36] (see also [16]) and the 1-form $\mathcal{H}$ was called the Cartan structure. In this situation the “space” argument belongs to the Poisson manifold itself, while the “time” argument belongs to the dual manifold, i.e., to a finite-dimensional pseudogroup. In the symplectic case this pseudogroup can be identified with the “space” manifold, and we again obtain (2.12), (2.15), (3.2), (3.3). Thus one can use those “Poisson” results in order to study the solutions of our basic equations in the symplectic case.

The following geometric lemma has general Poisson settings [37, 38] (see also [16]).

**Lemma 3.1.** Let $\mathcal{X}$ be a symplectic manifold. In a neighborhood of the zero section, the cotangent bundle $T^*\mathcal{X}$ admits a symplectic fibration $\ell$ over the base $\mathcal{X}$ such that

(i) $\ell$ is the identical map on the zero section $T_0\mathcal{X} \approx \mathcal{X}$, that is, $\ell(x, \eta)|_{\eta=0} = 0$;

(ii) the dual fibration $r$ given by the reflection in momenta:

$$ r(x, \eta) \overset{\text{def}}{=} \ell(x, -\eta), \quad \eta \in T_x^*\mathcal{X}, $$

is in involution with $\ell$, that is

$$ \{r^j, \ell^k\}_{T^*\mathcal{X}} = 0, \quad \forall j, k. $$

A symplectic fibration with properties (i), (ii) will be called reflective.

Now, let us construct the solution of the zero curvature equation (3.3). Note that in the symplectic case under study the matrix $\partial\ell/\partial\eta$ is not degenerate at least in a sufficiently small neighborhood of the zero section. Thus the following equation is solvable:

$$ \ell(x, \eta) = z \implies \eta = \mathcal{H}_x(z). \quad (3.5) $$

The symplecticity of $\ell$ means

$$ \{\ell^j, \ell^k\}_{T^*\mathcal{X}} = \Psi^{jk}(\ell), \quad \Psi \overset{\text{def}}{=} \omega^{-1}, $$

and so it is easy to check that the solution of (3.5) satisfies

$$ \partial_j \mathcal{H}_k - \partial_k \mathcal{H}_j + \{\mathcal{H}_j, \mathcal{H}_k\}_{\mathcal{X}} = 0. \quad (3.3a) $$
Here we use the notation $\partial_j = \partial/\partial x_j$ for the derivatives, and the Poisson brackets are taken with respect to $z \in \mathcal{X}$. The system (3.3a) coincides with (3.3).

From statements (i), (ii) of Lemma 3.1, we obtain

$$\frac{1}{2}D\mathcal{H}_x(z)\big|_{z=x} = \omega(x).$$  \hspace{1cm} (3.6)

Here we use the notation $D = \partial/\partial z$ for the derivatives.

For the second derivatives at $z = x$, from (3.3a) we obtain the formula

$$\frac{1}{2}D^2\mathcal{H}_x(z)\big|_{z=x} = \omega(x)\Gamma(x),$$  \hspace{1cm} (3.7)

where $\Gamma$ is a symplectic connection on $\mathcal{X}$, that is,

$$\partial\omega_{jk} - \Gamma^\ell_{js}\omega_{\ell k} - \Gamma^\ell_{ks}\omega_{j \ell} = 0, \quad \text{or} \quad \nabla \omega = 0.$$

This is a crucial place: the symplectic connection $\Gamma$ has appeared automatically from the symplectic fibration $\ell$ of the secondary phase space (local symplectic groupoid\footnote{The fibration $\ell$ is the target mapping (or the left restriction mapping) in the local symplectic groupoid over $\mathcal{X}$ which is realized as a neighborhood of the zero section of $T^*\mathcal{X}$ (§6, 39).}) over the original manifold $\mathcal{X}$.

Of course, instead of (3.6), (3.7), one can write direct formulas for $\Psi = \omega^{-1}$ and for $\Gamma$ via $\ell$:

$$\Psi(x) = 2\frac{\partial \ell}{\partial \eta}(x, \eta)\bigg|_{\eta=0}, \quad \Gamma(x) = 4\omega(x) \cdot \frac{\partial^2 \ell}{\partial \eta \partial \eta}(x, \eta)\bigg|_{\eta=0} \cdot \omega(x).$$

Let us fix a point $x \in \mathcal{X}$ and a tangent vector $v \in T_x\mathcal{X}$. Denote by $\text{Exp}_x(2v\tau)$ the trajectory of the Hamiltonian $v\mathcal{H}_x$ starting from the point $x$. Then

$$\mathcal{H}_x\left(\text{Exp}_x(-v)\right) = -\mathcal{H}_x\left(\text{Exp}_x(v)\right).$$  \hspace{1cm} (3.8)

**Theorem 3.2.** Let $(\mathcal{X}, \omega, \Gamma)$ be a symplectic manifold $\mathcal{X}$ with symplectic form $\omega$ and symplectic connection $\Gamma$. In a neighborhood of the diagonal $z \equiv x$, there is a unique solution of the zero curvature equation (3.3), which obeys conditions (3.6), (3.7), and (3.8). This solution can be obtained by (3.5) via a unique reflective symplectic fibration $\ell$ over $\mathcal{X}$.
We call $\mathcal{H}$ the classical Ether Hamiltonian. The mapping

$$\text{Exp}_x : T_x \mathcal{X} \to \mathcal{X}$$

will be called the Ether exponential mapping and the trajectory $\text{Exp}_x(2v\tau)$ will be called the Ether geodesics through the point $x$ with velocity $v$.

Note that Ether geodesics correspond to some vertical curves in the fiber $T^*_x \mathcal{X}$, namely, to the curves

$$\eta(\tau) \overset{\text{def}}{=} \mathcal{H}_x \big( \text{Exp}_x(2v\tau) \big). \quad (3.9)$$

They are perpendicular to the velocity $v \in T_x \mathcal{X}$ and obey the equations

$$\frac{d}{d\tau} \eta(\tau) = v \Omega^x[\eta(\tau)], \quad \eta(\tau) \big|_{\tau=0} = 0. \quad (3.10)$$

Here the tensor $\Omega^x$ is determined in a neighborhood of zero in $T^*_x \mathcal{X}$ by the following relations

$$\Omega^x_{lk}(\mathcal{H}_x) = \{\mathcal{H}_x, \mathcal{H}_x\}. \quad (3.11)$$

This tensor introduces a symplectic structure to the fiber $T^*_x \mathcal{X}$ so that the Ether mapping $\mathcal{H}_x$ is symplectic. With respect to this structure, system (3.10) is Hamiltonian and the related Hamilton function is just the linear function $v$ (i.e., the function $(v, \eta)$, $\eta \in T^*_x \mathcal{X}$).

Besides other properties, let us stress that the infinitesimal geometry of space, including its symplectic structure, connection, and curvature, is sitting inside the Ether Hamiltonian which can be considered as a “generating function” for this kinematic geometry. Using the Ether exponential coordinates $z = \text{Exp}_x(v)$, we represent the components of this “generating function” as follows:

$$\mathcal{H}_j = 2\omega_{jk}(x)v^k + 2\omega_{km}(x)R^m_{ljs}(x)v^k v^l v^s + O(v^5), \quad (3.12)$$

where $R$ is the curvature of the symplectic connection $\Gamma$. The skew-symmetry with respect to the tangent argument $v$ corresponds to the cyclicity condition (3.8). Higher terms in (3.9) contain derivatives of the curvature $R$.

Now we demonstrate an alternative way to obtain the Ether Hamiltonian.
Let \( x \in \mathcal{X} \). A symplectic mapping \( s_x : \mathcal{X} \to \mathcal{X} \) is called a reflection in \( x \) if it is an involution: \( s_x^2 = \text{id} \), and \( x \) is an isolated fixed point: \( s_x(x) = x \). A symplectic manifold \( \mathcal{X} \) with a smooth family of reflections \( \{ s_x \mid x \in \mathcal{X} \} \) will be called a reflective symplectic manifold.

**Theorem 3.3.** Let \( (\mathcal{X}, \omega) \) be a simply connected reflective symplectic manifold. Then there is a natural symplectic connection on \( \mathcal{X} \):

\[
\Gamma(z) = -\frac{1}{2} D^2 s_x(z) \bigg|_{z=x}, \quad z \in \mathcal{X}.
\]  

(3.13)

The Ether Hamiltonian corresponding to \( (\mathcal{X}, \omega, \Gamma) \) is given by

\[
\mathcal{H}_x(z) = \int_x^z \langle \omega(z) \partial s_x(s_x(z)), dz \rangle.
\]  

(3.14)

And vise versa: the Ether Hamiltonian corresponding to \( (\mathcal{X}, \omega, \Gamma) \) uniquely determines the reflective structure over \( \mathcal{X} \) by solving the Cauchy problem

\[
\partial s_x + \{ \mathcal{H}_x, s_x \} = 0, \quad s_x(z) \bigg|_{z=x} = z,
\]  

(3.15)

or just by solving the implicit equation

\[
\mathcal{H}_x(s_x(z)) = -\mathcal{H}_x(z)
\]  

(3.16)

(in a domain where \( DH \) not degenerate). This family of reflections is related to the connection \( \Gamma \) via formula (3.13). The Ether geodesics are symmetric with respect to reflections

\[
s_x \big( \text{Exp}_x(v) \big) = \text{Exp}_x(-v)
\]  

(3.17)

(but, in general, reflections \( s_x \) are not affine with respect to \( \Gamma \)).

In particular, this theorem can be applied to symplectic symmetric manifolds \([41]\), i.e., to the case where the curvature of \( \Gamma \) is covariantly constant \((\nabla R = 0)\). In this case the connection (3.13) is just the canonical Cartan–Loos connection, the axiom \( s_x s_y s_x = s_{s_x(y)} \) holds, all reflections \( s_x \) are affine, and the Ether geodesics coincide with the usual geodesics of the connection \( \Gamma \).

\(^5\)In \([40]\) such a mapping is called a symmetry under the additional condition that the fixed point is unique. However, the term “symmetry” in the classical theory of symmetric spaces carries the strong property \( s_x s_y s_x = s_{s_x(y)} \), which does not hold in our case.
As an example, let us consider the sphere $S^2 = \{ |x| = 1 \}$ embedded in $\mathbb{R}^3$ and endowed with standard symplectic form. In this case

$$\mathcal{H}_x(z) = 2[x \times z] \cdot dx \bigg|_{S^2},$$

(3.18)

where the brackets $[\cdot \times \cdot]$ stay for the vector-product in $\mathbb{R}^3$.

4 Intrinsic dynamical objects

Let us denote by $g_{x,y}$ the symplectic transformations

$$z \to g_{x,y}(z), \quad z \in \mathcal{X},$$

obtained by shifts along the Ether dynamical system (that is, the “time” derivative $\partial_x g_{x,y}(z)$ coincides with the Ether Hamiltonian vector field at the point $g_{x,y}(z)$, and the “initial” data are $g_{y,y}(z) = z$).

These transformations obey the groupoid rule

$$g_{x,y} \cdot g_{y,z} = g_{x,z},$$

(4.1)

and produce a certain group of intrinsic transformations of the phase space. This group, in general, has an infinite dimension, and we prefer to speak about the finite-dimensional groupoid of Ether translations.

In terms of symplectic reflections $s_x$ the symplectic transformations $g_{x,y}$ are expressed by the formula

$$g_{x,y} = s_x \cdot s_y,$$

(4.2)

and so

$$s_x(y) = g_{x,y}(y).$$

Other important transformations are related to the Ether exponential mapping. Let us fix $x \in \mathcal{X}$, and let $u \in T_x \mathcal{X}$. Recall that by $\text{Exp}_x(2vt)$ we denote the trajectory of the Hamiltonian $u \mathcal{H}_x$ starting from the point $x$. But there are other trajectories starting from other points. We consider all of them.

Denote by $e^t_x(y)$ the trajectory of the Hamiltonian $v \mathcal{H}_x$ starting from the point $y$. In particular, if $y = x$, then we have
Lemma 4.1. For each \( x \in X \) the pseudogroup of symplectic transformations \( \{ e^v_x \} \) obeys the identity

\[
s_x \cdot e^v_x = e^{-v}_x \cdot s_x.
\]  

We call \( e^v_x \) exponential transformations.

What is the quantum version of all these transformations?

Heuristically, we associate the reflections \( s_x \) with the operators \( S_x \), the translations \( g_{y,x} \) with the operators \( S_x S_y = \hat{K}_{x,y} \), and the exponential transformations \( e^v_x \) with the operators \( \exp\{ \frac{i}{\hbar} v \hat{H}_x \} \).

More precisely, we introduce quantum mappings \( \hat{s}^*_x \), \( \hat{g}^*_{y,x} \), and \( \hat{e}^*_v \), acting in the \(*\)-product algebra, by evaluating the Heisenberg transforms. Say, we could define the mapping \( f \to \hat{s}^*_x f \) in the following way:

\[
\hat{s}^*_x f = S_x f S_x^{-1},
\]

or

\[
(\hat{s}^*_x f)(z) = \text{tr} (S_x^{-1} S_x f).
\]

If one does not know the operator realization of the \(*\)-product algebra, then, instead of operators \( S_x \), it is possible to use the differential equations with respect to the parameters \( x, y, \ldots \). These equations follow form our basic relations (2.12) and (2.15).

The solution of the Cauchy problem

\[
\begin{align*}
\frac{\partial F}{\partial x} + \frac{i}{\hbar} [\hat{H}_x, F]_* &= 0, \\
F \bigg|_{\text{diagonal}} &= f,
\end{align*}
\]

we denote by \( F(x, z) = (\hat{s}^*_x f)(z) \), and we call the mappings \( \hat{s}^*_x \) quantum reflections.

In Eqs. (4.6) the unknown function \( F = F(x, z) \) depends on \( x, z \in X \), the differential \( \partial \) is taken with respect to \( x \), the quantum brackets \( [\cdot, \cdot]_* \).
are taken with respect to \( z \), and the Cauchy data are given on the diagonal \( x = z \).

If we consider the same equation (4.6) but for the function \( \mathcal{F} = \mathcal{F}(x, y, z) \) depending on the additional argument \( y \in \mathcal{X} \) and change the Cauchy data by the following ones:

\[
\mathcal{F}(x, y, z)\big|_{x=y} = f(z),
\]

then the solution \( \mathcal{F} \) determines the *quantum translations* \( \mathcal{g}_{y,x}^* \) as

\[
\mathcal{F}(x, y, z) = (\mathcal{g}_{y,x}^* f)(z).
\]

At last, the solution of the Cauchy problem

\[
\frac{\partial \mathcal{E}}{\partial t} + \frac{i}{\hbar} [\mathcal{E}, v \mathcal{H}_x^\hbar]_* = 0, \quad \mathcal{E}\big|_{t=0} = f,
\]

taken at \( t = 1/2 \), determines the *quantum exponentials*

\[
\mathcal{E}\big|_{t=1/2} = \mathcal{E}_{x,v}^\hbar f, \quad v \in T_x\mathcal{X}.
\]

The restriction of this function to the diagonal (as in (4.3)) can be called a *quantum Ether exponential mapping*:

\[
f \rightarrow f\big|_{\text{Exp}_x(v)} \overset{\text{def}}{=} (\mathcal{E}_{x,v}^\hbar f)(x).
\]

On the quantum level we have analogs of identity (4.4):

\[
\mathcal{E}_{x,v}^\hbar = \mathcal{E}_{x,-v}^\hbar
\]

and of identities (3.8), (3.16):

\[
\mathcal{E}_{x}^\hbar \mathcal{H}_x^\hbar = -\mathcal{H}_x^\hbar
\]

\[
\mathcal{H}_x^\hbar \big|_{\text{Exp}_x(-v)} = -\mathcal{H}_x^\hbar \big|_{\text{Exp}_x(v)}.
\]

All these formulas assume that one knows the *-product, and so it is possible to solve Eqs. (4.6) and (4.9) at least asymptotically as \( \hbar \to 0 \).
Under the usual assumption

\[ f \ast g \simeq fg + \sum_{k \geq 1} \hbar^k c_k(f, g), \quad (4.15) \]

where \( c_k \) are bidifferential operators of order \( k \),

\[ c_1 = -\frac{i}{2} \{ \cdot, \cdot \}, \quad c_k(f, g) = (-1)^k c_k(g, f) = \bar{c}_k(g, f), \quad (4.16) \]

the quantum mappings defined above correspond to their classical counterparts:

\begin{align*}
\hat{s}_x^* &= s_x^*(I + O(\hbar^2)), \\
\hat{e}_x^p &= e_x^p(I + O(\hbar^2)), \\
f \mid_{\text{Exp}_x(v)} &= (I + O(\hbar^2)) f \mid_{\text{Exp}_x(v)}. \quad (4.17)
\end{align*}

Here the quantum corrections \( O(\hbar^2) \) are formal \( \hbar^2 \)-series of the following type:

\[ O(\hbar^2) = \hbar^2 a_1 + \hbar^4 a_2 + \ldots; \]

their coefficients \( a_k \) are differential operators of order \( 2k + 1 \) easily derived from (4.6), (4.9) via the coefficients \( c_k \) (4.15).

In the next section we consider the procedure of constructing the coefficients \( c_k \).

5 Weak asymptotics and quantum zero curvature equation

We begin to study the basic equation (2.15) in order to construct the \( \ast \)-product over \( \mathcal{X} \) at least asymptotically as \( \hbar \to 0 \). There are two types of asymptotic expansions of \( \ast \)-products: weak and semiclassical. Both are useful and both follow from the exact fundamental nonlinear equation (2.15) for the \( \ast \)-product kernel given by the Ether Hamiltonian.

Note that the semiclassical view on the quantization problem over general symplectic manifolds has been developed from the first papers [42, 43, 4] (summarized in the book [16]). In [4] this method was called asymptotic
quantization. The \( \ast \)-product was constructed there by matching local semi-classical expansions, and the quantization condition

\[
\frac{1}{2\pi \hbar} \omega - \frac{1}{2} c_1 \in H^2(\mathcal{X}, \mathbb{Z})
\]  

(5.1)

first appeared as a necessary condition for the existence of an irreducible operator representation of the \( \ast \)-product \[43, 4, 16\].

The weak asymptotics approach to the quantization problem, pioneered in the fundamental papers \[1, 5\], was a basis of all those semiclassical developments. In \[4\] the name deformation quantization was proposed for the weak asymptotics method. The general results in this deformation direction were obtained in \[44\]–\[47\], \[7, 8\].

Now we apply our basic evolutionary equations to construct a unique \( \ast \)-product in the standard deformation form (4.15):

\[
(f \ast g)(z) = \left( 1 + \sum_{k \geq 1} \hbar^k c_k \right) f(x)g(y) \bigg|_{x=y=z}.
\]  

(5.2)

Here \( c_k \) are certain differential operators acting by the \( x,y \)-arguments and obeying conditions (4.16). The corresponding weak asymptotics of the integral kernel is

\[
K_{x,y} = (2\pi \hbar)^{2n} \left( 1 + \sum_{k \geq 1} \hbar^k c'_k \right) \delta_x \delta_y,
\]  

(5.3)

where the prime ‘ means the transposition with respect to the measure \( dm \).

The easiest way to find the coefficients \( c_k \) in (5.2) or (5.3) is to substitute expression (5.2) into (2.20) and (4.14). Then we uniquely derive all \( c_k \) in terms of the Taylor expansion (3.12) of the Ether Hamiltonian as follows:

\[
f \ast g = fg - \frac{i\hbar}{2} \leftarrow \nabla \Psi \rightarrow g - \frac{\hbar^2}{8} \leftarrow \nabla \Psi \rightarrow \left( \nabla \Psi \right)^2 g + \ldots
\]  

(5.4)

Here \( \Psi = \omega^{-1} \) is the Poisson tensor on the phase space, \( \nabla \) is the covariant derivative corresponding to the symplectic connection \( \Gamma \), the lower arrows (\( \leftarrow \) or \( \rightarrow \)) indicate the multiplier (\( f \) or \( g \)) on which the derivative acts. Higher terms of (5.4) contain the curvature of \( \Gamma \).

Note that, together and simultaneously with deriving formulas (5.2), (5.4), we have to satisfy the quantum zero curvature equation (3.2) in which
the same \(*\)-product (5.2) should be used. Thus we are looking for the quantum Ether Hamiltonian in the form

\[ \mathcal{H}^h = \mathcal{H} + h^2 \mathcal{C} + h^4 \mathcal{D} + \ldots, \]  

(5.5)

where \( \mathcal{H} \) is the classical Ether Hamiltonian and the quantum corrections \( \mathcal{C}, \mathcal{D}, \ldots \) obey variations of the zero curvature equation.

**Theorem 5.1.** Let \( (\mathcal{X}, \omega, \Gamma) \) be a symplectic simply-connected manifold, and let \( \omega^h = \omega + h^2 \alpha + h^4 \beta + \ldots \) be a closed deformation of \( \omega \). Then the Ether structure generates uniquely, explicitly, and geometrically (via \( \omega^h \) and \( \Gamma \)) the formal \( h \)-power expansions of the \(*\)-product (5.2), (5.4) and of the solution (5.5) of the quantum zero curvature equation near the diagonal. On the diagonal the boundary condition holds:

\[ \frac{1}{2} D\mathcal{H}_x^h(z) \bigg|_{z=x} = \mathcal{H}^h(x). \]

6 **Semiclassical asymptotics and quantum lift**

The weak asymptotics (5.2)–(5.4) loses some important information about the “wave function” \( K_{x,y} \). A more accurate approximation is the semiclassical one, say, the WKB-approximation

\[ K_{x,y} = e^{i \Phi_{x,y}} \phi_{x,y} + O(\hbar), \]  

(6.1)

which works well outside the focal set (in the semiclassically-simple domain). The difference between (6.1) and (5.3) is the same as the difference between (2.2) and (2.3).

The phase \( \Phi_{x,y} \) in (6.1) carries a dynamic geometry of the \(*\)-product. At the “initial moment” \( x = y \) this phase is zero, but its “time” derivatives are not zero:

\[ \Phi_{y,y}(z) = 0, \quad \partial_y \Phi_{x,y}(z) \bigg|_{x=y} = \mathcal{H}_y(z). \]  

(6.2)

Thus, in semiclassically-simple domains, the Ether structure is automatically generated by the WKB-phase of the \(*\)-product integral kernel.
The ∗-product itself, after the substitution of (6.1) into (2.7), reads
\[
(f ∗ g)(z) = \frac{1}{(2\pi\hbar)^2n} \iiint \left( e^{\frac{1}{\hbar} \Phi_{x,y}(z)} \varphi_{x,y}(z) + O(\hbar) \right) f(x)g(y) \, dm(x)dm(y).
\]
(6.3)

In order to compute the semiclassical approximation (6.1) of the ∗-product kernel, we use the Ether wave equation and Ether translations.

First, we resolve the nonlinearity of Eqs. (2.15), (2.25) following the scheme (2.26), (2.27). For this, one needs to know operators of the left regular representation. We compute them in the same way as in [15, 16].

**Lemma 6.1.** The left regular representation of the ∗-product algebra is given by the formula
\[
(f ∗ g)(x) = f^#(\frac{1}{2}, -i\hbar\partial_x)g(x),
\]
(6.4)
where the function $f^#$ on the secondary phase space $T^*\mathcal{X}$ is determined by the equation
\[
f^#(\frac{1}{2}, -i\hbar\partial_x + H^h_*)1 = f, \quad \forall x.
\]
(6.5)
Here the integers $1, 2, \ldots$ over the operators indicate (as in [49]) the mutual order of action of these operators.

**Corollary 6.2.** The asymptotic solution of Eq. (6.5) as $\hbar \to 0$ is given by
\[
f^#(x, \eta) = f(\ell(x, \eta)) - \frac{i\hbar}{2} \frac{\partial^2}{\partial x^k \partial \eta_k} f(\ell(x, \eta)) - \frac{i\hbar}{2} \frac{\partial}{\partial \eta_s} f(\ell(x, \eta)) \frac{\partial}{\partial \eta_k} \left( \partial_k \mathcal{H}_x(\ell(x, \eta)) \right) + O(\hbar^2).
\]
(6.6)
Thus the operation $\#$ in the classical limit $\hbar = 0$ is just the lift of functions from $\mathcal{X}$ to $T^*\mathcal{X}$ by means of the symplectic fibration $\ell$ (see Section 3 above). We denote $f^# = \hat{\ell}^*f$ and call the mapping $\hat{\ell}^*$ a left quantum lift.

Now let us return to our basic wave equations (2.25), (2.15) and to apply (6.4). The result is the following.
Theorem 6.3. (i) The integral kernel of the ∗-product satisfies the linear equation

\[ i\hbar \partial_z K_{x,y}(z) = L^h_x(z, -i\hbar \partial_z)K_{x,y}(z). \]  

(6.7)

Here the 1-form \( L^h \) over \( X \), taking values in functions over \( T^*X \), is defined by means of the left quantum lift of the quantum Ether Hamiltonian: \( L^h = \ell^*H^h \), i.e., by procedure (6.5):

\[ L^h_x(z, -i\hbar \partial_z + H^h z) = H^h_x, \quad \forall x, z. \]  

(6.8)

The asymptotics of \( L^h \) as \( \hbar \to 0 \) is

\[ L^h = L - \frac{i\hbar}{2} \frac{\partial^2 L}{\partial z^k \partial \xi^k} - \frac{i\hbar}{2} A + O(\hbar^2), \]  

(6.9)

where

\[ L_x = \ell^*H_x, \quad \text{or} \quad L_x(z, \xi) = H_x(\ell(z, \xi)), \quad \xi \in T^*_zX, \]  

(6.10)

and

\[ A_x(z, \xi)_j = \frac{\partial}{\partial \xi_s}L_x(z, \xi)_j \frac{\partial}{\partial \xi_k}H_x(\ell(z, \xi))_s. \]  

(6.11)

(ii) If the arguments \( x, y, z \) are considered all together without separation of “space” and “time” variables, then the linear equation for the kernel \( K = K_{x,y}(z) \) reads

\[ i\hbar \Delta^h K = \Delta^h K. \]  

(6.12)

Here

\[ \Delta^h = L^h_x(z, -i\hbar \partial_z) + L^h_y(z, -i\hbar \partial_x) + L^h_z(z, -i\hbar \partial_y), \]  

where the 1-form \( L^h \) is defined in (6.8) and (6.9).

So, for the integral kernel of the ∗-product we derived the linear(!) pseudodifferential equation (6.7) equipped with the Cauchy data (2.16) or the linear equation (6.12) with the Cauchy data

\[ K \bigg|_{x=y=z} = \mu^2. \]  

(6.13)

Now it is a routine exercise to compute the semiclassical asymptotics of the solution \( K \) in the semiclassically-simple domain without focal effects. We substitute (6.1) into (6.7) or (6.12) and derive a Hamilton–Jacobi equation for the phase \( \Phi = \Phi_{x,y}(z) \) and a transport equation for the amplitude \( \varphi = \varphi_{x,y}(z) \).
7 Intrinsic Hamilton–Jacobi and transport equations

To the pseudodifferential equation (6.7), one has to assign the following *intrinsic* Hamilton–Jacobi equation (with multidimensional “time”):

\[
\partial_x \Phi_{x,y}(z) + \mathcal{L}_x(z, \partial_z \Phi_{x,y}(z)) = 0, \quad \Phi \bigg|_{x=y} = 0, \tag{7.1}
\]

where \( \mathcal{L}_x \) is given by (6.10) via the classical Ether Hamiltonian.

The Hamilton system (with multidimensional “time”), which is related to (7.1), is

\[
\begin{cases}
\partial_x z = D\mathcal{L}_x / D\xi, & z \bigg|_{x=y} = b \in \mathcal{X}, \\
\partial_x \xi = -D\mathcal{L}_x / Dz, & \xi \bigg|_{x=y} = 0. \tag{7.2}
\end{cases}
\]

Recall that \( \xi \) is the momentum dual to \( z \) so that

\[ \xi = \partial_z \Phi_{x,y}(z), \tag{7.3} \]

and the zero Cauchy data for \( \xi \) in (7.2) follow from the zero Cauchy data for \( \Phi_{x,y} \) at \( x = y \).

Let us denote

\[ c = \ell(z, \xi). \tag{7.4} \]

Then for this equation we obtain from (7.2):

\[
\frac{\partial c^m}{\partial x^k} = \{\ell^n, \ell^m\} D_n \mathcal{H}_x(c)_k.
\]

Since \( \ell \) is symplectic \( \{\ell^n, \ell^m\} = \Psi^{nm}(\ell) \), then

\[
\frac{\partial c^m}{\partial x^k} = D_t \mathcal{H}_x(c)_k \Psi^{km}(c), \quad c \bigg|_{x=y} = b. \tag{7.5}
\]

So the point \( c \) is the Ether translation of the point \( b \):

\[ c = g_{x,y}(b). \tag{7.6} \]

Now from the first equation (7.2) it follows that

\[
\frac{\partial z^r}{\partial x^k} D_t \mathcal{H}_x(c)_r = D_t \mathcal{H}_x(c)_k,
\]
and from (6.16) we obtain
\[ \frac{\partial c^m}{\partial x^k} = \frac{\partial z^r}{\partial x^k} D_1 \mathcal{H}_z(c), \Psi^{lm}(c). \]

On the other hand, the reflection equation (3.15) implies
\[ \frac{\partial s^m}{\partial x^k} = \frac{\partial z^r}{\partial x^k} \frac{\partial s^m}{\partial z^r} = \frac{\partial z^r}{\partial x^k} D_1 \mathcal{H}_z(s_z), \Psi^{lm}(s_z). \]

Comparing this relation with the previous one, we see that \( c^m \) are equal to \( s^m \) if their initial data at \( x = y \) coincide. Thus we prove that
\[ c = s_z(b). \quad (7.7) \]

By definition (7.4), we also have
\[ \xi = \mathcal{H}_z(c) = \mathcal{H}_z(s_z(b)) = -\mathcal{H}_z(b). \quad (7.8) \]

From (7.3), (7.8), and (7.1) we get the derivatives of \( \Phi_{x,y}(z) \) with respect to \( z \) and with respect to \( x \). The derivative with respect to \( y \) can be computed, say, from the skew-symmetry condition (2.18) which implies \( \Phi_{x,y}(z) = -\Phi_{y,x}(z) \). So, we have \( \partial_y \Phi_{x,y}(z) = -\partial_y \Phi_{y,x}(z) = \mathcal{H}_y(\ell(z, \partial_z \Phi_{y,x}(z))) \). Here we have again used (7.3), (7.8), and (7.1).

Thus we have proved the following statements.

**Lemma 7.1.** (i) The pair of Eqs. (7.7), (7.8) determines the trajectories \( z = z(x') \), \( \xi = \xi(x') \) of the Hamilton system (7.2) in \( T^*X \) via the trajectories \( c = c(x') \) of the Ether system (7.5) in \( X \).

(ii) In the semiclassically-simple domain\(^6\) the phase of the *-product kernel is determined by the Ether Hamiltonian as follows:
\[ d(\Phi_{x,y}(z)) = \mathcal{H}_x(a) + \mathcal{H}_y(b) + \mathcal{H}_z(c). \quad (7.9) \]

Here, on the left-hand side, the differential is taken with respect to all variables \( x, y, z \) and, on the right-hand side, the points \( a, b, c \) are taken from the equations
\[ c = s_z(b), \quad b = s_y(a), \quad a = s_x(c). \quad (7.10) \]

\(^6\text{i.e., in a domain where a solution of (7.10) exists and is unique.}\)
These points are related to one another via the pseudogroup operation \( \ast \) on \( \mathcal{X} \) corresponding to the “Cartan structure” as in [36]:

\[
\begin{align*}
  z &= x \ast y, \\
  x &= z \ast y, \\
  y &= z \ast x.
\end{align*}
\] (7.11)

(iii) The Poincare–Cartan integral representation for the solution of the Hamilton–Jacobi equation (7.1) is

\[
\Phi_{x,y}(z) = \int_y^x \mathcal{H}(a) - \int_b^z \mathcal{H}(b).
\] (7.12)

Here the first integral is taken along an arbitrary path \( \gamma \) from \( y \) to \( x \), and the second integral is taken along the corresponding path \( \{z(x') \mid x' \in \gamma\} \) from \( b \) to \( z \) (see item (i) above).

**Corollary 7.2.** Let the triple \( x, y, z \) belong to the semiclassically-simple domain, and let \( \Delta(x,y,z) \) be the triangle membrane in \( \mathcal{X} \) bounded by the Ether geodesics through the points \( x, y, z \) connecting the vertices \( a, b, c \). The solution of the intrinsic Hamilton–Jacobi equation (7.1) is given by the symplectic area of this triangle:

\[
\Phi_{x,y}(z) = \int_{\Delta(x,y,z)} \omega.
\] (7.13)

This statement specifies the result of [40] by fixing the choice of sides of the triangle in (7.13), and proves that formula (7.13) actually presents the phase of the \( \ast \)-product kernel determined by Eqs. (2.15)–(2.19).

Now let us calculate the amplitude \( \varphi = \varphi_{x,y}(z) \) in the semiclassical representation (6.1). It follows from (6.7), (6.9), and (2.16) that this amplitude obeys the intrinsic transport equation

\[
\left( \frac{\partial}{\partial x} + \frac{\partial \mathcal{L}_x}{\partial \xi} \frac{\partial}{\partial z} + \frac{1}{2} \text{tr} \left( \frac{\partial^2 \mathcal{L}_x}{\partial \xi \partial z} \frac{\partial^2 \Phi}{\partial z \partial \xi} + \frac{\partial^2 \mathcal{L}_x}{\partial z \partial \xi} \right) + \frac{1}{2} \mathcal{A}_x \right) \varphi = 0,
\] (7.14)

where \( \mathcal{L}_x \) is determined by (6.10) via the classical Ether Hamiltonian. In this equation, instead of the argument \( \xi \), one has to substitute the derivative of the phase \( \Phi \) from (7.3).
Solving (7.14) directly, we obtain
\[ \varphi_{x,y}(z) = 2^n \mu^2 \det \left( I - D(s_z \circ s_x \circ s_y)(b) \right)^{-1/2}. \] (7.13)

In particular, we have proved the conjecture [40].

**Corollary 7.3.** (i) Let \( \mathcal{X} \) be a reflective simply connected symplectic manifold. In the semiclassically-simple domain the amplitude of the \(*\)-product kernel (6.1) over \( \mathcal{X} \) is given by formula (7.13) in which \( b \) is the fixed point\(^7\) of the mapping \( s_z \circ s_x \circ s_y \). The equivalent formula for the amplitude, which does not use reflections, is the following:

\[ \varphi_{x,y}(z) = 2^n \mu^2 \left( \frac{\det \partial_y D_z \Phi_{x,y}(z) \cdot \det \omega(b)}{\det D\mathcal{H}_y(b) \cdot \det D\mathcal{H}_z(b)} \right)^{1/2}. \]

(ii) The semiclassically multiple domain of Eq. (2.25) is formed by those triples \( x, y, z \in \mathcal{X} \) for which the mapping \( s_z \circ s_x \circ s_y \) has several isolated fixed points (say, on the sphere \( S^2 \) any nonfocal triple is of multiplicity 2). If the triple \( x, y, z \) belongs to a semiclassically-multiple domain, then in formula (6.1) one has to take a sum over all possible Ether geodesic triangles \( \Delta(x, y, z) \) and to multiply each summand by a suitable exponential \( \exp \{ -i \frac{\pi}{2} m \} \), where \( m \in \mathbb{Z} \) is the Maslov index on the graph of the groupoid multiplication.

(iii) The focal set of the basic equation (2.25), where the asymptotics of the solution is singular as \( \hbar \to 0 \), consists of those triples \( x, y, z \in \mathcal{X} \) for which the symplectic mapping \( s_z \circ s_x \circ s_y \) has nonisolated fixed points.

(iv) If the second Betti number of the manifold \( \mathcal{X} \) is nontrivial, then the existence of the global semiclassical solution of the basic equation (2.25) is ensured by the quantization condition (5.1).

Actually, the semiclassical approximation of the kernel \( K_{x,y}(z) \), globally in the phase space, including a neighborhood of the focal set, can also be obtained by applying some version of the “canonical operator” [14] to Eq. (6.7).

There are some interesting papers about the semiclassical approximation of the \(*\)-product kernel in the special symmetric case (the curvature \( R \) is covariantly constant) [21, 32, 50]. For instance, in [50] it was first demonstrated that the focal set, where the kernel \( K_{x,y}(z) \) is not of the WKB-type (6.1), exists not only on such topologically nontrivial manifolds like \( S^2 \), but even on the Lobachevsky plane with its standard symplectic form.

---

\(^7\)This is equivalent to Eqs. (7.10).
The formal deformation quantization in the symmetric symplectic case was first constructed in [41]. It would be interesting to check whether the explicit $\hbar$-power series for the $*$-product on symmetric spaces obtained in [41] follows as a particular case from our wave equations.

There is another special case, namely, Kählerian manifolds. In [51], the presence of the complex structure allowed deriving integral representations for $*$-products via coherent states under some additional conditions. The most critical condition is that the Liouville measure should be a reproducing measure (up to a constant multiplier; see [51]). This condition certainly holds on homogeneous manifolds. Schematically, this means that the integral $*$-product kernel has the form $K = \exp\{\frac{i}{\hbar} \Phi\} \cdot \text{const}$, i.e., the amplitude $\varphi$ is constant, assuming that the integration measure $dm$ is the Liouville one. Of course, on general Kählerian manifolds $\varphi$ is not constant and the entire series for the amplitude $\varphi + \hbar \varphi^1 + \ldots$ can be calculated explicitly [52]. Thus the asymptotic quantization is geometrically well defined and the semiclassical expansion of the integral $*$-kernel is explicitly known (without the focal set problem) over any Kählerian manifold. It is easy to see that the scheme (2.12), (2.15), (2.25) works in this polarized case, although the cyclic property (2.19) fails.

8 Exterior quantum dynamics

There is another opportunity to apply the intrinsic geometry generated by the Ether Hamiltonian. Let us consider a Hamilton function $H$ on the phase space $X$ and the corresponding “exterior” quantum equations

$$i\hbar \frac{d}{dt} G_t = H \ast G_t = G_t \ast H.$$  \hspace{1cm} (8.1)

If the operator representation (2.5), (2.8) of the $*$-product algebra is known, then

$$G_t(x) = \text{tr} \left( S_x \cdot \exp \left\{ - \frac{it}{\hbar} \hat{H} \right\} \right), \quad G_0 = 1.$$  \hspace{1cm} (8.2)

Here the exponential $\exp \left\{ - \frac{it}{\hbar} \hat{H} \right\}$ represents the solution of the “exterior” Schrödinger evolution equation. So, the function $G_t$ is the symbol of the quantum evolution operator.
To solve Eq. (8.1) means that one has to take the exponential $\exp\{-\frac{\hbar}{\hbar}H\}$ and then replace the classical multiplication determining this exponential by the quantum multiplication $\ast$. The last procedure can be made using the intrinsic Ether dynamics following (2.21).

**Theorem 8.1.** Let $X$ be a reflective simply connected symplectic manifold.

(i) The Schrödinger evolution equation (8.1) is equivalent to

$$i\hbar\frac{d}{dt}G_t = H^\#(\dot{x},-i\hbar\partial_x)G_t, \quad G_0 = 1,$$

where $H^\# = \ell^*H$ is the left quantum lift of $H$ defined by (6.5).

(ii) Denote by $\gamma^t_H$ the Hamilton flow generated by $H$ on $X$. Then the semiclassically-simple domain for problem (8.3) as $\hbar \to 0$ consists of those $x \in X$ for which the mapping $s_x \circ \gamma^t_H$ has a unique fixed point.

If this mapping has several (finitely many) fixed points, then $x$ belongs to the semiclassically-multiple domain.

The focal set of problem (8.3) consists of those $x$ for which the mapping $s_x \circ \gamma^t_H$ has nonisolated fixed points.

(iii) In the semiclassically-simple domain the asymptotics of $G_t$ is the following

$$G_t = \exp\left\{\frac{i}{\hbar} \int_{\Sigma_t} \omega - \frac{it}{\hbar}H\right\} \varphi_t + O(\hbar).$$

Here $\Sigma_t(x)$ is a certain dynamic segment in $X$ (a sickle-shaped membrane); the exterior arc of this segment is a Hamilton trajectory of $H$, whose time-length is $t$, and the other arc of the segment connecting its ends (spikes of the sickle) is given by Ether geodesics through the midpoint $x$. The value of $H$ in (8.4) is taken on the exterior arc of the segment.

The amplitude $\varphi_t$ in formula (8.4) is given by

$$\varphi_t(x) = 2^n \left(\det(I - D(s_x \circ \gamma^t_H)(x_0))\right)^{-1/2},$$

where $x_0$ is the fixed point of the mapping $s_x \circ \gamma^t_H$.

(iv) In the semiclassically-multiple domain on $X$, the asymptotics of $G_t$ is a sum of expressions like (8.4) over all fixed points of the mapping $s_x \circ \gamma^t_H$; each of the summands is multiplied by an exponential $\exp\{-\frac{\pi}{2}m\}$, where $m \in \mathbb{Z}$ is the Maslov index on the graph of $\gamma^t_H$. 

27
In the case of the Euclidean phase space $X = \mathbb{R}^{2n}$, formulas like (8.4) on membranes bounded by straight-line chords were first derived in [53, 54]. In [33] see also the case of the Euclidean space endowed with the “magnetic” symplectic form $\omega$ and with an additional “electric” form along space-time directions; in this case membranes were constructed by means of special “magnetic wings.”

References

[1] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, *Quantum mechanics as a deformation of classical mechanics*, Lett. Math. Phys. 1 (1975/77), 521–530.

[2] F. A. Berezin, *General concept of quantization*, Comm. Math. Phys. 40 (1975), 153–174.

[3] A. Connes, *Noncommutative Geometry*, Acad. Press, London, 1994.

[4] M. V. Karasev and V. P. Maslov, *Asymptotic and geometric quantization*, Uspekhi Mat. Nauk 39 (1984), no. 6, 115–173; English transl., Russian Math. Surveys 39 (1984), no. 6, 133–205.

[5] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, *Deformation theory and quantization*, Ann. Physics 111 (1978), 61–151.

[6] A. Lichnerowicz, *Deformation of quantification*, Lecture Notes in Phys. 106 (1979), 209–219.

[7] B. Fedosov, *Formal quantization*, in: *Some Topics of Modern Math. and Their Appl. to Problems of Math. Physics*, 1985, pp. 129–136. (Russian).

[8] B. Fedosov, *A simple geometrical construction of deformation quantization*, J. Diff. Geom. 40 (1994), 213–238.

[9] M. Kontsevich, *Deformation quantization of Poisson manifolds*, Preprint \(q\)-alg/9709040, 1997.

[10] A. Cattaneo and G. Felder, *Poisson sigma models and symplectic groupoids*, math.sg/0003023.
[11] A. Cattaneo and G. Felder, *A path integral approach to the Kontsevich quantization formula*, math/9902090.

[12] J. Klauder, *Quantization is geometry. After all*, Ann. Physics 188 (1988), 120–141.

[13] B. Kostant, *Quantization and representation theory*, London Math. Soc. Lecture Note Ser. 34 (1979), 287–316.

[14] V. Maslov, *Perturbation Theory and Asymptotic Methods*, Moscow Univ. Publ., 1965; French transl., Dunod, Paris, 1972.

[15] M. V. Karasev, *Quantization of nonlinear Lie-Poisson brackets in semiclassical approximation*, Inst. Theor. Phys., Kiev, Preprint N ITP-85-72P, 1985. (Russian).

[16] M. V. Karasev and V. P. Maslov, *Nonlinear Poisson Brackets. Geometry and Quantization*, Nauka, Moscow, 1991; English transl., Ser. Translations of Mathematical Monographs, Vol. 119, Amer. Math. Soc., Providence, RI, 1993.

[17] M. A. Rieffel, *Deformation quantization of Heisenberg manifold*, Comm. Math. Phys. 122 (1989), 531–562.

[18] M. V. Karasev, *Advances in quantization: quantum tensors, explicit star-products, and restriction to irreducible leaves*, Diff. Geometry and Its Appl. 9 (1998), 89–134.

[19] M. V. Karasev and E. M. Novikova, *Non-Lie permutation relations, coherent states, and quantum embedding*, in: Coherent Transform, Quantization, and Poisson Geometry (M. Karasev, ed.), Amer. Math. Soc., Providence, RI, 1998.

[20] J. M. Gracia-Bondia, *Generalized Moyal quantization on homogeneous symplectic spaces*, in: Deformation Theory and Quantum Groups (M. Gerstenhaber and J. Stasheff, eds.), Contemp. Math. 134 (1992), 93–114.

[21] P. Bieliavsky, *Strict quantization of solvable symmetric spaces*, math.qa/0010004.
[22] H. Weyl, *Reine Infinitesimalgeometrie*, Math. Z. 2 (1918), 384–411.

[23] E. Scholz, *Hermann Weyl’s “Purely Infinitesimal Geometry,”* in: *Proc. Intern. Congress of Math., Zürich, 1994*, Birkhäuser, Basel–Boston–Berlin, 1995, pp. 1592–1603.

[24] H. Weyl, *Selecta*, Birkhäuser, Basel–Boston–Berlin, 1956, p. 192.

[25] R. Stratonovich, *On distributions in representation space*, Zh. Éxper. Teoret. Fiz. 31 (1956), 1012–1020; English transl., Soviet Phys. JETP 4 (1957), no. 6, 891–898.

[26] C. Fronsdal, *Some ideas about quantization*, Rep. Math. Phys. 15 (1979), no. 1, 111–145.

[27] D. Arual, *The ∗-exponential*, in: *Quantum Theories and Geometry* (M. Cahen and M. Flato, eds.), Kluwer Akad, 1988, 23–51.

[28] A. Grossmann and P. Huguenin, *Group-theoretical aspects of the Wigner–Weyl isomorphism*, Helvetica Physica Acta 51 (1978), 252–261.

[29] F. A. Berezin *Quantization*, Izv. Akad. Nauk SSSR Ser. Mat. 38 (1974), no. 5; English transl., Math. USSR Izv. 8 (1974), no. 5, 1109–1165.

[30] I. Batalin and I. Tyutin, *Quantum geometry*, Nuclear Phys. B 345 (1990), 645–650.

[31] A. Weinstein, *Classical theta-functions and quantum tori*, Publ. RIMS, Kyoto Univ. 30 (1994), 327–333.

[32] Zhao-Hui Qian, *Groupoids, Midpoints, and Quantization*, Thesis Univ. of California, Berkeley, 1997.

[33] M. V. Karasev and T. Osborn, *Symplectic areas, quantization, and dynamics in electromagnetic fields*, J. Math. Phys. 43 (2002), no. 2, 756–788 [quant-ph/0002041].

[34] M. V. Karasev and T. Osborn, *Magnetic curvature of quantum phase space*, in: *Proc. A. Sakharov Conference, Phys. Inst. Russian Akad. Sci.*, Moscow, June 2002 (to appear).
[35] C. Emmrich and A. Weinstein, The differential geometry of Fedosov's quantization, in: Lie Theory and Geometry. In Honor of B. Kostant, Progr. Math. 123 (1994), Birkhäuser, New York, 217–240.

[36] M. V. Karasev, Analogues of objects of the theory of Lie groups for nonlinear Poisson brackets, Izv. Akad. Nauk SSSR 50 (1986), no. 3, 508–538; English transl., Math. USSR Izv. 28 (1987), no. 3, 497–527.

[37] M. V. Karasev, Maslov quantization conditions in higher cohomology and analogs of notions developed in Lie theory for canonical fiber bundles of symplectic manifolds. I, II. Preprint MIEM, 1981. Deposited at VINITI (March 12, 1982, No. 1092-82, 1093-82). Abstract: in “Ref. Zhurnal Matematika” (1982) No. 7A676, 7A677; English transl., Selecta Math. Soviet 8 (1989), no. 3, 213–258.

[38] A. Weinstein, The local structure of Poisson manifolds, J. Differential Geom. 18 (1983), 523–557.

[39] A. Weinstein, Symplectic groupoids and Poisson manifolds, Bull. Amer. Math. Soc. 16 (1987), 101–104.

[40] A. Weinstein, Traces and Triangles in Symmetric Symplectic Spaces, Contemp. Math. 179 (1994), 262–270.

[41] P. Bieliavsky, M. Cahen, and S. Gutt, Symmetric symplectic manifolds and deformation quantization, in: Modern Group Theor. Methods in Phys. (J. Bertrand et al., eds.), Kluwer Acad., 1995, pp. 63–73.

[42] M. V. Karasev and V. P. Maslov, Algebras with general permutation relations and their applications. II, Itogi Nauki i Tekhniki: Sovremennye Problemy Mat. 13 (1979), VINITI, Moscow, 145–267; English transl., J. Soviet Math. 15 (1981), no. 3, 273–368.

[43] M. V. Karasev and V. P. Maslov, Global asymptotic operators of regular representation, Dokl. Akad. Nauk SSSR 257 (1981), no. 1, 33–37; English transl., Soviet Math. Dokl. 23 (1981), 228–232.

[44] M. De Wilde and P. Lecomte, Existence of star products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds, Lett. Math. Phys. 7 (1983), 487–496.
[45] J. Huebschmann, *On the quantization of Poisson algebras*, in: *Symplectic Geometry and Mathematical Physics, Actes du colloque en l’honneur de J.-M.Souriau* (P. Donato et al., eds.) Birkhäuser, Basel–Boston, 1991, 204–233.

[46] H. Omori, Y. Maeda, and A. Yoshioka, *Weyl manifolds and deformation quantization*, Adv. Math. 85 (1991), no. 2, 224–255.

[47] H. Omori, Y. Maeda, and A. Yoshioka, *Deformation quantization of Poisson algebras*, Contemp. Math. 179 (1994), 213–240.

[48] B. Fedosov, *Deformation Quantization and Index Theory*, Akademie Verlag, Berlin, 1996.

[49] V. Maslov, *Operator Methods*, Nauka, Moscow, 1973; English transl., Mir, Moscow, 1976.

[50] G. Tuynman and P. Rios, *Weyl quantization from geometric quantization*, Preprint.

[51] M. Cahen, S. Gutt, and J. Rawnsley, *Quantization of Kähler manifolds*. I, J. Geom. Phys. 7 (1990), 45–62; II, Trans. Amer. Math. Soc. 337 (1993), 73–98; III, Lett. Math. Phys. 30 (1994), 291–305; IV, Lett. Math. Phys. 30 (1995), 159–168.

[52] M. V. Karasev, *Quantum surfaces, special functions, and the tunneling effect*, Lett. Math. Phys. 59 (2001), 229–269.

[53] M. Berry, *Semi-classical mechanics in phase space: a study of Wigner’s function*, Philos. Trans. Roy. Soc. London Ser. A 287 (1977), 237–271.

[54] M. Marinov, *An alternative to the Hamilton–Jacobi approach in classical mechanics*, J. Phys. A 12 (1979), no. 1, 31–47.