MOMENT MAPS AND GALOIS ORBITS FOR SIC-POVMS

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Abstract. The equations that define covariant SIC-POVMs are interpreted in terms of moment maps. Attention is focussed on orbits of a cyclic subgroup of a maximal torus and their images in the moment polytope. In the SIC case, these images lie in an intersection of quadrics, which we describe explicitly. We then develop the conjectural relationships with number field theory by describing the structure of Galois orbits of overlap phases.

1. Introduction

It is conjectured that, for every positive integer \( d \), there exist \( d^2 \) points in the complex projective space \( \mathbb{CP}^{d-1} \) that are pairwise equidistant with respect to the standard Fubini-Study metric. This is equivalent to asserting that, as an adjoint orbit in the Lie algebra \( \mathfrak{su}(d) \cong \mathbb{R}^{d^2-1} \) of Killing vector fields, projective space contains the \( d^2 \) vertices of a regular simplex. The common distance of separation depends only on \( d \) and the diameter of \( \mathbb{CP}^{d-1} \). Such sets of points were originally studied under the guise of equiangular lines \([12, 27, 20]\), and are related to topics such as tight frames in design theory.

Versions of the conjecture arose from the pioneering work of Gerard Zauner \([35]\), who revitalized the subject from the viewpoint of quantum information. Such a set of \( d^2 \) points defines a so-called symmetric informationally complete positive operator measure, for short SIC-POVM or (in this paper) ‘SIC set’. The concept of a POVM in quantum theory was introduced in \([10, 21, 11]\), though in the present context it is an entirely discrete object. The rank-one projections defined by the \( d^2 \) points provide an optimal way to measure a mixed state, and applications of SIC sets arise in quantum tomography, a topic advanced by Ugo Fano \([13]\).

The advent of computing has emphasized the importance of finite-dimensional Hilbert spaces in quantum theory, and in particular metric properties of \( \mathbb{CP}^{d-1} \). Virtually all known SIC sets arise as orbits of a discrete Heisenberg group, acting as \( \mathbb{Z}_d^2 = \mathbb{Z}_d \times \mathbb{Z}_d \) on \( \mathbb{CP}^{d-1} \) (\( \mathbb{Z}_d \) denotes \( \mathbb{Z}/d\mathbb{Z} \) throughout the paper). A vector \( z \in \mathbb{C}^d \) is called fiducial if the orbit containing \( [z] \) is a SIC set. Fiducial vectors for such orbits have been constructed exactly for all \( d \leq 21 \) and in a few higher dimensions. There is strong numerical evidence for their existence for all \( d \) up to a 3-figure value, and seminal papers on the subject include \([11, 6, 15, 18, 30, 32, 36]\).

\[^1\]On a historical note that places the discipline in a family context, his brother Robert worked on Shannon-Fano coding \([33]\), whilst their father was the algebraic geometer Gino Fano.
Representative unit vectors \( \mathbf{w}, \mathbf{z} \) of any two distinct points in a SIC set satisfy

\[
|\langle \mathbf{w} | \mathbf{z} \rangle| = \frac{1}{\sqrt{d+1}}
\]

with respect to the standard Hermitian product in \( \mathbb{C}^d \). The problem is stated in elementary terms, though its interpretation is much deeper. The inner product represents the overlap between pure quantum states described by \[ \mathbf{w} \] and \[ \mathbf{z} \], and its modulus a transition probability. The simplest way to describe a Heisenberg SIC set appears to be via the \( d \times d \) matrix \( \Phi_\mathbf{z} \) of so-called overlap phases indexed by \( \mathbb{Z}_2^d \), as in Definition 3.2. We shall often regard \( \Phi_\mathbf{z} \) as a mapping \( \mathbb{Z}_2^d \to \mathbb{C} \), and (1.1) implies that the image of any non-identity element lies on the circle of radius \( 1/(\sqrt{d+1}) \) centred at 0. The papers [2, 4, 26] establish strong links between such SIC sets and abelian extensions of the real quadratic field \( \mathbb{K} = \mathbb{Q}(\sqrt{D}) \) where \( D \) is the square-free part of \( (d-3)(d+1) \). In many examples, the phases actually lie in a ray class field extension of \( \mathbb{K} \) characterized by the ramification of primes dividing \( d \), though the reasons for these empirical observations remain largely unexplained.

The existence of SIC sets for \( d = 2 \) and 3 is elementary. For \( d = 2 \), \( \mathbb{CP}^1 \) coincides (as a Riemannian manifold) with the round 2-sphere. Any SIC set is an inscribed tetrahedron, and any two are related by an element of \( \text{SO}(3) \cong \text{SU}(2)/\mathbb{Z}_2 \). For \( d = 3 \), there is a one-parameter family of SIC sets on \( \mathbb{CP}^2 \) up to the action of the isometry group \( \text{SU}(3) \), any one of which is isometric to the orbit of a suitable fiducial vector. Nonetheless, it is much harder to prove that there are no other SIC sets on \( \mathbb{CP}^2 \), and both known proofs involve an element of computation [24, 34]. It is believed that there are only finitely many isometry classes of SIC sets in dimensions \( d > 3 \), though the current state of the art may be analogous to that of Calabi-Yau spaces twenty years after Yau’s proof of their existence.

One aim of this paper is to extend the moment map approach of [24] to arbitrary dimensions. This has led us independently to a formulation involving the discrete Fourier transform, which is known to be central in understanding the action of the Heisenberg group [17], though our emphasis is more on the underlying geometry. We fix a cyclic subgroup \( C \cong \mathbb{Z}_d \) of \( \mathbb{Z}_2^d \) that generates a maximal torus \( T^C \) of \( \text{SU}(d) \), and a moment map \( \mu^C \) from \( \mathbb{CP}^{d-1} \) onto a standard simplex \( \Delta \) in \( \mathbb{R}^{d-1} \). We relate \( \mu^C \) to the map \( [\mathbf{z}] \mapsto \Phi_\mathbf{z}|C \) using the discrete Fourier transform. In Theorem 4.6, we describe the image \( \mathcal{A} \) by \( \mu^C \) of points of \( \mathbb{CP}^{d-1} \) whose \( C \) orbits could hypothetically lie in a SIC set. \( \mathcal{A} \subset \Delta \) is universal in the sense that it does not depend on \( C \). For example when \( d = 2n + 1 \) is even, \( \mathcal{A} \) is a Clifford torus, suitably interpreted. This approach will not help in the quest for SIC sets in arbitrary dimensions, but it may conceivably lead to the construction of non-covariant examples. In any case, it provides a striking framework for understanding fiducial vectors in low dimensions. For the remainder of the paper, we work with Heisenberg SIC sets. Organizing the overlap phases as a collection of moment maps provides a useful guide for understanding how the phases relate to each other.

The other aim of this paper is to build on the conjectural relationship with number theory developed in [2, 3, 4, 5, 26]. It is conjectured that one can always find a fiducial that is strongly centred, see [3], an assumption that allows one to give an explicit description.
of the action of the Galois group on the overlap phases. We shall always adopt this assumption, and work exclusively with such a fiducial vector $z$. Following notation of Appleby [1], we define

$$\bar{d} = \begin{cases} d & \text{if } d \text{ is odd} \\ 2d & \text{if } d \text{ is even}. \end{cases}$$

When $d$ is even, there is a natural way to extend the domain of $\Phi_z$ from $\mathbb{Z}_d^3$ to $\mathbb{Z}_{\bar{d}}^3$, so that as a matrix its size is doubled. Let $\text{GL}_2\mathbb{Z}_{\bar{d}}$ denote the group of invertible $2 \times 2$ matrices with values in $\mathbb{Z}_{\bar{d}}$. As described in [2], there is a subgroup $M$ of $\text{GL}_2\mathbb{Z}_{\bar{d}}$ such that the action of $M$ corresponds to Galois conjugation in the following sense. Let $E_1$ denote the field extension of $K$ generated by the values of $\Phi_z$. For each $G \in M$, there exists an element $g$ of a Galois group $G$ acting on $E_1$ such that $\Phi_z \circ G = g \circ \Phi_z$, see Theorem 5.3 and its refinement (5.22). This will enable us to study the effect of $M$ on the overlap phases.

Recall that the ring $\mathcal{O}_K$ of integers of $K = \mathbb{Q}(\sqrt{D})$ is generated by $\frac{1}{2}(1 + \sqrt{D})$ if $D = (d - 3)(d + 1)$ is congruent to 1 modulo 4, otherwise by $\sqrt{D}$. In all the known examples, there is a strongly centred fiducial such that

$$M \cong (\mathcal{O}_K/(\bar{d}))^\times,$$

a property that we call algebraic. Known fiducial vectors are classified into types $z, a_4, a_6, a_8$, and (1.2) determines this type for algebraic fiducials, while every known non-algebraic fiducial is type-$z$. Previously it was not known whether it is possible to have type-$a_6$ fiducials, but this criterion suggests that these will occur for all $d$ congruent to 3 modulo 27.

In all the known fiducials where $E_1$ is minimal, it is a ray class field [4]. In this case we show that the subset $S$ of $M$ corresponding to the identity in the Galois group is the cyclic group generated by the fundamental unit in $\mathcal{O}_K$ modulo $\bar{d}$. We also describe the structure of $E_1$ in general.

The group structure of $(\mathcal{O}_K/(\bar{d}))^\times$ has been studied long ago [29], and it is this that allows us to investigate the action of $M$ on the overlap phases. In Theorem 6.5, we find a one-to-one correspondence between orbits of $M$ and factors of $\bar{d}$ in $\mathcal{O}_K$ (in the algebraic case, with suitable adjustments in the other case). Noting that, in our setting, prime factors of $\bar{d}$ in $\mathcal{O}_K$ that are rational must be congruent to 2 modulo 3, we find

**Corollary 1.1.** The set of overlap phases are all Galois conjugate if and only if $d$ is an odd prime congruent to 2 modulo 3.

This result relates to a recent conjecture [26], which states that in the case when $d$ is an odd prime congruent to 2 modulo 3, a solution to the SIC-POVM problem can be constructed from a Stark unit. This uses an ansatz derived from the overlap phases all being Galois conjugate. Theorem 6.6 describes in general which field each Galois orbit takes values in. It may be possible to use this result to generalize this ansatz.

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2. Fubini-Study distance and SIC sets

Complex projective space \( \mathbb{CP}^{d-1} \) is a compact Kähler manifold. Its Riemannian metric \( g \) arises from the standard Hermitian form

\[
h(w, z) = \langle w | z \rangle = \sum_{i=1}^{n} \bar{w}_i z_i
\]
on \( \mathbb{C}^d \), which is invariant by the unitary group \( U(d) \). Note that \( h \) converts any point \([w]\) of \( \mathbb{CP}^{d-1} \) into a hyperplane

\[H_w = \ker h([w], \cdot),\]
and \( h \) is determined up to a constant by the correspondence \([w] \mapsto H_w\).

The Riemannian distance \( \delta \), obtained by integrating \( g \), satisfies

\[
\cos^2 \left( \frac{1}{2} \delta ([w], [z]) \right) = p([w], [z]),
\]
where

\[
p([w], [z]) = \frac{|\langle w | z \rangle|^2}{||w||^2 ||z||^2} = \frac{\langle w | z \rangle \langle z | w \rangle}{\langle w | w \rangle \langle z | z \rangle}.
\]

Since any two points \([w], [z]\) are contained in a totally geodesic projective line \( \ell_{w,z} \cong \mathbb{CP}^1 \cong S^2 \), the formula can be proved by restricting to this 2-sphere. The normalization ensures that the diameter of \( \mathbb{CP}^{d-1} \) naturally equals \( \pi \).

The description above yields

**Lemma 2.1.** \( p([w], [z]) \) equals the cross ratio of the four points \([w], [z], [z]', [w]'\) in order, where \([w]' = \ell_{w,z} \cap H_w \) and \([z]' = \ell_{w,z} \cap H_z\).

Points \([w], [z]\) of \( \mathbb{CP}^{d-1} \) represent pure quantum states, and the projective line they span the set of all their possible suppositions. Observables correspond to Hermitian matrices whose eigenvalues are the results of measurements. If \( z \) is the eigenvector of such a matrix then \( p([w], [z]) \) is the probability that the state \([w]\) is observed to coincide with \([z]\).

Further links with Fubini-Study geometry are described in [8].

**Remark 2.2.** Wigner’s theorem states that any isometry \( \phi \) of \( \mathbb{CP}^{d-1} \) arises from a unitary or conjugate unitary transformation of \( \mathbb{C}^d \). A proof of this fact was given by Freed [16] using the notion of holonomy, which we outline briefly\(^2\). Consider the effect of \( \phi \) on the almost complex structure \( J \) of \( \mathbb{CP}^{d-1} \). The Levi-Civita connection has holonomy \( U(d-1) \), so \( \nabla J \) and \( \nabla (\phi^* J) \) both vanish and \( R(\phi^* J) = \nabla_1 \nabla (\phi^* J) = 0 \). It follows from the invariant nature of the Riemann tensor \( R \in S^2 \mathfrak{h} \) (where \( \mathfrak{h} = \mathfrak{u}(n-1) \) is the holonomy algebra) that \( \phi^* J = \pm J \) and \( \phi \) is holomorphic or anti-holomorphic. In the former case, it lifts to (the transpose of) the linear map

\[
\tilde{\phi}: H^0 (\mathbb{CP}^{d-1}, L) \longrightarrow H^0 (\phi^* L) \cong H^0 (L),
\]
where \( L = \mathcal{O}(1) \). Since \( \phi \) is an isometry, \( \tilde{\phi} \) is unitary, as required.

\(^{2}\)to further acknowledge our sponsor
Let
\[ \mathcal{D} = \{ A \in \mathbb{C}^{d,d} : A = A^*, \; \text{tr} A = 1 \} \]
be the set of density matrices, and consider the Lie algebra \( \mathfrak{su}(d) \) of trace-zero anti-Hermitian matrices. The map
\[
(2.3) \quad \alpha : \mathcal{D} \rightarrow \mathfrak{su}(d), \quad A \mapsto i \left( A - \frac{1}{d} \mathbf{1} \right)
\]
is an affine isomorphism between these two spaces of real dimension \( d^2 - 1 \). The Lie algebra has a natural inner product defined by minus the Killing form:
\[
\langle A, B \rangle = -\frac{1}{2d} \text{tr}(AB),
\]
and this enables us to identify \( \mathfrak{su}(d) \) with its dual \( \mathfrak{su}(d)^* \). The mapping \( f : S^{2d-1} \rightarrow \mathcal{D} \) for which
\[
(2.4) \quad f(z) = z^* z = \begin{pmatrix}
|z_1|^2 & \bar{z}_1 z_2 & \bar{z}_1 z_3 & \cdots \\
\bar{z}_2 z_1 & |z_2|^2 & \bar{z}_2 z_3 & \cdots \\
\bar{z}_3 z_1 & \bar{z}_3 z_2 & |z_3|^2 & \cdots \\
\cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\]
is SU(d)-equivariant and the composition
\[
(2.5) \quad \alpha \circ f : \mathbb{C}P^{d-1} \hookrightarrow \mathfrak{su}(d) \cong \mathfrak{su}(d)^*
\]
can be identified with the moment mapping defined by the action of \( SU(d) \) on the symplectic manifold \( \mathbb{C}P^{d-1} \) \([25]\). It follows that \([2.5]\) defines an embedding of \( \mathbb{C}P^{d-1} \) in \( \mathfrak{su}(d)^* \) as a coadjoint orbit that is (up to a universal constant) isometric.

Given \([2.5]\), \( \mathcal{D} \) is the convex hull of the set of pure states \( z^* z \). An arbitrary point of \( \mathcal{D} \) represents a mixed quantum state, whereas an observable is represented by an arbitrary Hermitian matrix \( A \). The expectation that the observable is in the state \( \rho \in \mathcal{D} \) is then given by \( \text{tr}(\rho A) \), so that \( \rho \) can be viewed as a probability density \([7]\).

The acronym SIC-POVM is an abbreviation of symmetric informationally complete positive operator valued measure. In the context of \( \mathbb{C}^d \), this is a finite set of \( d^2 \) points, and we favour the terminology ‘SIC set’. We present two definitions; the first reflects the etymology of the italicized words:

**Definition 2.3.** A SIC set is a subset \( \{ P_\alpha \} \) of \( d^2 \) rank-one projections in \( \mathcal{D} \) such that
\[
\sum_{\alpha=1}^{d^2} P_\alpha = d \mathbf{1}, \quad \text{tr}(P_\alpha P_\beta) = \begin{cases} 
1 & \alpha = \beta \\
\lambda & \alpha \neq \beta,
\end{cases}
\]
for some fixed \( \lambda \in (0, 1) \).

By setting \( P_\alpha = z_\alpha^* z_\alpha \), we obtain \( d^2 \) points \( [z_\alpha] \) in \( \mathbb{C}P^{d-1} \) with the property that
\[
p(z_\alpha, z_\beta) = \lambda, \quad \alpha \neq \beta.
\]
The points represent equiangular 1-dimensional subspaces in \( \mathbb{C}^d \). Since \( P_\alpha = \mu([z_\alpha]) \), we can record the equivalent
Definition 2.4. A SIC set consists of a regular simplex in su(d) whose $d^2$ vertices lie in the orbit $\mathbb{CP}^{d-1}$.

The fact that, for fixed $d$, one knows the distance between any two points of a hypothetical SIC set helps to make the existence problem tractable. This is really a consequence of the second definition:

Lemma 2.5 ([12]). $d^2$ is the maximum number of mutually equidistant points possible in $\mathbb{CP}^{d-1}$, and in this case $\lambda$ must equal $1/(d+1)$.

Proof. Given an equidistant set $\{P_\alpha\}$, set $Q_\beta = P_\beta - \lambda 1$. Then
\[
\text{tr}(P_\alpha Q_\beta) = \begin{cases}
1 - \lambda & \alpha = \beta \\
0 & \alpha \neq \beta
\end{cases}
\]
So $\{P_\alpha\}$ is linearly independent in $i u(d)$. Applying $\text{tr}(Q_\beta \cdot)$ to
\[
1 = \sum c_\alpha P_\alpha,
\]
gives
\[
1 - \lambda d = c_\alpha (1 - \lambda) \forall \alpha.
\]
Then $d = d^2 c_\alpha$, and so $d(1 - \lambda d) = 1 - \lambda$. □

This innocuous result allows us to record

Definition 2.6. Two points $[w], [z]$ of $\mathbb{CP}^{d-1}$ are correctly separated if $p([w], [z]) = 1/(d+1)$.

The point being that any two distinct points of a SIC set in $\mathbb{CP}^{d-1}$ are correctly separated.

Example 2.7. We now describe some SIC-POVMs in very low dimensions. Equidistance can be verified by hand.

- **Case $d = 2$.** As remarked in the Introduction, a SIC set is a regular tetrahedron inscribed in $S^2 = \mathbb{CP}^1$. An example consists of the four points
  \[
  [1+r, 1+i], \quad [1+r, -1-i], \quad [1+i, 1+r], \quad [1+i, -1-r],
  \]
  where $r = \sqrt{3}$.

- **Case $d = 3$.** An example consists of the nine points
  \[
  [0, 1, -1], \quad [0, 1, -\omega], \quad [0, 1, -\omega^2],
  [-1, 0, 1], \quad [-\omega, 0, 1], \quad [-\omega^2, 0, 1],
  [e^{i\theta}, -1, 0], \quad [e^{i\theta}, -\omega, 0], \quad [e^{i\theta}, -\omega^2, 0],
  \]
  where $\omega = e^{2\pi i/3}$ and $\theta$ is any fixed angle. Any unordered set of nine mutually equidistant points in $\mathbb{CP}^2$ is isometric to one of the above for some $\theta$ [24, 34]. When $\theta = 0$, the points are the flexes of the cubic $x^3 + y^3 + z^3 - 3czxz = 0$ with $c \notin \{1, \omega, \omega^3\}$, forming the Hesse configuration [23]. Other special orbits were described by Zhu in [36].

- **Case $d = 4$.** Let $s = \sqrt{2}$ and $t = \sqrt{2 + \sqrt{5}}$. A SIC set consists of the orbit of the point
  \[
  [-t - i(s+t), \ 1 - s + i, \ t + i(t-s), \ 1 + s + i]
  \]
of $\mathbb{CP}^4$ under the action of the group $\mathbb{Z}_d^2$ described in the next section. Incidentally, the twistor fibration $\mathbb{CP}^3 \rightarrow S^4$ is the Hopf map obtained by identifying $S^4$ with the quaternionic projective line $\mathbb{HP}^1$. It follows from Definition 2.6 that no SIC sets in $\mathbb{CP}^3$ can project to only 4 points in $S^4$. A separate problem is the determination of optimal configurations of mutually equidistant points in the Riemannian symmetric space $\mathbb{HP}^n$.

3. Review of overlap phases

Fix $d \geq 3$, and set $\omega = e^{2\pi i/d}$. Having chosen coordinates on $\mathbb{C}^d$, one can define two cyclic groups $H$ and $W$ each of size $d$. The former is generated by the ‘clock’ map $h: [z_0, z_1, \ldots, z_{d-1}] \mapsto [z_0, \omega z_1, \ldots, \omega^{d-1} z_{d-1}]$, whereas the latter is generated by the ‘shift’ map $w: [z_0, z_1, \ldots, z_{d-1}] \mapsto [z_{d-1}, z_0, \ldots, z_{d-2}]$.

Observe that $H = \langle h \rangle$ is a subgroup of the maximal torus $T$ of $\text{SU}(d)$ consisting of diagonal matrices. On the other hand, $W$ is a subgroup of $\text{SU}(d)$ (indeed, $\text{SO}(d)$) only if $d$ is odd. The two actions commute on $\mathbb{CP}^{d-1}$, but are remnants there of the action of the Heisenberg group of matrices with entries in $\mathbb{Z}_d$ (and size $d^3$) that we now describe.

Set $\tau = -e^{\pi i/d}$. The full Heisenberg action leads one to define a map

\[
D: \mathbb{Z}_d^2 \rightarrow \text{U}(d) \quad p = (p, q) \mapsto \tau^{pq}w^ph^q = D_p.
\]

Although $D$ is not a homomorphism, it satisfies

\[
D_pD_q = \tau^{(p,q)}D_{p+q},
\]

where $(p, q) = \det(q, p)$ is a symplectic pairing. Note that $D$ is $d$-periodic up to sign, since

\[
D_{p+da} = (-1)^{(p, q)}D_p
\]

if $d$ is even.

The normalizer $N(d)$ of $W \times H$ in $\text{U}(d)$ is the Clifford group. There is a representation $U: \text{GL}_2\mathbb{Z}_d \times \mathbb{Z}_d^2 \rightarrow N(d)/\text{U}(1)$ satisfying

\[
U_{F,q}D_pU_{F,q}^{-1} = \omega^{(q,Fp)}D_{Fp}.
\]

It is an isomorphism if $d$ is odd. This theory can be extended to include complex conjugation and $N(d)$ becomes a subgroup of index 2 in the so-called extended Clifford group. This is the natural symmetry group for investigating orbits for the action of $\mathbb{Z}_d$ on $\mathbb{C}^d$, though we shall not reply on knowledge of its exact structure in this paper. We shall be more concerned with the symmetries arising from Definition 3.2 below.
Example 3.1. The Clifford group for \( d = 5 \) is immortalized in the construction of the Horrocks-Mumford bundle \( \mathcal{F} \), which is a stable rank 2 holomorphic vector bundle over \( \mathbb{CP}^4 \) \cite{22}. If \( L = \mathcal{O}(1) \) denotes the line bundle over \( \mathbb{CP}^4 \) defining a hyperplane, then \( L^{-1} \) is a tautological subbundle of the trivial bundle \( V \) with fibre \( \mathbb{C}^5 \). It follows that \( L \otimes V \) contains a trivial line subbundle, and it is known that the quotient is isomorphic to the holomorphic tangent bundle \( T = T\mathbb{CP}^4 \). More generally there is a natural map \( \Lambda^k T \to L^{k+1} \otimes \Lambda^{k+1} V \). The resulting sequence

\[
L^2 \otimes \Lambda^2 V \xrightarrow{p_0} \Lambda^2 T \xrightarrow{q_0} L^3 \otimes \Lambda^3 V
\]

is a monad in the sense that \( q_0 \circ p_0 = 0 \), and \( \mathcal{F} = \ker q_0/\operatorname{im} p_0 \) is a holomorphic vector bundle of rank 2. The construction is invariant by the action of the full Heisenberg group \( \mathbb{Z}_3^2 \) (which acts on the \( \Lambda^k V \)) and the group \( N(5)/U(1) \) of order 3000, whence the 15000 symmetries in the paper’s title.

The well-known solution to the SIC-POVM problem in \( \mathbb{CP}^4 \) is likely to fit into the geometry underlying \( \mathcal{F} \), by analogy to \cite{23}. A generic section \( s \in H^0(\mathbb{CP}^4, \mathcal{F}) \) vanishes on a non-singular abelian surface, whilst the zero set of a generic quintic on \( \mathbb{CP}^4 \) is Calabi-Yau. These are related as follows: there is a 6-dimensional space of quintics invariant by the Heisenberg group, which can be identified with \( \Lambda^2(H^0(\mathbb{CP}^4, \mathcal{F})) \) \cite{22,4}.

Let \( z \) be a unit vector in \( \mathbb{C}^d \), we write \( z \in S^{2d-1} \). For each non-zero element \( p \in \mathbb{Z}_d^2 \), the quantity

\[
\operatorname{tr}(D_p z^* z) = \langle z | z D_p \rangle
\]

is called an overlap phase for the Heisenberg action.

Definition 3.2. Let \( z \in S^{2d-1} \). The overlap map associated to \( z \in S^{2d-1} \) is the mapping \( \Phi_z : \mathbb{Z}_d^2 \to \mathbb{C} \) defined by \( \Phi_z(p) = \langle z | z D_p \rangle \).

Of course, \( \Phi_z \) can be thought of as a square matrix whose rows and columns are each indexed by \( \mathbb{Z}_d^2 \), and we shall also refer to its values as ‘entries’. Beware that the top left entry \( \Phi_z(0, 0) = 1 \) is not an overlap phase. The point is that \( z \) generates a SIC set if and only if all the other entries of \( \Phi_z \) are complex numbers of modulus \( 1/\sqrt{d+1} \).

The set \( \{ D_p : p \in \mathbb{Z}_d^2 \} \) forms a basis for the complex vector space \( \mathfrak{gl}_d \mathbb{C} \) of complex \( d \times d \) matrices. Indeed, \( \{(1/\sqrt{d}) D_p\} \) is an orthonormal basis since \( \operatorname{tr}(D_p) = 0 \) provided \( p \neq 0 \). The entries of \( \Phi_z | \mathbb{Z}_d^2 \) are merely the coordinates of \( z^* z \) with respect to this basis, up to an overall scale. Indeed,

\[
z^* z = \frac{1}{d} \sum_{p \in \mathbb{Z}_d^2} \Phi_z(p) D_{-p},
\]

and \( [z] \) can be recovered from any row of this matrix. If \( D_p \) were itself Hermitian or skew-Hermitian, then \( \Phi_z(p) \) would be a component of the moment map \( \{2.3\} \). The problem is that (in the terminology of \cite{31}) the \( D_p \) form a unitary basis of \( \mathfrak{gl}_d \mathbb{C} \), rather than a basis of \( \mathfrak{u}(d) \). Nonetheless, we shall explain in the next section precisely how the overlap map encodes the moment maps associated to maximal tori of the isometry group of \( \mathbb{CP}^{d-1} \).
Example 3.3. The case $d = 4$ is well understood, and the relatively simple nature of its overlap map was thoroughly explained in [3]. Nonetheless, the underlying geometry of circles and golden ratios, described in [28], illustrates the moment map approach. Let $z = (z_0, z_1, z_2, z_3)$ be a unit vector in $\mathbb{C}^4$ whose Heisenberg orbit $\mathbb{Z}_4^2 \cdot [z]$ is a SIC set in $\mathbb{C}P^4$. Let $x_i = |z_i|^2$ and abbreviate $\Phi_z$ to $\Phi$. Then

\begin{align*}
1 &= \Phi(0, 0) = x_0 + x_1 + x_2 + x_3 \\
\frac{1}{\sqrt{5}} e^{i\theta} &= \Phi(0, 1) = x_0 + ix_1 - x_2 - ix_3 \\
\pm \frac{1}{\sqrt{5}} &= \Phi(0, 2) = x_0 - x_1 + x_2 - x_3,
\end{align*}

for some $\theta \in U(1)$ and choice of sign. It follows that

\begin{equation}
2\sqrt{5}x = \begin{cases} 
(\varphi + \cos \theta, \psi + \sin \theta, \varphi - \cos \theta, \psi - \sin \theta), & \text{or} \\
(\psi + \cos \theta, \varphi + \sin \theta, \varphi - \cos \theta, \psi - \sin \theta),
\end{cases}
\end{equation}

where $\varphi = \frac{1}{2}(\sqrt{5} + 1)$ and $\psi = \frac{1}{2}(\sqrt{5} - 1)$. This shows that $x$ must belong to the disjoint union of two circular arcs each of radii $1/(2\sqrt{5})$ ‘suspended’ in the hyperplane $\sum x_i = 1$ of $\mathbb{R}^4$, as in Figure 1. However, the circles themselves escape the confines of the moment polytope $\Delta$ (here a solid tetrahedron) that is the image of $\mathbb{C}P^3$, see (4.17) below.

Change notation so that $z = (ae^{i\alpha}, be^{i\beta}, ce^{i\gamma}, de^{i\delta})$ with $x_0 = a^2, x_1 = b^2, x_2 = c^2, x_3 = d^2$ for clarity (so $d$ is temporarily not a dimension). Then

\begin{equation}
0 = |\Phi(2, 0)|^2 - |\Phi(2, 2)|^2 = 16abcd \cos(\alpha - \gamma) \cos(\beta - \delta).
\end{equation}

None of $a, b, c, d$ can vanish, meaning that (in contrast to the case $d = 3$) points of the SIC set cannot project to the boundary of the polytope. If we fix the second circle in (3.11), we are furthermore forced to assume that $\alpha - \gamma = \pm \pi/2$. Without loss of generality, we can then set $\delta = 0$, which implies

\begin{equation}
\frac{1}{5} = \Phi(0, 2)^2 = 4b^2d^2 \cos^2 \beta,
\end{equation}

and $(\varphi^2 - \sin^2 \theta) \cos^2 \beta = 2$. This relationship can be interpreted as a link between moment maps arising from different maximal tori.

The equations

\begin{equation}
|\Phi(1, 0)|^2 - |\Phi(1, 2)|^2 = 0 = |\Phi(1, 1)|^2 - |\Phi(1, 3)|^2
\end{equation}

allow us to eliminate $\beta$ and deduce that

\begin{equation}
(ad - bc)(ad + bc)(ab - cd)(ab + cd) = 4a^2b^2c^2d^2 \cos^2 \beta.
\end{equation}

One can eliminate $\beta$ using (3.12) to find $\theta$, which takes on one of four values on each circle represented by the ‘beads’ in Figure 1. If we fix one of these solutions on the second circle, we have the following choices: 4 for $\alpha$, for each of these 2 for $\gamma$, and independently 4 for $\beta$. Choices of signs for $a, b, c, d$ are taken care of by the different angles and overall phase. This gives a total of 32 fiducial vectors lying over each bead, in accordance with the known results [1].
The solution described by Bengsston [5] with overlap map

$$\Phi|\mathbb{Z}^2_d = \frac{1}{\sqrt{5}} \begin{pmatrix} \sqrt{5} & u & -1 & \overline{u} \\ u & \overline{u} & -\overline{u} & \overline{u} \\ -1 & -u & -1 & \overline{u} \\ \overline{u} & u & u & u \end{pmatrix},$$

projects to a point on the second circle, where

$$u = \Phi(0, 2) = e^{i\theta} = \frac{1}{\sqrt{2}}(\psi + i\sqrt{5}).$$

With our conventions however, we needed to replace $\tau$ by $\overline{\tau}$ in the definition (3.6).

Let $\mathbb{K} = \mathbb{Q}(\sqrt{D}) = \mathbb{Q}(\sqrt{5})$ as in the Introduction, noting that this is a subfield of $\mathbb{E}_1 = \mathbb{Q}(e^{i\theta})$. The minimum polynomial of $e^{i\theta}$ over $\mathbb{Q}$ has degree 8, whereas its splitting field $\mathbb{E} = \mathbb{E}_1(i)$ has degree 16 over $\mathbb{Q}$. Indeed, the Galois group $\text{Gal}(\mathbb{E}/\mathbb{Q})$ is isomorphic to $\mathbb{Z}_2 \times D_8$, where $D_8$ is the dihedral group, and $\mathbb{K}$ is the fixed field of the normal subgroup $\mathbb{Z}_2 \times V_4$. Therefore, $\mathbb{E}$ is an abelian extension of $\mathbb{K}$, and $\mathbb{E}$ contains the fiducial vector $z$, which is projectively equivalent to the one in Example 2.7. However, all the information of the SIC set is provided by $e^{i\theta}$, which is a unit in the smaller field $\mathbb{E}_1$. In Figure 1, the 8 beads form an orbit of $D_8$ acting as rotations of the tetrahedron.
Let $p$ denote the natural homomorphism (doubling if $d$ is even, the identity if $d$ is odd).

**Definition 4.1.** Let $\mathbb{Z}_d^2$ denote the set of cyclic subgroups of $\mathbb{Z}_d^2$ of size $d$. When $d$ is even, the image of $\bar{C} \in \mathbb{Z}_d^2$ under $\bar{\pi}^2 : \mathbb{Z}_d^2 \to \mathbb{Z}_d^2$ is a cyclic subgroup $C$ of $\mathbb{Z}_d^2$ of order $d$, and we say that $C$ and $\bar{C}$ are associated.

For $\bar{C} \in \mathbb{Z}_d^2$, the restriction $D|_C$ is a homomorphism and $d$-periodic even when $d$ is even, as can be seen from (3.7). The $d$-periodicity allows us to define a map

$$\Phi_{\bar{C}} : C \to \mathbb{C}$$

satisfying $\Phi_{\bar{C}} \circ \bar{\pi}^2 = \Phi_z | \bar{C}$, where $C$ and $\bar{C}$ are associated. We shall refer to (4.14) as the restricted overlap map determined by $\bar{C}$.

When $d$ is odd, we take $\bar{C} = C$ and $\Phi_{\bar{C}} = \Phi_z | C$.

When $d$ is even, $d\mathbb{Z}_d^2 \cong \mathbb{Z}_d^2$. Thus if $p$ is a generator for $\bar{C}$, then $\{p + dq : q \in \mathbb{Z}_d^2\}$ has four points. Two of these (namely $p$ and $p + d(p)$) lie in $\bar{C}$, while the other two lie in a different subgroup $\bar{C}'$ associated to $C$. Choose $q \in \mathbb{Z}_d^2$ such that $r = p + dq$ generates $\bar{C}'$. Since $p$ and $q$ are linearly independent, $\tau(p, dq)$ must equal $-1$. Using (3.7), we have $D_{kr} = (-1)^kD_{kp}$ for every $k \in \mathbb{Z}_d$,

$$\Phi_{\bar{C}'}(p) = \begin{cases} \Phi_{\bar{C}}(p) & \text{if } p \in 2C, \\ -\Phi_{\bar{C}}(p) & \text{if } p \notin 2C. \end{cases}$$

To summarize, when $d$ is even, each $C \in \mathbb{Z}_d^2$ is associated to two distinct subgroups $\bar{C}, \bar{C}' \in \mathbb{Z}_d^2$, but $\Phi_{\bar{C}}$ and $\Phi_{\bar{C}'}$ agree up changes of sign on odd entries. However, we shall largely ignore this ambiguity as we shall ultimately be concerned only with the respective images of these mappings.

Since the restriction of $D$ to $\bar{C}$ is a homomorphism, we can identify the image of $\Phi_{\bar{C}}$ with an element of the affine space

$$\mathcal{T} = \left\{(\alpha_0, \ldots, \alpha_{d-1}) \in \mathbb{C}^d : \alpha_0 = 1, \, \alpha_i = \overline{\alpha_{d-i}} \right\}$$

(4.15)

$$\cong \begin{cases} \mathbb{C}^n & \text{if } d \text{ is odd} \\ \mathbb{C}^{n-1} \oplus \mathbb{R} & \text{if } d \text{ is even.} \end{cases}$$

Let $p \in \mathbb{Z}_d^2$ generate $\bar{C}$. Since $D|_C$ is $d$-periodic, the eigenvalues of $D_p$ are the $d$th roots of unity $1, \omega, \omega^2, \ldots, \omega^{d-1}$ with an ordered set of unit eigenvectors $(e_j)$ indexed by $\mathbb{Z}_d$. Note that if $p' \neq p$ with $\bar{\pi}^2(p) = \bar{\pi}^2(p')$, then $D_{p'} = -D_p$. Thus the unordered set of eigenvectors $\{e_j\}$ depends only on $\bar{\pi}^2(p)$ generating the associated $C \in \mathbb{Z}_d^2$. The corresponding projectors $\{e^*_j e_j : j \in Z_d\}$ are Hermitian, and we may identify them with elements of $\mathfrak{u}(d)$. They generate a maximal torus $\hat{T}^C$ in $U(d)$ that descends to a torus $T^C$ in the projective unitary group $U(d)/U(1)$. Recalling (2.3), its Lie algebra $\mathfrak{t}^C < \mathfrak{su}(d)$ contains trace $1$ elements in $\text{span}_\mathbb{R}\{e^*_j e_j : j \in \mathbb{Z}_d\}$.
We now can associate to $\bar{C}$ a moment map $\mu^{\bar{C}}: \mathbb{CP}^{d-1} \to \mathbb{R}^d$ by restricting (2.5) to $t^C$. This yields

$$\mu_j^{\bar{C}} := \mu^{\bar{C}} \cdot e^*_j e_j := \text{tr}(e^*_j e_j z^* z) = |\langle e_j | z \rangle|^2.$$ 

The dependence on $\bar{C}$ rather than $C$ comes from the ordering of the basis by eigenvalues. For any $i \in \mathbb{Z}_d$, we can now write

$$D_{ij} = \sum_{j \in \mathbb{Z}_d} \omega^{ij} e^*_j e_j.$$ 

In particular, the components of $\Phi^{\bar{C}}_z$ relative to (4.15) are given by

$$(\Phi^{\bar{C}}_z)_i = \text{tr}(D_{ij} z^* z) = \sum_{j \in \mathbb{Z}_d} \omega^{ij} \text{tr}(e^*_j e_j z^* z) = \sum_{j \in \mathbb{Z}_d} \omega^{ij} \mu_j^{\bar{C}}.$$ 

This is expressed succinctly by the equation

\begin{equation}
(4.16) \quad \Phi^{\bar{C}}_z = V \mu^{\bar{C}}([z])
\end{equation}

for $z \in S^{2d-1}$, where

$$V = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & \omega & \cdots & \omega^{d-1} \\
1 & \omega^2 & \cdots & \omega^{2(d-1)} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \omega^{d-1} & \cdots & \omega^{(d-1)^2}
\end{pmatrix}$$

is the Vandermonde matrix that represents the discrete Fourier transform. The $(i,j)$th entry of $V$ is $\omega^{ij}$ (we start indexing at 0), and $(1/\sqrt{d})V \in U(d)$. We can summarize the discussion by the following result, a version of which appears in [17]:

**Proposition 4.2.** For any $\bar{C} \in \mathbb{PZ}_d^2$, the restricted overlap map $\Phi^{\bar{C}}_z$ is a Fourier transform of the moment map $\mu^{\bar{C}}$.

**Example 4.3.** The argument above is rendered more concrete by taking $\bar{C} = \{0\} \times \mathbb{Z}_d$, which is associated to $H$, so that is the standard maximal torus in $SU(d)$. Its moment map is determined by the diagonal entries of (2.4), and

\begin{equation}
(4.17) \quad \Delta = \{ (x_0, \ldots, x_d) : \sum x_i = 1, \ x_i \geq 0 \}
\end{equation}

of $\mathbb{R}^d$. Define

\begin{equation}
(4.18) \quad \alpha_i = \langle z | z h^i \rangle = x_0 + \omega^i x_1 + \cdots + \omega^{i(d-1)} x_{d-1},
\end{equation}

where $h^i = (x_0, \ldots, x_d)$ is a unit vector.
so that

\[
\begin{pmatrix}
\alpha_0 \\
\alpha_1 \\
\vdots \\
\alpha_{d-1}
\end{pmatrix} = V
\begin{pmatrix}
x_0 \\
x_1 \\
\vdots \\
x_{d-1}
\end{pmatrix},
\]

and \((\alpha_0, \ldots, \alpha_{d-1}) \in T\).

For our next statement, let \(C\) be a cyclic subgroup of \(\mathbb{Z}_2^d\) associated to \(\overline{C} \in \mathbb{PZ}_2^d\), and let \(T^C\) be the corresponding maximal torus.

**Definition 4.4.** A point \(\overline{z}\) \(\in \mathbb{CP}^{d-1}\) is \(C\)-admissible if all points in its \(C\)-orbit are correctly separated. A point \(x \in \Delta\) is admissible if the same is true for any \(\overline{z} \in (\mu^C)^{-1}(x)\). Let \(\mathcal{A}\) denote the subset of \(\Delta\) consisting of \(C\)-admissible points.

The definition of \(\mathcal{A}\) makes sense for a given \(C\) because \(C\)-admissibility is a \(T^C\)-invariant concept. It does not depend on \(C\) because a different choice of finite subgroup will give rise to a conjugate maximal torus \(P^{-1}T^CP\) with \(P \in N(d)\), and \(\mu^C(\overline{z}P)\) is replaced by \(\mu^C(\overline{z}P)\), see (3.9) and (4.16).

**Example 4.5.** In practice, one can compute \(\mathcal{A}\) using \(H\) as in Example 4.3. For \(d = 4\), the set \(\mathcal{A}\) is the union of the two circular arcs \((3.11)\) illustrated in the Figure 1. For \(d = 3\), the simplex \(\Delta\) is a filled equilateral triangle with vertices \((1, 0, 0)\), \((0, 1, 0)\), \((0, 0, 1)\) in \(\mathbb{R}^3\), and

\[
\mathcal{A} = \left\{ \frac{2}{3} \left( \cos^2 \phi, \cos(\phi + \frac{2\pi}{3}), \cos^2(\frac{4\pi}{3}) \right) : \phi \in (-\frac{\pi}{2}, \frac{\pi}{2}) \right\}
\]

is its incircle. The inverse image of each midpoint \((0, 1, 1), (1, 0, 1), (1, 1, 0)\) contains three of the nine points in Example 2.7. Returning to (4.16), the Fourier transform converts \(\Delta\) into the convex hull of the third roots of unity, and \(\mathcal{A}\) into a circle of radius 1/2.

To complete the story for \(d = 3\), let \(S\) be any SIC set in \(\mathbb{CP}^2\). The latter is a two-point homogeneous space so, up to isometries, we are free to assume that \(S\) contains the two points \([0, 1, -\omega], [0, 1, -\omega^2]\). Any other point \([z]\) of \(S\) lies a distance \(2\pi/3\) apart from these two. It turns out that \([z]\) must line on the singular torus

\[
\left\{ \left[ e^{i\alpha} \cos \phi, \cos \left( \phi + \frac{2\pi}{3} \right), \cos \left( \phi + \frac{4\pi}{3} \right) \right] : \alpha \in (-\pi, \pi], \phi \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \right\}
\]

in \(\mathbb{CP}^2\), pinched at \(\phi = \pi/2\). It follows that its image is constrained to lie in \(\mathcal{A}\), and this is a key observation in [24]. The proof that any SIC set in \(\mathbb{CP}^2\) is equivalent to one of those in Example 2.7 now follows from an analysis of equilateral triangles in \(\mathbb{CP}^2\) with vertices lying on the singular torus. The case \(d = 3\) is exceptional, but one might hope that \(\mathcal{A}\) could play a role in the classification of more general SIC sets, at least for small \(d\).
For any $d$, the vertices of $V\Delta \subset \mathcal{T}$ are given by the rows of $V$. A $C$-admissible point $[z]$ is characterized by the condition that each component of (4.16) has norm $\frac{1}{\sqrt{d+1}}$, and

$$V\mathcal{A} = V\Delta \cap \begin{cases} \frac{1}{\sqrt{d+1}}T \subset \mathbb{C}^n & \text{if } d \text{ is odd} \\ \frac{1}{\sqrt{d+1}}(T \times \{\pm 1\}) \subset \mathbb{C}^{n-1} \oplus \mathbb{R} & \text{if } d \text{ is even,} \end{cases}$$

where $T$ denotes a Clifford torus in $\mathbb{C}^n$ or $\mathbb{C}^{n-1}$ respectively.

The complex numbers $\alpha_k$ were defined in (4.18) with reference to the action of $H$. Indeed, $[z]$ is $H$-admissible if and only if $|\alpha_k|^2 = \frac{1}{d+1}$ for all $1 \leq k \leq n$, so in particular $\alpha_n = \pm 1/\sqrt{d+1}$ if $d = 2n$. The proof of Proposition 4.2 established a bijection between $\mathcal{A}$ and products of circles parametrizing the $\alpha_k$. In order to describe more accurately the shape of $\mathcal{A}$, we first define a series of quadratic forms

$$f_j = \sum_{i=0}^{d-1} x_i x_{i+j}, \quad j = 0, \ldots, d - 1$$

derived from (4.18). To make sense of the right-hand side, the range of indices is extended cyclically, so that $x_i$ is defined to equal $x_{i-d}$ if $d \leq i \leq 2d - 1$. In particular,

$$f_0 = \sum_{i=0}^{d-1} x_i^2,$$

and $f_{d-j} = f_j$ for $1 \leq j \leq d - 1$. If $d$ is even then $f_n = 2f'_n$ where

$$f'_n = \sum_{i=0}^{n-1} x_i x_{i+n}.$$

The forms $f_0, \ldots, f_n$ constitute a basis of the space $S^2(\mathbb{R}^d)^*$ of bilinear forms invariant by the action of $\mathbb{Z}_d$ cyclically permuting the $x_i$.

We can now state

**Theorem 4.6.** Let $d \geq 3$. The set $\mathcal{A}$ is the intersection of $\Delta$ with $n$ quadrics in $\mathbb{R}^d$, and lies in a round sphere $S^{d-2}$ of radius $\sqrt{\frac{d}{d(d+1)}}$ centred in $\Delta$. Moreover,

- if $d = 2n + 1$, then $\mathcal{A} = \Delta \cap T$ where $T$ is a torus of revolution of dimension $n$ and maximal radius $\sqrt{\frac{2}{d(d+1)}}$ in $S^{d-2}$;
- if $d = 2n$ then $\mathcal{A} = \Delta \cap (T' \cup T'')$ where $T', T''$ are tori of revolution of dimension $n - 1$ and radii $\sqrt{\frac{2}{d(d+1)}}$ in parallel hyperspheres of $S^{d-2}$.

By a torus of revolution of radius $r$, we mean the orbit of a subtorus of the Euclidean group acting on $\mathbb{R}^d$ in which all circles have radius $r$. Figure 1 displays the sphere $S^{d-2}$ for $d = 4$, and this exits the tetrahedron (whose front face has been removed).
Proof. We will use \( C = H \) to compute \( A \). Since
\[
|\alpha_j|^2 = \sum_{i=0}^{d-1} \omega^j f_i,
\]
the \( H \)-admissible assumption also implies that
\[
V \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{d-1} \end{pmatrix} = \frac{1}{d+1} \begin{pmatrix} d+1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.
\]
The sum of the entries of the \((i + 1)\)st row of \( V \) equals
\[
\sum_{j=0}^{d-1} \omega^j = \frac{1 - (\omega^i)^d}{1 - \omega} = 0
\]
for \( 1 \leq i \leq d - 1 \). Therefore the unique solution to the matrix equation must be given by (4.20)
\[
\frac{1}{2} f_0 = f_1 = f_2 = \cdots = f_{d-1} = \frac{1}{d+1}.
\]
This shows that \( A \) lies in the intersection of the sphere \( S^{d-1} \) defined by \( f_0 = 2/(d+1) \) and the remaining quadrics \( f_i = 1/(d+1) \) for \( 1 \leq i \leq n \). It also lies on the intersection of \( S^{d-1} \) with the plane containing \( \Delta \), and this small hypersphere has radius \( r \) given by
\[
r^2 = \sum_{i=0}^{d} \left( x_i - \frac{1}{d} \right)^2 = \frac{2}{d+1} - \frac{2}{d} \sum_{i=0}^{d} x_i + \frac{1}{d} = \frac{d-1}{d(d+1)},
\]
in the notation of (4.17), as stated.

The values of \( f_i \) found above are consistent with the equation
\[
1 = \left( \sum_{i=0}^{d-1} x_i \right)^2 = \sum_{i=0}^{d-1} f_i.
\]
When \( d = 2n \) is even, there is an analogous equation that gives new information, namely
\[
\alpha_n^2 = \left( \sum_{i=0}^{d-1} (-1)^i x_i \right)^2 = \sum_{i=0}^{d-1} (-1)^i f_i.
\]
It follows that
\[
\sum_{i=0}^{d-1} (-1)^i x_i = \pm \frac{1}{\sqrt{d+1}},
\]
and \( A \) lies in the union of two hyperplanes.

Let \( C, S \) be the real and imaginary parts of \( V \), so that \( C \) is a matrix of cosines and \( S \) a matrix of sines. Recalling that \( \alpha_{d-k} = \overline{\alpha_k} \), set
\[
\sqrt{d+1} \alpha_k = \cos \theta_k + i \sin \theta_k, \quad 1 \leq k \leq n.
\]
If \( d \) is odd then the angles are unconstrained, but if \( d \) is even then \( \theta_n = 0, \pi \pmod{2\pi} \) to ensure that \( \alpha_n = \pm 1 \). Applying the inverse Fourier transform gives

\[
d\sqrt{d+1} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{d-1} \end{pmatrix} = C \begin{pmatrix} \sqrt{d+1} \\ \cos \theta_1 \\ \vdots \\ \cos \theta_1 \end{pmatrix} + S \begin{pmatrix} 0 \\ \sin \theta_1 \\ \vdots \\ -\sin \theta_1 \end{pmatrix}.
\]

Note that the last column vector has a zero sub-middle entry if \( d \) is even since \( \text{Im} \alpha_n = 0 \). It follows that

\[
(4.21) \quad d\sqrt{d+1} x_k = \sqrt{d+1} + \sum_{i=1}^{d-1} (c_{ki} \cos \theta_k + s_{ki} \sin \theta_k),
\]

where \( c_j = \cos\left(\frac{2\pi j}{d}\right) \) and \( s_j = \sin\left(\frac{2\pi j}{d}\right) \). There are two cases to consider, according to the parity of \( d \) and the properties of \( \cos \theta_k, \sin \theta_k \) that will reflect (4.15).

- **Case \( d = 2n + 1 \).** Let \( P \) be the \( d \times d \) matrix indexed with \( (i, j) \in \mathbb{Z}_d^2 \) and

\[
P_{ij} = \begin{cases} \sqrt{2} & \text{if } j = 0 \\ 2c_{ij} & \text{if } 0 < j \leq n \\ 2s_{i(j-n)} & \text{if } n < j. \end{cases}
\]

Thus, every entry in the first column of \( P \) is \( \sqrt{2} \), and the remaining entries in the first row of \( P \) are \( 2 \) (\( n \) times) followed by \( 0 \) (\( n \) times). Use of the Dirichlet kernel

\[
\sum_{k=-n}^{n} e^{k\theta} = \frac{\sin((n + \frac{1}{2})\theta)}{\sin(\frac{1}{2}\theta)}
\]

and elementary trigonometric identities imply that the rows of \( P \) are orthogonal, and that the norm squared of each one equals \( 2d \). Therefore, \( (1/\sqrt{2d})P \in O(d) \) and (4.21) implies that

\[
d\sqrt{d+1} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{d-1} \end{pmatrix} = P \begin{pmatrix} \sqrt{n+1} \\ \cos \theta_1 \\ \vdots \\ \cos \theta_n \\ \sin \theta_1 \\ \vdots \\ \sin \theta_n \end{pmatrix}.
\]

- **Case \( d = 2n \).** We again index \( P \) by \( \mathbb{Z}_d^2 \), but this time we set

\[
P_{ij} = \begin{cases} \sqrt{2} & \text{if } j = 0 \\ 2c_{ij} & \text{if } 0 < j < n \\ -(-1)^i \sqrt{2} & \text{if } n = j \\ 2s_{i(j-n)} & \text{if } n < j. \end{cases}
\]

\[
\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{d-1} \end{pmatrix} = P \begin{pmatrix} \sqrt{n+1} \\ \cos \theta_1 \\ \vdots \\ \cos \theta_n \\ \sin \theta_1 \\ \vdots \\ \sin \theta_n \end{pmatrix}.
\]
Once again, \((1/\sqrt{2d})P \in O(d)\). Equation (4.21) translates into

\[
\begin{pmatrix}
  x_0 \\
  x_1 \\
  \vdots \\
  x_{d-1}
\end{pmatrix}
\begin{pmatrix}
  \sqrt{n + \frac{1}{2}} \\
  \cos \theta_1 \\
  \cos \theta_{n-1} \\
  \pm \cos \theta_n / \sqrt{2} \\
  \sin \theta_1 \\
  \sin \theta_{n-1}
\end{pmatrix} = P
\begin{pmatrix}
  \sqrt{n + \frac{1}{2}} \\
  \cos \theta_1 \\
  \cos \theta_{n-1} \\
  \pm \cos \theta_n / \sqrt{2} \\
  \sin \theta_1 \\
  \sin \theta_{n-1}
\end{pmatrix}.
\]

In both cases, \(x\) parametrizes a Clifford type torus (or tori) of radius \(\sqrt{2d/(d\sqrt{d} + 1)}\). □

**Remark 4.7.** As a curiosity, let \(d = 2n+1\), and consider the \(n \times n\) submatrix \(C' = (c_{ij})_{i,j=1}^{n}\) of \(C\). It has rank 2, being the real part of a complex projection matrix. The sum of entries in every row of \(C\) equals \(-\frac{1}{2}\). The equations from the proof above tell us that

\[
2C' \begin{pmatrix}
  f_1 \\
  \vdots \\
  f_n
\end{pmatrix} = \frac{1}{d+1} \begin{pmatrix}
  1 \\
  \vdots \\
  1
\end{pmatrix},
\]

which is the equation that characterizes a simple Nash equilibrium for a zero-sum game with payoff matrix \(C'\). The fact that \(f_1 = \cdots = f_n\) tells us that the game has value \(1/(2n)\) and a solution in which each pure strategy is played with probability \(1/n\) [14].

To conclude this section, we note that the condition of being fiducial is equivalent to being \(C\)-admissible for every \(C \in \mathbb{PZ}_d^2\). Theorem 4.6 then helps one to grasp the essence of the SIC-POVM problem. Whether \(d\) is even or odd, the set of \(C\)-admissible points is characterized by the equations

\[
f_1 = \cdots = f_n = \frac{1}{d+1},
\]

derived from (4.20), assuming the normalization \(f_0 = 1\). It forms a real subvariety of \(\mathbb{CP}^{d-1}\) of codimension \(n\). The SIC-POVM existence question is whether these subvarieties have a non-empty intersection as \(C\) varies over \(\mathbb{PZ}_d^2\), whose size is determined by Lemmas 6.1 and 6.2 below. For example if \(d\) is prime then there are \(d+1\) subgroups and subvarieties to consider.

Given that \(\mathbb{CP}^{d-1}\) has real dimension at most \(4n\), one might conjecture that four subgroups can always be found to reduce the set of solutions to be finite provided \(d > 3\). Such a statement appears to be a weaker version of the 3d conjecture of [17]. An infinitesimal version of this setup is the following. For each \(z \in \mathbb{CP}^{d-1}\) and \(C \in \mathbb{PZ}_d^2\), one can in theory compute the tangent space to the level set of the \(T^C\)-invariant real polynomial \(f_i\) at \(z\). Knowledge of the configuration of these spaces as \(i\) and \(C\) vary would have a bearing on the finiteness question for Heisenberg SIC-POVMs, but is an independent problem that may be accessible for small values of \(d\).
5. Overlap symmetries and Galois conjugation

Suppose that \( \mathbf{z} \in \mathbb{C}^d \) is a fiducial vector for some Heisenberg SIC set. Throughout the ensuing discussion we fix \( \mathbf{z} \) and introduce fields and groups that depend implicitly on \( \mathbf{z} \).

Define the symmetry group \( S \) of \( \Phi_{\mathbf{z}} \) by
\[
S := \{ G \in \text{GL}_2 \mathbb{Z}_{\bar{d}} : \Phi_{\mathbf{z}} \circ G = \Phi_{\mathbf{z}} \}.
\]
Thus, \( S \) consists of automorphisms of \( \mathbb{Z}_{\bar{d}}^2 \) that have no effect on the overlap phases. Zauner conjectured that \( S \) always contains an element \( F \) of order 3 with trace \(-1\), an example being
\[
F_{\mathbf{z}} := \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.
\]

Remark 5.1. When \( d \) is even, the relations 3.8 show that \((d+1)I \in S\), and for this reason, the matrix
\[
\hat{F}_{\mathbf{z}} := \begin{pmatrix} 0 & d-1 \\ d+1 & d-1 \end{pmatrix}
\]
is often used instead of \( F_{\mathbf{z}} \). When \( d \) is odd, obviously \( F_{\mathbf{z}} = \hat{F}_{\mathbf{z}} \), but when \( d \) is even, \( \hat{F}_{\mathbf{z}} \) is order 6 in \( \text{GL}_2 \mathbb{Z}_{\bar{d}} \), with \( \hat{F}_{\mathbf{z}}^2 = F_{\mathbf{z}}^2 \) and \( \hat{F}_{\mathbf{z}}^3 = (d+1)I \).

Definition 5.2 ([2]). The fiducial \( \mathbf{z} \) is called centred if \( S \) contains an element \( F \) of order 3 with trace \(-1\) and is displacement-free, in the sense that the corresponding Clifford unitaries are of the form \( U_{F,0} \) for some \( F \in \text{GL}_2 \mathbb{Z}_{\bar{d}} \). A centred fiducial \( \mathbf{z} \) is said to be type-z if \( F \) is conjugate to \( F_{\mathbf{z}} \), and type-a otherwise.

To understand the different types, first note that by [19], \( \text{GL}_2 \mathbb{Z}_{\bar{d}} \cong \prod_{p \mid d} \text{GL}_2 \mathbb{Z}_{p^r} \), where \( \bar{d} = \prod_{p \mid d} p^r \). The image \( F_{p^r} \in \text{GL}_2 \mathbb{Z}_{p^r} \) of \( F \) under each of the projections must also have trace \(-1\) and order a factor of 3. When the order is 3, then it is conjugate to \( F_{\mathbf{z}} \). The only alternative is order 1, so that \( F_{p^r} = I \). The condition of having trace \(-1\) then forces \( p^r = 3 \).

To summarize this, \( \mathbf{z} \) is always type-z unless \( \gcd(9,d) = 3 \) and \( F \equiv I \pmod{3} \).

The centred condition allows us to understand the Galois action. Recall that \( \mathbb{E}_1 \) is the field extension of \( \mathbb{K} \) generated by the image of \( \Phi_{\mathbf{z}} \). Let \( \mathbb{E}_0 \) be the maximal subfield of \( \mathbb{E}_1 \) such that the Galois group \( G = \text{Gal}(\mathbb{E}_1/\mathbb{E}_0) \) fixes the image of \( \Phi_{\mathbf{z}} \) as an unordered set.

Theorem 5.3 ([2]). Let \( \mathbf{z} \) be a centred fiducial. For each \( g \in G \), there exists some \( G_g \in \text{GL}_2 \mathbb{Z}_{\bar{d}} \) and \( r_g \in \mathbb{Z}_{\bar{d}}^2 \) such that
\[
g \circ \Phi_{\mathbf{z}}(\mathbf{p}) = \begin{cases} 
\Phi_{\mathbf{z}}(G_g \mathbf{p}) & \text{if} \ 3 \nmid d \\
\sigma^{(r_g,\mathbf{p})} \Phi_{\mathbf{z}}(G_g \mathbf{p}) & \text{if} \ 3 \mid d
\end{cases},
\]
where \( \sigma \) is a third root of unity.

We will soon see that the above formula can be simplified if we assume a stronger condition, which we will now work to motivate. Let \( \mathbb{E} \) be the field extension of \( \mathbb{K} = \mathbb{Q}(\sqrt{D}) \) generated by the \( \bar{d} \)th root of unity \( \tau \) and the components of the projector \( \mathbf{z}^* \mathbf{z} \).
Lemma 5.4. \( E = E_1(\tau) \).

Proof. First note that each \( D_p \) is a matrix with entries in \( \mathbb{Q}(\tau) \). The formula \( \Phi_z(p) = \text{tr}(D_p z^* z) \) shows that \( \Phi_z(p) \) takes values in \( E \), so that \( E_1 \subseteq E \).

For the other inclusion, (3.10) implies that \( z^* z \in E_1(\tau) \), so that \( E \subseteq E_1(\tau) \). \( \square \)

Working more carefully, we note that \( z^* z \) being Hermitian means that \( \Phi_z(p) = \text{tr}(D_p z^* z) \in \mathbb{Q}(z^* z, \text{Re} \tau) \).

In all of the known solutions, we actually have \( \text{Re} \tau \in E_1 \). Moreover, if the square-free part \( \hat{d} \) of \( \bar{d} \) is congruent to 1 modulo 4, then \( E_1 \) contains \( \text{Im} \zeta_{\hat{d}} \), where \( \zeta_{\hat{d}} \) is a \( \hat{d} \)th root of unity.

Lemma 5.5. If \( E_1 \) contains \( \text{Im} \zeta_{\hat{d}} \) whenever \( \hat{d} \equiv 1 \pmod{4} \), then \( E \supseteq E_1(i\sqrt{\bar{d}}) \).

Proof. Since \( E = E(\tau) \), it suffices to show \( i\sqrt{\bar{d}} \in E \). By the Kronecker-Weber theorem, for any square-free integer \( n \), \( \mathbb{Q}(\sqrt{n}) \subseteq \mathbb{Q}(\zeta_{|\Delta|}) \), where

\[
\Delta = \begin{cases} 
  n & \text{if } n \equiv 1 \pmod{4} \\
  4n & \text{if } n \equiv 2, 3 \pmod{4}
\end{cases}
\]

is the discriminant of \( \mathbb{Q}(\sqrt{n}) \). We proceed by taking cases:

- **Case \( \hat{d} \equiv 3 \pmod{4} \).** Taking \( n = -\hat{d} \) gives \( i\sqrt{-\bar{d}} \in \mathbb{Q}(\zeta_{\hat{d}}) \subseteq \mathbb{Q}(\zeta_{\hat{d}}) \subseteq E \).

- **Case \( \hat{d} \equiv 1 \pmod{4} \).** Taking \( n = \hat{d} \) gives \( \sqrt{\bar{d}} \in E \). But we also have

  \( E \supseteq \mathbb{Q}(\tau) \supseteq \mathbb{Q}(\zeta_{\hat{d}}) \supseteq \mathbb{Q}(\text{Im} \zeta_{\hat{d}}) \),

  and \( E \supseteq E_1 \supseteq \mathbb{Q}(\text{Im} \zeta_{\hat{d}}) \),

  so that \( i \in E \). Thus \( i\sqrt{\bar{d}} \in E \).

- **Case \( 2|\bar{d} \).** Then \( 4|\bar{d} \). Taking \( n \) to be the negative of the square-free part of \( \bar{d}/4 \), we find that \( \mathbb{Q}(i\sqrt{\bar{d}}) = \mathbb{Q}(\sqrt{n}) \subseteq \mathbb{Q}(\zeta_{4n}) \subseteq \mathbb{Q}(\zeta_{\bar{d}}) \subseteq E \).

The proof is complete. \( \square \)

This motivates

**Definition 5.6** (3). A centred fiducial is called strongly centred if the image of \( \Phi_z \) generates a field extension \( E_1 \) of \( \mathbb{K} \) for which \( E = E_1(i\sqrt{\bar{d}}) \).

One consequence of being strongly centred is that the \( r_g \) from Theorem 5.3 vanishes. By [3], this gives a cleaner formula with no condition on \( d \):

\[
(5.22) \quad g \circ \Phi_z = \Phi_z \circ G_g.
\]

Every known fiducial is equivalent under the action of the Clifford group to a strongly centred one.

In Theorem 5.3, \( G_g \) is well defined up to multiplication by an element of \( S \). It follows that there exists a subgroup \( M \) of \( \text{GL}_2\mathbb{Z}_{\bar{d}} \) in the centralizer \( C(S) \) of \( S \) such that \( G \cong M/S \), with the isomorphism given by \( g \mapsto G_g S \). All known solutions satisfy the following
Conjecture 5.7. $M$ is a maximal abelian subgroup of $C(S)$.

For the remainder of the paper, we shall assume this conjecture and that $z$ is strongly centred.

By Theorem 5.3 and the definition of $S$, the right action of $M$ on $\Phi_z$ relates the overlap phases by (possibly trivial) Galois conjugation. For a fixed $d$, different fiducials may have different symmetry groups. However, $M$ is determined only by $d$ and the type of the fiducial.

Lemma 5.8. If $F$ is conjugate to $F_z$, then $M = C(S) = \mathbb{Z}_d[I, F]^\times$.

Here $\mathbb{Z}_d[I, F]^\times \leq GL_2\mathbb{Z}_d$ are the set of invertible elements in the algebra $\mathbb{Z}_d[I, F]$.

Let $u_f$ be the fundamental unit in $\mathcal{O}_K$ that is positive with respect to $\infty_1$, which is the real embedding of $\mathbb{K}$ for which $\sqrt{D}$ is positive.. Let $u_D$ be the smallest power of $u_f$ with norm 1 (so that $u_D$ is either $u_f$ or $u_f^2$, since the norm of $u_f$ is in $\{\pm 1\}$). By [4], there exists some $r \in \mathbb{N}$ such that the rational part of $u_D^r$ equals $(d-1)/2$, and $u_D$ (mod $\bar{d}$) has order $3r$ where $\ell = \bar{d}/d$.

Lemma 5.9. The homomorphism $j : \mathbb{Z}_d[I, F] \to \mathcal{O}_K/(\bar{d})$ that maps $I$ to $[1]$ and $F$ to $[u_D^r]$ is an isomorphism if and only if either $d$ is coprime to 3 or $3|D$ and $d \neq 3$ (mod 27).

Proof. Note that $j$ is well defined since $F$ and $u_D^r$ are both order 3. Recall that $\mathcal{O}_K = \mathbb{Z}[1, \omega]$, where $\omega \in \{\sqrt{D}, \frac{1+\sqrt{D}}{2}\}$. The surjectivity of $j$ is equivalent to $[\omega]$ being in the image of $j$. This is equivalent to the $\omega$-coefficient $y \in \mathbb{Z}$ of $u_D^r$ being invertible modulo $\bar{d}$. This equivalence is clear when $d$ is odd, since $u_D^r = j(F)$. When $d$ is even, then $j(F) = (u_D^r)^2$ has $\omega$-coefficient $y(d-1)$, since the rational part of $u_D^r$ is $(d-1)/2$. The equivalence then follows from $d-1$ being invertible modulo $\bar{d}$.

Since $y \neq 0$, $j$ can only have a kernel when $d$ and $y$ share a common factor, which is the condition for $j$ to not be surjective. Thus $j$ is bijective if and only if it is surjective.

Since $u_D^r$ has norm 1 and rational part $\frac{d-1}{2}$, the norm of $2u_D^r$ is $4 = (d-1)^2 - D\sigma^2 y^2$, where $\sigma^-1$ is the $\sqrt{D}$ coefficient of $\omega$. Modulo gcd($d, y$), this gives $4 \equiv 1$ (mod gcd($d, y$)), so that gcd($d, y$)$|3$. In particular, $j$ is not surjective if and only if gcd($d, y$) = 3.

If $3|\text{gcd}(D, y)$, then the norm of $2u_D^r$ modulo 27 gives $3 \equiv d^2 - 2d$ (mod 27), so that $d \equiv -1$ or 3 (mod 27). Thus $3 = \text{gcd}(d, y)$ if and only if $d|3$ and either $3 \not{|} D$ or $d \equiv 3$ (mod 27).

Remark 5.10. When $d$ is even, one may be tempted to use a variant of $j$ which instead maps $\tilde{F}$ to $u_D^r$. However, this may not give the same map as $j$, since $\tilde{F}^3$ is diagonal, while $[u_D^r]$ may not be rational. For example, when $d = 4$, $u_D^r \equiv 9 + 4\sqrt{5} \neq (d+1)$ (mod 8).

Remark 5.11. Since gcd($d, y$)$|3$, we see that ker $j$ is generated by $(\bar{d}/3)F \in (\bar{d}/3)M \cong M_3$. Here we write $M_n$ to mean the projection of $M$ onto GL$_2\mathbb{Z}_n$ when $n$ is a factor of $\bar{d}$ co-prime to $\bar{d}/n$. If $z$ is type-$a$, then the kernel is trivial, since the same is true of the projection of $F$ to $M_3$. 
The map $j$ restricts to a homomorphism $j^*: \mathbb{Z}_d[I, F]^x \to (\mathcal{O}_K/(\bar{d}))^x$ with the same isomorphism criteria as in Lemma 5.9. Whenever these criteria are satisfied there are only type-$z$ fiducials (in the known solutions). These satisfy $(\mathcal{O}_K/(\bar{d}))^x \cong \mathbb{Z}_d[I, F]^x \cong M$, where the last isomorphism is Lemma 5.8.

Consider the case when $z$ is a type-$a$ fiducial, so that $j$ is not an isomorphism. By the type-$a$ condition and Chinese remainder theorem, we have $\mathbb{Z}_d[I, F] = \mathbb{Z}_3 \times \mathbb{Z}_{d/3}[I, F]$. This gives

$$M \cong M_3 \times M_{d/3} \cong (M_3/\mathbb{Z}_3^3 \times \mathbb{Z}_3^3) \times \mathbb{Z}_{d/3}[I, F]^x \cong M_3/\mathbb{Z}_3^3 \times \mathbb{Z}_d[I, F]^x \cong M_3/\mathbb{Z}_3^3 \times (\mathcal{O}_K/(\bar{d}))/\text{coker} j^x,$$

where $\mathbb{Z}_3^3$ acts by scalar multiplication, and we used $M_{d/3} \cong \mathbb{Z}_{d/3}[I, F]^x$ from Lemma 5.8 since $F$ modulo $\bar{d}/3$ is congruent to $F_2$. This gives the exact sequence

$$1 \to M_3/\mathbb{Z}_3^3 \to M \to (\mathcal{O}_K/(\bar{d}))^x \to \text{coker} j^x \to 1.$$

In the known solutions, the type-$a$ fiducials satisfy $M \cong (\mathcal{O}_K/(\bar{d}))^x$, which by the above exact sequence is equivalent to $M_3/\mathbb{Z}_3^3 \cong \text{coker} j^x$ or equivalently $M_3 \cong (\mathcal{O}_K/(3))^x$.

This discussion motivates

**Definition 5.12.** A strongly centred fiducial is algebraic if $M \cong (\mathcal{O}_K/(\bar{d}))^x$.

To make this concrete, we will compute $(\mathcal{O}_K/(\bar{d}))^x$. This was essentially done in [29], but we will have to do a bit of work before applying this result to the SIC setting.

**Lemma 5.13.** Let $p$ be a prime factor of $d$. If $p \equiv 1 \pmod{3}$, then $p$ splits in $\mathbb{K}$. If $p \equiv 2 \pmod{3}$, then $p$ is prime in $\mathbb{K}$. If $p = 3$, then $p$ splits/ramifies/is prime in $\mathbb{K}$ if $D \equiv 1/0/-1 \pmod{3}$.

**Proof.** Since $\mathbb{K} = \mathbb{Q}(\sqrt{D})$ is a quadratic extension, $(p)$ is prime (respectively ramifies or is a product of two primes) in $\mathbb{K}$ if and only if the polynomial

$$f(x) := \begin{cases} x^2 - D & \text{if } D \not\equiv 1 \pmod{4} \\ (2x - 1)^2 - D & \text{if } D \equiv 1 \pmod{4} \end{cases}$$

is irreducible (respectively a square, or a product of different linear factors) modulo $p$ [9]. When $p = 3$, the claim is immediate. Otherwise, $p$ being prime is equivalent to $D$ not being a quadratic residue modulo $p$. Since $D$ is the square-free part of $(d + 1)(d - 3)$, this is equivalent to $(d + 1)(d - 3) \equiv -3 \pmod{p}$ not being a quadratic residue modulo $p$. By quadratic reciprocity, this is equivalent to $p$ not being a quadratic residue modulo $3$, so that $p \equiv 2 \pmod{3}$.

On the other hand, $p$ ramifies if and only if $p|D$, which can only happen if $p = 3$ since $D|(d + 1)(d - 3) \equiv -3 \pmod{p}$. Thus if $p \equiv 1 \pmod{3}$, then $p$ splits. □

**Lemma 5.14.** If $9|d$, then $D \not\equiv 3 \pmod{9}$.
Theorem 5.18. For ray class fiducials, $S$ is congruent to the cyclic group generated by the fundamental unit in $O_K$ modulo $d$.

Proof. There is a well-known five-term exact sequence

$$1 \to U_{m}(F) \to U(F) \to (O_{F}/m)^{\times} \to C_m \to C \to 1,$$

...
where \( \mathcal{C} \) is the class group of some field \( \mathbb{F} \), \( \mathcal{C}_m \) is the ray class group of some modulus \( m = m_0 m_{\infty} \), \( \mathcal{U}(\mathbb{F}) \) is the unit group, and
\[
\mathcal{U}_m(\mathbb{F}) = \{ \alpha \in \mathcal{U}(\mathbb{F}) : v_p(\alpha - 1) \geq v_p(m_0), \sigma_i(\alpha) > 0, \forall p | m_0, \sigma_i \in m_{\infty} \},
\]
where \( v_p(n) := \max \{ r \in \mathbb{N} : p^r | n \} \). The class groups have the property that \( \mathcal{C}_m \cong \text{Gal}(\mathbb{F}(m)/\mathbb{F}) \). Thus
\[
\mathcal{C}_m/\mathcal{C} \cong \text{Gal}(\mathbb{F}(m)/\mathbb{F})/\text{Gal}(\mathbb{F}(1)/\mathbb{F}) \cong \text{Gal}(\mathbb{F}(m)/\mathbb{F}(1)).
\]
Now consider a ray class fiducial, where \( \mathbb{F} = \mathbb{K} \) and \( m = m_1 = (\bar{d})_{\infty 1} \). We have the (not necessarily commutative) diagram of short exact sequences
\[
\begin{array}{ccccccc}
1 & \rightarrow & S & \rightarrow & M & \rightarrow & \mathbb{G} & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & \mathbb{U}/\mathbb{U}(\bar{d}) & \rightarrow & (\mathbb{O}_\mathbb{K}/(\bar{d}))^\times & \rightarrow & \mathcal{C}(\bar{d})/\mathcal{C} & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & 1 \\
\mathbb{Z}_2 & & & & & & & & 1 \\
\end{array}
\]
The \( \mathbb{Z}_2 \) term in the last column comes from
\[
\mathcal{G}/(\mathcal{C}_d/\mathcal{C}) \cong \text{Gal}(\mathbb{K}(m_1)/\mathbb{K}(1))/\text{Gal}(\mathbb{K}(d)/\mathbb{K}(1)) \cong \text{Gal}(\mathbb{K}(m_1)/\mathbb{K}(d)) \cong \mathbb{Z}_2,
\]
and the first column is deduced from the rest of the diagram. This \( \mathbb{Z}_2 \) is generated by complex conjugation \( \mathbb{I} \), which is the image of \(-I \in M\). \(-I\) gets mapped to \(-1\) in \((\mathbb{O}_\mathbb{K}/(\bar{d}))^\times\), which must generate the \( \mathbb{Z}_2 \) in the first column. In fact, \( \mathcal{U} \) is generated by \( u_f \) and \(-1\), so \( \mathcal{U}/\mathcal{U}_f \) is generated by \([u_f]\) and \([-1]\), where \([\cdot]\) indicates equivalence classes in \( \mathbb{K}/(\bar{d}) \). We can factor out the instances of \( \mathbb{Z}_2 \) from the diagram to get
\[
\begin{array}{ccccccc}
1 & \rightarrow & S & \rightarrow & M & \rightarrow & \mathcal{G} & \rightarrow & 1 \\
\downarrow & \cong & \downarrow & \cong & \downarrow & \cong & \downarrow & \cong & \downarrow \\
1 & \rightarrow & \langle [u_f](\bar{d}) \rangle & \rightarrow & (\mathbb{O}_\mathbb{K}/(\bar{d}))^\times & \rightarrow & \mathcal{C}_m/\mathcal{C} & \rightarrow & 1 \\
\end{array}
\]
and complete the proof.
\( \square \)
To generalize this result, we note that conjecturally ray class fiducials are minimal in the sense that every strongly centred fiducial has $E_1$ an extension of $K(m_1)$. Moreover, for algebraic fiducials, we observe that $E_0 \cap K(m_1) = K(1)$.

**Theorem 5.19.** For every strongly centred fiducial such that $K(1) \leq E_0, K(m_1) \leq E_1$, and $E_0 \cap K(m_1) = K(1)$, $G$ is a cyclic extension of $C_{m_1}/C$, and $S$ is isomorphic to a subgroup of the cyclic group in $(O_K/(\bar{d}))^\times$ generated by the fundamental unit in $O_K$ modulo $\bar{d}$.

**Proof.** Since $E_0 \cap K(m_1) = K(1)$, it follows that $\text{Gal}(E_0/K(1))$ is a subgroup of $\text{Gal}(E_1/K(1))$ with some fixed field $F$ containing $K(m_1)$. Thus we have $G = \text{Gal}(E_1/E_0) \cong \text{Gal}(F/K(1))$. This gives the diagram

\[
\begin{array}{cccccc}
1 & & & & & 1 \\
\downarrow & & & & & \downarrow \\
1 & \longrightarrow & S & \longrightarrow & M & \longrightarrow & G & \longrightarrow & 1 \\
\downarrow & & \downarrow \cong & & & & & & & & \downarrow \\
1 & \longrightarrow & \langle [u_f]\rangle & \longrightarrow & (O_K/(\bar{d}))^\times & \longrightarrow & C_{m_1}/C & \longrightarrow & 1 \\
\downarrow & & & & & & & & & & \downarrow \\
1 & & & & & & & & & & 1
\end{array}
\]

In particular, $S$ is isomorphic to a subgroup of $\langle [u_f]\rangle$, which is cyclic. Hence the quotient $\langle [u_f]\rangle/S$ is also cyclic. By the diagram, this quotient is isomorphic to $\text{Gal}(F/K(m_1))$, giving the claim about $G$. \hfill \Box

Note that in the non-algebraic case there are examples where for a given $d$ there are fiducials $\alpha$ and $\zeta$ of types $a_{2k}$ and $z$ respectively for some $k \in \{2, 3, 4\}$, where

\[E_0^\alpha \leq E_0^\zeta \leq E_1^\alpha \leq E_1^\zeta\]

and $[E_0^\zeta : E_0^\alpha] = k$.

6. **Orbits of $M$**

Now that we have more understanding of the structure of $M$, we will study its action on $\mathbb{Z}_d^2$. First note that the center $Z$ of $\text{GL}_2\mathbb{Z}_d$ is contained in $M$. Its action preserves each $C \in \mathbb{P}\mathbb{Z}_d^2$, so we get an induced action of $M/Z$ on $\mathbb{P}\mathbb{Z}_d^2$. Before we study this action, first note that we have a Chinese remainder theorem for $\mathbb{P}\mathbb{Z}_d^2$.

**Lemma 6.1.** If $2 \geq n, m \in \mathbb{N}$ are coprime then $\mathbb{P}\mathbb{Z}_{mn}^2 \cong \mathbb{P}\mathbb{Z}_m^2 \times \mathbb{P}\mathbb{Z}_n^2$. 

Proof. Note that \( \mathbb{P}Z_{mn}^2 \) consists of cyclic subgroups whose generators lie in the set

\[
Z_{mn}^{2x} := \left\{ \left( \frac{p}{q} \right) \in \mathbb{Z}_{mn}^2 : \langle p, q \rangle = \mathbb{Z}_{mn} \right\}.
\]

Using the Chinese remainder theorem \((\pi_m, \pi_n) : \mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n\), we find

\[
\langle p, q \rangle = \mathbb{Z}_{mn} \iff \langle \pi_mp, \pi_nq \rangle = \mathbb{Z}_m \quad \text{and} \quad \langle \piMp, \piNq \rangle = \mathbb{Z}_n.
\]

Thus \( \mathbb{Z}_{mn}^{2x} \cong \mathbb{Z}_m^{2x} \times \mathbb{Z}_n^{2x} \). Since \( \mathbb{Z}_m^{2x} \cong \mathbb{Z}_m^x \times \mathbb{Z}_m^x \), where the \(^x\) in the exponent denotes the group of units, we have

\[
\mathbb{P}Z_{mn}^2 = \mathbb{Z}_{mn}^{2x} / \mathbb{Z}_{mn} \cong \mathbb{Z}_m^{2x} / \mathbb{Z}_m^x \times \mathbb{Z}_n^{2x} / \mathbb{Z}_n^x = \mathbb{P}Z_m^2 \times \mathbb{P}Z_n^2
\]
as stated. □

This allows us to reduce to the case when \( \bar{d} = p^k \), which one can easily count:

**Lemma 6.2.** \( |\mathbb{P}Z_{p^k}^2| = p^{k-1}(p + 1) \).

*Proof.* If \( \left( \frac{a}{b} \right) \) generates some \( C \in \mathbb{P}Z_{p^k}^2 \), then either \( a, b \), or both lie in \( \mathbb{Z}_{p^k}^x \). This gives

\[
\mathbb{Z}_{p^k}^{2x} = (\mathbb{Z}_{p^k}^x \times \mathbb{Z}_{p^k}^x) \cup (\overline{\mathbb{Z}_{p^k}^x} \times \mathbb{Z}_{p^k}^x) \cup \left( \frac{\mathbb{Z}_{p^k}^x \times \mathbb{Z}_{p^k}^x}{\mathbb{Z}_{p^k}^x} \right),
\]
where \( \overline{\mathbb{Z}_{p^k}^x} = \mathbb{Z}_{p^k}^x \setminus \mathbb{Z}_{p^k}^x \). Thus

\[
|\mathbb{P}Z_{p^k}^2| = 2|\mathbb{Z}_{p^k}^x| + |\mathbb{Z}_{p^k}^x| = 2|\mathbb{Z}_{p^k}| - |\mathbb{Z}_{p^k}^x| = 2p^k - \phi(p^k) = 2p^k - p^{k-1}(p - 1),
\]
as required. □

Using the Chinese remainder theorem for \( M \), it suffices to consider the action of \( M_{p^k} / Z_{p^k} \) on \( \mathbb{P}Z_{p^k}^2 \).

**Lemma 6.3.** For algebraic fiducials,

\[
M_{p^k} / Z_{p^k} \cong \begin{cases} 
\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1} & \text{if } p \text{ splits} \\
\mathbb{Z}_{p+1} \times \mathbb{Z}_{p+1} & \text{if } p > 2 \text{ is prime} \\
\mathbb{Z}_6 \times \mathbb{Z}_{2k-2} & \text{if } p = 2 \\
\mathbb{Z}_3 \times \mathbb{Z}_{3k-1} & \text{if } p = 3 \text{ ramifies}.
\end{cases}
\]

For non-algebraic fiducials, the result is the same except when \( p = 3 \), where \( M_3 / Z_3 \cong \mathbb{Z}_3 \) (as if \( 3 \) ramifies).

*Proof.* It is well known that

\[
\mathbb{Z}_{p^k} \cong \mathbb{Z}_{p^k}^x \cong \begin{cases} 
\mathbb{Z}_{p-1} \times \mathbb{Z}_{p^k} & \text{if } p \neq 2 \\
\mathbb{Z}_2 \times \mathbb{Z}_{2k-2} & \text{if } p = 2, \ k > 1.
\end{cases}
\]

Aside from the case \( p = 2 < k \), the result follows directly from Theorem 5.15. For \( p = 2 < k \), there are two possible quotients of \( M_{2^k} \) by different embeddings of \( \mathbb{Z}_2 \times \mathbb{Z}_{2k-2} \).
By Theorem 5.15, $M_{2^k} \cong \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_{2^{k-1}} \times \mathbb{Z}_{2^{k-2}}$. Note that $(I + 2F)^2 = -3I$ generates $Z_{2^k}/\pm 1 \cong Z_{2^{k-2}}$. Thus $I + 2F$ generates the $Z_{2^{k-1}}$ factor of $M_{2^k}$. It follows that

$$M_{2^k}/Z_{2^k} \cong \mathbb{Z}_3 \times \mathbb{Z}_{2^{k-1}}/Z_{2^{k-2}} \times \mathbb{Z}_{2^{k-2}} \cong \mathbb{Z}_6 \times \mathbb{Z}_{2^{k-2}}.$$  

Lemma 6.4. The (cyclic) action of $M_{p^k}/Z_{p^k}$ on $\mathbb{P}Z_{p^k}^2$ has one free orbit and $s$ orbits of size $p^k/\bar{p}$, where

$$s = \begin{cases} 
0 & \text{if } p \equiv 2 \pmod{3} \text{ or } p = 3 \text{ is type-$a_8$}, \\
1 & \text{if } p = 3 \text{ is type-$z$ or type-$a_6$}, \\
2 & \text{if } p \equiv 1 \pmod{3} \text{ or } p = 3 \text{ is type-$a_4$}.
\end{cases}$$

Note that in the algebraic case, $s$ is the number of proper factors of $p$ in $\mathcal{O}_K$.

Proof. Combining the two previous lemmas gives that $M_{p^k}/Z_{p^k} \cong Z_{|\mathbb{P}Z_{p^k}^2| - s} \times Z_{p^k/\bar{p}}$. We first consider the case $p^k = \bar{p}$. Assume that there exists a non-free orbit $o$ with more than one element. Since the fixed points of the action of some $gZ_{\bar{p}} \in M_{\bar{p}}/Z_{\bar{p}}$ correspond to eigenspaces of $g$, there can be at most 2 of them unless $g \in Z_{\bar{p}}$.

Since $o$ must be fixed by some non-trivial element of $M_{\bar{p}}/Z_{\bar{p}}$, this means that $o$ must have 2 elements.

- **Case $s < 2$.** Note that $|M_{\bar{p}}/Z_{\bar{p}}| + |o| > |\mathbb{P}Z_{\bar{p}}^2|$, so there is not enough room for any free orbits. Thus every orbit has size 1 or 2. This contradicts $M_{\bar{p}}/Z_{\bar{p}}$ being a cyclic group of order $> 2$ acting effectively.

- **Case $s = 2$.** From [1], we know that $M_{\bar{p}}$ is diagonalizable. The claim easily follows.

Now consider the case when $p^k > \bar{p}$. The multiplication map $\mathbb{Z}_p^k \to (p^k/\bar{p})Z_{p^k} \cong Z_{\bar{p}}$ induces a map $\rho: \mathbb{P}Z_{p^k}^2 \to \mathbb{P}Z_{\bar{p}}^2$ where all of the fibers have the same size, which must be $p^k/\bar{p}$ by Lemma 6.2.

The previous lemma shows that $M_{p^k}/Z_{p^k} \cong M_{\bar{p}}/Z_{\bar{p}} \times Z_{p^k/\bar{p}}$. One can easily show that the second factor is generated by $I + \bar{p}F$. Since this is congruent to $I$ modulo $\bar{p}$, $I + \bar{p}F$ acts trivially on $\mathbb{P}Z_{\bar{p}}^2$. Thus it restricts to an action on each fibre of $\rho$, which have size $p^k/\bar{p}$. Since this number is a prime power and the same as the order of $I + \bar{p}F$, it must act transitively.

Theorem 6.5. For algebraic fiducials, each $M$-orbit in $Z_d^2$ has stabilizer $j^{-1}(\ker \pi_n)$ for some factor $n$ of $\bar{d}$ in $\mathcal{O}_K$, where $j: M \cong (\mathcal{O}_K/(\bar{d}))^\times$ comes from the algebraic condition, and $\pi_n: (\mathcal{O}_K/(\bar{d}))^\times \to (\mathcal{O}_K/(n))^\times$ is the map which takes elements modulo $n$. This gives a one-to-one correspondence between orbits of $M$ and factors of $\bar{d}$ in $\mathcal{O}_K$. For non-algebraic fiducials, the same is true if 3 is treated as if it ramifies in $\mathcal{O}_K$.

Proof. The action of $\text{GL}_2\mathbb{Z}_d$ on $Z_d^2$ preserves the function

$$f: Z_d^2 \to \mathbb{N}, \quad \left(\frac{p}{q}\right) \mapsto \gcd(p, q, \bar{d}).$$
The action of $Z$ restricted to each $C \in \mathbb{P}Z_2^3$ is equivalent to the action of $Z^*_d$ on $Z_d$, whose orbits are the fibres of $f|_C$. Thus the orbits of $M$ correspond to the intersection of the orbits of $M/Z$ with the fibres of $f$.

For any $n|d$, there is a minimal $n \in \mathbb{N}$ such that $n|n$. We find that $j^{-1}(\text{ker} \pi_n) \subseteq j^{-1}(\text{ker} \pi_n)$ stabilizes $f^{-1}(\bar{d}/n) = (\bar{d}/n)Z_2^3 \cong Z_2^3$.

Using Chinese remainder results, we can reduce to the case when $n = p^k$ is a prime power. Using the previous lemma, we see that if $p$ is prime, then $n = n$ and there is only one orbit in $f^{-1}(n)$, so this must have stabilizer $j^{-1}(\text{ker} \pi_n)$.

If $p = \prod_{i=1}^2 p_i$ factors into different primes $\{p\}_{i=1}^2$, then $M$ is diagonalizable, with $j$ mapping each factor of the diagonalization to a factor of $(\mathcal{O}_K/(p^k)) \cong \prod_{i=1}^2 (\mathcal{O}_K/(p_i^k))$. By Chinese remainder results, we can consider each factor separately. Each factor is equivalent the action of $Z^*_d$ on $Z_{p^k}$, whose orbits are $\{p^\ell Z_{p^k}\}_{\ell=0}$. The result follows since the stabilizer of the action of $Z^*_d$ on $p^\ell Z_{p^k}$ is ker(mod: $Z_{p^k} \rightarrow Z_{p^k}$).

If $p$ ramifies with square root $p$, then $(\mathcal{O}_K/(p)) \cong Z_p$. Thus $j^{-1}(\text{ker} \pi_p) \cong M_3/Z_3$ stabilizes the exceptional orbit from the previous lemma. More generally, the free and exceptional orbits in $f^{-1}(n)$ have stabilizers $j^{-1}(\text{ker} \pi_n)$ and $j^{-1}(\text{ker} \pi_{n/p})$ respectively. \(\square\)

**Theorem 6.6.** For an algebraic fiducial, for each $\mathcal{O}_K \ni n|\bar{d}$, an overlap phase contained in the orbit of $M$ labelled by $n$ takes values in the fixed field of a Galois group isomorphic to $\text{Gal}(K(m_1)/K((n)\infty_1))$.

**Proof.** Using the isomorphism $M \cong (\mathcal{O}_K/(\bar{d}))^\times$, the elements of the orbit labelled by $n$ are those which are stabilized by the subgroup $\text{stab}(n)$ whose elements which are congruent to 1 modulo $n$. The quotient $(\mathcal{O}_K/(\bar{d}))^\times/\text{stab}(n) \cong (\bar{d}/n)(\mathcal{O}_K/(\bar{d}))^\times \cong (\mathcal{O}_K/n)^\times$. This gives the diagram of short exact sequences

\[
\begin{array}{cccccc}
1 & 1 \\
\downarrow & \downarrow \\
\text{stab}(n) & \mathcal{C}_{m_1}/\mathcal{C}_{n\infty_1} \\
\downarrow & \downarrow \\
1 & \langle [u_f(\bar{d})] \rangle & (\mathcal{O}_K/(\bar{d}))^\times & \mathcal{C}_{m_1}/\mathcal{C} & \rightarrow 1. \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & \langle [u_f(n)] \rangle & (\mathcal{O}_K/(n))^\times & \mathcal{C}_{n\infty_1}/\mathcal{C} & \rightarrow 1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 1 & 1 & 1 & 1
\end{array}
\]

The projection modulo $n$ gives a surjection $\langle [u_f(\bar{d})] \rangle \rightarrow \langle [u_f(n)] \rangle$ with some kernel $K$. We deduce from the commutativity of the diagram that $K$ injects into $\text{stab}(n)$ with $\text{stab}(n)/K \cong \mathcal{C}_{m_1}/\mathcal{C}_{n\infty_1} \cong \text{Gal}(K(m_1)/K(n\infty_1))$. \(\square\)
This suggests that for ray-class fiducials, the Galois orbit labelled by $n$ takes values in the field $\mathbb{K}(n\infty_1)$.

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