Lefschetz pencils and the canonical class for symplectic 4-manifolds

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Abstract
We present a new proof of a result due to Taubes: if \((X, \omega)\) is a closed symplectic four-manifold with \(b^+(X) > 1 + b_1(X)\) and \(\lambda[\omega] \in H^2(X; \mathbb{Q})\) for some \(\lambda \in \mathbb{R}^+\), then the Poincaré dual of \(K_X\) may be represented by an embedded symplectic submanifold. The result builds on the existence of Lefschetz pencils on symplectic four-manifolds. We approach the topological problem of constructing submanifolds with locally positive intersections via almost-complex geometry. The crux of the argument is that a Gromov invariant counting pseudoholomorphic sections of an associated bundle of symmetric products is non-zero.

1 Introduction

Taubes’ renowned work on the Seiberg-Witten invariants of symplectic 4-manifolds shone light on an area which had, before then, been largely the domain of conjecture and speculation. The conclusions of Taubes’ work can be divided into two categories. First, results relating the existence of a symplectic structure to the differential topology of the underlying 4-manifold; second, results which stay within the framework of symplectic geometry. The results in the first category hinge on the fact that the Seiberg-Witten theory yields invariants of the smooth structure, and the Seiberg-Witten theory, or something equivalent, seems unavoidable. For the results in the second category, on the other hand, it is natural to ask if there are alternative proofs, avoiding the Seiberg-Witten theory and staying more within the realm of symplectic geometry. The purpose of this paper is to take a step in this direction. Specifically, we will give a new proof of the

(1.1) Theorem: [Taubes] Let \((X, \omega)\) be a closed symplectic four-manifold with \(b^+(X) > 1 + b_1(X)\) and such that some positive real multiple of the cohomology class \([\omega] \in H^2(X; \mathbb{R})\) is rational. Then there is a smooth embedded symplectic surface in \(X\) which represents the homology class Poincaré dual to \(K_X\).

Here, as usual, \(b^+(X)\) is the dimension of a maximal positive subspace for the intersection form on \(X\), endowed with the conventional orientation, while \(b_1\) is the first Betti number.
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The symplectic structure defines, up to homotopy, an almost-complex structure on $X$, and $K_X \in H^2(X; \mathbb{Z})$ is the first Chern class of the cotangent bundle. A surface $\Sigma \subset X$ is symplectic if the restriction of $\omega$ to $\Sigma$ is nondegenerate. The statement of (1.1) is weaker than the result proved by Taubes (Tan95, Tan96, Tan00) in two respects. First, Taubes’ proof only requires the hypothesis that $b_+ > 1$. As we explain later, the tighter constraint simplifies the proof but does not seem essential, and one can hope to obtain the stronger statement by a suitable refinement of the techniques we develop in this paper. Second, we impose the requirement that (a multiple of) $[\omega]$ be a rational class, which is not required by Taubes. It is not possible to remove this without some significant extension of the arguments, as we explain in more detail below (and see Smi); however, the restriction does not seem significant for most applications, since every symplectic form is deformation equivalent to a rational symplectic form.

Our proof of (1.1) will avoid any use of the Seiberg-Witten equations - although see the remarks below - and is in general “softer” (in Gromov’s terminology (Gro87)) than Taubes’. Thus we avoid delicate arguments with elliptic partial differential equations, although elliptic theory does play a vital role in our argument: at one point through the index theory for linear operators and at another point through pseudo-holomorphic curves and a “Gromov invariant”.

Symplectic 4-manifolds occupy the ground between, on one side, general smooth 4-manifolds and, on the other side, complex algebraic surfaces. In this spirit, one path by which to approach Taubes’ result is to recall the proof in the case when $X$ is a complex algebraic surface and $\omega$ is a Kähler form. Then one can simply argue as follows. The Hodge index formula states that $b_+ = 1 + 2p_g$ so the hypothesis $b_+ > 1$ says $p_g > 0$, i.e. there is a non-trivial holomorphic 2-form $\Theta$ on $X$. The zero-set of $\Theta$ is a complex curve representing $K_X$ and, at least if this curve is non-singular, this immediately gives the result. (In the rare event that, for all $\Theta$, the curve is singular, one would need to consider a $C^\infty$ perturbation, in the manner of Proposition (2.11) below.) Our proof - and, to some extent, that of Taubes - can be seen as a translation of this simple argument into the symplectic setting. In this case we always have compatible almost-complex structures, and the crux of the matter is to get around the lack of integrability.

The starting point for our proof is the existence, proved in [Don99], of Lefschetz pencils on symplectic 4-manifolds. This means that a blow-up $X'$ of $X$ is the total space of a Lefschetz fibration $\pi : X' \to S^2$. Here the fibres are Riemann surfaces, so both the base and fibre are complex and the deviation from integrability is confined to the twisting of the fibration. In the classical case, with a complex surface and holomorphic Lefschetz fibration, standard techniques allow us to study the geometry of the total space, and in particular the holomorphic 2-forms, in terms of data on the base and fibres—in algebro-geometric language we consider direct image sheaves over the Riemann sphere. Our proof can be understood, from one point of view, as an adaptation of these arguments to the non-integrable case, and we obtain our symplectic surface via suitable sections of a bundle $X_r(f)$, with $r = 2g - 2$, of symmetric products along the fibres.

A second point of view is to try to trace the parallel between our arguments and those of Taubes, using the Seiberg-Witten equations. Thus we ask what might be said about the solutions of the Seiberg-Witten equations over the total space of a Lefschetz fibration.

Here we make contact with a programme which is being developed by D. Salamon and his collaborators involving the “adiabatic limit” of the Seiberg-Witten equations. In that programme one would consider a family of metrics $g_\epsilon$ on $X'$ in which the fibre has diameter $O(\epsilon)$, and study the asymptotics as $\epsilon \to 0$. One would expect, by analogy with earlier work of Dostoglou and Salamon [DS94], that when $\epsilon$ is small the Seiberg-Witten solutions are modelled on holomorphic sections of an associated bundle of “vortices” on the fibres (the solutions of the Seiberg-Witten equations on a Riemann surface times $\mathbb{R}^2$, invariant under translations in the $\mathbb{R}^2$ variables). It is a well-known and simple result that the moduli spaces of vortices can be identified with the symmetric products of the Riemann surface, so one would expect to arrive at a similar picture to that studied in the present paper. (Indeed in the introduction to $Gr \Rightarrow SW$, which appears as the third paper in [Tau00], Taubes introduces an elliptic operator which looks at vortices in the normal bundle fibres to a fixed holomorphic curve, and which he describes as interpolating between the Gromov and Seiberg-Witten theories. Related work has also been done by Hong, Jost and Struwe [HJS96].) Put another way, the almost-complex structure on $X'$ defines a natural almost-complex structure on $X_r(f)$: the tangent space to $X_r(f)$ at a tuple $(p_1 + \cdots + p_r)$ is an extension of the direct sum $\oplus T_{p_i} \Sigma$ by a tangent direction to the base, and each component has a given anti-involution $J$. This almost-complex structure is $C^0$ but not $C^1$; nonetheless, it is the limit of smooth structures (this is exploited in [Smi]). The parameter $\epsilon$ plays a similar role to the deformation parameter used by Taubes, and in both cases the main difficulties have to do with the details of the asymptotic analysis with respect to the parameter.

The advantage of our argument, in which the connection with the Seiberg-Witten theory is kept in the background, is that we do not need to contend with such problems. From this perspective, the simplification has two fundamental aspects. Just as Taubes, we regard a surface in a four-manifold as the zero set of equations $f(z, w) = 0$. Since the situation is not integrable, if $z$ and $w$ are local almost-complex co-ordinates, then we cannot expect $f$ to be holomorphic in both variables. However, by first imposing a Lefschetz pencil, we can ensure that it is holomorphic in $z$ (along the fibres of a pencil) and smooth in $w$ (varying the element of the pencil). Second, we replace the non-compact deformation of a metric in Taubes proof with a compact deformation of almost-complex structures, in the definition of the Gromov invariant. Some additional consideration of the degenerating family of smooth almost complex structures mentioned above, or the Salamon programme, would be needed to remove the hypothesis that the symplectic form be rational from our approach.

Taubes’ proof of (1.1) can be seen as falling into two parts. One part is the identification of the Seiberg-Witten invariant of a line bundle $L \to X$ with the Gromov invariant counting pseudo-holomorphic curves homologous to $(K_X/2) + c_1(L)$ (assuming for simplicity that $K_X$ is even). This uses Taubes’ perturbation of the Seiberg-Witten equations. The second part is the statement that the invariants for $L$ and $L^*$ are equal. This symmetry is completely trivial using the original, unperturbed, Seiberg-Witten equations, but in conjunction with the first part gives a highly non-obvious assertion about pseudo-holomorphic curves:

The Gromov invariants defined by counting holomorphic curves in the homology classes $(K_X/2) + \lambda$ and $(K_X/2) - \lambda$ are equal.

Taking $\lambda = (K_X/2)$, one of the pair of homology classes is zero which trivially - or by definition - has Gromov invariant 1 (the empty set is the unique pseudo-holomorphic repre-
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sentative). Hence this symmetry includes as a special case Taubes’ result that the Gromov invariant of \( K_X \) is 1, which (again up to the issue of perhaps smoothing singular pseudoholomorphic curves) gives \([L.1]\). In our setting the fundamental symmetry appears through the duality between divisors \( D \) and \( K_C - D \) on a Riemann surface \( C \). A detailed treatment of this, and of some constructions of symplectic surfaces in four-manifolds with \( b_+ = 1 \), can be found in the paper \([Smi]\).

Outline of the paper:

1. As indicated above, we will produce the desired symplectic surface from a section of a (compactified) bundle of symmetric products. The proof hinges on the availability of two almost complex structures on this space - one which allows the computation of a Gromov invariant, and one for which the resulting pseudoholomorphic sections will yield embedded symplectic surfaces.

2. After a quick review of Lefschetz pencils, we introduce “positive symplectic divisors” and prove a smoothing lemma which gives sufficient conditions to deform such to embedded symplectic surfaces (section 2).

3. Given a Lefschetz fibration \( f : X \to \mathbb{S}^2 \) there is an associated fibration \( X_r(f) \) whose fibre over a smooth point \( t \in \mathbb{S}^2 \) is the \( r \)-th symmetric product of the fibre of the given pencil. This bundle of symmetric products can be naturally compactified to a smooth space (section 3 and the Appendix).

4. Write \( 2g - 2 = r \) henceforth. The fibres of the Lefschetz pencil are Riemann surfaces. The vector spaces \( H^0(\Sigma_t; \mathbb{K}_{\Sigma_t}) \) patch to give a vector bundle \( V \) over \( \mathbb{S}^2 \) whose projectivisation naturally embeds into \( X_r(f) \), avoiding the locus of critical values of \( X_r(f) \to \mathbb{S}^2 \) (section 3 and the Appendix).

5. A nowhere zero section of \( V \) defines a two-dimensional homology class \([\psi_V]\) in \( X_r(f) \) via the embedding of \( \mathbb{P}(V) \). \( X_r(f) \) admits symplectic structures and there is a well-defined Gromov invariant which counts pseudoholomorphic sections of \( X_r(f) \) in the class \([\psi_V]\), a problem of virtual index zero (section 4).

6. Fixing a generic \( \overline{\partial} \)-operator on \( V \) gives a distinguished \( N \)-dimensional space of holomorphic sections, for \( N = [b_+(X) - 1 - b_1(X)] / 2 \) (section 5). For a suitable almost-complex structure, the moduli space of holomorphic sections of \( X_r(f) \) in the distinguished homology class is the projective space \( \mathbb{P}^{N-1} \). To prove this one considers a natural map from \( X_r(f) \) to a fibration of Picard varieties \( P_r(f) \), for which sections are generically precluded by the existence of sections of \( V \) (section 5).

7. The Gromov invariant above can be evaluated as the Euler class of an obstruction bundle over the projective space; the obstruction bundle is the dual of the quotient bundle, and the Gromov invariant is \( \pm 1 \) (section 5).

8. The symmetric product fibration contains a number of natural diagonal strata; although singular, these have “smooth models”, and there are almost-complex structures on \( X_r(f) \) which are compatible with the natural inclusions of the strata (section 6).
9. For such an almost-complex structure on \(X_r(f)\), a pseudoholomorphic section \(\phi\) gives rise to a positive symplectic divisor \(C_\phi\) inside \(X'\). If the original Lefschetz pencil is of sufficiently high degree, \(\phi\) has no bubble components, and the divisor \(C_\phi\) descends to \(X\), where it may be smoothed to give an embedded symplectic surface in the class \(PD[K_X]\) (section 7).

It may be worth stressing that the arguments of the paper go through with comparatively little machinery from the pseudoholomorphic curves industry. Whilst the “virtual class” methods have brought the theory of Gromov-Witten invariants to a very satisfactory status, the more direct arguments of [MS94] or Gromov’s original [Gro85] suffice in this paper.

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2 Lefschetz pencils and standard surfaces

Let \((X, \omega_X)\) be a closed symplectic four-manifold. As in [Don99], a system of local complex co-ordinates \((z_1, z_2)\) centred at a point \(x \in X\) is compatible with \(\omega\) if the symplectic form at \(z = 0\) is positive and of type \((1, 1)\). Local co-ordinates are always assumed compatible with the given global orientation on \(X\).

**(2.1) Definition:** A topological Lefschetz pencil of degree \(k\) on \(X\) comprises a map \(f : X\setminus\{b_1, \ldots, b_d\} \to S^2\) defined on the complement of finitely many points in \(X\) to the two-sphere, which is a submersion outside of finitely many critical points \(\{p_1, \ldots, p_r\}\), all in distinct fibres of \(f\). We demand this data conforms to local models:

1. at each \(p_i\) there are compatible local complex co-ordinates with respect to which \(f\) has the form \((z_1, z_2) \mapsto z_1^2 + z_2^2\);

2. at each \(b_i\) there are compatible local complex co-ordinates with respect to which \(f\) has the form \((z_1, z_2) \mapsto z_1/z_2\).

The closures of the (open) fibres of \(f\) are given by including the points \(\{b_1, \ldots, b_d\}\); it follows from the local models that the fibres are then manifolds near the \(b_i \in X\). The results of this paper start with a general existence theorem for Lefschetz pencils [Don96, Don99]:

**(2.2) Theorem:** Let \((X, \omega)\) be a symplectic four-manifold for which \([\omega/2\pi] \in H^2(X; \mathbb{Z}) \subset H^2(X; \mathbb{R})\). Then for any sufficiently large \(k\) there are Lefschetz pencils on \(X\) for which the closed fibres are Poincaré dual to \(k[\omega]\) and are symplectic submanifolds away from their singularities.
The adjunction formula implies that the genus \( g \) of a smooth fibre increases with \( k \). In fact the pencils have an asymptotic uniqueness property, but we shall not make use of that in this paper. However, the choice of degree does enter into the argument:

\[ (2.3) \text{ Remark: At various stages of the proof, it will be important that we fix on the given } (X,\omega) \text{ a Lefschetz pencil of sufficiently high degree } k; \text{ for instance, it would be sufficient to ensure that } g > \max\{b_+(X)/2,2\} \text{ and } k > 2\pi(\omega \cdot \omega)/(\omega \cdot K_X). \text{ We will explain why these conditions arise at the relevant points.} \]

\[ \square \]

The singular fibres of a pencil are nodal curves, and \( a \) \text{ priori the node could separate the fibre. We note that you can always avoid these reducible fibres, although the proof does not rely on this in a crucial fashion:} \]

\[ (2.4) \text{ Proposition: Every symplectic four-manifold admits a Lefschetz pencil in which every fibre is irreducible.} \]

One route to proving this is by stabilisation, as in [Smi01]; it also follows from the formulae of Auroux and Katzarkov [AK]. (If \( K_X \) or the intersection form of \( X \) is even it is purely elementary.) Now fix compatible almost-complex structures on the tangent spaces to \( X \) at the points \( b_i \). We can then define the “blow-up” \( X' \) of \( X \) at these \( d \) points, as a set, in the usual way by replacing the points by the complex projectivisations of their tangent spaces. Again in a familiar way we can endow this with the structure of a smooth 4-manifold with a smooth blowing-down map \( p: X' \rightarrow X \). On the other hand, the map \( f \) clearly induces a map (which we still call) \( f: X' \rightarrow S^2 \). The exceptional spheres \( E_i \) in the blow-up appear as sections of \( f \).

Now for any \( N > 0 \) we can define a 2-form \( \omega_{(N)} = p^*(\omega_X) + Nf^*(\omega_{st}) \) on \( X' \), where \( \omega_{st} \) is the standard area form on \( S^2 \) with total area \( 2\pi \) : that is, \( \int_{S^2} \omega_{st} = 2\pi \).

\[ (2.5) \text{ Lemma: The form } \omega_{(N)} \text{ is a symplectic form on } X'. \text{ There are disjoint embedded balls } B_i \subset X \text{ containing } b_i \text{ such that the manifold } X'\setminus \bigcup E_i, \text{ with the symplectic form } \omega_{(N)}, \text{ is symplectomorphic to } X\setminus \bigcup B_i \text{ with the symplectic form } (1+kN)\omega_X. \]

\[ \text{Proof: } \text{Clearly } \omega_{(N)} \text{ is closed so we need to see it is non-degenerate. We have} \]

\[ \omega_{(N)}^2 = p^*(\omega_X^2) + 2Np^*(\omega_X) \wedge f^*(\omega_{st}). \]

The first term is non-negative, and strictly positive away from the \( E_i \), while the second term is strictly positive away from the points \( p_j \), so \( \omega_{(N)}^2 \) is strictly positive everywhere.

For the second part we identify \( X'\setminus \bigcup E_i \) with \( X\setminus \{b_1,\ldots,b_d\} \) via the blow down map \( p \). Clearly \( H^2(X\setminus \{b_1,\ldots,b_d\}) \) is isomorphic to \( H^2(X) \); we claim the de Rham cohomology class represented by the restriction of \( f^*(\omega_{st}) \) is \( k[\omega_X] \). It is clear that it is some linear multiple of \( [\omega_X] \), and the constant follows by integrating over a fibre:

\[ \int_{\mu} \omega_X = [\omega_X] \cdot [k/(2\pi)\omega_X] = (2\pi/k)^{\# \text{Basepoints}} \]
as compared with
\[ \int_F \omega_{st} = \text{Area}(S^2) \cdot \sharp \{ \text{Basepoints} \}. \]

Thus we can write
\[ \omega_{(N)} = (1 + kN)\omega_X + Nda \quad (2.6) \]
on this punctured manifold. The 1-form \( a \) will not extend over the punctures; in fact we may suppose that, in standard complex co-ordinates centred on a puncture,
\[ a = \frac{1}{4\pi(|z_1|^2 + |z_2|^2)}(\overline{z}_1 dz_1 - z_1 d\overline{z}_1 + \overline{z}_2 dz_2 - z_2 d\overline{z}_2). \]

Rewrite equation (2.6) as
\[ \frac{1}{kN + 1} \omega_{(N)} = \omega_X + tda, \]
where \( t = N/(kN+1) \). If we were working on a compact manifold we could immediately apply Moser’s theorem to obtain a symplectomorphism between \( \frac{1}{kN + 1} \omega_{(N)} \) and \( \omega_X \), integrating a time-dependent vector field. The argument does not apply immediately as things stand, since one may not be able to integrate vector fields on a non-compact manifold. However closer examination shows that we can carry through the argument since the relevant vector field points into the manifold near the puncture, so we can integrate for positive time. The basic model is the case of \( \mathbb{C}^2 \) when the map
\[ F_t(z_1, z_2) = \sqrt{4\pi t + 1/|z|^2}(z_1, z_2) \]
gives a diffeomorphism from \( \mathbb{C}^2 \setminus \{0\} \) to the complement of the ball of radius \( \sqrt{4\pi t} \), and
\[ F_t^*(\Omega_{\mathbb{C}^2}) = \Omega_{\mathbb{C}^2} + tda_{\mathbb{C}^2}, \]
where \( a \) is the 1-form above (cf. Lemma 6.40 in [MS99]).

Using this Lemma, we see that to obtain a symplectic surface in \( X \) it suffices to find a surface in \( X' \) which does not meet the exceptional curves \( E_i \) and which is symplectic with respect to the form \( \omega_{(N)} \) for some \( N \).

(2.7) DEFINITION: A standard surface \( \Sigma \) in \( X' \) is an embedded, oriented surface disjoint from the critical points \( p^{-1}(p_j) \) and such that the restriction \( f|_{\Sigma} \) satisfies the following.

1. The map defines a branched covering from \( \Sigma \) to \( S^2 \) which has positive degree on each component of \( \Sigma \) and which has only simple branch points.

2. At each branch point \( q \), \( (df)_q \) defines an isomorphism from the normal bundle \( (\nu_{\Sigma/X'})_q \) to \( T_{f(q)}S^2 \) which is orientation-preserving (with respect to the orientation induced by the given orientations on \( X' \) and \( \Sigma \)).
If $\Sigma$ is a standard surface, with branch points $\{q_j\}$ the restriction of $f$ to $\Sigma\setminus\{q_j\}$ is a covering map, and near each branch point $q_j$ there are (compatibly oriented) co-ordinates which identify the triple $(X', \Sigma, f)$ with the standard model
\[
\{(z, w) \in \mathbb{C}^2 : z = w^2\} ; \quad (z, w) \mapsto z \in \mathbb{C}.
\]
In particular the tangent space of $\Sigma$ at $q_j$ is the same as the tangent space of the fibre, and the orientations match up.

**Lemma (2.8):** A standard surface $\Sigma$ in $X'$ is symplectic with respect to the form $\omega(N)$ for large enough $N$.

This is elementary. The restriction of $p^*(\omega_X)$ to $\Sigma$ is positive near the branch points, since it is positive on the fibres of $f$, and the restriction of $f^*(\omega_{st})$ is non-negative everywhere and strictly positive away from the branch points.

In sum we see that a standard surface in $X$ which does not meet the exceptional curves $E_i$ yields a symplectic surface in the original manifold $X$. There is a converse to this statement which we will state here, although it is not directly relevant in the present paper.

**Proposition (2.9):** Let $\Sigma$ be any symplectic surface in $X$. Replacing $\omega_X$ by a sufficiently high multiple $k\omega_X$, there is a Lefschetz pencil on $X$ such that $\Sigma$ arises from a standard surface in the blow-up $X'$, by the correspondence above.

The proof of this Proposition is a straightforward modification of the proof of the existence of Lefschetz pencils in [Don99] (branched covering maps are the analogue, in real dimension 2, of Lefschetz pencils in real dimension 4). The upshot of this discussion is that the study of symplectic surfaces in symplectic 4-manifolds is reduced, in principle, to the study of standard surfaces which can in turn be described in a rather obvious way by “monodromy” data.

In the proof of our main theorem we will combine the discussion above with a simple smoothing criterion. Suppose $\Sigma_1, \ldots, \Sigma_n$ are symplectic surfaces in a compact symplectic 4-manifold $X$ which intersect transversally with locally positive intersection numbers (and no triple intersections).

**Definition (2.10):** For positive integers $a_i$ we call the formal sum $\sum_{i=1}^n a_i \Sigma_i$ a positive symplectic divisor in $X$.

The prototypical construction is then :

**Proposition (2.11):** Suppose $\sum a_i \Sigma_i$ is a positive symplectic divisor in $X$ and that
\[
(\sum a_i [\Sigma_i]) \cdot [\Sigma_j] \geq 0
\]
for each \( j \). Then in an arbitrarily small neighbourhood of \( \bigcup_i \Sigma_i \) there is a smooth symplectic surface representing the homology class \( \sum_i a_i[\Sigma_i] \).

PROOF: Recall, by assumption, that there are no triple intersections. Our proof will appeal to a local complex model for the union of the surfaces, but as things stand such may not exist. Let \( x \) be one of the transverse intersection points for branches \( \Sigma_1 \) and \( \Sigma_2 \) of the divisor. Then \( T_x X \) contains a pair of transverse symplectic two-planes, and the delicacy arises since such a pair has non-trivial moduli (we have not assumed that the planes are mutually symplectically orthogonal). Let \( S^\perp \) denote the symplectic orthogonal complement to a symplectic subspace \( S \) of \( T_x X \). We can linearise the situation near \( x \) and write \( \Sigma_2 \) as the graph of a matrix \( A : \mathbb{R}^2 \to \mathbb{R}^2 \) over the surface \( (T_x \Sigma_1)^\perp \cong \mathbb{R}^2 \), viewed as a standard two-plane in \( \mathbb{R}^4 \). The condition that \( \Sigma_2 \) be symplectic asserts that the determinant \( \det(A) > -1 \), whereas a sufficient condition for a local complex model to exist (near the transverse intersection point) is that \( \det(A) > 0 \).

However, we can restore the situation with a small perturbation near \( x \) as follows. Consider a map \( u : \mathbb{R}^2 \to \mathbb{R}^2 \) for which \( \det(\text{Jac}(u)) > -1 \) everywhere and which co-incides with the linear map \( A \) outside of a large ball. The graph of any such \( u \) defines a symplectic surface in \( \mathbb{R}^4 \) which can be glued back in place of \( \Sigma_2 \) in a neighbourhood of the point \( x \). It follows that we want to find such a map \( u \) satisfying the additional condition that the determinant of the Jacobian of \( u \) is positive at the origin. By choosing appropriate bases, we can suppose that the matrix \( A \) is diagonal \( A = \text{diag}(\lambda x, -\mu y) \) where \( \lambda > 0 \) and \( \mu < 0 \); the symplectic condition ensures \( \lambda \mu < 1 \). Choose some \( \varepsilon > 0 \) such that \( \lambda \mu < 1 - \varepsilon \). We will work with deformations of the shape

\[
u(x, y) = (\lambda x, f(x, y))\]

where \( f(x, y) = -\mu y \) for \( |(x, y)| \gg 0 \) and \( (\partial f/\partial y)|_{(0,0)} > 0 \). We can introduce a "kink" into the graph \( (y \mapsto \mu y) \) which gives a function \( f_0 : \mathbb{R} \to \mathbb{R} \) with the properties: (i) \( f_0(y) = 0 \) only at \( y = 0 \); (ii) \( f_0(y) = -\mu y \) for \( |y| \) sufficiently large; (iii) for every \( y \) the derivative \( (df_0/\partial y) > -\mu - (\varepsilon/\lambda) \); (iv) at the origin \( (df_0/\partial y)|_{y=0} > 0 \). Moreover, a suitable real one-parameter family of functions \( f_x(x) \) which interpolates between this kink and the linear map \( y \mapsto \mu y \), gives a function \( f(x, y) = f_x(y) \) with \( f(0, y) = f_0(y) \) and \( (\partial f/\partial y)|_{y=0} > -\mu - (\varepsilon/\lambda) \) everywhere. For the associated map \( u \), the graph defines a new surface \( (\Sigma_2)' \) which meets \( \Sigma_1 \) only at the origin \( x \) of the co-ordinate system, co-incides with \( \Sigma \) outside a ball \( B \) and which at the intersection point is given by a linear map \( \text{Jac}(u) \) whose determinant is \( \lambda (df_0/\partial y)|_{y=0} > 0 \). Moreover, the new surface is again symplectic, since at every point of \( B \) we see that \( \det(\text{Jac}(u)) > -\lambda \mu - \varepsilon > -1 \), where the last equality uses the fact that \( \lambda \mu < 1 - \varepsilon \) by choice of \( \varepsilon \).

Hence we reduce to the situation where all the isolated intersection points are locally given by graphs of complex matrices \( A \in \mathbb{C}^* \) (this remaining parameter is a "Kähler angle"). We can now appeal to a version of Weinstein’s extension of the Darboux Theorem [Wei71]. This implies that the symplectic structure in a neighbourhood of \( \bigcup_i \Sigma_i \) is uniquely determined by the areas of the \( \Sigma_i \), their normal bundles and the angle parameter at each intersection point. That is, if \( \Sigma_i' \) are symplectic surfaces in any other symplectic 4-manifold \( Y \), meeting transversally with locally positive intersections and at the same angles, such that \( \Sigma_i' \) is dif-
feomorphic to $\Sigma_i$, of equal symplectic area and with the same self-intersection numbers then suitable neighbourhoods of $\bigcup \Sigma_i' \subset Y$ and $\bigcup \Sigma_i \subset X$ are symplectomorphic. For any values of the parameters, we may construct such a $Y$ by “plumbing” holomorphic line bundles over $\Sigma_i$, so it suffices to prove the Lemma in the case when the $\Sigma_j$ are complex curves in a Kähler manifold $X$. In this case, let $L \to X$ be the holomorphic line bundle corresponding to the divisor $\sum a_i \Sigma_i$, so there is a holomorphic section $\gamma$ of $L$ with zero-set $\sum a_i \Sigma_i$. The hypothesis asserts that the degree of $L$ over each Riemann surface $\Sigma_i$ is nonnegative. It is clear then that we can find a $C^\infty$ section $s$ of $L$ over a neighbourhood of $\bigcup \Sigma_i$ with the following properties.

1. The section $s$ is holomorphic and non-vanishing in a neighbourhood of the intersections $\Sigma_i \cap \Sigma_j$. In fact we can arrange that $s$ defines a local holomorphic trivialisation of $L$ such that, in suitable local complex co-ordinates $z_1, z_2$, the section $\gamma = z_1^{\alpha_1} z_2^{\alpha_2} s$.

2. The section $s$ is transverse to zero and holomorphic in a neighbourhood of its zero set.

In fact we can arrange that around any zero of the section $s$ on $\Sigma_i$ there is a local holomorphic trivialisation of $L$ and local complex co-ordinates in which $s$ and $\gamma$ are represented by the functions $z_2$ and $z_1^{\alpha_1}$ respectively.

Now, for small non-zero $\epsilon$, let $\Sigma_{\epsilon}$ be the zero-set of the section $\gamma - \epsilon s$ of $L$. It is clear that, if $\epsilon$ is small enough, $\Sigma_{\epsilon}$ is compact and, provided it is cut out transversally, represents the homology class $\sum a_i [\Sigma_i]$. We need to prove that $\Sigma_{\epsilon}$ is a symplectic surface, cut out transversally. To do this it suffices to show that for any point $p$ of $\bigcup \Sigma_i$ there is a neighbourhood $N_p$ of $p$ and a $\delta_p > 0$ such that $\Sigma_{\epsilon} \cap N_p$ is a symplectic surface, cut out transversally, provided $0 < |\epsilon| < \delta_p$.

Relabelling the $\Sigma_i$ appropriately, there are three cases to consider:

- $p$ is an intersection point of two surfaces, $\Sigma_1$ and $\Sigma_2$;
- $p$ lies on a single surface $\Sigma_1$, and $s$ does not vanish at $p$;
- $p$ is a zero of $s$ on $\Sigma_1$.

The first and third cases are immediate. In these cases the section $s$ is holomorphic near $p$ and $\Sigma_1$ is locally a complex variety defined by equations

$$z_1^{a_1} z_2^{a_2} = \epsilon \quad \text{or} \quad z_1^{a_1} = \epsilon z_2,$$

respectively. Since these equations cut out transversally smooth curves in $\mathbb{C}^2$, for any non-zero $\epsilon$, the result follows. In the second case, we can choose local complex co-ordinates $z_1, z_2$ and a local holomorphic trivialisation of $L$ so that $\gamma$ is represented by the function $z_1^{a_1}$, while $s$ is represented by a smooth function $F(z_1, z_2)$, with $F(0, 0) = C \neq 0$. If we put $\eta = \epsilon^{1/a_1}$, then $\Sigma_{\epsilon}$ is locally defined by an equation

$$z_1^{a_1} = \eta^{a_1} F(z_1, z_2).$$

Now let $w_1, w_2$ be rescaled co-ordinates $w_i = \eta^{-1} z_i$. The equation becomes

$$w_1^{a_1} = F(\eta w_1, \eta w_2).$$

We view this as a perturbation of the equation $\{ w_1^{a_1} = C \}$ whose solution set is the union of $a_1$ complex lines parallel to the $w_2$ axis. It is then easy to see that the zero-set of the
perturbed equation has the desired properties over a ball \( \| (w_1, w_2) \| \leq r \eta^{-1} \) provided that \( r \) and \( \eta \) are sufficiently small, so we can take \( N_p \) to be the ball \( \| (z_1, z_2) \| < r \). ■

Note that the same proof can be extended to obtain a more general result, in which the surfaces \( \Sigma_i \) are allowed to have singularities modelled on singular points of complex curves. A particular variant we shall use allows us to separate out \((-1)\)-sphere components: here is the simplest case.

\[ (2.12) \text{Lemma: Let } C + 2E \text{ be a positive symplectic divisor comprising an embedded symplectic sphere } E \text{ of square } (-1) \text{ with multiplicity 2 and a reduced component } C \text{ which meets } E \text{ transversely once. Then in an arbitrarily small neighbourhood of } C \cup E \text{ there is a smooth symplectic surface which comprises a multiplicity one copy of } E \text{ and a disjoint surface representing } C + E. \]

**Proof:** By plumbing, we can again assume that the surfaces \( C \) and \( E \) are complex curves in a Kähler surface. The divisor \( C + 2E \) is cut out by a section \( \nu \) of a line bundle \( \mathcal{L} \); if we take the reduced divisor \( (C + 2E)_{\text{red}} \) we obtain a holomorphic section \( \eta \) of the line bundle \( \mathcal{L} \) with first Chern class \( C + E \). Observe that \( (C + E) \cdot E = 0 \). We can choose a smooth section \( s \) of \( \mathcal{L} \) which is nowhere vanishing on \( \mathcal{L}|_E \), which is holomorphic near its zeroes and which near the intersection point \( p \) of \( C \) and \( E \) trivialises \( \mathcal{L} \) and such that in local co-ordinates the section \( \eta \) is given by \( \eta = z_1 z_2 s \). Then, for sufficiently small \( \epsilon \), the zero-set of \( \eta - \epsilon s \) will be a symplectic surface of the required structure by arguments as above. ■

Again there is an obvious generalisation, allowing one to separate out a number of embedded exceptional curves which may themselves have higher multiplicities.

We close the section with a (perhaps helpful) analogy. It follows from (2.1) that we can always choose a compatible almost-complex structure \( j \) on \( X' \) such that

- the projection map to \( S^2 \) is pseudoholomorphic;
- the structure is integrable in a sufficiently small tubular neighbourhood of each singular fibre, i.e. over discs \( D_\epsilon \subset S^2 \) centred on the \( f(p_i) \).

We thus induce a map from the smooth part of the base \( S^2 \setminus \{ f(p_1), \ldots, f(p_r) \} \) to the Deligne-Mumford coarse moduli space of curves \( M_g \); from the local models, this map extends to a map of the closed sphere \( \phi_f : S^2 \to \overline{M}_g \) into the projective moduli space of stable curves. (For a generic choice of \( j \), and a pencil of curves of genus \( g > 3 \), the sphere \( \phi_f(S^2) \) will be disjoint from the orbifold loci of curves with automorphisms. Away from these loci the moduli space is fine, and if we wish we can invoke the existence of a “universal curve”. ) This sphere has transverse locally positive intersections with the divisor of stable curves; transversality provides nodal singular fibres, and the positivity shows the local complex co-ordinates have the correct orientations to construct, directly from the topology, a symplectic structure on \( X' \). Similarly, standard surfaces can be obtained from sections of symmetric product fibrations which have transverse locally positive intersections with natural diagonal strata: the transversality ensures the surfaces are smooth, and the positivity fixes the orientations to be compatible with the symplectic structure produced above.
3 The Abel-Jacobi map

The rest of the proof shall involve the relationship between three fibre bundles which can be associated to a smooth surface fibration. We begin by reviewing some of the standard theory of "divisors" on a Riemann surface $S$. We write $\text{Sym}^r(Z)$ for the $r$-fold "symmetric product" of a space $Z$: the quotient of the product of $r$ copies of $Z$ by the action of the permutation group on $r$ letters. Elements of $\text{Sym}^r(Z)$ can also be regarded as formal sums $\sum a_i p_i$ where the $a_i$ are positive integers with $\sum a_i = r$ and the $p_i$ are points of $Z$. The starting point is the fact that $\text{Sym}^r(\mathbb{C})$ can be identified with $\mathbb{C}^r$ via the elementary symmetric functions. An $r$-tuple $(z_1, \ldots, z_r)$ maps to $\sigma(z_1, \ldots z_r) = (\sigma_1, \ldots \sigma_r)$ where

$$\sigma_j = \sum_{u \in \text{Sym}(j)} \prod_{i=1}^j z_{u(i)}; \quad \text{so} \quad \sigma_1 = \sum z_i, \quad \sigma_2 = \sum_{i \neq j} z_i z_j, \ldots, \sigma_r = z_1 \cdots z_r. \quad (3.1)$$

Suppose we have a holomorphic diffeomorphism $\alpha : U \to V$ between open sets $U, V$ in $\mathbb{C}$. Write, for any such $U$, $\sigma(U)$ for the image of $\sigma : \text{Sym}^r(U) \to \mathbb{C}^r$. Then we get an induced holomorphic diffeomorphism

$$\text{Sym}^r(\alpha) : \sigma(U) \to \sigma(V).$$

It follows from this that if $S$ is any Riemann surface the symmetric product $\text{Sym}^r(S)$ has a natural structure of an $r$-dimensional complex manifold. Now suppose $S$ is compact of genus $g$. The Picard variety $\text{Pic}_r(S)$ of line bundles of degree $r$ is a $g$-dimensional complex torus which, up to a choice of origin, can be identified with

$$\text{Pic}_0(S) = H^1(S; \mathcal{O})/H^1(S; \mathbb{Z}).$$

The complex vector space $H^1(S; \mathcal{O})$ is the dual of the space of holomorphic 1-forms $H^0(\Omega^1_S) = H^0(K_S)$. There is a canonical holomorphic map - the Abel-Jacobi map -

$$\tau_r : \text{Sym}^r(S) \to \text{Pic}_r(S),$$

which can be thought of either as the map assigning a line bundle to a divisor or as the map induced by integration of holomorphic 1-forms along paths. The fibres of $\tau_r$ are complex projective spaces: the fibre over $L$ is $\mathbb{P}(H^0(S; L))$. For future reference, here is an easy but important topological lemma, which will play a role in our considerations of bubbling (4.8):

1) A proof can be found in [BT01].

(3.2) Lemma: For every $r > 1$ (and every genus) the Hurewicz homomorphism

$$\pi_2(\text{Sym}^r(\Sigma)) \to H_2(\text{Sym}^r(\Sigma))$$

has rank precisely one.

Now consider in particular the case when $r = 2g - 2$. The canonical line bundle $K_S$ gives a preferred point in $\text{Pic}_{2g-2}$. The fibre of $\tau_r$ over $K_S$ is a copy of $\mathbb{CP}^{g-1}$, whereas all the

1 We will sometimes refer to this point as "zero", and the section of a bundle of Picard varieties defined by the points representing the canonical line bundles as the "zero-section".
other fibres are copies of $\mathbb{CP}^{g-2}$. Thus, away from $K_S$, the map $\tau_{2g-2}$ is the projection of a holomorphic $\mathbb{CP}^{g-2}$ bundle; in general, at a divisor $p_1 + \cdots + p_r$ the image of $d\tau_r$ is dual to the subspace

$$(\text{im}(d\tau_r)\sum p_i)^* = \{ \nu \in H^0(\Sigma, K_\Sigma) \mid \text{div}(\nu) \geq p_1 + \cdots + p_r \}.$$ 

Here $\text{div}(\nu) \geq D$, by definition of notation, if the zero-set of $\nu$ contains the divisor $D$. Hence, if $\sum p_i$ is not canonical then necessarily $\nu = 0$ and $d\tau_r$ is surjective: if $\sum p_i = \text{Zeroes}(\nu)$ for $\nu \in H^0(K_\Sigma)$ then $d\tau_r$ has a one-dimensional cokernel.

Families of elliptic operators can be understood through their linearisations, and the structure of the map around the special fibre can be identified with a standard model. Let $V$ be a complex vector space of dimension $g$ and let $M$ be the subset of $V^* \times \mathbb{P}(V)$ defined by

$$M = \{ (\theta, [x]) : \theta(x) = 0 \}.$$  \hspace{1cm} (3.3)

Let $A : M \to V^*$ be the projection map onto the first factor. The fibre of $A$ over 0 is $\mathbb{P}(V) \cong \mathbb{CP}^{g-1}$, while the fibre over a non-zero $\theta$ is $\mathbb{P}(\ker(\theta)) \cong \mathbb{CP}^{g-2}$. Now take $V = H^0(S; K_S)$ and identify a neighbourhood of $K_S$ in $\text{Pic}_{2g-2}(S)$ with a neighbourhood of 0 in $V^*$ via the canonical flat structure on the torus.

\textbf{(3.4) Lemma:} Let $S$ be a compact Riemann surface; write $r = 2g - 2$ as usual. Then $(\text{Sym}^r(S), \tau_r)$ is holomorphically equivalent to $(M, A)$ in neighbourhoods of the fibres over $K_S$ and over 0 respectively.

\textbf{Proof:} Points in a neighbourhood of $0 \in V^*$ correspond to line bundles whose holonomies around loops are all small; we can therefore regard the holonomy of such a point as a real-valued, rather than circle-valued, one-form $a$. (Via "holonomy difference", we can parametrise a neighbourhood of $K \in \text{Pic}_r(S)$ in the same fashion.) On $S$ there is a canonical isomorphism between the real vector space $H^1(S; \mathbb{R})$ and the complex vector space of $(0, 1)$-forms $H^{0,1}(S)$. Under this isomorphism, $a$ maps to a form $\nu$ which defines a local holomorphic parameter on the Jacobian and hence Picard varieties. To see this directly, regard the space of holomorphic bundles as a space of unitary connexions on a fixed Hermitian line bundle. Two such connexions differ precisely by a $(0, 1)$-form, unique up to gauge; in a small disc around $\overline{\partial}_K$, the $\overline{\partial}$-bar operator defined by the canonical line bundle, we can fix the gauge consistently. It follows that $\overline{\partial}_K + \nu$ is the $\overline{\partial}$-operator on the holomorphic line bundle defined by $a \in V^*$. The holomorphic sections $\phi$ of $L_a$ by definition satisfy

$$\overline{\partial}_a(\phi) = \overline{\partial}_K \phi + \nu \wedge \phi = 0.$$ 

We have an embedding into this space of those sections $s \in H^0(S; K)$ which satisfy (by definition) $\overline{\partial}_K(s) = 0$ and for which $\langle \nu, s \rangle = 0$. Since the two vector spaces have the same dimension, this embedding is an isomorphism, and identifies the linear system of sections of $L_a$ with the fibre of the map $A$. Since the (scheme-theoretic) fibres of $\tau_r$ are precisely linear systems, the result follows. \hfill \blacksquare
Return to the Lefschetz fibration $$X' \to \mathbb{S}^2$$. Fix an almost-complex structure $$J$$ on $$X'$$ for which the projection map is holomorphic; write $$X^*$$ for the complement in $$X'$$ of the singular fibres. The restriction of $$f$$ to $$X^*$$ is a genuine $$C^\infty$$ fibration and each fibre has the structure of a Riemann surface. We define the fibrewise symmetric product $$\text{Sym}_r^f(X^*) = X_r^*(f)$$, as a topological manifold, in the obvious way; there is a projection map from $$X_r^*(f)$$ to $$\mathbb{S}^2$$ whose fibres are the symmetric products of the fibres of $$f$$. (Since a homeomorphism of a space defines a homeomorphism of its symmetric products the construction at this level can just be seen as the ordinary construction of an associated bundle, associated in this case to the group of self-homeomorphisms of a surface.)

To put a smooth manifold structure on $$X_r^*(f)$$ requires more care, since a diffeomorphism of a surface does not induce diffeomorphisms of its symmetric products.

**Definition:** A restricted chart on $$X^*$$ comprises a diffeomorphism $$\chi : D \times D \to X^*$$ (where $$D$$ is the unit disc in $$\mathbb{C}$$) such that

1. there is a holomorphic diffeomorphism $$\theta$$ from $$D$$ to an open set in $$\mathbb{S}^2$$ such that the diagram below commutes:

   $$
   \begin{array}{ccc}
   D \times D & \xrightarrow{\chi} & X^* \\
   \downarrow{\text{pr}_1} & & \downarrow{f} \\
   D & \xrightarrow{\theta} & \mathbb{S}^2
   \end{array}
   $$

2. for each fixed $$\tau \in D$$ the map $$\chi_\tau$$ defined by $$\chi_\tau(z) = \chi(\tau, z)$$ gives a holomorphic diffeomorphism from $$D$$ to an open set in the fibre $$f^{-1}(\theta(\tau))$$, where the latter is endowed with complex structure coming from $$J$$ on $$X'$$. 

It is true, but not completely trivial, that any point of $$X^*$$ lies in the image of a restricted chart. (This follows from the Riemann mapping theorem, as in [AB60], for example, with smooth dependence on parameters.) Two restricted charts compare, on the intersection of their images, by a smooth family of holomorphic diffeomorphisms of open sets in $$\mathbb{C}$$. Since these holomorphic diffeomorphisms induce holomorphic diffeomorphisms of the symmetric products as after (3.1) above, and the introduction of smooth parameters introduces no difficulties, these restricted charts induce a system of charts on $$X_r^*(f)$$ which differ on their overlaps by smooth maps. In sum we define $$X_r^*(f)$$ as a smooth manifold, with a smooth fibration over $$\mathbb{S}^2 \setminus \{f(p_i)\}$$, each fibre in $$X_r^*(f)$$ has a natural complex structure, and these vary smoothly in the obvious sense.

All of this discussion is compatible with the construction of the Picard varieties. We now specialise to the case when $$d = 2g - 2$$, where $$g$$ is the genus of the fibres of $$f$$. There is one subtlety, which we introduce now and clarify at the end of the section. If $$Z \to B$$ is any smooth fibration of curves, with smoothly varying holomorphic structures on the fibres in the above sense, there is a vector bundle $$W \to B$$ whose fibres are canonically identified with the spaces of holomorphic one-forms on the fibres. On the other hand, if $$Z$$ is holomorphic then
$V = f_*K_Z$ and $W$ are not equal, but differ by twisting by the canonical bundle of the base: $f_*K_Z = W \otimes K_B$. In constructing submanifolds representing the homology class $K_Z$, for a given manifold $Z$, it is sections of this twisted bundle $V$ that will be of primary importance.

Thus, for the smooth part of a Lefschetz fibration $f : X' \to \mathbb{S}^2 \backslash \{\text{Crit}\}$, we have:

1. A rank $g$ complex vector bundle $W$ over $\mathbb{S}^2 \backslash \{f(p_i)\}$ whose fibres are canonically identified with the holomorphic 1-forms on the fibres of $f$.

2. A bundle of complex tori $\text{Pic}_r^f(X^*) = P_r^* (f)$ over $\mathbb{S}^2 \backslash \{f(p_i)\}$, with a “zero-section”, whose fibres are quotients of the fibres of $W^*$ by the integer lattices $H^1(f^{-1}(r); \mathbb{Z})$.

3. A smooth map $\tau_f : X^*_r(f) \to P^*_r(f)$, commuting with the projection maps, which is equal to the Abel-Jacobi map $\tau_r$ on each fibre.

4. A diffeomorphism, compatible with the projection maps, between a neighbourhood of $\tau_f^{-1}(0) \subset X^*_r(f)$ and a neighbourhood of $A^{-1}(0)$ in $\mathcal{M}(V)$, which agrees with the map considered above on each fibre.

Here $\mathcal{M}(V)$ is the space, fibring over $\mathbb{S}^2 \backslash \{f(p_i)\} = \Delta$, constructed from the vector bundle $V = W \otimes K_{\Delta}$ in the obvious way. Precisely, inside the fibre product of $W^* \to \Delta$ and $\mathbb{P}(V) \to \Delta$ we take the subvariety defined by (3.3) in each fibre. Note that a $\partial$-operator (for instance from a connexion) on the complex vector bundle $V$ will induce an integrable holomorphic structure on the total space of this model $\mathcal{M}(V)$.

Before turning to the singular fibres, we need one more ingredient. Recall that we have chosen a genuine complex structure on $X'$ in the neighbourhood of each singular fibre, and the map $f$ is holomorphic. Thus the discussion can take place entirely in the realm of complex geometry. We need, then, a suitable extension of the discussion of divisors and line bundles to singular curves. Such a theory is well established in algebraic geometry; we will give a “users’ guide” to this in the Appendix, quoting the substantial theory from the literature. For now we simply summarise the result that we need.

\[ \text{(3.6) Theorem: Suppose } f : \mathcal{X} \to D \text{ is a holomorphic map from a smooth complex surface to the disc such that the fibres over } t \neq 0 \text{ are smooth but } f^{-1}(0) \text{ is an irreducible curve with} \]

\[ \text{This is well-defined, and smoothly equal to } V^* \times_{\pi} \mathbb{P}(V), \text{ as the spaces } V \text{ and } W \text{ are isomorphic up to scale.} \]
a single quadratic singularity \( Q \). Let \( E_i \) be sections of \( f \). Then there are holomorphic vector bundles \( W \) and \( V = W \otimes K_D \) over \( D \), and smooth complex manifolds \( X_r(f) \) and \( P_r(f) \), with commuting maps

\[
\begin{array}{ccc}
P(V) & \xrightarrow{\iota} & X_r(f) \xrightarrow{\tau} P_r(f) \\
\downarrow F & & \downarrow \pi \\
D & = & D
\end{array}
\]

and a relatively ample holomorphic line bundle \( \Lambda_r(\pi) \) over \( X_r(\pi) \) such that:

- when restricted to the punctured disc the data agrees with the symmetric product construction above, giving on each fibre the maps

\[
P(H^0(K_\Sigma)) \xrightarrow{\iota} \text{Sym}^r(\Sigma) \rightarrow \text{Pic}_r(\Sigma);
\]

- the line bundle \( \Lambda_r(\pi) \) restricts on each smooth fibre to the line bundle defined by the sum of the divisors in \( \text{Sym}^r(\Sigma) \) of \( r \)-tuples of points meeting one of the \( E_i \);

- the spaces \( (X_r(f))_0 \) and \( (P_r(f))_0 \) over 0 are irreducible and have simple normal crossing singularities: in suitable co-ordinates around the singularities the vertical maps in the diagram above are given by the standard model \((z_1, \ldots, z_n) \rightarrow z_1z_2\);

- the image of the map \( \iota \) does not meet the set of critical values of the map \( F \) and there is a holomorphic diffeomorphism between a neighbourhood of this image and a neighbourhood of the exceptional fibres in the complex manifold \( M(V) \);

- the canonical isomorphism, over \( f^{-1}(D \setminus \{0\}) \), between the tangent space along the fibres of \( \pi \) and the pull-back of the bundle \( W^* \), extends to the smooth part of \( f^{-1}(0) \);

- a Zariski-open subset of \( F^{-1}(0) \) can be identified with the open complex manifold \( \text{Sym}^r(X_0^0 \setminus \{Q\}) \).

Using this we can immediately extend our constructions over the whole space \( X' \). We obtain a vector bundle \( V \rightarrow S^2 \), spaces \( X_r(f) \), \( P_r(f) \) and a line bundle \( \Lambda \rightarrow X_r(f) \), with a diagram of maps

\[
\Lambda \rightarrow X_r(f) \rightarrow P_r(f) \leftarrow M(V),
\]

where all the data fibres over \( S^2 \). Although the difference between the bundles \( V \) and \( W \) seems rather artificial above, where the canonical bundles of the (open) base surfaces were trivial, it now plays an important role. There is an embedding \( P(V) \rightarrow X_r(f) \) but a neighbourhood of the section of \( P_r(f) \) defined by the canonical lines is naturally identified with a neighbourhood of zero in the bundle

\[
W^* = (V \otimes K_{S^2}^{-1})^* = V^* \otimes O(-2).
\]

Thus although \( P(V) = P(W) \), sections of the projective bundle arising from nowhere zero sections of \( V \) and \( W \) lie in different homotopy classes.
4 Symplectic structures and the Gromov invariant

In order to count pseudoholomorphic sections of the map $X_r(f) \to S^2$ we will need to discuss almost-complex and symplectic structures on the total space. We begin with a general discussion of almost-complex structures on fibre bundles. Suppose we have an exact sequence of $\mathbb{R}$-linear maps

$$0 \to U \xrightarrow{i} V \xrightarrow{\pi} W \to 0 \quad (4.1)$$

where $U$ and $W$ are complex vector spaces but $V$ is initially only a real vector space. We want to consider the set $\mathcal{S}$ of complex structures on $V$ such that the given maps are complex linear. We can regard a complex structure on $V$ as a complex subspace $V'$ of the complexification $V_C = V \otimes \mathbb{C}$ such that $V_C = V' \oplus V''$, where $V''$ is the conjugate of $V'$. From this one sees that the compatible complex structures correspond to splittings of the sequence of complex linear maps $U \to V_C/i(U) \to W$ obtained from $(4.1)$ by complexifying and then writing $U_C = U \oplus \overline{U}$ and $W_C = W \oplus \overline{W}$. This means that there is a natural affine structure on $\mathcal{S}$: two elements of $\mathcal{S}$ differ by a map in $\text{Hom}(W,U)$.

Now consider a manifold $Z$ which is the total space of a smooth fibre bundle $g: Z \to B$ and suppose that we are given almost-complex structures on the base $B$ and on each fibre of $g$, varying smoothly in the obvious sense. Then the compatible almost-complex structures on $Z$ are sections of a bundle over $Z$ whose fibres are affine copies of $\text{Hom}(g^*TB,T^v_vZ)$, where $T^v_vZ$ denotes the “vertical” tangent spaces along the fibres of $g$. We see at once from this that such almost-complex structures exist, since we can take affine linear combinations using a partition of unity. More generally if we are given a structure over an open set $U \subset Z$ and we have a closed set $K \subset U$ we may find a structure over the whole of $Z$ which agrees with the given one over a neighbourhood of $K$. Taking $K$ to be some closed neighbourhood of the singular fibres of $X_r(f) \to S^2$, equipped with its integrable complex structure, gives the

$(4.2)$ **Lemma:** Let $\mathcal{J}$ denote the class of smooth almost complex structures on $X_r(f)$ which agree with the given integrable structures on the fibres and in some (not necessarily fixed) open neighbourhoods of the singular fibres, and for which the projection map to the sphere is holomorphic. Then $\mathcal{J}$ is a non-empty smooth infinite-dimensional manifold.

We should stress that we will always deal with almost complex structures drawn from this class. To make this discussion more concrete, return to $g: Z \to B$ and suppose that the base $B$ has complex dimension 1; in fact let us suppose that $B$ is an open set in $\mathbb{C}$. Then two compatible almost-complex structures differ by a smooth vector field along the fibres. Explicitly, suppose we have chosen preferred co-ordinates $(z_i, \tau)$ in a neighbourhood of a point of $Z$ in the manner of $(3.5)$. So the $z_i$ are holomorphic co-ordinates along the fibres and $\tau$ is the canonical co-ordinate in the base. In these co-ordinates we have a distinguished almost-complex structure given by the identification with an open set in $\mathbb{C}^n$. Any other almost-
complex structure is represented by a vector field $v_i(z_j;\tau)$. For example the $\bar{\partial}$-operator of this structure is just

$$\bar{\partial}v_i(f) = \left(\frac{\partial f}{\partial z_i}, \frac{\partial f}{\partial \tau} + \sum v_i \frac{\partial f}{\partial \tau}\right).$$

Similarly a section $z_i = \phi_i(\tau)$ defines a pseudoholomorphic curve in this structure if

$$\frac{\partial \phi_i}{\partial \tau} = -v_i.$$

If our structure is given by a vector field $v_i$ in one set of co-ordinates, and we consider a set differing by a family of holomorphic maps $g_\tau$, the same structure is represented in the new co-ordinates by

$$(Dg_\tau)v + \frac{\partial g_\tau}{\partial \tau}.$$

We will use these explicit presentations later to obtain almost-complex structures on $X_r(f)$ with helpful “genericity” properties. For now, we turn to symplectic structures. According to Gompf [Gom01], given any map of compact almost-complex manifolds $\pi: (Z; J_Z) \to (B; J_B)$ for which

- $B$ admits a symplectic structure;
- for each $b \in B$ there is a neighbourhood $W_b$ of $\pi^{-1}(b)$ in $Z$ with a closed two-form $\eta_b$ which tames the almost-complex structure $J_Z|_{\ker d\pi}$;
- the $\eta_b$ are all induced by a single non-zero cohomology class $c \in H^2(Z; \mathbb{Z})$,

then $Z$ admits a symplectic structure. The proof uses a patching argument to obtain vertical non-degeneracy and Thurston’s trick of pulling back a large multiple of the base form to obtain global non-degeneracy. In our situation, the subvariety of the symmetric product of a Riemann surface consisting of divisors containing a given point $e$ is dual to a Kähler form. (To see that line bundle is ample, one can use the Nakai-Moishezon criterion, for instance.) This determines a class $c$ and the positivity properties of the bundle $\Lambda_r(\pi)$ in (3.6) take care of the singular fibres, providing a Kähler form near the ends with which we can patch the symplectic forms on $X_r^*(f)$. Putting this together, we obtain symplectic forms $\Omega$ which have the shape

$$\Omega = \mu_{\text{Sym}} + Rf^*\omega_{\text{st}}; \quad R \gg 0 \quad (4.3)$$

where $\mu_{\text{Sym}}$ is some closed two-form on $X_r(f)$ which is symplectic on the fibres and $\omega_{\text{st}}$ is the standard symplectic form on $S^2$ as before. Note that all such symplectic forms $\Omega$ are deformation equivalent, once we have chosen $c$ (which for us comes from the exceptional sections of $f$). It is easy to see that a given almost-complex structure $J \in \mathcal{J}$, as provided by (4.2), is tamed by the forms (4.3) once $R > R(J)$ is sufficiently large (where the precise value will depend on $J$).
(4.4) Lemma: $X_r(f)$ admits symplectic structures $\Omega$ which restrict on each fibre to the usual Kähler structure induced from the integrable complex structures on the fibres of $f$.

It seems likely that a more precise statement holds. For a space $Z$ equipped with a map $Z \to S^2$ let $\Gamma(Z)$ denote the set of homotopy classes of sections. We have natural maps

$$H^2(X';\mathbb{Z}) \xrightarrow{\mu} H^2(X_r(f);\mathbb{Z});$$

(4.5)

$$\Gamma(X_r(f)) \xrightarrow{\nu} H_2(X';\mathbb{Z});$$

(4.6)

defined as follows. For the first, represent the class $c$ as the first Chern class of a line bundle $L_c \to X'$. This defines a topological line bundle on $X_r(f)$ whose fibre at a tuple $p_1 + \cdots + p_r$ is precisely the quotient of $\otimes_i (L_c)_p$ by the symmetric group; take the first Chern class to define $\mu$. For the second, choose a smooth section $\psi$ in the homotopy class and define $\nu(\psi)$ by taking the $r$-tuples of points in each fibre $f^{-1}(t)$ of $X'$ designated by the value $\psi(t)$. (Thus $\psi$ defines a closed subset $C_{\psi}$ of $X'$ and $\nu(\psi) = [C_{\psi}]$). Then we have the

(4.7) Question: If $\omega$ is an integral symplectic form on $X'$ does $\mu(\omega)$ contain symplectic forms on $X_r(f)$?

If this is true, one would expect that for a section $\psi \in \Gamma(X_r(f))$, we have an identity $\langle \mu(\omega), [\psi] \rangle = \langle \omega, [\nu(\psi)] \rangle$. This would give another approach to the question of bubbling that we will tackle, by different means, later on. (The reason is simply that identifying a particular symplectic form on $X_r(f)$ enables us to estimate the area of bubbles in terms of data on $X'$.) Before computing the Gromov invariant of relevance to us, we recall the general theory. There is now a rigorous definition of Gromov-Witten invariants, valid for any taming almost-complex structure $J$ on any closed symplectic manifold and independent of regularity or monotonicity hypotheses. (Invariants counting holomorphic sections were also considered by Seidel [Sei97], who arranged the invariants for different homology classes of section into an element of a suitable Novikov ring.) One can proceed in two directions: develop a theory for “generic” almost-complex structures, and prove a lemma on how to compute the invariants given a non-generic structure for which the moduli space is compact and smooth but of the wrong dimension (cf. 4.11), or develop a theory via “virtual classes” which works at once for any almost-complex structure, at the cost of laying heavier analytic foundations. Although our computations use obstruction bundles, we are in the former “elementary” case above, and our proof does not require analysis beyond that presented in [Gro85] to produce the desired invariant.

Recall $\mathcal{J}$ denotes the set of smooth almost-complex structures on $X_r(f)$ described in (4.2). Write $[\psi_V]$ for the homology class of sections of $X_r(f)$ defined by a smooth nowhere zero section of the vector bundle $V \to S^2$ and the embedding $\mathbb{F}(V) \to X_r(f)$. (That such sections exist is trivial if the degree of the pencil $k$ is large, and hence the rank $g$ of $V$ is at least three.) To define the Gromov invariant, we will need to understand bubbling, and so we begin with the relevant technical Lemma: suppose our Lefschetz pencil is by surfaces of genus $g > 2$. 


4 SYMPLECTIC STRUCTURES AND THE GROMOV INVARIANT

(4.8) Lemma: Two-spheres inside the fibres of $F : X_r(f) \to S^2$ are governed by the following constraints:

1. The second homotopy group $\pi_2(\text{Sym}^{2g-2}(\Sigma)) = \mathbb{Z}$. A generator $h$ for this group can be given by a line - a rational curve of degree 1 - inside a projective space fibre of the Abel-Jacobi map.

2. Let $(\phi_j)$ be a family of holomorphic sections of $F$ in the homology class $[\psi_V]$, and suppose that the $\phi_j \to \phi_\infty$ converge to a cusp curve in the sense of Gromov. Each bubble component is homologous to a multiple of $h$.

3. To each bubble $\phi$ in a fibre of $X_r(f)$ we can associate a closed subset $C_\phi$ of $X'$. If the bubble lies in the homology class $N h$ then $[C_\phi] = N[Fibre]$.

Proof: The first statement is a sharper version of (3.2), and can be checked for instance by mapping $\text{Sym}^{2g-2} \hookrightarrow \text{Sym}^{2g-1}$ as an ample divisor and noticing that the right hand side is a smooth fibre bundle. It then follows by the Lefschetz hyperplane theorem. For the smooth fibres of $F$, the second statement is an immediate consequence of the first, whilst for the singular fibres we defer a proof to the Appendix (8.11). Finally, the third statement can be checked by picking a particular complex structure on $\Sigma$ and a particular rational curve representing $h$. For instance, if $\Sigma$ is hyperelliptic there is a natural $\mathbb{P}^1 \subset \text{Sym}^2(\Sigma)$ which arises from the double covering. Adding to this a fixed set of $2g-4$ points of $\Sigma$ defines a rational curve $\phi : \mathbb{P}^1 \to \text{Sym}^{2g-2}(\Sigma)$ whose image is homologous to $h$. By construction, the associated closed subset $C_\phi$ covers $\Sigma$ and contains a generic point of $\Sigma$ with multiplicity one. The result follows. ■

The Lemma will give us control on bubbles both for a generic almost complex structure, in the next Theorem, and for a particular non-generic almost complex structure employed in the last two sections of the paper. In any case, with this in hand, the proof of the following is standard:

(4.9) Theorem: Fix some generic $J \in \mathcal{J}$. There is an integer-valued invariant $I_r(f)$ which counts $J$-holomorphic cusp sections of $F : X_r(f) \to S^2$ in the homology class $[\psi_V] \in H_2(X_r(f); \mathbb{Z})$. It is independent of the choice of generic $J \in \mathcal{J}$ and of (deformation equivalences of) the symplectic form $\Omega$.

Proof: [Sketch] The key point will be that for generic $J \in \mathcal{J}$ the moduli spaces of $J$-holomorphic sections of $F$ will be smooth, compact manifolds of the correct (virtual) dimension.

Begin by fixing some $J \in \mathcal{J}$ and a symplectic form $\Omega$ which tames $J$. Recall that the defining condition for a map $\phi : S^2 \to X_r(f)$ to be pseudoholomorphic is that

$$\overline{\partial}_\phi = d\phi \circ j - J \circ d\phi = 0.$$ (4.10)
Here $\phi$ belongs to a suitable Banach space of sections (those of Hölder class $C^{k,a}$ for $k \geq 1$ and $a \in (0,1)$ for instance):

$$\{ \phi \in C^{k,a}(\text{Maps}(S^2; X_r(f))) \mid dF \circ d\phi = \text{id} \}.$$ 

There is a naturally induced almost-complex structure on $\phi^* T^v T_{vt}(X_r(f))$, where the superscript denotes the vertical tangent bundle. (This bundle is well-defined on the image of the section, since the condition that $dF \circ d\phi = \text{id}$ implies that im($\phi$) is disjoint from the locus of critical values of $F$.) The linearisation of (4.10) defines a section of Banach bundles

$$D(\overline{\partial}_\phi) : C^{k,a}(S^2; \phi^* T^v T_{vt}(f)) \to C^{k-1,a}(S^2; \phi^* T^v T_{vt}(f) \otimes \Lambda^{0,1} T^*(S^2)).$$

In the usual way, this operator differs from an integrable $\overline{\partial}$-operator by an order zero operator, and is hence Fredholm with index

$$\text{Index}(\overline{\partial}_\phi) = 2\langle c_1(T^v T_{vt}(f); \Omega), [\psi_V] \rangle + 2 \dim_{\mathbb{C}}(F^{-1}(\text{point})).$$

In our situation this index is zero. A more general version of the relevant computation is given in [Smi]; the key point is the existence of a sequence (well-defined over the image of the section)

$$0 \to V \to T^v T_{vt}(f) \to W^* \to L \to 0$$

coming from the differential of the projection $\tau$ of the Abel-Jacobi map. The cokernel line bundle $L$ can be shown to be isomorphic to $K_{\mathbb{P}1}$; the result then follows by considering the associated exact sequences in cohomology, exactly as in the obstruction computations given in full in section 5. Recall a cusp section is any cusp curve [MS94] for which the principal component is a section of $F$. According to Gromov’s compactness theorem, and since the condition that a curve contain a section component is closed, the moduli space $M$ of cusp sections holomorphic with respect to $(j,J)$ is a compact Hausdorff space (though not necessarily smooth). Note that for our class of complex structures $J$ all bubble components will lie in fibres of $F$; otherwise consider the restriction of $F$ to the component and obtain a contradiction to the homological intersection number with a fibre.

For “regular” $J$ the moduli space of smooth sections will be smooth and of the expected dimension. Even though we fix the complex structures to be integrable over small discs, there is still a dense notion of regularity as surjectivity of $D(\overline{\partial}_\phi)$ will hold whenever $J$ is generic on the bundle restricted to the section away from these discs. See (4.11) and, for proofs, [MS94] and ([Sei97], Section 7).

We now claim that, again for a generic - and dense - choice of the almost complex structure $J$, the moduli space will also be compact, that is there can be no cusp curves (this is a "fibred monotonicity" property). We have already remarked that any bubbles are vertical. By (4.8) we know that in fact any bubble is homologous to $N h$ for some $N \geq 1$. Any cusp curve has a unique component with non-trivial intersection number 1 with the fibre of $F$. We claim this component is in fact a smooth section. For it is $J$-holomorphic with respect to a smooth $J$, and hence by elliptic regularity the map is everywhere smooth. Hence it is a section away from the singular fibres. Near these fibres, the map $S^2 \to X_r(f) \to S^2$ is holomorphic with respect to an integrable almost complex structure and hence a branched
covering over its image. Since it has local degree one, it must be a local diffeomorphism here also, and hence the differential of the projection is invertible over the whole image. Thus, any non-smooth cusp curve gives rise to a smooth section of \( X_r(f) \). By the second part of (4.8), we see that the homology class of this section is \([\psi_V - Nh]\). But now an index computation shows that the virtual dimension for the space of such sections is negative, and so for generic \( J \) such moduli spaces will be empty. The point is that for sections of \( X_r(f) \) which give rise to cycles in \( X' \) in the homology class \( A \), the virtual dimension for the space of \( J \)-holomorphic sections is \((A^2 - K_{X'} \cdot A)/2\). By the third part of (4.8), splitting off a bubble corresponds to changing \( A \rightarrow A - [\text{Fibre}] \). Taking \( A = K_{X'} \) gives the required negativity in our case. The details of this general index result can be found in [Sm], and again involve studying an exact cohomology sequence coming from the differential of the Abel-Jacobi map. An analogous, but harder, argument will be given in the last section.

Our compact, zero-dimensional moduli space is naturally oriented, and our Gromov invariant is the signed count of its number of points. Picking a point \( P \in S^2 \setminus \{\text{Crit}\} \) defines an evaluation map

\[
ev : M \to F^{-1}(P) \cong \text{Sym}^r(\Sigma_g)
\]

and hence a map \( \ev_* : H_0(M) \to \mathbb{Z} \). Then we define \( \mathcal{I}_r(f) = \ev_*([M]) \). That this is independent of choices of \( J \) and deformations of \( \Omega \) is a familiar cobordism argument, for one-parameter families of moduli spaces, which asserts that in fact the fundamental homology class \([M]\) is independent of such choices. (Since \( J \)-holomorphic curves are smooth whenever \( J \) is smooth, we also have independence of the choice of Sobolev spaces used in constructing the invariant.)

In fact, in the rest of the paper, we shall predominantly work with two almost complex structures from \( J \) that are not regular, and we will provide specific arguments in each case to obtain smoothness and compactness of the relevant moduli spaces of holomorphic curves. We shall make use of the following standard results: proofs can be found in [Sa] and [MS94] respectively.

(4.11) Proposition:

- Let \( J \in J \) be such that the moduli space \( M \) is a compact smooth manifold of positive dimension \( r \) whose tangent space at every point \( \phi \) is given by \( \ker(\partial_\phi) \). Then the Gromov invariant (virtual class \([M]^{\text{virt}}\)) is given by the Poincaré dual of the Euler class of the obstruction bundle over \( M \) with fibre \( \text{coker}(\partial_\phi) \).

- Let \( U \subset X_r(f) \) be any open set. By a generic perturbation of the almost-complex structure \( J \) supported in \( U \), we can ensure that moduli spaces of all pseudoholomorphic curves passing through \( U \) are regular. A similar comment applies to structures generic on an open set in \( P_r(f) \).

In our framework in which the invariant is only defined for “regular” almost-complex structures, the first statement above gives a method for computing \( \mathcal{I} \) starting with an irregular but
sufficiently well-behaved $J$. (The proof is a prototype for the definition of virtual fundamental classes more generally.)

5 Computing the invariant

In this section, by employing the first result of [4.11], we show that the Gromov invariant introduced above is non-vanishing under a suitable linear constraint on the topology of the given four-manifold $X$.

(5.1) Proposition: There is an almost-complex structure $J_V \in J$ on $X_r(f)$ for which the moduli space of cusp sections in the class $[\psi_V]$ is a projective space of dimension $N - 1$, where $N = [b_+(X) - 1 - b_1(X)]/2$. Moreover no sections contain any bubbles.

We start with the following. Recall that any $\overline{\partial}$-operator on a vector bundle over a Riemann surface induces a holomorphic structure. (There is an essentially equivalent discussion of the following in terms of connexions rather than $\overline{\partial}$-operators, if the reader prefers.) Recall also our notation: $W$ denotes the “relative dualising sheaf” of the fibration, with fibre canonically identified with the space of holomorphic one-forms on the fibres, whilst $V$ denotes $W \otimes O(-2)$.

(5.2) Lemma: For a generic $\overline{\partial}$-operator on $V \to S^2$ the space of holomorphic sections of the vector bundle has dimension $[b_+(X) - 1 - b_1(X)]/2$. Every non-zero section defines a smooth section of the projective bundle $P(V)$ in the class $[\psi_V]$.

Proof: The result is almost an application of standard machinery: for a family of $\overline{\partial}$-operators down the fibres of a smooth fibre bundle, the first Chern class of the index bundle can be computed using the Atiyah-Singer index theorem. For a holomorphic family with singular fibres, this is replaced by the Grothendieck-Riemann-Roch theorem. In our case, neither quite apply; however, in [Smi99] an excision argument was used to compute the first Chern class of the relative dualising sheaf $W = f_*(\omega_{X'/\mathbb{S}^2})$. From the formulae

$$c_1(V) = c_1(f_*(\omega_{X'/\mathbb{S}^2} \otimes O(-2))); \quad \text{rk}(V) = g$$

and, from [Smi99]

$$c_1(f_*(\omega_{X'/\mathbb{S}^2})) = [\sigma(X') + \sharp\{\text{Crit}(f)\}]/4$$

it follows that $c_1(V) = [\sigma(X') + \delta]/4 - 2g$, writing $\delta$ for the number of singular fibres. On the other hand, the Euler characteristic $e(X') = 4 - 4g + \delta$, and hence $c_1(V) = [\sigma(X') + e(X')]/4 + (g - 1) - 2g$. Since the index is the sum of the rank and first Chern class, and since the sum of signature and Euler characteristic for a four-manifold is invariant under blowing up and down, we arrive at the formula (which is independent of the degree $k$ of the original Lefschetz pencil):

\[^3\text{In [Smi99] the determinant of this bundle was denoted by } \lambda.\]
\[ \text{Index } (\partial) = \sigma + e - 1 = \left[ b_+(X) - 1 - b_1(X) \right] / 2. \]

Now on the space of connexions \( \{ \partial \} \) on a complex vector bundle, the discrete-valued function \( h^1(V; J_{\partial}) \) is upper semicontinuous and jumps on a subvariety of positive codimension. Hence for generic \( \partial \) we have \( H^1 = 0 \) and \( H^0 = \text{Index} \); this is a stability result, in the lines of Atiyah and Bott’s paper [AB82].

According to Grothendieck, every holomorphic vector bundle over \( \mathbb{P}^1 \) is a direct sum of line bundles. Hence \( \partial \) induces a splitting of \( V \). By the first assumption in (2.3), we know that the rank of \( V \) is larger than the index; we deduce \( c_1(V) < 0 \). In this case the generic (most stable) splitting is given by

\[
(V, J_{\partial}) \cong \bigoplus_i O(n_i)
\]

with \( n_i \in \{0, -1\} \) for every \( i \). For such a \( d \)-bar operator, the holomorphic sections of \( V \) are exactly the constant sections of the \( O \)-factors, and hence no non-zero sections have any zeroes. It follows that every non-trivial section of \( V \) defines a section of the projective bundle \( \mathbb{P}(V) \), in the fixed homology class \( [\psi_V] \), which has no bubbles, since these arise from isolated zeroes of sections of \( V \). (This is clear, for instance, from a computation in local co-ordinates near the zero.)

The discrepancy between the linear constraint \( b_+ > 1 + b_1 \) of (1.1) and \( b_+ > 1 \), as required by Taubes [Tau95], arises here. There is a distinguished trivial topological subbundle of \( V \) coming from the first homology of the four-manifold \( X \) when \( b_1(X) \neq 0 \), and replacing generic \( d \)-bar-operators on \( V \) by ones adapted to this subbundle should give the sharper result. We will suppress the issue in this paper.

Recall that a choice of a \( d \)-bar operator, and hence holomorphic structure, on \( V \) induces one on the manifold \( \mathbb{M}(V) \). This in turn defines an integrable complex structure on a neighbourhood of the image of the embedding \( \mathbb{P}(V) \hookrightarrow X_r(f) \), and also on \( W^* \) and hence a neighbourhood of the zero-section (defined by the canonical lines on each fibre) of the Picard fibration \( P_r(f) \).

(5.3) Definition: Almost-complex structures \( J \in \mathcal{J} \) on \( X_r(f) \) and \( j \) on \( P_r(f) \) are standard near the zero-sections if they agree with the integrable complex structures on neighbourhoods of \( \mathbb{P}(V) \) and its image, induced as above from a choice of generic \( d \)-bar operator on \( V \).

In our discussion of almost-complex structures on fibre bundles, we noted the freedom to extend structures which are prescribed over neighbourhoods of fixed closed sets.

(5.4) Lemma: Given an almost-complex structure \( j \) on \( P_r(f) \) which is standard near the zero section, there is an almost-complex structure \( J \in \mathcal{J} \) on \( X_r(f) \) which is standard near the zero section and for which the map \( X_r(f) \to P_r(f) \) is \( (J, j) \)-holomorphic.
For away from $\mathbb{P}(V)$ the map is a fibration and we can just use the existence assertion before (4.2).

(5.5) Lemma: There is an almost-complex structure on $P_r(f)$ which is standard near the zero-section, and for which any holomorphic section homologous to the zero-section is itself zero.

Proof: Along the zero-section, the vertical tangent bundle to $P_r(f)$ is, by (3.6) and the constructions of the Appendix, canonically isomorphic to the bundle $W^*$. Recalling that $W = V \otimes K_{\mathbb{P}^1}$ we have an identity

$$\text{Index}_\partial(V) = -\text{Index}_\partial(W^*)$$

(5.6)

from Serre duality for any fixed $d$-bar operators on $V$ and $W$. If we choose a generic $d$-bar operator on $V$ as above, and induce the almost-complex structure on $P_r(f)$ near the zero-section as in (5.3), it follows that $W^*$ has no holomorphic sections at all, and hence $P_r(f)$ can have no non-zero sections with image near the zero-section. But then for a generic extension of the almost-complex structure from a neighbourhood of the zero-section, moduli spaces of holomorphic curves in $P_r(f)$ away from zero will be regular by (4.11). Now (5.6) shows that the virtual dimensions are negative provided $X$ satisfies $b_+ > 1 + b_1$, and hence the moduli spaces of sections of $P_r(f)$ away from zero will be empty. ■

If we combine the three results (5.4, 5.5 and 5.2) we have a proof of the proposition (5.1) stated at the beginning of the section. That is, choose almost complex structures $j$ on $P_r(f)$ and $J \in \mathcal{J}$ on $X_r(f)$ satisfying (5.5) and (5.4) respectively. Any $J$-holomorphic section of $X_r(f)$ projects to a $j$-holomorphic section of $P_r(f)$, which must be the zero-section. Hence the moduli space of smooth $J$-holomorphic sections of $X_r(f)$ projects to the moduli space of smooth $J$-holomorphic sections of $\mathbb{P}(V)$, the preimage of the zero-section of $P_r(f)$ under the Abel-Jacobi map. Now by (5.2) this last moduli space of holomorphic sections is a projective space $\mathbb{P}^N$; this is already compact, and the space $\mathcal{M}$ of $J$-holomorphic cusp sections of $X_r(f)$ is the same projective space. This is just the required result.

Appealing again to (4.11), to compute the Gromov invariant for sections of $X_r(f)$ in the distinguished class $[\psi_V]$ we must compute the Euler class of the obstruction bundle $\mathcal{E} \to \mathbb{P}^{N-1}$ (where $N = (b_+ - b_1 - 1)/2$). All the points of the moduli space correspond to sections with image inside the neighbourhood of $\mathbb{P}(V) \subset X_r(f)$ which we have identified with a universal, holomorphic local model $\mathcal{M}(V)$, and it follows that we can perform the obstruction computation here. By definition, the fibre $\mathcal{E}_\phi$ at a section $\phi$ is given by $H^1(\nu_\phi)$ where $\nu_\phi$ denotes the normal bundle to the image. All the sections $\phi$ are simple, with no bubbles, and the normal bundle can be identified with the vertical tangent bundle. As a piece of notation, we refer to the cokernel of the natural inclusion $\mathcal{O}_{\text{taut}} \to \mathbb{C}^{N+1}$ of the tautological line bundle over projective space $\mathbb{P}^N$ into the trivial rank $N + 1$ bundle as the quotient bundle. Note that this has Euler class $(-1)^N$.

(5.7) Proposition: The obstruction bundle $\mathcal{E} \to \mathcal{M} \cong \mathbb{P}(H^0(V))$ is topologically the dual
of the quotient bundle over projective space. In particular, $I_r(f) = \pm 1$ for any symplectic four-manifold $X$ satisfying $b_+(X) > 1 + b_1(X)$.

**Proof:** Recall the definition of the model $M(V)$; we take the subvariety of $W^* \times \pi \mathbb{P}(V)$, where $\times \pi$ denotes the fibre product over $S^2$, defined by

$$\{ (\theta, [x]) \mid \theta(x) = 0 \}.$$ 

Since the vector bundles $V$ and $W$ can be identified up to scale, this subvariety is well-defined, and has a projection $\tau$ to $W^*$ whose fibre jumps precisely over $0 \in W^*$. If we take the differential of this projection, and recall that the fibre over $0$ is exactly $\mathbb{P}(V)$, we obtain a sequence (of holomorphic bundles)

$$0 \to TP(V) \to T^{\text{ref}} M(V) \xrightarrow{\tau} T(W^*) \to \text{cok}(d\tau) \to 0.$$ 

We restrict this sequence to a sphere $\phi(S^2)$ corresponding to a point $\phi \in M$. It splits into two short exact sequences

$$0 \to TP(V)|_{\text{im}(\phi)} \to (T^{\text{ref}} M(V))|_{\text{im}(\phi)} \to \text{im}(d\tau)|_{\text{im}(\phi)} \to 0;$$

$$0 \to \text{im}(d\tau)|_{\text{im}(\phi)} \to T(W^*)|_{\text{im}(\phi)} \to \text{cok}(d\tau)|_{\text{im}(\phi)} \to 0.$$

Take the long exact sequences in cohomology: from the first,

$$H^1(TP(V)) = 0 \to E_\phi \to H^1(\text{im} d\tau) \to 0$$

and from the second

$$H^0(W^*) = 0 \to H^0(\text{cok}(d\tau)) \to E_\phi \to H^0(V^*) \to H^1(\text{cok}(d\tau)) \to 0.$$ 

The term $H^0(V^*) = H^1(W^*)$ arises by Serre duality. We claim that over the image $\phi(S^2)$, for $\phi \in M$, the bundle $\text{cok}(d\tau)$ defines a copy of the canonical bundle of $S^2$; hence $H^0 = 0$ and $H^1 \cong \mathbb{C}$ is one-dimensional. To see this, note that the cokernel of $d\tau$ is generated, for each $t \in S^2$, by a vector in $H^1(\Sigma_t, O) = W^*$ on which $\phi(t)$ is non-zero. If the vector spaces $V$ and $W$ were dual, we could choose a metric on $V$ and then the non-zero vector $\phi(t)$ would trivialise the cokernel bundle; by the same token, the failure of this is measured by the discrepancy $V = W^* \otimes K_{S^2}$. Our second sequence above now has the form

$$0 \to E_\phi \to H^0(V^*) \to L_\phi \to 0$$

where $\mathcal{M} = \mathbb{P}(H^0(V))$. The remarks above show that, after choosing a metric, the section $\phi$ defines a generator of $L_\phi$, and this line bundle is therefore dual to the tautological line bundle. Thus we have the dual of the standard exact sequence, and the obstruction bundle is the dual of the quotient bundle as claimed. 

It may be helpful to point out a translation of the above if we work not in the model $M(V)$ but on the fibration $X_r(f) \to P_r(f)$ itself. The cokernel of the canonical projection

\footnote{Such a perspective is important in [Smi], where no analogue of $M(V)$ exists.}
from \( \text{Sym}^r(\Sigma) \) to \( \text{Pic}^r(\Sigma) \) at a divisor \( p_1 + \cdots + p_r \) is canonically equal to \( H^1(\Sigma; \mathcal{O}_\Sigma(\oplus p_i)) \).

It follows that the cokernel bundle, for our situation, is canonically the pullback from the universal curve \( C_g \to M_g \) of the bundle \( R^1 \pi_* K_C \). Now Grothendieck's relative duality \([Har77]\) asserts that for a holomorphic fibre bundle \( \pi : Z \to B \) and locally free sheaf \( \eta \) on \( Z \) there is a natural morphism

\[
\pi^*(\eta^* \otimes \omega_{Z/B}) \to (R^1 \pi_* \eta)^*
\]

(5.8)

which is moreover an isomorphism for flat families of curves. Taking \( \eta = K \), and pulling back the identity of Chern classes induced by (5.8), again shows that \( \text{cok}(d\tau) \) gives a copy of the canonical bundle of the sphere for each \( \phi \in M \). Then one proceeds as above.

## 6 Strata in the symmetric product

Our goal is now to construct almost-complex structures on \( X_r(f) \) for which the section provided by the above computation of the Gromov invariant will yield a symplectic surface.

Our first task is to review certain natural strata inside the symmetric product of a Riemann surface. Fix a single complex curve \( (\Sigma, j) \) and a partition \( \pi : r = \sum a_i n_i \) for integers \( a_i, n_i \geq 1 \). Moreover suppose at least one \( a_i > 1 \) (to obtain a stratum which is not the entire space). Inside \( \text{Sym}^r(\Sigma, j) \) there is a diagonal stratum \( \chi_\pi \) indexed by \( \pi \) comprising the image of the map

\[
\text{Sym}^n_1(\Sigma, j) \times \cdots \times \text{Sym}^n_t(\Sigma, j) \xrightarrow{p \mapsto} \text{Sym}^r(\Sigma, j)
\]

which takes tuples \( (D_1, \ldots, D_t) \) to \( \sum a_i D_i \) (viewed additively as an effective divisor of degree \( r \) on \( \Sigma \)). An example: writing \( 10 = 3 + 2 + 2 + 2 + 1 \) gives rise to the mapping

\[
\Sigma \times \text{Sym}^3(\Sigma) \times \Sigma \to \text{Sym}^{10}(\Sigma) : (p, D, q) \mapsto 3p + 2D + q.
\]

For each partition \( \pi \) of \( r \) of the above form we obtain a stratum of complex codimension \( r - \sum n_i \). The union of all the strata, which is just the image of \( \Sigma \times \text{Sym}^{r-2}(\Sigma) \) under the map \( (p, D) \mapsto 2p+D \), is the diagonal locus inside \( \text{Sym}^r(\Sigma) \). The various partitions give rise to a stratification of the diagonal whose top open stratum comprises the locus where precisely two and no more points of the tuple have coalesced, and whose open strata parametrise effective divisors with prescribed multiplicities of points (their closures allowing multiplicities greater than or equal to those values). Although these strata, as closed topological subsets of the topological symmetric product, are independent of the complex structure \( j \), they are not smoothly embedded. However, each stratum \( \chi_\pi \) has a smooth model \( Y_\pi \), which is just the domain of the mapping \( p_\pi \) defined above. This map is finite and holomorphic, and defines an isomorphism over the dense open subset of points \( (D_1, \ldots, D_t) \in \prod_{i=1}^t \text{Sym}^{n_i}(\Sigma) \) for which the supports of the \( D_i \) are all disjoint. The above discussion is clearly compatible with smooth variations of complex structure in the fibres, so at least over \( S^2 \setminus \{\text{Crit}\} \) we have smooth fibre spaces \( Y_\pi \to S^2 \) together with maps, which are fibrewise holomorphic, to the strata \( \chi_\pi \to S^2 \). (The situation is rather similar to the case of the map \( t \mapsto (t^2, t^3) \) which takes \( \mathbb{C} \) homeomorphically onto a singular complex curve in \( \mathbb{C}^2 \).)
Along with the diagonal strata, we need to consider strata arising from the exceptional spheres of the fibration $f : X' \to S^2$. Here, for a fixed point $e \in \Sigma$ and multiplicity $a(e) \in \mathbb{Z}_{>0}$, we have a stratum $\chi_{a(e)} \subset \Sym^r(\Sigma)$ defined as the image of the map

$$\Sym^{r-a(e)}(\Sigma) \xrightarrow{p_a(e)} \Sym^r(\Sigma) : D \mapsto D + a(e)e.$$ 

The image of the map is again not a smooth subset of the $r$-th symmetric product, but we again regard the domain of $p_a(e)$ as a smooth model for $\chi_{a(e)}$, and again $p_a(e)$ is holomorphic, a homeomorphism on a dense open set, and compatible with passing to smooth families. Combining the two discussions, we define strata $\chi_{\pi, \mathcal{N}} \subset \Sym^r(\Sigma)$ as we vary over partitions $\pi$ of the integer $r - |\mathcal{N}|$ and ordered subsets $\mathcal{N} \subset \{1, \ldots, k^2 \omega^2_X\}$ of the set of exceptional spheres (which may be taken with multiplicities, so $\mathcal{N}$ may have repeated elements). There are smooth models

$$Y_{\pi, \mathcal{N}} \to \chi_{\pi, \mathcal{N}}$$

where if $\pi = \sum_{i=1}^t a_i n_i = r - |\mathcal{N}|$ we map

$$\prod_i \Sym^{n_i}(\Sigma) \to \Sym^r(\Sigma); \quad (D_1, \ldots, D_t) \mapsto \sum a_i D_i + \sum_{j \in \mathcal{N}} e_j.$$

Here $e_j$ denotes the point of intersection of the exceptional sphere $E_j \subset X'$ with the fibre corresponding to $\Sigma$. The complex codimension of the stratum is $r - \sum n_i - |\mathcal{N}|$ where the elements of $\mathcal{N}$ must be counted with their multiplicities. As we vary in families, we obtain smooth fibre bundles $Y_{\pi, \mathcal{N}}$ over $S^2$. The importance of these for the proof of (1.1) is explained by the following remarks (which rely on a lemma at the start of the next section but may better motivate the present discussion than be deferred).

Suppose we have a smooth section $\phi$ of $X_r(f)$. There is then an obvious way to associate a closed set $C_\phi \subset X'$ to $\phi$. Note first that this cannot meet the singular set in the special fibres. Suppose for simplicity that $\phi$ also lies in the dense open set $\Sym^r(\Sigma_0 \setminus \{Q\})$ at each of the singular fibres, where $Q$ denotes the node, and that $\phi$ is disjoint from all the strata $\chi$ of real codimension greater than two. Clearly $C_\phi$ is a smooth embedded surface away from the points where $\phi$ meets the diagonals.

Furthermore, we claim that $C_\phi$ is a standard surface (2.7) if $\phi$ meets the (real) codimension 2-strata transversally, with locally positive intersection. (Away from the strata of higher codimension, the diagonals are smoothly embedded, and so transversality here makes good sense.) The only real codimension two strata correspond to the partition $r = 2 + 1 + \cdots + 1$ and to the multiplicity one loci $\chi_{a(E)=1}$. In each case, by throwing away local smooth sections, we can reduce to a model inside the second symmetric product. For the first case, after a diffeomorphism the model is as follows. The curve is given by

$$\{x^2 + y = 0\} \subset \mathbb{C}^2 \to \mathbb{C};$$

the map to $\mathbb{C}$ is projection on the second factor, and the associated map from the base to the second symmetric product of some fixed fibre $F$ is given by $t \mapsto [ivt, -iv^2 t]$. (The equivalence relation $\sim$ is that of the natural covering $F \times F \to \Sym^2(F)$.) The image of the map to $\Sym^2(F)$ has intersection number 1 with the diagonal, and hence since the model is
holomorphic this must be a transverse intersection. By contrast, if for instance $C_\phi$ contains a nodal point we can take a neighbourhood of the node:

$$\{x^2 + y^2 = 0\} \subset \mathbb{C}^2 \to \mathbb{C}$$

with the fibration again being second projection. Then the map from a disc to the second symmetric product of the fibre $F$ is locally two-to-one; $\pm t \mapsto [it, -it]$. Now the intersection with the diagonal has multiplicity two, and amounts to a smooth tangency. We can always perturb this second model to two transverse intersections with the diagonal; for instance, $\{x = 0\} \subset \mathbb{C}^2$ passes through the node of a local map $(x, y) \mapsto x^2 + y^2$ but the perturbation $\{x = \varepsilon\}$ is instead tangent to the fibres at two points. Similarly, intersections of $\phi$ with the stratum $\chi_\pi(E) = 1$ at a point $F(e) \in \mathbb{S}^2$ correspond to transverse intersections of the surface $C_\phi$ with the fixed section $E \ni e$. Thus if the section is transverse to all the strata of the diagonal it defines a smooth symplectic curve which meets $E$ locally positively. More generally we have

**6.1 Proposition:** Suppose $\phi$ lies inside a stratum $\chi_{\pi, N}$, does not meet any stratum of (real) codimension bigger than 2 in $\chi_{\pi, N}$ and cuts the codimension 2 strata transversally and with locally positive intersections. Then $C_\phi$ is a union of standard surfaces in $X$, with positive transverse intersections and no triple intersections. Moreover, at an intersection point, either one of the components must be an exceptional sphere, or the two components must have strictly different multiplicities.

It is not quite clear how to interpret transversality here (consider, again, the case of a disc in $\mathbb{C}^2$ lying inside the cuspidal cubic curve $\{y^2 = x^3\}$ and the meaning of transversality of the disc to the origin). However, as we shall explain at the start of the next section, $\phi$ defines an associated section $\tilde{\phi}$ of a unique minimal smooth model $Y_{\pi, N}$ in which the hypotheses assert that it is disjoint from the strata of real codimension at least four. In this case, transversality makes sense for the smooth open loci in the codimension two strata. (This amounts to pulling a disc back to the smooth domain $\mathbb{C}$ of the cuspidal cubic in our analogy, and asking for transversality to the origin here.) Clearly $C_{\tilde{\phi}}$ is naturally partitioned by collecting together the components belonging to each distinct factor in the partition $\pi$ which indexes its associated stratum. This exactly groups the components by their multiplicities. The discussion before the Proposition implies that the union of the components of any given multiplicity is a smooth symplectic surface, and moreover that the components coming from different factors can meet only transversally. By construction $C_\phi$ is given, as a set, by the components of $C_{\tilde{\phi}}$ and various copies of exceptional curves. The result follows.

**6.2 Remark:** It is worth noting that although the two kinds of strata $\chi_\pi$ and $\chi_{\alpha(E)}$ play similar roles in the proof, conceptually they enter with different flavours: we are trying to find sections which will be transverse to all the strata of the first sort, but which for topological reasons will necessarily lie in all the strata $\chi_{\alpha(E)} = 1$. Many of the subsequent technical difficulties arise because we cannot exclude the possibility that in fact the sections do also lie inside various of the $\chi_\pi$ or $\chi_{\alpha(E)} > 1$. □
We can also attach a multiplicity $a_i$ to each component $C_i$ of $C_\phi$, derived from its multiplicity in the symmetric product (so for the exceptional sections, these multiplicities are the $\alpha(E)$ we have encountered before). This associates to $\phi$ a positive symplectic divisor in the sense of (2.10). We associate the homology class $[C_\phi] = \sum a_i [C_i] \in H_2(X')$ to $\phi$, and if $\phi$ is homotopic to a section of $P(V) \subset X_\tau(f)$ in the homotopy class $[\psi_V]$ of sections coming from $V$, then $[C_\phi] = K_{X'}$.

The smooth models $Y_{\pi,R}$ form smooth fibre bundles as we vary the Riemann surface over $S^2$, and these admit almost-complex structures for which the projection map to the sphere is pseudoholomorphic and which agree with the standard integrable structures on the fibres. The proof of this precedes as in (4.2), again building on results of the Appendix.

(6.3) **Definition:** An almost-complex structure $J \in J$ on $X_\tau(f) \to S^2$ is compatible with the strata if there are almost-complex structures on all the $Y_{\pi,R}$ as above for which the canonical projection maps $Y_{\pi,R} \to X_\tau(f)$ are holomorphic.

Again, such a $J$ will tame the symplectic forms $\Omega_R$ for $R > R(J)$. Then the important ingredient for us is the

(6.4) **Proposition:** There exist almost-complex structures on $X^*_\tau(f)$ which are compatible with the strata and which agree with the given (integrable) structures near the ends.

We shall give the proof for a stratum $\chi_\pi$ with $|R| = 0$; the general case is complicated only by notation.

**Proof:** Recall that for an open set $U$ in $\mathbb{C}$ we have the map $\sigma : \text{Sym}^r(U) \to \mathbb{C}^r$ defined by the elementary symmetric functions. Let $\pi = \sum a_i n_i$ be a partition of $r$; write $\sum n_i = s$. We then have sequences of maps

$$
Y_\pi(U) \to \sigma_\pi(U) \subset \mathbb{C}^s \\
Y_\pi(U) \to \text{Sym}^r(U) \to \sigma(U) \subset \mathbb{C}^r.
$$

Here the first map is defined by using the identification with the appropriate product of smaller symmetric products. The key point is that if we have a holomorphic diffeomorphism $\alpha : U \to V \subset \mathbb{C}$ we get a diagram of holomorphic maps connecting the above $U$-sequences and $V$-sequences (where all the vertical maps are induced by $\alpha$):

$$
\begin{array}{ccc}
\mathbb{C}^s & \supset & \sigma_\pi(U) \\
\downarrow & & \downarrow \\
\mathbb{C}^s & \supset & \sigma_\pi(V)
\end{array} \quad \begin{array}{ccc}
Y_\pi(U) & \to & \text{Sym}^r(U) & \to & \sigma(U) \\
\downarrow & & \downarrow & & \downarrow \\
Y_\pi(V) & \to & \text{Sym}^r(V) & \to & \sigma(V)
\end{array} \quad \begin{array}{c}
\subset \mathbb{C}^r \\
\subset \mathbb{C}^r
\end{array}
$$

Here the vertical map on the left is just the product of the maps induced on the smaller symmetric products. Now apply this to the charts on $X^*_\tau(f)$ defined by the restricted charts on $X^*$. These charts map sets $G \times D$ into $X^*_\tau(f)$, where $G$ is a product of open sets of the form $\sigma(U_\alpha)$, and $U_\alpha$ are local holomorphic charts along the fibres of $X^*$. In one of these charts, say $G_1 \times D$, we have a preferred almost-complex structure $J_1$ given by the product structure. Let $G_2 \times D$ be another such chart, with the same image. These charts compare by
a smooth family of holomorphic maps \( g_\tau : G_1 \to G_2 \). Now these maps \( g_\tau \) are not arbitrary holomorphic maps; they are all induced in the manner above by holomorphic diffeomorphisms between open sets in \( \mathbb{C} \). Hence these maps fit into a diagram:

\[
\begin{array}{ccc}
G_{1,\pi} \times D & \to & G_1 \times D \\
\downarrow & & \downarrow \\
G_{2,\pi} \times D & \to & G_2 \times D 
\end{array}
\]

where \( G_{1,\pi}, G_{2,\pi} \) are open sets in some \( \mathbb{C}^n \) which give local charts on the smooth model \( Y_\pi \) of the stratum \( \chi_\pi \). The almost-complex structure \( J_1 \) is represented in the second chart \( G_2 \times D \) by a vector field along the fibres \( \xi = \frac{\partial g_\tau}{\partial \tau} \). The product structure in the first chart also gives a local almost-complex structure \( J_{1,\pi} \) on the smooth model of the stratum. Thus there is another vector field \( \xi_\pi \) along the fibres of \( G_{2,\pi} \) and clearly the derivative of the map \( \mu : G_{2,\pi} \times D \to G_2 \times D \) takes \( \xi_\pi \) to \( \xi \).

Now for any smooth function \( \beta \) on \( G_2 \times D \), the product \( \beta \xi \) defines a vector field and hence another almost-complex structure on \( G_2 \times D \). The composite of \( \beta \) with the map \( \mu \) is a smooth function \( \beta_\pi \) on \( G_{2,\pi} \times D \). So \( \beta \xi \) and \( \beta_\pi \xi_\pi \) define local almost-complex structures on \( X^*_r(f) \) and \( Y_\pi \), respectively, which are compatible, in the sense that the natural projection map is holomorphic.

With these remarks in place we can carry through the usual procedure to construct a compatible almost-complex structure on the whole of \( X^*_r(f) \): we take the trivial structures in charts given by the product structure and glue these together using a partition of unity. Then the key observation is merely that a smooth partition of unity on \( X^*_r(f) \) induces, by pullback, a smooth partition of unity on the smooth models of the strata. We can arrange that the complex structure agrees with the standard ones near the ends of \( S^2 \) by taking the original restricted charts on \( X^r \) to be holomorphic near the singular fibres.

We now need to discuss how to extend these strata over the singular fibres. From one perspective, the diagonal strata arise from the semigroup structure on \( \Pi_n \text{Sym}^n(\Sigma) \), for which there is no analogue for the union of the zero-fibres of the \( X_n(f) \). This union, as observed in the Appendix, is just \( \Pi_n \text{Hilb}^{[n]}(\Sigma_0) \), where \( \Sigma_0 \) is the fibre of \( f \) over the critical value. On the other hand, we do have a canonical projection map \( \text{Hilb}^{[n]}(Z) \to \text{Sym}^n(Z) \) for any space \( Z \) \cite{Nak99}, and using this we can pull back the strata defined by the semigroup action on the singular symmetric product spaces. In the usual way, the diagonal strata will be (singular) algebraic varieties in the total space of \( X_r(f) \) given a holomorphic projection from a smooth complex surface \( X \) to the disc \( D \) as in \( \text{(3.6)} \). The point of importance for us is the following

\( \text{(6.5) Lemma:} \) For every partition \( \pi \) of \( r \), the stratum \( \chi_\pi \) of the zero-fibre \( (X_r(f))_0 \) meets \( \text{Sym}^r(\Sigma_0 \setminus \{Q\}) \) in a Zariski dense set. In particular, the complement of this set has codimension at least two in every stratum.

This follows, for instance, from the discussion in Nakajima’s notes \( \text{[Nak99]} \) or the Appendix) of the Hilbert-Chow morphism, and the fact that the analogous statement for the strata in \( \text{Sym}^r \) does hold.
7 Constructing the symplectic surface

In this section we assemble the pieces already established to complete the proof. The delicacy will be that although we consider sections which have index zero in \( X_r(f) \), their indices as sections of the strata \( Y_{\pi, \alpha} \) (in which a priori they may lie) are not known. To get around this, we will establish a regularity result for almost complex structures compatible with the strata which will have the following consequence. Fix some \( J \) compatible with the strata and a holomorphic section \( \phi \) provided by the non-vanishing of the Gromov invariant. (Suppose there are no bubbles, a fact we will prove later in the section.) If the section lies inside some stratum \( \chi \) we pass to an associated smooth section of a smooth model \( Y \). If this section has negative index, our regularity result will allow us to perturb \( J \) and assume no such section in fact existed. Hence, we move \( \phi \) outside of the stratum \( \chi \). If, on the other hand, the index for the associated section inside \( Y \) is non-negative, our regularity result on \( Y \) will allow us to assume that \( \phi \) is transverse to all the smaller strata. An obvious finite induction (successively pushing off any strata of negative index), coupled with the fact that an intersection of dense sets is dense, will enable us to conclude there are "enough" almost complex structures \( J \) compatible with the strata; then, as usual, Sard's theorem provides such for which smooth holomorphic sections satisfy all the conditions of proposition (6.1).

Before turning to this programme, let us review more carefully the way we associate the sections of the smooth models. We begin with a discussion of pseudoholomorphic maps to stratified spaces. Let \( B \) be an open ball in \( \mathbb{C}^n \) and \( A \) be a complex analytic subvariety in \( B \). Thus \( A \) has a stratification by subsets which are locally-closed complex manifolds in \( B \). Suppose \( \mu \) is a smooth almost-complex structure on \( B \). We say that \( A \) is a \( \mu \)-subvariety if all the strata of \( A \) are complex submanifolds with respect to the almost-complex structure \( \mu \). Equip the unit complex disc \( D \) with its standard complex structure.

(7.1) Lemma: Let \( A \subset (B, \mu) \) be a \( \mu \)-subvariety in the above sense. If \( f : D \to B \) is pseudoholomorphic with respect to the structure \( \mu \) then the set \( f^{-1}(A) \) is either the whole of \( D \) or a discrete subset of \( D \).

Proof: By considering the stratification of a singular variety we can reduce to the case when \( A \) is a submanifold. By applying a suitable holomorphic diffeomorphism we can then suppose that \( A \) is a linear subspace in \( \mathbb{C}^n \). In the case when \( A \) is a complex line the result is proven by McDuff in ([AL94], Chapter 6). For the general case one finds that the proof of ([AL94]) goes over unchanged. ■

An obvious extension of the above gives

(7.2) Corollary: Suppose \( A_1, A_2 \) are \( \mu \)-subvarieties in \( B \) as above, neither of which is contained in the other. Then if \( f : D \to B \) is a pseudoholomorphic map whose image lies in \( A_1 \cup A_2 \) then either \( f \) maps into \( A_1 \cap A_2 \) or \( f \) maps into precisely one of the sets \( A_1 \) and \( A_2 \).

We can apply these results to our subsets \( \chi_{\pi, \alpha} \), since these are represented in local charts by complex varieties. We deduce that there is a unique minimal stratum \( \chi_{\pi, \alpha} \) associated
to a pseudoholomorphic section \( \phi \) of \( X_r(f) \). That is, \( \phi \) lies in \( \chi_{\pi, R} \) but not in any smaller stratum. Moreover \( \phi \) meets the smaller strata in discrete (hence finite) sets.

(7.3) Lemma: If \( \chi \) is the “minimal” stratum associated to a section \( \phi \) as above, then there is a unique holomorphic section \( \tilde{\phi} \) of \( Y_{\chi} \) which maps to \( \phi \) under the canonical map \( Y_{\chi} \to X_r(f) \).

The map \( Y_{\chi} \to X_r(f) \) is a homeomorphism on a dense open set, so \( \tilde{\phi} \) is uniquely defined as a continuous section on a dense set. Moreover, it is smooth and pseudoholomorphic away from the lower strata, i.e., where the derivative of the projection of \( Y_{\chi} \) is injective. In a local chart near one of the finitely many intersection points with the lower strata, \( \tilde{\phi} \) is bounded and hence extends to a smooth map on the entire sphere. Thus the assertion follows from the fact that a continuous map which satisfies a pseudoholomorphic mapping equation outside a discrete set is actually pseudoholomorphic everywhere. This follows from elliptic regularity as in [SU81], for example (since we know \( \tilde{\phi} \) is a priori continuous we are in the easy case).

In order to achieve the transversality of (6.1) we will establish a regularity result for almost-complex structures which are compatible with the strata. Let \( S \subseteq J \) denote the set of such almost-complex structures on \( X_r(f) \); this is an affine space in the familiar way. For each \( j \in S \) we have an almost-complex structure \( J_{\pi, R} \) on \( Y_{\pi, R} \) for which the projection \( Y_{\pi, R} \to X_r(f) \) is holomorphic. Let \( H_{\pi, R} \) be the set of homotopy classes of sections of \( Y_{\pi, R} \), so for each \( h \in H \) and \( j \in S \) we have a moduli space \( M_{h,j} \) of pseudoholomorphic sections of \( Y_{\pi, R} \) which do not all lie in the closure of any proper stratum in \( Y_{\pi, R} \). Let \( M'_{h,j} \) denote the subset of sections which are “good”, that is which are (i) transverse to all the lower strata in \( Y_{\pi, R} \) and (ii) lie in the dense open set \( p_{\pi, R}(\prod_i \text{Sym}^n(\Sigma_0 \setminus \{Q\})) \) over each critical value of \( f \). The vanishing of the index for our original problem does not allow us to conclude anything about the dimensions of these spaces \( M_{h,j} \). However:

(7.4) Proposition: For generic \( j \in S \) all pseudoholomorphic sections of \( (Y_{\pi, R}, J_{\pi, R}) \) are regular, so \( M_{h,j} \) is a manifold of the expected dimension. Moreover, for generic \( j \) and all \( h \), the set \( M'_{h,j} \) is dense in \( M_{h,j} \).

The corresponding assertion where \( j \) varies in the set of all almost-complex structures on \( Y_{\pi, R} \) (or even all those structures compatible with its fibration) is standard. The point here is that we are only allowed to consider the restricted set of almost-complex structures which arise from the compatible structures on \( X_r(f) \). As usual, the proof proceeds by constructing a smooth ”universal” moduli space; then the appropriate class of generic \( j \) is given by the regular values of a projection map to the space \( S \). We will ignore the Sobolev spaces, which are standard, and focus on the key geometric feature of the argument. Fix some reference structure \( j^{(0)} \in S \), inducing an almost-complex structure \( J_{\pi, R}^{(0)} \) on \( Y_{\pi, R} \). Recall that any other almost-complex structure on \( Y_{\pi, R} \) differs by a vertical vector field on \( Y_{\pi, R} \), as in the discussion of section 4. Let \( \phi \) be a \( J_{\pi, R}^{(0)} \)-holomorphic section of \( Y = Y_{\pi, R} \).

Let \( \overline{\partial} \) denote the operator \( (s, J) \mapsto \overline{\partial}_J(s) \) viewed as a map of Banach manifolds. For the universal moduli space of holomorphic curves to be smooth, we need to know that the cokernel of the linearised operator \( D(\overline{\partial}) \) is everywhere zero. If there is some non-zero element of such
a cokernel, then there is a non-zero element $\eta \in (\Lambda^{0,1}T^*\mathbb{S}^2) \otimes \phi^*TY$ of the kernel of the adjoint map. By the Hahn-Banach theorem, this is possible only if $\eta$ is orthogonal to the image of $D(\partial)$. Clearly it is enough to prove that in fact all such $\eta$ vanish on a dense set, and so we can assume for contradiction that $\eta$ is non-vanishing at $q \in \Delta_\phi$, where $\Delta_\phi$ is the open set in $\mathbb{S}^2$ obtained by removing the critical values of $f$ and the points $p$ where $\phi(p)$ lies in some smaller stratum of $Y$. Following the argument of [MS94, p.35] the crucial point is to construct a tangent vector $v \in \text{End}(TY, j^{(0)}_{\pi, r})$ to the space of almost complex structures for which $(\eta, v \circ d\phi \circ j^2)$ is non-vanishing. (The RHS of the inner product is one term in $D(\partial)$ evaluated at a tangent vector of the shape $(d\phi, v)$, and the non-vanishing contradicts the orthogonality to the image of $D(\partial)$.)
Hence it is certainly enough to show that all tangent vectors to $(T_qY, (j^{(0)}_{\pi, r})_q)$, viewed as a point of the space of almost complex structures on $Y$ at $q$ arising from structures compatible with the fibration, can be generated by perturbations inside the space $S$. This is the content of the following:

**Lemma (7.5)**: There is an open neighbourhood $\Delta \subset \mathbb{S}^2$ of $q$ such that for any compactly supported section $v$ of $\phi^*(T^1Y)$ over $\Delta$, there is a $j \in S$ such that $j_{\pi, r}$ differs from $j^{(0)}_{\pi, r}$ by a vertical vector field $\xi$ on $Y$ which pulls back to $v$ over $\Delta$.

**Proof**: We return to the charts for $X_r(f)$ and $Y$ obtained from restricted charts on $X'$. We have seen that a holomorphic diffeomorphism $\alpha : U \rightarrow V$ of open sets in $\mathbb{C}$ induces holomorphic diffeomorphisms from $g(U)$ to $\sigma(V)$ and from $\sigma(U)$ to $\sigma_*(V)$. In the same way a holomorphic vector field $\eta$ on $U$ induces holomorphic vector fields $\eta_\tau$ on $g(U)$ and $\eta_\tau$ on $\sigma(U)$. A smooth family $(\eta_\tau)$ of holomorphic vector fields on $U$, smooth in the base parameter $\tau$, then gives vertical vector fields over $g(U) \times D$ and $\sigma_*(U) \times D$ which we can use to deform the given almost-complex structure $j^{(0)}$. Suppose we have a point $x$ of $\sigma_*(U)$ which does not lie in any lower stratum. Then it is easy to see that for any tangent vector $w$ to $\sigma_*(U)$ at $x$ there is a holomorphic vector field $\eta$ on $U$ such that $\eta_\tau$ is equal to $w$ at $x$. When one unwinds the definitions this just amounts to finding a vector field on $U$ taking prescribed values at a finite set of points. More generally, suppose $g$ is a smooth map from the disc into $\sigma_*(U)$ with $g(0) = x$. Then one can find a neighbourhood $\Delta$ of $0 \in D$ and smoothly varying families of vector fields $(\eta_1, \ldots, \eta_\Delta)$ on $U$, depending on the parameter $\tau \in \Delta$, such that for each $\tau$ the corresponding vector fields $(g^\tau_1, \ldots, g^\tau_\Delta)$ on $\sigma_*(U)$ evaluated at $g(\tau)$ give a basis for the tangent space of $\sigma_*(U)$ at $g(\tau)$. This construction leads immediately to the proof of the Lemma, when we take a co-ordinate chart around the point $\phi(q)$ in $X_r(f)$, and extend using cut-off functions.

Both statements of (7.4) follow from the above. Using the affine structure on $S$ induced by vector fields, we can superimpose these local deformations. As a consequence, we have the statement:

**Lemma (7.6)**: Let $\phi$ be a $j^{(0)}_{\pi, r}$-holomorphic section of $Y = Y_{\pi, r}$. Then every section of $Y$ which co-incides with $\phi$ outside a compact subset of $\Delta_\phi$ is $j_{\pi, r}$-holomorphic for some $j \in S$.

The relevance of this result is the following. Fix some symplectic form $\Omega$ on $X_r(f)$ and
choose a compatible \( J \in \mathcal{S} \). If we are given a \( J \)-holomorphic section \( \phi \) then any sufficiently \( C^k \)-small perturbation \( \phi' \) of \( \phi \) (which co-incides with \( \phi \) outside of \( \Delta_\phi \)) will be holomorphic for some perturbed \( J' \in \mathcal{S} \) which still tames the fixed \( \Omega \). Using this, it follows that the evaluation map from the space of \( J \)-holomorphic sections to a fibre of \( X_r(f) \) is submersive; this is standard for regular \( J \), and important for defining the invariants coming from higher dimensional moduli spaces (some of these are computed in \([\text{Smil}]\)).

To complete the proof, we must provide a discussion of bubbling for the almost complex structures compatible with the strata. This shall hinge entirely on the assumption (2.3) that we originally fixed a Lefschetz pencil of (sufficiently) high degree \( k \). Fix a generic almost-complex structure \( J \in \mathcal{S} \) on \( X_r(f) \) which is compatible with the strata. From the non-triviality of the Gromov invariant, we know the moduli space \( \overline{\mathcal{M}}[\psi_V] \) of holomorphic cusp sections in the homology class \([\psi_V]\) is non-empty. Fix such a cusp section \( \bar{\phi} \), and denote by \( \phi \) its unique section component (which is smooth by the arguments of the previous section). This section \( \phi \) has an associated minimal stratum \( Y \), where it defines a point \( \phi \) in a moduli space \( \mathcal{M}_Y \) of pseudoholomorphic sections. Since \( J \) is generic, there is a dense set in \( \mathcal{M}_Y \) corresponding to sections which are “good” over the singular fibres and which meet all lower strata transversely. Suppose \((\phi, \bar{\phi})\) are good in this sense.

\[(7.7) \text{Lemma:} \quad \text{The section } \bar{\phi} \text{ cannot contain any bubbles, so } \bar{\phi} = \phi. \]

\text{Proof:} \quad \text{We have already established (1.8) that all bubble components are homologous to multiples of the standard projective line \( h \) inside the projective space } \mathbb{P}(V). \text{ It follows that if there are any bubbles, then the section component } \phi \text{ of the curve lies inside a moduli space of curves } \mathcal{M}[\psi_V - Nh] \text{ in the homology class } [\psi_V] - Nh \text{ for some positive integer } N.

The section \( \phi \) of \( X_r(f) \) defines a section \( \bar{\phi} \) of some associated minimal stratum \( Y \). We claim that in fact \( C_\phi \) must contain all of the exceptional sections of the Lefschetz fibration, so in particular \( Y \) is a smooth model of a stratum lying inside \( \cap_i x_{\alpha(E_i)} = 1 \). By symmetry, it is enough to prove this for some fixed exceptional curve \( E \). If \( C_\phi \) does not contain \( E \), then \( \bar{\phi} \) is transverse to \( Y \cap \cap_i x_{\alpha(E_i)} = 1 \), where the last term denotes the obvious real codimension two stratum of \( Y \). According to the discussion before (1.1), in this case \( C_\phi \) has locally positive intersections with the exceptional curve \( E \), and in particular \( C_\phi \cdot E \geq 0 \). But this is a contradiction: we know from (1.8) that \( C_\phi \) represents the homology class \( PD[K_X'] - N[\text{Fibre}] \) in \( H_2(X') \), and this has intersection number \( -(N + 1) \) with \( E \).

By combining (1.1) and (1.8), we see that the section component \( \bar{\phi} \) of the associated minimal stratum defines a smooth symplectic surface, and after we pass back to \( C_\phi \) and assign multiplicities we obtain a positive symplectic divisor on \( X' \) in the homology class \( PD[K_X'] - N[\text{Fibre}] \). Moreover this contains all of the exceptional components to multiplicity at least one. On the other hand, if it contains some exceptional curve \( E \) to multiplicity greater than one, then - running the same intersection argument as before and using (1.1) again - we see that the other components meet \( E \) transversely and positively. We can now apply our long-ignored smoothing lemma (2.12) to separate out the exceptional curves and obtain a smooth symplectic surface inside \( X' \) which (i) contains the exceptional curves to multiplicity precisely one and (ii) represents the homology class \( K_X' - N[\text{Fibre}] \). Throwing out the exceptional curves, we can push the resulting surface down to \( X \). This remains symplectic,
by (2.3) and the ensuing discussion; but now we have a contradiction, for the surface in $X$ represents the homology class $K_X - N[(k/2\pi)\omega]$. But by our initial choice of $k$ in (2.3) the evaluation of $\omega$ against this class is negative, and so the existence of such a symplectic surface is precluded.

This essentially completes the proof. The non-vanishing of the Gromov invariant for sections in the class $[\psi_V]$, together with the regularity results above and the absence of bubbles, means that we have a smooth holomorphic section $\phi$ for which the associated positive symplectic divisor $C_\phi$ is in the class $K_X'$. By the same argument as above, this divisor contains all of the exceptional sections; however, it may contain some exceptional curves with multiplicity greater than one. In this case we can apply (2.12) to decrease their total multiplicities. (From another perspective, the virtual dimension for holomorphic sections of $\mathcal{X}_2(f)$, in the homology class defined by the section $[\mu]$ of divisors of multiplicity two supported along the exceptional curve $E \subset X'$, is negative. Hence for generic $J$ we do not expect the situation to arise, although we do not need to prove that here.) An easy induction, pushing off multiple exceptional curves, yields a positive symplectic divisor $C_\phi$ which does satisfy the hypothesis of (2.11). Accordingly, we can smooth the components of $C_\phi \cup E_i$ in a small neighbourhood of their union. The resulting surface remains disjoint from the exceptional sections $E_i$, and hence we can identify the smooth symplectic submanifold that results with a symplectic submanifold inside $X$, by (2.4). This represents $K_X - \sum E_i = K_X$ in homology, and the proof is complete.

To end, it may be worth mentioning that the hypothesis that the pencil has high degree is indeed playing a definite role here in excluding bubbles.

(7.9) Example: There is a genus two Lefschetz pencil, with total space $X$, leading to a fibration $X'$ with mapping class group word $(\delta_1 \delta_2 \delta_3 \delta_4)^{10} = 1$ in standard generators. (This pencil is described in [Smi99] for instance.) The four-manifold underlying the pencil is a minimal complex surface of general type on the Noether line. This surface is simply connected and has $b_3 > 1$ and hence our basic index problem has solutions. For the symplectic form on $X$ dual to one of the genus two curves in the pencil, we have $\omega^2 = 1$, $K_X \cdot \omega = 1$. By the obstruction computation, we have a section $\phi$ of a bundle of second symmetric products over $S^2$; one of the two points on each fibre is moreover that defined by the unique exceptional section $E$. 

\[ K \cdot C_i + (C_i)^2 = 2g(C_i) - 2 = (1 + a_i)(C_i)^2 + \sum_{j \neq i} a_j (C_i \cdot C_j). \]
Now if the other component of the cycle $C_\phi$ defined by $\phi$ is a section of $f$ disjoint from $E$, then we have represented $K_X$ by a symplectic sphere, which violates the adjunction inequality. It follows that $[C_\phi] = [2E + F]$ for a fibre $[F] = [p^*K_X - E]$; indeed for the underlying minimal surface, $K_X = [\omega]$ can indeed be represented by a genus two curve, but the pencil of genus two curves does not extend to a web. The section of $X_2(f)$ is a cusp section, and the curve $C_{\text{bubble}} \subset X'$ pushes down to give a surface in $X$, in the class $K_X$, which passes through the basepoint of the pencil.

The point here is that $K_X = [\omega/2\pi]$ for this pencil of low degree $k = 1$. It follows that there is a holomorphic representative for $K_X - [\text{Fibre}]$ - the empty holomorphic curve - and this, stabilised by the exceptional section, is trapping the section component of our cusp curve. The bubble defines a symplectic surface in the four-manifold $X'$ which in fact projects to a smooth complex curve in the class $K_X$. □

8 Appendix

Our purpose here is to give a “users’ guide”, for non-specialists, to the theory of divisors and line bundles on a nodal curve. There is a large literature on these topics to which we defer for careful proofs. One construction of the compactified Jacobian of a nodal curve, and of the relative Hilbert scheme (which gives the smooth compactification of the family of symmetric products), involves geometric invariant theory. Standard (but inexhaustive) references include [OS79] for Jacobians of stable curves, following on from work of Mumford [Mum72]; the relative Hilbert scheme is treated carefully in [HL97]. A fine overview of Hilbert schemes on complex surfaces is Nakajima’s book [Nak99]. Finally, a technical survey covering all that we quote and more is Kleiman’s summary [Kle84]. To fit more closely with these references, in this Appendix we introduce $X_r(f)$ and $P_r(f)$ as pullbacks of the “universal” families over the moduli space of curves; the smooth structures one obtains this way are of course the same as those induced by the restricted charts of (3.5).

8.1 The projective bundle

Let $\pi : Z \to B$ be any holomorphic family of Riemann surfaces. Then there is a unique vector bundle $W \to B$ whose fibre over $b \in B$ is canonically identified with the space of holomorphic sections $H^0(K_{\pi^{-1}(b)})$. For we can define a line bundle - the dualising sheaf - on $Z$ by

$$\omega_{Z/B} = K_Z \otimes (\pi^*K_B)^{-1}$$

and then set $W = \pi_*\omega_{Z/B}$. The bundle $W \to B$ is called the relative dualising sheaf. If we apply this construction to the universal curve $\pi_g : C_g \to M_g$ then the bundle $(\pi_g)_*\omega$ extends over the stable compactification $\overline{M}_g$ [HM98].

5The notation $\omega$ for dualising sheaves is as established as the identical notation for symplectic forms; we hope no confusion will arise.
(8.1) Definition: Let $f : X' \to \mathbb{S}^2$ be a Lefschetz fibration inducing a map $\phi_f : \mathbb{S}^2 \to \overline{\mathcal{M}}_g$. Let $V = \phi_f^\ast((\pi_g)_\ast\omega) \otimes \mathcal{O}(-2)$ be the bundle of fibrewise canonical forms associated to $f$. This is a rank $g$ complex vector bundle over $\mathbb{S}^2$.

The vector bundle $V \to \mathbb{S}^2$ has the following property. A section of $V$ defines a cycle in $X'$ as follows: for each $b \in \mathbb{S}^2$ we have a one-form on $f^{-1}(b)$ defined up to scale; the zeroes of this one-form are well-defined and give a collection of $2g - 2$ points, counted to multiplicity, in $\pi^{-1}(b)$. If the section of $V$ has no zeroes we obtain a cycle in the class $\text{PD}[K_{X'}]$. We make a few remarks about this construction at the critical values of $f$. If $F \subset Z$ is a smooth complex curve in a complex surface $Z$ (not necessarily compact) then there is an adjunction formula for $K_F$:

$$K_F = K_Z|_F \otimes \nu_{F/Z}$$

where the last term denotes the normal bundle. If $F$ is now a nodal complex curve, it still defines an effective divisor and hence line bundle $\mathcal{O}_Z(F)$ on $Z$, and we may formally define a normal bundle $\nu_{F/Z} = \mathcal{O}_Z(F)|_F$ and hence a canonical bundle by (8.2). The resulting locally free sheaf is independent of the choice of $Z$ and of embedding $F \subset Z$, as in the smooth case.

This defines an extension of the dualising sheaf over $\overline{\mathcal{M}}_g$. To give a geometric picture of the elements of the fibre of $V$ over critical values of $f$ we can look to the normalisation of the nodal fibre $\tilde{\Sigma}_0 \to \Sigma_0$. Recall that this is a naturally associated Riemann surface in which the two sheets which meet at the node are separated. If the node does not separate $\Sigma_0$ then the normalisation has genus smaller than that of a smooth deformation (connect sum) at the node, and if the node separates then the normalisation is a disconnected surface. There are two distinguished points $\alpha, \beta \in \tilde{\Sigma}_0$ given by the preimages of the node.

(8.3) Proposition: The elements of $H^0(\Sigma_0, \omega_{\Sigma_0})$ can be identified with the meromorphic sections of $K_{\tilde{\Sigma}_0}$ over $\tilde{\Sigma}_0$ which are smooth away from $\alpha, \beta$ and have at worst simple poles, with opposite residues, at each of $\alpha, \beta$.

This is clear from the explicit “residue map” $[BPV84]$; for a local section of $K_Z \otimes \mathcal{O}_Z(\Sigma_0)$, say $h(du \wedge dv)/f$ with $u,v$ local complex co-ordinates on $Z$, $h$ a local section of $K_Z$ and $f$ a local defining equation for $\Sigma_0$, we write

$$(\text{res}) \left( \frac{h \, du \wedge dv}{f} \right) = (\text{norm})^\ast \left( \frac{h \, dv}{\partial f/\partial u} \right)$$

where the partial derivative is chosen not to vanish on any open set in $\Sigma_0$. Then (res) identifies sections of the canonical sheaf of $\Sigma_0$ with meromorphic forms on the normalisation, and the poles arise from the zeroes of the derivative $\partial f/\partial u$; these are at worst simple when $f$ is quadratic, as near a node. (Indeed the condition on the residues being opposite is forced in this case, since the sum of residues must be trivial by Cauchy’s theorem.) For a smooth Riemann surface, the canonical bundle has degree $2g - 2$ and any holomorphic section defines a distinguished set of $2g - 2$ points (to multiplicity) via its zeroes. The same is true for a nodal Riemann surface; a section $s \in H^0(\Sigma_0, \omega_{\Sigma_0})$, viewed as a meromorphic section on the normalisation, defines the points.
1. which are zeroes of the meromorphic section on \( \tilde{\Sigma}_0 \) if it has poles at the points \( \alpha, \beta \);

2. which are zeroes of the meromorphic section and the nodal point of \( \Sigma_0 \) if the section is actually holomorphic upstairs.

This is again clear from the residue; if the residue form is smooth at the node, then the canonical section \( h \) must vanish there. It follows that in the case of a separating node, all the elements of \( H^0(K_{\tilde{\Sigma}_0}) \) give rise to tuples of points including the node; in algebraic geometry, the nodal points in reducible curves form base-points for the canonical linear system. This explains why some arguments are simpler in the absence of reducible fibres.

### 8.2 The Picard fibration

There is a fibre bundle over \( M_g \) with fibre the Picard torus of degree \( r \) line bundles on the associated Riemann surface. (The Picard variety of a complex curve identifies, after choosing an origin, with the Jacobian of the curve.) This also extends to the stable compactification, but no longer as an orbifold. For curves with a single node the situation is simpler than for arbitrary stable curves. The precise notion in algebraic geometry is the relative moduli scheme for rank one torsion free coherent sheaves with fixed Euler characteristic. This is reduced for stable curves ([DO79], Cor. 13.3); when there is a unique node, the compactified Jacobian can be described explicitly (below). The papers of Igusa [Igu56] gave an early construction of compactified Jacobians using linear systems, taking closures under suitable projective embeddings. Mumford [Mum72] showed that the resulting spaces are indeed (components of) parameter spaces for torsion free sheaves.

**8.4 Definition:** Given a Lefschetz fibration \( f : X' \to \mathbb{S}^2 \) write \( P_r(f) \) for the pullback by \( \phi_f : \mathbb{S}^2 \to \overline{M}_g \) of the relative moduli scheme of rank one torsion free sheaves of fixed Euler characteristic \( r - g + 1 \).

Here is a description of this object near the singular fibres of \( f \). A line bundle of degree \( r \) on a nodal curve \( \Sigma_0 \) is given by a line bundle of degree \( r \) on the normalisation \( \tilde{\Sigma}_0 \) together with a choice of identification of the complex lines \( L_\alpha \) and \( L_\beta \) over the preimages of the node \( Q \in \Sigma_0 \). The bundle upstairs is given by pullback: writing \( (\text{norm}) : \tilde{\Sigma}_0 \to \Sigma_0 \) then take

\[
\tilde{L} = (\text{norm})^* L \otimes \mathcal{O}(\tilde{\Sigma}_0) \mathcal{O}(\Sigma_0)/(\text{Torsion});
\]

dividing by torsion ensures the final sheaf is locally free. This yields a non-compact space, which is a \( \mathbb{C}^* \) fibre bundle over the Picard torus of the normalisation. The natural compactification of this to a \( \mathbb{P}^1 \)-bundle, allowing the degenerate gluing maps of the two complex lines by the 0 and \( \infty \) multiplications, is no longer a moduli space for a natural class of objects. However, if we parametrise rank one torsion free sheaves on \( \Sigma_0 \), then we find that a quotient of this \( \mathbb{P}^1 \) bundle is the required moduli space. Precisely, we glue together the 0-section and \( \infty \)-section of the \( \mathbb{P}^1 \) bundle over an automorphism of the base torus which is a translation

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\( ^6 \)Recall that any torsion free sheaf on a smooth curve is locally free, so the fibres do reproduce the Picard varieties over the smooth locus \( M_g \).
by \( \mathcal{O}(\alpha - \beta) \) in the group action of degree zero line bundles on the Picard. This is derived in ([OS79], Example p.83) and also ([Igu56], Supplement p.187).

One can see the degeneration of the Picard / Jacobian fibration explicitly in terms of periods. Suppose we have a smooth complex surface \( f : \mathcal{X} \to D \) with a nodal fibre \( \Sigma_0 \) over 0, with normalisation \( \tilde{\Sigma}_0 \) as usual. Fix a basis of loops for \( H_1(\Sigma_t) \) in the obvious way: \( \gamma_3, \ldots, \gamma_{2g} \) correspond to a standard basis on \( \tilde{\Sigma}_0 \), \( \gamma_1 \) is the vanishing cycle, \( \gamma_2 \) a loop with \( \gamma_2 \cdot \gamma_1 = 1 \).

Note that because of the topological monodromy we can only define \( \gamma_2 \) consistently on a covering or cut plane. Now let \( \omega_\alpha \) be a basis for the “1-forms” on the singular fibre, which we can obtain by the residue map from holomorphic 1-forms on \( \mathcal{X} \). We can suppose that the \( \omega_\alpha \), for \( \alpha > 1 \), are holomorphic on the normalisation while \( \omega_1 \) has residue 1. Moreover, by the discussion of the residue map above, we can regard \( \omega_\alpha \) as being defined on all the curves \( \Sigma_t \). We consider the periods

\[
\int_{\gamma_i} \omega_\alpha.
\]

It is not hard to see that these are all holomorphic functions of \( t \) except that

\[
\int_{\gamma_2} \omega_1 = \log t \int_{\gamma_1} \omega_1 + \text{holomorphic}.
\]

Moreover at \( t = 0 \) the integrals \( \int_{\gamma_i} \omega_\alpha \) for \( i \geq 3 \) and \( \alpha \geq 2 \) give the periods of \( \tilde{\Sigma}_0 \), while \( \int_{\gamma_1} \omega_1 = 2\pi \) and \( \int_{\gamma_1} \omega_\alpha = 0 \) for \( \alpha \geq 2 \).

This gives an explicit description of the total space of \( P_r(f) \) (induced from \( \mathcal{X} \to D \)) minus the normal crossing divisor, as a quotient of \( \mathbb{C} \times D \). Then one can construct the compactification of this explicitly in local co-ordinates. The basic model is to consider the quotient of \( \mathbb{C} \times D \) by the equivalence relation

\[
(z, t) \sim (z + 2\pi n_1 + n_2 i \log t, t),
\]

if \( t \neq 0 \) and

\[
(z, 0) \sim (z + 2\pi n_1, 0).
\]

Consider the regions \( 0 < \Im(z) < -\Re(\log t) \) and \( \Re(\log t) < \Im(z) < 0 \), where \( \Re \) and \( \Im \) denote real and imaginary parts respectively. We map the first region to \( \mathbb{C}^2 \) by \( (z, t) \mapsto (e^{iz}, te^{-iz}) \) and the second by \( (z, t) \mapsto (te^{iz}, e^{-iz}) \). These induce a map from a neighbourhood of the end of the quotient to \( \mathbb{C}^2 \setminus \{0\} \), and we compactify by adding 0.

(8.5) Proposition: For a Lefschetz fibration \( f : X' \to \mathbb{S}^2 \) with irreducible fibres, the Picard fibration \( P_r(f) \) has smooth symplectic total space. The critical points of the natural map \( P_r(f) \to \mathbb{S}^2 \) are precisely the normal crossings divisors described above in the fibres \( \text{Pic}_r(\Sigma_{f(p_i)}) \), for \( \{f(p_1), \ldots, f(p_n)\} \) the critical values of \( f \).

The global smoothness will follow from the discussion for symmetric products below, or can be proven directly in an analogous fashion. For irreducible curves with one node, the
natural tensor product action of degree zero line bundles on the compactified Picard has a single orbit. Under the map given by $\otimes^\omega_{\Sigma_0}$ from Pic$_0$ to Pic$_{2g-2}$ the point corresponding to $(\mathcal{O}(\tilde{\Sigma}_0), 1 \in \mathbb{C}^*)$ is mapped to the canonical sheaf of the nodal curve. The locus of critical values for the natural projection to $\mathbb{S}^2$ corresponds to torsion free non-locally free sheaves. The canonical sheaf of a nodal curve is locally free; hence the natural section of $P_r(f) \to \mathbb{S}^2$ defined by taking a point $t \in \mathbb{S}^2$ to the canonical sheaf $K_{f^{-1}(t)}$ is well-defined and smooth.

8.3 The relative Hilbert scheme

The symmetric product of a smooth curve is equal to the Hilbert scheme, parametrising fixed length quotients of the structure sheaf. For background on Hilbert schemes and Quot schemes see [HL97], where relative Hilbert schemes are shown to exist as projective schemes in great generality. (From the algebraic perspective, after fixing appropriate discrete data, symmetric products are moduli spaces of structure sheaves whilst Hilbert schemes are moduli spaces of ideal sheaves.) For a family of curves with isolated nodal members, this gives a compactification of the fibre bundle of symmetric products. The important points for us will be that the total space is smooth and there is still a well-defined and global Abel-Jacobi map. A careful treatment of the latter assertion can be found in Altman and Kleiman ([AK80, Section 8]), and we shall provide a proof of the former (8.8). Some additional information on these spaces is given in [Sm].

(8.6) Definition: Given a Lefschetz fibration $f : X' \to \mathbb{S}^2$ with irreducible fibres, write $F : X_r(f) \to \mathbb{S}^2$ for the total space of the relative Hilbert scheme. This is the $\phi_f$-pullback of the scheme $\text{Hilb}^r(\mathcal{C}_g/M_g)$ which parameterises length $r$ subschemes of the fibres of the universal curve.

Given $f : X' \to \mathbb{S}^2$ smooth over a locus $\mathbb{S}^2 \ast$, construct $P_r^*(f)$ over $\mathbb{S}^2 \ast$. By the previous remarks this extends naturally to the entire sphere. As for the smooth fibres, we can identify a family of projective spaces over the Picard torus of the nodal curve. Let $P \in \text{Pic}_r(\Sigma_0)$ be a point of the smooth locus, arising from a line bundle $L_P \to \Sigma_0$ on the normalisation together with a gluing parameter $\lambda \in \mathbb{C}^*$ to identify the $\alpha$ and $\beta$ fibres of $L_P$. Suppose for definiteness that the normalisation is connected of genus $g - 1$. Then $L_P$, having degree $2g - 2$, generically has a space of holomorphic sections of dimension

$$(2g - 2) - (g - 1) + 1 = g.$$  

We take the $\lambda$-hyperplane in the space of all holomorphic sections; that is, restrict to sections whose values at $\alpha$ and $\beta$ are transformed by the gluing $\lambda : (L_P)_\alpha \to (L_P)_\beta$. Hence for a generic point $P$ and line $L_P$ the projective space of sections is a copy of $\mathbb{P}^{g-2}$. A similar analysis applies along the normal crossing divisor. This changes, however, at a unique point in the smooth locus of $\text{Pic}_r(\Sigma_0)$, corresponding to the canonical line of the normalisation. Here we see a projective space of dimension $g - 1$ as the space of sections. Note this description of the space of sections $H^0(\omega_{\Sigma_0})$ differs from that given above in terms of meromorphic differentials: $(\text{norm})^*\omega_{\Sigma_0}/\langle\text{Tors}\rangle$ is not the canonical sheaf of the normalisation, it has the wrong degree.
Following work of Nakajima, we can give a more down-to-earth description of the relative Hilbert scheme for the local model $\pi : \mathbb{C}^2 \to \mathbb{C}$ taking $(z, w) \mapsto zw$. For $\text{Hilb}^r[\mathbb{C}^2]$, the fibre over $t$ is the set of ideal sheaves in the local ring $\mathbb{C}[z, w]/(zw - t)$ whose quotient is of length $r$. Equivalently we want the ideals in $\mathbb{C}[z, w]$ which contain $(zw - t)$ and which have quotient length $r$. The Hilbert scheme of the complex plane has an elementary description $[\text{Nak99}]$:

$$\text{Hilb}^r[\mathbb{C}^2] = \{(B, B', v) \in M_r(\mathbb{C}) \times M_r(\mathbb{C}) \times \mathbb{C}^r \mid [B, B'] = 0, \ (\ast) \}/GL_r(\mathbb{C})$$

where $(\ast)$ is a stability condition for a geometric invariant theory construction: no subspace $S \subset \mathbb{C}^r$ invariant under each of $B$ and $B'$ can contain the vector $v$. The gauge group $GL_r(\mathbb{C})$ acts by simultaneous conjugation on $B$ and $B'$ and by left multiplication on $v$. Given an ideal $I$, the commuting matrices $B$ and $B'$ represent multiplication by $z$ and $w$ respectively on the $r$-dimensional vector space $\mathbb{C}[z, w]/I$, and $v$ arises as the image of $1 \in \mathbb{C}[z, w]$. Conversely, given a triple as above, we define a map $\phi : \mathbb{C}[z, w] \to \mathbb{C}^r$ by $f \mapsto f(B, B')v$; the stability condition ensures that $\phi$ is surjective and then the kernel defines an ideal $I_\phi$. This gives a set-theoretic description of the Hilbert scheme for length $r$ quotients, and $[\text{Nak99}]$ shows that this is a holomorphic isomorphism. Again by stability, the ideal associated to $(B, B', v)$ is just $\{f \in \mathbb{C}[z, w] \mid f(B, B') = 0\}$. This contains $zw - t$, for given $t \in \mathbb{C}$, just when $BB' = tI_r$. In particular, this gives a fairly explicit description of the singular fibre $\text{Hilb}^r[\{(zw = 0)\}]:$

$$\text{Hilb}^r(\pi^{-1}(0)) = \{(B, B', v) \mid [B, B'] = 0, \ BB' = 0, \ (\ast)\}/GL_r(\mathbb{C})$$

From this perspective, a general point of the symmetric product of this fibre $\pi^{-1}(0)$ can be given by a pair of diagonal matrices $\text{diag}(\lambda_1, \ldots, \lambda_r)$ and $\text{diag}(\mu_1, \ldots, \mu_r)$ with $\lambda_i \mu_i = 0$ for each $i$; a general point of the Hilbert scheme can be given by a suitable pair of upper triangular matrices $(B, B')$ whose diagonal entries satisfy the same condition. The Hilbert-Chow morphism from $\text{Hilb}^r[\mathbb{C}^2] \to \text{Sym}^r$ takes $(B, B')$ to the set of pairs $(\lambda_i, \mu_i)$ viewed as points of $\mathbb{C}^2$ lying on $\pi^{-1}(0)$. Where the eigenvalues of $B$ and $B'$ all remain distinct, this map is an isomorphism; one can use this explicit form of the map to prove $[\text{B2}]$. The Hilbert scheme is stratified by the number of supports $(\lambda, \mu)$ lying at the node of $\{zw = 0\}$, and as points collapse into the node configurations of off-diagonal matrix entries become (projective) co-ordinates over the stratum.

**Example:** Suppose $r = 2$. If either $B$ or $B'$ has distinct eigenvalues then we can simultaneously diagonalise both with $B = \text{diag}(\lambda_1, \lambda_2)$ and $B' = \text{diag}(\mu, \mu')$ and with $(\lambda_1, \mu_1) \neq (\lambda_2, \mu_2)$. This point of $\text{Hilb}^2[\mathbb{C}^2]$ belongs to the relative Hilbert scheme of $\pi$ precisely when $\lambda_1 \mu_1 = \lambda_2 \mu_2$. If, however, $B$ and $B'$ have only one eigenvalue each, then they cannot be simultaneously diagonalised by the stability condition $(\ast)$ (otherwise take $S = \langle v \rangle$). Hence we can write

$$B = \begin{pmatrix} \lambda & \alpha \\ 0 & \lambda \end{pmatrix}; \quad B' = \begin{pmatrix} \mu & \beta \\ 0 & \mu \end{pmatrix}$$

for $(\alpha, \beta) \in \mathbb{C}^2 \backslash \{0\}$. The associated ideal is generated by

$$\langle \beta(z - \lambda) - \alpha(w - \mu), (z - \lambda)^2, (w - \mu)^2 \rangle.$$
This subscheme is supported on the fibre $\pi^{-1}(0)$ when $\lambda = 0$ or $\mu = 0$. However, if $\lambda = 0$ and $\mu \neq 0$ then the ideal does not contain $\langle zw \rangle$, and the corresponding quotient is not a point of the relative Hilbert scheme. (Thus the relative Hilbert scheme is not just the preimage of the relative symmetric product under the Hilbert-Chow morphism.) We will interpret the parameters $\alpha$ and $\beta$ geometrically below. □

From this description, one can prove the smoothness of all the relative Hilbert schemes $X_r(f)$, which in addition implies the smoothness of the fibre product spaces $Y_{w, R}$. The details of this are carried out in [Smi], but do not illuminate how the strata of the relative Hilbert scheme fit together. Instead, here is a more direct and geometric argument for smoothness of $X_g$: which can be adapted to the general case - based upon our “normal crossing” picture:

\[ (8.8) \text{Proposition: Let } f : \mathcal{X} \to D \text{ be a smooth complex surface which fibres over the disc with an irreducible nodal fibre } \Sigma_0 \text{ over } 0, \text{ as usual. The total space of the relative Hilbert scheme } X_r(f) \to D \text{ is smooth.} \]

\[ \text{Proof: From the discussion above it follows that the total space has normal crossing singularities in the fibre over } 0 \in D \text{ whilst all the other fibres are given by the (smooth) symmetric products of the fibres of } f. \text{ The general theory of deformations of spaces with normal crossings was carefully described by Friedman in [Fri83]. Be given a flat proper map } Y \to D \text{ over the disc, with central fibre } Y_0 = Y \text{ having normal crossing singularities. There is a Kodaira-Spencer map } \theta : (TD)_0 \to \text{Ext}^1_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y) \text{ to the global Ext-group, which is formally the tangent space to the deformation space of } Y. \text{ The class } \theta(\partial/\partial t) \text{ defines an extension which is just the conormal sequence} \]

\[ 0 \to \mathcal{I}_Y/\mathcal{I}_Y^2 \to (\Omega^1_Y)|_Y \to \Omega^1_Y \to 0. \]

Since $Y$ appears as a fibre in a flat family, the first term above is $\mathcal{O}_Y$. Now Friedman’s theorem asserts that $Y$ is smooth at a point $y \in Y$ precisely where $\theta(\partial/\partial t)$ generates the local group $\text{Ext}^1_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y)$. (Note that the conormal sequence also defines an element of $H^0(\text{Ext})$.) In our situation, this local Ext-group is exactly $\mathcal{O}_R$, where $R$ denotes the singular locus of $Y$, and where the sheaf $\mathcal{O}_R$ denotes the normal bundle of $Y$ in $Y$ restricted to $R$.

More concretely, the germ of the family $\mathcal{X}_r(f) \to D$ over 0 defines a germ of a map $D \to \text{Def}(\mathcal{X}_0) = \text{Ext}$ into the deformation space. The derivative of this deformation at 0 defines a section $\epsilon$ over $\mathcal{X}_0$ of the line bundle $N_1^* \otimes N_2^*$, where the $N_i$ are the normal bundles to the two branches of the normal crossing divisor. Friedman’s theorem [Fri83] amounts to saying that it suffices to check that the section $\epsilon$ is nowhere zero.

In our case, the normal bundles $N_1$ and $N_2$ can be identified canonically with the tangent lines $(T\Sigma_0)|_a$ and $(T\Sigma_0)|_b$ to the normalisation of the fibre of $f$ at the marked points. (A similar statement for curves is presented, in down-to-earth fashion, in [HM98].) It follows that $\epsilon$ gives a constant section of a trivial line bundle, and we need to see that we obtain a
non-zero element of the line. Conversely, if the section is trivial then the total space will be singular along the entire normal crossing divisor, so it suffices to check smoothness at a single point. Choose \(2g - 3\) local sections of \(X \to D\); using the semigroup structure on \(\oplus_d \text{Sym}^d\) we obtain a map \(X_1(f) \to X_{2g-2}(f)\) which adds twice the first co-ordinate to the given sections. However, \(X_1(f) = X\) is exactly the complex surface we start with. It follows that we have a factorisation

\[
X \xrightarrow{\phi} X_{2g-2}(f) \to D.
\]  

(8.9)

Let \(x, y\) be co-ordinates near the node in \(X\) so that locally the map \(f\) is given by \((x, y) \mapsto xy\); let \(z_1, \ldots, z_n\) be co-ordinates near a point of the normal crossing divisor in \(X_r(f)\), so the projection and section are given by

\[
(z_1, \ldots, z_n) \mapsto z_1 z_2; \quad z_1 z_2 = \epsilon(z_3, \ldots, z_n)
\]

respectively. Then (8.8) implies that locally we can write the product

\[
z_1(x, y) z_2(x, y) = F(xy)
\]

where \(\frac{\partial z_1}{\partial x}\) and \(\frac{\partial z_2}{\partial y}\) are both non-zero. It follows that \(F'(0) \neq 0\), and hence that \(X_r(f)\) is smooth at points in the normal crossing divisor in the image of \(\phi\), and hence smooth globally. 

The existence of a relatively ample holomorphic line bundle on the total space of \(X_r(f)\), extending that induced over the punctured disc by a given collection of sections, follows from ([HL97], 2.2.5). Roughly, the line bundle we want is given by a twist of the determinant line bundle of an appropriate “universal” line. The geometric invariant theory construction of Hilbert schemes proceeds by embedding the required set of quotients of a sheaf \(H\) over a space \(X\) into a Grassmannian of subspaces of \(H^0(H \otimes L_X^m)\), where we have twisted \(H\) by some ample sheaf \(L_X\) on the space \(X\). The determinant of the universal line is ample since it is the pullback of the \(O(1)\) bundle on some \(P^n\) under the Plücker (determinant) embedding of the Grassmannian. In the relative case, we twist by a bundle which is ample on the fibres of the given map of schemes \(X \to B\), which certainly holds for the line bundle on a Lefschetz fibration defined by the exceptional sections. This completes our treatment of the material underlying Theorem (3.6).

(8.10) Example: For a smooth genus two curve, the second symmetric product maps to the Picard by blowing down a single \(\mathbb{CP}^1\) which is the fibre over the canonical line bundle. Given a genus two curve \(\Sigma_0\) with a single non-separating node, the second symmetric product is singular along a copy of \(\Sigma_0\) parametrising all pairs \((p, \text{node})\) with \(p \in \Sigma_0\). The Hilbert scheme for two points on \(\Sigma_0\) is given by blowing up the second symmetric product at the point \((\text{node}, \text{node})\) which creates an exceptional divisor \(E \cong \mathbb{CP}^1\). (In the description above by ideals, the point \([\alpha : \beta]\) defines a co-ordinate on this copy of \(\mathbb{P}^1\).) The singular locus of \(\text{Hilb}^2(\Sigma_0)\) is a copy of the normalisation \(\tilde{\Sigma}_0\); the generic point on the exceptional divisor \(E\) is smooth. The two singular points on \(E\) correspond to subschemes with the two points lying at the node, with an infinitesimal deformation (showing the direction in which they collided)
also being tangent to the node. Locally, the blow-up serves to give a crepant resolution of the
(globally singular) fibrewise second symmetric product of $\pi: \mathbb{C}^2 \to \mathbb{C}$ taking $(z, w) \mapsto zw$. □

We end the Appendix with a proof of the technical result (4.8) asserting that bubbles in the
singular fibres of $F$ realise no more homology classes than bubbles in the smooth fibres.

(8.11) Lemma: Let $\theta: \mathbb{S}^2 \to \operatorname{Hilb}^{|r|}(\Sigma_0)$ be a non-constant holomorphic map from the
two-sphere to the singular fibre of $F: X_r(f) \to D$ over $0 \in D$ which is a bubble component of
some cusp curve. Then $[\operatorname{im}(\theta)] \equiv Nh$: the homology class of the image is equal to a multiple
of the class $h$ defined by a projective line inside $\mathbb{P}(H^0(K_{\Sigma_t})) \subset F^{-1}(t)$.

Proof: Compose $\theta$ with the map $\tau$ from $X_r(f)$ to $P_r(f)$; we get a map into $\operatorname{Pic}_r(\Sigma_0)$. This
lifts to the normalisation of $\operatorname{Pic}$ which we have seen fibres over $\operatorname{Pic}(\tilde{\Sigma}_0)$. Hence the composite
$\mathbb{S}^2 \to \operatorname{Pic}(\Sigma_0)$ is constant. The only possibilities are that either the map $u = \tau \circ \theta: \mathbb{S}^2 \to \operatorname{Pic}_r(\Sigma_0)$ is constant or it is maps non-trivially onto one of the $\mathbb{S}^2$ fibres of $\operatorname{Pic}_r(\Sigma_0) \to \operatorname{Pic}(\Sigma_0)$.
In the first case the map $u$ is clearly homotopic to the standard sphere so we have to rule out
the second case. In this case $\theta$ must meet the singular locus of $\operatorname{Hilb}^{|r|}(\Sigma_0)$ at least twice. We
claim that this means that $\theta$ cannot arise as the bubble component of a sequence of sections
$\phi_i$ of the fibration $X_r(f)$. If it did there would be pairs of disjoint discs $A_i, B_i \subset D$ and
disks $A, B \subset \mathbb{S}^2$ centred on points $a, b$ such that $\theta(a), \theta(b) \in \{\text{Singular locus}\}$, and such that
after rescaling $\phi_i|_{A_i}$ converges in $C^\infty$ to $\theta|_A$ (respectively for the $B$'s). Now consider the
situation in local co-ordinates around $\theta(a)$. From our identification of the singularities as
normal crossing divisors in the fibre, the projection map is given by

$$(z_1, z_2, \ldots, z_n) \mapsto z_1 z_2.$$

The map $\theta$ must map locally into one of the branches $z_1 = 0, z_2 = 0$ of the singular fibre, say
into the first branch. This means that $\phi_i$ has intersection number $\geq 1$ with the hyperplane
$z_1 = 0$ so there is a point $\alpha_i$ in $A_i$ such that $\phi_i(\alpha_i)$ lies in the fibre of $X_r(f)$ over 0. Similarly
there is another point in $B_i$ with the same property. Hence $\phi_i$ meets this fibre at least twice,
and cannot be a section, which gives the contradiction we require. □

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