A NOTE ON ALMOST RIEHMANN SOLITON AND GRADIENT ALMOST RIEHMANN SOLITON

KRISHNENDU DE AND UDAY CHAND DE

Abstract. The quest of the offering article is to investigate almost Riemann soliton and gradient almost Riemann soliton in a non-cosymplectic normal almost contact metric manifold $M^3$. Before all else, it is proved that if the metric of $M^3$ is Riemann soliton with divergence-free potential vector field $Z$, then the manifold is quasi-Sasakian and is of constant sectional curvature $-\lambda$, provided $\alpha, \beta = \text{constant}$. Other than this, it is shown that if the metric of $M^3$ is ARS and $Z$ is pointwise collinear with $\xi$ and has constant divergence, then $Z$ is a constant multiple of $\xi$ and the ARS reduces to a Riemann soliton, provided $\alpha, \beta = \text{constant}$. Additionally, it is established that if $M^3$ with $\alpha, \beta = \text{constant}$ admits a gradient ARS $(\gamma, \xi, \lambda)$, then the manifold is either quasi-Sasakian or is of constant sectional curvature $-(\alpha^2 - \beta^2)$. At long last, we develop an example of $M^3$ conceding a Riemann soliton.

1. Introduction

Since Einstein manifolds play out a huge job in Mathematics and material science, the examination of Einstein manifolds and their speculations is an intriguing point in Riemannian and contact geometry. Lately, various generalizations of Einstein manifolds such as Ricci soliton, gradient Einstein soliton, gradient Ricci soliton, gradient $m$-quasi Einstein soliton etc. have researched. The notion of Ricci flow was introduced by Hamilton [12] and defined by $\frac{\partial}{\partial t} g(t) = -2S(t)$, where $S$ denotes the Ricci tensor.

As a spontaneous generalization, the idea of Riemann flow ([19], [20]) is defined by $\frac{\partial}{\partial t} G(t) = -2Rg(t), \ G = \frac{1}{2}g \otimes g$, where $R$ is the Riemann curvature tensor and $\otimes$

\hspace{1cm} 0AMS 2010 Mathematics Subject Classification : 53C15, 53C25 53D15.

Key words and phrases: 3-dimensional normal almost contact metric manifold, Almost Riemann soliton, Gradient almost Riemann soliton.
is Kulkarni-Nomizu product (executed as (see Besse [1], p. 47),

\[(P \otimes Q)(E, F, Z, W) = P(E, W)Q(F, U) + P(F, U)Q(E, W)
- P(E, U)Q(F, W) - P(F, W)Q(E, U))\].

Analogous to Ricci soliton, the entrancing thought of Riemann soliton was promoted by Hirica and Udriste [13]. As per Hirica and Udriste [13], a Riemannian metric \(g\) on a Riemannian manifold \(M\) is called a Riemann solitons if there exists a \(C^\infty\) vector field \(Z\) and a real scalar \(\lambda\) such that

\[(1.1) \quad 2R + \lambda g \otimes g + g \otimes \mathcal{L}_Z g = 0.\]

Here we should see that, this new thought of Riemann soliton is nothing but a generalization of the space of constant sectional curvature. The soliton will be termed as expanding (if \(\lambda > 0\)), steady (if \(\lambda = 0\)) or shrinking (if \(\lambda < 0\)), respectively. The manifold is said to be gradient Riemann soliton if the vector field \(Z\) is gradient of the potential function \(\gamma\). For this situation the forerunner condition can be composed as

\[(1.2) \quad 2R + \lambda g \otimes g + g \otimes \nabla^2 \gamma = 0,\]

where \(\nabla^2 \gamma\) denotes the Hessian of \(\gamma\). In the event that we fix the condition on the parameter \(\lambda\) to be a variable function, then the equation (1.1) and (1.2) turns in to ARS and gradient ARS respectively. All through this paper the terminology “almost Riemann solitons” is composed as ARS.

Riemann solitons and gradient Riemann solitons on Sasakian manifolds have been investigated in detail by Hirica and Udriste (see, [13]). Furthermore, Riemann’s soliton concerning infinitesimal harmonic transformation was studied in [18]. In this association, we notice that Sharma in [16] explored almost Ricci soliton in \(K\)-contact geometry and in [17], with divergence-free soliton vector field. In [6], Riemann soliton under the context of contact manifold has been studied and demonstrated a few intriguing outcomes.

Quite a long while prior, in [14], Olszak explored the three dimensional normal almost contact metric (briefly, \(acm\)) manifolds mentioning several examples. After the citation of [14], in recent years normal \(acm\) manifolds have been studied by numerous eminent geometers (see, [7],[8],[9],[10] and references contained in those).
The above studies motivate us to investigate an ARS and the gradient ARS in a 3-dimensional normal acm manifolds, since 3-dimensional normal acm manifold covers Sasakian manifold, Cosymplectic manifold, Kenmotsu manifold and Quasi-Sasakian manifold.

The forthcoming article is structured as:

In section 2, we reminisce about some facts and formulas of normal acm manifolds, which we will require in later sections. Beginning from Section 3, after giving the proof, we will engrave our main Theorems. After that, we develop an example of a 3-dimensional normal acm manifold admitting a Riemann soliton. This exposition terminates with a concise bibliography that has been utilized during the formulation of the article.

2. Preliminaries

Let $M^3$ be an acm manifold endowed with a triplet of almost contact structure $(\eta, \xi, \phi)$. In details, $M^3$ is an odd-dimensional differentiable manifold equipped with a global 1-form $\eta$, a unique characteristic vector field $\xi$, and a $(1,1)$-type tensor field $\phi$, respectively, such that

\begin{equation}
\phi^2 E = -E + \eta(E)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0.
\end{equation}

A structure, named almost complex structure $J$ on $M \times \mathbb{R}$ is defined by

\begin{equation}
J(E, \frac{d}{ds}) = (\phi E - \lambda \xi, \eta(E) \frac{d}{ds}),
\end{equation}

where $(E, \lambda \frac{d}{ds})$ indicates a tangent vector on $M \times \mathbb{R}$, $E$ and $\lambda \frac{d}{ds}$ being tangent to $M$ and $\mathbb{R}$ respectively. After fulfilling the condition, the structure $J$ is integrable, $M(\eta, \xi, \phi)$ is said to be normal (see, [2],[3]).

The Nijenhuis torsion is defined by

\begin{equation}
[\phi, \phi](E, F) = \phi^2[E, F] + [\phi E, \phi F] - \phi[\phi E, F] - \phi[E, \phi F].
\end{equation}

The structure $(\eta, \xi, \phi)$ is said to be normal if and only if

\begin{equation}
[\phi, \phi] + 2d\eta \otimes \xi = 0.
\end{equation}

The Riemannian metric $g$ on $M^3$ is said to be compatible with $(\eta, \xi, \phi)$ if the condition

\begin{equation}
g(\phi E, \phi F) = g(E, F) - \eta(E) \eta(F),
\end{equation}

where $g(E, F)$ is a Riemannian metric on $M^3$. This exposition terminates with a concise bibliography that has been utilized during the formulation of the article.
holds for any $E, F \in \mathfrak{X}(M)$. In such case, the quadruple $(\eta, \xi, \phi, g)$ is termed as an acm structure on $M^3$ and $M^3$ is an acm manifold. The equation

$$
(2.5) \quad \eta(E) = g(E, \xi),
$$

is withal valid on such a manifold.

Certainly, we can define the fundamental 2-form $\Phi$ by

$$
(2.6) \quad \Phi(E, F) = g(E, \phi F),
$$

where $E, F \in \mathfrak{X}(M)$.

For a normal acm manifold, we can write [14]:

$$
(2.7) \quad (\nabla_E \phi)(F) = g(\phi \nabla_E \xi, F) - \eta(F) \phi \nabla_E \xi,
$$

$$
(2.8) \quad \nabla_E \xi = \alpha[E - \eta(E) \xi] - \beta \phi E,
$$

where $\alpha = \frac{1}{2} \text{div} \xi$ and $\beta = \frac{1}{2} \text{tr}(\phi \nabla \xi)$, $\text{div} \xi$ is the divergent of $\xi$ defined by $\text{div} \xi = \text{trace}\{E \rightarrow \nabla_E \xi\}$ and $\text{tr}(\phi \nabla \xi) = \text{trace}\{E \rightarrow \phi \nabla_E \xi\}$. Utilizing (2.8) in (2.7) we lead

$$
(2.9) \quad (\nabla_E \phi)(F) = \alpha[g(\phi E, F) \xi - \eta(F) \phi E] + \beta[g(\xi, F) \phi E - \eta(F) E].
$$

Also in this manifold the subsequent relations hold [14]:

$$
(2.10) \quad R(E, F) \xi = [F \alpha + (\alpha^2 - \beta^2) \eta(F)] \phi^2 E - [E \alpha + (\alpha^2 - \beta^2) \eta(E)] \phi^2 F + [F \beta + 2 \alpha \beta \eta(F)] \phi E - [E \beta + 2 \alpha \beta \eta(E)] \phi F,
$$

$$
(2.11) \quad S(E, \xi) = -E \alpha - (\phi E) \beta - [\xi \alpha + 2 (\alpha^2 - \beta^2)] \eta(E),
$$

$$
(2.12) \quad \xi \beta + 2 \alpha \beta = 0,
$$

$$
(2.13) \quad (\nabla_E \eta)(F) = \alpha g(\phi E, \phi F) - \beta g(\phi E, F).
$$
It is well admitted that in a 3-dimensional Riemannian manifold the Riemann curvature tensor is always satisfies

\begin{align}
R(E, F)Z &= S(F, Z)E - S(E, Z)F + g(F, Z)QE - g(E, Z)QF \\
&\quad - \frac{r}{2}[g(F, Z)E - g(E, Z)F].
\end{align}

(2.14)

By (2.10), (2.11) and (2.14) we infer

\begin{align}
S(E, F) &= \left(\frac{r}{2} + \xi + \alpha^2 - \beta^2\right)g(\phi E, \phi F) \\
&\quad - \eta(E)(F\phi + (\phi F)\beta) - \eta(F)(E\phi + (\phi E)\beta) \\
&\quad - 2(\alpha^2 - \beta^2)\eta(E)\eta(F).
\end{align}

(2.15)

For \(\alpha, \beta = \text{constant}\), it follows from the above equation that a 3-dimensional normal acm manifold becomes an \(\eta\)-Einstein manifold.

From (2.9) we conclude that the manifold is either \(\alpha\)-Kenmotsu \(\text{[11]}\) or cosymplectic \(\text{[2]}\) or \(\beta\)-Sasakian, provided \(\alpha, \beta = \text{constant}\). Also it is well known that a 3-dimensional normal acm manifold reduces to a quasi-Sasakian manifold if and only if \(\alpha = 0\) (see, \(\text{[14]}, \text{[15]}\)).

3. Riemann Soliton

In this segment, we first write the subsequent result \(\text{([4]),[5]}\):

**Lemma 3.1.** In a Riemannian manifold if \((g, Z)\) is a Ricci soliton, then we have

\begin{equation}
\frac{1}{2}\|\mathcal{L}_Z g\|^2 = dr(Z) + 2\text{div}(\lambda Z - QZ).
\end{equation}

(3.1)

Now, because of (2.8) we obtain

\begin{equation}
(\mathcal{L}_\xi g)(E, F) = 2\alpha\{g(E, F) - \eta(E)\eta(F)\}.
\end{equation}

(3.2)

We consider a normal acm manifold \(M^3\) with \(\alpha, \beta = \text{constants}\) admitting a Riemann soliton defined by (1.1). Using Kulkarni-Nomizu product in (1.1) we write

\begin{align}
2R(E, F, W, X) + 2\lambda\{g(E, X)g(F, W) - g(E, W)g(F, X)\} \\
&+ \{g(E, X)\mathcal{L}_Z g(F, W) + g(F, W)\mathcal{L}_Z g(U, E) \\
&- g(E, W)\mathcal{L}_Z g(F, X) - g(F, X)\mathcal{L}_Z g(E, W)\} = 0.
\end{align}

(3.3)

Contracting (3.3) over \(E\) and \(X\), we infer
(3.4) \((\mathcal{L}_Z g)(F, W) + 2S(F, W) + (4\lambda + 2\text{div}Z)g(F, W) = 0\).

Thus Riemann soliton whose potential vector field is of vanishing divergence reduces to Ricci soliton.

Hence we have

(3.5) \((\mathcal{L}_Z g)(F, W) + 2S(F, W) + 4\lambda g(F, W) = 0\).

Setting \(Z = \xi\) and utilizing (3.2) we lead

\[2\alpha\{g(F, W) - \eta(F)\eta(W)\} + 2S(F, W) + 4\lambda g(F, W) = 0,\]

which implies that

(3.6) \(\alpha\{F - \eta(F)\xi\} + QF + 2\lambda F = 0\).

Putting \(F = W = e_i\) and taking \(\text{div}Z = 0\) from (3.5) we get \(r = -6\lambda\). Hence in our case (3.1) takes the form

(3.7) \(\frac{1}{2}\|\mathcal{L}_Z g\|^2 = dr(Z) + 2\text{div}(-2\lambda Z - QZ)\).

From (3.6) we get \(Q\xi = -2\lambda \xi\). Therefore utilizing \(r = -6\lambda\) and \(Q\xi = -2\lambda \xi\) we obtain from (3.7) \(\xi\) is a Killing vector. Hence (3.2) implies \(\alpha = 0\) that is the manifold is quasi-Sasakian.

Utilizing \(\alpha = 0\) in (3.6), we infer

\(QF = -2\lambda F\),

. Hence from (2.14) we can write that the manifold is of constant sectional curvature \(-\lambda\). Therefore we write:

**Theorem 3.1.** If the metric of a non-cosymplectic normal acm manifold \(M^3\) is Riemann soliton with a divergence-free potential vector field, then the manifold is quasi-Sasakian and is of constant sectional curvature \(-\lambda\), provided \(\alpha, \beta =\text{constant}\).

4. **Almost Riemann Soliton**

Here we consider a normal acm manifold \(M^3\) with \(\alpha, \beta =\text{constants}\) admitting an ARS defined by(1.1).
In particular, let the potential vector field $Z$ be point-wise collinear with $\xi$ (i.e., $Z = c\xi$, where $c$ is a function on $M$) and has constant divergence. Then from (3.4) we lead

\[(4.1) \quad g(\nabla_{E}c\xi, F) + g(\nabla_{F}c\xi, E) + 2S(E, F) + (4\lambda + 2\text{div}Z)g(E, F) = 0.\]

Utilizing (2.5) and (2.8) in (4.1), we obtain

\[(4.2) \quad 2\alpha c[g(E, F) - \eta(E)\eta(F)] + (Ec)\eta(F) + (Fc)\eta(E) + 2S(E, F) + (4\lambda + 2\text{div}Z)g(E, F) = 0.\]

Replacing $F$ by $\xi$ in (4.2) and utilizing (2.1), (2.5) and (2.11) gives

\[(4.3) \quad (Ec) + (\xi c)\eta(E) - 4(\alpha^{2} - \beta^{2})\eta(E) + (4\lambda + 2\text{div}Z)\eta(E) = 0.\]

Putting $E = \xi$ in (4.3) and utilizing (2.1) yields

\[(4.4) \quad \xi c = [2(\alpha^{2} - \beta^{2}) - 2\lambda - 2\text{div}Z].\]

Putting the value of $\xi c$ in (4.3) we infer

\[(4.5) \quad dc = [2(\alpha^{2} - \beta^{2}) - 2\lambda - 2\text{div}Z]\eta.\]

Applying $d$ on (4.5) and using Poincare lemma $d^2 \equiv 0$, we lead

\[(4.6) \quad [2(\alpha^{2} - \beta^{2}) - 2\lambda - 2\text{div}Z]d\eta + (d\lambda)\eta = 0.\]

Taking wedge product of (4.6) with $\eta$, we obtain

\[(4.7) \quad [2(\alpha^{2} - \beta^{2}) - 2\lambda - 2\text{div}Z]\eta \wedge d\eta = 0.\]

Since $\eta \wedge d\eta \neq 0$, we infer

\[(4.8) \quad [2(\alpha^{2} - \beta^{2}) - 2\lambda - 2\text{div}Z] = 0.\]

Utilizing (4.8) in (4.5) gives $dc = 0$ i.e., $c =$constant. Also from (4.8) we have

\[(4.9) \quad \lambda = [(\alpha^{2} - \beta^{2}) - \text{div}Z] = \text{constant}.\]
Hence we write the following:

**Theorem 4.1.** If the metric of a non-cosymplectic normal acm manifold $M^3$ is ARS and $Z$ is pointwise collinear with $\xi$ and has constant divergence, then $Z$ is constant multiple of $\xi$ and the ARS reduces to a Riemann soliton, provided $\alpha, \beta$ =constant.

**Corollary 4.1.** If a non-cosymplectic normal acm manifold $M^3$ with $\alpha, \beta$ =constant admits an ARS of type $(g, \xi)$, then the ARS reduces to a Riemann soliton.

5. **Gradient Almost Riemann Soliton**

In this section we investigate a non-cosymplectic normal acm manifold $M^3$ with $\alpha, \beta$ =constant, admitting gradient ARS. Now we prove the subsequent results:

**Lemma 5.1.** For a non-cosymplectic normal acm manifold $M^3$ with $\alpha, \beta$ =constant, we have

\[(\nabla E Q)\xi = -\left\{\frac{r}{2} + 3(\alpha^2 - \beta^2)\right\}[\alpha E - \eta(E)\xi] - \beta \phi E].\]

**Proof.** For $\alpha, \beta$ =constants, we get from (2.15)

\[QF = \left\{\frac{r}{2} + \alpha^2 - \beta^2\right\}F - \left\{\frac{r}{2} + 3(\alpha^2 - \beta^2)\right\}\eta(F)\xi.\]

Differentiating (5.2) covariantly in the direction of $E$ and using (2.8) and (2.13), we get

\[(\nabla E Q)F = \frac{dr(E)}{2}(F - \eta(F)\xi) - \left\{\frac{r}{2} + 3(\alpha^2 - \beta^2)\right\}[\alpha g(E, F)\xi - 2\alpha \eta(E)\eta(F)\xi + \alpha \eta(F)E - \beta g(\phi E, F)\xi - \beta \eta(F)\phi E].\]

Replacing $F$ by $\xi$ in (5.3) and utilizing (2.8), we get

\[(\nabla E Q)\xi = -\left\{\frac{r}{2} + 3(\alpha^2 - \beta^2)\right\}[\alpha E - \eta(E)\xi] - \beta \phi E].\]
Lemma 5.2. Let $M^3(\eta, \xi, \phi, g)$ be a non-cosymplectic normal acm manifold with $\alpha, \beta$ = constant. Then we have

\begin{equation}
\xi r = -4\alpha \left\{ \frac{r}{2} + 3(\alpha^2 - \beta^2) \right\}
\end{equation}

Proof. Recalling (5.3), we can write

\begin{align*}
(5.5)\ g((\nabla F)Q)F, Z) &= \frac{dr(E)}{2} [g(F, Z) - \eta(F)\eta(Z)] \\
&\quad - \left\{ \frac{r}{2} + 3(\alpha^2 - \beta^2) \right\} \left[ \alpha g(E, F)\eta(Z) - 2\alpha \eta(E)\eta(F)\eta(Z) \\
&\quad + \alpha \eta(F)g(E, Z) - \beta g(\phi E, F)\eta(Z) - \beta \eta(F)g(\phi E, Z) \right] \\
&\quad - \left\{ \frac{r}{2} + 3(\alpha^2 - \beta^2) \right\} \left[ \alpha g(E, F)\eta(Z) - 2\alpha \eta(E)\eta(F)\eta(Z) \\
&\quad + \alpha \eta(F)g(E, Z) - \beta g(\phi E, F)\eta(Z) - \beta \eta(F)g(\phi E, Z) \right].
\end{align*}

Putting $E = Z = e_i$ (where $\{e_i\}$ be the orthonormal basis for the tangent space of $M$ and taking $\sum_i 1 \leq i \leq 3$ ) in the foregoing equation and using the so called formula of Riemannian manifolds $\text{div}Q = \frac{1}{2}\text{grad} r$, we obtain

\begin{equation}
(\xi r)\eta(F) = -4\alpha \left\{ \frac{r}{2} + 3(\alpha^2 - \beta^2) \right\} \eta(F).
\end{equation}

Replacing $F = \xi$ in the previous equation we have the required result. □

Lemma 5.3. (Lemma. 3.8 of [6]) For any vector fields $E, F$ on $M^3$, in a gradient ARS $(M, g, \gamma, m, \lambda)$, we infer

\begin{align*}
R(E, F)D\gamma &= (\nabla F)E - (\nabla E)F + \{F(2\lambda + \triangle \gamma)E - E(2\lambda + \triangle \gamma)F\},
\end{align*}

where $\triangle \gamma = \text{div} D\gamma$, $\triangle$ is the Laplacian operator.

Superseding $F$ by $\xi$ in (5.7) and utilizing Lemma 5.1, we get

\begin{align*}
R(E, \xi)D\gamma &= \frac{dr(\xi)}{2} [E - \eta(E)\xi] - \left\{ \frac{r}{2} + 3(\alpha^2 - \beta^2) \right\} \left[ 2\alpha E - 2\alpha \eta(E)\xi - \beta \phi E \right] \\
&\quad + \{\xi(2\lambda + \triangle \gamma)E - E(2\lambda + \triangle \gamma)\xi\}.
\end{align*}

Then utilizing (2.10), we infer

\begin{align*}
g(E, (\alpha^2 - \beta^2)D\gamma + D(2\lambda + \triangle \gamma))\xi &= \frac{dr(\xi)}{2} [E - \eta(E)\xi] \\
&\quad - \left\{ \frac{r}{2} + 3(\alpha^2 - \beta^2) \right\} \left[ 2\alpha E - 2\alpha \eta(E)\xi - \beta \phi E \right] \\
&\quad + \{(\alpha^2 - \beta^2)(\xi\gamma) + \xi(2\lambda + \triangle \gamma)\} E.
\end{align*}

Executing inner product of the foregoing equation with $\xi$ yields

\begin{equation}
(5.10)\ E((\alpha^2 - \beta^2)\gamma + (2\lambda + \triangle \gamma)) = \{(\alpha^2 - \beta^2)(\xi\gamma) + \xi(2\lambda + \triangle \gamma)\} \eta(E),
\end{equation}
from which easily we lead
\begin{equation}
\frac{\partial}{\partial t}(\alpha^2 - \beta^2)\gamma + (2\lambda + \triangle \gamma) = \{(\alpha^2 - \beta^2)(\xi\gamma) + \xi(2\lambda + \triangle \gamma)\} \theta,
\end{equation}

where the exterior derivative is denoted by \(d\). From the above equation we conclude that \((\alpha^2 - \beta^2)\gamma + (2\lambda + \triangle \gamma)\) is invariant along the distribution \(D\). In other terms, \(E((\alpha^2 - \beta^2)\gamma + (2\lambda + \triangle \gamma)) = 0\) for any \(E \in D\). Hence utilizing (5.10) in (5.9), we get
\begin{equation}
\frac{\partial}{\partial t}(\xi) - \left[\frac{r}{2} + 3(\alpha^2 - \beta^2)\right][2\alpha E - 2\alpha\eta(E)\xi - \beta\phi E] = 0.
\end{equation}

Contracting the previous equation and using (5.4), we lead
\begin{equation}
\alpha\left\{\frac{r}{2} + 3(\alpha^2 - \beta^2)\right\} = 0.
\end{equation}

Now we split our study in the following cases:

Case (i): If \(\alpha = 0\), then the manifold reduces to a quasi-Sasakian manifold.

Case (ii): If \(r = -6(\alpha^2 - \beta^2)\), then from (2.15) we get \(S = -2(\alpha^2 - \beta^2)g\), that is the manifold is an Einstein manifold and hence from (2.14) it follows that the manifold is of constant sectional curvature \(-\alpha^2 - \beta^2\). Hence we write:

**Theorem 5.1.** If a non-cosymplectic normal acm manifold \(M^3\) with \(\alpha, \beta = \text{constant}\) admits a gradient \(ARS(\gamma, \xi, \lambda)\), then the manifold is either quasi-Sasakian or is of constant sectional curvature \(-\alpha^2 - \beta^2\).
Then utilizing the linearity of $\phi$ and $g$, we infer
\[ \eta(u_3) = 1, \]
\[ \phi^2 E = -E + \eta(E)u_3, \]
\[ g(\phi E, \phi F) = g(E, F) - \eta(E)\eta(F), \]
for any $E, F \in \mathfrak{X}(M)$. Obviously, the structure $(\eta, \xi, \phi, g)$ admits an acm structure on $M^3$ for $u_3 = \xi$. Then we lead
\[ [u_1, u_3] = u_1u_3 - u_3u_1 \]
\[ = z\frac{\partial}{\partial x}(z\frac{\partial}{\partial z}) - z\frac{\partial}{\partial z}(z\frac{\partial}{\partial x}) \]
\[ = z^2 \frac{\partial}{\partial x\partial z} - z^2 \frac{\partial^2}{\partial z\partial x} - z \frac{\partial}{\partial x} \]
\[ = -u_1. \tag{6.1} \]

Similarly
\[ [u_1, u_2] = 0 \quad \text{and} \quad [u_2, u_3] = -u_2. \]

Utilizing Koszul’s formula for the Riemannian metric $g$, we can calculate
\[ \nabla_{u_1}u_3 = -u_1, \quad \nabla_{u_1}u_2 = 0, \quad \nabla_{u_1}u_1 = u_3, \]
\[ \nabla_{u_2}u_3 = -u_2, \quad \nabla_{u_2}u_2 = u_3, \quad \nabla_{u_2}u_1 = 0, \]
\[ \nabla_{u_3}u_3 = 0, \quad \nabla_{u_3}u_2 = 0, \quad \nabla_{u_3}u_1 = 0. \tag{6.2} \]

From the above expression it is obvious that the manifold under consideration is a normal acm manifold with $\alpha, \beta=$-constants, since it satisfies \((2.3)\) for $\alpha = -1$ and $\beta = 0$ and $\xi = e_3$.

It can be easily verified that
\[ R(u_1, u_2)u_3 = 0, \quad R(u_2, u_3)u_3 = -u_2, \quad R(u_1, u_3)u_3 = -u_1, \]
\[ R(u_1, u_2)u_2 = -u_1, \quad R(u_2, u_3)u_2 = u_3, \quad R(u_1, u_3)u_2 = 0, \]
\[ R(u_1, u_2)u_1 = u_2, \quad R(u_2, u_3)u_1 = 0, \quad R(u_1, u_3)u_1 = u_3. \]

In this example, it is easy to verify that the characteristic vector field $\xi$ has constant divergence and obviously $\mathcal{L}_\xi g = 0$. Then equation \((3.3)\) reduces to
\[ 2R(E, F)W + 2\lambda\{g(F, W)E - g(E, W)Y\} = 0, \tag{6.3} \]
for all vector field $E, F, W$. Also equation (6.3) holds for $\lambda = 1$. Thus the manifold under consideration admits a Riemann soliton $(g, \xi, \lambda)$.

REFERENCES

[1] Besse, A., *Einstein Manifolds*, Springer, Berlin, 1987. https://doi.org/10.1007/978-3-540-74311-8.

[2] Blair, D. E., *Contact manifolds in Riemannian geometry*, Lecture notes in math., 509 (1976), Springer-Verlag, Berlin-New York.

[3] Blair, D. E., *Riemannian geometry of contact and symplectic manifolds*, Progress in Math., 203 (2002), Birkhäuser Boston, Inc., Boston.

[4] Cho, J.T., *Notes on contact Ricci soliton*, Proc. Edinb. Math. Soc. 54 (2011), 47-53.

[5] Cho, J.T., *Almost contact 3-Manifolds and Ricci solitons*, Int. J. Geom. Methods Mod. Phys. 10 (2013), 1220022(7 pages).

[6] Devaraja, M.N., Kumara, H.A. and Venkatesha, V., *Riemann soliton within the framework of contact geometry*, Quaestiones Mathematicae, (2020) DOI:10.2989/16073606.2020.1732495

[7] De, U. C. and Mondal, A. K., *On 3-dimensional normal almost contact metric manifolds satisfying certain curvature conditions*, Commun. Korean Math. Soc., 24 (2009), 265 – 275.

[8] De, U. C., Turan, M., Yildiz, A. and De, A., *Ricci solitons and gradient Ricci solitons on 3-dimensional normal almost contact metric manifolds*, Publ. Math. Debrecen, 80 (2012), 127-142.

[9] De, U. C., Yildiz, A. and Sarkar A., *Isometric immersion of three dimensional quasi-Sasakian manifolds*, Math. Balkanica (N.S.), 22 (2008), 297 – 306.

[10] De, U. C., Yildiz, A. and Yalnz, A. F., *Locally $\phi$-symmetric normal almost contact metric manifolds of dimension 3*, Appl. Math. Lett., 22 (2009), 723 – 727.

[11] Janssen, D. and Vanhecke, L., *Almost contact structures and curvature tensors*, Kodai Math. J., 4 (1981), 1 – 27.

[12] Hamilton, R. S., *The Ricci flow on surfaces*, Math. gen. relativ. (Santa Cruz, CA, 1986), 237–262, Contemp. Math. 71, (1988).

[13] Hirica, I.E. and Udriste, C., *Ricci and Riemann solitons*, Balkan J. Geom. Applications. 21 (2016), 35-44.

[14] Olszak, Z., *Normal almost contact manifolds of dimension three*, Ann. Polon. Math., 47 (1986), 41-50.
[15] Olszak, Z., *On three dimensional conformally flat quasi-Sasakian manifolds*, Period, Math. Hung., 33 (1996), 105 – 113.

[16] Sharma, R., *Almost Ricci solitons and K-contact geometry*, Monatsh Math. 175 (2014), 621–628.

[17] Sharma, R., *Some results on almost Ricci solitons and geodesic vector fields*, Beitr. Algebra Geom. 59 (2018), 289–294.

[18] Stepanov, S.E. and Tsyganok, I.I., *The theory of infinitesimal harmonic trans-formations and its applications to the global geometry of Riemann solitons*, Balk. J.Geom. Appl. 24 (2019), 113-121.

[19] Udriste, C., *Riemann flow and Riemann wave*, Ann. Univ. Vest, Timisoara. Ser.Mat.-Inf. 48 (2010), 265-274.

[20] Udriste, C., *Riemann flow and Riemann wave via bialternate product Riemannian metric*, preprint, arXiv.org/math.DG/1112.4279v4 (2012).

**Krishnendu De,**
**Assistant Professor of Mathematics,**
**Kabi Sukanta Mahavidyalaya,**
**Bhadreswar, P.O.-Angus, Hooghly,**
**Pin 712221, West Bengal, India.**
**E-mail address:** krishnendu.de@outlook.in

**Uday Chand De**
**Department of Pure Mathematics**
**University of Calcutta**
**35, Ballygunge Circular Road**
**Kol- 700019, West Bengal, India.**
**E-mail address:** uc_de@yahoo.com