Free boundary value problems for abstract elliptic equations and applications

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Abstract

Free boundary value problem for elliptic differential-operator equations with variable coefficients is studied. The uniform maximal regularity properties and Fredholmness of this problem are obtained in vector-valued Holder spaces.

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1. Introduction, notations and background

In last years, the maximal regularity properties of boundary value problems (BVPs) for differential-operator equations (DOEs) have found many applications in PDE, pseudo DE and in the different physical process (see for references [1 – 4], [6], [8], [10], [12 – 23], [27 – 28]).

Let Ω be a domain in \( \mathbb{R}^n \) and \( E \) is a Banach space. \( C_b^{(m)}(\Omega; E) \) will denote the spaces of \( E \)-valued bounded uniformly strongly continuous and \( m \)-times continuously differentiable functions on \( \Omega \). For \( m = 0 \) it denotes by \( C_b(\Omega; E) \). Let \( \mathbb{C} \) denote the set of complex numbers. For \( E = \mathbb{C} \) the space \( C_b^{(m)}(\Omega; \mathbb{C}) \) will be denoted by \( C_b^{(m)}(\Omega) \). Moreover, \( C_b^{\infty}(\Omega; E) \) denotes spaces of \( E \)-valued bounded strongly continuously differentiable functions of arbitrary order. We put \( \mathbb{R} = (-\infty, \infty) \) and \( \mathbb{R}_+ = (0, \infty) \). Let \( f(x) \) is a \( E \)-valued function and \( f(x) \neq 0 \). Consider

\[ \Omega_f = \{(x,y) \in \mathbb{R} \times \mathbb{R}_+, \ f \in C_b(\mathbb{R}; E), \ 0 < y < \| f(x) \| \}. \]

The boundaries of \( \Omega_f \) are given by

\[ \Gamma_0 = \mathbb{R} \times 0, \ \Gamma_f = \{(x,y) \in \mathbb{R} \times \mathbb{R}_+, \ y = \| f(x) \| \}. \]
Consider the following problem: Given \( f_0, \nu \in C_b^{(2)} (\mathbb{R}; E) \). Find a pair of functions \((u, f)\) possessing the regularity
\[
f \in C_b^{(1)} \left([0, T]; C_b^{(1)} (\mathbb{R}; E)\right),
\]
and satisfying the following equations a.e.
\[
-\Delta u (t, z) + A(x) u (t, z) = 0, \quad t \in J, \quad z \in \Omega (t),
\]
\[
\frac{\partial u}{\partial y} = 0, \quad t \in J, \quad z \in \Omega (t),
\]
\[
u (t, z) = f (t, x), \quad t \in J, \quad z \in \Gamma (t),
\]
\[
\lim_{z \to \infty} u (t, z) = \nu (t), \quad t \in [0, T),
\]
\[
L_1 u = u |_{\Gamma} = 0,
\]
where \( A \) is a linear operator in a Banach space \( E \) and \( z = (x, y) \) represents a generic point in \( \Omega \). Moreover, \( \Delta \) denotes the Laplace operator with respect to the Euclidean metric, \( \frac{\partial}{\partial n} \) denotes the derivative in direction of the outer unit normal \( n \) at \( \Gamma \).

Maximal regularity properties of partial DOEs in \( L_p \) spaces were studied in \([1], [4], [7], [18-23]\). The results in \([4] \) and \([18-23]\) were restricted to rectangular domain and equations that were not contained mixed derivatives in leading part. Moreover, problems investigated in \([1]\) and \([8]\) involve only bounded operator coefficients. In \([18]\) the Dirichlet problem for the elliptic differential-operator equation of the second order in general domain was studied.

In contrast to all above we study general BVP \((1.1)\) for equation with unbounded operator coefficients in the general domain.

Consider the BVP
\[
L u = \sum_{i,j=1}^{n} a_{ij} (x) \frac{\partial^2 u}{\partial x_i \partial x_j} - A(x) u (x) = F(x),
\]
\[
L_1 u = u |_{\Gamma} = 0,
\]
where \( \Gamma \) is a boundary of region \( G \subset \mathbb{R}^n \) and \( a_{ij} \) are real-valued functions on \( G \).

We say that the problem \((1.3)\) is maximal \( H \)-regular (or separable in Holder space \( C^{\gamma} \)) if:

1. for all \( F \in C^{\gamma} (G; E) \) there exists a unique solution \( u \in C^{2,\gamma} (G; E(A), E) \) satisfying \((1.3)\) a.e. on \( G \).
(2) there exists a positive constant $C$ independent of $F$ such that

$$
\sum_{i,j=1}^{n} \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{C^\gamma(G;E)} + \| Au \|_{C^\gamma(G;E)} \leq C \| F \|_{C^\gamma(G;E)}.
$$

Let $G$ denote the operator generated by the problem (1.3) for $\lambda = 0$, i.e.,

$$
D(G) = C_0^{2,\gamma} (G; E(A), E) = \{ u \in C^{2,\gamma} (G; E) \cap C(G; E(A)) \},
$$

$u |_{\Gamma} = 0$, $Gu = Lu$.

The paper is organized as follows: Section 1 collects definitions and background materials, embedding theorems of Sobolev-Lions spaces.

Let $C$ be the set of complex numbers and

$$
S(\varphi) = \{ \lambda \in C, |\arg \lambda| \leq \varphi \} \cup \{ 0 \}, 0 \leq \varphi < \pi.
$$

Let $E_1$ and $E_2$ be two Banach spaces. $L(E_1, E_2)$ denotes the space of all bounded linear operators from $E_1$ to $E_2$. For $E_1 = E_2 = E$ it will be denoted by $L(E)$.

A linear operator $A$ is said to be $\varphi$-positive in a Banach space $E$ with bound $M > 0$ if $D(A)$ is dense on $E$ and

$$\left\| (A + \lambda I)^{-1} \right\|_{B(E)} \leq M (1 + |\lambda|)^{-1}$$

with $\lambda \in S(\varphi)$, $\varphi \in [0, \pi)$, where $I$ is an identity operator in $E$.

Sometimes instead of $A + \lambda I$ will be written $A + \lambda$ and will denoted by $A_\lambda$. It is known that ([25, §1.15.1]) there exist fractional powers $A^\theta$ of positive operator $A$. Let $E(A^\theta)$ denote the space $D(A^\theta)$ with graphical norm

$$
\| u \|_{E(A^\theta)} = \left( \| u \|^{p} + \| A^\theta u \|^{p} \right)^{\frac{1}{p}}, 1 \leq p < \infty, -\infty < \theta < \infty.
$$

A linear operator $A(x)$ is said to be positive in $E$ uniformly in $x$ if $D(A(x))$ is independent of $x$, $D(A(x))$ is dense in $E$ and

$$\left\| (A(x) + \lambda I)^{-1} \right\| \leq M (1 + |\lambda|)^{-1}$$

for all $\lambda \in S(\varphi)$ and $\varphi \in [0, \pi)$.

Let $\Omega$ be a domain in $R^n$. $C(\Omega, E)$ and $C_m(\Omega; E)$ will denote the spaces of $E$-valued bounded uniformly strongly continuous and $m$-times continuously differentiable functions on $\Omega$, respectively.

Let $0 < \gamma \leq 1$. $C^\gamma(\Omega; E)$ denotes the space of $E$-valued strongly bounded continuous functions that are defined on $\Omega \subset R^n$ with the norm

$$
\| f \|_{C^\gamma(\Omega; E)} = \| f \|_{C^\gamma(\Omega; E)} + [f]^\gamma (E),
$$
where

\[ [f]_E^\gamma = \sup_{x \neq y, x, y \in \Omega} \frac{\| f(x) - f(y) \|_E}{|x - y|^\gamma}. \]

\( C^{\gamma,m}(\Omega; E) \) denotes the space of \( E \)-valued strongly bounded continuous functions that are defined on \( \Omega \subset \mathbb{R}^n \) with the norm

\[ \| f \|_{C^{\gamma,m}(\Omega; E)} = \| f \|_{C^m(\Omega; E)} + \| f^{(m)} \|_{C^\gamma(\Omega; E)} < \infty. \]

Let \( E_0 \) and \( E \) be two Banach spaces and \( E_0 \) is continuously and densely embedded into \( E \). Let \( m \) be a natural number.

Let \( \alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \) are \( n \) tuples of nonnegative integer numbers and

\[ D^\alpha = \frac{\partial^{\alpha}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}}. \]

\( C^{m,\gamma}(\Omega; E_0, E) \) denote the space of \( E_0 \)-valued bounded uniformly strongly continuous and \( m \)-times continuously differentiable functions on \( \Omega \) with norm

\[ \| f \|_{C^{m,\gamma}(\Omega; E_0, E)} = \| f \|_{C^{m,\gamma}(\Omega; E)} + \| f \|_{C^\gamma(\Omega; E_0)} < \infty. \]

For \( E_0 = E \) the space \( C^{m,\gamma}(\Omega; E_0, E) \) will denoted by \( C^{m,\gamma}(\Omega; E) \).

Let \( A \) be a linear operator in a Banach space \( E \) so that is a generator of analytic semigroup \( U(t) = U_A(t) \). Let

\[ D_A(\theta, p) = \left\{ u \in E, \| u \|_{D_A(\theta, p)} = \left\| t^{1-\theta} \frac{1}{\theta} A U(t) u \right\|_{L^p(0,1; E)} < \infty \right\} \]

for \( 1 \leq p < \infty \) and \( \theta \in (0,1) \);

\[ D_A(\theta, \infty) = \left\{ u \in E, \| u \|_{D_A(\theta, \infty)} = \left\| t^{1-\theta} A U(t) u \right\|_{L^\infty(0,1; E)} < \infty \right\} \]

for \( p = \infty \) and \( 0 < \theta \leq 1 \);

\[ D_A(\theta) = \left\{ u \in D_A(\theta, \infty), \lim_{t \to 0} t^{1-\theta} A U(t) u = 0 \right\}. \]

From [15, Proposition 2.2.2] we obtain the following

\[ D_A(\theta, p) = (E, D(A))_{\theta,p}, \text{ for } 1 \leq p < \infty, \theta \in (0,1), \]

\[ D_A(\theta, \infty) = (E, D(A))_{\theta, \infty} \text{ for } 0 < \theta \leq 1, \]

\[ D_A(\theta) = (E, D(A))_{\theta}, \text{ for } \theta \in (0,1). \]

\( H(E_0, E) \) denotes the class of linear operators that are isomorphism from \( E_0 \) onto \( E \) and are negative generators of strong continuous and analytic semigroups.
Let $S'(R^n; E)$ denote the space of all continuous linear operators $L : S(R^n; E) \to E$, equipped with the bounded convergence topology. Recall $S(R^n; E)$ is norm dense in $L^p(R^n; E)$ when $1 \leq p < \infty$.

Let $L^r_+(E)$ denote the space of all $E$-valued functions $u(t)$ such that

$$
\|u\|_{L^r_+(E)} = \left( \int_0^\infty \|u(t)\|_E^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty, \ 1 \leq q < \infty, \ \|u\|_{L^r_+(E)} = \sup_{0 < t < \infty} \|u(t)\|_E.
$$

Let $F$ denote the Fourier transform. Fourier-analytic definition of $E$-valued Besov space on $R^n$ are defined as in [25 § 3], i.e.,

$$
B^s_{p,q} (R^n) = \left\{ u \in S'(R^n; E), \ \|u\|_{B^s_{p,q} (R^n)} = \left\| F^{-1} \sum_{k=1}^n t^{\kappa_k - s_k} (1 + |\xi_k|^\kappa_k) e^{-t|\xi|^2} F u \right\|_{L^r_+(L^p(R^n; E))} < \infty \right\}.
$$

For appropriate domain $\Omega \subset R^n$ the space $B^s_{p,q} (\Omega; E)$ is defined as usual restriction of the space $B^s_{p,q} (R^n; E)$.

For $E = \mathbb{C}$ the space $B^s_{p,q} (\Omega; \mathbb{C})$ will be denoted by $B^s_{p,q} (\Omega)$.

Let $h^s = h^s (R^n; E)$ denote the closure of $S(R^n; E)$ in $B^s_{\infty, \infty} (R^n; E)$. Assume that $\Omega$ is an open subset of $R^n$ and let $r_{\Omega}$ denote the restriction operator with respect to $\Omega$, i.e., $r_{\Omega} u = u |_{\Omega}$ for $B^s_{\infty, \infty} (R^n; E)$. Here, $h^s (\Omega; E)$ is defined as the closure of $r_{\Omega} (S(R^n; E))$ in $B^s_{\infty, \infty} (\Omega; E)$ and $h^{m, \gamma} (\Omega; E)$ is defined as the closure of $r_{\Omega} (S(\Omega; E))$ in $C^{m, \gamma} (\Omega; E)$. For $E = \mathbb{C}$ the spaces $h^s (\Omega; E), h^{m, \gamma} (\Omega; E)$ will be denoted by $h^s$ and $h^{m, \gamma}$, respectively. Moreover, let $C^s_0 (\Omega; E)$ denote the closure of $C^\infty (\Omega; E)$ in $C^s (\Omega; E)$.

Let

$$
h^s_+ = h^s_+ (\mathbb{R}; E) = \{ g \in h^s, g(x) \neq 0 \},
$$

$$
X = h^s (Q; E), \ X_k = h^{k, \alpha} (Q; E), \ Y = h^{2, \alpha} (Q; E (A), E) = \{ u \in h^\alpha (Q; D (A)) \cap h^{2, \alpha} (Q; E) \}
$$

with the norm

$$
\|u\|_Y = \|u\|_{h^\alpha (Q; D (A))} + \|u^{(2)}\|_{h^{2, \alpha} (Q; E)} < \infty.
$$

Here,

$$
B^{2+\alpha}_+ = \{ u \in B^{2+\alpha}_{\infty, \infty} (R^n; E), g(x) \neq 0 \}
$$

and

$$
h^{m, \alpha} (A) = h^{m, \alpha} (\mathbb{R}; D_A (\alpha, \infty)), \ \alpha \in (0, 1).
$$
**Remark 1.1.** In order to formulate our result, let

\[ h^s_\nu = h^s_\nu (E) = \{ \nu + g; \ g \in h^s (\mathbb{R}; E) \}; \ h^{s,\alpha}_\nu = \{ \nu + g; \ g \in h^{s,\alpha} (\mathbb{R}; E) \} \]

and \( f \in h^{2,\alpha}_\nu \) given. Let \( u_f \) denote the unique solution of the BVP

\[ -\Delta u + Au = 0, \ \partial_y u = 0 \text{ on } \Gamma_0, \ u = f \text{ on } \Gamma_f, \]

where \( A \) is a linear operator in a Banach space \( E \).

\[ k_f = \frac{\|f\|^2}{\left(1 + \|f\|^2 + \|f_x\|^2 \right) \left(1 + \|f_x\|^2 \right)} \]

and define

\[ V_\nu = \{ f \in C^2_0 (\mathbb{R}; E) \}, \ f (x) \neq 0, \ \partial_y u_f (x, f (x)) < k_f \text{ for } x \in \mathbb{R}. \]

It is clear to see that \( u_\nu \equiv \nu \). Hence, \( \nu \in V_\nu \). More precisely, by following [9, Lemma 5.10] it can be shown that \( V_\nu \) is a open neighborhood of \( \nu \) in \( h^{s,2}_\nu (E) \) and that

\[ \text{diam}_{h^{s,2}} (V_\nu) = \sup_{g, \nu \in V_\nu} \|g - \nu\|_{h^{s,2}} = \infty. \]

Suppose now that \((u, f)\) is a classical solution of (1.1)–(1.2). We call \((u, f)\) a classical H"older solution on \( J \) if it possesses the additional regularity

\[ f \in C (J; V_\nu) \cap C^1 (J; h^{1,\alpha}_\nu), \ u (t, \cdot) \in h^{2,\alpha}_\nu (\mathbb{R}; E), \ t \in J. \]

We will prove the following main result

**Theorem 1.** Given \( f_0 \in V_\nu \), there exist \( t^+ = t^+ (f_0) \) and a unique maximal classical H"older solution \((u, f)\) of problem (1.1)–(1.2) on \([0, t^+)\). Moreover, the mapping \((t, f_0) \to f\) defines a local \( C^\infty\)-semiflow on \( V_\nu \). If \( t^+ < \infty \) and \( f : [0, t^+) \to V_\nu \) is uniformly continuous then either

\[ \lim_{t \to t^+ \atop t \in \mathbb{R}} \|f (t, \cdot)\|_{h^{s,\alpha}} = \infty, \ \lim_{t \to t^+ \atop t \in \mathbb{R}} \inf_{v \in V_\nu} \|f (t, \cdot) - v\|_{h^{s,\alpha}} = 0. \]

In the first stage, we transform problem (1.1)–(1.2) into a nonlinear problem on a fixed domain

\[ \frac{df}{dt} + O (f) = 0, \ f (0) = f_0 \]

with respect to only the unknown function \( f \), which determines the free boundary \( \Gamma_f \), where \( O \) is a nonlinear operator in \( E \).

Then, by using the solution of the above problem we will show the existence of regular solution of the free BVP (1.1)–(1.2).

2. Transformed problem
Let $\nu = \nu(t) > 0$ be fixed. Define
\[ G_\nu = \left\{ g \in C^2_b(\mathbb{R}; E), \nu(t) I + g(x) \neq 0 \right\}, \]
where $I$ is an identity element in the Banach space $E$.

Consider the following transformation
\[(x, y) = \varphi(x', y') = \varphi_g(x', y') = \left(x', 1 - \frac{y'}{\nu + g(x')}\right), \text{ for } (x', y') \in \Omega_f. \quad (2.1)\]

It is easily verified that $\varphi_g$ is a diffeomorphism of class $C^2$ which maps $\Omega_g$ onto the strip $Q = \mathbb{R} \times (0, 1)$. Moreover,
\[(x', y') = \varphi^{-1}(x, y) = \varphi_g^{-1}(x, y) = (x, (1 - y)g(x)) \text{ for } (x, y) \in Q. \]

\[ \varphi_* u = \varphi^g_* u = u(\varphi^{-1}_g(x, y)) \text{ for } u \in W^{2,p}_p(\Omega_g; E(A); E). \quad (2.2) \]

Let $u$ be an $E$-valued function defined on $Q$. Here, $u_{|\Gamma_i}$ denote the restriction of $u$ on $\Gamma_i$, where
\[ \Gamma_i = \mathbb{R} \times \{i\}, \gamma_i u = u_{|\Gamma_i}, \ i = 0, 1. \]

**Lemma 2.1.** Given $g \in \Phi_\eta$ and $\nu \in C^{2,\gamma}(Q; E(A), E)$; under the map (2.2) the operators in (1.1) are transformed into the following:
\[ B(g) u = -\varphi^g_\nu(\Delta + A)(\varphi_g \ast v) \text{ on } [0, T) \times Q, \quad (2.3) \]
\[ B_1(g) u = \varphi^g_\nu \left(\nabla (\varphi^g_\nu) \cdot n_i\right) \text{ on } (0, T) \times \Gamma_i, \]
where $n_0 = (-g_x, 1)$, $n_1 = (0, -1)$ denote the outer normals according to $\Gamma_f$ and $\Gamma_0$, i.e.,
\[ B_0(g) u = \varphi^g_\nu \left(\frac{\partial}{\partial x}\varphi^g_\nu v + \frac{\partial}{\partial y}\varphi^g_\nu v\right) \text{ on } (0, T) \times \Gamma_0, \quad (2.4) \]
\[ B_1(f) u = \varphi^g_\nu \left(-\frac{\partial}{\partial y}\varphi^g_\nu v\right) \text{ on } [0, T) \times \Gamma_1. \]

**Lemma 2.2.** Given $g \in \Phi_\nu$ and $\nu \in C^{2,\gamma}(Q; E(A), E)$. Under the map (2.2) the problem (1.1) is transformed into the following:
\[ B(f) u = -\varphi^g_\nu(\Delta + A)(\varphi_g \ast v) = 0 \text{ on } [0, T) \times Q, \]
\[ v = f \text{ on } [0, T) \times \Gamma_0, \]
\[ B_1(f) u = 0 \text{ on } [0, T) \times \Gamma_1, \quad (2.5) \]
\[ \lim_{z \to \infty} v(t, z) = 0, \text{ on } [0, T), \]
\[ \frac{\partial g}{\partial t} + B_0(g) u = 0 \text{ on } (0, T) \times \Gamma_0, \]
\[ g(0, \cdot) = g_0(x) \text{ on } \mathbb{R}, \]

A pair \((v, g)\) is called a solution of the problem (2.5) if

\[ g \in C_b([0, T); \Phi) \cap C_b^{(1)}((0, T); C_b^{(1)}[\mathbb{R}]), \tag{2.5} \]

\[ u(t, \cdot) \in W^2_p(\Omega_f; E), \quad t \in [0, T) \]

and \((v, g)\) satisfies (2.5) a.e. on \([0, T) \times Q\).

**Condition 2.1.** Assume the following conditions are satisfied:

(1) \(\sum_{i,j=1}^2 a_{ij} \xi_i \xi_j \geq C |\xi|^2\), for \(\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n\) and \(C > 0\);

(2) operator \(A\) is a positive operator in a Banach space \(E\) for some \(\varphi \in (0, \pi]\).

In a similar way as in [9, Lemma 2.2] we obtain

**Lemma 2.3.** Assume the Condition 2.1 are satisfied. Then for given \(g \in \Phi\), we have

\[ B(g) u = \sum_{i,j=1}^2 -a_{ij}(g) \frac{\partial^2 u}{\partial x_i \partial x_j} + a_2(g) \frac{\partial u}{\partial x_2} + A_g u, \tag{2.6} \]

\[ B_1(g) u = \sum_{j=1}^2 b_{ij}(g) \gamma_j \frac{\partial u}{\partial x_j}, \quad i = 0, 1, \]

and

\[ \sum_{i,j=1}^2 a_{ij}(g) \xi_i \xi_j \geq \alpha(g) |\xi|^2, \text{ for } \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n, \]

where \(\alpha(g) > 0, \gamma_i\) are trace operators from \(Q\) to \(\Gamma_i, \ i = 0, 1,\)

\[ a_{11}(g) = 1, \ a_{12}(g) = a_{21}(g) = \frac{\beta g_{x_1}}{\nu + g}, \ a_{22}(g) = \frac{1 + \beta^2 g^2_{x_1}}{\nu + g}, \ \beta = 1 - x_1, \]

\[ a_2(g) = \frac{\beta}{\nu + g} \left[ \frac{2 g_{x_1}^2}{\nu + g} - g_{x_1 x_1} \right], \quad b_{10}(g) = -g_{x_1}, \ b_{20}(g) = -\frac{1 + g^2_{x_1}}{\nu + g}, \tag{2.7} \]

\[ b_{11}(g) = 0, \ b_{21}(g) = \frac{1}{\nu + g}, \ \alpha(g) = \frac{1}{1 + (\nu + g)^2 + \beta^2 g^2_{x_1}}, \quad A(g) = A(\varphi_g). \]

3. **Abstract elliptic equation in the fixed domain**

In this section we study the elliptic BVP

\[ B(g) u = -\sum_{i,j=1}^2 a_{ij}(g) \frac{\partial^2 u}{\partial x_i \partial x_j} + a_2(g) \frac{\partial u}{\partial x_2} + A(x) u = f, \tag{3.1} \]
\[
B_i(g) u = \sum_{j=1}^{2} b_{ij}(g) \gamma_i \frac{\partial u}{\partial x_j} = f_i, \ i = 0, 1, \tag{3.2}
\]

where \(B(g)\) and \(B_i(g)\) are differential operators defined by (2.6).

We will derive a priori estimates as well as isomorphism properties in framework of abstract Hölder spaces.

**Condition 3.1.** Assume the following conditions are satisfied:

1. \(a_{ij} \in C^{0, \alpha}(\bar{G})\), \(a_{ij} = a_{ji}\);
2. \[\sum_{i,j=1}^{2} a_{ij}(g) \xi_i \xi_j \geq C(g) |\xi|^2, \text{ for } \xi = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n \text{ and } C(g) > 0;\]
3. operator \(A(g)\) is uniformly positive in a Banach algebra \(E\) for some \(\varphi \in (0, \pi]\).

Here \(\partial B(g)[\psi, \nu]\) denotes the Gateaux derivative of operator function \(B(g)\) at \(\psi\) in the direction of \(\nu\).

**Lemma 3.1.** Suppose the Condition 3.1 is satisfied and \(A(g)\) is Gateaux differentiable for \(g \in h_{\Phi}^{2, \alpha}\). Then the map \(g \rightarrow O_0(g) = \{B(g), B_0(g)\}\) is bounded linear operator-function from \(Y\) into \(X \times h_{\Phi}^{1, \alpha}(A)\) and have continuous derivatives of all order with respect to \(g \in h_{\Phi}^{2, \alpha}\), i.e.

\[B(.) \in C^\infty \left(h_{\Phi}^{2, \alpha}; L(Y, X)\right), \ B_0(.) \in C^\infty \left(h_{\Phi}^{2, \alpha}; L(Y, E)\right)\]

and

\[\partial B(g)[u, v] = (\partial B(g) u) v = \frac{2\beta}{\nu + g} \left\{ \left( \frac{g_x u}{\nu + g} - u_x \right) v_{x_1 x_2} + \frac{u}{(\nu + g)^2} \left( \frac{1}{\beta} + \beta g_x^2 \right) \right\} - \frac{\beta}{\nu + g} g_x u_x v_{x_2 x_2} - \left( \frac{g_x u}{(\nu + g)^2} - \frac{g_{xx} u + 4g_x u_x}{2(\nu + g)} + \frac{u_{xx}}{2} \right) v_{x_2} \] + \partial A(g)[u, v],

\[\partial B_0(g)[u, v] = -u_x v_{x_1} + \frac{\beta}{\nu + g} \left[ \frac{u(1 + g_x^2)}{\nu + g} - 2g_x u_x \right] v_{x_2}\]

for \(g \in h_{\Phi}^{2, \alpha}\) and \(u, v \in Y\).

**Proof.** It is clear to see that \((\varphi, v) \rightarrow \varphi v\) is bilinear and continuous from \(h^s\) into \(Y\). Moreover, the mapping

\[g \rightarrow \frac{1}{\nu + g} u\]

are continuous and are infinitely many differentiable from \(h^s\) into \(Y\). By using the definition of the space \(Y\) and Lemma 2.3 we get that for all fixed \(g \in h_{\Phi}^{2, \alpha}\) the operator \(u \rightarrow B(g)\) is bounded linear operator from \(Y\) into \(X\). So, we obtain that

\[B(.) \in C^\infty \left(h_{\Phi}^{2, \alpha}; L(Y, X)\right).\]
Hence, in view of Lemma 2.3 we obtain

\[ B_0 (.) \in C^\infty \left( \mathcal{H}^{2,\alpha}_\Phi, L (Y, E) \right). \]

By using [4, Theorem 2] we obtain the following:

**Theorem 3.1.** Suppose the Condition 3.1 is satisfied and \( \alpha \in (0, 1) \). Then for \( \lambda \in S(\phi) \) and for sufficiently large \(|\lambda|\):

(a) the operator \( u \to \tilde{O} (g) u = \{ (B (g) + \lambda) u, \gamma_0 u, (\eta + g) B_1 (g) \} \) is isomorphism from \( Y \) onto \( X \times \mathcal{H}^{2,\alpha} (A) \times \mathcal{H}^{1,\alpha} (A) \);

(b) for \( \mu > 0 \) the operator \( u \to \{ (B (g) + \lambda) u, \mu \gamma_0 u, (\eta + g) B_1 (g) \} \) is isomorphism from \( Y \) onto \( X \times \mathcal{H}^{2,\alpha} (A) \times \mathcal{H}^{1,\alpha} (A) \);

(c) For \( u \in Y \) there exists a positive constant \( C \), depending only on \( g, \eta, p \) and \( E \) such that the coercive estimate holds:

\[ \| u \|_Y \leq C \left( \| (B (g) + \lambda) u \|_X + \| \gamma_0 u \|_{\mathcal{H}^{2,\alpha}(E)} + \| (\eta + g) B_1 u \|_{\mathcal{H}^{1,\alpha}(E)} \right). \] (3.3)

**Proof.** Indeed, since the domain \( Q \) is a strip, the functions \( g, \eta \) are fixed smooth functions, by virtue of trace theorem in Hölder space [15, § 2] we obtain the assertion.

Consider now, the following BVPs

\[ -\sum_{i,j=1}^{2} a_{ij} (g) \frac{\partial^2 u}{\partial x_i \partial x_j} + a_2 (g) \frac{\partial u}{\partial x_2} + A (g) u = F, \] (3.4)

\[ B_i u = \sum_{j=1}^{2} b_{ji} (g) \gamma_j \frac{\partial u}{\partial x_j} = 0, \quad i = 0, 1, \] (3.5)

\[ -\sum_{i,j=1}^{2} a_{ij} (g) \frac{\partial^2 u}{\partial x_i \partial x_j} + a_2 (g) \frac{\partial u}{\partial x_2} + A (g) u = 0, \] (3.6)

\[ \sum_{j=1}^{2} b_{j0} (g) \gamma_j \frac{\partial u}{\partial x_j} = \psi, \quad \sum_{j=1}^{2} b_{j1} (g) \gamma_1 \frac{\partial u}{\partial x_j} = 0. \] (3.7)

Consider the operators \( \tilde{S} (g) \) and \( \tilde{K} (g) \) generated by problems (3.4) – (3.5) and (3.6) – (3.7), respectively, i.e.

\[ D \left( \tilde{S} (g) \right) = \{ u \in Y, \ B_i u = 0, \ i = 0, 1 \}, \]

\[ \tilde{S} (g) u = -\sum_{i,j=1}^{2} a_{ij} (g) \frac{\partial^2 u}{\partial x_i \partial x_j} + a_2 (g) \frac{\partial u}{\partial x_2} + A (g) u \]

and

\[ D \left( \tilde{K} (g) \right) = \{ u \in Y, \ B u = 0, \ B_1 u = 0 \}, \tilde{K} (g) u = B_0 u \in B_{0p}. \]
From the Theorem 3.5 we obtain that the inverse operators \( \tilde{O}^{-1}(g) \), \( \tilde{S}^{-1}(g) \), \( \tilde{K}^{-1}(g) \) are bounded from \( X \times h^{2,\alpha}(A) \times h^{1,\alpha}(A) \), \( X \), \( h^{2,\alpha}(E) \) into \( Y \), respectively.

Here,

\[
O(g) = \tilde{O}^{-1}(g), \quad S(g) = \tilde{S}^{-1}(g), \quad K(g) = \tilde{K}^{-1}(g).
\]

Assume \( g \in h^{2,\alpha}_{+}(\mathbb{R}; E) \), \( \psi \in D_{A}(\alpha) \), and put \( u = K(g)v \). Then \( u \) is the solution of the BVP (3.6) – (3.7).

**Condition 3.1.** Assume \( A(g) \) is Gateaux differentiable for \( g \in h^{2,\alpha}_{+}, \alpha \in (0,1) \) and the operator \( \partial A(g) \) is uniformly \( R \)-positive in UMD (see e.g. [8] for definitions) Banach algebra \( E \).

**Lemma 3.2.** Suppose the conditions 3.1 and 3.2 are satisfied. Then we have

\[
K(.) \in C^{\infty}(h^{2+\alpha}_{+}; L(E,Y))
\]

and

\[
\partial K(g)[v,\psi] = -S(g)\partial B(g)[v, K(g)\psi]
\]

for \( g \in h^{2,\alpha}_{+}, \psi \in E \) and \( v \in Y \).

**Proof.** It follows from Lemma 3.1 and Theorem 3.1 that the map \( g \rightarrow \tilde{O}(g) \) is isomorphism from \( Y \) onto \( X \times h^{2,\alpha}(A) \times h^{1,\alpha}(A) \) and have continous derivatives of all order with respect to \( g \in h^{2,\alpha}_{+} \), i.e.

\[
\tilde{O} \in C^{\infty}(h^{2,\alpha}_{+}; \text{IsomY}, X \times h^{2,\alpha}(A) \times h^{1,\alpha}(A))
\]

and

\[
\partial \tilde{O}(g)\psi = \{ \partial B(g)[\psi, .], 0, 0 \} \text{ for } \psi \in h^{2,\alpha}(A).
\]

Then by reasoning as the Lemma 3.4 in [6] we obtain the assertion.

### 4. The nonlinear operator for free BVPs

In this section we introduce the basic nonlinear operator and we derive some properties of it.

Moreover, we show that the corresponding evolution problem involving this operator is equivalent to the original problem (1.2). Given \( g \in h^{2,\alpha}_{+} \), we define the following operator

\[
O(g) = B_{0}(g)K(g)g.
\]

From lemmas 3.1 and 3.2 we get that

\[
O \in C^{\infty}(h^{2,\alpha}_{+}, h^{1,\alpha}(E)).
\]

Assume that \( g_{0} \in h^{2,\alpha}_{+} \), and let \( \sigma = [0, T] \). A function \( \sigma \rightarrow B_{10} \) is a classical solution of

\[
\frac{dg}{dt} + O(g) = 0, \ g(0) = g_{0}
\]
iff $g \in C \left( \sigma; h_+^{2,\alpha} \right) \cap C^1 \left( \sigma; h^{2,\alpha} (E) \right)$ and $g$ satisfies \eqref{eq:4.2} pointwise.

**Lemma 4.1.** Suppose the Condition 3.1 is satisfied. Then for $g_0 \in h_+^{2,\alpha}$:

(a) Suppose that $g$ is a classical solution of problem \eqref{eq:4.2} on $\sigma$ and let $v (t, .) = K (g (t, .)) g (t, .)$. Then the pair $(v, g)$ is a classical solution of \eqref{eq:2.5} on $\sigma$, having the additional regularity

$$g \in C \left( \sigma; h_+^{2,\alpha} \right) \cap C^1 \left( \sigma; h^{1,\alpha} \right),$$

$$v (t, .) \in Y, \ t \in \sigma; \quad \text{(4.3)}$$

(b) Suppose that $(v, g)$ is a classical solution of \eqref{eq:2.5} on $\sigma$, having the regularity \eqref{eq:4.3}. Then $g$ is a classical solution of \eqref{eq:4.2} on $\sigma$.

**Proof.** The proof is obtained from Lemma 2.2 and definitions of the spaces $B_{ip}, i = 0, 1 \text{ and } Y$.

For fixed $g \in h_\Phi$ consider the operator

$$v \rightarrow B_0 (g) K (g) v. \quad \text{(4.4)}$$

In view of Theorem 3.1 we obtain that

$$B_0 (g) K (g) \in L \left( h_+^{2,\alpha} (A), h^{1,\alpha} (A) \right). \quad \text{(4.5)}$$

**Lemma 4.2.** Suppose the conditions 3.1 and 3.2 are satisfied. Then $O \in C^\infty \left( h_+^{2,\alpha}, h^{1,\alpha} (A) \right)$ and

$$\partial O (g) \psi = B_0 (g) K (g) \psi + \partial B_0 (g) [v, K (g) \psi] -$$

$$- B_0 (g) S (g) \partial B (g) [v, K (g) \psi]$$

for $g \in h_+^{2,\alpha}$, $\psi \in h^{2,\alpha} (A)$ and $v \in Y$.

**Proof.** The assertion is obtained from Lemma 3.1 and Lemma 3.2.

5. **Linear equation with constant coefficients**

We put

$$R_+^2 = \{ x = (x_1, x_2) \in R^2, \ x_2 > 0 \}.$$

Consider the problem

$$- \sum_{i,j=1}^2 a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + Au + \mu^2 u = 0, \quad \text{(5.1)}$$

$$u (x_1, 0) = \psi (x_1), \ x \in R_+^2, \quad \text{(5.2)}$$

12
where $A = A(g)(x_0, 0)$, $a_{ij} = a_{ij}(g)(x_0, 0)$, and $a_{ij}(g)$ are defined by (2.7) and $\psi \in h^{2,\alpha}(A)$. By applying the Fourier transform to the problem (5.1) – (5.2) with respect to $x_1$ we get

$$
-a_{22} \frac{d^2 \hat{u}}{d\eta^2} + 2a_{12i} \frac{d \hat{u}}{d\eta} + (A + \eta^2 + \mu^2) \hat{u} = 0, \quad (5.3)
$$

$$
\hat{u}(\eta, 0) = \hat{\psi}(\eta), \quad \eta \in \mathbb{R}, \quad y \in \mathbb{R}_+,
$$

where $x_2$ and $\hat{u}(\eta, x_2)$ are denoted by $y$ and $\hat{u}(\eta, y)$, respectively.

Let

$$
p(\xi) = \xi_1^2 + 2a_{12}\xi_1 \xi_2 + a_{22}\xi_2^2 \quad \text{for} \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2.
$$

There exists an $\alpha > 0$ with

$$
p(\xi) \geq \alpha |\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^2. \quad (5.5)
$$

The condition (5.5) implies that

$$
a_{12}^2 - a_{22} \geq \alpha. \quad (5.6)
$$

Moreover, we define

$$
q_0(\mu, \lambda) = -a_{22}\lambda^2 + 2a_{12i}\lambda + \mu^2 + \eta^2 + 1 = 0. \quad (5.7)
$$

**Remark 5.1.** Given $\eta \in \mathbb{R}$ and $z \in \mathbb{C}$, then in view of (5.5) there is exactly one root of the equation (5.7) with positive real part. It is given by

$$
\lambda(\eta, \mu) = ia(\eta) + b(\eta, \mu), \quad (5.8)
$$

where

$$
a(\eta) = -\frac{a_{12}}{a_{22}}, \quad b(\eta, \mu) = \frac{1}{a_{22}} \left[ a_{22}(1 + \mu^2) + (a_{12}^2 - a_{22})\eta^2 \right]^\frac{1}{2}.
$$

We put

$$
\tilde{X} = h^{\alpha}(\mathbb{R}_+^2; \mathcal{E}), \quad \tilde{Y} = h^{2,\alpha}(\mathbb{R}_+^2; \mathcal{E}(A) ; \mathcal{E}),
$$

$$
N_\mu(\eta, y) = \exp \left\{ -\lambda(\eta, \mu) A_{\mathbb{R}_+^2}(\eta) y \right\}.
$$

The main result of this section is the following:

**Theorem 5.1.** Assume the Condition 3.1 is satisfied. Then problem (5.1) – (5.2) has a unique solution $u \in \tilde{Y}$ for $\psi \in h^{2,\alpha}(A)$ and $u$ is represented by

$$
\hat{u}(x_1, x_2) = \left( K_0 + \mu^2 \right) \hat{\psi} = F^{-1} N_\mu(\eta, x_2) F \hat{\psi}(\eta).
$$

Moreover, the estimate holds

$$
\sum_{j=0}^2 |\mu|^{2-j} \left\| \frac{\partial^j u}{\partial x_1^j} \right\|_{\tilde{X}} + \sum_{i,j=1}^2 \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{\tilde{X}} + \|Au\|_{\tilde{X}} \leq C \|\psi\|_{h^{2,\alpha}(A)} \quad (5.9).
$$
For the proof we need some preparation. Here,
\[ A_\mu = A_\mu (\eta) = A + \mu^2 + \eta^2, \quad q_0 (\eta, y) = AN_\mu (\eta, y), \]
\[ q_j (\eta, y) = |\mu|^{2-j} \eta^j N_\mu (\eta, y). \]

We need the following lemmas:

**Lemma 5.1.** Assume the Condition 3.1 is satisfied. Then there exists a unique solution \( \hat{u} (\eta, y) \) of (5.3) – (5.4) expressing as
\[ \hat{u} (\eta, y) = N_\mu (\eta, y) \hat{\psi} (\eta) \]  

Moreover, the following uniformly in \( \eta \) estimate holds
\[ \| A \hat{u} \|_{h^{\alpha}(\mathbb{R}_+; E)} + \sum_{j=0}^{2} |\mu|^{2-j} \| \hat{u}^{(j)} \|_{h^{\alpha}(\mathbb{R}_+; E)} \leq C \| \hat{\psi} \|_{h^{2,\alpha}(A)}. \]  

**Proof.** In view of positivity of operator \( A \) we know that \( -A_{\mu}^\frac{1}{2} (\eta) \) is analytic semigroup in \( E \) (see e.g. [25, § 1.18]). The the equation (5.3) have a solution \( \hat{u} = U_{\eta,\mu} (y) \hat{\psi} (\eta) \) on \((0, \infty)\), where
\[ U_{\eta,\mu} (y) = \exp \left\{ -\lambda (\eta, \mu) A_{\mu}^\frac{1}{2} (\eta) y \right\} \hat{\psi} (\eta) \]
and \( \lambda (\eta, \mu) \) is a root of (5.7) with positive real part. Then from the above expression and the properties of analytic semigroups we get the uniform estimate
\[ \| A \hat{u} \|_{h^{\alpha}(\mathbb{R}_+; E)} + \sum_{j=0}^{2} |\mu|^{2-j} \| \hat{u}^{(j)} \|_{h^{\alpha}(\mathbb{R}_+; E)} \leq C \]
\[ C_1 \sup_{\eta \in [0,1]} \| \eta^{1-\alpha} AU (\eta) \hat{\psi} (\cdot, \eta) \|_{h^{\alpha}(\mathbb{R}_+; E)} \leq C \| \hat{\psi} \|_{h^{2,\alpha}(A)}, \]
where \( U (y) \) is a semigroup generated by \( -A \).

**Lemma 5.2.** Assume the Condition 3.1 is satisfied. Then operator functions \( q_0 (\eta, y) \) and \( q_j (\eta, y) \) are Fourier multipliers in \( h^{\alpha} (\mathbb{R}; E) \) uniformly with respect to \( y \in \mathbb{R}_+ \).

**Proof.** In view of (5.6) and the Remark 5.1 we get that
\[ \sup_{\eta \in \mathbb{R}, y \in \mathbb{R}_+} (1 + |\eta|) \frac{d}{d\eta} |g_0 (\eta, y)| \leq C_1, \]
\[ \sup_{\eta \in \mathbb{R}, y \in \mathbb{R}_+} (1 + |\eta|) \frac{d}{d\eta} |q_j (\eta, y)| \leq C_2, \]
Then by using the multiplier results in [5] we can prove that the operator function \( g_0 (\cdot, y), q_j (\cdot, y) \) are multipliers in \( h^{\alpha} (\mathbb{R}; E) \) uniformly in \( y \in \mathbb{R}_+ \) and \( \mu \in \mathbb{R} \).
Let
\[ \Phi_0 (y) = \| q_0 (., y) \|_{M(h^\alpha (R; E))}, \quad \Phi_j (y) = \| q_j (., y) \|_{M(h^\alpha (R; E))}. \]

**Lemma 5.3.** Assume the Condition 3.1 is satisfied. Then
\[ \Phi_0 (y) \to 0, \quad \Phi_j (y) \to 0 \text{ as } y \to \infty. \]

**Proof.** Indeed, by properties of Fourier multiplier operators from in \( h^\alpha (R; E) \) and the theory of analytic semigroups there exists \( \omega > 0 \) such that we have
\[
\| q_0 (., y) \|_{M(h^\alpha (R; E))} \leq C_1 (| \lambda (\eta, \mu) | y)^{-1} \exp \{ -\omega | \lambda (\eta, \mu) | y \},
\]
\[
\| q_j (., y) \|_{M(h^\alpha (R; E))} \leq C_2 | \mu |^{2-j} (1 + y^2)^{\frac{3}{2}} \exp \{ -\omega | \lambda (\eta, \mu) | y \}. \quad (5.12)
\]
By estimates (5.12) and Remark 5.1 we obtain the assertion.

**Proof of Theorem 5.1.** By Lemma 5.1 the problem (5.1) – (5.2) has a solution
\[ u (x_1, x_2) = \left( \tilde{K}_0 + \mu^2 \right) \psi = F^{-1} N_\mu (\eta, x_2) \hat{\psi} (\eta). \]
By Lemmas 5.2, 5.3 the operator-functions \( q_0 (\eta, y) \) and \( q_j (\eta, y) \) are Fourier multipliers in \( h^\alpha (R; E) \) uniformly with respect to \( y \in R^+. \) Then from the estimate 5.11 we obtain the assertion.

Here \( K_{0\mu} = K_{0\mu} (x_0) \) denotes the inverse of operator \( \tilde{K}_0 (x_0) + \mu^2, \) i.e.
\[ K_{0\mu} (x_0) = \left( \tilde{K}_0 (x_0) + \mu^2 \right)^{-1}. \]

**Result 5.1.** From Theorem 5.1 we obtain that the operator \( u \to K_{0\mu} u \) is bounded from \( h^{2,\alpha} (A) \) into \( \check{Y} \) and the following estimate holds
\[
\sum_{j=0}^{2} | \mu |^{2-j} \left\| \frac{\partial^j K_{0\mu}}{\partial x_1^j} \right\|_{B(h^{2,\alpha} (A), \check{Y})} + \sum_{i,j=1}^{2} \left\| \frac{\partial^2 K_{0\mu}}{\partial x_i \partial x_j} \right\|_{B(h^{2,\alpha} (A), \check{Y})}
\]
\[ + \| A K_{0\mu} \|_{B(h^{2,\alpha} (A), \check{Y})} \leq C. \]

Consider now the BVP
\[
B (x_0) u = \sum_{i,j=1}^{2} -a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + Au + \mu^2 u = 0, \quad x \in R^2_+,
\]
\[
B_0 (x_0) u = \left[ b_1 \frac{\partial}{\partial x_1} u (x) + b_2 \frac{\partial}{\partial x_2} u (x) \right] |_{x_2=0} = \psi (x_1),
\]
where
\[ A = A (g) (x_0, 0), \quad a_{ij} = a_{ij} (g) (x_0, 0), \quad b_i = b_{ij} (g) (x_0, 0) \quad (5.13) \]
and \( a_{ij} (g) \) are defined by (2.7).

Here,
\[ a_1 (\eta, \mu) = ib_1 (x_0) \eta - b_2 (x_0) \lambda (\eta, \mu). \]
6. Regularity properties of abstract elliptic operator with constant coefficients

Consider the operator

\[ u \rightarrow O_1 u = u \rightarrow O_1 (g) u = B_0 (g) K (g) u \]

Here, \( O_{10} \) denotes the constant coefficients version of \( O_1 \) fixed in \((x_0, 0)\) by (5.13), i.e.

\[ u \rightarrow O_{10} u = O_1 (x_0, 0) = B_0 K (g) (x_0, 0) u. \]

Here,

\[ O_{10} u = (O_1 + \mu^2) u, \quad O_{10, u} = (O_{10} + \mu^2) u, \]

\[ K_0 = K (g (x_0, 0)), \quad K_\mu (g) = \left( \hat{K} (g) + \mu^2 \right)^{-1}. \]

We show the following result:

**Theorem 6.1.** Assume the Condition 3.1 is satisfied. Then the operator

\[ u \rightarrow (O_{10} + \mu^2) u \]

is an isomorphism from \( h^{2, \alpha} (A) \) onto \( h^{1, \alpha} (A) \). Moreover, the following uniform estimates hold

\[
\sum_{i=0}^{1} |\mu|^{2-i} \left\| \frac{\partial^i u}{\partial x_1^i} \right\|_{h^{1, \alpha} (A)} + \left\| Au \right\|_{h^{1, \alpha} (A)} \leq C \left\| (O_{10} + \mu^2) u \right\|_{h^{2, \alpha} (A)}, \quad (6.1)
\]

\[
C_1 \left\| u \right\|_{h^{2, \alpha} (A)} \leq \left\| O_{10} + \mu^2 u \right\|_{h^{1, \alpha} (A)} \leq C_2 \left\| u \right\|_{h^{2, \alpha} (A)}
\]

for all \( u \in h^{2, \alpha} (A) \) and \( \mu > 0 \) with sufficiently large \( \mu_0 \).

**Proof.** In view of Lemma 3.1 and Theorem 5.1 it is clear to see that the solution of the problem

\[ (O_{10} + \mu^2) u = \psi \]

has a unique solution \( u \) expressed as

\[ u = F^{-1} a_1 (\eta, \mu) N_\mu (\eta, x_2) F \psi (\eta). \]

By lemmas 5.1-5.3 and by multiplier result in [5] we get that the operator functions \( |\mu|^{2-i} \eta^i K (\eta, x_2) \) and \( AK (\eta, x_2) \) are multipliers in \( h^{\alpha} (\mathbb{R}^4; E) \) uniformly with respect to \( x_2 \in \mathbb{R}^4 \), where

\[ K (\eta, x_2) = a_1 (\eta, \mu) N_\mu (\eta, x_2). \]

Hence, we obtain the estimate

\[
\sum_{i=0}^{1} |\mu|^{2-i} \left\| \frac{\partial^i u}{\partial x_1^i} \right\|_{h^{1, \alpha} (A)} + \left\| A_0 u \right\|_{h^{1, \alpha} (A)} \leq C \left\| (O_{10} + \mu^2) u \right\|_{h^{2, \alpha} (A)}. \]

Moreover, by definitions of the space \( h^{2, \alpha} (A) \) we get (6.1) and the estimate

\[
\left\| O_{10} + \mu^2 u \right\|_{h^{1, \alpha} (A)} \leq C_2 \left\| u \right\|_{h^{2, \alpha} (A)}. \]
Then in view of the above estimate, by reasoning as in the Theorem 5.1 we get the assertion and corresponding estimates.

By reasoning as in Theorem 6.1 we obtain

**Theorem 6.2.** Assume the Condition 3.1 is satisfied. Then the operator $O_{10}$ is positive and $-O_{10}$ is a generator of an analytic semigroup in $h^{1,\alpha}(A)$.

**Proof.** Indeed, for positivity of the operator $O_{10}$ in $h^{1,\alpha}(A)$ we need to show the estimate

$$\left\| (O_{10} + \mu^2)^{-1} \right\|_{B(h^{1,\alpha}(A))} \leq C\mu^{-2},$$

i.e. we have to prove the estimate

$$\|u\|_{h^{2,\alpha}(A)} \leq C\mu \left\| (O_{10} + \mu^2)^{-1} u \right\|_{h^{1,\alpha}(A)}$$

for $u \in h^{2,\alpha}(A)$. By reasoning as the above we get that the function $\mu^2 \eta a_2(\eta, \mu)$ is a multiplier in $h^\alpha(\mathbb{R}; E)$ uniformly with respect to $x_2 \in \mathbb{R}^+$, where

$$a_2(\eta, x_2) = a_1(\eta, \mu) N_\mu(\eta, x_2).$$

So, the operator $O_{10}$ is positive in $h^{1,\alpha}(A)$. Then $-O_{10}$ is a generator of an analytic semigroup in $h^{1,\alpha}(A)$.

Here

$$Q_2 u = O_2 (g) u = \partial B_0 (g) \left[ u, K(g) g \right]$$

for $g \in h^{2,\alpha}(A)$.

Consider the operator $O_{20}$ that is a constant coefficients version of $O_2$ with fixed in $(x_0, 0)$ by (5.15), i.e.

$$O_{20} = O_2 (g) = \partial B_0 (g) \left[ ., K(g) g \right](x_0, 0)$$

for $g \in h^{2,\alpha}(A)$.

We prove the following result:

**Theorem 6.3.** Assume the Conditions 3.1 and 3.2 are satisfied. Then the operator $u \rightarrow (O_{20} + \mu^2) u$ is an isomorphism in $h^{2,\alpha}(A)$ onto $h^{1,\alpha}(A)$.

Moreover, the following estimate holds

$$\sum_{i=0}^{1} |\mu|^{2-i} \left\| \frac{\partial^i u}{\partial x_i^i} \right\|_{h^{1,\alpha}(A)} + \|A_0 u\|_{h^{1,\alpha}(A)} \leq C \left\| (O_{20} + \mu^2) u \right\|_{h^{2,\alpha}(A)} , \quad (6.4)$$

$$C_1 \|u\|_{h^{2,\alpha}(A)} \leq \left\| O_{20} + \mu_{0}^2 u \right\|_{h^{1,\alpha}(A)} \leq C_2 \|u\|_{h^{2,\alpha}(A)}$$

for all $u \in h^{2,\alpha}(A)$, for sufficiently large $\mu_0$ and $\mu > 0$.

**Proof.** By Lemma 3.1 we have

$$O_{20} u = -\frac{\partial v}{\partial x_1} u_{x_1} + \frac{\partial v}{\partial x_2} (A_1 u + A_2 u) ,$$
where

\[ \nu = v(g) = K(g) g, \quad \Lambda_1 u = \frac{(1 + g^2 z_1)}{\nu + g} u, \quad \Lambda_2 u = -\frac{1}{\nu + g} 2g x_1 u x_1, \quad (6.5) \]

\[ O_{20} u = -\frac{\partial v_0}{\partial x_1} u x_1 + \frac{\partial v_0}{\partial x_2} (\Lambda_1 u + \Lambda_2 u), \]

\[ v_0 = v_0(g) = K(g)(x_0, 0) g, \quad \Lambda_1 u = \frac{(1 + g^2 z_1)}{\nu + g} u, \quad \Lambda_2 u = -\frac{1}{\nu + g} 2g x_1 u x_1, \]

By using lemmas 5.1-5.3 we get that the operator function

\[ (\eta^2 + \mu^2 + A_0) [\eta + \Lambda(u) a_1(\eta, \mu)] N_{0\mu}(\eta, x_2) \]

is a multiplier in \( h^{\alpha} (\mathbb{R}; E) \) uniformly with respect to \( x_2 \in \mathbb{R}_+ \), where

\[ A_{0\mu} = A_{0\mu}(\eta) = A + \mu^2 + \eta^2, \quad N_{0\mu}(\eta, y) = A_{0\mu}^{-\frac{1}{2}}(\eta) \exp \left\{ -\lambda(\eta, \mu) A_{0\mu}^{-\frac{1}{2}}(\eta) y \right\}. \]

Then by reasoning as in the Theorem 6.1 we obtain the assertion.

From Theorem 6.3 we obtain

**Result 6.1.** Assume the conditions 3.1 and 3.2 are satisfied. Then the operator \( O_{20} \) is positive and \( -O_{20} \) is a generator of an analytic semigroup in \( h^{1,\alpha} (A) \).

Consider first of all, the BVP

\[ -\sum_{k,j=1}^2 a_{kj} \frac{\partial^2 u}{\partial x_k \partial x_j} + (A + \mu^2) u + \mu^2 u = V(x), \quad (6.7) \]

\[ u(x_1, 0) = 0, \quad x = (x_1, x_2) \in \mathbb{R}_+^2, \quad (6.8) \]

where \( A = A(g)(x_0, 0) \), \( a_{kj} = a_{kj}(g)(x_0, 0) \) and \( a_{kj}(g) \) are defined by (5.15).

Let \( S_0 = S(g)(x_0, 0) \) denote the realization operator in \( \tilde{X} \) generated by (6.7) – (6.8) for \( \mu = 0 \), i.e.

\[ D(S_0) = \tilde{Y}, \quad S_0 u = -\sum_{k,j=1}^2 a_{kj} \frac{\partial^2 u}{\partial x_k \partial x_j} + Au. \]

From [4, Theorem 2] we obtain the following:

**Result 6.2.** Assume the Condition 3.1 is satisfied. Then:

1. problem (6.7) – (6.8) for sufficiently large \( \mu > 0 \) has a unique solution \( u \in \tilde{Y} \) for \( V \in \tilde{X} \);

2. the uniform coercive estimate holds

\[ \sum_{i=0}^1 |\mu|^{2-i} \left\| \frac{\partial^i u}{\partial x_1^i} \right\|_{\tilde{X}} + \sum_{k,j=1}^2 \left\| \frac{\partial^2 u}{\partial x_k \partial x_j} \right\|_{\tilde{X}} + \|Au\|_{\tilde{X}} \leq C \|V\|_{\tilde{X}} ; \quad (6.9) \]
The operator $S_0$ is a positive and $-S_0$ is a generator of an analytic semigroup in $h^{1,\alpha}(A)$.

The estimate (6.9) particularly, implies that $(S_0 + \mu^2)^{-1} \in B \left( \tilde{X}, \tilde{Y} \right)$.

Consider the inhomogenous problem

$$- \sum_{k,j=1}^{2} a_{ij} \frac{\partial^2 u}{\partial x_k \partial x_j} + Au + \mu^2 u = V(x), \quad (6.10)$$

$$\gamma u = u(x_1,0) = \psi(x_1), \quad x = (x_1, x_2) \in \mathbb{R}^2_+.$$  \hspace{1cm} (6.11)

**Theorem 6.4.** Assume the conditions 3.1 and 3.2 are satisfied. Then the operator $u \rightarrow G_0u = \{L_0u, \gamma u\}$ is an isomorphism from $\tilde{Y}$ onto $\tilde{X} \times h^{2,\alpha}(A)$.

**Proof.** From definition of $\tilde{X}, \tilde{Y}, h^{2,\alpha}(A)$, from expresion of $L_0$ and by virtue of trace result in $\tilde{Y}$ [15, Ch.2] we get that

$$\|G_0u\|_{\tilde{X} \times h^{1,\alpha}(A)} = \|L_0u\|_{\tilde{X}} + \|\gamma u\|_{h^{2,\alpha}(A)} \leq C \|u\|_{\tilde{Y}},$$

i.e. the operator $G_0$ is bounded linear from $\tilde{Y}$ into $\tilde{X} \times h^{2,\alpha}(A)$. Hence, in view of Banach theorem it is sufficient to show that the operator $G_0$ is inective and surjective from $\tilde{Y}$ onto $\tilde{X} \times h^{2,\alpha}(A)$. From Theorem 5.1 we obtain that the corresponding homogenous problem $L_0u = 0, \gamma u = 0$ has a zero solution, i.e. the operator $G_0$ is inective. So, it remain to show that this operator is surjective.

By Theorem 5.1 we obtain that problem $L_0u = V, \gamma u = \psi$ has a solution $u_1 \in \tilde{Y}$ for all $\psi \in h^{2,\alpha}(A)$. Moreover, from the Result 6.2 we get that problem $L_0u = V, \gamma u = 0$ has a solution $u_2 \in \tilde{Y}$ for all $V \in \tilde{X}$. Then $u = u_1 + u_2$ is a solution of (6.10) – (6.11) that belongs to $\tilde{Y}$, i.e. the operator $G_0$ is surjective from $\tilde{Y}$ onto $\tilde{X} \times h^{2,\alpha}(A)$.

From Theorems 5.1 and 6.4 we obtain the following

**Result 6.3.** The solution $u$ of the problem (6.10) – (6.11) is expressed as

$$u(x) = S_{1,\mu}V + S_{2,\mu}(\psi - \gamma u_1),$$

where

$$S_{1,\mu}V = r_+ F^{-1} \left(p(\xi) + A + \mu^2\right)^{-1} F\tilde{V}, \quad S_{2,\mu}V = F^{-1} N_{\mu}(\xi_1, x_2) Fv,$$

here $r_+$ is the restriction operator from $\mathbb{R}^2$ into $\mathbb{R}^2_+$ and $\tilde{V} = \tilde{V}(x_1, x_2)$ is an extension of $V(x_1, x_2)$ on $\mathbb{R}^2$, i.e

$$\tilde{V}(x_1, x_2) = \begin{cases} V(x_1, x_2), & \text{if } x_1, x_2 \in \mathbb{R}^2_+ \\ V(x_1, -x_2), & \text{if } x_1, x_2 \in \mathbb{R}^2_+ \end{cases}$$

$N_{\mu}(\xi_1, x_2)$ is operator function defined by (5.8) and $u_1$ is a solution of the equation

$$- \sum_{k,j=1}^{2} a_{ij} \frac{\partial^2 u}{\partial x_k \partial x_j} + Au + \mu^2 u = \tilde{V}(x), \quad x \in \mathbb{R}^2.$$
Let $O_3 = O_3 (g)$ be the operator in (4.6) defined by

$$O_3 u = B_0 (g) S (g) \partial B (g) [u, K (g) g]$$

(6.12)

for $g \in h^{2, \alpha} (A)$ and $u \in Y$.

In view of Lemma 3.1 and Lemma 3.2 we get

$$O_3 u = B_0 (g) S (g) \partial B (g) [u, v], \ v = v_\gamma (x) = (K (g) g) (x),$$

where

$$\partial B (g) [u, v] = \frac{2 \beta}{\nu + g} \left\{ \left( \frac{g x_1 u}{\nu + g} - u x_1 \right) \right\} v_{x_1 x_2} +$$

$$\frac{1}{(\nu + g)^2} \left[ \left( \frac{1}{\beta} + \beta g_{x_1}^2 \right) u - \frac{\beta}{\nu + g} g_{x_1} u x_1 \right] v_{x_2 x_2} + \partial A (g) [u, v] -$$

(6.13)

\[
\begin{align*}
G_1 (g) &= G_1 (g) [u, v], \ G_1 (g) = \frac{2 \beta}{\nu + g} \left\{ \left( \frac{g x_1 u}{\nu + g} - u x_1 \right) \right\} v_{x_1 x_2}, \\
G_2 (g) &= \frac{u}{(\nu + g)^2} \left( \frac{1}{\beta} + \beta g_{x_1}^2 \right) - \frac{\beta}{\nu + g} g_{x_1} u x_1 \right\} v_{x_2 x_2}, \\
G_3 (g) &= \frac{2 \beta}{\nu + g} \partial A (g) [u, v], \\
G_4 (g) &= - \left[ \left( \frac{g_{x_1}^2}{(\nu + g)^2} - \frac{g x_1 u x_1}{2 (\nu + g)} \right) u - \frac{4 g_{x_1} u x_1}{2 (\nu + g)} + \frac{u x_1 x_1}{2} \right] v_{x_2}. 
\end{align*}
\]

Consider the operator $O_{30}$ that is the constant coefficients version of $O_3$ fixed in $(x_0, 0)$ which defined by (5.16), i.e. from the above equality and from (6.13) we get $O_{30k} = B_0 (g) S (g) G_k (g) (x_0, 0)$,

$$O_{30} u = O_{30} (g) u = O_{301} u + O_{302} u + O_{303} u + O_{304} u, \quad (6.15)$$

where

$$O_{30i} = O_{3i} (g (x_0, 0)) u$$

\[
\begin{align*}
w (x) &= w_\gamma (x) = \frac{\beta}{\nu + g} \frac{\partial}{\partial x_2} v_\gamma (x_1, x_2), \ (x_1, x_2) \in \Omega, \ w_0 = w_\gamma (x^0, 0) 
\end{align*}
\]
For $u \in H^{2,\alpha}(A)$ define the operator by
\[
P_1 u(x_1, x_2) = (P_1 (g, x_1^0) u)(x_1, x_2) = w_g(x) u_{x_1 x_1} e^{-x_2}.
\]

For later purposes we need the following technical lemmas:

**Lemma 6.1.** Assume the conditions 3.1 and 3.2 are satisfied. Then:
(a) $P_1 \in B(H^{2,\alpha}(A), Y)$;
(b) There exists a positive constant $C = C(g)$ such that
\[
\|P_1 u - w u_{x_1 x_1}\|_{H^0(U;E)} \leq C r\|u\|_{H^{2,\alpha}(A)}
\]
for all $u \in H^{2,\alpha}(A)$, $r \in (0, 1)$, where
\[
\tilde{U}_r = \mathbb{R}^2_+ \cap \tilde{U}_r(x_1^0, 0),
\]
here, $\tilde{U}_r(x_1^0, 0)$ denotes the two dimensional ball with radius $r$ centered at $(x_1^0, 0)$.

**Proof.** Indeed, from the expression (6.14) in view of the Theorem 5.1 and by virtue of trace theorem in $Y$ we get (a); Then by using the integral mean value theorem and the trace theorem we obtain
\[
\|P_1 u - w u_{x_1 x_1}\|_{H^0(U;E)} \leq \|(w e^{-x_2} - w) u_{x_1 x_1}\|_{H^0(U;E)} \leq C r\|u\|_{H^{2,\alpha}(A)}.
\]

Let
\[
\partial A = \partial A(x) = \partial A(g)(x), \quad A_0 = \partial A(g)(x_{10}, 0).
\]

**Lemma 6.2.** Assume the conditions 3.1 and 3.2 are satisfied. Then the operator $u \to (O_{304} + \mu^2)u$ is an isomorphism from $H^{2,\alpha}(A)$ onto $H^{1,\alpha}(A)$. Moreover, the following estimate holds
\[
\sum_{i=0}^{1} |\mu^{2-i} \left\| \frac{\partial^i u}{\partial x_1^i} \right\|_{H^{1,\alpha}(A)} + \|A_0 u\|_{H^{1,\alpha}(A)} \leq C\|(O_{304} + \mu^2) u\|_{H^{2,\alpha}(A)},
\]
\[
C_1\|u\|_{H^{2,\alpha}(A)} \leq \|(O_{304} + \mu_0^2) u\|_{H^{1,\alpha}(A)} \leq C_2\|u\|_{H^{2,\alpha}(A)}
\]
for all $u \in H^{2,\alpha}(A)$ and for sufficiently large $\mu_0, \mu > 0$.

**Proof.** From the expression (6.14) and Lemma 6.1 by reasoning as in [9, lemma 5.3] we get that the operator $u \to (P_1 + \mu^2) u$ is an isomorphism from $Y$ onto $X$. The Theorem 6.4 implies that the operator $u \to [S_{x_0}(g) + \mu^2] G_{304} u$ is an isomorphism from $\tilde{Y}$ onto $\tilde{Y}$ for sufficiently large $\mu > 0$. Then in view of trace theorem in $\tilde{Y}$ we get that the operator $u \to [O_{304} + \mu^2] u$ is an isomorphism from $\tilde{Y}$ onto $H^{1,\alpha}(A)$. Moreover, in view of Result 6.3 by reasoning as in the Theorem 6.1 we obtain the estimates (6.16) for all $u \in H^{2,\alpha}(A)$ and for $\mu > 0$ with sufficiently large $\mu_0$.
Lemma 6.3. Assume the conditions 3.1 and 3.2 are satisfied. Then the operator \( u \rightarrow (O_{303} + \mu^2) u \) is an isomorphism from \( h^{2,\alpha}(A) \) onto \( h^{1,\alpha}(A) \). Moreover, the following estimate holds

\[
\sum_{i=0}^{1} |\mu|^{2-i} \left\| \frac{\partial^i u}{\partial x^i} \right\|_{h^{1,\alpha}(A)} + \| A_0 u \|_{h^{1,\alpha}(A)} \leq C \left\| (O_{303} + \mu^2) u \right\|_{h^{2,\alpha}(A)},
\]

\[
C_1 \| u \|_{h^{2,\alpha}(A)} \leq \| O_{303} + \mu^2 u \|_{h^{1,\alpha}(A)} \leq C_2 \| u \|_{h^{2,\alpha}(A)}
\]

for all \( u \in h^{2,\alpha}(A) \) and for sufficiently large \( \mu_0, \mu > 0 \).

Proof. From the expression (6.14) by properties of positive operators we get that the map \( u \rightarrow (G_{30} + \mu^2) u \) is an isomorphism from \( \tilde{Y} \) onto \( X \). The Theorem 6.4 implies that the operator \( u \rightarrow [S_{x_0} (g) + \mu^2] G_{30} u \) is an isomorphism from \( \tilde{Y} \) onto \( \tilde{Y} \) for sufficiently large \( \mu > 0 \). Then in view of trace theorem in \( \tilde{Y} \) we get that the operator \( u \rightarrow [O_{303} + \mu^2] u \) is an isomorphism from \( \tilde{Y} \) onto \( h^{1,\alpha}(A) \). Moreover, in view of Result 6.3 by reasoning as in the Theorem 6.1 we obtain the estimates (6.17) for all \( u \in h^{2,\alpha}(A) \) and for \( \mu > 0 \) with sufficiently large \( \mu_0 \).

In a similar way as Lemma 6.1 and by reasoning as in [9, lemma 5.4] we obtain

Lemma 6.4. Assume the conditions 3.1 and 3.2 are satisfied. Then the operator \( u \rightarrow (O_{30k} + \mu^2) u \) is an isomorphism from \( h^{2,\alpha}(A) \) onto \( h^{1,\alpha}(A) \). Moreover, the following estimate holds

\[
\sum_{i=0}^{1} |\mu|^{2-i} \left\| \frac{\partial^i u}{\partial x^i} \right\|_{h^{1,\alpha}(A)} + \| A_0 u \|_{h^{1,\alpha}(A)} \leq C \left\| (O_{30k} + \mu^2) u \right\|_{h^{2,\alpha}(A)},
\]

\[
C_1 \| u \|_{h^{2,\alpha}(A)} \leq \| O_{30k} + \mu^2 u \|_{h^{1,\alpha}(A)} \leq C_2 \| u \|_{h^{2,\alpha}(A)}, \ k = 1, 2,
\]

for all \( u \in h^{2,\alpha}(A) \) and \( \mu > 0 \).

Theorem 6.5. Assume the conditions 3.1 and 3.2 are satisfied. Then the operator \( u \rightarrow (O_{30} + \mu^2) u \) is an isomorphism from \( h^{2,\alpha}(A) \) onto \( h^{1,\alpha}(A) \). Moreover, the following estimate holds

\[
\sum_{i=0}^{1} |\mu|^{2-i} \left\| \frac{\partial^i u}{\partial x^i} \right\|_{h^{1,\alpha}(A)} + \| A_0 u \|_{h^{1,\alpha}(A)} \leq C \left\| (O_{30} + \mu^2) u \right\|_{h^{2,\alpha}(A)},
\]

\[
C_1 \| u \|_{h^{2,\alpha}(A)} \leq \| O_{30} + \mu^2 u \|_{h^{1,\alpha}(A)} \leq C_2 \| u \|_{h^{2,\alpha}(A)}
\]

for all \( u \in h^{2,\alpha}(A) \) and \( \mu > 0 \).

Proof. From the expressions (6.13) – (6.14) and from lemmas 6.2-6.4 we get that the operator \( u \rightarrow K_{1x_0} (g) u \) is an isomorphism from \( \tilde{Y} \) onto \( X \). The Theorem 6.4 implies that the operator \( u \rightarrow [S_{x_0} (g) + \mu^2] K_{1x_0} (g) u \) is an isomorphism from \( \tilde{Y} \) onto \( \tilde{Y} \) for sufficiently large \( \mu > 0 \). Then in view of trace
theorem in $\tilde{Y}$ we get that the operator $u \rightarrow [O_{30} + \mu^2] u$ is an isomorphism from $\tilde{Y}$ onto $h^{1,\alpha}(A)$ for sufficiently large $\mu_0$ and $\mu > 0$.

From Theorem 6.5 we obtain

**Result 6.4.** Assume the Conditions 3.1 and 3.2 are satisfied. Then the operator $O_{30}$ is positive and $-O_{30}$ is a generator of an analytic semigroup in $h^{1,\alpha}(A)$.

From Theorems 6.1, 6.3 and 6.5 we obtain

**Result 6.5.** Assume the Conditions 3.1 and 3.2 are satisfied. Then

$$O_{k0} + \mu_0 \in H \left(h^{2,\alpha}(A), h^{1,\alpha}(A)\right), \quad k = 1, 2, 3$$

for sufficiently large $\mu_0 > 0$.

In view of theorems 6.3, 6.5 and by lemmas 2.3, 3.1 and Theorem 3.1 we obtain

**Result 6.6.** Assume the Conditions 3.1 and 3.2 are satisfied. The following estimate holds

$$\|O_{20}(g) + O_{30}(g) + \mu_0\|_{H \left(h^{2,\alpha}(A), h^{1,\alpha}(A)\right)} \leq C \|g\|_{h^{2,\alpha}(A)}$$

for all $g \in B_0 \subset h^{2,\alpha}(A)$.

Let

$$O_0 = O_{01} + O_{02} + O_{03}.$$

Now, we will show that

$$O_0 \in H \left(h^{2,\alpha}(A), h^{1,\alpha}(A)\right).$$

**Theorem 6.6.** Assume the Conditions 3.1 and 3.2 are satisfied. Suppose

$$\|u_0\|_{D_A(\alpha)} \leq \frac{\alpha(g)(x)}{a_{22}(g)(x,0)} = \frac{\alpha_0}{a_{22}}.$$ 

Then the operator $u \rightarrow (Q_0 + \mu^2) u$ is an isomorphism from $h^{2,\alpha}(A)$ onto $h^{1,\alpha}(A)$. Moreover, the following estimate holds

$$\sum_{i=0}^1 |\mu|^{2-i} \left\|\partial_i^j u\right\|_{h^{1,\alpha}(A)} + \|A_0 u\|_{h^{1,\alpha}(A)} \leq C \left\|(O_0 + \mu^2) u\right\|_{h^{2,\alpha}(A)},$$

$$C_1 \|u\|_{h^{2,\alpha}(A)} \leq \|O_0 + \mu_0^2 u\|_{h^{1,\alpha}(A)} \leq C_2 \|u\|_{h^{2,\alpha}(A)} \quad (6.20)$$

for all $u \in h^{2,\alpha}(A)$ and for sufficiently large $\mu_0, \mu > 0$. Particularly, the operator $O_0$ is positive and $-O_0$ is a generator of an analytic semigroup in $h^{1,\alpha}(A)$.

**Proof.** Indeed, by using Theorems 6.1, 6.3, 6.5, the Results 6.3, 6.5, by reasoning as in lemma 5.4, Theorem 5.6, Corollary 5.7 in [2] and the perturbation results for space $H \left(h^{2,\alpha}(A), h^{1,\alpha}(A)\right)$ we obtain the assertion.
Let
\[ W_t = \left\{ g \in h_+^{2,\alpha}, \inf_{x \in (-\infty, \infty)} \left[ \frac{t}{\nu + g \partial_x^2} v_g (x, 0) + k_g (x) > 0 \right] \right\}, \]

\[ k_g (x) = \frac{\alpha (g) (x)}{a_{22} (g) (x, 0)}, \quad t \in [0, 1]. \]

**Remark 6.1.** Suppose that \( g \in W_1 \). Then
\[ w_\pi = w_g (x, 0) < \frac{\alpha_0}{a_{22}} \text{ for } x \in (-\infty, \infty). \]

From Theorem 5.1 we know that there is a constant \( M > 0 \) such that
\[ \| K (g) \|_{B (h^{2,\alpha} (\Gamma; H), h^{2,\beta} (\Omega; H))} \leq M, \quad \beta \in (0, \alpha) \]
for all \( g \in h^{2,\beta} \) satisfying \( \| g \|_{h^{2,\beta}} \leq \chi \). Now define
\[ k = \frac{(\eta - \chi)^3 \chi}{2M \left[ 1 + (\eta + \chi)^2 + \chi^2 \right] (1 + \chi)^2}. \]

**7. Coercive estimates for the linearization**

It is clear that \( O_0 = O_{10} + O_{20} + O_{30} \) may be viewed as a principal part of the linearization of \( \partial O (g) \) with coefficients fixed in \( (x^0, 0) \). Our main goal in this section is to prove that the operator \( O \) belongs to the class
\[ H \left( h^{2,\alpha} (A), h^{1,\alpha} (A) \right). \]

We use the estimates of local operators in the preceding section to derive coercive estimates for the linearization operator \( \partial O (g) \). Let
\[ \partial O (g) = O_1 (g) + O_2 (g) + O_3 (g), \]
where
\[ O_1 (g) = B_0 (g) K (g), \quad O_2 (g) = \partial B_0 (g) [., K (g) g], \]
\[ O_3 (g) = -B_0 (g) S (g) \partial B (g) [., K (g) g]. \]

Given \( g \in h_+^{2,\alpha} \) and \( t \in [0, 1], \) set
\[ \partial O^t (g) = O_1 (g) + t\partial B_0 (g) [., K (g) g] - tB_0 (g) S (g) \partial B (g) [., K (g) g] \]
and observe that
\[ \partial O^1 (g) = \partial O (g). \]

Let \( \delta > 0 \) be given and let \( \{ V_j, \varphi_j, j \in \mathbb{N} \} \) denote \( \delta \)-localization sequence for \( S = \mathbb{R} \times \left( -\frac{3}{2}, \frac{3}{2} \right) \), the covering \( \{ V_j, j \in \mathbb{N} \} \) has finite multiplicity, \( \text{diam } U_j < \delta, \)
and \( \{ V_j, \varphi_j, j \in \mathbb{N} \} \) is a partition of unity on \( S \) with \( \sum_{j \in \mathbb{N}} \varphi_j(x) = 1 \). Moreover, we fix \( x_{1j} \in \mathbb{R} \) such that \( (x_{1j}, 0) \in V_j, j \in \mathbb{N} \).

Here, we will prove the following result.

**Theorem 7.1.** Assume the conditions 3.1 and 3.2 are satisfied. Suppose that \( W_0 \subset W_1 \) is compact, \( \beta \in (0, \alpha) \) and that \( k > 0 \). Then there exist \( \delta \in (0, 1] \), a \( \delta \)-localization sequence \( \{ V_j, \varphi_j, j \in \mathbb{N} \} \), \( \beta \in (0, \alpha) \) and a positive constant \( C = C(W_0, M, \delta) \) such that

\[
\| \varphi_j \partial O \rho (g) - O \pi (g, x_{1j}) \varphi_j \|_{h, 1, \alpha(A)} \leq k \| \varphi_j v \|_{h, 2, \alpha(A)} + C \| v \|_{h, 2, \alpha(A)}
\]

for all \( v \in h^{2, \alpha}(A), j \in \mathbb{N}, t \in [0, 1] \) and \( g \in W_0 \).

For proving Theorem 7.1 we need some preparation. Let

\[
A_j = \partial A (g) (x_{1j}, 0).
\]

Consider the following equation

\[
(O (g) + \mu) u = f \tag{7.2}
\]

for

\[
f = [O (g) + \mu] u \in B_{3p}.
\]

From (7.2) for \( u_j = u \varphi_j, u \in D_A (\alpha) \) we get

\[
(O (g) + \mu) u_j = \sum_{k=1}^{3} O_k (g) u_j + \mu u_j = \sum_{k=1}^{3} F_{kj} + f \varphi_j, \tag{7.3}
\]

where

\[
F_{1j} = \gamma \left[ b_1 \frac{\partial \varphi_j}{\partial x_1} + b_2 \frac{\partial \varphi_j}{\partial x_2} \right] U_1 (g), U_1 (g) = K (g) u,
\]

\[
F_{2j} = \gamma \left[ - \frac{\partial v \partial \varphi_j}{\partial x_1} + 2 g x_2 \frac{\partial v}{\partial x_1} \partial \varphi_j \right] u, \quad v = v (g) = K (g) g,
\]

\[
F_{3j} = \gamma \left[ b_1 \frac{\partial \varphi_j}{\partial x_1} + b_2 \frac{\partial \varphi_j}{\partial x_2} \right] U_3 (g), U_3 (g) = S (g) \partial B (g) [u, K (g) g].
\]

By freezing in (7.3) coefficients at points \((x_{1j}, 0)\) we have localized equations

\[
(O (x_{1j}) + \mu) u_j = F_j, \tag{7.5}
\]

where

\[
F_j = \sum_{k=1}^{3} F_{kj} + f \varphi_j + \sum_{k=1}^{3} (O_{kj} - O_j) u_j,
\]

\[
(O_{1j} - O_1) u_j = B_0 (g) [K_j - K (g)] u_j + [B_0 - B_0 (g)] K_j u_j,
\]

\[
(O_{2j} - O_2) u_j = - \left( \frac{\partial v_j}{\partial x_1} - \frac{\partial v}{\partial x_1} \right) \frac{\partial}{\partial x_1} u_j + \left( \frac{\partial v_j}{\partial x_2} - \frac{\partial v}{\partial x_2} \right) \left( \Lambda_1 + \Lambda_2 \right) u_j. \tag{7.6}
\]
\[(O_{3j} - O_3) u_j = \sum_{i=1}^{4} (O_{3ij} - O_{3i}) u_j\]

here

\[O_{3ij} = B_0 (g) S (g) G_i (g) (x_{1j}, 0),\]

and \(O_{3ij}, G_i\) are defined as in (6.14); moreover,

\[O (x_{1j}) = O (x_{1j}, 0) = \sum_{k=1}^{3} O_k (g) (x_{1j}), \quad O_k (x_{1j}) = O_k (g) (x_{1j}, 0),\]

\[B_{0j} = B_0 (g) (x_{1j}, 0), \quad K_j = K (g) (x_{1j}, 0), \quad v_j = v_j (g) = K (g) (x_{1j}, 0) g,\]

\[S_j = S (g) \partial B (g) [. , K (g) g] (x_{1j}, 0),\]

\[O_k (x_{1j})\] are local operators fixed at points \((x_{1j}, 0)\) defined by equalities (6.3), (6.5), (6.14), respectively. From expressions of \(F_j\), \(K (g) (x_{1j}, 0)\) by using (6.3), (6.5), (6.14), (6.15) we get that \(F_j \in h^{1, \alpha} (A)\).

For proving the Theorem 7.1 we need the following lemmas:

**Lemma 7.1.** The operator \(u \to (O (x_{1j}) + \mu^2) u\) is an isomorphism from \(h^{2, \alpha} (A)\) onto \(h^{1, \alpha} (A)\). Moreover, the following estimate holds

\[
\sum_{i=0}^{1} |\mu|^{2-i} \left\| \frac{\partial^i u}{\partial x_i} \right\|_{h^{1, \alpha}(A)} + \|A_j u\|_{h^{1, \alpha}(A)} \leq C \left\| (O (x_{1j}) + \mu^2) u \right\|_{h^{1, \alpha}(A)} \tag{7.7}
\]

for all \(u \in h^{2, \alpha} (A)\) and for sufficiently large \(\mu > 0\).

**Proof.** Consider the equation

\[(O (x_{1j}) + \mu^2) u = f.\]

Then, by virtue of Theorem 6.6 we obtain that the operator \(u \to (O (x_{1j}) + \mu^2) u\) is an isomorphism from \(h^{2, \alpha} (A)\) onto \(h^{1, \alpha} (A)\) and the estimate (7.7) holds.

**Lemma 7.2.** There is a positive \(\varepsilon \in (0, 1)\) such that the following local estimate holds

\[
\| (O_{1j} - O_1) u_j \|_{h^{1, \alpha}(A)} \leq \varepsilon \| K_j u_j \|_{h^{1, \alpha}(A)} \cdot \tag{7.8}
\]

**Proof.** From (7.6) we get

\[
\| (O_{1j} - O_1) u_j \|_{h^{1, \alpha}(A)} \leq \| B_0 (g) [K_j - K (g)] u_j \|_{h^{1, \alpha}(A)} + \| [B_{0j} - B_0 (g)] K_j u_j \|_{h^{1, \alpha}(A)}.
\]

Then by taking into account of expressions \(B_0 (g), B_{0j}, K (g), K_j\) and in view of smoothness of coefficients, choosing \(\delta\) sufficiently small we have

\[
\| (O_{1j} - O_1) u_j \|_{h^{1, \alpha}(A)} \leq \| B_0 (g) [K_j - K (g)] u_j \|_{h^{1, \alpha}(A)} +
\]

26
\[ \left\| (b_1 - b_{1j}) \frac{\partial}{\partial x_1} K_j u_j \right\|_{h^{1,\alpha}(A)} + \left\| (b_2 - b_{2j}) \frac{\partial}{\partial x_2} K_j u_j \right\|_{h^{1,\alpha}(A)} \leq \varepsilon \left( \left\| \frac{\partial}{\partial x_1} K_j u_j \right\|_{h^{1,\alpha}(A)} + \left\| \frac{\partial}{\partial x_2} K_j u_j \right\|_{h^{1,\alpha}(A)} \right) \leq \varepsilon \left\| K_j u_j \right\|_{h^{2,\alpha}(A)}. \]

**Lemma 7.3.** There is a positive \( \varepsilon \in (0,1) \) such that the following local estimate holds
\[
\left\| (O_{2j} - O_2) u_j \right\|_{h^{1,\alpha}(A)} \leq \varepsilon \left\| u_j \right\|_{h^{1,\alpha}(A)}. \tag{7.9}\]

**Proof.** From the (7.6) we get
\[
\left\| (O_{2j} - O_2) u_j \right\|_{h^{1,\alpha}(A)} \leq \left\| \left( \frac{\partial v_j}{\partial x_1} - \frac{\partial v}{\partial x_1} \right) \frac{\partial}{\partial x_1} u_j \right\|_{h^{1,\alpha}(A)} + \left\| \left( \frac{\partial v}{\partial x_1} - \frac{\partial v}{\partial x_2} \right) (\Lambda_1 + \Lambda_2) u_j \right\|_{h^{1,\alpha}(A)}.
\]

Then, by using the smoothness of coefficients and choosing \( \delta \) sufficiently small, we get the estimate (7.9).

**Lemma 7.3.** There is a positive \( \varepsilon \in (0,1) \) such that the following local estimate holds
\[
\left\| (O_{3j} - O_3) u_j \right\|_{h^{1,\alpha}(A)} \leq \varepsilon \left\| u_j \right\|_{h^{1,\alpha}(A)}. \tag{7.10}\]

**Proof.** From expressions (6.13) – (6.14) we get
\[
\left\| (O_{3j} - O_3) u_j \right\|_{h^{1,\alpha}(A)} \leq \sum_{i=1}^{4} \left\| (O_{3ij} - O_{3i}) u_j \right\|_{h^{1,\alpha}(A)}. \tag{7.11}\]

Moreover, from expressions \( O_{3i} \) and \( G_1 \) in (6.14) by boundedness of functions \( g, \nu, \beta \) we have
\[
\left\| (O_{3ij} - O_{3i}) u_j \right\|_{h^{1,\alpha}(A)} \leq C \left\| (B_0 (g) S (g) (x_{1j}, 0) - B_0 (g) S (g) (x)) \left( u_j + \frac{\partial u_j}{\partial x_1} \right) \right\|_{h^{1,\alpha}(A)} \leq \left\| (B_0 (g) S (g) (x_{1j}, 0) - B_0 (g) S (g) (x)) u_j \right\|_{h^{1,\alpha}(A)} + \left\| B_0 (g) S (g) (x) [u_j (x) - u_j (x_{1j}, 0)] \right\|_{h^{1,\alpha}(A)} + \left\| B_0 (g) S (g) (x_{1j}, 0) - B_0 (g) S (g) (x) \frac{\partial u_j}{\partial x_1} \right\|_{h^{1,\alpha}(A)} + \left\| B_0 (g) S (g) (x) \left[ \frac{\partial u_j}{\partial x_1} - \frac{\partial u_j}{\partial x_1} (x_{1j}, 0) \right] \right\|_{h^{1,\alpha}(A)}.
\]
Then by boundedness of operator $B_0(\gamma)S(\gamma)$, smoothness of coefficients, choosing $\delta$ sufficiently we obtain from the above
\[
\| (O_{31} - O_{31}) u_j \|_{h^{1,\alpha}(A)} \leq \varepsilon \| u_j \|_{h^{2,\alpha}(A)}. \tag{7.12}
\]

In a similar way, from expressions $O_{32}$ and $G_2$ in (6.14) we have
\[
\| (O_{32} - O_{32}) u_j \|_{h^{1,\alpha}(A)} \leq \varepsilon \| u_j \|_{h^{2,\alpha}(A)}. \tag{7.13}
\]

In view of the condition on the operator $\partial A(\gamma)$, boundedness of operator $B_0(\gamma)S(\gamma)$, smoothness of coefficients, choosing $\delta$ sufficiently we obtain
\[
\| (O_{33} - O_{33}) u_j \|_{D_A(\alpha)} \leq \varepsilon \| u_j \|_{D_A(\alpha)}. \tag{7.14}
\]

Finally, from expressions $O_{34}$ an $G_4$ in (6.14) by boundedness of functions $g, \nu, \beta$ we have
\[
\| (O_{34} - O_{34}) u_j \|_{h^{1,\alpha}(A)} \leq C \left\{ \begin{array}{l}
\left\| (B_0(\gamma)S(\gamma)(x_{1j}, 0) - B_0(\gamma)S(\gamma)(x)) u_j \left( u_j + \frac{\partial u_j}{\partial x_1} + \frac{\partial^2 u_j}{\partial x_1^2} \right) \right\|_{h^{1,\alpha}(A)} \\
\left\| [B_0(\gamma)S(\gamma)(x_{1j}, 0) - B_0(\gamma)S(\gamma)(x)] u_j \right\|_{h^{1,\alpha}(A)} \\
\left\| [B_0(\gamma)S(\gamma)(x_{1j}, 0) - B_0(\gamma)S(\gamma)(x)] \left( \frac{\partial^2 u_j}{\partial x_1^2} \right) \right\|_{h^{1,\alpha}(A)} + \\
\left\| [B_0(\gamma)S(\gamma)(x_{10}, 0) - B_0(\gamma)S(\gamma)(x)] \left( \frac{\partial^2 u_j}{\partial x_1^2} \right) \right\|_{D_A(\alpha)} + \\
\left\| [B_0(\gamma)S(\gamma)(x_{10}, 0) - B_0(\gamma)S(\gamma)(x)] \left( \frac{\partial^2 u_j}{\partial x_1^2} \right) \right\|_{B_1p} \leq \varepsilon \| u_j \|_{h^{1,\alpha}(A)}.
\end{array} \right.
\]

Then the estimate (7.10) is obtained from (7.12) - (7.15).

Now, we can prove the Theorem 7.1.

**Proof of Theorem 7.1.** By virtue of Theorem 6.6 the following estimate holds
\[
\sum_{i=0}^{1} |\mu|^{2-i} \left\| \frac{\partial^i u_j}{\partial x_1^i} \right\|_{h^{1,\alpha}(A)} + \| A_0 u_j \|_{h^{1,\alpha}(A)} \leq C \| F_j \|_{h^{1,\alpha}(A)}, \tag{7.16}
\]
for all solution $u_j \in h^{2,\alpha}(A)$ of the equation (7.5).
Whence, using smoothness of coefficients of equations (7.3) – (7.6), in view of Lemmas 7.1-7.3 for \( \mu \) with sufficiently large \( \text{Re} \mu > 0 \) we get
\[
\| F_j \|_{h^{1,\alpha}(A)} \leq \varepsilon \left[ \sum_{i=0}^{1} \mu^{2-i} \left\| \frac{\partial^i u_j}{\partial x_1^i} \right\|_{h^{1,\alpha}(A)} + \| A_j u_j \|_{h^{1,\alpha}(A)} + \| f \varphi_j \|_{h^{1,\alpha}(A)} \right].
\]

(7.17)

Moreover, by applying the microlocal analysis reasoning as in theorem 1, 2 in [21] and in theorem 6.2 and as in [9, Lemma 6.5-6.7] we obtain the same estimates for corresponding commutators operators. Then from this and from (7.16), (7.17) we get the assertion.

From the Theorem 7.1 by microlocal analysis reasoning as in theorem 6.2 and corollary 6.3 in [9] we obtain

**Corollary 7.1.** Assume the conditions 3.1, 3.2 are satisfied and \( K \subset W \) is compact. Then there exist positive constants \( \mu_0 \) and \( C = C(K) \) such that
\[
\| v \|_{h^{2,\alpha}(A)} + |\mu| \| v \|_{h^{1,\alpha}(A)} \leq C \| (\mu + \partial O_t (g)) v \|_{h^{1,\alpha}(A)}
\]
for all \( v \in h^{2,\alpha} (A), g \in K, t \in [0, 1] \) and \( \mu \in \{ z : \text{Re} z > \mu_0 \} \).

**Corollary 7.2.** Let \( g \in W \) and \( t \in [0, 1] \) be given. Then
\[
\partial O_t (g) \in H (h^{2,\alpha}, h^{1,\alpha}).
\]

Now, by using Theorem 7.1 we can prove Theorem 1.

**Proof of Theorem 1:** Let \( f_0 \in V_\nu \) be given and set \( g_0 = f_0 - \nu \). Observe that \( g_0 \in W_{\alpha,1} = W \). It follows from Lemma 4.1 that we only have to prove that there exist \( t_+ > 0 \) and a unique maximal classical solution of (1.1) – (1.2) on \([0, t_+])\) satisfying
\[
\lim_{t \to t_+} \| g(t, .) \|_{h^{2,\alpha}} = \infty, \lim_{t \to t_+} \inf_{v \in \partial W} \| f(t, .) - v \|_{h^{2,\alpha}} = 0 \quad (7.18)
\]
if \( t_+ < \infty \) and \( g \in C_b ([0, t_+); W) \).

By reasoning as [9, Lemma 5.10] it follows that \( W \) is an open subset of \( h^{2,\alpha}_+ \). Hence, thanks to Lemma 4.2 and Corollary 7.2, we know that \( O \in H (W, h^{1,\alpha}) \) and that
\[
\partial O (g) \in H (h^{2,\alpha}, h^{1,\alpha}), g \in W. \quad (7.19)
\]

Let now \( \beta \in (0, \alpha) \) be fixed and observe that \( W \subset W_{\beta,1} \). Thus the very same arguments as above also ensure that
\[
\partial O (g) \in H (h^{2,\beta}, h^{1,\beta}), g \in W.
\]

It is not difficult to see that the maximal \( h^{1,\alpha} \)-realization of \( \partial O (g) \) belongs to \( B (h^{2,\beta}, h^{1,\beta}) \) for \( g \in W \), is just the linear operator in (7.19). Note that
\[
(h^{1,\beta}, h^{2,\beta})_{\alpha-\beta,\infty} = h^{1,\alpha}
\]
where $(\cdot, \cdot)_{\alpha-\beta, \infty}$ denotes the real interpolation. Consequently, invoking Theorem 2.3 in [24], we find that

$$\partial O(g) \in M_{1}\left(h^{2-\alpha}, h^{1-\alpha}\right), g \in W, \quad (7.20)$$

where $M(E_1, E_2)$ denotes the class of all operators in $s$ having the property of maximal regularity in the sense of Da Prato and Grisvard [6]. The existence of a unique maximal classical solution of $(E)g$ and the property of a smooth semiflow on $W$ can now be obtained along the lines of the proofs of Proposition 3.5 and Theorem 3.2 in [24].

Finally suppose that $t_+ < \infty$, $g \in C_0([0, t_+); W)$ and that (7.18) is not true. Then $g_1 = \lim_{t \to t_+} g(t)$ exists in $W$. Hence taking $g_1$ as initial value in (4.2) one easily constructs a solution $\bar{g}$ of (4.2) for initial date $g_1$ extending $g$. This contradicts the maximality of $g$.

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