THE EXISTENCE OF TWO NON-CONTRACTIBLE CLOSED GEODESICS ON EVERY BUMPY FINSLER COMPACT SPACE FORM

HUI LIU 1
School of Mathematics and Statistics, Wuhan University
Wuhan 430072, Hubei, China

YIMING LONG 2
Chern Institute of Mathematics and LPMC, Nankai University
Tianjin 300071, China
and
Beijing Advanced Innovation Center for Imaging Technology
Capital Normal University, Beijing 100048, China

YUMING XIAO 3
School of Mathematics, Sichuan University
Chengdu 610064, China

(Communicated by Kuo-Chang Chen)

Abstract. Let $M = S^n/\Gamma$ and $h$ be a nontrivial element of finite order $p$ in $\pi_1(M)$, where the integer $n \geq 2$, $\Gamma$ is a finite group which acts freely and isometrically on the $n$-sphere and therefore $M$ is diffeomorphic to a compact space form. In this paper, we establish first the resonance identity for non-contractible homologically visible minimal closed geodesics of the class $[h]$ on every Finsler compact space form $(M, F)$ when there exist only finitely many distinct non-contractible closed geodesics of the class $[h]$ on $(M, F)$. Then as an application of this resonance identity, we prove the existence of at least two distinct non-contractible closed geodesics of the class $[h]$ on $(M, F)$ with a bumpy Finsler metric, which improves a result of Taimanov in [39] by removing some additional conditions. Also our results extend the resonance identity and multiplicity results on $\mathbb{R}P^n$ in [25] to general compact space forms.

1. Introduction. Let $M = S^n/\Gamma$ and $h$ be a nontrivial element of finite order $p$ in $\pi_1(M)$, where the integer $n \geq 2$, $\Gamma$ is a finite group which acts freely and isometrically on the $n$-sphere and therefore $M$ is diffeomorphic to a compact space form which is typically a non-simply connected manifold. In particular, if $\Gamma = \mathbb{Z}_2$, then $S^n/\Gamma$ is the $n$-dimensional real projective space $\mathbb{R}P^n$. Motivated by the works

2010 Mathematics Subject Classification. 53C22, 58E05, 58E10.

Key words and phrases. Non-contractible closed geodesics, resonance identity, compact space forms, morse theory, index iteration theory, systems of irrational numbers.

1 Partially supported by NSFC (Nos. 11401555, 11771341), Anhui Provincial Natural Science Foundation (No. 1608085QA01).

2 Partially supported by NSFC (Nos. 11131004, 11671215 and 11790271), MCME and LPMC of MOE of China, Nankai University and BAICIT of Capital Normal University.

3 Corresponding author: Yuming Xiao. Supported by the Sichuan Science and Technology Program (No. 2018JY0140).
[45], [12] and [25] about closed geodesics on Finsler $\mathbb{R}P^n$, and based on Taimanov’s work [39] on rational equivariant cohomology of non-contractible loops on $S^n/\Gamma$, this paper is concerned with the multiplicity of closed geodesics on Finsler $S^n/\Gamma$.

Let $(M, F)$ be a Finsler manifold and $\Lambda M$ be the free loop space on $M$ defined by

$$\Lambda M = \left\{ \gamma : S^1 \to M \mid \gamma \text{ is absolutely continuous and } \int_0^1 F(\gamma, \dot{\gamma})^2 dt < +\infty \right\},$$

endowed with a natural structure of Riemannian Hilbert manifold on which the group $S^1 = \mathbb{R}/\mathbb{Z}$ acts continuously by isometries (cf. Shen [37]). A closed geodesic $c : S^1 = \mathbb{R}/\mathbb{Z} \to M$ is prime if it is not a multiple covering (i.e., iteration) of any other closed geodesics. Here the $m$-th iteration $c^m$ of $c$ is defined by $c^m(t) = c(mt)$. The inverse curve $c^{-1}$ of $c$ is defined by $c^{-1}(t) = c(1-t)$ for $t \in \mathbb{R}$. Note that unlike Riemannian manifolds, the inverse curve $c^{-1}$ of a closed geodesic $c$ on an irreversible Finsler manifold need not be a geodesic. We call two prime closed geodesics $c$ and $d$ distinct if there is no $\theta \in (0, 1)$ such that $c(t) = d(t + \theta)$ for all $t \in \mathbb{R}$. For a closed geodesic $c$ on $(M, F)$, denote by $P_c$ the linearized Poincaré map of $c$. Recall that a Finsler metric $F$ is bumpy if all the closed geodesics on $(M, F)$ are non-degenerate, i.e., $1 \notin \sigma(P_c)$ for any closed geodesic $c$.

It is well known (cf. Chapter 1 of Klingenberg [20]) that $c$ is a closed geodesic or a constant curve on $(M, F)$ if and only if $c$ is a critical point of the energy functional

$$E(\gamma) = \frac{1}{2} \int_0^1 F(\gamma, \dot{\gamma})^2 dt.$$

Based on it, many important results on this subject have been obtained (cf. [1], [16]-[17], [33]-[34]). In particular, in 1969 Gromoll and Meyer [15] used Morse theory and Bott’s index iteration formulae [6] to establish the existence of infinitely many distinct closed geodesics on $M$, when the Betti number sequence $\{\beta_k(\Lambda M; \mathbb{Q})\}_{k \in \mathbb{Z}}$ is unbounded. Then Vigué-Poirrier and Sullivan [40] further proved in 1976 that for a compact simply connected manifold $M$, the Gromoll-Meyer condition holds if and only if $H^*(M; \mathbb{Q})$ is generated by more than one element.

However, when $\{\beta_k(\Lambda M; \mathbb{Q})\}_{k \in \mathbb{Z}}$ is bounded, the problem is quite complicated. In 1973, Katok [19] endowed some irreversible Finsler metrics to the compact rank one symmetric spaces

$$S^n, \mathbb{R}P^n, CP^n, \mathbb{H}P^n \text{ and } CaP^2,$$

each of which possesses only finitely many distinct prime closed geodesics (cf. also Ziller [46],[47]). On the other hand, Franks [13] and Bangert [3] together proved that there are always infinitely many distinct closed geodesics on every Riemannian sphere $S^2$ (cf. also Hingston [17], Klingenberg [21]). These results imply that the metrics play an important role in the multiplicity of closed geodesics on those manifolds.

In 2004, Bangert and Long [5] (published in 2010) proved the existence of at least two distinct closed geodesics on every Finsler $S^2$. Subsequently, such a multiplicity result for $S^n$ with a bumpy Finsler metric was proved by Duan and Long [8] and Rademacher [36] independently. Furthermore in a recent paper [10], Duan, Long and Wang proved the same conclusion for any compact simply-connected bumpy Finsler manifold. We refer the readers to [9]-[11], [18], [30], [35][41]-[42] and the references therein for more interesting results and the survey papers of Long [29],
Taimanov [38], Burns and Matveev [7] and Oancea [32] for more recent progresses on this subject.

Motivated by the studies on simply connected manifolds, in particular, the resonance identity proved by Rademacher [33], and based on Westerland’s works [43], [44] on loop homology of $\mathbb{R}P^n$, Xiao and Long [45] in 2015 investigated the topological structure of the non-contractible loop space and established the resonance identity for the non-contractible closed geodesics on $\mathbb{R}P^{2n+1}$ by use of $\mathbb{Z}_2$ coefficient homology. As an application, Duan, Long and Xiao [12] proved the existence of at least two distinct non-contractible closed geodesics on $\mathbb{R}P^3$ endowed with a bumpy and irreversible Finsler metric. Subsequently in [39], Taimanov used a quite different method from [45] to compute the rational equivariant cohomology of the non-contractible loop spaces in compact space forms $S^n/T$ and proved the existence of at least two distinct non-contractible closed geodesics on $\mathbb{R}P^2$ endowed with a bumpy and irreversible Finsler metric. Then in [24], Liu combined Fadell-Rabinowitz index theory with Taimanov’s topological results to get many multiplicity results of non-contractible closed geodesics on positively curved Finsler $\mathbb{R}P^n$. Very recently, Liu and Xiao [25] established the resonance identity for the non-contractible closed geodesics on $\mathbb{R}P^n$, and together with [12] and [39] proved the existence of at least two distinct non-contractible closed geodesics on every bumpy $\mathbb{R}P^n$ with $n \geq 2$.

Based on the works of [10] and [25], it is natural to ask whether every bumpy Finsler compact space form possesses two distinct closed geodesics on each of its nontrivial classes. This paper gives a positive answer to this question. To this end, we first establish the following resonance identity in Section 2. Comparing with Theorem 1.1 of [25], the difficulties mainly lie in that the parity of the order $p$ of the nontrivial element $h$ in $\pi_1(M)$ is unknown. This unknown parity causes that the computations of critical modules of non-contractible closed geodesics become very complicated (cf. Lemma 2.1 below), and the parity of $i(c^{p+1}) - i(c)$ becomes unknown for a closed geodesic $c$. Note that here the proof of the positivity of mean index of a non-contractible homologically visible minimal closed geodesic on a compact space form (cf. Lemma 2.2 below) becomes non-trivial, and a non-contractible minimal closed geodesic $c$ of the class $[h]$ may be some iteration of a closed geodesic $c$ which is not in the class $[h]$. Recall that $\Gamma$ is a finite group which acts freely and isometrically on the $n$-sphere.

**Theorem 1.1.** Let $M = S^n/T$ and $h$ be a nontrivial element of finite order $p$ in $\pi_1(M)$. Suppose the Finsler manifold $(M, F)$ possesses only finitely many distinct non-contractible minimal closed geodesics of the class $[h]$, among which we denote the distinct non-contractible homologically visible minimal closed geodesics by $c_1, \ldots, c_r$ for some integer $r > 0$, where $n \geq 2$ and a closed geodesic $c$ of the class $[h]$ is called minimal if it is not an iteration of any other closed geodesics in class $[h]$. Then we have

$$\sum_{j=1}^{r} \hat{\chi}(c_j) i(c_j) = \hat{B}(A_h M; \mathbb{Q}) = \begin{cases} \frac{n+1}{2(n-1)}, & \text{if } n \in 2\mathbb{N} - 1, \\ \frac{n}{2(n-1)}, & \text{if } n \in 2\mathbb{N}. \end{cases} \quad (1.1)$$

where the mean Euler number $\hat{\chi}(c_j)$ of $c_j$ is defined by

$$\hat{\chi}(c_j) = \frac{1}{n_j} \sum_{m=1}^{n_j/p} \sum_{l=0}^{2n-2} (-1)^l d_i(c_j^{(p-1)+1}) K_{l+i(c_j^{p(m-1)+1})} c_j^{p(m-1)+1} \in \mathbb{Q},$$
and \( n_j = n_{c_j} \) is the analytical period of \( c_j \), \( k_0^{c_j^{p(m-1)+1}}(c_j^{p(m-1)+1}) \) is the local homological type number of \( c_j^{p(m-1)+1} \), \( i(c_j) \) and \( \hat{i}(c_j) \) are the Morse index and mean index of \( c_j \) respectively.

In particular, if the Finsler metric \( F \) on \( M = S^n/\Gamma \) is bumpy, then (1.1) has the following simple form

\[
\sum_{j=1}^{r} \left( (-1)^{i(c_j)} k_0^{c_j} (c_j) + (-1)^{i(c_j^{p+1})} k_0^{c_j^{p+1}} (c_j^{p+1}) \right) \frac{1}{i(c_j)}
= \begin{cases} 
\frac{p(n+1)}{n-1}, & \text{if } n \in 2\mathbb{N} - 1, \\
\frac{p}{n-1}, & \text{if } n \in 2\mathbb{N}.
\end{cases}
\] (1.2)

Based on Theorem 1.1, we use Morse theory and the well known Kronecker approximation theorem to prove our main multiplicity result of non-contractible closed geodesics on \((S^{2n+1}/\Gamma, F)\).

**Theorem 1.2.** Let \( M = S^{2n+1}/\Gamma \) and \( h \) be a nontrivial element of finite order \( p \) in \( \pi_1(M) \). Then every bumpy Finsler metric \( F \) on \( M \) has at least two distinct non-contractible closed geodesics of the class \([h]\).

Note that the only non-trivial group which acts freely on \( S^{2n} \) is \( \mathbb{Z}_2 \) and \( S^{2n}/\mathbb{Z}_2 = \mathbb{R}P^{2n} \) (cf. P.5 of [39]). Since we have proved the same result as the above Theorem 1.2 for \( \mathbb{R}P^{2n} \) in Theorem 1.2 and Corollary 1.1 of [25], then we have

**Theorem 1.3.** Let \( M = S^n/\Gamma \) and \( h \) be a nontrivial element of finite order \( p \) in \( \pi_1(M) \), where \( n \geq 2 \). Then every bumpy Finsler metric \( F \) on \( M \) has at least two distinct non-contractible closed geodesics of the class \([h]\).

**Remark 1.1.** (i) In Theorem 5 of [39], Taimanov proved the same result as Theorem 1.2 under the conditions that \( \pi_1(\Lambda_h(M))_{SO(2)} \neq 1 \), \( h \) has an even order in \( \pi_1(M) \) and the centralizer of \( h \) are pairwise non-conjugate, our Theorem 1.2 improves Taimanov’s result by removing these additional conditions.

(ii) When \( \Gamma = \mathbb{Z}_2 \), then \( S^n/\Gamma \) is the \( n \)-dimensional real projective space \( \mathbb{R}P^n \) and \( p = 2 \), one can easily check that for \( \mathbb{R}P^n \), the results of the above Theorems 1.1-1.3 are just the results of Theorems 1.1-1.2 and Corollary 1.1 of [25]. So the main results of this paper are generalizations of those of [25]. Note that the only non-trivial group which acts freely on \( S^n \) with even \( n = 2k \) is \( \mathbb{Z}_2 \) and \( S^{2k}/\mathbb{Z}_2 = \mathbb{R}P^{2k} \) (cf. P.5 of [39]), then we only need to prove Theorem 1.1 for the case when \( n \) is odd.

This paper is organized as follows. In Section 2, we apply Morse theory to the non-contractible loops of the class \([h]\) and establish the resonance identity of Theorem 1.1. Then in Section 3, we firstly recall the precise iteration formulae of Morse indices for orientable closed geodesics, and combine it with Theorem 1.1 to investigate the Morse indices for closed geodesics on \( S^n/\Gamma \) and build a bridge between the Morse indices and a division of an interval, then our problem are reduced to a problem in Number Theory and we review some theories about a special system of irrational numbers associated to our problem developed in [25]. In Section 4, we use the well known Kronecker’s approximation theorem and other techniques in Number theory to give the proof of Theorem 1.2. Finally in Section 5, for the reader’s convenience, we give the proof of Theorem 3.2 about a special system of irrational numbers as an appendix.
In this paper, let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{Q}^c$ denote the sets of natural integers, non-negative integers, integers, rational numbers and irrational numbers respectively. We also use notations $E(a) = \min\{k \in \mathbb{Z} \mid k \geq a\}$, $[a] = \max\{k \in \mathbb{Z} \mid k \leq a\}$, $\varphi(a) = E(a) - [a]$ and $\{a\} = a - [a]$ for any $a \in \mathbb{R}$. Throughout this paper, we use $\mathbb{Q}$ coefficients for all homological and cohomological modules.

2. Resonance identity of non-contractible closed geodesics on $(S^n/\Gamma, F)$. Let $M = (M, F)$ be a compact Finsler manifold, the space $\Lambda = \Lambda M$ of $H^1$-maps $\gamma : S^1 \to M$ has a natural structure of Riemannian Hilbert manifolds on which the group $S^1 = \mathbb{R}/\mathbb{Z}$ acts continuously by isometries. This action is defined by $(s \cdot \gamma)(t) = \gamma(t + s)$ for all $\gamma \in \Lambda$ and $s, t \in S^1$. For any $\gamma \in \Lambda$, the energy functional is defined by

$$E(\gamma) = \frac{1}{2} \int_{S^1} F(\gamma(t), \dot{\gamma}(t))^2 dt. \quad (2.1)$$

It is $C^{1,1}$ and invariant under the $S^1$-action. The critical points of $E$ with positive energies are precisely the closed geodesics $\gamma : S^1 \to M$. The index form of the functional $E$ is well defined at any closed geodesic $c$ on $M$, which we denote by $E''(c)$. As usual, we denote by $i(c)$ and $\nu(c)$ the Morse index and nullity of $E$ at $c$. In the following, we denote by

$$\Lambda^\kappa = \{d \in \Lambda \mid E(d) \leq \kappa\}, \quad \Lambda^\kappa_\cdot = \{d \in \Lambda \mid E(d) < \kappa\}, \quad \forall \kappa \geq 0. \quad (2.2)$$

For a closed geodesic $c$ we set $\Lambda(c) = \{\gamma \in \Lambda \mid E(\gamma) < E(c)\}$.

For $m \in \mathbb{N}$ we denote the $m$-fold iteration map $\phi_m : \Lambda \to \Lambda$ by $\phi_m(\gamma)(t) = \gamma(mt)$, for all $\gamma \in \Lambda, t \in S^1$, as well as $\gamma^m = \phi_m(\gamma)$. If $\gamma \in \Lambda$ is not constant then the multiplicity $m(\gamma)$ of $\gamma$ is the order of the isotropy group $\{s \in S^1 \mid s \cdot \gamma = \gamma\}$. For a closed geodesic $c$, the mean index $\bar{i}(c)$ is defined as usual by $\bar{i}(c) = \lim_{m \to \infty} i(c^m)/m$.

Using singular homology with rational coefficients we consider the following critical $\mathbb{Q}$-module of a closed geodesic $c \in \Lambda$:

$$\mathbb{C}_* (E, c) = H_* \left( (\Lambda(c) \cup S^1 \cdot c)/S^1, \Lambda(c)/S^1; \mathbb{Q} \right). \quad (2.3)$$

In the following we let $M = S^n/\Gamma$ and $h$ be a nontrivial element of finite order $p$ in $\pi_1(M)$, where the integer $n \geq 2$, $\Gamma$ acts freely and isometrically on the $n$-sphere and therefore $M$ is diffeomorphic to a compact space form. Then the free loop space $\Lambda M$ possesses a natural decomposition

$$\Lambda M = \bigsqcup_{g \in \pi_1(M)} \Lambda_g M,$$

where $\Lambda_g M$ is the connected component of $\Lambda M$ whose elements are homotopic to $g$. We set $\Lambda_h(c) = \{\gamma \in \Lambda_h M \mid E(\gamma) < E(c)\}$. Note that for a non-contractible minimal closed geodesic $c$ of class $[h], c^m \in \Lambda_h M$ if and only if $m \equiv 1 \pmod{p}$.

We call a non-contractible minimal closed geodesic $c$ of class $[h]$ satisfying the isolation condition, if the following holds:

**Iso** For all $m \in \mathbb{N}$ the orbit $S^1 \cdot c^{p(m-1)+1}$ is an isolated critical orbit of $E$.

Note that if the number of non-contractible minimal closed geodesics of class $[h]$ on $M$ is finite, then all the non-contractible minimal closed geodesics of class $[h]$ satisfy (Iso).
For a non-contractible closed geodesic \( d \) of class \([h]\), we have \( d = c^p(m-1)+1 \) for some \( m \in \mathbb{N} \), where \( c \) is a minimal closed geodesic of class \([h]\) and \( c = \gamma^t \) for a prime closed geodesic \( \gamma \) with \( t \in \mathbb{N} \). Then \( d \) has multiplicity \( tp(m-1)+t \), the subgroup

\[
\mathbb{Z}_{tp(m-1)+t} \{ l \leq tp(m-1)+t \}
\]

of \( S^1 \) acts on \( T_q(E, d) \). As studied in p.59 of [34], for all \( m \in \mathbb{N} \), let

\[
H^*_c(X, A) \equiv \{ [\xi] \in H^*_c(X, A) \mid T_*[\xi] = \pm[\xi] \},
\]

where \( T \) is a generator of the \( \mathbb{Z}_{tp(m-1)+t} \) action. On \( S^1 \)-critical modules of \( c^p(m-1)+1 \), the following lemma holds:

**Lemma 2.1.** (cf. Satz 6.11 of [34] and [5]) Suppose \( c \) is a non-contractible minimal closed geodesic of class \([h]\) on a Finsler manifold \( M = S^n/\Gamma \) satisfying \((Iso)\). Then there exist \( U_{\gamma^{tp(m-1)+t}} \) and \( N_{\gamma^{tp(m-1)+t}} \), the so-called local negative disk and the local characteristic manifold at \( c^p(m-1)+1 \) respectively, such that \( \nu(c^p(m-1)+1) = \dim N_{\gamma^{tp(m-1)+t}} \) and

\[
\mathcal{C}_q(E, c^p(m-1)+1) = \begin{cases} \mathbb{Q} & \text{if } i(c^p(m-1)+1) - i(\gamma) \in 2\mathbb{Z} \text{ and } q = i(c^p(m-1)+1), \\ 0 & \text{otherwise}. \end{cases}
\]

(i) When \( \nu(c^p(m-1)+1) = 0 \), there holds

\[
\mathcal{C}_q(E, c^p(m-1)+1) = \frac{Q_1}{0}, \quad \text{if } i(c^p(m-1)+1) - i(\gamma) \in 2\mathbb{Z} \text{ and } q = i(c^p(m-1)+1),
\]

(ii) When \( \nu(c^p(m-1)+1) > 0 \), there holds

\[
\mathcal{C}_q(E, c^p(m-1)+1) = H_q \left( \mathcal{H}_q(c^p(m-1)+1) / S^1, \mathcal{H}_q(c^p(m-1)+1) / S^1 \right)
\]

where \( e(c^p(m-1)+1) = (-1)^i(c^p(m-1)+1) - i(\gamma) \).

As usual, for \( m \in \mathbb{N} \) and \( l \in \mathbb{Z} \) we define the local homological type numbers of \( c^p(m-1)+1 \) by

\[
k^j_e(c^p(m-1)+1) = \dim H_j \left( \mathcal{H}_j(c^p(m-1)+1) / S^1 \right).
\]

Based on works of Rademacher in [33], Long and Duan in [30] and [9], we define the \textit{analytical period} \( n_c \) of the closed geodesic \( c \) by

\[
n_c = \min \{ j \in 2p\mathbb{N} \mid \nu(c^j) = \max_{m \geq 1} \nu(c^m) \}.
\]

Note that here in order to simplify the study for non-contractible closed geodesics of class \([h]\) on \( M = S^n/\Gamma \), we have slightly modified the definition in [30] and [9].
by requiring the analytical period to be integral multiple of 2\( p \). Then by the same proofs in [30] and [9], we have

\[
k_l^{e^{(p(m-1)+1+kn_c)}}(e^{(p(m-1)+1+kn_c)}) = k_l^{e^{(p(m-1)+1)}}(e^{p(m-1)+1}), \quad \forall \, m, \, k \in \mathbb{N}, \; l \in \mathbb{Z}.
\]

(2.6)

For more detailed properties of the analytical period \( n_c \) of a closed geodesic \( c \), we refer readers to the two Sections 3s in [30] and [9].

As in [4], we have

**Definition 2.1.** Let \((M, F)\) be a compact Finsler manifold. A closed geodesic \( c \) on \( M \) is homologically visible, if there exists an integer \( M \) such that \( \bar{Z} \) is homologically visible, if there exists an integer \( M \) such that \( \bar{Z} \) is homologically visible.

**Lemma 2.2.** Suppose the Finsler manifold \( M = S^n / \Gamma \) possesses only finitely many distinct non-contractible minimal closed geodesics of the class \([h] \), among which we denote the distinct non-contractible homologically visible minimal closed geodesics by \( c_1, \ldots, c_r \). Then we have

\[
i^*(c_i) > 0, \quad \forall \, 1 \leq i \leq r.
\]

(2.7)

**Proof.** First, we claim that Theorem 3 in [4] for \( M = S^n / \Gamma \) can be stated as:

"Let \( c \) be a closed geodesic in \( \Lambda_h M \) such that \( i^*(c^m) = 0 \) for all \( m \in \mathbb{N} \). Suppose \( c \) is neither homologically invisible nor an absolute minimum of \( E \) in \( \Lambda_h M \). Then there exist infinitely many closed geodesics in \( \Lambda_h M \)."

Indeed, one can focus the proofs of Theorem 3 in [4] on \( \Lambda_h M \) with some obvious modifications. Assume by contradiction. Similarly as in [4], we can choose a different \( c \in \Lambda_h M \), if necessary, and find \( p \in \mathbb{N} \) such that \( \psi_0(\Lambda_h(c) \cup S \cdot c, \Lambda_h(c)) \neq 0 \) and \( \psi_{q}(\Lambda_h(c) \cup S \cdot c, \Lambda_h(c)) = 0 \) for every \( q > p \) and every closed geodesic \( d \in \Lambda_h M \) with \( i^*(d^m) \equiv 0 \).

Consider the following commutative diagram

\[
\begin{array}{ccc}
H_p(\Lambda_h(c) \cup S \cdot c, \Lambda_h(c)) & \xrightarrow{\psi^m} & H_p(\Lambda_h(c^m) \\ \downarrow i^* & & \downarrow i^* \\
H_p(\Lambda_h M, \Lambda_h(c)) & \xrightarrow{\psi^m} & H_p(\Lambda_h M, \Lambda_h(c^m)),
\end{array}
\]

(2.8)

where \( m \equiv 1(\mod \, p) \) and \( \psi^m : \Lambda_h M \rightarrow \Lambda_h M \) is the \( m \)-fold iteration map. By similar arguments as those in [4], there is \( A > 0 \) such that the map \( i^* \circ \psi^m \) is one-to-one, if \( E(c^m) > A \) and none of the \( k_i \in K_0 \) divides \( m \), where

\[
K_0 = \{ k_0, k_1, k_2, \ldots, k_s \},
\]

with \( k_0 = p \) and \( k_1, k_2, \ldots, k_s \) therein. Here note that the required \( m \equiv 1(\mod \, p) \) and so \( c^m \in \Lambda_h(M) \) for \( c \in \Lambda_h M \).

On the other hand, we define

\[
K = \{ m \geq 2 \mid E(c^m) \leq A \} \cup K_0.
\]

Then by Corollary 1 of [4], there exists \( m \in \mathbb{N} \setminus \{1\} \) such that no \( k \in K \) divides \( \bar{m} \) and \( \psi^m \circ i^* \) vanishes. In particular, \( E(c^m) > A \) and none of the \( k_i \in K_0 \) divides \( \bar{m} \). Due to \( \psi^m \circ i^* = i^* \circ \psi^m \) in (2.8), this yields a contradiction. Hence there exist infinitely many closed geodesics in \( \Lambda_h M \).

Accordingly, Corollary 2 in [4] for \( M = S^n / \Gamma \) can be stated as:

"Suppose there exists a closed geodesic \( c \in \Lambda_h M \) such that \( c^m \) is a local minimum of \( E \) in \( \Lambda_h M \) for infinitely many \( m \equiv 1(\mod \, p) \). Then there exist infinitely many closed geodesics in \( \Lambda_h M \)."
Based on the above two variants of Theorem 3 and Corollary 2 in [4], we can prove our Lemma 2.2 as follows.

It is well known that every closed geodesic \( c \) on \( M \) must have mean index \( \hat{i}(c) \geq 0 \). Assume by contradiction that there is a non-contractible homologically visible minimal closed geodesic \( c \) of the class \([h]\) on \( M \) satisfying \( \hat{i}(c) = 0 \). Then \( \hat{i}(c^m) = 0 \) for all \( m \in \mathbb{N} \) by Bott iteration formula and \( c \) must be an absolute minimum of \( E \) in \( \Lambda_h M \), since otherwise there would exist infinitely many distinct non-contractible closed geodesics of the class \([h]\) on \( M \) by the above variant of Theorem 3 on p.385 of [4].

On the other hand, by Lemma 7.1 of [34], there exists a \( k(c) \in p\mathbb{N} \) such that \( \nu(c^{m+k(c)}) = \nu(c^m) \) for all \( m \in \mathbb{N} \). Specially we obtain \( \nu(c^{mk(c)+1}) = \nu(c) \) for all \( m \in \mathbb{N} \) and then elements of \( \ker(E''(c^{mk(c)+1})) \) are precisely \( mk(c) + 1 \)st iterates of elements of \( \ker(E''(c)) \). Thus by the Gromoll-Meyer theorem in [14], the behavior of the restriction of \( E \) to \( \ker(E''(c^{mk(c)+1})) \) is the same as that of the restriction of \( E \) to \( \ker(E''(c)) \). Then together with the fact \( \hat{i}(c^m) = 0 \) for all \( m \in \mathbb{N} \), we obtain that \( c^{mk(c)+1} \) is a local minimum of \( E \) in \( \Lambda_h M \) for every \( m \in \mathbb{N} \). Because \( M \) is compact and possessing finite fundamental group (\( \pi_1(M) \) is finite for the spherical space forms!), there must exist infinitely many distinct non-contractible closed geodesics of the class \([h]\) on \( M \) by the above variant of Corollary 2 on p.386 of [4]. Then it yields a contradiction and proves (2.7).

In [39], Taimanov calculated the rational equivariant cohomology of the spaces of non-contractible loops in compact space forms which is crucial for our proof of Theorem 1.1 and can be stated as follows.

**Lemma 2.3.** (cf. Theorem 3 of [39]) For \( M = S^n/\Gamma \), we have

(i) When \( n = 2k + 1 \) is odd, the \( S^1 \)-cohomology ring of \( \Lambda_h M \) has the form

\[
H^{S^1,*}(\Lambda_h M; \mathbb{Q}) = \mathbb{Q}[w, z]/\{w^{k+1} = 0\}, \quad \deg(w) = 2, \quad \deg(z) = 2k.
\]

Then the \( S^1 \)-equivariant Poincaré series of \( \Lambda_h M \) is given by

\[
P^{S^1}(\Lambda_h M; \mathbb{Q})(t) = \frac{1 - t^{2k+2}}{(1 - t^2)(1 - t^{2k})} = \frac{1}{1 - t^2} + \frac{t^{2k}}{1 - t^{2k}} = (1 + t^2 + t^4 + \cdots + t^{2k} + \cdots) + (t^{2k} + t^{4k} + t^{6k} + \cdots),
\]

which yields Betti numbers

\[
\bar{\beta}_q = \text{rank} H_q^{S^1}(\Lambda_h M; \mathbb{Q}) = \begin{cases} 2, & \text{if } q \in \{j(n - 1) \mid j \in \mathbb{N}\}, \\ 1, & \text{if } q \in (2N_0 \setminus \{j(n - 1) \mid j \in \mathbb{N}\}, \\ 0, & \text{otherwise}, \end{cases}
\]

and the average \( S^1 \)-equivariant Betti number of \( \Lambda_h M \) satisfies

\[
\bar{B}(\Lambda_h M; \mathbb{Q}) = \lim_{q \to +\infty} \frac{1}{q} \sum_{k=0}^{q} (-1)^k \bar{\beta}_k = \frac{n + 1}{2(n - 1)}.
\]

(ii) When \( n = 2k \) is even, the \( S^1 \)-cohomology ring of \( \Lambda_h M \) has the form

\[
H^{S^1,*}(\Lambda_h M; \mathbb{Q}) = \mathbb{Q}[w, z]/\{w^{2k} = 0\}, \quad \deg(w) = 2, \quad \deg(z) = 4k - 2.
\]
Then the $S^1$-equivariant Poincaré series of $\Lambda_h M$ is given by
\[
P^{S^1}(\Lambda_h M; \mathbb{Q})(t) = \frac{1 - t^{4k}}{(1 - t^2)(1 - t^{4k-2})}
= \frac{1}{1 - t^2} + \frac{t^{4k-2}}{1 - t^{4k-2}}
= (1 + t^2 + t^4 + \cdots + t^{2k} + \cdots)
+ (t^{4k-2} + t^{2(4k-2)} + t^{3(4k-2)} + \cdots),
\]
which yields Betti numbers
\[
\bar{\beta}_q = \text{rank} H^q_{S^1}(\Lambda_h M; \mathbb{Q}) = \begin{cases} 2, & \text{if } q \in \{2j(n - 1) \mid j \in \mathbb{N}\}, \\ 1, & \text{if } q \in (2\mathbb{N}_0) \setminus \{2j(n - 1) \mid j \in \mathbb{N}\}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.11}
\]
and the average $S^1$-equivariant Betti number of $\Lambda_h M$ satisfies
\[
\bar{B}(\Lambda_h M; \mathbb{Q}) = \lim_{q \to +\infty} \frac{1}{q^{k}} \frac{1}{q^{k}} \sum_{k=0}^{q} (1 - t) \bar{\beta}_k = \frac{n}{2(n - 1)}. \tag{2.12}
\]

Now we give the proof of the resonance identity in Theorem 1.1.

**Proof of Theorem 1.1.** Recall that we denote the non-contractible homologically visible minimal closed geodesics of the class $[h]$ by $CG_{hv}^{[h]}(M) = \{c_1, \ldots, c_r\}$ for some integer $r > 0$, when the number of distinct non-contractible minimal closed geodesics of the class $[h]$ on $M = S^n / \Gamma$ is finite. Note also that by Lemma 2.2 we have $\hat{i}(c_j) > 0$ for all $1 \leq j \leq r$. In the sequel, we assume $n = 2k + 1$ for $k \in \mathbb{N}$ by Remark 1.1 (iii), and thus $M$ is orientable.

Let
\[
m_q = M_q(\Lambda_h M) = \sum_{1 \leq j \leq r, \, m \geq 1} \dim C_q(E, c_j^{p(m-1)+1}), \quad q \in \mathbb{Z}.
\]

The Morse series of $\Lambda_h M$ is defined by
\[
M(t) = \sum_{q=0}^{+\infty} m_q t^q. \tag{2.13}
\]

**Claim 1.** $\{m_q\}$ is a bounded sequence.

In fact, by (2.6), we have
\[
m_q = \sum_{j=1}^{r} \sum_{m=1}^{n_j/p} \sum_{l=0}^{2n_j-2} \bar{c}(c_j^{p(m-1)+1}) \langle c_j^{p(m-1)+1} \rangle
\times \# \left\{ s \in \mathbb{N}_0 \mid q - i(c_j^{p(m-1)+1} + s n_j) = l \right\}, \tag{2.14}
\]
by Theorem 10.1.2 of [27] and Lemma 3.1 below, we have
\[
|i(c_j^{p(m-1)+1} + s n_j) - (p(m - 1) + 1 + s n_j)\hat{i}(c_j)| \leq n - 1.
\]
For notational simplicity, let $T = p(m - 1) + 1$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then,
\[
\# \left\{ s \in \mathbb{N}_0 \mid q - i(c_j^{T + s n_j}) = l \right\}
= \# \left\{ s \in \mathbb{N}_0 \mid l + i(c_j^{T + s n_j}) = q, \, |i(c_j^{T + s n_j}) - (T + s n_j)\hat{i}(c_j)| \leq n - 1 \right\}
\]
and
\[
\]
Hence Claim 1 follows from (2.14) and (2.15).

On the other hand, we have

\[
M = \sum_{j=1}^{r} \sum_{m=1}^{n_j} \sum_{l=0}^{2n_j-2} \sum_{k=0}^{q} (-1)^k k_l^{c_j} (c_j^T) \# \left\{ s \in \mathbb{N}_0 \mid k - i(c_j^T+s_n_j) = l \right\}
\]

\[
= \sum_{j=1}^{r} \sum_{m=1}^{n_j} \sum_{l=0}^{2n_j-2} \sum_{k=0}^{q} (-1)^k k_l^{c_j} (c_j^T) \# \left\{ s \in \mathbb{N}_0 \mid l + i(c_j^T+s_n_j) \leq q \right\}.
\]

By (2.14), Lemma 3.1 below and the fact that \( n_j \in 2\mathbb{N} \), we obtain

\[
M = \sum_{k=0}^{q} m_k (-1)^k
\]

\[
M = \sum_{k=0}^{q} m_k(-1)^k
\]

\[
= \sum_{j=1}^{r} \sum_{m=1}^{n_j} \sum_{l=0}^{2n_j-2} \sum_{k=0}^{q} (-1)^k k_l^{c_j} (c_j^T) \# \left\{ s \in \mathbb{N}_0 \mid k - i(c_j^T+s_n_j) = l \right\}
\]

\[
= \sum_{j=1}^{r} \sum_{m=1}^{n_j} \sum_{l=0}^{2n_j-2} \sum_{k=0}^{q} (-1)^k k_l^{c_j} (c_j^T) \# \left\{ s \in \mathbb{N}_0 \mid l + i(c_j^T+s_n_j) \leq q \right\}.
\]

On the one hand, we have

\[
\# \left\{ s \in \mathbb{N}_0 \mid l + i(c_j^T+s_n_j) \leq q \right\}
\]

\[
= \# \left\{ s \in \mathbb{N}_0 \mid l + i(c_j^T+s_n_j) \leq q, |i(c_j^T+s_n_j) - (T+s_n_j)i(c_j)| \leq n - 1 \right\}
\]

\[
\leq \# \left\{ s \in \mathbb{N}_0 \mid 0 \leq (T+s_n_j)i(c_j) \leq q - l + n - 1 \right\}
\]

\[
= \# \left\{ s \in \mathbb{N}_0 \mid 0 \leq q - l + n - 1 - T_i(c_j) \right\}
\]

\[
\leq \frac{q - l + n - 1}{n_j i(c_j)} + 1.
\]

On the other hand, we have

\[
\# \left\{ s \in \mathbb{N}_0 \mid l + i(c_j^T+s_n_j) \leq q \right\}
\]

\[
= \# \left\{ s \in \mathbb{N}_0 \mid l + i(c_j^T+s_n_j) \leq q, |i(c_j^T+s_n_j) - (T+s_n_j)i(c_j)| \leq n - 1 \right\}
\]

\[
\geq \# \left\{ s \in \mathbb{N}_0 \mid i(c_j^T+s_n_j) \leq (T+s_n_j)i(c_j) + n - 1 \leq q - l \right\}
\]

\[
\geq \# \left\{ s \in \mathbb{N}_0 \mid 0 \leq s \leq \frac{q - l - n + 1 - T_i(c_j)}{n_j i(c_j)} \right\}
\]

\[
\geq \frac{q - l - n + 1}{n_j i(c_j)} - 1.
\]
Thus we obtain
\[
\lim_{q \to +\infty} \frac{1}{q} M^q(-1) = \sum_{j=1}^{n_q} \sum_{n=1}^{\frac{n_q}{2n-2}} \sum_{l=0}^{\frac{1}{n_j}} (-1)^{l+i(c_j)} k_l^{\epsilon(c_j)}(c_j^T) \frac{1}{n_j \hat{i}(c_j)} = \sum_{j=1}^{r} \hat{\chi}(c_j).
\]
Since \( n_q \) is bounded, we then obtain
\[
\lim_{q \to +\infty} \frac{1}{q} M^q(-1) = \lim_{q \to +\infty} \frac{1}{q} P^{S^1, q}(\Lambda_h M; \mathbb{Q})(-1)
\]
\[
= \lim_{q \to +\infty} \frac{1}{q} \sum_{k=0}^{q} (-1)^k \beta_k = B(\Lambda_h M; \mathbb{Q}),
\]
where \( P^{S^1, q}(\Lambda_h M; \mathbb{Q})(t) \) is the truncated polynomial of \( P^{S^1}(\Lambda_h M; \mathbb{Q})(t) \) with terms of degree less than or equal to \( q \). Thus by (2.10) we get
\[
\sum_{j=1}^{r} \frac{\hat{\chi}(c_j)}{\hat{i}(c_j)} = \frac{n+1}{2(n-1)} \forall \, n \in 2\mathbb{N} + 1.
\]
which proves (1.1) of Theorem 1.1. For the special case when each \( c_j^T \) is non-degenerate with \( 1 \leq j \leq r \) and \( m \in \mathbb{N} \), we have \( n_j = 2p \) and \( k_l^{\epsilon(c_j)}(c_j^T) = 1 \) when \( l = 0 \) and \( i(c_j^T) - i(\gamma_j) \in 2\mathbb{Z} \), where \( c_j \) is some iteration of a prime closed geodesic \( \gamma_j \), and \( k_l^{\epsilon(c_j)}(c_j^T) = 0 \) for all other \( l \in \mathbb{Z} \). Then (1.1) has the following simple form
\[
\sum_{j=1}^{r} \left( (-1)^{i(c_j)} k_0^{\epsilon(c_j)}(c_j) + (-1)^{i(c_j+1)} k_0^{\epsilon(c_j+1)}(c_j+1) \right) \frac{1}{\hat{i}(c_j)} = \frac{p(n+1)}{n-1}, \forall \, n \in 2\mathbb{N} + 1.
\]
This proves (1.2) of Theorem 1.1. \( \square \)

3. Preliminary for the proof of Theorem 1.2.

3.1. Index iteration formulae for closed geodesics. In [26] of 1999, Y. Long established the basic normal form decomposition of symplectic matrices. Based on it, he further established the precise iteration formulae of Maslov \( \omega \)-indices for symplectic paths in [27], which can be related to Morse indices of either orientable or non-orientable closed geodesics in a slightly different way (cf. [22], [23] and Chap. 12 of [28]). Roughly speaking, the orientable (resp. non-orientable) case corresponds to \( i_1 \) (resp. \( i_{-1} \)) index, where \( i_1 \) and \( i_{-1} \) denote the cases of \( \omega \)-index with \( \omega = 1 \) and \( \omega = -1 \) respectively (cf. Chap. 5 of [28]). Since we have assumed the manifold \( M = S^n/\Gamma \) to be odd dimensional in Theorem 1.2, then \( M \) is orientable and we only state the precise index iteration formulae of orientable closed geodesics in the following. Throughout this section we write \( i_1(\gamma) \) as \( i(\gamma) \) for short.

For the reader's convenience, we briefly review some basic materials in Long's book [28].

Let \( P \) be a symplectic matrix in \( Sp(2N-2) \) and \( \Omega^0(P) \) be the path connected component of its homotopy set \( \Omega(P) \) which contains \( P \). Then there is a path \( f \in C([0,1], \Omega^0(P)) \) such that \( f(0) = P \) and
\[
f(1) = N_1(1,1)^{op-} \circ I_{2p_0} \circ N_1(1,-1)^{op+} \circ N_1(-1,1)^{qp-} \circ (-I_{2p_0}) \circ N_1(-1,1)^{qp+} \circ R(\theta_1) \circ \cdots \circ R(\theta_{r'}) \circ R(\theta_{r'+1}) \circ \cdots \circ R(\theta_r)
\] (3.1)
\[ \circ N_2(e^{i\alpha_1}, A_1) \circ \cdots \circ N_2(e^{i\alpha_r}, A_r) \\
\circ N_2(e^{i\beta_1}, B_1) \circ \cdots \circ N_2(e^{i\beta_0}, B_{r_0}) \\
\circ H((\pm 2)^{\circ h}), \]

where \( N_1(\lambda, \chi) = \begin{pmatrix} \lambda & \chi \\ 0 & \lambda \end{pmatrix} \) with \( \lambda = \pm 1 \) and \( \chi = 0, \pm 1; \)
\( H(b) = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \)

with \( b = \pm 2; \)
\( R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \) with \( \theta \in (0, 2\pi) \setminus \{\pi\} \)
and we suppose that \( \pi < \theta_j < 2\pi \) iff \( 1 \leq j \leq r'; \)

\[ N_2(e^{i\alpha_j}, A_j) = \begin{pmatrix} R(\alpha_j) A_j & 0 \\ 0 & R(\alpha_j) \end{pmatrix} \]
and \[ N_2(e^{i\beta_j}, B_j) = \begin{pmatrix} R(\beta_j) B_j & 0 \\ 0 & R(\beta_j) \end{pmatrix} \]

with \( \alpha_j, \beta_j \in (0, 2\pi) \setminus \{\pi\} \) are non-trivial and trivial basic normal forms respectively.

Let \( \gamma_0 \) and \( \gamma_1 \) be two symplectic paths in \( \text{Sp}(2N - 2) \) connecting the identity matrix \( I \) to \( P \) and \( f(1) \) satisfying \( \gamma_0 \sim \omega \gamma_1. \)
Then it has been shown that \( i_\omega(\gamma_0^m) = i_\omega(\gamma_1^m) \) for any \( \omega \in \mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}. \)
Based on this fact, we always assume without loss of generality that each \( P_c \) appearing in the sequel has the form (3.1).

\textbf{Lemma 3.1.} (cf. Theorem 8.3.1 and Chap. 12 of [28]) Let \( c \) be an orientable closed geodesic on an \( N \)-dimensional Finsler manifold with its Poincaré map \( P_c. \)
Then, there exists a continuous symplectic path \( \Psi \) with \( \Psi(0) = I \) and \( \Psi(1) = P_c \) such that

\[ i(e^m) = i(\Psi^m) = m(i(\Psi) + p_- + p_0 + r) - (p_- + p_0 + r) - \frac{1 + (-1)^m}{2}(q_0 + q_+) \]
\[ + 2 \sum_{j=1}^r E \left( \frac{m\theta_j}{2\pi} \right) + 2 \sum_{j=1}^{r_x} \varphi \left( \frac{m\alpha_j}{2\pi} \right) - 2r_x, \quad (3.2) \]

and

\[ \nu(e^m) = \nu(\Psi^m) = \nu(\Psi) + \frac{1 + (-1)^m}{2}(q_- + 2q_0 + q_+) + 2\varsigma(c, m), \quad (3.3) \]

where we denote by

\[ \varsigma(c, m) = \left( r - \sum_{j=1}^r \varphi \left( \frac{m\theta_j}{2\pi} \right) \right) + \left( r_x - \sum_{j=1}^{r_x} \varphi \left( \frac{m\alpha_j}{2\pi} \right) \right) + \left( r_0 - \sum_{j=1}^{r_0} \varphi \left( \frac{m\beta_j}{2\pi} \right) \right). \]

\textbf{3.2. A variant of the precise index iteration formulae.} In this subsection, we give a variant of the precise index iteration formulae in Subsection 3.1 which makes them more intuitive and enables us to apply the Kronecker’s approximation theorem to study the multiplicity of non-contractible closed geodesics of the class \([h].\)

To prove Theorem 1.2, in the following of this paper, we always assume that there exists only one non-contractible minimal closed geodesic \( c \) of the class \([h]\) on \( S^{2n+1}/\Gamma \) with a bumpy irreversible metric \( F, \) which is then just the well known minimal point of the energy functional \( E \) on \( \Lambda_h M \) satisfying \( i(c) = 0. \) We can suppose that \( c = \gamma^t \) for some prime closed geodesic \( \gamma \) and \( t \in \mathbb{N}, \) then we also have \( i(\gamma) = 0 \) since \( \gamma \) is also a local minimal point of the energy functional \( E. \)

Now the Morse-type number is given by

\[ m_q \equiv M_q(\Lambda_h M) = \sum_{m \geq 1} \dim \mathcal{C}_q(E, c^{(m-1)+1}), \quad \forall q \in \mathbb{N}_0. \]
Since the Finsler metric $F$ is bumpy, for the Poincaré map $P_c$ of $c$, there is a path $f \in C([0, 1], \Omega^1(P_c))$ such that $f(0) = P_c$ and
\[
f(1) = R(\theta_1) \circ \cdots \circ R(\theta_k) \circ N_2(e^{i\alpha_1}, A_1) \circ \cdots \circ N_2(e^{i\alpha_r}, A_r) \circ N_2(e^{i\beta_1}, B_1) \circ \cdots \circ N_2(e^{i\beta_{r_0}}, B_{r_0}) \circ H(\pm 2)^{\phi_k},
\]
where $\frac{\theta_i}{2\pi}, \frac{\alpha_i}{2\pi}$'s and $\frac{\beta_j}{2\pi}$'s are all in $\mathbb{Q} \cap (0, 1)$, $k + h + 2r_s + 2r_0 = 2\pi$. Then by (3.2) in Lemma 3.1 we obtain
\[
i(c^m) = -mk - k + 2 \sum_{j=1}^{k} E \left( \frac{m\theta_j}{2\pi} \right), \quad (3.4)
Then we have
\[
\textbf{Lemma 3.2. Assuming the existence of only one non-contractible minimal closed geodesic $c$ of the class $[h]$ on $S^{2n+1}/\Gamma$ with a bumpy irreversible metric $F$, where the order of $h$ is $p$ with $p \geq 2$, there hold}
\[
m_{2q+1} = \bar{\beta}_{2q+1} = 0 \quad \text{and} \quad m_{2q} = \bar{\beta}_{2q}, \quad \forall \ q \in \mathbb{N}_0. \quad (3.5)
\]
\[
\textbf{Proof. We carry out the proof into two cases according to the parity of $i(c^{p+1}) - i(c)$.}
\]
\textbf{Case 1. $i(c^{p+1}) - i(c)$ is even.}
In this case, due to (3.4) and $i(c) = 0$, we have $pk \in 2\mathbb{N}$ and $i(c^{(p(m-1)+1)}) - i(c) \in 2\mathbb{Z}$ for any $m \in \mathbb{N}$, and so $i(c^{(p(m-1)+1)}) \in 2\mathbb{N}_0$. Thus by Lemma 2.1(i), the definition of $m_q$, and recalling again $i(\gamma) = 0$, we obtain $m_{2q+1} = 0$ for every $q \in \mathbb{N}_0$. By (2.9) of Lemma 2.3 we get $m_{2q+1} = \bar{\beta}_{2q+1} = 0$ for every $q \in \mathbb{N}_0$, and $m_{2q} = \bar{\beta}_{2q}$ then follows from the following Morse inequality
\[
m_q - m_{q-1} + \cdots + (-1)^{q}m_0 \geq \bar{\beta}_q - \bar{\beta}_{q-1} + \cdots + (-1)^q\bar{\beta}_0, \quad \forall q \in \mathbb{N}_0.
\]
\textbf{Case 2. $i(c^{p+1}) - i(c)$ is odd.}
Similarly, due to (3.4) and $i(c) = 0$ we have $pk \in 2\mathbb{N} - 1$ and $i(c^{(p(m-1)+1)}) - i(c) \in 2\mathbb{Z}$ if and only if $m \in 2\mathbb{N} - 1$, and so $i(c^{(p(m-1)+1)}) \in 2\mathbb{N}_0$ if and only if $m \in 2\mathbb{N} - 1$. Then by Lemma 2.1(i) we get
\[
\overline{C}_q(E, c^{(p(m-1)+1)}) = \begin{cases} \mathbb{Q}, & \text{if } m \in 2\mathbb{N} - 1 \text{ and } q = i(c^{(p(m-1)+1)}), \\ 0, & \text{otherwise}, \end{cases}
\]
where the fact $i(\gamma) = 0$ is used. Thus by the definition of $m_q$, we obtain $m_{2q+1} = 0$ for each $q \in \mathbb{N}_0$. Then by (2.9) of Lemma 2.3 it yields $m_{2q+1} = \bar{\beta}_{2q+1} = 0$ for every $q \in \mathbb{N}_0$ while $m_{2q} = \bar{\beta}_{2q}$ follows from the following Morse inequality
\[
m_q - m_{q-1} + \cdots + (-1)^{q}m_0 \geq \bar{\beta}_q - \bar{\beta}_{q-1} + \cdots + (-1)^q\bar{\beta}_0, \quad \forall q \in \mathbb{N}_0.
\]
The proof is complete. \hfill \Box

Now we prove Theorem 1.2 for $M = S^{2n+1}/\Gamma$ with a bumpy reversible Finsler metric $F$.

\textbf{Theorem 3.1. Let $M = S^{2n+1}/\Gamma$ and $h$ be a nontrivial element of finite order $p$ in $\pi_1(M)$. Then every bumpy reversible Finsler metric $F$ on $M$ has at least two distinct non-contractible closed geodesics of the class $[h]$.}

\textbf{Proof. Assume by contradiction that there exists only one non-contractible minimal closed geodesic $c$ of the class $[h]$ on $M$ which is just the minimum point of the energy functional $E$ on $\Lambda_h M$. Since the metric $F$ on $M$ is reversible, the inverse curve $c^{-1}$ of a closed geodesic $c$ of the class $[h]$ plays the same role in the variational setting of the energy functional $E$ on $\Lambda_h M$ as $c$. In particular, the $m$-th iterates $c^m$ and $c^{-m}$...}
have precisely the same Morse indices, nullities and critical modules. Since Lemma 3.2 also holds for bumpy reversible Finsler metrics, the equalities in (3.5), together with (2.9) of Lemma 2.3, then yield

\[ m_0 = \tilde{\beta}_0 = 1. \]  

(3.6)

However, by the fact \( i(c^{\pm 1}) = i(\gamma) = 0 \) and Lemma 2.1(i), we get

\[ m_0 = \sum_{m \geq 1} \dim C_0(E, c^{\pm (p(m-1)+1)}) \geq \dim C_0(E, c) + \dim C_0(E, c^{-1}) = 2; \]

which contradicts to (3.6) and the proof of Theorem 3.1 is complete. \( \square \)

By Theorem 3.1, we only need to prove Theorem 1.2 for \( M = S^{2n+1}/\Gamma \) with a bumpy irreversible Finsler metric \( F \) in the sequel.

**Lemma 3.3.** Suppose \( c \) is the only one non-contractible minimal closed geodesic \( c \) of the class \([h]\) on \( S^{2n+1}/\Gamma \) with a bumpy irreversible metric \( F \), where the order of \( h \) is \( p \) with \( p \geq 2 \). Then there exist an integer \( \bar{p} \geq 2 \) and \( \hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_k \) in \( \mathbb{Q}^c \) with \( 2 \leq k \leq 2n \) such that \( k \bar{p} \in 2\mathbb{N} \) and

\[
\sum_{j=1}^{k} \hat{\theta}_j = \frac{1}{2} \left( k + \frac{2n}{p(n+1)} \right), \tag{3.7}
\]

\[
i(c^m) = m \left( \frac{2n}{p(n+1)} \right) + k - 2 \sum_{j=1}^{k} \left\{ m\hat{\theta}_j \right\}, \quad \forall \ m \geq 1. \tag{3.8}
\]

Moreover, \( c^m \) contributes to the Morse-type number \( \{m_q \mid q \in \mathbb{N}_0\} \) if and only if \( m \equiv 1 \pmod{\bar{p}} \).

**Proof.** From (3.4), we have

\[
\tilde{i}(c) = -k + \sum_{j=1}^{k} \frac{\theta_j}{\pi}. \tag{3.9}
\]

We carry out the proof into two cases according to the parity of \( i(c^{p+1}) - i(c) \).

**Case 1.** \( i(c^{p+1}) - i(c) \) is even.

For this case, by (3.4) we have \( pk \in 2\mathbb{N} \). From (1.2) of Theorem 1.1 and Lemma 2.1(i), we get

\[
\frac{(-1)^{i(c)}}{i(c)} = \frac{p(n+1)}{2n},
\]

which, together with (3.9) and the fact \( i(c) = 0 \), yield

\[
\sum_{j=1}^{k} \frac{\theta_j}{\pi} = k + \frac{2n}{p(n+1)}. \tag{3.10}
\]

Then by (3.4) and (3.10), we have

\[
i(c^m) = -mk + k + 2 \sum_{j=1}^{k} \left\lfloor \frac{m\theta_j}{2\pi} \right\rfloor = -mk + k + 2 \sum_{j=1}^{k} \left( \frac{m\theta_j}{2\pi} - \left\{ \frac{m\theta_j}{2\pi} \right\} \right).
\]
\[ i(c) = m \left( \frac{2n}{p(n+1)} \right) + k - 2 \sum_{j=1}^{k} \left\{ \frac{m\theta_j}{2\pi} \right\}. \] (3.11)

Let \( \tilde{p} = p \) and \( \tilde{\theta}_j = \frac{\theta_j}{2\pi} \) for \( j = 1, 2, \cdots, k \), then \( k\tilde{p} \in 2\mathbb{N} \) and (3.7)-(3.8) hold by (3.10)-(3.11). From the proof for Case 1 of Lemma 3.2, it yields that \( c^m \) contributes to the Morse-type numbers \( \{ m_q \mid q \in \mathbb{N}_0 \} \) if and only if \( m \equiv 1 \pmod{p} \).

**Case 2.** \( i(c^{p+1}) - i(c) \) is odd.

Similarly, by (1.2) of Theorem 1.1 and Lemma 2.1(i), we get

\[ (-1)^{i(c)} \frac{i(c)}{i(c)} = \frac{p(n+1)}{n}, \]

which, together with (3.9) and the fact \( i(c) = 0 \), yield

\[ \sum_{j=1}^{k} \frac{\theta_j}{\pi} = k + \frac{n}{p(n+1)}. \] (3.12)

Then by (3.4) and (3.12), we have

\[ i(c^m) = m \left( \frac{n}{\tilde{p}(n+1)} \right) + k - 2 \sum_{j=1}^{k} \left\{ \frac{m\hat{\theta}_j}{2\pi} \right\}. \] (3.13)

Let \( \tilde{p} = 2\tilde{p}^2 \) and \( \tilde{\theta}_j = \frac{\theta_j}{\tilde{p}(n+1)} \) for \( j = 1, 2, \cdots, k \), then \( k\tilde{p} \in 2\mathbb{N} \) and (3.7)-(3.8) hold by (3.12)-(3.13). From the proof of Case 2 of Lemma 3.2, it yields that \( c^m \) contributes to the Morse-type numbers \( \{ m_q \mid q \in \mathbb{N}_0 \} \) if and only if \( m \equiv 1 \pmod{\tilde{p}} \). The proof is complete. \( \square \)

Now we give a variant of the precise index iteration formulae (3.8) specially for our purpose. Let \( m = \tilde{p}(n+1)l + \tilde{p}L + 1 \) with \( l \in \mathbb{N} \) and \( L \in \mathbb{Z} \). By (3.7) and (3.8) we obtain

\[ i(c^m) = 2nl + k \left( \tilde{p}L + 1 \right) \frac{2n}{\tilde{p}(n+1)} \]

\[ -2 \left( \left\{ \frac{k}{2} + \frac{(\tilde{p}L + 1)n}{\tilde{p}(n+1)} \right\} - \sum_{j=2}^{k} \left\{ m\hat{\theta}_j \right\} + \sum_{j=2}^{k} \left\{ m\hat{\theta}_j \right\} \right) \]

\[ = 2nl + 2 \left\{ \frac{k}{2} + \frac{(\tilde{p}L + 1)n}{\tilde{p}(n+1)} \right\} + 2 \left\{ \frac{k}{2} + \frac{(\tilde{p}L + 1)n}{\tilde{p}(n+1)} \right\} \]

\[ -2 \left( \left\{ \frac{k}{2} + \frac{(\tilde{p}L + 1)n}{\tilde{p}(n+1)} \right\} - \sum_{j=2}^{k} \left\{ m\hat{\theta}_j \right\} + \sum_{j=2}^{k} \left\{ m\hat{\theta}_j \right\} \right) \]

\[ = 2nl + 2 |Q_L| + 2 \left\{ Q_L \right\} \]

\[ -2 \left( \left\{ Q_L \right\} - \sum_{j=2}^{k} \left\{ m\hat{\theta}_j \right\} + \sum_{j=2}^{k} \left\{ m\hat{\theta}_j \right\} \right), \] (3.14)

where the fact \( k\tilde{p} \in 2\mathbb{N} \) is used in the first identity, and for notational simplicity, in the last identity we denote by

\[ Q_L = \frac{k}{2} + \frac{(\tilde{p}L + 1)n}{\tilde{p}(n+1)}. \] (3.15)
Since \( \sum_{j=2}^{k} \{ m \hat{\theta}_j \} \in \mathbb{Q}^c \), we obtain by (3.14) that for \( 1 \leq i \leq k - 2 \),

\[
i(c^m) = \begin{cases} 
2nl + 2 \lfloor Q_L \rfloor, & \text{iff } \sum_{j=2}^{k} \{ m \hat{\theta}_j \} \in (0, \{ Q_L \}), \\
2nl + 2 \lfloor Q_L \rfloor - 2i, & \text{iff } \sum_{j=2}^{k} \{ m \hat{\theta}_j \} \in (i - 1 + \{ Q_L \}, i + \{ Q_L \}), \\
2nl + 2 \lfloor Q_L \rfloor - 2(k - 1), & \text{iff } \sum_{j=2}^{k} \{ m \hat{\theta}_j \} \in (k - 2 + \{ Q_L \}, k - 1). 
\end{cases}
\]

(3.16)

Let

\[
\begin{align*}
I_0(L) &= (0, \{ Q_L \}), \\
I_{i-1}(L) &= (k - 2 + \{ Q_L \} , k - 1), \\
I_i(L) &= (i - 1 + \{ Q_L \} , i + \{ Q_L \}) \quad \text{for } 1 \leq i \leq k - 2.
\end{align*}
\]

(3.17)

Then, (3.16) can be stated in short as that for any integers \( m = \bar{p}(n + 1)l + \bar{p}L + 1 \) and \( 0 \leq i \leq k - 1 \),

\[
i(c^m) = 2nl + 2 \lfloor Q_L \rfloor - 2i \quad \text{if and only if } \sum_{j=2}^{k} \{ m \hat{\theta}_j \} \in I_i(L).
\]

(3.18)

**Remark 3.1.** Let \( (\tau(1), \tau(2), \ldots, \tau(k)) \) be an arbitrary permutation of \( (1, 2, \ldots, k) \). Then the same conclusion as (3.18) with \( j \) ranging through \( \{ \tau(1), \tau(2), \ldots, \tau(k - 1) \} \) instead is still valid.

The following lemma will be also needed in the proof of Theorem 1.2 for \( S^{2n+1}/\Gamma \) in Section 4.

**Lemma 3.4.** Under the assumption of Lemma 3.3, for any positive integers \( l \) and \( m \), we have

\[
|i(c^m) - 2nl| > 2n \quad \text{holds whenever } |m - \bar{p}(n + 1)|l > 2\bar{p}(n + 1).
\]

**Proof.** From (3.8), we have

\[
i(c^m) = 2nl + (m - \bar{p}(n + 1)l) \cdot \frac{2n}{\bar{p}(n + 1)} + k - 2 \sum_{j=1}^{k} \{ m \hat{\theta}_j \},
\]

which yields immediately that

\[
|i(c^m) - 2nl| \geq |m - \bar{p}(n + 1)l| \cdot \frac{2n}{\bar{p}(n + 1)} - |k - 2 \sum_{j=1}^{k} \{ m \hat{\theta}_j \}|
\]

\[
> 4n - k \geq 4n - 2n = 2n,
\]

where the fact \( k \leq 2n \) is used. \( \square \)

### 3.3. The system of irrational numbers

In this subsection, we review some properties of a system of irrational numbers associated to our proof of Theorem 1.2, all the details can be found in Section 4 of [25]. Let \( \alpha = \{ \alpha_1, \alpha_2, \ldots, \alpha_m \} \) be a set of \( m \) irrational numbers. As usual, we have

**Definition 3.1.** The set \( \alpha \) of irrational numbers is linearly independent over \( \mathbb{Q} \), if there do not exist \( c_1, c_2, \ldots, c_m \) in \( \mathbb{Q} \) such that \( \sum_{j=1}^{m} |c_j| > 0 \) and

\[
\sum_{j=1}^{m} c_j \alpha_j \in \mathbb{Q},
\]

(3.19)
and is linearly dependent over $\mathbb{Q}$ otherwise. The rank of $\alpha$ is defined to be the number of elements in a maximal linearly independent subset of $\alpha$, which we denote by $\text{rank}(\alpha)$.

**Lemma 3.5.** Let $r = \text{rank}(\alpha)$. Then there exist $p_{jl} \in \mathbb{Z}$, $\beta_l \in \mathbb{Q}^c$ and $\xi_j \in \mathbb{Q}$ for $1 \leq l \leq r$ and $1 \leq j \leq m$ such that

$$\alpha_j = \sum_{l=1}^{r} p_{jl}\beta_l + \xi_j, \quad \forall 1 \leq j \leq m. \quad (3.20)$$

In order to study the multiplicity of closed geodesics on $S^{2n+1}/\Gamma$ with a bumpy irreversible metric $F$, we are particularly interested in the irrational system $\{\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_k\}$ with rank 1 satisfying (3.7). Then by Lemma 3.5, it can be reduced to the following system

$$\hat{\theta}_j = p_j\theta + \xi_j, \quad \forall 1 \leq j \leq k, \quad (3.21)$$

with $\theta \in \mathbb{Q}^c$, $p_j \in \mathbb{Z}\setminus\{0\}$, $\xi_j \in \mathbb{Q} \cap [0, 1)$ satisfying

$$p_1 + p_2 + \cdots + p_k = 0, \quad (3.22)$$

$$\{\xi_1 + \xi_2 + \cdots + \xi_k\} \in (0, 1) \setminus \{1/2\}, \quad (3.23)$$

where to get $\xi_j \in [0, 1)$, if necessary, we can replace $\hat{\theta}_j$ and $\xi_j$ by $\bar{\theta}_j = \hat{\theta}_j - [\xi_j]$ and $\bar{\xi}_j = \{\xi_j\}$.

Next we prove (3.22) and (3.23). In fact, by (3.21) we get

$$\sum_{j=1}^{k} \hat{\theta}_j = (\sum_{j=1}^{k} p_j)\theta + (\sum_{j=1}^{k} \xi_j),$$

where $\sum_{j=1}^{k} p_j \in \mathbb{Z}$, $\theta \in \mathbb{Q}^c$, and $\sum_{j=1}^{k} \xi_j \in \mathbb{Q}$. On the other hand, by (3.7) we have $\sum_{j=1}^{k} \hat{\theta}_j \in \mathbb{Q}$ and $\{\hat{\theta}_1 + \hat{\theta}_2 + \cdots + \hat{\theta}_k\} \neq \frac{1}{2}$ or 0 due to $\frac{2n}{p(n+1)} \notin \mathbb{Z}$. Thus (3.22) and (3.23) hold.

Take an arbitrarily $\eta \in \mathbb{Q}$ and make the following natural $\eta$-action to the system (3.21):

$$\eta(\theta) = \theta + \eta, \quad \eta(\hat{\theta}_j) = \hat{\theta}_j - [\xi_j - p_j\eta], \quad \text{and} \quad \eta(\xi_j) = \{\xi_j - p_j\eta\}, \quad \forall 1 \leq j \leq k, \quad (3.24)$$

which is obviously induced by the transformation $\eta(\theta) = \theta + \eta$. Then, we get a new system

$$\eta(\hat{\theta}_j) = p_j\eta(\theta) + \eta(\xi_j), \quad \forall 1 \leq j \leq k, \quad (3.25)$$

with

$$\{\eta(\xi_1) + \eta(\xi_2) + \cdots + \eta(\xi_k)\} = \{\{\xi_1 - p_1\eta\} + \{\xi_2 - p_2\eta\} + \cdots + \{\xi_k - p_k\eta\}\}$$

$$= \{\xi_1 + \xi_2 + \cdots + \xi_k - (p_1 + p_2 + \cdots + p_k)\eta\}$$

$$= \{\xi_1 + \xi_2 + \cdots + \xi_k\}, \quad (3.26)$$

where in the third equality we have used the condition (3.22). For simplicity, we also denote the new system (3.25) by $\eta(3.21)$ meaning that it comes from (3.21) by an $\eta$-action.
For the system \((3.21)_\eta\) with \(\eta \in \mathbb{Q}\), we divide the set \(\{1 \leq j \leq k\}\) into the following three parts:

\[
K^+_0(\eta) = \{1 \leq j \leq k \mid \eta(\xi_j) = 0, \, p_j > 0\}, \\
K^-_0(\eta) = \{1 \leq j \leq k \mid \eta(\xi_j) = 0, \, p_j < 0\}, \\
K_1(\eta) = \{1 \leq j \leq k \mid \eta(\xi_j) \neq 0\}.
\] (3.27)

Denote by \(k^+_0(\eta), k^-_0(\eta)\) and \(k_1(\eta)\) the numbers \#\(K^+_0(\eta)\), \#\(K^-_0(\eta)\) and \#\(K_1(\eta)\) respectively. For the case of \(\eta = 0\), we write them for short as \(k^+_0\), \(k^-_0\) and \(k_1\). It follows immediately that

\[
k^+_0(\eta) + k^-_0(\eta) + k_1(\eta) = k.
\]

By (3.23) and (3.26), it is obvious that \(k_1(\eta) \geq 1\) for every \(\eta \in \mathbb{Q}\).

**Definition 3.2.** For every \(\eta \in \mathbb{Q}\), the **absolute difference number** of \((3.21)_\eta\) is defined to be the non-negative number \(|k^+_0(\eta) - k^-_0(\eta)|\). The **effective difference number** of \((3.21)\) is defined by

\[
\max\{|k^+_0(\eta) - k^-_0(\eta)| \mid \eta \in \mathbb{Q}\}.
\]

Two systems of irrational numbers with rank 1 are called to be **equivalent**, if their effective difference numbers are the same.

**Remark 3.2.** By the definition of an \(\eta\)-action in (3.24), it can be checked directly that \(\eta_1 \circ \eta_2 = \eta_1 + \eta_2\) for every \(\eta_1\) and \(\eta_2\) in \(\mathbb{Q}\). So every system of irrational numbers with rank 1 is equivalent to the one which comes from itself by an \(\eta\)-action.

The following theorem is concerned with the lower estimate on the effective difference number of (3.21) and will play a crucial role in our proof of Theorem 1.2 in Section 4. For the reader’s convenience, we give its proof as an appendix in Section 5.

**Theorem 3.2.** For every system of irrational numbers \((3.21)\) satisfying the conditions (3.22) and (3.23), it holds that

\[
\max\{|k^+_0(\eta) - k^-_0(\eta)| \mid \eta \in \mathbb{Q}\} \geq 1. \tag{3.28}
\]

4. **Proof of Theorem 1.2.** In this section, we prove our main Theorem 1.2. By Theorem 3.1, we only need to prove it for \(M = S^{2n+1}/\Gamma\) with a bumpy irreversible Finsler metric \(F\) which is involved in the irrational system \(\{\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_k\}\) with \(2 \leq k \leq 2n\) satisfying (3.7). For sake of readability, we carry out the proof into two cases according to whether \(\text{rank}(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_k) = 1\) or not. We will give in details the proof for the first case. Based on the well known Kronecker’s approximation theorem in Number theory, the second one can be then proved quite similarly and so we only sketch it.

**Proof of Theorem 1.2.** We carry out the proof in two cases.

**Case 1.** \(r = \text{rank}(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_k) = 1\).

As we have mentioned in Subsection 3.3, the irrational system (3.7) with \(r = 1\) can be seen as a special case of (3.21) satisfying (3.22) and (3.23).

Since any \(\eta\)-action with \(\eta \in \mathbb{Q}\) to (3.21), if necessary, does no substantive effect on our following arguments, by Theorem 3.2 and Remark 3.1 we can assume without loss of generality that

\[
|k^+_0 - k^-_0| \geq 1 \text{ and } K_1 = \{1, 2, \ldots, k_1\},
\]
with $k_1 \geq 1$ due to (3.23), and denote by $\xi_j = \frac{r_j}{q_j}$ for $1 \leq j \leq k_1$. Let $q = q_1 q_2 \cdots q_{k_1}$ and $m_l = \tilde{p}(n + 1)q_l + 1$ with $l \in \mathbb{N}$, where $\tilde{p}$ is given by Lemma 3.3. Then by (3.21) we have

$$\sum_{j=2}^{k_1} \{m_l \hat{\theta}_j\} = \sum_{j=2}^{k_1} \{m_l \hat{\theta}_j\} + \sum_{j=k_1+1}^{k} \{m_l \hat{\theta}_j\} = \sum_{j=2}^{k_1} \{p_j \{m_l \theta\} + \xi_j\} + \sum_{j=k_1+1}^{k} \{p_j \{m_l \theta\}\}, \tag{4.1}$$

for some $\theta \in \mathbb{Q}^c$. Then the set \{\{m_l \theta\} | l \in \mathbb{N}\} is dense in $[0, 1]$. For every $L \in \mathbb{Z}$, we introduce the auxiliary function

$$f_L(x) = \sum_{j=2}^{k_1} \{p_j x + \xi_j\} + \sum_{j=k_1+1}^{k} \{p_j x\} + \bar{p}L \hat{\theta}_j, \quad \forall x \in [0, 1], \tag{4.2}$$

and denote for simplicity by $f = f_0$, which contains only finitely many discontinuous points.

Let $a$ and $b$ in $(0, 1)$ be two real numbers sufficiently close to 0 and 1 respectively. Then,

$$f(a) = \sum_{j=2}^{k_1} (p_j a + \xi_j) + \sum_{j=k_1+1}^{k} p_j a = \sum_{j=2}^{k_1} (p_j a + \xi_j) + \sum_{j \in K_0^{\pm}} p_j a + \sum_{j \in K_0^{-}} (1 + p_j a) = k_0^- + \sum_{j=2}^{k} p_j a + \sum_{j=2}^{k} \xi_j, \tag{4.3}$$

and by similar computation,

$$f(b) = k_0^+ + \sum_{j=2}^{k} p_j (b - 1) + \sum_{j=2}^{k} \xi_j. \tag{4.4}$$

It follows by (4.3) and (4.4) that

$$|f(b) - f(a)| = |k_0^+ - k_0^- + \sum_{j=2}^{k} p_j (b - a)| = |k_0^+ - k_0^- + p_1 (-b + 1 + a)|, \tag{4.5}$$

where in the second identity we have used $\sum_{j=1}^{k} p_j = 0$.

**Lemma 4.1.** Given $N \in \mathbb{N}$, for any $a$ and $b$ in $(0, 1)$ sufficiently close to 0 and 1 respectively, we have

(i) $f(a)$ and $f(b)$ lie in different intervals of (3.17) with $L = 0$,

(ii) $f_L(a)$ and $f_L(b)$ lie in the same interval of (3.17) for any $1 \leq |L| \leq N$, including $f_L(0)$.

**Proof.** (i) By (4.5) and the assumption, $|f(b) - f(a)| \approx |k_0^+ - k_0^-|$. Here and below, we write $A \approx B$, if $A$ and $B$ can be chosen to be as close to each other as we want. Since the length of each interval in (3.17) with $L = 0$ is less than or equal to 1, so $f(a)$ and $f(b)$ must lie in different ones, provided $|k_0^+ - k_0^-| \geq 2$. 


If $|k_0^+ - k_0^-| = 1$, then $|f(b) - f(a)| \approx 1$. For the case of $k = 2$, since the length of each interval of (3.17) with $L = 0$ is less than 1, (i) follows immediately. The rest case is $k \geq 3$, which still contains three subcases.

1° If $k_1 \geq 2$, by (3.21)-(3.22), (3.7) and (3.15), we have

$$\left\{ \sum_{j=1}^{k_1} \xi_j \right\} = \left\{ \sum_{j=1}^{k} \xi_j \right\} = \left\{ \sum_{j=1}^{k} \bar{\theta}_j \right\} = \left\{ \frac{k}{2} + \frac{n}{p(n+1)} \right\} = \{Q_0\},$$

and so

$$\left\{ \frac{k_1}{2} + \sum_{j=2}^{k_1} \xi_j \right\} = \left\{ -\xi_1 + \sum_{j=1}^{k_1} \xi_j \right\} = \left\{ \sum_{j=1}^{k_1} \xi_j - \xi_1 \right\} = \{\{Q_0\} - \xi_1\}.$$ It then follows from (4.3) that $\{f(a)\} \approx \{\{Q_0\} - \xi_1\}$ or 1 and $f(a)$ is not equal to these two numbers since $\sum_{j=2}^{k} p_j = -p_1 \neq 0$ by (3.23) and the fact that $p_1 \neq 0$. Notice that the dividing points of the intervals in (3.17) with $L = 0$ are

$$0, \{Q_0\}, 1 + \{Q_0\}, 2 + \{Q_0\}, \ldots, k - 2 + \{Q_0\}, k - 1.$$ Therefore $f(a)$ must be an interior point of these intervals. It then yields that $f(a)$ and $f(b)$ must lie in two different intervals.

2° If $k_1 = 1$ and $k_0^+ \geq 1$, then $f(a) \approx k_0^-$ is also an interior point and (i) follows.

3° If $k_1 = 1$ and $k_0^- = 0$, then $f(a) = \sum_{j=2}^{k} p_j a_j$ lies in the first interval whose length is $\{Q_0\} < 1$ and so $f(b)$ must lie in another one.

(ii) It can be checked directly that $\lim_{a \to 0} f_L(a) = \lim_{b \to 1} f_L(b) = f_L(0) \in \mathbb{Q}^c$, since $\xi_j \in \mathbb{Q}$ for $1 \leq j \leq k$ and $\sum_{j=2}^{k} p_j \bar{\theta}_j \in \mathbb{Q}^c$ by (3.7). But the dividing points of these intervals in (3.17) with $1 \leq |L| \leq N$ are finitely many rational numbers, so $f_L(0)$ is an interior point of these intervals and (ii) follows.

Notice that $f$ contains only finitely many discontinuous points on $(0,1)$. Without loss of generality, we assume $a$ and $b$ to be two continuous points of $f$ and choose $l_1$, $l_2 \in \mathbb{N}$ with $l_2 - l_1$ sufficiently large such that $\{m_{l_1} \theta\} \approx a$ and $\{m_{l_2} \theta\} \approx b$. Then by (4.1), (4.2) and (i) of Lemma 4.1, we get $\sum_{j=2}^{k} \{m_{l_1} \bar{\theta}_j\}$ and $\sum_{j=2}^{k} \{m_{l_2} \bar{\theta}_j\}$ lie in different intervals of (3.17) with $L = 0$. Suppose that

$$\sum_{j=2}^{k} \{m_{l_1} \bar{\theta}_j\} \in I_{l'} \text{ and } \sum_{j=2}^{k} \{m_{l_2} \bar{\theta}_j\} \in I_{l''},$$

with $\{l', l''\} \subseteq \{0, 1, 2, \ldots, k - 1\}$ and $l' \neq l''$. By (3.18) we have $i(e^{m_{l_1} \theta}) = 2nq\ell_1 + 2|Q_0| - 2l'$ and

$$i(e^{m_{l_2} \theta}) = 2nq\ell_2 + 2|Q_0| - 2l''.$$

Since $2n \mid (2nq\ell_1 + 2|Q_0| - 2l')$ if and only if $2n \mid (2nq\ell_2 + 2|Q_0| - 2l'')$, we get by (2.9) with $n$ therein replaced by $2n + 1$ that

$$\beta_{2nq\ell_1 - 2l'} = \beta_{2nq\ell_2 + 2|Q_0| - 2l''} = \beta.$$

Take $N > 2(n+1)$ in (ii) of Lemma 4.1 and observe that

$$|2|Q_0| - 2l''| = 2 \left| \frac{k}{2} + \frac{n}{p(n+1)} \right| - 2l''$$

$$\leq \max \left\{ 2 \left( \frac{k}{2} + \frac{n}{p(n+1)} \right), 2(k-1) - 2 \left( \frac{k}{2} + \frac{n}{p(n+1)} \right) \right\} \leq k \leq 2n.$$
Claim. There are $L_i \in \mathbb{Z}$ with $1 \leq i \leq \beta$ satisfying $1 \leq |L_i| \leq \bar{N}$ and

$$i(e^{m_i + \bar{p}L_i}) = 2nq\bar{l}_1 + 2|Q_0| - 2l''.$$  

Indeed, by (4.7) there exist $\bar{m}_i$ with $1 \leq i \leq \beta$ such that $i(e^{\bar{m}_i}) = 2nq\bar{l}_1 + 2|Q_0| - 2l''$. According to Lemma 3.3, it also holds $\bar{m}_i \equiv \bar{l}_1 \pmod{\bar{p}}$. Recalling that $m_i = \bar{p}(n + 1)q\bar{l}_1 + 1 \equiv 1 \pmod{\bar{p}}$, we obtain $m_i - m_i \equiv 0 \pmod{\bar{p}}$ and so there is $L_i \in \mathbb{Z}$ such that $m_i = m_i + \bar{p}L_i$. Moreover, it follows from (4.8) and Lemma 3.4 that $|L_i| \leq \bar{N}$. Due to $i' \neq i''$, we have $L_i \neq 0$ and therefore $1 \leq |L_i| \leq \bar{N}$. The claim is proved.

Since $\{m_i\theta\} \approx a$ and $\{m_i\theta\} \approx b$, by (4.2) it holds that $\sum_{j=2}^{k} \{ (m_i + \bar{p}L_i) \bar{\theta}_j \} \approx f_{L_i}(a)$ and $\sum_{j=2}^{k} \{ (m_i + \bar{p}L_i) \bar{\theta}_j \} \approx f_{L_i}(b)$. Then (ii) of Lemma 4.1 yields that $\sum_{j=2}^{k} \{ (m_i + \bar{p}L_i) \bar{\theta}_j \} \approx f_{L_i}(b)$ and $\sum_{j=2}^{k} \{ (m_i + \bar{p}L_i) \bar{\theta}_j \} \approx f_{L_i}(b)$ are in the same interval of (3.17) with $1 \leq |L_i| \leq \bar{N}$. As a result, we get by (3.18) that

$$i(e^{m_i + \bar{p}L_i}) = 2nq\bar{l}_2 + 2|Q_0| - 2l'', \quad \forall 1 \leq i \leq \beta \quad (4.9)$$

Then it follows immediately from (4.6) and (4.9) that $\hat{\beta}2nq\bar{l}_2 + 2|Q_0| - 2l'' \geq \beta + 1$, which contradicts to (4.7).

Case 2. $r = \text{rank}(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_k) \geq 2$.

By Lemma 3.5, there are $p_{jl} \in \mathbb{Z}$, $\theta_{ki} \in \mathbb{Q}$ and $\xi_j \in \mathbb{Q}$ with $1 \leq l \leq r$ and $1 \leq j \leq k$ such that

$$\hat{\theta}_j = \sum_{l=1}^{r} p_{jl} \theta_{ki} + \xi_j, \quad \forall 1 \leq j \leq k. \quad (4.10)$$

Moreover, $\theta_{k_1}, \theta_{k_2}, \ldots, \theta_{k_r}$ are linearly independent over $\mathbb{Q}$. Due to (3.7), it follows

$$\sum_{j=1}^{k} p_{jl} = 0, \quad \forall 1 \leq l \leq r. \quad (4.11)$$

Then we have

$$\{\xi_1 + \xi_2 + \cdots + \xi_k\} \in (0, 1) \setminus \{1/2\}. \quad (4.12)$$

In fact, by (4.10) and (4.11) we have $\sum_{j=1}^{k} \hat{\theta}_j = \sum_{l=1}^{r} (\sum_{j=1}^{k} p_{jl}) \theta_{ki} + \sum_{j=1}^{k} \xi_j$, and from (3.7) we have $\{\hat{\theta}_1 + \hat{\theta}_2 + \cdots + \hat{\theta}_k\} \neq 1/2$ or 0, since $2n/\bar{p}(n+1) \notin \mathbb{Z}$. Thus (4.12) holds.

Our basic idea for proving Case 2 is to construct an irrational system with rank 1 associated to (4.10), which plays the essential role in our sequel arguments due to the following result.

Kronecker’s approximation theorem (cf. Theorem 7.10 in [2]): If $\theta_1, \theta_2, \ldots, \theta_r$ are linearly independent over $\mathbb{Q}$, then the set of all vectors of the form $(\{m\theta_1\}, \{m\theta_2\}, \ldots, \{m\theta_r\})$ for all $m \in \mathbb{N}$ is dense in $[0, 1]^r = \left[0, 1\right] \times [0, 1] \times \cdots \times [0, 1]$.

Lemma 4.2. For the integers $p_{jl}$’s in (4.10)-(4.11), there are $s_2, s_3, \ldots, s_r \in \mathbb{Z}$ such that

$$p_{j1} + \sum_{l=2}^{r} s_l p_{jl} \in \mathbb{Z} \setminus \{0\}, \quad \forall 1 \leq j \leq k, \quad (4.13)$$
Proof. Let $J_0 = \{1 \leq j \leq k \mid p_{j1} = 0\}$. If $J_0 = \emptyset$, we need only take $s_2 = s_3 = \cdots = s_r = 0$. If $J_0 \neq \emptyset$, we claim that $(p_{j2}, p_{j3}, \ldots, p_{jr}) \neq (0, 0, \ldots, 0)$ for each $j \in J_0$. Otherwise, then (4.10) yields that $\tilde{\theta}_j = \xi_j \in \mathbb{Q}$, which contradicts $\tilde{\theta}_j \in \mathbb{Q}^c$. So the set

$$X_j \equiv \{(x_2, x_3, \ldots, x_r) \mid p_{j2}x_2 + p_{j3}x_3 + \cdots + p_{jr}x_r = 0\},$$

is a subspace of dimension $r - 2$ in $\mathbb{R}^{r-1}$ which yields that $X = \bigcup_{j \in J_0} X_j$ is a proper subset of $\mathbb{R}^{r-1}$. Pick up an arbitrary integral point $(\bar{s}_2, \bar{s}_3, \ldots, \bar{s}_r) \in \mathbb{R}^{r-1}\setminus X$. Then for every $N \in \mathbb{N}$ we have

$$|p_{j1} + \sum_{i=2}^{r} \bar{N} \bar{s}_i p_{ji}| = \begin{cases} \bar{N} \sum_{i=2}^{r} |\bar{s}_i p_{ji}| \neq 0, & \text{if } j \in J_0, \\ |p_{j1}| \neq 0, & \text{if } j \notin J_0 \text{ and } \sum_{i=2}^{r} \bar{s}_i p_{ji} = 0, \\ |p_{j1} + \bar{N} \sum_{i=2}^{r} \bar{s}_i p_{ji}|, & \text{if } j \notin J_0 \text{ and } \sum_{i=2}^{r} \bar{s}_i p_{ji} \neq 0. \end{cases}$$

(4.14)

For the third case in the righthand side of (4.14), we can take $\bar{N} \in \mathbb{N}$ sufficiently large so that $|p_{j1} + \bar{N} \sum_{i=2}^{r} \bar{s}_i p_{ji}| \neq 0$ for all these $j$’s therein. Finally let $s_l = \bar{N} \bar{s}_l$ and (4.13) follows. \hfill \Box

Let $\tilde{p}_{j1} = p_{j1} + \sum_{i=2}^{r} \bar{s}_i p_{ji} \in \mathbb{Z}\setminus\{0\}$ and $\tilde{p}_{jl} = p_{jl}$ if $2 \leq l \leq r$. By Lemma 4.2, we can make the change of variables $\theta_{k_1} = \tilde{\theta}_{k_1}$, and $\theta_{k_l} = \tilde{\theta}_{k_l} - s_l\theta_{k_1}$ for $2 \leq l \leq r$. Then the system (4.10) is transformed to

$$\tilde{\theta}_j = \sum_{l=1}^{r} \tilde{p}_{jl} \tilde{\theta}_{k_l} + \xi_j, \ \forall 1 \leq j \leq k,$$

(4.15)

and by (4.11) we have

$$\sum_{j=1}^{k} \tilde{p}_{j1} = \sum_{j=1}^{k} p_{j1} + \sum_{j=1}^{k} \sum_{i=2}^{r} s_i p_{ji} = 0 + \sum_{i=2}^{r} \bar{s}_i \left( \sum_{j=1}^{k} p_{ji} \right) = 0.$$

Since $\theta_{k_1}, \theta_{k_2}, \ldots, \theta_{k_r}$ are linearly independent over $\mathbb{Q}$, so do $\tilde{\theta}_{k_1}, \tilde{\theta}_{k_2}, \ldots, \tilde{\theta}_{k_r}$.

Consider the following irrational system with rank 1 associated to (4.15)

$$\hat{\alpha}_j = \tilde{p}_{j1} \tilde{\theta}_{k_1} + \xi_j, \ \forall 1 \leq j \leq k.$$

(4.16)

By Theorem 3.2 and the properties of $\xi_j$ in (4.10) and (4.12), without loss of generality we can assume for (4.16) that $|\hat{k}_1 – \hat{k}_0| \geq 1$ and denote the corresponding integer set in (3.27) by $K_1(0) = \{1, 2, \ldots, \hat{k}_1\}$, and denote $\xi_j$ by $\xi_j = \frac{r_j}{\hat{q}_j}$ with $(r_j, q_j) = 1$ for $1 \leq j \leq \hat{k}_1$.

Let $\hat{q} = q_1q_2\cdots q_{\hat{k}_1}$, and $\hat{m}_l = \hat{p}(n+1)\hat{q}l + 1$ for $l \in \mathbb{N}$, where $\hat{p}$ is given by Lemma 3.3. Then, we get by (4.15) that

$$\sum_{j=2}^{k} \left\{ \hat{m}_j \hat{\theta}_j \right\} = \sum_{j=2}^{k} \left\{ \hat{m}_j \tilde{\theta}_j \right\} + \sum_{j=k_1+1}^{k} \left\{ \hat{m}_j \tilde{\theta}_j \right\}$$

$$= \sum_{j=2}^{k_1} \left\{ \sum_{l=1}^{r} \tilde{p}_{jl} \hat{m}_j \tilde{\theta}_{k_l} + \xi_j \right\} + \sum_{j=k_1+1}^{k} \left\{ \sum_{l=1}^{r} \tilde{p}_{jl} \hat{m}_j \tilde{\theta}_{k_l} \right\},$$

(4.17)

where we have used the fact that $\xi_j = 0$ when $\hat{k}_1 + 1 \leq j \leq k$. By Kronecker’s approximation theorem, the set $\{(\hat{m}_1 \tilde{\theta}_{k_1}), (\hat{m}_1 \tilde{\theta}_{k_2}), \ldots, (\hat{m}_l \tilde{\theta}_{k_r}) \mid l \in \mathbb{N}\}$ is dense
in $[0, 1]^r$. For every $L \in \mathbb{Z}$, similarly to (4.2) we can introduce the auxiliary multivariable function on $[0, 1]^r$,
\[
g_L(x_1, x_2, \ldots, x_r) = \sum_{j=2}^{k_1} \left\{ \sum_{i=1}^{r} \tilde{p}_{ji} x_i + \xi_j + \tilde{p} L \tilde{\theta}_j \right\} + \sum_{j=k_1+1}^{k} \left\{ \sum_{i=1}^{r} \tilde{p}_{ji} x_i + \tilde{p} L \tilde{\theta}_j \right\},
\]
and denote for simplicity by $g = g_0$. Similarly as before, we have

**Lemma 4.3.** Given $\bar{N} \in \mathbb{N}$, let $(a_1, a_2, \ldots, a_r)$ and $(b_1, b_2, \ldots, b_r)$ in $(0, 1)^r$ be sufficiently close to $(0, 0, 0, \ldots, 0)$ and $(1, 0, 0, \ldots, 0)$ respectively. Then

(i) $g(a_1, a_2, \ldots, a_r)$ and $g(b_1, b_2, \ldots, b_r)$ lie in different intervals of (3.17) with $L = 0$, if we further require $\frac{a_2 + \cdots + a_r}{a_1}$ and $\frac{b_2 + \cdots + b_r}{1 - b_1}$ are sufficiently small.

(ii) $g_L(a_1, a_2, \ldots, a_r)$ and $g_L(b_1, b_2, \ldots, b_r)$ lie in the same interval of (3.17) for any $1 \leq |L| \leq \bar{N}$, including $g_L(0, 0, \ldots, 0)$.

**Proof.** (i) Since $a_1, a_2, \ldots, a_r$ (resp. $b_1, b_2, \ldots, b_r$) are independent, we can select them by such a way that the decimal functions in $g(a_1, a_2, \ldots, a_r)$ and $g(b_1, b_2, \ldots, b_r)$ are mainly determined by $a_1$ and $b_1$ respectively. For instance, this can be realized by requiring $a_1$ (resp. $b_1$) with $2 \leq l \leq r$ to be much smaller than $a_1$ (resp. $1 - b_1$). The rest proof is then similar to that in Lemma 4.1-(i), with the function there replaced by the current $g$.

(ii) follows from the proof of Lemma 4.1-(ii) without the choices on $\frac{a_2 + \cdots + a_r}{a_1}$ and $\frac{b_2 + \cdots + b_r}{1 - b_1}$ there.

Due to Lemma 4.3, the rest proof is then almost word by word as that in Case 1 and thus is omitted. The proof of Theorem 1.2 is complete. \hfill \Box

5. **Appendix.** For the reader’s convenience, we give the proof of Theorem 3.2 as an appendix in this section.

**Lemma 5.1.** Assume that
\[
\begin{align*}
\hat{\theta}_j &= p_j \theta + \xi_j, & \forall 1 \leq j \leq k - 1, \\
\hat{\theta}_k &= p_k \theta,
\end{align*}
\]
with $\sum_{j=1}^{k} p_k = 0$ and $\left\{ \sum_{j=1}^{k} \xi_k \right\} \in (0, 1) \setminus \{1/2\}$.

Then, (5.1) is equivalent to
\[
\begin{align*}
\hat{\theta}_j &= p_j \theta + \xi_j, & \forall 1 \leq j \leq k - 1, \\
\hat{\theta}_{k,l} &= \text{sgn}(p_k) \theta + \frac{l}{|p_k|}, & \forall 0 \leq l \leq |p_k| - 1,
\end{align*}
\]
where as usual we define $\text{sgn}(a) = \pm 1$ for $a \in \mathbb{R} \setminus \{0\}$ when $\pm a > 0$.

**Proof.** Take $\eta \in \mathbb{Q}$ arbitrarily and recall the definition of $\eta$-action in (3.24). Then the equation $\hat{\theta}_k = p_k \theta$ contributes $\text{sgn}(p_k)$ to the absolute difference number of (5.1)$_\eta$ if and only if
\[
\eta(0) = \{0 - p_k \eta\} = \{-p_k \eta\} = 0,
\]
that is $\eta \in \mathbb{Z}_{|p_k|}$, which is also the sufficient and necessary condition such that the equations
\[
\hat{\theta}_{k,l} = \text{sgn}(p_k) \theta + \frac{l}{|p_k|}, & \forall 0 \leq l \leq |p_k| - 1,
\]
contribute $\text{sgn}(p_k)$ to the absolute difference number of (5.2)$_\eta$. Since the other equations with $1 \leq j \leq k - 1$ in (5.1) and (5.2) are the same, so do their contributions to the absolute difference numbers of (5.1)$_\eta$ and (5.2)$_\eta$. As a result, the absolute
difference numbers of \((5.1)_\eta\) and \((5.2)_\eta\) are equal for any \(\eta \in \mathbb{Q}\) which yields that the effective difference numbers of \((5.1)\) and \((5.2)\) are the same and so they are equivalent.

**Remark 5.1.** For the system \((5.2)\), we have

\[
\begin{aligned}
\left\{ \sum_{j=1}^{k-1} \xi_j + \sum_{l=0}^{\frac{|p_k|}{2p_k}} \frac{l}{|p_k|} \right\} = \left\{ \begin{array}{ll}
\sum_{j=1}^{k-1} \xi_j + \frac{1}{2}, & \text{if } p_k \text{ is even,} \\
\sum_{j=1}^{k-1} \xi_j, & \text{if } p_k \text{ is odd,}
\end{array} \right.
\end{aligned}
\]

By the assumption of \(\eta\)
difference numbers of \((5.1)\) and \((5.2)\) are the same and so they are equivalent.

**Lemma 5.2.** If there exist \(1 \leq j' < j'' \leq k\) satisfying that \(p_{j'} \cdot p_{j''} = -1\) and \(\{\xi_{j'} + \xi_{j''}\} = 0\) in

\[
\theta_j = p_j \theta \pm \xi_j, \quad \forall 1 \leq j \leq k, \tag{5.3}
\]
then \((5.3)\) is equivalent to the system

\[
\theta_j = p_j \theta \pm \xi_j, \quad \forall j \in \{1, 2, \ldots, k\} \setminus \{j', j''\}. \tag{5.4}
\]

**Proof.** Assume without loss of generality that \(p_{j'} = -p_{j''} = 1\) and take \(\eta \in \mathbb{Q}\) arbitrarily. Then by (3.24) and the given condition, we have

\[
\{\eta(\xi_{j'}) + \eta(\xi_{j''})\} = \{\{\xi_{j'} - \eta\} + \{\xi_{j''} + \eta\}\} = \{\xi_{j'} + \xi_{j''}\} = 0.
\]
Thus, \(\eta(\xi_{j'}) = 0\) if and only if \(\eta(\xi_{j''}) = 0\), that is, \(j' \in K\eta_{\mathbb{Q}}^+(\eta)\) and only if \(j'' \in K\eta_{\mathbb{Q}}^- (\eta)\). As a result, \(p_{j'}\) and \(p_{j''}\) together contribute nothing to the absolute difference number of \((5.3)_\eta\) for any \(\eta \in \mathbb{Q}\). It then follows immediately that \((5.3)\) is equivalent to \((5.4)\). \(\square\)

**Proof of Theorem 3.2.** We carry out the proof in two steps.

**Step 1.** First, letting \(\eta_k = \frac{\xi_k}{p_k}\) and making \(\eta_k\)-action to the original system (3.21), we obtain by (3.24) that

\[
\begin{aligned}
\{ \eta_k(\theta_j) = p_j \eta_k(\theta) + \eta_k(\xi_j), \quad \forall 1 \leq j \leq k - 1,
\eta_k(\theta_k) = p_k \eta_k(\theta). \quad \forall 1 \leq j \leq k - 1\},
\end{aligned}
\]

Then by Lemma 5.1, the system \((5.5)\) is equivalent to

\[
\begin{aligned}
\eta_k(\theta_j) = p_j \eta_k(\theta) + \eta_k(\xi_j), \quad \forall 1 \leq j \leq k - 1,
\theta_{k,l'} = \text{sgn}(p_k) \eta_k(\theta) + \frac{l'}{|p_k|}, \quad \forall 0 \leq l' \leq |p_k| - 1, \quad \forall 1 \leq j \leq k - 1\},
\end{aligned}
\]

Secondly, taking \(\eta_{k-1} \in \mathbb{Q}\) such that \(\eta_{k-1} \circ \eta_k(\xi_{k-1}) = 0\) and making \(\eta_{k-1}\)-action to the system \((5.6)\), we get

\[
\begin{aligned}
\eta_{k-1} \circ \eta_k(\theta_j) = p_j \eta_{k-1} \circ \eta_k(\theta) + \eta_{k-1} \circ \eta_k(\xi_j), \quad \forall 1 \leq j \leq k - 2,
\eta_{k-1} \circ \eta_k(\theta_{k-1}) = p_{k-1} \eta_{k-1} \circ \eta_k(\theta),
\eta_{k-1} \circ \theta_{k,l'} = \text{sgn}(p_k) \eta_{k-1} \circ \eta_k(\theta) + \eta_{k-1}(\frac{l'}{|p_k|}), \quad \forall 0 \leq l' \leq |p_k| - 1, \quad \forall 1 \leq j \leq k - 2, \quad \forall 1 \leq j \leq k - 1\},
\end{aligned}
\]

Again by Lemma 5.1, the system \((5.7)\) is equivalent to

\[
\begin{aligned}
\eta_{k-1} \circ \eta_k(\theta_j) = p_j \eta_{k-1} \circ \eta_k(\theta) + \eta_{k-1} \circ \eta_k(\xi_j), \quad \forall 1 \leq j \leq k - 2,
\theta_{k-1,l''} = \text{sgn}(p_{k-1}) \eta_{k-1} \circ \eta_k(\theta) + \frac{l''}{|p_{k-1}|}, \quad \forall 0 \leq l'' \leq |p_{k-1}| - 1, \quad \forall 1 \leq j \leq k - 2, \quad \forall 1 \leq j \leq k - 1\},
\end{aligned}
\]

We conclude the proof. \(\square\)
Repeating the above procedure for the rest equations with \( j = k - 2, k - 3, \ldots, 2, 1 \) one at a time in order, we can finally get a system equivalent to the original system (3.21) which can be written in a simple form such as
\[
\tilde{\alpha}_{jl} = \text{sgn}(p_j) \alpha + \xi_{jl}, \ \forall \ 1 \leq j \leq k \text{ and } 0 \leq l \leq |p_j| - 1,
\]
with \( \alpha \in \mathbb{Q}^c \) and \( \xi_{jl} \in \mathbb{Q} \cap [0, 1) \). Moreover, by (3.26) and Remark 5.1 we have
\[
\left\{ \sum_{j=1}^{k} \sum_{l=0}^{|p_j|-1} \xi_{jl} \right\} \in (0, 1) \backslash \{1/2\}.
\]

**Step 2.** We can cut off all the superfluous equations of the system (5.9), if there are such pairs as that in Lemma 5.2. That is, (5.9) is equivalent to the system
\[
\theta_i' = p_i' \alpha + \xi'_i, \ \forall \ 1 \leq i \leq \bar{k},
\]
with \( |p_i'| = 1, \sum_{i=1}^{\bar{k}} p_i' = 0 \) and
\[
\left\{ \sum_{i=1}^{\bar{k}} \xi'_i \right\} \in (0, 1) \backslash \{1/2\}.
\]

Here note that \( \bar{k} \geq 1 \) is ensured by the condition (5.12).

Since all the superfluous equations have been cut off, it follows that \( \bar{k}_0^+ \cdot \bar{k}_0^- = 0 \). Assume without loss of generality that \( \bar{k}_0^+ = \bar{k}_0^- = 0 \), otherwise we have nothing to do. Since \( \sum_{i=1}^{\bar{k}} p_i' = 0 \), we get
\[
\# \{1 \leq i \leq \bar{k} \mid p_i' = 1\} = \# \{1 \leq i \leq \bar{k} \mid p_i' = -1\}.
\]

Pick up an arbitrarily \( i_1 \in \{1 \leq i \leq \bar{k} \mid p_i' = 1\} \). Let \( \bar{\eta} = \xi'_{i_1} \) and make the \( \bar{\eta} \)-action to (5.11). Then it follows immediately that \( \bar{k}_0^+ (\bar{\eta}) \geq 1 \). Recalling again that all the superfluous equations have been cut off at the beginning of Step 2, we obtain \( \bar{\eta}(\xi'_{i}) = \{\xi'_{i_1} + \xi'_{i}\} \neq 0 \) for every \( i \in \{1 \leq i \leq \bar{k} \mid p_i' = -1\} \) which yields \( \bar{k}_0^+ (\bar{\eta}) = 0 \).

As a result, we get
\[
\max \{ |\bar{k}_0^+ (\bar{\eta}) - \bar{k}_0^- (\eta)| \mid \eta \in \mathbb{Q} \} \geq |\bar{k}_0^+ (\bar{\eta}) - \bar{k}_0^- (\bar{\eta})| = \bar{k}_0^+ (\bar{\eta}) \geq 1.
\]

Since the original system (3.21) is equivalent to (5.11), the estimate (3.28) follows immediately.

**Acknowledgments.** We would like to sincerely thank the referee for her/his careful reading of the manuscript, valuable suggestion and comments on improving the exposition.

**REFERENCES**

[1] D. V. Anosov, Geodesics in Finsler geometry, Proc. I.C.M. (Vancouver, B.C. 1974), 2 (1975), 293–297; Montreal (Russian), Amer. Math. Soc. Transl., 109 (1977), 81–85.

[2] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, GTM 41, 1990.

[3] V. Bangert, On the existence of closed geodesics on two-spheres, Internat. J. Math., 4 (1993), 1–10.

[4] V. Bangert and W. Klingenberg, Homology generated by iterated closed geodesics, Topology., 22 (1983), 379–388.

[5] V. Bangert and Y. Long, The existence of two closed geodesics on every Finsler 2-sphere, Math. Ann., 346 (2010), 335-366.

[6] R. Bott, On the iteration of closed geodesics and the Sturm intersection theory, Comm. Pure Appl. Math., 9 (1956), 171–206.
[7] K. Burns and S. Matveev, Open problems and questions about closed geodesics, arXiv:1308.5417v2, 2014.

[8] H. Duan and Y. Long, Multiple closed geodesics on bumpy Finsler n-spheres, J. Diff. Equa., 233 (2007), 221–240.

[9] H. Duan and Y. Long, The index growth and multiplicity of closed geodesics, J. Funct. Anal., 259 (2010), 1850–1913.

[10] H. Duan, Y. Long and W. Wang, Two closed geodesics on compact simply-connected bumpy Finsler manifolds, J. Differ. Geom., 104 (2016), 275–289.

[11] H. Duan, Y. Long and W. Wang, The enhanced common index jump theorem for symplectic paths and non-hyperbolic closed geodesics on Finsler manifolds, Calc. Var. and PDEs, 55 (2016), Art. 145, 28 pp.

[12] H. Duan, Y. Long and Y. Xiao, Two closed geodesics on $RP^n$ with a bumpy Finsler metric, Calc. Var. and PDEs, 54 (2015), 2883–2894.

[13] J. Franks, Geodesics on $S^2$ and periodic points of annulus homeomorphisms, Invent. Math., 108 (1992), 403–418.

[14] D. Gromoll and W. Meyer, On differentiable functions with isolated critical points, Topology, 8 (1969), 361–369.

[15] D. Gromoll and W. Meyer, Periodic geodesics on compact Riemannian manifolds, J. Diff. Geom., 3 (1969), 493–510.

[16] N. Hingston, Equivariant Morse theory and closed geodesics, J. Diff. Geom., 19 (1984), 85–116.

[17] N. Hingston, On the growth of the number of closed geodesics on the two-sphere, Inter. Math. Research Notices, 9 (1993), 253–262.

[18] N. Hingston and H.-B. Rademacher, Resonance for loop homology of spheres, J. Differ. Geom., 93 (2013), 133–174.

[19] A. B. Katok, Ergodic properties of degenerate integrable Hamiltonian systems. Izv. Akad. Nauk. SSSR, 37 (1973), 539–576; [Russian]; Math. USSR-Izv., 7 (1973), 535–571.

[20] W. Klingenberg, Lectures on Closed Geodesics, Springer-Verlag, Berlin, heidelberg, New York, 1978.

[21] W. Klingenberg, Riemannian Geometry, De Gruyter, 2nd Rev ed. edition, 1995.

[22] C. Liu, The relation of the morse index of closed geodesics with the maslov-type index of symplectic paths, Acta Math. Sinica, 21 (2005), 237–248.

[23] C. Liu and Y. Long, Iterated index formulae for closed geodesics with applications, Science in China., 45 (2002), 9–28.

[24] H. Liu, The Fadell-Rabinowitz index and multiplicity of non-contractible closed geodesics on Finsler $RP^n$, J. Differential Equations, 262 (2017), 2540–2553.

[25] H. Liu and Y. Xiao, Resonance identity and multiplicity of non-contractible closed geodesics on Finsler $RP^n$, Advances in Math., 318 (2017), 158–190.

[26] Y. Long, Bott formula of the Maslov-type index theory, Pacific J. Math., 187 (1999), 113–149.

[27] Y. Long, Precise iteration formulae of the Maslov-type index theory and ellipticity of closed characteristics, Advances in Math., 154 (2000), 76–131.

[28] Y. Long, Index Theory for Symplectic Paths with Applications, Progress in Math. 207, Birkhäuser, 2002.

[29] Y. Long, Multiplicity and stability of closed geodesics on Finsler 2-spheres, J. Eur. Math. Soc., 8 (2006), 341–353.

[30] Y. Long and H. Duan, Multiple closed geodesics on 3-spheres, Advances in Math., 221 (2009), 1757–1803.

[31] Y. Long and W. Wang, Multiple closed geodesics on Riemannian 3-spheres, Calc. Var. and PDEs, 30 (2007), 183–214.

[32] A. Oancea, Morse theory, closed geodesics, and the homology of free loop spaces, With an appendix by Umberto Hryniewicz. IRMA Lect. Math. Theor. Phys., 24, Free loop spaces in geometry and topology, 67–109, Eur. Math. Soc., Zürich, 2015. arXiv:1406.3107, 2014.

[33] H.-B. Rademacher, On the average indices of closed geodesics, J. Diff. Geom., 29 (1989), 65–83.

[34] H.-B. Rademacher, Morse Theorie Und Geschlossene Geodatische, Bonner Math. Schr., 1992.

[35] H.-B. Rademacher, Existence of closed geodesics on positively curved Finsler manifolds, Ergodic Theory Dynam. System., 27 (2007), 957–969.

[36] H.-B. Rademacher, The second closed geodesic on Finsler spheres of dimension $n > 2$, Trans. Amer. Math. Soc., 362 (2010), 1413–1421.
Z. Shen, *Lectures on Finsler Geometry*, World Scientific, Singapore, 2001.
[38] I. A. Taimanov, The type numbers of closed geodesics, *Regul. Chaotic Dyn.*, 15 (2010), 84–100.
[39] I. A. Taimanov, The spaces of non-contractible closed curves in compact space forms, *Mat. Sb.*, 207 (2016), 105–118.
[40] M. Vigué-Poirrier and D. Sullivan, The homology theory of the closed geodesic problem, *J. Diff. Geom.*, 11 (1976), 663–644.
[41] W. Wang, Closed geodesics on positively curved Finsler spheres, *Advances in Math.*, 218 (2008), 1566–1603.
[42] W. Wang, On a conjecture of Anosov, *Advances in Math.*, 230 (2012), 1597–1617.
[43] C. Westerland, Dyer-Lashof operations in the string topology of spheres and projective spaces, *Math. Z.*, 250 (2005), 711–727.
[44] C. Westerland, String Homology of Spheres and Projective Spaces, *Algebr. Geom. Topol.*, 7 (2007), 309–325.
[45] Y. Xiao and Y. Long, Topological structure of non-contractible loop space and closed geodesics on real projective spaces with odd dimensions, *Advances in Math.*, 279 (2015), 159–200.
[46] W. Ziller, The free loop space of globally symmetric spaces, *Invent. Math.*, 41 (1977), 1–22.
[47] W. Ziller, Geometry of the Katok examples, *Ergod. Th. Dyn. Sys.*, 3 (1983), 135–157.