The local metric dimension of edge corona and corona product of cycle graph and path graph

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Abstract. Let $G$ be a nontrivial connected graph with vertex set $V(G)$. For an ordered set $W=\{w_1, w_2, \ldots, w_n\}$ of $n$ distinct vertices in $G$, the representation of a vertex $v \in V(G)$ with respect to $W$ is an ordered value of distance between $v$ and every vertex of $W$. The set $W$ is a local metric set of $G$ if the representations of every pair of adjacent vertices with respect to $W$ are different. The local metric set with minimum cardinality is called local metric basis and its cardinality is the local metric dimension of $G$ and denoted by $\dim_l(G)$. The edge corona product of cycle graph and path graph denoted by $C_m \circ P_n$, this graph is obtained from a cycle graph $C_m$ and $m$ copies of path graph $P_n$, and then joining two end-vertices of $i^{th}$ edge of $C_m$ to every vertex in the $i^{th}$ copy of $P_n$, where $1 \leq i \leq m$. The corona product of cycle graph and path graph denoted by $C_m \odot P_n$, this graph is obtained from a cycle graph $C_m$ and $m$ copies of path graph $P_n$, and then joining by an edge each vertex from the $i^{th}$ copy of $P_n$ with $i^{th}$ vertex of $C_m$. In this paper, we determine the local metric dimension of edge corona and corona product of cycle graph and path graph for positive integer $m \geq 3$ and $n \geq 1$. We obtain the local metric dimension of edge corona product of cycle and path graphs is $\dim_l(C_m \circ P_n)=2$ for $n=1$ and $\dim_l(C_m \circ P_n)=m\lceil\frac{n-1}{2}\rceil$ for $n \geq 2$. The local metric dimension of corona product of cycle and path graphs is $\dim_l(C_m \odot P_n)=1$ for $n=1$ and even positive integer $m \geq 3$, $\dim_l(C_m \odot P_n)=2$ for $n=1$ and odd positive integer $m \geq 3$, and $\dim_l(C_m \odot P_n)= m\lceil\frac{n-1}{2}\rceil$ for $n \geq 2$.

1. Introduction
In 1975, Slater [12] introduced the concept of metric dimension of a graph, where metric generator was called locating set. Then in 1976, Harary and Melter [6] independently introduced the same concept, where metric generator was called resolving set. Let $G$ be a nontrivial connected graph with vertex set $V(G)$. A set $W \subset V(G)$ is called metric generator $G$ if $d(u, x) \neq d(v, x)$ for any pair of vertices $u, v \in V(G)$ and some vertices $x \in W$, where $d(u, x)$ is the distance between $u$ and $x$. The metric set with minimum cardinality is called metric basis and its cardinality is the metric dimension. Until now, there are several variations of metric generator, and one of them is a concept of local metric dimension that introduced by Okamoto et al. [8] in 2010. For a set $W=\{w_1, w_2, \ldots, w_n\}$ of $n$ distinct vertices in $G$, the representation of a vertex $v$ of $G$ with respect to $W$ is $n$-vector $r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_n))$. The set $W$ is a local metric set of $G$ if the representations of every pair of adjacent vertices with respect to $W$ are different. The local metric set with minimum cardinality is called local metric basis and its cardinality is the local metric dimension.

Some authors have investigated the local metric dimension of some graph classes. In 2016, Rodríguez-Velázquez et al. [11] found the local metric dimension of corona product graphs. In
the same year, Cynthia and Ramya [4] determined the local metric dimension of mesh related architectures. Cahyabudi and Kusmayadi [2], in 2017 determined the local metric dimension of a lollipop graph, a web graph, and a friendship graph. Rimadhany and Darmaji [9], in the same year found the local metric dimension of circulant graph \( \text{circ}(n : 1, 2, \ldots, \frac{n+1}{2}) \). Also in 2017, Rinurwati et al. [10] determined the local metric dimension of graphs with \( m \)-pendant points. In 2018, Budianto and Kusmayadi [1] found the local metric dimension of starbarbell graph, \( K^m \odot P_n \) graph, and Möbius ladder graph. In this paper, we determine the local metric dimension of edge corona and corona product of cycle and path graphs.

2. Main Results

2.1. The Local Metric Dimension of Edge Corona Product of Cycle Graph and Path Graph

Hou and Shiu [7] defined the edge corona product \( C_m \odot P_n \) is a graph obtained from \( C_m \) and \( m \) copies of \( P_n \), and then joining two end-vertices of the \( i \)th edge of \( C_m \) to every vertex in the \( i \)th copy of \( P_n \). The \( C_m \odot P_n \) graph is represented as in Figure 1.

![Figure 1. \( C_m \odot P_n \) graph](image)

The following theorem given by Okamoto et al. [8] used to prove Theorem 2.3.

**Theorem 2.1** Let \( G \) be a nontrivial connected graph of order \( n \). Then \( \dim_l(G) = n - 1 \) if and only if \( G = K_n \) and \( \dim_l(G) = 1 \) if and only if \( G \) is bipartite.

**Theorem 2.2** For all positive integer \( m \geq 3 \) and \( n \geq 1 \),

\[
\dim_l(C_m \odot P_n) = \begin{cases} 
2, & n = 1; \\
\frac{n-1}{m}, & 2 \leq n \leq 5; \\
\frac{n}{m} & n \geq 6.
\end{cases}
\]

**Proof.** We consider three cases based on the values of \( m \) and \( n \).

**Case 1.** \( n = 1 \).

Based on Chartrand et al. [3], a nontrivial graph is bipartite if and only if there are no odd cycle. The edge corona product \( C_m \odot P_n \) graph for \( n = 1 \) is a graph which contains odd cycle, thus \( C_m \odot P_n \) is not a bipartite graph and \( \dim_l(C_m \odot P_n) \neq 1 \). Assume \( W = \{v_1, v_1^{[\frac{n}{m}]}, \ldots, v_1^{[\frac{n-1}{m}]}, u_1^{[\frac{n}{m}]} \} \), then every pair of adjacent vertices have distinct representations with respect to \( W \). Therefore, \( W \) is a local metric basis and \( \dim_l(C_m \odot P_n) = 2 \) for \( m \geq 3 \) and \( n = 1 \).

**Case 2.** \( 2 \leq n \leq 5 \).

By using the same reason with Case 1, we have \( \dim_l(C_m \odot P_n) \neq 1 \) for \( 2 \leq n \leq 5 \).
Assume $W = \{v^i_n\}$ for $2 \leq n \leq 5$ and $1 \leq i \leq m$, then every pair of adjacent vertices have distinct representations with respect to $W$. Therefore, $W$ is a local metric basis and $\dim_l(C_m \circ P_n) = m\left\lceil \frac{n-1}{4} \right\rceil$ for $m \geq 3$ and $2 \leq n \leq 5$.

**Case 3.** $n \geq 6$.

Using the same reason with Case 1, we have $\dim_l(C_m \circ P_n) \neq 1$ for $n \geq 6$.

(i) If $v^1_1, v^2_1, v^3_1 \notin W$ then $d(v^1_1, u_i) = d(v^2_1, u_i) = 1$ and $d(v^1_1, v^3_1) = d(v^2_1, v^3_1) = 2$ for $1 \leq i \leq m$ and $4 \leq r \leq n$, thus $r(d(v^1_1, W) = d(v^2_1, W)$. We know that $v^1_1$ and $v^2_1$ are adjacent vertices, hence $W$ is not a local metric set. It can be said, if $W$ is a local metric set then at least one of three vertices $v^1_1, v^2_1$ or $v^3_1$ belongs to $W$.

(ii) According to the reason on point (i), if $W$ is a local metric set then at least one of three vertices $v^3_{n-2}, v^4_{n-1}$ or $v^5_n$ belongs to $W$.

(iii) Assume vertex $v^t_i$, for $1 \leq i \leq m$ and $1 \leq t \leq n - 4$ belongs to $W$. If all of the vertices $v^t_{i-1}, v^t_{i+2}, v^t_{i+3}$ and $v^t_{i+4}$ do not belongs to $W$, then $d(v^t_{i+2}, u_i) = d(v^t_{i+3}, u_i) = 1$ and $d(v^t_{i+2}, v^t_i) = d(v^t_{i+3}, v^t_i) = 2$, with $1 \leq r \leq t$, thus $r(v^t_{i+2}, W) = r(v^t_{i+3}, W)$. We know that $v^t_{i+2}$ is adjacent with $v^t_{i+3}$, then $W$ is not a local metric set. In other words, if $W$ is a local metric set then at least one of four vertices $v^t_{i+1}, v^t_{i+2}, v^t_{i+3}$ or $v^t_{i+4}$ belongs to $W$.

Based on three conditions (i), (ii), and (iii), the construction of $W$ such that $W$ is a local metric basis is by choosing every vertex $v^t_{i+4r}$ with $r = 0, 1, 2, \ldots, \left\lceil \frac{n-3}{4} \right\rceil$ for $1 \leq i \leq m$, as the elements of $W$. If $n - (3 + 4r_{\max}) = 3$, we have to choose one of the vertices $v^t_{n-2}, v^t_{n-1}$ or $v^t_n$ as the element of $W$. Therefore, the cardinality of $W$ is $m\left\lceil \frac{n-1}{4} \right\rceil$.

2.2. The Local Metric Dimension of Corona Product of Cycle Graph and Path Graph

Based on the definition from Frucht and Harary [5], the corona product $C_m \circ P_n$ is a graph obtained from $C_m$ and $P_n$ by taking one copy of $C_m$ and $m$ copies of $P_n$, and then joining by an edge each vertex from the $i^{th}$ copy of $P_n$ with the $i^{th}$ vertex of $C_m$. The $C_m \circ P_n$ graph can be depicted as in Figure 2.

![Figure 2. $C_m \circ P_n$ graph](image-url)
The representations of every vertex of $C$ are

$$\dim_i(C \circ P_n) = \begin{cases} 
1, & n = 1, m \text{ even}; \\
2, & n = 1, m \text{ odd}; \\
m\left\lceil \frac{n-1}{4} \right\rceil, & 2 \leq n \leq 5; \\
\infty, & n \geq 6.
\end{cases}$$

**Proof.** We consider four cases based on the values of $m$ and $n$.  

**Case 1.** $n = 1$ and $m$ even.

The corona product $C_n \circ P_n$ graph for $n = 1$ and even positive integer $m \geq 3$ is a bipartite graph because there is no odd cycle. Then by using Theorem 2.1, we have $\dim_i(C_n \circ P_n) = 1$, for $n = 1$ and even positive integer $m \geq 3$.  

**Case 2.** $n = 1$ and $m$ odd.

By using the same reason with Case 1 on Theorem 2.2, we have $\dim_i(C_n \circ P_n) \neq 1$ for $n = 1$ and odd positive integer $m \geq 3$. Assume $W = \{u_1, u_2\}$, then representations of every vertex of $C_n \circ P_n$ for $m = 3$ with respect to $W$ are

$$r(u_1|W) = (0,1), \quad r(v_1^1|W) = (1,2),$$
$$r(u_2|W) = (1,0), \quad r(v_2^1|W) = (2,1),$$
$$r(u_3|W) = (1,1), \quad r(v_3^1|W) = (2,2).$$

The representations of every vertex of $C_n \circ P_n$ for $m = 5$ with respect to $W$ are

$$r(u_1|W) = (0,1), \quad r(v_1^1|W) = (1,2),$$
$$r(u_2|W) = (1,0), \quad r(v_2^1|W) = (2,1),$$
$$r(u_3|W) = (2,1), \quad r(v_3^1|W) = (3,2),$$
$$r(u_4|W) = (2,2), \quad r(v_4^1|W) = (3,3),$$
$$r(u_5|W) = (1,2), \quad r(v_5^1|W) = (2,3).$$

The representations of every vertex of $C_n \circ P_n$ for $m \geq 7$ with respect to $W$ are

$$r(u_1|W) = (0,1), \quad r(v_1^1|W) = (1,2),$$
$$r(u_2|W) = (1,0), \quad r(v_2^1|W) = (2,1),$$
$$r(u_3|W) = (2,1), \quad r(v_3^1|W) = (3,2),$$
$$r(u_4|W) = (3,2), \quad r(v_4^1|W) = (4,3),$$
$$\vdots$$
$$r(u_{m-1}|W) = \left(\frac{m-1}{2}, \frac{m-3}{2}\right), \quad r(v_{m-1}^1|W) = \left(\frac{m+1}{2}, \frac{m-1}{2}\right),$$
$$r(u_{m-2}|W) = \left(\frac{m-1}{2}, \frac{m-1}{2}\right), \quad r(v_{m-2}^1|W) = \left(\frac{m+1}{2}, \frac{m+1}{2}\right),$$
$$r(u_{m-3}|W) = \left(\frac{m-3}{2}, \frac{m-1}{2}\right), \quad r(v_{m-3}^1|W) = \left(\frac{m-1}{2}, \frac{m+1}{2}\right),$$
$$r(u_{m-4}|W) = \left(\frac{m-3}{2}, \frac{m-3}{2}\right), \quad r(v_{m-4}^1|W) = \left(\frac{m-3}{2}, \frac{m-1}{2}\right),$$
$$\vdots$$
$$r(u_{m}|W) = (1,2), \quad r(v_{m}^1|W) = (2,3).$$

Every pair of adjacent vertices have distinct representation, thus $W$ is a local metric basis and $\dim_i(C_n \circ P_n) = 2$, for $n = 1$ and odd positive integer $m \geq 3$.  

**Theorem 2.3** For all positive integer $m \geq 3$ and $n \geq 1$,
Case 3. \(2 \leq n \leq 5\).

By using the same reason with Case 1 on Theorem 2.2, we have \(\dim(C_m \circ P_n) \neq 1\) for \(2 \leq n \leq 5\). Assume \(W = \{v^i|_{\frac{1}{2}}\}\) for \(2 \leq n \leq 5\) and \(1 \leq i \leq m\), then every pair of adjacent vertices have distinct representations with respect to \(W\). Therefore, \(W\) is a local metric basis and \(\dim(C_m \circ P_n) = \lceil \frac{n+1}{2} \rceil m\) for \(m \geq 3\) and \(2 \leq n \leq 5\).

Case 4. \(n \geq 6\).

Using the same reason with Case 1 on Theorem 2.2, we have \(\dim(C_m \circ P_n) \neq 1\) for \(n \geq 6\).

(i) If \(v^i_{n-2}, v^i_{n-1}, v^i_n \notin W\) then \(d(v^i_{n-1}, u_i) = d(v^i_n, u_i) = 1\) and \(d(v^i_{n-1}, v^i_r) = d(v^i_n, v^i_r) = 2\) for \(1 \leq i \leq m\) and \(4 \leq r \leq n\), thus \(r(v^i_{n-1}|W) = d(v^i_n|W)\). We know that \(v^i_{n-1}\) and \(v^i_n\) are adjacent vertices, hence \(W\) is not a local metric set. It can be said, if \(W\) is a local metric set then at least one of three vertices \(v^i_{n-2}, v^i_{n-1}\) or \(v^i_n\) belongs to \(W\).

(ii) Based on the reason on point (i), if \(W\) is a local metric set then at least one of three vertices \(v^i_1, v^i_2\) or \(v^i_3\) belongs to \(W\).

(iii) Assume vertex \(v^i_t\), for \(1 \leq i \leq m\) and \(5 \leq t \leq n\) belongs to \(W\). If all of the vertices \(v^i_{t-1}, v^i_{t-2}, v^i_{t-3}\) and \(v^i_{t-4}\) do not belongs to \(W\), then \(d(v^i_{t-1}, u_i) = d(v^i_{t-3}, u_i) = 1\) and \(d(v^i_{t-2}, v^i_t) = d(v^i_{t-3}, v^i_t) = 2\), with \(r = t\), thus \(r(v^i_{t-2}|W) = r(v^i_{t-3}|W)\). We know that \(v^i_{t-2}\) is adjacent with \(v^i_{t-3}\), then \(W\) is not a local metric set. In other words, if \(W\) is a local metric set then at least one of four vertices \(v^i_{t-1}, v^i_{t-2}, v^i_{t-3}\) or \(v^i_{t-4}\) belongs to \(W\).

Based on three conditions (i), (ii), and (iii), the construction of \(W\) such that \(W\) is a local metric basis is by choosing every vertex \(v^i_{n-2+4r}\) with \(r = 0, 1, 2, \ldots, \lceil \frac{n-2}{4} \rceil\) for \(1 \leq i \leq m\), as the elements of \(W\). If \(n - 2 - 4r_{\text{max}} = 4\), we have to choose one of the vertices \(v^i_1, v^i_2\) or \(v^i_3\) as the element of \(W\). Therefore, the cardinality of \(W\) is \(\lceil \frac{n-2}{4} \rceil m\).}

\[ \square \]

3. Conclusion

It can be concluded that the local metric dimension of edge corona and corona product of cycle graph and path graph are as stated in Theorem 2.2 and Theorem 2.3, respectively.

Open Problem: Determine the local metric dimension of edge corona and corona product of path graph and cycle graph.

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