Quantum operational measurement of amplitude and phase parameters for SU(3)-symmetry optical field

A.P.Alodjants, A.Yu.Leksin, S.M.Arakelian
Department of Physics and Applied Mathematics, Vladimir State University
600000, Vladimir, Russia
E-mail: laser@vpti.vladimir.ru

Abstract. We consider a new approach to describe a quantum optical Bose-system with internal Gell-Mann symmetry by the SU(3)-symmetry polarization map in Hilbert space. The operational measurement in density (or coherency) matrix elements for the three mode optical field is discussed for the first time. We have introduced a set of operators that describes the quantum measurement procedure and the behavior of fluctuations for the amplitude and phase characteristics of three level system. The original twelve-port interferometer for parallel measurements of the Gell-Mann parameters is proposed. The quantum properties of W-qutrit states under the measurement procedure are examined.

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1. Introduction

At present, the properties of three level optical and atomic systems (qutrit states) evoked a great interest in modern problems of quantum information and communication [1-5]. In fact, implementation of qutrit states in quantum system is more preferable than qubit presentation in some cases - see also [3]. Establishment of qutrit states in quantum optics can be realized by different ways [2,4,5]. One of them is taking into account the three optical mode entanglement at multiport beam-splitter (tritter) [4]. Other possibility is based on exploiting nonlinear spontaneous down conversion (SPDC) process as a source of entangled linearly polarized photons [5]. In this case the polarization characteristics for biphoton state can be represented by three components of polarization vector.

Another keystone problem is to measure the qutrit state characteristics. Nowadays the quantum state tomography approach is being discussed for that [6-8]. The basic idea of the method is to reconstruct the density matrix elements by a set of measurements (projections). The quantum tomography for low dimensional systems,
Quantum operational measurement ...  

e.g. for spin states, has been firstly proposed in [9,10]. It has been shown that the density matrix elements can be reconstructed from marginal distributions - for especially prepared diagonal elements of density matrix with the help of unitary transformation. In quantum optics the density (or coherency) matrix elements can be expressed in terms of SU(2) observables - Stokes parameters of optical field [11,12]. In fact, the procedure of spin state tomography is very close to the usual classical ellipsometric measurement technique [12,13]. For last case the measured intensity of light $I(\chi, \delta) = \frac{1}{2} (S_0 + S_1 \cos (\chi) + S_2 \sin (2\chi) \cos (\delta) + S_3 \sin (2\chi) \sin (\delta))$ depends on $\delta$ and $\chi$ parameters introduced by linear optical elements, i.e., phase plates, polarizers. Thus, we need at least four measurements of the Stokes parameters $S_j$ ($j = 0, 1, 2, 3$) for complete determination of the light polarization state. In quantum domain the Stokes operators $S_{1,2,3}$ do not commute with each other, and therefore the procedure of the measurement of these parameters should be clarified. In general, for quantum optical field we are able to carry out the measurements consecutively with any desired accuracy. But in this case the number of copies of initial quantum state of the system is absolutely necessary in order to do the measurements [6,7].

According to alternative operational approach, the measuring apparatus operates with initial quantum state of the system and performs all measurements for non-commuting observables simultaneously with some accuracy determined by uncertainty relations. The approach becomes more preferable in some cases, and results in obtaining unique information about quantum characteristics of the system, e.g. for the problem of optical phase measurements [14] and quantum polarization phase characteristics determination [15]. In particular, in [15] we consider a special multi-port interferometer for simultaneous measurements of all polarization Stokes parameters of light. The quantum error of the measurement is determined by vacuum field fluctuations, and plays a principal role in the case as it can not be avoided due to quantum nature of optical field. Such an approach permits to obtain the complete information about the density matrix elements (i.e. coherency) and to evaluate the degree of polarization for light as well.

In this paper we are developing a quantum theory for the SU(3)-polarization symmetry systems. The approach is based on three mode interaction in quantum optics in general. We analyze the quantum phase and amplitude properties in three level system. In section 2 the mathematical description of the SU(3)-polarization states for optical fields is presented. Section 3 is devoted to the problem of operational determination of non-diagonal elements of the coherency matrix and degree of polarization with the help of the Gell-Mann parameters of optical field by simultaneous measurement by means of the twelve-port interferometer. The measurement of amplitude characteristics of three mode optical field is considered in section 4.
2. Quantum description of SU(3)-polarization for Bose-systems

The quantum three-mode Bose-system with SU(3)-symmetry can be described in the Schwinger representation by Hermitian Gell-Mann operators $\lambda_j$ ($j = 0, 1, ..., 8$) (cf. [8]):

\[\lambda_0 = a_1^+ a_1 + a_2^+ a_2 + a_3^+ a_3,\]  
\[\lambda_1 = a_1^+ a_2 + a_2^+ a_1,\]  
\[\lambda_2 = i \left( a_2^+ a_1 - a_1^+ a_2 \right),\]  
\[\lambda_3 = a_1^+ a_1 - a_2^+ a_2,\]  
\[\lambda_4 = a_1^+ a_3 + a_3^+ a_1,\]  
\[\lambda_5 = i \left( a_3^+ a_1 - a_1^+ a_3 \right),\]  
\[\lambda_6 = a_2^+ a_3 + a_3^+ a_2,\]  
\[\lambda_7 = i \left( a_3^+ a_2 - a_2^+ a_3 \right),\]  
\[\lambda_8 = \frac{1}{\sqrt{3}} \left( a_1^+ a_1 + a_2^+ a_2 - 2a_3^+ a_3 \right),\]

where $a_j$ ($a_j^+$), $j = 1, 2, 3$ are the photon annihilation (creation) operators. The operators $\lambda_j$ defined above in Eq. (1) obey standard commutation relations for SU(3)-algebra - see e.g. [8].

In expression (1a) the operator $\lambda_0$ determines the total number of photons; the operators $\lambda_{1,2,3}$ present the SU(2) sub-group of the SU(3)-algebra. In quantum optics this sub-group corresponds to the polarization Stokes parameters $S_{1,2,3}$ for the optical field, where the modes 1 and 2 are two linear (circular) polarization components of light [11]. The operators $\lambda_{4,5}$ and $\lambda_{6,7}$ in expressions (1c,d) characterize coupling between first two modes ($j = 1, 2$) and the third one ($j = 3$), respectively. The operator $\lambda_8$ in Eq. (1e) is described by a combination of photon numbers for all modes of quantum field.

Let us introduce the unit vector $\vec{e}$ for a three mode system in Hilbert space:

\[\vec{e} a = \vec{e}_1 a_1 + \vec{e}_2 a_2 + \vec{e}_3 a_3\]

where $a$ is annihilation operator for a three-mode field; $\vec{e}_j$ ($j = 1, 2, 3$) are orthogonal vectors under the condition

\[\sum_{j=1}^{3} |\vec{e}_j|^2 = 1.\]

The relation (2) can be written as:

\[a = e_1^* a_1 + e_2^* a_2 + e_3^* a_3,\]

where $e_j^* = e^* \vec{e}_j$ are the projections of vector $\vec{e}$.

The expressions (2)–(4) determine decomposition of a three-mode optical field by the analogy of usual decomposition of elliptically polarized light in respect of two orthogonal (linearly or circularly) polarization components in quantum optics – cf. [13]. However only two orthogonal vectors fulfil to transversality condition for plane waves. Therefore the physical meaning of $e_j$ parameters in expressions (2)–(4) should be clarified for each of specific polarization problems. For example, we also refer here to some
problems of quantum optics when additional (longitudinal) component of polarization is presented (see e.g. [16]) and our description can be useful.

Thus, we rewrite expression \((1)\) for the three-mode problem in terms of the four parameters \(\theta, \phi, \psi_1, \psi_2\) according to the SU(3) symmetry approach (see e.g. [8]):

\[
e_1 = e^{i\psi_1} \sin \theta \cos \phi, \quad e_2 = e^{i\psi_2} \sin \theta \sin \phi, \quad e_3 = \cos \theta,
\]

where parameters \(\theta, \phi \in [0; \pi/2]\) and \(\psi_{1,2} \in [0; 2\pi]\).

The parameters \(\theta\) and \(\phi\) describe amplitude characteristics of the three level quantum system. The phase properties of quantum state of optical field with SU(3)-symmetry are determined by parameters \(\psi_1\) and \(\psi_2\) respectively.

Let us consider the SU(3)-polarization state for coherent optical field:

\[
|\alpha\rangle = |\alpha_1\rangle |\alpha_2\rangle |\alpha_3\rangle, \quad j = 1, 2, 3
\]

where \(|\alpha\rangle = |\alpha_1\rangle |\alpha_2\rangle |\alpha_3\rangle\) is the coherent state of the three-mode field. With the help of expressions \((1)-(6)\) we obtain:

\[
\alpha = \sum_{j=1}^{3} e_j^* \alpha_j, \quad \alpha_j = e_j \alpha, \quad j = 1, 2, 3.
\]

In the paper we also examine the following tripartite state that can be established as (cf. [17]):

\[
|\Psi\rangle_N = \frac{1}{\sqrt{N!}} \left(e_1 a_1^+ + e_2 a_2^+ + e_3 a_3^+\right)^N |0\rangle.
\]

where \(|0\rangle \equiv |0\rangle_1 |0\rangle_2 |0\rangle_3\) is a vacuum state; \(N = \langle \lambda_0 \rangle\) is the total number of particles. For microscopic limit, when \(N = 1\), we have a tripartite entangled qutrit W-state for a three level system:

\[
|\Psi\rangle_q = e_1 |1\rangle_1 |0\rangle_2 |0\rangle_3 + e_2 |0\rangle_1 |1\rangle_2 |0\rangle_3 + e_3 |0\rangle_1 |0\rangle_2 |1\rangle_3
\]

The state \(|\Psi\rangle_q\) can be produced by using a tritter [4]. The maximally entangled state is realized for \(e_j = 1/\sqrt{3} (\theta = \arccos \left(1/\sqrt{3}\right))\), but for some problems of quantum information the maximally-entangled qutrits are not optimal - see e.g. [1]. In the paper we also consider non-symmetric qutrit W-states that can be obtained from Eq.\((9)\) for \(\theta = \phi = \pi/4\).

For \(e_i = 0\) and \(e_j \neq 0 (i, j = 1, 2, 3, \ i \neq j)\) the qutrit state \(|\Psi\rangle_q\) in Eq.\((9)\) reduces to one of three qubit states \(|\Psi\rangle_{ij}\).

With the help of definitions \((1)\) it is easy to obtain the following relations for the Gell-Mann parameter variances for the optical field in state \(|\Psi\rangle_N\):

\[
N \langle \Psi |(\Delta \lambda_0)^2 |\Psi\rangle_N = 0, \quad N \langle \Psi |(\Delta \lambda_j)^2 |\Psi\rangle_N \leq \langle \alpha |(\Delta \lambda_j)^2 |\alpha\rangle, \quad j = 1, \ldots, 8
\]

where the expressions \(\langle \alpha |(\Delta \lambda_j)^2 |\alpha\rangle\) represent the variances of Gell-Mann parameters \((1)\) for coherent state \(|\alpha\rangle\). The inequality in \((10)\) characterizes non-classical properties of entangled states \(|\alpha\rangle\). When the optical field fluctuations that correspond to the Gell-Mann parameter variances are suppressed below the level of fluctuations for coherent states, i.e. the effect of squeezing occurs.
3. Degree of polarization; the SU(3)-interferometer

Let us consider for the first time the problem of degree of polarization for optical field with SU(3) symmetry.

We start from classical definition for a two-mode optical system. In particular in the case of stochastic plane waves the degree of polarization $P_2$ can be represented as – see e.g. [13]:

$$ P_2 = \left( 1 - \frac{4 \det(J_2)}{\text{Tr}(J_2)^2} \right)^{1/2}, \quad (11) $$

where $J_2$ is coherency matrix of size $2 \times 2$. It is important that $P_2$ parameter could be expressed in terms of scalar invariants $\text{Tr}(J_2), \det(J_2)$ and $\text{Tr}(J_2^2) = (\text{Tr}(J_2))^2 - 2 \det(J_2)$ respectively.

Alternatively the degree of polarization $P_2$ defined in Eq. (11) can be rewritten in terms of Stokes parameters $\langle S_j \rangle (j = 0, 1, 2, 3)$ for two-mode optical field as:

$$ P_2 = \frac{(\langle S_1 \rangle^2 + \langle S_2 \rangle^2 + \langle S_3 \rangle^2)^{1/2}}{\langle S_0 \rangle} \quad (12) $$

Although there are various definitions of degree of polarization in quantum optics (we do not discuss them in the paper – see e.g. [18–20]) the expression (12) can be used in quantum domain as well. In this case the $\langle S_j \rangle$ variables in Eq. (12) represent expectation values of the Stokes operators $S_j$.

Now we switch our attention to the case of three-mode optical system (non-plane waves – cf. [13]).

Quantum properties of optical field with SU(3)-symmetry can be described in this case by the following density (or coherency) matrix $J_3$:

$$ J_3 = \begin{pmatrix} \langle a_1^+ a_1 \rangle & \langle a_1^+ a_2 \rangle & \langle a_1^+ a_3 \rangle \\ \langle a_2^+ a_1 \rangle & \langle a_2^+ a_2 \rangle & \langle a_2^+ a_3 \rangle \\ \langle a_3^+ a_1 \rangle & \langle a_3^+ a_2 \rangle & \langle a_3^+ a_3 \rangle \end{pmatrix} \quad (13) $$

In general, scalar invariants $\text{Tr}(J_3^3)$ and $\text{Tr}(J_3^2)$ take place in this case. In particular they can be expressed as:

$$ \text{Tr}(J_3^3) = P_3^2 (\text{Tr}(J_3))^3 + 3 \det(J_3), \quad \text{Tr}(J_3^2) = \frac{(\text{Tr}(J_3))^2}{3} \left( 1 + 2P_3^2 \right), \quad (14) $$

where $P_3$ represents the degree of polarization for a three-mode optical field. With the help of the definition of Gell-Mann operators (1) the $P_3$ quantity can be represented as:

$$ P_3 = \frac{\sqrt{3}}{2} \left( \frac{\sum_{j=1}^{8} \langle \lambda_j \rangle^2}{\langle \lambda_0 \rangle} \right)^{1/2}. \quad (15) $$

Note that expression (15) for the degree of polarization $P_3$ for a three-mode optical system can be also considered as generalization of the well known definition of polarization degree $P_2$ in Eq. (12).
The quantum system with SU(3)-symmetry is completely polarized if and only if:

$$\det(J_3) = 0, \quad P_3 = 1.$$  \hspace{1cm} (16)

It is easy to check that the states (6), (8) and (9) are fulfilled conditions (16).

Let us consider the procedure of measurement of non-diagonal matrix elements of $J_3$ (see Eq. (13)). Schematic set-up for measurement of all phase dependent Gell-Mann parameters is shown in Fig.1. At the input of the system we have three modes $b_j$ ($j = 1, 2, 3$). The boxes $B_j$ represent the balanced beam splitters or symmetric cloning machines (see e.g. [21]) to produce the three modes $a_j$ ($j = 1, 2, 3$) and their copies (clones) $a'_j$ at the output. Then, each of the three mode sets is transformed in two physically identical star-like interferometers denoted as $I_1$ and $I_2$ respectively. Thus, we obtain six Gell-Mann parameters at the output of the device in Fig.1.

Let us precisely analyze the SU(3) twelve-port interferometer scheme (Fig.2) for operational (simultaneous) measurement of the three Gell-Mann parameters $\lambda_j$. The 100% efficiency detectors $D$ are the devices for measurement of the photon numbers $N_{ij} = d_{ij}^+ d_{ij}$, where $d_{ij}$ ($d_{ij}^+$) are annihilation (creation) operators for the modes at the output of the interferometer. They can be represented as linear combination of input fields $a_j$ and vacuum modes $V_j$ after beam splitters (BS) in Fig.2. The measurement procedure results in the detection of photon number differences $(N_{ij}^{(-)})_{i,j = 1, 2, 3, \quad i < j}$:

\begin{align}
N_{12}^{(-)} &= N_{12} - N_{21} = \frac{1}{2} \lambda_{12} + M_{12}, \quad (17a) \\
N_{13}^{(-)} &= N_{13} - N_{31} = \frac{1}{2} \lambda_{13} + M_{13}, \quad (17b) \\
N_{23}^{(-)} &= N_{32} - N_{23} = \frac{1}{2} \lambda_{23} + M_{23} \quad (17c)
\end{align}

where Gell-Mann parameters $\lambda_{ij}$ are represented as (cf. Eqs.(1)):

\begin{align}
\lambda_{12} &= a_1^+ a_2 e^{i\phi_2} + a_2^+ a_1 e^{-i\phi_2}, \quad (18a) \\
\lambda_{13} &= a_1^+ a_3 e^{-i\phi_3} + a_3^+ a_1 e^{i\phi_3}, \quad (18b) \\
\lambda_{23} &= a_2^+ a_3 e^{i\phi_3} + a_3^+ a_2 e^{-i\phi_3} \quad (18c)
\end{align}

The normally ordered operators $M_{ij}$ in Eq. (17) are proportional to the operators $a_j$, ($a_j^+$) and vacuum modes $V_j$ , $j = 1, 2, 3$ at the input of the interferometer. The average values of these operators fulfill the condition $\langle M_{ij} \rangle = 0$. From Eqs. (18) it is easy to see that with the phase shifts $\phi_j = 0$ for interferometer $I_1$ we measure $\lambda_1$, $\lambda_4$ and $\lambda_6$ Gell-Mann parameters - Eqs. (1), and if $\phi_1 = \frac{\pi}{2}$, $\phi_{2,3} = -\frac{\pi}{2}$ the measurement $\lambda_2$, $\lambda_5$ and $\lambda_7$ is realized by interferometer $I_2$ - see Fig.1.

From Eqs.(17) for the average values of the photon number difference $\langle N_{ij}^{(-)} \rangle$ and for variances $\langle (\Delta N_{ij}^{(-)})^2 \rangle$ we obtain:

\begin{align}
\langle N_{ij}^{(-)} \rangle &= \frac{1}{2} \langle \lambda_{ij} \rangle, \quad (19a) \\
\langle (\Delta N_{ij}^{(-)})^2 \rangle &= \frac{1}{4} \langle (\Delta \lambda_{ij})^2 \rangle + \frac{1}{4} \left( \langle a_i^+ a_i \rangle + \langle a_j^+ a_j \rangle \right), \quad i,j = 1, 2, 3, \quad i < j \quad (19b)
\end{align}
Quantum operational measurement ... where the condition $\langle M_{ij} \rangle = 0$ is taken into account.

The last two terms in Eq. (19b) are determined by vacuum fluctuations contribution of the modes $V_j$ and characterize the lowest possible level of the considered variances when $\left\langle \left( \Delta \lambda_{ij} \right)^2 \right\rangle = 0$.

Let consider an ultimate case for Fig. 2. We assume $a_3$-mode to be the control field in coherent state $|\alpha_3\rangle$ with complex amplitude $\alpha_3 = |\alpha_3| e^{i\varphi}$ ($\varphi$ is the phase). The measured mean photon number differences $\langle N^{(-)}_{ij} \rangle$ are:

\begin{align}
\langle N^{(-)}_{12} \rangle &= \frac{1}{2} (\langle \lambda_1 \rangle \cos (\phi_2) - \langle \lambda_2 \rangle \sin (\phi_2)), \\
\langle N^{(-)}_{13} \rangle &= \frac{1}{2} |\alpha_3| (\langle q_1 \rangle \cos (\phi_1) - \langle p_1 \rangle \sin (\phi_1)), \\
\langle N^{(-)}_{23} \rangle &= \frac{1}{2} |\alpha_3| (\langle q_2 \rangle \cos (\phi_3) + \langle p_2 \rangle \sin (\phi_3))
\end{align}

where $q_j = a_j e^{-i\varphi} + a^*_j e^{i\varphi}$, $p_j = i (a^*_j e^{i\varphi} - a_j e^{-i\varphi})$, ($j = 1, 2$) are the Hermitian quadratures for two other modes.

Let us briefly discuss the properties of polarization degree $P_3$ in Eq. (15) in the case under consideration. In the limit of the weak control field $|\alpha_3|^2 \ll 1$ we can measure quantum polarization properties for the two-mode optical field (for bipartite qutrit state as well - cf. [5]) by means of the scheme in Fig. 2. Note that the relation (15) for $P_3$ in this case will differ from the usual definition of degree polarization for two-mode system – see Eq. (12) and [11]. In the other limit when $|\alpha_3|^2 \gg \langle a^+_1 a_1 a^+_2 a_2 \rangle$ the control field $a_3$ plays the role of strong local oscillator field for simultaneous homodyne measurement of quadratures of two-mode optical system according to Eqs. (20b,c). In this case we have $P_3 \simeq 1$ from Eq. (15).

4. Measurements of the $\theta$ and $\phi$ parameters

In this section we consider the problem of simultaneous measurement for diagonal elements of matrix $J_3$ (Eq. (13)). Namely, we focus our attention on the procedure of measurements for the $\theta$ and $\phi$ amplitude parameters.

In classical optics sine and cosine of $\theta$ and $\phi$ amplitude parameters (see Eqs. (5)) can be obtained from simultaneous measurement of the light intensities $I_j \sim \langle a^+_j a_j \rangle$ ($j = 1, 2, 3$) of three mode optical field.

Now we define $S_{\phi, \theta}$ and $C_{\phi, \theta}$ – operators in quantum domain for the $\theta$ and $\phi$ parameters:

\begin{align}
S_\phi &= \sqrt{\frac{\hat{n}_2}{\hat{n}_1 + \hat{n}_2}}, & C_\phi &= \sqrt{\frac{\hat{n}_1}{\hat{n}_1 + \hat{n}_2}}, \\
S_\theta &= \sqrt{\frac{\hat{n}_1 + \hat{n}_2}{\hat{n}_1 + \hat{n}_2 + \hat{n}_3}}, & C_\theta &= \sqrt{\frac{\hat{n}_3}{\hat{n}_1 + \hat{n}_2 + \hat{n}_3}}, \tag{21}
\end{align}

where $\hat{n}_j = a^+_j a_j$ is the photon number operator and we use Eq. (7) as well. To calculate the expectation values and variances for amplitude operators we represent the operators
Quantum operational measurement ...

\( \hat{n}_j \) as \( \hat{n}_j = \bar{n}_j + \Delta \hat{n}_j \) \((j = 1, 2, 3)\), where \( \bar{n}_j = \langle \hat{n}_j \rangle \) is the mean (classical) value of the j-th photon number, \( \Delta \hat{n}_j \) is the small fluctuation part of the corresponding operator with the properties \( \langle \Delta \hat{n}_j \rangle = 0, \langle \Delta \hat{n}_i \Delta \hat{n}_j \rangle = 0 \) \((i, j = 1, 2, 3, i \neq j)\).

After some mathematical calculations for variances of detected relative amplitudes \( \langle (\Delta S_{\phi, \theta})^2 \rangle \), \( \langle (\Delta C_{\phi, \theta})^2 \rangle \) we obtain:

\[
\langle (\Delta S_{\phi})^2 \rangle = \frac{\bar{n}_1}{4(\bar{n}_1 + \bar{n}_2)^3} \left( \frac{\bar{n}_2}{\bar{n}_1} \sigma_1^2 + \frac{\bar{n}_1}{\bar{n}_2} \sigma_2^2 \right),
\]

\[
\langle (\Delta C_{\phi})^2 \rangle = \frac{\bar{n}_2}{4(\bar{n}_1 + \bar{n}_2)^3} \left( \frac{\bar{n}_2}{\bar{n}_1} \sigma_1^2 + \frac{\bar{n}_1}{\bar{n}_2} \sigma_2^2 \right),
\]

\[
\langle (\Delta S_{\theta})^2 \rangle = \frac{1}{4N^3} \left( \frac{\bar{n}_3^2}{(\bar{n}_1 + \bar{n}_2)} (\sigma_1^2 + \sigma_2^2) + (\bar{n}_1 + \bar{n}_2) \sigma_3^2 \right),
\]

\[
\langle (\Delta C_{\theta})^2 \rangle = \frac{1}{4N^3} \left( \bar{n}_3 (\sigma_1^2 + \sigma_2^2) + \frac{(\bar{n}_1 + \bar{n}_2)^2}{\bar{n}_3} \sigma_3^2 \right)
\]

where \( N = \bar{n}_1 + \bar{n}_2 + \bar{n}_3 \) is the total average number of photons. In relations (22) the variances in the photon number \( \sigma_j^2 \equiv \langle (\Delta \hat{n}_j)^2 \rangle \) \((j = 1, 2, 3)\) characterize the additional terms to the mean values of the amplitude parameters in the quantum domain. In a quasi-classical limit (when \( N \gg 1 \)) from Eqs.(22) we have:

\[
\langle (\Delta S_j)^2 \rangle \simeq \langle (\Delta C_j)^2 \rangle \simeq 0, \quad j = \phi, \theta,
\]

i.e. the role of fluctuations is not essential. For the three mode Fock state we have \( \sigma_j^2 = 0 \) and the conditions \[23\] are satisfied as well. The minimal (zero) value of the relative amplitude fluctuations can be also achieved for qubit states when one of the parameters \( e_j = 0 \).

The Eqs. (22) are violated for a small average number of photons \( \bar{n}_j \ll 1 \). In the case of the photon counting measurement the average value of some amplitude operator \( f (\{\hat{n} \}) \) is represented as:

\[
\langle f (\{\hat{n} \}) \rangle = \sum_{\{n \}} f (\{n \}) W (\{n \})
\]

where \( W (\{n \}) \equiv W (n_1, n_2, n_3) \) is joint probability of the set of eigenvalues \( n_1, n_2, n_3 \) that have the form:

\[
W (\{n \}) = \left\langle : \prod_{j=1}^{3} \frac{(a_j^+ a_j)^{n_j} e^{-a_j^+ a_j}}{n_j!} : \right\rangle
\]

For weak field \( \bar{n}_j \ll 1 \) the term with \( n_1 = n_2 = n_3 = 0 \) should be discarded in Eq.\(24\) and the sum in relation \(24\) should be renormalized to the expression \( 1 - \left\langle : \exp \left( - \sum_j a_j^+ a_j \right) : \right\rangle \) - cf. [14]. Finally, for the variances \( \langle (\Delta S_j)^2 \rangle, \langle (\Delta C_j)^2 \rangle \) we obtain:

\[
\langle (\Delta S_{\phi})^2 \rangle = \langle (\Delta C_{\phi})^2 \rangle = \frac{1}{4} \sin^2 (2\phi), \quad \langle (\Delta S_{\theta})^2 \rangle = \langle (\Delta C_{\theta})^2 \rangle = \frac{1}{4} \sin^2 (2\theta)
\]

(26)
The minimal (i.e. zero) value of the fluctuations corresponds in Eqs. (26) to qubit state when some $\bar{n}_j = 0$ (but not the denominator). The maximal value of the variances is $\langle (\Delta S_j)^2 \rangle = \langle (\Delta C_j)^2 \rangle = \frac{1}{4}$ for non-symmetric qutrits ($\theta = \frac{\pi}{4}, \phi = \frac{\pi}{4}$). The variances are $\langle (\Delta S_\theta)^2 \rangle = \langle (\Delta C_\theta)^2 \rangle \approx 0.22$ for maximally entangled qutrit state (9) when $\bar{n}_1 = \bar{n}_2 = \bar{n}_3$.

5. Conclusion

In the paper we presented the quantum theory of SU(3)-polarization for a three mode optical field. In particular we define the degree of polarization that can be connected with scalar invariants in respect of unitary transformations. We offer special twelve-port interferometer for operational reconstruction of non-diagonal elements of coherency matrix $J_3$ and for the measurement of polarization degree $P_3$ as well. It is important that the scheme in Fig. 2 contains only linear optical elements – balanced beam splitters – as input ports for splitting the state for each input mode $a_j$.

To describe amplitude properties of optical field with SU(3) symmetry we introduce special variables ($S_j$ – and $C_j$ – operators, $j = \theta, \phi$) that can be measured by direct simultaneous photodetection procedure. We have shown that qutrit states are characterized by non-vanishing variances of these variables.

In this paper we do not present the measurement procedure for the phases $\psi_1$ and $\psi_2$ - see Eq. $[15]$. The operators for the phase parameters $\psi_{1,2}$ can be introduced operationally by using the interferometer in Fig.2 as well – cf.[15].

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Figure captions

**Figure 1.** Schematic set-up for simultaneous measurement of all phase depending Gell-Mann parameters. Boxes $B_1$, $B_2$ and $B_3$ transform three input optical modes $b_1$, $b_2$ and $b_3$ into six modes $a_j$ and $a_j'$ ($j = 1, 2, 3$) respectively; $V_{j0}$ are vacuum modes at the input of boxes $B_j$. Two other boxes $I_1$ and $I_2$ correspond to two interferometers for simultaneous measurement of three parameters $\lambda_j$ at the output ports.

**Figure 2.** The SU(3)-interferometer for simultaneous measurement of the phase depending Gell-Mann parameters and the phases $\psi_{1,2}$, $\psi_1 - \psi_2$. The input (output) modes are denoted as $a_j$ ($d_{ij}$), ($i, j = 1, 2, 3$, $i \neq j$). Modes $V_j$ at the input ports are in vacuum states; BS’s are balanced beam splitters, $\phi_j$ are phase shifts.