Power–law mass inflation in Einstein–Yang–Mills–Higgs black holes

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Abstract

Analytical formulas are presented describing a generic singularity inside the static spherically symmetric black holes in the $SU(2)$ Einstein–Yang–Mills–Higgs theories with triplet or doublet Higgs field. The singularity is spacelike and exhibits a ‘power–low mass inflation’. Alternatively this asymptotic may be interpreted as a pointlike singularity with a non–vanishing shear in the Kantowski–Sachs anisotropic cosmology.

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Nature of singularity in ‘generic’ black holes remains one of the unsettled issues in General Relativity. Exact solutions available demonstrate possibility of both spacelike (Schwarzschild) and timelike (Reissner–Nordström and Kerr) strong singularities, while the mass–inflation scenario [1] predicts strong spacelike, null (weak or strong), or combination of the two. Recently new insights into the problem were gained in the study of non–abelian black holes [2, 3]. Such black holes are known to be endowed with external hair violating the naive no–hair conjecture [4]. Similar hair exists inside the event horizon and may play crucial role in determining the structure of singularity. In the framework of the Einstein–Yang–Mills (EYM) theory a generic singularity inside a static spherical black hole was shown to be spacelike and of the oscillatory nature [2]. The singularity is dominated by both kinetic and ‘potential’ (arising due to dimensional reduction) Yang–Mills matter terms. In the case of the dilatonic version of the same theory (EYMD) a generic black hole singularity was shown to be dominated by the dilaton [2]. The corresponding mass function diverges in the singularity according to the power–law. Here we want to show that the same is true for the typical gravity coupled gauge theories such as SU(2) Einstein–Yang–Mills–Higgs (EYMH) models with triplet and doublet Higgs. The singularity in generic static spherical black holes in these theories is strong, spacelike, and dominated by the kinetic terms of the scalar fields. It can be interpreted as power–law homogeneous mass–inflationary singularity, or, alternatively, as a pointlike singularity of the corresponding Kantowski–Sachs anisotropic cosmological model. Our analysis is purely analytical. Recently the EYMH system with triplet Higgs was studied numerically [5], but the conclusion made about an exponential mass inflation is incorrect.

We consider the SU(2) EYMH theory

\[
S = \frac{1}{16\pi} \int \left( -R - F^2 + 2|D\Phi|^2 - \frac{\lambda}{2}(|\Phi|^2 - \eta^2)^2 \right) \sqrt{-g} \, d^4x ,
\]

where \( F \) is the YM field strength, \( \Phi \) is a Higgs field in either vector (real triplet) or fundamental (complex doublet) representation, \( D\Phi \) is the corresponding YM covariant derivative (in the doublet case \( |D\Phi|^2 = (D\Phi)^\dagger(D\Phi) \), \( |\Phi|^2 = \Phi^\dagger\Phi \)) and without loss of generality both the Planck mass and the gauge coupling constant are set to unity. In the flat space–time the triplet version of the theory gives rise to regular magnetic monopoles, the doublet version — to sphalerons. New physically interesting configurations emerge when gravity is coupled in a self–consistent way [6]. In particular, static spherical black holes exist in both cases: monopole [7] and sphaleron [8]. Our aim here is to reveal the generic singularity structure in static spherically symmetric black hole solutions to these theories.

From rather general considerations it follows that no Cauchy horizons may form inside generic black holes with non–linear interior hair (although some non–generic solutions with such horizons still may exists) [2]. Then it is convenient to use the standard curvature coordinates

\[
ds^2 = \left( 1 - \frac{2m(r)}{r} \right) \sigma^2(r) dt^2 - \left( 1 - \frac{2m(r)}{r} \right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2).
\]

taking into account that \( 2m/r > 1 \) in the whole interior region.
The purely magnetic Yang–Mills connection relevant to the cases of interest reads

\[ A^\mu T_a dx^\mu = (f(r) - 1)(L_\phi d\theta - L_\theta d\varphi), \]

where \( L_r = T_a n^a, L_\theta = \partial_\theta L_r, L_\phi = (\sin \theta)^{-1} \partial_\varphi L_r, n^a = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta), T_a \)

are generators of \( SU(2) \) in the corresponding representation. The Higgs field in the triplet case is

\[ \Phi^a T_a = \phi(r) L_r, \]

while for the doublet \( \Phi = \phi(r)v \), where \( v \) is some (here irrelevant) spinor depending only on angle variables. In both cases \( \phi(r) \) is the only real scalar function of the radial variable.

The system of equations following from the action (1) with this ansatze may be presented as a set of three decoupled equations for \( f, \phi \) and \( \Delta = r^2 - 2mr \):

\[
\left( \frac{\Delta}{r^2} f' \right)' + \frac{\Delta}{r} f' \phi'^2 = \frac{1}{2} \frac{\partial V}{\partial f} f' - Q f',
\]

\[
\left( \frac{\Delta}{r} \right)' + \Delta \phi'^2 = 1 - 2V - Q,
\]

\[
(\Delta \phi')' + \Delta r \phi'^3 = \frac{\partial V}{\partial \phi} - Q r \phi',
\]

where \( Q = 2\Delta f'^2/r^2 \), and

\[ V \equiv V(f, \phi, r) = \frac{(f^2 - 1)^2}{2r^2} + \frac{\lambda r^2}{8} (\phi^2 - \eta^2)^2 + W^2, \]

with \( W = f \phi \) in the triplet case and \( W = (f + 1) \phi \) in the doublet one. An equation for \( \sigma \) decouples from the system

\[ (\ln \sigma)' = \frac{2}{r} f'^2 + r \phi'^2, \]

and can be easily integrated once \( f, \phi \) and \( \Delta \) are found.

The point \( r = 0 \) is a degenerate singular point of the system (5-7), which is a non–autonomous system of the fifth order. It can be shown to give rise to four different solution branches. Three local branches were presented in [2] for the EYM theory (Higgs field can be added rather trivially [3]), so we do not give them explicitly here. Physically, two asymptotics correspond to the Schwarzschild and Reissner–Nordström (RN) type singularities, while the third looks like a RN one with imaginary charge [3] which can also be interpreted as describing a ‘homogeneous mass inflation’ (HMI) model [4]. [5]. Genericity of local solutions may be explored by counting free parameters; from the above three only RN has a sufficient number of parameters to be locally generic, but it fails to be globally generic due to the ‘second quantization’, as discussed in [2]. So we look for another local branch which should contain five free parameters and satisfy an assumption \( \Delta < 0 \), necessary condition for the absence of Cauchy horizons. (Since the solution we are looking for is supposed to be valid only in the vicinity of \( r = 0 \), this does not guarantee in general the absence of Cauchy horizons outside this region, but numerical integration shows that a continuation without Cauchy horizons exists indeed for some values of parameters.)
This fourth local branch can be found as follows. Assume the solution to be dominated by the scalar gradient (kinetic) term, then the truncated system can be integrated analytically resulting in a five–parameter solution family. Substituting such a solution into the full system, one finds consistency conditions for its validity, it turns out that the convergence radius is non–zero. The assumption made allows one to neglect the right hand side of the Eqs. (5-7) retaining the terms at the left hand side. Thus, the truncated system reads

\[
\left( \frac{\Delta}{r^2} f' \right)' + \frac{\Delta}{r} f' \phi'^2 = 0, \tag{10}
\]

\[
\left( \frac{\Delta}{r} \right)' + \Delta \phi'^2 = 0, \tag{11}
\]

\[
(\Delta \phi')' + \Delta r \phi'^3 = 0. \tag{12}
\]

It can be easily disentangled leading to the following decoupled equations for the YM and scalar fields:

\[
f'' - \frac{f'}{r} = 0, \tag{13}
\]

\[
\phi'' + \frac{\phi'}{r} = 0. \tag{14}
\]

¿From here the following first integral is obvious, which may be used to detect the onset of the asymptotic region numerically:

\[
\frac{d}{dr} (f' \phi') = 0. \tag{15}
\]

Once \( f \) and \( \phi \) are found, the metric function then can be obtained from the simple equation

\[
\frac{\Delta'}{\Delta} = \frac{1}{r^2} - r \phi'^2. \tag{16}
\]

General solution to the Eqs. (13-14) contains four constant parameters \( f_0, \phi_0, b, k \):

\[
f = f_0 + br^2, \tag{17}
\]

\[
\phi = \phi_0 + k \ln r, \tag{18}
\]

it shows that the Higgs field logarithmically diverges towards the singularity, while the Yang–Mills function has a finite limit. Integrating (16) one finds

\[
\Delta = -2m_0 r^{(1-k^2)}, \tag{19}
\]

with the fifth (positive) constant \( m_0 \). Hence, by counting free parameters, this is a generic solution with non–positive \( \Delta \). Now, to find whether the truncation (10-12) of the full system (5-7) is consistent, one has to substitute there the solution (17-19) and to check whether the terms at the right hand side of Eqs. (5-7) are small indeed with respect to accounted terms. One finds the following condition on the parameter \( k \):

\[
k^2 > 1. \tag{20}
\]
This means that the metric function $\Delta$ diverges at the singularity. The corresponding mass function is also divergent according to the power-law

$$m = \frac{m_0}{r^{k_0^2}}. \quad (21)$$

From the Eq. (9) one can see that the behavior of another metric function $\sigma$ is dominated by the scalar term, explicitly one obtains

$$\sigma = \sigma_0 r^{k_2}, \quad (22)$$

with (positive) constant $\sigma_0$.

This local solution can be interpreted as exhibiting a ‘power–law mass inflation’. Usually mass inflation phenomenon is discussed in more general situation when the stress–energy tensor depends on both $r$ (time) and $t$ (space) coordinates in the $T$–region inside the horizon. A simplified model known as ‘homogeneous mass inflation’ [9, 10] deals with space–independent matter distributions, i.e. depending only on $r$, as in our case. Note that our definition of the mass function is different from the standard definition in the mass inflation scenario where the RN–like contribution due to the charge is singled out explicitly. In our case there is no Coulomb component of the YM field, so such contribution is absent. Also, we have assumed from the very beginning that no Cauchy horizon form and therefore $\Delta(0) \leq 0$ (with possible equality only in the singularity). Hence the $1/r$ behavior of the mass function would mean the RN type asymptotic with imaginary charge [2]. As me already noted, such solution was encountered indeed within the context of the mass inflation (HMI solution [10]). However, it was shown that, at least in the pure EYM theory [4], HMI interior does not correspond to an asymptotically flat exterior spacetime (no proof has been done so far, however, for the EYMH theory). In view of (20), the actual divergence of the mass function found here is stronger than $1/r$ and it is likely that the non–zero threshold $k_0^2$ exists below which our generic local interior solution do not meet asymptotically flat exterior ones. Our analytical solution seems to agree with numerical results of [5], though we disagree with an interpretation given there as an ‘exponential inflation of mass’. Unlike the pure EYM theory, where at some stage of each oscillation cycle $m(r)$ exhibits a typical for the standard mass–inflation scenario exponential rise, here one observes a monotonous power–law behavior.

As it was suggested in [4], another useful interpretation of the static spherical black hole interiors may be given in terms of the anisotropic cosmology. Indeed, the metric (2) in the $T$–region inside the horizon may be rewritten as a Kantowski–Sachs metric

$$ds^2 = -d\tau^2 - X^2(\tau)d\rho^2 - Y^2(\tau)(d\theta^2 + \sin^2 \theta d\varphi^2). \quad (23)$$

The correspondence with our initial paramatrization (2) is non–pointlike:

$$r = Y, \quad \Delta = -(Y\dot{Y})^2, \quad \sigma = X/\dot{Y}, \quad (24)$$

where a dot denotes the derivative with respect to $\tau$ and $t = \rho$.

Field equations for YM and Higgs functions look fairly simple

$$\frac{d}{d\tau}(fX) = -\frac{X}{2} \frac{\partial V}{\partial f}, \quad (25)$$

$$\frac{d}{d\tau}(\phi XY^2) = -X \frac{\partial V}{\partial \phi}, \quad (26)$$
where $V$ is given by (8) with $r$ replaced by $Y$. One of the Einstein equations contains only $Y$ and matter fields:

$$2Y\dddot{Y} + \ddot{Y}^2 + 1 + 2f^2 + \dot{\phi}^2 - 2V = 0,$$

(27)

while for $X$ the following first order equation holds

$$\dot{X} = XYZ^{-1} \left( \dddot{Y} + Y \dot{\phi}^2 + 2f^2Y^{-1} \right).$$

(28)

The solution above can be obtained neglecting the right sides of Eqs. (25,26) and omitting in the Einstein equations all matter terms except for those containing $\dot{\phi}$. Kantowski–Sachs metric variables in terms of $\tau$ will read

$$X \sim \tau^{(k^2-1)/(k^2+3)}, \quad Y \sim \tau^{2/(k^2+3)}.$$  

(29)

With account for (20), this corresponds to a pointlike anisotropic singularity. It is common to characterize cosmological expansion using an expansion parameter $\theta$ and a shear $\tilde{\sigma}$. Using (29) one obtains

$$\theta = \frac{\dot{X}}{X} + \frac{2\dot{Y}}{Y} = \frac{1}{\tau},$$

(30)

$$\tilde{\sigma} = \frac{1}{\sqrt{3}} \left( \frac{\dot{X}}{X} - \frac{\dot{Y}}{Y} \right) = \frac{k^2 - 3}{\sqrt{3}(k^2 + 3)\tau}.$$  

(31)

Comparing this power–law mass–inflationary solution with the corresponding solution to the EYMD system [3], one finds that they are essentially the same (apart from the different range of parameters). This is not surprising since in both cases the main matter contribution to the Einstein equations comes from the gradient of the scalar field. However, in the dilaton case $k^2 < 1$ is allowed, such solutions were found numerically in [3]. This corresponds to a ‘cigar’ singularity.

Thus, scalar–dominated singularity is rather typical for non–Abelian gravity coupled theories including scalar fields. The singularity is spacelike as it is generally beleived. Unlike the pure EYM theory it is not oscillating. Its nature is the most transparent in the Kantowski–Sachs interpretation where it looks like a point or cigar singularity usually associated with the ‘stiff matter’ equation of state. It is easy to check that the scalar kinetic dominance leads indeed this kind of an effective state equation. Asymptotic similar to the present one was found by Paul, Datta and Mukherjee in the simpler case of a single scalar field [11]. Kantowski–Sachs cosmology with dilaton–axion matter (without vector fields) as a source was studied recently by Barrow and Dabrowski [12].

Alternative interpretation of the generic EYMH singularity can be given within the framework of the ‘mass inflation’: mass function diverges towards the singularity as a negative power of the (curvature) radial coordinate.

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