ON THE $\infty$-STACK OF COMPLEXES OVER A SCHEME

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Abstract. We study fppf descent for enhanced derived categories. We revisit the work of [HS] and [TV08] in a lax context. More precisely, we construct a Cartesian and coCartesian fibration $\mathcal{D}_S^+ \to N(\text{Sch}_S)$ whose fibre over an $S$-scheme $T$ is the opposite $\mathcal{D}(T)^{op}$ of the quasi-category of bounded below complexes of $\mathcal{O}_T$-modules. We show that this fibration satisfies fppf-descent for schemes. The main components in the proof are limit formulas for the mapping spaces in the section quasi-category $\Gamma(K, \mathcal{X})$ and its subcategory of Cartesian sections $\Gamma_{\text{Cart}}(K, \mathcal{X})$ of a Cartesian fibration over a quasi-category $\mathcal{X} \to K$. These formulas are of independent interest. Since our construction gives a functor of quasi-categories of complexes, it yields $\mathcal{RHom}$ $\infty$-stacks with natural composition maps. The final section gives an explicit description of the $\infty$-group structure of the automorphism $\infty$-group of a complex.

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1. Introduction

In this article we study gluing of complexes up to quasi-isomorphism over fppf covers of schemes. In trying to do so, we find that homotopies and higher homotopies between quasi-isomorphisms naturally enter into the picture and we find ourselves in the world of higher categories.

Descent for quasi-coherent sheaves has its origins in [Gro71]. The stack of quasi-coherent sheaves was first constructed here. It is important to note that we do not naturally have a functor of quasi-coherent sheaves, but rather a pseudo-functor. In other words, for a pair of composable morphisms of schemes

\[ U \xrightarrow{f} T \xrightarrow{g} S, \]

the functors \((g \circ f)^*\) and \(f^* \circ g^*\) are not the same but are canonically isomorphic. The canonical isomorphisms satisfy certain other compatibilities. For this reason it is easier to state descent in terms of fibred categories \(\text{QCoh} \rightarrow \text{Sch}\).

The first descent statements in derived categories appeared in [AGV72, Exp. Vbis]. In this article, given a fppf morphism \(f : T \rightarrow S\) one expands it to its Cech nerve \(T^\bullet/S\). One can then consider the Grothendieck category of abelian sheaves on the nerve, and form its derived category. A descent statement for sheaves and cohomology to \(S\) can be proved in this context. Note that one could consider the diagram of derived categories associated to the Cech nerve. This category, lacks enough information to prove descent and is markedly different from the category considered in [AGV72]. It is this gap, by suitably enhancing the derived category, that this paper seeks to address. Furthermore, it is not shown in loc. cit. that the collection of derived categories forms a stack.

This question was taken up in [HS]. This work introduces the notion of a Quillen presheaf. A strictification theorem is proved in the context of Quillen preseaves. The final section of this paper proves descent in a very general setting for derived categories. These methods were applied and extended in [TV08].

These results described above consider various enhanced derived categories. What if we work with ordinary derived categories? A gluable complex is a complex whose negative self extensions vanish. These negative extension groups are precisely the homotopy groups of the mapping spaces in the stack that we consider below, see §5.4. Descent for universally gluable complexes was first proved in [Be˘ı82], building upon the theory of cohomological descent in [AGV72]. Using this result, Lieblich has shown that universally gluable complexes form an algebraic stack, see [Lie06].

In this article we will construct a fibration of enhanced derived categories in the form of a relative dg-nerve, see §5. We will prove some basic properties of it, such as, it is a presentable fibration (§5.10), and it satisfies descent (§6.6). In a nutshell we carry out the constructions of [Gro71] for suitably enhanced derived categories.

It has been long known that the category perfect complexes is a well behaved generalisation of the category of vector bundles, see for instance [Tho90]. A first step towards studying moduli of perfect complexes is to define and prove descent for this category, in other words show that it forms a (higher) stack. To do so, we need to construct a sheaf of perfect complexes. The difficulty, just
as in the case of quasi-coherent sheaves, is that naturally we can only construct a pseudofunctor as we will now explain. Let \( T \) be an \( S \)-scheme. Then the category of perfect complexes on \( T \) is a homotopical category, that is it can be equipped with a notion of weak equivalence, the quasi-isomorphisms. Then one can take the simplicial localization \( \text{Perf}(T) \). Let \( V \xrightarrow{h} U \xrightarrow{g} T \) be morphisms of \( S \)-schemes. We then have derived pullback functors giving a diagram of simplicial sets

\[
\begin{array}{ccc}
\text{Perf}(V) & \xrightarrow{Lh^*} & \text{Perf}(U) \\
\downarrow{Lh^*} & & \downarrow{Lg^*} \\
\text{Perf}(T) & \xleftarrow{L(hg)^*} & \text{Perf}(T).
\end{array}
\]

But this diagram is only commutative up to a homotopy \((Lh^*)(Lg^*) \rightarrow L(hg)^*\). Therefore, to get an actual presheaf of simplicial sets

\[
\text{Sch}_S \xrightarrow{\text{Perf}} \text{Set}_{\Delta},
\]

one needs to take a strictification. This is a strict functor, which is only equivalent to the natural lax functor. Usually this construction is only made for affine \( S \), from which the stack of perfect complexes on a general scheme is obtained by taking the homotopy limit along an affine open cover. This makes the structure even more inexplicit.

We get a natural explicit description using the quasi-categorical generalization of the Grothendieck construction, which is recalled in §2.3. On fibres, we can take the dg-nerve, the simplices of which are by construction diagrams in the dg-category of complexes. This is recalled in §3.1. Then using functorial dg-flat resolutions, we get derived pullback maps with a functorial choice of homotopies making the diagrams above commutative (§3.2). Homotopy limits and the statement of descent in a higher categorical setting is recalled in §2.4. Descent for a Cartesian fibration over the nerve of a site can be formulated using the quasi-category of Cartesian sections. In §4 we describe mapping spaces in section quasi-categories and their subcategories on Cartesian sections as homotopy limits of certain diagrams. These formulas can be thought of as an extension of the description in [GHN17]. This section is of independent interest and it is likely that these homotopy limit descriptions will have other applications. The main object of study is introduced in §5. We construct a presentable fibration

\[
\mathcal{S}_S \xrightarrow{p} \mathcal{N} \text{(Sch)}_S
\]

with simplices explicitly given as diagrams in the dg-category of complexes (§5). We can then formulate and check fppf descent in this setting (§6). The main theorem is proved in §6. The theorem has various applications, for example we obtain a description of \( \text{QC}(T) \) for an ordinary scheme. This object plays the role of the category of quasi-coherent sheaves in derived algebraic geometry, see [BZFN10]. The final section gives an explicit description of the \( \infty \)-group structure of the automorphism \( \infty \)-group of a complex. This topic will be expanded upon in future work.

**Notation 1.1.** In this paper, every scheme will be assumed to be quasi-compact and quasi-separated. Let’s fix once and for all a morphism of schemes \( X \xrightarrow{f} S \).
We will work with quasi-categories and largely follow the notation of [Lur09]. By an $\infty$-category we will mean a quasi-category. See 2.1 for a brief introduction. By an ordinary category we will mean a small quasi-category that is equivalent to the nerve of an ordinary category, that is a set of objects and morphisms subject to the usual conditions.

We might suppress the nerve of a 1-category from notation. That is, for example $\text{Sch}_S$ as a simplicial set is $N(\text{Sch}_S)$.

The $\infty$-category of spaces will be denoted by $\mathcal{S}$. This is just the simplicial nerve of the simplicial category of Kan complexes.

2. Background on quasi-categories

2.1. Quasi-categories as $\infty$-categories. For a more detailed exposition, we refer the reader to [Lur09, §1]. A quasi-category $X$ is a type of simplicial set which can model an $\infty$-category. Its vertices $\Delta^0 \to X$ are objects, and its edges $\Delta^1 \to X$ are 1-morphisms. The 2-simplices $y \xrightarrow{f} x \xleftarrow{g} z$. We can to think of as composition diagrams. By abuse of notation, we will write $g f = h$ if $X$ has a 2-simplex as above. Note that this means that we can have $g f = h_1$ and $g f = h_2$ with $h_1 \neq h_2$, that is composition is not unique. But one can check that if $X$ is a quasi-category, then we get $h_1 \simeq h_2$ in this case.

For example, in the main example of the quasi-category $\mathcal{D}(T)$ which is recalled in §3.1, a 2-simplex $\Delta^2 \to \mathcal{D}(T)$ is given by

1. 3 complexes of injective $\mathcal{O}_T$-modules $I_0, I_1, I_2$,
2. 3 morphisms of complexes $I_0 \xrightarrow{f_{01}} I_1$, $I_1 \xrightarrow{f_{12}} I_2$, $I_0 \xrightarrow{f_0} I_2$,
3. and a homotopy $I_0 \xrightarrow{f_{012}} I_2[-1]$ such that $d f_{012} = f_{12} f_{01} - f_{02}$.

Note that an inner horn $\Lambda^2_1 \to X$ given by $f_{01}$ and $f_{12}$ can be completed to a 2-simplex in many ways.

For a simplicial set $X$ to be a quasi-category, we need to be able to compose morphisms, and higher morphisms also. The relative notion is that of an inner fibration. A map of simplicial sets $X \xrightarrow{p} S$ is an inner fibration, if it satisfies the right lifting property with respect to all inner horn inclusions $\Lambda^m_k \subset \Delta^n$ for $n \geq 2$ and $0 < k < n$. A simplicial set $X$ is a quasi-category, if the canonical map $X \to *$ is an inner fibration. If $X$ is a quasi-category, then by $x \in X$ we mean that $x$ is a vertex: $x \in X_0$. Let $x \xrightarrow{f} y \xleftarrow{g} z$ be a $\Lambda^2_1$-diagram in $X$. Then by the lifting property, it can be complexes to a 2-simplex. We will write $x \xrightarrow{gf} z$ for some composite we get this way.

As we said, composition is not unique in a quasi-category. But there are still ways to get mapping spaces and composition maps. Let $K$ and $L$ be simplicial sets. Then their join $K \star L$ has as set of $n$-simplices

$$(K \star L)_n = \bigsqcup_{-1 \leq i \leq n} (K_{[0,i]} \times L_{[i+1,n]}),$$
where we set $K_0 = L_0 = \ast$. Let $K$ be a simplicial set, $X$ a quasi-category, and $K \overset{\tau}{\to} X$ a diagram. Then the overcategory $X/\tau$ has as set of $n$-simplices

$$(X/\tau)_n = [\Delta^n \star K \overset{\tau}{\to} X : \sigma|K = k].$$

Let $y \in X$ be an object and $\Delta^0 \overset{\tau}{\to} X$ be its inclusion map. Then we define $X/\tau_y = X/\tau$. Let $x \in X$ be another object. Then we define the right Hom space $\text{Hom}^R_X(x, y) = \{x\} \times_X X/\tau_y$. Note that its set of $n$-simplices is

$$\text{Hom}^R_X(x, y)_n = [\Delta^{n+1} \overset{\tau}{\to} X : \sigma|\Delta^{[0,n]} = \{x\}, \sigma|\Delta^{[n+1]} = \{y\}].$$

One can check that $\text{Hom}^R_X(x, y)$ is a Kan complex. It is one of the ways to define a mapping space in a quasi-category.

Let $\Delta^1 \overset{y}{\to} X$ be a morphism, and $x \in X$ another object. Then one can check that the restriction map $X/\tau_y \to X/\tau$ is a trivial fibration. Therefore, its pullback $\{x\} \times_X X/\tau_y \overset{\tau}{\to} \{x\} \times_X X/\tau_y$ is also a trivial fibration. Thus, it has a section $\{x\} \times_X X/\tau_y \overset{s}{\to} \{x\} \times_X X/\tau_y$. We can get a postcomposition by $f$ map as the composite

$$f \circ : \text{Hom}^R_X(x, y) = \{x\} \times_X X/\tau_y \overset{s}{\to} \{x\} \times_X X/\tau_y \to \{x\} \times_X X/z = \text{Hom}^R_X(x, z).$$

Note that this map is not unique as it depends on the choice of the section $s$. Dually, we can define undercategories $X_{k/}$, left Hom spaces $\text{Hom}^L_X(x, y)$, and precomposition by $f$ maps. There is a third version $\text{Hom}_X(x, y)$ of the mapping space, with set of $n$-simplices

$$\text{Hom}_X(x, y)_n = [\Delta^1 \times \Delta^n \overset{\tau}{\to} X : \sigma|\Delta^{[0]} \times \Delta^n = \{x\}, \sigma|\Delta^{[1]} \times \Delta^n = \{y\}].$$

One can show that the natural inclusions

$$\text{Hom}^L_X(x, y) \to \text{Hom}_X(x, y) \leftarrow \text{Hom}^R_X(x, y)$$

are homotopy equivalences of Kan complexes [Lur09 Corollary 4.2.1.8]. Because of these equivalences, we write $\text{Map}_X(x, y)$ to mean any Kan complex homotopy equivalent to any of these.

Let $C$ be a category. Then its categorical nerve $N(C)$ is the simplicial set with set of $n$-simplices composable chains of morphisms of length $n$:

$$N(C)_n = \{c_0 \overset{f_0}{\to} c_1 \overset{f_1}{\to} \cdots \overset{f_{n-1}}{\to} c_n\},$$

the face maps are given by composition, and the degeneracy maps are given by identity maps. One can show that this gives a fully faithful functor

$$\text{Cat} \overset{N}{\to} \text{Set}_\Delta.$$}

We denote its left adjoint by $\tau_1$. Let $X$ be a quasi-category. Then we can describe the category $\tau_1(X)$ as follows. The objects of $\tau_1(X)$ are the objects of $X$. Let $f, g : x \to y$ be two morphisms. Then we say that $f$ and $g$ are homotopic, if there exists a 2-simplex in $X$ of the form

\[
\begin{array}{ccc}
& y \\
\downarrow & & \downarrow \text{id}_y \\
\leftarrow x & \overset{f}{\to} & y, \\
\downarrow & & \downarrow \text{id}_y \\
& y .
\end{array}
\]
One can show that this relation is an equivalence, and that letting $\text{Hom}_{\tau_1(X)}(x, y)$ be the set of morphisms modulo this relation, we can give a category structure to $\pi_1(X)$ using composition diagrams [Lur09 Proposition 1.2.3.8].

Let $X \xrightarrow{f} Y$ be a map of simplicial sets. Then $f$ is essentially surjective, if $\tau_1(f)$ is an essentially surjective functor of categories. The map $f$ is fully faithful, if for all $x, y \in X$, the induced map $\text{Map}_X(x, y) \xrightarrow{F_{x,y}} \text{Map}_Y(f(x), f(y))$ is a homotopy equivalence. The map $f$ is a categorical equivalence, if it is both essentially surjective and fully faithful.

Using this notion, we can define the Joyal model structure on the category $\text{Set}_\Delta$. In it,

1. cofibrations are monomorphisms, and
2. weak equivalences are categorical equivalences.

One can show that the fibrant objects in the Joyal model structure are precisely the quasi-categories [Lur09 Theorem 2.4.6.1].

We have seen that we can get composition maps in a quasi-category, even if they are not unique. One can go further with strictification, and from a quasi-category $X$ get a simplicial category $\mathbb{C}[X]$ with the same object set, and equivalent mapping spaces and composition maps. The statement uses the Bergner model structure on the category $\text{Cat}_\Delta$ on simplicial categories [Ber07 Theorem 1.1]. In it,

1. Weak equivalences are DK-equivalences, that is maps of simplicial categories $C \xrightarrow{F} D$ such that
   a. for all $x, y \in C$, the induced map $\text{Map}_C(x, y) \xrightarrow{F_{x,y}} \text{Map}_D(Fx, Fy)$ is a weak equivalence, and
   b. the induced map on the underlying categories $\pi_0C \xrightarrow{\pi_0 F} \pi_0D$ is an equivalence of categories.
2. Fibrations are local fibrations, that is maps of simplicial categories $C \xrightarrow{F} D$ such that
   a. for all $x, y \in C$, the induced map $\text{Map}_C(x, y) \xrightarrow{F_{x,y}} \text{Map}_D(Fx, Fy)$ is a Kan fibrations, and
   b. for all $x \in C$ and homotopy equivalence $Fx \xrightarrow{f} y'$ in $D$ there exists a homotopy equivalence $x \xrightarrow{f'} y$ in $C$ such that $Ff = f'$.

Then there exists a Quillen equivalence $\text{Set}_\Delta \xrightarrow{\text{N}_\Delta} \text{Cat}_\Delta$ [Lur09 Theorem 2.2.5.1]. See [Lur09 §1.1.5] for the construction of $\mathbb{C}$. Let $C$ be a simplicial category. We call $\text{N}_\Delta(C)$ its homotopy coherent nerve. Let $X$ be a quasi-category. Then for all $x, y \in X$, we have a weak equivalence of simplicial sets $\text{Hom}^R_X(x, y) \simeq \text{Map}_{\mathbb{C}[X]}(x, y)$ [Lur09 Corollary 2.2.2.10, Proposition 2.2.4.1].

2.2. Straightening-unstraightening between right fibrations and presheaves of spaces. Let $\text{Kan} \subseteq \text{Set}_\Delta$ denote the full simplicial subcategory on Kan complexes. Then the quasi-category of spaces $\mathcal{S}$ is its coherent nerve: $\mathcal{S} = N_{\Delta} \text{Kan}$. Thus, a presheaf of Kan complexes on a quasi-category $\mathcal{C}$ can be given as a map of simplicial sets $\mathcal{C}^{\text{op}} \to \mathcal{S}$. As this includes pseudofunctors, it is very difficult to define presheaves like this. Therefore, we employ the quasi-categorical generalization of the Grothendieck construction. The quasi-categorical generalization of the notion of a fibred
category is the notion of a right fibration. A morphism of simplicial sets \( X \to S \) is a right fibration, if it satisfies the right lifting property with respect to the horn inclusions \( \Lambda^n_k \subset \Delta^n \) for \( n \geq 1 \) and \( 0 < k \leq n \).

Let \( S \) be a simplicial set. The overcategory \( (\text{Set}_\Lambda)_S \) can be equipped with the contravariant model structure. This is a left proper, combinatorial, simplicial model category \([Lur09\text{, Propositions 2.1.4.7 and 2.1.4.8}]\) in which

1. A cofibration is a monomorphism.
2. An \( S \)-morphism of simplicial sets \( X \to Y \) is a contravariant equivalence, if the induced map

\[
S \sqcup_X X^c \to S \sqcup_Y Y^c
\]

is a categorical equivalence.

One can show that every contravariant fibration is a right fibration, and moreover the fibrant objects of \( (\text{Set}_\Delta)_S \) are precisely the right fibrations over \( S \), see \([Lur09\text{, Proposition 2.1.4.9}]\).

The main result \([Lur09\text{, 2.2.1.2}]\) is that equipping \( (\text{Set}_\Delta)_S \) with the projective model structure induced by the Quillen model structure, we obtain a Quillen equivalence

\[
(\text{Set}_\Lambda)_S \xleftarrow{\text{St}} \xrightarrow{\text{Un}} (\text{Set}_\Delta)^{[S]}.
\]

The functors \( \text{St} \) and \( \text{Un} \) are called the straightening and unstraightening functors. Let \( p : X \to S \) be an inner fibration. Let \( x \to y \) be an edge in \( X \). We say that \( e \) is a \( p \)-Cartesian edge, if the canonical map

\[
X/e \to X/y \times_{S/p(y)} S/p(e)
\]

is a trivial fibration. This definition makes sense because of the following. As described in \( \S 2.1 \), we can get postcomposition maps \( \text{Hom}^R_X(z, x) \xrightarrow{e^o} \text{Hom}^R_X(x, y) \) and \( \text{Hom}^R_S(p(z), p(x)) \xrightarrow{p(e)^o} \text{Hom}^R_S(p(z), p(y)) \). Then \( e \) is \( p \)-Cartesian if and only if for all \( z \in X \), the diagram

\[
\text{Hom}^R_X(z, x) \xrightarrow{e^o} \text{Hom}^R_X(x, y) \xrightarrow{e^o} \text{Hom}^R_S(p(z), p(y))
\]
The following are equivalent [Lur09, Proposition 3.1.3.3]. Note that if \( S = * \), then \( e \) is a \( p \)-Cartesian edge if and only if it is an equivalence. The inner fibration \( X \to S \) is a Cartesian fibration, if for all vertices \( y \in X \) and edges \( \bar{e} : x \to p(y) \) in \( S \), there exists a \( p \)-Cartesian edge \( e \) such that \( p(e) = \bar{e} \).

To get a straightening-unstraightening construction between Cartesian fibrations and presheaves in quasi-categories, we need to keep track of where morphisms take Cartesian edges. Therefore, it needs to be formulated using marked simplicial sets. A marked simplicial set is a pair \((X, \mathcal{E})\) where

1. \( X \) is a simplicial set, and
2. \( \mathcal{E} \) is a collection of edges of \( X \) containing the degenerate edges.

For a simplicial set \( X \), we have the two extreme cases.

1. The marked edges in the marked simplicial set \( X^\flat \) are only the degenerate edges.
2. In the marked simplicial set \( X^\sharp \), every edge is marked.

Let \( \text{Set}_{\Delta}^a \) denote the category with

1. objects the marked simplicial sets, and
2. morphisms the morphisms of simplicial sets which take marked edges to marked edges.

Let \( X, Y \in \text{Set}_{\Delta}^a \) be marked simplicial sets. Then we denote by \( \text{Map}^\flat(X, Y) \) and \( \text{Map}^\sharp(X, Y) \) the simplicial sets with

\[
\text{Hom}_{\text{Set}_{\Delta}}(\Delta^n, \text{Map}^\flat(X, Y)) = \text{Hom}_{\text{Set}_{\Delta}^a}(X \times (\Delta^n)^\flat, Y), \text{ and } \\
\text{Hom}_{\text{Set}_{\Delta}}(\Delta^n, \text{Map}^\sharp(X, Y)) = \text{Hom}_{\text{Set}_{\Delta}^a}(X \times (\Delta^n)^\sharp, Y).
\]

Let \( S \) be a simplicial set. Then a marked \( S \)-simplicial set is a marked simplicial set with a morphism to \( S^\sharp \). We denote their category by \((\text{Set}_{\Delta}^a)_S\). Let \( X, Y \in (\text{Set}_{\Delta}^a)_S \) be marked \( S \)-simplicial sets. Then \( \text{Map}^\sharp_S(X, Y) \subseteq \text{Map}^\flat(X, Y) \) is the simplicial subset on simplices \( X \times (\Delta^n)^\flat \to Y \) such that the postcomposite with the structure map \( Y \to S \) is the composite of the projection map and the structure map \( X \times (\Delta^n)^\flat \to X \to S^\sharp \). We define \( \text{Map}^\flat_S(X, Y) \) similarly.

Let \( Z \to S \) be a Cartesian fibration. Then the marked \( S \)-simplicial set \( Z^\flat \) has as marked edges the \( p \)-Cartesian edges. Let \( X \) be a marked \( S \)-simplicial set. Then \( \text{Map}^\flat_S(X, Z^\flat) \) is a quasi-category, and \( \text{Map}^\sharp_S(X, Z^\flat) \) is its interior. Let \( X \to Y \) be a morphism of marked \( S \)-simplicial sets. Then the following are equivalent [Lur09, Proposition 3.1.3.3].

1. For every Cartesian fibration \( Z \to S \), the precomposition map

\[
\text{Map}^\flat_S(Y, Z^\flat) \to \text{Map}^\flat_S(X, Z^\flat)
\]

is an equivalence of quasi-categories.

2. For every Cartesian fibration \( Z \to S \), the precomposition map

\[
\text{Map}^\sharp_S(Y, Z^\flat) \to \text{Map}^\sharp_S(X, Z^\flat)
\]
is a homotopy equivalence of Kan complexes.

If these equivalent conditions are satisfied, then we say that \( f \) is a Cartesian equivalence. Now using the \( \text{Map}_S^\# \) as mapping spaces, we can equip \((\text{Set}_\Delta^+)_S\) with a left proper, combinatorial, simplicial model structure as follows [Lur09, Proposition 3.1.3.7, Corollary 3.1.4.4].

1. The cofibrations are the maps which are monomorphisms on the underlying simplicial sets.
2. The weak equivalences are the Cartesian equivalences.

This is called the \textit{Cartesian model structure}. Let \( X \to S \) be a marked \( S \)-simplicial set. Then it is a fibrant object if and only if \( X \to Y \) for some Cartesian fibration \( Y \to S \) [Lur09, Proposition 3.1.4.1].

Let \( X \to Y \) be a morphism of Cartesian fibrations over \( S \). Then \( f \) is a Cartesian equivalence if and only if the fibres \( X_s \to Y_s \) are categorical equivalences for all \( s \in S \) [Lur09, Proposition 3.1.3.5].

Let \( C \) be a quasi-category. Let \( \mathcal{E} \) be a collection of edges containing the degenerate edges. Then \((C, \mathcal{E})\) is a marked simplicial set. Let \((C, \mathcal{E}) \to \mathcal{D}^\#\) be a fibrant replacement. Then we call \( \mathcal{D} \) the simplicial localization of \( C \) with respect to \( S \), and write \( \mathcal{D} = C[\mathcal{E}^{-1}] \).

Note that the full simplicial subcategory of fibrant objects of \( \text{Set}_\Delta^+ \) is precisely \( \text{Cat}_{\infty}^\Delta \). Equipping \((\text{Set}_\Delta^+)^{\mathcal{E}[S]}\) with the projective model structure induced by the Cartesian model structure, we get a Quillen equivalence [Lur09, Theorem 3.2.0.1]

\[
\begin{array}{ccc}
(\text{Set}_\Delta^+)_S & \xrightarrow{\text{St}^+} & (\text{Set}_\Delta^+)^{\mathcal{E}[S]} \\
\downarrow & & \downarrow \\
\text{Un}^+ & \xrightarrow{\text{Un}^+} & (\text{Set}_\Delta^+)^{\mathcal{E}[S]} \\
\end{array}
\]

We refer to the functors \( \text{St}^+ \) and \( \text{Un}^+ \) as the straightening and unstraightening functors. Let \( X \to S \) be a Cartesian fibration, and \( S^{\text{op}} \to \text{Cat}_{\infty} \) a presheaf. Then by the \((\mathcal{C}, N_\Lambda)\)-adjunction, \( f \) corresponds to a simplicial functor \( C[S^{\text{op}}] \to \text{Cat}_{\infty} \subseteq \text{Set}_\Delta^+ \). We say that \( p \) is classified by \( f \), if \( p \cong \text{Un}^+(f^\#) \).

2.4. \textbf{Homotopy limits and descent}. Let \( I \) and \( C \) be 1-categories. Consider the category of diagrams \( \text{Fun}(I, C) \). There is a constant functor

\[
\text{const} : C \to \text{Fun}(I, C).
\]

The right adjoint of the constant functor is the limit functor, when it exists. If \( C \) is a model category, we may take its right derived functor. In order to do this, we need to equip \( \text{Fun}(I, C) \) with a model structure. We would like to do this so that the adjunction \( \text{const} \dashv \lim \) becomes a Quillen pair. This requires that the constant functor preserves cofibrations and acyclic cofibrations. With this in mind we can try to put a model structure on \( \text{Fun}(I, C) \) where the cofibrations and weak equivalences are defined objectwise on \( I \). The fibrations are then determined. When it exists this is the injective model structure on the functor category and we may use it define a right derived functor of the limit which is known as the homotopy limit, written \( \text{holim} \).

We will be interested in the case where \( C \) is \( \text{Set}_\Delta \) with its Quillen model structure.

**Definition 2.1.** Let \( C \) be a Grothendieck site with products. Given a cover \( U \to T \) we may form the \( \text{Cech nerve} \ U^* \to T \). A simplicial presheaf \( F \) is a sheaf if for every cover \( U \to T \) we have

\[
F(T) \cong \text{holim} \ F(U^*).
\]
Remark 2.2. This reduces to the ordinary definition in the case of a presheaf of sets. To reconcile this with the notion of stack, see [Hol] 6.5.

Warning 2.3. Note that this notion does not include stacks, since it only includes functors, not pseudofunctors.

We want to study descent conditions for presheaves of ∞-categories, which in our case will be maps \( N(C)^{op} \rightarrow \text{Cat}_\infty \). As above, they can be formulated via homotopy limits. In the language of quasi-categories, it is easy to define these. Let \( X \) be a quasi-category. Then \( x \in X \) is a final object, if for all \( y \in X \), the mapping space \( \text{Map}_X(y,x) \) is contractible. This happens if and only if \( x \in X \) is strongly final, that is the restriction map \( X/_{/x} \rightarrow X \) is a trivial fibration [Lur09, Corollary 1.2.12.5].

Now let \( K \) be a simplicial set, and \( K \rightarrow X \) a diagram. Then the limit \( \lim k \) is simply a final object in the overcategory \( X/_{/k} \). Note that these are automatically homotopy limits. Dually, we can define initial objects and colimits.

Following the straightening-unstraightening construction recalled in §2.3, a presheaf \( N(C)^{op} \rightarrow \text{Cat}_\infty \) classifies a Cartesian fibration \( \mathcal{X} \rightarrow \text{N}(C) \). More generally, we can replace \( N(C) \) with some simplicial set \( K \). In this case, there is a generalization of the description of the homotopy limit of a fibrant cosimplicial space [BK72, X,§3] in terms of the quasi-category of Cartesian sections which we will define now.

**Notation 2.4.** Let \( \mathcal{K} \rightarrow K \) be an inner fibration. The quasi-category of sections, denoted \( \Gamma(K, \mathcal{X}) \), is the pullback

\[
\begin{array}{ccc}
\Gamma(K, \mathcal{X}) & \rightarrow & \text{Fun}(K, \mathcal{X}) \\
\downarrow & & \downarrow \text{id}_K \\
\{ \text{id}_K \} & \rightarrow & \text{Fun}(K, K).
\end{array}
\]

We will say that a section \( \sigma \), that is a 0-simplex \( \sigma \in \Gamma(K, \mathcal{X}) \), is Cartesian if \( \sigma(e) \in \Gamma(K, \mathcal{X}) \) for each edge \( e \in K_1 \).

Let us denote by \( \Gamma_{\text{Cart}}(K, \mathcal{X}) \subset \Gamma(K, \mathcal{X}) \) the full subcategory on Cartesian sections. Then the limit result is the following. Let \( \mathcal{X} \rightarrow K \) be a Cartesian fibration classified by a map \( K^{op} \rightarrow \text{Cat}_\infty \). Then we have [Lur09, Corollary 3.3.3.2]

\[
\lim k = \Gamma_{\text{Cart}}(K, \mathcal{X}).
\]

Let \( L \rightarrow K \) be a map of simplicial sets. Then we write the pullback \( L \times_K \mathcal{X} \) as \( p|k \), and we write \( \Gamma(k, \mathcal{X}) = \Gamma(L, p|k) \) and \( \Gamma_{\text{Cart}}(k, \mathcal{X}) = \Gamma_{\text{Cart}}(L, p|k) \).

Suppose that \( S \in L \) is a final object. We will usually notation such as \( L \rightarrow \mathcal{X} \) to denote sections so that we can let \( X_S = X \).

Recall [Lur09 4.1.1.1] that a morphism \( L \rightarrow K \) of simplicial sets is cofinal if for any right fibration \( X \rightarrow K \) we have a homotopy equivalence

\[\Gamma(K, X) \rightarrow \Gamma(L, X|k).\]
An example of a cofinal morphism is the inclusion $\{S\} \hookrightarrow C$ of a final object of a quasi-category. This follows from the quasi-categorical version of Quillen’s theorem A, see loc. cit 4.1.3.1.

**Lemma 2.5.** Let $X \to K$ be a Cartesian fibration over a quasi-category. Let $S \in K$ be a final object. Then the natural restriction

$$\Gamma_{\text{Cart}}(K, X) \to X_S$$

is a homotopy equivalence.

**Proof.** Let $X_{\text{Cart}} \subseteq X$ be the subcategory generated by Cartesian edges. In other words $X_{\text{Cart}}$ and $X$ have the same 0-simplicies. For $n > 0$, the $n$-simplicies of $X_{\text{Cart}}$ are those $n$-simplicies of $X$ whose edges are Cartesian. Then

$$X_{\text{Cart}} \to K$$

is a right fibration by [Lur09, 2.4.2.5]. The result follows from definitions now. □

**Definition 2.6.** Let $p : \mathcal{X} \to K$ be a Cartesian fibration, and $\Delta^\text{op}_+ \xrightarrow{k_+} K$ an augmented simplicial object. Let us denote the restriction $k_+|\Delta^\text{op}$ by $k$. The restriction map $\Gamma_{\text{Cart}}(k_+, \mathcal{X}) \to \Gamma(k_+(-1), \mathcal{X})$ is a trivial fibration by Lemma 2.5. Therefore the zigzag

$$\Gamma(k_+(-1), \mathcal{X}) \xleftarrow{\text{La}^*} \Gamma_{\text{Cart}}(k_+, \mathcal{X}) \to \Gamma_{\text{Cart}}(k, \mathcal{X})$$

gives a map $\Gamma(k_+(-1), \mathcal{X}) \xrightarrow{\text{La}^*} \Gamma_{\text{Cart}}(k, \mathcal{X})$. We say that $p$ satisfies descent along $k_+$, if the functor $\text{La}^*$ is an equivalence of quasi-categories. Let $U \xrightarrow{g} T$ be an edge in $K$. Then we say that $p$ satisfies descent along $g$, if $p$ satisfies descent along the augmented Čech nerve $\check{\mathcal{C}}(g)_+$. We will reconcile this definition with Definition 2.1 in Remark 2.7 below. Note also, that as per our conventions, we have omitted the nerve in our notation in various places, for example $N(\Delta^\text{op})$.

**Remark 2.7.** Let $\Delta^\text{op}_+ \xrightarrow{k_+} K$ be an augmented simplicial object. Let $\Delta \xrightarrow{\text{St}(p|k)} \text{Cat}_{\omega}$ be the straightening of $p|k$. Then the straightening $\Delta \xrightarrow{\text{St}(p|k)} \text{Cat}_{\omega}$ is a cone over $\text{St}(p|k)$, that is a point of $(\text{Cat}_{\omega})/\text{St}(p|k)$. We obtain a functor of quasi-categories

$$\Gamma(k_+(-1), \mathcal{X}) \xrightarrow{(\text{La}^*)' \text{holim}} \text{holim} \text{St}(p|k).$$

Descent is usually phrased by asserting that $(\text{La}^*)'$ is a weak equivalence. It is equivalent to Definition 2.6 as $\text{holim} \text{St}(p|k) = \Gamma_{\text{Cart}}(k, p)$ [Lur09, Corollary 3.3.3.2].

### 3. Background on quasi-categories of complexes

3.1. **The dg-nerve and the bounded below derived category.** This construction is from [Lur16 Ch. 1]. We will work exclusively with complexes whose differential has degree +1, so it is worth recalling the definition in our context here. One can pass from a cohomological complex $C^\bullet$ to a homological complex $C_\bullet$ by setting $C_n = C^{-n}$.

Given a dg-category $\mathcal{C}$ we will denote the (cohomological) mapping complex between a pair of objects by $\text{Hom}_\mathcal{C}^\bullet(x, y)$. We may apply our reindexing construction to $\mathcal{C}$ to obtain a dg-category with homological mapping complexes. Lurie’s dg-nerve construction may then be applied to
this category to obtain a quasi-category. Let’s unwind definitions to see what we obtain. The $n$-simplicies are pairs $((X_i)_{0 \leq i \leq n}, \{f_i\}_{I \in \text{str}(n)})$ where

- $X_i$ are objects of $C$
- $\text{str}(n)$ is the collection of subsets of $[n]$ length at least 2
- $f_i \in \text{Hom}^{2-||}(X_{\min(I)}, X_{\max(I)})$.

This data is subject to the condition that for each $I \in \text{str}[n]$ of the form $I = \{i_\min < i_1 < \ldots < i_m < i_\max\}$ we have

$$df_i = \sum_{1 \leq j \leq m} (-1)^j (f_{i_\min < \ldots < i_m < i_\max} \circ f_{i_\min < \ldots < i_j}).$$

These collections acquire the structure of a simplicial set by defining

$$\alpha^*((X_i)_{0 \leq i \leq n}, \{f_i\}_{I \in \text{str}(n)}) = (X_{\alpha(I)})_{0 \leq I \leq m}, \{g_I\}_{I \in \text{str}(m)}$$

for an order preserving function $\alpha : [m] \to [n]$ where

$$g_I = \begin{cases} f_{\alpha(I)} & \text{if } \alpha|_I \text{ is injective} \\ 1_{X_i} & |I| = 2 \text{ and } \alpha(I) = \{i\} \\ 0 & \text{otherwise.} \end{cases}$$

Consider now a Grothendieck abelian category $A$. We may consider the dg-category $\text{Ch}^+(A)$ whose objects are bounded below chain complexes.

For a pair of chain complexes $A$ and $B$ let’s write $\text{Hom}_{\text{dg}}(A, B)$ for the chain complex of maps between them. In degree $p$ it is

$$\text{Hom}_{\text{dg}}(A, B)^p = \prod_n \text{Hom}(A^n, B^{n+p}).$$

Given $f \in \text{Hom}(A^n, B^{n+p})$, the differential is given by the formula $d(f) = d_B \circ f - (-1)^p f \circ d_A$. The choice of sign insures that the 0-cycles are chain maps.

Applying the dg-nerve we obtain a quasi-category $N_{\text{dg}}(\text{Ch}^+(A))$. We will be interested in the full subcategory on complexes of injectives, $\mathcal{D}^+(A) = N_{\text{dg}}(\text{Ch}^+(A - \text{inj}))$ Let’s recall what this simplicial set looks like.

The $n$-simplicies consist of pairs $((K_i), \{f_i\})$ where

1. for $0 \leq i \leq n$ we have a bounded below chain complex $K_i$ of injectives
2. for each subset $I \subseteq \{0, \ldots, n\}$ of the form $I = \{i_- < i_1 < \cdots < i_m < i_+\}$ where $m > 0$ we have $f_i \in \text{Hom}_{\text{dg}}(K_{i_-}, K_{i_+})$ satisfying the equation

$$df_i = \sum_{1 \leq j \leq m} (-1)^j (f_{i_- < \ldots < i_m < i_+} \circ f_{i_- < \ldots < i_j}).$$

Now the homotopy category of this simplicial set, [Lur09, page 29], is exactly the homotopy category of bounded below complexes of injectives. This is the ordinary derived category. For this reason we call $\mathcal{D}^+(A)$ the bounded below derived quasi-category.

There is a useful description of mapping spaces in a dg-nerve using Dold–Kan complexes, which we recall now. Let $A$ be a simplicial abelian group. Then its Moore complex is the strictly connective chain complex with $(\text{CA})_n = A_n$ and $d_{CA} = \sum_{k=0}^{n} (-1)^k d_A$ for $n \geq 0$. The normalized chain complex is the
subcomplex with \((NA)_n = \cap_{k=1}^n \ker d_k^A A_n \subseteq CA_n\) for \(n \geq 0\). It turns out that the functor \(\text{Ab}^{\text{sep}} \xrightarrow{A \mapsto NA} \text{Ch}^{\geq 0}(\mathbb{Z})\) is an equivalence of categories \([GJ99\text{ Corollary III.2.3}]\). It has a canonical quasi-inverse, the Dold–Kan complex functor. One of its descriptions is to make \((\text{DK} A)_n \xrightarrow{\psi} \bigoplus_{[k] \rightarrow [n]} A_k\) \([GJ99\text{ Proposition III.2.2}]\), but its inverse needs to be constructed via a recursive process, not explicitly. We propose the following alternative description, based on the fact that, letting \((\text{DA} A)_n \subset (\text{CA} A)_n\) denote the subcomplex on degenerate simplices, the inclusion map induces an isomorphism \(\text{NA} \xrightarrow{\psi} \text{CA}/\text{DA}\) \([GJ99\text{ Theorem III.2.1}]\).

**Definition 3.1.** For \(n \geq 0\), the Koszul complex \(K\Delta^n\) is the subcomplex \(K\Delta^n \subseteq C\Delta^n\) on nondegenerate simplices.

**Remark 3.2.** We call this the Koszul complex, since \(K\Delta^n\) is the naïve truncation and shift to the positive part of the Koszul complex \(K(\mathbb{Z}^n)\).

**Proposition 3.3.** (1) The inclusion map induces an isomorphism \(K\Delta^n \rightarrow C\Delta^n/\text{DA}^n\).

(2) Let \(A\) be a strictly connective chain complex. Then for any \(n \geq 0\), restriction to simplices of the form \(\{0\} \cup \sigma\) for \(\sigma \in \{1, \ldots, n\}\) gives an isomorphism \(\text{Hom}_{\text{Ch} (\mathbb{Z})} (K\Delta^n, A) \rightarrow \bigoplus_{\sigma \in \{1, \ldots, n\}} A_{|\sigma|}\).

**Lemma 3.4.** (1) Let \(C\) be a dg-category and \(X\) and \(Y\) objects of \(C\). Then \(\pi_n(\text{Map}_C(X, Y)) = H^{-n}(\text{Hom}^\bullet(X, Y))\).

(2) We denote by \(\mathcal{D}^{\geq 0}(A)\) the full subcategory on complexes that are acyclic in negative degrees. Let \(K \in \mathcal{D}^{\geq 0}(A)\) and \(J \in \mathcal{D}^+(A)\). The the canonical morphism to the good truncation \(J \rightarrow \tau_{\geq 0}J\) induces a weak equivalence \(\text{Map}(\tau_{\geq 0}J, K) \rightarrow \text{Map}(J, K)\).

**Proof.** The proof of \([\text{Lur}16\text{ Proposition 1.3.1.17}]\) shows that \(\pi_n(\text{Map}_C(X, Y)) = \pi_n(\text{DK} \tau_{\geq 0} \text{Hom}^\bullet(X, Y))\).

Here DK is the Dold–Kan functor. The result follows from the fact that the Dold–Kan functor identifies cohomology with homotopy groups.

For the second part, as the mapping space is invariant under equivalence, we may assume that \(K\) is concentrated in non-negative degrees. The result follows easily from the previous part. \(\square\)

The most important feature of the quasi-category \(\mathcal{D}^+(A)\) is that it is in fact a stable quasi-category, \([\text{Lur}16\text{ 1.3}]\).

---

1By abuse of notation, for a simplicial set \(K\), we will also denote the free simplicial abelian group \(\mathbb{Z}[K]\) by \(K\).
3.2. **Cotorsion pairs and unbounded complexes.** We want to get a presentable fibration \( \mathcal{D}_S \to \text{Sch}_S \), because they have a nice theory which we want to use. That is, we want the fibres \( \mathcal{D}(T)^{op} \to \text{Sch}_S \) to be presentable quasi-categories. Presentable quasi-categories have all small limits and colimits. Therefore, we will need to consider unbounded complexes. For a morphism of \( S \)-schemes \( U \to T \), we will also want to get an adjoint pair \( \mathcal{D}(T) \overset{Lg^*}{\leftarrow} \mathcal{D}(U) \). To define these functors, we will need to restrict to complexes on which the functors \( g^* \) resp. \( g_* \) are homotopical, that is they take equivalences (which in this setting are exactly the quasi-isomorphisms) to equivalences. That is, we will need to get functorial dg-flat resp. dg-injective resolutions. To get these, we will employ two model structures given in [Gil07].

Let \( \mathcal{G} \) be an abelian category. Let \( \mathcal{A}, \mathcal{B} \subseteq \mathcal{G} \). Let

\[
\mathcal{A}^\perp = \{ X \in \mathcal{G} : \text{Ext}^1(A, X) = 0 \text{ for all } A \in \mathcal{A} \}, \quad \mathcal{B}^\perp = \{ X \in \mathcal{G} : \text{Ext}^1(X, B) = 0 \text{ for all } B \in \mathcal{B} \}.
\]

Then \((\mathcal{A}, \mathcal{B})\) is a cotorsion pair, if \( \mathcal{A}^\perp = \mathcal{B} \), and \( \mathcal{A} = \mathcal{B}^\perp \).

Let \( T \) be an \( S \)-scheme and let \( \mathcal{G} = \mathcal{O}_T\text{-Mod} \). Then we have two important cotorsion pairs:

- \((\mathcal{O}_T\text{-Mod}, \mathcal{I})\), where \( \mathcal{I} \) is the class of injective \( \mathcal{O}_T \)-modules, and
- \((\mathcal{F}, \mathcal{C})\), where \( \mathcal{F} \) is the class of flat \( \mathcal{O}_T \)-modules, and \( \mathcal{C} \) is the class of cotorsion \( \mathcal{O}_T \)-modules.

Let \( X \) be a cochain complex in \( \mathcal{G} \). Then

1. \( X \) is an \( \mathcal{A} \)-complex, if it is exact, and \( Z^n X \in \mathcal{A} \) for all \( n \). The collection of \( \mathcal{A} \)-complexes is denoted by \( \mathcal{A}^\perp \).
2. \( X \) is a \( \mathcal{B} \)-complex, if it is exact, and \( Z^n X \in \mathcal{B} \) for all \( n \). The collection of \( \mathcal{B} \)-complexes is denoted by \( \mathcal{B}^\perp \).
3. \( X \) is a dg-\( \mathcal{A} \)-complex, if \( X^n \in \mathcal{A} \) for all \( n \), and for every map \( X \to B \), if \( B \) is a \( \mathcal{B} \)-complex, then \( f \) is nullhomotopic. The collection of dg-\( \mathcal{A} \)-complexes is denoted by \( \text{dg} \mathcal{A}^\perp \).
4. \( X \) is a dg-\( \mathcal{B} \)-complex, if \( X^n \in \mathcal{B} \) for all \( n \), and for every map \( A \to X \), if \( A \) is a \( \mathcal{A} \)-complex, then \( f \) is nullhomotopic. The collection of dg-\( \mathcal{B} \)-complexes is denoted by \( \text{dg} \mathcal{B}^\perp \).

A complex \( I \) of \( \mathcal{O}_T \)-modules is called dg-injective, if it is a dg-\( \mathcal{I} \)-complex. A complex \( P \) of \( \mathcal{O}_T \)-modules is called dg-flat, if it is a dg-\( \mathcal{F} \)-complex.

If certain conditions are satisfied [Gil07, Theorem 4.12], then we get a model structure on \( \text{Ch} \mathcal{G} \) such that

- the weak equivalences are the quasi-isomorphisms,
- the cofibrations (resp. trivial cofibrations) are the monomorphisms whose cokernels are in \( \text{dg} \mathcal{A}^\perp \) (resp. \( \mathcal{A}^\perp \)), and
- the fibrations (resp. trivial fibrations) are the epimorphisms whose kernels are in \( \text{dg} \mathcal{B}^\perp \) (resp. \( \mathcal{B}^\perp \)).

In the case \( \mathcal{G} = \mathcal{O}_T\text{-Mod} \), the cotorsion pairs \((\mathcal{O}_T\text{-Mod}, \mathcal{I})\) and \((\mathcal{F}, \mathcal{C})\) satisfy these conditions [Gil07, Corollaries 7.1 and 7.8]. The model structures on \( \text{Ch}(T) \) we get we call the injective model structure, and the flat model structure, respectively.
**Notation 3.5.** Let \( T \) be an \( S \)-scheme. We denote by \( \mathcal{Ch}_q(T) \subseteq \mathcal{Ch}(T) \) the full sub-dg-category on complexes of \( \mathcal{O}_T \)-modules with quasi-coherent cohomology. Let \( \mathcal{Ch}_{q,\text{inj}}(T), \mathcal{Ch}_{q,\text{fl}}(T) \) be the full sub-dg-categories of dg-injective and dg-flat complexes respectively. Let’s denote their dg-nerves by \( \mathcal{D}(T) \) and \( \mathcal{D}_\emptyset(T) \), respectively. Let’s denote \( N_{dg} \mathcal{Ch}_q(T) \) by \( \mathcal{C}(T) \).

A pair of functors \( U : \mathcal{C} \to \mathcal{D} \) and \( F : \mathcal{D} \to \mathcal{C} \) between quasi-categories are adjoint, if there is a unit transformation \( u : id_{\mathcal{D}} \to U \circ F \) such that the composition

\[
\text{Map}_\mathcal{C}(F \circ x, y) \to \text{Map}_\mathcal{D}(U(F \circ x), U(y)) \overset{u}{\to} \text{Map}_\mathcal{D}(x, U(y))
\]

is a weak equivalence, see [Lur09, 5.2.7.5.2.8].

The dg-injective resolution functor \( \mathcal{C}(T) \overset{I_T}{\to} \mathcal{D}(T) \) will be a left adjoint to the inclusion \( \mathcal{D}(T) \to \mathcal{C}(T) \). That is, \( I_T \) will be a localization functor. To show that such a functor exists, it is enough to show [Lur09, Proposition 5.2.7.8] that every complex \( E \in \mathcal{C}(T) \) admits a \( \mathcal{D}(T) \)-localization, that is there exists a dg-injective complex \( I \in \mathcal{D}(T) \) and a morphism \( E \overset{q}{\to} I \) such that for all dg-injective complexes \( J \in \mathcal{D}(T) \), the precomposition map

\[
\text{Map}_T(I, J) \overset{q}{\to} \text{Map}_T(E, J)
\]

is a weak equivalence.

Similarly, the dg-flat resolution functor \( \mathcal{C}(T) \overset{P_C}{\to} \mathcal{D}_\emptyset(T) \) will be a right adjoint to the inclusion \( \mathcal{D}_\emptyset(T) \to \mathcal{C}(T) \). To get it, we will need to show that every complex \( E \in \mathcal{C}(T) \) admits a \( \mathcal{D}_\emptyset(T) \)-colocalization, that is there exists a dg-flat complex \( P \in \mathcal{D}_\emptyset(T) \) and a morphism \( P \overset{r}{\leftarrow} E \) such that for all dg-flat complexes \( Q \in \mathcal{D}_\emptyset(T) \), the postcomposition map

\[
\text{Map}_T(Q, P) \overset{r}{\to} \text{Map}_T(Q, E)
\]

is a weak equivalence.

**Proposition 3.6.** Let \( T \) be an \( S \)-scheme. Let \( E \in \mathcal{C}(T) \). Then \( E \) admits a \( \mathcal{D}(T) \)-localization, and a \( \mathcal{D}_\emptyset(T) \)-colocalization.

**Proof.** Applying Lemma [3.7] in the situation of [Gil07, Corollary 7.8] gives a \( \mathcal{D}_\emptyset(T) \)-colocalization. Applying the dual argument of Lemma [3.7] in the situation of [Gil07, Corollary 7.1] gives a \( \mathcal{D}(T) \)-localization. \( \square \)

**Lemma 3.7.** Let \( \mathcal{A} \) be a class of objects in a Grothendieck category \( \mathcal{I} \) satisfying the conditions of [Gil07, Theorem 4.12]. Let \( \mathcal{B} = \mathcal{A}^- \). Let \( E \in \mathcal{Ch}(\mathcal{I}) \). Then there exists a fibrant resolution \( P \overset{r}{\to} E \). For any dg-\( \mathcal{A} \) complex \( Q \), the postcomposition map between the derived Hom complexes

\[
\text{RHom}(Q, P) \to \text{RHom}(Q, E)
\]

is a quasi-isomorphism.

**Proof.** Let \( K \) be the kernel of \( r \). Then we get a distinguished triangle \( K \to P \to E \to \text{in} \mathcal{D}(T) \) [Wei94, Example 10.4.9]. Therefore, we get a long exact sequence

\[
\cdots \to \text{Ext}^p(Q, K) \to \text{Ext}^p(Q, P) \to \text{Ext}^p(Q, E) \to \text{Ext}^{p+1}(Q, K) \to \cdots
\]
By construction, $K$ is a $\mathcal{B}$-complex. Therefore, every map $Q \to K$ is nullhomotopic. This implies that $\text{Ext}^m(Q,K) = 0$ for all $m$. This proves the claim.

\[ \square \]

**Notation 3.8.** Let $T$ be an $S$-scheme. By Proposition 3.6, we get an injective resolution functor $\mathcal{C}(T) \xrightarrow{I_T} \mathcal{P}(T)$ with unit map $\mathcal{C}(T) \times \Delta^1 \xrightarrow{q_T} \mathcal{C}(T)$, and a flat resolution functor $\mathcal{C}(T) \xrightarrow{P_T} \mathcal{R}(T)$ with counit map $\mathcal{C}(T) \times \Delta^1 \xrightarrow{r_T} \mathcal{C}(T)$. We will often drop the subscript $T$ from the notation.

Let $U \xrightarrow{g} T$ be a morphism of $S$-schemes. Then we get a derived pullback functor $\mathcal{D}(U) \xrightarrow{Lg} \mathcal{D}(T)$ and a derived pushforward functor $\mathcal{D}(U) \xrightarrow{Rg} \mathcal{D}(T)$.

**Proposition 3.9.** Let $U \xrightarrow{g} T$ be a morphism of schemes.

1. The derived pullback functor $\mathcal{D}(T) \xrightarrow{Lg} \mathcal{D}(U)$ is left adjoint to the derived pushforward functor $\mathcal{D}(U) \xrightarrow{Rg} \mathcal{D}(T)$.

2. The functor $Rg_*$ restricts to a functor $\mathcal{D}^{ge}(U) \to \mathcal{D}^{ge}(T)$.

**Proof.** (1) We have a natural quasi-isomorphism $\text{RHom}_U(Lg^*I, J) \xrightarrow{\alpha} \text{RHom}_V(I, Rg_*J)$ [Lip09, Proposition 3.2.3], which gives a natural equivalence $\text{Map}_U(Lg^*I, J) \xrightarrow{\alpha} \text{Map}_V(I, Rg_*J)$ [SS03, §4.1].

This shows that $(Lg^*, Rg_*)$ gives an adjoint pair of simplicial categories $\mathcal{C}\mathcal{D}(U) \xrightarrow{\perp} \mathcal{C}\mathcal{D}(T)$ [Lur16, Proposition 1.3.1.17]. Therefore, a functor equivalent to $Lg^*$ is left adjoint to a functor equivalent to $Rg_*$, [Lur09, Corollary 5.2.4.5], and that is enough [Lur09, Proposition 5.2.1.4].

(2) Let $I \in \mathcal{D}^{ge}(U)$. Then $I \cong \tau_{\geq n} I$. By the Cartan–Eilenberg resolution, we get an injective complex $J$ such that $I \cong \tau_{\geq n} I$, and $J^m = 0$ for $m < n$. Since we have $I(I) \cong I$, a zigzag of quasi-isomorphisms between dg-injective complexes, we get $Rg_*I = g_*I(I) \cong g_*I$. Here, $g_*I \in \mathcal{D}^{ge}(T)$ by construction. Therefore, we get $Rg_*I \in \mathcal{D}^{ge}(T)$.

\[ \square \]

4. Twisted arrow categories and mapping spaces

Let $C$ be an ordinary category. The twisted arrow category of $C$ is the category denoted $\text{Tw}(C)$, whose objects are arrows in $C$. A morphism in $\text{Tw}(C)$ from $m' \xrightarrow{\mu} n'$ to $m \xrightarrow{\alpha} n$ amounts to a commutative diagram in $C$ of the form

\[
\begin{array}{ccc}
  m' & \xrightarrow{\mu} & m \\
  \downarrow{\alpha'} & & \downarrow{\alpha} \\
  n' & \xleftarrow{\nu} & n.
\end{array}
\]

Notice that morphisms $\mu$ and $\nu$ are in opposite directions, so that we have a functor $\text{Tw}(C) \to C \times C^{\text{op}}$.

This construction has been made for quasi-categories, see [Lur16, 5.2]. Let us briefly recall it. If $I$ and $J$ are finite ordered sets we can form the ordered set $I \star J$. The underlying set of $I \star J$ is the disjoint union of $I$ and $J$ and the elements of $I$ are placed before those of $J$ in the ordering. If $S$ is
a simplicial set, the \( n \) simplices of \( \text{Tw}(S) \) are \( S([n] \star [n]^{\text{op}}) \). The \( \text{op} \) is important for the simplicial structure.

The main application of the twisted arrow category is that it is a right fibration classified by the mapping space functor. Let \( C \) be a quasi-category. Then the canonical map \( \text{Tw} C \xrightarrow{\lambda} C \times C^{\text{op}} \) is a right fibration [Lur16, Proposition 5.2.1.3]. That is, it is classified by a presheaf \( C^{\text{op}} \times C \rightarrow \mathcal{S} \).

Then the corresponding map \( C \rightarrow \mathcal{P}(C) \) is equivalent to the Yoneda embedding [Lur16, 5.2.1.11]. Therefore, we will denote the map \( C \rightarrow \mathcal{P}(C) \) by \( \text{Map} C \).

This can be used to give formulas for mapping spaces in functor categories. Let \( F, G: K \Rightarrow X \) be functors of quasi-categories. Then we have

\[
\text{Map}_{\text{Fun}(K,X)}(F, G) = \lim_{\text{op}}(\text{Tw} K) \xrightarrow{(m \rightarrow n)} \text{Map}_{X}(F_m, G_n) \rightarrow \mathcal{S}.
\]

Based on this result, we will give formulas for mapping spaces in section categories, Proposition 4.4:

\[
\text{Map}_{\Gamma(K, X)}(F, G) = \lim_{\text{op}}(\text{Tw} K) \xrightarrow{(m \rightarrow n)} \text{Map}_{X}(F_m, G_n) \rightarrow \mathcal{S},
\]

and Cartesian section categories, Proposition 4.12:

\[
\text{Map}_{\Gamma_{\text{Cart}}(K, X)}(F, G) = \lim_{\text{op}}(\text{Tw} K) \xrightarrow{m} \text{Map}_{X}(F_m, G_m) \rightarrow \mathcal{S}.
\]

### 4.1. Mapping spaces in the section quasi-category.

**Remark** 4.1. Let \( L \xrightarrow{k} K \) be a morphism of simplicial sets, and \( X \rightarrow K \) an inner fibration. In the notation of [Lur09], we have \( \Gamma(k, X) = \text{Map}_K(L, X) \).

**Lemma 4.2.** Let \( X \xrightarrow{p} K \) be an inner fibration and \( F, G \in \Gamma(K, X) \) two sections. Then we can form the fibre product of the maps

\[
\text{Tw} K \xrightarrow{\lambda} K \times K^{\text{op}} \xrightarrow{F \times G^{\text{op}}} X \times X^{\text{op}} \quad \text{and} \quad \text{Tw} X \xrightarrow{\lambda} X \times X^{\text{op}}.
\]

Then the induced map

\[
\text{Tw} K \times X \times X^{\text{op}} \xrightarrow{\text{Tw} p \times \text{Tw} X^{\text{op}}} \text{Tw} K \xrightarrow{\text{Tw} F \times \text{Tw} G^{\text{op}}} \text{Tw} K \quad \text{is a right fibration.}
\]

**Proof.** Let \( n \geq 1 \) and \( 0 < k \leq n \). Let \( \Delta^{2n+1} \xrightarrow{\min} (\Delta^{2n+1})^{\text{op}} \) denote the mirror map \( j \mapsto 2n + 1 - j \). We need to solve the lifting problem

\[
\begin{array}{ccc}
\Lambda^n_k & \xrightarrow{(\tau_1, \Delta^n)} & \text{Tw} K \times X \times X^{\text{op}} \xrightarrow{\text{Tw} p} \text{Tw} K \\
\downarrow & & \downarrow \\
\Delta^n & \xrightarrow{(\tau, 0)} & \text{Tw} K \times X^{\text{op}} \xrightarrow{\text{Tw} K} \text{Tw} K,
\end{array}
\]

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that is we need to find $\Delta^n \overset{\delta}{\to} \text{Tw } \mathcal{X}$ such that $\delta|\Lambda^n_k = \sigma$, $\text{Tw } p \circ \delta = \delta$ and $\lambda_{\mathcal{X}} \circ \delta = (F \times G^\text{op}) \circ \lambda_k \circ \tau = (F \times G^\text{op}) \circ \lambda_K \circ \delta$. Let’s rephrase this as a lifting problem

$$\begin{array}{ccc}
\Delta^{2n+1} & \overset{\lambda}{\to} & \mathcal{X} \\
\downarrow & & \downarrow \\
& K,
\end{array}$$

that is let’s find the simplicial subset $P \subseteq \Delta^{2n+1}$ on which the value in $\mathcal{X}$ has been fixed. The map $\Lambda^n_k \overset{\delta}{\to} \text{Tw } \mathcal{X}$ corresponds by construction to a map $(\Lambda^n_k) \star (\Lambda^n_k)^\text{op} \to \mathcal{X}$. The map $\Delta^n \overset{(F \times G^\text{op}) \circ \lambda_k \circ \delta}{\to} \mathcal{X} \times \mathcal{X}^\text{op}$ corresponds to a map $\Delta^{[0, n]} \cup \Delta^{[n+1, 2n+1]} \to \mathcal{X}$. Therefore, we have $P = ((\Lambda^n_k) \star (\Lambda^n_k)^\text{op}) \cup \Delta^{[0, n]} \cup \Delta^{[n+1, 2n+1]}$. Let $T$ be the vertex set of a face of $P$. Then it is one of the following two types.

1. We have $([0, n] \setminus \{k\}) \not\subseteq T$ and $([n+1, 2n+1] \setminus \{2n+1-k\}) \not\subseteq T$.
2. We have $T \subseteq [0, n]$ or $T \subseteq [n+1, 2n+1]$.

That is, $P \subseteq \Delta^{2n+1}$ is the largest simplicial subset which does not have any of the following faces.

- Faces with vertex set $([0, n] \setminus \{k\}) \cup S'$ for a nonempty subset $S' \subseteq [n+1, 2n+1]$.
- Faces with vertex set $S'' \cup ([n+1, 2n+1] \setminus \{2n+1-k\})$ for a nonempty subset $S'' \subseteq [0, n]$.

Consider the chain of inclusions

$$P = P_0 \subseteq P_1 \subseteq \cdots \subseteq P_{n+1} = \Delta^{2n+1},$$

where $P_\ell \subseteq \Delta^{2n+1}$ is the largest sub-simplicial set which does not have any of the following faces.

- Faces with vertex set $([0, n] \setminus \{k\}) \cup S'$ for a subset $S' \subseteq [n+1, 2n+1]$ of size larger than $\ell$.
- Faces with vertex set $S'' \cup ([n+1, 2n+1] \setminus \{2n+1-k\})$ for a subset $S'' \subseteq [0, n]$ of size larger than $\ell$.

We will solve the lifting problem by ascending this chain one by one. Suppose that we have already lifted to a map $P_\ell \to \mathcal{X}$ for some $0 \leq \ell \leq n$. Let

$$\{S' \subseteq [n+1, 2n+1] : |S'| = \ell + 1\} = \{S'_1, \ldots, S'_{(\ell+1)}\} \text{ and } \{S'' \subseteq [0, n] : |S''| = \ell + 1\} = \{S''_1, \ldots, S''_{(\ell+1)}\}.$$

Consider the chain of inclusions

$$P_\ell = P'_{\ell,0} \subseteq P'_{\ell,1} \subseteq \cdots \subseteq P'_{\ell,(\ell+1)} = P''_{\ell,0} \subseteq P''_{\ell,1} \subseteq \cdots \subseteq P''_{\ell,(\ell+1)} = P_{\ell+1},$$

where $P'_{\ell,i} \subseteq \Delta^{2n+1}$ is the largest sub-simplicial set which does not have any of the following faces.

- Faces with vertex set $([0, n] \setminus \{k\}) \cup S'_i$ for $i < j \leq (\ell+1)$.
- Faces with vertex set $([0, n] \setminus \{k\}) \cup S'_j$ for a subset $S' \subseteq [n+1, 2n+1]$ of size larger than $\ell + 1$.
- Faces with vertex set $S''_i \cup ([n+1, 2n+1] \setminus \{2n+1-k\})$ for $i < j \leq (\ell+1)$.
- Faces with vertex set $S''_j \cup ([n+1, 2n+1] \setminus \{2n+1-k\})$ for a subset $S'' \subseteq [0, n]$ of size larger than $\ell + 1$.

and $P''_{\ell,i} \subseteq \Delta^{2n+1}$ is the largest sub-simplicial set which does not have any of the following faces.

- Faces with vertex set $([0, n] \setminus \{k\}) \cup S'_j$ for a subset $S' \subseteq [n+1, 2n+1]$ of size larger than $\ell + 1$.
- Faces with vertex set $S''_i \cup ([n+1, 2n+1] \setminus \{2n+1-k\})$ for $i < j \leq (\ell+1)$.
- Faces with vertex set $S''_j \cup ([n+1, 2n+1] \setminus \{2n+1-k\})$ for a subset $S'' \subseteq [0, n]$ of size larger than $\ell + 1$. 

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Since the inclusions
\[ P_{ℓ,i} ′ \subset P′_{ℓ,i+1} = P_{ℓ,i} \sqcup Δ^k_{1,2n+1} \] (Δ^k_{0,n} \star Δ^{k+1}_{n+1}) and
\[ P′_{ℓ,i} ⊂ P″_{ℓ,i+1} = P″_{ℓ,i} \sqcup Δ^{k+1}_{n+1} \sqcup Δ^k_{1,2n+1} \] (Δ^{k+1}_{0,n} \star Δ_{n+1,2n+1})
are inner anodyne, the induction step is proven.

\[ \Box \]

**Lemma 4.3.** Let \( \mathcal{Y} \to T \) be a right fibration. Then \( \Gamma(T, \mathcal{Y}) \) is a Kan complex.

**Proof.** It is easy to see that \( \mathcal{Y}^T \to T^T \) is a right fibration. Hence, \( \Gamma(T, \mathcal{Y}) \) is a right fibration over a point. Hence a Kan complex, [Lur09, Lemma 2.1.3.3]. \[ \Box \]

**Proposition 4.4.** Let \( \mathcal{X} \xrightarrow{p} K \) be an inner fibration between quasi-categories. Let \( F, G \in \Gamma(K, \mathcal{X}) \) be two sections. Then by Lemma 4.2, the induced map \( TwK \times \mathcal{X} \times \mathcal{X} \xrightarrow{p} TwK \) is a right fibration. Therefore, its pullback \( Z \xrightarrow{zh} TwK \) along the diagonal \( TwK \xrightarrow{ΔTwK} TwK \times TwK \) is also a right fibration. The right fibration \( q \) is classified by the diagram

\[ (TwK)^{op} \xrightarrow{(m \to n) \mapsto Map_q(F(m), G(n))} \mathcal{X}. \]

Then the (homotopy) limit of this diagram is \( Map_{Γ(K, \mathcal{X})}(F, G) \).

**Proof.** First, we claim that the induced map \( \text{Hom}^1_{Fun(K, \mathcal{X})}(F, G) \xrightarrow{p_{FG}} \text{Hom}^1_{Fun(K, K)}(id_K, id_K) \) is a right fibration. That is, for \( 0 < k \leq n \), any lifting problem

\[ \Lambda^n_k \xrightarrow{\text{id}_k} \text{Hom}^1_{Fun(K, \mathcal{X})}(F, G) \]

needs to have a solution. But this corresponds to a lifting problem

\[ (\Lambda^n_k \star Δ^0) \cup (Δ^n \star 0) \xrightarrow{\text{id}_k} \text{Fun}(K, \mathcal{X}) \]

As the inclusion \( (\Lambda^n_k \star Δ^0) \cup (Δ^n \star 0) = Δ^{n+1}_k \subset Δ^{n+1} \) is inner anodyne, the lifting problems have solutions.

Thus, the map \( p_{FG} \) is a right fibration between Kan complexes and therefore a Kan fibration [Lur09, Lemma 2.1.3.3 op]. Therefore the strict Cartesian square

\[ \xymatrix{ \text{Hom}^1_{Γ(K, \mathcal{X})}(F, G) \ar[r] \ar[d] & \text{Hom}^1_{Fun(K, \mathcal{X})}(F, G) \ar[d] \quad {\{id, id\}} \ar[r] \ar[d] & \text{Hom}^1_{Fun(K, K)}(id_K, id_K). } \]
is moreover homotopy Cartesian. We have
\[
\text{Map}_{\text{Fun}(K, X)}(F, G) = \lim_{(\text{T}w K)^{\text{op}} \to K^{\text{op}} \times K} \xrightarrow{\text{F}^{\text{op}} \times \text{G}} \mathcal{J}^{\text{op}} \times \mathcal{J}^{\text{op}} \xrightarrow{\text{Map}_{\mathcal{J}}(\text{F})} \mathcal{J}
\]

[GHN17, Definition 2.1, Proposition 5.1]. By construction, the presheaf \((\text{T}w K)^{\text{op}} \to K^{\text{op}} \times K \xrightarrow{\text{F}^{\text{op}} \times \text{G}} \mathcal{J}^{\text{op}} \times \mathcal{J}^{\text{op}} \xrightarrow{\text{Map}_{\mathcal{J}}(\text{F})} \mathcal{J}\) classifies the right fibration \(\text{T}w \mathcal{J} \times \mathcal{J} \times \mathcal{J}^{\text{op}} \xrightarrow{\text{Tw K}} \text{Tw K} \to \text{Tw K}\) where the pullback is taken along the composite \(\text{Tw K} \to K \times K^{\text{op}} \xrightarrow{\text{F}^{\text{op}} \times \text{G}} \mathcal{J}^{\text{op}} \times \mathcal{J}^{\text{op}} \). Therefore, we get
\[
\text{Map}_{\text{Fun}(K, X)}(F, G) = \Gamma(\text{T}w K, \text{Tw} \mathcal{J} \times \mathcal{J} \times \mathcal{J}^{\text{op}} \xrightarrow{\text{Tw K}})
\]

[Lur09, Corollary 3.3.3.4\textsuperscript{op}]. This combines to produce an equivalence
\[
\text{Hom}^{\text{L}}_{\text{Fun}(K, X)}(F, G) \simeq \Gamma(\text{T}w K, \text{Tw} \mathcal{J} \times \mathcal{J} \times \mathcal{J}^{\text{op}} \xrightarrow{\text{Tw K}}).
\]

We may apply the same argument with \(p\) replaced with \(\text{id}_K\) to produce an equivalence
\[
\text{Hom}^{\text{L}}_{\text{Fun}(K, K)}(\text{id}_K, \text{id}_K) \simeq \Gamma(\text{T}w K, \text{Tw} \mathcal{J} \times \mathcal{J} \times \mathcal{J}^{\text{op}} \xrightarrow{\text{Tw K}}).
\]

Hence we have a homotopy pullback diagram
\[
\begin{array}{ccc}
\text{Map}_{\Gamma(K, \mathcal{J})}(F, G) & \rightarrow & \Gamma(\text{T}w K, \text{Tw} \mathcal{J} \times \mathcal{J} \times \mathcal{J}^{\text{op}} \xrightarrow{\text{Tw K}}) \\
\downarrow & & \downarrow \\
\{\Delta_{\text{T}w K}\} & \rightarrow & \Gamma(\text{T}w K, \text{Tw} K \times K \times K^{\text{op}} \xrightarrow{\text{Tw K}}).
\end{array}
\]

But we also have the strict pullback diagram
\[
\begin{array}{ccc}
\Gamma(\text{T}w K, Z) & \rightarrow & \Gamma(\text{T}w K, \text{Tw} \mathcal{J} \times \mathcal{J} \times \mathcal{J}^{\text{op}} \xrightarrow{\text{Tw K}}) \\
\downarrow & & \downarrow \\
\{\Delta_{\text{T}w K}\} & \rightarrow & \Gamma(\text{T}w K, \text{Tw} K \times K \times K^{\text{op}} \xrightarrow{\text{Tw K}}).
\end{array}
\]

It suffices to show that the map
\[
\Gamma(\text{T}w K, \text{Tw} K \times \mathcal{J} \times \mathcal{J}^{\text{op}} \xrightarrow{\text{Tw K}}) \rightarrow \Gamma(\text{T}w K, \text{Tw} K \times K \times K^{\text{op}} \xrightarrow{\text{Tw K}})
\]
is a right fibration. This is because the section categories are Kan complexes by the previous lemma and [Lur09, 2.1.3.3].

The fact that it is a right fibration follow from follows from Lemma 4.2 and that for \(0 < k \leq n\), the inclusion \(\Lambda^n_k \times \text{Tw K} \rightarrow \Delta^n \times \text{Tw K}\) is right anodyne [Lur09, Corollary 2.1.2.7].

\[\square\]

**Corollary 4.5.** Let \(\mathcal{J} \xrightarrow{p} K\) be an inner fibration between quasi-categories. Let \(F, G \in \Gamma_{\text{Cart}}(K, \mathcal{J})\) be two Cartesian sections. Suppose that \(\text{Map}_{\mathcal{J}}(F(k), G(k))\) is contractible for all \(k \in K\). Then \(\text{Map}_{\Gamma(K, \mathcal{J})}(F, G)\) is contractible too.

**Proof.** Let \(k \xrightarrow{c} \ell\) be an edge of \(K\). By assumption, \(G_c\) is a \(p\)-Cartesian edge. Therefore, postcomposition with it gives an equivalence \(\text{Map}_{\mathcal{J}}(F(k), G(k)) \rightarrow \text{Map}_{\mathcal{J}}(F(k), G(\ell))\). Then by the formula of Proposition 4.4 \(\text{Map}_{\Gamma(K, \mathcal{J})}(F, G)\) is a limit of contractible spaces, thus contractible itself.

\[\square\]
4.2. Mapping spaces in the Cartesian section quasi-category. Let $X \to S$ be a coCartesian fibration. In this subsection, $X^\natural$ will denote the marked simplicial set with marked edges the coCartesian edges.

**Lemma 4.6.** Let $k \xrightarrow{e} \ell$ be an edge in a quasi-category $K$. Then $e$ is an equivalence if and only if it is an initial object of $K_{k/}$.

**Proof.** An $n$-simplex of $(K_{k/})_{e/}$ is by definition a morphism $\Delta^{n+2} = \Delta^0 \star \Delta^0 \star \Delta^n \to K$ whose restriction to $\Delta^1 = \Delta^0 \star \Delta^0$ is $e$. Hence we have a commutative triangle

$$
\begin{array}{ccc}
(K_{k/})_{e/} & \xrightarrow{s} & K_{e/} \\
\downarrow r & & \downarrow s \\
K_{k/} & &
\end{array}
$$

The vertical maps are obtained by restricting to $\Delta^{[n+2]}[1]$. The map $e$ is an initial object of $K_{k/}$ if and only if it is a strongly initial object [Lur09, Corollary 1.2.12.5], that is the map $r$ is a trivial fibration. Unwinding the definitions, the map $s$ has the right lifting property with respect to $\partial \Delta^n \subset \Delta^n$ if and only if every diagram $\Delta^{n+2}_{0+1} \xrightarrow{e} K$ with $e|\Delta^{0+1} = e$ extends to $\Delta^{n+2}$. The latter lifting property is equivalent to $e$ being an equivalence by [Lur09, Proposition 1.2.4.3].

□

**Lemma 4.7.** Let $X$ be a quasi-category. Let $X^\# \xrightarrow{q} Y^\#$ be a coCartesian trivial cofibration in $\text{Set}_\Delta^+$. Then $Y$ is a Kan complex.

**Proof.** It is automatic that $Y$ is a quasi-category hence it suffices to show that all its edges are equivalences. First note that the coCartesian edges in $Y$ (over a point), are exactly the equivalences. It follows that $\text{Map}^\#(X^\#, (Y^\#)^\natural) = \text{Map}^\#(X^\#, Y^\natural)$. Therefore, since $q \in \text{Map}^\#(X^\#, Y^\natural) = \text{Map}^\#(X^\#, (Y^\#)^\natural)$, and the precomposition map $\text{Map}^\#(Y^\#, (Y^\#)^\natural) \xrightarrow{q^\#} \text{Map}^\#(X^\#, (Y^\#)^\natural)$ is a homotopy equivalence of Kan complexes, there exists $Y \xrightarrow{f} Y^\#$ together with a homotopy $f \sim q$. Since the precomposition map $\text{Map}^\#(Y^\#, Y) \xrightarrow{q^\#} \text{Map}^\#(X^\#, Y^\natural)$ is a homotopy equivalence, this implies that there exists a homotopy $f \xrightarrow{H} \text{id}_Y$ in $\text{Map}^\#(Y^\#, Y^\natural)$. But then for all edges $x \xrightarrow{e} y$ in $Y$, we get a homotopy commutative square in $Y$:

$$
\begin{array}{ccc}
x & \xrightarrow{H_x} & f(x) \\
\downarrow e & & \downarrow f(e) \\
y & \xrightarrow{H_y} & f(y)
\end{array}
$$

showing that $e$ is an equivalence.

□

**Lemma 4.8.** Let $y \in Y$ be a vertex in a Kan complex. Then the undercategory $Y_{y/}$ is contractible.

**Proof.** Consider a lifting problem

---
It corresponds to a lifting problem

\[
\begin{array}{ccc}
\partial \Delta^n & \rightarrow & Y_{y/} \\
\downarrow & & \downarrow \\
\Delta^n & \rightarrow & Y_{Y/}
\end{array}
\]

which has a solution as \( Y \) is a Kan complex.

\[\Box\]

**Lemma 4.9.** Let \( X \) be a quasi-category. Suppose that it has an initial object \( x \in X \). Let \( X^\sharp \xrightarrow{q} Y^\sharp \) be a coCartesian trivial cofibration in \( \text{Set}^+_\Lambda \). Then \( q(x) \in Y \) is an initial object.

**Proof.** Since \( x \in X \) is initial, it is strongly initial [Lur09, Corollary 1.2.15.5], that is the restriction map \( X_x/ \xrightarrow{\tau_X} X \) is a trivial fibration. Therefore, it has a section \( X \xrightarrow{s_X} X_x/ \) such that \( \text{id}_{X_x/} \sim s_X r_X \). Since the quasi-category \( Y \) is a Kan complex by Lemma 4.7, so is the undercategory \( Y_{q(x)/} \). Therefore, the induced map \( \xrightarrow{s_Y} Y_{q(x)/} \) takes all edges into equivalences, and therefore it induces a map \( (X_x^\sharp)^\# \xrightarrow{q_x^\#} (Y_{q(x)/})^\# \). Therefore, as the postcomposition map \( \xrightarrow{\text{Map}^\#(Y^\sharp, (Y_{q(x)/})^\#)} \xrightarrow{q} \text{Map}^\#(X^\sharp, (Y_{q(x)/})^\#) \) is a homotopy equivalence, there exists a map \( Y \xrightarrow{s_Y} Y_{q(x)/} \) and a homotopy \( s_Y q \sim q_x^\#(s_X)^\# \). Since we have

\[ r_Y s_Y q \sim r_Y q_x^\#(s_X)^\# = q(r_X)^\#(s_X)^\# = q, \]

we get a homotopy \( r_Y s_Y \sim \text{id}_Y \). Moreover, as the space \( Y_{q(x)/} \) is contractible by Lemma 4.8, the canonical map \( \xrightarrow{p} \) has a section \( \xrightarrow{i} Y_{q(x)/} \) together with a homotopy \( \text{id}_{Y_{q(x)/}} \sim ip \). Therefore, we have a homotopy

\[ s_Y r_Y \sim ip s_Y r_Y = ip \sim \text{id}_{Y_{q(x)/}}. \]

Therefore, \( r_Y \) is a homotopy equivalence, and thus it is a weak equivalence. Since it is moreover a left fibration, it is a trivial fibration. This shows that \( q(x) \in Y \) is strongly initial, and thus it is an initial object.

\[\Box\]

**Lemma 4.10.** Let \( K \) be a quasi-category. Let \( \tilde{\nu} \) denote the collection of edges of \( (\text{Tw}K)^\text{op} \) given by diagrams in \( K \) of the form

\[
\begin{array}{ccc}
m & \xrightarrow{id_m} & m \\
\downarrow{\alpha'} & & \downarrow{\alpha} \\
n' & \xleftarrow{\nu} & n.
\end{array}
\]
Let \(((\text{Tw } K)^{\text{op}}, \bar{v}) \xrightarrow{q} H\) be a coCartesian trivial cofibration in \((\text{Set}^+_\Delta)/_{K^{\text{op}}}\) with source the restriction map \(((\text{Tw } K)^{\text{op}}, \bar{v}) \xrightarrow{r} (K^{\text{op}})^\#\) and target a coCartesian fibration \(H^\# \xrightarrow{p} (K^{\text{op}})^\#\). Then \(H^\# \xrightarrow{p} (K^{\text{op}})^\#\) is a trivial coCartesian fibration.

Proof. It will be enough to show that the fibres of \(H^\# \xrightarrow{p} K^{\text{op}}\) are contractible [Lur09, Proposition 3.1.3.5]. Fix \(m \in K^{\text{op}}\). Note that the fibre \(r^{-1}(m) = ((\text{Tw } K)^{\text{op}}, \bar{v})_m = (K_m)^\#\) by construction. The fibre \((K_m)^\# \xrightarrow{\eta_m} H_m\) is a trivial coCartesian cofibration in \(\text{Set}^+_\Delta\). By Lemma 4.6, \(\text{id}_m \in K_m/\) is an initial object. Then by Lemma 4.9, \(H_m\) has an initial object \(y \in H_m\). That is, the restriction map \((H_m)^y \xrightarrow{\alpha} H_m\) is a trivial fibration. But by Lemma 4.7, \(H_m\) is a Kan complex. Therefore, so is \((H_m)^y\).

Then by Lemma 4.8, \((H_m)^y\) is contractible. Therefore, \(H_m\) is contractible as claimed. \(\square\)

Lemma 4.11. Let \(\mathcal{E} \xrightarrow{p} K\) be an inner fibration over a quasi-category. Let \(F, G \in \Gamma(K, \mathcal{E})\) be two sections. Suppose that \(G\) is a Cartesian section. Then the map \((\text{Tw } K)^{\text{op}} \xrightarrow{f,\alpha \mapsto \text{Map}_\alpha(F_m, G_m)} \mathcal{E}\) takes a diagram \(\nu' \in \bar{v}\):

\[
\begin{array}{ccc}
m & \xrightarrow{\text{id}_m} & m \\
\downarrow{\alpha'} & & \downarrow{\alpha} \\
n' & \xleftarrow{\nu} & n.
\end{array}
\]

to an equivalence.

Proof. As shown in the Proof of Proposition 4.4, the map \(f\) classifies the right fibration \(Z \to \text{Tw } K\) that is the pullback of \(\text{Tw } K \times \mathcal{E} \times \mathcal{E}^{\text{op}} \xrightarrow{q} \text{Tw } K \times K \times K^{\text{op}} \xrightarrow{\Delta} \text{Tw } K \times K \times K^{\text{op}} \xrightarrow{\text{Tw } K}, \mathcal{E} \times \mathcal{E}^{\text{op}}\), where \(q\) itself is the pullback of \(\mathcal{E}^{\text{op}} \xrightarrow{\lambda} \mathcal{E} \times \mathcal{E}^{\text{op}}\) along \(\text{Tw } K \xrightarrow{\text{Tw } K \times K K^{\text{op}}} \text{Tw } K^{\text{op}}\). Since \(\lambda\) corresponds to \(\mathcal{E}^{\text{op}} \xrightarrow{\text{Fun}(\mathcal{E}, \mathcal{E})} \mathcal{E}^{\text{op}}\) [Lur16, Proposition 5.2.1.11], the map \((\text{Tw } K)^{\text{op}} \xrightarrow{\alpha \mapsto \text{Map}_\alpha(F_m, G_m)} \mathcal{E}\) takes \(\nu'\) to the postcomposition map \(\text{Map}_\mathcal{E}(F_m, G_n) \xrightarrow{G_{\nu'}} \text{Map}_\mathcal{E}(F_m, G_{n'})\). Therefore, its restriction \(f\) takes \(\nu'\) to the postcomposition map \(\text{Map}_\mathcal{E}(F_m, G_n) \xrightarrow{G_{\nu'}} \text{Map}_{\text{Fun}}(F_m, G_{n'})\), which is an equivalence as \(G\) is a Cartesian fibration. \(\square\)

Proposition 4.12. Let \(\mathcal{E} \xrightarrow{p} K\) be an inner fibration over a quasi-category. Let \(F, G \in \Gamma_{\text{Cart}}(K, \mathcal{E})\) be two Cartesian sections. Then we have

\[
\text{Map}_{\Gamma_{\text{Cart}}(K, \mathcal{E})}(F, G) \simeq \text{lim}(K^{\text{op}} \xrightarrow{m \mapsto \text{Map}_m(F_m, G_m)} \mathcal{E}).
\]

Proof. Consider the diagram
Here, $i$ is the map given by the identity, and $q$ is a coCartesian trivial cofibration in $(\text{Set}_\Delta^*)_{/K^{\text{op}}}$ between the restriction map $((\text{Tw} K)^{\text{op}}, \tilde{\nu}) \to (K^{\text{op}})^\sharp$ and a coCartesian fibration $H \xrightarrow{\tilde{\nu}} (K^{\text{op}})^\sharp$. Since by Lemma 4.11, $f$ takes the $\nu' \in \tilde{\nu}$ to equivalences, we have $f = if'$. Since $p$ is a coCartesian trivial fibration by Lemma 4.10, it has a section $t$. This shows that $t$ is co-marked anodyne, and thus left anodyne. We claim that $qi$ is also left anodyne. Let $X \xrightarrow{\pi} S$ be a left fibration. Then it is a coCartesian fibration. Therefore, $X^\sharp = X^\flat \xrightarrow{\pi} S^\sharp$ is a coCartesian fibration. Consider a lifting diagram

$$
\begin{array}{ccc}
\text{(Tw} K)^{\text{op}} & \xrightarrow{a} & X^\sharp \\
\downarrow i & & \downarrow \pi^\sharp \\
\text{(Tw} K)^{\text{op}}[\tilde{\nu}]_{/c} & \xrightarrow{q} & S^\sharp \\
\downarrow q & & \downarrow b \\
H & \xrightarrow{t} & S^\sharp.
\end{array}
$$

Since every edge of $X^\sharp$ is marked, $a$ gives $a'$. Since $q$ is co-marked anodyne, we get $c$ such that $cq = a'$ and $\pi^\sharp c = b$. But then $q_i = a$ and $\pi^\sharp c = b$ shows that $qi$ has the left lifting property with respect to $\pi$.

Since the precomposition map $\text{Map}^\sharp(H^\flat, (\mathcal{S} \times K^{\text{op}})^\sharp) \xrightarrow{cq} \text{Map}^\sharp(((\text{Tw} K)^{\text{op}}, \tilde{\nu}), (\mathcal{S} \times K^{\text{op}})^\sharp)$ is a homotopy equivalence, there exists a map $H \xrightarrow{(g', p)} \mathcal{S} \times K^{\text{op}}$ together with a homotopy $(g', p)q \sim (f', r)$. Let $g = g't$. We know that a map is left anodyne if and only if it is final [Lur09] Proposition 4.1.1.3 (4). That is, the maps $qi$ and $t$ are final. Then, using Proposition 4.4, we get

$$
\text{Map}_{\Gamma(K, \mathcal{S})}(F, G) = \lim(f) = \lim(f'i) = \lim(g'qi) = \lim(g') = \lim(g).
$$

Remark 4.13. Let $m \in K^{\text{op}}$. Then the homotopy $(f', pr) \sim (g', p)q$ gives a homotopy $f' \sim g'q$. Moreover, as $t$ is a section of the coCartesian trivial fibration $p$, we get a homotopy $pt \sim \text{id}_H$. Therefore, from $m = r(\text{id}_m) = p(q(\text{id}_m))$, we get a homotopy $t(m) \sim q(\text{id}_m)$. This gives

$$
g(m) = g'(t(m)) \simeq g'(q(\text{id}_m)) \simeq f'(\text{id}_m) = \text{Map}_m(F_m, G_m).
$$
5. The construction of $\text{op} \mathcal{D}_S$

In this section, we relativize the dg-nerve construction [Lur16, Construction 1.3.1.6] to give a presentable fibration over $\text{Sch}_S$, which is classified by the functor $\text{Sch}_S^{\text{op}} \xrightarrow{T \mapsto \mathcal{D}(T)^{\text{op}} \Rightarrow \text{Cat}_\infty}$.

Construction 5.1. Let $\text{op} \mathcal{D}_S$ denote the simplicial set with $n$-simplices tuples $(\sigma, (K_i)_{i \in [n]}, (f_i)_{i \in [n]})$, where

1. $\sigma = (T_i, t_{ij})$ is an $n$-simplex in the nerve of the category $\text{Sch}_S$. That is, for each $0 \leq i \leq n$, $T_i$ is an $S$-scheme, and for each $0 \leq i < j \leq n$, $t_{ij}$ is a morphism of $S$-schemes $T_i \rightarrow T_j$ such that for each $0 \leq i < j < k \leq n$, we have $t_{ij} = t_{jk}t_{ij}$.
2. For each $0 \leq i \leq n$, $K_i$ is a complex of injective $\mathcal{O}_T$-modules with quasi-coherent cohomology sheaves.
3. For each $I = \{i_0 < i_1 < \cdots < i_m < i_n\} \subseteq [n]$ with $m \geq 0$, we have $f_i \in \text{RHom}^n_{L_{T_i}}(L_{T_i}^*, K_{i_0}, K_{i_n})$ such that

$$
df_i = \sum_{1 \leq j \leq m} (-1)^j (f_{1i-1}) \circ (t_{1i-1, j}^* \cdot r_{1i-1j} \cdot K_{i_0}) - f_{1i-1} \circ (L_{T_i} f_{1i-1j} \circ (L_{T_i} f_{1i-1j}))
$$

Let $[m] \xrightarrow{\alpha} [n]$ be a morphism in $\Delta$. Then the corresponding map $(\text{op} \mathcal{D}_S)_n \xrightarrow{\alpha^*} (\text{op} \mathcal{D}_S)_m$ is defined as

$$(\sigma, (K_i)_{i \in [n]}, (f_i)_{i \in [n]}) \mapsto (\sigma \circ \alpha, (K_{\alpha(i)})_{i \in [m]}, (g_j)_{i \in [m]}),$$

where

$$g_j = \begin{cases} f_{\alpha(j)} & \text{if } \alpha(j) \text{ is injective,} \\ \text{id}_{f_i} & |j| = 2 \text{ and } \alpha(j) = \{i\}, \\ 0 & \text{else.} \end{cases}$$

We need to check that this makes sense, in other words the third condition in the constructions holds. To see this, observe that any $\alpha$ can be factored into coface and codegeneracy maps, hence it suffices to check the condition for those. In the case of a coface map, we will always be in the situation where $\alpha(j)$ is injective, hence the condition for $g_j$ boils down to the same condition for $f_j$.

Now consider the case where $\alpha : [n + 1] \rightarrow [n]$ is a degeneracy with $\alpha(i) = \alpha(i + 1)$. We may assume that both $i, i + 1 \in J$ otherwise we will be in the injective situation. Then $g_j = 0$. In the sum, all terms of the form

$$g_{\{i, \ldots, i\}} \circ (L_{T_i} f_{\{i, \ldots, i\}})$$

will vanish due to non-injectivity of the restricted morphism. Exactly two of the terms $g_{\{i, \ldots, i\}}$ will not vanish but will occur with opposite sign.

Let $\text{op} \mathcal{D}_S \xrightarrow{\mathcal{D}_S} \text{Sch}_S$ denote the forgetful map.

Remark 5.2. We put the op in the notation to avoid confusion as the fibres $\text{op} \mathcal{D}(T)$ are indeed the opposite categories of the derived category $\mathcal{D}(T)$. To see this, note that the dg-nerve functor [Lur16, Proposition 1.3.1.20] commutes with opposites. We will write $\mathcal{D}_S = (\text{op} \mathcal{D}_S)^{\text{op}}$.

Notation 5.3. For each integer $n$ we can consider full subcategories $\text{op} \mathcal{D}_S^n$ (resp. $\text{op} \mathcal{D}_S^{\leq n}$) of $\text{op} \mathcal{D}_S$ on complexes with cohomology in degrees at most (resp. at least) $n$. 

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Given a simplicial set $S$ and a pair of 0-simplicies $s,t \in S_0$ recall the definition of the right mapping space $\text{Hom}^R(s,t)$ from [Lur09 page 27]

**Proposition 5.4.** Let $(U, J), (T, I) \in \text{op} \mathcal{S}$ be 0-simplices. Then we have

$$\text{Hom}^R_{\text{op} \mathcal{S}}((U, J), (T, I)) \cong \bigsqcup_{U \xrightarrow{g} T} \text{DK} \tau_{\leq 0} \text{RHom}_U(Lg^*I, J).$$

**Proof.** This is a direct generalization of [Lur16 Remark 1.3.1.12]. Let $\Delta^{n+1} \xrightarrow{\lambda} \text{op} \mathcal{S}$ be an $n$-simplex of $\text{Hom}^R_{\text{op} \mathcal{S}}((U, J), (T, I))$.

Given a pair of integers $0 \leq i < j \leq n+1$ there is an induced morphism $\delta_{ij} : \Delta^1 \hookrightarrow \Delta^{n+1}$ induced by the inclusion $[1] \hookrightarrow [n+1]$ with image $[i, j]$. Let $g = p \circ \lambda \circ \delta_{n,n+1}$, a morphism of schemes. Then by construction, for $0 \leq i < j \leq n+1$, we have

$$p \circ \lambda \circ \delta_{ij} = \begin{cases} \text{id}_U & j < n + 1 \\ g & j = n + 1 \end{cases}.$$

This shows that by construction we have

$$\text{Hom}_{\text{Set}_U}(\Delta^n, \text{Hom}^R_{\text{op} \mathcal{S}}((U, J), (T, I))) \cong \bigsqcup_{U \xrightarrow{g} T} \text{Hom}_{\text{Ch}(\mathbb{Z})}(NZ\Delta^n, \text{RHom}_U(Lg^*I, J))$$

$$\cong \bigsqcup_{U \xrightarrow{g} T} \text{Hom}_{\text{Ab}_S}(\mathbb{Z}\Delta^n, \text{DK} \tau_{\leq 0} \text{RHom}_U(Lg^*I, J))$$

$$\cong \bigsqcup_{U \xrightarrow{g} T} \text{Hom}_{\text{Set}_U}(\Delta^n, \text{DK} \tau_{\leq 0} \text{RHom}_U(Lg^*I, J)).$$

□

**Notation 5.5.** Let $U \xrightarrow{g} T$ be a morphism of schemes, $I \in \mathcal{T}(T)$, and $J \in \mathcal{T}(U)$. Then we let

$$\text{Hom}^R_{\text{op} \mathcal{S}, S}(J, I) = \text{Hom}^R_{\text{op} \mathcal{S}}(J, I) \times_{\text{Hom}_{\text{Sch}_S}(U, T)} \{g\}.$$

If $\mathcal{C}$ is an $\infty$-category, the largest sub-Kan complex of $\mathcal{C}$, denoted $\mathcal{C}^\circ$ is called the interior of $\mathcal{C}$. If $\mathcal{C}$ is the nerve of an ordinary category, then its interior is the nerve of the largest subgroupoid in $\mathcal{C}$. In the case of a Cartesian fibration $\mathcal{D} \to N$ we may take the subcategory on Cartesian edges $\mathcal{D}_{\text{Cart}} \to N$ and obtain a right fibration. The fibers are now Kan complexes, see [Lur09 2.1.3].

**Corollary 5.6.** Let $X \xrightarrow{f} S$ be a flat morphism of schemes. Then by construction, the 1-category $\mathcal{D}_{\text{plg}}(X/S)$ [Lie06 §2.1] is equivalent to the 1-truncation of the interior of the full subcategory of $f_*\mathcal{D}_X$ on universally glueable $S$-perfect complexes.

We will call a map $f : X \to Y$ of simplicial sets a presentable fibration if it is an inner fibration that is both Cartesian and coCartesian, see [Lur09 Ch. 2], and its fibres are presentable quasi-categories.

**Lemma 5.7.** The map $\text{op} \mathcal{S} \xrightarrow{p} \text{Sch}_S$ is an inner fibration.

**Proof.** For $n \geq 2$ and $0 < k < n$, consider a lifting problem
determined by the factorisation $\Delta$ in (5.1). There are two possibilities, either 

\[ \Delta \text{ by mapping } \]

\[ \text{such a problem has a solution by [Lur16, Proposition 1.3.1.10].} \]

\[ \text{Lemma 5.8. Let } U \xrightarrow{g} T \text{ be a morphism of } S\text{-schemes, and } I \text{ a complex of injective } \mathcal{O}_T \text{-modules. Then an edge in } \text{op } \mathcal{D}_S \text{ corresponding to a morphism } Lg^*I \xrightarrow{q} f \text{ is Cartesian precisely when } \tau_{\leq 0}q \text{ is a quasi-isomorphism.} \]

\[ \text{Proof. Let } V \xrightarrow{h} U \text{ be a morphism of schemes and } K \in \mathcal{D}(V). \text{ Then as the precomposition map} \]

\[ \text{RHom}_V(Lh^*Lg^*I, K) \leftrightarrow \text{RHom}_V(L(gh)^*I, K) \text{ is a quasi-isomorphism between dg-injective complexes, it has a section s. Let us define a section } t \text{ of the restriction map} \]

\[ \text{Hom}^g_{\text{op } \mathcal{D}_S}(K, j) = (\text{op } \mathcal{D}_S)_j \times_{\text{op } \mathcal{D}_S \times (\text{Sch}_S)/I} \{(K, h)\} \leftarrow (\text{op } \mathcal{D}_S)_q \times_{\text{op } \mathcal{D}_S \times (\text{Sch}_S)/I} \{(K, (V \xrightarrow{h} U \xrightarrow{g} T))\} \]

by mapping $\Delta^n \to \text{Hom}^g_{\text{op } \mathcal{D}_S}(K, j)$ defined by $\Delta^{n+1} \xrightarrow{(\alpha(K)_i, f_i)} \text{op } \mathcal{D}_S \xrightarrow{\alpha(K)} \text{op } \mathcal{D}_S$, which is defined as follows.

1. We have $\delta = (V \xrightarrow{id} V \to \cdots \to V \xrightarrow{h} U \xrightarrow{g} T)$.
2. We have $\delta_0 = \cdots = \delta_n = K$, $\delta_{n+1} = f$, and $\delta_{n+2} = I$.
3. (a) If $I \subseteq [0, n+1)$, then $\tilde{f}_i = f_i$.
   (b) We have $\tilde{f}_{[n+1, n+2]} = q$.
   (c) If $[n, n+2] \subseteq I$, then $\tilde{f}_i = 0$.
   (d) If $[n+2] \subseteq I \subseteq [0, n] \cup [n+2]$, then $\tilde{f} = s(f_{[0, n] \cup [n+1] \setminus [n+2]} \circ Lh^*q)$.

Then we get a commutative diagram
This implies that the restriction map \( \tau_{\leq 0} q \) is a quasi-isomorphism, then so is \( R\operatorname{Hom}_V (L h^* I, K) \xrightarrow{\partial L h q} R\operatorname{Hom}_V (L h^* L g I, K) \). Thus the map \( \operatorname{Map}_V (L h^* I, K) \xrightarrow{\partial L h q} \operatorname{Map}_V (L h^* L g I, K) \) is an equivalence \cite{SS03} 4.1. This in turn implies that the restriction map \( (\mathcal{D}_S)^{op} \times \mathcal{D}_S \times \mathcal{S}(\mathcal{S}ch_S)^{op} \to \mathcal{S}(\mathcal{S}ch_S)^{op} \) is an equivalence too. This shows that \( q \) gives a \( p \)-Cartesian edge of \( \mathcal{D}_S \) \cite[Lur09, Proposition 2.4.4.3]{Lur09}.

On the other hand, suppose that \( q \) gives a \( p \)-Cartesian edge of \( \mathcal{D}_S \). Then for all \( K \in \mathcal{D}(U) \), the map \( \operatorname{Map}_V (I, K) \xrightarrow{\partial L h q} \operatorname{Map}_V (L g^* I, K) \) is an equivalence. Therefore, \( \tau_{\leq 0} q \) is an equivalence. This implies that \( \tau_{\leq 0} q \) is a quasi-isomorphism.

Let \( U \xrightarrow{g} T \) be a morphism of \( S \)-schemes. Since \( \mathcal{D}_S \xrightarrow{p} \mathcal{S}ch_S \) is a Cartesian fibration, its restriction \( \mathcal{D}_S \mid g \to \Delta^1 \) is classified by a functor of quasi-categories \( \mathcal{D}(T)^{op} \xrightarrow{\mathcal{D}_S \mid g} \mathcal{D}(U)^{op} \). We want to show that this functor is equivalent to the opposite \( \mathcal{D}(T)^{op} \xrightarrow{\mathcal{D}_S \mid g} \mathcal{D}(U)^{op} \) of the derived pullback functor we have constructed in Notation \ref{not:derived-pullback}. We will show this by showing that the Cartesian fibration \( p \mid g \) is equivalent to the opposite of the relative nerve of \( (L g)^{op} \). That is a Cartesian fibration over \( \Delta^1 \), which is classified by \( \mathcal{S}(\mathcal{S}ch_S)^{op} \) \cite[Lur09, Corollary 3.2.5.20]{Lur09}.

Let \( C \) be a 1-category. Let’s recall the construction of the opposite \( N_F(C)^{op} \) of the relative nerve of a functor \( C^{op} \xrightarrow{F} \mathcal{S}et_\Delta \). For a finite linearly ordered set \( J \), a simplex \( \Delta^J \to N_F(C)^{op} \) is given by the following data.

1. A diagram \( J \xrightarrow{\sigma} C \),
2. and for each \( 0 \neq J' \subseteq J \) with \( \min J' = j' \), a map of simplicial sets \( \Delta^{J'} \xrightarrow{\tau_{j'}} F(\sigma(j')) \), such that
3. for each \( 0 \neq J'' \subseteq J' \subseteq J \) with \( \min J' = j' \) and \( \min J'' = j'' \), the diagram

\[
\begin{array}{ccc}
\Delta^{J''} & \xrightarrow{\tau_{j''}} & F(\sigma(j'')) \\
\downarrow & & \downarrow F(\sigma(j' \to j'')) \\
\Delta^{J'} & \xrightarrow{\tau_{j'}} & F(\sigma(j')) \\
\end{array}
\]

is commutative.

The face and degeneracy maps can be given by precomposition.

**Lemma 5.9.** Let \( U \xrightarrow{g} T \) be a morphism of \( S \)-schemes. Then the restriction \( \mathcal{D}_S \mid g \) is classified by \( \mathcal{D}(T)^{op} \xrightarrow{(L g)^{op}} \mathcal{D}(U)^{op} \).

**Proof.** Let’s write down the opposite relative nerve of \( F: (\Delta^J)^{op} \xrightarrow{\mathcal{D}(T)^{op} \xrightarrow{(L g)^{op}} \mathcal{D}(U)^{op}} \mathcal{S}et_\Delta \). A simplex \( \Delta^J \to (N_F \Delta^J)^{op} \) is given by the following.
(1) A map \( J \xrightarrow{\sigma} [1] \),
(2) and for \( \emptyset \neq J' \subseteq J \),
(2a) a simplex \( \Delta' \xrightarrow{\tau(J')} \mathcal{D}(U) \) if \( 0 \in \sigma(J') \), and
(2b) a simplex \( \Delta' \xrightarrow{\tau(J')} \mathcal{D}(T) \) if \( 0 \notin \sigma(J') \), such that
(3) for \( \emptyset \neq J' \subseteq J \),
(3a) if \( 0 \in \sigma(J') \) or \( 0 \notin \sigma(J') \), then we have \( \tau(J'') = \tau(J')|\Delta'' \), and
(3b) if \( 0 \in \sigma(J') \setminus \sigma(J'') \), then we have \( Lg^* \circ \tau(J'') = \tau(J')|\Delta'' \).

We need to show that these two concepts agree, see [Lur16, Ch. 1].

**Proposition 5.10.** The map \( \mathcal{D}_S \xrightarrow{p} N(S\text{Sch}_S) \) is a presentable fibration.

**Proof.** By Lemma 5.7, \( p \) is an inner fibration. Moreover, by Lemma 5.8, it is a Cartesian fibration. Let \( U \xrightarrow{\eta} T \) be a morphism of \( S \)-schemes. Then by Lemma 5.9, the pullback map over \( g \) is equivalent to \( Lg^* \). This functor admits a right adjoint by Proposition 3.9. Moreover, the fibres are indeed presentable [Lur16 Proposition 1.3.5.21]. Therefore, \( p \) is a presentable fibration [Lur09 Proposition 5.5.3.3].

**Corollary 5.11.** The simplicial set \( \mathcal{D}_S \) is a quasi-category.

**Proof.** We know that \( N(S\text{Sch}_S) \) is a quasi-category, indeed it is a category. The relevant lifting problem is then solved by first lifting to schemes then applying the proposition.

6. DESCENT AND THE STACK \( \mathbf{RH} \text{om} \)

**Proposition 6.1.** Let \( K \) be a small simplicial set. Let \( k \xrightarrow{\varphi} S\text{ch}_S \) be a diagram of \( S \)-schemes. Then \( \Gamma(k, \mathcal{D}_S) \) and \( \Gamma_{\text{Cart}}(k, \mathcal{D}_S) \) are presentable stable quasi-categories.

**Proof.** These section categories are in fact quasi-categories via 5.11.

Recall that \( \mathcal{D}_S \xrightarrow{p} N(S\text{ch}_S) \) is a presentable fibration, see Proposition 5.10. If follows that the category of functors \( \text{Fun}(K, \mathcal{D}_S) \) is presentable by [Lur09 5.5.3.6], for any small simplicial set \( K \). It follows that the category of sections \( \Gamma(k, \mathcal{D}_S) \) is presentable by [Lur09 5.5.3.17]. The subquasi-category \( \Gamma_{\text{Cart}}(k, \mathcal{D}_S) \) is an accessible localization of a presentable quasi-category [Lur09 Proposition 5.5.3.17].

It remains to show that the sections are in fact stable.

Firstly observe that there is a 0-section, denoted \( 0 \in \Gamma(k, \mathcal{D}_S) \) It sends a simplex \( \sigma = (T_i, t_i) \in N(S\text{ch}_S) \) to the data \( (\sigma, 0, (0)_I, 0) \) where \( 0_i \) is the zero complex on \( T_i \) and \( 0_I \) is the zero morphism of degree \( |I| - 2 \).

It follows from 5.8 that this section lies inside \( \Gamma_{\text{Cart}}(k, \mathcal{D}_S) \). The zero section is a zero object by Corollary 4.3.

As these sections are presentable they have small colimits. By [Lur09 5.5.2.5] these quasi-categories have small limits. It follows that these section categories have fibers and cofibers. We need to show that these two concepts agree, see [Lur16 Ch. 1].
The fibers of $p : \text{op} \mathcal{D}_S \to N(\text{Sch}_S)$ are the usual derived quasi-categories, see 1.3 loc. cit, which are known to be stable. As fibers and cofibers agree pointwise they agree in the section categories by [Lur09] 5.1.2.2.

The looping and delooping functors amount to shifts. Hence Cartesian sections are closed under delooping and looping.

□

If $X$ is a stable quasi-category then a $t$-structure on $X$ amounts to a $t$-structure on its homotopy category. In other words a $t$-structure on $X$ amounts to two full subcategories $X_{\geq 0}$ and $X_{\leq 0}$ that produce $t$-structures on $\tau_1 X$.

Recall in (5.3) we introduced subcategories $\text{op} \mathcal{D}_S^{\geq 0}$ and $\text{op} \mathcal{D}_S^{\leq 0}$ of $\text{op} \mathcal{D}_S$. We would like to show that taking Cartesian sections into these subcategories indeed produces a $t$-structure on $\Gamma_{\text{Cart}}(k, \text{op} \mathcal{D}_S)$ for appropriate choice of diagram $k$.

As we do not have a good description of the homotopy category of $\Gamma_{\text{Cart}}(k, \text{op} \mathcal{D}_S)$ we need some other tools to obtain a $t$-structure on it.

The subcategories $X_{\geq n}$ are localisations of $X$. Localisations are characterised by sets of morphisms with respect to which we are localising, see [Lur09] 5.5.4.2, 5.5.4.15. Lets briefly recall some of these ideas.

Let $A \subseteq X$ be a set of morphisms (i.e edges). An object $Z \in X_0$ is said to be $A$-local if for each $a : X \to Y$ in $Z$ the induced morphism $\text{Map}_X(Y, Z) \to \text{Map}_X(X, Z)$ is a weak equivalence. Let $X'$ be the full subcategory on the $A$-local object. Then $X'$ is a localisation of $X$, in other words the inclusion $X' \hookrightarrow X$ has a left adjoint.

For each $S$-scheme $T$, the fiber of $\text{op} \mathcal{D}_S$ over $T$ is the opposite of the usual derived $\infty$-category which has a $t$-structure. Let $A$ be the collection of edges of the form $I \to \tau^{\geq 0} I$ as $I$ and $T$ vary.

Recall that $\text{op} \mathcal{D}_S^{\geq 0}$ is the full subcategory of $\text{op} \mathcal{D}_S$ spanned by those complexes whose negative cohomology vanishes.

Lemma 6.2. In the notation above, a complex $K^\bullet$ is $A$-local if and only if $K \in \text{op} \mathcal{D}_S^{\geq 0}$. Hence the inclusion $\text{op} \mathcal{D}_S^{\geq 0} \hookrightarrow \text{op} \mathcal{D}_S$ has a left adjoint.

Proof. Let's start by assuming that $K \in \text{op} \mathcal{D}_S^{\geq 0}$ is complex of injectives on a scheme $U$. We need to show $K$ is $A$-local.

In view of this, fix the truncation map $I \to \tau^{\geq 0} I$ of a complex of injective $\mathcal{O}_T$-modules with quasi-coherent cohomology on a scheme $T$ and a morphism $g : U \to T$. Then

$$\text{Map}(Lg^* \tau^{\geq 0} I, K) \simeq \text{Map}(\tau^{\geq 0} I, Rg_* K) \quad \text{see (3.9)}$$

$$\simeq \text{Map}(I, Rg_* K)$$

$$\simeq \text{Map}(Lg^* I, K).$$

To complete the proof we need to show that every complex with a negative cohomology sheaf is not $A$-local. Suppose that $I \in \mathcal{D}(T)$ has $H^i I \neq 0$ for some $i < 0$. Let $J$ be an injective resolution of $\mathcal{O}_T$. Then the map

$$H^i I = \pi_0 \text{Map}(J[i], I) \xrightarrow{\pi_0(\delta \cdot^{\geq 0})} \pi_0 \text{Map}(\tau^{\geq 0} J[i], I) = 0$$
is not a bijection, thus $I$ is not $A$-local.

We are now in a position to equip the category of Cartesian sections with a $t$-structure. As we do not have a good handle on the homotopy category of the category of Cartesian sections, we will make use of the following proposition from [Lur16].

**Proposition 6.3.** Let $\mathcal{C}$ be a stable $\infty$-category. Let $i: \mathcal{C}' \hookrightarrow \mathcal{C}$ be a full subcategory with localisation functor $L: \mathcal{C} \rightarrow \mathcal{C}'$. Then the following conditions are equivalent.

1. $\mathcal{C}'$ is closed under extensions.
2. For each $A, B \in \mathcal{C}_0$, the natural map
   $$\text{Ext}^1(LA, B) \rightarrow \text{Ext}^1(A, B)$$
   is injective.
3. The full subcategories $\mathcal{C}^{\geq 0} = \{A | LA \simeq 0\}$ and $\mathcal{C}^{\leq -1} = \{A | LA \simeq A\}$ determine a $t$-structure on $\mathcal{C}$.

*Proof.* This is proposition 1.2.1.16 of loc. cit. □

**Proposition 6.4.** Let $K \rightarrow \mathbf{Sch}_S$ be a diagram of $S$-schemes. Then $\Gamma_{\text{Cart}}(k, \text{op} D_S)$ is a presentable stable category. Furthermore $(\Gamma_{\text{Cart}}(k, \text{op} D_S^{\geq 0}), \Gamma_{\text{Cart}}(k, \text{op} D_S^{\leq 0}))$ is an accessible left complete $t$-structure.

*Proof.* First note that $\Gamma(k, \text{op} D_S) \supseteq \Gamma_{\text{Cart}}(k, \text{op} D_S)$ is a localisation by [Lur09, 5.5.3.17]. We claim that the localization map $\Gamma(k, \text{op} D_S) \rightarrow \Gamma_{\text{Cart}}(k, \text{op} D_S)$ is left exact. It will be enough to show that $\Gamma_{\text{Cart}}(k, \text{op} D_S) \subset \Gamma(k, \text{op} D_S)$ is closed under delooping [Lur16, Lemma 1.1.3.3 and Proposition 1.4.4.9]. But as we have seen in 5.8, an edge in $\text{op} D_S$ is Cartesian precisely when it is given by a quasi-isomorphism $Lg^*I \rightarrow I$. This is stable under translation, which proves the claim.

Given $I \in \Gamma_{\text{Cart}}(k, \text{op} D_S^{\geq 0})$ there is an edge $I \rightarrow \tau^{\geq 0}I$ which we can complete to a fiber sequence

$$I' \rightarrow I \xrightarrow{\tau^{\geq 0}} I''$$

as the Cartesian section category is stable, (6.1). Now by Corollary 4.5 $\text{Map}_{\Gamma_{\text{Cart}}(k, \text{op} D_S)}(I', I'')$ is contractible. The second criterion of (6.3) is now verified. We thus have a $t$-structure.

To show left completeness, it is enough to show that $\cap_n \Gamma_{\text{Cart}}(k, \text{op} D_S^{\geq n})$ only has zero objects. It only contains Cartesian diagrams of cohomologically trivial sheaves, thus pointwise zero objects, which proves the claim. □

Given a stable infinity category with a $t$-structure $(\mathcal{D}, \mathcal{D}^{\geq 0}, \mathcal{D}^{\leq 0})$ we can form its heart, $\mathcal{D}^\heartsuit = \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$. It is equivalent to the nerve of an ordinary abelian category. In the case where our diagram $k$ is in fact the Cech nerve of a cover $U \rightarrow T$, so $k: N(U^*/T) \rightarrow \text{Sch}_S$, it is in fact the category of descent data for a quasi-coherent sheaf on $T$. Hence it is equivalent to the category of quasi-coherent sheaves on $T$, denoted $\text{QCoh}(T)$.

In the situation where the heart is Grothendieck abelian category, we can now ask if the original stable quasi-category is the derived quasi-category of this abelian category. This is answered by the following result:
Proposition 6.5. Let $(\mathcal{D}, \mathcal{D}^{\geq 0}, \mathcal{D}^{< 0})$ be a stable $\infty$-category with a right complete $t$-structure. Suppose further that $\mathcal{D}^{\geq 0}$ is a Grothendieck abelian category. Then there is a functor $F : \mathcal{D}^{+}(\mathcal{D}^{\geq 0}) \to \mathcal{D}$ extending the inclusion of the heart inside $\mathcal{D}$, which is unique up to a contractible space of equivalences. Moreover, the following are equivalent.

1. The functor $F$ is fully faithful.
2. For every pair of objects $X, I \in \mathcal{D}^{\geq 0}$, if $I$ is injective, then $\operatorname{Ext}^i_{\mathcal{D}^{+}}(X, I) = 0$ for $i > 0$.

Further, if these conditions are satisfied the essential image of $F$ is the full subcategory $\cup_{n \in \mathbb{Z}} \mathcal{D}_{\geq -n}$.

Proof. This is [Lur16, 1.3.3.7].

For a cosimplicial abelian group $A^\bullet$, we let $\pi^s(A) = H^s(A, \sum (-1)^i d^i)$.

Theorem 6.6. The Cartesian fibration $\operatorname{op} \mathcal{D}^+_S \to \text{Sch}_S$ satisfies fppf descent.

Proof. Let $U \rightarrow T$ be an fppf covering over $S$. Let $N(\Delta^{op}) \rightarrow \text{Sch}_S$ denote its Čech nerve. Let $\mathcal{C} = \Gamma_{\text{cart}}(k, \operatorname{op} \mathcal{D}_S)$. By Proposition 6.4 we can apply Proposition 6.5. By descent for quasi-coherent sheaves, we have $(\mathcal{C}^{op})^{\geq 0} \simeq \text{QCoh}(T)$. By applying Proposition 6.5 to $\mathcal{C}^{op}$, it is enough to show that for $M, I \in \text{QCoh}(T)$ with $I$ injective, we have $\operatorname{Ext}^i_{\mathcal{C}^{op}}(M, I) = 0$ for $i > 0$. By Proposition 4.12, $\operatorname{Map}_{\mathcal{C}^{op}}(M, I)$ is the total complex of a cosimplicial space $m \mapsto \operatorname{Map}_{U_m}(M_{U_m}, I_{U_m})$. Therefore, we can apply the Bousfield–Kan spectral sequence to calculate $\operatorname{Ext}^i_{\mathcal{C}^{op}}(M, I) = \pi_0 \operatorname{Map}_{\mathcal{C}^{op}}(M, I[i])$. We have [GJ99, VIII, Proposition 1.15]

$$E^{s,t}_2 = \pi^s \operatorname{Map}_{U_\bullet}(M_{U_\bullet}, I_{U_\bullet}[i]) = \pi^s \operatorname{Ext}^0_{U_\bullet}(M_{U_\bullet}, I_{U_\bullet}[i - t]).$$

We can see that these terms vanish for $i \neq t$. Therefore, we have

$$E^{s,t}_2 = \begin{cases} \pi^s \operatorname{Ext}^0_{U_\bullet}(M_{U_\bullet}, I_{U_\bullet}) & t = i \\ 0 & t \neq i. \end{cases}$$

But we have $E^{s,t}_2 \Rightarrow \pi_{t-s} \text{Tot} \operatorname{Map}_{U_\bullet}(M_{U_\bullet}, I_{U_\bullet}[i])$. Therefore, $\operatorname{Ext}^i_{\mathcal{C}^{op}}(M, I) = \pi_0 \text{Tot} \operatorname{Map}_{U_\bullet}(M_{U_\bullet}, I_{U_\bullet}[i]) = 0$ as we wanted.

□

Corollary 6.7. For any $S$-scheme $T$, we have a natural equivalence $\mathcal{D}^{+}(T) \to \text{QC}^{+}(T)$. [BZF10, §3.1]

We refer the reader to loc. cit. for a definition of $\text{QC}^{+}(T)$.

Construction 6.8. Let $\mathcal{X} \to K$ be a Cartesian fibration. Suppose that $K$ is a quasi-category with a final object $S \in K$. Let $I, J \in \mathcal{X}(S)$. Then by Lemma 2.5 there exists a Cartesian section $K \xrightarrow{\eta} \mathcal{X}$ with $I_S = I$. We can then form the mapping prestack as the left vertical arrow in the pullback diagram

$$\operatorname{Map}_{\mathcal{D}}(I, J) \to \mathcal{D}^{+}$$

$$\downarrow \\
K \xrightarrow{\eta} \mathcal{X}.$$
Note that as $\mathcal{M}\text{ap}_X(I,J) \to K$ is a right fibration, it can be straightened to the presheaf
\[ T \mapsto \text{Map}_X(IT, J) \cong \text{Map}_{\mathcal{X}(T)}(IT, JT). \]
In the case of the Cartesian fibration $\text{op}\mathcal{D}_S \to \text{Sch}_S$, we write $\text{opRHom}_S(I, J) = \mathcal{M}\text{ap}_{\text{op}\mathcal{D}_S}(I, J)$. By Theorem [6.6] it is a stack.

**Remark 6.9.** Note that by Lemma 2.5, the restrict to $S$ map $\Gamma_{\text{Cart}}(K, \mathcal{X}) \to \mathcal{X}(S)$ is a trivial fibration. Therefore, $\mathcal{M}\text{ap}_X(I,J)$ does not depend on the choice of $I_*$ up to equivalence.

7. An explicit construction of loop groups in $\infty$-topoi

Let $E \in \mathcal{D}(S)^+$ be a bounded below complex of $\mathcal{O}_S$-modules. In this subsection, we give a description of the loop group $\Omega(E, (\mathcal{D}_S^+)^\times)$. We will give a general construction for a pointed object $* \to X$ in an $\infty$-topos $\mathcal{X}$.

By definition, an $\infty$-topos is a left exact localization of a presheaf category $\mathcal{P}(C)$ of a small quasi-category $C$ [Lur09 §6.1]. We will use the following equivalent description [Lur09 Proposition 5.1.1.1]. The simplicial overcategory $(\text{Set}_\Delta)^{/C}$ can be equipped with the contravariant model structure, the fibrant objects of which are exactly the right fibrations [Lur09 Corollary 2.2.3.12]. Then the presheaf quasi-category $\mathcal{P}(C) = \text{Fun}(C^{\text{op}}, \mathcal{X})$ is equivalent to the categorical nerve $\mathcal{P}'(C)$ of the full simplicial subcategory of $(\text{Set}_\Delta)^{/C}$ on fibrant objects. Therefore, we will represent the objects in the $\infty$-topos $\mathcal{X}$ as right fibrations $X \to C$. In this description, a pointed object is a section $C \times \to X$.

The loop group $\Omega(x, \mathcal{X})$ is the Čech nerve of $C \times \to X$ [Lur09 below Proposition 6.1.2.11]. Let’s recall this notion. A groupoid object in $\mathcal{X}$ is a simplicial object $\Delta^{\text{op}} \to \mathcal{X}$ such that for all $n \geq 0$ and all partitions $[n] = S \cup S'$ such that $S \cap S' = \{s\}$, the square
\[
\begin{array}{ccc}
G_n & \to & G_{S'} \\
\downarrow & & \downarrow \\
G_S & \to & G_{\{s\}}
\end{array}
\]
is homotopy Cartesian. Let $x \to y$ be a morphism in $\mathcal{X}$. Then its Čech nerve is a groupoid object $\tilde{G}$ such that there exists an augmented simplicial object $\Delta^{\text{op}} \to \mathcal{X}$, such that

1. $\tilde{G}|_{\Delta^{\text{op}}} = G$, and
2. the square
\[
\begin{array}{ccc}
\tilde{G}_1 & \to & \tilde{G}_{[1]} \\
\downarrow & & \downarrow \\
\tilde{G}_{[0]} & \to & \tilde{G}_{-1}
\end{array}
\]
is homotopy Cartesian.

As this involves a lot of homotopy fibre products, the loop group structure is highly inexplicit. Our construction makes the composition, associativity, etc. diagrams explicit in one bisimplicial set.

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Construction 7.1. Let $G$ be the following simplicial object of simplicial sets over $\mathcal{C}$. Let $G_0 = \mathcal{C}$. If $m > 0$, then the $(m, n)$-simplices are the maps $\Delta^m \times \Delta^n \to X$ such that their restriction to $\text{sk}_0 \Delta^m \times \Delta^n$ factors through $x$. Both the horizontal and vertical face and degeneracy maps can be given by restriction. Unless we state otherwise, we will think of $G$ as a vertical bisimplicial set, that is as a simplicial object $\Delta^{op} \to X$.

Theorem 7.2. The bisimplicial set $G$ is a simplicial right fibration over $\mathcal{C}$, and it is the Čech nerve of the classifying map $\mathcal{C} \xrightarrow{x} X$.

Proof. Since $\mathcal{P}'(\mathcal{C}) \to \mathcal{X}$ is a left exact localization, that is it commutes with finite limits, it is enough to show that $G$ is the Čech nerve of $x$ in $\mathcal{P}'(\mathcal{C})$. First of all, let’s prove that the $G_m$ are right fibrations. Since $G_0 = \mathcal{C}$, let’s assume $m > 0$. Then we need to show that for all $0 < k \leq m$, any lifting problem

$$
\begin{array}{ccc}
\Lambda^n_k & \to & G_m \\
\downarrow & & \downarrow \\
\Delta^n & \to & \mathcal{C}
\end{array}
$$

has a solution. Since $\sigma|(\text{sk}_0 \Delta^m \times \Delta^n)$ needs to factor through $x$, we don’t have to worry about the projection to $\mathcal{C}$. Therefore, it is enough to show that the inclusion $(\text{sk}_0 \Delta^m \times \Delta^n) \cup (\Delta^m \times \Lambda^n_k) \subset \Delta^m \times \Delta^n$ is right anodyne. That in turn holds by [Lur09, Corollary 2.1.2.7].

We next claim that the square

$$
\begin{array}{ccc}
G_1 & \to & \mathcal{C} \\
\downarrow & & \downarrow x \\
\mathcal{C} & \to & X
\end{array}
$$

is homotopy Cartesian. The square

$$
\begin{array}{ccc}
G_1 & \to & X/x \\
\downarrow & & \downarrow \text{res} \\
\mathcal{C} & \to & X
\end{array}
$$

is strict Cartesian, and as the restriction map $X/x \to X$ is a right fibration, it is moreover homotopy Cartesian. Therefore, it is enough to show that the projection $X/x \to \mathcal{C}$ is a trivial fibration, since that will imply that $X/x \to X$ is a right fibration, since it’s a composite of such [Lur09, Proposition 4.2.1.6], therefore it is enough to show that it’s a contravariant equivalence. That can be checked fibrewise [Lur09, Corollary 2.2.3.13], that is, assuming that $X$ is a Kan complex and $x \in X$, we need to show that $X/x$ is contractible. The restriction map $X/x \to X$ is a right fibration [Lur09 Proposition 4.2.1.6], therefore as $X$ is a Kan complex, $X/x$ is one too. We have a categorical equivalence $X_{/x} \to X/x$.
[Lur09] Proposition 4.2.1.5] between Kan complexes, therefore it is a weak homotopy equivalence [Lur09] Lemma 3.1.3.2]. Finally, \(X/\cdot\) is contractible: for all \(n \geq 0\), a lifting problem

\[
\begin{array}{ccc}
\partial \Delta^n & \longrightarrow & X_{/x} \\
\downarrow & & \downarrow \\
\Delta^n & \rightarrow & *
\end{array}
\]

is a lifting problem

\[
\begin{array}{ccc}
\Lambda^{n+1}_{n+1} & \longrightarrow & X \\
\downarrow & & \downarrow \\
\Delta^{n+1} & \rightarrow & *
\end{array}
\]

therefore it has a solution.

Let us finally show that \(G\) is a groupoid object of right fibrations over \(\mathcal{C}\). That is, we need to show that for all \(m \geq 2\) and all partitions \(I \cup J = [m]\) such that \(I \cap J = \{i\}\), the diagram

\[
\begin{array}{ccc}
G_m & \longrightarrow & G_I \\
\downarrow & & \downarrow \\
G_I & \longrightarrow & G_{[i]}
\end{array}
\]

is homotopy Cartesian, or equivalently, the canonical map \(G_m \xleftarrow{f} G_I \times_{G_{[i]}} G_J\) is a contravariant equivalence. Since \(G_J \to G_{[i]} = \mathcal{C}\) is a right fibration, we have \(G_I \times_{G_{[i]}} G_J = G_I \times_{G_{[i]}} G_J\). Therefore, it is enough to show that for all \(c \in \mathcal{C}\), the fibre \(f_c\) is a weak homotopy equivalence [Lur09] Corollary 2.2.3.13]. That is, we can assume that \(\mathcal{C} = \{c\}\). We claim that in this case \(f\) is a trivial fibration. That is, we need to show that for every \(n \geq 1\), every lifting problem

\[
\begin{array}{ccc}
\partial \Delta^n & \longrightarrow & G_m \\
\downarrow & & \downarrow \\
\Delta^n & \rightarrow & G_I \times_{G_{[i]}} G_J
\end{array}
\]

has a solution. Unwinding the construction, it is enough to show that the inclusion

\[
(\text{sk}_0 \Delta^n \times \Delta^n) \cup ((\Delta^I \cup \Delta^J) \times \Delta^n) \cup (\Delta^m \times \partial \Delta^n) \subset \Delta^m \times \Delta^n
\]

is anodyne. Since \(I \cup J = [m]\), we have \(\text{sk}_0 \Delta^n \subseteq \Delta^I \cup \Delta^J\), therefore we can drop the first component of the union. Then we are done by Lemma 7.3 [GJ99 Corollary I.4.6].

\[\square\]

**Lemma 7.3.** Let \(m \geq 2\) and consider a partition \(I \cup J = [m]\) such that \(I \cap J \neq \emptyset\). Then the inclusion \(\Delta^I \cup \Delta^J \subseteq \Delta^m\) is anodyne.
Proof. We can assume neither $I$ or $J$ is $[m]$. Let us use induction on $k = |(I \cup J) \setminus (I \cap J)| \geq 2$. The $k = 2$ case is proven by Lemma 7.4. Suppose that there are $i \neq j \in I \setminus J$. Then as we have a pushforward diagram

$$
\begin{array}{c}
\Delta^I \cup \Delta^{[i]} \leftarrow \Delta^{I \setminus [i]} \cup \Delta^{[j]}
\end{array}
$$

the statement is proven by the induction hypothesis \cite[Proposition 4.2]{GJ99}. The $i \neq j \in J \setminus I$ case can be proven the same way.

\[ \square \]

Lemma 7.4. Let $S \hookrightarrow \Delta^n$ be the inclusion of the union of a proper subset of the set of facets of $\Delta^n$. Then $i$ is anodyne.

Proof. Let us use induction on $n \geq 1$, the $n = 1$ case being satisfied by definition. Suppose $n > 1$. Let us use induction on the number $m \geq 1$ of facets missing from $S$. If $m = 1$, then $S$ is a horn, so $i$ is anodyne by definition. Suppose that $m > 1$, and that the facet $\Delta^{[n \setminus [i]]}$ is missing from $S$. By the induction hypothesis on $n$, the inclusion $S \cap \Delta^{[n \setminus [i]]} \subset \Delta^{[n \setminus [i]]}$ is anodyne, thus so is its pushout $S \subset S \cup \Delta^{[n \setminus [i]]}$. By the induction hypothesis on $m$, this in turn shows that $i$ is the composite of anodyne maps $S \subset S \cup \Delta^{[n \setminus [i]]} \subset \Delta^n$.

\[ \square \]

References

\cite{AGV72} Michael Artin, Alexander Grothendieck, and Jean-Louis Verdier, Théorie des topos et cohomologie étale des schémas. Tome 2, Lecture Notes in Mathematics, Vol. 270, Springer-Verlag, Berlin-New York, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

\cite{Be˘ı82} A. A. and Bernstein Be˘ılinson J. and Deligne, Faisceaux pervers, Analysis and topology on singular spaces, I (Luminy, 1981), 1982, pp. 5–171.

\cite{BZFN10} David Ben-Zvi, John Francis, and David Nadler, Integral transforms and Drinfeld centers in derived algebraic geometry, J. Amer. Math. Soc. 23 (2010), no. 4, 909–966, DOI 10.1090/S0894-0347-10-00669-7. MR2669705

\cite{Ber07} Julia E. Bergner, A model category structure on the category of simplicial categories, Trans. Amer. Math. Soc. 359 (2007), no. 5, 2043–2058, DOI 10.1090/S0002-9947-06-03987-0. MR2276611

\cite{BK72} A. K. Bousfield and D. M. Kan, Homotopy limits, completions and localizations, Lecture Notes in Mathematics, Vol. 304, Springer-Verlag, Berlin-New York, 1972. MR0365573

\cite{GHN17} David Gepner, Rune Haugseng, and Thomas Nikolaus, Lax colimits and free fibrations in \(\infty\)-categories, Doc. Math. 22 (2017), 1225–1266. MR3690268

\cite{Gil07} James Gillespie, Kaplansky classes and derived categories, Math. Z. 257 (2007), no. 4, 811–843. MR2342555

\cite{GJ99} Paul G. Goerss and John F. Jardine, Simplicial homotopy theory, Progress in Mathematics, vol. 174, Birkhäuser Verlag, Basel, 1999. MR1711612

\cite{Gro71} Alexander Grothendieck, Revêtements étalés et groupe fondamental (SGA 1), Lecture notes in mathematics, vol. 224, Springer-Verlag, 1971.
[HS] Andre Hirschowitz and Carlos Simpson, *Descente pour les n-champs*. arXiv:math/9807049 [math.AG]. ↑1, 2

[Hol] Sharon Hollander, *A homotopy theory for stacks*. arXiv:math/0110247. ↑10

[Lie06] Max Lieblich, *Moduli of complexes on a proper morphism*, J. Algebraic Geom. 15 (2006), no. 1, 175–206. MR2177199 ↑2, 26

[Lip09] Joseph Lipman, *Notes on derived functors and Grothendieck duality*, Foundations of Grothendieck duality for diagrams of schemes, Lecture Notes in Math., vol. 1960, Springer, Berlin, 2009, pp. 1–259. MR2490557 ↑16

[Lur09] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR2522659(2010j:18001) ↑4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16, 17, 19, 20, 21, 22, 23, 24, 26, 28, 29, 30, 31, 33, 34, 35

[Lur16] ______, *Higher algebra*, 2016. ↑11, 13, 16, 17, 23, 25, 26, 27, 29, 31, 32

[SS03] Stefan Schwede and Brooke Shipley, *Equivalences of monoidal model categories*, Algebr. Geom. Topol. 3 (2003), 287–334 (electronic), DOI 10.2140/agt.2003.3.287. MR1997322 ↑16, 28

[Tho90] R. W. and Trobaugh Thomas, *Higher algebraic K-theory of schemes and of derived categories*, The Grothendieck Festschrift, Vol. III, 1990, pp. 247–435. ↑12

[TV08] Bertrand Toën and Gabriele Vezzosi, *Homotopical algebraic geometry. II. Geometric stacks and applications*, Mem. Amer. Math. Soc. 193 (2008), no. 902, x+224, DOI 10.1090/memo/0902, available at http://perso.math.univ-toulouse.fr/btoen/files/2012/04/HAGII.pdf MR2394633(2009h:14004) ↑1, 2

[Wei94] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR1269324(95f:18001) ↑15

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