ON WEAKLY HAUSDORFF SPACES AND LOCALLY STRONGLY SOBER SPACES

JEAN GOUBAULT-LARRECQ

Abstract. We show that the locally strongly sober spaces are exactly the coherent sober spaces that are weakly Hausdorff in the sense of Keimel and Lawson. This allows us to describe their Stone duals explicitly. As another application, we show that weak Hausdorffness is a sufficient condition for lenses and of quasi-lenses to form homeomorphic spaces, generalizing previously known results.

1. Introduction

In their study of measure extension theorems for $T_0$ spaces, Keimel and Lawson introduced so-called weakly Hausdorff spaces [14, Lemma 6.6]. The notion does not seem to have been investigated much. We will show that these spaces are related to the more well-known locally strongly sober spaces: in Section 3 we show that the latter are exactly the weakly Hausdorff, coherent sober spaces. This allows us to elucidate their Stone duals in Section 5, as those frames in which the filter-theoretic join of any two Scott-open filters is Scott-open. The notion of weakly Hausdorff spaces also helps us generalize a few results on variants of the so-called Plotkin powerdomain, namely on spaces of lenses and of quasi-lenses, with shorter proofs, as we will see in Section 6. An intermediate Section 4 will be an opportunity of making remarks and giving examples and counterexamples. We start off immediately with Section 2, giving the required preliminaries.

2. Preliminaries

In general, we refer to [4, 6] for material on domain theory and topology.

A subset $A$ of a preordered set $(P, \leq)$ is upwards-closed if and only if every element larger than or equal to an element of $A$ is in $A$. The upward closure $\uparrow A$ of $A$ is the collection of points that are larger than...
or equal to some point of $A$. We write $\uparrow x$ instead of $\uparrow \{x\}$. We define downwards-closed sets, downward closures $\downarrow A$, and $\downarrow x$ similarly.

Every topological space $X$ has a specialization preordering $\leq$, defined by $x \leq y$ if and only if every open neighborhood of $x$ contains $y$. A space is $T_0$ if and only if the specialization preordering $\leq$ is antisymmetric.

Conversely, every preordered set $(P, \leq)$ can be given the Alexandroff topology, whose open subsets are the upwards-closed subsets of $P$.

The following is due to Keimel and Lawson [14, Lemma 6.6].

**Definition 2.1.** A topological space $X$ is weakly Hausdorff if and only if for all $x, y \in X$, every open neighborhood $W$ of $\uparrow x \cap \uparrow y$ contains an intersection $U \cap V$ of an open neighborhood $U$ of $x$ and of an open neighborhood $V$ of $y$.

This should not be confused with the many other notions with similar names [9], in particular with McCord’s weak Hausdorff spaces [15].

The following enumerates a few kinds of weakly Hausdorff spaces. Items (1) and (3) are obvious, and item (2) is Lemma 8.1 in [14].

**Proposition 2.2.** The following are weakly Hausdorff spaces:

1. all Hausdorff spaces; in fact, the Hausdorff spaces are exactly the $T_1$ weakly Hausdorff spaces;
2. all stably locally compact spaces (see below) [14];
3. all preordered sets in their Alexandroff topology.

We write $cl(A)$ for the closure of $A$ in $X$. For any $x \in X$, $cl(\{x\})$ is the downward closure $\downarrow x$ of $x$ in the specialization preordering $\leq$.

A closed subset $C$ of a space $X$ is irreducible if and only if $C \neq \emptyset$ and whenever $C$ intersects two open sets, it also intersects their intersection. A sober space is a space in which every irreducible closed set is the closure $\downarrow x$ of a unique point $x$.

A subset $Q$ of $X$ is compact if and only if one can extract a finite cover from any open cover of $Q$. We assume no separation property. $X$ is locally compact if and only if every point has a base of compact neighborhoods.

A subset $A$ of a topological space is saturated if and only if it is equal to the intersection of its open neighborhoods, or equivalently if and only if it is upwards-closed in the specialization preordering $\leq$. A space $X$ is coherent if and only if the intersection $Q \cap Q'$ of any two compact saturated subsets $Q$ and $Q'$ is compact (and saturated). A space is stably locally compact if and only if it is locally compact, coherent, and sober.

A subset $D$ of a partially ordered set $(P, \leq)$ is directed if and only if every finite subfamily of $D$ has an upper bound in $D$. (In particular,
D is non-empty.) $P$ is **directed complete**, or a **dcpo**, if and only if every directed family $D$ in $P$ has a supremum $\sup^1 D$. A **Scott-open** subset of $P$ is an upwards-closed subset $U$ of $P$ such that for every directed family $D$, $\sup^1 D \in U$ implies that some element of $D$ is in $U$. The Scott-open sets form the **Scott topology**.

Every sober space is a **monotone convergence** space [4, Exercise O-5.15], namely a $T_0$ space such that every directed subset $D$ has a supremum, and in which every open set is Scott-open.

A subset $F$ of a partially ordered set $(P, \leq)$ is **filtered** if and only if it is directed in the opposite ordering. A **filter** on $P$ is an upwards-closed filtered subset of $P$, and it is a **proper filter** if different from $P$.

Given a topological space $X$, let us write $\mathcal{O}_X$ for its set of open subsets, ordered by inclusion. A **limit** of a filter $\mathcal{F}$ on $\mathcal{O}_X$ is any point $x$ such that every open neighborhood of $x$ is in $\mathcal{F}$. We write $\lim \mathcal{F}$ for the set of limits of $\mathcal{F}$. This is always a closed set. Additionally, given any closed subset $C$ of $X$ in $\mathcal{F}$, every limit of $\mathcal{F}$ is in $C$.

A **filter base** is any filtered collection of non-empty subsets of $X$. By a standard use of Zorn’s Lemma, every filtered base can be completed to a maximal proper filter, namely to an **ultrafilter**.

3. **Locally Strongly Sober**$=$**Weakly Hausdorff**$+$**Coherent**$+$**Sober**

A space $X$ is **locally strongly sober** if and only if the set $\lim U$ of limits of every ultrafilter $U$ on $X$ is either empty or the closure $\downarrow x$ of a unique point $x$ [4, Definition VI-6.12].

**Lemma 3.1.** Every locally strongly sober space is weakly Hausdorff.

**Proof.** Let $X$ be a locally strongly sober space, $x, y \in X$, and $W$ be an open neighborhood of $\uparrow x \cap \uparrow y$. Let also $C \overset{\text{def}}{=} X \setminus W$. For the sake of contradiction, we assume that there is no pair of an open neighborhood $U$ of $x$ and of an open neighborhood $V$ of $y$ such that $U \cap V \subseteq W$. For all those pairs, $U \cap V \cap C$ is non-empty, so those sets form a filter base. Let us extend it to an ultrafilter $\mathcal{U}$. Every open neighborhood of $x$ is in $\mathcal{U}$, so $x \in \lim \mathcal{U}$. Similarly, $y \in \lim \mathcal{U}$. Since $\lim \mathcal{U}$ is non-empty, by assumption there is a point $z \in X$ such that $\lim \mathcal{U} = \downarrow z$. Now, on the one hand, $x$ and $y$ are in $\lim \mathcal{U} = \downarrow z$, so $z \in \uparrow x \cap \uparrow y \subseteq W$. On the other hand, $C$ is closed and in $\mathcal{U}$, so every limit of $\mathcal{U}$ is in $C$. In particular $z$ is in $C$, and that is impossible since $z$ is in $W$. □

We write $\bigcap_{i \in J}^\downarrow C_i$ for the intersection of any filtered family $(C_i)_{i \in I}$. A subset $Q$ of $X$ is compact if and only if for every filtered family
\((C_i)_{i \in I}\) of closed subsets of \(X\), if \(Q \cap \bigcap_{i \in I} C_i = \emptyset\) then \(Q \cap C_i = \emptyset\) for some \(i \in I\).

**Lemma 3.2.** For every ultrafilter \(\mathcal{U}\) on a space \(X\), \(\lim \mathcal{U} = \bigcap_{A \in \mathcal{U}} \text{cl}(A)\).

**Proof.** For every \(A \in \mathcal{U}\), \(\text{cl}(A)\) is a closed set in \(\mathcal{U}\), so every limit of \(\mathcal{U}\) is in \(\text{cl}(A)\). Conversely, we show that every point \(x \in X \setminus \lim \mathcal{U}\) is in the complement of \(\bigcap_{A \in \mathcal{U}} \text{cl}(A)\). Since \(x\) is not a limit of \(\mathcal{U}\), there is an open neighborhood \(U\) of \(x\) that is not in \(\mathcal{U}\). Its complement \(A \overset{\text{def}}{=} X \setminus U\) is in \(\mathcal{U}\), since \(\mathcal{U}\) is an ultrafilter, and we conclude since \(x\) is not in \(\text{cl}(A)\) = \(A\).

Let us introduce the following weakening of the notion of coherence.

**Definition 3.3.** A space \(X\) is weakly coherent if and only if \(\uparrow x \cap \uparrow y\) is compact for all points \(x, y \in X\).

Every coherent space, every \(T_1\) space is weakly coherent.

**Lemma 3.4.** Every weakly Hausdorff, weakly coherent, monotone convergence space \(X\) is locally strongly sober.

**Proof.** Let \(\mathcal{U}\) be an ultrafilter on \(X\), with \(\lim \mathcal{U} \neq \emptyset\). For any two points \(x, y \in \lim \mathcal{U}\), \(\uparrow x \cap \uparrow y\) is compact since \(X\) is weakly coherent. Let us assume that \(\uparrow x \cap \uparrow y\) does not intersect \(\lim \mathcal{U}\). Since \(\lim \mathcal{U} = \bigcap_{A \in \mathcal{U}} \text{cl}(A)\) by Lemma 3.2, \(\uparrow x \cap \uparrow y \cap \text{cl}(A) = \emptyset\) for some \(A \in \mathcal{U}\). Let \(W \overset{\text{def}}{=} X \setminus \text{cl}(A)\). By weak Hausdorffness, there are an open neighborhood \(U\) of \(x\) and an open neighborhood \(V\) of \(y\) such that \(U \cap V \subseteq W\). Since \(x \in \lim \mathcal{U}\), \(U\) is in \(\mathcal{U}\), and similarly \(V\) is in \(\mathcal{U}\); therefore \(U \cap V\), and then also \(W\), is in \(\mathcal{U}\). But \(A \in \mathcal{U}\), so \(W \cap A\) is in \(\mathcal{U}\); that is impossible, since \(W \cap A = \emptyset\).

This shows that \(\lim \mathcal{U}\) is directed. Since \(X\) is a monotone convergence space, \(\lim \mathcal{U}\) has a supremum \(z\), and \(z \in \lim \mathcal{U}\) because \(\lim \mathcal{U}\) is closed. Every closed set is downwards-closed, so \(\lim \mathcal{U} = \downarrow z\). The uniqueness of \(z\) follows from the fact that every monotone convergence space is \(T_0\). \(\square\)

Every locally strongly sober space is sober [1, Lemma VI-6.13] and coherent [1, Lemma VI-6.14]. This, together with Lemma 3.1 and Lemma 3.4, leads to the following.

**Theorem 3.5.** For every topological space \(X\), it is equivalent that \(X\) be:

1. locally strongly sober;
2. a weakly Hausdorff, weakly coherent, monotone convergence space;
3. or a weakly Hausdorff, coherent, sober space.
4. Remarks and Examples

**Remark 4.1.** Theorem 6.8 of [14] states that given any weakly Hausdorff coherent sober space \( X \), every locally finite inner regular valuation on the lattice of open subsets of \( X \) extends to a \( \sigma \)-additive measure on a \( \sigma \)-algebra containing all the open sets and all the compact saturated sets. By Theorem 3.5 the assumption on \( X \) can be read as “given any locally strongly sober space \( X \)."

**Remark 4.2.** A space is stably locally compact if and only if it is locally compact and locally strongly sober [4, Corollary VI-6.16]. With Theorem 3.5 this allows us to give an equivalent definition as a conjunction of apparently weaker properties: a space is stably locally compact if and only if it is a core-compact, weakly Hausdorff, weakly coherent, monotone convergence space. (A space is core-compact if and only if its lattice of open sets is continuous. A dcpo is continuous if and only if every element is the supremum of a directed family of elements way-below it; the way-below relation \( \ll \) is defined by \( u \ll v \) if and only if every directed family \( D \) such that \( v \leq \bigvee D \) contains an element larger than or equal to \( u \) [4, Section I-1]. Every locally compact space is core-compact [4, Example I-1.7(5)], but not conversely [10, Section 7]. But every core-compact sober space is locally compact [4, Theorem V-5.6].) I said “apparently weaker”: in the presence of weak Hausdorffness, the following shows that monotone convergence equals sobriety, and that weak coherence equals coherence.

**Proposition 4.3.** Every weakly Hausdorff monotone convergence space \( X \) is sober.

**Proof.** Let \( C \) be an irreducible closed subset of \( X \). For any two points \( x, y \) of \( C \), if \( \uparrow x \cap \uparrow y \) were included in the complement \( W \) of \( C \), then \( W \) would contain an intersection \( U \cap V \) of an open neighborhood \( U \) of \( x \) and of an open neighborhood \( V \) of \( y \). Since \( C \) is irreducible, \( U \cap V \), and therefore also \( W \), would intersect \( C \), but that is impossible. Hence \( \uparrow x \cap \uparrow y \) intersects \( C \). This shows that \( C \) is directed. Since \( X \) is a monotone convergence space, \( z \overset{\text{def}}{=} \sup \uparrow C \) exists and is in \( C \), so \( C = \downarrow z \). Since \( X \) is \( T_0 \), \( z \) is unique, and therefore \( X \) is sober. \( \square \)

**Proposition 4.4.** Every weakly Hausdorff, weakly coherent space \( X \) is coherent.

**Proof.** Let \( Q_1, Q_2 \) be two compact saturated subsets of \( X \), and let \((W_i)_{i \in I}\) be an open cover of \( Q_1 \cap Q_2 \). For each pair \((x,y) \in Q_1 \times Q_2\), \( \uparrow x \cap \uparrow y \) is included in \( Q_1 \cap Q_2 \), hence in \( \bigcup_{i \in I} W_i \). Since \( X \) is weakly coherent, there is a finite subset \( J_{xy} \) of \( I \) such that \( \uparrow x \cap \uparrow y \subseteq \bigcup_{i \in J_{xy}} W_i \).
Since $X$ is weakly Hausdorff, $x$ has an open neighborhood $U_x$ and $y$ has an open neighborhood $V_y$ such that $U_x \cap V_y \subseteq \bigcup_{i \in J_{xy}} W_i$. The sets $U_x$, $x \in Q_1$, form an open cover of $Q_1$, so there is a finite subset $E_1$ of $Q_1$ such that $Q_1 \subseteq \bigcup_{x \in E_1} U_x$. Similarly, there is a finite subset $E_2$ of $Q_2$ such that $Q_2 \subseteq \bigcup_{y \in E_2} V_y$. Then the sets $W_i$, where $i \in \bigcup_{x \in E_1, y \in E_2} J_{xy}$, form a finite subcover of $Q_1 \cap Q_2$. □

Remark 4.5. Jia, Jung and Li show that every well-filtered weakly coherent dcpo is coherent in its Scott topology [12]. A space is well-filtered if and only if given any filtered family $(Q_i)_{i \in I}$ of compact saturated subsets, and any open subset $U$, $\bigcap_{i \in I} Q_i \subseteq U$ if and only if some $Q_i$ is included in $U$. All sober spaces are well-filtered. Proposition 4.4 states a similar result, generalizing from dcpos to all topological spaces, and replacing well-filteredness by weak Hausdorffness. Nonetheless, it is no generalization of the Jia-Jung-Li theorem, see Example 4.10.

The main purpose of the following examples is to show that the terms “weakly Hausdorff”, “(weakly) coherent”, and “sober” (or “monotone convergence space”) in Theorem 3.5 (3) are irredundant. They can more generally be used to get a better grasp of the notions.

Example 4.6 (Weakly Hausdorff, sober $\not\Rightarrow$ weakly coherent). Let us consider the dcpo $\mathbb{N} \cup \{a, b\}$, where all the natural numbers are pairwise incomparable, and $a$ and $b$ are themselves incomparable and less than all natural numbers. The Scott topology coincides with the Alexandroff topology, hence this dcpo is weakly Hausdorff. It is also sober, but it is not weakly coherent, as $\uparrow a \cap \uparrow b = \mathbb{N}$ is not compact.

Example 4.7 (Weakly Hausdorff, coherent $\not\Rightarrow$ monotone convergence, sober). Let us consider any dcpo $P$ in its Alexandroff (not Scott) topology. If $P$ has an infinite ascending chain $x_0 < x_1 < \cdots < x_n < \cdots$, then $P$ is not a monotone convergence space, since the open set $\uparrow \sup_{n \in \mathbb{N}} x_n$ is not Scott-open. However, $P$ is weakly Hausdorff, and is coherent if and only if $\uparrow x \cap \uparrow y$ can be written as $\uparrow E$ for some finite set $E$, for all points $x$ and $y$. (Indeed, the compact subsets of $P$ are exactly the upward closures of finite sets.) Hence, for example, $\mathbb{N} \cup \{\omega\}$ (with $\omega > n$ for every $n \in \mathbb{N}$) is weakly Hausdorff, coherent, but not a monotone convergence space in the Alexandroff topology of its natural ordering.

Example 4.8 (Coherent $\not\Rightarrow$ sober, weakly Hausdorff). The space $\mathbb{N}$ with the cofinite topology is Noetherian (every subspace is compact), hence coherent, but neither sober nor weakly Hausdorff. It is not sober because $\mathbb{N}$ itself is irreducible closed. It is not weakly Hausdorff because
for any two distinct points \( x, y, \uparrow x \cap \uparrow y = \{x\} \cap \{y\} \) is empty, but all open neighborhoods of \( x \) and of \( y \) intersect.

**Example 4.9** (Sober, coherent \( \not\Rightarrow \) weakly Hausdorff). A *KC-space* is a space in which every compact set is closed. All KC-spaces are coherent, and also \( T_1 \) since one-element sets are compact, hence closed. Let \( \alpha(\mathbb{Q}) \) denote \( \mathbb{Q} \cup \{\infty\} \), the one-point compactification of the rational numbers. Its open subsets are the open subsets of \( \mathbb{Q} \), plus the sets \( \alpha(\mathbb{Q}) \setminus K \), where \( K \) is compact in \( \mathbb{Q} \). This is a \( T_1 \), non-Hausdorff space \( [16, \text{Counterexample 35}] \), hence is not weakly Hausdorff. We argue that it is sober as follows. Let \( C \) be irreducible closed. \( C \) cannot contain two distinct rational numbers \( x \) and \( y \), otherwise there would be an open neighborhood \( U \) of \( x \) in \( \mathbb{Q} \) and an open neighborhood \( V \) of \( y \) in \( \mathbb{Q} \) whose intersection is empty. If \( C \) contains a rational number \( x \) and \( \infty \), therefore, \( C \) must be equal to \( \{x, \infty\} \); then \( C \) intersects \( [x - 1, x + 1] \cap \mathbb{Q} \) and \( \alpha(\mathbb{Q}) \setminus \{x\} \), but not their intersection. Hence \( C \) contains only one point. A result of Alas and Wilson states that the one-point compactification of a countable KC-space \( X \) is a KC-space if and only if \( X \) is sequential \( [1, \text{Corollary 7}] \). It follows that \( \alpha(\mathbb{Q}) \) is a KC-space, hence is coherent.

**Example 4.10** (Coherent, well-filtered \( \not\Rightarrow \) sober, weakly Hausdorff). Isbell constructed a complete lattice \( L \) that is not sober in its Scott topology \( [11] \). \( L \) is well-filtered, because every complete lattice is \( [17] \). It is also weakly coherent, because every complete lattice is, trivially. Hence it is coherent, by the Jia-Jung-Li theorem cited in Remark 4.5. However, it is not weakly Hausdorff, otherwise it would be sober by Theorem 3.5.

5. **The Stone duals of locally strongly sober spaces**

For every topological space \( X \), the set \( \mathcal{O}X \) of open subsets of \( X \) ordered by inclusion is a frame, namely a complete lattice such that \( u \wedge \bigvee_{i \in I} v_i = \bigvee_{i \in I} (u \wedge v_i) \) for all elements \( u \) and \( v_i, i \in I \). This extends to a functor from the category \( \textbf{Top} \) of topological spaces and continuous maps to the opposite \( \textbf{Loc} \) of the category of frames \( \textbf{Frm} \) (whose morphisms are the maps that preserve all suprema and finite infima): for a continuous map \( f: X \to Y \), \( \mathcal{O}f \) maps \( V \in \mathcal{O}Y \) to \( f^{-1}(V) \).

Conversely, given any frame \( L \), a filter \( \Phi \) on \( L \) is completely prime if and only if \( \bigvee_{i \in I} u_i \in \Phi \) implies that \( u_i \in \Phi \) for some \( i \in I \). There is a sober space \( \text{Spec} L \), whose points are the completely prime filters, and whose open subsets are \( \mathcal{O}_u \overset{\text{def}}{=} \{x \in \text{Spec} L \mid u \in x\}, u \in L \). This
extends to a functor $\text{Spec} : \text{Loc} \to \text{Top}$; for every frame morphism $\varphi : L \to M$, $\text{Spec} \varphi : \text{Spec} M \to \text{Spec} L$ maps every $y \in \text{Spec} M$ to $\varphi^{-1}(y)$. Additionally, $\mathcal{O}$ is left-adjoint to $\text{Spec}$, and this adjunction and cuts down to an equivalence between the full subcategories of sober spaces and so-called spatial frames on the other side. (See Section V-4 of [4], for example.)

We will identify the class of spatial frames that are related to the locally strongly sober spaces through this equivalence. In order to do so, for any two filters $F, G$ on a frame $L$, let us define the join $F \lor G$ as

\[
\{ u \land v \mid u \in F, v \in G \}.
\]

This is a filter on $L$, and in fact the supremum of $F$ and $G$ in the poset of all filters on $L$, ordered by inclusion.

**Definition 5.1.** A frame $L$ is **locally temperate** if and only if the join $F \lor G$ of any two Scott-open filters on $L$ is Scott-open. It is **temperate** if and only if it is locally temperate and the smallest filter $\{\top\}$ (where $\top$ is the largest element of $L$) is Scott-open.

Alternatively, a temperate frame is a frame in which the join of any finite family of Scott-open filters is Scott-open. We note that $\{\top\}$ is Scott-open in $\mathcal{O}X$ if and only if $X$ is compact.

We will rely on the Hofmann-Mislove theorem [4, Theorem II-1.20]: on a sober space $X$, the map $F \mapsto \bigcap F$ defines an order-isomorphism between the space of Scott-open filters on $\mathcal{O}X$ and the space of compact saturated subsets of $X$ ordered by reverse inclusion. The inverse map maps every compact saturated subset $Q$ to its set of open neighborhoods $\blacksquare Q$. We also rely on the following.

**Fact 5.2 (Lemma 6.6 of [14]).** A space $X$ is weakly Hausdorff if and only if for every pair of compact saturated subsets $Q_1$ and $Q_2$, for every open neighborhood $W$ of $Q_1 \cap Q_2$, there is an open neighborhood $U$ of $Q_1$ and and open neighborhood $V$ of $Q_2$ such that $U \cap V \subseteq W$.

**Lemma 5.3.** The following are equivalent for every sober space $X$:

1. $\mathcal{O}X$ is locally temperate;
2. $X$ is weakly Hausdorff and coherent;
3. $X$ is locally strongly sober.

**Proof.** (2) $\Rightarrow$ (1). Let $F$ and $G$ be two Scott-open filters on $\mathcal{O}X$. By the Hofmann-Mislove theorem, $F = \blacksquare Q_1$ and $G = \blacksquare Q_2$, for some compact saturated sets $Q_1$ and $Q_2$. We claim that $F \lor G = \blacksquare (Q_1 \cap Q_2)$. Every element of $F \lor G$ is of the form $U \cap V$ where $U \in \blacksquare Q_1$ and $V \in \blacksquare Q_2$, hence belongs to $\blacksquare (Q_1 \cap Q_2)$. Conversely, let $W \in \blacksquare (Q_1 \cap Q_2)$. By Fact 5.2, there is an open neighborhood $U$ of $Q_1$ and and open neighborhood $V$ of $Q_2$ such that $U \cap V \subseteq W$. In other words, $U \cap V$,
and therefore also the larger set \( W \), in in \( F \lor G \). Finally, since \( X \) is coherent, \( Q_1 \cap Q_2 \) is compact, so \( F \lor G = \ll (Q_1 \cap Q_2) \) is Scott-open.

(1) \( \Rightarrow \) (2). Since \( OX \) is locally temperate, for all compact saturated subsets \( Q_1 \) and \( Q_2 \) of \( X \), \( \ll Q_1 \lor \ll Q_2 \) is Scott-open, hence is equal to \( \ll Q \) for some compact saturated set \( Q \). Then \( Q \) is the intersection of its open neighborhoods \( W \in \ll Q \), hence is the intersection of the sets \( U \cap V \) with \( U \in \ll Q_1 \) and \( V \in \ll Q_2 \), which is equal to \( Q_1 \cap Q_2 \). It follows that \( Q_1 \cap Q_2 \) is compact. Therefore \( X \) is coherent. Using the same construction, we obtain that every open neighborhood \( W \) of \( Q_1 \cap Q_2 \) can be written as \( U \cap V \) with \( U \in \ll Q_1 \) and \( V \in \ll Q_2 \). By Fact 5.2, \( X \) is weakly Hausdorff.

(2) \( \iff \) (3). This is by Theorem 3.5 \( \square \)

**Theorem 5.4.** The \( O \dashv \text{Spec} \) adjunction cuts down to an adjoint equivalence between the full subcategories of (locally) strongly sober spaces and of (locally) temperate spatial frames.

Not all locally temperate frames are spatial. In order to see this, we first prove the following lemma.

**Lemma 5.5.** Every complete Boolean algebra is weakly temperate.

**Proof.** Let \( L \) be a complete Boolean algebra, and \( F \) and \( G \) be two Scott-open filters in \( L \). Let \( (u_i)_{i \in I} \) be a directed family whose supremum is in \( F \lor G \). We can write this supremum as \( v \land w \) where \( v \in F \) and \( w \in G \). The family \( (u_i \lor (v \land \neg w))_{i \in I} \) is directed, and its supremum is \( (v \land w) \lor (v \land \neg w) = v \). Since \( v \in F \) and \( F \) is Scott-open, \( u_i \lor (v \land \neg w) \) is in \( F \) for some \( i \in I \). Similarly, \( u_j \lor (w \land \neg v) \) is in \( G \) for some \( j \in I \). By directedness, we may assume that \( i = j \). Then \( (u_i \lor (v \land \neg w)) \land (u_i \lor (w \land \neg v)) \) is in \( F \lor G \). But \( (u_i \lor (v \land \neg w)) \land (u_i \lor (w \land \neg v)) = u_i \lor ((v \land \neg w) \land (w \land \neg v)) = u_i \), so \( u_i \in F \lor G \). \( \square \)

**Remark 5.6.** Not all locally temperate frames are spatial. For a counterexample, consider the complete Boolean algebra of regular open subsets of \( \mathbb{R} \). This is weakly temperate by Lemma 5.5, but does not have any point, hence is not spatial [6, Exercise 8.1.25].

**Remark 5.7.** Let us write \( \uparrow u \) for \( \{ v \in L \mid u \ll v \} \). A frame is stable if and only if \( u \ll v, w \) implies \( u \ll v \land w \) [3, Definition 2.12]. A sober space \( X \) is stably locally compact if and only if \( OX \) is continuous and stable. A compact, continuous, stable frame is called a stably continuous frame by Banaschewski and Brümmer [2], or a stably compact frame by Johnstone [13]. We prefer the former name, since the latter seems to imply that the frame would be stably compact in its Scott topology, which would be wrong. Using Theorem 5.4, one can show
the following localic analogue of the characterization of stably (locally) compact spaces given in Remark 4.2. We give an elementary proof, which, as for several other results in locale theory, does not require the axiom of choice.

**Proposition 5.8.** A continuous frame $L$ is stable if and only if it is locally temperate. The stably continuous frames are exactly the continuous, temperate frames.

**Proof.** Let $L$ be a continuous frame. We only need to show that it is stable if and only if it is locally temperate. Since $L$ is continuous, every Scott-open subset $U$ is the union of sets $\uparrow\uparrow u$, $u \in U$, and those sets are themselves Scott-open [4, Proposition II-1.10]. Given any two Scott-open filters $F$ and $G$ of $L$, $F \lor G = \bigcup_{u \in F} \uparrow\uparrow u \lor \bigcup_{v \in G} \uparrow\uparrow v = \{u' \land v' \mid \exists u \in F, v \in G, u \ll u', v \ll v'\}$. For every directed family $(w_i)_{i \in I}$ whose supremum is in $F \lor G$, namely such that $\bigvee_{i \in I} w_i = u' \land v'$ for some $u \ll u'$ in $F$ and some $v \ll v'$ in $G$, we have $u \land v \ll u', v'$. If $L$ is stable, then $u \land v \ll u' \land v'$, so $u \land v \leq w_i$ for some $i \in I$, and therefore $F \lor G$ is Scott-open.

Conversely, if $L$ is locally temperate, we show that $L$ is stable as follows. We use the following trick: since $L$ is continuous, for any two elements $u, v \in L$ such that $u \ll v$, $v$ belongs to a Scott-open filter $F$ included in $\uparrow u$, hence in $\uparrow\uparrow u$ [4, Proposition I-3.3]. Given another element $w$ such that $u \ll w$, similarly, we find a Scott-open filter $G$ containing $w$ and included in $\uparrow u$. Since $L$ is locally temperate, $F \lor G$ is Scott-open. It is included in $\uparrow u$, and contains $v \land w$. It follows easily that $u \ll v \land w$. □

6. An application: on lenses and quasi-lenses

One of the standard powerdomains considered in domain theory is the so-called Plotkin powerdomain [4, Definition IV-8.11]. In specific situations, this coincides with various other, more concrete constructions. Let us cite three, parameterized by a base space $X$.

- The space $\mathcal{P}_V X$ of all lenses, with the Vietoris topology. A lens is a non-empty set of the form $Q \cap C$ where $Q$ is compact saturated and $C$ is closed in $X$. The Vietoris topology is the familiar one on hyperspaces, and has subbasic open subsets of the form $\Box U$ (the set of lenses included in $U$) and $\Diamond U$ (the set of lenses that intersect $U$), for each open subset $U$ of $X$.
- The space $\mathcal{P}^A X$ of $A$-valuations [8]. We omit the definition.
- The space $\mathcal{P}_V^{\text{quasi}} X$ of all quasi-lenses, with a topology which we will again call the Vietoris topology. A quasi-lens is a pair...
(\(Q, C\)) where \(Q\) is compact saturated, \(C\) is closed, and the following three conditions are met:

1. \(Q\) intersects \(C\);
2. \(Q \subseteq \uparrow (Q \cap C)\);
3. for every open neighborhood \(U\) of \(Q\), \(C \subseteq \text{cl}(U \cap C)\).

The subbasic open subsets are \(\square^{\text{quasi}} U\) (the set of quasi-lenses \((Q, C)\) such that \(Q \subseteq U\)) and \(\diamond^{\text{quasi}} U\) (the set of quasi-lenses \((Q, C)\) such that \(C\) intersects \(U\)), for all open subsets \(U\) of \(X\).

Quasi-lenses originate from Heckmann’s work [7, Theorem 9.6]. If \(X\) is sober, then \(\mathcal{P}_V^{\text{quasi}} X\) is homeomorphic to \(\mathcal{P}_\Lambda X\) [5, Fact 5.2]. It is known that \(\mathcal{P}_V^{\text{quasi}} X\) (equivalently, \(\mathcal{P}_\Lambda X\)) is homeomorphic to \(\mathcal{P}_V X\) when \(X\) is Hausdorff [8, Theorem 5.1], or when \(X\) is stably compact [5, Proposition 5.3]. We show that both of these results stem from a more general statement on weakly Hausdorff spaces. The proof is also simpler than the one given in the stably compact case [5].

The key to the new proof is the following lemma.

**Lemma 6.1.** For every compact saturated subset \(Q\) of a weakly Hausdorff space \(X\) and for every closed subset \(C\) of \(X\), if \(C \subseteq \text{cl}(U \cap C)\) for every open neighborhood \(U\) of \(Q\), then \(C \subseteq \text{cl}(Q \cap C)\).

**Proof.** Let us imagine that \(C\) is not included in \(\text{cl}(Q \cap C)\), and let \(y\) be a point that in \(C\) and not in \(\text{cl}(Q \cap C)\). The intersection \(Q \cap \uparrow y \cap C\) is empty, otherwise it would contain a point \(z \in Q \cap C \subseteq \text{cl}(Q \cap C)\) such that \(y \leq z\), and that would entail that \(y\) is in \(\text{cl}(Q \cap C)\). Therefore \(Q \cap \uparrow y \subseteq W\), where \(W\) is the complement of \(C\). Since \(X\) is weakly Hausdorff, and using Fact 5.2, there is an open neighborhood \(U\) of \(Q\) and an open neighborhood \(V\) of \(\uparrow y\) such that \(U \cap V \subseteq W\). Since \(Q \subseteq U\), by assumption \(C \subseteq \text{cl}(U \cap C)\). Now \(y\) is both in \(C\) and in \(V\), so \(V\) intersects \(C\), and therefore also the larger set \(\text{cl}(U \cap C)\). Since \(V\) is open, it must also intersect \(U \cap C\), so \(U \cap V \cap C\) is non-empty. Then \(W \cap C\) is non-empty, which is impossible since \(W\) is the complement of \(C\). \(\square\)

**Proposition 6.2.** Let \(X\) be a topological space.

1. For every lens \(L\), \(\iota(L) \overset{\text{def}}{=} (\uparrow L, \text{cl}(L))\) is a quasi-lens.
2. For every quasi-lens \((Q, C)\), \(\varrho(Q, C) \overset{\text{def}}{=} Q \cap C\) is a lens.

Additionally, \(\varrho \circ \iota\) is the identity map, and \(\iota \circ \varrho\) is the identity map if the conclusion of Lemma 6.1 holds, in particular if \(X\) is weakly Hausdorff.

**Proof.** (1) Let \(L \overset{\text{def}}{=} Q \cap C\) be a lens. In particular, \(L\) is compact, so \(Q' \overset{\text{def}}{=} \uparrow L\) is compact saturated. \(C' \overset{\text{def}}{=} \text{cl}(L)\) is clearly closed, and \(Q' \cap C'\)
contains $L$, hence is non-empty. For every $x \in Q'$, there is a $y \in L$ such that $y \leq x$. Then $y$ is both in $Q'$ and in $C'$, so $x$ is in $\uparrow(Q' \cap C')$. For every open neighborhood $U$ of $Q'$, $U$ contains $L$, so $U \cap C'$ also contains $L$, hence $\text{cl}(U \cap C')$ contains $\text{cl}(L) = C'$. Therefore $(Q', C')$ is a quasi-lens.

(2) For every quasi-lens $(Q, C)$, $L \overset{\text{def}}{=} Q \cap C$ is clearly a lens.

For every lens $L$, $\varrho(\iota(L)) = \uparrow L \cap \text{cl}(L)$ is equal to $L$; see for example [4, Definition IV-8.15].

For the final part of the proposition, let $(Q, C)$ be any quasi-lens, $L \overset{\text{def}}{=} \varrho(Q, C) = Q \cap C$, and $(Q', C')$ be $\iota(L)$, namely $Q' \overset{\text{def}}{=} \uparrow L$ and $C' \overset{\text{def}}{=} \text{cl}(L)$. Since $L \subseteq C$ and $C$ is closed, $C' \subseteq C$. The reverse inclusion is the conclusion of Lemma 6.1, so $C' = C$. Since $L \subseteq Q$ and $Q$ is saturated, $Q' \subseteq Q$; but we also have $Q \subseteq \uparrow(Q \cap C) = \uparrow L = Q'$, by definition of a quasi-lens, so $Q' = Q$. Therefore $\iota \circ \varrho$ is the identity map. \hfill $\Box$

**Theorem 6.3.** For every topological space $X$, the map $\iota$ is a topological embedding of $\mathcal{P}_{\vartheta}X$ into $\mathcal{P}_{\text{quasi}}^\vartheta X$. For every open subset $U$ of $X$, $\iota^{-1}(\square_{\text{quasi}} U) = \square U$ and $\iota^{-1}(\Diamond_{\text{quasi}} U) = \Diamond U$.

If the conclusion of Lemma 6.1 holds, notably if $X$ is weakly Hausdorff, then $\iota$ is a homeomorphism of $\mathcal{P}_{\vartheta}X$ onto $\mathcal{P}_{\text{quasi}}^\vartheta X$, with inverse $\varrho$.

**Proof.** For every lens $L$, $\iota(L) \in \square_{\text{quasi}} U$ if and only if $\uparrow L \subseteq U$, if and only if $L \in \square U$. Therefore $\iota^{-1}(\square_{\text{quasi}} U) = \square U$. Also, $\iota(L) \in \Diamond_{\text{quasi}} U$ if and only if $\text{cl}(L)$ intersects $U$, if and only if $L$ intersects $U$, if and only if $L \in \Diamond U$. Hence $\iota^{-1}(\Diamond_{\text{quasi}} U) = \Diamond U$.

The map $\iota$ is injective by Remark IV-8.16 of [4], and what we have just shown then implies that $\iota$ is a topological embedding. The final claim follows from Proposition 6.2. \hfill $\Box$

The specialization ordering of $\mathcal{P}_{\vartheta}X$ is the so-called topological Egli-Milner ordering: $L \overset{\text{TEM}}{\subseteq} L'$ if and only if $\uparrow L \supseteq \uparrow L'$ and $\text{cl}(L) \subseteq \text{cl}(L')$ [5, Discussion before Fact 4.1]. Instead, the Egli-Milner ordering is defined by $L \overset{\text{EM}}{\subseteq} L'$ if and only if $\uparrow L \supseteq \uparrow L'$ and $\downarrow L \subseteq \downarrow L'$. When $X$ is stably compact, those two orderings coincide [5, Lemma 4.2], and in fact $\downarrow L = \text{cl}(L)$ for every lens $L$. We show that this holds, more generally, on all weakly Hausdorff spaces. The proof, once again, is elementary.

**Theorem 6.4.** For every lens $L$ in a weakly Hausdorff space $X$, $\downarrow L$ is closed, and so $\text{cl}(L) = \downarrow L$. The orderings $\subseteq_{\text{TEM}}$ and $\subseteq_{\text{EM}}$ coincide.
Proof. Let us write \( L \) as \( Q \cap C \) where \( Q \) is compact saturated and \( C \) is closed. We show that \( \downarrow L \) is closed by showing that every point \( x \) outside \( \downarrow L \) lies in some open set disjoint from \( \downarrow L \). Since \( x \notin \downarrow L \), \( \uparrow x \) is disjoint from \( L \), so \( \uparrow x \cap Q \cap C \) is empty. Let \( W \) be the complement of \( C \). Then \( \uparrow x \cap Q \) is included in \( W \). Since \( W \) is weakly Hausdorff, there is an open neighborhood \( U \) of \( x \) and an open neighborhood \( V \) of \( Q \) such that \( U \cap V \subseteq W \). Then \( U \cap L = U \cap Q \cap C \subseteq U \cap V \cap C \subseteq W \cap C = \emptyset \), so \( U \) is disjoint from \( L \), as desired.

Finally, \( \downarrow L \subseteq \text{cl}(L) \) since every closed set is downwards-closed. We have just shown that \( \downarrow L \) is closed. It contains \( L \), so it contains \( \text{cl}(L) \).

\( \square \)

Weak Hausdorffness is not necessary in those results, as the following example shows; but something is needed, see Example 6.6.

**Example 6.5.** In any \( T_1 \) space, the lenses are exactly the non-empty compact subsets. Although \( \alpha(Q) \) is not weakly Hausdorff (see Exercise 4.9), \( \downarrow L \) is closed for every lens \( L \) of \( \alpha(Q) \), because \( \downarrow L = L \), and every compact set is closed. Given any quasi-lens \( (Q,C) \) in \( \alpha(Q) \), \( Q \subseteq \uparrow(Q \cap C) \) means that \( Q \subseteq C \). We claim that \( Q = C \). This trivially entails that the conclusion of Lemma 6.1 holds, although \( \alpha(Q) \) is not weakly Hausdorff.

In order to show the claim, let us assume \( x \in C \setminus Q \). If \( \infty \notin Q \), then \( Q \) is compact in \( Q \), and we can cover it by a finite union \( U \) of open balls of radius \( \epsilon > 0 \), where \( \epsilon \) is strictly less than the distance of \( x \) to the closed set \( Q \). Let \( K \) be the union of the corresponding closed balls. The condition \( C \subseteq \text{cl}(U \cap C) \) then entails \( C \subseteq K \cap C \); this is impossible since \( x \in C \) but \( x \notin K \). If \( \infty \) is in \( Q \), then \( x \neq \infty \), and we use a similar argument. We pick \( n > |x| \). Then \( Q \cap [-n,n] \) is closed and bounded, hence compact in \( Q \). We cover it by a finite union \( U \) of open balls of radius \( \epsilon > 0 \), where \( \epsilon \) is strictly less than the distance of \( x \) to \( Q \cap [-n,n] \) (if that set is non-empty, otherwise \( \epsilon \) is unconstrained).

We let \( K \) be the union of the corresponding closed balls. We note that \( Q \subseteq V \) where \( V \overset{\text{def}}{=} U \cup (\alpha(Q) \setminus [-n,n]) \). The condition \( C \subseteq \text{cl}(V \cap C) \) entails \( C \subseteq K \cup [-\infty,-n] \cup [n,\infty] \cup \{\infty\} \); but \( x \) is in \( C \) and not in the right-hand side.

**Example 6.6.** Let us consider \( \mathbb{N} \) with the cofinite topology (see Example 4.8). As in every \( T_1 \) space, the lenses \( L \) are the non-empty compact subsets, and here this means the non-empty subsets. Then \( \text{cl}(L) = \downarrow L \) if and only if \( L \) is finite or \( L = \mathbb{N} \), so \( \text{cl}(L) \neq \downarrow L \) for every infinite proper subset \( L \) of \( \mathbb{N} \). We have \( L \subseteq_{\text{TEM}} L' \) if and only if \( L = L' \) is finite, or \( L' \) is infinite and is included in \( L \); while \( L \subseteq_{\text{EM}} L' \) if and only if
\(L = L'\). Therefore the relations \(\sqsubseteq_{\text{TEM}}\) and \(\sqsubseteq_{\text{EM}}\) differ. The quasi-lenses are exactly the pairs \((Q, C)\) where \(Q = C\) is finite and non-empty, or where \(Q \neq \emptyset\) and \(C = \mathbb{N}\). However, the image of \(\iota\) consists of all pairs \((Q, C)\) such that \(Q = C\) is finite and non-empty, or \(Q\) is infinite and \(C = \mathbb{N}\), so \(\iota\) is not surjective. It also follows that \(\iota \circ \varrho\) is not the identity map.

7. Open Questions

In relation to Section 5: (1) There are non-spatial locally temperate frames, but is there a non-spatial temperate frame? (2) Is \(\text{Spec } L\) locally strong sober for every weakly temperate frame? (3) Considering that every Hausdorff space is locally strongly sober, is every \(T_2\) frame locally temperate (whatever the notion of being \(T_2\) for frames you may wish to consider)?

Acknowledgments

I thank Xiaodong Jia for suggesting Example 4.6 and for finding a mistake in the original proof of Proposition 4.4.

References

[1] Ofélia Teresa Alas and Richard Gordon Wilson. Spaces in which compact subsets are closed and the lattice of \(T_1\)-topologies on a set. *Comment. Math. Univ. Carol.*, 43(4):641–652, 2002.
[2] Bernard Banaschewski and Guillaume C. L. Brümmer. Stably continuous frames. *Math. Proc. Cambridge Philos. Soc.*, 104(1):7–19, 1988.
[3] Guram Bezhanishvili and John Harding. Stable compactifications of frames. *Cah. Topol. Geom. Differ. Categ.*, LV-1:37–65, 2014.
[4] Gerhard Gierz, Karl Heinrich Hofmann, Klaus Keimel, Jimmie D. Lawson, Michael Mislove, and Dana Stewart Scott. *Continuous Lattices and Domains*, volume 93 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 2003.
[5] Jean Goubault-Larrecq. De Groot duality and models of choice: Angels, demons and nature. *Math. Struct. Comput. Sci.*, 20(2):169–237, 2010.
[6] Jean Goubault-Larrecq. *Non-Hausdorff Topology and Domain Theory—Selected Topics in Point-Set Topology*, volume 22 of *New Mathematical Monographs*. Cambridge University Press, 2013.
[7] Reinhold Heckmann. Power domains and second-order predicates. *Theor. Comput. Sci.*, 111:59–88, 1993.
[8] Reinhold Heckmann. Abstract valuations: A novel representation of Plotkin power domain and Vietoris hyperspace. *Electron. Notes Theor. Comput. Sci.*, 6, 1997. Proc. MFPS’97.
[9] Rudolf-Eberhart Hoffmann. On weak Hausdorff spaces. *Arch. Math.*, 32:487–504, 1979.
[10] Karl Heinrich Hofmann and Jimmie D. Lawson. The spectral theory of distributive continuous lattices. *Trans. Am. Math. Soc.*, 246:285–310, 1978.
[11] John Isbell. Completion of a construction of Johnstone. *Proc. Am. Math. Soc.*, 85:333–334, 1982.
[12] Xiaodong Jia, Achim Jung, and Qingguo Li. A note on the coherence of dcpo’s. *Topol. Appl.*, 209:235–238, 2016.
[13] Klaus Keimel and Jimmie Lawson. Measure extension theorems for $T_0$-spaces. *Topol. Appl.*, 149(1–3):57–83, 2005.
[14] Peter T. Johnstone. *Stone Spaces*. Cambridge University Press, 1982.
[15] Michael Campbell McCord. Classifying spaces and infinite symmetric products. *Trans. Am. Math. Soc.*, 146, 1969.
[16] Lynn Arthur Steen and J. Arthur Seebach, Jr. *Counterexamples in Topology*. Springer-Verlag, second edition, 1978.
[17] Xiaoyong Xi and Jimmie Lawson. On well-filtered spaces and ordered sets. *Topol. Appl.*, 228(1):139–144, 2017.

*Université Paris-Saclay, CNRS, ENS Paris-Saclay, Laboratoire Méthodes Formelles, 91190, Gif-sur-Yvette, France.*

*Email address: goubault@lsv.fr*