BERGMAN-HÖLDER FUNCTIONS, AREA INTEGRAL MEANS AND EXTREMAL PROBLEMS

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Abstract. We study certain weighted area integral means of analytic functions in the unit disc. We relate the growth of these means to the property of being mean Hölder continuous with respect to the Bergman space norm. In contrast with earlier work, we use the second iterated difference quotient instead of the first. We then give applications to Bergman space extremal problems.

This paper deals with mean Hölder type smoothness conditions for functions in Bergman spaces on the unit disc \( \mathbb{D} \), and the relation of these conditions to extremal problems in Bergman spaces.

The first main topic is area integral means and smoothness conditions. It is well known (due to Hardy and Littlewood) that \( f \) is analytic in the unit disc and \( |f'(re^{i\theta})| \leq C(1-r)^{-1+\beta} \) for \( 0 < \beta \leq 1 \) if and only if \( f \) is continuous in the closed unit disc and \( |f(e^{i\theta}+it) - f(e^{i\theta})| \leq C'|t|^\beta \). This result can be thought of as dealing with the \( H^\infty \) norm of the boundary function. There is a similar result for \( 1 \leq p < \infty \), also due to Hardy and Littlewood, that states that for an analytic function \( f \), the integral means \( M_p(r, f') \leq C(1-r)^{-1+\beta} \) if and only if \( f \in H^p \) and \( \|f(\cdot) - f(e^{it}\cdot)\|_{H^p} \leq C'|t|^\beta \). (See chapter 5 of [2]).

Zygmund [18] obtained similar results for the second iterated difference. For example, he proved that for an analytic function \( f \), one has that \( f \) is continuous in \( |z| \leq 1 \) and \( |f(e^{it}z) + f(e^{-it}z) - 2f(z)| \leq C|t| \) if and only if \( |f''(z)| \leq C(1-r)^{-1} \) (see [2]). Similar results hold for powers of \( |t| \) greater than 0 and at most 2, and for integral means.

Analogous properties hold for harmonic functions in higher dimensions (see e.g. Chapter V of [14]).

In [8], the authors give results relating growth of area integral means of analytic functions to mean Hölder regularity of these functions. In this article, we prove similar results, but instead use the second iterated difference, like in the result of Zygmund. Also, we work on the standard weighted Bergman spaces instead of just the unweighted case. For example, we prove that if \( 0 < \beta \leq 2 \) and \(-1 < \alpha < \infty\), and if \( A_\alpha \) denotes the standard weighted Bergman space, then \( \|f(e^{it}\cdot) + f(e^{-it}\cdot) - \)}
$2f(\cdot)\|_{A^p_\alpha} \leq C|t|^\beta$ if and only if the weighted area integral means of $f$ are $O((1 - r)^{\beta-2})$. We let $\Lambda^*_{\beta, A^p_\alpha}$ denote the space of all such $f$. The advantage of working with the second iterated difference is that it can be used to characterize higher regularity than the difference $f(z) - f(e^{it}z)$.

Related to this, we prove various results about the growth of weighted area integral means of analytic functions and how this relates to the growth of area integral means of integrals and derivatives of analytic functions. We also relate growth of area integral means to growth of classical integral means.

The second main topic is the relation of smoothness conditions to extremal functions. In [10], the authors give a result about mean smoothness of the solution of an extremal problem for Bergman spaces. Their work is based on [13], where a similar result is given for Hardy spaces. Actually, many techniques in these papers that are relevant to this paper are very general and only use the fact that the spaces in question are closed subspaces of $L^p$ that are invariant under translations of the form $z \mapsto ze^{it}$.

We derive a result similar to the one in [10] for another type of extremal problem in weighted Bergman spaces. In particular, given a $k \in A^p_\alpha$, for $1 < q < \infty$, the extremal problem in question is to find $F \in A^p_\alpha$ such that $\|F\| = 1$ and $\text{Re} \int D f dA_\alpha$ is as large as possible, where $1/p + 1/q = 1$. Because of the uniform convexity of $L^p(dA_\alpha)$, such an $F$ always exists.

Several results are known that allow one to deduce regularity properties of $F$ from regularity properties of $k$, and vice-versa. See for example [5, 7, 12]. Our result is of this type and says that if $0 < \beta < 2$ and $\|k(e^{it} \cdot) + k(e^{-it} \cdot) - 2k(\cdot)\|_{A^p_\alpha} \leq C|t|^\beta$ then $\|F(e^{it} \cdot) + F(e^{-it} \cdot) - 2F(\cdot)\|_{A^p_\alpha} \leq C'|t|^{\beta/p}$ for $p > 2$ and $\|F(e^{it} \cdot) + F(e^{-it} \cdot) - 2F(\cdot)\|_{A^p_\alpha} \leq C''|t|^{\beta/2}$ for $1 < p < 2$. (There is no need to consider the case $p = 2$ because then $F$ is a multiple of $k$.) It is helpful to use the second iterated difference here because if we used only the first difference we would be restricted to $0 < \beta < 1$.

Notice the exponent on the $|t|$ is $\beta/2$ for $p < 2$. This comes from an improvement to the techniques of [10], which yield $|t|^{\beta/q}$. We obtain this improvement by using an inequality from [1] instead of Clarkson’s inequalities. The inequality we use gives a worse constant but a better power on $|t|$. The inequality is related to the fact that $L^p$ is 2-uniformly convex for $1 < p < 2$, but Clarkson’s inequalities only show that it is $q$-uniformly convex (see [1]). This improvement is crucial for applying our results later in the paper.
We then combine our results on extremal functions with our results on growth of area integral means. We find two notable results. One is Corollary 4.3, which applies to $A^p_\alpha$ extremal problems and says that if $k \in \Lambda^*_2 A^\infty$ then $F$ has Hölder continuous boundary values if $2 \leq p < \infty$ and $-1 < \alpha < 0$ or if $1 < p < 2$ and $-1 < \alpha < p - 2$.

The other result is Theorem 4.2, which applies to extremal problems in unweighted Bergman spaces and says that if $k \in \Lambda A^p$ and $1 < p < \infty$ then $|F|^{p-1}F' \in L^1$. Also $F' \in L^s$ for some $s > 1$. (This applies to the unweighted case). This is important because it allows for a more elementary proof of the results of [5], without using results from [11] which rely on deep results from the theory of partial differential equations. (In fact, that was the original motivation for this paper).

We now discuss a subtlety that arises when dealing with area integral means of weighted Bergman spaces, since there seem to be two different types of definition of integral means possible. Perhaps the most obvious definition for the area integral mean of $f$ at radius $r$ is as

$$
\left( (\alpha + 1) \int_{|z|<r} |f(z)|^p (1 - |z|^2)\alpha \frac{dA(z)}{\pi} \right)^{1/p},
$$
or equivalently (except for an unimportant factor of $r^{2/p}$) as

$$
\left( (\alpha + 1) \int_{|z|<1} |f(rz)|^p (1 - |rz|^2)\alpha \frac{dA(z)}{\pi} \right)^{1/p},
$$

where $dA$ is normalized area measure. On the other hand, we could define the integral mean as

$$
\|f_r\|_{A^p_\alpha} = \left( (\alpha + 1) \int_{|z|<1} |f(rz)|^p (1 - |z|^2)\alpha \frac{dA(z)}{\pi} \right)^{1/p},
$$

where $f_r(z) = f(rz)$. It seems likely that there are analytic functions for which these two types of quantities have different orders of growth. However, we prove that if one of them has growth in $O((1 - r)^\gamma)$ for $\gamma \leq 1$, then the so does the other.

Throughout this paper, we often keep track of constants in inequalities. We do not investigate whether these constants are the best possible. However, in future work, we expect to make use of the fact that explicit values for these constants are known (even if they are not the best possible values).
1. Integral Means and Area Integral Means

This paper deals with Hardy and Bergman spaces. See [2] for information on Hardy spaces and [3] and [9] for information on Bergman spaces.

Let \( dA_\alpha(z) = \frac{(\alpha + 1)(1 - |z|^2)^\alpha dA(z)}{\pi} \) be the standard weighted area measure, where \(-1 < \alpha < \infty\). Let the Bergman space \( A_p^\alpha \) be the space of all functions analytic in the unit disc such that
\[
\|f\|_{A_p^\alpha} = \left( \int_D |f(z)|^p dA_\alpha(z) \right)^{1/p} < \infty.
\]
We deal here mainly with the case \( 1 < p < \infty \). For \( 1 < p < \infty \), the dual of \( A_p^\alpha \) is isomorphic to \( A_q^\alpha \), where \( q \) is the conjugate exponent of \( p \) (see [9]).

If \( f \) is analytic in the unit disc, we define the integral mean of order \( p \) as
\[
M_p(r, f) = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}
\]
for \( 0 < p < \infty \). If \( p = \infty \), we define \( M_\infty(r, f) = \sup_{0 \leq \theta < 2\pi} |f(re^{i\theta})| \). It is known that the integral means increase with \( r \) (see [2]). A function is said to be in \( H_p \) if its integral means of order \( p \) are bounded, and we define \( \|f\|_{H_p} = \sup_{0 \leq r < 1} M_p(r, f) \).

We now formally define the area integral means.

**Definition 1.1.** Let \( f \) be in \( L^p_{\text{loc}}(\mathbb{D}) \). Define
\[
A_p,\alpha(r, f) = \left( (\alpha + 1) \int_D |f(rz)|^p (1 - |rz|^2)^\alpha \frac{dA(z)}{\pi} \right)^{1/p}
\]
and define
\[
\tilde{A}_p,\alpha(r, f) = \left( (\alpha + 1) \int_{rD} |f(z)|^p (1 - |z|^2)^\alpha \frac{dA(z)}{\pi} \right)^{1/p}.
\]
We will also define
\[
\hat{A}_p,\alpha(r, f) = \left( (\alpha + 1) \int_D |f(rz)|^p (1 - |z|^2)^\alpha \frac{dA(z)}{\pi} \right)^{1/p}.
\]
and
\[
\hat{\tilde{A}}_p,\alpha(r, f) = \left( (\alpha + 1) \int_{rD} |f(z)|^p (1 - |z/r|^2)^\alpha \frac{dA(z)}{\pi} \right)^{1/p}.
\]
A change of variables shows that $\tilde{A}_{p,\alpha}(r, f) = r^{2/p} A_{p,\alpha}(r, f)$, so that
$A_{p,\alpha}(r, f) \simeq \tilde{A}_{p,\alpha}(r, f)$ as $r \to 1^-$. Similarly, $\tilde{A}_{p,\alpha}(r, f) = r^{2/p} \hat{A}_{p,\alpha}(r, f)$. We also have that $\hat{A}_{p,\alpha}(r, f) = \|f_r\|_{A_p^\alpha}$, where $f_r(z) = f(rz)$.

In [15–17], the authors defined area integral means and studied their convexity properties. The area integral means they define are equal to $\tilde{A}_{p,\alpha}(r, f)/\tilde{A}_{p,\alpha}(r, 1)$. These integral means have the same order of growth as $\tilde{A}_{p,\alpha}$ for $-1 < \alpha < \infty$, so we do not consider them separately. It could be interesting to see if convexity results also hold for analogues of $\hat{A}_{p,\alpha}$.

The following lemma is sometimes useful in analyzing the case $\alpha > 0$.

**Lemma 1.1.** If $0 \leq \rho \leq r^2$ then $(1 - \rho^2) \leq 2(1 - \rho^2/r^2)$.

This lemma is true because for fixed $0 < r < 1$, the ratio $(1 - \rho^2)/(1 - \rho^2/r^2)$ is increasing.

The following theorem gives information about the growth of integral means of functions when information about the growth of their area integral means is known. We exclude the case $\beta > 0$ because no function other than the zero function satisfies $A_{p,\alpha}(r, f) \leq C(1 - r)^\beta$ for $\beta > 0$.

**Theorem 1.2.** Suppose $\beta \leq 0$. Suppose first that $\alpha \geq 0$. If $A_{p,\alpha}(r, f) \leq B(1 - r)^\beta$ then

$$M_p(r, f) \leq B(\alpha + 1)^{-1}\frac{2^{-\beta+(1+\alpha)/p}(1-r)^{\beta-(1+\alpha)/p}}{(1 + \sqrt{r})^{(1+\alpha)/p}}$$

and if $A_{p,\alpha}(r, f) \leq B|\log(1-r)|$ then

$$M_p(r, f) \leq B(\alpha + 1)^{-1}\frac{2^{(1+\alpha)/p}(1-r)^{-1+(1+\alpha)/p}|\log(1-r)| + \log(2)}}{1 + \sqrt{r})^{(1+\alpha)/p}}.$$  

If we suppose instead that $\alpha \leq 0$, then if $A_{p,\alpha}(r, f) \leq B(1 - r)^\beta$ then

$$M_p(r, f) \leq (\alpha + 1)^{-1}\frac{2^{-\beta+(1+\alpha)/p}(1-r)^{\beta-(1+\alpha)/p}}{(1 + \sqrt{r})^{1/p}(1 + \sqrt{r + \sqrt{r^3})^\alpha/p}}.$$  

If $A_{p,\alpha}(r, f) \leq B|\log(1-r)|$ then

$$M_p(r, f) \leq B(\alpha + 1)^{-1}\frac{2^{(1+\alpha)/p}(1-r)^{-1+(1+\alpha)/p}|\log(1-r)| + \log(2)}}{1 + \sqrt{r})^{1/p}(1 + \sqrt{r + \sqrt{r^3})^\alpha/p}}.$$  

If $\alpha \leq 0$, then the same conclusion also holds if in the hypothesis $A_{p,\alpha}$ is replaced with $\hat{A}_{p,\alpha}$. Further, if the hypothesis of the theorem holds for all $r > R$, then the conclusion holds for all $r > R^2$. 
Proof. First note that if the hypothesis holds for $\alpha \leq 0$ and with $\tilde{A}_{p,\alpha}$, it holds for $A_{p,\alpha}$ since then $A_{p,\alpha}(r, f) \leq \tilde{A}_{p,\alpha}(r, f)$. First assume that $A_{p,\alpha}(r, f) \leq C(1 - r)^{-1+\beta}$. Now
\[
(\alpha + 1) \int_{r^2}^{r^2} M_p^p(\sqrt{u}, f)(1-u)^\alpha \, du = 
\]
\[
(\alpha + 1) \int_{r^2}^{r^2} M_p^p(t, f)(1-t^2)^\alpha 2t \, dt \leq \tilde{A}_{p,\alpha}(r, f) = r^2 A_{p,\alpha}(r, f) 
\]
Suppose first that $\alpha \leq 0$ and that $A_{p,\alpha}(r, f) \leq B(1 - r)^\beta$. To simplify notation we may assume $B = 1$. Since the integral means are increasing we have
\[
(r^2 - r^4)(1 - r^4)\alpha M_p^p(r^2, f) \leq (\alpha + 1)^{-1} r^2 A_{p,\alpha}(r, f). 
\]
And thus, we have
\[
M_p^p(r^2, f) \leq (\alpha + 1)^{-1}(1 - r)^{-1+\beta p} A_{p,\alpha}(r, f) 
\]
and so
\[
M_p^p(r, f) \leq (\alpha + 1)^{-1}(1 - \sqrt{r})^{-1+\beta p} \frac{(1 - \sqrt{r})^{\beta p}}{(1 + \sqrt{r})(1 + \sqrt{r} + r + \sqrt{r^3})^\alpha} 
\]
\[
\leq (\alpha + 1)^{-1} \frac{2^{-\beta p+1+\alpha}(1-r)^{\beta p-1+\alpha}}{(1 + \sqrt{r})(1 + \sqrt{r} + r + \sqrt{r^3})^\alpha} 
\]
The proofs of the other assertions are similar. \qed

The following theorem provides a partial converse to Theorem 1.2.

**Theorem 1.3.** If $M_p(r, f) \leq B(1 - r^2)^{\beta-(1+\alpha)/p}$ and $\beta < 0$ then $\tilde{A}_{p,\alpha}(r, f) \leq B(\alpha + 1)(-\beta p)^{-1/p}(1 - r^2)^\beta$. If $\beta = 0$ then $A_{p,\alpha} \leq C(\alpha + 1)\log(1-r^2)$. If instead $\beta > 0$ then $f \in A_{p,\alpha}^p$.

**Proof.** If $\beta < 1$ note that
\[
(\alpha + 1)^{-1} \tilde{A}_{p,\alpha}^p(r, f) = \int_0^r M_p^p(t, f) 2t(1-t^2)^\alpha \, dt 
\]
\[
\leq \int_0^r B^p(1 - t^2)^{\beta p-1-\alpha}(1 - t^2)^\alpha 2t \, dt 
\]
\[
= \int_0^r B^p(1 - t^2)^{\beta p-1} 2t \, dt 
\]
\[
\leq \frac{B^p}{-\beta p}(1 - r^2)^{\beta p}. 
\]
In the cases $\beta = 0$ or $\beta > 1$, a similar computation gives the result. \qed
The following two lemmas are used for certain estimates later in the paper. They are from [4], and can be proven by using hypergeometric functions.

**Lemma 1.4.** Suppose that \( s < 1 \) and \( m + s > 1 \) and that \( k > -1 \). Let \( 0 \leq x < 1 \). Then

\[
\int_0^1 \frac{(1-y)^{-s}}{(1-xy)^m} y^k \, dy \leq C_1(s, m, k)(1-x)^{1-s-m}
\]

where \( C_1(s, m, k) < \infty \) is defined by

\[
C_1(s, m, k) = \frac{\Gamma(k + 1)\Gamma(1-s)}{\Gamma(2 + k - s)} \max_{0 \leq x \leq 1/2} {}_2F_1(2+k-s-m, 1-s; 2+k-s; x).
\]

If \( 2 + k > s + m \) and \( 2 + k > s \), then

\[
C_1(s, m, k) = \frac{\Gamma(s + m - 1)\Gamma(1-s)}{\Gamma(m)}.
\]

There is another way to prove this lemma with worse constants. We break the integral into three pieces, one from 0 to 1/2 and one from 1/2 to \( x \) and one from \( x \) to 1. The first integral is bounded by a constant. Now, use the fact that if \( 0 \leq y \leq x \) then \((1/2)(1-y^2) \leq 1-xy \leq 1-y^2 \) and \( 1-y \leq 1-y^2 \leq 2(1-y)\) to bound the second integral by

\[
C \int_{1/2}^x (1-y)^{-s-m} \, dy.
\]

Now use the fact that \( 1-x \leq 1-xy \) to bound the third integral by

\[
C(1-x)^{-m} \int_x^1 (1-y)^{-s} \, dy.
\]

**Lemma 1.5.** Let \( p > 1 \) and \( 0 < r < 1 \). Then

\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1-re^{i\theta}|^p} \, d\theta = (1-r^2)^{1-p} {}_2F_1\left(1 - \frac{p}{2}, 1 - \frac{p}{2}; 1; r^2\right) \leq \frac{\Gamma(p-1)}{\Gamma(p/2)^2} (1-r^2)^{1-p}.
\]

If \( p = 2 \) we have

\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1-re^{i\theta}|^2} \, d\theta = (1-r^2)^{-1}.
\]

We are now in a position to prove that \( A_{\rho, \alpha} = O((1-r)^{\beta}) \) if and only if \( \hat{A}_{\rho, \alpha} = O((1-r)^{\beta}) \).
Lemma 1.6. Let $\beta < 0$ and let $\alpha < 0$. If $M_p(r, f) \leq C(1 - r^2)^{\beta - (1+\alpha)/p}$ then $\widetilde{A}_{p,\alpha}(r, f) \leq C(\alpha + 1)C_1(-\alpha, 1 + \alpha - \beta p, 0)^{1/p}(1 - r^2)^\beta$, where $C_1$ is defined in Lemma 1.4.

Proof. Let $\gamma = -((\beta - (1 + \alpha)/p)p = 1 + \alpha - \beta p$. Then we have

$$(\alpha + 1)^{-1} \widetilde{A}_p(r, f) \leq C \int_0^1 \frac{(1 - t^2)^\alpha}{(1 - r^2 t^2)^\gamma} 2t \, dt = C \int_0^1 \frac{(1 - u)^\alpha}{(1 - r^2 u)^\gamma} \, du$$

But by Lemma 1.4 the integral is at most $CC_1(-\alpha, \gamma, 0)(1 - r^2)^{1+\alpha-\gamma} = CC_1(-\alpha, \gamma, 0)(1 - r^2)^{-1+\beta}$.

Theorem 1.7. Suppose that $\beta \leq 0$. There exists a constant $C > 0$ such that $A_p(r, f) \leq C(1 - r^2)^\beta$ if and only if there exists a constant $C > 0$ such that $\widetilde{A}_p(r, f) \leq C(1 - r^2)^\beta$.

Proof. The proof is clear in the case $\alpha = 0$ since the two integral means are equal.

Consider now $\alpha < 0$. For the case $\beta = 0$, note that since the weight and the integral means are both increasing, both $A_p(r, f)$ and $\widetilde{A}_p(r, f)$ are increasing and the monotone convergence theorem shows that both approach $\|f\|_{A_p}$, which proves the theorem in this case.

Now consider $\beta < 0$. One direction is clear because $\widetilde{A}_p(r, f) \geq A_p(r, f)$. Note that if $A_p(r, f) \leq C(1 - r^2)^\beta$ then $M_p(r, f) \leq C(1 - r^2)^{\beta - (1+\alpha)/p}$, and $\widetilde{A}_p(r, f) \leq C(1 - r^2)^\beta$ by the lemma.

Now, for $\alpha > 0$, we have that $\widetilde{A}_{p,\alpha}(r, f) \leq A_{p,\alpha}(r, f)$ because the weights are decreasing. Also note that for fixed $0 < r < 1$, Lemma 1.1 shows that if $0 \leq \rho \leq r^2$ then $(1 - \rho^2) \leq 2(1 - \rho^2/r^2)$. Thus

$$\widetilde{A}_{p,\alpha}(r^2, f) = \int_0^{r^2} M_p^p(\rho, f)(1 - \rho^2)^{\alpha}\rho(\alpha + 1) \, d\rho$$

$$\leq 2^\alpha \int_0^{r^2} M_p^p(\rho, f)(1 - \rho^2/r^2)^{\alpha}\rho(\alpha + 1) \, d\rho$$

$$\leq 2^\alpha \int_0^r M_p^p(\rho, f)(1 - \rho^2/r^2)^{\alpha}\rho(\alpha + 1) \, d\rho$$

$$= 2^\alpha \widetilde{A}_{p,\alpha}(r, f).$$

So if $\widetilde{A}_{p,\alpha}(r, f) \leq C(1 - r^2)^\beta$ then

$$A_{p,\alpha}(r, f) \leq 2^\alpha \widetilde{A}_{p,\alpha}(\sqrt{r}, f) \leq 2^{\alpha-\beta}(1 - r)^\beta \leq 2^{\alpha+1-2\beta}(1 - r^2)^{-1+\beta}.$$
2. BERGMAN INTEGRAL MEANS AND DERIVATIVES

We now discuss the relation between the area integral means of a function and the area integral means of its derivative and antiderivative. Many of the results in this section can be proved by changing to classical integral means and using the corresponding results for classical integral means and the theorems of the previous section, but we provide direct proofs.

We let \( f_s \) denote the dilation of \( f \), defined by \( f_s(z) = f(sz) \). Thus, \( (f')_s(z) = f'(sz) \) and \( (f_s)'(z) = sf'(sz) = s(f')_s(z) \).

**Theorem 2.1.** Let \( f \) be analytic in the disc. Then

\[
\hat{A}_{p,\alpha}(r, f - f(0)) \leq r \int_0^1 \hat{A}_{p,\alpha}(r, (f')_s) \, ds.
\]

If also \( \beta < -1 \) and \( \hat{A}_{p,\alpha}(r, f') = O((1 - r)^\beta) \) then \( \hat{A}_{p,\alpha}(r, f) = O((1 - r)^{1+\beta}) \).

If also \( \beta > -1 \) and \( \hat{A}_{p,\alpha}(r, f') = O((1 - r)^\beta) \) then \( f \in A_p^\alpha \).

All the above statements are true if \( \hat{A} \) is replaced with any of the other integral means.

**Proof.** Let \( z = te^{i\theta} \) and suppose that \( t < r \). Assume without loss of generality that \( f(0) = 0 \). Note that

\[
|f(z)| \leq \int_0^t |f'(\rho e^{i\theta})|d\rho = \int_0^1 |f'(sz)|t \, ds \leq r \int_0^1 |f'(sz)| \, ds.
\]

Now use Minkowski’s inequality to conclude that

\[
\hat{A}_{p,\alpha}(r, f) \leq r \int_0^1 \hat{A}_{p,\alpha}(r, (f')_s) \, ds
\]

The same proof words for the other integral means. The results about order of growth follow immediately. \( \Box \)

**Theorem 2.2.** Let \( \beta \geq 0 \). If \( \hat{A}_{p,\alpha}(r, f') = O((1-r)^{-\beta}) \) then \( \hat{A}_{p,\alpha}(r, f) = O((1-r)^{-1-\beta}) \).

**Proof.** We may assume without loss of generality that \( f(0) = 0 \). Let \( z = re^{i\theta} \), where \( 0 < r < 1 \). Note that

\[
f'(\rho z) = \frac{1}{2\pi i} \int_{|\zeta|=\rho} \frac{f(\zeta)}{(\zeta - \rho z)^2} \, d\zeta.
\]
Now parametrize the integral with \( \zeta = \rho e^{i(t+\theta)} \) to see that
\[
f'(\rho e^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} f(\rho e^{i(t+\theta)}) \frac{\rho e^{i(t-\theta)}}{(\rho e^{it} - \rho r)^2} \, dt
\]
\[
= \frac{1}{2\pi} \int_0^{2\pi} f(\rho e^{i(t+\theta)}) \frac{\rho^{-1}e^{i(t-\theta)}}{(e^{it} - r)^2} \, dt
\]

Let \( g(z) = f(z)/z \). Apply the integral \( \int_0^r \int_0^{2\pi} 2\rho (1 - \rho^2/r^2)^\alpha \, d\theta \, d\rho \) to both sides of the above equation and use Minkowski’s inequality to conclude that
\[
\hat{A}_{p,\alpha}(r, (f')_r) \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{\hat{A}_{p,\alpha}(r, g)}{|e^{it} - r|^2} \, dt \leq \hat{A}_{p,\alpha}(r, g)(1 - r^2)^{-1}
\]
\[= r^{2p} \hat{A}_{p,\alpha}(r, g)(1 - r^2)^{-1}.
\]
The last inequality follows from Lemma 1.5.

If \( h \in A_\alpha^p \) with norm 1, then because the integral means increase, we have that \( \|zh\|_{A_\alpha^p} \) is minimized when \( h = 1 \). Let \( M_{\alpha,p} = \|z\|_{A_\alpha^p} \). Then the above argument implies that \( \hat{A}_{p,\alpha}(r, g) = \|f(rz)/(rz)\|_{A_\alpha^p} \leq M_{\alpha,p}^{-1} ||f(rz)|| \). Thus for large enough \( r \) we have that \( \hat{A}_{p,\alpha}(r, (f')_r) \leq C\hat{A}_{p,\alpha}(r, f)(1 - r^2)^{-1} \) for some constant \( C \).

A change of variables shows that \( \hat{A}_{p,\alpha}(r, (f')_r) = r^{-2}\hat{A}_{p,\alpha}(r^2, f') \).

Thus
\[
\hat{A}_{p,\alpha}(r^2, f') \leq C\hat{A}_{p,\alpha}(r, f)(1 - r^2)^{-1}.
\]

So if \( \hat{A}_{p,\alpha}(r, f) \leq C(1 - r^2)^{-\beta} \), then \( \hat{A}_{p,\alpha}(r^2, f') \leq C(1 - r^2)^{-1-\beta} \). \( \square \)

3. Bergman Mean Hölder Functions

We now come to the relation between mean smoothness of functions and the growth of their area integral means. The results in this section are similar to some of those in [8], except we use the second iterated difference instead of the first. The proof techniques are a combination of those in [8] and the techniques of Zygmund in [18] (see also Chapter 5 of [2]).

**Definition 3.1.** Let \( f \in A_\alpha^p \). Suppose
\[
\|f(e^{it} \cdot) + f(e^{-it} \cdot) - 2f(\cdot)\|_{p,\alpha} \leq C|t|^\beta
\]
for some constant \( C \). We then say that \( f \in \Lambda_{\beta,A_\alpha^p}^* \). Furthermore, we define \( \|f\|_{\Lambda_{\beta,A_\alpha^p}^*} \) to be the infimum of the constants \( C \) such that the above inequality holds.

We define \( \Lambda_{\beta,H^p}^* \) similarly, but instead use the \( H^p \) norm.
We now prove Theorems 3.1 and 3.4, which identify the classes \( \Lambda^*_{\beta,A_p^\alpha} \) with the classes of functions whose second derivatives have area integral means with a certain order of growth. Both of the theorems are known to be true if we replace \( \Lambda^*_{\beta,A_p^\alpha} \) with \( \Lambda^*_{\beta,H_p^\alpha} \) and area integral means with classical integral means. (For further information and corresponding results in higher dimensions, see Chapter V of [14]).

The proof of this theorem and the one following are similar to the proof of Theorem 5.3 in [2], and also bear some resemblance to techniques from [8].

**Theorem 3.1.** Let \( f \in A_p^\alpha \). Suppose that \( \|f\|_{\Lambda^*_{\beta,A_p^\alpha}} < B \) for some \( 0 < \beta \leq 2 \). Then \( A_p^\alpha(r,f'') = \leq CB(1-r)^{\beta-2} \) where \( C \) depends only on \( \alpha \), but not on \( \beta \).

If in addition we have \( 0 < \beta < 1 \) then \( A_{p,\alpha}(r,f') = O((1-r)^{\beta-1}) \) and if also \( -1 < \alpha \leq 0 \) then \( \tilde{A}_{p,\alpha}(r,f') \leq 382.5B(1-\beta)^{-1}(1-r)^{-1+\beta} \) for \( R < r < 1 \), where \( R \) is some universal constant.

If \( 1 < \beta < 2 \) we have \( f' \in A_p^\alpha \).

If \( \beta = 1 \) we have \( A_{p,\alpha}(r,f') = O(|\log(1-r)|) \) and if also \( -1 < \alpha \leq 0 \) we have \( \tilde{A}_{p,\alpha}(r,f') \leq 382.5B|\log(1-r)| \) for \( R < r < 1 \), where \( R \) is some universal constant.

**Proof.** By the Poisson integral formula,

\[
f(\rho z) = \frac{1}{2\pi} \int_0^{2\pi} P(\rho, \theta - t) f(re^{it}) \, dt
\]

where \( P \) is the Poisson kernel and \( 0 < \rho < 1 \), and \( z = re^{i\theta} \). Thus

\[
f_{\theta\theta}(\rho z) = \frac{1}{2\pi} \int_0^{2\pi} P_{22}(\rho, \theta - t) f(re^{it}) \, dt
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} P_{22}(\rho, -t) f(re^{i(\theta+t)}) \, dt
\]

\[
= \frac{1}{2\pi} \int_0^{\pi} P_{22}(\rho, t) \left( f(re^{i(\theta+t)}) + f(re^{i(\theta-t)}) - 2f(re^{i\theta}) \right) \, dt.
\]

In the last step, we have used the fact that \( P_{22} \) is even, and that \( f(re^{i\theta}) \) is constant with respect to \( t \) and thus has second \( \theta \) derivative of 0.

Let \( \delta = 1 - \rho \). Since \( P(r,t) = (1-r^2)/(1-2r \cos(t)+r^2) \), we have

\[
P_{tt} = \frac{8r^2(1-r^2)\sin^2(t)}{(1-2r \cos(t)+r^2)^3} - \frac{2r(1-r^2)\cos(t)}{(1-2r \cos(t)+r^2)^2}.
\]
Now for $0 \leq t \leq \pi$ we have $1 - 2r \cos(t) + r^2 = (1 - r)^2 + 4r \sin^2(t/2) \geq (1 - r)^2 + 4rt^2/\pi^2$, and thus

$$|P_{tt}| \leq \frac{16(1 - r)t^2}{[(1 - r)^2 + 4\pi^{-2}rt^2]^3} + \frac{4(1 - r)}{[(1 - r)^2 + 4\pi^{-2}rt^2]^2}. $$

Thus we have that

$$|P_{tt}| \leq 4^{-1}r^{-3} \pi^6 (1 - r)t^{-4} + 4^{-1}r^{-2} \pi^4 (1 - r)t^{-4} \leq 4^{-1}(\pi^6 + \pi^4 r)(1 - r)^{-3}t^{-4}. $$

Let $C_1(r) = (3\pi^6 + 3\pi^4 r)/(24\pi^3)$.

We also wish to find the maximum for all $t$. Note that the second term of in the estimate of $|P_{tt}|$ is maximized if $t = 0$, and the maximum is $4(1 - r)^{-3}$. The first term is more difficult to maximize. However, calculus shows that for fixed $y > 0$ the maximum of $x/(y + x)^3$ for $x \geq 0$ occurs at $x = y/2$. Taking $y = (1 - r)^2$ and $x = 4\pi^{-2}rt^2$ shows that the maximum in the first term occurs for $4\pi^{-2}rt^2 = (1 - r)^2/2$, and the maximum is

$$\frac{2}{3} \frac{16(1 - r)(2^{-3}\pi^2(1 - r)^2r^{-1})}{(1 - r)^6} \leq \frac{4\pi^2}{3} r^{-1}(1 - r)^{-3}. $$

So

$$|P_{tt}| \leq \frac{4\pi^2 r^2 + 12r^3}{3r^3(1 - r)^{-1}}. $$

Let $C_2(r) = (2\pi^2 r^2 + 6r^3)/(3\pi r^3)$.

Now Minkowski’s inequality gives

$$\|f_{\theta\theta}(r)\|_{A^p} \leq \frac{B}{2\pi} \int_0^\pi |P_{22}(\rho, t)| |t|^\beta \ dt$$

$$\leq BC_2(r) \int_0^\delta \delta^{-3}t^\beta \ dt + BC_1(\rho)r^3 \int_\delta^\pi \delta t^{-4}t^\beta \ dt$$

$$\leq BC_3(\rho)\delta^{-2+\beta}$$

where $C_1(\rho)/(3 - \beta) + C_2(\rho)/(\beta + 1) \leq C_3(\rho) = C_1(\rho) + C_2(\rho)/2 = (3\pi^6 + 3\pi^4 \rho + 8\pi^2 \rho^2 + 24\rho^3)/(24\pi\rho^3)$. Thus

$$\|f_{\theta\theta}(r)\|_{A^p} \leq BC_3(\rho)(1 - r)^{-2+\beta}. $$

This bound goes to $\infty$ as $r \to 0$, but we can then use the fact that $\|f_{\theta\theta}(r)\|_{A^p}$ is increasing to see that $\|f_{\theta\theta}(r)\|_{A^p}$ is bounded by

$$BC_3(\max(r, 1/2))(1 - r)^{-2+\beta}, $$

where the constant $C$ is independent of $r$. 
Now note that because $f$ is analytic we have $f_\theta = izf'(z)$, which is itself an analytic function, so $f_\theta = iz(izf'(z))' = -z^2f''(z) - zf'(z)$. Thus

$$f''(z) = -z^{-2}(f_\theta + zf'(z))$$

and

$$f'(z) = \left(\frac{1}{iz}\right)\int_0^z f_\theta(\zeta)/(i\zeta)\,d\zeta.$$  

Let $g \in A^p_\alpha$ with norm 1. Because the integral means increase, we have that $\|gz\|_{A^p_\alpha}$ is minimized when $g = 1$. Call this value $M_{\alpha,p}$.

Since $\|f_\theta(r\cdot)||_{A^p_\alpha} = O((1-r)^{-2+\beta})$, equation (3.2), the finiteness of $M_{\alpha,p}$, and Theorem 2.1 shows that $|f'(re^{i\theta})|$ is $O((1-r)^{-1.5+\beta})$ if $0 < \beta < 1$ and $|f'(re^{i\theta})|$ is $O(1)$ if $1 < \beta < 2$. (The number 1.5 is not essential here, as any number between 1 and 2 would work.) Note also that the implied constant can be chosen independently of $\beta$ by using either the estimate $\|f_\theta(r\cdot)||_{A^p_\alpha} = O((1-r)^{-2+\beta})$ or the estimate $\|f_\theta(r\cdot)||_{A^p_\alpha} = O((1-r)^{-2.5+\beta})$.

By equation (3.1), we have that $\|f''(r\cdot)||_{A^p_\alpha} \leq C(1-r)^{-2+\beta}$ for large enough $r$, where $C$ is some constant, which again may be chosen independently of $\beta$. The statement about the order of growth of $f'$ follows from Theorem 2.1.

We now compute the constants more explicitly in the case $-1 < \alpha \leq 0$. Note that for fixed $p$, $M_{\alpha,p}$ is minimized when $\alpha = 0$. For $\alpha = 0$, the quantity is minimized when $p = 1$, and it is then $2/3$. So $\|f_\theta(rz)/(rz)||_{A^p_\alpha} \leq M_{\alpha,p}C_3(r)B(1-r)^{-2+\beta}$ for large enough $r$. Also note that $C_3(r) \to C_3(1)$ as $r \to 1$, and that $C_3(1) < 42.47$. The fact that $\|f_\theta(rz)/(rz)||_{A^p_\alpha}$ is increasing now shows that $\|f_\theta(rz)/(rz)||_{A^p_\alpha} \leq M_{\alpha,p}C_3(1)B(1-r)^{-2+\beta} + o((1-r)^{-2+\beta})$. Also, the implied constant in the $o((1-r)^{-2+\beta})$ does not depend on $\alpha$, $\beta$, or $f$.

By Theorem 2.1 we have

$$\tilde{A}_{\alpha,\alpha}(r,izf') \leq o((1-r)^{-1+\beta}) + \int_0^1 (3/2)C_3(1)B(1-r \rho)^{-2+\beta}\,d\rho$$

$$\leq (3/2)C_3(1)B(-1+\beta)^{-1}(1-r)^{-1+\beta} + o((1-r)^{-1+\beta})$$

for $0 < \beta < 1$ and similarly $\tilde{A}_{\alpha,\alpha}(r,izf') \leq (3/2)C_3(1)B|\log(1-r)| + o(|\log(1-r)|)$ for $\beta = 1$.

Thus, since $M_{\alpha,p} \leq 3/2$, we have

$$\tilde{A}_{\alpha,\alpha}(r,f') \leq 95.6B(-1+\beta)^{-1}(1-r)^{-1+\beta} + o((1-r)^{-1+\beta})$$

for $0 < \beta < 1$ and similarly $\tilde{A}_{\alpha,\alpha}(r,f') \leq 95.6B|\log(1-r)| + o(|\log(1-r)|)$ for $\beta = 1$. \qed
Corollary 3.2. Let \( f \in A^p_\alpha \) for \( \alpha \leq 0 \). Suppose \( \|f(e^{it}) + f(e^{-it}) - 2f(\cdot)\|_p \leq B|t|^\beta \) for some \( 0 < \beta < 1 \). Then for all sufficiently large \( r \), one has

\[
M_p(r, f') \leq \frac{2^{1-\beta-\alpha/p}}{1-\beta} 95.6C(1-r)^{-1+\beta-(1+\alpha)/p} + o((1-r)^{-1+\beta-(1+\alpha)/p})
\]

\[
\leq 383(1-\beta)^{-1}B(1-r)^{-1+\beta-(1+\alpha)/p} + o((1-r)^{-1+\beta-(1+\alpha)/p}).
\]

If instead \( \beta = 1 \) we have

\[
M_p(r, f') \leq \frac{192}{1-\beta} B\log(1-r)(1-r)^{(-1+\alpha)/p} + o(\log(1-r)(1-r)^{(-1+\alpha)/p}).
\]

Proof. This follows from the above theorem and Theorem 1.2. \(\square\)

The next corollary is interesting because it is known that the condition \( |\phi(x+t) + \phi(x-t) - 2\phi(x)| \leq C|t|^\beta \) is not enough to guarantee that a function is measurable, much less Hölder continuous of some order (see [2], page 72).

Corollary 3.3. If we exclude the condition \( f \in A^p_\alpha \) in the definition of \( \Lambda^*_{\beta,A^p_\alpha} \), it makes no difference.

Proof. Suppose \( \|f(e^{it}) + f(e^{-it}) - 2f(\cdot)\|_{p,\alpha} \leq B|t|^\beta \). Since for each \( r \), the dilation \( f_r \in A^p_\alpha \) and satisfies \( \|f_r(e^{it}) + f_r(e^{-it}) - 2f_r(\cdot)\|_{p,\alpha} \leq B|t|^\beta \), we can apply the above theorem to it to see that \( \Lambda^*_{p,\alpha}(r, f_r) \leq CB(1-r)^{\beta-2} \) where \( C \) is independent of \( r \). But \( \Lambda^*_{p,\alpha}(r, f_r) = r^2\Lambda^*_{p,\alpha}(r^2, f_r) \). So then \( \Lambda^*_{p,\alpha}(r, f_r) \leq CB(1-r)^{\beta-2} \). Thus \( \Lambda^*_{p,\alpha}(r) \leq CB(1-r)^{\beta-2} \), which implies that \( f \in A^p_\alpha \). \(\square\)

The next theorem is the converse of Theorem 3.1. The proof is very similar to that of Theorem 5.3 of [2] (see also [18]).

Theorem 3.4. Suppose \( 0 < \beta \leq 2 \). If \( \Lambda^*_{p,\alpha}(f''(r) \leq B(1-r)^{-2+\beta} \) then \( f \in \Lambda^*_{\beta,A^p_\alpha} \). If we assume that \( f''(0) = 0 \) then

\[
\|f\|_{\Lambda^*_{\beta,A^p_\alpha}} \leq \left( 48\pi^2 + 12\pi + \frac{1}{\beta} \right) B.
\]

Proof. We may assume without loss of generality that \( f(0) = 0 \), since it does not affect any bounds in the statement of the theorem. (Note however that \( e^{it} + e^{-it} - 2 = 2(\cos(t) - 1) \) so the value of \( f'(0) \) does affect \( \|f\|_{\Lambda^*_{p,\alpha}} \).) To simplify notation, we will assume without loss of generality that \( B = 1 \). Let \( (\Delta_\beta f)(z) = f(e^{iz}) - f(z) \). So we are required to show that \( \|\Delta_\beta \Delta_\beta f\|_{p,\alpha} = O(h^\beta) \).

First write \( \Delta_{-\beta} \Delta_\beta f = \Delta_{-\beta} (f - f_\rho) + \Delta_{-\beta} f_\rho \) where \( f_\rho(z) = f(\rho z) \), and \( 0 < \rho < 1 \) is a positive number that will be chosen later.
Now note that integration by parts shows that
\[ f(ze^{it}) - f(\rho ze^{it}) = (1 - \rho)ze^{it}f'(\rho ze^{it}) + \int_\rho^1 ze^{it}(1 - r)f''(rze^{it})ze^{it}dr \]

Notice that if we apply the \( A^p_\alpha \) norm to the integral (with respect to \( z \)) and use Minkowski’s inequality, we can bound the integral by
\[ \int_\rho^1 (1 - r)(1 - r)^{-2+\beta} dr \leq \beta^{-1}(1 - \rho)^\beta. \]

We let \( \Delta \) apply to the variable \( t \). Note that
\[ \Delta_h ze^{it}f'(\rho ze^{it}) = \Delta_h(ze^{it})f'(\rho ze^{it}) + ze^{i(t+h)}\Delta_h f'(\rho ze^{it}) \]
\[ = \Delta_h(ze^{it})f'(\rho ze^{it}) + ze^{i(t+h)} \int_0^h f''(\rho z e^{it} e^{i\theta})i\rho z e^{it} e^{i\theta} d\theta \]
\[ = I + II. \]

The term I is bounded by \( \|h f'(\rho ze^{it})\| \). Note that \( \|f''(\rho z)\|_{p,\alpha} \leq (1 - \rho)^{-3+\beta} \) so \( \|f'(\rho ze^{it})\|_{p,\alpha} \leq (2 - \beta)^{-1}(1 - \rho)^{-2+\beta} \) by Theorem 2.1. But we also have that \( \|f''(\rho z)\|_{p,\alpha} \leq (1 - \rho)^{-2+\beta} \) so by Theorem 2.1, we have \( \|f'(\rho ze^{it})\|_{p,\alpha} \leq (\beta - 1)^{-1}(1 - \rho)^{-1+\beta} \). Thus in either case \( \|f'(\rho ze^{it})\|_{p,\alpha} \leq 2(1 - \rho)^{-2+\beta} \).

Now if we take the \( A^p_\alpha \) norm of term II with respect to \( z \) and use Minkowski’s inequality we see that it is bounded in absolute value by
\[ \int_0^h (1 - \rho)^{-2+\beta} d\theta = (1 - \rho)^{-2+\beta}h. \]

Putting this all together shows that
\[ \|\Delta_h[f(ze^{it}) - f(\rho ze^{it})]\|_{p,\alpha} \leq 2h(1 - \rho)^{-1+\beta} + h(1 - \rho)^{-1+\beta} + \beta^{-1}(1 - \rho)^\beta. \]

Now note that
\[ -\Delta_{-h} \Delta_h f(zpe^{it}) = irpe^{i\theta}e^{it} \int_0^h f'(zpe^{it} e^{i\theta})e^{is} - f'(zpe^{it} e^{-i\theta})e^{-is} ds. \]

But the above integrand equals
\[ (e^{is} - e^{-is})f'(zpe^{it} e^{i\theta}) + [f'(zpe^{it} e^{i\theta}) - f'(zpe^{it} e^{-i\theta})]e^{-is} \]

Now the \( A^p_\alpha \) norm of the first term is bounded by \( 4s(1 - \rho)^{-2+\beta} \), as above. And the second term equals
\[ \int_{-s}^s f''(zpe^{it} e^{iu})izpe^{it} e^{iu} du \]
But applying Minkowski’s inequality shows that the $A^p_\alpha$ norm of the above integral is bounded by $\int_{-s}^{s} (1-\rho)^{-2+\beta} \leq 2s(1-\rho)^{-2+\beta}$. Thus the expression in equation (3.3) is bounded by $6s(1-\rho)^{-2+\beta}$. Therefore

$$\|\Delta_h f(z\rho e^{it})\|_{p,\alpha} \leq \int_{0}^{h} 6s(1-\rho)^{-2+\beta} ds \leq 3h^2(1-\rho)^{-2+\beta}.$$ 

Putting all of this together shows that

$$\|\Delta_h f(z\rho e^{it})\|_{p,\alpha} \leq \int_{0}^{h} 6s(1-\rho)^{-2+\beta} ds \leq 3h^2(1-\rho)^{-2+\beta}.$$ 

Now, we need to choose $\rho$ in terms of $h$ so that $1-\rho = O(h)$ but $1-\rho \geq 0$ for $0 \leq h < 2\pi$. We may take $\rho = 1 - h/(4\pi)$, which gives the result. □

We now have the following corollary, which relates functions that are mean Hölder continuous with respect to the Bergman space norm with functions that are mean Hölder continuous with respect to the Hardy space norm.

**Corollary 3.5.** Let $f$ be analytic in $\mathbb{D}$, and let $(1+\alpha)/p < \beta < 2$ and $1 < p < \infty$. Then $f \in \Lambda^{\beta,A^p_\alpha}$ if and only if $f \in \Lambda^{\beta/2,A^{p/2}_{\alpha}}$. The “only if” part of the statement also holds if $\beta = 2$.

**Proof.** This follows from Theorems 1.2, 1.3, 3.1 and 3.4, and the fact that the latter two theorems hold when all area integral means and Bergman spaces are replaced by classical integral means and Hardy spaces. □

### 4. Extremal Problems

Let $k \in A^q_\alpha$ be given, where $1 < q < \infty$. Let $F \in A^p_\alpha$ be such that $\|F\| = 1$ and $\Re \int_{\mathbb{D}} F \overline{k} dA_\alpha$ is as large as possible. There is always a unique function $F$ with this property because $L^p(dA_\alpha)$ is uniformly convex, see for example [6]. The next theorem allows us to obtain knowledge about regularity of $F$ from knowledge about the regularity of $k$. For similar results that give regularity results about $k$ from knowledge of the regularity of $F$, see [1].

Note the assumption $\int_{\mathbb{D}} F \overline{k} dA_\alpha = 11$ in the statement of the next theorem. Choosing any scalar multiple of $k$ gives the same function extremal function $F$. However, this assumption simplifies the notation in the proof. Also, it is clear that for bounding $\|F\|_{A^\alpha_{\beta,A^p_\alpha}}$ in terms of $B$, we must have some lower bound on the size of $k$.

**Theorem 4.1.** Suppose that $k \in \Lambda^{\alpha/2}$, and let $F$ be the extremal function for $k$. Then if $2 \leq p < \infty$ we have $F \in \Lambda^{\beta/2,A^p_\alpha}$ while if $1 < p \leq 2$ we have $F \in \Lambda^{\beta/2,A^p_\alpha}$.
Furthermore, suppose that \( \int_D \| k(e^{it}) + k(e^{-it}) - 2k(\cdot) \|_{q, \alpha} \leq B|t|^\beta \). If \( p \geq 2 \) then \( \| F \|^{\alpha, \beta}_{\Lambda^p} \leq 2e^{1/e}(B/2)^{1/p} \) whereas if \( 1 < p < 2 \) then \( \| F \|^{\alpha, \beta}_{\Lambda^p} \leq 2(p-1)^{-1/2}(B/2)^{1/2} \).

**Proof.** Suppose that \( \| k(e^{it}) + k(e^{-it}) - 2k(\cdot) \|_{q, \alpha} \leq B|t|^\beta \). Then if we define \( \phi_t \) to be the functional associated with \( k(e^{it}) \), and let \( \phi = \phi_0 \), we have \( \| \phi_t + \phi_{-t} - 2\phi \|_{(A^p)^*} \leq B|t|^\beta \). Now let \( \tilde{\phi} = (\phi_t + \phi_{-t})/2 \). Also let \( \tilde{F} = (F(e^{it}) + F(e^{-it}))/2 \), where \( F \) is the extremal function for \( \phi \).

Thus \( \| \tilde{\phi} - \phi \|_{(A^p)^*} \leq C|t|^\beta \) and \( \| \tilde{F} \|_p \leq 1 \). Note that

\[
\int_D F(z)^2k(e^{it})z dA_{\alpha}(z) = \int_D F(e^{-it})k(z) dA_{\alpha}(z)
\]

so \( \phi(\tilde{F}) = \tilde{\phi}(F) \). But \( |\tilde{\phi}(F)| \geq 1 - \| \phi - \tilde{\phi} \| \geq 1 - B|t|^\beta \). Thus \( |\phi(F + \tilde{F})| \geq 2 - B|t|^\beta \) so \( \| F + \tilde{F} \| \geq 2 - B|t|^p \).

Now let \( p \geq 2 \). Clarkson’s inequality states that

\[
\|(F + \tilde{F})/2\|^p + \|(F - \tilde{F})/2\|^p \leq (\|F\|^q + \|\tilde{F}\|^q)^{p/q}/2^{p/q}.
\]

Let \( B' = B/2 \). This shows that, if \( |t| > B'^{-1/\beta} \), then

\[
(1 - B'|t|^\beta)^p + \|(F - \tilde{F})/2\|^p \leq (\|F\|^q + \|\tilde{F}\|^q)^{p/q}/2^{p/q} \leq 1.
\]

Thus \( \|(F - \tilde{F})/2\|^p \leq 1 - (1 - B'|t|^\beta)^p \). But since \( (1 - x)^p \) is convex one has \( (1 - x)^p \geq 1 - px \) so

\[
\|(F - \tilde{F})/2\|^p \leq 1 - (1 - B'p|t|^\beta) = B'p|t|^\beta.
\]

Thus \( \| F - \tilde{F} \| \leq 2p^{1/p}B^{1/p}|t|^\beta/p \leq 2e^{1/e}B^{1/p}|t|^\beta/p \) for \( \| t \| > B'^{-1/\beta} \). And one always has \( \| F - \tilde{F} \| \leq 2 \), so \( \| F - \tilde{F} \| \leq 2B^{1/p}|t|^\beta/p \) for \( \| t \| > B'^{-1/\beta} \).

But \( e^{1/e} > 1 \) so we always have \( \| F - \tilde{F} \| \leq 2e^{1/e}B^{1/p}|t|^\beta/p \).

The proof for \( 1 < p < 2 \) is similar, but we use the inequality

\[
\|(f + g)/2\|^2 + (p-1)\|(f - g)/2\|^2 \leq (\|f\|^2 + \|g\|^2)/2
\]

from [1]. (Note that in the reference the authors give the inequality in Proposition 3 as \( (\|x + y\|^2 + \|x - y\|^2)/2 \geq \|x\|^2 + (p-1)\|y\|^2 \), which gives the one we use by setting \( x = f + g \) and \( y = f - g \) and dividing by 4. One could also use their inequality from Theorem 1, namely \( (\|x + y\|^2 + \|x - y\|^2)/2 \geq \|x\|^2 + (p-1)\|y\|^2 \) and set \( x = (F + \tilde{F})/2 \) and \( y = (F - \tilde{F})/2 \), which also gives \( \|(F + \tilde{F})/2\|^p + (p-1)\|(F - \tilde{F})/2\|^p \leq 1 \), which is the same result as we get below.)

Letting \( f = F \) and \( g = \tilde{F} \) in the displayed inequality above gives

\[
\|(F + \tilde{F})/2\|^2 + (p-1)\|(F - \tilde{F})/2\|^2 \leq (\|F\|^2 + \|\tilde{F}\|^2)/2 \leq 1.
\]
As above this yields
\[(1 - C|t|^\beta)^2 + (p - 1)\|(F - \tilde{F})/2\|^2 \leq 1\]
for \(|t| \leq C^{-1/\beta}\). Thus \((p - 1)\|(F - \tilde{F})/2\|^2 \leq 1 - (1 - C|t|^\beta)^2\). As above, this shows that
\[
\|(F - \tilde{F})/2\|^2 = \frac{2C}{p - 1}|t|^\beta.
\]
Thus \(\|F - \tilde{F}\| \leq \sqrt{2}(p - 1)^{-1/2}C^{1/2}|t|^{\beta/2}\) for \(|t| \leq C^{-1/\beta}\). But since we always have \(\|F - \tilde{F}\| \leq 2C^{1/2}|t|^{\beta/2}\) for \(|t| > C^{-1/\beta}\). So in any event, \(\|F - \tilde{F}\| \leq 2(p - 1)^{-1/2}C^{1/2}|t|^{\beta/2}\).

**Theorem 4.2.** Let \(\alpha = 0\). If \(k \in \Lambda_{2, Ap}\) and \(1 < p < \infty\) then \(|F|^{p-1}F' \in L^1\). Also \(F' \in L^s\) for some \(s > 1\).

**Proof.** First let \(2 < p < \infty\). If \(k \in \Lambda_{2, Ap}\) then \(F \in \Lambda_{2/p, Ap}\) by Theorem 4.1. But this shows that \(A_p(r, F') \leq C(1 - r)^{2/p - 1}\) by Theorem 3.1. Thus \(M_p(r, F') \leq C/(1 - r)^{2/p - 1 - 1/p} = C/(1 - r)^{1/p - 1}\). Then for small \(\delta > 0\) we can use the fact that integral means increase with \(p\) to see that \(\|F'\|_{A^{1+\delta}} \leq \int_0^1 2r(1 - r)^{(1/p - 1)(1+\delta)} dr < \infty\).

Also, \(F'\) is in \(H^p\) by [2], Theorem 5.4 (one may also apply Ryabykh’s theorem to see this). Then \(M_1(r, |F|^{p-1}F') \leq M_2(r, |F|^{p-1})M_p(r, F') \leq \|F\|_{H^p}^{p-1}C/(1 - r)^{1-1/p}\). But \(\|F^{p-1}|F'|_{L^1} = \int_0^1 M_1(r, |F|^{p-1}F') 2r dr \leq C\).

Now let \(1 < p \leq 2\). The function \(F \in \Lambda_{1, Ap}^s\) by Theorem 4.1. But this shows that \(A_p(r, F') \leq C(1 - r)^{-\epsilon}\) for any \(\epsilon > 0\), by Theorem 3.1. Thus \(M_p(r, F') \leq C/(1 - r)^{-1/p + \epsilon}\). And we may choose \(\epsilon\) so that \(\epsilon + 1/p < 1\). The same reasoning as above shows that \(F' \in A^{1+\delta}\) for small enough delta.

Also, \(F'\) is in \(H^p\) as above. Then
\[
M_1(r, |F|^{p-1}F') \leq M_2(r, |F|^{p-1})M_p(r, F') \leq \|F\|_{H^p}^{p-1}C/(1 - r)^{-\epsilon - 1/p}.
\]
But
\[
\|F^{p-1}|F'|_{L^1} = \int_0^1 M_1(r, |F|^{p-1}F') 2r dr \leq C.
\]

In fact the same method combined with the theorem of Hardy and Littlewood on the comparative growth of integral means shows that \(|F|^{p-1}F'\) is in \(L^s\) for some \(s > 1\).

This allows us to give an alternate proof of the results of [S], by providing an alternative proof of Lemma 1.1 which avoids using the regularity results of Khavinson and Stessin from [11]. To give more detail: Using the above result, we can prove Theorem 2.1 of [S] in
The next corollary is similar to the result of Khavinson and Stessin from [11] about the Hölder continuity of extremal functions in the unweighted Bergman space, given enough regularity on \( k \). Our corollary applies to certain weighted Bergman spaces, however, and its method of proof is completely different. It uses two well known lemmas, which we state after the proof.

**Corollary 4.3.** Let \( 1 < p < \infty \) and let \( p \) and \( q \) be conjugate exponents. Suppose \( k \in \Lambda_{2,A_p}^* \). Then \( M_p(r, f') \leq C(1-r)^{-1+2/\nu-(1+\alpha)/p} \) where \( \nu \) is any number greater than 2 for \( 1 < p < 2 \) and \( \nu = p \) for \( 2 \leq p < \infty \). If \( 2 \leq p < \infty \) and \( -1 < \alpha < 0 \), then \( f \) has Hölder continuous boundary values. If \( 1 < p < 2 \) and \( -1 < \alpha < p - 2 \), the same conclusion holds.

**Proof.** Suppose \( B > \|k\|_{\Lambda_{2,A_p}^*} \).

First let \( p > 2 \). First apply Theorem 4.1 to see that \( F \in \Lambda_{2,A_p}^* \). Then apply Theorem 3.1 to see that \( A_{p,\alpha}(r, f') \leq C(1-r)^{2/p-1} \). Then apply Theorem 1.2 to see that \( M_p(r, f') \leq C(1-r)^{2/p-1-1/p-\alpha/p} \). Then apply the Lemma 4.5 to see that \( M_{\infty}(r, f') \leq C(1-r)^{2/p-1-1/p-\alpha/p-1/p} = C(1-r)^{-1-\alpha/p} \). If \( -1 < \alpha < 0 \) then \( -1 - \alpha/p > -1 \), so we have that \( f \) is Hölder continuous in the disc by Lemma 4.4. The Hölder constant is bounded above by

\[
2e^{1/\epsilon}(B/2)^{1/p} \cdot 383 \left(1 - \frac{2}{p}\right)^{-1} \cdot 2 \left( \frac{\Gamma(q-1)}{\Gamma(q/2)^2} \right)^{1/q} \cdot \left(1 - \frac{2p}{\alpha}\right).
\]

For \( p < 2 \), apply Theorem 4.1 to see that \( F \in \Lambda_{1,A_p}^* \). Then apply Theorem 3.1 to see that \( A_{p,\alpha}(r, f') \leq C|\log(1-r)| \leq C(1-r)^{-\epsilon} \) for any \( \epsilon > 0 \). Then apply Theorem 1.2 to see that \( M_p(r, f') \leq C(1-r)^{-\epsilon-1/p-\alpha/p} \). Then apply Lemma 4.5 to see that \( M_{\infty}(r, f') \leq C(1-r)^{-\epsilon-2/p-\alpha/p} \). But \( -2/p - \alpha/p > -1 \) if \( \alpha < p - 2 \), so we have that \( f \) is Hölder continuous in the disc by Lemma 4.4. The Hölder exponent is \( 1 - 2/p - \alpha/p - \epsilon \). The constant is bounded.
above by
\[
2(p - 1)^{-1/2}(B/2)^{1/2} \cdot 192\left(1 - \frac{2}{p}\right)^{-1} \cdot 2\left(\frac{\Gamma(q - 1)}{\Gamma(q/2)^2}\right)^{1/q} 
\cdot \left(1 - \frac{2}{1 - 2/p - \alpha/p - \epsilon}\right).
\]

\[\square\]

The lemmas that follow are used in the proof of the above theorem. Both of them are due to Hardy and Littlewood.

**Lemma 4.4** (see [2], Theorem 5.1). If \(|f'(re^{i\theta})| \leq C(1 - r)^{1-\beta}\) for all sufficiently large \(r\) then \(f\) is continuous in the closed unit disc and
\[
|f(e^{i\phi}) - f(e^{i\theta})| \leq \left(1 + \frac{2}{\beta}\right) C|\phi - \theta|^\beta.
\]

The constant in the next lemma follows from applying Hölder’s inequality to the Cauchy Integral formula, and using Lemma 1.5 to see that
\[
\frac{1}{(2\pi)} \int_0^{2\pi} |\rho e^{it} - r|^{-p} dt \leq \rho^{-p}\Gamma(p - 1)/\Gamma(p/2)^2(1 - (r/\rho))^{1-p}
\leq 2^{1-p}(1 + \epsilon)\Gamma(p - 1)/\Gamma(p/2)^2(1 - r)^{1-p}.
\]

for large enough \(r\). (See the proof in [2]).

**Lemma 4.5** (see [2], Theorem 5.9). Let \(1 < p < \infty\). If for sufficiently large \(r\) we have \(M_p(r, f) \leq K(1 - r)^{-a}\) then given any \(\epsilon > 0\) there is an \(R\) such that for \(R < r < 1\) we have \(M_\infty(r, f) \leq CK(1 - r)^{-(a + 1/p)}\) where \(C = (2(1 + \epsilon)\Gamma(p - 1)/\Gamma(p/2)^2)^{1/p}\).

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