Equivalence of Diagonal Contractions to Generalized IW-Contractions with Integer Exponents

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We present a simple and rigorous proof of the claim by Weimar-Woods [Rev. Math. Phys., 2000, 12, 1505–1529] that any diagonal contraction (e.g., a generalized Inönü–Wigner contraction) is equivalent to a generalized Inönü–Wigner contraction with integer parameter powers.

1 Introduction

Usual or generalized Inönü–Wigner contractions (IW-contractions) is a conventional way for realizing contractions of Lie algebras. The most known examples on contractions of Lie algebras arising in physics (contractions from the Poincaré algebra to the Galilean one and from the Heisenberg algebras to the Abelian ones of the same dimensions, forming a symmetry background of limit processes from relativistic and quantum mechanics to classical mechanics) are represented by usual IW-contractions. The second of the above examples is a trivial contraction. Any Lie algebra is contracted to the Abelian algebra of the same dimension via the IW-contraction corresponding to the zero subalgebra.

The concept of contractions of Lie algebras introduced by Segal [20] in a heuristic way became well known only after the invention of IW-contractions in [11, 12]. Saletan [19] gave the first rigorous general definition of contractions and investigated the whole class of one-parametric contractions whose matrices are first-order polynomials with respect to contraction parameters. Later contractions of Lie algebras appeared in different areas of physics and mathematics, e.g., in the study of representations, invariants and special functions.

The name “generalized Inönü–Wigner contraction” were first used in [9] for so-called $p$-contractions by Doebner and Melsheimer [7]. Generalizing IW-contractions, they studied contractions whose matrices become diagonal after choosing appropriate bases of initial and contracted algebras, and diagonal elements being powers of the contraction parameter with real exponents. In the algebraic literature, similar contractions with integer exponents are called one-parametric subgroup degenerations [1, 2, 4, 8]. The notion of degenerations of Lie algebras extends the notion of contractions to the case of an arbitrary algebraically closed field and is defined in terms of orbit closures with respect to the Zariski topology. Note that in fact a one-parametric subgroup degeneration is induced by a one-parametric matrix group only under an agreed choice of bases in the corresponding initial and contracted algebras.

All continuous contractions arising in the physical literature are generalized IW ones. The question whether every contraction can be realized by a generalized IW-contraction was posed in [23]. Later it was conjectured that the answer is positive [24]. The attempt of proving the conjecture in [24] was not successful since, as shown in [17], the proof contains an unavoidable incorrectness at the initial step. In fact, contrary instances on this conjecture, involving characteristically nilpotent Lie algebras, were earlier constructed by Burde [1, 2] for all dimensions not less than seven but they are not well-known among the physical community. Since each
proper generalized IW-contraction induces a proper grading on the contracted algebra and each characteristically nilpotent Lie algebra possesses only nilpotent derivations and hence has no proper gradings then any contraction to characteristically nilpotent Lie algebras obviously is inequivalent to a generalized IW-contraction. This fact cannot be used for lower dimensions in view of the absence of characteristically nilpotent Lie algebras in dimensions less than seven.

Examples of another kind on non-universality of generalized IW-contractions were recently presented in [18]. There exist exactly two (resp. one) well-defined contractions of four-dimensional Lie algebras over the field of real (resp. complex) numbers, which are inequivalent to generalized IW-contractions in spite of that the contracted algebras possess a wide range of proper gradings. These examples are important since they establish new bounds for applicability of generalized IW-contractions, showing that a well-defined contraction may be inequivalent to a generalized IW-contraction even if the contracted algebra possesses proper gradings. The other contractions of four-dimensional Lie algebras were realized in [5, 16, 17] by generalized IW-contractions involving nonnegative integer parameter exponents not greater than three, and the upper bound proved to be exact [18]. Uniting these results gives the exhaustive description of generalized IW-contractions in dimension four. It is expected that a similar answer may be true for dimensions five and six. Therefore, generalized IW-contractions seem to be universal realizations only for contractions of Lie algebras of dimensions not greater than three [17].

Considering different subclasses of Lie algebras closed with respect to contractions or imposing restrictions on contraction matrices, we can pose the problem on partial universality of generalized IW-contractions in specific subsets of contractions. Thus, generalized IW-contractions of nilpotent algebras are studied in [3]. Diagonal contractions arose in [24] as an intermediate step in realizing general contractions via generalized IW-contractions. It was indirectly claimed as a part of a more general incorrect theorem on universality of generalized IW-contractions that every diagonal contraction is equivalent to a generalized IW-contraction and every generalized IW-contraction is equivalent to a generalized IW-contraction with integer exponents. Although the claim is correct and important for the theory of contractions, the initial step of the proof presented in [24] has to be corrected to avoid an essential inconvenience (especially for the case of complex Lie algebras) partially induced by incorrectness of preliminary results and the general theorem.

In this paper we rigorously prove two statements. The proof that integer exponents are sufficient for generalized IW-contractions is rather geometrical. The second theorem which states equivalence of every diagonal contraction to a generalized Inönü–Wigner contraction involving integer powers of a parameter is proved in a more algebraic way. The first statement obviously follows from the second one but it is of independent interest and hence is separately formulated and proved. In particular, it connects the investigations of generalized IW-contractions (admitting real exponents) in the physical literature and one-parametric subgroup degenerations (whose parameter exponents are necessarily integer) in the algebraic literature. The proofs are essentially simpler than those from [24] and so much algorithmic that the described algorithms can be directly realized in symbolic calculation programs. This completely solves the problem on universality of generalized IW-contractions in the set of diagonal contractions.

2 Contractions and generalized IW-contractions

The notion of contraction is defined for arbitrary algebraically closed fields in terms of orbit closures in the variety of Lie algebras [1, 2, 4, 8, 13]. Let $V$ be an $n$-dimensional vector space over an algebraically closed field $\mathbb{F}$, $n < \infty$, and $\mathcal{L}_n = \mathcal{L}_n(\mathbb{F})$ denote the set of all possible Lie brackets on $V$. We identify $\mu \in \mathcal{L}_n$ with the corresponding Lie algebra $\mathfrak{g} = (V, \mu)$. $\mathcal{L}_n$ is an algebraic subset of the variety $V^* \otimes V^* \otimes V$ of bilinear maps from $V \times V$ to $V$. Indeed, under
setting a basis \( \{ e_1, \ldots, e_n \} \) of \( V \) there is the one-to-one correspondence between \( \mathcal{L}_n \) and

\[
\mathcal{C}_n = \{ (c^k_{ij}) \in \mathbb{F}^{n^3} \mid c^k_{ij} + c^k_{ji} = 0, c^k_{ij} c^l_{k\ell} + c^l_{ki} c^k_{\ell j} + c^k_{jk} c^l_{i\ell} = 0 \},
\]

which is determined for any Lie bracket \( \mu \in \mathcal{L}_n \) and any structure constant tuple \((c^k_{ij}) \in \mathcal{C}_n \) by the formula \( \mu(e_i, e_j) = c^k_{ij} e_k \). Throughout the indices \( i, j, k, i', j' \) and \( k' \) run from 1 to \( n \) and the summation convention over repeated indices is used. \( \mathcal{L}_n \) is called the variety of \( n \)-dimensional Lie algebras (over the field \( \mathbb{F} \)) or, more precisely, the variety of possible Lie brackets on \( V \). The group \( \text{GL}(V) \) acts on \( \mathcal{L}_n \) in the following way:

\[
(U \cdot \mu)(x, y) = U^{-1}(\mu(Ux, Uy)) \quad \forall U \in \text{GL}(V), \forall \mu \in \mathcal{L}_n, \forall x, y \in V.
\]

(This is the right action conventional for the physical contraction theory. In the algebraic literature, the left action defined by the formula \((U \cdot \mu)(x, y) = U(\mu(U^{-1}x, U^{-1}y)) \) is used that is not of fundamental importance.) Denote the orbit of \( \mu \in \mathcal{L}_n \) under the action of \( \text{GL}(V) \) by \( \mathcal{O}(\mu) \) and the closure of it with respect to the Zariski topology on \( \mathcal{L}_n \) by \( \overline{\mathcal{O}(\mu)} \).

**Definition 1.** The Lie algebra \( \mathfrak{g}_0 = (V, \mu_0) \) is called a contraction (or degeneration) of the Lie algebra \( \mathfrak{g} = (V, \mu) \) if \( \mu_0 \in \overline{\mathcal{O}(\mu)} \). The contraction is proper if \( \mu_0 \in \overline{\mathcal{O}(\mu)} \setminus \mathcal{O}(\mu) \). The contraction is nontrivial if \( \mu_0 \neq 0 \).

In the case \( \mathbb{F} = \mathbb{C} \) the orbit closures with respect to the Zariski topology coincide with the orbit closures with respect to the Euclidean topology and Definition 1 is reduced to the usual definition of contractions which is also suitable for the case \( \mathbb{F} = \mathbb{R} \).

**Definition 2.** Consider a parameterized family of the Lie algebra \( \mathfrak{g}_\varepsilon = (V, \mu_\varepsilon) \) isomorphic to \( \mathfrak{g} = (V, \mu) \). The family of the new Lie brackets \( \mu_\varepsilon, \varepsilon \in (0, 1] \), is defined via the Lie bracket \( \mu \) with a continuous function \( U: (0, 1] \to \text{GL}(V) \) by the rule \( \mu_\varepsilon(x, y) = U^{-1}_\varepsilon \mu(U_\varepsilon x, U_\varepsilon y) \) \( \forall x, y \in V \). If for any \( x, y \in V \) there exists the limit

\[
\lim_{\varepsilon \to +0} \mu_\varepsilon(x, y) = \lim_{\varepsilon \to +0} U_\varepsilon^{-1} \mu(U_\varepsilon x, U_\varepsilon y) =: \mu_0(x, y)
\]

then \( \mu_0 \) is a well-defined Lie bracket. The Lie algebra \( \mathfrak{g}_0 = (V, \mu_0) \) is called a one-parametric continuous contraction (or simply a contraction) of the Lie algebra \( \mathfrak{g} \). The procedure \( \mathfrak{g} \to \mathfrak{g}_0 \) providing \( \mathfrak{g}_0 \) from \( \mathfrak{g} \) is also called a contraction.

If a basis of \( V \) is fixed, the operator \( U_\varepsilon \) is defined by the corresponding matrix \( U_\varepsilon \in \text{GL}_n(\mathbb{F}) \) and Definition 2 can be reformulated in terms of structure constants. Let \( c^k_{ij} \) be the structure constants of the algebra \( \mathfrak{g} \) in the fixed basis \( \{ e_1, \ldots, e_n \} \). Then Definition 2 is equivalent to that the limit

\[
\lim_{\varepsilon \to +0} (U_\varepsilon)^{i'}_{j'}(U_\varepsilon)^{j}_{k'}(U_\varepsilon^{-1})^{k'}_k c^k_{ij} =: c^k_{i'j'}
\]

exists for all values of \( i', j' \) and \( k' \) and, therefore, \( c^k_{i'j'} \) are components of the well-defined structure constant tensor of a Lie algebra \( \mathfrak{g}_0 \). The parameter \( \varepsilon \) and the matrix-function \( U_\varepsilon \) are called a contraction parameter and a contraction matrix, respectively.

The contraction \( \mathfrak{g} \to \mathfrak{g}_0 \) is called trivial if \( \mathfrak{g}_0 \) is Abelian and improper if \( \mathfrak{g}_0 \) is isomorphic to \( \mathfrak{g} \).

**Definition 3.** The contractions \( \mathfrak{g} \to \mathfrak{g}_0 \) and \( \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}_0 \) are called (weakly) equivalent if the algebras \( \tilde{\mathfrak{g}} \) and \( \tilde{\mathfrak{g}}_0 \) are isomorphic to \( \mathfrak{g} \) and \( \mathfrak{g}_0 \), respectively.

Using the weak equivalence concentrates one’s attention on existence and results of contractions and neglects differences in ways of contractions. To take into account such ways, we can introduce different notions of stronger equivalence [17].
Lemma 1. If the matrix $U_\varepsilon$ of a contraction $g \to g_0$ can be represented in the form $U_\varepsilon = \bar{U}, \bar{U}_\varepsilon$, where $\bar{U}$ and $\bar{U}_\varepsilon$ are continuous functions from $(0, 1]$ to $\text{GL}_n(\mathbb{F})$ and $\exists \lim_{\varepsilon \to +0} \bar{U}_\varepsilon =: \bar{U}_0 \in \text{GL}_n(\mathbb{F})$ then $\bar{U}_\varepsilon \bar{U}_0$ also is a matrix of the contraction $g \to g_0$ and the matrix $\bar{U}_\varepsilon$ leads to an equivalent contraction.

Generalized Inönü–Wigner contractions is defined as a specific way for realizations of general contractions.

Definition 4. The contraction $g \to g_0$ (over $\mathbb{C}$ or $\mathbb{R}$) is called a generalized Inönü–Wigner contraction if its matrix $U_\varepsilon$ can be represented in the form $U_\varepsilon = AW_\varepsilon P$, where $A$ and $P$ are constant nonsingular matrices and $W_\varepsilon = \text{diag}(\varepsilon^{\alpha_1}, \ldots, \varepsilon^{\alpha_n})$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. The tuple of exponents $(\alpha_1, \ldots, \alpha_n)$ is called the signature of the generalized IW-contraction $g \to g_0$.

In fact, the signature of a generalized IW-contraction $\mathcal{C}$ is defined up to a positive multiplier since the reparametrization $\varepsilon = \tilde{\varepsilon}^\beta$, where $\beta > 0$, leads to a generalized IW-contraction strongly equivalent to $\mathcal{C}$.

Due to the possibility of changing bases in the initial and contracted algebras, we can set $A$ and $P$ equal to the unit matrix. This is appropriate for some theoretical considerations but not for working with specific Lie algebras. For $U_\varepsilon = \text{diag}(\varepsilon^{\alpha_1}, \ldots, \varepsilon^{\alpha_n})$ the structure constants of the resulting algebra $g_0$ are calculated by the formula $c_{\alpha, \beta, \gamma}^\varepsilon = \lim_{\varepsilon \to 0} c_{ij}^\varepsilon \varepsilon^{\alpha_1+\gamma_j-\alpha_i}$ with no summation with respect to the repeated indices. Therefore, the constraints $\alpha_i + \alpha_j \geq \alpha_k$ if $c_{i,j}^{\alpha,\beta} \neq 0$ are necessary and sufficient for the existence of the well-defined generalized IW-contraction with the contraction matrix $U_\varepsilon$, and $c_{i,j}^{\alpha,\beta} = 0$ if $\alpha_i + \alpha_j = \alpha_k$ and $c_{i,j}^{\alpha,\beta} = 0$ otherwise. This obviously implies that the conditions of existence of generalized IW-contractions and the structure of contracted algebras can be reformulated in the basis-independent terms of gradings of contracted algebras associated with filtrations on initial algebras [8, 14]. (Probably, this was a motivation for introducing and developing the purely algebraic notion of graded contractions [10, 15].) In particular, the contracted algebra $g_0$ has to admit a derivation whose matrix is diagonalizable to $\text{diag}(\alpha_1, \ldots, \alpha_n)$.

The following statement is known as a conjecture for a long time (see, e.g., [24]).

Theorem 1. Any generalized IW-contraction is equivalent to a generalized IW-contraction with an integer signature (and the same associated constant matrices).

Proof. Let the contraction $g \to g_0$ have the matrix $U_\varepsilon = AW_\varepsilon P$, where $A$ and $P$ are constant nonsingular matrices and $W_\varepsilon = \text{diag}(\varepsilon^{\alpha_1}, \ldots, \varepsilon^{\alpha_n})$ for some $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$. We introduce the notations

$$\mathcal{E} = \{(i, j, k) \mid c_{ij}^k \neq 0, \alpha_i + \alpha_j = \alpha_k\}, \quad \mathcal{N} = \{(i, j, k) \mid c_{ij}^k \neq 0, \alpha_i + \alpha_j > \alpha_k\}.$$

We can assume that $\mathcal{N} \neq \emptyset$ since otherwise the contraction $g \to g_0$ is improper and therefore, equivalent to a generalized IW-contraction with the zero signature. The system $S$ of the equations $x_i + x_j = x_k$, $(i, j, k) \in \mathcal{E}$, and the inequalities $x_i + x_j > x_k$, $(i, j, k) \in \mathcal{N}$, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is compatible since it has the solution $x = \alpha$. If $x$’s satisfy $S$ then $\lambda x$ is a solution of $S$ for any positive real $\lambda$. Therefore, the solution set of $S$ is a nonempty conus in $\mathbb{R}^n$. We express a maximal subset of $x$’s via the other $x$’s using the equations $x_i + x_j = x_k$, $(i, j, k) \in \mathcal{E}$. Denote by $I$ (resp. $\bar{I}$) the set of numbers of the expressed $x$’s (resp. unconstrained $x$’s); $\bar{I}$ is complimentary to $I$ in $\{1, \ldots, n\}$. The expressions for $x_i$, $i \in I$, are linear in $x_j$, $j \in \bar{I}$, and have rational coefficients. After substituting these expressions into the inequalities $x_i + x_j > x_k$, $(i, j, k) \in \mathcal{N}$, we obtained a system $S'$ of strict inequalities for $x_j$, $j \in \bar{I}$, which defines an open nonempty conus in $\mathbb{R}^{\bar{I}}$. The conus necessarily contains points with rational coordinates. This means that the system $S$ possesses rational solutions. Therefore, it also has integer solutions. Any solution of $S$ is the signature of a well-defined generalized IW-contraction $g \to g_0$ with the same associated constant matrices $A$ and $P$. \qed
Theorem 1 is a consequence of a more general statement on diagonal contractions, discussed in the next section.

Note 1. In fact, the proof of Theorem 1 gives a constructive way for finding an integer signature via solving the system $S$, e.g., by the Gaussian elimination [21]. (See also [6] for different methods of solving linear systems of inequalities.) At first the system $S$ is reduced to the system $S'$ by the Gaussian elimination of $x_i$, $i \in I$, due to the equations $x_i + x_j = x_k$, $(i, j, k) \in E$. Then the Gaussian elimination is applied to the (compatible) system $S'$ of strict inequalities. After the final step of the elimination we take rational (e.g., zero) values for the residual components of $x$ and go back with the elimination conditions step-by-step, choosing rational values for the steps when the corresponding components of $x$ are constrained by inequalities.

The similar remark is true for the proof of Theorem 2.

The notion of sequential contractions is introduced similarly to continuous contractions [20]. Namely, consider a sequence of the Lie algebra $g_p = (V, \mu_p)$, $p \in \mathbb{N}$, isomorphic to $g = (V, \mu)$. The sequence of the new Lie brackets $\{\mu_p, p \in \mathbb{N}\}$, is defined via the Lie bracket $\mu$ with a sequence $\{U_p, p \in \mathbb{N}\} \subset \text{GL}(V)$ by the rule $\mu_p(x, y) = U_p^{-1}\mu(U_px, U_py) \forall x, y \in V$. If for any $x, y \in V$ there exists the limit

$$\lim_{p \to \infty} \mu_p(x, y) = \lim_{p \to \infty} U_p^{-1}\mu(U_px, U_py) =: \mu_0(x, y)$$

then $\mu_0$ is a well-defined Lie bracket. The Lie algebra $g_0 = (V, \mu_0)$ is called a sequential contraction of the Lie algebra $g$.

Any continuous contraction from $g$ to $g_0$ gives an infinite family of matrix sequences resulting in sequential contractions from $g$ to $g_0$. More precisely, if $U_\epsilon$ is the matrix of the continuous contraction and the sequence $\{\epsilon_p, p \in \mathbb{N}\}$ satisfies the conditions $\epsilon_p \in (0, 1]$, $\epsilon_p \to +0$, $p \to \infty$, then $\{U_\epsilon_p, p \in \mathbb{N}\}$ is a matrix sequence generating a sequential contraction from $g$ to $g_0$.

Conversely, if a sequence $\{U_p, p \in \mathbb{N}\} \subset \text{GL}(V)$ defines a sequential contraction from $g$ to $g_0$ (and the sign of $\text{det} U_p$ is the same for all $p \in \mathbb{N}$ if $F = \mathbb{R}$) then there exists a one-parametric continuous contraction from $g$ to $g_0$ with the associated continuous function $\tilde{U} : (0, 1] \to \text{GL}(V)$ such that $\tilde{U}_{1/p} = U_p$ for any $p \in \mathbb{N}$. The simple proof of this fact involves logarithms and exponents of matrices and, additionally, the polar decomposition in the real case.

Definitions of special types of contractions, statements on properties and their proofs in the case of sequential contractions can be easily obtained via reformulation of those for the case of continuous contractions. It is enough to replace continuous parametrization by discrete one. The replacement is invertible.

3 Equivalence of diagonal contractions to generalized IW-contractions

There exists a class of contractions, which is wider than the class of generalized IW-contractions, and, at the same time, any contraction from this class is equivalent to a generalized IW-contraction involving only integer parameter powers. Similar to generalized IW-contractions, this class is singled out by restrictions on contraction matrices instead of restrictions on algebra structure.

Definition 5. The contraction $g \to g_0$ (over $F = \mathbb{C}$ or $\mathbb{R}$) is called diagonal if its matrix $U_\epsilon$ can be represented in the form $U_\epsilon = AW_\epsilon P$, where $A$ and $P$ are constant nonsingular matrices and $W_\epsilon = \text{diag}(f_1(\epsilon), \ldots, f_n(\epsilon))$ for some continuous functions $f_i : (0, 1] \to F \setminus \{0\}$.

Theorem 2. Any diagonal contraction is equivalent to a generalized IW-contraction with an integer signature.
Proof. Let the contraction $g \to g_0$ have the matrix $U_\varepsilon$ of the form from Definition 5. Due to possibility of changing bases in the initial and contracted algebras, we can set $A$ and $P$ equal to the unit matrix. If $U_\varepsilon = W_\varepsilon$, the structure constants of the contracted algebra $g_0$ are calculated by the formula

$$c_{0,ij}^k = \lim_{\varepsilon \to +0} c_{ij}^k \frac{f_if_j}{f_k}$$

with no summation with respect to the repeated indices. Therefore, the condition

$$\exists \lim_{\varepsilon \to +0} \frac{f_if_j}{f_k} =: F_{ij}^k \in \mathbb{F} \text{ if } c_{ij}^k \neq 0$$

are necessary and sufficient for the existence of the well-defined diagonal contraction with the contraction matrix $U_\varepsilon$, and $c_{0,ij}^k = c_{ij}^k F_{ij}^k$ if $F_{ij}^k$ is defined and belongs to $\mathbb{F}\{0\}$ and $c_{0,ij}^k = 0$ otherwise.

Introducing the notations

$$\mathcal{E} = \{(i,j,k) \mid i < j, c_{ij}^k \neq 0, F_{ij}^k \neq 0\}, \quad \mathcal{N} = \{(i,j,k) \mid i < j, c_{ij}^k \neq 0, F_{ij}^k = 0\},$$

we associate the set of the limits $F_{ij}^k$, $(i,j,k) \in \mathcal{E} \cup \mathcal{N}$, with two systems.

1) the system $\Sigma$ of the equations $y_i y_j/y_k = F_{ij}^k$, $(i,j,k) \in \mathcal{E}$, for $y = (y_1, \ldots, y_n) \in (\mathbb{F}\{0\})^n$ and

2) the mixed system $S$ of the equations $x_i + x_j - x_k = 0$, $(i,j,k) \in \mathcal{E}$, and the inequalities $x_i + x_j - x_k > 0$, $(i,j,k) \in \mathcal{N}$, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

We will prove using the rule of contraries that the existence of the nonzero limits $F_{ij}^k$ for $(i,j,k) \in \mathcal{E}$ (resp. these limits and, additionally, the zero limits for $(i,j,k) \in \mathcal{N}$) implies the compatibility of the system $\Sigma$ (resp. $S$). The principal observation is that the solving operations with equations and inequalities of the systems are equivalent to similar operations with the limits.

Suppose that the system $\Sigma$ is incompatible. Then $\mathcal{E} \neq \emptyset$. We use the multiplicative version of the Gaussian elimination, involving only integer powers. It is reduced to iterating the following procedure. Let $\Sigma_{\nu}$ denote the set $\{Y_s = G_s, \ s = 1, \ldots, \sigma\}$ of consequences of $\Sigma$ obtained after the $\nu$th iteration, $\Sigma_0 := \Sigma$. Here $Y_s$ (resp. $G_s$) are products of integer powers of a selection of $y$'s (resp. $F$'s), the numbers $|\Sigma_{\nu}|$ and $|\Sigma|$ of equations of the systems $\Sigma_{\nu}$ and $\Sigma$ coincides, $|\Sigma_{\nu}| = |\Sigma| =: \sigma$. We choose $i$ such that $y_i$ is in the system $\Sigma_{\nu}$ and denote by $\beta_s$ the degree of $y_i$ with respect to $y_i$. Let $\beta = \gcd(\beta_1, \ldots, \beta_\sigma)$ be the greatest common divisor of $\beta_1, \ldots, \beta_\sigma$. According the generalized Bézout's identity, we have the representation $\beta = \delta_s \beta_s$ with integer coefficients $\delta_s$. Consider the consequence $\bar{Y} = G$ of $\Sigma_{\nu}$, where $\bar{Y} = Y_{1}^{\delta_1} \cdots Y_{\sigma}^{\delta_\sigma}$ and $G = G_1^{\delta_1} \cdots G_{\sigma}^{\delta_{\sigma}}$. The degree of $\bar{Y}$ with respect to $y_i$ equals $\beta$. The equations $Y_s \bar{Y} - \beta_s / \beta = G_s G - \beta_s / \beta$, $s = 1, \ldots, \sigma$, form the system $\Sigma_{\nu+1}$. By the construction, the number of unknowns in $\Sigma_{\nu+1}$ is less by 1 than that in $\Sigma_{\nu}$. The incompatibility of the system $\Sigma_{\nu}$ is equivalent to the incompatibility of the system $\Sigma_{\nu+1}$. In view of the incompatibility of $\Sigma$, after iterations we have to obtain an inconsistent consequence of the form

$$1 = \prod_{(i,j,k) \in \mathcal{E}} (F_{ij}^k)^{m_{ij}^k} \quad \text{or} \quad Y^2 = \prod_{(i,j,k) \in \mathcal{E}} (F_{ij}^k)^{m_{ij}^k},$$

where the right-hand side does not equal the unity (resp. is negative), $m_{ij}^k \in \mathbb{Z}$, $(i,j,k) \in \mathcal{E}$, and $Y$ is a product of integer powers of $y$'s. Inconsistent consequences of the second form are

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1Here we also can assume that $\mathcal{N} \neq \emptyset$ since otherwise the contraction $g \to g_0$ is improper and therefore, equivalent to a generalized IW-contraction with the zero signature.
related only to the case of real numbers. The same combination of operations is well defined for
the limits with \((i, j, k) \in \mathcal{E}\) and, applied to them, results in the same (resp. similar) inconsistent
equality for limits that contradicts the existence of the diagonal contraction.

Suppose that the system \(S\) is incompatible. The subsystem of the equations \(x_i + x_j - x_k = 0,\)
\((i, j, k) \in \mathcal{E},\) should have solutions. (At least, it has a zero solution.) Applying the Gaussian
elimination over \(\mathbb{Z}\) to this subsystem, we present it in the form

\[
a_i x_i = \sum_{j \in \bar{I}} b_j^i x_j, \quad i \in I,
\]

where \(I \subset \{1, \ldots, n\}, \bar{I} = \{1, \ldots, n\} \setminus I, a_i \in \mathbb{N}, b_j^i \in \mathbb{Z}, i \in I\) and \(j \in \bar{I}.\) We eliminate \(x_i, i \in I,\)
from the inequalities \(x_{i'} + x_{j'} - x_{k'} > 0, (i', j', k') \in \mathcal{N},\) multiplying them by appropriate products
of some of \(a_i, i \in I.\) As a result, we obtain a system \(S'\) of strict homogenous linear inequalities
for \(x_j, j \in \bar{I},\) with integer coefficients. Since the system \(S\) is incompatible, there exists an
identically vanishing linear combination with natural coefficients, composed of left-hand sides of
inequalities from \(S'[2].\) This combination gives the inconsistent inequality \(0 > 0.\)

The above chain of additive operations with equations and inequalities of \(S\) is associated with
a chain of well-defined multiplicative operations with the limits \(F_{ij}^k, (i, j, k) \in \mathcal{E} \cup \mathcal{N}.\) Under
this association adding, subtracting and multiplying by integers are replaced by multiplying,
dividing and raising to the corresponding powers, respectively. Only multiplying and raising to
natural powers can be applied to the zero limits \(F_{ij}^k, (i, j, k) \in \mathcal{N},\) that agrees with restrictions
on operations with inequalities. The chain of operations with the limits leads to the inconsistent
equality \(1 = 0\) that contradicts the existence of the diagonal contraction.

Let \((\gamma_1, \ldots, \gamma_n)\) and \((\alpha_1, \ldots, \alpha_n)\) be solutions of the systems \(\Sigma\) and \(S,\) respectively. It is
obvious that \(\gamma_1 \cdots \gamma_n \neq 0.\) The system \(S'\) possesses rational solutions since the solution set
of \(S'\) is open and nonempty. This implies that the system \(S\) has rational solutions and hence
has integer solutions, i.e., we can assume then \(\alpha_1, \ldots, \alpha_n \in \mathbb{Z}.\) Then the matrix \(\tilde{U}_\varepsilon = \tilde{A} W_\varepsilon P,\)
where \(\tilde{A} = A \text{diag}(\gamma_1, \ldots, \gamma_n)\) and \(W_\varepsilon = \text{diag}(\varepsilon^{\alpha_1}, \ldots, \varepsilon^{\alpha_n}),\) generates a well-defined generalized
IW-contraction from \(g\) to \(g_0\) with integer exponents. \(\square\)

In other words, Theorem 2 says that generalized IW-contractions are universal in the class
of diagonal contractions.

**Note 2.** Under constructing a generalized IW-contraction equivalent to a diagonal contraction
in the way described in the proof, the constant matrix factors of the associated contraction
matrix are in fact known and coincide, up to a multiplier, with the ones of the diagonal
contraction. The multiplier is a constant diagonal matrix whose diagonal entries give
a solution of the system \(\Sigma.\) Only solving the system \(S\) of linear equations and inequalities
with respect to parameter exponents is of a significant value. We can choose an integer
solution of \(S\) which is optimal in some sense, e.g., the maximum of the absolute values
of whose components is minimal. In general, the chosen solution may be non-optimal, in the
same sense, in the entire set of integer signatures of generalized IW-contractions from \(g\) to \(g_0.\)
To choose a totally optimal signature, for each tested tuple of exponents from a number
of preliminary selected ones we have either to find a solution of a cumbersome nonlinear
system of algebraic equations on coefficients of the constant matrix factors or to prove in-
compatibility of this system. This is much more complicated problem than that discussed in
Theorem 2.

**Corollary 1.** Any diagonal contraction whose matrix possesses a finite limit at \(\varepsilon \rightarrow +0\) is
equivalent to a generalized IW-contraction with nonnegative integer exponents.

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²This statement is a modification of the well-known Voronoy theorem [22] (see also [6, p. 10]) to the case of
homogenous strict linear inequalities with integer (or rational) coefficients.
Proof. Since the functions \( f_i \) possess finite limits at \( \varepsilon \to +0 \), we can attach the additional equations \( x_i > 0 \) to the system \( S \) and prove using the same way as in the proof of Theorem [2] that the extended system has integer solutions. 

**Note 3.** Other additional restrictions on exponents of generalized IW-contractions which are equivalent to diagonal contractions with certain properties can be set in a similar way. In particular, it obviously follows from the proof of Theorem [2] that for any fixed \( j \) the \( j \)th exponent can be chosen nonnegative (negative) if there exists a finite (infinite) limit of \( f_j \) at \( \varepsilon \to +0 \).

**Note 4.** The same theorem is true for diagonal sequential contractions, and its proof completely coincides with the proof of Theorem [2].

**Note 5.** Theorem [2] is obviously extended to the class of contractions wider than the class of diagonal contractions. In particular, it implies that any contraction \( g \to g_0 \) whose matrix can be represented in the form \( U_\varepsilon = \hat{U}_\varepsilon AW_\varepsilon \tilde{U}_\varepsilon \) is equivalent to a generalized IW-contraction with an integer signature. Here \( \hat{U}_\varepsilon \) is an automorphism matrix of the algebra \( g \), \( \hat{U}_\varepsilon \) is a nonsingular matrix with the well-defined componentwise limit \( \lim_{\varepsilon \to +0} \hat{U}_\varepsilon =: P \), both the matrices \( \hat{U}_\varepsilon \) and \( \tilde{U}_\varepsilon \) are continuously parameterized by \( \varepsilon \in (0, 1] \), \( A \) and \( P \) are constant nonsingular matrices and \( W_\varepsilon = \text{diag}(f_1(\varepsilon), \ldots, f_n(\varepsilon)) \) for some continuous functions \( f_i : (0, 1] \to \mathbb{F}\backslash\{0\} \). The problem on the widest class of parameterized matrices which are associated with contractions equivalent to generalized IW-contractions is still open.

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