VARIATIONAL APPROACH TO GAUSSIAN APPROXIMATE COHERENT STATES: QUANTUM MECHANICS AND MINISUPERSPACE FIELD THEORY

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ABSTRACT

This paper has a dual purpose. One aim is to study the evolution of coherent states in ordinary quantum mechanics. This is done by means of a Hamiltonian approach to the evolution of the parameters that define the state. The stability of the solutions is studied. The second aim is to apply these techniques to the study of the stability of minisuperspace solutions in field theory. For a $\lambda \varphi^4$ theory we show, both by means of perturbation theory and rigorously, by means of theorems of the K.A.M. type, that the homogeneous minisuperspace sector is indeed stable for positive values of the parameters that define the field theory.
I. INTRODUCTION

This paper has a dual purpose. It has its origin in the study of superspaces in field theory, that is, the function spaces of solutions of any field theory (even though the name superspace originated in the study of the gravitational field). However, we would like to emphasize that the techniques we will use here may be applied to the study of coherent states in ordinary quantum mechanics, and the examples we will give are, in fact, equivalent to one- and two-dimensional nonrelativistic quantum mechanics.

The simplest example of a superspace is that of a one-dimensional real scalar field $\varphi(z, t)$. If we expand $\varphi$ in a real Fourier series (assuming the domain of $\varphi$ to be confined to $-L/2 < z < L/2$ with the end points identified)

$$\varphi(z, t) = \varphi_0(t) + \sum_{n=1}^{\infty} \left\{ \varphi_n(t) \cos \left( \frac{2n\pi z}{L} \right) + \varphi_{-n}(t) \sin \left( \frac{2n\pi z}{L} \right) \right\},$$

the evolution of $\varphi$ (independent of the action that generates the evolution) is nothing more than a curve in the space of countably infinite dimension defined by the “coordinates” $\varphi_n$, $-\infty \leq n \leq +\infty$. Of course, classically any nonlinear action for $\varphi$ gives an extremely complicated infinite set of coupled ODE’s for the $\varphi_n$, and this approach to field theory is rarely used in direct calculations.

Nevertheless, such Fourier expansions were the basis for field quantization for many years, and are still used in many contexts. It is not difficult to show that for a real Fourier series such as (1.1) the quantum evolution of $\varphi$ is just the quantum mechanics of a particle moving in $\varphi_n$-space under the influence of what may be a very complicated potential (even in the simplest cases of nonlinear actions for $\varphi$). There have been a number of studies of nonlinear field theories which have attempted to glean information about the behavior of fields such as $\varphi(z, t)$ by studying model theories where the configuration space of the system is reduced by putting all but a finite number of the $\varphi_n$ equal to zero [1, 2, 3, 4]. These are called “minisuperspace” field theories. Notice that such theories are equivalent to one-particle quantum mechanics in a space of dimension of the surviving $\varphi_n$. One may
ask whether such theories are just models or whether they could be approximate solutions to the full field theory. One method for answering this question is to study approximate “coherent states” whose center is supposed to move on a track in $\varphi_n$-space that is centered on some classical path which is restricted to the reduced configuration space.

This problem leads to the study of the simpler quantum mechanical problem of finding consistent approximations to the time-dependent evolution of localized solutions to the Schrödinger equation in dimensions corresponding to the reduced configuration space. Of course, a problem of this sort is completely independent of any field theory, and we feel that our results are useful in the general study of coherent states.

An interesting method for the study of these states is based on the ideas of average variational principles. This approach is widely used in the context of nonlinear waves and oscillations [5] and more recently in the problem of soliton propagation under the influence of various perturbing effects [6]. The emphasis here will be on obtaining approximate and rigorous stability results rather than on detailed calculations of the evolution of the solution which is the purpose of many works on wave propagation. These stability arguments are of much more interest for field theory, since what has been called the “quantum stability” of minisuperspace solutions is directly related to the problem of the use of quantum minisuperspace solutions as approximations, but we will find evolution equations for coherent state solutions which could be of use in the study of coherent states in ordinary quantum mechanics.

The technique we will use is based on the consideration of time-dependent trial functions in the Lagrangian of the Schrödinger equation. The Lagrangian is then averaged over the space variables to obtain an effective action which involves only the time-dependent parameters of the trial function. The Euler equations of this new Lagrangian give us the evolution of the parameters. This procedure gives a consistent way to approximate the infinitely many degrees of freedom of the wave function by means of a finite number of parameters while preserving the Lagrangian structure. On the other hand, pointwise ap-
proximations could, in principle (and often in practice), produce in the truncation spurious non-conservative terms. Since we are interested in a Hamiltonian formulation of the problem, its stability is not determined by damping terms but rather by the nature of of the Hamiltonian. Because of this the approximation must be consistent with the underlying Hamiltonian structure. We would like to emphasize once more that this idea gives us a way to construct approximate coherent states in quantum mechanics and may be very useful in what has become a very active field.

In Ref. [4] a pointwise approximation to the motion of “Gaussian” wave packets in minisuperspaces of a $\lambda\varphi^4$ theory was presented where the action was

$$L = \int \left[ \frac{1}{2} \left( \frac{\partial \varphi}{\partial t} \right)^2 - \frac{1}{2} \left( \frac{\partial \varphi}{\partial z} \right)^2 - \frac{\mu^2}{2} \varphi^2 - \lambda \varphi^4 \right] dz dt, \quad (1.2)$$

and an $S^1$ topology was assumed for $t = \text{const}.$ slices, identifying the end points $z = \pm L/2$. The field $\varphi$ was then expanded in the same Fourier series as (1.1). One- and two-dimensional minisuperspaces were studied, with special attention to the quantum stability of solutions of the one-dimensional sector as imbedded in the two-dimensional sector. The one-dimensional superspace had a state function of the form $\Psi(x,t)$, where $x$ was equal to $\varphi_0$ of (1.1). This function obeyed a non-relativistic Schrödinger equation of the form

$$i \frac{\partial \Psi}{\partial t} = -\frac{1}{2} \frac{\partial^2 \Psi}{\partial x^2} + \left( \frac{1}{2} \mu_0^2 x^2 + \varepsilon x^4 \right) \Psi, \quad (1.3)$$

with $\varepsilon = \lambda/L$. Notice that this is nothing more than a one-particle Schrödinger equation for a particular anharmonic potential.

The behavior of localized packets of the form $\Psi = e^{-S}$ was then considered, where $S$ was taken to be

$$S = \alpha(t)x^4 + \beta(t)x^3 + \gamma(t)x^2 + \sigma(t)x^2 + \frac{\mu_0}{2} [x - g(t)]^2 + iB(t)x^3 + iC(t)x^2 + iP(t) + D(t)]x. \quad (1.4)$$

To derive the evolution of $S$ a pointwise approximation resulting from inserting $\Psi = e^{-S}$ in (1.3) was used. The parameters $\alpha, \beta, \gamma, \sigma, B, C, D$ were assumed to be of order $\varepsilon$, and a consistent set of coupled linear ODE’s for these parameters was obtained.
The two-dimensional minisuperspace was one where \( \varphi_0 \) and one of the \( \varphi_n \) were taken to be non-zero. This will be discussed in more detail in Section IV.

The equations for the parameters in both the one- and two-dimensional minisuperspaces could be solved recursively, but the theory predicted secular growth of both the “center of mass” of the state and the localization parameter (controlling “spreading”). In this paper we will use the method outlined above to show that the motions are modulated and, hence, stable. Since the motion is Hamiltonian, it is not possible to have asymptotic stability, and the averaging results are not conclusive for all time. However, it is possible to show rigorously, by applying the Kolmogorov, Arnold, Moser (K.A.M.) theorem [7], that special motions are indeed stable for all time.

The paper is organized as follows. In Section II we study the approximate evolution of localized states in ordinary quantum mechanics by means of a variational approximation where we use (1.3) as an example. In Section III we present perturbation equations for the parameters of the localized state of Sec. II and show that their solutions are bounded, and then use the K.A.M. theorem to show stability for the full equations. In Section IV we study the two-dimensional minisuperspace field theory using a theorem by Melnikov, Poesch, and Kuskin [8] to show that the solutions are stable for a set of full measure in the parameter space. Section V contains conclusions and suggestions for further research. In Appendix A we present the equations for a pointwise approximation for \( \Psi = e^{-S} \), using an \( S \) similar to (1.4). Appendix B contains the full action for the parameters of the two-dimensional minisuperspace solutions.
II. VARIATIONAL APPROXIMATION

In this section we will study the behavior of an approximate coherent state in ordinary quantum mechanics for a particle moving in the potential \( V(x) = (\mu_0^2/2)x^2 + \varepsilon x^4 \), where \( \varepsilon = \lambda/L \) gives us the one-dimensional minisuperspace theory. Since \( L \) will be something like a “radius of the universe” in the minisuperspace case, we will assume that it is very large, and thus concern ourselves with potentials where \( \varepsilon \) is small. In order to find the behavior of our state, we will appeal to various approximations, but of course there is nothing to say that an exact “coherent state” does not exist for this \( V(x) \). In fact, depending on the definition one takes for a coherent state, one should be able to find such exact solutions.

For example, if we take a one-dimensional system moving in an arbitrary potential \( \tilde{V}(x) \), and assume that a solution of the form \( \Psi = e^{-S(x,t)} \) exists, where \( S = S[x, x - w(t)] \), and define \( u \equiv x, v \equiv x - w(t) \), \( S \) will satisfy the nonlinear partial differential equation

\[
\begin{align*}
    i\dot{w}[w^{-1}(u - v)] \frac{\partial S}{\partial v} + & \frac{1}{2} \frac{\partial^2 S}{\partial u^2} + \frac{\partial^2 S}{\partial u \partial v} + \frac{1}{2} \frac{\partial^2 S}{\partial v^2} - \\
    & - \frac{1}{2} \left( \frac{\partial S}{\partial u} \right)^2 - \frac{\partial S}{\partial u} \frac{\partial S}{\partial v} - \frac{1}{2} \left( \frac{\partial S}{\partial v} \right)^2 + \tilde{V}(u) = 0. \quad (2.1)
\end{align*}
\]

The boundary conditions must be taken such that \( S \to \infty \) as \( x \to \pm\infty \). The maximum value of \( \Psi \) will occur where \( (\partial S/\partial u) \pm (\partial S/\partial v) = 0 \), which can be solved for \( x = F(t) \). The function \( w \) can be chosen to make the peak value follow the classical motion of a particle moving in \( \tilde{V}(x) \). There should be no a priori reason why (2.1) would be impossible to solve for these boundary conditions for many potentials \( \tilde{V}(x) \). However, for most \( \tilde{V}(u) \) an analytic solution for \( S \) will be difficult to find, and most solutions for coherent states are only approximate.

The pointwise approximation mentioned in the Introduction consisted in using \( \Psi = e^{-S} \), with \( S \) from (1.4), in (1.3). However, there are a number of other ways to approximate such a state. One that is frequently used in the study of solitonic solutions to nonlinear differential equations is to insert a simple Gaussian trial function with time-dependent
parameters into the action for the theory. In our case we can do the same, inserting such a function into the usual action for the Schrödinger equation

\[
L = \int_{t_0}^{t_1} dt \int_{-\infty}^{+\infty} dx [i\Psi^* \dot{\Psi} - \frac{1}{2} \Psi^*_x \dot{\Psi}_x - V(x) \Psi^* \Psi].
\] (2.2)

If we take a Gaussian ansatz for \( \Psi, \Psi \equiv W e^{-S} \), where \( W = W(t) \) and

\[
S = \frac{\mu(t)}{2} [x - g(t)]^2 + iP(t)x + iC(t)x^2 + i\phi(t),
\] (2.3)

the \( g(t) \) gives the center of the probability packet \( \Psi^* \Psi \), and \( \mu(t) \) allows for “breathing” or “spreading” of the packet. Plugging this expansion into (2.2) and doing the \( x \)-integrations, we find

\[
I = \sqrt{\pi} \int_{t_0}^{t_1} \frac{W^2}{\sqrt{\mu}} \left\{ \dot{P}g + \dot{C} \left(g^2 + \frac{1}{2\mu}\right) + \dot{\phi} - \left[ \frac{P^2}{2} + \frac{\mu g^2}{2} + \varepsilon g^4 + \frac{3\varepsilon}{\mu} g^2 + 2C^2 \left(g^2 + \frac{1}{2\mu}\right) + 2PCg + \frac{\mu}{4} \left(1 + \frac{\mu g^2}{\mu^2}\right) + \frac{3\varepsilon}{4\mu^2}\right]\right\} dt
\] (2.4)

In the \( S \) given in (1.4) one had to assume that many of the parameters were of order \( \varepsilon \) so that a consistent set of equations could be found. Here we are no longer forced to suppose that the deviations of these quantities from their harmonic-oscillator values are small, and we will see that a much simpler set of equations emerge.

The Euler equations for (2.4) are

\[
\left(\frac{W^2}{\sqrt{\mu}}\right)^{\cdot} = 0,
\] (2.5a)

\[
-\dot{g} - P - 2Cg = 0,
\] (2.5b)

\[
\dot{P} - \mu g^2 - 4\varepsilon g^3 - \frac{6\varepsilon}{\mu} g + 2\dot{C}g - 4C^2 g - 2PC = 0,
\] (2.5c)

\[
-2g\dot{g} + \frac{\mu}{2\mu^2} - 4C \left(g^2 + \frac{1}{2\mu}\right) - 2Pg = 0,
\] (2.5d)

\[
\frac{\dot{C}}{2\mu^2} + \frac{C^2}{\mu^2} - \frac{3\varepsilon}{\mu^2} g^2 - \frac{1}{4} + \frac{\mu g^2}{4\mu^2} - \frac{3\varepsilon}{2\mu^3} = 0,
\] (2.5e)
together with an equation which results from varying $W$ that gives $\phi$ as a quadrature once $g$, $P$, $\mu$, and $C$ are known. Equations (2.5) can be solved recursively. Using (2.5b) and (2.5d) we obtain

$$C = \frac{\dot{\mu}}{4\mu}.$$  \hspace{1cm} (2.6)

From (2.5c) and (2.5e) we find the equation of motion for the center of the packet,

$$\ddot{g} + \mu_0^2 g + 4\varepsilon g^3 + \frac{6\varepsilon}{\mu} g = 0.$$  \hspace{1cm} (2.7)

Finally, using (2.6) in (2.5e) we find

$$-\frac{\ddot{\mu}}{\mu} + \frac{3}{2} \left( \frac{\dot{\mu}}{\mu} \right)^2 - 2\mu^2 + 2\mu_0^2 - \frac{12\varepsilon}{\mu} + 24\varepsilon g^2 = 0.$$  \hspace{1cm} (2.8)

If we introduce the change of variables $\mu \equiv 1/a^2$, we find the final system

$$\ddot{g} + \mu_0^2 g + 4\varepsilon g^3 + 6\varepsilon a^2 g = 0,$$  \hspace{1cm} (2.9a)

$$\ddot{\sigma} + \mu_0^2 \sigma = -2g_0^3 - 3g_0 \frac{\sigma}{\mu_0},$$  \hspace{1cm} (2.9b)

for the position $g$ and the variance (“breathing”) of our approximate coherent state.

In order to compare these equations for $\mu$ and $g$ with those in Ref. [4], we can see that if we write, as above, $\mu = \mu_0 + 2\varepsilon \gamma$, and $g = g_0 + 2\varepsilon \sigma$ and keep terms to order $\varepsilon$, we find

$$\ddot{g}_0 + \mu_0^2 g_0 = 0,$$  \hspace{1cm} (2.10a)

$$\ddot{\sigma} + \mu_0^2 \sigma = -2g_0^3 - 3g_0 \frac{\sigma}{\mu_0},$$  \hspace{1cm} (2.10b)

and

$$\ddot{\gamma} + 4\mu_0^2 \gamma = 6 + 12g_0^2.$$  \hspace{1cm} (2.11)

Instead of comparing these equations directly with those of Ref. [4], it is more useful to use the parametrization used in this article and apply it to the pointwise approximation of Ref [4]. To do this we write

$$S = \alpha(t)x^4 + \beta(t)x^3 + \frac{\mu(t)}{2}[x - g(t)]^2 + iB(t)x^3 + iC(t)x^2 + iP(t)x + i\phi(t).$$  \hspace{1cm} (2.12)
(leaving out a trivial real $x$-independent term which appears in Ref. [4]). We have to assume that $\alpha$, $\beta$, and $B$ are small enough that we only keep terms to linear order in them or we will have terms in $x$ of order higher than $x^4$, and the ansatz will be inconsistent. We can substitute this $S$ in the Schrödinger equation (1.3) with $\Psi = e^{-S}$ and find a series of coupled equations for $\alpha$, $\beta$, $\mu$, $g$, $B$, $C$, and $P$ (we will ignore the equation for $\phi$). These are given in Appendix A. The equations for $\alpha$, $\beta$, and $B$ are the same as in Ref. [4]. If we take $\mu = \mu_0 + 2\varepsilon\gamma$ and $g = g_0 + 2\varepsilon\sigma$, and assume that $\alpha$, $\beta$, and $B$ are of order $\varepsilon$, we find $\dot{\alpha} = 0$, $\alpha = \varepsilon/4\mu_0$. Equations (A3) and (A4) imply

$$\dot{\beta} + 9\mu_0^2\beta - 4\varepsilon\mu_0g_0 = 0,$$

(2.13)

$$B = \frac{\dot{\beta}}{3\mu_0} - \frac{\varepsilon}{3\mu_0^2} + \frac{\varepsilon\dot{g}_0}{3\mu_0^2},$$

(2.14)

and the solutions are (taking $g_0 = Q \cos \mu_0 t$)

$$\beta = \varepsilon x_3 \cos 3\mu_0 t + \frac{\varepsilon Q}{2\mu_0} \cos \mu_0 t,$$

(2.15)

$$B = \varepsilon x_3 \sin 3\mu_0 t - \frac{Q\varepsilon}{2\mu_0} \sin \mu_0 t - \frac{\varepsilon}{3\mu_0^2},$$

(2.16)

where $x_3$ is a constant. Now, Eqs. (A6) and (A8) can be solved for $C$ and $P$ respectively. Taking the derivative of the equation for $C$ and inserting the result in (A5) gives

$$-\frac{\ddot{\mu}}{\mu} + \frac{3}{2} \left( \frac{\dot{\mu}}{\mu} \right)^2 - 2\mu^2 + 2\mu_0^2 + \varepsilon \left( -6g^2 + \frac{6}{\mu} + \frac{6}{\mu^2} \dot{g}^2 + \frac{6\dot{\mu}^2}{\mu^4}g^2 + 12\frac{\dot{\mu}}{\mu^3} \dot{g}g \right) + 36\mu g \beta + B \left( -36\dot{g} - 33\frac{\dot{\mu}}{\mu}g \right) = 0.$$  

(2.17)

Substituting the result for $\dot{P}$ in (A7) gives

$$\ddot{g} + \mu_0^2 g + \varepsilon \left( 6g^3 + \frac{6\dot{g}^2 g}{\mu^2} + \frac{6\dot{\mu}^2}{\mu^4} \dot{g}g \right) +$$

$$+ \beta \left( 30\mu g^2 + 12 - \frac{9}{2} \frac{\dot{\mu}^2}{\mu^3} g^2 - \frac{21}{2} \frac{\dot{\mu}}{\mu^2} \dot{g}g - \frac{6\dot{g}^2 g}{\mu} \right) +$$

8
\[ +B \left( -36\dot{g}g - \frac{63\dot{\mu}}{2\mu}g^2 \right) = 0. \] (2.18)

These two equations can be compared to (2.7) and (2.8). Note that they are the same except for differences in the order-\(\varepsilon\) “driving terms” that modify the second order ODE’s that determine \(g\) and \(\mu\).

In order to compare these equations with (2.10b) and (2.11), we will again take \(\mu = \mu_0 + 2\varepsilon\gamma\), \(g = g_0 + 2\varepsilon\sigma\). To order \(\varepsilon\) we again get (2.10a) for \(g_0\) and

\[
\ddot{\sigma} + \mu_0^2\sigma + \left(3g_0^3 + \frac{3\dot{g}_0^2g_0}{\mu_0^2}\right) + \left(\frac{\beta}{\varepsilon}\right) \left(15\mu_0g_0^2 + 12 - \frac{6\dot{g}_0^2}{\mu_0}\right) - 18\left(\frac{B}{\varepsilon}\right)\dot{g}_0g_0 = 0,
\]

(2.19)

\[
\ddot{\gamma} - 4\mu_0^2\gamma + \left(3\mu_0g_0^2 + 3 + \frac{3\dot{g}_0^2}{\mu_0}\right) + 18\left(\frac{\beta}{\varepsilon}\right)\mu_0^2g_0 - 18\left(\frac{B}{\varepsilon}\right)\mu_0\dot{g}_0 = 0.
\]

(2.20)

Once more these differ from (2.10b) and (2.11) in the “driving terms”, where the term proportional to \(\varepsilon\) is slightly different and there are added terms in \(B\) and \(\beta\).

Clearly, the pointwise approximation, aside from being more complicated, does not add much to the simpler equations from the variational approximation. In Ref. [4] the main problem with the equations for \(\gamma\) and \(\sigma\) (i.e. \(\mu\) and \(g\)) is that they generated terms with secular growth, which made the predictions of the theory only valid for a limited time. Both (2.17) and (2.18) have terms that drive \(\sigma\) and \(\gamma\) at resonance, so to order \(\varepsilon\) in perturbation they will also have secular terms. In Ref. [4] it was assumed that better perturbation methods would remove such terms. In the next section we will apply such methods to the variational approximation as well as provide a rigorous demonstration that the solutions for \(\mu\) and \(g\) are indeed bounded.
III. ABSENCE OF SECULAR GROWTH

A. PERTURBATION SOLUTION

In the previous section we constructed the action and obtained the equations for \( \mu \) (i.e. \( a = 1/\sqrt{\mu} \)) and \( g \). From (2.10b) and (2.11) one can see that the order-\( \varepsilon \) perturbation theory leads to secular growth for both the \( g \) and \( a \) perturbations. However, there are more sophisticated perturbation methods which remove these terms and provide a uniform approximation to the solution.

If we look at Eqs. (2.9) and make certain assumptions about the size of \( a \) and \( g \), we can readily show that \( g \) must be bounded. If in the last term on the LHS of (2.9a) we take \( a \) to be little enough different from \( 1/\sqrt{\mu_0} \) so that \( \varepsilon a^2 \approx \varepsilon/\mu_0 \), (2.9a) then becomes (with no assumption about the size of \( g \))

\[
\ddot{g} = -\left( \frac{\mu_0^2}{2} + \frac{6\varepsilon}{\mu_0} \right) g - 4\varepsilon g^3. \tag{3.1}
\]

This is the equation of a particle moving in the potential

\[
V(g) = \left( \frac{\mu_0^2}{2} + \frac{3\varepsilon}{\mu_0} \right) g^2 + \varepsilon g^4, \tag{3.2}
\]

and \( g \) can be solved for in terms of elliptic functions. The form of \( V \) shows that the motion of \( g \) is always bounded. Here, of course, we are assuming that \( a \) is bounded (and small enough) so that there are no large excursions of the \( 3\varepsilon g^2/\mu_0 \) term in the potential.

In order to show that \( a \) is bounded we can take \( a \) to be \( (1/\sqrt{\mu_0}) + \tilde{a} \), where \( \tilde{a} \) is small compared to \( 1/\sqrt{\mu_0} \) (but not necessarily of order \( \varepsilon \)). As long as \( \varepsilon \tilde{a} \) is small enough to be ignored, we need only take \( g_0 \) in the last term of Eq. (2.9b). Taking \( g_0(t) = Q \cos \mu_0 t \) we have that the equation for \( \tilde{a} \) is, to second order,

\[
\ddot{\tilde{a}} + 4\mu_0^2 \tilde{a} - 6\mu_0^{5/2} \tilde{a}^2 = -12 \left( \frac{1 + Q^2 \mu_0}{\mu_0^{3/2}} \right) \varepsilon - 12 \frac{\varepsilon}{\mu_0^{1/2}} Q^2 \cos 2\mu_0 t. \tag{3.3}
\]

To remove the constant forcing to order \( \varepsilon^2 \) we take \( \tilde{a} = \rho - \frac{3(1+Q^2 \mu_0)}{\mu_0^{1/2}} \varepsilon \) and obtain the equation

\[
\dot{\rho} + (4\mu_0^2 - \varepsilon \kappa \mu_1) \rho - \kappa \rho^2 = \varepsilon F \cos 2\mu_0 t, \tag{3.4}
\]
where $\kappa = 6\mu_0^{5/2}$, $\mu_1 = 6\left(\frac{1+Q^2\mu_0}{\mu_0^{3/2}}\right)$, $F = -12\frac{Q^2}{\mu_0^{1/2}}$. This equation, for $\kappa = 0$, is the equation for an oscillator forced at resonance. The equation will now be solved to show that the nonlinear terms prevent the growth of the solution. To obtain the approximate explicit solution we will use the method of averaging for the Lagrangian

$$L = \int_{t_0}^{t_1} \left\{ -\frac{1}{2}\dot{\rho}^2 + \frac{(4\mu_0^2 + \varepsilon \mu_1)}{2}\rho^2 - \frac{\kappa}{3}\rho^3 - \varepsilon F \rho \cos 2\mu_0 t \right\} dt,$$  \hspace{1cm} (3.5)$$

and use the trial function

$$\rho = \varepsilon^{1/2}[A(\varepsilon t) \cos 2\mu_0 t + B(\varepsilon t) \sin 2\mu_0 t] + \varepsilon D(\varepsilon t),$$  \hspace{1cm} (3.6)$$

where the mean value $D$ has to be introduced due to the quadratic nature of the nonlinearity. Averaging over the fast time $2\mu_0 t$, we obtain

$$\bar{L} = \int_{t_0}^{t_1} \left[ \varepsilon^2 \left\{ \mu_0 (A'B - AB') + \frac{\mu_1}{4}(A^2 + B^2) + D^2 - \frac{\kappa}{2}D(A^2 + B^2) \right\} - \frac{1}{2}\varepsilon^{3/2} FA \right] dt.$$

\hspace{1cm} (3.6)$$

We now assume the so far arbitrary (but small) amplitude $F$ to be $F = \varepsilon^{1/2}F_0$ to balance the resonant forcing and the nonlinear effect. With this, $\varepsilon^{3/2}F \approx O(\varepsilon^2)$ becomes of the same order as the modulated terms (the argument can, of course, be reversed, assuming that $F \approx O(1)$ and rescaling $\rho$ and the time appropriately).

Variation with respect to $D$ gives the usual algebraic equation for the mean value as

$$2D - \frac{\kappa}{2}(A^2 + B^2) = 0,$$

\hspace{1cm} (3.7)$$

i.e.

$$D = \frac{\kappa}{4}(A^2 + B^2).$$

\hspace{1cm} (3.8)$$

Using this result, and integrating by parts, we obtain the final form of $\bar{L}$ as

$$\bar{L} = 2\varepsilon^2\mu_0 \int_{t_0}^{t_1} \left[ AB' - \left( \frac{\mu_1}{8\mu_0}(A^2 + B^2) + \frac{\kappa^2}{32\mu_0}(A^2 + B^2)^2 + \frac{F}{4\mu_0}A \right) \right] dt.$$  \hspace{1cm} (3.9)$$
Clearly the solutions of the Euler equations are the level curves of the Hamiltonian

\[ H = \frac{\mu_1}{8\mu_0} (A^2 + B^2) + \frac{\kappa^2}{32\mu_0} (A^2 + B^2)^2 + \frac{F}{4\mu_0} A. \]  

(3.10)

It is easy to see that the critical points of the system are the critical points of \( H \). They satisfy \( \partial H / \partial A = \partial H / \partial B = 0 \). The local minima are centers and the local maxima are saddles. Also for large \( A^2 + B^2 \) all curves are closed. It is then clear that all the motions are bounded. This shows that the resonant forcing is balanced by the nonlinearity, giving a bounded motion for \( \rho \). This in turn implies that the coherent state does not spread indefinitely away from its original spreading \((\mu_0)^{-1/2}\). To complete the description of the motion we now examine the central points. They satisfy

\[ \frac{\partial H}{\partial A} = 0 = \frac{F}{4\mu_0} + \frac{\mu_1 A}{4\mu_0} + \frac{\kappa^2}{8\mu_0} A(A^2 + B^2), \]  

(3.11a)

\[ \frac{\partial H}{\partial B} = 0 = B \left[ \left( \frac{\mu_1}{4\mu_0} \right) + \frac{\kappa^2}{8\mu_0} (A^2 + B^2) \right]. \]  

(3.11b)

The only solution for the second equation is \( B = 0 \), since \( \mu_1 \) is always greater than zero. When \( B = 0 \) the equation for \( A \) is the cubic

\[ F + \mu_1 A + \frac{\kappa^2}{2} A^3 = 0. \]  

(3.12)

There is only one critical point \((A^*, 0)\) where \( A^* \) is the only solution of (3.12). Clearly this is a minimum of \( H \). This represents a periodic solution of finite amplitude \( A^* \) with period \( 2\mu_0 \). In this case all level lines of \( H \) are concentric curves. They give periodic solutions for \( A \) and \( B \), which in turn represent quasiperiodic motions of \( \rho \).

This perturbation solution shows that small perturbations remain small. Interestingly enough some of these perturbation results can be proved rigorously as we will now show.

**B. RIGOROUS RESULTS**

We now establish the boundedness of the motion for not necessarily small initial conditions of the amplitude \( g \) of these oscillations. We consider again the Eq. (2.9b). Here
we will assume that \( g_0 = A \cos \mu_0 t \), where \( A \) is not small. Moreover, assume the initial conditions for \( a \) to be arbitrary, that is, not necessarily close to \( 1/\sqrt{\mu_0} \). In this case we will show that for sufficiently small \( \varepsilon \) the solution for \( a(t) \) is always bounded. To do this we appeal to the classical results of periodic perturbations of Hamiltonian systems with one degree of freedom [7, 9]. This perturbation theory is based on the Poincaré map. Since the equation is periodic with period \( 2\pi/2\mu_0 \) this map is defined as follows. Consider the map:

\[
[a(0), \dot{a}(0)] \rightarrow [a(2\pi/2\mu_0), \dot{a}(2\pi/2\mu_0)],
\]

where \((a, \dot{a})\) is the solution of (2.9b). Clearly this map is area-preserving since the system is Hamiltonian. An invariant curve of this map represents a two-dimensional invariant torus in the space \((a, \dot{a}, t); (t \mod 2\pi/2\mu_0)\). Since two-dimensional tori divide three-dimensional space, we have that initial values inside an invariant torus remain there. This bounds the solutions. Thus the problem of boundedness of solutions for all time is transformed into the problem of finding invariant curves for the Poincaré map of (2.9b).

The existence of invariant curves is proved using action-angle variables for Eq. (2.9b). To study this map it is convenient to first introduce action-angle variables for the unperturbed oscillator. The transformation is not expressed in terms of elementary functions, but an explicit representation is not necessary in order to derive the results.

The generating function for the canonical formulation is given by

\[
W(a, E) = \int_0^a \sqrt{E - \left( \frac{\mu_0^2}{2} \xi^2 + \frac{1}{2\xi^2} \right)} \, d\xi, \tag{3.13}
\]

where the variable \( E \) is expressed in terms of the action (the area enclosed by the orbit)

\[
I(E) = 2 \int_{a_1(E)}^{a_2(E)} \left\{ E - \left( \frac{\mu_0^2}{2} \xi^2 + \frac{1}{2\xi^2} \right) \right\}^{1/2} \, d\xi. \tag{3.14}
\]

Clearly \( I'(E) \geq 0 \) and \( E \) can be given as \( E(I) \). The generating function becomes \( W(a, I) \). We now have

\[
p = \dot{a} = \frac{\partial W}{\partial a}, \quad \dot{\theta} = \frac{\partial W}{\partial I} \tag{3.15}
\]

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in the new canonical variables \( I \) and \( \theta \). In these new variables the Hamiltonian is independent of \( \theta \).

The perturbed problem in these variables takes the form

\[
\dot{I} = -\varepsilon \frac{\partial H^{(1)}}{\partial \theta}(\theta, I, 2\mu_0 t),
\]

(3.16a)

\[
\dot{\theta} = \frac{\partial W(I)}{\partial I} + \varepsilon \frac{\partial H^{(1)}}{\partial I}(\theta, I, 2\mu_0 t)
\]

(3.16b)

The Hamiltonian \( H^{(1)} \) is the image of \((a^2/12)g^2\) under the action-angle change of variables. The explicit form is not needed, just note that \( H^{(1)} \) is a smooth function of the new variables. The Poincare map in these variables, if \((I_0, \theta_0)\) is the initial value and \((I^{(1)}, \theta^{(1)})\) the final value, takes the form

\[
I^{(1)} = I_0 - \varepsilon \int_0^{2\pi/2\mu_0} \frac{\partial H^{(1)}}{\partial \theta}[\theta(\xi, I_0, \theta_0), I(\xi, I_0, \theta_0), 2\mu_0 \xi] d\xi,
\]

(3.17a)

\[
\theta^{(1)} = \theta_0 + \int_0^{2\pi/2\mu_0} \frac{\partial W(I)}{\partial I}[I(\xi, I_0, \theta_0)] + \varepsilon \int_0^{2\pi/2\mu_0} \frac{\partial H^{(1)}}{\partial I}[\theta(\xi, I_0, \theta_0), I(\xi, I_0, \theta_0), 2\mu_0 \xi] d\xi.
\]

(3.17b)

The integrated terms are smooth functions of \( I_0, \theta_0 \) since the solutions depend smoothly on the initial values \((I_0, \theta_0)\) and the integration interval is finite. Thus the map takes the form

\[
I^{(1)} = I_0 + \varepsilon F(\theta_0, I_0),
\]

(3.18a)

\[
\theta^{(1)} = \theta_0 + \frac{\pi}{\mu_0} \omega(I_0) + \varepsilon G(\theta_0, I_0),
\]

(3.18b)

where \( \omega(I_0) \equiv \partial W/\partial I|_{I=I_0} \).

Since \( I \) can be expanded in a power series in \( \varepsilon \) (the function is analytic in \( \varepsilon \) because the equation is integrated over a finite interval). The integrated terms of \( \omega(I) \) contribute to the term \( G \) to order \( \varepsilon \). The contribution to the leading order just comes from \( I_0 \).

This is the canonical form of the map for the application of the K.A.M. theorem [10]. The map for \( \varepsilon = 0 \) has as invariant curves the circles \( I = \text{constant}, \ 0 \leq \theta \leq 2\pi \).
Moreover on each circle the rate of advance is $\pi \omega(I)/\mu_0$ which is action dependent. (In this simple case the invariant curves of the map are just the level lines of the unperturbed Hamiltonian.) In this setting the K.A.M. theorem guarantees that for $E$ sufficiently small and $\omega(I)$ sufficiently irrational the invariant curves persist [10]. It is also known that this set of “sufficiently irrational” numbers has full measure. Thus, since the function $\omega(I)$ is increasing (and tends to infinity as $I \to \infty$) it is clear that there is a set of full measure in the variable $I$ for which invariant curves persist. It then follows that the motion of $a(t)$, $\dot{a}(t)$ is bounded for all times, provided that the initial conditions fall within the perturbed invariant tori. This gives a rigorous verification of the perturbation results obtained in the previous section.

Up to this point we have made no reference to field theory. The results in this section are actually a convenient way to study approximate coherent states in one-dimensional quantum mechanics. They can be applied to the motion of coherent states with “breathing” or “spreading” such as those studied by Guth and Pi [11]. There are a number of ways to extend our results to field theory, some of which will be mentioned briefly in Sec. V. As mentioned in the Introduction, one of the main objectives of this article was to use the variational approach to investigate how close a microsuperspace solution is to a larger minisuperspace solution over a long period of time. In order to do this, in the next section we will apply the methods used above to the same minisuperspace-microsuperspace problem used in Ref. [4].
IV. MINISUPERSPACE FIELD THEORY

In Ref. [4] coherent states peaked around some small sector of a field theory were used to investigate whether they could be approximated by a state restricted to that sector and then quantized. As discussed in the Introduction, the system was a $\lambda \varphi^4$ model in an $S^1$ topology with only two modes “unfrozen”. That is, if in (1.1) we put all but $\varphi_0$ and one of the $\varphi_n$ equal to zero, we get a minisuperspace field theory by plugging this ansatz into the $\lambda \varphi^4$ action. Putting the $\varphi^{-n}$ equal to zero and dropping third and fourth order cross terms among the $\varphi_n$ in the action yielded a two-dimensional classical theory that could be quantized. Moreover, using the definitions $\varphi_0 \equiv x$ and $\varphi_n \equiv y$ it was shown that the minisuperspace state function obeyed the Schrödinger equation

$$-\frac{1}{2} \frac{\partial^2 \Psi}{\partial x^2} - \frac{1}{2} \frac{\partial^2 \Psi}{\partial y^2} + \frac{\mu_0^2}{2} x^2 \Psi + \epsilon x^4 \Psi + \frac{m_0^2}{2} y^2 \Psi + 6 \epsilon x^2 y^2 \Psi = i \frac{\partial \Psi}{\partial t},$$

(4.1)

where $\epsilon = \lambda / L$ and $m_0^2 = \mu_0^2 + (2\pi n / L)^2$. In Ref. [4] a pointwise approximation for $\Psi = e^{-S}$ with $S$ similar to (2.12) was used. Here we will use the variational approach with $\Psi = W(t) e^{-S}$, where

$$S = \frac{\mu^2}{2} (x - g_1)^2 + \frac{m^2}{2} (y - g_2)^2 + \theta xy + iP_1 x + iP_2 y + iC_1 x^2 + iC_2 y^2 + iM xy + i\phi(t),$$

(4.2)

and $\mu, m, g_1, g_2, \theta, P_1, P_2, C_1, C_2, M$ are all functions of $t$. If we insert this $S$ into

$$L = \int_{t_0}^{t_1} dt \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy [i\Psi^* \Psi_t - \frac{1}{2} \Psi^*_x \Psi_x - \frac{1}{2} \Psi^*_y \Psi_y - V(x, y) \Psi^* \Psi],$$

(4.3)

with

$$V(x, y) = \frac{\mu_0^2}{2} x^2 + \epsilon x^4 + \frac{m_0^2}{2} y^2 + 6 \epsilon x^2 y^2,$$

(4.4)

we find a very large action for $\mu, m, g_1, g_2, \theta, P_1, P_2, C_1, C_2$, and $M$ which is given in Appendix B. Of course, we are only interested in states that can be approximated by microsuperspace states with $y = 0$, i.e. those with $g_2 = 0$. If we now look at the form of $\Psi^* \Psi$, we see that a $\Psi^* \Psi = \text{const.}$ surface is an ellipse centered on $y = 0, x = g_1$.
with its semi-major and semi-minor axes given by \( \mu \) and \( m \) (which axis is associated with which parameter depends on their relative size). The parameter \( \theta \) gives the angle that this ellipse makes with the \( x \)-axis. There is some freedom in defining what one means by a minisuperspace state “close” to a microsuperspace state (here the \( y = 0 \) state). For simplicity we would like to take \( \theta = 0 \) and \( g_2 = 0 \). If one varies the action (B1) with respect to \( g_2, P_2, \theta, \) and \( M \), one finds a series of equations which, for the initial conditions \( \dot{g}_2 = g_2 = 0, \dot{P}_2 = P_2 = 0, \dot{\theta} = \theta = 0, \dot{M} = M = 0 \), maintain \( \theta, g_2, P_2, \) and \( M \) zero for all time. For this solution the reduced action

\[
L = \int_{t_0}^{t_1} \frac{\pi W^2}{\sqrt{m\mu}} \left[ \dot{P}_1 g_1 + \dot{C}_1 \left( g_1^2 + \frac{1}{2\mu} \right) + \frac{\dot{C}_2}{2m} + \dot{\phi} - \left\{ \frac{P_2^2}{2} + \frac{\mu_0^2}{2} g_1^2 + \epsilon g_1^4 + \frac{3\epsilon g_1^2}{m} + 2P_1 C_1 g_1 + 2C_2^2 \left( g_1^2 + \frac{1}{2\mu} \right) + \frac{C_2^2}{m} + \frac{\mu}{4} + \frac{\mu_0^2}{4\mu^2} + \frac{3\epsilon}{4\mu^2} + \frac{m}{4} + \frac{m_0^2}{4m} + \frac{3\epsilon}{m} \left( g_1^2 + \frac{1}{2\mu} \right) \right\} dt \right. (4.5)
\]

gives the correct equations of motion.

The Euler equations for this action are

\[
\begin{align*}
-\frac{2C_2}{m} + \frac{\dot{m}}{2m^2} &= 0, \\
-\frac{\dot{C}_2}{2m} + \frac{C_2^2}{m^2} - \frac{1}{4} + \frac{m_0^2}{4m^2} + \frac{3\epsilon}{m^2} \left( g_1^2 + \frac{1}{2\mu} \right) &= 0, \\
-\dot{g}_1 - P_1 - 2C_1 g_1 &= 0, \\
\dot{P}_1 + 2g_1 \dot{C}_1 - \mu_0^2 g_1 - 4\epsilon g_1^3 - \frac{6\epsilon g_1}{m} - 2P_1 C_1 &= 0, \\
-4C_1^2 g_1 - \frac{6\epsilon}{m} g_1 &= 0, \\
-2g_1 \dot{g}_1 + \frac{1}{2} \frac{\dot{\mu}}{\mu^2} - 2P_1 g_1 - 4C_1 \left( g_1^2 + \frac{1}{2\mu} \right) &= 0, \\
-\frac{\dot{C}_2^2}{2\mu^2} + \frac{C_2^2}{\mu^2} - \frac{\mu_0^2}{4\mu^2} + \frac{3\epsilon g_1^2}{\mu^2} + \frac{3\epsilon}{2m} \frac{1}{\mu} &= 0,
\end{align*}
\]
together with the normalization condition \( W^2 / \sqrt{m \mu} \cdot = 0 \) and an equation for the trivial phase \( \phi \). These equations give

\[
-\ddot{g}_1 - \mu_0^2 g_1 - 4 \varepsilon g_1^3 - \frac{6 \varepsilon g_1}{\mu} - \frac{6 \varepsilon g_1}{m} = 0,
\]

(4.7a)

\[
-\ddot{\mu} + \frac{3}{2} \left( \frac{\dot{\mu}}{\mu} \right)^2 - 2 \mu^2 + 2 \mu_0^2 + 24 \varepsilon g_1^2 + \frac{12 \varepsilon}{m} + \frac{12 \varepsilon}{\mu} = 0,
\]

(4.7b)

\[
-\ddot{m} + \frac{3}{2} \left( \frac{\dot{m}}{m} \right)^2 - 2m^2 + 2m_0^2 + 24 \varepsilon \left( g_1^2 + \frac{1}{2 \mu} \right) = 0.
\]

(4.7c)

In this case we can also appeal to a K.A.M. type of result in order to obtain qualitative information about the nature of the solutions. For this we observe that the unperturbed system

\[
\ddot{g}_1 + \mu_0^2 g_1 = 0,
\]

(4.8a)

\[
\ddot{a} + \mu_0^2 a - \frac{1}{a^3} = 0,
\]

(4.8b)

\[
\ddot{b} + \mu_0^2 b - \frac{1}{b^3} = 0,
\]

(4.8c)

can be written in terms of action-angle variables. The equations for \( a \) and \( b \) have the variables given in (3.13) and (3.15). The equation for \( g_1 \) has the polar coordinates as action-angle variables. Denoting the actions by \( I_g, I_a, I_b \) and the angles by \( \theta_g, \theta_a, \theta_b \), the perturbed problem takes the form

\[
\dot{I}_g = -\varepsilon \frac{\partial H^{(1)}}{\partial \theta_g}, \quad \dot{\theta}_g = \mu_0 + \varepsilon \frac{\partial H^{(1)}}{\partial I_g},
\]

(4.9a)

\[
\dot{I}_a = -\varepsilon \frac{\partial H^{(1)}}{\partial \theta_a}, \quad \dot{\theta}_a = \omega_a(I_a) + \varepsilon \frac{\partial H^{(1)}}{\partial I_a},
\]

(4.9b)

\[
\dot{I}_b = -\varepsilon \frac{\partial H^{(1)}}{\partial \theta_b}, \quad \dot{\theta}_b = \omega_b(I_b) + \varepsilon \frac{\partial H^{(1)}}{\partial I_b},
\]

(4.9c)

Since the frequency \( \mu_0 \) does not depend on the actions, the usual K.A.M. theorem cannot be applied to guarantee the persistence of the invariant tori \( I_g = \text{const.}, I_a = \text{const.}, I_b = \)
const., instead a modification due to Melnikov, Poesch and Kuskin [8] can be used. The result is as follows. If the unperturbed Hamiltonian system is of the form

\[ \dot{\Theta} = \Lambda, \quad \dot{I} = 0, \quad (4.10) \]

where \( \Lambda \) is a constant vector, then small Hamiltonian perturbations preserve the invariant tori \( I = \text{const.} \). \( \Theta = \Lambda t + \Phi \) for most values of the vector \( \Lambda \). In this case we choose

\[ \Lambda = [\mu_0, \omega_a(I^0_a), \omega_b(I^0_b)], \quad (4.11) \]

and thus the remainder is small provided that we search for tori close to \( I^0_a, I^0_b \). The result guarantees that for most values (a set of full measure) of \( \mu_0, I^0_a, I^0_b \) the unperturbed tori persist.

This shows the existence of quasiperiodic, and thus bounded, motions for most initial conditions \( I^0_a, I^0_b \). In this case the existence of invariant tori does not prove stability because three dimensional tori do not separate the six-dimensional phase space. Moreover, a universal instability, known as Arnold diffusion is present. This implies that for sufficiently long times, that is, \( t_{AD} = O(e^{1/\varepsilon}) \) particular initial values always leave any bounded region of phase space. In general this time is very long compared to other time scales in the problem. In order to see whether this instability is relevant for our problem, it is necessary to rewrite our equations in terms of more conventional units. The expression for the Arnold diffusion time mentioned above is actually \( t_{AD} = (1/\omega_0)\varepsilon^{1/\varepsilon} \), where \( \varepsilon \) is a constant which estimates the ratio of a perturbation to a Hamiltonian to the unperturbed Hamiltonian, and \( \omega_0 \) is the frequency of oscillation associated with the unperturbed Hamiltonian.

The units of \( \varphi \) in (1.2) are \( Q\ell \), where \( \ell \) is the unit of length and \( Q \) is a “charge” associated with \( \varphi \). The fact that we are working in one space dimension rather than three means that \( Q^2 \) has units of force. The constant \( \mu_0 \) has units of one over length, and the constant \( \lambda \) must have units of the reciprocal of energy times length cubed, and \( \varepsilon \) has units of one over energy times length to the fourth power. This, along with the fact that \( \partial/\partial t \)
is really $\partial/\partial ct$, allows us to determine the units of $g$ and $a$ which are both $Q\ell^{3/2}$. We can define the dimensionless variables, $u = g/QL^{3/2}$ and $b = a/QL^{3/2}$, and Eqs. (2.8) become

$$\ddot{u} + c^2 \mu_0^2 u + 4\varepsilon_0 \left( \frac{c}{L} \right)^2 u^2 + 6\varepsilon_0 \left( \frac{c}{L} \right)^2 b^2 u = 0,$$

$$\ddot{b} + c^2 \mu_0^2 b - \left( \frac{c^4 h^2}{Q^4 L^6} \right) \frac{1}{b^3} - 12\varepsilon_0 \left( \frac{c}{L} \right)^2 b^3 + 24\varepsilon_0 \left( \frac{c}{L} \right)^2 bu^2 = 0,$$

where $\varepsilon_0$ is a small dimensionless parameter. By adjusting $Q$ we can put $\varepsilon_0$ equal to one.

We need only give $\mu_0$ and $Q$ to find $t_{AD}$. These quantities depend on the type of field that $\varphi$ is supposed to model. Since $\varphi$ is one-dimensional, these quantities are somewhat different from those one would expect in the three-dimensional case. For $\mu_0$, a classical quantity that has the proper units is $mc^2/e^2$, where we can take $e$ and $m$ the charge and the mass of the proton, which gives $\omega_0 = mc^3/e^2$. A quantity $Q$ associated with the charge that has the proper units is $Q = mc^2/e$ (note that the dimensionality forces us to take a reciprocal of $e$). This implies that $c^4 h^2/Q^4 L^6$ is $(1/\alpha_f^2)(\lambda_c L^2 c^2)^2$, where $\lambda_c$ is the Compton wavelength of the proton and $\alpha_f$ is the fine structure constant. This quantity is small with respect to $c/L$ for $L > \lambda_c$, so $H_1$ is proportional to $(c/L)^2$, while $H_0$ is proportional to $m^2 e^6/e^4$, so $t_{AD}$ is roughly $t_{AD} = (e^2/mc^3) \exp[(Lmc^2/e^2)^2]$. In principle, Arnold diffusion could be important when $L < e^2/mc^2$. However, we can take a cosmological model which, for simplicity, has the time behavior of a $k = 0$ Robertson-Walker universe, but the mass of a $k = 1$ model, which gives the age of the universe for any $L$ to be

$$t_u = \frac{L^{3/2}}{\sqrt{6\pi GM_0}},$$

where $M_0 = 5 \times 10^{56}\text{g}$.

If we now take $L = \beta(t)(e^2/mc^2)$, then the ratio of $t_{AD}$ to the age of the universe at any time would be

$$\frac{t_{AD}}{t_u} = \frac{1}{\beta^{3/2} e^{\beta^2}} \left( \frac{6\pi GM_0}{e^2/m} \right)^{1/2} = \frac{1}{\beta^{3/2} e^{\beta^2}} (3.5 \times 10^{20}).$$
This quantity has a minimum of order one at $\beta$ of order one and grows for all other values of $\beta$, so $t_{AD}$ is always much larger than the age of the universe.

If $\varphi$ is supposed to model the gravitational field itself, we can take $Q^2 = c^4/G$ (again the reciprocal of the usual “charge”). Here $c^4\hbar^2/Q^4L^6$ is $(L_p/L)^4(c/L)^2$, where $L_p$ is the Planck length. Since the field is massless, we can take $\mu_0 = 0$. Here, then, the $(c/L)^2$ terms are $H_0$, and the term in $c^4\hbar^2/Q^4L^6$ is $H_1$, so $\epsilon$ is of order $(L_p/L)^4$, so $t_{AD}$ is roughly $t_{AD} = (L/c)\exp[(L/L_p)^4]$, and the ratio of $t_{AD}$ to the age of the universe (the age of the universe does not change) is (now taking $L = \beta(t)L_p$)

$$t_{AD} = \frac{1}{\sqrt{\beta}} e^{\beta^4 \left( \frac{6\pi GM_0}{L_p c^2} \right)^{1/2}} = \frac{1}{\sqrt{\beta}} e^{\beta^4 (7 \times 10^{31})},$$

which is, again, always much larger than one.

We thus conclude that the approximation based on coherent states is good in the sense that classical trajectories of the center of mass are not qualitatively modified by the existence of small numbers of higher modes in field theory. However, when infinitely many modes are present it is not possible to draw any rigorous conclusions about the limit. This case has to be investigated independently. Even simple finite-mode systems exhibit new features in the limit.
V. CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

The twofold purpose of this paper has been addressed by means of the approximation techniques given in Section II. We have shown that it is possible to obtain time-dependent consistent approximations to the motion and breathing of coherent states using average Lagrangians. These Lagrangians have allowed us to explore not only the evolution of these states in quantum mechanics, but have allowed us to study the stability of such solutions in minisuperspace field theory. The most severe potential instability arose from the resonant interaction of the center of mass of the state and the width parameter. We showed, using a technique of averaging of ordinary differential equations, how nonlinear effects saturate the resonance amplitude and change the solution into a modulated solution. In special cases the stability results can be proved rigorously by means of the K.A.M. theorem. This formalism suggests a new potential instability when many modes of comparable size are present in the coherent state. In fact, a finite but high dimensional Hamiltonian system is obtained. In this case Arnold diffusion could have a destabilizing effect over very long times. Clearly the same instability will be present in any Hamiltonian truncation of more realistic field theoretic models, and its relevance in practice will have to be studied in each case.

Finally, we would like to emphasize the fact that this variational idea could be applied to functional Lagrangians. The parameters will now be functions of both position and time. The averaged action will have as its Euler equations nonlinear partial differential equations which will describe the spread and the position of the coherent state in function space. This is currently under investigation for a simple $\varphi^4$ functional model.
APPENDIX A. POINTWISE APPROXIMATION EQUATIONS

If we write $\Psi = e^{-S}$ with $S$ given by (2.12), then $S$ obeys

$$\frac{i}{\partial t} + \frac{1}{2} \frac{\partial^2 S}{\partial x^2} - \frac{1}{2} \left(\frac{\partial S}{\partial x}\right)^2 + V(x) = 0.$$ 

Inserting (2.12) in this equation and assuming that $\alpha^2$, $\beta^2$ and $B^2$ are small enough to be ignored, we can equate the real and imaginary coefficients of each power of $x$ on the LHS equal to zero. We find

\begin{align*}
x^4 & : \
&\quad -4\alpha\mu + \varepsilon = 0 \quad \text{(real)} \quad (A1) \\
&\dot{\alpha} - 8\alpha C = 0 \quad \text{(Imag.)} \quad (A2) \\
x^3 & : \
&\quad 4\alpha\mu g - 3\beta\mu + 6BC - \dot{B} = 0 \quad \text{(Real)} \quad (A3) \\
&\dot{\beta} - 4\alpha P - 6C\beta - 3B\mu = 0 \quad \text{(Imag.)} \quad (A4) \\
x^2 & : \
&\quad 6\alpha - \dot{C} - \frac{\mu^2}{2} + 2C^2 + 3\beta\mu g + 3P\beta + \frac{\mu_0^2}{2} = 0 \quad \text{(Real)} \quad (A5) \\
&\frac{\dot{\mu}}{2} - 3\beta P + 3B\mu g - 2\mu C = 0 \quad \text{(Imag.)} \quad (A6) \\
x & : \
&\quad 3\beta - \dot{P} + \mu^2 g + 2PC = 0 \quad \text{(Real)} \quad (A7) \\
&\quad -\dot{\mu} g - \mu \dot{g} + 3B + 2\mu C g - \mu P = 0 \quad \text{(Imag.)} \quad (A8)
\end{align*}
APPENDIX B. COMPLETE TWO-DIMENSIONAL ACTION

The action (4.3) for $\Psi = W(t)e^S$ with $S$ given by (4.2) is, after integration over $x$ and $y$,

$$
\int_{t_0}^{t_1} \pi W^2e^{A} \left[ \dot{P}_1 \left( g_1 - \frac{m_\theta g_2 - \theta^2 g_1}{m_\mu - \theta^2} \right) + \dot{P}_2 \left( g_2 - \frac{\mu \theta g_1 - \theta^2 g_2}{m_\mu - \theta^2} \right) + 
\right] +
$$

$$
+ \dot{C}_1 \left( g_1^2 + \frac{1}{2\mu} - 2\theta g_1 \left( \frac{mg_2 - \theta g_1}{m_\mu - \theta^2} \right) + \frac{m\theta^2 - \theta^4/\mu + 2(mg_2 - \theta g_1)^2\theta^2}{2(m_\mu - \theta^2)^2} \right) +
$$

$$
+ \dot{C}_2 \left( g_2^2 + \frac{1}{2m} - 2\theta^2 g_2 - 2\theta g_2 \left( \frac{2m_\mu g_2 - 2\theta^2 g_2}{m_\mu - \theta^2} \right) + \frac{\mu \theta^2 - \theta^4/m + 2(\mu g_1 - g_2)^2\theta^2}{2(m_\mu - \theta^2)^2} \right) +
$$

$$
\dot{M} \left( \frac{\mu mg_1 g_2 - \mu \theta g_1^2}{m_\mu - \theta^2} - \frac{\theta}{2(m_\mu - \theta^2)} - \frac{\mu \theta (mg_2 - \theta g_1)^2}{2(m_\mu - \theta^2)^2} \right) -
$$

$$
- \frac{1}{2\mu^2} \left( \frac{1}{2\mu} + \frac{m\theta^2 - \theta^4/\mu + 2(mg_2 - \theta g_1)^2\theta^2}{2(m_\mu - \theta^2)^2} \right) -
$$

$$
- \mu \theta \left( \frac{\mu mg_1 g_2 - \mu \theta g_1^2}{m_\mu - \theta^2} - \frac{\theta}{2(m_\mu - \theta^2)} - \frac{\mu \theta (mg_2 - \theta g_1)^2}{(m_\mu - \theta^2)^2} - g_1 \left[ g_2 - \frac{\mu \theta g_1 - \theta^2 g_2}{m_\mu - \theta^2} \right] \right) -
$$

$$
- \frac{\theta^2}{2} \left( g_2^2 + \frac{1}{2m} - 2\mu \theta g_1 g_2 - 2\theta^2 g_2 \left( \frac{2m_\mu g_2 - 2\theta^2 g_2}{m_\mu - \theta^2} \right) + \frac{\mu \theta^2 - \theta^4/m + 2(\mu g_1 - g_2)^2\theta^2}{2(m_\mu - \theta^2)^2} \right) -
$$

$$
- \frac{P^2}{2} - 2C_1 \left( g_1^2 + \frac{1}{2\mu} - \frac{2m \theta g_1 g_2 - 2\theta^2 g_1^2}{m_\mu - \theta^2} + \frac{m\theta^2 - \theta^4/\mu + 2(mg_2 - \theta g_1)^2\theta^2}{2(m_\mu - \theta^2)^2} \right) -
$$

$$
- \frac{M^2}{2} \left( g_2^2 + \frac{1}{2m} - \frac{2\mu \theta g_1 g_2 - 2\theta^2 g_2^2}{m_\mu - \theta^2} + \frac{\mu \theta^2 - \theta^4/m + 2(\mu g_1 - g_2)^2\theta^2}{2(m_\mu - \theta^2)^2} \right) -
$$

$$
- 2P_1 C_1 \left( g_1 - \frac{m \theta g_2 - \theta^2 g_1}{m_\mu - \theta^2} \right) - P_1 M \left( g_2 - \frac{m \theta g_1 - \theta^2 g_2}{m_\mu - \theta^2} \right) -
$$

$$
- 2C_1 M \left( \frac{\mu mg_1 g_2 - \mu \theta g_1^2}{m_\mu - \theta^2} - \frac{\theta}{2(m_\mu - \theta^2)^2} - \frac{\mu \theta (mg_2 - \theta g_1)^2}{(m_\mu - \theta^2)^2} \right) -
$$

$$
- \frac{m^2}{2} \left( \frac{1}{2m} + \frac{\mu \theta^2 - \theta^4/m + 2(\mu g_1 - g_2)^2\theta^2}{2(m_\mu - \theta^2)^2} \right) -
$$

$$
- m \theta \left[ \frac{\mu mg_1 g_2 - \mu \theta g_1^2}{m_\mu - \theta^2} - \frac{\theta}{2(m_\mu - \theta^2)} - \frac{\mu \theta (mg_2 - \theta g_1)^2}{(m_\mu - \theta^2)^2} - g_2 \left( g_1 - \frac{m \theta g_2 - \theta^2 g_1}{m_\mu - \theta^2} \right) \right] -
$$

$$
- \frac{\theta^2}{2} \left( g_1^2 + \frac{1}{2\mu} - \frac{2m \theta g_1 g_2 - 2\theta^2 g_1^2}{m_\mu - \theta^2} + \frac{m \theta^2 - \theta^4/\mu + 2(mg_2 - \theta g_1)^2\theta^2}{2(m_\mu - \theta^2)^2} \right) -
$$
\[-\frac{1}{2} P_2^2 - 2C_2^2 \left( g_2 + \frac{1}{2m} - \frac{2\mu \theta_1 g_2 - 2\theta^2 g_2^2}{m\mu - \theta^2} + \frac{\mu \theta^2 - \theta^4/m + 2(\mu g_1 - \theta g_2)^2\theta^2}{2(m\mu - \theta^2)^2} \right) - \]

\[-\frac{1}{2} M^2 \left( g_2^2 + \frac{1}{2\mu} - \frac{2m\theta_1 g_2 - 2\theta^2 g_1^2}{m\mu - \theta^2} + \frac{m\theta^2 - \theta^4/\mu + 2(mg_2 - \theta g_1)^2\theta^2}{2(m\mu - \theta^2)^2} \right) - \]

\[-2P_2C_2 \left( g_2 - \frac{\mu \theta_1 g_2 - \theta^2 g_2^2}{m\mu - \theta^2} \right) - P_2M \left( g_1 - \frac{m\theta_1 g_2 - \theta^2 g_1^2}{m\mu - \theta^2} \right) - \]

\[-2C_2M \left( \frac{m\theta_1 g_2 - \theta^2 g_1^2}{m\mu - \theta^2} - \frac{\theta}{2(m\mu - \theta^2)} - \frac{\theta}{(m\mu - \theta^2)^2} \right) - \]

\[-\frac{\mu_0^2}{2} \left( g_1^2 + \frac{1}{2\mu} - \frac{2m\theta_1 g_1 - 2\theta^2 g_1^2}{m\mu - \theta^2} + \frac{m\theta^2 - \theta^4/\mu + 2(mg_1 - \theta g_1)^2\theta^2}{2(m\mu - \theta^2)^2} \right) - \]

\[-\varepsilon \left[ g_1^4 + \frac{3g_1^2}{\mu} + \frac{3}{4\mu^2} - \theta \left( \frac{4g_1^3 + 6g_1/\mu}{m\mu - \theta^2} \right) + \right.\]

\[+ \left. \frac{3\theta^2}{(m\mu - \theta^2)^2} \left( \frac{2\mu^2 g_1^2 + \frac{1}{\mu}}{\mu} \left( \frac{1}{2} \frac{m\mu - \theta^2}{m\mu - \theta^2} + \frac{m\mu - \theta^2}{m\mu - \theta^2} \right)^2 \right) \right] - \]

\[= 4\theta^3 g_1 \left( \frac{[mg_2 - \theta g_1]^3}{[m\mu - \theta^2]^3} + \frac{3}{2\mu} \frac{[mg_2 - \theta g_1]}{[m\mu - \theta^2]^2} \right) + \]

\[+ \theta^4 \left( \frac{[mg_2 - \theta g_1]^4}{[m\mu - \theta^2]^4} + \frac{3}{\mu} \frac{[mg_2 - \theta g_1]^2}{[m\mu - \theta^2]^3} + \frac{3}{4\mu^2} \frac{1}{[m\mu - \theta^2]^2} \right) \right] - \]

\[-\frac{m_0^2}{2} \left( g_2^2 + \frac{1}{2m} - \frac{2m\theta_1 g_2 - 2\theta^2 g_2^2}{m\mu - \theta^2} + \frac{m\theta^2 - \theta^4/m + 2(\mu g_1 - \theta g_2)^2\theta^2}{2(m\mu - \theta^2)^2} \right) - \]

\[-6\varepsilon \left[ \left( \frac{1}{2\mu} + g_1^2 \right) \left( \frac{\mu}{2(m\mu - \theta^2)} + \frac{\mu^2 (mg_2 - \theta g_1)^2}{(m\mu - \theta^2)^2} \right) - \right.\]

\[= 2\theta g_1 \left( \frac{\mu^3 [mg_2 - \theta g_1]^3}{[m\mu - \theta^2]^3} + \frac{3}{2} \frac{\mu^2 [mg_2 - \theta g_1]}{[m\mu - \theta^2]^2} \right) + \]

\[= \frac{\theta^2}{\mu^2} \left( \frac{\mu^4 [mg_2 - \theta g_1]^4}{[m\mu - \theta^2]^4} + \frac{3\mu^3 [mg_2 - \theta g_1]^2}{[m\mu - \theta^2]^3} + \frac{3}{4} \frac{\mu^2}{[m\mu - \theta^2]^2} \right) \right] \right] \right] \right] dt, \]

where \( A = (\theta^2[\mu g_1^2 + mg_2^2] - 2m\mu g_1 g_2)/(m\mu - \theta^2) \).
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