Quantum phase transitions of the Majorana toric code in the presence of finite Cooper-pair tunneling

Ananda Roy,* Barbara M. Terhal, and Fabian Hassler

JARA Institute for Quantum Information, RWTH Aachen University, 52056 Aachen, Germany

The toric code based on Majorana fermions on mesoscopic superconducting islands is a promising candidate for quantum information processing. In the limit of vanishing Cooper-pair tunneling, it has been argued that the phase transition separating the topologically ordered phase of the toric code from the trivial one is in the universality class of (2+1)D-XY. On the other hand, in the limit of infinitely large Cooper-pair tunneling, the phase transition is in the universality class of (2+1)D-Ising. In this work, we treat the case of finite Cooper-pair tunneling and address the question of how the continuous XY symmetry breaking phase transition turns into a discrete \( Z_2 \) symmetry breaking one when the Cooper-pair tunneling rate is increased. We show that this happens through a couple of tricritical points and first order phase transitions. Using a Jordan-Wigner transformation, we map the problem to that of spins coupled to quantum rotors and subsequently, propose a Landau field theory for this model that matches the known results in the respective limits. We calculate the effective field theories and provide the relevant critical exponents for the different phase transitions. Our results are relevant for predicting the stability of the topological phase in realistic experimental implementations.

The toric code [1, 2] is a promising candidate for fault-tolerant quantum computation [3, 4]. It describes a topologically ordered system whose four-fold degenerate ground state is protected from local perturbations. In contrast to standard realizations of the toric code using qubits, an alternative approach has been proposed using interacting Majorana fermions [5–10]. This approach considers a 2D array of Majorana fermions on mesoscopic superconducting islands (see Fig. 1). Each island has a charging energy \( E_C = \epsilon^2/2C \), where \( C \) is the capacitance of each island to a ground plane. The Majorana fermions enable tunneling of single electrons between two neighboring islands [11], in addition to Cooper-pair tunneling. The rates of single-electron and Cooper-pair tunneling are denoted by \( E_M \) and \( E_J \), respectively. In the limit of vanishing Cooper-pair tunneling rate and large charging energy, the system supports a topologically ordered phase described by an effective toric code Hamiltonian [5, 8]. Furthermore, in this limit, upon increasing the single-electron tunneling rate, the system goes through a zero-temperature, topological phase transition in the universality class of (2+1)D-XY [5]. On the other hand, in the limit of infinite Cooper-pair tunneling rate, the system also shows the topologically ordered toric code phase and the transition to the trivial phase is in the universality class of (2+1)D-Ising [6]. In this work, we address the question how the topological phase transition that breaks continuous XY symmetry gets transformed into one that breaks discrete \( Z_2 \) symmetry when the Cooper-pair tunneling rate is increased. This question is, in particular, nontrivial as the system undergoes a Mott insulator-superconductor transition when increasing the Cooper-pair tunneling rate with respect to the charging energy [12–16]. Starting from the microscopic Hamiltonian, we present a symmetry-based, phenomenological, Ginzburg-Landau field-theoretic description that captures the critical behavior of the system in the presence of both single-electron and Cooper-pair tunneling. We show that the XY phase transition transforms to an Ising phase transition through a couple of tricritical points and first order phase transitions.

The paper is outlined as follows. From the microscopic Hamiltonian of the system, after a Jordan-Wigner transformation, we map the problem to coupled spins and rotors, with nearest-neighbor interactions. From symmetry considerations, we propose a field theory that describes the critical behavior of the system. Subsequently, we derive effective field theories for the different phase transitions. Finally, we discuss experimental signatures of the phase transitions. Throughout this work, we restrict...
ourselves to zero temperatures.

The Hamiltonian of the system is given by

\[ H = H_C + H_J + H_M,\]

where

\[ H_C = 4E_C \sum_i n_i^2, \quad H_J = -E_J \sum_{\langle i,j \rangle} \cos(\phi_i - \phi_j), \]

\[ H_M = -E_M \sum_{\langle i,j \rangle} V_{ij} \cos \left( \frac{\phi_i - \phi_j}{2} \right). \]  

(1)

Here, the superconducting phase \( \phi_i \) and the excess charge \( n_i \) (in units of Cooper pairs) on the \( i \)-th island are canonically conjugate. In this work, we treat the idealized case of zero offset charges in the absence of disorder. The Majorana tunneling operator \( V_{ij} \) between the two neighboring islands \( i \) is given by (see Fig. 1) [6]

\[ V_{\mu,\mu',\tilde{z}} = i\gamma^\mu_0 \gamma^\tilde{z} \gamma^\mu_0 \gamma^\mu', \]

\[ V_{-\mu,\mu',\tilde{z}} = i\gamma^\mu_0 \gamma^\mu_0 \gamma^\mu_0 \gamma^\mu', \]

where \( \gamma^\alpha_0 \) are Hermitian, Majorana fermion operators obeying \( \{\gamma^\alpha_0, \gamma^\beta_0\} = \delta_{\alpha\beta} \delta_{ij} \). The fermion parity on the \( i \)-th island is given by the operator \( \mathcal{P}_i = -\gamma^\alpha_0 \gamma^\beta_0 \gamma^\gamma_0 \gamma^\delta_0 \). As the charge is constraint by the fermion parity, the (physical) Hilbert-space for the Hamiltonian \( H \) is spanned by the wavefunctions satisfying \( \psi(\phi_i + 2\pi) = (-1)^{(1-P_i)/2}\psi(\phi_i) \) [11]. At finite charging energy, the ground state is in the even parity sector on each island (\( \mathcal{P}_i \equiv +1 \)). In this sector, the four Majorana fermions on each island encode one qubit [17] and, neglecting \( H_J \), a perturbation calculation in \( E_M/E_C \) yields the toric code Hamiltonian [5, 8].

Going beyond this perturbation analysis, first, we perform a gauge transformation in order to simplify the Hilbert-space to \( 2\pi \)-periodic functions [18]. Then, we map the Majorana fermions to spins using a Jordan-Wigner transformation (for details, see Supplement of [6]). As a result, we arrive at the Hamiltonian

\[ H_C = 4E_C \sum_i \left( n_i + \frac{1 + \sigma_z^i}{4} \right)^2, \]  

(3)

\[ H_M = -\frac{E_M}{2} \sum_{\langle i,j \rangle} s_{ij} \left\{ \sigma_i^- \sigma_j^- (e^{i\phi_i} + e^{i\phi_j}) + \sigma_i^+ \sigma_j^+ (1 + e^{i(\phi_i - \phi_j)}) \right\} + \text{H.c.}, \]

while \( H_J \) remains invariant [19]. Here, the sign of the interaction is determined by gauge bits \( s_{ij} = \pm 1 \) [20]. This form of the Hamiltonian is most useful for numerical analysis and makes the symmetries of the model explicit. Most importantly, the Hamiltonian is invariant under the simultaneous transformations \( U_\theta \) of \( e^{i\phi_i} \mapsto e^{i\theta}e^{i\phi_i} \), \( \sigma_i^z \mapsto \sigma_i^z e^{i\theta/2} \). Physically, this global symmetry originates from the fact that the spins correspond to single-electrons that carry half of the charge of the Cooper pairs.

The three terms in the Hamiltonian give rise to phases which can be classified according to how they break the \( U_\theta \) symmetry. In the phase where the Cooper-pair tunneling \( H_J \) aligns the rotors \( e^{i\phi_i} \), the \( U_\theta \) symmetry is spontaneously broken. The ground state is only invariant under \( U_\theta \) with \( \theta \) being a multiple of \( 2\pi \). We denote this phase by \( \{2\pi\} \). The single-electron tunneling \( H_M \), on the other hand, orders the spins, with \( \sigma_i^z \) obtaining a finite expectation value, such that the ground state is only invariant with \( \theta \) being a multiple of \( 4\pi \). We denote this phase by \( \{4\pi\} \). The Coulomb interaction \( H_C \) disorders both the rotors and the spins and restores the full symmetry of the ground state under \( U_\theta \) for all \( \theta \). We denote this phase by \( \{\theta\} \).

Next, we discuss the signatures of the three phases in the charge sector and the relation to the \( \mathbb{Z}_2 \) spin liquids. In the phase \( \{\theta\} \), the strong Coulomb interactions localize all charges and turn the system into a Mott insulator. Going over to the phase \( \{2\pi\} \), the charges condense into Cooper pairs turning the system into a superconductor of charge \( 2e \). In the phase \( \{4\pi\} \) where the spins are ordered, the condensate is comprised of charge \( e \)-bosons (also called ‘holons’ or ‘chargons’) [21–23]. In the language of \( \mathbb{Z}_2 \) spin liquids, in our model, the ‘vison’ excitation on the plaquette, \( U \), and the ‘vison’ excitation on the plaquette, \( \theta \), correspond to the deconfined phase where visons can be separated from each other without energy cost. On the other hand, in the confined phase \( \{4\pi\} \), the energy associated with two vison excitations increases with the spatial separation between them [24]. The system is in the toric code phase when the visons are deconfined [21, 25] or equivalently, when the spin sector is ordered [6]. In what follows, we derive an effective Landau field theory for the model. We drop explicit reference to vison degrees of freedom and infer topological ordering from the ordering in the spin sector.

We consider complex fields \( \psi_i(\mathbf{r}, \tau) \) and \( \psi_s(\mathbf{r}, \tau) \) which correspond to the coarse-grained expectation values of \( e^{i\phi_i} \) and \( \sigma_i^z \) in imaginary time \( \tau \) respectively. Interested in the behavior of the system close to the point where all the three phases meet, we expect the relevant degrees of freedom to be given by the low-frequency, long-wavelength behavior of these complex fields. The microscopic symmetry \( U_\theta \) is elevated to the symmetry \( \psi \mapsto e^{i\theta} \psi \), \( \psi \mapsto e^{i\theta/2} \psi \) on the coarse-grained variable that has to be respected in the effective field theory. Close to the phase transition, the fields are small. Thus, we perform a Taylor and gradient expansion in \( \psi \), \( \psi_s \).

The partition function at zero temperature is given by

\[ Z = \int D\psi_s D\psi^*_s D\psi D\psi^* e^{-S} \]

with the Euclidean action

\[ S = \int d^2 r d\tau \left[ \partial_\tau |\psi|_s^2 + |\partial_\tau \psi|_s^2 + K_s |\nabla \psi_s|^2 + K_i |\nabla \psi|^2 + r_M |\psi_s|^2 + r_J |\psi|^2 + u_s |\psi_s|^4 + u_i |\psi|^4 + \beta |\psi_s|^2 |\psi|^2 - \alpha(\psi_s^* \psi + \psi_s \psi^*_s) \right]. \]  

(4)
We assert that the terms in Eq. (4) exhaust the relevant terms up to quartic order in the fields, consistent with the symmetries of the Hamiltonian, that can appear in the action. Note that only modulus-square of the first-order imaginary-time derivatives of the fields appear in the action [5, 26]. The first two lines of Eq. (4) is the theory of the tetracritical point (see Chap. 4 of [27]). The cubic term in the third line has, to our knowledge, not been investigated and is crucial for the prediction of the phase-diagram of the system. In order to have a stable theory, \( u_4, u_2, \beta \) must be positive and we choose \( \alpha > 0 \) without loss of generality. The parameter \( r_\sigma \) is used to tune through the phase transition and corresponds to \(-E_x/E_C\), where \( x = M, J \). The phase diagram of the model is given in Fig. 2 and the phase transitions will be analyzed below.

Before deriving effective field theories for the different phase transitions, we qualitatively explain how the field theory [Eq. (4)] describes the different phases of the system. For positive \( r_J \) and \( r_M \), the fields \( \psi_s, \psi_t \) vanish and the system is in the \( \{\theta\} \) phase. When \( r_M \) changes sign [across the line (a) in Fig. 2], the spin orders and \( \{\psi_s\} \) attains a finite value resulting in the phase \( \{4\pi\} \). In this transition, \( \psi_s \) is slaved to \( \psi_t \) since, due to the cubic term, \( \psi_s^3 \) acts as a ‘magnetic field’ for \( \psi_t \). Note that the phases \( \theta_s, \theta_t \) of the complex order parameters \( \psi_s, \psi_t \) are locked via \( \theta_t = 2\theta_s \). On the other hand, when \( r_J \) changes sign [across the line (b) in Fig. 2] for \( r_M \) large enough, only the field \( \{\psi_t\} \) becomes finite and the system is in the phase \( \{2\pi\} \). Lowering \( r_M \) reduces the stability of the disordered phase in the spin sector until the spin orders. The system enters the phase \( \{4\pi\} \) via the line (c) in Fig. 2 with the spin and the rotor order parameters phase locked as described above. In the following, we derive effective field theories for each of the transitions. This will provide information about the nature of the phase-transitions and the various associated critical exponents.

First, we analyze the phase transition between the phases \( \{\theta\} \) and \( \{4\pi\} \). To get an effective theory for \( \psi_s \), we integrate out the rotor field \( \psi_t \) by considering small fluctuations \( \psi_t = \bar{\psi}_t + \delta \psi_t \) around the saddle point \( \bar{\psi}_t \) with \( \delta S = 0 \). In the vicinity of the phase transition, to leading order, the saddle point solution is given by \( \bar{\psi}_t \propto \psi_s^2 \). Substituting \( \psi_s \) in Eq. (4) and integrating out the Gaussian fluctuations \( \delta \psi_t \) keeping the lowest order terms in \( \alpha, \beta \) in the cumulant expansion, the partition function assumes the form \( Z^{(a)} = \int \mathcal{D} \psi_s \psi_s^* e^{-S^{(a)}} \), where the action is given by

\[
S^{(a)} = \int d^2r dt \left\{ \left[ \partial_t \psi_s \right]^2 + K_s \left\{ \nabla \psi_s \right\}^2 + r_M |\psi_s|^2 \\
+ \left( u_4 - \frac{\alpha^2}{r_J} \right) |\psi_s|^4 + \frac{\alpha^2 \beta}{r_J} |\psi_s|^6 \right\}.
\]

As long as the prefactor of the quartic term is positive, i.e., for \( r_J > \alpha^2/u_4 \), the sextic term is irrelevant and the phase transition at \( r_M = 0 \) is a second order \((2+1)\)D-X-Y transition. The phase transition line [marked by (a) in Fig. 2] terminates at the tricritical point (TP\(_1\)) given by \( r_J = \alpha^2/u_4 \). Lowering \( r_J \) further, the quartic term changes sign and the transition becomes first order (see Chap. 4 of [27]).

Next, we analyze the phase transition between the phases \( \{\theta\} \) and \( \{2\pi\} \) [marked by (b)]. Across this transition, \( \bar{\psi}_t \) stays zero, while \( |\psi_s| \) turns finite. As before, integrating over small fluctuations \( \delta \psi_s \) around the saddle point \( \bar{\psi}_s = 0 \), we get an effective partition function \( Z^{(b)} = \int \mathcal{D} \psi_t \psi_t^* e^{-S^{(b)}} \) with

\[
S^{(b)} = \int d^2r dt \left\{ \left[ \partial_t \psi_t \right]^2 + K_t \left\{ \nabla \psi_t \right\}^2 + r_J |\psi_t|^2 + u_t |\psi_t|^4 \right\}.
\]

Thus, this transition is the Bose-Hubbard phase transition [12, 15, 16], and the phase transition line is given by \( r_J = 0 \). This phase transition line terminates at the first order line coming out of TP\(_1\) [29].

Now, we analyze the phase transition [marked by (c)] between the phases \( \{2\pi\} \) and \( \{4\pi\} \). To get the effective field theory, we use the parametrization \( \psi_t = (\tilde{\rho}_t + \delta \rho_t) e^{i\theta_t/\tilde{\rho}_t} \), \( \psi_s = (\sigma + iw) e^{i\theta_s/2\tilde{\rho}_s} \), where \( \tilde{\rho}_t \) is the saddle point value of \( |\psi_t| \) and the real fields \( \delta \rho_t, \theta_t, \sigma, w \) denote the fluctuations of \( \delta \psi_t \) and \( \delta \psi_s \).
The fluctuations in $\theta_i$ correspond to the massless Goldstone mode associated with the symmetry breaking in the rotor sector. They decouple from the rest. Integrating over $v, w$, we arrive at the partition function $Z^{(c)} = \int D\theta_x e^{-\int d^3r \left\{ (\partial_\tau \theta_x)^2 + K_\psi (\nabla \theta_x)^2 + t_c \sigma^2 + u_c \sigma^4 + \bar{u}_c \sigma^6 \right\}}$.

We see that the phase transition is described by the emergent Ising degree of freedom $\sigma$. In particular, the field $\sigma$ acquires a finite value when $t_c$ changes sign. The Ising degree of freedom corresponds to the two possibilities $\theta = \pm \theta_0$, and $\theta_0 \pm \pi$ of phase-locking of the spin order parameter with the rotor field. To lowest order, the parameters in the action $S^{(c)}$ are related to the parameters of the original field via

$$t_c = r_M - \alpha \sqrt{-2r_j \frac{\alpha_\beta}{\alpha u_r}},$$

$$u_c = u_s + \alpha^2 \frac{\beta^2}{2r_j} - \frac{\alpha \beta}{\sqrt{-2u_r r_j}} - \frac{\beta^2}{4u_r}.$$

For $u_c > 0$, i.e., $r_j < r_j^* = -2u_c \alpha_\beta^2/(2\sqrt{u_r u_r + \beta^2}$, the quartic term is positive and the phase transition at $t_c = 0$ is of second order $(2+1)$-D-Ising type. The phase transition line (c) terminates at a tricritical point (TP) when $u_c = 0$ after which, i.e., for $r_j > r_j^*$, the phase transition turns first order (see Chap. 4 of [27]).

Finally, we comment on the line of first order phase transition that connects the two tricritical points TP1 and TP2 in Fig. 2. From the analysis of the field theories $S^{(a)}$ and $S^{(c)}$ close to the tricritical points, we know that the lines emanate tangential to the second order lines. Across the first order line separating $\{\theta\}$ and $\{4\pi\}$, an XY symmetry breaking occurs. This results in both $\psi_s, \psi_i$ being discontinuous. On the other hand, only an Ising symmetry is broken between the phases $\{\theta\}$ and $\{4\pi\}$ resulting in only $\psi_i$ being discontinuous. The nature of the symmetry breaking changes exactly at the point where the line (b) meets the first order line. Note that this point is not a tricritical point (see also [29]).

Before concluding, we comment on experimental signatures of the proposed phases and phase transitions. As already discussed before, the three phases $\{\theta\}$, $\{2\pi\}$, and $\{4\pi\}$ correspond to a Mott insulator, a 2$e$-superconductor, and an e-superconductor. Thus, they can be distinguished by their current-voltage characteristic. For instance, the frequency of the Josephson radiation under a dc voltage bias determines the charge of the condensate in the superconducting phases while the insulator has no charge response [30–32]. In fact, the Mott-insulator-superconductor transition in Josephson junction arrays in absence of Majorana fermions [transition across (b)] has already been observed [33]. Measurement of the superfluid densities given by $|\psi_s|^2$ and $|\psi_i|^2$ can be used to determine the critical exponents $\beta_r$ and $\beta_a$ describing the behavior of the order parameters close to the phase transition. The critical exponent $\nu$ that determines the divergence of the correlation length can be accessed by electromagnetic correlation measurements as the fields are charged; for example, one can imagine probing the system by measuring the low-frequency conductivity through a pair of spatially separated capacitive contacts. Close to the phase transition, the conductance will obtain a finite value for arbitrary distances [34]. For convenience, we provide the theoretical values for the different critical exponents for the three second order phase transitions and the two tricritical points (using Chap. 5 of [27]) in Table I.

To summarize, we have analyzed the different phases and the phase transitions occurring in the Majorana toric code in the presence of Cooper-pair tunneling. Starting from the microscopic model, we have performed a Jordan-Wigner transformation and mapped the problem to that of spins coupled to rotors with nearest neighbor interaction. Subsequently, based on symmetry considerations, we have proposed a Landau field theory to analyze the critical behavior of the system at zero temperature. We have shown that as one changes the Cooper-pair tunneling rate, the topological phase transition separating the toric code phase from the trivial phase changes from a $(2+1)$D-XY type to a $(2+1)$D-Ising type through a couple of tricritical points and first order transitions. Our results match the known results in the limiting cases. In particular, we have provided evidence that the topological order survives for any finite Cooper-pair tunneling. We have derived an effective field theory for each of the transitions and commented on the experimental signatures of the phases and the phase transitions. The present work provides a starting point for further numerical and field-theoretical investigations of the rich phase diagram of the Majorana toric code in the presence of Cooper-pair tunneling. Moreover, with the recent developments in detecting Majorana bound states in solid state systems [35, 36], we are optimistic of experimental

| phase transition | type     | $\nu$  | $\beta_r$ | $\beta_a$ |
|------------------|----------|--------|-----------|-----------|
| (a)              | $(2+1)$-XY | 0.67   | 0.35      | 0.70      |
| (b)              | $(2+1)$-XY | 0.67   | 0.35      | 0.70      |
| (c)              | $(2+1)$-Ising | 0.63   | 0.32      | –         |
| TP1              | XY       | 0.50   | 0.25      | 0.5       |
| TP2              | Ising    | 0.50   | 0.25      | –         |

TABLE I. Table summarizing the different phase transitions and tricritical points occurring in the model, their types and the critical exponents $\nu$ for the correlation length and $\beta_r, x = s, r$ for the order parameters for the spin and rotor sectors. For transitions (a) and TP1, $\beta_3 = 2\beta_s$, since in the vicinity of the phase transition, to leading order, $\psi_s \propto \psi_i^2$. For the tricritical points, mean field exponents are exact since $(2+1)$D is above the upper critical dimension for the sextic term in $S^{(a)}$ and $S^{(c)}$. |
verifications of the field theory predictions.

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*roy@physik.rwth-aachen.de

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