FLAT MANIFOLDS AND REDUCIBILITY

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Abstract. Hiss and Szczepański proved in 1991 that the holonomy group of any compact flat Riemannian manifold, of dimension at least two, acts reducibly on the rational span of the Euclidean lattice associated with the manifold via the first Bieberbach theorem. Geometrically, their result states that such a manifold must admit a nonzero proper parallel distribution with compact leaves. We study algebraic and geometric properties of the sublattice-spanned holonomy-invariant subspaces that exist due to the above theorem, and of the resulting compact-leaf foliations of compact flat manifolds. The class consisting of the former subspaces, in addition to being closed under spans and intersections, also turns out to admit (possibly nonorthogonal) complements. For the latter foliations, we provide descriptions, first, of the intrinsic geometry of their generic leaves in terms of that of the original flat manifold and, secondly, of the leaf-space orbifold, illustrating the general conclusions by examples in the form of generalized Klein bottles.

1. Introduction

As shown by Hiss and Szczepański [7, the corollary in Sect. 1], on any compact flat Riemannian manifold $M$ with $\dim M \geq 2$ there exists a parallel distribution $D$ of dimension $k$, where $0 < k < n$, such that the leaves of $D$ are all compact. Their result, in its original algebraic phrasing (see the Appendix), stated that the holonomy group $H$ of $M$ must acts reducibly on $L \otimes \mathbb{Q}$, for the Euclidean lattice $L$ corresponding to $M$ (that is, the maximal Abelian subgroup of the fundamental group $\Pi$ of $M$).

The present paper explores the algebraic context and geometric consequences of this fact. We view $L$ as an additive subgroup of a Euclidean vector space $V$ (so that $L \otimes \mathbb{Q}$ becomes identified with the rational span of $L$ in $V$), and use the term $L$-subspace when referring to a vector subspace of $V$ spanned by some subset of $L$.

Hiss and Szczepański’s theorem amounts to the existence a nonzero proper $H$-invariant $L$-subspace $V' \subseteq V$. We begin by observing that the class of $H$-invariant $L$-subspaces of $V$ is closed under the span and intersection operations applied to arbitrary subclasses (Lemma 4.3), while every $H$-invariant $L$-subspace of $V$ has an $H$-invariant $L$-subspace complementary to it (Theorem 4.7).

The Bieberbach group of a given compact flat Riemannian manifold $M$ is its fundamental group $\Pi$ treated as the deck transformation group acting via affine isometries on the Euclidean affine space $E$ that constitutes the Riemannian universal covering space of

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\( \mathcal{M} \). The space \( \mathcal{V} \) mentioned above is associated with \( \mathcal{E} \) be being its translation vector space, that is, the space of parallel vector fields on \( \mathcal{E} \), and the roles of the lattice \( L \) and holonomy group \( H \) are summarized by the short exact sequence \( L \to \Pi \to H \). See Section 6. We proceed to describe, in Sections 7–10, the constituents \( L', \Pi', H' \) appearing in the analog \( L' \to \Pi' \to H' \) of this short exact sequence for any (compact, flat) leaf \( \mathcal{M}' \) a parallel distribution \( D \), guaranteed to exist on \( \mathcal{M} \) by the aforementioned result of [7]. Specifically, \( \Pi' \) (or, \( L' \)) may be identified with a subgroup of \( \Pi \) (or, \( \mathcal{V} \)), and \( H \) with a homomorphic image of a subgroup of \( H \). This description becomes particularly simple in the case of leaves \( \mathcal{M}' \) which we call generic (Theorem 10.1): their union is an open dense subset of \( \mathcal{M} \), they all have the same triple \( L', \Pi', H' \), and are mutually isometric. When all leaves of \( D \) happen to be generic, they form a locally trivial bundle with compact flat manifolds serving both as the base and the fibre (the fibration case).

Aside from the holonomy group \( H' \) of each individual leaf \( \mathcal{M}' \) of \( D \), forming a part of its intrinsic (submanifold) geometry, \( \mathcal{M}' \) also gives rise to two “extrinsic” holonomy groups, one arising since \( \mathcal{M}' \) is a leaf of the foliation \( F_M \) of \( \mathcal{M} \) tangent to \( D \), the other coming from the normal connection of \( \mathcal{M}' \). Due to flatness of the normal connection, the two extrinsic holonomy groups coincide, and are trivial for all generic leaves. In Section 11 we briefly discuss the leaf space \( \mathcal{M}/F_M \), noting that

\[
(1.1) \quad \mathcal{M}/F_M \text{ forms a flat compact orbifold, canonically identified with the quotient } [\mathcal{E}/L]/H \text{ of the torus } \mathcal{E}/L \text{ under the isometric action of the finite group } H.
\]

In the fibration case (see above), \( \mathcal{M}/F_M = [\mathcal{E}/L]/H \) is the base manifold of the bundle.

We illustrate the above conclusions by examples (generalized Klein bottles, Section 13), where both the fibration and non-fibration cases occur, depending on the choice of \( D \).

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2. Preliminaries

Given an integer \( n \geq 2 \) and vectors \( e_1, \ldots, e_n \) in a real vector space, one has

\[
(2.1) \quad (e_1 + \ldots + e_n) \land (e_2 - e_1) \land \ldots \land (e_n - e_1) = ne_1 \land \ldots \land e_n.
\]

Namely, denoting the left-hand side by \( \zeta_n \) we see that \( \zeta_2 = 2e_1 \land e_2 \). For the induction step we get \( \zeta_n = \zeta_{n-1} \land (e_n - e_1) + e_n \land (e_2 - e_1) \land \ldots \land (e_n - e_1) \) by writing \( e_1 + \ldots + e_n = (e_1 + \ldots + e_{n-1}) + e_n \), while the last term equals \( e_n \land (e_2 - e_1) \land \ldots \land (e_{n-1} - e_1) \land (-e_1) = e_n \land e_2 \land \ldots \land e_{n-1} \land (-e_1) = e_1 \land \ldots \land e_n \).

Manifolds, mappings and tensor fields, such as bundle and covering projections, submanifold inclusions, and Riemannian metrics, are by definition of class \( C^\infty \). Submanifolds need not carry the subset topology, and a manifold may be disconnected (although, being required to satisfy the second countability axiom, it must have at most countably many
connected components). Connectedness/compactness of a submanifold always refer to its own topology, and imply the same for its underlying set within the ambient manifold. Thus, a compact submanifold is always endowed with the subset topology. By a *distribution* on a manifold $\mathcal{N}$ we mean, as usual, a (smooth) vector subbundle $D$ of the tangent bundle $T\mathcal{N}$. An *integral manifold* of $D$ is any submanifold $L$ of $\mathcal{N}$ with $T_xL = D_x$ for all $x \in L$. The maximal connected integral manifolds of $D$ will also be referred to as the *leaves* of $D$. In the case where $D$ is integrable, its leaves form the foliation associated with $D$. We call $D$ *projectable* under a mapping $\psi : \mathcal{N} \to \mathcal{\hat{N}}$ onto a distribution $\mathcal{\hat{D}}$ on the target manifold $\mathcal{\hat{N}}$ if $d\psi_x(D_x) = \mathcal{\hat{D}}_{\psi(x)}$ whenever $x \in \mathcal{N}$.

**Remark 2.1.** Given manifolds $\mathcal{N}, \mathcal{\hat{N}}$ and a covering projection $\mu : \mathcal{\hat{N}} \to \mathcal{N}$, projectability of a distribution under $\mu$ (onto some distribution on $\mathcal{N}$) is clearly equivalent to its invariance under deck transformations.

**Remark 2.2.** We will use the following well-known facts.

(a) Free diffeomorphic action of finite groups on manifolds are properly discontinuous and thus give rise to covering projections onto the resulting quotient manifolds.

(b) Any locally-diffeomorphic mapping from a compact manifold into a connected manifold is a (surjective) finite covering projection. More generally, the phrases ‘locally-diffeomorphic mapping’ and ‘finite covering projection’ might be replaced here with *submersion* and *fibration*.

**Lemma 2.3.** Let a distribution $\mathcal{\hat{D}}$ on $\mathcal{\hat{M}}$ be projectable, under a locally diffeomorphic surjective mapping $\psi : \mathcal{\hat{M}} \to \mathcal{M}$ between manifolds, onto a distribution $D$ on $\mathcal{M}$.

(i) The $\psi$-image of any leaf of $\mathcal{\hat{D}}$ is a connected integral manifold of $D$.

(ii) Integrability of $\mathcal{\hat{D}}$ implies that of $D$.

(iii) For any compact leaf $\mathcal{L}$ of $\mathcal{\hat{D}}$, the image $\mathcal{L}' = \psi(\mathcal{L})$ is a compact leaf of $D$, and the restriction $\psi : \mathcal{L} \to \mathcal{L}'$ constitutes a covering projection.

(iv) If the leaves of $\mathcal{\hat{D}}$ are all compact, so are those of $D$.

**Proof.** Assertion (i) is immediate from the definitions of a leaf and projectability, while (i) implies (ii) since integrability amounts to the existence of an integral manifold through every point. Remark 2.2(b) combined with (i) yields (iii), and (iv) follows.

**Lemma 2.4.** Suppose that $F$ is a mapping from a manifold $\mathcal{W}$ into any set. If for every $x \in \mathcal{W}$ there exists a diffeomorphic identification of a neighborhood $B_x$ of $x$ in $\mathcal{W}$ with a unit open Euclidean ball centered at 0 under which $x$ corresponds to 0 and $F$ becomes constant on each open straight-line interval of length 1 in the open ball having 0 as an endpoint, then $F$ is locally constant on some open dense subset of $\mathcal{W}$.

**Proof.** We use induction on $n = \dim \mathcal{W}$. The case $n = 1$ being trivial, we now assume the assertion to be valid in dimension $n - 1$ and consider a function $F$ on...
an $n$-dimensional manifold $W$, satisfying our hypothesis, along with an embedded open Euclidean ball $B_x \subseteq M$ “centered” at a given point $x$, as in the statement of the lemma. Due to constancy of $F$ along the fibres of the normalization projection $\mu : B_x \setminus \{x\} \to S$ onto the unit $(n-1)$-sphere $S$, we may view $F$ as a mapping $G$ with the domain $S$. Next, let us fix $y \in B_x \setminus \{x\}$ with an embedded open Euclidean ball $B_y$ “centered” at $y$, such that $F$ is constant on each radial open interval in $B_y$. The obvious submersion property of $\mu$ allows us to pass from $B_y$ to a smaller concentric ball and then choose a codimension-one open Euclidean ball $B'_y$ arising as a union of radial intervals within this new $B_y$, for which $\mu : B'_y \to S$ is an embedding. The assumption of the lemma thus holds with $W$ and $F$ replaced by $S$ and $G$, leading to local constancy of $G$ (and $F$) on a dense open set in $S$ (and, respectively, in $B_x \setminus \{x\}$). The union of the latter sets over all $x$ being obviously dense in $W$, our claim follows. □

Remark 2.5. It is well known that, as a consequence of the inverse mapping theorem combined with the Gauss lemma for submanifolds, given a compact sub manifold $M'$ of a Riemannian manifold $M$ there exists $\delta \in (0, \infty)$ with the following properties.

(a) The normal exponential mapping restricted to the radius $\delta$ open-disk subbundle $N_\delta$ of the normal bundle of $M'$ forms a diffeomorphism $\text{Exp}^\perp : N_\delta \to M_\delta$ onto the open submanifold $M_\delta$ of $M$ equal to the preimage of $[0,\delta)$ under the function $\text{dist}(M', \cdot)$ of metric distance from $M'$.

(b) Every $x \in M_\delta$ has a unique point $y \in M'$ nearest $x$, which is also the unique point of $M'$ joined to $x$ by a geodesic in $M_\delta$ normal to $M'$, and the resulting assignment $M_\delta \ni x \mapsto y \in M'$ coincides with the composite mapping of the inverse diffeomorphism of $\text{Exp}^\perp : N_\delta \to M_\delta$ followed by the normal-bundle projection.

(c) The $\text{Exp}^\perp$ images of length $\delta$ radial line segments in the fibres of $N_\delta$ are precisely the length $\delta$ minimizing geodesic segments in $M_\delta$ emanating from $M'$ as well as normal to the levels of $\text{dist}(M', \cdot)$, and they realize the minimum distance between any two such levels within $M_\delta$.

Remark 2.6. Recall Baire’s theorem [6, p. 187]: in a complete metric space, the intersection of countably many dense open subsets is dense. Equivalently, any countable union of closed sets with empty interiors has an empty interior.

Remark 2.7. We will use the fact that, if a linear endomorphisms of a finite-dimensional real/complex vector space is diagonalizable, then each of its invariant subspaces must be spanned by some subspaces of its eigenspaces.

Remark 2.8. The unique irreducible factorization over $\mathbb{Z}$ of the polynomial $t^n - 1$ in the variable $t$ expresses it as the product of cyclotomic polynomials $\Phi_d$, for all positive divisors $d$ of $n$. The roots of $\Phi_d$ consist of those $d$th roots of unity which are primitive in the sense of not being of order less than $d$ or, equivalently, of arising as powers of $e^{2\pi i/d}$
for positive exponents $k \leq d$, relatively prime with $d$. Consequently, $\deg \Phi_d = \varphi(d)$, where $\varphi$ denotes Euler’s totient function. See [10].

3. Free Abelian groups

The following well-known facts, cf. [1], are gathered here for easy reference.

For a finitely generated Abelian group $G$, being torsion-free amounts to being free, in the sense of having a $\mathbb{Z}$-basis, by which one means an ordered $n$-tuple $e_1, \ldots, e_n$ of elements of $G$ such that every $x \in G$ can be uniquely expressed as an integer combination of $e_1, \ldots, e_n$. The integer $n \geq 0$, also denoted by $\text{dim}_\mathbb{Z} G$, is an algebraic invariant of $G$, called its Betti number or $\mathbb{Z}$-dimension.

Any finitely generated Abelian group $G$ is isomorphic to the direct sum of its (necessarily finite) torsion subgroup $S$ and the free group $G/S$, and we set $\text{dim}_\mathbb{Z} G = \text{dim}_\mathbb{Z}[G/S]$. A subgroup $G'$ (or, a homomorphic image $G'$) of such $G$, in addition to being again finitely generated and Abelian, also satisfies the inequality $\text{dim}_\mathbb{Z} G' \leq \text{dim}_\mathbb{Z} G$, strict unless $G/G'$ is finite (or, respectively, the homomorphism involved has a finite kernel).

Remark 3.1. Given Abelian groups $P, Q$, a surjective homomorphism $\varphi : P \to Q$, and elements $x_j, y_a$ (with $j, a$ ranging over finite sets), such that $x_j$ and $\varphi(y_a)$ form $\mathbb{Z}$-bases of $\ker \varphi$ and, respectively, of $Q$, the system consisting of all $x_j$ and $y_a$ is a $\mathbb{Z}$-basis of $P$. (In fact, one trivially sees that every element of $P$ can be uniquely expressed as an integer combination of $x_j$ and $y_a$.) Consequently,

(i) a subgroup $G'$ of a free Abelian group $G$ constitutes a direct summand of $G$ if and only if the quotient group $G/G'$ is torsion-free.

(ii) Due to (i), for any finitely generated subgroup $G$ of the additive group of a finite-dimensional real vector space $V$, the intersection $G \cap W$ with any vector subspace $W \subseteq V$ forms a direct-summand subgroup of $G$. Also, by (i), the class of direct-summand subgroups of $G$ is closed under intersections (finite or not).

4. Lattices and vector subspaces

Throughout this section $V$ denotes a fixed finite-dimensional real vector space. We call subspaces $V', V''$ of $V$ complementary to each other if $V = V' \oplus V''$.

As usual, a (full) lattice in $V$ is defined to be any subgroup $L$ of the additive group of $V$ generated by a basis of $V$ (which then must also be a $\mathbb{Z}$-basis of $L$).

The quotient $T = V/L$ then is a torus, and we use the term subtori when referring to its compact connected Lie subgroups. From Remark 2.1 applied to the projection $V \to T$, every parallel distribution on $V$ is projectable onto the torus $T = V/L$.

Definition 4.1. Given a lattice $L$ in $V$, by an $L$-subspace of $V$ we will mean any vector subspace $V'$ of $V$ spanned by $L \cap V'$. One may equivalently require $V'$ to be the span of a subset of $L$, rather specifically of $L \cap V'$. 
Remark 4.2. The parallel distribution on $\mathcal{V}$ tangent to any prescribed vector subspace $\mathcal{V}'$ projects, by (4.1), onto a parallel distribution $D$ on the flat affine torus $\mathcal{T} = \mathcal{V}/L$. The leaves of $D$ must be either all compact, or all noncompact, and they are compact if and only if $\mathcal{V}'$ is an $L$-subspace.

Namely, the first claim follows as the leaves are one another’s translation images. For the second, let $\mathcal{N}$ be the leaf of $D$ through zero. Requiring $\mathcal{V}'$ to be (or, not to be) an $L$-subspace makes $L \cap \mathcal{V}'$, by Remark 3.1(ii), a direct-summand subgroup of $L$ or, respectively, yields the existence of a nonzero linear functional $f$ on $\mathcal{V}'$, the kernel of which contains $L \cap \mathcal{V}'$. In the former case, $\mathcal{N}$ is a factor of a product-of-tori decomposition of $\mathcal{T}$, in the latter $f$ descends to an unbounded function on $\mathcal{N}$.

The following lemma will obviously remain valid if one replaces the phrase ‘$L$-subspaces’ with $H$-invariant $L$-subspaces, $H$ being any fixed group of $L$-preserving linear automorphisms of $\mathcal{V}$.

Lemma 4.3. For a lattice $L$ in $\mathcal{V}$, the span of finitely many $L$-subspaces, as well as the intersection of any family of $L$-subspaces, are again $L$-subspaces.

Proof. The assertion about spans follows from the case of two $L$-subspaces, obvious in turn due to the sentence preceding Remark 4.2. Next, the intersection of any family of subtori in $\mathcal{T} = \mathcal{V}/L$ constitutes a compact Lie subgroup of $\mathcal{T}$, so that it is the union of finitely many cosets of a subtorus $\mathcal{N}$. Since subtori are totally geodesic relative to the flat affine connection on $\mathcal{T}$, while the projection $\mathcal{V} \to \mathcal{T}$ is locally diffeomorphic, the tangent space of $\mathcal{N}$ at zero equals the intersection of the tangent spaces of subtori forming the family, and each tangent space corresponds to a vector subspace of $\mathcal{V}$ generating a parallel distribution on $\mathcal{T}$. Our conclusion is now immediate from Remark 4.2. □

Remark 4.4. If a lattice $L$ in $\mathcal{V}$ is generated by a basis $e_1, \ldots, e_n$ of $\mathcal{V}$, the translational action of $L$ on $\mathcal{V}$ has an obvious compact fundamental domain (a compact subset of $\mathcal{V}$ intersecting all orbits of $L$): the parallelepiped $\{t_1 e_1 + \ldots + t_n e_n : t_1, \ldots, t_n \in [0, 1]\}$.

Remark 4.5. We need the well-known fact [3] that

(a) lattices in $\mathcal{V}$ are the same as discrete subgroups of $\mathcal{V}$, spanning $\mathcal{V}$.

Given a lattice $L$ in $\mathcal{V}$ and a vector subspace $\mathcal{V}' \subseteq \mathcal{V}$, let $L' = L \cap \mathcal{V}'$. Then

(b) $L'$ forms a lattice in the vector subspace spanned by it, and

(c) $L'$ constitutes a direct-summand subgroup of $L$,

as one sees using (a) and Remark 3.1(ii).

Lemma 4.6. Let $\mathcal{W}$ be the rational vector subspace of a finite-dimensional real vector space $\mathcal{V}$, spanned by a fixed lattice $L$ in $\mathcal{V}$. The four sets formed, respectively, by

(i) $L$-subspaces $\mathcal{V}'$ of $\mathcal{V}$,

(ii) direct-summand subgroups $L'$ of $L$,
Finally, (iv) follows from Theorem 4.2, and the dimension equalities become obvious if one, again, chooses a \( Z \)-basis of \( L' \) and every rational vector subspace \( V' \), \( W/\mathbb{R}^n = \text{span}_n \) as in Theorem 4.7, and any \( \mathbb{Q} \)-basis of \( L' \), the kernel of which corresponds via Lemma 4.6 to our \( V' \)-subspace \( W \cap V' \). Similarly for (iii) \( \to \) (i) \( \to \) (iii) \( \to \) (i), as long as one replaces the letters \( L \) and \( \mathbb{Z} \) with \( W \) and \( \mathbb{Q} \), using (4.2) and the line following it. Finally, (iv) \( \to \) (i) \( \to \) (iv) \( \to \) (i) are the identity mappings as a consequence of Remark 4.2 and the dimension equalities become obvious if one, again, chooses a \( \mathbb{Z} \)-basis of \( L \) containing a \( \mathbb{Z} \)-basis of \( L' \).

\( \square \)

**Theorem 4.7.** For a lattice \( L \) in a finite-dimensional real vector space \( V \), a finite group \( H \) of \( L \)-preserving linear automorphisms of \( V \), and an \( H \)-invariant \( L \)-subspace \( V' \) of \( V \), there exists an \( H \)-invariant \( L \)-subspace \( V'' \) of \( V \), complementary to \( V' \) in the sense defined above.

**Proof.** Let \( V'' = W \cap V' \), for the rational span \( W \) of \( L \) (see Lemma 4.6). Restricted to \( W \), elements of \( H \) act by conjugation on the rational affine space \( \mathcal{P} \) of all \( \mathbb{Q} \)linear projections \( W \to W' \) (by which we mean linear operators \( W \to W' \) equal to the identity on \( W' \)). Averaging any orbit of the action of \( H \) on \( \mathcal{P} \), we obtain an \( H \)-invariant projection \( W \to W' \), the kernel of which corresponds via Lemma 4.6 to our \( V'' \).

\( \square \)

**Remark 4.8.** Given \( L, V, H, V', V'' \) as in Theorem 4.7 and any \( A \in H \), denoting by \( P, P', P'' \) the characteristic polynomials of \( A \) and, respectively, of the restrictions of \( A \) to \( V' \) and \( V'' \), one has the factorization \( P = P'P'' \), and \( P, P', P'' \) all have integer coefficients.
Namely, the correspondence (i) $\leftrightarrow$ (ii) in Lemma 4.6 allows us to extend a $\mathbb{Z}$-basis of $L' = L \cap V'$ to a $\mathbb{Z}$-basis of $L$ and, as $AL \subseteq L$, the matrices representing, in these bases, $A$ and its restriction to $V'$, have integer entries. (Also, being of finite order, $A$ has the determinant $\pm 1$, and $AL = L$.) The block form of the former matrix gives $P = P'P''$ for our $P, P'$ and the characteristic polynomial $P''$ of the endomorphism of $V'/V'$ induced by $A$, along with integrality of the coefficients of $P$ and $P'$. The $H$-equivariant identification $V'/V' \cong V''$ shows that the last definition of $P''$ agrees with the preceding one. Switching the roles of $V'$ and $V''$ we see that $P''$ has integer coefficients as well.

**Corollary 4.9.** If $L, V, H$ satisfy the hypotheses of Theorem 4.7, every nonzero $H$-invariant $L$-subspace $V_0'$ of $V$ can be decomposed into a direct sum of one or more nonzero $H$-invariant $L$-subspaces, each of which is minimal in the sense of not containing any further nonzero proper $H$-invariant $L$-subspace.

**Proof.** Induction on the possible values of $\dim V_0'$. Assuming the claim true for subspaces of dimensions less than $\dim V_0'$, along with non-minimality of $V_0'$, we fix a nonzero proper $H$-invariant $L$-subspace $V'$ of $V$, contained in $V_0'$, and choose a complement $V''$ of $V'$, guaranteed to exist by Theorem 4.7. Since $V''$ intersects every coset of $V'$ in $V$, including cosets within $V_0'$, the subspace $V_0' \cap V''$ is an $H$-invariant complement of $V'$ in $V_0'$, as well as an $L$-subspace (due to Lemma 4.3). We may now apply the induction assumption to both $V'$ and $V_0' \cap V''$. □

**Remark 4.10.** Given a lattice $L$ in a finite-dimensional real vector space $V$ and an $L$-subspace $V'$ of $V$, the restriction to $L$ of the quotient-space projection $V \to V'/V'$ has the kernel $L' = L \cap V'$, and so it descends to an injective group homomorphism $L/L' \to V/V'$, the image of which is a (full) lattice in an $V/V'$ (which follows if one uses a $\mathbb{Z}$-basis of $L$ containing a $\mathbb{Z}$-basis of $L'$). From now on we will treat $L/L'$ as a subset of $V/V'$. Discreteness of the lattice $L/L' \subseteq V/V'$ clearly implies the existence of an open subset $U'$ of $V$ containing $V'$ and forming a union of cosets of $V'$, such that $L \cap U' = L'$.

5. **Affine spaces**

We denote by $\text{End} \ V$ the space of linear endomorphisms of a given real vector space $V$. Scalars stand for the corresponding multiples of identity, so that the identity itself becomes $1 \in \text{End} \ V$. Given a real affine space $\mathcal{E}$ of any dimension $n$, with the translation vector space $V$, let $\text{Aff} \ \mathcal{E}$ be the group of all affine transformations of $\mathcal{E}$. The inclusion $V \subseteq \text{Aff} \ \mathcal{E}$ expresses the fact that $\text{Aff} \ \mathcal{E}$ contains the normal subgroup consisting of all translations. Any vector subspace $V'$ of $V$ gives rise to a foliation of $\mathcal{E}$, with the leaves formed by affine subspaces $\mathcal{E}'$ parallel to $V'$, that is, orbits of the translational action of $V'$ on $\mathcal{E}$ (which we may also refer to as cosets of $V'$ in $\mathcal{E}$). The resulting leaf (quotient) space $\mathcal{E}/V'$ constitutes an affine space with the translation vector space $V/V'$. 
A fixed inner product in $\mathcal{V}$ turns $\mathcal{E}$ into a Euclidean affine space, with the isometry group $\text{Iso} \mathcal{E} \subseteq \text{Aff} \mathcal{E}$. If $\delta \in (0, \infty)$, we define the $\delta$-neighborhood of an affine subspace $\mathcal{E}'$ of $\mathcal{E}$ to be the set of points in $\mathcal{E}$ lying at distances less that $\delta$ from $\mathcal{E}'$. Clearly, the $\delta$-neighborhood of $\mathcal{E}'$ is a union of cosets of a vector subspace $\mathcal{V}'$ of $\mathcal{V}$ (one of them being $\mathcal{E}'$ itself), as well as the preimage, under the projection $\mathcal{E} \rightarrow \mathcal{E}/\mathcal{V}'$, of the radius $\delta$ open ball centered at the point $\mathcal{E}'$ in the quotient Euclidean affine space $\mathcal{E}/\mathcal{V}'$ (for the obvious inner product on $\mathcal{V}/\mathcal{V}'$).

**Remark 5.1.** For a Euclidean affine space $\mathcal{E}$, and an affine subspace $\mathcal{E}'$ parallel to a vector subspace $\mathcal{V}'$ of the translation vector space $\mathcal{V}$ of $\mathcal{E}$, (affine) self-isometries $\zeta$ of $\mathcal{E}$ leaving $\mathcal{E}'$ invariant and equal to the identity on $\mathcal{E}'$ are in an obvious one-to-one correspondence with linear self-isometries $A$ of the orthogonal complement of $\mathcal{V}'$. In this case $\zeta$ is referred to as an affine extension of $A$, depending on $\mathcal{E}'$.

**Remark 5.2.** Any choice of an origin $o \in \mathcal{E}$ in an affine space $\mathcal{E}$ leads to the obvious identification of $\mathcal{E}$ with its translation vector space $\mathcal{V}$, under which a vector $v \in \mathcal{V}$ corresponds to the point $x = o + v \in \mathcal{E}$. Affine mappings $\gamma \in \text{Aff} \mathcal{E}$ are then represented by pairs $(A,b)$ consisting of $A \in \text{End} \mathcal{V}$ and $b \in \mathcal{E}$, so that $\gamma(o + v) = o + Av + b$. The pair associated in this way with $\gamma$ and a new origin $o + w$ is, obviously, $(A, c)$, for the same $A$ (the linear part of $\gamma$) and $c = b + (A - 1)w$. Thus, the coset $b + \mathcal{V} \subseteq \mathcal{V}$, where $\mathcal{V}$ denotes the image of $A - 1$, forms an invariant of $\gamma$ (while $b$ itself does not, except in the case of translations $\gamma$, having $A = 1$). For any fixed vector subspace $\mathcal{V}'$ of $\mathcal{V}$ and any $\gamma \in \text{Aff} \mathcal{E}$ with a linear part $A$ leaving $\mathcal{V}'$ invariant, it now makes sense to require that $A$ descend to the identity transformation of $\mathcal{V}/\mathcal{V}'$ (i.e., $(A - 1)(\mathcal{V}) \subseteq \mathcal{V}'$) and, simultaneously, the “translational part” $b$ of $\gamma$ lie in $\mathcal{V}'$. More precisely, such a property of $\gamma$ does not depend on the origin used to represent $\gamma$ as a pair $(A,b)$.

**Remark 5.3.** Given $\mathcal{E}, \mathcal{V}$ and $\mathcal{V}'$ as in Remark 5.2, the affine transformations $\gamma$ of $\mathcal{E}$ with linear parts leaving $\mathcal{V}'$ invariant and descending to the identity transformation of $\mathcal{V}/\mathcal{V}'$ obviously form a subgroup of $\text{Aff} \mathcal{E}$ containing, as a normal subgroup, the set of such $\gamma$ which have “translational parts” in $\mathcal{V}'$. This follows since the latter set is the kernel of the obvious homomorphism from the original subgroup into $\text{Aff} \mathcal{E}/\mathcal{V}'$. In fact, it is clear that $\gamma$ represented by the pair $(A,b)$ (see Remark 5.2) preserves each element of $\mathcal{E}/\mathcal{V}'$ if and only if $Av + b$ differs from $v$, for every $v \in \mathcal{V}$, by an element $\mathcal{V}'$ or, equivalently (as one sees setting $v = 0$), $\mathcal{V}'$ contains both $b$ and the image of $A - 1$.

**Lemma 5.4.** Remark 2.3 has the following additional conclusions when $\mathcal{M}'$ is a compact leaf of a parallel distribution $\mathcal{D}$ on a complete flat Riemannian manifold $\mathcal{M}$.

(a) Every level of $\text{dist}(\mathcal{M}', \cdot)$ in $\mathcal{M}_0$, and $\mathcal{M}_0$ itself, is a union of leaves of $\mathcal{D}$.

(b) Restrictions of $\mathcal{M}_0 \ni x \mapsto y \in \mathcal{M}'$ to leaves of $\mathcal{D}$ in $\mathcal{M}_0$ are locally isometric.

(c) The local inverses of all the above locally-isometric restrictions correspond via the diffeomorphism $\text{Exp}^\perp$ to all local sections the normal bundle of $\mathcal{M}'$ obtained by restricting
to $\mathcal{M}'$ local parallel vector fields of lengths $r \in [0, \delta)$ that are tangent to $\mathcal{M}$ and normal to $\mathcal{M}'$, with $r$ equal to the value of $\text{dist}(\mathcal{M}', \cdot)$ on the leaf.

This trivially follows from the fact the pullback of $D$ to the Euclidean affine space $\mathcal{E}$ constituting the Riemannian universal covering space of $\mathcal{M}$ is a distribution with the leaves provided by affine subspaces parallel to $\mathcal{V}'$, for some vector subspace $\mathcal{V}'$ of the translation vector space $\mathcal{V}$ of $\mathcal{E}$.

6. Bieberbach groups and flat manifolds

Let $\mathcal{E}$ be a Euclidean affine $n$-space (Section 5), with the translation vector space $\mathcal{V}$. By a Bieberbach group in $\mathcal{E}$ one means any torsion-free discrete subgroup $\Pi$ of $\text{Iso} \mathcal{E}$ for which there exists a compact fundamental domain (Remark 4.4). The lattice subgroup $L$ of $\Pi$, and its holonomy group $H \subseteq \text{Iso} \mathcal{V} \cong O(n)$ then are defined by

$$L = \Pi \cap \mathcal{V}, \quad H = \lambda(\Pi),$$

$\lambda : \text{Aff} \mathcal{E} \to \text{Aut} \mathcal{V} \cong \text{GL}(n, \mathbb{R})$ being the linear-part homomorphism. Thus, $L$ is the set of all translations lying in $\Pi$ (which also makes it the kernel of the restriction $\lambda : \Pi \to H$), and $H$ consists of the linear parts of elements of $\Pi$. Note that $L \subseteq \mathcal{V}$ is a (full) lattice in the usual sense, cf. Section 4. The relations involving $\Pi, L$ and $H$ are conveniently summarized by the short exact sequence

$$L \to \Pi \to H,$$

where the arrows are the inclusion homomorphism and $\lambda$.

Remark 6.1. The action of a Bieberbach group $\Pi$ on the Euclidean affine space $\mathcal{E}$ being always free and properly discontinuous, the quotient $\mathcal{M} = \mathcal{E}/\Pi$, with the projected metric, forms a compact flat Riemannian manifold, while $H$ must be finite [3].

Remark 6.2. As the normal subgroup $L$ of $\Pi$ is Abelian, the action of $\Pi$ on $L$ by conjugation descends to an action on $L$ of the quotient group $\Pi/L$, identified via (6.2) with $H$. This last action is clearly nothing else than the ordinary linear action of $H$ on $\mathcal{V}$, restricted to the lattice $L \subseteq \mathcal{V}$ (and so, in particular, $L$ must be $H$-invariant).

Remark 6.3. The assignment of $\mathcal{M} = \mathcal{E}/\Pi$ to $\Pi$ establishes a well-known bijective correspondence [8] between equivalence classes of Bieberbach groups and isometry types of compact flat Riemannian manifolds. Bieberbach groups $\Pi$ and $\tilde{\Pi}$ in Euclidean affine spaces $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are called equivalent here if some affine isometry $\mathcal{E} \to \tilde{\mathcal{E}}$ conjugates $\Pi$ onto $\tilde{\Pi}$. Furthermore, $\Pi$ and $H$ in (6.2) serve as the fundamental and holonomy groups of $\mathcal{M}$, while $\Pi$ also acts via deck transformations on the Riemannian universal covering space of $\mathcal{M}$, isometrically identified with $\mathcal{E}$. 
7. Lattice-reducibility

A Bieberbach group $\Pi$ in a Euclidean affine space $E$ (or, the compact flat Riemannian manifold $M = E/\Pi$ corresponding to $\Pi$, cf. Remark 6.3) will be called lattice-reducible if, for $V, H$ and $L$ associated with $E$ and $\Pi$ as in Section 6, there exists $V'$ such that

$$V' \text{ is a nonzero proper } H\text{-invariant } L\text{-subspace of } V.$$  \hfill (7.1)

To emphasize the role of $V'$ in (7.1), we also say that

$$\text{the lattice-reducibility condition (7.1) holds for the quadruple } (V, H, L, V').$$  \hfill (7.2)

As shown by Hiss and Szczepański [7], every compact flat Riemannian manifold of dimension greater than one is lattice-reducible. For more details, see the Appendix.

Under the assumption (7.2) with fixed $V'$, where $\Pi$ is a given Bieberbach group in a Euclidean affine space $E$, we denote by $\sigma'$ the stabilizer subgroup of $E'$ relative to the action of $\Pi$, meaning that

$$\sigma' \text{ consists of all the elements of } \Pi \text{ mapping } E' \text{ into itself},$$  \hfill (7.3)

**Theorem 7.1.** Given a lattice-reducible Bieberbach group $\Pi$ in a Euclidean affine space $E$ and a vector subspace $V'$ of $V$ with (7.2), the following three conclusions hold.

(i) The affine subspaces of dimension $\dim V'$ in $E$, parallel to $V'$, are the leaves of a foliation $F_E$ on $E$, projectable under the covering projection $pr : E \to M = E/\Pi$ onto a foliation $F_M$ of $M$ with compact totally geodesic leaves, tangent to a parallel distribution on $M$.

(ii) The leaves $M'$ of $F_M$ coincide with the $pr$-images of leaves $E'$ of $F_E$, and the restrictions $pr : E' \to M'$ are covering projections. Any such $M'$, being a compact flat Riemannian manifold, corresponds via Remark 6.3 to a Bieberbach group $\Pi'$ in the Euclidean affine space $E'$. For $L, H$ appearing in the analog $L' \to \Pi' \to H'$ of (6.2), this $\Pi'$, and $\sigma'$ defined by (7.3),

(a) $\Pi'$ consists of the restrictions to $E'$ of all the elements of $\sigma'$,
(b) $H'$ is formed by the restrictions to $V'$ of the linear parts of elements of $\sigma'$,
(c) $L' = \Pi' \cap V'$, as in (6.1), and $L \cap V' \subseteq L'$.

For a proof of Theorem 7.1 see the next section.

**Remark 7.2.** The restriction homomorphism $\sigma' \to \Pi'$ implicitly mentioned in (ii-a) above is an isomorphism: nontrivial elements of $\sigma'$, being fixed-point free (Remark 6.1), have nontrivial restrictions to $E'$. The last inclusion of (ii-c) may be proper (Remark 13.2).
8. Proof of Theorem 7.1

Projectability of the foliation $\mathcal{F}_E$ under the covering projection $\text{pr} : \mathcal{E} \to \mathcal{M} = \mathcal{E}/\Pi$ follows as a trivial consequence of the fact that, due to $H$-invariance of $\mathcal{V}'$,

\[(8.1) \quad \mathcal{F}_E \text{ is } \Pi\text{-invariant,}\]

while Lemma 2.3(ii) implies integrability of the image distribution. Next,

\[(8.2) \quad \text{pr is the composite } \mathcal{E} \to \mathcal{E}/L \to \mathcal{M} \text{ of two mappings:}\]

\[(8.3) \quad \text{pr} : \mathcal{E}' \to \mathcal{M}' \text{ is a (surjective) covering projection,}\]

since (8.2) decomposes $\text{pr} : \mathcal{E}' \to \mathcal{M}'$ into the composition $\mathcal{E}' \to \mathcal{E}'/L' \to \mathcal{M}'$, in which the first mapping is the universal-covering projection of the torus $\mathcal{E}'/L'$, and the second one must be a covering due to Remark 2.2(b).

In view of (8.1), two points of $\mathcal{E}'$ have the same $\text{pr}$-image if and only if one is transformed into the other by an element of the group $\Pi'$ described in assertion (ii). (Specifically, the ‘only if’ part follows since, given $x, y \in \mathcal{E}'$ with $\text{pr}(x) = \text{pr}(y)$ in $\mathcal{M} = \mathcal{E}/\Pi$, the element of $\Pi$ sending $x$ to $y$ must lie in $\Pi'$, or else it would map $\mathcal{E}'$ onto a different leaf of the foliation $\mathcal{F}_E$.) Furthermore, the action of $\Pi'$ on $\mathcal{E}'$ is free and properly discontinuous according to Remarks 7.2 and 6.1. Thus, $\Pi'$ coincides with the deck transformation group for the universal-covering projection (8.3). Now (ii) is a consequence of Remark 6.3 and the definitions of the data (6.2) for any Bieberbach group $\Pi$, applied to our $\Pi'$.

9. Geometries of individual leaves

Throughout this section we adopt the assumptions and notations of Theorem 7.1. The $\Pi$-invariance of the foliation $\mathcal{F}_E$, cf. (8.1), trivially gives rise to

\[(9.1) \quad \text{the obvious isometric action of } \Pi \text{ on the quotient Euclidean affine space } \mathcal{E}/\mathcal{V}'\]

(that is, on the leaf space of $\mathcal{F}_E$, the points of which coincide with the affine subspaces $\mathcal{E}'$ of $\mathcal{E}$ parallel to $\mathcal{V}'$). Whenever $\mathcal{E}' \in \mathcal{E}/\mathcal{V}'$ is fixed, $\sigma'$ in (17.3) obviously coincides with the isotropy group of $\mathcal{E}'$ for (9.1). The action (9.1) is not effective, as the kernel of the corresponding homomorphism $\Pi \to \text{Iso} \mathcal{E}/\mathcal{V}'$ clearly contains the group $L' = \ldots$
$L \cap \mathcal{V}'$ forming a lattice in $\mathcal{V}'$, cf. Definition 1.1 and Remark 2.5(b). The final clause of Remark 5.2 combined with $H$-invariance of $\mathcal{V}'$, shows that $L'$ is a normal subgroup of $\Pi$, which leads to a further homomorphism $\Pi/L' \to \text{Iso} \mathcal{E}/\mathcal{V}'$ (still in general noninjective; see Remark 10.2 below).

**Remark 9.1.** Given $\mathcal{E}' \in \mathcal{E}/\mathcal{V}'$ and a vector $v \in \mathcal{V}$ orthogonal to $\mathcal{V}'$, let us set $\mathcal{M}'_o = \text{pr}(\mathcal{E}' + v)$, so that $\mathcal{M}'_o = \mathcal{M}'$, cf. (8.3). By (8.3), $\mathcal{M}'_o$ must be a (compact) leaf of $F_{\mathcal{M}'}$, and $\text{pr} : \mathcal{E}' + v \to \mathcal{M}'_o$ is a locally-isometric universal-covering projection. Also, we choose $\delta$ as in Remark 2.5 and Lemma 5.1 for the submanifold

$$\mathcal{M}' = \text{pr}(\mathcal{E}')$$

of the compact flat manifold $\mathcal{M} = \mathcal{E}/\Pi$, and denote by $\sigma'_v \subseteq \Pi$ the isotropy group of $\mathcal{E}' + v$, as in (7.3).

**Lemma 9.2.** Under the above hypotheses, for any $\mathcal{E}' \in \mathcal{E}/\mathcal{V}'$ there exists $\delta \in (0, \infty)$ such that, whenever $u \in \mathcal{V}$ is a unit vector orthogonal to $\mathcal{V}'$ and $r, s \in (0, \delta)$, the isometries $\mathcal{E}' + ru \to \mathcal{E}' + su$ and $\mathcal{E}' + ru \to \mathcal{E}'$ of translations by the vectors $(s - r)u$ and, respectively, $-ru$, descend under the universal-covering projections of Remark 9.1 with $u$ equal to $ru$, $su$ or $0$, to an isometry $\mathcal{M}'_r \to \mathcal{M}'_s$ or, respectively, a $k$-fold covering projection $\mathcal{M}'_{ru} \to \mathcal{M}'$, where the integer $k = k(u) \geq 1$ may depend on $u$, but not on $r$.

**Proof.** With the choice of $\delta$ made in (9.2), on the level of pr-images, under the $\text{Exp}^+$-diffeomorphic identification of Remark 2.5(a), our isometries $\mathcal{E}' + ru \to \mathcal{E}' + su$ and $\mathcal{E}' + ru \to \mathcal{E}'$ become multiplications of vectors normal to $\mathcal{M}'$ by the scalars $s/r$ and 0, and so they are well-defined, that is, descend – by (8.3) – to locally-isometric mappings $\mathcal{M}'_r \to \mathcal{M}'_s$ and $\mathcal{M}'_r \to \mathcal{M}'$, which also constitute finite coverings (Remark 2.2(b)). The former must in addition be bijective, with the inverse arising when one switches $r$ and $s$. As the composite $\mathcal{M}'_s \to \mathcal{M}'_r \to \mathcal{M}'$ clearly equals the analogous covering projection $\mathcal{M}'_{ru} \to \mathcal{M}'$ (with $s$ rather than $r$), the coverings $\mathcal{M}'_{ru} \to \mathcal{M}'$ and $\mathcal{M}'_s \to \mathcal{M}'$ have the same multiplicity, which completes the proof. $\square$

**Remark 9.3.** Replacing $\delta$ of (9.2) with 1/4 times its original value, we can also require it to have the following property: if $\gamma \in \Pi$ and $x \in \mathcal{E}$ are such that both $x$ and $\gamma(x)$ lie in the $\delta$-neighborhood of $\mathcal{E}'$, cf. Section 5 then $\gamma \in \sigma'$ for the stabilizer group $\sigma'$ of $\mathcal{E}'$ defined by (7.3). In fact, letting $\mathcal{E}''$ be the leaf of $F_\mathcal{E}$ through $x$, we see from (8.1) that its $\gamma$-image $\gamma(\mathcal{E}'')$ is also a leaf of $F_\mathcal{E}$, while both leaves are within the distance $\delta$ from $\mathcal{E}'$, which gives $\text{dist}(\mathcal{E}'', \gamma(\mathcal{E}'')) < 2\delta$ and so, due to the triangle inequality, $\text{dist}(\mathcal{E}', \gamma(\mathcal{E}'')) \leq \text{dist}(\mathcal{E}', \mathcal{E}'') + \text{dist}(\mathcal{E}''', \gamma(\mathcal{E}'')) < \delta + 2\delta + \delta = 4\delta$. Thus, $x + ru \in \gamma(\mathcal{E}')$ for some $x \in \mathcal{E}'$, some unit vector $u \in \mathcal{V}$ orthogonal to $\mathcal{V}'$, and $r = \text{dist}(\mathcal{E}', \gamma(\mathcal{E}')) \in [0, 4\delta)$. Assuming now (9.2) with $\delta$ replaced by $4\delta$, one gets $r = 0$, that is, $\gamma(\mathcal{E}') = \mathcal{E}'$ and $\gamma \in \sigma'$. Namely, the pr-image of the curve $[0, 4\delta) \ni t \mapsto x + tu$ is a geodesic in the image of the diffeomorphism $\text{Exp}^+$ of Remark 2.5(a), which intersects $\mathcal{M}'$ only at $t = 0$, while $\mathcal{M}' = \text{pr}(\mathcal{E}') = \text{pr}(\gamma(\mathcal{E}'))$, since $\mathcal{M} = \mathcal{E}/\Pi$. 


Lemma 9.4. Let there be given $\mathcal{E}'$ as in Lemma 9.2, $\delta$ having the additional property of Remark 9.3 any $r \in (0, \delta)$, and any unit vector $u \in \mathcal{V}$ orthogonal to $\mathcal{V}'$.

(a) The isotropy group $\sigma'_r$, cf. (9.2), does not depend on $r \in (0, \delta)$.
(b) The linear part of each element of $\sigma'_r$ keeps $u$ fixed.
(c) $\sigma'_r$ is a subgroup of $\sigma'_0$ with the finite index $k = k(u) \geq 1$ of Lemma 9.2.
(d) $\text{pr} : \mathcal{E} \to \mathcal{M}$ maps the $\delta$-neighborhood $\mathcal{E}_\delta$ of $\mathcal{E}'$ in $\mathcal{E}$ onto $\mathcal{M}_\delta$ of Remark 2.5(a).
(e) $\mathcal{E}_\delta$ and $\mathcal{M}_\delta$ are unions of leaves of, respectively, $F_\mathcal{E}$ and $F_{\mathcal{M}}$.
(f) The preimage under $\text{pr} : \mathcal{E}_\delta \to \mathcal{M}_\delta$ of the leaf $\mathcal{M}'_{ru} = \text{pr}(\mathcal{E}' + ru)$ of $F_{\mathcal{M}}$ equals the union of the orbit of $\mathcal{E}' + ru$ for the action of $\sigma'_0$ on the leaf space of $F_\mathcal{E}$.

Proof. By (8.3) and (9.2), $\mathcal{M}'_r = (\mathcal{E}' + v)/\Pi'_r$, if one lets $\Pi'_r$ denote the image of $\sigma'_r$ under the injective homomorphism of restriction to $\mathcal{E}'$, cf. Remark 7.2. Fixing $s \in [0, \delta)$ and $r \in (0, \delta)$ we therefore conclude from Lemma 9.2 that, whenever $x \in \mathcal{E}' + ru$ and $\gamma \in \sigma'_ru$, there exists $\hat{\gamma} \in \sigma'_su$ satisfying the condition
\begin{equation}
(9.3) \quad \gamma(x) + v = \hat{\gamma}(x + v), \quad \text{where } v = (s - r)u, \quad \text{and } \hat{\gamma} = \gamma \text{ when } s = r,
\end{equation}
the last clause being obvious since $\gamma, \hat{\gamma} \in \Pi$ and the action of $\Pi$ is free. With $u$ and $\gamma$ fixed as well, for each given $\hat{\gamma} \in \sigma'_su$ the set of all $x \in \mathcal{E}' + ru$ having the property (9.3) is closed in $\mathcal{E}' + ru$ while, as we just saw, the union of these sets over all $\hat{\gamma} \in \sigma'_su$ equals $\mathcal{E}' + ru$. Thus, by Baire’s theorem (Remark 2.6), some $\hat{\gamma} \in \sigma'_su$ satisfies (9.3) with all $x$ from some nonempty open subset of $\mathcal{E}' + ru$, and hence $- \text{ by real-analyticity$- for all } x \in \mathcal{E}' + ru$. In terms of the translation $\tau_v$ by the vector $v$, we consequently have $\hat{\gamma} = \tau_v \circ \gamma \circ \tau_v^{-1}$ on $\mathcal{E}' + su$, so that $\gamma$ uniquely determines $\hat{\gamma}$ (due to the injectivity claim of Remark 7.2), the assignment $\gamma \mapsto \hat{\gamma}$ is a homomorphism $\sigma'_ru \to \sigma'_su \subseteq \Pi$, and $\zeta = \hat{\gamma} \circ \tau_v \circ \gamma^{-1} \circ \tau_v^{-1}$ equals the identity on $\mathcal{E}' + su$. If we now allow $s$ to vary from $r$ to $0$, the resulting curve $s \mapsto \zeta$ consists, due to Remark 5.4 of affine extensions of linear self-isometries of the orthogonal complement of $\mathcal{V}'$, and $\hat{\gamma} = \zeta \circ \tau_v \circ \gamma \circ \tau_v^{-1}$ on $\mathcal{E}$. As $\Pi$ is discrete, the curve $s \mapsto \hat{\gamma} \in \Pi$, with $v = (s - r)u$, must be constant, and can be evaluated by setting $s = r$ (or, $v = 0$). Thus, $\hat{\gamma} = \gamma$ on $\mathcal{E}$ from the last clause of (9.3), and so $\sigma'_ru \subseteq \sigma'_su$. For $s > 0$, switching $r$ with $s$ we get the opposite inclusion, and (a) follows. Also, taking the linear part of the resulting relation $\gamma = \zeta \circ \tau_v \circ \gamma \circ \tau_v^{-1}$, we see that $\zeta$ equals the identity, for all $s$. Hence $\gamma = \tau_v \circ \gamma \circ \tau_v^{-1}$ commutes with $\tau_v$, which amounts to (b). Setting $s = 0$, we obtain the first part of (c): $\sigma'_ru \subseteq \sigma'_0$. Assertion (d) is clear as $\text{pr}$, being locally isometric, maps line segments onto geodesic segments. Lemma 5.4(a) for $D = F_{\mathcal{M}}$ and (8.3) yield (e). Since $\text{pr} : \mathcal{E} \to \mathcal{M} = \mathcal{E}/\Pi$, the additional property of $\delta$ (see Remark 9.3) implies (f). Finally, for $k = k(u)$, the geodesic $[0, r] \ni t \mapsto \text{pr}(x + tu)$, normal to $\mathcal{M}'$ at $y = \text{pr}(x)$, is one of $k$ such geodesics $[0, r] \ni t \mapsto \text{pr}(x + tw)$, joining $y$ to points of its preimage under the projection $\mathcal{M}'_{ru} \to \mathcal{M}'$ of Lemma 9.2 where $w$ ranges over a $k$-element set $\mathcal{R}$ of unit vectors in $\mathcal{V}$, orthogonal to $\mathcal{V}'$. The union of the corresponding subset $C = \{\mathcal{E}' + rw : w \in \mathcal{R}\}$ of the leaf space of $F_\mathcal{E}$ equals the preimage
in (f) – and hence an orbit for the action of $\sigma'_0$ – as every leaf in the preimage contains a point nearest $x$. Due to the already-established inclusion $\sigma'_0 \subseteq \sigma'_0$ and (6.2), $\sigma''_r$ is the isotropy group of $E'$ relative to the transitive action of $\sigma'_0$ on $C$, and so $k$, the cardinality of $C$, equals the index of $\sigma''_r$ in $\sigma'_0$, which proves the second part of (c). □

10. The generic isotropy group

Given a Bieberbach group $\Pi$ in a Euclidean affine space $E$, let us fix a vector subspace $V'$ of $V$ satisfying (7.2). As long as $\dim E \geq 2$, such $V'$ always exists (Section 7). An element $E'$ of $E/V'$, that is, a coset of $V'$ in $E$, will be called generic if its stabilizer (isotropy) subgroup $\sigma' \subseteq \Pi$, defined by (7.3), equals

$$\text{(10.1) the kernel of the homomorphism } \Pi \rightarrow \text{Iso} E/V' \text{ corresponding to (9.1).}$$

Still using the symbols pr, $L$ and $H$ for the universal-covering projection $E \rightarrow M = E/\Pi$ and the groups appearing in (6.1) – (6.2), we let $K' \subseteq H$ and $U' \subseteq E/V'$ denote the normal subgroup consisting of all elements of $H$ that act on the orthogonal complement of $V'$ as the identity, and the set of all generic elements of $E/V'$.

**Theorem 10.1.** The above assumptions yield the following conclusions.

(i) $U'$ constitutes an open dense subset of $E/V'$.

(ii) The normal subgroup of $\Pi$ is contained as a finite-index subgroup in the isotropy group of every $E' \in E/V'$ for the action (9.1), and equal to this isotropy group if $E' \in U'$.

(iii) The the pr-images $M', M''$ of any $E', E'' \in U'$ are isometric to each other.

(iv) If one identifies $E$ with its translation vector space $V$ via a choice of an origin, $\sigma'$ becomes the set of all elements of $\Pi$ having the form

$$\text{(10.2) } V \ni x \mapsto Ax + b \in V, \text{ in which } b \in V' \text{ and the linear part } A \text{ lies in } K'.$$

(v) Whenever $E' \in U'$, the homomorphism which restricts elements of the generic isotropy group $\sigma'$ to $E'$ is injective, and the resulting isomorphic image $\Pi'$ of $\sigma'$ constitutes a Bieberbach group in the Euclidean affine space $E'$. The lattice subgroup $K'$ and its holonomy group $H'$ are the intersection $L' = L \cap V'$ and the image $H'$ of the group $K'$ appearing in (10.2) under the injective homomorphism of restriction to $V'$.

**Proof.** Lemma 9.4(a) states that the assumptions of Lemma 2.4 are satisfied by the Euclidean affine space $W = E/V'$ and the mapping $F$ that sends $E' \in E/V'$ to its isotropy group $\sigma'$ with (7.3). The assignment $E' \mapsto \sigma'$ is thus locally constant on some open dense set $U' \subseteq E/V'$. Letting $\sigma'$ be the constant value of this assignment assumed on a nonempty connected open subset $W'$ of $U'$, and fixing $\gamma \in \sigma'$, we obtain $\gamma(E') = E'$ for all $E' \in W'$, and hence, from real-analyticity, for all $E' \in E/V'$. Thus, our $\sigma'$ is contained in the isotropy group of every $E' \in E/V'$. Since the same applies also to another constant value $\sigma''$ assumed on a nonempty connected open set, $\sigma'' = \sigma'$ and the phrase ‘locally
constant’ may be replaced with constant. By Lemma 9.4(c), such \( \sigma' \) must be a finite-index subgroup of each isotropy group. As \( \sigma' \) consists of the elements of \( \Pi \) preserving every \( E' \in \mathcal{U}' \), real-analyticity implies that they preserve all \( E' \in \mathcal{E}' / \mathcal{V}' \), and so \( \sigma' \) coincides with \( (10.1) \), which also shows that \( \sigma' \) is a normal subgroup of \( \Pi \), and (i) – (ii) follow.

Assertions (iv) – (v) are in turn immediate from Remark 5.3 and, respectively, Theorem 7.1 combined with Remark 7.2 while (v) implies (iii) via Remark 6.3. □

Remark 10.2. An element of \( \Pi \) acting trivially on \( \mathcal{E}' \) need not lie in \( L' \). An example arises when the compact flat manifold \( \mathcal{M} = \mathcal{E} / \Pi \) is a Riemannian product \( \mathcal{M} = \mathcal{M}' \times \mathcal{M}'' \) with \( \mathcal{E} = \mathcal{E}' \times \mathcal{E}'' \) and \( \Pi = \Pi' \times \Pi'' \) for two Bieberbach groups \( \Pi' \), \( \Pi'' \) in Euclidean affine spaces \( \mathcal{E}' \), \( \mathcal{E}'' \) having the translation vector spaces \( \mathcal{V}' \), \( \mathcal{V}'' \), while \( \mathcal{M}' \) is not a torus. The \( H \)-invariant subspace \( \mathcal{V}' \times \{ 0 \} \) then gives rise to the \( \mathcal{M} \)' factor foliation \( F_\mathcal{M} \) of the product manifold \( \mathcal{M} \), and the action of the group \( \Pi' \times \{ 1 \} \) on its leaf space is obviously trivial, even though \( \Pi' \times \{ 1 \} \) contains some elements that are not translations.

11. The leaf space

By a crystallographic group \( [11] \) in a Euclidean affine space one means a discrete group of isometries having a compact fundamental domain, cf. Remark 4.4.

Proposition 11.1. Under the assumptions listed at the beginning of Section 10 with \( \sigma' \) denoting the normal subgroup \( (10.1) \) of \( \Pi \), the quotient group \( \Pi / \sigma' \) acts effectively by isometries on the quotient Euclidean affine space \( \mathcal{E} / \mathcal{V}' \) and, when identified a subgroup of \( \text{Iso} \mathcal{E} / \mathcal{V}' \), it constitutes a crystallographic group.

Proof. A compact fundamental domain exists by Remark 4.4 since, according to Remark 4.10, \( \Pi / \sigma' \) contains the lattice subgroup \( L / L' \) of \( \mathcal{E} / \mathcal{V}' \). To verify discreteness of \( \Pi / \sigma' \), suppose that, on the contrary, some sequence \( \gamma_k \in \Pi, k = 1, 2 \ldots \), has terms representing mutually distinct elements of \( \Pi / \sigma' \) which converge in \( \text{Iso} \mathcal{E} / \mathcal{V}' \). As \( L' \) is a lattice in \( \mathcal{V}' \), fixing \( x \in \mathcal{E} \) and suitably choosing \( v_k \in L' \) we achieve boundedness of the sequence \( \tilde{\gamma}_k(x) = \gamma_k(x) + v_k \), while \( \tilde{\gamma}_k \) represent the same (distinct) elements of \( \Pi / \sigma' \) as \( \gamma_k \). The ensuing convergence of a subsequence of \( \tilde{\gamma}_k \) contradicts discreteness of \( \Pi \). □

The resulting quotient of \( \mathcal{E} / \mathcal{V}' \) under the action of \( \Pi / \sigma' \) is thus a flat compact orbifold \( [4] \), which may clearly be identified both with the leaf space \( \mathcal{M} / \mathcal{F}_\mathcal{M} \), and with the quotient \( [\mathcal{E} / L] / H \) mentioned in \( [11] \). The latter identification clearly implies the Hausdorff property the leaf space \( \mathcal{M} / \mathcal{F}_\mathcal{M} \).

On the other hand, for an \( H \)-invariant subspace \( \mathcal{V}' \) of \( \mathcal{V} \) not assumed to be an \( L \)-subspace, there exists an \( L \)-closure of \( \mathcal{V}'' \), meaning the smallest \( L \)-subspace \( \mathcal{V}' \) of \( \mathcal{V} \) containing \( \mathcal{V}'' \), which is obviously obtained by intersecting all such \( L \)-subspaces (Lemma 4.3). The leaf space \( \mathcal{M} / \mathcal{F}_\mathcal{M} \) corresponding to \( \mathcal{V} \) then forms a natural “Hausdorffization” of the leaf space of \( \mathcal{V}'' \), and may also be described in terms of Hausdorff-Gromov limits. See the forthcoming paper \([2]\).
12. Examples

For the linear operator \( A : \mathbb{R}^n \to \mathbb{R}^n, n \geq 2 \), acting on the standard basis \( e_1, \ldots, e_n \) via \( A e_j = e_{j-1} \) when \( j > 1 \), and \( A e_1 = e_n \), each of the two subspaces: \( \text{span}_{\mathbb{R}} u \), where \( u = (1, \ldots, 1) \), and its orthogonal complement \( u^\perp \), is the only \( A \)-invariant subspace complementary to the other in the sense of Section 4. In fact, with \( \mathbb{R}^n \subseteq \mathbb{C}^n \) and \( A \) extended complex-linearly to \( \mathbb{C}^n \), we have

\[
Au_q = qu_q \quad \text{for} \quad u_q = qe_1 + q^2e_2 + \ldots + q^{n-1}e_{n-1} + e_n \quad \text{and any} \quad n \text{th root of unity} \quad q \notin \mathbb{C}, \quad \text{such that} \quad q \text{ being the (simple) complex eigenvalues of} \; A,
\]

and so the characteristic polynomial of \( A \) is given by \( P(t) = 1 - t^n \).

This makes \( \text{span}_{\mathbb{R}} u = \text{span}_{\mathbb{R}} u_1 \) the only one-dimensional \( A \)-invariant subspace. On the other hand, any \( A \)-invariant subspace complementary to \( \text{span}_{\mathbb{R}} u \) has an orthogonal complement equal, by its \( A \)-invariance, to \( \text{span}_{\mathbb{R}} u \).

As in Theorem 4.7, let us consider a lattice \( L \) in a finite-dimensional real vector space \( V \) and two \( L \)-subspaces \( \mathcal{V}, \mathcal{V}'' \) of \( V \) complementary to each other. We define their \textit{L-intersection-index} \( \text{ind}_L(\mathcal{V}, \mathcal{V}'') \) to be the index in \( L \) of the subgroup generated by the union of \( L' = L \cap \mathcal{V}' \) and \( L'' = L \cap \mathcal{V}'' \). Thus, \( \text{ind}_L(\mathcal{V}, \mathcal{V}'') \) is a positive integer (see the lines preceding Remark 3.1), and \( \text{ind}_L(\mathcal{V}, \mathcal{V}'') = 1 \) if and only if \( L = L' \oplus L'' \).

For \( \mathcal{V}' \) satisfying the hypotheses of Theorem 4.7, there need not exist an \( H \)-invariant complementary \( L \)-subspace \( \mathcal{V}'' \) with \( \text{ind}_L(\mathcal{V}', \mathcal{V}'') = 1 \). Namely, let \( H \cong \mathbb{Z}_n \) be generated by \( A \) defined above, so that each of the two subspaces, consisting of all \( x_1e_1 + \ldots + x_ne_n \) such that \( x_1 = \ldots = x_n \) or, respectively, \( x_1 + \ldots + x_n = 0 \), is the only \( H \)-invariant subspace complementary to the other. Both \( \mathcal{V}', \mathcal{V}'' \) are \( L \)-subspaces, with the \( \mathbb{Z} \)-bases \( e_1 + \ldots + e_n \) for \( L \cap \mathcal{V}' \) and \( e_2 - e_1, \ldots, e_n - e_1 \) for \( L \cap \mathcal{V}'' \). (If \( x_1, \ldots, x_n \in \mathbb{Z} \) and \( x_1 + \ldots + x_n = 0 \), one has \( x_1e_1 + \ldots + x_ne_n = x_2(e_2 - e_1) + \ldots + x_n(e_n - e_1) \).) By (2.1), \( \text{ind}_L(\mathcal{V}', \mathcal{V}'') = n \).

(This \( H \) is the holonomy group of a generalized Klein bottle, cf. Section 13.)

Next, for \textit{any} finite group \( H \), we set \( V = \mathbb{R}^H \) and \( L = \mathbb{Z}^H \), with the set-theoretical convention of denoting by \( Y^X \) the set of all mappings \( X \to Y \). Thus, \( L \) is a lattice in the real vector space \( V \), while \( V \cong \mathbb{R}^n \) and \( L \cong \mathbb{Z}^n \) for \( n = |H| \), the isomorphic identifications \( \cong \) coming from a fixed bijection \( \{1, \ldots, n\} \to H \). Letting \( \tau_a : H \to H \) be the left translation by \( a \in H \), we define a right action \( V \times H \ni (f, a) \mapsto f \circ \tau_a \in V \) of the group \( H \) on \( V \), preserving \( L \). The action being obviously effective, we may view \( H \) as a finite group of \( L \)-preserving linear automorphisms of \( V \). A \( k \)-element normal subgroup \( \hat{H} \) of \( H \), \( k = 1, \ldots, n \), now gives rise to two \( H \)-invariant \( L \)-subspaces \( \mathcal{V}', \mathcal{V}'' \) of \( V \), of dimensions \( n/k \) and \( n - n/k \), with \( \mathcal{V}' \) formed by all \( f \in V \) constant on each left coset \( a\hat{H} \), \( a \in H \) (that is, \( \hat{H} \)-invariant), and \( \mathcal{V}'' \) by those \( f \in V \) having the sum of values over every coset \( a\hat{H} \) equal to zero. Note that \( \mathcal{V}' \) and \( \mathcal{V}'' \) constitute each other’s \( \ell^2 \)-orthogonal complements. They both are \( L \)-subspaces (Definition 4.1): a subset of \( L \) spanning \( \mathcal{V}' \) (or, \( \mathcal{V}'' \) if \( k > 1 \) and \( \mathcal{V}'' \neq \{0\} \)) consists of functions equal to 1 on one coset and to 0
on the others (or, respectively, of functions assuming the values 1 and \(-1\) at two points lying in the same coset, and vanishing everywhere else).

For \(n, A, e_1, \ldots, e_n, u\) as at the beginning of this section, and the lattice \(L \subseteq \mathbb{R}^n\) with the \(\mathbb{Z}\)-basis \(e_1, \ldots, e_n\), let \(H \cong \mathbb{Z}_n\) be the group generated by \(A\), so that the simply transitive action of \(H\) on the set \(\{e_1, \ldots, e_n\}\) allows us to identify \(\mathbb{R}^n\) with \(\mathbb{R}^H\), cf. the last paragraph. (To be specific, we may view \((x_1, \ldots, x_n)\) as the function \(H \ni A^{-k} \mapsto x_k, k = 1, \ldots, n\).) Any positive divisor \(d\) of \(n\) now gives rise to a subgroup \(H_d\) of \(H\) and a vector subspace \(V_d\) of \(\mathbb{R}^n\). Namely, \(H_d \cong \mathbb{Z}_{n/d}\) is generated by \(A^{d}\), while \(V_d\) equals the intersection of the family of subspaces consisting of \(V'\) associated as in the last paragraph with the (normal) subgroup \(H/H_d\) which run through all subgroups of \(H\) containing \(H_d\) as a proper subgroup (that is, over \(H_k \cong \mathbb{Z}_{n/k}\) for all divisors \(k\) of \(d\) with \(1 \leq k < d\)). Note that \(V_d\) then must be an \(H\)-invariant \(L\)-subspace of \(\mathbb{R}^n\), since so are all the subspaces constructed in the preceding paragraph and – by Lemma 12.3 – their intersections. Also, since elements of \(V'\) (in other words, \(H_d\)-invariant vectors in \(\mathbb{R}^n\)) are nothing else than functions on the quotient group \(H/H_d \cong \mathbb{Z}_d\),

\[
(12.2) \quad V_d \text{ coincides with the orthogonal complement, in the space of all functions } H/H_d \rightarrow \mathbb{R}, \text{ of the subspace spanned by } H'-\text{invariant functions, } H' \text{ ranging over all nontrivial subgroups of } H/H_d \cong \mathbb{Z}_d.
\]

**Remark 12.1.** For any positive divisor \(d\) of \(n\), let \(V'_d\) be the \(H\)-invariant \(L\)-subspace \(V' \subseteq \mathbb{R}^n\) mentioned immediately before (12.2), so that \(V'_d \cong \mathbb{R}^d\) consists of all \(A^d\)-invariant vectors in \(\mathbb{R}^n\). The restriction of \(A\) to \(V'_d\) then has the characteristic polynomial \(P_d\) given by \(P_d(t) = 1 - t^d\). This follows from the last line in (12.1), since the assumptions of (12.1) still hold if we identify \(V'_d\) with \(\mathbb{R}^d\) using the basis \(e_j = (1 + A^d + A^{2d} + \ldots + A^{n-d})e_j, j = 1, \ldots, d\), replacing \(n\) and \(e_j\) by \(d\) and \(e_j\), the sum of the \(A^d\)-orbit of \(e_j\).

**Lemma 12.2.** Under the above assumptions, \(\mathbb{R}^n\) equals the orthogonal direct sum of the \(H\)-invariant \(L\)-subspaces \(V_d\), for all positive divisors \(d\) of \(n\), and \(\dim V_d = \varphi(d)\), where \(\varphi\) denotes Euler’s totient function. Every \(H\)-invariant \(L\)-subspace of \(\mathbb{R}^n\) is spanned by some of the subspaces \(V_d\). Furthermore, with minimality defined as in Corollary 12.9.

\[
(12.3) \quad V_d \text{ are precisely all the minimal nonzero } H\text{-invariant } L\text{-subspaces of } \mathbb{R}^n.
\]

**Proof.** The generator \(A\) of \(H\) is a linear isometry of \(\mathbb{R}^n\) having, in view of (12.1), simple complex eigenvalues provided by the \(n\)th roots of unity, which gives rise to an orthogonal decomposition of \(\mathbb{R}^n\) into
two-dimensional eigenspaces of \(A\) for the eigenvalues \(1, -1\)

\[
(12.4) \quad \text{(the latter for even } n \text{ only), and two-dimensional } H\text{-invariant subspaces corresponding to pairs } q, \overline{q} \text{ of nonreal eigenvalues, where } q^n = 1, \text{ so that a suitable identification of the subspace with } \mathbb{C} \text{ makes } A \text{ act in it via multiplication by } q. \text{ Any } H\text{-invariant subspace } V' \subseteq \mathbb{R}^n \text{ is the span of some of} \]
the orthogonal summands just mentioned, as a consequence of diagonalizability of the 
$C$-linear extension of $A$ to $C^n$ (see Remark 2.7) and the fact that the summands are the
real-part projection images of its one-dimensional complex eigenspaces. This results in

\[(12.5)\]
obs

one consisting of all $H$-invariant subspaces of $\mathbb{R}^n$, the other of subsets, closed under
conjugation, of the group $\mathbb{Z}_n$ of $n$th complex roots of unity, and the third one formed
by all monic real polynomials in the variable $t$, dividing $1 - t^n$ (the monic real divisors
of $1 - t^n$ being, up to a sign, the characteristic polynomials of the restrictions of $A$ to
invariant subspaces).

If a nonzero $H$-invariant subspace $V'$ happens to be an $L$-subspace, the monic real
polynomial representing it via (12.5) is, by Corollary 4.9, a factor in a factorization of
$1 - t^n$ over $\mathbb{Z}_n$, and so – due to the uniqueness assertion of Remark 2.8 – it equals, up to a
sign, the product of several (one or more) cyclotomic polynomials $\Phi_k$, for positive divisors
$k$ of $n$. Applying the last conclusion to $V' = V'_d$ of Remark 12.1, with the characteristic
polynomial $1 - t^d$, we see that $V'_d$ corresponds, again via (12.5), to the subset of $\mathbb{Z}_n$
formed by all $d$th roots of unity, and so $V'_d$ is the orthogonal direct sum of the subspaces
(12.4) corresponding to the $d$th roots of unity. At the same time, $V'_d$ contains $V'_k$, for all
divisors $k$ of $d$ with $1 \leq k < d$, and $V_d$ is, by (12.2), the orthogonal complement in $V'_d$
of the span of all such $V'_k$. Thus, the decomposition of $V'_d$ into the selected summands
(12.4) will not include ones associated with $k$th roots of unity, for $k < d$ dividing $d$, and
so it will only involve just the primitive $d$th roots of unity, implying that $A$ restricted
to $V_d$ has the characteristic polynomial $-\Phi_d$, and $\dim V_d = \varphi(d)$ (see Remark 2.8), while
irreducibility of $\Phi_d$ combined with (12.5), Remark 4.8 and Theorem 4.7, proves (12.3).
Our assertion now follows from Corollary 4.9. □

13. Generalized Klein bottles

This section presents some known examples \[3, p. 163\] to illustrate our discussion.

Let $\Sigma$ and $r_\theta : \Sigma \to \Sigma$ denote the unit circle in $C$ and the rotation by angle $\theta$
(multiplication by $e^{i\theta}$). For $(t, \psi) \in \mathbb{R} \times \mathbb{Z}^\Sigma$ and $f \in \mathbb{R}^{\Sigma}$ (cf. Section 12),
the assignment

\[(13.1)\]

$((t, \psi), f) \mapsto f \circ r_{2\pi t} + t + \psi$,

defines a left action on $\mathbb{R}^{\Sigma}$ of the group $\mathbb{R} \times \mathbb{Z}^\Sigma$, with the group operation

\[(13.2)\]

$(t, \psi)(t', \psi') = (t + t', \psi' \circ r_{2\pi t} + \psi)$.

The term $t$ in (13.1) is the constant function $t : \Sigma \to \mathbb{R}$, and one has the obvious short
exact sequence $\mathbb{Z}^\Sigma \to \mathbb{R} \times \mathbb{Z}^\Sigma \to \mathbb{R}$, the arrows being $\psi \mapsto (0, \psi)$ and $(t, \psi) \mapsto t$.

The functions $f : \Sigma \to \mathbb{R}$ are not assumed continuous and, whenever $H \subseteq \Sigma$, we treat
$\mathbb{R}^H$ (and $\mathbb{Z}^H$) as subsets of $\mathbb{R}^\Sigma$ (and $\mathbb{Z}^\Sigma$) via the zero extension of functions $H \to \mathbb{R}$ to
$\Sigma$. For $n \geq 2$ and the group $H = \mathbb{Z}_n \subseteq \Sigma$ of $n$th roots of unity, $\mathbb{Z}^H \cong \mathbb{Z}^n$ is a lattice.
in the Euclidean space \( \mathcal{V} = \mathbb{R}^H = \mathbb{R}^n \), and the action (13.1) has a restriction to an affine isometric action of the subgroup \( \Pi = (1/n)\mathbb{Z} \times \mathbb{Z}_0^H \subseteq \mathbb{R} \times \mathbb{Z}^\Sigma \) on \( \mathcal{V} \), with the subgroup \( \mathbb{Z}_0^H \cong \mathbb{Z}^{n-1} \) of \( \mathbb{Z}^H \) given by \( \{ \psi \in \mathbb{Z}^H : \psi_{\text{avg}} = 0 \} \), where \( (\cdot)_{\text{avg}} \) denotes the averaging functional \( \mathcal{V} \to \mathbb{R} \). Note that, in the right-hand side of (13.1),

\[
(13.3) \quad t_{\text{avg}} = t, \quad \psi_{\text{avg}} = 0, \quad (f \circ r_{2\pi t})_{\text{avg}} = f_{\text{avg}}.
\]

**Lemma 13.1.** The action of \( \Pi \) on \( \mathcal{V} \) is effective and \( \Pi \) constitutes a Bieberbach group in the underlying Euclidean affine \( n \)-space of \( \mathcal{V} \), its lattice subgroup and holonomy group being \( L = \{ \psi \in \mathbb{Z}^H : \psi_{\text{avg}} \in \mathbb{Z} \} \cong \mathbb{Z}^n \) embedded in \( \Pi \) via \( \psi \mapsto (0, \psi) \), and our \( H \cong \mathbb{Z}_n \).

**Proof.** First, \( \Pi \) acts on \( \mathcal{V} \) freely: if \( f \circ r_{2\pi t} + t + \psi = f \), cf. (13.1), with \( f : \Pi \to \mathbb{R} \), applying \( (\cdot)_{\text{avg}} \) to both sides, and using (13.3), we obtain \( t = 0 \), and hence \( f \circ r_{2\pi t} = f \), which turns the equality \( f \circ r_{2\pi t} + t + \psi = f \) into \( \psi = 0 \). Secondly, both our \( H \) and \( L = \{ \psi \in \mathbb{Z}^H : \psi_{\text{avg}} \in \mathbb{Z} \} \) arise from \( \Pi \) as required in (6.1): the claim about \( H \) and the inclusion \( L \subseteq \Pi \cap \mathcal{V} \) are obvious, the latter being due to the fact that the translation by any \( \psi' \in \mathbb{Z}^H \) with \( \psi'_{\text{avg}} \in \mathbb{Z} \) has the form (13.1) for \( t = \psi'_{\text{avg}} \) and \( \psi = \psi' - t \). Conversely, \( \Pi \cap \mathcal{V} \subseteq L \). To see this, suppose that \( f \circ r_{2\pi t} + t + \psi = f + \psi' \) for all \( f \in \mathcal{V} = \mathbb{R}^H \), some \( (t, \psi) \in \Pi \), and some \( \psi' \in \mathbb{Z}^H \). Taking the linear parts of both sides, we get \( t \in \mathbb{Z} \), and so \( t + \psi = \psi' \) which, by (13.3), yields \( \psi'_{\text{avg}} = t \in \mathbb{Z} \), that is, \( \psi' \in L \). Being a lattice in \( \mathcal{V} \), our \( L \subseteq \Pi \) has a compact fundamental domain, and hence so does \( \Pi \). Also, \( \Pi \) must be torsion-free: as \( \Pi \ni (t, \psi) \mapsto t \in \mathbb{R} \) is a group homomorphism, any finite-order element \( (t, \psi) \) of \( \Pi \) has \( t = 0 \), and so it lies in the embedded copy of the lattice \( \mathbb{Z}^H \). Finally, to establish discreteness of \( \Pi \) as a subset of \( \text{Iso} \mathcal{V} \), suppose that a sequence \( (t_k, \psi_k) \in \Pi \) with pairwise distinct terms yields, via (13.1), a sequence convergent in \( \text{Iso} \mathcal{V} \). Evaluating (13.1) on \( f = 0 \), we get \( (t_k, \psi_k) \to (t, \psi) \) as \( k \to \infty \) in the vector space \( \mathbb{R} \times \mathbb{R}^H \), for some \( (t, \psi) \) and, since \( (t_k, \psi_k) \in (1/n)\mathbb{Z} \times \mathbb{Z}_0^H \), the sequence \( (t_k, \psi_k) \) becomes eventually constant, contrary to its terms’ being pairwise distinct. \( \square \)

The compact flat Riemannian manifold \( \mathcal{V}/\Pi \), for our Bieberbach group \( \Pi \) (see Section 6) is called the \( n \)-dimensional generalized Klein bottle [3], p. 163. The linear functional \( \mathcal{V} \ni f \mapsto f_{\text{avg}} \in \mathbb{R} \) is equivariant, due to (13.3), with respect to the actions of \( \Pi \) and \( \mathbb{Z} \) (the latter, on \( \mathbb{R} \), via translations by multiples of \( 1/n \)), relative to the homomorphism \( \Pi \ni (t, \psi) \mapsto t \in (1/n)\mathbb{Z} \). Thus, it descends, in view of the final clause of Remark 2.2(b), to a bundle projection \( \mathcal{V}/\Pi \to \mathbb{R}/[(1/n)\mathbb{Z}] \), making \( \mathcal{V}/\Pi \) a bundle of tori over the circle.

**Remark 13.2.** Given any \( n \geq 2 \), the \( n \)-dimensional generalized Klein bottle shows that the last inclusion of Theorem 7.1(ii-c) may be proper. Specifically, choosing the \( H \)-invariant \( L \)-subspace \( \mathcal{V}' \), as well as its coset \( \mathcal{E}' \), to be the line of constant functions \( H = \mathbb{Z}_n \to \mathbb{R} \), that is, \( \mathcal{V}_d \) of (12.2) for \( d = 1 \), we see that elements of the group \( \Pi \) (defined before Lemma 13.1) preserving \( \mathcal{E}' \) have the form \( (t, \psi) \) with constant functions
ψ : H → Z, and hence act on constant functions H =⇒ R via translations by (1/n)Z, while restrictions to E' of elements of L ∩ V' are integer translations.

14. Remarks on holonomy

The correspondence (see Remark 6.3) between Bieberbach groups and compact flat manifolds has an extension to almost-Bieberbach groups and infra-nilmanifolds [5] obtained by replacing the translation vector space of a Euclidean affine space with a connected, simply connected nilpotent Lie group G acting simply transitively on a manifold E, and the Bieberbach group with a torsion-free uniform discrete subgroup Π of Diff E contained in a semidirect product, canonically transplanted into E, of G and a maximal compact subgroup of Aut G. Here 'uniform' means admitting a compact fundamental domain, cf. Remark 4.4. The analogs of (6.2) and (8.2) remain valid, reflecting the fact that any infra-nilmanifold is the quotient of a nilmanifold under the action of a finite group H.

A somewhat similar picture may arise in some cases where G is not assumed nilpotent. As an example, one has G ∼= Spin(m, 1), the universal covering group of the identity component G/Z₂ ∼= SO⁺(m, 1) of the pseudo-orthogonal group in an (m+1)-dimensional Lorentzian vector space L, m ≥ 3. Here E is the (two-fold) universal covering manifold of the orthonormal-frame bundle of the future unit pseudosphere Y ⊆ L, isometric to the hyperbolic m-space, and G/Z₂ acts on Y via hyperbolic isometries, leading to an action of G on E. The orthonormal-frame bundles of compact hyperbolic manifolds obtained as quotients of Y give rise to the required torsion-free uniform discrete subgroups Π.

The resulting compact quotient manifolds M = E/Π can be endowed with various interesting Riemannian metrics coming from Π-invariant metrics on E. For Π and E of the preceding paragraph, a particularly natural choice of an invariant indefinite metric is provided by the Killing form of G, turning M into a compact locally symmetric pseudo-Riemannian Einstein manifold.

Outside of the Bieberbach-group case, however, these metrics are not flat, and finite groups H such as mentioned above cannot serve as their holonomy groups. The holonomy interpretation of H still makes sense, though, if – instead of metrics – one uses Π-invariant flat connections, with (parallel) torsion, on E. Two such standard connections are naturally induced by bi-invariant connections on G, characterized by the property of making all left-invariant (or, right-invariant) vector fields parallel. Both of these connections are, due to their naturality, invariant under all Lie-group automorphisms of G.

Appendix: Hiss and Szczepański’s reducibility theorem

Let us consider an abstract Bieberbach group, that is, any torsion-free group Π containing a finitely generated free Abelian normal subgroup L of a finite index, which is at the same time a maximal Abelian subgroup of Π. As shown by Zassenhaus [12], up to
isomorphisms these groups are the same as the Bieberbach groups of Section 6 and one can again summarize their structure using the short exact sequence

\[(A.1) \quad L \to \Pi \to H, \quad \text{where} \quad H = \Pi/L.\]

For the tensor product \(G \otimes G'\) of Abelian groups \(G, G'\) one has canonical isomorphisms

\[(A.2) \quad \mathbb{Z} \otimes G \cong G, \quad (G_1 \oplus G_2) \otimes G' \cong (G_1 \otimes G') \oplus (G_2 \otimes G'), \quad L \otimes \mathbb{Q} \cong \text{Hom}(L^*, \mathbb{Q}),\]

where \(L^* = \text{Hom}(L, \mathbb{Z})\) and, for simplicity, \(L\) is assumed to be finitely generated and free. In the last case, with a suitable integer \(n \geq 0\), there are noncanonical isomorphisms

\[(A.3) \quad \begin{cases} a) \ L \cong \mathbb{Z}^n, & b) \ L \otimes \mathbb{Q} \cong \mathbb{Q}^n, \end{cases}\]

while, using the injective homomorphism \(L \ni \lambda \mapsto \lambda \otimes 1 \in L \otimes \mathbb{Q}\) to treat \(L\) as a subgroup of \(L \otimes \mathbb{Q}\), we see that, under suitably chosen identifications \((A.3)\),

\[(A.4) \quad \text{the inclusion} \ L \subseteq L \otimes \mathbb{Q} \quad \text{corresponds to the standard inclusion} \quad \mathbb{Z}^n \subseteq \mathbb{Q}^n.\]

Finally, if \(L\) as above is a (full) lattice in a finite-dimensional real vector space \(V\) (cf. Remark 4.5), a further canonical isomorphic identification arises:

\[(A.5) \quad L \otimes \mathbb{Q} \cong \text{Span}_\mathbb{Q} L,\]

that is, we may view \(L \otimes \mathbb{Q}\) as the rational vector subspace of \(V\) spanned by \(L\).

Let \(\Pi\) now be an abstract Bieberbach group. Hiss and Szczepański [7, the corollary in Sect. 1] proved that, if \(L\) in \((A.1)\) satisfies \((A.3) a)\) with \(n \geq 2\), then the (obviously \(\mathbb{Q}\)-linear) action of \(H\) on \(L \otimes \mathbb{Q}\) must be reducible, in the sense of admitting a nonzero proper invariant rational vector subspace \(W\).

Next, using \((A.4)\), we may write \(L = \mathbb{Z}^n \subseteq \mathbb{Q}^n = L \otimes \mathbb{Q}\), so that \(W \subseteq \mathbb{Q}^n \subseteq \mathbb{R}^n\), and the closure \(V'\) of \(W\) in \(\mathbb{R}^n\) has the real dimension \(\dim_\mathbb{Q} W\) (any \(\mathbb{Q}\)-basis of \(W\) being, obviously, an \(\mathbb{R}\)-basis of \(V')\). By clearing denominators, one can replace such a \(\mathbb{Q}\)-basis with one consisting of vectors in \(L = \mathbb{Z}^n\), and so, by Remark 4.5(b), the intersection \(L' = L \cap W = L \cap V'\) is a lattice in \(V'\). In other words, we obtain \((7.2)\).

A stronger version of Hiss and Szczepański’s reducibility theorem was more recently established by Lutowski [9].

**References**

[1] D. M. Arnold, *Finite Rank Torsion-Free Abelian Groups and Rings*, Lecture Notes in Mathematics 931, Springer-Verlag, New York, 1982.

[2] R. G. Bettiol, A. Derdzinski, R. Mossa, P. Piccione, *On the collapse of compact flat manifolds*, in preparation.

[3] L. S. Charlap, *Bieberbach Groups and Flat Manifolds*, Universitext, Springer-Verlag, New York, 1986.

[4] M. W. Davis, *Lectures on orbifolds and reflection groups*, in: Advanced Lectures in Mathematics 16: Transformation Groups and Moduli Spaces of Curves (L. Ji and S.-T. Yau, eds.), International Press, Somerville, MA, USA/Higher Education Press, Beijing, China, 2011, 63–93.
[5] K. Dekimpe, *Almost-Bieberbach Groups: Affine and Polynomial Structures*, Lecture Notes in Mathematics 1639. Springer-Verlag, Berlin, 1996.

[6] R. E. Edwards, *Fourier Series: A Modern Introduction, Volume 1*, 2nd ed., Springer-Verlag, New York, 1979.

[7] G. Hiss, A. Szczepański, *On torsion free crystallographic groups*. J. Pure Appl. Algebra 74 (1991), 39–56.

[8] S. Lang, *Algebra*, revised 3rd ed., Graduate Texts in Mathematics 211. Springer-Verlag, New York, 2002.

[9] R. Lutowski, *Flat manifolds with homogeneous holonomy representation*. arXiv:1803.07177v2.

[10] H. Maier, *Anatomy of integers and cyclotomic polynomials*, in: Anatomy of Integers (J.-M. de Koninck, A. Granville, and F. Luca, eds.), CRM Proceedings & Lecture Notes, vol. 46, American Mathematical Society, Providence, RI, 2008, pp. 89–95.

[11] A. Szczepański, *Geometry of Crystallographic Groups*. Algebra and Discrete Mathematics, 4. World Scientific, Hackensack, NJ, 2012.

[12] H. Zassenhaus, *Beweis eines Satzes über diskrete Gruppen*. Abh. Math. Sem. Univ. Hamburg 12 (1938), 289–312.