Fluctuation Theorem for an Arbitrary Quantum Process

Hyukjoon Kwon and M. S. Kim
QOLS, Blackett Laboratory, Imperial College London, London SW7 2AZ, United Kingdom

Fluctuation theorems are at the heart of thermodynamics. We introduce fluctuation theorems from a quantum information theoretic viewpoint by establishing a relationship between the transition amplitudes of a quantum channel and its recovery map. The reference state, which can be fully recovered by a rotated Petz recovery map, is utilised to define the complex-valued information exchange during the quantum process. We derive a quantum Crooks relation through the ratio between the entropy production amplitudes of the forward and backward (recovery) quantum processes. The Jarzynski equality of the quantum process can be directly obtained from our fluctuation relation, and the generalised second law of thermodynamics is obtained in the form of the quantum relative entropy. Our fluctuation theorem can be applied to a various range of quantum processes to investigate quantum thermodynamics involving coherence and entanglement.

I. INTRODUCTION

Fluctuation theorem (FTs) in classical thermodynamics establish the connection between the forward and backward processes when system’s microscopic states have time-reversal symmetry. The Crooks fluctuation relation \(1 \) states that the entropy production under non-equilibrium processes satisfies balanced equality and leads to a refined version of work statistics, known as the Jarzynski equality \(2 \). FTs derived from microscopic reversibility lead to macroscopic irreversibility \(2 \); the second law of thermodynamics. Classical FTs have been experimentally well-verified in microscopic systems \(3 \, 4 \), and it has been attempted to extend FTs to the quantum regime \(5 \, 6 \). The fluctuation in a classical work has been tested in quantum systems \(10 \, 11 \), yet the difficulty in defining the quantum observable of work \(12 \) made it wonder how to figure out the nonclassical characteristics of thermodynamic fluctuations.

Quantum FTs (QFTs) should be able to consider non-equilibrium quantum processes which are represented by quantum superpositions and coherences \(13 \, 27 \). In the presence of coherence, quantum thermodynamic free energy \(28 \, 30 \) can be larger than its classical counterpart which only concerns with classical energetic values. The role of quantum coherences and correlations is noticed as a resource that can be utilised for work extraction \(31 \, 32 \) or time referencing tasks \(33 \, 34 \). These nonclassical features stemming from quantum coherences not only affect thermodynamic quantities on average but also bring the outcome probability distributions differ from the classical theory. In particular, a no-go theorem \(22 \) has recently been found that it is impossible to reconstruct a positive joint work distribution of a two-point measurement (TPM), preserving the marginal distributions of the both first and second energy measurements, and this is equivalent to the measurements without disturbing a quantum system in a similar vein to the violation of macrorealism introduced by Leggett and Garg \(35 \). The no-go theorem implies that the negative work quasi-probability \(14 \, 22 \) is an intrinsic characteristic of quantum statistics closely related to quantum contextuality \(23 \).

Meanwhile, in quantum information theory, reversibility of a quantum process can be described by how closely a quantum state is recovered by a certain reverse process, known as the Petz recovery map \(57 \). In quantum thermodynamics there have been attempts to construct a framework for thermodynamic processes in the quantum regime \(28 \, 30 \, 38 \), while the time-reversal trajectory can be described in the form of the recovery map \(22 \). It has been studied that we can obtain the QFT between a quantum thermodynamic process and its time-reversal. The fluctuation relations for entanglement \(39 \) and coherences \(40 \) have also been proposed in a similar vein. However, the complete relationship between the FT and the Petz recovery map has yet to be discovered.

In this paper, we introduce FTs from the quantum information theoretic point of view by establishing the connection between a quantum channel and its recovery map. We first establish the pure state fluctuation relation between the transition amplitudes of the forward and backward quantum processes via reference state rescaling. We introduce the transition amplitude of the TPM and a generalised quantum information exchange (QIE) during the quantum process, based on the observation that for every quantum channel there exists a family of recovery maps which fully reinstate a given reference state. The QIE has a complex value so as the transition amplitude, while it becomes real when considering a covariant quantum process, which preserves time-translation symmetry. We note that our transition amplitude keeps the marginalisation property by recovering the probabilities of the initial and final states. In our QFT, the von Neumann entropy and the QIE play a central role replacing the Shannon entropy and heat respectively, in the entropy production during the quantum process. We show that the transition amplitudes of the entropy production obey balanced equality between the forward and backward processes for an arbitrary mixed state. Our result provides a unified framework to construct the statistical equality for general quantum processes bridging quantum information theory to thermodynamics.
II. RECOVERY MAPS FOR A QUANTUM PROCESS

We first briefly review basic concepts of quantum channels and their recovery maps. The general quantum channel acting on a quantum state $\rho$ can be expressed by a completely positive trace-preserving (CPTP) map. Using the Kraus representation, $\mathcal{N}(\rho) = \sum_{\omega} K_{\omega} \rho K_{\omega}^\dagger$, the adjoint map of $\mathcal{N}$ is defined as $\mathcal{N}^\dagger(A) := \sum_{\omega} K_{\omega}^\dagger A K_{\omega}$ satisfying $\text{Tr}[\mathcal{N}(\rho)A] = \text{Tr}[\rho \mathcal{N}^\dagger(A)]$. In terms of the adjoint map $\mathcal{N}^\dagger$ and the reference state $\gamma$ with evolution $\tilde{\gamma} = \mathcal{N}(\gamma)$, we can construct the CPTP map, the so called Petz recovery map as follows:

**Definition 1** (Petz recovery map). For a given reference state $\gamma$ and CPTP map $\mathcal{N}$, the Petz recovery map $\mathcal{R}_\gamma$ is defined as

$$\mathcal{R}_\gamma(\chi) := \gamma^{\frac{1}{2}} \mathcal{N}^\dagger(\mathcal{N}(\gamma)^{-\frac{1}{2}} \chi \mathcal{N}(\gamma)^{-\frac{1}{2}}) \gamma^{\frac{1}{2}},$$

which has been generalised to the following form of the rotated recovery map $\mathcal{R}_\gamma^R$,

$$\mathcal{R}_\gamma^R(\chi) := (\mathcal{U}_{t,-t} \circ \mathcal{R}_\gamma \circ \mathcal{U}_{\gamma (\gamma), t})(\chi),$$

where $\mathcal{U}_{t}(\rho) := \tau^t \rho \tau^{-t}$. The ordinary Petz recovery map is a special case with $t = 0$.

Every recovery map is a CPTP map and the reference state fully recovers by the recovery map, i.e., $(\mathcal{R}_\gamma^R \circ \mathcal{N})(\gamma) = \gamma$; yet the full recovery of an arbitrary state is not possible in general, i.e., for a given $\mathcal{N}$ there will be a quantum state $\rho$ such that $(\mathcal{R}_\gamma^R \circ \mathcal{N})(\rho) \neq \rho$. The difference between an original and its recovered state is closely related to the monotonicity of the relative entropy under a CPTP map $S(\mathcal{N}(\rho)||\mathcal{N}(\gamma)) \leq S(\rho||\gamma)$. The equality “$=$” holds if and only if $\rho$ is fully recovered by every rotated Petz recovery map $\mathcal{R}_\gamma^R$, i.e., $(\mathcal{R}_\gamma^R \circ \mathcal{N})(\rho) = \rho$ for $t \in (-\infty, \infty)$. Here, the relative entropy between the two quantum states $\rho$ and $\sigma$ is given by $S(\rho||\sigma) = \text{Tr} [\rho \log \rho - \log \sigma]$. Figure 1 shows the relationship between a quantum channel $\mathcal{N}$ and its recovery map $\mathcal{R}_\gamma^R$.

![FIG. 1: Rotated Petz recovery maps. Reference state $\gamma$ is fully recovered by the Petz recovery map, i.e. $(\mathcal{R}_\gamma \circ \mathcal{N})(\gamma) = \gamma$. The distances between the reference states $\gamma$ or $\tilde{\gamma}$ and $\rho$ or $\mathcal{N}(\rho)$ represent the relative entropy $S(\rho||\gamma)$ or $S(\mathcal{N}(\rho)||\tilde{\gamma})$.](image)

III. FLUCTUATION RELATIONS FOR THE RECOVERY MAPS

The fluctuation relation establishes a direct connection between forward and backward processes in a standard TPM scheme. Here, we consider an arbitrary CPTP map $\mathcal{N}$ as a forward process, while its corresponding backward process would be the rotated Petz recovery map $\mathcal{R}_\gamma^R$, with a fully reversible reference state $\gamma$ satisfying $(\mathcal{R}_\gamma^R \circ \mathcal{N})(\gamma) = \gamma$. In order to construct the fluctuation relation for the rotated Petz recovery map, we consider reference bases $|i\rangle$ using the eigenvalue decomposition of the reference state $\gamma = \sum_i r_i |i\rangle \langle i|$ and its evolution under a CPTP map $\mathcal{N}(\gamma) = \tilde{\gamma} = \sum_k \tilde{r}_k |k\rangle \langle k|$. We define the transition amplitude of the forward and backward processes

$$T_{ij}^{kl} := (\tilde{k} | \mathcal{N}(|i\rangle \langle j|) | \tilde{l})$$

and

$$\tilde{T}_{ij}^{kl} := (i | \mathcal{R}_\gamma^R(|\tilde{k}\rangle \langle \tilde{l}|) | j),$$

respectively. Note that $(T_{ij}^{kl})^* = T_{ji}^{lk}$ and $(\tilde{T}_{ij}^{kl})^* = \tilde{T}_{ji}^{lk}$, where taking a conjugate of the transition amplitude is equivalent to the transpose operation satisfying $|i\rangle \langle j| \leftrightarrow |j\rangle \langle i|$ and $|\tilde{k}\rangle \langle \tilde{l}| \leftrightarrow |\tilde{l}\rangle \langle \tilde{k}|$. Taking conjugate of the transition amplitude or transposing with respect to the reference can be compared to the time-reversal operation in the thermodynamic FT. For example, in a harmonic oscillator system having a Hamiltonian $H = \frac{p^2}{2m} + \omega^2 x^2 / 2$ with $[x, p] = i \hbar$, the transpose operation $\Theta$ with respect to the energy eigenstates is identical to the reflection in phase space $p \leftrightarrow -p$, which is exactly the time-reversal operation. We then obtain the following relation between the quantum process $\mathcal{N}$ and its Petz recovery map $\mathcal{R}_\gamma^R$:

**Proposition 1** (Connection between a CPTP map and its Petz recovery map).

$$T_{ij}^{kl} = \frac{\tilde{r}_k \tilde{r}_l}{r_i r_j} e^{it \log \frac{r_i r_j}{\tilde{r}_k \tilde{r}_l}} (\tilde{T}_{ij}^{kl})^*$$

(1)

For $i = j$ and $k = l$, $T_{ii}^{kk} = (\tilde{k} | \mathcal{N}(|i\rangle \langle i|) | \tilde{k}) = T(i \rightarrow \tilde{k})$ has the meaning of transition probability from $|i\rangle$ to $|\tilde{k}\rangle$ under $\mathcal{N}$, and we have the diagonal fluctuation relation

$$\frac{T(i \rightarrow \tilde{k})}{T(i \rightarrow i)} = \frac{\tilde{r}_k}{r_i},$$

(2)

regardless of $t$. On the other hand, a quantum state can have coherences (i.e., superposition) between the eigenstates of the reference state. In order to extend our fluctuation relation, we consider the following decomposition
of the transition amplitude for $|\psi\rangle \xrightarrow{N} |\phi\rangle$ as
\[
T(\psi \rightarrow \phi) := \langle \phi | N(|\psi\rangle |\psi\rangle) |\phi\rangle = \sum_{i,j,k,l} T(\psi \rightarrow \phi)_{ij}^{kl},
\]
where $T(\psi \rightarrow \phi)_{ij}^{kl} := \langle \phi | \mathbb{P}_k | N(\Pi_i |\psi\rangle |\psi\rangle \Pi_j) |\phi\rangle = \langle \phi | \langle j| \Pi_k \rangle |\phi\rangle \langle i| \Pi_l |\phi\rangle T(\psi \rightarrow \phi)_{ij}^{kl}$. Here, $\Pi_i = |i\rangle \langle i|$ and $\Pi_k = |k\rangle \langle k|$ are the projections onto the eigenstates of the reference states. The backward transition amplitude can be similarly defined using the Petz recovery map $R_\gamma^t$. Then by using Proposition 1 we can show the following relation between the forward and backward transition amplitudes:

**Theorem 1** (Pure state fluctuation relation). The transition amplitude from $|\psi\rangle$ to $|\phi\rangle$ under a CPTP map $N$ and its backward process under $R_\gamma^t$ obeys the following relation:
\[
T(\psi \rightarrow \phi) = \bar{T}^{(t)}(\mathcal{J}_{\gamma,t}(|\phi\rangle \rightarrow \mathcal{J}^{-1}_{\gamma,t}|\psi\rangle)),
\]
where $\mathcal{J}_{\gamma,t}$ is a reference-rescaling map converting $|\psi\rangle$ into $\gamma^{1/2+it} |\psi\rangle$, which is a generalisation of the Gibbs rescaling map $24$, $25$.

Note that the reference-rescaling map $\mathcal{J}_{\gamma,t}$ does not change the eigenstate of the reference state $|i\rangle$, but only gives a weight proportional to $\sqrt{\gamma}t$, i.e., $\mathcal{J}_{\gamma,t}(|i\rangle) = \sqrt{\gamma} |i\rangle$. This fact leads to the diagonal fluctuation relation Eq. (2) from Theorem 1.

Now we are ready to write the general state transformation under the CPTP map $N$. Suppose that $\rho = \sum_{\mu} p_{\mu} |\psi_\mu\rangle \langle \psi_\mu|$ is mapped to $\mathcal{N}(\rho) = \sum_{\nu} q_{\nu} |\phi_\nu\rangle \langle \phi_\nu|$ by the CPTP map $\mathcal{N}$. Here $\{p_{\mu}, |\psi_\mu\rangle\}$ and $\{q_{\nu}, |\phi_\nu\rangle\}$ are the eigenvalue decompositions of $\rho$ and $\sigma$, respectively, and $|\psi_\mu\rangle$ and $|\phi_\nu\rangle$ are not necessarily orthogonal. In this case, we note that the following joint transition amplitude
\[
p_{\mu} T(\psi_\mu \rightarrow \phi_\nu)_{ij}^{kl}
\]
lets us access the marginal probability distributions of the both initial and final states. More precisely, we observe that
\[
\sum_{\nu \text{ or } \mu} \left[ \sum_{i,j,k,l} p_{\mu} T(\psi_\mu \rightarrow \phi_\nu)_{ij}^{kl} \right] = p_{\mu} \text{ or } q_{\nu},
\]
\[
\sum_{k \text{ or } l} \left[ \sum_{\mu,\nu,j,l} p_{\mu} T(\psi_\mu \rightarrow \phi_\nu)_{ij}^{kl} \right] = \text{Tr}[\Pi_i \rho] \text{ or } \text{Tr}[\Pi_k \mathcal{N}(\rho)].
\]

**IV. QUANTUM CROOKS FLUCTUATION THEOREM**

**A. Crooks fluctuation relation**

The Crooks fluctuation theorem (CFT) relates the work done on the system during a non-equilibrium process with the free energy difference. We construct the quantum version of CFT for a general quantum process by applying Theorem 1 to the joint transition amplitude given by Eq. (1). To this end, we define a general single-shot entropy production of the extended transition $T(\psi_\mu \rightarrow \phi_\nu)_{ij}^{kl}$ as
\[
\delta\sigma := \delta s - \delta q,
\]
where the entropy change is given by
\[
\delta s := -\log q_\nu + \log p_\mu.
\]
We define the information exchange during the quantum process $\delta q$ as a QIE. Here we note that the QIE is complex $\delta q = \delta q_R + i \delta q_I$ with
\[
\delta q_R := \log \sqrt{r_i r_j} \frac{r_k r_l}{r_i r_j} \text{ and } \delta q_I := \log \sqrt{r_i r_l} \frac{r_j r_k}{r_i r_j}.
\]
This complex-valued QIE satisfies following relation
\[
\begin{cases}
\delta q_R + \delta q_I = \log (r_i r_j), \\
\delta q_R - \delta q_I = \log (r_j r_i),
\end{cases}
\]
and the detailed meaning of QIE will be discussed with relevant physical examples in the following sections. We are now ready to derive our main result—“quantum CFT of the entropy production” between the transition amplitudes of the forward and backward processes. Using the joint transition amplitude $p_\mu T(\psi_\mu \rightarrow \phi_\nu)_{ij}^{kl}$, the probability to get $\sigma$ amount of entropy production is given by
\[
P(\sigma) := \sum_{\mu,\nu} \sum_{i,j,k,l} p_{\mu} T(\psi_\mu \rightarrow \phi_\nu)_{ij}^{kl} \times \delta \left( \sigma_R - \log \left[ \frac{p_{\mu} \sqrt{r_i r_j}}{q_{\nu} \sqrt{r_k r_l}} \right] \right) \delta \left( \sigma_I - \log \left[ \frac{r_i r_l}{r_j r_k} \right] \right)
\]
and $\bar{T}^{(t)}(\sigma)$ can be similarly defined using the recovery map $R_{\gamma}^t$. Analogous to the classical CFT, we can establish the link between the forward and backward quantum processes.

**Theorem 2.** The transition probability generating the entropy production $\sigma$ by a CPTP map $\mathcal{N}$ is connected to generate $-\sigma^*$ by its reverse process $R_{\gamma}^{t/2}$:
\[
\frac{P(\sigma)}{P(t/2)(-\sigma^*)} = e^{\gamma \sigma - i \sigma \gamma},
\]
where $\sigma = \sigma_R + i \sigma_I$.

Note that $P(\sigma)$ and $\bar{P}(t/2)(-\sigma^*)$ in Eq. (7) cannot be interpreted as the probabilities since they are complex-valued unless $t = 0$. However, we point out that the imaginary part of the QIE only comes from the rotations $U_{\gamma,t}$ and $U_{\tilde{\gamma},t}$ in the recovery map, and each does not have any effects to the eigenstates of $\gamma$ or $\tilde{\gamma}$. Thus, the imaginary part of quantum CFT with $t \neq 0$ can be interpreted to describe the fluctuation of coherences during the quantum process. In the following section, we will discuss when the imaginary part of QIE vanishes.
B. Covariant quantum process

We first study the case in which the imaginary part of QIE vanishes. A simple choice would be considering a diagonal to diagonal transformation $\rho = \sum_i p_i \rho^i$ for $N(\rho) = \sum_k q_k \rho^k$, which can be referred as an incoherent process. In this case, $\delta q_R = -\log(\tilde{r}_k/r_i)$. Note that $T_{ii}^{kk}$ is the transition probability from $|i\rangle$ to $|k\rangle$ and the quantum CFT can be written as $P(\sigma) = \sum_n i_k T_{ii}^{kk} \delta(\sigma - \log(p_i \tilde{r}_k / q_k r_i)) = \tilde{P}(-\sigma) e^\sigma$.

More generally, we introduce a covariant quantum process with respect to a pair of references $(\gamma, \tilde{\gamma})$ satisfying $\mathcal{U}_{\gamma,-t} \circ \mathcal{N} \circ \mathcal{U}_{\gamma,t} = \mathcal{N}$. This covariant process has the physical meaning of translational symmetry as every rotation of $t$ with respect to the reference state $\gamma$ given by $\mathcal{U}_{\gamma,t}$ leads to the effect of the same $t$ after applying the quantum channel $\mathcal{N}$, i.e., $\mathcal{N} \circ \mathcal{U}_{\gamma,t} = \mathcal{U}_{\gamma,t} \circ \mathcal{N}$ (see Fig. 2). By assuming the quantum channel $\mathcal{N}$ to be covariant, we obtain a simple and familiar relation as follows:

**Theorem 3 (QFT for a covariant quantum process).** Suppose that a covariant CPTP map $\mathcal{N}$, satisfying $\mathcal{U}_{\gamma,-t} \circ \mathcal{N} \circ \mathcal{U}_{\gamma,t} = \mathcal{N}$ with respect to the reference state $\gamma$ and its evolution $\tilde{\gamma} = N(\gamma)$. Then $\mathcal{R}_\gamma^t = \mathcal{R}_\gamma$ for every $t$, and the imaginary part of the QIE vanishes. In this case, a real-valued production $\sigma$ by the forward process $\mathcal{N}$ is connected to its reverse process $\mathcal{R}_\gamma$ as

$$\frac{P(\sigma)}{P(-\sigma)} = e^\sigma.$$  

Figure 2 describes how every rotated Petz recovery map is reduced into the ordinary Petz recovery map. The incoherent quantum process mentioned above is an example of covariant quantum processes. This type of incoherent processes, which map an incoherent state to another incoherent state, can be characterised as $N_{\text{incoh}} = D_\gamma \circ M \circ D_\gamma$, where $M$ is an arbitrary CPTP map, and $D_\gamma(\chi) = \lim_{\Delta t \to \infty} \frac{1}{\Delta t} \int_{-\Delta t/2}^{+\Delta t/2} d\tau \mathcal{U}_{\gamma,t}(\chi)$ is a decohering operation with respect to the eigenstates of $\tau$. More nontrivial covariant operations are noisy operations and thermal operations which will be discussed in the following section.

C. Quantum Jarzynski equality and the second law

Using the quantum version of the Crooks relation given by Eq. (4), it is straightforward to derive the quantum Jarzynski equality as

$$\langle e^{-\sigma_R - it\sigma_I} \rangle = 1,$$  

for every $t$. where $\langle \cdot \rangle$ is the expectation over every transition given by the matrix $p_{\mu} T(\psi_{\mu} \to \phi_{\nu})_{ij}$. Note that both the transition amplitude and the quantum entropy production have complex values, so one may expect that we cannot obtain the second law—non-increasing of the first-order moment—from the quantum Jarzynski equality by directly applying Jensen’s inequality. Interestingly, however, we can obtain the expectation of the entropy production

$$\langle \sigma \rangle = \langle \sigma_R \rangle = S(\rho || \gamma) - S(\mathcal{N}(\rho) || \mathcal{N}(\gamma)) \geq 0,$$  

by using Eq. (6) and noting that $\langle \sigma_I \rangle = 0$. The last inequality is given by the monotonicity of the relative entropy under the CPTP map. This non-increasing of the first-order moment of the entropy production can be interpreted as the generalised second law of thermodynamics from the quantum FT in the form of the relative entropy difference.

In particular, in the case of a covariant quantum process, a complete condition of reversibility is given by the equality condition of Eq. (10), which can be deduced from Ref. [11].

**Theorem 4 (Reversibility of a covariant quantum process).** For a covariant process $\mathcal{N}$ with respect to a reference state and its evolution $\gamma, \tilde{\gamma}$,

$$S(\rho || \gamma) - S(\mathcal{N}(\rho) || \mathcal{N}(\gamma)) \geq -\log F(\rho, (\mathcal{R}_\gamma \circ \mathcal{N})(\rho)),$$

where $F(\rho, \tau) = ||\sqrt{\gamma} \sqrt{\tau}||^2$ is the quantum fidelity. Furthermore, a transformation $\rho \xrightarrow{\mathcal{R}_\gamma} N(\rho)$ is reversible by the Petz recovery map $\mathcal{R}_\gamma$, i.e., $(\mathcal{R}_\gamma \circ \mathcal{N})(\rho) = \rho$, if and only if $S(\rho || \gamma) = S(\mathcal{N}(\rho) || \mathcal{N}(\gamma))$.

Meanwhile, from the quantum Jarzynski equality given by Eq. (9), the expectation values of higher-order momenta should obey

$$\sum_{k=1}^{\infty} \frac{\langle \sigma_R - it\sigma_I \rangle^k}{k!} = 0$$

for any value of $t$. 

FIG. 2: Petz recovery maps for a covariant quantum process satisfying $\mathcal{U}_{\gamma,-t} \circ \mathcal{N} \circ \mathcal{U}_{\gamma,t} = \mathcal{N}$. Note that all the rotated Petz recovery maps are equivalent, i.e. $\mathcal{R}_\gamma^t = \mathcal{R}_\gamma$. 

V. APPLICATIONS TO QUANTUM PROCESSES

A. Noisy operation

We first apply the QFT to a noisy operation given by

$$\mathcal{N}(\rho) = \text{Tr}_E \left[ U(\rho \otimes \frac{1}{d_E}) U^\dagger \right],$$

where $d_E$ is the dimension of the environment and $U$ represents a unitary operation. Note that $\frac{1}{d}$ for a $d$-dimensional system is the fixed point of every noisy operation since $\mathcal{N}(\frac{1}{d}) = \frac{1}{d}$. Thus we may choose the reference state to be $\gamma = \frac{1}{d}$, then $\tilde{\gamma} = \frac{1}{d} = \gamma$ and $\mathcal{N}$ is obviously covariant with respect to the reference state. Now we consider a transformation from $\rho = \sum_n p_n |\psi_n\rangle \langle \psi_n|$ to $\sigma = \sum_n q_n |\phi_n\rangle \langle \phi_n|$ by a noisy operation. We can additionally choose the basis of the reference and its evolution to be $\{ i \} = \{ |\psi_n\rangle \}$ and $\{ k \} = \{ |\phi_n\rangle \}$, respectively because they span the Hilbert space of the system. In this case, we can obtain the diagonal FT $P(s) = \tilde{P}(s)e^s$ with $\delta \sigma = -\log q_n + \log p_n = \delta s$ since $r_i = 1/d = r_k$ for any $i$ and $k$. This relation leads to the single-shot entropy difference $\delta s = -\log q_n + \log p_n$, which connects the transition probability of the forward and backward processes between the transition $|\psi_n\rangle \rightarrow |\phi_n\rangle$. Moreover, Eq. 10 implies that the entropy of a quantum state $S(\rho) = -\text{Tr} \rho \log \rho = -S(\rho || \frac{1}{d}) + \log d$ is non-decreasing under a noisy operation, since $\langle s \rangle = S(\mathcal{N}(\rho)) - S(\rho || \frac{1}{d}) \leq 0$.

B. Quantum thermodynamic process

Now we apply our QFT to quantum thermodynamic processes. Analogous to classical thermodynamics, we can define the equilibrium state, as known as the Gibbs state by which the maximum von Neumann entropy can be obtained for a fixed mean energy. For a given system Hamiltonian $H$, the Gibbs state is defined as $\gamma = Z^{-1}e^{-\beta H}$, where $Z = \text{Tr} e^{-\beta H}$ with the inverse temperature $\beta = (k_B T)^{-1}$. We demonstrate that the Gibbs state plays its role as the reference state of quantum thermodynamic processes, resulting in the heat and free energy consistently defined.

Suppose that a system is interacting with a thermal bath $\gamma_B$ in equilibrium, and their Hamiltonians are given by $H_S$ and $H_B$, respectively. The general thermal operation can be written as [25, 29, 30]

$$\mathcal{N}(\rho_S) = \text{Tr}_B \left[ U(\rho_S \otimes \gamma_B) U^\dagger \right],$$

with the condition of the energy conservation $[U, H_S + H_B] = 0$ when the system’s Hamiltonian remains unchanged during the evolution, or more generally, $U(H_S + H_B)U^\dagger = \tilde{H}_S + \tilde{H}_B$. Analogous to a noisy operation, we can find a fixed point of thermal operations such that $\mathcal{N}(\gamma_S) = \tilde{\gamma}_S$ when both $\gamma_S$ and $\tilde{\gamma}_S$ are in equilibrium. In this case $r_i = Z_S^{-1}e^{-\beta E_i}$ and $\tilde{r}_k = \tilde{Z}_S^{-1}e^{-\beta \tilde{E}_k}$, where the initial and final Hamiltonians of the system are given by $H_S = \sum_i E_i |i\rangle \langle i|$ and $H_B = \sum_k \tilde{E}_k |k\rangle \langle k|$, respectively. It is important to note that a general thermodynamic process is a covariant quantum operation with respect to the Gibbs state, which is related to time-translation symmetry [42] since $U_{\text{eq}}(\rho) = e^{-iH_S t}pe^{iH_S t}$ is the time-evolution operation. In this case the imaginary part of QIE $\delta q$ vanishes and the real part of QIE is given by $\delta q = \log(r_i/r_k) = \beta(\tilde{E}_k - E_i) + \log(\tilde{Z}_S/Z_S)$ in terms of the energy difference of the system. Then for an arbitrary state transformation from $\rho = \sum_n p_n |\psi_n\rangle \langle \psi_n|$ to $\mathcal{N}(\rho) = \sum_n q_n |\phi_n\rangle \langle \phi_n|$ by a quantum thermal process, the single-shot entropy production can be written as

$$\delta \sigma = -\log q_n + \log p_n - \beta(\tilde{E}_k - E_i) - \log(\tilde{Z}_S/Z_S).$$

Now the physical meaning of QIE becomes clearer because $\tilde{E}_k - E_i = \tilde{E}_k - E_i$ is the heat gain of the system from the bath. We also note that $\delta F = (\tilde{E}_k - E_i) - k_B T(-\log q_n + \log p_n)$ is a single-shot free energy difference, then we achieve the following thermodynamic FT. The quantum Jarzynski equality in this case can be written as

$$\langle e^{\beta \Delta F} \rangle = \tilde{Z}_S/Z_S,$$

where $\langle \Delta F \rangle = \Delta E - k_B T \Delta S$ is the quantum thermodynamic free energy difference on average. Using Theorem 3 we observe that the quantum free energy difference is bounded by the recoverability of $\rho$ by the reverse
process $\mathcal{R}_\gamma$:

$$\langle \Delta F \rangle \leq k_B T \log \frac{\bar{Z}_S}{Z_S} F(\rho, (\mathcal{R}_\gamma \circ \mathcal{N})(\rho))$$

$$\leq k_B T \log (\bar{Z}_S/Z_S).$$

We generalise this result to involve work done on the system by extending the definition of work and free energy in a more general sense. Suppose that the QIE $\delta q$ by a quantum process with reference state $\gamma$. During the same process the system gains the energy $\delta \epsilon = E_k - E_l$. Then a quantum work done on the system can be defined as

$$w := \delta \epsilon - \beta^{-1} \delta q$$

with some balancing coefficient $\beta$. Furthermore, a generalised single-shot free energy can be defined as

$$\delta f := \delta \epsilon - \beta^{-1} \delta s.$$  

Figure 4 describes the conversion of quantum information into entropy for the reference state $\mathcal{N}$ and the fluctuation relation for work and free energy exchange for a general quantum state. Here we derive the FT for quantum work

$$\langle e^{-\beta(w-\delta f)} \rangle = 1.$$  

In particular, when we consider a quantum thermodynamic process transforming a Gibbs state to another Gibbs state $\gamma \to \tilde{\gamma}$, we get the following fluctuation relation:

$$\frac{P(W)e^{-\beta W}}{P(-W)} = \frac{\bar{Z}_S}{Z_S} = e^{-\beta \Delta F_{eq}},$$

which leads to the classical work fluctuation relation $\langle e^{-\beta W} \rangle = e^{-\beta \Delta F_{eq}}$ in terms of the equilibrium free energy $F_{eq} = -k_B T \log Z_S$.

**C. Coherence merging and self referencing of covariant quantum processes**

By using the fluctuation relation of a covariant quantum process, we can prove that the off-diagonal element of a density matrix satisfies the following coherence merging inequality:

**Theorem 5** (Coherence merging bound for a covariant process).

$$|\mathcal{N}(\rho)|_{kl} \leq \sum_{\Omega^+} |\rho_{ij}| e^{-\delta q_R} + \sum_{\Omega^-} |\rho_{ij}|,$$

where $\Omega^+ = \{(i, j) | \delta q_R \geq 0 \text{ and } \delta q_l = 0\}$ and $\Omega^- = \{(i, j) | \delta q_R < 0 \text{ and } \delta q_l = 0\}$.  

The coherence merging inequality is a generalisation of coherence merging inequality in quantum thermodynamics, which provides restrictions on the evolution of each off-diagonal element under the covariant quantum process.

We also note that we can reconstruct the resource theory of asymmetry/coherence under a covariant operation $\mathcal{N}$ satisfying

$$\mathcal{U}_{r,t} \circ \mathcal{N} \circ \mathcal{U}_{r,t} = \mathcal{N},$$

which constructs a $U(1)$ asymmetry theory with respect to the generator $\log \gamma$. In the language of the resource theory of coherences, the degree of coherence can be quantified as

$$C_\gamma(\rho) := S(\rho|\mathcal{D}_\gamma(\rho)),$$

which is the relative entropy between a quantum state and its diagonalised state. In this case, even if $\gamma$ does not operate as a fixed reference state (i.e., $\mathcal{N}(\gamma) \neq \gamma$), we have freedom to choose a new reference state $\gamma = \mathcal{D}_\gamma(\tau)$ and its evolution $\tilde{\gamma} = \mathcal{N}(\chi) = (\mathcal{N} \circ \mathcal{D}_\gamma)(\tau) = (\mathcal{D}_\gamma \circ \mathcal{N})(\tau)$ for an arbitrary quantum state $\tau$. Especially when $\gamma$ has a set of eigenvalues $\{|i\rangle\}$ with non-degenerate eigenvalues, a QIE for the new reference can be defined as

$$\delta q' = \log \sqrt{r'_i r'_j} + i \log \sqrt{r'_i r'_k},$$

where $r'_i = |\langle i|\tau|i\rangle|$ and $r'_k = |\langle k|\mathcal{N}(\tau)|k\rangle|$, respectively. Then a fluctuation relation for this new references can be written as

$$\langle e^{-\delta s + \delta q'_R + i \delta q'_l} \rangle = 1. \quad (11)$$

One interesting remark is that one can choose a reference state based on the target state $\rho$ as $\gamma = \mathcal{D}_\gamma(\rho)$ then the first moments imply the monotonicity of the coherence measure

$$C_\gamma(\rho) \geq C_\gamma(\mathcal{N}(\rho)).$$

**D. Fluctuation of entanglement and coherent information by local measurements**

Finally, we apply the QFT to a stochastic entanglement generation by a local measurement on a pure quantum state, which has been studied in Ref. 40 and its generalisation in terms of the coherent information. Suppose that a pure bitartite state $|\Psi\rangle_{AB}$ is measured by a local measurement, say on $B$ and a local unitary operation is applied on the other party, say $A$ based on the measurement outcome as $M_\omega(|\Psi\rangle_{AB}) = (V_\omega \otimes K_\omega) |\Psi\rangle_{AB}$, where $V_\omega$ is a unitary operator acting on $A$ and $K_\omega$ is a generalised positive operator-valued measurement on $B$ satisfying $\sum_\omega K_{\omega}^\dagger K_\omega = I$. The resulting state would be another bipartite pure state $|\Psi_\omega\rangle_{AB}$.
with some probability $P_\omega$ to happen. Using the Schmidt decomposition of bipartite pure states, this process can be described as follows:

$$|\Psi\rangle_{AB} = \sum_\mu \sqrt{\rho_\mu} |\psi_\mu\rangle_{AB} \xrightarrow{\mathcal{M}_\omega} \sqrt{P_\omega}|\Phi_\omega\rangle_{AB},$$

where $|\Phi_\omega\rangle_{AB} = \sum_\nu \sqrt{q_\nu^\omega} |\phi_\nu^\omega\rangle_{AB}$. The quantum channel $\mathcal{N}$ for this process can be described as

$$|\Psi\rangle_{AB} (\omega) \xrightarrow{\mathcal{N}} \sum_\omega P_\omega |\Phi_\omega\rangle_{AB} |\Phi_\omega\rangle \otimes |\omega\rangle_M (\omega),$$

where $|\omega\rangle_M$ are orthogonal states which indicate the measurement outcomes $\omega$. During this process, the degree of entanglement change by a certain measurement outcome $\omega$ is given by $\Delta E_S^\omega = E_S(|\Phi_\omega\rangle_{AB}) - E_S(|\Psi\rangle_{AB})$ where $E_S(|\Psi\rangle_{AB}) = S(\text{Tr}_B |\Psi\rangle_{AB} |\Psi\rangle)$ is the entanglement entropy with $S(\rho) = -\text{Tr} \rho \log \rho$.

Since the local unitary operation $V_\omega$ does not change the entanglement of the pure bipartite system, the entanglement difference is determined by the entropy change of $B$ under the local measurement $K_\omega$. When we investigate the local party of $B$, the process can be simplified as $\rho_B \xrightarrow{\mathcal{M}_\omega} \{P_\omega, \rho_B^\omega\}$, where $\rho_B = \text{Tr}_A |\Psi\rangle_{AB} |\Psi\rangle = \sum_\mu P_\mu |\psi_\mu\rangle_B |\psi_\mu\rangle_B$ and $\rho_B^\omega = \text{Tr}_A |\Phi_\omega\rangle_{AB} |\Phi_\omega\rangle = \sum_\nu q_\nu^\omega |\phi_\nu^\omega\rangle_B |\phi_\nu^\omega\rangle_B$, respectively.

Now we take the reference state to be $\gamma = \frac{d_A}{d_A} \otimes \rho_B$, the evolution of which is given by $\tilde{\gamma} = \mathcal{N}(\gamma) = \frac{d_A}{d_A} \otimes \sum_\omega P_\omega \rho_B^\omega \otimes |\omega\rangle_M (\omega)$, where $d_A$ is the dimension of the party $A$. Then we can apply the quantum Jarzynski equality for entanglement given by

$$e^{-\log P_\omega + \log (P_\omega \rho_B^\omega)} = 1,$$

while the monotonicity of the first-order moment,

$$S(|\Psi\rangle |\Psi\rangle |\gamma\rangle) - S(\mathcal{N}(|\Psi\rangle |\Psi\rangle)) = S\left(|\Psi\rangle |\Psi\rangle \frac{1_A}{d_A} \otimes \rho_B \right) - S\left(\sum_\omega P_\omega |\Phi_\omega\rangle_{AB} |\Phi_\omega\rangle \otimes |\omega\rangle_M (\omega) \frac{1_A}{d_A} \otimes \sum_\omega P_\omega \rho_B^\omega \otimes |\omega\rangle M (\omega)\right),$$

leads to the non-increasing of entanglement by local measurements with local quantum feedbacks [40].

It is worth noting that Eq. (12) also holds for an arbitrary mixed bipartite state $\rho_{AB}$ and the monotonicity of the first-order moment becomes

$$I(A)B_{\rho} \geq \sum_\omega P_\omega I(A)B_{\rho_\omega}.$$

where $I(A)B_{\rho} = S(\rho_B) - S(\rho_{AB})$ is the coherent information [47, 48]. In particular, this results can be utilised to investigate information fluctuation relation with feedback controls [43, 50], by which thermodynamic quantities such as quantum heat and work can be coherently combined to both quantum and classical information quantities involved in Maxwell’s demon or Laundauer’s erasure in a unified viewpoint.

**VI. DISCUSSIONS**

We have demonstrated that for every quantum process it is possible to construct the QFT with respect to a reference state by replacing microscopic reversibility in classical thermodynamics with the reference state recoverability. In this generalised QFT, every marginal probability distributions of a TPM is preserved by introducing the transition amplitude rather than transition probability. In our QFT, quantum information quantities of the system state and the reference state play an important role as entropy and the QIE in the entropy production during the quantum process. We have demonstrated that the ratio between the probabilities of entropy production by forward and backward quantum processes obeys a fluctuation relation in a similar form in the classical FT, yet this entropy production has a complex value in the QFT. Moreover, our work has shown that a covariant process provides only a real-valued QIE corresponding to classical physics. The observation of an imaginary valued QIE may imply the broken time-translational symmetry, yet finding the deeper physical meaning of it would be an interesting question to be answered in the future research. Our approach provides a unified understanding of the fluctuations of both thermodynamic and information theoretic quantities in a single framework for applications to a wide range of physics, besides important examples including thermodynamics and coherence and entangle-
ment fluctuations.

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Appendix A: Proof of quantum fluctuation relation

We first prove Proposition 1

\[ T_{ij}^{kl} = \sqrt{\frac{\tilde{r}_i\tilde{r}_j}{\tilde{r}_i\tilde{r}_j}} \frac{i\hbar \log (\frac{\tilde{r}_i\tilde{r}_j}{\gamma_{ij}})}{\gamma_{ij}} \left( T(t) \right)_{ij}^{kl} \]

Proof. Suppose that quantum channel \( \mathcal{N} \) has a Kraus representation \( \mathcal{N}(\rho) = \sum \kappa_i \rho \). Then the matrix element of a rotated Petz recovery map \( \mathcal{R}_{ij} \) is given by

\[
\left( T(t) \right)_{ij}^{kl} = \langle j | \mathcal{R}_{ij}^t (|i\rangle \langle \tilde{k}|) |i\rangle \\
= \sqrt{\frac{\tilde{r}_i\tilde{r}_j}{\tilde{r}_i\tilde{r}_j}} \sum \omega \langle j | \mathcal{K}_i^j (|\tilde{k}\rangle |K_\omega i\rangle |i\rangle \rangle \frac{i\hbar \log (\frac{\tilde{r}_i\tilde{r}_j}{\gamma_{ij}})}{\gamma_{ij}} \\
= \sqrt{\frac{\tilde{r}_i\tilde{r}_j}{\tilde{r}_i\tilde{r}_j}} \sigma_{ij}^{kl} T_{ij}^{kl},
\]

which completes the proof.

This directly leads to the following:

\[ T(\psi)_{ij}^{kl} = \langle i | \psi \rangle \langle j | \tilde{\tilde{u}} \rangle \langle \tilde{\tilde{v}} | \psi \rangle T_{ij}^{kl} \]

\[ = \langle i | \psi \rangle \langle j | \tilde{\tilde{u}} \rangle \langle \tilde{\tilde{v}} | \psi \rangle \frac{\tilde{r}_i\tilde{r}_j}{\gamma_{ij}} \frac{i\hbar \log (\frac{\tilde{r}_i\tilde{r}_j}{\gamma_{ij}})}{\gamma_{ij}} \left( T(t) \right)_{ij}^{kl} \]

\[ = \frac{\tilde{r}_i\tilde{r}_j}{\gamma_{ij}} \frac{i\hbar \log (\frac{\tilde{r}_i\tilde{r}_j}{\gamma_{ij}})}{\gamma_{ij}} \left( T(t) \right)_{ij}^{kl} \]

Finally, by taking summation over \( \{i,j,k,l\} \),

\[ T(\psi) = \sum_{i,j,k,l} T(\psi)_{ij}^{kl} \]

\[ = \sum_{i,j,k,l} \frac{\tilde{r}_i\tilde{r}_j}{\gamma_{ij}} \frac{i\hbar \log (\frac{\tilde{r}_i\tilde{r}_j}{\gamma_{ij}})}{\gamma_{ij}} \left( T(t) \right)_{ij}^{kl} \]

\[ = \left( T(t) \right)_{ij}^{kl} \left( J_{ij} \right)_{kl} \]

leads to Theorem 1.
Appendix B: Proof of the quantum Crooks fluctuation relation

Proof. We provide the proof of Theorem 2 as follows:

\[ P(\sigma) e^{-\sigma R + i\sigma I} = \sum_{\mu,\nu} \sum_{ij,kl} p_{\mu} T(\psi_{\mu} \rightarrow \phi_{\nu})_{ij} e^{-\log \left[ \frac{p_{\mu} \sqrt{r_{ij} r_{ik} r_{jk} r_{kl}}}{\sqrt{r_{kl} r_{ij}} \sqrt{r_{jk} r_{ik}}} \right]} \]

\[ \times \delta \left( \sigma_{R} - \log \left[ \frac{p_{\mu} \sqrt{r_{ij} r_{ik} r_{jk} r_{kl}}}{\sqrt{r_{kl} r_{ij}} \sqrt{r_{jk} r_{ik}}} \right] \right) \delta \left( \sigma_{I} - \log \left[ \frac{r_{ij} r_{ik} r_{jk} r_{kl}}{r_{kl} r_{ij} r_{jk} r_{ik}} \right] \right) \]

\[ = \sum_{\mu,\nu} \sum_{ij,kl} q_{\nu} \sqrt{r_{ij} r_{ik} r_{jk} r_{kl}} e^{i t \log \left[ \frac{r_{ij} r_{ik} r_{jk} r_{kl}}{r_{kl} r_{ij} r_{jk} r_{ik}} \right]} \sum_{\mu,\nu} \sum_{ij,kl} \frac{p_{\mu}}{p_{\nu}} T(\psi_{\mu} \rightarrow \phi_{\nu})_{ij} \]

\[ \times \delta \left( \sigma_{R} - \log \left[ \frac{p_{\mu} \sqrt{r_{ij} r_{ik} r_{jk} r_{kl}}}{\sqrt{r_{kl} r_{ij}} \sqrt{r_{jk} r_{ik}}} \right] \right) \delta \left( \sigma_{I} - \log \left[ \frac{r_{ij} r_{ik} r_{jk} r_{kl}}{r_{kl} r_{ij} r_{jk} r_{ik}} \right] \right) \]

\[ = \sum_{\mu,\nu} \sum_{ij,kl} q_{\nu} \tilde{T}^{(t/2)}(\phi_{\nu} \rightarrow \psi_{\nu})_{ij} \]

\[ \times \delta \left( \sigma_{R} - \log \left[ \frac{p_{\mu} \sqrt{r_{ij} r_{ik} r_{jk} r_{kl}}}{\sqrt{r_{kl} r_{ij}} \sqrt{r_{jk} r_{ik}}} \right] \right) \delta \left( \sigma_{I} - \log \left[ \frac{r_{ij} r_{ik} r_{jk} r_{kl}}{r_{kl} r_{ij} r_{jk} r_{ik}} \right] \right) \]

\[ = \sum_{\mu,\nu} \sum_{ij,kl} q_{\nu} \tilde{T}^{(t/2)}(\phi_{\nu} \rightarrow \psi_{\nu})_{ij} \]

\[ \times \delta \left( \sigma_{R} - \log \left[ \frac{p_{\mu} \sqrt{r_{ij} r_{ik} r_{jk} r_{kl}}}{\sqrt{r_{kl} r_{ij}} \sqrt{r_{jk} r_{ik}}} \right] \right) \delta \left( \sigma_{I} - \log \left[ \frac{r_{ij} r_{ik} r_{jk} r_{kl}}{r_{kl} r_{ij} r_{jk} r_{ik}} \right] \right) \]

\[ = \tilde{P}^{(t/2)}(-\sigma^{*}), \]

where the third equality comes from Theorem 1 and entropy production by the backward process \( \tilde{T}^{(t/2)}(\phi_{\nu} \rightarrow \psi_{\nu})_{ij} \) is given by \( \delta \tilde{\sigma} = \log \left[ \frac{\sqrt{r_{ij} r_{ik} r_{jk} r_{kl}}}{p_{\nu} \sqrt{r_{kl} r_{ij} r_{jk} r_{ik}}} \right] + i \log \sqrt{r_{ij} r_{ik} r_{jk} r_{kl}} \) for any values of \( t \).

Appendix C: Proofs for covariant quantum processes

We first show that if a quantum channel \( \mathcal{N} \) is covariant with respect to the reference state, i.e.,

\[ \mathcal{U}_{\gamma_{-t}} \circ \mathcal{N} \circ \mathcal{U}_{\gamma_{t}} = \mathcal{N}, \]

then all of its rotated Petz recovery maps \( \mathcal{R}^{\gamma}_{t} \) are the same with \( \mathcal{R}_{\gamma} \). Note that \( \mathcal{U}_{\gamma_{t}} \) and \( \mathcal{U}_{\gamma_{-t}} \) are unitary processes, therefore the adjoint map \( \mathcal{N}^{\dagger} \) satisfies

\[ \mathcal{N}^{\dagger} = (\mathcal{U}_{\gamma_{-t}} \circ \mathcal{N} \circ \mathcal{U}_{\gamma_{t}})^{\dagger} = \mathcal{U}_{\gamma_{t}} \circ \mathcal{N}^{\dagger} \circ \mathcal{U}_{\gamma_{-t}}. \]

Therefore, the rotated Petz recovery map is given by

\[ \mathcal{R}^{\gamma}_{t}(\chi) = (\mathcal{U}_{\gamma_{-t}} \circ \mathcal{R}_{\gamma} \circ \mathcal{U}_{\gamma_{t}})(\chi) \]

\[ = \mathcal{U}_{\gamma_{-t}} \left[ \frac{1}{2} \mathcal{N}^{\dagger}(\tilde{\gamma}^{-1/2} \mathcal{U}_{\gamma_{t}}(\chi) \tilde{\gamma}^{-1/2}) \frac{1}{2} \right] \]

\[ = \gamma^{1/2} \left[ (\mathcal{U}_{\gamma_{-t}} \circ \mathcal{N}^{\dagger} \circ \mathcal{U}_{\gamma_{t}})(\tilde{\gamma}^{-1/2} \chi \tilde{\gamma}^{-1/2}) \right] \gamma^{-1/2} \]

\[ = \gamma^{1/2} \mathcal{N}^{\dagger}(\tilde{\gamma}^{-1/2} \chi \tilde{\gamma}^{-1/2}) \gamma^{-1/2} \]

\[ = \mathcal{R}_{\gamma}(\chi), \]

since \( \mathcal{U}_{\gamma_{t}} \) commutes with \( \gamma \).

Now we prove the remaining part of Theorem 3. In terms of the elements of a transition matrix, every covariant quantum process obeys

\[ T^{kl}_{ij} e^{i t \log \left( \frac{r_{ij} r_{kl}}{r_{ik} r_{jl}} \right)} = T^{kl}_{ij}. \]

Multiplying the both side by \( e^{-i t \sigma} \) followed by the integration over \( t \) leads to

\[ T^{kl}_{ij} \delta \left( \sigma_{I} - \log \left( \frac{r_{ij} r_{kl}}{r_{ik} r_{jl}} \right) \right) = T^{kl}_{ij} \delta(\sigma_{I}). \quad (C1) \]

Therefore, the probability distribution of entropy production is given by

\[ P(\sigma_{R} + i \sigma_{I}) = P(\sigma_{R}) \delta(\sigma_{I}) \]

and the quantum CFT with real values of \( \sigma \) is given by

\[ \frac{P(\sigma)}{P(-\sigma)} = e^{\sigma}, \]

where the parameter \( t \) is omitted since every rotated recovery map is identical to the Petz recovery map \( \mathcal{R}_{\gamma} \).

Theorem 4 can be directly proven by the relationship between the relative entropy and reversibility of a quantum channel [11]

\[ S(\rho|\gamma) - S(\mathcal{N}(\rho)||\mathcal{N}(\gamma)) \geq - \int_{-\infty}^{\infty} dt g_{0}(t) \log \left[ \mathcal{F}(\rho, (\mathcal{R}^{t/2}_{\gamma} \circ \mathcal{N})(\rho)) \right], \]

where \( g_{0}(t) = (\pi/2)[\cosh(\pi t) + 1]^{-1} \) satisfying \( \int_{-\infty}^{\infty} dt g_{0}(t) dt = 1 \). The righthand side of the above inequality becomes \( \log \left[ \mathcal{F}(\rho, (\mathcal{R}_{\gamma} \circ \mathcal{N})(\rho)) \right] \) for every \( t \) [11], which is equivalent to \( \rho = (\mathcal{R}_{\gamma} \circ \mathcal{N})(\rho) \) for covariant quantum processes.

Also, the equality condition \( S(\rho|\gamma) = S(\mathcal{N}(\rho)||\mathcal{N}(\gamma)) \) is achieved if and only if \( \rho = (\mathcal{R}^{t/2}_{\gamma} \circ \mathcal{N})(\rho) \) for every \( t \) [11], which is equivalent to \( \rho = (\mathcal{R}_{\gamma} \circ \mathcal{N})(\rho) \) for covariant quantum processes.

Appendix D: Proof of coherence merging bound

Finally, we prove the bound for coherence merging (Theorem 5) under a covariant quantum process. By the previous arguments, \( T^{kl}_{ij} \) has a non-vanishing value only
if \( q_l = \log \sqrt{\frac{r_i \hat{r}_j}{r_j \hat{r}_k}} = 0 \). Then we have

\[
|N(\rho)_{kl}| = \left| \sum_{i,j} \rho_{ij} T_{ij}^{kl} \right| \\
\quad \leq \sum_{\Omega^+} \rho_{ij} |T_{ij}^{kl}| + \sum_{\Omega^-} \rho_{ij} |T_{ij}^{kl}| \\
\quad = \sum_{\Omega^+} \rho_{ij} \sqrt{\frac{\hat{r}_i \hat{r}_j}{r_i r_j}} \left| \hat{T}^{(t)}_{ij} \right|^{(t)} + \sum_{\Omega^-} \rho_{ij} |T_{ij}^{kl}| \\
\quad \leq \sum_{\Omega^+} \rho_{ij} e^{-q_R} \left| T_{ij}^{kl} \right| + \sum_{\Omega^-} \rho_{ij} |T_{ij}^{kl}| \\
\quad \leq \sum_{\Omega^+} \rho_{ij} e^{-q_R} + \sum_{\Omega^-} \rho_{ij},
\]

by using that both \( |T_{ij}^{kl}| \) and \( |\hat{T}^{(t)}_{ij}^{(t)}| \) are less than 1 and

\[q_R = -\log \sqrt{\frac{r_i \hat{r}_j}{r_j \hat{r}_k}}.\]