Abstract. In this paper we state and prove the statement that there are finitely many m-maximal green sequences of tame hereditary algebras.

1. Introduction

Maximal green sequences were invented by Bernhard Keller [6]. In Brüstle-Dupont-Péro tin [1] and the paper by the first author together with Brüstle, Hermes and Todorov [2] it is proven, using representation theory, that there are finitely many maximal green sequences for a valued quiver of finite or tame type or the quiver is mutation equivalent to a quiver of finite or tame types. Furthermore in [2] it is proven that any tame valued quiver has finitely many k-reddening sequences.

Since we have maximal green sequences in cluster theory it is reasonable to look at the generalization of this concept in m-cluster theory, namely m-maximal green sequences.

We recall that, for Λ a finite dimensional hereditary algebra over any field, the indecomposable objects of the bounded derived category $D^b(\Lambda)$ of $\text{mod-} \Lambda$ are $M[k]$ where $M$ is an indecomposable $\Lambda$-module. Such an object is rigid if $\text{Ext}^1(M, M) = 0$. Two rigid objects $M[i], N[j]$ with $i \leq j$ are called compatible if either

1. $i = j$ and $\text{Ext}^1(M, N) = 0 = \text{Ext}^1(N, M)$ or
2. $i < j$ and $\text{Hom}(N, M) = 0 = \text{Ext}^1(N, M)$.

An object of $D^b(\Lambda)$ is called pre-silting if its components are pairwise compatible rigid objects. A pre-silting object is called silting if it has the maximum number of nonisomorphic rigid objects which is $n$, the number of simple $\Lambda$-modules.

Definition 1.1. An m-maximal green sequence is defined as a finite sequence $\{T_i\}$ of silting objects from $T_0 = \Lambda$ to $T_N = \Lambda[m]$ such that $T_{i+1}$ is obtained by a forward Iyama-Yoshino mutation from $T_i$. (See [5])

Theorem 1.2. Any tame hereditary algebra has finitely many m-maximal green sequences for any $m \geq 1$.

To prove this theorem we only need to prove that only finitely many indecomposable objects can appear as summands of those silting objects that appear in m-maximal green sequences of tame hereditary algebras Λ. It is well-known that all indecomposable objects of $D^b(\Lambda)$ are either transjective or regular and there are only finitely many rigid regular objects between $\Lambda$ and $\Lambda[m]$ in $D^b(\Lambda)$ for $\Lambda$ tame. Hence the problem is reduced to proving that only finitely many indecomposable

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transjective objects between $\Lambda$ and $\Lambda[m]$ can appear in $m$-maximal green sequences.

To prove this theorem we need two lemmas.

**Lemma 1.3.** For a tame hereditary algebra $\Lambda$ any silting object in $D^b(\Lambda)$ contains at most $n - 2$ regular summands. In other words, at least 2 summands have to be transjective.

**Lemma 1.4.** For a tame hereditary algebra $\Lambda$ there is a uniform bound, depending only on $\Lambda$ and $m$, on the transjective degree of any transjective summand in any silting object in any $m$-maximal green sequence $D^b(\Lambda)$.

It is easy to see why Lemma 1.4 implies the theorem. Here the transjective degree of an indecomposable transjective object $\tau^i P_j[k]$ is defined as $\deg(\tau^i P_j[k]) = i$. The maximal transjective degree and minimal transjective degree of a silting object are defined as the highest/lowest transjective degree of its indecomposable transjective summands respectively.

In Section 2 we prove Lemma 1.3. In Section 3 we prove Lemma 1.4. In Section 4 we further generalize the theorem to arbitrary finite mutation sequences with finitely many forward/green or backward/red mutations.

**2. Lemma 1.3: at least 2 transjective summands**

To prove Lemma 1.3 we recall that regular components of Auslander-Reiten quivers of tame hereditary algebras are all standard stable tubes with at most three tubes which are nonhomogeneous (see [4] and Chapter X of [9]). Note that objects in a homogeneous tube are not rigid so cannot appear in a silting object of $D^b(\Lambda)$. Hence we only need to discuss the nonhomogeneous tubes.

It is clear that two shifts of the same indecomposable object $M$ are not compatible. Hence the regular indecomposable components of any silting object are distinct regular modules in various degrees. We say that a family of rigid indecomposable $\Lambda$-modules $\{M_i\}_{i \in I}$ is silting-incompatible if $\bigoplus_{i \in I} M_i[k_i]$ is not pre-silting for any $\{k_i\}_{i \in I}$. Otherwise we say that the family of modules is silting-compatible.

Let $M_i$ be the quasi-simple modules in a tube of size $s$ such that $\tau M_i = M_{i-1}$ where the indices are understood to be modulo $s$. We call a regular module in $D^b(\Lambda)$ regular sincere if its composition series contains all regular simples (the quasi-simple modules). Indecomposable regular sincere modules and their shifts cannot appear as summands in any silting object because they are not rigid. (See Corollary X.2.7 of [9]). The remaining $s(s-1)$ indecomposable objects in the tube are rigid and we can unambiguously label them as $M_{ij}$ if the regular top and regular socle of the object are $M_j$ and $M_i$ respectively. Note that $M_i = M_i$. It is clear that $\tau M_{ij} = M_{i+1,j-1}$ and $\tau^{-1} M_{ij} = M_{i+1,j+1}$ with indices taken modulo $s$.

Now let’s prove two easy lemmas on what can not appear in a pre-silting object in a regular component of the Auslander-Reiten quiver of $D^b(\Lambda)$.

**Lemma 2.1.**

1. If $M$ and $N$ are regular modules in a nonhomogeneous tube in the Auslander-Reiten quiver of $kQ$. If $\text{Hom}(M, N) \neq 0$ and $\text{Ext}^1(N, M) \neq 0$, then $M$ and $N$ are silting-incompatible.

2. Let $X_1, \ldots, X_k$ be regular modules in a nonhomogeneous tube in the Auslander-Reiten quiver of $\Lambda$. If $\text{Hom}(X_i, \tau X_{i+1}) \neq 0$ for any $1 \leq i < k$ and $\text{Hom}(X_k, \tau X_1) \neq 0$, then $\{X_i\}$ is silting-incompatible.
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Proof: For (1) since $\text{Hom}(M, N) \neq 0$, $\text{Ext}^{i-j}(M[i], N[j]) \neq 0$ if $i > j$. Since $\text{Ext}^{i}(N, M) \neq 0$ $\text{Ext}^{j-i+1}(N[j], M[i]) \neq 0$ if $i \leq j$. Hence $M[i] \oplus N[j]$ is not pre-silting for any $i$ and $j$.

For (2) for arbitrary $n_1, \cdots, n_k$, by a similar argument we see that, if $\oplus_{i=1}^{k} X_i[n_i]$ is pre-silting, then $n_1 < n_2 < \cdots < n_k < n_1$ which is impossible. Hence $\{X_i\}$ is silting-incompatible.

Lemma 2.2. Any pre-silting object in a standard stable tube of size $s$ contains at most $s - 1$ summands.

To prove this lemma we need the following lemma.

Lemma 2.3. Any pre-silting object in a standard stable tube of size $s$ cannot be regular sincere.

Proof. Assume that a pre-silting object $T$ in a standard stable tube of size $s$ is regular sincere. For each component $X = M_{ij}$ of $T$, there is another component $Y$ of $T$ having $M_{j+1}$ in its composition series. Then $\tau Y$ has $M_j$ in its composition series. So, $\text{Hom}(X, \tau Y) \neq 0$. Continuing in this way, we find a sequence of components of $T$ so that each maps to $\tau$ of the next. This sequence eventually repeats giving a contradiction to Lemma 2.1(2). Therefore, $T$ cannot be regular sincere.

Now we can prove Lemma 2.2.

Proof of Lemma 2.2. Since any pre-silting object in a standard stable tube of size $s$ cannot be regular sincere, without loss of generality it is a pre-silting object in the exact subcategory of $T$ closed under extensions such that $M_1, \cdots, M_{s-1}$ are the only simple objects. This category is equivalent to the module category of $KA_{s-1}$ with linear orientation and as a result any pre-silting object in it has at most $s - 1$ summands.

Finally we can prove Lemma 1.4.

Proof of Lemma 1.3. Due to Lemma 2.2 and [4] there are at most $n - 2$ regular components in $D^b(\Lambda)$ when $\Lambda$ is a tame hereditary algebra. This is true for each type so this is true in all cases.

3. Lemma 1.4 Uniform bound on transjective degree

To prove Lemma 1.4 we need to rephrase an argument in [1] using degrees.

Lemma 3.1. ([1], Lemma 10.1) Let $H$ be a representation-infinite connected hereditary algebra. Then there exists $N \geq 0$ such that for any $k \geq N$, for any projective $H$-module $P$, the $H$-modules $\tau^{-k} P$ and $\tau^{k+1} P[1]$ are sincere.

For example, if $n = 2$, then $N = 1$.

Lemma 3.2. ([1]) Let $\Lambda$ be a tame hereditary algebra and $M_1, M_2$ two transjective $\Lambda$-modules. If $\{M_1, M_2\}$ is silting-compatible, then $|\text{deg}(M_1) - \text{deg}(M_2)| \leq N$

Proof. For $k - \ell > N$ we need to prove that $\tau^k P_a$ and $\tau^{-k} P_b$ are silting-incompatible. If $i \leq j$, $\text{Ext}^{j-i+1}(\tau^k P_a[i], \tau^{-k} P_b[j]) = \text{Ext}^{i}(\tau^k P_a, \tau^{-k} P_b) = \text{DHom}(\tau^k P_a, \tau^{-k} P_b) = 0$ since $\tau^{-k-1} P_b$ is a sincere preprojective module. If $i > j$, $\text{Ext}^{j-i}(\tau^k P_a[i], \tau^{-k} P_b[j]) = \text{Hom}(\tau^k P_a, \tau^{-k} P_b) = \text{Hom}(P_a, \tau^{k-1} P_b) = 0$ since $\tau^{k-1} P_b$ is also sincere. Hence $\tau^k P_a$ and $\tau^{-k} P_b$ are silting-incompatible. Exchange the objects if $k - \ell < -N$. Hence the lemma has been proven.
We now prove Lemma 1.4 following a modified version of the argument in [1].

**Proof of Lemma 1.4.** We will prove that there is a lower bound on the minimal transjective degree of any silting object that can appear in an $m$-maximal green sequence. Let $R_j$ denote the set of objects $R[j]$ where $R$ is regular and let $P_j$ denote the set of all objects $\tau^k P[j]$. Given a silting object $T$, let $n_j$ be the number of components of $T$ in $P_j \cup R_j - 1$. Then we claim:

$$\min \deg T \geq -m \sum_{j=0}^{m} n_j (m - j)N = -nmN + \sum_{j=0}^{m} jn_j N$$

The proof will be by induction on the number of green mutations from $T$ to $\Lambda[m]$. If this number is zero then $T = \Lambda[m]$ and both sides of the inequality are zero. So, the inequality holds in this case.

Suppose that (3.1) holds for $T$ and $T'$ is obtained from $T$ by one red mutation, say $T' = \mu_i T$. So, $T, T'$ differ in their $i$th components $T_i, T'_i$. Then we will show that the inequality also holds for $T'$.

By Lemma 1.3 $T$ has at least 2 transjective components. So, $T/T_i = T'/T'_i$ has a transjective component and $\min \deg (T/T_i) \geq \min \deg T$. By Lemma 3.2,

$$\min \deg T'_i \geq \min \deg (T'/T'_i) - N \geq \min \deg T - N.$$

However, the only way that $\min \deg T'$ can be less than $\min \deg T$ is if $T'_i \in P_k$ and $T_i \in P_j \cup R_{j-1}$ for some $j > k$. But then the RHS of (3.1) decreases by at least $N$. So, the inequality will hold for $T'$ in that case. Finally, if $\min \deg T' \geq \min \deg T$ then the inequality clearly holds for $T'$ since the RHS of (3.1) does not increase under a red mutation.

Thus (3.1) holds for any silting object $T$ in an $m$-maximal green sequence. In particular each transjective component of $T$ has degree at least $-nmN$.

Similarly, silting objects in $m$-maximal green sequences can not have maximal transjective degree higher than $nmN$ or it can not start from $\Lambda$. \hfill \Box

## 4. $m$-RED SEQUENCES

Using the same method we can prove a stronger result.

**Definition 4.1.** Let $\Lambda$ be a finite dimensional hereditary algebra. A mutation sequence in $D^b(\Lambda)$ is $m$-red if it contains $m$ backward mutations with the rest being forward mutations and it is $m$-green if it contains exactly $m$ forward mutations.

A 0-red sequence is just a green one. A 0-green sequence is just a red one.

**Theorem 4.2.** If $\Lambda$ is a hereditary algebra of finite or tame type and $T_1, T_2$ are silting objects of $D^b(\Lambda)$ then there are only finitely many $m$-red and $m$-green mutation sequences from $T_1$ to $T_2$ for any $m$.

The main purpose of this theorem is to show, assuming the “$m$-Rotation Lemma”, that there are only finitely many $m$-maximal green sequences for any cluster-tilted algebra $A$ of tame type. The idea is that $A$ is the endomorphism ring of a silting object $T$ for a hereditary algebra $\Lambda$ and an $m$-cluster version of the Rotation Lemma from [2] is expected to state that $m$-maximal green sequences for $A$ are in bijection with 0-red, i.e., green sequences from $T$ to $T[m]$ in $D^b(\Lambda)$. The idea that this holds comes from [7].
Note that an \(m\)-green sequence from \(T_1\) to \(T_2\) is equivalent to an \(m\)-red sequence from \(T_2\) to \(T_1\). So, we only need to prove that part of the statement about \(m\)-red sequences. For this, we first need to prove the following lemma which is a generalization of Lemma 4.4.2 in [2].

**Lemma 4.3.** Any \(m\)-red sequence from \(T_1\) to \(T_2\) can go through any silting object at most \(m + 1\) times.

**Proof.** It is clear from the definition of mutations that a green sequence can go through any silting object at most once. (See [3] and [8] for more details.) Let’s define a maximal green arm of a mutation sequence as a maximal subsequence \(X_1, X_2, \ldots, X_k\) of silting objects in the mutation sequence so that each mutation \(X_i \rightarrow X_{i+1}\) is green. Since there are \(r\) red mutations, the entire mutation sequence is a disjoint union of exactly \(m + 1\) maximal green arms, some of which can have length one. Since each maximal green arm can go through a silting object at most once, the union of these arms can pass through the same silting object at most \(m + 1\) times. □

By repeating the same mutation \(T \leftrightarrow T'\) \(2m\) times we see that the bounds established in the lemma are optimal. Now we can prove the theorem. Note that Lemma 4.3 above implies that, in the tame case, if we can prove that for any \(m\) there are only finitely many rigid objects that can appear as summands of silting objects in \(m\)-red sequences, the theorem will be proven.

**Proof of Theorem 4.2.** As we said above we will only prove the part about \(m\)-red sequences. Assume that all indecomposable summands of \(T_1\) and \(T_2\) are in \((\text{mod-}\Lambda)[k]\) for \(i \leq k < j\). Then, all indecomposable summands that appear in \(m\)-red sequences from \(T_1\) to \(T_2\) have to be in \((\text{mod-}\Lambda)[k]\) for \(i - m \leq k < j + m\), i.e., between \(\Lambda[i - m]\) and \(\Lambda[j + m]\).

For \(\Lambda\) of finite type, there are only finitely many indecomposable objects in this range and hence only finitely many silting objects can exist on an \(m\)-red sequence. Due to Lemma 4.3, there are finitely many \(m\)-red sequences.

From now on we assume that \(\Lambda\) is tame. There are only finitely many regular rigid indecomposable objects between \(\Lambda[i - m]\) and \(\Lambda[j + m]\) so the problem has been reduced to proving that only finitely many transjective indecomposable components can appear in silting objects in \(m\)-red sequences.

Let the minimal degree of \(T_2\) be \(L\). Note that a red mutation can increase the minimal degree of a silting object by at most \(N\). Use an argument similar to that one used to prove Theorem 1.2 we can prove that no indecomposable transjective object with degree less than \(L - nN(2m + j - i) - mN\) can appear in any \(m\)-red sequences from \(T_1\) to \(T_2\). Similarly let the maximal degree of \(T_1\) be \(H\). No indecomposable transjective object with degree greater than \(H + nN(2m + j - i) + mN\) can appear in any \(m\)-red sequence from \(T_1\) to \(T_2\). Hence, only finitely many indecomposable transjective objects can appear in any \(m\)-red sequence from \(T_1\) to \(T_2\) and the theorem is proven. □

The bounds on transjective degrees in the proofs of Theorems 1.2 and 4.2 above are very crude. In the future we will try to find better bounds. We also hope to prove the \(m\)-Rotation Lemma.
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