Higher spin currents in Wolf space: Part II

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Abstract

By calculating the operator product expansions (OPEs) between the 16 higher spin currents of spins $\left(\frac{1}{2}, \frac{3}{2}, 2\right)$, $\left(\frac{3}{2}, 2, 2, \frac{3}{2}\right)$, $\left(\frac{3}{2}, 2, 2, \frac{5}{2}\right)$, and $\left(2, 2, 2, \frac{5}{2}\right)$ in the $\mathcal{N} = 4$ superconformal Wolf space coset $SU(3) \times SU(2) \times U(1)$, realized by $SU(3) \times SU(2) \times U(1)$ WZW affine currents, the next 16 higher spin currents of spins $\left(2, \frac{5}{2}, 2, \frac{5}{2}\right)$, $\left(2, \frac{5}{2}, 2, \frac{5}{2}\right)$, and $\left(2, \frac{5}{2}, 2, \frac{5}{2}\right)$ are determined from the right hand sides of these OPEs. Moreover, the composite fields consisting of both the 11 currents in the large $\mathcal{N} = 4$ nonlinear superconformal algebra and the above 16 lowest higher spin currents also occur in the right hand sides of these OPEs. The latter appears quadratically (and linearly) in the fusion rules together with large $\mathcal{N} = 4$ nonlinear superconformal family of the identity operator.

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1. Introduction

For the $\mathcal{N} = 4$ superconformal coset theory (in two dimensions) described by the Wolf space $SU(N + 2) / SU(N) \times SU(2) \times U(1)$ with $N = 3$, the $\mathcal{N} = 2$ WZW affine current algebra with constraints was obtained in [1]. The 16 generators of the large $\mathcal{N} = 4$ (linear) superconformal algebra were described by these $\mathcal{N} = 2$ WZW affine currents explicitly. By factoring out both four spin-$\frac{1}{2}$ currents and the spin-$1$ current from these 16 generators, the remaining 11 generators (i.e., spin-$2$ current, four spin-$\frac{3}{2}$ currents, and six spin-$1$ currents) leading to the large $\mathcal{N} = 4$ (nonlinear) superconformal algebra were obtained. Based on the large $\mathcal{N} = 4$ holography [2], the extra 16 higher spin currents, with spin contents

\begin{equation}
\begin{pmatrix}
1, \frac{3}{2}, \frac{3}{2}, 2 \\
\frac{3}{2}, 2, 2, \frac{5}{2} \\
\frac{3}{2}, 2, 2, \frac{5}{2} \\
2, \frac{5}{2}, \frac{5}{2}, 3
\end{pmatrix}
\end{equation}
described in terms of four \( \mathcal{N} = 2 \) multiplets, were found and realized by the above \( \mathcal{N} = 2 \) WZW affine currents. Then the operator product expansions (OPEs) between the above 11 currents and these extra 16 higher spin currents (1.1), which are a part of \( \mathcal{N} = 4 \) \( \mathcal{W}_\infty \) algebra, were described explicitly. It turned out that the composite \( \mathcal{WZW} \) af multiplets, were found and realized by the above \( \mathcal{N} = 2 \) affine currents in the complex basis (or Cartan–Weyl basis).

One can vary \( N \) from \( N = 3 \) in order to observe the \( N \) dependence of the structure constants of OPEs. Then one can obtain the higher spin currents for \( N = 5, 7, 9 \) and 11 where the corresponding \( SU(N + 2) \) group has even dimensions. One expects that the structure constants appearing all the OPEs in [1] and in this paper depend on \( N \) explicitly and one hopes that one determines the structure constants for general \( N \) from the above low \( N \) values results. Then in principle, the higher spin (Casimir) currents for general \( N \) can be written in terms of multiple products of \( \mathcal{N} = 2 \) WZW affine currents. For example, it is already known that the 11 currents of large \( \mathcal{N} = 4 \) nonlinear superconformal algebra are given in the [41] of [1]. It is an open problem to construct the lowest 16 higher spin Casimir currents (which will include the expressions for \( N = 3 \) results in [1] and in this paper) explicitly. One has also arbitrary \( k \) dependence in the higher spin currents. One can take the large \( (N, k) \) limit for any quantity which has an explicit finite \( (N, k) \) dependence, as in the purely bosonic duality case.

On the other hand, the \( (0, f) \) representation of Wolf space coset is of the form \( Q^A \langle 0 \rangle \), where \( Q^A(z) \) is a fermionic spin-\( \frac{1}{2} \) WZW affine current (the \( f \); 0) representation corresponds to the ground state representation) in [2]. The zero mode for each current acts on this representation. The immediate question is how to obtain the eigenvalue equations for the zero mode acting on the above states. It is an open problem to determine the zero modes for the above \( (11 + 16) \) currents explicitly.

Therefore, the implications for the results of the present work are the intermediate and necessary steps to obtain the low higher spin currents for general \( N \) and construct the three-point functions with two scalars for low higher spins of \( s \leq 4 \) with arbitrary \( (N, k) \). One hopes that this will give some hints for the general spin \( s \)-dependence of the three-point function which is dual to the corresponding three-point function in the bulk.

As suggested in [1], it is natural to calculate the OPEs between the higher spin currents in (1.1) and themselves in order to determine next 16 higher spin currents. The spin contents are characterized by [2]

\[
\left( \frac{2}{2}, \frac{5}{2}, \frac{5}{2}, 3 \right), \left( \frac{5}{2}, 3, 3, \frac{7}{2} \right), \left( \frac{5}{2}, 3, 3, \frac{7}{2} \right), \left( 3, \frac{7}{2}, \frac{7}{2}, 4 \right).
\]

(1.2)

How does one determine these 16 higher spin currents (1.2) in terms of \( \mathcal{N} = 2 \) WZW affine currents? First of all, by recalling that the four higher spin-\( \frac{3}{2} \) currents in (1.1) were obtained in [1] from the OPEs between the higher spin-1 current and four spin-\( \frac{3}{2} \) currents in the large \( \mathcal{N} = 4 \) nonlinear algebra, the four higher spin-\( \frac{5}{2} \) currents in (1.2) can be determined by the
OPEs between the higher spin-1 current in (1.1) and the four higher spin-$\frac{5}{2}$ currents in (1.1). The next step is to determine the six higher spin-3 currents in (1.2). One way to obtain these currents is to use the spin-$\frac{3}{2}$ currents in the large $\mathcal{N} = 4$ nonlinear superconformal algebra appropriately and calculate the OPEs between them and the above four higher spin-$\frac{5}{2}$ currents in (1.2). This motivation comes from the observations of how to determine the six higher spin-2 currents in (1.1) in [1]. During this calculation, the presence of the higher spin-2 current in (1.2) is observed also. For the last higher spin-3 current locating at the last $\mathcal{N} = 2$ multiplet in (1.2), one can take, for example, the corresponding higher spin-$\frac{5}{2}$ current is the one in the second $\mathcal{N} = 2$ multiplet. What about four higher spin-$\frac{3}{2}$ currents in (1.2)? Once again, by recalling how one has determined four higher spin-$\frac{5}{2}$ currents in (1.1) [1], the four higher spin-$\frac{3}{2}$ currents in (1.2) can be determined by acting spin-$\frac{3}{2}$ currents of the large $\mathcal{N} = 4$ nonlinear superconformal algebra on the above higher spin-3 currents living in the last three $\mathcal{N} = 2$ multiplets of (1.2). Finally, the higher spin-4 current in (1.2) can be fixed by calculating the OPE between one of the spin-$\frac{3}{2}$ currents and one of the higher spin-$\frac{3}{2}$ currents living in the last $\mathcal{N} = 2$ multiplet of (1.2).

From the results of [1] and this paper, one can construct an extension of large $\mathcal{N} = 4$ linear superconformal algebra. For the first 16 currents in the large $\mathcal{N} = 4$ linear superconformal algebra, one can express them for $N = 3$ case from [1]. What about for general $N$? First of all, one should obtain the spin-1 Casimir current and four spin-$\frac{1}{2}$ Casimir currents for general $N$. As before, the remaining $11 (= 16 - 1 - 4)$ (nonlinear) Casimir currents can be obtained from the [41] of [1]. Now using the relations (3.2), (3.6), (3.10) and (3.13) of [1], one obtains the corresponding 11 Casimir currents for general $N$ in the linear version.

How one can determine the higher spin currents generalizing the large $\mathcal{N} = 4$ linear superconformal algebra for general $N$? For the lowest spin-1 current $T^{(1)}(z)$ in [1], one can take it in the linear version also without any change: $T^{(1)}_{\text{lin}}(z) = T^{(1)}(z)$. For the next higher spin currents, one obtains the following results, by calculating the OPEs between the spin-$\frac{3}{2}$ currents $G_{\alpha}(z)$ of large $\mathcal{N} = 4$ linear superconformal algebra in [1] and the above $T^{(1)}(w)$ and rewriting the expressions in terms of $\mathcal{N} = 2$ WZW affine currents as the (higher spin) currents and the spin-1 current $U(z)$ and and the four spin-$\frac{1}{2}$ currents $F_{\alpha}(z)$

\[
T^{(2)}_{+, \text{lin}}(z) = T^{(2)}_{+}(z) + \frac{1}{(5 + k)} \left( UF_{21} + \frac{2}{(5 + k)} F_{21} F_{11} F_{22} - i F_{21} \hat{A}_3 + i F_{11} \hat{A}_- - i F_{22} \hat{B}_- + i F_{21} \hat{B}_3 \right)(z),
\]

\[
T^{(2)}_{-, \text{lin}}(z) = T^{(2)}_{-}(z) + \frac{1}{(5 + k)} \left( -U F_{12} + \frac{2}{(5 + k)} F_{12} F_{11} F_{22} - i F_{12} \hat{A}_3 + i F_{22} \hat{A}_+ - i F_{11} \hat{B}_+ + i F_{12} \hat{B}_3 \right)(z),
\]

\[
U^{(2)}_{\text{lin}}(z) = U^{(2)}(z) + \frac{1}{(5 + k)} \left( U F_{11} - \frac{2}{(5 + k)} F_{12} F_{21} F_{11} + i F_{11} \hat{A}_3 + i F_{21} \hat{A}_+ + i F_{12} \hat{B}_- + i F_{11} \hat{B}_3 \right)(z),
\]
\[ V_{\text{lin}}^{(\ell)}(z) = V^{(\ell)}(z) + \frac{1}{(5 + k)} \times \left( -UF_{22} + \frac{2}{(5 + k)} F_{21}F_{12}F_{22} + iF_{22}\hat{A}_3 + iF_{21}\hat{A}_- + iF_{21}\hat{B}_+ + iF_{22}\hat{B}_3 \right)(z). \]

In the right hand side of these equations, there are \( U(z) \) and \( F_\alpha(z) \) (which commute with the (higher spin) currents in the nonlinear version) as well as the (higher spin) currents from the nonlinear version. It is an open problem to obtain all the other higher spin currents in the linear version for general \( N \), as in above higher spin-2 currents which can be generalized to the general \( N \) case. Therefore, the OPEs between the higher spin currents in the linear version can be obtained from the OPEs in the nonlinear version. The nontrivial thing is to express all the OPEs in terms of the (higher spin) currents in the linear version. Note that all the higher spin currents in the nonlinear version can be expressed in terms of those in the linear version for general \( N \).

On the other hand, the results of [1] and this paper (and their \( N \) generalizations) should provide the asymptotic symmetry algebra in AdS\(_3\) bulk theory. In [2], the explicit construction using the oscillator formalism for spin-1 and four spin-2 currents living in the lowest \( \mathcal{N} = 4 \) multiplet (as well as some currents of large \( \mathcal{N} = 4 \) nonlinear superconformal algebra) was found. In the present case, it is nontrivial to obtain the six spin-2 currents, four spin-\( \frac{5}{2} \) currents and spin-3 current using the method of higher spin algebra in the bulk. After obtaining the \( N \) generalization for the result of [1] and this paper, the extension of large \( \mathcal{N} = 4 \) nonlinear superconformal algebra should match with the corresponding higher spin algebra in the bulk. Furthermore, it is an open problem to construct the next 16 higher spin currents we have constructed in this paper using the oscillator formalism in the context of higher spin algebra. One expects that it is nontrivial to express them in the primary basis where all the higher spin fields are primary under the stress energy tensor.

In this paper, we calculate the OPEs between the lowest 16 higher spin currents (1.1). The explicit expressions for these higher spin currents in terms of \( \mathcal{N} = 2 \) WZW affine currents were given in [1]. As a result, all the OPEs for each singular term are written in terms of the multiple products of these \( \mathcal{N} = 2 \) WZW affine currents. The nontrivial task is to extract the correct multiple products (with definite \( U(1) \) charges) between the above 11 currents of large \( \mathcal{N} = 4 \) nonlinear superconformal algebra and the lowest 16 higher spin currents (1.1) from each singular term. When one cannot succeed in writing the particular pole term in the given OPE in terms of above (11 + 16) currents, this implies that there should exist a new primary current with definite \( U(1) \) charge. One can exhaust all the next 16 higher spin currents in (1.2) (in terms of \( \mathcal{N} = 2 \) WZW affine currents) by examining the OPEs between the lowest 16 higher spin currents (1.1) carefully. It turns out that the lowest 16 higher spin currents appear in the fusion rules (of the OPEs between the lowest 16 higher spin currents) quadratically and linearly, together with large \( \mathcal{N} = 4 \) nonlinear superconformal family of the identity operator.

In section 2, the new 16 higher spin currents in (1.2) are obtained explicitly. In section 3, the OPEs between the higher spin-1 current in (1.1) and the remaining higher spin currents are obtained. Then all the other OPEs between the higher spin currents in (1.1) are obtained similarly.
In section 4, we summarize the main results of this paper with the fusion rules. In section 5, the comments on the future works are given. In appendices A–M, all the OPEs between the higher spin currents in (1.1) and themselves are written explicitly.

2. The 16 second lowest higher spin currents in the Wolf space coset

Let us recall that the 16 lowest higher spin currents are described by

\[
\begin{align*}
\left(1, \frac{3}{2}, \frac{3}{2}, 2\right): & \quad \left(T^{(1)}, T^{(2)}_{+}, T^{(2)}_{-}, T^{(2)}\right), \\
\left(\frac{3}{2}, 2, 2, \frac{5}{2}\right): & \quad \left(U^{(1)}, U^{(2)}_{+}, U^{(2)}_{-}, U^{(2)}\right), \\
\left(\frac{3}{2}, 2, 2, \frac{5}{2}\right): & \quad \left(V^{(1)}, V^{(2)}_{+}, V^{(2)}_{-}, V^{(2)}\right), \\
\left(2, \frac{5}{2}, \frac{5}{2}, 3\right): & \quad \left(W^{(2)}, W^{(2)}_{+}, W^{(2)}_{-}, W^{(2)}\right).
\end{align*}
\]

(2.1)

where the spins are specified in each primary field. By adding these (2.1) to spin with one unit, one obtains the following 16 next higher spin currents

\[
\begin{align*}
\left(2, \frac{5}{2}, \frac{5}{2}, 3\right): & \quad \left(P^{(1)}, P^{(2)}_{+}, P^{(2)}_{-}, P^{(2)}\right), \\
\left(\frac{5}{2}, 3, 3, \frac{7}{2}\right): & \quad \left(Q^{(1)}, Q^{(2)}_{+}, Q^{(2)}_{-}, Q^{(2)}\right), \\
\left(\frac{5}{2}, 3, 3, \frac{7}{2}\right): & \quad \left(R^{(1)}, R^{(2)}_{+}, R^{(2)}_{-}, R^{(2)}\right), \\
\left(3, \frac{7}{2}, \frac{7}{2}, 4\right): & \quad \left(S^{(1)}, S^{(2)}_{+}, S^{(2)}_{-}, S^{(2)}\right).
\end{align*}
\]

(2.2)

It turned out that the \(U(1)\) charge (defined in (3.19) of [1]) of each field in (2.2) is exactly the same as the one of each field in (2.1).

\[1\] The asymptotic symmetry of the \(\text{AdS}_3\) higher spin theory based on super higher spin algebra \(\mathfrak{shs}_{s}\) has been studied further and it matches with those of the two-dimensional CFT Wolf space coset in the \(\text{'t Hooft limit}\) [3]. The extension of large \(\mathcal{N} = 4\) superconformal algebra which contains one \(\mathcal{N} = 4\) multiplet for each integer spin \(s = 1, 2, \cdots\) as well as the currents of large \(\mathcal{N} = 4\) superconformal algebra has been studied in [4]. The first two \(\mathcal{N} = 4\) multiplets with \(s = 1, 2\) correspond to (1.1) and (1.2) respectively. For the particular level at the Kazama–Suzuki model, the \(\mathcal{N} = 3\) (enhanced) supersymmetry is observed in [5]. The full spectrum of the tensionless string theory (in \(\text{AdS}_3 \times S^1 \times \mathbb{T}^2\)) can be reorganized in terms of representations of the \(\mathcal{N} = 4\) super \(\mathfrak{psu}\) algebra where the large level limit is taken [6].
We would like to construct the 16 next lowest higher spin currents of spins in (2.2). Because there are 11 currents from the large $\mathcal{N} = 4$ nonlinear algebra and 16 lowest higher spin currents given by (2.1), one expects that the next lowest higher spin currents (2.2) will occur in the various OPEs between these 27 ($= 11 + 16$) currents. In particular, the spin-$\frac{3}{2}$ currents, $\hat{G}_{21}(z)$ and $\hat{G}_{12}(z)$, which are the generators of large $\mathcal{N} = 4$ nonlinear algebra, and the higher spin-1 current, $T^{(1)}(z)$, which is one of the generators in (2.1), will play an important role in this construction. Of course, when the higher spin-3 current $S^{(3)}(w)$ is determined, other spin-$\frac{3}{2}$ current $\hat{G}_{22}(z)$ is also useful.

2.1. Construction of four higher spin-$\frac{3}{2}$ currents: $P^{(\frac{3}{2})}_{\pm}(z), Q^{(\frac{3}{2})}_{\pm}(z)$ and $R^{(\frac{3}{2})}(z)$

Let us construct the four lowest fermionic higher spin-$\frac{3}{2}$ currents contained in (2.2). Recall that the four lowest fermionic higher spin-$\frac{3}{2}$ currents in (2.1) were obtained from the OPEs between the higher spin-1 current, $T^{(1)}(z)$, and four spin-$\frac{3}{2}$ currents, $G^{\hat{A}}(\hat{B})(a)$, where

\[^2\text{It is also useful to write down the spin-1 affine current in terms of purely bosonic spin-1 current and spin-1/2 current. One has the following expressions (in the [34 of [1]])}

\[\begin{align*}
\mathcal{D}K^{\alpha|\beta|0}(z) &= V^{\alpha}(z) - \frac{1}{5 + k} \left( f^{\alpha}_{\mu \nu} K^{\mu \nu} + f_{ab}^{\alpha} J^{ab}(z) \right), \\
\mathcal{D}K^{\alpha|\beta|0}(z) &= V^{\alpha}(z) - \frac{1}{5 + k} \left( f^{\alpha}_{\mu \nu} K^{\mu \nu} + f_{ab}^{\alpha} J^{ab}(z) \right), \\
\mathcal{D}J^{(1)}(z) &= V^{(1)}(z) - \frac{1}{5 + k} \left( f_{ab}^{(1)} J^{ab}(z) \right), \\
\mathcal{D}J^{(1)}(z) &= V^{(1)}(z) - \frac{1}{5 + k} \left( f_{ab}^{(1)} J^{ab}(z) \right).
\end{align*}\]

The following OPEs between the purely bosonic spin-1 currents satisfy

\[\begin{align*}
V^{A}(z) V^{B}(w) &= -\frac{1}{(z - w)} f_{AB}^{\gamma} V^{\gamma}(w) + \cdots, \\
V^{A}(z) V^{B}(w) &= -\frac{1}{(z - w)} f_{AB}^{\gamma} V^{\gamma}(w) + \cdots, \\
V^{A}(z) V^{B}(w) &= \frac{1}{(z - w)^2} k \delta^{AB} - \frac{1}{(z - w)} \left( f_{AB}^{\gamma} V^{C}(w) + f_{AB}^{\gamma} V^{C}(w) \right) + \cdots,
\end{align*}\]

where the indices $A, B, \cdots$ and $\hat{A}, \hat{B}, \cdots$ stand for the indices of the group $SU(N + 2 = 5)$ in the complex basis and furthermore the following regularity holds

\[\begin{align*}
V^{A}(z) K^{B}(w) &= V^{A}(z) K^{B}(w) = V^{A}(z) J^{(1)}(w) = V^{A}(z) J^{(2)}(w) = \cdots, \\
V^{A}(z) K^{B}(w) &= V^{A}(z) K^{B}(w) = V^{A}(z) J^{(1)}(w) = V^{A}(z) J^{(2)}(w) = \cdots.
\end{align*}\]

The OPEs between the spin-$\frac{3}{2}$ currents are given in the reference in [1] (with the footnote 11). The reason why one should express all the higher spin currents in this basis is due to the fact that the number of independent affine currents for given higher spin current is significantly reduced. For example, the number of independent terms in the higher spin-3 current $W^{(3)}(z)$ in the basis of [1] is given by 3438 while those in the present basis is given by 681.

\[^3\text{Note that in the convention of [4], the last component of fourth $N = 2$ multiplet is not a primary but quasiprimary field under the stress energy tensor. In our case, all the fields, $T^{(1)}(w), U^{(1)}(w), V^{(1)}(w), W^{(3)}(w), P^{(3)}(w), Q^{(3)}(w)$, $R^{(3)}(w)$ and $S^{(3)}(w)$ are primary fields under the $\hat{T}(z)$. Furthermore in [3], some higher spin currents in the extension of large $\mathcal{N} = 4$ superconformal algebra in terms of affine currents were found for arbitrary $N$ but only up to higher spin-2 currents. For example, the higher spin-$\frac{3}{2}$, $\cdots$, $\frac{7}{2}$, 4 currents are not known for general $N$ so far.}\]
\[ a = 11, 12, 21, 22, \text{via Appendices (C.1) and (C.2) of [1]. Then it is natural to consider the OPEs between the higher spin-1 current, } T^{(1)}(z), \text{ and the four fermionic higher spin-}^{\frac{5}{2}} \text{ currents in (2.1).}

It turns out that the OPEs between the higher spin-1 current, \( T^{(1)}(z) \), and the spin-\( \frac{5}{2} \) currents living in the last components of the second and third \( \mathcal{N} = 2 \) multiplets of (2.1) respectively, from (4.7), (4.32) and (4.46) of [1], are summarized by

\[
T^{(1)}(z) \left( \frac{U^{(z)}}{V^{(z)}} \right)(w) = \frac{4}{(z - w)^2} \left[ \frac{1}{3(5 + k)} \left[ \mp (3 + k) \left( \hat{G}_{11} \right) + \left( \frac{U^{(z)}}{V^{(z)}} \right)(w) \right] + \frac{1}{(z-w)} \left[ \pm W^{(z)}_{\pm} + P_{\pm}^{(z)} \right] + \cdots \right).
\]

From the \( U(1) \) charge in the table 2 of [1], one can easily see that the right hand side of (2.3) preserves the \( U(1) \) charges, \( \pm \frac{(-3 + k)}{(3 + k)} \), of the left-hand side of (2.3). The \( U(1) \) charge of \( Q^{(2)}(z)[R^{(2)}(z)] \) is the same as the one of \( U^{(2)}(z)[V^{(2)}(z)] \). Note that the factor \((k - 3)\) appears in the first term of the second order pole in (2.3) which will vanish at \( k = 3 \).

Similarly, the OPEs between the higher spin-1 current, \( T^{(1)}(z) \), and the higher spin-\( \frac{5}{2} \) currents living in the second and third components of last \( \mathcal{N} = 2 \) multiplet of (2.1) are described by

\[
T^{(1)}(z) \left( \frac{W^{(z)}}{V^{(z)}} \right)(w) = \frac{4}{(z - w)^2} \left[ \frac{\mp (-4 + k)}{3(5 + k)} \left[ \hat{G}_{21} \right] + \left( \frac{2T^{(z)}}{3} \right)(w) \right] + \frac{1}{(z-w)} \left[ \pm W^{(z)}_{\pm} + P_{\pm}^{(z)} \right] + \cdots,
\]

where the equations (4.7), (4.53) and (4.56) of [1] are used. The \( U(1) \) charges, \( \pm \frac{(3 + k)}{(5 + k)} \), are preserved in (2.4) respectively. The \( U(1) \) charge of \( P^{(2)}_{\pm}(z) \) is the same as the one of \( W^{(2)}_{\pm}(z) \).

The OPEs (2.3) and (2.4) imply that the higher spin-1 current, which is the lowest component of \( \mathcal{N} = 4 \) multiplet in (2.1), generates the higher spin-\( \frac{5}{2} \) currents in other \( \mathcal{N} = 4 \) multiplet (2.2) by acting with those currents with same spins in (different) \( \mathcal{N} = 4 \) multiplet (2.1). Note that the four higher spin-\( \frac{5}{2} \) currents, \( P_{\pm}^{(3)}(z)[W_{\pm}^{(3)}(z)] \), \( P_{\pm}^{(2)}(z)[W_{\pm}^{(2)}(z)] \), \( Q^{(4)}(z)[U^{(4)}(z)] \) and \( R^{(4)}(z)[V^{(4)}(z)] \), transform as (2, 2) under the \( SU(2) \times SU(2) \) [2]. The descendant fields of the spin-\( \frac{5}{2} \) currents in the second-order pole in (2.3) and (2.4) do not occur in the first-order pole because the difference of spins in the left hand side, \( 1 - \frac{5}{2} = -\frac{3}{2} \), is equal to the minus spin for the primary current appearing in the second-order pole of the

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\footnote{One can explicitly check that the OPE between the \( U(1) \) current given in (5.2) of [1] and \( Q^{(2)}(w)[R^{(2)}(w)] \) provides these \( U(1) \) charges.}

\footnote{Similarly, the four spin-\( \frac{3}{2} \) currents, \( \hat{G}_{21}(z)[T_{\pm}^{(3)}(z)], \hat{G}_{12}(z)[T_{\pm}^{(4)}(z)] \), \( \hat{G}_{11}(z)[U_{\mp}^{(4)}(z)] \) and \( \hat{G}_{22}(z)[V_{\mp}^{(4)}(z)] \), transform as (2, 2) under the \( SU(2) \times SU(2) \).}
right hand side. See the footnote 51 of [1]. Furthermore, one can imagine the \( \mathcal{N} = 2 \) fusion rule from the two results (2.3) and (2.4). In an appropriate basis, for example, the OPE between the first \( \mathcal{N} = 2 \) multiplet and the second \( \mathcal{N} = 2 \) multiplet of (2.1) will produce the second \( \mathcal{N} = 2 \) multiplet of (2.2) in the right hand side of the OPE. We will come back this issue in next section 4.

2.2. Construction of six higher spin-3 currents and one higher spin-2 current: \( P^{(3)}(z) \), \( Q_{\pm}^{(3)}(z) \), \( R_{\pm}^{(3)}(z) \), \( S^{(3)}(z) \) and \( P^{(2)}(z) \)

Based on the previous four higher spin-\( \frac{5}{2} \) currents, we would like to construct six higher spin-3 currents in (2.2). As a byproduct, one also determines the higher spin-2 current, which is the lowest component of \( \mathcal{N} = 4 \) multiplet (2.2).

Recall that the OPE between the second component of \( \mathcal{N} = 2 \) super stress energy tensor, corresponding to \( \hat{G}_{21}(z) \) (after factoring out four spin-\( \frac{1}{2} \) and one spin-1 currents), and the third component of any \( \mathcal{N} = 2 \) super primary current provides its first and fourth components.

One calculates the following OPE between \( \hat{G}_{21}(z) \) and \( P^{(2)}(w) \) (whose explicit expression was obtained in previous subsection via (2.4)) and the right hand side can be summarized as follows:

\[
\hat{G}_{21}(z) P^{(2)}(w) = \frac{1}{(z - w)^3} \left[ \frac{16(-4 + k)}{(5 + k)^2} iA_1 + \frac{16k(8 + k)}{3(5 + k)^2} iB_3 - \frac{8(-4 + k)}{3(5 + k)} T^{(1)} \right]\]

\[
+ \frac{1}{(z - w)^2} P^{(2)}(w)\]

\[
+ \frac{1}{(z - w)} \left[ \frac{1}{4} \partial P^{(2)} + \frac{8(-4 + k)}{(19 + 23k)} \left( \frac{\partial T^{(1)}}{2} - \frac{1}{2} \partial^2 T^{(1)} \right) \right] \]

\[
+ \frac{1}{(5 + k)^2(19 + 23k)} \left( \frac{\partial A_1}{2} \right) \]

\[
+ \frac{48 \left( \frac{-20 + k + k^2}{(5 + k)(19 + 23k)} \right) i \left( \frac{\partial A_1}{2} \right) \left( \frac{\partial B_3}{2} \right) + P^{(3)}(w) + \ldots , \right. (2.5)
\]

where the equation (3.17) of [1] is used. The second-order pole in (2.5) can give the lowest component of the first \( \mathcal{N} = 2 \) multiplet in (2.2) while the first-order pole provides the higher spin-3 current of same \( \mathcal{N} = 2 \) multiplet. No descendant fields for the spin-1 current appearing in the third-order pole arise in the second-order pole. The higher spin-3 current can be extracted from the first-order pole after subtracting the descendant field for the spin-2 current with correct coefficient and the three quasi-primary fields of spin 3. Similarly, the OPE between the third component of \( \mathcal{N} = 2 \) super stress energy tensor, corresponding to \( \hat{G}_{21}(z) \), and the second component of any \( \mathcal{N} = 2 \) super primary current provides its first and fourth components.

This higher spin-3 current is not to be confused with the expression \( P^{(3)}(w) \) in Appendix (C.26) of [1].
$P_{(2)}^{(1)}(z)$ is equal to $-\frac{(3 + k)}{(5 + k)}$ as explained before. Then the explicit form for the first $\mathcal{N} = 2$ multiplet in (2.2) is determined completely in terms of $\mathcal{N} = 2$ WZW affine currents $^8$.

Recall that the OPE between the second(third) component of $\mathcal{N} = 2$ super stress energy tensor, corresponding to $\hat{G}_{21}(z)[\hat{G}_{12}(z)]$, and the first component of any $\mathcal{N} = 2$ super primary current provides its second(third) component. Because the first components of the second and third $\mathcal{N} = 2$ multiplets in (2.2) are determined, one can calculate the following OPEs in order to obtain other kind of higher spin-3 currents appearing in those $\mathcal{N} = 2$ multiplets. It turns out that

$$
\begin{align*}
\left( \hat{G}_{21}(z) \right) \left( Q_i^{(2)}(z) \right) \left( R_j^{(2)}(z) \right) (w) &= \frac{1}{(z - w)^3} \left[ \frac{16k(3 + k)}{3(5 + k)^2} i\hat{B}_z \right] (w) \\
&+ \frac{1}{(z - w)^2} \left[ \frac{4(9 + 4k)}{3(5 + k)} \frac{U_+(2)}{V^{(2)}} \pm \frac{8(9 + 2k)}{3(5 + k)^2} \hat{A}_i \hat{B}_x \right] \\
&\mp \frac{8k}{(5 + k)^2} \hat{B}_x \hat{B}_3 - \frac{4k}{(5 + k)^2} \hat{A}_x \hat{B}_x \mp \frac{4}{(5 + k)} \hat{i}T^{(1)}(z) \hat{B}_x (w)
\end{align*}
$$

$^8$ One has the following OPEs

$$
\begin{align*}
\left( \hat{G}_{21}(z) \right) P^{(2)}(z) &= \mp \frac{1}{(z - w)^2} \left[ \frac{8(11 + k)}{3(5 + k)^2} T_+^{(2)} \right] (w) \\
&\pm \frac{24}{(5 + k)^2} \hat{A}_i \frac{\hat{G}_{11}}{\hat{G}_{12}} \pm \frac{8}{3(5 + k)} \hat{B}_x \left( \frac{\hat{G}_{21}}{\hat{G}_{12}} \right) \\
&\pm \frac{8(9 + 2k)}{3(5 + k)^2} \hat{B}_x \left( \frac{U^{(2)}}{V^{(2)}} \right) \pm \frac{8(8 + k)}{3(5 + k)^2} \hat{B}_x \left( \frac{\hat{G}_{21}}{\hat{G}_{12}} \right) + \frac{16}{(5 + k)} \hat{B}_x T_+^{(2)} \\
&\pm \frac{4}{(5 + k)} \hat{i}T^{(1)} \left( \frac{\hat{G}_{21}}{\hat{G}_{12}} \right) + \frac{8(8 + k)}{3(5 + k)^2} \hat{A}_i \left( \frac{\hat{G}_{21}}{\hat{G}_{12}} \right) \pm \frac{8(9 + k)}{3(5 + k)^2} \hat{B}_x T_+^{(2)} (w).
\end{align*}
$$

(2.6)

Now the first-order pole of these OPEs give the second (third) component of the first $\mathcal{N} = 2$ multiplet in (2.2). The second order pole of this OPE can be removed by introducing other higher spin-2 current $\hat{R}^{(2)}(z)$ correctly. Let us write down each primary field in terms of $\mathcal{N} = 2$ WZW affine currents as follows:

$$
\begin{align*}
&\mathcal{P}^{(2)}(z) = \frac{16(8 + k)}{3(5 + k)} \left( \hat{v}^i \hat{v}^i + \hat{v}^2 \hat{v}^2 + \hat{v}^3 \hat{v}^3 \right) (z) + \text{other 650 terms}, \\
&\mathcal{P}^{(2)}(z) = \frac{4\sqrt{2}}{(5 + k)^2} \left( \hat{k}^i \hat{v}^i \hat{v}^i + \hat{k}^i \hat{v}^i \hat{v}^i + \hat{k}^i \hat{v}^i \hat{v}^i \right) + \text{other 33 terms}, \\
&\mathcal{P}^{(2)}(z) = \frac{-4i\sqrt{2}}{(5 + k)^2} \left( \hat{k}^i \hat{v}^i \hat{v}^i - \hat{k}^i \hat{v}^i \hat{v}^i \right) + \text{other 33 terms}, \\
&\mathcal{P}^{(2)}(z) = \frac{8}{(5 + k)^2} \left( \hat{v}^i \hat{v}^i \hat{v}^i + \hat{v}^i \hat{v}^i \hat{v}^i \right) (z) + \text{other 75 terms}.
\end{align*}
$$
where the explicit form for $Q_\xi^3(w)$ and $R_\xi^2(w)$ via (2.3) is used again. Once again, the higher spin-3 current occurs in the first-order pole in (2.7) after subtracting the descendant field with coefficient $\frac{1}{4}$ for the spin-2 field appearing in the second-order pole and the quasi-primary field which is the only unique quasi primary field for spin-3 with given U(1) charge\(^9\). The U(1) charge of $Q_\xi^3(z)$ is equal to $\frac{2k}{(5+k)}$ while the one of $R_\xi^3(z)$ is equal to $\frac{2k}{(5+k)}$. No descendant field for the spin-1 in the third order pole because the spin difference of left hand side is given by $\frac{k}{2} - \frac{5}{2} = -1$ which is equal to the minus spin for this spin-1 current. Note that the nonlinear terms arise in the second-order pole of (2.7) and all the possible five composite fields of spin-2 in table 3 of [1] with given U(1) charge occur. In this case, the third order pole can be removed in an appropriate basis by changing the higher spin-$\frac{5}{2}$ current.

Now one calculates the different combination of OPEs between the spin-$\frac{5}{2}$ currents and the higher spin-$\frac{5}{2}$ currents with different total U(1) charges as follows:

\[
\begin{aligned}
\left( \hat{G}_{12}(z) \left[ Q_\xi^3(z) \right] \hat{G}_{21}(z) \right)^{[w]} &= \frac{1}{(z-w)^3} \left[ \frac{8(9 + k)}{(5 + k)^2} \beta_2 \right]^{[w]} \\
&+ \frac{1}{(z-w)^2} \left[ \frac{4(15 + 2k)}{3(5 + k)} \left( U_2^{(0)} \right) \left( V_2^{(0)} \right) \right] \pm \frac{24}{(5 + k)^2} \beta_3 \beta_3 \\
&+ \frac{8(12 + k)}{3(5 + k)^2} \hat{A}_z \hat{B}_3 - \frac{12}{(5 + k)^2} i \partial \hat{A}_z \mp \frac{4}{(5 + k)} \left( T^{(1)} \hat{A}_z \right) \\
&+ \frac{1}{(z-w)^3} \left[ \frac{1}{4} \left( \hat{G}_{12} Q_\xi^3(z) \right) \hat{G}_{21}(z) \right]^{[w]} \\
&+ \frac{24(9 + k)}{(5 + k)(19 + 23k)} i \left( \hat{F} \hat{A}_z - \frac{1}{2} \partial \hat{A}_z \right) + \left( Q_\xi^3(z) \right) \left( R_\xi^3(z) \right) ^{[w]} + \cdots , \tag{2.8}
\end{aligned}
\]

where the explicit form for $Q_\xi^3(w)$ and $R_\xi^3(w)$ via (2.3) is used again. Compared to the previous case (2.7), the right hand side of (2.8) contains $\hat{A}_z(w)$ dependent terms corresponding to $\hat{B}_3(w)$ in (2.7). The U(1) charge of $Q_\xi^3(z)$ is equal to $\frac{6}{(5+k)}$ and the one of $R_\xi^3(z)$ is equal to $\frac{6}{(5+k)}$. The fusion rule from the three results (2.5), (2.7) and (2.8) can be described as follows. The OPE between the first $\mathcal{N} = 2$ multiplet, (2.44) of [1], of large $\mathcal{N} = 4$ nonlinear algebra and the first (second) (third) $\mathcal{N} = 2$ multiplet of (2.2) will

\(^9\) For the nth order pole in the OPE $\Phi(z) \Phi_r(w)$, one denotes by $\{ \Phi_r \} _n \Phi(z)$.\(^{10}\)
produce the sum of the first (second) (third) \(\mathcal{N} = 2\) multiplet in (2.1) and the first (second) (third) multiplet of (2.2) as well as other \(\mathcal{N} = 2\) multiplets of large \(\mathcal{N} = 4\) nonlinear superconformal algebra respectively in the right hand side of the OPE.

How does one determine the last undetermined higher spin-3 current appearing in the fourth \(\mathcal{N} = 2\) multiplet in (2.2)? Recall that the higher spin-2 current \(W^{(2)}(w)\) in (2.1) was obtained from the OPE between \(\hat{G}_z(z)\) and \(\hat{U}_w(z)\) via (4.48) of [1]. Then one expects that the undetermined higher spin-3 current can be obtained from the OPE between \(\hat{G}_z(z)\) and \(\hat{Q}_w(w)\) and it turns out that

\[
\hat{G}_{22}(z) Q^{(2)}(w) = \frac{1}{(z-w)^3} \left[ \frac{16(3+2k)}{(5+k)^2} \hat{A}_3 + \frac{16(6+k)}{3(5+k)^2} i\hat{B}_3 + \frac{8(-3+k)}{3(5+k)^2} T^{(1)}(w) \right]
\]

where the explicit form for \(Q^{(2)}(w)\) can be found in (2.3) and the equation (3.15) of [1] is used. Compared to the case (2.5), the same three quasi-primary fields in the first-order pole of (2.9) occur but the second-order pole contains other spin-2 currents as well as \(P^{(2)}(w)\) itself. The spin-3 current \(S^{(3)}(w)\) has vanishing \(U(1)\) charge. One can rewrite the quadratic expression appearing in the second order pole \((\hat{A}_1 A_1 + A_2 A_2)(w) = (A_1 A_1 + iA_3)(w)\) and similarly one has the relation \((\hat{B}_1 \hat{B}_1 + \hat{B}_2 \hat{B}_2)(w) = (\hat{B}_1 \hat{B}_1 + i\hat{B}_3)(w)\) with definite \(U(1)\) charge. Note the appearance of the factor \((k-3)\) in the last term of third order pole (and the term containing \(T^{(1)}(w)\) in the first order pole) in (2.9).

Then the explicit forms for the higher spin-3 currents in (2.2), from (2.5), (2.7), (2.8) and (2.9), are determined completely in terms of \(\mathcal{N} = 2\) WZW affine currents.\(^\text{10}\)

\(^{10}\) Note that the six higher spin-3 currents, \(P^{(3)}(z)[T^{(2)}(z)]\), \(Q^{(3)}(z)[U^{(2)}(z)]\), \(Q^{(3)}(z)[U^{(2)}(z)]\), \(R^{(3)}(z)[V^{(2)}(z)]\), \(R^{(3)}(z)[V^{(2)}(z)]\), and \(S^{(3)}(z)[W^{(2)}(z)]\), transform as \((3, 1) \oplus (1, 3)\) under the \(SU(2) \times SU(2)\) [2].
2.3. Construction of four higher spin-$\frac{7}{2}$ currents: $Q^{(7/2)}(z)$, $R^{(7/2)}(z)$ and $S^{(7/2)}(z)$

Based on the previous six higher spin-3 currents, we would like to construct four higher spin-$\frac{7}{2}$ currents in $\mathcal{N} = 4$ multiplet in (2.2).

As done in (2.5) for the first $\mathcal{N} = 2$ multiplet in (2.2), one can calculate, from (3.17) of [1] and (2.8), the following OPE

\[
\hat{G}_{21}(z) \cdot Q^{(3)}(w) = -\frac{1}{(z-w)^3} \left[ 8 \left( -174 + 114k + 187k^2 + 23k^3 \right) \right] (5 + k)^2 \hat{G}_{11} + \frac{20(-1 + k)U(\bar{z})}{(5 + k)^2} \left( w \right)
\]

\[
+ \frac{1}{(z-w)^2} \left[ 6(3 + k)Q(\bar{z}) + \frac{17 + 6k}{(5 + k)} U(\bar{z}) \right]
\]

\[
- 2 \left( \frac{924 + 409k + 41k^2}{(5 + k)^2} \right) i\hat{A}_+ \hat{G}_{21}
\]

\[
+ \frac{2(9 + 5k)}{(5 + k)^2} i\hat{A}_+ T^{(2)}_{+} + \frac{18}{(5 + k)^2} i\hat{A}_3 \hat{G}_{11} + \frac{16(3 + k)}{(5 + k)^2} i\hat{A}_3 U(\bar{z})
\]

\[
+ \frac{8(-1 + k)}{(5 + k)^2} i\hat{B}_- \hat{G}_{12} - \frac{8(-1 + k)}{(5 + k)^2} i\hat{B}_- T^{(2)}_{-} - \frac{2(12 + k)}{(5 + k)^2} i\hat{B}_3 \hat{G}_{11}
\]

\[
- \frac{16}{(5 + k)} i\hat{B}_3 U(\bar{z}) + \frac{3}{(5 + k)} T^{(1)}\hat{G}_{11} + \frac{2}{3(5 + k)^2(19 + 23k)} \left[ \hat{G}_{21} \cdot R^{(3)} \right] \left( w \right)
\]

\[
- \frac{4(11 + 7k)}{3(5 + k)^2} \partial U(\bar{z}) \left( w \right) + \frac{1}{(z-w)} \left[ \frac{1}{5} \partial \left\{ \frac{\hat{G}_{21}}{Q^{(3)}} \right\} \right] \left( w \right)
\]

\[
- \frac{32}{(5 + k)^2(19 + 23k)(47 + 35k)} \left( \hat{T}\hat{G}_{11} - \frac{3}{8} \partial^2 \hat{G}_{11} \right)
\]

\[
- \frac{80(-1 + k)}{(5 + k)(47 + 35k)} \left( \hat{T}U(\bar{z}) - \frac{3}{8} \partial^2 U(\bar{z}) \right) + Q(\bar{z}) \left( w \right) + \cdots. \tag{2.10}
\]

All the composite fields with spin-$\frac{5}{2}$ except $T^{(1)}U(\bar{z})(w)$ appearing in table 4 of [1] arise in the second order pole in (2.10). In other words, this implies that the OPEs between the 11 currents and the second lowest higher spin currents will do not contain any quadratic expressions in the lowest higher spin currents. As explained before, the first and last components of the second $\mathcal{N} = 2$ multiplet in (2.2) occur in the second order and first order poles in (2.10) respectively. The higher spin-$\frac{7}{2}$ current occurs in the first-order pole in (2.10) after subtracting the descendant field with coefficient $\frac{1}{5}$ for the spin-$\frac{5}{2}$ field appearing in the second-order pole and the two quasi-primary fields where the numerical factor $\frac{3}{8}$ is fixed. Recall that the two structure constants appearing in two quasi primary fields can be fixed by using the property that the above reduced first order pole subtracted by descendant terms should transform as a primary field.
Similarly, the first and last components of the third $\mathcal{N} = 2$ multiplet in (2.2) can be obtained from the following OPE

$$\hat{G}_{21}(z) R^{(3)}(w) = \frac{1}{(z-w)^3} \left[ - \frac{8 \left( 684 + 1035k + 361k^2 + 14k^3 \right)}{3(5+k)^2(19 + 23k)} \hat{G}_{22} \\
+ \frac{8(-1 + 2k)}{(5+k)^2} V^{(\tilde{z})}(w) \\
+ \frac{1}{(z-w)^2} \left[ \frac{4(6 + k)}{(5+k)} R^{(\tilde{z})} - \frac{23 + 4k}{5+k} V^{(\tilde{z})} \\
+ \frac{16}{(5+k)^2} \hat{A}_1 T^{(\tilde{z})} - \frac{2(9 + 2k)}{(5+k)^2} i\hat{A}_3 \hat{G}_{22} + \frac{16}{(5+k)} i\hat{A}_3 V^{(\tilde{z})} \\
+ \frac{6}{(5+k)^2} i\hat{B}_1 \hat{G}_{22} - \frac{1}{(5+k)^2} i\hat{B}_3 V^{(\tilde{z})} - \frac{3}{(5+k)} T^{(1)}(\hat{G}_{22}) \\
+ \frac{2}{3(5+k)^2(19 + 23k)} \partial \hat{G}_{22} + \frac{4(23 + 3k)}{3(5+k)^2} \partial V^{(\tilde{z})} \right] + \cdots \right]$$

In this case, all the composite fields with spin-$\frac{5}{2}$ except $\hat{A}_1 \hat{G}_{12}(w)$ and $T^{(1)}V^{(\tilde{z})}(w)$ appearing in table 4 of [1] arise in the second order pole in (2.11). As observed in previous case, in the OPE of (2.11), the only linear terms in the lowest higher spin currents can occur. The two structure constants appearing in two quasi primary fields can be fixed by using the fact that the first order pole subtracted by two descendant terms containing $\hat{G}_{22}(w)$ and $V^{(\tilde{z})}(w)$ terms should behave as a primary field under the stress energy tensor $\hat{T}(z)$.

Motivated by (2.6) or the equation (4.52) of [1], one calculates the following OPE

$$\hat{G}_{21}(z) S^{(3)}(w) = \frac{1}{(z-w)^3} \left[ \frac{4 \left( 1809 + 1382k - 63k^2 + 28k^3 \right)}{3(5+k)^2(19 + 23k)} \hat{G}_{21} \\
- \frac{4 \left(-2433 - 964k + 731k^2 + 110k^3 \right)}{3(5+k)^2(19 + 23k)} T^{(1)}(w) \right]$$
\[
+ \frac{1}{(z-w)^2} \left[ \frac{(3-k) P_k^{(2)}}{(5+k)} - \frac{6(3+k) W_k^{(2)}}{(5+k)} - \frac{4(-2+k)}{(5+k)^2} i\hat{A}_+ \hat{G}_{11} \right] \\
- \frac{4(7+4k)}{(5+k)^2} i\hat{A}_- U_k^{(2)} - \frac{4(327 + 322k + 59k^2)}{(5+k)^2(19 + 23k)} i\hat{A}_3 \hat{G}_{21} \\
+ \frac{4}{(5+k)} i\hat{A}_1 T_k^{(2)} - \frac{12}{(5+k)^2} iB \hat{G}_{22} + \frac{4(17+2k)}{(5+k)^2} iB_+ V_k^{(2)} \\
- \frac{4(171 + 368k + 43k^2 + 6k^3)}{(5+k)^2(19 + 23k)} iB_3 \hat{G}_{21} \\
+ \frac{20(1+k)}{(5+k)^2} iB_2 T_k^{(2)} - \frac{4(-26 + 29k + 3k^2)}{(5+k)(19 + 23k)} T^{(1)} \hat{G}_{21} \\
- \frac{4(423 + 851k + 118k^2 + 6k^3)}{3(5+k)^2(19 + 23k)} \partial \hat{G}_{21} \\
- \frac{4(-55 + 86k - 9k^2 + 2k^3)}{(5+k)^2(19 + 23k)} \partial T_k^{(2)} \bigg|_w \\
+ \frac{1}{(z-w)} \left[ \frac{1}{5} \partial \left\{ \hat{G}_{21} S_k^{(3)} \right\} \right]_{-2} \\
+ \frac{16}{3(5+k)(19 + 23k)(47 + 35k)} \left( \hat{T} \hat{G}_{21} - \frac{3}{8} \partial^2 \hat{G}_{21} \right) \\
- \frac{16}{3(5+k)(19 + 23k)(47 + 35k)} \left( \hat{T} T_k^{(2)} - \frac{3}{8} \partial^2 T_k^{(2)} \right) + S_k^{(2)} \bigg|_w \\
+ \cdots. \tag{2.12}
\]

All the composite fields with spin-$\frac{5}{2}$ except $T^{(1)} T_k^{(2)} (w)$ appearing in table 4 of [1] arise in the second order pole in (2.12). The linear terms in the lowest higher spin currents arise in (2.12). The factor $(k-3)$ appears in the $P_k^{(2)} (w)$ term in the second order pole of (2.12). The analysis for the two quasi primary fields in the first order pole can be done as before.

Similarly, by taking other spin-$\frac{3}{2}$ current with same higher spin current

\[
\hat{G}_{12}(z) S_k^{(3)}(w) = - \frac{1}{(z-w)^3} \left[ \frac{4 \left( 1695 + 1586k + 351k^2 + 28k^3 \right)}{3(5+k)^2(19 + 23k)} \hat{G}_{12} \\
- \frac{4 \left( -2205 - 1372k - 97k^2 + 110k^3 \right)}{3(5+k)^2(19 + 23k)} T_k^{(2)} \bigg|_w \right] 
\]
The composite field with spin $\frac{5}{2}$, $\tilde{T}_{\frac{1}{2}}$, appearing in table 4 of [1] does not arise in the second order pole in (2.13). The factor $(k - 3)$ appears in the $P_{\frac{1}{2}}(w)$ term in the second order pole of (2.13). The structure constants appearing in the two quasi primary fields in the first order pole can be determined as before. No quadratic terms in the lowest higher spin current in (2.13) appear.
The first order poles in (2.10)–(2.13) have common feature in the sense that the two quasi primary fields for fixed $U(1)$ charge occur\footnote{One has the following expressions for the second and third $\mathcal{N} = 2$ multiplets in (2.2)

\begin{align*}
Q_{\mathcal{Z}}^{(2)}(z) &= -\frac{4i\sqrt{2}}{(5 + k)^2}(K^1 V^4 V'^4 + K^1 V^5 V'^5)(z) + \text{other 27 terms}, \\
Q_{\mathcal{Z}}^{(3)}(z) &= -\frac{16k(3 + k)}{(5 + k)^4(9 + 23k)}(K^{ij} V^i V'^j + K^{ij} V^j V'^i)(z) + \text{other 307 terms}, \\
Q_{\mathcal{Z}}^{(4)}(z) &= \frac{8i}{(5 + k)^4}(J^{ij} V^i V'^j + J^{ij} V'^i V^j)(z) + \text{other 143 terms}, \\
Q_{\mathcal{Z}}^{(5)}(z) &= \frac{8i\sqrt{2}}{(5 + k)^2}(763 + 1582k + 623k^2 + 92k^3)(K^{ij} V^i V'^j V^j V'^i)(z) + \text{other 850 terms}, \\
R_{\mathcal{Z}}^{(2)}(z) &= -\frac{4i\sqrt{2}}{(5 + k)^2}(K^V V^V + iK^V V^V V^V)(z) + \text{other 27 terms}, \\
R_{\mathcal{Z}}^{(3)}(z) &= -\frac{24i(9 + k)}{(5 + k)^4(9 + 23k)}(V^i V'^i V^9 V'^9 + V^8 V'^8 V^8 V'^8)(z) + \text{other 143 terms}, \\
R_{\mathcal{Z}}^{(4)}(z) &= -\frac{16k(3 + k)}{(5 + k)^4(9 + 23k)}(K^{ij} V^i V'^j + K^{ij} V^j V'^i)(z) + \text{other 307 terms}, \\
R_{\mathcal{Z}}^{(5)}(z) &= \frac{16i\sqrt{2}}{3(5 + k)^2(9 + 23k)}(1482 + 2403k + 902k^2 + 133k^3)(J^{ij} V^i V'^j + J^{ij} V'^i V^j)(z) + \text{other 859 terms},
\end{align*}

The four higher spin-$\frac{3}{2}$ currents, $\hat{Q}_{\mathcal{Z}}^{(2)}(z)$, $\hat{R}_{\mathcal{Z}}^{(2)}(z)$, and $\hat{S}_{\mathcal{Z}}^{(2)}(z)$, transform as (2, 2) under the $SU(2) \times SU(2)$ as before [2].}

The first order poles in (2.10)–(2.13) have common feature in the sense that the two quasi primary fields for fixed $U(1)$ charge occur\footnote{One has the following expressions for the second and third $\mathcal{N} = 2$ multiplets in (2.2)

\begin{align*}
Q_{\mathcal{Z}}^{(2)}(z) &= -\frac{4i\sqrt{2}}{(5 + k)^2}(K^1 V^4 V'^4 + K^1 V^5 V'^5)(z) + \text{other 27 terms}, \\
Q_{\mathcal{Z}}^{(3)}(z) &= -\frac{16k(3 + k)}{(5 + k)^4(9 + 23k)}(K^{ij} V^i V'^j + K^{ij} V^j V'^i)(z) + \text{other 307 terms}, \\
Q_{\mathcal{Z}}^{(4)}(z) &= \frac{8i}{(5 + k)^4}(J^{ij} V^i V'^j + J^{ij} V'^i V^j)(z) + \text{other 143 terms}, \\
Q_{\mathcal{Z}}^{(5)}(z) &= \frac{8i\sqrt{2}}{(5 + k)^2}(763 + 1582k + 623k^2 + 92k^3)(K^{ij} V^i V'^j V^j V'^i)(z) + \text{other 850 terms}, \\
R_{\mathcal{Z}}^{(2)}(z) &= -\frac{4i\sqrt{2}}{(5 + k)^2}(K^V V^V + iK^V V^V V^V)(z) + \text{other 27 terms}, \\
R_{\mathcal{Z}}^{(3)}(z) &= -\frac{24i(9 + k)}{(5 + k)^4(9 + 23k)}(V^i V'^i V^9 V'^9 + V^8 V'^8 V^8 V'^8)(z) + \text{other 143 terms}, \\
R_{\mathcal{Z}}^{(4)}(z) &= -\frac{16k(3 + k)}{(5 + k)^4(9 + 23k)}(K^{ij} V^i V'^j + K^{ij} V^j V'^i)(z) + \text{other 307 terms}, \\
R_{\mathcal{Z}}^{(5)}(z) &= \frac{16i\sqrt{2}}{3(5 + k)^2(9 + 23k)}(1482 + 2403k + 902k^2 + 133k^3)(J^{ij} V^i V'^j + J^{ij} V'^i V^j)(z) + \text{other 859 terms},
\end{align*}

The four higher spin-$\frac{3}{2}$ currents, $\hat{Q}_{\mathcal{Z}}^{(2)}(z)$, $\hat{R}_{\mathcal{Z}}^{(2)}(z)$, and $\hat{S}_{\mathcal{Z}}^{(2)}(z)$, transform as (2, 2) under the $SU(2) \times SU(2)$ as before [2].}

2.4. Construction of one higher spin-4 current: $S^{(4)}(z)$

How does one determine the highest component of the last $\mathcal{N} = 2$ multiplet in (2.2)? One way to obtain the highest spin-4 current is to consider the following OPE

\begin{align*}
\hat{G}_{21}(z) S_{(z)}^{(4)}(w) &= \frac{1}{(z-w)^2}\left[c_1 \hat{A}_3 + c_2 \hat{B}_3 + c_3 T^{(1)}(w)\right] + \frac{1}{(z-w)^3}\left[c_4 P^{(2)} + c_5 W^{(2)} + c_6 \hat{\cal T} + c_7 T^{(2)} \right. \\
&+ c_8 (\hat{A}_1 \hat{A}_1 + \hat{A}_2 \hat{A}_2) + c_9 \hat{A}_3 \hat{A}_3 \\
&+ c_{10} \hat{A}_3 \hat{B}_3 + c_{11} (\hat{B}_1 \hat{B}_1 + \hat{B}_2 \hat{B}_2) + c_{12} \hat{B}_3 \hat{B}_3 \\
&+ c_{13} \partial \hat{A}_3 + c_{14} \partial \hat{B}_3 + c_{15} \partial T^{(1)} \\
&+ c_{16} T^{(1)} \hat{A}_3 + c_{17} T^{(1)} \hat{B}_3 \right] \bigg| (w)
\end{align*}
\[ + c_{30} \hat{A}_- \partial \hat{A}_+ + c_{31} \hat{A}_3 \hat{\dot{T}} + c_{32} \hat{A}_3 \partial \hat{A}_3 + c_{33} \hat{A}_3 \mathcal{W}^{(2)} + c_{34} \hat{A}_3 \partial \hat{B}_3 + c_{35} \hat{A}_3 \partial T^{(1)} + c_{36} \hat{\dot{B}}_4 \mathcal{W}^{(2)} + c_{37} \hat{\dot{B}}_- \mathcal{V}^{(2)} + c_{38} \hat{\dot{B}}_4 \partial \hat{B}_- + c_{39} \hat{B}_- \partial \hat{B}_+ + c_{40} \hat{B}_3 \hat{\dot{T}} + c_{41} \hat{B}_3 T^{(2)} + c_{42} \hat{B}_3 \mathcal{W}^{(2)} + c_{43} \hat{B}_3 \partial T^{(1)} + c_{44} \hat{B}_3 \partial \hat{A}_3 + c_{45} \hat{B}_3 \partial \hat{B}_3 + c_{46} \partial \hat{T} + c_{47} \partial T^{(2)} + c_{48} \partial \mathcal{W}^{(2)} + c_{49} \hat{G}_{11} \hat{G}_{22} + c_{50} \hat{G}_{11} \mathcal{V}^{(2)} + c_{51} \partial^2 \hat{A}_3 + c_{52} \partial^2 \hat{B}_3 + c_{53} \partial^2 T^{(1)} + c_{54} \hat{G}_{12} \hat{G}_{21} + c_{55} \hat{G}_{12} T^{(2)} + c_{56} \hat{G}_{21} T^{(2)} + c_{57} \hat{A}_4 \hat{A}_- \hat{B}_3 + c_{58} \hat{G}_{22} \mathcal{U}^{(2)} + c_{59} \hat{A}_3 \hat{A}_3 \hat{A}_3 + c_{60} \hat{A}_4 \hat{A}_- \hat{A}_3 + c_{61} \hat{A}_3 \hat{A}_3 \hat{B}_3 + c_{62} \hat{A}_3 \hat{\dot{B}}_4 \hat{\dot{B}}_- + c_{63} \hat{A}_3 \hat{\dot{B}}_3 \hat{\dot{B}}_3 + c_{64} T^{(1)} \hat{A}_1 \hat{\dot{A}}_3 + c_{65} \hat{\dot{B}}_3 \hat{\dot{B}}_3 \hat{\dot{B}}_3 + c_{66} T^{(1)} \hat{A}_1 \hat{\dot{B}}_3 + c_{67} \hat{B}_3 \hat{\dot{B}}_3 \hat{\dot{B}}_3 + c_{68} T^{(1)} \hat{\dot{B}}_3 \hat{\dot{B}}_3 \hat{\dot{B}}_- + c_{69} T^{(1)} \hat{\dot{B}}_3 \hat{\dot{B}}_3 + c_{70} \hat{A}_- T^{(1)} \hat{A}_+ \]
\[ + c_{80}\left( \hat{T} \partial \hat{\Lambda}_3 - \frac{1}{2} \partial \hat{T} \hat{\Lambda}_3 - \frac{1}{4} \partial^3 \hat{\Lambda}_3 \right) \]
\[ + c_{81}\left( \hat{T} \partial \hat{B}_3 - \frac{1}{2} \partial \hat{T} \hat{B}_3 - \frac{1}{4} \partial^3 \hat{B}_3 \right) \]
\[ + c_{82}\left( \hat{T} \partial T^{(1)} - \frac{1}{2} \partial \hat{T} T^{(1)} - \frac{1}{4} \partial^3 T^{(1)} \right) \]
\[ + c_{83}\left( \hat{T} \hat{\Lambda}_4 + \hat{\Lambda}_4 - \frac{1}{2} \partial^3 \hat{\Lambda}_4 + \hat{\Lambda}_4 - \frac{1}{2} \partial^3 \hat{\Lambda}_4 \right) \]
\[ - \frac{1}{12} \partial^3 \hat{\Lambda}_3 + \frac{1}{2} \partial \hat{T} \hat{\Lambda}_3 \]
\[ + c_{84}\left( \hat{T} \hat{B}_4 + \hat{B}_4 - \frac{1}{2} \partial^3 \hat{B}_4 + \hat{B}_4 - \frac{1}{2} \partial^3 \hat{B}_4 \right) \]
\[ - \frac{1}{12} \partial^3 \hat{B}_3 + \frac{1}{2} \partial \hat{T} \hat{B}_3 \]
\[ + S^{(4)}(w) + \ldots, \quad (2.14) \]

with the coefficient functions in (M.1). All the composite fields with spin-2 except \( T^{(1)}T^{(1)}(w) \) appearing in table 3 of [1] arise in the third order pole in (2.14). No quadratic terms in the lowest higher spin current in (2.14) appear. The \((k - 3)\) factor appears in \( P^{(2)}(w) \) term of third order pole, \( P^{(3)}(w) \) term of second order pole, and \((\hat{T}P^{(2)} - \frac{3}{10} \partial^3 P^{(2)})w) \) term of the first order pole. One can reexpress the third order pole in terms of the descendant field of spin-1 current appearing in the fourth order pole plus other (quasi) primary fields as describe before. There is no descendant field for spin-2 fields appearing in the third order pole in the second order pole because the difference between the two fields in the left hand side is equal to \( \frac{3}{2} - \frac{1}{2} = -2 \). Furthermore, there is no descendant field in the second order pole for spin-1 fields in the fourth order pole. Note the presence of higher spin-3 current, \( S^{(3)}(w) \) in the second order pole. In the first order pole, all the expressions except the first term and last term are expressed in terms of 14 quasi primary fields. The first four quasi primary fields have standard quadratic expressions with the coefficient \( \frac{3}{10} \). The next five quasi primary fields are cubic in their expressions. This feature is rather special because all the previous constructions on the quasi primary fields are restricted to the quadratic case. See for example [54] of [1].

One can easily check that \( \hat{\Lambda}_3 \hat{\Lambda}_3(w) \{ \hat{B}_3 \hat{B}_3(w) \} \) is a quasi primary field. The next three quasi primary fields are again quadratic in their expressions. The last two quasi primary fields are cubic in their expressions. Let us emphasize that when one calculates the OPE between the stress energy tensor \( \hat{T}(z) \) and the first order pole of (2.14) subtracted by \( \frac{1}{6} \partial \hat{G}_{21} S^{(2)} \), the fourth order pole of this OPE contains nontrivial expressions with spin-2 with vanishing \( U(1) \) charge. One can rewrite these expressions in terms of 14 independent terms appearing in the third order pole of (2.14). This implies that the quasi primary fields look like as \( \hat{T} \Phi(w) \) plus other terms where \( \Phi(w) \) is above 14 independent terms: the field contents with vanishing \( U(1) \) charge in table 3 of [1] where \( T^{(1)}T^{(1)}(w) \) is replaced by \( P^{(2)}(w) \).
Therefore, one has the complete expressions for the last $\mathcal{N} = 2$ multiplet in (2.2).

3. The OPEs between the 16 lowest higher spin currents in the Wolf space coset

We would like to construct the OPEs between the $\mathcal{N} = 4$ multiplet in (2.1) and itself and to express them in terms of the 11 generators of large $\mathcal{N} = 4$ nonlinear superconformal algebra, 16 generators of the $\mathcal{N} = 4$ multiplet in (2.1) and 16 generators of the other $\mathcal{N} = 4$ multiplet in (2.2) (and their composite operators with possible derivatives).

Because the pole structures are written in terms of $\mathcal{N} = 2$ WZW affine currents, the main step is to write them in terms of above $\mathcal{N} = 4$ affine currents. The $\mathcal{N} = 4$ generators in terms of $\mathcal{N} = 2$ affine currents are given in [1] and the other 16 generators are given in the footnotes 8, 11 and 12. Basically we would like to calculate the following OPEs

$$
\begin{pmatrix}
T^{(1)} & T_{+}^{(2)} & T_{-}^{(2)} \\
U^{(2)} & U^{(2)}_{+} & U^{(2)}_{-} \\
V^{(2)} & V_{+}^{(2)} & V_{-}^{(2)} \\
W^{(2)} & W_{+}^{(2)} & W_{-}^{(2)} & W^{(3)}
\end{pmatrix}
\begin{pmatrix}
T^{(1)} & T_{+}^{(2)} & T_{-}^{(2)} \\
U^{(2)} & U^{(2)}_{+} & U^{(2)}_{-} \\
V^{(2)} & V_{+}^{(2)} & V_{-}^{(2)} \\
W^{(2)} & W_{+}^{(2)} & W_{-}^{(2)} & W^{(3)}
\end{pmatrix}^{(w)}.
$$

(3.1)

In the tables 2, 3, ..., 7 in [1], the composite fields with spins, $s = 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}$ are listed according to their $U(1)$ charges. In the OPEs (3.1), the composite fields with maximum spin 5 appear in the first order pole in the OPE $W^{(3)}(z) W^{(3)}(w)$. Then one should find the composite fields with spins $s = 4, \frac{9}{2}$ and 5. Furthermore, by including the extra 16 generators in (2.2), the tables of [1] have more composite fields and the extra tables with above spins $s = 4, \frac{9}{2}$ and 5 can be obtained. We will present the OPEs (3.1) completely except the first order pole of $W^{(3)}(z) W^{(3)}(w)$.

12 Also the last $\mathcal{N} = 2$ multiplet in (2.2) has the following expressions

$$
S^{(3)}(z) = \frac{8}{(5 + k)^2} \left( V^4 V^{11} V^{10} + V^5 V^{12} V^{10} \right)(z) + \text{other 633 terms},
$$

$$
S^{(2)}_{11}(z) = \frac{8\sqrt{2}}{(5 + k)} \left( J^{10} V^{11} V^{10} + J^{10} V^{12} V^{10} \right)(z) + \text{other 879 terms},
$$

$$
S^{(2)}_{22}(z) = \frac{8\sqrt{2}}{(5 + k)} \left( J^{10} V^{11} V^{10} + J^{10} V^{12} V^{10} \right)(z) + \text{other 876 terms},
$$

$$
S^{(4)}_{1}(z) = \frac{16}{5(5 + k)^2(19 + 23k)(29 + 25k)(47 + 35k)} \times (584 940 052 5 + 219 016 549 54k + 318 228 834 78k^2
$$

$$
+ 225 707 054 36k^3 + 801 666 157 7k^4 + 132 708 747 0k^5 + 937 336 60k^6)
$$

$$
\times \left( V^4 V^{11} V^{10} + V^5 V^{12} V^{10} \right)(z)
$$

+ \text{other 4926 terms}.

Note that the higher spin-2 current, $P^{(2)}(z)$, and the higher spin-4 current, $S^{(4)}(z)$, transform as $(1, 1)$ under the $SU(2) \times USp(2)$ respectively [2].

The first order pole of this OPE is very special in the sense that there is no new primary field in this singular term. Suppose that there exists such a singular term. After reversing the arguments $z$ and $w$ and expanding around $w$, the same term will appear with opposite sign. This implies that it is identically zero. See also appendix in [63] of [1].
One can calculate the OPEs (3.1) step by step. Let us take \( T^{(1)}(z) \) in the first operator and \( T^{(1)}(z) \) in the second operator. The nontrivial four OPEs are given in (2.3) and (2.4). Now let us compute the remaining nontrivial OPEs as follows.

- The OPEs between the higher spin-1 current and the first \( \mathcal{N} = 2 \) multiplet in (1.1).
  
  We use the equations (4.7), (4.10), (4.14), and (4.17) of [1] where the expressions for the \( \mathcal{N} = 2 \) WZW affine currents are known. It turns out that

\[
T^{(1)}(z) T^{(1)}(w) = \frac{1}{(z - w)^2} \left[ \frac{6k}{(5 + k)^2} + \ldots \right],
\]

\[
T^{(1)}(z) T^{(2)}(w) = \pm \frac{1}{(z - w)} T^{(2)}(w) + \ldots,
\]

\[
T^{(1)}(z) T^{(2)}(w) = \frac{1}{(z - w)^2} \left[ - \frac{6i}{(5 + k)} \hat{A}_3 - \frac{2ik}{(5 + k)} \hat{B}_3 + \frac{(3 + k)}{(3 + 7k)} T^{(1)}(w) + \ldots \right].
\]

- The OPEs between the higher spin-1 current and some components in the second and third \( \mathcal{N} = 2 \) multiplets in (2.1).
  
  We use the equations (4.21), (4.24), (4.28), (4.32), (4.35), (4.38), (4.42), and (4.46) of [1]. The results are as follows:

\[
T^{(1)}(z) \left[ \begin{array}{c} U^{(2)}(z) \\ V^{(2)}(z) \end{array} \right] (w) = \frac{1}{(z - w)} \left[ \begin{array}{c} U^{(2)}(z) \\ -V^{(2)}(z) \end{array} \right] (w) + \ldots,
\]

\[
T^{(1)}(z) \left[ \begin{array}{c} U^{(2)}(z) \\ V^{(2)}(z) \end{array} \right] (w) = \pm \frac{1}{(z - w)^2} \left[ \frac{2k}{(5 + k)} i \hat{B}_3 \right] (w) + \ldots,
\]

\[
T^{(1)}(z) \left[ \begin{array}{c} U^{(2)}(z) \\ V^{(2)}(z) \end{array} \right] (w) = \mp \frac{1}{(z - w)^2} \left[ \frac{6i}{(5 + k)} \hat{A}_3 \right] (w) + \ldots.
\]

- The OPEs between the higher spin-1 current and the fourth \( \mathcal{N} = 2 \) multiplet in (2.1).
  
  We use the equations (4.49), (4.53), (4.56), and (4.60) of [1]. One has the following OPEs

\[
T^{(1)}(z) W^{(2)}(w) = \frac{1}{(z - w)^2} \left[ \frac{6i}{(5 + k)} \hat{A}_3 - \frac{2ik}{(5 + k)} \hat{B}_3 + T^{(1)}(w) \right] + \ldots,
\]

\[
T^{(1)}(z) W^{(3)}(w) = \frac{1}{(z - w)^3} \left[ \frac{i}{(5 + k)^2} \left[ 12(1 + k) \hat{A}_3 - 16k \hat{B}_3 \right] (w)
\right.
\]

\[
\left. + \frac{1}{(z - w)^2} \left[ -P^{(2)} - \frac{4(741 + 2530k + 1873k^2 + 436k^3)}{3(3 + 7k)(19 + 23k)(5 + k)} \hat{T} \right] \right].
\]
Compared to the previous expression for the third order pole in (2.14), the composite fields with spin 2 are the same as those in the third order pole in (2.14) except \( \partial_{w} T^{(1)} \). Also the quadratic expression \( T^{(1)} T^{(1)} (w) \) occurs in (3.3). In the first line of the second order pole in (3.3), each term is a primary field. In the second line, the first three terms is a quasi primary field and the last term is a primary field. In the third line, the whole expression is a quasi primary field. In the fourth line, the first two terms are the descendant fields with relative coefficient \(-\frac{1}{2}\) coming from the third order pole in (3.3) and the last term is a primary field. In the last line of the second order pole in (3.3), the first term is a quasi primary field and each field in the last two terms is a primary field. One can check that the first order pole in (3.3) is a primary field. Note that there is no descendant field coming from the second order pole in (3.3) because the spin difference of the left hand side of this OPE is given by \(-3 = -2\).

Now one can move to other OPEs. Let us take \( T_{z} T^{(2)}(z) \) and 15 remaining higher spin currents except the higher spin-1 current from (2.1). Note that the OPE \( T^{(2)}(z) T^{(1)} (w) \) can be read off from the second equation in (3.2). The full expression is given by appendix A. Next, one can calculate the OPEs between \( T^{(2)}(z) \) and other remaining higher spin currents and they are given also in appendix A. Furthermore, the OPEs between the higher spin-2 current \( T^{(3)}(z) \) and other 13 remaining higher spin currents are given in appendix B. Then at the moment, the OPEs between the first \( \mathcal{N} = 2 \) multiplet and other four \( \mathcal{N} = 2 \) multiplets in (2.1) are listed in appendices A and B as well as the section 2.1.

Let us move the second \( \mathcal{N} = 2 \) multiplet in (2.1). The OPEs between the higher spin-\( \frac{3}{2} \) current \( U^{(2)}(z) \) and other 12 higher spin currents can be found in appendix C. Similarly, the OPEs between the higher spin-2 current \( U^{(2)}(z) \) and others are in appendix D. Furthermore, the remaining appendices E, F have the OPEs corresponding to the higher spin-2 current \( U^{(3)}(z) \) and the higher spin-\( \frac{5}{2} \) current \( U^{(2)}(z) \) respectively. Then the OPEs between the second \( \mathcal{N} = 2 \) multiplet and other three \( \mathcal{N} = 2 \) multiplets in (2.1) are listed in appendices C–F.

Appendices G–J contain the OPEs between the third \( \mathcal{N} = 2 \) multiplet and other two \( \mathcal{N} = 2 \) multiplets in (2.1).

Appendix K lists the OPEs between the fourth \( \mathcal{N} = 2 \) multiplet and itself in (2.1).
The final appendix L contains the composite spin-$\frac{9}{2}$ fields in the first order poles of the OPEs, $U^{(2)}(z) W^{(3)}(w)$, $V^{(2)}(z) W^{(3)}(w)$, $W^{(2)}(z) W^{(3)}(w)$ and $W^{(2)}(z) W^{(3)}(w)$. In next section, we will summarize the main results with the fusion rules.

4. The fusion rules in the OPEs between the lowest 16 higher spin currents

Let us describe the fusion rules from the main results of this paper. Among the OPEs in (3.1), we would like to focus on the right hand sides of these OPEs which contain any components appearing in the second $\mathcal{N} = 4$ multiplet in (2.2). There are four $\mathcal{N} = 2$ multiplets in (2.1). Then there exist $10(= 4 + 3 + 2 + 1)$ $\mathcal{N} = 2$ superfusion rules. From the sections 2, 3, and appendices appendix A, appendix B, one has the following fusion rules between the multiplets in (2.1) in component approach\(^{14}\)

\[
\begin{align*}
[T^{(1)}], U^{(2)}] &= [Q^{(2)}] + \ldots, \\
[T^{(2)}], U^{(2)}] &= [Q^{(3)}] + \ldots, \\
[T^{(2)}], U^{(2)}] &= [Q^{(3)}] + \ldots, \\
[T^{(2)}], U^{(2)}] &= [Q^{(3)}] + \ldots, \\
[T^{(2)}], U^{(2)}] &= [P^{(2)}] + [Q^{(2)}] + [Q^{(3)}] + \ldots. \tag{4.1}
\end{align*}
\]

One denotes the large $\mathcal{N} = 4$ nonlinear superconformal family of the identity operator by $[I]$. In the first OPE of (4.1), there exist other terms, $[U^{(2)}], U^{(2)}] and [I]$ (we ignore) by looking at the equation (2.3). The abbreviated part in the upper case in the second OPE of (4.1) contains $U^{(2)}] and [U^{(2)}] and those in the lower case has $U^{(2)}] and [I]. One has

\[
\begin{align*}
[U^{(2)}], T^{(1)}] and [I] in the upper case of the second OPE. For the lower case, there are $U^{(2)}], [U^{(2)}], W^{(2)}], T^{(2)}], T^{(2)}] U^{(2)}], [U^{(2)}] and [I] if one specifies the complete form. Then the quadratic fields appear in this OPE. Similarly, the fourth OPE contains $U^{(2)}] and $U^{(2)}]. The fifth OPE with upper case has $U^{(2)}], [T^{(1)}], [T^{(2)}], T^{(2)}] U^{(2)}] and [I]. For the lower case, one has the following extra terms $U^{(2)}], [T^{(1)}], W^{(2)}], U^{(2)}], T^{(2)}], T^{(2)}] U^{(2)}] and [I] if one describes the full result. For the final OPE of (4.1), one also has $U^{(2)}], [U^{(2)}], [T^{(2)}] U^{(2)}] and [I] if one writes down the full expression. Then one realizes that the quadratic terms (as well as linear ones) in the lowest higher spin currents can arise in the fusion rules.

\(^{14}\) There are no second higher spin currents in the OPE between the first $\mathcal{N} = 2$ multiplet in (2.1) by collecting the results of (3.2) and appendices A and B. Then the total number of $\mathcal{N} = 2$ superfusion rule is given by 9. They will appear in the equations (4.1)–(4.9) below.
It would be interesting to see whether the above fusion rules can be written in terms of \( \mathcal{N} = 2 \) superspace by adding these six equations. It is obvious to see that there is a second \( \mathcal{N} = 2 \) multiplet in (2.2). Moreover, there is also \( P_{(2)}^{(2)} \) coming from the last OPE of (4.1). It is an open problem whether one can remove this unwanted term (or one can generate other three partners, \( P_{(2)}^{(2)}, P_{(2)}^{(4)} \) and \( P_{(3)}^{(3)} \) in the above fusion rules) by redefining the primary fields correctly. Note that according to \( U(1) \) charge conservation in (4.1), the left hand side of last fusion rule has \( \frac{3 + k}{3 + k} \) while the \( U(1) \) charge of \( P_{(2)}^{(2)}(w) \) is given by \( \frac{3 + k}{3 + k} \). In the notation of \( P_{(2)}^{(2)} \), this is actually given by \( P_{(1)}^{(2)} \hat{B}_-(w) \) in appendix B which gives the correct \( U(1) \) charge.

Now one can summarize the following fusion rules between the first and third \( \mathcal{N} = 2 \) multiplets in (2.1)

\[
\begin{align*}
T^{(1)} & \cdot V^{(2)} = R^{(2)} + \ldots, \\
T^{(2)} \cdot V^{(2)} & = R^{(2)} + \ldots, \\
T^{(2)} \cdot V^{(4)} & = R^{(3)} + \ldots, \\
T^{(2)} \cdot V^{(4)} & = R^{(3)} + \ldots, \\
T^{(2)} \cdot V^{(2)} & = P_{(2)}^{(2)} + R^{(2)} + \ldots. \\
\end{align*}
\]

(4.2)

These are very similar to the previous fusion rules (4.1). We can see the extra structures we ignored in (4.2) from sections 2 and 3 and appendices A and B as done in previous considerations on (4.1). Simply adding these fusion rules gives the third \( \mathcal{N} = 2 \) multiplet in (2.2) and there exists a \( P_{(2)}^{(2)} \) which will disappear in an appropriate basis.

Next one can select, from appendices A and B, the following fusion rules between the first and fourth \( \mathcal{N} = 2 \) multiplets in (2.1)

\[
\begin{align*}
T^{(1)} \cdot W^{(2)} = P_{(2)}^{(2)} + \ldots, \\
T^{(2)} \cdot W^{(2)} = P_{(2)}^{(2)} + \ldots, \\
T^{(2)} \cdot W^{(3)} = P_{(2)}^{(2)} + Q^{(2)} + R^{(2)} + \ldots, \\
T^{(2)} \cdot W^{(3)} = P_{(2)}^{(2)} + \ldots, \\
T^{(2)} \cdot W^{(2)} = P_{(2)}^{(2)} + \ldots, \\
T^{(2)} \cdot W^{(3)} = P_{(2)}^{(2)} + P_{(3)}^{(3)} + Q_{(3)}^{(3)} + \ldots. \\
\end{align*}
\]

(4.3)
One can see that by adding these fusion rules (4.3) there is no \( Q(\frac{2}{3}) \) dependence we want to include and there exists \( S(\frac{3}{5}) \) we want to remove in the context of \( \mathcal{N} = 2 \) supersymmetric fusion rule. The extra structures we ignored can be found in appendices A and B.

For the fusion rules between the second \( \mathcal{N} = 2 \) multiplet in (2.1), one obtains, from appendices D and F

\[
[U^{(2)}_{\pm} \cdot U^{(2)}] = \left[ Q^{(2)}_{\pm} \right] + \ldots,
[U^{(2)} \cdot U^{(2)}] = \left[ Q^{(3)} \right] + \ldots.
\]

(4.4)

In the first equation of (4.4), there exist \( U^{(2)}_{\pm}, U^{(2)}_{\mp} \) and \( [I] \). If one specifies the complete expression, the extra structures \( U^{(2)} U^{(2)}_{\pm}, U^{(2)} U^{(2)}_{\mp}, U^{(2)} U^{(2)}_{\mp}, W^{(2)} \), \( W^{(2)} \), \( T^{(1)} \), \( T^{(1)} U^{(2)} \), \( T^{(1)} U^{(2)} \), \( U^{(2)} \) and \( [I] \) appear in the second equation of (4.4). Again, the quadratic terms (as well as linear ones) in the lowest higher spin currents occur in these fusion rules.

From appendices C–F, one obtains the following fusion rules between the second and third \( \mathcal{N} = 2 \) multiplets in (2.1)

\[
[U^{(3)}] \cdot V_{\pm}^{(2)} = \left[ P_{\pm}^{(2)} \right] + \ldots,
[U^{(3)}] \cdot V^{(2)} = \left[ P^{(2)} \right] + \left[ S^{(3)} \right] + \ldots,
[U^{(2)}_{\pm} \cdot V^{(2)}] = \left[ P^{(2)}_{\pm} \right] + \ldots,
[U^{(2)}_{\pm} \cdot V^{(2)}] = \left[ P^{(2)}_{\pm} \right] + \left[ S^{(3)}_{\pm} \right] + \ldots,
[U^{(2)} \cdot V^{(2)}] = \left[ P^{(2)} \right] + \left[ S^{(2)} \right] + \ldots,
[U^{(3)}] \cdot V^{(2)} = \left[ P^{(3)} \right] + \left[ S^{(3)} \right] + \ldots,
[U^{(3)}] \cdot V^{(2)} = \left[ P^{(2)} \right] + \left[ S^{(3)} \right] + \ldots.
\]

(4.5)

One expects that the right hand side of \( \mathcal{N} = 2 \) superfusion rule contains the first and fourth \( \mathcal{N} = 2 \) multiplets in (2.2) by adding (4.5) each other. It is nontrivial to find how one can remove the unwanted terms \( Q^{(2)} \) and \( R^{(2)} \).

Once again, one obtains the following fusion rules between the second and fourth \( \mathcal{N} = 2 \) multiplets in (2.1), from appendices C–F

\[
[U^{(2)}] \cdot W^{(2)} = \left[ Q^{(3)} \right] + \ldots,
[U^{(2)}] \cdot W^{(2)} = \left[ Q^{(2)} \right] + \left[ Q^{(4)} \right] + \ldots,
[U^{(2)}_{\pm} \cdot W^{(2)}] = \left[ Q^{(3)}_{\pm} \right] + \ldots,
[U^{(2)}_{\pm} \cdot W^{(2)}] = \left[ Q^{(3)}_{\pm} \right] + \ldots,
[U^{(2)}_{\pm} \cdot W^{(2)}] = \left[ Q^{(3)}_{\pm} \right] + \left[ Q^{(4)} \right] + \ldots.
\]
\[
\begin{align*}
\left[ U_+^{(2)} \right] \cdot \left[ W^{(3)} \right] &= \left[ P^{(2)} \right] + \left[ P^{(3)} \right] + \left[ Q^{(3)}_+ \right] + \left[ S^{(3)} \right] + \ldots, \\
\left[ U_-^{(2)} \right] \cdot \left[ W_+^{(2)} \right] &= \left[ P^{(1)} \right] + \left[ Q^{(1)}_+ \right] + \left[ Q^{(2)}_+ \right] + \ldots, \\
\left[ U^{(2)}_\bar{\tau} \right] \cdot \left[ W^{(3)} \right] &= \left[ Q^{(3)}_+ \right] + \ldots, \\
\left[ U^{(2)} \right] \cdot \left[ W^{(2)}^{(3)} \right] &= \left[ P^{(2)}_+ \right] + \left[ Q^{(3)}_+ \right] + \left[ Q^{(2)}_+ \right] + \ldots, \\
\left[ U^{(2)} \right] \cdot \left[ W_+^{(2)} \right] &= \left[ P^{(2)}_+ \right] + \left[ P^{(3)}_+ \right] + \left[ Q^{(2)}_+ \right] + \ldots, \\
\left[ U^{(2)}_\bar{\tau} \right] \cdot \left[ W^{(2)}_+ \right] &= \left[ Q^{(2)}_+ \right] + \left[ Q^{(2)}_+ \right] + \left[ Q^{(2)}_+ \right] + \ldots.
\end{align*}
\]  

(4.6)

By adding the fusion rules (4.6), one sees the first and second \( \mathcal{N} = 2 \) multiplets in (2.1). It would be interesting to see whether the other \( \mathcal{N} = 2 \) multiplets can appear in \( \mathcal{N} = 2 \) superfusion rules. In the right hand side of last fusion rule of (4.6), the fermion currents are multiplied by the bosonic currents, \( \left[ P^{(2)}(w) \right] \) and so on.

For the fusion rules between the third \( \mathcal{N} = 2 \) multiplet in (2.1), one obtains, from appendices H and J

\[
\begin{align*}
\left[ V_+^{(2)} \right] \cdot \left[ V^{(2)}_+ \right] &= \left[ R^{(2)}_+ \right] + \ldots, \\
\left[ V^{(2)}_+ \right] \cdot \left[ V^{(1)}_+ \right] &= \left[ R^{(3)}_+ \right] + \ldots. 
\end{align*}
\]  

(4.7)

In the first equation of (4.7), there exist \( \left[ V^{(1)}_+ \right], \left[ T^{(1)}_+ \right], \left[ V_+^{(2)} \right] \) and \( [I] \). The extra structures, if one writes down them explicitly, \( \left[ V^{(2)}_+ V^{(2)}_+ \right], \left[ V_+^{(3)} V^{(1)}_+ \right], \left[ V^{(3)}_+ V^{(2)}_+ \right], \left[ W^{(2)}_+ \right], \left[ V^{(2)}_+ \right], \left[ T^{(1)}_+ \right], \left[ T^{(2)}_+ V^{(2)}_+ \right], \left[ T^{(2)}_+ V^{(2)}_+ \right], \left[ V^{(2)}_+ \right] \) and \( [I] \) appear in the second equation of (4.4). The quadratic terms (as well as linear ones) in the lowest higher spin currents can be seen from these fusion rules.

One obtains the following fusion rules between the third and fourth \( \mathcal{N} = 2 \) multiplets in (2.1) from appendices G–J

\[
\begin{align*}
\left[ V^{(2)}_+ \right] \cdot \left[ W^{(2)}_+ \right] &= \left[ R^{(3)}_+ \right] + \ldots, \\
\left[ V^{(2)}_+ \right] \cdot \left[ W^{(3)}_+ \right] &= \left[ P^{(3)}_+ \right] + \left[ R^{(2)}_+ \right] + \ldots, \\
\left[ V^{(2)}_+ \right] \cdot \left[ W^{(2)}_+ \right] &= \left[ R^{(3)}_+ \right] + \ldots, \\
\left[ V^{(2)}_+ \right] \cdot \left[ W^{(3)}_+ \right] &= \left[ P^{(3)}_+ \right] + \left[ P^{(3)}_+ \right] + \left[ R^{(3)}_+ \right] + \left[ S^{(3)}_+ \right] + \ldots, \\
\left[ V^{(2)}_+ \right] \cdot \left[ W^{(2)}_+ \right] &= \left[ P^{(3)}_+ \right] + \left[ R^{(2)}_+ \right] + \left[ R^{(2)}_+ \right] + \ldots. 
\end{align*}
\]
By adding the fusion rules (4.8), one sees the first and third $\mathcal{N} = 2$ multiplets in (2.1). It is an open problem to see whether the other $\mathcal{N} = 2$ multiplets can appear in $\mathcal{N} = 2$ superfusion rules.

Finally, from appendix K, one obtains the following fusion rules between the last $\mathcal{N} = 2$ multiplet in (2.1)

\[
\begin{align*}
[ V^{(2)} ] \cdot [ W^{(3)} ] & = [ R^{(3)} ] + \ldots, \\
[ V^{(2)} ] \cdot [ W^{(2)} ] & = [ P^{(2)} ] + [ R^{(2)} ] + \ldots, \\
[ V^{(2)} ] \cdot [ W^{(2)} ] & = [ P^{(2)} ] + [ P^{(3)} ] + [ R^{(2)} ] + \ldots, \\
[ V^{(2)} ] \cdot [ W^{(2)} ] & = [ P^{(2)} ] + [ R^{(2)} ] + \ldots, \\
[ V^{(2)} ] \cdot [ W^{(2)} ] & = [ P^{(2)} ] + [ P^{(3)} ] + [ S^{(2)} ] + \ldots.
\end{align*}
\]  

(4.8)

By adding the fusion rules (4.9), one sees the first, second and fourth $\mathcal{N} = 2$ multiplets in (2.1) in the $\mathcal{N} = 2$ supersymmetric fusion rule. It would be interesting to see whether the other remaining third $\mathcal{N} = 2$ multiplet can join this $\mathcal{N} = 2$ superfusion rule.

Then one has the following fusion rules from (4.1)-(4.9)

\[
[ \Phi_i ] \cdot [ \Phi_j ] = | I | + \sum_k C_{ijk} [ \Phi_k ] \\
+ \sum_{i,m} C_{ijm} [ \Phi_l \Phi_m ] + \sum_a C_{ija} [ \Psi_a ] , \quad i, j, k, l, m, a = 1, 2, \ldots, 16,
\]
where $\Phi_i$ stands for any 16 components in $\mathcal{N} = 4$ multiplet in (2.1) and $\Psi_a$ stands for any 16 components in $\mathcal{N} = 4$ multiplet in (2.2). All the structure constants, $C_{ijk}$, $C_{ijlm}$ and $C_{ija}$, are known\(^{15}\).

5. Conclusions and outlook

As in an abstract, we have constructed the complete OPEs between the lowest 16 higher spin currents.

Let us comment on future directions as follows.

- The complete OPEs between the 11 nonlinear currents and the second higher spin $\mathcal{N} = 4$ multiplet.
  In (2.5)–(2.14), we have seen some of these OPEs before. For the first $\mathcal{N} = 4$ multiplet, the complete OPEs with 11 currents were given and the right hand sides of these OPEs consist of above 11 currents and the first $\mathcal{N} = 4$ multiplet (and their composite fields). One expects that the right hand sides of the complete OPEs will consist of above 11 currents and the first and second $\mathcal{N} = 4$ multiplets (and their composite fields).

- The OPEs between the first higher spin $\mathcal{N} = 4$ multiplet and the second higher spin $\mathcal{N} = 4$ multiplet.
  We have seen some of these OPEs in (E.3), (H.3) and (H.4). In order to find the complete algebra, it is necessary to calculate these OPEs.

- The OPEs between the second higher spin $\mathcal{N} = 4$ multiplet and itself.
  Furthermore, it is natural to complete these OPEs and to check whether there exist new primary fields in the right hand sides of these OPEs.

- The $\mathcal{N} = 2$ and $\mathcal{N} = 4$ superfusion rules.
  In (4.1)–(4.9) we presented the fusion rules in the component approach. It is nontrivial to write them in terms of $\mathcal{N} = 2$ supercurrents and furthermore, it would be interesting to see whether these $\mathcal{N} = 2$ superspace approach can be generalized to $\mathcal{N} = 4$ superspace approach. For example, one expects the following $\mathcal{N} = 4$ superfusion rules $[\Omega^\cdot] \cdot [\Omega^\cdot] = \Omega^\cdot \cdot \Omega^\cdot + [\Omega^\cdot \cdot \Omega^\cdot] + [\Omega^\cdot \cdot \Omega^\cdot] + [\Omega^\cdot \cdot \Omega^\cdot]$ where $\Omega^\cdot$ stands for the whole 16 higher spin currents in (2.1) and $\Omega^\cdot$ stands for the whole 16 higher spin currents in (2.2).

- An extension of small $\mathcal{N} = 4$ superconformal algebra.
  It is well-known that the small (or regular) $\mathcal{N} = 4$ superconformal algebra can be obtained by taking one of the level as zero and the other level as an infinity in the large $\mathcal{N} = 4$ linear superconformal algebra. Therefore, an extension of small $\mathcal{N} = 4$ superconformal algebra, which will be useful in [6], can be obtained by taking these limits to the results of this paper as well as the one in [1].

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\(^{15}\) In [1], the main results in the fusion rules can be written in terms of $|I| \cdot |I| = |I|$ and $|I| \cdot \left[ \Phi_i \right] = |I| + \sum_i C_{ij} \left[ \Phi_j \right]$.
Appendix A. The nontrivial OPEs between the higher spin-$\frac{3}{2}$ currents, $T_{\pm}^{(3)}(z)$, and other 15 higher spin currents

We list the main OPE results case by case.

- The OPEs between the higher spin-$\frac{3}{2}$ currents and the first $\mathcal{N} = 2$ multiplet of (2.1)

$$T_{\pm}^{(3)}(z) T^{(3)}(w) = -\frac{1}{(z-w)^3} \left[ \frac{6k}{5+k} \right]$$

$$+ \frac{1}{(z-w)^2} \left[ -\frac{6i}{5+k} \hat{A}_3 + \frac{2ik}{5+k} \hat{B}_3 - T^{(1)} \right](w)$$

$$+ \frac{1}{(z-w)} \left[ \frac{1}{2} \partial \left\{ T_{\pm}^{(3)} T^{(3)}(w) \right\}_{-2} - \frac{6k}{(3+7k)} T \right](w)$$

$$+ \cdots, \quad T_{\pm}^{(3)}(z) T^{(2)}(w) = \frac{1}{(z-w)^2} \left[ \frac{12(1+k)(3+k)}{(5+k)(3+7k)} T_{\pm}^{(2)}(w) \right]$$

$$+ \frac{1}{(z-w)} \left[ \frac{1}{3} \partial \left\{ T_{\pm}^{(2)} T^{(2)} \right\}_{-2}(w) + \cdots. \right.$$  

Note that the relative coefficients, $\frac{1}{2}$ and $\frac{1}{5}$, can be easily determined by counting the spins in both the left and right hand sides. See also the footnote 51 of [1].

- The OPEs between the higher spin-$\frac{3}{2}$ currents and the second and third $\mathcal{N} = 2$ multiplets of (2.1)

$$T_{\pm}^{(3)}(z) \left( \begin{array}{c} V^{(3)} \\ U^{(3)} \end{array} \right)(w) = \mp \frac{1}{(z-w)^2} \left[ \frac{6}{5+k} i\hat{A}_+ \right](w)$$

$$+ \frac{1}{(z-w)} \left[ \left( -\frac{V^{(2)}}{U^{(2)}} \right) + \frac{1}{2} \partial \left\{ T_{\pm}^{(3)} \left( \begin{array}{c} V^{(3)} \\ U^{(3)} \end{array} \right) \right\}_{-2} \right](w) + \cdots,$n

$$T_{\pm}^{(3)}(z) \left( \begin{array}{c} U^{(2)} \\ V^{(2)} \end{array} \right)(w) = \frac{1}{(z-w)^2} \left[ \frac{2}{5+k} i\hat{B}_+ T_{\pm}^{(2)} \right](w) + \cdots,$n

$$T_{+}^{(2)}(z) U^{(2)}(w) = -\frac{1}{(z-w)^2} \left[ \frac{5+2k}{5+k} U^{(2)} \right](w)$$

$$+ \frac{1}{(z-w)} \left[ -\frac{1}{2} Q^{(2)} - U^{(2)} + \frac{1}{3} \partial \left\{ T_{+}^{(2)} U^{(2)} \right\}_{-2} \right](w) + \cdots,$n

$$T_{-}^{(2)}(z) V^{(2)}(w) = \frac{1}{(z-w)^2} \left[ \frac{5+2k}{5+k} V^{(2)} \right](w)$$

$$+ \frac{1}{(z-w)} \left[ \frac{1}{2} R^{(2)} - V^{(2)} + \frac{2}{5+k} i\hat{A}_- T^{(2)} + \frac{4}{5+k} i\hat{A}_1 V^{(2)} \right]$$

Note that the relative coefficients, $\frac{1}{2}$ and $\frac{1}{5}$, can be easily determined by counting the spins in both the left and right hand sides. See also the footnote 51 of [1].
\[-\frac{2}{(5 + k)} \hat{b}_z \hat{g}_{21} - \frac{2}{(5 + k)} \hat{b}_z \hat{T}^{(2)}_z - \frac{4}{(5 + k)} \hat{b}_z V^{(2)}\]
\[+ \left( \frac{13 + 2k}{3(5 + k)} \right) V^{(2)}_z \] 

\[T^{(2)}_z (z) \left\{ \left( T^{(2)}_z V^{(2)}_z \right) \right\}_w = \frac{1}{(z - w)^2} \left\{ \frac{6 + k}{(5 + k)} \left[ \hat{g}_{12} + \left( - \hat{v}^{(2)}_z \right) \right] \right\}_w \]
\[+ \frac{1}{(z - w)} \left\{ \frac{1}{2} \left( - \hat{r}^{(2)}_z \right) + \frac{1}{3} \left( \frac{T^{(2)}_z}{U^{(2)}_z} \right) \right\}_w \]
\[\cdots, \]

\[T^{(2)}_z (z) U^{(2)}_z (w) = \frac{1}{(z - w)^2} \left\{ \frac{8k(3 + k)}{(5 + k)^2} \hat{b}_z \right\}_w \]
\[+ \frac{1}{(z - w)^2} \left\{ \frac{2(9 + 4k)}{3(5 + k)} U^{(2)}_z + \frac{4(9 + 2k)}{3(5 + k)^2} \hat{a}_1 \hat{B}_z - \frac{4k}{(5 + k)^2} \hat{B}_z \right\}_w \]
\[\cdots, \]

\[T^{(2)}_z (z) \left\{ \left( T^{(2)}_z V^{(2)}_z \right) \right\}_w = \frac{1}{(z - w)^2} \left\{ \frac{4}{(5 + k)} \hat{a}_- \right\}_w \]
\[+ \frac{1}{(z - w)^2} \left\{ \frac{2(13 + 2k)}{3(5 + k)} V^{(2)}_z + \frac{12}{(5 + k)^2} \hat{a}_- \hat{A}_- + \frac{4(8 + k)}{(5 + k)^2} \hat{A}_- \right\}_w \]
\[\cdots, \]

\[-\frac{2}{(5 + k)} \hat{a}_- \hat{T}^{(2)} - \frac{2}{(5 + k)^2} \hat{T}^{(1)} \hat{A}_- \] 
\[+ \frac{1}{(z - w)^2} \left\{ \frac{1}{2} \hat{r}^{(3)} - \frac{3}{2(5 + k)} \hat{T}^{(1)} \hat{A}_- \right\}_w \]
\[\cdots, \]

\[-\frac{2}{(5 + k)} \hat{a}_- \hat{W}^{(2)} - \frac{15}{(5 + k)^2} \hat{A}_- \hat{A}_- + \frac{(16 + 2k)}{(5 + k)^2} \hat{A}_- \hat{B}_z \] 
\[+ \frac{1}{2(5 + k)} \hat{a}_- \hat{D}^{(1)} + \frac{9}{(5 + k)^2} \hat{A}_- \hat{A}_- + \frac{(13 + 2k)}{6(5 + k)} \hat{W}^{(2)}_z \] 
\[+ \frac{4}{(5 + k)^2} \hat{a}_- \hat{A}_- \hat{A}_- + \frac{4}{(5 + k)^2} \hat{a}_- \hat{A}_- \hat{B}_z - \frac{12 + k}{(5 + k)^2} \hat{A}_- \hat{A}_- \hat{B}_z \] 
\[+ \frac{8}{(5 + k)^2} \hat{a}_- \hat{B}_z \hat{B}_z + \frac{4}{(5 + k)^2} \hat{a}_- \hat{B}_z \hat{B}_z + \frac{4 + k}{(5 + k)^2} \hat{A}_- \hat{A}_- \hat{B}_z \] 
\[-\frac{8}{(5 + k)^2} \hat{a}_- \hat{A}_- \] 

\[\cdots, \]
The composite fields $\hat{B}_\xi T_{\xi}^{(2)}(w)$ appearing in the second OPE of (A.1) are primary fields. In the third order OPE of (A.1), there exists a field with boldface $^{16}$ In the fourth OPE of (A.1), the last term in the first order pole can be written in terms of two terms with the coefficient \( \frac{(13 + 2k)}{3(5 + k)^2} \), \( \frac{8}{3(5 + k)^2} \). Then the first factor comes from the the higher spin current living in the second order pole. One can easily check that the five composite (quadratic) fields in the first order pole plus the derivative term with above.

\[ T^{(2)}(z)U^{(2)}(w) = \frac{1}{(z - w)^3} \left[ \frac{8(5 + 2k)}{3(5 + k)^2} i\hat{A}_+ \right](w) \]

\[ + \frac{1}{(z - w)^3} \left[ \frac{2 \mathcal{O}^3}{3(5 + k)} + \frac{2}{3(5 + k)^2} IT^{(1)} \hat{A}_+ + \frac{8(23 + 13k)}{(5 + k)(19 + 23k)} i\hat{A}_+ \hat{T} \right] \]

\[ + \frac{1}{(z - w)^3} \left[ \frac{2}{(5 + k)^2} i\hat{A}_+ W^{(2)} + \frac{9}{(5 + k)^2} \hat{A}_+ \hat{A}_3 \right] \]

\[ - \frac{2}{(5 + k)^2} i\hat{A}_+ \hat{A}_3 \hat{T}^{(1)} - \frac{4}{(5 + k)^2} \hat{A}_3 U^{(2)} + \frac{3}{(5 + k)^2} \hat{A}_3 \hat{A}_+ \]

\[ + \frac{4}{(5 + k)^2} i\hat{B}_3 U^{(2)} + \frac{(-8 + k)}{(5 + k)^2} \hat{B}_3 \hat{A}_+ \]

\[ + \frac{2}{(5 + k)^2} \hat{A}_3 \hat{A}_3 \hat{T}^{(1)} - \frac{4}{(5 + k)^2} \hat{A}_3 \hat{A}_3 \hat{A}_3 \]

\[ + \frac{4}{(5 + k)^2} i\hat{A}_+ \hat{A}_3 \hat{B}_3 + \frac{4}{(5 + k)^2} i\hat{A}_+ \hat{A}_+ \hat{A}_+ \]

\[ + \left[ \frac{2}{(5 + k)^2} i\hat{A}_+ \hat{B}_3 \hat{B}_3 + \frac{4}{(5 + k)^2} i\hat{A}_+ \hat{A}_3 \hat{A}_3 \right](w) + \cdots, \]

\[ T^{(2)}(z) V^{(2)}(w) = \frac{1}{(z - w)^3} \left[ \frac{8k(3 + k)}{3(5 + k)^2} i\hat{B}_+ \right](w) \]

\[ + \frac{1}{(z - w)^3} \left[ \frac{2(9 + 4k)}{3(5 + k)} V^{(2)} - \frac{4(9 + 2k)}{3(5 + k)^2} \hat{A}_+ \hat{B}_3 + \frac{4k}{(5 + k)^2} \hat{B}_3 \hat{B}_3 \right] \]

\[ - \frac{2k}{(5 + k)^2} i\hat{B}_+ \right] \]

\[ + \frac{1}{(z - w)^3} \left[ \frac{1}{4} \left\{ T^{(2)} \right\} V^{(2)} \right]_{\xi} + \frac{1}{2} \mathcal{O}^3 + \frac{8k(3 + k)}{(5 + k)(19 + 23k)} i\hat{B}_+ \hat{T} \]

\[ - \frac{2}{(5 + k)^2} T^{(2)} \right] \]

The composite fields $\hat{B}_\xi T_{\xi}^{(2)}(w)$ appearing in the second OPE of (A.1) are primary fields. In the third order OPE of (A.1), there exists a field with boldface $^{16}$ In the fourth OPE of (A.1), the last term in the first order pole can be written in terms of two terms with the coefficient \( \frac{(13 + 2k)}{3(5 + k)^2} \), \( \frac{8}{3(5 + k)^2} \). Then the first factor comes from the the higher spin current living in the second order pole. One can easily check that the five composite (quadratic) fields in the first order pole plus the derivative term with above.

\[ \text{In appendices we use a boldface notation for the second } \mathcal{N} = 4 \text{ multiplet in (2.2), } P^{(2)}, P^{(1)}_{\xi}, \cdots, S^{(2)}, \text{ in order to see them more clearly in the right hand side of all the OPEs. They will appear at the beginning of each singular term.} \]
The coefficient \( \frac{8}{3(5 + k)} \) is a primary field. In the sixth OPE, the second order pole subtracted by \( U^{(2)}_w \) term is a primary field. Similarly, the second order pole subtracted by \( V^{(2)}_w \) term in the seventh OPE is a primary field. Furthermore, one can check that the first order pole, subtracted by \( R^{(1)}_w \) term and the descendant terms coming from the primary spin-2 field located at the second singular term, behaves as a quasi primary field. In the eighth OPE, the first order pole, subtracted by \( Q^{(1)}_w \) term and the descendant terms coming from the primary spin-2 field located at the second singular term, is a quasi primary field. In the ninth OPE, the second order pole is a primary field and the last two terms in the first order pole is a quasi primary field.

- The OPEs between the higher spin-\( \frac{3}{2} \) currents and the fourth \( \mathcal{N} = 2 \) multiplet of (2.1)

\[
T^{(2)}_\pm(z) W^{(2)}(w) = \frac{1}{(z - w)^2} \left[ \frac{(7 + k) T^{(2)}_\pm}{(5 + k)} \right](w)
+ \frac{1}{(z - w)^3} \left[ -\frac{1}{2} \mathcal{P}^{(2)}_\pm \mp W^{(2)}_\pm + \frac{1}{3} \partial \left\{ T^{(2)}_\pm W^{(2)} \right\}_{-2} \right]_{-2} + \cdots,
\]

\[
T^{(2)}_\pm(z) W^{(2)}_\pm(w) = \pm \frac{1}{(z - w)^2} \left[ \frac{4(11 + k) \hat{A}_\pm \hat{B}_\pm}{3(5 + k)^2} \hat{A}_\pm \hat{B}_\pm \right](w)
+ \frac{1}{(z - w)^3} \left[ \frac{2(8 + k)}{(5 + k)^2} \left( \hat{G}_{21} \hat{G}_{21} - \hat{G}_{12} \hat{G}_{12} \right) \mp \frac{2}{(5 + k)^2} \hat{A}_\pm \left( \frac{U^{(2)}_w}{V^{(2)}_w} \right) \right.
+ \frac{2(k + k)}{(5 + k)^2} \hat{A}_\pm \hat{B}_\pm + \frac{2}{(5 + k)^2} \left( \hat{G}_{21} \hat{G}_{12} \right) T^{(2)}_\pm(w) + \cdots,
\]

\[
T^{(1)}_\pm(z) W^{(2)}_\pm(w) = \frac{1}{(z - w)^2} \left[ -\frac{8(5 + 2k)}{(5 + k)^2} i\hat{A}_3 + \frac{8k(-2 + k)}{3(5 + k)^2} i\hat{B}_3 \right.
- \frac{4(-2 + k)}{3(5 + k)} T^{(1)}_w \right]_{-2} + \left[ \frac{4}{3(5 + k)} \right](w)
+ \frac{1}{(z - w)^3} \left[ \frac{1}{2} \mathcal{P}^{(2)}_\pm + \frac{4}{3(5 + k)} T^{(1)}_w \right]_{-2}
+ \frac{4(4 + k)}{(5 + k)} W^{(2)}_\pm + \frac{16(-1 + k)}{(5 + k)^2} i\hat{A}_3 \hat{B}_3 + \frac{16(-1 + k)}{(5 + k)^2} i\hat{A}_2 \hat{B}_2
+ \frac{4(-13 + 4k)}{3(5 + k)^2} i\hat{A}_3 \hat{B}_3 + \frac{8(-4 + k)}{3(5 + k)^2} i\hat{A}_3 \hat{B}_3 + \frac{4}{3(5 + k)^2} i\hat{A}_3 \hat{B}_3
+ \frac{4}{3(5 + k)^2} i\hat{A}_2 \hat{B}_2 + \frac{4(-5 + 2k)}{(5 + k)^2} i\hat{A}_2 \hat{B}_2
+ \frac{4}{3(5 + k)^2} iT^{(1)}_w \hat{A}_3 = \frac{4}{(5 + k)^2} iT^{(1)}_w \hat{B}_3 \right]_{-2}
+ \frac{1}{(z - w)^3} \left[ \frac{1}{4} \partial \left\{ T^{(2)}_\pm W^{(2)}_\pm \right\}_{-2} + \frac{1}{2} \mathcal{P}^{(2)}_\pm \right]_{-2} - W^{(3)}
\]
\begin{align*}
- \frac{12k}{(5 + k)(3 + 7k)} \partial T & - \frac{4(-2 + k)}{(19 + 23k)} \left( \hat{T}^{(1)} - \frac{1}{2} \partial^2 T^{(1)} \right) \\
- \frac{24(25 + 15k + 2k^2)}{(5 + k)^2(19 + 23k)} \left( \hat{T} A_3 - \frac{1}{2} \partial^2 A_3 \right) \\
+ \frac{8k(-2 + k)}{(5 + k)(19 + 23k)} \left( \hat{T} B_3 - \frac{1}{2} \partial^2 B_3 \right) - \frac{2}{(5 + k)} \partial T^{(2)} \\
- \frac{2}{(5 + k)} \hat{C}_{12} T^{(2)} + \frac{2}{(5 + k)} \hat{C}_{21} T^{(2)} \right)_{(w)} + \ldots .
\end{align*}

\begin{align*}
T_{(z)}^{(2)} W^{(2)}_{(w)} = \frac{1}{(z - w)^3} \left[ \frac{8(5 + 2k)}{(5 + k)^2} \partial A_3 + \frac{8k(-2 + k)}{(5 + k)^2} iB_3 \\
- \frac{4(-2 + k)}{(5 + k)^2} T^{(1)} \right]_{(w)} \\
+ \frac{1}{(z - w)^3} \left[ \frac{1}{2} \partial^2 - \frac{4}{(5 + k)^2} \left( \frac{75 + 340k + 175k^2 + 22k^3}{3(5 + k)} \right) \hat{T} \\
+ \frac{4(-3 + k)}{3(5 + k)} T^{(2)} + \frac{4(4 + k)}{(5 + k)} W^{(2)} - \frac{16(-1 + k)}{3(5 + k)^2} \hat{A}_3 \hat{A}_3 \\
- \frac{4(-13 + 4k)}{3(5 + k)^2} \hat{A}_3 \hat{A}_3 + \frac{8(-4 + k)}{3(5 + k)^2} \hat{A}_3 \hat{B}_3 - \frac{4}{3(5 + k)} \hat{B}_3 \hat{B}_3 \\
+ \frac{4(-5 + 2k)}{3(5 + k)^2} \hat{B}_3 \hat{B}_3 - \frac{16(-1 + k)}{3(5 + k)^2} i\partial A_3 \\
- \frac{4}{3(5 + k)} i\partial B_3 - \frac{1}{(5 + k)^2} \partial T^{(1)} A_3 + \frac{2}{(5 + k)} \left( \frac{75 + 340k + 175k^2 + 22k^3}{3(5 + k)} \right) \hat{T} \\
+ \frac{1}{(z - w)^3} \left[ \frac{1}{2} \partial^2 - \frac{8}{(5 + k)^2} W^{(3)} - \frac{4}{(5 + k)} \left( \frac{75 + 340k + 175k^2 + 22k^3}{3(5 + k)} \right) \hat{T} \\
- \frac{3}{(5 + k)^2} \partial T^{(1)} A_3 - \frac{2}{(5 + k)} \partial A_3 V^{(2)} - \frac{4(-1 + k)}{3(5 + k)^2} \hat{A}_3 \hat{A}_3 \\
- \frac{2}{(5 + k)^2} \hat{A}_3 \hat{A}_3 U^{(2)} - \frac{24(25 + 15k + 2k^2)}{(5 + k)^2(19 + 23k)} \hat{A}_3 \hat{T} + \frac{2(13 - 4k)}{3(5 + k)^2} \hat{A}_3 \hat{A}_3 \\
+ \frac{2(-4 + 7k)}{3(5 + k)^2} \hat{A}_3 \hat{B}_3 + \frac{1}{(5 + k)} i\partial \hat{A}_3 \partial T^{(1)} - \frac{1}{3(5 + k)} \hat{B}_3 \partial B_3 \\
- \frac{1}{3(5 + k)} \hat{B}_3 \partial B_3 + \frac{8k(-2 + k)}{(5 + k)(19 + 23k)} i\partial T^{(1)} \\
- \frac{2(4 + 5k)}{3(5 + k)^2} \hat{B}_3 \partial A_3 + \frac{2(-5 + 2k)}{3(5 + k)^2} \hat{B}_3 \partial B_3 \\
+ \frac{1}{(5 + k)} i\partial \hat{B}_3 \partial T^{(1)} - \frac{1}{3(5 + k)} \left( \frac{15 + 83k + 22k^2}{3(5 + k)^2} \right) \partial T^{(1)} \hat{B}_3 \hat{B}_3 \\
+ \frac{1}{3(5 + k)} \hat{B}_3 \partial T^{(1)} - \frac{4(-1 + k)}{3(5 + k)^2} \hat{A}_3 \hat{A}_3 \right)_{(w)} + \ldots .
\end{align*}
\[ T_+^{(\frac{1}{2})}(z) \, W^{(3)}(w) = \frac{1}{(z - w)^2} \left[ \frac{4(-1 + 3k)}{(5 + k)^2} G_{21} \right. \\
+ \frac{4 \left( -1170 - 918k + 119k^2 + 55k^3 \right)}{3(5 + k)^2(19 + 23k)} T_+^{(\frac{1}{2})} \bigg|_{(w)} \right. \\
+ \frac{1}{(z - w)^2} \left[ \frac{2(24 + 5k)}{2(5 + k)} \mathbf{P}_+^{(\frac{1}{2})} + \frac{24 + 5k}{(5 + k)} W_+^{(\frac{1}{2})} - \frac{4}{(5 + k)^2} i\hat{A}_- \hat{G}_{11} \right. \\
+ \frac{4(11 + 3k)}{(5 + k)^2} i\hat{A}_- U^{(\frac{1}{2})} - \frac{12}{(5 + k)^2} i\hat{A}_3 \hat{G}_{21} \\
- \frac{2}{(5 + k)^2(19 + 23k)} \left( 413 + 774k + 169k^2 \right) i\hat{A}_1 T_+^{(\frac{1}{2})} + \left( 1 + k \right) \frac{(1 + k)}{(5 + k)^2} i\hat{B}_- \hat{G}_{22} \\
- \frac{1}{(5 + k)} i\hat{B}_- V^{(\frac{1}{2})} - \frac{4k}{(5 + k)^2} i\hat{B}_3 \hat{G}_{21} - \frac{2}{(5 + k)^2(19 + 23k)} \left( 57 + 350k + 37k^3 \right) i\hat{B}_3 T_+^{(\frac{1}{2})} \\
+ \frac{2}{(5 + k)^2} T^{(1)} \hat{G}_{21} - \frac{12}{(19 + 23k)} \left( -3 + k \right) T^{(1)} T_+^{(\frac{1}{2})} + \frac{4}{(5 + k)^2} \partial \hat{G}_{21} \\
+ \left( -1731 - 1754k - 283k^2 + 12k^3 \right) \frac{3(5 + k)^2}{(19 + 23k)} \partial T_+^{(\frac{1}{2})} \bigg|_{(w)} \right. \\
+ \frac{1}{(z - w)^2} \left[ - \frac{1}{(5 + k)} i\mathbf{R}_+^{(\frac{1}{2})} \hat{B}_- + \frac{8 + k}{2(5 + k)} \mathbf{P}_+^{(\frac{1}{2})} \right. \\
+ \frac{1}{(5 + k)} i\mathbf{Q}_+^{(\frac{1}{2})} \hat{A}_- + \ldots \bigg|_{(w)} \right. \\
+ \ldots, \\
\]
In the second OPE of (A.2), the first order pole subtracted by the correct descendant field leads to a primary field. Note that one has a relation \( \hat{G}_{12}(z) = \hat{T}(z) \). In the third OPE, the second order pole containing the nonlinear terms is a quasi primary field. The first order term consisting of last three terms with \( \partial \hat{T}(w) \) term leads to a primary field. In the fourth OPE, the second order pole starting from \( \hat{A}_+ \hat{A}_+(w) \) gives a quasi primary field. The first order term starting from \( T^{(1)}T(w) \) term by subtracting the correct descendant field provides a quasi primary field. In the fifth OPE, the second order pole is a primary field and the first order pole is a quasi primary field after subtracting the descendant field. In the sixth OPE, the second order pole is a primary field\(^{17}\).

### Appendix B. The nontrivial OPEs between the higher spin-2 current, \( T^{(2)}(z) \), and other 13 higher currents

We present the OPEs between the last component in the first \( \mathcal{N} = 2 \) multiplet and the remaining higher spin currents.

- The OPEs between the higher spin-2 current and the first \( \mathcal{N} = 2 \) multiplet of (2.1)

\[
T^{(2)}(z) T^{(2)}(w) = \frac{1}{(z - w)^2} \left[ \frac{72k(3 + 4k + k^2)}{(5 + k)^2(3 + 7k)} \right] + \frac{1}{(z - w)^2} \left[ \frac{96k(1 + k)(3 + k)}{(5 + k)(3 + 7k)^2} \right] + \frac{4}{(5 + k)(3 + 7k)} \hat{T}(w) \]

- The OPEs between the higher spin-2 current and the second and third \( \mathcal{N} = 2 \) multiplet of (2.1)

\(^{17}\) In the published version of this paper, the detailed expressions in some singular terms in appendices (for example, in the first order poles in the last two OPEs of (A.2)) are ignored due to the space limitations and they can be found in the arXiv version.
\[ T^{(2)}(z) U^{(2)}(w) = -\frac{1}{(z - w)^2} \left\{ \left( \frac{-21 + 7k + 2k^2}{(5 + k)(3 + 7k)} U^{(2)} \right)_{(w)} \right. \\
+ \frac{1}{(z - w)} \left\{ \frac{2}{3} \partial \left\{ T^{(2)} U^{(2)} \right\} \right. \\
\left. - \frac{1}{2} Q^{(2)} + U^{(2)} \right\} \right\} \right\} + \cdots, \]

\[ T^{(2)}(z) V^{(2)}(w) = -\frac{1}{(z - w)^2} \left\{ \left( \frac{-21 + 7k + 2k^2}{(5 + k)(3 + 7k)} V^{(2)} \right)_{(w)} \right. \\
+ \frac{1}{(z - w)} \left\{ \frac{1}{2} R^{(2)} - V^{(2)} + \frac{2}{(5 + k)} i\Lambda V^{(2)} \right. \\
\left. - \frac{2}{(5 + k)} i\bar{B}_3 T^{(2)} + \frac{4}{(5 + k)} i\bar{B}_3 V^{(2)} - \frac{2}{3(5 + k)(3 + 7k)} \partial V^{(2)} \right\} \right\} \right\} + \cdots, \]

\[ T^{(2)}(z) U^{(2)}(w) = \frac{1}{(z - w)^3} \left\{ \frac{2k(6 + k)}{(5 + k)^2} i\bar{B}_+ \right\} \right\} \right\} + \cdots, \]

\[ T^{(2)}(z) V^{(2)}(w) = \frac{1}{(z - w)^3} \left\{ \frac{2k(6 + k)}{(5 + k)^2} i\bar{B}_+ \right\} \right\} \right\} + \cdots, \]
\[
\begin{align*}
&+ \frac{4k}{(5 + k)^2} \hat{B}_1 \hat{B}_1 + \frac{k(4 + k)}{(5 + k)^2} i\hat{a} \hat{B}_3 + \frac{2}{(5 + k)^2} iT(1) \hat{B}_1 \\
&+ \frac{1}{(z - w)} \left[ \frac{1}{2} R^{(3)} + \frac{1}{2(5 + k)} iT(1) \hat{a} \hat{B}_3 + \frac{11 + 2k}{(5 + k)^2} i\hat{A}_3 \hat{B}_3 \hat{B}_3 \right] (w) \\
&+ \frac{4k}{(5 + k)(3 + 7k)(19 + 23k)} i\hat{B}_1 \hat{T} + \frac{2}{(5 + k)^2} iT(2) \\
&- \frac{k(6 + k)}{3(5 + k)^2} i\hat{B}_1 \hat{B}_3 \hat{B}_3 + \frac{k(33 + 4k)}{3(5 + k)^2} \hat{B}_1 \hat{B}_3 \hat{B}_3 + \frac{3}{2(5 + k)} i\hat{B}_3 \hat{a} \hat{T} \\
&+ \frac{k}{(5 + k)^2} i\hat{B}_1 \hat{a} \hat{B}_3 + \frac{21 + 5k + 2k^2}{2(5 + k)(3 + 7k)} \hat{a} \hat{U}^{(2)} - \frac{5 + 2k}{2(5 + k)^2} i\hat{A}_3 \hat{A}_3 \hat{B}_3 \\
&+ \frac{(5 + 2k)}{2(5 + k)^2} i\hat{A}_3 \hat{A}_3 \hat{B}_3 - \frac{2}{(5 + k)^2} \hat{G}_{12} \hat{T}(2) - \frac{2}{(5 + k)^2} \hat{T}(2) \hat{V}(2) \\
&+ \frac{k(6 + k)}{3(5 + k)^2} i\hat{B}_3 \hat{B}_3 \hat{B}_3 - \frac{11 - 2k}{(5 + k)^2} i\hat{B}_3 \hat{A}_3 \hat{B}_3 \hat{B}_3 \right] (w) + \ldots, \\
T^{(2)}(z) U^{(2)}(w) &= \frac{1}{(z - w)} \left[ \frac{6(5 + 2k)}{(5 + k)^2} i\hat{A}_3 \right] (w) \\
&+ \frac{1}{(z - w)} \left[ \frac{2}{(5 + k)^2} i\hat{U}^{(2)} \right] + \frac{3(5 + 2k)}{(5 + k)^2} i\hat{A}_3 \hat{U}^{(2)} \\
&+ \frac{2}{(5 + k)(19 + 23k)} i\hat{A}_3 \hat{T} + \frac{2}{(5 + k)^2} i\hat{A}_3 \hat{W}^{(2)} \\
&+ \frac{(29 + 8k)}{(5 + k)^2} i\hat{A}_3 \hat{A}_3 \hat{B}_3 - \frac{3}{2(5 + k)} i\hat{A}_3 \hat{a} \hat{T} \\
&+ \frac{3}{(5 + k)^2} i\hat{A}_3 \hat{a} \hat{A}_3 + \frac{4}{(5 + k)^2} i\hat{B}_3 \hat{U}^{(2)} \\
&+ \frac{39 + 37k + 2k^2}{2(5 + k)(3 + 7k)} \hat{U}^{(2)} - \frac{2}{(5 + k)} \hat{G}_{12} \hat{G}_{12} + \frac{2}{(5 + k)} \hat{G}_{12} \hat{U}(2) \\
&+ \frac{2}{(5 + k)^2} \hat{G}_{12} \hat{T}(2) - \frac{2}{(5 + k)^2} \hat{T}(2) \hat{U}(2) - \frac{(5 + 2k)}{(5 + k)^2} i\hat{A}_3 \hat{A}_3 \hat{B}_3 \\
&+ \frac{4}{(5 + k)^2} i\hat{A}_3 \hat{A}_3 \hat{B}_3 - \frac{k}{(5 + k)^2} i\hat{A}_3 \hat{B}_3 \hat{B}_3 + \frac{4}{(5 + k)^2} i\hat{A}_3 \hat{B}_3 \hat{B}_3 \\
&+ \frac{(9 + 2k)}{(5 + k)^2} i\hat{A}_3 \hat{A}_3 \hat{B}_3 + \frac{(9 + 2k)}{(5 + k)^2} i\hat{A}_3 \hat{A}_3 \hat{B}_3 \\
&+ \frac{(9)}{(5 + k)^2} i\hat{A}_3 \hat{A}_3 \hat{B}_3 \right] (w) + \ldots.
\end{align*}
\]
\[
T_{(2)} (z) \ V_{(2)}^+(w) = \frac{1}{(z-w)^{3}} \left[ \frac{6(5+2k)}{(5+k)^2} iA_- \right](w) \\
+ \frac{1}{(z-w)^{2}} \left[ \frac{2(12+k+k^2)}{(5+k)(3+7k)} V_{(2)}^+(w) + \frac{3(5+2k)}{(5+k)^2} iA_- \right](w) \\
+ \frac{1}{(z-w)} \left[ \frac{1}{2} R_{+}^{(3)} - \frac{3}{2(5+k)} iT_{(1)}^{(1)} A_- + \frac{4}{(5+k)} iA_3 V_{(2)}^+ \right] \\
+ \frac{8(23+13k)}{(5+k)(19+23k)} iA_- \tilde{T} - \frac{2}{(5+k)} iA_\omega W_{(2)}^+ - \frac{9}{(5+k)^2} A_\omega \tilde{A}_3 \\
+ \frac{(-8+k)}{(5+k)^2} A_\omega \tilde{B}_3 + \frac{3}{2(5+k)} iA_- \tilde{T}^{(1)} - \frac{3}{(5+k)^2} A_\omega \tilde{A}_3 \\
+ \frac{1}{2(5+k)(3+7k)} \partial V_{(2)}^+ - \frac{2}{(5+k)} \tilde{G}_{21} \tilde{G}_{22} + \frac{2}{(5+k)} \tilde{G}_{21} V_{(2)}^+ \\
+ \frac{2}{(5+k)} \tilde{G}_{21} T_{(2)}^+ + \frac{2}{(5+k)} t_{(2)} V_{(1)}^+ + \frac{4}{(5+k)^2} iA_\omega A_\omega \\
+ \frac{5+2k}{(5+k)^2} iA_\omega A_\omega + \frac{4}{(5+k)^2} iA_- \tilde{A}_3 \tilde{A}_3 - \frac{8}{(5+k)^2} iA_- \tilde{A}_3 \tilde{B}_3 \\
+ \frac{4}{(5+k)^2} iA_- \tilde{B}_3 \tilde{B}_3 + \frac{4}{(5+k)^2} iA_- \tilde{B}_3 \tilde{B}_3 - \frac{(-16+k)}{(5+k)^2} \tilde{B}_3 \tilde{A}_- \\
- \frac{4}{(5+k)} iB_3 V_{(2)}^+ \right](w) + \ldots.
\]
The OPEs between the higher spin-2 current and the fourth $\mathcal{N} = 2$ multiplet of (2.1)

\[
T^{(2)}(z) V^{(2)}(w) = \frac{1}{(z - w)^3} \left[ -\frac{8(2 + k)(6 + k)}{3(5 + k)^2} \hat{G}_{22} + \frac{4(11 + k)}{3(5 + k)^2} V^{(2)}(w) \right] + \frac{1}{(z - w)^2} \left[ \frac{1}{6(5 + k)} R^{(2)}(z) + \frac{105 + 41k + 4k^2}{3(5 + k)(3 + 7k)} V^{(2)}(z) \right]
\]

\[
- \frac{4(9 + 2k)}{3(5 + k)^2} i\hat{A}_1 \hat{G}_{22} - \frac{4(5 + 2k)}{3(5 + k)^2} i\hat{A}_3 V^{(2)}(z) + \frac{4(19 + 4k)}{3(5 + k)^2} i\hat{B}_3 T^{(2)}(z)
\]

\[
+ \frac{4k}{(5 + k)^2} i\hat{B}_3 \hat{G}_{22} + \frac{4(8 + k)}{3(5 + k)^2} i\hat{B}_3 V^{(2)}(z) - \frac{4(18 + 15k + 2k^2)}{9(5 + k)^2} \partial \hat{G}_{22}
\]

\[
- \frac{4(14 + 5k)}{9(5 + k)^2} \partial V^{(2)}(z) - \frac{2}{5 + k} T^{(1)} T^{(2)} + \frac{2}{(5 + k)} T^{(1)} V^{(2)}(z)
\]

\[
+ \frac{1}{(z - w)} \left[ \frac{1}{15(5 + k)} \partial R^{(2)}(z) - \frac{1}{(5 + k)} i\mathcal{P}^{(2)}(z) \hat{A}_- \right]
\]

\[
+ \frac{1}{2} R^{(2)} + \ldots \right] (w) + \ldots
\]

\[
T^{(2)}(z) W^{(2)}(w) = \frac{1}{(z - w)^2} \left[ -\frac{1}{2} P^{(2)} - \frac{4(15 + 23k + 16k^2)}{3(5 + k)(3 + 7k)} \hat{T} + \frac{8(2 + k)}{3(5 + k)} T^{(2)} \right]
\]

\[
+ \frac{16(3 + 4k + k^2)}{(5 + k)(3 + 7k)} W^{(2)} - \frac{16(-1 + k)}{(5 + k)^2} \hat{A}_+ \hat{A}_-
\]

\[
- \frac{4(-13 + 4k)}{3(5 + k)^2} \hat{A}_3 \hat{A}_3 + \frac{8(-4 + k)}{3(5 + k)^2} \hat{A}_3 \hat{B}_3 - \frac{4}{3(5 + k)^2} \hat{B}_+ \hat{B}_-
\]

\[
+ \frac{4(-5 + 2k)}{3(5 + k)^2} \hat{B}_3 \hat{B}_3 - \frac{16(-1 + k)}{3(5 + k)^2} i\partial \hat{A}_3 - \frac{4}{3(5 + k)^2} i\partial \hat{B}_3
\]

\[
- \frac{4}{(5 + k)} i\mathcal{T}^{(1)} \hat{A}_3 + \frac{4}{(5 + k)} i\mathcal{T}^{(1)} \hat{B}_3 \right] (w)
\]

\[
+ \frac{1}{(z - w)} \left[ -\frac{1}{4} \partial P^{(2)} - \frac{4}{(5 + k)} i\mathcal{T}^{(1)} \partial \hat{A}_3 + \frac{2}{(5 + k)} i\mathcal{T}^{(1)} \partial \hat{B}_3
\]

\[
- \frac{2}{(5 + k)} i\hat{A}_+ V^{(2)} + \frac{8(-1 + k)}{3(5 + k)^2} \hat{A}_+ \partial \hat{A}_- - \frac{2}{(5 + k)} i\partial \hat{U}^{(2)}
\]

\[
- \frac{4(-13 + 4k)}{3(5 + k)^2} \hat{A}_3 \partial \hat{A}_3 + \frac{16(-1 + k)}{3(5 + k)^2} \hat{A}_3 \partial \hat{B}_3 - \frac{2}{3(5 + k)} \hat{B}_+ \partial \hat{B}_-
\]

\[
\]
\[- \frac{2}{3(5 + k)} \dddot{B}_- \dot{B}_+ - \frac{8(2 + k)}{3(5 + k)^2} \dddot{B}_3 \dot{A}_3 + \frac{4(-5 + 2k)}{3(5 + k)^2} \dddot{B}_3 \dot{B}_3 \\
- \frac{2(15 + 32k + 16k^2)}{3(5 + k)(3 + 7k)} \dddot{T} - \frac{8(-1 + k)}{3(5 + k)^2} \dddot{A}_- \dot{A}_+ \\
+ \frac{2}{(5 + k)} i \dddot{B}_3 \dddot{T}^{(1)} + \frac{(5 + 4k)}{3(5 + k)} \dddot{A}_+ \dddot{A}_+ \\
+ \frac{(27 + 39k + 8k^2)}{(5 + k)(3 + 7k)} \dddot{W}^{(2)} \right] (w) + \ldots. \]

\[
T^{(2)}(z) \ W^{(2)}_Z (w) = -\frac{1}{(z - w)^3} \left[ \frac{4(11 + k)}{3(5 + k)^2} \dddot{G}_{21} + \frac{4(11 + k)}{3(5 + k)^2} \dddot{T}^{(2)}_Z \right] (w) \\
+ \frac{1}{(z - w)^2} \left[ \frac{(21 + 5k)}{2(5 + k)} \dddot{P}_Z^{(2)} + \frac{2(33 + 47k + 10k^2)}{(5 + k)(3 + 7k)} \right] W^{(2)}_Z \\
+ \frac{4(8 + k)}{3(5 + k)^2} \dddot{A}_- \dot{G}_{11} + \frac{4(23 + 7k)}{3(5 + k)^2} \dddot{A}_- \dot{U}^{(2)} - \frac{12}{(5 + k)^2} \dddot{A}_3 \dot{G}_{21} \\
- \frac{8(4 + k)}{(5 + k)^2} \dddot{A}_3 T^{(2)}_Z + \frac{4(8 + k)}{3(5 + k)^2} i \dddot{B}_- \dot{V}^{(2)} - \frac{4k}{(5 + k)^2} i \dddot{B}_3 \dot{G}_{21} \\
+ \frac{2}{(5 + k)} T^{(1)} \dot{G}_{21} - \frac{4(8 + k)}{3(5 + k)^2} \dddot{G}_{21} - \frac{4(13 + 3k)}{3(5 + k)^2} \dddot{T}^{(2)}_Z \right] (w) \\
+ \frac{1}{(z - w)} \left[ \dddot{P}_Z^{(2)} + \ldots \right] (w) + \ldots. \]

\[
T^{(2)}(z) \ W^{(2)}_Z (w) = \frac{1}{(z - w)^3} \left[ -\frac{4(11 + k)}{3(5 + k)^2} \dddot{G}_{12} + \frac{4(11 + k)}{3(5 + k)^2} \dddot{T}^{(2)}_Z \right] (w) \\
+ \frac{1}{(z - w)^2} \left[ -\frac{(21 + 5k)}{2(5 + k)} \dddot{P}_Z^{(2)} + \frac{2(33 + 47k + 10k^2)}{(5 + k)(3 + 7k)} \right] W^{(2)}_Z \\
+ \frac{4(23 + 7k)}{3(5 + k)^2} \dddot{A}_+ \dot{V}^{(2)} + \frac{12}{(5 + k)^2} \dddot{A}_3 \dot{G}_{12} - \frac{8(4 + k)}{(5 + k)^2} \dddot{A}_3 T^{(2)}_Z \\
+ \frac{4(8 + k)}{3(5 + k)^2} \dddot{B}_- \dot{U}^{(2)} + \frac{4k}{(5 + k)^2} i \dddot{B}_3 \dot{G}_{12} - \frac{2}{(5 + k)} T^{(1)} \dot{G}_{12} \\
- \frac{4(8 + k)}{3(5 + k)^2} \dddot{G}_{12} + \frac{4(13 + 3k)}{3(5 + k)^2} \dddot{T}^{(2)}_Z - \frac{4(8 + k)}{3(5 + k)^2} i \dddot{A}_+ \dot{G}_{22} \right] (w) \\
+ \frac{1}{(z - w)} \left[ -\dddot{P}_Z^{(2)} + \ldots \right] (w) + \ldots. \]
\[
T^{(2)}(z) W^{(3)}(w) = \frac{1}{(z - w)^4} \left[ \frac{24 \left( 24 + 83k + 21k^2 \right)}{(5 + k)^2(19 + 23k)} i\hat{A}_3 \\
- 8k \left( 32 + 435k + 160k^2 + 21k^3 \right) i\hat{B}_3 \\
+ \frac{4 \left( -82 + 117k + 34k^2 + 3k^3 \right)}{(5 + k)^2(19 + 23k)} T^{(1)} \right] (w) \\
+ \frac{1}{(z - w)^3} \left[ \frac{1}{(5 + k)} \left( p^{(2)} + \frac{4 \left( -33 - 11k + 20k^2 \right)}{3(5 + k)^2(3 + 7k)} \right) \hat{T} \\
- \frac{2(-17 + 5k)T^{(2)}}{3(5 + k)^2} - \frac{2(3 + k)W^{(2)}}{(5 + k)^3} + \frac{4(25 + 23k)}{3(5 + k)^3} \hat{A}_+ \hat{A}_- \\
+ \frac{4(-29 + 23k)}{3(5 + k)^3} \hat{A}_+ \hat{A}_3 + \frac{8(-4 + k)(1 + 3k)}{3(5 + k)^3} \hat{A}_3 \hat{B}_3 \\
- \frac{4 \left( 11 + 22k + 3k^2 \right)}{3(5 + k)^3} \hat{B}_+ \hat{B}_- - \frac{4 \left( 11 + k + 6k^2 \right)}{3(5 + k)^3} \hat{B}_3 \hat{B}_3 \\
+ \frac{4(25 + 23k)}{3(5 + k)^3} i\hat{A}_3 - \frac{4 \left( 11 + 22k + 3k^2 \right)}{3(5 + k)^3} i\hat{B}_3 + \frac{4(3 + k)}{(5 + k)^2} i\hat{T}^{(1)} \hat{A}_3 \\
- \frac{4(1 + k)}{(5 + k)^2} i\hat{T}^{(1)} \hat{B}_3 \right] (w) \\
+ \frac{1}{(z - w)^2} \left[ -\frac{(13 + 3k)}{(5 + k)} p^{(3)} + \frac{1}{4(5 + k)} \partial p^{(2)} \\
+ \frac{4 \left( 21 + 31k + 6k^2 \right)}{(5 + k)^2(3 + 7k)} W^{(3)} + \frac{8(5 + k)(-3 + 2k)}{(3 + 7k)(19 + 23k)} T^{(1)} \hat{T} \\
- \frac{16(-3 + k)T^{(1)} \hat{T}^{(2)}}{(19 + 23k)} - \frac{2(15 + 2k)}{(5 + k)^2} i\hat{T}^{(1)} \hat{A}_3 \\
+ \frac{2(1 + k)}{(5 + k)^2} i\hat{T}^{(1)} \hat{B}_3 + \frac{2(22 + 5k)}{(5 + k)^2} i\hat{A}_+ V^{(2)} - \frac{4(17 + k)}{3(5 + k)^3} \hat{A}_+ \partial \hat{A}_- \\
- \frac{2(6 + k)}{(5 + k)^2} i\hat{A}_- U^{(2)} - \frac{32 \left( 237 + 604k + 179k^2 + 12k^3 \right)}{(5 + k)^2(3 + 7k)(19 + 23k)} i\hat{A}_3 \hat{T} \\
- \frac{8(-7 + k)}{3(5 + k)^3} \hat{A}_- \partial \hat{A}_+ + \frac{28(-1 + k)}{3(5 + k)^3} \hat{A}_3 \partial \hat{A}_3 \\
- \frac{4k}{(5 + k)^2} \hat{B}_3 \partial T^{(1)} + \frac{8 \left( 87 + 184k + 41k^2 \right)}{(5 + k)(95 + 134k + 23k^2)} i\hat{A}_3 T^{(2)} \right].
\]
\[
\begin{align*}
&+ \frac{8}{(5 + k)^2} i \hat{A}_3 W^{(2)} - \frac{2(12 + k)}{(5 + k)^2} i \hat{A}_3 \delta T^{(1)} + \frac{(9 + k)}{(5 + k)^2} i \hat{B}_+ U^{(2)} \\
&- \left( \frac{88 + 107k + 15k^2}{3(5 + k)^3} \right) \hat{B}_+ \partial \hat{B}_- + \frac{5(1 + k)}{(5 + k)^2} i \hat{B}_- V^{(2)} \\
&+ \frac{3k}{(5 + k)^2} \hat{B}_- \partial \hat{B}_+ - \frac{8 (19 + 123k + 20k^2)}{(5 + k)^2 (19 + 23k)} i \hat{B}_3 T^{(2)} \\
&+ \frac{4 \left( 10 + 17k + 6k^2 \right)}{3(5 + k)^3} \hat{B}_3 \partial \hat{A}_3 + \frac{2 \left( -66 + 203k + 64k^2 \right)}{3(5 + k)^2 (3 + 7k)} \partial \hat{F} \\
&- \frac{4 (121 + 26k)}{3(5 + k)^2} \partial T^{(2)} + \frac{5(4 + k)}{(5 + k)^2} \partial W^{(2)} + \frac{4(20 + 13k)}{3(5 + k)^3} i \partial^2 \hat{A}_3 \\
&- \frac{2 \left( 22 + 77k + 13k^2 \right)}{3(5 + k)^3} i \hat{A}_3 \hat{B}_3 + \frac{2(17 + 3k)}{3(5 + k)} \partial^2 T^{(1)} + \frac{4}{(5 + k)^2} \hat{G}_{11} \hat{G}_{22} \\
&- \frac{4}{(5 + k)^2} \hat{G}_{11} U^{(2)} + \frac{2(11 + 3k)}{(5 + k)^2} \hat{G}_{12} T^{(2)} - \frac{6}{(5 + k)} \hat{G}_{21} T^{(2)} \\
&- \frac{64}{(5 + k)^2} \hat{A}_+ \hat{A}_- \hat{B}_3 + \frac{32}{(5 + k)^3} \hat{A}_+ \hat{A}_+ \hat{A}_3 \\
&+ \frac{8}{(5 + k)^3} \hat{A}_3 \hat{A}_3 \hat{A}_3 + \frac{8(2 + k)}{(5 + k)^3} i \hat{A}_3 \hat{A}_3 \hat{B}_3 \\
&- \frac{2 \left( -61 + 19k + 3k^2 \right)}{3(5 + k)^3} i \hat{A}_3 \hat{B}_3 \hat{B}_- + \frac{2 \left( -25 + 19k + 3k^2 \right)}{3(5 + k)^3} i \hat{B}_- \hat{A}_3 \hat{B}_3 \\
&+ \frac{2 \left( 22 + 11k + 3k^2 \right)}{3(5 + k)^3} i \hat{B}_- \hat{B}_- \hat{B}_3 - \frac{2 \left( 22 - k + 3k^2 \right)}{3(5 + k)^3} i \hat{B}_+ \hat{B}_+ \hat{B}_3 \\
&+ \frac{8k}{(5 + k)^3} i \hat{B}_3 \hat{B}_3 \hat{B}_3 - \frac{4}{(5 + k)^2} T^{(1)} \hat{A}_3 \hat{A}_3 \\
&+ \frac{8}{(5 + k)^2} T^{(1)} \hat{A}_3 \hat{B}_3 - \frac{4}{(5 + k)^2} T^{(1)} \hat{B}_3 \hat{B}_3 \\
&- \frac{4}{(5 + k)^2} \hat{B}_- T^{(1)} \hat{B}_+ = \frac{16k}{(5 + k)^2 (3 + 7k)(19 + 23k)} \left( -30 + 175k + 87k^2 + 14k^3 \right) i \hat{B}_+ \hat{T} \\
&- \frac{8(-1 + 2k)}{(5 + k)^3} i \hat{A}_3 \hat{B}_3 \hat{B}_3 \right) (w) \\
&+ \frac{1}{(z - w)} \left[ - \frac{1}{(5 + k)} i \hat{A}_- Q^{(3)} - \frac{4}{(5 + k)} i \hat{B}_3 S^{(3)} \right]
\end{align*}
\]
\[-\frac{4}{(5 + k)}i\hat{B}_3 \hat{P}^{(3)} + \frac{2}{(5 + k)}\hat{B}_3 \hat{B}_3 \hat{P}^{(2)} + \frac{3}{(5 + k)}i\hat{B}_3 \hat{R}^{(3)} \]
\[+ \frac{2}{(5 + k)}i\hat{B}_4 \hat{Q}_4^{(3)} - \frac{1}{(5 + k)}\hat{B}_4 \hat{B}_4 \hat{P}^{(2)} - \frac{1}{(5 + k)}(\partial \hat{B}_3) \hat{P}^{(2)} \]
\[-\frac{3 + k}{(5 + k)}\partial \hat{P}^{(3)} - \frac{1}{4(5 + k)} \hat{P}^{(2)} \]
\[+ \cdots] \]
\text{(B.1)}

The first order pole in the last OPE of (B.1) contains composite field with spin-4 with vanishing \(U(1)\) charge. Of course, the analysis done in appendix A can be done here similarly. For given any OPE, one can determine the nonzero descendant fields with correct coefficients. Then any singular terms consist of these descendant fields and a couple of (quasi) primary fields. The singular terms subtracted by the descendant fields should behave as a quasi primary or primary field. So one should calculate the OPE between the stress energy tensor \(\hat{T}(z)\) and the above reduced singular terms. Then the third order singular terms of this OPE vanish. If the fourth or higher singular terms of this OPE are not vanishing, then it will provide a quasi primary field. If they are vanishing, then one has a primary field as usual.

Appendix C. The nontrivial OPEs between higher spin-\(\frac{3}{2}\) current, \(U^{(\frac{3}{2})}(z)\), and other 12 higher spin currents

Now we move to other \(\mathcal{N} = 2\) multiplet and list the OPEs as follows

\[U^{(\hat{2})}(z) U^{(\hat{2})}(w) = \frac{1}{(z - w)^2} \left\{ -\frac{4(-3 + k)}{3(5 + k)^2} \hat{A}_4 \hat{B}_4 \right\}(w) \]
\[+ \frac{1}{(z - w)^2} \left\{ \frac{2}{(5 + k)} \hat{B}_4 U^{(2)} - \frac{2(7 + k)}{(5 + k)^2} \hat{B}_2 \partial \hat{A}_4 \right\}(w) \]
\[+ \frac{2}{(5 + k)} \hat{A}_4 U^{(2)} - \frac{(12 + k)}{3(5 + k)} \hat{G}_{11} \hat{G}_{11} \right\}(w) + \cdots, \]

\[U^{(\hat{2})}(z) V^{(\hat{2})}(w) = -\frac{1}{(z - w)^2} \left\{ \frac{6k}{(5 + k)} \right\}(w) \]
\[+ \frac{1}{(z - w)^2} \left\{ \frac{6}{(5 + k)} i\hat{A}_3 + \frac{2k}{(5 + k)} i\hat{B}_3 - T^{(1)} \right\}(w) \]
\[+ \frac{1}{(z - w)^2} \left\{ \frac{1}{2} \partial \left\{ U^{(\hat{2})} V^{(\hat{2})} \right\}_{-2} - W^{(2)} \right\}(w) + \cdots, \]

\[U^{(\hat{2})}(z) V^{(2)}_{+}(w) = \frac{1}{(z - w)^2} \left\{ \frac{(5 + 2k) T^{(2)}_{+}}{(5 + k)} \right\}(w) \]
\[+ \frac{1}{(z - w)^2} \left\{ \frac{1}{3} \partial \left\{ U^{(\hat{2})} V^{(2)}_{+} \right\}_{-2} + \frac{1}{2} P^{(2)}_{+} + W^{(2)} \right\}(w) + \cdots, \]

\[U^{(\hat{2})}(z) V^{(2)}_{-}(w) = \frac{1}{(z - w)^2} \left\{ \frac{(8 + k)}{(5 + k)} \hat{G}_{12} + \frac{(8 + k) T^{(2)}_{-}}{(5 + k)} \right\}(w) \]
\[+ \frac{1}{(z - w)^2} \left\{ \frac{1}{3} \partial \left\{ U^{(\hat{2})} V^{(2)}_{-} \right\}_{-2} + \frac{1}{2} P^{(2)}_{-} \right\}(w) + \cdots,\]
$$U(\tilde{z})(z) \ V(\tilde{z})(w) = \frac{1}{(z - w)^3} \left[ - \frac{8(-4 + k)}{(5 + k)^2} i\tilde{A}_- + \frac{8k(8 + k)}{3(5 + k)^2} i\tilde{B}_- - \frac{4(-4 + k)}{3(5 + k)} T^{(1)}(w) \right] + \frac{1}{(z - w)^2} \left[ \frac{1}{2} \left( \frac{16(30 + 61k + 31k^2 + 4k^3)}{3(5 + k)^2(3 + 7k)} \right) \tilde{T} - \frac{8(2 + k)}{3(5 + k)} T^{(2)}(w) \right]$$

$$- \frac{8(2 + k)}{3(5 + k)} W^{(2)}(w) + \frac{16(2 + k)}{3(5 + k)^2} \tilde{A}_+ \tilde{A}_- + \frac{16(2 + k)}{3(5 + k)^2} \tilde{A}_3 \tilde{A}_3$$

$$- \frac{32(2 + k)}{3(5 + k)^2} \tilde{A}_3 \tilde{B}_3 + \frac{16(2 + k)}{3(5 + k)^2} \tilde{B}_3 \tilde{B}_3 + \frac{16(2 + k)}{3(5 + k)^2} \tilde{B}_3 \tilde{B}_3$$

$$+ \frac{16(2 + k)}{3(5 + k)^2} i\partial \tilde{A}_3 + \frac{16(2 + k)}{3(5 + k)^2} i\partial \tilde{B}_3 \right]$$

$$+ \frac{1}{(z - w)^3} \left[ \frac{1}{2} S^{(3)}(w) + \frac{1}{8} i\partial \tilde{B}_3 - \frac{4(-3 + k)}{(19 + 23k)} T^{(1)}(w) \right]$$

$$- \frac{2}{(5 + k)} i\tilde{A}_+ \tilde{V}_+^{(2)} + \frac{(-7 + 4k)}{3(5 + k)^2} \tilde{A}_+ \tilde{A}_- + \frac{(23 + 4k)}{3(5 + k)^2} \tilde{A}_- \tilde{A}_+$$

$$+ \frac{2}{(5 + k)} \tilde{A}_3 \partial \tilde{B}_3 - \frac{4}{(5 + k)} i\tilde{A}_3 T^{(2)}(w) - \frac{4}{(5 + k)} i\tilde{A}_3 W^{(2)}(w)$$

$$+ \frac{8(-1 + k)}{3(5 + k)^2} \tilde{A}_3 \partial \tilde{A}_3 + \frac{2}{(8 + 7k)} \tilde{B}_3 \partial \tilde{B}_3 - \frac{2}{(5 + k)^2} \tilde{B}_3 \partial \tilde{B}_3$$

$$+ \frac{8}{(3 + 7k)(19 + 23k)} i\tilde{B}_3 \tilde{T} + \frac{4}{(5 + k)} i\tilde{B}_3 T^{(2)} - \frac{2}{(19 + 23k)} \tilde{B}_3 \partial \tilde{A}_3$$

$$+ \frac{1}{(5 + k)} i\tilde{B}_3 \partial T^{(1)} + \frac{1}{6(5 + k)} i\partial T^{(1)}(w)$$

$$- \frac{4(1 + 2k)}{6(5 + k)} (17 + 4k) \tilde{T} - \frac{6(5 + k)}{4(5 + k)} \partial T^{(2)}$$

$$- \frac{2}{3(5 + k)^2} \partial T^{(1)} + \frac{2}{(5 + k)} U(\tilde{z})(z)$$

$$+ \frac{8}{(5 + k)^2} i\tilde{A}_+ \tilde{A}_- \tilde{A}_3 + \frac{8}{(5 + k)^2} i\tilde{A}_3 \tilde{A}_3 \tilde{A}_3$$

$$- \frac{16}{(5 + k)^2} i\tilde{A}_3 \tilde{B}_3 + \frac{8}{(5 + k)^2} i\tilde{A}_3 \tilde{B}_3 \tilde{B}_3 + \frac{8}{(5 + k)^2} i\tilde{A}_3 \tilde{B}_3 \tilde{B}_3$$

$$+ \frac{4(2 + k)}{3(5 + k)^2} \tilde{B}_- \tilde{B}_3 \tilde{B}_3 + \frac{4(2 + k)}{3(5 + k)^2} \tilde{B}_- \tilde{B}_3 \tilde{B}_3 - \frac{1}{2(5 + k)} T^{(1)} \tilde{B}_3 \tilde{B}_3$$

$$+ \frac{1}{2(5 + k)} \tilde{B}_- T^{(1)}(w) \tilde{B}_+$$

$$- \frac{2}{(5 + k)} T^{(2)}(w) \tilde{T}$$
\[
U(\hat{z}) W^{(2)}(w) = \left. \frac{1}{(z - w)^2} \left[ \frac{3(4 + k)}{(5 + k)} U^{(2)}(\hat{z}) \right] \right|_{w}(w) \\
+ \left. \frac{1}{(z - w)} \frac{1}{3} \left\{ U^{(2)}(\hat{z}) \right\} \right|_{-2}(w) + \ldots,
\]
\[
U(\hat{z}) W^{(2)}_-(w) = -\left. \frac{1}{(z - w)^2} \left[ \frac{8k(8 + k)}{3(5 + k)^2} i\hat{B}_- \right] \right|_{w}(w) \\
- \left. \frac{1}{(z - w)^2} \left[ \frac{8(2 + k)}{3(5 + k)^2} U^{(2)}_+ \right] \right|_{w}(w) \\
+ \left. \frac{1}{(z - w)} \left\{ -\frac{1}{2} Q^{(3)}_+ + \frac{1}{2(5 + k)} iT^{(1)} \partial \hat{B}_- - \frac{4}{(5 + k)} i\hat{A}_3 U^{(2)}_+ \right\} \right|_{w}(w) \\
- \left. \frac{1}{(5 + k)^2} \hat{A}_3 \partial \hat{B}_- = -\frac{4k}{(5 + k)(3 + 7k)(19 + 23k)} i\hat{B}_- \hat{T} \right|_{w}(w) \\
+ \left. \frac{1}{(5 + k)^2} \hat{B}_- \partial \hat{A}_3 \right|_{w}(w) = -\frac{k}{(5 + k)^2} \hat{B}_- \partial \hat{B}_- - \frac{1}{2(5 + k)} i\hat{B}_- \partial T^{(1)} \right|_{w}(w) \\
+ \left. \frac{2}{(5 + k)} \hat{G}_1 T^{(2)}_+ = -\frac{2}{(5 + k)} T^{(2)}_+ U(\hat{z}) \right|_{w}(w) \\
- \left. \frac{2}{(5 + k)} i\hat{B}_- U^{(2)}_+ \right|_{w}(w) + \ldots,
\]
\[
U(\hat{z}) W^{(2)}_+(w) = \left. \frac{1}{(z - w)^3} \left[ \frac{8(5 + 2k)}{(5 + k)^2} i\hat{A}_+ \right] \right|_{w}(w) \\
- \left. \frac{1}{(z - w)^2} \left[ \frac{4(7 + k)}{3(5 + k)} U^{(2)}_+ \right] \right|_{w}(w) \\
+ \left. \frac{1}{(z - w)} \frac{1}{2} Q^{(3)}_+ + \frac{4}{(5 + k)(3 + 7k)(19 + 23k)} i\hat{A}_+ \hat{T} \right|_{w}(w) \\
- \left. \frac{2}{(5 + k)} i\hat{A}_+ T^{(2)} \right|_{w}(w) = -\frac{7}{(5 + k)^2} \hat{A}_+ \partial \hat{A}_3 + \frac{8 + 3k}{(5 + k)^2} i\hat{A}_+ \partial \hat{B}_- \\
+ \left. \frac{1}{2(5 + k)} i\hat{A}_+ \partial T^{(1)} \right|_{w}(w) + \frac{3}{(5 + k)^2} \hat{A}_3 \partial \hat{A}_+ \right|_{w}(w) = -\frac{4}{(5 + k)} i\hat{B}_+ U^{(2)}_+ \right|_{w}(w) \\
- \left. \frac{8 + 3k}{(5 + k)^2} \hat{B}_3 \partial \hat{A}_+ \right|_{w}(w) = -\frac{(11 + 2k)}{6(5 + k)} \partial U^{(2)}_+ + \frac{4}{(5 + k)^2} i\hat{A}_+ \partial \hat{A}_+ \\
- \left. \frac{2}{(5 + k)} \hat{G}_1 \hat{G}_{12} \right|_{w}(w) + \frac{2}{(5 + k)} \hat{G}_1 T^{(2)} \right|_{w}(w) + \frac{2}{(5 + k)} \hat{G}_{12} U(\hat{z}) \right|_{w}(w) + \ldots
\]
\[- \frac{2}{(5 + k)} T^{(2)}_{2} U^{(\natural)}(\hat{\natural}) + \frac{4}{(5 + k)^2} i \hat{A}_{+} \hat{A}_{+} \hat{A}_{-} + \frac{4}{(5 + k)^2} i \hat{A}_{+} \hat{A}_{1} \hat{A}_{3} \]
\[- \frac{8}{(5 + k)^2} i \hat{A}_{+} \hat{A}_{1} \hat{B}_{3} + \frac{4}{(5 + k)^2} i \hat{A}_{+} \hat{B}_{+} \hat{B}_{-} + \frac{4}{(5 + k)^2} i \hat{A}_{+} \hat{B}_{1} \hat{B}_{3} \]
\[- \frac{1}{2(5 + k)} T^{(1)}_{2} \hat{A}_{+} \hat{A}_{3} + \frac{1}{2(5 + k)} \hat{A}_{3} T^{(1)}_{2} \hat{A}_{+} \bigg] (w) + \ldots, \]

\[U^{(\hat{\natural})}(z) W^{(3)}(w) = \frac{1}{(z - w)^3} - \frac{4(-3 + k)}{(5 + k)^2} \hat{G}_{11} \]
\[+ \frac{4(-3 + k) \left(345 + 296k + 55k^2\right)}{3(5 + k)^2(19 + 23k)} U^{(\hat{\natural})} \bigg] (w) \]
\[+ \frac{1}{(z - w)^2} \left[ \frac{(19 + 5k)}{2(5 + k)} \hat{G}_{21} + \frac{(21 + 5k)}{(5 + k)} U^{(\hat{\natural})} \right] \]
\[+ \frac{12}{(5 + k)^2} i \hat{A}_{+} \hat{G}_{21} + \frac{4(9 + 2k)}{(5 + k)^2} \hat{A}_{+} T^{(2)}_{1} \]
\[+ \frac{2}{(5 + k)^2(19 + 23k)} \left(404 + 443k + 107k^2\right) i \hat{A}_{+} U^{(\hat{\natural})} \]
\[+ \frac{(-5 + 3k)}{(5 + k)^2} i \hat{B}_{-} \hat{G}_{12} + \frac{1}{(5 + k)} i \hat{B}_{+} T^{(2)}_{1} \]
\[+ \frac{(-199 + k + 12k^2)}{(5 + k)^2(19 + 23k)} i \hat{B}_{+} U^{(\hat{\natural})} \]
\[- \frac{(-199 + k + 12k^2)}{(95 + 134k + 23k^2)} T^{(1)}_{2} U^{(\hat{\natural})} + \frac{4(-3 + k)}{3(5 + k)^2} \partial \hat{G}_{11} \]
\[+ \frac{(-2121 - 2020k - 359k^2 + 12k^3)}{3(5 + k)^2(19 + 23k)} \partial U^{(\hat{\natural})} \bigg) (w) \]
\[+ \frac{1}{(z - w)} \left[ \frac{(11 + k)}{2(5 + k)} \partial \hat{Q}(\hat{\natural}) + \frac{1}{(5 + k)} i \hat{P}(\hat{\natural}) \hat{B}_{-} \right] \]
\[+ \frac{2}{(5 + k)} \hat{Q}(\hat{\natural}) \hat{A}_{3} - \frac{2}{(5 + k)} i \hat{Q}(\hat{\natural}) \hat{B}_{3} + \ldots \bigg] (w) + \ldots, \quad (C.1) \]

The first OPE of (C.1) has \(\hat{G}_{11} \hat{G}_{11}(w)\) which can be written in terms of a derivative of \(\hat{A}_{+} \hat{B}_{-}(w)\). Note that there exists the \((k - 3)\) factor in the third order pole of the last OPE of (C.1).
Appendix D. The nontrivial OPEs between higher spin-2 current, $U_+^{(2)}(z)$, and other 11 higher spin currents

The corresponding OPEs are as follows

\begin{align*}
U_+^{(2)}(z) U_+^{(2)}(w) &= \frac{1}{(z - w)^2} \left[ \frac{2(3 + k)}{(5 + k)^2} \hat{A}_+ \hat{B}_- \right](w) \\
&\quad + \frac{1}{(z - w)} \left[ \frac{2}{5 + k} i \hat{B}_- U_+^{(2)} \right] - \frac{2(7 + k)}{(5 + k)^2} \hat{B}_- \hat{A}_+ + \frac{2}{(5 + k)} i \hat{A}_+ U_+^{(2)} - \hat{G}_{11} \hat{G}_{11} \right](w) + \cdots, \\
U_+^{(2)}(z) U_+^{(2)}(w) &= \frac{1}{(z - w)^2} \left[ \frac{2(6 + k)}{3(5 + k)^2} i \hat{B}_- \hat{G}_1 \right] - \frac{16(3 + k)}{3(5 + k)^2} i \hat{B}_- U_+^{(2)} \right](w) \\
&\quad + \frac{1}{(z - w)} \left[ \frac{8}{5 + k} \hat{A}_+ \hat{B}_- \hat{G}_1 - 4(9 + 4k) \hat{B}_1 \hat{B}_- U_+^{(2)} \right] - \frac{4(12 + k)}{3(5 + k)^2} i \hat{B}_- \hat{G}_11 \\
&\quad + \frac{4(9 + 4k)}{3(5 + k)^2} \hat{B}_1 \hat{B}_- U_+^{(2)} \right] - \frac{1}{5 + k} i \hat{B}_- \hat{Q}(\hat{r}) - \frac{2}{5 + k} \hat{G}_{11} U_+^{(2)} \\
&\quad - \frac{8}{5 + k} \hat{G}_{11} \hat{A}_+ \hat{B}_- \right](w) + \cdots, \\
U_+^{(2)}(z) V_+^{(2)}(w) &= -\frac{1}{(z - w)^2} \left[ \frac{(8 + k)}{(5 + k)} \hat{G}_{21} + \frac{(8 + k)}{(5 + k)} \right](w) \\
&\quad + \frac{1}{(z - w)} \left[ \frac{1}{2} p_+^{(2)}(\hat{r}) + \frac{2}{5 + k} \left\{ U_+^{(2)} \right\}_+^{(2)} \right](w) + \cdots, \\
U_+^{(2)}(z) V_+^{(2)}(w) &= -\frac{1}{(z - w)^2} \left[ \frac{2(7 + k)}{(5 + k)^2} \hat{B}_- \hat{A}_+ \right](w) \\
&\quad + \frac{1}{(z - w)} \left[ \frac{2}{(5 + k)} i \hat{A}_- U_+^{(2)} \right] - \frac{2(3 + k)}{(5 + k)^2} \hat{A}_- \hat{B}_- \\
&\quad - \frac{2}{(5 + k)} i \hat{B}_- V_+^{(2)} \right] - \frac{2}{5 + k} \hat{G}_{21} \hat{G}_{21} \right](w) + \cdots, \\
U_+^{(2)}(z) V_+^{(2)}(w) &= \frac{1}{(z - w)^2} \left[ \frac{4k(8 + k)}{(5 + k)^2} \hat{B}_3 \right](w) \\
&\quad + \frac{1}{(z - w)^2} \left[ \frac{1}{2} p_+^{(2)} + \frac{4\left( -60 - 77k + 7k^2 + 4k^3 \right)}{3(5 + k)^2(3 + 7k)} \hat{f} \right] \\
&\quad - \frac{2(4 + k)}{3(5 + k)} T^{(2)} \right] - \frac{2(4 + k)}{(5 + k)} W^{(2)} + \frac{4(-4 + k)}{3(5 + k)^2} \hat{A}_+ \hat{A}_- \\
&\quad - \frac{1}{46} \\
\end{align*}
\[
\begin{align*}
+ \frac{4(-4 + k)}{3(5 + k)^2} \dot{\lambda}_A \dot{\lambda}_3 &= \frac{8(-4 + k)}{3(5 + k)^2} \dot{\lambda}_3 \dot{B}_3 \\
+ \frac{4(-4 + k)}{3(5 + k)^2} \dot{B}_3 \dot{B}_- &= \frac{4(-4 + k)}{3(5 + k)^2} \dot{B}_3 \dot{B}_3 \\
+ \frac{4(-4 + k)}{3(5 + k)^2} i\partial \dot{\lambda}_3 + \frac{2(-8 + 26k + 3k^2)}{3(5 + k)^2} i\partial \dot{B}_3 \\
+ \frac{1}{(z - w)} \left[ \frac{1}{2} \mathbf{S}^{(2)} + \frac{1}{2} \mathbf{P}^{(2)} + \frac{1}{4} \delta \mathbf{P}^{(2)} - W^{(3)} - \frac{4}{(19 + 23k)} T^{(1)} \right]
\end{align*}
\]
\[
- \frac{2}{(5 + k)} i\dot{\lambda}_A \dot{v}^{(2)} + \frac{(-23 + 2k)}{3(5 + k)^2} \dot{\lambda}_A \dot{\lambda}_- \\
+ \frac{(7 + 2k)}{3(5 + k)^2} \dot{\lambda}_- \dot{\lambda}_A + \frac{8(195 + 554k + 563k^2 + 92k^3)}{(5 + k)^2(3 + 7k)(19 + 23k)} i\dot{\lambda}_3 \dot{\bar{T}} \\
- \frac{4}{(5 + k)} i\dot{\lambda}_A \dot{T}^{(2)} = \frac{4}{(5 + k)} i\dot{\lambda}_A \dot{W}^{(2)} + \frac{4(-10 + k)}{3(5 + k)^2} \dot{\lambda}_A \dot{\lambda}_3 \\
+ \frac{2(4 + 10k + k^2)}{3(5 + k)^2} \dot{\lambda}_A \dot{B}_3 - \frac{8k}{(5 + k)(3 + 7k)(19 + 23k)} i\dot{B}_3 \dot{\bar{T}} \\
+ \frac{2}{3(5 + k)^2} \dot{B}_3 \dot{B}_+ + \frac{8}{(5 + k)^2} \dot{B}_3 \dot{B}_- \\
+ \frac{4}{3(5 + k)^2} i\dot{B}_3 \dot{v}^{(2)} - \frac{i}{(5 + k)} \dot{B}_3 \dot{\bar{T}}^{(1)} \\
- \frac{2}{3(5 + k)} \dot{\bar{T}}^{(1)} = - \frac{2}{(5 + k)} U^{(1)} \dot{v}^{(2)} \\
+ \frac{8}{(5 + k)^2} i\dot{\lambda}_A \dot{\lambda}_- \dot{\lambda}_3 + \frac{8}{(5 + k)^2} i\dot{\lambda}_A \dot{\lambda}_3 \dot{\lambda}_3 \\
- \frac{16}{(5 + k)^2} i\dot{\lambda}_A \dot{\lambda}_3 \dot{B}_3 + \frac{8}{(5 + k)^2} i\dot{\lambda}_A \dot{\lambda}_3 \dot{B}_- + \frac{8}{(5 + k)^2} i\dot{\lambda}_A \dot{\lambda}_3 \dot{B}_3 \\
+ \frac{2k(14 + k)}{3(5 + k)^2} i\dot{B}_- \dot{B}_3 \dot{B}_3 - \frac{2k(14 + k)}{3(5 + k)^2} i\dot{B}_- \dot{B}_3 \dot{B}_- \\
- \frac{1}{2(5 + k)} T^{(1)} \dot{B}_- \dot{B}_- + \frac{1}{2(5 + k)} \dot{B}_- T^{(1)} \dot{B}_- \\
- \frac{2}{(5 + k)} T^{(2)} \dot{v}^{(2)} \right] (w) + \ldots,
\[ U_{+}^{(2)}(z) V^{(2)}(w) = \frac{1}{(z - w)^{2}} \left[ \frac{8(2 + k)(8 + k)}{3(5 + k)^{2}} \hat{G}_{21} + \frac{8(4 + k)}{3(5 + k)^{2}} T_{++}^{(2)} \right](w) \]

\[ + \frac{1}{(z - w)^{2}} \left[ - \frac{(34 + 7k)}{3(5 + k)} p_{++}^{(2)} - \frac{(30 + 7k)}{3(5 + k)} W_{++}^{(2)} \right] \]

\[ - \frac{2(30 + 7k)}{3(5 + k)^{2}} i\hat{A}_{-} U^{(2)} - \frac{12}{(5 + k)^{2}} i\hat{A}_{3} \hat{G}_{21} - \frac{12}{(5 + k)^{2}} i\hat{A}_{3} T_{++}^{(2)} \]

\[ + \frac{2(-2 + k)}{(5 + k)^{2}} i\hat{B}_{-} \hat{G}_{22} - \frac{2(14 + 3k)}{(5 + k)^{2}} i\hat{B}_{-} V^{(2)} + \frac{4(8 + k)}{(5 + k)^{2}} i\hat{B}_{3} \hat{G}_{21} \]

\[ + \frac{4(36 + 7k)}{3(5 + k)^{2}} i\hat{B}_{-} T_{++}^{(2)} - \frac{2}{(5 + k)^{2}} T_{++}^{(1)} \hat{G}_{21} = \frac{2}{(5 + k)^{2}} T_{++}^{(1)} T_{++}^{(2)} \]

\[ + \frac{4}{9(5 + k)^{2}} \frac{2(36 + 7k)}{3(5 + k)^{2}} i\hat{B}_{-} T_{++}^{(2)} - \frac{2}{(5 + k)^{2}} T_{++}^{(1)} \hat{G}_{21} = \frac{2}{(5 + k)^{2}} T_{++}^{(1)} T_{++}^{(2)} \]

\[ + \frac{1}{(5 + k)^{2}} \left[ - \frac{14(7 + k)}{15(5 + k)} p_{++}^{(2)} + \frac{2}{2} S_{++}^{(2)} \right] \]

\[ - \frac{1}{(5 + k)^{2}} iQ_{++}^{(2)} \hat{A}_{-} + \ldots \right] (w) + \ldots \]

\[ U_{+}^{(2)}(z) W^{(2)}(w) = - \frac{1}{(z - w)^{2}} \left[ \frac{2k(8 + k)}{(5 + k)^{2}} i\hat{B}_{-} \right](w) \]

\[ + \frac{1}{(z - w)^{2}} \left[ \frac{2(4 + k)}{(5 + k)^{2}} U_{++}^{(2)} + \frac{1}{2} \partial \left( U_{+}^{(2)} W^{(2)} \right)_{-3} \right] (w) \]

\[ + \frac{1}{(z - w)^{2}} \left[ - \frac{1}{2} Q_{++}^{(2)} + \frac{1}{2} T_{++}^{(1)} \partial \hat{B}_{-} - \frac{4}{(5 + k)^{2}} i\hat{A}_{3} U_{++}^{(2)} \right] \]

\[ - \frac{(11 + 2k)}{(5 + k)^{2}} i\hat{A}_{3} \partial \hat{B}_{-} = \frac{4k}{(5 + k)(3 + 7k)(19 + 23k)} i\hat{B}_{-} \hat{F} \]

\[ - \frac{2}{(5 + k)} i\hat{B}_{-} T^{(2)} = \frac{k}{(5 + k)^{2}} \hat{B}_{-} \partial \hat{B}_{3} - \frac{1}{2(5 + k)^{2}} i\hat{B}_{-} \partial T^{(1)} \]

\[ + \frac{k}{(5 + k)^{2}} \hat{B}_{3} \partial \hat{B}_{-} + \frac{(9 + 2k)}{2(5 + k)} \partial U_{++}^{(2)} \]

\[ - \frac{k}{(5 + k)^{2}} \partial^{2} \hat{B}_{-} = \frac{2}{(5 + k)^{2}} T_{++}^{(2)} U_{++}^{(2)} \]

\[ + \frac{15 + 2k}{2(5 + k)^{2}} \hat{A}_{+} \hat{A}_{-} \hat{B}_{-} = \frac{(15 + 2k)}{2(5 + k)^{2}} i\hat{A}_{+} \hat{A}_{-} \hat{B}_{-} \]

\[ + \frac{2}{(5 + k)^{2}} \hat{G}_{11} T_{++}^{(2)} \right] (w) + \ldots , \]
\[ U_+^{(2)}(z) W_{+}^{(2)}(w) = -\frac{1}{(z-w)^2} \left[ \frac{2(16+k)}{3(5+k)^2} \hat{B}_- \hat{T}^{(2)}_+ \right](w) \\
+ \frac{1}{(z-w)} \left[ -\frac{8(8+k)}{3(5+k)^2} \hat{B}_+ \hat{G}_1 \hat{G}_2 - \frac{2}{(5+k)} \hat{B}_- \hat{W}^{(2)}_+ \right] \\
- \frac{4(11+k)}{3(5+k)^2} \hat{B}_- \partial T^{(2)}_+ + \frac{14(8+k)}{3(5+k)^2} i\hat{B}_- \hat{G}_1 + \frac{4(7+k)}{3(5+k)^2} i\hat{B}_- \hat{T}^{(2)}_+ \\
- \frac{2}{(5+k)} \hat{G}_2 \hat{U}^{(2)}_+ + \frac{8(8+k)}{3(5+k)^2} \hat{G}_2 \hat{B}_- \hat{B}_3 \right](w) + \ldots, \]

\[ U_+^{(2)}(z) W_{+}^{(3)}(w) = \frac{1}{(z-w)^2} \left[ -\frac{4(7+k)(3+2k)}{3(5+k)^2} \hat{G}_{11} + \frac{4(11+3k)}{3(5+k)^2} \hat{U}^{(3)}_+ \right](w) \\
+ \frac{1}{(z-w)} \left[ \frac{(30+7k)}{3(5+k)} \hat{Q}^{(2)}_+ + \frac{(38+7k)}{3(5+k)} \hat{U}^{(3)}_+ + \frac{2(6+k)}{(5+k)^2} i\hat{A}_+ \hat{G}_1 \hat{G}_2 \hat{G}_3 + \frac{4(28+5k)}{3(5+k)^2} i\hat{A}_+ \hat{T}^{(2)}_+ + \frac{4(47+7k)}{3(5+k)^2} i\hat{A}_+ \hat{U}^{(3)}_+ + \frac{2(-10+k)}{(5+k)^2} i\hat{B}_- \hat{G}_1 \hat{G}_2 \hat{G}_3 + \frac{2}{(5+k)} \hat{T}^{(1)} \hat{U}^{(3)}_+ \\
- \frac{4(30+17k+2k^2)}{9(5+k)^2} \partial \hat{G}_{11} - \frac{8(25+4k)}{9(5+k)^2} \partial \hat{U}^{(3)}_+ \right](w) \\
+ \frac{1}{(z-w)} \left[ \frac{2(51+7k)}{15(5+k)} \hat{Q}^{(2)}_+ + \frac{1}{(5+k)} \hat{P}^{(2)}_+ \hat{B}_- \hat{B}_+ \hat{B}_+ + \frac{1}{2} \hat{Q}^{(2)}_+ + \frac{2}{(5+k)} \hat{Q}^{(2)}_+ \hat{A}_+ \hat{A}_+ \hat{A}_+ \hat{A}_+ \hat{A}_+ \hat{A}_+ + \frac{2}{(5+k)} \hat{Q}^{(2)}_+ \hat{B}_3 + \ldots \right](w) + \ldots, \]

\[ U_+^{(2)}(z) W_{+}^{(3)}(w) = \frac{1}{(z-w)^4} \left[ \frac{2k(1509+1724k+835k^2+84k^3)}{(5+k)^3(19+23k)} \hat{B}_- \hat{B}_- \right](w) \\
+ \frac{1}{(z-w)^3} \left[ \frac{2(114-853k+13k^2)}{(5+k)(285+402k+69k^2)} \hat{U}^{(2)}_+ \right] \\
- \frac{4(1+23k+3k^2)}{(5+k)^3} \hat{A}_+ \hat{B}_- + \frac{12k}{(5+k)^3} \hat{B}_- \hat{B}_3 + \frac{6k}{(5+k)^3} i\hat{B}_- \hat{B}_- \\
- \frac{10}{(5+k)^2} \hat{T}^{(1)} \hat{B}_- \hat{B}_- \right](w) + \frac{1}{(z-w)^2} \left[ \frac{25+6k}{2(5+k)} \hat{Q}^{(3)} \right] \\
+ \frac{1}{(5+k)} \hat{P}^{(2)}_+ \hat{B}_- + \frac{25}{2(5+k)^2} \hat{T}^{(1)} \hat{B}_-. \]
\begin{align*}
+ & \frac{2(602 + 623k + 89k^2)}{(5 + k)^2(19 + 23k)} i\hat{A}_1 U^{(1)}(\hat{z}) \\
+ & \frac{93 + 30k + 4k^2}{(5 + k)^3} \hat{A}_3 \partial \hat{B}_- + \frac{259 - 9k - 16k^2}{95 + 134k + 23k^2} T^{(1)} U(\hat{z}) \\
+ & \frac{2(285 + 12662k + 23609k^2 + 6896k^3 + 336k^4)}{3(5 + k)^2(3 + 7k)(19 + 23k)} i\hat{B}_- \hat{T} \\
+ & \frac{(121 + 17k)}{3(5 + k)^2} i\hat{B}_- T^{(2)} - \frac{(59 + 13k)}{(5 + k)^2} i\hat{B}_- W^{(2)} = - \frac{27}{2(5 + k)^2} i\hat{B}_- \partial T^{(1)} \\
- & \frac{2(209 + 691k + 126k^2)}{(5 + k)(19 + 23k)} i\hat{B}_1 U^{(1)}(\hat{z}) + \frac{k(35 - 2k)}{(5 + k)^3} \hat{B}_3 \partial \hat{B}_- \\
- & \frac{6175 + 11357k + 2274k^2}{6(5 + k)^2(19 + 23k)} \partial U^{(1)}(\hat{z}) \\
- & \frac{(10 + 193k + 42k^2)}{6(5 + k)^3} i\partial \hat{B}_- - \frac{2(6 + k)}{(5 + k)^2} \hat{G}_{11} \hat{G}_{21} \\
- & \frac{2(23 + 6k)}{(5 + k)^2} \hat{G}_{11} T^{(2)} + \frac{4(7 + k)}{(5 + k)^2} \hat{G}_{21} U(\hat{z}) + \frac{2(25 + 6k)}{(5 + k)^2} T^{(2)} U(\hat{z}) \\
- & \frac{(479 + 250k + 30k^2)}{6(5 + k)^3} i\hat{A}_4 \hat{A}_3 \hat{B}_- + \frac{(547 + 350k + 30k^2)}{6(5 + k)^3} i\hat{A}_4 \hat{A}_3 \hat{B}_- \\
+ & \frac{2(-67 + 19k)}{3(5 + k)^3} i\hat{A}_3 \hat{A}_1 \hat{B}_- - \frac{4(-31 + 7k)}{3(5 + k)^3} i\hat{A}_3 \hat{B}_- \hat{B}_3 \\
+ & \frac{4(-10 - 77k + 6k^2)}{6(5 + k)^3} i\hat{B}_- \hat{B}_- \hat{B}_- + \frac{(-10 - 35k + 2k^2)}{2(5 + k)^3} i\hat{B}_- \hat{B}_- \hat{B}_- \\
- & \frac{10(-1 + k)}{3(5 + k)^3} i\hat{B}_- \hat{B}_- \hat{B}_- = \frac{8}{(5 + k)^2} T^{(1)} \hat{A}_3 \hat{B}_- \\
+ & \frac{8}{(5 + k)^2} T^{(1)} \hat{B}_- \hat{B}_- \hat{B}_- (w) + \frac{1}{(z - w)} \left[ 2 \right. \frac{(5 + k)}{i\hat{A}_1 Q^{(3)}} \\
- & \frac{5(21 + 4k)}{6(5 + k)} i\hat{B}_1 Q^{(3)} = \frac{81 + 20k}{12(5 + k)} i\hat{B}_- S^{(3)} \\
- & \frac{(81 + 20k)}{12(5 + k)} \hat{B}_- \hat{B}_1 \hat{S}^{(2)} + \frac{93 + 20k}{12(5 + k)} \hat{B}_\hat{B}_1 \hat{S}^{(2)} \\
+ & \frac{93 + 20k}{24(5 + k)} i\hat{B}_- \hat{P}^{(2)} + \frac{1}{2(5 + k)} i\hat{B}_- \partial \hat{P}^{(2)} \\
+ & \frac{(5 + 2k)}{2(5 + k)} \partial \hat{Q}^{(3)} + \ldots \right] + \ldots \tag{D.1}
\end{align*}

The first OPE of (D.1) has \( \hat{G}_{11} \hat{G}_{11}(w) \) which can be written in terms of a derivative of \( \hat{A}_3 \hat{B}_-(w) \). Similarly, The fourth OPE of (D.1) has \( \hat{G}_{21} \hat{G}_{21}(w) \) which can be written in terms of
a derivative of $\hat{\mathcal{A}}_+\hat{\mathcal{B}}_-(w)$. The first order pole in the last OPE of (D.1) contains composite field with spin-4 with $U(1)$ charge of $\frac{2k}{(5+k)}$. The second order pole contains a composite field of spin-3 and each term can be seen in the table 5 of [1]. There is no $T^{(1)}T^{(1)}\hat{B}_-(w)$ term in the second order pole.

Appendix E. The nontrivial OPEs between higher spin-2 current, $U^{(2)}_-(z)$, and other 10 higher spin currents

We present the corresponding OPEs as follows:

$$U^{(2)}_-(z)U^{(2)}(w) = \frac{1}{(z-w)^2} \left[ \frac{2(3+2k)}{3(5+k)^2} i\hat{A}_+ \left( \hat{G}_{11} + U^{(2)} \right) \right](w)$$

$$+ \frac{1}{(z-w)} \left[ -\frac{2}{(5+k)} i\hat{A}_+ U^{(2)} - \frac{4(3+2k)}{3(5+k)^2} \hat{A}_+ \hat{A}_1 U^{(2)} \right]$$

$$- \frac{4}{(5+k)^2} \hat{A}_+ \hat{B}_- \hat{G}_{12} + \frac{8(1+k)}{3(5+k)^2} i\hat{A}_+ \partial U^{(2)}$$

$$+ \frac{4(3+2k)}{3(5+k)^2} \hat{A}_3 \hat{A}_+ U^{(2)} + \frac{2(9+2k)}{3(5+k)^2} i\hat{A}_+ \hat{G}_{11}$$

$$+ \frac{4}{(5+k)^2} \hat{G}_{12} \hat{A}_+ \hat{B}_- + \frac{2}{(5+k)} \hat{G}_{11} U^{(2)}$$

$$+ \frac{4}{(5+k)} U^{(2)} U^{(2)}](w) + \ldots.$$ 

$$U^{(2)}_-(z)V^{(2)}(w) = -\frac{1}{(z-w)^2} \left[ \frac{(5+2k)}{(5+k)} T^{(2)} \right](w)$$

$$+ \frac{1}{(z-w)} \left[ \frac{1}{2} p^{(2)} - W^{(2)} \right] - \frac{2(5+2k)}{3(5+k)} \partial T^{(2)} \right](w) + \ldots,$$

$$U^{(2)}_-(z)V^{(2)}_+(w) = -\frac{1}{(z-w)} \left[ \frac{6k(5+2k)}{(5+k)^2} \right]$$

$$- \frac{1}{(z-w)^2} \left[ \frac{12(5+2k)}{(5+k)^2} \hat{A}_3 \right](w)$$

$$+ \frac{1}{(z-w)^2} \left[ -\frac{1}{2} p^{(2)} - \frac{4}{3(5+k)^2} \left( 75 + 310k + 184k^2 + 25k^3 \right) \right] - T^{(2)}$$

$$+ \frac{2(-4+k)}{(5+k)} T^{(2)} + \frac{2(4+k)}{(5+k)} W^{(2)} - \frac{16(-1+k)}{(5+k)^2} \hat{A}_1 \hat{A}_1$$

$$\hat{A}_2 \hat{A}_1 = \frac{4(-13+4k)}{(5+k)^2} \hat{A}_3 \hat{A}_3$$

$$+ \frac{8(-4+k)}{(5+k)^2} \hat{A}_3 \hat{B}_1 - \frac{4(5+k)}{(5+k)^2} \hat{B}_1 \hat{B}_1$$

$$- \frac{4(5+k)}{(5+k)^2} \hat{B}_2 \hat{B}_2 + \frac{4(-5+2k)}{(5+k)^2} \hat{B}_3 \hat{B}_3$$
\[
\begin{align*}
&- \frac{6(5 + 2k)}{(5 + k)^2} \partial_\nu \hat{A}_3 - \frac{4}{(5 + k)} i T^{(1)} \hat{A}_3 + \frac{4}{(5 + k)} i T^{(1)} \hat{B}_3 \right] (w) \\
&+ \frac{1}{(z - w)} \left[ - \frac{1}{2} S^{(3)} + \frac{1}{2} p^{(3)} - \frac{1}{4} \partial \mathbf{p}^{(2)} - \frac{4}{(19 + 23k)} T^{(1)} \hat{T} \\
&- \frac{3}{(5 + k)} i T^{(1)} \partial_\nu \hat{A}_3 + \frac{(11 - 8k)}{(5 + k)^2} \hat{A}_+ \partial_\nu \hat{A}_- - \frac{(1 + 8k)}{(5 + k)^2} \hat{A}_- \partial_\nu \hat{A}_+ \\
&8 \left( 600 + 1625k + 866k^2 + 113k^3 \right) \hat{A}_3 \hat{T} + \frac{4}{(5 + k)} \partial_\nu \hat{A}_3 T^{(2)} \\
&\frac{2}{(3(5 + k)^2)} \hat{A}_3 \partial_\nu \hat{A}_3 - \frac{1}{(5 + k)} i \hat{A}_3 \partial T^{(1)} - \frac{2}{(5 + k)} \hat{B}_+ U^{(2)}_+ \\
&\frac{(26 + k)}{3(5 + k)^2} \hat{B}_+ \partial_\nu \hat{B}_+ - \frac{k}{(5 + k)^2} \hat{B}_- \partial_\nu \hat{B}_+ \\
&\frac{8k(51 + 55k)}{(5 + k)(3 + 7k)(19 + 23k)} i \hat{B}_3 \hat{T} - \frac{4}{(5 + k)} i \hat{B}_3 T^{(2)} \\
&+ \frac{4}{(5 + k)} i \hat{B}_3 W^{(2)} + \frac{10(-1 + k)}{3(5 + k)^2} \hat{B}_3 \partial_\nu \hat{A}_3 + \frac{4}{(5 + k)} i \hat{B}_3 \partial T^{(1)} \\
&- \frac{3(5 + k)(3 + 7k)}{6(5 + k)} \hat{A}_3 \partial T^{(2)} \\
&\frac{(5 + 2k)}{2(5 + k)} \partial W^{(2)} = \frac{2(7 + 2k)}{(5 + k)^2} i \partial^2 \hat{A}_3 \\
&\frac{13}{3(5 + k)^2} i \partial^2 \hat{B}_3 - \frac{2}{3(5 + k)^2} \partial^2 T^{(1)} \\
&+ \frac{2}{(5 + k)} \hat{G}_{12} \hat{G}_{21} + \frac{4}{(5 + k)} \hat{G}_{21} T^{(2)} \\
&- \frac{2}{(5 + k)} U^{(2)} \hat{V}^{(2)} + \frac{8}{(5 + k)^2} i \hat{A}_+ \hat{A}_- \hat{A}_3 \\
&\frac{8}{(5 + k)^2} i \hat{A}_+ \hat{A}_3 \hat{A}_3 + \frac{16}{(5 + k)^2} i \hat{A}_+ \hat{A}_3 \hat{B}_3 - \frac{(17 + k)}{3(5 + k)^2} i \hat{A}_3 \hat{B}_+ \hat{B}_+ \\
&\frac{8}{(5 + k)^2} i \hat{A}_3 \hat{B}_+ \hat{B}_+ + \frac{(-7 + k)}{3(5 + k)^2} i \hat{B}_- \hat{A}_3 \hat{B}_+ \\
&\frac{(-13 + 4k)}{3(5 + k)^2} i \hat{B}_- \hat{B}_+ \hat{B}_+ + \frac{(-13 + 4k)}{3(5 + k)^2} i \hat{B}_3 \hat{B}_+ \hat{B}_+ \\
&+ \frac{2}{(5 + k)} T^{(2)} \hat{T}^{(2)} (w) + \cdots ,
\end{align*}
\]

\[
U^{(2)}_-(z) V^{(2)}_+(w) = -\frac{1}{(z - w)^*} \left[ \frac{2(7 + k)}{(5 + k)^2} i \hat{A}_3 \hat{B}_+ \right] (w) \\
+ \frac{1}{(z - w)} \left[ \frac{2}{(5 + k)} i \hat{A}_+ V^{(2)}_+ + \frac{2(3 + k)}{(5 + k)^2} \hat{A}_3 \partial \hat{B}_+ + \frac{2}{(5 + k)} i \hat{B}_+ U^{(2)}_+ \\
+ \hat{G}_{12} \hat{G}_{12} \right] (w) + \cdots ,
\]

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\[ U^{(2)}(z) V^{(2)}(w) = \frac{1}{(z-w)^2} \left[ -8 \left( \frac{12 + 9k + k^2}{3(5+k)} \right) \hat{G}_{12} + \frac{4(11 + 3k)}{3(5+k)^2} T^{(4)}(w) \right] \]

\[ + \frac{1}{(z-w)^2} \left[ \frac{2(13 + 4k)}{3(5+k)} \hat{p}^{(5)} - \frac{1}{3(5+k)} \hat{A}_+ \hat{G}_{22} \right] \]

\[ - \frac{2(19 + 4k)}{(5+k)^2} \hat{A}_+ V^{(2)}(w) - \frac{8(9 + k)}{3(5+k)^2} \hat{A}_3 \hat{G}_{12} + \frac{8(9 + k)}{3(5+k)^2} \hat{A}_3 T^{(2)}(w) \]

\[ - \frac{2(5 + 2k)}{(5+k)^2} \hat{B}_+ \hat{G}_{11} - \frac{28(3 + k)}{3(5+k)^2} \hat{B}_+ U^{(4)}(w) + \frac{4(5 + 2k)}{(5+k)^2} \hat{B}_+ T^{(2)}(w) \]

\[ - \frac{4k(13 + 2k)}{9(5+k)^2} \hat{A}_+ \hat{G}_{12} - \frac{4(55 + 14k)}{9(5+k)^2} \partial T^{(2)}(w) \]

\[ + \frac{1}{(z-w)} \left[ \frac{4(13 + 4k)}{15(5+k)} \hat{p}^{(5)} + \frac{1}{2} \hat{s}^{(3)} + \frac{2}{(5+k)} \hat{p}^{(5)} \hat{A}_3 \right] \]

\[ - \frac{2}{(5+k)} \hat{p}^{(5)} \hat{B}_3 + \ldots \]

\[ U_{-}^{(2)}(z) W^{(2)}(w) = \frac{1}{(z-w)^2} \left[ \frac{6(5+2k)}{(5+k)^2} \hat{A}_+ \right] \]

\[ + \frac{1}{(z-w)^2} \left[ \frac{2(4+k)}{(5+k)} U^{(2)}(w) + \frac{3(5+2k)}{(5+k)^2} \hat{A}_+ \right] \]

\[ + \frac{1}{(z-w)} \left[ \frac{1}{2} Q^{(3)} + \frac{4}{(5+k)(3+7k)(19+23k)} \hat{A}_+ \hat{T} \right] \]

\[ - \frac{2}{(5+k)} \hat{A}_+ T^{(2)}(w) - \frac{7}{(5+k)^2} \hat{A}_+ \partial \hat{A}_3 + \frac{(8 + 3k)}{(5+k)^2} \hat{A}_+ \hat{B}_3 \]

\[ + \frac{1}{2(5+k)} \hat{A}_+ \partial T^{(4)}(w) + \frac{4}{(5+k)} \hat{B}_+ U^{(2)}(w) - \frac{(8 + 3k)}{(5+k)^2} \hat{B}_+ \partial \hat{A}_3 \]

\[ + \frac{9 + 2k}{2(5+k)} \partial U^{(2)}(w) - \frac{2}{(5+k)} \hat{G}_{11} \hat{G}_{12} + \frac{2}{(5+k)} \hat{G}_{11} T^{(2)}(w) \]

\[ + \frac{2}{(5+k)} \hat{G}_{12} U^{(4)}(w) - \frac{2}{(5+k)} T^{(2)} U^{(4)}(w) + \frac{4}{(5+k)^2} \hat{A}_+ \hat{A}_+ \hat{A}_- \]

\[ - \frac{2(7 + 2k)}{(5+k)^2} \hat{A}_+ \hat{A}_3 \hat{A}_3 - \frac{8}{(5+k)^2} \hat{A}_+ \hat{A}_3 \hat{B}_3 + \frac{4}{(5+k)^2} \hat{A}_+ \hat{B}_3 \hat{B}_- \]

\[ + \frac{4}{(5+k)^2} \hat{A}_+ \hat{B}_3 \hat{B}_3 + \frac{2(9 + 2k)}{(5+k)^2} \hat{A}_3 \hat{A}_+ \hat{A}_3 - \frac{(15 + 4k)}{(5+k)^2} \hat{A}_3 \partial \hat{A}_+ \]

\[ - \frac{1}{2(5+k)} T^{(1)} \hat{A}_+ \hat{A}_3 + \frac{1}{2(5+k)} \hat{A}_3 T^{(1)} \hat{A}_+ \]
\( U^{(2)}_- (z) \) \( W^2_+ (w) = \frac{1}{(z - w)^3} \left[ - \frac{8(2 + k)(6 + k)}{3(5 + k)^2} \dot{G}_{11} + \frac{4(11 + 3k)}{3(5 + k)^2} U^{(2)}(w) \right] \)

\[ + \frac{1}{(z - w)^2} \left[ - \frac{(27 + 8k)}{3(5 + k)} \dot{Q}^{(2)}(w) - \frac{35 + 8k}{3(5 + k)} U^{(2)}(w) \right] \]

\[ - \frac{6}{(5 + k)^2} i\dot{A} \dot{A}_+ \dot{G}_{21} - \frac{2(13 + 4k)}{(5 + k)^2} i\dot{A}_+ T^{(2)}(w) - \frac{4(5 + 2k)}{(5 + k)^2} i\dot{A}_+ U^{(2)}(w) \]

\[ - \frac{2(3 + 2k)}{(5 + k)^2} i\dot{B}_+ \dot{U}^{(2)}(w) - \frac{4k}{(5 + k)^2} i\dot{B}_+ U^{(2)}(w) + \frac{2}{(5 + k)} T^{(1)} U^{(2)}(w) \]

\[ - \frac{4(8 + k)(3 + 2k)}{9(5 + k)^2} \dot{a}_{11} + \frac{8(13 + 3k)}{9(5 + k)^2} \dot{a} U^{(2)}(w) \]

\[ + \frac{1}{(z - w)} \left[ \frac{1052 + 331k + 18k^2}{15(5 + k)(14 + k)} \dot{a} Q^2 + \frac{1}{(5 + k)} \dot{p}^{(2)} \dot{B}_- \right] \]

\[ + \frac{1}{2} \dot{Q}^{(2)}(w) - \frac{(20 + k)}{10(14 + k)} \dot{p}^{(2)} U^{(2)}(w) \]

\[ + \frac{20 + k}{10(14 + k)} U^{(2)}(w) \dot{p}^{(2)} + \ldots \right](w) + \ldots, \]  \[ (E.1) \]

\( U^{(2)}_- (z) \) \( W^{(3)}_+ (w) = \frac{1}{(z - w)^3} \left[ - \frac{2(13 + 2k)}{3(5 + k)^2} i\dot{A} \dot{A}_+ \dot{G}_{12} + \frac{2(13 + 2k)}{3(5 + k)^2} i\dot{A}_+ T^{(2)}(w) \right] \]

\[ + \frac{1}{(z - w)^2} \left[ - \frac{2(13 + 2k)}{3(5 + k)^2} i\dot{A} \dot{A}_+ \dot{G}_{12} + \frac{2(13 + 2k)}{3(5 + k)^2} i\dot{A}_+ T^{(2)}(w) \right] \]

\[ - \frac{8(2 + k)}{3(5 + k)^2} \dot{A}_+ \dot{A}_+ \dot{G}_{12} \dot{T}^{(2)}(w) + \frac{2(7 + 2k)}{3(5 + k)^2} \dot{A}_+ \dot{A}_+ \dot{G}_{12} \dot{T}^{(2)}(w) \]

\[ + \frac{8(2 + k)}{3(5 + k)^2} \dot{A}_+ \dot{A}_+ \dot{G}_{12} \dot{T}^{(2)}(w) - \frac{27 + 10k}{3(5 + k)^2} \dot{G}_{12} \dot{A}_+ \dot{B}_+ \]

\[ - \frac{8(4 + k)}{3(5 + k)^2} \dot{U}^{(2)} \dot{A}_+ \dot{B}_+ + \frac{2}{(5 + k)} i\dot{A}_+ W^{(2)}(w) - \frac{2}{(5 + k)} \dot{G}_{12} U^{(2)}(w) \]

\[ - \frac{2(7 + 2k)}{3(5 + k)^2} \dot{G}_{12} \dot{A}_+ \dot{A}_3 \right](w) + \ldots, \]

\( U^{(2)}_- (z) \) \( W^{(3)}_+ (w) = \frac{1}{(z - w)^3} \left[ - \frac{12(289 + 1049k + 636k^2 + 296k^3)}{(5 + k)^3(19 + 23k)} i\dot{A}_+ \right] \]

\[ + \frac{1}{(z - w)^3} \left[ - \frac{2(-127 + 85k + 72k^2)}{(5 + k)^3(19 + 23k)} U^{(2)} - \frac{12(5 + 2k)}{(5 + k)^3} \dot{A}_+ \dot{A}_3 \right] \]

\[ + \frac{44k}{(5 + k)^3} i\dot{A}_+ \dot{A}_3 + \frac{6(5 + 2k)}{(5 + k)^3} i\dot{A}_+ + \frac{6}{(5 + k)^3} i\dot{A}_+ T^{(1)} \dot{A}_+ \right](w) \]
In (E.1), the fourth and fifth terms in the first order pole have relative minus sign. It is easy to calculate the following OPE

\[ U(\bar{\partial})_{(z)} P^{(2)}(w) = \frac{1}{(z - w)^2} \left[ \frac{16(2 + k)(8 + k)}{3(5 + k)^2} U(\bar{\partial}) \right]_{(w)} + \frac{1}{(z - w)} \left[ \frac{4(19 + 2k)}{3(5 + k)^3} U(\bar{\partial}) + \frac{8(19 + 2k)}{3(5 + k)^3} T^{(1)}_{\partial} \right]_{(w)} + \cdots. \]
From this OPE (E.3), one obtains the following identity (See also [63 of [1])

\[
\left[ U(\hat{z}), P^{(2)} \right](w) = -\frac{1}{2} \partial^2 \left\{ U(\hat{z}) P^{(2)} \right\}_{-2}(w) + \theta \left\{ U(\hat{z}) P^{(2)} \right\} _{-1}(w).
\]

Then the fourth and fifth terms in the first order pole of (E.1) can be written in terms of derivatives of the composite fields appearing in the singular terms in (E.3).

The fourth OPE of (E.2) has \( \hat{G}_{ij1} \hat{G}_{ij2}(w) \) which can be written in terms of a derivative of \( \hat{A}_+ \hat{B}_+(w) \). The first order pole in the last OPE of (E.2) contains composite field with spin-4 with \( U(1) \) charge of \(-\frac{6}{(5+k)}\). The second order pole contains a composite field of spin-3 which can be seen from the table 5 of [1].

**Appendix F. The nontrivial OPEs between higher spin-\( \frac{5}{2} \) current, \( U^{(\frac{5}{2})}(z) \), and other 9 higher spin currents**

We continue to present the corresponding OPEs as follows:

\[
U(\hat{z})(z) U(\hat{t})(w) = \frac{1}{(z - w)^3} \left[ \frac{16(-3 + 13k + 4k^2)}{9(5 + k)^3} \hat{A}_+ \hat{B}_- \right](w) - \frac{1}{(z - w)^2} \left[ \frac{4(-3 + 13k + 4k^2)}{9(5 + k)^2} \hat{G}_{ij1} \hat{G}_{ij1} \right](w) + \frac{1}{(z - w)} \left[ \frac{2}{(5 + k)} i \hat{B}_- Q^{(3)} + \ldots \right](w) + \ldots,
\]

\[
U(\hat{z})(z) V(\hat{t})(w) = \frac{1}{(z - w)^3} \left[ \frac{8(5 + 2k)}{(5 + k)^2} i \hat{A}_3 + \frac{8k(-2 + k)}{3(5 + k)^2} i \hat{B}_3 \right] - \frac{4(-2 + k)}{3(5 + k)} T^{(1)}(w) + \frac{1}{(z - w)^2} \left[ \frac{1}{2} P^{(2)} + \frac{4(15 + 23k + 16k^2)}{3(5 + k)(3 + 7k)} \hat{T} \right].
\]
\[- \frac{8(2 + k)}{3(5 + k)} T^{(2)} - \frac{4(5 + 2k)}{3(5 + k)} W^{(2)}
+ \frac{16(-1 + k)}{3(5 + k)^2} \dot{A}_1 \dot{A}_1 + \frac{16(-1 + k)}{3(5 + k)^2} \dot{A}_2 \dot{A}_2
+ \frac{4(-13 + 4k)}{3(5 + k)^2} \dot{A}_3 \dot{A}_3
- \frac{8(-4 + k)}{3(5 + k)^2} \dot{A}_3 \dot{B}_3 + \frac{4(5 + k)}{3(5 + k)^2} \dot{B}_1 \dot{B}_1
+ \frac{4(5 + k)}{3(5 + k)^2} \dot{B}_2 \dot{B}_2 - \frac{4(-5 + 2k)}{3(5 + k)^2} \dot{B}_3 \dot{B}_3
+ \frac{8(5 + 2k)}{(5 + k)^2} iA_3 \dot{A}_3 + \frac{8(-2 + k)}{(5 + k)^2} iA_3 \dot{A}_3
- \frac{4(-2 + k)}{3(5 + k)} iA_3 \dot{A}_3 + \frac{4}{(5 + k)} iT^{(1)} \dot{A}_3 - \frac{4}{(5 + k)} iT^{(1)} \dot{B}_3 \right\}
+ \frac{1}{(z - w)} \frac{3}{8} \partial P^{(2)} - W^{(3)} - \frac{4(-3 + k)}{(19 + 23k)} T^{(1)}
+ \frac{4}{(5 + k)} iT^{(1)} \dot{A}_3 + \frac{(-5 + 4k)}{(5 + k)^2} \dot{A}_3 \dot{A}_3 + \frac{(-1 + 4k)}{(5 + k)^2} \dot{A}_3 \dot{A}_3
+ \frac{8\left(75 + 178k + 71k^2\right)}{(5 + k)(3 + 7k)(19 + 23k)} \dot{A}_3 \dot{T}
- \frac{4}{(5 + k)} iA_3 T^{(2)} + \frac{8(-4 + k)}{(5 + k)^2} \dot{A}_3 \dot{A}_3 + \frac{2}{(5 + k)} iA_3 \partial T^{(1)}
+ \frac{2}{(5 + k)} iB_3 U^{(2)} + \frac{2\left(-9 - 5k + 2k^2\right)}{3(5 + k)^2} \dot{B}_3 \dot{B}_3
+ \frac{8k(9 + 7k)}{(3 + 7k)(19 + 23k)} iB_3 \dot{T}
+ \frac{2}{(5 + k)^2} \dot{B}_3 \dot{B}_3 + \frac{2}{(5 + k)^2} \dot{B}_3 \dot{B}_3
+ \frac{4}{(5 + k)} iB_3 W^{(2)} - \frac{4}{(5 + k)} iB_3 W^{(2)}
- \frac{2(-3 + 2k)}{(5 + k)^2} \dot{B}_3 \dot{A}_3 + \frac{4\left(9 - 5k + 2k^2\right)}{3(5 + k)^2} \dot{B}_3 \dot{B}_3 - \frac{5}{(5 + k)} iB_3 \partial T^{(1)}
+ \frac{(18 + 39k + 16k^2)}{(5 + k)(3 + 7k)(19 + 23k)} \dot{T} - \frac{(5 + 4k)}{2(5 + k)} \partial T^{(2)} - \frac{(7 + 4k)}{2(5 + k)} \partial W^{(2)}
+ \frac{8(3 + 2k)}{(5 + k)^2} iA_3 \partial A_3 - \frac{2(-3 + k)}{3(5 + k)} \partial A_3 T^{(1)}
- \frac{2}{(5 + k)} \dot{G}_{12} \dot{G}_{21} - \frac{4}{(5 + k)} \dot{G}_{21} T^{(2)} + \frac{2}{(5 + k)} U^{(2)} V^{(2)}}
\[ U(\tilde{z}) V_+^{(2)}(w) = \frac{1}{(z - w)^2} \left[ \frac{8(12 + 9k + k^2)}{3(5 + k)^2} \tilde{G}_{21} + \frac{4(11 + k)}{3(5 + k)^2} T_+^{(2)}(\tilde{z}) \right] \left( w \right) \\
+ \frac{1}{(z - w)^2} \left[ \frac{(31 + 8k)}{3(5 + k)} P_+^{(2)}(w) + \frac{(35 + 8k)}{3(5 + k)} W_+^{(2)}(w) + \frac{2}{(5 + k)^2} i\tilde{A}_- \tilde{G}_{11} \right] \left( w \right) \\
+ \frac{2(11 + 4k)}{(5 + k)^2} i\tilde{A}_- U(\tilde{z}) - \frac{8(9 + k)}{3(5 + k)^2} i\tilde{A}_3 \tilde{G}_{21} - \frac{4(33 + 8k)}{3(5 + k)^2} i\tilde{A}_3 T_+^{(2)}(\tilde{z}) \\
+ \frac{2(27 + 8k)}{3(5 + k)^2} i\tilde{B}_- V(\tilde{z}) + \frac{4k}{(5 + k)^2} i\tilde{B}_1 T_+^{(2)}(\tilde{z}) - \frac{2}{(5 + k)^2} T^{(1)} T_+^{(2)}(\tilde{z}) \\
- \frac{4(8 + k)(9 + 4k)}{9(5 + k)^2} \partial \tilde{G}_{21} - \frac{4(17 + 9k)}{9(5 + k)^2} \partial T_+^{(2)}(\tilde{z}) \right] \left( w \right) \\
+ \frac{1}{(z - w)} \left[ \frac{1}{(5 + k)} \left( \frac{21 + 8k}{5(5 + k)} \partial P_+^{(2)}(w) \right) \right] \left( w \right) \\
- \frac{1}{2} S_+^{(2)}(w) + \ldots \right] \left( w \right) + \ldots, \]

\[ U(\tilde{z}) V_-^{(2)}(w) = \frac{1}{(z - w)^2} \left[ \frac{4(1 + k)(13 + 2k)}{3(5 + k)^2} \tilde{G}_{12} + \frac{4(11 + 3k)}{3(5 + k)^2} T_-^{(2)}(\tilde{z}) \right] \left( w \right) \\
+ \frac{1}{(z - w)^2} \left[ \frac{(29 + 7k)}{3(5 + k)} P_-^{(2)}(w) + \frac{(30 + 7k)}{3(5 + k)} W_-^{(2)}(w) \right] \left( w \right) \\
- \frac{2(8 + k)}{(5 + k)^2} i\tilde{A}_+ \tilde{G}_{22} + \frac{4(27 + 5k)}{3(5 + k)^2} i\tilde{A}_+ V(\tilde{z}) \\
+ \frac{4(8 + k)}{(5 + k)^2} i\tilde{A}_3 \tilde{G}_{12} - \frac{4(48 + k)}{(5 + k)^2} i\tilde{A}_3 T_-^{(2)}(\tilde{z}) \\
+ \frac{2(-2 + k)}{(5 + k)^2} i\tilde{B}_1 \tilde{G}_{11} + \frac{2(16 + 5k)}{(5 + k)^2} i\tilde{B}_1 U(\tilde{z}) - \frac{16(3 + k)}{3(5 + k)^2} i\tilde{B}_1 T_-^{(2)}(\tilde{z}) \\
+ \frac{4(-1 + k)(25 + 4k)}{9(5 + k)^2} \partial \tilde{G}_{12} + \frac{4(91 + 22k)}{9(5 + k)^2} \partial T_-^{(2)}(\tilde{z}) \right] \left( w \right) \]
\[
U^{(2)}(z) \mathcal{V}^{(2)}(w) = -\frac{1}{(z - w)^2} \left[ \frac{8k(37 + 33k + 4k^2)}{(5 + k)^3} \right] \\
+ \frac{1}{(z - w)^2} \left[ \frac{2}{(5 + k)^3} \partial p^{(2)} \hat{A}_3 + \frac{2}{(5 + k)^3} i \partial p^{(2)} \hat{B}_3 + \cdots \right] (w) + \cdots \\
- \frac{2}{(5 + k)^3} i \partial p^{(2)} \hat{A}_3 + \frac{2}{(5 + k)^3} i \partial p^{(2)} \hat{B}_3 + \cdots \\
+ \frac{8k(37 + 33k + 4k^2)}{3(5 + k)^3} i \hat{B}_3 + \left( \frac{4(11 + 3k)}{3(5 + k)^2} T^{(1)} \right) (w) \\
+ \frac{1}{(z - w)^3} \left[ \frac{2(-3 + k)}{3(5 + k)} p^{(2)} - \frac{8}{9}(315 + 1038k + 587k^2 + 48k^3) \hat{T} \right] \\
- \frac{4(-3 + k)(-5 + 2k)}{9(5 + k)^2} T^{(2)} - \frac{8}{9}(33 + 77k + 6k^2) \hat{A}_1 \hat{A}_1 - \frac{8}{9}(33 + 77k + 6k^2) \hat{A}_2 \hat{A}_2 \\
- \frac{8}{9}(3 + k)(35 + 6k) \hat{A}_3 \hat{B}_3 - \frac{8}{9}(3 + k)(35 + 6k) \hat{B}_2 \hat{B}_2 \\
- \frac{56(15 + 8k)}{9(5 + k)^3} \hat{B}_1 \hat{B}_1 + \frac{4}{(5 + k)^3} T^{(1)} T^{(1)} - \frac{4(37 + 33k + 4k^2)}{(5 + k)^3} i \partial \hat{A}_3 \\
+ \frac{4k(37 + 33k + 4k^2)}{3(5 + k)^3} i \partial \hat{B}_3 + \frac{2(11 + 3k)}{3(5 + k)^2} \partial T^{(1)} \\
+ \frac{8}{3(5 + k)^2} i T^{(1)} \hat{A}_3 - \frac{8}{3(5 + k)^2} i T^{(1)} \hat{B}_3 \right] (w) \\
+ \frac{1}{(z - w)^2} \left[ \frac{(-3 + k)}{3(5 + k)} S^{(3)} + \frac{(11 + 3k)}{(5 + k)} p^{(3)} \right] \\
+ \frac{(3 + k)}{3(5 + k)} i \partial p^{(2)} - \frac{10(3 + k)}{3(5 + k)} W^{(3)} - \frac{4(41 + 29k)}{(5 + k)(19 + 23k)} T^{(1)} \hat{T} \\
+ \frac{4}{(5 + k)} T^{(1)} W^{(2)} - \frac{76}{3(5 + k)^2} i T^{(1)} i \partial \hat{A}_3 - \frac{2(-15 + 11k)}{3(5 + k)^2} i T^{(1)} i \partial \hat{B}_3 \right]
\]
\[
\begin{align*}
&+ \frac{4}{(5 + k)} T^{(1)} \partial T^{(1)} - \frac{4(31 + 7k)}{3(5 + k)^2} \hat{A}_+ V^{(2)}_+ \\
&- \frac{2\left(-399 + 73k + 12k^2\right)}{9(5 + k)^3} \hat{A}_+ \partial \hat{A}_- - \frac{2\left(447 + 175k + 12k^2\right)}{9(5 + k)^3} \hat{A}_- \partial \hat{A}_+ \\
&- \frac{8\left(4437 + 12102k + 5819k^2 + 352k^3\right)}{3(5 + k)^2(3 + 7k)(19 + 23k)} \hat{A}_3 \hat{T} \\
&+ \frac{8(9 + k)}{3(5 + k)^2} i \hat{A}_3 T^{(2)} - \frac{8(-2 + k)}{3(5 + k)^2} i \hat{A}_3 W^{(2)} \\
&- \frac{8\left(-168 + 65k + 6k^2\right)}{9(5 + k)^3} \hat{A}_3 \partial \hat{A}_3 + \frac{20}{(5 + k)^2} i \hat{A}_3 \partial T^{(1)} \\
&- \frac{16(7 + 2k)}{3(5 + k)^2} i \hat{B}_3 T^{(2)} - \frac{8\left(42 + 43k + 3k^2\right)}{9(5 + k)^3} \hat{B}_+ \partial \hat{B}_- \\
&- \frac{8\left(42 - 5k + 3k^2\right)}{9(5 + k)^3} \hat{B}_- \partial \hat{B}_+ \\
&+ \frac{8k}{3(5 + k)^2(3 + 7k)(19 + 23k)} i \hat{B}_3 \hat{T} \\
&+ \frac{16(3 + k)}{3(5 + k)^2} i \hat{B}_3 T^{(2)} - \frac{8\left(-4 + k\right)}{3(5 + k)^2} i \hat{B}_3 W^{(2)} \\
&- \frac{8(84 + 23k)}{9(5 + k)^3} \hat{B}_3 \partial \hat{B}_3 + \frac{2\left(17 + 3k\right)}{3(5 + k)^2} i \hat{B}_3 \partial T^{(1)} \\
&- \frac{2\left(504 + 2691k + 1207k^2 + 96k^3\right)}{9(5 + k)^3} \partial \hat{T} \\
&- \frac{\left(363 + 59k + 4k^2\right)}{9(5 + k)^2} \partial T^{(2)} - \frac{\left(-55 + 21k + 12k^2\right)}{9(5 + k)^2} \partial W^{(2)} \\
&- \frac{8(7 + k)(1 + 2k)}{3(5 + k)^3} i \partial^2 \hat{A}_3 + \frac{16k}{9(5 + k)^3} i \partial^2 \hat{B}_3 \\
&- \frac{4(37 + 9k)}{9(5 + k)^2} \partial^2 T^{(1)} - \frac{4(3 + k)}{(5 + k)^2} \hat{G}_{11} \hat{G}_{22} \\
&+ \frac{4(27 + 5k)}{3(5 + k)^2} \hat{G}_{11} V^{(2)} + \frac{8(8 + k)}{3(5 + k)^2} \hat{G}_{12} \hat{G}_{21} \\
&+ \frac{4(29 + 5k)}{3(5 + k)^2} \hat{G}_{21} T^{(2)} + \frac{28(3 + k)}{3(5 + k)^2} \hat{G}_{22} U^{(2)} - \frac{4(11 + 3k)}{3(5 + k)^2} U^{(2)} V^{(2)}
\end{align*}
\]
\[-\frac{16(18 + k)}{3(5 + k)^3} i\hat{A}_4 \hat{A}_3 \hat{A}_3 = \frac{2(-465 - 193k + 2k^2)}{9(5 + k)^3} i\hat{A}_4 \hat{A}_3 \hat{A}_3 \]
\[+ \frac{2(-177 - 121k + 2k^2)}{9(5 + k)^3} \hat{A}_4 \hat{A}_3 \hat{B}_3 - \frac{8(27 + 2k)}{3(5 + k)} i\hat{A}_3 \hat{A}_3 \hat{A}_3 \]
\[+ \frac{8(54 + 7k)}{3(5 + k)^3} i\hat{A}_3 \hat{A}_3 \hat{B}_3 + \frac{2(-201 + 115k + 10k^2)}{9(5 + k)^3} i\hat{A}_3 \hat{B}_3 \hat{A}_3 \]
\[-\frac{8(27 + 8k)}{3(5 + k)^3} i\hat{A}_3 \hat{B}_3 \hat{B}_3 = \frac{2(303 + 211k + 10k^2)}{9(5 + k)^3} i\hat{B}_3 \hat{A}_3 \hat{A}_3 \]
\[+ \frac{16k}{(5 + k)^2} i\hat{B}_3 \hat{B}_a \hat{B}_3 + \frac{8k}{(5 + k)^2} i\hat{B}_3 \hat{B}_a \hat{B}_3 - \frac{7 + k}{3(5 + k)} T^{(1)} \hat{A}_4 \hat{A}_3 \]
\[+ \frac{8}{(5 + k)^2} T^{(1)} \hat{A}_3 \hat{B}_3 - \frac{4(-3 + k)}{3(5 + k)^2} T^{(1)} \hat{B}_a T^{(2)} \]
\[+ \frac{1}{(z - w)} \left[ \frac{1}{2} \left( \hat{G}_{21} T^{(2)} \right) \right] \left( \hat{P}(z) \right) \]
\[+ \frac{2}{(5 + k)} i\hat{B}_3 \hat{P}(z) - \frac{2}{(5 + k)} i\hat{B}_3 \hat{P}(z) - \frac{7 + k}{3(5 + k)} \partial S^{(1)} \]
\[+ \frac{(34 + 7k)}{5(5 + k)} \hat{P}(z) + \frac{(-1 + 2k)}{20(5 + k)} \hat{P}(z) \]
\[1 \left( z - w \right) \left[ \frac{4(-3 + k)}{3(5 + k)^2} \partial G_{11} + \frac{4(-3 + k)}{3(5 + k)^2} U^{(2)} \right] \]
\[+ \frac{1}{(z - w)^2} \left[ \frac{1}{2} \left( 19 + 5k \right) Q^{(1)} \right] + \frac{1}{(z - w)^2} \left[ \frac{22 + 5k}{(5 + k)} \right] U^{(2)} + \frac{4(6 + k)}{(5 + k)^2} i\hat{A}_4 \hat{G}_{21} \]
\[+ \frac{28(3 + k)}{(5 + k)^2} i\hat{A}_4 T^{(2)} + \frac{4(5 + 2k)}{(5 + k)^2} i\hat{A}_3 U^{(2)} + \frac{4(3 + 2k)}{(5 + k)^2} i\hat{B}_3 T^{(2)} \]
\[+ \frac{4k}{(5 + k)^2} i\hat{B}_3 U^{(2)} - \frac{2}{(5 + k)} T^{(1)} U^{(2)} + \frac{4(-3 + k)}{3(5 + k)^2} \partial G_{11} \]
\[+ \frac{4(8 + k)}{3(5 + k)^2} \partial U^{(2)} \left( w \right) + \frac{1}{(z - w)} \left[ \frac{1}{2} \left( 3 + k \right) \partial Q^{(1)} \right] \]
\[+ \frac{1}{(5 + k)} \left[ \frac{4}{(5 + k)^2} \hat{B}_a + \ldots \right] w \ldots, \]
\[U^{(2)}(z) W^{(2)}(w) = - \left[ \frac{1}{(z - w)^3} \left[ \frac{32k}{3(5 + k)^3} i\hat{B}_a \right] + \ldots \right] \]
\[
\begin{align*}
-16k(-4 + k) \hat{B}_3 \hat{B}_3 \\
-8k \left( 16 + 17k + 2k^2 \right) \frac{i\partial \hat{B}_3}{3(5 + k)^3} - \frac{8(-4 + k)}{3(5 + k)^2}T^{(1)} \hat{B}_3 \\
+ \frac{1}{(z - w)^2} \left[ \frac{(31 + 10k)}{3(5 + k)} Q^3_+ - \frac{1}{(5 + k)} \right] (w) \\
+ \frac{4}{(5 + k)} T^{(1)} U^{(2)} - \frac{8(7 + 4k)}{3(5 + k)^2} i\hat{A}_3 U^{(2)}_+ \\
+ 2 \left( 483 + 292k + 52k^2 \right) \hat{A}_3 \hat{B}_3 \\
-9(5 + k)^3 \\
-8 \left( 285 + 3563k + 5819k^2 + 2041k^3 + 140k^4 \right) \frac{i\hat{B}_- \hat{T}}{3(5 + k)^2(3 + 7k)(19 + 23k)} \\
-4(31 + 7k) \frac{i\hat{B}_- T^{(2)}}{3(5 + k)^2} + \frac{4(32 + 7k)}{3(5 + k)^2} \frac{i\hat{B}_- W^{(2)}}{3(5 + k)^2} \\
+ 2 \left( 315 + 236k + 20k^2 \right) \frac{i\hat{B}_- \hat{A}_3}{3(5 + k)} + \frac{2 \left( 60 + 85k + 2k^2 \right)}{3(5 + k)^3} \hat{B}_- \hat{B}_3 \\
+ \frac{(73 + 2k)}{3(5 + k)^2} \frac{i\hat{B}_- \hat{D}^{(1)}}{3(5 + k)^2} - \frac{8k}{(5 + k)^2} i\hat{B}_3 U^{(2)}_+ \\
-2k(53 + 10k) \frac{i\hat{B}_3 \hat{D}^{(1)}}{3(5 + k)^2} + \frac{(253 + 106k)}{9(5 + k)^2} \frac{i\partial U^{(2)}_+}{3(5 + k)^2} \\
+ \frac{4(25 + 9k)}{3(5 + k)^2} \frac{\hat{G}_{11} T^{(1)}_+}{3(5 + k)^2} - \frac{8 \left( -15 + 16k + 21k^2 + 2k^3 \right)}{9(5 + k)^3} i\partial^2 \hat{B}_- \\
+ \frac{4(6 + k)}{3(5 + k)^2} \frac{\hat{G}_{11} \hat{G}_{21}}{3(5 + k)^2} - \frac{4k}{(5 + k)^2} \frac{i\hat{G}_{21} U^{(2)}}{3(5 + k)^2} \\
+ \frac{4(11 + k)}{3(5 + k)^2} \frac{T^{(1)}_+ U^{(2)}}{3(5 + k)^2} - \frac{8 \left( 13 + 6k \right)}{3(5 + k)^3} i\hat{A}_+ \hat{A}_- \hat{B}_- \\
-8 \left( -13 + 4k \right) \frac{i\hat{A}_3 \hat{A}_3 \hat{B}_-}{3(5 + k)^3} + \frac{16 \left( -4 + k \right)}{3(5 + k)^3} i\hat{A}_3 \hat{B}_- \hat{B}_3 \\
-\frac{8}{3(5 + k)^2} i\hat{B}_- \hat{B}_- \hat{B}_- \\
-8 \left( -5 + 2k \right) \frac{i\hat{B}_- \hat{B}_2 \hat{B}_3}{3(5 + k)^3} + \frac{8}{(5 + k)^2} T^{(1)} \hat{A}_3 \hat{B}_- \\
+ \frac{29 + 10k}{3(5 + k)^2} T^{(1)} \hat{B}_- \hat{B}_3 - \frac{(53 + 10k)}{3(5 + k)^2} \hat{B}_3 T^{(1)} \hat{B}_- \right] (w)
\end{align*}
\]
\[ U^{(2)}(z) W^{(2)}(w) = - \frac{1}{(z - w)^2} \left[ \frac{263 + 108k + 8k^2}{3(5 + k)^2} \hat{\Lambda}_+ \hat{T} \right] + \frac{4(29 + 8k)}{3(5 + k)^2} i\hat{\Lambda}_+ \hat{T}^{(2)} \]

\[ = - \frac{1}{(z - w)^2} \left[ T^{(1)} \hat{\Lambda}_+ \right] (w) + \frac{1}{(z - w)^2} \left[ \frac{2(19 + 4k)}{3(5 + k)} \hat{\Lambda}_+ \hat{A}_3 + \frac{4}{(5 + k)} T^{(1)} U^{(2)} \right] \]

\[ - \frac{1}{(z - w)^2} \left[ \frac{2(19 + 4k)}{3(5 + k)} \hat{\Lambda}_+ \hat{A}_3 + \frac{4}{(5 + k)} T^{(1)} U^{(2)} \right] \]

\[ - \frac{8(5 + 2k)}{3(5 + k)^2} i\hat{A}_3 U^{(2)} + \frac{4}{3(5 + k)^3} \hat{A}_3 \partial T^{(1)} \]

\[ - \frac{8(31 + 4k)}{3(5 + k)^2} \hat{B}_3 U^{(2)} - \frac{2(151 + 66k)}{9(5 + k)^2} \hat{A}_3 \partial U^{(2)} \]

\[ - \frac{4}{3(5 + k)^2} \hat{G}_{11} \hat{G}_{12} - \frac{32}{3(5 + k)} \hat{G}_{11} \hat{T}^{(2)} \]

\[ - \frac{8}{3(5 + k)^2} \hat{G}_{12} U^{(2)} + \frac{8(19 + 4k)}{3(5 + k)^2} T^{(1)} U^{(2)} \]

\[ - \frac{8(31 + 10k)}{3(5 + k)^3} i\hat{A}_+ \hat{A}_+ \hat{A}_- + \frac{16}{3(5 + k)^3} (48 + 25k + 2k^2) i\hat{A}_+ \hat{A}_3 \hat{A}_3 \]
\begin{align*}
&- \frac{4(66 + 13k + 20k^2)}{9(5 + k)^3} i\hat{A}_* \hat{A}_3 \hat{B}_3 - \frac{2(1020 + 317k + 4k^2)}{9(5 + k)^3} i\hat{A}_* \hat{B}_3 \hat{B}_-
&- \frac{32(10 + k)}{3(5 + k)^3} i\hat{A}_* \hat{B}_3 \hat{B}_3 - \frac{16(59 + 30k + 2k^2)}{3(5 + k)^3} i\hat{A}_3 \hat{A}_* \hat{A}_3
&+ \frac{4(438 + 97k + 20k^2)}{9(5 + k)^3} i\hat{A}_3 \hat{A}_* \hat{B}_3 + \frac{2(468 + 209k + 4k^2)}{9(5 + k)^3} i\hat{B}_* \hat{A}_3 \hat{B}_3
&+ \frac{8}{(5 + k)^2} T^{(1)}(\hat{A}_* \hat{A}_3) - \frac{8}{(5 + k)^2} T^{(1)}(\hat{A}_* \hat{B}_3) \left( \begin{array}{c}
w \\
(\hat{A}_* \hat{B}_3) Q^{(3)} + \ldots \end{array} \right) (w) + \ldots
\end{align*}

\begin{align*}
U(\hat{z})(z) W^{(3)}(w) &= \frac{1}{(z - w)^3} \left[ - \frac{4(218k^4 + 2688k^3 + 10195k^2 + 13278k + 3585)}{3(5 + k)^3(19 + 23k)} \hat{G}_{11} \right]
- \frac{4(7k^3 - 639k^2 - 1937k - 1091)}{(5 + k)^3(19 + 23k)} U(\hat{z}) \left( \begin{array}{c}
w \\
(\hat{A}_* \hat{A}_3) Q^{(3)} \end{array} \right) (w)
+ \frac{1}{(z - w)^3} \left[ \frac{1}{3(5 + k)^2(19 + 23k)} \hat{Q}(\hat{z}) 
+ \frac{2}{3(5 + k)^2(19 + 23k)} U(\hat{z}) \right]
+ \frac{8}{3(5 + k)^3} i\hat{A}_* \hat{G}_{21}
+ \frac{8k(4 + k)}{3(5 + k)^3} i\hat{A}_* T^{(2)}(\hat{z}) = \frac{8(8 + k)(3 + 2k)}{3(5 + k)^3} i\hat{A}_3 \hat{G}_{11}
+ \frac{4(7 + 13k + 2k^2)}{3(5 + k)^3} i\hat{B}_- \hat{G}_{12} - \frac{8(-1 + 8k + 2k^2)}{3(5 + k)^3} i\hat{B}_- U(\hat{z})
+ \frac{8(12 + 23k + 2k^2)}{3(5 + k)^3} i\hat{B}_3 \hat{G}_{11} - \frac{8(-17 - 8k + 2k^2)}{3(5 + k)^3} i\hat{B}_3 U(\hat{z})
- \frac{4(-10 + k)(7 + 2k)}{3(5 + k)^3} i\hat{A}_3 U(\hat{z})
+ \frac{8(-10 + k)(7 + 2k)}{3(5 + k)^3} i\hat{A}_3 U(\hat{z})
\end{align*}
\[- \frac{4(218k^4 + 2734k^3 + 9727k^2 + 14102k + 4611)}{9(5 + k)^3(19 + 23k)} \partial \hat{G}_{11} \]
\[- \frac{4(53k^3 - 785k^2 - 3193k - 2003)}{3(5 + k)^3(19 + 23k)} \partial U(\hat{\tau}) \bigg|_{(w)} \]
\[+ \frac{1}{(z - w)^2} \left[ \frac{(-21233 - 11116k + 377k^2 + 364k^3)}{15(5 + k)^2(19 + 23k)} \partial Q(\hat{\tau}) \right] \]
\[- \frac{(43 + 8k)}{3(5 + k)^2} P^{(3)} \hat{B}_+ + \frac{(29 + 7k)}{2(5 + k)} Q(\hat{\tau}) + \frac{(13 + 5k)}{3(5 + k)^2} \hat{Q}(\hat{\tau}) \hat{A}_3 \]
\[- \frac{(13 + 5k)}{3(5 + k)^2} \hat{Q}(\hat{\tau}) \hat{B}_3 + \ldots \bigg|_{(w)} \]
\[+ \frac{1}{(z - w)} \left\{ U(\hat{\tau}) W^{(3)} \right\}_{-1}(w) + \ldots. \quad \text{(F.1)} \]

The first order pole in the first OPE of (F.1) contains composite field with spin-4 with \( U(1) \) charge of \( \frac{2(3 + k)}{5 + k} \). The \( \hat{G}_{11} \hat{G}_{11} \) (w) can be written in terms of a derivative of \( \hat{A}_+ \hat{B}_- \) (w). The first order pole in the fifth OPE has a composite field with spin-4 with vanishing \( U(1) \) charge. Also the fifth OPE has a term of the first order pole in (2.14) where the higher spin-4 current was constructed. The \( (k - 3) \) factor appears in \( P^{(3)} \) (w), \( T^{(3)} \) (w) and \( \hat{S}^{(3)} \) (w) terms. The composite fields of spin-2 except the first term can be seen from the table 3 of [1]. The \( (k - 3) \) factor appears in the third order pole in the sixth OPE. The first order pole in the seventh OPE has a composite field with spin-4 with \( U(1) \) charge of \( \frac{2k}{(5 + k)} \). The first order pole in the eighth OPE has a composite field with spin-4 with \( U(1) \) charge of \( \frac{6}{(5 + k)} \). The first order pole in the last OPE has a composite field with spin-\( 2 \) with \( U(1) \) charge of \( \frac{2(3 + k)}{5 + k} \) and this will appear in appendix L separately. The 13 composite fields of spin-\( \frac{5}{2} \) in the third order pole can be found in table 4 of [1].

Appendix G. The nontrivial OPEs between higher spin-\( \frac{5}{2} \) current, \( V^{(3)}(z) \), and other eight higher spin currents

Now the OPEs containing third \( \mathcal{N} = 2 \) multiplet are given by

\[ V^{(3)}(z) V^{(2)}(w) = \frac{1}{(z - w)^2} \left[ \frac{4(3 + k)}{3(5 + k)^2} \hat{A}_+ \hat{B}_- \right] \bigg|_{(w)} \]
\[+ \frac{1}{(z - w)} \left[ - \frac{2}{(5 + k)} i \hat{B}_+ V^{(2)} + \frac{2(7 + k)}{(5 + k)^2} \hat{B}_+ \hat{A}_- - \frac{2}{(5 + k)} i \hat{A}_- V^{(2)} \right] \]
\[+ \frac{(12 + k)}{3(5 + k)} \hat{G}_{22} \hat{G}_{22} \bigg|_{(w)} + \ldots, \]

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\[ V(\hat{z})(z) W(2)_{+}(w) = \frac{1}{(z-w)^2} \left[ \frac{3(4+k)}{(5+k)} V(\hat{z}) \right](w) + \frac{1}{(z-w)} \left[ \frac{(4+k)}{(5+k)} \partial V(\hat{z}) \right](w) + \ldots, \]

\[ V(\hat{z})(z) W(2)_{+}^{(2)}(w) = \frac{1}{(z-w)^3} \left[ \frac{8(5+2k)}{(5+k)^2} \hat{A}_{\hat{A}} \right](w) \]

\[ \quad - \frac{1}{(z-w)^2} \left[ \frac{4(7+k)}{(5+k)} V_{+}^{(2)} \right](w) \]

\[ \quad + \frac{1}{(z-w)} \left[ \frac{1}{2} \mathcal{R}_{+}^{(3)} + \frac{4}{(5+k)(3+7k)(19+23k)} \right. \]

\[ \quad \left. \hat{A}_{\hat{A}} \partial \hat{T}^{(2)} \right] \]

\[ \quad - \frac{2}{(5+k)} \hat{A}_{\hat{A}} \partial \hat{T}^{(2)} \]

\[ \quad - \frac{(8+3k)}{(5+k)^2} \hat{A}_{\hat{A}} \partial \hat{B}_3 \]

\[ \quad + \frac{1}{2(5+k)} \hat{A}_{\hat{A}} \partial \hat{T}^{(1)} \]

\[ \quad - \frac{3}{(5+k)^2} \hat{A}_{\hat{A}} \partial \hat{B}_3 \]

\[ \quad + \frac{4}{(5+k)} \hat{A}_{\hat{A}} \partial \hat{B}_{3}^{(2)} \]

\[ \quad + \frac{1}{(5+k)} \hat{G}_{22} \hat{G}_{21} + \frac{2}{(5+k)} \hat{G}_{22} T_{+(2)}^{(2)} \]

\[ \quad + \frac{2}{(5+k)} \hat{G}_{21} V(\hat{z}) + \frac{2}{(5+k)} \hat{T}^{(2)} V(\hat{z}) \]

\[ \quad + \frac{4}{(5+k)^2} \hat{A}_{\hat{A}} \hat{A}_{\hat{A}} \hat{A}_{\hat{A}} \]

\[ \quad - \frac{8}{(5+k)^2} \hat{A}_{\hat{A}} \hat{A}_{\hat{A}} \hat{B}_3 \]

\[ \quad + \frac{4}{(5+k)^2} \hat{A}_{\hat{A}} \hat{B} \hat{B} + \frac{4}{(5+k)^2} \hat{A}_{\hat{A}} \hat{B}_3 \hat{B}_3 \]

\[ \quad + \frac{4}{(5+k)^2} \hat{A}_{\hat{A}} \hat{A}_{\hat{A}} \hat{A}_{\hat{A}} \]

\[ \quad - \frac{1}{2(5+k)} \hat{T}^{(1)} \hat{A}_{\hat{A}} \hat{A}_{\hat{A}} \]

\[ \quad + \frac{1}{2(5+k)} \hat{A}_{\hat{T}^{(1)} \hat{A}_{\hat{A}}} \]

\[ \quad + \ldots, \]

\[ V(\hat{z})(z) W(2)_{+}^{(2)}(w) = - \frac{1}{(z-w)^3} \left[ \frac{8k(8+k)}{3(5+k)^2} \hat{B}_+ \right](w) \]

\[ \quad - \frac{1}{(z-w)^2} \left[ \frac{8(2+k)}{(5+k)^2} V_{+}^{(2)} \right](w) \]

\[ \quad + \frac{1}{(z-w)} \left[ \frac{1}{2} \mathcal{R}_{+}^{(3)} - \frac{1}{2(5+k)} \hat{T}^{(1)} \partial \hat{B}_+ + \frac{4}{(5+k)} \hat{A}_{\hat{A}} V_{+}^{(2)} \right] \]

\[ \quad + \frac{1}{(5+k)^2} \hat{A}_{\hat{A}} \partial \hat{B}_+ \]

\[ \quad - \frac{4k}{(5+k)(3+7k)(19+23k)} \hat{B}_+ \hat{T} \]
\[
- \frac{2}{(5 + k)} i\hat{B}_+ T^{(2)} + \frac{1}{2(5 + k)} i\hat{B}_+ \partial T^{(1)} - \frac{k}{(5 + k)^2} \hat{B}_3 \partial \hat{B}_+ \\
- \frac{(5 + 4k)}{6(5 + k)} \partial V^{(2)} - \frac{5k}{6(5 + k)^2 i\partial^2 \hat{B}_+} \\
+ \frac{2}{(5 + k)} \hat{G}_{22} T^{(2)} + \frac{2}{(5 + k)} T^{(1)} T^{(2)} V^{(3)} \\
+ \frac{(15 + 2k)}{2(5 + k)^2} i\hat{A}_- \hat{A}_+ \hat{B}_+ - \frac{(15 + 2k)}{2(5 + k)^2} i\hat{A}_+ \hat{A}_- \hat{B}_+ \\
- \frac{k}{2(5 + k)^2} i\hat{B}_- \hat{B}_+ \hat{B}_+ + \frac{k}{2(5 + k)^2} i\hat{B}_- \hat{B}_- \hat{B}_+ \right] (w) + \cdots,
\]

\[
V^{(2)}(z) W^{(3)}(w) = - \frac{1}{(z - w)^2} \left[ \frac{4(-3 + k)}{(5 + k)^2} \hat{G}_{22} \\
+ \frac{4(-3 + k)(345 + 296k + 55k^2)}{3(5 + k)^2(19 + 23k)} V^{(2)}(w) \\
+ \frac{1}{(z - w)^2} \left[ - \frac{(19 + 5k)}{2(5 + k)} \mathbf{R}^{(2)} \\
+ \frac{(21 + 5k)}{(5 + k)} V^{(2)} - \frac{12}{(5 + k)^2} i\hat{A}_- \hat{G}_{12} \\
+ \frac{2(1 + k)}{(5 + k)^2} i\hat{A}_- T^{(2)} - \frac{2}{(5 + k)^2(19 + 23k)} \hat{A}_3 V^{(2)}(w) \\
+ \frac{33 + 5k}{(5 + k)^2} i\hat{B}_+ \hat{G}_{21} + \frac{3(11 + 3k)}{(5 + k)^2} i\hat{B}_+ T^{(2)}(w) \\
+ \frac{2(551 + 481k + 78k^2)}{(5 + k)^2(19 + 23k)} i\hat{B}_3 V^{(2)}(w) \\
+ \frac{(161 - 47k - 12k^2)}{(5 + k)(19 + 23k)} T^{(1)} V^{(2)} + \frac{4(-3 + k)}{3(5 + k)^2} \partial \hat{G}_{22} \\
- \frac{243 + 692k + 141k^2 + 4k^3}{(5 + k)^2(19 + 23k)} \partial V^{(2)}(w) \right] (w) + \cdots.
\]

In the first OPE of (G.1), the \((k - 3)\) factor appears in the second order pole. Also the third order pole of last OPE contains the \((k - 3)\) factor.
Appendix H. The nontrivial OPEs between higher spin-2 current, $V_+^{(2)}(z)$, and other seven higher spin currents

We present the following OPEs

\[
V_+^{(2)}(z) V_-(w) = \frac{1}{(z-w)^2} \left[ \frac{2(3 + k)}{(5 + k)^2} \hat{A}_- \hat{B}_+ \right](w) \\
+ \frac{1}{(z-w)} \left[ - \frac{2}{(5 + k)} i\hat{B}_+ V_+^{(2)} \right] + \frac{2(7 + k)}{(5 + k)^2} \hat{B}_- \partial \hat{A}_- \\
+ \frac{2}{(5 + k)} \hat{A}_- V_+^{(2)} - \frac{2}{(5 + k)} \hat{G}_{22} \hat{G}_{22}(w) + \ldots ,
\]

\[
V_+^{(2)}(z) V(\hat{z})(w) = \frac{1}{(z-w)^2} \left[ \frac{2(3 + 2k)}{3(5 + k)^2} i\hat{A}_- \hat{G}_{22} + \frac{8(9 + k)}{3(5 + k)^2} i\hat{A}_- V(\hat{z}) \right](w) \\
+ \frac{1}{(z-w)} \left[ - 4(15 + 2k) \hat{A}_- \hat{R}(\hat{z}) \right] + \frac{2}{(5 + k)^2} i\hat{A}_- T(1) \hat{G}_{22} + \frac{4(15 + 2k)}{3(5 + k)^2} \hat{A}_- V(\hat{z}) \\
+ \frac{2}{(5 + k)^2} \hat{G}_{21} \hat{A}_- \hat{B}_+ + \frac{2}{(5 + k)^2} i\hat{G}_{22} T(1) \hat{A}_- \\
+ \frac{2}{(5 + k)} \hat{G}_{22} V_+^{(2)}(w) + \ldots ,
\]

\[
V_+^{(2)}(z) W(\hat{w})(w) = \frac{1}{(z-w)^2} \left[ \frac{6(5 + 2k)}{(5 + k)^2} i\hat{A}_- \right](w) \\
+ \frac{1}{(z-w)^2} \left[ \frac{3(5 + 2k)}{(5 + k)^2} i\hat{A}_- \right] + \frac{2(4 + k)}{(5 + k)} V_+^{(2)}(w) \\
+ \frac{1}{(z-w)} \left[ \frac{1}{2} R_+^{(3)} \right] + \frac{1}{2(5 + k)} i\hat{A}_- T(1) \hat{A}_- \\
+ \frac{4 \left( 138 + 343k + 113k^2 \right)}{(5 + k)(3 + 7k)(19 + 23k)} \hat{A}_- \hat{T} - \frac{2}{(5 + k)} i\hat{A}_- T(2) \\
- \frac{9}{(5 + k)^2} \hat{A}_- \partial \hat{A}_3 - \frac{8 + 3k}{(5 + k)^2} \hat{A}_- \partial \hat{B}_3 \\
- \frac{1}{2(5 + k)} i\hat{A}_- T(1) - \frac{3}{(5 + k)^2} \hat{A}_3 \partial \hat{A}_- \\
+ \frac{(8 + 3k)}{(5 + k)^2} \hat{B}_3 \hat{A}_- + \frac{(9 + 2k)}{2(5 + k)} \partial V_+^{(2)} \\
+ \frac{(5 + 2k)}{(5 + k)^2} \hat{A}_- \hat{T} = \frac{2}{(5 + k)} \hat{G}_{22} \hat{G}_{22}.
\]
\[ + \frac{2}{(5 + k)} \hat{G}_{21} V(\tilde{z}) + \frac{2}{(5 + k)} \hat{G}_{22} T(\tilde{z})^{(z)} \\
+ \frac{2}{(5 + k)} T(\tilde{z})^{(z)} V(\tilde{z}) + \frac{4}{(5 + k)^2} i\hat{A}_4 \hat{A}_3 \hat{A}_\_ \\
+ \frac{4}{(5 + k)^2} i\hat{A}_1 \hat{A}_2 \hat{A}_3 + \frac{4}{(5 + k)^2} i\hat{A}_1 \hat{B}_4 \hat{B}_\_ \\
+ \frac{4}{(5 + k)^2} i\hat{A}_1 \hat{B}_3 \hat{B}_3 - \frac{8}{(5 + k)^2} i\hat{A}_3 \hat{B}_3 \\
- \frac{4}{(5 + k)} i\hat{B}_3 V(\tilde{z}) \right] (w) + \ldots, \]

\[ V^{(2)}_+(z) W(z) = \frac{1}{(z - w)^2} \left[ - \frac{2(13 + 2k)}{(5 + k)^2} i\hat{A}_4 \hat{G}_{21} - \frac{2(13 + 2k)}{(5 + k)^2} i\hat{A}_4 T(\tilde{z})^{(z)} \right] (w) \\
+ \frac{1}{(z - w)^2} \left[ \frac{2}{(5 + k)} i\hat{A}_4 W(\tilde{z}) - \frac{4}{(5 + k)^2} i\hat{A}_4 \partial \hat{G}_{21} \\
- \frac{8(4 + k)}{3(5 + k)^2} i\hat{A}_4 \partial T(\tilde{z})^{(z)} - \frac{4}{3(5 + k)^2} i\hat{A}_4 \hat{G}_{21} \\
+ \frac{8(2 + k)}{3(5 + k)^2} i\hat{A}_4 \hat{T}(\tilde{z})^{(z)} - \frac{2}{(5 + k)} \hat{G}_{21} V(\tilde{z}) \right] (w) + \ldots, \]

\[ V^{(2)}_+(z) W(z) = \frac{1}{(z - w)^2} \left[ \frac{8(2 + k)(6 + k)}{(5 + k)^2} \hat{G}_{22} + \frac{4(11 + 3k)}{(5 + k)^2} \hat{V}(\tilde{z}) \right] (w) \\
+ \frac{1}{(z - w)^2} \left[ - \frac{(27 + 8k)}{(5 + k)^2} \hat{R}(\tilde{z}) \\
+ \frac{8(1 + k)}{(5 + k)^2} i\hat{A}_4 \hat{T}(\tilde{z})^{(z)} - \frac{8(10 + k)}{(5 + k)^2} i\hat{A}_3 \hat{V}(\tilde{z}) \\
+ \frac{2(35 + 8k)}{(5 + k)^2} i\hat{B}_3 \hat{G}_{21} + \frac{4(22 + 7k)}{(5 + k)^2} i\hat{B}_3 \hat{T}(\tilde{z})^{(z)} \\
+ \frac{4(35 + 11k)}{(5 + k)^2} i\hat{B}_3 \hat{V}(\tilde{z}) - \frac{2}{(5 + k)} T^{(1)}(\tilde{z}) \hat{V}(\tilde{z}) \\
+ \frac{4(24 + 19k + 2k^2)}{9(5 + k)^2} \partial \hat{G}_{22} - \frac{8(22 + 5k)}{9(5 + k)^2} \partial \hat{V}(\tilde{z}) \right] (w) \\
+ \frac{1}{(z - w)^2} \left[ - \frac{1}{2} \hat{R}(\tilde{z}) + \frac{(-25 + 4k)}{90} \hat{P}(\tilde{z}) \hat{V}(\tilde{z}) \\
+ \frac{(25 - 4k)}{90} \hat{V}(\tilde{z}) \hat{P}(\tilde{z}) - \frac{2}{(5 + k)} i\hat{R}(\tilde{z}) \hat{A}_3 \right] (w) \]
\[
- \frac{2}{(5 + k)} \text{Re} \epsilon^{(2)} \hat{B}_3 - \left( \frac{1339 - 118 k + 8 k^2}{135(5 + k)} \right) \text{Re} \epsilon^{(2)} \hat{T}^{(1)} + \frac{1339 - 118 k + 8 k^2}{135(5 + k)} T^{(1)} \text{Re} \epsilon^{(2)} \\
+ \frac{1}{(5 + k)} i \text{P}^{(2)} \hat{A}_- \] + \ldots \right) \right) (w) + \ldots. \tag{H.1}
\]

\[
V_{+}^{(2)} (z) W^{(3)} (w) = - \frac{1}{(z - w)^4} \left[ \frac{12}{(5 + k)^3 (19 + 23 k)} \frac{1270 + 893 k + 437 k^2 + 50 k^3}{(5 + k)^3 (19 + 23 k)} i \hat{A}_- \right] (w) + \left[ \frac{2}{(5 + k)^2 (19 + 23 k)} \right] \frac{12(-3 + 2k)}{(5 + k)^3} \hat{A}_- \hat{A}_3 \\
+ \frac{4(40 + 19k)(5 + k)^3}{(5 + k)^3} \hat{A}_- \hat{B}_3 - \frac{6(-3 + 2k)}{(5 + k)^2} i \hat{A}_- \hat{A}_3 \\
+ \frac{10}{(5 + k)^2} i \hat{A}_- \hat{T}^{(1)} V_{+}^{(2)} + \frac{3(-3 + 2k)}{(5 + k)^2} i \hat{T}^{(1)} \hat{A}_- \\
+ \frac{4(7476 + 20801k + 11060k^2 + 1343k^3)}{(5 + k)^3} i \hat{A}_- \hat{T}^{(1)} V_{+}^{(2)} \\
+ \frac{2(47 + 13k)(5 + k)^3}{(5 + k)^3} i \hat{A}_- \hat{T}^{(2)} + \frac{6(4 + k)(5 + k)^3}{(5 + k)^3} i \hat{A}_- \hat{W}^{(2)} \\
+ \frac{(251 + 58k)(5 + k)^3}{(5 + k)^3} i \hat{A}_- \hat{d} \hat{A}_3 + \frac{(448 + 251k + 18k^2)(5 + k)^3}{(5 + k)^3} i \hat{A}_- \hat{d} \hat{B}_3 \\
+ \frac{(25 + 6k)(5 + k)^3}{(5 + k)^3} i \hat{A}_- \hat{d} \hat{T}^{(1)} + \frac{2(329 + 675k + 118k^2)(5 + k)^3}{(5 + k)^3} i \hat{A}_3 \hat{V}^{(2)} \\
+ \frac{3(17 - 10k)(5 + k)^3}{(5 + k)^3} i \hat{A}_3 \hat{d} \hat{A}_3 + \frac{(724 + 185k + 18k^2)(5 + k)^3}{(5 + k)^3} i \hat{B}_3 \hat{d} \hat{A}_- \\
+ \frac{(2107 + 4933k + 1582k^2)(5 + k)^3}{(5 + k)^3} i \hat{A}_3 \hat{d} \hat{V}^{(2)} + \frac{10(-1 + 4k)(5 + k)^3}{(5 + k)^3} i \hat{A}_3 \hat{d} \hat{A}_- \\
+ \frac{8(5 + k)^3}{(5 + k)^3} \hat{G}_{21} \hat{G}_{22} - \frac{2(17 + 2k)(5 + k)^3}{(5 + k)^3} \hat{G}_{21} \hat{V}^{(2)} + \frac{2(23 + 6k)(5 + k)^3}{(5 + k)^3} \hat{G}_{22} \hat{T}^{(2)} \\
+ \frac{2(25 + 6k)(5 + k)^3}{(5 + k)^3} \hat{V}^{(2)} + \frac{4(53 + 10k)(5 + k)^3}{(5 + k)^3} i \hat{A}_+ \hat{A}_- \hat{A}_- \\
\]
As explained before, the combination of the second and third terms in the first order pole of (H.1) can be written in terms of derivatives of known composite fields. One has the following OPE

\[
V(\hat{z}) P^2(\hat{w}) = \frac{1}{(z - w)^2} \left[ \frac{16(2 + k)(8 + k)}{3(5 + k)^2} V(\hat{z}) \right](w) + \frac{1}{(z - w)^2} \left[ \frac{-4(19 + 2k)}{3(5 + k)^2} V(\hat{z}) - \frac{8(8 + k)}{(5 + k)^2} i\bar{B}_4 T^2(\hat{z}) \right] + \frac{4}{(5 + k)^2} i\bar{B}_4 V(\hat{z}) - \frac{8(8 + k)}{(5 + k)^2} T^{(1)} V(\hat{z}) + \frac{8(8 + k)(7 + 2k)}{9(5 + k)^2} \partial V(\hat{z}) + \frac{2}{3(5 + k)^2} T^{(1)} V(\hat{z}) \right] + \ldots. \tag{H.3}
\]

From (H.3), the following commutator can be expressed as follows:

\[
\left[ V(\hat{z}), P^2 \right](w) = -\frac{1}{2} \partial^2 \left\{ V(\hat{z}), P^2 \right\}_{w>2}(w) + \partial \left\{ V(\hat{z}), P^2 \right\}_{w<1}(w).
\]

Therefore, one does not see any mixture between different \( N = 4 \) multiplets in (H.1). Similarly, one has the following OPE

\[
T^{(1)}(\hat{z}) R^2(\hat{w}) = \frac{1}{(z - w)^2} \left[ \frac{8(-3 + k)}{3(5 + k)} \left( \bar{G}_{22} - V(\hat{z}) \right) \right](w) + \frac{1}{(z - w)^2} R(\hat{z}) + \ldots.
\]

Then one can reexpress the commutator as follows:

\[
\left[ T^{(1)}, R^2 \right](w) = -\frac{1}{2} \partial^2 \left\{ T^{(1)}, R^2 \right\}_{w>2}(w) + \partial \left\{ T^{(1)}, R^2 \right\}_{w<1}(w). \tag{H.4}
\]

The combination of the sixth and seventh terms in the first order pole of (H.1) can be written in terms of derivatives of known composite fields.
In the first OPE of (H.2), the expression $\hat{G}_{22} \hat{G}_{22}(w)$ can be expressed as a derivative of $\hat{A}_- \hat{B}_+(w)$. The first order pole in the last OPE of (H.2) contains composite field with spin-4 with $U(1)$ charge of $\frac{6}{(5 + k)}$. The second order pole contains a composite field of spin-3 which can be seen from the table 5 of [1].

**Appendix I. The nontrivial OPEs between higher spin-2 current, $V^{(2)}_-(z)$, and other seven higher spin currents**

We present the following OPEs

$$V^{(2)}_-(z) V^{(2)}_-(w) = \frac{1}{(z-w)^2} \left[ \frac{2(6+k)}{3(5+k)^2} i \hat{B}_+ \hat{G}_{22} - \frac{2(6+k)}{3(5+k)^2} i \hat{B}_+ V^{(2)}_-(w) \right]$$

$$+ \frac{1}{(z-w)^2} \left[ \frac{4(6+k)}{3(5+k)^2} i \hat{B}_+ V^{(2)}_-(w) - \frac{4}{3(5+k)^2} i \hat{B}_+ \partial V^{(2)}_-(w) \right]$$

$$- \frac{2(12+k)}{3(5+k)^2} i \hat{B}_+ \partial \hat{G}_{22} - \frac{4}{(5+k)^2} i \hat{B}_+ \partial \hat{G}_{22}$$

$$+ \frac{2}{(5+k)} i \hat{B}_+ V^{(2)}_-(w) - \frac{2}{(5+k)} \hat{G}_{22} V^{(2)}_-(w)$$

$$+ \frac{4}{(5+k)} V^{(2)}_-(w) + \ldots$$

$$V^{(2)}_-(z) W^{(2)}_-(w) = -\frac{1}{(z-w)^3} \left[ \frac{2k(8+k)}{(5+k)^2} i \hat{B}_+ \right]$$

$$+ \frac{1}{(z-w)^3} \left[ \frac{2(4+k)}{(5+k)} V^{(2)}_-(w) - \frac{k(8+k)}{(5+k)^2} i \hat{B}_+ \right]$$

$$+ \frac{1}{(z-w)^3} \left[ -\frac{1}{2} R^{(3)} + \frac{4}{(5+k)^2} i \hat{B}_+ V^{(2)}_-(w) + \frac{(11+2k)}{(5+k)^2} \hat{A}_3 \partial \hat{B}_+ \right]$$

$$- \frac{4k}{(5+k)(3+7k)(19+23k)} i \hat{B}_+ \hat{T} - \frac{2}{(5+k)} i \hat{B}_+ T^{(2)}$$

$$- \frac{1}{(15+2k)} \hat{B}_+ \partial \hat{A}_3 + \frac{k}{(5+k)^2} \hat{B}_+ \partial \hat{B}_3$$

$$+ \frac{1}{2(5+k)} i \hat{B}_+ \partial T^{(1)} - \frac{k}{(5+k)^2} \hat{B}_3 \partial \hat{B}_+$$

$$+ \frac{1}{2(5+k)} i \hat{B}_+ \partial V^{(2)}_-(w) - \frac{k(12+k)}{(3(5+k)^2)} i \hat{B}_+^2$$

$$+ \frac{2}{(5+k)} i \hat{B}_+ \hat{T}^{(2)} + \frac{2}{(5+k)} \hat{T}^{(2)} V^{(2)}_-(w)$$

$$- \frac{1}{2(5+k)} T^{(1)} \hat{B}_+ T^{(1)} + \frac{1}{2(5+k)} T^{(1)} \hat{B}_+ V^{(2)}_-(w)$$
\[ V_{-}^{(2)} (z) W_{+}^{(2)} (w) = \frac{1}{(z - w)^3} \left[ \frac{4(7 + k)(3 + 2k)}{3(5 + k)^2} \hat{c}_{22} + \frac{4(11 + 3k)}{3(5 + k)^2} V(z) \right] (w) \\
+ \frac{1}{(z - w)^2} \left[ \frac{(30 + 7k)}{3(5 + k)} \mathbf{R}^{(2)}(z) - \frac{(38 + 7k)}{3(5 + k)} V(z) + \frac{2(6 + k)}{(5 + k)^2} i \hat{A}_- \hat{G}_{12} \right] + \frac{2(6 + k)}{(5 + k)^2} i \hat{A}_- T^{(2)}(z) + \frac{12}{(5 + k)^2} i \hat{A}_- V(z) - \frac{4(8 + k)}{(5 + k)^2} i \hat{B}_4 \hat{G}_{21} \\
- \frac{2(16 + 3k)}{(5 + k)^2} i \hat{B}_4 T^{(2)}(z) - \frac{4(8 + k)}{(5 + k)^2} i \hat{B}_3 V(z) - \frac{2}{(5 + k)} T^{(1)} V(z) \\
+ \frac{4(30 + 17k + 2k^2)}{9(5 + k)^2} i \hat{G}_{22} + \frac{8(13 + 3k)}{9(5 + k)^2} \theta V(z) \left] (w) \right. \\
+ \frac{1}{(z - w)} \left[ \frac{2(27 + 7k)}{15(5 + k)} \theta \mathbf{R}^{(2)}(z) - \frac{1}{2} \mathbf{R}^{(2)}(z) \right] \\
+ \frac{1}{(5 + k)} \left[ \mathbf{P}^{(2)}(z) \hat{A}_- + \cdots \right] (w) + \cdots, \\
\]

\[ V_{-}^{(2)} (z) W_{+}^{(2)} (w) = \frac{1}{(z - w)^2} \left[ \frac{2(16 + k)}{3(5 + k)^2} i \hat{B}_4 T^{(2)}(z) \right] (w) \\
+ \frac{1}{(z - w)} \left[ \frac{8(8 + k)}{3(5 + k)^2} \hat{A}_3 \hat{B}_4 \hat{G}_{12} - \frac{2(8 + k)}{(5 + k)^2} \hat{B}_3 \hat{B}_4 \hat{G}_{12} \right] \\
- \frac{2}{(5 + k)} i \hat{B}_4 W(z) + \frac{2(8 + k)}{(5 + k)^2} i \hat{B}_3 \hat{B}_4 \hat{G}_{12} \\
- \frac{2}{(5 + k)} \hat{G}_{12} V^{(2)}(z) + \frac{8(8 + k)}{3(5 + k)^2} \hat{G}_{12} \hat{A}_3 \hat{B}_4 \\
+ \frac{4(7 + k)}{3(5 + k)^2} \hat{B}_3 \hat{B}_4 T^{(2)}(z) - \frac{4(7 + k)}{3(5 + k)^2} \hat{B}_4 \hat{B}_3 T^{(2)}(z) \\
+ \frac{4(11 + k)}{3(5 + k)^2} \hat{B}_4 \theta T^{(2)}(z) \right] (w) + \cdots, \\
\]

\[ V_{-}^{(2)} (z) W^{(3)} (w) = - \frac{1}{(z - w)^4} \left[ \frac{2k \left( 2687 + 3264k + 973k^2 + 84k^3 \right)}{(5 + k)^4(19 + 23k)} \right] i \hat{B}_+ \right] (w) \\
+ \frac{1}{(z - w)^3} \left[ \frac{-2 \left( 114 - 853k + 13k^2 \right)}{3(5 + k)^2(19 + 23k)} \theta V^{(2)}(w) \right. \\
- \frac{4 \left( -9 + 11k + k^2 \right)}{(5 + k)^3} \hat{A}_3 \hat{B}_+ + \frac{4k(5 + 2k)}{(5 + k)^3} \hat{B}_i \hat{B}_3 \\
- \frac{2k(5 + 2k)}{(5 + k)^3} i \hat{B}_+ + \frac{2(-3 + 2k)}{(5 + k)^2} i T^{(1)} \hat{B}_+ \right] (w) \\
+ \cdots. \\
\]

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The first order pole in the last OPE of (I.1) contains composite field with spin-4 with $U(1)$ charge of $-\frac{2k}{5+k}$. The second order pole has a composite field of spin-3 starting from the second term. The field contents can be seen from the table 5 in [1]. Note that there is no $T^{(1)} T^{(1)} \hat{B}_s (w)$ term in the second order pole. One can also analyze each singular term in terms of descendant fields plus (quasi) primary fields as done before.
Appendix J. The nontrivial OPEs between higher spin-$\frac{5}{2}$ current, $V^{(\frac{5}{2})}(z)$, and other seven higher spin currents

The corresponding OPEs are given by

\[
V^{(\hat{z})}(z) V^{(\hat{z})}(w) = -\frac{1}{(z - w)^3} \left[ \frac{16(-111 + 7k + 2k^2)}{9(5 + k)^3} \hat{A}_- \hat{B}_3 \right](w) \\
+ \frac{1}{(z - w)^2} \frac{1}{2} \left\{ V^{(\hat{z})} V^{(\hat{z})} \right\}(w) \\
+ \frac{1}{(z - w)} \left[ \frac{2}{(5 + k)} i\hat{A}_- \hat{R}^{(3)} + \ldots \right](w) + \ldots,
\]

\[
V^{(\hat{z})}(z) W^{(2)}(w) = -\frac{1}{(z - w)^3} \left[ \frac{4(-3 + k)}{3(5 + k)^2} \hat{G}_{22} - \frac{4(-3 + k)}{3(5 + k)^2} V^{(\hat{z})}(w) \right] \\
+ \frac{1}{(z - w)^2} \left[ -\frac{(19 + 5k)}{2(5 + k)} \hat{R}^{(2)} + \frac{(22 + 5k)}{(5 + k)} V^{(\hat{z})} - \frac{4(6 + k)}{(5 + k)^2} i\hat{A}_- \hat{G}_{12} \right] \\
+ \frac{4(6 + k)}{3(5 + k)^2} i\hat{A}_- T^{(2)} + \frac{12}{(5 + k)^2} i\hat{A}_1 V^{(\hat{z})} + \frac{4(8 + k)}{(5 + k)^2} i\hat{B}_4 \hat{G}_{21} \\
+ \frac{4(27 + 5k)}{(5 + k)^2} i\hat{B}_4 T^{(2)} + \frac{2}{(5 + k)} T^{(1)} V^{(\hat{z})} + \frac{4(-3 + k)}{(5 + k)^2} \partial \hat{G}_{22} \\
- \frac{4(5 + 2k)}{(5 + k)^2} \partial V^{(\hat{z})}(w) \\
+ \frac{1}{(z - w)} \left[ -\frac{3(3 + k)}{2(5 + k)} \partial \hat{R}^{(2)} - \frac{1}{(5 + k)} i\hat{P}^{(2)} \hat{A}_- + \ldots \right](w) + \ldots,
\]

\[
V^{(\hat{z})}(z) W^{(2)}(w) = -\frac{1}{(z - w)^4} \left[ \frac{16(17 + 17k + 2k^2)}{(5 + k)^3} i\hat{A}_- \right](w) \\
+ \frac{1}{(z - w)^3} \left[ \frac{8(13 + 2k)}{9(5 + k)^2} V^{(2)} + \frac{16(-2 + k)}{(5 + k)^3} \hat{A}_- \hat{A}_3 \right] \\
+ \frac{16(108 + 20k + k^2)}{(5 + k)^3} \hat{A}_- \hat{B}_3 + \frac{8(19 + 16k + 2k^2)}{(5 + k)^3} i\hat{A}_- \\
+ \frac{8(-2 + k)}{(5 + k)^2} iT^{(1)} \hat{A}_- \right](w) + \frac{1}{(z - w)^2} \left[ \frac{37 + 8k}{3(5 + k)} \hat{R}^{(3)} \right] \\
- \frac{1}{(5 + k)} i\hat{P}^{(2)} \hat{A}_- - \frac{4}{(5 + k)} T^{(1)} V^{(2)} \\
- \frac{(35 + 8k)}{(5 + k)^2} iT^{(1)} \partial \hat{A}_- \\
+ \frac{8(4935 + 13246k + 6359k^2 + 720k^3)}{3(5 + k)^2(3 + 7k)(19 + 23k)} i\hat{A}_- \hat{T}
\]


\[-8(17 + 3k) \frac{i\hat{A}_- T^{(2)}}{3(5 + k)^2} - \frac{4(5 + 2k)}{3(5 + k)^2} i\hat{A}_- W^{(2)}\]
\[-2(191 + 40k) \frac{\hat{A}_- \partial \hat{A}_3}{(5 + k)^3}\]
\[-\frac{2(-204 + 13k + 8k^2)}{9(5 + k)^3} \hat{A}_- \partial \hat{B}_3 + \frac{(55 + 16k)}{3(5 + k)^2} i\hat{A}_- \partial T^{(1)}\]
\[-\frac{24}{(5 + k)^2} i\hat{A}_3 V_+^{(2)}\]
\[+ \frac{2(35 + 8k)}{(5 + k)^3} \hat{A}_3 \partial \hat{A}_- - \frac{8(13 + 2k)}{3(5 + k)^2} i\hat{B}_3 V_+^{(2)}\]
\[+ \frac{2(1008 + 233k + 16k^2)}{9(5 + k)^3} \hat{B}_3 \partial \hat{A}_- + \frac{(25 + 14k)}{9(5 + k)^2} \partial V_+^{(2)}\]
\[+ \frac{4(-27 + 15k + 4k^2)}{3(5 + k)^3} i\partial^2 \hat{A}_- + \frac{4(31 + 4k)}{3(5 + k)^2} \hat{G}_{22} \hat{G}_{21}\]
\[+ \frac{8(17 + 3k)}{3(5 + k)^2} \hat{G}_{22} T_+^{(2)} - \frac{4(17 + 2k)}{3(5 + k)^2} \hat{G}_{21} V^{(2)}\]
\[-\frac{8(4 + k)}{3(5 + k)^2} T_+^{(2)} V^{(2)} + \frac{16(17 + 3k)}{3(5 + k)^3} i\hat{A}_- \hat{A}_3 \hat{A}_3\]
\[-\frac{32(17 + 3k)}{3(5 + k)^3} i\hat{A}_- \hat{A}_3 \hat{B}_3 + \frac{40(9 + 2k)}{3(5 + k)^3} i\hat{A}_- \hat{B}_3 \hat{B}_+\]
\[+ \frac{16(17 + 3k)}{3(5 + k)^3} i\hat{A}_- \hat{B}_3 \hat{B}_3 + \frac{16(17 + 3k)}{3(5 + k)^3} i\hat{A}_+ \hat{A}_- \hat{A}_-\]
\[\left(\frac{1}{z - w}\right) \left[\frac{1}{(5 + k)} i\hat{A}_- \partial \hat{P}^{(3)} + \frac{1}{(5 + k)} i\partial \hat{A}_- \hat{P}^{(2)}\right]\]
\[\left[\frac{1}{4(5 + k)} i\hat{A}_- \partial \hat{P}^{(2)} + \frac{(15 + 4k)}{3(5 + k)} \partial \hat{R}^{(4)} + \cdots\right] (w) + \cdots\]

\[V^{(2)}(z) W^{(2)}(w) = \frac{1}{(z - w)^4} \left[\frac{8k(59 + 37k + 4k^2)}{3(5 + k)^3} i\hat{B}_+\right] (w)\]
\[+ \frac{1}{(z - w)^3} \left[\frac{4(-61 + 9k)}{9(5 + k)^2} V^{(2)}_+ + \frac{16(-45 + 35k + 2k^2)}{9(5 + k)^3} \hat{A}_3 \hat{B}_+\right]\]
\[-\frac{32k(2 + k)}{3(5 + k)^3} \hat{B}_3 \hat{B}_3 + \frac{4k(67 + 41k + 4k^2)}{3(5 + k)^3} i\hat{B}_+\]
\[-\frac{16(-4 + k)}{3(5 + k)^2} i\hat{T}^{(1)} \hat{B}_+\]
\[
\frac{1}{(z - w)^2} \left[ \frac{2(16 + 5k)}{3(5 + k)} R^{(3)} + \frac{(-15 + 65k + 27k^2 + 2k^3)}{3k(5 + k)(8 + k)} i\mathbf{p}^{(2)} \hat{B}_s \\
+ \frac{(-15 + 65k + 27k^2 + 2k^3)}{3k(5 + k)(8 + k)} i\mathbf{p}^{(2)} \hat{B}_s \\
+ \frac{4}{(5 + k)} T^{(1) V^{(2)}} - \frac{2(10 + 9k)}{3(5 + k)^2} \partial \hat{B}_s \\
- \frac{8(19 + 8k)}{3(5 + k)^2} i\hat{\mathcal{A}}_1 V^{(2)} - \frac{4\left(402 + 95k + 26k^2\right)}{9(5 + k)^3} \hat{\mathcal{A}}_3 \partial \hat{B}_s \\
+ \frac{4(372 + 337k + 34k^2)}{9(5 + k)^3} \hat{B}_s \partial \hat{\mathcal{A}}_3 \\
+ \frac{8(285 + 3638k + 5936k^2 + 2055k^3 + 140k^4)}{3(5 + k)^2(3 + 7k)(19 + 23k)} \hat{B}_s \hat{T} \\
+ \frac{4(32 + 7k)}{3(5 + k)^3} i\hat{B}_s T^{(2)} - \frac{4(32 + 7k)}{3(5 + k)^2} i\hat{B}_s W^{(2)} \\
- \frac{4\left(-30 + 22k + 7k^2\right)}{3(5 + k)^3} \hat{B}_s \partial \hat{B}_3 + \frac{2(36 + k)}{3(5 + k)^2} i\hat{B}_s \partial T^{(1)} \\
- \frac{8(8 + k)}{3(5 + k)^2} i\hat{B}_3 V^{(2)} - \frac{4k(-6 + k)}{3(5 + k)^3} \hat{B}_3 \partial \hat{B}_e \\
+ \frac{2\left(-240 + 1080k + 438k^2 + 215k^3 + 16k^4\right)}{9k(5 + k)^2(8 + k)} dV^{(2)} \\
+ \frac{4(6 + k)}{3(5 + k)^2} \hat{G}_{12} \hat{G}_{12} - \frac{8(17 + 5k)}{3(5 + k)^2} \hat{G}_{12} T^{(2)} \\
+ \frac{4(7 + k)}{(5 + k)^2} \hat{G}_{12} V^{(2)} - \frac{8(16 + 5k)}{3(5 + k)^2} \hat{T}^{(2)} V^{(2)} \\
+ \frac{40(1 + k)}{3(5 + k)^3} i\hat{\mathcal{A}}_1 \hat{\mathcal{A}}_3 \hat{B}_e + \frac{8(-13 + 4k)}{3(5 + k)^3} i\hat{\mathcal{A}}_3 \hat{\mathcal{A}}_3 \hat{B}_e \\
+ \frac{8}{3(5 + k)^2} i\hat{B}_s \hat{B}_s \hat{B}_s - \frac{8(-5 + 2k)}{3(5 + k)^3} i\hat{B}_s \hat{B}_s \hat{B}_s \\
- \frac{16(-4 + k)}{3(5 + k)^3} i\hat{B}_s \hat{\mathcal{A}}_3 \hat{B}_s - \frac{8}{3(5 + k)^2} T^{(1) \hat{\mathcal{A}}_3 \hat{B}_s} \\
+ \frac{8}{(5 + k)^2} T^{(1) \hat{B}_s \hat{B}_3} \left[(w) + \frac{1}{(z - w)} \left[- \frac{2}{(5 + k)} i\hat{\mathcal{A}}_3 R^{(3)} \right] \\
+ \frac{2}{(5 + k)} i\hat{B}_3 R^{(3)} + \frac{(12 + 5k)}{3(5 + k)} \partial R^{(3)} + \cdots \right](w) \\
+ \cdots,
\]
\[ V^{(3)}(z) W^{(3)}(w) = \frac{1}{(z - w)^3} \left[ \frac{4(2559 + 12378k + 10609k^2 + 2688k^3 + 218k^4)}{3(5 + k)^3(19 + 23k)} \hat{G}_{22} \right. \\
+ \left. \frac{8(-1959 + 1029k + 1328k^2 + 208k^3)}{3(5 + k)^3(19 + 23k)} V^{(2)}(z) \right] (w) \\
+ \frac{1}{(z - w)^3} \left[ \frac{(-4367 + 108k + 795k^2 + 152k^3)}{3(5 + k)^2(19 + 23k)} R^{(2)}(z) \right. \\
- \left. \frac{2(-2139 + 24k + 647k^2 + 76k^3)}{3(5 + k)^2(19 + 23k)} V^{(2)}(z) \right] (w) \\
- \frac{8(-33 + 5k + k^2)}{3(5 + k)^3} i\hat{A}_\_\hat{G}_{22} - \frac{8(29 + 4k)}{3(5 + k)^3} i\hat{A}_\_ T^{(1)}(z) \\
- \frac{8(6 + 25k + 2k^2)}{3(5 + k)^3} i\hat{A}_3 \hat{G}_{22} + \frac{8(-1 + 8k)}{3(5 + k)^3} i\hat{A}_3 V^{(2)}(z) \\
- \frac{8(75 + k)}{3(5 + k)^3} i\hat{B}_4 \hat{G}_{21} + \frac{4(-117 + 7k + 6k^2)}{3(5 + k)^3} i\hat{B}_4 T^{(1)}(z) \\
- \frac{8(34 + 16k + k^2)}{3(5 + k)^3} i\hat{B}_3 V^{(2)}(z) - \frac{4(-3 + 4k)}{3(5 + k)^2} T^{(1)}(z) \hat{G}_{22} \\
- \frac{4(-3 + 4k)}{3(5 + k)^2} T^{(1)}(z) V^{(2)}(z) \\
+ \frac{4(3813 + 13402k + 10049k^2 + 2734k^3 + 218k^4)}{9(5 + k)^3(19 + 23k)} \partial \hat{G}_{22} \\
+ \frac{8(-135 + 3370k + 1413k^2 + 116k^3)}{9(5 + k)^3(19 + 23k)} \partial V^{(2)}(z) \\
+ \frac{8(12 + 17k + 4k^2)}{3(5 + k)^3} i\hat{B}_3 \hat{G}_{22} \right] (w) + \frac{1}{(z - w)^2} \\
\times \left[ \frac{(-11011 + 3804k + 3459k^2 + 364k^3)}{15(5 + k)^2(19 + 23k)} \partial R^{(4)}(z) - \frac{29(7k)}{2(5 + k)} R^{(4)}(z) \right. \\
+ \frac{(25 + k)}{3(5 + k)^2} \partial R^{(4)}(z) \hat{A}_3 - \frac{(25 + k)}{3(5 + k)^2} i\hat{R}^{(4)}(z) \hat{B}_3 \\
- \frac{(37 + 10k)}{3(5 + k)^2} \partial R^{(4)}(z) \hat{A}_3 + \ldots \right] (w) \\
+ \frac{1}{(z - w)} \left\{ V^{(2)}(z) W^{(3)}(w) \right\} \rightarrow (w) + \ldots \]
The first order pole in the first OPE of (J.1) contains composite field with spin-4 with $U(1)$ charge of $\frac{2(-3 + k)}{(5 + k)}$. Although $\hat{G}_{22} \hat{G}_{22}(w)$ can be written in terms of derivative of $\hat{A}_k \cdot \hat{B}_k(w)$, the expression $\hat{G}_{22} \partial \hat{G}_{22}(w)$ is an independent quantity we should consider. Note the $(k - 3)$ factor in the third order pole in the second OPE. The third OPE has a composite field with spin-4 with $U(1)$ charge of $\frac{6}{(5 + k)}$. For the fourth OPE, the corresponding $U(1)$ charge is given by $-\frac{2k}{(5 + k)}$. For the last OPE, the corresponding $U(1)$ charge of spin-$\frac{3}{2}$ is given by $\frac{(-3 + k)}{2}$. For the last OPE, the complete expression will appear in appendix L. The 13 composite fields of spin-$\frac{5}{2}$ in the third order pole can be seen from the table 4 of [1].

Appendix K. The nontrivial OPEs between the fourth $\mathcal{N} = 2$ multiplet in (2.1)

Now the last OPEs corresponding to the last $\mathcal{N} = 2$ multiplet in (2.1) are as follows:

$$ W^{(2)}(z) W_{(2)}(w) = \frac{1}{(z - w)^3} \left[ \frac{18k(4 + k)}{(5 + k)^2} \right] + \frac{1}{(z - w)^2} \left[ \frac{4(4 + k)}{(5 + k)} W^{(2)}(w) \right] + \frac{1}{(z - w)} \left[ \frac{2(4 + k)}{(5 + k)} \partial W^{(2)}(w) \right] + \cdots, $$

$$ W^{(2)}(z) W_{(2)}(w) = \frac{1}{(z - w)^3} \left[ -\frac{8(2 + k)(8 + k)}{3(5 + k)^2} \hat{G}_{21} - \frac{4(-3 + k)}{3(5 + k)^2} T_{(2)}^{(2)} \right] + \frac{1}{(z - w)^2} \left[ -\frac{(-3 + k)}{6(5 + k)} \mathcal{P}_{(2)}^{(2)} + \frac{(31 + 7k)}{3(5 + k)} W_{(2)}^{(2)} \right] - \frac{8(16 + 10k + k^2)}{9(5 + k)^2} \partial \hat{G}_{21} - \frac{4(-3 + k)}{9(5 + k)^2} \partial T_{(2)}^{(2)} + \frac{1}{(z - w)} \left[ -\frac{(-3 + k)}{15(5 + k)} \partial \mathcal{P}_{(2)}^{(2)} - \frac{1}{2} \mathcal{S}_{(2)}^{(2)} \right] + \cdots, $$

$$ W^{(2)}(z) W_{(2)}(w) = \frac{1}{(z - w)^3} \left[ -\frac{8(2 + k)(8 + k)}{3(5 + k)^2} \hat{G}_{12} + \frac{4(-3 + k)}{3(5 + k)^2} T_{(2)}^{(2)} \right] + \frac{1}{(z - w)^2} \left[ -\frac{(-3 + k)}{6(5 + k)} \mathcal{P}_{(2)}^{(2)} + \frac{(31 + 7k)}{3(5 + k)} W_{(2)}^{(2)} \right] - \frac{8(16 + 10k + k^2)}{9(5 + k)^2} \partial \hat{G}_{12} + \frac{4(-3 + k)}{9(5 + k)^2} \partial T_{(2)}^{(2)} + \frac{1}{(z - w)} \left[ -\frac{(-3 + k)}{15(5 + k)} \partial \mathcal{P}_{(2)}^{(2)} + \frac{1}{2} \mathcal{S}_{(2)}^{(2)} \right] + \cdots. $$

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\[
W^{(2)}(z) W^{(3)}(w) = \frac{1}{(z-w)^4} \left[ -\frac{24(169 + 553k + 341k^2 + 29k^3)}{(5+k)^3(19+23k)} \hat{A}_3 \\
- \frac{8k(268 + 775k + 280k^2 + 21k^3)}{(5+k)^3(19+23k)} \hat{B}_3 \\
+ \frac{12(-3+k)(4 - 7k + k^2)}{(5+k)^3(19+23k)} T^{(1)}(w) \right] \\
+ \frac{1}{(z-w)^3} \left[ -\frac{4(21 + 43k + 10k^2)}{(5+k)^2(7+7k)} \hat{T} \\
- \frac{2(-3+k)}{(5+k)^2} T^{(2)} + \frac{2}{(5+k)} W^{(2)} \\
- \frac{4(19 + 7k)}{(5+k)^3} \hat{A}_3 \hat{A}_3 = -\frac{28(1+k)}{(5+k)^3} \hat{A}_3 \hat{A}_3 \\
+ \frac{8(16 + k + k^2)}{(5+k)^3} \hat{A}_3 \hat{B}_3 = -\frac{4(1+k)(7+k)}{(5+k)^3} \hat{B}_3 \hat{B}_3 \\
- \frac{28(1+k)}{(5+k)^3} \hat{B}_3 \hat{B}_3 = -\frac{2}{(5+k)} T^{(1)} T^{(1)} \\
- \frac{4(19 + 7k)}{(5+k)^3} i\hat{A}_3 \hat{A}_3 = -\frac{4(1+k)(7+k)}{(5+k)^3} i\hat{B}_3 \hat{B}_3 \\
+ \frac{4(-3+k)}{(5+k)^2} T^{(1)} \hat{A}_3 + \frac{4(-3+k)}{(5+k)^2} i T^{(1)} \hat{B}_3 \right] (w) \\
+ \frac{1}{(z-w)^2} \left[ -\frac{(13 + 3k)}{(5+k)} \hat{T} = -\frac{2}{(5+k)} i P^{(2)} \hat{B}_3 \\
+ \frac{2}{(5+k)} i P^{(2)} \hat{A}_3 + W^{(3)} \\
+ \frac{8(-84 - 199k + 63k^2 + 14k^3)}{(5+k)(3+7k)(19+23k)} \hat{T}^{(1)} \hat{T} \\
- \frac{4}{(5+k)} T^{(1)} T^{(2)} - \frac{16(-3+k)}{(19+23k)} T^{(1)} W^{(2)} \\
- \frac{2(2+k)}{(5+k)^2} i T^{(1)} \hat{A}_3 = -\frac{2(23+4k)}{(5+k)^2} i T^{(1)} \hat{B}_3 \\
- \frac{1}{(5+k)} T^{(1)} \hat{A}_3 = -\frac{10}{(5+k)} i \hat{A}_+ V^{(2)} \\
- \frac{4(19 + 4k)}{(5+k)^3} \hat{A}_+ \hat{A}_- = -\frac{2(7+k)}{(5+k)^3} i \hat{A}_- U^{(2)} \right].
\]
\[-\frac{16 \left(1965 + 5170k + 2971k^2 + 334k^3\right)}{3(5 + k)^3(3 + 7k)(19 + 23k)} \hat{\mathcal{A}}_3 \hat{T} + \frac{40}{3(5 + k)} i \hat{\mathcal{A}}_3 T^{(2)} + \frac{8(13 + 5k)}{(5 + k)(19 + 23k)} i \hat{\mathcal{A}}_3 W^{(2)} \]
\[+ \frac{2(3 + 2k)}{(5 + k)^2} i \hat{\mathcal{A}}_3 \partial T^{(1)} - \frac{(47 + 11k)}{(5 + k)^2} i \hat{\mathcal{B}}_+ U^{(2)}_+ \]
\[- \frac{(116 + 181k + 27k^2)}{3(5 + k)^3} \hat{\mathcal{B}}_+ \partial \hat{\mathcal{B}}_- + \frac{5(1 + k)}{(5 + k)^2} i \hat{\mathcal{B}}_- V^{(2)}_- \]
\[+ \frac{(-40 + 127k + 21k^2)}{3(5 + k)^3} \hat{\mathcal{B}}_- \partial \hat{\mathcal{B}}_+ \]
\[- \frac{16 \left(285 + 2735k + 3143k^2 + 1031k^3 + 42k^4\right)}{3(5 + k)^3(3 + 7k)(19 + 23k)} i \hat{\mathcal{B}}_+ \hat{T} \]
\[- \frac{16(17 + k)}{3(5 + k)^3} i \hat{\mathcal{B}}_+ T^{(2)} + \frac{8 \left(247 + 256k + 49k^2\right)}{(5 + k)^2(19 + 23k)} i \hat{\mathcal{B}}_+ W^{(2)}_+ \]
\[4 \left(113 + 109k + 15k^2\right) i \hat{\mathcal{B}}_+ \partial \hat{\mathcal{A}}_3 \]
\[+ \frac{10(4 + k)}{(5 + k)^2} i \hat{\mathcal{B}}_+ \partial T^{(1)} - \frac{2(13 + k)}{(5 + k)^2} \partial T^{(1)} - \frac{2(121 + 15k)}{(5 + k)^3} i \hat{\mathcal{A}}_3 \]
\[- \frac{2 \left(19 + 182k + 22k^2\right)}{3(5 + k)^3} \partial^2 \hat{\mathcal{B}}_+ + \frac{2(-27 + k)}{3(5 + k)^2} g^2 T^{(1)} \]
\[+ \frac{4(10 + k)}{(5 + k)^2} \hat{G}_{12} \hat{G}_{21} - \frac{2(-3 + k)}{(5 + k)^2} \hat{G}_{12} T^{(2)}_+ \]
\[+ \frac{2(29 + 5k)}{(5 + k)^2} \hat{G}_{21} T^{(2)}_+ - \frac{2 \left(62 + 25k\right)}{3(5 + k)^3} i \hat{\mathcal{A}}_+ \hat{\mathcal{A}}_3 \hat{\mathcal{A}}_3 \]
\[- \frac{16(5 + 7k)}{3(5 + k)^3} i \hat{\mathcal{A}}_+ \hat{\mathcal{A}}_3 \hat{\mathcal{B}}_3 - \frac{2 \left(61 + 8k\right)}{3(5 + k)^3} i \hat{\mathcal{A}}_- \hat{\mathcal{A}}_+ \hat{\mathcal{A}}_3 \]
\[- \frac{14(11 + k)}{3(5 + k)^3} i \hat{\mathcal{A}}_3 \hat{\mathcal{A}}_+ \hat{\mathcal{A}}_- - \frac{8(59 + 10k)}{3(5 + k)^3} i \hat{\mathcal{A}}_3 \hat{\mathcal{A}}_3 \hat{\mathcal{A}}_3 \]
\[+ \frac{24(16 + k)}{(5 + k)^3} i \hat{\mathcal{A}}_3 \hat{\mathcal{A}}_3 \hat{\mathcal{B}}_3 \]
\[- \frac{2 \left(124 + 107k + 12k^2\right)}{3(5 + k)^3} i \hat{\mathcal{A}}_3 \hat{\mathcal{B}}_+ \hat{\mathcal{B}}_-]
\[
- \frac{200}{(5 + k)^3} \hat{\Lambda}_3 \hat{B}_3 \hat{B}_3 \\
+ 2 \left( -148 + 43k + 12k^2 \right) i \hat{B}_- \hat{\Lambda}_3 \hat{B}_+ \\
+ \frac{2(19 + 8k)}{3(5 + k)^3} i \hat{B}_- \hat{B}_+ \hat{B}_3 \\
- \frac{2(59 + 16k)}{3(5 + k)^3} i \hat{B}_3 \hat{B}_- \hat{B}_- + \frac{8(-10 + k)}{3(5 + k)^3} i \hat{B}_3 \hat{B}_3 \hat{B}_3 \\
+ \frac{8}{(5 + k)^2} T^{(1)} \hat{\Lambda}_+ \hat{\Lambda}_- + \frac{4}{(5 + k)^2} T^{(1)} \hat{\Lambda}_1 \hat{\Lambda}_3 \\
+ \frac{8}{(5 + k)^2} T^{(1)} \hat{\Lambda}_3 \hat{B}_3 - \frac{12}{(5 + k)^2} T^{(1)} \hat{B}_3 \hat{B}_3 \\
+ \frac{4(13 + 3k)}{(5 + k)^2} T^{(2)} T^{(2)} \\
+ \frac{1}{(z - w)} \left[ \frac{1}{(5 + k)} i \hat{\Lambda}_3 \partial \mathbf{P}^{(2)} \right] \\
+ \frac{1}{(5 + k)} i \hat{\Lambda}_3 \mathbf{Q}^{(3)} + \frac{2}{(5 + k)} i \hat{B}_3 \mathbf{S}^{(3)} \\
+ \frac{2}{(5 + k)} i \hat{B}_3 \mathbf{P}^{(3)} - \frac{1}{(5 + k)} \hat{B}_3 \hat{B}_3 \mathbf{P}^{(2)} \\
+ \frac{1}{(5 + k)} i \hat{B}_3 \partial \mathbf{P}^{(2)} - \frac{1}{(5 + k)} i \hat{B}_3 \mathbf{Q}^{(3)} \\
+ \frac{1}{2(5 + k)} \hat{B}_+ \hat{B}_- \mathbf{P}^{(2)} + \frac{1}{2(5 + k)} i \partial \hat{B}_+ \mathbf{P}^{(2)} \\
- \frac{(9 + 2k)}{2(5 + k)} \partial \mathbf{S}^{(3)} + \frac{3}{2(5 + k)} \partial \mathbf{P}^{(3)} + \ldots \right] + \ldots,
\]

\[
W_+^{(2)}(z) W_+^{(2)}(w) = - \frac{1}{(z - w)^3} \left[ \frac{16 \left( 131 + 51k + 4k^2 \right)}{9(5 + k)^3} \hat{\Lambda}_- \hat{B}_- \right] \left( w \right) \\
- \frac{1}{(z - w)^2} \left[ \frac{4 \left( 131 + 51k + 4k^2 \right)}{9(5 + k)^2} \hat{G}_{21} \hat{G}_{21} \right] \left( w \right) \\
+ \frac{1}{(z - w)^3} \frac{4 \left( 51 + 24k + 2k^2 \right) i \partial \hat{\Lambda}_- U_+^{(2)}}{3(5 + k)^2} + \frac{4(7 + 4k)}{3(5 + k)^2} i \partial \hat{\Lambda}_- U_+^{(2)} \\
- \frac{8(1 + k)}{3(5 + k)^2} i \hat{\Lambda}_- \partial U_+^{(2)} - \frac{4 \left( 51 + 24k + 2k^2 \right)}{3(5 + k)^3} \partial^2 \hat{\Lambda}_- \hat{B}_- 
\]
\[
\begin{align*}
&+ \frac{4(-27 + 8k + 2k^2)}{3(5 + k)^3} \partial\hat{\Lambda}_- \partial\hat{B}_- - \frac{4(48 + 19k + 2k^2)}{3(5 + k)^3} \hat{\Lambda}_- \partial^2 \hat{B}_- \\
&- \frac{4(7 + 2k)}{3(5 + k)^2} i\partial\hat{B}_- V_+^{(2)} + \frac{4(11 + k)}{3(5 + k)^2} i\hat{B}_- \partial V_+^{(2)} \\
&+ \frac{4}{(5 + k)} \hat{G}_{21} W_+^{(2)} - \frac{8(9 + k)}{3(5 + k)^2} \partial \hat{G}_{21} T_+^{(2)} \\
&+ \frac{8(9 + k)}{3(5 + k)^2} \hat{G}_{21} \partial T_+^{(2)} + \frac{8(8 + k)}{3(5 + k)^2} \hat{G}_{21} \partial \hat{G}_{21} \\
&+ \frac{8}{(5 + k)} T_+^{(4)} \partial T_+^{(2)}(w) + \cdots,
\end{align*}
\]

\[
W_+^{(2)}(z) W_+^{(2)}(w) = -\frac{1}{(z - w)^3} \left[ \frac{8k \left( 67 + 39k + 4k^2 \right)}{(5 + k)^3} \right]
\]

\[
+ \frac{1}{(z - w)^4} \left[ \frac{8 \left( 67 + 39k + 4k^2 \right)}{(5 + k)^3} \hat{\Lambda}_3 + \frac{8k \left( 67 + 39k + 4k^2 \right)}{3(5 + k)} \hat{B}_3 + \frac{8(-3 + k) T^{(1)}}{3(5 + k)^2} \right](w)
\]

\[
+ \frac{1}{(z - w)^3} \left[ \frac{2(-3 + k)}{3(5 + k)} P^{(2)} - \frac{8 \left( 321 + 1193k + 604k^2 + 48k^3 \right)}{9(5 + k)^2(3 + 7k)} \hat{T} \right]
\]

\[
- \frac{8(-9 + k)(-3 + k)}{9(5 + k)^2} T^{(2)} - \frac{4(-1 + 7k + 2k^2)}{3(5 + k)^2} W^{(2)}
\]

\[
- \frac{8(-19 + 49k + 6k^2)}{9(5 + k)^3} \hat{\Lambda}_4 \hat{\Lambda}_- - \frac{8(71 + 58k + 6k^2)}{9(5 + k)^3} \hat{\Lambda}_3 \hat{\Lambda}_3
\]

\[
+ \frac{16(8 + 97k)}{9(5 + k)^3} \hat{\Lambda}_3 \hat{B}_3 - \frac{8(107 + 25k)}{9(5 + k)^3} \hat{B}_4 \hat{B}_-
\]

\[
- \frac{8(107 + 28k + 12k^2)}{9(5 + k)^3} \hat{B}_3 \hat{B}_3 + \frac{4}{(5 + k)} T^{(1)} T^{(1)}
\]

\[
+ \frac{4 \left( 641 + 253k + 24k^2 \right)}{9(5 + k)^3} i\partial \hat{\Lambda}_3
\]

\[
+ \frac{4 \left( -214 + 151k + 117k^2 + 12k^3 \right)}{9(5 + k)^3} i\partial \hat{B}_3
\]

\[
+ \frac{2(-3 + k)}{3(5 + k)^2} \partial T^{(1)} - \frac{8(-7 + 2k)}{3(5 + k)^2} T^{(1)} \hat{\Lambda}_3
\]
\[-\frac{8(11 + 4k) i T^{(1)} \hat{B}_3}{3(5 + k)^2}\]
\[+ \frac{1}{(z-w)^2}\left(\frac{13 + 3k}{5 + k}\right) S^{(3)} + \frac{(-3 + k)}{3(5 + k)} p^{(3)} \]
\[+ \frac{(-3 + k)}{3(5 + k)} \partial p^{(2)} + \frac{2}{5 + k} p^{(2)} \hat{B}_3 \]
\[- \frac{2}{5 + k} \partial p^{(2)} \hat{A}_3 - \frac{2(26 + 5k)}{3(5 + k)} W^{(3)} \]
\[-\frac{4\left(-135 - 348k + 307k^2\right)}{3(5 + k)(3 + 7k)(19 + 23k)} T^{(1)}\hat{T} + \frac{4}{(5 + k)} T^{(1)} T^{(2)} \]
\[+ \frac{10}{(5 + k)^2} iT^{(1)} \hat{A}_3 + \frac{10(18 + k)}{3(5 + k)^2} iT^{(1)} \partial \hat{B}_3 \]
\[+ \frac{4}{(5 + k)} T^{(1)} \partial T^{(1)} - \frac{4(37 + 7k)}{3(5 + k)^2} i\hat{A}_\Lambda V^{(2)} \]
\[-\frac{4\left(-131 + 92k + 12k^2\right)}{9(5 + k)^3} \hat{A}_\Lambda \partial \hat{A}_\Lambda + \frac{4(11 + 2k)}{3(5 + k)^2} i\hat{A}_\_ U^{(2)} \]
\[+ \frac{8\left(5217 + 14234k + 8201k^2 + 1088k^3\right)}{3(5 + k)^2(3 + 7k)(19 + 23k)} i\hat{A}_\_ \hat{T} \]
\[-\frac{40}{3(5 + k)} i\hat{A}_\_ \hat{T}^{(2)} - \frac{8}{(5 + k)} i\hat{A}_\_ W^{(2)} \]
\[+ \frac{4(421 + 524k + 42k^2)}{9(5 + k)^3} \hat{A}_3 \partial \hat{B}_3 = \frac{2(-1 + 8k)}{3(5 + k)^2} i\hat{A}_3 \partial T^{(1)} \]
\[+ \frac{8(17 + 4k)}{3(5 + k)^2} i\hat{B}_4 U^{(2)} + \frac{2\left(-302 + 101k + 12k^2\right)}{9(5 + k)^3} \hat{B}_3 \partial \hat{B}_3 \]
\[-\frac{4(5 + 4k)}{3(5 + k)^2} i\hat{B}_- V^{(2)} = \frac{2\left(-20 + 63k + 4k^2\right)}{3(5 + k)^3} \hat{B}_- \partial \hat{B}_+ \]
\[+ \frac{8\left(570 + 6223k + 8340k^2 + 2779k^3 + 140k^4\right)}{3(5 + k)^2(3 + 7k)(19 + 23k)} i\hat{B}_4 \hat{T} \]
\[+ \frac{16(17 + k)}{3(5 + k)^2} i\hat{B}_3 T^{(2)} \]
\[-\frac{8\left(13 + 3k\right)}{(5 + k)^2} i\hat{B}_3 W^{(2)} = \frac{2(46 + 21k)}{3(5 + k)^2} - i\hat{B}_3 \partial T^{(1)} \]
\[-\frac{2\left(363 + 1636k + 1193k^2 + 96k^3\right)}{9(5 + k)^2(3 + 7k)} \partial \hat{T} \]
\[-\frac{(-3 + k)(-27 + 4k)}{9(5 + k)^2} \partial T^{(2)} \]

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\[- \frac{(-57 + 5k + 4k^2)}{3(5 + k)^3} \partial W^{(2)} + \frac{2 \left( 1451 + 241k + 12k^2 \right)}{9(5 + k)^3} i \partial^2 \hat{B}_3^k + \frac{2 \left( -181 + 594k + 120k^2 + 8k^3 \right)}{9(5 + k)^3} i \partial^2 \hat{B}_3^k - \frac{4 \left( -39 + k \right)}{9(5 + k)^2} \partial^2 T^{(1)} \]
\[- \frac{4 \left( 10 + k \right)}{(5 + k)^2} \hat{G}_{12} \hat{T}_{21} + \frac{4 \left( -3 + k \right)}{3(5 + k)^2} \hat{G}_{12} T^{(2)}_+ - \frac{4 \left( 45 + 7k \right)}{3(5 + k)^2} \hat{G}_{21} T^{(2)}_+ \]
\[+ \frac{2 \left( 20 + 19k \right)}{3(5 + k)^3} i \hat{A}_+ \hat{A}_- \hat{A}_3^k - \frac{2 \left( 107 - 32k + 42k^2 \right)}{9(5 + k)^3} i \hat{A}_+ \hat{A}_- \hat{B}_3^k \]
\[+ \frac{2 \left( 349 + 170k + 12k^2 \right)}{9(5 + k)^3} i \hat{A}_- \hat{A}_+ \hat{A}_3^k + \frac{2 \left( 227 + 136k + 42k^2 \right)}{9(5 + k)^3} i \hat{A}_- \hat{A}_+ \hat{B}_3^k \]
\[- \frac{2 \left( -191 + 107k + 12k^2 \right)}{9(5 + k)^3} i \hat{A}_+ \hat{A}_- \hat{A}_3^k + \frac{8 \left( 59 + 10k \right)}{3(5 + k)^3} i \hat{A}_+ \hat{A}_- \hat{A}_3^k \]
\[- \frac{24 \left( 16 + k \right)}{(5 + k)^3} i \hat{A}_+ \hat{A}_- \hat{B}_3^k + \frac{200}{(5 + k)^3} i \hat{A}_+ \hat{A}_- \hat{B}_3^k + \frac{32 \left( 17 + 4k \right)}{3(5 + k)^3} i \hat{B}_+ \hat{A}_- \hat{B}_+^k \]
\[+ \frac{2 \left( 181 + 62k + 24k^2 \right)}{9(5 + k)^3} i \hat{B}_- \hat{B}_+ \hat{B}_3^k - \frac{2 \left( 61 + 38k + 24k^2 \right)}{9(5 + k)^3} i \hat{B}_- \hat{B}_+ \hat{B}_3^k \]
\[- \frac{8 \left( -10 + k \right)}{3(5 + k)^3} i \hat{B}_- \hat{B}_+ \hat{B}_3^k - \frac{8}{(5 + k)^2} T^{(1)} \hat{A}_+ \hat{A}_- \hat{A}_3^k - \frac{4}{(5 + k)^2} T^{(1)} \hat{A}_+ \hat{A}_3^k \]
\[- \frac{8 \left( 13 + 3k \right)}{(5 + k)^2} T^{(2)}_+ T^{(2)}_+ \]
\[
W_z^{(2)}(z) W^3(w) = \frac{1}{(z-w)^2} \left[ -\frac{4(1833 + 15134k + 13873k^2 + 3214k^3 + 218k^4)}{3(5 + k)^3(19 + 23k)} \hat{G}_{21}^{(2)}(w) \right]
\]
\[
- \frac{4(-3+k)\left( 2249 + 2616k + 487k^2 \right)}{3(5 + k)^3(19 + 23k)} T_+^{(2)}(w)
\]
\[
+ \frac{1}{(z-w)^2} \left[ \frac{(-3+k)\left( 1861 + 1113k + 152k^2 \right)}{3(5 + k)^3(19 + 23k)} \hat{p}_+^{(2)} \right]
\]
\[
+ \frac{2(-2361 - 1292k + 193k^2 + 76k^3)}{3(5 + k)^3(19 + 23k)} W_+^{(2)}
\]
\[
- \frac{8(-9 + 6k + k^2)}{3(5 + k)^3} i\hat{A}_{-} \hat{G}_{11}^{(2)} - \frac{8\left( 13 + 11k + k^2 \right)}{3(5 + k)^3} i\hat{A}_{-} U^{(2)}
\]
\[
+ \frac{8(-22 + 3k)}{(5 + k)^3} i\hat{A}_1 \hat{G}_{21}^{(2)}
\]
\[
- \frac{4\left( -9 + 9k + 2k^2 \right)}{3(5 + k)^3} i\hat{B}_{-} \hat{G}_{22}^{(2)}
\]
\[
+ \frac{4\left( 41 + 17k + 2k^2 \right)}{3(5 + k)^3} i\hat{B}_{-} V^{(2)}
\]
\[
- \frac{8\left( 52 - 16k + k^2 \right)}{3(5 + k)^3} i\hat{B}_3 \hat{G}_{21}^{(2)}
\]
\[
+ \frac{8\left( -37 + 12k + 3k^2 \right)}{3(5 + k)^3} i\hat{B}_3 T_+^{(2)}
\]
\[
+ \frac{4(-36 + k)T_+^{(1)} \hat{G}_{21}^{(2)}}{3(5 + k)^3} - \frac{4\left( 13 + 3k \right) T_+^{(1)} T_+^{(2)}}{(5 + k)^2}
\]
\[
+ \frac{8\left( -59 - k + 4k^2 \right)}{3(5 + k)^3} i\hat{A}_1 T_+^{(2)}
\]
\[
- \frac{4\left( 8787 + 23590k + 13881k^2 + 3168k^3 + 218k^4 \right)}{9(5 + k)^3(19 + 23k)} \partial \hat{G}_{21}^{(2)}
\]
\[
- \frac{4(-3 + k)\left( 501 + 158k + 73k^2 \right)}{9(5 + k)^3(19 + 23k)} \partial T_+^{(2)}(w)
\]
+ \frac{1}{(z - w)^2} \left[ 2(33 + 7k) \mathfrak{i} \mathbf{R}(\xi) \mathbf{B}_- \\
+ \frac{34 + 7k}{2(5 + k)} \mathbf{S}_+^2 (\xi) - \frac{(37 + 9k)}{(5 + k)^2} \mathbf{P}^2_+ (\xi) \mathbf{A}_3 + \frac{(37 + 9k)}{(5 + k)^2} \mathbf{P}^2_+ (\xi) \mathbf{B}_3 \\
+ \frac{4(15 + 4k)}{3(5 + k)^2} \mathbf{Q}(\xi) \mathbf{A}_- \\
+ \frac{2(-3 + k)}{15(5 + k)^2(19 + 23k)} \partial \mathbf{P}^4_+ + \cdots \right] (w) \\
+ \frac{1}{(z - w)} \left\{ \mathbf{W}_+^{(2)} \mathbf{W}_-^{(3)} \right\}_{-1} (w) + \cdots, \quad \text{(K.1)}

\mathbf{W}_+^{(2)} (z) \mathbf{W}_-^{(2)} (w) = - \frac{1}{(z - w)^3} \left[ 16 \left( 131 + 51k + 4k^2 \right) \mathbf{A}_+ \mathbf{B}_+ \right] (w) \\
- \frac{1}{(z - w)^3} \left[ 4 \left( 131 + 51k + 4k^2 \right) \mathbf{G}_{12} \mathbf{G}_{12} \right] (w) \\
+ \frac{1}{(z - w)} \left[ \frac{4}{(5 + k)} \mathbf{U}^{(2)} \mathbf{V}^{(2)} + \frac{4(1 + 4k)}{3(5 + k)^2} i \partial \mathbf{A}_+ \mathbf{V}^{(2)} - \frac{8(4 + k)}{3(5 + k)^2} i \mathbf{A}_+ \partial \mathbf{V}^{(2)} \right] \\
\mathbf{A}_+ \partial \mathbf{B}_+ - \frac{4}{3(5 + k)^3} \mathbf{A}_+ \partial^2 \mathbf{B}_+ \\
- \frac{4(13 + 2k)}{3(5 + k)^2} \partial \mathbf{B}_+ \mathbf{U}^{(2)} \right] + \frac{4}{3(5 + k)} i \partial \mathbf{A}_+ \partial \mathbf{U}^{(2)} \\
+ \frac{8(9 + k)}{3(5 + k)^2} \mathbf{G}_{12} \mathbf{W}_+^{(2)} + \frac{8(9 + k)}{3(5 + k)^2} i \mathbf{G}_{12} \partial \mathbf{T}_+^{(2)} \\
- \frac{8(9 + k)}{3(5 + k)^2} \mathbf{G}_{12} \partial \mathbf{G}_{12} - \frac{8(9 + k)}{3(5 + k)^2} \mathbf{G}_{12} \mathbf{G}_{12} \\
+ \frac{8}{(5 + k)} \mathbf{T}_+^{(2)} \partial \mathbf{T}_+^{(2)} \right] (w) + \cdots.

\mathbf{W}_+^{(2)} (z) \mathbf{W}_-^{(3)} (w) = - \frac{1}{(z - w)^4} \left[ 4 \left( -1587 + 10766k + 13597k^2 + 3214k^3 + 218k^4 \right) \mathbf{G}_{12} \\
- \frac{4(-3 + k)}{3(5 + k)^3(19 + 23k)} \mathbf{T}_+^{(2)} \right] (w)
\[ + \frac{1}{(z - w)^2} \left[ \frac{(3 + k) \left( 1823 + 1067k + 152k^2 \right)}{3(5 + k)^2(19 + 23k)} \right] \mathbf{p}^{(2)} \]
\[ - \frac{2 \left( -2551 - 1598k + 101k^2 + 76k^3 \right)}{3(5 + k)^2(19 + 23k)} W^{(2)} \]
\[ + \frac{8 \left( 41 + 14k + k^2 \right)}{3(5 + k)^3} i\hat{A}_w \hat{G}_{22} \]
\[ + \frac{8 \left( 11 + 8k + k^2 \right)}{3(5 + k)^3} i\hat{A}_w V^{(w)} + \frac{40(-7 + k)}{3(5 + k)^3} i\hat{A}_3 \hat{G}_{12} \]
\[ + \frac{8(29 + k)}{3(5 + k)^3} \hat{A}_3 T^{(w)} \]
\[ + \frac{8(17 + 2k + k^2)}{3(5 + k)^3} i\hat{B}_3 T^{(w)} \]
\[ + \frac{4(9 + 17k + 2k^2)}{3(5 + k)^3} i\hat{B}_3 U^{(w)} \]
\[ - \frac{8(-12 - 3k + k^2)}{3(5 + k)^3} i\hat{B}_3 \hat{G}_{12} \]
\[ + \frac{4(19 + 17k + 2k^2)}{3(5 + k)^3} i\hat{B}_3 T^{(w)} \]
\[ + \frac{4(-15 + k)}{3(5 + k)^2} \tau^{(1)} \hat{G}_{12} \]
\[ + \frac{4}{(5 + k)} \tau^{(1)} T^{(w)} \]
\[ + \frac{4(3695 + 18604k + 15459k^2 + 3352k^3 + 218k^4)}{9(5 + k)^3(19 + 23k)} \partial \hat{G}_{12} \]
\[ - \frac{4(-3 + k)(2743 + 3328k + 625k^2)}{9(5 + k)^3(19 + 23k)} \tau^{(w)} \]
\[ + \frac{1}{(z - w)^2} \left[ \frac{2(-3 + k)(2611 + 1413k + 182k^2)}{15(5 + k)^2(19 + 23k)} \partial \mathbf{p}^{(2)} \right] \]
\[ - \frac{(32 + 7k)}{2(5 + k)} S^{(w)} + \frac{(27 + 7k)}{(5 + k)^2} \mathbf{p}^{(1)} \hat{A}_3 \]
\[ - \frac{(27 + 7k)}{(5 + k)^2} \mathbf{p}^{(1)} \hat{B}_3 + \ldots \{w\} \]
\[ + \frac{1}{(z - w)^2} \left\{ W^{(2)} W^{(3)} \right\}_{w-1} (w) + \ldots, \quad (K.2) \]
\begin{align*}
&+ \frac{4}{3(5 + k)^3(3 + 7k)(19 + 23k)^2} \\
&\left( 134764k^6 + 1883036k^5 + 9239691k^4 + 16022216k^3 \\
&+ 9216682k^2 - 412572k - 986841 \right) \tilde{T} \\
&+ \frac{8(-3 + k) \left( -1438 - 2187k - 414k^2 + 7k^3 \right)}{3(5 + k)^3(19 + 23k)} T^{(2)} \\
&+ \frac{8(3950 + 3052k + 1000k^2 + 133k^3 + 7k^4)}{(5 + k)^3(19 + 23k)} W^{(2)} \\
&+ \frac{64 \left( -1617 - 212k + 2034k^2 + 627k^3 + 50k^4 \right)}{3(5 + k)^4(19 + 23k)} \hat{A}_+ \hat{A}_- \\
&+ \frac{8 \left( 9200k^5 + 142885k^4 + 605694k^3 \\
&+ 881935k^2 + 661760k + 206826 \right)}{3(5 + k)^4(19 + 23k)^2} \hat{A}_3 \hat{A}_3 \\
&+ \frac{16 \left( 4922k^5 + 102226k^4 + 573765k^3 \\
&+ 878806k^2 + 262283k - 74670 \right)}{3(5 + k)^3(19 + 23k)^2} \hat{A}_+ \hat{B}_3 \\
&+ \frac{4 \left( 428k^4 + 6675k^3 + 34401k^2 + 28441k - 17313 \right)}{3(5 + k)^4(19 + 23k)} \hat{B}_+ \hat{B}_- \\
&+ \frac{4 \left( 25093k^5 + 277883k^4 + 135973k^3 \\
&+ 1631252k^2 + 304573k - 328947 \right)}{3(5 + k)^4(19 + 23k)^2} \\
&\times \hat{B}_1 \hat{B}_3 - \frac{\left( 608k^4 + 7295k^3 + 38991k^2 + 97249k + 24913 \right)}{3(5 + k)^3(19 + 23k)^2} T^{(1)} T^{(1)} \\
&\hat{A}_3 \\
&+ \frac{4 \left( 428k^4 + 6675k^3 + 34401k^2 + 28441k - 17313 \right)}{3(5 + k)^4(19 + 23k)} i\hat{A}_3 \\
&+ \frac{4 \left( 50k^4 + 627k^3 + 2034k^2 - 212k - 1617 \right)}{3(5 + k)^3(19 + 23k)} i\hat{B}_3 \\
&+ \frac{8 \left( 58k^4 + 3302k^3 - 9909k^2 - 67948k - 38075 \right)}{(5 + k)^3(19 + 23k)^2} \hat{A}_3 \\
&+ \frac{8 \left( 58k^4 + 3302k^3 - 9909k^2 - 67948k - 38075 \right)}{(5 + k)^3(19 + 23k)^2} \hat{B}_3 \\
&+ \frac{8 \left( 58k^4 + 3302k^3 - 9909k^2 - 67948k - 38075 \right)}{(5 + k)^3(19 + 23k)^2} \hat{A}_3 \\
&+ \frac{8 \left( 58k^4 + 3302k^3 - 9909k^2 - 67948k - 38075 \right)}{(5 + k)^3(19 + 23k)^2} \hat{B}_3
\end{align*}
In the second OPE of (K.3), the \((k - 3)\) factor appears in \(T^{(2)}_w\) and \(P^{(2)}_w\) (and their descendant fields). Similar behavior appears in the third OPE. One sees also \((k - 3)\) factor in the several places of singular terms. The first order pole in the seventh OPE of (K.3) contains a composite field with spin-4 with vanishing \(U(1)\) charge. The \((k - 3)\) factor appears in \(T^{(2)}_w\) and \(P^{(2)}_w\) (and their descendant fields). The first order pole in the eighth OPE contains a composite field with spin-4 with vanishing \(U(1)\) charge. The \((k - 3)\) factor appears in \(T^{(2)}_w\) and \(P^{(2)}_w\) (and their descendant fields). The first order pole in the ninth OPE contains a composite field with spin-4 with vanishing \(U(1)\) charge.
appendix L. Also the 13 composite fields of spin-$\frac{5}{2}$ in the third order pole are seen from the table 4 of [1]. The $(k - 3)$ factor appears in $T^{(2)}(w)$ and $P^{(2)}(w)$. The second order pole in the last OPE of (K.3) contains a composite field with spin-4 with vanishing $U(1)$ charge. Also the last OPE has a term of the first order pole in (2.14) where the higher spin-4 current was constructed. The $(k - 3)$ factor appears in $P^{(2)}(w)$ and $T^{(2)}(w)$ terms.

Appendix L. The composite fields with spin-$\frac{9}{2}$

The first order singular term in the last OPE of (F.1) can be summarized by the following complicated expression

$$
\left\{ U^{(2)} W^{(3)} \right\}_{-1}(w) = \left[ \frac{2i}{(k + 5)} \hat{A}_3 \hat{Q}^{(2)} \right]
+ \frac{8}{(k + 5)^2 d(k)} \left( 140k^4 + 35816k^3 + 133397k^2 
+ 128682k + 39105 \hat{A}_3 \hat{B}_3 \hat{Q}^{(2)} \right)
+ \frac{4}{(k + 5)^2 d(k)} \left( 140k^4 + 35816k^3 + 133397k^2 
+ 128682k + 39105 \hat{A}_4 \hat{B}_3 \hat{P}^{(2)} \right)
- \frac{2}{(k + 5)^2 d(k)} \left( 140k^4 + 35816k^3 + 133397k^2 
+ 128682k + 39105 \hat{A}_4 \hat{R}^{(2)} \right)
- \frac{4i}{(k + 5)^2 d(k)} \left( 140k^4 + 16496k^3 + 77003k^2 
+ 75822k + 23031 \hat{B}_3 \hat{Q}^{(2)} \right)
+ \frac{4}{(k + 5)^2 d(k)} \left( 140k^5 + 36376k^4 + 218701k^3 
+ 493088k^2 + 395253k + 108198 \right) \hat{B}_3 \hat{B}_3 \hat{Q}^{(2)}
+ \frac{i}{(k + 5)^2 d(k)} \left( 140k^4 + 35816k^3 + 133397k^2 
+ 128682k + 39105 \hat{B}_3 \hat{G}_{11} \hat{P}^{(2)} \right)
+ \frac{12i}{(k + 5)^2 d(k)} \left( 4k + 3)(23k + 19)(35k + 47) \hat{B}_3 \hat{S}^{(2)} \right)
+ \frac{16}{(k + 5)^2 d(k)} \left( 70k^4 + 3418k^3 + 24403k^2 
+ 24696k + 7497 \right) \hat{B}_3 \hat{B}_3 \hat{P}^{(2)}
$$
\[ - \frac{i}{2(k + 5)d(k)} \left( 140k^4 + 35816k^3 + 133397k^2 \right) + 128682k + 39105 \) \( \hat{B}_- \hat{G}_2 \mathcal{P}^{(2)} \) \\
\[ - \frac{2}{(k + 5)^2d(k)} \left( 140k^5 + 36516k^4 + 254517k^3 + 626485k^2 + 523935k + 147303 \right) \hat{B}_- \hat{B}_- \mathcal{Q}^{(2)} + 6 \] \\
\[ + \frac{1}{(k + 5)d(k)} \left( 140k^4 + 35816k^3 + 133397k^2 + 128682k + 39105 \right) \hat{G}_1 \mathcal{S}^{(3)} \] \\
\[ + \frac{1}{(k + 5)d(k)} \left( 140k^4 + 35816k^3 + 133397k^2 + 128682k + 39105 \right) \hat{G}_1 \mathcal{P}^{(3)} \] \\
\[ - \frac{1}{(k + 5)d(k)} \left( 140k^4 + 35816k^3 + 133397k^2 + 128682k + 39105 \right) \hat{G}_1 \mathcal{Q}^{(3)} + 8(k - 3) \frac{(23k + 19)}{5(k + 5)^2} \hat{A}_3 \partial \mathcal{Q}^{(2)} + i \frac{(7k + 11)}{5(k + 5)^2} \hat{A}_3 \partial \mathcal{Q}^{(2)} + \frac{2i}{(k + 5)^2d(k)} \left( 140k^4 + 35816k^3 + 133397k^2 + 128682k + 39105 \right) \hat{A}_+ \partial \mathcal{P}^{(2)} \] \\
\[ - \frac{i}{(k + 5)^2d(k)} \left( 140k^5 + 74876k^4 + 430913k^3 + 860169k^2 + 668739k + 181611 \right) \partial \hat{B}_3 \mathcal{Q}^{(2)} \] \\
\[ - \frac{i}{5(k + 5)^2d(k)} \left( 2660k^5 + 419404k^4 + 2560459k^3 + 6018209k^2 + 4805133k + 1301967 \right) \hat{B}_3 \partial \mathcal{Q}^{(2)} \] \\
\[ - \frac{i}{(k + 5)^2d(k)} \left( 280k^5 - 5228k^4 - 160454k^3 + 456189k^2 - 445800k - 139869 \right) \partial \hat{B}_- \mathcal{P}^{(2)} \]
where we introduce
\[ d(k) \equiv \left( 140k^4 - 2824k^3 + 20609k^2 + 22962k + 6957 \right). \]  
\[ (L.2) \]
Compared to the last three expressions below, the equation \((L.1)\) is rather complicated. The common denominator appearing in the most of the terms denoted by \(d(k)\) in \((L.2)\) is not a simple polynomial of \(k\). As seen in the fusion rules in section 4, this is the only case where the second higher spin currents in \((2.2)\) with integer or half integer spins appear. When one looks at the first couple of terms in \((L.1)\), the spin- \(3/2\) currents of the large \(\mathcal{N} = 4\) nonlinear superconformal algebra are multiplied by the higher spin currents with integer spin.

The first order singular term in the last OPE of \((J.1)\) can be summarized by the following expression
\[
\left\{ V^{(2)} W^{(1)} \right\}_{-1} (w) = \left[ \frac{2i}{(k + 5)} \hat{A}_3 R^{(2)} + \frac{i}{(k + 5)} \hat{A}_+ S^{(2)} - \frac{2i}{(k + 5)} \hat{B}_3 R^{(2)} \right. \\
- \frac{16}{3(k + 5)^2} \hat{B}_1 \hat{B}_1 R^{(2)} - \frac{16}{3(k + 5)^2} \hat{B}_1 \hat{B}_1 P^{(2)} + \frac{8}{3(k + 5)^2} \hat{B}_1 \hat{B}_1 R^{(2)} + \frac{8(k - 3)}{(23k + 19)} T R^{(2)} \\
- \frac{i}{(k + 5)} \hat{A}_1 R^{(2)} + \frac{i(3k + 23)}{5(k + 5)^2} \hat{A}_3 \partial R^{(2)} \\
- \frac{i(2k + 11)}{(k + 5)^2} \hat{A}_+ P^{(2)} - \frac{3i(2k + 9)}{5(k + 5)^2} \hat{A}_+ \partial P^{(2)} \\
+ \frac{i(3k + 23)}{3(k + 5)^2} \partial \hat{B}_3 R^{(2)} - \frac{i(9k + 149)}{15(k + 5)^2} \hat{B}_3 \partial R^{(2)} \\
+ \frac{8i}{3(k + 5)^2} \partial \hat{B}_3 P^{(2)} - \frac{8i}{3(k + 5)^2} \hat{B}_3 \partial P^{(2)} + \left( \frac{152k^3 - 171k^2 - 4370k - 7407}{30(k + 5)^2(23k + 19)} \right) \partial^2 R^{(2)} \\
- \frac{3(k + 3)}{2(k + 5)} \partial R^{(2)} + \cdots \right\} (w). \tag{L.3} 
\]
The \((k - 3)\) factor appears in seventh term in \((L.3)\).
The first order singular term in (K.1) can be summarized by

\[
\left\{ W^{(2)}_k W^{(3)}_k \right\}_{-1} (w) = \left[ \frac{i}{(k+5)} \right. \\
\left. \frac{8(k-3)}{(23k+19)} \right] \left[ \hat{A}_k Q^{(2)}_k + \frac{i}{(k+5)} \hat{B}_k R^{(2)}_k + \frac{8(k-3)}{(23k+19)} \hat{P}^{(2)}_k \right]
\]

The (k - 3) factor appears in the third term of (L.4).

Finally, the first order singular term in (K.2) can be summarized by

\[
\left\{ W^{(2)}_k W^{(3)}_k \right\}_{-1} (w) = \left[ \frac{8(k-3)}{(23k+19)} \right. \\
\left. \frac{152k^3 - 1459k^2 - 12058k - 12879}{30(k + 5)^2(23k + 19)} \right] \left[ \hat{P}^{(2)}_k \right]
\]

The (k - 3) factor appears in the first term in (L.5). Of course, one can reexpress this first order term in terms of several descendant fields with correct numerical relative coefficients coming from each higher singular terms plus several (quasi) primary fields as done in previous many examples.

Appendix M. The coefficients in the OPE (2.14) between the spin-\(3/2\) current and the spin-\(7/2\) current

The \(k\) dependent coefficients appearing in (2.14) are as follows:

\[
c_1 = - \frac{32 \left( -24198 - 29457k + 10669k^2 + 23557k^3 + 3085k^4 \right)}{5(5 + k)^5(19 + 23k)(47 + 35k)}
\]

\[
c_2 = \frac{32 \left( 120642 + 228177k + 152639k^2 + 36707k^3 + 3635k^4 \right)}{15(5 + k)^5(19 + 23k)(47 + 35k)}
\]
\[\begin{align*}
c_3 &= -\frac{16(-5859 - 84466k - 83502k^2 - 22818k^3 + 1885k^4)}{15(5 + k)^3(19 + 23k)(47 + 35k)}, \\
c_4 &= \frac{12(-3 + k)}{5(5 + k)}, \quad c_5 = -\frac{8(-57 + 11k + 6k^2)}{5(5 + k)^3}, \\
c_6 &= -\frac{32(-145224 - 597681k - 737779k^2 - 289037k^3 + 22531k^4 + 21350k^5)}{15(5 + k)^3(3 + 7k)(19 + 23k)(47 + 35k)}, \\
c_7 &= -\frac{8(-9327 + 13091k + 84711k^2 + 37305k^3 + 4900k^4)}{15(5 + k)^3(19 + 23k)(47 + 35k)}, \\
c_8 &= \frac{32(-40371 - 116053k - 94358k^2 - 24257k^3 + 35k^4)}{15(5 + k)^3(19 + 23k)(47 + 35k)}, \\
c_9 &= \frac{16(-136578 - 409469k - 456070k^2 - 172309k^3 + 70k^4)}{15(5 + k)^3(19 + 23k)(47 + 35k)}, \\
c_{10} &= -\frac{32(-51273 + 40960k + 212891k^2 + 122858k^3 + 8680k^4)}{15(5 + k)^3(19 + 23k)(47 + 35k)}, \\
c_{11} &= -\frac{32(48408 + 104977k + 57260k^2 + 15821k^3 + 7210k^4)}{15(5 + k)^3(19 + 23k)(47 + 35k)}, \\
c_{12} &= -\frac{16(96816 + 217991k + 148000k^2 + 45067k^3 + 10298k^4 + 840k^5)}{15(5 + k)^3(19 + 23k)(47 + 35k)}, \\
c_{13} &= -\frac{16(-24198 - 29457k + 10669k^2 + 23557k^3 + 3085k^4)}{5(5 + k)^3(19 + 23k)(47 + 35k)}, \\
c_{14} &= -\frac{16k(120642 + 228177k + 152639k^2 + 36707k^3 + 3635k^4)}{15(5 + k)^3(19 + 23k)(47 + 35k)}, \\
c_{15} &= -\frac{8(-5859 - 84466k - 83502k^2 - 22818k^3 + 1885k^4)}{15(5 + k)^2(19 + 23k)(47 + 35k)}, \\
c_{16} &= -\frac{16(-389 - 39k + 242k^2)}{5(5 + k)^2(19 + 23k)}, \\
c_{17} &= \frac{16(19 - 113k - 84k^2 + 4k^3)}{5(5 + k)^2(19 + 23k)}, \quad c_{18} = \frac{4}{(5 + k)}, \\
c_{19} &= \frac{4}{(5 + k)}, \quad c_{20} = \frac{2(13 + 3k)}{(5 + k)}.
\end{align*}\]
\[
c_{21} = \frac{6(-3 + k)}{5(5 + k)}, \quad c_{22} = -\frac{4(22 + 9k)}{5(5 + k)}, \quad 256\left(-201 - 613k - 199k^2 + 5k^3\right), \\
c_{23} = -\frac{15(5 + k)(19 + 23k)(47 + 35k)}{4\left(1493 + 1115k + 98k^2\right)}, \\
c_{24} = -\frac{4\left(1732 + 2121k + 365k^2 + 12k^3\right)}{5(5 + k)^2(19 + 23k)}i, \\
c_{25} = -\frac{8(8 + 7k)}{5(5 + k)^2}i, \quad c_{27} = -\frac{32}{(5 + k)^2}i, \quad c_{28} = -\frac{16(59 + 7k)}{15(5 + k)^2}i, \\
c_{29} = -\frac{4\left(-131009 - 210397k + 30389k^2 + 85501k^3 + 4620k^4\right)}{15(5 + k)^3(19 + 23k)(47 + 35k)}, \\
c_{30} = -\frac{4\left(535451 + 1125793k + 640141k^2 + 73811k^3 + 4620k^4\right)}{15(5 + k)^4(19 + 23k)(47 + 35k)}i, \\
c_{31} = -\frac{32\left(-213447 - 561982k - 242654k^2 + 62770k^3 + 30905k^4\right)}{15(5 + k)^3(3 + 7k)(19 + 23k)(47 + 35k)}i, \\
c_{32} = -\frac{8\left(43643 + 135346k + 182215k^2 + 78956k^3 + 4620k^4\right)}{15(5 + k)^3(19 + 23k)(47 + 35k)}i, \\
c_{33} = -\frac{16}{(5 + k)}i, \\
c_{34} = -\frac{8\left(-392947 - 520298k + 64729k^2 + 233948k^3 + 39900k^4\right)}{15(5 + k)^3(19 + 23k)(47 + 35k)}i, \\
c_{35} = -\frac{4\left(1091 + 709k + 102k^2\right)}{5(5 + k)^2(19 + 23k)}i, \\
c_{36} = -\frac{8(24 + k)}{5(5 + k)^2}i, \quad c_{37} = \frac{8(-1 + k)}{(5 + k)^2}i, \\
c_{38} = -\frac{4\left(-152213 - 467881k - 461995k^2 - 135887k^3 + 4200k^4\right)}{15(5 + k)^3(19 + 23k)(47 + 35k)}, \\
c_{39} = -\frac{4\left(252229 + 447515k + 208535k^2 + 23425k^3 + 13440k^4\right)}{15(5 + k)^3(19 + 23k)(47 + 35k)}, \\
c_{40} = -\frac{32\left(18753 + 292055k + 584767k^2 + 370207k^3 + 77464k^4 + 3010k^5\right)}{15(5 + k)^3(3 + 7k)(19 + 23k)(47 + 35k)}i.
\[
c_{41} = \frac{64(20 + 7k)}{15(5 + k)^2}, \quad c_{42} = -\frac{16(3 + k)}{(5 + k)^2}, \\
c_{43} = -\frac{12(570 + 725k + 123k^2 + 4k^3)}{5(5 + k)^2(19 + 23k)}, \\
c_{44} = -\frac{8\left(184361 + 676672k + 781117k^2 + 342086k^3 + 48720k^4\right)}{15(5 + k)^3(19 + 23k)(47 + 35k)}, \\
c_{45} = \frac{8\left(227225 + 470020k + 300589k^2 + 56762k^3 + 4200k^4\right)}{15(5 + k)^3(19 + 23k)(47 + 35k)}, \\
c_{46} = \frac{4\left(202221 + 1319940k + 2127646k^2 + 1300340k^3 + 238973k^4 + 10000k^5\right)}{5(5 + k)^2(19 + 23k)(47 + 35k)}, \\
c_{47} = -\frac{2\left(-130131 - 241473k - 146133k^2 - 16163k^3 + 260k^4\right)}{5(5 + k)^2(19 + 23k)(47 + 35k)}, \\
c_{48} = -\frac{2(143 + 37k)}{5(5 + k)^2}, \quad c_{49} = -\frac{8(-13 + 3k)}{5(5 + k)^2}, \quad c_{50} = -\frac{8(37 + 3k)}{5(5 + k)^2}, \\
c_{51} = \frac{16\left(232555 + 471215k + 264827k^2 + 41029k^3 + 2190k^4\right)}{15(5 + k)^3(19 + 23k)(47 + 35k)}, \\
c_{52} = \frac{32\left(-6251 + 55591k + 124930k^2 + 64288k^3 + 4989k^4 + 365k^5\right)}{15(5 + k)^3(19 + 23k)(47 + 35k)}, \\
c_{53} = \frac{8\left(66831 + 131711k + 62169k^2 - 2127k^3 + 40k^4\right)}{15(5 + k)^3(19 + 23k)(47 + 35k)}, \\
c_{54} = \frac{16\left(26898 + 57784k + 36995k^2 + 6638k^3 + 385k^4\right)}{5(5 + k)^2(19 + 23k)(47 + 35k)}, \\
c_{55} = \frac{8\left(-700 - 704k - 117k^2 + 3k^3\right)}{5(5 + k)^2(19 + 23k)}, \\
c_{56} = \frac{8\left(-36184 - 71452k - 47423k^2 - 4530k^3 + 25k^4\right)}{5(5 + k)^2(19 + 23k)(47 + 35k)}, \\
c_{57} = \frac{16\left(2014 + 2485k - 95k^2 + 6k^3\right)}{15(5 + k)^3(19 + 23k)}, \\
c_{58} = \frac{8(11 + 9k)}{5(5 + k)^2}, \quad c_{59} = \frac{32\left(2126 + 2639k + 5k^2\right)}{15(5 + k)^3(19 + 23k)}, \\
c_{60} = \frac{16\left(1687 + 2173k + 10k^2\right)}{15(5 + k)^3(19 + 23k)}.
\[
c_{61} = \frac{32\left(1183 + 1683k + 229k^2 + k^3\right)}{5(5 + k)^3(19 + 23k)},
\]
\[
c_{62} = \frac{16\left(889 + 1207k + 10k^2\right)}{15(5 + k)^3(19 + 23k)},
\]
\[
c_{63} = \frac{32\left(-330 - 153k + 223k^2 + 2k^3\right)}{5(5 + k)^3(19 + 23k)},
\]
\[
c_{64} = \frac{16\left(-243 - 274k + k^2\right)}{5(5 + k)^3(19 + 23k)},
\]
\[
c_{65} = \frac{16\left(-266 + 67k + 319k^2 + 6k^3\right)}{15(5 + k)^3(19 + 23k)},
\]
\[
c_{66} = \frac{32\left(-148 - 159k + k^2\right)}{5(5 + k)^3(19 + 23k)},
\]
\[
c_{67} = \frac{32\left(-133 - 109k - 13k^2 + 3k^3\right)}{15(5 + k)^3(19 + 23k)},
\]
\[
c_{68} = \frac{8\left(-11 + 27k + 2k^2\right)}{5(5 + k)^3(19 + 23k)},
\]
\[
c_{69} = \frac{16\left(-53 - 44k + k^2\right)}{5(5 + k)^2(19 + 23k)},
\]
\[
c_{70} = \frac{8\left(-201 - 203k + 2k^2\right)}{5(5 + k)^2(19 + 23k)},
\]
\[
c_{71} = \frac{24\left(-57 + 11k + 6k^2\right)}{(5 + k)(265 + 149k)},
\]
\[
c_{72} = \frac{8\left(-9327 + 13091k + 84711k^2 + 37305k^3 + 4900k^4\right)}{(5 + k)(19 + 23k)(47 + 35k)(265 + 149k)},
\]
\[
c_{73} = \frac{36\left(-3 + k\right)}{(265 + 149k)},
\]
\[
c_{74} = \frac{32}{(5 + k)(3 + 7k)(19 + 23k)(29 + 25k)(47 + 35k)(155 + 127k)(265 + 149k)}
\times ( - 705062520 - 4044532095k - 8746728739k^2 - 9146242164k^3
\quad - 4619929142k^4 - 717097199k^5 + 225222129k^6 + 67792130k^7 ),
\]
\[
c_{75} = \frac{16\left(-136578 - 409469k - 456070k^2 - 172309k^3 + 70k^4\right)}{(5 + k)^2(19 + 23k)(47 + 35k)(265 + 149k)}.
\]
\begin{align*}
\mathcal{C}_{26} & = - \frac{32(-51273 + 40960k + 212891k^2 + 122858k^3 + 8680k^4)}{(5 + k)^2(19 + 23k)(47 + 35k)(265 + 149k)}, \\
\mathcal{C}_{27} & = \frac{48(-389 - 39k + 242k^3)}{(5 + k)(19 + 23k)(265 + 149k)}, \\
\mathcal{C}_{28} & = - \frac{16(96816 + 217991k + 148000k^2 + 45067k^3 + 10298k^4 + 840k^5)}{(5 + k)^2(19 + 23k)(47 + 35k)(265 + 149k)}, \\
\mathcal{C}_{29} & = \frac{48(19 - 113k - 84k^2 + 4k^3)}{(5 + k)(19 + 23k)(265 + 149k)}, \\
\mathcal{C}_{30} & = \frac{8}{15(5 + k)^2(1 + 5k)(19 + 23k)(29 + 25k)(47 + 35k)} \left( -3993162 - 11110911k - 10230596k^2 \\
& \quad - 1203174k^3 + 1936950k^4 + 387725k^5 \right) i, \\
\mathcal{C}_{31} & = - \frac{45(5 + k)^2(1 + 5k)(19 + 23k)(29 + 25k)(47 + 35k)}{\times \left( 1742688 + 29985702k + 69123135k^2 \\
& \quad + 61531136k^3 + 27509730k^4 + 6413250k^5 + 454375k^6 \right) i, \\
\mathcal{C}_{32} & = - \frac{9(5 + k)(1 + 5k)(19 + 23k)(47 + 35k)}{\times \left( -5859 - 84466k - 83502k^2 - 22818k^3 + 1885k^4 \right)}, \\
\mathcal{C}_{33} & = - \frac{32(-40371 - 116053k - 94358k^2 - 24257k^3 + 35k^4)}{5(5 + k)^2(19 + 23k)(29 + 25k)(47 + 35k)}, \\
\mathcal{C}_{34} & = - \frac{32(48408 + 104977k + 57260k^2 + 15821k^3 + 7210k^4)}{5(5 + k)^2(19 + 23k)(29 + 25k)(47 + 35k)}. \tag{M.1}
\end{align*}

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