Restrictions of the Complementary Series of the Universal
Covering of the Symplectic Group

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Abstract

In this paper, we study the restrictions of the complementary series representation onto
a symplectic subgroup no bigger than half of the size of the original symplectic group.

1 Introduction

Let \( \widetilde{Sp}(n, \mathbb{R}) \) be the universal covering of \( Sp(n, \mathbb{R}) \). \( \widetilde{Sp}(n, \mathbb{R}) \) is a central extension of \( Sp(n, \mathbb{R}) \):

\[
1 \to C \to \widetilde{Sp}(n, \mathbb{R}) \to Sp(n, \mathbb{R}) \to 1,
\]

where \( C \cong \mathbb{Z} \). The unitary dual of \( C \) is parametrized by a torus \( T \). For each \( \kappa \in T \), denote the corresponding unitary character of \( C \) by \( \chi^\kappa \). We say that a representation \( \pi \) of \( \widetilde{Sp}(n, \mathbb{R}) \) is of class \( \kappa \) if \( \pi|_C = \chi^\kappa \). Since \( C \) commutes with \( \widetilde{Sp}(n, \mathbb{R}) \), for any irreducible representation \( \pi \) of \( \widetilde{Sp}(n, \mathbb{R}) \), \( \pi|_C = \chi^\kappa \) for some \( \kappa \).

Denote the projection \( \widetilde{Sp}(n, \mathbb{R}) \to Sp(n, \mathbb{R}) \) by \( p \). For any subgroup \( H \) of \( Sp(n, \mathbb{R}) \), denote the full inverse image \( p^{-1}(H) \) by \( \widetilde{H} \). We adopt the notation from [10]. Let \( P \) be the Siegel parabolic subgroup of \( Sp(n, \mathbb{R}) \). One dimensional characters of \( \widetilde{P} \) can be parametrized by \( (\epsilon, t) \) where \( \epsilon \in T \) and \( t \in \mathbb{C} \). Let \( I(\epsilon, t) \) be the representation of \( \widetilde{Sp}(n, \mathbb{R}) \) induced from the one dimensional character parametrized by \( (\epsilon, t) \) of \( \widetilde{P} \). If \( t \) is purely imaginary, \( I(\epsilon, t) \) is unitary and irreducible. If \( t \) is real, then \( I(\epsilon, t) \) has an invariant Hermitian form. Sahi gives a classification of all irreducible unitarizable \( I(\epsilon, t) \). Besides the unitary principal series, there are complementary series \( C(\epsilon, t) \) for \( t \) in a suitable interval ([10]).

Let \( (Sp(p, \mathbb{R}), Sp(n-p, \mathbb{R})) \) be a pair of symplectic groups diagonally embedded in \( Sp(n, \mathbb{R}) \). Suppose that \( p \leq n - p \). Let \( U(n) \) be a maximal compact subgroup such that \( Sp(n-p, \mathbb{R}) \cap U(n) \) is a maximal compact subgroup of \( Sp(n-p, \mathbb{R}) \). Denote \( Sp(n-p, \mathbb{R}) \cap U(n) \) by \( U(p) \).

**Theorem 1.1.** Suppose that \( p \leq n - p \) and \( C(\epsilon, t) \) is unitary. Then

\[
C(\epsilon, t)|_{\widetilde{U}(n-p)} \cong I(\epsilon, 0),
\]

\[
C(\epsilon, t)|_{\widetilde{U}(n-p)} \cong I(\epsilon, i\lambda),
\]

\( (\lambda \in \mathbb{R}) \).
p = [\frac{n}{2}] is the best possible value for such a statement. In particular, for \( \widetilde{Sp}(2m + 1, \mathbb{R}) \)

\[
I(\epsilon, 0)|_{\widetilde{Sp}(m+1, \mathbb{R})} \not\in C(\epsilon, t)|_{\widetilde{Sp}(m+1, \mathbb{R})}.
\]

To see this, let \( L^2(\widetilde{Sp}(n, \mathbb{R}))_\kappa \) be the set of functions with

\[
f(zg) = \mu^\kappa(z)f(g) \quad (z \in C, g \in \widetilde{Sp}(n, \mathbb{R}));
\]

\[
\|f\|^2 = \int_{Sp(n, \mathbb{R})} |f(g)|^2 d|g| \quad (g \in \widetilde{Sp}(n, \mathbb{R}), \mu \in Sp(n, \mathbb{R})).
\]

We say that a representation of class \( \kappa \) is tempered if it is weakly contained in \( L^2(\widetilde{Sp}(n, \mathbb{R}))_\kappa \).

By studying the leading exponents of \( I(\epsilon, 0) \) and \( C(\epsilon, t) \), it can be shown that \( I(\epsilon, 0)|_{\widetilde{Sp}(m+1, \mathbb{R})} \) is “tempered” and \( C(\epsilon, t)|_{\widetilde{Sp}(m+1, \mathbb{R})} \) is not “tempered”. Therefore

\[
I(\epsilon, 0)|_{\widetilde{Sp}(m+1, \mathbb{R})} \not\in C(\epsilon, t)|_{\widetilde{Sp}(m+1, \mathbb{R})}.
\]

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2 A Lemma on Friedrichs Extension

Let \( S \) be a semibounded densely defined symmetric operator on a Hilbert space \( H \). Suppose that \( (Su, u) > 0 \) for every nonzero \( u \in D(S) \). We call \( S \) positive. For \( u, v \in D(S) \), define

\[
(u, v)_S = (u, Sv),
\]

\[
\|u\|_S = (u, Su).
\]

Let \( H_S \) be the completion of \( D(S) \) under the norm \( \| \cdot \|_S \).

The operator \( S + I \) has a unique Friedrichs extension \( (S + I)_0 \) such that \( D((S + I)_0) \subseteq H_{S+I} \) and \( (u, v)_{S+I} = (u, (S + I)_0 v) \) for all \( u \in H_{S+I} \) and \( v \in D((S + I)_0) \) (see Theorem in Page 335 [\S]). Here \( H_{S+I} \subseteq H \) and \( (S + I)_0 \) is self-adjoint. Now consider \( (S + I)_0 - I \). It is an self-adjoint extension of \( S \). It is nonnegative. By the spectral decomposition and functional calculus, \( (S + I)_0 - I \) has unique square root \( T \) (See 127. 128. [\S]).

**Lemma 2.1.** Let \( S \) be a positive densely defined symmetric operator. Then the square root of \( (S + I)_0 - I \) extends to an isometry from \( H_S \) into \( H \).

Proof: Clearly, the spectrum of \( T \) is contained in the nonnegative part of the real line. By spectral decomposition \( D((S + I)_0 - I) = D((S + I)_0) \subseteq D(T) \) and \( TT = (S + I)_0 - I \).

In addition for any \( u, v \in D(S) \subseteq D((S + I)_0)_0 \),

\[
(Tu, Tv) = (u, TTV) = (u, (S + I)_0 v - v) = (u, Sv) = (u, v)_S.
\]

So \( T \) is an isometry from \( D(S) \) into \( H \). Since \( D(S) \) is dense in \( H_S \), \( T \) extends to an isometry from \( H_S \) into \( H \). \( \Box \)

We denote the isometry by \( I_S \). It is canonical.
3 Complementary Series of \( \widetilde{Sp}(n, \mathbb{R}) \)

Fix the Lie algebra \( \mathfrak{sp}(n, \mathbb{R}) \):

\[
\left\{ \begin{pmatrix} X & Y & Z \\ Z & -X^t & -Y^t \end{pmatrix} \mid Y^t = Y, Z^t = Z \right\}
\]

and the Siegel parabolic algebra \( \mathfrak{p} \):

\[
\left\{ \begin{pmatrix} X & Y \\ 0 & -X^t \end{pmatrix} \mid Y^t = Y \right\}.
\]

Fix the Levi decomposition \( \mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n} \) with

\[
\mathfrak{l} = \left\{ \begin{pmatrix} X & 0 & 0 \\ 0 & -X^t & 0 \end{pmatrix} \mid X \in \mathfrak{gl}(n, \mathbb{R}) \right\}
\]

and

\[
\mathfrak{n} = \left\{ \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix} \mid Y^t = Y \right\}.
\]

Fix a Cartan subalgebra \( \mathfrak{a} = \{ \text{diag}(H_1, H_2, \ldots, H_n, -H_1, -H_2, \ldots, -H_n) \mid H_i \in \mathbb{R} \} \).

Let \( \text{Sp}(n, \mathbb{R}) \) be the symplectic group and \( P \) be the Siegel parabolic subgroup. Let \( LN \) be the Levi decomposition and \( A \) be the analytic group generated by the Lie algebra \( \mathfrak{a} \). Clearly, \( L \cong GL(n, \mathbb{R}) \) and \( L \cap U(n) \cong O(n) \). On the covering group, we have \( \widetilde{L} \cap \widetilde{U}(n) = \widetilde{O}(n) \).

Recall that

\[
\text{det}(g) = \pm 1.
\]

We have the following exact sequence

\[
1 \to SO(n) \to \tilde{O}(n) \to \frac{1}{2} \mathbb{Z} \to 1.
\]

Consequently, we have

\[
1 \to GL_0(n, \mathbb{R}) \to \tilde{L} \to \frac{1}{2} \mathbb{Z} \to 1.
\]

In fact,

\[
\tilde{L} = \{ (x, g) \mid g \in L, \exp 2\pi ix = \frac{\det g}{|\det g|}, x \in \mathbb{R} \}.
\]

The one dimensional unitary characters of \( \frac{1}{2} \mathbb{Z} \) are parametrized by the one dimensional torus \( T \). Identify \( T \) with \([0, 1)\). Let \( \mu^\epsilon \) be the character of \( \frac{1}{2} \mathbb{Z} \) corresponding to \( \epsilon \in [0, 1) \).

Now each character \( \mu^\epsilon \) yields a character of \( \tilde{L} \), which in turn, yields a character of \( \tilde{P} \). For simplicity, we retain \( \mu^\epsilon \) to denote the character on \( \tilde{L} \) and \( \tilde{P} \). Let \( \nu \) be the det-character on \( \tilde{L}_0 \), i.e.,

\[
\nu(x, g) = |\det g| \quad (x, g) \in \tilde{L}.
\]
Let
\[ I(\epsilon, t) = \text{Ind}_{\tilde{P}}^{\tilde{\text{Sp}}(n, \mathbb{R})} \mu^\epsilon \otimes \nu^t \]
be the normalized induced representation with \( \epsilon \in [0, 1) \) and \( t \in \mathbb{C} \). This is Sahi’s notation applied to the universal covering of the symplectic group \([10]\). \( I(\epsilon, t) \) is a degenerate principal series representation. Clearly, \( I(\epsilon, t) \) is unitary when \( t \) is purely imaginary.

When \( t \) is real and \( I(\epsilon, t) \) is unitarizable, the unitary representation, often denoted by \( C(\epsilon, t) \), is called a complementary series representation. Various complementary series of \( \text{Sp}(n, \mathbb{R}) \) and its metaplectic covering was determined explicitly or implicitly by Kudla-Rallis, Ørsted-Zhang, Brason-Olafsson-Ørsted and others. See \([6], [2], [7]\) and the references therein.

The complete classification of the complementary series of the universal covering is due to Sahi (\([10]\)).

**Theorem 3.1** (Thm A, \([10]\)). Suppose that \( t \) is real. For \( n \) even, \( I(\epsilon, t) \) is irreducible and unitarizable if and only if \( 0 < |t| < \frac{1}{2} - |2\epsilon - 1| \). For \( n \) odd and \( n > 1 \), \( I(\epsilon, t) \) is irreducible and unitarizable if and only if \( 0 < |t| < \frac{1}{2} - \frac{1}{2} - |2\epsilon - 1| \).

![Figure 1: Complementary Parameters \((E, t)\)](image)

One can easily check that the complementary series exist if \( \epsilon \neq 0, \frac{1}{2} \) for \( n \) odd and \( n > 1 \); if \( \epsilon \neq \frac{1}{2}, \frac{3}{4} \) for \( n \) even. It is interesting to note that complementary series always exist unless \( I(\epsilon, t) \) descends into a representation of the metaplectic group. For the metaplectic group \( Mp(2n+1, \mathbb{R}) \), there are two complementary series \( I(\frac{1}{4}, t)(0 < t < \frac{1}{2}) \) and \( I(\frac{3}{4}, t)(0 < t < \frac{1}{2}) \). For the metaplectic group \( Mp(2n, \mathbb{R}) \), there are two complementary series \( I(0, t)(0 < t < \frac{1}{2}) \) and \( I(\frac{1}{2}, t)(0 < t < \frac{1}{2}) \). These four complementary series are the “longest.”

For \( n = 1 \), the situation is quite different. The difference was pointed out in \([6]\). For example, there are Bargmann’s complementary series representation for \( I(0, t)(t \in (0, \frac{1}{2})) \). The classification of the complementary series of \( \tilde{\text{Sp}}(1, \mathbb{R}) \) can be found in \([1], [9], [5]\).

Since our restriction theorem only makes sense for \( n \geq 2 \). We will assume \( n \geq 2 \) from now on. The parameters for the complementary series of \( \tilde{\text{Sp}}(n, \mathbb{R}) \) are illustrated in fig. \([1]\).

### 4 The generalized compact model and The Intertwining Operator

Recall that
\[ I^\infty(\epsilon, t) = \{ f \in C^\infty(\tilde{\text{Sp}}(n, \mathbb{R})) \mid f(gl_n) = (\mu^\epsilon \otimes \nu^{1+t})(l^{-1})f(l) (l \in \tilde{L}, n \in N) \} \]
where \( \rho = \frac{n+1}{2} \). Let \( X = \tilde{\text{Sp}}(n, \mathbb{R}) / \tilde{P} \). Then \( I^\infty(\epsilon, t) \) consists of smooth sections of the homogeneous vector bundle \( \mathcal{L}_{\epsilon, t} \)
\[ \tilde{\text{Sp}}(n, \mathbb{R}) \times_{\tilde{P}} C_{\mu^\epsilon \otimes \nu^{1+t}} \rightarrow X. \]
Since \( X \cong Sp(n, \mathbb{R})/P \cong \tilde{U}(n)/\tilde{O}(n) \), \( \tilde{U}(n) \) acts transitively on \( X \). \( f \in I^\infty(\epsilon, t) \) is uniquely determined by \( f|_{\tilde{U}(n)} \) and vice versa. Moreover, the homogeneous vector bundle \( L_{\epsilon,t} \) can be identified with \( K_{\epsilon,t} \)

\[
\tilde{U}(n) \times _{\tilde{O}(n)} C_{\mu \otimes \epsilon + r}|_{\tilde{O}(n)} \to X
\]

naturally. Notice that the homogeneous vector bundle \( K_{\epsilon,t} \) does not depend on the parameter \( t \). We denote it by \( K_\epsilon \). The representation \( I(\epsilon, t) \) can then be modeled on smooth sections of \( K_\epsilon \). This model will be called the generalized compact model.

The generalized compact model provides much convenience. First, it equips the smooth sections of \( K_{\epsilon,t} \) with a pre-Hilbert structure

\[
(f_1, f_2)_X = \int_{[k] \in X} f_1(k)\overline{f_2(k)}d[k],
\]

where \( k \in \tilde{U}(n) \) and \([k] \in X\). It is easy to verify that \( f_1(k)\overline{f_2(k)} \) is a function of \([k]\) and it does not depend on any particular choice of \( k \). Notice that our situation is different from the compact model since \( \tilde{U}(n) \) is not compact. We denote the completion of \( I^\infty \) with respect to \((,)_X \) by \( I_X(\epsilon, t) \). Secondly, the action of \( \tilde{U}(n) \) on \( K_\epsilon \) induces an orthogonal decomposition of \( I_X(\epsilon, t) \):

\[
I_X(\epsilon, t) = \oplus_{\alpha \in \mathbb{Z}^n} V(\alpha + \epsilon(2, 2, \ldots, 2)),
\]

where \( V(\alpha + \epsilon(2, 2, \ldots, 2)) \) is an irreducible representation of \( \tilde{U}(n) \) with highest weight \( \alpha + \epsilon(2, 2, \ldots, 2) \). Let

\[
V(\epsilon, t) = \oplus_{\alpha \in \mathbb{Z}^n} V(\alpha + \epsilon(2, 2, \ldots, 2)).
\]

\( V(\epsilon, t) \) possesses an action of the Lie algebra \( sp(n, \mathbb{R}) \). It is called the Harish-Chandra module of \( I(\epsilon, t) \). Clearly, \( V(\epsilon, t) \subset I^\infty(\epsilon, t) \subset I_X(\epsilon, t) \).

For each \( t \), there is an \( \tilde{Sp}(n, \mathbb{R}) \)-invariant sesquilinear pairing of \( I(\epsilon, t) \) and \( I(\epsilon, -\bar{t}) \), namely,

\[
(f_1, f_2) = \int_X f_1(k)\overline{f_2(k)}d[k],
\]

where \( f_1 \in I(\epsilon, t) \) and \( f_2 \in I(\epsilon, -\bar{t}) \). If \( t \) is purely imaginary, we obtain a \( \tilde{Sp}(n, \mathbb{R}) \)-invariant Hermitian form which is exactly \((,)_X \). Since \((,)_X \) is positive definite, \( I(\epsilon, t) \) is unitary.

For each real \( t \), the form \((,)_X \) gives an \( \tilde{Sp}(n, \mathbb{R}) \)-invariant sesquilinear pairing of \( I(\epsilon, t) \) and \( I(\epsilon, -t) \). There is an intertwining operator

\[
A(\epsilon, t) : V(\epsilon, t) \to V(\epsilon, -t)
\]

which preserves the action of \( sp(n, \mathbb{R}) \) (see for example [2]). Define a Hermitian structure \((,)_e,t \) on \( V(\epsilon, t) \) by

\[
(u, v)_{e,t} = (A(\epsilon, t)u, v), \quad (u, v \in V(\epsilon, t)).
\]

Clearly, \((,)_e,t \) is \( sp(n, \mathbb{R}) \)-invariant. So \( A(\epsilon, t) \) induces an invariant Hermitian form on \( V(\epsilon, t) \).

Now \( A(\epsilon, t) \) can also be realized as an unbounded operator on \( I_X(\epsilon, t) \) as follows. For each \( f \in V(\epsilon, t) \), define \( A_X(\epsilon, t)f \) to be the unique section of \( L_{\epsilon,t} \) such that

\[
(A_X(\epsilon, t)f)|_{\tilde{U}(n)} = \left(A(\epsilon, t)f\right)|_{\tilde{U}(n)}.
\]
Notice that $A_X(\epsilon, t)f \in I(\epsilon, t)$ and $A(\epsilon, t)f \in I(\epsilon, -t)$. They differ by a multiplier.

Now $A_X(\epsilon, t)$ is an unbounded operator on the Hilbert space $I_X(\epsilon, t)$. The following fact is well-known in many different forms. I state it in a way that is convenient for later use.

**Lemma 4.1.** Let $t \in \mathbb{R}$. $I(\epsilon, t)$ is unitarizable if and only if $A_X(\epsilon, t)$ extends to a self-adjoint operator on $I_X(\epsilon, t)$ with spectrum on the nonnegative part of the real axis.

The spectrum of $A_X(\epsilon, t)$ was computed in [2] and [17] explicitly for special cases and in [10] implicitly. In particular, $A_X(\epsilon, t)$ restricted onto each $\widetilde{U}(n)$-type is a scalar multiplication and the scalar is bounded by a polynomial on the highest weight. We obtain

**Lemma 4.2.** $A_X(\epsilon, t)$ extends to an unbounded operator from $I^\infty(\epsilon, t)$ to $I^\infty(\epsilon, t)$.

This lemma follows from a standard argument that the norm of each $\tilde{U}(n)$-component in the Peter-Weyl expansion of any smooth section of $K_\epsilon$ decays rapidly with respect to the highest weight.

## 5 The Mixed Model

Suppose that $p + q = n$ and $p \leq q$. Fix a subgroup $Sp(p, \mathbb{R}) \times Sp(q, \mathbb{R})$ in $Sp(n, \mathbb{R})$. Then we have a subgroup $\widetilde{Sp}(p, \mathbb{R})\tilde{Sp}(q, \mathbb{R})$ in $\tilde{Sp}(n, \mathbb{R})$. Notice that $\widetilde{Sp}(p, \mathbb{R}) \cap \tilde{Sp}(q, \mathbb{R}) \cong \mathbb{Z}$. So $\widetilde{Sp}(p, \mathbb{R})\tilde{Sp}(q, \mathbb{R})$ is not a direct product, but rather the product of the two groups as sets. Let $U(q) = \tilde{Sp}(q, \mathbb{R}) \cap U(n)$.

**Theorem 5.1** (Main Theorem). Suppose that $p + q = n$ and $p \leq q$. Given a complementary series representation $C(\epsilon, t)$, 

$$C(\epsilon, t)|_{\widetilde{Sp}(p, \mathbb{R})\tilde{U}(q)} \cong I(\epsilon, 0)|_{\tilde{Sp}(p, \mathbb{R})\tilde{U}(q)}.$$ 

In other words, there is an isometry between $C(\epsilon, t)$ and $I(\epsilon, 0)$ that intertwines the actions of $\tilde{U}(q)$ and of $\tilde{Sp}(p, \mathbb{R})$.

We begin by recall a result concerning the action of $Sp(p, \mathbb{R}) \times Sp(q, \mathbb{R})$ on $X$ (\[4\]).

**Lemma 5.1.** $Sp(p, \mathbb{R}) \times Sp(q, \mathbb{R})$ acts on $X$ with a unique open dense orbit $X_0$. Let $p \leq q$. Let $P_{p, 2q-2p}$ be a maximal parabolic subgroup of $Sp(q, \mathbb{R})$ preserving an $q - p$ dimensional isotropic subspace. Let $GL_{q-p}Sp(p, \mathbb{R})N_{p, 2q-2p}$ be the Langlands decomposition of $P_{p, 2q-2p}$. Let

$$H = \{(u, m_2 \, t^2 \, u^{-1} \, n_2) \mid m_2 \in GL(q - p, \mathbb{R}), u \in Sp(p, \mathbb{R}), n_2 \in N_{p, 2q-2p}\}.$$ 

Then $X_0 \cong Sp(p, \mathbb{R}) \times Sp(q, \mathbb{R})/H$.

Notice that $H \cong P_{p, 2q-2p}$. But the $Sp(p, \mathbb{R})$ factor in $P$ is diagonally embedded in $Sp(p, \mathbb{R}) \times Sp(q, \mathbb{R})$. In particular, there is a principal fibration 

$$Sp(p, \mathbb{R}) \to X_0 \to Sp(q, \mathbb{R})/P_{p, 2q-2p} \cong U(q)/O(q - p)U(p).$$

Here $O(q - p)U(p) = U(q) \cap P_{p, 2q-2p} \subset Sp(q, \mathbb{R})$. Let $M = Sp(p, \mathbb{R})U(q)$ and $\tilde{M} = \tilde{Sp}(p, \mathbb{R})\tilde{U}(q)$. Then $\mathcal{L}_{\epsilon, t}$ restricted onto $X_0$ becomes

$$\tilde{M} \times_{O(q-p)\tilde{U}(p)} C_{\mu^*} \to \tilde{M}/O(q-p)\tilde{U}(p) \cong X_0.$$
Let \( I^\infty_{c,X_0}(\epsilon,t) \) be the set of smooth section of \( \mathcal{L}_{c,t} \) that are compactly supported on \( X_0 \). Clearly
\[
I^\infty_{c,X_0}(\epsilon,t) \subset I^\infty(\epsilon,t).
\]
Consider the restriction of \((\cdot, \cdot)_X\) onto \( I^\infty_{c,X_0}(\epsilon,t) \). We are interested in expressing \((\cdot, \cdot)_X\) as an integral on \( \sim M/O(q-p)\sim U(p) \). This boils down to a change of variables from \( \sim U(n)/\sim O(n) \) to \( \sim M/O(q-p)\sim U(p) \).

Let \( dg_1 \) be the invariant measure on \( \sim Sp(p, \mathbb{R}) \) and \( d[k_2] \) be the invariant measure on \( \sim U(q)/O(q-p)\sim U(p) \). Every element in \( \sim Sp(n, \mathbb{R}) \) has a \( \sim U(n)\sim P_0 \) decomposition where \( P_0 \) is the identity component of \( \sim P \). For each \( g \in \sim Sp(n, \mathbb{R}) \), write \( g = u(g)p(g) \). For each \( g_1k_2 \in \sim M \), write \( g_1k_2 = u(g_1k_2)p(g_1k_2) \). Then \( u \) defines a map from \( \sim M \) to \( \sim \sim U(n) \). \( u \) induces a map from \( \sim M/O(q-p)\sim U(p) \) to \( \sim \sim U(n)/\sim O(n) \) which will be denoted by \( j \). Clearly, \( j \) is an injection from \( \sim M/O(q-p)\sim U(p) \) onto \( X_0 \). Change the variable on \( X \) from \( \sim M/O(q-p)\sim U(p) \) to \( \sim \sim U(n)/\sim O(n) \).

Let \( J([g_1], [k_2]) \) be the Jacobian:
\[
\frac{dj([g_1], [k_2])}{d[g_1]d[k_2]}
\]
We have

**Lemma 5.2.** Let
\[
\Delta_{\epsilon,t}(g_1, k_2) = \nu(p(g_1k_2))^{-t+2p}J([g_1], [k_2]).
\]
Then for every \( f_1, f_2 \in I^\infty(\epsilon,t) \) we have
\[
(f_1, f_2)_{X} = \int_{\sim M/O(q-p)\sim U(p)} f_1(g_1k_2)f_2(g_1k_2)\Delta_{\epsilon,t}(g_1, k_2)d[g_1]d[k_2]
\]
where \( g_1 \in \sim Sp(p, \mathbb{R}) \), \( k_2 \in \sim U(q) \), \( [g_1] \in Sp(p, \mathbb{R}) \) and \( [k_2] \in \sim U(q)/O(q-p)\sim U(p) \). Furthermore, \( \Delta_{\epsilon,t}(g_1, k_2) \) is a nonnegative right \( O(q-p)\sim U(p) \)-invariant function on \( \sim Sp(p, \mathbb{R}) \times \sim U(q) \).

**Proof:** We compute
\[
\int_{\sim M/O(q-p)\sim U(p)} f_1(g_1k_2)f_2(g_1k_2)\Delta_{\epsilon,t}(g_1, k_2)d[g_1]d[k_2]
\]
\[
= \int_{\sim M/O(q-p)\sim U(p)} f_1(\bar{u}(g_1k_2))f_2(\bar{u}(g_1k_2))\nu(p(g_1k_2))^{-t-2p}\Delta_{\epsilon,t}(g_1, k_2)d[\bar{u}]d[k_2]
\]
\[
= \int_{\sim M/O(q-p)\sim U(p)} f_1(\bar{u}(g_1k_2))f_2(\bar{u}(g_1k_2))\nu(p(g_1k_2))^{-t-2p}\Delta_{\epsilon,t}(g_1, k_2)J^{-1}(g_1, k_2)\nu(p(g_1k_2))^{-t-2p}d[\bar{u}]d[k_2]
\]
\[
= \int_{X_0} f_1(\bar{u})f_2(\bar{u})d[\bar{u}] = (f_1, f_2)_X.
\]
\[
(1)
\]
Since \( \nu(p(g_1k_2)) \) and \( J([g_1], [k_2]) \) remains the same when we multiply \( k_2 \) on the right by \( O(q-p)\sim U(p) \), \( \Delta_{\epsilon,t}(g_1, k_2) \) is a nonnegative right \( O(q-p)\sim U(p) \)-invariant function. \( \square \)

For each \( f_1, f_2 \in I^\infty_{c,M}(\epsilon,t) \), define
\[
(f_1, f_2)_{M,t} = \int_{M} f_1(g_1k_2)f_2(g_1k_2)\Delta_{\epsilon,t}(g_1k_2)d[g_1]d[k_2],
\]
\[ (f_1, f_2)_M = \int_M f_1(g_1, k_2) \overline{f_2(g_1, k_2)} d[g_1] d[k_2]. \]

Let \( I_M(\epsilon, t) \) be the completion of \( I^\infty_{C_M}(\epsilon, t) \) under \( (\cdot, \cdot)_M \). Clearly \( I_M(\epsilon, t) \cong I_X(\epsilon, t) \) as Hilbert representations of \( \widehat{Sp}(n, \mathbb{R}) \). \( I_M(\epsilon, t) \) is called the mixed model.

### 6 Mixed Model for Unitary Principal Series

**Lemma 6.1.** If \( t \) is purely imaginary, then \( \Delta_{\epsilon, t}(g_1, k_2) \) is a constant and \( (\cdot, \cdot)_M \) is a constant multiple of \( (\cdot, \cdot)_M \).

Proof: Let \( t \in i \mathbb{R} \). Let \( f_1, f_2 \in I^\infty(\epsilon, t) \) and \( h \in \widehat{Sp}(p, \mathbb{R}) \). We have

\[
(I(\epsilon, t)(h)f_1, I(\epsilon, t)(h)f_2)_X
= (I(\epsilon, t)(h)(f_1, f_2))_M
= \int_{M/O(\widetilde{q} - p)U(p)} f_1(h^{-1}g_1k_2) \overline{f_2(h^{-1}g_1k_2)} \Delta_{\epsilon, t}(g_1, k_2) d[g_1] d[k_2]
= \int_{M/O(\widetilde{q} - p)U(p)} f_1(g_1k_2) \overline{f_2(g_1k_2)} \Delta_{\epsilon, t}(h^gk_1k_2) d[g_1] d[k_2]
\]

Since \( I(\epsilon, t) \) is unitary,

\[
(I(\epsilon, t)(h)f_1, I(\epsilon, t)(h)f_2)_X = (f_1, f_2)_X.
\]

We have

\[
\int_{M/O(\widetilde{q} - p)U(p)} f_1(g_1k_2) \overline{f_2(g_1k_2)} \Delta_{\epsilon, t}(h^gk_1k_2) d[g_1] d[k_2] = \int_{M/O(\widetilde{q} - p)U(p)} f_1(g_1k_2) \overline{f_2(g_1k_2)} \Delta_{\epsilon, t}(g_1k_2) d[g_1] d[k_2].
\]

It follows that \( \Delta_{\epsilon, t}(h^gk_1k_2) = \Delta_{\epsilon, t}(g_1k_2) \) for any \( h \in \widehat{Sp}(p, \mathbb{R}) \). Similarly, we obtain \( \Delta_{\epsilon, t}(g_1k_2) = \Delta(g_1, k_2) \) for any \( k \in \tilde{U}(q) \). Hence, \( \Delta_{\epsilon, t}(g_1, k_2) \) is a constant for purely imaginary \( t \). \( \square \)

**Corollary 6.1.** \( J([g_1], [k_2]) = cv(p(g_1k_2))^{-2p} \) and \( \Delta_{\epsilon, t}(g_1, k_2) = cv(p(g_1k_2))^{t+\tau} \). Furthermore,

\[
I_M(\epsilon, t) \cong L^2(\tilde{M} \times O(\tilde{q} - p)U(p), \mu^t, \nu(p(g_1k_2))^{t+\tau}d[g_1] d[k_2]). \tag{3}
\]

From now on, identify \( L^2(M/O(q - p)U(p)) \) with \( I_M(\epsilon, t) \).

**Corollary 6.2.** \( \nu(p(g_1k_2))^{-p} \in L^2(M/O(q - p)U(p)) \) and \( \nu(p(g_1k_2))^{-1} \) is a bounded positive function.

Proof: Since \( X \) is compact, \( J([g_1], [k_2]) \in L^1(\tilde{M} / O(q - p)U(p)) \). Notice that \( \nu(p(g_1k_2))^{-2p} = cJ([g_1], [k_2]) \). It follows that \( \nu(p(g_1k_2))^{-p} \in L^2(\tilde{M} / O(q - p)U(p)) \). Since \( J([g_1], [k_2]) \) is continuous and \( [k_2] \) is a compact manifold, it suffice to show that for each \( k_2 \), \( J([g_1], [k_2]) \) is bounded. This is true because \( j_{k_2}(\tilde{Sp}(p, \mathbb{R})) \) is an analytic compactification of \( \tilde{Sp}(p, \mathbb{R}) \) and the Jacobian can be easily computed just like the way it is done in \[ \text{[3]} \] (see Theorem 2.3). In particular, it is always positive. By Cor. 6.1 \( \nu(p(g_1k_2))^{-1} \) is bounded and positive. \( \square \)

If \( f \in I_M(\epsilon, t_1) \) and \( h > 0 \), we have \( ||f||_{M,t_1} \geq C ||f||_{M,t_1-h} \). So \( I_M(\epsilon, t_1) \subset I_M(\epsilon, t_1-h) \).
Lemma 7.1. Suppose that $h > 0$. Then $I_M(\epsilon, t_1) \subset I_M(\epsilon, t_1 - h)$ under the Equation [99x90]

Notice that in the mixed model, the actions of $\overline{Sp(p, \mathbb{R})}$ and $\tilde{U}(q)$ does not depend on the parameter $t$. We obtain

Theorem 6.1. Let $t$ be purely imaginary. $I(\epsilon, t)$ can all be modeled on

$$L^2(\tilde{M} \times O(q-p)\tilde{U}(p)) \subset C_{\mu^*}, d[g_1|d[k_2]].$$

In particular, $I_M(\epsilon, t)|\overline{Sp(p, \mathbb{R})}\tilde{U}(q) \cong I_M(\epsilon, 0)|\overline{Sp(p, \mathbb{R})}\tilde{U}(q)$ and the identity operator intertwines $I_M(\epsilon, 0)|\overline{Sp(p, \mathbb{R})}\tilde{U}(q)$ with $I_M(\epsilon, t)|\overline{Sp(p, \mathbb{R})}\tilde{U}(q)$.

For $t$ a nonzero real number, $\Delta_{\epsilon,t}(g, k)$ is not a constant. So Theorem 6.1 does not hold for real nonzero $t$.

7 “Square Root ” of the Intertwining Operator

Suppose from now on $t \in \mathbb{R}$. For $f \in I^\infty(\epsilon, t)|\tilde{M}$, define a function on $\tilde{M}$,

$$(A_M(\epsilon, t)f)(g_1k_2) = A(\epsilon, t)f(g_1k_2) \quad (g_1 \in \overline{Sp(p, \mathbb{R})}, k_2 \in \tilde{U}(q)).$$

So $A_M(\epsilon, t)$ is the “restriction” of $A(\epsilon, t)$ onto $\tilde{M}$. $A_M(\epsilon, t)$ is not yet an unbounded operator on $I_M(\epsilon, t)$. In fact, for $t > 0$, $A_M(\epsilon, t)$ does not behave well. In this case, it is not clear whether $A_M(\epsilon, t)$ can be realized as an unbounded operator on $I_M(\epsilon, t)$. $A_M(\epsilon, t)f$ differs from $AX(\epsilon, t)f$ by a multiplier.

Lemma 7.1. For $t \in \mathbb{R}$ and $f \in I^\infty(\epsilon, t)$,

$$(A_M(\epsilon, t)f|\tilde{M})(g_1k_2) = (AX(\epsilon, t)f)(g_1k_2)\nu(p(g_1k_2))^{2t} = (AX(\epsilon, t)f)(g_1k_2)\Delta_{\epsilon,t}(g_1, k_2).$$

This Lemma is due to the fact that $AX(\epsilon, t)f \in I(\epsilon, t)$ but $A(\epsilon, t)f \notin I(\epsilon, t)$.

Let $f \in I^\infty(\epsilon, t)$. In terms of the mixed model, the invariant Hermitian form $(,)_{\epsilon,t}$ can be written as follows:

$$(f, f)_{\epsilon,t} = (AX(\epsilon, t)f, f)_{X} = \int_{\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)} A_M(\epsilon, t) f|\tilde{M} \tilde{F}_{\tilde{M}} d[g_1|d[k_2]].$$

We obtain

Lemma 7.2. For $f_1, f_2 \in I^\infty(\epsilon, t)$, $(f_1, f_2)_{\epsilon,t} = (A_M(\epsilon, t)f_1|\tilde{M}, f_2|\tilde{M})_M$.

Theorem 7.1. If $t < 0$ and $C(\epsilon, t)$ is a complementary series representation, then $A_M(\epsilon, t)$ is positive and densely defined symmetric operator. Its self-adjoint-extension $(A_M(\epsilon, t) + I)_0 - I$ has a unique square root which extends to an isometry from $C(\epsilon, t)$ onto

$$L^2(\tilde{M} \times O(q-p)\tilde{U}(p)) \subset C_{\mu^*}, d[g_1|d[k_2]].$$
Proof: Let \( t < 0 \). Put

\[
\mathcal{H} = L^2(M \times O(q-p) U(p), C_{\mu^\varepsilon}, d[g_1]d[k_2]).
\]

Let \( f \in I^\infty(\varepsilon, t) \). Then

\[
A_M(\varepsilon, t)(f|_{\tilde{\mathcal{M}}}) = \nu(\rho(g_1k_2))^{2t}A_X(\varepsilon, t)f(g_1k_2).
\]

By Lemma 2.1, Cor. 6.2 and Lemma 5.2, we have

\[
\begin{align*}
\int_{\tilde{\mathcal{M}}/\tilde{\partial}(q-p)U(p)} A_M(\varepsilon, t)(f|_{\tilde{\mathcal{M}}})A_M(\varepsilon, t)(f|_{\tilde{\mathcal{M}}})d[g_1]d[k_2] \\
= \int_{\tilde{\mathcal{M}}/\tilde{\partial}(q-p)U(p)} \nu(\rho(g_1k_2))^{2t}|(A_X(\varepsilon, t)f)(g_1k_2)|^2 \Delta_{\varepsilon,t}(g_1, k_2)d[g_1]d[k_2] \\
\leq C \int_{\tilde{\mathcal{M}}/\tilde{\partial}(q-p)U(p)} |A_X(\varepsilon, t)f(g_1k_2)|^2 \Delta_{\varepsilon,t}(g_1, k_2)d[g_1]d[k_2] \\
= C(A_X(\varepsilon, t)f, A_X(\varepsilon, t)f)_X < \infty.
\end{align*}
\]

Therefore, \( A_M(\varepsilon, t)(f|_{\tilde{\mathcal{M}}}) \in \mathcal{H} \). Let \( \mathcal{D} = I^\infty(\varepsilon, t)|_{\tilde{\mathcal{M}}} \). Clearly, \( \mathcal{D} \) is dense in \( \mathcal{H} \). So \( A_M(\varepsilon, t) \) is a densely defined unbounded operator. It is positive and symmetric by Lemma 2.1.

Now \( (f, g)_{\varepsilon,t} = (A_M(\varepsilon, t)f|_{\tilde{\mathcal{M}}}, g|_{\tilde{\mathcal{M}}})_M \) for any \( f, g \in I^\infty(\varepsilon, t) \). So \( C(\varepsilon, t) = \mathcal{H}_{A_M(\varepsilon, t)} \). By Lemma 2.1, there exists an isometry \( I_{A_M(\varepsilon, t)} \) mapping from \( C(\varepsilon, t) \) into \( \mathcal{H} \).

Suppose that \( I_{A_M(\varepsilon, t)} \) is not onto. Let \( f \in \mathcal{H} \) such that for any \( u \in \mathcal{D}(((A_M(\varepsilon, t)+I)_0 - I)^{\frac{1}{2}}) \),

\[
(f, ((A_M(\varepsilon, t)+I)_0 - I)^{\frac{1}{2}}u)_M = 0.
\]

Notice that

\[
I^\infty(\varepsilon, t)|_{\tilde{\mathcal{M}}} \subset \mathcal{D}((A_M(\varepsilon, t)+I)_0 - I) \subset \mathcal{D}(((A_M(\varepsilon, t)+I)_0 - I)^{\frac{1}{2}}),
\]

and

\[
(A_M(\varepsilon, t)+I)_0 - I)^{\frac{1}{2}}(A_M(\varepsilon, t)+I)_0 - I)^{\frac{1}{2}} = (A_M(\varepsilon, t)+I)_0 - I.
\]

In particular,

\[
((A_M(\varepsilon, t)+I)_0 - I)^{\frac{1}{2}}I^\infty(\varepsilon, t)|_{\tilde{\mathcal{M}}} \subset \mathcal{D}(((A_M(\varepsilon, t)+I)_0 - I)^{\frac{1}{2}}).
\]

It follows that

\[
(f, A_M(\varepsilon, t)I^\infty(\varepsilon, t)|_{\tilde{\mathcal{M}}})_M \\
= (f, ((A_M(\varepsilon, t)+I)_0 - I)^{\frac{1}{2}}I^\infty(\varepsilon, t)|_{\tilde{\mathcal{M}}})_M \\
= (f, ((A_M(\varepsilon, t)+I)_0 - I)^{\frac{1}{2}}((A_M(\varepsilon, t)+I)_0 - I)^{\frac{1}{2}}I^\infty(\varepsilon, t)|_{\tilde{\mathcal{M}}})_M \\
= 0.
\]

Let \( f_{\varepsilon,t} \) be a function such that \( f_{\varepsilon,t}|_{\tilde{\mathcal{M}}} = f \) and

\[
f_{\varepsilon,t}(gln) = (\mu^\varepsilon \otimes \nu^{t+\rho})(l^{-1}) f_{\varepsilon,t}(g) \quad (l \in \tilde{L}, n \in N).
\]
The function \( f_{\epsilon,t} \) is not necessarily in \( I(\epsilon,t) \). By Lemma 7.2, \( \forall u \in V(\epsilon,t) \),

\[
0 = (f, A_M(\epsilon,t)(u|_{\tilde{M}}))_M = (f_{\epsilon,t}, u)_{\epsilon,t} = (f_{\epsilon,t}, A_X(\epsilon,t)u)_X.
\]

This equality is to be interpreted as an equality of integrals according to the definitions of \( (\, , \,)_M \) and \( (\, , \,)_X \). Since \( A_X(\epsilon,t) \) acts on \( \tilde{U}(n) \)-type in \( V(\epsilon,t) \) as a scalar, \( A_X(\epsilon,t)V(\epsilon,t) = V(\epsilon,t) \). We now have

\[
(f_{\epsilon,t}, V(\epsilon,t))_X = 0.
\]

In particular, \( f_{\epsilon,t}|_{\tilde{U}(n)} \in L^1(X) \). Therefore \( f_{\epsilon,t} = 0 \). We see that \( I_{AM(\epsilon,t)} \) is an isometry from \( C(\epsilon,t) \) onto

\[
L^2(\tilde{M} \times \tilde{O}(q,p)\tilde{U}(p) \tilde{C}_{\mu'}, d\tilde{g}d[k_2]).
\]

\( \Box \)

The Hilbert space

\[
L^2(\tilde{M} \times \tilde{O}(q,p)\tilde{U}(p) \tilde{C}_{\mu'}, d\tilde{g}d[k_2])
\]

is the mixed model for \( I(\epsilon,0) \). We now obtain an isometry from \( C(\epsilon,t) \) onto \( I(\epsilon,0) \). Denote this isometry by \( \tilde{U}(\epsilon,t) \). Now, with in the mixed model, the action of \( I(\epsilon,t)(g_1k_2) \) is simply the left regular action and it is independent of \( t \). We obtain

**Lemma 7.3.** Suppose \( \epsilon < 0 \). Let \( g \in \tilde{Sp}(p,\mathbb{R}) \) or \( g \in \tilde{U}(q) \). Let \( L(g) \) be the left regular action on

\[
L^2(\tilde{M} \times \tilde{O}(q,p)\tilde{U}(p) \tilde{C}_{\mu'}, d\tilde{g}d[k_2]).
\]

As a operator on \( I^\infty(\epsilon,t)|_{\tilde{M}} \), \( L(g) \) commutes with \( A_M(\epsilon,t) \). Furthermore, \( L(g) \) commutes with \( (A_M(\epsilon,t) + I)_{0} - I \).

Proof: Let \( g \in \tilde{M} \). Both \( A_M(\epsilon,t) \) and \( L(g) \) are well-defined operator on \( I^\infty(\epsilon,t)|_{\tilde{M}} \). Restricting \( A(\epsilon,t)I(\epsilon,t)(g) = I(\epsilon,-t)(g)A(\epsilon,t) \) onto \( \tilde{M} \), we have

\[
A_M(\epsilon,t)L(g) = L(g)A_M(\epsilon,t).
\]

It follows that

\[
(A_M(\epsilon,t) + I)L(g) = L(g)(A_M(\epsilon,t) + I).
\]

Recall that \( (A_M(\epsilon,t) + I)_0 \) can be defined as the inverse of \( (A_M(\epsilon,t) + I)^{-1} \), which exists and is bounded. So \( L(g) \) commutes with both \( (A_M(\epsilon,t) + I)^{-1} \) and \( (A_M(\epsilon,t) + I)_0 \). \( \Box \)

**Lemma 7.4.** We have, for \( g \in \tilde{M} \), \( \tilde{U}(\epsilon,t)(I(\epsilon,t)(g) = I(\epsilon,0)(g)\tilde{U}(\epsilon,t) \).

Proof: It suffices to show that on the mixed model, \( ((A_M(\epsilon,t) + I)_0 - I)^\frac{1}{2} \) commutes with \( L(g) \). This follows from Lemma 7.3. \( \Box \)

Our main theorem is proved.

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Restrictions of Certain Degenerate Principal Series of the Universal Covering of the Symplectic Group

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Abstract

In this paper, we study the restrictions of degenerate unitary principal series \( I(\epsilon, t) \) of \( \widetilde{Sp}(n, \mathbb{R}) \), the universal covering of the symplectic group, onto \( \widetilde{Sp}(p, \mathbb{R}) \widetilde{Sp}(n-p, \mathbb{R}) \). We prove that if \( n \geq 2p \), \( I(\epsilon, t)\big|_{\widetilde{Sp}(p, \mathbb{R})\widetilde{Sp}(n-p, \mathbb{R})} \) is unitarily equivalent to an \( L^2 \)-space of a homogeneous line bundle \( L^2(\widetilde{Sp}(n-p, \mathbb{R}) \times \overline{GL(n-2p)} N \mathbb{C} \epsilon, t + \rho) \) (see Theorem 1.1). We further study the restriction of complementary series \( C(\epsilon, t) \) onto \( \widetilde{U}(n-p)\widetilde{Sp}(p, \mathbb{R}) \). We prove that this restriction is unitarily equivalent to \( I(\epsilon, t)\big|_{\widetilde{U}(n-p)\widetilde{Sp}(p, \mathbb{R})} \) for \( t \in i\mathbb{R} \). Our results suggest that the direct integral decomposition of \( C(\epsilon, t)\big|_{\widetilde{Sp}(p, \mathbb{R})\widetilde{Sp}(n-p, \mathbb{R})} \) will produce certain complementary series for \( \widetilde{Sp}(n-p, \mathbb{R}) \).

1 Introduction

Let \( \widetilde{Sp}(n, \mathbb{R}) \) be the universal covering of \( Sp(n, \mathbb{R}) \). \( \widetilde{Sp}(n, \mathbb{R}) \) is a central extension of \( Sp(n, \mathbb{R}) \):

\[ 1 \to C \to \widetilde{Sp}(n, \mathbb{R}) \to Sp(n, \mathbb{R}) \to 1, \]

where \( C \cong \mathbb{Z} \). The unitary dual of \( C \) is parametrized by a torus \( T \). For each \( \kappa \in T \), denote the corresponding unitary character of \( C \) by \( \chi^\kappa \). We say that a representation \( \pi \) of \( \widetilde{Sp}(n, \mathbb{R}) \) is of class \( \kappa \) if \( \pi|_C = \chi^\kappa \). Since \( C \) commutes with \( \widetilde{Sp}(n, \mathbb{R}) \), for any irreducible representation \( \pi \) of \( \widetilde{Sp}(n, \mathbb{R}) \), \( \pi|_C = \chi^\kappa \) for some \( \kappa \).

Denote the projection \( \widetilde{Sp}(n, \mathbb{R}) \to Sp(n, \mathbb{R}) \) by \( p \). For any subgroup \( H \) of \( Sp(n, \mathbb{R}) \), denote the full inverse image \( p^{-1}(H) \) by \( \hat{H} \). We adopt the notation from [14]. Let \( P \) be the Siegel parabolic subgroup of \( Sp(n, \mathbb{R}) \). One dimensional characters of \( \hat{P} \) can be parametrized by \( (\epsilon, t) \) where \( \epsilon \in T \) and \( t \in \mathbb{C} \). Let \( I(\epsilon, t) \) be the representation of \( \widetilde{Sp}(n, \mathbb{R}) \) induced from the one dimensional character \( \mathbb{C} \epsilon, t \) parametrized by \( (\epsilon, t) \) of \( \hat{P} \). If \( t \in i\mathbb{R} \) and \( t \neq 0 \), \( I(\epsilon, t) \) is unitary and irreducible. \( I(\epsilon, t) \) is called unitary degenerate principal series. If \( t \) is real, then

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$I(\epsilon, t)$ has a nontrivial invariant Hermitian form. Sahi gives a classification of all irreducible unitarizable $I(\epsilon, t)$. If $I(\epsilon, 0)$ is irreducible, there are complementary series $C(\epsilon, t)$ for $t$ in a suitable interval \((\Re)\). Let some of these complementary series are obtained by Kudla-Rallis \([9]\), Orsted-Zhang \([11]\), Branson-Orsted-Olafsson \([3]\), Lee \([10]\). Strictly speaking $C(\epsilon, t)$ should be called degenerate complementary series because there are complementary series associated with the principal series, which should be called complementary series \([8], [1]\). Throughout this paper, complementary series will mean $C(\epsilon, t)$.

Let \((Sp(p, \mathbb{R}), Sp(n - p, \mathbb{R}))\) be a pair of symplectic groups diagonally embedded in $Sp(n, \mathbb{R})$ (see Definition \([5.1]\). Let $U(n)$ be a maximal compact subgroup such that $Sp(n - p, \mathbb{R})$ and $Sp(p, \mathbb{R}) \cap U(n)$ are maximal compact subgroups of $Sp(n - p, \mathbb{R})$ and $Sp(p, \mathbb{R})$ respectively. Denote $Sp(n - p, \mathbb{R}) \cap U(n)$ by $U(n - p)$ and $Sp(p, \mathbb{R}) \cap U(n)$ by $U(p)$. The main results of this paper can be stated as follows.

**Theorem 1.1.** Suppose $p \leq n - p$ and $t \in i\mathbb{R}$. Let $P_{p,n-2p}$ be a maximal parabolic subgroup of $Sp(n - p, \mathbb{R})$ with Langlands decomposition $Sp(p, \mathbb{R})GL(n - 2p)N_{p,n-2p}$. Let $M_{\epsilon,t}$ be the homogeneous line bundle

$$\begin{align*}
\widetilde{Sp}(n - p, \mathbb{R}) \times_{GL(n - 2p)N_{p,n-2p}} C_{\epsilon,t+p} &\to Sp(n - p, \mathbb{R})/GL(n - 2p)N_{p,n-2p} \\
&\cong Sp(p, \mathbb{R})U(n - p)/U(p)O(n - 2p),
\end{align*}
$$

(1)

where $\rho = \frac{n+1}{2}$. Let $dg_1d[k_2]$ be an $Sp(p, \mathbb{R})U(n - p)$-invariant measure. Then

$$I(\epsilon, t)|_{\widetilde{Sp}(n - p, \mathbb{R})} \cong L^2(M_{\epsilon,t}, dg_1d[k_2]),$$

on which $\widetilde{Sp}(n - p, \mathbb{R})$ acts from the left and $\widetilde{Sp}(p, \mathbb{R})$ acts from the right.

**Theorem 1.2.** Let $C(\epsilon, t)$ be a complementary series representation. Suppose that $p \leq n - p$.

Then

$$C(\epsilon, t)|_{\widetilde{Sp}(n - p, \mathbb{R})} \cong I(\epsilon, 0)|_{\widetilde{Sp}(n - p, \mathbb{R})} \cong I(\epsilon, i\lambda)|_{\widetilde{Sp}(n - p, \mathbb{R})} \quad (\lambda \in \mathbb{R}).$$

$p = \left[\frac{n}{2}\right]$ is the best possible value for such a statement. In particular, for $\widetilde{Sp}(2m + 1, \mathbb{R})$

$$I(\epsilon, 0)|_{\widetilde{Sp}(m+1, \mathbb{R})} \neq C(\epsilon, t)|_{\widetilde{Sp}(m+1, \mathbb{R})}.$$

To see this, let $L^2(\widetilde{Sp}(n, \mathbb{R}))_{\kappa}$ be the set of functions with

$$f(zg) = \chi^\kappa(z)f(g) \quad (z \in C, g \in \widetilde{Sp}(n, \mathbb{R}));$$

$$||f||^2 = \int_{\widetilde{Sp}(n, \mathbb{R})} |f(g)|^2 d[g] < \infty \quad (g \in \widetilde{Sp}(n, \mathbb{R}), [g] \in Sp(n, \mathbb{R})).$$

We say that a representation of class $\kappa$ is tempered if it is weakly contained in $L^2(\widetilde{Sp}(n, \mathbb{R}))_{\kappa}$. By studying the leading exponents of $I(\epsilon, 0)$ and $C(\epsilon, t)$, it can be shown that $I(\epsilon, 0)|_{\widetilde{Sp}(m+1, \mathbb{R})}$ is “tempered” and $C(\epsilon, t)|_{\widetilde{Sp}(m+1, \mathbb{R})}$ is not “tempered”. Therefore

$$I(\epsilon, 0)|_{\widetilde{Sp}(m+1, \mathbb{R})} \neq C(\epsilon, t)|_{\widetilde{Sp}(m+1, \mathbb{R})}.$$

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2 A Lemma on Friedrichs Extension

Let $S$ be a semibounded densely defined symmetric operator on a Hilbert space $H$. $S$ is said to be positive if $(Su, u) > 0$ for every nonzero $u \in \mathcal{D}(S)$. Suppose that $S$ is positive. For $u, v \in \mathcal{D}(S)$, define

$$(u, v)_S = (u, Sv),$$

$$\|u\|_S^2 = (u, Su).$$

Let $H_S$ be the completion of $\mathcal{D}(S)$ under the norm $\| \cdot \|_S$. Clearly $H_{S+I} \subseteq H$ and $H_{S+I} \subseteq H_S$.

The operator $S + I$ has a unique self-adjoint extension $(S + I)_0$ in $H$, the Friedrichs extension. $(S + I)_0$ has the following properties

- $\mathcal{D}(S) \subseteq \mathcal{D}((S + I)_0) \subseteq H_{S+I} \subseteq H$;
- $(u, v)_{S+I} = (u, (S + I)v)$ for all $u \in H_{S+I}$ and $v \in \mathcal{D}((S + I)_0)$

(see Theorem in Page 335 [12]). Now consider $(S + I)_0 - I$. It is an self-adjoint extension of $S$. It is nonnegative. By the spectral decomposition and functional calculus, $(S + I)_0 - I$ has a unique square root $T$ (See Pg. 127, 128 [12]).

Lemma 2.1. Let $S$ be a positive densely defined symmetric operator. Then the square root of $(S + I)_0 - I$ extends to an isometry from $H_S$ into $H$.

Proof: Clearly, the spectrum of $T$ is contained in the nonnegative part of the real line. By spectral decomposition $\mathcal{D}((S + I)_0 - I) = \mathcal{D}((S + I)_0) \subseteq \mathcal{D}(T)$ and $TT = (S + I)_0 - I$. In addition for any $u, v \in \mathcal{D}(S) \subseteq \mathcal{D}((S + I)_0)$,

$$(Tu, Tv) = (u, TTv) = (u, (S + I)_0v - v) = (u, Sv) = (u, v)_S.$$

So $T$ is an isometry from $\mathcal{D}(S)$ into $H$. Since $\mathcal{D}(S)$ is dense in $H_S$, $T$ extends to an isometry from $H_S$ into $H$. □

3 Degenerate Principal Series of $\tilde{Sp}(n, \mathbb{R})$

Fix the Lie algebra:

$$\mathfrak{sp}(n, \mathbb{R}) = \{ \begin{pmatrix} X & Y \\ Z & -X^t \end{pmatrix} \mid Y^t = Y, Z^t = Z \}$$

and the Siegel parabolic algebra:

$$\mathfrak{p} = \{ \begin{pmatrix} X & Y \\ 0 & -X^t \end{pmatrix} \mid Y^t = Y \}.$$

Fix the Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ with

$$\mathfrak{l} = \{ \begin{pmatrix} X & 0 \\ 0 & -X^t \end{pmatrix} \mid X \in \mathfrak{gl}(n, \mathbb{R}) \}, \quad \mathfrak{n} = \{ \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix} \mid Y^t = Y \}.$$

Fix a Cartan subalgebra

$$\mathfrak{a} = \{ \text{diag}(H_1, H_2, \ldots, H_n, -H_1, -H_2, \ldots, -H_n) \mid H_i \in \mathbb{R} \}.$$
Let $Sp(n, \mathbb{R})$ be the symplectic group and $P$ be the Siegel parabolic subgroup. Set $U(n) = Sp(n, \mathbb{R}) \cap O(2n)$ where $O(2n)$ is the standard orthogonal group. Let $LN$ be the Levi decomposition of $P$ and $A$ be the analytic group generated by the Lie algebra $a$. Clearly, $L \cong GL(n, \mathbb{R})$ and $L \cup U(n) \cong O(n)$. On the covering group, we have $\tilde{L} \cap \tilde{U}(n) = \tilde{O}(n)$.

Recall that

$$\tilde{U}(n) = \{(x, g) \mid g \in U(n), \exp 2\pi ix = \det g, x \in \mathbb{R}\}.$$ 

Therefore

$$\tilde{O}(n) = \{(x, g) \mid g \in O(n), \exp 2\pi ix = \det g, x \in \mathbb{R}\}.$$ 

Notice that for $g \in O(n)$, $\det g = \pm 1$ and $x \in \frac{1}{2}\mathbb{Z}$. We have the following exact sequence

$$1 \to SO(n) \to \tilde{O}(n) \to \frac{1}{2}\mathbb{Z} \to 1.$$ 

Consequently, we have

$$1 \to GL_0(n, \mathbb{R}) \to \tilde{L} \to \frac{1}{2}\mathbb{Z} \to 1.$$ 

In fact,

$$\tilde{L} = \{(x, g) \mid g \in L, \exp 2\pi ix = \frac{\det g}{|\det g|}, x \in \mathbb{R}\}.$$ 

The one dimensional unitary characters of $\frac{1}{2}\mathbb{Z}$ are parametrized by the one dimensional torus $T$. Identify $T$ with $[0, 1)$. Let $\mu^\epsilon$ be the character of $\frac{1}{2}\mathbb{Z}$ corresponding to $\epsilon \in [0, 1)$

Now each character $\mu^\epsilon$ yields a character of $\tilde{L}$, which in turn, yields a character of $\tilde{P}$. For simplicity, we retain $\mu^\epsilon$ to denote the character on $\tilde{L}$ and $\tilde{P}$. Let $\nu$ be the det-character on $L_0$, i.e.,

$$\nu(x, g) = |\det g| \quad (x, g) \in \tilde{L}. \quad (2)$$

Let

$$I(\epsilon, t) = \text{Ind}_{\tilde{P}}^{\tilde{Sp}(n, \mathbb{R})} \mu^\epsilon \otimes \nu^t$$

be the normalized induced representation with $\epsilon \in [0, 1)$ and $t \in \mathbb{C}$. This is Sahi’s notation in the case of the universal covering of the symplectic group ([14]). $I(\epsilon, t)$ is a degenerate principal series representation. Clearly, $I(\epsilon, t)$ is unitary when $t \in i\mathbb{R}$.

When $t$ is real and $I(\epsilon, t)$ is unitarizable, the unitary representation, often denoted by $C(\epsilon, t)$, is called a complementary series representation. Various complementary series of $Sp(n, \mathbb{R})$ and its metaplectic covering was determined explicitly or implicitly by Kudla-Rallis, Órsted-Zhang, Brason-Olafsson-Órsted and others. See [9], [3], [11] and the references therein. The complete classification of the complementary series of the universal covering is due to Sahi.

**Theorem 3.1 (Thm A, [14]).** Suppose that $t$ is real. For $n$ even, $I(\epsilon, t)$ is irreducible and unitarizable if and only if $0 < |t| < \frac{1}{2} - |2\epsilon - 1|$. For $n$ odd and $n > 1$, $I(\epsilon, t)$ is irreducible and unitarizable if and only if $0 < |t| < \frac{1}{2} - \frac{1}{2} - |2\epsilon - 1|$.

One can easily check that the complementary series exist if $\epsilon \neq 0, \frac{1}{2}$ for $n$ odd and $n > 1$; if $\epsilon \neq \frac{1}{4}, \frac{3}{4}$ for $n$ even. It is interesting to note that complementary series always exist unless $I(\epsilon, t)$ descends into a representation of the metaplectic group. For the metaplectic group $Mp(2n+1, \mathbb{R})$, there are two complementary series $I(\frac{1}{4}, t)(0 < t < \frac{1}{2})$ and $I(\frac{3}{4}, t)(0 < t < \frac{1}{2})$.

For the metaplectic group $Mp(2n, \mathbb{R})$, there are two complementary series $I(0, t)(0 < t < \frac{1}{2})$ and $I(\frac{1}{2}, t)(0 < t < \frac{1}{2})$. These four complementary series are the “longest”.

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For $n = 1$, the situation is quite different. The difference was pointed out in [9]. For example, there are Bargmann’s complementary series representation for $I(0, t)$ ($t \in (0, \frac{1}{2})$). The classification of the complementary series of $\tilde{Sp}(1, \mathbb{R})$ can be found in [2], [13], [7].

Since our restriction theorem only makes sense for $n \geq 2$, we will assume $n \geq 2$ from now on. The parameters for the complementary series of $\tilde{Sp}(n, \mathbb{R})$ are illustrated in fig.

4 The generalized compact model and The Intertwining Operator

Recall that

$$I^\infty(\epsilon, t) = \{ f \in C^\infty(\tilde{Sp}(n, \mathbb{R})) \mid f(gl_n) = (\mu^t \otimes \nu^t)(l^{-1})f(g), \quad (g \in \tilde{Sp}(n, \mathbb{R}), l \in \tilde{L}, n \in \mathbb{N}) \}$$

where $\rho = \frac{n+1}{2}$. Let $X = \tilde{Sp}(n, \mathbb{R})/\tilde{P}$. Then $I^\infty(\epsilon, t)$ consists of smooth sections of the homogeneous line bundle $L_{\epsilon, t}$

$$\tilde{Sp}(n, \mathbb{R}) \times \mathbb{C}_{\mu^t \otimes \nu^t} \to X.$$ 

Since $X \cong \tilde{U}(n)/\tilde{O}(n)$, $\tilde{U}(n)$ acts transitively on $X$. The function $f \in I^\infty(\epsilon, t)$ is uniquely determined by $f|_{\tilde{U}(n)}$ and vice versa. Moreover, the homogeneous vector bundle $L_{\epsilon, t}$ can be identified with $K_{\epsilon, t}$

$$\tilde{U}(n) \times \tilde{O}(n) \mathbb{C}_{\mu^t \otimes \nu^t}|_{\tilde{O}(n)} \to X$$

naturally. Notice that the homogeneous line bundle $K_{\epsilon, t}$ does not depend on the parameter $t$. We denote this line bundle by $K_{\epsilon}$. The representation $I^\infty(\epsilon, t)$ can then be modeled on smooth sections of $K_{\epsilon}$. This model will be called the generalized compact model.

Let $d[k]$ be the normalized $\tilde{U}(n)$-invariant measure on $X$. The generalized compact model equips the smooth sections of $K_{\epsilon, t}$ with a natural pre-Hilbert structure

$$(f_1, f_2)_{X} = \int_{[k] \in X} f_1(k)\overline{f}_2(k)d[k],$$

where $k \in \tilde{U}(n)$ and $[k] \in X$. It is easy to verify that $f_1(k)\overline{f}_2(k)$ is a function of $[k]$ and it does not depend on any particular choice of $k$. Notice that our situation is different from the compact model since $\tilde{U}(n)$ is not compact. We denote the completion of $I^\infty$ with respect to $(, )_X$ by $I_X(\epsilon, t)$.

Secondly, the action of $\tilde{U}(n)$ on $K_{\epsilon}$ induces an orthogonal decomposition of $I_X(\epsilon, t)$:

$$I_X(\epsilon, t) = \oplus_{\alpha \in \mathbb{Z}^n} V(\alpha + \epsilon(2, 2, \ldots, 2)),$$

where $V(\alpha + \epsilon(2, 2, \ldots, 2))$ is an irreducible finite dimensional representation of $\tilde{U}(n)$ with highest weight $\alpha + \epsilon(2, 2, \ldots, 2)$ and $\alpha$ satisfies

$$\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n.$$
This is essentially a consequence of Helgason’s theorem. Let

\[ V(\epsilon, t) = \bigoplus_{\alpha \in \mathbb{Z}^n} V(\alpha + \epsilon(2, 2, \ldots, 2)). \]

\( V(\epsilon, t) \) possesses an action of the Lie algebra \( \mathfrak{sp}(n, \mathbb{R}) \). It is the Harish-Chandra module of \( I(\epsilon, t) \). Clearly, \( V(\epsilon, t) \subset I^\infty(\epsilon, t) \subset I_X(\epsilon, t) \).

For each \( t \), there is an \( \tilde{Sp}(n, \mathbb{R}) \)-invariant sesquilinear pairing of \( I_X(\epsilon, t) \) and \( I_X(\epsilon, -\ell) \), namely,

\[ (f_1, f_2) = \int_X f_1(k)f_2(k)d[k], \]

where \( f_1 \in I_X(\epsilon, t) \) and \( f_2 \in I_X(\epsilon, -\ell) \). If \( t \in i\mathbb{R} \), we obtain a \( \tilde{Sp}(n, \mathbb{R}) \)-invariant Hermitian form which is exactly \( (,)_X \). Since \( (,)_X \) is positive definite, \( I_X(\epsilon, t) \) is a unitary representation of \( \tilde{Sp}(n, \mathbb{R}) \).

For each real \( t \), the form \((,)_X\) gives an \( \mathfrak{sp}(n, \mathbb{R}) \)-invariant sesquilinear pairing of \( V(\epsilon, t) \) and \( V(\epsilon, -t) \). In addition, there is an intertwining operator

\[ A(\epsilon, t) : V(\epsilon, t) \to V(\epsilon, -t) \]

which preserves the action of \( \mathfrak{sp}(n, \mathbb{R}) \) (see for example [8]). Define a Hermitian structure \((,)_\epsilon,t\) on \( V(\epsilon, t) \) by

\[ (u, v)_\epsilon,t = (A(\epsilon, t)u, v), \quad (u, v \in V(\epsilon, t)). \]

Clearly, \((,)_\epsilon,t\) is \( \mathfrak{sp}(n, \mathbb{R}) \)-invariant. So \( A(\epsilon, t) \) induces an invariant Hermitian form on \( V(\epsilon, t) \).

Now \( A(\epsilon, t) \) can also be realized as an unbounded operator on \( I_X(\epsilon, t) \) as follows. For each \( f \in V(\epsilon, t) \), define \( A_X(\epsilon, t)f \) to be the unique section of \( \mathcal{L}_{\epsilon,t} \) such that

\[ (A_X(\epsilon, t)f)|_{\tilde{U}(n)} = (A(\epsilon, t)f)|_{\tilde{U}(n)}. \]

Notice that \( A_X(\epsilon, t)f \in I(\epsilon, t) \) and \( A(\epsilon, t)f \in I(\epsilon, -t) \). They differ by a multiplier.

Now \( A_X(\epsilon, t) \) is an unbounded operator on the Hilbert space \( I_X(\epsilon, t) \). The following fact is well-known in many different forms. I state it in a way that is convenient for later use.

**Lemma 4.1.** Let \( \ell \in \mathbb{R} \). \( I(\epsilon, t) \) is unitarizable if and only if \( A_X(\epsilon, t) \) extends to a self-adjoint operator on \( I_X(\epsilon, t) \) with spectrum on the nonnegative part of the real axis.

The spectrum of \( A_X(\epsilon, t) \) was computed in [8] and [11] explicitly for special cases and in [14] implicitly. In particular, \( A_X(\epsilon, t) \) restricted onto each \( \tilde{U}(n) \)-type is a scalar multiplication and the scalar is bounded by a polynomial on the highest weight. We obtain

**Lemma 4.2 ([15]).** \( A_X(\epsilon, t) \) extends to an unbounded operator from \( I^\infty(\epsilon, t) \) to \( I^\infty(\epsilon, t) \).

This lemma follows from a standard argument that the norm of each \( \tilde{U}(n) \)-component in the Peter-Weyl expansion of any smooth section of \( \mathcal{K}_e \) decays rapidly with respect to the highest weight. It is true in general (see [15]).
5 Actions of \( \hat{Sp}(p, \mathbb{R})\hat{Sp}(q, \mathbb{R}) \)

Suppose that \( p + q = n \) and \( p \leq q \). Fix a standard basis
\[
\{e_1, e_2, \ldots, e_p; e_1^*, e_2^*, \ldots, e_p^*\}
\]
for the symplectic form \( \Omega_p \) on \( \mathbb{R}^{2p} \). Fix a standard basis
\[
\{f_1, f_2, \ldots, f_q; f_1^*, f_2^*, \ldots, f_q^*\}
\]
for the symplect form \( \Omega_q \) on \( \mathbb{R}^{2q} \).

**Definition 5.1.** Let \( Sp(p, \mathbb{R}) \) be the symplectic group preserving \( \Omega_p \) and \( Sp(q, \mathbb{R}) \) be the symplectic group preserving \( \Omega_q \). Let
\[
\Omega = \Omega_p - \Omega_q
\]
and \( Sp(n, \mathbb{R}) \) be the symplectic group preserving \( \Omega \). We say that \( (Sp(p, \mathbb{R}), Sp(q, \mathbb{R})) \) is diagonally embedded in \( Sp(n, \mathbb{R}) \).

We shall make a remark here. In [6], \( \Omega = \Omega_p + \Omega_q \). \( Sp(p, \mathbb{R})Sp(q, \mathbb{R}) \) is embedded differently there. The effect of this difference is an involution \( \tau \) on the representation level.

Let \( P_{p,q-p} \) be the subgroup of \( Sp(q, \mathbb{R}) \) that preserves the linear span of \( \{f_{p+1}, \ldots, f_q\} \). Choose the Levi factor \( GL(q-p)Sp(p, \mathbb{R}) \) to be the subgroup of \( P_{p,q-p} \) that preserves the span of \( \{f_{p+1}^*, \ldots, f_q^*\} \). In particular the \( Sp(p, \mathbb{R}) \) factor can be identified with the symplectic group of
\[
\text{span}\{f_1, \ldots, f_p; f_1^*, \ldots, f_p^*\},
\]
which will be identified with the standard \( Sp(p, \mathbb{R}) \). More precisely, for \( x \in Sp(p, \mathbb{R}) \), by identify \( e_i \) with \( f_i \) and \( e_i^* \) with \( f_i^* \) and extending \( x \) trivially on \( f_{p+1}, \ldots, f_q; f_{p+1}^*, \ldots, f_q^* \), we obtain the identification
\[
x \in Sp(p, \mathbb{R}) \rightarrow \hat{x} \in Sp(q, \mathbb{R}). \tag{3}
\]

Now fix a Lagrangian Grassmanian
\[
x_0 = \text{span}\{e_1 + f_1, \ldots, e_p + f_p, e_1^* + f_1^*, \ldots, e_p^* + f_p^*, f_{p+1}, \ldots, f_q\}.
\]

Then the stabilizer \( Sp(q, \mathbb{R})_{x_0} = GL(q-p)N_{p,q-p} \) where \( N_{p,q-p} \) is the nilradical of \( P_{p,q-p} \). Put
\[
\Delta(Sp(p, \mathbb{R})) = \{ (u, \hat{u}) \mid u \in Sp(p, \mathbb{R}) \} \subseteq Sp(p, \mathbb{R})Sp(q, \mathbb{R})
\]
and
\[
H = \Delta(Sp(p, \mathbb{R}))GL(q-p)N_{p,q-p}.
\]

**Lemma 5.1 ([6]).** Let \( p \leq q \) and \( p + q = n \). Let \( X_0 \) be the \( Sp(p, \mathbb{R}) \times Sp(q, \mathbb{R}) \)-orbit generated by \( x_0 \). Then \( X_0 \) is open and dense in \( X \) and \( [Sp(p, \mathbb{R})Sp(q, \mathbb{R})]_{x_0} = H \).

Notice here that \( X_0 \) depends on \( (p,q) \). Let \( P = Sp(n, \mathbb{R})_{x_0} \). The smooth representation \( \mathcal{I}(\epsilon, t) \) consists of smooth sections of \( \mathcal{L}_{\epsilon,t} : \)
\[
\hat{Sp}(n, \mathbb{R}) \times P \mathcal{C}_\mu \circ \mathcal{O}_{\mu + \eta} \rightarrow X.
\]

Consider the subgroup \( \hat{Sp}(p, \mathbb{R})\hat{Sp}(q, \mathbb{R}) \) in \( \hat{Sp}(n, \mathbb{R}) \). Notice that \( \hat{Sp}(p, \mathbb{R}) \cap \hat{Sp}(q, \mathbb{R}) \cong \mathbb{Z} \). So \( \hat{Sp}(p, \mathbb{R})\hat{Sp}(q, \mathbb{R}) \) is not a direct product, but rather the product of the two groups as sets.

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Definition 5.2. For any \( f \in I_X(\epsilon, t) \), define

\[ I_{X_0} = f|_{\tilde{S}(p, \mathbb{R})} \tilde{S}(q, \mathbb{R}) \to \mathcal{L}_{\epsilon, t} \]

Let \( I_{\epsilon, X_0}(\epsilon, t) \) be the set of smooth sections of \( \mathcal{L}_{\epsilon, t} \) that are compactly supported in \( X_0 \).

Clearly \( f_{X_0} \) is a smooth section of

\[ \tilde{S}(p, \mathbb{R}) \tilde{S}(q, \mathbb{R}) \times \tilde{H} \mathbb{C}_{\mu^* \otimes \nu^{t+p}} \to X_0. \]

Notice that \( \Delta(\tilde{S}(p, \mathbb{R})) \) sits inside of \( SL(n, \mathbb{R}) \subseteq GL(n, \mathbb{R}) \subseteq \mathbb{P} \). The universal covering of \( \tilde{S}(p, \mathbb{R}) \) splits over \( SL(n, \mathbb{R}) \subseteq \mathbb{P} \). Similarly the universal covering of \( \tilde{S}(q, \mathbb{R}) \) also splits over \( N_{p,q-p} \). So we have

\[ \tilde{H} \cong \Delta(\tilde{S}(p, \mathbb{R})) \tilde{GL}(q-p)N_{p,q-p}, \]

where \( \tilde{GL}(q-p)N_{p,q-p} \subseteq \tilde{S}(q, \mathbb{R}) \). In particular, \( \mu^* \otimes \nu^{t+p}|_{\Delta(\tilde{S}(p, \mathbb{R}))N_{p,q-p}} \) is trivial and \( \mu^* \otimes \nu^{t+p}|_{\tilde{GL}(q-p)} \) is essentially the restriction from \( GL(p + q) \) to \( GL(q-p) \). If \( p = q \), then \( GL(0) \) will be the identity element. So \( \tilde{GL}(0) \) is just \( C \). We have

Lemma 5.2. The identification \[ x \in S(p, \mathbb{R}) \to \tilde{x}S(q, \mathbb{R}) \]

lifts naturally to \( \tilde{S}(p, \mathbb{R}) \to \tilde{S}(q, \mathbb{R}) \). Let \( \phi \in I^\infty(\epsilon, t) \). Then

\[ \phi(g_1, g_2) = \phi(1, g_2g_1^{-1}) \quad (g_1 \in \tilde{S}(q, \mathbb{R}), g_2 \in \tilde{S}(q, \mathbb{R})). \]

In addition

\[ \phi(1, g_2h) = \mu^* \otimes \nu^{t+p}(h^{-1})\phi(1, g_2) \quad (h \in \tilde{GL}(q-p)N_{p,q-p}). \]

Now let us consider the action of \( \tilde{S}(p, \mathbb{R}) \) and \( \tilde{S}(q, \mathbb{R}) \) on \( I(\epsilon, t) \). By Lemma 5.2, we obtain

Lemma 5.3. Let \( \phi \in I^\infty(\epsilon, t) \) and \( h_1 \in \tilde{S}(p, \mathbb{R}) \) and \( g_2 \in \tilde{S}(p, \mathbb{R}) \). Then

\[ [I(\epsilon, t)(h_1)\phi](1, g_2) = f(1, g_2h_1). \]

In particular the restriction map

\[ \phi \in I^\infty(\epsilon, t) \to \phi|_{\tilde{S}(q, \mathbb{R})} \in C^\infty(\tilde{S}(q, \mathbb{R}) \times \tilde{GL}(q-p)N_{p,q-p} \mathbb{C}_{\mu^* \otimes \nu^{t+p}}) \]

intertwines the left regular action of \( \tilde{S}(p, \mathbb{R}) \) on \( I^\infty(\epsilon, t) \) with the right regular action of \( \tilde{S}(q, \mathbb{R}) \) on \( C^\infty(\tilde{S}(q, \mathbb{R}) \times \tilde{GL}(q-p)N_{p,q-p} \mathbb{C}_{\mu^* \otimes \nu^{t+p}}) \).

Obviously, the restriction map also intertwines the left regular actions of \( \tilde{S}(q, \mathbb{R}) \).
6 Mixed Model

Now fix complex structures on $\mathbb{R}^{2p}$ and $\mathbb{R}^{2q}$ and inner products $(\ ,)_p$, $(\ ,)_q$ such that

$$\Omega_p = \Im(\ ,)_p, \quad \Omega_q = -\Im(\ ,)_q.$$ 

Let $U(p)$ and $U(q)$ be the unitary groups preserving $(\ ,)_p$ and $(\ ,)_q$ respectively. $U(p)$ and $U(q)$ are maximal compact subgroups of $Sp(p, \mathbb{R})$ and $Sp(q, \mathbb{R})$. Let $U(n)$ be the unitary group preserving $(\ ,)_p + (\ ,)_q$. Then $U(n)$ is a maximal compact subgroup of $Sp(n, \mathbb{R})$. In addition,

$$U(p) = Sp(p, \mathbb{R}) \cap U(n) \quad U(q) = Sp(q, \mathbb{R}) \cap U(n).$$

Identify $U(q) \cap P_{p,q-p}$ with $O(q-p)U(p)$. Recall that $X_0 \cong Sp(q, \mathbb{R})/GL(q-p)N_{p,q-p}$. The group $Sp(p, \mathbb{R})$ acts on $X_0$ freely from the right. We obtain a principal fibration

$$Sp(p, \mathbb{R}) \to X_0 \to Sp(q, \mathbb{R})/P_{p,q-p} \cong U(q)/O(q-p)U(p).$$

Let $dg_1$ be a Haar measure on $Sp(p, \mathbb{R})$ and $d[k_2]$ be an invariant probability measure on $U(q)/O(q-p)U(p)$. Then $dg_1[d[k_2]]$ defines an $U(q)Sp(p, \mathbb{R})$ invariant measure on $X_0$.

**Definition 6.1.** Let $M = Sp(p, \mathbb{R})U(q) \subset Sp(p, \mathbb{R})Sp(q, \mathbb{R}) \subset Sp(n, \mathbb{R})$. Elements in $X_0$ are parametrized by a pair $(g_1, [k_2])$ for $(g_1, k_2) \in M$. For each $g \in \tilde{Sp}(n, \mathbb{R})$, write $g = \tilde{u}(q)p(g)$ where $\tilde{u}(g) \in \tilde{U}(n)$ and $p(g) \in P_0$, the identity component of $P$. For each $(g_1, k_2) \in (Sp(p, \mathbb{R}), U(q))$, we have

$$g_1k_2 = \tilde{u}(g_1k_2)p(g_1k_2) = k_2\tilde{u}(g_1)p(g_1).$$

The component $\tilde{u}$ defines a map from $\tilde{M}$ to $\tilde{U}(n)$. In particular, $\tilde{u}$ induces a map from $\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)$ to $\tilde{U}(n)/\tilde{O}(n)$ which will be denoted by $j$. The map $j$ parametrizes the open dense subset $X_0$ in $X$ by

$$([g_1], [k_2]) \in \tilde{Sp}(p, \mathbb{R})/C \times \tilde{U}(q)/\tilde{O}(q-p)\tilde{U}(p).$$

Change the variables on $X_0$ from $\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)$ to $\tilde{U}(n)/\tilde{O}(n)$. Let $J([g_1], [k_2])$ be the Jacobian:

$$\frac{dj([g_1], [k_2])}{d[g_1]d[k_2]}.$$ 

$J$ can be regarded as a function on $Sp(p, \mathbb{R})U(q)$ or $Sp(p, \mathbb{R})U(q)/U(p)O(q-p)$, even though it is defined as a function on the covering. Denote the line bundle

$$\tilde{Sp}(q, \mathbb{R}) \times_{GL(q-p)N_{p,q-p}} C_{\mu^* \otimes \mu^* + p} \to X_0.$$ 

by $\mathcal{M}_{\epsilon,t}$. Denote the line bundle

$$\tilde{M} \times_{\tilde{O}(q-p)\tilde{U}(p)} C_{\mu^*} \to \tilde{M}/\tilde{O}(q-p)\tilde{U}(p) \cong X_0.$$ 

by $\mathcal{M}_\epsilon$.

Clearly, $I_{c,X_0}^\infty(\epsilon,t) \subset I^\infty(\epsilon,t)$. Consider the restriction of $(\ ,)_X$ onto $I_{c,X_0}^\infty(\epsilon,t)$. We are interested in expressing $(\ ,)_X$ as an integral on $\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)$. This boils down to a change of variables from $U(n)/\tilde{O}(n)$ to $\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)$. We have
Lemma 6.1. Let $\Delta_t(g_1, k_2) = \nu(p(g_1))^{t+1-\rho/2}J([g_1], [k_2])$ (see Equ. (2)). Then for every $f_1, f_2 \in I^\infty(\epsilon, t)$ we have

$$(f_1, f_2)_X = \int_{\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)} f_1(g_1k_2)\overline{f_2(g_1k_2)}\Delta_t(g_1, k_2)d[g_1]d[k_2]$$

where $g_1 \in \tilde{S}p(p, \mathbb{R})$, $k_2 \in \tilde{U}(q)$, $[g_1] \in Sp(p, \mathbb{R})$ and $[k_2] \in \tilde{U}(q)/\tilde{O}(q-p)\tilde{U}(p)$. Furthermore, $\Delta_t(g_1, k_2)$ is a nonnegative right $\tilde{O}(q-p)\tilde{U}(p)$-invariant function on $M$.

Proof: We compute

$$
\int_{\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)} f_1(g_1k_2)\overline{f_2(g_1k_2)}\Delta_t(g_1, k_2)d[g_1]d[k_2]
= \int_{\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)} f_1(\tilde{u}(g_1k_2))\overline{f_2(\tilde{u}(g_1k_2))}\nu(p(g_1))^{t-1-\rho/2}\Delta_t(g_1, k_2)d[g_1]d[k_2]
= \int_{\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)} f_1(\tilde{u}(g_1k_2))\overline{f_2(\tilde{u}(g_1k_2))}\nu(p(g_1))^{t-1-\rho/2}\Delta_t(g_1, k_2)J^{-1}(g_1, k_2)J^{-1}([g_1], [k_2])
= \int_{X_\rho} f_1(\tilde{u})\overline{f_2(\tilde{u})}d[\tilde{u}] = (f_1, f_2)_X.
$$

Since $\nu(p(g_1))$ and $J([g_1], [k_2])$ remain the same when we multiply $k_2$ on the right by $\tilde{O}(q-p)\tilde{U}(p)$, $\Delta_t(g_1, k_2)$ is a nonnegative right $\tilde{O}(q-p)\tilde{U}(p)$-invariant function.

Combining with Lemma 5.3 we obtain

Corollary 6.1. As representations of $\tilde{S}p(p, \mathbb{R})\tilde{S}p(q, \mathbb{R})$,

$$I_X(\epsilon, t) \equiv L^2(\mathcal{M}_{\epsilon, t}, \Delta_t[d[g_1]d[k_2]]).$$

For each $f_1, f_2 \in I^\infty_{c, X_\rho}(\epsilon, t)$, define

$$(f_1, f_2)_{M, t} = \int_{\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)} f_1(g_1k_2)\overline{f_2(g_1k_2)}\Delta_t(g_1, k_2)d[g_1]d[k_2],$$

$$(f_1, f_2)_M = \int_{\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)} f_1(g_1k_2)\overline{f_2(g_1k_2)}d[g_1]d[k_2].$$

The completion of $I^\infty_{c, X_\rho}(\epsilon, t)$ under $(,)_M$ is $L^2(\mathcal{M}_{\epsilon, t}, \Delta_t[d[g_1]d[k_2]])$. We call $L^2(\mathcal{M}_{\epsilon, t}, \Delta_t[d[g_1]d[k_2]])$, the mixed model. We denote it by $I_M(\epsilon, t)$. On $I_M(\epsilon, t)$, the actions of $\tilde{S}p(p, \mathbb{R})$ and $\tilde{S}p(q, \mathbb{R})$ are easy to manipulate.

7 Mixed Model for Unitary Principal Series

Lemma 7.1. If $t \in i\mathbb{R}$, then $\Delta_t(g_1, k_2)$ is a constant and $(,)_M$ is a constant multiple of $(,)_M$.
Proof: Let \( t \in i \mathbb{R} \). Let \( f_1, f_2 \in L^\infty(\epsilon, t) \) and \( h \in \widetilde{Sp}(p, \mathbb{R}) \). Recall that \( X_0 \) is parametrized by a pair \([g_1] \in \widetilde{Sp}(p, \mathbb{R})/C\) and \([k_2] \in \tilde{U}(q)/\tilde{O}(q-p)\tilde{U}(p)\). By Lemma 6.1, we have

\[
(I(\epsilon, t)(h) f_1, I(\epsilon, t)(h) f_2)_X = \int_{X_0} f_1(h^{-1}g_1k_2)f_2(h^{-1}g_1k_2) \Delta_t(g_1, k_2)d[g_1]d[k_2]
\]

(5)

Since \( I(\epsilon, t) \) is unitary, \((I(\epsilon, t)(h) f_1, I(\epsilon, t)(h) f_2)_X = (f_1, f_2)_X\). We have

\[
\int_{X_0} f_1(g_1k_2)f_2(g_1k_2) \Delta_t(hg_1, k_2)d[g_1]d[k_2] = \int_{X_0} f_1(g_1k_2)f_2(g_1k_2) \Delta_t(g_1, k_2)d[g_1]d[k_2].
\]

It follows that \( \Delta_t(hg_1, k_2) = \Delta_t(g_1, k_2) \) for any \( h \in \widetilde{Sp}(p, \mathbb{R}) \). Similarly, we obtain \( \Delta_t(g_1, kk_2) = \Delta(g_1, k_2) \) for any \( k \in \tilde{U}(q) \). Hence, \( \Delta_t(g_1, k_2) \) is a constant for purely imaginary \( t \). \( \square \)

Combined with Cor. 6.1, we obtain

**Theorem 7.1.** Let \( t \in i \mathbb{R} \). The restriction map \( f \mapsto f_{X_0} \) induces an isometry between \( I(\epsilon, t) \) and \( L^2(M_{\epsilon, t}, d[g_1]d[k_2]) \). In addition, this isometry intertwines the actions of \( \widetilde{Sp}(p, \mathbb{R}) \) and \( \widetilde{Sp}(q, \mathbb{R}) \) representations,

\[ I(\epsilon, t) \cong L^2(M_{\epsilon, t}, d[g_1]d[k_2]); \]

and as \( \widetilde{Sp}(p, \mathbb{R}) \tilde{U}(q) \) representations,

\[ I(\epsilon, t) \cong L^2(M_{\epsilon, t}, d[g_1]d[k_2]). \]

Notice that \( L^2(M_{\epsilon, t}, d[g_1]d[k_2]) \) does not depend on the parameter \( t \). The following corollary is automatical.

**Corollary 7.1.** Suppose that \( p + q = n \) and \( p \leq q \). For \( t \) real,

\[ I(\epsilon, it)|_{\widetilde{Sp}(p, \mathbb{R})\tilde{U}(q)} \cong I(\epsilon, 0)|_{\widetilde{Sp}(p, \mathbb{R})\tilde{U}(q)} \cong L^2(M_{\epsilon, t}, d[g_1]d[k_2]). \]

For \( t \) a nonzero real number, \( \Delta_t(q, k) \) is not a constant. So \( C(\epsilon, t) \) cannot be modeled naturally on \( L^2(M_{\epsilon, t}, d[g_1]d[k_2]) \). Nevertheless, we have

**Theorem 7.2** (Main Theorem). Suppose that \( p + q = n \) and \( p \leq q \). Given a complementary series representation \( C(\epsilon, t) \),

\[ C(\epsilon, t)|_{\widetilde{Sp}(p, \mathbb{R})\tilde{U}(q)} \cong I(\epsilon, 0)|_{\widetilde{Sp}(p, \mathbb{R})\tilde{U}(q)} \cong L^2(M_{\epsilon, t}, d[g_1]d[k_2]). \]

In other words, there is an isometry between \( C(\epsilon, t) \) and \( I(\epsilon, 0) \) that intertwines the actions of \( \tilde{U}(q) \) and of \( \widetilde{Sp}(p, \mathbb{R}) \).

We shall postpone the proof of this theorem to the next section. We will first derive some corollaries from Lemma 7.1 concerning \( \Delta \) and \( \nu(g_1) \).

**Corollary 7.2.** \( J([g_1], [k_2]) = cv(p(g_1))^{-2p} \) for a constant \( c \) and \( \Delta_t(g_1, k_2) = cv(p(g_1))^{t\nu}. \) So both \( \Delta_t \) and \( J([g_1], [k_2]) \) do not depend on \( k_2 \). Furthermore,

\[ I(\epsilon, t) \cong L^2(M_{\epsilon, t}, \nu(p(g_1)))^{t\nu}d[g_1]d[k_2] = I_M(\epsilon, t). \]
\( \nu(p(g_1)) \) is a function on \( \widetilde{Sp}(p, \mathbb{R})/O \). So it can be regarded as a function on \( Sp(p, \mathbb{R}) \).

**Corollary 7.3.** \( \nu(p(g_1))^{-p} \in L^2(Sp(p, \mathbb{R})) \) and \( \nu(p(g_1))^{-1} \) is a bounded positive function.

Proof: Since \( X \) is compact, \( \int_{Sp(p, \mathbb{R})} \nu(p(g_1))^{-2p}dg_1 = C \int_{\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)} J([g_1], [k_2])d[g_1]d[k_2] = C \int_{\tilde{U}(n)/\tilde{O}(n)} 1d[k] < \infty. \) So \( \nu(p(g_1))^{-p} \in L^2(Sp(p, \mathbb{R})) \). Now we need to compute \( \nu(g_1) \). Recall that \( P \) is defined to be the stabilizer of

\[ x_0 = \text{span}\{e_1 + f_1, \ldots, e_p + f_p, e_1^* + f_1^*, \ldots, e_p^* + f_p^*, f_{p+1}, \ldots f_q\}. \]

So \( j(g_1, 1) \) is the following Lagrangian

\[ \text{span}\{g_1e_1 + f_1, \ldots, g_1e_p + f_p, g_1e_1^* + f_1^*, \ldots, g_1e_p^* + f_p^*, f_{p+1}, \ldots f_q\}. \]

The action of \( \tilde{U}(n) \) will not change the volume of the \( n \)-dimensional cube spanned by the basis above. So \( \nu(p(g_1)) \), as the determinant character, is equal to the volume of the \( n \)-dimensional cube, up to a constant. Hence

\[ \nu(p(g_1)) = [2^{-n} \det(g_1g_1^* + I)]^{1/2}. \]

Clearly, \( \nu(p(g_1))^{-1} \) is bounded and positive. \( \square \)

This corollary is easy to understand in terms of compactification. Notice that the map \( j \), without the covering,

\[ Sp(p, \mathbb{R})U(q)/U(p)O(q-p) \rightarrow U(n)/O(n) \]

is an analytic compactification. Hence the Jacobian \( J(g_1, [k_2]) \) should be positive and bounded above. Since \( J(g_1, [k_2]) = c\nu(p(g_1))^{-2p} \), \( \nu(p(g_1))^{-1} \) must also be positive and bounded above. The situation here is similar to \([4]\) (see Appendix) and \([5]\) (Theorem 2.3). It is not clear that \( j(g_1, 1) \) gets mapped onto \( U(2p)/O(2p) \) though.

If \( f \in I_M(\epsilon, t_1) \) and \( h > 0 \), by Cor. \([7,3]\) and Equation \([8]\), we have \( \|f\|_{M,t_1-h} \leq C\|f\|_{M,t_1} \). So \( I_M(\epsilon, t_1) \subset I_M(\epsilon, t_1-h) \).

**Corollary 7.4.** Suppose that \( h > 0 \). Then \( I_M(\epsilon, t_1) \subset I_M(\epsilon, t_1-h) \).

## 8 "Square Root" of the Intertwining Operator

Suppose from now on \( t \in \mathbb{R} \). For \( f \in I^\infty(\epsilon, t)|_{\tilde{M}} \), define a function on \( \tilde{M} \),

\[ (A_M(\epsilon, t)f)(g_1k_2) = A(\epsilon, t)f(g_1k_2) \quad (g_1 \in \widetilde{Sp}(p, \mathbb{R}), k_2 \in \tilde{U}(q)). \]

So \( A_M(\epsilon, t) \) is the “restriction” of \( A(\epsilon, t) \) onto \( \tilde{M} \). \( A_M(\epsilon, t) \) is not yet an unbounded operator on \( I_M(\epsilon, t) \). In fact, for \( t > 0 \), \( A_M(\epsilon, t) \) does not behave well and it is not clear whether \( A_M(\epsilon, t) \) can be realized as an unbounded operator on \( I_M(\epsilon, t) \). The function \( A_M(\epsilon, t)f \) differs from \( A_X(\epsilon, t)f \).
Lemma 8.1. For $t \in \mathbb{R}$ and $f \in I^{\infty}(\epsilon,t)$,

$$(A_M(\epsilon,t)f|_{\tilde{M}})(g_1 k_2) = (A_X(\epsilon,t)f)(g_1 k_2)\nu(p(g_1))^{2t} = (A_X(\epsilon,t)f)(g_1 k_2)\Delta_t(g_1, k_2).$$

This Lemma is due to the fact that $A_X(\epsilon,t)f \in I(\epsilon,t)$ but $A(\epsilon,t)f \in I(\epsilon, -t)$.

Let $f \in I^{\infty}(\epsilon,t)$. In terms of the mixed model, the invariant Hermitian form $(\cdot, \cdot)_{\epsilon,t}$ can be written as follows:

$$(f, f)_{\epsilon,t} = (Ax(\epsilon,t)f, f)_{X} = \int_{\tilde{M}/\bar{O}(\nu-p)U(p)} A_M(\epsilon,t)f|_{\tilde{M}} \tilde{\tau}_{\bar{M}} d[g_1]d[k_2].$$

This follows from Lemma 8.1 and Lemma 6.1. We obtain

Lemma 8.2. For $f_1, f_2 \in I^{\infty}(\epsilon,t)$, $(f_1, f_2)_{\epsilon,t} = (A_M(\epsilon,t)f_1|_{\tilde{M}}, f_2|_{\tilde{M}})_M$.

Theorem 8.1. If $t < 0$ and $C(\epsilon,t)$ is a complementary series representation, then $A_M(\epsilon,t)$ is a positive and densely defined symmetric operator. Its self-adjoint-extension $(A_M(\epsilon,t) + I)_0 - I$ has a unique square root which extends to an isometry from $C(\epsilon,t)$ onto

$L^2(M, d[g_1]d[k_2])$.

Proof: Let $t < 0$. Put

$$\mathcal{H} = L^2(M, d[g_1]d[k_2]).$$

Let $f \in I^{\infty}(\epsilon,t)$. Then $A_M(\epsilon,t)(f|_{\tilde{M}})(g_1 k_2) = \nu(p(g_1))^{2t}A_X(\epsilon,t)f(g_1 k_2)$. By Lemma 8.1 Cor. 7.3 and Lemma 6.1 we have

$$\int_{\tilde{M}/\bar{O}(\nu-p)U(p)} A_M(\epsilon,t)f|_{\tilde{M}}A_M(\epsilon,t)(f|_{\tilde{M}})d[g_1]d[k_2] = \int_{\tilde{M}/\bar{O}(\nu-p)U(p)} \nu(p(g_1))^{2t}||A_X(\epsilon,t)f)(g_1 k_2)||^2\nu(p(g_1))^{2t}d[g_1]d[k_2]
\geq C \int_{\tilde{M}/\bar{O}(\nu-p)U(p)} |A_X(\epsilon,t)f(g_1 k_2)|^2\Delta_t(g_1, k_2)d[g_1]d[k_2]
= C(A_X(\epsilon,t)f, A_X(\epsilon,t)f)_X < \infty.$$

Therefore, $A_M(\epsilon,t)(f|_{\tilde{M}}) \in \mathcal{H}$. Let $D = I^{\infty}(\epsilon,t)|_{\tilde{M}}$. Clearly, $D$ is dense in $\mathcal{H}$. So $A_M(\epsilon,t)$ is a densely defined unbounded operator. It is positive and symmetric by Lemma 8.2.

Definition 8.1. Define $\mathcal{U}(\epsilon,t) = ((A_M(\epsilon,t) + I)_0 - I)^{\frac{1}{2}}$.

Now $(f, g)_{\epsilon,t} = (A_M(\epsilon,t)f|_{\tilde{M}}, g|_{\tilde{M}})_M$ for any $f, g \in I^{\infty}(\epsilon,t)$. So $C(\epsilon,t) = \mathcal{H}_{A_M(\epsilon,t)}$. By Lemma 2.1, $\mathcal{U}(\epsilon,t)$, mapping from $C(\epsilon,t)$ into $\mathcal{H}$, is an isometry.

Suppose that $\mathcal{U}(\epsilon,t)$ is not onto. Let $f \in \mathcal{H}$ such that for any $u \in D(\mathcal{U}(\epsilon,t))$,

$$(f, \mathcal{U}(\epsilon,t)u)_M = 0.$$ 

Notice that

$I^{\infty}(\epsilon,t)|_{\tilde{M}} \subset D((A_M(\epsilon,t) + I)_0 - I) \subset D(\mathcal{U}(\epsilon,t))$.
Since \( L \) can be defined as the inverse of \( \langle A \rangle \),

\[ U(\epsilon, t) U(\epsilon, t) = (A_M(\epsilon, t) + I)_0 - I. \]

In particular,

\[ U(\epsilon, t) I^\infty(\epsilon, t)|_{\tilde{M}} \subset D(U(\epsilon, t)). \]

It follows that

\[ \langle f, A_M(\epsilon, t) I^\infty(\epsilon, t)|_{\tilde{M}} \rangle_M = \langle f, (A_M(\epsilon, t) + I)_0 - I I^\infty(\epsilon, t)|_{\tilde{M}} \rangle_M = \langle f, U(\epsilon, t) U(\epsilon, t) I^\infty(\epsilon, t)|_{\tilde{M}} \rangle_M = 0. \]

Let \( f_{\epsilon, t} \) be a function such that \( f_{\epsilon, t}|_{\tilde{M}} = f \) and

\[ f_{\epsilon, t}(g l n) = (\mu^\epsilon \otimes \nu^l + r)(l^{-1}) f_{\epsilon, t}(g) \quad (l \in \tilde{L}, n \in N). \]

By Lemma 8.2, \( \forall u \in V(\epsilon, t) \),

\[ 0 = \langle f, A_M(\epsilon, t)(u|_{\tilde{M}}) \rangle_M = \langle f_{\epsilon, t}, A_X(\epsilon, t) u \rangle_X = \langle f_{\epsilon, t}, u \rangle_{\epsilon, t}. \]

This equality is to be interpreted as an equality of integrals according to the definitions of \( \langle , \rangle_M \) and \( \langle , \rangle_X \). Since \( A_X(\epsilon, t) \) acts on \( U(n) \)-types in \( V(\epsilon, t) \) as scalars, \( A_X(\epsilon, t)V(\epsilon, t) = V(\epsilon, t) \). We now have

\[ \langle f_{\epsilon, t}, V(\epsilon, t) \rangle_X = 0. \]

In particular, \( f_{\epsilon, t}|_{U(n)} \in L^1(X) \). By Peter-Weyl Theorem, \( f_{\epsilon, t} = 0 \). We see that \( U(\epsilon, t) \) is an isometry from \( C(\epsilon, t) \) onto \( L^2(\mathcal{M}_c, d[g_1]d[k_2]) \). \( \Box \)

The Hilbert space \( L^2(\mathcal{M}_c, d[g_1]d[k_2]) \) is the mixed model for \( I(\epsilon, 0) \) restricted to \( \tilde{M} \). We now obtain an isometry from \( C(\epsilon, t) \) onto \( I(\epsilon, 0) \). Within the mixed model, the action of \( I(\epsilon, t)(g|_{k_1}k_2) \) is simply the left regular action and it is independent of \( t \). We obtain

**Lemma 8.3.** Suppose \( t < 0 \). Let \( g \in \tilde{U}(g) \). Let \( L(g) \) be the left regular action on \( L^2(\mathcal{M}_c, d[g_1]d[k_2]) \). As an operator on \( I^\infty(\epsilon, t)|_{\tilde{M}} \), \( L(g) \) commutes with \( A_M(\epsilon, t) \). Furthermore, \( L(g) \) commutes with \( (A_M(\epsilon, t) + I)_0 - I \). Similar statement holds for \( g \in \tilde{Sp}(p, \mathbb{R}) \).

**Proof:** Let \( g \in \tilde{M} \). Both \( A_M(\epsilon, t) \) and \( L(g) \) are well-defined operator on \( I^\infty(\epsilon, t)|_{\tilde{M}} \). Regarding \( A(\epsilon, t)I(\epsilon, t)(g) = I(\epsilon, -t)(g)A(\epsilon, t) \) as operators on the mixed model \( L^2(\mathcal{M}_c, d[g_1]d[k_2]) \), we have

\[ A_M(\epsilon, t)L(g) = L(g)A_M(\epsilon, t). \]

It follows that

\[ L(g)^{-1}(A_M(\epsilon, t) + I)L(g) = (A_M(\epsilon, t) + I). \]

Since \( L(g) \) is unitary, \( L(g)^{-1}(A_M(\epsilon, t) + I)_0 L(g) = (A_M(\epsilon, t) + I)_0 \). In fact, \( (A_M(\epsilon, t) + I)_0 \) can be defined as the inverse of \( (A_M(\epsilon, t) + I)^{-1} \), which exists and is bounded. So \( L(g) \) commutes with both \( (A_M(\epsilon, t) + I)^{-1} \) and \( (A_M(\epsilon, t) + I)_0 \). \( \Box \)

**Lemma 8.4.** We have, for \( g \in \tilde{M} \), \( U(\epsilon, t)I(\epsilon, t)(g) = I(\epsilon, 0)(g)U(\epsilon, t) \).
Proof: Recall from Theorem 7.1 that the action of $\tilde{M}$ on the mixed model is independent of $t$. It suffices to show that on the mixed model, $\mathcal{U}(\epsilon, t)$ commutes with $L(g)$ for any $g \in \tilde{M}$. By Lemma 8.3,

$$L(g)^{-1}[(A_M(\epsilon, t) + I) - I]L(g) = (A_M(\epsilon, t) + I) - I.$$ 

Since $L(g)$ is unitary on $L^2(\mathcal{M}_\epsilon, d[g_1]d[k_2])$, both sides are positive self-adjoint operators. Taking square roots, we obtain $L(g)^{-1}\mathcal{U}(\epsilon, t)L(g) = \mathcal{U}(\epsilon, t)$. \[\square\]

Theorem 7.1 is proved.

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