Some Arithmetic Properties of the $q$-Euler Numbers and $q$-Salié Numbers

Victor J. W. Guo$^1$ and Jiang Zeng$^2$

Institut Camille Jordan, Université Claude Bernard (Lyon I)
F-69622, Villeurbanne Cedex, France
$^1$jwguo@eyou.com, $^2$zeng@igd.univ-lyon1.fr

Abstract. For $m > n ≥ 0$ and $1 ≤ d ≤ m$, it is shown that the $q$-Euler number $E_{2m}(q)$ is congruent to $q^{m-n}E_{2n}(q) \mod (1 + q^d)$ if and only if $m \equiv n \mod d$. The $q$-Salié number $S_{2n}(q)$ is shown to be divisible by $(1 + q^{2r+1})\lfloor \frac{2n}{2r+1} \rfloor$ for any $r ≥ 0$. Furthermore, similar congruences for the generalized $q$-Euler numbers are also obtained, and some conjectures are formulated.

AMS Subject Classifications (2000): Primary 05A30, 05A15; Secondary 11A07.

1 Introduction

The Euler numbers $E_{2n}$ may be defined as the coefficients in the Taylor expansion of $2/(e^x + e^{-x})$:

\[
\sum_{n=0}^{\infty} E_{2n} \frac{x^{2n}}{(2n)!} = \left( \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \right)^{-1}.
\]

A classical result due to Stern [13] asserts that

\[ E_{2m} \equiv E_{2n} \pmod{2^n} \text{ if and only if } 2m \equiv 2n \pmod{2^n}. \]

The so-called Salié numbers $S_{2n}$ [7, p. 242] are defined as

\[
\sum_{n=0}^{\infty} S_{2n} \frac{x^{2n}}{(2n)!} = \frac{\cosh x}{\cos x}.
\]

Carlitz [3] first proved that the Salié numbers $S_{2n}$ are divisible by $2^n$.

Motivated by the work of Andrews-Gessel [2], Andrews-Foata [11], Désarménien [4], and Foata [5], we are about to study a $q$-analogue of Stern’s result and a $q$-analogue of Carlitz’s result for Salié numbers. A natural $q$-analogue of the Euler numbers is given by

\[
\sum_{n=0}^{\infty} E_{2n}(q) \frac{x^{2n}}{(q; q)_{2n}} = \left( \sum_{n=0}^{\infty} \frac{x^{2n}}{(q; q)_{2n}} \right)^{-1},
\]

*European J. Combin. 27 (2006), 884–895.
where \((a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})\) for \(n \geq 1\) and \((a; q)_0 = 1\).

A recent arithmetic study of Euler numbers and more general \(q\)-Euler numbers can be found in \([14]\) and \([11]\). Note that, in order to coincide with the Euler numbers in \([14, 15]\), our definition of \(E_{2n}(q)\) differs by a factor \((-1)^n\) from that in \([12, 14]\).

**Theorem 1.1** Let \(m > n \geq 0\) and \(1 \leq d \leq m\). Then

\[
E_{2m}(q) \equiv q^{m-n}E_{2n}(q) \pmod{1 + q^d} \quad \text{if and only if} \quad m \equiv n \pmod{d}.
\]

Since the polynomials \(1 + q^{2d}\) and \(1 + q^{2d} \neq b\) are relatively prime, we derive immediately from the above theorem the following

**Corollary 1.2** Let \(m > n \geq 0\) and \(2m - 2n = 2^r s\) with \(r\) odd. Then

\[
E_{2m}(q) \equiv q^{m-n}E_{2n}(q) \pmod{1 + q^{2^r}}.
\]

Define the \(q\)-Salié numbers by

\[
S_{2n}(q) = \sum_{n=0}^{\infty} \frac{q^n x^{2n}}{(q; q)_{2n}} = \sum_{n=0}^{\infty} \frac{q^n x^{2n}}{(q; q)_{2n}} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(q; q)_{2n}}.
\]

(1.3)

For each positive integer \(n\), write \(n = 2^s (2r + 1)\) with \(r \geq 0\) (so \(s\) is the 2-adic valuation of \(n\)), and set \(p_n(q) = 1 + q^{2r+1}\). Define

\[
P_n(q) = \prod_{k=1}^{\infty} p_k(q) = \prod_{r \geq 0} (1 + q^{2r+1})^{a_{n,r}},
\]

where \(a_{n,r}\) is the number of positive integers of the form \(2^s (2r + 1)\) less than or equal to \(n\). The first values of \(P_n(q)\) are given in Table 1

| \(n\) | \(P_n(q)\) | \(1 + q\) | \(1 + q + q^2\) | \(1 + q)(1 + q^2)(1 + q^3)\) | \(1 + q)(1 + q^2)(1 + q^3)(1 + q^4)\) |
|------|------------|----------|----------------|-----------------------------|----------------------------------|
| \(n\) | \(P_n(q)\) | \(2\) | \(4\) | \(6\) | \(8\) |

Note that \(P_n(1) = 2^n\). The following is a \(q\)-analogue of Carlitz’s result for Salié numbers:

**Theorem 1.3** For every \(n \geq 1\), the polynomial \(S_{2n}(q)\) is divisible by \(P_n(q)\). In particular, \(S_{2n}(q)\) is divisible by \((1 + q^{2r+1})^{\lfloor \log_q n \rfloor}\) for any \(r \geq 0\).

We shall collect some arithmetic properties of Gaussian polynomials or \(q\)-binomial coefficients in the next section. The proofs of Theorems 1.1 and 1.3 are given in Sections 3 and 4, respectively. We will give some similar arithmetic properties of the generalized \(q\)-Euler numbers in Section 5. Some combinatorial remarks and open problems are given in Section 6.
2 Two properties of Gaussian polynomials

The Gaussian polynomial \([M \atop N]_q\) may be defined by

\[
[M \atop N]_q = \begin{cases} 
\frac{(q; q)_M}{(q; q)_N(q; q)_{M-N}}, & \text{if } 0 \leq N \leq M, \\
0, & \text{otherwise.}
\end{cases}
\]

The following result is equivalent to the so-called \(q\)-Lucas theorem (see Olive [10] and Désarménien [4, Proposition 2.2]).

**Proposition 2.1** Let \(m, k, d\) be positive integers, and write \(m = ad + b\) and \(k = rd + s\), where \(0 \leq b, s \leq d - 1\). Let \(\omega\) be a primitive \(d\)-th root of unity. Then

\[
[M \atop k]_{\omega} = (a \atop r)(b \atop s)_{\omega}.
\]

Indeed, we have

\[
[M \atop k]_{q} = \prod_{j=1}^{rd+s} \frac{1 - q^{(a-r)d+b-s+j}}{1 - q^j} = \left( \prod_{j=1}^{s} \frac{1 - q^{(a-r)d+b-s+j}}{1 - q^j} \right) \left( \prod_{j=1}^{rd} \frac{1 - q^{(a-r)d+b+j}}{1 - q^{s+j}} \right).
\]

By definition, we have \(\omega^d = 1\) and \(\omega^j \neq 1\) for \(0 < j < d\). Hence,

\[
\lim_{q \to \omega} \prod_{j=1}^{s} \frac{1 - q^{(a-r)d+b-s+j}}{1 - q^j} = \prod_{j=1}^{s} \frac{1 - \omega^{b-s+j}}{1 - \omega^j} = (b \atop s)_{\omega}.
\]

Notice that, for any integer \(k\), the set \(\{k + j : j = 1, \ldots, rd\}\) is a complete system of residues modulo \(rd\). Therefore,

\[
\lim_{q \to \omega} \prod_{j=1}^{rd} \frac{1 - q^{(a-r)d+b+j}}{1 - q^{s+j}} = \lim_{q \to \omega} \frac{(1 - q^{(a-r+1)d})(1 - q^{(a-r+2)d}) \cdots (1 - q^{ad})}{(1 - q^d)(1 - q^{2d}) \cdots (1 - q^{rd})}
\]

\[
= (a \atop r).
\]

Let \(\Phi_n(x)\) be the \(n\)-th cyclotomic polynomial. The following easily proved result can be found in [8, Equation (10)].

**Proposition 2.2** The Gaussian polynomial \([m \atop k]_q\) can be factorized into

\[
[M \atop k]_q = \prod_d \Phi_d(q),
\]

where the product is over all positive integers \(d \leq m\) such that \([k/d] + [(m-k)/d] < [m/d]\).
Indeed, using the factorization \( q^n - 1 = \prod_{d|n} \Phi_d(q) \), we have

\[
(q; q)_m = (-1)^m \prod_{k=1}^{m} \Phi_d(q) = (-1)^m \prod_{d=1}^{m} \Phi_d(q)^{[m/d]},
\]

and so

\[
\left[ \frac{m}{k} \right]_q = \frac{(q; q)_m}{(q; q)_k(q; q)_{m-k}} = \prod_{d=1}^{m} \Phi_d(q)^{[m/d]-[k/d]-[(m-k)/d]}.
\]

Proposition 2.2 now follows from the obvious fact that

\[
| \alpha + \beta | - | \alpha | - | \beta | = 0 \text{ or } 1, \quad \text{for } \alpha, \beta \in \mathbb{R}.
\]

### 3 Proof of Theorem 1.1

Multiplying both sides of (1.2) by \( \sum_{n=0}^{\infty} x^{2n} / (q; q)_{2n} \) and equating coefficients of \( x^{2m} \), we see that \( E_{2m}(q) \) satisfies the following recurrence relation:

\[
E_{2m}(q) = -\sum_{k=0}^{m-1} \left[ \frac{2m}{2k} \right]_q E_{2k}(q).
\]  
(3.1)

This enables us to obtain the first values of the \( q \)-Euler numbers:

\[
E_0(q) = -E_2(q) = 1,
\]
\[
E_4(q) = q(1 + q)(1 + q^2) + q^2,
\]
\[
E_6(q) = -q^2(1 + q^3)(1 + 4q + 5q^2 + 7q^3 + 6q^4 + 5q^5 + 2q^6 + q^7) + q^3.
\]

We first establish the following result.

**Lemma 3.1** Let \( m > n \geq 0 \) and \( 1 \leq d \leq m \). Then

\[
E_{2m}(q) \equiv q^{m-n} E_{2n}(q) \pmod{\Phi_{2d}(q)} \quad \text{if and only if} \quad m \equiv n \pmod{d}.
\]  
(3.2)

**Proof.** It is easy to see that Lemma 3.1 is equivalent to

\[
E_{2m}(\zeta) = \zeta^{m-n} E_{2n}(\zeta) \quad \text{if and only if} \quad m \equiv n \pmod{d},
\]  
(3.3)

where \( \zeta \in \mathbb{C} \) is a \( 2d \)-th primitive root of unity.

We proceed by induction on \( m \). Statement (3.3) is trivial for \( m = 1 \). Suppose it holds for every number less than \( m \). Let \( n < m \) be fixed. Write \( m = ad + b \) with \( 0 \leq b \leq d - 1 \), then \( 2m = a(2d) + 2b \). By Proposition 2.1, we see that

\[
\left[ \frac{2m}{2k} \right]_{\zeta} = \binom{a}{r} \left[ \frac{2b}{2s} \right]_{\zeta}, \quad \text{where} \quad k = rd + s, \quad 0 \leq s \leq d - 1.
\]  
(3.4)
Hence, by (3.1) and (3.4), we have

\[
E_{2m}(\zeta) = -\sum_{k=0}^{m-1} \left[\frac{2m}{2k}\right]_{\zeta} E_{2k}(\zeta)
\]

\[
= -\sum_{r=0}^{a} \sum_{s=0}^{b-d_{a}l_{r}} \left(\frac{a}{r}\right)_{\zeta} \left[\frac{2b}{2s}\right]_{\zeta} E_{2rd+2s}(\zeta),
\]

\[
= -\sum_{s=0}^{b} \sum_{r=0}^{a-d_{b}l_{s}} \left(\frac{a}{r}\right)_{\zeta} \left[\frac{2b}{2s}\right]_{\zeta} E_{2rd+2s}(\zeta),
\]

where \(\delta_{i,j}\) equals 1 if \(i = j\) and 0 otherwise.

By the induction hypothesis, we have

\[
E_{2rd+2s}(\zeta) = \zeta^{rd} E_{2s}(\zeta) = (-1)^r E_{2s}(\zeta). \tag{3.6}
\]

Thus,

\[
\sum_{r=0}^{a} \left(\frac{a}{r}\right)_{\zeta} \left[\frac{2b}{2s}\right]_{\zeta} E_{2rd+2s}(\zeta) = \left[\frac{2b}{2s}\right]_{\zeta} E_{2s}(\zeta) \sum_{r=0}^{a} \left(\frac{a}{r}\right)(-1)^r = 0.
\]

Therefore, Equation (3.5) implies that

\[
E_{2m}(\zeta) = (-1)^a E_{2b}(\zeta) = \zeta^{m-b} E_{2b}(\zeta). \tag{3.7}
\]

From (3.7) we see that

\[
E_{2m}(\zeta) = \zeta^{m-n} E_{2n}(\zeta) \iff E_{2n}(\zeta) = \zeta^{n-b} E_{2b}(\zeta).
\]

By the induction hypothesis, the latter equality is also equivalent to

\[
n \equiv b \pmod{d} \iff m \equiv n \pmod{d}.
\]

This completes the proof.

Since

\[
1 + q^d = \frac{q^{2d} - 1}{q^d - 1} = \prod_{k \mid 2d} \Phi_k(q) = \prod_{k \mid 2d} \Phi_{2k}(q),
\]

and any two different cyclotomic polynomials are relatively prime, Theorem 3.1 follows from Lemma 3.1.

Remark. The sufficiency part of (3.2) is equivalent to Désarménien’s result [4]:

\[
E_{2km+2n}(q) \equiv (-1)^m E_{2n}(q) \pmod{\Phi_{2k}(q)}.
\]

5
4 Proof of Theorem 1.3

Recall that the q-tangent numbers $T_{2n+1}(q)$ are defined by

$$
\sum_{n=0}^{\infty} T_{2n+1}(q) \frac{x^{2n+1}}{(q; q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(q; q)_{2n+1}} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(q; q)_{2n}}.
$$

Foata [5] proved that $T_{2n+1}(q)$ is divisible by $D_n(q)$, where

$$
D_n(q) = \begin{cases} 
\prod_{k=1}^{n} Ev_k(q), & \text{if } n \text{ is odd}, \\
(1 + q^2) \prod_{k=1}^{n} Ev_k(q), & \text{if } n \text{ is even},
\end{cases}
$$

and

$$
Ev_n(q) = \prod_{j=0}^{s} (1 + q^{2r}), \quad \text{where } n = 2^s r \text{ with } r \text{ odd}.
$$

Notice that this implies that $T_{2n+1}(q)$ is divisible by both $(1 + q)^n$ and $(-q; q)_n$, a result due to Andrews and Gessel [2].

To prove our theorem we need the following relation relating $S_{2n}(q)$ to $T_{2n+1}(q)$.

**Lemma 4.1** For every $n \geq 1$, we have

$$
\sum_{k=0}^{n} (-1)^k q^{k} \binom{2n}{2k} q \quad S_{2k}(q) S_{2n-2k}(q) = T_{2n-1}(q) (1 - q^{2n}). \quad (4.1)
$$

**Proof.** Replacing $x$ by $q^{1/2} i x \ (i = \sqrt{-1})$ in (1.3), we obtain

$$
\sum_{n=0}^{\infty} S_{2n}(q) \frac{(-1)^n q^n x^{2n}}{(q; q)_{2n}} = \sum_{n=0}^{\infty} (-1)^n q^{2n} x^{2n} \sum_{n=0}^{\infty} \frac{q^n x^{2n}}{(q; q)_{2n}}. \quad (4.2)
$$

Multiplying (1.3) with (4.2), we get

$$
\left( \sum_{n=0}^{\infty} S_{2n}(q) \frac{x^{2n}}{(q; q)_{2n}} \right) \left( \sum_{n=0}^{\infty} S_{2n}(q) \frac{(-1)^n q^n x^{2n}}{(q; q)_{2n}} \right) \quad (4.3)
$$

$$
= \sum_{n=0}^{\infty} (-1)^n q^{2n} x^{2n} \sum_{n=0}^{\infty} \frac{x^{2n}}{(q; q)_{2n}}
$$

$$
= 1 + x \sum_{n=1}^{\infty} (-1)^{n-1} q^{2n-1} x^{2n-1} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(q; q)_{2n}}
$$

$$
= 1 + x \sum_{n=0}^{\infty} T_{2n+1}(q) \frac{x^{2n+1}}{(q; q)_{2n+1}}. \quad (4.4)
$$
Equating the coefficients of $x^{2n}$ in (4.3) and (4.4), we are led to (4.1).

It is easily seen that $P_n(q)$ is the least common multiple of the polynomials $(1 + q^{2r+1})^\left\lfloor \frac{n^2}{2r+1} \right\rfloor$ ($r \geq 0$). For any $r \geq 0$, there holds

$$1 + q^{2r+1} = \frac{q^{4r+2} - 1}{q^{2r+1} - 1} = \prod_{d|(4r+2)} \Phi_d(q) = \prod_{d|(2r+1)} \Phi_{2d}(q).$$

It follows that

$$P_n(q) = \prod_{r \geq 0} \Phi_{4r+2}(q)^{\left\lfloor \frac{n^2}{2r+1} \right\rfloor}.$$  

Theorem 1.3 is trivial for $n = 1$. Suppose it holds for all integers less than $n$. In the summation of the left-hand side of (4.1), combining the first and last terms, we can rewrite Equation (4.1) as follows:

$$(1 + (−1)^n q^n) S_{2n}(q) + \sum_{k=1}^{n-1} (-1)^k q^{\left\lfloor \frac{2n}{2k} \right\rfloor} S_{2k}(q) S_{2n-2k}(q) = T_{2n-1}(q)(1 - q^{2n}). \quad (4.5)$$

For every $k$ ($1 \leq k \leq n-1$), by the induction hypothesis, the polynomial $S_{2k}(q) S_{2n-2k}(q)$ is divisible by

$$P_k(q) P_{n-k}(q) = \prod_{r \geq 0} \Phi_{4r+2}(q)^{\left\lfloor \frac{k}{2r+1} \right\rfloor + \left\lfloor \frac{n-k}{2r+1} \right\rfloor}.$$  

And by Proposition 2.2, we have

$$\left\lfloor \frac{2n}{2k} \right\rfloor q = \prod_{d=1}^{2n} \Phi_d(q)^{\left\lfloor \frac{2n}{d} \right\rfloor - \left\lfloor \frac{2k}{d} \right\rfloor - \left\lfloor \frac{(2n-2k)}{d} \right\rfloor},$$

which is clearly divisible by

$$\prod_{r \geq 0} \Phi_{4r+2}(q)^{\left\lfloor \frac{n}{2r+1} \right\rfloor - \left\lfloor \frac{k}{2r+1} \right\rfloor - \left\lfloor \frac{n-k}{2r+1} \right\rfloor}.$$  

Hence, the product $\left\lfloor \frac{2n}{2k} \right\rfloor q S_{2k}(q) S_{2n-2k}(q)$ is divisible by

$$\prod_{r \geq 0} \Phi_{4r+2}(q)^{\left\lfloor \frac{n}{2r+1} \right\rfloor} = P_n(q).$$

Note that $P_{n-1}(q) \mid D_{n-1}(q)$ and $p_n(q) \mid (1 - q^{2n})$. Therefore, by (4.5) and the aforementioned result of Foata, we immediately have

$$P_n(q) \mid (1 + (−1)^n q^n) S_{2n}(q).$$

Since $P_n(q)$ is relatively prime to $(1 + (−1)^n q^n)$, we obtain $P_n(q) \mid S_{2n}(q)$.

Remark. Since $S_0(q) = 1$ and $S_2(q) = 1 + q$, using (4.5) and the divisibility of $T_{2n+1}(q)$, we can prove by induction that $S_{2n}(q)$ is divisible by $(1 + q)^n$ without using the divisibility property of Gaussian polynomials.
5 The generalized $q$-Euler numbers

The generalized Euler numbers may be defined by

$$\sum_{n=0}^{\infty} E_{kn}(x) \frac{x^{kn}}{(kn)!} = \left(\sum_{n=0}^{\infty} \frac{x^{kn}}{(kn)!}\right)^{-1}.$$ 

Some congruences for these numbers are given in [6,9]. A $q$-analogue of generalized Euler numbers is given by

$$\sum_{n=0}^{\infty} E_{kn}^{(k)}(x) \frac{x^{kn}}{(q; q)_{kn}} = \left(\sum_{n=0}^{\infty} \frac{x^{kn}}{(q; q)_{kn}}\right)^{-1},$$

or, recurrently,

$$E_{0}^{(k)}(q) = 1, \quad E_{kn}^{(k)}(q) = -\sum_{j=0}^{n-1} \binom{kn}{kj} q E_{kj}^{(k)}(q), \quad n \geq 1. \quad (5.1)$$

Note that $E_{kn}^{(k)}(q)$ is equal to $(-1)^{n} f_{nk,k}(q)$ studied by Stanley [12, p. 148, Equation (57)].

**Theorem 5.1** Let $m > n \geq 0$ and $1 \leq d \leq m$. Let $k \geq 1$, and let $\zeta \in \mathbb{C}$ be a $2kd$-th primitive root of unity. Then

$$E_{km}^{(k)}(\zeta^2) = \zeta^{k(m-n)} E_{kn}^{(k)}(\zeta^2) \quad (5.2)$$

if and only if

$$m \equiv n \pmod{d}.$$ 

The proof is by induction on $m$ and using the recurrence definition (5.1). Since it is analogous to the proof of (5.3), we omit it here. Note that $\zeta^2$ in Theorem 5.1 is a $kd$-th primitive root of unity. Therefore, when $k$ is even or $m \equiv n \pmod{2}$, Equation (5.2) is equivalent to

$$E_{km}^{(k)}(q) \equiv q^{\frac{k(m-n)}{2}} E_{kn}^{(k)}(q) \pmod{\Phi_{kd}(q)}.$$ 

As mentioned before,

$$1 + q^{2kd} = \prod_{i|2^k d, 2i \neq 2^k d} \Phi_{2i}(q),$$

and we obtain the following theorem and its corollaries.

**Theorem 5.2** Let $k \geq 1$. Let $m > n \geq 0$ and $1 \leq d \leq m$. Then

$$E_{2m}^{(2k)}(q) \equiv q^{k-1(m-n)} E_{2n}^{(2k)}(q) \pmod{1 + q^{2k-1,d}} \quad \text{if and only if} \quad m \equiv n \pmod{d}.$$
Corollary 5.3 Let $k \geq 1$. Let $m > n \geq 0$ and $m - n = 2^{s-1}r$ with $r$ odd. Then
\[
E_{2k}^{(2k)}(q) \equiv q^{k-1(m-n)} E_{2k}^{(2k)}(q) \pmod{\prod_{i=0}^{s-1}(1 + q^{2^{k+i-1}r})}.
\]

Corollary 5.4 Let $k, m, n, s$ be as above. Then
\[
E_{2k}^{(2k)} \equiv E_{2k}^{(2k)} (\mod 2^s).
\]

Furthermore, numerical evidence seems to suggest the following congruence conjecture for generalized Euler numbers.

Conjecture 5.5 Let $k \geq 1$. Let $m > n \geq 0$ and $m - n = 2^{s-1}r$ with $r$ odd. Then
\[
E_{2k}^{(2k)} \equiv E_{2k}^{(2k)} + 2^s (\mod 2^{s+1}).
\]

This conjecture is clearly a generalization of Stern’s result, which corresponds to the $k = 1$ case.

6 Concluding remarks

We can also consider the following variants of the $q$-Salié numbers:

\[
\sum_{n=0}^{\infty} \overline{S}_{2n}(q) \frac{x^{2n}}{(q; q)_{2n}} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(q; q)_{2n}} / \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(q; q)_{2n}}, \quad (6.1)
\]

\[
\sum_{n=0}^{\infty} \hat{S}_{2n}(q) \frac{x^{2n}}{(q; q)_{2n}} = \sum_{n=0}^{\infty} q^n \frac{x^{2n}}{(q; q)_{2n}} / \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(q; q)_{2n}}, \quad (6.2)
\]

\[
\sum_{n=0}^{\infty} \tilde{S}_{2n}(q) \frac{x^{2n}}{(q; q)_{2n}} = \sum_{n=0}^{\infty} q^{n^2} \frac{x^{2n}}{(q; q)_{2n}} / \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(q; q)_{2n}}. \quad (6.3)
\]

Multiplying both sides of (6.1)–(6.3) by $\sum_{n=0}^{\infty} (-1)^n x^{2n}/(q; q)_{2n}$ and equating coefficients of $x^{2n}$, we obtain

\[
\overline{S}_{2n}(q) = 1 - \sum_{k=0}^{n-1} (-1)^{n-k} \left[ \begin{array}{c} 2n \\ 2k \end{array} \right] q \overline{S}_{2k}(q), \quad (6.4)
\]

\[
\hat{S}_{2n}(q) = q^{2n} - \sum_{k=0}^{n-1} (-1)^{n-k} \left[ \begin{array}{c} 2n \\ 2k \end{array} \right] q \hat{S}_{2k}(q), \quad (6.5)
\]

\[
\tilde{S}_{2n}(q) = q^{n^2} - \sum_{k=0}^{n-1} (-1)^{n-k} \left[ \begin{array}{c} 2n \\ 2k \end{array} \right] q \tilde{S}_{2k}(q). \quad (6.6)
\]
This gives
\[
\tilde{S}_0(q) = 1, \quad \tilde{S}_2(q) = 1 + q^2, \quad \tilde{S}_4(q) = q(1 + q^2)(1 + q + q^2),
\]
\[
\hat{S}_0(q) = 1, \quad \hat{S}_2(q) = 1 + q^2, \quad \hat{S}_4(q) = q(1 + q^2)(1 + 3q + q^2 + q^3),
\]
\[
\hat{S}_0(q) = 1, \quad \hat{S}_2(q) = 1 + q, \quad \hat{S}_4(q) = q(1 + q)(1 + q^2)(2 + q),
\]
and
\[
\tilde{S}_0(q) = 1, \quad \tilde{S}_2(q) = 1 + q^2, \quad \tilde{S}_4(q) = q^2(1 + q^2)(1 + q + 2q^2 + q^3 + q^4 + q^5 + 4q^6 + 5q^7 + 2q^8 + 4q^9 + q^{10}),
\]
\[
\hat{S}_0(q) = q^2(1 + q^2)(1 + 4q + 7q^2 + 6q^3 + 6q^4 + 5q^5 + 2q^7 + q^8),
\]
\[
\hat{S}_0(q) = q^2(1 + q)(1 + q^2)(1 + q^3)(2 + 4q + 5q^2 + 4q^3 + 3q^4 + q^5).
\]

For \( n \geq 1 \) define three sequences of polynomials:

\[
\overline{Q}_n(q) := \prod_{r \geq 1} \Phi_{4r}(q)^{\lfloor n/2r \rfloor},
\]

\[
\hat{Q}_n(q) := \begin{cases} 
\overline{Q}_n(q), & \text{if } n \text{ is even}, \\
(1 + q^2)\overline{Q}_n(q), & \text{if } n \text{ is odd},
\end{cases}
\]

\[
\tilde{Q}_n(q) := (1 + q)(1 + q^2) \cdots (1 + q^n).
\]

Note that \( \overline{Q}_n(q) \) is the least common multiple of the polynomials \( (1 + q^{2r})^{\lfloor n/2r \rfloor}, \) \( r \geq 1 \) (see Table 2).

**Table 2: Table of \( \overline{Q}_n(q) \).**

| \( n \) | 1 | 3 | 5 | 7 |
|---|---|---|---|---|
| \( Q_n(q) \) | 1 | \( 1 + q^2 \) | \( (1 + q^2)^2(1 + q^4) \) | \( (1 + q^2)^2(1 + q^4)(1 + q^6) \) |
| \( Q_n(q) \) | 2 | 4 | 6 | 8 |
| \( \overline{Q}_n(q) \) | \( 1 + q^2 \) | \( (1 + q^2)^2(1 + q^4) \) | \( (1 + q^2)^2(1 + q^4)^2(1 + q^6) \) | \( (1 + q^2)^2(1 + q^4)^2(1 + q^6)(1 + q^8) \) |

From (6.4)–(6.6), it is easy to derive by induction that for \( n \geq 1 \),

\[
2 \mid \overline{S}_{2n}(q), \quad (1 + q^2) \mid \hat{S}_{2n}(q), \quad (1 + q) \mid \tilde{S}_{2n}(q).
\]

Moreover, the computation of the first values of these polynomials seems to suggest the following stronger result.

**Conjecture 6.1** For \( n \geq 1 \), we have the following divisibility properties:

\[
\overline{Q}_n(q) \mid \overline{S}_{2n}(q), \quad \hat{Q}_n(q) \mid \hat{S}_{2n}(q), \quad \tilde{Q}_n(q) \mid \tilde{S}_{2n}(q).
\]
Similarly to the proof of Lemma 4.1 we can obtain

\[
\left( \sum_{n=0}^{\infty} \tilde{S}_{2n}(q) \frac{x^{2n}}{(q; q)_{2n}} \right) \left( \sum_{n=0}^{\infty} \tilde{S}_{2n}(q) \frac{(-1)^n q^{2n} x^{2n}}{(q; q)_{2n}} \right) = 1 - qx^2 + (1 + q) x \sum_{n=0}^{\infty} T_{2n+1}(q) \frac{x^{2n+1}}{(q; q)_{2n+1}},
\]

which yields

\[
\sum_{k=0}^{n} (-1)^k q^{\left[ \frac{2n}{2k} \right]} \tilde{S}_{2k}(q) \tilde{S}_{2n-2k}(q) = T_{2n-1}(q)(1 + q)(1 - q^{2n}), \quad n \geq 2. \tag{6.7}
\]

However, it seems difficult to use (6.7) to prove directly the divisibility of \( \tilde{S}_{2n}(q) \) by \( \tilde{Q}_n(q) \), because when \( n \) is even \( 1 + (-1)^n q^{2n} \) is in general not relatively prime to \( \tilde{Q}_n(q) \).

Finally it is well-known that \( E_{2n}(q) \) has a nice combinatorial interpretation in terms of generating functions of alternating permutations. Recall that a permutation \( x_1 x_2 \cdots x_{2n} \) of \([2n] := \{1, 2, \ldots, 2n\} \) is called alternating, if \( x_1 < x_2 > x_3 < \cdots > x_{2n-1} < x_{2n} \). As usual, the number of inversions of a permutation \( x = x_1 x_2 \cdots x_{2n} \), denoted \( \text{inv}(x) \), is defined to the number of pairs \( (i, j) \) such that \( i < j \) and \( x_i > x_j \). It is known (see [12] p. 148, Proposition 3.16.4) that

\[
(-1)^n E_{2n}(q) = \sum_{\pi} q^{\text{inv}(\pi)},
\]

where \( \pi \) ranges over all the alternating permutations of \([2n] \). It would be interesting to find a combinatorial proof of Theorem 1 within the alternating permutations model.

A permutation \( x = x_1 x_2 \cdots x_{2n} \) of \([2n] \) is said to be a Salié permutation, if there exists an even index \( 2k \) such that \( x_1 x_2 \cdots x_{2k} \) is alternating and \( x_{2k} < x_{2k+1} < \cdots < x_{2n} \), and \( x_{2k-1} \) is called the last valley of \( x \). It is known (see [17] p. 242, Exercise 4.2.13) that \( \frac{1}{2} \tilde{S}_{2n} \) is the number of Salié permutations of \([2n] \).

**Proposition 6.2** For every \( n \geq 1 \) the polynomial \( \frac{1}{2} \tilde{S}_{2n}(q) \) is the generating function for Salié permutations of \([2n] \) by number of inversions.

**Proof.** Substituting (1.2) into (6.1) and comparing coefficients of \( x^{2n} \) on both sides, we obtain

\[
\tilde{S}_{2n}(q) = \sum_{k=0}^{n} \left[ \frac{2n}{2k} \right] q^{\left[ \frac{2n}{2k} \right]} (-1)^k E_{2k}(q). \tag{6.8}
\]

As \( \left[ \frac{2n}{2k} \right] q \) is the generating function for the permutations of \( 1^{2k} 2^{2n-2k} \) by number of inversions (see e.g. [12] p. 26, Proposition 1.3.17]), it is easily seen that \( \left[ \frac{2n}{2k} \right] q (-1)^k E_{2k}(q) \) is the generating function for permutations \( x = x_1 x_2 \cdots x_{2n} \) of \([2n] \) such that \( x_1 x_2 \cdots x_{2k} \) is alternating and \( x_{2k+1} \cdots x_{2n} \) is increasing with respect to number of inversions. Notice that such a permutation \( x \) is a Salié permutation with the last valley \( x_{2k-1} \) if \( x_{2k} < x_{2k+1} \) or \( x_{2k+1} \) if \( x_{2k} > x_{2k+1} \). Therefore, the right-hand side of (6.8) is twice the generating function for Salié permutations of \([2n] \) by number of inversions. This completes the proof.
It is also possible to find similar combinatorial interpretations for the other $q$-Salié numbers, which are left to the interested readers.

**Acknowledgment.** The second author was supported by EC’s IHRP Programme, within Research Training Network “Algebraic Combinatorics in Europe,” grant HPRN-CT-2001-00272.

**References**

[1] G. E. Andrews and D. Foata, Congruences for the $q$-Euler numbers, European J. Combin. 1 (1980), 283–287.

[2] G. E. Andrews and I. Gessel, Divisibility properties of the $q$-tangent numbers, Proc. Amer. Math. Soc. 68 (1978), 380–384.

[3] L. Carlitz, The coefficients of $\cosh x/\cos x$, Monatsh. Math. 69 (1965), 129–135.

[4] J. Désarménien, Un analogue des congruences de Kummer pour les $q$- nombres d’Euler, European J. Combin. 3 (1982), 19–28.

[5] D. Foata, Further divisibility properties of the $q$-tangent numbers, Proc. Amer. Math. Soc. 81 (1981), 143–148.

[6] I. M. Gessel, Some congruences for generalized Euler numbers, Canad. J. Math. 35 (1983), 687–709.

[7] I. P. Goulden and D. M. Jackson, Combinatorial Enumeration, reprint of the 1983 original, Dover Publications, Inc., Mineola, NY, 2004.

[8] D. Knuth and H. Wilf, The power of a prime that divides a generalized binomial coefficient, J. Reine Angew. Math. 396 (1989), 212–219.

[9] D. J. Leemng and R. A. MacLeod, Some properties of generalized Euler numbers, Canad. J. Math. 33 (1981), 606–617.

[10] G. Olive, Generalized powers, Amer. Math. Monthly 72 (1965) 619–627.

[11] B. E. Sagan and P. Zhang, Arithmetic properties of generalized Euler numbers, Southeast Asian Bull. Math. 21 (1997), 73–78.

[12] R. P. Stanley, Enumerative Combinatorics, Vol. 1, Cambridge Studies in Advanced Mathematics, 49, Cambridge University Press, Cambridge, 1997.

[13] M. A. Stern, Zur Theorie der Eulerschen Zahlen, J. Reine Angew. Math. 79 (1875), 67–98.

[14] Z.-W. Sun, On Euler numbers modulo powers of two, J. Number Theory, to appear.

[15] S. S. Wagstaff, Jr., Prime divisors of the Bernoulli and Euler numbers, In: Number Theory for the Millennium, III (Urbana, IL, 2000), A K Peters, Natick, MA, 2002, pp. 357–374.