On the cuspidal representations of $GL_2(F)$ of level 1 or $1/2$

in the cohomology of the Lubin-Tate space $\mathcal{X}(\pi^2)$

September 27, 2011

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Abstract

In [T2], we explicitly compute the stable reduction of the Lubin-Tate space $\mathcal{X}(\pi^2)$, in the equal characteristic case. This paper is a continuation of [T2]. In this paper, we compute defining equations of irreducible components which appear in the Lubin-Tate space $\mathcal{X}(\pi^2)$ in the mixed characteristic case. We also determine the action of $GL_2$, the action of the central division algebra of invariant $1/2$ and the inertia action on the components in the stable reduction of $\mathcal{X}(\pi^2)$. As a result, in a sense, we observe that the local Jacquet-Langlands correspondence and the local Langlands correspondence for cuspidal representations of $GL_2(F)$ of level 1 or $1/2$ are realized in the cohomology of the Lubin-Tate space $\mathcal{X}(\pi^2)$.

1 Introduction

Let $F$ be a non-archimedean local field with ring of integers $\mathcal{O}_F$, uniformizer $\pi$ and residue field $k$ of characteristic $p > 0$. Let $|k| = q$. We fix an algebraic closure $F^{ac}$ of $F$. Its completion is denoted by $C$. We write $k^{ac}$ for the residue field of $C$. Let $v(\cdot)$ denote the valuation of $C$ normalized such that $v(\pi) = 1$. Let $F^{nr}$ denote the maximal unramified extension of $F$ in $F^{ac}$ and $F_0$ denote its completion. Let $\Sigma$ denote the unique, up to isomorphism, one-dimensional formal $\mathcal{O}_F$-module of height $h$ over $k^{ac}$. Then, the Lubin-Tate space $\mathcal{X}(\pi^n)/F_0$ of level $n$ and height $h$ means a generic fiber of a formal scheme, which is a deformation space of $\Sigma$ equipped with Drinfeld level $\pi^n$-structure. Then, the space $\mathcal{X}(\pi^n)$ is a rigid analytic variety of dimension $h - 1$ over $F_0$. In the remainder of this introduction, we assume $p \neq 2$ and $h = 2$. The existence of the stable model of a curve with genus greater than 1 is guaranteed by the work of Deligne-Mumford in [DM]. In [W3], Jared Weinstein determines irreducible components in the stable reduction of $\mathcal{X}(\pi^n)$ up to purely inseparable map, when $F = \mathbb{Q}_p$ and char $F = p > 0$, by using the Deligne-Carayol theorem for $GL_2$ and the Bushnell-Kutzko type theory. See also a series of papers [W]-[W4].

In this paper, we compute irreducible components in the stable reduction of $\mathcal{X}(\pi^2)$ explicitly for any $F$, by using blow-up. The stable reduction of $\mathcal{X}(\pi^2)$ has actions of $G_2^F := GL_2(\mathcal{O}_F/\pi^2)$, the central division algebra $\mathcal{O}_D^\times$ of invariant $1/2$ and the inertia group $I_F$. Then, we analyze étale cohomologies of the components in the stable reduction of $\mathcal{X}(\pi^2)$ as a $G_2^F \times \mathcal{O}_D^\times \times I_F$-representation. As a result, in a sense, we observe that the local Jacquet-Langlands correspondence and the local Langlands correspondence for cuspidal representations of $GL_2(F)$ of level 1.
or $1/2$ are realized in the cohomology of the Lubin-Tate space $\mathcal{X}(\pi^2)$. These are done without depending on Deligne-Carayol’s theorem for $GL_2$. Rather, to check Deligne-Carayol’s theorem for $GL_2$ purely locally in a lower level our aim is in this paper. In [12], in a case char $F = p > 0$, we completely compute the stable reduction of $\mathcal{X}(\pi^2)$ explicitly. This paper is a sequence of loc. cit. Similar computations of the stable models are found in [CM], [CM2], [E], [E2] and [T]. In particular, in [CM], very significant and stimulus developments are done to compute the stable model of a curve. See also [DR] and [KM]. We write $\mathcal{X}(\pi^2)$ for the stable reduction of $\mathcal{X}(\pi^2)$.

We define several subspaces $Y^*_{3,1,\ast}$, $Y^*_{2,2}$ and $Z^*_{1,1,\ast}$ with $\ast \in \mathbb{P}^1(F_q)$ of $\mathcal{X}(\pi^2)$, and compute their reduction in the mixed characteristic case, in section 3. We identify $\mathcal{X}(1)$ with an open unit ball $B(1) \ni u$ appropriately. Then, under the natural projection $\mathcal{X}(\pi^2) \to \mathcal{X}(1) \cong B(1)$, the subspaces $\bigcup_{* \in \mathbb{P}^1(F_q)} Y^*_{3,1,\ast}$, $Y^*_{2,2}$ and $\bigcup_{* \in \mathbb{P}^1(F_q)} Z^*_{1,1,\ast}$ in $\mathcal{X}(\pi^2)$ are over the loci $v(u) = 1/(q + 1)$, $v(u) \geq q/(q + 1)$ and $v(u) = 1/2$ respectively. To compute their reduction, we choose some simple model of the universal formal $\mathcal{O}_F$-module over Spf $\mathcal{O}_F[[u]]$, which is given in [GH]. See subsection 2.2 for more details on this model. It is well-known that the set $\pi_0(\mathcal{X}(\pi^n))_{\ast}$ of geometrically connected components of $\mathcal{X}(\pi^n)$ is identified with $(\mathcal{O}_F/\mathcal{O}_F^{\pi^n})^\times$. Furthermore, all geometrically connected components are defined over the classical Lubin-Tate extension $F_\pi/F_\tor$. See subsection 2.3 for the definition of $F_\pi$. Now, we fix an identification $\pi_0(\mathcal{X}(\pi^n)) \simeq (\mathcal{O}_F/\mathcal{O}_F^{\pi^n})^\times$ appropriately. See Theorem 2.23 for the identification. We choose one connected component $\mathcal{X}(\pi^2) \subset \mathcal{X}(\pi^2) \times F_\pi F_2$ with $i \in (\mathcal{O}_F/\mathcal{O}_F^{\pi^2})^\times$. Let $\bar{\imath}$ be the image of $i$ by the canonical map $(\mathcal{O}_F/\mathcal{O}_F^{\pi^2})^\times \to k^\times$. For a subspace $W \subset \mathcal{X}(\pi^2)$, we write $\bar{W}$ for the intersection $(W \times F_\pi F_2) \cap \mathcal{X}(\pi^2)$. For an affine curve $X$ over $k^{ac}$, we write $X^\circ$ for the smooth compactification of the normalization of $X$. The genus of $X$ means the genus of $X^\circ$. Then, for each $\ast \in \mathbb{P}^1(F_q)$, the reduction $\mathbf{Y}_{3,1,\ast}$ is defined by $x^q y - xy^q = 1$ with genus $q(q - 1)/2$ and $\mathbf{Y}_{3,1,\ast}$ appears in $\mathcal{X}(\pi^2)$. On the other hand, for each $\ast \in \mathbb{P}^1(F_q)$, the reduction $\bar{Z}_{1,1,\ast}$ is defined by $Z^q = X^q - 1 + X^{-1}$ with genus $0$. This curve has singularities at $X \in \mu_{2(q^2 - 1)}$. Then, by analyzing these singular points, we find $2(q^2 - 1)$ components $\{W_{i,c}^1\}_{i \in \mu_{2(q^2 - 1)}}$ having an affine model $a^q - a = s^2$ with genus $(q - 1)/2$. Then, the components $\{Z_* : = \bar{Z}_{1,1,\ast}\}_{* \in \mathbb{P}^1(F_q)}$ appear in $\mathcal{X}(\pi^2)$. In $\mathcal{X}(\pi^2)$, the components $\{W_{i,c}^1\}_{i \in \mu_{2(q^2 - 1)}}$ attach to $Z_*$ at distinct $2(q^2 - 1)$ points of $Z_*$ over the singular points of $\bar{Z}_{1,1,\ast}$. Furthermore, the reduction $\mathbf{Y}^*_{2,2}$ is defined by the following equations

$$x^q y - xy^q = \bar{\imath}, \quad Z^q = x^q y - xy^q.$$ 

This affine curve has singularities at

$$S^i_{00} := \{(x_0, y_0, Z_0) \in \mathbf{Y}^*_{2,2}(k^{ac}) \mid x_0^q - y_0^q = z_0^q - 1\}.$$ 

We have $|S^i_{00}| = |\mathbb{S}|(F_q) = q(q^2 - 1)$. By analyzing these singular points, we find $q(q^2 - 1)$ components $\{X^i_{j,c}\}_{j \in S^i_{00}}$ each having an affine model $X^q = X = Y^q + 1$ with genus $q(q - 1)/2$.

Then, in $\mathcal{X}(\pi^2)$, $\mathbf{Y}_{2,2}$ appears and $\{X^i_{j,c}\}_{i \in S^i_{00}}$ attach to $\mathbf{Y}_{2,2}$ at distinct $q(q^2 - 1)$ points above the points $S^i_{00}$.

In section 3[16] by using the explicit computation of components in $\mathcal{X}(\pi^2)$ explained above, we also determine the (right) action of $G_F^c \times \mathcal{O}_F^\times \times I_F$ on the components explicitly. To determine the action of $\mathcal{O}_F^\times$, we use the description of the action of $\mathcal{O}_D^\times$ on the Lubin-Tate space in [GH]. See 2.1 for more details. Let $\varphi$ denote a prime element of $D$, and $U_{\varphi}^n \subset \mathcal{O}_D^\times$ a subgroup $1 + (\varphi^n)$. Then, the group $\mathcal{O}_D^n$ acts on the stable reduction $\mathcal{X}(\pi^2)$ by factoring through the finite quotient $\mathcal{O}_D^\times := \mathcal{O}_D^\times/U_{\varphi}^n$. On the other hand, we compute the inertia action on the basis of [CM2].

Let $l \neq p$ be a prime number. In section 7 we analyze the following étale cohomologies, by
using the explicit description of the action of $\mathbf{G} := G_F^\mathcal{O} \times \mathcal{O}_F^\times \times I_F$,

$$W := \bigoplus_{i \in (\mathcal{O}_F/\mathfrak{p})^\times} H^1(X^{i,c}_j, \overline{\mathbb{Q}}_l), \quad W' := \bigoplus_{(i, s, y_0) \in (\mathcal{O}_F/\mathfrak{p})^\times \times \mathbb{P}^1(q_4) \times \mathbb{P}(\mathfrak{m}^2, \mathfrak{q}^2)} H^1(W^{i,c}_{s, y_0}, \overline{\mathbb{Q}}_l)$$

as $\mathbf{G}$-representations. Since the dual graph of the stable reduction $\mathcal{X}(\pi^n)$ is a tree, the spaces $W$ and $W'$ are $\mathbf{G}$-subrepresentations of the étale cohomology $H^1(\mathcal{X}(\pi^n)_{C, \overline{\mathbb{Q}}_l})$. Note that the group $\mathbf{G}$ acts on $H^1(\mathcal{X}(\pi^n)_{C, \overline{\mathbb{Q}}_l})$ on the left. By investigating $W$ (resp. $W'$) as in Corollary 7.3 (resp. Corollary 7.9) as a $\mathbf{G}$-representation, we show that the local Jacquet-Langlands correspondence and the $\ell$-adic local Langlands correspondence for unramified (resp. ramified) cuspidal representations of $GL_2(F)$ of level 1 (resp. level $1/2$) are realized in the cohomology group $W$. (resp. $W'$.) To check these things, we heavily depend on BH. See [BH] for more details. This is a part of Carayol’s program in [Ca]. See also [Bo] and [HT]. For general $h$, T. Yoshida computes the stable model of $\mathcal{X}(\pi)$ and studies its cohomology in [Y].

In a subsequent paper [MT], we will investigate a relationship between $W$ and the cohomology $H^2_F$ of some Lusztig surface for $G_F^\mathcal{O}$. The third author thanks Professor T. Saito for helpful comments. He also thanks to N. Imai for very fruitful discussions around topics in this paper.

**Notation.** We fix some notations in this paper. Let $F$ be a non-archimedean local field with the ring of integers $\mathcal{O}_F$, the uniformizer $\pi$ and the residue field $k = \mathbb{F}_q$ of characteristic $p > 0$. We fix an algebraic closure $\overline{\mathbb{F}}$ of $F$. The completion of $\overline{\mathbb{F}}$ is denoted by $\mathbf{C}$. We write $\mathbf{C}_n$ for the ring of integers of $\mathbf{C}$. We write $k^{ac}$ for the residue field of $\mathbf{C}$. For an element $a \in \mathbf{C}_n$, we denote by $\bar{a}$ the image of $a$ by the reduction map $\mathbf{C}_n \rightarrow k^{ac}$. Let $v(\cdot)$ denote the normalized valuation of $\mathbf{C}$ such that $v(\pi) = 1$ and $|\cdot|$ the absolute value given by $|x| = p^{-v(x)}$. Let $F^{ac}$ denote the maximal unramified extension of $F$ in $\overline{\mathbb{F}}$ and $F_0$ denote its completion. Let $E/F$ denote the quadratic unramified extension. Let $\mathcal{R} = \mathbf{C}^* = p^{Q}$. We let $L$ be a complete subfield of $\mathbf{C}$. For $r \in \mathcal{R}$, we let $B_L[r]$ and $B_L(r)$ denote the closed and open disks over $L$ of radius $r$ around 0, i.e., the rigid spaces over $L$ whose $\mathbf{C}$-valued points are $\{x \in \mathbf{C} : |x| \leq r\}$ and $\{x \in \mathbf{C} : |x| < r\}$ respectively. If $r, s \in \mathcal{R}$ and $r \leq s$, let $A_L[r, s]$ and $A_L(r, s)$ be the rigid spaces over $L$ whose $\mathbf{C}$-valued points are $\{x \in \mathbf{C} : r \leq |x| \leq s\}$ and $\{x \in \mathbf{C} : r < |x| < s\}$, which we call a closed annulus and an open annulus respectively. Furthermore, we write $C_L[r]$ for $A_L[r, r]$, which we call a circle. We simply write $G_F^\mathcal{O}$ for $GL_2(\mathcal{O}_F/\mathfrak{p}^n)$. Let $D$ be an $F$-algebra which is a division algebra, with center $F$ and dimension 4. Let $\mathcal{O}_D$ be the ring of integers of $D$ and $\varphi$ a prime element of $D$ such that $\varphi^2 = \pi$. We set $U_D^0 := 1 + (\varphi^{i-1})(\mathcal{O}_D)$ the group $U_D^0$ is a compact open normal subgroup of $D^\times$. We put $\mathcal{O}_D^0 := \mathcal{O}_D^0/U_D^0$. Let $\text{Nrd}_{D/F} : D^\times \rightarrow F^\times$ be the reduced norm map, and $\text{Trd}_{D/F} : D \rightarrow F$ the reduced trace map. For an element $x \in (\mathcal{O}_F/\mathfrak{p}^2)^x$, we write $x = a_0 + a_1\pi$ with $a_0 \in (\mathcal{O}_F)$ and $a_1 \in (\mathcal{O}_F/\mathfrak{p}) \cup \{0\}$. Then, we identify $(\mathcal{O}_F/\mathfrak{p}^2)^x$ with $\mathbb{F}_q^\times \times \mathbb{F}_q$ by the following map $x = a_0 + a_1\pi \mapsto (\bar{a}_0, \frac{a_1}{a_0})$. Let $W_F$ denote the Weil group of $F$ and $I_F \subset W_F$ denote the inertia subgroup. For a prime number $l \neq p$ and a finite abelian group $A$, we set $A^\times := \text{Hom}(A, \overline{\mathbb{Q}}_l^\times)$. If we have $v(f - g) > \alpha$ with $\alpha \in \mathbb{Q}_{\geq 0}$, we write $f \equiv g \pmod{\alpha}$. For an affine curve $X/k^{ac}$, the genus of $X$ means the genus of the smooth compactification $X^c$ of the normalization of $X$.

## 2 Preliminaries from [W2], [W3] and [GH]

In this section, we collect several things and facts around the Lubin-Tate space $\mathcal{X}(\pi^n)$, for example, formal $\mathcal{O}_F$-module, good model of the universal $\mathcal{O}_F$-module $\mathcal{F}^{\text{univ}}$, a set of geometrically connected components of $\mathcal{X}(\pi^n)$, the definition of the action of $\mathcal{O}_F^\mathcal{O}$ on $\mathcal{X}(\pi^n)$ etc. from [W2], [W3] and [GH]. Furthermore, we define several subspaces in $\mathcal{X}(\pi^n)$, whose reduction will be computed in the proceeding section.
2.1 definition of formal modules

We begin with the definition of formal \( \mathcal{O}_F \)-modules.

**Definition 2.1.** Let \( R \) be a commutative \( \mathcal{O}_F \)-algebra, with structure map \( i : \mathcal{O}_F \to R \). A formal one-dimensional \( \mathcal{O}_F \)-module \( \mathcal{F} \) is a power series \( \mathcal{F}(X,Y) = X + Y + \cdots \in R[[X,Y]] \) which is commutative, associative, admits 0 as an identity, together with a power series \([a]_{\mathcal{F}}(X) \in R[[X]]\) for each \( a \in \mathcal{O}_F \) satisfying \([a]_{\mathcal{F}}(X) \equiv i(a)X \mod X^2 \) and \( \mathcal{F}([a]_{\mathcal{F}}(X), [a]_{\mathcal{F}}(Y)) = [a]_{\mathcal{F}}(\mathcal{F}(X,Y)) \).

The addition law on a formal \( \mathcal{O}_F \)-module \( \mathcal{F} \) will usually be written \( X + Y \). If \( R \) is a \( k \)-algebra, we either have \([\pi]_{\mathcal{F}}(X) = 0\) or else \([\pi]_{\mathcal{F}}(X) = f(X^h)\) for some power series \( f(X) \) with \( f'(0) \neq 0 \). In the latter case, we say \( \mathcal{F} \) has height \( h \) over \( R \). Let \( \Sigma \) be a one-dimensional formal \( \mathcal{O}_F \)-module over \( k^{ac} \) of height \( h \). The formal \( \mathcal{O}_F \)-module \( \Sigma \) is unique up to isomorphism. Furthermore, a model for \( \Sigma \) is given by the simple rules when \( \text{char} F > 0 \)

\[
X + \Sigma Y = X + Y, \quad [\zeta]_{\Sigma}(X) = \zeta X (\zeta \in k), \quad [\pi]_{\Sigma}(X) = X^h.
\]

The functor of deformations of \( \Sigma \) to complete local Noetherian \( \mathcal{O}_{F_0} \)-algebra is representable by a universal deformation \( \mathcal{F}^{\text{univ}} \) over an algebra \( A(1) \) which is isomorphic to the power series ring \( \mathcal{O}_{F_0}[[u_1, \ldots, u_{h-1}]] \) in \( (h-1) \) variables, cf [Dr]. That is, if \( \mathcal{A} \) is a complete local \( \mathcal{O}_{F_0} \)-algebra with maximal ideal \( P \), then, the isomorphism classes of deformations of \( \Sigma \) to \( \mathcal{A} \) are given exactly by specializing each \( u_i \) to an element of \( P \) in \( \mathcal{F}^{\text{univ}} \).

2.2 Moduli of deformations with level structure

Let \( A \) be a complete local \( \mathcal{O}_F \)-algebra with maximal ideal \( M \), and let \( \mathcal{F} \) be a one-dimensional \( \mathcal{O}_F \)-module over \( A \), and let \( h \geq 1 \) be the height of \( \mathcal{F} \otimes (A/M) \).

**Definition 2.2.** Let \( n \geq 1 \). A Drinfeld level \( \pi^n \)-structure on \( \mathcal{F} \) is an \( \mathcal{O}_F \)-module homomorphism

\[
\phi : (\pi^{-n}\mathcal{O}_F/\mathcal{O}_F)^h \to M
\]

for which the relation

\[
\prod_{x \in (\pi^{-1}\mathcal{O}_F/\mathcal{O}_F)^h} (X - \phi(x)) \mid [\pi]_{\mathcal{F}}(X)
\]

holds in \( A[[X]] \). If \( \phi \) is a Drinfeld level \( \pi^n \)-structure, the image of \( \phi \) of the standard basis elements \((\pi^{-n}, 0, \ldots, 0), \ldots, (0, 0, \ldots, \pi^{-n})\) of \((\pi^{-n}\mathcal{O}_F/\mathcal{O}_F)^h\) form a Drinfeld basis of \( \mathcal{F}[\pi^n] \).

Fix a formal \( \mathcal{O}_F \)-module \( \Sigma \) of height \( h \) over \( k^{ac} \). Let \( A \) be a noetherian local \( \mathcal{O}_{F_0} \)-algebra such that the structure morphism \( \mathcal{O}_{F_0} \to A \) induces an isomorphism between residue fields. A deformation of \( \Sigma \) with level \( \pi^n \)-structure over \( A \) is a triple \((\mathcal{F}, \eta, \phi)\) where \( \mathcal{F} \) is a formal \( \mathcal{O}_F \)-module over \( A \), \( \eta : \Sigma \to \mathcal{F} \otimes k^{ac} \) is an isomorphism of \( \mathcal{O}_F \)-modules over \( k^{ac} \) and \( \phi \) is a Drinfeld level \( \pi^n \)-structure on \( \mathcal{F} \).

**Proposition 2.3.** ([Dr]) The functor which assigns to each \( A \) as above the set of deformations of \( \Sigma \) with Drinfeld level \( \pi^n \)-structure over \( A \) is represented by a regular local ring \( A(\pi^n) \) of relative dimension \( h - 1 \) over \( \mathcal{O}_{F_0} \). Let \( X_1^{(n)}, \ldots, X_h^{(n)} \) be the corresponding Drinfeld basis for \( \mathcal{F}^{\text{univ}}[\pi^n] \). Then, these elements form a set of regular parameters for \( A(\pi^n) \).
There is a finite injection of $\mathcal{O}_F$-algebras $[\pi]_w : \mathcal{A}(\pi^n) \hookrightarrow \mathcal{A}(\pi^{n+1})$ corresponding to the obvious degeneration map of functors. We therefore may consider $\mathcal{A}(\pi^n)$ as a subalgebra of $\mathcal{A}(\pi^{n+1})$, with the equation $[\pi]_w(X_i^{(n)}) = X_i^{(n+1)}$ holding in $\mathcal{A}(\pi^{n+1})$. Let $X(\pi^n) = \text{Spf} \mathcal{A}(\pi^n)$, so that $X(\pi^n) \to \text{Spf} \mathcal{O}_F$ is formally smooth of relative dimension $h - 1$. Let $\mathcal{X}(\pi^n)$ be the generic fiber of $X(\pi^n)$. Then, $\mathcal{X}(\pi^n)$ is a rigid analytic variety of dimension $h - 1$ over $F_0$. We call $\mathcal{X}(\pi^n)$ the Lubin-Tate space of level $n$. The coordinates $X_i^{(n)}$ are then analytic functions on $\mathcal{X}(\pi^n)$ with values in the open unit disc. We have that $\mathcal{X}(1)$ is the rigid analytic open unit polydisc of dimension $h - 1$. The group $\text{GL}_h(\mathcal{O}_F/\pi^n\mathcal{O}_F)$ acts on the right on $\mathcal{X}(\pi^n)$ and on the left on $A(\pi^n)$. The degeneration map $\mathcal{X}(\pi^n) \to \mathcal{X}(1)$ is Galois with group $\text{GL}_h(\mathcal{O}_F/\pi^n\mathcal{O}_F)$. For an element $M \in \text{GL}_h(\mathcal{O}_F/\pi^n\mathcal{O}_F)$ and an analytic function $f$ on $\mathcal{X}(\pi^n)$, we write $M(f)$ for the translated function $z \mapsto f(zM)$. When $f$ happens to be one of the parameters $X_i^{(n)}$, there is a natural definition of $M(X_i^{(n)})$ when $M \in M_h(\mathcal{O}_F/\pi^n\mathcal{O}_F)$ is an arbitrary matrix: if $M = (a_{ij})$, then

\[ M(X_i^{(n)}) = [a_{j1}]_{f, \text{univ}}(X_1^{(n)}) + [a_{j2}]_{f, \text{univ}}(X_2^{(n)}) + \cdots + [a_{jh}]_{f, \text{univ}}(X_h^{(n)}). \]

### 2.3 The universal deformation in the equal characteristic case

We briefly recall a simple model of $\mathcal{F}^{\text{univ}}$ given in [W2, 2.2], [W4, 3.8] and in [St2, Proposition 5.1.1]. Assume $\text{char } F = p > 0$, so that $F = k((\pi))$ is the field of Laurent series over $k$ in one variable, with the ring of integers $\mathcal{O}_F = k[[\pi]]$.

The universal deformation $\mathcal{F}^{\text{univ}}$ of $\Sigma$ has a simple model over $\mathcal{A}(1) \simeq \mathcal{O}_{F_0}[[u_1, \ldots, u_{h-1}]]$:

\[ X^+_{\mathcal{F}^{\text{univ}}} Y = X + Y \]

\[ [\zeta]_{\mathcal{F}^{\text{univ}}}(X) = \zeta X, \zeta \in k \]

\[ [\pi]_{\mathcal{F}^{\text{univ}}}(X) = \pi X + u_1 X^q + \cdots + u_{h-1} X^{q^{h-1}} + X^{q^h}. \quad (2.1) \]

In particular, we have the following, in the case $h = 2$,

\[ [\pi]_{\mathcal{F}^{\text{univ}}}(X) = X^{q^2} + uX^q + \pi X \]

\[ \text{over } X(1) \simeq \text{Spf } \mathcal{O}_{F_0}[[u]]. \]

### 2.4 The universal deformation in the mixed characteristic case

From now on, we assume $h = 2$. In this subsection, we fix an identification $X(1) \simeq \text{Spf } \mathcal{O}_{F_0}[[u]]$ over which the universal formal $\mathcal{O}_F$-module $\mathcal{F}^{\text{univ}}$ has a simple form ("$\mathcal{O}_F$-typical") in the mixed characteristic case.

Let $\mathcal{F}$ be a formal $\mathcal{O}_F$-module of dimension 1 over an $\mathcal{O}_F$-algebra $R$. Let $w$ be an invariant differential on $\mathcal{F}$. See [GH][section 3] for more details. We assume that $R$ is flat over $\mathcal{O}_F$. Then, a logarithm of $\mathcal{F}$ means a unique isomorphism $\mathcal{F} \cong \mathcal{G}_a$ over $R \otimes \overline{F}$ with $df = w$. In the following, we consider a logarithm $f$ which has the following form

\[ f(X) = X + \sum_{k \geq 1} b_k X^{q^k} \]

with $b_k \in R \otimes \overline{F}$. By [GH] (5.3) and (12.3), the universal logarithm $f(X)$ over $\mathcal{O}_F[[u]]$ satisfies the following Hazewinkel’s “functional equation” in [Haz, 21.5]

\[ f(X) = X + \frac{f(uX^q)}{\pi} + \frac{f(X^{q^2})}{\pi}. \quad (2.3) \]
If we write \( f(X) = \sum_{k \geq 0} b_k X^q^k \in F[[u, X]] \), the coefficients \( \{b_k\}_{k \geq 0} \) satisfy the following

\[
b_0 = 1, \quad \pi b_k = \sum_{0 \leq j \leq k-1} b_j v^{q^j}
\]

where \( v_1 = u, v_2 = 1, v_k = 0 \) \( (k \geq 3) \). For example, we have the followings

\[
b_0 = 1, \quad b_1 = \frac{u}{\pi}, \quad b_2 = \frac{1}{\pi} \left( 1 + \frac{u^{q+1}}{\pi} \right), \quad b_3 = \frac{1}{\pi^2} \left( u + u^q + \frac{u^{q^2+q+1}}{\pi} \right),
\]

\[
b_4 = \frac{1}{\pi^2} \left( 1 + \frac{u^{q+1} + u^{q^2+1} + u^q (q+1)}{\pi} + \frac{u (q+1) (q^2+1)}{\pi^2} \right), \ldots.
\]

(2.4)

By \cite{GH} Proposition 5.7 or \cite{Haz} 21.5, if we set as follows

\[
\mathcal{F}^{\text{univ}}(X, Y) := f^{-1}(f(X) + f(Y)) \tag{2.5}
\]

\[
[a]_{\mathcal{F}^{\text{univ}}}(X) := f^{-1}(af(X)) \tag{2.6}
\]

for \( a \in \mathcal{O}_F \), it is known that these power series have coefficients in \( \mathcal{O}_F[[u]] \) and define the universal formal \( \mathcal{O}_F \)-module \( \mathcal{F}^{\text{univ}} \) over \( \mathcal{O}_F[[u]] \) with logarithm \( f(X) \). In the following, we simply write \([a]_u\) for \([a]_{\mathcal{F}^{\text{univ}}} \) and \( X + u Y \) for \( \mathcal{F}^{\text{univ}}(X, Y) \). Then, we have \([\zeta]_u(X) = \zeta X \) for \( \zeta \in \mu_{q-1}(\mathcal{O}_F) \). Note that the above model \( \mathcal{F}^{\text{univ}} \) exists for general \( h \), which is given in \cite{GH}.

We have the following approximation formula for \([\pi]_u\), which plays a key role when we compute irreducible components in the stable reduction of \( \mathcal{X}(\pi^2) \).

**Lemma 2.4.** We assume that \( \text{char } F = 0 \). Let \( e_{F/\mathbb{Q}_p} = v(p) \) be the absolute ramification index.

1. Then, we have the following congruence

\[
[\pi]_u(X) \equiv \pi X + uX^q + X^{q^2} - \left( \frac{q}{\pi} \right) u^2 X^{q(q^2-q+1)} \pmod{(\pi^3, \pi^2 X^{q^2-q+1}, \pi u X^{q^2-q+1}, \pi u^2 X^q, \pi^2 u X^q, \pi^3 u X^{q^2}, \pi^2 u^2 X^{q^2})},
\]

2. If \( e_{F/\mathbb{Q}_p} \geq 2 \), we have the following congruence

\[
[\pi]_u(X) \equiv \pi X + uX^q (1 - \pi^{q-1}) + X^{q^2} \pmod{(X^{q^2}, u^{2q+1}, \pi^3, \pi^2 u X^{q+1}, \pi u^2 X^{q(q^2-q+1)})}.
\]

3. If \( e_{F/\mathbb{Q}_p} \geq 3 \), we have the following congruence

\[
[\pi]_u(X) \equiv \pi X + uX^q (1 - \pi^{q-1}) + X^{q^2} \pmod{(u^{q^2+q+1}, X^{q^6}, \pi^4, \pi^3 u X^{q+1}, \pi^2 u^2 X^{q(q^2-q+1)})}.
\]

**Proof.** These assertions follow from direct computations by using the relationship \( f([\pi]_u(X)) = \pi f(X) \) and (2.4). \( \square \)

Throughout the paper, \( \mathcal{F}^{\text{univ}}/X(1) \) means the universal formal \( \mathcal{O}_F \)-module given in (2.2) in the equal characteristic case, and (2.3) in the mixed characteristic case with the identification \( \mathcal{X}(1) \simeq \text{Spf } \mathcal{O}_{F_0}[[u]] \) respectively. By \( X(1) \simeq \text{Spf } \mathcal{O}_{F_0}[[u]] \), we also identify \( \mathcal{X}(1) \simeq B(1) \ni u. \)
2.5 Geometrically connected components of the Lubin-Tate space $\mathcal{X}(\pi^n)$ in [W3]

We collect some facts on the geometrically connected components of $\mathcal{X}(\pi^n)$ from [W3] subsection 3.6.

Let $LT$ be a one-dimensional formal $\mathcal{O}_F$-module over $\mathcal{O}_{F_0}$ for which $LT \otimes k^{ac}$ has height one; this is unique up to isomorphism. In the following, we choose a model of $LT$ such that

$$[\pi]_{LT}(X) = \pi X - X^q.$$ See [W2] Section 3. For $n \geq 1$, let $F_n := F_0(\mathbb{LT}[\pi^n](C))$. Then, by the classical Lubin-Tate theory, $F_n/F_0$ is an abelian extension with Galois group $(\mathcal{O}_F/\pi^n\mathcal{O}_F)^\times$, and the union $\bigcup_{n \geq 1} F_n$ is the completion of the maximal abelian extension of $F$.

Let $\Sigma$ be a one-dimensional formal $\mathcal{O}_F$-module over $k^{ac}$ of height 2 as in subsection 2.2. Then, we have $\mathcal{O}_D = \text{End}_{k^{ac}}(\Sigma)$. We have the following right action of $\mathcal{O}_D^\times$ on $\mathcal{X}(\pi^n)$. Let $b \in \mathcal{O}_D^\times$. Then, $b$ acts on $\mathcal{X}(\pi^n)$ by $(\mathcal{F}, \eta, \phi) \mapsto (\mathcal{F}, \eta \circ b, \phi)$. We will introduce another description of the above $\mathcal{O}_D^\times$-action in the proceeding subsection.

All geometrically connected components of $\mathcal{X}(\pi^n)$ are defined over $F_n$, and they are in bijection with primitive elements in the free of rank 1 $((\mathcal{O}_F/\pi^n\mathcal{O}_F)^\times$, and the union $\bigcup_{n \geq 1} F_n$ of the maximal abelian extension of $F$.

Theorem 2.5. ([S7]) For each $n \geq 1$, there exists a locally constant rigid analytic morphism $\Delta^{(n)} : \mathcal{X}(\pi^n) \rightarrow \mathcal{LT}[\pi^n]$ which surjects onto the subset of $\mathcal{LT}[\pi^n]$ which are primitive. For a triple $(g, b, \tau) \in \text{GL}_2(\mathcal{O}_F) \times \mathcal{O}_D^\times \times I_F$, we have

$$\Delta^{(n)} \circ (g, b, \tau) = [\delta(g, b, \tau)]_{LT}(\Delta^{(n)}),$$

where $\delta$ is the homomorphism

$$\delta : \text{GL}_2(F) \times D^\times \times W_F \rightarrow F^\times; (g, b, w) \mapsto \text{det}(g) \times \text{Nrd}_{D/F}(b)^{-1} \times a_F(w)^{-1} \quad (2.7)$$

and $a_F : W_F^{ab} \rightarrow F^\times$ is the reciprocity map from local class field theory, normalized so that $a_F$ sends a geometric Frobenius element to a prime element. The geometric fiber of $\Delta^{(n)}$ are connected.

Remark 2.6. In Theorem 2.5, we consider the right action of $\text{GL}_2(\mathcal{O}_F) \times \mathcal{O}_D^\times \times I_F$ on the Lubin-Tate space $\mathcal{X}(\pi^n) \times F_0 \mathbb{C}$. See [Ca] 1.3 for more detail on group action on the Lubin-Tate tower.

We write $\pi_0(\mathcal{X}(\pi^n))$ for the set of geometrically connected components of $\mathcal{X}(\pi^n)$. Then, by Theorem 2.5, we can identify $\pi_0(\mathcal{X}(\pi^n)) = (\mathcal{O}_F/\pi^n)^\times$, and know the action of $\text{GL}_2(\mathcal{O}_F) \times \mathcal{O}_D^\times \times I_F$ on it. For $i \in (\mathcal{O}_F/\pi^n)^\times$, let $\mathcal{X}(\pi^n) \subset \mathcal{X}(\pi^n) \times F_0 \mathcal{F}_n$ denote the connected component corresponding to $i$ under the above identification $\pi_0(\mathcal{X}(\pi^n)) = (\mathcal{O}_F/\pi^n)^\times$. Then, for a subspace $W \subset \mathcal{X}(\pi^n)$, we write $W^i$ for the intersection $(W \times F_0 F_n) \cap \mathcal{X}(\pi^n)$.

2.6 Action of the division algebra $\mathcal{O}_D^\times$ on the Lubin-Tate space $\mathcal{X}(\pi^n)$ ($n \geq 0$)

In this subsection, we recall a description of the right action of $\mathcal{O}_D^\times$ on the Lubin-Tate space $\mathcal{X}(\pi^n)$ from [GH]. Recall that we set $X(\pi^n) := \text{Spf} \mathcal{A}(\pi^n)$ and $\mathcal{X}(\pi^n)$ is the generic fiber of
For a formal scheme $X$ over $\text{Spf } O_{F_0}$ and a finite extension $L/F$, we simply write $X \times_F L$ for the base change $X \times_{\text{Spf } O_{F_0}} \text{Spf } O_{L^{ur}}$. Let $E/F$ be the unramified quadratic extension. Let $\sigma \neq 1 \in \text{Gal}(E/F)$. The ring of integers $O_E$ has the following description $O_E \simeq O_F \oplus \varphi O_E$ with $a \varphi = \varphi a^\sigma$ for $a \in O_E$. Let $b = \alpha + \varphi \beta \in O_E^\times$ with $\alpha \in O_E^\times$ and $\beta \in O_E$. By [GH, Proposition 14.7], we have the following map

$$b : X(1) \times_F E \to X(1) \times_F E.$$ (2.8)

In the following, we recall the definition of the map (2.8).

For an element $z \in O_E$, we denote by $\bar{z}$ for the image of $z$ by the canonical map $O_E \to \mathbb{F}_{q^2}$. By [GH, Section 13], $b = \alpha + \varphi \beta \in O_E^\times \simeq \text{Aut}(\Sigma)$ is written as follows as an element of $\mathbb{F}_{q^2}[[X]]$,

$$b(X) \equiv \bar{a}_0 X + \bar{a}_1 X^q + (\bar{\beta} X)^q \pmod{(X^q)}$$ (2.9)

where we write $\alpha = \sum_{i \geq 0} a_i \pi^i$ with $a_i \in \mu_{q-1}(O_E) \cup \{0\}$. Hence, we acquire the following congruence

$$\bar{b}^{-1}(X) \equiv \frac{X - (\bar{\beta}/\bar{a}_0) q X^q + (-\bar{a}_1/\bar{a}_0) + (\bar{\beta}/\bar{a}_0)^{q+1}) X^q}{\bar{a}_0} \pmod{(X^q)}.$$ (2.10)

Let $\bar{b} \in O_E[[X]]$ denote a lifting of $b \in \mathbb{F}_{q^2}[[X]]$. Let $\mathcal{F}_{\bar{b}}/\mathcal{X}(1) \times_F E$ denote the universal formal $O_F$-module whose multiplications are defined as follows

$$\mathcal{F}_{\bar{b}}(X, Y) = \bar{b}^{-1} \circ \mathcal{F}(\tilde{b}(X), \tilde{b}(Y)), \ [\pi]_{\mathcal{F}_{\bar{b}}}(X) = \bar{b}^{-1} \circ [\pi]_{\mathcal{F}} \circ \tilde{b}(X).$$

We simply write $[\pi]_{\bar{b}}$ for $[\pi]_{\mathcal{F}_{\bar{b}}}$. Then, clearly we have the following isomorphism as in [GH (14.4)]

$$\bar{b}^{-1} : \mathcal{F}_{\bar{b}} \rightarrow \mathcal{F}_{\bar{b}} : (X, u) \mapsto (\bar{b}^{-1}(X), u)$$

and the following equality

$$[\pi]_{\bar{b}}(\bar{b}^{-1}(X)) = \bar{b}^{-1}([\pi]_{\mathcal{F}}(X)).$$ (2.11)

Since $\mathcal{F}_{\bar{b}}$ is the universal formal $O_F$-module over $X(1) \times_F E$, we have the following map

$$b : X(1) \times_F E \to X(1)$$ (2.12)

as in [GH (14.5)]. This map depends only on $b$. The map (2.12) clearly extends to the following map as in [GH Proposition 14.7]

$$b : X(1) \times_F E \to X(1) \times_F E.$$ (2.13)

This map is the map (2.8) which we want. The map $b$ is an automorphism, which is proved in loc. cit. Let $b^* \mathcal{F}_{\bar{b}} / X(1) \times_F E$ denote the pull-back of the universal formal $O_F$-module $\mathcal{F}_{\bar{b}} / X(1)$ by the map (2.12). Then, we have the following unique isomorphism by [GH (14.6)]

$$j : b^* \mathcal{F}_{\bar{b}} \simeq \mathcal{F}_{\bar{b}} : (X, u) \mapsto (j(X), u)$$

such that $j(X) \equiv X \pmod{(\pi, u)}$. Hence, we acquire the following equality

$$[\pi]_{b^* \mathcal{F}_{\bar{b}}}(j^{-1}(X)) = j^{-1}([\pi]_{\bar{b}}(X)).$$ (2.14)

On the other hand, we have the following isomorphism

$$b^* \mathcal{F} \simeq \mathcal{F} : (X', u) \mapsto (X', b(u)).$$
We consider the following embedding
\[ X(\pi^n) \hookrightarrow \mathcal{F}^{\text{univ}}[\pi^n] \times_{X(1)} \mathcal{F}^{\text{univ}}[\pi^n] \ni (X_n, Y_n, u). \tag{2.15} \]

The image of this embedding is determined by the Drinfeld condition for \((X_n, Y_n)\). In the remainder of this paper, we always consider the space \(X(\pi^n)\) as a subspace of the product \(\mathcal{F}^{\text{univ}}[\pi^n] \times_{X(1)} \mathcal{F}^{\text{univ}}[\pi^n]\). Then, we have the following well-defined map
\[ b : X(\pi^n) \times_{\mathcal{F}} E \to X(\pi^n) \times_{\mathcal{F}} E : (X_n, Y_n, u) \mapsto (j^{-1} \circ \tilde{b}^{-1}(X_n), j^{-1} \circ \tilde{b}^{-1}(Y_n), b(u)). \tag{2.16} \]

This is the action of \(b \in \mathcal{O}_\mathcal{D}^b\) on \(X(\pi^n)\). This action also induces the action of \(\mathcal{O}_\mathcal{D}^\times\) on \(\mathcal{X}(\pi^n)\). We set
\[ b^*(X) := j^{-1} \circ \tilde{b}^{-1}(X). \tag{2.17} \]

### 2.7 Several subspaces in \(\mathcal{X}(\pi^n)\)

We will define several subspaces in the Lubin-Tate space \(\mathcal{X}(\pi^n)\). We identify \(\mathcal{X}(1)\) with an open unit ball \(B(1) \supset u\) as in subsection 2.4. Let \(n \geq 1\) be a positive integer. Let
\[ p_n : \mathcal{X}(\pi^n) \to \mathcal{X}(1) : (\mathcal{F}, \eta, \phi) \mapsto (\mathcal{F}, \eta) \]
be the canonical projection. Let \(\mathcal{T} \mathcal{S}^0\) be the closed disc \(B[p^{-\epsilon/\eta} \mathcal{S}] \subset \mathcal{X}(1)\). This is called the "too-supersingular locus." We define subspaces \(Y_{2n-m,m}(1 \leq m \leq n)\) as follows;
\[ Y_{n,n} := p_n^{-1}(\mathcal{T} \mathcal{S}^0) \subset \mathcal{X}(\pi^n), \]
\[ Y_{2n-m,m} := p_n^{-1}(C[p^{-\epsilon/\eta}] \subset \mathcal{X}(\pi^n) \quad (1 \leq m \leq n-1). \]

Let \(n \geq 2\) be a positive integer. For \(1 \leq m \leq n-1\), we define subspaces \(Z_{2(n-1)-m,m}\) as follows:
\[ Z_{2(n-1)-m,m} := p_n^{-1}(C[p^{-\epsilon/\eta}] \subset \mathcal{X}(\pi^n) \quad (1 \leq m \leq n-1). \]

We introduce several subspaces in the spaces \(Y_{3,1}\) and \(Z_{1,1}\), whose reduction appears in the stable reduction of the Lubin-Tate space \(\mathcal{X}(\pi^2)\).

As in \(2.15\), we consider \(X(\pi^n)\) as a formal subscheme of \(\mathcal{F}^{\text{univ}}[\pi^n] \times_{X(1)} \mathcal{F}^{\text{univ}}[\pi^n] \ni (X_n, Y_n, u)\).

**Definition 2.7.** 1. Let \((X_2, Y_2, u) \in Y_{3,1,0} \subset Y_{3,1}\) be a subspace defined by the following conditions;
\[ v(u) = \frac{1}{q+1}, \quad v(X_1) = \frac{q}{q^2 - 1}, \quad v(X_2) = \frac{1}{q(q^2 - 1)}, \quad v(Y_1) = \frac{1}{q(q^2 - 1)}, \quad v(Y_2) = \frac{1}{q^3(q^2 - 1)}. \]

2. Let \((X_2, Y_2, u) \in Y_{3,1,\infty} \subset Y_{3,1}\) be a subspace defined by the following condition; \((X_2, Y_2, u) \in Y_{3,1,\infty}\) is equivalent to \((Y_2, X_2, u) \in Y_{3,1,0}\).

3. Let \((X_2, Y_2, u) \in Y_{3,1,c} \subset Y_{3,1}\) be a subspace defined by the following conditions;
\[ v(u) = \frac{1}{q+1}, \quad v(X_1) = v(Y_1) = \frac{1}{q(q^2 - 1)}, \quad v(X_2) = v(Y_2) = \frac{1}{q^3(q^2 - 1)}. \]

4. Let \((X_2, Y_2, u) \in Z_{1,1,0} \subset Z_{1,1}\) be a subspace defined by the following conditions;
\[ v(u) = \frac{1}{2}, \quad v(X_1) = \frac{1}{2(q-1)}, \quad v(X_2) = \frac{1}{2q^2(q-1)}, \quad v(Y_1) = \frac{1}{2q(q-1)}, \quad v(Y_2) = \frac{1}{2q^3(q-1)}. \]
5. Let \((X_2, Y_2, u) \in Z_{1,1,\infty} \subset Z_{1,1}\) be a subspace defined by the following condition; \((X_2, Y_2, u) \in Z_{1,1,\infty}\) is equivalent to \((Y_2, X_2, u) \in Z_{1,1,0}\).

6. Let \((X_2, Y_2, u) \in Z_{1,1,c} \subset Z_{1,1}\) be a subspace defined by the following conditions;

\[
v(u) = \frac{1}{2}, \quad v(X_1) = v(Y_1) = \frac{1}{2q(q-1)}, \quad v(X_2) = v(Y_2) = \frac{1}{2q^2(q-1)}.
\]

**Lemma 2.8.** Let the notation be as above.

1. The space \(Y_{3,1}\) has the following description

\[
Y_{3,1} = Y_{3,1,0} \bigsqcup Y_{3,1,\infty} \bigsqcup Y_{3,1,c}.
\]

2. The space \(Z_{1,1}\) has the following description

\[
Z_{1,1} = Z_{1,1,0} \bigsqcup Z_{1,1,\infty} \bigsqcup Z_{1,1,c}.
\]

3 **Computation of irreducible components in the stable reduction of** \(\mathcal{X}(\pi^2)\)

From this section until the end of this paper, we assume \(p \neq 2\). In this section, we will compute the reduction of the spaces \(Y_{2,2}, Y_{3,1}\) and \(Z_{1,1}\) by using the approximation formula for \([\pi]^\mu\) in Lemma 2.7. In the equal characteristic case, we have already computed the reduction of them in [2], hence we assume \(\text{char } F = 0\) in this section. However, the computation in this section also holds in the equal characteristic case. In the equal characteristic case, the computation in loc. cit. is easier than the one in this section, because the rigid analytic morphism \(\Delta^{(n)} : \mathcal{X}(\pi^n) \to \operatorname{LT}[\pi^n]\) in subsection 2.5 is given by the Moore determinant, which has a very simple form. See [W2] for more details on the Moore determinant. We briefly give a summary of results in this section. The reduction of the space \(Y_{3,1,0}\) has \(q(q-1)\) connected components and each component is defined by \(x^qy - xy^q = \zeta\) with some \(\zeta \in \mathbb{F}_q^\times\). Similarly, the reduction of the space \(Y_{3,1,c}\) has \(q(q-1)^2\) connected components and each is defined by \(x^qy - xy^q = \zeta\) with some \(\zeta \in \mathbb{F}_q^\times\). The reduction of the space \(Z_{1,1,0}\) has \(q(q-1)\) connected components and each is defined by

\[
Z_q = y^{q^2-1} + y^{-(q^2-1)}.
\]

This affine curve with genus 0 has singular points at \(y \in \mu_{2(q^2-1)}\). By blowing-up these points, we obtain \(2(q^2-1)\) irreducible components with genus \((q-1)/2\), defined by \(a^q - a = s^2\). Similar things happen for the reduction of the space \(Z_{1,1,c}\).

The reduction of the space \(Y_{2,2}\) has \(q(q-1)\) connected components and each component is defined by

\[
Z_q = x^qy - xy^q, \quad x^qy - xy^q = \zeta
\]

with some \(\zeta \in \mathbb{F}_q^\times\). This affine curve with genus \(q(q-1)/2\) has \(q(q^2-1)\) singular points at \((x, y) = (x_0, y_0) \in (k^\times)^2\) with \(x_0^{q^2-1} = y_0^{q^2-1} = -1\). Then, by blowing-up these points, we find \(q(q^2-1)\) irreducible components defined by \(a^q - a = t^{q+1}\).

By [W3, Proposition 5.1], it is guaranteed that the reduction of the spaces, explained above, really appears in the stable reduction \(\overline{\mathcal{X}(\pi^2)}\). See also [Liu, Proposition 4.37] for an analogous statement in the language of schemes.
3.1 Computation of the reduction of $Y_{2,2}$

In this subsection, we compute the reduction of the space $Y_{2,2}$ by using the approximation formula for $[\pi]_u$ in Lemma 2.3.1. Let $\pi_0(Y_{2,2})$ denote the set of geometrically connected components of $Y_{2,2}$. First, we check that the reduction $Y_{2,2}$ has $q(q-1)$ connected components and fix an identification $\pi_0(Y_{2,2}) \cong (\mathcal{O}_F/\pi^2)\times$. Furthermore, we show that each component is defined by the following equations

$$Z^q = x^q y - xy^q, \quad x^q y - xy^q = \xi$$

with some $\xi \in \mathbb{F}_q^\times$.

Let $(X_2, Y_2)$ denote the Drinfeld $\pi^2$-basis of $\mathcal{F}^{\text{univ}}$. We set $X_1 := [\pi]_u(X_2)$ and $Y_1 := [\pi]_u(Y_2)$. On the space $(X_2, Y_2, u) \in Y_{2,2}$, we have the followings as in subsection 2.7.

$$v(u) \geq \frac{q}{q+1}, \quad v(X_1) = v(Y_1) = \frac{1}{q^2 - 1}, \quad v(X_2) = v(Y_2) = \frac{1}{q^2(q^2 - 1)}.$$ 

We choose an element $\kappa_1$ such that $\kappa_1^{q^2(q^2 - 1)} = \pi$ with $v(\kappa_1) = 1/q^3(q^2 - 1)$. We set $\kappa := \kappa_1^q$ and put $\gamma := \kappa^{q(q-1)(q^2-1)}$ with $v(\gamma) = (q-1)/q^2$. We write $\gamma^{q(q^2-1)}$ for an element $\kappa_1^{q^2-1}$.

Then, we change variables as follows $u = \kappa^{q(q-1)} u_0$, $X_1 = \kappa^q x_1$, $Y_1 = \kappa^q y_1$, $X_2 = \kappa x$ and $Y_2 = \kappa y$. By $0 = [\pi]_u(X_1) = [\pi]_u(Y_1)$ and Lemma 2.4.1, we acquire the following congruence

$$u_0 = -x_1^{q(q-1)} - \frac{1}{x_1^{q(q-1)}} - y_1^{q(q-1)} - \frac{1}{y_1^{q(q-1)}} \pmod{1+}.$$ (3.1)

Let $f(X, Y)$ be a polynomial with coefficients in $\mathcal{O}_F$ such that $X^q - Y^q = (X - Y)^q + \pi f(X, Y)$. By the Drinfeld condition for $(X_1, Y_1)$, we have $x_1 \neq y_1$. Hence, we acquire

$$(x_1^q y_1 - x_1 y_1^q)^q - 1 = -\pi \frac{f(x_1^q y_1, x_1 y_1^q)}{x_1^q y_1 - x_1 y_1^q} \pmod{1+}.$$ 

This splits to $(q-1)$ congruences as follows

$$x_1^q y_1 - x_1 y_1^q = \zeta + \pi f(x_1^q y_1, x_1 y_1^q) \pmod{1+}$$ (3.2)

with $\zeta \in \mu_q - 1(\mathcal{O}_F)$.

By $X_1 = [\pi]_u(X_2)$ and $Y_1 = [\pi]_u(Y_2)$ and Lemma 2.4.1, we acquire the following congruences

$$x_1 \equiv x_1^q + \gamma^q u_0 x^q + \gamma^{q+1} x, \quad y_1 \equiv y^q + \gamma^q u_0 y^q + \gamma^{q+1} y \pmod{1+}.$$ (3.3)

We set

$$Z := (x^q y - xy^q)^q - \gamma(x^q y - xy^q).$$ (3.4)

Substituting (3.3) to (3.2), we acquire the following congruence by using (3.1)

$$Z^q - \gamma Z = \zeta + \pi (f^q - f) \pmod{1+}$$ (3.5)

where we write $f$ for $f(x^q y, x^q y^q)$. We choose an element $\gamma_0$ such that $\gamma_0^q - \gamma^q \gamma_0 = \zeta$. We set as follows

$$Z = (x^q y - xy^q)^q - \gamma(x^q y - xy^q) = \gamma_0 - \gamma \pi^{-1} c.$$ (3.6)

We set $\mu := c + f$. Then, by substituting (3.6) to (3.5), and dividing it by $\pi$, we acquire $\mu^q \equiv \mu \pmod{0+}$. Therefore, the set $\pi_0(Y_{2,2})$ is identified with $(\bar{\zeta}, \bar{\mu}) \in \mathbb{F}_q^\times \times \mathbb{F}_q$. Hence, we fix the following identification

$$\pi_0(Y_{2,2}) \cong (\mathcal{O}_F/\pi^2)\times \cong \mathbb{F}_q^\times \times \mathbb{F}_q \ni (\bar{\zeta}, \bar{\mu}^q)$$ (3.7)
We choose an element \( \tilde{\gamma}_0 \) such that \( \tilde{\gamma}_0^q = \gamma_0 \). We set as follows
\[
x^q y - xy^q = \tilde{\gamma}_0 + \gamma^{1/q} Z_1. \tag{3.8}
\]
By substituting (3.8) to (3.9) and dividing it by \( \gamma \), we obtain the following congruence
\[
Z_1^q \equiv x^q y - xy^q - \gamma^{1/(q-1)} c \pmod{(1/q^2^2)}. \tag{3.9}
\]
Hence, we have proved the following proposition by (3.8) and (3.9).

**Proposition 3.1.** The reduction \( Y_{2,2} \) has \( q(q-1) \) connected components, and each component is defined by the following equations
\[
x^q y - xy^q = \zeta, \quad Z^q = x^q y - xy^q^3
\]
with some \( \zeta \in \mathbb{F}_q^\times \).

### 3.2 Analysis of the singular residue classes in \( Y_{2,2} \)

In this subsection, we analyze the singular residue classes of \( Y_{2,2} \). Let \( \{ Y_{2,2} \}_{i=\langle \zeta, \mu \rangle \in \mathbb{F}_q^\times \times \mathbb{F}_q} \) denote the connected components of the reduction \( Y_{2,2} \). Let \( i = \langle \zeta, \mu \rangle \in \mathbb{F}_q^\times \times \mathbb{F}_q \). As in the previous subsection, the reduction \( Y_{2,2} \) is defined by the following equations
\[
x^q y - xy^q = \zeta, \quad Z^q = x^q y - xy^q^3
\]
with \( \zeta \in \mathbb{F}_q^\times \). This affine curve has singular points at the following set
\[
S_{00} := \{(x_0, y_0, Z_0) \in Y_{2,2}(k^{ac}) \mid x_0^q y_0 - x_0 y_0^q = 0\}.
\]
Note that we have \( |S_{00}| = q(q^2 - 1) = |\text{SL}_2(\mathbb{F}_q)| \). This set \( S_{00} \) is identified with the following set \( \{(x_0, y_0) \in (k^{ac})^2 \mid x_0^q y_0 - x_0 y_0^q = \zeta, x_0^{q-1} = y_0^{q-1} = -1\} \). For \( i = \langle x_0, y_0 \rangle \in S_{00} \), we denote by \( X^i \) the underlying affinoid of the singular residue class of \((x, y) = i \) in the space \( Y_{2,2} \). Furthermore, we write \( X^i_j \) for the reduction of the affinoid \( X^i \). In this subsection, we compute the reduction \( X^i_j \).

By using (3.8), the congruence (3.9) is rewritten as follows
\[
Z_1^q \equiv \tilde{\gamma}_0 \left( \frac{\zeta_{0}^{q-1} + y^{q-1}}{y^{q-1}} \right)^q + y^{q(q^2-1)}(\tilde{\gamma}_0 + \gamma^{1/q} Z_1) - \gamma^{1/(q-1)} c \pmod{(1/q^2^2)}. \tag{3.10}
\]
We choose an element \( \tilde{\gamma}_1 \) such that \( \tilde{\gamma}_1^q + \tilde{\gamma}_0 + \gamma^{1/q} \tilde{\gamma}_1 = 0 \). Furthermore, choose \( y_0, x_0 \) such that \( y_0^{q-1} + \tilde{\gamma}_0^{q-1} = 0 \) and \( x_0^q y_0 - x_0 y_0^q = \tilde{\gamma}_0 + \gamma^{1/q} \tilde{\gamma}_1 \). We set \( w := \gamma^{1/q(q-1)} \) and \( w_1 := \tilde{\gamma}_0 \gamma^{1/(q^2-1)} \). Then, we have \( w(w) = 1/q^3 \) and \( w(w_1) = 1/q^2(q+1) \).

We change variables as follows
\[
Z_1 = \tilde{\gamma}_1 + w a, \quad y = y_0 + w_1 z_1. \tag{3.11}
\]
Substituting them to (3.10), we acquire the following congruence
\[
w^q(a^q + a) \equiv -w_1^{q+1} \zeta \left( \frac{z_1}{y_0} \right)^{q+1} - \gamma^{1/(q-1)} c_0 \pmod{(1/q^2^2)}. \tag{3.12}
\]
where we set $c_0 = \mu - f(x_0^q y_0^q, x_0^q y_0^q)$ with some $\mu \in \mu_{q-1}(O_F) \cup \{0\}$. Hence, by dividing by $w^q$, the following congruence holds
\[ a^q + a = \zeta z_1^{q+1} - c_0 \pmod{0+}. \]  

(3.13)

Hence, the reduction $X_j^i$ is defined by the following equation $a^q + a = \zeta z_1^{q+1} - c_0$. Therefore, we have proved the following Proposition 3.2. We set
\[ \pi := \{(\zeta, \mu), (x_0, y_0) \} \in (\mathbb{F}_q^\times \times \mathbb{F}_q^\times) \times \mathbb{Z}_{\geq 2} \mid x_0^q y_0 - x_0 y_0^q = \zeta, x_0^q \gamma - y_0^q = -1 \} \]

For an element $((\zeta, \mu), (x_0, y_0)) \in \mathcal{S}$, we set $i := (\zeta, \mu)$ and $j := (x_0, y_0) \in \mathcal{S}_{00}$.

**Proposition 3.2.** Let the notation be as above. Then, in the stable reduction of the Lubin-Tate space $X(\pi^2)$, there exist $q(q-1) \times q(q-1)$ irreducible components $\{X_j^i\}_{(i,j) \in \mathcal{S}}$ such that each $X_j^i$ has an affine model with an equation $a^q + a = \zeta z_1^{q+1} - c$ with some $c \in \mathbb{F}_q$. These components attach to the connected component of $\mathcal{Y}_{2,2}$ at $q(q-1)$ singular points.

### 3.3 Computation of the reduction of the space $Z_{1,1,\ast}$ ($\ast = 0, \infty$)

In this subsection, we calculate the reduction of the space $Z_{1,1,\ast}$ with $\ast = 0, \infty$ by using Lemma 2.4.1. First, we check that the reduction $Z_{1,1,0}$ has $q(q-1)$ connected components. Secondly, we show that each component is defined by the following equation
\[ Z_1^q = X^{q^2-1} + X^{-(q^2-1)} \]

with genus 0. Since we have
\[(X_2, Y_2, u) \in Z_{1,1,0} \iff (Y_2, X_2, u) \in Z_{1,1,\infty} \]
by Definition 2.7.5, the same things happen for $Z_{1,1,\infty}$.

Let $(X_2, Y_2)$ be a Drinfeld $\pi^2$-basis of the universal formal $O_F$-module $\mathcal{F}_{\text{univ}}$. We set $X_1 := [\pi]_u(X_2)$ and $Y_1 := [\pi]_u(Y_2)$. Recall that we have the followings on the space $(X_2, Y_2, u) \in Z_{1,1,0}$ as in Definition 2.7.7
\[ v(u) = \frac{1}{2}, \quad v(X_1) = \frac{1}{2(q-1)} \quad v(Y_1) = \frac{1}{2q(q-1)} \quad v(X_2) = \frac{1}{2q^2(q-1)} \quad v(Y_2) = \frac{1}{2q^3(q-1)} \]

We choose an element $\kappa_1$ such that $\kappa_1^{q^2(q-1)} = \pi$ with $v(\kappa_1) = 1/2q^4(q-1)$. We set $\kappa := \kappa_1^q$. We set $\gamma := \kappa^{q(q-1)^2}$ with $v(\gamma) = (q-1)/2q^2$. We write $\gamma^{q(q-1)}$ for $\kappa_1^{-q-1}$. We change variables as follows $u = \kappa^q(q-1)u_0$, $X_1 = \kappa^q x_1$, $Y_1 = \kappa^q y_1$, $X_2 = \kappa^q x$ and $Y_2 = \kappa y$. Considering $[\pi]_u(X_1) = [\pi]_u(Y_1) = 0$, we acquire the following congruence by using Lemma 2.4.1
\[ u_0 \equiv - \frac{1}{x_1^{q+1}} \equiv -y_1^{q(q-1)} - \frac{\gamma y}{y_1^{q-1}} \pmod{1+}. \]  

(3.14)

By (3.14), the following congruence holds
\[ x_1 \equiv \frac{\zeta}{y_1} \left( 1 + \frac{\gamma y}{y_1^{q-1}} \right) \pmod{1+} \]  

(3.15)

with $\zeta \in \mu_{q-1}(O_F)$. By considering $X_1 = [\pi]_u(X_2)$ and $Y_1 = [\pi]_u(Y_2)$, we obtain the following congruences again by Lemma 2.4.1
\[ x_1 \equiv x^{q^2} + \gamma y u_0 x^{q^2} + \gamma^{q+1} x (\pmod{1+}) \quad y_1 \equiv y^{q^2} + \gamma^{q+1} y_0 y^{q^2} (\pmod{(q+1)/2q+}). \]  

(3.16)
We set
\[ Z := (xy^q)^q - \gamma \left( xy^q + \frac{\zeta}{y^{q(q^2-1)}} \right). \] (3.17)

By substituting (3.19) to (3.15) \times y^q$, we acquire the following congruence by using (3.14)
\[ Z^q - \gamma^2 q Z = \zeta \pmod{1+}. \] (3.18)

We choose an element $\gamma_0$ such that $\gamma_0 - \gamma^{2q} \gamma_0 = \zeta$. Then, by (3.15), if we set $Z = \gamma_0 - \gamma^{2q} \gamma_0$, we obtain $\mu \equiv \mu \pmod{0+}$. Similarly as (3.7), we fix the following identification
\[ \pi_0(Z_{1,1,0}) \simeq (\mathcal{O}_F/\pi^2)^* \simeq \mathbb{F}_q^* \times \mathbb{F}_q \in (\zeta, \frac{\mu}{\zeta}). \] (3.19)

Now, we choose an element $\mu \in \mu_{q-1}(\mathcal{O}_F) \cup \{0\}$. We have the following congruence by (3.17)
\[ Z = (xy^q)^q - \gamma \left( xy^q + \frac{\zeta}{y^{q(q^2-1)}} \right) \equiv \gamma_0 + \gamma^{2q} \gamma_0 \pmod{(1/q)}. \] (3.20)

We choose an element $\gamma_0$ such that $\gamma_0 = \gamma + \gamma^{2q} \gamma_0$. Then, we set
\[ xy^q = \gamma_0 + \gamma^{1/q} Z_1. \] (3.21)

Substituting this to (3.20), and dividing it by $\gamma$, the following congruence holds
\[ Z_1^q \equiv xy^q + \zeta y^{-q(q^2-1)} \pmod{(1/2q^2)}. \] (3.22)

Substituting $x = \frac{\gamma_0 + \gamma^{1/q} Z_1}{y^q}$ to the right hand side of the congruence (3.22), we obtain the following congruence
\[ (Z_1 - y^{q^2-1} \gamma_0^{1/q} - \zeta y^{-(q^2-1)})^q \equiv \gamma^{1/q} y^{-q(q^2-1)} Z_1 \pmod{(1/2q^2)}. \] (3.23)

We introduce a new parameter $Z_2$ such that
\[ y^{q^2-1} \gamma_0^{1/q} Z_2 = Z_1 - y^{q^2-1} \gamma_0^{1/q} - \zeta y^{-(q^2-1)}. \] (3.24)

Substituting this to (3.23), and dividing it by $\gamma^{1/q} y^{q^2-1}$, we obtain the following congruence
\[ Z_2^q \equiv Z_1 \pmod{(1/2q^3)}. \] (3.25)

By substituting this to (3.24), the following congruence holds
\[ Z_2^q \equiv \zeta (y^{q^2-1} + y^{-(q^2-1)}) + y^{q^2-1} \gamma_0^{1/q} Z_2 \pmod{(1/2q^3)}. \] (3.25)

Note that we have $\gamma_0^{1/q} = \gamma^{1/q} = \zeta \pmod{(1/2q^3)}$. Therefore, we have proved the following proposition.

**Proposition 3.3.** The reduction $\mathbb{Z}_{1,1,0}$ and $\mathbb{Z}_{1,1,\infty}$ has $q(q-1)$ connected components, and each component is defined by the following equation
\[ Z^q = \zeta (y^{q^2-1} + y^{-(q^2-1)}). \]

**Proof.** The required assertion for $\mathbb{Z}_{1,1,0}$ follows from (3.25). Switching the roles of $X_2$ and $Y_2$, we obtain the required assertion for $\mathbb{Z}_{1,1,\infty}$. \qed
3.4 Analysis of the singular residue classes of $Z_{1,1,*}$ ($*=0,\infty$)

In this subsection, we analyze the singular residue classes in the space $Z_{1,1,*}$ with $*=0,\infty$. We keep the same notation as in the previous subsection. Let $\{Z_{1,1,*}\}_{i\in F_{\mathcal{Z}}}$ denote the connected components of the reduction $Z_{1,1,*}$. Recall that the reduction $Z_{1,1,*}$ is defined by the following equation

$$Z^q = \zeta(y^2-1+y^{-1}(q^2-1)).$$

This affine curve has singular points at $y \in \mu_2(q^2-1)$. We set

$$S_0 := (O_{\mathcal{F}}/\pi^2)^{\times} \times \mu_2(q^2-1) \ni (i,j).$$

For $(i,j) \in S_0$, we denote by $W_{0,i,j}$ (resp. $W_{i,j}$) the underlying affinoid of the singular residue class at $y=j$ of the space $Z_{1,1,0}$ (resp. $Z_{1,1,\infty}$). Furthermore, for $*=0,\infty$, we denote by $W_{*,i,j}$ the reduction of the affinoid $W_{i,j}$.

We choose elements $\gamma_1$ and $y_0$ such that $\gamma_1 = i2\zeta\{1 + \gamma^{1/q^2}(\zeta/\zeta)^{1/2}\}$ and $y_0^2-1 = (i/2)\{1 + \gamma^{1/q^2}(\zeta/\zeta)^{1/2}\}$ with $i \in \{\pm 1\}$. Set $w := y_0^{p+1}\gamma^{1/(q-1)}$. Furthermore, we choose an element $w_1$ such that $y_0^{q^2-3}(\zeta + \gamma^{1/q^2}\gamma_1)w_1^2 = w^q$. Then, we have $v(w) = 1/2q^4$ and $v(w_1) = 1/4q^3$.

We change variables as follows

$$Z_2 = \gamma_1 + wa, \ y = y_0 + w_1y_1.$$ (3.26)

Substituting them to (3.25), we acquire the following

$$w^q(a^q-a) \equiv y_0^{q^2-3}(\zeta + \gamma^{1/q^2}\gamma_1)w_1^2y_1^2 \pmod{(1/2q^3)+}.$$ Hence, by dividing this by $w^q$, we acquire the following $a^q-a = y_0^q$ (mod 0+). Hence, the reduction $W_{0,i,j}$ is defined by $a^q-a = y_0^q$. Therefore, we have proved the following proposition.

**Proposition 3.4.** In the stable reduction of the Lubin-Tate space $X(\pi^2)$, there exist $q(q-1) \times 2(q^2-1)$ irreducible components $\{W_{0,i,j}\}_{(i,j)\in S_0}$ (resp. $\{W_{i,j}\}_{(i,j)\in S_0}$) with an affine model $a^q-a = s^2$. For each $i \in (O_{\mathcal{F}}/\pi^2)^{\times}$, the components $\{W_{0,i,j}\}_{j\in \mu_2(q^2-1)}$ (resp. $\{W_{i,j}\}_{j\in \mu_2(q^2-1)}$) attach to the component $Z_{1,1,0}$ (resp. $Z_{1,1,\infty}$) at $2(q^2-1)$ singular points.

3.5 Computation of the reduction of the space $Z_{1,1,c}$

In this subsection, we compute the reduction of the space $Z_{1,1,c}$. We prove that the space $Z_{1,1,c}$ has $q(q-1)^2$ connected components, and each component is defined by $Z^q = X^{q^2-1} + X^{-1}(q^2-1)$.

Recall that we have the followings on the space $Z_{1,1,c}$ as in Definition 2.14

$$v(u) = \frac{1}{2}, \ v(X_1) = v(Y_1) = \frac{1}{2q(q-1)}, \ v(X_2) = v(Y_2) = \frac{1}{2q^2(q-1)}.$$ Let $\kappa$ and $\gamma^{1/q^2}(q-1)$ be as in subsection 3.3. We change variables as follows $u = \kappa^{q^2(q-1)}u_0$, $X_1 = \kappa^{q^2}x_1$, $Y_1 = \kappa^{q^2}y_1$, $X_2 = \kappa x$ and $Y_2 = \kappa y$. We put $i := (2q+1)/2q$.

By $[\pi]_a(X_1) = [\pi]_a(Y_1) = 0$ and Lemma 2.41, we acquire the following congruence

$$u_0 = -x_1^{q(q-1)} - \frac{\gamma^q}{x_1^{q-1}} \equiv -y_1^{q(q-1)} - \frac{\gamma^q}{y_1^{q-1}} \pmod{(3/2)+}.$$ (3.27)
Note that we have $v(\gamma^q) + i = 3/2$. Let $f(X,Y)$ be a polynomial such that $X^q - Y^q = (X - Y)^q + pf(X,Y)$. By considering \((\ref{3.27}) \times (x_1 y_1)^q\), we acquire the following congruence

$$x_1^q y_1 - x_1 y_1^q \equiv \zeta \frac{y}{y^q} + \left(\frac{p}{\gamma^q}\right) f(x_1^q y_1, x_1 y_1^q) \pmod{i^+}$$

(3.28)

with some $\zeta \in \mu_{q-1}(\mathcal{O}_F)$. Note that we have $(x_1^q y_1 - x_1 y_1^q) \vert f(x_1^q y_1, x_1 y_1^q)$ and $v(x_1^q y_1 - x_1 y_1^q) = 1/2q > 0$. Furthermore, by $X_1 = [\pi]_u(X_2), Y_1 = [\pi]_u(Y_2)$ and Lemma 2.3.1, we obtain the following congruence

$$x_1 \equiv x^q + \gamma^{q+1} u_0 x^q + \gamma \frac{2x^q + 2x_{q+1}^q}{q} x - q\gamma u_0^2 x^q (q^2 - q^1) \pmod{i^+}.$$  

(3.29)

The same congruence as \((\ref{3.29})\) holds for $(y, y_1)$. On the right hand side of \((\ref{3.28})\), we have the following congruence

$$\left(\frac{p}{\gamma^q}\right) f(x_1^q y_1, x_1 y_1^q) \equiv \left(\frac{p}{\gamma^q}\right) \{f(x^q y^q, x^q y^q) - (q/p)\gamma^{q+1} u_0 (xy)^q (q-1)(x^q y - y^q)^q\} \pmod{i^+}.$$ 

(3.30)

Note that, if $e_{F/q_0} \geq 2$, this term \((\ref{3.30})\) is congruent to zero modulo $i^+$. We set

$$Z \equiv (x^q y - y^q)^q - \gamma \frac{2x^q + 2x_{q+1}^q}{q} (x^q y - y^q)^q - (p/q)^{1/q} f(x^q y^q, x^q y^q).$$

(3.31)

On the other hand, we have the following congruence, by substituting \((\ref{3.27})\) and \((\ref{3.29})\) to $x_1^q y_1 - x_1 y_1^q$,

$$x_1^q y_1 - x_1 y_1^q \equiv Z^q - \gamma^{q+1} Z + (p/q^q) f(x^q y^q, x^q y^q) - q\gamma u_0^2 (xy)^q (q-1)(x^q y - y^q)^q \pmod{i^+}.$$ 

(3.32)

Hence, \((\ref{3.28})\) induces the following congruence by \((\ref{3.30})\) and \((\ref{3.32})\)

$$Z^q - \gamma^{q+1} Z \equiv \gamma \frac{y}{y^q} \zeta \pmod{i^+}.$$ 

(3.33)

We choose an element $\gamma_0$ such that $\gamma_0^q - \gamma^{q+1} \gamma_0 = \gamma \frac{y}{y^q} \zeta$. Clearly, we have $v(\gamma_0) = 1/2q^2$. Then, if we set $Z = \gamma_0 + \gamma \frac{2x^q + 2x_{q+1}^q}{q} \mu$, we acquire $\mu^q \equiv \mu \pmod{0^+}$ by \((\ref{3.33})\). Hence, by \((\ref{3.31})\) and $v(f(x^q y^q, x^q y^q)) = 1/2q^2 > 0$, we obtain the following congruence

$$Z \equiv (x^q y - y^q)^q - \gamma \frac{2x^q + 2x_{q+1}^q}{q} (x^q y - y^q)^q \equiv \gamma_0 + \gamma \frac{2x^q + 2x_{q+1}^q}{q} \mu \pmod{\left(\frac{q + 1}{2q^2}\right)^{+}}.$$ 

(3.34)

We choose an element $\tilde{\gamma}_0$ such that $\tilde{\gamma}_0^q = \gamma_0 + \gamma \frac{2x^q + 2x_{q+1}^q}{q} \mu$. Then, we introduce a new parameter $Z$ as follows

$$x^q y - y^q = \tilde{\gamma}_0 + y^q - \gamma \frac{2x^q + 2x_{q+1}^q}{q} \zeta_{01/q} + y^q - \gamma \frac{2x^q + 2x_{q+1}^q}{q} \zeta_{(q-1)q^2/2}.$$ 

(3.35)

By substituting \((\ref{3.35})\) to the left hand side of the congruence \((\ref{3.34})\), we obtain the following congruence in the same way as in \([11. (4.21), (4.25)]\)

$$Z^q \equiv \zeta (y^q - y^{(q-1)}) + \gamma_1 y^q Z \pmod{(1/2q^2)^{+}}.$$ 

(3.36)

By \((\ref{3.35})\), we have $x^q \equiv y^q \pmod{0^+}$. Therefore, we acquire $x \equiv \zeta_1 y \pmod{0^+}$ with some $\zeta_1 \in \mu_{q-1}(\mathcal{O}_F)$. Hence, the reduction $\mathbb{Z}_{1,1,c}$ has $q(q - 1)^2$ connected components, which are parametrized by $(\zeta, (\bar{\mu}, \zeta_1)) \in \mathbb{P}_q^1 \times \mathbb{P}_q^1$. Furthermore, each component is defined by $Z^q = \zeta (y^q - y^{(q-1)})$ by \((\ref{3.36})\). Therefore, we obtain the following propositions.

**Proposition 3.5.** The space $\mathbb{Z}_{1,1,c}$ has $q(q - 1)^2$ connected components and each component reduces to the following curve

$$Z^q = \zeta (y^q - y^{(q-1)})$$ 

(3.37)

with some $\zeta \in \mathbb{P}_q^1$. This affine curve has $2(q^2 - 1)$ singular points at $y \in \mu_{2(q^2-1)}$. 


Let \( i \in (\mathcal{O}_F/\pi^2)^\times \). We write \( \{Z_{i,1,j}\}_{j \in \mathbb{F}_q^\times} \) for the connected components of \( Z_{i,1,c} \). For each \( k \in \mu_2(q^2-1) \), let \( W_{i,k} \subset Z_{i,1,j} \) denote the underlying affinoid of the singular residue class at \( y = k \). Then, by the same computations as the ones in subsection 3.3, we acquire the following proposition.

**Proposition 3.6.** In the stable reduction of \( X(\pi^2) \), there exist \( q(q-1)^2 \times 2(q^2-1) \) irreducible components \( \{W_{i,k}\}_{(i,j,k) \in (\mathcal{O}_F/\pi^2)^\times \times \mathbb{F}_q^\times \times \mu_2(q^2-1)} \), with an affine model \( a^3 - a = s^2 \). For each \( (i,j) \in (\mathcal{O}_F/\pi^2)^\times \times \mathbb{F}_q^\times \), the components \( \{W_{i,k}\}_{k \in \mu_2(q^2-1)} \) attach to the component \( Z_{i,1,j} \) at each singular point.

### 3.6 Computation of the reduction of \( Y_{3,1,0} \)

In this subsection, under a technical assumption \( e_{F/Q_p} \geq 2 \), we compute the reduction of the space \( Y_{3,1,0} \). The reduction \( Y_{3,1,0} \) has \( q(q-1) \) connected components, and each component is defined by \( x^qy - xy^q = \zeta \) with some \( \zeta \in \mathbb{F}_q^\times \). In the proceeding sections, we do not use results in subsections 3.6 and 3.7.

Recall that we have the following on the space \( Y_{3,1,0} \) in Definition 2.3

\[
v(u) = \frac{1}{q+1}, \quad v(X_1) = \frac{q}{q^2-1}, \quad v(Y_1) = \frac{1}{q(q^2-1)}, \quad v(X_2) = \frac{1}{q(q^2-1)}, \quad v(Y_2) = \frac{1}{q^2(q^2-1)}.
\]

We choose an element \( \kappa \) such that \( \kappa^3(q^2-1) = \pi \). We set \( \gamma := \kappa^q(q-1)(q^2-1) \). Then, we have \( v(\kappa) = 1/\kappa^3(q^2-1) \) and \( v(\gamma) = (q-1)/q^2 \). Furthermore, we set \( \gamma_{(q-1)u} := \kappa^{q-1} \). We change variables as follows \( u = \kappa^q(q-1)u_0, \quad X_1 = \kappa^q x_1, \quad Y_1 = \kappa^q y_1, \quad X_2 = \kappa^q x \) and \( Y_2 = \kappa y \). From now on, we assume \( e_{F,Q_p} \geq 2 \), then we acquire the following congruences by \( [\pi]_u(X_1) = [\pi]_u(Y_1) = 0 \) and Lemma 2.3

\[
u_0 \equiv -\frac{1}{x_1^{q-1}} = -y_1^{q(q-1)} - \frac{\gamma^q}{y_1^{q-1}} \pmod{1+}. \tag{3.38}
\]

Therefore, we acquire the following congruence

\[
x_1y_1^q \equiv \zeta \left( 1 + \frac{\gamma^q}{y_1^{q-1}} \right) \pmod{1+}. \tag{3.39}
\]

with some \( \zeta \in \mu_{q-1}(\mathcal{O}_F) \). On the other hand, by \( Y_1 = [\pi]_u(Y_2), \quad X_1 = [\pi]_u(X_2) \) and Lemma 2.3, we acquire the followings

\[
y_1 \equiv y^q + \gamma u_0 y^q \pmod{(1/q)+}, \quad x_1 \equiv x^q + u_0 x^q + \gamma^q x \pmod{1+}. \tag{3.40}
\]

By substituting (3.40) to the right hand side of (3.38), we acquire the following

\[
u_0 \equiv -y^q(q-1) - \gamma \left( y^q(q^2-2q+1) + \frac{1}{y^q(q-1)} \right) - \gamma^{q+1} y^q - 2q^2q^2 + q \pmod{1+}. \tag{3.41}
\]

We set

\[
Z := (x^q y^q - xy^q) - \gamma \left( x^q y^q - q^2 + \zeta/y^q(q^2-1) \right).
\]

Substituting (3.40) and (3.41) to the term \( x_1y_1^q \) in the left hand side of the congruence (3.39), we acquire the following by \( e_{F,Q_p} \geq 2 \)

\[
Z^q - \gamma^q Z \equiv \zeta \pmod{1+}. \tag{3.42}
\]
In this subsection, we calculate the reduction \( Z \) defined by the following congruence on, we assume \( e \in \mathbb{F}_q \). Therefore, we obtain \( \tilde{c} \in \mathbb{F}_q^* \). Hence, \( \Psi_{3,1,0} \) has \( q(q-1) \) connected components, which are parametrized by \( (\zeta, \tilde{c}) \in \mathbb{F}_q^* \times \mathbb{F}_q^* \). Furthermore, each component is defined by \( x^q y^2 - xy^q = \tilde{\zeta} \), because we have \( x^q y^2 - xy^q \equiv Z \equiv \gamma_0 \equiv \zeta \pmod{0+} \) by (3.43).

By setting as follows \( z := \frac{xy^2 - (1+y^2y^{-1})}{1+y^2y^{-1}} \), we acquire \( z^q y - z y^q = \tilde{\zeta} \). Hence, we have proved the following lemma.

**Lemma 3.7.** We assume \( e_{F/Q_p} \geq 2 \). Then, the reduction \( \Psi_{3,1,0} \) has \( q(q-1) \) connected components and each component is defined by \( x^q y - xy^q = \tilde{\zeta} \) with some \( \tilde{\zeta} \in \mathbb{F}_q^* \).

### 3.7 Computation of the reduction of the space \( Y_{3,1,c} \)

In this subsection, we calculate the reduction \( \Psi_{3,1,c} \). By a technical reason, only in this subsection, we assume that the absolute ramification index \( e_{F/Q_p} \geq 3 \). Under this assumption, we show that the reduction \( \Psi_{3,1,c} \) has \( q(q-1)^2 \) connected components, and each component is defined by \( x^q y - xy^q = \zeta \) with some \( \zeta \in \mathbb{F}_q^* \). However, we believe that the same phenomenon happens when \( e_{F/Q_p} \leq 2 \).

Recall that we have on the space \( Y_{3,1,c} \) in Definition [247]

\[
\begin{align*}
\nu(u) &= \frac{1}{q+1}, \quad \nu(X_1) = \nu(Y_1) = \frac{1}{q(q^2 - 1)}, \quad \nu(X_2) = \nu(Y_2) = \frac{1}{q^3(q^2 - 1)}.
\end{align*}
\]

Let \( \kappa, \gamma \) and \( \gamma \equiv \frac{1}{\gamma} \pmod{0+} \) be as in the previous subsection. We change variables as follows \( u = \kappa^q(u_0), \quad X_1 = \kappa^q x_1, \quad Y_1 = \kappa^q y_1, \quad X_2 = \kappa x, \quad Y_2 = \kappa y \). We set \( k = (q+1)/q \). From now on, we assume \( e_{F/Q_p} \geq 3 \). Then, we acquire the following, by \( \nu_u(X_1) = \nu_u(Y_1) = 0 \) and Lemma [2.4.3],

\[
(1 - \pi^{q-1})u_0 \equiv -y_1^{q(q-1)} - \frac{\gamma^q}{x_1^{q(q-1)}} \equiv -y_1^{q(q-1)} + \frac{y^q}{y_1^{q(q-1)}} \pmod{2+}.
\]

Note that we have \( \pi^{q-1}u_0 \equiv 0 \pmod{2+} \) on the left hand side of (3.44) unless \( q = 3 \). By considering \( (3.44) \times (x_1 y_1)^q \), we obtain \( \nu(x_1^q y_1 - x_1 y_1^q) = 1/2q \). Furthermore, we acquire the following congruence

\[
x_1^q y_1 - x_1 y_1^q \equiv \zeta \gamma \pmod{0+} \quad (\text{mod } k+)(3.45)
\]

with some \( \zeta \in \mu_{q-1}(\mathcal{O}_F) \). On the other hand, we have the following congruences, by \( X_1 = [\pi]_u(X_2) \) and \( Y_1 = [\pi]_u(Y_2) \) and Lemma [2.4.3],

\[
x_1 \equiv x^q + \gamma u_0 x^q + \gamma \frac{x^{q+q+1}}{q} x, \quad y_1 \equiv y^q + \gamma u_0 y^q + \gamma \frac{y^{q+q+1}}{q} y \pmod{k+}.
\]

Substituting (3.46) to the term \( x_1^q y_1 - x_1 y_1^q \) in the left hand side of the congruence (3.45), we acquire the following congruence, by the assumption \( e_{F/Q_p} \geq 3 \),

\[
x_1^q y_1 - x_1 y_1^q \equiv (x^q y - xy^q)^q + \gamma u_0 (x^q y - xy^q)^q + \gamma^{q+1} u_0^{q+1} (x^q y - xy^q)^q + \gamma \frac{x^{q+q+1}}{q} (x^q y - xy^q)^q \pmod{k+}.
\]
(mod k+). By (3.44) and (3.40), we obtain the following congruence
\[ u_0 \equiv -x^q(q-1) + \gamma q u_0^q y^q(q-1)^2 - \frac{\gamma q}{xy^q(q-1)} - \frac{\gamma q+1 u_0}{xy^q(q-1)} \] (3.48)
\[ \equiv -y^q(q-1) + \gamma q u_0^q y^q(q-1)^2 - \frac{\gamma q}{y^q(q-1)} - \frac{\gamma q+1 u_0}{y^q(q-1)} \pmod{k} \] (mod 3.49)

Hence, we acquire the following congruence on the right hand side of the congruence (3.47)
\[ \gamma u_0(x^q y - xy^q)^q + \gamma q+1 u_0^q (x^q y - xy^q)^q \equiv -\gamma(x^q y - xy^q)^q - \gamma q+1(x^q y - xy^q)^q \] (3.50)

modulo k+. We set as follows
\[ Z := (x^q y - xy^q)^q - \gamma^{1/q}(x^q y - xy^q) \] (3.51)

Then, by (3.45), (3.47) and (3.50), the following congruence holds
\[ \zeta^{1/q} \equiv x_1^q y_1 - x_1^q y_1 \equiv Z^q - y^{q+1}Z \pmod{k,+} \] (3.52)

We choose an element \( \gamma_0 \) such that \( \gamma_0^q - \gamma q+1 \gamma_0 = \zeta^{1/q} \). Then, we have \( v(\gamma_0) = 1/q^2 \). If we set
\[ Z = \gamma_0 + \gamma^{1/q} c, \] (3.53)
by (3.52), \( e^q \equiv c \pmod{0+} \). Therefore, we have \( \tilde{c} \in \mathbb{F}_q \). Set \( Z_1 := x^q y - xy^q \). Note that we have \( v(Z_1) = 1/q^2 \) by (3.49). Then, the congruence (3.53) induces the following congruence
\[ Z_1^q - \gamma^{1/q} y^{q(q-1)} Z_1 = \zeta^{1/(q-1)} (1/q^2)+. \] (3.54)

We set as follows
\[ Z_1 = x^q y - xy^q = \gamma^{1/(q-1)} z. \] (3.55)

By substituting (3.55) to (3.54) and dividing it by \( \gamma^{1/(q-1)} \), we acquire the following
\[ z^q - y^{q(q-1)} z = \zeta \pmod{0+}. \] (3.56)

If we set \( z := \tilde{c}(1 + y^{q-1}) + y^{q^2+q-1} w_1 \), the curve (3.56) is isomorphic to a curve defined by the following \( g w_1^q - y^q w_1 = \tilde{c} \), because the function \( y \pmod{0+} \) is an invertible function. Furthermore, we have \( (x/y) \in \mathbb{F}_q^{\times} \), because we have \( Z_1 = x^q y - xy^q = 0 \) modulo 0+ by (3.55). Therefore, the reduction \( Y_{3,1,c} \) has \( q(q-1)^2 \) connected components by (3.53) and each component is defined by \( x^q y - xy^q = \tilde{c} \) with some \( \tilde{c} \in \mathbb{F}_q^{\times} \). Hence, we have proved the following lemma.

**Lemma 3.8.** We assume \( e_{F/Q_p} \geq 3 \). Then, the reduction of the space \( Y_{3,1,c} \) has \( q(q-1)^2 \) connected components and each component is defined by \( x^q y - xy^q = \zeta \) with some \( \zeta \in \mathbb{F}_q^{\times} \).

### 4 Action of the division algebra \( \mathcal{O}_D^\times \) on the components in the stable reduction of \( \mathcal{X}(\pi^2) \)

Recall that we set \( S_0 = (\mathcal{O}_F/\pi^2)^{\times} \times \mathbb{P}^1(\mathbb{F}_q) \in (i,j) \) in (3.4). Moreover, we set \( S_1 := S_0 \times \mu_{2(q^2-1)} \supseteq (i,j,k) \). In this section, we determine the right action of the division algebra \( \mathcal{O}_D^\times \) on the components \( Y_{2,2}, X_j^s \) for \((i,j) \in S, \mathbb{Z}_{i,k} (s = e_1, c), W^j_{i,k} \) for \((i,j,k) \in S_1 \), which appear in the stable reduction of \( \mathcal{X}(\pi^2) \). To compute the \( \mathcal{O}_D^\times \)-action, we use the description of the
action of $\mathcal{O}_D^\times$ on the Lubin-Tate space given in (2.10). Then, the group $\mathcal{O}_D^\times$ acts on the stable reduction of $\mathcal{X}(\pi^2)$ by factoring through $\mathcal{O}_3^\times$.

First, we prepare some notations. The reduced norm $\text{Nrd}_{D/F} : \mathcal{O}_D^\times \to \mathcal{O}_F^\times$ induces $\text{Nrd}_{D/F} : \mathcal{O}_3^\times \to (\mathcal{O}_F/\pi^2)^\times$. For $i \in (\mathcal{O}_F/\pi^2)^\times$ and $b \in \mathcal{O}_3^\times$, we set $ib := \text{Nrd}_{D/F}(b) \times i \in (\mathcal{O}_F/\pi^2)^\times$. It is well-known that, under some identification $\pi_0(\mathcal{X}(\pi^2)) \simeq (\mathcal{O}_F/\pi^2)^\times$, the group $\mathcal{O}_3^\times \ni b$ acts on $\pi_0(\mathcal{X}(\pi^2))$ by $i \mapsto ib$. See Theorem 2.4. For an element $b \in \mathcal{O}_3^\times$, we write $b = a_0 + \varphi b_0 + \pi a_1$ with $a_0 \in \mu_{q-1}(OE)$ and $b_0, a_1 \in \mu_{q^2-1}(OE) \cup \{0\}$.

4.1 The action of $\mathcal{O}_D^\times$ on the reduction $\mathbf{Y}_{2,2}$ and $\{X_j^i\}_{(i,j) \in S}$

In this subsection, first, we write down the action of $\mathcal{O}_D^\times$ on each connected component $\mathbf{Y}_{2,2}$ with $i \in (\mathcal{O}_F/\pi^2)^\times$ in Proposition 4.4. Secondly, we determine the action of $\mathcal{O}_D^\times$ on each connected component $\mathbf{Y}_{2,2}$ with $i \in (\mathcal{O}_F/\pi^2)^\times$ in Proposition 4.4. Thirdly, we calculate the $\mathcal{O}_3^\times$-action on the components $\{X_j^i\}_{(i,j) \in S}$ with each defined by $X^q + X = Y^{q+1}$ in Proposition 4.4. In the following computations, we freely use the notations in subsections 3.1 and 3.2.

Recall that we have the following on the space $\mathbf{Y}_{2,2}$

$$v(u) \geq \frac{q}{q + 1}, \quad v(X_1) = v(Y_1) = \frac{1}{q^2 - 1}, \quad v(X_2) = v(Y_2) = \frac{1}{q^2(q^2 - 1)}.$$  

We choose an element $\kappa_1$ such that $\kappa_1^{q(q^2 - 1)} = \pi$ with $v(\kappa_1) = 1/q^3(q^2 - 1)$. We set $\kappa := \kappa_1^q$ and $\gamma := \kappa(q-1)(q^2-1)$ as in subsection 3.2. We write $\gamma^{q(q^2 - 1)}$ for $\kappa_1^{-1}$. In subsection 3.2, we change variables as follows $u = \kappa^q (q-1)u_0$, $X_1 = \kappa^q x_1$, $Y_1 = \kappa^q y_1$, $X_2 = \kappa x$ and $Y_2 = \kappa y$.

Recall that the reduction $\mathbf{Y}_{2,2}$ has $q(q - 1)$ connected components. Each component is defined by

$$Z_1^q = x^3 y - xy^3, \quad x^q y - xy^q = \zeta$$

with some $\zeta \in \mathbb{F}_q^\times$. See subsection 3.2 for more details. In the following, we determine the action of $\mathcal{O}_D^\times$ on the parameters $(x, y, Z_1)$.

Let $b = a_0 + \varphi b_0 + \pi a_1 \in \mathcal{O}_3^\times$. Recall the definition of $b^*$ in (2.10). Then, we have the following by (2.10) and $j^{-1}(X) \equiv X \pmod{\pi, u}$

$$b^*(x) \equiv x - (b_0/a_0)^q x^2 \gamma^{1/(q^2 - 1)} - ((a_1/a_0) - (b_0/a_0)^q + 1) x^q \gamma^{1/(q^2 - 2)} - a_0 (\text{mod } 1/q^2 + ). \quad (4.1)$$

The same formula as (4.1) holds for $y$. Recall that we have $x_1^q y_1 - x_1 y_1^q \in \mu_{q-1}(OE)$ modulo $1^+$ as proved in (3.2). We choose an element $\zeta \in \mu_{q-1}(OE)$. In the following, we assume that $x_1^2 y_1 - x_1 y_1^2 = \zeta$ modulo $1^+$. We calculate $b^*(x_1)^q b^*(y_1) - b^*(x_1) b^*(y_1)^q$ in the following. We acquire the following congruence, by $b^*(x_1) \equiv x_1/a_0$ and $b^*(y_1) \equiv y_1/a_0$ modulo $0^+$,

$$b^*(x_1)^q b^*(y_1) - b^*(x_1) b^*(y_1)^q \equiv \frac{x_1^3 y_1 - x_1 y_1^3}{a_0^q + 1} \equiv \frac{\zeta}{a_0^{q^2 - 1}} (\text{mod } 0^+). \quad (4.2)$$

Recall that we set in (3.3)

$$Z := (x^q y - xy^q)^q - \gamma(x^q y - xy^q). \quad (4.3)$$
We acquire the following congruences by using (4.1). We fix an element \( \gamma_0 \) such that \( \gamma_0 = 0 \pmod{a + f} \) and \( \mu := c + f \). Then, we obtain \( \mu^q \equiv \mu \pmod{a + f} \). Now, we have chosen the following identification in (3.7)

\[
\pi_0(Y_{2, 2}) \simeq F_q \times \mathbb{F}_q \ni (\zeta, \bar{\mu}).
\] (4.4)

Then, we choose a pair \((\bar{\zeta}, \bar{\mu}) \in (\mathcal{O}_F/\pi^2)^\times\). We put

\[
\b(Z) := (\b(x)\b(y) - \b(x)^2)^q - \gamma(\b(x)^3 \b(y) - \b(x)^2 \b(y)^3).
\] (4.5)

We acquire the following congruences by using (4.1)

\[
\b(x)^q \b(y) - \b(x)^2 \b(y)^q \equiv \left( x^q - y^q \right)^q - \left( b_0/a_0 \right)^q \gamma q - \gamma(\b(x)^3 \b(y) - \b(x)^2 \b(y)^3)
\] (4.6)

\[
\b(x)^q \b(y) - \b(x)^2 \b(y)^q \equiv x^q - y^q - \left( b_0/a_0 \right)^q \gamma q - \gamma(\b(x)^3 \b(y) - \b(x)^2 \b(y)^3)
\] (4.7)

modulo \((1/q^2)^+\). Hence, we obtain the following by (4.3), (4.4), (4.6) and (4.7)

\[
\b(Z) = \left( \b(x)^q \b(y) - \b(x)^2 \b(y)^q \right) - \gamma(\b(x)^3 \b(y) - \b(x)^2 \b(y)^3)
\] (4.8)

\[
\equiv \frac{Z}{a_0^{q+1}} + \gamma q \left( \frac{a_0^q}{a_0^{q+1}} \right)^q (x^q - y^q)^2 \pmod{(1/q^2)^+}.
\]

We set \( f_0 := f((\b(x)^q \b(y))/(\b(x)^q \b(y)^q)). \) Then, we obtain the following \( \pi f_0 \equiv \pi(\frac{f}{a_0^q}) \pmod{1+}. \) Therefore, we acquire \( (\b(Z))^q - \gamma q(b(Z)^2) = \zeta/a_0^{q+1} + \pi(f_0^q) \pmod{1+}. \) We set \( h := -(a_1/a_0)^q - (a_1/a_0) \pmod{1+}. \)

Note that we have \( h \in \mathbb{F}_q. \) If we write \( Z = \gamma_0 - \gamma q^{-1} c \), we acquire the following by (4.8)

\[
\b(Z) = \frac{\gamma_0}{a_0^{q+1}} + \frac{\gamma q^{-1} c}{a_0^{q+1}} \pmod{(1/q^2)^+}.
\] (4.9)

We write \( \b(Z) = (\gamma_0/a_0^{q+1}) - \gamma q^{-1} \b c \) with \( \b c = (c + h \zeta)/a_0^{q+1}. \) Hence, we have proved the following lemma.

**Lemma 4.1.** Let \( b = a_0 + \varphi b_0 + \pi a_1 \in \mathcal{O}_E^\times \) with \( a_0 \in \mu_{q^2-1}(\mathcal{O}_E) \) and \( b_0, a_1 \in \mu_{q^2-1}(\mathcal{O}_E) \cap \{0\}. \) We set \( h := -(a_1/a_0)^q - (a_1/a_0) \pmod{1+}. \) We consider the identification (4.7). Then, the element \( b \) acts on the group \( \pi_0(Y_{2, 2}) \) as follows

\[
b : \pi_0(Y_{2, 2}) \rightarrow \pi_0(Y_{2, 2}) ; i := (\zeta, \bar{\mu}) \mapsto ib := (\zeta a_0^{q+1}, \bar{\mu} + h).
\]

**Proof.** The required assertion follows from (4.9). \( \square \)

**Remark 4.2.** The group \( \mathcal{O}_E^\times \) acts on the set of connected components of the Lubin-Tate space according to the inverse of the reduced norm. See subsection 2.5 for more details. Let \( \sigma \neq 1 \in \text{Gal}(E/F). \) Then, we have \( \text{Nrd}_{D/F}(a + \varphi \beta) = a_0 \sigma^{-1} - \sigma \beta \sigma. \) Let \( b = a_0 + \varphi b_0 + \pi a_1 \in \mathcal{O}_E^\times \) with \( a_0 \in \mu_{q^2-1}(\mathcal{O}_E) \) and \( b_0, a_1 \in \mu_{q^2-1}(\mathcal{O}_E) \cap \{0\}. \) Then, we have \( \text{Nrd}_{D/F}(b) = \frac{1}{a_0^{q+1}} (1 + \pi h) \in \left( \mathcal{O}_F/\pi^2 \mathcal{O}_F \right)^\times. \) Hence, as observed in Lemma 4.1, \( \mathcal{O}_D^\times \) acts on \( \pi_0(Y_{2, 2}) \) according to the inverse of the reduced norm.
Secondly, we write down the action of $O_D^\gamma$ on the reduction $\mathbf{Y}_{2,2}$. We write $\{\mathbf{Y}_{2,2}\}_{i=(\zeta, \mu) \in \mathbb{P}^\times_\mathbb{F} \times \mathbb{P}^\times_\mathbb{F}}$ for the connected components of $\mathbf{Y}_{2,2}$. Now, we compute the action of $b$ on the component $\mathbf{Y}_{2,2}^b$ with $i = (\tilde{\zeta}, \tilde{\mu}) \in (O_F/\pi^2)^\times$.

Let $\gamma_0, \gamma'_0$ be elements such that $\gamma_0^q = \gamma_0$ and $\gamma'_0 = \gamma_0^{a_0^{-(q+1)}}$. Recall that we introduce new parameters $Z_1$ and $b^*Z_1$ as in (4.8),

\[ x^q y - xy^q = \tilde{\gamma}_0 + \gamma^1/q Z_1, \quad b^*(x)^q b^*(y) - b^*(x)b^*(y)^q = \gamma'_0 + \gamma^1/q(b^*Z_1). \quad (4.10) \]

As computed in subsection 3.1 the components $\mathbf{Y}_{2,2}^b$ and $\mathbf{Y}_{2,2}^{b'}$ are defined by the following equations respectively

\[ x^q y - xy^q = \tilde{\gamma}, \quad Z_1^q = x^q y - xy^q, \]
\[ b^*(x)^q b^*(y) - b^*(x)b^*(y)^q = \tilde{\gamma}a_0^{-(q+1)}, \quad (b^*Z_1)^q = b^*(x)^q b^*(y)^q - b^*(x)b^*(y)^q. \quad (4.11) \]

**Proposition 4.3.** Let $b \in O_D^\gamma$. We write $\tilde{b}$ for the image of $b$ by $O_D^\gamma \to \mathbb{P}^\times_\mathbb{F}$. We choose an element $i \in (O_F/\pi^2)^\times$. See (4.11) for the defining equations of $\mathbf{Y}_{2,2}^b$ and $\mathbf{Y}_{2,2}^{\tilde{b}}$. Then, $b$ induces the following morphism

\[ b : \mathbf{Y}_{2,2} \to \mathbf{Y}_{2,2}^b ; (x, y, Z_1) \mapsto (\tilde{b}^{-1}x, \tilde{b}^{-1}y, \tilde{b}^{-1}(q+1)Z_1). \]

**Proof.** By (4.11) and (4.10), we acquire $b^*Z_1 \equiv Z_1/\tilde{b}^{q+1}$ modulo 0+. Hence, the required assertion follows.

Thirdly, we write down the action of $O_D^\gamma$ on the irreducible components $\{X^i_j\}_{(i=(\zeta, \mu), j) \in S}$.

Let $y_0$ be an element such that $y_0^q - y_0 + 1 = 0$. We choose an element $\gamma_1$ such that $\gamma_0^q + \gamma_0 - q = 0$. Set $\gamma_1 := \gamma_1/\gamma_0^{q+1}$. Let $x_0$ be an element such that $x_0^q y_0 - x_0 y_0^q = \gamma_0 + \gamma^1/q\gamma_1$. We choose elements $w, w_1$ such that $w = \gamma^1/q(a-1)$ and $w_1 = y_0^{q+1}/(q-2)$. Then, we have $v(w) = 1/q^3$ and $v(w_1) = 1/q^2(q+1)$. We set $b^*(w_1) = w_1/a_0^{q+1}$. Furthermore, for $j := (\tilde{x}_0, \tilde{y}_0) \in S^\times_0$, we set $j^b := (\tilde{a}_0^{-1}\tilde{x}_0, \tilde{a}_0^{-1}\tilde{y}_0) \in S^\times_{0,0}$. Now, we determine a morphism $X^i_j \to X^i_{j^b}$ which is induced by $b$.

We change variables as follows as in (3.11)

- $Z_1 = \tilde{\gamma}_1 + wa$, $y = y_0 + w_1 z_1$
- $b^*Z_1 = \gamma_1' + w(b^*a)$, $b^*y = x_0 + w_1(b^*z_1)$.

See subsection 3.2 for the definition of $c_0$. We set $c_0' := (c_0 + h\zeta)a_0^{-(q+1)}$. Then, we acquire the following by (3.13)

\[ a^q + a \equiv \zeta z_1^{q+1} - c_0, \quad (b^*a)^q + b^*a \equiv a_0^{-(q+1)}\zeta(b^*z_1)\zeta^{q+1} - c_0' \pmod{0+}. \quad (4.12) \]

These equations define the components $X^i_j$ and $X^i_{j^b}$ respectively. On the term $x^q y - xy^q$ in the right hand side of the congruence (4.12), we acquire the following congruence

\[ x^q y - xy^q \equiv \zeta \left( \frac{y_0^{q-1} + 1}{y_0^{q-1}} \right) \equiv \frac{\zeta}{y_0^{q-1}} w_1 z_1 \pmod{(1/q^2(q+1))}. \quad (4.13) \]
Therefore, we acquire the following by (4.10)
\[
b^*(x)^y b^*(y) - b^*(x) b^*(y)^y = b^*(x)^y b^*(y) - b^*(x) b^*(y)^y = \gamma_1^q + \gamma_1^q w a - \gamma_2^q (b_0/a_0)(w_1 z_1/y_0^q) + \gamma_2^q (a_1/a_0) z_1
\]

(4.14)
modulo \(1/q^2 + \). On the other hand, we have the following by (4.10)
\[
b^*(x)^y b^*(y) - b^*(x) b^*(y)^y = \gamma_1^q + \gamma_1^q b^* Z_1 = \gamma_1^q + \gamma_1^q w (b^* a).
\]

(4.15)
Hence, by (4.13) and (4.15), we acquire the following congruence
\[
b^*a \equiv a - (b_0/a_0) z_1 + (a_1/a_0) z_1 \pmod{a_0^{q+1}}.
\]

(4.16)

On the other hand, by considering the definitions of \(z_1\) and \(b^* z_1\), we obtain the following congruence by (4.11)
\[
b^* z_1 \equiv a_0^{-q^2} (z_1 - (b_0/a_0)^q) \pmod{a_0^{q+1}}.
\]

(4.17)

In the following proposition, we describe the action of \(b\) on the irreducible components \(\{X^i_j\}_{(i,j) \in S}\) in subsection 3.3.

**Proposition 4.4.** Let \(b = a_0 + \varphi b_0 + \pi a_1 \in \mathcal{O}^\times_D\). We choose elements \(i = (\zeta, \mu) \in (\mathcal{O}_F/\pi^2)^\times\) and \(j = (x_0, y_0) \in S_0\). We set \(j^b := (x_0 a_0^{-1}, y_0 a_0^{-1}) \in S_0^b\). See (4.12) for the defining equations of \(X^i\) and \(X_{j}^{b^*}\). Then, the element \(b\) induces the following morphism

\[
b : X^i_j \to X_{j^b}^{b^*} ; (a, z_1) \mapsto \left(a_0^{-(q+1)} (a - (b_0/a_0)^q a_0^{-1} (z_1 - (b_0/a_0)^q))\right).
\]

**Proof.** The required assertion follows from (4.10) and (4.17).

\[
\]

**4.2 The action of \(\mathcal{O}^\times_D\) on the reduction \(\mathcal{Z}_{1,1,*}\) and \(\{W^i_{*,j}\}_{(i,j) \in S_0^*} (\ast = 0, \infty)\)**

In this subsection, we compute the action of \(\mathcal{O}^\times_D\) on the reduction \(\mathcal{Z}_{1,1,*}\) \((\ast = 0, \infty)\) and \(\{W^i_{*,j}\}_{(i,j) \in S_0^*}\). See Lemma 4.3 and Proposition 4.4 for precise statements. In the following, we use freely the notations in subsections 3.3 and 3.4. But, we briefly recall them used in this subsection. Recall that we have the followings on the space \(Z_{1,1,0}\)

\[
v(u) = \frac{1}{2}, \quad v(X_1) = \frac{1}{2(q-1)}, \quad v(X_2) = \frac{1}{2q^2(q-1)}, \quad v(Y_1) = \frac{1}{2q(q-1)}, \quad v(Y_2) = \frac{1}{2q^2(q-1)}.
\]

We choose an element \(\kappa_1\) such that \(\kappa_1^{2q^2(q-1)} = \pi\). We set \(\kappa := \kappa_1^q\) and \(\gamma := \kappa^{q(q-1)^2}\). We write \(\gamma^{q(q-1)^2}\) for \(\kappa^{q^{-1}}\). Then, we have \(v(\kappa_1) = 1/2q^4(q-1)\) and \(v(\gamma) = (q-1)/2q^2\). We change variables as follows \(u = \kappa^{q(q-1)^2} u_0, \quad X_1 = \kappa^q x_1, \quad Y_1 = \kappa^q y_1\) and \(X_2 = \kappa^q x, \quad Y_2 = \kappa y\). We choose an element \(\zeta \in \mu_{q-1}(\mathcal{O}_F)\). We set \(r := (q-1)/2q^2\).

Let \(b = a_0 + \varphi b_0 + \pi a_1 \in \mathcal{O}^\times_D\). Recall the definition of \(b^*\) in (2.17). Then, we have the following congruences by (2.10) and \(j^{-1}(X) \equiv X \pmod{(\pi, u)}\)
\[
b^* \equiv x - (b_0/a_0)^q \gamma^{1/(q-1), x} + (-a_1/a_0) + (b_0/a_0)^{q+1} \gamma^{q^2} x^2 (\pmod{r^+}),
\]

(4.18)
\[ b^* (y) \equiv \frac{y - (b_0/a_0)^q \gamma \frac{y^{q+1}}{a_0} + (-a_1/a_0) + (b_0/a_0)^{q+1}) \gamma \frac{y^{q+1}}{a_0} y^q}{a_0} \pmod{(r/q^+}). \] (4.19)

Then, the congruences (4.18) induce the following congruence

\[ b^*(x)b^*(y)^q \equiv \frac{xy^q - ((b_0/a_0)^q(xy)^q + (b_0/a_0)xy^q) \gamma \frac{y^{q+1}}{a_0} + (b_0/a_0)^{q+1} \gamma \frac{y^{q+1}}{a_0} (xy^q)^q}{a_0^{q+1}} \] (4.20)

modulo \((1/q^2)\). Recall that we set as in (3.17)

\[ Z = (xy^q)^q - \gamma \left( \frac{xy^q}{y^{q(q^2-1)}} \right), \]

\[ b^* Z = (b^*(x)b^*(y)^q)^q - \gamma \left( b^*(x)b^*(y)^q + \frac{\zeta}{a_0^{q+1}b^*(y)q(q+q^2-1)} \right). \] (4.21)

By (4.18), we have the following congruences on the right hand side of the above equality (4.21)

\[ b^*(x)b^*(y)^q \equiv \frac{xy^q - (b_0/a_0)^q \gamma^{1/(q-1)} (xy^q)^q + (-a_1/a_0) + (b_0/a_0)^{q+1}) \gamma \frac{y^{q+1}}{a_0^{q+1}} (xy^q)^q}{a_0^{q+1}} \] (4.22)

and

\[ \frac{\zeta}{a_0^{q+1}b^*(y)^{q(q^2-1)}} \equiv \frac{\zeta y^q - (b_0/a_0)^q \gamma^{1/(q-1)} y^q + (-a_1/a_0)^q + (b_0/a_0)^{q+1} \gamma \frac{y^{q+1}}{a_0^{q+1}} y^q}{a_0^{q+1} y^q} \] (4.23)

modulo \(r^+\). By substituting the congruences (4.20), (4.22), and (4.23) to the right hand side of the equality (4.21) above, \(b^* Z\) is congruent to the following

\[ \frac{Z}{a_0^{q+1}} + \gamma^{\frac{q}{q-1}} \frac{b_0}{a_0^{q+2}} \left( \frac{\zeta - xy^q}{y^{q-1}} \right) + \gamma^{\frac{2q}{q-1}} \left( \frac{b_0}{a_0^{q}} \right)^{q+1} (xy^q)^q + (a_1/a_0)^q + (a_1/a_0)^{q+1} - 2(b_0/a_0)^{q+1} \zeta \] (4.24)

modulo \((1/q^+)^+\). Since we set \(xy^q = \gamma_0 + \gamma^{1/q} Z_1\) in (3.21), we have \((\zeta - xy^q)^q \equiv 0 \pmod{(1/2q^+)^+}\) by the choice of \(\gamma_0\) in (3.3). Note that we have \(\gamma_0 \equiv \zeta \pmod{0^+}\). Hence, (4.24) has the following form

\[ b^* Z \equiv \frac{Z}{a_0^{q+1}} + \gamma^{\frac{q}{q-1}} \left( \frac{a_1}{a_0} + (a_1/a_0)^q + (b_0/a_0)^{q+1} \right) \zeta \pmod{(1/q^+)^+}. \] (4.25)

Recall that we have set \(Z^q - \gamma^2 q Z = \zeta \pmod{1^+}\) in (3.18). On the other hand, we have \((b^* Z)^q - \gamma^2 b^* Z = \zeta / a_0^{q+1} \pmod{1^+}\). We choose an element \(\gamma_0\) such that \(\gamma_0^q - \gamma^2 \gamma_0 = \zeta\). Then, we set \(\gamma_0' := \gamma_0 / a_0^{q+1}\). Then, we set as follows \(Z := \gamma_0 - \gamma^{\frac{1}{q-1}} \mu\) and \(b^* Z := \gamma_0' - \gamma^{\frac{1}{q-1}} b^* \mu\). Hence, by (4.25), we acquire \(b^* Z = \frac{\gamma_0 - \gamma^{\frac{1}{q-1}} \mu}{a_0^{q+1}} - (a_1/a_0)^q + (b_0/a_0)^{q+1} \zeta \pmod{0^+}\). Therefore, \(O_0^\gamma\) acts on \(\pi_0(Z_{1,1,0})\) according to the inverse of the reduced norm, under the identification (3.19).

Now, fixing \(i = (\zeta, \bar{\mu}) \in (O_F/\pi^2)^\times = \pi_0(Z_{1,1,0})\), we will determine the morphism \(\mathbb{Z}_{1,1,0}^\gamma \to \mathbb{Z}_{1,1,0}^\gamma\), which is induced by \(b\) in Lemma 4.5.5 below.

Let \(\gamma_0, \bar{\gamma}_0\) be elements such that \(\gamma_0' = \gamma_0 - \gamma^{\frac{1}{q-1}} \mu\) and \(\gamma_0' = \gamma_0' - \gamma^{\frac{1}{q-1}} b^* \mu\). See subsection 3.3 for more details. By definitions of \(\gamma_0\) and \(\gamma_0'\) and (4.25), we obtain \(v(\gamma_0 - \gamma_0 a_0^{-(q+1)}) \geq 1/q^2\).

Recall that, in (3.21), we set as follows

\[ xy^q = \gamma_0 + \gamma^{1/q} Z_1, \quad b^*(x)b^*(y)^q = \gamma_0' + \gamma^{1/q} b^* Z_1. \] (4.26)
Furthermore, we put
\[ h(x, y) := \frac{b_0^2 a_0 (xy)^q + a_0^2 b_0 xy^{q^2}}{a_0^{q(q+1)}}. \]

Then, by (4.20) and (4.26), we acquire the following congruence
\[ b^*Z_1 \equiv \frac{Z_1}{a_0^{q+1}} - \gamma^{\frac{1}{q(q+1)}} h(x, y) \pmod{1/2q^3}. \] (4.27)

Recall that we have introduced new parameters \( Z_2 \) and \( b^*Z_2 \) as follows, as in (3.23).

- \( \gamma^1/y^q-1 Z_2 = Z_1 - \zeta(y^q-1 + y^{-1}(q^2-1)), \)
- \( \gamma^1/y^q(b^*y)q^2-1 b^*Z_2 = b^*Z_1 - \frac{\zeta}{a_0^{q+1}}((b^*y)^q-1 + (b^*y)^{-1}(q-1)). \)

By considering (4.27) and the definitions of \( Z_2 \) and \( b^*Z_2 \) above, we acquire the following congruence
\[ b^*Z_2 \equiv \frac{Z_2}{a_0^{q+1}} - \gamma^{1/y(q-1)} h_1(y) \pmod{1/2q^4}. \] (4.28)

We set as follows
\[ h_1(y) = \zeta \{(b_0^2/a_0^{q+1})y^{-q(q-1)} + (b_0/a_0^{q+2})y^q(q-1)\}. \] (4.29)

Since we have \( Z_1 \equiv \zeta(y^q-1 + y^{-1}(q^2-1)) \pmod{0+} \), the congruence (4.28) has the following form
\[ b^*Z_2 \equiv \frac{Z_2}{a_0^{q+1}} - \gamma^{1/y(q-1)} h_1(y) \pmod{1/2q^4}. \] (4.30)

As computed in subsection 3.3, the components \( Z_{1,1,0}^b \) and \( Z_{1,0}^{ib} \) are defined by the following equations respectively
\[ Z_2^b = \tilde{\zeta}(y^{q^2-1} + y^{-1}(q^2-1)), \quad (b^*Z_2)^q = \frac{\tilde{\zeta}}{a_0^{q+1}}((b^*y)^q-1 + (b^*y)^{-1}(q^2-1)). \]

Then, we have the following lemma.

**Lemma 4.5.** Let \( b \in \mathcal{O}_D^X \) and \( \tilde{b} \) the image of \( b \) by \( \mathcal{O}_D^X \rightarrow \mathbb{F}_q^X \). We choose an element \( i \in (\mathcal{O}_F/\pi^2) \). Let \( * = 0, \infty \). Then, \( b \) induces the following morphism
\[ b : Z_{1,1,0}^b \rightarrow Z_{1,1,0}^{ib} ; \quad (y, Z_2) \mapsto \left( \frac{y}{b}, \frac{Z_2}{b^{q+1}} \right). \]

**Proof.** The required assertion follows from (4.18) and (4.30) immediately. \( \square \)

In the following, we determine the action of \( \mathcal{O}_D^X \) on the components \( \{ W_{j,j'} \}_{(j',j) \in \mathcal{S}_0} \) where each \( W_{j,j'} \) is defined by the Artin-Schreier equation \( a^q = a + s^2 \).

Let \( \epsilon \in \{ \pm 1 \} \). Recall that we choose \( \tilde{\gamma}_1 \) such that \( \gamma_1 = \epsilon 2\zeta \{1 + \gamma^{1/q^2}(\tilde{\gamma}_1/\zeta)\}^{1/2} \). Similarly, we choose \( b^*\tilde{\gamma}_1 \) such that
\[ b^*\tilde{\gamma}_1 = \frac{2\zeta}{a_0^{q+1}} \{1 + \gamma^{1/q^2}(a_0^{q+1}b^*\tilde{\gamma}_1/\zeta)\}^{1/2}. \] (4.31)
Let \( y_0 \) and \( x_0 \) be elements such that \( y_0^{q^2-1} = 1/(1 + \gamma^{1/q^2} \xi_1 \zeta^{-1})^{1/2} \) and \( x_0 y_0^q = \xi_0 + \gamma^{1/q^2} \xi_1 \). We set \( w := y_0^{q+1} \chi^{(q-1)/q} \) and \( b^w := w/a_0^{q+1} \). Then, we have \( \xi_0 \in \mu_{2(q^2-1)} \). We set \( y_0 b := \xi_0/a_0 \in \mu_{2(q^2-1)} \). Now, we determine the morphism \( W_{y_0} \to W_{\bar{y}_0 b} \) which is induced by \( b \) in Proposition 4.6.

In (3.20), we have changed variables as follows
\[
Z_2 = \xi_1 + w a, \quad b^* Z_2 = b^* \xi_1 + b^* w (b^* a).
\] (4.32)
Then, by (4.30) and (4.32), we acquire the following congruence
\[
b^*(a) \equiv a - a_0^{q+1} \chi^{-1} (q+1) h_1(y_0) \pmod{0+}.
\] (4.33)
We choose an element \( w_1 \) such that \( y_0^{q^2-3} (\xi + \gamma^{1/q^2} \xi_1) w_1^2 = w^9 \). We set \( b^* y_0 := y_0/a_0 \) and \( b^* w_1 := w_1/a_0 \). As in (3.20), we change variables as follows
\[
y = y_0 + w_1 y_1, \quad b^*(y) = b^* y_0 + b^* w_1 b^*(y_1).
\]
Then, the congruence (4.31) induces the following congruence
\[
b^*(y_1) \equiv y_1 \pmod{0+}.
\] (4.34)
As proved in subsection 3.3, the components \( W_{0,j}^i \) and \( W_{0,jb}^i \) are defined by the following equations respectively
\[
a^q - a = y_1^2, \quad b^*(a)^q - b^*(a) = b^*(y_1)^2.
\] (4.35)
Then, we acquire the following proposition.

**Proposition 4.6.** Let \( b = a_0 + \varphi b_0 + \pi a_1 \in \mathcal{O}_D^\times \). We choose elements \( i = (\tilde{\xi}, \tilde{\mu}) \in \mathbb{F}_q^\times \times \mathbb{F}_q \) and \( \bar{y}_0 \in \mu_{2(q^2-1)} \). Let \( s = 0, \infty \). See (4.30) for the defining equations of \( W_{s,\bar{y}_0}^i \) and \( W_{s,\bar{y}_0 b}^i \). Then, the element \( b \) induces the following morphism
\[
b : W_{s,\bar{y}_0}^i \to W_{s,\bar{y}_0 b}^i ; (a, y_1) \mapsto \left(a - \Tr_{\mathbb{F}_q^2/\mathbb{F}_q} \left( \frac{\tilde{b}_0}{\tilde{a}_0} \gamma \right) \bar{\zeta}, y_1 \right).
\]

**Proof.** By (4.29), we have the following
\[
a_0^{q+1} \xi_0^{(q+1)} h_1(\hat{y}_0) \equiv \Tr_{\mathbb{F}_{q^2}/\mathbb{F}_q} \left( \tilde{b}_0 (\tilde{a}_0 y_0^{2q})^{-1} \right) \bar{\zeta} \pmod{0+}.
\]
Hence, the required assertion follows from (4.32) and (4.33).

4.3 Action of \( \mathcal{O}_D^\times \) on the components \( \mathcal{Z}_{1,1,c} \) and \( \{W_{k,j}^i\}_{(i,j) \in S_0, k \in \mathbb{F}_q^\times} \)

In this subsection, we compute the action of \( \mathcal{O}_D^\times \) on the reduction \( \mathcal{Z}_{1,1,c} \) and \( \{W_{k,j}^i\}_{(i,j) \in S_0, k \in \mathbb{F}_q^\times} \) explicitly. Then, we obtain Propositions 4.7 and 4.8 similar to Lemma 3.3 and Proposition 4.6. In the following computations, we freely use the notations in subsection 3.5.

First, recall that we have the following on the space \( \mathcal{Z}_{1,1,c} \)
\[
v(u) = \frac{1}{2}, \quad v(X_1) = v(Y_1) = \frac{1}{2q(q-1)}, \quad v(X_2) = v(Y_2) = \frac{1}{2q^2(q-1)}.
\]
We choose an element \( \kappa_1 \) such that \( \kappa_1^{2q(q-1)} = \pi \). We set \( \kappa := \kappa_1^q \) and \( \gamma := \kappa^{q(q-1)^2} \). We write \( \gamma^{q(q-1)^2} \) for an element \( \kappa_1^{q(q-1)} \). Then, we have \( v(\kappa) = 1/2q^3(q-1) \) and \( v(\gamma) = (q-1)/2q^2 \). We
Recall the definition of modulo $m$. Recall that we have $\mathbb{Z}$. Therefore, we acquire the following by (4.37) and $j^{-1}(X) \equiv X \ (\text{mod} \ (\pi, u))$

$$b^*(x) \equiv \frac{x - (b_0/a_0)^q x^q - ((a_1/a_0) - (b_0/a_0)^q 1) x^q + c(b)\kappa^{q-1}x^q}{a_0^{q+1}} \ (\text{mod} \ m+) \hspace{1cm} (4.36)$$

with some element $c(b) \in \mathcal{O}_E$. The same congruence holds for $y$. The congruence (4.36) induces the following congruence

$$b^*(x)^q b^*(y) - b^*(x) b^*(y)^q \equiv \frac{x^q y - x y^q - (b_0/a_0)\gamma^{1/(q-1)}(x^q y - x y^q) + (a_1/a_0)\gamma \gamma^{q+1} x^q y - x y^q) q}{a_0^{q+1}} \hspace{1cm} (4.37)$$

modulo $(2q + 1)/2q^3 +$. On the other hand, we acquire the following by (4.36)

$$b^*(x)^q b^*(y) - b^*(x) b^*(y)^q \equiv \frac{(x^q y - x y^q) - (b_0/a_0)\gamma^{1/(q-1)}(x^q y - x y^q) - (a_1/a_0)\gamma \gamma^{q+1} x^q y - x y^q) q}{a_0^{q+1}} \hspace{1cm} (4.38)$$

modulo $m +$. We set as follows

$$Z := (x^q y - x y^q) q - \gamma^{q+1} (x^q y - x y^q) \hspace{1cm} (4.39)$$

and

$$b^* Z := ((b^* x)^q b^* y - b^* x(b^* y)^q) q - \gamma^{q+1} ((b^* x)^q b^* y - b^* x(b^* y)^q). \hspace{1cm} (4.40)$$

Recall that we have $Z^q - \gamma^{2q+1} Z \equiv \gamma^{q+1} \zeta (\text{mod} \ (2q + 1)/2q^2)$. Hence, by (4.37) and (4.38), the following holds

$$b^* Z \equiv \frac{Z + ((a_1/a_0)\gamma + (a_1/a_0) - (b_0/a_0)\gamma^{q+1}) \gamma^{q+1} x^q y - x y^q) q}{a_0^{q+1}} (\text{mod} \ \frac{(2q + 1)}{2q^2}) \hspace{1cm} (4.41)$$

Therefore, we have $(b^* Z)^q - \gamma^{2q+1} b^* Z \equiv \gamma^{q+1} \zeta (\text{mod} \ (2q + 1)/2q^2)$. Then, we have $v(b^* Z) = v(Z) = 1/2q^2$.

We choose an element $\tilde{\gamma}_0$ such that $\tilde{\gamma}_0 - \gamma^{(2q+1)/q} \tilde{\gamma}_0 = \gamma^{1/(q-1)} \zeta$. We set $\tilde{\gamma}'_0 = \tilde{\gamma}_0/a_0^{q+1}$. We write $\tilde{\gamma}'_0 = Z$ and $\tilde{\gamma}'_0 = b^* Z$. We set

$$Z_1 = x^q y - x y^q, \ b^* Z_1 = b^* (x)^q b^* (y) - b^* (x) b^* (y)^q.$$ 

Note that we have $v(b^* Z_1) = v(Z_1) = 1/2q^3$ by (4.39) and (4.40), because of $v(Z) = v(b^* Z) = 1/2q^2$. We have the following

$$x^q y - x y^q \equiv \frac{Z_1^{q}}{y^{q+1}} + y^{q(q-1)} Z_1 \equiv y^{q(q-1)} Z_1 (\text{mod} \ (1/2q^3) +).$$

Therefore, we acquire the following by (4.37)

$$b^* Z_1 \equiv \frac{(1 - (b_0/a_0)\gamma^{1/(q-1)} y^{q(q-1)}) Z_1}{a_0^{q+1}} (\text{mod} \ ((q + 1)/2q^3) +). \hspace{1cm} (4.42)$$

change variables as follows $u = \kappa^{q(q-1)} u_0$, $X_1 = \kappa^{q} x_1$, $Y_1 = \kappa^{q} y_1$, $X_2 = \kappa x$, $Y_2 = \kappa y$. We set $m = (q^2 + q + 1)/2q^3$. As in the previous subsection, we write $b = a_0 + \varphi b_0 + \pi a_1 \in \mathcal{O}_E$. Recall the definition of $b^*$ in (2.17). Then, we acquire the following congruences by (2.10) and $j^{-1}(X) \equiv X \ (\text{mod} \ (\pi, u))$.
Furthermore, we introduce new parameters $Z$ and $b^* Z$ as follows

$$Z_1 = \tilde{\gamma}_0 + y^{q^2-1} \gamma \frac{a + b}{a_0 + 1} Z^{1/q} + y^{q^2-1} \gamma \frac{a + b}{a_0 + 1} Z$$

(4.43)

and

$$b^* Z_1 = \tilde{\gamma}_0' + (b^* y)^{q^2-1} \gamma \frac{a + b}{a_0 + 1} \gamma' \frac{a + b}{a_0 + 1} + (b^* y)^{q^2-1} \gamma \frac{a + b}{a_0 + 1} b^* Z.$$ 

(4.44)

Then, by (4.36), on the right hand side of (4.44), we acquire the following congruence

$$\gamma(q+1)/q \gamma_0^{1/q} (b^* y)^{q^2-1} \equiv \frac{\tilde{\gamma}_0(q+1)/q y^{q^2-1}}{a_0 + 1} + \gamma \frac{a + b}{a_0 + 1} \left( \frac{b_0}{a_0 + 1} \right) y^{q^2+q-2} \pmod{(q+1)/2q^3}. \quad (4.45)$$

Hence, by (4.42), we obtain

$$b^* Z \equiv \frac{Z}{a_0 + 1} - \gamma^{1/q} y^{q^2(q-1)} \left( \frac{y^{q^2-1} b_0 \gamma}{a_0 + 1} + \frac{b_0 \gamma}{a_0 + 1} \right) \pmod{(1/2q^4)}.$$ 

Since $Z_1 = x^q y - xy^q \equiv 0 \pmod{q+1}$ by (4.43), we acquire $x = \tilde{\gamma} y$ with $\tilde{\gamma} \in \mathbb{F}_q^\times$. Note that $\mathbf{Z}_{1,1,c}$ has $q(q-1)^2$ connected components. Let $\{ \mathbf{Z}^j_{1,1,j} \}_{(i,j) \in (\mathbb{F}_q^\times \times \mathbb{F}_q) \times \mathbb{F}_q}$ denote the connected components of $\mathbf{Z}_{1,1,c}$. The group $\mathcal{O}_{D}^\times$ acts on $j \in \mathbb{F}_q^\times$ trivially. The group $\mathcal{O}_{D}^\times$ acts on $i = (\zeta, \mu) \in \mathbb{F}_q^\times \times \mathbb{F}_q$ according to the inverse of the reduced norm. Assume that we have $x = \tilde{\gamma} y$ with $\tilde{\gamma} \in \mathbb{F}_q^\times$. As computed in subsection 3.5, the components $\mathbf{Z}^j_{1,1,j}$ and $\mathbf{Z}^{ib}_{1,1,j}$ are defined by the following equations

$$Z^j = \tilde{\gamma} y^{q^2-1} y^{-(q^2-1)} \left. \right|_{y^{q^2-1} + y^{-(q^2-1)}}$$

respectively. Therefore, we acquire the following proposition by (4.45).

**Proposition 4.7.** Let $b \in \mathcal{O}_{D}^\times$ and $\tilde{b}$ the image of $b$ by $\mathcal{O}_{D}^\times \rightarrow \mathbb{F}_q^\times$. We choose elements $i \in (\mathcal{O}_F/\pi^2)^\times$ and $j \in \mathbb{F}_q^\times$. Then, the element $b$ induces the following morphism

$$b: \mathbf{Z}^j_{1,1,j} \rightarrow \mathbf{Z}^{ib}_{1,1,j} : \left( y, Z \right) \mapsto \left( \frac{y}{b}, \frac{Z}{b^{q+1}} \right).$$

Let $\{W^i_{j,y} \}_{i \in \mathbb{F}_q^\times \times \mathbb{F}_q, j \in \mathbb{F}_q^\times, i \in \mathcal{O}_{D}^\times}$ be the irreducible components defined by the Artin-Schreier equation $a^q - a = s^2$. In the same way as Proposition 4.6, we acquire the following proposition by (4.36) and (4.45).

**Proposition 4.8.** Let $b = a_0 + \varphi b_0 + a_1 \pi \in \mathcal{O}_{D}^\times$. We choose elements $j \in \mathbb{F}_q^\times$, $i = (\zeta, \mu) \in \mathbb{F}_q^\times \times \mathbb{F}_q$ and $\tilde{y}_0 \in \mathcal{O}_{D}^\times$. We set $\tilde{y}_0 : = \tilde{y}_0 / a_0$. Then, the element $b$ induces the following morphism

$$b: W^i_{j,\tilde{y}_0} \rightarrow W^{ib}_{j,\tilde{y}_0} : \left( a, y_1 \right) \mapsto \left( a - \text{Tr}_{\mathbb{F}_q^\times/\mathbb{F}_q}(\frac{\tilde{b}_0}{a_0 y_0}) \tilde{\gamma}, y_1 \right).$$

### 5 The action of $G^F_{t}$ on the components in the stable reduction of $\mathcal{X}(\pi^2)$

In this section, we determine the right action of $G^F_{t}$ on the components $\mathbf{Y}_{2,2, j} \left( (i, j) \in \mathcal{S} \right)$, $\mathbf{Z}_{1,1,0}, W^i_{j,k}$ for $(i, j, k) \in \mathcal{S}_1$. First, we prepare some notations used through this section. For $i \in (\mathcal{O}_F/\pi^2)^\times$ and $g \in \mathcal{O}_F^\times$, we set $ig := \text{det}(g) \times i$. It is well-known that, under some identification $\pi_0(\mathcal{X}(\pi^2)) \simeq (\mathcal{O}_F/\pi^2)^\times$, the group $G^F_{t} \supseteq g$ acts on $\pi_0(\mathcal{X}(\pi^2))$ by $i \mapsto ig$. See Theorem 2.5. For an element $\alpha \in \mathcal{O}_F/\pi^2$, we write $\alpha = a_0 + a_1 \pi$ with $a_0 \in \mathcal{O}_{F,1}$, $a_1 \in \mathcal{O}_{F,1} \cup \{0\}$.
5.1 Action of $G^F_2$ on $\mathbf{Y}_{2,2}$ and $\{X^i_j\}_{(i,j)\in S}$

In this subsection, for each $i \in (\mathcal{O}_F/\pi^2)^\times$, we determine the action of $G^F_2$ on the components $\mathbf{Y}_{2,2}$ and $\{X^i_j\}_{j \in S_{i\alpha}}$ explicitly. See Lemma 5.2 and Proposition 6.3 for precise statements. In the following, we use freely the notations in subsections 3.1 and 9.2.

Let $g := \left( \begin{array}{cc} a_0 + a_1 \pi & b_0 + b_1 \pi \\ c_0 + c_1 \pi & d_0 + d_1 \pi \end{array} \right) \in G^F_2$. Let $(X_2, Y_2, u) \in \mathbf{Y}_{2,2}$ and $X_1 := [\pi]u(X_2), Y_1 := [\pi]u(Y_2)$. Then, the element $g$ acts on the Drinfeld basis $(X_2, Y_2)$ as follows

$$X_2 \mapsto [a_0]u(X_2) + u [c_0]u(Y_2) + u [a_1]u(X_1) + u [c_1]u(Y_1)$$

$$Y_2 \mapsto [b_0]u(X_2) + u [d_0]u(Y_2) + u [b_1]u(X_1) + u [d_1]u(Y_1).$$

We choose an element $\kappa_1$ such that $\kappa_1^q(q^2-1) = \pi$ with $v(\kappa_1) = 1/q^3(q^2 - 1)$. We set $\kappa := \kappa_1^q$ and $\gamma := \kappa^{q(q-1)/q^2}$ with $v(\gamma) = (q - 1)/q^2$. We write $\gamma_t$ for an element $\kappa_t$. In subsection 2.3, we change variables as follows $u = \kappa^q(q-1)u_0$, $X_1 = \kappa^q x_1$, $Y_1 = \kappa^q y_1$, $X_2 = \kappa x$ and $Y_2 = \kappa y$. For $a, b \in \mathcal{O}_F$, we set as follows

$$f_{a,b}(X,Y) := \frac{aX^q + bY^q - (aX + bY)^q}{\pi} \in \mathcal{O}_F[X,Y].$$

Then, we acquire the followings by (5.1)

$$g^*(x) \equiv a_0 x + c_0 y + \kappa^q x \{ (a_1 x + c_1 y)^q + f_{a_0,c_0}(x, y) \} \pmod{(1/q^2) +},$$

$$g^*(y) \equiv b_0 x + d_0 y + \kappa^q y \{ (b_1 x + d_1 y)^q + f_{b_0,d_0}(x, y) \} \pmod{(1/q^2) +}. \quad (5.1)$$

Note that if char $F = p > 0$, or char $F = 0$ and $e_{F/q_p} \geq 2$, the terms $f_{*,*}(x, y)$ in (5.1) do not appear. We set as follows

$$g(x, y) := \{ (a_0 x + c_0 y)(b_1 x + d_1 y)^q - (a_1 x + c_1 y)^q(b_0 x + d_0 y)^q \} \in \mathcal{O}_F[x, y]$$

and

$$h(x, y) := (a_0 x + c_0 y)^q(b_1 x + d_1 y)^q - (a_1 x + c_1 y)^q(b_0 x + d_0 y)^q \in \mathcal{O}_F[x, y].$$

Furthermore, we put

$$G_0(x, y) := (a_0 x + c_0 y)^q f_{b_0,d_0}(x, y) - (b_0 x + d_0 y)^q f_{a_0,c_0}(x, y) \in \mathcal{O}_F[x, y];$$

$$G_1(x, y) := (a_0 x + c_0 y)^q f_{b_0,d_0}(x, y) - (b_0 x + d_0 y)^q f_{a_0,c_0}(x, y), \quad g_1(x, y) := G_0(x, y)^q - G_1(x, y)$$

$$f_0(x, y) := \frac{x^q y^q - x^q y^q - (x^q y - y^q)^q}{\pi}, \quad \det(\bar{g}) := a_0d_0 - b_0c_0, \quad k_0 := \frac{\det(\bar{g}) - \det(\bar{g})^q}{p}.$$}

Then, we acquire the following congruence by (5.1)

$$g^*(x)^q g^*(y)^q - g^*(x) g^*(y)^q \equiv \det(\bar{g})(x^q y - y^q)^q + \kappa^{q(q-1)/q^2} \{ (g(x, y) + G_0(x, y) \} \pmod{(1/q^2) +}. \quad (5.2)$$

We choose an element $\zeta \in \mu_{q-1}(\mathcal{O}_F)$. Recall that we set in (3.1)

$$\mathcal{Z} := (x^q y - y^q)^q - \gamma(x^q y - y^q)^q.$$}

Then, we acquire the following $\mathcal{Z}^q - \gamma^q \mathcal{Z} = \zeta + l(f_0(x^q, y^q)^q - f_0(x^q, y^q)) \pmod{1 +}$. We put $\mathcal{Z} = \gamma_0 - \gamma^{q(q-1)c}$. Furthermore, we set $\mu := f_0(x^q, y^q) + c$. Then, we obtain $\mu^q \equiv \mu \pmod{0 +}$. Similarly as above, we set

$$g^* \mathcal{Z} := (g^*(x)^q g^*(y)^q - g^*(x) g^*(y)^q)^q - \gamma(g^*(x)^q g^*(y)^q - g^*(x) g^*(y)^q)^q$$
and \( g^* Z := \det(g)\gamma_0 - \gamma^{q/(q-1)} g^* c \). Moreover, we put \( g^* \mu := g^* c + f_0((a_0 x + c_0 y)^q, (b_0 x + d_0 y)^q) \). Then, we obtain \((g^* \mu)^q \equiv g^* \mu \pmod{0+} \). By (5.1) and (5.2), we acquire the following by a direct computation

\[
g^* Z = \det(g) Z + \gamma^{q/(q-1)} \{ g(x, y)^q - h(x, y) + g_1(x, y) \} \pmod{1/q+}. \tag{5.3}
\]

We set

\[
\gamma(g) := -a_1 d_0 - a_0 d_1 + b_1 c_0 + b_0 c_1.
\]

We can easily check that

\[
g(x, y)^q - h(x, y) \equiv \gamma(g)(x^q y - xy^q)^q \equiv \gamma(g) \zeta \pmod{0+} \tag{5.4}
\]

and

\[
g_1(x, y) = f_0((a_0 x + c_0 y)^q, (b_0 x + d_0 y)^q) - f_0(x^q, y^q) - f_0(x^q y - xy^q)^q (\pmod{0+}). \tag{5.5}
\]

Hence, we acquire by (5.3), (5.4) and (5.5)

\[
g^* \mu \equiv \det(g) \mu - \{ \gamma(g) - k_0 \} \zeta \pmod{0+}. \tag{5.6}
\]

**Lemma 5.1.** W choose the identification \( \pi_0(Y_{2,2}) \simeq (\mathcal{O}_F/\pi^2)^\times \) in (3.7). Then, the group \( G_2^F \) acts on the set \( \pi_0(Y_{2,2}) \) by \( \det : G_2^F \to (\mathcal{O}_F/\pi^2)^\times \).

**Proof.** The required assertion follows from (5.6) and the identification (3.7) immediately. \( \square \)

Now, fixing an element \( i = (\tilde{\zeta}, \tilde{\mu}) \in (\mathcal{O}_F/\pi^2)^\times \), we determine the morphism \( \Xi_{2,2}^i \to \Xi_{2,2}^g \) which is induced by \( g \) in Lemma 5.2.

We choose an element \( \gamma_0 \) such that \( \gamma_0^q - \gamma^q \gamma_0 = \zeta \). Furthermore, we choose an element \( \tilde{\gamma}_0 \) such that \( \tilde{\gamma}_0^q = \gamma_0 \). We recall that we set in (3.8)

\[
x^q y - xy^q = \gamma_0 + \gamma^{1/q} Z_1, 
\]

\[
g^* (x^q y - xy^q) = \gamma_0 + \gamma^{1/q} g^* Z_1.
\]

Then, by (5.2), we obtain the following congruence

\[
g^* Z_1 \equiv \det(g) Z_1 + \gamma^{1/q(q-1)} (g(x, y) + G_0(x, y)) \pmod{1/q^3+}. \tag{5.7}
\]

Recall that the components \( \Xi_{2,2}^i \) and \( \Xi_{2,2}^g \) are defined by the following equations respectively

- \( x^q y - xy^q = \tilde{\zeta}, \ Z_1^q = x^q y - xy^q, \)
- \( g^* (x^q y - xy^q) = \det(g) \tilde{\zeta}, \ (g^* Z_1)^q = g^* (x^q y - xy^q)^q. \)

Then, we have the following lemma.

**Lemma 5.2.** Let \( g = \left( \begin{array}{cc} a_0 + a_1 \pi & b_0 + b_1 \pi \\ c_0 + c_1 \pi & d_0 + d_1 \pi \end{array} \right) \in G_2^F \). We choose an element \( i \in (\mathcal{O}_F/\pi^2)^\times \).

Then, \( g \) induces the following morphism

\[
\Xi_{2,2}^i \to \Xi_{2,2}^g : (x, y, Z_1) \mapsto (a_0 x + c_0 y, b_0 x + d_0 y, \det(g) Z_1).
\]

**Proof.** The required assertion follows from (5.1) and (5.7) immediately. \( \square \)
We choose an element $\tilde{\gamma}_1$ such that $\tilde{\gamma}_1^q + \tilde{\gamma}_0 + \tilde{\gamma}_1^{1/q} = 0$. Let $y_0$ be an element such that $y_0^2 - 1 + \tilde{\gamma}_0^{\frac{1}{q-1}} = 0$. We choose an element $x_0$ such that $x_0^q y_0 - x_0 y_0^q = \tilde{\gamma}_0 + x_0^{1/q} \tilde{\gamma}_1$. Recall that we set $w := \gamma^{1/q(q-1)}$ and $w_1 := \gamma^{1/(q-1)}$. Then, we have $v(w) = 1/q^3$, $v(w_1) = 1/q(q + 1)$. We set $j := (\bar{a}_0, \bar{y}_0) \in S_{\bar{a}_0}$ and $j_2 := (\bar{a}_0 \bar{x}_0 + c_0 \bar{y}_0, b_0 \bar{x}_0 + d_0 \bar{y}_0) \in S_{\bar{a}_0}^{\bar{a}_0}$. Now, we determine the morphism $X^i_j \to X^i_{j_2}$, which is induced by $g$ in Proposition 5.3 below.

As in (3.11), we change variables $Z_1 = \tilde{\gamma}_1 + w_1$, $y = y_0 + w_1 z_1$. Then, we acquire the following defining equation of $X^i_j$ by (5.8)

$$a^q + a = \tilde{\gamma}_1^{q+1} - c_0$$

where $c_0 := -f(x_0^q, y_0^q)$.

Let $g^\ast(y_0) := b_0 x_0 + d_0 y_0$ and $g^\ast w_1 := g^\ast(y_0)^q w_1$. Furthermore, we set $g^\ast Z_1 = \det(\tilde{\gamma}_1 + w \gamma^\ast(a))$ and $g^\ast(y) := g^\ast(y_0) + g^\ast w_1 g^\ast(z_1)$. Then, similarly as (5.8), we acquire the following defining equation of $X^i_{j_2}$

$$g^\ast(a)^q + g^\ast(a) = \overline{\det(\tilde{\gamma}_1)^q(g^\ast z_1)^{q+1}} - \overline{g^\ast c_0}.$$  

Here, note that we have the following

$$\overline{g^\ast c_0} = \overline{\det(\tilde{\gamma}_1) c_0} + \overline{\det(\tilde{\gamma}_1) f_0(x_0^q, y_0^q)} - \overline{f_0((\bar{a}_0 \bar{x}_0 + c_0 \bar{y}_0)^q, (\bar{b}_0 \bar{x}_0 + d_0 \bar{y}_0)^q) - (\gamma(y) - \tilde{\bar{\gamma}}_0)^q}.$$

By (5.1) and (5.7), we acquire the following congruence

$$g^\ast(a) \equiv \det(\tilde{\gamma}_1) a + g(x_0, y_0) + G_0(x_0, y_0), \qquad g^\ast(z_1) \equiv z_1 \pmod{0^+}.$$  

Note that we have $g(x_0, y_0) \equiv -h(x_0, y_0)$ and hence $\gamma(y) \equiv g(x_0, y_0) + g(x_0, y_0)$ modulo $0^+$. We also have $G_1(x_0, y_0) \equiv -G_0(x_0, y_0)$ and hence $G_0^q(x_0, y_0) + G_0(x_0, y_0) = -b_0 \gamma + f_0((\bar{a}_0 \bar{x}_0 + c_0 \bar{y}_0)^q, (\bar{b}_0 \bar{x}_0 + d_0 \bar{y}_0)^q) - f_0(x_0^q, y_0^q)$ (mod $0^+$) by (5.3). Hence, by (5.10), we easily check that $(g^\ast(a), g^\ast(z_1))$ satisfies (5.10).

As a result of the above computations, we acquire the following proposition.

**Proposition 5.3.** Let $g := \begin{pmatrix} a_0 + a_1 \pi & b_0 + b_1 \pi \\ c_0 + c_1 \pi & d_0 + d_1 \pi \end{pmatrix} \in G_F^2$. Let $\bar{g}(x, y) := \{(\bar{a}_0 \bar{x} + c_0 \bar{y})(\bar{b}_1 \bar{x} + \bar{d}_1 \bar{y}) - (\bar{a}_1 \bar{x} + \bar{c}_1 \bar{y})(\bar{b}_0 \bar{x} + \bar{d}_0 \bar{y})\} \in \mathbb{F}_q[x, y]$. We define two polynomials with coefficients in $\mathcal{O}_F$

$$f_{a,b}(X, Y) := \frac{a X^2 + b Y^2 - (aX + bY)^2}{\pi}, \quad G_0(x, y) := (a_0 x + c_0 y)^q f_{b_0, d_0}(x, y) - (b_0 x + d_0 y)^q f_{a_0, c_0}(x, y)$$

for $a, b \in \mathcal{O}_F$. Let $j := (x_0, y_0) \in S_{\bar{a}_0}$. We set $j(j_2) := (\bar{a}_0 x_0 + c_0 y_0, b_0 x_0 + d_0 y_0) \in S_{\bar{a}_0}^{\bar{a}_0}$. See (5.8) and (5.7) for the defining equations of $X^i_j$ and $X^i_{j_2}$. We set $f(j, g) := \bar{g}(x_0, y_0) + \bar{G}_0(x_0, y_0)$. Then, $g$ induces the following morphism

$$g : X^i_j \to X^i_{j_2} : (a, z_1) \mapsto (\overline{\det(\bar{g}) a + f(j, g)}, z_1).$$

In particular, we have $\overline{G}_0(x, y) = 0$ if $\text{char } F > 0$, or $\text{char } F = 0$ and $e_{F/q_0} \geq 2$.

**Proof.** The required assertion follows from (5.10) immediately.

**Remark 5.4.** For $f(j, g)$, we have $f(j, g_1 g_2) = \overline{\det(\bar{g}_2) f(j, g_1)} + f(j g_1, g_2)$ for $g_1, g_2 \in G_F^2$.  

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5.2 Action of $G_2^F$ on $\mathbf{Z}_{1,1,0}$ and $\{W_{0,j}^i\}_{(i,j)\in S_0}$

In this subsection, we compute the action of $G_2^F$ on the components $\mathbf{Z}_{1,1,0}$ and $W_{0,j}^i$ for $(i,j)\in S_0$ explicitly in Lemma 5.3 and Proposition 5.0. In the following computations, we freely use the notations in 3.3 and 3.4. Recall that we have on the space $\mathbf{Z}_{1,1,0}$

\[ v(u) = \frac{1}{q}, \quad v(X_1) = \frac{1}{2(q-1)}, \quad v(X_2) = \frac{1}{2q^2(q-1)}, \quad v(Y_1) = \frac{1}{2q(q-1)}, \quad v(Y_2) = \frac{1}{2q^3(q-1)}. \]

We choose an element $\kappa_1$ such that $\kappa_1^2q^4(q-1) = \pi$. We set $\kappa := \kappa_1^q$ and $\gamma := \kappa_1^3(q-1)^2$. We have $v(\kappa_1) = 1/2q^4(q-1)$ and $v(\gamma) = (q-1)/2q^2$. We write $\gamma \equiv \kappa^{-1}$ for $\kappa_1$. We change variables as follows $u = \kappa_1^2(q-1)u_0$, $X_1 = \kappa^2 x_1$, $Y_1 = \kappa^2 y_1$ and $X_2 = \kappa^q x$, $Y_2 = \kappa^q y$. We choose an element $\zeta \in \mu_{q-1}(O_F)$. See subsection 5.4 for an element $\tilde{\zeta}$. Let $B \subset G_0^F$ be a subgroup of upper triangular matrices. Let $\mathbf{T}_1^\times \subset G_2^F$ be the inverse image of $B$ by $G_2^F \to G_0^F$. The stabilizer of $\mathbf{Z}_{1,1,0}$ in $G_2^F$ is equal to $\mathbf{T}_1^\times$.

In the following, for an element $i = (\zeta, \tilde{\mu}) \in (O_F/\pi^2)^\times$, we determine the morphism $\mathbf{Z}_{1,1,0} \to \mathbf{Z}_{1,1,0}^q$ which is induced by $g \in \mathbf{T}_1^\times$ in Lemma 5.3 below.

Let $\left( \begin{array}{ccc} a_0 + a_1 \pi & b_0 + b_1 \pi \\ c_1 \pi \\ d_0 + d_1 \pi \end{array} \right) \in \mathbf{T}_1^\times$. Then, we acquire the following congruences

\[ g^*(x) \equiv a_0 x + c_1 \gamma^{1/(q-1)} y^q \pmod{1/(2q^2)} , \quad g^*(y) \equiv d_0 y + b_0 \gamma^{1/(q-1)} x \pmod{1/(2q^3)} . \]  

The following congruence holds by (5.11)

\[ g^*(x)g^*(y)^q \equiv a_0 d_0 xy^q + \gamma^{1/(q-1)} (c_1 d_0 y^{q(q+1)} + a_0 b_0 x^{q+1}) \pmod{1/(2q^2)} . \]  

Recall that we have set $xy^q = \tilde{z}_0 + \gamma^{1/q} Z_1$ and $g^*(x)g^*(y)^q = a_0 d_0 \tilde{z}_0 + \gamma^{1/(q+1)} (g^*Z_1)$. Moreover, we put $h(x,y) := c_1 d_0 y^{q(q+1)} + a_0 b_0 x^{q+1}$. Then, the congruence (5.12) induces the following congruence

\[ g^*Z_1 \equiv a_0 d_0 Z_1 + c_1 \gamma^{1/(q-1)} h(x,y) \pmod{1/(2q^3)} . \]  

We recall the following equality in (3.21)

\[ g^q \gamma^{1/(q^2)} Z_2 = Z_1 - \tilde{z}_0 ^{1/q} y^{q^2-1} - \zeta y^{-/(q^2-1)} . \]  

We have a similar equality for $(g^*Z_1, g^*Z_2, g^*(y))$. By (5.11), (5.13) and (5.14), we obtain the following

\[ g^*Z_2 \equiv a_0 d_0 Z_2 + \gamma^{1/q} z^{q-1} g(x,y) \pmod{1/(2q^4)} \]  

where we set $g(x,y) := y^{-(q^2-1)} \{ h(x,y) + \frac{\zeta b_0 x}{d_0 y} (y^{q^2-1} - y^{-(q^2-1)}) \}$. As computed in 5.3, the components $\mathbf{Z}_{1,1,0}^q$ and $\mathbf{Z}_{1,1,0}^g$ are defined by the following equations respectively

\[ Z_2^q = \tilde{\zeta}(g^q y^{q^2-1} + y^{-(q^2-1)}), \quad (g^*Z_2)^q = a_0 d_0 \tilde{\zeta}((g^*y)^q y^{q^2-1} + (g^*y)^{-/(q^2-1)}) \].

Then, we have the following lemma.
Lemma 5.5. Let \( g = \left( \begin{array}{cc} a_0 + a_1 \pi & b_0 + b_1 \pi \\ c_1 \pi & d_0 + d_1 \pi \end{array} \right) \in \overline{H}^\times \). We choose an element \( i \in (\mathcal{O}_F/\pi^2)^\times \). Then, \( g \) induces the following morphism

\[
\mathcal{Z}_{1,1,0}^{i} \to \mathcal{Z}_{1,1,0}^{a} : (y, Z_2) \mapsto (\bar{d}_0 y, \bar{a}_0 \bar{d}_0 Z_2).
\]

Proof. The required assertion follows from (5.11) and (5.15) immediately.

We choose elements \( \tilde{\gamma}_1, y_0 \) such that \( \tilde{\gamma}_1^d = \ell 2\zeta \{ 1 + \gamma^{1/4} (\tilde{\gamma}_1/\zeta) \}^{1/2} \) and \( y_0^{d-1} = (1 + \gamma^{1/4} (\tilde{\gamma}_1/\zeta))^{1/2} \) where \( \ell \in \{ \pm 1 \} \). We set \( w := y_0^{q+1} \zeta^{q-1}/q \) and choose \( w_1 \) such that \( y_0^{d-3} (\zeta + \gamma^{1/4} \tilde{\gamma}_1) w_1^2 = w^q \). Then, we have \( v(w) = 1/2q^2 \) and \( v(w_1) = 1/4q^3 \). Set \( x_0 := \tilde{\gamma}_1/y_0^q \). We choose a 2-th root \((a_0/d_0)^{1/2} \) of \( a_0/d_0 \). Furthermore, we put \( g^* w := d_0^2 w \) and \( g^* w_1 := d_0 (d_0/a_0)^{1/2} w_1 \). We have \( \bar{y}_0 \in \mu_2(q^2-1) \). Put \( \bar{y}_0 g := \bar{d}_0 \bar{y}_0 \in \mu_2(q^2-1) \). Now, we determine the morphism \( W_i^{\bar{y}_0} \to W_i^{\bar{y}_0 g} \) which is induced by \( g \) in Proposition 5.5 below.

As in (3.20), we change variables as follows

\[
Z_2 = \tilde{\gamma}_1 + wa, \ y = y_0 + w_1 s_1.
\]

(resp.,

\[
g^* Z_2 = a_0 d_0 \tilde{\gamma}_1 + g^* w(g^* a), \ g^* (y) = d_0 y_0 + g^* w_1 (g^* s_1).\]

By (5.11) and (5.15), the following congruences hold

\[
g^* a \equiv \left( \frac{a_0}{d_0} \right) a + \frac{g(x_0, y_0)}{d_0^2 y_0^{q+1}} (\mod 0+), \ g^* s_1 \equiv (a_0/d_0)^{1/2} s_1 (\mod 0+). \quad (5.16)
\]

Furthermore, we obtain the following \( g(x_0, y_0) \equiv c_1 d_0 y_0^{q+1} + a_0 b_0 y_0^{-(q+1)} \zeta^2 \) (mod 0+). Hence, we acquire the following by (5.10)

\[
g^* a \equiv (a_0/d_0) (a + (c_1/a_0) + (b_0/d_0)(\zeta/y_0^{q+1}))^2, \ g^* s_1 \equiv (a_0/d_0)^{1/2} s_1 \ (\mod 0+). \quad (5.17)
\]

As computed in subsection 3.4, the components \( W_{i, \bar{y}_0} \) and \( W_{i, \bar{y}_0 g} \) with \( \bar{y}_0 g = \bar{d}_0 \bar{y}_0 \) are defined by the following equations

\[
a^q - a = s^2, \ (g^* a)^q - g^* a = (g^* s_1)^2 \quad (5.18)
\]

respectively. Then, we have the following proposition.

Proposition 5.6. Let \( g = \left( \begin{array}{cc} a_0 + a_1 \pi & b_0 + b_1 \pi \\ c_1 \pi & d_0 + d_1 \pi \end{array} \right) \in \overline{H}^\times \). We consider the components \( \{ W_{i, \bar{y}_0}^{\ell, u} \}_{(\ell, \bar{u}) \in S_0} \). We choose elements \( i = (\zeta, \bar{u}) \in \mathbb{F}_q^\times \times \mathbb{F}_q^\times \) and \( \bar{y}_0 \in \mu_2(q^2-1) \). Furthermore, we set \( \bar{y}_0 g := \bar{d}_0 \bar{y}_0 \). See (7.18) for the defining equations of \( W_{i, \bar{y}_0} \) and \( W_{i, \bar{y}_0 g} \). Then, \( g \) induces the following morphism

\[
g : W_{i, \bar{y}_0} \to W_{i, \bar{y}_0 g} : (a, s) \mapsto \left( (\bar{a}_0/\bar{d}_0) (a + (\bar{c}_1/\bar{a}_0) + (\bar{b}_0/\bar{d}_0)(\zeta/y_0^{q+1})), (\bar{a}_0/\bar{d}_0)^{1/2} s_1 \right).
\]

Proof. The required assertion follows from (5.17).

Remark 5.7. The canonical map \( G_2^F \to G_1^F \) induces the following bijective \( G_2^F/\overline{H}^\times \simeq G_1^F/B \). Let \( g := \left( \begin{array}{cc} a_0 & b_0 \\ c_0 & d_0 \end{array} \right) \in G_2^F \). Then, we consider the right \( G_2^F \)-action on \( \mathbb{P}^1(\mathbb{F}_q) \in \{ x : y \} \) by
\[ x : y \mapsto [a_0 x + c_0 y : b_0 x + d_0 y]. \] The stabilizer of \([0 : 1]\) in \(G_1^F\) is equal to \(B\). Hence, we obtain bijective \(G_2^F / \Gamma \simeq G_1^F / B \simeq \mathbb{P}^1(\mathbb{F}_q)\) by \([g] \mapsto [0 : 1] g = [c_0 : d_0]\). We set \(S_2 := (\mathcal{O}_F / \pi^2)^\times \times \mu_{21(q^2 - 1)}\). In the following, we consider the components \(\mathbb{Z}_{1,1,*}\) and \(\{W_{i,0,y_0}\}_{(i,y_0) \in S_2}\) for \(i \in \mathbb{P}^1(\mathbb{F}_q)\) in 3.3, 3.4 and 3.5. Then, \(G_2^F\) acts on the index set \(i \in \mathbb{P}^1(\mathbb{F}_q)\) by \(i \mapsto i g\). Clearly, this action is transitive. As mentioned at the beginning of this subsection, for each \(i \in (\mathcal{O}_F / \pi^2)^\times\), the stabilizer of the components \(\mathbb{Z}_{1,1,0}\) and \(\{W_{i,0,y_0}\}_{(i,y_0) \in S_2}\) in \(G_2^F\) is equal to \(\Gamma\). Hence, we acquire the following isomorphism as a \(G_2^F\)-representation

\[
\bigoplus_{(i,j,0,y) \in S_1} H^1(W_{i,j,0,\hat{y}}^c, \mathbb{Q}_l) \simeq \bigoplus_{(i,y_0) \in S_2} \text{Ind}_{\Gamma}^{G_2^F} H^1(W_{i,0,y_0}^c, \mathbb{Q}_l) \tag{5.19}
\]

with \(l \neq p\). Hence, to understand the étale cohomology group in the left hand side of (5.19), it suffices to understand \(H^1(W_{0,0,0,\hat{y}}^c, \mathbb{Q}_l)\) as a \(\Gamma\)-representation.

### 6 Inertia action on the components in the stable reduction of \(\mathcal{X}(\pi^2)\)

In this section, we determine the right action of inertia on the components \(\mathbb{Y}_{2,2, X_j^i ((i,j) \in S), \mathbb{Z}_{1,1,0, W_{j,k}^b}\) for \((i,j,k) \in S_1\). First, we prepare some notations used through this section. For a finite extension \(L/F\), we have the Artin reciprocity map \(a_L : W_{L}^{ab} \rightarrow L^\times\), which is normalized such that the geometric Frobenius is sent to a prime element \(\pi_L\) by this map. Then, for an extension \(L/K\), we have \(a_K = N_{L/K} \circ a_L\). Let \(LT_L\) denote the formal \(\mathcal{O}_L\)-module over \(\mathcal{O}_L\), with \(LT_L \otimes \mathbb{K}^{ac}\) of height 1. For \(n \geq 1\), let \(\pi_{n,L} \in LT_L[\pi_L]\) be primitive \(\pi^n\)-torsion points. Then, by the classical Lubin-Tate theory, we have the following equality

\[
\sigma(\pi_{n,L}) = [a_L(\sigma)]|LT_L(\pi_{n,L}) \tag{6.1}
\]

for any \(n \geq 1, \sigma \in I_{F}^{ab}\).

We consider a case \(L = F\). We denote by the same letter \(a_{L}\) for the composite

\[
a_{L} : \ I_{F}^{ab} \rightarrow (\mathcal{O}_F / \pi^2)^\times \rightarrow (\mathcal{O}_F / \pi^2)^\times. \tag{6.2}
\]

We write \(a_{L}(\sigma) = (\zeta_{o}(\sigma), \lambda_{0}(\sigma)) \in (\mathcal{O}_F / \pi^2)^\times \simeq \mathbb{F}_q^\times \times \mathbb{F}_q\). Then, by (6.1), we have the followings

\[
\zeta_{o}(\sigma) = \sigma(\pi_1) / \pi_1, \quad \lambda_{0}(\sigma) = \left( \frac{\pi_1 \sigma(\pi_2) - \sigma(\pi_1) \pi_2}{\sigma(\pi_1) \pi_2} \right).
\]

For \(i \in (\mathcal{O}_F / \pi^2)^\times\) and \(\sigma \in I_F\), we set \(i \sigma := a_{L}(\sigma)^{-1} \times i\). It is well-known that, under some identification \(\pi_0(\mathcal{X}(\pi^2)) \simeq (\mathcal{O}_F / \pi^2)^\times\), the inertia \(I_F \supset \sigma\) acts on \(\pi_0(\mathcal{X}(\pi^2)) \) by \(i \mapsto i \sigma\). See Theorem 6.

#### 6.1 Action of Inertia

We will recall the right action of inertia on the stable model of a curve over \(F\) from [CM2, Section 6]. Note that the inertia action considered in loc. cit. is a left action. In this paper, we want to consider the right action. Hence, the (right) action of inertia is characterized as in 6.
If $Y/F$ is a curve, and $\mathcal{Y}$ its stable model over $C$, there is a homomorphism $w_Y$

$$w_Y : I_F = \text{Gal}(C/F^w) \to \text{Aut}(\mathcal{Y}).$$

(6.3)

It is characterized by the fact that for each $P \in Y(C)$ and $\sigma \in I_F$,

$$\overline{\sigma^{-1}(P)} = (P)w_Y(\sigma).$$

(6.4)

We have something similar if $Y$ is a reduced affinoid over $F$. Namely, we have a homomorphism $w_Y : I_F \to \text{Aut}(\overline{Y}_C)$ characterized by (6.4). This follows from the fact that $I_F$ preserves $A(Y_C)^0$ (power bounded elements of $A(Y_C)$) and $A(Y_C)^\infty$ (topologically nilpotent elements of $A(Y_C)$).

6.2 Inertia action on $\overline{Y}_{2,2}$ and $\{X_{ij}^\sigma\}_{(i,j) \in S}$

We determine the right action of the inertia on the components $\overline{Y}_{2,2}$ and $\{X_{ij}^\sigma\}_{(i,j) \in S}$ using (6.4). These components are computed in subsections 5.1 and 5.2.

In the following, we use the notations in 5.1 and 5.2. We briefly recall them. Let $\kappa_1$ be an element such that $\kappa_1^q(q^2-1) = \pi$. We set $\kappa := \kappa_1^q$ and $\gamma := \kappa^{(q-1)(q^2-1)}$. We write $\gamma = \sqrt{\gamma}$ for $\kappa_1^{-1}$. We fix an element $\gamma \in \mu_{q-1}(O_F)$. Moreover, we choose elements $\gamma_0$, $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ such that $\gamma_0^q - \gamma q \gamma_0 = \zeta$, $\gamma_0^q = \gamma_0$ and $\tilde{\gamma}_0^q + \gamma + 1/\tilde{\gamma}_1 = 0$. Let $y_0$ be an element such that $y_0^q - 1 + \gamma_0^q = 0$. We set $w := \gamma^q(q-1)$ and $w_1 := y_0^q(q-1)$. Then, we have $e(w) = 1/q^3$, $e(w_1) = 1/q^2(q+1)$. Let $x_0$ be an element such that $x_0^q y_0 - x_0 y_0 = \gamma + 1/\tilde{\gamma}_1$.

Lemma 6.1. Let the notation be as above. Let $\sigma \in I_F$. We write $\sigma(\kappa) = \zeta_\sigma \kappa$ with some $\zeta_\sigma \in \mu_{q^2(q^2-1)}$. We choose elements $i = (\tilde{\gamma}, \tilde{\mu}) \in (O_F/\pi^2)^\times$ and $j = (\tilde{x}_0, \tilde{y}_0) \in S_{00}$. We set $j \sigma := (\tilde{\zeta}^{-1}_\sigma \tilde{x}_0, \tilde{\zeta}_\sigma^{-1} \tilde{y}_0) \in S_{00}^\sigma$. Furthermore, we set as follows

$$a_0 := \frac{\sigma(\tilde{\gamma}_1) - \tilde{\gamma}_1}{w}, \quad d_0 := \frac{\sigma(\tilde{\gamma}_0) - \tilde{\gamma}_0}{\gamma^q(q-1)}.$$ 

Note that we have $\tilde{a}_0 \in F_q^2$, $\tilde{d}_0 \in F_q$ and $\tilde{a}_0^q + \tilde{a}_0 + \tilde{d}_0 = 0$.

1. The element $\sigma$ induces the following morphism

$$\overline{Y}_{2,2} \to \overline{Y}_{2,2}^\sigma : (x, y, Z_1) \mapsto \left(\tilde{\zeta}_\sigma^{-1} x, \tilde{\zeta}_\sigma^{-1} y, \tilde{\zeta}_\sigma^{-q+1} Z_1\right).$$

2. The element $\sigma \in I_F$ also induces the following morphism

$$X_{ij} \to X_{ij}^\sigma : (a, z_1) \mapsto \left(\tilde{\zeta}_\sigma^{-q+1}(a - a_0 - d_0), z_1\right).$$

Proof. Let $\sigma \in I_F$ and $P \in \overline{Y}_{2,2}(C)$. First, note that we have $X_2(\sigma^{-1}(P)) = \sigma^{-1}(X_2(P))$. Since we have $X_2 = \kappa x$, we have $x(\sigma^{-1}(P)) = \zeta_\sigma^{-1} \sigma^{-1}(x(P))$. The same thing holds for $y$. We consider the followings by (6.3).

- $x(P) y(P) - x(P) y(P) = \tilde{\gamma}_0 + \gamma^{1/q} Z_1(P),$

- $x(\sigma^{-1}(P))^q y(\sigma^{-1}(P)) - x(\sigma^{-1}(P))^q y(\sigma^{-1}(P)) = \zeta_\sigma^{-q+1} \tilde{\gamma}_0 + \gamma^{1/q} Z_1(\sigma^{-1}(P)).$
By applying $\sigma^{-1}$ to the first equality, we acquire
\begin{equation}
Z_1(\sigma^{-1}(P)) = \zeta_{\sigma}^{(q+1)}(\sigma^{-1}(Z_1(P)) - wd_0).
\end{equation}
(6.5)

Note that we have $\tilde{d}_0 = (\tilde{\gamma}_0 - \sigma^{-1}(\tilde{\gamma}_0))/\gamma_1^{1/(q+1)}$. Since we have $y(\sigma^{-1}(P)) = \zeta_{\sigma}^{-1}\sigma^{-1}(y(P)) \equiv \zeta_{\sigma}^{-1} y(P) \pmod{0+}$ and $\sigma^{-1}(Z(P)) \equiv Z(P) \pmod{0+}$, the required assertion 1 follows from (6.5). We prove the assertion 2. We consider the following composite quadratic extension. We choose a model $LT_E$ to the first equality in (6.6), we acquire $\tilde{d}_1 + wa = \tilde{d}_1 = \tilde{d}_0$. By applying $\sigma^{-1}$ to the first equality, we obtain the following equality by (6.5)
\begin{equation}
\phi(\sigma^{-1}(P)) = \zeta_{\sigma}^{(q+1)}(\sigma^{-1}(\phi(P)) - a_0 - d_0).
\end{equation}
Hence, by $\sigma^{-1}(\phi(P)) \equiv \phi(P) \pmod{0+}$, we acquire $\phi(\sigma(a)) = \zeta_{\sigma}^{(q+1)}(a - a_0 - d_0) \pmod{0+}$. On the other hand, we consider the following equalities induced by (6.11)
\begin{equation}
y(P) = y_0 + w_1 z_1(P), \ y(\sigma^{-1}(P)) = \zeta_{\sigma}^{-1}y_0 + \zeta_{\sigma}^{-1} w_1 (\sigma^{-1}(P)).
\end{equation}
(6.6)
We easily check $y(\sigma^{-1}(P)) = \zeta_{\sigma}^{-1}\sigma^{-1}(y(P)), \ \sigma^{-1}(z_1(P)) \equiv z_1(P) \pmod{0+}$ and $\sigma^{-1}w_1 \equiv \zeta_{\sigma}^{-1} w_1 \pmod{0+}$. Hence, by applying $\sigma^{-1}$ to the first equality in (6.6), we acquire
\begin{equation}
z_1(\sigma^{-1}(P)) = \zeta_{\sigma}^{q+1}(\sigma^{-1}(w_1)/w_1) \sigma(z_1(P)) \equiv z_1(P) \pmod{0+}.
\end{equation}
Hence, we obtain the required assertion.

In the following, we rewrite Lemma 6.1 as in Corollary 6.2. Let $E/F$ denote the unramified quadratic extension. We choose a model $LT_E$ such that $[\pi]_{LT_E}(X) = \pi X - X^{q^2}$.

We consider the following composite
\begin{equation}
\begin{array}{ccc}
a_E : I_F^{ab} & \xrightarrow{\mathcal{O}_E^\times} (\mathcal{O}_E/\pi^2)^\times & \simeq \mathbb{F}_q^\times \\
\sigma \mapsto (\zeta(\sigma), \lambda(\sigma)) & = & \left( \begin{array}{c}
\sigma(1) \\
\pi_1.E
\end{array} \right), \\
\sigma = (\zeta(\sigma), \lambda(\sigma)) & = & \left( \begin{array}{c}
\sigma(1) - \pi_1.E \\
\pi_1.E
\end{array} \right).
\end{array}
\end{equation}
By $a_E = N_{E/F} \circ a_E$, we have $\zeta(\sigma)^{q+1} = \zeta_0(\sigma)$ and $\lambda(\sigma)^{q+1} + \lambda(\sigma) = \lambda_0(\sigma)$ for $\sigma \in I_F$.

**Corollary 6.2.** Let $\sigma \in I_F$. Then, the element $\sigma$ acts on the set of the connected components $\pi_0(Y_{2,2})$ as follows
\begin{equation}
i := (\zeta, \mu) \mapsto i\sigma = (\zeta_0(\sigma)^{-1})\zeta, \mu - \lambda_0(\sigma) \zeta.
\end{equation}
The element $\sigma \in I_F$ acts on $S_{20}$ as follows
\begin{equation}j := (x_0, y_0) \mapsto j\sigma := (\zeta(\sigma)^{-1}x_0, (\zeta(\sigma)^{-1}y_0).
\end{equation}Moreover, $\sigma$ acts on the components $\{X_{ij}^I(i,j) \in S_{20} \}$ as follows
\begin{equation}\sigma : X_{ij}^I \to X_{ij}^{I\sigma} : (a, z_1) \mapsto (\zeta(\sigma)^{-1}(a + \lambda(\sigma) \zeta), z_1).
\end{equation}

*Proof. Clearly, we have $\tilde{d}_0 = \zeta(\sigma)$ in $F_q^\times$. By $\gamma_1^q - \gamma_0 = \zeta, \gamma_0^q = \gamma_0$ and $\tilde{d}_0^q + \gamma_1^q \tilde{d}_1 + \gamma_0 = 0$, we easily check that $\tilde{d}_0 = -\lambda_0(\sigma) \zeta$ and $\tilde{d}_0^q = \lambda(\sigma) \zeta$ in $F_q$. Hence, by $\lambda(\sigma)^{q+1} + \lambda(\sigma) = \lambda_0(\sigma)$, we acquire $\tilde{d}_0 + d_0 = -\lambda(\sigma) \zeta$. Therefore, the required assertion follows from Lemma 6.1. 

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6.3 Inertial action on $\bar{Z}_{1,1,0}$ and $\{W^j_{0,i}\}_{(i,j)\in S_0}$

In this subsection, we determine the action of inertia on the components $\bar{Z}_{1,1,0}$ and $W^j_{0,i}$ for $(i, j) \in S_0$ in the same way as in the previous subsection. We use the notations in subsections 3.3 and 3.4 freely.

We briefly recall the notations in subsections 3.3 and 3.4. We choose an element $\kappa_1$ such that $\kappa_1^{2q^4(q-1)} = \pi$ with $v(\kappa_1) = 1/2q^4(q-1)$. We set $\kappa := \kappa_1^{q}$. We set $\gamma := \kappa^{q_1}$. We write $\gamma = \kappa^{q_1}$ for $\kappa_1^{-1}$. Let $\gamma \in \mu_{q-1}(\mathcal{O}_F)$. We choose elements $\tilde{\gamma}_1$ and $y_0$ such that $\tilde{\gamma}_1^q = \epsilon 2\zeta \left(1 + \gamma^{1/q_1} \left(\frac{\gamma}{\zeta}\right)^{1/2}\right)$ and $y_0^{q_2-1} = \epsilon \left(1 + \gamma^{1/q_1} \left(\frac{\gamma}{\zeta}\right)^{1/2}\right)$ with $\epsilon \in \{\pm 1\}$. Set $w := y_0^{q_2} \gamma^{q_1} (\epsilon 1)$. Furthermore, we choose an element $\omega_1$ such that $y_0^{2(q_2-3)}(\zeta + \gamma^{1/q_1} \tilde{\gamma}_1)\omega_1^2 = w^2$. Then, we have $v(w) = 1/4q^4$ and $v(w_1) = 1/4q^2$.

**Lemma 6.3.** Let $\sigma \in I_F$. We write $\sigma(\kappa) = \zeta_1 \kappa$ with $\zeta_1 \in \mu_{2q^4(q-1)}$. We choose elements $i = (\zeta, \mu) \in (\mathcal{O}_F/\pi^2)^\times$ and $\tilde{y}_0 \in \mu_{2(q^2-1)}$. We put $\tilde{y}_0 \sigma := \zeta_1^{-1} \tilde{y}_0$. Furthermore, we set

\[
\begin{align*}
a_0 := \frac{\sigma(\zeta_1) - \zeta_1}{w}, & \quad b_0 := \sigma(w)/w, & \quad c_0 := \sigma(w_1)/w_1.
\end{align*}
\]

Then, we have $a_0 \in F_q$, $b_0 \in \{\pm 1\}$ and $c_0^2 = b_0$.

1. Then, $\sigma$ induces the following morphism

\[\bar{Z}_{1,1,0} \to \bar{Z}_{1,1,0}^{i\sigma} : (Z_2, y) \mapsto (Z_2, \zeta_1^{-1} y).\]

2. The element $\sigma$ induces the following morphism

\[W^i_{0,\tilde{y}_0} \to W^i_{0,\tilde{y}_0\sigma} : (a, s) \mapsto (b_0^{-1} a - a_0, c_0^{-1} s).\]

**Proof.** We prove the assertion in the same way as in Lemma 6.1. We omit the detail. \(\square\)

We prepare some notations. Let $\tilde{\zeta} \in F_q^\times$ and $\tilde{y}_0 \in \mu_{2(q^2-1)}$. We consider an element $a := (\tilde{\zeta}/\tilde{y}_0^{q+1})^2 \in F_q^\times$. Let $\tilde{a} \in \mu_{q-1}(\mathcal{O}_F)$ denote the unique lifting of $a$. Let $\tilde{a}_1$ be an element such that $\tilde{a}_1^q = \tilde{a}$. We set $t := \tilde{a}_1 \gamma^2 \pi$ and $E_1 := F[t]$. Then, $E_1/F$ is a totally ramified quadratic extension and $t$ is a uniformizer of $E_1$. Moreover, we have $t^2 = \tilde{a}_1 \pi$. Clearly, we have $v(t) = 1/2$. We choose a model of $L_{E_1}$ such that

\[v(t)_{L_{E_1}}(X) = tX - X^q.\]

We consider the following composite

\[\begin{array}{c}
a_{E_1} : r_{E_1} \overset{a_{E_1}}{\longrightarrow} \mathcal{O}_{E_1}^\times \to (\mathcal{O}_{E_1}/\pi)^\times \simeq \mathbb{F}_q^\times \times \mathbb{F}_q \quad (6.7)
\end{array}\]

\[\sigma \mapsto (\zeta(\sigma), \lambda(\sigma)) := \left(\frac{\pi_{1, E_1} \sigma(\pi_{2, E_1}) - \pi_{2, E_1} \sigma(\pi_{1, E_1})}{\pi_{1, E_1} \sigma(\pi_{1, E_1})}\right).
\]

Then, by Lemma 6.3 we obtain the following proposition.

**Corollary 6.4.** Let $\sigma \in I_{E_1}$. We choose elements $i = (\tilde{\zeta}, \mu) \in (\mathcal{O}_F/\pi^2)^\times$ and $\tilde{y}_0 \in \mu_{2(q^2-1)}$. We set $\tilde{y}_0 \sigma := \zeta(\sigma)^{-1} \tilde{y}_0 \in \mu_{2(q^2-1)}$. Furthermore, we set

\[a(\sigma) := \frac{\tilde{y}_0^{q-1} \cdot 2 \tilde{\zeta}^2}{\tilde{y}_0^{q+1}} \lambda(\sigma) \in \mathbb{F}_q.
\]

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Then, the element $\sigma$ acts on the components $\{W_{0,\bar{y}_0}^i\}_{(i,\bar{y}_0)\in S_0}$ as follows

$$W_{0,\bar{y}_0}^i \rightarrow W_{0,\bar{y}_0}^{i\sigma} \cdot (a, s) \mapsto (a - a(\sigma), \zeta(\sigma)^{-\langle q-1 \rangle/2} s).$$

**Proof.** We have the following congruence by the definitions of $\pi, E_1$ for $i = 1, 2$

$$\pi_{1,E_i}(\pi_{2,E_i}^q - \pi_{2,E_i}^q \pi_{1,E_i}) \equiv t(\sigma(\pi_{2,E_i})\pi_{1,E_i} - \sigma(\pi_{1,E_i})\pi_{2,E_i}) \pmod{\langle q+1 \rangle +}.$$  

By this, we check the following by the definition of $\gamma$

$$a_0 \equiv -\bar{y}_0^{q-1} 2\zeta \left( \frac{\sigma(\pi_{2,E_i})\pi_{1,E_i} - \pi_{2,E_i}^q \pi_{1,E_i}}{w^{q-1} \sigma(\pi_{1,E_i})^q}\right) \equiv \left( \frac{q_0^{q-1} 2\zeta \lambda(\sigma) \equiv a(\sigma) \pmod{0+}}{y_0^{q+1} \gamma^{q+1}} \right).$$

Hence, the required assertion follows. 

\[ \square \]

7 Analysis of cuspidal part in the étale cohomology of the Lubin-Tate space $\mathcal{X}(\pi^2)$

In this section, we investigate the cohomologies of the curves in the stable reduction of $\mathcal{X}(\pi^2)$ using the explicit descriptions of the action of $\mathcal{O}_\mathbb{F}_l$ on $\mathcal{G}_2$ and $I_F$ given in the previous sections. Recall that the components $\{X_{j,c}^{i,E}\}_{(i,j)\in S}$, with each having an affine model $X^q + X = Y^{q+1}$, and the components $\{W_{j,k}^{i,c}\}_{(i,j,k)\in S_1}$, with each having an affine model $a^q - a = s^2$, appear in the stable reduction of $\mathcal{X}(\pi^2)$. See Propositions 3.2, 3.3 and 3.6. We set

$$W := \bigoplus_{(i,j)\in S} H^1(X_{j,c}^{i,E}, \mathbb{Q}_l), \quad W' := \bigoplus_{(i,j,k)\in S_1} H^1(W_{j,k}^{i,c}, \mathbb{Q}_l)$$

with $p \neq l$. The dual graph of the stable reduction $\mathcal{X}(\pi^n)$ is known to be a tree. For example, see [W3] Proposition 3.4 for this fact. Because of the fact, these spaces $W$ and $W'$ are direct summands of $H^1(\mathcal{X}(\pi^n)^c, \mathbb{Q}_l)$. Furthermore, we can easily check that these subspaces $W$ and $W'$ are $G := \mathcal{G}_2^F \times \mathcal{O}_\mathbb{F}_l^\times \times I_F$-stable. Note that the group $G$ acts on the spaces $W$ and $W'$ on the left. In this section, we analyze these representations by using the explicit action of $G$ on $W$ and $W'$. See Proposition 7.3.2, Corollary 7.3 and Corollary 7.9 for precise statements. As a result, we know that the local Jacquet-Langlands correspondence and the $\ell$-adic local Langlands correspondence for unramified (resp. ramified) cuspidal representations of $GL_2(F)$ of level 1 (resp. of level 1/2) are realized in $W$. (resp. $W'$.) See [7.3] for more details.

7.1 Analysis of the étale cohomology of the components $\{X_{j,c}^{i,c}\}_{(i,j)\in S}$

In this subsection, we analyze the following étale cohomology group of the components $\{X_{j,c}^{i,c}\}_{(i,j)\in S}$ computed in subsection 6.2.

$$W := \bigoplus_{(i,j)\in S} H^1(X_{j,c}^{i,c}, \mathbb{Q}_l)$$

as a $G$-representation. As a result, for unramified cuspidal representations of $GL_2(F)$ of level 1, we will check that the local Jacquet-Langlands correspondence and the local Langlands correspondence are realized in $W$. See 7.3 for a precise meaning of this.
Moreover, let $\mathcal{S}_{00}^{q^x} = \{(x_0, y_0) \in (k^{ac})^2| \xi := x_0^q y_0 - x_0 y_0^q \in \mathbb{F}_q^\times, y_0^q - 1 = -1\}$. Then, we have $S = \mathbb{F}_q \times \mathcal{S}_{00}^{q^x}$. Let $X_i$ be a projective smooth curve with an affine equation $X^2 - X = \xi(Y^q(q+1) - Y^q+1)$. The curve $X_i$ has $q$ connected components, and each component has an affine model defined by $X^q + X = \xi Y^q+1 - c$ with some $c \in \mathbb{F}_q$. Then, $W$ is written as follows

$$W \simeq \bigoplus_{i \in \mathcal{S}_{00}^{q^x}} H^1(X_i, \overline{\mathcal{O}}_i) \quad (7.1)$$

as a $\overline{\mathcal{O}}_i$-vector space. Since we have $|\mathcal{S}_{00}^{q^x}| = |\text{GL}_2(\mathbb{F}_q)| = q(q^2 - 1)(q - 1)$ and $\dim H^1(X_i, \overline{\mathcal{O}}_i) = q^2(q - 1)$ for each $i$, we have $\dim W = q^3(q - 1)^2(q^2 - 1)$.

Now, we write down the right action of $G$ on the components $\{X_i\}_{i \in \mathcal{S}_{00}^{q^x}}$. This action induces the left action on the right hand side of (7.1). Then, it is not difficult to check the isomorphism (7.1) is an isomorphism as a $G$-representation.

$G_2^F$-action \ First, we recall the action of $G_2^F$ given in Proposition 5.3. Let $g = \begin{pmatrix} a_0 + a_1 \pi & b_0 + b_1 \pi \\ c_0 + c_1 \pi & d_0 + d_1 \pi \end{pmatrix} \in G_2^F$ with $a_j, b_j, c_j, d_j \in \mu_{q-1}(O_F)$ $(j = 0, 1)$. Then, $g \in G_2^F$ acts on $\mathcal{S}_{00}^{q^x}$ as follows, factoring through $G_1^F$,

$$g : i := (x_0, y_0) \mapsto ig := (\bar{a}_0 x_0 + \bar{c}_0 y_0, \bar{b}_0 x_0 + \bar{d}_0 y_0). \quad (7.2)$$

Of course, this action of $G_1^F$ on $\mathcal{S}_{00}^{q^x}$ is simply transitive. Furthermore, $g$ induces

$$g : X_i \to X_{ig} : (X, Y) \mapsto (\det(g)X + f(i, g), Y). \quad (7.3)$$

$O_3^x$-action \ We recall the action of $O_3^x$ given in Proposition 4.4. Let $E/F$ denote the unramified quadratic extension. Let $b = a_0 + \varphi b_0 + \pi a_1 \in O_3^x$ with $a_0 \in \mu_{q^2 - 1}(O_E)$ and $a_1, b_0 \in \mu_{q^2 - 1}(O_E) \cup \{0\}$. Then, $b$ acts on $\mathcal{S}_{00}^{q^x}$ as follows

$$i = (x_0, y_0) \mapsto ib := (\bar{a}_0^{-1} x_0, \bar{a}_0^{-1} y_0). \quad (7.4)$$

Moreover, $b$ induces a morphism

$$X_i \to X_{ib} : (X, Y) \mapsto (a_0^{-(q+1)}(X - (\bar{b}_0 \bar{a}_0^{-1})Y + (\bar{a}_0^{-1} - \bar{a}_0^{-1})\xi), a_0^{-1}(Y - (\bar{b}_0 \bar{a}_0^{-1})^q)). \quad (7.5)$$

Let $t := a_0 + \pi a_1 \in \Gamma := (O_E/\pi^2)^x \subset O_3^x$. Then, $t$ induces the following by (7.5)

$$X_i \to X_{it} : (X, Y) \mapsto (\bar{a}_0^{-(q+1)}(X + (\bar{a}_0^{-1} - \bar{a}_0^{-1})\xi), \bar{a}_0^{-1} - 1Y). \quad (7.6)$$

Inertial action \ We recall the inertia action $I_E$ given in Corollary 6.2. Let $LT_E$ be the formal $O_E$-module over $E^{ur}$, with $LT_E \otimes k^{ac}$ of height 1. Recall that we have chosen a model $LT_E$ such that $[\pi]_{LT_E}(X) = \pi X - X^{q^2}$.

Let $\pi_{i,E} \in LT_E[\pi]^{i}$ for $i \geq 1$ be primitive elements. We define

$$a_E : I_E \to I_E^{ab} \simeq I_E^{ab} \to (O_E/\pi^2)^x \simeq \mathbb{F}_q^x \times \mathbb{F}_q^2$$

by

$$\sigma \mapsto (\zeta(\sigma), \lambda(\sigma)) = \left(\frac{\sigma(\pi_{1,E})}{\pi_{1,E}}, \frac{\pi_{1,E}\sigma(\pi_{2,E}) - \sigma(\pi_{1,E})\pi_{2,E}}{\pi_{1,E}\sigma(\pi_{1,E})}\right).$$
Moreover, $\sigma$ induces a morphism
\[
X \to X_{i\sigma} : (X,Y) \mapsto (\zeta(\sigma)^{-1}x_0, \zeta(\sigma)^{-1}y_0).
\]
(7.7)

Let $X$ be a projective smooth curve with an affine model $X^g + X = Y^{q+1}$ with genus $q(q-1)/2$. Then, we have $\dim \, H^1(X, \mathcal{O}_X) = q(q-1)$. In the following, to investigate $W$, we prove some elementary facts on $H^1(X, \mathcal{O}_X)$ in Lemma 7.1 and Corollary 7.2.

Let $\mathcal{I} := \ker \operatorname{Tr}_{\mathbb{F}_q^2/F_q}$. Then, the group $\mathcal{I} \supseteq a_1$ acts on $X$ by $(X,Y) \mapsto (X + a_1, Y)$. On the other hand, $\mu_{q+1} \ni \zeta$ acts on $X$ by $(X,Y) \mapsto (X, \zeta Y)$. Therefore, we consider $H^1(X, \mathcal{O}_X)$ as a $\mathcal{O}_X[\mathcal{I} \times \mu_{q+1}]$-module.

**Lemma 7.1.** Let the notation be as above. Then, we have the following isomorphism
\[
H^1(X, \mathcal{O}_X) \cong \bigoplus_{\psi \in \mathcal{I}'} \bigoplus_{\chi \in \mathbb{G}_m} \psi \otimes \chi \quad (7.8)
\]
as a $\mathcal{O}_X[\mathcal{I} \times \mu_{q+1}]$-module.

**Proof.** We have the following short exact sequence
\[
0 \to \bigoplus_{\psi \in \mathcal{I}'} \psi \to H^1_c(X \setminus X(\mathbb{F}_q), \mathcal{O}_X) \to H^1(X, \mathcal{O}_X) \to 0 \quad (7.9)
\]
as a $\mathcal{O}_X[\mathcal{I} \times \mu_{q+1}]$-module.

Let $\mathcal{L}_\psi(t)$ denote the smooth $\mathcal{O}_X$-sheaf associated to the finite Galois étale covering $a^g + a = t$ of $\mathbb{A}^1 \ni t$ and a character $\psi \in \mathcal{I}'$. Let $\mathcal{K}_\chi(t)$ denote the smooth $\mathcal{O}_X$-sheaf associated to the Kummer covering $y^{q+1} = t$ of $\mathbb{G}_m \ni t$ and a character $\chi \in \mu_{q+1}$. Since $X \setminus X(\mathbb{F}_q) \to \mathbb{G}_m; (a,s) \mapsto a^{q+1}$ is a finite Galois étale covering of Galois group $\mathcal{I} \times \mu_{q+1}$, the group $H^1_c(X \setminus X(\mathbb{F}_q), \mathcal{O}_X)$ is isomorphic to
\[
\bigoplus_{\psi \in \mathcal{I}'} \bigoplus_{\chi \in \mathbb{G}_m, \mu_{q+1}} H^1_c(\mathbb{G}_m, \mathcal{L}_\psi(t) \otimes \mathcal{K}_\chi(t)) \quad (7.10)
\]
as a $\mathcal{O}_X[\mathcal{I} \times \mu_{q+1}]$-module. Note that we have $\dim \, H^1_c(\mathbb{G}_m, \mathcal{L}_\psi(t) \otimes \mathcal{K}_\chi(t))$ is equal to 1 if $\psi \neq 1$ and 0 otherwise by the Grothendieck-Ogg-Shafarevich formula. Furthermore, we have
\[
\bigoplus_{\psi \in \mathcal{I}'} \bigoplus_{\chi \in \mathbb{G}_m, \mu_{q+1}} H^1_c(\mathbb{G}_m, \mathcal{L}_\psi(t) \otimes \mathcal{K}_\chi(t)) \cong \bigoplus_{\psi \neq 1 \in \mathcal{I}'} \bigoplus_{\chi \neq 1 \in \mu_{q+1}} H^1_c(\mathbb{G}_m, \mathcal{L}_\psi(t) \otimes \mathcal{K}_\chi(t)) \oplus \bigoplus_{\psi \in \mathcal{I}'} \psi \quad (7.11)
\]
as a $\mathcal{O}_X[\mathcal{I} \times \mu_{q+1}]$-module. By (7.9), (7.10) and (7.11), the required assertion follows.

**Corollary 7.2.** Let $X$ be a projective smooth curve with an affine model $X^g + X = Y^{q+1} - Y^{q+1}$. The group $(a, \zeta) \in \mathbb{F}_q^2 \times \mu_{q+1}$ acts on $X$ by $(X,Y) \mapsto (X + a, \zeta Y)$. We consider $\mathbb{F}_q^\vee$ as a subgroup of $\mathbb{F}_q^\vee$ by $\operatorname{Tr}_{\mathbb{F}_q^2/\mathbb{F}_q}$. Then, we have the following isomorphism
\[
H^1(X, \mathcal{O}_X) \cong \bigoplus_{(\psi, \chi) \in (\mathbb{F}_q^\vee)^{2}(\mu_{q+1}) \setminus \{1\}} \psi \otimes \chi \quad (7.12)
\]
as a $\mathcal{O}_X[\mathbb{F}_q^2 \times \mu_{q+1}]$-module.
We obtain the following isomorphism as a $\text{Sta}$ and $\text{Sta}_2$. Clearly, we have $\dim \pi \chi$ denote by the same letter $w$.

To do so, in the following, for each $\pi, \chi, \psi$ we define a representation $\pi_w$, which is called (strongly) cuspidal representation in [AOPS, 5.2] and [Onn]. Now, we fix an element $\zeta_0 \in \mu_{q^2-1}(O_E) \backslash \mu_{q-1}(O_F)$. We fix the following embedding

$$\Gamma := (O_E/\pi^2) \hookrightarrow G_2^F$$

(7.12)

$$a + b\zeta_0 \mapsto a12 + b \begin{pmatrix} \zeta_0^2 + \zeta_0 & 1 \\ -\zeta_0^{q+1} & 0 \end{pmatrix}$$

with $a, b \in O_E/\pi^2$.

We identify $\Gamma \simeq F_q^\times \times F_q^\times$ by $a_0 + a_1 \pi \mapsto (\bar{a}_0, (\bar{a}_1/\bar{a}_0))$. For a character $\psi \in F_q^\times \backslash F_q$ and an element $\zeta \in F_q^\times \backslash F_q$, we define a character $\tilde{\psi} \zeta$ of $N$ by the following

$$\tilde{\psi} \zeta : N \ni \begin{pmatrix} 1 + \pi a_1 & \pi b_1 \\ \pi c_1 & 1 + \pi d_1 \end{pmatrix} \mapsto \psi \left( -\bar{a}_1 \zeta + \bar{c}_1 - \bar{a}_1 \zeta + \bar{d}_1 \right).$$

(7.13)

Note that the restriction of $\tilde{\psi} \zeta_0 \in N^\times$ to a subgroup $F_q^\times \simeq \Gamma \cap N \subset N$ is equal to $\psi$. For a strongly primitive character $w = (\chi, \psi) \in \Gamma^\times \simeq (F_q^\times)^\times \times F_q^\times$ and an element $\zeta \in F_q^\times \backslash F_q$, we define a character $w$ of $\Gamma N = \mu_{q^2-1}(O_E) \backslash F_q$ by

$$w(xu) = \chi(x) \tilde{\psi} \zeta_0(u)$$

(7.14)

for all $x \in \mu_{q^2-1}(O_E)$ and $u \in N$. We set

$$\pi_w := \text{Ind}_{\Gamma \cap N}^{G_2^F}(w).$$

Then, $\pi_w$ is called a (strongly) cuspidal representation of $G_2^F$ in [AOPS Section 5.2], [Onn], [Sta] and [Sta2]. Clearly, we have $\dim \pi_w = q(q-1)$.

Let us set $H := G_2^F \times \Gamma \times I_F \subset G$.

To analyze $W$ as a $G$-representation, we will investigate the restriction $W|_{H}$ in Proposition 7.3.

To do so, in the following, for each $w \in \Gamma^\times_{\text{st}}$, we define a $H$-subrepresentation $W_w \subset W|_{H}$. For a character $\psi \in F_q^\times$ and an element $a \in F_q^\times$, we write $\psi_a \in F_q^\times$ for the character $x \mapsto \psi(ax)$. For $i \in S_{00}^{v_{\infty}}$ we write $i = (x_0, y_0)$. Let $\nu : \Gamma \to \mu_{q^2+1}$, $a_0 + a_1 \pi \mapsto a_0^{q+1}$. For $\chi_0 \in \mu_{q^2+1}$, we denote by the same letter $\chi_0$ for the composite $\chi_0 \circ \nu \in \Gamma^\times$. Then, by (7.11) and Corollary 7.2, we obtain the following isomorphism as a $\mathbb{Q}_l$-vector space

$$W \simeq \bigoplus_{i \in S_{01}^{v_{\infty}}} \bigoplus_{(\psi, \chi_0) \in (F_q^\times \backslash F_q^\times) \times (\mu_{q^2+1}(\{1\}))} \mathbb{Q}_l e_{i, \psi, \chi_0}.$$  

(7.15)

The isomorphism (7.15) and the action of $H$ on $W$ induce the $H$-action on the right hand side of (7.15). We write down the action of $H$ on the basis $\{e_{i, \psi, \chi_0}\}$ in (7.15) below. By (7.2) and (7.3), we have the following $G_2^F$-action

$$G_2^F \ni g^{-1} : e_{i, \psi, \chi_0} \mapsto \psi_{\det(g)^{-1}}(f(i, g))e_{i, g \psi_{\det(g)^{-1}}, \chi_0}$$

(7.16)
and the following $\Gamma$-action by (7.6)

$$\Gamma \ni t = a_0 + a_1\pi : e_{i,\psi} : \psi(-\overline{(a_1/\bar{a}_0)}\xi)\chi_0^{-1}(t)e_{i\psi^{-1},\psi_0^{-1}(q+1)\chi_0}$$

(7.17)

where we set $\xi := x_0^q y_0 - x_0 y_0^q \in F_q^\times$. Furthermore, we have the following $I_F$-action by (7.7)

$$I_F \ni \sigma : e_{i,\psi} : \psi(-\chi(\sigma))\chi^{-1}(\sigma)e_{i\psi^{-1},\psi_0^{-1}(q+1)\chi_0}.$$  

(7.18)

Now, we choose an element $y_0 \in \mu_2(q^2 - 1)$ such that $y_0^q - 1 = -1$. For $w = (\psi, \chi) \in (F_{q^2}^\times)^\vee \times E_{q^2}$, $\zeta \in F_{q^2}/F_q$ and $\chi_0 \in \mu_{q^2+1}\setminus\{1\}$, we define a vector of $W$ as follows

$$e_{\zeta,\chi_0}^w := \sum_{\mu \in \mathbb{F}_{q^2}} \chi^{-1}(\mu)e_{(\zeta,\mu_0),\psi_0^{-1}(\chi_0^{-1})(\chi q - \chi)} \chi_0 \in W.$$

For $w \in \Gamma_{\text{stp}}$, we define a subspace of $W$ as follows

$$W_w := \bigoplus_{\chi_0 \in \mu_{q^2+1}\setminus\{1\}} \bigoplus_{\zeta \in \mathbb{F}_{q^2}/\mathbb{F}_q} \mathbb{Z}_q e_{\zeta,\chi_0}^w.$$

Note that we have $\dim W_w = q^2(q - 1)$. For $g = \begin{pmatrix} a_0 + a_1\pi & b_0 + b_1\pi \\ c_0 + c_1\pi & d_0 + d_1\pi \end{pmatrix} \in G_2$, we obtain the following by (7.16)

$$g^{-1}e_{\zeta,\chi_0}^w = \chi(\overline{b_0}\zeta + \overline{d_0})\psi_0^{-1}(g^{-1}) e_{\zeta,\chi_0}^w = \chi(\overline{b_0}\zeta + \overline{d_0})\psi_0^{-1}(g^{-1}) e_{\zeta,\chi_0}^w.$$

(7.19)

The element $-f((\zeta,\mu_0,\mu_0),g)/(\mu_0)^{q+1}(\chi q - \chi)$ does not depend on $\mu$. If $g \in N$, clearly we have $f((\zeta,\mu_0,\mu_0),g) = \overline{\chi}(\mu_0,\mu_0)$ by $G_0(\chi,\mu_0,\mu_0) \equiv 0$. See Proposition 5.3 for the notations. Hence, for $g \in N$, (7.19) has the following form by (7.13)

$$g^{-1}e_{\zeta,\chi_0}^w = \psi_0^{-1}(g)e_{\zeta,\chi_0}^w.$$

(7.20)

Furthermore, by (7.14) and (7.18), we obtain

$$te_{\zeta,\chi_0}^w = w(t)\chi_0^{-1}(t)e_{\zeta,\chi_0}^w, \quad \sigma e_{\zeta,\chi_0}^w = \chi(\zeta(\sigma))\psi(\chi(\sigma))e_{\zeta,\chi_0}^w = w \circ \mathbb{a}(\sigma)e_{\zeta,\chi_0}^w$$

(7.21)

for $t \in \Gamma \subset \mathcal{O}_3^\times$ and $\sigma \in I_F$. Hence, $W_w$ is a $H$-subrepresentation of $W|_H$.

Now, by decomposing the $H$-representation $W|_H$ to a direct sum of irreducible components $\{W_w\}_{w \in \Gamma_{\text{stp}}}$, we have the following proposition.

**Proposition 7.3.** Let the notation be as above. Then, we have the followings

1. The following isomorphism as a $H$-representation holds

$$W_w \simeq \pi_w^\vee \bigotimes_{\chi_0 \in \mu_{q^2+1}\setminus\{1\}} \chi_0^{-1}(w) \otimes (w \circ \mathbb{a}(\sigma)).$$

(7.22)

2. We have the following isomorphism as a $H$-representation

$$W \simeq \bigoplus_{w \in \Gamma_{\text{stp}}} \left( \pi_w^\vee \bigotimes_{\chi_0 \in \mu_{q^2+1}\setminus\{1\}} \chi_0^{-1}(w) \otimes (w \circ \mathbb{a}(\sigma)) \right).$$
Proof. We prove the first assertion. We set $H_1 := \Gamma N \times \Gamma \times I_F \subset H$ and $W^\psi_w := \mathbb{C}[e_{\xi,\chi_0}] \subset W_w$. By (7.19) and (7.21), the stabilizer in $H$ of a subspace $W^\psi_{w_0}$ is equal to $H_1$. For $t \in \mu_{q^2-1}(O_E) \subset \Gamma \subset G_2^F$, we have $t e_{\overline{w},\chi_0} = \chi^{-1}(t) e_{w,\chi_0}$. Therefore, by (7.20), the subgroup $\Gamma N \subset G_2^F$ acts on $W^\psi_{w_0}$ via the character $w$ in (7.14). Hence, by (7.21), we acquire

$$W^\psi_{w_0} \simeq w^{-1} \otimes \chi_0^{-1} w \otimes (w \circ a_E)$$

as a $H_1$-representation. Since $G_2^F \hookrightarrow H$ permutes the subspaces $\{W^\psi_{w_0} \}_{\xi \in F_q^2 \setminus F_q}$ transitively by (7.19), we obtain the following isomorphism

$$W_w \simeq \bigoplus_{\chi_0 \in \mu_{q^2-1} \setminus \{1\}} \text{Ind}_{H_1}^H (w^{-1} \otimes \chi_0^{-1} w \otimes w \circ a_E) \simeq \bigoplus_{\chi_0 \in \mu_{q^2-1} \setminus \{1\}} \pi_w^\psi \otimes \bigoplus_{\chi_0 \in \mu_{q^2-1} \setminus \{1\}} \chi_0^{-1} w \otimes (w \circ a_E)$$

as a $H$-representation. Hence, the first assertion follows.

The second assertion follows from the first one and the following isomorphism

$$W|_H \simeq \bigoplus_{w \in \Gamma_{stp}} W_w$$

as a $H$-representation. Hence, the required assertions follow. \[\square\]

Let $w = (\chi, \psi) \in \Gamma_{stp}^\psi$. We define a $O_3^X$-representation

$$\rho_w := \text{Hom}_{G_2^F \times I_F} (\pi_w^\psi \otimes (w \circ a_E), W).$$

Then, we have

$$W \simeq \bigoplus_{w \in \Gamma_{stp}^\psi} \pi_w^\psi \otimes \rho_w \otimes (w \circ a_E) \quad (7.23)$$

as a $G$-representation.

By combining Proposition 7.3 with some fact in the representation theory of a finite group in [BH] Lemma 16.2, we understand $W$ as a $G$-representation by the following corollary.

**Corollary 7.4.** Let the notation be as above. Let $U := U_2^1/U_2^3 \subset \Gamma \subset O_3^X$. Note that $U \simeq F_q^\times$. The additive character $\psi \in F_q^\times \setminus F_q^\times$ is considered as a character of $U$. Then, we have the followings:

1. The $O_3^X$-representation $\rho_w$ is irreducible. Moreover, we have the following isomorphism

$$\rho_w|_\Gamma \simeq \bigoplus_{\chi_0 \in \mu_{q^2-1} \setminus \{1\}} \chi_0^{-1} w \quad (7.24)$$

as a $\Gamma$-representation.

2. We have the followings

$$\dim \rho_w = q, \quad \rho_w|_U \simeq \psi^\otimes q, \quad \text{Tr} \rho_w(\zeta) = -\chi(\overline{\zeta}) \quad (7.25)$$

for $\zeta \in \mu_{q^2-1}(O_E) \setminus \mu_{q^2-1}(O_F)$.

**Proof.** The isomorphism (7.24) follows from Proposition 7.3 and (7.25) follows from (7.24) immediately. Hence, it suffices to prove that $\rho_w$ is irreducible. Let $H := U_2^1/U_2^3 \subset O_3^X$ be a $p$-Sylow subgroup of order $q^4$. Then, we have $U \subset H$. By applying [BH] Lemma 16.2 to the situation $G = H/\ker \psi, N = U/\ker \psi$, we know that there exists a unique irreducible $H$-representation $\rho$ of degree $q$ such that $\rho|_U$ is a multiple of $\psi$. Hence, by $\dim \rho_w = q$, the $\rho_w$ must be isomorphic to $\rho$, and hence irreducible. Thereby, we have proved the required assertion. \[\square\]
7.2 Analysis of $\bigoplus_{(i,j,0)} H^1(W^{i,c}_{j,0}, \mathbb{Q}L)$

Recall that, in subsections 3.4 and 3.5, we have shown that the following curves

$$\{W^{i,c}_{j,0}\}_{(i,j,0)\in S_1}$$

with each having an affine model $a^q - a = s^2$, appear in the stable reduction of $X(\pi^2)$. In this subsection, we analyze the following cohomology group

$$W' := \bigoplus_{(i,j,0)\in S_1} H^1(W^{i,c}_{j,0}, \mathbb{Q}L).$$

(7.26)

Then, we understand $W'$ as a $G$-representation very explicitly in Corollary 7.9. The $G$-representation $W'$ is related to ramified representation of $GL_2(F)$ of normalized level $1/2$. See [7.3] for a precise statement.

Let $Y$ be the smooth compactification of an affine curve $Y_0 : a^q - a = s^2$. In the following, to analyze $W'$, we will show some elementary facts on $H^1(Y, \mathbb{Q}L)$ in Lemma 7.5. We have $\dim H^1(Y, \mathbb{Q}L) = q - 1$, because the genus of $Y$ is $(q - 1)/2$. The complement $Y\setminus Y_0$ consists of one point. Furthermore, we have $H^1(Y, \mathbb{Q}L) \cong H^1(Y_0, \mathbb{Q}L)$. The curve $Y_0$ is a finite Galois étale covering of an affine curve $F_q$ by $(a, s) \mapsto s$. Then, we consider $H^1(Y, \mathbb{Q}L)$ as a $\mathbb{Q}[F_q]$-module. Let $\alpha$ be an automorphism of $Y_0$ such that $(a, s) \mapsto (a + 1, s)$. On the other hand, a group $\mu_{2(q-1)} \ni b$ acts on $Y_0$ as follows $\beta_b : (a, s) \mapsto (b^a, bs)$. Note that the automorphism group of $Y_0$ is generated by $\alpha$ and $\beta_b$ for $b \in \mu_{2(q-1)}$. See also [CM2, Lemma 6.12] for the automorphism of the Artin-Schreier curve $a^q - a = s^2$. For $\psi \in F_q^\times$, let $L_\psi(s^2)$ be the smooth $\mathbb{Q}$-sheaf on $A^1$ defined by the covering $Y_0$ and $\psi$.

We introduce the following elementary lemma.

**Lemma 7.5.** Let the notation be as above.

1. Then, we have

$$H^1(Y, \mathbb{Q}L) \cong \bigoplus_{\psi \in F_q^\times \setminus \{0\}} \psi =: V$$

as a $\mathbb{Q}[F_q]$-module. We write $\{e_\psi\}_{\psi \in F_q^\times \setminus \{0\}}$ for the basis of $V$ above.

2. Let $b \in \mu_{2(q-1)}$. For a character $\psi \in F_q^\times$ and $x \in F_q^\times$, we write $\psi_x \in F_q^\times$ for a character $y \mapsto \psi(xy)$. Then, the automorphism $\beta_b$ of $Y_0$ induces the following action on $V$

$$\beta_b : e_\psi \mapsto c_{\psi,b} e_{\psi_{b^{-1}}}$$

with some constant $c_{\psi,b} \in \mathbb{Q}_l^\times$. Furthermore, we have $c_{\psi,-1} = 1$.

**Proof.** We have $H^1_c(Y_0, \mathbb{Q}L) = \bigoplus_{\psi \in F_q^\times \setminus \{0\}} H^1_c(A^1, L_\psi(s^2))$ as a $\mathbb{Q}[F_q]$-module. By the Grothendieck-Ogg-Shafarevich formula, we have $\dim H^1_c(A^1, L_\psi(s^2)) = 1$ and $H^1_c(A^1, L_\psi(s^2)) \cong \psi$ as a $\mathbb{Q}[F_q]$-module. Hence, the first assertion follows.

The second assertion follows from $\beta_b\alpha\beta_b^{-1} = \alpha^b$ for any $b \in \mu_{2(q-1)}$. The assertion $c_{\psi,-1} = 1$ follows from the Lefschetz trace formula. Hence, we have proved the required assertions.

By $\dim H^1(W^{i,c}_{j,0}, \mathbb{Q}L) = q - 1$, we have $\dim W' = 2q(q-1)(q^2-1)^2$. We set

$$\mathcal{S} := S_1 \times (F_q^\times \setminus \{0\}) = F_q^\times \times F_q^\times \times F_q^\times \times \mu_{2(q-1)} \times (F_q^\times \setminus \{0\})$$

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Then, by Lemma 7.1 and the identification $(\mathcal{O}_F/\pi^2)\times \cong \mathbb{F}_q^\times \times \mathbb{F}_q$, we have the following isomorphism
\[
W' \cong \bigoplus_{((\zeta,\tilde{\mu}),j,y_0,\psi) \in T} \mathbb{Q}_l e_{(\zeta,\tilde{\mu}),j,y_0,\psi} \tag{7.27}
\]
as a $\mathbb{Q}_l$-vector space. In the following, by the $G$-action on $W'$, we consider the right hand side in (7.27) as a $G$-representation.

In the following, we define $G$-subrepresentations $W^w_a \subset W'$ for $a \in \mathbb{F}_q^\times$ and $w \in ((\mathcal{O}_F/\pi^2)\times)^\vee$, and investigate a shape of $W^w_a$ as a $G$-representation in Proposition 7.8. As a result, we understand $W'$ very explicitly in Corollary 7.9.

Let $w = (w_1, w_2) \in ((\mathcal{O}_F/\pi^2)\times)^\vee \cong (\mathbb{F}_q^\times)^\vee \times \mathbb{F}_q^\vee$, $\psi \in \mathbb{F}_q^\vee \setminus \{0\}$, $\zeta \in \mathbb{F}_q^\times$, $j \in \mathbb{P}^1(\mathbb{F}_q)$ and $y_0 \in \mu_2(q^2-1)$. We define a vector of $W'$ under the isomorphism (7.27)
\[
e_{\zeta,y_0,j,\psi}^w := \sum_{(\mu,\tilde{\mu}) \in \mathbb{F}_q^\times \times \mathbb{F}_q} w_1^{-1}(\mu)w_2^{-1}(\tilde{\mu})e_{(\mu^2\zeta,\tilde{\mu}),j,\mu y_0,\psi}.
\]
Then, clearly we have
\[
e_{\mu^2\zeta,\mu y_0,j,\psi}^w = w_1(\mu_1)e_{\zeta,y_0,j,\psi}^w \tag{7.28}
\]
for any $\mu_1 \in \mathbb{F}_q^\times$.

We consider a set $\mathbb{F}_q^\times \times \mu_2(q^2-1)$ and the following equivalence relation on the set:
\[
(\zeta,y_0) \sim (\zeta',y_0') \iff (\mu^2\zeta,\mu y_0) = (\zeta',y_0')
\]
for some $\mu \in \mathbb{F}_q^\times$. Let $\mathcal{U} := (\mathbb{F}_q^\times \times \mu_2(q^2-1))/\sim$. Then, we have $|\mathcal{U}| = 2(q^2 - 1)$. We write $[(\zeta,y_0)] \in \mathcal{U}$. For each $a \in \mathbb{F}_q^\times$, we set
\[
\mathcal{K}_a := \{[(\zeta,y_0)] \in \mathcal{U} \mid a = \zeta^2 y_0^{-2(q+1)}\} \subset \mathcal{U}.
\]
Then, we have $|\mathcal{K}_a| = 2(q+1)$. For $w \in ((\mathcal{O}_F/\pi^2)\times)^\vee$, $[(\zeta,y_0)] \in \mathcal{U}$, $j \in \mathbb{P}^1(\mathbb{F}_q)$ and $\psi \in \mathbb{F}_q^\vee \setminus \{0\}$, we define $W^w_{[(\zeta,y_0)],j,\psi} := \bigoplus_{\mathbb{Q}_l e_{(\zeta,y_0),j,\psi}^w} \subset W'$. This one-dimensional $\mathbb{Q}_l$-vector subspace depends on $(w,[(\zeta,y_0)],j,\psi)$ by (7.28).

We fix a character $\psi \in \mathbb{F}_q^\vee \setminus \{0\}$. We set $\mathcal{T}_1 := \mathbb{F}_q^\times \times ((\mathcal{O}_F/\pi^2)\times)^\vee$. Obviously, we have $|\mathcal{T}_1| = q(q-1)^2$. Let $(a,w) \in \mathcal{T}_1$. We write $w = (w_1, w_2) \in ((\mathcal{O}_F/\pi^2)\times)^\vee \cong (\mathbb{F}_q^\times)^\vee \times \mathbb{F}_q^\vee$. Then, we define a $G$-subrepresentation of $W'$ as follows
\[
W^w_a := \bigoplus_{[(\zeta,y_0)] \in \mathcal{K}_a} \bigoplus_{j \in \mathbb{P}^1(\mathbb{F}_q)} \bigoplus_{\zeta \in \mathbb{F}_q^\times} W^w_{[(\zeta,y_0)],j,\psi} \subset W'. \tag{7.29}
\]
Then, we have $\text{dim} \ W^w_a = 2(q+1)(q^2-1)$. Then, by (7.27), we easily check the following isomorphism
\[
W' \cong \bigoplus_{(a,w) \in \mathcal{T}_1} W^w_a \tag{7.30}
\]
as a $\mathbb{Q}_l$-vector space. By the action of $G$ on $W'$, we consider the right hand side of (7.30) as a $G$-representation. Then, the subspace $W^w_a$ is $G$-stable for each $(a,w) \in \mathcal{T}_1$. In the following, we will explicitly write down the action of $G$ on $W^w_a$. To do so, we prepare some notations.

We fix elements $(a,(w_1, w_2)) \in \mathcal{T}_1$ and $[(\zeta, y_0)] \in \mathcal{K}_a$. Let $\tilde{\zeta}, \tilde{\mu} \in \mu_{q-1}(\mathcal{O}_F)$ be the unique liftings of $\zeta, \mu \in \mathbb{F}_q^\times$ respectively. Let $E/F$ be the unramified quadratic extension as before. Let
\[
\alpha := \begin{pmatrix} 0 & 1 \\ \pi \tilde{\mu} & 0 \end{pmatrix} \in \text{GL}_2(F). \tag{7.31}
\]
Of course, we have $\alpha^2 = \pi \hat{u}$. We set $E_1 := F(\alpha) \subset \text{GL}_2(F)$. Then, $E_1$ is a quadratic ramified extension of $F$. We choose an embedding

$$E_1 \hookrightarrow D : a_0 + b_0 \alpha \mapsto a_0 + b_0 \alpha'$$

for $a_0, b_0 \in F$. Note that we have $\alpha' / \varphi \equiv \zeta / y_0^2 \pmod{\varphi}$. Let

$$M_2(F) \supset \mathfrak{U} := \left( \begin{array}{cc} \mathcal{O}_F & \mathcal{O}_F \\ \mathfrak{p}_F & \mathfrak{p}_F \end{array} \right) \supset \mathfrak{B}_\mathfrak{U} := \left( \begin{array}{cc} \mathfrak{p}_F & \mathcal{O}_F \\ \mathfrak{p}_F & \mathfrak{p}_F \end{array} \right), \ U^n_\mathfrak{U} := 1 + \mathfrak{B}_\mathfrak{U} \subset \text{GL}_2(\mathcal{O}_F)$$

for $n \geq 1$. Note that $\mathfrak{U} \subset M_2(F)$ is a chain order and $\mathfrak{B}_\mathfrak{U}$ is the Jacobson radical of the order. See [BH, Section 12] for more details. Note that $U^n_\mathfrak{U}$ is a compact open subgroup of $\text{GL}_2(F)$. The image of $\mathfrak{U}$ and $U^n_\mathfrak{U}$ under $\text{GL}_2(\mathcal{O}_F) \rightarrow \mathcal{G}_2^F$ is denoted by $\overline{\mathfrak{U}}$ and $\overline{U^n_\mathfrak{U}}$ respectively. Similarly, we denote by $\overline{\mathfrak{O}}_{\mathfrak{E}_1}$ the image of $\mathcal{O}_{\mathfrak{E}_1} \subset \text{GL}_2(\mathcal{O}_F)$ under $\text{GL}_2(\mathcal{O}_F) \rightarrow \mathcal{G}_2^F$. The subgroup $\overline{\mathfrak{O}}_{\mathfrak{E}_1} \overline{\mathfrak{U}}^{\mathfrak{U}} \subset \overline{\mathfrak{U}}$ is a normal subgroup and its index is equal to $q - 1$. Thereby, we have $[\mathcal{G}_2^F : \overline{\mathfrak{O}}_{\mathfrak{E}_1} \overline{\mathfrak{U}}^{\mathfrak{U}}] = q^2 - 1$. Similarly, let $\overline{\mathfrak{O}}_{\mathfrak{D}_1} \subset \mathcal{O}_\mathfrak{D}_1$ be the images of $\mathcal{O}_{\mathfrak{D}_1} \hookrightarrow \mathcal{O}_\mathfrak{D}_1$ and $U^n_\mathfrak{D} \subset \mathcal{O}_\mathfrak{D}_1$ under the canonical map $\mathcal{O}_{\mathfrak{D}_1} \rightarrow \mathcal{O}_\mathfrak{D}_1$ respectively. Then, we have $[\mathcal{O}_\mathfrak{D}_1 : \overline{\mathfrak{O}}_{\mathfrak{E}_1} \overline{\mathfrak{U}}^{\mathfrak{U}}] = q + 1$.

Now, we describe the $\mathbf{G}$-action on $\mathcal{W}_a^w$.

$\mathcal{G}_2^F$-action First, consider the action of $\mathcal{G}_2^F$. Let $g = \left( \begin{array}{cc} a_0 + a_1 \pi & b_0 + b_1 \pi \\ c_1 \pi & d_0 + d_1 \pi \end{array} \right) \in \overline{\mathfrak{U}}$ with $a_i, b_i, c_i, d_i \in \mu_{q-1}(\mathcal{O}_F) \cup \{0\}$ ($i = 0, 1$). Let $\tilde{w}_2 : G_2^F \rightarrow \overline{\mathfrak{U}}^{\mathfrak{U}}$ be a character defined by the composite of $\det : G_2^F \rightarrow (\mathcal{O}_F / \pi^2)^\times$ and $(\mathcal{O}_F / \pi^2)^\times \simeq \mathbb{F}_q^{\times} \times \mathbb{F}_q^{\times} \rightarrow \mathbb{F}_q^{\times} \rightarrow \overline{\mathfrak{U}}^{\mathfrak{U}}$. Let $b \in \mu_2(q^2 - 1)$ be an element such that $b^2 = a_0 / d_0$. Let $[(\zeta', y_0')] \in \mathcal{K}_a$. Then, $g^{-1}$ acts on $\mathcal{W}_a^w$ as follows by Proposition [7.5] and Lemma [7.3](2)

$$g^{-1} : e_{\zeta', y_0', \zeta'_{-1}} \mapsto c_{\zeta'_{-1}} \cdot w_1((a_0d_0)\tilde{w}_2(g)(\zeta_1a_0^{-1} + b_0(\tilde{a}_0^{-1}\zeta_1) - d_0))e_{\zeta_1, y_0, \zeta'_{-1}}.$$  

By (7.32), the quotient $\overline{\mathfrak{U}} / \overline{\mathfrak{O}}_{\mathfrak{E}_1} \overline{\mathfrak{U}}^{\mathfrak{U}}$ acts on the index set $\zeta_1 \in \mathbb{F}_q^{\times}$ of $\mathcal{W}_a^w$ simply transitively. In particular, let $g = \left( \begin{array}{cc} a_0 + a_1 \pi & b_0 + b_1 \pi \\ c_1 \pi & d_0 + d_1 \pi \end{array} \right) \in \overline{\mathfrak{O}}_{\mathfrak{E}_1} \overline{\mathfrak{U}}^{\mathfrak{U}} \subset \overline{\mathfrak{U}}$. Then, $g^{-1}$ acts on $\mathcal{W}_a^w$ as follows by (7.32) and Lemma [7.3](2)

$$g^{-1} : e_{\zeta', y_0, \zeta'_{-1}} \mapsto w_1((a_0)\tilde{w}_2(g)(\zeta_1a_0^{-1}(\tilde{a}_0^{-1}\zeta_1 + b_0a_0^{-1}))e_{\zeta', y_0, \zeta'_{-1}}.$$  

Thereby, the restriction $\mathcal{W}_a^w \mid \overline{\mathfrak{O}}_{\mathfrak{E}_1} \overline{\mathfrak{U}}^{\mathfrak{U}} \times \{1\} \times \{1\}$ is a direct sum of characters. Let us define characters

$$\Lambda_{w_1, a} : \overline{\mathfrak{O}}_{\mathfrak{E}_1} \overline{\mathfrak{U}}^{\mathfrak{U}} \rightarrow \mathbb{Q}^{\times}_l : g \mapsto w_1(a_0)(\tilde{a}_0^{-1}(\tilde{a}_0^{-1} + b_0a_0^{-1}))$$

and

$$\tilde{w}_2 \Lambda_{w, a} : \overline{\mathfrak{O}}_{\mathfrak{E}_1} \overline{\mathfrak{U}}^{\mathfrak{U}} \rightarrow \mathbb{Q}^{\times}_l : g \mapsto \Lambda_{w_1, a}(g)\tilde{w}_2(g).$$

$\mathcal{O}_\mathfrak{D}_1$-action Secondly, consider the action of $\mathcal{O}_\mathfrak{D}_1$ on $\mathcal{W}_a^w$. Let $b := a_0 + \varphi b_0 + \pi a_1 \in \mathcal{O}_\mathfrak{D}_1$ with $a_0 \in \mu_{q-1}(\mathcal{O}_F)$ and $a_1, b_0 \in \mu_{q-2}(\mathcal{O}_F) \cup \{0\}$. Let $\tilde{w}_2^D : \mathcal{O}_\mathfrak{D}_1 \rightarrow \overline{\mathfrak{U}}^{\mathfrak{U}}$ be the composite of $\text{Nrd}_{\mathfrak{D}_1/F} : \mathcal{O}_\mathfrak{D}_1 \rightarrow (\mathcal{O}_F / \pi^2)^\times$ and $(\mathcal{O}_F / \pi^2)^\times \rightarrow \mathbb{F}_q \rightarrow \overline{\mathfrak{U}}^{\mathfrak{U}}$. Then, $b$ acts on $\mathcal{W}_a^w$ as follows by Proposition 1.6

$$b : e_{\zeta', y_0, \zeta'_{-1}} \mapsto \tilde{w}_2^D(b)(\zeta_1)(\text{Tr}_{\mathfrak{D}_1/F}((\tilde{b}_0a_0^{-1}y_0^{-2g}))\zeta_1')e_{\zeta', y_0, \zeta'_{-1}}.$$  

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In particular, if \( b = a_0 + \varphi b_2 + \pi a_1 \in \overline{\mathbb{O}}_{E_1}^{m} \), then \( b \) acts on \( W^w_a \) as follows by (7.28):
\[
b : e^w_{\zeta_1 \zeta_0, y_0/j, \psi_{\zeta_1-1}} \mapsto w_1(\tilde{a}_0)\tilde{w}^D_2(b)\psi_{\zeta_1-1}(\text{Tr}_{F/E}(\tilde{b}_0\tilde{a}_0^{-1}y_0^{-2q}))\zeta'_{\zeta_1}e^w_{\zeta_1 \zeta_0, y_0/j, \psi_{\zeta_1-1}}.
\]
(7.36)
Hence, \( W^w_a \mid_{\{1\} \times \overline{\mathbb{O}}_{E_1}^{m} \times \{1\}} \) is a direct sum of characters. Let us define characters
\[
\Lambda^D_{w_1,a} : \overline{\mathbb{O}}_{E_1}^{m} \rightarrow \mathbb{O}^\times_1 : b \mapsto w_1(\tilde{a}_0)\psi(\text{Tr}_{F/E}(\tilde{b}_0\tilde{a}_0^{-1}y_0^{-2q}))
\]
(7.37)
and
\[
\tilde{w}_2^D \Lambda^D_{w_1,a} : \overline{\mathbb{O}}_{E_1}^{m} \rightarrow \mathbb{O}^\times_1 : b \mapsto \tilde{w}_2^D(b)\Lambda^D_{w_1,a}(b).
\]
(7.38)

**Inertial action** Finally, we consider the action of inertia on \( W^w_a \). Recall the following map in (6.7)
\[
a_{E_1} : I^b_{E_1} \rightarrow (\mathbb{O}E_1/\pi)^\times \simeq \mathbb{F}_q^\times \times \mathbb{F}_q : \sigma \mapsto (\zeta(\sigma), \lambda(\sigma)).
\]
Let \( \tilde{w}_2 : (\mathbb{O}E_1/\pi)^\times \rightarrow \mathbb{O}^\times_1 \) be the composite \((\mathbb{O}E_1/\pi)^\times \rightarrow \mathbb{O}^\times_1 \rightarrow (\mathbb{O}E_1/\pi)^\times \rightarrow \mathbb{O}^\times_1 \). We write \( \tilde{w}_2 \circ a_{E_1} \) for the composite of \( I^b_{E_1} \rightarrow (\mathbb{O}E_1/\pi)^\times \) and \( \tilde{w}_2 \). Let
\[
a_{E} : I^b_{E} \rightarrow (\mathbb{O}E/\pi)^\times \simeq \mathbb{F}_q^\times \times \mathbb{F}_q
\]
\[
\sigma \mapsto (\zeta(\sigma), \lambda(\sigma))
\]
be the reciprocity map in (6.2). Let \( \sigma \in I_F \) and \( \kappa \) an element such that \( \kappa^{2q(q-1)} = \pi \). We write \( \sigma(\kappa) = \zeta(\sigma) \kappa \) and \( \zeta(\sigma) \in \mu_{2q(q-1)} \). Let the notation be as in Lemma 6.3. Then, \( \sigma \) acts on \( W^w_a \) as follows by Lemma 6.3
\[
\sigma : e^w_{\zeta_1 \zeta_0, y_0/j, \psi_{\zeta_1-1}} \mapsto w_1(\zeta_0(\sigma))\psi_{\zeta_1-1}(a_0b_0)c_{\psi_{\zeta_1-1}},c_0e^w_{\zeta_1 \zeta_0, y_0/j, \psi_{\zeta_1-1}}.
\]
Note that we have \( \zeta_1(\sigma)^2 = \zeta_0(\sigma) \). If \( \sigma \in I_{E_1} \), then, we have \( \zeta(\sigma) = \zeta_1(\sigma) \) in \( \mathbb{F}_q^\times \) and \( c_0 \in \{1, -1\} \). Hence, in particular, \( \sigma \in I_{E_1} \) acts on \( W^w_a \) as follows by Corollary 6.3 Lemma 7.5 and (7.28)
\[
\sigma : e^w_{\zeta_1 \zeta_0, y_0/j, \psi_{\zeta_1-1}} \mapsto w_1(\zeta_0(\sigma))\zeta_0^{1/2} \circ a_{E_1}(\sigma) \psi_{\zeta_1^{-1}}(-y_0^{-2}a_0a_0^{-1}b_0^{-1}c_0\lambda(\sigma)^2)e^w_{\zeta_1 \zeta_0, y_0/j, \psi_{\zeta_1^{-1}}}
\]
(7.39)
Note that we have \( y_0^{-2} \in \{\pm 1\} \). Let us define characters
\[
\Lambda'_{w_1,a} : (\mathbb{O}E_1/\pi)^\times \rightarrow \mathbb{O}^\times_1 : \zeta_0 + \lambda_0 \alpha \mapsto w_1(\zeta_0)\psi(2a \lambda_0 \alpha)
\]
(7.40)
with \( \zeta_0 \in \mu_{q-1}(\mathbb{O}E_1), \lambda_0 \in \mu_{q-1}(\mathbb{O}E_1) \cup \{0\} \) and
\[
(w_2^D \Lambda'_{w_1,a}) \circ a_{E_1} : I^b_{E_1} \rightarrow \mathbb{O}^\times_1 : \tilde{w}_2^D \circ a_{E_1}(\sigma) \Lambda_{w_1,a} \circ a_{E_1}(\sigma)
\]
(7.41)

**Lemma 7.6.** Let \( \Lambda_{w_1,a}, \Lambda_{w_1,a}^D \) and \( \Lambda'_{w_1,a} \) be the characters defined in (7.37), (7.37) and (7.40) respectively. Then, we have the followings;
1. The restriction \( \Lambda_1 \) of \( \Lambda_{w_1,a} \) to a subgroup \( \overline{\mathbb{O}}_{E_1}^m \) and the restriction \( \Lambda_2 \) of \( \Lambda_{w_1,a}^D \) to a subgroup \( \overline{\mathbb{O}}_{E_1}^m \) factor through \( (\mathbb{O}E_1/\pi)^\times \). Moreover, we have \( \Lambda_1 = \Lambda_2 = \Lambda_{w_1,a} \).
2. The restriction of the character \( \Lambda_{w_1,a} \) to a subgroup \( \overline{\mathbb{O}}_{1}^{m} \) is given by \( g \mapsto \psi(\text{Tr}((\frac{a}{\pi}(g - 1))) \)), where \( \text{Tr} \) is the composite \( \mathbb{O}^\times_1 \rightarrow \mathbb{O}E_1/\pi^2 \rightarrow \mathbb{F}_q \). Similarly, the restriction of the character \( \Lambda_{w_1,a}^D \) to a subgroup \( \overline{\mathbb{O}}_{1}^{m} \) is given by \( b \mapsto \psi(\text{Tr}d_{D/F}(\frac{a}{\pi}(b - 1))) \), where \( \text{Tr}d_{D/F} \) is the composite \( \mathbb{O}^\times_1 \rightarrow \mathbb{O}E_1/\pi^2 \rightarrow \mathbb{F}_q \). The restriction of \( \Lambda'_{w_1,a} \) to a subgroup \( x \in \overline{\mathbb{U}}^1_{E_1} \) is given by \( \psi \circ \text{Tr}_{E_1/F}(x) \)).
Proof. The required assertions are checked by direct computations. □

Remark 7.7. For the meaning of the above lemma, see [BH] 19.2 and 19.3 and [BH] 56.5.

Proposition 7.8. Let the notation be as in (7.29), (7.35), (7.38) and (7.41). We set
\[ \pi_{w,a} := \text{Ind}_{E_1}^{G_2} (\tilde{w}_2 \Lambda_{w_1,a}), \quad \rho_{w,a} := \text{Ind}_{E_1}^{G_2} (\tilde{w}_2 D_{\Lambda_{w_1,a}}), \quad \pi'_{w,a} := \text{Ind}_{E_1/F} ((\tilde{w}_2' \Lambda_{w_1,a}) \circ a_{E_1}). \]

1. Then, \( \pi_{w,a} \) and \( \pi'_{w,a} \) are irreducible. On the other hand, \( \rho_{w,a} \) is not irreducible. We have \( \dim \pi_{w,a} = q^2 - 1 \), \( \dim \rho_{w,a} = q + 1 \) and \( \dim \pi'_{w,a} = 2 \).

2. The following isomorphism as a \( G \)-representation holds
\[ W^w_a \cong \pi_{w,a} \otimes \rho_{w,a} \otimes \pi'_{w,a}. \]

Proof. We set \( \xi := (\tilde{w}_2' \Lambda'_{w_1,a}) \circ a_{E_1}. \) Let \( \tau \neq 1 \in \text{Gal}(E_1/F) \). Then, we easily check \( \xi^\tau \neq \xi. \) Hence, \( \pi'_{w,a} \) is irreducible. By Mackey’s irreducibility criterion in [Sc] Proposition 7.23, we check that \( \pi_{w,a} \) is irreducible.

We show that \( \rho_{w,a} \) is not irreducible. We consider the character (7.38). For simplicity, we set \( \tilde{w} := \tilde{w}_2 D_{\Lambda_{w_1,a}}. \) The group \( \mathcal{O}^\times_E \) is contained in the stabilizer of \( \mathcal{O}^\times_{E_1} \mathcal{U}_D \) in \( \mathcal{O}_E^\times. \) We choose an element \( s \in \mathcal{O}^\times_E \backslash \mathcal{O}^\times_{E_1} \mathcal{U}_D. \) Then, we have \( \tilde{w}^s(x) := \tilde{w}(s^{-1}xs) = \tilde{w}(x) \) for \( x \in \mathcal{O}^\times_{E_1} \mathcal{U}_D. \) Hence, again by Mackey’s irreducibility criterion, \( \rho_{w,a} \) is not irreducible.

We prove the second assertion. We consider the subspace \( \tilde{W} := W^w_a \). The stabilizer of \( \tilde{W} \) in \( G \) is equal to \( H_1 := \mathcal{O}^\times_{E_1} \mathcal{U}_D \times \mathcal{O}^\times_{E_1} \mathcal{U}_D \times I_{E_1}. \) Furthermore, we have an isomorphism by (7.33), (7.36) and (7.39)
\[ \tilde{W} \cong (\tilde{w}_2 \Lambda_{w_1,a}) \otimes (\tilde{w}_2 D_{\Lambda_{w_1,a}}) \otimes ((\tilde{w}_2' \Lambda'_{w_1,a}) \tau \circ a_{E_1}) \]
as a \( H_1 \)-representation. On the other hand, by (7.19), we easily check that \( W^w_a \) is a direct sum
\[ \bigoplus_{(\xi,y) \in K_a} \bigoplus_{j \in \mathbb{P}^1(F_\xi)} \bigoplus_{\psi \in \mathcal{P}_e} W^w_{[(\xi,y),j,\psi]} \]
of subspaces permuted transitively by the action of \( G. \) Hence, the required assertion follows. □

Corollary 7.9. Let the notation be as in Proposition 7.8. Then, we have the following isomorphism
\[ W' \cong \bigoplus_{(w,a) \in T_1} \pi^w_{w,a} \otimes \rho_{w,a} \otimes \pi'_{w,a} \]
as a \( G \)-representation.

Proof. This follows from Proposition 7.8 and (7.30) immediately. □

7.3 Conclusion

As mentioned in the introduction, we investigate a relationship between our \( G \)-representations \( W \) and \( W' \), and the local Jacquet-Langlands correspondence and the local Langlands correspondence. In the following, we often quote several facts from the book [BH]. First, we briefly recall
the \(\ell\)-adic local Langlands correspondence and the local Jacquet-Langlands correspondence for \(\text{GL}_2\). For example, see \[BH\] Sections 34 and 35.

Let \(\mathcal{G}_2(F,\overline{\mathbb{Q}}_l)\) denote the set of equivalence classes of 2-dimensional, semisimple, Deligne representations of the Weil group \(W_F\), over \(\overline{\mathbb{Q}}_l\). See \[BH\] p.200 and p.221 for more details. Let \(\mathcal{A}_2(F,\overline{\mathbb{Q}}_l)\) denote the set of equivalence classes of irreducible smooth representations of \(\text{GL}_2(F)\) over \(\overline{\mathbb{Q}}_l\). See \[BH\] p.212 for more details.

We recall the \(\ell\)-adic local Langlands correspondence in \[BH\] p.222 and p.223. There is a unique bijection

\[
\text{LL}_\ell : \mathcal{G}_2(F,\overline{\mathbb{Q}}_l) \rightarrow \mathcal{A}_2(F,\overline{\mathbb{Q}}_l)
\]

which commutes with automorphisms of \(\overline{\mathbb{Q}}_l\). Furthermore, for any isomorphism \(\iota : \overline{\mathbb{Q}}_l \simeq \mathbb{C}\), the correspondence \(\text{LL}_\ell\) satisfies

\[
L(\chi \text{LL}_\ell(\sigma)^\iota, s) = L(\chi \sigma^\iota, s - \frac{1}{2})
\]

\[
\epsilon(\chi \text{LL}_\ell(\sigma)^\iota, s, \psi) = \epsilon(\chi \sigma^\iota, s - \frac{1}{2}, \psi)
\]

for all \(\sigma \in \mathcal{G}_2(F,\overline{\mathbb{Q}}_l)\), all \(\chi \in (F^\times)^\vee\) and all \(\psi \in F^\vee\). See \[BH\] Section 6 for L-function and local constants.

Let \(\mathcal{A}_2^\nu(F,\overline{\mathbb{Q}}_l)\) denote the set of equivalence classes of irreducible smooth representations of \(\text{GL}_2(F)\) which are essentially square-integrable, over \(\overline{\mathbb{Q}}_l\). See \[BH\] 17.4 and 56.1. We write \(\mathcal{A}_1(D,\overline{\mathbb{Q}}_l)\) for the set of equivalence classes of irreducible smooth representations of \(D^\times\) over \(\overline{\mathbb{Q}}_l\). The local Jacquet-Langlands correspondence is a bijection

\[
\text{JL} : \mathcal{A}_2^\nu(F,\overline{\mathbb{Q}}_l) \rightarrow \mathcal{A}_1(D,\overline{\mathbb{Q}}_l)
\]

satisfying the appropriate trace identity. See \[BH\] p.334 and 56.9 and [?] for more details.

**unramified case**: We consider the \(G\)-representation \(W\) in subsection 7.3.1. As in (7.23), we have the following isomorphism

\[
W \simeq \bigoplus_{w \in F^\times_{\text{stp}}} \pi_w^\vee \otimes \rho_w \otimes \chi \circ a_E
\]

as a \(G\)-representation.

Let \(\chi : \mathbb{Z} \rightarrow \overline{\mathbb{Q}}_l^\times\) be a character. In the following, we identify \(F^\times \simeq \mathbb{Z} \otimes \mathcal{O}_E^\times\) by \(x = \pi^{\pi(x)} u \mapsto (v(x), u)\). We also identify \(E^\times \simeq \mathbb{Z} \otimes \mathcal{O}_E^\times\) in the same manner as \(F^\times\). Then, we define an irreducible and cuspidal representation \(\pi_{w,\chi}\) of \(\text{GL}_2(F)\) of level 1 in the following. The level means the normalized level in \[BH\] p.91. First, we also write \(\pi_w\) for the inflation to \(\text{GL}_2(\mathcal{O}_F)\) of the \(G_2^1\)-representation \(\pi_w\). We extend \(\pi_w\) to a representation of \(F^\times \text{GL}_2(\mathcal{O}_F)\) by using \(\chi\), which we denote by \(\pi_w \otimes \chi\). Then, we define \(\pi_{w,\chi} := c - \text{Ind}_{\text{GL}_2(F)}^{\text{GL}_2(\mathcal{O}_F)}(\pi_w \otimes \chi)\). Then, \(\pi_{w,\chi}\) is an irreducible and cuspidal representation of \(\text{GL}_2(F)\) as in \[AOPS\] 5.4. We are able to construct \(\pi_{w,\chi}\) in another way. Recall that we fixed the embedding \(\Gamma \rightarrow G_2^F\) in (7.12). Let \(U_{\mathfrak{m}}\) be a compact open normal subgroup \(1 + \pi^{\nu} \mathfrak{m}_2(\mathcal{O}_F) \subset \text{GL}_2(\mathcal{O}_F)\). We inflate a character \(w\) (7.11) of \(\Gamma N / \mathcal{O}_E U_{\mathfrak{m}} \subset \text{GL}_2(F)\), for which we write \(\hat{w}\). Then, we extend \(\hat{w}\) to a character \(\hat{w}_\chi\) of \(J_{\mathfrak{m}}^1 := E^\times U_{\mathfrak{m}}^1\) by using \(\chi\). We consider \(\pi_{w,\chi}^0 := c - \text{Ind}_{\mathfrak{m}}^{\text{GL}_2(F)}(\hat{w}_\chi)\). We easily check that \(\pi_{w,\chi}^0\) is isomorphic to \(\pi_{w,\chi}\). The fact that the representation \(\pi_{w,\chi}^0 \simeq \pi_{w,\chi}\) is an irreducible and cuspidal representation is also verified by \[BH\] Theorem 15.3. It is not difficult to check that
the representation \( \pi_{w,\chi} \) contains an unramified simple stratum. This follows from Definition ?? almost immediately. Hence, by [BH] Lemma 20.3, the representation \( \pi_{w,\chi} \) is an unramified irreducible cuspidal representation in a sense of [BH] 20.1. (i.e. there exists an unramified character \( \phi \neq 1 \) of \( F^\times \) such that \( \pi_{w,\chi} \phi \cong \pi_{w,\chi} \).

Secondly, we define a smooth representation \( \rho_{w,\chi} \) of \( D^\times \) in the following. We have the reduction map \( \mathcal{O}_D^\times \to \mathcal{O}_3^\times \). If this map is restricted to a subgroup \( \mathcal{O}_E^\times U_D^1 \), it induces a surjection \( \mathcal{O}_E^\times U_D^1 \to \mathcal{O}_3^\times \) First, we write \( \rho_w \) for the inflation to \( \mathcal{O}_E^\times U_D^1 \) of \( \rho_w \). Then, we extend \( \rho_w \) to a representation of \( J_D^1 := E^\times U_D^1 \) by using \( \chi \), which we denote by \( \rho_w \otimes \chi \). Then, we define a \( D^\times \)-representation \( \rho_{w,\chi} \) by

\[
\rho_{w,\chi} := c - \text{Ind}_{J_D^1}^{\mathcal{O}_D^\times} (\rho_w \otimes \chi).
\]

Thirdly, we define a 2-dimensional representation \( \pi'_{w,\chi} \) of \( W_F \) in the following. Let \( w \) be a character of \( E^\times \) defined by \( \chi \) and \( w \). Moreover, let \( \Delta \) be the unramified character of \( E^\times \) of order 2 as in [BH] 34.4. We write \( v_F : W_F \to \mathbb{Z} \) for the canonical map taking a geometric Frobenius element to 1. We set \( ||x|| := q^{-v_F(x)} \) for \( x \in W_F \). We define \( \pi'_{w,\chi} \) by

\[
\pi'_{w,\chi} := || \cdot ||^{-\frac{1}{2}} \text{Ind}_{F/F} (\Delta^w \circ a_E).
\]

See [BH] p.219 and p.222 for this normalization.

**ramified case:** We consider the \( G \)-representation \( W' \) in subsection [7.2]. Now, we recall the following isomorphism

\[
W' \cong \bigoplus_{(w,a) \in \mathcal{T}_1} \pi^\vee_{w,a} \otimes \rho_{w,a} \otimes \pi'_{w,a}
\]

as a \( G \)-representation in Corollary [7.9]

As in the unramified case, we define representations of \( \text{GL}_2(F) \), \( D^\times \) and \( W_F \) one by one. We choose a character \( Z \to \overline{Q}_l^\times \). First, we define a ramified cuspidal representation of \( \text{GL}_2(F) \) of level 1/2. We inflate the character \( \tilde{w}_2 : \mathbb{O}_2^F \to \overline{Q}_l^\times \) to \( \text{GL}_2(\mathcal{O}_F) \), and extend it to a character of \( \text{GL}_2(F) \) by using \( \chi \), which we denote by \( w_2,\chi \). We inflate the character \( \Lambda_{w,1,a} : \overline{\mathcal{O}}_E^\times U_{E_1}^1 \to \overline{Q}_l^\times \) to \( \mathcal{O}_E^\times U_{E_1}^1 \subset \text{GL}_2(\mathcal{O}_F) \), which we denote by the same letter \( \Lambda_{w,1,a} \). Then, we extend it to a character of \( J_{E_1}^1 := E^\times U_{E_1}^1 \) by using \( \chi \), which we denote by \( \Lambda_{w,1,a,\chi} \). We set as follows

\[
\pi_{w,1,a,\chi} := c - \text{Ind}_{J_{E_1}^1}^{\text{GL}_2(F)} \Lambda_{w,1,a,\chi}.
\]

Recall the definition of the element \( \alpha \in \mathfrak{U} \) in [7.3]. Since \( \frac{\pi}{\pi} \in E_1 \) is minimal in a sense of [BH] Definition 13.4], a triple \( (\mathfrak{U},1,\hat{\varphi}) \) is a ramified simple stratum by [BH] Proposition 13.5.] See [BH] p.96 for a definition of ramified simple stratum. Since \( \pi_{w,1,a,\chi} \) contains \( (\mathfrak{U},1,\hat{\varphi}) \), \( \pi_{w,1,a,\chi} \) is an irreducible and cuspidal representation of \( \text{GL}_2(F) \) of level 1/2. Furthermore, \( \pi_{w,1,a,\chi} \) is minimal i.e. \( l(\pi_{w,1,a,\chi}) \leq l(\pi_{w,1,a,\chi} \phi') \) for any character \( \phi' \) of \( F^\times \), where \( l(\cdot) \) means the normalized level. We have \( l(w_2,\chi \pi_{w,1,a,\chi}) = 1/2 \).

Secondly, we define a smooth \( D^\times \)-representation. Let \( \chi' \) denote the character of \( Z \) defined by \( n \mapsto (-1)^n \chi(n) \). We inflate the character \( \tilde{w}_2 : \mathbb{O}_3^\times \to \overline{Q}_l^\times \) to \( \mathcal{O}_D^\times \), and extend it to a character of \( D^\times \) by using \( \chi \), which we denote by \( w_2,\chi \). We inflate the character \( \Lambda_{D,w,1,a} : \overline{\mathcal{O}}_E^\times U_{D}^1 \to \overline{Q}_l^\times \) to \( \mathcal{O}_E^\times U_{D}^1 \), which we denote by the same letter \( \Lambda_{D,w,1,a} \). Then, we extend it to a character of a group \( J_D^1 := E^\times U_D^1 \subset D^\times \) by using \( \chi' \), which we denote by \( \Lambda_{D,w,1,a,\chi} \). See [BH] 56.5]. Now, we define

\[
\rho_{w,1,a,\chi} := c - \text{Ind}_{J_D^1}^{\mathcal{O}_D^\times} \Lambda_{D,w,1,a,\chi}.
\]
Thirdly, we define a 2-dimensional \( W_F \)-representation. We inflate the character \( w_2 : (O_F/\pi^2)^\times \to \overline{\mathbb{Q}}_l^\times \) to \( O_F^\times \), and extend it to a character \( F^\times \) by using \( \chi \), which we denote by \( w_2,\chi \). We inflate a character \( \Lambda'_{w_1,a} : (O_{E_1}/\pi)^\times \to \overline{\mathbb{Q}}_l^\times \) to \( O_{E_1}^\times \), and extend it to a character \( E_1^\times \) by using \( \chi \), which we denote by \( \Lambda'_{w_1,a}\chi \). The local class field theory, we obtain the character \( \Lambda'_{w_1,a}\chi \circ a_{E_1} : W_{ab}^{E_1} \simeq E_1^\times \to \overline{\mathbb{Q}}_l^\times \). For a pair \((E_1/F, \Lambda'_{w_1,a}\chi)\), there exists a character \( \Delta \) of \( E_1^\times \) of level zero, which is defined in [BH, 34.4]. We define

\[
\pi'_{w_1,a}\chi := || \cdot ||^{-\frac{d}{2}} \text{Ind}_{E_1/F}(\Delta \Lambda'_{w_1,a}\chi) \circ a_{E_1}.
\]

Then, we have the following main theorem in this paper.

**Theorem 7.10.** Let the notation be as above.

1. Then, we have the following, for the unramified case,

\[
\rho_{w,\chi} = JL(\pi_{w,\chi}), \quad \pi_{w,\chi} = \text{LL}_\ell(\pi'_{w,\chi}).
\]

2. We consider the ramified case. We set

\[
\pi_{w,a,\chi} := w_{2,\chi} \pi_{w_1,a,\chi}, \quad \rho_{w,a,\chi} := w_{2,\chi}^D \rho_{w_1,a,\chi}, \quad \pi'_{w,a,\chi} := w_{2,\chi}' \pi'_{w_1,a,\chi}.
\]

Then, we have the following

\[
\rho_{w,a,\chi} = JL(\pi_{w,a,\chi}), \quad \pi_{w,a} = \text{LL}_\ell(\pi'_{w,a,\chi}).
\]

**Proof.** We prove the assertion 1. The first (resp. second ) equality follows from the description of \( JL \) (resp. \( \text{LL}_\ell \)) given in [BH 56.6] and Corollary 7.3 (resp. [BH p.219 and p.223]). The required assertion 2 is proved by Lemma 7.6 and [BH 34,35] for \( \text{LL}_\ell \), and [BH 56.5] for \( JL \).

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