Second moments of work and heat for a single particle stochastic heat engine in a breathing harmonic potential

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Abstract

We consider a simple model of a stochastic heat engine, which consists of a single Brownian particle moving in a one-dimensional periodically breathing harmonic potential. Overdamped limit is assumed. Expressions of second moments (variances and covariances) of heat and work are obtained in the form of integrals, whose integrands contain functions satisfying certain differential equations. The results in the quasi-static limit are simple functions of temperatures of hot and cold thermal baths. The coefficient of variation of the work is suggested to give an approximate probability for the work to exceed a given threshold. During derivation, we get the expression of the cumulant-generating function.

I. INTRODUCTION

With the progress of microfabrication and measurement technology, there has arisen much interest in the properties of artificial or biological molecular machines. A number of research has been made theoretically and experimentally (see Refs. [1, 2] and references therein). Especially a single particle stochastic heat engine, that was realized experimentally by an optically trapped colloidal particle (see, e.g., Ref. [3]), has invoked a lot of work.

For example, the efficiency of a heat engine at maximal power was discussed in Refs. [4, 5]. There the efficiency of a stochastic heat engine was defined as the ratio of the average of work to the average of heat. Since fluctuations often dominate averages in such small-scale systems [6, 7], it is important to understand what kind of role these fluctuations play in the performance of stochastic heat engines. Toward this goal, we present in this paper second moments (variances and covariances) of heat and work for a simple model of a stochastic heat engine used in Ref. [5].

Statistical properties of a stochastic heat engine have been analyzed in Refs. [6, 7], where probability distributions of heat and work were obtained by numerical simulations. Further a distribution of the efficiency was calculated and compared with the analytical expression obtained in Ref. [8]. The variance of work was analytically calculated with using of Green's function in Ref. [9]. We give the analytical results of all the second moments of heat and work which still seem absent in literature.

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Refs. [10–13] have discussed statistical properties of work done on a Brownian particle immersed in a single heat bath. Here in this paper, we consider a heat engine which contacts with two heat baths at different temperatures.

This paper is organized as follows: Sect. II describes the model. The joint probability density is given in Sect. III. There, heat and work are defined. Sect. IV gives the expression of a cumulant-generating function. Differentiating this function, we calculate second moments in Sect. V. Sect. VI gives figures which show how these quantities approach their quasi-static limits. Sect. VII summarizes our results and comments on further study.

II. MODEL

We use a model of a stochastic heat engine introduced in Ref. 5: a Brownian particle moving along $x$-axis is trapped in a harmonic potential,

$$U(x, t) = \frac{\lambda(t)}{2} x^2.$$  

The stiffness of the potential $\lambda(t)$ varies periodically in time. The equation of motion for the particle is given by

$$m\ddot{x}(t) = -\frac{\partial U}{\partial x}(x, t) - \gamma \dot{x}(t) + \xi(t)$$

where $m$ represents the mass and $\gamma$ is the friction coefficient. Here and hereafter a dot means time-derivative. The noise $\xi(t)$ is Gaussian with zero mean and is delta correlated:

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t)\xi(t') \rangle = 2\gamma k_B T \delta(t - t')$$

where $T$ is the temperature of the thermal bath and the ensemble average is denoted by $\langle \ldots \rangle$. For the sake of simplicity, we hereafter set the Boltzmann constant $k_B$, to be unity.

We further assume the overdamped limit and neglect the inertia term $m\ddot{x}$ in Eq. (2) and start with the following overdamped Langevin equation:

$$\gamma \dot{x}(t) = -\lambda(t) x + \xi(t).$$

Although $\gamma$ can be eliminated from the equation by rescaling the time as $t = \gamma \tau$, we keep this symbol in order to consider the quasi-static limit which is equivalent to taking the limit $\gamma \to 0$.

The cycle consists of two isothermal steps and two adiabatic steps:
1. Isothermal transition at temperature $T_h$ during time interval $0 < t < t_1$. $\lambda(t)$ decreases and the system expands.

2. Adiabatic transition instantaneously from temperature $T_h$ to temperature $T_c$ at time $t = t_1$. $\lambda(t)$ decreases discontinuously.

3. Isothermal transition at temperature $T_c$ ($< T_h$) during time interval $t_1 < t < t_1 + t_3$. $\lambda(t)$ increases and the system is compressed.

4. Adiabatic transition from temperature $T_c$ to temperature $T_h$ at time $t = t_1 + t_3$. $\lambda(t)$ increases discontinuously.

The system repeats these four steps.

In Ref. [5], the probability distribution for the particle to be found at position $x$ at time $t$, $p(x,t)$ is shown to remain Gaussian with 0 mean and the variance $w(t) = \langle X^2(t) \rangle$ for all times $t$ if it is so initially:

$$p(x,t) = \frac{1}{\sqrt{2\pi w(t)}} \exp \left( -\frac{x^2}{2w(t)} \right),$$

where $X(t)$ denotes a stochastic variable representing the position of the particle at the time $t$.

The time evolution of $w(t)$ is given by

$$\dot{w}(t) = \frac{2}{\gamma} \left( T(t) - \lambda(t) w(t) \right).$$

For the numerical calculation in Sect. [VI] we use the protocol for the variance $w(t)$:

$$w(t) = \begin{cases} 
  w_a \left( 1 + \left( \frac{w_b}{w_a} - 1 \right) \frac{t}{t_1} \right)^2 ; & 0 < t < t_1 \\
  w_b \left( 1 + \left( \frac{w_b}{w_a} - 1 \right) \frac{t-t_1}{t_3} \right)^2 ; & t_1 < t < t_1 + t_3
\end{cases}.$$

If the time-interval spent in the isothermal transitions, $t_1$ and $t_3$, are set to

$$t_1^* = t_3^* = \frac{8\gamma \left( \sqrt{w_b} - \sqrt{w_a} \right)^2}{(T_h - T_c) \log(w_b/w_a)},$$

the system gives a maximal power output for given $T_h$, $T_c$, $w_a$, and $w_b$. See Fig.2. in Ref. [5] for the profile of $w(t)$ and $\lambda(t)$.

In order to analyze the quasi-static limit, we use

$$t_1 = s t_1^*, \quad t_2 = s t_2^*,$$

and vary the value of $s$. 
III. PROBABILITY DENSITY

In order to make the presentation simple, we first divide the time-interval into \( n \) segments whose length is \( \Delta t \), and consider the limit \( \Delta t \to 0 \) \( (n \to \infty) \) in the final stage.

\( X_j \) denotes a stochastic variable representing the position of the particle at \( t_j = j\Delta t \). Since \( \lambda(t) \) is discontinuous at \( t = 0, t = t_1, \) and \( t = t_1 + t_3 \), we use the notation

\[
 t_0 = +0, \quad t_{n_1} = t_1 + 0, \quad t_n = t_1 + t_3 + 0
\]

in order to avoid ambiguity. We also use the following notations:

\[
 \lambda_j = \lambda(t_j), \quad T_j = T(t_j), \quad w_j = w(t_j)
\]

and

\[
 w_a = w_0 = w(0), \quad w_b = w_{n_1} = w(t_1).
\]

Note \( w(t) \) is a continuous function.

The stochastic differential equation corresponding to Eq. (4) becomes

\[
 \gamma (X_{j+1} - X_j) = -\lambda_j X_j \Delta t + \sqrt{2\gamma T_j} \Delta B_j
\]

where \( \Delta B_j \) obeys a Gaussian distribution:

\[
 p(\Delta B_t) = \frac{1}{\sqrt{2\pi \Delta t}} \exp \left( -\frac{(\Delta B_t)^2}{2\Delta t} \right).
\]

The conditional probability density for \( X_{j+1} = x_{j+1} \) with the condition \( X_j = x_j \) is

\[
 \begin{align*}
 p(x_{j+1}|x_j) &= \frac{1}{\sqrt{2\pi \Delta t}} \exp \left( -\frac{(\Delta B_j)^2}{2\Delta t} \right) \cdot \sqrt{\frac{\gamma}{2T_j}} \\
 &= \sqrt{\frac{\gamma}{4\pi T_j \Delta t}} \exp \left( -\frac{(\gamma(x_{j+1} - x_j) + \lambda_j x_j \Delta t)^2}{4\Delta t \gamma T_j} \right).
\end{align*}
\]

The joint probability density for \( \{X_j = x_j; j = 0, \cdots, n\} \) is

\[
 \begin{align*}
 p(x_0, x_1, \cdots, x_n) &= p(x_n|x_{n-1})p(x_{n-1}|x_{n-2}) \cdots p(x_1|x_0)p(x_0, t_0) \\
 &= \Pi_{j=0}^{n-1} \sqrt{\frac{\gamma}{4\pi T_j \Delta t}} \exp \left( -\frac{(\gamma(x_{j+1} - x_j) + \lambda_j x_j \Delta t)^2}{4\Delta t \gamma T_j} \right) \frac{\exp\left(-\frac{x_0^2}{2w_0}\right)}{\sqrt{2\pi w_0}}.
\end{align*}
\]

Thermodynamic quantities such as work, heat, and internal energy can be defined on a single stochastic trajectory according to the formalism of stochastic thermodynamics \[14–16\]. We define the heat \( Q_h \) \( (Q_c) \) uptake from the hotter \( (\text{cooler}) \) heat bath at temperature
The change in internal energy during one cycle is
\[ \Delta U = \lambda_n \frac{n}{2} X_n^2 - \lambda_0 \frac{n}{2} X_0^2. \] (19)

The work done by the particle during one cycle is
\[ W = -\frac{1}{2} \sum_{j=0}^{n-1} X_{j+1}^2 (\lambda_{j+1} - \lambda_j). \] (20)

The energy conservation holds for an individual stochastic trajectory:
\[ \Delta U = Q_h + Q_c - W. \] (21)

IV. MOMENT- AND CUMULANT-GENERATING FUNCTION

With the use of Eqs. (16)∼(19), the moment-generating function of \( Q_h, Q_c, \) and \( \Delta U \) is defined as
\[
M(S_h, S_c, S_u) = \langle e^{S_h Q_h + S_c Q_c + S_u \Delta U} \rangle = \prod_{i=0}^{n-1} \int_{-\infty}^{\infty} dx_i \sqrt{\frac{\gamma}{4\pi T_i \Delta t}} \exp \left( -\frac{(\gamma(x_{i+1} - x_i) + \lambda_i x_i \Delta t)^2}{4\Delta t \gamma T_i} \right) \frac{e^{-\frac{x_i^2}{2 \sigma^2}}}{\sqrt{2\pi \sigma^2}} e^{S_h Q_h + S_c Q_c + S_u \Delta U}. \]

In order to calculate \( M(S_h, S_c, S_u) \), we first define, for \( j = 0, \cdots, n-1 \),
\[
h_j(x_{j+1}) = \Pi_{i=0}^{j} \int_{-\infty}^{\infty} dx_i \sqrt{\frac{\gamma}{4\pi T_i \Delta t}} \exp \left( -\frac{(\gamma(x_{i+1} - x_i) + \lambda_i x_i \Delta t)^2}{4\Delta t \gamma T_i} \right) \frac{e^{-\frac{x_i^2}{2 \sigma^2}}}{\sqrt{2\pi \sigma^2}} e^{S_h Q_h + S_c Q_c + S_u \Delta U}.
\]

where
\[
g(t) = \begin{cases} 
S_h & 0 < t < t_1 \\
S_c & t_1 < t < t_1 + t_3 
\end{cases}
\]
(24)
with

\[ h_0(x_0) = \frac{1}{\sqrt{2\pi w_0}} \exp \left( -\frac{x_0^2}{2w_0} - \frac{\lambda_0}{2} x_0^2 S_u \right). \] \tag{25}

By induction, we can see that \( h_j(x_j) \) has the following form:

\[ h_j(x_j) = \frac{1}{\sqrt{2\pi a_j}} \exp \left( -\frac{x_j^2}{2b_j} \right). \] \tag{26}

Eq. (25) gives

\[ a_0 = w_0 = w_a, \quad b_0 = w_a(1 + S_u \lambda_0 w_a)^{-1}. \] \tag{27}

Then the moment-generating function is expressed as

\[
M(S_h, S_c, S_u) = \int_{-\infty}^{\infty} dx_n h_n(x_n) \exp \left( \frac{\lambda_n}{2} x_n^2 S_u \right)
= \frac{1}{\sqrt{2\pi a_n}} \int_{-\infty}^{\infty} dx_n \exp \left( -\frac{1}{2b_n} x_n^2 \right) \exp \left( \frac{\lambda_n}{2} x_n^2 S_u \right)
= \sqrt{b_n (1 - b_n \lambda_n S_u)^{-1}}. \tag{28}
\]

From the following equation,

\[
h_{j+1}(x_{j+1}) = \sqrt{\frac{\gamma}{4\pi T_j \Delta t}} \int_{-\infty}^{\infty} dx_j h_j(x_j)
\exp \left( -\frac{(\gamma(x_{j+1} - x_j) + \lambda_j x_j \Delta t)^2}{4\Delta t \gamma T_j} \right) + \frac{\lambda_j}{2} \left( x_{j+1}^2 - x_j^2 \right) g(t_j)
= \sqrt{\frac{\gamma}{4\pi T_j \Delta t}} \frac{1}{\sqrt{2\pi a_j}} \int_{-\infty}^{\infty} dx_j \exp \left( -\frac{x_j^2}{2b_j} \right)
\exp \left( -\frac{(\gamma(x_{j+1} - x_j) + \lambda_j x_j \Delta t)^2}{4\Delta t \gamma T_j} \right) + \frac{\lambda_j}{2} \left( x_{j+1}^2 - x_j^2 \right) g(t_j)
, \tag{29}
\]

we can obtain recurrence relations for \( a_j \) and \( b_j \):

\[
a_{j+1} = \frac{2\Delta t T_j}{\gamma} \left( \frac{1}{b_j} + \lambda_j g(t_j) + \frac{(\gamma - \lambda_j \Delta t)^2}{2\Delta t \gamma T_j} \right) a_j, \tag{30}
\]

\[
\frac{1}{b_{j+1}} = -\lambda_j g(t_j) + \frac{\gamma}{2\Delta t T_j} - \frac{(\gamma - \lambda_j \Delta t)^2}{4(\Delta t T_j)^2} \left( \frac{1}{b_j} + \lambda_j g(t_j) + \frac{(\gamma - \lambda_j \Delta t)^2}{2\Delta t \gamma T_j} \right)^{-1}. \tag{31}
\]

Considering the limit \( \Delta t \to 0 \), we can get equations

\[
\dot{a} = \frac{2a}{\gamma} \frac{(1 + g\lambda b) T - \lambda b}{b}, \quad \dot{b} = \frac{2}{\gamma} (1 + g\lambda b) ((1 + g\lambda b) T - \lambda b) \tag{32}
\]
with initial conditions

\[ a(0) = w_a, \quad b(0) = w_a(1 + S_u \lambda(0)w_a)^{-1}. \]  

(33)

The moment-generating function, Eq. (28), becomes

\[ M(S_h, S_c, S_u) = \sqrt{\frac{b(t)(1 - b(t)\lambda(t)S_u)^{-1}}{a(t)}} \].  

(34)

We note that if we set \( S = (S_h, S_c, S_u) = 0 \), both \( a(t) \) and \( b(t) \) become \( w(t) \), and \( h_j(x_j) \) reduces to the probability distribution of the particle, \( p(x_j, t_j) \).

The cumulant generating function is

\[ \log M(S_h, S_c, S_u) = \log(\hat{M}(t_1 + t_3 + 0)) - \frac{1}{2} \log (1 - b(t_1 + t_3 + 0)\lambda(t_1 + t_3 + 0)S_u) \]  

(35)

where

\[ \hat{M}(t) = \sqrt{\frac{b(t)}{a(t)}}. \]  

(36)

Eqs. (32) give

\[ \frac{d}{dt} \log \hat{M} = \frac{1}{2} \left( \frac{\frac{db}{dt}}{b} - \frac{\frac{da}{dt}}{a} \right) = \frac{1}{\gamma} g \lambda ((1 + g \lambda b)T - \lambda b) \]  

(37)

which leads to

\[ \log \hat{M}(t_1 + t_3 + 0) = \log \left( \frac{\hat{M}(t_1 + t_3 + 0)}{\hat{M}(+0)} \right) + \log \hat{M}(+0) \]

\[ = \frac{1}{\gamma} \int_{0}^{t_1+t_3} g(t')\lambda(t') \left( T(t') + \lambda(t') b(t') \left( g(t')T(t') - 1 \right) \right) dt' \]

\[ - \frac{1}{2} \log (1 + S_u \lambda(+0)w_a) \].  

(38)

Eqs. (35) and (38) give the expression of the cumulant-generating function:

\[ \log M(S_h, S_c, S_u) = -\frac{1}{2} \log(1 + S_u \lambda(+0)w_a) - \frac{1}{2} \log(1 - b(t_1 + t_3 + 0)\lambda(t_1 + t_3 + 0)S_u) \]

\[ + \frac{S_h T_h}{\gamma} \int_{0}^{t_1} \lambda(t') dt' + \frac{S_h^2 T_h - S_h}{\gamma} \int_{0}^{t_1} \lambda^2(t') b(t') dt' \]

\[ + \frac{S_c T_c}{\gamma} \int_{t_1}^{t_1+t_3} \lambda(t') dt' + \frac{S_c^2 T_c - S_c}{\gamma} \int_{t_1}^{t_1+t_3} \lambda^2(t') b(t') dt', \]  

(39)

which, together with Eqs. (32) and (33), is our first result.
We would like to make two comments: First, the parameter setting \( \{ S_h = 1/T_h, S_c = S_u = 0 \} \) gives a kind of an integral fluctuation relation \([1, 2]\):

\[
\left\langle \exp \left( \frac{Q_h}{T_h} \right) \right\rangle = \exp \left( \frac{1}{\gamma} \int_0^{t_1} \lambda(t')dt' \right).
\] (40)

Secondly, when we consider the case \( T_h = T_c = T \) and set \( \{ S_h = S_c = S_w, S_u = -S_w \} \), \( M \) reduces to a moment generating function of \( W \), \( \langle e^{S_w W} \rangle \), which can be shown to be equivalent to the result obtained in Ref. \([10]\).

V. VARIANCES AND COVARIANCES

We can calculate averages, variances and covariances of \( Q_h, Q_c \) and \( \Delta U \) by differentiating \( \log M \) with respect to \( S = \{ S_h, S_c, S_u \} \) and then setting \( S = 0 \).

We first calculate averages and check whether they agree with previously obtained results \([5]\):

\[
\langle Q_h \rangle = \frac{\partial \log M}{\partial S_h} \bigg|_{S=0} = \frac{T_h}{\gamma} \int_0^{t_1} \lambda(t')dt' - \frac{1}{\gamma} \int_0^{t_1} \lambda^2(t')w(t')dt',
\]

\[
\langle Q_c \rangle = \frac{\partial \log M}{\partial S_c} \bigg|_{S=0} = \frac{T_c}{2} \log \frac{w_c}{w_a} - \frac{\gamma}{4} \int_0^{t_1} \frac{\dot{w}_c^2(t')}{w(t')}dt',
\]

\[
\langle W \rangle = \langle Q_h + Q_c - \Delta U \rangle = \langle Q_h + Q_c \rangle
\]

\[
= \frac{(T_h - T_c)}{2} \log \frac{w_h}{w_a} - \frac{\gamma}{4} \int_0^{t_1 + t_3} \frac{\dot{w}_a^2(t')}{w(t')}dt'.
\] (43)

In the 2nd line of Eq. (41), by using Eq. (6), we express the integrand in terms of \( w \) and \( \dot{w} \) rather than \( \lambda \) because the protocol of the system is given by \( w(t) \) and not by \( \lambda(t) \). The first term in the 2nd line gives the quasi-static limit.

All these results, Eqs. \((41) \sim (43)\), agree with those obtained in Ref. \([5]\). We note that in the quasistatic limit, \( \frac{\langle W \rangle}{\langle Q_h \rangle} \) becomes the Carnot efficiency:

\[
\frac{\langle W \rangle}{\langle Q_h \rangle} = 1 - \frac{T_c}{T_h} + \cdots.
\] (44)

As for second moments, we need to differentiate \( b(t) \) with respect to \( S = \{ S_h, S_c, S_u \} \):

\[
b_\alpha(t) = \frac{\partial b(t)}{\partial S_\alpha} \bigg|_{S=0}, \quad \alpha = h, c, u.
\] (45)
which we use when considering the quasi-static limit.

The formal solutions of the above equations are given as follows:

\[
\begin{align*}
\frac{db_h(t)}{dt} &= -\frac{2}{\gamma} \lambda(t) b_h(t) + \frac{2}{\gamma} \theta(t_1 - t) \lambda(t) w(t) \left( 2T_h - \lambda(t) w(t) \right), \quad b_h(0) = 0, \quad (46) \\
\frac{db_c(t)}{dt} &= -\frac{2}{\gamma} \lambda(t) b_c(t) + \frac{2}{\gamma} \theta(t - t_1) \lambda(t) w(t) \left( 2T_c - \lambda(t) w(t) \right), \quad b_c(0) = 0, \quad (47) \\
\frac{db_u(t)}{dt} &= -\frac{2}{\gamma} \lambda(t) b_u(t), \quad b_u(0) = -\lambda(+0) w_a^2. \quad (48)
\end{align*}
\]

The formal solutions of the above equations are given as follows:

\[
\begin{align*}
b_h(t) &= \left\{ \begin{array}{ll}
\frac{w(t)}{t_h} \int_0^t ds \left( T_h^2 - \frac{\gamma^2}{4} \dot{w}(s) \right) \frac{dt}{ds} \left( e^{\frac{-2T_h}{\gamma} \int_s^t \frac{dt'}{w(t')}} \right), & 0 < t < t_1 \\
\frac{b_h(t_1)}{w(t_1)} w(t) e^{\frac{-2T_h}{\gamma} \int_{t_1}^t \frac{dt'}{w(t')}} , & t_1 < t < t_1 + t_3
\end{array} \right. \quad (49)
\end{align*}
\]

\[
\begin{align*}
b_c(t) &= \left\{ \begin{array}{ll}
0 , & 0 < t < t_1 \\
\frac{w(t)}{T_c} \int_{t_1}^t ds \left( T_c^2 - \frac{\gamma^2}{4} \dot{w}(s) \right) \frac{dt}{ds} \left( e^{\frac{-2T_c}{\gamma} \int_s^t \frac{dt'}{w(t')}} \right), & t_1 < t < t_1 + t_3
\end{array} \right. \quad (50)
\end{align*}
\]

\[
b_u(t) = -\left( T_h - \frac{\gamma^2}{2} \dot{w}(+0) \right) w(t) e^{\frac{-2T_h}{\gamma} \int_0^t \frac{dt'}{w(t')}} , \quad (51)
\]

which we use when considering the quasi-static limit.

Using these functions, we get the following expressions for 2nd moments;

\[
\begin{align*}
\text{Var}(Q_h) &= \frac{\partial^2 \log M}{\partial S_h^2} \bigg|_{S=0} = \frac{2T_h}{\gamma} \int_0^{t_1} \lambda^2(t') w(t') dt' - \frac{2}{\gamma} \int_0^{t_1} \lambda^2(t') b_h(t') dt' \\
&= \frac{2}{\gamma} \int_0^{t_1} \frac{1}{w^2(t')} (T_h - \frac{\gamma^2}{2} \dot{w}(t')) (T_h w(t') - b_h(t')) dt' \\
&= (T_h)^2 + \ldots . \quad (52)
\end{align*}
\]

\[
\begin{align*}
\text{Var}(Q_c) &= \frac{\partial^2 \log M}{\partial S_c^2} \bigg|_{S=0} = \frac{2}{\gamma} \int_{t_1}^{t_1+t_3} \frac{1}{w^2(t')} (T_c - \frac{\gamma^2}{2} \dot{w}(t')) (T_c w(t') - b_c(t')) dt' \\
&= (T_c)^2 + \ldots . \quad (53)
\end{align*}
\]

\[
\begin{align*}
\text{Var}(\Delta U) &= \frac{\partial^2 \log M}{\partial S_u^2} \bigg|_{S=0} = (\lambda(+0) w_a)^2 + b_u(t_1 + t_3 + 0) \lambda(+0) \\
&= (T_h - \frac{\gamma^2}{2} \dot{w}(+0))^2 \left( 1 - e^{\frac{-2T_h}{\gamma} \int_0^{t_1+t_3} \frac{dt'}{w(t')}} \right) = (T_h)^2 + \ldots . \quad (54)
\end{align*}
\]

\[
\begin{align*}
\text{Cov}(Q_h, Q_c) &= \frac{\partial^2 \log M}{\partial S_h \partial S_c} \bigg|_{S=0} = -\frac{1}{\gamma} \int_{t_1}^{t_1+t_3} \lambda^2(t') b_h(t') dt' \\
&= -\frac{1}{\gamma} \frac{b_h(t_1)}{w(t_1)} \int_{t_1}^{t_1+t_3} dt' \frac{(T_c - \gamma \dot{w}(t')/2)^2}{w(t')} e^{\frac{-2T_h}{\gamma} \int_{t_1}^{t_1+t_3} \frac{dt'}{w(t')}} \\
&= -\frac{T_h T_c}{2} + \ldots . \quad (55)
\end{align*}
\]
\[ \text{Cov}(\Delta U, Q_h) = \frac{\partial^2 \log M}{\partial S_h \partial S_u} \bigg|_{S=0} \]
\[ = -\frac{1}{\gamma} \int_0^{t_1} \lambda^2(t') b_u(t') dt' + \frac{1}{2} \lambda(t_1 + t_2 + 0) b_c(t_1 + t_2 + 0) \]
\[ = -\frac{1}{2} \left( 1 - \frac{\gamma \hat{\omega}(+0)}{2 T_h} \right) \int_0^{t_1} dt' \left( T_h - \frac{\gamma \hat{\omega}(t')}{2} \right)^2 \frac{d}{dt'} e^{-\frac{\gamma T_h}{\gamma} \int_0^{t_1} \frac{dt'}{n(s)}} \]
\[ + \left( T_h - \frac{\gamma \hat{\omega}(+0)}{2} \right) \frac{b_h(t_1)}{2w_b} e^{-\frac{\gamma T_h}{\gamma} \int_1^{t_1+\tau} \frac{ds}{n(s)}} = \frac{(T_h)^2}{2} + \ldots. \]  
\[ (56) \]

\[ \text{Cov}(\Delta U, Q_c) = \frac{\partial^2 \log M}{\partial S_c \partial S_u} \bigg|_{S=0} \]
\[ = -\frac{1}{\gamma} \int_0^{t_1+\tau} \lambda^2(t') b_u(t') dt' + \frac{1}{2} \lambda(t_1 + t_2 + 0) b_c(t_1 + t_2 + 0) \]
\[ = -\frac{T_h - \frac{\gamma \hat{\omega}(+0)}{2 T_c} \int_0^{t_1+\tau} \frac{dt'}{n(s)}} {2 T_c} \]
\[ + \frac{T_h - \frac{\gamma \hat{\omega}(+0)}{2 T_c} \int_0^{t_1+\tau} \frac{ds}{n(s)}} {2 T_c} \]
\[ = \frac{T_h T_c}{2} + \ldots. \]  
\[ (57) \]

Eqs. (52)∼(57) are our second results. The term in the rightmost side of each equation shows the quasi-static limit where terms of the order of \( O(\gamma) \), \( O \left( \exp \left( -\frac{2T_h}{\gamma} \int_0^{t_1+\tau} \frac{ds}{n(s)} \right) \right) \), \( O \left( \exp \left( -\frac{2T_c}{\gamma} \int_1^{t_1+\tau+\tau} \frac{ds}{n(s)} \right) \right) \) are neglected. We note that the quasi-static limit depends only on \( T_h \) and \( T_c \).

The variance and covariance of \( W \) are obtained from the above results with the use of Eq. (21):

\[ \text{Var}(W) = \text{Var}(Q_h) + \text{Var}(Q_c) + \text{Var}(\Delta U) \]
\[ + 2 \left( \text{Cov}(Q_h, Q_c) - \text{Cov}(\Delta U, Q_c) - \text{Cov}(\Delta U, Q_h) \right) \]
\[ = (T_h - T_c)^2 + \ldots. \]
\[ (58) \]

\[ \text{Cov}(Q_h, W) = \text{Var}(Q_h) + \text{Cov}(Q_h, Q_c) - \text{Cov}(\Delta U, Q_h) \]
\[ = \frac{T_h(T_h - T_c)}{2} + \ldots. \]
\[ (59) \]

\[ \text{Cov}(Q_c, W) = \text{Var}(Q_c) + \text{Cov}(Q_h, Q_c) - \text{Cov}(\Delta U, Q_c) \]
\[ = -T_c(T_h - T_c) + \ldots. \]
\[ (60) \]
We here write down only the expressions in the quasi-static limit. We note that the quasi-static limit of $\text{Var}(W)$ is consistent with the previously obtained result (see Eq. (S12) in the supplemental material of Ref. [9]).

VI. DISCUSSION

Figs. 1 ∼ 3, show how $\text{Var}(W)$, $\text{Var}(Q_h)$, and $\text{Cov}(Q_h, W)$ (denoted by solid lines) approach their quasi-static limit (denoted by horizontal dashed lines), respectively. The horizontal axis shows the time interval of a cycle of the heat engine. The following values are used:

$$\gamma = 1, \ w_a = 0.5, \ w_b = 1, \ T_c = 1, \ T_h = 2.$$  \hspace{1cm} (61)

The dots show results obtained from the ensemble of 10000 trajectories which are calculated with the use of Eq. (13) with $n = 1001$. The agreement between solid lines and dots demonstrates the correctness of the procedure in Sect. IV.

![Graph of Var(W) plotted against s defined in (9)](image)

FIG. 1. $\text{Var}(W)$ plotted against $s$ defined in (9) is denoted by the solid line. The horizontal dashed line shows the quasi-static limit. Dots are calculated from the ensemble of 10000 trajectories.

What can we say about the performance of a stochastic heat engine from the results obtained in this paper? Ref. [6] stated that the full probability distribution is necessary for the proper analysis of the system because fluctuations dominate the mean values. Lacking the full probability distribution, we have sought possible candidates which can be calculated using 2nd moments. We suggest the approximate probability for $W$ to exceed a given
FIG. 2. \( \text{Var}(Q_h) \) plotted against \( s \) defined in (9) is denoted by the solid line. The horizontal dashed line shows the quasi-static limit. Dots are calculated from the ensemble of 10000 trajectories.

FIG. 3. \( \text{Cov}(Q_h, W) \) plotted against \( s \) defined in (9) is denoted by the solid line. The horizontal dashed line shows the quasi-static limit. Dots are calculated from the ensemble of 10000 trajectories.

threshold \( W_0 \):

\[
\Pr (W \geq W_0) = \Pr \left( \frac{W - \langle W \rangle}{\sqrt{\text{Var}(W)}} \geq -\frac{1}{k} + \frac{W_0}{\sqrt{\text{Var}(W)}} \right) \\
\approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} \, dz \\
\approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{W_0}{\sqrt{\text{Var}(W)}} + \frac{1}{k}\right)^2} \, dw \\
\approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{W_0}{\sqrt{\text{Var}(W)}} + \frac{1}{k}\right)^2} \, dw
\]

(62)

where

\[
k = \frac{\sqrt{\text{Var}(W)}}{\langle W \rangle} = 2 \log \left( \frac{\nu_b}{\nu_a} \right) + \ldots
\]

(63)

is the coefficient of variation of \( W \). In the rightmost side of Eq. (62), we approximate the distribution of \( W \) by Gaussian. We know it is not Gaussian but hope the details (e.g. skewness or long tails etc.) may not significantly affect the integrated value. ( As for the
FIG. 4. The approximate value of $\Pr(W \geq 0)$ plotted against $s$ defined in (9) is denoted by the solid line. The horizontal dashed line shows the quasi-static limit. Dots are calculated from the ensemble of 10000 trajectories.

distribution of $W$, see Fig.22 in Ref. [6] though the protocol of $\lambda(t)$ is different from that used here.)

Fig.4 shows the results for $W_0 = 0$. Solid line denotes the rightmost side of Eq. (62). Dots show the ratio of trajectories with $W \geq 0$ obtained from the ensemble. We can see that the approximation reproduces the overall tendency though the agreement is moderate and further investigation is certainly necessary.

VII. SUMMARY

In this paper, we consider a simple model of a stochastic heat engine, which consists of a single Brownian particle moving in a one-dimensional periodically breathing harmonic potential. Overdamped limit is assumed (see Eq. (4)). Expressions of second moments (variances and covariances) of heat and work are obtained in the form of integrals (see Eqs. (52)∼(60)), whose integrands contain functions satisfying certain differential equations (see Eqs. (46)∼(48)). The results in the quasi-static limit are simple functions of temperatures of hot and cold thermal baths. The coefficient of variation of the work is suggested to give an approximate probability for the work to exceed a given threshold, which may serve as an index of the performance of a stochastic heat engine. In the course of derivation, we get the expression of the cumulant-generating function (see Eq. (39)).

Comments on further study are in order; Including a linear term $c(t)x$ in $U(x,t)$ could
be addressed within the present formulation. Considering the underdamped case seems also interesting. To consider $U(x, t)$ other than a harmonic potential would require a different formulation. It may be worth investigating to what extent do quasi-static limits obtained here hold.

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