Dynamical supersymmetry of the spin particle–magnetic field interaction

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Abstract
We study the dynamical and supersymmetries of a fermion in a $D = d = 3$-dimensional monopole background. The Hamiltonian also involves an additional spin–orbit coupling term, which is parameterized by the gyromagnetic ratio. We construct the superinvariants associated with the system using a SUSY extension of a previously proposed algorithm, based on Grassmann-valued Killing tensors. Conserved quantities arise for certain definite values of the gyromagnetic factor: $\mathcal{N} = 1$ SUSY requires $g = 2$; a Kepler-type dynamical symmetry only arises, however, for the anomalous values $g = 0$ and $g = 4$. The two anomalous systems can be unified into an $\mathcal{N} = 2$ SUSY system built by doubling the number of Grassmann variables. For $D = d = 2$, the planar system also exhibits an $\mathcal{N} = 2$ supersymmetry without Grassmann variable doubling.

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1. Introduction

Following a classical result of D’Hoker and Vinet [1] (see also [2, 4–7]) a non-relativistic spin-$\frac{1}{2}$ charged particle with gyromagnetic ratio $g = 2$, interacting with a point magnetic monopole, admits an $osp(1|2)$ supersymmetry. It has no Runge–Lenz-type dynamical symmetry, though [8].

Another, surprising, result of D’Hoker and Vinet [9] says, however, that a non-relativistic spin-$\frac{1}{2}$ charged particle with anomalous gyromagnetic ratio $g = 4$, interacting with a point magnetic monopole plus a Coulomb plus a fine-tuned inverse-square potential, does have such a dynamical symmetry. This is to be compared with the one about the $O(4)$ symmetry of a scalar particle in such a combined field [10]. Replacing the scalar particle by a spin 1/2 particle with gyromagnetic ratio $g = 0$, one can prove that two anomalous systems, the one
with $g = 4$ and the one with $g = 0$, are, in fact, superpartners [11]. Note that for both particular $g$-values, one also has an additional $\alpha(3)$ ‘spin’ symmetry.

On the other hand, it has been shown by Spector [12] that the $\mathcal{N} = 1$ supersymmetry only allows $g = 2$ and no scalar potential. Runge–Lenz and SUSY appear, hence, inconsistent.

In this paper, we investigate the bosonic as well as supersymmetries of the Pauli-type Hamiltonian,

$$\mathcal{H}_g = \frac{\vec{\Pi}^2}{2} - \frac{eg}{2} \vec{S} \cdot \vec{B} + V(r),$$  \hspace{1cm} (1)

which describes the motion of a fermion with spin $\vec{S}$ and electric charge $e$, in the combined magnetic field, $\vec{B}$, plus a spherically symmetric scalar field $V(r)$, which also includes a Coulomb term (a ‘dyon’ in what follows). In (1), $\vec{\Pi} = \vec{p} - eA$ denotes the gauge covariant momentum and the constant $g$ represents the gyromagnetic ratio of the spinning particle. Except in section 7, the gauge field is taken that of an Abelian monopole.

We derive the (super)invariants by considering the Grassmannian extension of the algorithm proposed before by one of us [13].

The main ingredients are Killing tensors, determined by a linear system of first-order partial differential equations.

Our recipe has already been used successfully to derive bosonic symmetries [13–16]; in this paper we systematically extend these results to supersymmetries associated with Grassmann-algebra-valued Killing tensors [2, 3, 13, 17].

The plan of this paper is as follows: in section 2 we derive the equations of the motion of the system. In section 3, we present the general formalism and we analyze the conditions under which conserved quantities are generated. In sections 4 and 5 we investigate the super- and bosonic symmetries respectively of the fermion–monopole system. Our investigations confirm Spector’s theorem.

In section 6, we show, however, that the obstruction can be overcome by a *dimensional extension of fermionic space* [18–20]. Working with two, rather than just one Grassmann variable allows us to combine the two anomalous systems into one with $\mathcal{N} = 2$ supersymmetry.

In section 7, we investigate the SUSY of the spinning particle coupled with a static magnetic field in the plane.

2. Hamiltonian dynamics of the spinning system

Let us consider a charged spin-$\frac{1}{2}$ particle moving in a flat manifold $\mathcal{M}^{D+d}$ which is the extension of the bosonic configuration space $\mathcal{M}^{D}$ by a $d$-dimensional internal space carrying the fermionic degrees of freedom. The $(D + d)$-dimensional space $\mathcal{M}^{D+d}$ is described by the local coordinates $(x^\mu, \psi^a)$ where $\mu = 1, \ldots, D$ and $a = 1, \ldots, d$. The motion of the spin particle is, therefore, described by the curve $\tau \rightarrow (x(\tau), \psi(\tau)) \in \mathcal{M}^{D+d}$. We choose $D = d = 3$ and we focus our attention to the spin-$\frac{1}{2}$ charged particle interacting with the static $U(1)$ monopole background, $\vec{B} = \vec{\nabla} \times \vec{A} = \vec{e} (\vec{x}/r^3)$, such that the system is described by the Hamiltonian (1). In order to deduce, in a classical framework, the supersymmetries and conservation laws, we introduce the covariant Hamiltonian formalism, with basic phase-space variables $(x^\mu, \Pi^\mu, \psi^a)$. Here the variables $\psi^a$ transform as tangent vectors and satisfy the Grassmann algebra, $\psi^i \psi^j + \psi^j \psi^i = 0$. The internal angular momentum of the particle can also be described in terms of vector-like Grassmann variables:

$$S^i = -\frac{i}{2} \varepsilon^{ijkl} \psi^k \psi^l.$$  \hspace{1cm} (2)

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We define the covariant Poisson–Dirac brackets for the functions $f$ and $h$ of the phase-space as
\[
\{ f, h \} = \partial_j f \partial_{h_j} - \partial_j h \partial_{f_j} + e F_{ij} \partial_j \partial_{f_i} + i(-1)^{a_f} \frac{\partial f}{\partial \psi a} \frac{\partial h}{\partial \psi^a},
\]
(3)
where $a_f = (0, 1)$ is the Grassmann parity of the function $f$ and the magnetic field reads $B_i = (1/2) \epsilon_{ijk} F_{jk}$. It is straightforward to obtain the non-vanishing fundamental brackets
\[
\{ \pi^i, \Pi_j \} = \delta^i_j,
\]
\[
\{ \Pi_i, \Pi_j \} = e F_{ij},
\]
\[
\{ \psi^i, \psi^j \} = -i \delta^{ij},
\]
(4)
(5)
It follows that, away from the monopole’s location, the Jacobi identities are verified \[21, 22\].

The equations of the motion can be obtained in this covariant Hamiltonian framework:\[3\]:
\[
\dot{\vec{G}} = \frac{e g}{2} \vec{G} \times \vec{B},
\]
(6)
\[
\dot{\vec{\Pi}} = e \vec{\Pi} \times \vec{B} - \vec{\nabla} V(\vec{r}) + \frac{eg}{2} \vec{\nabla} (\vec{S} \cdot \vec{B}).
\]
(7)
Equation (6) shows that the fermionic vectors $\vec{S}$ and $\vec{\psi}$ are conserved when the spin and the magnetic field are uncoupled, i.e. for the vanishing gyromagnetic ratio, $g = 0$. Note that, in addition to the magnetic field term, the Lorentz equation (7) also involves a potential term augmented with a spin–field interaction term.

3. Killing tensors for the fermion–monopole system

Now we outline the algorithm developed in [13] to construct constants of the motion. First, a phase-space function associated with a (super)symmetry can be expanded in powers of the covariant momenta:
\[
Q(\vec{x}, \vec{\pi}, \vec{\psi}) = C(\vec{x}, \vec{\psi}) + \sum_{k=1}^{p-1} \frac{1}{k!} C^{(k)}(\vec{x}, \vec{\psi}) \Pi_{i_1} \cdots \Pi_{i_k}.
\]
(8)
Requiring that $Q$ Poisson commutes with the Hamiltonian, $\{ H, Q \} = 0$, implies the series of constraints:
\[
C^i \partial_i V + \frac{ie g}{4} \psi^m C^i \partial_i F_{am} = 0,
\]
order 0
\[
\partial_j C = C^{jk} \partial_k V + e F_{jk} C^k + \frac{ie g}{4} \psi^m C^{jk} \partial_k F_{am},
\]
order 1
\[
\partial_j C^k + \partial_k C^j = C^{km} \partial_m V + e (F_{jm} C^{mk} + F_{km} C^{mj}) + \frac{ie g}{4} \psi^m C^{ijk} \partial_i F_{am},
\]
order 2
(9)
\[
\partial_j C^{kl} + \partial_k C^{lj} + \partial_l C^{jk} = C^{klm} \partial_m V + e (F_{jm} C^{mkl} + F_{km} C^{mlj} + F_{km} C^{mlj}),
\]
order 3
\[
\vdots = \vdots
\]
\[
\dot{C}^{kl} + \dot{C}^{lj} + \dot{C}^{jk} = C^{klm} \partial_m V + e (F_{jm} C^{mkl} + F_{km} C^{mlj} + F_{km} C^{mlj}) + \frac{ie g}{4} \psi^m \psi^n C^{ijkl} \partial_i F_{mn} - \frac{eg}{2} \psi^m \psi^n \partial C^{ijkl} F_{am},
\]
order 3
\[
\vdots
\]
\[
3 \text{ The dot means derivative wrt the evolution parameter } \frac{d}{d\tau}.
\]
3
This series can be truncated at a finite order, $p$, provided the constraint of order $p$ becomes a Killing equation. The zeroth-order equation can be interpreted as a consistency condition between the potential and the (super)invariant. Apart from the zeroth-order constants of the motion, i.e. such that they do not depend on the momentum, all other order-$n$ (super)invariants are deduced by the systematic method (9) implying rank-$n$ Killing tensors. Each Killing tensor solves the higher order constraint of (9) and can generate a conserved quantity.

In this paper, we are interested in (super)invariants which are linear or quadratic in the momenta. Thus, we have to determine generic Grassmann-valued Killing tensors of rank 1 and rank 2.

Let us first investigate the Killing equation
\[
\partial_j C^k(\vec{x}, \vec{\psi}) + \partial_k C^j(\vec{x}, \vec{\psi}) = 0. \tag{10}
\]
Following Berezin and Marinov [23], any tensor which takes its values in the Grassmann algebra may be represented as a finite sum of homogeneous monomials:
\[
C^i(\vec{x}, \vec{\psi}) = \sum_{k \geq 0} c^i_{a_1 \cdots a_k} (\vec{x}) \psi^{a_1} \cdots \psi^{a_k}, \tag{11}
\]
where the coefficients tensors, $c^i_{a_1 \cdots a_k}$, are completely anti-symmetric in the fermionic indices $\{a_k\}$. The tensors (11) satisfy (10), from which we deduce that their (tensor) coefficients satisfy
\[
\partial_j c^k_{a_1 \cdots a_k} (\vec{x}) + \partial_k c^j_{a_1 \cdots a_k} (\vec{x}) = 0 \implies \partial_j \partial_k c^k_{a_1 \cdots a_k} (\vec{x}) = 0, \tag{12}
\]
providing us with the most general rank-1 Grassmann-valued Killing tensor
\[
C^i(\vec{x}, \vec{\psi}) = \sum_{k \geq 0} \left( M^{ij} x^j + N^i \right)_{a_1 \cdots a_k} \psi^{a_1} \cdots \psi^{a_k}, \quad M^{ij} = -M^{ji}. \tag{13}
\]
Here $N^i$ and the antisymmetric $M^{ij}$ are the constant tensors.

Let us now construct the rank-2 Killing tensors which solve the Killing equation
\[
\partial_j C^{kl}(\vec{x}, \vec{\psi}) + \partial_k C^{jl}(\vec{x}, \vec{\psi}) + \partial_l C^{jk}(\vec{x}, \vec{\psi}) = 0. \tag{14}
\]
We consider the expansion in terms of Grassmann degrees of freedom [23] and the coefficients $c^{ij}_{a_1 \cdots a_k}$ are constructed as symmetrized products [24] of Yano-type Killing tensors, $C^i_j(\vec{x})$, associated with the rank-1 Killing tensors $C^i(\vec{x})$:
\[
c^{ij}_{a_1 \cdots a_k} (\vec{x}) = \frac{1}{2} (C^i_j C^j_i + \tilde{C}^i_j C^j_i)_{a_1 \cdots a_k}. \tag{15}
\]
It is worth noting that the Killing tensor (15) is symmetric in its bosonic indices and anti-symmetric in the fermionic indices. Thus, we obtain
\[
C^{ij}(\vec{x}, \vec{\psi}) = \sum_{k \geq 0} \left( M^{ij}_{kl} \tilde{M}^{jm}_{mk} x^j x^m + M^{ij}_{ln} \tilde{N}^{jm}_{mk} x^j x^m + N^{ij}_{ln} \tilde{M}^{jm}_{mk} x^m + N^{ij}_{ln} \tilde{N}^{jm}_{mk} \right)_{a_1 \cdots a_k} \psi^{a_1} \cdots \psi^{a_k}, \tag{16}
\]
where $M^{ij}_{kl}$, $\tilde{M}^{ij}_{kl}$, $N^{ij}_{ln}$ and $\tilde{N}^{ij}_{ln}$ are the skew-symmetric constants tensors. Then one can verify with direct calculations that (13) and (16) satisfy the Killing equations.
4. SUSY of fermion in the magnetic monopole field

Having constructed the generic Killing tensors (13) and (16) generating constants of the motion, we now describe the supersymmetries of the Pauli-like Hamiltonian (1). To start, we search for momentum-independent invariants, i.e. which are not derived from a Killing tensor, $C^i = C^{ij} = \cdots = 0$. In this case, the system of equations (9) reduces to the constraints

\begin{align}
&\frac{g}{\psi^a} \frac{\partial Q_0(x, \psi)}{\partial \psi^a} F_{am} = 0, \quad \text{order 0} \\
&\partial_j Q_0(x, \tilde{\psi}) = 0 \quad \text{order 1}.
\end{align}

(17)

For $g = 0$ (which means no spin-gauge field coupling), it is straightforward to see that the spin vector, together with an arbitrary function $f(\psi)$ which only depends on the Grassmann variables, is conserved.

For the non-vanishing gyromagnetic ratio $g$, only the ‘chiral’ charge $Q_0 = \tilde{\psi} \cdot \tilde{S}$ remains conserved. Hence, the charge $Q_0$ can be considered as the projection of the internal angular momentum, $\tilde{S}$, onto the internal trajectory $\psi(\tau)$. Thus $Q_0$ can be viewed as the internal analog of the projection of the angular momentum, in bosonic sector, onto the classical trajectory $x(\tau)$.

Let us now search for superinvariants which are linear in the covariant momentum.

\begin{align}
&\frac{g}{\psi^a} \frac{\partial Q_0(x, \psi)}{\partial \psi^a} F_{am} = 0, \quad \text{order 0} \\
&\partial_j C(x, \tilde{\psi}) = eF_{jk} C_k(x, \tilde{\psi}) - \frac{eg}{2} \psi^a \frac{\partial C(l, \tilde{\psi})}{\partial \psi^a} F_{am}, \quad \text{order 1} \\
&\partial_j C^l(x, \tilde{\psi}) + \partial_l C^j(x, \tilde{\psi}) = 0 \quad \text{order 2}.
\end{align}

(18)

We choose the non-vanishing $N^j_a = \delta^j_a$ in the general rank-1 Killing tensor (13). This provides us with the rank-1 Killing tensor generating the supersymmetry transformation:

\[ C^l(x, \tilde{\psi}) = \delta^l_a \psi^a. \]

(19)

By substitution of this Grassmann-valued Killing tensor into the first-order equation of (18) we get

\[ \tilde{\nabla} C(x, \tilde{\psi}) = \frac{q}{2} (g - 2) \tilde{\psi} \times \frac{x}{r^3}. \]

(20)

Consequently, a solution $C(x, \tilde{\psi}) = 0$ of (20) is only obtained for a fermion with an ordinary gyromagnetic ratio

\[ g = 2. \]

(21)

Thus we obtain, for $V(r) = 0$, the Grassmann-odd supercharge generating the $\mathcal{N} = 1$ supersymmetry of the spin-monopole field system:

\[ \mathcal{Q} = \tilde{\psi} \cdot \tilde{\nabla}, \quad \{\mathcal{Q}, \mathcal{Q}\} = -2i\mathcal{H}_2. \]

(22)

For the non-vanishing potential, $V(r) \neq 0$, the zeroth-order consistency condition of (18) is expressed as $4 \langle V'(r)/r \rangle \tilde{\psi} \cdot \tilde{x} = 0$. Consequently, adding any spherically symmetric potential $V(r)$ breaks the supersymmetry generated by the Killing tensor $C^l = \delta^l_a \psi^a : \mathcal{N} = 1$ SUSY requires an ordinary gyromagnetic factor, and no additional radial potential is allowed [12].

\[ ^4 \text{We use the identity} \ S^l G^j \partial_j B^k = \psi^l \psi^a G^j \partial_j F_{am} = 0. \]
Another Killing tensor (13) is obtained by considering the particular case with the non-null
tensor \( N_{i_1 i_2} = \epsilon_{i_1 i_2} \). This leads to the rank-1 Killing tensor
\[
C^j (\vec{x}, \vec{\psi}) = \epsilon_{a b} \psi^a \psi^b .
\] (23)
The first-order constraint of (18) is solved with \( C(\vec{x}, \vec{\psi}) = 0 \), provided the gyromagnetic ratio
takes the value \( g = 2 \). For vanishing potential, it is straightforward to verify the zeroth-order
consistency constraint and to obtain the Grassmann-even supercharge,
\[
Q_1 = \vec{S} \cdot \vec{P} ,
\] (24)
defining the ‘helicity’ of the spinning particle. As expected, the consistency condition of
superinvariance under (24) is again violated for \( V(r) \neq 0 \), breaking the supersymmetry of the
Hamiltonian \( \mathcal{H}_2 \) in (22).

Let us now consider the rank-1 Killing vector,
\[
C^j (\vec{x}, \vec{\psi}) = (\vec{S} \times \vec{x})^j ,
\] (25)
obtained by putting \( M_{i j a} = (i/2) \epsilon_{i j a} k_{i j a} \) into the generic rank-1 Killing tensor (13).
The first-order constraint is satisfied with \( C(\vec{x}, \vec{\psi}) = 0 \), provided the particle carries the
gyromagnetic ratio \( g = 2 \). Thus, we obtain the supercharge,
\[
Q_2 = (\vec{S} \times \vec{P}) \cdot \vec{\psi} ,
\] (26)
which, just like those in (22) and (24), only appears when the potential is absent, \( V = 0 \).

We consider the SUSY given when \( M_{i j a} = \epsilon_{i j a} \) so that the Killing tensor (13) reduces to
\[
C^j (\vec{x}, \vec{\psi}) = - \epsilon^j_{a b} \psi^a \psi^b .
\] (27)
The first-order constraint of (18) is solved with \( C(\vec{x}, \vec{\psi}) = q (g - 2) \frac{\vec{\psi} \cdot \vec{x}}{r} \). The zeroth-order
consistency condition is, in this case, identically satisfied for an arbitrary radial potential. We
have thus constructed the Grassmann-odd supercharge,
\[
Q_3 = (\vec{S} \times \vec{P}) \cdot \vec{\psi} + q (g - 2) \frac{\vec{\psi} \cdot \vec{x}}{r} ,
\] (28)
which is still conserved for a particle carrying an arbitrary gyroscopic ratio \( g \); see also [4].

Now we turn to superinvariants which are quadratic in the covariant momentum. For this,
we solve the reduced series of constraints:
\[
\begin{align*}
C^i \partial_i V + & \frac{ie g}{4} \psi^l \psi^m C^l \partial_j F_{lm} - \frac{eg}{2} \psi^m \partial C^j \partial a F_{am} = 0, \quad \text{order 0} \\
\partial_j C = & C^k \partial_k V + e F_{jk} C^k + \frac{ie g}{4} \psi^l \psi^m C^l \partial_j F_{lm} - \frac{eg}{2} \psi^m \partial C^j \partial a F_{am}, \quad \text{order 1} \\
\partial_j C^l + & \partial_k C^j = e (F_{jm} C^m + F_{km} C^m) - \frac{eg}{2} \psi^m \partial C^k \partial a F_{am}, \quad \text{order 2} \\
\partial_j C^{lm} + & \partial_m C^{kj} + \partial_k C^{mj} = 0 \quad \text{order 3}.
\end{align*}
\] (29)
We first observe that \( C^{ij} (\vec{x}, \vec{\psi}) = \delta^{ij} \) is a constant Killing tensor. Solving the second- and
first-order constraints of (29), we obtain \( C^i (\vec{x}, \vec{\psi}) = 0 \) and \( C(\vec{x}, \vec{\psi}) = V(r) - \frac{eg}{2} \vec{S} \cdot \vec{B} \),
respectively. The zeroth-order consistency condition is identically satisfied and we obtain the
energy of the spinning particle:
\[
\mathcal{E} = \frac{1}{2} \vec{P}^2 - \frac{eg}{2} \vec{S} \cdot \vec{B} + V(r) .
\] (30)

Next, introducing the non-vanishing constants tensors, \( M^{ijk} = \epsilon^{ijk} \), \( \vec{N}^{ij} = - \epsilon_{i a} \), into
(16), we derive the rank-2 Killing tensor with the property
\[
C^{jk} (\vec{x}, \vec{\psi}) = 2 \delta^{jk} \vec{x} \cdot \vec{\psi} = x^l \psi^k - x^k \psi^l .
\] (31)
Using the Killing tensor (31), we solve the second-order constraints of (29) with \( \mathcal{C}(\vec{x}, \vec{\psi}) = (q/2)(2-g)(\vec{\psi} \times \vec{x})/r \). In order to deduce the integrability condition of the first-order constraint of (29), we require the vanishing of the commutator:

\[
[q, \vec{x}] = 0 \implies \Delta \left( V(r) - (2-g)^2 \frac{q^2}{8r^2} \right) = 0. \tag{32}
\]

Then the Laplace equation (32) provides us with the most general form of the potential admitting a Grassmann-odd supercharge quadratic in the velocity, namely with

\[
V(r) = (2-g)^2 \frac{q^2}{8r^2} + \frac{\alpha}{r} + \beta. \tag{33}
\]

Thus, we solve the first-order constraint with

\[
\mathcal{C}(\vec{x}, \vec{\psi}) = \left( \frac{\alpha}{r} - eg \vec{S} \cdot \vec{B} \right) \vec{x} \cdot \vec{\psi}, \tag{34}
\]

so that the zeroth-order consistency constraint is identically satisfied. Collecting our results leads to the Grassmann-odd supercharge quadratic in the velocity:

\[
Q_4 = (\vec{\Pi} \times (\vec{x} \times \vec{\Pi})) \cdot \vec{\psi} + \frac{q}{2} (2-g) \frac{\vec{x} \times \vec{\Pi} \cdot \vec{\psi}}{r} + \left( \frac{\alpha}{r} - eg \vec{S} \cdot \vec{B} \right) \vec{x} \cdot \vec{\psi}. \tag{35}
\]

This supercharge is not a square root of the Hamiltonian \( \mathcal{H}_g \), and \( Q_4 \) is conserved without restriction on the gyromagnetic factor, \( g \). We can also remark that for \( g = 0 \), the supercharge coincides with the scalar product of the separately conserved Runge–Lenz vector for a scalar particle [10] by the Grassmann-odd vector:

\[
Q_{4|g=0} = \vec{K}_{x=0} \cdot \vec{\psi}. \tag{36}
\]

The supercharges \( Q \) and \( Q_j \) with \( j = 0, \ldots, 3 \), previously determined, form together, for ordinary gyromagnetic ratio, the classical superalgebra

\[
\begin{align*}
\{ Q_0, Q_0 \} &= \{ Q_0, Q_1 \} = \{ Q_2, Q_2 \} = 0, \\
\{ Q_0, Q_2 \} &= i Q_1, \\
\{ Q_0, Q_3 \} &= i Q_2, \\
\{ Q, Q \} &= -2i \mathcal{H}_2, \\
\{ Q, Q_j \} &= \{ Q_1, Q_3 \} = Q_4, \\
\{ Q_1, Q_2 \} &= i Q_3 Q, \\
\{ Q_3, Q_3 \} &= 2i Q_1, \\
\{ Q_2, Q_3 \} &= i Q_2 Q, \\
\{ Q_3, Q_3 \} &= i(2 Q_2 - Q_3), \\
\end{align*}
\]

where \( Q_3 \) is the supercharge constructed in section 5, cf (43). From these results it follows that the linear combination \( Q_4 = Q_3 - 2Q_0 \) has the special property that its bracket with the standard supercharge \( Q \) vanishes:

\[
\{ Q_4, Q \} = 0. \tag{38}
\]

Indeed, \( Q_4 \) is precisely the Killing–Yano supercharge constructed in [4].

**5. Bosonic symmetries of the spinning particle**

Let us investigate the bosonic symmetries of the Pauli-like Hamiltonian (1). We use the generic Killing tensors constructed in section 3 to derive the associated constants of the motion. Firstly, we describe the rotationally invariance of the system by solving the reduced series of constraints (18). For this, we consider the Killing vector provided by the replacement \( M^j = -\epsilon^{ij} n^k \) into (13). Thus we obtain for any unit vector \( \vec{n} \), the generator of space rotations around \( \vec{n} \):

\[
\mathcal{C}(\vec{x}, \vec{\psi}) = \vec{n} \times \vec{x}. \tag{39}
\]
Inserting the previous Killing vector into the first-order equation of (18) yields \( C(\vec{x}, \vec{\psi}) = -q(\vec{n} \cdot \vec{x})/r + c(\vec{\psi}) \). The zeroth-order consistency condition of (18) requires, for arbitrary radial potential, \( c(\vec{\psi}) = \vec{S} \cdot \vec{n} \). Collecting our results provides us with the total angular momentum (which is plainly conserved for arbitrary gyromagnetic ratio)

\[
\vec{J} = \vec{L} + \vec{S} = \vec{x} \times \vec{\Pi} - q\frac{\vec{x}}{r} + \vec{S}. \tag{40}
\]

In addition to the typical monopole term, \( \vec{J} \) also involves the spin vector, \( \vec{S} \). It generates an \( o(3) \) rotations bosonic symmetry algebra, \( \{J^i, J^j\} = \epsilon^{ijk} J^k \).

In the case of the vanishing gyromagnetic factor \( g = 0 \), the orbital part \( \vec{L} \) and the spin angular momentum \( \vec{S} \) are separately conserved involving an \( o(3) \) rotations \( \oplus o(3) \) spin symmetry algebra.

Now we turn to invariants which are quadratic in the velocity. Then, we have to solve the series of constraints (29). We first observe that for \( M^{jm} = \tilde{M}^{jm} = \epsilon^{jm} \), the Killing tensor (16) reduces to the rank-2 Killing–Stäckel tensor

\[
C^{ij}(\vec{x}, \vec{\psi}) = 2\delta^{ij} - 2x^ix^j. \tag{41}
\]

Inserting (41) into the second- and first-order constraints of (29), we get, for any gyromagnetic factor and for any arbitrary radial potential,

\[
\vec{C}(\vec{x}, \vec{\psi}) = 0 \quad \text{and} \quad C(\vec{x}, \vec{\psi}) = -gq\frac{\vec{x} \cdot \vec{S}}{r}. \tag{42}
\]

Thus, we obtain the Casimir

\[
Q_5 = J^2 - q^2 + (g - 2)\vec{J} \cdot \vec{S} - gQ_2. \tag{43}
\]

The bosonic supercharge \( Q_5 \) is, as expected, the square of the total angular momentum, augmented with another separately conserved term. Indeed, for \( g = 0 \), it is straightforward to see that the spin and hence \( \vec{J} \) are separately conserved. For \( g = 2 \), we recover the conservation of \( Q_2 \), cf (26). For the anomalous gyromagnetic ratio \( g = 4 \) we obtain that \( \vec{J} \cdot \vec{S} - 2gQ_2 \) is a constant of the motion.

Now we are interested in the hidden symmetry generated by conserved Laplace–Runge–Lenz-type vectors; therefore we introduce into algorithm (29) the generator,

\[
C^{ij}(\vec{x}, \vec{\psi}) = 2\delta^{ij} - n^i x^j - n^j x^i, \tag{44}
\]

easily obtained by choosing the non-vanishing \( \tilde{N}^{ij} = \epsilon^{ijm}n^m \) and \( M^{jm} = \epsilon^{jm} \) into the generic rank-2 Killing tensor (16). Inserting (44) into the second-order constraint of (29), we get

\[
\vec{C}(\vec{x}, \vec{\psi}) = q\frac{\vec{n} \times \vec{x}}{r} + \vec{C}(\vec{\psi}). \tag{45}
\]

In order to solve the first-order constraint of (29) we write the expansion [23] in terms of Grassmann variables:

\[
C(\vec{x}, \vec{\psi}) = C(\vec{x}) + \sum_{k \geq 1} C_{a_1 \cdots a_k}(\vec{x})\psi^{a_1} \cdots \psi^{a_k}. \tag{46}
\]

Consequently, the first- and zeroth-order equations of (29) can be classified order-by-order in Grassmann-odd variables. Thus, inserting (45) into the first-order equation, and requiring again the vanishing of the commutator,

\[
[\partial_i, \partial_j]C(\vec{x}) = 0 \implies \Delta \left( V(r) - \frac{q^2}{2r^2} \right) = 0, \tag{47}
\]

8
we deduce the most general radial potential admitting a conserved Laplace–Runge–Lenz vector in the fermion–monopole interaction, namely

\[ V(r) = \frac{g^2}{2r^2} + \frac{\mu}{r} + \gamma, \quad \mu, \gamma \in \mathbb{R}. \] (48)

We can now find the first term on the rhs of (46), \( C(\vec{\gamma}) = \mu \frac{(\vec{n} \cdot \vec{x})}{r_c} \). Introducing (45) and (48) into the first-order constraint of (29) leads to (16). Inserting (52) into the second-order constraint of (29) gives us with

\[ \sum_{k \geq 1} C_{\mu_1 \cdots \mu_k}(\vec{\gamma}) \psi^{\mu_1} \cdots \psi^{\mu_k} = -\frac{eg}{2} \left( \vec{S} \cdot \vec{B} \right) (\vec{n} \cdot \vec{x}) - \frac{gq}{2} \left( 1 - \frac{g}{2} \right) \frac{\vec{n} \cdot \vec{S}}{r} + C(\vec{\gamma}), \]

\[ \text{with} \quad g(g - 4) = 0. \] (49)

The zeroth-order consistency condition of (29) is only satisfied for \( C(\vec{\gamma}) = \frac{g^2}{q} \vec{S} \cdot \vec{n} \). Collecting our results, (44), (45), (48) and (49), we get a conserved Runge–Lenz vector if and only if

\[ g = 0 \quad \text{or} \quad g = 4; \] (50)

we obtain namely

\[ \vec{K}_g = \vec{\Pi} \times \vec{J} + \frac{\vec{g}}{r} + \left( 1 - \frac{g}{2} \right) \vec{S} \times \vec{\Pi} - \frac{eg}{2} \left( \vec{S} \cdot \vec{B} \right) \vec{x} - \frac{gq}{2} \left( 1 - \frac{g}{2} \right) \frac{\vec{S}}{r} + \frac{\mu}{q} \vec{S}. \] (51)

Note that the spin angular momentum which generates the extra ‘spin’ symmetry for the vanishing gyromagnetic ratio is not more separately conserved for \( g = 4 \). Then, an interesting question is to know if the extra ‘spin’ symmetry of \( g = 0 \) is still present for the anomalous superpartner \( g = 4 \), cf section 6, in some ‘hidden’ way.

Let us consider the ‘spin’ transformation generated by the rank-2 Killing tensor with the property

\[ C^{mk}(\vec{x}, \vec{\gamma}) = 2S^{mk} \vec{S} \cdot \vec{n} - \frac{g}{2} (S^m n^k + S^k n^m). \] (52)

The previous rank-2 Killing tensor, \( C^{mk} = C_{+}^{mk} + C_{-}^{mk} \), cf (52), is obtained by putting

\[ N_{+}^{l} = (g/2)\epsilon^{l}k n^{j}, \quad \vec{N}_{+}^{l} = -i/2\epsilon^{l}k n^{j}, \]

\[ N_{-}^{j} = (1 - (g/2))\epsilon^{j}k n^{l}, \quad \vec{N}_{-}^{j} = -i/2\epsilon^{j}k n^{l} \]

into the general rank-2 Killing tensor (16). Inserting (52) into the second-order constraint of (29) provides us with

\[ \vec{C}(\vec{x}, \vec{\gamma}) = -\frac{gq}{2} \left( \vec{S} \times \vec{n} \right) + \vec{C}(\vec{\gamma}) \quad \text{and} \quad g(g - 4) = 0. \] (53)

We use potential (48) to solve the first-order equation of (29):

\[ C(\vec{x}, \vec{\gamma}) = \left( 2V(r) - \frac{g^2 S^2}{8r^2} - \frac{\mu g^2}{4r} \right) \vec{S} \cdot \vec{n} + c(\psi), \]

\[ \vec{C}(\vec{\gamma}) = \frac{\mu g}{2q} \vec{n} \times \vec{S} \quad \text{and} \quad g(g - 4) = 0. \] (54)

The zeroth-order consistency condition is satisfied with \( c(\psi) = -\frac{g^2 q^2}{8q} \vec{S} \cdot \vec{n} \), so that collecting our results leads to the conserved vector

\[ \vec{\Omega}_g = \left( \vec{\Pi}^2 + \left( 2 - \frac{g^2}{4} \right) V(r) \right) \vec{S} - \frac{g}{2} (\vec{\Pi} \cdot \vec{S}) \vec{\Pi} + \frac{g}{2} \left( \frac{q}{r} + \frac{\mu}{q} \right) \vec{S} \times \vec{\Pi} - \frac{g^2}{4} \left( \frac{\mu^2}{2q^2} - \gamma \right) \vec{S} \quad \text{with} \quad g(g - 4) = 0. \] (55)

In conclusion, the additional \( o(3)_{\text{spin}} \) ‘spin’ symmetry is recovered in the same particular cases of the anomalous gyromagnetic ratios 0 and 4, cf (50).
For $g = 0$, in particular,
\[ \vec{\Omega}_0 = 2\vec{\omega}. \] (56)

For $g = 4$, we find an expression equivalent to that of D’Hoker and Vinet [9], namely
\[ \vec{\Omega}_4 = (\vec{\Pi}^2 - 2V(r))\vec{S} - 2(\vec{\Pi} \cdot \vec{S})\vec{\Pi} + 2\left( \frac{q}{r} + \frac{\mu}{q} \right) \vec{S} \times \vec{\Pi} - 4\left( \frac{\mu^2}{2q^2} - \gamma \right) \vec{S}. \] (57)

Note that this extra symmetry is generated by a Killing tensor, rather than a Killing vector, as for an ‘ordinary’ angular momentum. Thus, for sufficiently low energy, the motions are bounded and the conserved vectors $J$, $K_g$ and $\vec{\Omega}_g$ generate an $o(4) \oplus o(3)_{\text{spin}}$ bosonic symmetry algebra.

6. $\mathcal{N} = 2$ supersymmetry of the fermion–monopole system

So far we have seen that, for a spinning particle with a single Grassmann variable, SUSY and dynamical symmetry are inconsistent, since they require different values for the $g$-factor. Now, adapting the idea of D’Hoker and Vinet to our framework, we show that the two contradictory conditions can be conciliated by doubling the odd degrees of freedom. The systems with $g = 0$ and $g = 4$ will then become superpartners inside a unified system [11].

We consider, hence, a charged spin-$\frac{1}{2}$ particle moving in a flat manifold $\mathcal{M}^{D+2d}$, interacting with a static magnetic field $\vec{B}$. The fermionic degrees of freedom are now carried by a $2d$-dimensional internal space. This is to be compared with the $d$-dimensional internal space sufficient to describe the $\mathcal{N} = 1$ SUSY of the monopole. In terms of the Grassmann-odd variables $\psi_{1,2}$, the local coordinates of the fermionic extension $\mathcal{M}^{2d}$ read $(\psi^1, \psi^2_\alpha)$ with $a, b = 1, \ldots, d$. The system is still described by the Pauli-like Hamiltonian (1). Choosing $d = 3$, we consider the fermion $\xi_a$ which is a two-component spinor, $\xi_a = (\psi^a_\alpha)$, and whose conjugate is $\bar{\xi}^a$. Thus, we have a representation of the spin angular momentum
\[ S^k = \frac{1}{2} \bar{\xi}^{[\alpha} \sigma_{\alpha \beta}^{k\beta} \xi^\beta \quad \text{with} \quad \alpha, \beta = 1, 2, \] (58)
and $\sigma^{k\beta}$ with $k = 1, 2, 3$ are the standard Pauli matrices. Defining the covariant Poisson–Dirac brackets as
\[ \{ f, h \} = \partial_j f \frac{\partial h}{\partial \Pi_j} - \partial_j h \frac{\partial f}{\partial \Pi_j} + \epsilon_{ijk} B_k \frac{\partial f}{\partial \Pi_i} \frac{\partial h}{\partial \Pi_j} + i(-1)^{\nu/2} \left( \frac{\partial f}{\partial \xi_{\alpha}} \frac{\partial h}{\partial \bar{\xi}^\alpha} + \frac{\partial f}{\partial \bar{\xi}^\alpha} \frac{\partial h}{\partial \xi_{\alpha}} \right), \] (59)
we deduce the non-vanishing fundamental brackets
\[ \{ x^i, \Pi_j \} = \delta^i_j, \quad \{ \Pi_i, \Pi_j \} = \epsilon_{ijk} B^k, \quad \{ \xi_{\alpha}, \bar{\xi}^\beta \} = -i\delta_{\alpha}^\beta, \quad \{ S^k, \xi_{\alpha} \} = -\frac{i}{2} \bar{\xi}^{[\alpha} \sigma_{\alpha \beta}^{k\beta} \xi^\beta, \quad \{ S^k, \bar{\xi}^\beta \} = i \frac{1}{2} \sigma_{\alpha \beta}^{k\mu} \xi_{\alpha}. \] (60)

We also introduce an auxiliary scalar field, $\Phi(r)$, satisfying the ‘self-duality’ or ‘Bogomolny’ relation
\[ \{ \Pi^k, \Phi(r) \} = \pm e B^k. \] (61)

This auxiliary scalar field also defines a square root of the external potential of the system so that $\frac{1}{2} \Phi^2(r) = V(r)$. As an example we obtain the potential in (48) by considering the auxiliary field, $\Phi(r) = \pm \left( \frac{q}{r} + \frac{\mu}{q} \right)$.

5 See [11] to justify terminology.

6 The constant is $\gamma = \frac{\mu^2}{2q^2}$. 

In order to investigate the $N = 2$ supersymmetry of the Pauli-like Hamiltonian (1), we outline the algorithm developed that we used to construct supercharges linear in the gauge covariant momentum:

$$\begin{align*}
\left\{ \begin{array}{l}
\mp e\Phi(r)B^j C^j + \frac{i e}{4} B^k \left( \xi^\mu \sigma^\nu_{\mu} \frac{\partial C}{\partial \xi^\nu} - \frac{\partial C}{\partial \xi^\nu} \sigma^\nu_{\mu} \xi^\nu \right) - \frac{e g}{4} \xi^\mu \sigma^\nu_{\mu} \xi^\nu C^j \partial_j B^k = 0, \quad \text{order 0} \\
\partial_m C = e \epsilon_{mjk} B^k C^j + \frac{i e}{4} B^k \left( \xi^\mu \sigma^\nu_{\mu} \frac{\partial C^m}{\partial \xi^\nu} - \frac{\partial C^m}{\partial \xi^\nu} \sigma^\nu_{\mu} \xi^\nu \right), \quad \text{order 1} \\
\partial_j C^k(x, \xi, \bar{\xi}) + \partial_k C^j(x, \xi, \bar{\xi}) = 0 \quad \text{order 2}
\end{array}\right. \\
\right. \\
\end{align*}$$

(62)

Let us first consider the Killing spinor

$$C^j_\beta = \frac{1}{2} \sigma^j_{\alpha \beta} \xi^\alpha.$$

Inserting this Killing spinor into the first-order equation of (62) provides us with

$$\partial_m C_\beta = - \frac{i}{2} e B_m \xi_\beta$$

and $g = 4$,

(64)

which can be solved using the self-duality relation (61). We get $C_\beta(x, \xi, \bar{\xi}) = \pm \frac{i}{2} \Phi(r) \xi_\beta$. The zeroth-order constraint of (62) is identically satisfied so that collecting our results provides us with the supercharge

$$Q_\beta = \frac{1}{2} \Pi_j \sigma^j_{\beta \alpha} \xi^\alpha \pm \frac{i}{2} \Phi(r) \xi_\beta.$$

(65)

To obtain the supercharge conjugate to (65), we consider the Killing spinor

$$\bar{C}^k_\beta = \frac{1}{2} \bar{\xi}^k_{\alpha \beta} \sigma^k_{\alpha \beta}.$$

We solve the first-order equation of (62) for the anomalous value of the gyromagnetic ratio $g = 4$ using the Bogomolny equation (61). This leads to the conjugate $\bar{C}^\beta(x, \xi, \bar{\xi}) = \mp \frac{i}{2} \Phi(r) \bar{\xi}^\beta$. The zeroth-order consistency constraint is still satisfied and we obtain the odd-supercharge

$$\bar{Q}^\beta = \frac{1}{2} \bar{\xi}^k_{\alpha \beta} \Pi_k \mp \frac{1}{2} \Phi(r) \bar{\xi}^\beta.$$

(67)

The supercharges $Q_\beta$ and $\bar{Q}^\beta$ are, both, square roots of the Pauli-like Hamiltonian $H_4$ and therefore generate the $N = 2$ supersymmetry of the spin-monopole field system

$$\{ \bar{Q}^\beta, Q_\beta \} = -i H_4.$$

(68)

It is worth noting that by defining the rescaled $\bar{U}^\beta = \bar{Q}^\beta / \sqrt{H_4}$ and $U_\beta = 1 / \sqrt{H_4} Q_\beta$ it is straightforward to get

$$H_0 = \bar{U}^\beta H_4 U_\beta.$$

(69)

which make manifest the fact that the two anomalous cases $g = 0$ and $g = 4$ can be viewed as superpartners, cf [11]. Moreover, in our enlarged system, the following bosonic charges,

$$\begin{align*}
\vec{J} &= \vec{x} \times \vec{\Pi} - q \vec{x} - \vec{S}, \\
\vec{K} &= \vec{\Pi} \times \vec{J} + \mu \vec{x} - \vec{S} \times \vec{\Pi} - 2e(\vec{S} \cdot \vec{B})\vec{x} + 2q \vec{S} + \frac{\mu}{q} \vec{S}, \\
\vec{\Omega} &= \vec{Q}^\beta \sigma^k_{\beta \mu} Q_\mu = \frac{1}{2} (\Phi^2(r) - \vec{\Pi}^2) \vec{S} + (\vec{\Pi} \cdot \vec{S}) \vec{\Pi} \mp \Phi(r) \vec{S} \times \vec{\Pi},
\end{align*}$$

(70)

with the scalar $\bar{\xi}^\beta \xi_\beta = 2$. 

\footnote{With the scalar $\bar{\xi}^\beta \xi_\beta = 2$.}
remain conserved such that they form, together with the supercharges \( Q_\beta \) and \( \bar{Q}^\beta \), the classical symmetry superalgebra \([9, 11]\)

\[
\{\bar{Q}_\beta, Q_\gamma\} = \left\{\bar{Q}_\beta, \bar{Q}^\gamma\right\} = 0, \quad \{Q_\beta, J^k\} = \frac{i}{4} \sigma^a \sigma^b a^b_\beta, \\
\{Q_\beta, K^j\} = -\frac{i}{4} \sigma^a \sigma^b a^a_\beta q^b_\alpha Q_{\alpha}, \\
\{Q_\beta, \Omega^k\} = -i \tau a^a_\beta q^b_\alpha Q_{\alpha}.
\]

\[
\{\bar{Q}_\beta, J^k\} = \delta^{jk} \bar{Q}_\beta, \quad \{\bar{Q}_\beta, K^j\} = \delta^{jk} \bar{Q}^\beta, \quad \{\bar{Q}_\beta, \Omega^k\} = i \tau \bar{Q}_\beta, \\
\{Q_\beta, \Omega^k\} = i \tau Q_\beta.
\]

7. Planar system

In two dimensions the models simplify. The magnetic field is \( F_{ij} = \epsilon_{ij} B = \partial_i A_j - \partial_j A_i \) and the spin tensor is actually a scalar

\[
S = \frac{i}{2} \epsilon_{ij} \psi_i \psi_j.
\]

The Hamiltonian takes the form

\[
H = \frac{1}{2} \pi^2 - \frac{e g}{2} SB + V(r).
\]

The fundamental brackets remain the same as in (3). The dynamical quantities (8) become constants of motion if the constraints (9) are satisfied:

\[
\partial_i C_j + \partial_j C_i = e (F_{ik} C_{kj} - C_{ik} F_{kj}) + \frac{\partial H}{\partial \psi_k} \partial C_{ij} \partial \psi_k + C_{ijk} \partial_k H, \quad \text{order 2}
\]

Using

\[
\frac{\partial H}{\partial \psi_i} = -\frac{e g}{2} F_{ij} \psi_j = -\frac{e g}{2} B \epsilon_{ij} \psi_j,
\]

the first (zeroth-order) constraint becomes

\[
\frac{e g}{2} B \epsilon_{ij} \psi_j \frac{\partial C_{ij}}{\partial \psi_i} = C_{ij} \left( \partial_i V - \frac{e g}{2} S \partial_j B \right),
\]

complemented by the first-order equation

\[
\partial_i C_j = e B \left( \epsilon_{ij} C_j + \frac{g}{2} \epsilon_{jk} \psi_j \frac{\partial C_{ij}}{\partial \psi_k} \right) + C_{ij} \left( \partial_j V - \frac{e g}{2} S \partial_i B \right).
\]

Similarly the second- and higher-order equations take the form

\[
\partial_i C_j + \partial_j C_i = e B \left( \epsilon_{ik} C_{kj} + \frac{g}{2} \epsilon_{jk} \psi_j \frac{\partial C_{ij}}{\partial \psi_k} \right) + C_{ijk} \left( \partial_k V - \frac{e g}{2} S \partial_i B \right).
\]
etc. For the radial functions $V(r)$ and $B(r)$: $\partial_i V = \frac{(x_i/r)V'}{\partial_i B} = \frac{C_{\ldots j} S j}{r} (V' - \frac{e^2}{2} S B')$. 

Let us now consider some specific cases. Universal generalized Killing vectors are

$$C_i = (\gamma_i, \epsilon_{ij} x_j, \psi_i, \epsilon_{ij} \psi_j),$$

with $\gamma_i$ a constant vector. Observe that $S$ is a constant of motion itself:

$$\{H, S\} = 0,$$

and all quantities quadratic in the Grassmann variables are proportional to $S$.

- A constant Killing vector $\gamma_i$ gives a constant of motion only if we can find solutions for the equations

$$\partial_i C = e B \epsilon_{ij} \gamma_j, \quad B \epsilon_{ij} \frac{\partial C}{\partial \psi_j} = \gamma_l \left( \frac{2}{e^2} \partial_i V - S \partial_i B \right).$$

Now for a Grassmann-even function $C = c_0 + c_2 S$, the left-hand side of the second equation vanishes; therefore we must require $B$ and $V$ to be constant. Taking $V = 0$, this leads to the solution

$$C = -e B \epsilon_{ij} \gamma_j x_j, \quad V = 0, \quad B = \text{constant}.$$  

The corresponding constant of motion is $\gamma_l P_l$, with

$$P_l = \Pi_l - e B \epsilon_{ij} x_j,$$

identified with 'magnetic translations' [25].

- Next we consider the linear Killing vector $C_i = \epsilon_{ij} x_j$, with all higher-order coefficients $C_{ij\ldots} = 0$. Again for the Grassmann-even $C$ the left-hand side of equation (75) vanishes, and we get the condition

$$\epsilon_{ij} x_j \partial_i B = \epsilon_{ij} x_j \partial_j V = 0,$$

which is automatically satisfied for the radial functions $B(r)$ and $V(r)$. Therefore we only have to solve equation (76):

$$\partial_i C = -e B x_i = -\frac{e x_i}{r} (rB).$$

We infer that $C(r)$ is a radial function, with $C' = -e r B$. Therefore $C$ is given by the magnetic flux through the disk $D_r$ centered at the origin with radius $r$:

$$C = -\frac{e}{2\pi} \int_{D_r} B(r) \, d^2 x \equiv -\frac{e}{2\pi} \Phi_B(r).$$

We then find the constant of motion representing angular momentum:

$$L = \epsilon_{ij} x_j \Pi_j + \frac{e}{2\pi} \Phi_B(r).$$

- There are two Grassmann-odd Killing vectors, the first one being $C_i = \psi_i$. With this ansatz, we get for the scalar contribution to the constant of motion the constraints

$$\frac{e^2}{2} B \epsilon_{ij} \psi_j \frac{\partial C}{\partial \psi_i} = \psi_i \partial_i V, \quad \partial_i C = \frac{e B}{2} (2 - g) \epsilon_{ij} \psi_j.$$  

It follows that either $g = 2$ and $(C, V)$ are constant (in which case one may take $C = V = 0$), or $g \neq 2$ and $C$ is of the form

$$C = \epsilon_{ij} K_j(r) \psi_j \quad \text{with} \quad \partial_i V = -\frac{e^2}{2} B K_i, \quad \partial_i K_j = \frac{(2 - g) e B}{2} \delta_{ij}.$$
This is possible only if $B$ is constant and

$$K_i = \frac{eB(2-g)}{2} x_i \equiv \kappa x_i, \quad V(r) = \frac{g(g-2)}{8} e^2 B^2 r^2 = -\frac{eg\kappa}{4\pi} \Phi_B(r). \quad (90)$$

It follows that we have a conserved supercharge of the form [25]

$$Q = \psi_i (\Pi_i - \kappa \epsilon_{ij} x_j). \quad (91)$$

The bracket algebra of this supercharge takes the form

$$i\{Q, Q\} = 2H + (2-g) eBJ, \quad J = L + S. \quad (92)$$

Of course, as $S$ and $L$ are separately conserved, $J$ is a constant of motion as well.

Finally we consider the dual Grassmann-odd Killing vector $C_i = \epsilon_{ij} \psi_j$. Then the constraints (75) and (76) become

$$\frac{eg}{2} B \frac{\partial C}{\partial \psi_i} = \partial_i V, \quad \partial_i C = \frac{(g-2)eB}{2} \psi_i, \quad (93)$$

implying that $C = N_i(x) \psi_i$ and

$$\frac{eg}{2} B N_i = \partial_i V, \quad \partial_j N_j = \frac{(g-2)eB}{2} \delta_{ij}. \quad (94)$$

As before, $B$ must be constant and the potential is identical to (90):

$$N_i = -\kappa x_i, \quad V = -\frac{eg\kappa}{4\pi} \Phi_B(r) = \frac{g(g-2)}{8} e^2 B^2 r^2. \quad (95)$$

Thus we find the dual conserved supercharge [7],

$$\tilde{Q} = \epsilon_{ij} \psi_i (\Pi_j - \kappa \epsilon_{jk} x_k) = \psi_i (\epsilon_{ij} \Pi_j + \kappa x_i), \quad (96)$$

which satisfies the bracket relations

$$i\{\tilde{Q}, \tilde{Q}\} = 2H + (2-g) eBJ, \quad i\{Q, \tilde{Q}\} = 0. \quad (97)$$

Thus the harmonic potential (90) with constant magnetic field $B$ allows a classical $N = 2$ supersymmetry with supercharges $(Q, \tilde{Q})$, whilst the special conditions $g = 2$ and $V = 0$ allow for $N = 2$ supersymmetry for any $B(r)$.

8. Discussion

In this paper we studied, in the framework of the covariant Hamiltonian dynamics proposed in [13], the symmetries and the supersymmetries of a spinning particle coupled to a magnetic monopole field. The gyromagnetic ratio determines the type of (super)symmetry the system can admit: for the Pauli-like Hamiltonian (1) $N = 1$ supersymmetry only arises for the gyromagnetic ratio $g = 2$ and with no potential, $V = 0$, confirming Spector’s observation [12]. We also derived additional supercharges, which are not square roots of the Hamiltonian of the system, though.

A Runge–Lenz-type dynamical symmetry requires instead an anomalous gyromagnetic ratio,

$$g = 0 \quad \text{or} \quad g = 4,$$

with the additional bonus of an extra ‘spin’ symmetry. These particular values of $g$ come from the effective coupling of the form $F_{ij} \equiv \epsilon_{ijk} D_k \Phi$, which add or cancel for self-dual fields, $F_{ij} = \epsilon_{ijk} D_k \Phi$ [11].
The super- and the bosonic symmetry can be combined; the price to pay is, however, to enlarge the fermionic space, as proposed by D’Hoker and Vinet [9] (see also [11]); this provides us with an $\mathcal{N} = 2$ SUSY.

Our recipe also applies to a planar fermion in any planar magnetic field (i.e. one perpendicular to the plane). As an illustration, we have shown, for ordinary gyromagnetic, that in addition to the usual supercharge $(91)$ generating the supersymmetry, the system also admits another square root of the Pauli Hamiltonian $H$. This happens due to the existence of a dual Killing tensor.

At last, we remark that confining the spinning particle onto a sphere of fixed radius $\rho$ implies the set of constraints [4]
\[
\vec{x}^2 = \rho^2, \quad \vec{x} \cdot \vec{\psi} = 0 \quad \text{and} \quad \vec{x} \cdot \vec{\Pi} = 0. \tag{98}
\]
This freezes the radial potential to a constant, and we recover the $\mathcal{N} = 1$ SUSY described by the supercharges $Q$, $Q_1$ and $Q_2$ for the ordinary gyromagnetic factor $g = 2$.

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