Splitting of macroscopic fundamental strings in flat space and holographic hadron decays

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Abstract

In this review article we present the calculation of the splitting rate in flat space of a macroscopic fundamental string either intersecting at a generic angle a D-p-brane or lying on it. The result is then applied, in the context of the string/gauge theory correspondence, to the study of exclusive decay rates of large spin mesons into mesons. As examples, we discuss the cases of $\mathcal{N} = 4$ SYM with a small number of flavors, and of QCD-like theories in the quenched approximation. In the latter context, explicit analytic formulas are given for decay rates of mesons formed either by heavy quarks or by massless quarks.

1 Introduction

Since the mid–seventies, when it was discovered [1] that quantized string theory incorporates gravity, this fact has been the main agent pushing forward the development of the theory, even if quantum gravity has not yet come within the realm of tomorrow’s experiments. Whereas some very surprising structural properties were uncovered, these have not (yet) brought string theory to the level of presently testable predictions as applied to gravity or the universe. The discovery of, most notably, branes [2], and the string/gauge theory correspondence [3] have however also
revitalized the more ancient arena for the theory of strings, namely strong interaction physics. Under the accepted paradigm of quantum chromodynamics with color confinement, the latter is realized by the formation of a color flux tube. String theory provides a model for such color flux tubes. In this way, developments in ‘pure’ string theory alluded to above can be put to use for the ‘applied’ string theory of hadronic physics. At present, no single standard model of strings (and branes) for strongly interacting particles has been agreed upon: string/brane models do not exactly correspond to the physical world, and parameters in manageable computations are not always in the measured physical ranges, to put it mildly. Nevertheless we think that hadronic physics is a natural arena to put the tricks and tools of string technology to use.

A variety of hadronic properties have been approached from this perspective. Using the string/gauge theory correspondences, various brane settings have been proposed that mimic at least partly a $SU(N)$ color gauge theory with quarks. Spectra (see for example [4]) and scattering processes, both inclusive [5] and exclusive [6] were investigated. On a perhaps more model independent level, the question of the color flux tube model itself (thin strings vs. fat strings) received attention in [7].

In this paper, we review our treatment [8, 9] of meson decays, where mesons are pictured as a quark and an antiquark at opposite ends of a color tube. The way that quarks and flux tubes are represented differs, depending on the concrete incorporation of QCD (or a QCD-like theory) into a brane model. Always the decay of the meson (into other mesons) is governed by the splitting of the string that represents the color tube. The splitting happens as a consequence of the string crossing another brane, that is required to incorporate different flavors into the model. Therefore, in section 2, we start by discussing the general question of splitting rates when it intersects a brane, following the pattern set in [10, 11]. In section 3, after the general treatment is introduced, a first application is made in which mesons are represented as open strings attached to D7-branes, placed in the $AdS_5 \times S^5$ gravity background sourced by $N \gg 1$ D3-branes, at positions that are correlated with the quark masses of the different flavors. The calculation (unfortunately) requires an approximation where the number of flavors is much smaller than the number of colors. It applies to the decay of high spin mesons $\bar{Q}Q$ into a pair of mesons $\bar{q}q$ and $\bar{q}Q$, where the quarks $q$ and $\bar{q}$ created in the decay have mass $m_q \ll m_Q$. In a second model, a stack of $(N \gg 1)$ D4−D3−branes provides the gauge theory background to which $N_f D6−[12]$ or $D8−[13]$ branes add flavor. For both cases, we model the decay $\bar{Q}Q \to \bar{Q}q + \bar{q}Q$ and $\bar{q}q \to \bar{q}q + \bar{q}q$ respectively, where the latter involves light quarks only.

2 Splitting of macroscopic strings in flat space

In this section we provide the formulas for the splitting rate of a macroscopic open string either intersecting at generic angle $\theta$ a generic Dp-brane, or lying on a generic Dp-brane, in flat space [8, 9]. The two computations are very similar and can be carried out at the same time.

A very useful trick to set up this computation is, following [10, 11, 14], to use the S-matrix formalism and an optical theorem, deriving thus the rate from the forward amplitude for the propagation of the string. The latter can be obtained from a vertex operator matrix element
after compactifying some space dimensions on a (very large) torus. In the case where the string intersects the Dp-brane, we compactify on $R_t \times T^2_\theta \times T_{p-1} \times T_{p-1}^\perp$, where $R_t$ refers to the time direction. The $T^2_\theta$ is parameterized by $(x^1, x^2) \simeq (x^1 + n_1 l_1 + n_2 l_2 \cos \theta, x^2 + n_2 l_2 \sin \theta)$, with $n_1, n_2 \in \mathbb{Z}$, and contains the nontrivial geometrical information concerning the angle of incidence. It is the compactification of the plane where the interaction takes place. The Dp-brane is then wrapping $T^p_{\parallel}$ and filling the direction $x^1$ inside $T^2_\theta$, and these directions will be decompactified at the end of the calculation. Note that in the actual models of decaying mesons, the $x^2$ direction will be in the transverse geometry and not in the Minkowski part of the metric, which will be accounted for by $x^1$ and by some dimensions of the (decompactification of the) $T^p_{\parallel}$ factor. The macroscopic string is winding along the other periodic direction on $T^2_\theta$, namely $x^2$, see figure 1.

Figure 1: The two torus $T^2_\theta$. The thick line is the brane, the slim line is the string. On the right, the string after the splitting.

In the case of the string lying on a Dp-brane along a direction $X$, the compact space is $R_t \times X \times T^p_{\parallel} \times T^\perp_{p-1}$ where $X$ has the length of the string $L$.

In order to avoid unnecessary complications in writing down the vertex operators, we adopt the trick [14] of choosing the periodic direction along which the string is wrapped as a “temperature” direction, giving the opposite GSO projection of the usual one, for which the ground states are scalars. Since we are considering very long and therefore very massive strings, one expects that the difference with respect to the usual GSO projection is irrelevant, because it involves a finite number of excitations only.

As to the final states after the splitting, one expects them to be very excited, kinked strings, so their vertex operators are presumably quite complicated. We are going to avoid the need of writing down such operators by adopting an optical theorem, that allows to sum over all the possible final states of the splitting, giving the total decay rate as the properly normalized imaginary part of the ‘forward’ amplitude, see figure 2. To leading order in $g_s$, to which we limit ourselves, the total decay rate is just the one for the simple splitting, giving then the desired result. The advantage of computing just the ‘forward’ amplitude is that it involves the same simple vertex operators for the “in” states and the “out” states. In our case it can be derived from the following correlator of two closed string vertex operators on the disk

$$A = \langle \mathcal{V}_{(0,0)}(p_L, p_R) \mathcal{V}_{(-1,-1)}(p'_L, p'_R) \rangle .$$

(1)

Even if this is an open string process on the disk, the states are closed strings since the open
The volume factor is \( V^{\theta} = \sin \theta l_1 l_2 V_{\perp} V_{\parallel} \) for the case of the string intersecting the brane, and \( V^L = LV_{\parallel} V_{\perp} \) for the string lying on the brane, with \( V_{\parallel} = \text{Vol}(T_{\parallel}) \) and \( V_{\perp} = \text{Vol}(T_{\perp}) \). It comes from the normalization of the amplitude with respect to the zero modes in the compact dimension. In the formulas above, \( \kappa \) is the gravitational constant (in the small energy limit the correlator (1) gives the propagation of the graviton), \( \phi, \tilde{\phi} \) are the bosonized superghosts and \( X, \tilde{X}, \psi, \tilde{\psi} \) the world-sheet bosons and fermions. We need one fixed and one integrated vertex because the amplitude is on the disk.

The left and right momenta, on shell at the tachyon mass, have to satisfy

\[
p^2_L = p^2_R = 2 \alpha',
\]

\( p_{L,R} = p \pm \vec{L}/2\pi\alpha' \), with \( \vec{L}^\theta = (0, l_2 \cos \theta, l_2 \sin \theta, \vec{0}_\parallel, \vec{0}_\perp) \) or \( \vec{L}^L = (0, L, \vec{0}_\parallel, \vec{0}_\perp) \). In the case of the string intersecting the Dp-brane, we also allow for a possible (almost continuous) velocity of the string along \( T_{\parallel} \), that is in the directions parallel to the brane and orthogonal to the string world-sheet. The string momentum has then the form

\[
p^\theta = \frac{m}{\sqrt{1 - v^2}}(1, 0, 0, \vec{0}_\parallel, \vec{0}_\perp), \quad \text{with} \quad m^2 = \left( \frac{l_2}{2\pi\alpha'} \right)^2 - \frac{2}{\alpha'},
\]

\[ m \text{ being the mass of the state, } \vec{v} \in T_{\parallel} \text{ and } \vec{0} \in T_{\perp}. \]

At leading order in \( g_s \), the Dp-brane is a fixed background object and does not recoil, so in (1) one can take \( |\vec{v}| = |\vec{v}'| \). Instead, for the string lying on the Dp-brane, we work in the rest frame of the string itself, so that

\[
p^L = m(1, 0, \vec{0}_\parallel, \vec{0}_\perp), \quad m^2 = \left( \frac{L}{2\pi\alpha'} \right)^2 - \frac{2}{\alpha'}.\]

The amplitude can be obtained by contracting all the fields in (1), using the usual formulas

\[
\langle X^\mu(z) X^\nu(z') \rangle = -\frac{\alpha'}{2} \eta^\mu\nu \log(z - z') , \quad \langle X^\mu(z) \tilde{X}^\nu(z') \rangle = -\frac{\alpha'}{2} G^{\mu\nu} \log(z - z')
\]

Figure 2: The optical theorem: the imaginary part of the forward amplitude is expressed as a the sum on the final states of the decay.
and the analogous ones for the fermions $\psi$ and the ghosts $\phi$. The open string metrics read in the two cases
\[ G^{\mu\nu, \theta} = \text{diag}(-1_I, 1, -1_{\parallel}, -1_{\perp}) , \quad G^{\mu\nu, L} = \text{diag}(-1_I, 1, 1_{\parallel}, -1_{\perp}) . \]

The invariants that appear in the amplitude are then
\[ -\sigma = \frac{\alpha'}{2} p_L \cdot G \cdot p_R = \frac{\alpha'}{2} p_L' \cdot G \cdot p_R' , \]
\[ -1 - \frac{\alpha' t}{4} = \frac{\alpha'}{2} p_L p_L' = \frac{\alpha'}{2} p_R p_R' , \]
\[ \sigma - 1 = \frac{\alpha'}{2} p_L \cdot G \cdot p_R' = \frac{\alpha'}{2} p_L' \cdot G \cdot p_R , \]
(note that $p_L'_{R} = -p_{L,R}$) with $t = 0$, $\sigma^L = -1 + \alpha'(l_2^2/2\pi\alpha')^2 \cos^2 \theta$, $\sigma^L = -1 + \alpha'(L^2/2\pi\alpha')^2$.

The calculation concerns macroscopically long strings, so the relevant limit is that of large $l_2$ and $L$, hence large $\sigma^\theta \simeq \alpha'(l_2^2/2\pi\alpha')^2 \cos^2 \theta$ (unless $\theta = \pi/2$) and large $\sigma^L \simeq \alpha'(L^2/2\pi\alpha')^2$.

Although $t = 0$, we keep $t \neq 0$ as a regulator for the divergence in the real part of the amplitude as $t \to 0$.

After the contractions are done, in order to obtain the amplitude $\mathcal{M}$ one faces the integral
\[ \int_0^1 dx \ (1 - x)^{-1 - \alpha' t/2} (1 + x)^{1 + 2\sigma - \alpha' t/2} - 1 - \sigma \sim 2^{2\sigma} \frac{\Gamma(-\alpha' t/4) \Gamma(-\sigma)}{\Gamma(-\alpha' t/4 - \sigma)} , \]
where the approximate expression is valid as $t \to 0$. The large $\sigma$ limit fluctuates wildly on the real axis, as one can see from the approximate expression, since it contains the closed string state poles at integer values of $\sigma$ with zero width. These fluctuations are averaged by taking the limit in a direction in the complex $\sigma$ plane at a small angle. The infinitely narrow poles will then contribute with the proper weight to the imaginary part. Practically, this amounts to applying Stirling’s formula with the proper choice of phase, and results in
\[ \mathcal{M} \simeq -N_{D^2} \frac{k^2}{(2\pi)^2 \bar{V}} \frac{4(\sigma)^{1+\alpha'/4}}{\alpha' t} e^{-i\pi t \alpha'/4} , \]
where the normalization $N_{D^2}$ can be obtained by T-duality from the standard partition function normalization $2\pi^2 V_0 \tau_0$, giving $N_{D^2} = 2\pi^2 l_1 V_\parallel \tau_p = 2\pi^2 l_1 V_\parallel/(2\pi)^p (\alpha')^{(p-1)/2} g_s$, $N_{D^2} = 2\pi^2 L V_\parallel/(2\pi)^p (\alpha')^{(p+1)/2} g_s$.

Using the optical theorem in order to extract the decay rate $\Gamma$ taking the imaginary part of (9), $\Gamma = \frac{1}{m} \text{Im} \mathcal{M}$, the singularity at $t = 0$ is resolved. Thus, the final results of the computation are
\[ \Gamma^\theta = \frac{g_s}{16\pi \sqrt{\alpha'}} \cdot \frac{(2\pi \sqrt{\alpha'})^{(8-\rho)}}{V_\perp} \cdot \frac{\cos^2 \theta}{\sin \theta} , \]
for the splitting rate of a string intersecting at an angle $\theta$ a generic Dp-brane and
\[ \Gamma^L = \frac{g_s}{32\pi^2 \alpha'} \cdot \frac{(2\pi \sqrt{\alpha'})^{(9-\rho)}}{V_\perp} \cdot L , \]

\footnote{Note that this quantity is finite and does not depend on the (transversal) velocity of the string, since it is computed in the rest frame of the latter.}
for the splitting rate of a string of length \( L \) lying on a generic Dp-brane.

The interpretation of the decay rates (10), (11) is the following. First of all, we have the natural \((2\pi\sqrt{\alpha'})^{(8-p)}/V_{\perp}\) or \((2\pi\sqrt{\alpha'})^{(9-p)}/V_{\perp}\) suppression given by the transversal torus. It is due to the quantum delocalization of the string in the directions transverse to the brane and it just states that the distance between the string and the brane in the transverse directions should be of order \( \alpha' \) in order for the interaction to take place. Second, in the rate (10) we have the factor \( 1/\sin \theta \) that describes the fact that, when the string becomes more and more parallel to the brane, the breaking probability increases, since the tension of the string creates a bigger transversal force which helps the string splitting. We also have the \( \cos^2 \theta \) term which is the natural term symmetric as \( \theta \to -\theta \) that vanishes for the supersymmetric configuration \( \theta = \pi/2 \), for which the string does not split. In the rate (11) we have instead the \( L \) factor, which is the phase space term, due to the fact that since the string is entirely on the brane, it can split at any point, so the rate is proportional to its length \( L \).

Note that the calculation giving the rate (10) is not valid, strictly speaking, for the extreme values \( \theta = 0 \) (by construction: the torus used for the calculation becomes singular) and \( \theta = \pi/2 \) (when we are no more in the Regge regime of large \( \sigma \)). In the latter case the behavior of the resulting rate is nevertheless the expected one. For \( \theta \sim 0 \), instead, we observe that if we impose that the vanishing torus direction, of length \( L \sin \theta \), in the limit becomes one of the transverse directions, we can write \( V_{\perp(8-p)} = V_{\perp(9-p)}/L \sin \theta \). By making this substitution in (11) we get the interpolating rate

\[
\Gamma^{\text{int}} = \frac{g_s}{32\pi^2\alpha'} \cdot \frac{(2\pi\sqrt{\alpha'})^{9-p}}{V_{\perp(9-p)}} \cdot L \cdot \cos^2 \theta ,
\]

which for \( \theta \to 0 \) exactly gives the rate (11) as it should, since the string is ultimately lying on the Dp-brane for \( \theta = 0 \).

### 3 Holographic hadron decays

In the previous section we have obtained very general results about the decay rate for the splitting of fundamental strings in flat space. We are now going to apply them in the study of some dynamical process involving mesonic states in the context of the gauge/string duality.

The basic idea is the following. In the gauge/gravity correspondence one usually starts from a gauge theory engineered using a stack of \( N \) D-branes in some background. By taking the large \( N \) limit, the strong 't Hooft coupling regime of the gauge theory is expected to be described by the near horizon limit of the geometry created by the D-branes. Even if matter in the fundamental representation can be in principle incorporated in this picture, this can be difficult to achieve in practice if the number of flavors is arbitrary. However, as discussed in [13], when the number of flavors is small (with respect to the number of colors \( N \)), we can study them from the holographic point of view by adding appropriate probe branes in the supergravity background dual to the theory without flavors. In this dual description, mesons are described by open strings attached to the flavor probe-branes. In particular one can have mesonic states of
high spin/energy which admit a semiclassical description as spinning macroscopic fundamental strings. We are going to study decay processes involving these kinds of states.

Let us discuss the general setting. The simplest supergravity backgrounds dual to 4d field theories have metrics of the form

$$ds^2 = e^{A(r)}(-dt^2 + dr^2 + \rho^2 d\eta^2 + dx_3^2) + e^{B(r)}d\phi_i d\phi_j + G_{ij}(r,\phi)d\phi^i d\phi^j,$$

(13)

where $r$ and $\phi^i$, $i = 1, \ldots, 5$, describe respectively the radial and the angular coordinates of the six dimensional internal space, and $r$ is associated (in a model dependent way) to the energy scale of the dual theory whose UV and IR regimes correspond to large and small $r$ respectively.

Quite generally, we picture the probe D-brane associated to the addition of a flavor $Q$ as partly filling some internal angular directions $\chi^a$, while it is located at a fixed value of the remaining angles $\psi^I_Q$. Furthermore, it fills the radial coordinate from $r = \infty$ up to a point defined by fixed angles $\chi^a_Q$ and a minimal radius $r_Q$. This minimal radius is then holographically associated to the flavor mass $m_Q$ in a model dependent way. The fluctuations of the brane describe low mass mesonic states, while mesonic states with very large spin can be described by semiclassical spinning open strings with end-points attached to the flavor D-brane.

We will now focus on the high spin mesons associated to rigid spinning strings whose world-sheet is of the form

$$t = \tau, \quad \eta = \omega \tau, \quad r = r(\sigma), \quad \rho = \rho(\sigma), \quad \chi^a = \chi^a_Q, \quad \psi^I = \psi^I_Q.$$

(14)

The relevant equations of motion can be easily derived from the effective action

$$S = -\frac{1}{2\pi\alpha'} \int d\tau d\sigma e^{A(r)} \sqrt{(1 - \omega^2 \rho^2)[(\rho')^2 + e^{B(r)-A(r)}(r')^2]},$$

(15)

supplemented by the boundary conditions $r|_{\partial \Sigma} = r_Q$ and $\rho'|_{\partial \Sigma} = 0$ [4]. For our purposes, it is convenient to fix the remaining reparameterization invariance by choosing the gauge $\sigma = r$. Then, if $r_0$ indicates the minimal radius reached by the string, the energy and the spin of the meson are given [4] by $E = 2F[r_0, r_Q]$ and $J = 2H[r_0, r_Q]$ where

$$F[a, b] = \frac{1}{2\pi\alpha'} \int_a^b d\sigma e^{A(\sigma)} \sqrt{\frac{(\rho')^2 + e^{B(\sigma)-A(\sigma)}}{1 - \omega^2 \rho^2}},$$

$$H[a, b] = \frac{\omega}{2\pi\alpha'} \int_a^b d\sigma \rho^2 e^{A(\sigma)} \sqrt{\frac{(\rho')^2 + e^{B(\sigma)-A(\sigma)}}{1 - \omega^2 \rho^2}}.$$ 

(16)

One can in principle invert these relations in order to get for example $E$ and $r_0$ as functions of the spin (and $r_Q$).

Let us consider now the effect of the introduction of another D-brane associated to a lighter flavor $q$ of mass $m_q < m_Q$, i.e. with $r_q < r_Q$. We want to study the possible decay of the above string, associated to a meson $\bar{Q}Q$, into a couple of strings representing the mesons $\bar{Q}q$ and $\bar{q}Q$, see figure 3.

2Decays of small spin mesons have been discussed in [16, 17].
We will focus on decay rates that can be described within the semiclassical picture, where the string classically intersects the $q$-brane and then can split in the semiclassical regime. For other kinds of mesons, whose D-brane is not aligned with the brane corresponding to the heavier meson in such a way that the spinning string intersects it, the meson decay involves world-sheet instantonic transitions and then is exponentially suppressed in the semiclassical regime. In order not to have this exponential suppression we must then fulfill the conditions that $r_0[J] \leq r_q$ and $\psi_q^I = \psi_Q^I$. If these conditions are satisfied, our classical string can split into two open strings with end-points attached to different branes which indeed correspond to mesons of the kind $\bar{Q}q$ and $\bar{q}Q$. Rigid spinning strings of this kind were studied in [18] and in our case we expect our states to be some excited version of these rigidly rotating strings, with also some linear momentum.

Even if we will not determine the explicit form of the outcoming strings, it is important to note that their energies and total angular momenta (computed with respect to the rest frame of the initial meson) are completely determined by the classical picture. Indeed we can immediately conclude that the lightest outcoming meson will have energy $E_1 = F[r_q, r_Q]$ and total angular momentum $J_1 = H[r_q, r_Q]$, while the heavier meson will have energy $E_2 = E - E_1$ and angular momentum $J_2 = J - J_1$. The outcoming states will also have definite and opposite linear momenta. If for example $P^1$ and $P^2$ denote the linear momenta of the lightest outcoming meson in the directions $x_1 = \rho \sin \omega \tau$ and $x_2 = \rho \cos \omega \tau$, we have that

$$P^1(t) = \frac{\omega}{2\pi \alpha'} \int_{r_q}^{r_Q} d\sigma \rho \cos \omega \tau e^{A(\sigma)} \sqrt{\frac{(\rho')^2 + e^{B(\sigma)} - A(\sigma)}{1 - \omega^2 \rho^2}}, \quad (17)$$

and $P^2$ can be obtained by the same expression by replacing $\cos$ with $-\sin$.

Let us now see what we can say in general on the rate for such a decay using the results obtained previously. Two basic ingredients are the velocity $v$ of the string in the point where

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Figure 3: (a) A large spin meson, bound state of two quarks of large mass $m_Q$, described by a string with both end-points on the same brane. The string intersects a second brane, corresponding to lighter quark masses $m_q$. (b) The strings after the splitting, representing two meson bound states of a heavy quark and a light quark.
it splits and its angle $\theta$ with the brane. These are given by

$$v = \omega \rho(r_q), \quad \cos^2 \theta = \frac{(\rho'(r_q))^2}{e^{B(r_q) - A(r_q)} + (\rho'(r_q))^2}.$$  \hspace{1cm} (18)

In order to determine the decay rate, we have also to take into account the suppression due to the effective transverse volume. This is a delicate point since such a transversal quantum delocalization of the string can be infinite \cite{7}. In fact, if the string is free to sit at a generic point of a transverse dimension, quantum mechanically it is fully delocalized and the effective transverse volume in that direction is the whole length of the direction, which is infinite in the non compact case. The situation is different if the string is classically at a fixed point of a direction, that is it sits at a minimum of a potential. In this case the quantum delocalization, and so the effective length of the dimension, can be smaller. The estimate of this effective size, which can be performed explicitly in the study of cosmic strings \cite{14}, is a non-trivial task in the present setting.

Finally, the decay rate is computed in flat space. When we go to (weakly) curved spaces, one has to replace $\alpha'$ with an effective $\alpha'_{eff}$ which depends on the warp factors of the metric.

3.1 Meson decay in $N = 4$ Super Yang-Mills

Of course, the equation of motion for $\rho(r)$ obtained from (15) is in general not analytically solvable and one must use some numerical or approximated method to evaluate it. We will now consider the most simple example where we can give an approximate analytical estimate of the above observable quantities, namely the maximally supersymmetric case $AdS_5 \times S^5$ with mesons of spin $J \gg \sqrt{\lambda}$ \cite{4}, where $\lambda = g_s N$ represents the 't Hooft coupling of the dual theory. In this case the flavor branes are D7-branes and it is convenient to use a different radial coordinate $z = R^2/r$, with $R^4 = 4\pi \alpha'^2 \lambda$ such that the relevant part of the metric is given by

$$ds^2 = \frac{R^2}{z^2}(-dt^2 + d\rho^2 + \rho^2 d\eta^2 + dz^2) + \ldots.$$  \hspace{1cm} (19)

In this case the mass-radius relation is unambiguous and is given by $m_Q = R^2/2\pi \alpha' z_Q$. As discussed in \cite{4}, in the case $J \gg \sqrt{\lambda}$, the spinning string solution is well approximated by a Wilson loop string \cite{19} slowly spinning around its center of mass, i.e. $\rho(z) \simeq \rho_{st}(z) + \delta \rho(z)$ with very small $\delta \rho(z)$ and

$$\rho_{st}(z) = \int_z^{z_0} dx \frac{x^2}{\sqrt{z_0^4 - x^4}}.$$  \hspace{1cm} (20)

In this case $\omega \ll 1$. Also, it is possible to show that in this limit

$$\omega^2 \simeq \frac{64 C^8 m_Q^2}{\pi^2 \lambda} \left( \frac{\lambda}{J^2} \right)^3, \quad \omega_0^2 \simeq \frac{\pi J^4}{16 C^6 m_Q^2 \lambda},$$  \hspace{1cm} (21)

where $C = \sqrt{2}\pi^{3/2}/\Gamma(1/4)^2 \simeq 0.599$. 

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If we now introduce a lighter flavor by placing a second D7-brane (we will call the original brane “Q-brane” and the second one “q-brane”) at a position \( z_q < z_0 \), the spinning string does intersect it and thus it can split. The condition that the decay is at all possible, which in string theory terms is the fact that the q-brane does intersect the string, \( z_Q < z_q < z_0 \), can be expressed in terms of the particle properties as

\[
1 > \frac{m_q}{m_Q} > \frac{4C^3\lambda}{\pi J^2} .
\]

The existence of a limiting minimal value of \( m_q \) in order for the decay to happen can be understood in field theory as follows. Since the theory is in a Coulomb phase, the binding energy of the heavy quarks decreases as their distance and so their spin, increases. The total energy is \( E_{QQ} \sim 2m_Q - \alpha m_Q\lambda/J^2 \), with \( \alpha \) being some constant \([4]\). On the other hand, the binding energy \( E^{\text{bind}} \) of the meson formed by a heavy and a light quark in the limit of large quark separation is proportional to the mass \( m_q \) of the light quark and can be larger than \( m_q \) in modulus \([20] [18]\). The total energy of the two mesons produced in the decay would be \( 2E_{Qq} \sim 2m_Q - E^{\text{res}} \) with positive \( E^{\text{res}} \). This is the strong coupling effect that makes it possible for the heavy quark meson to decay, since for large \( J^2/m_Q\lambda \) it is possible that \( E^{\text{res}} > \alpha m_Q\lambda/J^2 \), so that the total energy of the two produced mesons is smaller than the one of the heavy quark meson \([20]\). This is the regime where the string does intersect the q-brane in the dual setting.

But, crucially, since \( E^{\text{res}} \) is proportional to \( m_q \), for any fixed value of \( m_Q\lambda/J^2 \) there exists a minimal value of \( m_q \) below which \( E^{\text{res}} \) is smaller than \( \alpha m_Q\lambda/J^2 \), forbidding the decay. This critical value is precisely the one in (22).

As we said before, in order for the decay not to be exponentially suppressed, we must also require that the angular position of the two branes in the transverse direction are equal, \( \psi_q = \psi_Q \). In the dual field theory, an unequal angle \( \Delta\psi = \psi_q - \psi_Q \neq 0 \) would enter as a phase in the coupling of one type of quark, let us say q, with the complex scalar of \( \mathcal{N} = 4 \) SYM charged under \( \psi \), schematically in the superpotential as \( W = e^{i\Delta\psi} q\Phi_q \). This phase suppresses the decay channel mediated by \( \Phi \) and ultimately should be responsible for the exponential suppression of the decay rate. We do not venture at present to give a precise and explicit explanation of the suppression in field theory at strong coupling.

Coming back to the string side of the duality, note that the velocity of the string at \( z_q \) is of order \( \sqrt{\lambda}/J \) and can be neglected in first approximation. Furthermore, if we restrict to the case in which the q D7-brane is not “too close” to the Q D7-brane (like for example if \( z_Q/z_q = m_q/m_Q \sim \lambda/J^2 \)), then the decay rate is not completely suppressed since the angle \( \theta \) between the string and the brane is not too close to the value \( \pi/2 \) and can be evaluated to be

\[
\theta \simeq \arctg \left( \frac{\pi m_q J^2}{4C^3 m_Q \lambda} \right)^{1/4} - 1 .
\]

The effective slope is given by \( \alpha_{eff}' = \frac{\pi^{-3/2} \sqrt{\lambda}}{2m_q^2} \).

Finally, in order to extract the total decay rate we need the transversal volume \( V_L \), that in this case is one-dimensional. As we have already said, this is possibly the most subtle point
of the whole derivation. The string is classically at a point of the transverse dimension, so its quantum delocalization can be smaller than the size of the latter. Contrary to what is done in [14], we cannot estimate the delocalization with a local calculation around the intersection point, since ultimately what generates the classical localization are the boundary conditions on the Q-brane, which fix the value of the angle $\psi_Q$ (locally, there is no potential). So, since the complete calculation of the quantum fluctuations around the classical string embedding seems unfeasable at present, we will adopt a prudent choice that gives as a natural (maximal) estimate a transversal length of order $2\pi R \sim \sqrt{\alpha'}/\lambda^{1/4}$. We expect the actual value of the delocalization to be of the same order. Then, after taking into account the fact the the string can split at two distinct points, we obtain the following minimal estimate of the decay rate

$$\Gamma_{QQ\rightarrow q\bar{q}} = \frac{m_q \sqrt{\lambda}}{8\sqrt{\pi N} \left( \frac{\pi m_q}{4c'' m_Q} \right)^2 \left( \frac{J^2}{\pi} \right)^2 \sqrt{\left( \frac{\pi m_q}{4c'' m_Q} \right)^4 \left( \frac{J^2}{\pi} \right)^4} - 1}.$$  

(24)

The decay rate has precisely the expected behavior from the field theory point of view. It describes a $1/N$ process that increases as the coupling $\lambda$ increases. As the difference between the mass $m_q$ of the light quark and the mass $m_Q$ of the heavy quark becomes larger and larger, the decay is more and more probable. However, there is a lower bound on this difference, below which the rate looses its meaning due to the square root. This lower bound is precisely the point at which the $q$ D7-brane reaches the lowest point of the string, below which there is no more intersection and therefore no decay. Finally, the rate decreases as the spin $J$ of the heavy meson increases. In fact, increasing $J$ means increasing the distance between the two heavy quarks and since the theory is non confining, this reduces the binding energy and ultimately the energy density, making the decay process more and more disfavored.

### 3.2 Meson decay in QCD-like theories

Let us now try and put our general setting at work for models which are a bit much closer to quenched QCD than the one discussed above. Shortly after the $AdS/CFT$ correspondence was formulated, a non singular gravity dual to a confining, non supersymmetric 4d Yang-Mills theory was found in [21]. The gauge theory describes the low energy dynamics of a stack of $N \gg 1$ D4-branes wrapped on a supersymmetry breaking circle and, in the limit where the dual gravity description is reliable, it is coupled with adjoint Kaluza-Klein fields. The background metric and dilaton sourced by the D4-branes read

$$\begin{align*}
\text{d}s^2 &= \left( \frac{u}{R} \right)^{3/2} (dx)_\mu dx^\mu + \frac{4R^3}{9u_h} f(u)d\theta^2 + \left( \frac{R}{u} \right)^{3/2} \frac{du^2}{f(u)} + R^{3/2} u^{1/2} d\Omega_4^2, \\
e^\phi &= g_s \left( \frac{u}{R} \right)^{3/4},
\end{align*}$$

(25)

where $f(u) = (u^3 - u_h^3)/u^3$. The radial coordinate $u$ is bounded from below by its “horizon” value $u_h$. String and field theory quantities are connected by the following relations [22, 23]: $3u_h = \lambda m_0 \alpha'$, $\lambda g_s^{-1} = 3\pi N_c m_0 \sqrt{\alpha'}$, $R^3 = \lambda \alpha'/(3m_0)$, $6\pi T = \lambda m_0^2$. Here $\lambda = g_Y^2 N_c$ is the
't Hooft coupling at the UV cut-off and it has to be taken much greater than one in order for the gravity approximation to be valid. Differently from pure Yang-Mills, the theory has two different energy scales: $T$, the Yang-Mills string tension, and $m_0$, the glueball and Kaluza-Klein mass scale.

The addition of $N_f \ll N$ flavors to the model above was realized in [12] by means of $N_f$ D6-brane probes and in [13] by means of $N_f$ D8 probes. In the first model a generic D6 probe, embedded in the geometry (25), extends in the radial direction from a value $u_Q$ up to infinity; the corresponding quark has a constituent mass which depends on $u_Q$ via the relation

$$m_Q = \frac{T}{m_0} \int_{1}^{u_Q/u_h} dz \left[ 1 - \frac{1}{z^3} \right]^{-\frac{1}{2}},$$

which is nothing but the energy of an hypothetical string stretching from the horizon at $u = u_h$ to $u = u_Q$. It turns out that in this model the constituent quark mass cannot be zero, as the possible values which $u_Q$ can take are bounded from below by a certain $u_{min} > u_h$. The flavor symmetry in the model is $U(N_f)$ by construction and the spontaneous breaking of the chiral $U(1)_A$ symmetry is accounted for by the bending of the flavor branes.

In the second model the D8-branes are curved, orthogonal to the circle $S^1(\theta_2)$ and extend up to the horizon $u = u_h$; at large $u$ each curved D8-brane looks like a brane-antibrane pair. This picture provides a nice realization of the dynamical UV restoration of the $U(N_f) \times U(N_f)$ chiral symmetry. The model describes massless quarks.

We are now going to consider mesons with very high spin $J$ in the two models just introduced. If $J \gg \lambda$ the strings associated to the mesons can be studied semiclassically and the general expressions for their splitting rates, as deduced above, can be used to extract the corresponding exclusive meson decay rates. To study the physics of mesons built up by heavy (light) quarks we will use the model with D6-brane (D8-brane) probes.

Heavy quarkonia with very high spin in the setup of [12] are described by macroscopic open strings, with the extrema on a D6-brane at $u = u_Q$, which spin in the Minkowski directions and hang down with a $U$ shape up to a minimal radial position $u = u_0$. The exclusive decay $Q\bar{Q} \rightarrow Q\bar{q} + q\bar{Q}$ is described by the splitting of the string on a second type of flavor D6 brane whose minimum is at a lower position $u_q < u_Q$. The splitting can happen only at one of the two intersection points and the decay rate will be obtained by making use of our general formula (10). The decay is “asymmetric”, in that the decay products are a high spin ($J_1 \gg \lambda$) meson corresponding [18] to a string which bends down close to the horizon, and a meson with much smaller spin ($1 \ll J_2 \ll \lambda$), corresponding to a string hanging down from one brane to the other without approaching the horizon.

In the model of [13], the open strings corresponding to very light mesons extend and spin in the Minkowski directions, thus lying on the flavor branes. This implies that there are infinitely many points where the string can split, all along its length. In this case the decay rate for a process like $q\bar{q} \rightarrow q\bar{q} + q\bar{q}$ will be obtained by making use of our formula (11).

Let us start by estimating the decay rate for the process $Q\bar{Q} \rightarrow Q\bar{q} + q\bar{Q}$ in the D6 model. First of all we must put in eq. (10) the corrected string tension and dilaton to take care of
the fact that we are not in flat space but on the background \([25]\): \(\alpha' \to \alpha'_{eff} = \alpha' \left( \frac{R}{u_q} \right)^{3/2}\), and 
\(g_s \to e^{\Phi(u_q)} = g_s \left( \frac{R}{\alpha'} \right)^{3/4}\). Moreover we shall put \(p = 6\) and estimate the transverse volume as

\[
V_\perp = 2\pi R_{\theta_2} \cdot 2\pi R_{S^4} = \frac{8\pi^2 u_q}{3u_h^{1/2}} \frac{R^{3/2}}{f^{1/2}(u_q)}. \tag{27}
\]

In order to evaluate the \(\theta\)-dependent part in the decay rate, we need to know (see eq. \([18]\) )
the slope of the string profile at the intersection point. For this we do not have an analytic expression in general. However, for high spin mesons, provided \(u_q \gg u_h\) and so (see formula \([26]\) \(m_Q \gg T/m_0\), it is possible \([24, 18]\) to approximate the profile of the decaying open string with that corresponding to a small perturbation of a static, almost-\(U\) shaped Wilson line \([18]\). Using the fact that, in the semiclassical regime we work in \((J \gg \lambda)\), the minimal radial distance \(u = u_0\) reached by the “\(\bar{Q}Q\) string” can be taken equal to \(u_h\) up to exponentially suppressed terms \(^4\) we can give the following expression for \(r'(u) = r'_{st}(u) + \delta r'(u)\)

\[
\begin{align*}
\text{r'}_{st}(u) & = \left(R u_h\right)^{3/2} \frac{1}{(u^3 - u_h^3)}, \\
r'(u) & \approx \left(R u_h\right)^{3/2} \frac{1}{u_h^3 (x^3 - 1)} \left[1 - \frac{x^3 (x - 1)}{y (x^3 - 1)}\right], \quad x \equiv \frac{u_q}{u_h}, \quad y \equiv \frac{u_Q}{u_h}. \tag{28}\end{align*}
\]

We have now all the data to put in our general formulas \([10]\) and \([18]\). The \(\bar{Q}Q \to \bar{Q}q + q\bar{Q}\) decay rate of a large spin meson made up of heavy quarks reads

\[
\Gamma_{D6} = \frac{\lambda m_0}{16\pi^2 N} \sqrt{x} \left[1 + 1 \left(\frac{x - 1}{y (x^3 - 1)}\right)\right]. \tag{29}\]

It is possible to give a clear interpretation of this formula, rewriting it in terms of the constituent quark masses \([20]\). Let us focus on two special limits where we can have an analytic control of our expressions. The first amounts to taking \(u \gg u_h\) (large \(x\)), with \(m_q \approx u_q/(2\pi\alpha') \gg T/m_0\). The resulting rate, expressed in terms of the quark masses, reads

\[
\Gamma_{D6} \sim \frac{\lambda}{16\pi^2 N} \left(\frac{T}{m_0}\right)^{5/2} m_0^{3/2} \left[1 - 2\frac{m_q}{m_Q}\right]. \tag{30}\]

This expression depends on the two scales of the theory. In order to imagine how this could read in a QCD-like theory, let us consider a limit where we identify the two scales taking \(T \sim m_0^2 \sim \Lambda_{QCD}^2\); this way \(\Gamma_{D6} \sim \frac{\lambda \Lambda_{QCD}^{7/2}}{N^2 m_q^{1/2}} \left[1 - 2\frac{m_q}{m_Q}\right]\). We will refer to this formal limit as the “QCD limit”.

The second limit on the masses amounts on taking \(x \approx x_{min}(\approx 1.04, \text{ see } [24])\). This is the small mass limit where

\[
m_q \approx \left(\frac{T}{m_0}\right) \frac{2}{\sqrt{3}} \sqrt[3]{\frac{u - u_h}{u_h}}. \tag{31}\]

\(^4\)More precisely \([25]\) \(u_0 \sim u_h [1 + \exp(-3m_0 L/2)]\). In the semiclassical regime the inter-quark distance \(L\), which increase with \(J\), is very large.
The decay rate now goes as
\[ \Gamma_{D6} \sim \frac{\lambda}{36\pi^2 N \left( \frac{T}{m_0} \right)^2 \frac{m_0}{m_q^2} \left[ 1 - \frac{T}{3m_0m_Q} \right]}, \] (32)
that in the “QCD limit” would read \( \Gamma_{D6} \sim \frac{\Lambda_{QCD}^3}{N \left( m_0 \right)^3} \left[ 1 - \frac{\Lambda_{QCD}}{m_Q} \right]. \)

To get the decay rates in the rest frame of the laboratory, we must multiply the obtained expressions by the relativistic time dilation factor \( \sqrt{1 - v^2}. \) As the decay can happen only around the heavy quarks one can approximate with \( L/2 \) the distance of the splitting point from the center of rotation, so that \( \sqrt{1 - v^2} = \sqrt{1 - (\omega L/2)^2}. \) Then [26, 9] this factor reads \( \sqrt{\frac{2m_0/L}{1 + 2m_0/L}}, \) with \( L \) proportional to some power of \( J. \)

Let us now comment on the decay rates we have found. They are suppressed by \( 1/N, \) grow with the coupling \( \lambda \) and increase as the mass of the produced quarks \( m_q \) decreases: this is indeed an expected behavior. Moreover the leading order suppression of the rate with the mass \( m_q \) is power-like, so the one we have considered is the leading decay channel in the QCD-like strongly coupled gauge theory at hand [5]. The rates are mildly dependent on the mass of the decaying mesons, which indirectly enters in the formulas through the constituent quark mass \( m_Q. \) The rates increase with this mass and go to a constant in the case it is very large.

Let us now shift to the D8 model of [13] to study light meson decays. In order to translate formula (11) to our case, note that
\[ \frac{g_s}{\alpha'} e^\Phi \rightarrow \frac{g_s}{\alpha'_{\text{eff}}} \left( \frac{u_h}{R} \right)^{\frac{3}{2}} = \frac{\lambda m_0^2 \lambda^{3/2}}{N \left( \frac{3\sqrt{2}/2\pi} \right)} \] (33)
and that the strings are on the leading Regge trajectory
\[ L = \sqrt{\frac{8J}{\pi T}} = \frac{2M}{\pi T}, \] (34)
where \( M \) is the meson mass (the energy of the string). We then need to estimate the suppression due to the transverse dimension. The procedure proposed in [14] consists on evaluating the quantum delocalization of the string due to the quadratic fluctuations of the world-sheet massive field associated to the transverse direction. Taking the near horizon limit of the metric in [25] one can study the world sheet sigma model for the transverse directions and discover that

\[ \text{Other processes possibly involve instantonic world-sheet transitions and are exponentially suppressed with } m_q. \]
their fluctuations create a broadening \( \omega = \log[1 + (4R^{3/2}u_h^{1/2})/(9\alpha')] \). In terms of field theory quantities one can thus write \[ \frac{(2\pi\sqrt{\alpha'})^{9-p}}{V_\perp} = \frac{2\pi}{\log^{1/2}(1 + \frac{8\pi T m_0}{9m_0})} . \] (35)

We can now put everything in (11), getting

\[ \Gamma_{D8} = \frac{\lambda}{N} \frac{1}{6\pi} \frac{1}{\log^{1/2}(1 + \frac{8\pi T m_0}{9m_0})} \frac{T \sqrt{J}}{m_0} , \] (36)

or equivalently

\[ \Gamma_{D8} = \frac{\lambda}{N} \frac{1}{6\pi} \frac{1}{\sqrt{2\pi}} \frac{1}{\log^{1/2}(1 + \frac{8\pi T m_0}{9m_0})} \frac{\sqrt{T}}{m_0} M . \] (37)

To get the rate per unit length \( L \) in the meson rest frame, we have to multiply the expression above by the time dilation factor \( \sqrt{1 - v^2} \) and then integrate along the length of the string. This only amounts on multiplying the rate by a constant \( \pi/4 \) factor. In the “QCD limit” one just finds \( \Gamma_{D8} \sim \lambda M/N \).

The result we have obtained for the rate has the expected scaling with \( 1/N \), and with the mass \( M \) of the decaying meson.

4 Discussion

We have reviewed our attempts at coming to grips with experimentally accessible predictions of string theory technology as applied to models that view color tubes of QCD as strings, and meson decays as a splitting of this string. It is encouraging that rather specific results can be obtained, even if they are not yet situated in completely realistic models. From the phenomenological side, this field is dominated by the Lund model \[28\], which has penetrated successfully into widely used event generators. An important ingredient of this model, the Gaussian form of the decay constant as a function of the mass of the quark pair produced in the decay, does not seem to find an easy confirmation in the theory. In this connection the attempt by \[29\] to provide such basis, in models that are similar to the ones used in this paper, can be mentioned. There, it is linked to string fluctuations, and in fact results from a Gaussian fit. We leave to the future the task to resolve this issue, either by providing a perhaps more profound string theory explanation, or possibly by showing that phenomenology can be equally successful with a wider range of functional dependencies.

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