Reconstructions in Ultrasound Modulated Optical Tomography

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Abstract. We describe a mathematical model for ultrasound modulated optical tomography and present a simple reconstruction algorithm and numerical simulations for this model. The computational results show that stable reconstruction of sharp features of the absorption coefficient is possible. A formal linearization of the model leads to an equation with a Fredholm operator, which explains the stability observed in our numerical experiments.

1. Introduction

During the last two decades, optical tomography (OT) has received significant attention as a biomedical imaging modality. This can be attributed, in particular, to the fact that light of optical frequencies is harmless to the human organism and that optical properties of tissues reveal various important biological information, e.g. angiogenesis and hypermetabolism, both of which are well-known indicators of cancer (e.g., [22]). Unfortunately, reconstruction in OT is also known to be severely ill-posed, and consequently the recovery of sharp features is all but impossible. Various attempts to address this problem have been made. In this text, we are interested in a hybrid imaging method (so called Ultrasound Modulated Optical Tomography [22], which we will abbreviate as UOT) that attempts to combine the OT procedure with a simultaneous ultrasound modulation in order to alleviate the instability of OT reconstructions. The idea is to combine the good tumor specificity of OT with the high spatial resolution of ultrasound imaging. This approach utilizes the experimentally observed interaction between ultrasound and light propagation in tissue [13, 22]. As in OT, a coherent light source irradiates the tissue sample and causes speckle interference patterns in the camera’s view of the object’s surface. Simultaneously, a narrowly focused ultrasound wave is scanned throughout the tissue, influencing its optical properties and thus modulating the speckle pattern with ultrasound frequency. By picking up this modulated signal up at the boundary, information about the incident light intensity at the origin of these “tagged” photons, i.e. at the focus of the ultrasound beam, can be
obtained. In other words, by scanning the focus of the ultrasound wave, a measure of the light intensity in the body’s interior can be determined, not only at its boundary. In pure OT, this type of information is usually lost, due to multiple scattering of photons, although there are other variants of optical tomography that also strive to recover this information (see, e.g., [19]). It can be expected that this additional knowledge can help in stabilizing the inversion process and render it substantially less ill-posed than the original OT problem. The numerical experiments and initial analysis that we present in this paper suggest that this intuition is justified.

The literature contains a number of models that address the UOT technique, see for example [10, 13, 17, 20–22]. Most of them describe the coupling between ultrasound and light in terms of stochastic quantities, which permits particle-based simulations of the light intensity modulation effect caused by the ultrasound wave. However, for optical imaging at a depth of several centimeters, photon intensities can be accurately modeled by the diffusion limit, which allows us to formulate a model for the UOT procedure based on a parameter identification problem for a set of coupled diffusion-type partial differential equations. This model, along with a description of the measurements is presented in Section 2. In Section 3, we outline an algorithm that can be used to reconstruct the spatially varying absorption coefficient from the UOT measurements. Examples of the resulting reconstructions for numerical phantom data are provided in Section 4. In Section 5, we (formally) linearize the model and show that then the absorption coefficient is related to the measurements by a Fredholm operator acting between appropriate Sobolev spaces. This explains the stability of the reconstruction observed in numerical experiments. The last section contains final remarks and conclusions.

2. Mathematical model

A detailed description of the physical underpinnings of the UOT procedure can be found, for instance, in [22, Ch. 13]. We give a brief description of the set-up here.

Let the object of interest occupy the domain $\Omega \subset \mathbb{R}^3$. The internal optical properties in the diffusion limit are described by the reduced scattering coefficient $\mu'_s$ and the absorption coefficient $\mu_a$. For imaging soft tissues, it is common to assume $\mu'_s$ roughly equal to a known constant throughout $\Omega$, while the spatially varying absorption $\mu_a(x)$, $x \in \Omega$, represents the target of reconstruction. It is also assumed that the tissue of interest is turbid (highly scattering), so that $\mu_a(x) \ll \mu'_s$. It is known that in such media, the light intensity $u(x)$ inside $\Omega$ can be accurately described by the diffusion approximation (e.g., [5,19]):

$$-\nabla \cdot D\nabla u(x) + \mu_a(x)u(x) = 0 \quad \text{in } \Omega,$$

(1)

where

$$D = D(x) = \frac{1}{3(\mu_a(x) + \mu'_s(x))}.$$

(2)
is the diffusion coefficient. Due to the above conditions imposed on the medium, $D \approx \frac{1}{3a_s} \approx \text{const}$. To simplify the notation, we set $\mu := \mu_a$ in the rest of the text.

Equation (1) needs to be completed by boundary conditions. For tissue in contact with a surrounding medium, Robin-type boundary conditions are typically chosen [8]:

$$2D \frac{\partial u(x)}{\partial n} + \gamma u(x) = S(x) \quad \text{on } \partial \Omega. \quad (3)$$

Here $n$ denotes the outward normal to the surface $\partial \Omega$ and $\gamma > 0$ is a constant describing the optical refractive index mismatch at the boundary. The right hand side $S(x) \geq 0$ represents the intensity of the external light source illuminating the object.

We now assume that, concurrently with light irradiation, the tissue is subjected to scanning by a time harmonic ultrasound pressure wave with amplitude $p(x)$. Photons passing through the affected regions create a speckle pattern on the boundary that is modulated with the ultrasound frequency. The amplitude of this modulation is the signal that can be picked up at the boundary. It has been argued (e.g., [17, 22]) that the effect of such modulation can be considered as creation of a virtual optical source at the affected location $x$, with the strength of the source proportional to the fluence $u(x)$ of the original light field, as well as to the square $|p(x)|^2$ of the ultrasound amplitude. Denoting the fluence created by this virtual optical signal by $v(x)$, we arrive at the following boundary value problem for $v$:

$$\begin{align*}
-\nabla \cdot D \nabla v(x) + \mu(x) v(x) &= \alpha |p(x)|^2 u(x) \quad \text{in } \Omega, \\
2D \frac{\partial v(x)}{\partial n} + \gamma v(x) &= 0 \quad \text{on } \partial \Omega,
\end{align*} \quad (4)$$

where $\alpha$ is a physical yield factor, which will be assumed to be constant.

2.1. Focusing the ultrasound wave

We assume that the ultrasound pressure function $p$ is known and we have some control over it, in the sense that it can be focused to a particular location within the region of interest. In what follows, we will make the assumption that ultrasound signals can be sharply focused to arbitrary points $\xi$ in $\Omega$, such that $|p(x)|^2$ approximates the delta-function $\delta(x - \xi)$. It is known that in practice, even when the material is acoustically homogeneous, ultrasound waves can only be focused to a certain extent (e.g., [9]), and that such focusing is always limited by the acoustic setup and the bandwidth of the ultrasound transducer. However, since (4) is linear in $|p|^2$, the case of $|p|^2 \approx \delta$ will help us understand the behavior of the model even for other, non-focused ultrasound signals. Indeed, as suggested in [12], one can in principle use non-focused ultrasound waves to synthesize the measurements we will assume here. This issue is further discussed in Section 6.2.

2.2. Measurements

Photon detectors are placed along a part of the boundary $\Gamma \subset \partial \Omega$ to measure the modulated light intensity $v$. To simplify the discussion, we will assume in the following
that only a single photon detector is used, so that $\Gamma = \{ \eta \}$ for some $\eta \in \partial \Omega$. In practice, however, the measurements are polluted by noise, and collecting information from sensors at multiple locations $\eta \in \Gamma$, where $\Gamma$ is a part of the boundary $\partial \Omega$, would help reducing the influence of noise on the reconstruction. One may, for example, consider the points $\eta$ to be the centers of the field of view of the individual pixels of a CCD camera.

We also assume that the absorption coefficient $\mu$ is known in a vicinity of $\partial \Omega$, so that we are only reconstructing $\mu$ on a sub-domain $U \subset \Omega$ with $\overline{U} \subset \Omega$.

2.3. The inverse problem

We can now formulate the inverse problem addressed in this work:

**Inverse problem formulation:** Assuming that for a given point $\eta \in \partial \Omega$ and any point $\xi \in U$, the values

$$h(\xi) := v^\xi(\eta)$$

are known in the coupled system of equations

$$\begin{cases}
-\nabla \cdot D \nabla u(x) + \mu(x)u(x) = 0 & \text{in } \Omega, \\
2D \frac{\partial u(x)}{\partial n} + \gamma u(x) = S(x) & \text{on } \partial \Omega, \\
-\nabla \cdot D \nabla v^\xi(x) + \mu(x)v^\xi(x) = \alpha \delta(x - \xi)u(x) & \text{in } \Omega, \\
2D \frac{\partial v^\xi(x)}{\partial n} + \gamma v^\xi(x) = 0 & \text{on } \partial \Omega,
\end{cases}$$

then recover the absorption coefficient $\mu$ inside $U$.

3. Reconstruction algorithm

In this section, we introduce a simple algorithm that can be used to compute numerical reconstructions for the above inverse problem.

Let $G(x, y)$ be the Green’s function for the diffusion model (1), i.e. the solution of

$$\begin{cases}
-\nabla_x \cdot D \nabla_x G(x, y) + \mu(x)G(x, y) = \delta(x - y) & x \in \Omega, \\
2D \frac{\partial G(x, y)}{\partial n} + \gamma G(x, y) = 0 & x \in \partial \Omega.
\end{cases}$$

Then, (6) implies

$$v^\xi(x) = \alpha G(x, \xi)u(\xi),$$

and thus,

$$h(\xi) = \alpha G(\eta, \xi)u(\xi), \quad u(\xi) = \frac{h(\xi)}{\alpha G(\eta, \xi)}.$$  

Substituting this expression for $u$ into the first equation of (6), we obtain an equation for recovering $\mu$:

$$\mu(\xi) = \frac{[\nabla_\xi \cdot D \nabla_\xi] (h(\xi)G^{-1}(\eta, \xi))}{h(\xi)G^{-1}(\eta, \xi)}.$$  

(8)
(Notice the convenient disappearance of the yield factor $\alpha$ which will in general be unknown in practice.) The apparent difficulty in using this formula for reconstruction is that without knowing $\mu$ in advance, neither $D$ nor the Green’s function $G$ are known. However, we can apply the following natural iterative scheme for (8):

(i) **Initial step:** Using an initial guess $\mu^0$ for the absorption coefficient (e.g. constant absorption), compute the corresponding Green’s function numerically, and apply formula (8) to find a new approximation $\mu^1$ for the absorption.

(ii) **Iterative step:** Using the current approximation $\mu^k$, re-compute Green’s function and $D$ and apply formula (8) to find an updated absorption coefficient $\mu^{k+1}$.

(iii) **Iterate** till numerical convergence.

Although we can not prove convergence, this algorithm converges nicely in the numerical tests presented below.

4. Numerical implementation

Implementation of the algorithm outlined above requires the following steps:

(i) generate the measurements, and

(ii) evaluate the Green’s function $G(x, y)$ solving Eq. (7).

These steps are discussed in the following subsections. In this work, we only consider measurements obtained by forward calculations from mathematical phantoms, rather than actual experimental data. All computations were done in 2D, although they can be readily carried over to 3D. For the finite element calculations involved in the reconstruction scheme, the Open Source finite element library deal.II [2,3] was used.

4.1. Forward simulations

In order to generate the measurements $h(\xi)$ (see (5)), we need to compute the solution $u(x), v^\xi(x)$ of the forward problem (6) for a set of given data $D, \mu, S$ (diffusion coefficient, absorption coefficient, incoming light flux) and an ultrasound signal focused at the point $\xi \in U$. Then, evaluating $v^\xi$ at the detector location $\eta$, we obtain the measurement value $h(\xi)$.

4.1.1. **Computational setting.** We take $\Omega$ to be the square [0,5cm]$^2$, which approximately corresponds to the relevant dimensions in practical applications. For the boundary light source $S$ in (3), $\partial \Omega$ is split into $\partial \Omega_1 = \{ x \in \partial \Omega : x_1 = 0 \}$ and $\partial \Omega_2 = \partial \Omega \setminus \partial \Omega_1$. Constant illumination is assumed on $\partial \Omega_1$ and no light is injected on $\partial \Omega_2$:

$$S(x) = \begin{cases} 1 & \text{for } x \in \partial \Omega_1, \\ 0 & \text{for } x \in \partial \Omega_2. \end{cases}$$

The modulated light intensity is measured at a single detector location $\eta = (5\text{cm}, 2.5\text{cm})$. This layout is depicted in Fig. 1.
4.1.2. Incident light field. Since in our model the incident light intensity $u$ is independent of the shape and location of the ultrasound waves in the tissue, $u$ only needs to be computed once. For this computation, a finite element approximation to $u$ is constructed on a regular rectangular grid using $Q_1$ finite elements [4], solving equations (1)–(3). The left panel of Fig. 2 shows $u$ for the case of a constant absorption coefficient $\mu$.

4.1.3. Ultrasound field. In our numerical examples, we use Gaussian-shaped synthetic ultrasound signals:

$$p(x) = C e^{-\sum_{j=1}^{d} \frac{|x_j|^2}{\sigma_j^2}},$$

where $C$ is a normalization constant. By choosing different variances $\sigma_j^2$, we can model both well focused, or unfocused pressure fields.

To simulate scanning of the ultrasound focus, focusing points $\{\xi_i, \ i = 1, \ldots, N\}$ are placed at the vertices of a square grid covering the area of interest, here chosen as the square $U = [0.5\text{cm}, 4.5\text{cm}]^2 \subset \Omega$. For each $i$ we then construct a signal $p^{\xi_i}(x)$ focused at $\xi_i$ by setting

$$p^{\xi_i}(x) := p(x - \xi_i).$$

In the following we use the notations $v^i := v^{\xi_i}, p^i := p^{\xi_i}$.

4.1.4. Modulated light field and measurements. Given $u$ and $|p^i|^2$, we compute the intensity of the modulated light $v^i(x)$, using equations (6). The equations are solved using $Q_1$ finite elements. Two examples for $v^i$ are shown in Fig. 2 for two different focus positions. The modulated light intensities $v^i$ are then evaluated at the sensor location $\eta$ to yield the measurements $h(\xi_i) = v^i(\eta)$.

4.2. Green’s function and reconstruction

The reconstruction algorithm requires knowledge of the Green’s function $G(x, y)$, which, given the absorption coefficient $\mu$ and resulting diffusion coefficient $D$, solves (7). As before, this is done using a finite element scheme and employing a suitable approximation for the delta function on the right hand side.
4.3. Numerical phantoms

To test our algorithms, we use three test cases in which the true absorption coefficients have the following form:

- A disk-shaped inclusion \( K \subset \Omega \) with midpoint \((2.5\text{cm}, 2.5\text{cm})\) and radius 0.5cm. The absorption coefficient is assumed to be equal to \( \bar{\mu} \) outside the inclusion and slightly higher inside:

\[
\mu^*(x) = \begin{cases} 
\bar{\mu}, & x \in \Omega \setminus K \\
1.2 \bar{\mu}, & x \in K.
\end{cases}
\]

- For the same inclusion \( K \), a much higher absorption coefficient contrast

\[
\mu^*(x) = \begin{cases} 
\bar{\mu}, & x \in \Omega \setminus K \\
10 \bar{\mu}, & x \in K.
\end{cases}
\]

- A more complicated coefficient with multiple inclusions of different magnitude between 1.2 \( \bar{\mu} \) and 2.0 \( \bar{\mu} \). Their exact shape is shown in Fig. 3. This case tests the ability of our algorithms to resolve several nearby objects.

For actual numerical values, we used \( \bar{\mu} = 0.023\text{cm}^{-1} \), \( \mu'_s = 10.74\text{cm}^{-1} \) and \( \gamma = 0.431\text{cm}^{-1} \) in our computations. These values represent typical optical properties of soft tissue [16].
4.4. Reconstructions using a strongly focused ultrasound signal

For the computations shown in this section, we use the assumption that the ultrasound signal is sharply focused, and thus choose small variances $\sigma_1 = \sigma_2 = 0.1\text{cm}$ in the Gaussian (10), resulting in sharp focusing in each direction (Fig. 5, middle).

Fig. 4 shows reconstructions of the three different absorption coefficients for scanning the ultrasound focus $\xi^i$ on a $100 \times 100$ mesh of points inside the area of interest $U$.

![Figure 4. Reconstruction results for the three coefficient cases: after the first step of the algorithm (top) and after $N = 40, 70$ and $40$ iterations, respectively (bottom).](image)

These results show that under the main assumptions of the model, i.e. turbid medium (and thus $\mu \ll \mu'_s$), virtual light source, and strong focusing, stable reconstructions are possible, which in particular can recover sharp interfaces and quantitatively correct values of absorption.

5. Stability of the linearized problem

The quality of reconstructions shown above, especially the recovery of sharp singularities, is at first surprising, given that the standard OT problem is strongly ill-posed. In this section, we will make a first step towards understanding the stability of the UOT procedure.

Note that even though equations (6) defining $u$ and $v$ are linear, the relation between the absorption coefficient $\mu$ and the measurements $h$ is highly nonlinear. In this section, we consider a (formal) linearization of the system (6) that will allow us to gain some insight into the local properties of the inverse problem.
Let \( \Omega \subset \mathbb{R}^d \) with \( d = 2 \) or \( d = 3 \) be an open bounded domain with \( C^2 \)-boundary. We use a formal linearization, assuming that \( \mu \) is a small perturbation of a known absorption \( \mu_0 > 0, \mu_0 \in C^{0,1}(\overline{\Omega}) \), and then applying the formal asymptotic expansions

\[
\begin{align*}
\mu(x) &= \mu_0(x) + \varepsilon \mu_1(x) + o(\varepsilon), \\
u(x) &= u_0(x) + \varepsilon u_1(x) + o(\varepsilon), \\
v^\xi(x) &= v^\xi_0(x) + \varepsilon v^\xi_1(x) + o(\varepsilon),
\end{align*}
\]

where \( \varepsilon \to 0 \). Our goal is to relate the first order perturbations of the absorption coefficient \( \mu_1 \) and the measurements \( h_1(\xi) := v^\xi_1(\eta) \), where \( \eta \in \partial \Omega \) is the location of the detector.

By inserting the above expansions into equations (6) and sorting terms according to powers of \( \varepsilon \), we get the zeroth order perturbation system

\[
-\nabla \cdot D \nabla u_0(x) + \mu_0(x)u_0(x) = 0, \tag{11}
\]

\[
-\nabla \cdot D \nabla v_0^\xi(x) + \mu_0(x)v_0^\xi(x) = \alpha \delta(x - \xi)u_0(x), \tag{12}
\]

and the first order perturbation system

\[
-\nabla \cdot D \nabla u_1(x) + \mu_0(x)u_1(x) = -\mu_1(x)u_0(x), \tag{13}
\]

\[
-\nabla \cdot D \nabla v_1^\xi(x) + \mu_0(x)v_1^\xi(x) = \alpha \delta(x - \xi)u_1(x) - \mu_1(x)v_0^\xi(x) \tag{14}
\]

for all \( x \in \Omega \), complemented by inhomogeneous Robin boundary conditions as in (3) for \( u_0 \) and homogeneous Robin boundary conditions for \( v_0, u_1 \) and \( v_1^\xi \). Here we neglected the (weak) dependence of \( D \) on \( \mu \) and instead set \( D \equiv \text{const} > 0 \) for the rest of this section.

Equations (11)–(12) imply that \( u_0 \) and \( v_0^\xi \) are solutions to the forward model for absorption coefficient \( \mu_0 \). The standard elliptic regularity theorems (e.g., [7]) imply \( u_0 \in H^3(\Omega) \), and by Sobolev embeddings [6] \( u_0 \in C^1(\overline{\Omega}) \).

As before, we assume the absorption coefficient to be known near the boundary, so that it suffices to consider perturbations \( \mu_1 \) supported in an open set \( U \) with \( C^2 \)-boundary such that \( \overline{U} \subset \Omega \). We assume the data \( h_1(\xi) \) to be given for all \( \xi \in U \). In what follows, we will derive an explicit formula for the dependence of \( \mu_1 \) on \( h_1 \) and then investigate under which conditions this linear relation is continuous.

Let us denote by \( G_0(x, y) \) the Green’s function as defined in (7) corresponding to the background absorption coefficient \( \mu_0 \). Equation (12) implies that for all \( x \in \Omega \) and \( \xi \in U \),

\[
v_0^\xi(x) = \int_{\Omega} \alpha G_0(x, z) \delta(z - \xi)u_0(z) \, dz = \alpha G_0(x, \xi)u_0(\xi).
\]

From (14) we can now deduce that

\[
v_1^\xi(x) = \int_{\Omega} G_0(x, z) \left[ \alpha \delta(z - \xi)u_1(z) - \mu_1(z)v_0^\xi(z) \right] \, dz
\]

\[
= \alpha G_0(x, \xi)u_1(\xi) - \alpha u_0(\xi) \int_{\Omega} G_0(x, z)G_0(z, \xi)\mu_1(z) \, dz.
\]
Evaluating at \( x = \eta \) and solving for \( u_1 \) yields
\[
u_1(\xi) = \frac{h_1(\xi)}{\alpha G_0(\eta, \xi)} + \frac{u_0(\xi)}{G_0(\eta, \xi)} \int_\Omega G_0(\eta, z) G_0(z, \xi) \mu_1(z) \, dz.
\]

We now use this expression to eliminate \( u_1 \) from (13). Noting that the differential operators now act on \( \xi \) and that
\[
[-\nabla_\xi \cdot D\nabla_\xi + \mu_0(\xi)] G_0(x, \xi) = \delta(x - \xi),
\]
we get
\[
0 = u_0(\xi) \mu_1(\xi) + [-\nabla_\xi \cdot D\nabla_\xi + \mu_0(\xi)] \left( \frac{h_1(\xi)}{\alpha G_0(\eta, \xi)} \right) + [-\nabla_\xi \cdot D\nabla_\xi + \mu_0(\xi)] \left( \frac{u_0(\xi)}{G_0(\eta, \xi)} \int_\Omega G_0(\eta, z) G_0(z, \xi) \mu_1(z) \, dz \right)
\]
\[
= u_0(\xi) \mu_1(\xi) + [-\nabla_\xi \cdot D\nabla_\xi + \mu_0(\xi)] \left( \frac{h_1(\xi)}{\alpha G_0(\eta, \xi)} \right) + \left( [-\nabla_\xi \cdot D\nabla_\xi] \left[ \frac{u_0(\xi)}{G_0(\eta, \xi)} \right] \right) \int_\Omega G_0(\eta, z) G_0(z, \xi) \mu_1(z) \, dz
\]
\[
- 2D \left[ \nabla_\xi \left( \frac{u_0(\xi)}{G_0(\eta, \xi)} \right) \right] \cdot \left[ \nabla_\xi \int_\Omega G_0(\eta, z) G_0(z, \xi) \mu_1(z) \, dz \right] + \frac{u_0(\xi)}{G_0(\eta, \xi)} G_0(\eta, \xi) \mu_1(\xi).
\]

We will frequently view \( G_0(\eta, y) \) as a function of \( y \) in the following and hence introduce the notation
\[
G_0^n(\eta, y) := G_0(\eta, y) \quad \text{for } y \in \overline{U}.
\]

Note that since \( \eta \in \partial \Omega \), \( G_0^n \) has no singularities on \( \overline{U} \) and hence is a regular solution to (11) there. The elliptic regularity and Sobolev embeddings imply \( G_0^n \in C^1(\overline{U}) \).

Let us define the following operators acting on functions \( g \) defined on \( U \):

\[
K_1 g(\xi) := -\frac{1}{2u_0(\xi)} \left( [-\nabla_\xi \cdot D\nabla_\xi] \left[ \frac{u_0(\xi)}{G_0^n(\xi)} \right] \right) \int_U G_0^n(z) G_0(z, \xi) g(z) \, dz, \quad (15)
\]
\[
K_2 g(\xi) := \frac{D}{u_0(\xi)} \left[ \nabla_\xi \left( \frac{u_0(\xi)}{G_0^n(\xi)} \right) \right] \cdot \left[ \nabla_\xi \int_U G_0^n(z) G_0(z, \xi) g(z) \, dz \right], \quad (16)
\]

and
\[
F := 1 - K_1 - K_2.
\]

In terms of these operators, \( \mu_1 \) is a solution to the following linear equation:
\[
F \mu_1(\xi) = -\frac{1}{2u_0(\xi)} [-\nabla_\xi \cdot D\nabla_\xi + \mu_0(\xi)] \left( \frac{h_1(\xi)}{\alpha G_0^n(\xi)} \right).
\]

In order for the above expressions to be well-defined, we have to make sure that \( u_0 \) and \( G_0^n \) are bounded away from zero on \( \overline{U} \), which is stated in the following lemma, which follows immediately from the Hopf Lemma (e.g., [15, 18]).

**Lemma 1** There is a constant \( c > 0 \) such that \( u_0 \geq c \) and \( G_0^n \geq c \) on \( \overline{U} \).
Next we consider the properties of the integral term involved in $K_1$ and $K_2$. The important observation here is the following:

**Lemma 2** The mapping

$$\int_U G_0(z, \cdot) G_0^0(z) g(z) \, dz$$

is a bounded linear operator from $L^2(U)$ to $H^2(U)$.

**Proof:** Let us assume that $g \in L^2(U)$. Since $G_0^0 \in C(U)$, multiplication by $G_0^0$ is a bounded linear operator on $L^2(U)$. The following integration against $G_0^0$ results in the solution to the diffusion equation with homogeneous Robin boundary condition and right hand side $G_0^0 g \in L^2(U)$. Elliptic regularity theory (e.g., [6, 7]) implies that this is a continuous operator from $L^2(U)$ into $H^2(U)$. □

Because of the compact embedding of $H^2(U)$ in $L^2(U)$, the operator defined by (18), viewed as a mapping from $L^2(U)$ to $L^2(U)$, is compact. In (15), this operator is multiplied by the factor

$$-\frac{1}{2u_0^0} \left( [-\nabla_\xi \cdot D\nabla_\xi] \left[ \frac{u_0^0(\xi)}{G_0^0(\xi)} \right] \right).$$

The functions $u_0, \nabla u_0, G_0^0$ and $\nabla G_0^0$ are all bounded on $U$ because $u_0, G_0^0 \in C^1(U)$. Since $u_0$ and $G_0^0$ satisfy (11), the terms $\nabla \cdot D\nabla u_0$ and $\nabla \cdot D\nabla G_0^0$ are bounded on $U$ as well, and $u_0^{-1}$ and $(G_0^0)^{-1}$ are bounded due to Lemma 1. Consequently, multiplication by (19) represents a bounded linear operation on $L^2(U)$, and so $K_1$ is a compact operator in $L^2(U)$. Similarly, $K_2$ is a compact operator in $L^2(U)$. This leads us to the main result of this section:

**Theorem 3** $F : L^2(U) \to L^2(U)$ is a Fredholm operator of index zero.

Thus, the kernel $\mathcal{N}(F)$ of $F$ has finite dimension and the range $\mathcal{R}(F)$ is closed and of finite codimension, equal to the dimension of the kernel. This immediately implies the following result:

**Corollary 4** $F$ as an operator from the quotient space $L^2(U)/\mathcal{N}(F)$ to $\mathcal{R}(F)$ has bounded inverse, and the following norm equivalence holds:

$$c_1 \|F f\|_{L^2(U)} \leq \|f\|_{L^2(U)/\mathcal{N}(F)} \leq c_2 \|F f\|_{L^2(U)}.$$  

The $L^2$-norm of the right hand side expression in (17) can be estimated in terms of the $H^2$-norm of the measured perturbation $h_1$, so that we obtain the following stability result:

**Theorem 5** Under the stated assumptions, there is a constant $C > 0$ such that the following relation holds:

$$\|\mu_1\|_{L^2(U)/\mathcal{N}(F)} \leq C \|h_1\|_{H^2(U)}.$$
One would conjecture that the kernel $\mathcal{N}(F)$ is in fact trivial, and thus the operator $F$ is invertible. This would imply that $\mu_1$ is uniquely determined by the measured perturbation $h_1$, and allow us to replace the quotient space norms in (20) and (21) with the regular $L^2$ norms. However, we have not been able to prove the absence of the kernel yet.

Smaller norm coercive estimates for the absorption can be obtained if more is assumed about the unperturbed absorption $\mu_0$ and the domain. For instance, if $\mu_0 \in C^\infty(\Omega)$, $S \in C^\infty(\partial\Omega)$, and $\Omega$ has smooth boundary, the operators $K_1$ and $K_2$ defined in (15)–(16), are of order $-2$ and $-1$, respectively, in the Sobolev scale:

$$
K_1 : H^s(U) \to H^{s+2}(U),
$$
$$
K_2 : H^s(U) \to H^{s+1}(U).
$$

This and the Sobolev embedding theorem [1] imply that for any $s \geq 0$, $F$ is Fredholm as an operator

$$
F : H^s(U) \to H^s(U).
$$

This, in turn, leads to the estimate

$$
\|f\|_{H^s(U)} \leq c \left( \|f\|_{L^2(U)} + \|Ff\|_{H^s(U)} \right)
$$

for all $f \in H^s(U)$. Thus, we have the following result:

**Theorem 6** Under the stated assumptions, for any $s > 0$ there is a constant $C$ such that

$$
\|\mu_1\|_{H^s(U)} \leq C \left( \|\mu_1\|_{L^2(U)} + \|h_1\|_{H^{s+2}(U)} \right).
$$

Clearly, if only a specific value of $s$ is of interest, the smoothness assumptions on $\mu_0, S$ and $\partial\Omega$ can be relaxed appropriately.

### 6. Final remarks and conclusions

#### 6.1. Detector locations

In the discussion of stability above, as well as in our numerical reconstructions, we have chosen a single detector point $\eta$. However, using detectors distributed over a part $\Gamma$ of the boundary $\partial\Omega$ should help to suppress the effect of noise in the measured data.

#### 6.2. Ultrasound signal with elongated focus

In practice, perfect focusing of ultrasound waves is not a realistic assumption, and thus a lot of attention has been paid recently to this issue (e.g., [9, 12] and references therein). How well an ultrasound signal can be focused depends, in particular, on the geometry and bandwidth of the transducer. It is known (e.g., [14]) that focused ultrasound signals have an intensity profile similar to the one shown in Fig. 5 (left).

This signal has significantly sharper focus in the direction transverse to the transducer lens, while the well-focused Gaussian signal used in our results does not reflect
this behavior. Hence, we computed reconstructions for the case where the ultrasound intensity is a Gaussian signal with sharp focus in $x$-direction and elongated focus in $y$-direction (Fig. 5, right). As in the previous section, the ultrasound focus $\xi^i$ is scanned on a 100×100 mesh, and the reconstruction results are shown in Fig. 6. No compensation for the elongated focusing has been used.

![Figure 5](image)

**Figure 5.** Left: Simulated ultrasound pressure field $|p|^2$ with transducer at the bottom. Middle: Gaussian ultrasound signal $|p|^2$ with $\sigma_1 = \sigma_2 = 0.1$. Right: Gaussian signal with $\sigma_1 = 0.1$, $\sigma_2 = 0.3$.

![Figure 6](image)

**Figure 6.** Reconstruction results for ultrasound signal with elongated focus: after the first step of the algorithm (top) and after $N$ iterations (bottom).

The deterioration of the reconstruction – in particular in the direction of the ultrasound beam – is obvious. A more sophisticated model might be needed to treat the non-perfect focusing in these calculations.

### 6.3. Synthetic focusing

A *synthetic focusing* approach that would remove the need of good focusing was suggested in [12]. It is proposed to use various basis sets of non-focused ultrasound
waves (e.g., spherical or monochromatic planar ones), with a post-processing that synthesizes the would-be response to the focused illumination. In particular, in the case of spherical waves, the post-processing (= synthetic focusing) is essentially equivalent to the thermoacoustic tomography inversion (see [11] about TAT). Applications of this approach to UOT are planned to be considered elsewhere.

6.4. Uniqueness of reconstruction

Proving uniqueness of reconstruction, both in the non-linear and linearized versions, still remains a challenge. In particular, we conjecture that the operator $F$ in (17) is in fact invertible, and thus there is uniqueness of solution of the linearized problem, which would replace the quotient space norms in (20) and (21) with the regular $L^2$ norms. We plan to address this, as well as a justification for the formal linearization used, in future work.

6.5. Conclusions

In this paper, a virtual light source model is provided for the ultrasound modulated optical tomography procedure using well focused ultrasound waves. An iterative algorithm is suggested. The provided numerical results show feasibility of the algorithm and possibility of good reconstructions, both with regard to locating sharp interfaces, as well as recovering correct numerical values of the absorption coefficient. Such stability and resolution are impossible to achieve in standard optical tomography. The stability of reconstructions is explained by the derived in Theorems 5 and 6 stability estimates for a linearized model.

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