STRUCTURE OF RELATIVELY BI-EXACT GROUP VON NEUMANN ALGEBRAS

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Abstract. Using computations in the bidual of $B(L^2M)$ we develop a new technique at the von Neumann algebra level to upgrade relative proper proximality to full proper proximality. This is used to classify subalgebras of $L\Gamma$ where $\Gamma$ is an infinite group that is biexact relative to a finite family of subgroups $\{\Lambda_i\}_{i \in I}$ such that each $\Lambda_i$ is almost malnormal in $\Gamma$. This generalizes the result of [DKEP22] which classifies subalgebras of von Neumann algebras of biexact groups. By developing a combination with techniques from Popa’s deformation-rigidity theory we obtain a new structural absorption theorem for free products and a generalized Kurosh type theorem in the setting of properly proximal von Neumann algebras.

1. Introduction

Recently the authors and J. Peterson in [DKEP22] developed the theory of small at infinity compactifications a la Ozawa ([BO08]), in the setting of tracial von Neumann algebras. At the foundation of this work lies the theory of operator $M$-bimodules and the several natural topologies that arise in this setting (see [ER88], [Mag97, Mag98, Mag05]). The small at infinity compactification is a canonical strong operator bimodule (in the sense of Magajna [Mag97]) containing the compact operators. By using the noncommutative Grothendieck inequality (similar to Ozawa in [Oza10a]) it was seen that this strong operator bimodule coincides with $K^{\infty,1}(M)$, the closure $K^{\infty,1}(M)$ of $K(L^2M)$ with respect to the $\| \cdot \|_{\infty,1}$-norm on $B(L^2M)$ given by $\|T\|_{\infty,1} = \sup_{x \in M, \|x\| \leq 1} \|Tx\|_1$. The small at infinity compactification of a tracial von Neumann algebra $M$ is then given by

$$S(M) = \{ T \in B(L^2M) \mid [T, JxJ] \in K^{\infty,1}(M), \text{ for all } x \in M \}.$$  

It is easy to see that this operator $M$-system $S(M)$ contains $M$ and $K(L^2M)$, and is an $M$-bimodule. The advantage of the strong operator bimodule perspective is that it to identify an operator $T \in S(M)$ suffices to check that $[T, JxJ] \in K^{\infty,1}(M)$ for all $x$ in some weakly closed subset of $M$. This is what allows for the passage between the group and the von Neumann algebra settings. Using this technology [DKEP22] defined the notion of proper proximality for finite von Neumann algebras, extending the dynamical notion for groups [BIP18]: A finite von Neumann algebra $(M, \tau)$ is properly proximal if there does not exist an $M$-central state $\varphi$ on $S(M)$ such that $\varphi|_M = \tau$. By identifying and studying this property in various examples, the authors of [DKEP22] obtained applications to the structure theory of $\text{II}_1$-factors. The goal of the present paper is to add to the list of applications.
The machinery underlying the results in this paper is built on the notion of an $M$-boundary piece developed in [DKEP22], as an analogue of the group theoretic notion introduced in [BIP18]. The motivation for considering this notion is that it allows for one to exploit the dynamics that is available only on certain locations of the Stone-Cech boundary of the group. For a group $\Gamma$, a boundary piece is a closed left and right invariant subset of $\beta(\Gamma) \setminus \Gamma$, whereas in the von Neumann algebra setting, it is denoted by $X$ typically and is a certain hereditary $C^*$-subalgebra of $\mathcal{B}(L^2M)$ containing the compact operators (see Section 3.1). One then considers the small at infinity compactification relative to a boundary piece $S_X(M)$ where $K^\infty,1(M)$ is replaced with $K^\infty,1_X(M)$, a suitable analogue for the boundary piece. Then one can define the notion of proper proximality relative to $X$, demanding that there be no $M$-central state restricting to the trace on $S_X(M)$. The main example we will be working with is a boundary piece generated by a finite family of von Neumann subalgebras $\{M_i\}_{i=1}^n$ (see Example 3.1), which is adapted from the construction for a finite family of subgroups (see Example 3.3 in [BIP18]).

In [DKE22], the authors demonstrated an instance where relative proper proximality can be lifted to full proper proximality, i.e., when the boundary piece arises from subgroups that are almost malnormal and not co-amenable (see Lemma 3.3 in [DKE22]). The authors used this idea to classify proper proximality for wreath product groups. In this paper, we develop an analogue of this idea in the setting of von Neumann algebras (Theorem 1.1). In both cases, one has to work in the bidual of the small at infinity compactification for technical reasons, and this brings about an extra layer of subtlety especially in the von Neumann setting. More specifically we show that one can map the basic construction into the bidual version of the relative small at infinity compactification, provided the boundary piece arises from a mixing subalgebra. Composing with an appropriate state on this space, we get the link with relative amenability in the von Neumann setting. This upgrading theorem is the main new technical tool we develop in the present work:

**Theorem 1.1.** Let $M$ be a diffuse finite von Neumann algebra, $M_i \subset M$, $i = 1, \ldots, n$ diffuse von Neumann subalgebras such that the Jones projections $e_M$, pairwise commute, $M_i \subset M$ admits a bounded Pimsner-Popa basis (see Definition 2.1), and $A \subset pMp$ is a von Neumann subalgebra, for some $p \in \mathcal{P}(M)$. Suppose that $A$ is properly proximal relative to $X$ inside $M$, where $X$ is the boundary piece associated with $\{M_i\}_{i=1}^n$, and $M_i \subset M$ is mixing for each $i = 1, \ldots, n$. Then there exist projections $f_0 \in Z(A)$ and $f_i \in Z(A_i' \cap pMp)$, $1 \leq i \leq n$, such that $Af_0$ is properly proximal and $Af_i$ is amenable relative to $M_i$ inside $M$ for each $1 \leq i \leq n$, and $\sum_{i=0}^n f_i = p$.

Using these ideas we are interested in classifying subalgebras of group von Neumann algebras arising from groups that are biexact relative to a family of subgroups (see e.g. [BOO3, Chapter 15]). The first result of this kind was obtained in Theorem 7.2 of [DKEP22] where it was shown that every subalgebra of the von Neumann algebra of a biexact group either has an amenable direct summand or is properly proximal. As essentially observed there, what relative biexactness buys us is the relative proper proximality for any subalgebra, relative to the boundary piece arising from the subgroups. Combining this with our upgrading result above, we obtain our main result below which is a structure theorem for von

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1 A subgroup $H < G$ is almost malnormal if for all $g \in G \setminus H$, $gHg^{-1} \cap H$ is finite.
Neumann subalgebras of group von Neumann algebras that are biexact relative to a family of subgroups where each subgroup is almost malnormal.

**Theorem 1.2.** Let $\Gamma$ be a countable group with a family of almost malnormal subgroups $\{\Lambda_i\}_{i=1}^n$. If $\Gamma$ is biexact relative to $\{\Lambda_i\}_{i=1}^n$, then for any von Neumann subalgebra $A \subset L\Gamma$, there exists $p \in Z(A)$ and projections $p_j \in Z((Ap)\cap pL(\Gamma)p)$ such that $\bigvee_{j=1}^n p_j = p$ and $Ap_i$ is amenable relative to $L\Lambda_i$ inside $L\Gamma$, for each $i = 1, \ldots, n$ and $Ap^\perp$ is properly proximal.

There are two natural instances where such a phenomenon (a countable group $\Gamma$ with a family of almost malnormal subgroups $\{\Lambda_i\}_{i=1}^n$ where $\Gamma$ is biexact relative to $\{\Lambda_i\}_{i=1}^n$) is observed: First is in the setting of free products, which we deal with in the present paper. Second is in the setting of wreath products, which is investigated in a follow-up work by the first author [Din22]. There is conjecturally a third setting of relative hyperbolicity, which we comment on in the end of the introduction.

Thanks to Bass-Serre theory [Ser80] we have a complete understanding of subgroups of a free product of groups. As a result, one can derive results of the following nature: If $H < G_1 * G_2$ such that $|H \cap G_1| \geq 3$, then $H$ is amenable only if $H < G_1$. This phenomenon is referred to as amenable absorption. Interestingly, the situation for von Neumann algebras is much more complicated. There is comparatively a very limited understanding of von Neumann subalgebras of free products. Whether every self adjoint operator in any finite von Neumann algebra is contained in a copy of the hyperfinite $I_1$-factor was itself an open problem for many years. Popa settled it in the negative in [Pop83] by discovering a surprising amenable absorption theorem for free product von Neumann algebras, thereby showing that a generator masa in $L\mathbb{F}_2$ is maximally amenable.

Popa’s ideas been used to show maximal amenability in other situations (See for instance [CFRW10], [Wen16], [PSW18], [BW16]). In the past decade there have been other new ideas that have been used to prove absorption theorems: Boutonnet-Carderi’s approach [BC15] relies on elementary computations in a crossed-product $C^*$-algebra; Boutonnet-Houdayer [BH18] use the study of non normal conditional expectations; [UJNS21] used a free probabilistic approach to study absorption. Ozawa in [Oza10b] then gave a short proof of amenable absorption in tracial free products. There have also been a variety of important free product absorption results which are of a different flavor, and are structural in nature. See for example [IPP08] and [CH10].

By applying our Theorem 1.2 in the setting of free products and using machinery from Popa’s deformation-rigidity theory (specifically work of Ioana [Ioana18]), we obtain a generalized structural absorption theorem below:

**Corollary 1.3.** Let $(M_1, \tau_1)$ and $(M_2, \tau_2)$ be such that $M_i \cong L\Gamma_i$ where $\Gamma_i$ are countable exact groups and $M = M_1 * M_2$ be the tracial free product. Let $A \subset M$ be a von Neumann subalgebra with $A \cap M_1$ diffuse. If $A \subset M$ has no properly proximal direct summand, then $A \subset M_1$.

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2 This is a question of Kadison, Problem 7 from ‘Problems on von Neumann algebras, Baton Rouge Conference’
Remark 1.4. Using results of the upcoming work [DP22], one can relax the assumption on $M_i$ from being infinite group von Neumann algebras of exact groups, to just that they are diffuse weakly exact von Neumann algebras. We do not comment more on this at the moment because for the sake of examples, the above setting already provides many.

The authors of [IPR19] showed that there are examples of groups that are neither inner amenable nor properly proximal. All of these group von Neumann algebras fit into the setting of the above corollary. Note that Vaes constructed in [Vae12] plenty of groups that are inner amenable, yet their group von Neumann algebras lack Property (Gamma). Hence our results give a strict generalization of (Gamma) absorption (see Houdayer’s Theorem 4.1 in [Hou14] and see also Theorem A in [HJNS21]) in these examples.

Remark 1.5. The above result is false if one considers amalgamated free products. For instance, take $M_1 = M_2 = L^R F_2 \otimes R$ and $A = (L^R Z \ast L^R Z) \otimes R \subset M_1 \ast_R M_2$, where two copies of $L^R Z$ in $A$ are from $M_1$ and $M_2$, respectively.

Remark 1.6. Shortly before the posting of this paper, Drimbe announced a paper (see [Dri22]) where he shows using Popa’s deformation-rigidity theory that for any nonamenable inner amenable group $\Gamma$, if $L(\Gamma) \subset M_1 \ast M_2$, then $L(\Gamma)$ intertwines into $M_i$ for some $i = 1, 2$. This in particular generalizes Corollary 1.3 in the case that $A \cong L\Gamma$ for some inner amenable group $\Gamma$, because he doesn’t require any assumptions for $M_i$.

Our techniques also reveal the following new Kurosh type structure theorem for free products in the setting of proper proximality, (partially generalizing Corollary 8.1 in [Dri22]). See also [Oza06, IPP08, Pet09, HU16] for other important Kurosh type theorems.

Corollary 1.7. Let $M = L\Gamma_1 \ast \cdots \ast L\Gamma_m = L\Lambda_1 \ast \cdots \ast L\Lambda_n$, where all groups $\Gamma_i$ and $\Lambda_j$ are countable exact nonamenable non-properly proximal i.c.c. groups. Then $m = n$ and after a permutation of indices $L\Gamma_i$ is unitarily conjugate to $L\Lambda_i$.

We conclude by state the following folklore conjecture (also stated in [Oya22]), which would provide another family of examples for applying Theorem 1.2. Indeed the peripheral subgroups below are almost malnormal (see Theorem 1.4 in [Osi06]).

Conjecture 1 ([Oya22]). If $G$ is exact and hyperbolic relative to a family of peripheral subgroups $\{H_i\}_{i=1}^n$, then $G$ is biexact relative to $\{H_i\}_{i=1}^n$.

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2. Preliminaries

2.1. The basic construction and Pimsner-Popa orthogonal bases. Let $M$ be a finite von Neumann algebra and $Q \subset M$ be a von Neumann subalgebra. The basic construction $\langle M, e_Q \rangle$ is defined as the von Neumann subalgebra of $\mathcal{B}(L^2 M)$ generated by $M$ and the orthogonal projection $e_Q$ from $L^2(M)$ onto $L^2(Q)$. There is a semifinite faithful normal trace on $\langle M, e_Q \rangle$ satisfying $\text{Tr}(xe_Qy) = \tau(xy)$, for every $x, y \in M$. 

Let $N \subset M$ be a von Neumann subalgebra. Then a Pimsner-Popa basis (see [PP86]) of $M$ over $N$ is a family of elements denoted $M/N = \{m_j\}_{j \in J} \subset M$ such that

1. $E_N(m_j^*m_k) = \delta_{j,k}p_j$, where $p_j \in \mathcal{P}(N)$ is a projection.
2. $L^2(M) = \bigoplus_{j \in J} m_j L^2(N)$ and every $x \in M$ has a unique decomposition $x = \sum_j m_j E_N(m_j^*x)$.

In the case that $N = L(\Lambda)$ and $M = L(\Gamma)$ where $\Lambda < \Gamma$, we can identify a Pimsner-Popa basis in $M$ from a choice of coset-representatives i.e, $\Gamma = \bigsqcup_{k \geq 0} t_k \Lambda$, and $m_k := \lambda_k \in U(L\Gamma)$: $M/N = \{u_j\}_{j \in J}$.

For technical reasons, we will need the existence of the following type of Pimsner-Popa basis for our results:

**Definition 2.1.** Say that an inclusion of separable finite von Neumann algebras $N \subset M$ admits a bounded Pimsner-Popa basis in $M$ if there exists a Pimsner-Popa basis $\{m_k\}_{k \in \mathbb{N}}$ for the inclusion $N \subset M$ such that $\sup_{k \in \mathbb{N}} \|m_k\| < \infty$.

Note that if $M/N$ consists of unitaries in $M$, then it clearly also satisfies that it is a bounded Pimsner-Popa basis. This is a technical property considered by Ceccherini-Silberstein [CS04], called the U-property. It is a well known open problem if such bases always exist.

Such a Pimsner-Popa basis satisfying the above U-property always exists for the inclusion $L\Lambda \subset L\Gamma$ where $\Lambda < \Gamma$ is a subgroup of a countable group $\Gamma$.

### 2.2. Popa’s intertwining-by-bimodules.

**Theorem 2.2 ([Pop06]).** Let $(M, \tau)$ be a tracial von Neumann algebra and $P \subset pMp, Q \subset M$ be von Neumann subalgebras. Then the following are equivalent:

1. There exist projections $p_0 \in P, q_0 \in Q$, a $*$-homomorphism $\theta : p_0 P p_0 \to q_0 Q q_0$ and a non-zero partial isometry $v \in q_0 M p_0$ such that $\theta(x)v = vx$, for all $x \in p_0 P p_0$.
2. There is no sequence $u_n \in U(P)$ satisfying $\|E_Q(x^* u_n y)\|_2 \to 0$, for all $x, y \in pM$.

If one of these equivalent conditions holds, we write $P \prec_M Q$, and say that $a corner of P embeds into Q inside M$.

### 2.3. Relative amenability.

Let $P \subset M$ and $Q \subset M$ be a von Neumann subalgebras. We say that $P$ is amenable relative to $Q inside M$ if there exists a sequence $\xi_n \in L^2(\langle M, e_Q \rangle)$ such that $\langle x \xi_n, \xi_n \rangle \to \tau(x)$, for every $x \in M$, and $\|y \xi_n - \xi_n y\|_2 \to 0$, for every $y \in P$. By [OP10], Theorem 2.1 $P$ is amenable relative to $Q inside M$ if and only if there exists a $P$-central state in the basic construction $\langle M, e_Q \rangle$ that is normal when restricted to $M$, and faithful on $Z(P' \cap M)$.
2.4. Mixing subalgebras and free products of finite von Neumann algebras. Let $M$ be a finite von Neumann algebra and $N \subset M$ a von Neumann subalgebra. Recall the inclusion $N \subset M$ is mixing if $L^2(M \ominus N)$ is mixing as an $N-N$ bimodule, i.e., for any sequence $u_n \in U(N)$ converging to 0 weakly, one has $\|E_N(xu_ny)\|_2 \to 0$ for any $x, y \in M \ominus N$. When $M$ and $N$ are both diffuse, we may replace sequence of unitaries with any sequence in $N$ converging to 0 weakly [DKEP22 Theorem 5.9].

Remark 2.3. Let $M$ be a diffuse finite von Neumann algebra and $N \subset M$ a diffuse von Neumann subalgebra. If $N \subset M$ is mixing, then it is easy to check that $e_Nxe_N \in B(L^2M)$ is a compact operator from $M$ to $L^2M$ assuming $x$ or $y \in M \ominus N$.

Examples of mixing subalgebras include $M_1$ and $M_2 \subset M_1 \ast M_2$, where $M_1$ and $M_2$ are diffuse [Jol12 Proposition 1.6] and $\Lambda \subset L\Gamma$, where $\Lambda < \Gamma$ is almost malnormal (see Proposition 2.4 in [BC17]).

The following [Ioana Corollary 2.12] is crucial to the proof of Theorem 5.1.

Lemma 2.4 (Ioana). Let $M_1, M_2$ be two diffuse tracial von Neumann algebras and $M = M_1 \ast M_2$ be the tracial free product. Let $A \subset M$ be a subalgebra such that $A$ is amenable relative to $M_1$ in $M$. Then either $A \triangleright_M M_1$ or $A$ is amenable.

We also need the following case of the main result of [BH18]:

Theorem 2.5 (Boutonnet-Houdayer). Let $M = M_1 \ast M_2$, where $M_i$ are diffuse tracial von Neumann algebras. If $A \subset M$ is a von Neumann subalgebra that satisfies $A \cap M_1$ is diffuse and $A$ is amenable relative to $M_1$ inside $M$, then $A \subset M_1$.

3. Proper proximality for von Neumann algebras and boundary pieces

3.1. Boundary pieces from von Neumann subalgebras. Let $M$ be a finite von Neumann algebra. An $M$-boundary piece is a hereditary $C^*$-subalgebra $X \subset B(L^2M)$ such that $M(X) \cap M$ and $M(X) \cap JMJ$ are weakly dense in $M$ and $JMJ$, respectively, where $M(X)$ is the multiplier algebra of $X$. To avoid pathological examples, we will always assume that $X \neq \{0\}$, and it follows that $K(L^2M) \subset X$, by the assumption on $M(X)$.

The main example of an $M$-boundary piece we use in this paper is one generated by von Neumann subalgebras. We recall some facts about hereditary $C^*$-algebras for what follows (see e.g. [Bla06 II.5]).

Let $A$ be a $C^*$-algebra. There is a one-to-one correspondence between the set of hereditary $C^*$-subalgebras of $A$ and the set of closed left ideals in $A$: given a hereditary $C^*$-subalgebra $H \subset A$, $L_H := AH = \{ah \mid a \in A, h \in H\}$ is a closed left ideal; and for a closed left ideal $L \subset A$, $H_L = L \cap L^*$ is a hereditary $C^*$-subalgebra of $A$. Given a subset of operators $\{b_i\}_{i \in I} \subset A$, the hereditary $C^*$-subalgebra generated by $\{b_i\}_{i \in I}$ is $BAB = \{bab \mid b \in B_+, a \in A\}$, where $B$ is the $C^*$-subalgebra generated by $\{b_i\}_{i \in I}$.

Example 3.1. [Boundary piece generated by subalgebras] Let $M$ be a finite von Neumann algebra. Suppose $M_i \subset M$, $i = 1, \ldots, n$ are von Neumann subalgebras and denote by $e_{M_i} \in B(L^2M)$ the orthogonal projection from $L^2M$ onto the space $L^2M_i$. The $M$-boundary
Lemma 3.2. Let $M$ be a finite von Neumann algebra and $M_i \subset M$, $i = 1, \ldots, n$ von Neumann subalgebras such that the projections $\{e_{M_i}\}_{i=1}^n$ are pairwise commuting. Let $X$ be the hereditary $C^*$-subalgebra in $B(L^2 M)$ generated by operators of the form $xJyJe_{M_i}$ with $x, y \in M$, $i = 1, \ldots, n$, and it is clear that $M$ and $MJ$ are contained in its multiplier algebra.

Proof. First note that $e_{M_i} \in X$ for each $i$ since $0 \leq e_{M_i} \leq \vee_{i=1}^n e_{M_i}$. We also have $\vee_{i=1}^n e_{M_i} \in Y$. In fact, for each pair $i, j, e_{M_i} \wedge e_{M_j} \in Y$ as $0 \leq e_{M_i} \wedge e_{M_j} \leq e_{M_i}$, and $e_{M_i} \vee e_{M_j} = e_{M_i} + e_{M_j} \in Y$ as $[e_{M_i}, e_{M_j}] = 0$. To see that $X \subset Y$, note that $L = B(L^2 M)X$ is contained in $K = B(L^2 M)Y$. Indeed, for any $x, y \in M$ and $T \in B(L^2 M)$, we have $T(\vee_{i=1}^n e_{M_i})xJyJe_{M_i} = B(L^2 M)Y$ as $M$ and $MJ$ are in the multiplier algebra of $Y$. By a similar argument we see that $Y \subset X$. \hfill $\Box$

Lemma 3.3. Under the above assumption, $\{\vee_{u,v \in F} uJvJ(\vee_{i=1}^n e_{M_i})Jv^*Ju^*\}_{F \in F}$ is an approximate unit for $X$, where $F$ is the collection of finite subsets of $U(M)$ ordered by inclusion.

Fix an $M$-boundary piece $X$ and let $K_X^2(M) \subset B(L^2 M)$ denote the $\| \cdot \|_{\infty,2}$ closure of the closed left ideal $B(L^2 M)X$, i.e., $K_X^2(M) = \overline{B(L^2 M)X}^{\| \cdot \|_{\infty,2}}$, where $\| \cdot \|_{\infty,2}$ on $B(L^2 M)$ is given by $\|T\|_{\infty,2} = \sup_{x\in(M),1} \|T x\|_2$ for $T \in B(L^2 M)$.

We let $K_X(M) = (K_X^2(M))^* \cap (K_X^2(L^2 M))$, which is a hereditary $C^*$-subalgebra of $B(L^2 M)$ with $M$ and $MJ$ contained in $M(K_X^2(M))$ [DKEP22, Section 3]. Denote by $K_X^\infty(M)$ the $\| \cdot \|_{\infty,1}$ closure of $K_X(M)$ in $B(L^2 M)$, $\|T\|_{\infty,1} = \sup_{x,y \in (M),y} \langle T x, y \rangle$ for $T \in B(L^2 M)$ and it coincides with $K_X^{\infty,1}$. Now put $S_X(M) \subset B(L^2 M)$ to be $S_X(M) = \{T \in B(L^2 M) \mid [T, JxJ] \in K_X^{\infty,1}(M) \text{ for all } x \in M\}$, which is an operator system that contains $M$. In the case when $X = K(L^2 M)$, we write $S(M)$ instead of $S_X(K(L^2 M))(M)$.

Recall from [DKEP22, Theorem 6.2] that for a finite von Neumann subalgebra $N \subset M$ and an $M$-boundary piece $X$, we say $N$ is properly proximal relative to $X$ in $M$ if there is no $N$-central state $\varphi$ on $S_X(M)$ that is normal on $M$. And we say $M$ is properly proximal if $M$ is properly proximal relative to $K(L^2 M)$ in $M$.

Remark 3.4. Let $M$ and $Q$ be finite von Neumann algebras, $X$ an $M$-boundary piece, and $N \subset pMp$ be a von Neumann subalgebra, where $0 \neq p \in P(M)$.
1. Consider the u.c.p. map $\mathcal{E}_N := \text{Ad}(e_N) \circ \text{Ad}(pJp) : \mathcal{B}(L^2M) \to \mathcal{B}(L^2N)$. Then by [DKEP22, Remark 6.3] that $\mathcal{E}_N(K_X(M)) \subset \mathcal{B}(L^2N)$ forms an $N$-boundary piece. And we say $\mathcal{E}_N(K_X(M))$ is the induced $N$-boundary piece, which will be denoted by $X^N$.

2. If $N$ is properly proximal relative to $X$ inside $M$, then $zN$ is also properly proximal relative to $X$ inside $M$ for any $0 \neq z \in \mathcal{Z}(\mathcal{P}(N))$, since $\text{Ad}(z) \circ \mathbb{S}_X(M) \subset \mathbb{S}_X(M)$.

3. If $N$ is properly proximal relative to $X$ inside $M$, then $N$ has no amenable direct summand. To see this, suppose $qN$ is amenable for some $0 \neq q \in \mathcal{Z}(\mathcal{P}(N))$ and let $\varphi$ be a $qN$-central state on $\mathcal{B}(L^2(qN))$. Consider $\mu := \varphi \circ \text{Ad}(q) \circ \text{Ad}(e_N) : \mathcal{B}(L^2M) \to \mathbb{C}$, and one checks that $\mu$ is a $N$-central state with $\mu_{|M}$ being normal.

4. Notice that from the definition it follows that proper proximality is stable under taking direct sum. Thus we may take $f \in \mathcal{Z}(\mathcal{P}(Q))$ so that $Qf$ is the maximal properly proximal direct summand of $Q$.

3.2. Bidual formulation of proper proximality. Given a finite von Neumann algebra $M$ and a C*-subalgebra $A \subset \mathcal{B}(L^2M)$ such that $M$ and $JMJ$ are contained in $\mathcal{M}(A)$, we recall that $A^{M^2M}_M$ (resp. $A^{JMJ^2JMJ}_M$) denotes the space of $\varphi \in A^*$ such that for each $T \in A$ the map $X \times M \ni (a,b) \mapsto \varphi(aTb)$ (resp. $JMJ \times JMJ \ni (a,b) \mapsto \varphi(aTb)$) is separately normal in each variable and set $A^2_J = A^{M^2M}_J \cap A^{JMJ^2JMJ}_J$. Moreover, we may view $(A^2_J)^*$ as a von Neumann algebra in the following way, as shown in [DKEP22, Section 2]. Denote by $p_{nor} \in \mathcal{B}(L^2M)^{**}$ the supremum of support projections of states in $\mathcal{B}(L^2M)^*$ that restrict to normal states on $M$ and $JMJ$, so that $M$ and $JMJ$ may be viewed as von Neumann subalgebras of $\mathcal{P}(M(A))^{**}$. Note that $p_{nor}$ lies in $\mathcal{M}(A)^{**}$ and $p_{nor}M(A)^{**}p_{nor}$ is canonically identified with $(\mathcal{M}(A)^{**}_J)^*$. Let $q_A \in \mathcal{P}(\mathcal{M}(A))^{**}$ be the central projection such that $q_A(M(A))^{**} = A^{**}$ and we may then identify $(A^2_J)^*$ with $q_A p_{nor}M(A)^{**} = p_{nor}A^{**}p_{nor}$, which is also a von Neumann algebra. Furthermore, if $B \subset A$ is another C*-subalgebra with $M, JMJ \subset \mathcal{M}(B)$, we may identify $(B^2_J)^*$ with $q_B p_{nor}A^{**}p_{nor}q_B$, which is a non-unital subalgebra of $(A^2_J)^*$.

We will need the following bidual characterization of properly proximal.

Lemma 3.5. [DKEP22, Lemma 8.5] Let $M$ be a separable tracial von Neumann algebra with an $M$-boundary piece $X$. Then $M$ is properly proximal relative to $X$ if and only if there is no $M$-central state $\varphi$ on

$$\overline{\mathbb{S}_X(M)} := \left\{ T \in (\mathcal{B}(L^2M)_{f^*})^* \mid [T,a] \in (K_X(M)_{f^*})^* \text{ for all } a \in JMJ \right\}$$

such that $\varphi_{|M}$ is normal.

Using the above notations, we observe that we may identify $\overline{\mathbb{S}_X(M)}$ in the following way:

$$\overline{\mathbb{S}_X(M)} = \left\{ T \in (\mathcal{B}(L^2M)_{f^*})^* \mid [T,a] \in (\mathcal{K}_X(M)_{f^*})^* \text{ for any } a \in JMJ \right\}$$

$$= \left\{ T \in p_{nor}\mathcal{B}(L^2M)^{**}p_{nor} \mid [T,a] \in q_X p_{nor}M(\mathcal{K}_X(M))^{**}p_{nor}q_X, \text{ for any } a \in JMJ \right\},$$

where $q_X$ is the identity of $\mathcal{K}_X(M)^{**} \subset (M(\mathcal{K}_X(M)))^{**}$. If we set $q_X = q_X(L^2M)$ to be the identity of $K(L^2M)^{**} \subset \mathcal{B}(L^2M)^{**}$, then using the above description of $\overline{\mathbb{S}_X(M)}$, we have $q_X \overline{\mathbb{S}_X(M)}q_X^\perp \subset q_X\overline{\mathbb{S}_M}$, as $q_X$ commutes with $JMJ$. 


Remark 3.6. Recall that we may embed $\mathbb{B}(L^2 M)$ into $(\mathbb{B}(L^2 M)^{\sharp}_J)^*$ through the u.c.p. map $\iota_{\text{nor}}$, which is given by $\iota_{\text{nor}} = \text{Ad}(p_{\text{nor}}) \circ \iota$, where $\iota : \mathbb{B}(L^2 M) \to \mathbb{B}(L^2 M)^{**}$ is the canonical $*$-homomorphism into the universal envelope, and $p_{\text{nor}}$ is the projection in $\mathbb{B}(L^2 M)^{**}$ such that $p_{\text{nor}} \mathbb{B}(L^2 M)^{**} (p_{\text{nor}}) = (\mathbb{B}(L^2 M)^{\sharp}_J)^*$. Restricting $\iota_{\text{nor}}$ to $C^*$-subalgebra $A \subset \mathbb{B}(L^2 M)$ satisfying $M, JM J \subset \mathcal{M}(A)$ give rise to the embedding of $A$ into $(A_J^\sharp)^*$, and $(\iota_{\text{nor}}|_M, (\iota_{\text{nor}})_J$ are faithful normal representations of $M$ and $JM J$, respectively. Furthermore, although $\iota_{\text{nor}}$ is not a $*$-homomorphism, $sp M \ni B M$ (that is, the span of elements $x e B y$ where $x, y \in M$) is in the multiplicative domain of $\varphi_0$.

Lemma 3.7. Let $M$ be a finite von Neumann algebra and $X$ an $M$-boundary piece. Let $X_0 \subset \mathbb{K}_X(M)$ be a $C^*$-subalgebra and $\{ e_n \}_{n \in I}$ an approximate unit of $X_0$. If $X_0 \subset \mathbb{K}_X^{\infty, 1}(M)$ is dense in $\| \cdot \|_{\infty, 1}$ and $\iota(e_n)$ commutes with $p_{\text{nor}}$ for each $n \in I$, then $\lim_n \iota_{\text{nor}}(e_n) \in (\mathbb{K}_X(M)^{\sharp}_J)^*$ is the identity, where the limit is in the weak* topology.

Proof. Since $\iota_{\text{nor}}(\mathbb{K}_X(M)) \subset (\mathbb{K}_X(M)^{\sharp}_J)^*$ is weak* dense and functionals in $\mathbb{K}_X(M)^{\sharp}_J$ are continuous in $\| \cdot \|_{\infty, 1}$ topology by [DKEP22 Proposition 3.1], we have $\iota_{\text{nor}}(X_0) \subset (\mathbb{K}_X(M)^{\sharp}_J)^*$ is also weak* dense. Let $e = \lim_n \iota_{\text{nor}}(e_n) \in (\mathbb{K}_X(M)^{\sharp}_J)^*$ be a weak* limit point and for any $T \in X_0$, we have

$$e \iota_{\text{nor}} (T) = \lim_n p_{\text{nor}} \iota(e_n) \iota(T) p_{\text{nor}} = \lim_n p_{\text{nor}} \iota(e_n T) p_{\text{nor}} = \iota_{\text{nor}} (T),$$

and similarly $\iota_{\text{nor}} (T) e = \iota_{\text{nor}} (T)$. By density of $\iota_{\text{nor}}(X_0) \subset (\mathbb{K}_X(M)^{\sharp}_J)^*$, we conclude that $e$ is the identity in $(\mathbb{K}_X(M)^{\sharp}_J)^*$. □

Lemma 3.8. Let $M$ be a finite von Neumann algebra and $N \subset M$ a von Neumann subalgebra. Let $e_N \in \mathbb{B}(L^2 M)$ be the orthogonal projection onto $L^2 N$. Then $\iota(e_N) \in \mathbb{B}(L^2 M)^{**}$ commutes with $p_{\text{nor}}$.

Proof. Suppose $\mathbb{B}(L^2 M)^{**} \subset \mathcal{B}(\mathcal{H})$ and notice that $\xi \mathcal{H}$ is in the range of $p_{\text{nor}}$ if and only if $M \ni x \to \langle \iota(x) \xi, \xi \rangle$ and $JM J \ni x \to \langle \iota(x) \xi, \xi \rangle$ are normal. For $\xi \in p_{\text{nor}} \mathcal{H}$, we have $\varphi(x) := \langle \iota(x) \iota(e_B) \xi, \iota(e_N) \xi \rangle = \langle \iota(E_N(x)) \xi, \xi \rangle$ is also normal for $x \in M$ and $JM J$, which implies that $\iota(e_N) p_{\text{nor}} = p_{\text{nor}} \iota(e_N) p_{\text{nor}}$. It follows that $\iota(e_N)$ and $p_{\text{nor}}$ commutes. □

Lemma 3.9. Let $N \subset M$ be a mixing von Neumann subalgebra admitting a Pimsner-Popa basis $\{ m_k \}$ where $m_k \in M$. Let $X_N$ be the associated boundary piece (see Example 3.1), and $q_N \in (\mathbb{K}(L^2 M)^{\sharp}_J)^*$, $q_{X_N} \in (\mathbb{K}_{X_N}(M)^{\sharp}_J)^*$ be the respective identity elements. Then

$$\sum_{k,l} q_{X_N}^{1/2} \iota_{\text{nor}}(m_k J m_l^* J e_N J m_l J m_k^*) = q_N^{1/2} q_{X_N}.$$

Proof. For notational simplicity, denote by $p_{k,l} = q_N^{1/2} \iota_{\text{nor}}(m_k J m_l^* J e_N J m_l J m_k^*)$. By mixing property of the inclusion $N \subset M$, we see that $p_{k,l}$ are pairwise orthogonal projections. Indeed, if $N \subset M$ is mixing, we have $e_N x J y J e_N - e_N E_N(x) J e_N(y) J e_N \in \mathbb{K}(M)$, i.e., is a compact operator when viewed as a bounded operator from the normed space $M$ to $L^2(M)$.
Now we compute
\[ p_{k,l}p_{k',l'} = q_{k,l}^1 \iota_{nor}(m_k J_{m_k} e_N J_m J_{m'} e_N J_{m'}^* J_{m'}_{k'}) = q_{k,l}^1 \iota_{nor}(m_k J_{m_k} e_N J_m J_{m'} e_N J_{m'}^* J_{m'}_{k'}) \delta_{k,k'} \delta_{l,l'} = q_{k,l}^1 \iota_{nor}(m_k J_{m_k} e_N J_m J_{m'} e_N J_{m'}^* J_{m'}_{k'}) \delta_{k,k'} \delta_{l,l'} \]
where \( q_l \in \mathcal{P}(N) \) such that \( q_l = E_N(m_l^* m_l) \) and automatically satisfies \( m_l q_l = m_l \) (see Section [2.1]).

Denote by \( X_0 \subset \mathbb{B}(L^2 M) \) the hereditary C*-subalgebra generated by \( x_i y_i J_e N \) for \( x, y \) in the C*-algbera \( A \) generated by \( \{m_k a\}_{a \in A, k \in \mathbb{N}} \). It is clear that \( X_0 \) is an \( M \)-boundary piece and note that \( A \) is weakly dense (see Section 2.1, [2]) in \( M \).

Observe that \( K^{\infty,1}_{X_0} = K^{\infty,1}(M) \), where \( K^{\infty,1}(M) \) is obtained from \( X_0 \). Notice that \( \mathbb{B}(L^2 M) X_0 \subset K^{\infty,1}_{X_0} \) is dense in \( \| \cdot \|_{\infty,2} \). Indeed, for any contractions \( T \in \mathbb{B}(L^2 M) \) and \( x, y \in M \), we may find a net of contractions \( T_i \in \mathbb{B}(L^2 M) X_0 \) such that \( T_i \to T e_N x J y J e_N \) in \( \| \cdot \|_{\infty,2} \), as it follows directly from [DKEP22, Proposition 3.1]. It then follows that \( K^{\infty,1}_{X_0} \subset K^{\infty,1}(M) \) is dense in \( \| \cdot \|_{\infty,1} \) and hence \( K^{\infty,1} = K^{\infty,1}(M) = K^{\infty,1}(M) \) by [DKEP22] Proposition 3.6. Note that \( p_{k,l} \in (K^{\infty,1}(M))_{ij}^* = (K^{\infty,1}(M))_{ij}^* \) and \( p_{k,l} \leq q_{k,l}^1 q_{X_0} \).

By the above paragraph it suffices to check the following: \( \sum_{k',l'} q_{k,l}^1 \iota_{nor}(m_k J_{m_k} e_N J_m J_{m} e_N) = q_{k,l}^1 \iota_{nor}(m_k J_{m_k} e_N J_m J_{m} e_N) = q_{k,l}^1 \iota_{nor}(e_N J_{m} J_{m} e_N) \) for all \( a, b \in N \) and \( k, l \in \mathbb{N} \). Indeed, every element in \( X_0 \) can be written as a norm limit of linear spans consisting of elements of the form \( x_i J_y J_e N x_i J_y J_e N x_i J_y J_e N \). Let \( k, l \in \mathbb{N} \). Further we can assume \( x_i = m_k a \) with \( a \in A \) from density. Then we will get that for all \( z \in X_0 \), \( \sum_{k,l} q_{k,l}^1 \iota_{nor}(m_k J_{m_k} e_N J_m J_{m} e_N) \) is weak* dense in \( (K^{\infty,1}_{X_0} (M))_{ij}^* = (K^{\infty,1}_{X_0} (M))_{ij}^* \) by the previous paragraph, so we get that \( \sum_{k,l} q_{k,l}^1 \iota_{nor}(m_k J_{m_k} e_N J_m J_{m} e_N) = q_{k,l}^1 q_{X_0} \).

The above equality holds by a simple computation
\[ p_{k',l'} \iota_{nor}(m_k J_{m_k} e_N J_m J_{m} e_N) = \delta_{k,k'} \delta_{l,l'} q_{k,l}^1 \iota_{nor}(m_k J_{m_k} e_N J_m J_{m} e_N) \]
as in the beginning of this proof wherein we verified that \( p_{k,l} \) are projections. \( \square \)

**Lemma 3.10.** Let \( M \) be a finite von Neumann algebra and \( M_i \subset M, i = 1, \ldots, n \) be von Neumann subalgebras such that \( e_M \) are pairwise commuting. Let \( X \) denote the boundary piece associated to \( \{M_i\}_{i=1}^n \) as in Example [3.1]. Let \( X_i \) denote the boundary pieces associated to \( M_i \). Let \( q_i \) denote the identities of the von Neumann algebras \( (K^{\infty,1}_{X_i} (M))_{ij}^* \) and \( q_{X} \) denote the identity of \( (K^{\infty,1}_{X} (M))_{ij}^* \). Then we have that \( q_X = \bigvee_{i=1}^n q_i \).

**Proof.** Recall from the beginning of this section that \( (K^{\infty,1}_{X} (M))_{ij}^* \) is a von Neumann algebra, as \( M, J M J \) are in the multiplier algebra of \( M(K^{\infty,1}(M)) \). It is easy to see that \( q_{X} \geq q_i \) for each \( i \). Now we show that \( q_{X} \leq \bigvee_{i=1}^n q_i \). Fix an increasing family of finite subsets of unitaries in \( M, F_n \) such that \( \bigcup_n F_n = M \). Let \( e_n = \bigvee_{u,v \in F_n} u J v J (\bigvee_{i \in M_i} J v J)^* \). Clearly we have that \( \iota_{nor}(e_n) \leq \bigvee_{i=1}^n q_i \). Indeed, see that
\[ \iota_{nor}(\bigvee_{u,v \in F_n} u J v J (\bigvee_{i \in M_i} J v J)^*) < q_i \]
and then $t_{\text{nor}}(e_n) = t_{\text{nor}}(v_i \vee u, \forall u \in F_n, uJvJ)e_M, Jv^*Ju^*) \leq \vee_{i=1}^n q_i$. From Lemmas 3.3 and 3.7 we see that $q_X = \lim_{n} t_{\text{nor}}(e_n) \leq \vee_{i=1}^n q_i$ as required. \hfill $\square$

3.3. Induced boundary pieces in the bidual.

Lemma 3.11. Let $M$ be a finite von Neumann algebra, $X$ an $M$-boundary piece, and $N \subset pMp$ a von Neumann subalgebra for some $0 \neq p \in P(M)$. Set $E := \text{Ad}(e_N) \circ \text{Ad}(pMp) : \mathcal{B}(L^2M) \to \mathcal{B}(L^2N)$. Then its restriction $E|_{S_X(M)}$ maps $S_X(M)$ to $S(N)$. Moreover, there exists a u.c.p. map $\tilde{E} : \tilde{S}(M) \to \tilde{S}(N)$ such that $\tilde{E}|_M$ agrees with the conditional expectation from $M$ to $N$.

Proof. To see $E|_{S_X(M)} : S(M) \to S(N)$, note that $pMpJe_NaJ_N = JaMpJe_N$ for any $a \in N$. and $E : \mathcal{B}(L^2M) \to \mathcal{B}(L^2N)$ is $\| \cdot \|_{\infty, 1}$-continuous. Thus for any $T \in S_X(M)$ and any $a \in N$, we have

$$[E(T), J_NaJ_N] = E([T, JaJ]) \in E(\mathcal{K}(M)^{\infty, 1}) = \mathcal{K}(N)^{\infty, 1} = \mathcal{K}^{\infty, 1}(N),$$

i.e., $E(T) \in S(N)$.

Note that $E^* : \mathcal{B}(L^2N)^* \to \mathcal{B}(L^2M)^*$ maps $\mathcal{B}(L^2N)_J^\perp$ to $\mathcal{B}(L^2M)_J^\perp$ by [DKEP22, Lemma 5.3], and similarly $E^* : (\mathcal{K}(L^2N))_J^\perp \to (\mathcal{K}(L^2M))_J^\perp$. Therefore $\tilde{E} : = (E^*|_{\mathcal{B}(L^2N)_J^\perp})^* : \mathcal{B}(L^2M)_J^\perp \to (\mathcal{K}(L^2N))_J^\perp$ and $\tilde{E}|_{(\mathcal{K}(L^2M))_J^\perp} : (\mathcal{K}(L^2M))_J^\perp \to (\mathcal{K}(L^2N))_J^\perp$. Hence we conclude that $\tilde{E} : \tilde{S}(M) \to \tilde{S}(N)$ with $\tilde{E}|_M$ agrees with the conditional expectation from $M$ to $N$. \hfill $\square$

3.4. Relative biexactness and relative proper proximality. Given a countable discrete group $\Gamma$, a boundary piece $I$ is a $\Gamma \times \Gamma$ invariant closed ideal such that $c_0\Gamma \subset I \subset \ell^\infty \Gamma$ [BIP18]. The small at infinity compactification of $\Gamma$ relative to $I$ is the spectrum of the $C^*$-algebra $S_I(\Gamma) = \{ f \in \ell^\infty \Gamma \mid f - R_I f \in I, \forall t \in \Gamma \}$. Recall that $\Gamma$ is said to be biexact relative to $X$ if $\Gamma \cap S_I(\Gamma)/I$ is topologically amenable [Oza04], [BO08, Chapter 15], [BIP18]. We remark that this is equivalent to $\Gamma \cap S_I(\Gamma)$ is amenable. Indeed, since we may embed $\ell^\infty \Gamma \to I^*$ in a $\Gamma$-equivariant way, we have $\Gamma \cap I^* \oplus (S_I(\Gamma)/I)^\text{**} = S_I(\Gamma)^\text{**}$ is amenable, and it follows that $\Gamma \cap S_I(\Gamma)$ is an amenable action [BEW19, Proposition 2.7].

The following is a general version of [DKEP22, Theorem 7.1], whose proof follows similarly. For the convenience of the reader we include the proof sketch below. A more general version of this is obtained in the upcoming work [DP22].

Theorem 3.12. Let $M = \ell^2 \Gamma$ where $\Gamma$ is an nonamenable group that is biexact relative to a finite family of subgroups $\{ \Lambda_i \}_{i \in I}$. Denote by $X$ the $M$-boundary piece associated with $\{ L\Lambda_i \}_{i \in I}$. If $A \subset pMp$ for some $0 \neq p \in P(M)$ such that $A$ has no amenable direct summands, then $A$ is properly proximal relative to $X^A$, where $X^A$ is the induced $A$-boundary piece as in Remark 3.7.

Proof. Consider the $\Gamma$-equivariant diagonal embedding $\ell^\infty(\Gamma) \subset \mathcal{B}(\ell^2 \Gamma)$. Note that under this embedding $c_0(\Gamma, \{ \Lambda_i \}_{i \in I})$ is sent to $X$. Denote by $S_X(\Gamma) = \{ f \in \ell^\infty(\Gamma) \mid f - fg \in c_0(\Gamma, \{ \Lambda_i \}_{i \in I}), \forall g \in \Gamma \}$, the relative small at infinity compactification at the group level.
Restricting this embedding to $S_X(\Gamma)$ then gives a $\Gamma$-equivariant embedding into $S_X(M)$. Therefore we obtain a $*$-homomorphism from $S_X(\Gamma) \rtimes r, \Gamma \to \mathbb{B}(l^2(\Gamma))$ whose image is contained in $S_X(M)$. Composing this with the map $E$ from Lemma 3.11 we obtain a u.c.p map $\phi : S_X(\Gamma) \rtimes r, \Gamma \to S_X(A)$. By hypothesis we have a projection $p_0 \in Z(A)$ and an $A_{p_0}$ bimodular u.c.p map $\Phi : S_{X,A}(A) \to A_{p_0}$. Further composing with this map we obtain a u.c.p map from $\phi : S_X(\Gamma) \rtimes r, \Gamma \to A_{p_0}$.

Now set $\varphi : S_X(\Gamma) \rtimes r, \Gamma \to \mathbb{C}$, by $\varphi(x) := \langle x\rho_0, \rho_0 \rangle_{\pi(p)}$. We then get a representation $\pi_{\varphi} : S_X(\Gamma) \rtimes r, \Gamma \to \mathcal{H}_{\varphi}$ and a state $\tilde{\varphi} \in \mathbb{B}(\mathcal{H}_{\varphi})_*$ such that $\varphi = \tilde{\varphi} \circ \pi_{\varphi}$. Since $C^*_r(\Gamma)$ is weakly dense in $M$, we see by an argument of Boutonnet-Carderi (see Proposition 4.1 in [BC15]) that there is a projection $q \in (\pi_{\varphi}(S_X(\Gamma) \rtimes r, \Gamma))^\prime\prime$ such that $\tilde{\varphi}(q) = 1$ and there exists a normal unital $*$-homomorphism $\iota : L(\Gamma) \to q\pi_{\varphi}(S_X(\Gamma) \rtimes r, \Gamma)^\prime\prime q$.

Since $\Gamma$ is biexact relative to $X$, we have that $S_X(\Gamma) \rtimes r, \Gamma$ is a nuclear $C^*$-algebra. Therefore there is a u.c.p map $\tilde{\iota} : \mathbb{B}(l^2(\Gamma)) \to q(S_X(\Gamma) \rtimes r, \Gamma)^\prime\prime q$ extending $\iota$. Now we see that $\tilde{\varphi} \circ \tilde{\iota}$ is an $A_{p_0}$ central state on $\mathbb{B}(l^2(\Gamma))$ showing that $A$ has an amenable direct summand, which is a contradiction. \hfill \Box

In the case of general free products of finite von Neumann algebras $M = M_1 \ast M_2$ it ought to be the case that that if $A \subset M$ such that $A$ has no amenable direct summand, then $A \subset M$ is properly proximal relative to the boundary piece generated by $M_1$ and $M_2$. However, currently we are only able to obtain this with an additional technical assumption that $M_i \cong L\Gamma_i$ where $\Gamma_i$ are exact, so that $\Gamma_1 \ast \Gamma_2$ is biexact relative to $\{\Gamma_1, \Gamma_2\}$ [BO08, Proposition 15.3.12]. We record below a general result about subalgebras in free products which follows essentially from Theorem 9.1 in [DKEP22], however we do not get the boundary piece associated to the subalgebras $M_i$. We instead get the boundary piece associated to the word length:

Let $M_1, M_2$ be two finite von Neumann algebras and $M = M_1 \ast M_2$ be the tracial free product. Let $A \subset M$ be a nonamenable subalgebra. Consider the free product deformation from [PP08], i.e., $M = M \ast LF_2$, $\theta_t = Ad(u_t^1) \ast Ad(u_t^2) \in \text{Aut}(M)$, with $u_t^1 = \exp(it\alpha_1)$, $u_t^2 = \exp(it\alpha_2)$, where $\alpha_1, \alpha_2$ are selfadjoint element in $LF_2$ such that $\exp(i\alpha_1) = u_1$, $\exp(i\alpha_2) = u_2$ and $u_1 u_2$ are Haar unitaries in $LF_2$. For $t > 0$, we have $E_M \circ \alpha_t = P_0 + \sum_{n=1}^{\infty} (\sin(\pi t)/\pi t)^{2n} P_n$ (see Section 2.5 in [IOA]), where $P_n$ is the orthogonal projection to $H_n = \oplus_{(i_1, \cdots, i_n) \in S_n} L^2(M_{i_1} \otimes \mathbb{C}) \cdots \otimes L^2(M_{i_n} \otimes \mathbb{C})$ and $S_n$ is the set of alternating sequences of length $n$. Consider the hereditary $C^*$-algebra $X_F$ generated by $\{P_n\}_{n \geq 0}$.

**Proposition 3.13.** In the above setup, there exists a projection $p \in A$ such that $Ap$ is amenable and $Ap^\perp$ is properly proximal relative to $X_F$.

**Proof.** It follows from the proof of [DKEP22] Proposition 9.1 that there exists an $M$-bimodular u.c.p. map $\phi : (M^{op})' \cap \mathbb{B}(L^2M \otimes L^2M) \to S_{X_F}(M)$. Moreover, since $L^2M \otimes L^2M \cong L^2M \otimes \mathcal{K}$ as $M$-$M$ bimodule for some right $M$ module $\mathcal{K}$ [IOA, Lemma 2.10], we may restrict $\phi$ to $\mathbb{B}(L^2M) \otimes \text{id}_\mathcal{K}$. Take $p \in Z(A)$ to be the maximal projection such that $Ap$ is amenable and $p \neq 1$ as $A$ is nonamenable. If $Ap^\perp$ is not properly proximal relative to $X$ inside $M$, i.e., there exists an $A$-central state $\varphi$ on $S_X(A)$ which is normal when
restricted to \( p^1 Mp^1 \). Then pick \( q \in \mathcal{Z}(Ap^1) \) be the support projection of \( (\varphi \circ \phi)|_{Ap^1} \) and we have \( Aq \) is amenable, which contradicts the maximality of \( p \).

\[ \square \]

4. The Upgrading Theorem

Proof of Theorem 4.1. First notice that since \( A \) is properly proximal relative to \( X \) inside \( M \), it has no amenable direct summand by (3) of Remark 3.4. Let \( f \in \mathcal{Z}(A) \) be the projection such that \( Af^1 \) is the maximal properly proximal direct summand of \( A \) by (4) of Remark 3.4 and we may assume \( f \neq 0 \) since otherwise \( A \) would be properly proximal. Therefore \( Af \) has no amenable direct summand, is properly proximal relative to \( X \) inside \( M \) by (2) of Remark 3.4 and has no properly proximal direct summand. It follows from Lemma 3.5 that there exists an \( Af \)-central state \( \mu \) on \( \tilde{S}(Af) \) such that \( \mu |_Af \) is normal. Moreover, by a maximal argument, we may assume \( \mu |_{\mathcal{Z}(Af)} \) is faithful, as \( Af \) has no proper proximal direct summand.

Let \( \tilde{E} : \tilde{S}(M) \to \tilde{S}(Af) \) be the u.c.p. map as in Lemma 3.11. Define a state \( \varphi = \mu \circ \tilde{E} : \tilde{S}(M) \to \mathbb{C} \), and it follows that \( \varphi \) is \( Af \)-central and \( \varphi|_{fAf} \) is a faithful normal state. Let \( q_X \) be the identity of the von Neumann algebra \( (\mathbb{K}(L^2M)^*_j)^* \subset (\mathbb{B}(L^2M)^*_j)^* \), \( q_X \) the identity of von Neumann algebra \( (\mathbb{K}_X(M)^*_j)^* \subset (\mathbb{B}(L^2M)^*_j)^* \). Note that \( q_X \leq q_X \) as \( \mathbb{K}(L^2M) \subset \mathbb{K}_X(M) \).

First we analyze the support of \( \varphi \). Observe that \( \varphi(q_X^1) = 1 \). Indeed, if \( \varphi(q_X) > 0 \), i.e., \( \varphi \) does not vanish on \( (\mathbb{K}(L^2M)^*_j)^* \), then we may restrict \( \varphi \) to \( \mathbb{B}(L^2M) \), which embeds into \( (\mathbb{K}(L^2M)^*_j)^* \) as a normal operator \( M \)-system [DKEP22, Section 8], and this shows that \( Af \) would have an amenable direct summand. Moreover, we have \( \varphi(q_X) = 1 \). Indeed, if \( \varphi(q_X^1) > 0 \), then

\[
\frac{1}{\mu(q_X^1)} \varphi \circ \text{Ad}(q_X^1) : \tilde{S}_X(M) \to \mathbb{C}
\]

would be an \( Af \)-central that restricts to a normal state on \( fAf \). Since \( S_X(M) \) naturally embeds into \( \tilde{S}_X(M) \), this contradicts that \( Af \) is properly proximal relative to \( X \) inside \( M \). Therefore we conclude that \( \varphi(q_X^1) = 1 \).

For each \( 1 \leq i \leq n \), denote by \( X_i := X_{M_i} \subset \mathbb{B}(L^2M) \) the \( M \)-boundary piece associated with \( M_i \) and \( q_i \in (\mathbb{K}_{X_i}(M)^*_j)^* \) the identity. Since \( \varphi^\oplus_{i=1} q_i = q_X \) by Lemma 3.10 we have \( \varphi(q_jq_X^1) > 0 \) for some \( 1 \leq j \leq n \).

Claim: there exists a u.c.p. map \( \phi : \langle M, e_{M_j} \rangle \to q_X^1 q_j \tilde{S}(M) q_j \) such that \( \phi(x) = q_X^1 q_j x \) for any \( x \in M \).

Proof of the claim. Denote by \( \{ m_k \}_{k \geq 0} \subset M \) a bounded Pimsner-Popa basis of \( M \) over \( M_i \). For each \( n \geq 0 \), consider the u.c.p. map \( \psi_n : \langle M, e_{M_j} \rangle \to \langle M, e_{M_j} \rangle \) given by

\[
\psi_n(x) = (\sum_{k \leq n} m_k e_{M_j} m_k^*) x (\sum_{\ell \leq n} m_{\ell} e_{M_j} m_{\ell}^*),
\]

and notice that \( \psi_n \) maps \( \langle M, e_{M_j} \rangle \) into the \( * \)-subalgebra \( A_0 := \text{sp}\{ m_k a e_{M_j} m_{\ell}^* \mid a \in M_j, k, \ell \geq 0 \} \).
Recall notations from Remark 3.6. By Lemma 3.8 we have

\[ \{ \iota_{\text{nor}}(Jm_kJe_M, Jm_k^*) \}_{k \geq 0} \subset (B(L^2 M)^J)^* \]

is a family of pairwise orthogonal projections. Set

\[ e_j = \sum_{k \geq 0} \iota_{\text{nor}}(Jm_kJe_M, Jm_k^*) \in (B(L^2 M)^J)^* \]

and define the map

\[ \phi_0 : A_0 \to q_k^\perp (B(L^2 M)^J)^* \]

\[ m_r a e_M, m_\ell^* \mapsto q_k^\perp \iota_{\text{nor}}(m_r a) e_j \iota_{\text{nor}}(m_\ell^*) \]

It is easy to check that \( \phi_0 \) is well-defined. We then check that \( \phi_0 \) is a *-homomorphism. It suffices to show that for any \( x \in M \), we have

(1) \[ q_k^\perp e_j \iota_{\text{nor}}(x) e_j = q_k^\perp \iota_{\text{nor}}(E_M(x)) e_j. \]

Now we compute,

\[
q_k^\perp e_j \iota_{\text{nor}}(x) e_j
= q_k^\perp \sum_{k, \ell \geq 0} \iota_{\text{nor}}((Jm_kJe_M, Jm_k^*) x (Jm_\ell Je_M, Jm_\ell^*))
= q_k^\perp \sum_{k \geq 0} \iota_{\text{nor}}((Jm_kJe_M, Jm_k^*) x (Jm_\ell Je_M, Jm_\ell^*)) + \sum_{k \neq \ell} \iota_{\text{nor}}((Jm_kJe_M, Jm_k^*) x (Jm_\ell Je_M, Jm_\ell^*)).
\]

By Remark 2.3 we have \((Jm_kJe_M, Jm_k^*) x (Jm_\ell Je_M, Jm_\ell^*) \in B(L^2 M)\) is a compact operator from \( M \) to \( L^2 M \) for \( k \neq \ell \). Since

\[
(Jm_kJe_M, Jm_k^*) x (Jm_\ell Je_M, Jm_\ell^*) = 0 \quad \text{if} \quad \ell \neq k,
\]

we have \( \sum_{k \neq \ell} q_k^\perp \iota_{\text{nor}}((Jm_kJe_M, Jm_k^*) x (Jm_\ell Je_M, Jm_\ell^*)) = 0 \). Similarly, one checks that

\[
q_k^\perp \iota_{\text{nor}}((Jm_kJe_M, Jm_k^*) x (Jm_\ell Je_M, Jm_\ell^*)) = q_k^\perp \iota_{\text{nor}}(E_M(x) (Jm_kJe_M, Jm_k^*)).
\]

It then follows from (1) that \( \phi_0 \) is a *-homomorphism. Now we verify that \( \phi_0 \) is norm continuous.

Given \( \sum_{i=1}^d m_k a_i e_M, m_i^* \in A_0 \), we may assume that \( k_i \neq k_j \) and \( \ell_i \neq \ell_j \) if \( i \neq j \). Consider

\[
P_k = q_k^\perp \sum_{i=1}^d \iota_{\text{nor}}(Jm_{k_i}Je_M, m_{k_i}^* Jm_{k_i}^*) \text{ and } Q_k = q_k^\perp \sum_{i=1}^d \iota_{\text{nor}}(Jm_{k_i}Je_M, m_{k_i}^* Jm_{k_i}^*) \text{ as well as}
\]

\[
\iota_{\text{nor}}(E_M, m_{k_i} m_{k_i}^*) Q_k = q_k^\perp \iota_{\text{nor}}(E_M, m_{k_i}^* Jm_{k_i}^*) \text{ for each } 1 \leq i \leq d. \]

Let \( H \) be the Hilbert
space where \((B(L^2M)^*_0)^*\) is represented on. For \(\xi, \eta \in \mathcal{H}_1\), we compute

\[
|\langle \phi_0(\sum_{i=1}^{d} m_k a_i e_{M_j} m_{k_i}^*), \xi, \eta \rangle| \leq \sum_{k \geq 0} \sum_{i=1}^{d} |\langle q_k^{\ell} \tau_\text{nor}(e_{M_j} m_{k_i}^* J m_{k}^* J), \xi, \tau_\text{nor}(J m_k J m_k e_{M_j} a_i)^* \eta \rangle| \\
= \sum_{k \geq 0} \sum_{i=1}^{d} |\langle \tau_\text{nor}(e_{M_j} m_{k_i}^* J m_{k}^* J) P_k \xi, \tau_\text{nor}(J m_k J m_k e_{M_j} a_i)^* Q_k \eta \rangle| \\
\leq \sum_{k \geq 0} \| \tau_\text{nor}(J m_k J (\sum_{i=1}^{d} m_k a_i e_{M_j} m_{k_i}^*)) M_{k}^* J) \| \|P_k \xi\| \|Q_k \eta\| \\
\leq (\sup_{k \in \mathbb{N}} \|m_k\|^2) \| \sum_{i=1}^{d} m_k a_i e_{M_j} m_{k_i}^* \| (\sum_{k \geq 0} \|P_k \xi\|^2)^{1/2} (\sum_{k \geq 0} \|Q_k \xi\|^2)^{1/2} \\
\leq (\sup_{k \in \mathbb{N}} \|m_k\|^2) \| \sum_{i=1}^{d} m_k a_i e_{M_j} m_{k_i}^* \|.
\]

This shows that \(\phi_0\) is norm continuous as required.

Lastly we show that \(\phi_0\) maps into \(q_k^{\ell} S(M)\). It suffices to show that \(e_{\ell}, \tau_\text{nor}(J m_k u J) = 0\) for all \(\ell \in \mathbb{N}\) and \(u \in \mathcal{U}(M_j)\), since \(\phi_0(A_0)\) commutes with \(\tau_\text{nor}(J M J)\).

Without loss of generality, we may assume that \(m_0 = 1\). We compute

\[
[e_j, \tau_\text{nor}(J m_k u J)] = q_k^{\ell} (\sum_{k \geq 0} \tau_\text{nor}(J m_k J e_{M_j} J m_{k_i}^* e_{M_j} J m_k^* J) - \tau_\text{nor}(J m_k m_k J e_{M_j} J m_k^* J) \\
= q_k^{\ell} (\sum_{k \geq 0} \tau_\text{nor}(J m_k e_{M_j} m_{k_i}^* m_{k_i} e_{M_j} J) - \tau_\text{nor}(J m_k e_{M_j} m_{k_i} e_{M_j} m_{k_i}^* J)) \\
= q_k^{\ell} (\tau_\text{nor}(J m_k u J e_{M_j} J) - \tau_\text{nor}(J m_k u J e_{M_j})) = 0.
\]

Combining all the above arguments, we may extend \(\phi_0 : A \to q_k^{\ell} S(M)\) to a \(*\)-homomorphism on \(A\), where \(A = A_0^\delta\perp\) is a C*-algebra.

The next step is to define the map \(\phi\). For each \(n \geq 0\), set \(\phi_n := \phi_0 \circ \psi_n : \langle M, e_{M_j} \rangle \to q_k^{\ell} S(M)\), which is c.p. and subunital by construction. We may then pick \(\phi \in \text{CB}(\langle M, e_{M_j} \rangle, q_k^{\ell} S(M))\) a weak* limit point of \(\{\phi_n\}_{n \in \mathbb{N}}\), which exists as \(q_k^{\ell} S(M)\) is a von Neumann algebra.

We claim that

\[
\text{Ad}(q_j) \circ \phi : \langle M, e_{M_j} \rangle \to q_k^{\ell} q_j q_k^\ell S(M) q_j
\]

is an \(M\)-bimodular u.c.p. map, which amounts to showing \(\phi(x) = q_k^{\ell} q_j \tau_\text{nor}(x)\) for any \(x \in M\).
In fact, for any \( x \in M \), we have
\[
\phi(x) = \lim_{n \to \infty} \phi_0 \left( \sum_{0 \leq k, \ell \leq n} (m_k E_{M_j} (m_k^* x m_\ell) e_{M_j} m_\ell^*) \right)
\]
\[
= q_k^\perp \lim_{n \to \infty} \sum_{0 \leq k, \ell \leq n} t_{\text{nor}} (m_k E_{M_j} (m_k^* x m_\ell)) e_{\ell} t_{\text{nor}} (m_\ell^*)
\]
\[
= q_k^\perp \lim_{n \to \infty} \sum_{0 \leq k, \ell \leq n} (t_{\text{nor}} (m_k) e_j t_{\text{nor}} (m_\ell^*)) (t_{\text{nor}} (m_\ell^*) e_{\ell} t_{\text{nor}} (m_k^*)
\]
where the last equation follows from (1). Finally, note that \( \{p_k\}_{k \geq 0} \) is a family of pairwise orthogonal projections by Remark 2.3, where
\[
p_k := q_k^\perp t_{\text{nor}} (m_k) e_j t_{\text{nor}} (m_k^*) = q_k^\perp \sum_{r \geq 0} t_{\text{nor}} (Jm_r Jm_k e_{M_j} m_k^* Jm_r^* J),
\]
and \( \sum_{k \geq 0} p_k = \sum_{k, r \geq 0} q_k^\perp t_{\text{nor}} (Jm_r Jm_k e_{M_j} m_k^* Jm_r^* J) = q_k^\perp q_j \) by Lemma 3.9. Therefore, we conclude that \( \phi(x) = q_k^\perp q_j t_{\text{nor}} (x) \), as desired. \( \square \)

Now consider \( \nu = \phi \circ \phi \in (M, e_{M_j})^\ast \) and notice that \( \frac{1}{\phi(q_k^\perp)} \nu \) is an \( \text{Af} \)-central state, which is a normal state when restricted to \( f M f \). Let \( f_j \in \mathcal{Z}((Af)^\prime \cap f M f) \) be the support projection of \( \nu|_{\mathcal{Z}((Af)^\prime \cap f M f)} \) and then we have \( Af_j \) is amenable relative to \( M_j \) inside \( M \) [OP10 Theorem 2.1]. Apply the same argument for each \( i \) with \( \phi(q_k^\perp) > 0 \), we then obtain projections \( f_i \in f M f \) (possibly 0) such that \( Af_i \) is amenable relative to \( M_i \) inside \( M \).

Finally, to show \( \vee_{i=1}^n f_i = f \), note that \( \phi(q_i f_i) = \phi(q_i) \) as
\[
\phi(q_i f_i^\perp) = \phi(q_k^\perp q_i f_i^\perp) = \phi(f_i^\perp) = \nu(f_i^\perp) = 0.
\]
Consequently we have
\[
\phi(\vee_{i=1}^n f_i) \geq \phi(\vee_{i=1}^n q_i f_i) \geq \phi(\vee_{i=1}^n q_i) = 1,
\]
and hence \( \vee_{i=1}^n f_i = f \) by the faithfulness of \( \phi | f M f \). Since \( f_i \in \mathcal{Z}((Af)^\prime \cap f M f) \), we may rearrange these projections so that \( \sum_{i=1}^n f_i = f \). \( \square \)

5. Proofs of main theorems

Proof of Theorem 1.2. This follows from noticing that the Jones projections \( e_{L \Lambda_1} \) pairwise commute, and then applying Theorem 3.12 and Theorem 1.1. \( \square \)

Theorem 5.1. Let \( (M_1, \tau_1) \) and \( (M_2, \tau_2) \) be such that \( M_i \cong L \Gamma_i \) where \( \Gamma_i \) are countable exact groups and \( M = M_1 * M_2 \) be the tracial free product. Let \( A \subset M \) be von Neumann subalgebra, then there exists projections \( \{p_i\}_{i=1}^3 \in \mathcal{Z}(A^\prime \cap M) \) such that \( A p_i \prec_M M_i \) for each \( i = 1 \) and 2, \( A p_3 \) is amenable and \( A(\vee_{i=1}^3 p_i)^\perp \) is properly proximal.

Proof of Theorem 5.1. First note that the free products of the exact groups \( \Gamma_i \) are biexact relative to \( \{\Gamma_1, \Gamma_2\} \) [BO08 Proposition 15.3.12] and \( [e_{M_1}, e_{M_2}] = 0 \). Then by Theorem 3.12 we may take \( f_1 \) and \( f_2 \) from Theorem 1.1 and let \( p_i^0 \in \mathcal{Z}(Af_i) \) be the maximal projection
such that $A_{p_i'}$ is amenable for each $i = 1, 2$. Set $p_i = f_i - p_i'$ for $i = 1$ and 2, and $p_3 = p_1' + p_2'$ and the rest follows from Lemma 2.4.

Proof of Corollary 1.3. Since $A \subset M$ has no properly proximal direct summand, it follows from Theorem 1.1 and Theorem 3.12 that there exists central projections $f_1$ and $f_2$ in $\mathcal{Z}(A' \cap M)$ such that $A_{f_i}$ is amenable relative to $M_i$ inside $M$ for each $i$, and $f_1 + f_2 = 1$. If $A_{f_2}$ is not amenable, then by Lemma 2.4 we have that $A_{f_2} \prec_M M_2$. However, since $A \cap M_1$ is diffuse, we may pick a sequence of trace zero unitaries $\{u_n\}$ in $A \cap M_1$ converging to 0. One then checks that $\|E_{M_2}(xu_nf_2y)\|_2 \to 0$ for any $x, y \in M$, which is a contradiction. Therefore $A$ is amenable relative to $M_1$ inside $M$. And then it follows from Theorem 2.5 that $A \subset M_1$.

Proof of Corollary 1.7. Note that in the case of $A = L\Gamma_1$, we have $\mathcal{Z}(A' \cap M) = C$ and hence Theorem 5.1 implies that either $L\Gamma_1 \prec_M LA_1$ or $L\Gamma_1 \prec_M LA_2 \ast \cdots \ast LA_m$. The same argument as in [Dri22, Corollary 8.1] deduces the desired result.

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