Bosonization of Fermionic Fields and
Fermionization of Bosonic Fields

Waldyr A. Rodrigues Jr.
Institute of Mathematics, Statistics and Scientific Computation
IMECC-UNICAMP
walrod@ime.unicamp.br

November 15 2016

Abstract

In this paper using the Clifford and spin-Clifford bundles formalism we show how Weyl and Dirac equations formulated in the spin-Clifford bundle may be written in an equivalent form as generalized Maxwell like form formulated in the Clifford bundle. Moreover, we show how Maxwell equation formulated in the Clifford bundle formalism can be written as an equivalent equation for a spinor field in the spin-Clifford bundle. Investigating the details of such equivalences this exercise shows explicitly that a fermionic field is equivalent (in a precise sense) to an equivalence class of well defined boson fields and that a bosonic field is equivalent to a well defined equivalence class of fermionic fields. These equivalences may be called the bosonization of fermionic fields and the fermionization of bosonic fields.

1 Introduction

The idea to bosonize fermion fields and fermionize boson fields is probably a very old one and has produced in our opinion a considerable amount of misunderstandings. Here, we show how using the Clifford and spin-Clifford bundle formalisms we can easily show how to associate to a spin 1/2 fermionic field an equivalence class of bosonic fields and also how to associate to a bosonic field an equivalence class of fermionic fields. In particular the last exercise shows in a trivial way the origin of the many and many proposed Dirac like representations of the Maxwell field that appeared in the literature. And before you continue the reading and advertisement is in order: the mathematical equivalences proved in this paper, of course, do not change the statistics of sets of particles described by the different classes (fermions and bosons) of fields.
2 The Necessary Mathematical Tools

2.1 Forget Matrices

In order to prove in an easy way the results mentioned in the title of this paper the first thing to be done is to forget the matrix representation of spinor fields. So, we start recalling the necessary tools.

1. In Minkowski spacetime\(^1\) with manifold \(M \simeq \mathbb{R}^4\) introduce global coordinates \(\{x^\mu\}\). Write \(\{\partial_\mu = \frac{\partial}{\partial x^\mu}\}\) and \(\{\gamma^\mu := dx^\mu\}\) for the basis of \(TM\) (tangent bundle) and \(T^*M \simeq \bigwedge^1T^*M\) (cotangent bundle)

2. The metrics of the tangent and cotangent bundles are:

\[
\eta = \eta_{\mu\nu}\gamma^\mu \otimes \gamma^\nu, \quad \eta = \eta^{\mu\nu}\partial_\mu \otimes \partial_\nu
\]  \hspace{1cm} (1)

3. The bundle of (non homogeneous) differential form fields is denoted \(\bigwedge T^*M\) and \(\bigwedge^r T^*M = \bigoplus_{i=0}^{4-r} \bigwedge^i T^*M\), \hspace{1cm} (2)

where \(\bigwedge^1 T^*M\) is the bundle of \(r\)-form fields.

4. For each \(x \in M\) the \(2^4\)-dimensional vector space \(\bigwedge T^*x M\) is isomorphic to the basic vector space of the Clifford bundle of differential forms \(\mathcal{C}^\ell(M, \eta)\) for each \(x \in M\). This means that

\[
\bigwedge T^*M \hookrightarrow \mathcal{C}^\ell(M, \eta)
\]  \hspace{1cm} (3)

5. Then, if we suppose that the \(\gamma^\mu \in \sec \bigwedge^1 T^*M \hookrightarrow \sec\mathcal{C}^\ell(M, \eta)\) these objects satisfy the basic relation (where the Clifford product is simply denoted by juxtaposition)

\[
\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}
\]  \hspace{1cm} (4)

6. In arbitrary coordinates \(\{x^\mu\}\) for \(U \subset M\) the metrics of the tangent and cotangent bundles read:

\[
\eta = g_{\mu\nu}dx^\mu \otimes dx^\nu, \quad \eta = g^{\mu\nu}\frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu}
\]  \hspace{1cm} (5)

and for the Levi-Civita connection \(D\) of \(\eta\) we have

\[
D_{\frac{\partial}{\partial x^\rho}}\frac{\partial}{\partial x^\mu} = \Gamma^\alpha_{\mu\nu}\frac{\partial}{\partial x^\alpha}, \quad D_{\frac{\partial}{\partial x^\rho}}dx^\alpha = -\Gamma^\alpha_{\mu\nu}dx^\nu,
\]  \hspace{1cm} (6)

where in general the Christoffel symbols \(\Gamma^\alpha_{\mu\nu}\) are not null. However take into account that in the coordinates \(\{x^\mu\}\) it is

\[
D_{\partial_\mu}\partial_\nu = 0, \quad D_{\partial_\mu}dx^\alpha = 0.
\]  \hspace{1cm} (7)

\(^1\)Minkowski spacetime is the structure \((M, \eta, D, \tau_\eta, \uparrow)\), where \(M \simeq \mathbb{R}^4\). Moreover \(\eta\) is a metric field of signature (1, 3), \(D\) is the Levi-Civita connection of \(\eta\) and such that its associate Riemann curvature tensor is null. The structure is oriented by \(\tau_\eta \in \sec \bigwedge^4 T^*M\) and time oriented by \(\uparrow\). Details in [7].
7. It is very important to take in mind that even if we use general coordinates \( \{x^\mu\} \) for \( U \subset M \) it is always possible to introduce in \( U \) a set of orthonormal basis for \( TM \) and \( T^*M \simeq \bigwedge^1 T^*M \). Indeed, defining
e_{a} = h_{\mu}^{a} \frac{\partial}{\partial x^{\mu}}, \quad g^{a} = h_{\mu}^{a} dx^{\mu} \quad h_{\mu}^{a} h_{\nu}^{b} = \delta_{\nu}^{a}, \quad h_{\mu}^{a} h_{\nu}^{b} = \delta_{\nu}^{b} \tag{8}
we immediately see that if
g_{\mu\nu} = \eta_{ab} h_{\mu}^{a} h_{\nu}^{b}, \quad g^{\mu\nu} = \eta^{ab} h_{\mu}^{a} h_{\nu}^{b} \tag{9}
it is
\[ \eta = \eta_{ab} g^{a} \otimes g^{b}, \quad \eta = \eta^{ab} e_{a} \otimes e_{b}. \tag{10} \]
and we have
\[ D e_{a} e_{b} = \omega_{ab}^{c} e_{c}, \quad D e_{a} g^{c} = -\omega_{ab}^{c} g^{b}, \tag{11} \]
where the coefficients \( \omega_{ab}^{c} \) of the connection in the basis \( \{e_{a}\}, \{g^{b}\} \) are called by physicists the spin connection. The reason for that name will become clear in a while.

8. To proceed we recall that defining the objects\(^2\)
\[ \omega_{c} := \frac{1}{2} \omega_{ab}^{c} g^{a} \otimes g^{b} \in \text{sec} \bigwedge^2 T^*M \hookrightarrow \text{sec} \mathcal{C}(M, \eta) \tag{12} \]
we can easily show that for any \( C \in \text{sec} \mathcal{C}(M, \eta) \) it is
\[ D e_{a} C = \partial_{c} C + \frac{1}{2} [\omega_{c}, C], \tag{13} \]
where \( \partial_{c} \) is the so-called Pfaff derivative of form fields. We have
\[ \partial_{c} C := h_{\mu}^{c} \frac{\partial}{\partial x^{\mu}} C. \tag{14} \]

2.2 Dirac Operator Acting on \( \mathcal{C}(M, \eta) \)

9. The Dirac operator acting on the bundle of differential forms is the invariant first order operator
\[ \partial := g^{a} D e_{a} = \gamma^{\mu} \partial_{\mu} : \text{sec} \mathcal{C}(M, \eta) \to \text{sec} \mathcal{C}(M, \eta), \]
\[ \partial \mathcal{C} = \partial \wedge \mathcal{C} + \partial \cdot \mathcal{C} \tag{15} \]
where we can easily show that
\[ \partial \wedge \mathcal{C} = d \mathcal{C}, \quad \partial \cdot \mathcal{C} = -\delta \mathcal{C}, \tag{16} \]
where \( d \) is the differential operator and \( \delta \) is the Hodge codifferential operator.

\(^2\)Take notice that one can show that \( \omega_{c}^{ab} = -\omega_{c}^{ba} \).
2.3 Structure of the Clifford and spin-Clifford bundles

10. The Clifford bundle (an algebra bundle) in a spin manifold (which is the case of Minkowski spacetime and more generally of parallelizable Lorentzian manifolds) is the vector bundle

\[ \mathcal{C}(M, \eta) = P_{\text{Spin}_{1,3}} \times \text{Ad} \mathbb{R}_{1,3} \]  

(17)

where \( \mathbb{R}_{1,3} \simeq \mathbb{H}(2) \) is the so-called spacetime algebra. Also, \( \text{Ad} : \text{Spin}_{1,3} \rightarrow \text{End}(\mathbb{R}_{1,3}) \) is such that \( \text{Ad}(u)a = uau^{-1} \). And \( \rho : \text{SO}_{1,3} \rightarrow \text{End}(\mathbb{R}_{1,3}) \) is the natural action of \( \text{SO}_{1,3} \) on \( \mathbb{R}_{1,3} \).

Take notice that the Dirac algebra is \( \mathbb{R}_{4,1} \simeq \mathbb{C}(4) \) and that \( \mathbb{R}_{4,1} \simeq \mathbb{C} \otimes \mathbb{R}_{1,3} \simeq \mathbb{C}(4) \).

Now, the bundle of Dirac spinor fields is isomorphic to the bundle\(^3\)

\[ S(M) = P_{\text{Spin}_{1,3}} \times \ell I \]  

(18)

called the spin-Clifford bundle where \( I \) is the minimal left ideal

\[ I = (\mathbb{C} \otimes \mathbb{R}_{1,3}) f \]  

(19)

generated by the idempotent

\[ f = \frac{1}{2}(1 + \gamma^0) \frac{1}{2}(1 + i\gamma^2 \gamma^1). \]  

(20)

Indeed, as well know, the bundle of Dirac spinor fields (as usually employed in Physics texts books and papers) is the bundle\(^4\)

\[ S_{D}(M) = P_{\text{Spin}_{1,3}} \times \mathbb{L} I \]  

(21)

where \( I \) is the minimal left ideal

\[ I = \mathbb{C}_4 f \]  

(22)

generated by the idempotent

\[ f = \frac{1}{2}(1 + \gamma^0) \frac{1}{2}(1 + i\gamma^2 \gamma^1). \]  

(23)

with \( \gamma^a \) being the standard representation of the Dirac gamma matrices. These objects are the matrix representations in \( \mathbb{C}_4 \) of the (orthonormal) 1-form fields \( g^a \).

Now, recalling that the \( S(M) \) is a module over \( \mathcal{C}(M, \eta) \) we observe that it is a nontrivial fact that once we fix a spin-frame (i.e., an element of \( P_{\text{Spin}_{1,3}} \)), say \( \Xi_0 \), then any section of \( S(M) \) can be written any \( \Psi \in \text{sec} S(M) \) can be written as

\[ \Psi_{\Xi_0} = \psi_{\Xi_0} \frac{1}{2}(1 + \gamma^0) \frac{1}{2}(1 + i\gamma^2 \gamma^1) \]  

(24)

---

\(^3\)In Eq. (18) \( \ell \) is the representation of \( \text{Spin}_{1,3} \simeq \text{Sl}(2, \mathbb{C}) \) on \( I \) by left multiplication.

\(^4\)In Eq. (21) \( I \) refers to the \( D^{1/2.0} \oplus D^{1/2.0} \) representation of \( \text{Spin}_{1,3} \) on the ideal \( I \).
where $\psi \in \text{sec} \mathcal{C} \ell^0(M, \eta)$, with $\mathcal{C} \ell^0(M, \eta)$ the even subalgebra of $\mathcal{C} \ell(M, \eta)$. Then, a general $\psi$ can be (conveniently written, for what follows) as

$$\psi_{\Xi_0} = -S + F - \gamma^5 P,$$  \hspace{1cm} (25)

with

$$S \in \text{sec} \bigwedge^0 T^* M \hookrightarrow \text{sec} \mathcal{C} \ell^0(M, \eta),$$

$$F \in \text{sec} \bigwedge^2 T^* M \hookrightarrow \text{sec} \mathcal{C} \ell^0(M, \eta),$$

$$\gamma^5 P \in \text{sec} \bigwedge^4 T^* M \hookrightarrow \text{sec} \mathcal{C} \ell^0(M, \eta)$$ \hspace{1cm} (26)

where we have use the notable fact that for any $C \in \text{sec} \mathcal{C} \ell(M, \eta)$ we can write the Hodge star operator as

$$\star C = \tilde{C} \gamma^5 \hspace{1cm} (27)$$

with $\gamma^5 = g^0 \wedge g^1 \wedge g^1 \wedge g^3 = g^0 g^1 g^1 g^3 \in \text{sec} \bigwedge^4 T^* M \hookrightarrow \text{sec} \mathcal{C} \ell^0(M, \eta)$ is the volume element.

**Remark 1** The object $\psi_{\Xi_0}$ is said to be a representative in the Clifford bundle and in the spin frame $\Xi_0$ of a Dirac-Hestenes spinor. Thus a Dirac-Hestenes spinor field is a certain equivalence class of even sections of the Clifford bundle. If you want to see details, please consult [7].

**Remark 2** To save notation in what follows we will simply write $\Psi$ and $\psi$ for $\Psi_{\Xi_0}$ and $\psi_{\Xi_0}$.

### 2.4 The spin-Dirac Operator

11. The spin-Dirac operator is the first order differential operator

$$\partial^s := g^a D^s_{e a} = \gamma^\mu \partial_\mu : \text{sec} S(M) \to \text{sec} S(M),$$  \hspace{1cm} (28)

where $D^s$ is the spinor covariant derivative (spin connection) such that for each $\Psi \in \text{sec} S(M)$ it is

$$D^s_{e a} \Psi := e_a(\Psi) + \frac{1}{2} \omega_a \Psi \hspace{1cm} (29)$$

Note that the cobasis $\{\gamma^\mu\}$ is orthonormal and for that basis $\omega_\mu = 0$ and then in this basis

$$\partial^s = \gamma^\mu D^s_{\partial \mu} = \gamma^\mu \partial_\mu. \hspace{1cm} (30)$$

Now $D^s_{e a}$, the representative of $D^s_{e a}$ in the Clifford bundle acts on representatives of Dirac-Hestenes spinor fields $\psi \in \text{sec} \mathcal{C} \ell^0(M, \eta)$ as

$$D^s_{e a} \psi := e_a(\psi) + \frac{1}{2} \omega_a \psi \hspace{1cm} (31)$$

Of course, we also have a representative $\partial^{(s)}$ of the spin-Dirac operator acting on the representative of Dirac-Hestenes spinor fields $\psi \in \text{sec} \mathcal{C} \ell^0(M, \eta)$. It is:

$$\partial^{(s)} \psi = g^a D^{(s)}_{e a} \psi$$
Remark 3 So, using the cobasis $\{\gamma^\mu\}$ the expressions for the Dirac operator $\partial$ and the spin-Dirac operator are the same, namely $\partial = \gamma^\mu \partial_\mu$ and $\partial^s = \gamma^\mu \partial_\mu$. So, in what follows since we are only going to use the orthonormal cobasis $\{\gamma^\mu\}$ we will denote both as $\partial$ when this does not cause any misunderstanding.

2.5 The Dirac-Hestenes Equation

12. The Dirac equation is represented in the Clifford bundle by a representative of a Dirac-Hestenes spinor field once a spin frame is fixed. We have, as well known \[ \partial \psi \gamma^{21} - m \psi \gamma^0 = 0. \] (32)

If we multiply this equation on the right by the idempotent $f$ we immediately get the following equation for $\Psi$,

\[ i \gamma^\mu \partial_\mu \Psi - m \Psi = 0 \] (33)

This equation in the bundle $S_D(M)$ is for $\Psi \in \sec S_D(M)$

\[ i \gamma^\mu \partial_\mu \Psi - m \Psi = 0 \] (34)

where since $S_D(M)$ is trivial we can take (once a spin frame is chosen)

\[ \Psi : \mathbb{R}^4 \rightarrow \mathbb{C}^4. \] (35)

3 Generalized Maxwell Equation in the Clifford Bundle

The generalized Maxwell equation (GME) for $F = \frac{1}{2} F_{\mu \nu} \gamma^\mu \wedge \gamma^\nu \in \sec \bigwedge^2 T^* M \hookrightarrow \sec \mathcal{Cl}(M, \eta)$ generated by an electric current $J_e \in \sec \bigwedge^1 T^* M \hookrightarrow \sec \mathcal{Cl}(M, \eta)$ and an magnetic current $J_m \in \sec \bigwedge^1 T^* M \hookrightarrow \sec \mathcal{Cl}(M, \eta)$ can be written in the Clifford bundle formalism as

\[ \partial F = J_e + \gamma^5 J_m. \] (36)

Indeed, taking into account that $\star J_m = J_m \gamma^5 = -\gamma^5 J_m$ and that $\partial = d - \delta$ Eq. (36) is equivalent to the following pair of equations

\[ \delta F = -J_e, \quad d F = \gamma^5 J_m = -\star J_m \] (37)

Note that the equation $d F = \gamma^5 J_m = -\star J_m$ can be written as

\[ \delta \star F = J_m. \] (38)

Obviously, a field $F \in \sec \bigwedge^2 T^* M \hookrightarrow \sec \mathcal{Cl}(M, \eta)$ satisfying Eq. (36) cannot be derived from a potential $A \in \sec \bigwedge^1 T^* M \hookrightarrow \sec \mathcal{Cl}(M, \eta)$. However, $F$ can be
derived from the superpotential:
\[
A := A + \gamma^5 B, \quad (39)
\]
\[
A, B \in \sec \Lambda^1 T^* M \hookrightarrow \sec \mathcal{C}(M, \eta) \quad (40)
\]
if we impose the Lorenz gauge to A and B, i.e.
\[
\delta A = -\partial \cdot A = 0, \quad \delta B = -\partial \cdot B = 0. \quad (41)
\]
Indeed, in this case
\[
F = \partial A = \partial \wedge A + \partial \cdot A + \partial \wedge (\gamma^5 B) + \partial \cdot (\gamma^5 B). \quad (42)
\]
Now,
\[
\partial \wedge (\gamma^5 B) = \gamma^\mu \wedge \partial_\mu (\gamma^5 B) = \gamma^\mu \wedge (\gamma^5 \partial_\mu B) = \frac{1}{2} (\gamma^\mu \gamma^5 \partial_\mu B - \gamma^5 \partial_\mu B \gamma^\mu)
\]
\[
= -\gamma^5 \frac{1}{2} (\gamma^\mu \partial_\mu B + \partial_\mu B \gamma^\mu) = -\gamma^5 \partial \wedge B = -\star \partial \wedge B = \star \delta B. \quad (43)
\]
On the other hand,
\[
\partial \cdot (\gamma^5 B) = \gamma^\mu \partial_\mu (\gamma^5 B) = \gamma^\mu \cdot (\gamma^5 \partial_\mu B) = \frac{1}{2} (\gamma^\mu \gamma^5 \partial_\mu B + \gamma^5 \partial_\mu B \gamma^\mu)
\]
\[
= -\gamma^5 \frac{1}{2} (\gamma^\mu \partial_\mu B - \partial_\mu B \gamma^\mu) = -\gamma^5 \partial \cdot B = \star \partial \cdot B = \star dB. \quad (44)
\]
So, with conditions given in (41) we have
\[
F = \partial A = dA + \star dB \quad (45)
\]
Then,
\[
\partial F = (d - \delta)(dA + \star dB) = d \star dB - \delta dA \quad (46)
\]
and we must have
\[
J_e = -\delta dA, \quad \star J_m = d \star dB. \quad (47)
\]
Moreover, since A and B satisfy the Lorenz condition we have that they obey nonhomogeneous wave equations. Indeed,
\[
-\delta dA - \delta \delta A = (d - \delta)^2 A = \partial^2 A = \partial \wedge \partial A + \partial \cdot \partial A \quad (48)
\]
\[
\partial \cdot \partial A = J_e \quad (49)
\]
since \(\partial \wedge \partial A = 0\). Also, it is
\[
\partial \cdot \partial B = J_e. \quad (50)
\]
\[\text{Known as Cabibo-Ferrari potential. See more details in [3].}\]
4 The Neutrino Equation in Maxwell Like Form

A massless neutrino is represented by a Weyl spinor field. In the Clifford bundle formalism a representative of a Weyl spinor field is

$$\phi = \frac{1}{2} \psi (1 + \gamma^5)$$

(51)

where $\psi$ is a representative of a Dirac-Hestenes spinor field in $\mathcal{C}\ell(M, \eta)$. Now, if

$$\partial \psi = 0,$$

(52)

(massless Dirac-Hestenes Equation) $\phi$ satisfy the Weyl equation

$$\partial \phi = 0,$$

(53)

So, it is enough to show how to write the Eq. (52) as a GME. Indeed, recalling Eq. (25) we immediately see that we can write

$$\partial \psi = -\partial S + \partial F - \partial \gamma^5 P = 0$$

(54)

and thus we have

$$\partial F = \partial S + \gamma^5 \partial P$$

(55)

and calling

$$J_e := \partial S \in \text{sec} \bigwedge^1 T^* M \hookrightarrow \text{sec} \mathcal{C}\ell(M, \eta),$$

$$J_m := \partial P \in \text{sec} \bigwedge^1 T^* M \hookrightarrow \text{sec} \mathcal{C}\ell(M, \eta),$$

(56)

we see that Eq. (55) can be written as

$$\partial F = J_e + \gamma^5 J_m$$

(57)

which is formally identical to Eq. (56), the GME generated by electric and magnetic currents.

5 Bosonization of Fermionic Fields

Does the steps we use to go from Eq. (52) to Eq. (57) means that we “bosonized” a spinor field?

The answer is yes in the following sense. First recall that the spin-Dirac operator and the Dirac operator (which act on very different bundles) have the same form $\gamma^\mu \partial_\mu$ only because we used coordinates for $M$ in Einstein-Lorentz-Poincaré gauge and Eqs. (52) and (53) are only representatives of equivalent differential equations for legitimate spinor fields. This means that for

$$\Psi = \psi f \in \text{sec} S(M)$$

(58)

we have that

$$\partial \Psi = 0$$

(59)
which using the decomposition given by Eq. (25) we get
\[ \partial F = J_e f + \gamma^5 J_m f \] (60)

where \( F, J_e f, \gamma^5 J_m f \in \text{sec} S(M) \), i.e., they are spinor fields.

However, we already remarked that Eq. (60) is the expression of the Weyl equation once a spin frame is fixed. For different spin frames the object that represents the intrinsic object called a spinor field in the Clifford bundle is represented by different sections of the Clifford bundle. Indeed, if \( \psi = \psi_{\Xi_0} \) is the representative of a spinor field in the spin frame \( \Xi_0 \) and \( \psi_{\Xi} \) is the representative of the same spinor field in the spin frame \( \Xi \), then we have
\[ \psi_{\Xi_0} u_0 = \psi_{\Xi} u \] (61)
i.e., a spinor field, (a fermion field) which is a section of a spinor bundle \( S(M) \) may be represented by an equivalence class of nonhomogeneous even sections of the Clifford bundle (room of boson fields). For the particular case of the Weyl spinor field satisfying Weyl equation we can say that the Weyl spinor field is equivalent to an equivalent class of scalar, plus 2-forms and plus 4-forms fields, namely, \( S, F, \gamma^5 P \), coupled through Eq. (55).

6 The Electron Equation in Maxwell Form

Introduce a Hertz potential field \[ 0 \leq \Pi \in \text{sec} \bigwedge^2 T^* M \rightarrow \text{sec} Cl(M, \eta) \] satisfying the following equation
\[ \partial \Pi = (\partial \Phi + m \Phi \gamma_3 + m(\Pi_{\gamma_012})_1) + \gamma_5(\partial \Psi + m \Psi \gamma_3 - \gamma_5(m\Pi_{\gamma_012})_3) \] (62)

where \( \Phi, \Psi \in \text{sec} \bigwedge^0 T^* M \rightarrow \text{sec} Cl(M, \eta) \), and \( m \) is a constant. Under these conditions, the electromagnetic and Stratton potentials \[ A = \partial \Phi + m \Phi \gamma_3 + m(\Pi_{\gamma_012})_1, \] (63)
\[ \gamma_5 S = \gamma_5(\partial \Psi + m \Psi \gamma_3 - \gamma_5(m\Pi_{\gamma_012})_3), \] (64)
and must thus satisfy the following subsidiary conditions,
\[ \hat{\diamond}(\partial \Phi + m \Phi \gamma_3 + m(\Pi_{\gamma_012})_1) = J_e, \] (65)
\[ \hat{\diamond}(\gamma_5(\partial \Psi + m \Psi \gamma_3 - \gamma_5(m\Pi_{\gamma_012})_3)) = 0, \] (66)
\[ \hat{\diamond}\Phi + m \Phi \cdot (\Pi_{\gamma_012})_1 = 0, \] (67)
\[ \hat{\diamond}\Psi - m \Psi \cdot (\gamma_5(\Pi_{\gamma_012})_3) = 0, \] (68)

where \( \hat{\diamond} = -(d\delta + \delta d) = \partial^2 = \partial \wedge \partial + \partial_{\perp} \partial \).

6 Details may be found in [5, 6, 7].
Now, in the Clifford bundle formalism, the following sum is a legitimate operation

\[ \psi = -G + \Pi + \gamma_5 \mathcal{P} \]  

(69)

and according to previous results Eq. (69) defines \( \psi \) as a representative of some Dirac-Hestenes spinor field in a given spin frame. Now, we can verify that \( \psi \) satisfies the equation

\[ \partial \psi \gamma_{21} - m \psi \gamma_0 = 0 \]  

(70)

which is as we already know a representative of the standard Dirac equation (for a free electron) in the Clifford bundle (Eq. (32)). Once again we can say that we bosonized the electron field.

7 The Fermionization of Maxwell Field

As we already said, in the Clifford bundle the (generalized) Maxwell equation satisfied by \( F \in \sec \bigwedge^2 T^* M \hookrightarrow \sec \mathcal{C} \ell(M, \eta) \) is

\[ \partial F = J_e + \gamma^5 J_m. \]  

(71)

with electric current \( J_e \in \sec \bigwedge^1 T^* M \hookrightarrow \sec \mathcal{C} \ell(M, \eta) \) and an magnetic current \( J_m \in \sec \bigwedge^1 T^* M \hookrightarrow \sec \mathcal{C} \ell(M, \eta) \). Now, recall that the spin-Clifford bundle is module over the Clifford bundle[1, 7]. Then, choosing a spin frame, say \( \Xi_0 \) and an idempotent field \( f \in \sec S(M) \) and multiplying Eq. (71) on the right by \( f \) we get for \( \Psi_{\Xi_0} = F f \in \sec S(M) \) the equation

\[ \partial \Psi_{\Xi_0} = \mathcal{J}_e \Xi_0 + \gamma^5 \mathcal{J}_m \Xi_0, \]  

(72)

where \( \mathcal{J}_e \Xi_0 = J_e f, \mathcal{J}_m \Xi_0 = J_m f \in \sec S(M) \). In this way we can say that the Maxwell field satisfying Maxwell equation is an equivalence class of fermion fields \( \Psi_{\Xi_0} \) and we can say that we fermionize a bosonic field!!.

Remark 4 Of, course, there several different spinor like representations of the Maxwell field, since there are many non equivalent idempotent fields in \( S(M) \). By choosing appropriately these idempotents we can reproduce all Dirac like representations that appeared in the literature. Details in [4].

8 Conclusions

In this brief note we showed how using the Clifford and spin-Clifford bundles formalism we can give a rigorous mathematical meaning to the meaning of the sentences: (i) bosonization of fermionic fields and (ii) fermionization of bosonic fields. Each object in one class is represented by a well defined equivalence class of objects in the other class. It is also important to take in mind that the mathematical equivalences proved above do not imply, of course, any change in the statistics satisfied by each class (fermion or boson) of fields. We finally
recall that the above formalism can be trivially generalized for fields living in a
general Lorentzian spacetime structure, but in this case care must be taken in
distinguishing explicitly the Clifford and spin-Clifford Dirac operators.

Acknowledgement 5 The author thanks Professor L.C.L. Botelho for his sug-
gestion to write this paper and his very important comments.

References

[1] Crumeyrolle, A., Orthogonal and Symplectic Clifford Algebras. Spinor Structures. Kluwer Acad. Publ., Dordrecht, 1990.

[2] Hestenes, D., Space-Time Algebra (second revised edition), Birkhäuser, Basel, 2015.

[3] Maia, A. Jr., Recami E., Rodrigues, W. A. Jr., and Rosa, M. A. F., Magnetic Monopoles without String in the Kähler-Clifford Algebra Bundle: A Geometrical Interpretation, J. Math. Phys. 31, 502-505 (1990).

[4] Rodrigues, W. A. Jr. and Capelas de Oliveira, E., Dirac and Maxwell Equations in the Clifford and Spin-Clifford bundles, Int. J. Theor. Phys. 29, 397-412 (1990).

[5] Mosna, R. A., and Rodrigues, W. A. Jr., The Bundles of Algebraic and Dirac-Hestenes Spinor Fields, J. Math. Phys. 45, 2945-2966 (2004).

[6] Rodrigues, W. A. Jr., Algebraic and Dirac-Hestenes Spinors and Spinor Fields, J. Math. Phys. 45, 2908-2994 (2004).

[7] Rodrigues, W. A. Jr. and Capelas de Oliveira, E., The Many Faces of Maxwell Dirac and Einstein Equations. A Clifford Bundle Approach, Lecture Notes in Physics 922 (second edition revised and enlarged), Springer, Heidelberg, 2016 (first published as Lecture Notes in Physics 722, 2007).

[8] Stratton, J. A., Electromagnetic Theory, McGraw-Hill Book Co., New York, 1941.