Towards multigrid methods for propagators of staggered fermions with improved averaging and interpolation operators

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A Dirac choice for the averaging kernel $C$ is implemented numerically. This improved kernel will be needed in gauge covariant multigrid computations for propagators of staggered fermions. Results for $C$ and the variational coarse grid operator will be given in 2-$d$ $SU(2)$ gauge fields. C++ is advocated for future algorithm development.

Big efforts have been undertaken to find efficient multigrid (MG) methods for the computation of propagators in gauge fields; see [1] for a list of references. Up to now no practical method has been found which is competitive for fermionic propagators in 4-$d$ non-abelian gauge fields. However, it was proven that (geometric) MG works in principle without critical slowing down (CSD) in arbitrarily disordered systems, both for bosons [2,3] and for staggered fermions [4]. Unfortunately, this “ideal” algorithm is not practical because the interpolation kernels have exponential tails whose truncation brings CSD back [3]. Nevertheless, the proof that CSD can be eliminated by MG raises hope that there exist also practical – though sophisticated – competitive methods.

We consider the squared Dirac equation,

$$ (-\slashed{D}^2 + m^2) \chi = f , \quad (1) $$

where $f$ may be a pseudofermion field, for instance. We will focus on staggered fermions and a gauge covariant “Dirac choice” [5] (no gauge fixing required) for the averaging kernel $C$. The $C$-kernel should map smooth functions on a fine grid onto functions which are smooth on a coarser scale. In [6,7] it was proposed to adopt an algebraic definition of smoothness in disordered systems.

Given a differential operator $D$, according to [6,7] a function $\chi$ is smooth on length scale $a$ when

$$ \| D \chi \| \ll \| \chi \| \quad (\text{in units } a = 1) . \quad (2) $$

Figure 1. Norm of the ground-state of $-\slashed{D}^2$ on the even sublattice of a $24^4$ lattice in a quenched $SU(2)$ gauge field at $\beta = 2.5$. (Two-dimensional cut through the site with maximal norm.)

Smooth functions in the sense of Eq. (2) are localized in disordered systems. Bäker [8] found in quenched 2-$d$ $SU(2)$ lattice gauge theory that the eigenfunctions corresponding to the 2 or 3 lowest eigenvalues of the negative gauge covariant Laplacian are localized for $\beta = O(1)$. A fermionic example in 4-$d$ $SU(2)$ is given in Fig. 1.

The localization property of the lowest mode of $-\slashed{D}^2$ could be taken into account in a geometric MG algorithm by shifting the site with maximal norm such that it coincides with a block center.
Let us now turn to the improved averaging kernel $C$. In Ref. [5] two qualitatively different proposals were made for the choice of $C$ for staggered fermions in non-abelian gauge fields. The first one, the “Laplace choice” (LC), was used successfully in the “ideal” algorithm [4], but it has proven to be poor in practical variational MG methods [1,3]. Therefore we will test the second proposal which is called the “Dirac choice” (DC) of $C$.

In the DC, as in the LC, $C$ is defined by a ground-state projection method. This means that the adjoint $C^*$ of $C$ fulfills a gauge covariant eigenvalue equation:

$$(-\psi_N^2 \psi^2 C^*)(z, x) = \lambda_0(x) C^*(z, x),$$  

(3)

where $-\psi_N^2 \psi^2$ is a block-local approximation of $-\psi^2$ specified below, and $\lambda_0(x)$ is its lowest (gauge invariant) eigenvalue. $z$ denotes a site in a fine lattice, and $x$ is a coarse grid site (“a block”). $C(x, z)$ is an element of the linear span of the gauge group.

The solution of Eq. (3) is made unique by imposing a normalization and a covariance condition. The normalization condition reads

$$\frac{1}{L_b^d} \sum_{z \in \mathbb{Z}^d} \|C(x, z)\|^2 = 1$$  

(4)

for all $x$, where $L_b$ is the blocking factor ($= 3$), summation is over all fine grid sites within a block $x$, and the norm used is the trace norm. The covariance condition reads (for gauge group $U(1)$ or $SU(2)$)

$$C(x, \hat{x}) = r(x) \mathbb{1}$$  

(5)

with $\hat{x}$ being the block center, and $r(x) > 0$.

We use a blocking procedure for staggered fermions which is consistent with the lattice symmetries of free fermions [5]. This forces us to choose $L_b = 3$. Even $L_b$ are not allowed.

Fig. 2 illustrates our choice of blocks in case of the DC. The different fermionic degrees of freedom are called “pseudoflavor” [5]. Different pseudoflavors are distinguished by different symbols in Fig. 2. Block centers $\hat{x}$ are encircled. They are spaced by $L_b$, which means that the coarser grid has $1/L_b^d$ times the number of sites of the finer grid. The boundaries of seven blocks $x$ are marked.

The operator $\psi_N^2 \psi^2$ in (3) equals $\psi^2$ with the following boundary conditions (BC). For the sites which have the same pseudoflavor as the central site $\hat{x}$ we impose Neumann BC on the boundary of $x$, for all other boundary sites we use Dirichlet BC. Details can be found in [5,3].

The DC (3) has the following properties. In the limiting case of a pure gauge, we regain the LC with nonoverlapping blocks. The $C$-kernel for free fermions is piecewise constant on sublattices of given pseudoflavor. In nontrivial gauge fields — contrarily to the LC — the averaging kernel $C(x, z)$ of the DC is nonvanishing also when $x$ and

\footnote{In pure gauges the variational MG algorithm with these kernels is successful in eliminating CSD [5].}
z carry different pseudoflavor (but have the same parity, i.e. are both even or both odd). Therefore the blocks in Fig. 2 overlap in a nontrivial way. This overlap takes the field strength term $\sigma_{\mu\nu} F_{\mu\nu}$ in $\mathcal{F}$ into account. $\sigma_{\mu\nu} F_{\mu\nu}$ had been disregarded in the LC.

The DC for $C$ has been implemented in C++ [9,10]. The author advocates this language for future algorithm development. It requires some effort to define suitable C++ classes, but once this is done, all advantages of C++ [10, Sec. 1] are at your disposal.

The eigenvalue equation (3) was solved by the efficient algorithm of [11]. All computer programs were checked for gauge covariance (i.e. without gauge fixing corresponding results were found for different points on gauge orbits).

First tests have been performed in quenched 2-d $SU(2)$ gauge fields. An encouraging result is that $\lambda_0(x) = 0$ for all blocks $x$ and for arbitrary $\beta$. Global lowest eigenvalues are collected in Table 1. Examples for $C$, for the variational coarse grid operator $-C \mathcal{P}^2 C^*$, and for the non-diagonal(!) mass term $CC^*$ are in Table 2. Further results in $d = 2, 4$, including computations of propagators, will be available shortly [12].

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### Table 1

| $\beta$ | $-\mathcal{F}$ | $C \mathcal{P}^2 C^*$ |
|---------|----------------|---------------------|
| $\infty$ | 0              | 0                   |
| 40      | $24^2$         | 0.04513             |
| 1.0     | $24^2$         | 0.008173            |

### Table 2

Examples of kernels in 2-d $SU(2)$ at $\beta = 40$ ($\xi \approx 5.2$). The bold entries vanish in pure gauges.

| $C(0, z)$ | $C(0, z)$ |
|-----------|-----------|
| 0.997     | 0.002     |
| 1.00      | 0.203     |
| 0.129     | 0.164     |

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