Particle creation in the presence of minimal length: The time dependent gauge

M. Bouali and S. Haouat

LPTh, Department of Physics, University of Jijel,
BP 98, Ouled Aissa, Jijel 18000, Algeria.

(Dated:)

Abstract

In this paper we have studied the problem of scalar particles pair creation by a constant electric field in the presence of a minimal length. A closed expression for the corresponding Green’s function is obtained via path integral approach. Then by projecting this function on the outgoing particle and antiparticle states we have calculated the probability to create a pair of particles and the number density of created particles. From this, we have deduced the modifications brought by the minimal length to Hawking temperature and black hole entropy. It is shown that the first correction is a logarithmic term with a negative numerical factor. We have also examined the semiclassical WKB approximation in the calculation of the pair production rate. The result is that, unlike the ordinary case, the WKB approximation in the presence of a minimal length does not give the exact rate even for the constant electric field.

PACS numbers: 03.70.+k, 04.60.Bc, 04.62.+v, 04.70.Dy

Keywords: Schwinger Effect; Minimal Length; GUP; Green’s Function; Black Hole Radiation
I. INTRODUCTION

The existence of a minimal measurable length is a common feature to many phenomenological approaches to quantum gravity [1, 2]. In 1964, Mead has shown, through gedanken experiments, that the gravitational interactions make it impossible to measure the position of a particle with an uncertainty less than

\[ l_P = \sqrt{\frac{G\hbar}{c^3}} \approx 1.61605 \times 10^{-35} \text{ m} \]

where \( c \) is the speed of light in vacuum, \( \hbar \) is the Planck constant and \( G \) is the gravitational constant (the Newton constant). This fundamental scale existed, indeed, long before the theory of general relativity; in 1899, Planck used the universal constants \( c, \hbar \) and \( G \) to form, amongst others, a fundamental unit of length, which refers to as the Planck length. Nevertheless, at that date the Planck length, despite its universality, was not immediately admitted as an important scale. After the theory of general relativity was established, this fundamental length has found its significance. It can be understood as the scale at which the Schwarzschild radius of a black hole is equal to its Compton wavelength and therefore, the scale at which both gravitational and quantum effects become significant. Having taken on this quite meaning, the Planck length has become the indispensable ingredient in many phenomenological approaches to quantum gravity [4–7]. It arises in string theory [8–11], loop quantum gravity [12], black hole physics [13–15] and in non-commutative field theories [16–18].

The existence of such a minimal length in nature would change to some extent our understanding of short distance physics and hence very high energy physics. This explains why various physical problems are reconsidered by taking into account the minimal length either at quantum or classical level [19, 20]. Besides the large number of simple quantum mechanical systems [21–29], such as the harmonic oscillator and the Hydrogen atom, the effect of minimal length have been the subject of many studies. As example, we cite the influence of the minimal length on the Casimir effect [30, 31], the Unruh effect [32], the Hawking radiation and black hole thermodynamics [33–38] and the Schwinger effect [39–41]. Elsewhere, the quantum gravity corrections to the mean field theory of nucleons have been, recently, communicated [42].

In this paper, we propose to study the particle-antiparticle pair creation from vacuum
by an electric field in the presence of a minimal length. This effect, which refers to as the Schwinger effect, has been predicted in quantum electrodynamics several decades ago [43–45]. It can be interpreted as follows: In Minkowski space-time, the free quantum field has an invariant vacuum state under Poincaré symmetry, which defines a unique set of creation and annihilation operators and thence makes possible the interpretation of the field in terms of particles and antiparticles. This, together with relativistic covariance, implies that all inertial observers agree on the same number of particles contained in a quantum state. In the presence of an external electric field, the vacuum state is no longer invariant under space-time translations and, therefore, it is no longer possible to define a state that would be identified by all observers as the vacuum state. Because of this, the interpretation of the quantum field in terms of particles might be ambiguous since, in such a case, the vacuum is instable. In particular, the particle creation occurs spontaneously when the electric field breaks the invariance under space-time translations, implying instantaneous vacuum state. In other words, the state initially recognized as the vacuum state evolves in time and could contain outgoing particles at a very distant time.

On the other hand, the minimal length is, generally, implemented in the quantum theory through a generalized uncertainty principle (GUP) which implies a deformation of the Poincaré group. Hence, in order to formulate a quantum field theory with a minimal length, it is necessary to carefully retrace all steps of the standard quantization scheme [46]. However, to our knowledge, there is no well-established quantum field theory that consistently includes such a minimal length. It is thus of great interest to examine the accuracy of usual methods in the study of several physical processes in the presence of minimal length. This includes the Unruh effect, the Hawking radiation, the particle creation and many other phenomena. In this regard, the Schwinger effect is a good example to clarify several issues. It could give a clear answer to the following questions:

- When a minimal length is present, is the Schwinger effect still consistent with gauge invariance principle? In Schwinger effect, particles and antiparticles are produced in pairs, i.e. for each created particle an antiparticle is created simultaneously. Consequently, it is expected that the Schwinger effect preserves the electric charge conservation and, thus, the gauge invariance. A constant electric field can be described by the space dependent gauge, where \( A_0 = -eEx \), or by a time-dependent gauge;
\[ A_z = -eEt. \] In the ordinary case, without minimal length, we know how the creation of particles in these two gauges are related to each other [47]. However, in the presence of a minimal length, it is not clear how the two gauges give the same results.

- The second question is about the validity of the semiclassical approximation: It is well-known that the use of the semiclassical WKB approximation reproduce the exact pair production rate by constant electric field in Minkowski space-time. Indeed, the sum of all contributions from higher order corrections cancels. In the presence of a minimal length existing studies shows that the semiclassical approximation does not give the same results for the pair creation rate as the exact treatment. See for instance [39] and [40].

- How the Schwinger effect is connected to the Hawking radiation and Black hole entropy? The black hole radiation in the presence of minimal length and corrected black hole thermodynamics have been much written about, and remain attracting much attentions. The most used method in this context, besides some heuristic derivations, is the semiclassical WKB method, which has been considered as the method yielding more accurate results. However, since the accuracy of this method on the study of the Schwinger effect is called into question, it would be useful to discuss the possible link between Schwinger effect and Hawking radiation. The comparison between the two effects could lead us to many important findings.

We notice that in reference [39], that one of us was involved in, the creation of scalar particles by an electric field in the presence of a minimal length is investigated. In that work, the authors considered a charged scalar particle subjected to a constant electric field expressed in space dependent gauge. Then, since the corresponding Klein Gordon equation is exactly soluble in momentum space, they were able to calculate the pair production probability and the number density of created particles by the use of the Bogoliubov transformation connecting the "in" with the "out" states. Recently, the particle creation in the presence of minimal length is reconsidered in the context of WKB approximation for a deformed Schrodinger-like equation [40]. Otherwise, the problem has been discussed in different scenarios such as GUP models involving minimal length and/or maximal momentum [41, 48–50] and noncommutative geometry [51, 52]. This is because the Schwinger effect has become an interdisciplinary research area due to its considerable number of applications in
various fields of physics, from heavy nuclei to black hole physics [53]. In this paper, in order to satisfactorily answer the questions evoked above, we reconsider the Schwinger effect in the presence of a minimum length by using the time-dependent gauge.

The rest of this paper proceeds as follows. In Sec. 2, we provide some arguments that lead to estimate heuristically the effect of the minimal length on the creation of particles. In Sec. 3, we introduce a generalized Heisenberg uncertainty principle to derive the Klein Gordon equation with minimal length for a charged particle in an electric field. In Sec. 4, we use the path integral method to calculate the corresponding Green’s function. In Sec. 5, we calculate the probability to create one pair of particles. In Sec. 6, we discuss the application of the Schwinger effect with minimal length in black holes radiation. In Sec. 7, we examine the validity of semiclassical WKB approximation. Sec. 8 is reserved for concluding remarks.

II. THE MINIMAL LENGTH AND PARTICLE CREATION

In ordinary quantum electrodynamics, it has been shown that the pair production rate in constant electric field of strength $E$ is given by [54]

$$W \sim \exp\left(-\pi \frac{m^2 c^3}{e E \hbar}\right),$$

what confirms the nonperturbative nature of the process and stipulates a critical value above which the effect becomes appreciable

$$E_c = \frac{m^2 c^3}{e \hbar} \sim 10^{16} \text{V/cm.}$$

In this section, we try to find an adequate interpretation to the above formula by considering some intuitive arguments, which could be generalized straightforwardly to the case when a minimal length exists.

Let us first recall that the creation and annihilation of particle pairs, is related to the difficulties encountered when unifying quantum mechanics and special relativity in a single-particle theory. In relativistic quantum mechanics, it is the Heisenberg’s uncertainty principle that leads to the creation of particles. Indeed, when a particle is confined to a region of size $\Delta x$, it has, according to Heisenberg’s uncertainty relation, a momentum width $\Delta p \geq \frac{\hbar}{2 \Delta x}$. Consequently, if $\Delta x \lesssim \lambda_C$, with $\lambda_C = \frac{\hbar}{mc}$, the particle’s momentum uncertainty becomes

$$\Delta p \geq (\Delta p)_0 = \frac{1}{2} mc,$$
where \((\Delta p)_0 = \frac{1}{2}mc\) is the smaller momentum uncertainty that accompanies the localization of a mass \(m\) particle with precision equal to its Compton wavelength. Therefore, the energy uncertainty of a relativistic particle becomes of the same order as its rest mass, \(\Delta E \sim \Delta pc \sim mc^2\). This gives rise to the creation of virtual particle-antiparticle pairs. However, since the lifetime of these pairs is very short, they finish by conversely annihilating.

Nevertheless, when a virtual pair is subjected to a constant electric field of strength \(E\), the particle and its antiparticle move apart from each other and they will gain the energy \(eEL\) after covering a relative distance \(L\). If the gained energy exceeds the rest mass of the two particles, \(eEL \geq 2mc^2\), the pair will become real and the particles will continue to move apart. However, since the typical separation of the virtual pair is of order of the Compton wavelength, \(L \sim \frac{\hbar}{mc}\), it follows that

\[eE \gtrsim \frac{m^2c^3}{\hbar}.\]  

(4)

This explains why there is a critical value from which the effect becomes susceptible.

The aim now is to seek an intuitive interpretation for the pair production rate. To this aim, we first write the exponent in equation (1) in the form

\[
\pi \frac{m^2c^3}{eE \hbar} = \pi \frac{mc^2}{eE \times \frac{\hbar}{mc}}.
\]

(5)

This can be viewed as the ratio of the particle rest energy to the electric field energy over a distance equal to the particle Compton wavelength \([55]\). At first glance, this observation does not seem in any way generalizable to the case of minimal length. Instead, if we write

\[
\pi \frac{m^2c^3}{eE \hbar} = 2\pi \frac{mc^2}{eE} \times \frac{mc}{2} = 2\pi \frac{l \times (\Delta p)_0}{\hbar}
\]

where \((\Delta p)_0\) is given by equation (3) and \(l = \frac{mc^2}{eE}\) is the minimal distance travelled by each particle of the pair in the electric field to reach the energy \(2mc^2\), we obtain

\[\mathcal{W} \sim \exp \left( -2\pi \frac{l \times (\Delta p)_0}{\hbar} \right),\]

(6)

which can be immediately extended to case of minimal length. To show this, we consider the GUP

\[
\Delta P \Delta X \geq \frac{\hbar}{2} \left[1 + \beta (\Delta P)^2 + \ldots \right],
\]

(7)
where $\beta$ is very small positive parameter. Such a GUP leads to a nonzero minimal length given by

$$
(\Delta X)_{\text{min}} = \hbar \sqrt{\beta}.
$$

(8)

From (7), we can see that for a particle localized with precision $\Delta X$, we have

$$
\Delta P \geq \frac{\Delta X}{\beta \hbar} \left( 1 - \sqrt{1 - \frac{\beta \hbar^2}{(\Delta X)^2}} \right),
$$

(9)

Therefore, if $\Delta X$ is smaller than the Compton wave length of the particle, i.e. $\Delta X \leq \frac{\hbar}{mc}$, the uncertainty on momentum becomes

$$
\Delta P \geq (\Delta P)_0 = \frac{1}{\beta mc} \left( 1 - \sqrt{1 - \beta m^2 c^2} \right).
$$

As in ordinary case (without minimal length), we can guess the probability of particle creation to be

$$
W = \exp \left( -2\pi \frac{l \times (\Delta P)_0}{\hbar} \right) = \exp \left[ -2\pi \frac{c}{eE\beta \hbar} \left( 1 - \sqrt{1 - \beta m^2 c^2} \right) \right]
$$

(10)

This result, although heuristic handwaving, is exactly the same as that obtained by semi-classical method [40]. However, by taking only the correction to the first order

$$
W = \exp \left[ -2\pi \frac{c}{eE\beta \hbar} \left( 1 - \sqrt{1 - \beta m^2 c^2} \right) \right] \simeq \exp \left[ -\pi \frac{m^2 c^3}{eE \hbar} \left( 1 + \frac{\beta}{4} m^2 c^2 + ... \right) \right].
$$

(11)

we can see that this result differs from that of the reference [39], which is believed to have been obtained by an exact approach,

$$
W = \exp \left[ -\pi \frac{m^2 c^3}{eE \hbar} \left( 1 + \frac{1}{4} \beta m^2 c^2 \left( 1 - \frac{e^2 E^2 \hbar^2}{m^4 c^6} \right) \right) \right].
$$

(12)

In light of the above review which emphasizes clearly the discrepancy between the mentioned results, the following remarks are in order. The first one is that the authors of [39] expressed the Klein Gordon equation in the momentum space representation,

$$
\left[ (\omega + eE \hat{X})^2 - \hat{P}^2 - m^2 \right] \varphi = 0,
$$

(13)

where the action of the operators $\hat{X}$ and $\hat{P}$ is defined by

$$
\hat{P} = p,
$$

(14)

$$
\hat{X} = i\hbar \left[ (1 + \beta p^2) \frac{\partial}{\partial p} \right].
$$

(15)
Besides this momentum representation, we find for the operators $\hat{X}$ and $\hat{P}$ many other representations. However, it is not yet known if these representations are equivalent and how they are related to one another.

Secondly, we notice that in both [39] and [40], the authors consider a space dependent gauge. Actually, it is not clear whether the use of time dependent gauge will give the same results. This question should be quite investigated because any result for the pair creation rate does not make sense as long as it does not satisfy the gauge invariance principle.

In response to these questions it is of great interest to reconsider the problem by using a different approach. In this context, one can consider a position representation for the $\hat{P}$ operator and a time dependent gauge for the electric field. One can look also for a method beyond the Bogoliubov transformation and the WKB approximation. This is the main goal of rest of the paper.

In what follows, we shall use the system of units with $c = \hbar = 1$ but we shall show explicitly $G$. In addition, for the sake of simplicity, we shall consider the Schwinger effect in (1+1) dimensional space-time. It is well-known that, apart from the unimportant prefactor, the pair production rate does not depend on the space-time dimension.

**III. THE KLEIN GORDON EQUATION WITH MINIMAL LENGTH**

In the ordinary quantum mechanics, the momentum and the wave vector are related to one another by the linear relation $p = k$ (note that $\hbar$ is taken to be 1). This means that the physical moment is the same as the canonical moment. In order to reproduce the GUP in (7), we consider a nonlinear relation of the form $p = f(k)$, where $f$ is an injective function so that $f^{-1}(p) = k$ is well defined [1]. This requires a redefinition of physical operators in terms of canonical operators. The physical operators $\hat{X}$ and $\hat{P}$ are now defined by

\begin{align}
\hat{X} &= \hat{x}, \\
\hat{P} &= f (\hat{p}),
\end{align}

where $\hat{x}$ and $\hat{p}$ are the canonical operators obeying the usual commutation relation $[\hat{x}, \hat{p}] = i$.

Of course, there are infinitely many functions $f$ that could give rise to a minimal length. The simpler case is to consider the expansion

\[ f (\hat{p}) = \hat{p} \left( 1 + \frac{\beta}{3} \hat{p}^2 + ... \right). \]
Taking into account that \([\hat{x}, f(\hat{p})] = if'(\hat{p})\) and \(\hat{p} \approx \hat{P}(1 - \frac{2}{3}\hat{P}^2 + \ldots)\), we obtain the modified commutation relation

\[
[\hat{X}, \hat{P}] = i \left( 1 + \beta \hat{P}^2 \right),
\tag{19}
\]
that leads to equation (7).

From equations (16), (17) and (18), we can define for the operators \(\hat{X}\) and \(\hat{P}\) the following position space representation

\[
\hat{X} = x, \tag{20}
\]
\[
\hat{P} = \left( 1 + \frac{1}{3} \beta p^2 \right) \hat{p}, \tag{21}
\]
with

\[
\hat{p} = -i \frac{\partial}{\partial x}, \tag{22}
\]

In the position space representation, the free Klein Gordon equation to the first order in \(\beta\) is given by

\[
\left( -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - \frac{2\beta}{3} \frac{\partial^4}{\partial x^4} - m^2 \right) \psi(t, x) = 0, \tag{23}
\]

With this formulation that includes the gravitational minimal length, it is straightforward to show that the obtained Klein Gordon equation is consistent with the gauge invariance. In effect, assuming that a charged scalar particle couples to the electromagnetic field following the minimal coupling principle which consists on making the change

\[
i \frac{\partial}{\partial t} \rightarrow i \frac{\partial}{\partial t} - eA_0
\]
\[
i \frac{\partial}{\partial x} \rightarrow i \frac{\partial}{\partial x} - eA_1, \tag{25}
\]
we obtain, all to the first order in \(\beta\), the following Klein Gordon equation

\[
\left[ \left( i \frac{\partial}{\partial t} - eA_0 \right)^2 - \left( i \frac{\partial}{\partial x} - eA_1 \right)^2 - \frac{2\beta}{3} \left( i \frac{\partial}{\partial x} - eA_1 \right)^4 - m^2 \right] \psi(t, x) = 0. \tag{26}
\]

This equation is invariant under the gauge field transformation

\[
A'_\mu(t, x) = A_\mu(t, x) + \partial_\mu \chi, \tag{27}
\]
with

\[
\psi'(t, x) = e^{ie\alpha} \psi(t, x)
\]
Let us, now, consider a scalar particle of mass \( m \) and charge \( e \) subjected to a constant electric field \( E \). In the presence of a minimal length the use of the time-dependent gauge, with the assumption that \( \psi (t, x) = e^{-ipx} \varphi (t) \), leads to the following differential equation

\[
\left[ \frac{\partial^2}{\partial t^2} + (p + eEt)^2 + \frac{2\beta}{3} (p + eEt)^4 + m^2 \right] \varphi (t) = 0. \tag{28}
\]

To our knowledge, because of the quartic term \((p + eEt)^4\), there is no exact solution for this differential equation. However, as is shown in [56], one can find approximate solutions (to the first order on \( \beta \)) by making the change \( t \to \xi \) with

\[
\xi = \frac{1}{2} - \sqrt{\frac{\beta}{12}} (eEt + p). \tag{29}
\]

The resulting equation can be put in the form of hypergeometric equation

\[
\left[ \xi (1 - \xi) \frac{\partial^2}{\partial \xi^2} + (C - (A + B + 1) \xi) \frac{\partial}{\partial \xi} - AB \right] F (\xi) = 0. \tag{30}
\]

where the constants \( A, B \) and \( C \) are given by

\[
A = \frac{1}{2} + i \frac{1}{\alpha} \sqrt{1 - \lambda \alpha - \alpha^2} + \frac{i}{\alpha} \sqrt{1 - 2\lambda \alpha - \frac{1}{4} \alpha^2} \tag{31}
\]

\[
B = \frac{1}{2} + i \frac{1}{\alpha} \sqrt{1 - \lambda \alpha - \alpha^2} - \frac{i}{\alpha} \sqrt{1 - 2\lambda \alpha - \frac{1}{4} \alpha^2} \tag{32}
\]

\[
C = 1 + i \frac{1}{\alpha} \sqrt{1 - \lambda \alpha - \alpha^2} \tag{33}
\]

with

\[
\alpha = \frac{\beta}{3} eE, \tag{34}
\]

\[
\lambda = \frac{m^2}{eE}. \tag{35}
\]

The solution of the original equation is related to the new solution by

\[
\varphi (t) = \xi^a (1 - \xi)^b F (\xi) \tag{36}
\]

where \( a \) and \( b \) are given by

\[
a = b = \frac{1}{2} + i \frac{1}{2 \alpha} \sqrt{1 - \lambda \alpha - \alpha^2}. \tag{37}
\]

The two independent solutions for the hypergeometric equation (30) are given by [57]

\[
F_1 (\xi) = F (A, B; C; \xi) \tag{38}
\]

\[
F_2 (\xi) = \xi^{1-C} F (A - C + 1, B - C + 1; 2 - C; \xi) \tag{39}
\]
Then, the outgoing particle state is given by

$$\varphi_{\text{out}}^+ (t) = N \xi^{\alpha_1} (1 - \xi)^{\alpha_1} F (2 \alpha_1 + \alpha_2, 2 \alpha_1 - \alpha_2 - 1, 2 \alpha_1; \xi)$$  \hspace{1cm} (40)

with the normalization constant

$$N = 2 \left( \frac{\alpha}{eE \left( 1 - \lambda \alpha - \alpha^2 \right)} \right)^{\frac{1}{2}}. \hspace{1cm} (41)$$

Having shown how to investigate the Klein Gordon equation in the presence of a minimal length, we are now able to study the creation of particle pairs. Because of the nonperturbative nature of this effect, several practical techniques have been developed in order to calculate the pair production rate. Among these methods we cite the Schwinger effective action method \cite{58–61} and related instantons calculus \cite{62–65}, the Hamiltonian diagonalization technique \cite{66, 67}, the "in" and "out" states formalism \cite{68–71}, the quantum kinetic approach \cite{72–74} as well as the method based on the projection of Green's function on the particle states \cite{75–80} and the semiclassical WKB approximation \cite{81–83}.

In this work, we consider the Green’s function method. This method has been proved to be most fruitful in finding the probability to create a pair of particles either in external electromagnetic fields or in cosmological backgrounds.

We start with the calculation of the corresponding Green’s function using Feynman path integrals.

**IV. PATH INTEGRAL DERIVATION OF THE GREEN’S FUNCTION**

In the presence of a minimal length and the propagator of a relativistic particle subjected to an electric field described by the gauge $A^\mu = (A^0, A_x) = (0, -eEt)$ is the causal Green’s function solution of the equation

$$\hat{O}_{KG} G (x_f, x_i, t_f, t_i) = \delta (x_f - x_i) \delta (t_f - t_i)$$  \hspace{1cm} (42)

where $\hat{O}_{KG}$ is the Klein Gordon operator

$$\hat{O}_{KG} = (\hat{p}^0)^2 - (\hat{p} + eEt)^2 - \frac{2}{3} \beta (\hat{p} + eEt)^4 - m^2,$$  \hspace{1cm} (43)

where $\hat{p}^0 = i \frac{\partial}{\partial t}$ and $\hat{p} = -i \frac{\partial}{\partial x}$. It is well-known that $G (x_f, x_i, t_f, t_i)$ can be represented as a matrix element of an operator $F^{-1}$

$$G (x_f, x_i, t_f, t_i) = \langle x_f, t_f | F^{-1} | x_i, t_i \rangle$$  \hspace{1cm} (44)
where the operator $F$ is given by

$$F = - (\hat{p}^0)^2 + (\hat{p} + eE\hat{t})^2 + \frac{2}{3} \beta (\hat{p} + eE\hat{t})^4 + m^2$$

(45)

and $|x, t\rangle$ are eigenvectors for the operators $\hat{t}$ and $\hat{x}$;

$$\hat{x} |x, t\rangle = x |x, t\rangle \quad ; \quad \hat{t} |x, t\rangle = t |x, t\rangle$$

For the corresponding canonical-conjugated operators of momenta $\hat{p}^0$ and $\hat{p}$, we have

$$\hat{p}^0 |p, p^0\rangle = p^0 |p, p^0\rangle \quad ; \quad \hat{p} |p, p^0\rangle = p |p, p^0\rangle$$

Since these operators satisfy the usual commutation relations $[\hat{x}, \hat{p}] = i$ and $[\hat{t}, \hat{p}^0] = i$, we have for the eigenvectors the usual normalization and completeness relations

$$\int dx dt \langle x, t | x, t \rangle = 1 \quad ; \quad \langle x, t | x', t' \rangle = \delta (x - x') \delta (t - t')$$

and

$$\int dp^0 dp |p, p^0\rangle \langle p, p^0 | = 1 \quad ; \quad \langle p, p^0 | p', p^0 \rangle = \delta (p^0 - p^0) \delta (p - p') .$$

We define also a plane wave with the scalar product

$$\langle p, p^0 | x, t \rangle = \frac{1}{2\pi} e^{-i(p^0 t - px)}.$$ 

(46)

By the use of the integral representation

$$F^{-1} = i \int_0^\infty dT \exp \left[ -i (F - i\epsilon) T \right]$$ 

(47)

where $T$ is the Schwinger proper-time and $i\epsilon$ is an infinitesimal imaginary number that can be included to $m^2$ and has to be put to zero at the end of calculations, we obtain the proper time representation of the Green function $G(x_f, x_i, t_f, t_i)$

$$G(x_f, x_i, t_f, t_i) = i \int_0^\infty dT \ K(x_f, x_i; t_f, t_i; T) ,$$

(48)

where the kernel $K(x_f, x_i; t_f, t_i; T)$ is given by

$$K(x_f, x_i; t_f, t_i; T) = \langle x_f, t_f | \exp (-iFT) | x_i, t_i \rangle .$$

(49)

Here $K(x_f, x_i; t_f, t_i; T)$ is similar to the quantum-mechanical amplitude for the transition between an initial state $|x_i, t_i\rangle$ at the proper time 0 and the final state $|x_f, t_f\rangle$ at proper time.
To present $K(x_f, x_i; t_f, t_i; T)$ by means of path integrals we first write $\exp(-iFT) = [\exp(-iF\varepsilon)]^{N+1}$, with $\varepsilon = 1/(N + 1)$, and we insert $N$ identities $\int dx dt |x,t\rangle \langle x,t| = 1$ between all the operators $\exp(-iF\varepsilon)$. We obtain

$$K(x_f, x_i; t_f, t_i; T) = \int dx_1 dx_2 \ldots dx_N \int dt_1 dt_2 \ldots dt_N \times \prod_{n=1}^{N+1} \langle x_n, t_n| \exp(-iF\varepsilon)|x_{n-1}, t_{n-1}\rangle.$$  \hfill (50)

Now, we have to express the matrix elements $\langle x_n, t_n| \exp(-iF\varepsilon)|x_{n-1}, t_{n-1}\rangle$ through path integral. As $\varepsilon$ is small, we can write

$$\langle x_n, t_n| \exp(-iF\varepsilon)|x_{n-1}, t_{n-1}\rangle \approx \langle x_n, t_n| (1 - i\varepsilon F)|x_{n-1}, t_{n-1}\rangle.$$  \hfill (51)

Then, we insert in (51) the integral identity $\int dp_0 dp_n |p_n, p_0\rangle \langle p_n, p_0| = 1$. By taking into account that $F$ has no product of the noncommuting operators $\hat{p}^0$ and $\hat{t}$, and by using the plane wave definition (46) the matrix element (51) can be expressed in the middle point $\bar{t}_n = (t_n + t_{n-1})/2$ as follows

$$\langle x_n, t_n| \exp(-iF\varepsilon)|x_{n-1}, t_{n-1}\rangle = \int \frac{dp^0 dp_n}{(2\pi)^2} \exp \left\{ i \left[ p_n^0 (\Delta t_n) - p_n (\Delta x_n) - \varepsilon \left( (p_n^0)^2 - (p + eE\bar{t}_n)^2 - \frac{2}{3} \beta (p_n + eE\bar{t}_n)^4 - m^2 \right) \right] \right\}. \hfill (52)$$

where $\Delta q_n = q_n - q_{n-1}$ for $q \equiv x, t$. We have then

$$K(x_f, x_i; t_f, t_i; T) = \int \prod_{n=1}^{N} dx_n \prod_{n=1}^{N} dt_n \prod_{n=1}^{N+1} dp_n \prod_{n=1}^{N+1} \frac{d(p_n)_n}{2\pi} \times \prod_{n=1}^{N+1} \exp \left\{ i \left[ p_n^0 (\Delta t_n) - p_n (\Delta x_n) - \varepsilon (p_n^0)^2 + \varepsilon \left( (p_n + eE\bar{t}_n)^2 + \frac{2}{3} \beta (p_n + eE\bar{t}_n)^4 + m^2 \right) \right] \right\}. \hfill (53)$$

By doing integrations over $p_n^0, p_n$ and $x_n$, we obtain

$$K(x_f, x_i; t_f, t_i; T) = \int \frac{dp}{2\pi} e^{-ip(x_f-x_i)} K_p(t_f, t_i; T)$$  \hfill (54)
where the kernel $K_p(t_f, t_i; T)$ is given by

$$K_p(t_f, t_i; T) = \int \prod_{n=1}^N dt_n \prod_{n=1}^{N+1} \sqrt{\frac{1}{4i\pi\varepsilon}} \exp \left\{ i\varepsilon \sum_{n=1}^{N+1} \left[ \frac{(\Delta t_n)^2}{4\varepsilon^2} + (p + eE\tilde{t}_n)^2 + \frac{2}{3} \beta (p + eE\tilde{t}_n)^4 + m^2 \right] \right\}. \tag{55}$$

Then the Green’s function $G(x_f, x_i, t_f, t_i)$ can be written as

$$G(x_f, x_i, t_f, t_i) = \int \frac{dp}{2\pi} e^{-ip(x_f-x_i)} G_p(t_f, t_i) \tag{56}$$

with

$$G_p(t_f, t_i) = i \int_0^\infty dT \ K_p(t_f, t_i; T). \tag{57}$$

The aim now is to calculate $G_p(t_f, t_i)$, for this purpose we write $K_p(t_f, t_i; T)$ in its standard form

$$K_p(t_f, t_i; T) = \int Dt \exp \left[ i \int_0^T d\tau \left[ \frac{i^2}{4} + (p + eE\tau)^2 + \frac{2}{3} \beta (p + eE\tau)^4 + m^2 \right] \right] \tag{58}$$

with the measure

$$Dt = \prod_{n=1}^N dt_n \prod_{n=1}^{N+1} \sqrt{\frac{1}{4i\pi\varepsilon}} \tag{59}$$

It is well-known that, due to the quartic term $(p + eE\tau)^4$, the path integral in (58) has no exact solution. The use of approximations is therefore absolutely necessary. This is possible as long as $\beta$ is a small parameter. Let us first introduce the transformation

$$d\tilde{\tau} = \frac{d\tau}{\rho^2(t)} \tag{60}$$

with

$$\rho(t) = 1 - \frac{\beta}{3} (p + eE\tau)^2. \tag{61}$$

This transformation eliminates the term with $(p + eE\tau)^4$ and makes hence the problem approximately soluble. To show this we have first to express the slicing parameter $\varepsilon$ in terms of the new parameter $\tilde{\varepsilon}_n$ defined by

$$\tilde{\varepsilon}_n = \tilde{\tau}_n - \tilde{\tau}_{n-1} = \int_{\tilde{\tau}_{n-1}}^{\tilde{\tau}_n} \frac{d\tau}{\rho^2(t)}. \tag{62}$$
Developing the last integral as
\[
\tilde{\varepsilon}_n = \frac{\Delta \tau}{\rho^2(t_{n-1})} \left[ 1 - \frac{\dot{\rho}(t_{n-1})}{\rho(t_{n-1})} (\Delta \tau) \right]
\] (63)
and writing
\[
1 - \frac{\dot{\rho}(t_{n-1})}{\rho(t_{n-1})} (\Delta \tau) \simeq \exp \left( -\frac{\dot{\rho}(t_{n-1})}{\rho(t_{n-1})} (\Delta \tau) \right) = \exp \left( - \int_{t_{n-1}}^{t_n} \frac{\dot{\rho}(t)}{\rho(t)} \, dt \right),
\]
we obtain
\[
\tilde{\varepsilon}_n = \frac{\Delta \tau}{\rho^2(t_{n-1})} \exp \left( - \ln \frac{\rho(t_n)}{\rho(t_{n-1})} \right) = \frac{\varepsilon}{\rho(t_n) \rho(t_{n-1})}.
\] (64)

Next, by taking into account that
\[
\prod_{n=1}^{N+1} \frac{1}{\rho(t_n) \rho(t_{n-1})} = \prod_{n=1}^{N} \frac{1}{\sqrt{\rho(t_f) \rho(t_i)}} \frac{1}{\rho(t_n)}
\] (65)
we can rearrange the kernel $K_p(t_f, t_i; T)$ as follows
\[
K_p(t_f, t_i; T) = \sqrt{\frac{1}{\rho(t_f) \rho(t_i)}} \int_0^\infty \frac{dt_n}{\rho(t_n)} \prod_{n=1}^{N+1} \frac{1}{4i\pi \tilde{\varepsilon}_n} \exp \left\{ i \sum_{n=1}^{N+1} \left[ \frac{(\Delta t_n)^2}{4\tilde{\varepsilon}_n \rho(t_n) \rho(t_{n-1})} + \tilde{\varepsilon}_n \left( 1 - \frac{2}{3} \beta m^2 \right) \left( p + eEt_n \right)^2 + m^2 \right] \right\}
\] (66)

The transformation from $\tau$ to $\tilde{\tau}$, implies a change on the proper time from $T$ to $S$ with
\[
T = \int_{S_i}^{S_f} \rho^2(t) \, d\tau
\] (67)
and $S = S_f - S_i$. Therefore, in order to incorporate the novel proper time we use the identity
\[
\rho(t_f) \rho(t_i) \int_0^\infty dS \delta \left( T - \int_0^S \rho^2(t) \, d\tau \right) = 1.
\] (68)

As a result, we obtain for the Green’s function $G_p(t_f, t_i)$
\[
G_p(t_f, t_i) = i \sqrt{\rho(t_f) \rho(t_i)} \int_0^\infty dS \prod_{n=1}^{N} \rho(t_n) \prod_{n=1}^{N+1} \frac{1}{4i\pi \tilde{\varepsilon}_n} \exp \left\{ i \sum_{n=1}^{N+1} \left[ \frac{(\Delta t_n)^2}{4\tilde{\varepsilon}_n \rho(t_n) \rho(t_{n-1})} + \tilde{\varepsilon}_n \left( 1 - \frac{2}{3} \beta m^2 \right) \left( p + eEt_n \right)^2 + m^2 \right] \right\}
\] (69)

Here we remark an inconvenient time dependence in both the measure $\frac{dt_n}{\rho(t_n)}$ and the kinetic term $\frac{(\Delta t_n)^2}{4\tilde{\varepsilon}_n \rho(t_n) \rho(t_{n-1})}$. It is then necessary to eliminate this dependence by introducing the transformation $t_n \to u_n$, with
\[
du_n = \frac{dt_n}{\rho(t_n)}
\] (70)
or, equivalently,

\[ t_n + \frac{p}{eE} = \frac{1}{eE\sqrt{\frac{\beta}{3}}} \tanh \left( eE\sqrt{\frac{\beta}{3}} u_n \right). \]  

(71)

In this case we have

\[ \frac{(\Delta t_n)^2}{\rho(t_n) \rho(t_{n-1})} = (\Delta u_n)^2 + \frac{1}{9} e^2 E^2 \beta (\Delta u_n)^4 + \ldots \]  

(72)

and consequently, the Green’s function \( G_p(t_f, t_i) \) will be given by

\[ G_p(t_f, t_i) = i \sqrt{\frac{\rho(t_f) \rho(t_i)}{\rho(t_f) \rho(t_{i-1})}} \int_0^\infty dS \int_0^{N} du_n \prod_{n=1}^{N+1} \sqrt{\frac{1}{4i\pi\tilde{\varepsilon}_n}} \exp \left\{ i \sum_{n=1}^{N+1} \left[ \frac{1}{4\tilde{\varepsilon}_n} \left( (\Delta u_n)^2 + \frac{1}{9} e^2 E^2 \beta (\Delta u_n)^4 \right) + \tilde{\varepsilon}_n \left( \frac{3\beta}{\beta} - 2m^2 \right) \tanh \left( eE\sqrt{\frac{\beta}{3}} \bar{u}_n \right) + m^2 \right] \right\}. \]  

(73)

According to McLaughlin-Schulman procedure [84], the sum \( i \sum_{n=1}^{N+1} \tilde{\varepsilon}_n Q_n \), where \( Q_n \) can be calculated with the help of the property

\[ \int_{-\infty}^{\infty} \exp (-ax^2 - bx^4) \, dx = \int_{-\infty}^{\infty} \exp (-ax^2 - \frac{3b}{4a^2}) \, dx + O \left( \frac{1}{a^3} \right). \]  

(74)

This means that the term with \( (\Delta u_n)^4 \) leads to a quantum correction of the form

\[ \frac{i}{4\tilde{\varepsilon}_n} \frac{e^2 E^2 \beta}{9} \langle (\Delta u_n)^4 \rangle = -i\tilde{\varepsilon}_n \frac{1}{3} e^2 E^2 \beta. \]  

(75)

By incorporating this correction in the path integral (73) and by taking into account that

\[ \tanh^2 \left( eE\sqrt{\frac{\beta}{3}} u_n \right) = \frac{1}{\cosh^2 \left( eE\sqrt{\frac{\beta}{3}} u_n \right)} \]  

(76)

we obtain the path integral

\[ G_p(t_f, t_i) = i \sqrt{\frac{\rho(t_f) \rho(t_i)}{\rho(t_f) \rho(t_{i-1})}} \int_0^\infty dS \int_0^{N} du_n \prod_{n=1}^{N+1} \sqrt{\frac{1}{4i\pi\tilde{\varepsilon}_n}} \exp \left\{ i \sum_{n=1}^{N+1} \left[ \frac{1}{4\tilde{\varepsilon}_n} \left( (\Delta u_n)^2 + \tilde{\varepsilon}_n \left( \frac{3\beta}{\beta} - 2m^2 \right) \cosh^2 \left( eE\sqrt{\frac{\beta}{3}} \bar{u}_n \right) - \frac{3}{\beta} m^2 + \frac{1}{3} e^2 E^2 \right) \right] \right\}. \]  

(77)
which can be put in the well-known form
\[ G_p(t_f, t_i) = i \sqrt{\frac{3}{\beta e^2 E^2}} \sqrt{\rho(t_f) \rho(t_i)} \int_0^{+\infty} d\tilde{T} \int D\xi \exp \left\{ i \int_0^{\tilde{T}} d\sigma \left[ \frac{\xi^2}{4} - m_1^2 + \frac{l(l + 1)}{\cosh^2 \xi} \right] \right\} \]

by making the rescaling
\[ eE \sqrt{\frac{\beta}{3}} u \to \xi \]
\[ d\tilde{T} \to \frac{3}{\beta e^2 E^2} d\sigma \]
\[ \tilde{T} \to \frac{3}{\beta e^2 E^2} S. \]

The parameters \( m_1 \) and \( l \) are given by
\[ m_1 = \frac{i}{\alpha} \sqrt{1 - \lambda \alpha - \alpha^2} = i \mu \]
\[ l = -\frac{1}{2} + \frac{i}{\alpha} \sqrt{1 - 2\lambda \alpha - \frac{1}{4} \alpha^2} = -\frac{1}{2} + iv. \]

The problem is then reduced to a quantum mechanical system with a particular case of the generalized Poschl-Teller potential whose solution is presented by
\[ G_p(t_f, t_i) = i2 \sqrt{\frac{3}{\beta e^2 E^2}} \sqrt{\rho(t_f) \rho(t_i)} \frac{\Gamma(m_1 - l) \Gamma(m_1 + l + 1)}{\Gamma(m_1 + 1) \Gamma(m_1 + 1)} \left( \frac{1}{4} - \frac{\beta}{12} (p + eE t_f)^2 \right)^{\frac{1}{2} + m_1^2} \left( \frac{1}{4} - \frac{\beta}{12} (p + eE t_i)^2 \right)^{\frac{1}{2} + m_1^2} F \left( m_1 - l, m_1 + l + 1, m_1 + 1; \frac{1}{2} + \frac{\sqrt{\beta}}{12} (p + eE t_f) \right) \]
\[ F \left( m_1 - l, m_1 + l + 1, m_1 + 1; \frac{1}{2} - \frac{\sqrt{\beta}}{12} (p + eE t_i) \right). \]

This is an approximate but rigorous path integral derivation of the causal Green’s function associated with the Klein Gordon equation with minimal length in the presence of a constant electric field, described by the time dependent gauge.

V. PAIR CREATION AMPLITUDE

The one-pair creation amplitude \( A(p_f, p_i) \) is obtained by projecting the causal Green’s function on an outgoing particle and anti-particle states. We then have
\[ A(p_f, p_i) = A_0 \int dx f \int dx_i \psi_{p_i, out}^+ (x_f, t_f) \chi_{out}^+ (t_f) \hat{\partial}_{t_f} G (x_f, t_f; x_i, t_i) \hat{\partial}_{t_i} \psi_{p_i, out}^+ (x_i, t_i) \]
\[ \quad + A_0 \int dx f \int dx_i \psi_{p_i, out} (x_f, t_f) \chi_{out} (t_f) \hat{\partial}_{t_f} G (x_f, t_f; x_i, t_i) \hat{\partial}_{t_i} \psi_{p_i, out} (x_i, t_i) \]
where $A_0$ is the amplitude for no particle production and the derivative $\vec{\partial}_t$ is defined by

$$f \vec{\partial}_t g = f \frac{\partial}{\partial t} g - g \frac{\partial}{\partial t} f$$  \hspace{1cm} (83)

The outgoing particle state is given by

$$\psi_{p,\text{out}}^+ (x,t) = \frac{1}{\sqrt{2\pi}} e^{ipx} \varphi_{p,\text{out}}^+ (t).$$  \hspace{1cm} (84)

By doing integration over $x_i$ and $x_f$, we obtain

$$A(p_f,p_i) = \delta (p_f + p_i) S(p_i)$$  \hspace{1cm} (85)

where $S(p)$ is given by

$$S(p) = A_0 \varphi_{p,\text{out}}^{++} (t_f) \vec{\partial}_{t_f} G_p (t_f,t_i) \vec{\partial}_{t_i} \varphi_{p,\text{out}}^{++} (t_i).$$  \hspace{1cm} (86)

The delta function appearing in equation (85) is due to the spatial part of the outgoing states. It shows that the particle and its anti-particle are created with opposite momenta. However, when this delta function is squared, it leads to an inconvenient divergence of the form $\delta(0)$. Usually this divergence can be eliminated by employing the following trick

$$[\delta (p_f + p_i)]^2 = \lim_{L \to \infty} \int_{-L}^{+L} \frac{dx}{2\pi} e^{-ix(p_f + p_i)} \delta (p_f + p_i) = \lim_{L \to \infty} \frac{L}{2\pi} \delta (p_f + p_i)$$  \hspace{1cm} (87)

We have then the total probability to create a pair of particles

$$\int |A(p_f,p_i)|^2 dp_i dp_f = \int dp_i \frac{L}{2\pi} |S(p_i)|^2 = \int \frac{dxdp}{2\pi} |S(p)|^2$$  \hspace{1cm} (88)

Since $\frac{dxdp}{2\pi}$ is the number of states in the volume element $dxdp$ of the phase space, the quantity $|S(p)|^2$ is therefore interpreted as the probability to create one pair in the state $p$.

To calculate $S(p)$, we take into account that the particle and its antiparticle are created simultaneously at a given time $t$. Then we must take the limit $t_f \to t_i \to t$ after doing derivations. In addition, outgoing particles are well-defined only at very late times after any transient behavior has disappeared. Being aware of this condition and taking into account that $\beta$ is so small, we take $t$ so that

$$p + eEt \sim \frac{\sqrt{3}}{\beta}.$$  \hspace{1cm} (89)

Besides the fact that this limit facilitates the calculation, it is consistent with the general case of particle creation by a time dependent source, in which particle states are determined
at very late time where the source vanishes. In effect, if we write the Klein Gordon equation in the form

\[ \left( \frac{\partial^2}{\partial t^2} + \omega^2 \right) \chi(t) = 0, \tag{90} \]

with

\[ \omega^2 = (p + eEt)^2 + \frac{2\beta}{3} (p + eEt)^4 + m^2, \tag{91} \]

we can see that, in the limit \( p + eEt \sim \sqrt{\frac{3}{\beta}} \),

\[ \frac{\dot{\omega}}{\omega^2} \sim eE\beta \left( 1 - \frac{1}{6}\beta m^2 \right), \tag{92} \]

which implies that the adiabatic condition is verified and consequently the particles are well-defined.

In this limit the state \( \varphi_{p,\text{out}}^+(t) \) behaves like

\[ \varphi_{p,\text{out}}^+(t) = \frac{N}{2} e^{-\frac{m_1}{2} \ln 2} \left( 1 - \sqrt{\frac{\beta}{3}} (p + eEt) \right)^{\frac{1}{2} + \frac{m_1}{2}}, \tag{93} \]

and the Green’s function \( G_p(t_f, t_i) \) takes the form

\[ G_p(t_f, t_i) = ie^{-\frac{m_1}{2} \ln 2} \sqrt{\frac{3}{\beta e^2E^2}} \frac{\Gamma(m_1-l) \Gamma(m_1+l+1)}{\Gamma(m_1+1) \Gamma(m_1+1)} F(m_1-l, m_1+l+1, m_1+1; 1) \]

\[ \left( 1 - \sqrt{\frac{\beta}{3}} (p + eEt_f) \right)^{\frac{1}{2} + \frac{m_1}{2}} \times \left( 1 - \sqrt{\frac{\beta}{3}} (p + eEt_i) \right)^{\frac{1}{2} + \frac{m_1}{2}}. \tag{94} \]

Then the pair creation amplitude \( S(p) \) can be easily calculated. The result is

\[ S(p) = A_0 m_1 \frac{\Gamma(m_1-l) \Gamma(m_1+l+1)}{\Gamma(m_1+1) \Gamma(m_1+1)} F(m_1-l, m_1+l+1, m_1+1; 1). \]

By taking into account that

\[ F(A, B; C; 1) = \frac{\Gamma(C) \Gamma(C - A - B)}{\Gamma(C - A) \Gamma(C - B)} \]

and

\[ \Gamma(m_1 + 1) = m_1 \Gamma(m_1), \]

we obtain

\[ S(p) = A_0 \frac{\Gamma(-m_1) \Gamma(m_1-l) \Gamma(m_1+l+1)}{\Gamma(m_1) \Gamma(l+1) \Gamma(-l)}. \]
and consequently,
\[ |S(p)|^2 = |A_0|^2 \left| \frac{\Gamma \left( \frac{1}{2} + i\mu - i\nu \right) \Gamma \left( \frac{1}{2} + i\mu + i\nu \right)}{\Gamma \left( \frac{1}{2} + i\nu \right) \Gamma \left( \frac{1}{2} - i\nu \right)} \right|^2 \]
where \( \mu \) and \( \nu \) are defined in (79) and (80). By taking into account that
\[ \left| \Gamma \left( \frac{1}{2} + ix \right) \right|^2 = \frac{\pi}{\cosh \pi x} \]  
(95)
we obtain
\[ |S(p)|^2 = |A_0|^2 \frac{\cosh^2 \pi \nu}{\cosh \pi (\mu + \nu) \cosh \pi (\mu - \nu)}. \]
which can be written as
\[ |S(p)|^2 = |A_0|^2 \frac{1}{e^{\pi \Lambda} + 1} \]
where
\[ e^{\pi \Lambda} = \frac{\sinh^2 (\pi \mu)}{\cosh^2 (\pi \nu)}. \]

We now have to determine the constant \( A_0 \). To this aim, we define the probability to create \( n \) pairs of particles in the state \( p \) by
\[ P_n = |A_0|^2 \mathcal{P}^n \]
where \( \mathcal{P} \) is defined by
\[ \mathcal{P} = \frac{1}{e^{\pi \Lambda} + 1}. \]
We then have
\[ \sum_{n=0}^{\infty} P_n = \sum_{n=0}^{\infty} \mathcal{P}^n |A_0|^2 = \frac{|A_0|^2}{1 - \mathcal{P}} = 1 \]
which implies that \( |A_0|^2 \) is given by
\[ |A_0|^2 = \frac{1}{1 + e^{-\pi \Lambda}} \]
(96)

Another important result is that the average number of particles created in the state \( p \) is
\[ \bar{n} = \sum_{n=0}^{\infty} nP_n \]
(97)
\[ = |A_0|^2 \sum_{n=1}^{\infty} n\mathcal{P}^n \]
(98)
\[ = |A_0|^2 \frac{\mathcal{P}}{(1 - \mathcal{P})^2}, \]
(99)
It follows that
\[ \pi = e^{-\pi \Lambda} = \frac{\cosh^2 (\pi \nu)}{\sinh^2 (\pi \mu)}, \] (100)

Let us, now, calculate the total number of created particles by doing summation over all states. The total number of created particles is given by
\[ N = \int \frac{dx dp}{2\pi} e^{-\pi \Lambda}. \] (101)

Here we note that the minimal length as introduced in (19) has no influence on the number of states \( \frac{dx dp}{2\pi} \) because the time and energy operators in the Klein Gordon equation satisfy the ordinary canonical commutation relation.

Equation (101) can in turn be related to the rate of the Schwinger pair production. Just replace the integration over \( p \) by
\[ \int dp = eE \int dt. \] (102)

The total number of created particles is then given by
\[ N = \int dt dx \frac{e E}{2 \pi} e^{-\pi \Lambda}. \] (103)

On the other hand, if we write \( N \) as
\[ N = \int dN = \int \frac{dN}{dt dx} dt dx, \] (104)

we interpret \( \frac{dN}{dt dx} \equiv \mathcal{N} \) as the number of created particles per unit of time per unit of length.

It follows from equations (103) and (104) that
\[ \mathcal{N} = \frac{dN}{dt dx} = \frac{e E}{2\pi} e^{-\pi \Lambda}. \] (105)

Since \( \beta \) is small, the constants \( \mu \) and \( \nu \) are large and
\[ e^{-\pi \Lambda} = \frac{\cosh^2 (\pi \nu)}{\sinh^2 (\pi \mu)} \simeq \frac{\exp (2\pi \nu)}{\exp (2\pi \mu)} = e^{-2\pi (\mu - \nu)}, \] (106)

what implies that
\[ \Lambda = 2 (\mu - \nu) \] (107)

Thus, to the first order on \( \alpha \), we have
\[ \Lambda = \lambda + \alpha \left( \frac{3}{8} \lambda^2 - \frac{3}{8} \right) = \frac{m^2}{e E} \left[ 1 + \frac{\beta}{4} \left( m^2 - \frac{e^2 E^2}{m^2} \right) \right]. \] (108)
and consequently, the number of created particles per unit of time per unit of length is finally

\[ \mathcal{N} = \frac{eE}{2\pi} \exp \left[ -\pi \frac{m^2}{eE} \left( 1 + \frac{1}{4} \beta m^2 \left( 1 - \frac{e^2 E^2}{m^4} \right) \right) \right]. \]  

(109)

This is exactly equation (96) of [39].

In comparison with [39], we have used in the present work a different gauge for the electric field, a different method to derive the pair production rate and a different representation for the physical operators \( \hat{X} \) and \( \hat{P} \). However, this did not prevent us from obtaining the same results as [39].

VI. HAWKING RADIATION IN THE PRESENCE OF MINIMAL LENGTH

Interestingly, it is possible to make an immediate and clear link with some interesting applications of particle creation in general relativity by the use of simple arguments. The first application in this regard is the Unruh effect, which concerns accelerated particle detectors in vacuum [86]. Although all matter fields are in their vacuum states, the accelerated detector will find a thermal distribution of particles with the Unruh temperature

\[ T_U = \frac{a}{2\pi}, \]  

(110)

where \( a \) is the acceleration of the particle detector. Certainly, in Schwinger effect, we don’t have an accelerating detector. Instead, we have a fixed detector and a particle of charge \( e \) and mass \( m \) subjected to a constant electric field \( E_0 \). In such a case, the particle acceleration \( a = \frac{eE}{m} \) plays the same role as the detector acceleration and the particle creation probability can be written as follows

\[ \mathcal{W} = \exp \left( -\pi \frac{m^2}{eE} \right) = \exp \left( -2\pi \frac{\mu}{a} \right), \]  

(111)

where \( \mu = \frac{m}{2} \) is the reduced mass of the produced pair. Therefore we identify the Unruh temperature \( T_U = \frac{m}{2\pi eE} = \frac{a}{2\pi} \).

The second interesting application concerns the black hole radiation due to the particle creation. In black hole physics the situation is a bit different since a static gravitational field without an event horizon can’t create particles. It was in the seventies when Hawking has, for the first time, shown that a black hole can emit thermal spectrum with the Hawking
temperature

\[ T_H = \frac{1}{8\pi GM}, \]  

(112)

where \( M \) is the black hole mass. We can outline a qualitative picture of the Hawking radiation considering the scenario of creating virtual particle-antiparticle pairs. Then one particle of the pair may happen to be just outside of the black hole horizon while the other particle is inside it. The particle inside the horizon inevitably falls onto the black hole center, while the other particle can escape and may be detected by stationary observers far from the black hole \[87\].

Now, using the Unruh effect we can anticipate the Hawking result from the constant electric field case by considering the gravitational acceleration at the event horizon \( r_H = 2GM \) of the black hole

\[ a = \frac{GM}{r_H^2} = \frac{1}{4GM}. \]  

(113)

This reproduce the Hawking temperature from the Unruh one by replacing the electric field acceleration \( a = \frac{eE}{m} \) in Unruh’s result by \( a = \frac{1}{4GM} \). This correspondence between Schwinger effect and Hawking radiation can be, straightforwardly generalized to the case where a minimal length exists. For our case, by writing

\[ \left( 1 + \frac{1}{4} \beta m^2 \left( 1 - \frac{e^2 E^2}{m^4} \right) \right) = \left( 1 + \frac{1}{4} \beta m^2 \right) \left( 1 - \frac{1}{4} \beta \frac{e^2 E^2}{m^2} \right) \]  

(114)

we can see that

\[ \mathcal{W} \sim \exp \left( -2\pi \frac{\tilde{\mu}}{a} \left( 1 - \frac{\beta}{4} a^2 \right) \right) \]  

(115)

where \( \tilde{\mu} = \frac{m}{2} \left( 1 + \frac{1}{4} \beta m^2 \right) \). This implies a modification on the Unruh temperature due to the minimal length

\[ \frac{1}{T_U^{GUP}} = \frac{2\pi}{a} \left( 1 - \frac{\beta}{4} a^2 \right). \]  

(116)

The correspondence between Unruh and Hawking effects, allows us to find Hawking temperature corrected by the presence of a minimal length

\[ \frac{1}{T_H^{GUP}} = 8\pi GM \left( 1 - \frac{\beta}{64G^2 M^2} \frac{1}{M} \right). \]  

(117)

This is in good agreement with the corrected temperature obtained by considering some scenarios which implies a generalized uncertainty principle and preserve the relativistic dispersion relation (see for example \[88\]). It should be noted that in some studies we remark
that the temperature correction is 4 times greater than ours. This is because the authors of these papers start from the uncertainty relation $\Delta P \Delta X \geq \hbar (1 + \beta (\Delta P)^2 + ...)$ instead of equation (7). The missing $\frac{1}{2}$ factor is the only reason of this difference.

Furthermore, by the use of the first law of the black hole thermodynamics

$$dM = TdS$$

we obtain

$$dS = \frac{dM}{T_H} = 8\pi GM \left( 1 - \frac{\beta}{64G^2} \frac{1}{M^2} \right) dM,$$

which leads to

$$S = 4\pi GM^2 - \pi \frac{\beta}{8G} \ln M + C$$

where $C$ is a real constant and the logarithmic term is the well known correction from quantum gravity to the classical Bekenstein–Hawking entropy, which appears in different studies of GUP modified thermodynamics of black holes. This result is, to the first order in $\beta$, exactly the same as that recently communicated [89]

$$S_A = \frac{\pi \alpha}{16} \left[ \frac{2}{1 - \sqrt{1 - \frac{\alpha M^2_p}{16 M^2}}} \right] + \ln \left( 1 - \sqrt{1 - \frac{\alpha M^2_p}{16 M^2}} \right) - \ln \left( 1 + \sqrt{1 - \frac{\alpha M^2_p}{16 M^2}} \right)$$

which can be developed to the first order in $\alpha$ to be

$$S_A = 4\pi M^2 - \pi \frac{\alpha M^2_p}{16} + ...$$

Here $M^2_p$ is the Planck mass $M_p = l_p^{-1} = \sqrt{\frac{G}{\hbar}}$, and $\beta = \alpha M^2_p$.

Let us note that the corrected entropy for the Schwarzschild black hole can be expressed as

$$S = \frac{A}{4G} - \pi \frac{\beta}{16G} \ln \frac{A}{16\pi},$$

where $A$ is the surface area of the black hole. This implies that the Hawking radiation ceases before the black hole evaporates completely.

We also note that this finding can be generalized to many cases with the spherically symmetric metric

$$ds^2 = G(r) dt^2 - \frac{1}{G(r)} dr^2 + r^2 d\Omega^2$$
where $G(r)$ is some real function that allows the existence of an event Horizon. For this space-time metric we have the standard Hawking temperature

$$T_H = \frac{|G'(r_H)|}{4\pi}$$

where $r_H$ is the event horizon defined by $G(r) = 0$. Then according to the modification of the Unruh temperature we get

$$T_{H}^{GUP} = \frac{|G'(r_H)|}{4\pi} \left( 1 + \frac{\beta}{16} |G'(r_H)|^2 \right).$$

(126)

As example we consider the case of the Rindler metric with $G(r) = 1 - 2ar$. In this case we have $r_H = \frac{1}{2a}$ and $G'(r_H) = -2a$. This gives

$$T_{R}^{GUP} = \frac{a^2}{2\pi} \left( 1 + \frac{\beta}{4a^2} \right).$$

(127)

The second example is the static de-Sitter metric with $G(r) = 1 - H^2 r^2$. For this metric, we have $r_H = \frac{1}{H}$ and $G'(r_H) = -2H^2 r_H = -2H$. We then obtain

$$T_{dS}^{GUP} = \frac{H^2}{2\pi} \left( 1 + \frac{\beta}{4H^2} \right).$$

(128)

VII. EXACTNESS OF THE SEMICLASSICAL COMPUTATION

As is mentioned above, it is remarkable that semi-classical computations, using WKB method [81] produce the exact amplitude for the probability of pair production by constant electric field in ordinary Minkowski space-time. In this section, we examine this WKB approximation in the presence of a minimal length.

A. Time dependent gauge

Let us first consider the time dependent gauge, for which the pair creation probability is, in the WKB approximation, given by [82]

$$\mathcal{W} \approx \exp \left[ -2 \text{Im} \int \sqrt{m^2 + (p + eEt)^2} + \frac{2\beta}{3} (p + eEt)^4 dt \right].$$

(129)

By making the change $t \rightarrow z$, with

$$z = \sqrt{eEt} + \frac{p}{\sqrt{eE}}$$

(130)
we can express \( P \) as
\[
W \approx \exp \left[ -2 \text{Im} \int_{z_0^*}^{z_0} \sqrt{\lambda + z^2 + 2\alpha z^4} \, dz \right].
\] (131)
where \( z_0 \) and \( z_0^* \) are the turning points solutions of the equation \( \lambda + z^2 + 2\alpha z^4 = 0 \).

Then by writing \( \lambda + z^2 + 2\alpha z^4 \) in the form
\[
\lambda + z^2 + 2\alpha z^4 = 2\alpha (z^2 + b^2) (z^2 + a^2)
\] (132)
where
\[
b^2 = -\frac{1}{4\alpha} + \frac{1}{4\alpha} \sqrt{1 - 8\alpha \lambda} = - \left( \lambda + 2\alpha \lambda^2 + 8\alpha^2 \lambda^3 + ... \right)
\] (133)
\[
a^2 = -\frac{1}{4\alpha} - \frac{1}{4\alpha} \sqrt{1 - 8\alpha \lambda} = - \left( \frac{1}{2\alpha} - \lambda - 2\alpha \lambda^2 - 8\alpha^2 \lambda^3 + ... \right)
\] (134)
with
\[
\frac{b}{a} = \sqrt{2\alpha \lambda} \left( 1 + 2\alpha \lambda + 8\alpha^2 \lambda^2 \right).
\] (135)
We can see that we have two pairs of turning points. In addition to the expected two turning points that tend to the usual ones in the limit \( \beta \to 0 \), we have two extraturning points that will be rejected to infinity in the limit \( \beta \to 0 \). In this case we should studied the contribution of each pair of turning points. It is also possible to have interference between these turning points [90]. To this aim we first make the change \( z \to ix \) and next we use of the following integrals
\[
\int_b^a \sqrt{(x^2 - b^2)(x^2 - a^2)} \, dx = \frac{a}{3} \left[ (a^2 + b^2) E \left( \frac{b}{a} \right) - 2b^2 K \left( \frac{b}{a} \right) \right],
\] (136)
and
\[
\int_0^b \sqrt{(x^2 - b^2)(x^2 - a^2)} \, dx = \frac{a}{3} \left[ (a^2 + b^2) E \left( \frac{b}{a} \right) - (a^2 - b^2) K \left( \frac{b}{a} \right) \right],
\] (137)
where \( K (k) \) and \( E (k) \) are, respectively, the Elliptic integrals of first and second kind which have the following expansions for small values of \( k \),
\[
K (k) = \frac{\pi}{2} \left( 1 + \frac{1}{4} k^2 + \frac{9}{64} k^4 + ... \right)
\] (138)
\[
E (k) = \frac{\pi}{2} \left( 1 - \frac{1}{4} k^2 - \frac{3}{64} k^4 + ... \right).
\] (139)
It is therefore easy to show that the contribution of the two ordinary turning points
\[ W_1 \approx \exp \left[ -4\sqrt{2\alpha} \int_0^b \sqrt{(x^2 - b^2)(x^2 - a^2)} \, dx \right] = \exp \left[ -\pi\lambda \left( 1 + \frac{3}{4}\alpha\lambda \right) \right] \tag{140} \]
is very large compared to the contribution of the second pair of turning points
\[ W_2 \approx \exp \left( -\frac{\pi}{\beta eE} \right). \tag{141} \]
Thus, we obtain
\[ W \approx \exp \left[ -\pi\lambda \left( 1 + \frac{3}{4}\alpha\lambda \right) \right] = \exp \left[ -\frac{\pi m^2}{eE} \left( 1 + \frac{1}{4}\beta m^2 \right) \right] \tag{142} \]
This shows that unlike the ordinary case, the semiclassical WKB approximation does not give the exact results. The missing term is very important in the strong field regime.

**B. Space dependent gauge**

In space dependent gauge, the study of particle creation reduces to the study the tunneling effect. In the semiclassical approximation, it consists in calculating the transmission coefficient \( T \) which can be determined starting from the Klein Gordon equation
\[ \left[ (\omega + eEx)^2 + \frac{\partial^2}{\partial x^2} - \frac{2\beta}{3} \frac{\partial^4}{\partial x^4} - m^2 \right] \psi(x) = 0, \tag{143} \]
which is of fourth order. To the best of our knowledge, it does not admit exact solutions. Nevertheless, we can obtain a more convenient equation by introducing an auxiliary wave function \( \varphi \), so that
\[ \psi(x) = \left( 1 + \frac{2\beta}{3} \frac{\partial^2}{\partial x^2} \right) \varphi(x). \tag{144} \]
By substitute (144) into (143) and neglecting terms of higher order on \( \beta \), we obtain
\[ \left[ \left( 1 + \frac{2\beta}{3} ((\omega + eEx)^2 - m^2) \right) \frac{\partial^2}{\partial x^2} + (\omega + eEx)^2 - m^2 \right] \varphi(x) = 0, \tag{145} \]
which is an effective Schrödinger-like equation involving minimal length corrections. In the limit \( \beta \to 0 \), equation (145) reduces to the ordinary Klein Gordon equation. Since this equation is of second order, we can now apply the WKB method. First we write (145) in the form
\[ \left[ \frac{\partial^2}{\partial x^2} + \tilde{p}^2(x) \right] \psi(x) = 0, \tag{146} \]
with
\[ \tilde{p}(x) = \frac{\left(\omega + eEx\right)^2 - m^2}{1 + \frac{2\beta}{3} \left(\left(\omega + eEx\right)^2 - m^2\right)} \]

Using the WKB approximation, the probability of transition from a negative energy state to a positive energy state is given by the transmission coefficient \[ \text{(147)} \]

\[ T = \exp \left(-2\gamma\right) \]

where \( \gamma \) is given by
\[ \gamma = \int_{x_1}^{x_2} |p(x)| \, dx, \quad \text{(148)} \]

with
\[ |\tilde{p}(x)| = \frac{\sqrt{m^2 - (\omega + eEx)^2}}{1 + \frac{\beta}{3} \left(m^2 - (\omega + eEx)^2\right)}. \quad \text{(149)} \]

In \[ \text{(148)} \] \( x_1 \) and \( x_2 \) are the turning points solutions of the equation \( \tilde{p}(x) = 0 \). We have
\[ x_{1,2} = -\frac{\omega}{eE} \pm \frac{m}{eE} \quad \text{(150)} \]

and
\[ \gamma = \lambda \int_{-1}^{1} \sqrt{1 - y^2} \, dy + \alpha \lambda^2 \int_{-1}^{1} (1 - y^2) \sqrt{1 - y^2} \, dy. \quad \text{(151)} \]

\[ = \frac{\pi}{2} \lambda + \frac{3\pi}{8} \alpha \lambda^2 \quad \text{(152)} \]

Therefore, the factor \( T \) takes the form
\[ \mathcal{W} = T = \exp \left(-\pi \lambda \left(1 + \frac{3}{4} \alpha \lambda\right)\right) = \exp \left[-\pi \frac{m^2}{eE} \left(1 + \frac{1}{4} \beta m^2\right)\right]. \quad \text{(153)} \]

Although different gauges are considered in this section, the final results are the same. These results however are not exact. We have thus proven that even at the semi-classical level the Schwinger effect preserves the gauge invariance.

**VIII. CONCLUSION**

In this paper we have studied the Schwinger effect, concerning the creation of scalar particles by an electric field, in the presence of a minimal length. The concept of minimum length is incorporated in the quantum mechanics of the scalar particle through a specific generalization of the uncertainty principle that necessitates a new definition of the momentum.
operator in position representation. Consequently, the corresponding Klein Gordon equation contains an inconvenient quartic term which impedes any attempt to exact solution. However since the deformation parameter is so small, we were able to obtain an approximate expression for the Green’s function by introducing a spacio-temporal transformation. Then by projecting this Green’s function on the outgoing particle and antiparticle states, we have calculated the pair creation probability. Our results on the one hand confirm the results of [40] and on the other hand, suggests the consistency of the Schwinger effect with the gauge invariance principle of electrodynamics.

The close analogy between Schwinger effect in a constant electric field and the Unruh effect for accelerating observers enabled us to find the correction brought by the minimal length on the Unruh temperature. This result, applied to the Hawking radiation, has lead us to deduce the modified Hawking temperature due to minimal length. Then from the first law of black hole thermodynamics, we were able to extract the corrected black hole entropy. It is shown that the first correction is a logarithmic term with a negative numerical factor. This agrees with many anterior studies using different approaches.

As a last stage, we have examined the accuracy of the semiclassical WKB approximation in the calculation of the pair production rate. Our results show that, unlike the ordinary case, the WKB approximation in the presence of a minimal length does not give the exact rate even for the constant electric field.

At the end of this work, let us make the following remarks:

- Semiclassical treatments are widely used in Schwinger effect and black hole radiation. However, as we have seen in this work, it does not give the exact results and one has to ask whether this can indeed be accurate in computations of more complicated scenarios such as pair production by space or time dependant electric fields or pair production in dynamical black holes [93].

- In addition, the position-momentum uncertainty relation considered in this paper is not consistent with the usual Poincaré covariance, and consequently, the corresponding minimal length is frame dependent. To deal with this problem we have two possible paths; the first one is to consider, like in [92], a GUP model that is invariant under the usual Poincaré group. The second is to search for a deformation of the Poincaré group that makes the present minimal length invariant.

29
We leave these issues that demand more detailed investigations to future works. Indeed, the problem of particle creation in dynamical black holes with minimal length is under consideration.

Acknowledgement 1 M Bouali would like to thank L Cheriet for useful conversations. This work is partially supported by Algerian Ministry of High Education and Scientific Research and DGRSDT under the PRFU project: B00L02UN180120200001.

[1] S. Hossenfelder, Living Rev. Relativity 16 (2013) 2.
[2] A. Tawfik and A. Diab, Int. J. Mod. Phys. D 23 (2014) 1430025
[3] C. A. Mead, Phys. Rev. B 135, 849 (1964)
[4] M. Maggiore, Phys. Lett. B 319 (1993) 83.
[5] S. Hossenfelder, Mod. Phys. Lett. A 19 (2004) 2727.
[6] S. Hossenfelder, Phys. Rev. D 70 (2004) 105003.
[7] S. Hossenfelder, Phys. Lett. B 598 (2004) 92.
[8] G. Veneziano, Europhys. Lett. 2 (1986) 199.
[9] D. Amati, M. Ciafaloni, and G. Veneziano, Phys. Lett. B 197 (1987) 81
[10] K. Konishi, G. Paffuti, and P. Provero, Phys. Lett. B 234 (1990) 276.
[11] M. Kato, Phys. Lett. B 245 (1990) 43
[12] L. J. Garay, Int. J. Mod. Phys. A 10 (1995) 145
[13] F. Scardigli, Phys. Lett. B 452 (1999) 39.
[14] F. Scardigli and R. Casadio, Class. Quant. Grav. 20 (2003) 3915.
[15] R. J. Adler and D. I. Santiago, Mod. Phys. Lett. A 14 (1999) 1371.
[16] M. R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. 73 (2001) 977.
[17] S. Minwalla, M. Van Raamsdonk and N. Seiberg, JHEP 02 (2000) 020.
[18] R. J. Szabo, Phys. Rep. 378 (2003) 207.
[19] R. Casadio and F. Scardigli, Phys. Lett. B 807 (2020) 135558
[20] F. Scardigli, R. Casadio, Eur. Phys. J. C 75 (2015) 425
[21] L. N. Chang, D. Minic, N. Okamura and T. Takeuchi, Phys. Rev. D 65 (2002) 125027.
[22] R. Akhoury and Y-P. Yao, Phys. Lett. B 572 (2003) 37.
[23] K. Nouicer, J. Phys. A: Math. Gen. 38 (2005) 10027.
[24] M. M. Stetsko and V. M. Tkachuk, Phys. Rev. A 74 (2006) 012101.
[25] D. Bouaziz and M. Bawin, Phys. Rev. A 78 (2008) 032110.
[26] D. Bouaziz and N. Ferkous, Phys. Rev. A 80 (2010).
[27] M. Asghari, P. Pedram and K. Nozari, Phys. Lett. B 725 (2013) 451
[28] A. Kempf, J. Math. Phys. 35 (1994) 4483.
[29] A. Kempf, G. Mangano and R. B. Mann, Phys. Rev. D 52 (1995) 1108.
[30] K. Nouicer, J. Phys. A: Math. Gen. 39 (2006) 5125.
[31] U. Harbach and S. Hossenfelder, Phys. Lett. B 632 (2006) 379.
[32] P. Nicolini and M. Rinaldi, Phys. Lett. B 695 (2011) 303
[33] K. Nouicer, Phys. Lett. B 646 (2007), 63-71
[34] R. V. Maluf and J. C. S. Neves, Phys. Rev. D 97, 104015 (2018)
[35] M. Cavaglia, S. Das, Class.Quant.Grav. 21 (2004) 4511-4522
[36] B. Majumder, Phys. Lett. B 703 (2011) 402-405
[37] A. Alonso-Serrano and M. Liska, Phys. Rev. D 104, (2021) 084043
[38] S. Das, P. Majumdar, R. K. Bhaduri, Class.Quant.Grav.19 (2002) 2355-2368,
[39] S. Haouat and K. Nouicer, Phys. Rev. D 89, (2014) 105030
[40] F. Lu, B. Lv, P. Wang, and H. Yang, Nuclear Physics B 937 (2018) pp. 502–532.
[41] B-R. Mu, P. Wang, H.T. Yang, Commun. Theor. Phys. 63 (2015) 715
[42] A. A. Naqash, B. Majumder, S. Mitra, M. M. Bangle, M. Faizal, Eur. Phys. J. C 81 (2021) 870
[43] W. Heisenberg and H. Euler, Z. Phys. 98 (1936) 714.
[44] J. Schwinger, Phys. Rev. 82 (1951) 664
[45] F. Gelis and N. Tanji, Prog. Part. Nucl. Phys. 87 (2016) 1.
[46] S. Hossenfelder, Phys. Rev. D 73 (2006) 105013
[47] T Padmanabhan, Pramana - J. Phys. 37, (1991) 179
[48] Y.C. Ong, Eur. Phys. J. C 80, 777 (2020)
[49] B. Hamil, M. Merad and T. Birkandan, Int. J. Mod. Phys. A 35 (2020) 2050014
[50] B. Hamil and M. Merad, Int. J. Mod. Phys. A 33, 1850177 (2018).
[51] N. Chair and M. Sheikh-Jabbari, Phys. Lett. B 504 (2001) 141
[52] N. Mehdaoui, L. Khodja and S. Haouat, Int. J. Mod. Phys. A 36, (2021) 2150011
[53] R. Ruffini, G. Vereshchagin, S-S. Xue, Phys. Rep. 487 (2010)
[54] S. P. Gavrilov, D. M. Gitman, Phys. Rev. D 53 (1996) 7162
[55] I. J. R. Aitchison, Contemp. Phys. 26 (1985) 333-391
[56] L. Cheriet and S Haouat, Ermakov-Penny equation and particle creation: semi-classical treatment, To be submitted.
[57] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series and Products (Academic Press, New York 1979)
[58] G. Dunne and T. M. Hall, Phys. Lett. B 419, 322 (1998)
[59] G. Dunne, G.V. Dunne, Heisenberg-Euler effective Lagrangians: basics and extensions, in From fields to strings: circumnavigating theoretical physics, M. Shifman et al. eds., World Scientific, Singapore (2005), p. 445.
[60] S. Haouat and L. Chetouani, Phys. Scr. 75, 759 (2007).
[61] S. P. Kim, H. K. Lee, Y. Yoon, Phys. Rev. D 78, 105013 (2008)
[62] G. Dunne and C. Schubert, Phys. Rev. D 72, 105004 (2005)
[63] G. Dunne, Q. Wang, H. Gies, and C. Schubert, Phys. Rev. D 73, 065028 (2006)
[64] S. P. Kim and D. N. Page, Phys. Rev. D 65, 105002 (2002)
[65] S. P. Kim and D. N. Page, Phys. Rev. D 73, 065020 (2006).
[66] A. A. Grib, S. G. Mamayev and V.M. Mostepanenko, Vacuum Quantum Effects in Strong Fields ( Friedmann Lab. Publ., St. Petersburg 1994)
[67] A. A. Grib, S. G. Mamayev and V. M. Mostepanenko, Gen. Rel. Grav. 7 (1976) 535.
[68] S. Haouat and R. Chekireb, Mod. Phys. Lett. A 26, (2011) 2639
[69] S. Haouat and R. Chekireb, Int. J. Theor. Phys. 51 (2012) 1704
[70] S. Haouat and R. Chekireb, Eur. Phys. J. C 72 (2012) 2034
[71] S. Haouat and R. Chekireb, Int J Mod Phys A 30 (2015) 1550081
[72] Y. Kluger, J. M. Eisenberg, B. Svetitsky, F. Cooper and E. Mottola, Phys. Rev. Lett. 67 (1991) 2427.
[73] D. B. Blaschke, A. V. Prozorkevich, G. Roepke, C. D. Roberts, S. M. Schmidt, D. S. Shkirmanov, S. A. Smolyansky, Eur. Phys. J. D 55 (2009) 341
[74] C. K. Dumlu, Phys. Rev. D 79 (2009) 065027
[75] S. W. Hawking and J. B. Hartle, Phys. Rev. D 13 (1976) 2188
[76] D. M. Chitre and J. B. Hartle, Phys. Rev D 16 (1977) 251.
[77] A. O. Barut and I. H. Duru, Phys. Rev. D 41 (1990) 1312.
[78] I. H. Duru and N. Unal, Phys. Rev. D 34 (1986) 959.
[79] I. H. Duru, Gen. Rel. Grav. 26 (1994) 969.
[80] H. Aoyama, M. Kobayashi, Prog. Theor. Phys. 64 (1980) 1045.
[81] I. B. Khriplovich, Phys. Rep. 320 (1999) 37
[82] V.S. Popov, JETP Lett. 13 (1971) 185.
[83] S. Biswas, J. Guha and N. G. Sarkar; Class. Quantum Grav. 12 (1995) 1591
[84] D. McLaughlin and L. S. Schulman, J. Math. Phys. 12 (1971) 2520
[85] C. Grosch and F. Steiner, Handbook of Feynman Path Integrals, Springer Tracts in Modern Physics 145 (Springer, Berlin, Heidelberg 1998)
[86] B.R. Holstein, Am. J. Phys. 67 (1999) 499
[87] V. F. Mukhanov and S. Winitzki, Introduction to Quantum Effects in Gravity. Cambridge University Press, Cambridge (2007)
[88] G. Amelino-Camelia, M. Arzano, Yi Ling and G. Mandanici, Class. Quant. Grav 23 (2006) 2585
[89] X-D Du and C-Y. Long, JCAP 04 (2022) 031
[90] C.K. Dumlu and G.V. Dunne, Phys. Rev. D 84 (2011) 125023
[91] S. Haouat, Phys. Lett. B 729 (2014) 33
[92] C. Quesne and V. M. Tkachuk, J. Phys. A 39 (2006) 10909
[93] J. T. Firouzjaee, G. F. R. Ellis, Eur. Phys. J. C 76 (2016) 620