Striped quantum Hall state in a half-filled Landau level

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Nature of the fractional quantum Hall state at Landau level filling factor $5/2$ remains elusive despite intensive experimental and theoretical work. While the leading theoretical candidates are Moore-Read Pfaffian (Pf) and its particle-hole conjugate anti-Pfaffian (APf), neither received unambiguous experimental support. We show that a state that is intermediate between them, made of alternating stripes of Pf and APf in the bulk, is a viable candidate. Such a state is shown to be incompressible and thus a charge insulator in the bulk, but a heat conductor due to the presence of gapless neutral bulk modes. We argue that properties of such a state is consistent with existing numerical and experimental work, and discuss possible experimental probes of its presence.

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The fractional quantum Hall (FQH) state at Landau level (LL) filling factor $\nu = 5/2$ has been one of the main focuses in experimental and theoretical studies of FQH effect for well over a decade. The intense interest in this state lies in its likely non-Abelian nature and potential application for topological quantum computation \[2\]. Leading theoretical candidates include Moore-Read Pfaffian (Pf) state \[3\], which is the very first example of a non-Abelian state of matter, and its closely related particle-hole conjugate, anti-Pfaffian (APf) state \[4, 5\]. Determining the nature of the 5/2 FQH state experimentally would not only resolve a long-standing issue in the field of quantum Hall effect, but also be of tremendous interest to the physics community in general, in particular if the outcome is an unambiguous demonstration of a non-Abelian topological state of matter for the first time.

The Pf and APf states have identical bulk excitation spectra, and cannot be distinguished by bulk thermoelectric and thermodynamical probes \[6–9\] (result of an initial attempt appears to be consistent with both \[10\]). Qualitative differences are in their edge properties, which in principle allow for distinction between them as well as other candidate states through electron and quasiparticle tunneling. Recent experiments \[11–13\] appear to be more consistent with the Pf state, but cannot rule out the Pf state due to possible edge reconstruction \[14\] or the 331 and other Abelian states \[13, 15\]. Numerically, Pf and APf states are exactly degenerate in energy in the absence of LL mixing due to particle-hole symmetry for a half-filled LL. LL mixing breaks particle-hole symmetry and is expected to lift this degeneracy, but it remains controversial which state would be favored energetically at this point \[14, 21\].

The purpose of the present work is to point out that a new state made of alternating stripes of Pf and APf states in the bulk, is a possible candidate of the FQH state at 5/2 [see Fig. 1(a) for an illustration]. We show that such a striped state, unlike those realized in high LLs \[22–26\], is incompressible and thus a FQH state. On the other hand it supports gapless neutral modes in the bulk, and thus has the novel property of being a bulk heat conductor. We argue that its properties are qualitatively consistent with existing experiments and numerical studies, and suggest new experiments to probe its presence.

While the FQH state at 5/2 was discovered experi-
mentally in 1987 \[\text{[1]}\] and the Pf state first written down in 1991 \[\text{[3]}\], the connection between them started to be taken seriously only after numerical work by Morf in 1998 \[\text{[27]}\] and Rezayi and Haldane (RH) in 2000 \[\text{[28]}\]. At the time of these seminal numerical works the APf state was not known. In Morf’s work a shift that specifically favors the Pf state was chosen on the sphere, while in that of RH the Pf state was particle-hole symmetrized by hand on the torus when comparing with numerical ground state obtained by exactly diagonalizing the Coulomb Hamiltonian of a half-filled first-excited LL. Intriguingly, RH also found that the (particle-hole symmetrized) Pf state is actually stable only in a very narrow region in their phase diagram, sandwiched between the (compressible) composite fermion Fermi liquid-like state and a striped state. With the additional insight of the existence of the APf state, we now propose that the striped state is nothing but a mixture made of alternating stripes of Pf and APf states. It thus appears very naturally in the phase diagram very close to the (particle-hole symmetrized) Pf state. More importantly, the incompressible nature (as we will demonstrate below) of this striped FQH state explains the robustness of the experimentally observed WE will demonstrate below) of this striped FQH state. With the additional insight of the existence of the (particle-hole symmetrized) Pf state, we now propose that the striped state is not only supports neutral modes but the charge modes are gapped due to presence of electron tunneling between different modes and strong Coulomb interaction. As a result we arrive at a novel incompressible state that supports gapless neutral modes in the bulk.

The Lagrangian density of the Pf-APf interface includes terms corresponding to bosonic and fermionic modes \[\text{[29]}\]:

\[
L = \frac{1}{4\pi} [\partial_t \phi_l \partial_x \phi_l - 2(\partial_t \phi_p \partial_x \phi_p + \partial_t \phi_a \partial_x \phi_a)] - [i\bar{\psi}_p \partial_t \psi_p + i\bar{\psi}_a \partial_t \psi_a] - H[\phi, \psi],
\]  

where \(\phi_l\) is the (left moving) bosonic edge field of the \(\nu = 1\) background in which the embedded holes forming the Pf state (thus resulting in the APf state), \(\phi_p\) and \(\phi_a\) are the edge bosonic fields of the Pf state for the electrons and holes respectively, and \(\psi_p\) and \(\psi_a\) are the corresponding edge Majorana fermion fields, as illustrated in Fig. \[\text{[1]}\]. \(H\) is the Hamiltonian density.

We now show that in the present case the domain wall between Pf and APf states is incompressible (just like striped states in higher LLs). With the additional insight of the existence of the APf state, we now propose that the striped state is not only supports neutral modes but the charge modes are gapped due to presence of electron tunneling between different modes and strong Coulomb interaction. As a result we arrive at a novel incompressible state that supports gapless neutral modes in the bulk.

Domain wall between Pf and APf states — Pf and APf states are both incompressible. The striped state [as illustrated in Fig. \[\text{[1]}\(a)\] contains domain walls separating Pf and APf states. Such domain walls are expected to support gapless modes, just like edges. One thus might expect there are gapless charge modes in the bulk due to the presence of these domain walls, and the state would be incompressible (just like striped states in higher LLs). We now show that in the present case the domain wall only supports neutral modes but the charge modes are gapped due to presence of electron tunneling between different modes and strong Coulomb interaction. As a result we arrive at a novel incompressible state that supports gapless neutral modes in the bulk.

The charge density along the interface

\[
\rho(x) = \frac{1}{2\pi} \partial_x (\phi_l + \phi_p + \phi_a) = \frac{1}{2\pi} \partial_x \phi_c, \quad (2)
\]

where

\[
\phi_c = \phi_l + \phi_p + \phi_a \quad (3)
\]

is the total charge field. This naturally suggests the following combination of \(\phi_p\) and \(\phi_a\):

\[
\phi_{r,n} = \phi_p \pm \phi_a, \quad (4)
\]

where \(\phi_r\) is a right-moving charge field and \(\phi_n\) is a right-moving neutral bosonic field. In terms of these combinations the dynamical terms involving bosonic fields in \(L\) [Eq. \[\text{[1]}\]] become

\[
\frac{1}{4\pi} [(\partial_t \phi_l \partial_x \phi_l - \partial_t \phi_p \partial_x \phi_p - \partial_t \phi_a \partial_x \phi_a), \quad (5)
\]

as illustrated in Fig. \[\text{[1]}\(c)\]. Note in particular the two terms combined in the bracket above is the same as those of left and right movers in an ordinary (or non-chiral) Luttinger liquid.

All terms allowed by symmetry show up in \(H\) with varying magnitudes. Of particular importance to the physics we discuss here are the following.

(i) A pair of electrons tunnel from the \(\nu = 1\) edge into the Pf edges of electrons and holes, respectively.

\[
T_p \propto \psi_p \psi_a e^{2i(\phi_l + \phi_p + \phi_a)} + \text{h.c.} = 2\psi_p \psi_a \cos(2\phi_c). \quad (6)
\]

(ii) Strong Coulomb interaction

\[
V_c = v_c (\partial_x \phi_c)^2. \quad (7)
\]

In the absence of screening (say due to nearby metallic gates) \(v_c\) diverges logarithmically in the long-distance
limit, due to the long-range nature of Coulomb interaction. In practice this renders $v_c$ much larger than kinetic energy and other interaction terms involving neutral fields. This significantly reduces the scaling dimension of $\cos(2\phi_c)$, making it

$$\Delta_c \ll 1.$$  \hfill (8)

As a result of this we expect the scaling dimension of $T_p$

$$\Delta T_p = 2\Delta \phi + \Delta c = 1 + \Delta c < 2.$$  \hfill (9)

This means $T_p$ is a relevant perturbation under renormalization group, and develops an expectation value via the usual mechanism as in the sine-Gordon model (or in ordinary Luttinger liquid with back-scattering). This opens up a gap for the charge modes $\phi_1$ and $\phi_r$, leaving us with a single gapless neutral bosonic mode $\phi_n$ and two fermionic modes, as illustrated in Fig. 1(d). Note that they are propagating along the same direction and the total central charge is $c = 1 + 1/2 + 1/2 = 2$.

**Numerical evidence.** Now we revisit the RH work in sphere and torus geometries. The study showed that there is a first-order transition from the symmetrized Pf state to a compressible striped phase. The single Slater determinant that dominates the striped phase has the occupation pattern $00001111000000111100001111$, which can be regarded as alternating unit cells of the Pf root configuration 1100 and its particle-hole conjugation 0011. While RH originally interpreted the striped state as being compressible, in analogy to its higher LL counterparts, we note their numerical results only indicate broken translation symmetry but do not provide information on compressibility or charge gap, thus do not contradict the possibility of an incompressible state here.

In disk geometry, where there is a boundary, the agreement of the ground state to the Pf or APf is not as good as in a boundaryless geometry in terms of wave function overlap. However, the present authors and co-workers showed that a robust Pf state can be identified by the total angular momentum, edge excitations, and, more importantly, the possible induction of both Abelian and non-Abelian quasiholes in a model with realistic confining potential $30$. When the confining potential becomes much weaker, the ground state can be identified as the APf state based on the total angular momentum and the response of the system to the confining potential tuning $31$. The transition from Pf to APf states driven by background potential, which breaks particle-hole symmetry, has also been confirmed in the presence of LL mixing and finite layer thickness $21$.

Interestingly, there is an additional microscopic state separating the Pf and APf phases $21, 31$, as illustrated in Fig. 3. This intermediate state was identified to be a striped phase $31$. More recently, Zhang et al. found that as the edge confining potential is weakened, the Pf state is destabilized by softening of the (neutral) fermionic edge mode $32$. This suggests the difference between the resultant striped and Pf states lie in their low-energy neutral modes. This is clearly consistent with our assessment of the nature of the striped phase. While there is no numerical evidence for it, it is not unreasonable to speculate that the instability of the APf state driven by decreasing $d$ is similarly triggered by softening of neutral mode(s).

**Experimental Consequences.** It has been difficult to understand that while the theoretical modelings with long-range Coulomb interaction often show that the Pf or APf state survives in only a small parameter space $28$, the FQH effect at $\nu = 5/2$ is observed in essentially all high mobility samples. With the understanding that the adjacent striped phase observed numerically is indeed the striped FQH phase with a charge gap proposed here, the quantized plateau is then expected to be observable in a much larger parameter space, which is clearly consistent with experiments.

As discussed earlier, LL mixing induced by Coulomb interaction breaks the particle-hole symmetry, and can thus drive a Pf-APf transition $32, 34$, if they are the only two competing phases. No such transition, however, has been observed as the level of LL mixing is varied by changing the electron density $37, 38$, or varying other parameters like magnetic field $39$. This can again be easily understood in terms of the striped quantum Hall phase proposed here, which would respond to the changing LL mixing and other changes by varying the relative weight of Pf and APf domains, without having a phase transition. The only exception to this is a recent experiment that claimed to have observed a transition with

![FIG. 3: (Color online) Phase diagram of the Pf, striped, and APf. The striped phase is associated with the smallness of the neutral fermionic edge-mode velocity $v_n$, when the energies of the Pf and APf states are comparable. The transition can be tuned by the confining potential strength, controlled by the setback distance $d$ in the numerical model between the two-dimensional electron layer and the background charge layer. Note that on the APf side we illustrate by a positive $v_n$, but the neutral fermionic mode propagates along the opposite direction to its Pf counterpart.](Image 326x604 to 553x740)
unusually large Landau level mixing [40]. In this experiment, a drastic change of the energy gap dependence on the electron density was interpreted as the signature of a transition, although there is no direct method to identify either the Pf state or the APf state. We believe it is likely that this transition is actually between the striped state to either the Pf or APf state, with the latter being stabilized by unusually large LL mixing due to the low electron density.

More importantly, a distinctive novel property of the striped FQH phase is the presence of gapless neutral modes (but not the charge modes) in the bulk, in sharp contrast to all known quantum Hall or compressible states. A direct experimental observable consequence is that this electronic system is a bulk thermal conductor and a bulk charge insulator at the same time. This contrasts the known quantum Hall states that transport heat and charge along the edge but not in the bulk, and compressible states in which the bulk conducts both charge and heat. More specifically, in the limit of weak disorder when we expect the stripes are orderly aligned along a particular direction, we expect an anisotropic state that conducts heat along the direction of the stripes but not in the perpendicular direction, in analogy to the anisotropic conducting state near half-filled higher LLs [22–26]. Observation of such anisotropic heat conduction on a quantum Hall plate can be considered definite evidence of striped quantum Hall state proposed here. Stronger disorder may disorder the stripes, rendering the system an isotropic heat conductor, but still a FQH state. We note existence of bulk neutral modes was found in a very recent experiment [41], in which highly sensitive noise measurement revealed the unexpected heat propagation through incompressible FQH bulk at various filling factors in the lowest LL. The state proposed in this paper is a direct experimental observable consequence is that this electronic system is a bulk thermal conductor and a bulk charge insulator at the same time. This contrasts the known quantum Hall states that transport heat and charge along the edge but not in the bulk, and compressible states in which the bulk conducts both charge and heat.

The striped FQH state has a more complicated edge structure. Depending on how the state terminates, it could either have a Pf edge, APf edge, or more generically, a hybrid between them. The latter would occur when the edge intersects with the domain walls in the bulk, and is illustrated in Fig. 4. In this case there is a single charged mode that propagates along the edge, while the neutral modes of the Pf and APf edges merge into the bulk neutral modes at their intersections. The rich variety of edge structures may well be responsible for the lack of consistency in the results of existing edge tunneling experiments, and can have profound implications in interferometry experiments. These will be explored in future work.

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**Note added.** While the paper was being written up we became aware of a recent preprint [12] on the theory of particle-hole conjugated composite fermion Fermi liquid states, motivated by a recent experiment that hinted at a possible particle-hole symmetry breaking near $\nu = 1/2$ [43]. In Ref. [12] the possibility of a phase in which the charge modes being gapped at an interface between Pf and APf states was pointed out as a corollary of the authors’ detailed analysis of an interface between particle-hole conjugated composite fermion Fermi liquid states, although its physical consequences were not discussed.

**FIG. 4:** (Color online) Illustration of the edge structure of the alternating Pf and APf stripes. There is a single charged mode (solid red line) that propagates along the edge, while the neutral modes (dashed and dotted lines) of the Pf and APf edges merge into the bulk neutral modes at their intersections.

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SUPPLEMENTAL MATERIAL

In this section we review the numerical calculation of the FQH state at $\nu = 5/2$ in the disk geometry with long-range Coulomb interaction and realistic confining potential. We focus on the evolution of the ground state as the confining potential becomes weaker and their compatible theories. Our goal is to show that the numerical results available is consistent with the existence of a striped FQH state at $\nu = 5/2$ discussed in the main text.

Realistic Models

To determine the ground state at $\nu = 5/2$ by numerical calculation is a formidable task, because one needs to be able to resolve the subtle energy differences among various competing states, including the Pf and APf states, in experimentally relevant parameter space. Ideally, in a realistic model, we may want to consider the following factors:

1. Coulomb interaction,
2. neutralizing charge background, which gives rise to the confinement of electron near the edge,
3. LL mixing, which induces an effective three-body interaction for electrons in the same Landau level,
4. electron layer thickness, which softens the electron-electron interaction,
5. electron spin, and
6. disorder.

In the disk geometry, a minimum model that includes factors 1 and 2, is illustrated in Fig. 5(a). In the absence of LL mixing, we only need to consider the valence electrons, or the electrons in the half-filled first-excited LL (1LL). In this minimum model, the Hamiltonian takes the form

$$H_C = \sum_{mnl} V_{mn} c_m^\dagger c_n^\dagger c_l c_m + \sum_m U_m c_m^\dagger c_m,$$  \hspace{1cm} (10)

where $c_m^\dagger$ creates a valence electron with angular momentum $m$. The Coulomb interaction matrix elements $V_{mn}^l$ in the symmetric gauge can be expressed conveniently in terms of pseudopotentials for the 1LL. The confining potential matrix elements $U_m$ can be calculated by integrating the attraction from the background charge at a setback distance $d$ away from the electron layer. In principle, the effect of disorder can be considered by averaging over the random positional distribution of the ionized impurities. Practically, one can assume a uniformly distributed background charge for simplicity. This is the microscopic model used in Refs. 1-3.

High-mobility two-dimensional electron gases are commonly confined in GaAs quantum wells. The effect of the finite well thickness can be considered by multiplying to the LL wave functions an additional factor describing the lowest subband in an infinite rectangular quantum well. The smearing of the electrons in the quantum well tends to soften the Coulomb interaction between them. Because the Moore-Read Pf state is the exact ground state of a three-body interaction, LL mixing, which generates high-order corrections (including three-body terms) to the Coulomb interaction due to finite LL splitting, is important. For numerical work, one can include the corrections in the form of two-body and three-body pseudopotentials 3. The minimal model, now with LL mixing and finite well thickness, is considered in Ref. 4. Even before the perturbative LL mixing corrections were worked out, numerical models that include the Coulomb interaction with three-body interaction on phenomenological ground had been used 1.

In all of these numerical works, electrons are considered to be spin polarized. This assumption is supported by the density matrix renormalization group calculation in...
the spherical geometry \[6\] in the absence of LL mixing, and the truncated Hilbert space calculation in the torus geometry in the presence of LL mixing and finite layer thickness \[5\].

**Ground States and Edge Modes**

The total angular momentum \(M\) of the 12-electron ground state of the minimum model with an extra phenomenological three-body interaction

\[
H = (1 - \lambda)H_C + \lambda H_{3B}
\]

is mapped in Fig. 5(b) as a function of both the setback distance \(d\), in units of magnetic length \(l_B\), and the three-body content \(\lambda\), as was previously reported in Ref. 2. The ground state with \(M = 126\) has been unambiguously identified as the Pf phase for the following reasons.

1. Its momentum \(M\) matches that of the 12-electron Moore-Read Pf state \(M_p\) \[1\].

2. Increasing three-body interaction strength enhances the state \[2\].

3. Its edge spectrum matches that of the Pf state, with one branch of bosonic mode and one branch of fermionic mode \[1, 2\].

4. A local potential trap with different strength can stabilize charge \(e/2\) quasihole and charge \(e/4\) quasihole. In the latter case, the change of the edge-mode counting confirms that the charge \(e/4\) quasihole is a non-Abelian anyon \[1\].

Note that we intentionally downplay the role of the wave function overlap that is often used in a closed geometry as an indicator; the existence of the edge significantly lowers the overlap, but it provides extra information in return through the bulk-edge correspondence as we have already seen. In particular, the energetics of the edge spectrum can be used to extract information on the dispersion relation of the charged bosonic mode and the neutral fermionic mode, as illustrated in Fig. 5(e) for \(d = 0.6l_B\) and \(\lambda = 0.5\) (see Refs. 2, 3 for more detailed illustrations). As a result, the fermionic edge mode is found to have significantly smaller velocity than that of the bosonic mode \[2\]. Both are propagating along the same direction, so we can represent the edge spectrum by two straight lines with different slopes in the cartoon plot Fig. 5(c), which is meant to illustrate how energies of the bosonic or fermionic edge excitations grows with momentum around the value \(M_p\) for the Pf state in the long-wavelength limit.

For comparison, the ground state with \(M = 146\) is identified to be the APf state for the following reasons.

1. Its momentum \(M\) matches that of the 12-electron APf state \(M_n\) \[1\].

2. Increasing three-body interaction strength suppresses the state \[2\].

3. The state is robust with the inclusion of two extra orbitals \[2\].

4. Smoother confining potential can induce a state that is consistent with the APf state with a charge \(e/4\) quasihole \[2\].

5. There exists a low-energy excitation at \(\Delta M = -2\), corresponding to the smallest-momentum fermionic edge mode. As \(d\) decreases, its energy decreases toward zero; for comparison, as \(d\) increases in the Pf phase, the energy of the lowest \(\Delta M = 2\) state also decreases toward zero.

No further exploration on the APf state was carried out in Ref. 2 due to the complicated edge structure, as illustrated by the cartoon in Fig. 5(d).

Reason 5 in the APf case hints a transition from the Pf state to the APf state around \(d = l_B\) and an approximate mirror symmetry between the two phases around the transition. However, numerical calculation found that there is an intermediate state (\(M = 136\)), which appears to be of stripe nature \[2\]. The presence of the intermediate state is robust even when LL mixing and finite layer thickness are considered \[4\].

**The Transition from Pf to APf**

It is interesting to ask what emerges to be the ground state when one tunes the confining potential of the Pf state weaker and weaker. We already pointed out in the previous subsection that the APf state is not emerging directly. Two simplest possibilities, then, are as follows.

1. The Pf state with a quasihole in the bulk. Due to the rotational symmetry, such a quasihole can only appear at the center. However, the total angular momentum of the observed state (\(M = 136\)) matches neither the case of an Abelian \(e/2\) quasihole (\(M = 138\)) nor the case of a non-Abelian \(e/4\) quasihole (\(M = 132\)) \[1\].

2. Bosonic edge reconstructed Pf state. Bosonic edge mode at small \(\Delta M\) shows signature of bending down, as for the Laughlin state \[8, 9\]. If this persists to larger \(\Delta M\), the chiral boson edge mode becomes soft and the bosons condense. This is the microscopic interpretation of the edge reconstruction, which means that we expect to see charge density modulation around the edge. The fermionic mode is not necessary to be involved.
FIG. 6: (Color online) Possible profile of the low-energy states at \( \nu \sim \frac{1}{2} \). The ground state with the angular momentum in between the Pf and APf values has a lower energy than either the Pf or APf state and is possibly of either (a) bosonic nature or (b) fermionic nature. The red lines schematically represent the lowest fermionic excitations, while the black ones the lowest bosonic ones. In (a) the bosonic mode reconstructs. The energy dispersion curves of the edge modes obtained by Zhang et al. [3], on the other hand, can reproduce the profile in the circled area in (b).

If the bosonic edge reconstruction were the origin of the striped phase, we would expect to observe a crossing of the bosonic and fermionic modes at a sufficiently large \( \Delta M \), because at small \( \Delta M \) the bosonic mode has a larger velocity. Note that the new ground state has an angular momentum in between those of the Pf and the APf states. We can schematically illustrate the profile of the low-energy states, including the edge excitations of the Pf and APf states, as in Fig. 6(a).

Very recently, Zhang et al. studied the possibility of edge reconstruction at \( \nu = \frac{5}{2} \) [3]. In the minimum model they obtained the edge states of the Pf state by diagonalization in the restricted Pf basis, and resolved them by applying a density operator for bosonic edge excitations and trial wave functions for fermionic edge excitations, in addition to the self-consistency conditions of the edge-mode dispersions. They found that the bosonic edge mode persists to have a higher energy scale than the fermionic mode, up to the capability of their resolution [3]. Their findings indicate that the usual bosonic edge reconstruction is less likely to be responsible for the edge instability of the Pf state; instead, they proposed a scenario of Majorana-driven edge reconstruction [3]: this requires that the dispersion of the fermionic mode develop a roton-like structure, which is pushed down, presumably, by the bending down bosonic mode. Therefore, the profile of the low-energy excitations resembles the sketch in Fig. 6(b), in the green circle of which the discovery of Zhang et al. [3] is illustrated. We note the restricted basis used there limits the possible types of instabilities the Pf state (as we discuss below); but this work suggests that the instabilities are likely triggered by softening of neutral degree of freedom.

Fig. 7: (Color online) Evolution of a FQH droplet at \( \nu = \frac{5}{2} \). (a) Homogeneous Pf state is stable at a sharp confining potential. There are one charged bosonic mode and one neutral fermionic mode at the edge propagating along the same direction. (b) Pf-APf striped phase, driven by edge reconstruction, emerges at smoother confining potential. At the Pf-APf interface the bosonic modes gapped out and there are only neutral modes propagating along the same direction. (c) Homogeneous APf state is stable at a smooth confining potential. In (b) and (c) we assume the APf edge modes are strongly coupled such that there are only one charge mode, one counterpropagating neutral bosonic mode, and one counterpropagating neutral fermionic mode.

Proposed in the present work is an instability appearing at the edge, but the edge piece is an APf stripe. Naturally, higher electron density at the APf edge, as oppose to the Pf edge, is energetically favorable where the confining potential is smooth. The scenario is illustrated in Fig. 6 in disk geometry. When the confining potential is sharp, the Pf state is favored and there are one charged bosonic mode and one neutral fermionic mode at the edge [Fig. 7(a)]. Smoother confining potential drives edge reconstruction and induces an APf ring around the Pf bulk [Fig. 7(b)]. At the Pf-APf interface the bosonic modes are gapped out and there are neutral modes only as discussed in the main text; this is also consistent with the numerical results in Zhang et al. [3]. The outer edge is an APf edge; here we illustrate with the case in which the two counter-propagating charge modes are coupled, leading to one charge mode and one neutral mode. To see that this transition is due to edge reconstruction, we can compare the low-energy modes in terms of chiral central charges for left- and right-going modes. The chiral central charges are \( (c_+ + c_-) = (1.5, 0) \) for the Pf state, and \( (3, 1.5) \) for the striped state. Note the difference \( c_+ - c_- = 1.5 \) is the same for them, allowing an edge instability to connect them. As the confining potential smooths further, the Pf core shrinks to the center and
eventually an homogeneous APf state is left [Fig. 7(c)]. Therefore, the existence and evolution between the three phases are consistent with all available numerical results in disk geometry. We note in such a small system sizes there is only one interface between Pf and APf state in the striped phase. For large systems we expect many interfaces that are separated by distance(s) of order several magnetic length, thus forming a true bulk phase as discussed in the main text.

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