CANONICAL MODELS AND STABLE REDUCTION FOR PLURIFIBERED VARIETIES

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1. Introduction

All schemes considered here are over some noetherian base scheme \( S \) of pure characteristic 0.

1.1. Prologue: semistable reduction over a base of higher dimension. One result we prove here is the following weak variant of the main theorem of [X-Q]:

**Theorem 1.1.1.** Let \( X \to B \) be a morphism of projective varieties, with geometrically integral generic fiber. Then there is an alteration \( B_1 \to B \), a modification of the main component \( Y \to X \times_B B_1 \), and a divisor \( D \subset Y \) such that

1. both \( Y \to B_1 \) and \( D \to B_1 \) are flat,
2. the singularities of \((Y,D)\) are log-canonical, and
3. for every \( b \in B_1 \), the fiber \((Y_b,D_b)\) has at most semi-log-canonical singularities.

This result is weaker than [X-Q], Theorem 1, since the singularities of \((Y,D)\) are only log-canonical, and we do not claim that the singularities of \( Y \) itself are Gorenstein, or toroidal, or canonical. However, as we will see, the procedure that leads to \((Y,D)\) is more canonical than that of [X-Q], since the only data required is a rational plurifibration, and once this is chosen everything is canonically defined. The main idea is here inspired by A.J. de Jong’s work on alterations [J].

The contents of this note are centered around stable models and stable reduction for plurifibrations. This is an expansion of remarks which appeared in [X-V3] and [X-V2], on natural extensions of the results of [X-V1]. Related ideas, but with quite a different bent, were independently discovered by Mochizuki [Mo].

1.2. The minimal model program. One could say that the basic goal of the minimal model program (MMP) is to find, in the birational equivalence class of each variety \( X \) of general type, a canonical model

\[
X \dashrightarrow X^\text{can},
\]

where \( X^\text{can} \) has canonical singularities and ample canonical divisor. A more ambitious goal proposed by V. Alexeev is that of stable maps: given a triple

\[
(X, D, f : X \to \mathbb{P}),
\]

where \( X \) is, say, a smooth projective variety, \( D \) a normal-crossings divisors, and \( f : X \to \mathbb{P} \) is a morphism to a fixed projective space, with the “pre-stability assumption”

\[
L_{(X, D, f : X \to \mathbb{P})}, n := K_X(D) \otimes f^*\mathcal{O}_\mathbb{P}(n) \quad \text{is a big line bundle for } n \gg 0,
\]

one expects to find a suitable birational representative

\[
(X^\text{st}, D^\text{st}, f^\text{st} : X \to \mathbb{P})
\]

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which is a stable map, that is, \((X^{st}, D^{st})\) log-canonical and
\[ K_{X^{st}}(D^{st}) \otimes (f^{st})^*O_P(n) \] is an ample line bundle for \( n >> 0 \).

Following Deligne-Mumford [D-M], Alexeev [Al], Kollár, Shepherd-Barron [Ko, K-SB] and Viehweg [Vi], one also considers moduli. One defines certain degenerate versions of stable \((X, D, f)\) above, where \((X, D)\) becomes semi-log-canonical. Then one conjectures that, fixing suitable numerical invariants, stable maps admit a proper moduli stack having a projective coarse moduli space. It should be pointed out that, following Kollár [Ko], Alexeev [Al] and Karu [Ka], the projectivity of moduli in a sense follows from a strong enough version of the MMP in one dimension higher.

1.3. **Tweaking with the data.** The stable models above are supposed to arise by a prescribed procedure: \( X^{st} \) is the projective spectrum of the algebra of sections
\[ \bigotimes_{i \geq 0} L^{\otimes i}(X, D, f : X \to \mathbb{P}), n \]
and the divisor \( D^{st} \) and morphism \( f^{st} \) are induced. Without this requirement, one can easily cheat, i.e. by throwing in a sufficiently ample divisor \( D^{st} \), or by replacing \( f \) by an embedding. Moreover, such “cheat solutions” are far from canonical, are not birational in nature, and certainly are not sufficient in finding a good compactification of moduli. Nevertheless, the idea of temporarily “tweaking” with the given data has been quite fruitful: methods of changing the divisor \( D \) a little bit are fundamental in the work of an illustrious list of authors, specifically Shokurov; and the addition of a map \( f : X \to Z \) (specifically a small contraction of an extremal ray) puts the study of flips right into this framework.

In this note, we propose a situation where the MMP has a complete solution - stable models and complete moduli included - which is cheap given existing theory (though the existing theory is far from cheap). It is still a “cheat” solution, but in a milder sense: the additional data we add to the given variety is birational in nature, and the final maps and divisors arise canonically from the data, no further choices needed. As with other “cheat” solutions, one hopes that this can serve as a stepping stone for further understanding of the MMP. A situation where this can be done is studied in the paper [La].

1.4. **The setup.** We are given a function field \( K \) of transcendence degree \( d \) over a field \( k \) of characteristic 0. A rational plurifibration \((K_i)\) is a sequence
\[ K = K_0 \supset K_1 \supset \cdots \supset K_d = k \]
where each \( K_i \supset K_{i+1} \) is an extension of transcendence degree 1 such that \( K_{i+1} \) is algebraically closed in \( K_i \). We denote by \( C_i \to \text{Spec} K_{i+1} \) the smooth projective curve associated to the function field extension \( K_i \supset K_{i+1} \).

Suppose in addition we are given a projective variety \( M_0 \). A rational plurifibered map with target \( M_0 \) is a rational plurifibration \((K_i)\) and a morphism \( \text{Spec} K_0 \to M_0 \).

A slightly more general notion is the following: suppose \( M_0 \) is a proper Deligne–Mumford stack having \( M_0 \) as its coarse moduli space. Then a rational plurifibered map with target \( M_0 \) is a marked rational plurifibration \((K_i)\) and an object \( M_0(\text{Spec} K_0) \) (equivalently, a morphism \( \text{Spec} K_0 \to M_0 \)).

We need to impose a pre-stability condition. The most general situation allowed is a bit subtle to describe - essentially it is tautologically the situation where the main result works. In order to be more specific we will use a sufficient condition for pre-stability: we say that \((K_i, \text{Spec} K_0 \to M_0)\) is sufficiently good if one of the following conditions holds:

**G1.** the morphism \( \text{Spec} K_0 \to M_0 \) is finite to its image, or
G2. the genus \( g(C_i) > 1 \), for \( i = 0, \ldots, d - 2 \), and either \( g(C_{d-1}) > 1 \) or \( C_{d-2} \to \text{Spec } K_{d-1} \) is nonisotrivial, or

G3. the function field \( K_0 \) is of general type over \( K_d \).

There are some natural extensions, where \( C_i \) are allowed to have a-priori markings and where more maps \( \text{Spec } K_i \to \mathcal{M}_i \) are a-priori given. We delay facing these to a later version.

1.5. Twisted stable maps. Our main tool is the moduli stack of twisted stable maps. Recall that a twisted curve is a marked Deligne–Mumford stack of dimension 1 with simple local structure ([R-V3], Definition 4.1.2). A twisted stable map \( f : C \to \mathcal{M} \) is a representable morphism from a twisted curve \( C \) to a Deligne–Mumford stack \( \mathcal{M} \) such that the resulting map \( C \to \mathcal{M} \) of course moduli spaces is stable ([R-V3], Definition 4.3.1). It is proven in [R-V3], Theorem 1.4.1 that families of twisted stable with fixed genus \( g \), number of markings \( n \), and image class \( \beta \) form a Deligne–Mumford stack \( K_{g,n}(\mathcal{M}, d) \) having a projective coarse moduli space \( K_{g,n}(\mathcal{M}, d) \). In our discussions below we will restrict attention to the stack of balanced twisted stable maps \( K_{g,n}^{\text{bal}}(\mathcal{M}, d) \). From now on we will assume this and suppress the superscript “bal”. In addition, we will ignore the ordering of the markings, thus consider the stack

\[
K_{g,\{n\}}(\mathcal{M}, d) := [K_{g,n}(\mathcal{M}, d)/\mathcal{S}_n],
\]

where the symmetric group \( \mathcal{S}_n \) acts by permuting the numbers of the markings. Thus the stack \( K_{g,\{n\}}(\mathcal{M}, d) \) parametrizes maps where the twisted curves \( C \) are marked by \( \Sigma^C \), which is a reduced Cartier divisor which is \( n \)-sheeted finite étale over the base. In most of what follows the marking \( \Sigma^C \) will all be twisted, i.e. the inertia group at any point of \( \Sigma^C \) is nontrivial.

1.6. The outcome: twisted version. We now describe the type of objects that arise as the stable models in a certain sense of a rational plurifibered map.

By a twisted stable plurifibered map over a scheme \( S \) we mean a gadget

\[
(\mathcal{X}_i \to \mathcal{X}_{i+1}, \Sigma_i, \mathcal{X}_i \to \mathcal{M}_i)
\]

where

1. \( \mathcal{X}_i \) are Deligne–Mumford stacks, \( i = 0, \ldots, d - 1 \), and \( \mathcal{X}_d = S \)
2. \((\pi_i : \mathcal{X}_i \to \mathcal{X}_{i+1}, \Sigma_i \subset \mathcal{X}_i, f_i : \mathcal{X}_i \to \mathcal{M}_i)\) is a family of twisted stable maps of certain genus \( g_i \), unordered twisted marking of degree \( n_i \), and image class \( \beta_i \) for each \( i = 0, \ldots, d - 1 \), and
3. \( \mathcal{M}_{i+1} = K_{g_i,\{n_i\},\beta_i} \) for \( i = 0, \ldots, d - 2 \).

We remark that, almost by definition of twisted stable maps, each \( \mathcal{X}_i \) has a representable dense open substack, and the coarse moduli space \( \mathcal{X}_i \) is projective over \( S \).

We use the notation \( q_i : \mathcal{X}_i \to \mathcal{X}_i \) for the canonical map. Also, for each \( 0 \leq i \leq j \leq d \) we denote by \( \pi_{i,j} : \mathcal{X}_i \to \mathcal{X}_j \) the composite \( \pi_{j-1} \circ \cdots \circ \pi_i \).

1.7. Main results: twisted version. The existence of stable models is given by the following theorem:

Theorem 1.7.1. Let \((K_f, f : \text{Spec } K_0 \to \mathcal{M}_0)\) be a sufficiently good rational plurifibered map. Then there exists a canonical twisted stable plurifibered map \((\pi_1 : \mathcal{X}_1 \to \mathcal{X}_{i+1}, \Sigma_i \subset \mathcal{X}_i, f_i : \mathcal{X}_i \to \mathcal{M}_i)\) over \( S = \text{Spec } K_d \), where \( \mathcal{X}_i \) is irreducible with function field \( K_i \), with \( \pi_i \) corresponding to \( K_i \supset K_{i+1} \).

Completeness of moduli is immediately given:
Theorem 1.7.2. The category of families of twisted stable plurifibered maps with fixed data $\mathcal{M}_0, g_i, n_i, \beta_i$ is a proper Deligne–Mumford stack having a projective coarse moduli space.

Proof. This category is equivalent to $\mathcal{K}_{g_{d-1}, (n_{d-1})}(\mathcal{M}_{d-1}, \beta_{d-1})$, for which the claim follows from \([R-V3]\), Theorem 1.4.1.

1.8. Main results: coarse version. Let $(\pi_i : \mathcal{X}_i \to \mathcal{X}_{i+1}, \Sigma_i \subset \mathcal{X}_i, f_i : \mathcal{X}_i \to \mathcal{M}_i)$ be a twisted stable plurifibered variety.

We define a divisor on $X_0 = X$: for each $i$ we denote by $\Sigma^{X_i}$ the marking on $X_i$ thought of as the coarse curve of $\mathcal{X}_i \to \mathcal{X}_{i+1}$. Let $D \subset X$ be given by

$$D = \sum_{i=0}^{d-1} (\pi_{i,0}^* \Sigma^{X_i})_{\text{red}}.$$

We define a projective variety $\mathbf{M}$ and a morphism $f : X \to \mathbf{M}$: let $\mathbf{M}_i$ be the coarse moduli space of $\mathcal{M}_i$ and let

$$\mathbf{M} = \prod_{i=0}^{d-1} \mathbf{M}_i.$$

We define $X \to \mathbf{M}$ to be the product of the morphisms obtained by composing $X \to X_i$ with $X_i \to \mathbf{M}_i$.

Proposition 1.8.1. The pair $(X, D)$ is semi-log-canonical, and the triple $(X, D, f : X \to \mathbf{M})$ is a stable map in the sense of Alexeev.

We call $(X, D, f : X \to \mathbf{M})$ the Alexeev stable map associated to $(\pi_i : \mathcal{X}_i \to \mathcal{X}_{i+1}, \Sigma_i \subset \mathcal{X}_i, f_i : \mathcal{X}_i \to \mathcal{M}_i)$.

If $(K_i, D_i, f : \text{Spec} K_0 \to \mathcal{M}_0)$ is a sufficiently good rational plurifibered map, and $(\pi_i : \mathcal{X}_i \to \mathcal{X}_{i+1}, \Sigma_i \subset \mathcal{X}_i, f_i : \mathcal{X}_i \to \mathcal{M}_i)$ the associated twisted stable plurifibered map, then we obtain:

Corollary 1.8.2. Given a sufficiently good rational plurifibered map, with $K_0$ of general type over $K_d$, there is a canonical Alexeev stable map $(X, D, f : X \to \mathbf{M})$ associated to the corresponding twisted stable plurifibered map.

2. Proofs

2.1. Proof of Theorem [1.7.1]. The proof proceeds by induction on $d$. For $d = 0$ there is nothing to prove. Assume the claim is proven for $d-1$, and denote the stable twisted plurifibered map over $K_{d-1}$ associated to

$$K_0 \supset \cdots \supset K_{d-1}$$

by

$$(\pi_i^{d-1} : X_i^{d-1} \to X_{i+1}^{d-1}, \Sigma_i^{d-1}, f_i^{d-1} : X_i^{d-1} \to \mathcal{M}_i).$$

Construction of target. Denoting $\mathcal{M}_{d-1} = \mathcal{K}_{g_{d-2}, (n_{d-2})}(\mathcal{M}_{d-2}, \beta_{d-2})$, we have by definition a canonical morphism $\text{Spec} K_{d-1} \to \mathcal{M}_{d-1}$.

Construction of twisted curve and map. Essentially by \([R-V3]\), Lemma 7.2.6 there exists a unique smooth twisted curve $X_{d-1} \to \text{Spec} K_{d-1}$ having function field $K_{d-1}$, with a representable morphism $f_{d-1} : X_{d-1} \to \mathcal{M}_{d-1}$ and all markings twisted. Since that lemma had extra assumptions, we recall the construction: the morphism $\text{Spec} K_{d-1} \to \mathcal{M}_{d-1}$ extends uniquely to $C_{d-1} = X_{d-1} \to \mathcal{M}_{d-1}$, since the latter is proper. We have a morphism

$$\psi : \text{Spec} K_{d-1} \to C_{d-1} \times_{\mathcal{M}_{d-1}} \mathcal{M}_{d-1}. $$
Let

$$X_{d-1} = (\psi(\text{Spec } K_{d-1}))^{\text{norm}},$$

i.e. the normalization of the closure of the image of $\psi$. This is the required twisted curve.

**Numerical invariants.** We set $g_{d-1} = g(C_{d-1})$; $n_{d-1}$ is the degree of the marking on $X_{d-1}$, and $\beta_{d-1}$ is the class of $f_{d-1}. [C_{d-1}]$.

**Construction of $X_i \to M_i$.** We construct $X_i$ by descending recursion, $X_{d-1}$ being already given. Assume $X_{i+1} \to M_{i+1}$ given, we take $(X_i \to X_{i+1}, \Sigma_i, X_i \to M_i)$ to be the universal object corresponding to $M_i = K_{g_i.(n_i)}(M_i, \beta_i)$. We are also given canonical isomorphisms

$$(X_i \to X_{i+1}, \Sigma_i, X_i \to M_i)_{\text{Spec } K_{d-1}} \cong (X_{i+1}^{d-1} \to X_{i+1}, \Sigma_{i+1}^{d-1}, X_{i+1}^{d-1} \to M_i),$$

as required.

**Stability of map.** We claim that $X_{d-1} \to M_{d-1}$ is stable. We need to show:

1. if $X_{d-1} \to M_{d-1}$ is constant, then $2g_{d-1} - 2 + n_{d-1} > 0$,
   - claim 1: if $g_{d-1} = 1$ then $n_{d-1} > 0$, and
   - claim 2: if $g_{d-1} = 0$ then $n_{d-1} > 2$.

The claims are obvious under the “sufficiently good” assumptions (G1) and (G2). We need to prove claims (1),(2) under assumption (G3) that $K_0$ is of general type over $K_d$. The claims, as well as the general type assumption, are geometric, so we may replace $K_d$ by its algebraic closure. We prove the claim by contradiction. We now construct trivializations of $X_{d-1} \to M_{d-1}$ after finite étale covers.

Assuming we have $g_{d-1} = 1$ and $n_{d-1} = 0$, then $X_{d-1} = X_{d-1}$ is a smooth elliptic curve. Since $X_{d-1} \to M_{d-1}$ is constant, the object corresponding to $X_{d-1} \to M_{d-1}$ is isotrivial, and becomes trivial after a finite étale cover $E \to X_{d-1}$, and therefore $E$ is an elliptic curve.

Assume, on the other hand, we have $g_{d-1} = 0$ and $n_{d-1} < 2$. Then $X_{d-1}$ is rational and has two marked points. Since we have made the base field algebraically closed, we may choose coordinates so that the points are 0 and $\infty$. The map $X_{d-1} \to \{0, \infty\} \to M_{d-1}$ is isotrivial, which means that there is a map $E = \mathbb{P}^1 \to X_{d-1}$ (branched only at 0 and $\infty$) with a lifting $\mathbb{P}^1 \to M_{d-1}$. It actually follows that this map can be chosen étale over $X_{d-1}$, but this will not be essential.

In both cases we got a trivialization over a curve $E$ with genus $\leq 1$. By decending induction and the construction of $E$ and $X_i$ we obtain that $E \times X_{d-1} \to E$ is a trivial family $\mathbb{P}^1 \to E$.

We have a finite morphism $E \times X_{d-1} \to X_0$, and since the source is covered by curves $E$ of genus $\leq 1$ we get a contradiction to the assumption that $X_0$ is of general type. This completes claims (1),(2) and proves that $X_{d-1} \to M_{d-1}$ is stable.

Thus the theorem is proven.

2.2. **Proof of Proposition 1.8.1.** We need to show that $(X, D)$ has semi-log-canonical singularities and that a certain line bundle is ample. It is convenient to study first the situation on $X = X_0$, which has an analogous divisor

$$D = \sum_{i=0}^{d-1} \pi_{0,i}^* \Sigma_i.$$

This divisor is automatically reduced.

**Lemma 2.2.1.** The pair $(X, D)$ has semi-log-canonical singularities. Moreover, when $X$ is normal, the pair has log-canonical singularities.
Proof. We use induction on $d$. Since $\mathcal{X}$ is a composite of a fibration by nodal curves it is Gorenstein, in particular Cohen-Macaulay. Thus to prove the claim it suffices to look at the normalization, as follows:

Let $\mathcal{X}_0^{\text{norm}} \to \mathcal{X}_0$ be the normalization, and let $\mathcal{E} \subset \mathcal{X}_0^{\text{norm}}$ be the conductor divisor. Consider an irreducible component $\mathcal{X}' \subset \mathcal{X}_0^{\text{norm}}$. Denote $\mathcal{D}' = \mathcal{D}|_{\mathcal{X}'} + \mathcal{E}$. It suffices to show that the pair $(\mathcal{X}', \mathcal{D}')$ has log-canonical singularities.

By definition $\mathcal{X}'$ is a composite of fibrations by twisted curves $(\mathcal{X}_i' \to \mathcal{X}_{i+1}', \Sigma_i')$, each with smooth generic fiber (In particular $\mathcal{X}_i' \to \mathcal{X}_{i+1}'$ is a smooth curve). By \cite{K}, Corollary 4.5, $\mathcal{X}'$ has rational Gorenstein singularities, in particular canonical. We are going to show that $(\mathcal{X}', \mathcal{D}')$ is log-canonical for every positive integer $m$. Taking the limit as $m \to \infty$ we get that the pair $(\mathcal{X}', \mathcal{D}')$ is log-canonical.

So fix an integer $m$. Let $p_0$ be a geometric point on $\mathcal{X}_i'$ and $p_i$ its image in $\mathcal{X}_i'$. Passing to an étale neighborhood of $p_0$ we may replace $\mathcal{X}'$ by a scheme, which we still denote by $\mathcal{X}'$. For each $i$ such that $p_i$ lies on $\Sigma_i'$ we have that $\mathcal{X}_i' \to \mathcal{X}_{i+1}'$ is smooth and $\Sigma_i'$ is a section. Passing to a neighborhood again, we may assume there exists a cyclic cover $Y_i \to \mathcal{X}_i'$ of degree $m$ totally branched along $\Sigma_i'$ and étale elsewhere. Pulling back and taking the compositum, we get a branched cover $\psi_i : Z_i \to \mathcal{X}_i'$, totally branched along all the components of $\mathcal{D}_i$ with index $m$, and each $Z_i \to \mathcal{X}_{i+1}'$ is a family of nodal curves. Again by \cite{K}, Corollary 4.5, $Z_0$ has canonical singularities.

Note that $K_{Z_0} = \psi_0^*(K_{\mathcal{X}'} + \frac{m-1}{m}\mathcal{D}')$. Since $Z_0$ has canonical singularities, a result of Kollár (see \cite{K}, 20.3 (2)) says that the pair $(\mathcal{X}', \mathcal{D}')$ is log-canonical.

We continue with the proof of the proposition. The next step is

Lemma 2.2.2. The pair $(X, D)$ has semi-log-canonical singularities. Moreover, for a positive integer $m$ such that $O_X(m(K_X + D))$ is Cartier, we have $p^*(O_X(m(K_X + D))) = (\omega_X(\mathcal{D}))^m$.

Proof. Recall that when $\mathcal{X}$ is a Deligne–Mumford stack with coarse moduli space $X$, there is an étale covering $\cup X_j \to X$, schemes $V_j$, and finite groups $\Gamma_j$ such that $\mathcal{X} \times_X X_j \cong [V_j/\Gamma_j]$. In our case, we already have $(\mathcal{X}, \mathcal{D})$ semi-log-canonical, so $(V_j, \mathcal{D}_j)$ are semi-log-canonical. We also have that $\mathcal{D}_j$ contains the divisorial part of the fixed-point-logus of $\Gamma_j$. Denote by $D_j$ the image of $\mathcal{D}_j$ in $V_j/\Gamma_j$. The aforementioned result of Kollár (see \cite{K}, 20.3 (2)) implies that $(V_j/\Gamma_j, D_j)$ has semi-log-canonical singularities, and the lemma follows.

Let $\mathcal{O}_M(1)$ be an ample line bundle. The proposition follows once we prove the following lemma:

Lemma 2.2.3. For any large enough integer $n$ the sheaf $\omega_X(\mathcal{D}) \otimes f^*\mathcal{O}_M(n)$ is ample.

Proof. There is an integer $n$ such that $L_i := \omega_{\mathcal{X}_i/\mathcal{X}_{i+1}}(\Sigma_i) \otimes f_i^*\mathcal{O}_M(n)$ is $\pi_i$-ample. It follows from a result of Kollár (\cite{K}, Proposition 4.7) that $L_i$ is nef on $\mathcal{X}_i/S$: indeed, $L_i^n$ is $\pi_i$-ample and $\pi_i$-acyclic for large $m$; Kollár shows that the resulting vector bundle $\pi_i* L_i^n$ is semipositive, therefore the tautological line-bundle on its projectivization is nef, and therefore the restriction of this tautological bundle to $\mathcal{X}_i$, which is $L_i^n$, is nef.

Let $F \subset \mathcal{X}$ be closed and irreducible of dimension $l$. We need to show that

$$\deg_F c_1(L)^l > 0.$$  

We use induction on $d$ (the case $d = 0$ being trivial).

If $\dim(\pi_0(F) \subset \mathcal{X}_1) = l$ then we can use the inductive assumption on $d$ applied to $\mathcal{X}_1 \to S$ and the fact that $L_0$ is nef to conclude that $\deg_F c_1(L)^l > 0$. 

2.3. Proof of Theorem 1.1.1. Let \( X \rightarrow B \) be as in the theorem. We set \( S = B \) as base scheme, and \( \mathcal{M}_0 = X \) as target scheme. We set \( K_0 = K(X), K_d = K(B) \), and write \( \eta = \text{Spec} K_d \). Now define \( K_i, i = 1, \ldots, d - 1 \) by descending recursion: given \( K_i \) choose a general pencil of hypersurfaces \( H_i \hookrightarrow \mathbb{P}^1_{K_i} \) of \( X_{K_i} \). We have \( K(H) = K_0 \), and we set \( K_{i-1} = K(\mathbb{P}^1_{K_i}) \).

(The connectedness theorem implies that \( X_{K_i} \) is geometrically integral for all \( i \).)

Since the morphism
\[
\text{Spec} K_0 \rightarrow \mathcal{M} = X
\]
is an embedding, the rational plurifibered map \( (K_i, \text{Spec} K_0 \rightarrow \mathcal{M}_0) \) is sufficiently good. We therefore have a twisted stable plurifibered map over \( \eta \)
\[
(X^n_i, \Sigma^n_i, f^n_i : X^n_i \rightarrow \mathcal{M}_i).
\]

This gives a morphism
\[
\eta \rightarrow \mathcal{K}_{g_{d-1}, \{n_{d-1}\}}(\mathcal{M}_{d-1}, \beta_{d-1}).
\]
Since the latter is proper, there exists an alteration \( B_1 \rightarrow B \) and a lifting
\[
B_1 \rightarrow \mathcal{K}_{g_{d-1}, \{n_{d-1}\}}(\mathcal{M}_{d-1}, \beta_{d-1}),
\]
in other words, a family of stable plurifibered maps over \( B_1 \)
\[
(X^1_i, \Sigma^1_i, f^1_i : X^1_i \rightarrow \mathcal{M}_i)
\]
(extending the pullback of the family over \( \eta \) to the generic point of \( B_1 \)). In particular we have a rational map \( X_0 \rightarrow X \).

We now have an associated Alexeev stable map
\[
(Y, D, Y \rightarrow \mathcal{M}),
\]
where by construction \( \mathcal{M} = \mathcal{M}_0 \times \mathcal{M}' \) for some projective scheme \( \mathcal{M}' \). A-priori we have a rational map \( Y \rightarrow X \), but the composition \( Y \rightarrow \mathcal{M} \rightarrow \mathcal{M}_0 = X \) shows that this is regular.

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References

[8] D. Abramovich, A high fibered power of a family of varieties of general type dominates a variety of general type. Invent. Math. 128 (1997), no. 3, 481–494.

[8-K] D. Abramovich and K. Karu, Weak semistable reduction in characteristic 0, Invent. math. 139 (2000) 2, p. 241-273.

[8-V1] D. Abramovich and A. Vistoli, Complete moduli for fibered surfaces, Recent progress in intersection theory (Bologna, 1997), 1–31, Trends Math., Birkhäuser Boston, Boston, MA, 2000.

[8-V2] – , Complete moduli for families over semistable curves, preprint, math.AG/9811059
Compactifying the space of stable maps, J. Amer. Math. Soc. 15 (2002), no. 1, 27–75.

V. Alexeev, Moduli spaces $M_{g,n}(W)$ for surfaces, in Higher-dimensional complex varieties (Trento, 1994), 1–22, de Gruyter, Berlin.

P. Deligne and D. Mumford, The irreducibility of the space of curves of given genus, Inst. Hautes Études Sci. Publ. Math. No. 36 (1969), 75–109.

A. J. de Jong, Families of curves and alterations, Ann. Inst. Fourier (Grenoble) 47 (1997), no. 2, 599–621.

K. Karu, Minimal models and boundedness of stable varieties. J. Algebraic Geom. 9 (2000), no. 1, 93–109.

J. Kollár, Projectivity of complete moduli. J. Differential Geom. 32 (1990), no. 1, 235–268.

J. Kollár and N. Shepherd-Barron, Threefolds and deformations of surface singularities. Invent. Math. 91 (1988), no. 2, 299–338.

J. Kollár et al., Flips and abundance for algebraic threefolds. Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991. Astérisque No. 211 (1992). Société Mathématique de France, Paris, 1992. pp. 1–258.

G. LaNave, Explicit stable models of elliptic surfaces with sections, preprint. [math.AG/0205035]

S. Mochizuki, Extending families of curves over log regular schemes. J. Reine Angew. Math. 511 (1999), 43–71.

E. Viehweg, Quasi-projective moduli for polarized manifolds. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 30. Springer-Verlag, Berlin, 1995.

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