ON THE DUALITY OF F-THEORY AND THE CHL STRING IN EIGHT DIMENSIONS

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Abstract. We show that the duality between F-theory with discrete flux and the CHL string in eight dimensions defines algebraic correspondences between K3 surfaces polarized by the rank-ten lattices $H \oplus N$ and $H \oplus E_8(-2)$. In the special case when the F-theory admits an additional anti-symplectic involution or, equivalently, the CHL string admits a symplectic one, both moduli spaces coincide. In this case, we derive an explicit parametrization for the F-theory compactifications dual to the CHL string, using an auxiliary genus-one curve, based on a construction given by André Weil.

1. Introduction

By standard lattice-theoretic observations \cite{60}, one has the following lattice isomorphism:

\begin{equation}
H \oplus E_8(-2) \cong H(2) \oplus N.
\end{equation}

Here, $E_8$ is the positive definite root lattices associated with the $E_8$ root systems, $H$ is the unique even unimodular hyperbolic rank-two lattice and $N$ is the negative definite rank-eight Nikulin lattice (see \cite{54}, Def. 5.3, for a definition). The notation $L(\lambda)$ refers to the lattice obtained from $L$ after scaling of its bilinear form by $\lambda \in \mathbb{Z}$. Moreover, as proved in \cite{6}, the lattice isomorphism (1.1) implies that the lattice $H(2) \oplus E_8(-2)$ admits two special overlattices, namely the lattice $H \oplus N$, necessarily of index four, and $H \oplus E_8(-2)$ of index two.

In the above contexts, we shall consider $\mathcal{M}_{H \oplus N}$ and $\mathcal{M}_{H \oplus E_8(-2)}$, as moduli spaces of complex algebraic K3 surfaces with lattice polarizations of type $H \oplus N$ and $H \oplus E_8(-2)$. Both these moduli spaces are 10-dimensional. And the K3 surfaces classified by them can be described explicitly, via Weierstrass models. The first K3 family was studied by van Geemen and Sarti \cite{66}. These K3 surfaces carry a canonical Jacobian elliptic fibration with an element of order two in its Mordell-Weil group, which, in turn, determine a special class of K3 involutions referred to in the literature as van Geemen-Sarti involutions. The K3 surfaces in the second family, associated with $H \oplus E_8(-2)$ polarizations, also carry canonical Jacobian elliptic fibrations, but in this case one has compatible anti-symplectic involutions, with the property that the minimal resolution of the associated $\mathbb{Z}/2\mathbb{Z}$ quotient is a rational elliptic surface \cite{6}.
The lattice isomorphism (1.1), via Hodge theory, implies then that the two K3 families from above are related via special algebraic correspondences. These correspondences are governed by markings of even-eight configurations in the Néron-Severi group NS(\(\mathcal{X}\)) for \(\mathcal{X} \in \mathcal{M}_{H \oplus \mathcal{N}}\) and pairs of Brauer group elements \(\{\pm \theta\} \subset \text{Br}(\mathcal{Y})\) for \(\mathcal{Y} \in \mathcal{M}_{H \oplus \mathcal{E}_8(-2)}\). These algebraic correspondences stem from classical constructions in the (mathematics) literature [6, 20, 25, 37, 65].

The above mentioned construction has a remarkable consequence in string theory—it provides a mathematical framework for a class of string dualities linked to the so-called CHL string, named after Chaudhuri, Hockney, and Lykken [8]. In eight dimensions, the CHL string is obtained from the more familiar \(E_8 \times E_8\) heterotic string on a torus \(T^2\), as a certain \(\mathbb{Z}/2\mathbb{Z}\) quotient. The Narain construction shows that the physical CHL moduli space is identical with the moduli space \(\mathcal{M}_{H \oplus \mathcal{E}_8(-2)}\) of K3 surfaces principally polarized by the lattice \(H \oplus \mathcal{E}_8\) [1, 2].

F-theory, i.e., compactifications of the type-IIB string theory in which the complex coupling varies over a base, is a powerful tool for analyzing the non-perturbative aspects of heterotic string compactifications [55, 56]. The simplest F-theory constructions are K3 surfaces that admit Jacobian elliptic fibrations over \(\mathbb{P}^1\). It is well known (see [69]) that, for F-theory backgrounds with non-zero flux given by a \(B\)-field along the base curve \(\mathbb{P}^1\), the value of this flux is quantized and fixed to half the Kähler class of \(\mathbb{P}^1\). In geometric language, this structure is equivalent to a Jacobian elliptic fibration supported on the K3 surface, admitting a 2-torsion section, i.e., a van Geemen-Sarti involution.

It follows from above that the K3 moduli spaces \(\mathcal{M}_{H \oplus \mathcal{N}}\) and \(\mathcal{M}_{H \oplus \mathcal{E}_8(-2)}\) are also the moduli spaces of F-theory models with discrete flux and the CHL string (heterotic string with CHL involution), respectively. In this article, we shall use classical geometrical notions - algebraic correspondences, even-eight configurations, and elements of the Brauer group - to give a precise mathematical framework for the F-theory/CHL string duality.

Particular situations of the above duality are also interesting mathematically. One may study F-theory vacua and CHL string backgrounds in the presence of additional structure. Our framework then allows us to give explicit parameterizations for F-theory vacua and CHL string backgrounds in the presence of an additional involution. For instance, a natural 6-dimensional subspace that is contained simultaneously in both aforementioned physical moduli spaces is the moduli space of K3 surfaces polarized by the lattice \(H \oplus N \oplus D_4(-2)\). This is a subspace where the F-theory admits an additional anti-symplectic involution, and, on the CHL string side, one has an additional a symplectic involution. We derive an explicit parametrization for elements of the moduli space using an auxiliary genus-one curve. This parametrization is based on a construction of André Weil [68], in which the Abel-Jacobi map is used to obtain embeddings of genus-one curves as symmetric divisors of bi-degree \((2, 2)\) in \(\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1\). In fact, we show that the algebraic correspondences for the F-theory backgrounds and vacua of the CHL string are double-quadrics surfaces obtained from double covers \(\mathbb{F}_0\), where the branching divisor consists of a symmetric divisor of bi-degree \((2, 2)\) and an additional collection of lines.
The F-theory moduli space has another natural 6-dimensional subspace, namely the moduli space of K3 surfaces polarized by the lattice $\langle 2 \rangle \oplus \langle -2 \rangle \oplus D_4 (-1)^{\otimes 3}$. This special situation corresponds to the case when, on the F-theory side, surfaces admit an additional symplectic involution and, on the CHL string side, a special anti-symplectic involution exist. The K3 surfaces associated with the CHL string in the situation above also carry a beautiful geometric description: they are special double-sixtic surfaces, i.e., they can be obtained as minimal resolutions from double covers of the projective plane branched over a configuration of three distinct lines coincident in a point and an additional generic cubic. The latter divisor gives rise to an elliptic curve capturing part of the K3 moduli coordinates. This fact relates the current work to previous work by the authors in [11].

It is important to note that, in the two special examples described above, both involving 6-dimensional subspaces of $\mathcal{M}_{H \oplus N}$, an elliptic curve naturally emerges. And this is not the elliptic curve from which the CHL string is constructed, but rather a Seiberg-Witten type curve that parameterizes certain moduli of the F-theory/CHL vacua under consideration.

This article is structured as follows: in Section 2 we review the physics of the duality between F-theory and the CHL string. In Section 3 we give a construction for families of lattice polarized K3 surfaces with canonical anti-symplectic and symplectic involution, respectively. In Section 4 we prove that the duality between F-theory with discrete flux and the CHL string in eight dimensions determines certain algebraic correspondences between pairs of K3 surfaces polarized by the rank-ten lattices $H \oplus N$ and $H \oplus E_8 (-2)$, respectively. In Section 5, we restrict our attention to the moduli space of K3 surfaces polarized by the lattice $H \oplus N \oplus D_4 (-2)$, which is contained simultaneously in both $\mathcal{M}_{H \oplus N}$ and $\mathcal{M}_{H \oplus E_8 (-2)}$. We derive an explicit classification of these surfaces, using an auxiliary genus-one curve. In Section 6 we investigate a special region of the F-theory moduli space corresponding to K3 surfaces principally polarized by the lattice $\langle 2 \rangle \oplus \langle -2 \rangle \oplus D_4 (-1)^{\otimes 3}$. We show that (a finite covering of) this 6-dimensional moduli space is canonically isomorphic to the moduli space of 8-dimensional simple abelian varieties with quaternionic endomorphism rings. Some concluding remarks are included in Section 7.

This article is based on several prior papers by the authors and their collaborators [5, 9, 10, 14–21, 28, 49–52], as well as several other works [11, 26, 27, 33–36, 38, 39, 45–47, 53, 54, 58, 61].

2. The duality between F-theory and the CHL string

Compactifications of the type-IIB string theory in which the complex coupling varies over a base are generically referred to as F-theory [55,56]. One of the simplest F-theory construction corresponds to K3 surfaces that are elliptically fibered over $\mathbb{P}^1$, in physics equivalent to a type-IIB string theory compactified on $\mathbb{P}^1$ and hence eight-dimensional in the presence of 7-branes and Wilson lines [50]. In this way, an elliptically fibered K3 surface with section and fiber $F = C/ (\mathbb{Z} \oplus \mathbb{Z} \tau)$ defines an F-theory vacuum in eight dimensions where the complex-valued scalar axio-dilaton
field $\tau$ of the type-IIB string theory is allowed to be multi-valued and undergo monodromy transformations in $\text{SL}(2, \mathbb{Z})$ when encircling defects of co-dimension one. The Kodaira-table of singular fibers \[43\] gives the precise dictionary between characteristics of the elliptic fibration and the content of the 7-branes present in the physical theory and the local monodromy of $\tau$. It is well-known that the moduli space of these F-theory models is isomorphic to the moduli space of the heterotic string compactified on a two-torus $T^2$ – equipped with a complex structure and complexified K"ahler modulus – together with a principal $G$-bundle where $G$ is the gauge group of the heterotic string, i.e., $G = E_8 \times E_8 \times \mathbb{Z}_2$ or $\text{Spin}(32)/\mathbb{Z}_2$ or a subgroup of these \[7, 29, 30, 64\]. In fact, the moduli spaces for these physical theories are given by the same Narain space which is the quotient of the symmetric space for $O(2,18)$ by a particular arithmetic group \[57\]. This is the basic form of the F-theory/heterotic string duality.

In eight dimensions one can also consider the CHL string \[4, 48, 69\]. The CHL string is obtained from an $E_8 \times E_8$-heterotic string as a $\mathbb{Z}_2$-quotient. The quotient is obtained from a specific involution, called the CHL involution: the CHL involution $\iota_{\text{CHL}}$ acts by a half-period shift on the elliptic curve (obtained from the two-torus $T^2$ equipped with the given complex structure), acts trivially on the complex K"ahler modulus, and permutes the two $E_8$’s of the gauge bundle. The moduli space for the CHL string compactified on the elliptic curve is then obtained as follows: first, the Narain construction yields the moduli space of the $E_8 \times E_8$-heterotic string compactified on $T^2$ as the double coset space

\[
(2.1) \quad \text{O(Nar)} \backslash \text{O(Nar} \oplus \mathbb{R})/K,
\]

where $\text{Nar} = H \oplus H \oplus E_8(-1) \oplus E_8(-1)$ is the Narain lattice and $K \subset \text{O(Nar} \oplus \mathbb{R})$ is a maximal compact subgroup. The CHL involution $\iota_{\text{CHL}} : \text{Nar} \to \text{Nar}$ acts on the Narain lattice as the identity on $H \oplus H$ while interchanging the two summands of $E_8(-1)$. It follows that the invariant sublattice of the Narain lattice is $\text{Nar}^{(\iota_{\text{CHL}})} \cong H \oplus H \oplus E_8(-2)$. The restriction of (2.1) to the corresponding $\iota_{\text{CHL}}$-invariant symmetric space thus gives rise to a moduli space of K3 surfaces, namely the moduli space of K3 surfaces with a transcendental lattice isometric to $H \oplus H \oplus E_8(-2)$; its complement in the K3 lattice $\Lambda_{K3} \cong H^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ is isomorphic to $H \oplus E_8(-2)$. This suggests, that the moduli space of the CHL string is the 10-dimensional moduli space of K3 surfaces polarized by the lattice $H \oplus E_8(-2)$.

Witten analyzed the F-theory compactifications dual to the CHL string in the case of an isotrivial elliptic fibration on the F-theory background \[69\]: he found that the CHL string in eight dimensions is dual to an F-theory with non-zero flux of an antisymmetric two-form field, or $B$-field, along the base curve $\mathbb{P}^1$. The value of this flux is quantized and fixed to half the K"ahler class of $\mathbb{P}^1$. This picture was extended in \[4\] to the interior of the Narain moduli space of toroidal compactifications where the elliptic fibrations of the F-theory models are no longer isotrivial, but the essential features of Witten’s description are still valid. The presence of the flux freezes eight of the moduli in the physical moduli space, leaving a 10-dimensional moduli space.

On this moduli space, the single-valued background of an antisymmetric two-form in the physical theory is not compatible with a monodromy of the half-periods of
an elliptic fiber in $\text{SL}(2,\mathbb{Z})$ around the defects. Rather it must be contained in a subgroup of $\text{SL}(2,\mathbb{Z})$ that keeps the flux of the $B$-field invariant, the biggest possible being the congruence subgroup $\Gamma_0(2)$. Thus, for a consistent model one has to start with an elliptically fibered K3 surface with section and a monodromy group contained in $\Gamma_0(2)$ that keeps one of the three half-periods of the elliptic fibers invariant. The corresponding Weierstrass model has eight fibers of type $I_2$ and two sections, the zero-section and a 2-torsion section, arranged in such a way that at each reducible fiber of type $A_1$ the two sections pass through two different components of the fiber. In physics this fibration is called the $\Gamma_0(2)$ elliptic fibration [4], since the monodromy group is $\Gamma_0(2)$. In mathematics, it is called the alternate fibration.

The identification of points in the physical moduli space corresponding to a specific non-abelian gauge group of the CHL string is then based on a second elliptic fibration with two fibers of type $I_0^*$, called the inherited elliptic fibration in physics [4]. The correct F-theory compactification does not admit a section but only a bi-section due to the precise nature of the duality with the CHL string. In fact, Witten argued that the moduli space of F-theory compactifications dual to the CHL string is naturally isomorphic to the moduli space of K3 surfaces obtained as genus-one fibrations with two fibers of type $I_0^*$ and a bisection [69].

3. Constructions of K3 surfaces

3.1. Notions for lattice polarized K3 surfaces. We will use the following standard notations: $L_1 \oplus L_2$ is orthogonal sum of the two lattices $L_1$ and $L_2$, $L(\lambda)$ is obtained from the lattice $L$ by multiplication of its form by $\lambda \in \mathbb{Z}$, $\langle R \rangle$ is a lattice with the matrix $R$ in some basis; $A_n$, $D_m$, and $E_k$ are the positive definite root lattices for the corresponding root systems, $H$ is the unique even unimodular hyperbolic rank-two lattice, and $N$ is the negative definite rank-eight Nikulin lattice (for a definition, see [54, Sec. 5]). For a lattice $L$ with a primitive embeddings $\iota : L \hookrightarrow \Lambda_{K3}$ into the K3 lattice $\Lambda_{K3} \cong H^{\otimes 3} \oplus E_8(-1)^{\otimes 2}$, Dolgachev proved that there exists a coarse moduli space $\mathcal{M}_L$ of pseudo-ample $L$-polarized K3 surfaces, i.e., a moduli space of algebraic K3 surfaces $\mathcal{X}$ that are polarized by the lattice $L$ such that $\iota(L)$ contains a numerically effective class of positive self-intersection in the Néron-Severi lattice $\text{NS}(\mathcal{X})$. Dolgachev also established a version of mirror symmetry for the moduli spaces of lattice polarized K3 surfaces in [23]: given a lattice $L$ with the property that for any two primitive embeddings $\iota_1, \iota_2 : L \hookrightarrow \Lambda_{K3}$ there is an isometry $g \in O(\Lambda_{K3})$ such that $\iota_2 = g \circ \iota_1$, the isomorphism class of $\bar{L}$ for a fixed splitting $L^\perp = H \oplus \bar{L}$ of the orthogonal complement $L^\perp \subset \Lambda_{K3}$ is well defined. The mirror moduli space of $\mathcal{M}_L$ is then taken to be $\mathcal{M}_{\bar{L}}$.

For a lattice $L$ we also denote the discriminant group by $A_L = L^*/L$ and its discriminant form by $q_L$. A lattice $L$ is called 2-elementary if $A_L$ is a 2-elementary abelian group, i.e., $A_L \cong (\mathbb{Z}/2\mathbb{Z})^\ell$ such that $\ell = \ell(L)$ is the minimal number of generators of the discriminant group $A_L$, also called the length of the lattice $L$. Even, indefinite, 2-elementary lattices $L$ are uniquely determined by the rank $\rho(L)$, the length $\ell(L)$, and the parity $\delta(L)$ – which equals 1 unless $q_L(x)$ takes values in $\mathbb{Z}/2\mathbb{Z} \subset \mathbb{Q}/2\mathbb{Z}$ for all $x \in A_L$ in which case it is 0; this is a result by Nikulin [60, Thm. 4.3.2].
An important tool in relating families of K3 surfaces are Nikulin involutions. Recall that a Nikulin involution \([54, 59]\) is an involution \(\iota_X : X \to X\) on a K3 surface \(X\) that satisfies \(\iota_X^* (\omega) = \omega\) for any holomorphic two-form \(\omega\) on \(X\). The minimal resolution of the quotient \(X/\langle \iota_X \rangle\) is another K3 surface \(X'\), and the quotient map induces a two-to-one rational map \(\Phi : X \dashrightarrow X'\) whose branch locus is an even set of eight rational curves on \(X'\). Recall that a set of eight rational curves on a K3 surface is called an even eight if the sum is divisible by two in the Néron-Severi lattice; see \([3]\).

For a Jacobian elliptic surface \(X\) we denote the projection by \(\pi_X : X \to \mathbb{P}^1\), the zero-section by \(\sigma_X\), and the Mordell-Weil group of sections by \(\text{MW}(X, \pi_X)\) with its torsion subgroup denoted by \(\text{MW}(X, \pi_X)_{\text{tor}}\). A complete list of the possible singular fibers in a Weierstrass model was determined by Kodaira \([43]\): it encompasses two infinite families \((I_n, I_n^*, n \geq 0)\) and six exceptional cases \((II, III, IV, II^*, III^*, IV^*)\). It is well known that the Mordell-Weil lattice \(\text{MW}(X, \pi_X)_{\text{tor}} \subset \text{NS}(X)\) is isomorphic to the dual lattice \(S^\vee\) of the narrow Mordell-Weil lattice \(S\) of all sections which meet the zero component of every fiber. In particular, we have a direct sum decomposition
\[
\text{MW}(X, \pi_X) \cong \text{MW}(X, \pi_X)_{\text{tor}} \oplus S^\vee,
\]
obtained with respect to the height pairing of sections. The span of the cohomology classes associated with the elliptic fiber and the section \(\sigma_X\) induces a sublattice of the Néron-Severi lattice \(\text{NS}(X)\) isomorphic to \(H\). This sublattice determines uniquely the Jacobian fibration. Conversely, a pseudo-ample lattice embedding \(H \hookrightarrow \text{NS}(X)\) determines – up to the action of Hodge isometries of \(H^2(X, \mathbb{Z})\) – a unique isomorphism class of a Jacobian elliptic fibration on \(X\). Let \(T\) be the trivial lattice generated by all components of the reducible fibers of the elliptic fibration \(\pi_X : X \to \mathbb{P}^1\) not meeting the zero section \(\sigma_X\) (which is unique up to permutation). We have the following:

**Lemma 3.1.** For a Jacobian elliptic surface \(\pi_X : X \to \mathbb{P}^1\) with trivial lattice \(T\) and
\[
\text{MW}(X, \pi_X) \cong \text{MW}(X, \pi_X)_{\text{tor}} \oplus S^\vee,
\]
there is an isomorphism
\[
\text{NS}(X) \cong H \oplus T' \oplus S^\vee,
\]
where \(T'\) is an overlattice of \(T\) with index equal to the order of the torsion subgroup \(\text{MW}(X, \pi_X)_{\text{tor}}\).

**Proof.** The result of Kodaira implies that the trivial lattice is the root lattice of the lattice \(W\) obtained as the orthogonal complement of \(H\) in \(\text{NS}(X)\). Shioda then proved that there is a canonical group isomorphism
\[
\text{MW}(X, \pi_X) \cong W/T,
\]
induced by taking the horizontal part of a divisor on an elliptic surface. Moreover, if \(n\) is the order of the torsion subgroup \(\text{MW}(X, \pi_X)_{\text{tor}}\) then \(n^2\) must divide \(\det q_r\). Thus, \(T'\) is an overlattice of \(T\) of an index \(n\). The claim then follows. \(\square\)

**3.2. K3 surfaces double covering a rational elliptic surface.** For two cubics \(p, q\) in \(\mathbb{P}^2 = \mathbb{P}(X, Y, Z)\) the cubic pencil
\[
R = \{ U p(X, Y, Z) + V q(X, Y, Z) = 0 \} \subset \mathbb{P}^2 \times \mathbb{P}^1
\]
defines an elliptic surface with section \( \pi_{\mathcal{R}} : \mathcal{R} \to \mathbb{P}^1 = \mathbb{P}(U, V) \) which is isomorphic to \( \mathbb{P}^2 \) blown up in the nine base points of the pencil. The exceptional divisors are not contained in the fibers but yield independent sections. Selecting one of them as the zero section \( \sigma_{\mathcal{R}} \), the remaining eight generate a lattice of type \( E_8 \). The minimal resolution gives rise to a rational elliptic surface (RES). By the Shioda-Tate formula the rank of MW(\( \mathcal{R}, \pi_{\mathcal{R}} \)) drops if there are reducible fibers. It turns out that this characterization is complete [22, Thm. 5.6.1], and every rational elliptic surface \( \mathcal{R} \) is isomorphic to the blow-up of \( \mathbb{P}^2 \) in the base points of a cubic pencil.

The general rational elliptic surface with section \( \pi_{\mathcal{R}} : \mathcal{R} \to \mathbb{P}^1 = \mathbb{P}(U, V) \) has the Weierstrass model

\[
\mathcal{R} : \quad Y^2 Z = X^3 + f(U, V) X Z^2 + g(U, V) Z^3,
\]

where \( f, g \) are homogenous of degree four and six, respectively, such that the fibration has twelve singular fibers of type \( I_1 \) (that is, \( 4f^3 + 27g^2 = 0 \) has twelve distinct roots), and the Mordell-Weil group is \( \text{MW}(\mathcal{R}, \pi_{\mathcal{R}}) \cong E_8 \). Here, the Mordell-Weil group can also read off in the classification of all rational elliptic surfaces given by Oguiso and Shioda in [62]. The rational elliptic surface \( \mathcal{R} \) with the singular fibers \( 12I_1 \) and \( \text{MW}(\mathcal{R}, \pi_{\mathcal{R}}) \cong E_8 \) is referred to as (1) in Oguiso-Shioda classification.

A family of K3 surfaces is obtained by a base change of order two on the rational elliptic fibration \( \pi_{\mathcal{R}} \). A double cover \( h_{(d_0, d_\infty)} : \mathbb{P}^1 = \mathbb{P}(u, v) \to \mathbb{P}^1 = \mathbb{P}(U, V) \) is determined by choosing two points \([u, v] = [0 : 1], [1, 0] \) as the branch points with images \([U, V] = [d_0 : 1], [1, d_\infty] \) with \( d_0 d_\infty \neq 1 \), and an additional point, say \([u, v] = [1 : 1] \) with image \([U, V] = [1 : 1] \). The double cover map is then given by

\[
(3.6) \quad h_{(d_0, d_\infty)} : \quad [u : v] \mapsto [U : V] = \left[\frac{(1-d_0)u^2+d_0(1-d_\infty)v^2}{d_\infty(1-d_0)u^2+(1-d_\infty)v^2} : \frac{d_\infty(1-d_0)u^2+(1-d_\infty)v^2}{(1-d_0)u^2+d_0(1-d_\infty)v^2}\right].
\]

Here, we are assuming that the fibers of (3.5) over the points \([d_0 : 1], [1, d_\infty], [1 : 1] \) are not singular. Geometrically, \( h_{(d_0, d_\infty)} \) determines a line bundle \( \mathcal{L}^2 = \mathcal{O}_{\mathbb{P}^1} \). Pulling back the Weierstrass model in Equation (3.5), we obtain a Jacobian elliptic K3 surfaces \( \pi_\mathcal{Y} : \mathcal{Y} \to \mathbb{P}^1 = \mathbb{P}(u, v) \) given by

\[
(3.7) \quad \mathcal{Y} : \quad y^2 z = x^3 + F(u, v) x z^2 + G(u, v) z^3,
\]

where we have set \( F = h^*_{(d_0, d_\infty)} f \) and \( G = h^*_{(d_0, d_\infty)} g \). We have the following:

**Lemma 3.2.** The general K3 surface \( \mathcal{Y} \) in Equation (3.7) has 24 singular fibers of type \( I_1 \) and a Mordell-Weil group \( \text{MW}(\mathcal{Y}, \pi_\mathcal{Y}) \cong E_8(2) \).

**Proof.** In the formula for the height pairing on an elliptically fibered surface with section all summands – the holomorphic Euler characteristic, the intersection number, the number of singular fibers – double when pulling back the Weierstrass equation via Equation (3.6).

In particular, the general K3 surface \( \mathcal{Y} \) has \( \text{NS}(\mathcal{Y}) \cong H \oplus E_8(-2) \). Conversely, it is known that every element in \( \mathfrak{M}_{H} \oplus E_8(-2) \) is represented by a family of double covers of the plane branched over the union of cubics, i.e., double covers of the rational elliptic surface obtained by the minimal resolution of a reducible double-sixt c blown-up in nine points [23, Sec. 9.1]. To find the moduli from the Weierstrass model in
Equation (3.7), we observe that $f$ and $g$ depend on $5 + 7 = 12$ parameters, and there are two additional parameters $d_0, d_\infty$. Using a transformation of the type
\begin{equation}
\left( (u : v), (x : y : z) \right) \mapsto \left( (\lambda u : \lambda v), (\lambda^2 x : \lambda^3 y : z) \right)
\end{equation}
with $\lambda \in \mathbb{C}^*$ and the automorphism group $\text{PGL}(2, \mathbb{C})$ of $\mathbb{P}^1$ we obtain $12 + 2 - 1 - 3 = 10$ as the number of moduli. In fact, the following was proved in [6, Prop. 4.17]:

**Proposition 3.3.** The K3 surfaces in Equation (3.7) form the 10-dimensional moduli space $\mathcal{M}_{H \oplus D_4(-2)}$.

Conversely, the rational elliptic surface $\mathcal{R}$ is the minimal resolution of the quotient $\mathcal{Y}/\langle k_\mathcal{Y} \rangle$ where $k_\mathcal{Y}$ is the antisymplectic involution preserving the Jacobian elliptic fibration induced by the deck transformation $[u : v] \mapsto [-u : v]$. A Nikulin involution $j_\mathcal{Y} = k_\mathcal{Y} \circ (-1)$ is given by the composition of $k_\mathcal{Y}$ with the (antisymplectic) hyperelliptic involution acting as $(-1)$ on the fibers. The minimal resolution of $\mathcal{Y}/\langle j_\mathcal{Y} \rangle$ then yields another Jacobian elliptic K3 surface $\pi_\mathcal{Y} : \tilde{\mathcal{Y}} \to \mathbb{P}^1 = \mathbb{P}(U, V)$ with the Weierstrass model
\begin{equation}
\tilde{\mathcal{Y}} : \quad Y^2Z = X^3 + (U - d_0V)^2(d_\infty U - V)^2 f(U, V) XZ^2 + (U - d_0V)^3(d_\infty U - V) g(U, V) Z^3.
\end{equation}
We have the following:

**Lemma 3.4.** The general K3 surface $\tilde{\mathcal{Y}}$ in Equation (3.9) has 2 singular fibers of type $I_0^+$, 12 singular fibers of type $I_1$, and a trivial Mordell-Weil group.

Conversely, starting with a K3 surface $\tilde{\mathcal{Y}}$ with the given singular fibers and Mordell-Weil group, a double cover is obtained as the pull-pack via the degree-2 rational map branched over an obvious even eight, namely the even eight given by the non-central components of the two reducible fibers of type $D_4$. We have the following:

**Proposition 3.5.** The K3 surfaces in Equation (3.9) form the 10-dimensional moduli space $\mathcal{M}_{H \oplus D_4(-1)^{\oplus 2}}$.

The results of Proposition 3.3 and Proposition 3.5 also hold (for suitably modified lattices) if we restrict $\mathcal{R}$ to a rational elliptic surface with reducible fibers given in the classification by Oguiso and Shioda in [62]. Geometrically, these rational elliptic surfaces arise if the base points of the pencil in Equation (3.4) are not distinct. We will focus on the cases when the Mordell-Weil lattice of the rational elliptic surface has half the rank and is isomorphic to $D_4$ (up to torsion). That is, we will consider cases (9) or (13) in the Oguiso-Shioda classification [62], i.e., the rational elliptic surfaces with the singular fibers and Mordell-Weil groups given by
\begin{equation}
I_0^+ + 6I_1, \quad \text{MW}(\mathcal{R}, \pi_\mathcal{R}) \cong D_4^+, \quad 4I_2 + 4I_1, \quad \text{MW}(\mathcal{R}, \pi_\mathcal{R}) \cong \mathbb{Z}/2\mathbb{Z} \oplus D_4^+,
\end{equation}
respectively. We have the following:

**Corollary 3.6.**

1. The 6-dimensional moduli spaces $\mathcal{M}_L$ for $L = H \oplus D_4(-1)^{\oplus 2} \oplus D_4(-2)$ and $L = H \oplus N \oplus D_4(-2)$ are given by K3 surfaces $\mathcal{Y}$ in Equation (3.7) obtained by a base change of order two on the rational elliptic surfaces $\mathcal{R}$ labelled (9) and (13), respectively, in the Oguiso-Shioda classification [62].
(2) The 6-dimensional moduli spaces $\mathcal{M}_L$ for $L = H \oplus D_4(-1)^{\otimes 3}$ and $L = H \oplus D_8(-1) \oplus A_1(-1)^{\otimes 4}$ are given by K3 surfaces $\tilde{Y}$ in Equation (3.9) obtained as the minimal resolution of $\mathcal{Y}/(j_\mathcal{Y})$ where $j_\mathcal{Y}$ is the Nikulin involution constructed above and $\mathcal{Y}$ are chosen from the first and second family in (1), respectively. These cases are summarized in Table 1.

**Proof.** The proof of (1) follows by restricting Proposition 3.3 to cases (9) or (13) in the Oguiso-Shioda classification and Lemma 3.1. Part (2) follows by explicit construction using Equation (3.16). The fact that the lattice polarization for the second family is given by $L = H \oplus D_8(-1) \oplus A_1(-1)^{\otimes 4}$ follows from results in [12]. □

The importance of the subfamilies above stems from their geometric interpretation: the two families in Corollary 3.6 (1) can be understood as imposing the existence of another compatible involution (in addition to the antisymplectic involution $k_\mathcal{Y}$ already constructed). We consider the two cases when this additional involution, compatible with the elliptic fibration, is antisymplectic or symplectic.

Let us assume that the Weierstrass model in Equation (3.7) admits a second commuting antisymplectic involution preserving the Jacobian elliptic fibration. We use a transformation in PGL$(2, \mathbb{C})$ to have $d_0 = d_\infty = 0$. We can also assume that the second antisymplectic involution is induced by the deck transformation $[u : v] \mapsto [v : u]$. Thus, without loss of generality, a family of K3 surfaces $\mathcal{Y}$ admitting two commuting antisymplectic involutions preserving the Jacobian elliptic fibration is given by the Weierstrass model

$$
\mathcal{Y}' : Y^2 Z = X^3 + f((u^2 - \bar{v})^2, (u^2 + \bar{v})^2) X Z^2 + g((u^2 - \bar{v})^2, (u^2 + \bar{v})^2) Z^3,
$$

where $f$ and $g$ are homogenous of degree 2 and 3, respectively, and we changed the variables from $(u, v)$ to $(\bar{u}, \bar{v})$ to avoid any conflict of notation. To find the moduli from the Weierstrass model in Equation (3.11), we observe that $f$ and $g$ depend on $3 + 4 = 7$ parameters. With only the freedom of a transformation as in Equation (3.8) remaining, we obtain $7 - 1 = 6$ as the number of moduli in Equation (3.11). Thus, we have the following:

**Proposition 3.7.** The K3 surfaces in Equation (3.11) form a 6-dimensional subvariety of $\mathcal{M}_{H \oplus E_8(-2)}$ whose general member admits two commuting antisymplectic involutions preserving the Jacobian elliptic fibration.

The minimal resolution of $\mathcal{Y}'/(j_1)$ (where $k_1$ is induced by $[\bar{u} : \bar{v}] \mapsto [\bar{u} : \bar{v}]$ and $j_1 = k_1 \circ (-1)$) yields the Jacobian elliptic K3 surface $\pi_\mathcal{Y} : \mathcal{Y} \rightarrow \mathbb{P}^1 = \mathbb{P}(u, v)$ with the Weierstrass model

$$
\mathcal{Y} : y^2 z = x^3 + (u^2 - \bar{v})^2 f(u^2, v^2) x z^2 + (u^2 - \bar{v})^3 g(u^2, v^2) z^3.
$$

The image of the (second) involution $k_2$ on $\mathcal{Y}'$ (where $k_2$ is induced by $[\bar{u} : \bar{v}] \mapsto [\bar{u} : \bar{u}]$) is an antisymplectic involution $k_\mathcal{Y}$ preserving the Jacobian elliptic fibration on $\mathcal{Y}$ and is induced by $[u : v] \mapsto [-u : v]$. For the Nikulin involution $j_\mathcal{Y} = k_\mathcal{Y} \circ (-1)$ (which is the image of $k_2 = k_\mathcal{Y} \circ (-1)$), the minimal resolution of $\mathcal{Y}'/(j_\mathcal{Y})$ yields another Jacobian elliptic K3 surface $\pi_{\tilde{\mathcal{Y}}} : \tilde{\mathcal{Y}} \rightarrow \mathbb{P}^1 = \mathbb{P}(U, V)$ with the Weierstrass model

$$
\tilde{\mathcal{Y}} : Y^2 Z = X^3 + U^2 V^2 (U - V)^2 f(U, V) X Z^2 + U^3 V^3 (U - V)^3 g(U, V) Z^3.
$$
Similarly, one checks that the minimal resolution of $\mathcal{Y}/\langle k_3 \rangle$ yields the rational elliptic surface with the Weierstrass model

\begin{equation}
Y^2Z = X^3 + (U - V)^2f(U, V)XZ^2 + (U - V)^3g(U, V)Z^3,
\end{equation}

which realizes the first case in (3.10). Thus, we have the following:

**Proposition 3.8.** The K3 surfaces in Equation (3.12) and (3.13) are the minimal resolutions of $\mathcal{Y}/\langle j_1 \rangle$ and $\mathcal{Y}/\langle j_1, j_2 \rangle$ and form the 6-dimensional moduli spaces $\mathfrak{M}_L$ for $L = H \oplus D_4(-1) \oplus D_4(-2)$ and $L = H \oplus D_4(-1) \oplus D_4(-3)$, respectively.

As for the second case in (3.10), we remark that the existence of a 2-torsion section on $R$ guarantees the existence of a 2-torsion section on $\mathcal{Y}$ in Equation (3.7). In turn, the 2-torsion section yields an additional symplectic involution on $\mathcal{Y}$ known as van Geemen-Sarti involution – we will explain the construction of a van Geemen-Sarti involution in detail in Section 3.3.

### 3.3. K3 surfaces with van Geemen-Sarti involution

When a K3 surface $\mathcal{X}$ admits a Jacobian elliptic fibration with a 2-torsion section, then it admits a special Nikulin involution, called a van Geemen-Sarti involution; see [66]. When quotienting by this involution, denoted by $j_{\mathcal{X}}$, and blowing up the fixed locus, one obtains a new K3 surface $\mathcal{X}'$ together with a rational double cover map $\Phi_{\mathcal{X}}: \mathcal{X} \to \mathcal{X}'$. In general, a van Geemen-Sarti involution $j_{\mathcal{X}}$ does not determine a Hodge isometry between the transcendental lattices $T_{\mathcal{X}}(2)$ and $T_{\mathcal{X}'}$. However, van Geemen-Sarti involutions always appear as fiber-wise translation by 2-torsion in a suitable Jacobian elliptic fibration $\pi_{\mathcal{X}}: \mathcal{X} \to \mathbb{P}^1$ which we call the alternate fibration; see [20] for the nomenclature. In the Mordell-Weil group $\text{MW}(\mathcal{X}, \pi_{\mathcal{X}})$ with identity element $\sigma_{\mathcal{X}}$, let $\tau_{\mathcal{X}}$ denote the 2-torsion section such that translation by $\tau_{\mathcal{X}}$ is the involution $j_{\mathcal{X}}$. Moreover, the construction induces a Jacobian elliptic fibration $\pi_{\mathcal{X}'}: \mathcal{X}' \to \mathbb{P}^1$ on $\mathcal{X}'$ which also admits a 2-torsion section. Thus, we obtain Figure 1. We will refer to the construction of Figure 1 as van Geemen-Sarti-Nikulin duality.

The K3 surface $\mathcal{X}$ has the Weierstrass equation

\begin{equation}
\mathcal{X}: \quad Y^2Z = X \left( X^2 - A(s, t)XZ + B(s, t)Z^2 \right),
\end{equation}

| $\rho$ | surface | polar. lattice $L$ | sing. fibers | Mordell-Weil grp. |
|-------|---------|--------------------|--------------|------------------|
| 10 K3 | $\mathcal{Y}$ | $H \oplus E_8(-2)$ | $24I_1$ | $E_8(2)$ |
| 10 K3 | $\mathcal{Y}$ | $H \oplus D_4(-1) \oplus \mathbb{Z}$ | $2I_0^* + 12I_1$ | $\{1\}$ |
| RES: (1) | $R$ | – | $12I_1$ | $E_8$ |
| 14 K3 | $\mathcal{Y}$ | $H \oplus D_4(-1) \oplus D_4(-2)$ | $2I_0^* + 12I_1$ | $D_4^*(2)$ |
| 14 K3 | $\mathcal{Y}$ | $H \oplus D_4(-1) \oplus D_4(-3)$ | $3I_0^* + 6I_1$ | $\{1\}$ |
| RES: (9) | $\mathcal{R}$ | – | $I_0^* + 6I_1$ | $D_4^*$ |
| 14 K3 | $\mathcal{Y}$ | $H \oplus D_4(-1) \oplus A_1(-1) \oplus D_4(-2)$ | $8I_2 + 8I_1$ | $\mathbb{Z}/2\mathbb{Z} \oplus D_4^*(2)$ |
| 14 K3 | $\mathcal{Y}$ | $H \oplus D_4(-1) \oplus A_1(-1) \oplus D_4(-3)$ | $2I_0^* + 4I_2 + 4I_1$ | $\mathbb{Z}/2\mathbb{Z}$ |
| RES: (13) | $\mathcal{R}$ | – | $4I_2 + 4I_1$ | $\mathbb{Z}/2\mathbb{Z} \oplus D_4^*$ |

**Table 1.** K3 lattices and Jacobian elliptic fibrations in Prop. 3.3-3.5
where \([s : t] \in \mathbb{P}^1\), \([X : Y : Z] \in \mathbb{P}^2\), \(A(s, t)\) and \(B(s, t)\) are homogeneous polynomials of degree four and eight, respectively, and the sections \(\sigma_{\mathcal{X}}, \tau_{\mathcal{X}}\) are given by the section at infinity and \([X : Y : Z] = [0 : 0 : 1]\). To find the moduli from the Weierstrass model in Equation (3.15), we note that \(A\) and \(B\) depend on \(5 + 9 = 14\) parameters. Using transformations of the type \((X, Y) \mapsto (\lambda^2 X, \lambda^3 Y)\) with \(\lambda \in \mathbb{C}^\times\) and the automorphism group \(\text{PGL}(2, \mathbb{C})\) of \(\mathbb{P}^1\) we get \(14 - 1 - 3 = 10\) moduli. Since we have identified coordinates according to

\[
\left( (s, t), (X, Y, Z) \right) \sim \left( (\lambda s, \lambda t), (\lambda^4 X, \lambda^6 Y, Z) \right),
\]

Equation (3.15) defines a double cover of the Hirzebruch surface \(\mathbb{F}_4\). Similarly, the K3 surface \(\mathcal{X}'\) has the Weierstrass model

\[
(3.16) \quad \mathcal{X}' : \quad y^2 z = x \left( x^2 + 2A(s, t) xz + (A(s, t)^2 - 4B(s, t)) z^2 \right).
\]

Explicit equations for the rational maps \(\Phi_{\mathcal{X}}\) and \(\Phi_{\mathcal{X}'}\) in Figure 1 were given in [20]. The discriminant functions of the two elliptic fibrations are as follows:

\[
(3.17) \quad \Delta_{\mathcal{X}} = B(s, t)^2 \left( A(s, t)^2 - 4B(s, t) \right), \quad \Delta_{\mathcal{X}'} = 16B(s, t) \left( A(s, t)^2 - 4B(s, t) \right)^2.
\]

**Lemma 3.9.** The general K3 surface \(\mathcal{X}\) in Equation (3.15) and \(\mathcal{X}'\) in Equation (3.16) have 8 fibers of type \(I_1\) over the zeroes of \(A^2 - 4B = 0\) (resp. \(B = 0\)) and 8 fibers of type \(I_2\) over the zeroes of \(B = 0\) (resp. \(A^2 - 4B = 0\)) with \(\text{MW}(\mathcal{X}, \pi_{\mathcal{X}}) \cong \text{MW}(\mathcal{X}', \pi_{\mathcal{X}'}) \cong \mathbb{Z}/2\mathbb{Z}\).

In particular, the general Jacobian elliptic K3 surfaces \(\mathcal{X}\) in Equation (3.15) and \(\mathcal{X}'\) in Equation (3.16) have

\[
(3.18) \quad \text{NS}(\mathcal{X}) \cong \text{NS}(\mathcal{X}') \cong H \oplus N, \quad T_{\mathcal{X}} \cong T_{\mathcal{X}'} \cong H^2 \oplus N. \quad \text{Thus, we have the following:}
\]

**Proposition 3.10.** The K3 surfaces in Equation (3.15) form the 10-dimensional moduli space \(\mathcal{M}_{H \oplus N}\).

Geometrically, the K3 surface \(\mathcal{X}\) is a double cover of \(\mathcal{X}'\) (via the rational map \(\Phi_{\mathcal{X}}\)) branched over the even eight that consists of the eight components of the fibers of type \(I_2\) that are not met by the zero section \(\sigma_{\mathcal{X}'}\), i.e., the eight exceptional curves in the corresponding reducible fibers of type \(A_1\). The other components, which meet \(\sigma_{\mathcal{X}'}\) map 2:1 to components in \(\mathcal{X}\) where they are interchanged and also the two singular points of the translation by \(\tau_{\mathcal{X}}\) are the eight nodes in the \(I_1\)-type fibers, blowing them up gives \(I_2\)-type fibers in \(\mathcal{X}'\). Similarly, the K3 surface \(\mathcal{X}'\) is a double cover of \(\mathcal{X}\) (via the
rational map \( \Phi_{X'} \). The eight exceptional curves \( C_1, C_2, \ldots, C_8 \) resulting on \( X \) from the singularity resolution form an even-eight configuration \([3]\), i.e.

\[
\frac{1}{2} \left( C_1 + C_2 + \cdots + C_8 \right) \in \text{NS}(X).
\]

This configuration of eight curves whose formal sum is in \( 2\text{NS}(X) \) is known to determine in full the double cover \( \Phi_{X'} : X' \to X \); see \([3]\). Concretely, each reducible fiber in the Jacobian elliptic fibration \( \pi : X \to \mathbb{P}(s, t) \) consists of two components \( F_{i0} \) and \( F_{i1} \) such that

\[
F_{i0} \circ \sigma_X = 1, \quad F_{i0} \circ \tau_X = 0, \quad F_{i1} \circ \sigma_X = 0, \quad F_{i1} \circ \tau_X = 1,
\]

for \( i = 1, \ldots, 8 \). The van Geemen-Sarti involution interchanges \( F_{i0} \) and \( F_{i1} \) for \( i = 1, \ldots, 8 \). The eight curves \( F_{i1} \) for \( i = 1, \ldots, 8 \) form an even eight that is not met by the zero section \( \sigma_X \); see Figure 2. Thus, the double cover \( X' \) obtained from the double cover branched on \( F_{11} + \cdots + F_{81} \) is elliptically fibered with section \( \sigma_{X'} \) and the two-torsion section \( \tau_{X'} \); the two sections form the preimage of \( \sigma_X \) under \( \Phi_{X'} \).

The results of Proposition 3.10 and Figure 1 also hold for suitable lattice extensions induced by the existence of a second involution:

**Corollary 3.11.**

1. The 6-dimensional moduli space \( \mathcal{M}_{(2)\oplus(-2)\oplus D_4(-1)^{\oplus 3}} \subset \mathcal{M}_{H\oplus N} \) is given by the K3 surfaces admitting two commuting van Geemen-Sarti involutions.

2. The 6-dimensional moduli space \( \mathcal{M}_{H\oplus N\oplus D_4(-2)} \) is given by the K3 surfaces admitting a canonical van Geemen-Sarti involution and a commuting antisymplectic involution preserving the Jacobian elliptic fibration.

**Proof.** Part (1) is proved as follows: the existence of a second van Geemen-Sarti involution means that there is a second 2-torsion section. Its existence implies that there is a homogeneous polynomial \( C(s, t) \) of degree four such that \( A^2 - 4B = C^2 \). One checks that then there are now 12 singular fibers of type \( I_2 \) on \( X \) and the Mordell-Weil group is \((\mathbb{Z}/2\mathbb{Z})^2\). It follows from results in \([12]\) that the family of K3 surfaces is polarized by the lattice \( (2) \oplus (-2) \oplus D_4(-1)^{\oplus 3} \). For (2) we can assume that the antisymplectic involution \( k_X \) is induced by the deck transformation \( [s : t] \mapsto [-s : t] \).
on $\pi_X : X \to \mathbb{P}^1 \cong \mathbb{P}(s,t)$. A Nikulin involution $j_X = k_X \circ (-1)$ is then given by the composition with the (antisymplectic) hyperelliptic involution which acts as $(-1)$ on the fibers. In turn, this implies that the set of singular fibers of type $I_2$ and $I_1$ on $X$, respectively, is invariant under $j_X$, and we obtain the family of $H \oplus N \oplus D_4(-2)$-polarized K3 surfaces in Proposition 3.5. \hfill \Box

Remark 3.12. The van Geemen-Sarti-Nikulin dual K3 surface $X'$ in case (1) of Corollary 3.11 has the singular fibers $4I_4 + 8I_2$ and the Mordell-Weil group $\mathbb{Z}/2\mathbb{Z}$. In particular, it is not polarized by $(2) \oplus (-2) \oplus D_4(-1)^{\oplus 3}$.  

3.4. Relation to mirror symmetry. We argued in Section 2 that the moduli space of the CHL string is the moduli space $M_{H\oplus E_8(-2)}$. We also derived a normal form for these K3 surfaces relating them to certain rational elliptic surfaces. Dolgachev proved that $L = H(2) \oplus E_8(-2)$ diagonally embeds into $H^2 \oplus E_8(-1) \oplus E_8(-1)$ such that $L^4 = H \oplus H(2) \oplus E_8(-2)$ and a splitting $L^4 = H \oplus L$ is admissible \cite{dolgachev}. Thus, the moduli space $M_{H(2)\oplus E_8(-2)}$ is a self-mirror with respect to this splitting. However, a second admissible splitting is given by $L^4 = H(2) \oplus L'$ with $L' = H \oplus E_8(-2)$ and yields $M_{H\oplus E_8(-2)}$ as the mirror moduli space.

As we have seen, a polarization by the lattice $H \oplus N$ is equivalent with the existence of a canonical van Geemen-Sarti involution $j_X : X \to X$ on the K3 surface $X$. The construction in Figure 1 induces an involution on the moduli space $\iota_{\text{VGS}} : M_{H\oplus N} \to M_{H\oplus N}$. In the situation of Equation (3.18) one checks that for $L = H \oplus N$ and $L^4 = H^2 \oplus N$ the splitting $L^4 = H \oplus \tilde{L}$ with $\tilde{L} \cong L$ is admissible in the sense of Dolgachev’s mirror symmetry. Thus, the moduli space of the K3 surfaces given by Equation (3.15) is a self-mirror, i.e., $M_L \cong M_{L'}$ and the van Geemen-Sarti-Nikulin duality acts as involution on this moduli space. In fact, the action of $\iota_{\text{VGS}}$ can be identified with the action of mirror symmetry on the corresponding F-theory models.

4. The geometric description of the duality

In this section we will prove that the F-theory/CHL string duality can be understood as a relation between certain moduli spaces of lattice polarized K3 surfaces. The key observation is the following isometry of lattices.

For the K3 surface $X$ in Equation (3.15) we group the base points of the 8 fibers of type $I_2$ into two unordered sets of 4 elements. Concretely, we group the fibers of type $I_2$ over $B = 0$ into two sets by writing $B(s,t) = C(s,t)D(s,t)$ where $C, D$ are homogenous polynomials of degree four. The symplectic transformation given by

\[(X,Y,Z) \mapsto \left(U,V,W\right) = \left(C(s,t)XZ, Z^2, C(s,t)Y\right),\]

yields the equation

\[\mathcal{F} : W^2 = UV \left(C(s,t)U - A(s,t)UV + D(s,t)V^2\right).\]

Equation (4.2) puts the (marked) K3 surface into the equivalent form of a double-quadrics surface, i.e., a double cover of the Hirzebruch surface $\mathbb{P}_0 = \mathbb{P}^1 \times \mathbb{P}^1 \ni ([s : t], [U : V])$ branched over a curve of bi-degree $(4, 4)$. The ruling given by the projection onto the first factor, denoted by $\pi_{\mathcal{F}} : \mathcal{F} \to \mathbb{P}^1 = \mathbb{P}(s,t)$, recovers the Jacobian elliptic fibration in Equation (3.15).
Changing the ruling one recovers another elliptic fibration $\pi'_F : \mathcal{F} \to \mathbb{P}^1 = \mathbb{P}(U, V)$ without section. Concretely, the elliptic fibration $\pi'_F$ is obtained by re-writing Equation (4.2) in the form
\[(4.3) \quad \mathcal{F} : \quad W^2 = UV \left( a_4(U, V) s^4 + a_3(U, V) s^3 t + \cdots + a_0(U, V) t^4 \right),\]
where $a_i(U, V)$ for $0 \leq i \leq 4$ are homogeneous polynomial of degree two, obtained by requiring that we have
\[(4.4) \quad C(s, t) U^2 - A(s, t) UV + D(s, t) V^2 = a_4(U, V) s^4 + a_3(U, V) s^3 t + \cdots + a_0(U, V) t^4 \]
for all $s, t, U, V$. The relative Jacobian (fibration) $\text{Jac}^0(\pi'_F)$ is the Jacobian elliptic K3 surface $\tilde{\mathcal{Y}} : \tilde{\mathcal{Y}} \to \mathbb{P}^1 = \mathbb{P}(U, V)$ with the Weierstrass model
\[(4.5) \quad \tilde{\mathcal{Y}} : \quad y^2z = x^3 + U^2V^2 f(U, V) xz^2 + U^3V^3 g(U, V) z^3,\]
where $f, g$ are the homogeneous polynomials of degree four and six, respectively, given by
\[(4.6) \quad f = -4a_0a_4 + a_1a_3 - \frac{1}{3}a_2^2, \quad g = \frac{8}{3}a_0a_2a_4 + a_0a_3^2 + a_1^2a_4 - \frac{1}{3}a_1a_2a_3 + \frac{2}{27}a_3^3.\]
For a review and details of these classical formulas; see [20, 65, 68]. The general Jacobian elliptic K3 surface $\tilde{\mathcal{Y}}$ in Equation (4.5) has two singular fibers of type $I_6^*$, twelve singular fibers of type $I_1$, and a trivial Mordell-Weil group.

We also construct a K3 surface $\mathcal{G}$ as the double cover of $\mathcal{X}'$ branched over the even eight that consists of eight components in the reducible fibers over the zeros of $B(s, t) = C(s, t) D(s, t) = 0$. Using the same notation as the one used in Figure 2, we denote by $F_{i0}$ and $F_{i1}$ for $i = 1, \ldots, 4$ the components over $C(s, t) = 0$ and by $F_{i0}$ and $F_{i1}$ for $i = 5, \ldots, 8$ the components over $D(s, t) = 0$. The even eight we choose this time is different from the one in (3.19) used in the construction of $\mathcal{X}'$ in Equation (3.16). Over the zeros of $D(s, t) = 0$ we choose the components $\{F_{11}, \ldots, F_{41}\}$ of the reducible fibers that are not met by the zero section $\sigma_{\mathcal{X}}$. Over the zeros of $C(s, t) = 0$ we choose the components $\{F_{50}, \ldots, F_{80}\}$ that are met by the zero section $\sigma_{\mathcal{X}}$. In this way, the double cover branched on $F_{11} + \cdots + F_{41} + F_{50} + \cdots + F_{80}$ is an elliptic fibration $\pi'_{\mathcal{G}} : \mathcal{G} \to \mathbb{P}^1 = \mathbb{P}(s, t)$ with the same singular fibers as $\mathcal{X}'$, but it does not admit a section since both $\sigma_{\mathcal{X}}$ and $\pi_{\mathcal{X}}$ now intersect the branch locus. Thus, the total spaces of $\mathcal{G}$ and $\mathcal{X}'$ are not isomorphic, but the relative Jacobian satisfies $\text{Jac}^0(\pi'_{\mathcal{G}}) = \pi_{\mathcal{X}'}$. Concretely, the new K3 surface is obtained as the minimal resolution of the double-quadratics surface
\[(4.7) \quad \mathcal{G} : \quad w^2 = C(s, t) u^4 - A(s, t) u^2 v^2 + D(s, t) v^4,\]
which is branched over a curve bi-degree $(4, 4)$ (which has genus 9) in $\mathbb{P}_0 = \mathbb{P}(s, t) \times \mathbb{P}(u, v)$. Moreover, acting upon the even eight $F_{11} + \cdots + F_{41} + F_{50} + \cdots + F_{80} \in \text{NS}(\mathcal{X})$ with the van Geemen-Sarti involution on $\mathcal{X}$ yields the even eight $F_{10} + \cdots + F_{40} + F_{51} + \cdots + F_{81}$. The double cover branched on the latter yields a genus-one fibration isomorphic to Equation (4.7), as it simply corresponds to the interchange $u \leftrightarrow v$. We call the two combinations of eight exceptional curves selected from the components of the reducible fibers, namely $\{F_{10}, \ldots, F_{40}, F_{51}, \ldots, F_{81}\}$ and $\{F_{11}, \ldots, F_{41}, F_{50}, \ldots, F_{80}\}$, whose induced double covers yield isomorphic genus-one fibrations a marked pair of even eights on $\mathcal{X}$. 
A change of ruling yields a second elliptic fibration \( \pi'_G : G \to \mathbb{P}^1 = \mathbb{P}(u, v) \) given by
\[
G : \quad w^2 = a_1(u^2, v^2)s^4 + a_2(u^2, v^2)s^3t + \cdots + a_0(u^2, v^2)t^4,
\]
where we assumed Equation (4.4). The elliptic fibration \( \pi'_G : G \to \mathbb{P}(u, v) \) does not admit a section. The relative Jacobian \( \text{Jac}^0(\pi'_G) \) is the Jacobian elliptic K3 surface \( \pi_Y : Y \to \mathbb{P}^1 = \mathbb{P}(u, v) \) with the Weierstrass model
\[
Y : \quad Y^2Z = X^3 + f(u^2, v^2)XZ^2 + g(u^2, v^2)Z^3,
\]
where the polynomials \( f, g \) are the same polynomials as in Equation (4.6). Thus, the K3 surfaces in Equation (4.9) and (4.5) coincides with K3 surfaces in Equation (3.7) and (3.9), respectively, obtained from the general rational elliptic surface \( \mathcal{R} \) in Equation (3.5), normalized so that \( d_0 = 0 \) and \( d_\infty = 0 \). The situation is depicted in Figure 3.

One can also construct a third kind of double covers from \( X \) by grouping the eight fibers of type \( I_2 \) over \( B = 0 \) into sets of two and six elements:

**Remark 4.1.** If we group the eight fibers of type \( I_2 \) over \( B = 0 \) into sets of 2 and 6 elements by writing \( B(s, t) = C'(s, t)D'(s, t) \) where \( C', D' \) are homogenous polynomials of degree 2 and 6, respectively, we obtain a K3 double cover as the minimal resolution of the double-sixtic surface
\[
w^2 = C'(s, t)u^4 - A(s, t)u^2 + D'(s, t),
\]
which is branched over the strict transform of a sextic in \( \mathbb{F}_1 = \mathbb{P}^2 = \mathbb{P}(s, t, u) \).

Returning to the K3 surface \( G \), we have the following: (For a definition of a multi-section index of a genus-one fibration; see [22].)

**Proposition 4.2.** The K3 surfaces \( G \) in Equation (4.7) form the 10-dimensional moduli space \( \mathcal{M}_{H(2)@E_8(-1)} \). The general K3 surface \( G \) admits two elliptic fibrations, \( \pi_G : G \to \mathbb{P}(s, t) \) and \( \pi'_G : G \to \mathbb{P}(u, v) \) with multi-section index 4 and 2, respectively.

**Proof.** The K3 surfaces \( X \) in Equation (3.15) are polarized by the 2-elementary lattice \( H \oplus N \) with determinant of the discriminant group given by \( \det q_{NS(X)} = 2^6 \). The same applies to the K3 surfaces \( X' \) in Equation (3.16). Using Proposition 3.3 it follows that the K3 surfaces \( Y \) in Equation (4.9) are polarized by the 2-elementary lattice \( H \oplus E_8(-2) \) with determinant of the discriminant group given by \( \det q_{NS(Y)} = 2^8 \). These Jacobian elliptic K3 surfaces are obtained from the double-quadrics surface \( G \to \mathbb{P}^1 \times \mathbb{P}^1 \) in Equation (4.7) using either of the two rulings and then taking the relative Jacobian, that is,
\[
\pi_{X'} \cong \text{Jac}^0(\pi_G), \quad \pi_Y \cong \text{Jac}^0(\pi'_G).
\]
Thus, the Néron-Severi lattice \( \text{NS}(G) \) is a sublattice of the lattice \( \text{NS}(X') \) of index \( l_1 \) and \( \text{NS}(Y) \) of index \( l_2 \) such that \( \det q_{NS(G)} = l_1^2 \det q_{NS(X')} = l_2^2 \det q_{NS(Y)} \), and we find \( l_1 = 4 \) and \( l_2 = 2 \) and the lattice \( H(2) @ E_8(-2) \) has naturally two overlattices, namely the lattice \( H @ N \), necessarily of index four, and \( H @ E_8(-2) \) of index two. In turn, the numbers \( l_1 \) and \( l_2 \) are the multi-section index of fibration (4.7) and (4.8), respectively; this follows from [40, Lemma 2.1].
As explained before, the K3 surface $\mathcal{F}$ was obtained as double covers of $\mathbb{F}_0$ branched over the vanishing divisor of a section in the line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{F}_0}(4,4)$. This branch locus is reducible: it decomposes as $L_0 \cup L_\infty \cup E$ into two lines $L_0, L_\infty$ and a curve $C$. The latter is a curve of genus three in the linear system $|\mathcal{L}|$. Let $\mathcal{W}$ be the moduli space of K3 surfaces that can be obtained as the minimal resolution of double-quadrics surfaces over $\mathbb{F}_0$ whose branch locus is the union of divisors of type $(1,0), (1,0),$ and $(2,4)$. We note that the moduli space $\mathcal{W}$ is precisely the moduli space of F-theory compactifications dual to the CHL string considered by Witten in [69]: each general element $\mathcal{F} \in \mathcal{W}$ admits the Jacobian elliptic fibration $\pi_{\mathcal{F}}$ and the elliptic fibration $\pi_{\mathcal{F}}'$ without section, corresponding to the $\Gamma_0(2)$ elliptic fibration and the inherited elliptic fibration, respectively. We have the following:

**Proposition 4.3.** $\mathcal{W}$ is a finite covering space of $\mathcal{M}_{H@N}$.

**Proof.** Every K3 surface obtained as double cover branched over $L_0 \cup L_\infty \cup C$ can also be transformed into the double cover of the Hirzebruch surface $\mathbb{F}_4$, with an equation similar to Equation (3.15) that represents an element in $\mathcal{M}_{H@N}$. Conversely, every marking for a K3 surface $\mathcal{X} \in \mathcal{M}_{H@N}$ allows to construct a K3 surface $\mathcal{F}$. \hfill $\square$

Because of Proposition 4.2 we also have the following:

**Corollary 4.4.** $\mathcal{M}_{H(2)@E_s(-2)}$ is a finite covering space of $\mathcal{M}_{H@N}$.

We also have the equation for a two-parameter family of double-quadrics surfaces given by

$$\mathcal{F}_{(d_0,d_\infty)} : \quad W^2 = \left(U - d_0V\right)\left(d_\infty U - V\right) \left(C(s,t)U^2 - A(s,t)UV + D(s,t)V^2\right),$$

where the two fibers of type $I_0^*$ are now at $[U : V] = [d_0 : 1]$ and $[1 : d_\infty]$. A base transform, similar to Equation (4.1), converts Equation (4.11) into a double cover of the Hirzebruch surface $\mathbb{F}_4$ by forgetting the marking, given by

$$\mathcal{X}_{(d_0,d_\infty)} : \quad Y^2Z = X^3 - \left(2d_0C + 2d_\infty D - (1 + d_0d_\infty)A\right)X^2Z + \left(C + d_\infty D - d_\infty A\right)(d_0^2C + D - d_0A)XZ^2,$$

with the discriminant

$$\Delta_{\mathcal{X}_{(d_0,d_\infty)}} = (d_0d_\infty - 1)^2\left(C + d_\infty^2D - d_\infty A\right)^2\left(d_0^2C + D - d_0A\right)^2\left(A^2 - 4CD\right).$$

A comparison with the discriminant of $\mathcal{X}$ in Equation (3.17) shows that only the location of the eight fibers of type $I_2$ has changed whereas the location of the fibers of type $I_1$ has remained fixed. Moreover, for $d_0 = d_\infty = 0$ the surfaces $\mathcal{F}_{(d_0,d_\infty)}$ and $\mathcal{F}$
shows that there are two isomorphism classes with isomorphic transcendental lattice $T_\langle$surfaces with isomorphic transcendental lattice $T_\langle$. This sublattice is given by $T_\langle$Y that is the transcendental lattice of another K3 surface.

Connection to the Brauer group. It follows from the proof of Proposition 4.2 that the Néron-Severi lattice $\text{NS}(G)$ is a sublattice of the lattice $\text{NS}(Y)$ of index two. We also consider the problem of reconstructing $G$ from $Y$, that is, constructing a polarized Hodge substructure of the transcendental lattice $T_Y$ with

$$\Gamma := T_Y = H \oplus H \oplus E_8(-2),$$

that is the transcendental lattice of another K3 surface.

Recall that an element $\theta$ of order $n$ in the Brauer group $\text{Br}(Y) = \text{Hom}(T_Y, \mathbb{Q}/\mathbb{Z})$ is a surjective homomorphism $\theta : T_Y \rightarrow \mathbb{Z}/n\mathbb{Z}$ and defines a sublattice of index $n$ of $\Gamma$. This sublattice is given by $T_{\langle\theta\rangle} = \ker(\theta : \Gamma \rightarrow \mathbb{Z}/n\mathbb{Z})$ where $\langle\theta\rangle$ denotes the cyclic subgroup generated by $\theta$. Conversely, any sublattice $\Gamma'$ of index $n$ in $\Gamma$ with cyclic quotient group $\Gamma/\Gamma'$ determines a subgroup of order $n$ in $\text{Br}(Y)$. If there exists a primitive lattice embedding of $T_{\langle\theta\rangle}$ into the K3 lattice $\Lambda_{K3}$, then the Hodge structure $T_{\langle\theta\rangle}$ is guaranteed to be the transcendental lattice of another K3 surface $G$. Since the lattice embedding is in general not unique, neither is the surface $G$, and two K3 surfaces with isomorphic transcendental lattice $T_{\langle\theta\rangle}$ are Fourier-Mukai partners.

In our situation, we are considering elements $\theta \in \text{Br}(Y)_2$ of order two such that the sublattice $T_{\langle\theta\rangle} = \ker(\theta : \Gamma \rightarrow \mathbb{Z}/2\mathbb{Z})$ has index two in $\Gamma$. The existence of a primitive lattice embedding $T_{\langle\theta\rangle} \hookrightarrow \Lambda_{K3}$ is a-priori guaranteed since the K3 surface $Y$ already admits an elliptic fibration with a section; see [65]. We have the following fact from lattice theory:

**Lemma 4.5.** Let $\Gamma'$ be a sublattice of index two in $\Gamma$. Then, $\Gamma' \cong \Lambda_{2,c} \oplus H \oplus E_8(-2)$ with $c = 0, 1$ where $\Lambda_{2,c}$ is the indefinite lattice of rank two with $\Lambda_{2,0} = H(2)$ or $\Lambda_{2,1} = \langle 2 \rangle \oplus \langle -2 \rangle$.

**Proof.** A sublattice of index two in $\Gamma$ is 2-elementary. Moreover, we already know that there is a primitive lattice embedding into $\Lambda_{K3}$. Taking the complement we obtain an even, indefinite, 2-elementary lattice. According to a classification result by Nikulin [60, Thm. 4.3.2] the lattice can then only be $\Lambda_{2,c} \oplus E_8(-2)$ for either $c = 0$ or $c = 1$. □

Here, we are using the same notation as in [65], for example, for the indefinite lattice $\Lambda_{b,c} = (0, b, 2c)$ of rank two. Lemma 4.5 shows that there are two isomorphism classes for elements in the Brauer group, with representatives $T_{\langle\theta\rangle} = \Lambda_{2,c} \oplus H \oplus E_8(-2)$ for $c = 0$ and $c = 1$. The well-developed theory of linear series on K3 surfaces (cf. [63]) allows one to construct projective models for the corresponding K3 surfaces $G_{2,c}$ with transcendental lattice $\Lambda_{2,c} \oplus H \oplus E_8(-2)$ or, equivalently, Néron-Severi lattice $\Lambda_{2,c} \oplus E_8(-2)$. It turns out that the a projective model for the K3 surface $G_{2,0}$ is that...
of a double cover of \( \mathbb{P}_9 \) branched over a curve of bi-degree \((4, 4)\), necessarily with two elliptic fibrations, \( \pi_G \) and \( \pi'_G \), associated with the two elliptic pencils. Neither fibration admits a section and the two relative Jacobian fibrations are not isomorphic; see [65, Sec. 5.5]. Similarly, \( G_{2,1} \) is obtained as the desingularization of a double-sixtix surface. In particular, \( G_{2,1} \) is a double cover of the ruled surface \( \mathbb{P}_1 \) branched over a sextic curve with a node (which has genus 9); see [65, Sec. 5.6].

From Ogg-Shafarevich theory it follows that \( G_{2,c} \) with \( c = 0, 1 \) admits a genus-one fibration \( \pi'_{G_{2,c}} : G_{2,c} \to \mathbb{P}^1 \) with a multi-section of index two such that the relative Jacobian fibration recovers \( \pi_Y : Y \to \mathbb{P}^1 \). (For \( G_{2,0} \) we take this elliptic fibration to coincide with the elliptic fibration \( \pi''_G \) introduced before.) Moreover, every pair of elements \( \pm \theta \in \text{Br}(Y)_2 \) uniquely determines such a genus-one fibration on \( G_{2,c} \); see [65, Sec. 4.1]. Concretely, each element \( \theta \in \text{Br}(Y)_2 \simeq \text{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{F}_2^{12} \) corresponds to a homomorphism \( x \mapsto \langle \gamma, x \rangle \) for some element \( \gamma \in \Gamma \). The quadratic form on \( \Gamma \) induces a non-degenerate quadratic form

\[
\Gamma/2\Gamma \simeq \mathbb{F}_2^{12} \quad \text{mod } 2\mathbb{Z}.
\]

This is an even quadratic form. In particular, it has \( 2^5(2^6 + 1) \) zeroes in \( \mathbb{F}_2^{12} \), including \( [\gamma] = 0 \). Thus, there are \( 2^5(2^6 + 1) - 1 \) elements in \( \text{Br}(Y)_2 \) with \( T(\theta) = \Gamma_{2,0} \) and \( 2^5(2^6 - 1) \) elements with \( T(\theta) = \Gamma_{2,1} \). We then have the following:

**Proposition 4.6.** \( \mathcal{M}_{H(2)\oplus E_8(-2)} \) is a finite covering space of \( \mathcal{M}_{H\oplus E_8(-2)} \). In particular, for a general K3 surface \( Y \in \mathcal{M}_{H\oplus E_8(-2)} \) and every pair \( \pm \theta \in \text{Br}(Y)_2 \) in the isomorphism class \( \Gamma_{2,0} \) the Hodge substructure \( T(\theta) \) is the transcendental lattice of a K3 surface \( \mathcal{G} = G_{2,0} \in \mathcal{M}_{H(2)\oplus E_8(-2)} \) in Equation (4.7).

**Proof.** Every pair of elements \( \pm \theta \in \text{Br}(Y)_2 \) in the isomorphism class \( \Gamma_{2,0} \) determines an equation for \( G_{2,0} \) as a double cover of \( \mathbb{P}_9 \) branched over a curve of bi-degree \((4, 4)\), with a genus-one fibration \( \pi'_{G_{2,0}} : G_{2,0} \to \mathbb{P}^1 = \mathbb{P}(u, v) \) with a multi-section of index two such that the relative Jacobian fibration is \( \pi_Y : Y \to \mathbb{P}(u, v) \). Because of Proposition 3.3 every K3 surface \( Y \) is obtained as the double cover of a rational elliptic surface. Hence, the equation for \( \mathcal{G} \) must be biquadratic in \( \mathbb{P}(u, v) \) (after a possible change of coordinates) and is precisely of the form of Equation (4.8). Then it also covers a K3 surface \( \mathcal{X} \); using Equation (4.4), we arrive at Equation (4.7) and, by a simple base transform, at Equation (3.15).

We also have an analogous statement for the K3 surfaces \( F \in \mathcal{M} \):

**Corollary 4.7.** \( \mathcal{M} \) is a finite covering space of \( \mathcal{M}_{H\oplus D_4(-1)^{\oplus 2}} \).

**Proof.** For the K3 surfaces \( F \) in Equation (4.2) we have

\[
\pi_F \cong \text{Jac}^0(\pi_{\mathcal{Y}}), \quad \pi'_{\mathcal{Y}} \cong \text{Jac}^0(\pi'_{\mathcal{F}}).
\]

Following the argument in the proof of Proposition 4.6 the statement follows.

We have the following result:

**Theorem 4.8.** In Figure 3 we have the following:

1. every K3 surface \( \mathcal{X} \in \mathcal{M}_{H\oplus N} \) has an algebraic correspondence with an element \( \mathcal{Y} \in \mathcal{M}_{H\oplus E_8(-2)} \) and vice versa,
(2) every K3 surface $\mathcal{X} \in \mathcal{M}_{H\oplus N}$ has an algebraic correspondence with an element $\mathcal{Y} \in \mathcal{M}_{H\oplus D_4(-1)^{\oplus 2}}$ and vice versa.

A correspondence in (1) is an element $\mathcal{G} \in \mathcal{M}_{H(2)\oplus E_8(-2)}$ and in (2) an element $\mathcal{F} \in \mathcal{M}$.

Proof. The proof follows from Propositions 4.2, 4.3, 4.6 and Corollary 4.4. $\square$

The discrete choices involved in Theorem 4.8 are as follows:

Remark 4.9. We have the following:

(1) for every $\mathcal{X} \in \mathcal{M}_{H\oplus N}$ an element $\mathcal{G} \in \mathcal{M}_{H(2)\oplus E_8(-2)}$ is determined by a marked pair of even eights $F_{i1} + \cdots + F_{i4} + F_{j1} + \cdots + F_{j4} \in \text{NS}(\mathcal{X})$ for $(i, j) = (0, 1), (1, 0)$,

(2) for every $\mathcal{Y} \in \mathcal{M}_{H\oplus E_8(-2)}$ an element $\mathcal{G} \in \mathcal{M}_{H(2)\oplus E_8(-2)}$ is determined by a pair $\pm \theta \in \text{Br}(\mathcal{Y})_2$ with isomorphism class $\Gamma_{2,0}$.

5. A NATURAL SUBSPACE OF DIMENSION SIX

The construction of the K3 surface $\mathcal{Y}$ in Equation (3.7), considered a CHL string background, is based on a rational elliptic surface $\mathcal{R}$; see Proposition 3.3. We recall that in the general case the existence of an antisymplectic involution $k_Y$ on $\mathcal{Y}$ (such that the minimal resolution of $\mathcal{Y}/(k_Y)$ is isomorphic to $\mathcal{R}$) is equivalent to $\text{NS}(\mathcal{Y}) \cong H \oplus E_8(-2)$. One can ask what happens when there also exists a 2-torsion section $\tau_{\mathcal{R}} \in \text{MW}(\mathcal{R}, \pi_{\mathcal{R}})$ on the rational elliptic surface. If there is such a section, then the K3 surface $\mathcal{Y}$ will inherit this property. In fact, one easily checks that there exists a 2-torsion section $\tau_{\mathcal{R}} \in \text{MW}(\mathcal{R}, \pi_{\mathcal{R}})$ if and only if there exists a 2-torsion section $\tau_{\mathcal{Y}} \in \text{MW}(\mathcal{Y}, \pi_{\mathcal{Y}})$. The physical relevance of this requirement for the CHL string was discussed in [20]. The classification of rational elliptic surfaces in [62] provides an answer for what this rational elliptic surface $\mathcal{R}$ with a 2-torsion section is: it must satisfy

$$\text{(5.1) } \text{MW}(\mathcal{R}, \pi_{\mathcal{R}}) \cong \mathbb{Z}/2\mathbb{Z} \oplus D_4^\vee.$$ 

Equation (5.1) describes the rational elliptic surface of highest rank, admitting a 2-torsion section. Moreover, there are no other rational elliptic surfaces with section and rank($\text{MW}$) $\geq$ 4 that contain a 2-torsion section. We note that Equation (5.1) is precisely the second case considered earlier in (3.10). It follows that there is a (canonical) van Geemen-Sarti involution on $\mathcal{Y}$, i.e., a symplectic involution obtained as the translation by the 2-torsion section. Corollary 3.6 proves that such K3 surfaces $\mathcal{Y}$ form the moduli space $\mathcal{M}_{H\oplus N\oplus D_4(-2)}$. On the other hand, Corollary 3.11 shows that the Jacobian elliptic K3 surfaces admitting a canonical van Geemen-Sarti involution and a commuting antisymplectic involution preserving the Jacobian elliptic fibration form the same moduli space. Thus, we have the following:

Corollary 5.1. The moduli space of F-theory models with discrete flux admitting an antisymplectic involution (whose quotient is a rational elliptic surface) is naturally isomorphic to the moduli space of the CHL string backgrounds admitting a (symplectic) van Geemen-Sarti involution. It is the 6-dimensional moduli space $\mathcal{M}_{H\oplus N\oplus D_4(-2)}$.

For a general CHL string background $\mathcal{Y}$ we also constructed the Jacobian elliptic K3 surface $\tilde{\mathcal{Y}}$ in Equation (3.9) as the minimal resolution of $\mathcal{Y}/(j_\mathcal{Y})$ using the Nikulin
involution \( j_Y = k_Y \circ (-1) \). The K3 surfaces \( \breve{\mathcal{Y}} \) form the 10-dimensional moduli space \( \mathcal{M}_{H \oplus D_4(-1)^{\oplus 2}} \); see Proposition 3.5. Theorem 4.8 proves that the algebraic correspondences between the K3 surfaces \( \mathcal{X} \) and \( \breve{\mathcal{Y}} \) form a moduli space \( \mathfrak{M} \). Here, \( \mathfrak{M} \) is the moduli space of K3 surfaces that can be obtained as the minimal resolution of double-quadrics surfaces over \( \mathbb{F}_0 \) whose branch locus is the union of divisors of type \((1,0)\), \((1,0)\), and \((2,4)\).

Corollary 3.6 proved that – as the lattice polarization of \( \mathcal{Y} \) extends from \( H \oplus E_8(-2) \) to \( H \oplus N \oplus D_4(-2) \) – the lattice polarization of \( \breve{\mathcal{Y}} \) extends from \( H \oplus D_4(-1)^{\oplus 2} \) to \( H \oplus D_4(-1) \oplus A_1(-1)^{\oplus 4} \). Figure 3 can then be extended into Figure 4 where now the Jacobian elliptic K3 surfaces \( \mathcal{X}, \mathcal{X}' \), considered F-theory backgrounds, also admit antisymplectic involutions (with rational quotient surfaces \( \mathcal{R}, \mathcal{R}' \) and K3 quotients \( \breve{\mathcal{X}}, \breve{\mathcal{X}}' \)) and the Jacobian elliptic K3 surface \( \mathcal{Y} \), considered a CHL background, also admits a van Geemen-Sarti-Nikulin dual \( \mathcal{Y}' \) which in turn inherits an antisymplectic involution with rational quotient surface \( \mathcal{R}' \). Antisymplectic involutions also lift to the double-quadrics surfaces \( \mathcal{F} \) and \( \mathcal{G} \) whose quotients (after composing with the hyperelliptic involution) are \( \breve{\mathcal{F}} \) and \( \breve{\mathcal{G}} \), respectively. For the K3 surface \( \mathcal{F} \) in Equation (4.2) the existence of a compatible antisymplectic involutions implies that the polynomials \( A, C, D \) satisfy

\[
A(s,t) = -\alpha(s^2,t^2), \quad C(s,t) = \gamma(s^2,t^2), \quad D(s,t) = \delta(s^2,t^2),
\]

for some homogeneous polynomials \( \alpha, \gamma, \delta \) of degree two. (Here, we introduced a minus sign for convenience.) We have the following:

**Proposition 5.2.** In the situation of Corollary 5.1 there exist the algebraic correspondences in Figure 4. The defining equations for the surfaces are given in Table 2. Here, \( \alpha, \gamma, \delta \) are general homogeneous polynomials of degree 2, and the polynomials \( a, c, d \) of degree 2 are defined by requiring

\[
\gamma(S,T)U^2 + \alpha(S,T)UV + \delta(S,T)V^2 = c(U,V)S^2 + a(U,V)ST + d(U,V)T^2
\]

for all \( S, T, U, V \). In particular, we have:

1. \( \mathcal{X}, \mathcal{X}', \mathcal{Y}, \mathcal{Y}' \in \mathcal{M}_{H \oplus N \oplus D_4(-2)} \),
2. \( \mathcal{X}, \mathcal{X}', \mathcal{Y}, \mathcal{Y}' \in \mathcal{M}_{H \oplus D_4(-1) \oplus A_1(-1)^{\oplus 4}} \),
3. \( \mathcal{R}, \mathcal{R}', \mathcal{\breve{R}}, \mathcal{\breve{R}}' \) are rational elliptic surfaces with \( \text{MW}(\mathcal{R}, \pi_\mathcal{R}) \cong \mathbb{Z}/2\mathbb{Z} \oplus D_4^\gamma \),
4. \( \mathcal{G}, \mathcal{F}, \mathcal{\breve{G}}, \mathcal{\breve{F}} \) are double-quadrics surfaces over \( \mathbb{F}_0 \).

**Proof.** One uses the explicit constructions for double covers in Section 3.2 and Section 3.3 to construct all surfaces in Figure 4 explicitly.

We make the following:

**Remark 5.3.** A marking of \( \mathcal{X} \), given by the factorization \( \gamma(s^2,t^2) \cdot \delta(s^2,t^2) \), determines the double cover \( \mathcal{G} \). The marking also induces a canonical marking of \( \mathcal{Y}' \), given by the factorization \( c(u^2,v^2) \cdot d(u^2,v^2) \), with the same double cover \( \mathcal{G} \).

**Remark 5.4.** The fundamental object in Figure 4 is the double-quadrics surface \( \breve{\mathcal{F}} \) which is a correspondence between the K3 surfaces \( \breve{\mathcal{X}}, \breve{\mathcal{Y}}' \in \mathcal{M}_{H \oplus D_4(-1) \oplus A_1(-1)^{\oplus 4}} \). All other K3 surfaces can be obtained from \( \breve{\mathcal{F}} \) using various the double covers.
Figure 4. Extension of the F-theory/CHL string duality (Case I)

| label  | defining equation                                                                 |
|--------|-----------------------------------------------------------------------------------|
| $\mathcal{X}$ | $Y^2 Z = X^3 + \alpha(s^2, t^2) X^2 Z + \gamma(s^2, t^2) \delta(s^2, t^2) XZ^2$ |
| $\mathcal{X}'$ | $y^2 z = x^3 - 2 \alpha(s^2, t^2) x^2 z + \left( \alpha(s^2, t^2)^2 - 4 \gamma(s^2, t^2) \delta(s^2, t^2) \right) x z^2$ |
| $\mathcal{Y}$ | $Y^2 Z = X^3 - 2 a(u^2, v^2) X^2 Z + \left( a(u^2, v^2)^2 - 4 c(u^2, v^2) d(u^2, v^2) \right) XZ^2$ |
| $\mathcal{Y}'$ | $y^2 z = x^3 + a(u^2, v^2) x^2 z + c(u^2, v^2) d(u^2, v^2) x z^2$ |
| $\tilde{\mathcal{X}}$ | $y^2 z = x^3 + ST \alpha(S, T) x^2 z + S^2 T^2 \gamma(S, T) \delta(S, T) x z^2$ |
| $\tilde{\mathcal{X}}'$ | $Y^2 Z = X^3 - 2 ST \alpha(S, T) X^2 Z + S^2 T^2 \alpha(S, T)^2 - 4 \gamma(S, T) \delta(S, T) XZ^2$ |
| $\tilde{\mathcal{Y}}$ | $y^2 z = X^3 - 2 UV a(U, V) x^2 z + U^2 V^2 \left( a(U, V)^2 - 4 c(U, V) d(U, V) \right) x z^2$ |
| $\tilde{\mathcal{Y}}'$ | $Y^2 Z = X^3 + UV a(U, V) X^2 Z + U^2 V^2 c(U, V) d(U, V) X Z^2$ |
| $\mathcal{R}$ | $y^2 z = x^3 + \alpha(S, T) x^2 z + \gamma(S, T) \delta(S, T) x z^2$ |
| $\mathcal{R}'$ | $Y^2 Z = X^3 - 2 \alpha(S, T) X^2 Z + \left( \alpha(S, T)^2 - 4 \gamma(S, T) \delta(S, T) \right) XZ^2$ |
| $\mathcal{G}$ | $w^2 = \left\{ \begin{array}{l} \gamma(s^2, t^2) u^4 + \alpha(s^2, t^2) u^2 v^2 + \delta(s^2, t^2) v^4 \\ c(u^2, v^2) s^4 + a(u^2, v^2) s^2 t^2 + d(u^2, v^2) t^4 \end{array} \right.$ |
| $\mathcal{F}$ | $W^2 = \left\{ \begin{array}{l} UV \left( \gamma(s^2, t^2) U^2 + \alpha(s^2, t^2) UV + \delta(s^2, t^2) V^3 \right) \\ UV \left( c(U, V) s^4 + a(U, V) s^2 t^2 + d(U, V) t^4 \right) \end{array} \right.$ |
| $\tilde{\mathcal{G}}$ | $W^2 = \left\{ \begin{array}{l} ST \gamma(S, T) u^4 + \alpha(S, T) u^2 v^2 + \delta(S, T) v^4 \\ ST \left( c(u^2, v^2) S^2 + a(u^2, v^2) ST + d(u^2, v^2) T^2 \right) \end{array} \right.$ |
| $\tilde{\mathcal{F}}$ | $w^2 = \left\{ \begin{array}{l} ST U V \left( \gamma(S, T) U^2 + \alpha(S, T) UV + \delta(S, T) V^2 \right) \\ ST U V \left( c(U, V) S^2 + a(U, V) ST + d(U, V) T^2 \right) \end{array} \right.$ |

Table 2. Defining equations for surfaces in Figure 4

We will now determine the space of correspondences between $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{Y}}'$. We consider the general double-quadratics surfaces given by

\begin{equation}
W^2 = \left( S - c_0 T \right) \left( \epsilon_\infty S - T \right) \left( U - d_0 V \right) \left( d_\infty U - V \right) \left( \gamma(U, V) S^2 + \alpha(U, V) ST + \delta(U, V) T^2 \right),
\end{equation}
branched over bi-degree \((4, 4)\)-curves in \(\mathbb{F}_0 = \mathbb{P}(S, T) \times \mathbb{P}(U, V)\). Here \(\alpha, \gamma, \delta\) are homogenous polynomials of degree two and \(c_0, c_\infty, d_0, d_\infty\) are complex parameters such that \(c_0c_\infty \neq 1\) and \(d_0d_\infty \neq 1\). The branch locus is reducible and consists of two curves of bi-degree \((1, 0)\) and \((0, 1)\), respectively, and a genus-one curve in the linear system \(|\mathcal{O}_{\mathbb{F}_0}(2, 2)|\) such that the curves intersect in ordinary rational double points. Let \(\overline{M}\) be the moduli space of the K3 surfaces that can be obtained as the minimal resolution of the surfaces in Equation \((5.4)\). We note that Equation \((5.4)\) depends on 12 parameters since one has \(\mathbb{P}H^0(\mathbb{F}_0, \mathcal{O}_{\mathbb{F}_0}(2, 2)) = 3 \cdot 3 - 1 = 8\) in addition to \(c_0, c_\infty, d_0, d_\infty\). Since \(\mathbb{F}_0\) has a 6-dimensional group of automorphisms, the moduli space \(\overline{M}\) is 6-dimensional. We have the following:

**Corollary 5.5.** The correspondences \(\tilde{\mathcal{F}}\) in Figure 4 form the moduli space \(\overline{M}\).

We now derive an explicit parametrization for the K3 surfaces in \(\overline{M}\) using an auxiliary genus-one curve. This parametrization of the curves in the branch locus is based on a construction given by André Weil [68] using the Abel-Jacobi map for genus-one curves.

### 5.1. The Abel-Jacobi map.

Let \(H\) be a smooth curve of genus one given by \(w^2 = P(x) = \sum a_i x^i\), using the affine coordinates \((x, w) \in \mathbb{C}^2\). For a point \((x_0, -w_0) \in H\) we consider the Abel-Jacobi map \(J_{(x_0, -w_0)}: H \rightarrow \text{Jac}(H)\) which relates the algebraic curve \(H\) to its Jacobian variety \(\text{Jac}(H)\), i.e., an elliptic curve. A classical result due to Hermite states that \(\text{Jac}(H) \cong E\) where \(E\) is the elliptic curve given by

\[
E: \quad \eta^2 = S(\xi) = \xi^3 + f \xi + g.
\]

Here, we are using the affine coordinates \((\xi, \eta) \in \mathbb{C}^2\) and Equations \((4.6)\), i.e.,

\[
f = -4a_0 a_4 + a_1 a_3 - \frac{1}{3} a_2^2, \quad g = \frac{8}{3} a_0 a_2 a_4 + a_0 a_3^2 + a_1^2 a_4 - \frac{1}{3} a_1 a_2 a_3 + \frac{2}{27} a_2^3.
\]

We introduce the polynomial

\[
R(x, x_0) = a_4 x^2 x_0^2 + \frac{a_3}{2} x x_0(x + x_0) + \frac{a_2}{6}(x^2 + x_0^2) + \frac{2a_2}{3} x x_0 + \frac{a_1}{2}(x + x_0) + a_0
\]

such that \(R(x, x) = P(x)\). It turns out that the polynomial \(P(x)P(x_0) - R(x, x_0)^2\) factors. There is a polynomial \(R_1(x, x_0)\) of bi-degree \((2, 2)\) such that

\[
\forall x, x_0: R(x, x_0)^2 + R_1(x, x_0)(x - x_0)^2 - P(x)P(x_0) = 0.
\]

We obtain

\[
R_1(x, x_0) = \frac{8a_2 - 3a_3^2}{12} x_0^2 x_0^2 + \frac{a_4 - a_2 a_3}{6} x x_0(x + x_0) + \frac{36a_0 - a_2^2}{36} (x^2 + x_0^2)
\]

\[
+ \frac{36a_0 + 9a_1 a_3 - 5a_2^2}{18} x x_0 + \frac{6a_2 a_3 - a_1 a_2}{6} (x + x_0) + \frac{8a_0 a_2 - 3a_2^2}{12},
\]

and set \(Q(x) = R_1(x, x)\). In particular, we have

\[
Q(x) = \frac{1}{3} P(x)P''(x) - \frac{1}{4} P'(x)^2.
\]
We denote the discriminants of $H$ and $E$ by $\Delta_H = \text{Discr}_x(P)$ and $\Delta_E = \text{Discr}_x(S)$, respectively, such that $\Delta_H = \Delta_E$ by construction. One also checks $\text{Discr}_x(Q) = S(0)^2 \text{Discr}_x(P)$. From now on, we will assume that

$$\text{Discr}_x(Q) = S(0)^2 \text{Discr}_x(P) \neq 0. \tag{5.11}$$

We also set $[P, Q] = \partial_x P \cdot Q - P \cdot \partial_x Q$. A tedious but straight-forward computation yields the following:

**Lemma 5.6.** For a smooth curve $H$ of genus one given by $w^2 = \sum_{i=0}^4 a_ix^{4-i}$, the Abel-Jacobi map $J_{(x_0, w_0)} : H \to E \cong \text{Jac}(H)$ maps $(x, y) \mapsto (\xi, \eta)$ with

$$\xi = 2R(x, x_0) - w_0 w_0 \quad \frac{(x - x_0)^2}{(x - x_0)^2}, \quad \eta = \frac{4w_0(w - w_0)}{(x - x_0)^2} - \frac{P'(x)w_0 + P'(x_0)w}{(x - x_0)^2} \quad \text{for } x \neq x_0, \tag{5.12}$$

the point $(x_0, -w_0) \in H$ to the point at infinity on $E$, and $(x_0, w_0)$ to the point with $\xi = -Q(x_0)/P(x_0)$, $\eta = [P, Q]_{x_0}/(2w_0^2)$ if $w_0 \neq 0$.

It follows from Equation (5.12) that the coordinates $x$ and $\xi$ in the Abel-Jacobi map $(\xi, \eta) = J_{(x_0, -y_0)}((x, y)$ are related by the bi-quadratic polynomial

$$\xi^2(x - x_0)^2 - 4\xi R(x, x_0) - 4R_1(x, x_0) = 0. \tag{5.13}$$

The equation defines an algebraic correspondence between points of the two projective lines with affine coordinates $\xi$ and $x$, respectively, where -- given a point $x$ -- there are two solutions for $\xi$ in Equation (5.13) and vice versa. Equivalently, we consider Equation (5.13) an affine equation of bi-degree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ with the affine variables $(x, x_0)$ (which has genus one). A direct computation yields the following:

**Lemma 5.7.** The $j$-invariants of the following genus-one curves are identical:

$$\eta^2 = S(\xi), \quad w^2 = P(x), \quad \xi^2(x - x_0)^2 - 4\xi R(x, x_0) - 4R_1(x, x_0) = 0. \tag{5.14}$$

In particular, Lemma 5.7 shows that the $j$-invariant of the curve in Equation (5.13) is independent of the variable $\xi$. We have the following:

**Proposition 5.8.** Equation (5.13) is an embedding $\iota_\xi : H \hookrightarrow H' \subset \mathbb{P}^1 \times \mathbb{P}^1$ of a genus-one curve $H$ as a curve of bi-degree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, given by

$$H' : \quad \xi^2(x - x_0)^2 - 4\xi R(x, x_0) - 4R_1(x, x_0) = 0. \tag{5.15}$$

In particular, it is a symmetric affine equation in the variables $(x, x_0)$.

Based on results in [32, 67, 68] we have the following:

**Theorem 5.9.** Let

$$0 = \gamma(S, T)U^2 + \alpha(S, T)UV + \delta(S, T)V^2 \tag{5.16}$$

be a curve of genus one in the linear system $|\mathcal{O}_{\mathbb{F}_0}(2, 2)|$ over $\mathbb{F}_0 \ni ([S : T], [U, V])$ with the same $j$-invariant as $H : w^2 = P(x)$. Then, there is a point in the Jacobian $\xi \in \text{Jac}(H)$ such that Equation (5.16) is isomorphic to $\iota_\xi(H)$.

**Proof.** Equation (5.16) depends on eight parameters since one has

$$\phi H^0(\mathbb{F}_0, \mathcal{O}_{\mathbb{F}_0}(2, 2)) = 3 \cdot 3 - 1 = 8. \tag{5.17}$$
Since \( \mathcal{F}_0 \) has a 6-dimensional group of automorphisms, we have two moduli. The first one is the \( j \)-invariant defining an elliptic curve \( E \). It follows that \( E \cong \text{Jac}(H) \). On the hand, the points in \( \text{Jac}(H) \) parameterize the automorphisms of \( H \). Thus, there is a point \( \xi \in \text{Jac}(H) \) such that Equation (5.16) is isomorphic to \( \nu_\xi(H) \).

5.2. A parametrization of K3 surfaces. We derive an explicit parametrization for the K3 surfaces in \( \mathcal{W} \). The central idea is that genus-one curves in the branch locus of K3 surfaces \( \tilde{\mathcal{F}} \in \mathcal{W} \) can be represented by equations of bi-degree \( (2, 2) \) that are symmetric under the interchange of \( (U, V) \) and \( (S, T) \) in \( \mathcal{F}_0 = \mathbb{P}(S, T) \times \mathbb{P}(U, V) \).

We start by using transformations in \( \text{PGL}(2, \mathbb{C}) \) to turn the first terms of Equation (5.4) into the monomial \( STUV \). We then use the transformation

\[
(U, V) \mapsto (\mu U, \lambda V), \quad (S, T) \mapsto (\lambda (T - \lambda^{-1}c_\infty S), S - \lambda c_0 T),
\]

with parameters \( \lambda, \mu \in \mathbb{C}^\times \) and \( c_0, c_\infty \) with \( c_0 c_\infty \neq 1 \), to transform an equation of the form

\[
W^2 = STUV \left( \gamma'(U, V)^2 + \alpha'(U, V) ST + \delta'(U, V) T^2 \right)
\]

for polynomials \( \alpha', \gamma', \delta' \) into the equation

\[
W^2 = (S - \lambda c_0 T)(T - \lambda^{-1}c_\infty S) UV \left( \left( \gamma_2 S^2 + \lambda \alpha_2 ST + \lambda^2 \gamma_0 T^2 \right) U^2 + \left( \lambda^2 \alpha_2 S^2 + \lambda^3 \alpha_0 T^2 \right) UV + \left( \lambda^2 \gamma_2 S^2 + \lambda^3 \alpha_0 ST + \lambda^4 \delta_0 T^2 \right) V^2 \right),
\]

where \( \mu, c_0, c_\infty \) have been chosen such that the new coefficients \( \alpha_2, \ldots, \gamma_0 \) do not depend on \( \lambda \) and satisfy \( \gamma_1 = \alpha_2, \delta_2 = \gamma_0, \delta_1 = \alpha_0 \). Thus, we have proved the following:

**Proposition 5.10.** The moduli space \( \mathcal{W} \) is given by the K3 surfaces obtained from the minimal resolution of the double-quadrics surfaces

\[
W^2 = (S - c_0 T)(T - c_\infty S) UV \left( \gamma(U, V)^2 + \alpha(U, V) ST + \delta(U, V) T^2 \right).
\]

Here, \( c_0, c_\infty \in \mathbb{C} \) with \( c_0 c_\infty \neq 1 \) and \( \alpha, \gamma, \delta \) are given by

\[
\alpha = \alpha_2 x^2 + \alpha_1 xy + \alpha_0 y^2, \quad \gamma = \gamma_2 x^2 + \alpha_2 xy + \gamma_0 y^2, \quad \delta = \gamma_0 x^2 + \alpha_0 xy + \delta_0 y^2.
\]

We make the following:

**Remark 5.11.** Proposition 5.10 implies that we can assume

\[
\gamma(S, T) U^2 + \alpha(S, T) UV + \delta(S, T) V^2 = \gamma(U, V)^2 + \alpha(U, V) ST + \delta(U, V) T^2
\]

for all \( S, T, U, V \).

By a slight abuse of notation we will also denote by \( \tilde{\mathcal{F}}(\delta_0, \alpha_0, \gamma_0, \alpha_1, \alpha_2, \gamma_2; c_0, c_\infty) \) the smooth complex surface obtained as the minimal resolution of Equation (5.21).

We have the following symmetries:

**Lemma 5.12.** One has the following isomorphisms of K3 surfaces:

(a) \( \tilde{\mathcal{F}}(\delta_0, \alpha_0, \gamma_0, \alpha_1, \alpha_2, \gamma_2; c_0, c_\infty) \cong \tilde{\mathcal{F}}(\lambda c_0, \lambda \alpha_0, \lambda \gamma_0, \lambda \alpha_1, \lambda \alpha_2, \lambda \gamma_2; c_0, c_\infty), \)

(b) \( \tilde{\mathcal{F}}(\delta_0, \alpha_0, \gamma_0, \alpha_1, \alpha_2, \gamma_2; c_0, c_\infty) \cong \tilde{\mathcal{F}}(\mu^4 \delta_0, \mu^3 \alpha_0, \mu^2 \gamma_0, \mu \alpha_1, \mu \alpha_2, \mu \gamma_2; \mu c_0, \mu^{-1} c_\infty), \)

(c) \( \tilde{\mathcal{F}}(\delta_0, \alpha_0, \gamma_0, \alpha_1, \alpha_2, \gamma_2; c_0, c_\infty) \cong \tilde{\mathcal{F}}(\gamma_2, \alpha_2, \gamma_0, \alpha_1, \alpha_0, \delta_0; c_0^1, c_\infty^1), \)

(d) \( \tilde{\mathcal{F}}(\delta_0, \alpha_0, \gamma_0, \alpha_1, \alpha_2, \gamma_2; c_0, c_\infty) \cong \tilde{\mathcal{F}}(\delta_0', \alpha_0', \gamma_0', \alpha_1', \alpha_2', \gamma_2'; -c_0, -c_\infty). \)
for \( \lambda, \mu \in \mathbb{C}^* \) and
\[
\delta_0' = \gamma_2 c_0^4 + 2a_2 c_0^3 + (\alpha_1 + 2\gamma_0) c_0^2 + 2a_0 c_0 + \delta_0,
\]
\[
\alpha_0' = (a_2 c_0 + 2\gamma_2) c_0^3 + ((\alpha_1 + 2\gamma_0) c_0 + 3\alpha_2) c_0^2 + (3a_0 c_0 + \alpha_1 + 2\gamma_0) c_0 + 2\delta_0 c_0 + \alpha_0,
\]
\[
\gamma_0' = (\gamma_0 c_0^2 + \alpha_2 c_0 + \gamma_2) c_0^2 + (\alpha_0 c_0^2 + \alpha_1 c_0 + \alpha_2) c_0 + \delta_0 c_0^2 + \alpha_0 c_0 + \gamma_0.
\]

\begin{equation}
(5.24) \quad \alpha_1' = (\alpha_1 c_0^2 + 4a_2 c_0 + 4\gamma_2) c_0^2 + (4a_0 c_0^2 + 2(\alpha_1 + 4\gamma_0) c_0 + 4a_2) c_0
\end{equation}
\[
+ 4\delta_0 c_0 + 4a_0 c_0 + \alpha_1,
\]
\[
\alpha_2' = (\alpha_2 c_0 + 2\delta_0) c_0^3 + ((\alpha_1 + 2\gamma_0) c_0 + 3\alpha_0) c_0^2 + (3\alpha_2 c_0 + \alpha_1 + 2\gamma_0) c_0 + 2\gamma_2 c_0 + \alpha_2,
\]
\[
\gamma_2' = \delta_0 c_0^4 + 2a_0 c_0^3 + (\alpha_1 + 2\gamma_0) c_0^2 + 2a_2 c_0 + \gamma_2.
\]

**Proof.** Part (1) follows from rescaling \( W \mapsto W/\sqrt{\gamma} \), similarly part (2) follows from the rescaling \((T,V) \mapsto (\mu T, \mu V)\). For the remaining parts, the proof follows by constructing a suitable automorphism of \( \mathbb{F}_0 \) in Equation (5.21) that gives to the change of parameters. For part (3) this is given by interchanging \( S \leftrightarrow T \) and \( U \leftrightarrow V \). For part (4) the transformation is given by
\begin{equation}
(5.25) \quad U = S' + c_0 T', \quad V = c_0 S' + T', \quad S = U' + c_0 V', \quad T = c_0 U' + V'.
\end{equation}

\( \square \)

We now give an explicit parametrization for the K3 surfaces in \( \mathfrak{M} \) using the Abel-Jacobi map from Section 5.1. We have the following:

**Theorem 5.13.** The moduli space \( \mathfrak{M} \) of correspondences \( \mathfrak{F} \) in Figure 4 is given by the K3 surfaces obtained as the minimal resolution of the double-quadratics surfaces
\begin{equation}
W^2 = (S - c_0 T)(T - c_\infty S) U V (\gamma(U, V) S^2 + \alpha(U, V) ST + \delta(U, V) T^2).
\end{equation}

Here, \( c_0, c_\infty \in \mathbb{C} \) with \( c_0 c_\infty \neq 1 \), the polynomials \( \alpha, \gamma, \delta \) are given by
\begin{equation}
(5.27) \quad \forall x, x_0 : \gamma(x, 1) x_0^2 + \alpha(x, 1) x_0 + \delta(x, 1) = \xi^2 (x - x_0)^2 - 4 \xi R(x, x_0) - 4 R_1(x, x_0),
\end{equation}
and the polynomials \( R(x, x_0) \) and \( R_1(x, x_0) \) are
\begin{equation}
(5.28) \quad R(x, x_0) = x^2 x_0^2 + \frac{a_2}{6} (x^2 + x_0^2) + \frac{2a_0}{3} xx_0 + \frac{a_1}{2} (x + x_0) + a_0,
\end{equation}
\begin{equation}
+ \frac{36a_0 - a_2^2}{36}(x^2 + x_0^2),
\end{equation}
\begin{equation}
R_1(x, x_0) = \frac{2a_2}{3} x^2 x_0 + a_1 xx_0(x + x_0) + \frac{36a_0 - a_2^2}{36}(x^2 + x_0^2)
\end{equation}
\begin{equation}
+ \frac{36a_0 - 5a_2^2}{18} xx_0 - \frac{a_1 a_2}{6} (x + x_0) + \frac{8a_0 a_2 - 3a_1^2}{12},
\end{equation}
for the smooth genus-1 curve \( H : w^2 = x^4 + \sum_{i=0}^2 a_i x^i \) and \( \xi \in \text{Jac}(H) \).

**Remark 5.14.** The parameterization in Theorem 5.13 is given by the six parameters \( c_0, c_\infty, a_0, a_1, a_2, \xi \) such that \( c_0 c_\infty \neq 1 \), the curve \( H : w^2 = x^4 + \sum_{i=0}^2 a_i x^i \) is smooth, and \( \xi \in \text{Jac}(H) \). The curve \( H \) is smooth if its discriminant does not vanish, i.e.,
\[ 16a_0 a_2^4 - 4a_1^2 a_2^2 - 128a_0^2 a_2^2 + 144a_0 a_1^2 a_2^2 - 27a_1^4 + 256a_0^3 = 0. \]

The elliptic curve \( E = \text{Jac}(H) \) in \( \mathbb{P}^2 = \mathbb{P}(Z_1, Z_2, Z_3) \) is given by
\begin{equation}
(5.29) \quad E : Z_2 Z_3 = Z_1^3 - \left(4a_0 + \frac{1}{3} a_2^2 \right) Z_1^2 Z_3 + \left(a_1^2 + \frac{2a_0}{27} a_2^3 - \frac{8}{3} a_0 a_2 \right) Z_3^3.
\end{equation}
Proof. We apply Proposition 5.10, Theorem 5.9, and Proposition 5.8. One then checks that in terms of the curve $H$ given by $w^2 = P(x) = \sum_{i=0}^{4} a_i x^i$ and the Jacobian $\text{Jac}(H)$ given by Equation (5.5) one has

\begin{equation}
2a_0\gamma_2 - a_1\alpha_2 + 2a_2\gamma_0 = -8a_3\eta^2.
\end{equation}

Using Lemma 5.12 (4) one checks that the equation

\begin{equation}
2a'_0\gamma'_2 - a'_1\alpha'_2 + 2a'_2\gamma'_0 = 0.
\end{equation}

is linear in $c_0$ and can be solved such that $c_0$ is a rational function in $c_\infty$ where both the numerator and denominator have degree 3 in $c_\infty$ with coefficients quadratic in $\mathbb{Z}[\delta_0, \alpha_0, \gamma_0, \alpha_1, \alpha_2, \gamma_2]$. Thus, for a smooth curve $H$ of genus one we can assume $a_4 \neq 0$ and then shift the coordinate $x$ to obtain $a_3 = 0$. Moreover, Lemma 5.12 (1) provides an overall scaling that can be used to have $a_4 = 1$. \hfill \square

6. A SECOND SUBSPACE OF DIMENSION SIX

The Jacobian elliptic $K3$ surfaces $\mathcal{X}$ with $\text{MW}(\mathcal{X}, \pi_\mathcal{X}) \cong \mathbb{Z}/2\mathbb{Z}$ in Equation (3.15) represent eight-dimensional F-theory backgrounds with discrete flux, or, equivalently, elements of the moduli space $\mathcal{M}_{\text{H}}\text{N}$. Based on Corollary 3.11, imposing the existence of a second (commuting) van Geemen-Sarti involution on $\mathcal{X}$ implies that the Jacobian elliptic fibration has 12 singular fibers of type $I_2$, a Mordell-Weil group $(\mathbb{Z}/2\mathbb{Z})^2$, and a lattice polarization that extends to

\begin{equation}
H \oplus D_4(-1)^{\oplus 2} \oplus A_1(-1)^{\oplus 4} \cong H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 2} \cong (2) \oplus (-2) \oplus D_4(-1)^{\oplus 3}.
\end{equation}

The equivalence of the lattices was proven in [12]. The existence of two commuting van Geemen-Sarti involutions is equivalent to the existence of two independent 2-torsion sections in $\text{MW}(\mathcal{X}, \pi_\mathcal{X})$ and implies that the polynomial $B(s, t)$ in Equation (3.15) satisfies $B = A^2 - C^2$ for some homogeneous polynomial $C(s, t)$ of degree 4; see proof of Corollary 3.11. In turn, the factorization $4B = (A+C)(A-C)$ determines a marking of the eight fibers of type $I_2$ over $B = 0$ on $\mathcal{X}$ with the double cover $\mathcal{G}$. Observing that the coefficient of $x^2z^2$ in Equation (3.16) then becomes a perfect square, we obtain a canonical marking of $\mathcal{X}$ with another double cover $\mathcal{G}'$. The double-quadratics surface $\mathcal{G}'$ has a particular simple form which we will determine presently. Changing the ruling and computing the relative Jacobian fibration we obtain from $\mathcal{G}'$ a second $K3$ surface $\mathcal{Y}'$ admitting two commuting antisymplectic involutions due to the symmetry of $\mathcal{G}'$. We already constructed the $K3$ surfaces $\mathcal{Y}'$ and their quotients $\mathcal{Y} \cong \mathcal{Y}'$ in Propositions 3.7 and 3.8. Thus, we have the following:

Corollary 6.1. The moduli space of F-theory models with discrete flux admitting two commuting van Geemen-Sarti involutions is dual to the moduli space of the CHL string admitting two commuting antisymplectic involutions. The moduli space are $\mathcal{M}_{(2)\oplus(-2)\oplus D_4(-1)^{\oplus 3}}$ and $\mathcal{M}_{H \oplus D_4(-1)^{\oplus 2} \oplus D_4(-2)}$, respectively.

Figure 3 can then be extended. We obtain Figure 5 where now the rational elliptic surface $\mathcal{R}$ – due to the extended symmetry of $\mathcal{Y}'$ – must satisfy

\begin{equation}
\text{MW}(\mathcal{R}, \pi_\mathcal{R}) \cong D_4'.
\end{equation}

We note that Equation (5.1) is precisely the first case considered earlier in (3.10). We have the following:
Proposition 6.2. In the situation of Corollary 6.1 there exist the algebraic correspondences in Figure 5. The defining equations for the surfaces in Figure 5 are given by Table 2. Here, $A, C$ are general homogeneous polynomials of degree four, and $a_0, \ldots, a_4$ of degree 1 are defined by requiring

$$A(s,t) \frac{U-V}{2} - C(s,t) \frac{U+V}{2} = a_0(U,V) s^4 + a_1(U,V) s^3 t + \cdots + a_4(U,V) t^4$$

for all $s,t,U,V$. The polynomials $f,g$ are given by

$$f = -4a_0a_4 + a_1a_3 - \frac{1}{3}a_2^2, \quad g = -\frac{8}{3}a_0a_2a_4 + a_0a_3^2 + a_1^2a_4 - \frac{1}{3}a_1a_2a_3 + \frac{2}{27}a_2^3.$$ 

In particular, we have:

1. $\mathcal{X} \in \mathcal{M}_{(2)(-2)@D_4(-1)^{93}}, \quad \mathcal{Y} \in \mathcal{M}_{H@D_4(-1)^{92}@D_4(-2)}, \quad \widetilde{\mathcal{Y}} \in \mathcal{M}_{H@D_4(-1)^{93}},$

2. $\mathcal{G}', \mathcal{G}, \mathcal{F}$ are double-quadrics surfaces over $\mathbb{F}_0$,

3. $\mathcal{R}$ is a rational elliptic surface with $\text{MW}(\mathcal{R}, \pi_\mathcal{R}) \cong D_4'$.

Proof. One uses the explicit constructions for double covers in Section 3.2 and Section 3.3 to construct all surfaces in Figure 5 explicitly. \hfill \qed

6.1. The double $4\mathcal{H}$-surfaces. We will now determine the space of correspondences $\mathcal{F}$ for the K3 surfaces $\mathcal{X}$ and $\widetilde{\mathcal{Y}}$ in Figure 5.

Let $\mathcal{F}$ be a double cover of the Hirzebruch surface $\mathbb{P}^1 \times \mathbb{P}^1$ branched over the union of four bi-degree $(1,1)$ curves satisfying a certain genericity condition. Such a surface $\mathcal{F}$ is also called a double $4\mathcal{H}$-surface. We construct a geometric model as follows: using the coordinates $U$ and $s$ on $\mathbb{P}^1_{(U)} \times \mathbb{P}^1_{(s)}$ a double cover is given by

$$y^2 = \prod_{k=1}^4 \left( \rho_1^{(k)} sU + \rho_2^{(k)} U + \rho_3^{(k)} s + \rho_4^{(k)} \right),$$

for complex parameters $\rho_j^{(i)}$ with $i,j \in \{1,2,3,4\}$. We denote by $H_1, \ldots, H_4$ the four different rational curves of bi-degree $(1,1)$ and impose the following genericity conditions: (1) every $H_i$ is irreducible, (2) $H_i \cap H_j$ consists of two different points for all $i \neq j$, and (3) for any three different indices $i,j,k$ we have $H_i \cap H_j \cap H_k = \emptyset$. This is precisely the family considered in [44], and it was proven there that under the above
\[
\begin{array}{|c|c|}
\hline
\text{label} & \text{defining equation} \\
\hline
\mathcal{X} & Y^3 = X^3 - A(s, t) X^2 Z + (A(s, t)^2 - C(s, t)^2) X Z^2/4 \\
\mathcal{X}' & y^2 = x^3 + 2 A(s, t) x^2 z + C(s, t) x z^2 \\
\mathcal{G}' & W^2 = \left\{ C(s, t) \bar{v}^4 + 2 A(s, t) \bar{w}^2 \bar{v}^2 + C(s, t) \bar{v}^4 \right. \\
& \left. a_4 \left( (\bar{u}^2 - \bar{v}^2_0), (\bar{u}^2 + \bar{v}^2_0) \right) s^4 + \cdots + a_0 \left( (\bar{u}^2 - \bar{v}^2_0), (\bar{u}^2 + \bar{v}^2_0) \right) t^4 \right. \\
\mathcal{G} & \bar{w}^2 = \left\{ (u^2 - v^2) \left( \frac{A(s, t) - C(s, t)}{2} \right) u^2 - \frac{A(s, t) + C(s, t)}{2} v^2 \right. \\
& \left. (u^2 - v^2) a_4 u^2 v^2 \right. \\
\mathcal{F} & W^2 = \left\{ UV(U - V) \left( \frac{A(s, t) - C(s, t)}{2} U - \frac{A(s, t) + C(s, t)}{2} V \right) \right. \\
& \left. \left( u^2 - v^2 \right) a_4 (U, V) s^4 + \cdots + a_0 (U, V) t^4 \right\} \\
\mathcal{Y}' & Y^2 Z = X^3 + f \left( (\bar{u}^2 - \bar{v}_0^2), (\bar{u}^2 + \bar{v}_0^2) \right) X Z^2 + g \left( (\bar{u}^2 - \bar{v}_0^2), (\bar{u}^2 + \bar{v}_0^2) \right) Z^3 \\
\mathcal{Y} & y^2 z = x^3 + (u^2 - v^2)^2 f(u^2, v^2) x z^2 + (u^2 - v^2)^3 g(u^2, v^2) z^3 \\
\widetilde{\mathcal{Y}} & Y^2 Z = X^3 + U^2 V^2 (U - V)^2 f(U, V) X Z^2 + U^3 V^3 (U - V)^3 g(U, V) Z^3 \\
\mathcal{R}' & y^2 = x^3 + f \left( u^2, v^2 \right) x z^2 + g \left( u^2, v^2 \right) z^3 \\
\mathcal{R} & Y^2 Z = X^3 + (U - V)^2 f(U, V) X Z^2 + (U - V)^3 g(U, V) Z^3 \\
\hline
\end{array}
\]

Table 3. Defining equations for surfaces in Figure 5.

conditions $\mathcal{F}$ is a K3 surface. If we define the quadratic polynomials
\[
P^{(i,j)} = \left( \frac{\rho_1^{(i)} \rho_3^{(i)}}{\rho_3^{(i)} \rho_1^{(i)}} \right) s^2 + \left( \frac{\rho_2^{(j)} \rho_4^{(j)}}{\rho_4^{(j)} \rho_2^{(j)}} \right) t^2 \\
+ \left( \frac{\rho_1^{(i)} \rho_4^{(j)}}{\rho_4^{(j)} \rho_1^{(i)}} + \rho_2^{(j)} \rho_3^{(i)} - \rho_3^{(i)} \rho_2^{(j)} - \rho_1^{(i)} \rho_1^{(j)} \right) s,
\]
then Equation (6.5) can be brought into the Weierstrass form
\[
Y^2 Z = X \left( X - P^{(1,2)} P^{(3,4)} Z \right) \left( X - P^{(1,3)} P^{(2,4)} Z \right),
\]
which coincides with the equation for $\mathcal{X}$ in Table 3 in the affine chart $t = 1$ with
\[
P^{(1,2)} P^{(3,4)} = \frac{A + C}{2}, \quad P^{(1,3)} P^{(2,4)} = \frac{A - C}{2}.
\]
The following is then immediate:

**Lemma 6.3.** The generic K3 surface $\mathcal{F}$ admits a Jacobian elliptic fibration with 12 singular fibers of type $I_2$ and a Mordell-Weil group of sections $(\mathbb{Z}/2\mathbb{Z})^2$.

**Proof.** Given the Weierstrass model in Equation (6.7) the statement is checked by explicit computation. \(\square\)

Given any two distinct complex parameters $\mu, \nu \in \mathbb{C}$ with $\mu \neq \nu$, Equation (6.7) can be brought into the form
\[
y^2 = \left( \xi + \mu \right) \left( \xi + \nu \right) \left( \left( P^{(1,2)} P^{(3,4)} - P^{(1,3)} P^{(2,4)} \right) \xi + \left( \mu P^{(1,2)} P^{(3,4)} - \nu P^{(1,3)} P^{(2,4)} \right) \right),
\]
which in turn can be re-written as
\[
y^2 = \sum_{i=0}^{4} (\xi + \mu)(\xi + \nu) A_i(\xi, \mu, \nu) w^i,
\]
with $A_i(\xi, \mu, \nu) = a_{i,1}\xi + a_{i,2}\mu + a_{i,3}\nu$ for $0 \leq i \leq 4$. The coefficients are of the form

$$a_{i,j} = \sum_{k_1, \ldots, k_4} \alpha_{i,j}(\vec{k}) \rho_{k_1}^{(1)} \rho_{k_2}^{(2)} \rho_{k_3}^{(3)} \rho_{k_4}^{(4)}$$

with $\alpha_{i,j}(\vec{k}) \in \{0, \pm 1\}$. Considering $\xi$ the affine coordinate of a projective line and $(u, y)$ the affine coordinates of a genus-one fiber in $\mathbb{P}^2$, it follows that Equation (6.8) induces a genus-one fibration on the K3 surface $F$. Using [20, Prop. 3.3] it follows immediately:

**Lemma 6.4.** The general K3 surface $F$ admits a genus-one fibration (without section and) with three singular fibers of type $I_0^*$ and six singular fibers of type $I_1$.

The existence of such a genus-one fibration with three singular fibers of type $I_0^*$ on the K3 surface $F$ allowed the authors in [44, Thm. 1] to compute the lattice polarization of the family. We have:

**Proposition 6.5.** For generic parameters $\rho_j^{(i)}$ with $i, j \in \{1, 2, 3, 4\}$ the K3 surface $F$ is endowed with a canonical polarization given by the rank-fourteen lattice

$$\langle 2 \rangle \oplus \langle -2 \rangle \oplus D_4(-1)^{\oplus 3}.$$ 

Moreover, the transcendental lattice is $H(2)^{\oplus 2} \oplus (-2)^{\oplus 4}$ of signature $(2, 6)$.

We have the immediate:

**Corollary 6.6.** The double $4\mathcal{H}$-surfaces $\mathcal{F}$ form the moduli space of correspondences between the K3 surfaces $\mathcal{X}$ and $\tilde{\mathcal{Y}}$ in Figure 5.

### 6.2. Double covers of a cubic and three lines.

In this section we describe the geometry of the K3 surfaces $\tilde{\mathcal{Y}}$ in Figure 5. They represent the CHL string backgrounds dual to the F-theory with additional symplectic involution.

Let $\tilde{\mathcal{Y}}$ be the double cover of the projective plane $\mathbb{P}^2 = \mathbb{P}(Z_1, Z_2, Z_3)$ branched over the union of three lines $\ell_1, \ell_2, \ell_3$ coincident in a point and a cubic $E$. We call such a configuration generic if the cubic is smooth and meets the three lines in nine distinct points. In particular, the cubic does not meet the point of coincidence of the three lines. We construct a geometric model as follows: we use a suitable projective transformation to move the line $\ell_3$ to $\ell_3 = V(Z_3)$. We then mark three distinct points $q_0$, $q_1$, and $q_\infty$ on $\ell_3$ and use a Möbius transformation to move these points to $[Z_1 : Z_2 : Z_3] = [0 : 1 : 0]$, $[1 : 1 : 0]$, and $[1 : 0 : 0]$. Let $q_1 : [1 : 1 : 0]$ be the point of coincidence of the three lines. Up to scaling, the three lines, coincident in $q_1$, are then given by

$$\ell_1 = V(Z_1 - Z_2 + \mu Z_3), \quad \ell_2 = V(Z_1 - Z_2 + \nu Z_3), \quad \ell_3 = V(Z_3),$$

for some parameters $\mu, \nu$ with $\mu \neq \nu$. Let the cubic $E = V(C(Z_1, Z_2, Z_3))$ intersect the line $\ell_3$ at $q_0$, $q_\infty$, and at the point $[-d_2 : c_1 : 0] \neq [1 : 1 : 0]$. Thus, we have

$$C = e_3 Z_3^2 + (d_0 Z_1 + e_1 Z_2) Z_3^2 + (c_0 Z_1^2 + d_1 Z_1 Z_2 + e_2 Z_2^2) Z_3 + Z_1 Z_2 (c_1 Z_1 + d_2 Z_2).$$

This can be written as

$$C = (c_1 Z_2 + c_0 Z_3) Z_1^2 + (d_2 Z_2^2 + d_1 Z_2 Z_3 + d_0 Z_3^2) Z_1 + (e_2 Z_2^2 + e_1 Z_2 Z_3 + e_0 Z_3^2) Z_3,$$
such that in \( \mathbb{WP}_{(1,1,1,3)} = \mathbb{P}(Z_1, Z_2, Z_3, Y) \) the surface \( \tilde{Y} \) is given by
\[
Y^2 = (Z_1 - Z_2 + \mu Z_3)(Z_1 - Z_2 + \nu Z_3)Z_3 C(Z_1, Z_2 Z_3),
\]
for parameters \( \mu, \nu, c_0, c_1, d_0, d_1, d_2, e_0, e_1, e_2 \) with \( c_1 \neq 0, c_1 + d_2 \neq 0, \mu \neq \nu, \) and a smooth cubic \( E \) that intersects each line \( \ell_1, \ell_2, \ell_3 \) in three distinct points. We have the following:

**Lemma 6.7.** The cubic \( E \) is tangent to the line \( \ell_3 \) at \( q_0 \) if and only if \( d_2 = 0 \) and the remaining parameters are generic. The cubic \( E \) is singular at \( q_0 \) if and only if \( d_2 = e_2 = 0 \) and the remaining parameters are generic.

In [11] the authors proved that the coefficients of the curves in the branch locus can be normalized as follows:

**Lemma 6.8.** Let \( \tilde{Y} \) be the double cover of the projective plane \( \mathbb{P}^2 = \mathbb{P}(Z_1, Z_2, Z_3) \) branched over three lines coincident in a point and a generic cubic. There are affine parameters \( (d_2, \mu, c_0, e_2, d_0, e_1, e_0) \in \mathbb{C}^7 \), unique up to the action given by
\[
(d_2, \mu, c_0, e_2, d_0, e_1, e_0) \mapsto (d_2, \Lambda \mu, \Lambda c_0, \Lambda e_2, \Lambda^2 d_0, \Lambda^2 e_1, \Lambda^3 e_0)
\]
with \( \Lambda \in \mathbb{C}^\times \), such that \( \tilde{Y} \) in \( \mathbb{WP}_{(1,1,1,3)} = \mathbb{P}(Z_1, Z_2, Z_3, Y) \) is obtained by
\[
Y^2 = (Z_1 - Z_2 + \mu Z_3)(Z_1 - Z_2 + \nu Z_3)Z_3
\]
\[
\times \left( (Z_2 + c_0 Z_3 Z_1^2 + (d_2 Z_3^2 + d_0 Z_3^2)Z_1 + (e_2 Z_2^2 + e_1 Z_2 Z_3 + e_0 Z_3^2)Z_3 \right)
\]
with \( \mu + \nu = (1 + d_2/2)(c_0 + e_2) \) and \( d_2 \neq -1 \).

We denote by \( \bar{Y} \) the surface obtained as the minimal resolution of \( \tilde{Y} \). Since \( \bar{Y} \) is the resolution of a double-sextic surface, it is a K3 surface. We will now construct a Jacobian elliptic fibrations on it to establish the connection with \( \bar{Y} \) in Figure 5:

**Lemma 6.9.** A generic K3 surface \( \bar{Y} \) admits a Jacobian elliptic fibration with the singular fibers \( 3I_0^* + 6I_1 \) and a trivial Mordell-Weil group.

**Proof.** The pencil of lines \( (Z_1 - Z_2) - t Z_3 = 0 \) for \( t \in \mathbb{C} \) through the point \( q_1 = [1 : 1 : 0] \) induces an elliptic fibration on \( \bar{Y} \). We refer to this fibration as the standard fibration. When substituting \( Z_1 = X, Z_2 = X - (c_1 + d_2)(t + \mu)(t + \nu) t, \) and \( Z_3 = (c_1 + d_2)(t + \mu)(t + \nu) \) into Equation (6.14) we obtain the Weierstrass model
\[
Y^2 = X^3 - (t + \mu)(t + \nu)((c_1 + 2d_2)t - (c_0 + d_1 + e_2))X^2 
\]
\[
+ (c_1 + d_2)(t + \mu)^2(t + \nu)^2(d_2t^2 - (d_1 + 2e_2)t + (d_0 + e_1))X 
\]
\[
+ (c_1 + d_2)^2(t + \mu)^3(t + \nu)^3(e_2t^2 - e_1t + e_0),
\]
with a discriminant function of the elliptic fibration \( \Delta = (t + \mu)^6(t + \nu)^6(c_1 + d_2)^2p(t), \) where \( p(t) = c_1^2 d_2^2 t^6 + \ldots \) is a polynomial of degree six. Given the Weierstrass model in Equation (6.17) the statement is checked by explicit computation.  

Since we always assume \( c_1 \neq 0 \) we have:
Corollary 6.10. The fibration in Lemma 6.9 has the singular fibers $I_1^* + 2I_0^* + 5I_1$ if and only if $d_2 = 0$ and the remaining parameters are generic. It has the singular fibers $I_2^* + 2I_0^* + 4I_1$ if and only if $d_2 = e_2 = 0$ and the remaining parameters are generic, and the singular fibers $I_3^* + 2I_0^* + 3I_1$ if and only if $d_2 = e_2 = e_1 = 0$ and the remaining parameters are generic.

We also have the converse statement of Lemma 6.9:

Proposition 6.11. A $K3$ surface admitting a Jacobian elliptic fibration with the singular fibers $3I_0^* + 6I_1$ and a trivial Mordell-Weil group arises as the double cover of the projective plane branched over three lines coincident in a point and a cubic.

Proof. Using a M"obius transformation we can move the base points of the three singular fibers of type $I_0^*$ to $\mu, \nu, \infty$. An elliptic surface admitting the given Jacobian elliptic fibration then has a Weierstrass model of the form

$$Y^2 = X^3 + (t + \mu)(t + \nu)\left(\tilde{c}_1 t + \tilde{c}_0\right)X^2 + \left(t + \mu\right)^2(t + \nu)^2\left(d_2 t^2 + d_1 t + d_0\right)X$$

(6.18)

$$+ \left(t + \mu\right)^3(t + \nu)^3\left(\tilde{e}_3 t^3 + \tilde{e}_2 t^2 + \tilde{e}_1 t + \tilde{e}_0\right).$$

A shift $X \mapsto X + \rho(t + \mu)(t + \nu)$ eliminates the coefficient $\tilde{e}_3$ in Equation (6.18) if $\rho$ is a solution of $\rho^3 + \tilde{c}_1 \rho^2 + \tilde{d}_2 \rho + \tilde{c}_3 = 0$. Thus, we can assume $\tilde{e}_3 = 0$. Next, let $c_1$ be a root of $c_1^2 = \tilde{c}_1^2 - 4\tilde{d}_2$. Then substituting

$$c_0 = \frac{2\tilde{d}_1}{c_1 - \tilde{c}_1} + \frac{4\tilde{c}_2}{(c_1 - \tilde{c}_1)^2} + \tilde{c}_0, \quad d_0 = \frac{2\tilde{d}_0}{c_1 - \tilde{c}_1} + \frac{4\tilde{c}_1}{(c_1 - \tilde{c}_1)^2}, \quad e_0 = \frac{4\tilde{e}_0}{(c_1 - \tilde{c}_1)^2},$$

(6.19)

$$d_1 = -\frac{2\tilde{d}_2}{c_1 - \tilde{c}_1} - \frac{8\tilde{e}_1}{(c_1 - \tilde{c}_1)^2}, \quad e_1 = -\frac{4\tilde{e}_1}{(c_1 - \tilde{c}_1)^2},$$

$$d_2 = -\frac{c_1 + \tilde{c}_1}{2}, \quad e_2 = \frac{4\tilde{e}_2}{(c_1 - \tilde{c}_1)^2},$$

into Equation (6.17) recovers Equation (6.18).

For the double 4H surface $F = \coprod_{\xi} F_{\xi}$ in Proposition 6.4 with the fibers $F_{\xi}$ of genus one, we construct the relative Jacobian fibration $\coprod_{\xi} \text{Jac}^0(F_{\xi})$. We have the following:

Theorem 6.12. The relative Jacobian fibration $\coprod_{\xi} \text{Jac}(F_{\xi})$ associated with a generic double 4H surface $F$ is a $K3$ surface $\tilde{F}$ obtained as the minimal resolution of the double-sixfold surface for a generic configuration of three lines coincident in a point and a cubic. The latter defines an elliptic curve $E$ in $\mathbb{P}^2 = \mathbb{P}(Z_1, Z_2, Z_3)$ given by

$$E: \quad 0 = Z_1^3 + f(Z_2, Z_3) Z_1 + g(Z_2, Z_3),$$

(6.20)

where $f, g$ were given by Equation (6.4).

Proof. It was shown in [20] how a Weierstrass model for $F' = \coprod_{\xi} \text{Jac}(F_{\xi})$ is constructed explicitly. Applied to Equation (6.8) we obtain a Weierstrass model for $F'$ given by

$$Y^2 = X^3 + (\xi + \mu)(\xi + \nu)A_2 X^2$$

(6.21)

$$+ (\xi + \mu)^2(\xi + \nu)^2 \left(A_1 A_3 - 4A_0 A_4\right)X$$

$$+ (\xi + \mu)^3(\xi + \nu)^3 \left(A_1^2 A_4 + A_0 A_3^2 - 4A_0 A_2 A_4\right).$$
This equation has the form of Equation (6.18) considered in Proposition 6.11. For the Weierstrass model in Equation (6.17) we then reconstruct the cubic in the branch locus by setting $Y = 0$, rescaling $X \mapsto (t + \mu)(t + \nu)X$, and extracting the irreducible cubic part. Using the defining equation for $\tilde{Y}$ in Table 3 yields Equation (6.20). □

7. Summary of results and discussion

The highly non-trivial connection between families of K3 surfaces and their polarizing lattices appears in string theory as the manifestation of the F-theory/heterotic string duality. This viewpoint has been studied in [13, 19, 31, 41, 42, 50, 51]. We proved in Theorem 4.8 that there are algebraic correspondences between the K3 surfaces polarized by the rank-ten lattices $H \oplus N$ and $H \oplus E_8(-2)$. Since the moduli spaces $\mathcal{M}_{H \oplus N}$ and $\mathcal{M}_{H \oplus E_8(-2)}$ are also the moduli spaces of F-theory models with discrete flux and the CHL string (heterotic string with CHL involution), respectively, Theorem 4.8 and Figure 3 provide a mathematical framework for the duality between F-theory with discrete flux and the CHL string in eight dimensions.

A natural 6-dimensional subspace that is contained simultaneously in both aforementioned physical moduli spaces is the moduli space of K3 surfaces polarized by the lattice $H \oplus N \oplus D_4(-2)$. This is a subspace where the F-theory admits an additional anti-symplectic involution, and on the CHL string side one has an additional symplectic involution. The duality diagram in Figure 3 then extends to Figure 4. In Theorem 5.13 we proved an explicit parametrization for elements $\tilde{\mathcal{F}}$ of the moduli space of correspondences. A general element $\tilde{\mathcal{F}}$ is a double-quadrics with a branch locus that is reducible and consists of two curves of bi-degree $(1, 0)$ and $(0, 1)$, respectively, and a genus-one curve in the linear system $|O_{\mathcal{F}_0}(2, 2)|$ such that the curves intersect in ordinary rational double points. The parametrization is then based on a construction of Andrè Weil [68], in which the Abel-Jacobi map is used to obtain embeddings of genus-one curves as symmetric divisors of bi-degree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$.

We also showed that the F-theory moduli space has another natural 6-dimensional subspace, namely the moduli space of K3 surfaces polarized by the lattice $(2) \oplus (-2) \oplus D_4(-1)^{\otimes 3}$. This special situation corresponds to the case when, on the F-theory side, surfaces admit an additional symplectic involution and, on the CHL string side, a special anti-symplectic involution exist. The duality diagram in Figure 3 then extends to Figure 5. The correspondences $\mathcal{F}$ turn out to then be precisely the double $4\mathcal{H}$-surfaces considered in [44]. In Theorem 6.12 we proved that the K3 surfaces $\tilde{\mathcal{Y}}$ associated with the CHL string also carry a beautiful geometric description: they are special double-sextic surfaces branched over a configuration of three distinct lines coincident in a point and an additional generic cubic. The latter divisor gives rise to an elliptic curve capturing part of the K3 moduli coordinates.

In the two special examples above, both involving 6-dimensional subspaces of $\mathcal{M}_{H \oplus N}$, an elliptic curve naturally emerges. This is not the elliptic curve upon which the CHL string is constructed. Rather, the elliptic curve underlying the heterotic string arises an an anti-canonical curve (cf. [4]), here, the anti-canonical curve in the rational elliptic surface $\mathcal{R}$ in Figure 4 and Figure 5, respectively. The role of the elliptic curve underlying the heterotic string and the rational elliptic surface was
investigated in previous work of the authors in [20]. In contrast, the elliptic curve that emerges in the parametrization of the moduli space of K3 surfaces polarized by the lattice $H \oplus N \oplus D_4(-2)$ and $(2) \oplus (-2) \oplus D_4(-1)^{\#3}$ is a Seiberg-Witten type curve and parameterizes certain moduli of the F-theory/CHL vacua under consideration. The relation between this Seiberg-Witten type curve and the twisted principal $E_8 \times E_8$ bundle over the elliptic curve defining the heterotic string will be investigated in future work by the authors.

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