New Fundamental Frequency Domain Formula of Image Restoration and Signal Processing

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Abstract

It is usually considered that time (or space) domain’s convolution is equivalent to frequency domain’s multiplication in signal processing. However, in image restoration, we prove that this rule doesn’t satisfy a symmetric property of PSF convolution, which is an intrinsic physical property of blur producing. The space domain circular convolution corresponding to the existing frequency domain formula violates that symmetric property. New modified frequency domain formula is given. Experiments are done to show that the existing formula causes visible artifacts while the modified one does not. Since the frequency domain formula is a fundamental theory in image restoration, algorithms related with that are all needed to be modified, such as the state of the art methods of Wiener filter and constrained least square estimation. For clarity of proving, the one-dimensional case is proved first, which may be useful in frequency domain related techniques of signal processing.

Keywords: Image restoration; Frequency domain formula; Symmetric property; Circular convolution; PSF; Signal processing.

1. Introduction

The basic mathematical model of image restoration is [1]

\[
g(x, y) = f(x, y) \ast h(x, y) + n(x, y) \\
= \sum_{m,n} f(n, m)h(x - n, y - m) + n(x, y),
\]

(1-1)

where \(g(x,y)\) is the degraded image, \(f(x,y)\) is the original image, \(h(x,y)\) is the convolution operator as well as the point spread function (PSF), \(n(x,y)\)
is the additive noise. The goal of image restoration is to reconstruct the original image $f(x, y)$ from the degraded observation image $g(x, y)$ by getting the solution $f(x, y)$ of (1-1).

The corresponding frequency domain form of (1-1) is

$$G(u, v) = F(u, v) \mathcal{H}(u, v) + N(u, v), \quad (1-2)$$

which is a fundamental formula in image restoration [2]. As a fundamental result, many image restoration algorithms are based on (1-2). For example, IBD [1, 3] is a classical algorithm in blind image restoration and its main operations are in frequency domain, which is a direct use of (1-2).

The Wiener filter [4] applied in image restoration is widely known and its frequency domain solution is

$$F = \frac{S_{ff} \mathcal{H} G}{S_{ff} \|\mathcal{H}\|^2 + S_{nn}},$$

which can be expressed as

$$G(u, v) = F(u, v) \mathcal{H}(u, v) + P(u, v),$$

where $P(u, v) = F S_{nn} \setminus \mathcal{H} S_{ff}(u, v)$. The transforming into frequency domain is necessary because this can avoid the inverse computations of large matrix and using FFT can get the DFT or IDFT of the signal quickly and efficiently.

Similarly, the frequency domain solution of constrained least square estimation [5] can also be written in the form of (1-2). From these examples of the state of art methods, we can see that (1-2) plays an important role in image restoration.

In this paper, we’ll rigorously prove that (1-2)’s corresponding spatial domain PSF circular convolution does not satisfy a physical property and then give the modified one. For clarity, the case of one-dimensional signal will be dealt with first, which is used to generalize to the two-dimensional image.

2. Symmetric property of the PSF convolution

The sources of image blur are mainly classified into three categories [6]: defocus blur, atmospheric turbulence blur and motion blur. All of their PSF can be modeled by masks (also referred to as filters, kernels, templates, or
windows). The weight values of the PSF mask determine the degradation type.

The symmetric property of the PSF convolution means that the image pixel to be processed should be aligned with the symmetric center of the PSF mask. Fig. 1(a) gives an example. The big white square is a PSF mask of size $3 \times 3$. The gray-filled small square in the big square is the symmetric center of the PSF mask whose size equals one pixel of the image (in the sense of principle). The location of gray-filled square is exactly the processed image pixel. This symmetric property has clear physical meaning in producing of defocus blur and atmospheric turbulence blur and also is required in the blind deconvolution of motion blur.

The PSF of defocus blur is [6]

$$h(x, y) = \begin{cases} \frac{1}{\pi R^2}, & \text{if } \sqrt{x^2 + y^2} \leq R, \\ 0, & \text{otherwise} \end{cases}$$

which is centrosymmetric and its symmetric center is $(0, 0)$. The pixel to be convolved should be aligned with this center and the new pixel value is the weighted sum of neighborhood pixels. So is the case of Atmospheric turbulence blur whose PSF is $h(x, y) = Ke^{-\frac{x^2+y^2}{2\sigma^2}}$ [6].

Motion blur occurs when there is a relative motion between the scene imaged and the camera during exposure. Despite non-symmetric in weight values, motion blur PSF can be modeled by a square mask with nonzero weight values in the motion direction. The PSF of horizontal motion blur
with uniform velocity is [7]:

\[ h(x, y) = \frac{1}{\alpha_0} \text{rect}(\frac{x}{\alpha_0} - \frac{1}{2})\delta(y), \]

where \( \text{rect}(x) = 1 \) for \( |x| \leq \frac{1}{2} \) and \( \text{rect}(x) = 0 \) otherwise, and \( \delta(y) \) is the Dirac delta function. As shown in Fig.1(b), it’s a PSF mask of horizontal motion blur whose weight values are nonzero in the horizontal white line, convolving with which could produce a motion blur image in horizontal direction. In the perspective of mask model, the convolution process of motion blur is the same as defocus or atmospheric turbulence blur. This symmetric mask model is useful in blind deconvolution of motion blur image, since the motion direction is unknown so that all possible directions should be taken into consideration.

This paper is to solve the problem that the space domain’s PSF convolution corresponding to frequency domain formula (1-2) doesn’t satisfy that symmetric property.

3. The one-dimensional case

We first discuss the one-dimensional case in order to make the proving as clear as possible. Meanwhile, the result of one-dimension may be useful in frequency domain related signal processing techniques.

We’ll mainly prove two results. The first is why the existing frequency domain formula doesn’t satisfy the symmetric property, which is in Theorem 3.1. The second is the modified formula given in Theorem 3.2.

3.1. Problems of the symmetric property

Let \( x(n) \) and \( y(n) \) be the original signal and the convolved signal respectively for \( n = 0, 1, \ldots, N - 1 \). \( h(n) \) is the PSF for \( n = 0, 1, \ldots, M - 1 \), where \( M < N \) and \( M \) is an odd integer. \( h_e(n) \) is the extended form of \( h(n) \) defined as [5]

\[
h_e(n) = \begin{cases} h(n) & \text{for } 0 < n \leq M - 1 \\ 0 & \text{for } M \leq n \leq N - 1 \end{cases}
\]

Denote the PSF circular convolution of one-dimension as

\[
y(n) = x(n) * h(n), \tag{3-1}
\]

and its existing frequency domain formula is

\[
\mathcal{Y}(k) = \mathcal{X}(k)\mathcal{H}(k), \tag{3-2}
\]
Figure 2: Symmetric property of one-dimensional case.

where $\mathcal{X}(k)$, $\mathcal{H}(k)$, and $\mathcal{Y}(k)$ are the DFT of $x(n)$, $h_e(n)$ and $y(n)$ respectively.

The symmetric property of one-dimension is similar to that of two-dimension. As in Fig. 2, the time varying curve is a signal and the rectangle is a PSF mask with its symmetric center gray filled. It shows that the signal element to be processed is aligned with the symmetric center of the PSF, which is the one-dimensional symmetric property.

**Lemma 3.1.** If the frequency domain formula of one-dimensional circular convolution is (3-2) and the matrix form of (3-1) is

$$y = Hx,$$  \hspace{1cm} (3-3)

then we must have

$$H = \begin{pmatrix}
h_e(0) & h_e(N-1) & \cdots & h_e(1) \\
h_e(1) & h_e(0) & \cdots & h_e(2) \\
\vdots & \ddots & \ddots & \vdots \\
h_e(N-1) & h_e(N-2) & \cdots & h_e(0)
\end{pmatrix},$$  \hspace{1cm} (3-4)

which is a circulant matrix [8].

**Proof.** Expanding (3-3) by (3-4), we have

$$y(n) = \sum_{k=0}^{N-1} x(k)h(n - k \mod N) = x(n) * h(n),$$

which means that (3-3) is the matrix form of circular convolution. So the statement of this lemma is equivalent to: if the frequency domain formula is
\[ Y(k) = X(k)H(k), \]
then the corresponding time domain formula is \( y(n) = x(n) * h(n) \), which is in fact the converse of the DFT’s frequency domain property of circular convolution [9]. What we need to do is to rewrite the proof of DFT’s this property conversely and the computation details of this proof are simple that can be found in literature [9].

**Theorem 3.1.** The PSF with respect to the frequency domain formula (3-2) is not \( h(n) \), but the reverse order of \( h(n) \):

\[ h'(n) = [h(M-1) \ h(M-2) \ \cdots \ h(0)]^T. \quad (3-5) \]

The time domain’s convolution corresponding to (3-2) doesn’t satisfy the symmetric property. The PSF element \( h'(M-1) \) instead of the symmetric center is aligned with the processed signal element.

**Proof.** According to Lemma 3.1, (3-2)’s time domain convolution matrix is \( H \) of (3-4) and the matrix form of the convolution operation is (3-3).

It’s essential to describe the “convolution physical meaning” of the convolution matrix as (3-4) first. In matrix \( H \) of (3-4), each row corresponds to the convolution of each signal element. The convolution structure, i.e., the way how to sum neighbourhood’s value, is determined by matrix \( H \)’s first row, which is the same for all signal elements due to the property of circulant matrix. Different rows of \( H \) are only different positions of the PSF.
Thus, we only need to prove the case of the first signal element. The first processed signal element is $x(0)$ and the convolved result is $y(0)$. By (3-3) and (3-4), we know

$$y(0) = \begin{bmatrix} h_e(0) & h_e(N-1) & \cdots & h_e(1) \end{bmatrix} \cdot \begin{bmatrix} x(0) & x(1) & \cdots & x(N-1) \end{bmatrix}^T.$$ 

Rearrange the elements order of the two multiplying vectors and make the sum $y(0)$ unchanged simultaneously in such a way

$$y(0) = \begin{bmatrix} h_e(N-1) & \cdots & h_e(1) & h_e(0) \end{bmatrix} \cdot \begin{bmatrix} x(1) & \cdots & x(N-1) & x(0) \end{bmatrix}^T.$$ 

By definition, $h_e(n)$ is zero when $n \geq M$ and equals $h(n)$ otherwise, which follows

$$y(0) = \begin{bmatrix} h(M-1) & \cdots & h(1) & h(0) \end{bmatrix} \cdot \begin{bmatrix} x(N-(M-1)) & \cdots & x(N-1) & x(0) \end{bmatrix}^T. \tag{3-6}$$ 

We can visualize (3-6) by specifying $N = 6$ and $M = 3$ to see an example. Fig.3(a) indicates that the elements of the PSF weighting neighborhoods are the reverse order of the original PSF. The symmetric center $h(1)$ of the PSF mask is not aligned with the processed signal element $x(0)$, but the third PSF element is. The general description of this paragraph is Theorem 3.1.

Note that (3.6) visualized in Fig.3(a) is actually the computing method of circular convolution operation [9], which is extremely familiar with us; however, it indeed violates the physical property of PSF convolution. Fig.3(b) is an example of Theorem 3.1 where the PSF is laid on a concrete signal.

3.2. The modified frequency domain formula

The first step is to move the PSF in Theorem 3.1 to make its center aligned with the signal element to be processed.

**Lemma 3.2.** The symmetric center of the PSF corresponding to the convolution matrix $H$ of (3-4) can be made aligned with the processed signal element by moving matrix $H$’s elements in terms of

$$H' = H \alpha^I,$$ \tag{3-7}
where

\[
\alpha = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]  (3-8)

is a \( N \times N \) permutation matrix \([8]\) and

\[
t = \frac{M - 1}{2}. \tag{3-9}
\]

**Proof.** As stated in literature \([8]\), when a vector such as \( h_e(n) \) multiplies permutation matrix \( \alpha \), it means that all the elements of \( h_e(n) \) move one position to the right and wrap around, which is actually a forward shift permutation

\[
h_e^T \alpha = \sigma(h_e^T) = [h_e(N - 1) \ h_e(0) \ \cdots \ h_e(N - 2)],
\]

where \( \sigma \) is a permutation operator. So multiplying \( \alpha^t \) means to do this operation by \( t \) times. Obviously, moving the elements is equivalent to moving the position of PSF.

Write matrix \( H \) as

\[
H = [H_1 \ H_2 \ \cdots \ H_N]^T.
\]

where \( H_i \) is the \( i \)th row of \( H \), by which (3-7) can be expressed as

\[
H' = [H_1 \alpha^t \ H_2 \alpha^t \ \cdots \ H_N \alpha^t]^T. \tag{3-10}
\]

From (3-10), in each row of matrix \( H \), if we move each element \( t \) positions to the right and wrap around, the result is \( H' \). Theorem 3.1 has proved that when the convolution matrix is \( H \), the PSF element \( h'(M-1) \) is aligned with the processed signal element. Because the index of PSF’s center element is \((M - 1)/2 \), if we move the PSF right by

\[
t = M - 1 - \frac{M - 1}{2} = \frac{M - 1}{2}
\]

steps, which is (3-9), the symmetric property will be satisfied.

Fig.4 shows an example. Fig.4(a) is the convolution of \( H \). After matrix \( \alpha \)’s permutation operation, Fig.4(b) of matrix \( H' \) satisfies the symmetric property.

Note that this operation doesn’t change the elements order of the PSF, so the PSF is still (3-5) instead of the original \( h(n) \).
What’s the frequency domain formula of $y = H'x$ with respect to $H'$? This looks like literally related to DFT’s circular shift property; nevertheless, they are different in details. To answer this question, we need a lemma as follows:

**Lemma 3.3.** Let $C$ be an order $N$ circulant matrix

$$C = \begin{pmatrix} c_0 & c_1 & \cdots & c_{N-1} \\ c_{N-1} & c_0 & \cdots & c_{N-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & \cdots & c_0 \end{pmatrix}.$$  \hspace{1cm} (3-11)

and $W$ is a unitary matrix

$$W = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ W_N^{1 \cdot 0} & W_N^{1 \cdot 1} & \cdots & W_N^{1 \cdot (N-1)} \\ W_N^{2 \cdot 0} & W_N^{2 \cdot 1} & \cdots & W_N^{2 \cdot (N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ W_N^{(N-1) \cdot 0} & W_N^{(N-1) \cdot 1} & \cdots & W_N^{(N-1) \cdot (N-1)} \end{pmatrix}.$$ \hspace{1cm} (3-12)

where $W_N = e^{i2\pi/N}$. Then $C$ can be diagonalized by

$$D = W^{-1}CW,$$ \hspace{1cm} (3-13)

where $D$ is a diagonal matrix with its diagonal elements being

$$D(k,k) = \sum_{n=0}^{N-1} c_n W_N^{kn}.$$ \hspace{1cm} (3-14)

(3-14) is very important for our demonstration and can be interpreted that the diagonal elements of $D$ are the linear combination of $W_N^{kn}$ weighted by the first row elements of $C$. 

Figure 4: The effect of moving PSF in Lemma 3.2.
Proof. The proof can be found in literature [10].

**Lemma 3.4.** The frequency domain formula corresponding to $y = H'x$ of Lemma 3.2 is

$$\mathcal{Y}(k) = \mathcal{X}(k)\mathcal{H}(k)e^{\frac{j2\pi k}{N}t}, \quad (3-15)$$

where $t$ is defined in (3-9).

Proof. We get the result from $y = Hx$ whose frequency domain form is $\mathcal{Y}(k) = \mathcal{X}(k)\mathcal{H}(k)$. Because $H$ is a circulant matrix, it can be diagonalized by $D = W^{-1}HW$. We shall prove that $D(k, k)$ equals $H(k)$, i.e., the DFT of $h_e(n)$. The first row of $H$ is

$$H_1 = [h_e(0) \ h_e(N-1) \ \cdots \ h_e(1)].$$

By (3-14) of Lemma 3.3, it follows

$$D(k, k) = h_e(0) + h_e(N-1)W_N^{k-1} + \cdots + h_e(1)W_N^{k(N-1)}, \quad (3-16)$$

while the DFT of $h_e(n)$ is

$$\mathcal{H}(k) = h_e(0) + h_e(1)W_N^{k-1} + \cdots + h_e(N-1)W_N^{-k(N-1)}. \quad (3-17)$$

Compared (3-16) with (3-17), because $W_N^{-k(N-1)} = W_N^{ki}$, $D(k, k) = \mathcal{H}(k)$ holds.

$H'$ can also be diagonalized by Lemma 3.3 as

$$D' = W^{-1}H'W \quad (3-18)$$

and

$$D'(k, k) = \sum_{n=0}^{N-1} H'_{1n} W_n^{kn}, \quad (3-19)$$

where $H'_{1n}$ is the first row element of matrix $H'$. (3-19) is corresponding to (3-16) with respect to different circulant matrix. Since by Lemma 3.2, the first row of $H'$ is $H$’s first row elements moving right $t$ positions, we have

$$D'(k, k) = D(k, k)W_N^{kt} = \mathcal{H}(k)e^{\frac{j2\pi k}{N}t}. \quad (3-20)$$
Now we turn to the frequency form of $y = H'x$. (3-18) implies $H' = W'D'W^{-1}$. Substituting this into $y = H'x$ yields

$$W^{-1}y = D'W^{-1}x,$$

where $W^{-1}y$ and $W^{-1}$ are just the DFT of $y(n)$ and $x(n)$ respectively, so that

$$\mathcal{Y}(k) = D'(k,k)\mathcal{X}(k).$$ \hspace{1cm} (3-21)

(3-20) and (3-21) imply the final result of (3-15).

The mathematical form of (3-15) is very simple, which seems to be deduced directly from DFT’s circular shift property [9]. However, the shifted sequence is $H_1$ of matrix $H$’s first row instead of the original PSF signal $h_e(n)$, so the proof is not trivial.

Based on the above lemmas, we present the modified frequency domain formula of (3-2).

**Theorem 3.2.** If the frequency domain formula is modified to

$$\mathcal{Y}(k) = \mathcal{X}(k)\mathcal{H}'(k)e^{\frac{2\pi i}{N}k_0},$$ \hspace{1cm} (3-22)

where $\mathcal{H}'(k)$ is the DFT of $h_e'(n)$, which is the extended form of $h'(n)$ of (3-5), then its corresponding time domain convolution satisfies the symmetric property and the PSF is the original $h(n)$.

**Proof.** (3-15) is the frequency domain formula of matrix $H'$ that satisfies the symmetric property as mentioned in Lemma 3.2; however, the PSF of $H'$ is not $h(n)$ but the reverse order of $h(n)$. Noting that $\mathcal{H}(k)$ of (3-15) is corresponding to the convolution of $h'(n)$, while $h'(n)$ is the reverse order of $h(n)$, if we modify $\mathcal{H}(k)$ to be the DFT of $h_e'(n)$, then its corresponding time domain PSF will be the reverse order of $h'(n)$, which is just $h(n)$. Therefore, (3-22) is the final result.

### 3.3. Summary

In this section, the one-dimensional case was discussed. Next we’ll generalize it to the two-dimension.
4. The two-dimensional case

Let’s give a more detailed explanation of the notations in Section 1. Denote the original image and degraded image by \( f(m, n) \) and \( g(m, n) \) respectively for \( 0 \leq m \leq M - 1 \) and \( 0 \leq n \leq N - 1 \). \( h(m, n) \) is the PSF for \( 0 \leq m \leq J - 1 \), \( 0 \leq n \leq K - 1 \), where \( J \) and \( K \) are odd integers. \( h_e(m, n) \) is the extended form of \( h(m, n) \):

\[
h_e(m, n) = \begin{cases} 
  h(m, n) & \text{for } 0 \leq m \leq J - 1 \text{ and } 0 \leq n \leq K - 1 \\
  0 & \text{for } J \leq m \leq M - 1 \text{ and } K \leq n \leq N - 1 
\end{cases}
\]

\( \mathcal{F}(u, v), \mathcal{G}(u, v) \) and \( \mathcal{H}(u, v) \) are the DFT of \( f(m, n) \), \( g(m, n) \) and \( h_e(m, n) \) respectively.

\( \vec{g} \) is the vector-matrix form of matrix \( g \), which represents the two-dimensional matrix by zigzag order. So is the \( \vec{f} \).

Rewrite the space domain and frequency domain formula of image restoration here (with noise ignored):

\[
g(m, n) = f(m, n) * h(m, n) \tag{4-1}
\]

and

\[
\mathcal{G}(u, v) = \mathcal{F}(u, v) \mathcal{H}(u, v). \tag{4-2}
\]

(4-1)’s matrix form is

\[
\vec{g} = H \vec{f}, \tag{4-3}
\]

where \( H \) is a block circulant matrix [8].

4.1. Symmetric problems of the two-dimensional case

**Lemma 4.1.** If the frequency domain formula of the two-dimensional circular convolution is (4-2), then its corresponding convolution block matrix must be

\[
H = \begin{pmatrix}
  H_0 & H_{M-1} & \cdots & H_1 \\
  H_1 & H_0 & \cdots & H_2 \\
  \vdots & & \ddots & \vdots \\
  H_{M-1} & H_{M-2} & \cdots & H_0
\end{pmatrix}, \tag{4-4}
\]
where $H_i(0 \leq i \leq M - 1)$ is

$$H_i = \begin{pmatrix}
h_e(i,0) & h_e(i,N-1) & \cdots & h_e(i,1) \\
h_e(i,1) & h_e(i,0) & \cdots & h_e(i,2) \\
\vdots & \ddots & \ddots & \vdots \\
h_e(i,N-1) & h_e(i,N-2) & \cdots & h_e(i,0)
\end{pmatrix}. \tag{4-5}$$

**Proof.** Just as the one-dimensional case, this lemma is the converse deduction of circular convolution property of the two-dimensional DFT, which can be easily generalized by Lemma 3.1.

**Theorem 4.1.** The corresponding space domain convolution of the existing frequency domain formula (4-2) doesn’t satisfy the symmetric property and the PSF is changed into

$$h'(m,n) = h(\tau(m), \tau'(n)), \tag{4-6}$$

where $\tau(m) = M - 1 - m$ and $\tau'(n) = N - 1 - n$ are both reverse permutation operations. Element $h'(M-1,N-1)$ of the PSF mask is aligned with the processed image pixel, instead of the symmetric center $h'(\frac{M-1}{2}, \frac{N-1}{2})$.

**Proof.** As long as we notice the convolution meaning of block circulant matrix $H$ of (4-4), the two-dimensional generalization of Theorem 3.1 is trivial. The blocks of $H$ correspond to the convolution of row-dimension, while each block element $H_i$ as (4-5) is related to the convolution of column-dimension. Only considering the blocks of $H$, using Theorem 3.1 can prove the row-dimensional case. The column-dimension can be done by applying Theorem 3.1 to each matrix of (4-5).

Fig. 5 is a visualization of this theorem.

### 4.2. The modified frequency domain formula of two-dimension

**Theorem 4.2.** If the frequency domain formula is modified to

$$G(u, v) = F(u, v)H'(u, v)e^{\frac{j2\pi}{N}st}e^{\frac{j2\pi}{M}us}, \tag{4-7}$$

where $H'(u, v)$ is the two-dimensional DFT of $h'_e(m,n)$ (the extended form of $h'(m,n)$ of (4-6)), $s$ and $t$ are as the one-dimensional case of (3-9)

$$s = \frac{J-1}{2}, t = \frac{K-1}{2}, \tag{4-8}$$
where $J \times K$ is the PSF size, then its corresponding space domain convolution satisfies the symmetric property and the PSF is the original $h(m, n)$.

**Proof.** We decompose the proof into two kinds of one-dimensional case by the two-dimensional DFT of $h_e(m, n)$:

$$H(u, v) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} h_e(m, n) e^{-i \frac{2\pi u}{M} m} e^{-i \frac{2\pi v}{N} n},$$

which can be written as

$$H(u, v) = \sum_{m=0}^{M-1} h_e(m, n) e^{-i \frac{2\pi u}{M} m} \sum_{n=0}^{N-1} e^{-i \frac{2\pi v}{N} n}.$$  \hspace{1cm} (4-9)

In (4-9), fixing $n$, for arbitrary $v$, the left sum of (4-9) is the DFT of row-dimension in column $n$, where we can use Theorem 3.2, resulting in the factor $e^{i \frac{2\pi v}{N} n}$ and the one-dimensional DFT of the signal obtained by reversing row’s order. So is the column-dimension. Combinations of row and column yield the final result of (4-7).

5. Applications of new formula

Constrained least square estimation [5] is a classical method of image restoration, whose frequency domain solution is

$$F(u, v) = \frac{H(u, v)G(u, v)}{H(u, v)H(u, v) + \gamma C(u, v)C(u, v)},$$  \hspace{1cm} (5-1)
where $\mathcal{H}$ is the complex conjugate of $\mathcal{H}$ and $\mathcal{C}(u, v)$ is the smooth regularization term. (5-1) can be expressed in the form of fundamental frequency domain formula

$$G(u, v) = F(u, v)\mathcal{H}(u, v) + \alpha F(u, v)\mathcal{C}(u, v),$$

where $\alpha = \gamma \mathcal{C}(u, v)/\mathcal{H}(u, v)$.

The smooth regularization is also operated by mask’s convolution, so $\mathcal{C}(u, v)$ should be considered as the frequency domain form of a PSF. Both of $\mathcal{H}(u, v)$ and $\mathcal{C}(u, v)$ in (5-1) need to be modified by this paper’s result of
Figure 7: Deconvolution example of motion blur: (a) Motion blur image. (b) Deblurred image by new formula (5-2). (c) Deblurred image by formula (5-1). (d) Normalized difference of (5-2) between deblurred image and original image. (e) Normalized difference of (5-1).

(4-7). The modified version of (5-1) is

\[ F(u, v) = \frac{\overline{H}_1(u, v)G(u, v)}{H_1(u, v)\overline{H}_1(u, v) + \gamma C_1(u, v)\overline{C}_1(u, v)}, \quad (5-2) \]

where

\[ H_1(u, v) = H'(u, v)e^{\frac{j\pi v}{N}vt}e^{\frac{j\pi}{M}us}, \quad C_1(u, v) = C'(u, v)e^{\frac{j\pi v}{N}vt}e^{\frac{j\pi}{M}us}. \]

We’ll give two examples to demonstrate the effects of new formula intuitively. Fig.6(a) is an original image and Fig.6(b) is the blurred image by
circular convolution of uniform 2-D blur \([4]\) \(PSF\) mask of size \(7 \times 7\). Fig. 6(c) is the deblurred image by new formula (5-2) and Fig. 6(d) is deblurred by literature \([5]\)’s formula (5-1). Fig. 6(c) and Fig. 6(d) seem to have no differences apparently, but when comparing the normalized difference between the deblurred image and the original image, as shown in Fig. 6(e) and Fig. 6(f), it’s obvious that Fig. 6(c) is much better than Fig. 6(d), since its normalized difference is more uniform and much closer to zero.

Fig. 7 is a deconvolution example of motion blur image. Fig. 7(a) is a horizontal motion blur image of Fig. 6(a), which is obtained by circular convolution of \(27 \times 27\) size \(PSF\) mask. Fig. 7(b) is the deblurred image by new formula (5-2) and Fig. 7(c) is the result of formula (5-1). Because the size of the \(PSF\) mask is \(27 \times 27\), much bigger than that of Fig. 6, the artifacts caused by the existing formula (5-1) are visible in the right and bottom part of the image. The normalized difference is much more obvious comparing Fig. 7(d) to Fig. 7(e).

6. Conclusions

The symmetric property of the \(PSF\) convolution is the intrinsic physical property of the producing of defocus blur and atmospheric turbulence blur. Although motion blur doesn’t belong to that case, in mathematical form of the \(PSF\) mask model, it can be treated in the same way, which is necessarily required in motion blur blind deconvolution.

The frequency domain formula (1-2) as mentioned in Section 1 is fundamental in image restoration and widely used in several kinds of forms such as Wiener filter method \([4]\) and constrained least square estimation \([5]\). However, its corresponding space domain \(PSF\) convolution doesn’t satisfy the symmetric property. We proved this problem rigorously and gave the modified formula. The problems discussed in this paper are important whenever the restoration algorithm is related with (1-2).

For clarity of proving, we also dealt with the one-dimensional case, whose results may be useful in frequency domain related techniques of signal processing, such as the design of filters in frequency domain \([9]\).

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