EXPLICIT INVESTMENT SETTING IN A KALDOR MACROECONOMIC MODEL WITH MACRO SHOCK

ZHENGHUI LI, SHUANGLIAN CHEN* AND ZHEHAO HUANG
Guangzhou International Institute of Finance and Guangzhou University
Guangzhou, 510405, China

Abstract. As the inevitable attributes of macro shocks on macroeconomic system, in this paper, we develop a Kaldor macroeconomic model with shock. The shock is due to the investment uncertainty. We then provide an approach for macroeconomic control by calibrating the evolvement of the shocked Kaldor macroeconomic model with some expected benchmark process. The calibration is realized through the setting for investment. The benchmark process is usually the reflection of decisions or policies. An optimal investment setting associated with a five-dimensional nonlinear system of ordinary differential equations is presented. Through a logical modification for the boundary conditions, the nonlinear system is simplified to be linear and a completely explicit formula for the optimal investment setting is achieved. The rationality of the modification is supported by some stability condition. To cope with the systematic risk caused by the macro shock, we define a dynamic Value-at-Risk (VaR) as the risk measure capturing the risk level of the shocked Kaldor macroeconomic model and introduce a risk constraint into the programming of calibration. Then a constrained investment setting is presented. Finally, we carry out an application of the theoretical results by calibrating the evolvement of the shocked Kaldor macroeconomic model with the business cycle generated from the classical Kaldor model through the investment setting.

1. Introduction. There has been much interest in the macroeconomic evolvement. In the framework of the Keynesian macroeconomic theory, the mechanism of cyclic fluctuations was proposed by Kaldor [10]. The Kaldor model captures the interaction between primary economic variables of gross production and capital stock. What Kaldor claimed in the model is the essential economic proxy towards to cyclic fluctuations led by the nonlinearity in the investment-saving mechanism. The work of Kaldor was formalized by Chang and Smyth [5] into a two-dimensional system of ordinary differential equations as

\[
\begin{align*}
\dot{Y} &= \alpha (I(Y, K) - S(Y, K)), \\
\dot{K} &= I(Y, K) - qK,
\end{align*}
\]

(1)

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* Corresponding author: Shuanglian Chen.
where $I$ is the investment function and $S$ is the saving function. The economic variable $Y$ represents the gross production and $K$ represents the capital stock. Parameter $\alpha$ is the adjustment coefficient in the good market and $q$ is the depreciation rate of the capital stock. Chang and Smyth [5] verified rigorously the generation of endogenous business cycle dynamics through the Poincare-Bendixon theorem. The result was consistent with the hypothesis from Kaldor. Subsequently the macroeconomic dynamics, especially the cyclic fluctuation behaviors, were paid much attention by many authors, see e.g. [3, 12] and references therein. They have clarified that the underlying reasons causing the business cycles are against the exogenous periodic actions but the intrinsically endogeneity associated with the nonlinearity emerged by primary economic relationships. One popular modification for the Kaldor model is the introduction of delay effects motivated by Kalecki [11], who impacted the lag between the investment decision and the implementation. The unified model was developed by Krawiec and Szydlowski [14] by means of delayed differential equations or equivalently an infinite-dimensional dynamic system, named the Kaldor-Kalecki model. In this model, time delay is responsible for the Hopf bifurcation to a limit cycle solution and for generating cycles. This is the primary difference between the Kaldor model and Kaldor-Kalecki model. In the former cycles are generated due to the nonlinear investment function. Nevertheless, the investment lag is addressed in the latter. Thereafter, various types of delay Kaldor model were pouring out. These works concentrated on the complex dynamics modeling macro economy, including the cycle and chaotic fluctuations led by delay effects and the transitions between them, see e.g. [9, 15] and references therein.

As representative macroeconomic models, both the Kaldor model and Kaldor-Kalecki model neglect the effects of macro shocks. In reality, externally macro shocks are essential attributes in a macroeconomic system. Greenwood et al [8] investigated the impacts of shocks on the degree of economic cycle fluctuations and suggested the importance of shocks when considering macro economy. Generally, in macroeconomic models, macro shocks were embodied by random perturbations or stochastic elements introduced into the systems. Authors such as Brunnermeier and Sannikov [4], Kilmenko et al [13] and Phelan [20] constructed some macroeconomic models with macro shocks and investigated the equilibrium dynamics. Volatility effects and systematic risks caused by essential shocks were cared for in their works. Kaldor type macroeconomic models with shocks have been paid much attention by many authors as well. Mircea et al [18] studied the dynamics depicting the mean and variance of the system. The result suggests that the cyclic fluctuation is inherited by the mean and variance although perfect cyclic dynamics of the system has gone due to the shock. Li et al [16] applied the stochastic Kaldor model to modeling the financial business cycle. They investigated the bifurcation phenomena and the cyclic length. Bashkirtseva et al overcame the technique difficulty directly applying the Fokker-Planck-Kolmogorov equation, which is the most comprehensive probabilistic description for stochastic dynamics, through stochastic sensitivity function technique and investigated the complex dynamics of the shocked Kaldor models, mainly focusing on chaotic behaviors depicting economic crisis stage (see [1] and reference therein). Summarily, works on Kaldor models with shocks mostly concentrate on the dynamics addressing the effects of shocks on the economic system.

Due to the inevitable occurrences of macro shocks on the macroeconomic system, in the current paper, the authors develop a Kaldor type macroeconomic model with
Investment Setting for Shocked Kaldor Model

Macro shock. Especially, the shock is interpreted as the investment uncertainty [19]. In reality, investment decisions usually depend on some external references, which makes it should not be fully endogenous. On the other hand, due to uncertainty and risk of investment, returns from investing should be also uncertain and the evocable volatilities rely on the investment scale. Overall, investment could not always contribute to the accumulation of wealth. Thus the investment function in the model is improved to be time dependent and the shock relies on the investment scale in this paper. Then the developed Kaldor macroeconomic model with shock is written as

\[
\dot{Y} = \alpha I(t,Y,K) + \alpha \xi I(t,Y,K) - \alpha S(Y,K),
\]

\[
\dot{K} = I(t,Y,K) + \epsilon \xi I(t,Y,K) - qK,
\]

which can be equivalently written as

\[
dY = \alpha I(t,Y,K)dt - \alpha S(Y,K)dt + \alpha \epsilon I(t,Y,K)dB_t,
\]

\[
dK = I(t,Y,K)dt - qKdt + \epsilon I(t,Y,K)dB_t,
\]

where \(\xi\) is a white noise, \(dB_t\) is a Brownian shock defined on a probability space \((\Omega, \mathbb{F}, \mathcal{F})\) and \(\epsilon\) is the intensity of the shock. The existing literatures working on Kaldor type models with shocks intensively show complexities such as mixed-mode oscillations and chaos. In practice, an economic system evolves stably according to some objectives or utilities from agencies or decision makers. Brunnermeier and Sannikov [4] interpreted that maximum utilities and market clearing lead to economic equilibrium. Accordingly, contrary to investigating the complexity of the shocked economic system (2), in this paper, the authors provide an approach for macroeconomic control through calibrating the evolution of the shocked Kaldor macroeconomic system with some expected benchmark processes. From the viewpoint of economic dynamics, decisions and policies on macro economy usually could be translated into some expected processes referred by economic evolution. Thus the calibration could make the economic system operate purposefully with supervision and keep stable. Some dynamic programming on benchmark processes can be referred to [7, 22] and references therein.

The investment function plays a key role in the Kaldor macroeconomic model and hence it is selected as the control variable in the programming in this paper. In the following argument, one could see that the optimal investment setting in the programming is associated with a five-dimensional nonlinear system of ordinary differential equations with singularity. Such a system might be solved by numeric methods. But it requires a smaller step size in order to achieve absolute stability, impairing computational efficiency. In practice, a completely explicit investment setting is more preferred and could be executed more expediently. To achieve this, according to some stability condition derived, the authors modify the utility function in the programming to realize the derivation of such an explicit formula for the investment function. The stability condition guarantees that the modification of utility function would not cause intrinsic effects in the programming.

Uncertain shocks on the economic system lead to the generation of risks. Risk management continues to be a primary task in macroeconomic operation. The premise for risk management is the definition of risk measures. VaR as a popular risk measure, acknowledged in the Basel Agreement, has been widely employed in many scopes such as portfolio, insurance and information science [2, 17]. On the other hand, the demand for measuring risk dynamically leads to a dynamic

\[
\text{VaR} = \alpha I(t,Y,K) + \alpha \xi I(t,Y,K) - \alpha S(Y,K),
\]

\[
\text{VaR} = I(t,Y,K) + \epsilon \xi I(t,Y,K) - qK,
\]

which can be equivalently written as

\[
d\text{VaR} = \alpha I(t,Y,K)dt - \alpha S(Y,K)dt + \alpha \epsilon I(t,Y,K)dB_t,
\]

\[
d\text{VaR} = I(t,Y,K)dt - qKdt + \epsilon I(t,Y,K)dB_t,
\]
generation of VaR. Chen et al [6] and Zhang et al [21] introduced dynamic VaR constraints into reinsurance problems. Zhang and Gao [22] considered portfolio problem with dynamic VaR constraint. In this paper, motivated by these works, the authors define a dynamic version of VaR for the shocked Kaldor macroeconomic model (2) and attach a risk constraint induced by the dynamic VaR to the utility function in the programming.

This paper is organized as follows. In Section 2, we consider the investment setting in the programming and devote to the derivation of the explicit formula for the optimal investment setting in the shocked Kaldor macroeconomic model. In Section 3, we define the dynamic version of VaR for the shocked Kaldor macroeconomic model and introduce the risk constraint to the optimization problem. In Section 4, an application of the theoretical results is carried out. In Section 5, we come to a conclusion.

2. **Investment setting in the model.** The investment function plays an important role in the Kaldor macroeconomic model. The evolvement of economic system is almost determined by the setting of investment. Thus, the investment is selected as the control variable in the programming and determined according to the utility function given subsequently. Indeed, the utility function is the formulation of the dynamic calibration for the shocked Kaldor model (2). Without loss of the generality, suppose that the saving function \( S \) in (2) depends only on the gross production \( Y \) and is linear satisfying \( S(Y) = \gamma Y \), where \( \gamma \) is the saving rate. We rewrite the system (2) as

\[
\begin{align*}
dY &= \alpha I(t, Y, K)dt - \alpha \gamma Y dt + \alpha \epsilon I(t, Y, K)dB_t, \\
dK &= I(t, Y, K)dt - qK dt + \epsilon I(t, Y, K)dB_t.
\end{align*}
\]

For convenient argument, replacing \( Y \) by \( \alpha Y \) in (3), the system could be rewritten as

\[
\begin{align*}
dY &= I(t, Y, K)dt - \alpha \gamma Y dt + \epsilon I(t, Y, K)dB_t, \\
dK &= I(t, Y, K)dt - qK dt + \epsilon I(t, Y, K)dB_t.
\end{align*}
\]

Here we remark that the investment functions in (3) and (4) might be slightly different. However, this difference does not cause any trouble in our following consideration and conclusion. We also remark that the investment function in the model is allowed to be negative. Indeed, the negativity of investment means the excessive saving.

2.1. **Optimal investment setting through calibration.** In this subsection, through the dynamic programming, we realize the macro control by calibrating the evolvement of the shocked Kaldor macroeconomic model (4) with some expected process as benchmark. The effect of calibration is to reduce the amplification of shocks such that the economic system would not seriously separate from the originally expected evolvement. We formally denote the benchmark process as \((y, k)\) and denote the solution of (4) as \((Y, K)\). The utility function in the dynamic programming is set as

\[
\Xi(t, Y, K) = \inf I E^{t,Y,K} \left[ \int_t^T F(s, Y(s), K(s))ds + G(T, Y(T), K(T)) \right],
\]

where

\[
F(t, Y(t), K(t)) = (Y(t) - y(t))^2 + (K(t) - k(t))^2.
\]
\[ G(T, Y(T), K(T)) = (Y(T) - y(T))^2 + (K(T) - k(T))^2. \]

\(T\) is a terminal time meaning the length of period. \(\mathbb{E}^{x,y,k}\) is the conditional expectation with respect to the probability measure \(\mathbb{P}\). The utility function (5) defines a metric of distance from the orbit of the shocked system (4) to the expected benchmark process. Alternatively, it could be interpreted as the deviation between the systematic orbit and the benchmark process. The first part in the utility function minimizes the difference between these two whole processes, which meets the holistic calibration. On the contrary, the second part in the utility function minimizes the difference between the status at the terminal \(T\) of these two processes, which focuses on the local gap at the end of the period. The following theorem claims that the optimal investment setting under the utility function (5) can be achieved by solving a nonlinearly singular system of ordinary differential equations.

**Theorem 2.1.** With the utility function given in (5), the evolvement of the shocked Kaldor macroeconomic model (4) can be calibrated with the benchmark process \((y, k)\) through the dynamic investment setting

\[
I^*(t, Y, K) = -\frac{\phi_1(t) + \phi_2(t) + (2\phi_{11}(t) + \phi_{12}(t))Y + (\phi_{12}(t) + 2\phi_{22}(t))K}{2\epsilon^2(\phi_{11}(t) + \phi_{12}(t) + \phi_{22}(t))},
\]

where \(\phi_{11}, \phi_{12}, \phi_{22}, \phi_1\) and \(\phi_2\) satisfy

\[
\begin{align*}
\phi_{11} &= -1 + 2\alpha\gamma\phi_{11} + \frac{(2\phi_{11} + \phi_{12})^2}{4\epsilon^2(\phi_{11} + \phi_{12} + \phi_{22})}, \phi_{11}(T) = 1, \\
\phi_{12} &= -1 + 2q\phi_{12} + \frac{(2\phi_{12} + \phi_{12})^2}{4\epsilon^2(\phi_{11} + \phi_{12} + \phi_{22})}, \phi_{12}(T) = 0, \\
\phi_1 &= 2g(t) + \alpha\gamma\phi_1 + \frac{(\phi_1 + \phi_2)(\phi_{12} + 2\phi_{11})}{2\epsilon^2(\phi_{11} + \phi_{12} + \phi_{22})}, \phi_1(T) = -2g(T), \\
\phi_2 &= 2k(t) + q\phi_2 + \frac{(\phi_1 + \phi_2)(\phi_{12} + 2\phi_{22})}{2\epsilon^2(\phi_{11} + \phi_{12} + \phi_{22})}, \phi_2(T) = -2k(T).
\end{align*}
\]

**Proof.** According to the stochastic control theory, the utility function \(\Xi\) should satisfy the following Hamilton-Jacobi-Bellman equation

\[
\inf_I \left\{ \mathbb{I}\Xi_Y + I\Xi_K + \frac{\epsilon^2 I^2}{2}\Xi_{YY} + \epsilon^2 I^2\Xi_{YK} + \frac{\epsilon^2 I^2}{2}\Xi_{KK} \right\}
\]

\[ -\alpha\gamma Y\Xi_Y - qK\Xi_K + \Xi_x + F(t, Y, K) = 0, \tag{11} \]

with boundary condition

\[
\Xi(T, Y, K) = G(T, Y, K).
\]

Suppose that \(\Xi\) admits the following polynomial form

\[
\Xi(t, Y, K) = \phi_0(t) + \phi_1(t)Y + \phi_2(t)K + \phi_{11}(t)Y^2 + \phi_{12}(t)YK + \phi_{22}(t)K^2. \tag{12}
\]

Substituting (12) into (11) gives

\[
\begin{align*}
\inf_I &\left\{ \left(\phi_1 + \phi_2 + 2\phi_{11}Y + \phi_{12}Y + \phi_{22}K + 2\phi_{22}K\right)Y + \phi_{12}(t)YK + \phi_{22}(t)K^2 \right\} \\
&+ F(t, Y, K) - \alpha\gamma Y(\phi_1 + 2\phi_{11}Y + \phi_{12}K) - qK(\phi_2 + \phi_{12}Y + 2\phi_{22}K) \\
&+ \phi_0 + \phi_1Y + \phi_2K + \phi_{11}Y^2 + \phi_{12}YK + \phi_{22}K^2 = 0,
\end{align*}
\]
with boundary condition

\[ \phi_0(T) = y^2(T) + k^2(T), \phi_1(T) = -2y(T), \phi_2(T) = -2k(T), \]
\[ \phi_{11}(T) = 1, \phi_{22}(T) = 1, \phi_{12}(T) = 0. \]

If \( \phi_{11}(t) + \phi_{12}(t) + \phi_{22}(t) > 0 \) for \( 0 \leq t \leq T \), then the optimal investment should satisfy

\[ I^*(t, Y, K) = -\frac{\phi_1(t) + \phi_2(t) + (2\phi_{11}(t) + \phi_{12}(t))Y + (\phi_{11}(t) + 2\phi_{22}(t))K}{2\epsilon^2(\phi_{11}(t) + \phi_{12}(t) + \phi_{22}(t))}. \] (13)

Substituting (13) into the Hamilton-Jacobi-Bellman equation (11) again gives

\[
(\dot{\phi}_{11} + f_{11})Y^2 + (\dot{\phi}_{22} + f_{22})K^2 + (\dot{\phi}_{12} + f_{12})YK
+ (\dot{\phi}_1 + f_1)Y + (\dot{\phi}_2 + f_2)K + (\dot{\phi}_0 + f_0) = 0,
\]
where \( f_{11}, f_{22}, f_{12}, f_1, f_2 \) and \( f_0 \) are given as

\[
\begin{align*}
    f_{11} &= 1 - 2\alpha\gamma \phi_{11} - \frac{(2\phi_{11} + \phi_{12})^2}{4\epsilon^2(\phi_{11} + \phi_{12} + \phi_{22})}, \\
    f_{22} &= 1 - 2q\phi_{22} - \frac{(2\phi_{22} + \phi_{12})^2}{4\epsilon^2(\phi_{11} + \phi_{12} + \phi_{22})}, \\
    f_{12} &= -\left(\frac{1}{\epsilon^2} + q + \alpha\gamma\right)\phi_{12} + \frac{\phi_{12}^2 - 4\phi_{11}\phi_{22}}{2\epsilon^2(\phi_{11} + \phi_{12} + \phi_{22})}, \\
    f_1 &= -2y(t) - \alpha\gamma \phi_1 - \frac{\left(\phi_1 + \phi_2\right)(\phi_{12} + 2\phi_{11})}{2\epsilon^2(\phi_{11} + \phi_{12} + \phi_{22})}, \\
    f_2 &= -2k(t) - q\phi_2 - \frac{\left(\phi_1 + \phi_2\right)(\phi_{12} + 2\phi_{22})}{2\epsilon^2(\phi_{11} + \phi_{12} + \phi_{22})}, \\
    f_0 &= y^2(t) + k^2(t) - \frac{(\phi_1 + \phi_2)^2}{4\epsilon^2(\phi_{11} + \phi_{12} + \phi_{22})},
\end{align*}
\]
which implies the system of ordinary differential equations (6)-(10). Now what we need to verify is the condition

\[ \phi_{11}(t) + \phi_{12}(t) + \phi_{22}(t) > 0 \quad (14) \]
for \( 0 \leq t \leq T \). A direct calculation shows that

\[
\begin{align*}
    \phi_{11} + \phi_{12} + \phi_{22} &= -2 + 2\alpha\gamma \phi_{11} + 2q\phi_{22} + \left(\frac{1}{\epsilon^2} + \alpha\gamma + q\right)\phi_{12} \\
    &\quad + \frac{4\phi_{11}^2 + 4\phi_{11}\phi_{12} + 4\phi_{22}^2 + 4\phi_{22}\phi_{12} + 8\phi_{11}\phi_{22}}{4\epsilon^2(\phi_{11} + \phi_{12} + \phi_{22})} \\
    &= \frac{(1 + 2\alpha\gamma\epsilon^2)\phi_{11} + (1 + (q + \alpha\gamma)\epsilon^2)\phi_{12} + (1 + 2q\epsilon^2)\phi_{22} - 2\epsilon^2}{\epsilon^2}.
\end{align*}
\]

To obtain a contradiction, we suppose that there exists a time \( 0 \leq T_0 < T \) such that

\[ \phi_{11}(T_0) + \phi_{12}(T_0) + \phi_{22}(T_0) \leq 0. \]

Since \( \phi_{11}(T) + \phi_{12}(T) + \phi_{22}(T) = 2 > 0 \), there exists a time \( T_1 \) satisfying \( T_0 \leq T_1 < T \) such that

\[ \phi_{11}(T_1) + \phi_{12}(T_1) + \phi_{22}(T_1) = 0 \]
and

\[ \phi_{11}'(T_1) + \phi_{12}'(T_1) + \phi_{22}'(T_1) \geq 0. \] (15)
Therefore, it holds that
\[
\phi_{11}(T_1) + \phi_{12}(T_1) + \phi_{22}(T_1)
= 2\alpha\gamma\phi_{11}(T_1) + (q + \alpha\gamma)\phi_{12}(T_1) + 2q\phi_{22}(T_1) - 2
= 2\alpha\gamma\phi_{11}(T_1) + (q + \alpha\gamma)(-\phi_{11}(T_1) - \phi_{22}(T_2)) + 2q\phi_{22}(T_1) - 2
= (\alpha\gamma - q)(\phi_{11}(T_1) - \phi_{22}(T_1)) - 2.
\]
If \( q \geq \alpha\gamma \), then
\[
\phi_{11} + 1 - 2q\phi_{11} - \frac{(2\phi_{11} + \phi_{12})^2}{4\epsilon^2(\phi_{11} + \phi_{12} + \phi_{22})}
= \phi_{11} + 1 - 2\alpha\gamma\phi_{11} - \frac{(2\phi_{11} + \phi_{12})^2}{4\epsilon^2(\phi_{11} + \phi_{12} + \phi_{22})} + 2(\alpha\gamma - q)\phi_{11}
= 2(\alpha\gamma - q)\phi_{11} \leq 0,
\]
namely that \( \phi_{11} \) is a sub solution of \( \phi_{22} \) and \( \phi_{11}(t) \geq \phi_{22}(t) \) for \( 0 \leq t \leq T \) when \( \phi_{11}(T) = \phi_{22}(T) \). Hence, it holds that
\[
\phi_{11}(T_1) + \phi_{12}(T_1) + \phi_{22}(T_1) \leq -2 < 0,
\]
which contradicts to (15). If \( q \leq \alpha\gamma \), then \( \phi_{11} \) is a super solution of \( \phi_{22} \) and \( \phi_{11}(t) \leq \phi_{22}(t) \) for \( 0 \leq t \leq T \) when \( \phi_{11}(T) = \phi_{22}(T) \), which implies (16) as well and then contradicts to (15). We conclude that the condition (14) holds and complete the proof of theorem.

2.2. Explicit formula for investment setting. In the Subsection 2.1, Theorem 2.1 has claimed that the optimal investment setting is associated with a five-dimensional system of ordinary differential equations. Such a system of ordinary differential equations could be solved by numerical solvers such as Runge-Kutta method. However using this method may raise a numerical issue. When the ordinary differential equations are stiff, a numerical solver requires a smaller step size in order to achieve absolute stability, impairing computational efficiency. Thus, if it is possible, the decision executants prefer a more operable and even explicit formula for the investment setting in practice. To achieve this, we first modify the term \( G \) depicting terminal status in the utility function (5) to be

\[
G_0(T, X(T), K(T)) = C_1(X(T) - x(T))^2 + C_2(K(T) - k(T))^2,
\]
where \( C_1, C_2 > 0 \) are constants determined subsequently. It is easy to understand that in spite of \( G \) is replaced by \( G_0 \) in the utility function, the effect of calibration would not be influenced. With \( G \) replaced by \( G_0 \) in the utility function, the boundary conditions in (6)-(10) are correspondingly modified to be

\[
\phi_{11}(T) = C_1, \phi_{22}(T) = C_2, \phi_{12}(T) = 0, \phi_{1}(T) = -2C_1y(T), \phi_{2}(T) = -2C_2k(T).
\]

Lemma 2.2. The subsystem of ordinary differential equations consist of (6)-(8) admits an equilibrium denoted as \((\phi^{*}_{11}, \phi^{*}_{12}, \phi^{*}_{22})\), where

\[
\phi^{*}_{11} = \frac{-c - \sqrt{c^2 - 4ab}}{2b},
\]
\[
\phi^{*}_{12} = \frac{2c^2 - 2(1 + 2\alpha\gamma^2)\phi^{*}_{11}}{1 + (q + \alpha\gamma)\epsilon^2},
\]
\[
\phi^{*}_{22} = \frac{1 + 2\alpha\gamma\epsilon^2}{1 + 2q\epsilon^2}\phi^{*}_{11}.
\]
Lemma 2.3. Denote (8) in the sense of (31). The following lemma claims that \( \phi \) equilibrium of the sub system consist of (6)-(8), which completes the proof of the

**Proof.** Denote \( \phi_{11}, \phi_{12}, \phi_{22} \) the solutions of (6)-(8) respectively. Letting

\[ \dot{\phi}_{11} = \dot{\phi}_{12} = \dot{\phi}_{22} = 0, \]

it holds that

\[
\begin{align*}
(4 + 8\alpha\gamma\epsilon^2)\phi_{11}^2 &+ (4 + 8\alpha\gamma\epsilon^2)\phi_{11}\phi_{12} + 8\alpha\gamma\epsilon^2\phi_{11}\phi_{22} + \phi_{12}^2 \\
- 4\epsilon^2\phi_{11} - 4\epsilon^2\phi_{12} - 4\epsilon^2\phi_{22} &= 0, \\
(4 + 8q\epsilon^2)\phi_{22}^2 &+ (4 + 8q\epsilon^2)\phi_{22}\phi_{12} + 8q\epsilon^2\phi_{11}\phi_{22} + \phi_{12}^2 \\
- 4\epsilon^2\phi_{11} - 4\epsilon^2\phi_{12} - 4\epsilon^2\phi_{22} &= 0
\end{align*}
\] (24)

and

\[
\begin{align*}
(1 + 2\epsilon^2\phi_{11} + 2\epsilon^2\alpha\gamma)\phi_{12}^2 &+ (2 + 2\epsilon^2\phi_{12} + 2\epsilon^2\alpha\gamma)\phi_{11}\phi_{12} \\
+ (2 + 2\epsilon^2\phi_{22} + 2\epsilon^2\alpha\gamma)\phi_{12}\phi_{22} + 4\phi_{11}\phi_{22} &= 0
\end{align*}
\] (25)

Associating (24) with (25) gives

\[
(1 + 2\alpha\gamma\epsilon^2)\phi_{11} = (1 + 2q\epsilon^2)\phi_{22}.
\] (27)

Associating (24), (25) with (26) gives

\[
(\phi_{11} + \phi_{12} + \phi_{22})(2\epsilon^2 - (1 + 2\alpha\gamma\epsilon^2)\phi_{11} - (1 + (q + \alpha\gamma)\epsilon^2)\phi_{12} - (1 + 2q\epsilon^2)\phi_{22}) = 0,
\]

which implies that

\[
2\epsilon^2 - (1 + 2\alpha\gamma\epsilon^2)\phi_{11} - (1 + q\epsilon^2 + \alpha\gamma\epsilon^2)\phi_{12} - (1 + 2q\epsilon^2)\phi_{22} = 0.
\] (28)

Together with (27), we get

\[
\phi_{12} = \frac{2\epsilon^2 - 2(1 + 2\alpha\gamma\epsilon^2)\phi_{11}}{1 + (q + \alpha\gamma)\epsilon^2}.
\] (29)

Substituting (27) and (29) into (26) gives

\[
b\phi_{11}^2 + c\phi_{11} + a = 0,
\] (30)

where \( a, b \) and \( c \) are given in (21)-(23) respectively. Since \( a > 0, b < 0 \), we can solve that \( \phi_{11}^*, \phi_{12}^*, \phi_{22}^* \) as (18)-(20). We then conclude that \( (\phi_{11}^*, \phi_{12}^*, \phi_{22}^*) \) is an equilibrium of the sub system consist of (6)-(8), which completes the proof of the lemma.

If we set \( C_1 = \phi_{11}^*, C_2 = \phi_{22}^* \), then \( \phi_{11} \equiv \phi_{11}^*, \phi_{22} \equiv \phi_{22}^* \) are solutions of (6) and (7). For (8), due to the boundary condition, \( \phi_{12}^* \) is not the solution. Nevertheless, the following lemma claims that \( \phi_{12}^* \) could be considered as a stable equilibrium of (8) in the sense of (31).

**Lemma 2.3.** Denote \( \phi_{12}(t; T), 0 \leq t \leq T \) the solution of (8). Then for given \( 0 \leq t < T \), it holds that

\[
\phi_{12}(t; T) \to \phi_{12}^* \text{ as } T \to \infty.
\] (31)
Proof. Let \( \hat{\phi}_{12} \) satisfy \( \hat{\phi}_{12}(t + T) = \phi_{12}(-t; T) \) for \(-T \leq t \leq 0\). Then \( \hat{\phi}_{12} \) satisfies the following equation

\[
\hat{\phi}_{12} = -\left( \frac{1}{\epsilon^2} + q + \alpha \gamma \right) \hat{\phi}_{12} + \frac{\hat{\phi}_{12}^2 - 4\hat{\phi}_{11}^* \hat{\phi}_{22}^*}{2\epsilon^2(\hat{\phi}_{11}^* + \hat{\phi}_{12} + \phi_{22}^*)}, \hat{\phi}_{12}(0) = 0. \tag{32}
\]

Define the Lyapunov function with respect to \( \hat{\phi}_{12} \) as

\[
V(\hat{\phi}_{12}) = \left( \hat{\phi}_{12} - \phi_{12}^* \right)^2.
\]

Then it holds that

\[
V'(\hat{\phi}_{12}) = \frac{(\hat{\phi}_{12} - \phi_{12}^*)(-2\epsilon^2(\epsilon^{-2} + q + \alpha \gamma) \hat{\phi}_{12}(\phi_{11}^* + \hat{\phi}_{12} + \phi_{22}^*) + \hat{\phi}_{12}^2 - 4\phi_{11}^* \phi_{22}^*)}{\epsilon^2(\phi_{11}^* + \phi_{12} + \phi_{22}^*)},
\]

where

\[
U(\hat{\phi}_{12}) = (1 + 2q\epsilon^2 + 2\alpha \gamma \epsilon^2)\hat{\phi}_{12}^2 + (2 + 2q\epsilon^2 + 2\alpha \gamma \epsilon^2)(\phi_{11}^* + \phi_{22}^*)\hat{\phi}_{12} + 4\phi_{11}^* \phi_{22}^*.
\]

By the formulations of \( \phi_{11}^* \) and \( \phi_{22}^* \) in the proof of Lemma 2.2, letting \( U(\hat{\phi}_{12}) = 0 \), we can solve that

\[
\hat{\phi}_{12}^+ = \frac{2\epsilon^2 \sqrt{2(q^2 + \alpha^2 \gamma^2)} + 4(q + \alpha \gamma)(q^2 + \alpha^2 \gamma^2)\epsilon^2 + (q + \alpha \gamma)^2 \epsilon^4}{(1 + 2q\epsilon^2 + 2\alpha \gamma \epsilon^2)(1 + 2q\epsilon^2)} \phi_{11}^*
\]

and

\[
\hat{\phi}_{12}^- = \frac{2\epsilon^2 \sqrt{2(q^2 + \alpha^2 \gamma^2)} + 4(q + \alpha \gamma)(q^2 + \alpha^2 \gamma^2)\epsilon^2 + (q + \alpha \gamma)^2 \epsilon^4}{(1 + 2q\epsilon^2 + 2\alpha \gamma \epsilon^2)(1 + 2q\epsilon^2)} \phi_{11}^*
\]

are two roots. By (30), it is not difficult to check that

\[
\frac{\epsilon^4(2(q^2 + \alpha^2 \gamma^2) + 4(q + \alpha \gamma)(q^2 + \alpha^2 \gamma^2)\epsilon^2 + (q + \alpha \gamma)^4 \epsilon^4)}{(1 + 2q\epsilon^2 + 2\alpha \gamma \epsilon^2)(1 + 2q\epsilon^2)} \phi_{11}^2
\]

\[
= \frac{((1 + q\epsilon^2 + \alpha \gamma \epsilon^2) - (1 + 2q\epsilon^2)(1 + 2q\epsilon^2)(1 + 2q\epsilon^2 + 2\alpha \gamma \epsilon^2))^2}{(1 + 2q\epsilon^2 + 2\alpha \gamma \epsilon^2)(1 + 2q\epsilon^2 + 2\alpha \gamma \epsilon^2)} \phi_{11}^2
\]

\[
+ \frac{2\epsilon^2(1 + q\epsilon^2 + \alpha \gamma \epsilon^2) - 2\epsilon^2(1 + 2q\epsilon^2)(1 + 2q\epsilon^2)(1 + 2q\epsilon^2 + 2\alpha \gamma \epsilon^2)}{(1 + 2q\epsilon^2)(1 + q\epsilon^2 + \alpha \gamma \epsilon^2)} \phi_{11}^2
\]

\[
+ \frac{\epsilon^4(1 + 2q\epsilon^2 + 2\alpha \gamma \epsilon^2)}{(1 + q\epsilon^2 + \alpha \gamma \epsilon^2)^2},
\]

which implies that \( \hat{\phi}_{12}^+ = \phi_{12}^* \). Note that \( \phi_{11}^* + \hat{\phi}_{12}(t) + \phi_{22}^* > 0 \) for \( 0 \leq t \leq T \), which has been proved in Lemma 2.2. Thus, if \( \hat{\phi}_{12} \geq \phi_{12}^* \), then \( U(\hat{\phi}_{12}) > 0 \) and
$\hat{V}(\phi_{12}) < 0$. Again note that for any $0 \leq t \leq T$, $\hat{\phi}_{12}(t) > -\phi_{11}^* - \phi_{22}^*$. We have

$$\dot{\hat{\phi}}_{12} + \phi_{11}^* + \phi_{22}^* = \frac{(2q\epsilon^2 + 2\alpha\gamma\epsilon^2)(\phi_{11}^* + \phi_{22}^*)}{2 + 4q\epsilon^2 + 4\alpha\gamma\epsilon^2} - \sqrt{(2 + 2q\epsilon^2 + 2\alpha\gamma\epsilon^2)^2(\phi_{11}^* + \phi_{22}^*)^2 - 16(1 + 2q\epsilon^2 + 2\alpha\gamma\epsilon^2)\phi_{11}^* \phi_{22}^*}{2 + 4q\epsilon^2 + 4\alpha\gamma\epsilon^2}.$$  

Meanwhile, it is easy to calculate that

$$(2q\epsilon^2 + 2\alpha\gamma\epsilon^2)^2(\phi_{11}^* + \phi_{22}^*)^2 + 16(1 + 2q\epsilon^2 + 2\alpha\gamma\epsilon^2)\phi_{11}^* \phi_{22}^* - (2 + 2\alpha\gamma\epsilon^2 + 2q\epsilon^2)^2(\phi_{11}^* + \phi_{22}^*)^2 = 16(1 + 2q\epsilon^2 + 2\alpha\gamma\epsilon^2)\phi_{11}^* \phi_{22}^* - 4(\phi_{11}^* + \phi_{22}^*)^2 - 8(\alpha\gamma\epsilon^2 + q\epsilon^2)(\phi_{11}^* + \phi_{22}^*)^2 = 8(1 + 2q\epsilon^2 + 2\alpha\gamma\epsilon^2)\phi_{11}^* \phi_{22}^* - 4(1 + 2\alpha\gamma\epsilon^2 + 2q\epsilon^2)\phi_{11}^2 - 4(1 + 2\alpha\gamma\epsilon^2 + 2q\epsilon^2)\phi_{22}^2 = -4(1 + 2\alpha\gamma\epsilon^2 + 2q\epsilon^2)(\phi_{11}^* - \phi_{22}^*)^2 < 0,$$

which implies that

$$\hat{\phi}_{12} + \phi_{11}^* + \phi_{22}^* < 0.$$  

Thus, if $\hat{\phi}_{12} \leq \phi_{12}^*$, then $U(\hat{\phi}_{12}) > 0$ and $\hat{V}(\phi_{12}) < 0$. We conclude that $\phi_{12}^*$ is a stable equilibrium of (32). Hence, for given $t \in [0, T]$, we have

$$\phi_{12}(t; T) = \hat{\phi}_{12}(-t + T) \rightarrow \phi_{12}^* as T \rightarrow \infty,$$

which completes the proof of lemma.

The proof of the lemma can be seen in Appendix C. Through the modification of $G$ in the utility function, from the result of Lemma 2.2, the system of ordinary differential equations in Theorem 2.1 could be reduced to be three dimensional by solving $\phi_{11} = \phi_{11}^*, \phi_{22} = \phi_{22}^*$. Actually, with $\phi_{11} = \phi_{11}^*, \phi_{22} = \phi_{22}^*$ in (8), $\phi_{12}$ could be solved implicitly and we might roughly claim that the system could be further reduced to be two dimensional. Nevertheless, an implicit formula would not be preferred in practice as well. Fortunately, the result of Lemma 2.3 tells that with $\phi_{11} = \phi_{11}^*, \phi_{22} = \phi_{22}^*$ in (8), $\phi_{12}$ attracts the solution of (8) in the sense of (31). If the length of period $T$ is sufficiently large, then $\phi_{12}^*$ could be considered as a logical substitution for the solution of (8). In practice, decisions and policies usually work in a long time and $T$ is indeed sufficiently large. Thus it is logical to modify the term $G$ in the utility function once more as

$$G_1(T, X(T), K(T)) = C_1X(T) - x(T))^2 + C_2(\frac{K(T)}{k(T)})^2 + C_3Y(T)K(T)$$

and we determine the constants as

$$C_1 = \phi_{11}^*, C_2 = \phi_{22}^*, C_3 = \phi_{12}^*.$$  

Then the utility function (5) is rewritten as

$$\Xi_t(t, Y, K) = \inf_{I_t} \mathbb{E}^{\hat{\omega}, Y, K}\left[ \int_t^T F(s, Y(s), K(s))ds + G_1(T, Y(T), K(T)) \right].$$

The boundary condition of (8) transforms into $\phi_{12}(T) = \phi_{12}^*$ and it can be solved as $\phi_{12} = \phi_{12}^*$. We impact that the modification of $G$ is supported by the result of Lemma 2.3. It would not cause substantial influence on the calibration. Nevertheless, such a modification leads to an explicit formula for the investment setting, seeing the following theorem.
Theorem 2.4. With the utility function given in (35), the evolution of the shocked Kaldor macroeconomic model (4) can be calibrated with the benchmark process \((y, k)\) through the explicitly dynamic investment setting

\[ I^*(t, Y, K) = -\phi_1(t) + \phi_2(t) + (2\phi_{11}^* + \phi_{12}^*)Y + (\phi_{12}^* + 2\phi_{22}^*)K, \]

where \(\phi_{11}^*, \phi_{12}^*, \phi_{22}^*\) are given in Lemma 2.2 and

\[ \phi_1(t) = A_{\phi_1}(t, T)\phi_{11}^*y(T) + B_{\phi_1}(t, T)\phi_{22}^*k(T) - \int_t^T (A_{\phi_1}(t, s)y(s) + B_{\phi_1}(t, s)k(s))ds, \]

\[ \phi_2(t) = A_{\phi_2}(t, T)\phi_{11}^*y(T) + B_{\phi_2}(t, T)\phi_{22}^*k(T) - \int_t^T (A_{\phi_2}(t, s)y(s) + B_{\phi_2}(t, s)k(s))ds, \]

where

\[ A_{\phi_1}(t, s) = M_1(t)\bar{M}_1(s) + M_2(t)\bar{M}_3(s), \]

\[ A_{\phi_2}(t, s) = M_3(t)\bar{M}_1(s) + M_4(t)\bar{M}_3(s), \]

\[ M_1(t) = \frac{\phi_{12}^* + 2\phi_{11}^*}{2c^2(\phi_{11}^* + \phi_{12}^* + \phi_{22}^*)}e^{\lambda_+t}, \]

\[ M_2(t) = \frac{\phi_{12}^* + 2\phi_{11}^*}{2c^2(\phi_{11}^* + \phi_{12}^* + \phi_{22}^*)}e^{-\lambda_-t}, \]

\[ M_3(t) = \left(\lambda_+ - \alpha\gamma - \frac{\phi_{12}^* + 2\phi_{11}^*}{2c^2(\phi_{11}^* + \phi_{12}^* + \phi_{22}^*)}\right)e^{\lambda_+t}, \]

\[ M_4(t) = \left(\lambda_- - \alpha\gamma - \frac{\phi_{12}^* + 2\phi_{11}^*}{2c^2(\phi_{11}^* + \phi_{12}^* + \phi_{22}^*)}\right)e^{-\lambda_-t}, \]

\[ \bar{M}_1(t) = \frac{2c^2(\phi_{11}^* + \phi_{12}^* + \phi_{22}^*)}{(\phi_{12}^* + 2\phi_{11}^*)}(\alpha\gamma - \lambda_- + \frac{\phi_{12}^* + 2\phi_{11}^*}{2c^2(\phi_{11}^* + \phi_{12}^* + \phi_{22}^*)})e^{-\lambda_-t}, \]

\[ \bar{M}_3(t) = \frac{2c^2(\phi_{11}^* + \phi_{12}^* + \phi_{22}^*)}{(\phi_{12}^* + 2\phi_{11}^*)(\lambda_+ - \lambda_-)}(\lambda_+ - \alpha\gamma - \frac{\phi_{12}^* + 2\phi_{11}^*}{2c^2(\phi_{11}^* + \phi_{12}^* + \phi_{22}^*)})e^{-\lambda_-t}, \]

\[ \bar{M}_2(t) = -\frac{e^{-\lambda_-t}}{\lambda_+ - \lambda_-}, \]

\[ \bar{M}_4(t) = \frac{e^{-\lambda_-t}}{\lambda_+ - \lambda_-}, \]

\[ \lambda_+ = \frac{1}{2}\left(1 + \alpha\gamma + q\right) \pm \frac{1}{2}\sqrt{\left(1 + \alpha\gamma + q\right)^2 - H(\phi_{11}^*, \phi_{12}^*, \phi_{22}^*)}, \]

\[ H(\phi_{11}^*, \phi_{12}^*, \phi_{22}^*) = \frac{4q(1 + \alpha\gamma e^2)\phi_{11}^* + 2(q + \alpha\gamma + 2\alpha\gamma qe^2)\phi_{12}^* + 4\alpha\gamma(1 + qe^2)\phi_{22}^*}{c^2(\phi_{11}^* + \phi_{12}^* + \phi_{22}^*)}. \]

Proof. With \(G_1\) given as (33) and the constants given as (34), the boundary conditions of (6)-(8) transform into

\[ \phi_{11}(T) = \phi_{11}^*, \phi_{22}(T) = \phi_{22}^*, \phi_{12}(T) = \phi_{12}^*, \]

respectively, which implies that (6)-(8) can be solved as

\[ \phi_{11} = \phi_{11}^*, \phi_{22} = \phi_{22}^*, \phi_{12} = \phi_{12}^*. \]

In this way, (9) and (10) degenerate to a linear system as

\[ \phi_1 = 2y(T) + \alpha\gamma\phi_1 + \frac{(\phi_1 + \phi_2)(\phi_{12}^* + 2\phi_{11}^*)}{2c^2(\phi_{11}^* + \phi_{12}^* + \phi_{22}^*)}, \phi_1(T) = -2\phi_{11}^*y(T), \]
\[
\dot{\phi}_2 = 2k(t) + q\phi_2 + \frac{(\phi_1 + \phi_2)(\phi_{12}^* + 2\phi_{22}^*)}{2\epsilon^2(\phi_{11}^* + \phi_{12}^* + \phi_{22}^*)}, \phi_2(T) = -2\phi_{22}^* k(T). \tag{37}
\]

The equation for eigenvalues of the system consist of (36) and (37) is given as
\[
\lambda^2 - \left(\frac{1}{\epsilon^2} + \alpha\gamma + q\right)\lambda + 2q(1 + \alpha\gamma e^2)\phi_{11}^* + (q + \alpha\gamma + 2\alpha\gamma q e^2)\phi_{12}^* + 2\alpha\gamma(1 + q e^2)\phi_{22}^* = 0,
\]

which can be solved as
\[
\lambda_{\pm} = \frac{1}{2} \left(\frac{1}{\epsilon^2} + \alpha\gamma + q\right) \pm \frac{1}{2} \sqrt{\left(\frac{1}{\epsilon^2} + \alpha\gamma + q\right)^2 - 4H(\phi_{11}^*, \phi_{12}^*, \phi_{22}^*)},
\]

where
\[
H(\phi_{11}^*, \phi_{12}^*, \phi_{22}^*) = \frac{4q(1 + \alpha\gamma e^2)\phi_{11}^* + 2(q + \alpha\gamma + 2\alpha\gamma q e^2)\phi_{12}^* + 4\alpha\gamma(1 + q e^2)\phi_{22}^*}{\epsilon^2(\phi_{11}^* + \phi_{12}^* + \phi_{22}^*)}.
\]

By (28), it is easy to see that \(H(\phi_{11}^*, \phi_{12}^*, \phi_{22}^*) \neq (1/\epsilon^2 + \alpha\gamma + q)^2\), which implies that \(\lambda_+ \neq \lambda_-\). The corresponding eigenvectors are derived as
\[
\xi_{\pm} = h_{\pm} \left[\frac{\phi_{12}^* + 2\phi_{11}^*}{2\epsilon^2(\phi_{11}^* + \phi_{12}^* + \phi_{22}^*)}, \lambda_{\pm} - \alpha\gamma - \frac{\phi_{12}^* + 2\phi_{11}^*}{2\epsilon^2(\phi_{11}^* + \phi_{12}^* + \phi_{22}^*)}\right]^T
\]
for any \(h_{\pm} \neq 0\). Then it is easy to obtain the fundamental matrix and calculate its inverse. According to the theory of ordinary differential equations, it is not difficult to derive the result of the theorem. \(\square\)

3. Investment setting with risk constraint. Macroeconomic system is accompanied with risk. Decision makers usually pay much attention to systematic risk when they make decisions. VaR as a popular risk measure has been employed in economy and finance around the world. In addition, the tendency that measuring risk in real time has developed in the recent years. In this section, to assess the risk level of the shocked Kaldor macroeconomic model (4), we define a dynamic version of VaR. To achieve this, we rewrite the system (4) as
\[
Y(t + h) = Y(t)e^{-\alpha\gamma h} + \int_t^{t+h} e^{-\alpha\gamma(t+s)} I(s) ds + \epsilon \int_t^{t+h} e^{-\alpha\gamma(t+s)} I(s) dB_s
\]
\[
= Y(t)e^{-\alpha\gamma h} + \epsilon I(t) \int_t^{t+h} e^{-\alpha\gamma(t+s)} dB_s,
\]
\[
(38)
\]
\[
K(t + h) = K(t)e^{-qh} + \int_t^{t+h} e^{-q(t+s)} I(s) ds + \epsilon \int_t^{t+h} e^{-q(t+s)} I(s) dB_s
\]
\[
= K(t)e^{-qh} + \epsilon I(t) \int_t^{t+h} e^{-q(t+s)} dB_s,
\]
where \(h > 0\) is sufficiently small. Here we can deem that the investment function \(I(s), s \in [t, t + h]\) will not change and remain at the present value \(I(t)\) throughout the time horizon \([t, t + h]\), which is reasonable and consistent with practice.
3.1. Risk measure: Dynamic version of VaR. According to the expression (38), we define the dynamic version of VaR as
$$\text{VaR}_t^Y = \inf\{ L : P(Y(t + h) - Y(t)e^{-\alpha\gamma h} \leq -L|\mathcal{F}_t) \leq 1 - \mu \}$$

and
$$\text{VaR}_t^K = \inf\{ L : P(K(t + h) - K(t)e^{-qh} \leq -L|\mathcal{F}_t) \leq 1 - \mu \},$$

where \( \mu \) is the confidence level. The macro shock leads to drastic volatility of the economic system. Regulators admit asymmetric utilities on upward and downward volatilities of economic variables. Both for the gross production \( Y \) and the capital stock \( K \), upward volatilities are usually preferred and downward volatilities are averse. Abruptly downward fluctuations might lead to the unbalance of economic system and it is difficult to recover. The abruptly downward fluctuations in the economic system are identified as the generation of risk. The definitions of the dynamic VaR above are induced in this way. The following proposition presents explicit forms for the risk measures \( \text{VaR}_t^Y \) and \( \text{VaR}_t^K \).

**Proposition 1.** From the definitions of the dynamic risk measures \( \text{VaR}_t^Y \) and \( \text{VaR}_t^K \), they can be written equivalently as the following forms

$$\text{VaR}^Y_t = \frac{(e^{-\alpha\gamma h} - 1)I(t)}{\alpha\gamma} - \epsilon I(t)\Phi^{-1}(1 - \mu)\sqrt{e^{2\alpha\gamma h} - 1}.$$

and

$$\text{VaR}^K_t = \frac{(e^{-qh} - 1)I(t)}{q} - \epsilon I(t)\Phi^{-1}(1 - \mu)\sqrt{e^{2qh} - 1},$$

where \( \Phi \) is the cumulative distribution function of standard normal distribution.

**Proof.** According to the definition, it is easy to derive that

$$\mathbb{P}(Y(t + h) - Y(t)e^{-\alpha\gamma h} \leq -L|\mathcal{F}_t)$$

$$= \mathbb{P}\left(\epsilon I(t)e^{-\alpha\gamma(t+h)} \int_t^{t+h} e^{\alpha\gamma s}dB_s \leq \frac{e^{-\alpha\gamma h} - 1}{\alpha\gamma} I(t) - L \big|\mathcal{F}_t\right)$$

$$= \mathbb{P}\left(\epsilon xe^{-\alpha\gamma(t+h)} \int_t^{t+h} e^{\alpha\gamma s}dB_s \leq \frac{e^{-\alpha\gamma h} - 1}{\alpha\gamma} x - L \big|\mathcal{F}_t\right)_{x=I(t)}$$

$$= \mathbb{P}\left(\int_t^{t+h} xe^{\alpha\gamma s}dB_s \leq \left(\frac{e^{-\alpha\gamma h} - 1}{\alpha\gamma} x - L \right) \frac{e^{\alpha\gamma(t+h)}}{\epsilon} \big|\mathcal{F}_t\right)_{x=I(t)}$$

$$= \mathbb{P}\left(\xi \leq \frac{\sqrt{2\alpha\gamma}((e^{-\alpha\gamma h} - 1)x - \alpha\gamma L)e^{\alpha\gamma h}}{\alpha\gamma \epsilon |x|\sqrt{e^{2\alpha\gamma h} - 1}}\right)_{x=I(t)}$$

$$= \Phi\left(\frac{\sqrt{2\alpha\gamma}((e^{-\alpha\gamma h} - 1)x - \alpha\gamma L)e^{\alpha\gamma h}}{\alpha\gamma \epsilon |x|\sqrt{e^{2\alpha\gamma h} - 1}}\right)_{x=I(t)},$$

where \( \xi \) is a standard normal random variable and \( \Phi \) is the cumulative distribution function. From the definition of \( \text{VaR}_t^Y \), it holds that

$$\frac{\sqrt{2\alpha\gamma}((e^{-\alpha\gamma h} - 1)I(t) - \alpha\gamma \text{VaR}_t^Y)e^{\alpha\gamma h}}{\alpha\gamma \epsilon |I(t)|\sqrt{e^{2\alpha\gamma h} - 1}} = \Phi^{-1}(1 - \mu).$$

A rearrangement gives

$$\text{VaR}^Y_t = \frac{(e^{-\alpha\gamma h} - 1)I(t)}{\alpha\gamma} - \epsilon I(t)\Phi^{-1}(1 - \mu)\sqrt{e^{2\alpha\gamma h} - 1}.$$
A similar derivation holds for $VaR^K_t$, which completes the proof of the proposition.

The aggregate risk of the system is defined as the sum of $VaR^Y_t$ and $VaR^K_t$, namely that

$$VaR_t = VaR^Y_t + VaR^K_t.$$ 

From the result of Proposition 1, the risk level of the economic system relies on the investment scale. If $I(t) < 0$, then $VaR_t > 0$. This could be interpreted that excessive savings lead to downward volatilities of economic variables. If $I(t) = 0$, then $VaR_t = 0$. This is consistent with the fact that volatility is caused by the uncertainty of investment. For the case $I(t) > 0$, we can see that if the intensity of shock satisfies

$$
\epsilon \leq \frac{2(qe^{-\alpha \gamma^2 h} + \alpha \gamma e^{-qh} - \alpha \gamma - q)}{\Phi^{-1}(1 - \mu)(\sqrt{q - qe^{-2\alpha \gamma^2 h}} + \sqrt{\alpha \gamma - \alpha \gamma e^{-2qh}})},
$$

then $VaR_t \leq 0$. This claims that if the shock is not too strong, smaller than the threshold value given above, the regulators need not worry about the risk. Otherwise, $VaR_t > 0$ and the risk should be cared for. In the following subsection, the explicit forms of the risk measure given in Proposition 1 are used to induce a risk constraint on the investment setting.

3.2. Investment setting constrained by dynamic VaR. In this subsection, we introduce the risk constraint induced by the dynamic VaR defined in the last subsection to the programming, namely that the utility function (35) subjects to the risk constraint

$$VaR_t \leq R,$$ 

where $R > 0$ is considered as the acceptably maximum risk level for the regulators. By Proposition 1, (17) is equivalent to

$$
\frac{qe^{-\alpha \gamma^2 h} + \alpha \gamma e^{-qh} - \alpha \gamma - q}{\alpha \gamma q} I(t) - \frac{\epsilon \Phi^{-1}(1 - \mu)(\sqrt{q - qe^{-2\alpha \gamma^2 h}} + \sqrt{\alpha \gamma - \alpha \gamma e^{-2qh}})}{2\sqrt{\alpha \gamma q}|I(t)|} \leq R.
$$

Theorem 3.1. Suppose that the utility function (35) subjects to the risk constraint (39). Denote $I_1, I_2$ as

$$I_1 = 2\alpha \gamma q R /$$

$$\left(2(qe^{-\alpha \gamma^2 h} + \alpha \gamma e^{-qh} - \alpha \gamma - q)ight)$$

$$- \epsilon \Phi^{-1}(1 - \mu)(\sqrt{q - qe^{-2\alpha \gamma^2 h}} + \sqrt{\alpha \gamma - \alpha \gamma e^{-2qh}})$$

and

$$I_2 = 2\alpha \gamma q R /$$

$$\left(2(qe^{-\alpha \gamma^2 h} + \alpha \gamma e^{-qh} - \alpha \gamma - q) + \epsilon \Phi^{-1}(1 - \mu)(\sqrt{q - qe^{-2\alpha \gamma^2 h}} + \sqrt{\alpha \gamma - \alpha \gamma e^{-2qh}})\right).$$

Then the evolvement of the shocked Kaldor macroeconomic model (4) can be calibrated with the benchmark process $(y, k)$ through the dynamic investment setting $I^*_R$.
given as follows. If the intensity of shock $\epsilon$ satisfies
\[
\epsilon > \frac{2(\epsilon e^{-\alpha\gamma h} + \alpha\gamma e^{-qh} - \alpha\gamma - q)}{\Phi^{-1}(1 - \mu)\sqrt{2\alpha\gamma q}(\sqrt{\frac{q}{q - \epsilon e^{-2\alpha\gamma h}} + \sqrt{\alpha\gamma - \epsilon e^{-2qh}})},
\]
then $I_R^*$ is given as
\[
I_R^*(t, Y, K) = \min\{\max\{I^*(t, Y, K), I_1\}, I_2\},
\]
where $I^*$ is given in Theorem 2.4. Otherwise, $I_R^*$ is given as
\[
I_R^*(t, Y, K) = \max\{I^*(t, Y, K), I_2\}.
\]
Proof. The utility function $\Xi$ is given as
\[
\frac{\partial\Phi}{\partial Y} - \frac{\partial Y}{\partial K} I + qe^{-\alpha h} + \alpha\gamma e^{-qh} - \alpha\gamma - q I - \eta R
\]
subject to constraint (40). Introduce the Lagrange equation
\[
L(I, \eta) = I \frac{\partial\Xi_1}{\partial Y} + I \frac{\partial\Xi_1}{\partial K} + \frac{\epsilon^2 I^2 \frac{\partial^2\Xi_1}{\partial Y^2} + \frac{\epsilon^2}{2} \frac{\partial^2\Xi_1}{\partial K^2}}{2} + \frac{\epsilon^2 I^2 \frac{\partial^2\Xi_1}{\partial Y\partial K} + \frac{\epsilon^2 I^2 \frac{\partial^2\Xi_1}{\partial K\partial Y}}{2}}{2} - \frac{\epsilon\Phi^{-1}(1 - \mu)(\sqrt{2q - 2e^{-2\alpha\gamma h}} + \sqrt{2\alpha\gamma - 2\alpha\gamma e^{-2qh}})}{2\sqrt{\alpha\gamma q}} |I|,
\]
where $\eta > 0$ is the Lagrange multiplier. The solution of (44) should satisfy the following first order necessary conditions
\[
\frac{\partial\Xi_1}{\partial Y} + I \frac{\partial\Xi_1}{\partial K} + \frac{\epsilon^2 I^2 \frac{\partial^2\Xi_1}{\partial K^2}}{2} I + 2\epsilon^2 I^2 \frac{\partial K^2}{\partial Y^2} I + 2\epsilon^2 I^2 + \frac{\epsilon^2 I^2 \frac{\partial^2\Xi_1}{\partial K\partial Y}}{2} + \frac{\epsilon^2 I^2 \frac{\partial^2\Xi_1}{\partial K\partial Y}}{2} - \frac{\epsilon\Phi^{-1}(1 - \mu)(\sqrt{2q - 2e^{-2\alpha\gamma h}} + \sqrt{2\alpha\gamma - 2\alpha\gamma e^{-2qh}})}{2\sqrt{\alpha\gamma q}} |I| - R \leq 0
\]
and
\[
\eta \frac{qe^{-\alpha h} + \alpha\gamma e^{-qh} - \alpha\gamma - q I}{\alpha\gamma q}
\]
\[
\eta \frac{qe^{-\alpha h} + \alpha\gamma e^{-qh} - \alpha\gamma - q I}{\alpha\gamma q}
\]

where $\chi$ is the indicative function. We can solve the expression of $I$ from (45) and denote it as $I_R^*$. Substituting $I_R^*$ into (46) gives
\[
\frac{qe^{-\alpha h} + \alpha\gamma e^{-qh} - \alpha\gamma - q I_R^*}{\alpha\gamma q}
\]
\[
\frac{qe^{-\alpha h} + \alpha\gamma e^{-qh} - \alpha\gamma - q I_R^*}{\alpha\gamma q}
\]
\[
\frac{qe^{-\alpha h} + \alpha\gamma e^{-qh} - \alpha\gamma - q I_R^*}{\alpha\gamma q}
\]
\[
\frac{qe^{-\alpha h} + \alpha\gamma e^{-qh} - \alpha\gamma - q I_R^*}{\alpha\gamma q}
\]
\[
\frac{qe^{-\alpha h} + \alpha\gamma e^{-qh} - \alpha\gamma - q I_R^*}{\alpha\gamma q}
\]
If the left side of (48) is strictly less than the acceptable risk level $R$, then by the complementary slackness condition (47), we have $\eta = 0$ and

$$I^*_R = -\frac{\partial \Xi_1}{\partial Y} + \frac{\partial \Xi_1}{\partial K}.$$

Repeating the proofs of Theorem 2.1 and 2.4, $I^*_R$ admits the explicit form given in Theorem 2.4. Now suppose that the left side of (48) is equal to $R$. If the condition on the intensity of shock (43) holds, then from (48) with "≤" replaced by "=",

we can solve $I^*_R$ as

$$I^*_R = I_1$$

or

$$I^*_R = I_2,$$

where $I_1$ and $I_2$ are given as (41) and (42) respectively. Otherwise, we can only solve $I^*_R$ as $I^*_R = I_2$. Therefore, if the condition (43) holds, then we claim that the optimal investment function is given as

$$I^*_R = \min \left\{ \max \left\{ -\left( \frac{\partial \Xi_1}{\partial Y} + \frac{\partial \Xi_1}{\partial K} \right) \varepsilon \frac{\partial^2 \Xi_1}{\partial Y^2} + 2\varepsilon^2 \frac{\partial^2 \Xi_1}{\partial Y \partial K} + \varepsilon^2 \frac{\partial^2 \Xi_1}{\partial K^2}, I_2 \right\}, I_1 \right\}.$$

Otherwise, the optimal investment function is given as

$$I^*_R = \max \left\{ -\left( \frac{\partial \Xi_1}{\partial Y} + \frac{\partial \Xi_1}{\partial K} \right) \varepsilon \frac{\partial^2 \Xi_1}{\partial Y^2} + 2\varepsilon^2 \frac{\partial^2 \Xi_1}{\partial Y \partial K} + \varepsilon^2 \frac{\partial^2 \Xi_1}{\partial K^2}, I_2 \right\}.$$

4. Application. In this section, we carry out an application of the theoretical results achieved in the last sections. To verify the effectiveness of investment setting, we calibrate the evolvement of the shocked Kaldor macroeconomic system (4) with the business cycle. The business cycle is generated from the classical Kaldor model with hypothesis on the model parameters. The numeric solver used in this section is the Milstein method.

It has been proved by Chang and Smyth [5] that the nonlinear mechanism of investment function leads to the generation of a stable limit cycle for the classical Kaldor model (1). Since we focus ourselves on the effect of calibration through investment setting on the evolvement of the shocked Kaldor macroeconomic model, without loss of generality, we assume that the origin $(0, 0)$ is the equilibrium generating the limit cycle and set the nonlinear investment function in the classical Kaldor model as

$$I(Y, K) = \tanh(Y) - \beta K,$$

where $\beta > 0$. Through the normal form and Hopf bifurcation theory of ordinary differential equations, we can prove the existence of a stable limit cycle if the parameters satisfy $\beta = 0.5$, $q = 0.1$, $\alpha = 1$, $\gamma = 0.2$. Nevertheless, an explicit expression for such a limit cycle is usually unavailable and we just desire numerical solution.

Now we present some numeric results. In the simulation, we set the length of period as 200 months. Since the evolvement of the gross production $Y$ and the capital stock $K$ are similar, we just show the figures of $Y$ in this section. Figure 1 and Figure 2 show the evolvement of the gross production $Y$ in the shocked Kaldor macroeconomic model (4) without calibration, namely that the investment function in the model is set as (49). The intensity of shock is set as $\epsilon = 0.5$ in Figure 1 and $\epsilon = 2$ in Figure 2. The blue lines in the figures depict the evolvement of the gross production in the shocked model and the red lines depict the business cycle from the classical Kaldor model. We can observe the separations of the orbits led by the macro shock.

To see the effectiveness of calibration through investment setting when coping with macro shocks on economic system, the investment function in the shocked
The Kaldor macroeconomic model (4) is set as the one in Theorem 2.4. The business cycle from the classical Kaldor model is selected as the benchmark process. Figure 3 and Figure 4 show the calibrated evolvement of the gross production in the shocked model (4). The blue lines in the figures depict the calibrated evolvement of the gross production in the shocked model and the red lines still depict the business cycle from the Kaldor model. A comparison between Figure 1 and Figure 3 (Figure 2 and Figure 4) could realize the effectiveness of investment setting. The gross production in the shocked model evolves along the periodic orbit of the Kaldor model. Moreover, Figure 5 and Figure 6 show the calibrated evolvement of the gross production in the shocked model (4) with risk constraint. The green lines depict the evolvement with risk constraint. Due to the definition of the risk measure and the risk constraint, we can see the restraint of downward volatility from the constrained evolvement of gross production comparing with the blue lines depicting the calibrated evolvement without risk constraint. Especially, in Figure 6, it seems that the green line does not follow the blue line well. Actually, the evolvement with risk constraint should follow the evolvement without risk constraint in overall trend, such as those from the beginning to 40th month, from 60th to 140th month, from 160th to 190th month. The remarkable separations from the blue line happened during 40th-60th month, 140th-160th month and 190th-200th month. Such phenomena are due to the persistent effect of the dynamic risk constraint. Since the intensity of shock in Figure 6 is set as $\epsilon = 2$, larger than the one, $\epsilon = 0.5$, in Figure 5, the evolvement in Figure 6 should be fluctuating more strongly compared with the evolvement in Figure 5. The dynamic risk constraint should be more effective and the incentive for upward volatilities should be more positive when facing large shock.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Evolvement of the gross production $Y$. The blue line depicts the evolvement in the shocked Kaldor model without calibration and the red line depicts the cycle fluctuation. The intensity of shock is set as $\epsilon = 0.5$.}
\end{figure}

5. **Conclusion.** Motivated by the work of Brunnermeier and Sannikov [4], we develop a Kaldor macroeconomic model with macro shock in the current paper. Macro shocks are consider as inevitable ingredients when working on macroeconomic models. Decision makers desire to control the amplification of shocks through effective approaches. In this paper, we consider calibration through the setting of investment as an approach for macroeconomic control. The investment is selected as the control variable due to its primary role in the economic system. The evolvement of the shocked system is calibrated with some expected process as benchmark. The
Figure 2. Evolvement of the gross production $Y$. The blue line depicts the evolvement in the shocked Kaldor model without calibration and the red line depicts the cycle fluctuation. The intensity of shock is set as $\epsilon = 2$.

Figure 3. Evolvement of the gross production $Y$. The blue line depicts the calibrated evolvement in the shocked Kaldor model and the red line depicts the cycle fluctuation. The intensity of shock is set as $\epsilon = 0.5$.

Figure 4. Evolvement of the gross production $Y$. The blue line depicts the calibrated evolvement in the shocked Kaldor model and the red line depicts the cycle fluctuation. The intensity of shock is set as $\epsilon = 2$.

Benchmark processes are usually the reflections of decisions or policies. In section 2, (5) is the utility function in the programming meeting the objective of calibration. Theorem 2.1 presents the optimal investment setting with this utility function. It is
semi-explicit since it is associated with a five-dimensional nonlinear system of ordinary differential equations. Although these equations could be solved via numeric methods, in practice, the decision executants prefer to completely explicit formula for the investment setting. An explicit formula might be executed more directly and expediently. Theorem 2.4 replies to this problem by presenting an explicit investment setting. Although some modification is needed, but it is proved logical. The effectiveness of calibration through investment setting on the macroeconomic system is tested by numeric results in Section 4.

Macro shocks cause systematic risks which should be paid attention. In this paper, we define a dynamic VaR as the risk measure for the shocked macroeconomic system and induce a risk constrain subjecting to the utility in the programming. The shock leads to strong volatility of the system and the risk measure defined in this paper captures the possibility of the abruptly downward volatility. The risk constraint introduced into the programming aims to control the extent of the downward volatility through the control on investment scale. The effects of the risk constraint on the evolvement of the economic system could be seen by the numeric results in Section 4.
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E-mail address: w.zhenzhen@mail.scut.edu.cn
E-mail address: lizh@gzhu.edu.cn
E-mail address: chensl@gzhu.edu.cn
E-mail address: zhehao.h@gzhu.edu.cn