Analysis of shaft selection in terms of stiffness and mass

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Abstract. The currently available computer aided design tools significantly accelerate the design process. They enable creation of 3D structures and elements and allow checking their strength or other properties with the use of the different analytical and simulation modules. Selection of appropriate elements in terms of strength and cost is important for any engineering project. The above-mentioned tools for designing and performing analyses were used to select an appropriate shaft to drive the toothed belt. Analyses were needed to verify the designed shaft length and mounting method. The tested shaft was 1200 mm long, supported at one end. Shafts with different geometries, i.e. solid and hollow with different diameters were selected, taking into the consideration the design assumptions. The weight and rigidity of the system were considered relevant and were investigated. The paper presents the input parameters of the analysis and the simulation outputs. These are compared with the results of analytical calculations. The research outputs include the conclusion that while offering a higher rigidity, hollow shafts involve mounting design challenges.

1. Introduction

The design process can be significantly accelerated with the use of structural modelling and simulation programs. Modelling enables visualisation of the engineering concept and provides a better overview of the designed structure. With the model in hand it is possible to check how the parts will interact, look for possible collisions, obtain the overall dimensions of the machine and the travel ranges of its working mechanisms. Simulations provide a much better representation of the actual behaviour of systems than that provided by schematics. Modelling and drawings which can be quickly modified saves time and resources. With parametric modelling of parts, it is possible to conduct indicative tests of structures with different dimensions and compare the results. Unfortunately, these programs are not perfect and logical thinking remains of essence. It is the user rather than the program that is responsible for correct modelling of the problem under analysis. Also defining appropriate operating conditions and verification of the output results are the user’s tasks [1]. No computer program will ever be able to verify whether a given value is safe or not unless it has been provided with the relevant input data defining, for example, the maximum allowable limit. Besides mechanical problems, these programs are often used for analysing fluid flow problems. Application of numerical methods for solving various problems is widely described in the research publications [2–4]. If necessary and possible analytical results are compared with the results of FEM analysis [2, 5, 6].

Machine engineering is not limited to specification of parts needed to build the machine capable of performing the tasks for which it is intended but includes also verification of the correctness and effectiveness of the selections made. All machines must be designed to perform their intended functions. Many a time there are some specific requirements to be met by the designed machines, such as weight,
consumption of energy or amount of available space requirement. For this reason, calculations and analyses are conducted to verify if a given component will satisfy the strength requirement with the self-weight reduced as far as practicable. Besides the overall dimensions the relevant factors include the material type and connections with other parts.

The main load acting on machine shafts is the torque load. Appropriately designed geometry, material specification and rigidity are all critical for correct operation of machine shafts. For this reason, a shaft of a conveyor belt system, supported on one end only, an uncommon solution by its own right, was chosen for analysis in this study. The issue of shafts optimization and other machine components is often mentioned in the literature [7–10]. A shaft supported at one end by a mounting bearing was chosen. The reason behind it is the need to find a design solution facilitating the belt replacement which is needed quite frequently.

2. Analyzed system
This study concerns optimum geometry of shafts transferring power to endless toothed belts. The analysed system comprises the drive shaft (1) and driven shaft (2), both supported on mounting bearings at one end. On the shafts toothed pulleys are fitted which set the toothed belt in motion. This system is represented schematically in figure 1.

![Figure 1](image)

*Figure 1.* System of shafts: 1- drive shaft, 2- driven shaft, 3- angular contact ball bearings, 4- narrow toothed pulley, 5- wider toothed pulley, 6- toothed belt.

The design principle is the same as in the case of belt conveyor, except that here the shaft is supported on mounting bearing at one end only, this to facilitate belt replacement operation. Although solid shafts are presented in figure 1, both solid and hollow shafts are considered in the further part hereof, the cross-sections of which are presented in figure 2.

![Figure 2](image)

*Figure 2.* Cross-sections of analysed shafts: a) solid, b) hollow.

The assumed length of both shafts was 1 200 mm and this is also marked in figure 1. Additionally, it was assumed that the shaft diameter is constant over the analysed length marked a in figures 1 and 3. The range of outside diameter of the analysed shafts shaft was limited by the maximum pulley hole diameter of 50 mm. The geometrical parameters of shafts should be designed in a way to satisfy the strength requirements, provide adequate rigidity to weight ratio and ensure assemblability of components. Reduction of weight is relevant to inertia and energy loss analyses. For comparing the weights of shafts with the same shape and diameter C45 steel for heat treatment was chosen with the
following specifications: density $\rho = 7850 \text{ kg m}^{-3}$, $E = 210 000 \text{ MPa}$, Poisson's ratio $\nu = 0.3$, allowable bending stress $k_g = 115 \text{ MPa}$. Besides their self-weight the shafts must also carry the load induced by the belt weight. As an input assumption, the conveyor will not carry any materials on the belt surface and, as such, no other external loads are taken into account. The weight of belt per one shaft will not exceed 25 kg. The model with the applied self-weight load of the shaft and the load imposed by the belt weight is presented in figure 3 [11]. Line load model was chosen for the analyses of this study.

![Figure 3. Shaft mounting and loading diagram, where the gravity force is represented by small arrows, belt weight by big arrows, $l = 35 \text{ mm or } 50 \text{ mm}$ depending on the case under analysis, $a = 1000 \text{ mm}$, and the overall length $L = 1200 \text{ mm}.$](image)

3. Preparation to design calculations

3.1. Analytical Calculations
The analysed system was represented by a line load model including two supports. Table 1 gives the formula for determining the relevant shaft parameters, based on the information given in [11]. The symbols used in these equations are explained in the legend under the table 1.

|                  | Solid shaft                                      | Hollow shaft                                    |
|------------------|--------------------------------------------------|-------------------------------------------------|
| Cross-section area | $A_p = \frac{\pi \cdot D^2}{4}$                   | $A_d = \frac{\pi \cdot (D^2 - d^2)}{4}$          |
| Volume            | $V_p = A_p \cdot L$                               | $V_d = A_d \cdot L$                             |
| Mass              | $m_p = \rho \cdot V_p$                            | $m_d = \rho \cdot V_d$                          |
| Shaft weight (Equivalent concentrated load) | $Q_p = m_p \cdot g$                             | $Q_d = m_d \cdot g$                             |
| Belt weight (Force of gravity)       | $Q_b = m_b \cdot g$                             | $Q_b = m_b \cdot g$                             |
| Line load         | $q_p = \frac{Q_p}{a}$                             | $q_d = \frac{Q_d}{a}$                           |
| Moment of inertia | $I_p = \frac{\pi \cdot D^4}{64}$                 | $I_d = \frac{\pi \cdot (D^4 - d^4)}{64}$       |
| Section modulus   | $W_p = \frac{\pi \cdot D^3}{32}$                 | $W_d = \frac{\pi \cdot (D^4 - d^4)}{32 \cdot D}$ |
| Bending moment    | $M_{gp} = \frac{q_p \cdot (a + \frac{l}{2})^2}{2}$ | $M_{gd} = \frac{q_d \cdot (a + \frac{l}{2})^2}{2}$ |
| Maximum absolute deflection | $f_p = \frac{(q_p + \frac{Q_b}{L}) \cdot a^3 \cdot (4 \cdot 1 + 3 \cdot a)}{24 \cdot E \cdot I_p}$ | $f_d = \frac{(q_d + \frac{Q_b}{L}) \cdot a^3 \cdot (4 \cdot 1 + 3 \cdot a)}{24 \cdot E \cdot I_d}$ |
| Bending stress    | $\sigma_p = \frac{M_{gp}}{W_p}$                   | $\sigma_p = \frac{M_{gd}}{W_d}$                  |
Explanation of symbols used in the above equations:

- \( Ad \) – cross-sectional area of hollow shaft
- \( Ap \) – cross-sectional area of solid shaft
- \( D \) – outside diameter
- \( d \) – inside diameter
- \( L \) – shaft length
- \( l \) – spacing between support centres
- \( Vd \) – hollow shaft volume
- \( Vp \) – solid shaft volume
- \( md \) – hollow shaft mass
- \( mp \) – solid shaft mass
- \( mb \) – belt mass
- \( \rho \) – density
- \( Jp \) – moment of inertia of circular solid section
- \( Jd \) – moment of inertia of circular hollow section
- \( f_d \) – maximum absolute deflection of hollow shaft
- \( fp \) – maximum absolute deflection of solid shaft
- \( Wp \) – section modulus of solid shaft
- \( Wd \) – section modulus of hollow shaft
- \( Mgp \) – bending moment of solid shaft
- \( Mgd \) – bending moment of hollow shaft
- \( a \) – lever arm
- \( Qp \) – solid shaft weight
- \( Qd \) – hollow shaft weight
- \( Qb \) – belt weight
- \( qp \) – concentrated load for solid shaft
- \( qd \) – concentrated load for hollow shaft
- \( \sigma_d \) – hollow shaft bending stress
- \( \sigma_p \) – solid shaft bending stress
- \( F \) – equivalent point force from line load
- \( G \) – force of gravity

3.2. Numerical calculations

For the purpose of numerical analysis it was necessary to develop a model. From the models available in ABAQUS beam model was chosen. The designed geometry with appropriate values input in the program is presented in figure 4. Circular and pipe shapes were chosen for the solid and hollow shafts respectively, the latter with thin-walled pipe formulation.

![Figure 4](image_url)

Figure 4. Geometry of the analysed element in ABAQUS, a) beam model (line), b) view of solid shaft, c) view of hollow shaft.

The technical parameters of the chosen steel where input. Then appropriate constraints and loads were imposed onto the model (figure 5). The constraints include roller support (R) restraining translation in \( z \) axis and rotation about \( y \) and \( z \) axes and fixed support (E) constraining translation and rotation movements in all directions. The forces are induced by self-weight, this meaning the gravity force (\( G \)) and force from the weight of belt are considered, the later represented by an equivalent concentrated load obtained from the line load and applied at the mid point of the load application section.
5. Model with applied loads: G – force of gravity, F – loading force (induced by the belt weight), E – fixed support, R – roller support, a) beam model, b) overview.

The mesh density was varied depending on the analysed section (figure 6). For the section loaded with the belt weight the mesh was built of a definite number of elements, namely 100 per 1000 mm length. In the section between the supports there are 5 elements. In the terminal section behind the supports the mesh density was reduced due to a lesser significance of this part.

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Figure 6. FEM mesh generated for the system under analysis.

4. Results
Both analytical and numerical calculations were conducted for four different outside diameters of shafts 32 mm, 38 mm, 45 mm and 50 mm and for 35 mm and 50 mm distances between bearings (table 2). Each shaft was 1200 mm long and the load imposed by the belt was applied over the length of 1000 mm starting from the roller support. The density of the shaft material was \( \rho = 7.8 \) kg cm\(^{-3} \). It was assumed that the load imposed by the belt will not exceed 25 kg per shaft. The inside diameters of hollow shafts were based on the commercially available materials.

The results of analytical and numerical calculations for different bearing distances and different outside diameters of shafts are given in tables 3 and 4.

The below table gives also the percent difference between the values obtained with analytical and numerical calculations. As it can be seen, in the case of maximum absolute deflection the error increases with the increase of the diameter and decreases in the case of stress. The results of numerical simulation are presented also in figure 7.
### Table 2. Results of calculations for solid shaft.

| Outside diameter (mm) | 32  | 38  | 45  | 50  |
|-----------------------|-----|-----|-----|-----|
| Solid shaft mass (kg) | 7.53| 10.62| 14.89| 18.38|
| 35 mm distance between bearings | | | |
| Maximum absolute deflection $f$ (mm) ANALYTICALLY | 3.6002 | 1.9408 | 1.0785 | 0.7568 |
| Maximum absolute deflection $f$ (mm) ABAQUS | 3.6060 | 1.9840 | 1.1290 | 0.8048 |
| Percent error (%) | | | |
| Bending stress (MPa) ANALYTICALLY | 49.4150 | 31.9495 | 21.2715 | 16.7184 |
| Bending stress (MPa) ABAQUS | 50.7600 | 32.8200 | 21.8500 | 17.1800 |
| Percent error (%) | 2.72 | 2.72 | 2.72 | 2.76 |
| 50 mm distance between bearings | | | |
| Maximum absolute deflection $f$ (mm) ANALYTICALLY | 3.6718 | 1.9799 | 1.1006 | 0.7724 |
| Maximum absolute deflection $f$ (mm) ABAQUS | 3.8400 | 2.1150 | 1.2030 | 0.8571 |
| Percent error (%) | 4.58 | 6.8 | 9.31 | 10.97 |
| Bending stress (MPa) ANALYTICALLY | 50.1312 | 32.4126 | 21.5798 | 16.9607 |
| Bending stress (MPa) ABAQUS | 52.3 | 33.82 | 21.85 | 17.18 |
| Percent error (%) | 4.33 | 4.34 | 1.25 | 1.29 |

### Table 3. Results of calculations for hollow shaft.

| Outside diameter (mm) | 32  | 38  | 45  | 50  |
|-----------------------|-----|-----|-----|-----|
| Inside diameter (mm)  | 20  | 25  | 32  | 36  |
| Hollow shaft mass (kg) | 3.85 | 5.05 | 6.17 | 7.42 |
| 35 mm distance between bearings | | | |
| Maximum absolute deflection $f$ (mm) ANALYTICALLY | 3.9573 | 2.1496 | 1.2320 | 0.8515 |
| Maximum absolute deflection $f$ (mm) ABAQUS | 4.0870 | 1.2720 | 0.8869 | |
| Percent error (%) | 3.28 | 3.46 | 3.25 | 4.16 |
| Bending stress (MPa) ANALYTICALLY | 51.9037 | 33.6678 | 22.9625 | 17.7219 |
| Bending stress (MPa) ABAQUS | 47.3800 | 31.0400 | 21.5500 | 16.6900 |
| Percent error (%) | 11.80 | 10.92 | 9.32 | 9.01 |
| 50 mm distance between bearings | | | |
| Maximum absolute deflection $f$ (mm) ANALYTICALLY | 4.0349 | 2.1921 | 1.2565 | 0.8685 |
| Maximum absolute deflection $f$ (mm) ABAQUS | 4.3630 | 2.3740 | 1.3570 | 0.9459 |
| Percent error (%) | 8.13 | 8.30 | 7.99 | 8.91 |
| Bending stress (MPa) ANALYTICALLY | 54.4989 | 35.3512 | 24.1106 | 18.6080 |
| Bending stress (MPa) ABAQUS | 48.6500 | 31.8700 | 22.1300 | 17.1400 |
| Percent error (%) | 10.73 | 9.85 | 8.21 | 7.88 |
Figure 7. Graphical representation of the numerical calculation results for 45 mm hollow shaft showing the magnitudes of: a) stress, b) deflection.

The graph in figure 8 presents the relationship between the maximum absolute deflection and the shaft weight for hollow and solid shafts for 35 mm spacing between bearings. The values obtained for 50 mm spacing were similar yet are not shown as this would compromise clarity of the chart.

Figure 8. Relationship between deflection and shaft weight.

Table 4 gives the percent differences of deflections and weights of shafts with equal outside diameters. The results obtained for 50 mm spacing were similar and, as such, they have been skipped.

Table 4. Percent differences of weights and deflections of solid and hollow shafts for different outside diameters with 35 mm spacing between bearings.

| Outside diameter of shaft (mm) | 32    | 38    | 45    | 50    |
|-------------------------------|-------|-------|-------|-------|
| Percent difference of weight between solid and hollow shafts (%) | 39.063 | 43.283 | 50.568 | 51.84 |
| Percent difference of deflection between solid and hollow shafts (%)  | 9.024  | 9.714  | 12.461 | 11.121 |
5. Analysis of results and conclusions

The analysed system is designed to set in motion toothed belts which require frequent replacement. For this reason, one-sided bearing system has been designed, enabling replacement of the belt at the other side of the mechanism. This is not a common engineering practice and, as such, the design has been subjected to analyses. These analyses covered in particular the strength and stiffness of shafts with specific geometrical parameters in relation to the shaft weights. Solid and hollow shafts of equal length were considered. The results of analytical and numerical calculations allow us to draw the following conclusions:

- There is a generally a few percent difference (max. 11.8) between analytical and numerical results and such discrepancy can be considered acceptable.
- Characteristic points along the shaft are most conspicuous in the graphical representation of results (figure 6). The calculated bending stresses are the greatest at the support R (bearing on the belt side) and the highest deflection is located at the shaft end which represents the actual situation and confirms correctness of numerically obtained results.
- With the increase of the shaft diameter the discrepancies between analytical and numerical results increase in the case of the maximum absolute deflection and, quite the opposite, decrease in the case of bending stress.
- As the bending stress is in each case smaller than the maximum allowed bending strength for the selected steel type, the weight to stiffness ratio can be used as the criterion for designing the shaft geometric parameters.
- Considering the relationship between the shaft weight and the maximum absolute deflections it can be seen that hollow shafts are much lighter with only slightly smaller stiffness. According to data given in table 4, an almost 40% reduction of weight of hollow shaft with 32 mm outside diameter and 24 mm bore in relation to solid shaft with the same diameter decreases the maximum absolute deflection by only 9%. As such, the relationship is not linear which can be seen also in the chart (figure 7), where the rate of decrease of deflection decreases with the increasing rate of weight increase.
- This means that oversizing of the shafts would not bring a considerable increase of its stiffness.
- The spacing of supports is also relevant to the maximum absolute deflection and bending strength yet it does not influence the system weight. The direct relationship between the spacing of supports and the amount of absolute deflection is defined by the equation where both the load application length and spacing of supports are in the numerator. Although in the case under analysis the differences were small, this should be borne in mind when analysing similar systems.

In view of the above facts, a hollow shafts should be considered a preferred option as it will reduce the overall weight of the system. According to the calculation results the best weight to deflection ratio is obtained for a hollow shaft with 38 mm outside diameter and 25 mm bore. Unfortunately, the amount of deflection is still considerable, which adversely affects the performance of the analysed mechanisms when operating under full load. A measure to reduce the deflection without compromising the ease of belt replacement could be a movable support fitted on the other side for the time when the system is when in the running mode. This support could be designed as a split unit with a sleeve bearing matching the shaft.

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