Dynamical Poincaré Symmetry
Realized by Field-dependent Diffeomorphisms

R. JACKIW

Center for Theoretical Physics, Massachusetts Institute of Technology
Cambridge, MA 02139–4307, USA

A.P. POLYCHRONAKOS

Theoretical Physics Department, Uppsala University
S-75108 Uppsala, Sweden

Faddeev Festschrift, Steklov Mathematical Institute Proceedings

Abstract

We present several Galileo invariant Lagrangians, which are invariant against Poincaré transformations defined in one higher (spatial) dimension. Thus these models, which arise in a variety of physical situations, provide a representation for a dynamical (hidden) Poincaré symmetry. The action of this symmetry transformation on the dynamical variables is nonlinear, and in one case involves a peculiar field-dependent diffeomorphism. Some of our models are completely integrable, and we exhibit explicit solutions.

I. INTRODUCTION

Our colleague and friend Ludwig Faddeev has brought mathematics to physics and physics to mathematics. Here we mention only his work on integrable systems and on group theory, because these two themes also inform our essay, which is dedicated to him on the occasion of a significant birthday.

We shall be concerned with Lagrangians containing fields that move in time on a d-dimensional space. The models arise from diverse physical systems: continuum description of free particle motion, isentropic fluid mechanics with irrotational velocity field, hydrodynamical description of quantum mechanics, free motion of membranes as well as higher-dimensional “d-branes”. Our models are Galileo invariant on their (d, 1) space-time. However, they possess a further hidden symmetry: in terms of the dynamical canonical variables with which these theories are formed one can construct quantities whose algebra, determined by canonical Poisson brackets, reproduces a (d + 1, 1) Poincaré algebra. Moreover, one finds that the (d + 1, 1) Poincaré group is a symmetry of the (d, 1) Galileo invariant theory, with various (d + 1, 1) Poincaré transformations acting non-linearly on the available (d, 1) coordinates. Indeed some
of these coordinate transformations make use of peculiar diffeomorphisms that involve the fields themselves. In summary we find that our models give a non-linear representation for a dynamical (hidden) Poincaré group.

Another interesting feature is that several of our models are completely integrable. This is seen by explicitly integrating the Euler-Lagrangian equations for some models, while for others – in one spatial dimension – by identifying an infinite number of conservation laws, and replacing the nonlinear equations by linear ones, whose solutions are explicitly given.

II. THE MODELS

The Lagrangians that we study involve two canonically conjugate fields, $\theta$ and $\rho$. When presented in first-order form, they read

$$L = \int d^d r \left( \dot{\theta} \dot{\rho} - \frac{1}{2} \rho \nabla \theta \cdot \nabla \theta - V(\rho) \right)$$  \hspace{1cm} (2.1)

$$L_0 = \int d^d r \left( \dot{\theta} \dot{\rho} - \frac{1}{2} \rho \nabla \theta \cdot \nabla \theta \right)$$  \hspace{1cm} (2.2)

Fields depend on time and space, $(t, r)$; the over-dot denotes differentiation with respect to the temporal argument; the gradient is with respect the spatial arguments. $L_0$ is the “free” Lagrangian (even though it is not quadratic), while $L$ includes an interaction potential $V(\rho)$, which we take to be $\theta$-independent.

The canonical structure implies the Poisson bracket

$$\{\rho(t, r), \theta(t, r')\} = \delta(r - r')$$  \hspace{1cm} (2.3)

and the Hamiltonians

$$H = \int d^d r \mathcal{E} \hspace{1cm} \mathcal{E} = \frac{1}{2} \rho \nabla \theta \cdot \nabla \theta + V(\rho)$$  \hspace{1cm} (2.4)

$$H_0 = \int d^d r \mathcal{E}_0 \hspace{1cm} \mathcal{E}_0 = \frac{1}{2} \rho \nabla \theta \cdot \nabla \theta$$  \hspace{1cm} (2.5)

(Here we are following the simplectic method for Lagrangians that are linear in the time derivative – this approach was advocated by Faddeev and one of the present authors (RJ). [4]) Evidently the Euler-Lagrange equations read

$$\dot{\rho} = -\nabla \cdot (\rho \nabla \theta)$$  \hspace{1cm} (2.6)

$$\dot{\theta} = -\frac{1}{2} (\nabla \theta)^2 + f(\rho)$$  \hspace{1cm} (2.7)

$$f(\rho) = -\frac{\delta}{\delta \rho} \int d^d r V(\rho)$$  \hspace{1cm} (2.8)

where the “force” term $f$ is absent in the free case.

We now describe several contexts that lead to this system.
A. Continuum Description of Free Particle Motion

Begin with the free Lagrangian for a collection of particles, with equal masses scaled to unity.

\[ L_{\text{free particle}} = \frac{1}{2} \sum_i v_i^2(t) \quad (m = 1) \]  

(2.9)

Let the particle counting index \( i \) become the continuous variable \( r \), and introduce the density \( \rho(t, r) \) and current \( j(t, r) \), with

\[
\dot{j} = v \rho
\]  

(2.10)

Then \( \frac{1}{2} \sum_i v_i^2 \) becomes \( \frac{1}{2} \int d^d r v^2 \rho = \frac{1}{2} \int d^d r j^2 / \rho \). We wish to link the density and current by a continuity equation

\[
\dot{\rho} + \nabla \cdot j = 0
\]  

(2.11)

which can be enforced with help of a Lagrange multiplier \( \theta \). We thus arrive at the continuum Lagrangian

\[ L_{\text{free particle}} \rightarrow L'_0 = \int d^d r \left\{ \frac{1}{2} j^2 / \rho + \theta (\dot{\rho} + \nabla \cdot j) \right\} \]  

(2.12)

Since \( j \) does not participate in the canonical 1-form, \( \int d^d r \theta \dot{\rho} dt = \int d^d r \theta d \rho \), we can eliminate \( j \) by solving the constraint equation that follows when \( j \) is varied in \( L'_0 \).

\[
\dot{j} = \rho \nabla \theta
\]  

(2.13)

Comparing (2.10) with (2.13) shows that the velocity is given by \( \nabla \theta \); hence it is irrotational, with \( \theta \) playing the role of a velocity potential.

\[
\nabla \times v = 0
\]  

(2.14)

\[
v = \nabla \theta
\]  

(2.15)

Substitution of (2.13) in (2.12) yields \( L_0 \) of (2.2).

\[ L'_0 \rightarrow L_0 = \int d^d r \left\{ \theta \dot{\rho} - \frac{1}{2} \rho \nabla \theta \cdot \nabla \theta \right\} \]  

(2.16)

Supplementing \( L_0 \) with a velocity-independent (\( \theta \)-independent) interaction term produces

\[
L = L_0 + \int d^d r V(\rho)
\]  

(2.17)
B. Fluid Mechanics

Equations (2.6), (2.7) are recognized as the equations of fluid mechanics when the velocity field $v$ is irrotational and the motion is isentropic. The equation of motion (2.6) is the continuity equation (2.11), once (2.13) is taken into account. Moreover, that equation (2.7) for the velocity potential produces the Euler equation of fluid mechanics

$$\dot{v} + v \cdot \nabla v = -\frac{1}{\rho} \nabla \text{(pressure)} \quad (2.18)$$

is established by taking the gradient of (2.7), and using (2.15).

$$\dot{v} + v \cdot \nabla v = f'(\rho) \nabla \rho \quad (2.19)$$

The right-hand sides of (2.18) and (2.19) coincide for isentropic motion, where pressure is a function only of $\rho$ [Kelvin’s theorem]. [In that case $\frac{\partial V(\rho)}{\partial \rho} = -f(\rho)$ is the enthalpy $w$, and $(\rho \frac{\partial^2 V(\rho)}{\partial \rho^2})^{1/2}$ is the sound speed $u$.]

C. Hydrodynamical Description of Quantum Mechanics

For another derivation, we consider the Lagrangian for the Schrödinger equation.

$$L_{\text{Schrödinger}} = \int d^d r \left\{ i\dot{\psi}^* \dot{\psi} - \frac{1}{2} (\nabla \psi)^* \cdot (\nabla \psi) - V(\psi^* \psi) \right\} \quad (2.20)$$

The first two terms correspond to the free, linear quantum mechanical equation, while $V$ allows for possible non-linearity. Upon setting in (2.20)

$$\psi = \frac{1}{\sqrt{2}} e^{i\theta} \quad (2.21)$$

we get [1]

$$L_{\text{Schrödinger}} \rightarrow L = \int d^d r \left\{ \dot{\theta} \dot{\rho} - \frac{1}{2} \rho \nabla \theta \cdot \nabla \theta - V(\rho) \right\} \quad (2.22)$$

where

$$V(\rho) = \nabla (\rho) + \frac{1}{8} \left( \frac{\nabla \rho}{\rho} \right)^2 \quad (2.23)$$

[A similar result is obtained when a scalar field theory in $(d + 1, 1)$ dimensions is reduced dimensionally in the light-cone variable $\frac{1}{\sqrt{2}} (x^{(0)} - x^{(d+1)})$ [3].]
D. d-branes

Following Hoppe, we consider a Nambu-type Lagrangian for a closed extended d-dimensional object moving in (d + 1, 1)-dimensional Minkowski space-time.

\[ L = - \int d^d \phi \sqrt{G} \] (2.24)

\( G \) is \((-)^d\) times the determinant of the induced metric

\[ G_{\alpha \beta} = \frac{\partial x^\mu}{\partial \phi^\alpha} \frac{\partial x^\mu}{\partial \phi^\beta} \] (2.25)

where \( x^\mu(\phi) \) are the coordinates of the “d-brane” in space-time \((\mu = 0, 1, \ldots, d + 1)\), depending on parameters \( \phi^\alpha (\alpha = 0, 1, \ldots, d) \). \( \phi^0 \) is the evolution parameter; \( \phi^i (i = 1, \ldots, d) \) parameterize the object at fixed time.] We introduce light-cone variables

\[ \tau = \frac{1}{\sqrt{2}} (x^{(0)} + x^{(d+1)}) \] (2.26)

\[ \theta = \frac{1}{\sqrt{2}} (x^{(0)} - x^{(d+1)}) \] (2.27)

and the transverse coordinates \( x^i \) \((i = 1, \ldots, d)\) (2.28)

We choose the parameterization \( \tau = \phi^0 \). It follows that

\[ G_{\alpha \beta} = \begin{pmatrix} G_{00} & G_{0s} \\ G_{r0} & -g_{rs} \end{pmatrix} = \begin{pmatrix} 2\partial_r \theta - (\partial_r x)^2 & \partial_r \theta - \partial_r x \cdot \partial_s x \\ \partial_r \theta - \partial_r x \cdot \partial_s x & -\partial_r x \cdot \partial_s x \end{pmatrix} \] (2.29)

\[ G = g \Gamma \] (2.30a)

\[ \Gamma = 2\partial_r \theta - (\partial_r x)^2 + g^{rs} u_r u_s \] (2.30b)

\[ g \equiv \det g_{rs} \] (2.30c)

\[ u_r \equiv \partial_r x \cdot \partial_s x - \partial_s \theta \] (2.30d)

Here \( \partial_r \equiv \frac{\partial}{\partial \phi^r} \); \( \partial_r \equiv \frac{\partial}{\partial \phi^r} \), \( r = 1, \ldots, d \); \( g^{rs} \) is the inverse of \( g_{rs} \equiv \partial_r x \cdot \partial_s x \) and will be used to move the \((r, s)\) indices. Note that the dimensions of the \( x \) space (indexed by \( i \)) and of the \( \phi \) parameter space (indexed by \( r \)) are both \( d \). The Euler-Lagrange equations are conveniently presented in canonical form

\[ \partial_r x = -p/\Pi + u^r \partial_r x \] (2.31)

\[ \partial_r \theta = \frac{1}{2\Pi^2} (p^2 + g) + u^r \partial_r \theta \] (2.32)

\[ \partial_r p = -\partial_r (\frac{1}{\Pi} gg^{rs} \partial_s x) + \partial_r (u^r p) \] (2.33)

\[ \partial_r \Pi = \partial_r (\Pi u^r) \] (2.34)
with \( \Pi \) and \( p \) satisfying the constraint

\[
p \cdot \partial_r x + \Pi \partial_r \theta = 0 \tag{2.35}
\]

Here \( u^r \) is as given by (2.30d) [this is a consequence of (2.31), (2.35)], and can be set to zero by appropriate choice of the parameterization. Thereupon it follows that \( \partial_r \Pi \) vanishes, so we choose \( \Pi \) to be \(-1\). The equations then reduce to

\[
\begin{aligned}
\partial_r x &= p \tag{2.36} \\
\partial_r \theta &= \frac{1}{2}(p^2 + g) \tag{2.37} \\
\partial_r p &= \partial_r (gg^r s \partial_s x) \tag{2.38}
\end{aligned}
\]

and the constraint reads

\[
\partial_r \theta = p \cdot \partial_r x \tag{2.39}
\]

The constraint is now solved by a transformation introduced by Bordemann and Hoppe. [7] Rather than viewing the (dependent) variables \( p \) and \( \theta \) as functions of the (independent) parameters \( \tau \) and \( \phi \), a change of variables is performed by inverting \( x(\tau, \phi) \) and expressing \( \phi \) in terms of \( \tau \) and \( x \), (renamed \( r \)), so that \( p \) and \( \theta \) become functions of \( \tau \) (renamed \( t \)) and \( r \). It then follows from the chain rule that the constraint (2.39) becomes

\[
\frac{\partial r^i}{\partial \phi^r} \frac{\partial \theta}{\partial r^i} = p^j \frac{\partial r^j}{\partial \phi^r} \tag{2.40}
\]

and is solved by

\[
p = \nabla \theta \tag{2.41}
\]

where the gradient is with respect to \( r \). Moreover, according to the implicit function theorem (see sidebar)

\[
\partial_r = \partial_t + \nabla \theta \cdot \nabla \tag{2.42}
\]

so that (2.36) reproduces (2.41), since \( \partial_t x^i = 0 \); while (2.37) and (2.38) read, respectively

\[
\dot{\theta} + \frac{1}{2} \nabla \theta \cdot \nabla \theta = \frac{1}{2} g \tag{2.43}
\]

\[
\dot{p} + \nabla \theta \cdot \nabla p = \nabla (\dot{\theta} + \frac{1}{2} \nabla \theta \cdot \nabla \theta) = \partial_r (gg^r s \partial_s r) \tag{2.44}
\]

The over dot signifies differentiation with respect to \( t \).

It remains to show that (2.43) and (2.44) are consistent with each other. This is achieved by recognizing that
\( gg^{rs} = \frac{1}{(d-1)!} \epsilon^{ri_2 \ldots i_d} \epsilon^{sj_2 \ldots j_d} g_{i_2 j_2} \ldots g_{i_d j_d} \)  

(2.45a)

But \( g_{ij} = \frac{\partial x^k}{\partial \phi^i} \frac{\partial x^k}{\partial \phi^j} \), so

\[
\begin{align*}
 gg^{rs} \partial_s x^i = & \frac{1}{(d-1)!} \epsilon^{ri_2 \ldots i_d} \frac{\partial x^{k_2}}{\partial \phi^{i_2}} \ldots \frac{\partial x^{k_d}}{\partial \phi^{i_d}} \epsilon^{sj_2 \ldots j_d} \frac{\partial x^{k_2}}{\partial \phi^{j_2}} \ldots \frac{\partial x^{k_d}}{\partial \phi^{j_d}} \\
= & \frac{1}{(d-1)!} \epsilon^{ri_2 \ldots i_d} \frac{\partial x^{k_2}}{\partial \phi^{i_2}} \ldots \frac{\partial x^{k_d}}{\partial \phi^{i_d}} \epsilon^{ik_2 \ldots k_d} \det \frac{\partial x}{\partial \phi^i} 
\end{align*}
\]

(2.45b)

Note that \( \det \frac{\partial x}{\partial \phi^j} = g^{\frac{1}{2}} \). It now follows that

\[
\partial_t (gg^{rs} \partial_s x^i) = \frac{1}{(d-1)!} \epsilon^{ri_2 \ldots i_d} \frac{\partial x^{k_2}}{\partial \phi^{i_2}} \ldots \frac{\partial x^{k_d}}{\partial \phi^{i_d}} \frac{1}{2} \frac{\partial g}{\partial x^i} \frac{1}{2} \frac{\partial g}{\partial x^j} \\
= \frac{1}{(d-1)!} \epsilon^{ik_2 \ldots k_d} \frac{1}{2} \frac{\partial g}{\partial x^i} \frac{1}{2} \frac{\partial g}{\partial x^j} \\
= \frac{1}{2} \frac{\partial g}{\partial x^i} 
\]

(2.45c)

Thus (2.44) implies (2.43), once an \( r \)-independent constant of integration is absorbed in \( \theta \).

It is seen that the equations for the “d-brane” collapse into (2.43). But we still need an equation for \( g \), which can be obtained by differentiating \( g \) with respect to \( \tau \) and using (2.42).

\[
gg^{rs} \frac{\partial}{\partial \tau} g_{rs} \equiv \partial_\tau g = \dot{g} + \nabla \theta \cdot \nabla g 
\]

(2.46)

But

\[
g^{rs} \frac{\partial}{\partial \tau} g_{rs} = 2g^{rs} \partial \tau x^i \partial_\tau x^i \\
= 2g^{rs} \partial \tau x^i \partial_\tau p^i 
\]

Since \( g^{rs} = \frac{\partial \phi^r}{\partial x^s} \frac{\partial \phi^s}{\partial x^r} \), it follows that the right side of (2.46) is \( 2g \frac{\partial \phi^r}{\partial x^s} \partial_s p^i = 2g \frac{\partial ^s \phi}{\partial x^i} = 2g \nabla^2 \theta \). Therefore if we define \( g = 2\lambda/\rho^2 \), (2.46) is equivalent to

\[
\dot{\rho} + \nabla \cdot (\rho \nabla \theta) = 0 
\]

(2.47)

We conclude therefore that the motion of a “d-brane”, moving in time on \( d + 1 \) spatial dimensions is governed by equations derivable from our Lagrangian (2.1), with

\[
V(\rho) = \lambda/\rho. 
\]

(2.48)
Sidebar: Derivation of (2.42)  Consider a function $f$ that depends on $\tau$ and $\phi$: $f = f(\tau, \phi)$. The $\tau$ derivative differentiates $f$ with respect to the first argument. Next express $\phi$ as a function of $\tau$ and $x$, renamed $r$. It is clear that

$$\partial_\tau f(\tau, \phi) = \frac{\partial}{\partial \tau} f(\tau, \phi(\tau, r)) - \frac{\partial}{\partial \phi^i} f(\tau, \phi(\tau, r)) \frac{\partial \phi^i(\tau, r)}{\partial \tau}$$ \hspace{1cm} (2.49)

Next observe that

$$d\phi(\tau, r) = \frac{\partial \phi}{\partial \tau} d\tau + \frac{\partial \phi}{\partial r^i} dr^i$$ \hspace{1cm} (2.50)

When $\phi$ is held constant, we have

$$\frac{\partial \phi}{\partial \tau} = - \frac{\partial \phi}{\partial r^i} \frac{\partial r^i}{\partial \tau} \bigg|_{\phi \, \text{constant}}$$ \hspace{1cm} (2.51)

But from (2.36) it follows that the $\tau$ derivative of $r$ is $p$, which according to (2.41) is $\nabla \theta$. Thus (2.51) becomes

$$\frac{\partial \phi}{\partial \tau} = - \frac{\partial \phi}{\partial r^i} \frac{\partial \theta}{\partial r^i}$$ \hspace{1cm} (2.52)

Substitution in (2.49) and use of the chain rule shows that (2.42) is true, once $\tau$ is renamed $t$.

$$\partial_\tau f\big|_{\tau=t} = \frac{\partial}{\partial t} f + \frac{\partial \theta}{\partial r^i} \frac{\partial \phi^i}{\partial r^j} \frac{\partial f}{\partial \phi^j}$$

$$= \frac{\partial}{\partial t} f + \nabla \theta \cdot \nabla f$$ \hspace{1cm} (2.53)

III. SYMMETRIES OF THE MODEL

A. Galileo Symmetries

From the derivations, it should be obvious that $L_0$ (2.2), and also $L$ (2.1) (with obvious restriction on $V$) possess the Galileo symmetry. Let us record the generators of the infinitesimal transformations as integrals of the appropriate densities; also we specify the action of the finite transformations (parameterized by $\omega$) on the fields: $\rho \rightarrow \rho_\omega$, $\theta \rightarrow \theta_\omega$, by presenting formulas for $\rho_\omega(t, r)$ and $\theta_\omega(t, r)$ in terms of $\rho(t, r)$ and $\theta(t, r)$. One verifies that the generators are time independent according to the equations of motion (2.6)–(2.8), and this further implies that the transformed fields $\rho_\omega$ and $\theta_\omega$ also solve (2.6)–(2.8), when $\rho$ and $\theta$ are solutions.
• Time, space translation
  – Energy
  \[ H = \int d^d r \mathcal{E}, \quad \mathcal{E} = \frac{1}{2} \rho \nabla \theta \cdot \nabla \theta + V(\rho) = \frac{1}{2} j^2 / \rho + V(\rho) \] (3.1)
  – Momentum
  \[ P = \int d^d r \mathcal{P}, \quad \mathcal{P} = \rho \nabla \theta = \mathbf{j} \] (3.2)

• Space rotation
  – Angular momentum
  \[ J^{ij} = \int d^d r (r^i \mathcal{P}^j - r^j \mathcal{P}^i) \] (3.3)

With these space-time transformations \( \rho_\omega, \theta_\omega \) are obtained from \( \rho, \theta \) by respectively translating the time, space arguments and by rotating the spatial argument. 

• Galileo boost
  – Boost generator
  \[ B = t \mathcal{P} - \int d^d r \mathbf{r} \rho \] (3.4a)

The boosted fields are
  \[ \rho_\omega(t, \mathbf{r}) = \rho(t, \mathbf{r} - \omega t) \] (3.4b)
  \[ \theta_\omega(t, \mathbf{r}) = \theta(t, \mathbf{r} - \omega t) + \omega \cdot \mathbf{r} - \omega^2 t / 2 \] (3.4c)

The inhomogeneous terms in \( \theta_\omega \) are recognized as the well-known Galileo 1-cocycle, compare (2.21). Also they ensure that the transformation law for \( \mathbf{v} = \nabla \theta \)
  \[ \mathbf{v}(t, \mathbf{r}) \rightarrow \mathbf{v}_\omega(t, \mathbf{r}) = \mathbf{v}(t, \mathbf{r} - \omega t) + \omega \] (3.4d)

is appropriate for a co-moving velocity. Furthermore, knowledge about the Galileo 2-cocycle leads us to examine the \( \mathcal{P}, \mathcal{B} \) bracket, and its extension exposes another conserved generator, arising from an invariance against translating \( \theta \) by a constant; this just reflects the phase arbitrariness in (2.21).

• Phase symmetry
  – Charge
  \[ N = \int d^d r \rho \] (3.5a)
  \[ \rho_\omega = \rho \] (3.5b)
  \[ \theta_\omega = \theta - \omega \] (3.5c)

9
B. Connection with Poincaré Symmetry

It is well known that a Poincaré group in \((d+1,1)\) dimensions possesses the above extended Galileo group as a subgroup. This is seen by identifying selected light-cone components of the Poincaré generators \(P^\mu, M^{\mu\nu}\) with the Galileo generators,

\[
\begin{align*}
P^\mu &= (P^-, P^+, P^i) \approx (H, N, P^i) \\
M^{\mu\nu} &= (M^{+-}, M^{-i}, M^{+i}, M^{ij}) \\
M^{+i} &\approx B^i, \quad M^{ij} \approx J^{ij}
\end{align*}
\]  

(3.6)

(3.7)

where the ± components of tensors are defined as in (2.26)–(2.27)

\[
T(\pm) = \frac{1}{\sqrt{2}}(T^{(0)} \pm T^{(d+1)}) .
\]  

(3.8)

(This fact is responsible for behavior in the “infinite momentum” frame.) \[9\] But the Lorentz generators \(M^{+-}\) and \(M^{-i}\) have no Galilean counterparts.

C. Additional Symmetries

We observe that the free action \(I_0 = \int dt L_0\), as well as the interacting one

\[
I_\lambda = \int dt \int d^d r (\dot{\theta} \rho - \frac{1}{2} \rho \nabla \theta \cdot \nabla \theta - \lambda / \rho)
\]  

(3.9)

are invariant against time rescaling \(t \rightarrow e^{\omega} t\), which is generated by

\[
D = tH - \int d^d r \rho \theta .
\]  

(3.10a)

Fields transform according to

\[
\rho(t, r) \rightarrow \rho_\omega(t, r) = e^{-\omega} \rho(e^{\omega} t, r)
\]  

(3.10b)

\[
\theta(t, r) \rightarrow \theta_\omega(t, r) = e^{\omega} \theta(e^{\omega} t, r) .
\]  

(3.10c)

The dilation generator \(D\) is identified with \(M^{+-}\). It is straightforward to verify from (2.6)–(2.8) that \(D\) is indeed time independent.
More intricate is a further, obscure symmetry whose generator can be identified with $M^{-i}$. Consider the field-dependent coordinate transformations, implicitly defined by

$$
t \rightarrow T(t, r) = t + \omega \cdot r + \frac{1}{2} \omega^2 \theta(T, R)
$$

$$
r \rightarrow R(t, r) = r + \omega \theta(T, R)
$$

(3.11)

with Jacobian $|J|$.

$$
J = \det \begin{pmatrix}
\frac{\partial T}{\partial t} & \frac{\partial T}{\partial r_j} \\
\frac{\partial R_i}{\partial t} & \frac{\partial R_i}{\partial r_j}
\end{pmatrix} = \left(1 - \omega \cdot \nabla \theta(T, R) - \frac{1}{2} \omega^2 \dot{\theta}(T, R)\right)^{-1}
$$

(3.12)

The transformation parameter $\omega$ has dimensions of inverse velocity. When fields are taken to transform according to

$$
\rho(t, r) \rightarrow \rho_\omega(t, r) = \rho(T, R) \frac{1}{|J|}
$$

$$
\theta(t, r) \rightarrow \theta_\omega(t, r) = \theta(T, R)
$$

(3.13a,b)

one verifies that $I_\lambda$ and $I_0$ are invariant. This is readily seen for the interaction term

$$
\lambda \int dt \, d^d r \, \frac{1}{\rho(t, r)} \rightarrow \lambda \int dt \, d^d r \, \frac{|J|}{\rho(T, R)} = \lambda \int dT \, d^d R \, \frac{1}{\rho(T, R)}.
$$

(3.14)

To establish invariance of $I_0$, it is useful to write it first as

$$
I_0 = - \int dt \, d^d r \, \rho(\dot{\theta} + \frac{1}{2} \nabla \theta \cdot \nabla \theta)
$$

$$
\rightarrow - \int dt \, d^d r \, \frac{\rho(T, R)}{|J|} \left\{ \frac{\partial}{\partial t} \theta(T, R) + \frac{1}{2} \frac{\partial}{\partial r^i} \theta(T, R) \frac{\partial}{\partial r^j} \theta(T, R) \right\}
$$

(3.15)

The desired result follows once it is realized that the quantity in curly brackets equals $J^2 \{ \dot{\theta}(T, R) + \frac{1}{2} (\nabla \theta(T, R))^2 \}$. The transformations (3.13) are generated by

$$
G = \int d^d r \, (r \mathcal{E} - \frac{1}{2} \rho \nabla \theta^2)
$$

$$
= \int d^d r \, (r \mathcal{E} - \theta \mathcal{P})
$$

(3.16)

which is time independent according to (2.6)–(2.8), and whose algebra with the other generators show that $G^i$ can be identified with $M^{-i}$.

While we have no insight about the geometric aspects to this peculiar symmetry, the following remarks may help achieve some transparency.

Observe that the Galileo generators can be expressed in terms of $\rho$ and $j$ or $\rho$ and $v = j/\rho = \nabla \theta$. Consequently, they are also defined for velocity fields with

$$
\rho(t, r) \rightarrow \rho_\omega(t, r) = \rho(T, R) \frac{1}{|J|}
$$

$$
\theta(t, r) \rightarrow \theta_\omega(t, r) = \theta(T, R)
$$

(3.13a,b)

one verifies that $I_\lambda$ and $I_0$ are invariant. This is readily seen for the interaction term

$$
\lambda \int dt \, d^d r \, \frac{1}{\rho(t, r)} \rightarrow \lambda \int dt \, d^d r \, \frac{|J|}{\rho(T, R)} = \lambda \int dT \, d^d R \, \frac{1}{\rho(T, R)}.
$$

(3.14)

To establish invariance of $I_0$, it is useful to write it first as

$$
I_0 = - \int dt \, d^d r \, \rho(\dot{\theta} + \frac{1}{2} \nabla \theta \cdot \nabla \theta)
$$

$$
\rightarrow - \int dt \, d^d r \, \frac{\rho(T, R)}{|J|} \left\{ \frac{\partial}{\partial t} \theta(T, R) + \frac{1}{2} \frac{\partial}{\partial r^i} \theta(T, R) \frac{\partial}{\partial r^j} \theta(T, R) \right\}
$$

(3.15)

The desired result follows once it is realized that the quantity in curly brackets equals $J^2 \{ \dot{\theta}(T, R) + \frac{1}{2} (\nabla \theta(T, R))^2 \}$. The transformations (3.13) are generated by

$$
G = \int d^d r \, (r \mathcal{E} - \frac{1}{2} \rho \nabla \theta^2)
$$

$$
= \int d^d r \, (r \mathcal{E} - \theta \mathcal{P})
$$

(3.16)

which is time independent according to (2.6)–(2.8), and whose algebra with the other generators show that $G^i$ can be identified with $M^{-i}$.

While we have no insight about the geometric aspects to this peculiar symmetry, the following remarks may help achieve some transparency.

Observe that the Galileo generators can be expressed in terms of $\rho$ and $j$ or $\rho$ and $v = j/\rho = \nabla \theta$. Consequently, they are also defined for velocity fields with
vorticity, \((\nabla \times \mathbf{v} \neq \mathbf{0})\), and provide well-known constants of motion for the (isentropic) Euler equations [3]. However, the velocity potential \(\theta\) is needed to form \(D\) and \(G\), which therefore have a role only in vortex-free motion (with a specific potential or no potential).

From the identification with Poincaré generators, we see from (3.4), (3.6), (3.10) and (3.16) that \(\rho\) is the \(\mathbb{P}^+\) density and that \(\theta\) plays the role of \(x^-\). The latter identification is further suggested by the fact that for all the transformations that are identified with Lorentz transformations, \textit{viz.} (3.3), (3.4), (3.10), (3.11), (3.13) it is true that

\[
2T \theta(\mathbf{T}, \mathbf{R}) - R^2 = 2t \theta_\omega(t, \mathbf{r}) - r^2
\]

where \(T\) and \(R\) are the appropriately transformed coordinates. The naturalness of this expression is appreciated when it is recognized that

\[
x^\mu x_\mu = 2x^+ x^- - x^i x^i
\]

In the “\(d\)-brane” development we saw that \(\theta\) is indeed \(x^-\), see (2.27). Moreover there \(1/\rho \sim \sqrt{g}\) is the Jacobian of the inversion transformation \((\tau, \phi) \rightarrow (t, r(t, \phi))\), so it is quite natural that under the further transformation \((t, r) \rightarrow (T(t, r), R(t, r))\), \(1/\rho\) acquires the Jacobian of that transformation. Presumably transformations (3.11), (3.13) reflect a residual invariance or gauge-fixed “\(d\)-brane” theory, but the reason for the specific form (3.11) of the transformation is not apparent.

In applications to fluid mechanics our transformation generates nontrivial solutions of Euler’s equations, as we now explain.

\section*{IV. EXPLICIT SOLUTIONS AND THEIR TRANSFORMS}

In order to gain insight into the peculiar diffeomorphism transformations (3.11), (3.13), we consider its effect on some explicit solutions to Eqs. (2.6)–(2.8), in the free \((V = 0)\) and interacting \((V = \lambda/\rho)\) cases.

\subsection*{A. No Interaction, \(V = 0\)}

\subsubsection*{1. Particular Example}

With \(V = 0\), eq. (2.7) decouples from (2.6), and is solved by

\[
\theta(t, r) = \frac{r^2}{2t}
\]

(4.1a)

which, apart from selecting an origin in time and space and presenting a rotation and boost invariant profile, is also invariant against time rescaling (3.10) and the unconventional diffeomorphism (3.11)–(3.13). The fluid moves with a velocity unaffected by boosts (3.4d).
The density is not determined, since the solution of the continuity equation (2.6) in \(d\) spatial dimensions involves an arbitrary function of \(t/r\), and of the angles specifying \(\mathbf{r}\) that are denoted by \(\hat{\mathbf{r}} \equiv \mathbf{r}/r\).

\[
\rho(t, r) = f(t/r, \hat{\mathbf{r}})
\]

With \(\theta\) as in (4.1a), the coordinate transformations (3.11) takes explicit form

\[
T(t, r) = tr\omega
\]
\[
\mathbf{R}(t, r) = r\mathbf{r}\omega
\]
\[
J = r\omega^2
\]
\[
\mathbf{r}\omega \equiv \hat{\mathbf{r}} + \frac{\omega r}{2t}
\]

so that, as stated, the transformed \(\theta(\omega)(t, \mathbf{r}) = \theta(T, \mathbf{R}) = R^2/2T\) coincides with \(\theta(t, \mathbf{r})\), while the transformed density retains the form (4.1c) but with a different function of \(t/r\),

\[
\rho(\omega)(t, \mathbf{r}) = \frac{f(tr\omega/r, \hat{\mathbf{r}}\omega)}{r^d\omega^{d+2}}
\]

This coincides with \(\rho(t, \mathbf{r})\) for the special choice \(f(t/r, \hat{\mathbf{r}}) \propto (t/r)^{2+d}\) in (4.1c), which provides a density profile that is invariant under the diffeomorphism (3.11)–(3.13).

One may construct other “free” solutions, for which \(\theta\) remains unchanged under (3.11), (3.12), (3.13b), while \(\rho\) involves arbitrary functions. Also there are free solutions that respond nontrivially to the transformations. We do not pursue specific solutions any further, because it is possible to give the general solution to the free problem, which we now present.

2. General Solution

Since the pair of equations (2.6), (2.7) is first-order in time, to give a general solution we need initial data at initial time \(t_o\); owing to time-translation invariance, \(t_o\) can be taken to be 0, without loss of generality.

\[
\rho(0, \mathbf{r}) = \rho_o(\mathbf{r})
\]
\[
\theta(0, \mathbf{r}) = \theta_o(\mathbf{r})
\]

The general solution to the free version of (2.6), (2.7), that is, at \(f = 0\), is given in terms of a vector-valued function of \(t\) and \(\mathbf{r}, \mathbf{q}(t, \mathbf{r})\), which satisfies
The solution to (2.6), (2.7) then reads

\[ \rho(t, r) = \rho_0(q) \left| \det \frac{\partial q^i}{\partial r^j} \right| \]  

(4.7)

\[ \theta(t, r) = \theta_0(q) + \frac{t}{2} (\nabla \theta(q))^2 \]  

(4.8)

Note that the velocity, which is the gradient of \( \theta \), is just the initial velocity, evaluated on \( q \).

\[ \mathbf{v}(t, r) \equiv \nabla_r \theta(t, r) = \nabla_q \theta_0(q) = \mathbf{v}_0(q) \]  

(4.9)

Consequently, (4.6) may also be presented as

\[ q + t \mathbf{v}_0(q) = r \]  

(4.10)

and one verifies that the free Euler equation, \textit{without} the irrotational condition on \( \mathbf{v} \), \( \nabla \times \mathbf{v} \neq 0 \),

\[ \dot{\mathbf{v}} + \mathbf{v} \cdot \nabla \mathbf{v} = 0 \]  

(4.11)

is solved by \( \mathbf{v}_0(q) \), where \( \mathbf{v}_0(r) \) is the initial velocity and \( q \) satisfies (4.10). This of course is nothing but the description of freely moving dust particles. Also, one verifies that the form (4.7) for \( \rho \) solves the continuity equation, even when \( \nabla \times \mathbf{v} \neq 0 \).

Note that the free motion of dust particles leads to the conserved quantities

\[ J_f = \int d^d r \rho(t, r) f(v^1(t, r), \ldots, v^d(t, r)) \]

\[ \dot{J}_f = 0 \]  

(4.12)

where \( f \) is an arbitrary function of the velocity components. This is an expression in the continuum formalism of the fact that velocities of the free theory are constant. At \( d = 1 \), there are additional conserved quantities

\[ I_f = \int dx f(v(t, x)) \]

\[ \dot{I}_f = 0 \]  

(4.13)

for arbitrary functions \( f \).
B. With Interaction, $V = \lambda/\rho$

1. Particular Example

With interaction, the equations of motion remain coupled.

$$\dot{\rho} = -\nabla \cdot (\rho \nabla \theta)$$  \hspace{1cm} (4.14)

$$\dot{\theta} = -\frac{1}{2} \nabla \theta \cdot \nabla \theta + \lambda/\rho^2$$ \hspace{1cm} (4.15)

Remarkably a solution exists that is similar to (4.1): demanding rotational and time-rescaling invariance, (4.14) leads to a second order equation in $r$ for $\theta$, once $\rho$ has been eliminated with the help of (4.15). A particular solution of that equation then gives

$$\theta(t, r) = -\frac{r^2}{2(d-1)t}$$ \hspace{1cm} (4.16)

$$\rho(t, r) = \sqrt{\frac{2\lambda}{d}} \frac{|t|}{r}$$ \hspace{1cm} (4.17)

The solution is not diffeomorphism invariant; see below. The velocity flow is

$$v = -\frac{r}{(d-1)t}$$ \hspace{1cm} (4.18)

while the current reads

$$j = \pm \sqrt{\frac{2\lambda}{d}} \hat{r}$$ \hspace{1cm} (4.19)

where the sign is determined by the sign of $t$. Evidently here $d$ must be greater than 1 and $\lambda$ must be positive. (The latter requirement is natural, since in the fluid mechanical application the sound speed $u$ for our model is $= \sqrt{2\lambda/\rho}$.)

2. General Solutions at $d = 1$

It turns out that this model, along with a much more general class of models with local interactions, is completely integrable in one dimension. The integrals of motion can be expressed in the form

$$I_{f}^{(\pm)} = \int dx \rho f \left( v \pm \frac{\sqrt{2\lambda}}{\rho} \right)$$ \hspace{1cm} (4.20)
for arbitrary functions \( f \). We obtain two ‘chiral’ sectors involving the velocity of the fluid plus (minus) the local velocity of sound.

The general solution can be obtained in this case through linearization. It is known the non-linear Euler and conservation equations can be mapped into a linear system. This is achieved by performing a double Legendre transform on \( \theta \), thereby replacing the independent variables \( t \) and \( x \), with \( v = \frac{\partial \theta}{\partial x} \) and with the enthalpy \( w = -\frac{\partial \theta}{\partial t} - \frac{1}{2}(\frac{\partial \theta}{\partial x})^2 \).

\[
\Psi(v, w) = \theta(t, x) - t \frac{\partial \theta(t, x)}{\partial t} - x \frac{\partial \theta(t, x)}{\partial x}
= \theta(t, x) + t(w + \frac{1}{2}v^2) - xv
\] (4.21)

It is then true that
\[
\frac{\partial \Psi}{\partial w} = t, \quad v \frac{\partial \Psi}{\partial w} - \frac{\partial \Psi}{\partial v} = x \] (4.22)

An equation for \( \Psi \) follows from the continuity equation for \( \rho \), when it is remembered that for isentropic motion \( \rho \) can be taken to be a function of \( w \). In our model \( w = -\lambda/\rho^2 \). One finds

\[
 u^2 \frac{\partial^2 \Psi}{\partial w^2} - \frac{\partial^2 \Psi}{\partial v^2} + \frac{\partial \Psi}{\partial w} = 0 \] (4.23)

where the sound speed \( u = (\rho \frac{\partial^2 V}{\partial \rho^2})^{1/2} \) is regarded as a function of \( w \). For our problem \( u^2 = 2\lambda/\rho^2 = -2w \). Note that the coupling strength \( \lambda \) does not occur in (4.23). It reenters the formalism through \( w = -\lambda/\rho^2, w \leq 0 \) for \( \lambda \geq 0 \).

The general solution of (4.23) can be expressed in terms of two general functions, \( F \) and \( G \), of one variable. It is straightforward to verify that the following expression solves (4.23), with \( u^2 = -2w \).

\[
\Psi(v, w) = F(v + \sqrt{-2w}) - \sqrt{-2w} F'(v + \sqrt{-2w})
+ G(v - \sqrt{-2w}) + \sqrt{-2w} G'(v - \sqrt{-2w})
\] (4.24)

Explicit solutions can be obtained from this expression for specific choices of \( F, G \).

We record the general time-rescaling invariant solutions. The time-rescaling Ansatz for \( \theta \) and \( \rho \) \([\theta \propto 1/t, \rho \propto t]\) lets us use (4.13) to express \( \rho \) in terms of \( \theta \) and its spatial derivative. Then (4.14) yields a second-order equation in \( x \) for \( \theta \), which is solved by expressions that involve two arbitrary integration constants. For \( \lambda > 0 \) we find

\[
\theta(t, x) = \frac{1}{2k^2 t} \sinh^2 kx, \quad -\frac{1}{2k^2 t} \cosh^2 kx \] (4.25a), (4.25b)

\[
\rho(t, x) = \frac{\sqrt{2\lambda k|x|}}{\sinh^2 kx}, \quad \frac{\sqrt{2\lambda k|x|}}{\cosh^2 kx} \] (4.26a), (4.26b)
while for \( \lambda < 0 \)

\[
\theta(t, x) = \frac{1}{2k^2 t} \sin^2 kx, \quad \frac{1}{2k^2 t} \cos^2 kx
\]

(4.27a), (4.27b)

\[
\rho(t, x) = \sqrt{2|\lambda| k|t|} \sin \frac{2kx}{k^2 t}, \quad \sqrt{2|\lambda| k|t|} \cos \frac{2kx}{k^2 t}
\]

(4.28a), (4.28b)

Here \( k \) is an arbitrary, positive integration constant; the second constant gives an origin to \( x \), and has been suppressed in the above by setting it to zero. Note that the current of the second solution at \( \lambda > 0 \) has the kink shape.

\[
j = \mp \sqrt{2\lambda} \tanh kx
\]

(4.29)

This profile puts one in mind of soliton phenomena, and hints at the integrability of Eqs. (4.14), (4.15) in one dimension.

In terms of the linearized formalism of (4.21)–(4.24), the \( \lambda > 0 \) solutions (4.25a) and (4.26a) correspond to

\[
F(z) = -G(z) = \frac{z}{2k}(1 - \ln z)
\]

(4.30)

while (4.25b) and (4.26b) have

\[
F'(z) = G(-z) = \frac{z}{2k}(1 - \ln z)
\]

(4.31)

The \( \lambda < 0 \) solution in (4.27) and (4.28) are the analytic continuation of the above.

3. Transforming the \( d = 2 \) Solution

We now exhibit the form of the solutions when the field-dependent, coordinate transformation (3.11)-(3.13) are performed on (4.16)-(4.19). For simplicity we discuss only the \( d = 2 \) (membrane) case, and take \( t > 0 \). The new coordinates are determined by the old ones by (3.11) and (4.16).

\[
T = \frac{3}{4}t + \frac{1}{2}\omega \cdot r \pm \frac{1}{4}\sqrt{(t + 2\omega \cdot r)^2 - 2\omega^2 r^2}
\]

(4.32a)

\[
R = r + \frac{\omega}{2\omega^2}[-t - 2\omega \cdot r \pm \sqrt{(t + 2\omega \cdot r^2 - 2\omega^2 r^2}]\]

(4.32b)

\[
\frac{1}{J} = 1 + \frac{\omega \cdot R}{T} - \frac{\omega^2 R^2}{4T^2}
\]

(4.32c)

After \( \theta \) and \( \rho \) are transformed according to the rules (3.13), it is noticed that expressions are simplified by performing the Galileo boost \( r \to r - \omega t/\omega^2 \), according to the rules (3.4). (This precludes taking the limit \( \omega \to 0 \).) Also, time is rescaled according to (3.10), with \( t \to \sqrt{2}t \). Finally, we define \( \omega/\omega^2 = c \), which has dimension
of velocity, and then the transformed profiles provide two solutions, depending on the sign of the square root

\[
\theta_c(t, r) = \pm \sqrt{\frac{2(c \cdot r)^2 - c^2 r^2 - c^4 t^2}{r^2 + 2t(\sqrt{(c \cdot r)^2} - c^2 r^2 - c^4 t^2)}}.
\]

(4.33a)

\[
\rho_c(t, r) = \frac{\sqrt{2\lambda}}{c^2} \left[ \frac{2(c \cdot r)^2 - c^2 r^2 - c^4 t^2}{r^2 + 2t(\sqrt{(c \cdot r)^2} - c^2 r^2 - c^4 t^2)} \right]^{1/2}.
\]

(4.33b)

The velocity is

\[
v_c(t, r) = \pm \frac{2c(c \cdot r) - rc^2}{\sqrt{2(c \cdot r)^2 - c^2 r^2 - c^4 t^2}}
\]

(4.33c)

and the current reads

\[
j_c(t, r) = \pm \frac{\sqrt{2\lambda}}{[r^2 + 2t(\sqrt{(c \cdot r)^2} - c^2 r^2 - c^4 t^2)]^{1/2}}.
\]

(4.33d)

Note that \(c\) may be replaced by \(ic\) and \(\rho_c\) by \(-\rho_c\), to obtain another solution.

In the figures we exhibit the profiles of the interacting solutions. We plot the original and transformed densities, and the transformed currents \(j_c = \rho_c \nabla \theta_c\), in terms of the variables \(r/t\) \((t > 0)\). Without loss of generality, \(c\) is taken along the \(x\)-axis, and its magnitude is incorporated in the dimensionless ratio \(r/ct\). The original density possesses a singularity at the origin; in the transformed solutions the singularity is present only with the upper (negative) sign in the bracketed expression of (4.33b), where its denominator vanishes at \(r^2 = (\hat{c} \cdot r)^2 = 2c^2 t^2\). The transformed currents exhibit a similar singularity. In the physical region the argument of the square root must be positive, \(2(\hat{c} \cdot r)^2 - r^2 - c^2 t^2 \geq 0\). This requirement creates an envelope of validity for some of the profiles.
FIG. 1. The original density $\rho(t, \vec{r})/\sqrt{2\lambda}$.

FIG. 2. The transformed density $\rho_c(t, \vec{r})/\sqrt{2\lambda}$, with the upper sign.
FIG. 3. The transformed density $\rho_c(t, \vec{r})/\sqrt{2\lambda}$, with the lower sign. The envelope defining the physical region is at $x^2 - y^2 = c^2 t^2$.

FIG. 4. The transformed current $\vec{j}_c(t, \vec{r})/\sqrt{2\lambda}$, with the upper signs. The envelope defining the physical region is at $x^2 - y^2 = c^2 t^2$. 
FIG. 5. The transformed current $\vec{j}_c(t, \vec{r})/\sqrt{2\lambda}$, with the lower signs. The envelope defining the physical region is at $x^2 - y^2 = c^2t^2$.

REFERENCES

* This work is supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under contract #DE-FC02-94ER40818. MIT-CTP-2773; hep-th/9809123

[1] L.D. Faddeev and R. Jackiw, Phys. Rev. Lett. 60, 1692 (1988).
[2] C. Eckart, Phys. Rev. 54, 920 (1938); for recent work see A. Schakel, preprint, cond-mat/9607164 and N. Ogawa, preprint, hep-th/9801115.
[3] L. Landau and E. Lifschitz, Fluid Mechanics, 2nd ed. (Pergamon, Oxford UK, 1987).
[4] E. Madelung, Z. Phys. 40, 322 (1926); E. Merzbacher, Quantum Mechanics, 3rd ed. (Wiley, New York, 1998).
[5] A. Jevicki, Phys. Rev. D 57, 5955 (1998).
[6] J. Hoppe, MIT PhD Thesis (unpublished, 1982).
[7] M. Bordemann and J. Hoppe, Phys. Lett. B317, 315 (1993). These authors treat only the case of a membrane, $d = 2$, and their derivation is organized differently from ours.
[8] C. Duval and P. Horvathy (in preparation) have shown that the $0(2,1)$ “Schrödinger group”, that is, time translation, dilation and conformal redefinition, are symmetries of $L_0$ [For an account of this group see R. Jackiw, Phys. Today 25, 23 (January 1972); C. Hagen, Phys. Rev. D 5, 377 (1972); U. Niederer, Helv. Phys. Acta 45, 802 (1972), 46, 191 (1973), 47, 167 (1974), 51, 220 (1978). Note: the time dilation of the Schrödinger group does not coincide with the time dilation discussed below in connection with $V(\rho) = \lambda/\rho$ – the “d-brane” case; indeed the Schrödinger group is not an invariance of that interaction.]
[9] L. Susskind, Phys. Rev. 165, 1535 (1968).
[10] D. Bazeia and R. Jackiw, Ann. Phys. (NY) (in press) hep-th/9803165
[11] R. Jackiw and A.P. Polychronakos, in preparation.