ZETA FUNCTIONS OF GRAPHS WITH Z ACTIONS

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Abstract. Suppose Y is a regular covering of a graph X with covering transformation group \( \pi = \mathbb{Z} \). This paper gives an explicit formula for the \( L^2 \) zeta function of Y and computes examples. When \( \pi = \mathbb{Z} \), the \( L^2 \) zeta function is an algebraic function. As a consequence it extends to a meromorphic function on a Riemann surface. The meromorphic extension provides a setting to generalize known properties of zeta functions of regular graphs, such as the location of singularities and the functional equation.

1. Introduction

Given a finite graph, there is a zeta function which encodes some of the combinatorics of the graph. The zeta function was defined by Ihara and extended by Hashimoto and then Bass. See [7, 6] for a fine introduction to the subject taking a geometric approach.

There is an analogous zeta function for any infinite graph with cofinite action of a discrete group. Let \( Y = (V_Y, E_Y) \) be a locally finite (but typically infinite) graph and suppose the group \( \pi \) acts freely on \( Y \) with finite quotient graph \( X \). Let \( P \) denote the set of free homotopy classes of primitive closed paths in \( Y \). For \( \gamma \in P \), \( \ell(\gamma) \) is the length of the shortest representative of \( \gamma \). The group \( \pi_\gamma \) is the stabilizer of \( \gamma \) under the action of \( \pi \). The \( L^2 \) zeta function of \( Y \) is the infinite product

\[
Z_{Y}^{(2)}(u)^{-1} = \prod_{\gamma \in \pi \setminus P} \left( 1 - u^{\ell(\gamma)} \right)^{\left| \pi_\gamma \right|}.
\]

This definition was first given in [2] as a specialization from a more general setting, but beware that the notation \( Z \) in [2] refers to the reciprocal of the zeta function considered here and elsewhere in the literature. See [5] for a more direct treatment of the case considered here.

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For finite graphs, the fundamental theorem is the Ihara-Hashimoto-Bass rationality formula, which says that the zeta function is the reciprocal of a polynomial. The analogous theorem for infinite graphs requires techniques of von Neumann algebras. The infinite graph result is formally similar, and implies convergence of \( 1.1 \), but the \( L^2 \) zeta function is not typically a rational function.

Let \( \delta \) be the adjacency operator of \( Y \) acting on \( l^2(VY) \). For \( f \in l^2(VY) \) let \( Qf(v) = q(v)f(v) \) where \( q(v) + 1 \) is the degree of the vertex \( v \). Put \( \Delta_u = I - \delta u + Qu^2 \). Then from [2, Theorem 0.3],

\[
Z_Y^{(2)}(u)^{-1} = (1 - u^2)^{-\chi(X)} \operatorname{Det}_\pi \Delta_u
\]

(1.2)

where \( \operatorname{Det}_\pi \) is a von Neumann determinant defined in [2]. In particular, the product in \( 1.1 \) converges for small \( u \) (which was not a priori obvious).

In this paper, the only group considered is \( \pi = \mathbb{Z} \), so that \( X = Y/\mathbb{Z} \). Theorem 2.2 computes \( Z_Y^{(2)}(u) \) in this case. The main difficulties to overcome are the evaluation of a particular definite integral and careful bookkeeping with branches of multi-valued complex functions.

The formula for \( Z_Y^{(2)}(u) \) is algebraic, and Theorem 2.3 takes advantage of this to extend \( Z_Y^{(2)}(u) \) to a meromorphic function \( \tilde{Z} \) defined on a compact Riemann surface \( S \) (which depends on \( Y \)). From another viewpoint, \( Z_Y^{(2)}(u) \) is naturally a multi-valued meromorphic function defined on all of \( \mathbb{C} \).

The surface \( S \) covers the Riemann sphere \( \mathbb{C}P^1 \) with branch points, and the branch points play a similar role for infinite graphs as the poles do for zeta functions of finite graphs. Specifically, Theorem 3.2 gives conditions for \( \tilde{Z} \) of a \( q + 1 \) regular graph \( Y \) to have all its branch points over the set

\[
C = \{ u \in \mathbb{C} : |u| = q^{-1/2} \} \cup [-1, -\frac{1}{q}] \cup [\frac{1}{q}, 1].
\]

This is exactly the set where poles may occur for zeta functions of finite \( q + 1 \) regular graphs.

The extension to \( \tilde{Z} \) gives a meaningful context for functional equations relating \( u \leftrightarrow \frac{1}{qu} \), and Section 3.1 explores these.

Finally, Section 4 gives a number of computations for specific \( Y \).

This paper is intended as a model for how one might attack more general \( \pi \neq \mathbb{Z} \). It is shown in [3] that for a \( q \)-regular graph, the \( L^2 \) zeta function always extends...
holomorphically to the interior of the set $C$. In the most optimistic scenario, the $L^2$ zeta function is always algebraic and therefore extends past $C$ to a compact Riemann surface. More likely, one may need to allow noncompact surfaces with infinitely many sheets over $\mathbb{C}P^1$. In the worst scenario, the “branch points” could spread out continuously over $C$ and prevent any further extension of domain. In any event, the explicit computation that provides the key here is not likely to unlock the more general case.

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1.1. Group Von Neumann Algebras. For completeness, here is a quick overview of relevant material from von Neumann algebras. For $\pi$ a countable discrete group, the von Neumann algebra of $\pi$ is the algebra $\mathcal{N}(\pi)$ of bounded $\pi$-equivariant operators from $l^2(\pi)$ to $l^2(\pi)$.

The von Neumann trace of an element $f \in \mathcal{N}(\pi)$ is defined by

$$\text{Tr}_\pi f = \langle f(e), e \rangle$$

for $e \in \pi$ the unit element. The group ring $\mathbb{C}[\pi]$ is contained in $\mathcal{N}(\pi)$, acting on $l^2(\pi)$ by right multiplication. It is a dense subspace. The trace of an element of the group ring is simply the coefficient of the identity.

For $H = \bigoplus_{i=1}^n l^2(\pi)$ and a bounded $\pi$-equivariant operator $f : H \to H$, define

$$\text{Tr}_\pi f = \sum_{i=1}^n \text{Tr}_\pi f_{ii}.$$ 

The trace as defined is independent of the decomposition of $H$. The determinant $\text{Det}_\pi \Delta(Y, u)$ is defined via formal power series as $(\text{Exp} \circ \text{Tr}_\pi \circ \text{Log})\Delta(Y, u)$ and converges for small $u$.

**Example 1.1.** When $\pi = \mathbb{Z}$, Fourier transform identifies $l^2(\mathbb{Z})$ with $L^2(S^1)$. An element $\sum_{n=-\infty}^{\infty} c_n t^n \in \mathcal{N}(\pi)$ transforms to multiplication by $f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$, and

$$\text{Tr}_\pi f = \langle f \cdot 1, 1 \rangle = \int_{S^1} f(\theta) d\theta = c_0.$$
2. Graphs with $\mathbb{Z}$ actions

We assume $\pi = \mathbb{Z} = \langle t \rangle$, the free abelian group on one generator $t$. Suppose $X = Y/\mathbb{Z}$ has $v$ vertices. Choosing lifts of these vertices to $Y$, we identify $l^2(V Y) = \bigoplus_v l^2(\mathbb{Z})$ and the adjacency operator $\delta$ is then a $v \times v$ matrix with entries in the group ring $\mathbb{Z}[[t]]$. Since $\delta$ is self-adjoint, it satisfies $\delta(t) = \delta(t^{-1})^T$. Similarly, $\Delta_u(t) = \Delta_u(t^{-1})^T$ (but beware that $\Delta_u$ is not generally self-adjoint). Therefore, $\text{Det}_\pi \Delta_u \in \mathbb{C}[[t]]$ is symmetric in $t$ and $t^{-1}$, and we can write $\text{Det} \Delta_u = P_u(t + t^{-1})$ for some polynomial $P_u$. The coefficients of $P_u$ are integer polynomials in $u$.

We know in general that $\text{Det}_\pi \Delta_u$ is independent of the choice of lifts of vertices, but here it is very clear, since choosing a different lift will multiply a row by $t^k$ and the corresponding column by $t^{-k}$ (for some $k$). In particular, $P_u$ depends only on $Y$ and the $\mathbb{Z}$ action.

Now, under Fourier transform, $\bigoplus_v l^2(\mathbb{Z}) = \bigoplus_v L^2(S^1)$. Here $S^1 = \{ e^{i\theta} | \theta \in (-\pi, \pi] \}$ with measure normalized to have total measure 1. Under Fourier transform, multiplication by $t$ becomes multiplication by the function $e^{i\theta}$, and hence $\Delta_u$ is represented by a $v \times v$ matrix which will be denoted $M_u(\theta)$. To compute the zeta function,

$$\text{Det}_\pi \Delta_u = \exp \text{Tr}_\pi \log(\Delta_u) \quad (2.1)$$
$$= \exp \int_{S^1} \text{Tr} \log(M_u(\theta)) d\theta \quad (2.2)$$
$$= \exp \int_{S^1} \log \det(M_u(\theta)) d\theta \quad (2.3)$$
$$= \exp \int_{S^1} \log P_u(\cos(\theta)) d\theta. \quad (2.4)$$

2.1. The Line. To proceed further, we work out a crucial example. Let $V Y = \mathbb{Z}$, and connect $n$ to $n + 1$ with an edge (so $Y$ is a line). Then

$$\Delta_u = 1 - (t + t^{-1})u + u^2,$$

$$M_u(\theta) = 1 - 2 \cos(\theta) u + u^2,$$

and

$$P_u(x) = 1 - 2ux + u^2.$$
Notice that for $|u| < 1$ and for all $\theta$, $M_u(\theta) \notin (-\infty, 0]$. In what follows, log will be the principal branch of the logarithm.

Now restrict to $|u| < 1$. Because $Y$ has no loops, the $L^2$ zeta function for $Y$ is identically 1. Therefore, by [2, Theorem 0.3]

$$1 = (1 - u^2)^0 \det_z \Delta_u$$

$$= \exp \int_{S^1} \log(1 - 2u \cos(\theta) + u^2) d\theta$$  \hspace{1cm} (2.5)

$$= \exp \int_{S^1} \log(2u) + \log\left(\frac{u + u^{-1}}{2} - \cos(\theta)\right) d\theta$$  \hspace{1cm} (2.6)

$$= 2u \exp \int_{S^1} \log(r - \cos(\theta)) d\theta.$$  \hspace{1cm} (2.7)

Here, we have assumed $u \neq 0$ and put $r = (u + u^{-1})/2$. Generally, some care must be taken when writing $\log(xy) = \log(x) + \log(y)$. If $u \in (-1, 0)$, then $r - \cos(\theta) < 0$ and the identity is off by $2\pi i$. However, the $2\pi i$ is washed out by the exp in front.

For other values of $u$ there is no problem, because the imaginary parts of $u$ and $r$ have opposite sign.

Notice that $r = \cosh(-\log(u))$, so that $u = e^{-\text{arccosh}(r)}$. Here, arccosh has a branch cut discontinuity on $(-\infty, 1]$ and range $\{a + bi | a > 0, b \in (-\pi, \pi]\} \cup [0, \pi]i$.

Proposition 2.1. Suppose $r \in \mathbb{C}$. Then

$$\int_{S^1} \log(r - \cos(\theta)) d\theta = \text{arccosh}(r) - \log(2).$$  \hspace{1cm} (2.8)

Proof. The discussion above proves that for $r \in \mathbb{C} - [-1, 1]$,

$$\exp \int_{S^1} \log(r - \cos(\theta)) d\theta = \frac{1}{2}e^{\text{arccosh}(r)}.$$  \hspace{1cm} (2.9)

Taking the log of both sides,

$$\int_{S^1} \log(r - \cos(\theta)) d\theta = \text{arccosh}(r) - \log(2)$$  \hspace{1cm} (2.9)

In principle, this is only true up to $2\pi ik$ for some $k \in \mathbb{Z}$. However, $k$ must be zero since both log and arccosh have imaginary part in the range $(-\pi, \pi]$.

Now we extend (2.9) to all of $\mathbb{C}$. We check the imaginary part explicitly. For $r \in [-1, 1]$, put $r = \cos(\phi)$, $\phi \in [0, \pi]$. Then $\Im(\text{arccosh}(r) - \log 2) = \phi$. On the
other hand, \( \arg(r - \cos(\theta)) = \pi \) when \( \cos(\theta) > r \) and is 0 otherwise. Therefore,
\[
\int_{S^1} \Im(\log(r - \cos(\theta))) \, d\theta = \int_{S^1} \arg(r - \cos(\theta)) \, d\theta
\]
\[
= \pi \cdot m \{ \theta \mid \cos(\theta) > \cos(\phi) \}
\]
\[
= \pi \cdot \frac{2\phi}{2\pi} = \phi. \quad (2.12)
\]

Next, consider the real part of (2.9). The real part of the left hand side is
\[
\int_{S^1} \log|r - \cos(\theta)| \, d\theta.
\]
It is not hard to see that this integral is finite even for \( r \in [-1, 1] \). On the other hand, \( \Re(\text{arccosh}(r) - \log 2) \) is a continuous function on all of \( \mathbb{C} \) (and equals \( -\log 2 \) on \( [-1, 1] \)). Thus the two sides are defined on all of \( \mathbb{C} \), equal on \( \mathbb{C} - [-1, 1] \), and the right side is continuous.

Now, for \( r \in [-1, 1] \), \( n = 1, 2, \ldots \) put
\[
f_n(\theta) = \log|r - \cos(\theta) + i/n|.
\]
The \( f_n \) are a decreasing sequence of functions, bounded above (by \( \sqrt{5} \)), and converging a.e. to \( \log|r - \cos(\theta)| \). By the Monotone Convergence Theorem,
\[
-\log 2 = \lim_{n \to \infty} \Re(\text{arccosh}(r + i/n) - \log 2)
\]
\[
= \lim_{n \to \infty} \int_{S^1} f_n(\theta) \, d\theta
\]
\[
= \int_{S^1} \log|r - \cos(\theta)| \, d\theta, \quad (2.15)
\]
\[\Box\]

Remark 1. Computing the integral in (2.8) is a good, difficult calculus exercise for \( r > 1 \). I know of no elementary way to compute it in general.

Remark 2. The inverse hyperbolic cosine function satisfies
\[
\text{arccosh}(r) = \log \left( r + \sqrt{r + 1}\sqrt{r - 1} \right)
\]
where the principal branches of \( \text{arccosh} \), \( \log \), and \( \sqrt{z} \) are used. In particular, \( \Re(\text{arccosh}(r)) = \log |r + \sqrt{r + 1}\sqrt{r - 1}| \) is a continuous function. Taking the real part of both sides of (2.9) gives the integral
\[
\int_{S^1} \log|r - \cos(\theta)| \, d\theta = \log \frac{1}{2} \left| r + \sqrt{r + 1}\sqrt{r - 1} \right|. \quad (2.16)
\]
for all $r \in \mathbb{C}$.

2.2. The explicit formula.

**Theorem 2.2.** Let $Y$ be a regular $\mathbb{Z}$ covering of a finite graph $X$. Let $P_u(x)$ be the degree $n$ polynomial so that

$$\det \Delta_u = P_u \left( \frac{t + t^{-1}}{2} \right).$$

There is $R > 0$ so that for all $0 < |u| < R$,

$$Z_Y^{(2)}(u) = (1 - u^2)^{-\chi(X)} \frac{\alpha(u)}{2^n} \prod_i (r_i + \sqrt{r_i + 1}) \sqrt{r_i - 1}). (2.17)$$

Here $P_u(x) = \alpha(u) \prod_{i=1}^n (r_i(u) - x)$, and $r_i(u)$ are the roots of $P_u$. The square roots are principal, in the sense that $\sqrt{z} = \exp(\frac{1}{2} \log(z))$.

**Proof.** The polynomial $(-1)^n \alpha(u)$ is the coefficient of the top degree term $x^n$ of $P_u$. Since $P_0 = 1$, 0 is a root of $\alpha$. There is $R_1 > 0$ with $\alpha(u) \neq 0$ on $0 < |u| < R_1$, and so one can write $P_u(x) = \alpha(u) \prod_{i=1}^n (r_i(u) - x)$.

From (1.2), one need only compute $\text{Det}_\pi \Delta_u$. There is a subtle point involving the log of a product, but the heart of the argument is the computation below, which begins with (2.4), and uses Proposition 2.1:

$$\text{Det}_\pi \Delta_u = \exp \int_{S^1} \log P_u(\cos(\theta)) d\theta \quad \text{(2.18)}$$

$$= \exp \int_{S^1} \log \alpha(u) \prod_{i=1}^n (r_i(u) - \cos(\theta)) d\theta \quad \text{(2.19)}$$

$$= \exp \left( \log \alpha(u) + \sum_{i=1}^n \int_{S^1} \log(r_i(u) - \cos(\theta)) d\theta \right) \quad \text{(2.20)}$$

$$= \exp \left( \log \alpha(u) + \sum_{i=1}^n (\arccosh(r_i) - \log(2)) \right) \quad \text{(2.21)}$$

$$= \frac{\alpha(u)}{2^n} \prod_i \exp(\arccosh(r_i)) \quad \text{(2.22)}$$

$$= \frac{\alpha(u)}{2^n} \prod_i (r_i + \sqrt{r_i + 1}) \sqrt{r_i - 1}). \quad \text{(2.23)}$$

It remains to justify the transition from (2.19) to (2.20).

Write

$$\log P_u(x) = \log \alpha(u) + \sum_{i=1}^n \log(r_i(u) - x) + 2\pi ik(u, x). \quad \text{(2.24)}$$
The function $k(u, x)$ is always an integer. We will show that $k(u, x) = k(u)$ is independent of $x \in [-1, 1]$ and therefore pulls through the integral in (2.19) to be eaten by the exp.

Since $\Delta_u = I - \delta u + Q u^2$, we can write $P_u(x) = 1 + u T_u(x)$ for some polynomial $T$. Then there is $R_2 > 0$ so that for $|u| < R_2$ and $x \in [-1, 1]$ we have $\Re(P_u(x)) > 0$. Therefore, $\log(P_u(x))$ is a continuous function of $x \in [-1, 1]$.

In addition, for $0 < |u| < R_2$, we see that $P_u(x)$ has no roots on $[-1, 1]$, i.e. $r_i(u) \notin [-1, 1]$. Therefore $\log(r_i(u) - x)$ is a continuous function of $x \in [-1, 1]$ (since we’re using the principal branch of the logarithm).

We have shown that all other terms in (2.24) are continuous functions of $x$, and therefore $k(u, x)$ is a continuous function of $x$ on $[-1, 1]$, hence constant in $x$.

Setting $R = \min\{R_1, R_2\}$ completes the proof. □

2.3. The meromorphic extension. From Theorem 2.2, it is apparent that $Z_Y^{(2)}(u)$ is an algebraic function of $u$. In this section, we make this more explicit and then explore the consequences.

Let $s_i = \sqrt{r_i + 1} \sqrt{r_i - 1}$, and for $I = (\iota_1, \ldots, \iota_n) \in \{\pm 1\}^n = \mathbb{Z}_2^n$, put

$$W_I = \prod_{i=1}^n r_i + \iota_i s_i.$$ 

Note that $(r_i + s_i)(r_i - s_i) = 1$ so that $W_I^{-1} = W_{-I}$. Theorem 2.2 then says that

$$Z_Y^{(2)}(u) = (1 - u^2)^{\chi(\lambda)} \frac{2^n}{\alpha(u)} W_{-1, -1, \ldots, -1}. \quad (2.25)$$

Let

$$\Omega(T) = \prod_{I \in \mathbb{Z}_2^n} (T - W_I). \quad (2.26)$$

Then $\Omega$ is a polynomial in $T$ of degree $2^n$. It is invariant under the transformation $s_i \rightarrow -s_i$, hence it is even degree in each $s_i$. We can replace $s_i^2$ with $r_i^2 - 1$ so that $\Omega$ is a degree $n$ polynomial in $r_i$, symmetric in the $r_i$. This means that $\Omega$ is in fact a polynomial in the elementary symmetric functions $\sigma_1, \ldots, \sigma_n$ of the $r_i$, for example:

$$\Omega(T) = \begin{cases} T^2 - 2 T \sigma_1 + 1 & \text{for } n = 1, \\ T^4 - 4 T^3 \sigma_2 + T^2 (-2 + 4 \sigma_1^2 - 8 \sigma_2) - 4 T \sigma_2 + 1 & \text{for } n = 2, \end{cases} \quad (2.27)$$
and when \( n = 3 \),
\[
\Omega(T) = T^8 - 8T^7\sigma_3 + T^6 \left(4 - 8\sigma_1^2 + 16\sigma_2 + 16\sigma_2^2 - 32\sigma_1\sigma_3\right)
\]
\[
- T^5 \left(-40\sigma_3 + 32\sigma_1^2\sigma_3 - 64\sigma_2\sigma_3\right)
\]
\[
+ T^4 \left(6 - 16\sigma_1^2 + 16\sigma_1^4 + 32\sigma_2 - 64\sigma_1^2\sigma_2 + 32\sigma_2^2 + 64\sigma_1\sigma_3 + 64\sigma_3^2\right)
\]
\[
- \cdots - 8T\sigma_3 + 1
\]
using the symmetry of coefficients to finish (roots of \( \Omega \) occur in reciprocal pairs).

Since the \( r_i \) are the roots of \( P_u \),
\[
\sigma_i = (-1)^{n-i} \left(\text{the } n-i\text{th coefficient of } \frac{P_u}{\alpha(u)}\right).
\]
Thus \( \sigma_i \) is a rational function of \( u \), and so \( \Omega \in \mathbb{C}(u)[T] \).

We have shown that \( W_I \) and therefore \( Z_Y^{(2)}(u) \) are algebraic functions of \( u \) of degree less than or equal to \( 2^n \).

**Theorem 2.3.** Let \( Y \) be a regular \( Z = \pi \) covering of a finite graph \( X \). Then \( Z_Y^{(2)}(u) \) extends uniquely to a meromorphic function on a Riemann surface.

More precisely, there exists a compact Riemann surface \( S \), a (branched) covering map \( \Pi : S \to \mathbb{C}P^1 \), and a meromorphic function \( \tilde{Z} \) on \( S \). There is a point \( z_0 \in \Pi^{-1}(0) \) and a neighborhood \( U \) of \( z_0 \) on which \( \Pi \) is biholomorphic such that \( \tilde{Z}(z) = Z_Y^{(2)}(\Pi(z)) \) for all \( z \in U \).

The triple \((S, \Pi, \tilde{Z})\) is unique in the following sense: If \((S', \Pi', \tilde{Z}')\) has the corresponding properties, then there exists exactly one fiber preserving biholomorphic mapping \( \tau : S \to S' \) such that \( \tilde{Z} = \tilde{Z}' \circ \tau \).

**Remark 3.** The number of sheets of \( \Pi \) is less than or equal to \( 2^n \), where \( n \) is the degree of the polynomial \( P_u \) defined earlier.

**Proof.** Define \( W_I \) and \( \Omega \) as above. The difficult work is finished, as we showed already that \( W_I \) is algebraic. Since \( W_{-1,-1,...,-1} \) is holomorphic in a neighborhood of \( 0 \) and \( \Omega(W_{-1,-1,...,-1}) = 0 \), there is a unique irreducible factor \( \Phi \in \mathbb{C}(u)[T] \) with
\[
\Phi(W_{-1,-1,...,-1}) = 0.
\]
in a neighborhood of \( 0 \).
The algebraic function defined by $\Phi(T)$ consists of $S$ and $\Pi$ as above, plus a meromorphic function $f$ on $S$ such that $(\Pi^*\Phi)(f) = 0$. It is unique in the sense of fiber preserving biholomorphic mappings as above (see [4, I.8] for details).

Since $W_{-1, -1, ..., -1}$ is holomorphic in a neighborhood of 0, there is a point $z_0 \in \Pi^{-1}(0)$ and a neighborhood $U$ of $z_0$ on which $\Pi$ is biholomorphic with $f(z) = W_{-1, -1, ..., -1}(\Pi(z))$ for $z \in U$.

For $z \in S$, let $u = \Pi(z)$ and put

$$\tilde{Z}(z) = (1 - u^2)^{\chi(X)} \frac{2^n}{\alpha(u)} f(z).$$

(2.30)

to complete the proof. □

3. Regular graphs

In this section, assume that $X$ is $q + 1$ regular.

3.1. Functional Equations. The zeta function for finite regular graphs satisfies a number of functional equations under the transformation

$$\tau : u \rightarrow \frac{1}{qu},$$

(see [7]). The situation for $L^2$ zeta functions is somewhat less simple.

First notice that

$$\Delta_{1/\text{qu}} = I - \delta \frac{1}{\text{qu}} + q\frac{1}{(\text{qu})^2} = \frac{1}{\text{qu}^2} (I - \delta u + qu^2) = \frac{1}{\text{qu}^2} \Delta_u.$$ 

Then the polynomial $P_{1/\text{qu}}(x)$ has the same roots $r_1, ..., r_n$ as $P_u(x)$. Since $\Omega$ and $W_{-1, -1, ..., -1}$ are symmetric functions of the $r$'s, they are invariant under $\tau$.

Suppose $\Omega$ is irreducible, so that the $L^2$ zeta function is defined on the Riemann surface $S$ for $\Omega$ by (2.30). Then the transformation $u \rightarrow \frac{1}{\text{qu}}$ induces a biholomorphic involution $\tilde{\tau} : S \rightarrow S$ so that $f \circ \tilde{\tau} = f$. It is then easy to find functional equations for $\tilde{Z}$. For example:

**Proposition 3.1.** Suppose $X$ is $q + 1$-regular and $\Omega$ is irreducible. For $z \in S$ put $u = \Pi(z)$. Then

$$\left( \tilde{Z} \circ \tilde{\tau} \right)(z) = q^{2e - v} u^{2e} \left( \frac{1 - u^2}{qu^2 - 1} \right)^{-\chi} \tilde{Z}(z).$$

(3.1)

Here $v$ and $e$ are the number of vertices and edges of $X$, and $\chi = \chi(X) = v - e$.

(Compare [7, Cor 3.10])
Proof. This is a straightforward calculation using \( f \circ \tilde{\tau} = f \), equation (2.30), and

\[
\alpha \left( \frac{1}{qu} \right) = \left( \frac{1}{qu^2} \right)^v \alpha(u).
\]

\( \square \)

If \( \Omega \) is reducible, one gets a collection of disjoint Riemann surfaces \( S_1, \ldots, S_k \) and the map \( \tilde{\tau} \) may permute them. We are interested in \( \tilde{Z} \) on a particular choice \( S \), and so it will not satisfy a functional equation in any traditional sense. The line (example 4.1) is a good example of this.

3.2. Location of branch points. The zeta function for a finite, \( q + 1 \) regular graph has all of its poles in the set

\[
C = \{ u \in \mathbb{C} : |u| = q^{-1/2} \} \cup [-1, -\frac{1}{q}] \cup [\frac{1}{q}, 1].
\]

For the \( L^2 \) zeta function, we can make a slightly weaker statement for branch points.

**Theorem 3.2.** Let \( Y \) be a regular \( \mathbb{Z} = \pi \) covering of a finite graph \( X \). Suppose that \( X \) is \( q + 1 \) regular. Let \( \Omega \) be the polynomial defined in (2.26), and assume \( \Omega \) is irreducible. If the field extension \( \mathbb{C}(u)[T]/(\Omega(T)) : \mathbb{C}(u) \) is Galois, then the covering \( \Pi \) from Theorem 2.3 has all of its branch points over \( C \).

Proof. Let \( D_0 \) and \( D_\infty \) be the connected components of \( \mathbb{C} - C \). From [3], the \( L^2 \) zeta function \( Z_Y^{(2)}(u) \) extends holomorphically to \( D_0 \), so the neighborhood \( U \) from Theorem 2.3 must also extend to cover \( D_0 \) with no branch points. The field extension \( \mathbb{C}(u)[T]/(\Phi(T)) : \mathbb{C}(u) \) is Galois if and only if the deck transformations of \( S \) over \( \mathbb{C}P^1 \) act transitively on the sheets of \( S \) ([11, pg. 57]). Then \( \Pi^{-1}(D_0) \) is a union of copies of \( U \) and has no branch points. The involution \( \tilde{\tau} \) from the functional equation biholomorphically interchanges \( \Pi^{-1}D_0 \) with \( \Pi^{-1}D_\infty \), so that \( \Pi \) can only be branched on \( C \).

\( \square \)

In example 4.3 we will see a graph for which the \( L^2 \) zeta function is branched over 0 and the deck transformations of \( S \) are not transitive.

The assumption that \( \Omega \) is irreducible is less well motivated. As in example 4.4, the zeta function for a graph with reducible \( \Omega \) will still satisfy a functional equation if \( \tilde{\tau} \) preserves \( S \).
The following argument gives hope for a close relationship between branch points of $\tilde{Z}$ for $Y$ and zeros of $Z(X)$. To compute the zeta function $Z(X)$ of the quotient graph $X = Y/\mathbb{Z}$, one takes the determinant of $\Delta_X(u) = I - \delta_X u + Qu^2$, where $\delta_X$ is the adjacency operator on $X$. Poles of $Z(X)$ occur when $\det \Delta_X(u) = 0$. But $\delta_X$ is equal to $\delta$ on $Y$ under $t \to 1$, and so poles of $Z(X)$ occur when $P_u(1) = 0$, or equivalently when some root $r_i(u) = 1$.

If $r_i(u) = 1$ then the terms $r_i \pm \sqrt{r_i + 1} \sqrt{r_i - 1}$ coincide. In other words, two roots of $\Omega$ coincide at any $u$ where $Z(X)$ has a pole – a necessary condition for $S$ to be branched over $u$.

Frequently, branch points of $\tilde{Z}$ do coincide with poles of $Z(X)$. However, examples in the next section show that both possible implications are false in general.

4. Examples

Example 4.1 (The Line). Let $Y$ be the line, as in Section 2.1. We saw earlier that

$$P_u(x) = 1 + u^2 - 2ux.$$ 

Then $\alpha(u) = 2u$ and $r(u) = \frac{1+u^2}{2u}$. From (2.27), we have

$$\Omega(T) = T^2 - T \frac{1+u^2}{u} + 1 = \frac{(T-u)(Tu-1)}{u}.$$ 

Here $\Omega$ is reducible. Some careful computation shows that

$$W_{-1}(u) = \frac{1+u^2}{2u} - \sqrt{\frac{1+u^2}{2u} + 1} \sqrt{\frac{1+u^2}{2u} - 1} = \begin{cases} u & \text{if } |u| < 1 \\ \frac{1}{u} & \text{if } |u| > 1 \end{cases} \quad (4.1)$$

so $\Phi(T) = T - u$, the Riemann surface $S$ is $\mathbb{C}P^1$, $f(u) = u$, and the zeta function is $2f/\alpha = 1$.

Notice that the transformation $\tau : u \to \frac{1}{qu}$ (here $q = 1$) interchanges the two irreducible surfaces. On the other surface, the analog of $\tilde{Z}$ is $u^2$, and in fact the functional equation (3.1) becomes

$$u^2 = 1^{2+1}u^{2+1} \cdot 1 \cdot 1.$$ 

Example 4.2 (Some degree 1 graphs). Let $Y$ be the first graph shown in Table 4 (all these graphs take the obvious $\mathbb{Z}$ action). $Y$ is 4-regular, so $q = 3$. It’s quotient graph $X$ is a vertex with two loops, and $\chi(X) = -1$. 

The adjacency matrix for $Y$ is the $1 \times 1$ matrix $(t^{-1} + 2 + t)$. Then $P(x) = -2ux + 1 - 2u + 3u^2$ which has the one root shown in the table. From \cite{227},
\[
\Omega(T) = T^2 - \left(1 - \frac{2u + 3u^2}{u}\right)T + 1
\]
which is irreducible.

The associated Riemann surface $S$ is a two sheeted branched cover of $\mathbb{C}P^1$. Possible branch points occur when the discriminant of $\Omega$ vanishes, which happens in this case at
\[
u = 1, u = \frac{1}{3}, u = \frac{i}{\sqrt{3}}, u = -\frac{i}{\sqrt{3}}.
\]
Here, all four are in fact branch points of multiplicity 2. The pattern of branch points is shown the table, and the set $C$ is also indicated.

The Riemann-Hurwitz formula gives the genus of a branched covering of $\mathbb{C}P^1$ as
\[
g = b/2 - d + 1 \quad \text{with} \quad d \text{ the number of sheets and } b \text{ the total branching order. For this graph the genus is 1 and } S \text{ is a torus.}
\]

Other lines of Table give the results of similar computations for different $Y$ with $n = 1$. In all cases, $\Omega(T) = T^2 - 2rT + 1$ is irreducible, $S$ is a two sheeted branch cover, and all branch points are multiplicity 2.

Graph #3 is an example in which poles of the zeta function for the quotient graph do not correspond to branch points of $S$. In this graph #3 of the table, $r\left(-\frac{1}{4} \pm \frac{1}{2}\sqrt{7}\right) = 1$, but these are not branch points of the $L^2$ zeta function.

Graphs #2, 4, and 5 are bipartite and have vertical bilateral symmetry. Graphs #4 and 5 have different zeta functions because they have different $\alpha$ and different $\chi$.

Graph #6 is non-regular. It’s branch points are shown with circles of radius $1/\sqrt{2}$ and $1/\sqrt{3}$ for scale.

**Example 4.3** (A regular graph with branch point off of $C$). Consider the 4-regular graph $Y$ with vertices $\mathbb{Z} \cup \mathbb{Z}$ as shown in Figure 4.3. The adjacency matrix of $Y$ is
\[
\delta = \begin{pmatrix} t + t^{-1} & 1 + t^{-1} \\ 1 + t & t + t^{-1} \end{pmatrix}.
\]
$P_u(x)$ is degree 2, and the two roots of $P_u$ are
\[
\nu = \frac{1}{4u} \left(2 + u + 6u^2 \pm \sqrt{u(4 + 9u + 12u^2)}\right).
\]
From \(\mathcal{CP}^1\), \(\Omega\) is the irreducible degree 4 polynomial

\[
\Omega(T) = T^4 - \frac{1 + 4u^2 + 9u^4}{u^2}T^3 + \frac{2 + 4u + 15u^2 + 12u^3 + 18u^4}{u^2}T^2
- \frac{1 + 4u^2 + 9u^4}{u^2}T + 1.
\]

The Riemann surface \(S\) has four sheets covering \(\mathcal{CP}^1\). Evaluating the discriminant of \(\Omega\), one has 10 points \(u\) where \(Z_Y^{(2)}\) has duplicate values. Checking the local behavior near those 10 points and additionally near \(u = 0, u = \infty\), one finds that \(S\) is unbranched at four of them. At

\[
u \in \left\{ 1, \frac{1}{3}, \pm \frac{i}{\sqrt{3}} \right\},
\]
sheets of $S$ come together in two pairs of multiplicity two branch points. At
\[ u = \left\{ 0, -\frac{9}{24} \pm \frac{i}{24} \sqrt{111}, \infty \right\}, \]
one pair of sheets come together in a multiplicity two branch point and the other
two sheets are unbranched. The pattern of branchpoints is shown in Figure 1.3 and
the genus of $S$ is 3.

The most interesting thing here is that the zeta function is branched over 0 and
$\infty$, which are not in the set $C$. Of course, the sheet corresponding to the original
unextended definition of $Z_Y^{(2)}$ is not one of the two sheets that come together at
$u = 0$. The group of deck transformations of $S$ is not transitive, and the field
extension $\mathbb{C}(u)[T]/(\Omega(T)) : \mathbb{C}(u)$ is not Galois.

**Example 4.4 (A nontrivial reducible graph).** Let $Y$ have vertices $\mathbb{Z} \cup \mathbb{Z} \cup \mathbb{Z}$
connected as shown in Figure 4.4. It is the graph Cartesian product of the line with a
triangle.

Here $\Omega$ factors into a fourth degree term and the square of a quadratic. The
factor $\Phi$ corresponding to $Z_Y^{(2)}$ is the fourth degree term, so $S$ is four sheeted.
There are twelve branch points:
\[ u \in \left\{ \frac{1}{3}, 1, \frac{\pm i}{\sqrt{3}}, \frac{-3 \pm i \sqrt{3}}{6}, \frac{\pm 1 \pm i \sqrt{11}}{6}, \frac{1}{4} \pm \frac{\sqrt{7}}{4} \pm \frac{i}{2} \sqrt{\frac{1}{2} + \frac{\sqrt{7}}{2}} \right\}, \]
shown in Figure 4.4. At each $u$, sheets come together in two pairs of multiplicity
two branch points, so the genus of $S$ is 9.
Even though $\Omega$ is reducible, all branch points still lie on the set $C$. Here $Z_{Y}^{(2)}$ must still satisfy the functional equation, because the involution $\tilde{\tau}$ preserves $S$ - the other two irreducible factors of $\Omega$ are degree 2 while $\Phi$ is degree 4.

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