Fixed points of Lyapunov integral operators and Gibbs measures

F. H. Haydarov

Abstract In this paper we shall consider the connections between Lyapunov integral operators and Gibbs measures for models with four competing interactions and uncountable (i.e. \([0, 1]\)) set of spin values on a Cayley tree. We prove the existence of fixed points of Lyapunov integral operators and give a condition of uniqueness of a fixed point.

Keywords Cayley tree · Gibbs measures · Lyapunov integral operator · Fixed point

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1 Preliminaries

A Cayley tree \(\Gamma^k = (V, L)\) of order \(k \in \mathbb{N}\) is an infinite homogeneous tree, i.e., a graph without cycles, with exactly \(k + 1\) edges incident to each vertices. Here \(V\) is the set of vertices and \(L\) that of edges (arcs). Two vertices \(x\) and \(y\) are called nearest neighbors if there exists an edge \(l \in L\) connecting them. We will use the notation \(l = \langle x, y \rangle\). The distance \(d(x, y), x, y \in V\) on the Cayley tree is defined by the formula

\[
d(x, y) = \min \{d| x = x_0, x_1, \ldots, x_{d-1}, x_d = y \in V \text{ such that the pairs } \langle x_0, x_1 \rangle, \ldots, \langle x_{d-1}, x_d \rangle \text{ are neighboring vertices} \}.\]
Let $x^0 \in V$ be a fixed and we set

$$W_n = \{ x \in V \mid d(x, x^0) = n \}, \quad V_n = \{ x \in V \mid d(x, x^0) \leq n \},$$

$$L_n = \{ l = (x, y) \in L \mid x, y \in V_n \}.$$ 

The set of the direct successors of $x$ is denoted by $S(x)$, i.e.

$$S(x) = \{ y \in W_{n+1} \mid d(x, y) = 1 \}, \quad x \in W_n.$$ 

We observe that for any vertex $x \neq x^0$, $x$ has $k$ direct successors and $x^0$ has $k + 1$. The vertices $x$ and $y$ are called second neighbor which is denoted by $\langle x, y \rangle$, if there exist a vertex $z \in V$ such that $x, z$ and $y, z$ are nearest neighbors. We will consider only second neighbors $\langle x, y \rangle$, for which there exist $n$ such that $x, y \in W_n$. Three vertices $x, y$ and $z$ are called a triple of neighbors and they are denoted by $\langle x, y, z \rangle$, if $\langle x, y \rangle$, $\langle y, z \rangle$ are nearest neighbors and $x, z \in W_n$, $y \in W_{n-1}$, for some $n \in \mathbb{N}$.

Now we consider models with four competing interactions where the spin takes values in the set $[0, 1]$. For some set $A \subset V$ an arbitrary function $\sigma_A : A \to [0, 1]$ is called a configuration and the set of all configurations on $A$ we denote by $\Omega_A = [0, 1]^A$. Let $\sigma(\cdot)$ belong to $\Omega_V = \Omega$ and $\xi_i : (t, u, v) \in [0, 1]^3 \to \xi_i(t, u, v) \in R$, $\xi_i : (u, v) \in [0, 1]^2 \to \xi_i(u, v) \in R$, $i \in \{2, 3\}$ are given bounded, measurable functions. Then we consider the model with four competing interactions on the Cayley tree which is defined by following Hamiltonian

$$H(\sigma) = -J_3 \sum_{\langle x, y, z \rangle} \xi_1(\sigma(x), \sigma(y), \sigma(z)) - J \sum_{\langle x, y \rangle} \xi_2(\sigma(x), \sigma(y)) - J_1 \sum_{\langle x, y \rangle} \xi_3(\sigma(x), \sigma(y)) - \alpha \sum_{x \in V} \sigma(x), \quad (1.1)$$

where the sum in the first term ranges all triples of neighbors, the second sum ranges all second neighbors, the third sum ranges all nearest neighbors and $J, J_1, J_3, \alpha \in R \setminus \{0\}$. Let $\xi : (t, x) \in [0, 1] \times V \setminus \{x^0\} \to h_{t, x} \in R$ and $|h_{t, x}| < C$ where $x^0$ is a root of Cayley tree and $C$ is a constant which does not depend on $t$. For some $n \in \mathbb{N}$, $\sigma_n : x \in V_n \mapsto \sigma(x)$ and $Z_n$ is the corresponding partition function we consider the probability distribution $\mu^{(n)}$ on $\Omega^{(n)}_V$ defined by

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp \left( -\beta H(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x), x} \right), \quad (1.2)$$

$$Z_n = \int_{\Omega^{(n)}_V} \ldots \int_{\Omega^{(n)}_{V_{n-1}}} \exp \left( -\beta H(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x), x} \right) \lambda^{(p)}_{V_{n-1}}(d\sigma_n), \quad (1.3)$$
where for a set $A \subset V$ we denoted
\[
\Omega_A \times \Omega_A \times \ldots \times \Omega_A = \Omega_A^{(p)}, \quad \lambda_A \times \lambda_A \times \ldots \times \lambda_A = \lambda_A^{(p)}, \quad n, p \in \mathbb{N},
\]

Let $\sigma_{n-1} \in \Omega_{V_{n-1}}$ and $\sigma_{n-1} \lor \omega_n \in \Omega_{V_n}$ is the concatenation of $\sigma_{n-1}$ and $\omega_n$. For $n \in \mathbb{N}$ we say that the probability distributions $\mu^{(n)}$ are compatible if $\mu^{(n)}$ satisfies the following condition:
\[
\int \int_{\Omega_{W_n} \times \Omega_{W_n}} \mu^{(n)}(\sigma_{n-1} \lor \omega_n)(\lambda_{W_n} \times \lambda_{W_n})(d\omega_n) = \mu^{(n-1)}(\sigma_{n-1}). \tag{1.4}
\]

By Kolmogorov’s extension theorem there exists a unique measure $\mu$ on $\Omega_V$ such that, for any $n$ and $\sigma_n \in \Omega_{V_n}$, $\mu \left( \left\{ \sigma | V_n = \sigma_n \right\} \right) = \mu^{(n)}(\sigma_n)$. The measure $\mu$ is called splitting Gibbs measure corresponding to Hamiltonian (1.1) and function $x \mapsto h_x = \{h_{x,t} \}$, $x \neq x^0$ (see [1,2,5,7]).

Denote
\[
K(t, u, v) = \exp \{ J_3 \beta \xi_1(t, u, v) + J_2 \beta \xi_2(u, v) + J_1 \beta (\xi_3(t, u) + \xi_3(t, v)) \\
+ \alpha \beta (u + v) \}, \tag{1.5}
\]
and
\[
f(t, x) = \exp(h_{t,x} - h_{0,x}), \quad (t, u, v) \in [0, 1]^3, \quad x \in V \setminus \{x^0\}.
\]

The following statement describes conditions on $h_x$ guaranteeing compatibility of the corresponding distributions $\mu^{(n)}(\sigma_n)$.

**Proposition 1.1** [6] Let $k = 2$. The measure $\mu^{(n)}(\sigma_n)$, $n = 1, 2, \ldots$ satisfies the consistency condition (1.4) iff for any $x \in V \setminus \{x^0\}$ the following equation holds:
\[
f(t, x) = \frac{\int_0^1 \int_0^1 K(t, u, v)f(u, y)f(v, z)dudv}{\int_0^1 \int_0^1 K(0, u, v)f(u, y)f(v, z)dudv}, \tag{1.6}
\]

where $S(x) = \{y, z\}$.

**2 Existence of a fixed point of the operator $L$**

Now we prove that there exist at least one fixed point of Lyapunov integral equation, namely there is a splitting Gibbs measure corresponding to Hamiltonian (1.1).

**Proposition 2.1** Let $k = 2$, $J_3 = J = \alpha = 0$ and $J_1 \neq 0$. Then (1.6) is equivalent to
\[
f(t, x) = \prod_{y \in S(x)} \frac{\int_0^1 \exp \{ J_1 \beta \xi_3(t, u) \} f(u, y)du}{\int_0^1 \exp \{ J_1 \beta \xi_3(0, u) \} f(u, y)du}, \tag{2.1}
\]
where $f(t, x) = \exp(h_{t,x} - h_{0,x})$, $t \in [0, 1]$, $x \in V$.

**Proof** For $J_3 = J = \alpha = 0$ and $J_1 \neq 0$ one gets $K(t, u, v) = \exp\{J_1 \beta \left(\xi_3(u, t) + \xi_3(v, t)\right)\}$. Then (1.6) can be written as

$$f(t, x) = \frac{\int_0^1 \int_0^1 \exp\{J_1 \beta \left(\xi_3(t, u) + \xi_3(t, v)\right)\} f(u, y) f(v, z) du dv}{\int_0^1 \int_0^1 \exp\{J_1 \beta \left(\xi_3(0, u) + \xi_3(0, v)\right)\} f(u, y) f(v, z) du dv}$$

$$= \frac{\int_0^1 \int_0^1 \exp\{J_1 \beta \xi_3(t, u)\} f(u, y) du \cdot \int_0^1 \int_0^1 \exp\{J_1 \beta \xi_3(t, v)\} f(v, z) dv}{\int_0^1 \int_0^1 \exp\{J_1 \beta \xi_3(0, u)\} f(u, y) du \cdot \int_0^1 \int_0^1 \exp\{J_1 \beta \xi_3(0, v)\} f(v, z) dv}.$$

Since $y, z \leq S(x)$ Eq. (2.2) is equivalent to (2.1). $\square$

Now we consider the model (1.1) in the class of translational-invariant functions $f(t, x)$ i.e. $f(t, x) = f(t)$, for any $x \in V$. For such functions Eq. (1.1) can be written as

$$f(t) = \frac{\int_0^1 \int_0^1 K(t, u, v) f(u) f(v) du dv}{\int_0^1 \int_0^1 K(0, u, v) f(u) f(v) du dv},$$

(2.3)

where $K(t, u, v) = \exp\{J_3 \beta \xi_1(t, u, v) + J \beta \xi_2(u, v) + J_1 \beta \left(\xi_3(t, u) + \xi_3(t, v)\right) + \alpha \beta (u + v)\}$, $f(t) > 0$, $t, u \in [0, 1]$.

We shall find positive continuous solutions to (2.3) i.e. such that $f \in C^+[0, 1] = \{f \in C[0, 1] : f(x) > 0\}$.

Define a nonlinear operator $H$ on the cone of positive continuous functions on $[0, 1]$

$$(Hf)(t) = \frac{\int_0^1 \int_0^1 K(t, s, u) f(s) f(u) ds du}{\int_0^1 \int_0^1 K(0, s, u) f(s) f(u) ds du}.$$

We’ll study the existence of positive fixed points for the nonlinear operator $H$ (i.e., solutions of the Eq. (2.3)).

We define the Lyapunov integral operator $L$ on $C[0, 1]$ by the equality (see [3])

$$Lf(t) = \int_0^1 K(t, s, u) f(s) f(u) ds du.$$

Put

$$\mathcal{M}_0 = \{f \in C^+[0, 1] : f(0) = 1\}.$$

**Lemma 2.2** The equation $Hf = f$ has a nontrivial positive solution iff the Lyapunov equation $Lg = g$ has a nontrivial positive solution.

**Proof** At first we shall prove that the equation

$$Hf = f, \quad f \in C_0^+[0, 1]$$

(2.4)
has a positive solution iff the Lyapunov equation
\[ \mathcal{L} g = \lambda g, \quad g \in C^+[0, 1] \]  
has a positive solution in \( M_0 \) for some \( \lambda > 0 \).

Let \( \lambda_0 \) be a positive eigenvalue of the Lyapunov operator \( \mathcal{L} \). Then there exists \( f_0 \in C^+_0[0, 1] \) such that \( \mathcal{L} f_0 = \lambda_0 f_0 \). Take \( \lambda \in (0, +\infty), \lambda \neq \lambda_0 \). Define the function \( h_0(t) \in C^+_0[0, 1] \) by \( h_0(t) = \frac{\lambda}{\lambda_0} f_0(t), \quad t \in [0, 1] \). Then \( \mathcal{L} h_0 = \lambda h_0 \), i.e., the number \( \lambda \) is an eigenvalue of Lyapunov operator \( \mathcal{L} \) corresponding the eigenfunction \( h_0(t) \).

It’s easy to check that if the number \( \lambda_0 > 0 \) is an eigenvalue of the operator \( \mathcal{L} \), then an arbitrary positive number is eigenvalue of the operator \( \mathcal{L} \). Now we shall prove the lemma. Let Eq. (2.4) holds then the function \( \frac{1}{\lambda} g(t) \) be a fixed point of the operator \( \mathcal{L} \). Analogously, since \( H \) is non-linear operator we can correspond to the fixed point if there exist any eigenvector.

\[ \square \]

**Proposition 2.3** The equation
\[ \mathcal{L} f = \lambda f, \quad \lambda > 0 \]  
has at least one solution in \( C^+_0[0, 1] \).

**Proof** Clearly, that the Lyapunov operator \( \mathcal{L} \) is a compact on the cone \( C^+[0, 1] \). By the other hand we have
\[ \mathcal{L} f(t) \geq m \left( \int_0^1 f(s) ds \right)^2, \]
for all \( f \in C^+[0, 1] \), where \( m = \min K(t, s, u) > 0 \).

Put \( \Gamma = \{ f : \| f \| = r, \quad f \in C[0, 1] \} \). We define the set \( \Gamma_+ \) by
\[ \Gamma_+ = \Gamma \cap C^+[0, 1]. \]

Then we obtain
\[ \inf_{f \in \Gamma_+} \| \mathcal{L} f \| > 0. \]

Then by Schauder’s theorem (see [4], p.20) there exists a number \( \lambda_0 > 0 \) and a function \( f_0 \in \Gamma_+ \) such that, \( \mathcal{L} f_0 = \lambda_0 f_0 \). \[ \square \]

Denote by \( N_{fix, p}(H) \) and \( N_{fix, p}(\mathcal{L}) \) the set of positive numbers of nontrivial positive fixed points of the operators \( H \) and \( L \), respectively. By Lemma 2.2 and Proposition 2.3 we can conclude that:

**Proposition 2.4** (a) The Eq. (2.4) has at least one solution in \( C^+_0[0, 1] \).

(b) The equality \( N_{fix, p}(H) = N_{fix, p}(\mathcal{L}) \) is hold.

From Propositions 1.1 and 2.4 we get the following theorem.

**Theorem 2.5** The set of splitting Gibbs measures corresponding to Hamiltonian (1.1) is non-empty.
3 The uniqueness of fixed point of the operator $L$

In this section we shall give a condition of the uniqueness of fixed point of the operator $L$.

**Theorem 3.1** Let the kernel $K(t, u, v)$ satisfies the condition

$$\max_{(t,u,v)\in[0,1]^3} K(t, u, v) < c \min_{(t,u,v)\in[0,1]^3} K(t, u, v), \quad c \in \left(1, \frac{1}{2} \sqrt{\sqrt{17} + 1}\right). \quad (3.1)$$

Then the operator $L$ has the unique fixed point in $C^+[0,1]$.

**Proof** Let $\max_{(t,u,v)\in[0,1]^3} K(t, u, v) = \mathcal{K}$ and $\min_{(t,u,v)\in[0,1]^3} K(t, u, v) = k$. At first we shall prove that if $g \in C^+[0,1]$ is a solution of the equation $Lf = f$ then $g \in G$ where

$$G = \left\{ f \in C[0,1] : \frac{k}{\mathcal{K}^2} \leq f(t) \leq \frac{\mathcal{K}}{k^2} \right\}.$$ 

Let $s \in \mathcal{L}(C^+[0,1])$ be an arbitrary function. Then there exists a function $h \in C^+[0,1]$ such that $s = Lh$. Since $s$ is continuous on $[0,1]$, there exists $t_1, t_2 \in [0,1]$ such that

$$s_{\min} = \min_{t \in [0,1]} s(t) = s(t_1) = (Lh)(t_1), \quad s_{\max} = \max_{t \in [0,1]} s(t) = s(t_2) = (Lh)(t_2).$$

Consequently we get

$$s_{\min} \geq k \int_0^1 \int_0^1 h(u)h(v) dudv \geq k \int_0^1 \int_0^1 \frac{K(t_2, u, v)}{\mathcal{K}} h(u)h(v) dudv = \frac{k}{\mathcal{K}} s_{\max}. \quad (3.2)$$

Since $g$ is a fixed point of the operator $L$ we have $\|g\| \leq \mathcal{K}\|g\|^2 \Rightarrow \|g\| \geq \frac{1}{\mathcal{K}}$.

From (3.2)

$$g(t) = (Lg)(t) \geq \frac{k}{\mathcal{K}} \|g\| \Rightarrow g(t) \geq \frac{k}{\mathcal{K}^2}. \ \ \ \ \ \ (3.3)$$

Similarly, $g(t) = (Lg)(t) \geq \frac{k}{\mathcal{K}} \|g\| \Rightarrow g(t) \geq \frac{k}{\mathcal{K}^2}. \ \ \ \ \ \ (3.4)$

Hence

$$g(t) \leq g_{\max} = \frac{\mathcal{K}}{k} g_{\min} \leq \frac{\mathcal{K}}{k^2}. \ \ \ \ \ \ (3.5)$$

Thus we have $g \in G$. 
Now we show that $L$ has the unique fixed point. By Proposition 2.4, $Lg = g$ has at least one solution. Assume that there are two solutions $g_1 \in C^+_0[0, 1]$ and $g_2 \in C^+_0[0, 1]$, i.e $Lg_i = g_i$, $i = 1, 2$.

Let a function $f \in C[0, 1]$ changes its sign on $[0, 1]$. Then it is easy to check that for every $a \in \mathbb{R}$ the following inequality holds: $\|f(t) - a\| \geq \frac{1}{2} \|f\|$.  

Put $\xi(t) = g_1(t) - g_2(t)$. Since $\xi(t)$ changes its sign on $[0, 1]$, we get

$$\max_{t \in [0, 1]} \left| \xi(t) - \left( \frac{k^2}{K^2} + \frac{K^2}{k^2} \right) \int_0^1 \xi(s) ds \right| \geq \frac{1}{2} \|\xi\|, \quad \xi(t) = 2 \int_0^1 \int_0^1 K(t, u, v) (g_1(u)g_1(v) - g_2(u)g_2(v)) du dv.$$

The last equation can be written as

$$\xi(t) = \int_0^1 \int_0^1 K(t, u, v) \eta(u, v) (|\xi(u) - \xi(v)| + \xi(u) + \xi(v)) du dv,$$

where

$$\min\{g_1(t), g_2(t)\} \leq \eta(u, v) \leq \max\{g_1(t), g_2(t)\}, \quad t \in [0, 1].$$

Since $g_i(t) \in G$, $i = 1, 2$ we get $\frac{k}{K^2} \leq \eta(u, v) \leq \frac{K}{k^2}$, $(u, v) \in [0, 1]^2$. Hence

$$\left| 2 \cdot K(t, u, v) \eta(u, v) - \left( \frac{K^2}{k^2} + \frac{k^2}{K^2} \right) \right| \leq \frac{K^2}{k^2} - \frac{k^2}{K^2}. $$

Then

$$\left| \xi(t) - \left( \frac{K^2}{k^2} + \frac{k^2}{K^2} \right) \int_0^1 \int_0^1 (|\xi(u) - \xi(v)| + \xi(u) + \xi(v)) du dv \right| \leq \left( \frac{K^2}{k^2} - \frac{k^2}{K^2} \right) \|\xi\|. \quad (3.3)$$

Assume the kernel $K(t, u, v)$ satisfies the condition (3.1). Then $K^4 - k^4 < (Kk)^2 \Rightarrow K < ck$ but it’s contradict to the following: if $\xi \in C[0, 1]$ changes its sign on $[0, 1]$ then for every $a \in \mathbb{R}$ the following inequality holds $\|\xi - a\| \geq \frac{1}{2} \|\xi\|$. This completes the proof. □

**Theorem 3.2** Let $k \geq 2$. If the function $K(t, u, v)$ which defined in (1.5) satisfies the condition (3.1), then the model (1.1) has the unique-translational invariant Gibbs measure.

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