On regularized trace formula of Gribov
semigroup generated by the Hamiltonian of
reggeon field theory in Bargmann
representation

Abdelkader INTISSAR

(*) Équipé d’Analyse spectrale, UMR-CNRS n° 6134, Université de Corse, Quartier Grossetti, 20 250
Corté-France
Tél: 00 33 (0) 4 95 45 00 33
Fax: 00 33 (0) 4 95 45 00 33
e.mail:intissar@univ-corse.fr

(**) Le Prador, 129 rue du commandant Rolland, 13008 Marseille-France

Abstract

In J. Math. Anal. App. 305. (2005), we have considered the Gribov operator

\[ H_{\lambda'} = \lambda' S + H_{\mu,\lambda} \]

acting on Bargmann space where \( S = a^*a^2 \) and \( H_{\mu,\lambda} = \mu a^*a + i\lambda a^*(a + a^*)a \) with \( i^2 = -1 \).

Here \( a \) and \( a^* \) are the standard Bose annihilation and creation operators satisfying the commutation relation \([a, a^*] = I\). In Reggeon field theory, the real parameters \( \lambda' \) is the four coupling of Pomeron, \( \mu \) is Pomeron intercept, \( \lambda \) is the triple coupling of Pomeron and \( i^2 = -1 \).

We have given an approximation of the semigroup \( e^{-tH_{\lambda'}} \) generated by the operator \( H_{\lambda'} \). In particular, we have obtained an estimate approximation in trace norm of this semigroup by the unperturbed semigroup \( e^{-t\lambda'S} \).

In [12], we have regularized the operator \( H_{\mu,\lambda} \) by \( \lambda''G \) where \( G = a^*a^3 \), i.e. we have considered \( H_{\lambda''} = \lambda''G + H_{\mu,\lambda} \) where \( \lambda'' \) is the magic coupling of Pomeron. In this case, we have established an exact relation between the degree of subordination of the non-self-adjoint perturbation operator \( H_{\mu,\lambda} \) to the unperturbed operator \( G \) and the number of corrections necessary for the existence of finite formula of the regularized trace.

The goal of this article consists to study the trace of the semigroup \( e^{-tH_{\lambda''}} \), in particular to give an asymptotic expansion of this trace as \( t \to 0^+ \).

1. Introduction

Usually, quantum Hamiltonians are constructed as self-adjoint operators; for certain situations, however, non-self-adjoint Hamiltonians are also of importance. In particular, the reggeon field theory (as invented (1967) by V.
Gribov [7]) for the high energy behaviour of soft processes is governed by the non-self-adjoint Gribov operator

\[ H_{\lambda''\lambda',\mu,\lambda} = \lambda''a^*a^3 + \lambda'a^2a^2 + \mu a^*a + i\lambda a^*(a + a^*)a \]

where \(a\) and its adjoint \(a^*\) are annihilation and creation operators, respectively, satisfying the canonical commutation relations \([a, a^*] = I\).

It is convenient to regard the above operators as acting on Bargmann space \(E\) [4]:

\[ E = \{ \phi : \mathcal{G} \to \mathcal{G}_{\text{entire}}; \int_{\mathcal{G}}|\phi(z)|^2 \, e^{-|z|^2} \, dx \, dy < \infty \} \]

The Bargmann space \(E\) with the paring:

\[ <\phi, \psi> = \int_{\mathcal{G}}\phi(z)\overline{\psi(z)}e^{-|z|^2} \, dx \, dy \]

is a Hilbert space and \(e_n(z) = \frac{z^n}{\sqrt{n!}}; n = 0, 1, ...\) is an orthonormal basis in \(E\).

In this representation, the standard Bose annihilation and creation operators are defined by

\[
\begin{cases}
  a\phi(z) = \phi'(z) \\
  \text{with maximal domain} \\
  D(a) = \{ \phi \in E; \ a\phi \in E \}
\end{cases}
\]

and

\[
\begin{cases}
  a^*\phi(z) = z\phi(z) \\
  \text{with maximal domain} \\
  D(a^*) = \{ \phi \in E; \ a^*\phi \in E \}
\end{cases}
\]

Notice that \(D(a) = D(a^*)\) and \(D(a) \hookrightarrow E\) is compact but \(\rho(a) = \mathcal{G}\).

It has been established in [8] that \(T_1 = aa^*\) is a chaotic operator and it has been shown in [9] that \(T_1 + T_1^*\) is symmetric but not selfadjoint.

Notice also that for \(H_{\mu,\lambda} = \mu a^*a + i\lambda a^*(a + a^*)a\) with domain \(\{ \phi \in E \text{ such that } H_{\mu,\lambda}\phi \in E \}\), we have:

i) For \(\mu \neq 0\) and \(\lambda \neq 0\), \(H_{\mu,\lambda}\) is very far from normal and not only its self-adjoint and skew-adjoint parts do not commute but there is no inclusion
in either way between their domains or with the domain of their commutator (see [13]).

ii) For \( \mu > 0 \) and \( \lambda \in \mathbb{R} \), \( e^{-tH_{\mu,\lambda}} \) is compact (see [10]).

iii) For \( \mu > 0 \) and \( \lambda \in \mathbb{R} \), it has been established in [1] that the resolvent of \( H_{\mu,\lambda} \) belongs to the class \( C_{1+\epsilon} \) \( \forall \epsilon > 0 \).

We recalling that a compact operator \( K \) acting on a complex Hilbert space \( \mathbb{E} \) belongs to the Carleman class Carleman \( C_p \) of order \( p \) if

\[
\sum_{n=1}^{\infty} s_n^p < \infty,
\]

where \( s_n \) are s-numbers of operator \( K \), i.e., the eigenvalues of the operator \( \sqrt{K^*K} \).

In particular, the operator \( K \) is called nuclear operator if \( K \in C_1 \) and Hilbert-Shmidt operator if \( K \in C_2 \).

For \( p \geq 1 \) the value \( \left( \sum_{n=1}^{\infty} s_n^p \right)^{\frac{1}{p}} \) is a norm denoted by \( \| \cdot \|_p \) and for \( p = 1 \) it is called nuclear norm or trace norm.

We can consult [6] for a systematic study of operators of Carleman class \( C_p \) of order \( p \).

Let \( H_\lambda = \lambda ' S + H_{\mu,\lambda} \) where \( S = a^*a^2 \), this operator is more regular that \( H_{\mu,\lambda} \), its semigroup \( e^{-tH_\lambda} \) is analytic and it has been established in [11] that the convergence of the usual Trotter product formula for \( H_\lambda \) is of classical type and can be lifted to trace-norm convergence.

Moreover, there exist \( t_0 > 0 \) and \( C > 0 \) such that:

\[
\| (e^{-\frac{\lambda ' S}{n}}e^{-\frac{1}{n}H_{\mu,\lambda}})^n - e^{-tH_\lambda} \|_1 \leq C \frac{\log n}{n}, \quad n = 2, 3, \ldots, \forall \ t \geq t_0.
\]

Now, if \( H_{\mu,\lambda} \) is regularized by \( \lambda '' G \) where \( G = a^*a^3 \) and \( \lambda '' \) is the magic coupling of Pomeron, we can consider:

\[
H_{\lambda ''} = \lambda '' a^3 a^3 + \mu a^*a + i\lambda a^*(a + a^*)a \quad (1.1)
\]

\[
\begin{cases}
H_{\lambda ''} \phi(z) = \lambda '' z^3 \phi'''(z) + i\lambda z \phi''(z) + (i\lambda z^2 + \mu z) \phi'(z) \\
\text{with maximal domain} \\
D(H_{\lambda ''}) = \{\phi \in E; \quad H_{\lambda ''} \phi \in E\}
\end{cases}
\]

In [2] (see theorem 3.3, p. 595), Aimar et al have shown that the spectrum of \( H_{\lambda ''} \) is discrete and that the system of generalized eigenvectors of this operator is an unconditional basis in Bargmann space \( E \).
Recently in [12], we have established a regularized trace formula for $H_{\lambda''}$.

More precisely, we have shown the following result:

**Theorem 1.1 (see [12])**

Let $E$ be the Bargmann space and $H_{\lambda''} = \lambda''G + H_{\mu,\lambda}$ acting on $E$

where

- $G = a^*a^3$ and $H_{\mu,\lambda} = \mu a^*a + i\lambda a^*(a + a^*)a$

- $a$ and $a^*$ are the standard Bose annihilation and creation operators satisfying

  
  
  the commutation relation $[a, a^*] = I$.

Then

There exists an increasing sequence of radius $r_n$ such that $r_n \to \infty$ as $n \to \infty$

and

$$
\lim_{n} \sum_{k=0}^{n} (\sigma_k - \lambda''\lambda_k) = 
-\lim_{n} \frac{1}{2i\pi} \int_{\gamma_n} \text{Tr}[\sum_{j=1}^{4} \frac{(-1)^{j-1}}{j} [H_{\mu,\lambda}(\lambda''G - \sigma I)^{-1}]^j] d\sigma; n \to \infty \tag{1.2}
$$

where

- $\sigma_k$ are the eigenvalues of the operator $H_{\lambda''} = \lambda''G + H_{\mu,\lambda}$

- $\lambda_k = k(k - 1)(k - 2)$ are the eigenvalues of the operator of $G$ associated to eigenvectors $e_k(z)$

- $(\lambda''G - \sigma I)^{-1}$ is the resolvent of the operator $\lambda''G$

and
- $\gamma_n$ is the circle of radius $r_n$ centered at zero in complex plane.

The goal of this article consists to study the trace of the semigroup $e^{-tH_{\lambda''}}$, in particular, to give an asymptotic expansion of this trace as $t \to 0^+$.

Our procedure consists to prove that

i) The semigroup $e^{-tG}$ generated by the operator $G$ is analytic and nuclear (Gibbs analytic semigroup).

ii) \[ \forall \epsilon > 0, \exists C_\epsilon > 0; \| H_{\mu,\lambda} \phi \| \leq \epsilon \| G \phi \| + C_\epsilon \| \phi \| \forall \phi \in D(G). \] (1.3)

where $D(G) = \{ \phi \in E, G\phi \in E \}$.

iii) The operator $H_{\mu,\lambda} G^{-\delta}$ is bounded $\forall \delta \geq \frac{1}{2}$.

Now, with the aid of the results of Angescu et al [3] or of Zagrebnov [17-18] with Ginibre-Gruber inequality [5], it easy to prove that the series of general term $S_k(t)_{k\in\mathbb{N}}$ defined by:

$S_0(t)\phi = e^{-t\lambda''G}\phi$

and

$S_{k+1}(t)\phi = -\int_0^t e^{-(t-s)\lambda''G} H_{\mu,\lambda} S_k(t)\phi ds$

converges to $e^{-tH_{\lambda''}}$ (nuclear norm).

By using the properties i), ii) and iii) we get:

\[ \| e^{-tH_{\lambda''}} - e^{-t\lambda''G} \|_1 = t \| e^{-t\lambda''G} H_{\mu,\lambda} \|_1 + \| (\lambda''G)^{\delta} e^{-\frac{1}{2}\lambda''G} \|_1 O(t^2); \] (1.4)

To establish the above results, we give in section 2 some spectral properties of semigroups $e^{-t\lambda''G}$ and $e^{-t(\lambda''G+H_{\mu,\lambda})}$ in $C_p$. In section 3, we give the proof of the above formula (1.4) for the trace of $e^{-tH_{\lambda''}}$ as $t \to 0^+$. 

A. Intissar
2. Some spectral properties of semigroups $e^{-t\lambda'' G}$ and $e^{-t(\lambda'' G + H_{\mu, \lambda})}$ in Carleman class $C_p$

We begin this section by given some elementary spectral properties of $G$ and $H_{\lambda''}$:

**Lemma 2.1**

1) The operator $G$ is self adjoint with compact resolvent.

2) The eigenvalues of $G$ are $\lambda_n = n(n-1)(n-2)$ for $n \geq 0$ associated to eigenvectors $e_n(z) = \frac{z^n}{\sqrt{n!}}$.

3) $\lim || Ge^{-tG} || = \frac{1}{e}$ as $t \to 0$.

4) the resolvent of $G$ belongs to $C_p \forall p > \frac{1}{3}$.

5) $e^{-tG}$ is nuclear semigroup and $|| e^{-tG} ||_1 \leq Ct^{-\frac{1}{2}}$.

where the constant $C$ does not depend on $t$

6) $e^{-tG} \in C_p \forall p > 0$.

**proof**

1) It follows easily from Rellich theorem:

**Theorem 2.2.** (see [15], p. 386)

Let $B$ be a self adjoint operator in a complex Hilbert space $E$ such that $< B\phi, \phi > \geq < \phi, \phi >$, $\phi \in D(B)$, where $D(B) = \{ \phi \in E; B\phi \in E \}$. Then $B$ is discrete if and only if $\{ \phi \in D(B); < B\phi, \phi > \leq 1 \}$ is pre-compact.

that the operator $G$ is discrete.

Also, we can prove the above result by using the following observation:

Since $D(a)$ is compactly embedded in Bargmann space $E$ and $D(a^*a^3) \hookrightarrow D(a)$ is continuous then $D(a^*a^3)$ is compactly embedded in Bargmann space $E$. 

A. Intissar
And since \((a^3a^3 + I)\) is invertible then the operator \(G\) is discrete.

2) It is evident.

3) As \(Ge^{-tG}e_n = n(n-1)(n-2)e^{-tn(n-1)(n-2)}e_n\) we get

\[
\| Ge^{-tG} \| = \frac{1}{t} \sup \; tn(n-1)(n-2)e^{-tn(n-1)(n-2)} = \frac{1}{e} \text{ this implies}
\]

\[
\lim_{t \to 0} \| Ge^{-tG} \| = \frac{1}{e} \text{ as } t \to 0. \tag{2.1}
\]

4) As \(G\) is self adjoint and its eigenvalues are:

\[
\lambda_n = n(n-1)(n-2) \sim n^3
\]

this implies that the series of general term \(\frac{1}{n^p}\) converges \(\forall p > \frac{1}{3}\) and consequently, the resolvent of \(G\) belongs to Carleman class \(C_p \forall p > \frac{1}{3}\).

5) \(e^{-tG}\) is self adjoint and it is of trace class, because the series of general term \(e^{-tp\lambda_n}\) converges \(\forall t > 0\).

Now let \(x \in [0, +\infty[\) and if \(t \in [0, +\infty[,\) consider the function \(f(x) = e^{-tx^3}\) and its derivative \(f'(x) = -3tx^2e^{-tx^3}\) which non positive then the function \(f(x)\) is decreasing and we have

\[
\sum_{n=1}^{\infty} e^{-tn^3} \leq \int_{0}^{\infty} e^{-tx^3} \, dx
\]

By a change of variable in the above integral, we obtain that

\[
\int_{0}^{\infty} e^{-tx^3} \, dx = Ct^{-\frac{1}{3}}
\]

where the constant \(C\) does not depend on \(t\)

This implies that

\[
\| e^{-tG} \|_1 \leq \sum_{n=1}^{\infty} e^{-t^3n} \leq Ct^{-\frac{1}{3}}
\]

6) \(e^{-tG}\) is Carleman class \(C_p \forall p > 0\), because the series of general term \(e^{-tp\lambda_n}\) converges \(\forall t > 0\) and \(\forall p > 0\). \(\diamond\)
Remark 2.3

We can derive the property 6) from the fact that $e^{-tG} \in C_1$. In fact, since $C_1 \subset C_p$ then $e^{-tG} \in C_p \forall p > 1$.

Now if $p < 1$, we choose an integer $n$ such that $\frac{1}{n} < p$, then $C_\frac{1}{n} \subset C_p$.

Since $e^{-\frac{t}{n}G} \in C_1 \forall \tau = \frac{t}{n} > 0$ then $(e^{-\frac{t}{n}G})^n \in C_\frac{1}{n}$

and we obtain that

$e^{-tG} \in C_p \forall p > 0$.

From the above remark, we are now ready to prove following lemma:

Lemma 2.4.

1) Let $T(t)$ be a semigroup on a complex Hilbert space $E$. We assume that $T(t)$ is selfadjoint and there exists $p_0 > 0$ such that $T(t) \in C_{p_0} \forall t > 0$.

Then $T(t) \in C_p \forall t > 0, \forall p > 0$.

2) Let $T(t)$ be a semigroup on a complex Hilbert space $E$. We assume that there exists $p_0 > 0$ such that $T(t) \in C_{p_0} \forall t > 0$.

Then $T(t) \in C_p \forall t > 0, \forall p > 0$.

Proof

1) let $t = \frac{\tau}{p_0}$ and $\hat{T}(\tau) = T(\frac{\tau}{p_0})$ then $T(t) \in C_{p_0}$ if and only if $\hat{T}(\tau) \in C_1$.

Since $C_1 \subset C_p$ for all $p > 1$ it follows that $\hat{T}(\tau) \in C_p$ for all $p > 1$.

Let $0 < p < 1$, Since $\hat{T}(\tau)$ is self adjoint, then for each fixed $\tau > 0$, there exist a positive decreasing sequence $s_n \in l_1$ and an orthonormal sequence $e_n$ such that $\hat{T}(\tau) = \sum_{n=0}^{\infty} s_n e_n \otimes e_n$.

Let $p = \frac{1}{\delta}$ with $\delta > 1$ and $n$ such that $n > \delta$, then for fixed $\frac{\tau}{n}$, there exist a positive decreasing sequence $r_k \in l_1$ and an orthonormal sequence $e_k$ such that $T(\frac{\tau}{n}) = \sum_{k=0}^{\infty} r_k e_k \otimes e_k$. 

A. Intissar
It follows that:

\[ T(t) = [T(\frac{t}{n})] = \sum_{k=0}^{\infty} r_k^n e_k \otimes e_k. \]

Since, \( r_k \in l_1 \) then \( r_k^n \in l_1^n \subset l_p \), i.e. \( T(t) \in C_p \).

2) If \( p > p_0 \) then \( T(t) \in C_p \) for all \( t > 0 \) and all \( p > p_0 \) because \( C_{p_0} \subset C_p \) for all \( p > p_0 \).

If \( p < p_0 \), we choose an integer \( n \) such that \( \frac{p_0}{n} < p \), since \( T(t) \in C_{p_0} \) for all \( t > 0 \) then \( T(\frac{t}{n}) \in C_{p_0} \) and consequently \( T(t) = [T(\frac{t}{n})]^n \in C_{\frac{p_0}{n}} \subset C_p \).

**Remark 2.5**

Let \( T \) be a compact operator, we assume that \( T \) is positif and \( \| T \| \leq 1 \) then there exist a positive sequence \( 0 < s_n < 1 \) and an orthonormal sequence \( e_n \) such that \( T = \sum_{n=0}^{\infty} s_n e_n \otimes e_n \).

Let \( T(t) = \sum_{n=0}^{\infty} s_n^t e_n \otimes e_n \), if we choose \( s_n \in \cap_{p=0}^{\infty} l_p \) where \( l_p \) is the space of \( p \)-summable sequences, then \( T(t) \in C_p \) for all \( t > 0 \) and all \( p > 0 \), it follows that

\[ \| T(t) \|_p = \left( \sum_{n=0}^{\infty} s_n^{tp} \right)^{\frac{1}{p}} \]

For \( s < t \) we have \( s_n^t < s_n^s \) then \( \| T(t) \|_p < \| T(s) \|_p \).

By using the Beppo-Levi’s theorem, we deduce that \( \| T(t) \|_p \to \infty \) as \( t \to 0 \).

**Proposition 2.6**

Let \( E_0 = \{ \phi \in E; \phi(0) = 0 \} \), \( P_0 = \{ p \in P; p(0) = 0 \} \) where \( P \) is the space of polynomials

and

\( H_{\lambda^r}^{min} \) with domain \( D_{min}(H_{\lambda^r}) \) is the closure of the restriction of \( H_{\lambda^r} \) on \( P_0 \).

Then we have:

(a) \( \forall \epsilon > 0 \), there exists \( C_\epsilon > 0 \) such that:
\[ \| H_{\mu,\lambda}\phi \| \leq \epsilon \| G\phi \| + C_\epsilon \| \phi \| \text{ for all } \phi \in D(G). \] (2.3)

(b) \( \forall \epsilon > 0, \) there exists \( C_\epsilon > 0 \) such that :
\[ |< H_{\mu,\lambda}\phi, \phi> - \epsilon < G\phi, \phi > + C_\epsilon \| \phi \|^2 \| \phi \| \leq \epsilon < G\phi, \phi > + C_\epsilon \| \phi \|^2 \forall \phi \in D(G). \] (2.4)

(c) For \( \lambda'' > 0 \) and \( \forall \epsilon ; 0 < \epsilon < \lambda'' \), there exists \( C_\epsilon > 0 \) such that :
\[ \Re < H_{\lambda''}\phi, \phi> \geq (\lambda'' - \epsilon) < G\phi, \phi > - C_\epsilon \| \phi \|^2 \forall \phi \in D(G) \] (2.5)

in particular, the range of \( H_{\lambda''} \) is closed.

(d) For \( \lambda'' \geq 0 \) and \( \mu > 0, \) \( H_{\lambda''} \) is accretive and maximal.

(e) \( D_{\max}(H_{\lambda''}) = D_{\min}(H_{\lambda''}) = D(G). \)

(f) \(-H_{\lambda''} \) generates an analytic semigroup \( e^{-tH_{\lambda''}}, \ t > 0. \)

(g) For \( \lambda'' \neq 0, \) the resolvent of \( H_{\lambda''} \) belongs to Carleman class \( C_p \) for all
\[ p > \frac{1}{3}. \]

\begin{proof}

a) Let \( \phi \in E_0, \) we have \( \phi(z) = \sum_{n=1}^{\infty} a_n e_n(z). \) Then
\[ H_{\mu,\lambda}\phi(z) = \sum_{n=1}^{\infty} [\mu n a_n + i\lambda(n-1)\sqrt{n} a_{n-1} + i\lambda n \sqrt{n+1} a_{n+1}] e_n(z) \]
and
\[ G\phi(z) = \sum_{n=1}^{\infty} n(n-1)(n-2) a_n e_n(z) \]
we remark that there exists \( C > 0 \) such that \( \| H_{\mu,\lambda}\phi \|^2 \leq C \sum_{n=1}^{\infty} n^3 |a_n|^2 \)
and
\[ \| G\phi \|^2 \geq \frac{1}{36} \sum_{n=1}^{\infty} n^6 |a_n|^2. \]
Now, by using the Young’s inequality, we get:
\[ \forall \epsilon > 0, \ k^3 \leq \epsilon k^6 + \frac{1}{\epsilon} \ \forall k \in \mathbb{N}. \]

this implies that:

\[ \forall \epsilon > 0, \text{ there exists } C_\epsilon > 0 \text{ such that } || H_{\mu,\lambda} \phi || \leq \epsilon || G\phi || + C_\epsilon || \phi || \text{ for all } \phi \in D(G). \]

b) \[ < H_{\mu,\lambda} \phi, \phi > = \mu || a\phi ||^2 + i\lambda < a^2 \phi, a\phi > + i\lambda < a\phi, a^2 \phi > \text{ for all } \phi \in D(H_{\mu,\lambda}). \]

Then

\[ |< H_{\mu,\lambda} \phi, \phi >| \leq \mu || a\phi ||^2 + 2 |\lambda|||| a\phi || . || a^2 \phi || \text{ pour tout } \phi \in D(H_{\mu,\lambda}). \]

With the aid of following inequalities:

i) \[ \forall \epsilon_1 > 0 || a\phi || . || a^2 \phi || \leq \epsilon_1 || a^2 \phi ||^2 + \frac{1}{\epsilon_1} || a\phi ||^2. \]

ii) \[ \forall \epsilon_2 > 0 \text{ there exists } C_{\epsilon_2} > 0 \text{ such that } || a\phi ||^2 \leq \epsilon_2 || a^3 \phi ||^2 + C_{\epsilon_2} || \phi ||^2 \]

iii) \[ || a^2 \phi ||^2 \leq || a^3 \phi ||^2 \]

we obtain:

\[ \forall \epsilon > 0, \text{ there } C_\epsilon > 0 \text{ such that } |< H_{\mu,\lambda} \phi, \phi >| \leq \epsilon < G\phi, \phi > + C_\epsilon || \phi ||^2 \text{ for all } \phi \in D(G). \]

c) Since \( \text{Re} < H_{\lambda''} \phi, \phi > = \lambda'' < G\phi, \phi > + < H_{\mu,\lambda} \phi, \phi > \) and \( \lambda'' > 0 \)

Then

\[ \text{Re} < H_{\lambda''} \phi, \phi > \geq \lambda'' < G\phi, \phi > - |< H_{\mu,\lambda} \phi, \phi >|. \]

By using the above property we get:

\[ \text{Re} < H_{\lambda''} \phi, \phi > \geq \lambda'' < G\phi, \phi > - \epsilon < G\phi, \phi > - C_\epsilon || \phi ||^2 = (\lambda'' - \epsilon) < G\phi, \phi > - C_\epsilon || \phi ||^2 \]

We choose \( 0 < \epsilon < \lambda'' \) to deduce that
$Re < H_{\lambda''} \phi, \phi > \geq -C_\epsilon \| \phi \|^2$

Consequently the range of $H_{\lambda''}$ is closed.

d) Since $a^* (a + a^*) a$ is symmetric operator then

$Re < H_{\lambda''} \phi, \phi > = \lambda'' \| a^3 \phi \|^2 + \mu \| a \phi \|^2 \geq \mu \| \phi \|^2, \forall \phi \in D_{\min}(H_{\lambda''})$

Now for $\lambda'' \geq 0$ and $\mu > 0$ we deduce that:

$Re < H_{\lambda''} \phi, \phi > \geq \mu \| \phi \|^2, \forall \phi \in D_{\min}(H_{\lambda''})$.

This inequality will be not verified if we kept constant functions in Bargmann space $E$.

Now, we would like to show that there exists $\beta_0 \in IR; H_{\lambda''} + \beta_0 I$ is invertible.

We rewrite $H_{\lambda''}$ in the following form:

$H_{\lambda''} = \lambda''(G + \frac{1}{\lambda'} H_{\mu,\lambda} \ et \ G + \beta I + \frac{1}{\lambda''} H_{\mu,\lambda} = [I + \frac{1}{\lambda''} H_{\mu,\lambda} (G + \beta I)^{-1}](G + \beta I)$.

Using the property a) to get:

$\| \frac{1}{\lambda''} H_{\mu,\lambda} (G + \beta I)^{-1} \psi \| \leq \epsilon \| G(G + \beta I)^{-1} \psi \| + C_\epsilon \| (G + \beta I)^{-1} \psi \|

\leq \epsilon \| (G + \beta I - \beta I)(G + \beta I)^{-1} \psi \| + C_\epsilon \| (G + \beta I)^{-1} \psi \|

\leq \epsilon \| \psi \| + (\epsilon \beta + C_\epsilon) \| (G + \beta I)^{-1} \psi \|

$\| (G + \beta I)^{-1} \| \leq \frac{1}{\beta}$ then

$\| \frac{1}{\lambda''} H_{\mu,\lambda} (G + \beta I)^{-1} \psi \| \leq (2\epsilon + \frac{C_\epsilon}{\beta})$.

Now, we choose $0 < \epsilon < \frac{1}{2}$ and $\beta > \frac{C_\epsilon}{1 - 2\epsilon}$ to obtain:

$\| \frac{1}{\lambda''} H_{\mu,\lambda} (G + \beta I)^{-1} \| < 1$.

this implies $H_{\lambda''}^{\min} + \beta I$ is invertible.

e) We begin to show that $D_{\max}(H_{\lambda''}) = D_{\min}(H_{\lambda''})$. 

A. Intissar
First, $D_{\min}(H_{\lambda''}) \subset D_{\max}(H_{\lambda''})$ is trivial.

To show that $D_{\max}(H_{\lambda''}) \subset D_{\min}(H_{\lambda''})$, $\phi \in D_{\max}(H_{\lambda''})$ then $(H_{\lambda''} + \beta I)\phi \in E_0$ for all $\beta$.

Since there exists $\beta_0 \in \mathbb{R}$ such that $H_{\lambda''}^{\min} + \beta_0 I$ is invertible of $D_{\min}(H_{\lambda''})$ on $E_0$, then there exists $\phi_1 \in D_{\min}(H_{\lambda''})$ such that:

$$(H_{\lambda''} + \beta_0 I)\phi = (H_{\lambda''}^{\min} + \beta_0 I)\phi_1,$$

in particular, we have $(H_{\lambda''} + \beta_0 I)(\phi - \phi_1) = 0$.

To deduce that $\phi = \phi_1$, we need that $\ker(H_{\lambda''} + \beta_0 I) = \{0\}$.

we recall that the range of $H_{\lambda''}^{\min} + \beta_0 I$ is closed and the formal adjoint of $H_{\lambda''}$ is $H_{\lambda''}$ where we substitute $\lambda$ by its opposite.

Now, since the adjoint of the minimal is formal adjoint of the maximal and $H_{\lambda''}^{\min} + \beta_0 I$ is invertible then $\ker(H_{\lambda''} + \beta_0 I) = \{0\}$ this implies that $\phi = \phi_1$ and $\phi \in D_{\min}(H_{\lambda''})$.

From the inequality a) and the theorem 111 in book’s Kato [4], we deduce $D_{\max}(H_{\lambda''}) = D(G)$.

f) From the inequality a) and the theorem 2.1 in book’s Pazy [16], we deduce that $-H_{\lambda''}$ generates analytic semigroup $e^{-tH_{\lambda''}} \quad t > 0$.

g) For $\lambda'' \neq 0$ the resolvent of $H_{\lambda''}$ is Carleman class $C_p$ for all $p > \frac{1}{3}$, this property is the lemma 4.1 of [2].

This ends the proof of this proposition.  

Now, the above properties d) and e) allow us to show the following theorem:

**Theorem 2.7**

$-H_{\lambda''}$ generates a semigroup $e^{-tH_{\lambda''}}$ of Carleman $C_p$ for all $p > 0$ and $t > 0$.  

**Proof**

Let $E_0 = \{\phi \in E; \phi(0) = 0\}$ then on $E_0$ we have:

$$\Re < H_{\lambda''}\phi, \phi > = \lambda'' \| A^3 \phi \|^2 + \mu \| A\phi \|^2 \geq \mu \| \phi \|^2$$

for $\lambda'' \geq 0$ and
\[ \mu > 0. \]

From this inequality we deduce that 0 belongs to resolvent set \( \rho(H_{\lambda''}) \) of the operator \( H_{\lambda''} \).

Let \( T(t) = \int_0^t e^{-sH_{\lambda''}} \phi ds \) then

\[ T(t) = H_{\lambda''}^{-1}(I - e^{-tH_{\lambda''}}) \]

and as the resolvante of \( H_{\lambda''} \) is Carleman class \( C_p \) for all \( p > \frac{1}{3} \) and the operator \( I - e^{-tH_{\lambda''}} \) is bounded then \( T(t) \) is Carleman \( C_p \) for all \( p > \frac{1}{3} \) and with the aid of lemma 2.4, we end the proof. \( \diamond \)

3. Asymptotic expansion of trace of \( e^{-tH_{\lambda''}} \) as \( t \to 0^+ \)

We put \( S(s) = e^{-(t-s)\lambda''G}e^{-sH_{\lambda''}} \) then for \( \phi \in D(G) \), the application \( \phi \to S(s)\phi \) is differentiable and \( S'(s)\phi = \lambda''Ge^{-(t-s)\lambda''G}e^{-sH_{\lambda''}} - e^{-(t-s)\lambda''G}H_{\lambda''}e^{-sH_{\lambda''}} = -e^{-(t-s)\lambda''G}H_{\mu,\lambda}e^{-sH_{\lambda''}} \).

On \( [0,t] \), we have \( \int_0^t S'(s)\phi ds = S(t)\phi - S(0)\phi = -\int_0^t e^{-(t-s)\lambda''G}H_{\mu,\lambda}e^{-sH_{\lambda''}}\phi ds \)

This implies that \( e^{-tH_{\lambda''}}\phi \) is solution of the following integral equation :

\[ e^{-tH_{\lambda''}}\phi - e^{-t\lambda''G}\phi = -\int_0^t e^{-(t-s)\lambda''G}H_{\mu,\lambda}e^{-sH_{\lambda''}}\phi ds \] (3.1)

or

\[ e^{-tH_{\lambda''}}\phi - e^{-t\lambda''G}\phi = -\int_0^t N(t,s)e^{-sH_{\lambda''}}\phi ds \] with \( N(t,s) = e^{-(t-s)\lambda''G}H_{\mu,\lambda} \) (3.2)

It is well known that the solution of equation (3.1) can be obtained by successive approximation method:

\[ e^{-tH_{\lambda''}} = \sum_{k=0}^{\infty} S_k(t) \] (3.3)

where

\[ S_0(t)\phi = e^{-t\lambda''G}\phi \]

and
Regularized trace formula on Gribov’s semigroup

\[ S_{k+1}(t)\phi = -\int_0^t e^{-(t-s)\lambda''G}H_{\mu,\lambda}S_k(t)\phi ds \]

the convergence of (3.3) is in operator norm.

Notice that:

\[ S_1(t)\phi = -\int_0^t e^{-(t-t_1)\lambda''G}H_{\mu,\lambda}e^{-t_1\lambda''G}\phi dt_1 \]

\[ S_2(t)\phi = \int_0^t \int_0^{t_1} e^{-(t-t_1)\lambda''G}H_{\mu,\lambda}e^{-(t_1-t_2)\lambda''G}H_{\mu,\lambda}e^{-t_2\lambda''G}\phi dt_2 dt_1 \]

\[ S_3(t)\phi = (-1)^3 \int_0^t \int_0^{t_1} \int_0^{t_2} e^{-(t-t_1)\lambda''G}H_{\mu,\lambda}e^{-(t_1-t_2)\lambda''G}H_{\mu,\lambda}e^{-(t_2-t_3)\lambda''G}H_{\mu,\lambda}e^{-t_3\lambda''G}\phi dt_3 dt_2 dt_1 \]

\[ S_k(t)\phi = (-1)^k \int_0^t \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{k-1}} e^{-(t-t_1)\lambda''G}H_{\mu,\lambda}e^{-(t_1-t_2)\lambda''G}H_{\mu,\lambda}e^{-(t_2-t_3)\lambda''G}H_{\mu,\lambda}e^{-(t_{k-1}-t_k)\lambda''G}\phi dt_k \cdots \int_0 dt_3 dt_2 dt_1. \]

**Lemma 3.1**

The series \( e^{-tH_{\lambda''}} = \sum_{k=0}^{\infty} S_k(t) \) converges in trace norm. \( \diamond \)

**Proof**

The convergence in trace norm is obtained by using the results of Angeleascu-Nenciu-Bundaru, in particular their proposition in [3] or the results of Zagrebnov in [17-18], in particular the theorem 2.1 in [18] with the aid of Ginibre-Gruber’s inequality in [5].

Now we are going to derive an asymptotic expansion of the trace of \( e^{-tH_{\lambda''}} \) as asymptotique lorsque \( t \to 0^+ \)

For \( s \geq 0 \) we have
\[ e^{-sH_{\lambda''}} \phi = e^{-s\lambda''G} \phi - \int_0^s N(s, s_1)e^{-s_1H_{\lambda''}} \phi ds_1 \] with \( N(s, s_1) = e^{-(s-s_1)\lambda''G} H_{\mu,\lambda} \)

and for \(0 \leq s \leq t\), we substitute the above expression of \(e^{-sH_{\lambda''}} \phi \) in (3.2) to get:

\[
e^{-tH_{\lambda''}} \phi - e^{-t\lambda''G} \phi = -\int_0^t N(t, s)[e^{-s\lambda''G} \phi - \int_0^s N(s, s_1)e^{-s_1H_{\lambda''}} \phi]ds_1 ds
\]

\[
= -\int_0^t N(t, s)e^{-s\lambda''G} \phi ds + \int_0^t \int_0^s N(t, s)N(s, s_1)e^{-s_1H_{\lambda''}} \phi ds_1 ds
\]

\[
= -\int_0^t e^{-(t-s)\lambda''G} H_{\mu,\lambda}e^{-s\lambda''G} \phi ds + \int_0^t \int_0^s e^{-(t-s)\lambda''G} H_{\mu,\lambda}e^{-(s-s_1)\lambda''G} H_{\mu,\lambda}e^{-s_1H_{\lambda''}} \phi ds_1 ds
\]

Then we have:

\[
e^{-tH_{\lambda''}} \phi - e^{-t\lambda''G} \phi = \int_0^t e^{-(t-s)\lambda''G} H_{\mu,\lambda}e^{-s\lambda''G} \phi ds + \int_\Delta e^{-(t-s)\lambda''G} H_{\mu,\lambda}e^{-(s-s_1)\lambda''G} H_{\mu,\lambda}e^{-s_1H_{\lambda''}} \phi ds_1 ds
\]

(3.3)

where \(\Delta\) is the triangle \(\{(s_1, s); 0 \leq s_1 \leq s \leq t\}\).

The matrix associated to \(H_{\mu,\lambda}\) in the basis \(\{e_n\}\) can be written in this form:

\[
H_{\mu,\lambda}e_n = i\lambda(n - 1)\sqrt{n}e_{n-1} + n\mu e_n + i\lambda n\sqrt{n} + 1 e_{n+1}
\]

(3.4)

- The family of infinite matrices associated to \(H_{\mu,\lambda}\) is tridiagonal of the form \(J + i\lambda H^{(i)}\), where the matrix \(J\) is diagonal with entries \(J_{nn} := n\mu\), and the matrix \(H^{(i)}\) is off-diagonal, with nonzero entries \(H^{(i)}_{n,n+1} = H^{(i)}_{n+1,n} := H^{(i)} = n\sqrt{n} + 1\).

- The family of infinite matrices associated to \(\lambda''G\) is diagonal with entries \(G_{nn} := \lambda''n(n - 1)(n - 2)\)

Let the infinite matrix \(\hat{H}_{\mu,\lambda}\) be obtained from \(H_{\mu,\lambda}\) by transposing of the elements and the infinite matrix \(\hat{H}_{\mu,\lambda}^\perp\) be obtained from \(H_{\mu,\lambda}\) by transposing and by taking complex conjugates of the elements. Then observe that

i) \(H_{\mu,\lambda}\) is symmetric complex matrix i.e. \(H_{\mu,\lambda} = \hat{H}_{\mu,\lambda}^\perp\).

ii) \(H_{\mu,\lambda} \neq H_{\mu,\lambda}^\perp\) (The symbol \(\perp\) represents Dirac Hermitian conjugation; that is, transpose and complex conjugate.)

iii) As \(H^{(i)} = O(n^\alpha)\) with \(\alpha = \frac{3}{2} > 1\) then the standard perturbation theory
is not applicable.

iv) As $G_{nn} = \lambda''n(n-1)(n-2)$ then $G_{nn} = O(n^3)$ as $n \to \infty$

v) For other properties on the matrix associated to $H_{\mu,\lambda}$, we can consult [13].

vi) from the above observations, we deduce that the operator $H_{\mu,\lambda}G^{-\delta}$ is bounded for all $\delta \geq \frac{1}{2}$.

Now, we present the aim result of this work in following theorem:

**Theorem 3.2**

Let $H_{\lambda''} = \lambda''G + H_{\mu,\lambda}$ the Gribov’s operator acting on Bargmann’s space.

where

$G = a^{\dagger}3a^{\dagger}$ and $H_{\mu,\lambda} = \mu a^{\dagger}a + i\lambda a^{\dagger}(a + a^{\dagger})a$

$[a, a^{\dagger}] = I$ and $(\lambda'', \mu, \lambda)$ are reel parameters and $i^2 = -1$.

Then

\[ || e^{-tH_{\lambda''}} - e^{-t\lambda''G} || \leq t \| e^{-t\lambda''G} H_{\mu,\lambda} \| + \| (\lambda''G)^{\frac{1}{2}} e^{-t\lambda''G} \| O(t^2). \]

**Proof**

a) We begin by computing the trace of the operator:

\[ I_1(t) = \int_t^0 e^{-(t-s)\lambda''G} H_{\mu,\lambda} e^{-s\lambda''G} ds \]

We have

\[ || I_1(t) || = \int_0^t \| e^{-(t-s)\lambda''G} H_{\mu,\lambda} e^{-s\lambda''G} \| ds \]

\[ = \int_0^t \sum_{n=1}^{\infty} < e^{-(t-s)\lambda''G} H_{\mu,\lambda} e^{-s\lambda''G} e_n, e_n > ds \]

\[ = \int_0^t \sum_{n=1}^{\infty} < e^{-t\lambda''G} H_{\mu,\lambda} e^{-s\lambda''G} e_n, e_n > ds \text{ because } e^{s\lambda''G} \text{ is self adjoint} \]
\[ = \int_0^t \sum_{n=1}^{\infty} < e^{-t\lambda''G} H_{\mu,\lambda} e_n, e_n > ds \]
\[ = \int_0^t || e^{-t\lambda''G} H_{\mu,\lambda} ||_1 ds \]
\[ = t || e^{-t\lambda''G} H_{\mu,\lambda} ||_1 \]

Then we deduce that:

\[ || I_1(t) ||_1 = t || e^{-t\lambda''G} H_{\mu,\lambda} ||_1 \] (3.5)

b) We begin to recall the symmetry property of the norm in Carleman class \( C_p \)

The symmetry of the norm in \( C_p \) means that

\[ || K_1 K_2 K_3 ||_p \leq || K_1 || . || K_2 || . || K_3 || . \] (3.6)

for any bounded operators \( K_1 \) and \( K_3 \) and \( K_2 \in C_p \)

Consider the trace of the operator

\[ I_2(t) = \int_\Delta e^{-(t-s)\lambda''G} H_{\mu,\lambda} e^{-(s-s_1)\lambda''G} H_{\mu,\lambda} e^{-s_1 H_{\lambda''}} ds_1 ds \]

and let \( \delta \geq \frac{1}{2} \) such that \( H_{\mu,\lambda} G^{-\delta} \) bounded, then

\[ || I_2(t) ||_1 = \int_\Delta || e^{-(t-s)\lambda''G} H_{\mu,\lambda} G^{-\delta} [G^\delta e^{-(s-s_1)\lambda''G}] H_{\mu,\lambda} G^{-\delta} [G^\delta e^{-s_1 H_{\lambda''}}] ||_1 ds_1 ds \]

As \( t \) can be written as sum of three positif numbers \( t = (t-s) + (s-s_1) + s_1 \). It follows that at least one of them is not less than \( \frac{t}{3} \); suppose, for example, that \( s-s_1 \geq \frac{t}{3} \).

Then

\[ || G^\delta e^{-(s-s_1)\lambda''G} ||_1 \leq || G^\delta e^{-\frac{t}{3} \lambda''G} ||_1 \]

By using the inequality (3.6) we deduce that

\[ || e^{-(t-s)\lambda''G} H_{\mu,\lambda} G^{-\delta} [G^\delta e^{-(s-s_1)\lambda''G}] H_{\mu,\lambda} G^{-\delta} [G^\delta e^{-s_1 H_{\lambda''}}] ||_1 \leq || e^{-(t-s)\lambda''G} H_{\mu,\lambda} G^{-\delta} \]
\[ \cdot || G^\delta e^{-(s-s_1)\lambda''G} ||_1 \cdot || H_{\mu,\lambda} G^{-\delta} [G^\delta e^{-s_1 H_{\lambda''}}] ||_1 \]
\[ \leq \|| H_{\mu,\lambda}G^{-\delta} ||^2 || Ge^{-\frac{t}{2}\lambda''G} ||_1 \]

Then we have
\[ \|| I_2(t) \|_1 \leq || H_{\mu,\lambda}G^{-\delta} ||^2 || Ge^{-\frac{t}{2}\lambda''G} ||_1 \int_{\Delta} dsds \]
\[ \leq || H_{\mu,\lambda}G^{-\delta} ||^2 || Ge^{-\frac{t}{2}\lambda''G} ||_1 t^2 \]

It follows that
\[ \|| I_2(t) \|_1 = || Ge^{-\frac{t}{2}\lambda''G} ||_1 O(t^2) \quad (3.7) \]

and consequently we have
\[ \|| e^{-tH_{\lambda''}} - e^{-t\lambda''G} \|_1 = t \|| e^{-t\lambda''G} H_{\mu,\lambda} \|_1 + \|| (\lambda''G) e^{-\frac{t}{2}\lambda''G} \|_1 O(t^2) \quad (3.8) \]

References

[1] M.T. Aimar, A. Intissar, J.-M. Paoli, Quelques nouvelles propriétés de régularité de l’opérateur de Gribov, Comm. Math. Phys. 172 (1995) 461-466.

[2] M.T. Aimar, A. Intissar, A. Jeribi, On an unconditional basis of generalized eigenvectors of the nonself-adjoint Gribov Operator in Bargmann Space, Journal of Mathematical Analysis and Applications 231, (1999), 588-602.

[3] N. Angelescu, G. Nenciu, M. Bundaru, On the perturbation of Gibbs semigroups, Comm. Math. Phys., 42 (1975).

[4] V. Bargmann, On a Hilbert space of analytic functions and an associated integral transform I, Comm. Pure Appl. Math. 14 (1962) 187-214.

[5] J. Ginibre, C. Gruber, Green functions of anisotropic Heisenberg model, Comm. Math. Phys. 11 (1969).

[6] I.C. Gohberg, M. Krein, Introduction to the theory of linear non-self adjoint operators, 18, Providence R.I: A.M.S., (1969).

[7] V. Gribov, A reggeon diagram technique, Soviet Phys. JETP 26 (1968), no. 2, 414-423.
[8] A. Intissar, On a chaotic weighted Shift $z^p d^{p+1} / dz^{p+1}$ of order $p$ in Bargmann space, Advances in Mathematical Physics, Article ID 471314, (2011).

[9] A. Intissar, Analyse de Scattering d’un opérateur cubique de Heun dans l’espace de Bargmann, Comm. Math. Phys., 199 (1998) 243-256.

[10] A. Intissar, Etude spectrale d’une famille d’opérateurs non-symétriques intervenant dans la théorie des champs de Reggeons, Comm. Math. Phys. 113 (1987) 263-297.

[11] A. Intissar, Approximation of the semigroup generated by the Hamiltonian of Reggeon field theory in Bargmann space, Journal of Mathematical Analysis and Applications, vol. 305, no. 2,(2005), pp. 669-689

[12] A. Intissar, Regularized trace of magic Gribov operator on Bargmann space, Journal of Mathematical Analysis and Applications (submitted), arXiv:1311.1394

[13] A. Intissar, Analyse Fonctionnelle et Théorie Spectrale pour les Opérateurs Compacts Non Auto-Adjoints, Editions Cepadues, Toulouse, (1997).

[14] T. Kato, Perturbation Theory for Linear Operators, Springer, Berlin, (1966).

[15] M.A. Naymark, Linear differential operators. Nauka, M. 528 (1969)

[16] A. Pazy, Semigroups of linear operators and applications to partial differential equations, Springer-Verlag New York, Inc. (1983)

[17] V.A. Zagrebnov, On the families of Gibbs semigroups, Commun. Math. Phys. 76 (1980) 269-276

[18] V.A. Zagrebnov, Perturbations of Gibbs semigroups, Commun. Math. Phys. 120 (1989) 653-664