Rational vs Polynomial Character of $W^n_l$-Algebras *

L. Fehér

Laboratoire de Physique Nucléaire
Université de Montréal
Montréal, Canada H3C 3J7

L. O’Raifeartaigh, P. Ruelle and I. Tsutsui

Dublin Institute for Advanced Studies
10 Burlington Road, Dublin 4, Ireland

Abstract

The constraints proposed recently by Bershadsky to produce $W^n_l$ algebras are a mixture of first and second class constraints and are degenerate. We show that they admit a first-class subsystem from which they can be recovered by gauge-fixing, and that the non-degenerate constraints can be handled by previous methods. The degenerate constraints present a new situation in which the natural primary field basis for the gauge-invariants is rational rather than polynomial. We give an algorithm for constructing the rational basis and converting the base elements to polynomials.

* Revised version of the preprint UdeM-LPN-TH-77/91, DIAS-STP-91-42, entitled "Polynomial and Primary Field Character of $W^n_l$-Algebras"
In recent years it has been found that the $W$-algebras of Zamolodchikov (polynomial
extensions of the Virasoro algebra by primary fields) occur naturally in the context of
linearly constrained Kac-Moody (KM) theory [1-3], and are the canonical symmetry al-
gebras of the associated constrained dynamical systems [3-5]. Most results to date have
been obtained for the cases in which the constraints are first class and non-degenerate,
where non-degenerate means that no element of the associated gauge-algebra commutes
with the constant (non-zero) component of the constrained current [3]. However, a system
of constraints proposed recently by Bershadsky [6], who developed the idea of Polyakov [7]
for the $sl(n, R)$ KM algebra, satisfies neither of the above two conditions and thus raises
the question as to whether the algebras associated with this new system of constraints,
called $W^l_n$-algebras, are of the Zamolodchikov kind. This has already been shown to be
the case for the simplest example $W^2_3$ [7] and further studied for $W^3_4$ [8] and $W^3_4$ [9] but,
as far as we know, there are no results for the general case. The purpose of the present
note is to provide some general results, as follows:

(i) The mixed set of first- and second-class constraints admits a first-class subset $\Gamma$ from
which they can be recovered by gauge-fixing.

(ii) The first-class constraint algebra $\Gamma$ differs from those previously encountered in that
it has a degenerate subalgebra, i.e. a subalgebra $D_0$ such that $[M, D_0] = 0$, where $M$
is the constant component of the current. Only for the algebras $W^2_n$ with odd $n$ is $D_0 = 0$.
In all other cases $\Gamma$ is a semi-direct sum of the form $\Gamma = D_0 \wedge \tilde{\Gamma}$.

(iii) The system has an $sl(2, R)$ symmetry similar to that encountered previously. This
symmetry allows one to handle the non-degenerate part $\tilde{\Gamma}$ of the algebra by the techniques
developed in [1] and [3] and produces a complete set of primary-field $\tilde{\Gamma}$-invariant polyno-
mials $j^{hw}(j_{\tilde{\Gamma}})$ in the $\tilde{\Gamma}$-constrained currents $j_{\tilde{\Gamma}}$, which are highest weights with respect to
$sl(2, R)$. The Poisson bracket algebra of these $j^{hw}$ is a Zamolodchikov algebra, which is
the generalization of those found for the $W^2_4$ and $W^3_4$ in [8,9] and is a special case of the
$\mathcal{W}^G_S$ algebras considered in [3].

(iv) The full set of constraints in [6] include, however, those corresponding to $D_0$. Their
inclusion amounts to eliminating some of the highest weight fields $j^{hw}$, leaving a subset
$j^{hw}_{D_0}$, say. But, because of their degeneracy the $D_0$-constraints introduce a new feature,
namely that the natural basis for the gauge-invariants is a set of rational $\dim W^l_n$ functions
This means that the natural Poisson-bracket algebra for these invariants is a rational, rather than a polynomial, extension of the Virasoro algebra and thus is not a Zamolodchikov algebra in the original sense of the word.

(v) Because the gauge-subalgebra $\mathcal{D}_0$ is scalar with respect to $sl(2, R)$ it is possible to convert the rational base-functions $j^{\text{red}}_{\mathcal{D}_0}$ into polynomials $p^{\text{red}}_{\mathcal{D}_0}$. But (as we show by examples) there exist gauge-invariant polynomials of the $j^{\text{hw}}_{\mathcal{D}_0}$ that are only rational, but not polynomial, functions of the $p^{\text{red}}_{\mathcal{D}_0}$. Thus the basis remains rational and there is no guarantee that the Poisson-bracket algebra of the $p^{\text{red}}_{\mathcal{D}_0}$ closes polynomially.

We first recall the general structure of linear constraints [3]. For this it is convenient to write the KM algebra in the form

$$\{ \langle a, J(x) \rangle, \langle b, J(y) \rangle \} = \langle [a, b], J(y) \rangle \delta(x - y) + \kappa \langle a, b \rangle \delta'(x - y),$$

where $a$ and $b$ are elements of the underlying finite dimensional simple Lie algebra $\mathcal{G}$, and $\kappa$, up to a normalization factor, is the KM level. (For notational simplicity, we henceforth set $\kappa = 1$ except in the formulae of Virasoro centre given in the end.) Letting $\Gamma$ denote any subalgebra of $\mathcal{G}$, and $M$ any element of $\mathcal{G}$, the linear constraints bring the current into the following form:

$$J_{\Gamma}(x) = M + j_{\Gamma}(x), \quad \text{with} \quad j_{\Gamma}(x) \in \Gamma^\perp. \quad (2)$$

A sufficient condition for the constraints (2) to be conformally invariant is the existence of an element $H$ in the Lie algebra $\mathcal{G}$ such that

$$[H, M] = -M, \quad \langle H, \gamma \rangle = 0 \quad \text{and} \quad [H, \gamma] \in \Gamma, \quad \forall \gamma \in \Gamma. \quad (3)$$

Indeed, if there exists such an $H$, one can verify from (1) that the following modified Virasoro density

$$L_H(x) = L_{KM}(x) - \langle H, J'(x) \rangle, \quad \text{where} \quad L_{KM}(x) = \frac{1}{2} \langle J(x), J(x) \rangle, \quad (4)$$

weakly commutes with the constraints in (2). The equation $[H, M] = -M$ implies that $M$ is nilpotent, and every nilpotent element of a real simple Lie algebra has an $sl(2, R)$ subalgebra, $\{M_-, M_0, M_+\}$ say, associated with $M \equiv M_-$. It turns out to be very convenient to use this $sl(2, R)$ algebra, in which the $M_0$ element can play the role of $H$, to describe the constrained system.
If we demand that the constraints in (2) be first class, they must satisfy the following two conditions (in addition to \( \Gamma \) being a subalgebra) \([3]\):

\[
\langle \gamma_i, \gamma_j \rangle = 0 \quad \text{and} \quad \omega_M(\gamma_i, \gamma_j) \equiv \langle M, [\gamma_i, \gamma_j] \rangle = 0, \quad \forall \gamma_i, \gamma_j \in \Gamma.
\] (5)

When the constraints are first class, they generate gauge symmetries on the constraint surface (2) through the KM Poisson bracket. For this reason, the subalgebra \( \Gamma \) will be called a *gauge algebra*. It is natural to look at the gauge invariant functions, namely those functions (weakly) commuting with the constraints. Under certain technical conditions, which include the non-degeneracy condition \([M, \gamma] \neq 0 \) for any \( \gamma \in \Gamma \), it has been shown in \([3]\) that the set of gauge invariant functions of the constrained current has a basis which is differential polynomial in the current components and consists of a Virasoro density and primary fields. Thus, when these technical conditions are satisfied, the gauge invariant functions form a Zamolodchikov W-algebra under the KM Poisson bracket (1).

In the above context the choice of constraints made by Bershadsky for \( sl(n,R) \) may be described as follows. Let \( e_{r,s} \) denote the usual one-entry generators of \( gl(n,R) \) and \( \Delta = \{ e_{r,s} \}_{r<s} \) the upper triangular, maximal nilpotent subalgebra of \( sl(n,R) \). Then the constraints read (\( 1 \leq l \leq n-1 \))

\[
J_{\Delta}(x) = M + j_{\Delta}(x), \quad \text{with} \quad j_{\Delta}(x) \in \Delta^\perp,
\] (6.a)

where

\[
M = e_{l+1,1} + e_{l+2,2} + \cdots + e_{n,n-l}.
\] (6.b)

That is, the entries of the matrix \( M \) are all zero except those on a line parallel to the diagonal and \( l \) steps below it, which are unity, and \( J(x) \) is constrained to be upper triangular apart from a strictly lower triangular constant piece equal to \( M \).

The constraints (6) are not preserved by the standard Virasoro density \( L_{\text{KM}}(x) \), but the modified Virasoro density \( L_H(x) \) in (4) does preserve them with \( H \) given by the diagonal matrix \( H_l = \frac{1}{2l} \sum_{i=1}^{n} (n+1-2i)e_{ii} \). In other words, \( H_l \) gives all the elements on the \( k \)-th slanted line above (or below) the diagonal a grade +\( \frac{k}{l} \) (or −\( \frac{k}{l} \)) and, hence, gives \( M \) a grade −1. This implies (3), that is, \( L_{H_l}(x) \) weakly commutes with all the constraints in (6), and thus whatever the reduction process is, the final reduced system is going to be conformally invariant.
The case \( l = 1 \) is the usual Toda case \([1]\), but for \( l \geq 2 \) there are two features not encountered in previous reductions, as we shall see shortly. First, the constraints are not all first class because, although they satisfy the first condition in (5), they violate the second one. Second, the operator \( \text{ad}_M = [M, \cdot] \) has a non-trivial kernel in \( \Delta \). Accordingly, the differential polynomial gauge fixing algorithm developed in \([3]\) for analyzing reductions by first class, non-degenerate constraints cannot be applied to the present situation. On the other hand, since \( M \) is nilpotent, the \( sl(2,R) \) structure is intact, i.e., there should exist a set of \( sl(2,R) \) generators in which \( M \) is identified with \( M - \). Parametrizing \( n = ml + r \) with \( m = \left\lceil \frac{n}{l} \right\rceil \) and \( 0 \leq r < l \), a convenient choice of the other two generators is

\[
M_0 = \text{diag}(\frac{m}{2}, \ldots, \frac{m-1}{2}, \ldots, \frac{m}{2}, \ldots),
\]

(7)

(the multiplicities, \( r \) and \( l - r \), occur alternately and end with \( r \)) and

\[
M_+ = a_1 e_{1,l+1} + a_2 e_{2,l+2} + \ldots + a_{n-l} e_{n-l,l},
\]

(8)

where the coefficients \( a_i \) in (8) are given by the first \( n - l \) terms in the following series \((k \geq 0)\)

\[
a_{kl+1} = a_{kl+2} = \cdots = a_{kl+r} = (k+1)(m-k),
\]

\[
a_{kl+r+1} = a_{kl+r+2} = \cdots = a_{(k+1)l} = (k+1)(m-k-1).
\]

(9)

The meaning of (7) is that the fundamental of \( sl(n,R) \) branches into \( l \) irreducible \( sl(2,R) \) representations (irreps.), \( r \) of spin \( \frac{m}{2} \) and \( (l - r) \) of spin \( \frac{m-1}{2} \). From this, we get that the adjoint of \( sl(n,R) \) contains \((l^2m + r^2 - 1)\) \( sl(2,R) \) irreps.

From (7), we see that all the generators of \( \Delta \) have an \( \text{ad}_{M_0} \)-eigenvalue greater than or equal to zero. Thus the only elements of \( \Delta \) in the kernel of \( \text{ad}_{M_-} \) are necessarily \( sl(2,R) \) scalars. Let us denote this part of \( \Delta \) by \( \mathcal{D}_0 \):

\[
\mathcal{D}_0 = \{ \sigma \in \Delta : [M_-, \sigma] = 0, [M_0, \sigma] = 0 \}.
\]

(10)

Using the expressions (6) and (7) of \( M_- \) and \( M_0 \), we find that, as a concrete \( n \times n \) matrix, any element \( \sigma \) of \( \mathcal{D}_0 \) takes the following block-diagonal form

\[
\sigma = \text{block-diag}\{\Sigma_0, \sigma_0, \ldots, \Sigma_0, \sigma_0, \Sigma_0\},
\]

(11)
where $\Sigma_0$ and $\sigma_0$ are strictly upper triangular $r \times r$ and $(l-r) \times (l-r)$ matrices, respectively, which repeat themselves alternately. From this it easy to compute that

$$\dim \mathcal{D}_0 = \frac{1}{4}[l(l-2) + (l-2r)^2],$$

(12)

which shows that for $l \geq 2$ the set $\mathcal{D}_0$ is non-empty, except in the special case $l = 2, r = 1$, that is, $W_n^2$ with $n$ odd.

From the previous remarks, we can decompose the subalgebra $\Delta$ according to the grading provided by the eigenvalues of $\text{ad}_{M_0}$:

$$\Delta = \Delta_0 + \mathcal{G}_{\frac{1}{2}} + \mathcal{G}_{\geq 1}, \quad \mathcal{G}_{\geq 1} = \sum_{i=1}^{m} \mathcal{G}_i,$$

(13)

where $\Delta_0$ is the grade 0 subspace in $\Delta$. This equation clearly shows that $\omega_{M_0}(\Delta, \Delta) = 0$ is not satisfied, which means that the constraints (6) defined by $\Delta$ are not first class. Therefore we should separate this system of constraints into first class and second class parts. In fact, it follows from (1) that the first class part, generating gauge transformations on the constraint surface (6), is the one associated to the maximal subalgebra $\mathcal{D} \subset \Delta$ subject to $\omega_{M_0}(\mathcal{D}, \Delta) = 0$. Explicitly, we obtain from (13) that

$$\mathcal{D} = \mathcal{D}_0 + \mathcal{D}_1 + \mathcal{G}_{\geq 1},$$

(14)

where $\mathcal{D}_1$ is that subspace of $\mathcal{G}_1$ for which $\omega_{M_0}(\mathcal{D}_1, \Delta_0) = 0$. The second class part then belongs to the complementary space $\mathcal{C}$ entering into the decomposition $\Delta = \mathcal{D} + \mathcal{C}$, since the restriction of $\omega_{M_0}$ to $\mathcal{C}$ is non-degenerate. By combining (13) and (14), we see that $\mathcal{C}$ naturally decomposes into

$$\mathcal{C} = \mathcal{C}_0 + \mathcal{G}_{\frac{1}{2}} + \mathcal{C}_1,$$

(15)

where $\Delta_0 = \mathcal{D}_0 + \mathcal{C}_0$ and $\mathcal{G}_1 = \mathcal{D}_1 + \mathcal{C}_1$.

Although $\mathcal{D} \subset \Delta$ defines the maximal set of first class constraints which weakly commute with all constraints belonging to $\Delta$, it may be enlarged to a bigger subspace $\Gamma$, which still defines a set of first class constraints, by discarding troublesome elements in $\Delta$ which do not comply with the condition (5). This procedure has already been used in [3], where it was called ‘the method of symplectic halving’, since the general idea is to find a gauge algebra of first class constraints in the form $\Gamma = \mathcal{D} + \mathcal{P}$, where $\mathcal{P}$ is defined in
terms of an appropriate direct sum decomposition ("halving") of the second class part $C$ into symplectically conjugate subspaces, $C = P + Q$.

To apply the above method, we note that, for $M_0$-grading reasons, the 2-form $\omega_{M_-}$ is actually separately non-degenerate on the two subspaces $G_\frac{1}{2}$ and $C_0 + C_1$ of $C$ (15). Thus we can take the $Q$ and $P$ subspaces of $C$ to be $Q = Q_0 + Q_\frac{1}{2}$ and $P = P_\frac{1}{2} + P_1$ where, by definition, $Q_0 = C_0$, $P_1 = C_1$ and $G_\frac{1}{2} = P_\frac{1}{2} + Q_\frac{1}{2}$. What is meant by this decomposition is that $\omega_{M_-}$ vanishes identically on each of the four subspaces $Q_\frac{1}{2}$, $P_\frac{1}{2}$, $Q_0$ and $P_1$. There is obviously a large freedom in choosing the symplectic halving of $G_\frac{1}{2}$, and it should be noted that at this stage we have not yet made a specific choice. Using these decompositions, we define $\Gamma \subset \Delta$ as

$$\Gamma = D + P_\frac{1}{2} + P_1 = D_0 + P_\frac{1}{2} + G_{\geq 1},$$

and we wish to identify this $\Gamma$ as a gauge algebra of first class constraints, the use of which is soon to be explained.

It is clear from the definition (16) that $\Gamma$ satisfies the two conditions in (5), however, this definition does not automatically make $\Gamma$ a Lie algebra, since, for this to be the case, we must have

$$[D_0, P_\frac{1}{2}] \subset P_\frac{1}{2}. $$

We now show that there exists (at least) one halving of $G_\frac{1}{2}$ such that $\Gamma$ is indeed a Lie algebra. For this, we introduce another grading operator $\tilde{H}$ defined as follows. As a diagonal matrix of $sl(n,R)$, $\tilde{H}$ is obtained out of $M_0$ by first adding $\frac{1}{2}$ to its half-integral eigenvalues, and then subtracting a multiple of the unit matrix so as to make the result traceless. Hence in the fundamental of $sl(n,R)$, $\tilde{H}$ is given by

$$\tilde{H} = M_0 - \lambda I_n + \begin{cases} 0 & \text{on tensor irreps,} \\ \frac{1}{2} & \text{on spinor irreps.} \end{cases} $$

The definition (18) makes clear that $\tilde{H}$ is an integral grading, commuting with $M_0$ and such that $[\tilde{H}, M_{\pm}] = \pm M_{\pm}$. In the adjoint of $sl(n,R)$, we then have $\text{ad}_{\tilde{H}} = \text{ad}_{M_0}$ on tensors, and $\text{ad}_{\tilde{H}} = \text{ad}_{M_0} \pm \frac{1}{2}$ on spinors. In the last case, $\text{ad}_{\tilde{H}} - \text{ad}_{M_0}$ equals $\frac{1}{2}$ as many times as $-\frac{1}{2}$. In particular, exactly half of $G_\frac{1}{2}$ has an $\tilde{H}$-grade equal to 0, the $\tilde{H}$-grade of the other half being 1. We then choose the following symplectic halving of $G_\frac{1}{2}$:

$$Q_\frac{1}{2} \equiv G_\frac{1}{2} \cap G_0^R, \quad \text{and} \quad P_\frac{1}{2} \equiv G_\frac{1}{2} \cap G_1^R, $$

7
Figure 1. Various processes reaching the same reduced phase space. Single-lined arrows represent the imposition of constraints, and double-lined arrows represent gauge fixings.

where the superscript $\tilde{H}$ means that the grades are with respect to $\tilde{H}$. Since the elements of $\mathcal{D}_0$, being tensor states, have an $\tilde{H}$-grade equal to 0 (same value as their $M_0$-grade), this choice of $\mathcal{P}_{\frac{1}{2}}$ clearly guarantees (17), that is, $\Gamma$ is a Lie algebra.

Hence the subalgebra $\Gamma \subset \Delta$ given by (16) and (19) satisfies all the conditions for defining first class constraints by means of eq.(2). Furthermore, it is easily seen that the constraints belonging to $\mathcal{Q}$, which are originally in $\Delta$ but are missing from $\Gamma$, can be recovered by regarding them as (partial) gauge fixing conditions associated with the piece $\mathcal{P}$ of $\Gamma$. From this observation one concludes that the reduced phase space, obtained by imposing the first class constraints (2) and quotienting by the $\Gamma$-gauge-transformations, is identical with the reduced phase space obtained by imposing Bershadsky’s constraints (6) and quotienting by the $\mathcal{D}$-gauge-transformations:

$$\text{reduced phase space} = \{ J_\Gamma = M_- + j_\Gamma(x) \} / \{ \Gamma\text{-KM transformations} \}$$

$$= \{ J_\Delta = M_- + j_\Delta(x) \} / \{ \mathcal{D}\text{-KM transformations} \}. \quad (20)$$

Correspondingly, the following elementary counting gives the dimension of the $W^l_n$-algebras, i.e., the number of the gauge invariant degrees of freedom:

$$\dim W^l_n = \dim \mathcal{G} - \dim \Gamma = \dim \mathcal{G} - \dim \Delta - \dim \mathcal{D}$$

$$= l(n + r + 1) - (l^2 + r^2 + 1). \quad (21)$$

As usual, the reduced phase space may be regarded as the space of gauge-invariant functions of the constrained currents and a $\dim W^l_n$-dimensional basis for the gauge-invariants may be obtained by gauge-fixing. However, the space $\Gamma$ of the first class constraints (16) contains a degenerate part $\mathcal{D}_0$, which means that it is impossible to perform
the gauge fixing in the previous manner [3]. Thus it is natural to first consider the non-degenerate part of $\Gamma$,

$$\tilde{\Gamma} = \mathcal{P}_{\frac{1}{2}} + \mathcal{G}_{\geq 1}, \quad (22)$$

by the usual procedure of gauge fixing, and then find a convenient way of gauge fixing for the degenerate part. In doing this, we are actually taking yet another path leading to the reduced phase space, the lower path in Fig.1. This is again allowed because, as we shall see below, the space of current components eliminated by the gauge fixing for the non-degenerate part $\tilde{\Gamma}$ can be chosen to be independent of the degenerate part $D_0$.

Now, imposing the constraints belonging to the non-degenerate part $\tilde{\Gamma}$,

$$J_{\tilde{\Gamma}}(x) = M_- + j_{\tilde{\Gamma}}(x), \quad \text{with} \quad j_{\tilde{\Gamma}}(x) \in \tilde{\Gamma}^\perp, \quad (23)$$

one sees from (22) that the space in which the current lies can be explicitly given by $\tilde{\Gamma}^\perp = \mathcal{G}_{\geq 0} + Q_{-\frac{1}{2}}$ with $Q_{-\frac{1}{2}} = [M_-, \mathcal{P}_{\frac{1}{2}}]$ being the subspace of $\mathcal{G}_{-\frac{1}{2}}$ orthogonal to $\mathcal{P}_{\frac{1}{2}}$. One then finds that $\tilde{\Gamma}^\perp$ can also be written in the form

$$\tilde{\Gamma}^\perp = [M_-, \tilde{\Gamma}] + \ker(\text{ad}_{M_-}). \quad (24)$$

The essential ingredient in fixing the current in the usual form (called ‘Drinfeld-Sokolov gauge’ in [3]) is the following. First, let us observe the gauge transformation,

$$j_{\tilde{\Gamma}}(x) \rightarrow e^{a(x)}(j_{\tilde{\Gamma}}(x) + M_-)e^{-a(x)} + (e^{a(x)})'e^{-a(x)} - M_- = j_{\tilde{\Gamma}}(x) + [a(x), M_-] + [a(x), j_{\tilde{\Gamma}}(x)] + a'(x) + \cdots, \quad (25)$$

where $a(x) \in \tilde{\Gamma}$ is a local parameter which may be decomposed into its grades, $a(x) = \sum_{i=1}^{m} a_i(x)$. Concentrating in particular on the lowest grade component of the gauge transformed current, i.e., on the component in the space $Q_{-\frac{1}{2}}$, we find that the only contribution to the transformation comes from the term $[a_{\frac{1}{2}}(x), M_-]$. It then follows that as far as this grade is concerned we can put the corresponding current component to zero by a suitable choice of the gauge parameter $a_{\frac{1}{2}}(x)$. By using the non-degeneracy, $\ker(\text{ad}_{M_-}) \cap \tilde{\Gamma} = \{0\}$, and the graded structure (22), it is also easy to see that, by going up to spaces of higher grades iteratively, all the current components belonging to the space $[M_-, \tilde{\Gamma}]$ can be set to zero by choosing the $a_i(x)$’s appropriately. As a result, the current components which survive this gauge fixing lie in a complementary space to $[M_-, \tilde{\Gamma}]$, which,
in view of the decomposition (24), we can take to be \( \text{Ker}(\text{ad}_{M^+}) \), the space of the \( sl(2,R) \)
highest weight states in the adjoint of \( sl(n,R) \). Clearly, the gauge parameter determined
in the above procedure is a differential polynomial in the original current \( j_{\tilde{\Gamma}}(x) \). This in
turn implies that the components \( j^{\text{hw}}(x) \) of the gauge transformed, highest weight, current
are also (gauge invariant) differential polynomials \( j^{\text{hw}}(j_{\tilde{\Gamma}}) \) when expressed in the original
current components.

As we have noted earlier, the conformal invariance of our system is guaranteed by
choosing \( H_l \) for the \( H \) to define the modified Virasoro density (4). However, such an \( H \)
is by no means unique, as we have already encountered another example \( \tilde{H} \) in (18). A
natural choice in dealing with the \( sl(2,R) \) embedding is to take \( M_0 \) itself to be the \( H \) (which
evidently satisfies (3)) and use \( L_{M_0}(x) \) to specify the conformal structure of the theory.
The virtue of this choice lies in the fact that, under the conformal transformation generated
by \( L_{M_0}(x) \), all the surviving current components, \( j^{\text{hw}}(x) \), except the Virasoro component
turn out to be primary fields. One can see this by explicitly computing the response of
the current components under the infinitesimal conformal transformation \( \delta x = -f(x) \),

\[
\delta j^{\text{hw}}(x) = f(x)(j^{\text{hw}}(x))' + f'(x)(j^{\text{hw}}(x) + [M_0, j^{\text{hw}}(x)]) - \frac{1}{2}f''(x)M_+,
\]

which shows that the spin \( s \) component of the highest weight current has conformal weight
\( s + 1 \), except for the \( M_+ \)-component. This component turns out to be, up to \( sl(2,R) \)
scalars, \( L_{M_0}(x) \) itself once reduced to the highest weight gauge.

Since for every \( \tilde{\Gamma} \)-gauge-invariant polynomial \( P(j_{\tilde{\Gamma}}) \) the invariance implies \( P(j_{\tilde{\Gamma}}) = P(j^{\text{hw}}(j_{\tilde{\Gamma}})) \) it is clear that the \( j^{\text{hw}}(j_{\tilde{\Gamma}}) \) constitute a basis for the \( \tilde{\Gamma} \)-invariant polynomials.
It follows that the Poisson bracket algebra of the \( j^{\text{hw}}(j_{\tilde{\Gamma}}) \) closes and since the \( j^{\text{hw}}(j_{\tilde{\Gamma}}) \) are
primary and include a Virasoro density, this is a Zamolodchikov algebra. In fact it is the
Zamolodchikov algebra considered for \( W_2^2 \) and \( W_3^3 \) in [8,9] and is a special case of the \( W_\mathcal{G} \)
algebras considered in [3].

The original constraints proposed by Bershadsky include, however, the constraints
Corresponding to the subalgebra \( \mathcal{D}_0 \) and we now have to consider the situation when these
constraints are imposed. It is not difficult to see that when these constraints are imposed
the current $j_{\mathcal{D}_0}^{\text{hw}}(x)$ takes the form

$$j_{\mathcal{D}_0}^{\text{hw}}(x) = \begin{pmatrix}
K_0 & K_{\frac{1}{2}} & K_1 & K_{\frac{3}{2}} & K_2 & \ldots & K_{m-\frac{1}{2}} & K_m \\
0 & k_0 & k_{\frac{1}{2}} & k_1 & k_{\frac{3}{2}} & \ldots & k_{m-1} & k_{m-\frac{1}{2}} \\
0 & 0 & K_0 & * & * & \ldots & * & * \\
0 & 0 & 0 & k_0 & * & \ldots & * & * \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & 0 & \ldots & k_0 & * \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & K_0
\end{pmatrix} \tag{27}$$

where the subscripts denote the spins with respect to $sl(2,R)$, the $r \times r$ and $(l - r) \times (l - r)$ scalar matrices $K_0(x)$ and $k_0(x)$ are upper triangular (including the diagonal) and are identical copies, and the blocks denoted with stars $*$ are proportional to the block-matrices $K_i(x)$ and $k_i(x)$ for $i \neq 0$, due to the highest weight condition $[M_+, j_{\mathcal{D}_0}^{\text{hw}}(x)] = 0$. The matrix (27) being in $sl(n,R)$, the blocks $K_0$ and $k_0$ are subjected to the condition $(m + 1) \text{Tr} K_0 + m \text{Tr} k_0 = 0$. All the entries in (27) are, of course, gauge invariant with respect to the non-degenerate (non-scalar) part $\tilde{\Gamma}$ of the gauge algebra. Note that the Virasoro operator $L_{M_0}(x)$ is now linear in the $M_+$-component and quadratic in the diagonal scalar entries of $j_{\mathcal{D}_0}^{\text{hw}}(x)$. In (27) all the $\Delta$ constraints have been recovered, some of them as first class constraints, the others as $\tilde{\Gamma}$-gauge-fixing conditions.

The problem now is to find a basis for the functions of the currents (27) which are gauge-invariant with respect to the residual gauge-algebra $\mathcal{D}_0$, i.e. with respect to the gauge-transformations $j_{\mathcal{D}_0}^{\text{hw}} \rightarrow S(x)(j_{\mathcal{D}_0}^{\text{hw}} - \partial)S^{-1}(x)$ where $S(x)$ is the gauge-group generated by $\mathcal{D}_0$. Note that, because the elements of $\mathcal{D}_0$ are scalars, the $\mathcal{D}_0$ gauge-transformations leave invariant both the set of highest weights and the set of non-highest weights and thus preserve the form (27). Furthermore they do not mix highest weights of different grades.

To find a basis for the set of gauge-invariant functions we follow the usual procedure of gauge-fixing. The problem is that, because $M_-$ does not appear in the above $\mathcal{D}_0$ gauge-transformation, the gauge-parameters $\alpha$ cannot be determined as polynomial functions of the current. The best one can do is to note that since $\mathcal{D}_0$ is nilpotent there are current components that transform \textit{linearly} in the $\alpha$’s, and use these to obtain the $\alpha$’s as simple fractions of current components. We shall call such gauges \textit{fractional} gauges. It
we proceed recursively by examining successively the first column of
the triangular matrix. It can be written as

\[ S_0 = S_0^{(2)} \cdot S_0^{(3)} \cdots S_0^{(r)}, \quad \text{with } S_0^{(i)} = \begin{pmatrix} 1 & 0 & \ldots & 0 & a_{1,i} & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & a_{2,i} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & a_{i-1,i} & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}. \] (29)

The elements of \( S_0^{(i)} \) form an \((i-1)\)-dimensional abelian subgroup of \( S_0 \) and is an invariant
subgroup of \( S_0^{(2)} \cdot S_0^{(3)} \cdots S_0^{(i)} \) \((2 \leq i \leq r)\). In order to fix the gauge parameters \( a_{i,j}(x) \),
\( i < j \), contained in \( S_0 \), we consider the transformation of the block \( K_0 \) and the first column
of \( K_0 \frac{1}{2} \) (which always exists if \( S_0 \) does), which we parametrize as

\[ (K_0 \mid K_0 \frac{1}{2}) = \begin{pmatrix} * & * & \ldots & * & \psi_1 & \chi_1 & \phi_1 & * & \ldots & * \\ 0 & * & \ldots & * & \psi_2 & \chi_2 & \phi_2 & * & \ldots & * \\ 0 & 0 & \ldots & * & \psi_3 & \chi_3 & \phi_3 & * & \ldots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & * & \psi_{r-2} & \chi_{r-2} & \phi_{r-2} & * & \ldots & * \\ 0 & 0 & \ldots & 0 & \psi_{r-1} & \chi_{r-1} & \phi_{r-1} & * & \ldots & * \\ 0 & 0 & \ldots & 0 & 0 & \chi_r & \phi_r & * & \ldots & * \end{pmatrix}. \] (30)

We proceed recursively by examining successively the first column of \( K_0 \frac{1}{2} \), the last column
of \( K_0 \), the second-last column of \( K_0 \), and so on, thereby fixing the parameters of \( S_0^{(r)} \), then
those of \( S_0^{(r-1)} \), \( S_0^{(r-2)} \), and so forth. By the \( S_0^{(r)} \) gauge transformation, the first column
of \( K_0 \frac{1}{2} \) becomes

\[ \phi_i \rightarrow \phi_i + a_{i,r} \phi_r, \quad \phi_r \rightarrow \phi_r, \] (31)
for $1 \leq i \leq r - 1$. As a consequence, the choice

$$a_{i,r} = -\frac{\phi_i}{\phi_r}, \quad 1 \leq i \leq r - 1$$

(32)

makes the components $\phi_i$, $1 \leq i \leq r - 1$, vanish. The other components of the current also undergo the $S_0^{(r)}$ transformations, and since the parameters (32) are rational, so will be the components of the $S_0^{(r)}$ gauge-fixed current. In general, they will contain a dependence in $a_{i,r}$ and the first derivatives $a'_{i,r}$. Although not polynomial, we note that these components are still primary fields. Indeed, from (32), all the parameters $a_{i,r}$ are conformal scalars, because all the $\phi_i$’s are conformal primary fields with the same conformal weight, namely, $\frac{3}{2}$. Therefore, as long as the non-derivative terms in the transformations (28) are concerned, the entries of the $K$ and $k$ blocks retain their primary field character. As to the derivative terms, only the entries of $K_0$, which are $sl(2,R)$ scalars and thus conformal vectors, can possibly pick up a linear dependence in $a'_{i,r}$. But the parameters $a_{i,r}$ being conformal scalars, their first derivatives are conformal vectors, and so the entries of $K_0$ also remain primary fields. (In fact, a gauge transformation always preserves the conformal weights if the charge of the transformation is conformally invariant. In the present case this is equivalent to the condition that the gauge parameter be a conformal scalar.) So at this stage the components of the $S_0^{(r)}$ gauge-fixed current are all primary fields (except the $M_+$-component) with their conformal weight still given by their $sl(2,R)$ grade plus one. However, they are rational rather than polynomial, their rational character coming from the denominator of the parameters $a_{i,r}$, which is the gauge invariant current component $\phi_r$.

We now go to the second step, and look at the $S_0^{(r-1)}$ transformation of the $S_0^{(r)}$ gauge fixed components of the last column of $K_0$. These, of course, have changed under the $S_0^{(r)}$ transformation, for example, $\chi_{r-1}$ has become

$$\tilde{\chi}_{r-1} = \chi_{r-1} + (\psi_{r-1} - \chi_r)\frac{\phi_{r-1}}{\phi_r} - \left(\frac{\phi_{r-1}}{\phi_r}\right)'.$$  

(33)

For brevity we denote the (gauge-invariant) $S_0^{(r)}$ transformed entries in the column by $(\tilde{\chi}_1, \tilde{\chi}_2, \ldots, \tilde{\chi}_r)^t$. The $S_0^{(r-1)}$ transformation yields

$$\tilde{\chi}_i \mapsto \tilde{\chi}_i + a_{i,r-1} \tilde{\chi}_{r-1}, \quad \tilde{\chi}_{r-1} \mapsto \tilde{\chi}_{r-1}, \quad \tilde{\chi}_r \mapsto \tilde{\chi}_r,$$  

(34)

for $1 \leq i \leq r - 2$. One thus finds that the first $(r - 2)$ components can be gauged away by choosing

$$a_{i,r-1} = -\frac{\tilde{\chi}_i}{\tilde{\chi}_{r-1}}, \quad 1 \leq i \leq r - 2.$$  

(35)
As in the first step, the parameters \( a_{i,r-1} \) given by (35) are conformal scalars and the denominator \( \tilde{\chi}_{r-1} \) is a gauge invariant primary field. Thus we obtain a basis of primary field rational functions which are invariant under both \( S_0^{(r)} \) and \( S_0^{(r-1)} \) transformations.

We proceed in the same way for the rest of the columns of \( K_0 \). Namely, we use \( S_0^{(r-2)} \) to remove the first \((r-3)\) entries in the \((r-1)\)-th column of \( K_0 \), then use \( S_0^{(r-3)} \) to remove the first \((r-4)\) entries in the \((r-2)\)-th column of \( K_0 \), and so on. After the last step, the fully constrained and \( S_0 \) gauge-fixed current blocks \( K_0 \) and \( K_\frac{1}{2} \) take the following form

\[
\begin{pmatrix}
K_0 & K_\frac{1}{2}
\end{pmatrix} =
\begin{pmatrix}
\times \otimes 0 ... 0 0 0 & 0 * * ... * \\
0 \times \otimes ... 0 0 0 & 0 * * ... * \\
0 0 \times ... 0 0 0 & 0 * * ... * \\
\vdots & \vdots \\
0 0 0 \times 0 \otimes 0 & 0 * * ... * \\
0 0 0 0 0 \times \otimes & 0 * * ... *
\end{pmatrix}
\]

(36)

Here the symbols \( \times \) in the diagonal line in \( K_0 \), as well as in the lower-left corner of \( K_\frac{1}{2} \) (which is \( \phi_r \)), stand for the components which are \( D_0 \) gauge invariant from the beginning (see eq.(28)). The rest of the components denoted by \( * \) are still to be made gauge invariant in the course of the gauge fixing of \( s_0 \) transformations. Exactly the same procedure can be used to fix the parameters of \( s_0 \). In the first step, we use the first column of \( k_\frac{1}{2} \) (or \( k_1 \) if \( r = 0 \) when \( k_\frac{1}{2} \) does not exist) and proceed to the last column of \( k_0 \), and so on. At the end of the procedure, when all the gauge parameters are fixed, we are left with a set of \( \text{dim} \ W_n \) primary-field gauge-invariant rational functions \( j^{\text{red}}(j_{D_0}^{\text{hw}}) \). Now, for every \( \Gamma \)-invariant rational function \( R(j_r) \) the gauge-invariance with respect to \( \tilde{\Gamma} \) and \( D_0 \) respectively imply that \( R(j_r) = R(j_{D_0}^{\text{hw}}(j_{\gamma})) = R(j_{D_0}^{\text{hw}}(j_{\gamma})) \), which shows that the \( j^{\text{red}} \) constitute a basis for the \( \Gamma \)-invariant rational functions. From this it follows that the Poisson bracket algebra of the \( j^{\text{red}} \) closes rationally and, since the \( j^{\text{red}} \) are primary, this algebra is a rational extension of the Virasoro algebra by primary fields. Thus it is not quite a Zamolodchikov algebra in the original sense of the word.

Although the basis is only rational one can improve the situation a little by using the fact that the denominators in the rational basis are separately gauge-invariant and are primary fields. This permits one to convert the basis of rational functions into a basis of polynomials without losing the primary field character, as follows: In the first step given by (32), some of the current components (such as \( \tilde{\chi}_{r-1} \) in (33)) become rational functions with either \( \phi_r \) or \( \phi_r^2 \) in the denominator, so to convert them to polynomials we simply
multiply them by suitable powers of $\phi_r$. The field $\phi_r$ being itself a primary field, the multiplication $\phi_r$ does not spoil the primary field property, but simply shifts the values of the conformal weights. Thus, for example, $\tilde{\chi}_{r-1}$ becomes

$$\tilde{\chi}_{r-1} \phi_r^2 = \chi_{r-1} \phi_r^2 + (\psi_{r-1} - \chi_r) \phi_{r-1} \phi_r + \phi_{r-1} \phi_r', \quad (37)$$

which is a gauge-invariant polynomial of conformal weight 4. In the next step the denominators are either $\tilde{\chi}_{r-1}$ or $\tilde{\chi}_r^2$ and we make the $j^{\text{red}}$ polynomial by multiplying across by suitable powers of $\tilde{\chi}_{r-1}$. Again this field is primary and does not spoil the primary field property of the current components. The later steps are taken in a similar manner. The denominators $\phi_r$, $\tilde{\chi}_{r-1}$ etc. used in the conversion are those marked by the symbol $\otimes$ on the slanted line just above the diagonal line in the matrix (36).

The problem is that the basis of polynomials so constructed is still only a rational basis, in the sense that there are gauge-invariant primary-field polynomials which cannot be expressed in terms of them using only polynomial coefficients. We illustrate this by considering the $W_{2m}^2$ algebras, for which $D_0$ is one-dimensional and only the blocks $k_i$ for integer $i$ in the matrix (27) survive. If we write

$$k_0 = x\sigma_3 + y\sigma_+ \quad \text{and} \quad k_i = d_i + c_i\sigma_+ + a_i\sigma_3 + b_i\sigma_+ , \quad 1 \leq i \leq m - 1 , \quad (38)$$

where the $\sigma$'s are the Pauli matrices, we see at once that $\{x, c_i, d_i\}$ are $D_0$-invariants and the transformation properties of the remaining fields are

$$y \rightarrow y - 2\alpha x + \alpha', \quad a_i \rightarrow a_i + \alpha c_i , \quad b_i \rightarrow b_i - 2\alpha a_i - \alpha^2 c_i , \quad (39)$$

where $\alpha$ is the gauge-parameter. Thus if we gauge-fix by choosing $\alpha = -a/c$ for the $\{a, c\}$ out of some particular block $k$, we obtain the rational basis

$$x, \quad y + 2\frac{ax}{c} - \left(\frac{a}{c}\right)', \quad c_i, \quad d_i, \quad a_i - \frac{a}{c} c_i , \quad b_i + 2\frac{aa_i}{c} - \left(\frac{a}{c}\right)^2 c_i . \quad (40)$$

On conversion to polynomials these become $\{x, c_i, d_i\}$ and

$$Y = c^2 y + 2acx + (c'a - a'c) , \quad A_i = ca_i - ac_i, \quad B_i = c^2 b_i + 2aca_i - a^2 c_i. \quad (41)$$

But (41) is by no means a basis in which any gauge-invariant polynomial can be expanded using only polynomial coefficients since, for example,

$$\text{Tr} \ k_i k_j = 2a_i a_j + b_i c_j + c_i b_j \quad \text{and} \quad \text{Tr} \ \sigma_+ k_i k_j = c_i a_j - c_j a_i . \quad (42)$$
for \( k_i, k_j \neq k \) is a set of \((m - 2)^2\) gauge-invariant polynomials which, since they can also be written as

\[
\text{Tr} k_i k_j = \frac{1}{c^2}(2A_iA_j + B_i c_j + c_i B_j) \quad \text{and} \quad \text{Tr} \sigma_+ k_i k_j = \frac{1}{c}(c_i A_j - c_j A_i),
\]

cannot be expanded in the basis (41) using only polynomial coefficients. Thus the Poisson-bracket algebra of the polynomial basis (41) is not guaranteed to close polynomially.

Let us finally consider the Virasoro centre. The ambiguity of the Virasoro density implies that the conformal structure of the theory is not uniquely determined. In particular, the Virasoro centre appearing in the (quantum) \( W_n^l \)-algebras may depend on the conformal structure, which we now examine. For this, let us consider the BRST system constructed from the constrained KM theory in which a set of ghosts are introduced associated with the gauge algebra \( \Gamma \) [3]. Then in the BRST system the Virasoro density acquires the ghost part, \( L_{\text{tot}}(x) = L_{M_0}(x) + L_{\text{ghost}}(x) \), and hence the total Virasoro centre comprises the three contributions

\[
c_{\text{tot}} = c_{\text{KM}} + c_{\text{mod}} + c_{\text{ghost}},
\]

where \( c_{\text{KM}} \) is the usual centre coming from \( L_{\text{KM}}(x) \), and \( c_{\text{mod}} \) from the modification in the Virasoro density. If we adopt \( L_{M_0}(x) \) in the present \( W_n^l \) case, they read

\[
c_{\text{KM}} = \frac{(n^2 - 1)k}{k + n}, \quad c_{\text{mod}} = -12k\langle M_0, M_0 \rangle = -km(m + 1)[3n - (2m + 1)l],
\]

with \( k \) being the KM level. The ghost centre \( c_{\text{ghost}} \) arising from \( L_{\text{ghost}}(x) \) is computed by using the standard formula, \( c(i) = -2[1 + 6i(i - 1)] \), which gives the centre from a pair of ghosts associated with a grade \( i \) element in \( \Gamma \). Taking into account the multiplicities of the grades in \( \Gamma \), one finds

\[
c_{\text{ghost}} = c(0) \dim D_0 + c(\frac{1}{2}) \dim P_\frac{1}{2} + \sum_{i=1}^m c(i) \dim G_i \\
= -(m^3 + 4m^2 + 3m + 1)l^2 + [n(2m^3 + 3m^2 + 6m + 2) + 1]l \\
- n^2(3m^2 + 2).
\]

As we have expected, the result does not agree with the one obtained by Bershadsky [6] who adopted \( L_{H_i}(x) \) in computing the Virasoro centre for \( W_n^2 \).

In this paper we have shown that the constraints introduced by Bershadsky to define \( W_n^l \)-algebras are equivalent to a set of first-class constraints with gauge-algebra of the form
\[ \Gamma = D_0 \land \tilde{\Gamma}, \] where \( D_0 \) is degenerate and \( \tilde{\Gamma} \) is non-degenerate. If \( D_0 \) is zero (as happens only for \( W^2_n \) with \( n \) odd) or is simply omitted, as in [8,9], the natural basis for the gauge-invariant functions of the constrained currents is a set of polynomial primary fields and these generate a Zamolodchikov algebra which is a special case of the \( W^G_S \) algebras discussed in [3]. If \( D_0 \) is included, so as to obtain the full set of Bershadsky constraints, then the natural basis for the gauge-invariant functions is a set of primary fields, which are rational functions of the constrained currents, and we have given an algorithm for their construction. The associated Poisson-bracket algebra is a rational, rather than polynomial, extension of the Virasoro algebra by primary fields, and is thus not quite a Zamolodchikov algebra. There is a natural mechanism for replacing the rational primary-field basis by a polynomial primary-field basis, but not every gauge-invariant polynomial can be polynomially-expanded in the new basis (i.e. expanded using polynomial coefficients) and thus the associated Poisson-bracket algebra is not guaranteed to close polynomially. One may, of course, extend the polynomial basis to a (much larger) one, in terms of which the gauge-invariant polynomials can be polynomially-expanded. But then the expansion is not unique, and it is an open question as to whether there exists any \( \dim W^l_n \) subset whose Poisson-brackets close to form a Zamolodchikov algebra. This question is technically difficult to answer because the current components \( j^{hw}_{D_0}(j_{\tilde{\Gamma}}) \) used for the \( D_0 \) reduction satisfy a \( W \) algebra (a \( W^G_S \) algebra) rather than a Kac-Moody algebra.
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