Tolman-Oppenheimer-Volkoff equations in non-local $f(R)$ gravity

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Non-local $f(R)$ gravity was proposed as a powerful alternative to general relativity (GR). This theory has potentially adverse implications for infrared (IR) regime as well as ultraviolet (UV) early epochs. However, there are a lot of powerful features, making it really user-friendly. A scalar-tensor frame comprising two auxiliary scalar fields, used to reduce complex action. However this is not the case for the modification complex which plays a distinct role in modified theories for gravity. In this work, we study the dynamics of a static, spherically symmetric object. The interior region of spacetime had rapidly filled the perfect fluid. However, it is possible to derive a physically based model which relates interior metric to non-local $f(R)$. The Tolman-Oppenheimer-Volkoff (TOV) equations would be a set of first order differential equations from which we can deduce all mathematical (physical) truths and derive all dynamical objects. This set of dynamical equations govern pressure $p$, density $\rho$, mass $m$ and auxiliary fields ($\psi, \xi$). The full conditional solutions are evaluated and inverted numerically to obtain exact forms of the compact stars Her X-1, SAX J 1808.4-3658 and 4U 1820-30 for non-local Starobinsky model of $f(\Box^{-1} R) = \Box^{-1} R + \alpha (\Box^{-1} R)^2$. The program solves the differential equations numerically using adaptive Gaussian quadrature. An ascription of correctness is supposed to be an empirical equation of state $\frac{P}{\rho} = a(1 - e^{-\frac{b}{\rho}})$ for star which is informative in so far as it excludes an alternative non-local approach to compact star formation.
model is most suited for astrophysical observation.

Keywords: Higher-dimensional gravity and other theories of gravity; neutron stars; thermodynamic processes; conduction; convection; equations of state.

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1. Introduction

Modern cosmology is based on the relativistic description of large scale Universe. The situation is made worse when the Universe contains expansion and so accelerating. We are restricted to obey different observational data in favor of an accelerating universe at very large scales.\(^1\)\(^2\)\(^3\) Acclerating expansion is largely responsible for the fact that each galaxy in large scale are expanding the capacity to expand themselves. They provide evidence of the accelerating expansion of Universe and hence for the existence of the mysterious dark energy which drives this process. There is no doubt that these data are modifying the classical description of gravitation, general relativity (GR). We will engage other theories in modifying our GR to finally understand this epoch. Universe may be accelerating due to matter and geometry of the full action. Two contrasting methods for accelerating scenario have been identified. At the first candidate, an exotic matter fluid was used in the full action of theory to realize a variety of accelerating and decelerating phases. They had to prove that GR respond to cosmological needs in a world of accelerating expansion that we observed earlier. At the next stage, a geometrical modification was used in the action to study a variety of accelerating and decelerating epochs.\(^4\)\(^5\)\(^6\)\(^7\)\(^8\)\(^9\) As individuals, there are things we can perform to improve the situation, firstly as geometrical modifications and secondly as fluids.

Firstly there is the belief that the Buchdahl was the founder of the \(f(R)\) modified gravity.\(^1\)\(^0\) A model is proposed to explore the viable range of parameters of cosmological background due to replacing the Ricci scalar \(R\) by an arbitrary function \(f(R)\). Proposed action by nonlinear higher terms should be applied to the large scale. Among the proposed models it has been considered that Universe expansion caused in the late time where low curvature corrections were formed as space time were filled. Similar difficulties may apply to the higher order corrections as Gauss-Bonnet (GB) term, \(G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}\). Indeed, in four dimensional Einstein-Hilbert action, these GB terms have no contribution if they appear in non-minimally coupled form. The reason backs to the topology and topological invariants. Coupling of GB term to matter fields provides more dynamical features than before. In comparison to \(f(R)\), the modified GB gravity was proposed to be the \(f(G)\) gravity as model for dark energy in\(^1\)\(^2\)\(^3\) Several interesting cosmological features of this type modified gravity were studied.\(^1\)\(^4\)\(^1\)\(^5\)\(^1\)\(^6\) One of the most important motivations of GB gravity is that it arises naturally from string theory. Also, it appears as a non-minimally coupling term to higher order scalar fields.\(^1\)\(^7\) The list of possible modifications is not limited upto the above mentioned models. The non-local models are also so interesting theories and enough popular as modified
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gravities. This last case, is studied in our paper. We'll present a short review of the subject in Sec. (2).

These mass densities that are localized then immediately collapse, releasing thermodynamics that can cause a hydrodynamic pressure impose on the mass distribution. Following the collapse of the matter distribution, mass transformed itself into a compact star. The compact star has a finite size and a huge amount of mass. It was an open competition between different massive objects that enabled the neutron star carriers the opportunity to exist in the space time. The neutron star has very strong surface gravity and thermodynamics. We can detect these massive objects by looking at the Doppler shift in spectral lines emitted by atoms in the surface. The major elements mass $M \sim 1.4M_\odot$ and radius $R \sim 10Km$ of inputs are used to control the whole dynamics. Specific surface gravity known as the relative surface gravity of the neutron star also may be studied. Not much gravity on the earth, you can find more on the compact star $\kappa = 2 \times 10^{11}$! The field of force associated with compact star have both electric and magnetic components and contains a definite amount of electromagnetic energy. This means there is a greater difference between individual plants which makes the species and neutron stars. For some basic parameters, for example emission above-ground level, electromagnetic effects were stronger than thermodynamic effects.

It was shown that the thermodynamic parameters, pressure $p$, energy density $\rho$, the mass function $M$ and radius $R$ are related according to a set of "state" equations, called as TOV equations in GR. It defines a set of first order differential equations for a spherically symmetric metric which is filled with the perfect fluid with pressure and energy density. One main goal of our paper is to derive explicitly, the forms of TOV equations for non-local $f(R)$ gravity. Recently TOV equations and dynamics of stars have been investigated for different types of modified gravity models from $f(R)$, $f(G)$ and $f(T)$ ($T$ is torsion) numerically and in a non-perturbative scheme.

Our plan in this letter is the following scheme: In Sec. (2) we present non-local $f(R)$ gravity as an alternative theory for gravity. In Sec. (3) we derive equations of motion for a spherically symmetric star. In Sec. (4) we pass to the dimensionless parameters and we redefine all functions to obtain TOV equations. In Sec. (5) we study an isotropic model of compact star using astrophysical data. We conclude and summarize in Sec. (6). Some preliminary formulae are presented in Sec. (??).

2. Non-local $f(R)$ gravity

Non-local corrections to the Einstein-Hilbert action proposed an attempt to obtain a "healthy" version of GR with added quantum loop corrections. The simple problem was how to explain the current acceleration expansion of the Universe and to get the large numbers from inverse differential operator(s). Indeed, this modification is referred as an IR non-local modification of General Relativity. After the original one, another model proposed as non-local $F(R)$ gravity. It
was essentially a viable IR modification of the original $f(R)$ gravity. Our study will be started from this motivated idea (for a discussion on singularities and cosmological aspects see [23]). Let us start by the appropriate form of action for non-local $f(R)$ gravity:

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} R (1 + f(\Box^{-1}R)) + L_{\text{matter}} \right\}. \quad (1)$$

Here $f$ is an arbitrary function of $R$, $\Box = \nabla_\mu \nabla^\mu = \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \right)$ stands for Laplace-Beltrami (dAlembertian) operator in equations of motion. We also adopted the commonly used signature of the metric $g_{\mu\nu}$ as $(+ - - -)$. With this signature, the curvelinear derivative operator and the Riemann tensor read as the following:

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\lambda_{\mu\nu} V_\lambda, \quad (2)$$

$$R^{\sigma}_{\mu\nu\rho} = \partial_\nu \Gamma^\sigma_{\mu\rho} - \partial_\rho \Gamma^\sigma_{\mu\nu} + \Gamma^\omega_{\mu\rho} \Gamma^\sigma_{\omega\nu} - \Gamma^\omega_{\mu\nu} \Gamma^\sigma_{\omega\rho} \quad (3)$$

The above action (1) may be recast to the following scalar-tensor form using a pair of auxiliary (may be unphysical) scalar fields $\{\psi, \xi\}$:

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} \left[ R (1 + f(\psi)) + \xi (\Box \psi - R) \right] + L_{\text{matter}} \right\}. \quad (4)$$

The equations of motion are supplemented by a set of Euler-Lagrange equations for $\xi$ in the following form:

$$\frac{\delta S}{\delta \xi} = 0, \quad \Box \psi = R \quad (5)$$

The above equation may be recast to the following form $\psi = \Box^{-1}R$. If we substitute this equation into (4), we reobtain (1). Equations of motion for metric tensor $g_{\mu\nu}$ is obtained from $\frac{\delta S}{\delta g_{\mu\nu}} = 0$:

$$0 = \frac{1}{2} g_{\mu\nu} \left\{ R (1 + f(\psi) - \xi) - \partial_\rho \psi \partial^\rho \psi \right\} - R_{\mu\nu} (1 + f(\psi) - \xi) \quad (6)$$

$$+ \frac{1}{2} \left( \partial_\mu \xi \partial_\nu \psi + \partial_\mu \psi \partial_\nu \xi \right) - g_{\mu\nu} \left[ \nabla_\mu \nabla_\nu \right] (f(\psi) - \xi) + \kappa^2 T_{\mu\nu}. \quad (6)$$

If we write the equation of motion for $\psi$ we get the following equation of motion:

$$0 = \Box \xi + f'(\psi) R. \quad (7)$$

Our aim here is to derive explicit forms of (5,6,7) for spherically symmetric static configuration of compact, neutron, quark stars.

Notes:

- Quark (q) is a fundamental fermion that has strong interactions.
3. Spherically symmetric model of compact stars

We suppose there must be a broadly believable compact star in static-spherically symmetric coordinates given by system $x^\mu = (ct, r, \theta, \varphi)$ in the following form:

$$\text{ds}^2 = c^2 e^{2\phi} \text{d}t^2 - e^{2\lambda} \text{dr}^2 - r^2 (\text{d}\theta^2 + \sin^2 \theta \text{d}\varphi^2). \quad (8)$$

We also assume that the matter fields are same which have been used in the interior parts and are in the comoving motion. The appropriate energy-momentum tensor is expressed as $T^\mu_\nu = \text{diag}(\rho c^2, -p, -p, -p)$. To keep the homogeneity, we assume that $\xi \equiv \xi(r)$, $\psi \equiv \psi(r)$. We insert (8) in (6) and using the formula given in Sec. (??), we obtain the diagonal components of (6) for $(\mu, \nu) = (ct, ct)$, and $(\mu, \nu) = (r, r)$ are given by the following differential equations:

$$\text{tt} : - \frac{1}{2} \xi' \psi' + (1 + f(\phi) - \xi) \left(- \frac{2\lambda}{r} + \frac{1 - e^{2\lambda}}{r^2}\right) - \left(\psi'' f_{\psi} + \psi' \psi_{\psi} - \xi''\right) \quad (9)$$

$$- \frac{2}{r} f_{\psi} = \kappa^2 \rho c^2 e^{2\lambda}. \quad (10)$$

For scalar field $\psi$ we rewrite (9):

$$\text{psi} : \psi'' + \left(\frac{2}{r} \phi' - \lambda'\right) \psi' - \phi'' - \phi' \psi' - \phi' \lambda' - \frac{2}{r} (\phi' - \lambda') + \frac{1 - e^{2\lambda}}{r^2} = 0. \quad (11)$$

and similarly for $\xi$ we obtain:

$$\text{xi} : \xi'' + \left(\frac{2}{r} \phi' - \lambda'\right) \xi' + 2 f_{\psi} \left(\phi'' + \phi' \psi' - \phi' \lambda' + 2 \phi' - \lambda' + \frac{1 - e^{2\lambda}}{r^2}\right) = 0 \quad (12)$$

The trace of the equation of motion (11) provides another useful equation:

$$R (1 + f(\psi) - \xi) - \partial_{\mu} \xi \partial^{\mu} \psi - 3 \square (f(\psi) - \xi) = -\kappa^2 (\rho c^2 - p). \quad (13)$$

We are licensed to write the equation on to metric (8):

$$\text{Trace} : 2 \left(\phi'' + \phi' \psi' + \phi' \lambda' - \frac{2}{r} (\phi' - \lambda') + \frac{1 - e^{2\lambda}}{r^2}\right) (1 + f(\psi) - \xi) - \psi' \xi'$$

$$- 3 \left[\psi'' f_{\psi} + \psi' \psi_{\psi} - \xi''\right] + \left(\frac{2}{r} \phi' - \lambda'(\psi' f_{\psi} - \xi')\right) = \kappa^2 e^{2\lambda}(\rho c^2 - p). \quad (14)$$

4. Tolman-Oppenheimer-Volkoff Equations

The gravitational equations of motion must be supported by an appropriate hydrostatic equation for the matter fields inside the star. This equation is nothing just the familiar continuity equation for energy-momentum tensor:

$$\nabla_{\mu} T^\mu_\nu = 0$$
If we put $\nu = r$ in this equation with the metric (8), we get:

$$\frac{dp}{dr} = -(p + \rho c^2)\phi'.$$

(15)

Now, keeping $\nu = t$ in mind, a question may arise that this completely vanishes the possibility of constructing new hydrodynamic equation. We will note that equations (9,10,11,12,14) are reduced to GR by any $f(R) = R$ model in the action.

We need to find the forms of the TOV equations. It is safe to replace the metric part $\lambda$ of the equations with a mass function $M = M(r)$:

$$e^{-2\lambda} = 1 - \frac{2GM}{c^2r} \Rightarrow \frac{GdM}{c^2dr} = \frac{1}{2}\left[1 - e^{-2\lambda}(1 - 2r\lambda')\right].$$

(16)

Now, we must rewrite (9,10,11,12,14) in terms of $\{\frac{dp}{dr}, \frac{dM}{dr}, \rho\}$. Also, it is adequate to write equations in the dimensionless forms. It is desired to introduce the next set of the dimensionless parameters for field equations,

$$M \rightarrow mM_\odot, \quad r \rightarrow r_gr, \quad \rho \rightarrow \rho M_\odot r^3, \quad p \rightarrow \rho M_\odot c^2 r^3.$$

(17)

where $r_g = \frac{GM_\odot}{c^2} = 1.47473 KM$ and $M_\odot$ Stands out for its mass of the central Sun. Using these dimensionless parameters we rewrite the continuity equation as the following:

$$\phi' = -\frac{p'}{(p + \rho)}.$$

(18)

But (16) converts to the following:

$$\lambda' = \frac{m}{r^2} - \frac{dM}{dr}.$$

(19)

Furthermore, we have metric function as an equation in terms of the mass:

$$e^{2\lambda} = \left(1 - \frac{2m}{r}\right)^{-1}, \quad 1 - e^{2\lambda} = \left(1 - \frac{r}{2m}\right)^{-1}.$$

(20)

A very widely expressed system, TOV equations, connecting metric to matter is also found in the following forms:

$$tt: \quad \frac{p'\xi' + (1 + f(\phi) - \xi)(-\frac{2m}{r^3} + \frac{dM}{m dr} - \frac{1}{r^2(1 - \frac{r}{2m})})}{2(p + \rho)} + (1 + f(\phi) - \xi)((-\frac{2m}{r^3} + \frac{dM}{m dr} - \frac{1}{r^2(1 - \frac{r}{2m})}) - (\psi' f_\psi + \psi'^2 f_{\psi\psi} - \xi'') - \left(\frac{2}{r} - \frac{m - r dM}{2m dr - 1}\right)(\psi' f_\psi - \xi')) = \frac{8\pi\rho}{1 - \frac{2m}{r}}$$

(21)

$$rr: \quad (1 + f(\phi) - \xi)(\frac{2p'}{r^2(p + \rho)} - \frac{1}{r^2(1 - \frac{r}{2m})}) + \frac{\xi' p'}{2(p + \rho)}$$

$$+ (\frac{2}{r} - \frac{p'}{p + \rho})(\psi' f_\psi - \xi') = \frac{8\pi p}{1 - \frac{2m}{r}}.$$

(22)
In previous section we derived the full set of TOV equations for a generic model of compact stars. By compact star, we mean some relativistic massive objects with tiny size and high density. These astronomical objects have the mass of order $M \sim M_\odot$ and the radius $R \sim 10K_M$. For three types of the astronomical candidates the metric functions $\{\lambda, \phi\}$ were obtained as simple quadratic functions of radial coordinate $r$:

$$2\lambda = Ar^2, \quad 2\phi = Br^2 + C,$$

where $A$, $B$ and $C$ are physical constants to be evaluated using astronomical data given in the table I. In GR, compact stars with these metric functions were studied. In modified gravity from $f(R)$ to the GB model there are some interesting features. In reference a model for neutron star constructed numerically for a class of viable models of $f(G)$ gravity. Our aim in this letter is to investigate physical properties of compact stars in this non local $f(R)$ gravity (see for some recently reported works). We will study the dynamical stability, energy conditions and red

\[ \psi'' + \left( \frac{2}{r} - \frac{p'}{p + \rho} - \frac{m}{r^2} \frac{1 - \frac{r}{m}}{2m - 1} \right) \psi' + \left( \frac{p'}{p + \rho} \right)' - \left( \frac{p'}{p + \rho} \right)^2 \]

\[ \frac{m}{r^2} \frac{p'}{p + \rho} \left( 1 - \frac{r}{m} \right) - \frac{2}{r} \left( \frac{p'}{p + \rho} + \frac{m}{r^2} \frac{1 - \frac{r}{m}}{2m - 1} \right) - \frac{1}{r^2 \left( 1 - \frac{r}{2m} \right)} = 0. \]
shift properties of an isotropic model of compact star. The model which we address here is the non-local form of the one proposed by Starobinsky for inflation:

\[ f(\Box^{-1} R) = \Box^{-1} R + \alpha \left( \Box^{-1} R \right)^2. \]  

(27)

Even the best known neutron stars often have great uncertainties in their masses and radii. In fact no neutron star has a really precise radius measurement, and it is rare to have even rough measurements of both radius and mass for one star. The mass of strange star, describes a fit with two minima of error, one is 0.\(8M_\odot\), the other being 1.\(8M_\odot\) which is far more believable as it is consistent with good measurements of other systems and both mass values are highly dependent on the details of this very complicated model, in contrast with the simpler methods that produce good mass measurements in other systems. This binary is complicated by x-ray heating of the primary, among other things. Since even the work say that the 0.8\(M_\odot\) measurement cannot be favored by their analysis, it is unreasonable to take it as definitive. The \(R \sim 7Km\) radius for SAX J1808 may extract from which it seems to be still popular in the particle literature although the astronomy literature shows it to be wrong, as in and others. We still cannot figure out how can we obtain the mass and radii for the star 4U 1820-30 (for a recent work see for example for a detailed and careful discussion, recent analysis including statistical and (numerous) systematic sources of error).

| Strange star candidate | \(A(km^{-2})\) | \(B(km^{-2})\) |
|------------------------|----------------|----------------|
| Her X-1               | 0.006906276428 | 0.00426734618 |
| SAX J 1808.4-3658      | 0.01823156974  | 0.01488011569  |
| 4U 1820-30            | 0.01090644119  | 0.00988095281  |

- **Exact solutions for auxiliary fields** \([\psi, \xi]\): the equations of motion (5,7) are completely integrable:

\[ \psi(r) = C_1 \int \frac{g_r dr}{\sqrt{-g}} + \int_0^r \sqrt{-g(y)} R(y)(r - y) dy + C_2, \]  

(28)

here \(R(y)\) is (11). For \(\xi\) using (7) we obtain:

\[ \xi(r) = C_3 \int \frac{g_r dr}{\sqrt{-g}} - \int_0^r \sqrt{-g(y)(r - y)} f_\psi(y) \Box \psi(y) dy + C_4, \]  

(29)

Where \(\Box \psi(y)\) was obtained from (12). However, it is their hard work, integrating and simplifying to achieve their fields solutions (28, 29). Fortunately, exact solutions have given the graphically monotonic forms. In Fig. (11), we observe that \(\psi(r)\) never falls into a linear plot, which remains the polynomial. So auxiliary field \(\psi(r)\) is to be the perfect polynomial plot against the radius. An example Her X-1 plot of field \(\chi(r)\) in an interior
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area is shown in Fig. 2. We conclude that $\xi(r)$ is always decreasing or remaining constant and never increasing, so both of $\psi(r), \xi(r)$ are monotonic-increasing (decreasing) functions. For Starobinsky model, this behavior is a functional dependency of auxiliary fields on scalar curvature $R$ of metric (26). You may increase or decrease the Ricci scalar $R$ (11) simply by extending or decreasing the $\{A, B, C\}$, or by increasing or decreasing the amount of fields $\psi, \xi$. However, the increasing value of $R$ points to the growing appearance of second derivative of the scalar fields $\{\Box \xi, \Box \psi\}$. At the same time both field(s) and growing Ricci scalar are increasing (decreasing) behavior.

**Stability-conditions:** with applying external radial perturbations at the inflow boundary, large radial structures develop naturally in the flow field due to sound effects. The velocity of sound drift, produced by spatially uniform perturbations is obtained by:

$$V_{rv}^2 = \frac{dp}{d\rho}.$$

We plot (30) in Fig. 3. The probability (velocity) for the radial perturbations will peak at order ten to the one. Generally, however, there are instabilities with either pressure radial perturbations or with density to astrophysical groups SAX J 1808.4-3658, Her X-1 and 4U 1820-30. One may increase or decrease the $V_{rv}^2$ simply by extending or decreasing the pressure, or by increasing or decreasing the amount of density. For SAX J 1808.4-
3658, numerical analysis showed a significant similar relationship between increasing proportions of sample SAX J 1808.4-3658, Her X-1 and 4U 1820-30. Initially, increasing the pressure perturbation to the density perturbation increases the velocity $V_{oc}$ of sample. The plot may have a singularity, or the perturbation scheme may be inappropriate. But an instability ought to come after the singular point. The earlier mentioned sample SAX J 1808.4-3658 of astrophysical object is such indicative case. However, looking back we should have instabilities in SAX J 1808.4-3658 a lot earlier than Her X-1 and 4U 1820-30. Thermal instability of the stars with large perturbations is a function of the empirical parameters $\{A, B, C\}$ of the sample.

- **Surface Redshift**: we’ve an increase in the wavelength of radiation emitted by a celestial body as a consequence of the gravitational field. The gravitational redshift $z$ of thermal spectrum detected at infinity can be computed as

$$z = e^{-\phi} - 1.$$  \hspace{1cm} (30)

We plot (30) in Fig. (4). It is observed that the largest redshift occurs for an emitter at the center of the star.

- **Energy conditions**: by looking at the field equations in modified gravities we are often able to arrange a mutually acceptable energy density $\rho_{eff}$ and pressure $p_{eff}$. These energy conditions have altered our understanding of the range of conditions under which energy transferring onto a region
Fig. 3. The probability (velocity) for the radial perturbations $\frac{dP}{P_c} = \frac{d\rho}{\rho_c}$.

Fig. 4. Redshift of photons (30) emitted from the center of the star as a function of $r$ for three different candidates SAX J 1808.4-3658, Her X-1 and 4U 1820-30.
occurs\textsuperscript{65,66}:

\begin{align}
\text{NEC} & \iff \rho_{\text{eff}} + p_{\text{eff}} \geq 0. \\
\text{WEC} & \iff \rho_{\text{eff}} \geq 0 \text{ and } \rho_{\text{eff}} + p_{\text{eff}} \geq 0. \\
\text{SEC} & \iff \rho_{\text{eff}} + 3p_{\text{eff}} \geq 0 \text{ and } \rho_{\text{eff}} + p_{\text{eff}} \geq 0. \\
\text{DEC} & \iff \rho_{\text{eff}} \geq 0 \text{ and } \rho_{\text{eff}} \pm p_{\text{eff}} \geq 0.
\end{align}

The essence of energy conditions is purely geometrical simple and, as Hawking said, self-evident\textsuperscript{67}. However, we also believe that energy conditions in their pure essence are effective. Fig. (5) showing the $\rho_{\text{eff}}$ changed in order of decrease of pressure $\rho_{\text{eff}}$. The resulting graph (5) shows the rapid decrease in $\rho_{\text{eff}}$ in the three different candidates SAX J 1808.4-3658, Her X-1 and 4U 1820-30 and the more gradual decrease through the 4U 1820-30 and SAX J 1808.4-3658. The resulting graph (6) shows the rapid decrease in scaled pressure $\rho_{\text{eff}}$ in three different candidates and the more gradual decrease through the 4U 1820-30 and SAX J 1808.4-3658s. All the pressures get vanished on the radius of star and none of the pressures is remained at all. For Her X-1, the pressure is vanished near the $r \sim 8.25$ and for SAX J 1808.4-3658, at $r \sim 6.5$ and 4U 1820-30 at $r \sim 11$. The precise distances for the vanished pressures can be seen in the\textsuperscript{61–63}.

The levels of radius found numerically are comparable to those found by astrophysical data\textsuperscript{61–63}. These two methods clearly produce comparable data. Information verified and indexed by data from the\textsuperscript{7,8,9} would be easily cross-referenced in WEC and SEC. The satisfactions of all energy conditions will be independently verified numerically. Then again we empirically verified that our non local model for compact star was correct. The role of non locality in compact stars was therefore verified in saturation of all energy conditions. However, despite strong interest in their usage, a lack of fundamental test data and verified structural non local theory guidance is inhibiting uptake.

- **An empirical equation of state:**

Fig. (10) shows the EoS of a compact star calculated numerically. The results suggest that the appropriate form of EoS is given by the following:

\[ \frac{P}{P_c} = a(1 - e^{-b\rho_{\text{eff}}}). \]  

Tiny density increases the pressure size and gave value to the hardenable of star. We claimed that increasing density $\rho$ ” will thwart the aspirations of future neutron star “. Quark stars are also required to attend a generic EoS which emphasizes linear relation $p = Ap + B$. However, if the density is decreased, you would be required to attend the quark star EoS at the non local $f(R)$ model. Normally, someone contemplating these low densities

\[ \rho_{\text{eff}} = \frac{M}{r^2}, p_{\text{eff}} = \frac{M_c r^2}{r^2} \]

\textsuperscript{b}In all of these graphs $\rho_{\text{eff}} = \frac{M}{r^2}, p_{\text{eff}} = \frac{M_c r^2}{r^2}$
Fig. 5. Scaled energy density $\frac{\rho}{\rho_c}$ of the star as a function of $r$ for three different candidates SAX J 1808.4-3658, Her X-1 and 4U 1820-30.

Fig. 6. Scaled pressure $\frac{p}{p_c}$ of the star as a function of $r$ for three different candidates SAX J 1808.4-3658, Her X-1 and 4U 1820-30.
Fig. 7. Scaled $\frac{P}{P^*} + \frac{\rho}{\rho^*}$. 

Fig. 8. Scaled $\frac{3P}{P^*} + \frac{\rho}{\rho^*}$. 

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\( \rho \ll \rho_c \) would attend quark star EoS. All parameters are shown in Table 2. Some parameters offer a selection from a sample to which you give an option. For some objects parameters, for example 4U 1820-30, high pressure effects are stronger than pressure effects in other samples. Finally, the EoS parameters must also be consistent with a linearity in the quark EoS which leads to the observed samples. These parameters are tested in both density and pressure initial conditions. We mention here that the EoS (35) can be addressed as a generalized exponential Virial EoS:

\[
\frac{p}{\rho k_B T} = \exp \left( \sum_{m=2}^{\infty} K_m \rho^{m-1} \right). \tag{36}
\]

where the coefficients \( K_m \) and the virial coefficients \( B_n \) are related, \( k_B \) is the Boltzmann constant and \( T \) temperature. There are significant differences in the EoS of (35) versus exponential Virial EoS (36). These differences have been categorized in terms of pressure of the background \( p_b \sim p_c \). For instance, for anomalous differences all the normal gases with exponential Virial EoS have an \( p_b \) of zero. These differences arise because one or more compact objects in our case has a \( p_b \) value or indeed should not be omit. However, it is debatable whether these structural differences make this EoS any easier for the astrophysical purposes to investigate the thermodynamic.

The differences between (35) and exponential Virial EoS (36) are: (35) is made from the gravitational field, whereas exponential Virial EoS (36) is made from the high-density fluid. In conclusion, it is of utmost importance
to reiterate the differences between the \cite{35} EoS and the exponential Virial EoS \cite{35}.

| Astrophysical strange star candidate | a        | $\sigma_a$ | b        | $\sigma_b$ |
|--------------------------------------|----------|------------|----------|------------|
| Her X-1                              | 0.45526  | 0.0198     | 0.0152   | 0.00124    |
| SAX J 1808.4-3658                     | 1.2242   | 0.0563     | 0.0191   | 0.00162    |
| 4U 1820-30                            | 1.1255   | 0.0408     | 0.0243   | 0.00163    |

6. Summary and conclusion

Einstein gravity is a gauge theory of gravity. It should be modified to have more effective predictions and implications for recently observational data. One of the most accepted modifications of Einstein gravity is $f(R)$ gravity and its extensions. It is assumed that we gain more information about gravity if we replace $R$ by an arbitrary function $f(R)$. Several cosmological aspects of this type of modified gravity have been investigated in literature. Specially the late and early time evolution. Non-local terms, induced by quantum effects can be considered as non-local higher order corrections to Einstein gravity. It is reasonable to consider both scenarios in a same context, as non-local $f(R)$ gravity, a scenario which we studied in this letter. We derived the equations of motion for this non-local theory using a pair of auxiliary
scalar fields. As a motivated idea, we studied stellar structure using the modified forms of TOV equations. We obtained the set of equations of motion for a star in non-local form of $f(R)$ gravity. At the same time the TOV equations were recast and some reconstruction took place in the system, when a non-local correction was inserted. It is asserted here that the dynamic can adequately describe, explain or understand such a naive relationship from the perspective of compact stars. The full conditional solutions are evaluated and inverted numerically to obtain exact forms of the compact stars Her X-1, SAX J 1808.4-3658 and 4U 1820-30 for model of $f(\Box^{-1}R) = \Box^{-1}R + \alpha \left(\Box^{-1}R\right)^2$. The program solves the differential equations numerically using adaptive Gaussian quadrature. An ascription of correctness is supposed to be an empirical equations of state $P = a(1 - e^{-b\rho}/\rho_c)$ for star which is informative in so far as it excludes an alternative non-local approach to compact star formation. The differences between (35) and exponential Virial EoS (36) are: (35) is made from the gravitational field, whereas exponential Virial EoS (36) is made from the high-density fluid. This model is most suited for astrophysical observation. A theoretical perspective proposed by us, TOV equations for non-local $f(R)$ theory, is helpful for understanding these non-local effects.

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7. Appendices

In this appendix we present different geometrical quantities which have been used in this letter. For metric (8) the following nonzero components of the symmetric connection are obtained:

\begin{align*}
\Gamma^1_{12} &= \phi', \\
\Gamma^2_{11} &= \phi' e^{2\phi - 2\lambda}, \\
\Gamma^2_{22} &= \lambda', \\
\Gamma^2_{23} &= -re^{-2\lambda}, \\
\Gamma^2_{44} &= -r \sin^2 \theta e^{-2\lambda}, \\
\Gamma^3_{23} &= \frac{1}{r}, \\
\Gamma^3_{44} &= -\sin \theta \cos \theta, \\
\Gamma^4_{24} &= \frac{1}{r}, \\
\Gamma^4_{34} &= \cot \theta.
\end{align*}

(37)

So, the nonzero $(tt), (rr)$ components of the Ricci tensor read:

\begin{align*}
R_{tt} &= e^{2\phi - 2\lambda} \left(-\phi'' - \phi'^2 + \phi' \lambda' - \frac{2\phi'}{r}\right), \\
R_{rr} &= \phi'' + \phi'^2 - \phi' \lambda' - \frac{2\lambda'}{r}.
\end{align*}

(39)

The Ricci scalar is as the following:

\[ R = -2e^{-2\lambda} \left(\phi'' + \phi'^2 - \phi' \lambda' + \frac{2(\phi' - \lambda')}{r} + \frac{1 - e^{2\lambda}}{r^2}\right). \]

(41)
The operator \( \square \) is given by:

\[
\square A(r) = -e^{-2\lambda} \left( A'' + \left( \frac{2}{r} + \phi' - \lambda' \right) A' \right).
\]  

(42)

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