A Beale–Kato–Majda criterion for the 3-D Compressible Nematic Liquid Crystal Flows with Vacuum

Qiao Liu † and Shangbin Cui ‡
Department of Mathematics, Sun Yat-sen University, Guangzhou, Guangdong 510275,
People’s Republic of China

Abstract

In this paper, we prove a Beale–Kato–Majda blow-up criterion in terms of the gradient of the velocity only for the strong solution to the 3-D compressible nematic liquid crystal flows with nonnegative initial densities. More precisely, the strong solution exists globally if the $L^1(0, T; L^\infty)$-norm of the gradient of the velocity $\nabla u$ is bounded. Our criterion improves the recent result of X. Liu and L. Liu (arXiv:1011.4399v2 [math-ph] 23 Nov. 2010).

Keywords: Compressible nematic liquid crystal flows; strong solution; blow-up criterion; Compressible Navier–Stokes equations

2010 AMS Subject Classification: 76A15, 76N10, 35B65, 35Q35

1 Introduction

The governing system of equations for the compressible nematic liquid (NLC) crystal flows is the following system of scalar or vector fields $\rho(t, x)$, $u(t, x)$ and $d(t, x)$ for $(t, x) \in (0, +\infty) \times \Omega$, for a bounded smooth domain $\Omega \subset \mathbb{R}^3$:

$$\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla P &= \mu \Delta u - \lambda \nabla \cdot (\nabla d \odot \nabla d - \frac{1}{2}(|\nabla d|^2 + F(d)I)), \\
\partial_t d + (u \cdot \nabla) d &= \nu(\Delta d - f(d))
\end{align*}$$

(1.1)

together with the initial value conditions:

$$\rho(0, x) = \rho_0(x) \geq 0, \quad u(0, x) = u_0(x), \quad d(0, x) = d_0(x), \quad \forall x \in \Omega,$$  \quad (1.2)

and the boundary value conditions:

$$u(t, x) = 0, \quad d(t, x) = d_0(x), \quad |d_0(x)| = 1, \quad \forall (t, x) \in [0, +\infty) \times \partial \Omega.$$  \quad (1.3)

*Research supported by the National Natural Science Foundation of China (11171357).
†E-mail address: liuqao2005@163.com.
‡E-mail address: cuisb3@yahoo.com.cn.
Here we denote by $\rho, u = (u_1, u_2, u_3), d = (d_1, d_2, d_3)$ the unknown density, velocity and orientation parameter of liquid crystal, respectively, and $P = P(\rho)$ is the pressure function. Besides, $\mu, \lambda$ and $\nu$ are positive viscosity coefficients. The non-standard term $\nabla d \odot \nabla d$ denotes the $3 \times 3$ matrix, whose $(i, j)$-th element is given by $\sum_{k=1}^{3} \partial_i d_k \partial_j d_k$. $I$ is the unit matrix. $f(d)$ is a polynomial function of $d$ which satisfies $f(d) = \partial_d F(d)$, where $F(d)$ is the bulk part of the elastic energy; usually we choose $F(d) = \frac{1}{2\sigma^2}(|d|^2 - 1)^2$ and $f(d) = \frac{1}{\sigma^2}(|d|^2 - 1)d$, where $\sigma$ is a positive constant. In what follows, we will assume $\sigma = 1$ since its specific value does not play a special role in our discussion. Besides, we assume that the pressure function $P$ satisfies

$$P = P(\cdot) \in C^1[0, \infty), \quad P(0) = 0.$$ (1.4)

The above system (1.1) is a simplified version of Ericksen–Leslie system modeling the flow of compressible nematic liquid crystals, and the hydrodynamic theory of liquid crystals was established by Ericksen [5, 6] and Leslie [17] in the 1960’s. When $d \equiv 0$, the system becomes to the compressible Navier–Stokes (CNS) equations. Matsumura and Nishida [27] obtained global existence of smooth solutions for the initial data is a small perturbation of a non–vacuum equilibrium. For the existence of solutions for arbitrary initial value, Lions [18] and Feireisl [9] established the global existence of weak solution to the CNS equations. Cho et al. [2, 3, 4] proved that the existence and uniqueness of local strong solutions of the CNS equations in the case where initial density need not to be positive and may vanish in an open set. Xin in [32] showed that there is no global smooth solution to the Cauchy problem of the CNS equations with a nontrivial compactly supported initial density. Hence, there are many works [3, 7, 8, 12, 13, 14, 30, 31] try to establish blow–up criterion for the strong solution to the CNS equations. In particular, it is proved in [14] by Huang, Li and Xin that the serrin’s blow–up criterion (see [28]) for the incompressible Navier–Stokes equations still holds for the CNS equations, i.e., if $T^*$ is the maximal time of existence strong solution, then

$$\lim_{T \to T^*} \left( \| \text{div } u \|_{L^1(0, T; L^\infty)} + \| \rho^{\frac{1}{2}} u \|_{L^s(0, T; L^r)} \right) = \infty$$ (1.5)

or

$$\lim_{T \to T^*} \left( \| \rho \|_{L^1(0, T; L^\infty)} + \| \rho^{\frac{1}{2}} u \|_{L^s(0, T; L^r)} \right) = \infty,$$ (1.6)

where $r$ and $s$ satisfy $\frac{2}{s} + \frac{2}{r} \leq 1$, $3 < r \leq \infty$. In [12, 13], Huang et al. established that the Beale–Kato–Majda criterion (see [1]) for the ideal incompressible flows still hold for the CNS equations, that is

$$\lim_{T \to T^*} \int_0^T \| \nabla u \|_{L^\infty} dt = \infty.$$

Sun, Wang and Zhang in [30] (see also [14]) obtained another Beale–Kato–Majda criterion in terms of the density, i.e.,

$$\lim_{T \to T^*} \sup_{0 \leq t \leq T} \| \rho \|_{L^\infty(0, T; L^\infty)} = \infty.$$
When ρ is a positive constant, the system (1.1) becomes to the incompressible nematic liquid crystal (INLC) equations, the global-in-time weak solutions and local-in-time strong solution have been studied by Lin and Liu [20, 21]. In [11], Hu and Wang established global existence of strong solutions and weak–strong uniqueness for initial data belonging to the Besov spaces of positive order under some smallness assumptions. Liu and Cui in [24] obtained that the blow–up criterion (1.5) or (1.6) still holds for the solution of the INLC equations. We also refer [10, 19, 22, 23, 29] and the reference cited therein for other related work on the INLC equations.

Inspired by the above mentioned works on blow–up criterion of strong solution of CNS and INLC equations, particularly the results of Huang et al. [12, 13] and Sun et al. [30, 31], we want to investigate a similar problem for the compressible nematic liquid crystal flow (1.1)–(1.3). Before stating the main result, we denote the following simplified notations of Sobolev spaces

\[ L^q := L^q(\Omega), \quad W^{k,p} := W^{k,p}(\Omega), \quad H^k := H^k(\Omega), \quad H^1_0 := H^1_0(\Omega). \]

When the initial vacuum is allowed, the well-posedness and blow–up criterion for strong solutions to the compressible nematic liquid crystal flows (1.1)–(1.3) have been investigated by Liu et al. in [25, 26]. Here, we write down the main results of Liu et al. [25, 26].

**Theorem 1.1** Suppose that the initial value \((\rho_0, u_0, d_0)\) satisfies the following regularity conditions

\[ 0 \leq \rho_0 \in W^{1,6}, \quad u_0 \in H^1_0 \cap H^2 \quad \text{and} \quad d_0 \in H^3, \]

and the compatibility condition

\[ \mu \Delta u_0 - \lambda \text{div}(\nabla d_0 \otimes \nabla d_0 - \frac{1}{2}(|\nabla d_0|^2 + F(d_0))) - \nabla P(\rho_0) = \sqrt{\rho}_g \text{ for some } g \in L^2. \quad (1.7) \]

Then there exist a small \(T \in (0, \infty)\) and a unique strong solution \((\rho, u, d)\) to the system (1.1) with initial boundary condition (1.2)–(1.3) such that

\[ 0 \leq \rho \in C([0,T); W^{1,6}), \quad \rho_t \in C([0,T); L^6), \]
\[ u \in C([0,T); H^1_0 \cap H^2) \cap L^2(0,T; W^{2,6}), \quad u_t \in L^2(0,T; H^1_0), \]
\[ d \in C([0,T); H^3), \quad d_t \in C([0,T); H^1_0) \cap L^2(0,T; H^2), \]
\[ d_{tt} \in L^2(0,T; L^2), \quad \sqrt{\rho} u_t \in C([0,T); L^2). \]

Moreover, let \(T^*\) be the maximal existence time of the solution. If \(T^* < \infty\), then there holds

\[ \lim_{T \to T^*} \int_0^T (\|\nabla u\|_{L^\infty}^\beta + \|u\|_{W^{1,\infty}}) \, dt = \infty, \quad (1.8) \]

where \(\alpha, \beta\) satisfying \(\frac{1}{\alpha} + \frac{2}{\beta} < 2\) and \(\beta \geq 4\).

**Remark 1.1** Another similar system of partial differential equations modeling compressible nematic liquid crystal flows has been studied by Huang, Wang and Wen in [15, 16]. They obtained the existence of local in time strong solution and two blow–up criteria under some suitable assumption condition \(u\) and \(d\) or \(\rho\) and \(d\).

The purpose of this paper is to obtain the Beale–Kato–Majda blow–up criterion only in terms of the gradient of the velocity still holds for the liquid crystal flows. Our main result is the following
Theorem 1.2 Assume that \((\rho, u, d)\) is the strong solution constructed in Theorem 1.1, and \(T^*\) be the maximal existence time of the solution. If \(T^* < \infty\), then we have

\[
\limsup_{T \to T^*} \|\nabla u\|_{L^1(0,T;L^\infty)} = \infty.
\] (1.9)

The proof of this theorem will be given in the next section. As a standard practice, we will show that if (1.9) does not hold then the strong solution \((\rho, u, d)\) can be extended beyond the time \(T^*\). To this end we will step-by-step establish a series of higher-order norm estimates for the strong solution \((\rho, u, d)\). The key fact used in this deduction is that the boundedness of the \(L^1(0,T;L^\infty)\)-norm of \(\nabla u\) implies both boundedness of the \(L^\infty(0,T;L^\infty)\)-norm of the density \(\rho\) and boundedness of the \(L^\infty(0,T;W^{1,q})\)-norm of \(d\) with \(2 \leq q \leq \infty\).

2 Proof of Theorem 1.2

Let \((\rho, u, d)\) be the unique strong solution to the system (1.1) with initial–boundary condition (1.2)–(1.3). We assume that the opposite to (1.9) holds, i.e.,

\[
\lim_{T \to T^*} \|\nabla u\|_{L^1(0,T;L^\infty)} \leq M < \infty.
\] (2.1)

In what follows, we note that \(C\) denotes a generic constant depending only on \(\mu, \lambda, \nu, M, T, \Omega\) and the initial data. By using the mass conservation equation (1.1) and the assumption (2.1), it is easy to obtain the \(L^\infty\)-norm bounds of the density,

Lemma 2.1 Assume that

\[
\int_0^T \|\text{div } u\|_{L^\infty} \, dt \leq C, \quad 0 \leq T < T^*,
\] (2.2)

then

\[
\|\rho\|_{L^\infty(0,T;L^\infty)} \leq C \quad \forall 0 \leq T < T^*.
\] (2.3)

Proof. The proof is essentially due to Huang and Xin [12], for reader’s convenience, we sketch it here.

Multiplying the mass conservation equation (1.1) by \(q\rho^{q-1}\) with \(q > 1\), it follows that

\[
\partial_t (\rho^q) + \text{div}(\rho^q u) + (q-1)\rho^q \text{ div } u = 0.
\]

Integrating the above equality over \(\Omega\) yields

\[
\partial_t \|\rho\|_{L^q}^q \leq (q-1)\|\text{div } u\|_{L^\infty} \|\rho\|_{L^q}^q,
\]

i.e.,

\[
\partial_t \|\rho\|_{L^q} \leq \frac{(q-1)}{q} \|\text{div } u\|_{L^\infty} \|\rho\|_{L^q}.
\] (2.4)
The condition (2.2) and the estimate (2.3) imply that
\[ \partial_t \| \rho \|_{L^q} \leq C \quad \text{for} \quad \forall q > 1, \]
where \( C \) is a positive constant independent of \( q \), letting \( q \to \infty \), we obtain (2.3), and this completes the proof of the lemma.

According to the assumption (1.4) on the pressure \( P \) and Lemma 2.1, it is easy to obtain
\[ \sup_{0 \leq t \leq T} \{ \| P(\rho) \|_{L^\infty}, \| P'(\rho) \|_{L^\infty} \} \leq C < \infty. \]
(2.5)

Now, let us derive the stand energy inequality.

**Lemma 2.2** There holds
\[ \sup_{0 \leq t \leq T} \int_{\Omega} (\rho |u|^2 + |\nabla d|^2 + 2F(d)) dx + \int_0^T \int_{\Omega} |\nabla u|^2 dx \, dt + \int_0^T \int_{\Omega} |\Delta d - f(d)|^2 dx \, dt \leq C. \]
(2.6)

**Proof.** Multiplying the momentum equation (1.1) by \( u \), integrating over \( \Omega \) and making use of the mass conversation equation (1.1), it follows that

\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |u|^2 dx + \mu \int_{\Omega} |\nabla u|^2 dx = - \int_{\Omega} u \nabla P dx - \lambda \int_{\Omega} (u \cdot \nabla) d \cdot (\Delta d - f(d)) dx, \]
(2.7)

where we have used the fact that \( \text{div}(\nabla d \circ \nabla d) = (\nabla d)^T \Delta d - \nabla : \frac{1}{2} \nabla d^2 \). Multiplying the liquid crystal equation (1.1) by \( \Delta d - f(d) \) and integrating over \( \Omega \), we obtain

\[ \frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} |\nabla d|^2 + F(d) \right) dx + \nu \int_{\Omega} \Delta d - f(d)^2 dx = \int_{\Omega} (u \cdot \nabla) d \cdot (\Delta d - f(d)) dx. \]
(2.8)

Combining (2.7) and (2.8) together

\[ \frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} \rho |u|^2 + \lambda |\nabla d|^2 \right) dx + \mu \int_{\Omega} |\nabla u|^2 dx + \lambda \nu \int_{\Omega} \Delta d - f(d)^2 dx \]
\[ = - \int_{\Omega} u \nabla P dx = \int_{\Omega} P \text{div} u dx \leq \varepsilon \int_{\Omega} |\nabla u|^2 dx + C \varepsilon^{-1}, \]
(2.9)

where we have used the estimates (2.3), (2.5) and the Young inequality. Taking \( \varepsilon \) small enough and applying the Gronwall’s inequality, we can establish the estimate (2.6) immediately.

In the next lemma, we will derive some estimates of \( d \).

**Lemma 2.3** Under the assumption (2.1), it holds that for \( 0 \leq T < T^\ast \)

\[ \sup_{0 \leq t \leq T} (\| d \|_{L^q} + \| \nabla d \|_{L^q}) \leq C \quad \text{for all} \quad 2 \leq q \leq \infty; \]
(2.10)

\[ \sup_{0 \leq t \leq T} \| \nabla d \|_{L^2}^2 + \int_0^T \| d_t \|_{L^2}^2 dt \leq C; \]
(2.11)
Proof. We first multiplying the liquid crystal equation (1.1) by $q|d|^{q-2}d$ with $q \geq 2$, and integrating over $\Omega$, then there holds
\[
\frac{d}{dt}\|d\|_{L^q}^q + \int_\Omega (q\nu |\nabla d|^2|d|^2 + q(q-2)\nu |d|^{q-2}|\nabla d|^2)dx = -\sum_{i=1}^3 \int_\Omega u_i \partial_i(|d|^q)dx - q\nu \int_\Omega |d|^{q+2}dx + q\nu \int_\Omega |d|^q dx
\]
\[
= -\sum_{i=1}^3 \int_\Omega \partial_i u_i |d|^q dx - q\nu \int_\Omega |d|^{q+2}dx + q\nu \int_\Omega |d|^q dx
\]
\[
\leq C(\|\nabla u\|_{L^\infty} + 1)\|d\|^q_{L^q}.
\]
By using the Gronwall’s inequality, one obtains the inequality
\[
\sup_{0 \leq t \leq T} \|d\|_{L^q} \leq C \quad \text{for all } q \geq 2.
\] (2.12)

By letting $q \to \infty$, we notice that the estimate (2.12) still holds.

Multiplying the gradient of the liquid crystal equation (1.1) by $q|\nabla d|^{q-2}\nabla d$ with $q \geq 2$, and integrating over $\Omega$, then there holds
\[
\frac{d}{dt}\|\nabla d\|_{L^q}^q + \int_\Omega (q\nu |\nabla (\nabla d)|^2|\nabla d|^{q-2} + q(q-2)\nu |\nabla d|^2|\nabla |\nabla d|^{q-2})dx = -\sum_{i=1}^3 \int_\Omega u_i \partial_i(|\nabla d|^q)dx - q\nu \int_\Omega |\nabla d|^{q+2}\nabla ddx - q\nu \int_\Omega |\nabla d|^q\nabla ddx + q\nu \int_\Omega |\nabla d|^q dx
\]
\[
\leq C\|\nabla u\|_{L^\infty}\|\nabla d\|_{L^q}^q + \nu q\|\nabla d\|_{L^q}^q - q\nu \int_\Omega \nabla ((|\nabla d|^2)\nabla d)^{q-2}\nabla ddx
\]
\[
= (C\|\nabla u\|_{L^\infty} + \nu q)\|\nabla d\|_{L^q}^q - q\nu \int_\Omega |\nabla d|^{q+2}\nabla ddx - q\nu \int_\Omega |\nabla d|^q\nabla ddx - q\nu \int_\Omega |\nabla d|^q dx
\]
\[
\leq C(\|\nabla u\|_{L^\infty} + 1)\|\nabla d\|_{L^q}^q
\]
where we have used the fact that $|\nabla d|^2 = 2|\nabla d| = 2|d|\frac{\nabla d}{|\nabla d|} = 2d\nabla d$ in the last equality. By using the Gronwall’s inequality again, we obtain
\[
\sup_{0 \leq t \leq T} \|\nabla d\|_{L^q} \leq C \quad \text{for all } q \geq 2.
\] (2.13)

Letting $q \to \infty$, estimate (2.13) still holds, and the inequalities (2.12) and (2.13) imply that estimate (2.10) holds.

To prove the estimate (2.11), we multiplying the liquid crystal equation (1.1) by $d_i$ and integrating over $\Omega$, then
\[
\|d_i\|_{L^2}^2 + \frac{\nu}{2} \frac{d}{dt} \int_\Omega |\nabla d|^2 dx = - \int_\Omega (\nabla u \cdot \nabla) d_i dx - \nu \int_\Omega f(d) d_i dx
\]
\[
\begin{align*}
&\leq C(\|u\|_{L^2}\|\nabla d\|_{L^\infty} + \|d\|_{L^2}^2 + \|d_t\|_{L^2} + \|\nabla d\|_{L^\infty} + \|d\|_{L^2} \|d_t\|_{L^2}) \\
&\leq \frac{1}{2}\|d\|_{L^2}^2 + C,
\end{align*}
\]

where we have used the estimates (2.6) and (2.10). Integrating the above inequality over \([0, T]\) gives the estimate (2.11).

For function \(f \in \Omega \times (0, T)\), let
\[
\dot{f} = f_t + u \cdot \nabla f
\]
denote the material derivative of the function \(f\). Then we have following lemma.

**Lemma 2.4** Under the assumption (2.1), it holds that for \(0 \leq T < T^*\)
\[
\sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 + \|d\|_{H^2}^2) + \int_0^T \int_\Omega (\rho|\dot{u}|^2 + |\nabla d_t|^2) dx dt \leq C; \tag{2.14}
\]
\[
\int_0^T \|\nabla d\|_{H^2}^2 dt \leq C. \tag{2.15}
\]

**Proof.** Noticing that the momentum equation (1.1) can be rewrote as
\[
\rho \dot{u} + \nabla P = \mu \Delta u - \lambda (\nabla d)^T (\Delta d - f(d)). \tag{2.16}
\]

Multiplying the equation (2.10) above by \(\dot{u}\) and integrating over \(\Omega\), one obtains the equality
\[
\frac{\mu}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \int_\Omega \rho|\dot{u}|^2 dx \\
= \mu \int_\Omega u \cdot \nabla u \Delta u dx + \int_\Omega P \text{div} u_t dx - \int_\Omega u \cdot \nabla u \nabla P dx \\
- \lambda \int_\Omega (u_t \cdot \nabla) d(\Delta d - f(d)) dx - \lambda \int_\Omega (u \cdot \nabla) u \cdot \nabla d(\Delta d - f(d)) dx \tag{2.17}
\]

Combining the mass conservation equation (1.1) and the assumption (2.1), it follows that the pressure \(P\) satisfies the following equation
\[
P_t + P'(\rho) \nabla \rho \cdot u + P'(\rho) \rho \text{div} u = 0. \tag{2.18}
\]

Hence, we have
\[
\int_\Omega P \text{div} u_t dx = \frac{d}{dt} \int_\Omega P \text{div} u dx - \int_\Omega P_t \text{div} u dx \\
= \frac{d}{dt} \int_\Omega P \text{div} u dx + \int_\Omega P'(\rho) (\nabla \rho \cdot u + \rho \text{div} u) \text{div} u dx. \tag{2.19}
\]

To estimate the term \(-\lambda \int_\Omega (u_t \cdot \nabla) d(\Delta d - f(d)) dx\), we have
\[
-\lambda \int_\Omega (u_t \cdot \nabla) d(\Delta d - f(d)) dx = \lambda \sum_{i,j=1}^3 \left( \int_\Omega \partial_i u_{it} \partial_j d \partial_j d dx + \int_\Omega u_{it} \partial_i \partial_j d \partial_j d dx \right) + \lambda \int_\Omega u_t \cdot \nabla df(d) dx
\]
\begin{align*}
&= \lambda \sum_{i,j=1}^{3} \left( \int_{\Omega} \partial_{j} u_{i} \partial_{i} \partial_{j} \partial_{j} dx - \frac{1}{2} \int_{\Omega} \partial_{i} u_{i} |\partial_{j} d|^2 dx \right) - \lambda \sum_{i=1}^{3} \int_{\Omega} \partial_{i} u_{i} \frac{|d|^4}{4} - \frac{|d|^2}{2} dx \\
&= \lambda \sum_{i,j=1}^{3} \left\{ \frac{d}{dt} \int_{\Omega} (\partial_{j} u_{i} \partial_{i} \partial_{j} d - \frac{1}{2} \partial_{i} u_{i} |\partial_{j} d|^2) dx - \int_{\Omega} \partial_{j} u_{i} \partial_{i} d \partial_{j} \partial_{j} dx - \int_{\Omega} \partial_{j} u_{i} \partial_{i} d \partial_{j} d \partial_{j} dx \\
&+ \int_{\Omega} \partial_{i} u_{i} \partial_{i} \partial_{j} \partial_{j} dx - \frac{d}{dt} \int_{\Omega} (\frac{1}{4} |\partial_{i} u_{i}|^4 - \frac{1}{2} \partial_{i} u_{i} |\partial_{i} d|^2) dx \right\} \\
&\leq \lambda \sum_{i,j=1}^{3} \frac{d}{dt} \int_{\Omega} (\partial_{j} u_{i} \partial_{i} \partial_{j} d - \frac{1}{2} \partial_{i} u_{i} |\partial_{j} d|^2 - \frac{1}{4} \partial_{i} u_{i} |\partial_{i} d|^4 + \frac{1}{2} \partial_{i} u_{i} |\partial_{i} d|^2) dx \\
&+ C \|\nabla u\|_{L^2} \|\nabla d\|_{L^2} + C \|\Delta \nabla d\|_{L^2} + C \|\nabla u\|_{L^2} \|\Delta d\|_{L^2} + C \|\partial_{j} u\|_{L^2} + C \|d\|_{L^2},
\end{align*}

where we have used estimate \((2.10)\) in the last inequality. Inserting \((2.19)\) and \((2.20)\) into \((2.17)\), and integrating over \([0,T]\) give that

\begin{align*}
\|\nabla u\|_{L^2}^2 + \int_{0}^{T} \int_{\Omega} \rho |\dot{u}|^2 dx dt \\
&\leq C + C \int_{0}^{T} \int_{\Omega} u \cdot \nabla u \Delta u dx dt + C \int_{\Omega} \int_{0}^{T} P(\rho) divergence d dx dt + C \int_{0}^{T} \int_{\Omega} P'(\rho)(\nabla \rho \cdot u + \rho \text{div} u) dx dt \\
&+ C \sum_{i,j=1}^{3} \int_{\Omega} (\partial_{j} u_{i} \partial_{i} \partial_{j} d - \frac{1}{2} \partial_{i} u_{i} |\partial_{j} d|^2 - \frac{1}{4} \partial_{i} u_{i} |\partial_{i} d|^4 + \frac{1}{2} \partial_{i} u_{i} |\partial_{i} d|^2) dx (T) \\
&+ \varepsilon \int_{0}^{T} \|\nabla d\|_{L^2}^2 dx dt + C \varepsilon^{-1} \int_{0}^{T} \|\nabla u\|_{L^2}^2 dx dt + C \int_{0}^{T} \int_{\Omega} \int_{0}^{T} \|\nabla u\|_{L^2}^2 + ||\partial_{i} d||_{L^2}^2 dx dt \\
&+ \int_{0}^{T} \int_{\Omega} |u| \|\nabla P\| dx dt + C \int_{0}^{T} \int_{\Omega} |u| \|\nabla d\| \|\Delta d\| + |f(d)| dx dt \\
&\leq C + \varepsilon \int_{0}^{T} \|\nabla d\|_{L^2}^2 dx dt + C \int_{0}^{T} \int_{\Omega} u \cdot \nabla u \Delta u dx dt + C \int_{\Omega} \int_{0}^{T} P(\rho) \text{div} u dx dt \\
&+ C \int_{0}^{T} \int_{\Omega} P'(\rho)(\nabla \rho \cdot u + \rho \text{div} u) dx dt \\
&+ C \sum_{i,j=1}^{3} \int_{\Omega} (\partial_{j} u_{i} \partial_{i} \partial_{j} d - \frac{1}{2} \partial_{i} u_{i} |\partial_{j} d|^2 - \frac{1}{4} \partial_{i} u_{i} |\partial_{i} d|^4 + \frac{1}{2} \partial_{i} u_{i} |\partial_{i} d|^2) dx (T) \\
&+ \int_{0}^{T} \int_{\Omega} |u| \|\nabla u\| \|\nabla P\| dx dt + C \int_{0}^{T} \int_{\Omega} |u| \|\nabla d\| \|\Delta d\| + |f(d)| dx dt,
\end{align*}

where we have used the estimate \((2.6)\) and \((2.11)\). To estimate the terms on the right side of \((2.21)\), by using Lemma \((2.1)\) the estimates \((2.6)\), \((2.10)\) and \((2.11)\), we get

\begin{align*}
\int_{0}^{T} \int_{\Omega} u \cdot \nabla u \Delta u dx dt = \sum_{i,j=1}^{3} \int_{0}^{T} \int_{\Omega} (-\partial_{j} u_{i} \partial_{i} \partial_{j} u - u_{i} \partial_{i} \partial_{j} \partial_{j} u) dx dt
\end{align*}
By using the stand elliptic regularity result to (2.16), we have

\[
\int_0^T \int_\Omega \left( -\partial_j u_i \partial_i u \partial_j u + \frac{1}{2} \partial_i u_i |\partial_j u|^2 \right) dx dt 
\leq C \int_0^T \|\nabla u\|_{L^4}^4 dx dt \leq C \int_0^T \|\nabla u\|_{L^\infty} \|\nabla u\|_{L^2}^2 dx dt 
\leq C \int_0^T \|\nabla u\|_{L^4}^4 dx dt + C \int_0^T \|\nabla u\|_{L^2}^2 dx dt 
\leq C \int_0^T \|\nabla u\|_{L^4}^4 dx dt + C \int_0^T \|\nabla u\|_{L^2}^2 dx dt 
\leq C \int_0^T \|\nabla u\|_{L^4}^4 dx dt + C ; 
\tag{2.22}
\]

\[
\int_\Omega P(\rho) \text{ div } u dx(T) \leq \frac{1}{4} \|\nabla u\|_{L^2}^2 + C \int_\Omega |P(\rho)|^2 dx \leq \frac{1}{4} \|\nabla u\|_{L^2}^2 + C ; 
\tag{2.23}
\]

\[
\int_0^T \int_\Omega P'(\rho)(\nabla \rho \cdot u) \text{ div } u dx dt \leq C \int_0^T \|\nabla \rho\|_{L^2} \|u\|_{L^2} \|\text{ div } u\|_{L^\infty} dx dt 
\leq C \int_0^T \|\nabla \rho\|_{L^2} \|u\|_{L^2} dx dt + C \int_0^T \|\nabla \rho\|_{L^2} dx dt 
\leq C \int_0^T \|\nabla \rho\|_{L^2} \|u\|_{L^2} dx dt + C \int_0^T \|\nabla \rho\|_{L^2} dx dt 
\leq C \int_0^T \|\nabla \rho\|_{L^2} \|u\|_{L^2} dx dt + C ; 
\tag{2.24}
\]

\[
\int_0^T \int_\Omega P'(\rho) \text{ div } u^2 dx dt \leq C \int_0^T \|\rho\|_{L^\infty} \|\nabla u\|_{L^2} dx dt \leq C \int_0^T \|\nabla u\|_{L^2}^2 dx dt \leq C ; 
\tag{2.25}
\]

\[
\sum_{i,j=1}^3 \int_\Omega (\partial_i u_i \partial_i u_i \partial_j u_j - \frac{1}{2} \partial_i u_i |\partial_j u|^2 - \frac{1}{4} \partial_i u_i |d|^2 + \frac{1}{2} \partial_i u_i |d|^2) dx(T) 
\leq C (\|\nabla u\|_{L^2} \|\nabla d\|_{L^2} \|\nabla d\|_{L^\infty} + \|\nabla u\|_{L^2} \|d\|_{L^\infty} \|\nabla d\|_{L^2} + \|\nabla u\|_{L^2} \|d\|_{L^\infty} \|\nabla d\|_{L^2}) 
\leq \frac{1}{4} \|\nabla u\|_{L^2}^2 + C \|\nabla d\|_{L^2}^2 \|\nabla d\|_{L^\infty} + C \|\nabla u\|_{L^2} \|d\|_{L^\infty} \|\nabla d\|_{L^2}^2 
\leq \frac{1}{4} \|\nabla u\|_{L^2}^2 + C ; 
\tag{2.26}
\]

\[
\int_0^T \int_\Omega |u| \|\nabla u\| \|\nabla P| dx dt \leq C \int_0^T \int_\Omega |u| \|\nabla u\| \|\nabla \rho| dx dt 
\leq C \int_0^T \|u\|_{L^4} \|\nabla u\|_{L^4} \|\nabla \rho\|_{L^2} dx dt \leq C \int_0^T \|\nabla u\|_{L^2} \|\nabla u\|_{L^\infty} \|\nabla \rho\|_{L^2} dx dt 
\leq C \int_0^T (\|\nabla u\|_{L^2}^2 \|\nabla \rho\|_{L^2} + \|\nabla u\|_{L^\infty} \|\nabla u\|_{L^2}^2) dt 
\leq C \int_0^T \|\nabla u\|_{L^2} \|\nabla \rho\|_{L^2} \|\nabla u\|_{L^2} + \|\nabla u\|_{L^\infty} \|\nabla u\|_{L^2}^2 + C ) dt ; 
\tag{2.27}
\]

\[
\int_0^T \int_\Omega |u| \|\nabla u\| \|\nabla (\Delta d + f(d))| dx dt \leq C \int_0^T \|u\|_{L^4} \|\nabla u\|_{L^4} \|\nabla d\|_{L^6} \|\Delta d\|_{L^2} + \|u\|_{L^6} \|\nabla u\|_{L^2} \|\Delta d\|_{L^6} \|\nabla d\|_{L^2} \|\Delta d\|_{L^2} + \|\nabla u\|_{L^2}^2 \|\Delta d\|_{L^2}^2 dt 
\leq C \int_0^T \|\nabla u\|_{L^2} \|\nabla u\|_{L^2} \|\Delta d\|_{L^2}^2 dt + \|\nabla u\|_{L^2}^2 \|\Delta d\|_{L^2}^2 dt 
\leq C \int_0^T \epsilon \|\nabla u\|_{L^2}^2 + C \epsilon^{-1} \|\nabla u\|_{L^2}^2 \|\Delta d\|_{L^2}^2 dt + C . 
\tag{2.28}
\]

By using the stand elliptic regularity result to (2.16), we have

\[
\|\nabla^2 u\|_{L^2}^2 \leq \|\Delta u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \leq C (\|\nabla u\|_{L^2}^2 + \|\rho\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 (\Delta d - f(d))\|_{L^2}^2) .
\]
\[
\leq C(\|\nabla u\|_{L^2}^2 + \|\rho\|^2_\infty \|\rho^{\frac{2}{3}} \hat{u}\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 (\|\Delta d\|_{L^2}^2 + \|f(d)\|_{L^2}^2))
\]
\[
\leq C(\|\nabla u\|_{L^2}^2 + \|\rho^{\frac{2}{3}} \hat{u}\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 + 1).
\]

Combining estimates (2.21)–(2.29) and taking \(\varepsilon\) small enough, we can get
\[
\|\nabla u\|_{L^2}^2 + \int_0^T \int_\Omega |\rho| \hat{u}^2 \, dx \, dt
\]
\[
\leq C + \varepsilon \int_0^T \|\nabla u\|_{L^2}^2 \, dt + C \int_0^T (\|\nabla \rho\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) (\|\nabla u\|_{L^\infty} + \|\nabla u\|_{L^2}^2 + 1) \, dt.
\]

To estimate the orientation parameter \(d\), by the standard elliptic regularity result to the liquid crystal equation \(1.1\), one obtains that
\[
\|\nabla^3 d\|_{L^2} \leq C(\|\nabla d\|_{L^2} + \|\nabla (u \cdot \nabla d)\|_{L^2} + \|\nabla f(d)\|_{L^2} + \|d_0\|_{H^1})
\]
\[
\leq C(\|\nabla d\|_{L^2} + \|\nabla u\|_{L^2} \|d\|_{L^\infty} + \|\nabla u\|_{L^2} \|\nabla^2 d\|_{L^2} + \|\nabla d\|_{L^2} (\|d\|_{L^2}^2 + \|d\|_{L^\infty} + \|d_0\|_{H^3})
\]
\[
\leq C(\|\nabla d\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla^2 d\|_{L^2} + C)
\]

Multiplying the liquid crystal equation \(1.1\) by \(\Delta d\), and integrating over \(\Omega\), then we have
\[
\frac{d}{dt} \int_\Omega \nu |\Delta d|^2 \, dx + \int_\Omega |\nabla d|^2 \, dx
\]
\[
= \int_\Omega u \cdot \nabla \Delta d \, dx + \nu \int_\Omega (|d|^2 - 1) d \Delta d \, dx
\]
\[
= \sum_{i,j=1}^3 \int_\Omega u_i \partial_i \partial_j d \, dx - \nu \int_\Omega |\nabla |d|^2 d| \, dx + \nu \int_\Omega \nabla d \, dx
\]
\[
= \sum_{i,j=1}^3 \int_\Omega \partial_i u_i \partial_i \partial_j d \, dx + \nu \int_\Omega \nabla |d|^2 d \, dx + \nu \int_\Omega \nabla d \, dx
\]
\[
\leq C(\|\nabla u\|_{L^2} \|\nabla d\|_{L^\infty} \|\Delta d\|_{L^2}
\]
\[
+ \|\nabla^2 d\|_{L^2} \|\nabla d\|_{L^2} + \|d\|_{L^2} \|\Delta d\|_{L^2} + \|\nabla d\|_{L^2} \|\Delta d\|_{L^2})
\]
\[
\leq \varepsilon \|\nabla d\|_{L^2}^2 + C \varepsilon^{-1} (\|\nabla u\|_{L^2}^2 + \int_\Omega |u|^2 |\nabla^2 d|^2 \, dx + 1),
\]

where we have used the Hölder inequality and estimates (2.10). For the term \(\int_\Omega |u|^2 |\nabla^2 d|^2 \, dx\), applying estimate (2.31), we have for \(\eta > 0\)
\[
\int_\Omega |u|^2 |\nabla^2 d|^2 \, dx \leq C \|u\|_{L^6} \|\nabla^2 d\|_{L^2} \leq C \|\nabla u\|_{L^2} \|\nabla d\|_{L^6} \|\nabla^3 d\|_{L^2}
\]
\[
\leq \eta \|\nabla^3 d\|_{L^2}^2 + C \eta^{-1} \|\nabla u\|_{L^2}
\]
\[
\leq \eta \|\nabla d\|_{L^2}^2 + \eta \int_\Omega |u|^2 |\nabla^2 d|^2 \, dx + \eta \|\nabla u\|_{L^2}^2 + C \eta^{-1} (\|\nabla u\|_{L^2}^4 + C).
\]

Hence, taking \(\eta\) small enough
\[
\int_\Omega |u|^2 |\nabla^2 d|^2 \, dx \leq 2 \eta \|\nabla d\|_{L^2}^2 + C \eta^{-1} (\|\nabla u\|_{L^2}^2 + 1) + C).
\]
Inserting (2.33) into (2.32), taking \( \varepsilon, \eta \) small enough and integrating above inequality over \( (0; T] \) ensure that
\[
\|\Delta d\|_{L^2}^2 + \int_0^T \int_\Omega |\nabla d_t|^2 \, dx \, dt \leq C \int_0^T \|\nabla u\|_{L^2}^2 (\|\nabla u\|_{L^2}^2 + 1) \, dt + C. \tag{2.34}
\]

Now, we will estimate the density \( \rho \). Applying the operator \( \nabla \) to the mass conservation equation (1.1), then multiplying it by \( \nabla \rho \) and integrating over \( \Omega \) yield
\[
\frac{d}{dt}\|\nabla \rho\|_{L^2}^2 = -\int_\Omega |\nabla \rho|^2 \text{div} u \, dx - 2 \int_\Omega \rho \nabla \rho \text{div} u \, dx - 2 \int_\Omega (\nabla \rho \cdot \nabla u) \nabla \rho \, dx \\
\leq C\|\nabla \rho\|_{L^2}^2 \|\nabla u\|_{L^\infty} + C\|\nabla \rho\|_{L^2} \|\nabla \Delta u\|_{L^2} \\
\leq \varepsilon \|\nabla^2 u\|_{L^2}^2 + C\varepsilon^{-1}\|\nabla \rho\|_{L^2}^2 (\|\nabla u\|_{L^\infty} + 1) \\
\leq \varepsilon (\|\rho\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 + 1) + C\varepsilon^{-1}\|\rho\|_{L^2}^2 (\|\nabla u\|_{L^\infty} + 1) \\
\leq \varepsilon (\|\rho\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 + \|\Delta d\|_{L^2}^2),
\]
where we have used the estimate (2.29) in the above inequality. Integrating the above estimate over \( (0, T] \) gives that
\[
\|\nabla \rho\|_{L^2}^2 \leq \varepsilon \int_0^T \|\rho\|_{L^2}^2 \, dt + \int_0^T (C\varepsilon^{-1}\|\nabla \rho\|_{L^2}^2 (\|\nabla u\|_{L^\infty} + 1) + C(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2)) \, dt \tag{2.35}
\]
Combining estimates (2.30), (2.34) and (2.35), and taking \( \varepsilon \) small enough, one obtains that
\[
\|\nabla u\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 + \int_0^T \int_\Omega (\rho \nabla u + |\nabla d_t|)^2 \, dx \, dt \\
\leq C + C \int_0^T (\|\nabla \rho\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \, dt. \tag{2.36}
\]
Since the energy estimate (2.6) implies that \( \int_0^T \|\nabla u\|_{L^2}^2 \, dt \leq C \). By using the Gronwall’s inequality, the elliptic regularity result \( \|\nabla^2 d\|_{L^2} \leq C(\|\Delta d\|_{L^2} + \|d_0\|_{H^2}) \) and noticing that the assumption (2.11), we deduce that the inequality (2.14) holds.

To prove the estimate (2.15), by using the standard elliptic regularity result on (1.1), we have
\[
\|\nabla^2 d\|_{L^2}^2 \leq C(\|d_t\|_{L^2}^2 + \|u \cdot \nabla d\|_{L^2}^2 + \|f(d)\|_{L^2}^2 + \|d_0\|_{H^2}) \\
\leq C(\|d_t\|_{L^2}^2 + \|d_0\|_{H^2} + \|u \cdot \nabla d\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 + \|d\|_{L^2}^2 (\|d\|_{L^2}^2 + \|d\|_{L^2}^2 + C)) \\
\leq C(\|d_t\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 + \|u \cdot \nabla d\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 + C) \\
\leq C(\|d_t\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 + C), \tag{2.37}
\]
where we have used the estimate (2.14) in the last inequality. Then by using the estimates (2.10), (2.11), (2.14) and the above inequality, we have
\[
\int_0^T \|\nabla d_t\|_{L^2}^2 \, dt \leq \int_0^T (\|\nabla^2 d\|_{L^2}^2 + \|\nabla d\|_{L^2}^2) \, dt \\
\leq C \int_0^T (\|\nabla d_t\|_{L^2}^2 + \|\nabla (u \cdot \nabla d)\|_{L^2}^2 + \|\nabla f(d)\|_{L^2}^2 + C) \, dt.
\]
Lemma 2.5 Under the assumption \((2.1)\), it holds that for \(0 \leq T < T^*\)

\[
\sup_{0 \leq t \leq T} (\|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2) + \int_0^T (\|\nabla u_t\|_{L^2}^2 + \|d_t\|_{L^2}^2) dt \leq C. \tag{2.38}
\]

**Proof.** Differentiating the momentum equation \((1.1)_2\) with respect to time, multiplying the resulting equation by \(u_t\), integrating it over \(\Omega\) and making use of the mass conservation equation \((1.1)_1\), one obtains that

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_\Omega \rho |u_t|^2 dx + \mu \int_\Omega |\nabla u_t|^2 dx &- \int_\Omega \partial_t \text{div} u_t dx \\
&= - \int_\Omega \rho u_t \cdot \nabla (\frac{|u_t|^2}{2}) dx + (u_t \cdot \nabla) u_{tt} + \rho (u_t \cdot \nabla) u_t dx - \lambda \int_\Omega (u_t \cdot \nabla) d_t (\Delta d - f(d)) dx \\
&\quad - \lambda \int_\Omega (u_t \cdot \nabla) d_t (\Delta d - f(d)) dx \\
&= \int_\Omega \nabla \rho \cdot \frac{|u_t|^2}{2} dx + \rho \text{div} u_t \frac{|u_t|^2}{2} - \rho u_t \cdot \nabla ((u_t \cdot \nabla) u_{tt}) - \rho (u_t \cdot \nabla) u_t dx \\
&\quad - \lambda \int_\Omega (u_t \cdot \nabla) d_t (\Delta d - f(d)) dx - \lambda \int_\Omega (u_t \cdot \nabla) d_t (\Delta d - f(d)) dx. \tag{2.39}
\end{align*}
\]

Differentiating the liquid crystal equation \((1.1)_3\) with respect to time gives

\[
(u_t \cdot \nabla)d = \nu (\Delta d - f(d))_t - d_{tt} - (u \cdot \nabla) d_t.
\]

Multiplying the above equality with \((\Delta d - f(d))_t\) and integrating over \(\Omega\), one obtains the equality

\[
\begin{align*}
\int_\Omega (u_t \cdot \nabla) d_t (\Delta d - f(d))_t dx &\quad = \int_\Omega (\nu (\Delta d - f(d))_t)^2 dx - d_{tt} \Delta d_t + d_{tt} f(d)_t - (u \cdot \nabla) d_t (\Delta d - f(d))_t dx \\
&\quad = \int_\Omega \nu (\Delta d - f(d))_t^2 dx + \frac{1}{2} \frac{d}{dt} \|\nabla d_t\|_{L^2}^2 - \int_\Omega ((u_t \cdot \nabla) d_t) f(d)_t dx \\
&\quad + \int_\Omega \nu f(d)_t (\Delta d - f(d))_t dx - \int_\Omega ((u \cdot \nabla) d_t) \Delta d_t dx \\
&\quad = \int_\Omega \nu (\Delta d - f(d))_t^2 dx + \frac{1}{2} \frac{d}{dt} \|\nabla d_t\|_{L^2}^2 - \int_\Omega ((u_t \cdot \nabla) d_t) f(d)_t dx \\
&\quad + \int_\Omega \nu f(d)_t (\Delta d - f(d))_t dx + \int_\Omega ((\nabla u \cdot \nabla) d_t \nabla d_t - \frac{1}{2} \text{div} u |\nabla d_t|^2) dx, \tag{2.40}
\end{align*}
\]

This completes the proof of Lemma 2.5. \(\square\)
where we have used the fact

$$-\int_{\Omega} ((u \cdot \nabla)dt \Delta dt dx = - \sum_{i,j=1}^{3} \int_{\Omega} u_i \partial_i dt \partial_j dt dx$$

$$= \sum_{i,j=1}^{3} \int_{\Omega} (\partial_j u_i \partial_j dt + u_i \partial_i (\frac{\partial_j dt}{2}))dx$$

$$= \sum_{i,j=1}^{3} (\int_{\Omega} (\partial_j u_i \partial_j dt dx - \frac{1}{2} \int_{\Omega} \partial_i u_j \partial_j dt dx)$$

$$= \int_{\Omega} ((\nabla u \cdot \nabla) dt \nabla dt - \frac{1}{2} \text{div} u \nabla dt^2 dx$$

in the last equality.

From the equation (2.13), we can derive

$$\int_{\Omega} P_i \text{div} u dx = - \int_{\Omega} P'(\rho)(\nabla \rho \cdot u + \rho \text{div} u) \text{div} u dx.$$ (2.41)

Inserting the equalities (2.40) and (2.41) into (2.39) derives

$$\frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} \rho |u|^2 + \lambda \nabla u^2 \right) dx + \mu \nabla u^2 dx + \lambda \nu \nabla u^2 dx + \frac{1}{2} \int_{\Omega} \text{div} u \nabla u^2 dx$$

$$+ C \int_{\Omega} (|\nabla \rho| u^2 + |\nabla d| (d + \nabla d)^2) dx + C \int_{\Omega} |\nabla \rho| u dx + \rho |P'(\rho)| \text{div} u dx = \sum_{j=1}^{13} J_j.$$ (2.42)

We will estimate $J_j$ term by term. In the following calculations, we will make extensive use of Sobolev embedding, Hölder inequality, Lemmas 2.1–2.4 and the estimate (2.5),

$$J_1 \leq C \|\nabla \rho\|_{L^2} \|u_t\|_{L^2}^2 \|u\|_{L^6} \leq C \|\nabla u_t\|_{L^2}^2 \|u\|_{L^2} \leq \epsilon \|\nabla u_t\|_{L^2}^2 + C \epsilon^{-1};$$

$$J_2 + J_5 \leq C \|\nabla u\|_{L^\infty} \|\nabla u_t\|_{L^2}^2;$$

$$J_7 \leq C \|u_t\|_{L^2} \||\nabla d|_{L^\infty} \|d\|_{L^\infty} \|d\|_{L^6} \leq C \|\nabla u_t\|_{L^2}^2 + C \|\nabla u_t\|_{L^2}^2;$$

$$J_8 \leq C \|\nabla (\Delta d - f d)\|_{L^2} \|f\|_{L^2} \|d\|_{L^2} \leq \epsilon \|\nabla (\Delta d - f d)\|_{L^2}^2 + C \|\nabla d\|_{L^2}^2;$$

$$J_9 + J_{10} = \int_{\Omega} |(\nabla u \cdot \nabla) dt \nabla dt dx + |\text{div} u\| \nabla dt^2 dx$$

$$\leq C \|\nabla u\|_{L^\infty} \|\nabla dt\|_{L^2}^2;$$

$$J_{11} \leq C \|u_t\|_{L^6} \||\nabla d|_{L^2} \||\Delta d|_{L^3} + C \|u_t\|_{L^2} \||\nabla d|_{L^2} \|d\|_{L^\infty} \|d\|_{L^\infty} \|d\|_{L^3} \leq C \|\nabla u_t\|_{L^2} \||\nabla dt|_{L^2} \|d\|_{L^2}^2 + C \|\nabla u_t\|_{L^2} \||\nabla d|_{L^2} \|^2.$$
Substituting all the estimates of $J$.

Applying the Gronwall’s inequality to estimate (2.44), we deduce

and

where we have used the estimates (2.6) and (2.10) in the last inequality. Taking $\sigma$ small enough, we obtain

Making use of estimates (2.14) and (2.43), we can estimate $J_3, J_4, J_5$ and $J_{12}$ as

Substituting all the estimates of $J_3$ into (2.42), and taking $\varepsilon$ small enough, we obtain

Applying the Gronwall’s inequality to estimate (2.44), we deduce

\[ \sup_{0 \leq t \leq T} \int_{\Omega} (\rho |u|^2 + |\nabla d|^2) dx + \int_{0}^{T} \|\nabla u_t\|_{L^2}^2 + ||(\Delta d - f(d))_t\|_{L^2}^2 dt \]
Proof. From estimates (2.14), (2.38) and (2.43), we have

\[
\text{that is }
\]

By using the Gronwall’s inequality to the above estimate gives

\[
\text{the proof of Lemma 2.5.}
\]

where we have used estimate (2.15) and the assumption (2.1) in the last inequality. This completes

\[
\text{ing equation by 6}
\]

Under the assumption estimates (2.48) and (2.49) above gives the estimate (2.46).

\[
\text{where we have used the estimates (2.10), (2.14) and (2.38) in the las t inequality. Combining the}
\]

The following lemma gives the higher order norm estimates of \( u, d \) and \( \rho \).

**Lemma 2.6** Under the assumption (2.1), it holds that for \( 0 \leq T < T^* \)

\[
\sup_{0 \leq t \leq T} (\|u\|_{H^2} + \|\nabla d\|_{H^2}) \leq C;
\]

\[
\sup_{0 \leq t \leq T} \|\nabla \rho\|_{L^6} + \int_0^T \|\nabla^2 u\|_{L^6} dt \leq C.
\]

**Proof.** From estimates (2.14), (2.38) and (2.43), we have

\[
\|u\|_{H^2} \leq C(|\rho|^{\frac{1}{2}} u_t)_{L^2} + \|\nabla u\|_{L^2} + \|\nabla \rho\|_{L^2} + \|\nabla d_t\|_{L^2} \leq C.
\]

By using estimates (2.31) and (2.33), we have

\[
\|\nabla d\|_{H^2} \leq C(\|\nabla^3 d\|_{L^2} + \|\nabla d\|_{L^2})
\]

\[
\leq C(\|\nabla d_t\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla^2 d\|_{L^2} + \|\nabla d\|_{L^2})
\]

\[
\leq C(\|\nabla d_t\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla^2 u\|_{L^2} + \|\nabla d\|_{L^2} + C) \leq C,
\]

where we have used the estimates (2.10), (2.14) and (2.38) in the last inequality. Combining the

\[
\text{Applying the standard elliptic regularity result } \|\nabla^2 u\|_{L^6} \leq C \|\Delta u\|_{L^6}, \text{ Hölder inequality, Sobolev}
\]

\[
\text{estimates (2.10) and (2.46), we have}
\]

\[
\|\nabla^2 u\|_{L^6} \leq C(\|\rho u_t\|_{L^6} + \|\rho u \cdot \nabla u\|_{L^6} + \|\nabla P\|_{L^6} + \|\nabla d\|^{T}(\Delta d - f(d))\|_{L^6})
\]

\[
\leq C\left(\|\rho u_t\|_{L^6} + \|\rho u \cdot \nabla u\|_{L^6} + \|\nabla P\|_{L^6} + \|\nabla d\|^{T}(\Delta d - f(d))\|_{L^6}\right)
\]
Inserting the estimate (2.52) into (2.51) yields

\[ \| \nabla \rho \|_{L^6} \leq C \int_0^T (\| \nabla u \|_{L^2} + \| \nabla \rho \|_{L^6} + 1)dt \leq C \int_0^T (\| \nabla u \|_{L^2}^2 + \| \nabla \rho \|_{L^6}^2 + 1)dt \leq C \int_0^T \| \nabla \rho \|_{L^6} dt + C, \]

where we have used the estimate (2.38), then applying the Gronwall’s inequality gives

\[ \sup_{0 \leq t \leq T} \| \nabla \rho \|_{L^6} \leq C. \]  

(2.53)

From (2.52) and (2.53), we have

\[ \int_0^T \| \nabla^2 u \|_{L^6}^2 dt \leq C \left( \int_0^T \| \nabla u \|_{L^2}^2 dt + \sup_{0 \leq t \leq T} \| \nabla \rho \|_{L^6}^2 + C \right) \leq C. \]  

(2.54)

It is easy to known that the estimate (2.48) follows (2.53) and (2.54) immediately. This completes the proof of Theorem 1.2.

We now give the proof of Theorem 1.2.

**Proof of Theorem 1.2.** From the existence result of Theorem 1.1, we know that \( \| u(t) \|_{H^2}, \| \rho(t) \|_{H^2}, \| d(t) \|_{H^2} \) and \( \| \rho^\frac{1}{2} u(t) \|_{L^2} \) are all continuous on the time interval \([0, T^*)\). From the above Lemmas 2.1–2.6, we see that for all \( T \in (0, T^*) \),

\[ (\| u \|_{H^2}, \| \rho \|_{W^{1,6}}, \| d \|_{H^3}, \| \rho^\frac{1}{2} u \|_{L^2})(T) \leq C. \]  

(2.55)

Furthermore, there hold

\[ \rho^\frac{1}{2} u_t + \rho^\frac{1}{2} u \cdot \nabla u \in L^\infty([0, T^*]; L^2), \]  

(2.56)

and for all \( T \in (0, T^*) \),

\[ (\mu \Delta u - \lambda \text{div}(\nabla d \circ \nabla d - \frac{1}{2} (|\nabla d|^2 + F(d))) - \nabla P)(T) = (\rho u_t + \rho u \cdot \nabla u)(T) = \sqrt{\rho} g, \]  

(2.57)

where \( g(T) \triangleq (\rho^\frac{1}{2} u_t + \rho^\frac{1}{2} u \cdot \nabla u)(\cdot, T) \in L^2 \). Therefore, from (2.50) and (2.57), we can take \( (\rho, u, d)|_{t=T} \) with any \( T \in (0, T^*) \) as the initial data and apply Theorem 1.1 to extend the local strong solution to a time interval \([T, T + \delta]\) for a uniform \( \delta > 0 \) which only depends on the bounds obtained in these lemmas, so that the solution can be extended to the time interval \([0, T^* + \delta]\). This contradicts with the maximality of \( T^* \). Hence, the assumption (2.1) cannot be true. This completes the proof of Theorem 1.2.
References

[1] J. Beale, T. Kato and A. Majda, Remarks on breakdown of smooth solutions for the 3D Euler equations. Commun. Math. Phys., 94 (1984), 61–66.

[2] H. Choe and H. Kim, Strong solutions of the Navier-Stokes equations for isentropic compressible fluids, J. Differential Equations, 190 (2003), 504–523.

[3] Y. Cho, H. Choe and H. Kim, Unique solvability of the initial boundary value problems for compressible viscous fluids, J. Math. Pures Appl., 83 (2004), 243–275.

[4] Y. Cho and H. Kim, Existence results for viscous polytropic fluids with vacuum, J. Differential Equations 228 (2006), 377–411.

[5] J. Ericksen, Conservation Laws For Liquid Crystal, Trans. Soc. Rheol. 5 (1961), 22–34.

[6] J. Ericksen, Continuum theory of nematic liquid crystals, Res. Mechanica, 21 (1987), 381–392.

[7] J. Fan and S. Jiang, Blow-up criteria for the Navier-Stokes equations of compressible fluids, J. Hyperbolic Differential Equations, 5 (2008), 167–185.

[8] J. Fan, S. Jiang and Y. Ou, A blow-up criterion for compressible viscous heat-conductive flows, Ann. I. H. poincaré–AN, 27 (2010), 337–350.

[9] E. Feireisl, Dynamics of viscous compressible fluids, Oxford University Press, Oxford, 2004.

[10] R. Hardt, D. Kinderleher and F. Lin, Existence and partial regularity of static liquid crystal configurations, Commun. Math. Phys., 105 (1986), 547–570.

[11] X. Hu and D. Wang, Global Solution to the Three-Dimensional Incompressible Flow of Liquid Crystals, Commun. Math. Phys., 296 (2010), 861–880.

[12] X. Huang and Z. Xin, A blow-up criterion for classical solutions to the compressible Navier-Stokes equations, arXiv: 0903.3090v2 [math-ph], 19 March 2009.

[13] X. Huang, J. Li and Z. Xin, Blowup Criterion Viscous Barotropic Flows with Vacuum States, Commun. Math. Phys., 301 (2011), 23–35.

[14] X. Huang, J. Li and Z. Xin, Serrin Type Criterion for the Three-Dimensional Viscous Compressible Flows, arXiv:1004.4748v1 [math-ph] 27 Apr. 2010.

[15] T. Huang, C. Wang and H. Wen, Strong solutions of the compressible nematic crystal flow, arXiv:104.5684v1 [math.AP] 20 Apr. 2011.

[16] T. Huang, C. Wang and H. Wen, Blow up criterion for compressible nematic crystal flows in dimension three, arXiv:104.5685v1 [math.AP] 20 Apr. 2011.

[17] F. Leslie, Theory of flow phenomenon in liquid crystals. In: The Theory of Liquid Crystals, London-New York: Academic Press, 4 (1979), 1–81.
[18] P. Lions, *Mathematical topic in fluid mechanics*, Vol. 2 copressible models. New York, Oxford University Press, 1998.

[19] F. Lin, Nonlinear theory of defects in nematic liquid crystals; phase transition and flow phenomena, Commun. Pure. Appl. Math., 42 (1989), 789–814.

[20] F. Lin and C. Liu, Nonparabolic dissipative systems modeling the flow of liquid crystals, Commun. Pure. Appl. Math., 48 (1995), 501–537.

[21] F. Lin and C. Liu, Partial regularities of the nonlinear dissipative systems modeling the flow of liquid crystals, Disc. Contin. Dyn. Syst., A 2 (1996), 1–23.

[22] F. Lin and C. Wang, On the uniqueness of heat flows of harmonic maps and hydrodynamic flow of nematic liquid crystals, Chinese Annals of Math. 31B 6 (2010), 921–938.

[23] C. Liu and N. Wakington, Approximation of Liquid Crystal Flows, SIAM J. Numer. Anal., 37 (2000), 725–741.

[24] Q. Liu and S. Cui, Regularity of solutions to 3-D nematic liquid crystal flows, Electro. J. Differ. Equ. 173 (2010), 1–5.

[25] X. Liu and L. Liu, A blow-up criterion for the compressible liquid crystals system, arXiv:1011.4399v2 [math-ph] 23 Nov. 2010.

[26] X. Liu, L. Liu and Y. Hao, Existence results for the flow of compressible liquid crystals system, arXiv:1106.6140v1 [math.FA] 30 Jun. 2011.

[27] A. Matsumura and T. Nishida, The initial value problem for the equations of viscous and heat–conductive gases. J. Math. Kyoto. Univ. 20 (1980), 67–104.

[28] J. Serrin, On the interior regularity of weak solutions of the Navier–Stokes equations, Arch. Rational Mech. Anal., 9 (1962), 187–195.

[29] H. Sun and C. Liu, On energetic variational approaches in modeling the nematic liquid crystal flows, Disc. Contin. Dyn. Syst., A 23 (2009), 455–475.

[30] Y. Sun, C. Wang and Z. Zhang, A Beale-Kato-Majda blow-up criterion for the 3-D compressible Navier–Stokes equaitons, J. Math. Pures Appl., 95 (2011), 36–47.

[31] Y. Sun, C. Wang and Z. Zhang, A Beale-Kato-Majda Criterion for Three Dimensional Compressible Viscous Heat–Conductive Flows, Arch. Rational Mech. Anal., 201 (2011), 727–742.

[32] Z. Xin, Blow up of smooth solutions to the compressible NavierCStokes equation with compact density, Commun. Pure Appl. Math., 51 (1998), 229–240.