EXPOSITIONS OF SOME ONE-DIMENSIONAL GAUSS-MANIN COHOMOLOGIES

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Abstract. In this note we prove a result about the exponents of a particular kind of one-dimensional Gauss-Manin systems and the vanishing of the cohomologies of certain Koszul complex. This can be related to a more general family of such a complex of \(\mathcal{D}\)-modules. As an application, we calculate a set of possible exponents of the Gauss-Manin system of some arrangements of hyperplanes in general position, relevant to Dwork families.

1. Introduction

Let \(k\) be an algebraically closed field of characteristic zero. An algebraic variety, or just variety, will mean for us a quasi-projective separated finite type scheme over \(k\), reducible or not; in any case we will assume that all of our varieties are either equidimensional or smooth. For any variety \(X\), we will denote by \(\pi_X\) the projection from \(X\) to a point and, if \(X\) is smooth, \(\mathcal{D}_X\) will denote the category of bounded complexes of \(\mathcal{D}_X\)-modules.

The exponents of a \(\mathcal{D}_X\)-module are very related to the monodromy of its algebraic or formal solutions. This notion is topological in nature when \(k = \mathbb{C}\), but we can manage to work in an algebraic way with a similar concept, and because of that, we will usually use both names, monodromy and exponents, to denote the phenomenon and the object of study. Although this theory can be constructed in any dimension thanks to the formalism of the \(V\)-filtration, the Bernstein-Sato polynomial and the vanishing cycles of Malgrange and Kashiwara (cf. [Ma], [Kas], [MM] or the appendix by Mebkhout and Sabbah at [Me, § III.4]), it is defined in a much more simple way in dimension one. In fact, we will follow the classical approach of [Kn²] § 2.11.

The aim of this note is to prove the following result:

**Theorem 1.1.** Let \(n\) be a fixed positive integer, and let \(g \in k[x_1, \ldots, x_n]\) be a nonzero polynomial. Let now \(R = k((t))[x_1, \ldots, x_n, g^{-1}]\), \(f \in k[x_1, \ldots, x_n, g^{-1}]\), and denote by \(f'_i\) the partial derivatives of \(f\) and \(G = \{g(x) = 0\} \subseteq \mathbb{A}^n\). Let \(\alpha \in k\) and let \(\varphi_\alpha = \partial_t - \alpha t^{-1}\) be an endomorphism of \(k((t))\). Denote also by \(f\) the associated morphism \(k^n - G \to \mathbb{A}^1\). Then, \(\alpha \mod \mathbb{Z}\) is not an exponent of any of the \(\mathcal{D}_{\mathbb{A}^1}\)-modules \(\mathcal{H}^i f_* \mathcal{O}_{\mathbb{A}^n - G}\) at the origin if and only if the morphism

\[
\Phi : R^{n+1} \longrightarrow R
\]

\[
(a, b^1, \ldots, b^n) \mapsto (f(t)a, (\partial_1 + f'_1\varphi_\alpha)b^1, \ldots, (\partial_n + f'_n\varphi_\alpha)b^n)
\]

is surjective. If it is not surjective, the number of Jordan blocks associated with \(\alpha\) of the zeroth cohomology is the dimension of the cokernel of \(\Phi\) as a \(k\)-vector space.

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Let us recall now the basic setting of one-dimensional Gauss-Manin systems, seen from the point of view of $\mathcal{D}$-module theory.

Fix a positive integer $n$, some variables $x_1, \ldots, x_n$ and a special one called $\lambda$. Consider an open set $U \subseteq \mathbb{A}^n = \text{Spec}(k[x_1, \ldots, x_n])$ and a smooth variety $X \subseteq U \times \mathbb{A}^1 = U \times \text{Spec}(k[\lambda])$, together with the second canonical projection $\pi_2 : X \to \mathbb{A}^1$. The Gauss-Manin cohomology, or system, of $X$ is just its relative cohomology with respect to the variety of parameters. In terms of $\mathcal{D}$-modules, that means the direct image of the structure sheaf $\pi_2^*\mathcal{O}_X$. It is a complex of $\mathcal{D}_{\mathbb{A}^1}$-modules, so we could be interested in knowing its behaviour at the origin, and in particular its exponents. In this paper we will focus in the case where $X$ is a hypersurface.

Going back to the theorem, it is clear than the direct image $f_+\mathcal{O}_{\mathbb{A}^n-G}$ can be seen as the Gauss-Manin cohomology of the graph of $f$ in $(\mathbb{A}^n-G) \times \mathbb{A}^1$. However, that is a rather concrete and simple example of a family of hypersurfaces. We will explain in section 3 how to relate this setting to a broader family of Gauss-Manin systems.

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### 2. Preliminaries

In this section we will recall the basic concepts from $\mathcal{D}$-module theory that we will need in the following and have not been mentioned yet.

**Definition 2.1.** Let $f : X \to Y$ be a morphism of smooth varieties. The direct image of complexes of $\mathcal{D}_X$-modules is the functor $f_+ : \mathbf{D}^b(\mathcal{D}_X) \to \mathbf{D}^b(\mathcal{D}_Y)$ given by

$$f_+\mathcal{M} := Rf_* (D_{Y \leftarrow X} \otimes_{\mathcal{D}_X} \mathcal{M}),$$

where $D_{Y \leftarrow X}$ is the $(f^{-1}\mathcal{D}_Y, \mathcal{D}_X)$-bimodule

$$D_{Y \leftarrow X} := \omega_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\text{Hom}_{\mathcal{O}_Y}(\omega_Y, \mathcal{D}_Y).$$

**Remark 2.2.** When $f : X = Y \times Z \to Z$ is a projection, $D_{Z \leftarrow X} \otimes_{\mathcal{D}_X} \mathcal{M}$ is nothing but a shifting by $\dim Y$ places to the left of the relative de Rham complex of $\mathcal{M}$

$$\text{DR}_f(\mathcal{M}) := 0 \to \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^1_{X/Z} \to \cdots \to \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^n_{X/Z} \to 0,$$

so we will have that $f_+ \cong Rf_* \text{DR}_f(\bullet)(\dim Y)$ (cf. [McI 1.5.2.2]).

When $f : X \to Y$ is a closed immersion of smooth varieties and $\mathcal{M}$ is the inverse image $f^+\mathcal{N}$ of some complex $\mathcal{N} \in \mathbf{D}^b(\mathcal{D}_Y)$ of quasi-coherent $\mathcal{O}_Y$-modules, then $f_+\mathcal{M} \cong \mathcal{N}(\ast X)/\mathcal{N}$ (cf. [HTT 1.7.1]).

We will focus on the case in which $X$ is an open subvariety of the affine line. From now on, we will denote by $D_x$ the product $x\partial_x$, omitting the variable as long as it is clear from the context.

**Definition 2.3.** A Kummer $\mathcal{D}$-module is the quotient $\mathcal{K}_\alpha = \mathcal{D}_{\mathbb{G}_m}/(D - \alpha)$, for any $\alpha \in k$.

**Remark 2.4.** Note that, by a twist by $x$, any two Kummer $\mathcal{D}$-modules $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$ are isomorphic if $\alpha - \beta$ is an integer. Then $\mathcal{K}_\alpha \cong \mathcal{O}_{\mathbb{G}_m}$ for any $\alpha \in \mathbb{Z}$. 

Proposition 2.5. Let $\mathcal{M}$ be a holonomic $D_X$-module, let $p$ be a point of $X$, and fix a formal parameter $x$ at $p$ such that $\hat{\mathcal{O}}_{X,p} \cong k[[x]]$. The tensor product $\mathcal{M} \otimes_{\mathcal{O}_X} k((x))$ can be decomposed as the direct sum of its regular and purely irregular parts.

Now suppose that $\mathcal{M} \otimes_{\mathcal{O}_X} k((x)) \cong k((x))[D]/(L)$, where

$$L = \sum_i x^i A_i(D) \in k[[x]][D],$$

with $\deg_D L = g \geq g_0 = \deg_D A_0$. Then, the rank of $(\mathcal{M} \otimes_{\mathcal{O}_X} k((x)))_{\text{reg}}$ is $g_0$, and if this last degree is positive and $A_0(t) = \gamma \prod_i (t - \alpha_i)^{n_i}$, its composition factors are $K_{\alpha_i,p}$ with multiplicity $n_i$, where $K_{\beta,p}$ is the tensor product over $k((x))$ with the isomorphic image of the Kummer $D$-module $K_{\beta}$ under the translation $0 \to p$.

Moreover, if the roots of $A_0(t)$ are not congruent modulo $\mathbb{Z}$, then

$$(\mathcal{M} \otimes_{\mathcal{O}_X} k((x)))_{\text{reg}} \cong k((x))[D] / (A_0(D)) \cong \bigoplus_i k((x))[D] / (D - \alpha_i)^{n_i}.$$  

Proof. The decomposition into regular and purely irregular parts of the tensor product $\mathcal{M} \otimes_{\mathcal{O}_X} k((x))$ is a well-known fact of the theory of integrable connections over a field (cf. [Ka1] 11.5, 11.9).

The rest is analogous to [Ka2] 2.11.7. This result can be stated in fact for any algebraically closed field of characteristic zero, as any previous result over which it lies. □

Proposition 2.6. (Formal Jordan decomposition lemma) Let $\mathcal{M}$, $p$ and $x$ as before, and suppose that $\mathcal{M}$ is regular at $p$. Then,

i) $\mathcal{M} \otimes_{\mathcal{O}_X} k((x))$ is the direct sum of regular indecomposable $k((x))[[D]]$-modules.

ii) Any regular indecomposable $k((x))[[D]]$-module is isomorphic to $\text{Loc}(\alpha, n_\alpha) := k((x))[D] / (D - \alpha)^{n_\alpha}$, where $\alpha$ is unique modulo the integers.

iii) For any two regular indecomposables $\text{Loc}(\alpha, n_\alpha)$ and $\text{Loc}(\beta, n_\beta)$, and $i = 0, 1$, the vector space $\text{Ext}^i_{D_X}(\text{Loc}(\alpha, n_\alpha), \text{Loc}(\beta, n_\beta))$ has dimension $\min(n_\alpha, n_\beta)$ if $\alpha - \beta \in \mathbb{Z}$, being zero otherwise.

iv) Given $\alpha \in k$, the number of indecomposables of type $\text{Loc}(\alpha, m)$ in the decomposition of $\mathcal{M} \otimes_{\mathcal{O}_X} k((x))$ is the dimension of the vector space $\text{Hom}_{D_X}(\mathcal{M} \otimes_{\mathcal{O}_X} K_{\alpha,p}, k((x)))$.

Proof. When $k = \mathbb{C}$, there is a topological proof as in [Ka2] 2.11.8. However, we can give a purely (linear) algebraic proof of that.

Since $\mathcal{M}$ is holonomic, it is a finitely generated torsion $D_{G_m}$-module, and so will be $\mathcal{M} \otimes k((x))$ over $k((x))[[D]]$. This ring is a noncommutative principal ideal domain, so by the structure theorem for finitely generated modules over such a ring [JL §3, theorem 19] we obtain that $\mathcal{M} \otimes k((x))$ is the direct sum of indecomposable $k((x))[[D]]$-modules. They must be regular since $\mathcal{M}$ is, and that proves point i.

Therefore, we can affirm that

$$\mathcal{M} \otimes k((x)) \cong \bigoplus_{i=1}^r k((x))[D] / (A_i(x,D)),$$

where $A_i(x,D) = \sum_{j \geq 0} x^j A_{ij}(D)$ and every $A_{i0}$ has its roots incongruent modulo the integers. By the previous proposition,

$$k((x))[D] / (A_i(x,D)) \cong k((x))[D] / (A_{i0}(D)).$$
Now, since \( k \) is algebraically closed and \( k((x))[D]/(A_{d0}(D)) \cong k[D]/(A_{d0}(D)) \otimes_k k((x)) \), applying again the structure theorem for finitely generated modules over a commutative pid this time, we get point ii.

Let now Loc(\(\alpha, n_\alpha\)) and Loc(\(\beta, n_\beta\)) be as in point iii. We can suppose that both \(\alpha\) and \(\beta\) belong to the same fundamental domain (exhaustive set of representatives without repetitions) of \( k/\mathbb{Z} \), up to isomorphism. Since Loc(\(\alpha, n_\alpha\)) is a flat \( k((x))\)-module, we can assume that \(\alpha = 0\). Now the vector spaces Ext^{i}_{D_{X}}(\text{Loc}(\alpha, n_\alpha), \text{Loc}(\beta, n_\beta))\) are just the kernel and the cokernel of \( D^{n_\alpha} \) over \( \text{Loc}(\beta, n_\beta) \). If \(\beta = 0\), then the statement is easy to check. And if \(\beta \neq 0\), both are zero for \(\text{Loc}(\beta, n_\beta)\) is a successive extension of \( K_{\beta,p} \) and \( D^{n_\alpha} \) is bijective over them. Point iv is just an easy consequence of the two preceding ones. \( \square \)

Those two propositions show that the equivalence classes modulo \( \mathbb{Z} \) of the numbers \(\alpha\) appearing in the decomposition of the tensor product of a holonomic \( D_{X}\)-module with \( k((x)) \), and their associated \( n_\alpha\), are intrinsic to the \( D_{X}\)-module and quite important, actually, to know its behaviour at a point, so that motivates the following definition.

**Definition 2.7.** Let \( M, p \) and \( x \) as in proposition 2.5. The exponents of \( M \) at \( p \) are the values \(\alpha_i \in k\) such that
\[
(M \otimes_{O_{X}} k((x)))_{\text{reg}} \cong \bigoplus_i \text{Loc}(\alpha_i, n_i),
\]
seen as elements of \( k/\mathbb{Z} \). For each exponent \(\alpha_i\) we define its multiplicity to be \(n_i\).

**Remark 2.8.** For the sake of simplicity, we will usually denote both an exponent and some of its representatives in \( k \) in the same way.

Exponents are considered unordered and eventually repeated. Note that, when \( k = \mathbb{C} \), that notion of multiplicity of an exponent \(\alpha\) is related to the size of the Jordan blocks of local monodromy associated with the eigenvalue \( e^{2\pi i \alpha} \), and not to its multiplicity as a root of the characteristic polynomial of the monodromy. However, these two notions are the same under some special conditions (cf. [Ka2 3.2.2, 3.7.2]). Nevertheless, in our algebraic setting, whenever we mention “Jordan block” we will mean a regular indecomposable \( \text{Loc}(\alpha, n_\alpha) \), in analogy with the complex analytic case.

3. **GAUSS-MANIN SYSTEMS, MAIN RESULT AND LAURENT SERIES**

Let us return to the context of Gauss-Manin systems and recover the notion of the end of the introduction. Even though the variable \( \lambda \) can lie in the equation for \( X \) in a very complicated way, we will assume that there exist two rational functions \( p(x) \) and \( q(x) \), defined on \( U \), such that \( X \) is defined by the equation \( p(x) - \lambda^d q(x) = 0 \), for certain \( d > 0 \). We will reduce ourselves to the case in which \( U \) is a basic open set of \( A^n \), say \( \{ r(x) \neq 0 \} \) for some polynomial \( r(x) \in k[x] \).

The first reduction we can do is to consider \( d = 1 \). Indeed, form the cartesian diagram

\[
\begin{array}{ccc}
X & \xrightarrow{id \times [d]} & X' \\
\pi_2 & \downarrow \quad \square \quad \downarrow \pi_2 \\
A^1 & \xrightarrow{[d]} & A^1
\end{array}
\]

where \([d]\) just means taking the \( d \)-th power of the argument and \( X' \) is the variety of \( U \times A^1 \) given by \( p(x) - \lambda q(x) = 0 \). It is easy to check that if \( X \) is smooth, so is \( X' \). Therefore, by the base change formula [HTT 1.7.3], \( \pi_{2,*} O_X \cong [d]^+ \pi_{2,*} O_{X'} \), so we could find the
exponents of the Gauss-Manin cohomology of \( X \) by finding those of \( \bar{X} \); the former will just be \( d \) times the latter.

We are a bit closer to the particular setting of our main result. In fact, now we only need to invert \( q \). Write both \( p \) and \( q \) as fractions with the same denominator \( p/r \) and \( q/r \), respectively. Then \( X \) is the vanishing locus of \( p(x) - \lambda q(x) \) in \( U \times \mathbb{A}^1 \). Call \( Z \) to the hypersurface of \( X \) with equation \( q(x) = 0 \), and take \( V := X - Z \). Now we can form the excision triangle (cf. \[Me\] § I.6.1)

\[
\mathbf{R} \Gamma Z \mathcal{O}_X \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(*V). 
\]

Note what happens when we apply \( \pi_{2,+} \) to the triangle. Since \( Z \) is a product of a subvariety \( Z' \) of \( U \) with \( \mathbb{A}^1 \), by extension of scalars

\[
\text{Hom}_{\mathcal{O}_{U \times \mathbb{A}^1}} \left( \mathcal{O}_{U \times \mathbb{A}^1} / \mathcal{J}_Z^k, \mathcal{O}_X \right) \cong \text{Hom}_{\mathcal{O}_U} \left( \mathcal{O}_U / \mathcal{J}_Z^k, \mathcal{O}_U \boxplus \mathcal{O}_{\mathbb{A}^1} \right),
\]

for every \( k \geq 0 \), \( \mathcal{J}_Y \) being the ideal of definition of a variety \( Y \) in its associated ambient space. It is easy to see that the last \( \text{Hom} \) is isomorphic to \( \text{Hom}_{\mathcal{O}_U} \left( \mathcal{O}_U / \mathcal{J}_Z^k, \mathcal{O}_U \right) \boxplus \mathcal{O}_{\mathbb{A}^1} \) using that the first argument is a cyclic \( \mathcal{O}_U \)-module. In conclusion, by taking direct limits of the isomorphisms above and applying the Künneth formula \cite[1.5.30]{HTT}, we can affirm that \( \pi_{2,+} \mathbf{R} \Gamma Z \mathcal{O}_X \) is just a bunch of copies of the structure sheaf \( \mathcal{O}_{\mathbb{A}^1} \), so the noninteger exponents and the integer ones with multiplicity greater than one of \( \pi_{2,+} \mathcal{O}_X \) are the same as those of \( \pi_{2,+} \mathcal{O}_X(*V) \), that is to say, apart from the purely constant part, the useful information about the exponents of \( \pi_{2,+} \mathcal{O}_X \) can be found within \( \pi_{2,+} \mathcal{O}_X(*V) \). But now this complex can be realized in the form of our theorem just by taking \( g = r\bar{q} \) and \( f = p/\bar{q} \).

Once we have explained the relation of the main result to more general Gauss-Manin systems, let us return to it.

Proof of theorem \[J.J.\] We will deal first with the zeroth cohomology of \( f_+ \mathcal{O}_{\mathbb{A}^n-G} \) and then we will justify the extension of the statement to all of them.

Let \( K = f_+ \mathcal{O}_{\mathbb{A}^n-G} \). Since \( K \) is a complex of regular holonomic \( \mathcal{D}_{\mathbb{A}^1} \)-modules, by proposition \[2.6\] we can claim that

\[
\mathcal{H}^0(K) \otimes_{\mathcal{O}_{\mathbb{A}^1}} \mathbb{k}(t) \cong \bigoplus_{i=1}^r \mathbb{k}(t)[D]/(D - \beta_i)^{m_i}
\]

Therefore, \( \alpha \) will be an exponent of \( \mathcal{H}^0(K) \) if and only if the endomorphism

\[
D - \alpha : \mathcal{H}^0(K) \otimes_{\mathcal{O}_{\mathbb{A}^1}} \mathbb{k}(t) \longrightarrow \mathcal{H}^0(K) \otimes_{\mathcal{O}_{\mathbb{A}^1}} \mathbb{k}(t)
\]

is surjective, for

\[
\text{Ext}^1_{\mathbb{k}(t)[D]} \left( \mathbb{k}(t)[D]/(D - \alpha), \mathbb{k}(t)[D]/(D - \beta)^k \right) = 0,
\]

whenever \( \alpha \neq \beta \pmod{\mathbb{Z}} \), for \( i = 0, 1 \) and any \( k \).

Now let us decompose the morphism \( f \) as the closed immersion into its graph \( i_{\Gamma} \) followed by the projection on the first coordinates \( \pi \), so that we have to prove that \( D - \alpha \) is surjective on

\[
\pi_+ \mathcal{O}_{(\mathbb{A}^n-G) \times \mathbb{A}^1}(*\Gamma)/\mathcal{O}_{(\mathbb{A}^n-G) \times \mathbb{A}^1} \otimes_{\mathcal{O}_{\mathbb{A}^1}} \mathbb{k}(t),
\]

for the graph of \( f \) is smooth in \( \mathbb{A}^{n+1} \).

Note that we are always dealing with affine morphisms and quasi-coherent \( \mathcal{O}_{(\mathbb{A}^n-G) \times \mathbb{A}^1} \)-modules and we are taking tensor products with \( \mathbb{k}(t) \), so it suffices to work from now on with the global sections of the objects involved in the proof.
Let us denote by $M = k[x, g^{-1}, t] \langle (t - f)^{-1} \rangle / k[x, g^{-1}, t]$. Recall that we are interested in the last cohomology of $\text{DR}_x(M)$ (for $\pi_+$ is just a relative de Rham). Since $k((t))$ is flat over $k[t]$, tensor products with the former over the latter commutes with cohomology, and thus we are going to deal with

$$M_{\text{loc}} := k((t))[x, g^{-1}] \langle (t - f)^{-1} \rangle / k((t))[x, g^{-1}];$$

it is a module over $R$ and $\hat{D} := R(\partial_1, \partial_1, \ldots, \partial_n)$.

Let us introduce just a bit more of notation that we are going to use. We will denote by $D_t$, $D_\mathcal{L}$ and $\hat{D}_\mathcal{L}$ the rings $k((t))(\partial_t)$, $k[x, g^{-1}]<\partial_1, \ldots, \partial_n>$ and $k((t))[x, g^{-1}]<\partial_1, \ldots, \partial_n>$, respectively.

Summing everything up, $\alpha$ mod $\mathbb{Z}$ is not an exponent of the $\mathcal{D}_{\mathcal{A}_1}$-module $\mathcal{H}^0 f_+ \mathcal{O}_{\mathcal{A}^n}$ at the origin if and only if

$$R^1 \text{Hom}_{D_t} \left(D_t/(D - \alpha), R^n \text{Hom}_{\hat{D}_\mathcal{L}}(R, M_{\text{loc}})\right) = 0.$$  

Note that $R \cong \hat{D}_\mathcal{L} \otimes_{D_\mathcal{L}} k[x, g^{-1}]$, so by extension of scalars

$$R^n \text{Hom}_{\hat{D}_\mathcal{L}}(R, M_{\text{loc}}) \cong R^n \text{Hom}_{D_\mathcal{L}}(k[x, g^{-1}], M_{\text{loc}}).$$

Now applying the derived tensor-hom adjunction,

$$R^1 \text{Hom}_{D_t} \left(D_t/(D - \alpha), R^n \text{Hom}_{D_\mathcal{L}}(k[x, g^{-1}], M_{\text{loc}})\right) \cong$$

$$\cong R^{n+1} \text{Hom}_{D_\mathcal{L}} \left(D_t/(D - \alpha) \otimes k[x, g^{-1}], M_{\text{loc}}\right) \cong$$

$$\cong R^{n+1} \text{Hom}_{\hat{D}} \left(\hat{D}/(D - \alpha, \partial_1, \ldots, \partial_n), M_{\text{loc}}\right),$$

the last isomorphism being by extension of scalars again.

Now note that $M_{\text{loc}}$ is autodual, being the direct image by a closed immersion of the autodual object $k[x, g^{-1}]$, so it is equivalent to prove that

$$R^{n+1} \text{Hom}_{\hat{D}} \left(M_{\text{loc}}, \hat{D}/(D + 1 + \alpha, \partial_1, \ldots, \partial_n)\right) = 0.$$  

The second $\hat{D}$-module above is nothing but $R \cdot t^{-1-\alpha}$, where $t^{-1-\alpha}$ should be understood as a symbol. The actions of the partial derivatives are the usual ones in $R$ of $\partial_1, \ldots, \partial_n$, and regarding $\partial_t$,

$$\partial_t (a \cdot t^{-1-\alpha}) = \partial_t(a) \cdot t^{-1-\alpha} + (-1 - \alpha)t^{-1}a \cdot t^{-1-\alpha}.$$  

In order to finish all this construction, take into account that the annihilator of the class of $(t - f)^{-1}$ in $M_{\text{loc}}$ is the left ideal $(f - t, \partial_1 + f'_1 \partial_1, \ldots, \partial_n + f'_n \partial_n)$; indeed, each of its generators make it vanish and the ideal is maximal. Therefore, $M_{\text{loc}}$ can be presented as

$$M_{\text{loc}} \cong \hat{D}/(f - t, \partial_1 + f'_1 \partial_1, \ldots, \partial_n + f'_n \partial_n).$$

Consequently, taking another representative of the exponent (actually subtracting one to it), $\alpha$ mod $\mathbb{Z}$ is not an exponent of the $\mathcal{D}_{\mathcal{A}_1}$-module $\mathcal{H}^0 f_+ \mathcal{O}_{\mathcal{A}^n}$ at the origin if and only if the $k$-linear homomorphism $\Phi : R^{n+1} \rightarrow R$ given by

$$\Phi = (f - t, \partial_1 + f'_1 \varphi_1, \ldots, \partial_n + f'_n \varphi_n)$$

is surjective.

The statement on the dimension of the cokernel follows easily by reversing the isomorphisms and applying point $iii$ of proposition 2.6.
Now note that the operators $f - t, \partial_1 + f_1^t \varphi_1, \ldots, \partial_n + f_n^t \varphi_n$ commute pairwise, so the Koszul complex $K^\bullet(R; f - t, \partial_1 + f_1^t \varphi_1, \ldots, \partial_n + f_n^t \varphi_n)$ is well defined. As a consequence, we can invoke [Bo] 9.1, Corollaire 1] and we are done. 

Although one cannot have the surjectivity of the morphism $\Phi$, if we pay attention to the proof, one can still have a partial result when dealing with the vanishing of the cohomologies of the whole Koszul complex. More concretely, we have the following:

**Corollary 3.1.** Under the same conditions as before, if the Koszul complex $K^\bullet(R; f - t, \partial_1 + f_1^t \varphi_1, \ldots, \partial_n + f_n^t \varphi_n)$ is acyclic in degrees $d_0$ to $d_1$ (eventually equal to zero or $n + 1$, respectively), then $\alpha \mod Z$ is not an exponent at the origin of any of the cohomologies $H^k f_+ O_{h^n-G}$ for $0 \leq 1 \leq k + n \leq d_1$.

**Proof.** If $K^\bullet(R; f - t, \partial_1 + f_1^t \varphi_1, \ldots, \partial_n + f_n^t \varphi_n)$ is acyclic at degree $k$, then

$$\mathcal{R}^k \text{Hom}_{\mathcal{D}} \left( M_{\text{loc}}, \mathcal{D} / (D + 1 + \alpha, \partial_1, \ldots, \partial_n) \right) = 0.$$ 

Reversing the isomorphisms given in the proof of the proposition, that object is the extension of

$$\mathcal{R}^j \text{Hom}_{\mathcal{D}} \left( D / (\alpha), \mathcal{R}^j \text{Hom}_{\mathcal{D}} \left( R, M_{\text{loc}} \right) \right)$$

with $j = k$ and $j = k - 1$. As a consequence, for every $i$ and $j$ with $d_0 - 1 \leq j \leq d_1$ such an object must vanish, and in conclusion, the endomorphism

$$D - \alpha : \mathcal{H}^j(K) \otimes \mathcal{k}(t) \rightarrow \mathcal{H}^j(K) \otimes \mathcal{k}(t)$$

is surjective for $d_0 - 1 \leq j + n \leq d_1$, so $\alpha \mod Z$ is not an exponent at the origin of any of the cohomologies $\mathcal{H}^j f_+ O_{h^n-G}$ for such values of $j$. 

We finish this section by providing several results or notions regarding the field of formal Laurent series that will be of interest later when we tackle a particular example.

**Lemma 3.2.** Let $\varphi : \mathcal{k}(t) \rightarrow \mathcal{k}(t)$ be a $\mathcal{k}$-linear automorphism of $\mathcal{k}(t)$ such that $\varphi(\mathcal{k}[\![t]\!]) \cdot t^k = \mathcal{k}[\![t]\!] \cdot t^k$. Then, for any $\mathcal{k}$-linear endomorphism $\psi$ of $\mathcal{k}(t)$ such that $\psi(\mathcal{k}[\![t]\!]) \cdot t^k \subseteq \mathcal{k}[\![t]\!] \cdot t^{k+1}$, the sum $\varphi + \psi$ is another automorphism of $\mathcal{k}(t)$.

**Proof.** Multiplying by $\varphi^{-1}$ we can assume that $\varphi = \text{id}$. We will write the elements of $\mathcal{k}(t)$ as $a = \sum a_k t^k$.

Let then $b$ be a fixed formal Laurent series and let us see if there exists an $a \in \mathcal{k}(t)$ such that $(\text{id} + \psi)(a) = b$. Evidently, the exponents of the least powers of $t$ (which is called the order) of both of $a$ and $b$ will be the same, so let us write

$$a = \sum_{k \geq m} a_k t^k, \psi(a) = \sum_{k \geq m+1} a'_k t^k \quad \text{and} \quad b = \sum_{k \geq m} b_k t^k.$$ 

From the equation $(\text{id} + \psi)(a) = b$ we deduce that $a_m = b_m$. Now call $a^1 = a - a_m t^m$ and $b^1 = b - (\text{id} + \psi)(a_m t^m)$; both of them have order equal to $m + 1$. We have that

$$(\text{id} + \psi)a^1 = (\text{id} + \psi)a - (\text{id} + \psi)(a_m t^m) = b^1.$$ 

Thus we can start over again the same process with $a_1$ and $b_1$. Since this can be continued for every power of $t$, we can deduce the surjectivity of $\text{id} + \psi$. Moreover, if we take $b_k = 0$ for every $k \in \mathbb{Z}$, it follows that every $a_k$ vanishes too, so $\text{id} + \psi$ is also injective. 

□
Definition 3.3. Let $r$ be an element of $k$. Then we can define the operators $D_{t,r} = t\partial_t + r$, and analogously $D_{t,i,r}$, for $i = 1, \ldots, n$. We will write $\varphi_r = \partial_t + rt^{-1} - t^{-1}D_{t,r}$ (note the change with respect to the previous section). They are $k$-linear endomorphisms of $k((t))$, so we can also consider them as operating within any $\mathbb{k}((t))$-algebra by extension of scalars.

Remark 3.4. It is easy to see that $D_{t,r}$ (and so $\varphi_r$) is an automorphism of $k((t))$ for every $r$ not an integer and only for them, for $D_{t,r}$ sends a power $t^k$ of $t$ to $(k+r)t^k$. In this case we can define another operator that will be of interest from now on:

Definition 3.5. Fix an element $\alpha$ of $k$, and let $r$ and $s$ be two other elements of $k$ such that $\alpha + s$ is not an integer. Then we can define the operator $A_{r,s} = t + r\varphi_{\alpha+s}^{-1}$.

Let $R_n = k((t))[x_1, \ldots, x_n]$ and $\beta \in k$. We can also define the $k$-linear endomorphisms of $R_n$ given by $A_{\beta D_{i,r},s} = t + \beta D_{i,r}\varphi_{\alpha+s}^{-1}$, where $i = 1, \ldots, n$.

In the following, for the sake of simplicity, we will denote by $A_r$ and $D_r$ the operators $A_{r,0}$ and $D_{t,r}$, respectively.

Remark 3.6. As before, $A_{r,s}$ is not always an automorphism of $k((t))$, as $A_{\beta D_{i,r},s}$ of $R_n$.

Since $A_{r,s}\varphi_{\alpha+s} = D_{t,\alpha+r+s}$, the former is bijective whenever $\alpha + r + s$ is not an integer. Analogously, $A_{\beta D_{i,r},s}\varphi_{\alpha+s} = \beta D_{i,r} + D_{t,\alpha+s}$. It sends $t^{k\frac{w_n}{\omega}}$ to $(\beta(u_i + r) + \alpha + k + s)t^{k\frac{w_n}{\omega}}$, so $A_{\beta D_{i,r},s}$ is bijective if and only if, for every integer $l$, we have that $\beta(l+r) + \alpha + s$ is not an integer.

Now we could wonder about the commutativity of those operators that we have just defined. We have the following lemma, whose proof is easy and left to the reader (for each relation, use some of the ones proved before and the Leibniz rule):

Lemma 3.7. Let $\alpha$ and $\beta$ be two elements of $k$, and $r$, $r'$, $s$ and $s'$ four another elements of $k$ such that neither $\alpha + s$ nor $\alpha + s'$ are integers. Then, the following relations hold:

- $A_{r,s}\varphi_{\alpha+s} = D_{t,\alpha+r+s}$, $\varphi_{\alpha+s}A_{r,s} = D_{t,\alpha+r+s+1}$.
- $\varphi_{\alpha} = t^{-1}D_{t,\alpha} = D_{t,\alpha+1}t^{-1}$, $t\varphi_{\alpha} = \varphi_{\alpha-1}t$, $D_{t,\alpha}\varphi_{\beta} = \varphi_{\alpha-1}D_{t,\beta}$.
- $D_{t,\alpha}D_{t,\beta} = D_{t,\alpha,\beta}D_{t,\alpha}$, $\varphi_{\alpha,\beta} = \varphi_{\beta+1}\varphi_{\alpha-1}$, $A_{r,s}A_{r',s'} = A_{r,s',-1}A_{r',s+1}$.
- $A_{r,s}t = tA_{r,s+1}$, $A_{\beta D_{i,r},s}x_i = x_iA_{\beta D_{i,r+1},s}$.

4. A concrete example

As we mentioned in the abstract, we conclude this note by giving an example of an application of theorem focusing on the case in which our morphism $f$ is defined by an arrangement of $n + 1$ hyperplanes of $\mathbb{H}^n$ in general position with multiplicities.

Let us set some notation of use from now on. Let $(w_0, \ldots, w_n) \in \mathbb{Z}_{\geq 0}^{n+1}$ be an $(n+1)$-uple of positive integers. Under a suitable linear change of variables, we will be able to work with the polynomial $\lambda = x_1^{w_1} \cdots x_n^{w_n}(1 - x_1 - \ldots - x_n)^{w_0}$. This case may seem rather simple, but it turns out to be interesting when we study the Gauss-Manin cohomology of a generalized Dwork family (cf. [Ca]).

One could wonder why we consider exactly $n + 1$ hyperplanes. This case is interesting enough because of the discussion above, and considering more might lead us to a unnecessary and rather difficult calculation.

On the other hand, working with less hyperplanes is quite easy: assume that for some $r \geq 0$ and every $i = 0, \ldots, r$ we have that $w_i = 0$. Note that we can restrict our morphism $\lambda$ (abusing a bit of the notation and calling it in the same way as the polynomial) from
The following conditions are equivalent:

everything is proved, so assume that $n > 2$ and take $r ≥ 2$ such that, up to a reordering of the indexes, $p$ divides $w_1, w_2, \ldots, w_r$. Evidently, if $r = n$ we are done, so we can assume that $r < n$ and $p$ does not divide $w_r + 1, \ldots, w_n$. For such values of $i$ and for $j = 1, \ldots, r$, we have that $a_j w_i = a_i w_j$, so $p$ must divide $a_j$. Therefore, we can simplify the fraction $a_i w_i$ dividing by $p$ both the numerator and the denominator. Renaming them as $a_i$ and $w_i$ we return to a new pair of $n$-uples satisfying the first condition of the statement, so we can assume from the beginning that $a_1$ and $w_1$ are prime to each other, reducing as before by all of its common prime divisors in case they are not.

Consequently, doing the same as before, we can claim that the first $r ≥ 2$ of the $w_i$ have a prime common divisor $p$. Since $a_1 w_i = a_i w_1$ for every $i = r + 1, \ldots, n$ and $p$ cannot divide $a_1$, then $p | w_i$ for every $i$ and then $\gcd(w_1, \ldots, w_n) > 1$. \hfill \Box

Let us deal then with the case of $n + 1$ hyperplanes; this is what the main result of this section will cover. However, we will not calculate the exponents of $\lambda^+ \mathcal{O}_{A^n}$, but stay with just a finite set of rational numbers as candidates (see Remark 4.3 after the following proposition).

**Proposition 4.2.** Keeping the previous notation, a rational number $α$ is an exponent of some cohomology of $\lambda^+ \mathcal{O}_{A^n}$ only if $w_i α$ is an integer for some $i = 0, \ldots, n$.

**Proof.** By virtue of theorem 1.3 we will prove the equivalent statement telling that for any rational $α$ such that $w_i α$ is not an integer for any $i = 0, \ldots, n$, the $k$-linear homomorphism
Let us assume that \( n \geq 2 \), but we will comment throughout the proof the changes we should do to treat the case in which \( n = 1 \).

For the sake of simplicity, let us denote by \( \sigma \) and \( d \), respectively, the sums \( x_1 + \ldots + x_n \) and \( \sum w_i \). In the following, \( l_i \) will mean \( w_i \sigma + w_0 x_i \) for each \( i = 1, \ldots, n \). Therefore, \( \lambda_i' = x^{w-e_i}(1-\sigma)^{w_0-1}(w_i-l_i) \) for every \( i \).

Let us pick an element \( c \) of \( R \), which we can assume without loss of generality to be homogeneous of degree \( m \geq 0 \), and let us say that there exist \( a \), and \( n \) polynomials \( b_i \) for every \( i = 1, \ldots, n \), so that \( \Phi(a,b_1^i, \ldots, b^n) = c \), and see which conditions we have to impose on them. For every \( r \geq 0 \), we will have that

\[
\sum_{j+k=r} \lambda_j a_k - ta_r + \sum_{i} \left( \lambda_i' \right)_i a_i b_i + \sum_i b_{i+1,i} = c_r.
\]

We will consider that \( a \) only has nonvanishing \( k \)-th homogeneous components for \( k = m, \ldots, m+d-1 \), and each of the \( b_i \) for \( k = m+1, \ldots, m+d \). Thus our general formula will be useful for us only for \( r = m, \ldots, m+2d-1 \).

Let us focus first in that the expression \( \lambda a + \sum_i \lambda_i' \varphi_{a} b_i = 0 \) holds in degrees between \( m+d \) and \( m+2d-1 \). From this fact we will obtain some additional and useful information about \( a \) and the \( b_i \).

If we take the common factors to every summand of the formula above, we get that

\[
(1-\sigma)^{w_0-1} \left( a x^w (1-\sigma) + \sum_i \varphi_{a} b_i x^{w-e_i}(w_i-l_i) \right) = 0,
\]

so \((a(1-\sigma), \varphi_{a} b_1(w_1-l_1), \ldots, \varphi_{a} b^n(w_n-l_n))\) is a syzygy of the sequence \((x^w, \partial_1 x^w, \ldots, \partial_N x^w)\).

Therefore, since it forms a monomial ideal, \( x_i \) divides \( \varphi_{a} b_i \) for every \( i \). Let us write \( \varphi_{a} b_i = x_i \tilde{b}_i \), so that we can divide by \( x^w \) in the last formula to obtain \( a(1-\sigma) + \sum_i \tilde{b}_i(w_i-l_i) = 0 \), which, recall, it will only be valid for degrees from \( m+1 \) to \( m+d \).

Let us start then in the formula above by \( r = m+d \). We have that

\[
a_{m+d-1} + \sum_i l_i \tilde{b}_{m+d-1} = \sum_i \left( a_{m+d-1} + \sum_j w_j \tilde{b}_j + w_0 \tilde{b}_{m+d-1} \right) x_i = 0.
\]

We could argue that the \( x_1, \ldots, x_n \) form a regular sequence in order to obtain an expression for their “coefficients” in terms of other polynomials. However, we will be apparently much stricter and directly assume that all of them vanish. Moreover, we will also assume that all of the \( \tilde{b}_i \) are equal. All these assumptions, and the subsequent ones, can be thought of altering the result of the proof as a stone alters a pond when thrown to it. However, it does not actually matters; if we started doing things in full generality, in the end we could note that not so much information is needed, so we could assume exactly the same restrictions, as the pond turns quieter again.

Once having tried to calm the reader, let us rename \( \tilde{b}_{m+d-1} = f \), for every \( i \); \( f \) is an homogeneous polynomial of degree \( m + d - 1 \). In the end we can write also that \( a_{m+d-1} = -df + \sum_{j=N+1}^n w_j h_j \).
Let us go on by taking $r = m + d - 1$. Our equation turns into

$$a_{m+d-2} - a_{m+d-1} + \sum_{i} l_i \bar{b}^i_{m+d-2} - \sum_{i} w_i \bar{b}^i_{m+d-1} = 0.$$ 

We can replace $a_{m+d-1}$ and the $\bar{b}^i_{m+d-1}$ by their values in terms of $f$, and get that

$$a_{m+d-2} + \sum_{i} l_i \bar{b}^i_{m+d-2} + w_0 f = 0.$$ 

Note that, since $f$ is homogeneous of degree $m + d - 1 > 0$, there exist $n$ homogeneous polynomials $f^{(1)}, \ldots, f^{(n)} \in R$ of degree $m + d - 2$ such that $f = \sum_{i} x_i f^{(i)}$. Replace $f$ by that sum in the formula above. In addition, as before, assume that every factor of $f$, and get that

$$a_{m+d-2} = -d f^{(1)} + \sum_{j=1}^{n} w_j f^{(j)}$$

$$\bar{b}^i_{m+d-2} = f^{(1)} - f^{(i)}, \quad i = 1, \ldots, n.$$ 

Let us move on and see what happens when $r = m + d - 2$. Here our favorite formula reads

$$a_{m+d-3} - a_{m+d-2} + \sum_{i} l_i \bar{b}^i_{m+d-3} - \sum_{i} w_i \bar{b}^i_{m+d-2} = 0.$$ 

Writing $a_{m+d-2}$ and the $\bar{b}^i_{m+d-2}$ like above and proceeding as in degree $m + d - 1$ yields that the terms in the $F^{(i)}$ vanish, so we can proceed exactly as the previous step.

More concretely, taking lower and lower values of $r$ as long as it is possible and renaming the subsequent $f^{(k)}$ that appear, we finally get that

$$a = \sum_{i=1}^{N} (-dx_i + w_i) F^{(i)} - dF$$

$$\varphi_x b^i = x_i \left( \sum_{j=1}^{N} x_j F^{(j)} - F^{(i)} + \bar{F} \right), \quad i = 1, \ldots, n, \quad i = 1, \ldots, n,$$

where the $F^{(i)}$ are polynomials of $R$ which only have nonvanishing $k$-th homogeneous components for $k = m, \ldots, m + d - 2$, and $\bar{F}$ is an homogeneous polynomial of degree $m$.

Summing up, we have been able to express our first unknowns, the polynomials $a$ and $b^i$, in terms of many other polynomials, and we do not know anything about them but their degrees. However, recall that we have other $d$ equations left arising from our general formula. Those will be the ones which will give us some information about our new unknowns, and are:

$$\begin{aligned}
\sum_{j+k=r} \lambda_{j} a_k - t a_r + \sum_{i} \sum_{j+k=r} (\lambda^i_{j}) b^i \cdot \sum_{i} b^{i}_{r+1, i} &= 0, \quad r = m + d - w_0, \ldots, m + d - 1, \\
-t a_r + \sum_{i} \partial_i b^{i}_{r+1} &= 0, \quad r = m + 1, \ldots, m + d - w_0 - 1, \\
-t a_m + \sum_{i} \partial_i b^{i}_{m+1} &= c_m
\end{aligned}$$

Let us find the expression for the system above. Keep into account the new expressions for $a$ and $b$ and that of $\lambda$ and its partial derivatives. If we put all that into the remaining
equations, we obtain the following system:

\[
\begin{aligned}
\sum dx_i A_{m+d+n-1} F_{(i),m+d-2} &+ (-1)^{w_0} w_0^w \sigma^{w_0-1} \tilde{F} = 0 \\
\vdots \\
\sum dx_i A_{m+d+n-r} F_{(i),m+d-r-1} &- \sum w_i A_{d_i+1} F_{(i),m+d-r} + \nonumber \\
&+ (-1)^{u_0-r+1} u_0 (w_0 - 1) \frac{D^w}{\sigma^{w_0-r}} \tilde{F} = 0, \quad r = 2, \ldots, w_0 - 1 \\
\vdots \\
\sum dx_i A_{m+d+n-m} F_{(i),m+d-m-1} &- \sum w_i A_{d_i+1} F_{(i),m+d-m} - w_0^w \tilde{F} = 0 \\
\vdots \\
\sum dx_i A_{m+d+r} F_{(i),m+r-1} &- \sum w_i A_{d_i+1} F_{(i),m+r} = 0, \quad r = 1, \ldots, d - w_0 - 1 \\
\vdots \\
dA_{m+n} \tilde{F} - \sum w_i A_{d_i+1} F_{(i),m} = c_m
\end{aligned}
\]

Let us try to prove that this system has a solution, and then, that \( \Phi \) is surjective. Let us denote by \( S_k \) the set \( \{ u \in \mathbb{N}^n : |u| = k \} \). For each \( k = 1, \ldots, m + d - 1 \) we obviously have that \( \text{supp}(F_{m+k}) = S_{m+k} \). Whenever that condition holds, we could choose the support of all of the \( F_{(i),m+k} \), up to a reordering on the set of exponents which would appear at each one. We will say that the support of a polynomial \( F_{(i),m+k} \) is maximal if it is the whole \( S_{m+k-1} \). Then, without loss of generality, let us assume the maximality of the supports of the following polynomials:

\[
F_{(1),m+k} \quad \text{for} \quad k = 0, \ldots, w_1 - 1,
\]

and for every \( i = 2, \ldots, n \),

\[
F_{(i),m+k} \quad \text{for} \quad k = w_1 + \ldots + w_{i-1}, \ldots, w_1 + \ldots + w_i - 1.
\]

(Obviously this definition of maximality and the assumptions on the \( F_{(i),m+k} \) are useless when \( n = 1 \).) Thanks to the choice of \( \alpha \) and remark \[3.7\] we know that each \( A_{d_i+1} \) is invertible, so we can solve any \( F_{(i),m+r} \) of maximal support in terms of \( F_{(i),m+r-1} \) and \( \tilde{F} \), for \( r = 0, \ldots, d - w_0 - 1 \), over all the possible support of the corresponding equation.

Now is when the choice on the supports of the \( F_{(i),m+k} \) makes sense. Start at the last equation by solving \( F_{(1),m} \) and replace its value in the preceding equation, and do this with the polynomial \( F_{(i),m+k} \) assumed to have a maximal support until we reach the \( w_0 \)-th equation. Assume that every unused polynomial \( F_{(i),m+k} \) vanishes (again this assumption does not simplify too much the system). As a consequence of all that, we reduce ourselves to deal with a newer system of only \( w_0 \) equations, consisting of the first \( w_0 - 1 \) ones of the preceding system and a new \( w_0 \)-th equation, namely

\[
\sum dx_i A_{m+d+n} F_{(i),m+d-n} A_{d_i+1} \tilde{F} - \sum w_i A_{d_i+1} F_{(i),m+d-n} = \sum dx_i \gamma c_m,
\]

where

\[
\gamma = d^d \prod_i w_i^{-w_0} A_{m+d+n-1} A_{d_1+1}^{-1} \cdots A_{m+d+n-w_0} A_{d_1+1}^{-1} \cdots A_{m+n+1} A_{d_1+1}^{-1} =
\]
Let us simplify the system once more; as before, despite it can make us to lose some of the generality needed to prove the proposition, this new assumption not only will preserve it, but leave the equations in a more manageable way. More concretely, assume that, for every $i = 1, \ldots, n$ and every $r = 2, \ldots, w_0$, the polynomials $F(i, m+d-r)$ coincide, and are divisible by $x^{w_0-r}$. Write $F(i, m+d-r) = x^{w_0-r}F_{m+d-r}$ for all those values of $i$ and $r$ (note that every polynomial $F_{m+d-r}$ is homogeneous of degree $m$). Thanks to this assumption, we can divide by $x^{w_0-r+1}$ each corresponding equation to get the simpler system of homogeneous polynomials of degree $m$

\[
\begin{align*}
&\begin{cases}
  dA_{m+n+1}F_{m+d-2} + (-1)^{w_0}w_0\tilde{F} = 0 \\
  \vdots \\
  dA_{m+n+r}F_{m+d-r-1} - (d - w_0)A_{m+n+r}F_{m+d-r} + \\
  +(-1)^{w_0-r+1}w_0 \left( w_0 - 1 \right) \tilde{F} = 0, \quad r = 2, \ldots, w_0 - 1 \\
  \vdots \\
  \left( \Upsilon A_{m+n} - w_0 \right) \tilde{F} - (d - w_0)A_{m+n-w_0}F_{m+d-w_0} = \Upsilon c_m
\end{cases}
\end{align*}
\]

This system is the one which we will prove to have a solution, so that finally we show that $\Phi$ is surjective.

The $A_r$ do not commute pairwise, so in principle we cannot deal with the determinant of the matrix of the system. However, under an easy change of variables, we can see the $A_r$ as elements of a commutative subring of the ring of endomorphisms of $R$. By our assumption on $\alpha$, the endomorphism $D_\alpha$ of $k((t))$ is invertible, so we can define a new operator $B_r$ as $A_r\alpha^{-1} = \alpha(1+rD_\alpha^{-1})$, which is an element of $k[D_\alpha^{-1}]$, a commutative ring whose action on $k((t))$ is defined by $D_\alpha^{-1}t^l = (l+\alpha)^{-1}t^l$. Now $\alpha^{-1}$ is an isomorphism of $R$, so we can rename the $F_k$ to mean $\alpha^{-1}F_k$, for each $k = m + d - w_0, \ldots, m + d - 2$.

Now every coefficient of the system lives in $k[D_\alpha^{-1}]$, except for $\Upsilon A_{m+n}$ in the last equation, which has degree $1$ in $t$. Nevertheless, note that that operator goes together with $-w_0$, so by lemma [3.2] its sum is an automorphism of $R$. In fact, having applied the change of variables of the previous paragraph, every operator has degree $0$ in $t$, and thanks to the same lemma, regarding just the existence of solutions to the system and not its actual form, we can restrict ourselves to work only with $-w_0$. In fact, if $w_0 = 1$, then we only have a single equation, from which we can solve $\tilde{F}$ and thus the system, showing the existence of solutions. In the following we will assume that $w_0 \geq 2$.

We have finally arrived at a point in which we have a matrix of coefficients in $k[D_\alpha^{-1}]$, so we just need to show that its determinant is an invertible endomorphism of $k((t))$.

That determinant is the following, expanding it along the last column:

\[
\begin{vmatrix}
  dA_{m+n+1} & 0 & \cdots & 0 & (-1)^{w_0}w_0 \\
  -(d - w_0)B_{m+n+1} & dB_{m+n+1} & \cdots & 0 & (-1)^{w_0-1}w_0(w_0 - 1) \\
  0 & -(d - w_0)B_{m+n+2} & \cdots & \vdots & \vdots \\
  \vdots & \vdots & \ddots & dB_{m+n+w_0+1} & w_0(w_0 - 1) \\
  0 & 0 & \cdots & -(d - w_0)B_{m+n-w_0} & -w_0
\end{vmatrix}
\]
work; it can be found at [Ca].

First of all rewrite the formula above as

\[ \text{Proof.} \]

\[ m \]

of the form

\[ n \]

we just need to show that for the values of \( \alpha \) under consideration, \( d_{\alpha,l} \) does not vanish for every \( l \in \mathbb{Z} \). It is easy to see that

\[
d_{\alpha,l} = \sum_{r=1}^{w_0} \left( \frac{w_0 - 1}{w_0 - r} \right) (-1)^{w_0-r} \prod_{k=1}^{r-1} \left( d + \alpha \frac{m + d + n - k}{l + \alpha} \right) \prod_{k=r}^{w_0-1} \left( (d - w_0) \alpha + \frac{m + d + n - k - 1}{l + \alpha} \right) = \left( \frac{\alpha}{l + \alpha} \right)^{w_0-1} \sum_{r=1}^{w_0} \left( \frac{w_0 - 1}{w_0 - r} \right) (-1)^{w_0-r} \prod_{k=1}^{r-1} (d(\alpha + l) + m + d + n - k) \prod_{k=r}^{w_0-1} ((d - w_0)(\alpha + l) + m + d + n - k - 1).\]

Up to the factor \( (\alpha/(l + \alpha))^{w_0-1} \), the expression above is a polynomial in \( \alpha \) of degree \( w_0 - 1 \), so there will be at most \( w_0 - 1 \) values of \( \alpha \) so that it vanishes. In fact, for a fixed \( l \), they are \( -l - a/w_0 \), for \( a = 1, \ldots, w_0 - 1 \).

Indeed, let \( a \) as above. Then \( d_{l-a/w_0,l} \) can be written as

\[
\prod_{k=1}^{w_0-a} \left( \frac{-d}{w_0} a + m + d + n - k \right) \sum_{r=1}^{w_0} \left( \frac{w_0 - 1}{w_0 - r} \right) (-1)^{w_0-r} \prod_{j=1}^{a-1} \left( \frac{-d}{w_0} a + m + n + d - r + j \right) = C_a \sum_{r=1}^{w_0} \left( \frac{w_0 - 1}{w_0 - r} \right) (-1)^{w_0-r} p_a(r),
\]

where \( C_a \) and \( p_a \) are, respectively, a constant and a polynomial of degree \( a - 1 \leq w_0 - 2 \).

Now thanks to the following lemma, we can deduce the vanishing of the determinant, so we finally found that for every \( \alpha \) such that \( w_i/\alpha \) is not an integer number the original system has a solution.

All this process could be made independently of the choice of \( m, n \), all of the \( w_i \) and \( c_m \), so it finally proves the surjectivity of \( \Phi \) and the proposition. \( \square \)

**Remark 4.3.** The converse of the proposition holds in a stronger way: every cohomology of \( \mathcal{L}_+ \mathcal{O}_{K^n} \) is constant, excepting the last one, \( H^0 \mathcal{L}_+ \mathcal{O}_{K^n} \), whose exponents are exactly those of the form \( j/w_i \), for \( j = 1, \ldots, w_i \) and \( i = 0, \ldots, n \). However, it needs much more previous work; it can be found at [Ca].

**Lemma 4.4.** Let \( m, n \) be two integers such that \( n > 1 \) and \( 0 \geq m < n - 1 \). Then,

\[
\sum_{k=1}^{n} \binom{n-1}{n-k} (-1)^{n-k} k^m = 0.
\]

**Proof.** First of all rewrite the formula above as

\[
\sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k (n - k)^m,
\]

which is obviously true if \( n = 2 \), so assume \( n > 2 \). If we show that \( \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k k^m = 0 \) for every \( n \) and \( m < n - 1 \) we will be done. Let us do it by induction on \( n \) and \( b \). If \( n = 3 \) or \( m = 0 \) it is also very easy to prove it.
Let us go therefore for a general \( n \), take some \( n-1 > m > 0 \) and assume that
\[
\sum_{k=0}^{a-1} \binom{a-1}{k} (-1)^k b = 0 \text{ for every } a \leq n \text{ and } b < \min(m, a - 1).
\]
Now note that
\[
\sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k m = (-1)^{n-1} (n - 1)^m + \sum_{k=1}^{n-2} \left( \binom{n-2}{k-1} + \binom{n-2}{k} \right) (-1)^k k^m = \sum_{k=1}^{n-2} \binom{n-2}{k} (-1)^k (k^m - (k+1)^m).
\]
Since \( k^m - (k+1)^m \) is a polynomial of degree \( m-1 \) in \( k \), we just need to apply the induction hypothesis to finish. \( \square \)

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