Chiral black hole in three-dimensional gravitational Chern-Simons

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Abstract

A chiral black hole can be defined from the three-dimensional pure gravitational Chern-Simons action as an independent gravitational theory. The third order derivative of the Cotton tensor gives a dimensional constant which plays a role of the cosmological constant. The handedness of angular momentum depends on the signature of the Chern-Simons coefficient. Even in the massless black hole which corresponds to the static black hole, it has a nonvanishing angular momentum. We also study statistical entropy and thermodynamic stability.

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It has been shown by Deser, Jackiw, and Templeton that the three-dimensional Einstein-Hilbert action together with the gravitational Chern-Simons (GCS) term called the topologically massive gravity has nontrivial excitations even in the lower dimensional gravity [1]. On the other hand, Bañados, Teitelboim, and Zanelli (BTZ) have shown that the Einstein-Hilbert action with a negative cosmological constant admits a black hole solution which has a well-defined vacuum [2]. Recently, the fully combined theory of cosmological topologically massive gravity (CTMG) has been intensively studied in Refs. [3, 4] to investigate (negative energy) graviton modes. It has been shown that the bulk modes do not vanish even at the critical value [5, 6, 7].

In this paper, we would like to find a black hole solution from the pure GCS action as an independent gravitational theory, which is in fact the well-known BTZ solution; however, the role of conserved quantities should be reversed in the sense that the mass and the angular momentum in the BTZ black hole correspond to the angular momentum and the mass, respectively. Moreover, the third order derivative in the GCS action gives integration constants, one of them is a cosmological constant and the others are two conserved charges. Then, we will calculate the statistical entropy and discuss its thermodynamic stability.

The GCS action with a coupling $1/\mu$ is

$$S_{GCS} = \frac{1}{4\kappa^2_3 \mu} \int d^3x \sqrt{-g} \epsilon^{\lambda\mu\nu} \Gamma^\alpha_{\lambda\sigma} \left[ \partial_\mu \Gamma^\sigma_{\nu\rho} + \frac{2}{3} \Gamma^\sigma_{\mu\tau} \Gamma^\tau_{\nu\rho} \right],$$

(1)

where $\kappa^2 = 8\pi G_3$ is a three-dimensional Newton constant, and $\epsilon^{\lambda\mu\nu}$ is a three-dimensional anti-symmetric tensor defined by $\epsilon^{012} = +1/\sqrt{-g}$. Now, varying the GCS action (1), an equation of motion is given by the vanishing Cotton tensor

$$C^\mu_{\nu} \equiv \epsilon^{\mu\alpha\beta} \nabla_\alpha \left[ R_{\nu\beta} - \frac{1}{4} g_{\nu\beta} R \right] = 0.$$  

(2)

It admits various solutions with a nontrivial curvature scalar; for instance, a solution lifted up from a two-dimensional kink solution [8], $ds^2 = - \text{sech}^4(\sqrt{c}x/2)dt^2 + dx^2 + [dy + \text{sech}^2(\sqrt{c}x/2)dt]^2$, whose curvature scalar depends on the coordinate, $R = -3c/2 + (5c/2) \text{sech}^2(\sqrt{c}x/2)$.

Here, we are going to focus on a solution satisfying the constraint of $R_{\nu\beta} = (1/3) g_{\nu\beta} R$. Then, the equation of motion (2) can be reduced to $(1/12) \epsilon^{\mu\alpha\beta} g_{\nu\beta} \partial_\alpha R = 0$. As a particular solution, we get $R = -6\ell^{-2}$ with an integration constant $\ell$. Then, we can write the equation of motion in the familiar form of $R_{\mu\nu} - (1/2) g_{\mu\nu} R - \ell^{-2} g_{\mu\nu} = 0$. The corresponding action to
this equation of motion is the Einstein-Hilbert action with a cosmological constant so that
its exact solution can be the BTZ black hole \cite{2}. So, the line element is naturally written as
\begin{align}
    ds^2 &= -N^2(r)dt^2 + \frac{dr^2}{N^2(r)} + r^2(d\theta - \Omega_0(r)dt)^2, \\
    N^2(r) &= -m + \frac{r^2}{\ell^2} + \frac{j^2}{4r^2} = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2\ell^2}, \\
    \Omega_0(r) &= \frac{j}{2r^2} = \frac{r_+ - r_-}{r^2\ell},
\end{align}

where the inner and outer horizon radii are given by
\begin{align}
    r_\pm &= \frac{\ell}{2} \left[ \sqrt{m + \frac{j}{\ell}} \pm \sqrt{m - \frac{j}{\ell}} \right],
\end{align}

and the two parameters \(m\) and \(j\) were written as \(m = \frac{r_+^2 + r_-^2}{\ell^2}\), \(j = \frac{2r_+ - r_-}{\ell}\). Then, the mass \(M\) and angular momentum \(J\) in the GCS action are given by \cite{9, 10}
\begin{align}
    M &= \frac{j}{\mu\ell^2}, \quad J = \frac{m}{\mu}.
\end{align}

Rewriting the metric functions and the two horizons \(r_\pm\) in terms of \(M\) and \(J\), we get
\begin{align}
    N^2(r) &= -\mu J + \frac{r^2}{\ell^2} + \frac{\mu^2 \ell^4 M^2}{4r^2}, \\
    \Omega_0(r) &= \frac{\mu \ell^2 M}{2r^2}, \\
    r_\pm &= \frac{\ell}{2} \left[ \sqrt{\mu(J + M\ell)} \pm \sqrt{\mu(J - M\ell)} \right].
\end{align}

We now define a chiral black hole which is either a right-handed rotating black hole of
\(0 \leq M\ell \leq J\) for the positive Chern-Simons coefficient or a left-handed rotating black hole
of \(0 \leq M\ell \leq -J\) for the negative one. We assumed that the black hole mass is positive,
\(M \geq 0\). So, the handedness of angular momentum and angular velocity are associated with
the signature of the Chern-Simons coefficient. As for the static black hole, it can be achieved
by \(M = 0\), which is actually massless. The black hole can surprisingly have a finite conserved
angular momentum of the black hole even in the massless case. It means that the massless
case is not the vacuum state. Note that the similar feature appears in the BTZ solution in
the CTMG in that the conserved angular momentum does not vanish although the ADM
mass is zero \cite{10, 11}. The AdS spacetime can be also defined by \(M = 0\), \(J = -1/\mu\) for each
chiral sector. In connection with the vacuum state, it is regarded as an empty space which
can be realized by making the black hole disappear, i.e., \( M = 0, J = 0 \). So, the Hawking temperature and angular velocity evaluated at the horizon \( r_+ \),

\[
T_H = \frac{r_+^2 - r_-^2}{2\pi r_+ \ell^2}, \quad \Omega_H = \frac{r_-}{r_+ \ell},
\]

vanish for the vacuum state.

Next, we study the statistical entropy by using the brick wall method of counting quantum modes near the horizon \( r_+ \), then the free energy of the quantum field is given by

\[
F = - \frac{\zeta(3)}{\beta^3} \left( \frac{r_+^2 \ell^4}{(r_+^2 - r_-^2)^2} \right) \frac{r_+}{\bar{h}},
\]

where \( \bar{h} \) is the brick wall cutoff in terms of the proper length and \( \beta \) is an inverse temperature \([13, 14]\). Then, fixing the cutoff universally to \( \bar{h} = 3\zeta(3)/16\pi^3 \) which is independent of the mass and the angular momentum, the entropy is obtained as

\[
S_{bw} = \beta^2 \frac{\partial F}{\partial \beta} \bigg|_{\beta = \beta_H} = 4\pi r_+,
\]

which shows that this black hole entropy satisfies the area law. Now the internal energy \( U \) and the angular momentum \( J_m \) from this free energy are easily calculated as

\[
U = \frac{4}{3}\mu J - \frac{2r_-^2}{3\ell^2}, \quad J_m = \mu \ell^2 M,
\]

respectively, then the thermodynamic first law of the quantum field reads

\[
\delta U = T_H \delta S_{bw} + \Omega_H \delta J_m + (1/3\pi \ell^2) \left[ \delta A_+ - \delta A_- \right],
\]

where \( A_\pm = \pi r_\pm^2 \) are two-dimensional volumes of the inner and outer horizons \( r_\pm \). Note that the thermodynamic first law can be satisfied as long as we consider the pressure which is responsible for the source that is nothing but the cosmological constant. The positive heat capacity can be also obtained as

\[
C_{bw} = \left( \frac{\partial U}{\partial r_+} \right)_{J_m} \left( \frac{\partial T_H}{\partial r_+} \right)_{J_m}^{-1} = \frac{8\pi r_+(2r_+^2 - r_-^2)}{3(r_+^2 + 3r_-^2)} > 0,
\]

where we have used a relation \( \partial r_-/\partial r_+ \bigg|_{J_m} = -r_-/r_+ \). As a result, we have obtained a thermodynamically stable chiral black hole.

In fact, one can analyze the statistical entropy through the dual conformal field theory (CFT) using the well-known Cardy formula \([15]\). The central charges and the conformal
weights can be read from the liming case of the CTMG in Ref. [10], \( c_{L,R} = \mp (3\ell/2G_3)(\mu\ell)^{-1} \), and \( h_{L,R} = (M\ell \mp J)/2 \), then the entropy becomes \( S_{\text{CFT}} = 2\pi \left[ \sqrt{c_L h_L/6} + \sqrt{c_R h_R/6} \right] = 4\pi r_+/|\mu\ell| \). If the entropy expression is accepted, then one can easily show the heat capacity is positive. It is noteworthy that the entropy can be written in the form of the outer horizon; however, one of the two central charges should be negative so that the corresponding CFT cannot possess the unitarity, and even more the conformal weight is negative. In spite of this effort to derive the Bekenstein-Hawking’s area law, it is satisfactory up to the constant, \( 1/\mu\ell \); more worse, it fails to satisfy the thermodynamic first law, since it gives \( \delta M - \Omega_H \delta J = [2(r_+^2 - r_-^2)/r_+\mu\ell^3]\delta r_- = T_H \delta S \), where the resulting entropy is \( S = 4\pi r_-/\mu\ell \).

It is coincident rather with the Wald’s formula [16, 17], because the first law is related to the conserved charges through the Noether theorem. It can be also shown that the entropy expressed by the Wald’s formula does not give the positive heat capacity. So, the statistical entropy from the dual CFT can be expressed by the outer horizon related to the Bekenstein-Hawking’s area law up to the constant and the heat capacity is positive; however, it does not satisfies the thermodynamic first law apart from the pathology of the negative central charge.

In conclusion, we have obtained a chiral black hole whose handedness of the angular momentum depends on the signature of the Chern-Simons coefficient. The third order derivative of the Cotton tensor gives a dimensional constant which plays a role of the cosmological constant. The conserved angular momentum and the mass of the chiral black hole correspond to the mass and angular momentum of the BTZ black hole so that the angular momentum can exist even though the angular velocity is zero. On the other hand, the statistical entropy calculated by using the brick wall method gives the Bekenstein-Hawking’s area law, thermodynamic first law, and the positive heat capacity while the CFT side shows that the entropy is slightly different and the thermodynamic first law fails. It deserves to further study what the genuine entropy is because the three expressions of 't Hooft (brick wall), Cardy (CFT), and Wald (Noether theorem) are different.

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