Abstract. We consider unbounded curves without endpoints. Isomorphism is equivalence up to translation. Self-avoiding plane-filling curves cannot be periodic, but they can satisfy the local isomorphism property: We obtain a set \( \Omega \) of coverings of the plane by sets of disjoint self-avoiding nonoriented curves, generalizing the Peano-Gosper curves, such that:

1) each \( C \in \Omega \) satisfies the local isomorphism property; any set of curves locally isomorphic to \( C \) belongs to \( \Omega \);
2) \( \Omega \) is the union of \( 2^\omega \) equivalence classes for the relation “\( C \) locally isomorphic to \( D \)”; each of them contains \( 2^\omega \) (resp. \( 2^2 \), \( 4 \), \( 0 \)) isomorphism classes of coverings by 1 (resp. 2, 3, \( \geq 4 \)) curves.

Each \( C \in \Omega \) gives exactly 2 coverings by sets of oriented curves which satisfy the local isomorphism property. They have opposite orientations.

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We denote by \( \mathbb{N}^* \) the set of strictly positive integers. For each \( n \in \mathbb{N}^* \), we consider \( \mathbb{R}^n \) equipped with a norm \( x \to \|x\| \). For each \( x \in \mathbb{R}^n \) and each \( r \in \mathbb{R}^+ \), we denote by \( B(x, r) \) the ball of center \( x \) and radius \( r \). For any \( E, F \subset \mathbb{R}^n \), an isomorphism from \( E \) to \( F \) is a translation \( \tau \) such that \( \tau(E) = F \).

We say that \( E \subset \mathbb{R}^n \) satisfies the local isomorphism property if, for each \( x \in \mathbb{R}^n \) and each \( r \in \mathbb{R}^+ \), there exists \( s \in \mathbb{R}^+ \) such that each \( B(y, s) \) contains some \( z \) with \( (B(z, r) \cap E, z) \cong (B(x, r) \cap E, x) \). We say that \( E, F \subset \mathbb{R}^n \) are locally isomorphic if, for each \( x \in \mathbb{R}^n \) (resp. \( y \in \mathbb{R}^n \)) and each \( r \in \mathbb{R}^+ \), there exists \( y \in \mathbb{R}^n \) (resp. \( x \in \mathbb{R}^n \)) such that \( (B(x, r) \cap E, x) \cong (B(y, r) \cap F, y) \). These notions generalize those in [3, p. 58].

For each integer \( n \geq 2 \), we say that \( C \subset \mathbb{R}^n \) is a self-avoiding curve if there exists a bicontinuous map from \( \mathbb{R} \) to \( C \).

It follows from the proposition below that a self-avoiding curve \( C \) cannot be invariant through a non-trivial translation if there exists \( \alpha \in \mathbb{R}^+ \) such that \( \sup_{x \in \mathbb{R}^n} \inf_{y \in C} \|x - y\| \leq \alpha \):

**Proposition 1.** For each integer \( n \geq 2 \), each self-avoiding curve \( C \subset \mathbb{R}^n \) and each \( w \in \mathbb{R}^n \) such that \( C = w + C \), there exists \( B \subset C \) bounded such that \( C \) is the disjoint union of the subsets \( kw + B \) for \( k \in \mathbb{Z} \).
Proof. Consider \( f : \mathbb{R} \to C \) bicontinuous. Then \( \tau_w : C \to C : y \to w + y \) and \( \varphi : \mathbb{R} \to \mathbb{R} : s \to f^{-1}(w + f(s)) \) satisfy \( \varphi = f^{-1} \circ \tau_w \circ f \). The map \( \varphi \) is bicontinuous because \( f, \tau_w \) and \( f^{-1} \) are bicontinuous. In particular, \( \varphi \) is strictly increasing or strictly decreasing.

Now consider \( x \in C \) and write \( t_k = f^{-1}(kw + x) \) for each \( k \in \mathbb{Z} \). The equalities \( \varphi(t_k) = t_{k+1} \) for \( k \in \mathbb{Z} \) imply \( \varphi \) strictly increasing, \( \varphi([t_k, t_{k+1}]) = [t_{k+1}, t_{k+2}] \) for each \( k \in \mathbb{Z} \) and \( w + f([t_k, t_{k+1}]) = f([t_{k+1}, t_{k+2}]) \) for each \( k \in \mathbb{Z} \). Consequently, \( C \) is the union of the disjoint subsets \( kw + f([t_0, t_1]) = f([t_k, t_{k+1}]) \) for \( k \in \mathbb{Z} \).

Now, we are going to see that generalizations of Peano-Gosper curves give examples of self-avoiding plane-filling curves which satisfy the local isomorphism property in a strong form. Other examples are given in [3] and [4] for paperfolding curves.

For each tiling \( \mathcal{P} \) of the plane by hexagons and each center \( x \) of a tile of \( \mathcal{P} \), we construct some tilings \( \mathcal{P}_{x \lambda_1 \cdots \lambda_n} \) for \( n \in \mathbb{N} \) and \( \lambda_1, \ldots, \lambda_n \in \{-, +\} \) such that, for each \( n \in \mathbb{N} \) and any \( \lambda_1, \ldots, \lambda_{n+1} \in \{-, +\} \), \( x \) is the center of a tile of \( \mathcal{P}_{x \lambda_1 \cdots \lambda_{n+1}} \) and each tile of \( \mathcal{P}_{x \lambda_1 \cdots \lambda_{n+1}} \) is the union of 7 nonoverlapping tiles of \( \mathcal{P}_{x \lambda_1 \cdots \lambda_n} \) with one of them surrounded by the 6 others.

We write \( \mathcal{P}_x = \mathcal{P} \) and we consider the tilings \( \mathcal{P}_{x+} \) and \( \mathcal{P}_{x-} \) respectively given by Figures 1 and 2. The points of the plane which are common to 3 tiles of \( \mathcal{P}_{x+} \) (resp. \( \mathcal{P}_{x-} \)) determine an hexagonal tiling \( Q_{x+} \) (resp. \( Q_{x-} \)).

We denote by \( \Delta_{x+} \) (resp. \( \Delta_{x-} \)) the bijection which associates to each tile of \( \mathcal{P}_{x+} \) (resp. \( \mathcal{P}_{x-} \)) the tile of \( Q_{x+} \) (resp. \( Q_{x-} \)) with the same center. We see from Figures 1 and 2 that, for each tile \( Q \) of \( Q_{x+} \) (resp. \( Q_{x-} \)), \( \Delta_{x+}^{-1}(Q) \) (resp. \( \Delta_{x-}^{-1}(Q) \)) is obtained by replacing each side \( S \) of \( Q \) with three consecutive

![Figure 1](image1.png) ![Figure 2](image2.png)

We write \( \mathcal{P}_x = \mathcal{P} \) and we consider the tilings \( \mathcal{P}_{x+} \) and \( \mathcal{P}_{x-} \) respectively given by Figures 1 and 2. The points of the plane which are common to 3 tiles of \( \mathcal{P}_{x+} \) (resp. \( \mathcal{P}_{x-} \)) determine an hexagonal tiling \( Q_{x+} \) (resp. \( Q_{x-} \)).
sides of tiles of \( \mathcal{P} \); the first side is at the right (resp. left) of \( S \) and the third side is at its left (resp. right).

For \( n \geq 2 \) and \( \lambda_1, \ldots, \lambda_n \in \{-, +\} \) we write \( \mathcal{P}_{x\lambda_1\ldots\lambda_n} = \Delta_{x\lambda_1}^{-1}((Q_{x\lambda_1})_{x\lambda_2\ldots\lambda_n}) \).

Each \( \mathcal{P}_{x\lambda_1\ldots\lambda_n} \) is invariant through a rotation of center \( x \) and angle \( \pi/3 \), and consists of isomorphic tiles with connected interior and connected exterior, which are invariant through rotations of angle \( \pi/3 \).

In each \( \mathcal{P}_{x\lambda_1\ldots\lambda_n} \), each tile \( P \) has nonempty frontiers with 6 others. We call \( \textit{sides} \) of \( P \) these frontiers, which are unions of \( 3^n \) sides of tiles of \( \mathcal{P} \). The \textit{vertices} of \( P \) are the endpoints of its sides.

For each \( n \in \mathbb{N} \), any \( \lambda_1, \ldots, \lambda_n \in \{-, +\} \) and any centers \( x, y \) of tiles of \( \mathcal{P} \), each tile of \( \mathcal{P}_{x\lambda_1\ldots\lambda_n} \) and each tile of \( \mathcal{P}_{y\lambda_1\ldots\lambda_n} \) are isomorphic.

\textbf{Remark.} Write \( \lambda_n = + \) for each \( n \in \mathbb{N}^* \). For each \( n \in \mathbb{N} \), denote by \( P_n \) the tile of center \( x \) in \( \mathcal{P}_{x\lambda_1\ldots\lambda_n} \), and \( Q_n \) the tile with the same vertices as \( P_0 \) which is the image of \( P_n \) under a similarity. Then the limit of the tiles \( Q_n \) is the Peano-Gosper island considered in [2, p. 46]. It is the union of 7 isomorphic nonoverlapping tiles which are similar to it, with one of them surrounded by the 6 others.

Now, for each \( \Lambda = (\lambda_n)_{n \in \mathbb{N}^*} \in \{-, +\}^{\mathbb{N}^*} \), we consider the sequences \( X = (x_n)_{n \in \mathbb{N}} \) with \( x_0 \) center of a tile of \( \mathcal{P} \) and, for each \( n \in \mathbb{N} \), \( x_{n+1} = x_n \) or \( x_n, x_{n+1} \) centers of adjacent tiles in \( \mathcal{P}_{x_n\lambda_1\ldots\lambda_n} \).

For each such sequence \( X \) and each \( n \in \mathbb{N} \), we write \( \mathcal{P}_{X\lambda}^n = \mathcal{P}_{x_n\lambda_1\ldots\lambda_n} \). We denote by \( \mathcal{P}_{X\lambda} \) the union of the sets \( \mathcal{P}_{X\lambda}^n \). If \( x_n = x \) for each \( n \in \mathbb{N} \), we write \( \mathcal{P}_{X\lambda}^n \) and \( \mathcal{P}_{X\lambda} \) instead of \( \mathcal{P}_{X\lambda}^n \).

For each \( n \in \mathbb{N} \), we denote by \( P_{X\lambda}^n \) the tile of \( \mathcal{P}_{X\lambda}^n \) which contains \( x_n \). We consider the \textit{region} \( R_{X\lambda} = \cup_{n \in \mathbb{N}} P_{X\lambda}^n \).

For any sequences \( X = (x_n)_{n \in \mathbb{N}} \) and \( Y = (y_n)_{n \in \mathbb{N}} \) such that \( \mathcal{P}_{X\lambda} = \mathcal{P}_{Y\lambda} \) and for each \( m \in \mathbb{N} \), \( P_{X\lambda}^m \) and \( P_{Y\lambda}^m \) are disjoint, or they have one common side, or they are equal. If the third possibility is realized for some integer \( m \), then we have \( x_n = y_n \) for \( n \geq m \) and \( R_{X\lambda} = R_{Y\lambda} \). Otherwise, the second possibility is realized for \( m \) large enough.

It follows that one of the three following properties is true for each \( \mathcal{P}_{X\lambda} \):

1) The plane consists of 1 region.
2) The plane is the union of 2 nonoverlapping regions; their frontier is a line without endpoint which consists of sides of hexagons of \( \mathcal{P} \).
3) The plane is the union of 3 nonoverlapping regions, with 1 common point which is a vertex of tiles of \( \mathcal{P}_{X\lambda}^n \), for each \( n \in \mathbb{N} \); the frontier of 2 regions is a line starting from that point which consists of sides of hexagons of \( \mathcal{P} \).

\textbf{Proposition 2.} For any \( \Lambda, X \) such that \( \mathcal{P}_{X\lambda} \) exists, there are countably many sequences \( Y \) such that \( \mathcal{P}_{X\lambda} \cong \mathcal{P}_{Y\lambda} \). For each \( \Lambda \), there are \( 2^\omega \) isomorphism classes of sets \( \mathcal{P}_{X\lambda} \) with 1 region, \( 2^\omega \) isomorphism classes of sets \( \mathcal{P}_{X\lambda} \) with 2
regions, and 2 isomorphism classes of sets $\mathcal{P}_{\Lambda}$ with 3 regions, obtained from each other by a rotation of angle $\pi/3$.

**Proof.** In order to show the first statement, we first prove that there are countably many sequences $Y$ such that $\mathcal{P}_{\Lambda} \cong \mathcal{P}_{\Lambda}$. As $\mathcal{P}_{\Lambda}$ has at most 3 regions, it suffices to show that, for each sequence $Y$ such that $\mathcal{P}_{\Lambda}$ exists, there are countably many sequences $Z$ such that $\mathcal{P}_{\Lambda} = \mathcal{P}_{\Lambda}$ and $R_{\Lambda} = R_{\Lambda}$, which is clear since $Y$ and $Z$ are ultimately equal if these two equalities are true.

It follows that there are countably many sequences $Y$ such that $\mathcal{P}_{\Lambda} \cong \mathcal{P}_{\Lambda}$: For each such sequence $Y$, there exists a sequence $Z$ and a translation $\tau$ such that $\mathcal{P}_{\Lambda} = \mathcal{P}_{Z}$ and $\tau(Z) = Y$. Any such translation satisfies $\tau(\mathcal{P}) = \mathcal{P}$ and only countably many translations satisfy that property.

Now we prove the second statement. Each $\mathcal{P}_{\Lambda}$ is completely determined by $(P_{\Lambda}^n)_{n \in \mathbb{N}}$.

First suppose that $\mathcal{P}_{\Lambda}$ has 3 regions and denote by $M$ their common point. Then there exists an integer $k$ such that $M$ is a vertex of $P_{\Lambda}^n$ for each $n \geq k$. There are 6 possible choices for $(P_{\Lambda}^n)_{n \geq k}$; 3 choices give the same $\mathcal{P}_{\Lambda}$ and the 3 others give $\mathcal{P}_{\Lambda}$ which is obtained from $\mathcal{P}_{\Lambda}$ by a rotation of center $M$ and angle $\pi/3$.

Now consider the sets $\mathcal{P}_{\Lambda}$ with 1 region. In order to obtain an appropriate sequence $X$, it suffices to choose $x_k$ for each $k \geq 2$ so that $P_{\Lambda}^k$ is contained in the interior of $P_{\Lambda}^k$. There are at least 3 possible choices at each step, and therefore $2^\omega$ possible sequences.

Presently, it suffices to prove that there exist $2^\omega$ isomorphism classes of sets $\mathcal{P}_{\Lambda}$ with 2 or 3 regions. We consider any side $S$ of a tile of $\mathcal{P}$. In order to obtain an appropriate sequence $X$, it suffices to choose each $x_k$ so that $S$ is contained in a side of $P_{\Lambda}^k$. There are 3 possible choices at each step, and therefore $2^\omega$ possible sequences. •

Now we define the curves associated to $\mathcal{P}$. An oriented **bounded curve** (resp. **half curve**, **complete curve**) is a sequence of segments $(A_k)_{0 \leq k \leq n}$ (resp. $(A_k)_{k \in \mathbb{N}}$, $(A_k)_{k \in \mathbb{Z}}$), each of them joining 2 vertices of an hexagon which are neither consecutive nor opposite, and such that, if $A_k$ and $A_{k+1}$ exist, then the terminal point of $A_k$ is the initial point of $A_{k+1}$ and the initial point of $A_k$ is not the terminal point of $A_{k+1}$.

For each oriented bounded curve $(A_k)_{0 \leq k \leq n}$, we consider the sequence $(a_k)_{1 \leq k \leq n}$ with $a_k$ equal to $+2, +1, 0, -1, -2$ if the angle between $A_{k-1}$ and $A_k$ is $+2\pi/3, +\pi/3, 0, -\pi/3, -2\pi/3$. For each sequence $S = (a_k)_{1 \leq k \leq n}$, we write $\overline{S} = (-a_{n-k+1})_{1 \leq k \leq n}$. We have $\overline{\overline{S}} = S$. The curves associated to $\overline{S}$ are obtained by changing the orientation of the curves associated to $S$.

The set $V$ of vertices of hexagons of $\mathcal{P}$ is the disjoint union of 2 subsets $V_1, V_2$ which contain no consecutive vertices of an hexagon. The endpoints
of the segments of a curve necessarily belong to the same $V_i$. From now on, we choose one $V_i$, which we denote by $W$, and we only consider curves with vertices in $W$. We denote by $w$ the center of a tile of $\mathcal{P}$.

For each $n \in \mathbb{N}$ and any $\lambda_1, \ldots, \lambda_n \in \{-,+,\}$, we say that a curve $C$ covers a tile $P \in \mathcal{P}_{\lambda_1\ldots\lambda_n}$ if, for each $m \in \{0, \ldots, n\}$ and each $Q \in \mathcal{P}_{\lambda_1\ldots\lambda_m}$ contained in $P$, the segments $A_i$ of $C$ contained in $Q$ are consecutive, and $Q$ contains exactly 1 segment if $Q \in \mathcal{P}$. Then we denote by $C \upharpoonright P$ the restriction of $C$ to $P$ and we say that $C \upharpoonright P$ is a covering of $P$.

For any integers $n \geq m \geq 1$, any $\lambda_1, \ldots, \lambda_n \in \{-,+,\}$ and each $P \in \mathcal{P}_{\lambda_1\ldots\lambda_n}$, we write $\Delta_{\lambda_1\ldots\lambda_m}(P) = \Delta_{\lambda_m}(\cdots(\Delta_{\lambda_1}(P))\cdots)$. We have $\Delta_{\lambda_1\ldots\lambda_m}(P_{\lambda_1\ldots\lambda_m}) = (\Delta_{\lambda_1\ldots\lambda_m}(P))_{\lambda_{m+1}\ldots\lambda_n}$.

For each covering $C$ of $P$, we obtain a covering $\Delta_{\lambda_1\ldots\lambda_m}(C)$ of $\Delta_{\lambda_1\ldots\lambda_m}(P)$ by replacing each $C \upharpoonright Q$ for $Q \in \mathcal{P}_{\lambda_1\ldots\lambda_m}$ contained in $P$ with the segment from its initial point to its terminal point. We have $\Delta_{\lambda_1\ldots\lambda_m}(C) = \Delta_{\lambda_m}(\cdots(\Delta_{\lambda_1}(C))\cdots)$.

Now we write $S = (+1, +2, -1, -2, 0, -1)$ and we define by induction on $n \in \mathbb{N}^*$ some sequences $S_{\lambda_1\ldots\lambda_n} \in \{-1, +1\}^{2^n-1}$ with $S_+ = S$, $S_- = -S$, $S_{++} = (S, +1, S, +1, S, -1, S, +1, S, +1, S, +1, S)$, $S_{--} = -S_{++}$, $S_{+-} = (S, -1, S, +1, S, +1, S, +1, S, +1, S, +1, S, -1, S, -1, S)$, $S_{-+} = -S_{+-}$
and, for each integer $n \geq 3$, $S_{\lambda_1\ldots\lambda_n}$ obtained from $S_{\lambda_2\ldots\lambda_n}$ by replacing each subsequence $(a_{7k+1}, \ldots, a_{7k+6})$ equal to $S_\lambda$ (resp. $\overline{S_\lambda}$) with $S_{\lambda_1\lambda_2}$ (resp. $S_{\overline{\lambda_1}\lambda_2}$).

Figure 3 (resp. 4, 5, 6) shows a covering of a tile of $\mathcal{P}_{w+}$ (resp. $\mathcal{P}_{w-}$, $\mathcal{P}_{w++}$, $\mathcal{P}_{w+-}$) by a curve associated to $S_+$ (resp. $S_-$, $S_{++}$, $S_{+-}$).
Proposition 3. Consider $n \in \mathbb{N}^*$, $\lambda_1, \ldots, \lambda_n \in \{-, +\}$, $P \in \mathcal{P}_{w\lambda_1 \cdots \lambda_n}$ and $x$ the center of $P$. Then $P$ has exactly 6 coverings by oriented curves, each of them determined by its initial and terminal points. Each covering is associated to $S_{\lambda_1 \cdots \lambda_n}$ or $\overline{S_{\lambda_1 \cdots \lambda_n}}$; the 3 coverings associated to $S_{\lambda_1 \cdots \lambda_n}$ are obtained from one of them by rotations of center $x$ and angles $2k\pi/3$, and the 3 others by changing their orientations.

Proof. We see from Figures 3 and 4 that Proposition 3 is true for $n = 1$, and from Figures 5 and 6 that it is true for $n = 2$. Now we show that it is true for $n \geq 3$ if it is true for $n - 1$.

By the induction hypothesis, there exists a covering $D$ of $\Delta_{w\lambda_1}(P)$ associated to $S_{\lambda_2 \cdots \lambda_n}$. Each segment $A$ of $D$ is a covering of an hexagon $H_A \in \Delta_{w\lambda_1}(\mathcal{P}_{w\lambda_1 \cdots \lambda_n})$ which is contained in $\Delta_{w\lambda_1}(P)$. We obtain a covering $C$ of $P$ by replacing each such segment $A$ with a covering of $\Delta_{w\lambda_1}^{-1}(H_A)$ which has the corresponding initial and terminal points.

As $P$ is invariant through a rotation of angle $2\pi/3$, it has 3 coverings obtained from $C$ by rotations of angles $0, 2\pi/3, 4\pi/3$, and 3 others obtained by changing their orientations. These coverings are associated to the 6 possible pairs of initial and terminal points.

Any other covering $B$ of $P$ has the same initial and terminal points as one of the 6 coverings above, say $C'$. Then $\Delta_{w\lambda_1}(B)$ and $\Delta_{w\lambda_1}(C')$ are coverings of $\Delta_{w\lambda_1}(P)$ with the same initial and terminal points. By the induction hypothesis, we have $\Delta_{w\lambda_1}(B) = \Delta_{w\lambda_1}(C')$. Consequently, for each $Q \in \mathcal{P}_{w\lambda_1}$ contained in $P$, $B \upharpoonright Q$ and $C' \upharpoonright Q$ are coverings of $Q$ with the same initial and terminal points, which implies $B \upharpoonright Q = C' \upharpoonright Q$. It follows $B = C'$.
Now it suffices to show that $C$ is associated to $S_{\lambda_1 \cdots \lambda_n}$. We write $C = (C_h)_{0 \leq h \leq 7^n - 1}$ and $D = (D_h)_{0 \leq h \leq 7^n - 1}$. We consider the associated sequences $(c_h)_{1 \leq h \leq 7^n - 1} \in \{-1, +1\}^{7^n - 1}$ and $(d_h)_{1 \leq h \leq 7^n - 1} \in \{-1, +1\}^{7^n - 1}$. For $0 \leq k \leq 7^n - 2 - 1$, we write $U_k = (c_{49k+1})_{1 \leq h \leq 49k+48}$ and $V_k = (d_{49k+1})_{1 \leq h \leq 49k+6}$. There exists $Q_k \in \mathcal{P}_{w_{\lambda_1} \lambda_2}$ contained in $P$ such that $U_k$ is associated to $C \upharpoonright Q_k$ and $V_k$ is associated to $D \upharpoonright \Delta w_{\lambda_1}(Q_k) = \Delta w_{\lambda_1}(C \upharpoonright Q_k)$. Consequently, we have $U_k = S_{\lambda_1 \lambda_2}$ (resp. $U_k = \overline{S_{\lambda_1 \lambda_2}}$) if and only if $V_k = \overline{S_{\lambda_1 \lambda_2}}$ (resp. $V_k = S_{\lambda_1 \lambda_2}$).

It remains to be proved that $c_{49k} = d_{7k}$ for $1 \leq k \leq 7^n - 2 - 1$. As $U_k = V_k$, there exists $T \in \{S_{\lambda_1 \lambda_2} \cup \overline{S_{\lambda_1 \lambda_2}}\}$ such that $U_k$ is ending with $T$ and $U_k$ is beginning with $T$. It follows that the angle between $C_{49k}$ and $C_{49k+1}$ is equal to the angle between $D_{7k}$ and $D_{7k+1}$, which implies $c_{49k} = d_{7k}$.  

**Example.** The Peano-Gosper curves considered in [2, p. 46] and [1, p. 63] are associated to the sequences $T_n = S_{\lambda_1 \cdots \lambda_n}$ with $\lambda_k = +$ for $1 \leq k \leq n$. We have $T_{n+1} = (T_n + 1, T_n - 1, T_n + 1, T_n - 1, T_n + 1, T_n - 1, T_n + 1)$ for each $n \in \mathbb{N}^+$. In [1] and several papers mentioned among its references, W. Kuhirum, D.H. Werner and P.L. Werner prove that an antenna with the shape of a Peano-Gosper curve has particular electromagnetic properties. We can imagine that similar properties exist for the other values of $(\lambda_k)_{1 \leq k \leq n}$.

**Corollary 4.** For each $n \in \mathbb{N}$, any $\lambda_1, \ldots, \lambda_{n+1} \in \{-, +\}$ (resp. $\lambda_1, \ldots, \lambda_{n+2} \in \{-, +\}$) and each $P \in \mathcal{P}_{w_{\lambda_1} \cdots \lambda_n}$, each covering of some $Q \in \mathcal{P}_{w_{\lambda_1} \cdots \lambda_{n+1}}$ (resp. $R \in \mathcal{P}_{w_{\lambda_1} \cdots \lambda_{n+2}}$) by a nonoriented (resp. oriented) curve contains copies of the 3 (resp. 6) coverings of $P$ by nonoriented (resp. oriented) curves.

**Proof.** By Proposition 3, it suffices to show that each covering of some $Q \in \mathcal{P}_{w_{\lambda_1} \cdots \lambda_{n+1}}$ (resp. $R \in \mathcal{P}_{w_{\lambda_1} \cdots \lambda_{n+2}}$) by a nonoriented (resp. oriented) curve contains 3 (resp. 6) nonisomorphic coverings of tiles $P \in \mathcal{P}_{w_{\lambda_1} \cdots \lambda_n}$ by nonoriented (resp. oriented) curves. For $n = 0$, we see from Figures 3 and 4 (resp. 5 and 6) that the statement for nonoriented (resp. oriented) curves is true.

For each $n \geq 1$ and each covering $C$ of some $Q \in \mathcal{P}_{w_{\lambda_1} \cdots \lambda_{n+1}}$ (resp. $R \in \mathcal{P}_{w_{\lambda_1} \cdots \lambda_{n+2}}$) by a nonoriented (resp. oriented) curve, we consider some hexagonal tiles $Q_1, Q_2, Q_3$ (resp. $R_1, \ldots, R_6$) contained in $\Delta w_{\lambda_1} \cdots \lambda_n(Q)$ (resp. $\Delta w_{\lambda_1} \cdots \lambda_n(R)$) such that the nonoriented (resp. oriented) segments $\Delta w_{\lambda_1} \cdots \lambda_n(C \upharpoonright Q_i)$ (resp. $\Delta w_{\lambda_1} \cdots \lambda_n(C \upharpoonright R_i)$) are nonisomorphic. Then the nonoriented (resp. oriented) curves $C \upharpoonright \Delta w_{\lambda_1} \cdots \lambda_n(Q_i)$ (resp. $C \upharpoonright \Delta w_{\lambda_1} \cdots \lambda_n(R_i)$) are nonisomorphic.

**Lemma 5.** Consider nonoriented curves. Let $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_{n+1} \in \{-, +\}$, $P \in \mathcal{P}_{w_{\lambda_1} \cdots \lambda_{n+1}}$ and $Q_1, \ldots, Q_7 \in \mathcal{P}_{w_{\lambda_1} \cdots \lambda_n}$ be such that $P = Q_1 \cup \cdots \cup Q_7$. Let $Q \in \{Q_1, \ldots, Q_7\}$. If $Q$ contains the center of $P$, then each of the 3 coverings...
of $Q$ extends into a covering of $P$. Otherwise, denote by $S_1$ (resp. $S_2$, $S_3$) the vertex of $Q$ belonging to $W$ which is a vertex of 1 (resp. 2, 3) $Q_i$. If $S_1$ is not a vertex of $P$, then the covering of $Q$ which joins $S_2$ and $S_3$ extends to 3 coverings of $P$ and the 2 other coverings of $Q$ do not extend. If $S_1$ is a vertex of $P$, then the covering of $Q$ which joins $S_1$ and $S_2$ (resp. $S_2$ and $S_3$, $S_1$ and $S_3$) extends to 2 (resp. 1, 0) coverings of $P$.

**Proof.** We see from Figures 5 and 6 that Lemma 5 is true for $n = 0$. As in the proof of Corollary 4, we use the map $\Delta_{\lambda_1,\cdots,\lambda_n}$ to show that it is also true for $n \geq 1$. 

If $C$ is a set of complete curves associated to $P$ and $W$, and if each hexagon contains exactly 1 segment of 1 curve, then each point of $W$ is the common endpoint of exactly 2 consecutive segments of 1 curve: $C$ is plane-filling, the curves of $C$ are nonoverlapping and each of them is self-avoiding.

For each $\mathcal{P}_{X\Lambda}$, we say that a self-avoiding curve $C$ covers a region $R$ of $\mathcal{P}_{X\Lambda}$, or that its restriction to $R$ is a covering of $R$, if $C$ covers each tile of $\mathcal{P}_{X\Lambda}$ contained in $R$.

We say that a set $C$ of disjoint self-avoiding complete curves is a covering of $\mathcal{P}_{X\Lambda}$ if each tile of $\mathcal{P}_{X\Lambda}$ is covered by a curve of $C$. Then each region of $\mathcal{P}_{X\Lambda}$ is covered by a curve of $C$ since it is the union of an increasing sequence of tiles and each of them is covered by a curve of $C$.

If a curve of $C$ covers 2 regions, then its restrictions to the 2 regions have 1 common endpoint $y \in W$ which is the common point of 3 tiles $P_1, P_2, P_3 \in \mathcal{P}$. Each of the 2 regions only contains one $P_i$ since it is the union of an increasing sequence of tiles of $\mathcal{P}_{X\Lambda}$ which have $y$ as a common vertex, and each of these tiles only contains one $P_i$. Consequently, the third $P_i$ belongs to a third region.

Conversely, if $\mathcal{P}_{X\Lambda}$ has 3 regions and if their common point belongs to $W$, then the curve of $C$ which contains that point covers 2 regions.

**Proposition 6.** Consider nonoriented curves. Let $R$ be a region of some $\mathcal{P}_{X\Lambda}$. Then $R$ has a covering by a complete curve. This covering is unique if $R \subset \mathbb{R}^2$. If $R$ is the union of an increasing sequence of tiles having 1 common vertex $y \in W$, then $R$ has exactly 1 covering by a half curve and $y$ is the endpoint of that curve; otherwise, no such covering exists.

**Proof.** Consider an increasing sequence $(P_n)_{n \in \mathbb{N}} \in \times_{n \in \mathbb{N}} \mathcal{P}_{X\Lambda}^n$ such that $R = \bigcup_{n \in \mathbb{N}} P_n$. For each $m \in \mathbb{N}$, denote by $E_m$ the nonempty set which consists of the coverings of $P_m$ whose endpoints are different from the common vertex of the tiles $P_n$ if it exists. Then, for any $m \leq n$, each element of $E_n$ gives by restriction an element of $E_m$. Consequently, by König’s lemma, there exists
an increasing sequence \((C_n)_{n \in \mathbb{N}} \subseteq \times_{n \in \mathbb{N}} E_n\). Then \(\bigcup_{n \in \mathbb{N}} C_n\) is a complete curve and a covering of \(R\).

If \(R\) has a covering by a half curve, then its endpoint belongs to the frontier of \(R\). So we can suppose \(R \subseteq \mathbb{R}^2\) for the remainder of the proof. Then there exists \(h \in \mathbb{N}\) such that the frontier of \(R\) contains 1 side of \(P_h\), and therefore contains 1 side of \(P_n\) for each \(n \geq h\).

For each \(n \geq h\), denote by \(y_n^1\) (resp. \(y_n^2, y_n^3\)) the vertex of \(P_n\) belonging to \(W\) which is a vertex of 1 (resp. 2, 3) tiles of \(\mathcal{P}_{X\Lambda}\) contained in \(P_{n+1}\). Then \((y_n^1)_{n \geq h}\) is ultimately constant if and only if \(R\) is the union of an increasing sequence of tiles with a common vertex belonging to \(W\).

Suppose that there exists \(k \geq h\) such that \(y_n^1 = y_k^1\) for each \(n \geq k\). For each \(n \geq k\), denote by \(F_n\) the nonempty set which consists of the coverings of \(P_n\) with endpoint \(y_k^1\). Then, for any \(k \leq m \leq n\), each element of \(F_n\) gives by restriction an element of \(F_m\). Consequently, by König’s lemma, there exists an increasing sequence \((D_n)_{n \geq k} \subseteq \times_{n \geq k} F_n\). Then \(\bigcup_{n \in \mathbb{N}} D_n\) is a covering of \(R\) and a half curve with endpoint \(y_k^1\).

Conversely, suppose that \(R\) has 2 distinct coverings \(C, D\). For each \(n \in \mathbb{N}\), write \(C_n = C \upharpoonright P_n\) and \(D_n = D \upharpoonright P_n\). Let \(k \geq h\) be an integer such that \(C_k \neq D_k\).

For each \(n \geq k\), \(y_n^1\) is an endpoint of \(C_n\) or \(D_n\) since \(C_n \neq D_n\). It follows that \(y_n^1\) is also an endpoint of \(C_{n+1}\) or \(D_{n+1}\), since \(P_n\) is the only tile of \(\mathcal{P}_{X\Lambda}\) contained in \(P_{n+1}\) which contains \(y_n^1\). In particular, \(y_n^1\) is a vertex of \(P_{n+1}\). As \(C_n\) and \(D_n\) extend to coverings of \(P_{n+1}\), it follows from Lemma 5 that one of them connects \(y_n^2\) and \(y_n^3\), while the other one connects \(y_n^2\) and \(y_n^3\).

Suppose that there exists \(n \geq k\) such that \(y_{n+1}^1 \neq y_n^1\). Then, as the frontier of \(R\) contains a side of \(P_n\), one of the tiles of \(\mathcal{P}_{X\Lambda}^{n+1}\) which have \(y_n^1\) as a vertex is not contained in \(P_{n+2}\), which implies \(y_{n+1}^1 = y_n^1\). Consequently, \(y_n^1\) is an endpoint of \(C_{n+1}\) and \(D_{n+1}\), and therefore an endpoint of \(C_n = C_{n+1} \upharpoonright P_n\) and \(D_n = D_{n+1} \upharpoonright P_n\), whence a contradiction.

Now we have \(y_n^1 = y_k^1\) for each \(n \geq k\). Consequently, one of the curves \(C, D\) is a complete curve, the other one is a half curve with endpoint \(y_k^1\) and there is only one possibility for each of them.

Now we can state and prove the main results:

**Theorem 7.** For each \(\Lambda\), consider the coverings of the sets \(\mathcal{P}_{X\Lambda}\) by sets of nonoriented curves:

1) If \(\mathcal{P}_{X\Lambda}\) has 1 region, then each covering of \(\mathcal{P}_{X\Lambda}\) consists of 1 curve. If \(X\) is ultimately constant, then \(\mathcal{P}_{X\Lambda}\) has 3 coverings, obtained from one of them by rotations of angles \(2k\pi/3\). Otherwise, \(\mathcal{P}_{X\Lambda}\) has 1 or 2 coverings, and each case is realized for \(2^\omega\) values of \(X\).

2) If \(\mathcal{P}_{X\Lambda}\) has 2 regions, then \(\mathcal{P}_{X\Lambda}\) has 1 covering. It consists of 2 curves. This case is realized for \(2^\omega\) values of \(X\).
3) If $\mathcal{P}_{X \Lambda}$ has 3 regions and if their common point $y$ does not belong to $W$, then $\mathcal{P}_{X \Lambda}$ has 1 covering. It consists of 3 curves obtained from one of them by rotations of center $y$ and angles $2k\pi/3$.

4) If $\mathcal{P}_{X \Lambda}$ has 3 regions and if their common point $y$ belongs to $W$, then $\mathcal{P}_{X \Lambda}$ has 3 coverings obtained from one of them by rotations of center $y$ and angles $2k\pi/3$. Each covering consists of 2 curves; one covers 1 region and one covers 2 regions; the second curve is the union of 2 half curves with endpoint $y$, which are equivalent modulo a rotation of center $y$ and angle $2\pi/3$.

Theorem 7 will be proved after the remarks and the example below:

**Remark.** It follows from Theorem 7 that, for each region $R \subset \mathbb{R}^2$ of some $\mathcal{P}_{X \Lambda}$, the covering of $R$ by a nonoriented complete curve can be extended into 1 covering of $\mathcal{P}_{X \Lambda}$. The covering of $R$ by a nonoriented half curve, if it exists, can be extended into 2 coverings of $\mathcal{P}_{X \Lambda}$, which are equivalent modulo a rotation of angle $2\pi/3$.

**Remark.** For each $\Lambda$, as each $\mathcal{P}_{X \Lambda}$ has finitely many coverings, Proposition 2 implies that each isomorphism class of coverings of sets $\mathcal{P}_{X \Lambda}$ is countable. Consequently, it follows from Theorem 7 that we have $2^\omega$ isomorphism classes of coverings for case 2) and for case 1) with $X$ not ultimately constant. On the other hand, we have 1 class for case 3), 3 classes for case 4) and 3 classes for case 1) with $X$ ultimately constant.
Example. The inductive limit of the Peano-Gosper curves considered above, associated to the inductive limit of the sequences $T_n$ with $T_n$ identified with its first copy in $T_{n+1}$ for each $n \in \mathbb{N}^*$, is a covering of a region by a half curve. It is contained in a covering of the plane by 2 complete curves, one covering 2 regions and one covering 1 region (see Figure 7 above). The last curve is associated to the inductive limit of the sequences $T_n$ with $T_n$ identified with its third copy in $T_{n+1}$ for each $n \in \mathbb{N}^*$.

Proof of Theorem 7. We write $X = (x_n)_{n \in \mathbb{N}}$. For each $n \in \mathbb{N}$, we consider the tile $P_n \in \mathcal{P}_{XA}^n$ which contains $x_n$.

Proof of 1). We have $\mathbb{R}^2 = \bigcup_{n \in \mathbb{N}} P_n$. Each covering of $\mathcal{P}_{XA}$ consists of 1 complete curve by Proposition 6.

First we suppose that there exists $h$ such that $x_n = x_h$ for each $n \geq h$. Then, by Proposition 3, for each $n \geq h$, the 3 coverings of $P_n$ are obtained from one of them by rotations of center $x_h$ and angles $2k\pi/3$. Moreover, by Lemma 5, for any $n \geq m \geq h$, the 3 coverings of $P_m$ are restrictions of the 3 coverings of $P_n$. It follows that $\mathcal{P}_{XA}$ has exactly 3 coverings, which are the inductive limits of the coverings of the tiles $P_n$ for $n \geq h$, and therefore obtained from one of them by rotations of center $x_h$ and angles $2k\pi/3$.

From now on, we suppose $X$ not ultimately constant. First we observe that $\mathcal{P}_{XA}$ cannot have 3 coverings. Otherwise, for $n$ large enough, these 3 coverings would give by restriction 3 distinct coverings of $P_n$, which is not possible when $x_{n+1} \neq x_n$ since, by Lemma 5, at most 2 coverings of $P_n$ extend to coverings of $P_{n+1}$.

It follows from Lemma 5 that, for each $n \in \mathbb{N}$ and any coverings $A, B$ of $P_n$, we have 2 coverings of $P_{n+1}$ which extend $A, B$ in 2 cases: first when $P_n$ contains the center of $P_{n+1}$, second when the common endpoint of $A$ and $B$ belongs to exactly 2 tiles of $\mathcal{P}_{XA}^n$ contained in $P_{n+1}$, since in that case the other endpoint of $A$ or the other endpoint of $B$ is a vertex of $P_{n+1}$.

It follows that, for any coverings $C_0 \neq D_0$ of some $P \in \mathcal{P}$, there exist $2^\omega$ different ways to choose $X$ so that $\mathcal{P}_{XA}$ has 1 region and so that $C_0, D_0$ extend to increasing sequences $(C_n)_{n \in \mathbb{N}}, (D_n)_{n \in \mathbb{N}}$ with $C_n, D_n$ coverings of $P_n$ for each $n \in \mathbb{N}$. For that purpose, it suffices to have $K \subset \mathbb{N}$ with $K$ and $\mathbb{N} - K$ infinite such that, for each $n \in K$, $P_n$ contains the center of $P_{n+1}$, and, for each $n \in \mathbb{N} - K$, the common endpoint of $C_n$ and $D_n$ belongs to exactly 2 tiles of $\mathcal{P}_{XA}^n$ contained in $P_{n+1}$. Then $\bigcup_{n \in \mathbb{N}} C_n$ and $\bigcup_{n \in \mathbb{N}} D_n$ are coverings of $\mathcal{P}_{XA}$.

Finally, we observe that, by Lemma 5, for each $n \in \mathbb{N}$, if the point of $P_n$ which is an endpoint of sides of $P_{n+1}$ does not belong to $W$, then the 3 coverings of $P_{n+1}$ give by restriction the same covering of $P_n$, so that only 1 covering of $P_n$ is a restriction of a covering of $\mathcal{P}_{XA}$. There are $2^\omega$ different ways to choose $X$ so that this property is true for infinitely many integers $n,$
and so that \( x_{n+1} = x_n \) is also true for infinitely many integers \( n \). For such an \( X \), \( \mathcal{P}_{\Lambda X} \) only has 1 covering and 1 region.

**Proof of 2), 3), 4).** Suppose that \( \mathcal{P}_{\Lambda X} \) has 2 or 3 regions and let \( R \) be one of them.

If \( R \) is not the union of an infinite sequence of tiles having 1 common vertex, then \( \mathcal{P}_{\Lambda X} \) has 2 regions and the second region satisfies the same property. By Proposition 6, each region has a unique covering and it is a complete curve. The union of these coverings is the unique covering of \( \mathcal{P}_{\Lambda X} \).

This case is realized if:

a) there exists a sequence \( (\Sigma_n)_{n \in \mathbb{N}} \) with \( \Sigma_n \) side of \( P_\Lambda \) and \( \Sigma_n \subset \Sigma_{n+1} \) for each \( n \in \mathbb{N} \);

b) there is no common vertex of the tiles \( P_n \) for \( n \) large.

For each \( n \in \mathbb{N} \) and each choice of \( x_1, \ldots, x_n \) compatible with a), there are 3 choices compatible with a) for \( x_{n+1} \). Consequently, there are \( 2^\omega \) sequences \( X \) which satisfy a), and also \( 2^\omega \) sequences which satisfy a) and b) since countably many sequences do not satisfy b).

If \( R \) is the union of an infinite sequence of tiles having one common vertex \( y \), then \( \mathcal{P}_{\Lambda X} \) has 3 regions \( R_1, R_2, R_3 \) obtained from \( R \) by rotations of center \( y \) and angles \( 2k\pi/3 \).

If \( y \) does not belong to \( W \), then, by Proposition 6, each \( R_i \) has a unique covering and it is a complete curve. The union of these coverings is the unique covering of \( \mathcal{P}_{\Lambda X} \).

If \( y \) belongs to \( W \), then, by Proposition 6, each \( R_i \) has 1 covering by a complete curve and 1 covering by a half curve with endpoint \( y \). Each of the 3 coverings of \( \mathcal{P}_{\Lambda X} \) is obtained by taking for 1 region a covering by a complete curve, and for the 2 other regions 2 coverings by half curves. The 2 half curves form a complete curve since \( y \) is their common endpoint. ■

Now, for each set \( C \) of oriented curves, we consider the following property:

(P) If 2 segments of curves of \( C \) are opposite sides of a rhombus, then they have opposite orientations.

We observe that (P) is satisfied if \( C \) consists of 1 self-avoiding complete curve or 1 curve which forms a covering of a tile.

**Theorem 8.** 1) Each covering of some \( \mathcal{P}_{\Lambda X} \) by a set of nonoriented curves satisfies the local isomorphism property. For each \( \Lambda \) and any \( X, Y \), any coverings of \( \mathcal{P}_{\Lambda X} \) and \( \mathcal{P}_{\Lambda Y} \) by sets of nonoriented curves are locally isomorphic.

2) Each covering of some \( \mathcal{P}_{\Lambda X} \) by a set of oriented curves satisfies the local isomorphism property if and only if it satisfies (P). For each \( \Lambda \) and any \( X, Y \), any coverings of \( \mathcal{P}_{\Lambda X} \) and \( \mathcal{P}_{\Lambda Y} \) by sets of oriented curves are locally isomorphic if they satisfy (P).
Proof. It suffices to show 2) since it implies 1). From now on, we consider coverings of sets \( \mathcal{P}_{X_\Lambda} \) by sets of oriented curves.

If such a covering \( \mathcal{C} \) satisfies the local isomorphism property, then it satisfies (P) since, for each rhombus \( A \) with 2 opposite sides which are segments of curves of \( \mathcal{C} \), there exist a translation \( \tau \) and a tile \( T \) of \( \mathcal{P}_{X_\Lambda} \) such that \( \tau(A) \) is contained in \( T \) and \( \tau(A \cap \mathcal{C}) = \tau(A) \cap C \), where \( C \) is the curve of \( \mathcal{C} \) which covers \( T \).

It remains to be proved that, for each \( \Lambda \), the coverings of the sets \( \mathcal{P}_{X_\Lambda} \) which satisfy (P) are locally isomorphic and satisfy the local isomorphism property.

Corollary 4 implies that it is true for the coverings of the sets \( \mathcal{P}_{X_\Lambda} \) with 1 region, including \( \mathcal{P}_{w_\Lambda} \), since each bounded part of the plane is contained in a tile of each such \( \mathcal{P}_{X_\Lambda} \).

If \( \mathcal{P}_{X_\Lambda} \) has 2 or 3 regions, then it follows from Theorem 7 that all the coverings of \( \mathcal{P}_{X_\Lambda} \) which satisfy (P) are obtained from one of them by a rotation of angle \( 2k \pi/3 \), or changing the orientation of all the curves, or a combination of the two operations. Consequently, it suffices to show that, for each covering \( D \) of \( \mathcal{P}_{w_\Lambda} \) and each set \( X \) such that \( \mathcal{P}_{X_\Lambda} \) has 2 or 3 regions, there exists a covering of \( \mathcal{P}_{X_\Lambda} \) which is locally isomorphic to \( D \) and satisfies the local isomorphism property.

If \( \mathcal{P}_{X_\Lambda} \) has 2 regions, then it contains 2 increasing sequences \((I_n)_{n \in \mathbb{N}}\), \((J_n)_{n \in \mathbb{N}}\) such that, for each \( n \in \mathbb{N} \), \( I_n \cap J_n \) is a side of \( I_n \) and \( J_n \), and \( I_{n+1} \cap J_{n+1} \) extends \( I_n \cap J_n \) at both endpoints. For each \( n \in \mathbb{N} \), we consider the set \( E_n \) which consists of the pairs \((\tau^{-1}(D \upharpoonright \tau(I_n)), \tau^{-1}(D \upharpoonright \tau(J_n)))\) where \( \tau \) is a translation such that \( \tau(I_n), \tau(J_n) \in \mathcal{P}_{w_\Lambda} \).

For any \( m \leq n \), each element of \( E_n \) gives by restriction an element of \( E_m \). Consequently, by König’s lemma, there exists an increasing sequence \((A_n, B_n)_{n \in \mathbb{N}} \in \times_{n \in \mathbb{N}} E_n \). Then \( \cup_{n \in \mathbb{N}} A_n \) and \( \cup_{n \in \mathbb{N}} B_n \) form a covering of \( \mathcal{P}_{X_\Lambda} \) by 2 curves.

If \( \mathcal{P}_{X_\Lambda} \) has 3 regions, then it contains 3 increasing sequences \((I_n)_{n \in \mathbb{N}}\), \((J_n)_{n \in \mathbb{N}}\), \((K_n)_{n \in \mathbb{N}}\) such that, for each \( n \in \mathbb{N} \), \( I_n \cap J_n \) (resp. \( I_n \cap K_n \), \( J_n \cap K_n \)) is a side of \( I_n \) and \( J_n \) (resp. \( I_n \) and \( K_n \), \( J_n \) and \( K_n \)), these 3 sides have a common endpoint \( y \), and \( I_{n+1} \cap J_{n+1} \) (resp. \( I_{n+1} \cap K_{n+1} \), \( J_{n+1} \cap K_{n+1} \)) extends \( I_n \cap J_n \) (resp. \( I_n \cap K_n \), \( J_n \cap K_n \)) at its other endpoint. For each \( n \in \mathbb{N} \), we consider the set \( F_n \) which consists of the triples \((\tau^{-1}(D \upharpoonright \tau(I_n)), \tau^{-1}(D \upharpoonright \tau(J_n)), \tau^{-1}(D \upharpoonright \tau(K_n)))\) where \( \tau \) is a translation such that \( \tau(I_n), \tau(J_n), \tau(K_n) \in \mathcal{P}_{w_\Lambda} \).

For any \( m \leq n \), each element of \( F_n \) gives by restriction an element of \( F_m \). Consequently, by König’s lemma, there exists an increasing sequence \((A_n, B_n, C_n)_{n \in \mathbb{N}} \in \times_{n \in \mathbb{N}} F_n \). Then \( \cup_{n \in \mathbb{N}} A_n \), \( \cup_{n \in \mathbb{N}} B_n \) and \( \cup_{n \in \mathbb{N}} C_n \) form a covering of \( \mathcal{P}_{X_\Lambda} \) by 2 or 3 curves.

As \( D \) satisfies the local isomorphism property, we obtain in both cases a covering of \( \mathcal{P}_{X_\Lambda} \) which is locally isomorphic to \( D \) and satisfies the local
isomorphism property. 

**Remark.** Theorem 8 implies that each covering $C$ of some $\mathcal{P}_{X\Lambda}$ by oriented curves satisfies the local isomorphism property or can be transformed into a covering which satisfies that property by changing the orientation of one of the curves.

Proposition 9 below implies that coverings satisfy the local isomorphism property in a strong form: there exist $h, k \in \mathbb{R}^+$ such that, for each covering $C$ of some $\mathcal{P}_{X\Lambda}$ by nonoriented curves (resp. by oriented curves which satisfies the local isomorphism property), for each $x \in \mathbb{R}^2$ and for each $r \in \mathbb{R}^+$, each $B(y, hr + k)$ contains some $z$ with $(B(z, r) \cap C, z) \cong (B(x, r) \cap C, x)$.

**Proposition 9.** Let $C$ be a covering of some $\mathcal{P}_{X\Lambda}$ by nonoriented curves (resp. by oriented curves which satisfies the local isomorphism property). For each $k \in \mathbb{N}$ and each $y \in \mathbb{R}^2$, consider the tile $P^k_y \in \mathcal{P}_{X\Lambda}^k$ with center $y$, if it exists, and the union $Q^n_y$ of the 3 tiles with common vertex $y$ belonging to $\mathcal{P}_{X\Lambda}^k$, if they exist. Then, for each $n \in \mathbb{N}$ and each $x \in \mathbb{R}^2$ such that $Q^n_x$ exists, each $P^{n+3}_y$ (resp. $P^{n+4}_y$) contains some $Q^n_z$ such that $C \upharpoonright Q^n_z \cong C \upharpoonright Q^n_x$.

**Proof.** By Corollary 4, for any $y, z \in \mathbb{R}^2$ such that $P^{n+2}_y$ and $P^{n+3}_y$ (resp. $P^{n+4}_y$) exist, each covering of $P^{n+3}_y$ (resp. $P^{n+4}_y$) by a nonoriented (resp. oriented) curve contains copies of the 3 (resp. 6) coverings of $P^{n+2}$ by nonoriented (resp. oriented) curves.

Consequently, it suffices to show that, for each $x \in \mathbb{R}^2$ such that $Q^n_x$ exists, each $P^{n+2}_y$ contains some $Q^n_z$ such that $C \upharpoonright Q^n_z$ and $C \upharpoonright Q^n_y$ are equivalent modulo a rotation of angle $2k\pi/3$ (resp. a rotation of angle $2k\pi/3$ and/or changing the orientation of all the curves).

We write $\Lambda = (\lambda_k)_{k \in \mathbb{N}^*}$ and $X = (x_k)_{k \in \mathbb{N}}$. We denote by $\Delta_{x_n\lambda_1\cdots\lambda_n}(C)$ the set of curves such that $\Delta_{x_n\lambda_1\cdots\lambda_n}(C) \upharpoonright \Delta_{x_n\lambda_1\cdots\lambda_n}(P) = \Delta_{x_n\lambda_1\cdots\lambda_n}(C \upharpoonright P)$ for each $P \in \mathcal{P}_{X\Lambda}^n$.

For any $y, z \in \mathbb{R}^2$, $C \upharpoonright Q^n_y$ and $C \upharpoonright Q^n_z$ are equivalent modulo a translation, or a rotation of angle $2k\pi/3$, or changing the orientation of the curves, or a combination of these operations, if and only if the same property is true for $\Delta_{x_n\lambda_1\cdots\lambda_n}(C) \upharpoonright \Delta_{x_n\lambda_1\cdots\lambda_n}(Q^n_y)$ and $\Delta_{x_n\lambda_1\cdots\lambda_n}(C) \upharpoonright \Delta_{x_n\lambda_1\cdots\lambda_n}(Q^n_z)$. Consequently, it suffices to prove the statement for $n = 0$.

We treat the case of oriented curves, since the case of nonoriented curves is similar and simpler. Figure 8 below gives 9 possible configurations for $C \upharpoonright Q^0_z$. Each other possible configuration is obtained from one of them by a rotation of angle $2k\pi/3$ and/or changing the orientation of all the curves. We see from Figures 5 and 6 that, for each covering $C$ of some $P^2_y$, each of the 9 configurations above is realized by $C \upharpoonright Q^0_z$ for some $Q^0_z \subset P^2_y$. possibly
modulo a rotation of angle $2k\pi/3$ and/or changing the orientation of all the curves.

![Figure 8]

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