The non-triviality of the vacuum in light-front quantization: An elementary treatment

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It is often stated that the vacuum is trivial when light-front (null-plane) quantization is applied to a quantum field theory, in contrast to the situation with equal-time quantization. In fact, it is has long been known that the statement is false, and that in certain cases the standard rules for light-front perturbation theory need modification. This paper gives an elementary review of these issues, including an explanation of how and when there is a failure of the elementary derivation of the rules for light-front perturbation theory.

I. INTRODUCTION

It is commonly asserted (e.g., [1–3]) that the vacuum is trivial in a quantum field theory constructed using light-front (null-plane) quantization, at least within light-front perturbation theory. This is unlike the situation with standard Feynman perturbation theory. Associated with vacuum triviality appear to be a number of important consequences. Among these is the possibility of making a useful and natural definition of wave functions for particle states in the interacting theory; the definition appears to arise directly from the property that the state space of the theory is a free-particle Fock space.

Perhaps the most striking consequence that is claimed in Ref. [3, 4] is a solution of the cosmological constant problem. This solution arises because vacuum energy bubbles appear to be zero in light-front perturbation theory instead of being power-law divergent as they are in standard calculations. The vacuum bubbles give the vacuum expectation value of the energy-momentum tensor, and hence a contribution to an effective cosmological constant. The divergence must be canceled by a corresponding counterterm, a bare cosmological constant. The cosmological constant problem is that the value of the counterterm must be extremely fine tuned.

However, it has been known for nearly 50 years, since the work of Chang and Ma [5] and of Yan [6], that the argument leading to the triviality of the light-front vacuum is in fact incorrect, as are the calculations of a zero value for vacuum bubbles. It was shown that the rules for light-front perturbation theory must be modified, and that then the results always agree between light-front and Feynman perturbation theory, including for vacuum bubbles. Further work, by Nakanishi and Yabuki [7], and by Nakanishi and Yamawaki [8], showed among other things that triviality of the light-front vacuum conflicts with the well-established theorem of Haag [9, 10]. This theorem shows that the representations of the commutation relations of field operators are unitarily inequivalent between free and interacting theories.

In view of the continuing and prominent assertions of the triviality of the light-front vacuum, the purpose of this paper is to give an elementary treatment of the primary issues, especially concerning actual calculations:

1. I review, using a very simple example, the demonstration that an inconsistency arises from the calculational method that gives vanishing of vacuum bubbles.

2. I provide a new analysis to locate the failure in the derivation of the rules for light-front perturbation theory. The failure is only for graphs (and subgraphs) for which the plus components of external momenta are constrained to be zero.

The problem is an unrestricted use of the standard theorem $\int dx e^{ixq} = 2\pi \delta(q)$ that implements momentum conservation in terms of a delta function. This formula is incorrect when integrated with a function that is discontinuous at $q = 0$, as happens in the situations where there is a failure of the derivation of the standard rules for light-front perturbation theory.

3. I explain that, nevertheless, the non-triviality of the light-front vacuum does not itself affect the possibility of defining light-front wave functions. (Other issues do intervene in a gauge theory or when a non-trivial ultra-violet field renormalization is needed.)

This paper should be regarded as an elementary complement to Refs. [5–8, 11, 12]. Some parts of the argument here were already presented in a similar form in Sec. 7.2 of my QCD book [13]. Most of the specific points made here are undoubtedly known to many experts. I hope the presentation here will be useful to give an overall picture also accessible to interested outsiders.

The discussion given here is in terms of the continuum theory. The issues appear in a different form when Discrete Light-Cone Quantization is used [14, 15]. These need a separate discussion that goes far beyond the scope of the present paper.
II. A PARADOX AND ITS RESOLUTION IN LIGHT-FRONT PERTURBATION THEORY

A. Light-front perturbation theory; conventions

We use light-front coordinates where components of a Lorentz vector are defined as \( x = (x^+, x^-, x_T) \), and where, given a choice of Cartesian coordinates, \( x^+ = (x^0 + x^3)/\sqrt{2}, x^- = (x^0 - x^3)/\sqrt{2} \), and \( x_T \) denotes the remaining transverse coordinates.

Graphs in light-front perturbation theory — e.g., [1] — are specified with particular orderings in the space-time coordinate \( x^+ \) for the vertices. There are integrals over the values of momentum components \( k^+_j \) and \( k_{j,T} \) for the lines, constrained by conservation of the plus and transverse components. The values of \( k^+ \) are restricted to physical positive values \( k^+_j > 0 \) for forward moving momenta. The integrand has an “energy denominator factor” of the following form for each intermediate state \( I \)

\[
\frac{i}{P^- - \sum_{j \in I} \left( k_{j,x}^2 + m_j^2 \right) + i\epsilon},
\]

where the sum is over particles in the intermediate state, corresponding to lines in the graph. Each denominator is the difference between the external minus-component of momentum and the on-shell plus-component of momentum for the intermediate state. In graphs, we use the convention that \( x^+ \) increases from left to right, and correspondingly the flow of positive \( k^+_j \) is from left to right. Note that plus momentum is conserved, so that \( P^+_{\text{ext}} = \sum_{j \in I} k^+_j \).

With the basic rules for light-front perturbation theory, vacuum bubbles like Fig. 1 are zero. This is because the external momentum is zero. Hence it has zero plus component: \( P^+_{\text{ext}} = 0 \), and there are no possible intermediate states: All the lines have positive plus momentum from left to right, and these can never sum to the zero external plus momentum. The vanishing of the vacuum bubbles is what leads to the statement of the triviality of the vacuum.

B. A paradox

In this section, I show how the basic rules for light-front perturbation theory applied at vanishing external plus momentum lead to an inconsistency with standard analyticity properties. To make a very simple situation, let us examine the connected part of the following momentum-space Green function of composite operators in the theory of a free scalar field of mass \( m \):

\[
\Pi(p^2) = \int d^2x \, e^{ip \cdot x} \langle 0 | \frac{1}{2} \phi^2(x) \frac{1}{2} \phi^2(0) | 0 \rangle_{\text{conn}}\,.
\]

The factors of \( 1/2 \) are to get a standard normalization convention, and the use of the connected part means that from each instance of the operator \( \phi^2 \) is subtracted its vacuum expectation value. To make the calculations maximally simple, while still exhibiting the principles at stake, we work in \( 1 + 1 \) space-time dimensions, without any transverse dimensions. A standard textbook property is that \( \Pi(p^2) \) is an analytic function whose only singularity is a branch point at the threshold point \( p^2 = 4m^2 \). If the second \( \phi^2 \) operator were at a general position \( y \), we would change \( x \) in the exponent to \( x - y \) without change in \( \Pi(p^2) \), by translation invariance.

The use of time-ordering of the operators in Eq. (2) might appear to suggest the use of equal-time quantization. However, it is known that the definition is fully covariant. When \( x \) is space-like, the operators commute, and then the ordering is irrelevant. When \( x \) is time-like, it is frame-independent as to which position is future-most, and then the corresponding operator is defined to be on the left. The situation with light-like separation is governed by the usual rules for analyticity and for the distributional nature of the fields.

A Green function of composite operators is used instead of an actual S-matrix element to give maximum simplicity. In QCD, we often use such Green functions to formulate the strong-interaction part of a scattering with electroweak particles, with the composite operators corresponding to the coupling between an electroweak field and QCD fields.

Given that free-field theory is used, there are exactly two \( x^+ \)-ordered graphs for \( \Pi(p^2) \), as in Fig. 2.

When the external momentum has a positive plus component, i.e., \( p^+ > 0 \), the rules for \( x^+ \)-ordered perturbation theory give a single allowed ordering in \( x^+ \), symbolized in Fig. 2(a). The value of the graph, including its symmetry factor \( 1/2 \), is
\[ \Pi(p^2) = \frac{1}{4\pi} \int_0^{p^+} \frac{dk^+}{4k^+(p^+ - k^+)} \frac{1}{p^- - \frac{m^2}{2k^+} - \frac{i}{2(p^+ - k^+) + i\epsilon}} \]

(3)

where \( \xi = k^+/p^+ \).

When \( p^+ \) is negative, the other \( x^+ \) ordering, Fig. 2(b), is used and gives the same value for \( \Pi(p^2) \).

But when \( p^+ = 0 \), the graphs appear to be zero, because the two internal lines are required to have a physical momentum, with positive plus momentum, and this is prohibited by momentum conservation. However, we also know that the Green function is analytic at \( p^2 = 0 \), and hence the value at \( p^+ = 0 \) is the limit as \( p^2 \to 0 \) of Eq. (3), which is \(-i/(8\pi^2m^2)\) and non-zero. Something is therefore wrong in the calculation at \( p^+ = 0 \).

Once we realize that in this example the standard rules for light-front perturbation theory fail when the external momentum is zero, we must then expect them to fail also in the computation of vacuum bubbles. Corresponding results also apply in a higher space-time dimensions where there is also a transverse momentum integral.

C. Diagnosis within Feynman perturbation theory

To diagnose the situation, consider the calculation of the same graph in Feynman perturbation theory:

\[ \Pi(p^2) = -\frac{1}{8\pi^2} \int d^2k \frac{1}{\sqrt{[k^2 - m^2 + i\epsilon][(p - k)^2 - m^2 + i\epsilon]}} \]

\[ = -\frac{1}{8\pi^2} \int dk^+ dk^- \frac{1}{2k^+k^- - m^2 + i\epsilon}[2(k^- + p^-)(k^- - p^-) - m^2 + i\epsilon]. \]

(4)

To get a result corresponding to light-front perturbation theory, we perform the \( k^- \) integral at fixed \( k^+ \).

We first take \( p^+ > 0 \), and use contour integration on \( k^- \). This gives zero unless \( 0 < k^- < p^+ \), and then closing the contour in the upper or lower half plane of \( k^- \) gives the same result as was calculated in light-front perturbation theory in Eq. (3). This is an example of the established result [5, 6] that in this case both methods of calculation agree.

Next set \( p^+ = 0 \). The integral is now

\[ \Pi(0) = -\frac{1}{8\pi^2} \int dk^+ dk^- \frac{1}{2k^+k^- - m^2 + i\epsilon}[2(k^- + p^-)(k^- - p^-) - m^2 + i\epsilon]. \]

(5)

If \( k^+ \) is positive, then both \( k^- \) poles are in the lower half plane. We can deform the contour to infinity in the opposite half plane to get zero. Similarly if \( k^+ \) is negative, both poles are in the upper half plane, and we again get zero.

Hence it would appear that the graph is zero when \( p^+ = 0 \), in agreement with the result of standard light-front perturbation theory. But the result is surely wrong. If nothing else, we could choose to perform the \( k^+ \) integral first, which would correspond to obtaining the result for the opposite kind of light-front perturbation theory, i.e., ordered in \( x^- \). Provided that \( p^- \) is still non-zero, we get the expected non-zero result for the graph, i.e., the limit of Eq. (3) as \( p^2 \to 0 \).

Evidently there is a mathematical error in evaluating the integral in Eq. (3) by first performing the \( k^- \) integral and blindly using the zero result. The correct result, as found by Chang and Ma [5] and Yan [6], is that the integral over \( k^- \) in Eq. (5) gives a delta function at \( k^+ \).

To understand better what has happened, notice that we deformed the \( k^- \)-contour infinitely far into the upper half plane when \( k^+ > 0 \) and infinitely far into the lower half plane when \( k^+ < 0 \). The two-dimensional contour over both of \( k^+ \) and \( k^- \) is now broken at \( k^+ = 0 \). To complete the contour, we must couple these pieces by a section that goes from \((k^+, k^-) = (0, +i\infty)\) to \((k^+, k^-) = (0, -i\infty)\), as in Fig. 3. In deforming the unbroken two-dimensional contour, we inevitably encounter a region where the integrand is unsuppressed.

Another way of seeing the issue is to observe that when \( k^+ \) is fixed and non-zero, and we deform \( k^- \) to infinitely large imaginary \( k^- \), the integrand is of order \( 1/(k^-)^2 \),
and thus the contour at infinity gives a zero integral. But the coefficient of this asymptote is $1/(k^+)^2$, and hence the convergence of the integrand to zero is not uniform in $k^+$. Instead we get an unsuppressed contribution if we take $|k^-|$ to infinity while keeping $|k^+|$ of order $1/|k^-|$. This problem does not arise in the integral for the case that $p^+$ non-zero, for then one of the $1/k^-$ factors has a coefficient $1/(k^+ - p^+) \to -1/p^+$ instead of $1/k^+$, and there is a suppression of the otherwise dangerous region.

This dangerous region corresponds to modes with extremely large negative rapidity which propagate almost within the surfaces of equal $x^-$ on which light-front quantization is applied. Thus they involve large distances on these surfaces in the $x^-$ direction within the surfaces.

Of course, the diagnosis just given relies on assuming that Feynman perturbation theory is correct, while the authors of at least [1, 3] give the impression that Feynman perturbation theory is correct, while others have doubts as to the validity of light-front methods; outsiders to the field can wonder what other errors are so far unperceived.

### A. Momentum-conservation condition

Textbook derivations of perturbation formalisms in quantum field normally start with a coordinate space formulation and then use Fourier transforms into momentum space. This is what we do here. In formulating light-front perturbation theory, the field operators are stratified by the values of their $x^+$ position coordinates. Within a surface of constant $x^+$ we perform the integral over $x^-$. We would also integrate over transverse coordinates if the problem were in a higher space-time dimension than our example.

We apply this procedure to the graphs of Fig. 2, including both orderings in $x^+$. Then, Eq. (3) is replaced by its effective predecessor in the derivation, which is

$$
\Pi(p^2) = \frac{1}{8\pi^2} \int_{-\infty}^{\infty} \frac{dk^-}{2k^-} \int_{-\infty}^{\infty} \frac{dk'^+}{2k'^+} \int_{-\infty}^{\infty} dx^- e^{i x^- (p^+ - k^+ - k'^+)} \left[ \frac{i}{p^+ - m^2/2k^- - m^2/2k'^+ + i\epsilon} \theta(k^+)\theta(k'^+) - \frac{i}{-p^+ + m^2/2k^- + m^2/2k'^+ + i\epsilon} \theta(-k^+)\theta(-k'^+) \right].
$$

This equation applies for any value of $p^+$. The integral over $x^+$ has already been performed to give the energy denominator factors. The integrals over $k^+$ and $k'^+$ are over basis states corresponding to the possible intermediate states. The $\theta$ functions implement the physical-state conditions for the lines’ momenta for each of the $x^+$-ordered graphs. The convention for the signs of $k^+$ and $k'^+$ is chosen to give the same imaginary exponent for both graphs. Thus the positive, forward-moving momenta in Fig. 2(b) are $-k^+$ and $-k'^+$.

We now evaluate the integral over $x^-$. Normally, we would obtain a momentum-conservation delta function, from the theorem that

$$
\int dx^- e^{i x^- (p^+ - k^+ - k'^+)} = 2\pi\delta(p^+ - k^+ - k'^+).
$$

This reproduces the previously given results for the graph in light-front perturbation theory. However, the theorem must be interpreted in the sense of distributions, not as a theorem about functions. That is, the theorem applies provided that the left- and right-hand sides are integrated with a function that is smooth enough. In particular, the function must be continuous at $k^+ + k'^+ = p^+$, other-
wise the result of integrating with the delta function is undefined.

Now, if \( p^+ \) is non-zero, the rest of the integrand in Eq. (6) is indeed continuous at the relevant points. Then Eq. (7) applies, and the standard result is correct in this case.

However when \( p^+ \) is zero, the integrand is not continuous on the relevant line, which is \( k^+ + k'^+ = 0 \). The integrand has discontinuities when one or both of \( k^+ \) and \( k'^+ \) is zero, and hence at the point \( k^+ = k'^+ = 0 \), on the line \( k^+ + k'^+ = 0 \). In fact, the integrand diverges as that point is approached from some directions. So we must investigate the situation in more detail.

We evaluate the integral by changing variables to \( k^+ + k'^+ \) and \( \xi \), with \( k^+ = \xi(k^+ + k'^+) \) and \( k'^+ = (1 - \xi)(k^+ + k'^+) \). The use of \( k^+ + k'^+ \) ensures that one of the independent integration variables is exactly the variable that appears in the exponential and hence in the argument of the would-be delta function. This gives

\[
\Pi(p^2) = \frac{1}{16\pi^2} \int_{-\infty}^{\infty} d(k^+ + k'^+) \int_{0}^{1} d\xi \int_{-\infty}^{\infty} dx e^{i(x-(p^+ - k^+-k'^+))} \frac{i}{2p^-(k^+ + k'^+)(1 - \xi - m^2 + i\epsilon)}.
\]  

(8)

The integrand is now smooth, even at \( k^+ + k'^+ = 0 \), and we can therefore always use Eq. (7) to give a delta function that gives exactly the same value as the bottom line in Eq. (6). But now the derivation is valid for any \( p^+ \), including zero, and this was arranged by using a coordinate-space expression as a starting point.

### B. Immediate implications

When \( p^+ \) is zero, the integral over \( x^- \) in Eq. (8) gives a delta function at \( k^+ + k'^+ = 0 \). Then because \( k^+ \) and \( k'^+ \) are restricted to have the same sign, and because \( \xi \) is bounded, the values of \( k^+ \) and \( k'^+ \) that are relevant are both zero. Thus we have a kind of zero-mode contribution.

When one or both of \( k^+ \) and \( k'^+ \) goes to zero, the energy denominators in (6) go to infinity. This would give a zero in the integrand were it not for a divergence in the factor \( 1/(4k^+ k'^+) \) that is associated with the Lorentz invariant integration over momenta of the lines. The combined limiting behavior from these factors and the Jacobian of the transformation of variables gives the non-zero final result. The distributional nature of the calculation with an integral over all plus-momenta as a starting point allows the calculation to involve a limiting behavior from nonzero values of plus-momenta.

Zero plus-momentum implies infinitely large minus-momentum, and hence infinitely large and negative rapidity for each of the lines. Thus the physical contribution concerns contributions from intermediate states with arbitrarily negative rapidities.

Smoothness (and analyticity, in fact) of the integrand in (8) is only obtained after summing both \( x^+ \)-ordered graphs in Fig. 2. This results from the relationship between the two energy denominators in (6), which is presumably a fundamental property related to the CPT invariance of relativistic quantum field theories. But I will leave that issue to others to investigate in generality.

There is a difference in the definition of \( \xi \) compared with the one used in Eq. (3). There \( \xi \) was defined as \( k^+/p^+ \), which does not work when \( p^+ \) is zero. Here it is defined as \( k^+/k^+ + k'^+ \), which does not depend on \( p^+ \). Of course, after application of the delta function the definitions agree, but only when \( p^+ \) is strictly positive.

### C. General case

The calculation just given is, of course, specific to one particular pair of graphs. But we can now draw some more general conclusions.

First, the problem does not arise if we use time-ordered instead of \( x^+ \)-ordered perturbation theory. This because it is only \( x^+ \)-ordered perturbation theory that has boundaries on the allowed range of physical momenta.

As for a general graph in the \( x^+ \)-ordered case, each vertex has a factor that is a simple generalization of the \( x^- \) integral in Eq. (6). To avoid a long technical discussion, let us leave to other work a general derivation of the final answer strictly using only light-front methods. Here we just appeal to Refs. [5][6]. There it was shown by derivation from Feynman graphs that the only cases when the basic rules fail to be valid is in graphs where the momentum-conservation conditions require that one or more lines are constrained to have vanishing plus-momenta, just as in our example graphs at \( p^+ = 0 \) and in vacuum bubbles. Those references showed how to obtain corrected rules for light-front perturbation theory.

### D. Overall summary

We can now summarize the results of this section:

The failure of the standard derivation of the basic rules for light-front perturbation theory occurs where there is a failure of theorems like Eq. (7) that give delta-functions for momentum conservation. Momentum conservation does continue to hold. But when
some lines are constrained to be at zero plus-
momentum, the standard delta function must be changed to a different kind of delta func-
tion that correctly takes into account the limit-
ing behavior as the zero mode configuration is approached.

IV. CONSEQUENCES, OR LACK THEREOF,
FOR LIGHT-FRONT WAVE FUNCTIONS

A conspicuous contrast between relativistic quantum field theory and non-relativistic quantum mechanics con-
cerns the status and use of wave functions. Their use is routine in atomic, molecular, and condensed matter physics, and gives a lot of useful information about the states. For example, crucial insights into new structures emerging for interacting electrons in condensed matter physics has arisen from thinking about wave functions, two prominent examples being superconductivity \[12\] and the fractional quantum Hall effect \[13\].

In contrast, in work with relativistic quantum field theory there is little systematic use of wave functions, and correspondingly a paucity of information about the detailed microscopic nature of the quantum mechanical states involved. This is a particular important issue for non-perturbative bound states in QCD, i.e., for hadrons and nuclei.

The one known exception is with light-front quantization. Interesting progress has been made in applications, e.g., \[3, 19–22\] and references therein. However, the standard formulation, as in \[14\], appears to rely on the vacuum being trivial, so that a given state can be expressed as a non-pathological sum and integral over basis states that are in the same Fock space as free particles, e.g., for a proton in QCD in terms of states of quarks and gluons.

In general, a wave function gives an expression for a quantum-mechanical state in terms of a set of basis states, which are fixed independently of the presence and nature of the interactions in a theory. Thus the basis is defined before one has found a solution of the theory. In a non-relativistic system the basis is commonly of eigenstates of the position operators, but a momentum-space basis could also be used. It gives a representation of the canonical relations for the fundamental operators, which are coordinates and canonical momenta in a standard non-relativistic system.

But in a relativistic theory, these natural ideas appear to conflict with Haag’s theorem. Inequivalent representations of the canonical commutation relations are needed in the free and interacting theories. In a sense, the states for the interacting theory are in a different space than those of a corresponding free theory. One appears to need to solve the theory before knowing what the basis states are; equivalently, finding a solution of the theory includes a construction of its state space.

In this section, I show how light-front quantization evades this issue sufficiently to allow a definition of wave functions even given non-triviality of the vacuum. In a sense the difficulties are all confined to the nature of the vacuum. Haag’s theorem itself cannot be prevented from applying, but its consequences can be limited.

A. Overall framework

To be able to easily relate different viewpoints, let us use the Heisenberg picture. We conceive of a theory being defined in terms of operators and commutation relations specified on a quantization surface (e.g., fixed time or fixed \(x^+\)), and then evolution in \(t\) or \(x^+\) is applied. In the Heisenberg picture, it is the operators that are evolved rather than the states.

We now work in a field theory framework. Suppose that we are able to define annihilation and creation operators \(a_k^\dagger\) and \(a_k\) from the Fourier transform of the fields on the quantization surface, that they obey the standard commutation relations for annihilation and creation operator \(^\dagger\) and that the annihilation operators give zero when acting on the vacuum:

\[
a_k|0\rangle = 0. \tag{9}
\]

For the purposes of this discussion we assume there is only one kind of each operator; the extension to multiple fields is elementary.

The most general state constructed by applying creation operators to the vacuum, has the form

\[
|\psi\rangle = \sum_{N} \frac{1}{N!} \int \prod_{j=1}^{N} \langle 0 | a_{k_j}^\dagger \psi_{k_1,\ldots,k_N}, \tag{10}
\]

with an integration measure appropriate to the momentum variables and the normalization of the operators. The numerical coefficient functions we call the momentum-space wave functions of the state. Coordinate-space wave functions are defined by Fourier transformation of the momentum-space wave functions.

Then the wave functions of a state can be obtained from a matrix element of annihilation operators between the vacuum and the state:

\[
\psi_{k_1,\ldots,k_N} = \langle 0 | \prod_{j=1}^{N} a_{k_j} | \psi \rangle. \tag{11}
\]

This is proved using the commutation relations for the annihilation and creation operators, together with the vacuum-annihilation condition.

In the case of non-relativistic systems, wave functions with the above definition are the same as ordinary

\[2\] Anticommutation relations in the case of fermionic fields.

\[3\] Derived from the canonical commutation relations of the fields on the quantization surface.
Schrödinger wave functions. This is shown by the equivalence, e.g., [23], between a theory of non-relativistic Schrödinger field(s) and a collection of Schrödinger wave function theories for arbitrarily many particles.

Given the expression for the annihilation operators in terms of the fields, the right-hand side of Eq. (11) can be expressed as a Fourier transform of a corresponding matrix element of field operators restricted to the quantization surface. Thus the formula can be applied independently of the method by which calculations are made. For example, with a surface of fixed \( x^+ \) one can equivalently use Feynman perturbation theory or \( x^+ \)-ordered perturbation theory.

Formulas like the above, or some equivalent, can be found in many places in the literature, e.g., in the review [1] for the case of light-front quantization. Our ability to use them now depends on whether or not we can find a useful definition of the annihilation and creation operators in terms of the field operators.

As regards the implications of Haag’s theorem, we now see an interesting change of status of a wave function between non-relativistic and relativistic theories. In a non-relativistic theory, the state space is the same with and without interactions, and the vacuum is the same state. In a relativistic theory, provided that the vacuum-annihilation condition is obeyed, we have a labeling of a standard set of basis states that is the same as in the free theory, as shown by Eq. (10). To avoid a conflict with Haag’s theorem, one could have the free and interacting vacua being different. Alternatively, as a mathematical device one could take the abstract state spaces to be the same in the free and interacting theories; but then some pathologies must arise in how the free and interacting fields act on this space; that is, in how one constructs fields obeying the equations of motion in the free and interacting theories. See [12] [24] for explanations of how this works out in terms of the field operators when light-front quantization is used: essentially the complications are avoided in light-front quantization by projecting out of the field operator its behavior at \( k^+ = 0 \).

For the case of fields specified on a surface of fixed time \( t \), there is no way of defining annihilation and creation operators by Fourier transformation of fields on the surface, such that they give the vacuum-annihilation conditions. Certainly none has been discovered; see the literature on light-front quantization for details.

**B. Wave functions and light-front quantization**

Compared with equal-time quantization, the situation radically changes in light-front quantization. On a surface of fixed \( x^+ \), we define annihilation and creation operators by Fourier transformation:

\[
\phi(x^+, x^-, x_T) = \int \frac{dk^+ d^2k_T}{16\pi^3 |k^+|} \theta(k^+) \left( a_{k^+, k_T}(x^+) e^{-ik^+x^- + ik_T \cdot x_T} + a^\dagger_{k^+, k_T}(x^+) e^{ik^+x^- - ik_T \cdot x_T} \right)
\]

\[
= \int \frac{dk^+ d^2k_T}{16\pi^3 |k^+|} e^{-ik^+x^- + ik_T \cdot x_T} \left( a_{k^+, k_T}(x^+) \theta(k^+) + a^\dagger_{-k^+, -k_T}(x^+) \theta(-k^+) \right).
\]

The key thing is that values of physical plus momenta are restricted to be positive, and we express this by the allowed values for \( k^+ \) in each term. In the second line, it is arranged to have a common exponential factor, and there we can distinguish annihilation and creation operators by the sign of \( k^+ \). (Observe the reversal of sign of the argument of \( a^\dagger \) between the two lines.) The explicit denominator factor of \( |k^+| \) gives a Lorentz-invariant form of the integral over momenta. The annihilation and creation operators are given dependence on \( x^+ \), which is determined by the solution of the theory.

It follows from Eq. (12) that annihilation and creation operators can be obtained from the field operator by a Fourier transformation on a surface of fixed \( x^+ \), given a non-zero value of \( k^+ \). The standard commutation relations for the annihilation and creation operators follow from the commutation relation of the field on a surface of fixed \( x^+ \). Furthermore, applying an annihilation operator (with nonzero \( k^+ \)) to the vacuum gives zero. This is simply because by the known properties of the field under translations (in the full theory including interactions), the state \( a_{k^+, k_T} |0\rangle \) would have negative plus momentum relative to the vacuum, which is not possible.

Thus all the conditions summarized in Sec. IV A for defining wave functions are obeyed, and we have not had to invoke triviality of the vacuum to do this.

An immediate complication is that in a field theory, the most general state is obtained by repeatedly applying field operators to the field and taking linear combinations. Normally it would be sufficient to restrict the field operators to the quantization surface, and this would show that the most general state is of the form \( a_{k^+, k_T} |0\rangle \). The field operators get integrated with coordinate-space wave functions. This is compatible with the property

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4 But note that there are some further complications that need to be handled in gauge theories — Sec. IV C.
that the fields are not actual operators but are operator-valued distributions; operators are only defined with an integration with a smooth test function. But in light-front quantization, the creation (and annihilation) operators are restricted to non-zero $k^+$. When the algebra of the operators is examined \[12\] \[24\], this corresponds to a restriction to using test functions whose Fourier transform is zero at zero $k^+$. Correspondingly, the wave functions $\psi(k^+, k_1, T; k^+, k_2, T; \ldots)$ have to vanish when one or more $k^+$ is zero.

There is the potential for extra states obtained by applying zero-mode operators on the vacuum. To implement this properly in terms of field operators one must go slightly off the light-front \[12\] \[24\]. In momentum-space, the use of the appropriate distribution-theoretic framework indicates that the implementation of the extra zero-mode contributions uses integrals over a neighborhood of $k^+ = 0$, rather like that found in our earlier calculation in Eq. (8) at $p^+ = 0$. These are high rapidity modes, and presumably susceptible to a systematic analysis like those used in factorization or Regge theory. Further examination is needed.

C. Complications

Beyond general zero-mode issues of the kind just discussed, there are two complications that require modifications of the framework. One is to deal with non-trivial field renormalization, and the other is for gauge theories. Neither of these applies in a super-renormalizable non-gauge model, e.g., Yukawa theory in $2+1$ space-time dimensions.

1. Renormalization

The commutation relations that give the standard interpretation of annihilation and creation operators are derived from the commutation relations of the fields on the quantization surface. But in a renormalizable theory, the fields need a renormalization factor:

$$\phi_0(x) = \sqrt{Z} \phi_R(x). \quad (13)$$

Here $\phi_0$ is the bare field, having the standard normalization for its commutation relations, and $\phi_R$ is the renormalized field. It is the renormalized field whose Green functions and matrix elements are finite. To define the theory an ultra-violet cut-off is applied. Order by order in perturbation theory, the renormalization factor $Z$ diverges. The divergences can be resummed by renormalization-group methods, at least if the theory is asymptotically free.

When an on-shell “physical” renormalization prescription is used to define the normalization of $Z$, the Källen-Lehmann representation for the two-point functions can be used to show that the exact value of $Z$ obeys $0 < Z < 1$, a fact reflected in the sign of its anomalous dimension, and in a negative value for the lowest-order correction to $Z$ in perturbation theory. In the case that the lowest-order divergence is at one-loop order, the renormalization group shows that the exact value obeys $Z \rightarrow 0$ as the cut-off is removed, in an asymptotically free theory.

To get finite wave functions, we must apply the same renormalization factor to the annihilation and creation operators, with the outcome that the renormalized annihilation operators used to calculate finite wave functions in Eq. (10) are each an infinite factor $1/\sqrt{Z}$ times the ones obey the standard commutation relations. This needs a modification of the formula (10) for a state. I am not aware of a systematic treatment of this issue.

It is possible to do all calculations with a cutoff that is not removed. But it is surely preferable to say that the theory itself is the renormalized theory defined in the limit that the cutoff is removed. Then Eq. (11) gives a valid definition of renormalized wave functions in terms of matrix elements of renormalized operators. In situations where perturbation theory applies, perturbative calculations of wave functions work. But the actual expansion of the quantum mechanical state needs modification from (10). This is presumably a purely technical problem, since ultra-violet renormalization is very well understood.

2. Rapidity divergences in gauge theory

Much more interesting is the problem with rapidity divergences in any gauge theory.

The natural gauge condition to use with quantization on a plane of constant $x^+$ is \[25\] the light-cone gauge $A^+ = 0$, which gives the simplest version of the formalism. Unfortunately, the wave functions defined using this method have divergences \[26\] beyond those associated with ultra-violet renormalization. These divergences have exactly the same cause as those that arise when the same annihilation and creation operators are used in the natural way to try to define transverse-momentum-dependent (TMD) parton densities. The divergences can be seen readily in perturbative calculations, when going beyond lowest order. The divergences arise from regions of integration where rapidities of the lines for the gauge fields go to negative infinity.

For example, Brodsky et al. \[27\] made calculations in QED of the light-front wave functions of a single electron. The one-electron component is given in their Eq. (25) in terms of a quantity they call $Z$ (not to be confused with the renormalization factor in Eq. (13) in the present paper). The one-loop value is given in their Eq. (29). It has an integral over a variable $x$ from 0 to 1, and the integral diverges logarithmically at $x = 1$. This contrasts with the statement just after (25) that $Z$ is finite when the theory is regulated in the ultraviolet and infrared.
Soper [28, 29] devised for the TMD parton densities and fragmentation functions. To regulate the divergences they changed to a definition [30] that uses the same formula in terms of field operators but with the use of space-like axial gauge \( n \cdot A = 0 \). The matrix elements depend on an extra parameter \( \zeta \). Collins and Soper derived an equation for the \( n \)-dependence and hence the \( \zeta \)-dependence that is phenomenologically important. When \( n \) becomes light-like, i.e., when \( n^\mu \rightarrow (0, 1, 0_T) = g^\mu \), the parameter \( \zeta \) goes to infinity. The Collins-Soper equation shows in full generality that there is a divergence in this limit.

An alternative method gives an explicitly gauge invariant definition. It starts by writing operator matrix elements such as the one in (11) in terms of integrals over gauge-invariant field operators. In the expression in terms of operators in the light-cone gauge, one inserts Wilson lines in the minus direction, to make the operators gauge invariant. In the \( A^+ = 0 \) gauge, the Wilson lines are unity, but we now have a definition that can be used with any gauge condition. Calculations reproduce the same rapidity divergences found earlier. A new method [13] was devised to give a kind of rapidity renormalization factor using vacuum matrix elements of certain specially chosen Wilson loops. The resulting matrix elements have a parameter \( \zeta \) with the same significance Collins and Soper's \( \zeta \), and the evolution equations are of the same form. For related definitions in soft-collinear effective theory (SCET), see [31, 33].

Although these methods were devised in the context of TMD parton densities, the principles apply equally to light-front wave functions, as was shown by Ma and Wang [26]. An appropriate version of the modern definitions was given by Li and Wang [34].

Note the these considerations apply to the kind of light-front wave function that has dependence on transverse momenta. In contrast, much phenomenology is done with a different kind of distribution amplitude that is integrated over transverse momentum. For these, rapidity divergences cancel, and the evolution equation is a kind of renormalization-group equation, the Efremov-Radyushkin-Brodsky-Lepage (ERBL) equation. This is the case at least when Radyushkin's definition [35, 36] is used. However, the Brodsky-Lepage definition [37, 38] is made in light-cone gauge with an explicit cutoff in transverse-momentum. If that definition is taken literally and its implications examined closely enough, it is expected to give rapidity divergences.

V. DISCUSSION

The use of light-front quantization has a number of important advantages [11, 2, 5], compared with equal-time quantization. Many are presented [3] as being directly related to the vacuum being trivial. An interesting consequence of vacuum triviality is that the effective cosmological constant caused by vacuum bubbles is zero [3, 4], thereby trivially solving the notorious cosmological constant problem. But this solution comes at the price that it implies an inequivalence between the solution of a quantum field theory by light-front quantization and by conventional methods.

However, it has long been known that the claim of vacuum triviality is wrong [5–8]. Furthermore the validity of the inconsistency between the results of different kinds of quantization was challenged by the demonstration [9, 10] that an actual paradox is produced by the method of calculation that gives the vanishing vacuum bubbles. Chang and Ma [7] and Yan [10] derived corrected rules for light-front perturbation theory. The corrections are confined to situations typified by the vacuum bubbles used to calculate the bare contribution to an effective cosmological constant. But their starting point was an assumption that Feynman perturbation theory is correct. In view of a possible inequivalence between different methods of quantization, this is not a sufficient argument.

Therefore, motivated by continuing assertions about vacuum triviality and the cosmological constant, the present paper tried to give an elementary account that I hope will lay this issue to rest. First, I gave an example of the paradox mentioned above, and explained why it indicates that there is almost certainly an error in the rules that lead to the vanishing of vacuum bubbles, etc. Then I located the error in the derivation of the rules for light-front perturbation theory. The flaw is in the unrestricted use of a standard theorem to get a delta-function to implement momentum conservation. In exactly the conditions needed for the cosmological constant calculation, the theorem needs to be changed. The change restores the equivalence between the results in different methods, in accordance with the old results of Chang and Ma and Yan, but without needing the starting assumption that it is Feynman perturbation theory that is correct. Equally, the results now support non-triviality of the vacuum.

As we saw in Sec. IV, non-triviality of the vacuum does not affect the ability to define light-front wave functions. However, in a gauge theory, the standard definitions have rapidity divergences, and modified definitions are compulsory to deal with this [28, 34], with complete similarity to corresponding issues in the definition and use of transverse-momentum-dependent parton densities. The divergences to be dealt with in this fashion arise from an integral over the rapidity of gluonic configurations, and give divergences when the plus momenta of these configurations go to zero.

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