Testing the decoupling theorem with IBM quantum computer

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Abstract: Decoupling is a well-studied phenomenon in the theory of quantum information: the correlation between two systems - possibly in the form of entanglement - disappears almost entirely when one undergoes a random unitary evolution. Previous studies have proven the mathematical formulas and general conditions for decoupling two systems. The aim of this work is to observe the effectiveness of a real-world implementation of the decoupling scheme with the IBM quantum computer. In particular, we apply Clifford circuits to the systems in initial state and observe how correlation varies with respect to different circuit lengths. A significant trend was discovered, according to the mathematical theorems: the correlation between the systems decreases as the size of Clifford circuit increases. This observation shows that the decoupling theorem can be implemented effectively with the IBM quantum computer. Under a global perspective, this work is intended to provide a result in support of the fundamental task of laying the groundwork for the development of a reliable quantum computing architecture.

1. Introduction

Given a system $A$, a second system $E$ may compass information about $A$ when the bipartite quantum state $\rho_{AE}$ contains classical or quantum correlation. The latter property is called entanglement: this is a form of correlation for quantum systems that each system or the group cannot be described independently of the state of the others, including when the systems are separated by a large distance. In other words, the joint quantum systems cannot be written as the product of two independent states.

Decoupling is known as the reverse process of entanglement. System $A$ is decoupled from system $E$ if the joint state can be written in the form $\rho_{AE} = \rho_A \otimes \rho_E$. In other words, there is no correlation between two systems and thus no information is shared between them. The general decoupling theorem is proven by Frédéric Dupuis [1], which characterises the entropy conditions required to achieve decoupling. In this theorem we consider an initial joint system $AE$, possibly containing a correlation between the two; system $A$ then undergoes a random unitary evolution, followed by a completely positive and trace preserving mapping which does not act upon system $E$.

Decoupling has become one of the most relevant concepts in information theory, since it has been proven to be useful in various applications. One of the distinguished applications is that decoupling can be used to test the entanglement between three systems as the theorem of monogamy of entanglement implies that a quantum system cannot be fully entangled with more than one system [2]. Assume that $M$, $N$ and $Q$ are three quantum systems with a certain level of entanglement. If we know that the system $M$ is fully entangled with the system $N$, then it can be inferred that the system $M$ must be decoupled with system $Q$ because of the monogamous feature of quantum systems. Moreover, in quantum computing, decoupling the quantum apparatus from the environment is of paramount
importance in order to maintain *coherence*, that is, the quantum properties of the quantum computing hardware[3].

Although the decoupling theorem has been well-characterized and alternative proofs have been obtained, the study on the practical application of decoupling theorem still has considerable room to be explored. Quantum computers offer speed advantages over classical computers when dealing with computation such as superposition. In order to perform reliable computation, qubits must not be affected by the environment and stay decoupled. Thus, in order to test the practical effectiveness of decoupling theorem, we have constructed different correlated systems with an IBM quantum computer, and particularly we applied Clifford circuits of various lengths to decouple the initial state. To measure the amount of correlation of the output state, we used Qutip library[4]. At the end, We have obtained positive results from our tests which supports the possibility of obtaining a decoupled state with an IBM quantum computer by applying the Clifford circuits.

2. Preliminaries

2.1. The density operator

Density operator is also known as density matrix, which describes the state of an $n$-dimensional quantum systems. Density operator is usually denoted by $\rho$ and has the following properties:

- $\rho$ is a $n \times n$ matrix
- $\rho$ is always positive semi-definite, that is, $v^\dagger \rho v \geq 0$ for all $n$-dimensional vector $v$. This implies that all the eigenvalues of $\rho$ are non-negative.
- the trace of $\rho$ is equal to 1, i.e., $\text{Tr}[\rho]=1$

Generally, the density operator framework provides an interpretation of quantum mechanics in the vector space, which is more convenient and widely adopted in the context of quantum computing.

A quantum state a mathematical object which describes the probability distribution for the outputs of measurement of the systems that it represents. A quantum state is said to be pure if the there exists a vector $\varphi$ such that $\rho = |\varphi\rangle \langle \varphi|$. Otherwise, it is said to be a mixed state.

Consider a quantum system that is a mixture of pure states $|\varphi_j\rangle$, each associated with a probability $p_j$. The formula for calculating the density operator is given by[5].

$$\rho = \sum_j p_j |\varphi_j\rangle \langle \varphi_j|.$$  \hspace{1cm} (1)

2.2. Partial trace

The partial trace is defined as the restriction of trace on a subspace, where the trace is the operator that sums the diagonal elements of a matrix. The partial traces respectively on subsystems $A$ and $B$ are denoted as $\rho_A = \text{Tr}_B[\rho_{AB}]$ and $\rho_B = \text{Tr}_A[\rho_{AB}]$. These operators represent the state $\rho_{AB}$ when restricting to one of its subsystems.

2.3. Quantum gates

The quantum gates can be represented by a $2 \times 2$ matrix. Consider a quantum system represented by the state $\alpha|0\rangle + \beta|1\rangle$, the required normalisation condition for this quantum system is $|\alpha|^2 + |\beta|^2 = 1$. Thus, the resulting quantum state $|\varphi\rangle = \alpha' |0\rangle + \beta' |1\rangle$ must be normalized as well after the quantum gates are applied. Recall that $U^\dagger U = I$, where $U$ is an unitary matrix and $U^\dagger$ is the adjoint of $U$ and $I$ is the $2 \times 2$ identity matrix. Any single or multi-qubit gate can be described by a unitary matrix $U$. Overall, a quantum gate is a reversible unitary operator which can be applied on one or more qubits.
2.4. Quantum circuits
A quantum circuit is a model for quantum computation that consists of operations on quantum state. A quantum circuit is a sequence of quantum gates acting on an input state, followed by a measurement.

The circuit should be read from left to right as corresponding to the passage of time. Each row in the circuit corresponds to a wire in the quantum circuit, where a quantum wire represents a single qubit. Measurements are read onto classical wire which is the wire at the bottom of the quantum circuit. The first column shows the input state, usually initialized as $|0\rangle^n$.

2.5. IBM Quantum Experience
IBM Quantum Experience is a cloud application for programming real quantum hardware and high performance simulators. IBM Quantum backends is a service that hosting world-class quantum systems and high performance simulations for research, industry and education. The backends for IBM Quantum Experienced are all denoted by the name which started with ibmq_. These backends will provide a configuration which includes all information that is required to perform the quantum circuits on the backend. Moreover, the backends will also return properties information with details about the properties of the qubits devices and the quantum gates that are acting on these qubits. This also includes the noise information in the systems which are obtained from device calibration scripts.

IBM Quantum Experience also includes a circuit composer. This is a graphical quantum programming tool that allows the user to set up quantum circuits and run them on the real quantum hardware. The circuit composer is able to visualise the qubit states of the quantum circuits which clearly shows how the operation affects the state vectors and its measurement probabilities. With the help of circuit composer, writing codes are no necessary required as the circuit composer will automatically generate OpenQASM code that behaves the same way as the circuit the user created with Circuit Composer. The testing of decoupling theorem in this paper generally bases on the circuit composer.[6]

2.6. Mutual entropy
Entropy is a quantity which gives measurement of information for any probability distribution. Claude Shannon(1916-2001) first proved the measure of entropy in classical physics which is known as the Shannon entropy, given by [7]

$$ S = -\sum_j P_j \log P_j. $$

Later, John Von Neumann derived the von Neumann entropy which is known as the quantum version of the Shannon entropy. The Von Neumann entropy of a density operator $\rho_A$ describing the state of a quantum system $A$ is defined as [5]

$$ H(A)_\rho = -\text{Tr}(\rho \log \rho). $$

The concept of entropy has been extended to define mutual information, which is a measure of correlation between two systems. To be more precise, let us consider a system $A$ which is partially correlated to system $B$. Then, the mutual entropy is how much is unknown about systems $A$ and when knowing $B$, and vice versa (as why it is called “mutual”). The mutual information between systems $A$ and $B$, denoted by $I(A:B)$, can be calculated as followed [8]:

$$ I(A:B) = H(A) + H(B) - H(A,B), $$

where $H$ is the von Neumann entropy defined in eq.(3).

2.7. Pauli group and Clifford group
The Pauli matrices on a single qubit are given by $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The $n$-qubit Pauli group is then defined as
\[ P_n := \{ e^{i \theta} \sigma_{j_1} \otimes \cdots \otimes | \theta = 0,1,2,3, j_k = 0,1,2,3 \}. \]  

The Clifford group \( C_n \) is the normalizer of the \( n \)-qubit Pauli group, that is, it is the set \[ C_n := \{ V \in U_2^n \mid VPV^\dagger = P_n \}. \]  

This group can be generated by the single-qubit Hadamard gate \( H \) and the single qubit phase gate \( P \) together with the controlled gate CNOT, respectively given by

\[
H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \]  

### 2.8. QuTIP

QuTIP is a quantum toolbox in python which can simulate the dynamics for an open quantum system. It provides a wide range of efficient simulations in quantum mechanics and in physics. In this paper, we mainly used the entropy function in QuTIP to calculate the mutual entropy for the outputs we obtained, based on the function \( \text{entropy}_\text{mutual}(\rho, \text{selA}, \text{selB}, \text{base} = e, \text{sparse} = \text{False}) \), where \( \rho \) is the density matrix for the composite systems, \( \text{selA} \) is the first selected matrix subspace, \( \text{selB} \) is the second matrix subspace, \( \text{base} \) is the base of logarithm and \( \text{sparse} \) is the use of sparse eigensolvers.

### 2.9. Decoupling theorem

Consider a system \( A \) which is possibly correlated to the system \( E \), which we usually identify with the environment. To decouple these two systems, we let system \( A \) undergo a random unitary evolution followed by a completely positive, trace preserving mapping \( \mathcal{T} \) from density operators to density operators which brings system \( A \) to an output system \( B \). Additionally, the evolution does not interact with system \( E \) at any moment. We then observe remaining correlation between the output system \( B \) and \( E \). A version of the decoupling theorem characterized by smooth entropy measures is given by the following [1].

Let \( \rho_{AE} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_E) \) and let \( \mathcal{T}_{A \to B} \) be a complete-positive mapping with Choi-Jamiołkowski representation \( \tau = J(\mathcal{T}) \). Then we have

\[
\int_U \left\| \mathcal{T}_{A \to B}(U_{A}\rho_{AE}U_A^\dagger - \tau_B \otimes \rho_E) \right\| \, dU \leq 2^{H_\sharp(A|E)_\rho - \frac{1}{2}H_\sharp(A|B)_\tau}. \]  

The Choi-Jamiołkowski isomorphism is a map which transforms quantum maps into density operators. The TPCPM \( \mathcal{T} \) is in the forms of \( \mathcal{T} = \mathcal{T} \circ \mathcal{U} \) with the unitary channel \( \mathcal{U} \), where \( \mathcal{U}(\cdot) = \mathcal{U}_A(\cdot) \mathcal{U}_E^\dagger \). The decoupling theorem bounds the quality of decoupling and it only depends on two collision entropies \( H_\sharp(A|E)_\rho \) and \( H_\sharp(A|B)_\tau \). The first is a measure for initial state of joint system \( AE \) which is represents by \( \rho_{AE} \). Properties of the TPCPM \( \mathcal{T} \) are instead described by \( \tau_B \). The tensor product \( \tau_B \otimes \rho_E \) indicates that quantum systems \( B \) and \( E \) are completely independent to each other. The decoupling theorem provides a bound on the distance between \( \mathcal{T}_{A \to B}(U_{A}\rho_{AE}U_A^\dagger) \) from a tensor product state.

### 3. Method and simulations

Generally, if we want to decouple two correlated systems \( A \) and \( E \), we need to apply a channel that randomly selects a unitary \( U_A \) and output the choices of \( U_A \) onto the system \( A \). In another words, we need to choose from a uniform measure over unitary operators. However this is very unphysical because the unitary group is dense and infinite. The Clifford group is also easily implementable into the
quantum circuits and it is a distribution over the unitary group which ‘approximates’ the uniform measure [10]. Therefore, we applied Clifford gate in our tests instead of taking an actual uniform measure.

To start the experiment, we first construct a random circuit that entangles two systems together by using the IBM quantum computer. Figure 1 is one of the tested circuits.

Consider the first two wires as system $A$ and the rest of wires as system $E$. We entangled two systems together by applying a $H$-gates and followed by $CNOT$-gates. Recall that $CNOT$-gate is one of the sources of entanglement. With the application of $H$-gate, the top qubit has been put into a superposition; this then acts as the input of the $CNOT$ where the target qubit only get inverted if the control qubit is $1$ [5]. Up to this, the preparation before the application of decoupling theorem has been completed.

Before we start to decouple the quantum systems, we take a measurement for the entangled systems and consider it as the point where the circuit length of Clifford group is 0. Next, we apply random circuits of Clifford group with respect to different lengths, generated by algorithm 1, onto the quantum state to decouple systems $A$ and $E$. In this experiment, we tested the systems with Clifford group of lengths 20, 30 and 40.

In order to apply a completely random circuit, we wrote a Python code to generate circuits of Clifford group. First, we generate qubits-pairs, in the form of (gate, wire number), where gate is chosen from $H$-gate, $S$-gate and $CNOT$-gate and 'wire number' is chosen from the wires in the system $A$. Then we used the random function to provides a list of pairs in different lengths. Finally, we applied the pairs onto the correlated circuits according to the list. In the qubit pairs, the gate represents the qubit gates that are going to be used and the 'wire number' corresponds to the wire the gate is applied upon.

After the application of the random circuits we perform a measurement for the quantum systems. Finally, we calculate the mutual entropy of the outputs by using Python. In particular, using the Python library, Numpy and Qutip. Since in IBM Quantum Experience the outputs are given in the form of probabilities of computational basis, we had to elaborate the probabilities and turn the outputs into density operators form. After this, we create a Qobj variable with the density matrix we obtained, where the Qobj is used as an argument for the mutual entropy function. We run the code and we retrieve the mutual entropy of the quantum systems. We tested three different systems in total to get an interpretation of our experiment.
Algorithm 1 Constructing random Clifford circuit

**Input:** circuit length $\ell$;  
**Output:** Clifford circuit of length $\ell$;

1: define a function for qubit pairs (gates, wire-number), where gates are randomly chosen from the Clifford group and wire-number is randomly selected from the quantum system.;
2: the wire-number will be 1 wire; **return** qubit pairs(gates, wire-number);
3: randomly select number of qubit pairs (the number is the same as the circuit length) **return** Apply Clifford gate on qubit pairs.

Algorithm 2 Calculating mutual entropy of the output state

**Input:** probabilities of the vector state;  
**Output:** mutual entropy of the joint systems;

1: convert the probabilities into density operator form;
2: define a Qobj group;
3: use the entropy_.mutual function to calculate the mutual entropy; return mutual entropy;

4. Result

The outputs of the quantum circuits are given in forms of vector space, figure 3 is one of the outputs we have obtained from IBM quantum Experience.

Table 1 displays the mutual entropy of different quantum systems with respect to different lengths of circuits of Clifford group. And figure 5 is a line graph that based on the data from table 1.
Table 1 The mutual entropy for different systems with the variation of circuits lengths of Clifford group

| Quantum systems | Circuit length | 00  | 20  | 30  | 40  |
|-----------------|----------------|-----|-----|-----|-----|
| [2,3] system    | 0.4260         | 0.2343 | 0.1802 | 0.1864 |
| [2,4] system    | 0.2523         | 0.1202 | 0.0910 | 0.0437 |
| [3,2] system    | 0.3734         | 0.1338 | 0.1331 | 0.0118 |

Figure 4 Mutual entropy against circuits lengths of the Clifford group

From the result, we can see that with the increase of circuit lengths the mutual entropies of the [2,4] system and the [3,2] system decrease: the mutual entropy for [2,4] system drops from 0.2523 to 0.0437 and the mutual entropy for [3,2] system decreases from 0.3734 to 0.0118.

The same also applies to [2,3] system with the circuit length 0, 20 and 30, where the mutual entropy decreases from 0.4260 to 0.1802. However, the mutual entropy for [2,3] system increases slightly when the circuit length is 30, where the mutual entropy for length 30 and length 40 are 0.1802 and 0.1864 respectively. From the graph, we can clearly see that, after applying random circuits of Clifford group with a length of 40, the mutual entropies of all systems are less than the half comparing to the initial state. From the plot we can also find that the lines for all systems are generally decreasing, that is, the mutual entropies are decreasing with the gradual increase of application of Clifford group. Regarding to the test for [2,3] system, we have tried at least ten different circuits with both circuit length 30 and 40 and take average of all outputs to calculate the mutual entropy. In theory, the mutual entropy when circuit length is 40 is less than the mutual entropy when circuit length is 30. After obtaining this result, we have also tested [2,3] systems again with Clifford group circuits of length 30 and 40. However, we observe the outputs for length 30 are always less correlated than the outputs for length 40, and the reason is unclear. Thus, we leave this to be further explored by future research.

The analysis of results suggests that the mutual entropies of the quantum systems decreases as the length of Clifford group circuits increases: the correlation between system $A$ and $E$ is gradually
decreasing as Clifford gates are applied, that is Clifford group is an effective uniform measure in Quantum computer.

5. Conclusion
The present research was designed to examine the effectiveness of the decoupling theorem with the IBM Quantum computer. In this work, we tested three different quantum systems with the decoupling theorem. We conclude with a remarkable finding that the correlation between two entangled systems decreases as the size of the Clifford group increases. This is established from the results of our experiments, where the mutual entropy for the quantum states decreases with the application of random Clifford circuits. Therefore, our result has shown that decoupling theorem can be applied effectively with the IBM Quantum computer. Our study has proven that the realization of decoupling with a quantum computer is feasible and valid. Looking into the future, this study helps to support the application of fundamental task of quantum mechanics for the use of a quantum computer, as decoupling theorem is one of the central notions in the information theory.

In the meantime, we also acknowledge that our method yet achieved maximum efficiency as the process of constructing the random circuits and calculating the mutual entropy consumes a considerable amount of time. For this reason, we are not able to test quantum circuits with longer sequences. Moreover, the stability of our method is vulnerable if there are other operations to apply after decoupling because the circuit for decoupling is long and complicated. Therefore, a worthy follow up research can investigate the development of a more efficient method to obtain a decoupled state with a quantum computer. Additionally, it will be worthwhile to test more basic theories in quantum mechanics with a quantum computer.

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