EVOLUTION AND JUMP IN A WALRASIAN FRAMEWORK

ELVIO ACCINELLI
Facultad de Economía de la UASLP
Avenida Pintores S/N
San Luis Potosí, 78280, México

ENRIQUE COVARRUBIAS
Facultad de Economía de la UASLP
Avenida Pintores S/N
San Luis Potosí, 78280, México

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Abstract. Lower profit rates play an important role in the evolution of an ownership private economy. We argue that if managers look to maximize profits rates, then the decision to change, to those branches, or technologies, that offer higher rates of profits, plays an important role in the characterization of economies. If managers choose to produce according to those technologies that promise higher profit rates, then along the time, the distribution of the firms over the set of available technologies change, and therefore the economic fundamentals change. Under conditions of imperfect information, the imitation of the most successful firms plays can a decisive role in deciding how to produce. Along a path of Walrasian equilibria, regular economies can become singular and if this occurs, big changes must be expected after decisions of the firms for the next period.

1. Introduction. The search for higher profit rates by owners or managers of firms, is undoubtedly an important engine for change in productivity and economic performance. This is an issue widely considered in modern literature, but so far, at least in the knowledge of the authors, has not been considered the role that this process plays in characterizing the evolution of the economies along an equilibrium path. This analysis is of no less importance given that possible changes in regular economies arising from small perturbations in its foundations are soft, but not so in the case of so-called singular economies, precisely these economies are characterized by the possibility of abrupt changes in response to small perturbations in its foundations.

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This paper contributes in the sense to analyze the economical evolution and the possibility of economical crisis, as the result of an endogenous mechanism of election made by maximizers owners or managers. The evolution is driven by the decisions of decision makers of firms with lower rates of profits. We assume that manager and owner have the same interest (there is no agency problem) and so, they choose to produce according with the technologies with higher profit rates.

As in [9] we consider a distribution of heterogeneous firms producing with heterogeneous technologies, but we consider the rates of profits as the real engine of the decisions of managers. Certainly, it is natural to assume that more developed technologies give place to higher rates of profits. So this evolution goes from less productive to more productive technologies.

On the other hand, unlike what is done in the work previously cited, we do not focus in growth, our main attention point is on the characteristic of the equilibrium manifold and the evolution along an equilibrium path on this manifold. We show that crisis happen as a result of these decisions.

Our approach shows that a shift in the distribution of the firms over the set of available technologies, has important repercussions on the characterization of the set of equilibria of a Walrasian economy, whose structure changes according with the changes in this distribution. Under imperfect information, when a firm looks for higher rates of profits, is natural to assume that firms will imitate the behavior of those firms showing the best results.

This paper is structured as follows. Section 2 presents a the model of a private-ownership economy, its parameters, and the definition of equilibrium. Section 3 defines the equilibrium set and studies its properties. In this section we introduce the definition of regular and singular economies. The dynamical behavior of pure system is introduced in section 4. We consider that imitation play a significant role when the decision makers have incomplete information. In section 5 we analyze the stability of the dynamical equilibrium using the Liapunov method. In section 6 we consider a numerical example and finally, some conclusions are given.

2. The economy. In this section, we introduce the model of our economy. We will consider an economy with two goods, two types of firms and two types of consumers. Within each type, firms will use identical technologies, while consumers will have identical preferences and endowments (of both goods and shares in companies). We will assume that markets are competitive in the sense that every good is traded in the market at a publicly known price, and consumers trade to maximize their own welfare, while firms produce to maximize profits. We will formalize the structure of consumers, firms and equilibria in this section.

Since there are two goods in the economy, the commodity space is \( \mathbb{R}^2 \) while the consumption set and price set is \( \mathbb{R}_+^2 = \{ x = (x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0 \} \). We will also use the notation \( \mathbb{R}_+^2 = \{ x = (x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 > 0 \} \).

2.1. Firms. We consider an economy with a finite but very large number \( N \) of firms, which will be indexed by \( \mathcal{F} = \{1, \ldots, N\} \). There will be two types \( j = \{1, 2\} \) of such firms. There are \( n_1 \) firms of type 1 and \( n_2 \) of type 2, with \( n_1 + n_2 = N \). We will index firms of each type by \( F_j = \{0, \ldots, n_j\} \), with at least one firm of one of the two types. The proportion of firms of type \( j \) is given by \( N_j = n_j/N \), and one should keep in mind that as \( N \to \infty \), \( N_j \) takes values in the continuum \([0, 1]\).

Firms are characterized by their technological set \( Y_1 \) or \( Y_2 \), according to whether they are of type 1 or 2, respectively. Without loss of generality, we suppose that
technology is costless to acquire. That is, in each period each manager must choose between producing according to the technology represented by $Y_1$ or $Y_2$ and this choice does not imply any additional cost or loss of profit. The set of available technologies will be denoted by $T = \{Y_1, Y_2\}$. We consider that there is no principal-agent problem, that is, that the interests of managers are the same of the owners, which is to maximize profits. We assume technological sets satisfy the following standard properties, for $j = \{1, 2\}$: (i) $Y_j$ is closed and convex; (ii) $Y_j \cap \mathbb{R}^2_+ = \{0\}$; (iii) $-\mathbb{R}^2_+ \subset Y_j$; and, (iv) $Y_j$ is bounded from above, i.e., there exists some $a_j \in \mathbb{R}^2_+$ satisfying $y \leq a_j$ for all $y \in Y_j$.  

If $Y_j$ is a production set and $p \in \mathbb{R}^2_+$ is a price vector, then the profit function at price level $p$ is the function $\pi^j : Y_j \rightarrow \mathbb{R}$ defined by $\pi^j(y) = py$. Thus, firm $j$’s goal is to solve the problem

$$\max_{y \in Y_j} py \in \{1, 2\}$$

The function $y^j : \mathbb{R}^2_+ \rightarrow \mathbb{R}^2$ that solves the problem (1) is the supply of a firm of type $j \in \{1, 2\}$. When summarizing the supply functions of all firms in the economy we will use the notation $y = (y^1, y^2) \in \mathbb{R}^{2n_1} \times \mathbb{R}^{2n_2}$, where $y^j$ is the vector of supply functions of the $n_j$ firms of type $j$.

### 2.2. Consumers. They are $M$ consumers in the economy distributed into type 1 or type 2. We will index consumers by $I = \{1, \ldots, M\}$. There are $m_i$ consumers of type $i$, $i = \{1, 2\}$, with $m_1 + m_2 = M$. Therefore, the proportion of consumers of type $i$ is $M_i = m_i/M$. We will index consumers in each type by $I_i = \{0, \ldots, m_i\}$ with at least one consumer of one type. All consumers of type $i$ have identical $C^\infty$ utilities $u_i : \mathbb{R}^2_+ \rightarrow \mathbb{R}$ and identical initial endowments of goods $w^i = (w^i_1, w^i_2) \in \mathbb{R}^2_+$. We assume utilities satisfy the following standard properties:

(i) The bordered Hessian of $u_i$ is non-zero at every $x \mathbb{R}^2_+$.

(ii) The set $\{x' : u(x') \geq u(x)\}$ is closed in $\mathbb{R}^2$ for every $x \in \mathbb{R}^2_+$. The wealth of consumers is derived from individual endowments of commodities but also from ownership claims (shares) of the profits of firms, which we also assume to be identical within types. Hence, each consumer $i \in I$ has a claim to a non-negative share $\theta_{ij}$ of the profit of each firm $j \in F$ in a way that $\sum_{i=1}^M \theta_{ij} = 1$. This implies that the individual wealth $W^i$ of a consumer of type $i$ is given by

$$W^i(n_1, n_2, p) = p \cdot w_i + n_1 \theta_{i1} \pi_1(p) + n_2 \theta_{i2} \pi_2(p).$$

It is important to highlight that the wealth of an individual depends on the distribution $(n_1, n_2)$ of firms in the economy. In the next section, when we introduce dynamics to our model, this distribution will change endogenously so that the actions of a manager looking to use the most profitable technologies will have an immediate impact over the wealth of agents.

Now, the budget set $B^i$ of a consumer of type $i$ is given by

$$B^i(n_1, n_2, p) = \{x^i \in \mathbb{R}^2_+ : p \cdot x^i \leq W^i(n_1, n_2, p)\}.$$
Thus, given the distribution of firms \((n_1, n_2)\) and price level \(p\), the optimization problem of a consumer of type \(i\) is given by

\[
\max_{x \in \mathbb{R}^2_+} u_i(x) \quad \text{s.t.} \quad x \in B^i(n_1, n_2, p).
\]  

The function \(x^i : \mathbb{R}^2_+ \to \mathbb{R}^2\) that solves optimization problem (2) is called the \textit{individual demand function} of a consumer of type \(i\). When summarizing the demand functions of all consumers in the economy, we will use the notation \(x = (x^1, x^2) \in \mathbb{R}^{2m_1} \times \mathbb{R}^{2m_2}\), where \(x^i\) is the vector of demand functions of the \(m_i\) consumers of type \(i\).

2.3. \textbf{Walrasian equilibrium.} As discussed above, a private-ownership economy is determined by the following factors: the number and distribution of firms \((n_1, n_2)\) and their technologies \((Y_1, Y_2)\); the number and distribution of consumers \((m_1, m_2)\), their preferences \((u_1, u_2)\), their initial endowments \((w_1, w_2)\), and their portfolios of shares \((\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22})\). This gives rise to the following definition.

\textbf{Definition 1} (Private-ownership economies). A \textit{private ownership economy} \(\mathcal{E}\) is a choice of parameters \((n_j, Y_j, m_i, u_i, w_i, \theta_{ij})\) for \(i, j = \{1, 2\}\). We will write

\[
\mathcal{E} = \{\mathbb{R}^2_+, n_j, Y_j, m_i, u_i, w_i, \theta_{ij}, i, j \in \{1, 2\}\}
\]

Now, the notion of a price-taking equilibrium for a competitive private ownership economy is that of a Walrasian equilibrium, which we now define.

\textbf{Definition 2} (Walrasian equilibrium). Consider a private-ownership economy

\[
\mathcal{E} = \{\mathbb{R}^2_+, n_j, Y_j, m_i, u_i, w_i, \theta_{ij}, i, j \in \{1, 2\}\}.
\]

We say that a vector of demand-supply functions \((x^*, y^*) = (x^{1*}, x^{2*}, y^{1*}, y^{2*})\) and a system of prices \(p^* = (p_1^*, p_2^*)\) constitute a \textit{Walrasian equilibrium} or \textit{competitive equilibrium} if:

1. For each firm in \(\mathcal{F}\), \(y_j^i\) solves the optimisation problem (1) of a firm of type \(j\);
2. For each agent in \(I\), \(x_i^*\) solves the optimisation problem (2) of a consumer of type \(i\);
3. Aggregate demand equals aggregate supply. That is,

\[
\sum_{i=1}^{m_1} x_{i}^{1*} + \sum_{i=1}^{m_2} x_{i}^{2*} = m_1 w^1 + m_2 w^2 + \sum_{j=1}^{n_1} y_{j}^{1*} + \sum_{j=1}^{n_2} y_{j}^{2*}.
\]

Note in definition (2), that the market-clearing condition can be written succinctly as

\[
m_1 \left(x^{1*} - w^1\right) + m_2 \left(x^{2*} - w^2\right) = n_1 y^{1*} + n_2 y^{2*}.
\]  

3. \textbf{Equilibrium set.} Through the rest of the paper, we will suppose \((Y_j, m_i, u_i, w_i, \theta_{ij})\) are all fixed parameters. That is, we will suppose that the only parameter of the economy is the distribution of firms \((n_1, n_2)\) according to its type. Furthermore, we will also suppose that the total number of firms \(N\) is fixed (albeit very large). With this assumption, \(n_2\) is determined once a choice of \(n_1\) is made since \(n_2 = N - n_1\). To highlight this choice of parameters, we will denote an economy by \(\mathcal{E}_{n_1, n_2} = \{\mathbb{R}^2_+, n_j, Y_j, m_i, u_i, w_i, \theta_{ij}, i, j \in \{1, 2\}\}\) or, if no confusion arises, simply by \(\mathcal{E}(n_1)\). In other words, speaking of an economy is equivalent to choosing a parameter \(n_1\).
In this section we will explore the equilibrium set of an economy $E(n_1)$. Recall that the Walrasian correspondence is the set-valued function that assigns to each economy $E(n_1)$ its set of competitive equilibria. This idea gives rise to the notion of the equilibrium set as the set of all pairs $(x, y) \in X \times Y$ and the corresponding equilibria prices $p \in \mathbb{R}^m_+$, where $X = \mathbb{R}^{2m_1}_+ \times \mathbb{R}^{2m_2}_+$, $Y \subset \mathbb{R}^{2n_1} \times \mathbb{R}^{2n_2}$. Recall that $m_1 + m_2 = M$ and $n_1 + n_2 = N$. We will show that studying this set is an important first step in our analysis. Thus, in this section we formalize the notion of excess demand functions, the equilibrium set and regular and critical economies. Intuitively, a regular economy $n_1$ is one where equilibrium prices will be determinate (that is, prices are locally continuous functions of $n_1$, while a critical economy is one where small perturbations of $n_1$ lead to large perturbations of its equilibrium price.

3.1. Excess demand functions. The notion of a competitive equilibrium can be rephrased in terms of zeroes of aggregate excess demand functions. To see this, consider the private-ownership economy $E(n_1)$. For a fixed $n_1$, we say that the individual excess demand function $\zeta^i_n : \mathbb{R}^2_+ \to \mathbb{R}$ of a consumer of type $i$ is given by

$$\zeta^i_n(p) = x^i_n(p) - w^i.$$  

Similarly, for a fixed economy $n_1$ we define the aggregate excess demand function of this economy, $\Psi_{n_1} = (\Psi^1_{n_1}, \Psi^2_{n_1}) : \mathbb{R}^2_+ \to \mathbb{R}^2$, by

$$\Psi_{n_1}(p) = m_1\zeta^1_{n_1}(p) + m_2\zeta^2_{n_2}(p) - n_1y^1(p) - n_2y^2(p).$$

3.2. Equilibrium set. With the introduction of excess demand functions above, we can see that the set of equilibrium prices of economy $E(n_1)$ are the vectors $p$ such that $\Psi_{n_1}(p) = 0$. We denote the equilibrium set of economy $E(n_1)$ by

$$\Gamma_{n_1} = \{p \in \mathbb{R}^m_+ : \Psi_{n_1}(p) = 0\}.$$  

Notice that for each equilibrium price, demand and supply functions $(x, y)$ are fully determined so, equilibrium prices also fully determine Walrasian equilibria. Whenever it is convenient to highlight that the parameter $n_1$ is changing, we will explicitly write this dependence in the excess demand function by $\Psi_{n_1}(p) = \Psi(n_1, p)$ and its corresponding equilibrium set by

$$\Gamma = \{(n_1, p) : \Psi(n_1, p) = 0\}.$$  

There is a simplification to the analysis that can be made from the following result.

**Proposition 1.** The aggregate excess demand function $\Psi_{n_1}$ of economy $E(n_1)$ satisfies the following properties:

- $\Psi_{n_1}$ is homogeneous of degree zero, i.e., $\Psi_{n_1}(\lambda p) = \Psi_{n_1}(p)$ for all $\lambda > 0$;
- $\Psi_{n_1}$ satisfies Walras law, i.e., $p \cdot \Psi_{n_1}(p) = 0$, for all $p$.

This result is straightforward from definitions.

The results of Proposition 1 will allow us to make useful simplifications. In particular, homogeneity of degree zero allows us to choose a price normalization; thus, we will let prices to be in the simplex

$$\Delta = \{(p_1, p_2) \in \mathbb{R}^2_+ : p_1 + p_2 = 1\}.$$  

Note that, if $p$ is in the simplex and $p_2$ is known, then $p_1$ is also known, since $p_1 = 1 - p_2$. Also notice that, from Walras law, $p_1\Psi^1_{n_1}(p) + p_2\Psi^2_{n_1}(p) = 0$ for all
Figure 1. The equilibrium set $\Gamma$.

$p$. Combined, these two facts suggest we can consider a restricted excess demand function $\Psi_{n_1} : [0,1] \rightarrow \mathbb{R}$ such that $\Psi_{n_1}(p_1) = \Psi^1_{n_1}(p)$. Hence, a price $p = (p_1, p_2)$ is an equilibrium for the economy $E(n_1)$ if and only if $\Psi_{n_1}(p_1) = 0$. Thus the set $\Gamma_{n_1}$ can also be represented as

$$\Gamma_{n_1} = \left\{ p \in \Delta : \Psi_{n_1}(p_1) = 0 \right\} \subset \Delta.$$

The figure above shows the intuition behind the equilibrium sets $\Gamma$. First, notice that the horizontal axis consists of the set of economies, that is, the set of parameters $n_1$ ranging from 0 to $N$. Similarly, the vertical axis shows the possible values of prices $p$ which is isomorphic to the set $[0,1]$. Now, for a fixed economy $n_1$ as depicted, its equilibria (three in this case) form the set $\Gamma_{n_1}$. All pairs economies-equilibria together form the set $\Gamma$.

3.3. Regular and critical private-ownership economies. The previous figure showed the intuition behind the equilibrium sets $\Gamma$. If this diagram were a true representation of our model, one could see that there are two types of economies. On the one hand, there are economies such as $n_1$ where each price equilibrium has the property that small (i.e. infinitesimal) perturbations to $n_1$ would be accompanied by small changes in this equilibrium price. A price with this property will be called a regular price. Notice that, under this intuition, all equilibrium prices of the economy $E_{n_1}$ are regular. When this is the case, the economy will be called a regular economy.

More precisely, the equilibrium set of the economy $E_{n_1}$, is $\Gamma_{n_1} = \{ p^R_1, p^R_2, p^R_3 \}$, given that all its elements are regular, the economy is called regular.

On the other hand, note that economy $E_{n_1}$ has a different structure. The equilibrium set is $\Gamma_{n_1} = \{ p^S_1, p^S_2 \}$. While the lower equilibrium is regular (since an infinitesimal perturbation of $n_1$ leads to a small perturbation in this equilibrium price), the upper equilibrium is different. In this case, a perturbation $n_1^S - \epsilon$ of $n_1^S$ for some $\epsilon > 0$, would be accompanied by a continuous change of the price along one of the branches, but a perturbation $n_1^S + \epsilon$ of $n_1^S$ would lead to a jump of the equilibrium price to the lower branch. An equilibrium price with these features will be called a critical price. An economy that has at least one critical price will be called a critical economy. In other words, an economy will be called regular if

$^{3}$See Accinelli and Puchet (2011) and Balasko (2009).
and only if all of its prices are regular prices, or otherwise a critical economy. We formalize these ideas in the following definition.

**Definition 3** (Regular and critical prices). Consider a private ownership economy $\mathcal{E}(n_1)$, with its corresponding excess demand function $\Psi_{n_1} : \Delta \to \mathbb{R}^2$, and equilibrium price set $\Gamma_{n_1} = \{p \in \Delta : \Psi_{n_1}(p) = 0\}$. Then, we say that an equilibrium price $p$ is a *regular price* if and only if the rank of the Jacobian of the excess demand function evaluated at $p$ is equal to 1. Otherwise, we say $p$ is a *critical price*.

**Definition 4** (Regular and critical economies). We say that an economy $\mathcal{E}(n_1)$ is regular if all of its equilibrium prices are regular. If at least one if its equilibrium prices is critical, we say the economy is critical.

As a consequence of the Sard’s theorem (see [10]), it follows that the regular economies is an open and dense subset in the set of economies. This means that if a regular economy suffers a perturbation on its fundamentals, then the perturbed economy continuous being regular. Such does not happen if the economy is singular. According with this theorem theorem this subset is meager in the set of economies, or from the point of view of the measure theory this is a subset of zero Lebesgue-measure.

3.4. **Continuous selectors.** Suppose that while the number of firms $N$ is finite is big enough so that the values $N_1 = n_1/N$ and $N_2 = n_2/N$ correspond to the interval $[0, 1]$. Obviously $N_1 + N_2 = 1$. Let $E_{N_1}$ be a private ownership economy, and let $\Gamma_{N_1}$ be the set of Walrasian equilibria for such economy. We symbolize by $\sharp \Gamma_{N_1}$ the cardinal of this set and by $\rho_d(\Gamma_{N_1}, \Gamma_{N_1'})$ the Hausdorff metric between $\Gamma_{N_1}$ and $\Gamma_{N_1'}$. Note that $\Gamma_{N_1} \subset \mathbb{R}^{2|N_1}$.

So, if $E_{N_1}$ is a regular economy, then for each $\epsilon > 0$ there exist $\delta > 0$ such that if $|N_1 - N_1'| < \delta$ then $\sharp \Gamma_{N_1} = \sharp \Gamma_{N_1'}$ and $\rho_d(\Gamma_{N_1}, \Gamma_{N_1'}) < \epsilon$. See [8].

A classical result show that for each distribution $(N_1, N_2)$ the set of economies whose agents have continuous and differentiable utilities in $\mathbb{R}^2_+$ and $Y_j \in \mathbb{R}^2$ is generically regular and consequently $\Gamma_{N_1}$ is finite for all $n_1$. See [8].

As consequence, if $E_{N_1}$ is regular there exists a continuous selector $\psi : V_{N_1} \in \Gamma_{N_1}$ where $V_{N_1}$ is an open neighborhood of $N_1$.

**Definition 5.** A *selector or selection* from a correspondence $\phi$ mapping $X$ into $Y$ is a function: $f : X \to Y$ that satisfies $f(x) \in \phi(x)$ for each $x \in X$.

If in addition $X$ and $Y$ are topological spaces, then we say that $f$ is a **continuous selector** if $f$ is a selector and is continuous.

The following theorem is a direct consequence of the Browder’s Selection Theorem. See [2].

**Theorem 1.** Suppose $\mathcal{E}(N_1)$ is a regular economy. Then, for each $p \in \Gamma_{N_1}$ there exists neighborhood $V_{N_1}$ around $N_1$ and $V_p$ around $p$, and a continuous function $p^* : V_{N_1} \to V_p \in \Gamma_{N_1}$ such that $p^*(N_1) = p_1$ is an equilibrium for each $N_1 \in V_{N_1}$.

**Proof.** Let $E_{N_1}$ be a regular economy, and let $\psi : V_{N_1}(\delta) \to \mathbb{R}^{2|N_1}$ be a correspondence such that $\psi(N_1) = \Gamma_{N_1}$ for all $N_1 \in V_{N_1}$ where $V_{N_1}(\delta) = (N_1 - \delta, N_1 + \delta)$. So, from $\delta$ small enough, from the Browder’s Selection Theorem, there exist a continuous function $p^* : V_{N_1} \to \Gamma_n$, such that $p^*(N_1^1) \in \Gamma_{N_1} \forall N_1^1 \in V_{N_1}$.
4. Dynamical behaviour. As we have already argued, in equilibrium profits depend on the distribution of firms on the available types of technology. At the time of deciding how to produce, managers do not necessarily know how many firms will choose to produce according to the different available technologies. That is, they do not know the future distribution of firms over the set of technologies and consequently do not know the future benefits associated with a decision. In this regard, some decision-makers may not have sufficient ability, information, or resources to establish the optimal choice every time. However, we will assume that agents learn over time so that a growing proportion of managers will make rational decisions.

Suppose that in equilibrium, at time \( t \), the distribution of the firms is given by \( N(t) = (N_1(t), N_2(t)) \) and that the share profits of a firm of type \( Y_i \) are revealed to be higher than profits of a firm of type \( Y_j \) so that

\[
\Pi_i(y^{*i}) = \frac{\pi_i(y^{*i})}{c_i(y^{*i})} > \Pi_j(y^{*j}) = \frac{\pi_j(y^{*j})}{c_j(y^{*j})} \quad i \neq j \in \{1, 2\},
\]

where \( y^{*i} \) is the supply of producing according with \( Y_i \) and \( c(y^{*i}) \) is the cost of producing the amount \( y^{*i} \). Then, rational behavior would suggest a manager to produce with technology \( Y_i \). But if, at any given time some managers decide to change the technology under which their firms produce, certain characteristics of markets associated with each technology -including profits- can also be modified. Moreover, it is possible that if a large enough number of managers decide to migrate to the branch that is currently offering higher benefits, this advantage can be reversed in the future and the branch that currently has the greatest benefits might become the one that will offer the lowest level of profits. In this case, the best individual response to the decision of others, would be to not change the technology or branch.

The key question is then as follows: Based on what arguments managers will decide to change or not the technologies used by them so far, if at the time of deciding they do not know what others will do. Regardless of the answer to this question, it seems natural to assume that the number of firms producing according to the technology that in each time appears to be the most successful increases. Equivalently, at each time, the share of firms producing according with the technology with the highest profits increases. This process can be considered as the imitation of the most successful.

Such evolutionary process can be modeled by replicating dynamics. In this way, this dynamical behaviour becomes a good tool to describe and understand the evolution of the economy. This section introduces this dynamical behaviour to a private-ownership economy, first as a static game, and then by an evolutionary process.

As in previous sections, we suppose there is a finite (albeit very large) number of firms \( N \). We suppose this number remains constant, although each firm will endogenously change its type through time. At each time period \( t \), the number of firms of type \( j \in \{1, 2\} \) is given by \( n_j \) so that \( N = n_1(t) + n_2(t), \forall t \). The proportion of firms of type \( j \) at each \( t \) is thus given by \( N_j(t) = n_j(t)/N \).

For a better understanding of the dynamics we propose begin considering some concepts from game theory.

4.1. Static game. Consider the private-ownership economy \( \mathcal{E}(n_1) \). The distribution of firms is thus given by \((N_1, N_2)\). In this situation, the manager of each company must choose between changing its existing technology or to keep it. It is possible to model this situation as a \( N \)-population normal form game, where each
firm has two available strategies \( T = \{ Y_1, Y_2 \} \). Recall that the supply of a firm of type \( j \in \{1, 2\} \) is the function \( y_j : \mathbb{R}_+^2 \to \mathbb{R}^2 \), and its profit function is given by \( \pi_j : Y_j \to \mathbb{R} \). The function \( \pi^*_i : \Delta \to \mathbb{R} \) defined by \( \pi^*_i(N_1, N_2) = \pi_i(y^*(N_1, N_2)) \) represents the profits in equilibrium of a firm operating according with the technology \( Y_j \). Consequently \( \Pi^*_i(N_1, N_2) = \Pi_i(y^*(N_1, N_2)) \) symbolize the rate of profits of a firm of type \( i \in \{1, 2\} \) under equilibrium conditions. The two definitions below will establish a firm’s best response. Indeed, a firm’s best response is naturally a choice of technology \( Y_i \) with higher rate of profits. Furthermore, a distribution of firms is an equilibrium distribution if and only if the rate of profits of all firms is the same.

**Definition 6.** We say that to produce according with \( Y_j \) is a best response given the distribution \( N = (N_1, N_2) \), if the rate of profits producing according with technology \( Y_j \) is greater than or equal to the percentage obtained when produced according to the technology \( Y_i \), i.e.:

\[
\Pi_j(y^j(N)) \geq \Pi_i(y^i(N)),
\]

**Definition 7.** A distribution \( N^* = (N^*_1, N^*_2) \) is an equilibrium distribution (or an equilibrium in mixed strategies) for this game if and only if

\[
\Pi_j(y^j(N^*)) = \Pi_i(y^i(N^*))
\]

for all \( i, j \in \{1, 2\} \).

So, a Nash equilibrium for the \( N - population \) game \( G = \{Y_i, N_i, i \in \{1, 2, \} \} \) is a distribution \( N^* \) verifying equation 4.

4.2. Evolution. The competitive economy makes life difficult for owners or managers. They know that, in time to time, they will have to take sudden decisions, even if they do not feel ready to act, because otherwise one or more competitors will come up with a better solution, which can lead to the firm by him managed out of the market. Certainly, if owners do not have complete information or sufficient time to perform calculations, some mistakes in decisions might happen. In this framework to imitate the behavior of the agents considered most successful can be a good alternative.

Imitation is presented in modern literature as an alternative to rationality in an environment of uncertainty or when, urgent decisions must be taken without time to analyze all possible solutions. To take an absolutely rational decision generally is required unlimited time, knowledge, and computational might. Most of the time a manager does not have all these possibilities. The “imitation of the most successful qualifies” as a “fast and frugal heuristics” for making decisions (see [5]). Certainly this option is not free of mistakes, and the vision of what is the most successful behavior can be different for different decision makers.

So, a bounded rational vision of the top competent team set the direction of the firm and define the competence of the owner or manager involved. One of the fundamental issues in evolutionary game theory concerns the relationship between predictions as consequence of myopic decisions made by simple agents and those provided by traditional rationality-based concepts. In our model, in order to obtain the highest rate of profits, firms managers or owners, must make decisions, pressed for time or lack of information.

We start this section by introducing a replicating dynamical system in Definition 8 below. Recall that equilibrium supply functions depend on the distribution of firms so that \( y^*_i = y^*_i(N_1, N_2) \).
4.2.1. The replicator dynamical system. The degree of competition in a given economy depends on the variance on the profit rates, the larger the spread the grater the degree of dynamic competition will be. In equilibrium essentially all the firms are obtaining the same result. The next definition we introduce a function that measures the degree of dynamical competition in an economy, from the difference in profit rates.

Definition 8. (Augmented replicating dynamical system) Consider an economy $n_1$ where prices are given by $p^*(n_1) \in \Gamma_{n_1}$. Let $\phi_i : \mathbb{R} \to \mathbb{R}$ be a continuous, increasing function such that $\phi_i(0) = 0$, and $i \in \{1, 2\}$. Then, using the convention that $N_j = 1 - N_i$, $j \neq i \in \{1, 2\}$, the (augmented) replicating dynamical system is given by

$$
\frac{d}{dt} N_i = \left\{ \phi_i \left( \Pi_i \left( y_i^* (N) \right) - \Pi_j \left( y_j^* (N) \right) \right) \right\} N_i, \quad (i)
$$

$$
\frac{d}{dt} N_j = - \frac{d}{dt} N_i, \quad (ii)
$$

$$
N(t_0) = (N_1(t_0), N_2(t_0)) >> 0, \quad (iii)
$$

A few remarks regarding definition 8 are in order:

- In condition (i), the function $\phi_i : \mathbb{R} \to \mathbb{R}$ represents the growth rate in the share of each technology per unit time,

$$
\frac{\dot{N}_i}{N_i} = \lim_{\Delta t \to 0} \frac{\Delta N_i}{\Delta t} = \phi_i(\Pi_i(y_i^*(N)) - \Pi_j(y_j^*(N))).
$$

Here by $\dot{N}_i(t)$ we symbolize the derivative of $N_i(t)$ we respect to time $t$ i.e., $\frac{d}{dt} N_i(t) = \dot{N}_i(t)$. To simplify the notation we do not write the variable $t$.

- To guarantee that this system of differential equations induces a well defined dynamics on the state-space given by the simplex, i.e,

$$
\Delta = \left\{ (N_1, N_2) \in \mathbb{R}^2_+ : N_1 + N_2 = 1 \right\},
$$

$\phi_i : \mathbb{R}_+ \to \mathbb{R}$, $i \in \{1, 2\}$ are Lipschitz continuous functions in $(0, \infty)$.

- Recall that $N_2(t) = 1 - N_1(t)$ at all $t$. Therefore, condition (ii) is automatically satisfied. This forces no creation or destruction of firms and instead allows us to focus on the redistribution of firms according to their type. Furthermore, as mentioned before, this fact also allows us to assume that $y^*_i$ depends only on $N_1$.

- Condition (iii) establishes initial conditions. We assume that at time $t = t_0$, the proportion of both types of firms is positive, since if at time $t_0$ either $n_1$ or $n_2 = 0$, then the evolution given by the system (5) makes no sense. Equivalently, we could have asked for $N(t_0)$ to be in the interior of the simplex $\Delta$.

- Finally, recall that under the assumptions of our model, the following identity is always verified,

$$
m_1(x^1(N_1(t)) - w^1) + m_2(x^2(N_1(t)) - w^2) = n_1(t)y^1(N_1(t)) + n_2y^2(N_2(t))
$$

which implies that market clearing conditions always hold.

Note that if equilibrium prices at time $t_0$ are regular, then immediately after a perturbation on the distribution of firms, these prices will not change much and therefore the benefits of firms will suffer only small modifications. Conversely, if these prices correspond to a critical economy, a small perturbation on the distribution of firms can give place to large and discontinuous changes in the behavior of the
Notice that a migration process of firms to technologies with higher expected benefits, will lead to changes in the excess demand function, and therefore also in equilibrium prices. Moreover, the whole economy $E_n$ is changing along this process; that is, for each $t$ there is a different economy. Furthermore, note that even at time $t = t_0$ the inequality $\Pi_i(y^*(N_1)) > \Pi_j(y^*(N_1))$ holds but this does not mean that every manager will choose the technology $Y_i$, because profits at time $t > t_0$ depend on the future distributions of the firms over technology types, and these future distributions are unknown at time $t_0$, or because not necessarily every manager will know the profits of other firms.

A distribution $(N_1^*, N_2^*)$ defines a dynamical economic equilibrium in the sense of the replicator dynamics (5), if and only if $\Pi(y^{*1}(N_1^*)) = \Pi(y^{*2}(N_1^*))$. Under the hypothesis of our model, a Walrasian equilibrium does not verify necessarily this equality. Moreover, a Walrasian equilibrium $(p, \bar{x}, \bar{y}) \in E_{q_{n_1}}$ is at the same time a dynamical equilibrium, if and only if the profit rates of the firms are the same regardless the technology used. So, if at $t = t_0$ the equality on the profit rates holds, then, the economy will not change in the future, unless it happens some perturbation in its fundamentals for reasons exogenous to the model. (The stability in the Lyapunov sense of this equilibrium will be considered in the next section).

In other case, the economy is evolving in a transition process corresponding to a trajectory defined by the dynamical system (5), along this trajectory the economy is always in a Walrasian equilibrium. Certainly these equilibria change according with the evolution of the distribution of the firms on the set of available technologies. This means that, in each time, prices, allocations and plans of production correspond to a Walrasian equilibrium. More specifically, for each $t > t_0$ and $n_1(t)$ such that $n_1(t) = nN_1(t)$, $p^*(n_1(t)) \in E_{q_{n_1}(t)}$, where by $E_{q_{n_1}(t)}$ we denote the set of equilibrium prices, when the distribution of firms is given by $N(t) = (N_1(t), N_2(t))$. If the dynamical equilibrium $(N_1^*, N_2^*)$ is an attractor and if in time $t = t_0$ the initial distribution $(N_1(t_0), N_2(t_0))$ is in the basin of attraction of this equilibrium, then $(N_1(t), N_2(t))_{t \to \infty} \to (N_1^*, N_2^*)$ and so

$$E_{n_1(t), n_2(t)} \to E_{n_1^*, n_2^*}.$$  

This evolution will takes place along a trajectory of economies $E_{n_1(t), n_2(t)}$ in (Walrasian) equilibrium. As managers make decisions, the distribution of firms changes. We assume that the economy reaches at all times one of its Walrasian equilibria, though not necessarily these correspond to steady states. In our model a Walrasian equilibrium is stationary or steady state, if and only if profit rates between different branches or sectors of production are equal. When this happens, i.e, when profit rates are the same throughout the economy, then there is no incentives to new changes. This situation is achieved only when a particular distribution of firms is reached, let call it $N^e$. If such distribution corresponds to a stable equilibrium for the dynamical system (5) and if, in a given time $t = t_0$ the distribution $N(t_0)$ of firms is located in the basin of attraction of $N^e$, then the economy will evolve to reach this steady state, i.e, $N(t) \to N^e$ when $t > t_0$.

---

4This was also the main reason behind the need for the lengthy discussion of continuous random selections in the previous section.
Along this trajectory only smooth changes can occur if and only if the economies \( E_{n(t)} \) are regular. Then the trajectory is a continuous path. As we have already seem big changes necessarily take place because of the existence of singular economies.

**Definition 9.** Let \( N(N_0, \cdot) : \mathbb{R}_{2^+} \to \Delta \) be the solution of the system (5) where \( N(N_0, t_0) = N_0 \), being \( N_0 \in \Delta^0 \) the initial distribution (percentage) of the firms, then a transition path will be defined by the trajectory:

\[
T(t) = \{(t, N(N_0, t)) \in [t_0, \infty) \times \Delta\}.
\]

This is a Walrasian equilibrium trajectory, meaning that for each \( t \) and \( N(N_0, t) \) for each \( n(t) = nN(N_0, t) \) the corresponding \( p^*(n(t)) \in E_{\eta_n(t)} \forall t \geq t_0 \).

To simplify the notation, from now on we write \( \Pi^*(N_1) \) to represent \( \Pi_i(y^{**}) \) \( i = 1, 2 \).

It is easy to see that the trajectory remains in the simplex if the condition

\[
\bar{N}_1 + \bar{N}_2 = \phi_1 (\Pi_1(N_1) - \Pi_2(N_1)) N_1 + (\phi_2 (\Pi_2(N_1) - \Pi_1(N_1)) (1 - N_1) = 0.
\]

is verified in all time \( t \geq t_0 \), this means that the simplex is invariant under this dynamic, and it is true because the population remains constant.

Let \( p^*(t) = (p^*_1(n_1), p^*_2(n_1)) \) be the equilibrium price of a regular economy \( E_{n_1, n_2} \). As we already shown in section (3.4), because the existence of a continuous selector, the equilibrium prices are smooth functions of the distribution of the firms. There exist a neighborhood \( V_{\tilde{n}_1, \tilde{n}_2} \subset \mathbb{R}^2 \) of \( (\bar{N}_1, \bar{N}_2) \) such that for all \( (N_1, N_2) \in V_{\tilde{n}_1, \tilde{n}_2} \cap \Delta \) \( p^*(\tilde{N}_1) \in \Gamma_{N_1} \). Let us now introduce de definition of regular transition path:

**Definition 10.** Let \( N(t) = N(N_0, t) \) be the solution of the dynamical system: (5). Let \( \Psi : (0,1) \times \Delta \to \mathbb{R}^2 \) be the generalized excess demand function, then a transition path, will be regular if and only if \( rank J_p \Psi(N_1(t), p) = 1 \) for all \( (N_1(t), p) : \Psi(N_1(t), p) = 0 \), where \( \frac{n_1(t)}{n} = N_1(N_0, t) \) i.e. if and only for each \( N_1(t) \) along a trajectory defined by a solution of the dynamical system (5) the economy \( E_{n_1(t), n_2(t)} \) is regular.

Note that if the economy at time \( t_0 \) is regular, then after a small perturbation on the distribution of the firms, it will continue being regular, so there exist some interval \([t_0, t_1]\) such that

\[
T(t_0, t_1) = \{(t, N(N_0, t)) \in [t_0, t_1] \times \Delta\}
\]

is regular restricted transition path, because all economy \( E_{n_1(t), n_2(t)} \) is regular. We can summarize this assertion by the next proposition:

**Proposition 2.** The restricted transition path in a neighborhood of a regular equilibrium price is smooth.

The existence of such continuous transition paths, is a local property verifiable only in a neighborhood of a regular equilibrium prices. Recall that there is a maximal interval \((\alpha, \beta)\) containing \( t_0 \) where a solution \( N(t) \) with \( N(t_0) = N_0 \) exists. The solutions corresponding to two intervals containing \( t_0 \) are the same in the intersection of both. In some cases \( \alpha = -\infty, \beta = \infty \) or both, however, in general, it is not guaranteed that the solution of a differential equation can be defined for all \( t \). For details see [6].
Contrarily, in the neighborhood of a singularity we can not ensure the existence of a solution for the dynamical system. The transition path in the neighborhood of a singularity can be discontinuous. If the economy is singular, then small perturbations in the distribution of the firms can give place to large changes in the behavior of the future economies, with respect to the behavior of the actual, i.e, some equilibrium prices and consequently the corresponding profits can suffer larger changes. Theses changes can affect strongly the distribution of the wealth between the consumers and so, their respective demands can be altered. Most of these changes will be unpredictable an the consequence of a small perturbation in de distribution of the firms. Certainly, after and before this perturbation occurs, we can describe the evolution of the economy by a dynamical system similar to the one given in (5), but it is not possible at the moment when the trajectory attain a singular equilibrium price. The future evolution will start in a (previously) unforeseeable initial conditions. So, if we understand an economic crisis like an abrupt and unexpected change in the behavior of the economy, as the result of arbitrarily small changes in its fundamentals, then we can say that singular economies are thresholds of the economic crises.

4.3. Evolution by an imitative process. Now, we introduce a model of economic evolution where the engine of this process, is the imitative behavior of the owners or managers of the firms, looking for the most profitable technology. We imagine that these agents, acting strategically, stick to a given technology for a while, and occasionally, looking for technologies with higher profits, at least some of them and from time to time, review their previous decisions of production.

There are two particular elements characterizing this process. The first one is the specification of the time rate at which managers or owners review the technology under which their firms are producing. The second element is the probability that a reviewer, producing according with the technology \(Y_i\), change effectively, to the technology \(Y_j\). This probability is written \(p_{ij}\). Certainly the probability that a reviewer change effectively, depends on his believes on the future behavior of the others. By \(p_{ii}\) we denote the probability that a reviewing manager using the \(i\)–technology does not change the technology.

In a finite but large population of firms, following [13], one may imagine that the review times of an agent (an owner or a manager) are the arrival time of a Poisson process with arrival time \(r_i(\Pi^*)\), where \(\Pi^* = (\Pi^*(N_1), \Pi^*(N_2))\). Then, if the agent becomes a reviewer, he selects the technology to produce in the next period, he decides to change from technology \(i\) to technology \(j\) with probability \(p_{ij}\) or he decides to maintain the previous election with probability \(p_{ii}\). Assuming that all agents’ are random variables, statistically independent, the aggregate of reviewing of each subset of firms is itself a Poisson process with arrival rate (normalized) \(N_i r_i(\Pi^*)\). So, the aggregate Poisson process of switches from technology \(Y_i\) to technology \(Y_j\) is \(N_i r_i(\Pi^*)p_{ij}\).

Now, imagine a large number of firms. By the law of large number we can model the aggregate stochastic process as a deterministic flow, where the outflow from the subset of firms using the technology \(Y_i\) is \(N_i r_i(\Pi^*)p_{ij}(\Pi^*)\), and the inflow is \(N_j r_j(\Pi^*)p_{ji}(\Pi^*)\), then we obtain:

\[
\dot{N}_i = N_j r_j(\Pi^*)p_{ji} - N_i r_i(\Pi^*)p_{ij}
\]  

(6)

Each individual actor looks at the world through his or her, ex-ante experience, so, it is natural to assume that less successful manager, on average, review their
behavior at a higher rate than manager using more successful technologies. Then we consider that \( r_i \) is a decreasing function with respect to profits \( \Pi_i^* \) obtained by the firm.

\[
  r_i(\Pi^*) = \rho_i(\Pi^*).
\]  

(7)

where \( \rho_i \) is a Lipschitz continuous function, decreasing in it argument. Note that this assumption does not presume that the agent knows the expected profits associated with the technology currently in use.

So equation (6) can be rewritten as:

\[
  \hat{N}_i = -N_i \left[ \rho_j(\Pi_i^*)p_{ji} + \rho_i(\Pi_i^*)p_{ij} \right] + \rho_j(\Pi_j^*)p_{ji}.
\]  

Moreover we can consider the average review rate is linearly decreasing in the profit rates.

\[
  \rho_i(\Pi_i^*) = \alpha - \beta \Pi_i^*(N_i).
\]  

(9)

Then equation (8) becomes:

\[
  \hat{N}_i = -N_i \left[ (\alpha - \beta \Pi_i^*(N_j))p_{ji} + (\alpha - \beta \Pi_i^*(N_i))(\Pi_i^*)p_{ij} \right] +
  \left[ \alpha - \beta \Pi_j^*(N_j) \right] p_{ji}.
\]  

(10)

Suppose that each reviewing manager or owner, observe profits difference between her own profit and the profit associate with the other technology with some noise, and that the reviewer agent switches the technology if and only if \( \Pi_i^*(N_i) > \Pi_j^*(N_j) + \epsilon \) where \( \epsilon \) is a random variable, with a probability distribution \( \phi_i : R \rightarrow [0, 1] \) continuous and differentiable. Then

\[
  p_{ij} = \phi_i(\Pi_i^*(N_i) - \Pi_j^*(N_j)).
\]  

(11)

As a special case, assume that the error term is uniformly distributed with a support containing the range of all possible profits differences. Then \( \phi_i \) is an affine function over the relevant interval, \( \phi_i(z) = a_i + b_i z \) for some \( a_i, b_i \in R \) and \( b_i > 0 \) then (10) becomes:

\[
  \hat{N}_i = -N_i \left[ (\alpha - \beta \Pi_j^*)(a_i - b_i(\Pi_j^* - \Pi_i^*)) + (\alpha + \beta \Pi_i^*)(a_j - b_j(\Pi_i^* - \Pi_j^*)) \right] +
  \left[ (\alpha - \beta \Pi_j^*)(a_j - b_j(\Pi_i^* - \Pi_j^*)) \right].
\]  

(12)

The imitative behavior: Under conditions of incomplete information, the evolution of the economy depends on the perception that owners or managers have, over the future benefits associated with each technology available. This perception depends on the skills and potential of managers to predict the future actions of others. We can assume that, under incomplete information, each owner or manager, to choose the future strategy, takes account the behavior of the nearest competitors or that followed by those he considered leaders or most successful. These types of behaviors can be called imitative, and each one gives place to a different dynamic.

Assuming that all review rates are constantly equal to one, i.e.; \( r_i(\Pi^*) = 1 \) we obtain the following dynamics:

\[
  \hat{N}_i = -N_i \left[ (a_j + a_i) + (b_j + b_i)(\pi_j^* - \pi_i^*) \right] + \left[ (a_j + b_j)(\pi_j^* - \pi_i^*) \right].
\]  

(13)

The difference \( \Pi_i^* - \Pi_j^* \) corresponds to the true difference, because

\[
  p_{ij} = P(\Pi_i^* + \epsilon < \Pi_j^*) = P(\epsilon < \Pi_j^* - \Pi_i^*) = \phi(\Pi_j^* - \Pi_i^*).$


If in addition, we consider that along the time the difference between profits remain constant, then the solution of this dynamical system is given by:

\[ N_i(t) = \frac{A}{B} + \left[ N_{i0} - \frac{A}{B} \right] e^{-B(t-t_0)} \quad \forall t \geq t_0, \]

where \( A = (a_j + b_j)(\Pi_j^* - \Pi_i^*) \), \( B = (a_j + a_i) + (b_j + b_i)(\Pi_j^* - \Pi_i^*) \) and \( N_{i0} = N_i(t_0) \).

Note that if the difference \( \Pi_j^* - \Pi_i^* \) remains constant and positive along the time, then the solution converges to the stationary state \( N_i = \frac{A}{B} \). This means that following this particular imitative process, even in the case when the profit rates associate with the technology \( Y_j \) remain, along the time, higher than the profits associate with the technology \( Y_i \), even in the long run, will be possible to find in the market some firms producing according with the technology \( Y_i \).

5. **Stability of the dynamical equilibrium.** In both the social or natural world, the only dynamic equilibria we can see are those that are stable. In the previous section, we describe the evolution of an economy along a path of Walrasian equilibria. Once that the modification of the economy (i.e., the modification of the distribution of firms in the set of available technology) is permanent, the modification of equilibria prices will be also permanent, unless that the distribution correspond to a stable steady state of the dynamical system, only in these cases prices will remain constant. These changes will be continuous, at least until such time that a singularity appears, if such is the case, then after this moment, a discontinuity in prices or, in general in the economic behavior, can be observed.

Let us now introduce some considerations on the stability of the stationary state of the dynamical system considered in (5). A solution \( N^e = (N_1^e, N_2^e) \in \Delta \) of this system (5) is a dynamical equilibrium, or steady state, if and only if

\[ \left| \varphi_i \left( \Pi_i(y_i^*(N_1^e)) - \Pi_j(y_j^*(N_1^e)) \right) \right| N_i^e = 0 \quad \forall i = 1, 2. \]

i.e if and only if the payoffs corresponding to each technology are the same:

1. \( \Pi_i(y_i^*) = \Pi_j(y_j^*) \) or if
2. \( N_i^e = 0 \) for some \( i \in \{1, 2\} \) because in this case all firms use the same technology and earn the same payoff.
3. In both cases \( N(t) = N^e, \forall t \) and so \( \dot{N} = (\dot{N}_1, \dot{N}_2) = 0 \).

Each dynamical equilibrium, have associated a Walrasian equilibrium price, denoted by

\[ p(N_1^e, N_2^e) = (p_1(N_1^e, N_2^e), p_2(N_1^e, N_2^e)) \in E_{\nu_1^e, \nu_2^e}. \]

The corresponding profits and the optimal plan for firms using the technology \( Y_i \) are respectively \( \pi_i^* \) and \( y_i^* \). The rates of profits are the same when they are evaluate at these prices. Recall that, \( N^e = \frac{1}{n}(n_1^e, n_2^e) \) then, in addition, we can say that, each dynamical equilibrium has associated an economy \( E_{\nu_1^e, \nu_2^e} \).

Let us consider the function \( g : \Delta^2 \rightarrow \mathbb{R}^2 \)

\[ g_i(N) = \varphi_i \left( \Pi_i(y_i^*(N_1)) - \Pi_j(y_j^*(N_1)) \right), \quad i = 1, 2. \]

The differential equations system (5) can be written as:

\[ \dot{N}_i = g_i(N)N_i, \quad i = 1, 2. \]

(14)

It is clear that a population state \( N \in \Delta^0 \) is an stationary state if and only if \( g_i(N) = 0, i = 1, 2 \).
The following theorem provides a sufficient condition for asymptotic stability and instability in the Liapunov sense.

**Theorem 2.** Let $N^e$ a fixed point of the system (14) then if there exists some neighborhood $U \subset \mathbb{R}^2$ of $N^e$ such that:

(a) $g(M)N^e = g_1(M)N^e_1 + g_2(M)N^e_2 > 0$ for all distribution $M \neq N^e$ in $U$ then $N^e$ is asymptotically stable in (14).

(b) $g(M)N^e = g_1(M)N^e_1 + g_2(M)N^e_2 < 0$ for all $M \neq N^e$, then $N^e$ is unstable.

**Proof.** *(Case a.)* We will show that the function

$$H_{N^e}(M) = N^e_1 \log \left( \frac{N^e_1}{M_1} \right) + N^e_2 \log \left( \frac{N^e_2}{M_2} \right)$$

is a Liapunov function for the system (14).

1. It is straightforward that $H_{N^e}(N^e) = \log(1) = 0$

2. If $M \in \Delta$ then the inequalities $H_{N^e}(M) \geq -N^e_1 \left( 1 - \frac{M_1}{N^e_1} \right) - N^e_2 \left( 1 - \frac{M_2}{N^e_2} \right) \geq 0$ hold. Thus we conclude that there is a neighborhood $U \cap \Delta$ of $N^e$ where the function $H_{N^e}(M)$ is positive.

3. Finally, since $H_{N^e}(M) \geq \dot{H}_{N^e}(M) = \left( \frac{N^e_1}{M_1} \cdot g_1(M)M_1 + \frac{N^e_2}{M_2} g_2(M)M_2 \right)$ from condition given in item (a) the inequality

$$\dot{H}_{N^e}(M) = - \left( \frac{N^e_1}{M_1} g_1(M)M_1 + \frac{N^e_2}{M_2} g_2(M)M_2 \right) = -g(M)N^e < 0$$

holds.

4. The claim set out in item (b), follows similarly.

**A geometrical interpretation of the asymptotic stability:** First note that, since

$$\dot{N}_1 + \dot{N}_2 = g_1(N)N_1 + g_2(N)N_2 = g(N)N = 0 \quad (15)$$

so, the vector field $g(N)$ remain orthogonal to $N$. Thus, the condition $g(M)N < 0$ used in the theorem (2) said that in a neighborhood of $N$ the vector field $g$ form an obtuse angle with $N$. So if $N^e$ is asymptotically stable equilibrium for the system (14) then $g(N)$ drift locally to $N^e$. On the other hand, note that the identity (15) show that if at time $t = t_0$ $N(t_0) = N_0 \in \Delta$ then the trajectory $N(N_0, t) \in \Delta \forall t \geq t_0$.

Under the assumption of regularity, equilibrium prices are continuous functions of the firms see theorem (1).

1. The analysis of the stability in the Liapunov sense of the first type of equilibria, give place to the following 4 different cases, see figure (2). Let $N' = (N'_1, N'_2)$ be the distribution after a perturbation in the equilibrium distribution of the firms:

(i) Consider the case where: $(N'_1, N'_2) >> 0$ if there exists a neighborhood $V_{n^e} \subset \mathbb{R}^2 \cap \Delta$ of $(n'_1, n'_2)$ such that $\Pi_i(y^*_i(n_1, n_2)) < \Pi_i(y^*_i(n'_1, n'_2)) \forall i \in \{1, 2\} \forall (n_1, n_2) \in V_{n^e}$ then the dynamical equilibriums is asymptotically stable, see theorem (2).

(ii) In the others three cases, where $\pi_i(y^*_i(n'_1, n'_2)) > \pi_i(y^*_i(n'_1, n'_2))$ for at least one $i \in \{1, 2\}$ the dynamical equilibrium is unstable, see theorem (2).

---

5 Recall that for $x > 0, \log x \leq (x - 1)$. 
2. For the cases on the second type, the dynamical equilibrium is asymptotically stable if, being

(iii) \((N^e_1, N^e_2) = (1, 0)\) the inequality \(\Pi_2(y^*_1(n'_1, n'_2)) < \Pi_1(y^*_1(n'_1, n'_2))\) is verified, or if,

(iv) being \((N^e_1, N^e_2) = (0, 1)\) the inequality \(\Pi_2(y^*_2(n'_1, n'_2)) > \Pi_1(y^*_1(n'_1, n'_2))\) is verified.

Note that for the last case considered in section (4.3) we have that \(g_j(N) = a_j + b_j(\pi^*_j(N) - \pi^*_i(N)), j \neq i \in \{1, 2\}\) and then if \(\pi^*_j(N) - \pi^*_i(N) > 0\) it follows that \(g(N)N^e > 0\). Then the asymptotical stability of the stationary point \(N^e_i = A, N^e_j = 1 - A\) holds.

6. An example. This example is inspired by ([2]).

Consider a neoclassical private ownership production economy with commodity space \(\mathbb{R}^2\) having \(m\) consumers divided in two types, \(m_1\) of type 1 and \(m_2\) of type 2, the set \(F_1 = \{1, ..., m_1\}\) symbolize the consumers of type 1, and \(F_2 = \{(m_1 + 1), ..., m\}\) we symbolize the consumers of type 2. Having \(n\) firms of two types \(n_1\) of type 1 and \(n_2\) of type 2, with the following characteristics:

- Consumers type 1: Initial endowments \(w_i = \frac{1}{m_1}(1, 3)\) and utility function \(u_i(x, y) = xy, i \in \{1, ..., m_1\}\)
- Consumers type 2: Initial endowments \(w_i = \frac{1}{m_2}(2, 3)\) and utility function \(u_i(x, y) = xy^2, i \in \{(m_1 + 1), ..., m\}\).
- Firms type 1: Production set \(Y_1 = \{(x, y) \in \mathbb{R}^2 : x < 1, \text{and } y < \frac{x}{x-1}\}\).
- Firms type 2: Production set \(Y_2 = \{(x, y) \in \mathbb{R}^2 : x < 1, \text{and } y \leq g(x)\}\) where:

\[
g(x) = \begin{cases} 
1 - e^x & \text{if } x \leq 0 \\
\ln(1 - x) & \text{if } 0 < x < 1
\end{cases}
\]

- Shares: Type 1 consumers have own one third of type 1 firms, and two thirds of type 2, the rest is owned by the consumers of type 2. Within each type, individuals have the same share of firms, i.e:

\[
\theta_{1i} = \frac{1}{2m_1}, \forall i \in F_1 \text{ and } \theta_{1i} = \frac{2}{3m_1}, \forall i \in F_2, i = 1, ..., m_1;
\]

\[
\theta_{1i} = \frac{m_1}{2m_2}, \forall i \in F_1 \text{ and } \theta_{1i} = \frac{1}{3m_2}, \forall i \in F_2, i = m_1 + 1, ..., m.
\]
We start by considering the supply function of a firm of type 1: The technological set \( Y_1 \) is strictly convex and the efficiency frontier\(^6\)

\[
\text{EFF}(Y_1) = \left\{ \left( x, \frac{x}{x-1} \right), \ x < 1 \right\}.
\]

So the profit function of a firm of this type is given by:

\[
\pi_1(x) = p_1 x + p_2 \frac{x}{x - 1}.
\]  
(16)

The supply of a firm of type 1, is:

\[
y_1(t) = \left( 1 - t, 1 - \frac{1}{t} \right).
\]

Substituting \( x = y_2(t) \) in (16) we obtain that the profit for a firm of type 1 is:

\[
\pi_1(t) = p_1 (1 - t)^2.
\]

**Remark 1.** Note that the rate of profits is given by

\[
\Pi_1(t) = \begin{cases} 
-1 + \frac{1}{t} & 0 < t < 1 \\
1 - t & 1 < t 
\end{cases}
\]

(17)

For a firm of type 2, we obtain

\[
\text{EFF}(Y_2) = \{(x, 1 - e^x), \ x \leq 0\} \cup \{(x, \ln(1 - x), \ 0 < x < 1\}.
\]

The profit function is given by:

\[
\pi_2(x) = \begin{cases} 
p_1 \left[ x + t^2 (1 - e^x) \right] & x \leq 0 \\
p_1 \left[ x + t^2 \ln(1 - x) \right] & 0 < x < 1 
\end{cases}
\]

(18)

since \( \pi''_2(x) < 0 \) in both cases, then the supply of a firm of type 2 is:

\[
y_2(t) = \begin{cases} 
(1 - t^2, 2 \ln t) & \text{if } 0 < t < 1, \\
(-2 \ln t, 1 - \frac{1}{t^2}) & \text{if } t \geq 1.
\end{cases}
\]

Substituting \( x = y_2(t) \) in (18) we obtain that the profit for a firm of type 2 is:

\[
\pi_2(t) = \begin{cases} 
p_1 \left[ (1 - t^2) + 2t^2 \ln t \right] & \text{if } 0 < t < 1, \\
p_1 \left[ -2 \ln t + t^2 - 1 \right] & \text{if } t > 1.
\end{cases}
\]

**Remark 2.** Note that the rate of profits of a firm of type 2 is given by:

\[
\Pi_2(t) = \begin{cases} 
\frac{-1 + t^2}{2t^2 \ln t} + 1 & \text{if } 0 < t < 1, \\
\frac{-2 \ln t + t^2 - 1}{2 \ln t} & \text{if } t > 1
\end{cases}
\]

(19)

\(^6\)Recall that, the efficiency frontier \( \text{EFF}(Y) \) of a technological set \( Y \), is given by the set of production vectors \( y \in Y \), such that, there is not \( y' \geq y \in Y \) and \( y \neq y' \).
EVOLUTION AND JUMP IN A WALRASIAN FRAMEWORK

The demand of each consumer of type 1, is given by:

\[ x_1(t) = \frac{W_1(t)}{2p_1} \left( \frac{1}{t^2} \right) \]  \hspace{1cm} (20)

and the demand of each consumer of type 2, is given by:

\[ x_2(t) = \frac{W_2(t)}{3p_1} \left( \frac{1}{t^2} \right) \]  \hspace{1cm} (21)

Where \( W_i(t) \) denotes the income of a consumer of type \( i \) at prices \( p = p_1(1, t^2) \). We shall compute:

\[ W_i(t) = pw_i + n_1\theta_i_1\pi_1(t) + n_2\theta_i_2\pi_2(t), \, i \in \{1, 2\} \]

\[ W_1(t) = \frac{p_1}{3m_1} \left\{ \begin{array}{ll}
(9 + n_1 + 2n_2)t^2 - 2n_1 t + 3 + n_1 - 2n_2 - 4n_2 \ln t & \text{if } t \geq 1 \\
(9 + n_1 - 2n_2)t^2 - 2n_1 t + 4n_2 t^2 \ln t + (3 + n_1 + 2n_2) & \text{if } 0 < t < 1
\end{array} \right. \]  \hspace{1cm} (22)

\[ W_2(t) = \frac{p_1}{3m_2} \left\{ \begin{array}{ll}
(9 + 2n_1 + n_2)t^2 - 4n_1 t + 6 + 2n_1 - n_2 - 2n_2 \ln t & \text{if } t \geq 1 \\
(9 + 2n_1 - n_2)t^2 - 4n_1 t + 2n_2 t^2 \ln t + 6 + 2n_1 + n_2 & \text{if } 0 < t < 1
\end{array} \right. \]  \hspace{1cm} (23)

Substituting in (20) we obtain the demand for each consumer of type 1:

\[ x_1(t) = \]

\[ = \left\{ \begin{array}{ll}
\frac{(9 + n_1 + 2n_2)t^2 - 2n_1 t + 3 + n_1 - 2n_2 - 4n_2 \ln t}{6m_1 t^2} & \text{if } t \geq 1, \\
\frac{(9 + n_1 - 2n_2)t^2 - 2n_1 t + 4n_2 t^2 \ln t + (3 + n_1 + 2n_2)}{6m_1 t^2} & \text{if } 0 < t < 1.
\end{array} \right. \]  \hspace{1cm} (24)

and substituting in (21) we obtain the demand for each consumer of type 2:

\[ x_2(t) = \]

\[ = \left\{ \begin{array}{ll}
\frac{(9 + 2n_1 + n_2)t^2 - 4n_1 t + 6 + 2n_1 - n_2 - 2n_2 \ln t}{9m_2} & \text{if } t \geq 1, \\
\frac{(9 + 2n_1 - n_2)t^2 - 4n_1 t + 2n_2 t^2 \ln t + 6 + 2n_1 + n_2}{9m_2 t^2} & \text{if } 0 < t < 1.
\end{array} \right. \]  \hspace{1cm} (25)

The aggregate excess demand function will be:

\[ \zeta(t) = m_1(x_1(t) - w_1) + m_2(x_2(t) - w_2) - n_1 y_1(t) - n_2 y_2(t). \]
Note that this means that is given by the following dynamical system:

\[
\begin{align*}
\zeta_1(t) &= \frac{1}{18} 
\begin{cases} 
(45 + 7n_1 + 8n_2) t^2 + 4n_1 t - 33 - 11n_1 - 8n_2 - 20n_2 \ln t \\
\quad \text{if } t \geq 1 \\
(45 + 7n_1 + 10n_2) t^2 + 4n_1 t - 54 + 16n_2 t^2 \ln t \\
\quad \text{if } 0 < t < 1
\end{cases}
\end{align*}
\]

\[
\zeta_2(t) = -\frac{1}{18 t^2} 
\begin{cases} 
(45 + 7n_1 + 8n_2) t^2 + 4n_1 t - 33 - 11n_1 - 8n_2 + 20n_2 \ln t \\
\quad \text{if } t \geq 1 \\
(45 + 7n_1 + 10n_2) t^2 + 4n_1 t - 54 + 16n_2 t^2 \ln t \\
\quad \text{if } 0 < t < 1
\end{cases}
\]

The equilibrium prices are the solution of the equation \( \zeta(t) = 0 \). To find the solutions of this equation we consider the following functions:

\[
f : (0, \infty) \to R \quad \text{and} \quad g : (0, 1) \to R
\]

defined by:

\[
f(t) = (45 + 7n_1 + 8n_2) t^2 + 4n_1 t - 33 - 11n_1 - 8n_2 - 20n_2 \ln t, \quad (a)
\]
\[
g(t) = (45 + 7n_1 + 10n_2) t^2 + 4n_1 t - 54 + 16n_2 t^2 \ln t. \quad (b)
\]

Note that \( f'(t) = 2(45 + 7n_1 + 8n_2) t + 4n_1 - n_2 \frac{20}{t} > 0 \) \( \forall t \geq 1 \) and \( n_1, n_2 \geq 0 \) this means that \( f(t) \) is strictly increasing in the interval \([1, \infty)\). Since \( f(1) = 12 \) it follows that \( \zeta(t) \neq 0 \) for each \( t > 1 \), and then there is not an equilibrium with \( t > 1 \).

On the other hand, since \( \lim_{t \to 0} g(t) < 0 \), and \( g(1) > 0 \), \( \forall (n_1, n_2) \in \Delta \) there exists at least one equilibrium such that \( t^* < 1 \).

Note that \( \Pi_1(t) = \Pi_2(t) \) if and only if \( t = \approx 0.516691803 \). This means that a Walrasian equilibrium is a dynamical equilibrium if and only if \( \sqrt{\frac{p_2}{p_1}} = t^* = \bar{t} = \approx 0.516691803 \). This equilibrium corresponds (approximatelly) for instance to an economy with \( n_1^2 = 1 \) and \( n_2^2 = 40 \).

6.1. The dynamics. Assuming that the number \( n \) of firms remains fixed, but they can choose technology, we obtain for this example, that the evolution of the economy is given by the following dynamical system:

\[
\begin{align*}
\dot{N}_1 &= \phi \left( \pi_1(y_1^1(N_1)) - \pi_2(y_2^1(N_1)) \right) N_1 \\
\dot{N}_2 &= -\dot{N}_1 \\
N(t_0) &\in \Delta^0.
\end{align*}
\]

Hereafter, to avoid confusions we use \( t \) to symbolize time, and \( p = \sqrt{\frac{p_2}{p_1}} \).

So the dynamics along an equilibrium path is given by the

\[
\dot{N}_1 = N_1 \phi(\Pi_1(N)) - \Pi_2(N), \quad 0 < t < 1
\]

in our case

\[
\dot{N}_1 = N_1 \phi \left( -2 + \frac{1 - p^2(N_1)}{2p^2(N_1) \ln p(N_1)} + \frac{1}{p(N_1)} \right)
\]

Note that \( N_1(t) \) is decreasing in the interval \((0, 1)\) and

\[
\phi(\bar{t}) \approx \phi(0.516691803) = 0.
\]
So, if for a distribution $N$ of the firms over the available technologies the inequality $t(N) < 0.516691803$ then $\Pi_1(N_1) - \Pi_2(N) > 0$ and the quantity of firms of type 1 increase and then this difference decrease. Contrarily, if $t(N) > 0.516691803$ then $\Pi_1(N_1) - \Pi_2(N) < 0$ and the quantity of firms of type 2 increase and then this difference increase.

6.2. The transition path. Let us now characterize the trajectory of the economy of our example along a Walrasian equilibrium trajectory.

Consider a regular economy such that in time $t = t_0$

$$n(t_0) = n_0 = (n_{10}, n_{20})$$

i.e, $E_{n_{10}, n_{20}}$. It is possible to choose a neighborhood and a correspondence

$$p^* : \Delta \cup U_0 \rightarrow 2^{\mathbb{R}^*_+},$$

such that $p^*(n_1, n_2) = Eq_{n_1}$.

Let $p^*(n_0) = (p^*_1(n_{10}), p^*_2(n_{20})) \in p^*(n_{10}, n_{20})$ such that

$$p_0 = \sqrt{\frac{p^*_1(n_{10})}{p^*_2(n_{10})}} < 0.516691803$$

From the correspondence $p^*$ choose a continuous selection $p^*$, such that $p^*(n_0) = p^*(n_{10}, n_{20})$.

There exists a neighborhood $V_{n_0} \subset \Delta \cap U_0$ such that for all for all $(n_1, n_2) \in V_{n_0}$, $p^*(n_1)$ verify that $p(n_1) = \sqrt{\frac{p^*_1(n_1)}{p^*_2(n_1)}} < 0.516691803$.

From the implicit function theorem $p : V_{n_0} \rightarrow U_{p_0}$ is a smooth function verifying

$$\frac{dp^*}{dn_1}(n_{10}) = p'_{n_1}(n_{10}) = \frac{-3p_0 + 4 - 16p_0\ln p_0}{-4n_1(p_0)^3 + 108p_0^{-2} + 16(n - n_{10})}.$$ 

The economy $E_{n_{10}, n_{20}}$ evolves along the transition path to an stationary state, $(n^*_1, n^*_2) \in D_0$ being

$$\lim_{(n_1, n_2) \rightarrow (n^*_1, n^*_2)} p^*(n_1, n_2) = p^*(n^*_1, n^*_2) \approx 0.516691803.$$ 

Along this transition path a economy $E_{n_1}$ the share of economies of type 1 increases and the economies of type 2, decreases in equal proportion.

An economy $E_{n_1}$ is singular if and only if for some $p_0$,

$$-4n_1(p_0)^3 + 108p_0^{-2} + 16(n - n_{10}) = 0.$$ 

Given that the initial economy is regular, then all economy in the neighborhood $V_{n_0}$ is regular.

Analogously, it follows that, if the equilibrium for the initial economy, verify that $0.516691803 < p_0 < 1$ then we will observe a contrary evolution, i.e, the share of firms of type 1 decreases, and increases the share of firms of type 2.

Note that, a Walrasian equilibrium price $p^*(n_1, n_2)$ will be, at the same time, an stationary equilibrium if and only if, there exists a distribution $n = (n_1, n_2)$ for which the equalities

$$\pi_1(p^*(n_1, n_2)) = \pi_2(p^*(n_1, n_2))$$

and

$$\zeta(p^*(n_1, n_2)) = 0$$

are verified.
7. Conclusions. In this paper, we have shown that a dynamical process based on imitation leads to the economy over a trajectory of Walrasian equilibria. Since the equilibrium prices and the corresponding profits depend on the distribution of the firms over the set of available technologies, the hypothesis of rationality implies that each manager (with identical interest than the owner), try to anticipate the behavior of their competitors. This occurs because the future profits will depend on the joint choice of the managers on the set of available technologies.

One of the main questions, that we attempt to answer in this paper, is how or based on what arguments, managers, looking for more profitable investments, decide to maintain or change the actual technology under which their firms are producing. We considered that, under incomplete information, managers follow an imitation process.

As this process imitative process continues, the economy changes, more specifically, the set of Walrasian equilibria changes. The economy evolves along a transition path, which reflects the evolution of the distribution of the firms over the set of available technologies. In this process, the wealth of consumers changes, because their shares on firms change, and thereby the excess demand functions change. Consequently, the economy changes. So, the set of Walrasian equilibria depends on the distribution of the firms, and so, this distribution determines the characteristics of the economies, i.e., if they are regular or singular. The changes in the economy along the transition path, will no longer be continuous in a neighborhood of a singular price. It is for these reasons, that we consider singular economies as the thresholds of economic crises. Only singular economies have associate singular equilibria. Big changes can be expected when the distribution of the firms determines that the economy is in a neighborhood of a singular economy. Therefore, the possibility that an economic crisis happens is a structural phenomenon, that lies at the basis of the model itself, particularly in the assumed rationality of the agents.

REFERENCES

[1] E. Accinelli and M. Puchet, A Classification of infinite dimensional Walrasian economies and the economic crisis, In Dynamic, Game and Science (in Honour of Mauricio Peixoto and David Rand) Series, Springer-Verlag, 2 (2011), 55–77.
[2] C. D. Aliprantis, D. J. Brown and O. Burkinshaw, Existence and Optimality of Competitive Equilibria, Springer Verlag, 1990.
[3] Y. Balasko, The Equilibrium Manifold Postmodern Developments in the Theory of General Economic Equilibrium, The Mit Press, 2009.
[4] A. Ben-Shohama, R. Serrano and O. Volij, The evolution of exchange, Journal of Economic Theory, 114 (2004), 310–328.
[5] G. Gigerenzer and D.G. Goldstein, Reasoning the fast and frugal way: Models of bounded rationality psychological review by the american psychological association, Inc, 103 (1996), 650–669.
[6] M. W. Hirsch, S. Smale and R. L. Devaney, Differential Equations, Dynamical Systems, and an Introduction to Chaos, Academic Press, 2004.
[7] M. Kandori, R. Serrano and O. Volij, Decentralized Trade, Random Utility and the Evolution of Social Welfare, Working Paper, Department of Economics, University of Tokyo, 2004.
[8] A. Mas-Colell, The Theory of General Equilibrium: A Diferentiable Approach, Cambridge University Press, 1989.
[9] J. Perla, Equilibrium Imitation and Growth, Journal of Political Economy, 2014.
[10] A. Sard, The measure of the critical values of differentiable maps, Bulletin of the American Mathematical Society, 48 (1942), 883–890.
[11] K. H. Schlag, Why imitate, and if so, how? A boundedly rational approach to multi-armed bandits, Journal of Economic Theory, 78 (1998), 130–156.
[12] F. Vega Redondo, The evolution of walrasian behavior, Econometrica, 65 (1997), 375–384.
[13] J. W. Weibull, *Evolutionary Game Theory*, The Mit Press, 1985.

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E-mail address: elvio.accinelli@eco.uaslp.mx
E-mail address: enriquecovarr@gmail.com