PARABOLIC TYPE SEMIGROUPS: ASYMPTOTICS AND ORDER OF CONTACT

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Abstract. We study the asymptotic behavior of parabolic type semigroups acting on the unit disk as well as those acting on the right half-plane. We use the asymptotic behavior to investigate the local geometry of the semigroup trajectories near the boundary Denjoy–Wolff point. The geometric content includes, in particular, the asymptotes to trajectories, the so-called limit curvature, the order of contact, and so on. We then establish asymptotic rigidity properties for a broad class of semigroups of parabolic type.

Key words and phrases: holomorphic mapping, asymptotic behavior, parabolic type semigroup, contact order, rigidity.

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1. Preliminaries

The theory of semigroups of holomorphic self-mappings of a given domain in the complex plane \( \mathbb{C} \) has been developed intensively over the last few decades. The study began with the basic work of E. Berkson and H. Porta \cite{1} (see, e.g., \cite{13} and \cite{9} for a recent state of this theory). This paper is devoted to the study of a wide class of parabolic type semigroups acting on the open unit disk and on the right half-plane.

Throughout the paper, \( \text{Hol}(D, \mathbb{C}) \) denotes the set of holomorphic functions on a domain \( D \subset \mathbb{C} \) and \( \text{Hol}(D) \) denotes the set of holomorphic self-mappings of \( D \). Recall that a one-parameter continuous semigroup (semigroup, for short) acting on \( D \) is a family \( S = \{F_t\}_{t \geq 0} \subset \text{Hol}(D) \) such that

\[
(i) \quad F_t(F_s(z)) = F_{t+s}(z) \text{ for all } t, s \geq 0 \text{ and } z \in D,
\]
(ii) \( \lim_{t \to 0^+} F_t(z) = z \) for all \( z \in D \).

Berkson and Porta proved that each semigroup acting on \( D \) when \( D \) is either the open unit disk \( \Delta = \{ z \in \mathbb{C} : |z| < 1 \} \) or the right half-plane \( \Pi = \{ z \in \mathbb{C} : \text{Re } z > 0 \} \) is differentiable with respect to \( t \in \mathbb{R}^+ = [0, \infty) \).

Thus, for each one-parameter continuous semigroup the limit

\[
\lim_{t \to 0^+} \frac{F_t(z) - z}{t} = f(z), \quad z \in D,
\]

exists and defines a holomorphic function \( f \in \text{Hol}(D, \mathbb{C}) \). This function \( f \) is called the \textit{(infinitesimal) generator of} \( S \). Moreover, the function \( u(t, z) := F_t(z), (t, z) \in \mathbb{R}^+ \times D \), is the unique solution of the Cauchy problem

\[
\begin{cases}
\frac{\partial u(t, z)}{\partial t} = f(u(t, z)), \\
u(0, z) = z, \quad z \in D.
\end{cases}
\]

In the same paper, Berkson and Porta proved that \( f \in \text{Hol}(\Delta, \mathbb{C}) \) is a semigroup generator if and only if there exist a point \( \tau \in \Delta \) and a function \( p \in \text{Hol}(\Delta, \mathbb{C}) \) with \( \text{Re } p(z) \geq 0 \), such that

\[f(z) = (\tau - z)(1 - z\bar{\tau})p(z) .\]

This representation is unique. Moreover, if \( S \) contains neither the identity mapping nor an elliptic automorphism of \( \Delta \), then \( \tau \) is a unique attractive fixed point of \( S \), i.e., \( \lim_{t \to \infty} F_t(z) = \tau \) for all \( z \in \Delta \), and \( \lim_{r \to 1^-} F_t(r\tau) = \tau \).

The point \( \tau \) is called the Denjoy–Wolff point of \( S \).

Recently the asymptotic behavior of semigroups including the local geometry of semigroup trajectories near their boundary Denjoy–Wolff point \( \tau \in \partial \Delta \) has attracted considerable attention. It was shown in [3] that if \( \tau \in \partial \Delta \), then the angular derivative of \( f \) at \( \tau \in \partial \Delta \) defined by

\[f'(\tau) = \angle \lim_{z \to \tau} \frac{f(z)}{z - \tau}\]

exists and is a non-positive real number.

There is an essential difference between semigroups whose generator \( f \) satisfies \( f'(\tau) < 0 \) (semigroups of hyperbolic type) and those whose
generator $f$ satisfies $f'(\tau) = 0$ (semigroups of parabolic type). For example, in the hyperbolic case, the rate of convergence of the semigroup to its Denjoy–Wolff point is exponential, while in the parabolic case, the convergence is slower. The main problem we address can be stated as follows.

*Determine the rate of convergence of parabolic type semigroups; more precisely, find the asymptotic expansion up to a term small enough.*

Obviously, every semigroup trajectory $\gamma_z = \{F_t(z), \ t \geq 0\}, \ z \in \Delta$, is an analytic curve. Thus, the tangent line and the circle of curvature at each its point $F_s(z)$ exist and move as $s$ increases. The following natural question arises.

*Do tangent lines and disks of curvature have, in some sense, a limit location as $s \to \infty$?*

It turns out that in the hyperbolic case, limit tangent lines always exist and depend on the initial point of the trajectory. On the other hand, for all studied classes of parabolic type semigroups, all trajectories have the same limit tangent line, but even its existence has not been proven in general. More precisely, M. D. Contreras and S. Díaz-Madrigal in [3] considered the set $\text{Slope}^+(\gamma_z)$ of accumulation points (as $t \to \infty$) of the function $t \mapsto \arg (1 - \bar{\tau}F_t(z))$ and proved that these sets do not depend on $z \in \Delta$. There are cases in which it is known that $\text{Slope}^+(\gamma_z)$ is a singleton. The question as to whether, in general, $\text{Slope}^+(\gamma_z)$ is a singleton is still open (see [3, 7, 11, 5, 10] for details).

To be more concrete, we henceforth assume without losing any generality, that $\tau = 1$. We mention (see [7]) that if the generator $f$ of a parabolic type semigroup $S = \{F_t\}_{t \geq 0}$ admits the representation

$$f(z) = a(1 - z)^2 + o((1 - z)^2),$$
then for each \(z \in \Delta\), the limit tangent line to the trajectory \(\gamma_z = \{F_t(z), t \geq 0\}\) exists, and
\[
\lim_{t \to \infty} \arg(1 - F_t(z)) = -\arg a.
\]
Hence, this limit depends on neither \(z \in \Delta\) nor the remainder \(o((z-1)^2)\).

This fact was generalized in [11] (see also [5]) for the case
\[
f(z) = a(1 - z)^{1+\alpha} + o((1 - z)^{1+\alpha})\quad \text{for some } \alpha > 0.
\]
Moreover, it was shown in [11] that
\[
\alpha \leq 2, \quad |\arg a| \leq \frac{\pi}{2} \min\{\alpha, 2 - \alpha\}, \quad \lim_{t \to \infty} \arg(1 - F_t(z)) = -\frac{1}{\alpha} \arg a.
\]
In particular, this implies that all the trajectories are tangent to the unit circle if and only if \(\alpha \leq 1\) and \(\arg a = \pm \frac{\pi}{2}\) (see [5] for more details).

Proposition 3.1 below completes these results. An advanced question in this study is the following.

**How close is a semigroup trajectory to its tangent line?**

Following [10], for each \(z \in \Delta\), we denote the curvature of the trajectory \(\gamma_z\) at the point \(F_s(z)\) by \(\kappa(z, s)\) and define the limit curvature of the trajectory by \(\kappa(z) := \lim_{s \to \infty} \kappa(z, s)\), if the limit exists. Therefore, the above question can be reduced to the following one.

**When is the limit curvature finite?**

This question was studied in [10], where it was shown that every trajectory of a hyperbolic type semigroup has a finite limit curvature, while the finiteness in the parabolic case is, in a sense, exceptional. Namely, it was proved in [10] that if a semigroup generator is \((3 + \varepsilon)\)-smooth at the Denjoy–Wolff point, in the sense that it admits the representation
\[
f(z) = a(1 - z)^2 + b(1 - z)^3 + R(z),
\]
where \(R \in \text{Hol}(\Delta, \mathbb{C})\), \(\lim_{z \to 1} \frac{R(z)}{(1 - z)^{3+\varepsilon}} = 0\), and \(a \neq 0\),
\[
(a) \text{ if } \text{Im} \left( \frac{b}{a^2} \right) \neq 0, \text{ then all of the trajectories have infinite limit curvature, i.e., } \kappa(z) = \infty \text{ for all } z \in \Delta;
\]
(b) if \( \text{Im} \frac{b}{a^2} = 0 \), the limit curvature of every trajectory \( \gamma_z \) is finite. The value \( \kappa(z) \) was calculated explicitly in [10].

Thus, under the above assumptions, if \( \kappa(z) \) is finite for some \( z \in \Delta \), then it must be finite for all \( z \in \Delta \).

Once again, we see that there is a cardinal difference between semigroups of hyperbolic and parabolic types. In the hyperbolic case under some smoothness conditions, the limit curvature is always finite; in the parabolic case, the limit curvature may be infinite. For the above reasons, for parabolic type semigroups, a more relevant question is

*find the contact order of a trajectory and the limit tangent line* (which is less than 2 when the limit curvature is infinite).

This problem leads to the so-called rigidity problem, which is that of finding the weakest conditions on two holomorphic mappings at a boundary point under which the mappings coincide. Beginning with the outstanding work of D. Burns and S. G. Krantz [2], this problem has attracted considerable interest (see [14, 6] and reference therein). As a rule, the rigidity problem for one-parameter semigroups is approached by looking for conditions on generators. Another approach is related to semigroup asymptotics. Here, we investigate the rigidity problem via contact order of the trajectories. In our setting, the next question is natural.

**What is the minimal contact order of trajectories of parabolic type semigroups required to ensure that the semigroups coincide?**

We solve the above problems for parabolic type semigroups \( S = \{ F_t \}_{t \geq 0} \) whose generators \( f \in \text{Hol}(\Delta, \mathbb{C}) \) admit the representation

\[
f(z) = a(1 - z)^{1+\alpha} + R(z),
\]

or the representation

\[
f(z) = a(1 - z)^{1+\alpha} + b(1 - z)^{1+\alpha+\beta} + R_1(z),
\]
where $\alpha \in (0, 2]$, $\beta > 0$, $a \neq 0$, and functions $R, R_1 \in \text{Hol}(\Delta, \mathbb{C})$ satisfy
\[
\lim_{z \to 1} \frac{R(z)}{(1 - z)^{1+\alpha}} = 0, \quad \text{and} \quad \lim_{z \to 1} \frac{R_1(z)}{(1 - z)^{1+\alpha+\beta}} = 0. \tag{1.3}
\]
As previously mentioned, if $|\arg a| < \frac{\pi \alpha}{2}$, the semigroup converges non-tangentially. Since, we use formulas (1.1) and (1.2) to expand $f(F_t(z))$, in the case $|\arg a| \neq \frac{\pi \alpha}{2}$, the limits in (1.3) can be replaced by angular limits.

In what follows, $\mathcal{G}_{\alpha,\beta}(\Delta)$ denotes the set of semigroup generators having the form (1.2) with $a \neq 0$ and function $R_1$ satisfying (1.3).

Also, we apply a linearization model given by Abel’s functional equation
\[
h(F_t(z)) = h(z) + t. \tag{1.4}
\]
It is rather easy to see that the function $h : \Delta \mapsto \mathbb{C}$ defined by
\[
h'(z)f(z) = 1, \quad h(0) = 0, \tag{1.5}
\]
solves functional equation (1.4). This function is univalent and, due to (1.4), is convex in the positive direction of the real axis. Sometimes $h$ is called the Kœnigs function for the semigroup (see [3, 7, 11, 15] and [9]).

The class of semigroups acting on $\Pi$ and the class acting on $\Delta$ are conjugated by $\Phi_t(w) = C \circ F_t \circ C^{-1}(w)$, where $C$ is the Cayley transform $C(z) = \frac{1+z}{1-z}$. For technical reasons, we first study the behavior of semigroups acting on $\Pi$. Whence $S = \{F_t\}_{t \geq 0}$ has Denjoy–Wolff point $\tau = 1$, semigroup $\Sigma = \{\Phi_t\}_{t \geq 0} \subset \text{Hol}(\Pi)$ has Denjoy–Wolff point $\infty$, its generator $\phi$ belongs to $\text{Hol}(\Pi, \overline{\Pi})$, and the semigroup $\Sigma = \{\Phi_t\}_{t \geq 0}$ satisfies the Cauchy problem
\[
\begin{cases}
\frac{\partial \Phi_t(w)}{\partial t} = \phi(\Phi_t(w)), \\
\Phi_t(w)|_{t=0} = w, \quad w \in \Pi.
\end{cases}
\]
We modify the Kœnigs function $h$ defined by (1.5) to $\sigma := h \circ C^{-1}$. Direct calculations show that for all $w \in \Pi$, this modified function satisfies
Abel’s functional equation
\[ \sigma(\Phi_t(w)) = \sigma(w) + t \]  \hspace{1cm} (1.6)
as well as the initial value problem
\[ \sigma'(w)\phi(w) = 1, \quad \sigma(1) = 0. \]  \hspace{1cm} (1.7)

It follows by from Julia’s Lemma (see, for example, [12, 13, 9]) that since the Denjoy-Wolff point of \( \Sigma \) is \( \infty \), hence \( \text{Re} \Phi_t(w) \) is an increasing function in \( t \) for \( t \geq 0 \). This prompts an additional question.

What conditions ensure the existence of asymptotes to semigroup trajectories?

Note in passing that a semigroup trajectory \( \gamma_z \subset \Delta \) has a finite limit curvature if and only if \( C(\gamma_z) \subset \Pi \) has an asymptote as \( t \to \infty \).

In Section 2, we study the asymptotic behavior of semigroups acting on \( \Pi \). These semigroups not only give us a machinery for our main results, but are of intrinsic interest. Despite the fact that these semigroups tend to \( \infty \), the asymptotic behavior which we describe enables us to distinguish those semigroups whose trajectories are either asymptotically parallel, or mutually convergent, or mutually divergent (see Definition 2.1 below). As an application, we deduce the rather surprising result that in the case \( \alpha < \min\{1, \beta\} \), the motion on each trajectory is accelerating. Consequently, the distance between two particles starting at different points of the same trajectory grows to \( \infty \) (see Corollary 2.2). In addition, we present a complete description of conditions for the existence of asymptotes to semigroup trajectories and their possible coincidence.

In Section 3, we turn to semigroups acting on \( \Delta \) generated by functions of the class \( G_{\alpha, \beta}(\Delta) \). Theorem 3.1 contains a full description of the asymptotic behavior of such semigroups. One of the phenomena discovered is that for semigroup generators of the form (1.2), if \( \beta \leq \alpha \), the first two terms of the asymptotic expansion of the generated semigroup do not depend on the initial point. On the other hand, if \( \beta > \alpha \), the
initial point affects as from the second term. Moreover, if \( \beta \leq \alpha \), it may happen that all the trajectories have the same contact order (see Definition 3.1 below), while if \( \beta > \alpha \), there exists a trajectory \( \gamma \) of maximal contact order. One of the geometric implications of this phenomenon is that there exists no semigroup trajectory lying between \( \gamma \) and the limit tangent line. Each trajectory starting from a point between \( \gamma \) and the tangent line must intersect the tangent and approach it from the side opposite from \( \gamma \) (see Remark 3.1 below). We also provide conditions under which the limit curvature is either zero, finite, or infinite.

In Section 4, we study the contact order of two trajectories and use results from earlier sections to establish rigidity criteria for parabolic type semigroups. As a bonus, we discover another interesting geometric phenomenon. In the case \( 0 < \beta < \alpha \), each trajectory is closer to all other trajectories than to their common limit tangent line. Thus, all the trajectories approach this tangent line from the same side (see Remark 4.1).

2. Semigroups on the right half-plane

In this section, we study parabolic type semigroups acting on the right half-plane \( \Pi \). We begin by assuming that only the first term in the asymptotic expansion of the generator is known.

**Lemma 2.1.** Let \( \{ \Phi_t \}_{t \geq 0} \in \text{Hol}(\Pi) \) be a semigroup of parabolic type with the Denjoy–Wolff point at \( \infty \) generated by mapping \( \phi \). Suppose that

\[
\phi(w) = A(w + 1)^{1-\alpha} + g(w),
\]

where \( g \in \text{Hol}(\Pi, \mathbb{C}) \), and \( \angle \lim_{w \to \infty} \frac{g(w)}{(w + 1)^{1-\alpha}} = 0 \). Then

\[
\Phi_t(w) = (\lambda t)^{\frac{1}{\alpha}} + \Gamma(w, t) \quad \text{with} \quad \lim_{t \to \infty} t^{-\frac{1}{\alpha}} \Gamma(w, t) = 0,
\]

and

\[
\lim_{t \to \infty} (\Phi_t^\alpha(w) - \Phi_t^\alpha(1)) = \lambda \sigma(w),
\]
where \( \sigma \) is defined by (1.7) and \( \lambda = \alpha A \).

As already mentioned, if a semigroup generator satisfies (2.1) then \( 0 < \alpha \leq 2 \). The case \( \alpha = 1 \) was considered in [10, Theorem 4.1(i)].

**Proof.** Fix \( w \in \Pi \) and consider \( \Phi_t(w) \) as a (complex valued) function of the real variable \( t \). Since \( \lim_{t \to \infty} \Phi_t(w) = \infty \), L'Hopital's rule gives

\[
\lim_{t \to \infty} \frac{\Phi_t(w) + 1}{t + 1} = \lim_{t \to \infty} \frac{\alpha (\Phi_t(w) + 1)^{\alpha - 1} \phi(\Phi_t(w))}{1} = \lim_{t \to \infty} \alpha \left( A + \frac{\rho(\Phi_t(w))}{\Phi_t^{1-\alpha}(w)} \right) = \lambda.
\]

Thus,

\[
\lim_{t \to \infty} \frac{\Phi_t(w)}{t^\frac{1}{\alpha}} = \lim_{t \to \infty} \frac{\Phi_t(w)}{t^\frac{1}{\alpha}} \frac{\Phi_t(w) + 1}{\Phi_t(w)} \left( \frac{t}{t + 1} \right)^{\frac{1}{\alpha}} = \lim_{t \to \infty} \frac{\Phi_t(w) + 1}{(t + 1)^{\frac{1}{\alpha}} = \lambda^{\frac{1}{\alpha}}.
\]

This proves (2.2). Furthermore,

\[
\lim_{t \to \infty} \left( \Phi_t^\alpha(w) - \Phi_t^\alpha(1) \right) = \lim_{t \to \infty} \int_1^w (\Phi_t^\alpha(z))' \, dz
\]

\[
= \lim_{t \to \infty} \int_1^w \alpha \Phi_t^{\alpha - 1}(z) \frac{\phi(\Phi_t(z))}{\phi(z)} \, dz
\]

\[
= \lim_{t \to \infty} \int_1^w \left( \frac{\Phi_t(z)}{\Phi_t(z) + 1} \right)^{\alpha - 1} \frac{\alpha}{\phi(z)} \left( A + \frac{\rho(\Phi_t(z))}{\Phi_t^{1-\alpha}(z)} \right) \, dz
\]

\[
= \lambda \int_1^w \frac{dz}{\phi(z)} = \lambda \sigma(w)
\]

by (2.4). \( \blacksquare \)

In the case in which the function \( \rho \) in (2.1) can be written as \( \rho(w) = B(w+1)^{1-\alpha-\beta} + \varrho_1(w) \) with \( \beta > 0 \) and \( \lim_{w \to \infty} \frac{\varrho_1(w)}{[w+1]^{1-\alpha-\beta}} = 0 \), we can obtain a more precise estimate for the asymptotic behavior of the generated semigroup. Denote the set of generators \( \phi \in \text{Hol}(\Pi, \Pi) \) of the form

\[
\phi(w) = A(w + 1)^{1-\alpha} + B(w + 1)^{1-\alpha-\beta} + \varrho_1(w)
\]

(2.5)
by $\mathcal{G}_{\alpha,\beta}(\Pi)$, where $\varrho_1 \in \text{Hol}(\Pi, \mathbb{C})$ satisfies $\lim_{w \to \infty} \frac{\varrho_1(w)}{(w+1)^{1-\alpha-\beta}} = 0$ and $A \neq 0$. For the remainder of this section, we deal with semigroups whose Denjoy–Wolff point is $\tau = \infty$ and whose infinitesimal generators lie in $\mathcal{G}_{\alpha,\beta}(\Pi)$. We also set

$$\lambda = \alpha A \quad \text{and} \quad \mu = \frac{B}{A}. \quad (2.6)$$

It turns out that semigroups have different asymptotic behavior depending on whether $\beta < \alpha$, $\beta = \alpha$, or $\beta > \alpha$. We start with the case $\beta < \alpha$.

**Theorem 2.1.** Let $\Sigma = \{\varPhi_t\}_{t \geq 0} \in \text{Hol}(\Pi)$ be a semigroup generated by a mapping $\phi \in \mathcal{G}_{\alpha,\beta}(\Pi)$ with $0 < \beta < \alpha \leq 2$. Then

$$\varPhi_t(w) + 1 = (\lambda t)^{\frac{1}{\alpha}} \left(1 + \frac{\mu}{\alpha - \beta}(\lambda t)^{\frac{\beta}{\alpha}} + \Gamma(w,t)\right), \quad (2.7)$$

where $\lim_{t \to \infty} t^{\frac{\beta}{\alpha}} \Gamma(w,t) = 0$.

**Proof.** First we show that

$$\lim_{t \to \infty} \frac{1}{(t+1)^{\frac{1}{\alpha}}} \left((\varPhi_t(w) + 1)^{\alpha} - \lambda t - \frac{\alpha \mu}{\alpha - \beta}(\lambda(t+1))^{1-\frac{\beta}{\alpha}}\right) = 0. \quad (2.8)$$

Using (2.5), we calculate

$$\frac{d}{ds} \left((\varPhi_s(w) + 1)^{\alpha} - \lambda s - \frac{\alpha \mu}{\alpha - \beta}(\lambda(s+1))^{1-\frac{\beta}{\alpha}}\right)$$

$$= \alpha \left(B \left((\varPhi_s(w) + 1)^{-\beta} - (\lambda(s+1))^{-\frac{\beta}{\alpha}}\right) + (\varPhi_s(w) + 1)^{\alpha-1} \varrho_1(\varPhi_s(w))\right)$$

$$= \frac{\alpha}{(s+1)^{\frac{1}{\alpha}}} \left[B \left(\mu(s,w) - \lambda^{-\frac{\beta}{\alpha}}\right) + \frac{\mu(s,w)\varrho_1(\varPhi_s(w))}{(\varPhi_s(w) + 1)^{1-\alpha-\beta}}\right], \quad (2.9)$$

where $\mu(s, w) = \left(\frac{(s+1)^{\frac{1}{\alpha}}}{\varPhi_s(w) + 1}\right)^{\beta}$. Also, by (2.4), $\lim_{s \to \infty} \mu(s, w) = \lambda^{-\frac{\beta}{\alpha}}$. By our assumption on $\varrho_1$ in (2.5), it follows that

$$\lim_{s \to \infty} \left[B \left(\mu(s, w) - \lambda^{-\frac{\beta}{\alpha}}\right) + \frac{\mu(s, w)\varrho_1(\varPhi_s(w))}{(\varPhi_s(w) + 1)^{1-\alpha-\beta}}\right] = 0.$$
Therefore, for each \( \varepsilon > 0 \), there exists \( t_0 \) such that for all \( s > t_0 \),

\[
\left| B \left( \mu(s, w) - \lambda \frac{\beta}{\alpha} \right) + \frac{\mu(s, w) \rho_1(\Phi_s(w))}{(\Phi_s(w) + 1)^{1-\alpha-\beta}} \right| < \varepsilon \frac{(\alpha - \beta)}{\alpha^2},
\]

while, for each \( 0 \leq s \leq t_0 \), there exists \( K > 0 \) such that

\[
\left| B \left( \mu(s, w) - \lambda \frac{\beta}{\alpha} \right) + \frac{\mu(s, w) \rho_1(\Phi_s(w))}{(\Phi_s(w) + 1)^{1-\alpha-\beta}} \right| < K \frac{(\alpha - \beta)}{\alpha^2}.
\]

Since

\[
(\Phi_t(w) + 1)^\alpha - \lambda t - \frac{\alpha \mu}{\alpha - \beta} (\lambda(t + 1))^{1-\frac{\beta}{\alpha}} - (w + 1)^\alpha + \frac{\alpha \mu}{\alpha - \beta} \lambda^{1-\frac{\beta}{\alpha}}
\]

\[
= \int_0^t \frac{d}{ds} \left( (\Phi_s(w) + 1)^\alpha - \lambda s - \frac{\alpha \mu}{\alpha - \beta} (\lambda(s + 1))^{1-\frac{\beta}{\alpha}} \right) ds,
\]

formula (2.9) implies

\[
\frac{1}{(t + 1)^{1-\frac{\beta}{\alpha}}} \left| (\Phi_t(w) + 1)^\alpha - \lambda t - \frac{\alpha \mu}{\alpha - \beta} (\lambda(t + 1))^{1-\frac{\beta}{\alpha}} \right|
\]

\[
\leq \frac{1}{(t + 1)^{1-\frac{\beta}{\alpha}}} \left[ \int_0^{t_0} K \frac{(\alpha - \beta)}{\alpha(s + 1)^{\frac{\beta}{\alpha}}} ds + \int_{t_0}^t \varepsilon \frac{(\alpha - \beta)}{\alpha(s + 1)^{\frac{\beta}{\alpha}}} ds \right.
\]

\[
+ \left. \left| (w + 1)^\alpha - \frac{\alpha \mu}{\alpha - \beta} \lambda^{1-\frac{\beta}{\alpha}} \right| \right]
\]

\[
= \varepsilon + (K - \varepsilon) \frac{(t_0 + 1)^{1-\frac{\beta}{\alpha}}}{(t + 1)^{1-\frac{\beta}{\alpha}}} + \frac{K + \left| (w + 1)^\alpha - \frac{\alpha \mu}{\alpha - \beta} \lambda^{1-\frac{\beta}{\alpha}} \right|}{(t + 1)^{1-\frac{\beta}{\alpha}}}.
\]

Since \( \varepsilon > 0 \) is arbitrary, (2.8) follows and

\[
(\Phi_t(w) + 1)^\alpha = \lambda t + \frac{\alpha \mu}{\alpha - \beta} (\lambda(t + 1))^{1-\frac{\beta}{\alpha}} + \Gamma_1(w, t)
\]

(2.11)

with \( \lim_{t \to \infty} (t + 1)^{\frac{\beta}{\alpha}} \Gamma_1(w, t) = 0 \).
To proceed we calculate
\[
\Phi_t(w) + 1 = \left( \lambda t + \frac{\alpha \mu}{\alpha - \beta} (\lambda (t + 1))^{\frac{\alpha - \beta}{\alpha}} + \Gamma_1(w, t) \right)^{\frac{1}{\alpha}} \\
= (\lambda t)^\frac{1}{\alpha} \left( 1 + \frac{\alpha \mu}{\alpha - \beta} \lambda t (\lambda (t + 1))^{\frac{\alpha - \beta}{\alpha}} + \frac{\Gamma_1(w, t)}{\lambda t} \right)^{\frac{1}{\alpha}} \\
= (\lambda t)^\frac{1}{\alpha} \left( 1 + \frac{\mu}{\alpha - \beta} \lambda t (\lambda (t + 1))^{\frac{\alpha - \beta}{\alpha}} + \tilde{\Gamma}_1(w, t) \right),
\]
where \( \lim_{t \to \infty} t^{\frac{\beta}{\alpha}} \tilde{\Gamma}_1(w, t) = 0 \). This proves the assertion. ■

The case \( \beta = \alpha \) can be treated similarly. We state the analogous result.

**Theorem 2.2** (cf., Theorem 4.1(ii) in [10]). Let \( \Sigma = \{ \Phi_t \}_{t \geq 0} \subset \text{Hol}(\Pi) \) be a semigroup generated by a mapping \( \phi \in G_{\alpha, \beta}(\Pi) \) with \( 0 < \beta = \alpha \leq 2 \), i.e.,
\[
\phi(w) = A(w + 1)^{1-\alpha} + B(w + 1)^{1-2\alpha} + \varrho_1(w), \quad \lim_{w \to \infty} \frac{\varrho_1(w)}{(w + 1)^{1-2\alpha}} = 0.
\]
Then
\[
\Phi_t(w) + 1 = (\lambda t)^\frac{1}{\alpha} \left( 1 + \frac{\mu}{\alpha - \beta} \frac{\log(t + 1)}{\lambda t} + \Gamma(w, t) \right), \quad (2.12)
\]
where \( \lim_{t \to \infty} t^{\frac{\beta}{\alpha}} \Gamma(w, t) = 0 \).

It turns out that the asymptotic behavior of a semigroup can be estimated more precisely when the remainder \( \varrho_1 \) in (2.5) satisfies a stronger condition. The next result generalizes [10, Theorem 4.1(iii)].

**Proposition 2.1.** Let \( \phi \in G_{\alpha, \beta}(\Pi) \) be given by (2.5), where \( \varrho_1 \in \text{Hol}(\Pi, \mathbb{C}) \) satisfies \( \lim_{w \to \infty} (w + 1)^{2\alpha - 1 + \varepsilon} \varrho_1(w) = 0 \) for some positive \( \varepsilon \), and let \( \Sigma = \{ \Phi_t \}_{t \geq 0} \subset \text{Hol}(\Pi) \) be a semigroup generated by \( \phi \).

(i) If \( \beta = \alpha \), then there exists a constant \( C \) such that for all \( w \in \Pi \),
\[
(\Phi_t(w) + 1)^\alpha = \lambda t + \mu \log(t + 1) + \lambda \sigma(w) + C + \Gamma(w, t),
\]
where \( \lim_{t \to 0} \Gamma(w, t) = 0 \).
(ii) If $\frac{\alpha}{2} < \beta < \alpha$ then there exists a constant $C$ such that for all $w \in \Pi$,

$$(\Phi_t(w) + 1)^\alpha = \lambda t + \frac{\mu \alpha}{\alpha - \beta} (\lambda t)^{1 - \frac{\beta}{\alpha}} + \lambda \sigma(w) + C + \Gamma(w, t),$$

where $\lim_{t \to \infty} \Gamma(w, t) = 0$.

**Proof.** Since the proofs of assertions (i) and (ii) are similar, we prove only assertion (ii). We first show that the limit $H(w) := \lim_{t \to \infty} \left( (\Phi_t(w) + 1)^\alpha - \lambda t - \frac{\alpha \mu}{\alpha - \beta} (\lambda(t + 1))^{1 - \frac{\beta}{\alpha}} \right)$ exists for each $w \in \Pi$. Indeed, by the calculations in (2.9) and (2.10), we have

$$(\Phi_t(w) + 1)^\alpha - \lambda t - \frac{\alpha \mu}{\alpha - \beta} (\lambda(t + 1))^{1 - \frac{\beta}{\alpha}} =$$

$$= \int_0^t \alpha B \left( (\Phi_s(w) + 1)^{-\beta} - (\lambda(s + 1))^{-\frac{\beta}{\alpha}} \right) ds$$

$$+ \int_0^t \alpha (\Phi_s(w) + 1)^{\alpha - 1} \rho_1(\Phi_s(w)) ds + (w + 1)^\alpha - \frac{\alpha \mu}{\alpha - \beta} (\lambda(t + 1))^{1 - \frac{\beta}{\alpha}}.$$

For the first integral, (2.11) implies

$$\frac{1}{(\Phi_s(w) + 1)^\beta} - \frac{1}{(\lambda(s + 1))^{\frac{\beta}{\alpha}}} = \frac{1}{(\lambda(s + 1))^{\frac{\beta}{\alpha}}} \left( (\lambda(s + 1))^{\frac{\beta}{\alpha}} - 1 \right)$$

$$= \frac{1}{(\lambda(s + 1))^{\frac{\beta}{\alpha}}} \left( 1 + \frac{\alpha \mu}{\alpha - \beta} (\lambda(s + 1))^{-\frac{\beta}{\alpha}} - \frac{1}{s + 1} + \frac{\Gamma_1(w, s)}{\lambda(s + 1)}^{-\frac{\beta}{\alpha}} - 1 \right).$$

Since this expression is $O\left( \frac{1}{(s + 1)^{\frac{2\beta}{\alpha}}} \right)$ with $\frac{2\beta}{\alpha} > 1$, the first integral converges. For the second integral, we have

$$\frac{\alpha \rho_1(\Phi_s(w))}{(\Phi_s(w) + 1)^{-\alpha}} = \alpha (\Phi_s(w) + 1)^{-\alpha - \epsilon} \frac{\rho_1(\Phi_s(w))}{(\Phi_s(w) + 1)^{1 - 2\alpha - \epsilon}}$$

$$= \frac{\alpha}{(s + 1)^{\frac{s + 1}{\alpha}}} \cdot \left( \frac{s + 1}{(\Phi_s(w) + 1)^{\alpha}} \right)^{\frac{s + 1}{\alpha - \epsilon}} \cdot \left( \frac{\rho_1(\Phi_s(w))}{(\Phi_s(w) + 1)^{1 - 2\alpha - \epsilon}} \right).$$
If \( \lim_{t \to \infty} \frac{\rho_1(\Phi_s(w))}{(\Phi_s(w) + 1)^{1-2\alpha}} = 0 \), then by (2.4) we have

\[
\alpha \left| \frac{\rho_1(\Phi_s(w))}{(\Phi_s(w) + 1)^{1-\alpha}} \right| = O \left( \frac{1}{(s + 1)^{1+\frac{\alpha}{\beta}}} \right),
\]
and so the second integral also converges. The proof of the equality

\[
H(w) - H(1) = \lim_{t \to \infty} ((\Phi_t(w) + 1)^\alpha - (\Phi_t(1) + 1)^\alpha) = \lambda \sigma(w)
\]
is similar to that of (2.3). This proves that

\[
(\Phi_t(w) + 1)^\alpha = \lambda t + \frac{\mu \alpha}{\alpha - \beta} (\lambda(t + 1))^{1-\frac{\beta}{\alpha}} + \lambda \sigma(w) + C + \Gamma(w, t),
\]
where \( C = H(1) \) and

\[
\Gamma(w, t) = (\Phi_t(w) + 1)^\alpha - \lambda t - \frac{\mu \alpha}{\alpha - \beta} (\lambda(t + 1))^{1-\frac{\beta}{\alpha}} - \lambda \sigma(w) - C.
\]
The result now follows, since \( \lim_{t \to \infty} \Gamma(w, t) = 0 \).

We now turn to the case \( \beta > \alpha \).

**Theorem 2.3.** Let \( \Sigma = \{ \Phi_t \}_{t \geq 0} \in \text{Hol}(\Pi) \) be a semigroup generated by \( \phi \in \mathcal{G}_{\alpha, \beta}(\Pi) \), where \( \beta \neq \alpha \) satisfies \( 0 < k\alpha \leq \beta < (k + 1)\alpha \) for some \( k \in \mathbb{N} \). Then

\[
\Phi_t(w) + 1 = (\lambda t)^\frac{1}{\alpha} \times \left[ 1 + \sum_{j=1}^{k} \left( \frac{1}{\alpha} \right) \left( \frac{\sigma_1(w)}{t} \right)^j + \frac{\mu}{\alpha - \beta} (\lambda t)^{-\frac{\beta}{\alpha}} + \Gamma(w, t) \right],
\]

(2.13)

where \( \lim_{t \to \infty} t^{\frac{\alpha}{\beta}} \Gamma(w, t) = 0 \) and \( \sigma_1(w) = \sigma(w) + \frac{2\alpha}{\lambda} \int_1^\infty \left( \sigma'(v) - \frac{1}{\alpha}(v + 1)^{\alpha-1} \right) dv. \)

**Proof.** We apply the Kœnigs function \( \sigma \) which satisfies (1.6). By (1.7),

\[
\sigma'(w) = \frac{1}{\phi(w)} = \frac{1}{A(w + 1)^{1-\alpha} + B(w + 1)^{1-\alpha-\beta} + g_1(w)}.
\]
A direct calculation gives

\[
\sigma'(w) = \frac{1}{A} (w + 1)^{\alpha-1} - \frac{B}{A^2} (w + 1)^{\alpha-1-\beta} + r(w),
\]
(2.14)
where \( \lim_{w \to \infty} r(w)(w+1)^{1-\alpha+\beta} = 0 \). In particular, this implies that the improper integral \( \int_1^\infty r(v)dv \) converges, say to \( C_1 \). An application of L'Hôpital's rule gives

\[
\lim_{w \to \infty} \frac{\int_1^w r(v)dv - C_1}{(w+1)^{\alpha-\beta}} = 0.
\]

It now follows from (2.14) that

\[
\sigma(w) = \int_1^w \sigma'(v)dv = \frac{1}{\lambda}(w+1)^{\alpha} + \frac{\alpha \mu}{\lambda(\beta - \alpha)}(w+1)^{\alpha - \beta} + C_2 + r_1(w),
\]

where \( \lim_{w \to \infty} r_1(w)(w+1)^{\beta-\alpha} = 0 \) and \( C_2 = \int_1^\infty (\sigma'(v) - \frac{1}{\lambda}(v+1)^{\alpha-1})dv - \frac{\alpha}{\lambda} \). Substituting this asymptotic expansion into Abel's functional equation (1.6) yields

\[
\frac{1}{\lambda}(\Phi_t(w) + 1)^{\alpha} + \frac{\alpha \mu}{\lambda(\beta - \alpha)}(\Phi_t(w) + 1)^{\alpha - \beta} + C_2 + r_1(\Phi_t(w)) = \sigma(w) + t.
\]

(2.15)

According to Lemma 2.1, \( \lim_{t \to \infty} \frac{(\Phi_t(w) + 1)^{\alpha - \beta}}{(t+1)^{1-\frac{\beta}{\alpha}}} = \lambda^{1-\frac{\beta}{\alpha}} \). Therefore,

\[
(\Phi_t(w) + 1)^{\alpha - \beta} = (\lambda(t+1))^{1-\frac{\beta}{\alpha}} + r_2(t, w),
\]

where \( \lim_{t \to \infty} (t+1)^{\frac{\beta}{\alpha}-1}r_2(t, w) = 0 \). Comparing the last relation with (2.15), we conclude that

\[
(\Phi_t(w) + 1)^{\alpha} = \lambda t + \lambda(\sigma(w) - C_2) + \frac{\alpha \mu}{\alpha - \beta}(\lambda(t+1))^{1-\frac{\beta}{\alpha}}
\]

\[
\quad - \frac{\alpha \mu}{\beta - \alpha}r_2(t, w) - \lambda r_1(\Phi_t(w))
\]

\[
\quad = \lambda t + \lambda \sigma_1(w) + \frac{\alpha \mu}{\alpha - \beta}(\lambda(t+1))^{1-\frac{\beta}{\alpha}} + R(t, w),
\]

where \( \lim_{t \to \infty} R(t, w)(t+1)^{\frac{\beta}{\alpha}-1} = 0 \) and \( \sigma_1(w) = \sigma(w) - C_2 \).
The formula for the sum of a binomial series gives

\[
\Phi_t(w) + 1 = \left(\lambda t + \lambda \sigma_1(w) + \frac{\alpha \mu}{\alpha - \beta} (\lambda(t + 1))^{1 - \frac{\beta}{\alpha}} + R(t, w)\right)^{\frac{1}{\alpha}}
\]

\[
= (\lambda t)^{\frac{1}{\alpha}} \left(1 + \frac{\sigma_1(w)}{t}\right)^{\frac{1}{\alpha}} \left(1 + \frac{\alpha \mu}{\lambda (\alpha - \beta)} \frac{(\lambda(t + 1))^{1 - \frac{\beta}{\alpha}}}{t + \sigma_1(w)} + \frac{R(t, w)}{t + \sigma_1(w)}\right)^{\frac{1}{\alpha}}
\]

\[
= (\lambda t)^{\frac{1}{\alpha}} \left(1 + \sum_{j=1}^{k} \left(\frac{1}{\alpha}\right)^j \left(\frac{\sigma_1(w)}{t}\right)^j + O\left(\frac{1}{t^{k+1}}\right)\right) \times
\]

\[
\left(1 + \frac{\mu}{\lambda (\alpha - \beta)} \frac{(\lambda(t + 1))^{1 - \frac{\beta}{\alpha}}}{t + \sigma_1(w)} + o\left(t^{-\frac{\beta}{\alpha}}\right)\right)
\]

\[
= (\lambda t)^{\frac{1}{\alpha}} \left(1 + \sum_{j=1}^{k} \left(\frac{1}{\alpha}\right)^j \left(\frac{\sigma_1(w)}{t}\right)^j + \frac{\mu (\lambda t)^{-\frac{\beta}{\alpha}}}{\alpha - \beta} + o\left(t^{-\frac{\beta}{\alpha}}\right)\right).
\]

The proof is complete. ■

Theorems 2.1–2.3 give more than asymptotic expansions of semigroups. Using standard methods of analysis we can deduce, on the basis of these theorems, interesting facts about the geometry of semigroup trajectories. For example, we give criteria on \(\alpha\) and \(\beta\) which ensure the existence/non-existence of asymptotes to semigroup trajectories. We also determine whether the asymptote exists for all initial points \(w \in \Pi\) or only for \(w\) from some subset of \(\Pi\), and whether the asymptote (if it exists) depends on the initial point. As we will see below, the cases in which the asymptote passes through \(-1\) are of special interest.

First, we decompose the set \(\Omega = \{ (\alpha, \beta) : \; 0 < \alpha \leq 2, \; \beta > 0\}\) of all possible pairs of the parameters into the following subsets:

\(\Omega_1 := \{ (\alpha, \beta) \in \Omega : \alpha, \beta > 1 \}\),

\(\Omega_2 := \{ (\alpha, \beta) \in \Omega : 1 = \alpha < \beta \}\),

\(\Omega_3 := \{ (\alpha, \beta) \in \Omega : \alpha < \min\{1, \beta\}\}\),

\(\Omega_4 := \{ (\alpha, \beta) \in \Omega : 1 = \beta < \alpha \leq 2 \}\),

\(\Omega_5 := \{ (\alpha, \beta) \in \Omega : \beta \leq \min\{1, \alpha\}\} \setminus \Omega_4\).
Obviously, these sets are pairwise disjoint and their union covers \( \Omega \) (see Fig. 1).

**Proposition 2.2.** Let \( \Sigma = \{ \Phi_t \}_{t \geq 0} \subset \text{Hol}(\Pi) \) be a semigroup generated by \( \phi \in G_{\alpha, \beta}(\Pi) \).

(i) If \((\alpha, \beta) \in \Omega_1\), then all the trajectories of \( \Sigma \) have the same asymptote. This asymptote passes through the point \(-1\).

(ii) If \((\alpha, \beta) \in \Omega_2\), then each trajectory has its own asymptote. The asymptote depends on the initial point.

(iii) If \((\alpha, \beta) \in \Omega_3\), then the only trajectory \( \gamma \) defined by the condition \( \text{Im} \sigma_1 |_\gamma = 0 \) has an asymptote. This asymptote passes through the point \(-1\).

(iv) If \((\alpha, \beta) \in \Omega_4\), then all the trajectories of \( \Sigma \) have the same asymptote. This asymptote passes through the point \(-1\) if and only if \( \text{Im} \left( BA^{-\alpha+\beta} \right) = 0 \).

(v) If \((\alpha, \beta) \in \Omega_5\) and \( \text{Im} \left( BA^{-\alpha+\beta} \right) = 0 \), then all the trajectories have the same asymptote. This asymptote passes through the point \(-1\).

(vi) If \((\alpha, \beta) \in \Omega_5\) and \( \text{Im} \left( BA^{-\alpha+\beta} \right) \neq 0 \), no trajectory of \( \Sigma \) has an asymptote.

**Proof.** The problem reduces to an examination of the limit
\[
\lim_{t \to \infty} \text{Im} \left( \frac{1}{\alpha} (\Phi_t(w) + 1) \right).
\]
Indeed, the trajectory $\{\Phi_t(w)\}_{t \geq 0}$ has an asymptote if and only if this limit exists finitely. Moreover, if this limit vanishes, the asymptote passes through the point $-1$. We determine the existence of this limit and its value (if it it exists) using asymptotic expansions (2.7), (2.12) and (2.13).

For the case $\beta < \alpha$, using formula (2.7) from Theorem 2.1 we obtain

$$
\lim_{t \to \infty} \text{Im} \left( \lambda^\frac{\beta}{\alpha} (\Phi_t(w) + 1) \right) = \lim_{t \to \infty} \frac{|\lambda|^\frac{\beta}{\alpha} t^\frac{\beta-1}{\alpha}}{\alpha - \beta} \text{Im} \left( \mu \lambda^\frac{\beta}{\alpha} \right).
$$

The limit on the right vanishes for all pairs $(\alpha, \beta)$ with $1 < \beta < \alpha$; hence the asymptote exists and passes through $-1$. If $1 = \beta < \alpha$, the same limit exists and the asymptote passes through the point $\frac{|\lambda|^\frac{\beta}{\alpha}}{\alpha - \beta} \text{Im} \left( \mu \lambda^\frac{\beta}{\alpha} \right) - 1$. Finally, if $\beta < \min\{1, \alpha\}$, then an asymptote exists if and only if $\text{Im} \mu \lambda^\frac{\beta}{\alpha} = 0$, and if it does, it passes through $-1$. This proves assertion (iv) and parts of assertions (i) and (v). The remaining parts of assertions (i) and (v) as well as assertions (ii), (iii) and (iv) follow from a similar argument using Theorems 2.2 [2.3]. ■

The particular case of assertion (v) for $\alpha = \beta = 1$ was treated in [10, Theorem 4.2(a),(b)].

Another interesting issue is to estimate how far are two trajectories of the same semigroup having different initial points. The theorems above immediately imply the following.

**Corollary 2.1.** Let $\Sigma = \{\Phi_t\}_{t \geq 0} \in \text{Hol}(\Pi)$ be a semigroup generated by $\phi \in G_{\alpha,\beta}(\Pi)$. For all $w \in \Pi$,

(i) if $\beta = \alpha$, then $\lim_{t \to \infty} t^{\frac{1}{\alpha} - \frac{1}{\alpha}} (\Phi_t(w) - \Phi_t(1)) = 0$;

(ii) if $\beta < \alpha$, then $\lim_{t \to \infty} t^{\frac{\beta-1}{\alpha}} (\Phi_t(w) - \Phi_t(1)) = 0$;

(iii) if $\beta > \alpha$, then $\lim_{t \to \infty} \left( t^{\frac{\beta-1}{\alpha}} (\Phi_t(w) - \Phi_t(1)) - \frac{1}{\alpha} \lambda^\frac{\beta}{\alpha} t^\frac{\beta-1}{\alpha} \sigma(w) \right) = 0$.

In turn, Corollary 2.1 yields a simple description of the relative position of the semigroup trajectories going to the Denjoy–Wolff point at infinity. To formulate it we introduce the following notions.
Definition 2.1. Let $\Sigma = \{\Phi_t\}_{t \geq 0} \subset \text{Hol}(\Pi)$ be a semigroup with the Denjoy–Wolff point at infinity. For $w_1, w_2 \in \Pi$, let

$$s(w_1, w_2) := \lim_{t \to \infty} (\Phi_t(w_1) - \Phi_t(w_2))$$

whenever the limit exists. We say that the semigroup trajectories are

(i) mutually convergent if $s \equiv 0$ on $\Pi \times \Pi$,

(ii) asymptotically parallel if $s$ is well defined on $\Pi \times \Pi$ and does not vanish on $\Pi \times \Pi \setminus \{w_1 = w_2\}$,

(iii) mutually divergent if $s(w_1, w_2) = \infty$ for all $w_1 \neq w_2$.

Note that if the trajectories are mutually convergent, then for every compact $K \subset \Pi$ and $\varepsilon > 0$ there exists $t_0$ such that for each $t > t_0$, the set $\{\Phi_t(w), w \in K\}$ is contained in a disk of radius $\varepsilon$.

Corollary 2.2 (see Fig. 2). Under conditions of Corollary 2.1, the following assertions hold.

(i) If $\alpha > 1$ and $\beta \geq 1$, then the trajectories of $\Sigma$ are mutually convergent.

(ii) If $\beta > \alpha = 1$, then all the trajectories of $\Sigma$ are asymptotically parallel. Moreover, the function $\frac{s(w_1, w_2)}{\sigma(w_1) - \sigma(w_2)}$ is constant.

(iii) If $\alpha < \min\{1, \beta\}$, then all the trajectories of $\Sigma$ are mutually divergent. In particular,

$$\lim_{t \to \infty} (\Phi_{t+1}(w) - \Phi_t(w)) = \infty \quad \text{for all} \quad w \in \Pi.$$

3. Semigroups on the unit disk

In this section, we study the asymptotic behavior of parabolic type semigroups. The conclusions derived in [11] and [10] are a specific case of the results below, which can be applied to a broader set of semigroups.

Let $S$ be a semigroup of holomorphic self-mappings of the open unit disk $\Delta$. Using the Cayley transform $C(z) = \frac{1+z}{1-z}$, we transfer the study of semigroups acting on $\Delta$ to that of those acting on $\Pi$. For a
given semigroup $S = \{F_t\}_{t \geq 0} \subset \text{Hol}(\Delta)$ with Denjoy–Wolff point $\tau = 1$, we construct the semigroup $\Sigma = \{\Phi_t\}_{t \geq 0} \subset \text{Hol}(\Pi)$ with Denjoy–Wolff point $\infty$ by the composition

$$\Phi_t(w) = C \circ F_t \circ C^{-1}(w).$$

(3.1)

Clearly, $\Phi_t \circ C(z) = C \circ F_t(z)$, and hence

$$\Phi_t(C(z)) + 1 = \frac{2}{1 - F_t(z)}.$$

(3.2)

If $S$ is continuous (hence, differentiable) in $t$, then so is $\Sigma$. Suppose that $p \in \text{Hol}(\Delta, \Pi)$ and $f(z) = (1 - z)^2 p(z)$, $z \in \Delta$, generates $S$. Differentiating $\Phi_t$ given by (3.1) at $t = 0^+$, we conclude that $\Sigma$ is generated by the mapping

$$\phi(w) = 2p \left( C^{-1}(w) \right).$$

(3.3)

Suppose, in addition, that $f$ is of the form (1.1) or (1.2). By formula (3.3), the function $\phi$ can be represented, respectively, by

$$\phi(w) = 2^\alpha a(w + 1)^{1-\alpha} + \rho(w)$$

(3.4)

with $\lim_{w \to \infty} \frac{\rho(w)}{(w + 1)^{1-\alpha}} = 0$, or by

$$\phi(w) = 2^\alpha a(w + 1)^{1-\alpha} + 2^{a+\beta} b(w + 1)^{1-\alpha-\beta} + \rho_1(w)$$

(3.5)
with \( \lim_{w \to \infty} \frac{\rho_1(w)}{(w+1)^{1-\alpha-\beta}} = 0 \). Thus, \( \phi \) has the form (2.1) or (2.5), respectively, with
\[
A = 2^\alpha a \quad \text{and} \quad B = 2^{\alpha+\beta} b. 
\tag{3.6}
\]
We also use (3.6) and modify (2.6) to \( \lambda = 2^\alpha \alpha a \) and \( \mu = 2^\beta \frac{b}{a} \).

The Cayley transform allows us to apply the results of the previous section for semigroups acting on \( \Pi \) to study semigroups acting on \( \Delta \). The next result is a generalization of Theorem 1.4(i) in [10].

**Proposition 3.1.** Let \( S = \{F_t\}_{t \geq 0} \) be a semigroup of holomorphic self-mappings of \( \Delta \) generated by
\[
f(z) = a(1-z)^{1+\alpha} + R(z), \tag{3.7}
\]
where \( R \in \text{Hol}(\Delta, \mathbb{C}) \), \( \lim_{z \to 1} \frac{R(z)}{(1-z)^{1+\alpha}} = 0 \). Then
\[
\frac{1}{1 - F_t(z)} = \frac{1}{2} (\lambda t)^{\frac{1}{\alpha}} + r(z, t) \quad \text{with} \quad \lim_{t \to \infty} t^{-\frac{1}{\alpha}} r(z, t) = 0 \tag{3.8}
\]
and
\[
\lim_{t \to \infty} \left( \frac{1}{(1 - F_t(z))^{\alpha}} - \frac{1}{(1 - F_t(0))^{\alpha}} \right) = \frac{\lambda h(z)}{2^\alpha},
\]
where \( h \) is the Kœnigs function defined by (1.5).

**Proof.** Substituting (3.2) into formula (2.2) yields
\[
\frac{1}{1 - F_t(z)} = \frac{1}{2} (\lambda t)^{\frac{1}{\alpha}} + r(z, t),
\]
with \( r(z, t) = \frac{1}{2} \Gamma(C^{-1}(z), t) \) and \( \lim_{t \to \infty} t^{-\frac{1}{\alpha}} r(z, t) = 0 \).

As in the proof of (2.3),
\[
\lim_{t \to \infty} ((\Phi_t(w) + 1)^{\alpha} - (\Phi_t(1) + 1)^{\alpha}) = \lim_{t \to \infty} \int_1^w ((\Phi_t(z) + 1)^{\alpha})' \, dz
\]
\[
= \lim_{t \to \infty} \int_1^w \frac{\alpha}{\phi(z)} \left( A + \frac{\theta(\Phi_t(z))}{\Phi_t^{1-\alpha}(z)} \right) \, dz = \lambda \int_1^w \frac{dz}{\phi(z)} = \lambda \sigma(w)
\]
with \( \sigma(w) = h(C^{-1}(w)) \). Again from formula (3.2), we conclude that
\[
\lim_{t \to \infty} \left( \frac{1}{(1 - F_t(z))^{\alpha}} - \frac{1}{(1 - F_t(0))^{\alpha}} \right) = \frac{\lambda h(z)}{2^\alpha}.
\]
Using Theorems 2.1–2.3, we can deduce the following asymptotic representation of parabolic type semigroups for all possible pairs \((\alpha, \beta) \in \Omega\) (for \(\alpha = \beta = 1\), cf., assertion (ii) of Theorem 1.4 in [10]).

**Theorem 3.1.** Let \(S = \{F_t\}_{t \geq 0}\) be a semigroup of holomorphic self-mappings of \(\Delta\) and let \(f \in G_{\alpha,\beta}(\Delta)\) be its generator.

(i) If \(0 < \beta < \alpha\), then
\[
\frac{1}{1 - F_t(z)} = \frac{1}{2} (\lambda t)^{\frac{1}{\alpha}} \left( 1 + \frac{\mu}{\alpha - \beta} (\lambda t)^{-\frac{\beta}{\alpha}} + r_1(z, t) \right)
\]
with \(\lim_{t \to \infty} t^{-\beta} r_1(z, t) = 0\).

(ii) If \(\beta = \alpha\), then
\[
\frac{1}{1 - F_t(z)} = \frac{1}{2} (\lambda t)^{\frac{1}{\alpha}} \left( 1 + \frac{\mu}{\alpha} \cdot \frac{\log(t + 1)}{\lambda t} + r_1(z, t) \right)
\]
with \(\lim_{t \to \infty} \frac{r_1(z, t)t}{\log(t + 1)} = 0\).

(iii) If \(\beta \neq \alpha\) and \(k\alpha \leq \beta < (k + 1)\alpha\) for some \(k \in \mathbb{N}\), then
\[
\frac{1}{1 - F_t(z)} = \frac{1}{2} (\lambda t)^{\frac{1}{\alpha}} \left[ 1 + \sum_{j=1}^{k} \left( \frac{1}{j} \right) \left( \frac{h_1(z)}{t} \right)^j \right]
\]
\[
+ \frac{\mu}{\alpha - \beta} (\lambda t)^{-\frac{\beta}{\alpha}} + r_1(z, t)
\]
with \(\lim_{t \to \infty} t^{-\beta} r_1(z, t) = 0\) and
\[
h_1(z) = h(z) + \frac{2^\alpha}{\lambda} - \int_0^1 \left( h'(s) - \frac{1}{a(1 - s)^{\alpha + 1}} \right) ds.
\]

**Proof.** To prove these assertions, we use Theorems 2.1–2.3. Substituting formulas (3.2) and (3.6) into (2.7), (2.12) and (2.13) gives assertions (i), (ii) and (iii), respectively. To complete the proof, we only note that (3.6) implies that \(\lambda = \alpha \lambda\) and \(\mu = \frac{B}{A} = \frac{2^b}{\alpha}\), and that the relation for the
Königs functions $\sigma(w) = h\left(\frac{w+1}{w-1}\right)$ implies $
abla \int \frac{1}{1} \left(\sigma'(v) - \frac{1}{4}(v+1)^{3-1}\right) dv = \int_0^1 \left(h'(s) - \frac{1}{a(1-s)^{\alpha+1}}\right) ds.$

Theorem 3.1 not only provides a specification of the asymptotic behavior of semigroups, but also enables us to study the local geometry of semigroup trajectories in more detail. As already mentioned, Proposition 3.1 (see also [11]) implies that all trajectories of a semigroup generated by a function of the form (3.7) have the same limit tangent line. A more detailed analysis requires the following notion.

**Definition 3.1.** Let $\gamma, \gamma^* : [0, \infty) \mapsto \Delta$ be smooth disjoint curves which satisfy $\lim_{t \to \infty} \gamma(t) = \lim_{t \to \infty} \gamma^*(t) = 1$. Denote by $d(t)$ the distance between $\gamma(t)$ and $\gamma^*$. We say that the contact order between $\gamma$ and $\gamma^*$ (at the point 1) is $\kappa$ ($\kappa \geq 0$), if the limit

$$\lim_{t \to \infty} \frac{d(t)}{|1 - \gamma(t)|^{1+\kappa}}$$

exists finitely and is different from zero. If this limit is zero, we say that the contact order is greater than $\kappa$. In the case $\gamma^*$ is the limit tangent line of $\gamma$, instead of “contact order between $\gamma$ and $\gamma^*$” we say “contact order of $\gamma$”.

Note that the existence of the limit tangent line guarantees that the contact order is greater than zero, while the contact order of a curve $\gamma$ is equal to or greater than 1 if and only if $\gamma$ has a finite limit curvature.

**Theorem 3.2.** Let $S = \{F_t\}_{t \geq 0}$ be a semigroup of holomorphic self-mappings of $\Delta$ whose generator $f$ is in $G_{\alpha,\beta}(\Delta)$.

(i) If $0 < \beta < \alpha$, then the contact order of all the trajectories is at least $\beta$. In the case $\text{Im} \left(\lambda^{-\frac{\alpha}{\beta}} \mu\right) \neq 0$, this order equals $\beta$.

(ii) If $\beta = \alpha$, then

$$\lim_{t \to \infty} \frac{d(t)}{|1 - F_t(z)|^{1+\alpha} \log |1 - F_t(z)|} = \left|\frac{\lambda^{\frac{\alpha}{\beta}}}{2a}\right| \cdot \text{Im} \left(\frac{\mu}{\alpha \lambda}\right):$$
hence, for any \( \varepsilon > 0 \), the contact order of all the trajectories is greater than \( \alpha - \varepsilon \).

(iii) If \( \beta > \alpha \), then for each \( z \in \Delta \) such that \( \text{Im} \ h_1(z) \neq 0 \), the trajectory passing through \( z \) has contact order \( \alpha \).

For the trajectory \( \gamma \) defined by \( \text{Im} \ h_1|_{\gamma} = 0 \), the contact order is at least \( \beta \). In particular, if \( \text{Im} \left( \lambda - \frac{\alpha}{\mu} \right) \neq 0 \), the contact order is \( \beta \).

Proof. By Proposition 3.1

\[ F_t(z) = 1 - \frac{1}{\frac{1}{2} (\lambda t)^{\frac{1}{\alpha}} + r(z, t)} \quad \text{with} \quad \lim_{t \to \infty} t^{-\frac{1}{\alpha}} r(z, t) = 0 \]

and \( \lim_{t \to \infty} t^{\frac{1}{\alpha}} (1 - F_t(z)) = 2 \lambda^{-\frac{1}{\alpha}} \) (see also [11]). Therefore, all the trajectories have the common limit tangent line \( \ell = \{ z = 1 - \frac{2x}{\lambda^\frac{1}{\alpha}}, x \in \mathbb{R} \} \).

Following Definition 3.1, given a point \( z \in \Delta \), we denote the distance between \( F_t(z) \) and \( \ell \) by \( d(t) \). Standard analysis yields

\[ d(t) = \left| \text{Im} \left( \frac{\lambda^\frac{1}{\alpha} r(z, t)}{\lambda^\frac{1}{\alpha}} \right) \right| \cdot \left| \lambda^\frac{1}{\alpha} \right| \cdot \left| \lambda^\frac{1}{\alpha} \right| \cdot \left| r(z, t) \right| \]

so that

\[ d(t) = \left| \lambda^\frac{1}{\alpha} \right| \cdot \left| \text{Im} \left( \frac{r(z, t)}{\lambda^\frac{1}{\alpha}} \right) \right| \cdot |1 - F_t(z)|^2. \quad (3.10) \]

Let \( 0 < \beta < \alpha \). Theorem 3.1 (i) implies that

\[ r(z, t) = \left( \frac{\lambda^\frac{1}{\alpha} r(z, t)}{\lambda^\frac{1}{\alpha}} \right) \cdot \left( \frac{\mu}{\alpha - \beta} + r_1(z, t) \right), \]

where \( \lim_{t \to \infty} r_1(z, t) = 0 \). Hence

\[ d(t) = \frac{1}{2} \left| \lambda^\frac{1}{\alpha} \right| \cdot \left| \text{Im} \left( \lambda^{-\frac{\alpha}{\beta}} \left( \frac{\mu}{\alpha - \beta} + r_1(z, t) \right) \right) \right| \cdot t^{-\frac{1}{\alpha}} |1 - F_t(z)|^2 \]

and

\[ \lim_{t \to \infty} \frac{d(t)}{|1 - F_t(z)|^{1+\beta}} = \frac{1}{2} \left| \lambda^\frac{1}{\alpha} \right| \cdot \left| \text{Im} \left( \lambda^{-\frac{\alpha}{\beta}} \frac{\mu}{\alpha - \beta} \right) \right|. \]

Assertion (i) follows.
Let $\beta = \alpha$. According to Theorem 3.1 (ii),

$$r(z, t) = \frac{1}{2} \lambda^{\frac{\alpha}{\alpha}} t^{\frac{1}{\alpha} - 1} \log(t + 1) \left( \frac{\mu}{\alpha} + r_1(z, t) \right)$$

with $\lim_{t \to \infty} r_1(z, t) = 0$. By formula (3.10),

$$d(t) = \frac{1}{2} \left| \lambda^{\frac{\alpha}{\alpha}} \right| \cdot \text{Im} \left( \frac{\mu}{\alpha \lambda} + r_1(z, t) \right) \cdot t^{\frac{1}{\alpha} - 1} \log(t + 1) |1 - F_t(z)|^2.$$

Furthermore,

$$\lim_{t \to \infty} \frac{\log(t + 1)}{\log(1 - F_t(z))} = \lim_{t \to \infty} \frac{(1 - F_t(z))^{1+\alpha}}{(t + 1)(1 - F_t(z))^{\alpha f(F_t(z))}} = \frac{\lambda}{2^\alpha a},$$

and consequently

$$\lim_{t \to \infty} \frac{d(t)}{|1 - F_t(z)|^{1+\alpha} \log |1 - F_t(z)|} = \frac{\lambda^{\frac{\alpha}{\alpha}}}{2^\alpha a} \cdot |\text{Im} \left( \frac{\mu}{\alpha \lambda} \right)|,$$

which implies assertion (ii).

Let us turn to the case $\beta > \alpha$. As above, $\beta \in [k\alpha, (k + 1)\alpha)$ for some $k \in \mathbb{N}$. By Theorem 3.1 (iii),

$$r(z, t) = \frac{\lambda^{\frac{\alpha}{\alpha}} t^{\frac{1}{\alpha} - 1}}{2} \left[ \sum_{j=1}^{k} \left( \frac{1}{\alpha} \right)^j (h_1(z))^j t^{1-j} + \frac{\mu \lambda^{\alpha} t^{1-\frac{\alpha}{\alpha}}}{\alpha - \beta} + r_1(z, t) \right],$$

where $\lim_{t \to \infty} t^{\frac{1}{\alpha} - 1} r_1(z, t) = 0$. Substituting this into (3.10) yields

$$d(t) = \left| \text{Im} \left( \sum_{j=1}^{k} \left( \frac{1}{\alpha} \right)^j (h_1(z))^j t^{1-j} + \frac{\mu \lambda^{\alpha} t^{1-\frac{\alpha}{\alpha}}}{\alpha - \beta} + r_1(z, t) \right) \right|$$

$$\cdot \frac{1}{2} \left| \lambda^{\frac{\alpha}{\alpha}} \right| t^{\frac{1}{\alpha} - 1} |1 - F_t(z)|^2.$$

For each $z \in \Delta$, there are now two possibilities. One is that $\text{Im} h_1(z) \neq 0$. In this case,

$$\lim_{t \to \infty} \frac{d(t)}{|1 - F_t(z)|^{1+\alpha}} = \frac{\left| \lambda^{\frac{\alpha}{\alpha}} \right| |\text{Im} h_1(z)|}{2^\alpha}.$$
from which it follows that
\[
\lim_{t \to \infty} \frac{d(t)}{|1 - F_t(z)|^{1+\beta}} = \frac{|\lambda^{\frac{\alpha}{\beta}}|}{2^\beta (\beta - \alpha)} \cdot |\text{Im} \left( \mu \lambda^{-\frac{\alpha}{\beta}} \right)|.
\]
This implies assertion (iii).

**Remark 3.1.** Theorem 3.2 shows that the manner of approaching of different trajectories to their common limit tangent line essentially depends on the relation between \(\alpha\) and \(\beta\). For instance, if \(\beta < \alpha\) and \(\text{Im} \left( \lambda^{-\frac{\alpha}{\beta}} \mu \right) \neq 0\), then by assertion (i) of Theorem 3.2, all the trajectories have the same contact order. If \(\beta > \alpha\), we see another phenomenon: by assertion (iii) of Theorem 3.2, there exists a unique trajectory of maximal contact order. This has an interesting geometric consequence. The trajectory \(\gamma\) which is the pre-image of the real half-axis under \(h_1\) has (by Theorem 3.2) contact order at least \(\beta\). Suppose that \(\gamma\) is disjoint from the limit tangent line \(\ell\). Since all other trajectories have order \(\alpha < \beta\), each trajectory starting from a point between \(\gamma\) and \(\ell\) intersects \(\ell\), and for large enough \(t\), lies on the opposite side of \(\ell\).

**Example 3.1.** Consider the semigroup \(S\) generated by \(f(z) = \frac{(1 - z)^2}{4 + i(1 - z)^2}\).
Since \(a = \frac{1}{4}\) is real, the limit tangent line \(\ell\) coincides with the real axis. A direct calculation yields \(h_1(z) = 8 + iz + \frac{4z}{1 - z}\). Hence, the trajectory defined by
\[
\gamma := \left\{ z \in \Delta : \text{Im} \left( i + iz + \frac{4z}{1 - z} \right) = 0 \right\}
\]
has maximal contact order, and each trajectory starting from a point between \(\gamma\) and the real axis intersects the real axis and eventually lies below it (see Fig. 3).

Another implication of Theorem 3.2 and Proposition 2.2 relates to the limit curvature of the semigroup trajectories. Namely, a trajectory \(\gamma_z = \{F_t(z), \ t \geq 0\} \subset \Delta\) has a finite limit curvature if and only if its image \(C \circ \gamma_z\) under the Cayley transform has an asymptote (cf., [10]).
Moreover, if that asymptote passes through the point $-1$, then the limit curvature vanishes. This implies the following.

**Corollary 3.1.** Let $S = \{F_t\}_{t\geq 0}$ be a semigroup of holomorphic self-mappings of $\Delta$ generated by $f \in G_{\alpha,\beta}(\Delta)$. Let $\{\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5\}$ be the partition of $\Omega = \{(\alpha, \beta) : 0 < \alpha \leq 2, \beta > 0\}$ as in Proposition 2.2.

(i) If $(\alpha, \beta) \in \Omega_1$, then all the trajectories of $S$ have null limit curvature.

(ii) If $(\alpha, \beta) \in \Omega_2$, then each trajectory has a finite limit curvature (distinct for different trajectories).

(iii) If $(\alpha, \beta) \in \Omega_3$, then the trajectory $\gamma$ defined by the condition $\text{Im} h_1|_{\gamma} = 0$ is the only trajectory which has a finite limit curvature. Moreover, this curvature vanishes.

(iv) If $(\alpha, \beta) \in \Omega_4$, then all the trajectories of $\Sigma$ have the same finite limit curvature. Moreover, this curvature vanishes if and only if $\text{Im}(\mu \lambda^{-\frac{\beta}{\alpha}}) = 0$.

(v) If $(\alpha, \beta) \in \Omega_5$, then there is the following dichotomy: in the case $\text{Im}(\mu \lambda^{-\frac{\beta}{\alpha}}) = 0$, all the trajectories have null limit curvature; in the case $\text{Im}(\mu \lambda^{-\frac{\beta}{\alpha}}) \neq 0$, the limit curvature is infinite.

If the remainder in the asymptotic expansion of the generator tends to zero faster than in (1.3), the asymptotics of a semigroup can be estimated.
more precisely. The next result, which generalizes [10, Theorem 1.4(iii)], follows from (3.1)–(3.6) and transforming Proposition 2.1.

**Proposition 3.2.** Let \( S = \{F_t\}_{t \geq 0} \) be a semigroup of holomorphic self-mappings of \( \Delta \) whose generator \( f \) has the form

\[
f(z) = a(1 - z)^{1+\alpha} + b(1 - z)^{1+\alpha+\beta} + R_1(z),
\]

where \( R_1 \in \text{Hol}(\Delta, \mathbb{C}) \) satisfies \( \lim_{z \to 1} \frac{R_1(z)}{(1 - z)^{2\alpha + 1 + \varepsilon}} = 0 \) for some positive \( \varepsilon \).

(i) If \( \beta = \alpha \), then there exists a constant \( C \) such that for all \( z \in \Delta \),

\[
\left( \frac{2}{1 - F_t(z)} \right)^\alpha = \lambda t + \mu \log(t + 1) + \lambda h(z) + C + r(z, t),
\]

where \( \lim_{t \to 0} r(z, t) = 0 \).

(ii) If \( \alpha < \beta < \alpha \), then there exists a constant \( C \) such that for all \( z \in \Delta \),

\[
\left( \frac{2}{1 - F_t(z)} \right)^\alpha = \lambda t + \frac{\mu \alpha}{\alpha - \beta} (\lambda t)^{1 - \frac{\beta}{\alpha}} + \lambda h(z) + C + r(z, t),
\]

where \( \lim_{t \to \infty} r(z, t) = 0 \).

4. Rigidity via order of contact

In this section, we consider two semigroups \( S = \{F_t\}_{t \geq 0} \) and \( S^* = \{F^*_t\}_{t \geq 0} \) acting on \( \Delta \). Let \( f \) be the generator of \( S \) and \( f^* \) the generator of \( S^* \). Suppose that both \( f \) and \( f^* \) can be represented by (1.2). For \( z_1, z_2 \in \Delta \), let \( \mathcal{F} = (\{F_t(z_1), t \geq 0\}, \{F^*_t(z_2), t \geq 0\}) \) be the pair of semigroups trajectories. We study the following question: how close can the trajectories of \( S \) and \( S^* \) become? Naturally, this question includes the rigidity problem, i.e., that of determining conditions which ensure that these semigroups coincide. For this study we need a modification of Definition 3.1.
Definition 4.1. Let semigroups $S = \{F_t\}_{t \geq 0}$, $S^* = \{F_t^*\}_{t \geq 0}$ (not necessarily different) have generators of the form $a(1-z)^{1+\alpha} + R(z)$ with $\lim_{t \to \infty} \frac{R(z)}{(1-z)^{1+\alpha}} = 0$. Let $z_1, z_2 \in \Delta$. We say that the parameter-related contact order of $\mathcal{F} = (\{F_t(z_1), \ t \geq 0\}, \{F_t^*(z_2), \ t \geq 0\})$ is greater than $\kappa \geq 0$, if

$$\lim_{t \to \infty} \frac{|F_t(z_1) - F_t^*(z_2)|}{|1 - F_t(z_1)|^{\kappa+1}} = 0.$$ 

By Proposition 3.1, for all $z \in \Delta$, the limits

$$\lim_{t \to \infty} t|1 - F_t(z)|^\alpha \quad \text{and} \quad \lim_{t \to \infty} t|1 - F_t^*(z)|^\alpha$$

exist and are finite and nonzero. Therefore, this definition is symmetric relative to $S$ and $S^*$. Obviously, $|F_t(z_1) - F_t^*(z_2)|$ is greater than the distance between $F_t(z_1)$ and the trajectory $\{F_s^*(z_2), s \geq 0\}$. Hence, if the parameter-related contact order of $\mathcal{F}$ is greater than $\kappa$, then the contact order between them in a regular sense (see Definition 3.1) is also greater than $\kappa$.

We consider the two particularly important cases: $S = S^*$ and $z_1 = z_2$. Regarding the case $S = S^*$, it is easy to see from Corollary 2.1 that

- if $\beta = \alpha$, then $\lim_{t \to \infty} \frac{F_t(z) - F_t(0)}{\log(t+1)(1-F_t(z))^{1+\alpha}} = 0$;
- if $\beta < \alpha$, then $\lim_{t \to \infty} \frac{F_t(z) - F_t(0)}{(1-F_t(z))^{1+\beta}} = 0$;
- if $\beta > \alpha$, then $\lim_{t \to \infty} t^{\beta\alpha-1} \left( \frac{F_t(z) - F_t(0)}{(1-F_t(z))^{1+\alpha}} - \frac{\lambda z \sigma(w)}{\alpha} \right) = 0$.

By Definition 4.1, this implies the following fact.

Proposition 4.1. Let $S = \{F_t\}_{t \geq 0} \subset \text{Hol}(\Delta)$ be a semigroup generated by a mapping $f \in G_{\alpha,\beta}(\Delta)$, and let $z_1, z_2 \in \Delta$.

(i) If $\beta < \alpha$, then the parameter-related contact order of $\mathcal{F}$ is greater than $\beta$.

(ii) If $\beta \geq \alpha$, then for any $\varepsilon > 0$, the parameter-related contact order of $\mathcal{F}$ is greater than $\alpha - \varepsilon$.
Remark 4.1. Just from the triangle inequality, it follows that the parameter-related contact order of each pair of trajectories of the same semigroup cannot be less than the contact order of any one of them with the limit tangent line. Comparing Proposition 4.1 with Theorem 3.2, we see that if $0 < \beta < \alpha$ and $\text{Im} \left( \lambda^\beta \mu \right) \neq 0$, then the parameter-related contact order of two trajectories is actually greater than the contact order of any of them with the limit tangent line. Roughly speaking, this means that each trajectory is closer to all other trajectories than to the tangent. This can be the case only when all trajectories approach their common limit tangent line from the same side.

Example 4.1. Consider $f \in G_{\alpha,\beta}(\Delta)$ defined by $f(z) := (1 - z)^2 + i(1 - z)^{2.5}$, i.e., $\alpha = 1$, $\beta = 0.5 < \alpha$, $a = 1$ and $b = i$. Since $\text{arg} a = 0$, the limit tangent line coincides with the real axis. In addition, $\lambda = 2$ and $\mu = i\sqrt{2}$, so that $\text{Im} \left( \lambda^{-\frac{\beta}{\alpha}} \mu \right) = 1 \neq 0$. Fig. 4 shows the direction field in the part of $\Delta$ bounded by $0.75 < \text{Re} z < 1$ and $-0.12 < \text{Im} z < 0.03$. All trajectories approach the real axis from the upper half-plane.

We now turn to the case $z_1 = z_2$. We are interested in applying contact order to the rigidity problem.

Theorem 4.1. Let $S$ be a semigroup generated by a mapping $f \in G_{\alpha,\beta}(\Delta)$ with $\beta \leq \alpha$, and let $S^*$ be a semigroup generated by $f^*(z) = f(z) +$
c(1 − z)^{1+\alpha+\beta}. If for some \( z \in \Delta \), the parameter-related contact order of \( \mathfrak{F} = \{ F_t(z), \ t \geq 0 \}, \{ F_t^*(z), \ t \geq 0 \} \) is greater than \( \beta \), then \( c = 0 \); so the semigroups coincide.

Proof. It follows from Proposition 3.1 that
\[
\lim_{t \to \infty} t |1 - F_t(z)|^\alpha = \lim_{t \to \infty} t |1 - F_t^*(z)|^\alpha = \frac{2^{\alpha}}{\lambda}. \tag{4.1}
\]
Let \( \beta < \alpha \). Consider the quotient
\[
\frac{F_t(z) - F_t^*(z)}{(1 - F_t(z))^{1+\beta}} = \left( \frac{1}{1 - F_t(z)} - \frac{1}{1 - F_t^*(z)} \right) \cdot \frac{1 - F_t^*(z)}{1 - F_t(z)} \cdot \frac{1}{(1 - F_t(z))^{\beta-1}}.
\]
Formula (4.1) implies that the last two factors have finite nonzero limits. In addition, by Theorem 3.1 (i),
\[
\frac{1}{1 - F_t(z)} = \frac{(\lambda t)^{\frac{1}{\alpha}}}{2} \left( 1 + \frac{\mu}{\alpha - \beta} (\lambda t)^{\frac{\beta}{\alpha}} + r_1(z,t) \right)
\]
and
\[
\frac{1}{1 - F_t^*(z)} = \frac{(\lambda t)^{\frac{1}{\alpha}}}{2} \left( 1 + \frac{\mu^*}{\alpha - \beta} (\lambda t)^{\frac{\beta}{\alpha}} + r_1^*(z,t) \right),
\]
where \( \mu^* = \frac{2^{\beta(b+c)}}{a} \) and \( \lim_{t \to \infty} t^{\frac{\beta}{\alpha}} r_1(z,t) = \lim_{t \to \infty} t^{\frac{\beta}{\alpha}} r_1^*(z,t) = 0 \). Therefore,
\[
\frac{1}{1 - F_t(z)} - \frac{1}{1 - F_t^*(z)} = \frac{(\lambda t)^{\frac{1-\beta}{\alpha}}}{2} \left( \frac{\mu - \mu^*}{\alpha - \beta} + (\lambda t)^{\frac{\beta}{\alpha}} (r_1(z,t) - r_1^*(z,t)) \right). \tag{4.2}
\]
Thus, if the parameter-related contact order of the pair \( \mathfrak{F} \) is greater than \( \beta \), then \( \mu - \mu^* = 0 \), and the assertion follows.

The case \( \beta = \alpha \) can be treated analogously using assertion (ii) of Theorem 3.1. \( \blacksquare \)

Our next result concerns the rigidity problem in the case \( \beta > \frac{\alpha}{2} \). We prove the coincidence of semigroups under an essentially weaker local condition.
Theorem 4.2. Let semigroups $S$ and $S^*$ be generated, respectively, by

$$f(z) = a(1 - z)^{1+\alpha} + b(1 - z)^{1+\alpha+\beta} + R_1(z)$$

and

$$f^*(z) = a(1 - z)^{1+\alpha} + b^*(1 - z)^{1+\alpha+\beta} + R_1^*(z)$$

with $a \neq 0$ and $R_1, R_1^* \in \operatorname{Hol}(\Delta, \mathbb{C})$. Suppose that either

(i) $0 < \beta < \alpha$ and $\lim_{z \to 1} \frac{R_1(z)}{(1 - z)^{1+\alpha+\beta}} = \lim_{z \to 1} \frac{R_1^*(z)}{(1 - z)^{1+\alpha+\beta}} = 0$ for some positive $\varepsilon$, or

(ii) $\beta > \alpha$ and $\lim_{z \to 1} \frac{R_1(z)}{(1 - z)^{1+\alpha+\beta}} = \lim_{z \to 1} \frac{R_1^*(z)}{(1 - z)^{1+\alpha+\beta}} = 0$.

If there exist $\theta \in [0, 2\pi]$ and an open set $U \subset \Delta$ such that

$$\lim_{t \to \infty} t^{1+\frac{1}{\alpha}} \operatorname{Re} e^{i\theta} (F_t(z) - F_t^*(z)) = 0 \quad (4.3)$$

for all $z \in U$, then the two semigroups coincide.

Proof. Consider the case $0 < \beta < \alpha$. By our assumptions, both semigroups satisfy the asymptotic expansion (3.9). Since

$$\frac{1}{1 - F_t(z)} - \frac{1}{1 - F_t^*(z)} = \frac{F_t(z) - F_t^*(z)}{(1 - F_t(z))(1 - F_t^*(z))},$$

the parameter-related contact order of $\mathfrak{F} = (\{F_t(z), \ t \geq 0\}, \{F_t^*(z), \ t \geq 0\})$ is positive by formula (4.2). Furthermore,

$$\left(\frac{2}{1 - F_t(z)}\right)^\alpha - \left(\frac{2}{1 - F_t^*(z)}\right)^\alpha$$

$$= \left(\frac{2}{1 - F_t(z)}\right)^\alpha \cdot \frac{F_t^*(z) - F_t(z)}{1 - F_t(z)} \cdot \frac{1 - \left(1 + \frac{F_t^*(z) - F_t(z)}{1 - F_t(z)}\right)^\alpha}{\frac{F_t^*(z) - F_t(z)}{1 - F_t(z)}},$$

where the last factor tends to $-\alpha$ as $t \to \infty$. On the other hand, by Proposition 3.2,

$$\left(\frac{2}{1 - F_t(z)}\right)^\alpha - \left(\frac{2}{1 - F_t^*(z)}\right)^\alpha$$

$$= \frac{(\mu - \mu^*)^\alpha}{\alpha - \beta} (\lambda t)^{1 - \frac{\beta}{\alpha}} + \lambda (h(z) - h^*(z)) + C^* + r(z, t) - r^*(z, t),$$
where \(C, C^*\) are constants and \(h, h^*\) are the Kœnigs functions for \(S\) and \(S^*\), respectively.

Assume condition (4.3). Combining the last two displayed formulas with (3.8), we conclude that

\[
\lim_{t \to \infty} \Re \left( \frac{e^{i\theta}}{\lambda^{1+\frac{\alpha}{\alpha}}} \left( \frac{\mu - \mu^*}{\alpha - \beta} (\lambda t)^{1-\frac{\beta}{\alpha}} + \lambda (h(z) - h^*(z)) + C - C^* \right) \right) = 0.
\]

This is possible only if the coefficient of \(t^{1-\frac{\beta}{\alpha}}\) vanishes, in which case

\[
\Re \left( \frac{e^{i\theta}}{\lambda^{1+\frac{\alpha}{\alpha}}} (\lambda (h(z) - h^*(z)) + C - C^*) \right) = 0.
\]

Therefore, the function \(h(z) - h^*(z)\) is constant. Since \(h(0) = h^*(0) = 0\), we get \(h(z) = h^*(z)\). Now by (1.5), we conclude that \(f \equiv f^*\).

The case \(\beta > \alpha\) can be treated similarly. ■

Suppose now that (4.3) holds for all \(\theta \in [0, 2\pi]\). By Proposition 3.1, \(t^{1+\frac{\alpha}{\alpha}} \sim \frac{1}{(1 - F_1(z))^{\alpha+1}}\). Thus, we get the following consequence.

**Corollary 4.1.** If under conditions of Theorem 4.2, the parameter-related contact order of \(\mathfrak{F}\) is greater than \(\min(\alpha, \beta)\), then the semigroups \(S\) and \(S^*\) coincide.

Let return to the formulation of Theorem 4.2. It seems that the requirement on the remainders in assertion (i) is too strong and should be replaced by \(\lim_{z \to 1} \frac{R_1(z)}{(1 - z)^{1+\alpha+\beta}} = \lim_{z \to 1} \frac{R_1^*(z)}{(1 - z)^{1+\alpha+\beta}} = 0\), as in assertion (ii). Moreover, the rigidity condition (4.3) does not include \(\beta\) at all. These considerations lead to the following natural conjecture.

**Conjecture 1.** Let \(S\) and \(S^*\) be semigroups generated by functions of the form \(a(1 - z)^{1+\alpha} + R(z)\). If for some \(\varepsilon > 0\) the remainders are \(O((1 - z)^{1+\alpha+\varepsilon})\), then condition (4.3) implies the coincidence of the semigroups.
5. Appendix

We complete our analysis with assertions which give more information about the asymptotic behavior of semigroups but are different in nature.

**Proposition 5.1.** Let $\Sigma = \{\Phi_t\}_{t \geq 0} \subset \text{Hol}(\Pi)$ be a semigroup generated by $\phi \in G_{\alpha, \beta}(\Pi)$. Then

$$\lim_{t \to \infty} (t + 1)^{\frac{\beta}{\alpha}} ((\Phi_t(w) + 1)^{\alpha} - (\Phi_t(1) + 1)^{\alpha} - \lambda \sigma(w)) = \mu \lambda^{1 - \frac{\alpha}{\beta}} \sigma(w).$$

**Proof.** We just calculate the limit:

$$\lim_{t \to \infty} (t + 1)^{\frac{\beta}{\alpha}} ((\Phi_t(w) + 1)^{\alpha} - (\Phi_t(1) + 1)^{\alpha} - \lambda \sigma(w)) = \mu \lambda^{1 - \frac{\alpha}{\beta}} \sigma(w).$$

By Proposition 5.1, we conclude that

$$\lim_{t \to \infty} (t + 1)^{\frac{\beta}{\alpha}} ((\Phi_t(w) + 1)^{\alpha} - (\Phi_t(1) + 1)^{\alpha} - \lambda \sigma(w)) = \mu \lambda^{1 - \frac{\alpha}{\beta}} \sigma(w),$$

which completes the proof. ■

The particular case $\alpha = \beta = 1$ is contained in [10, Theorem 4.1(ii)].

Transferring, as above, Proposition 5.1 to semigroups acting in $\Delta$ yields the following result.

**Corollary 5.1.** Let $S = \{F_t\}_{t \geq 0}$ be a semigroup of holomorphic self-mappings of $\Delta$ generated by $f \in G_{\alpha, \beta}(\Delta)$. Then

$$\lim_{t \to \infty} (t + 1)^{\frac{\beta}{\alpha}} \left( \frac{1}{(1 - F_t(z))^\alpha} - \frac{1}{(1 - F_t(0))^\alpha} - \frac{\lambda h(z)}{2^\alpha} \right) = \mu \lambda^{1 - \frac{\alpha}{\beta}} \frac{h(z)}{2^\alpha}.$$
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