THE MUIRHEAD-RADO INEQUALITY, 2: SYMMETRIC MEANS AND INEQUALITIES

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Abstract. This paper gives elementary proofs of the Muirhead and Rado inequalities.

1. Symmetric means of monomial functions

A nonnegative vector is a vector with nonnegative coordinates. The nonnegative octant in $\mathbb{R}^n$ is

$$\mathbb{R}^n_{\geq 0} = \left\{ a = \left( \begin{array}{c} a_1 \\ \vdots \\ a_n \end{array} \right) \in \mathbb{R}^n : a_i \geq 0 \text{ for all } i \in \{1, \ldots, n\} \right\}.$$

A positive vector is a vector with positive coordinates. The positive octant in $\mathbb{R}^n$ is

$$\mathbb{R}^n_{> 0} = \left\{ x = \left( \begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \right) \in \mathbb{R}^n : x_i > 0 \text{ for all } i \in \{1, \ldots, n\} \right\}.$$

Every nonnegative vector $a$ defines the monomial function $x^a$ on the positive octant as follows: For $a = \left( \begin{array}{c} a_1 \\ \vdots \\ a_n \end{array} \right) \in \mathbb{R}^n_{\geq 0}$ and $x = \left( \begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \right) \in \mathbb{R}^n_{> 0}$, let

$$x^a = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}.$$ 

We call $a$ the exponent vector of the monomial $x^a$. Note that $x^a > 0$ for all $x \in \mathbb{R}^n_{> 0}$.

Let $S_n$ be the symmetric group. The symmetric mean of the monomial function $x^a$ is the function

$$[x^a]_{S_n} = \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)}^{a_{\sigma(1)}} x_{\sigma(2)}^{a_{\sigma(2)}} \cdots x_{\sigma(n)}^{a_{\sigma(n)}}.$$

For every subgroup $G$ of $S_n$, the $G$-symmetric mean of the monomial $x^a$ is the function

$$[x^a]_G = \frac{1}{|G|} \sum_{\sigma \in G} x_{\sigma(1)}^{a_{\sigma(1)}} x_{\sigma(2)}^{a_{\sigma(2)}} \cdots x_{\sigma(n)}^{a_{\sigma(n)}}.$$ 

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Let $\sigma \in S_n$. If $i \in \{1, \ldots, n\}$ and $j = \sigma(i)$, then
\[ x_{\sigma(i)}^{a_i} = x_j^{a_{\sigma^{-1}(j)}} \]
and
\[ \prod_{i=1}^n x_{\sigma(i)}^{a_i} = \prod_{j=1}^n x_j^{a_{\sigma^{-1}(j)}}. \]
Because $\{\sigma \in G\} = \{\sigma^{-1} \in G\}$, we have
\[ [x^a]_G = \frac{1}{|G|} \sum_{\sigma \in G} \prod_{j=1}^n x_j^{a_{\sigma(j)}} = \frac{1}{|G|} \sum_{\sigma \in G} \prod_{j=1}^n x_j^{a_{\sigma^{-1}(j)}} = \frac{1}{|G|} \sum_{\sigma \in G} x_1^{a_{\sigma(1)}} x_2^{a_{\sigma(2)}} \cdots x_n^{a_{\sigma(n)}}. \]
If $x \in \mathbb{R}_{>0}^n$ is the constant vector with all coordinates equal to $x$, then
\[ (1) \quad [x^a]_G = \frac{1}{|G|} \sum_{\sigma \in G} \prod_{j=1}^n x_j^{a_{\sigma(j)}} = \frac{1}{|G|} \sum_{\sigma \in G} x_{\sum_{i=1}^n a_i} = x_{\sum_{i=1}^n a_i}. \]
Let $G$ be a subgroup of $S_n$ and let $a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ be nonnegative vectors such that
\[ \sum_{j=1}^n a_j = \sum_{j=1}^n b_j. \]
If
\[ (2) \quad [x^b]_G < [x^a]_G \]
for all vectors $x \in \mathbb{R}_{>0}^n$, then (2) is called a monomial inequality with respect to the subgroup $G$ determined by $a$ and $b$. A monomial inequality is a monomial inequality with respect to the symmetric group $S_n$. Here are some examples. If $a = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \in \mathbb{R}_{\geq 0}^2$, then $x^a = x_1^2 x_2^2$. The $S_2$-symmetric mean of $x^a$ is
\[ [x^a]_{S_2} = \frac{1}{2} \left( x_1^2 x_2^2 + x_2^2 x_1^2 \right) = \frac{1}{2} \left( x_1^2 x_2^2 + x_1^2 x_2^2 \right). \]
If $b = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix} \in \mathbb{R}_{\geq 0}^2$, then $x^b = x_1^6 x_2^4$ and the $S_2$-symmetric mean of $x^b$ is
\[ [x^b]_{S_2} = \frac{1}{2} \left( x_1^6 x_2^4 + x_2^6 x_1^4 \right). \]
It is straightforward to check that
\[ (3) \quad x_1^6 x_2^4 + x_1^4 x_2^6 < x_1^7 x_2^3 + x_1^3 x_2^7 \]
for all nonconstant positive vectors $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and so $[x^b]_{S_2} < [x^a]_{S_2}$ is a monomial inequality.

Another example. Associated to the vector $e_1 = \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}_{\geq 0}^n$ is the monomial function
\[ x^{e_1} = x_1^1 x_2^0 \cdots x_n^0 = x_1. \]
For every ordered pair of integers \((i, j)\) with \(i, j \in \{1, \ldots, n\}\), there are \((n - 1)!\) permutations \(\sigma \in S_n\) with \(\sigma(i) = j\). In particular, for all \(j \in \{1, \ldots, n\}\), there are \((n - 1)!\) permutations \(\sigma \in S_n\) with \(\sigma(1) = j\). For all \(x = (x_1, \ldots, x_n) \in \mathbb{R}_\geq 0^n\), we have

\[
[x^{e_1}]_{S_n} = \frac{1}{n!} \sum_{\sigma \in S_n} x^{1}_{\sigma(1)}x^{0}_{\sigma(2)} \cdots x^{0}_{\sigma(n)} = \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} = \frac{1}{n!} \sum_{j=1}^n (n - 1)!x_j = \frac{1}{n} \sum_{j=1}^n x_j.
\]

Thus, the symmetric mean of the monomial \(x^{e_1}\) is the arithmetic mean of the positive numbers \(x_1, \ldots, x_n\).

Associated to the vector \(f_n = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)\) \(\in \mathbb{R}_\geq 0^n\) is the monomial function

\[
x^{f_n} = x_1^{1/n}x_2^{1/n} \cdots x_n^{1/n} = (x_1x_2 \cdots x_n)^{1/n}
\]

which is the geometric mean of the positive numbers \(x_1, \ldots, x_n\). Because \(x_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(n)} = x_1x_2 \cdots x_n\) for all \(\sigma \in S_n\), the symmetric mean

\[
[x^{f_n}]_{S_n} = \frac{1}{n!} \sum_{\sigma \in S_n} \left(x_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(n)}\right)^{1/n} = \frac{1}{n!} \sum_{\sigma \in S_n} (x_1x_2 \cdots x_n)^{1/n} = (x_1x_2 \cdots x_n)^{1/n}
\]

is also the geometric mean of the numbers \(x_1, \ldots, x_n\). The arithmetic and geometric mean inequality is equivalent to the monomial inequality

\[
[x^{f_n}]_{S_n} < [x^{e_1}]_{S_n}
\]

for all nonconstant vectors \(x = (x_1, \ldots, x_n) \in \mathbb{R}_\geq 0^n\).

There are infinitely many ordered pairs \((a, b)\) of nonnegative vectors such that

\[
[x^b]_{S_n} < [x^a]_{S_n}
\]

for all nonconstant vectors \(x \in \mathbb{R}_\geq 0^n\). Muirhead’s inequality (Theorem 4) and its converse (Theorem 5) classify the pairs of vectors \(a\) and \(b\) that determine monomial inequalities. Richard Rado (Theorems 9 and 10) classified monomial inequalities with respect to a subgroup \(G\) of \(S_n\). The purpose of this paper is to prove these results.

2. Simple inequalities

Here is a trivial fact, from which we obtain infinitely many monomial inequalities.
Theorem 1. The product of two real numbers is positive if and only if both numbers are positive or both numbers are negative.

Corollary 1. Let \(a = \left(\frac{a_1}{a_2}\right) \in \mathbb{R}_{>0}^2\) and \(x = \left(\frac{x_1}{x_2}\right) \in \mathbb{R}_{>0}^2\). If \(x_1 \neq x_2\), then

\[
x_1^{a_1}x_2^{a_2} + x_1^{a_2}x_2^{a_1} < x_1^{a_1+a_2} + x_2^{a_1+a_2}.
\]

If \(x_1 = x_2\), then

\[
x_1^{a_1}x_2^{a_2} + x_1^{a_2}x_2^{a_1} = x_1^{a_1+a_2} + x_2^{a_1+a_2}.
\]

Proof. If \(0 < x_2 < x_1\), then \(x_1^{a_1} - x_2^{a_1} > 0\) and \(x_1^{a_2} - x_2^{a_2} > 0\). If \(0 < x_1 < x_2\), then \(x_1^{a_1} - x_2^{a_1} < 0\) and \(x_1^{a_2} - x_2^{a_2} < 0\). Thus, the numbers \(x_1^{a_i} - x_2^{a_i}\) and \(x_1^{a_2} - x_2^{a_2}\) are both positive or both negative. Applying Theorem 1 to these numbers gives

\[
x_1^{a_1+a_2} + x_2^{a_1+a_2} - x_1^{a_1}x_2^{a_2} - x_1^{a_2}x_2^{a_1} = (x_1^{a_1} - x_2^{a_1})(x_1^{a_2} - x_2^{a_2}) > 0.
\]

This proves (4). If \(x_1 = x_2\), then

\[
x_1^{a_1}x_2^{a_2} + x_1^{a_2}x_2^{a_1} = x_1^{a_1+a_2} + x_2^{a_1+a_2}.
\]

This completes the proof. \(\Box\)

Corollary 2. Let \(\rho, \delta, \text{ and } \Delta\) satisfy the inequalities

\[
\rho > 0 \quad \text{and} \quad |\delta| < \Delta.
\]

Let \(x = \left(\frac{x_1}{x_2}\right) \in \mathbb{R}_{>0}^2\). If \(x_1 \neq x_2\), then

\[
x_1^{\rho+\delta}x_2^{-\delta} + x_1^{-\delta}x_2^{\rho+\delta} < x_1^{\rho+\Delta}x_2^{-\Delta} + x_1^{-\Delta}x_2^{\rho+\Delta}.
\]

Proof. We have \(\Delta+\delta > 0\) and \(\Delta-\delta > 0\). Applying Corollary 1 with \(a = \left(\frac{\Delta+\delta}{\Delta-\delta}\right) \in \mathbb{R}_{>0}^2\) gives

\[
x_1^{\Delta+\delta}x_2^{-\delta} + x_1^{-\delta}x_2^{\Delta+\delta} < x_1^{2\Delta} + x_2^{2\Delta}.
\]

Multiplying this inequality by \((x_1x_2)^{-\Delta}\) gives

\[
x_1^{\Delta-\delta}x_2^{-\delta} + x_1^{-\delta}x_2^{\Delta+\delta} < x_1^{\Delta}x_2^{-\Delta} + x_1^{-\Delta}x_2^{\Delta}.
\]

Multiplying by \((x_1x_2)^\rho\) completes the proof. \(\Box\)

For example, letting \(\rho = 5\), \(\delta = 1\), and \(\Delta = 2\) in inequality (5) gives inequality (3). Similarly, for all nonconstant vectors \(x \in \mathbb{R}_{>0}^2\), we have the chain of inequalities

\[
2x_1^5x_2^5 < x_1^6x_2^4 + x_1^4x_2^6 < x_1^7x_2^3 + x_1^3x_2^7 < x_1^8x_2^2 + x_1^2x_2^8 < x_1^9x_2 + x_1x_2^9 < x_1^{10} + x_2^{10}.
\]

These inequalities fail if \(x_1 = 1\) and \(x_2 = -1\) or, more generally, if \(x_1x_2 \leq 0\).

Theorem 2. Let vectors \(a = \left(\frac{a_1}{a_2}\right) \in \mathbb{R}_{>0}^2\) and \(b = \left(\frac{b_1}{b_2}\right) \in \mathbb{R}_{>0}^2\) satisfy

\[
a_2 < b_2 \leq b_1 < a_1 \quad \text{and} \quad a_1 + a_2 = b_1 + b_2.
\]

If \(x = \left(\frac{x_1}{x_2}\right) \in \mathbb{R}_{>0}^2\) with \(x_1 \neq x_2\), then \(|x|^b_{S_2} < |x|^a_{S_2}\) or, equivalently,

\[
x_1^{b_1}x_2^{b_2} + x_1^{b_2}x_2^{b_1} < x_1^{a_1}x_2^{a_2} + x_1^{a_2}x_2^{a_1}.
\]
Proof. Let
\[ \rho = \frac{a_1 + a_2}{2} = \frac{b_1 + b_2}{2} \]
and
\[ \delta = \frac{b_1 - b_2}{2} \quad \text{and} \quad \Delta = \frac{a_1 - a_2}{2}. \]
We have
\[ \rho > 0 \quad \text{and} \quad 0 \leq \delta < \Delta. \]
Applying inequality (5) with
\[ \rho + \delta = b_1 \quad \text{and} \quad \rho - \delta = b_2 \]
and
\[ \rho + \Delta = a_1 \quad \text{and} \quad \rho - \Delta = a_2 \]
gives inequality (6). This completes the proof. \(\square\)

This is Muirhead’s inequality for monomials in two variables. Inequality (6) is the special case \(a = \left( \frac{7}{3} \right)\) and \(b = \left( \frac{6}{4} \right)\).

3. Symmetric means of functions of \(n\) variables

Recall the Kronecker delta
\[ \delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \]
The standard basis for \(\mathbb{R}^n\) is \(E = \{e_1, \ldots, e_n\}\), where \(e_j = \left( \delta_{1,j} \; \vdots \; \delta_{n,j} \right)\). Every permutation \(\sigma \in S_n\) defines a linear operator on \(\mathbb{R}^n\) by \(\sigma(e_j) = e_{\sigma(j)}\). If \(x = \sum_{j=1}^{n} x_j e_j \in \mathbb{R}^n\), then
\[ \sigma(x) = \sum_{j=1}^{n} x_j \sigma(e_j) = \sum_{j=1}^{n} x_j e_{\sigma(j)} = \sum_{j=1}^{n} x_{\sigma^{-1}(j)} e_j \]
and so
\[ \sigma \left( \begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \right) = \left( \begin{array}{c} x_{\sigma^{-1}(1)} \\ \vdots \\ x_{\sigma^{-1}(n)} \end{array} \right). \]

Let \(\Omega\) be a subset of \(\mathbb{R}^n\) that is closed under the action of \(S_n\), that is, \(x \in \Omega\) implies \(\sigma(x) \in \Omega\) for all \(\sigma \in S_n\). Let \(\mathcal{F}(\Omega)\) be the set of real-valued functions defined on \(\Omega\).

For every function \(f \in \mathcal{F}(\Omega)\) and every permutation \(\sigma\) in the symmetric group \(S_n\), define the function \(\sigma f \in \mathcal{F}(\Omega)\) by
\[ (\sigma f)(x) = f \left( \sigma^{-1}(x) \right). \]
For all \(\sigma, \tau \in S_n\) we have
\[ (\tau(\sigma f))(x) = (\sigma f) \left( \tau^{-1}(x) \right) = f \left( \sigma^{-1} \left( \tau^{-1}(x) \right) \right) \]
\[ = f \left( \left( \sigma^{-1} \tau^{-1} \right)(x) \right) = f \left( \left( \tau \sigma \right)^{-1}(x) \right) \]
\[ = ((\tau \sigma) f)(x). \]
Thus,\[(\tau\sigma)f = \tau(\sigma f)\]
and \((\text{S})\) defines an action of the group \(S_n\) on the function space \(\mathcal{F}(\Omega)\).

Let \(G\) be a subgroup of \(S_n\) of order \(|G|\). The \(G\)-symmetric mean of the function \(f\) is the function \([f]_G \in \mathcal{F}(\Omega)\) defined by

\[ [f]_G = \frac{1}{|G|} \sum_{\sigma \in G} \sigma f. \]

Because \(G = \{\sigma^{-1} : \sigma \in G\}\), we have

\[ [f]_G(x) = \frac{1}{|G|} \sum_{\sigma \in G} (\sigma f)(x) = \frac{1}{|G|} \sum_{\sigma \in G} f(\sigma^{-1}(x)) = \frac{1}{|G|} \sum_{\sigma \in G} f(\sigma(x)). \]

The symmetric mean of the function \(f\) is the \(S_n\)-symmetric mean.

**Lemma 1.** Let \(G\) be a subgroup of \(S_n\) and let \(\tau\) in \(G\). For all functions \(f \in \mathcal{F}(\Omega)\),

\[ [\tau f]_G = [f]_G. \]

**Proof.** From the identity

\[ G\tau = \{\sigma\tau : \sigma \in G\} = G \]

we obtain

\[ [\tau f]_G = \frac{1}{|G|} \sum_{\sigma \in G} \sigma(\tau f) = \frac{1}{|G|} \sum_{\sigma \in G} (\sigma \tau)f = \frac{1}{|G|} \sum_{\sigma \in G} \sigma f = [f]_G. \]

This completes the proof. \(\square\)

**Lemma 2.** Let \(G\) be a subgroup of \(S_n\). For every nonnegative vector \(a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}_{\geq 0}^n\) and every permutation \(\tau \in G\),

\[ [x^a]_G = [x^{\tau(a)}]_G. \]

**Proof.** From (7) we have

\[ \tau(a) = \begin{pmatrix} a_{\tau^{-1}(1)} \\ \vdots \\ a_{\tau^{-1}(n)} \end{pmatrix} \quad \text{and} \quad x^{\tau(a)} = x_1^{a_{\tau^{-1}(1)}} \cdots x_n^{a_{\tau^{-1}(n)}}. \]
Lemma 3. Let 

This completes the proof. □

Proof. The transposition 

Because \( G \tau = \{ \sigma \tau : \sigma \in G \} = G \) for all \( \tau \in G \), we have

This completes the proof. □

Lemma 3. Let \( G \) be a subgroup of \( S_n \) that contains the transposition \( \tau = (j,k) \) with \( j < k \). For every nonnegative vector \( \mathbf{a} = \left( \frac{a_1}{\vdots} \frac{a_n}{a_n} \right) \in \mathbb{R}_{\geq 0}^n \) and positive vector

\[
\mathbf{x} = \left( \frac{x_1}{\vdots} \frac{x_n}{x_n} \right) \in \mathbb{R}_{> 0}^n,
\]

\[
\left[ \mathbf{x}^\tau \right]_G = \frac{1}{2|G|} \sum_{\sigma \in G} \left( x_{\sigma(j)}^{a_j} x_{\sigma(k)}^{a_k} + x_{\sigma(j)}^{a_k} x_{\sigma(k)}^{a_j} \right) \prod_{i=1, i \neq j, k}^{n} x_{\sigma(i)}^{a_i}.
\]

Proof. The transposition \( \rho = (j,k) \) acts as follows:

\[
\text{if } \mathbf{a} = \left( \frac{a_1}{\vdots} \frac{a_j}{a_j} \frac{a_{j+1}}{\vdots} \frac{a_{k-1}}{a_k} \frac{a_k}{a_{k+1}} \frac{\vdots}{a_n} \right) \text{ then } \rho(\mathbf{a}) = \left( \frac{a_1}{\vdots} \frac{a_j}{a_k} \frac{a_{j+1}}{a_j} \frac{\vdots}{a_{k-1}} \frac{a_k}{a_{k+1}} \frac{\vdots}{a_n} \right).
\]

By Lemma 2

\[
2[\mathbf{x}^\tau]_G = [\mathbf{x}]_G + [\mathbf{x}^\tau(\mathbf{a})]_G
\]

\[
= \frac{1}{|G|} \sum_{\sigma \in G} x_{\sigma(j)}^{a_j} x_{\sigma(k)}^{a_k} \cdots x_{\sigma(n)}^{a_n} + \frac{1}{|G|} \sum_{\sigma \in G} x_{\sigma(j)}^{a_k} x_{\sigma(k)}^{a_j} \cdots x_{\sigma(n)}^{a_n}
\]

\[
= \frac{1}{|G|} \sum_{\sigma \in G} \left( x_{\sigma(j)}^{a_j} x_{\sigma(k)}^{a_k} + x_{\sigma(j)}^{a_k} x_{\sigma(k)}^{a_j} \right) \prod_{i=1, i \neq j, k}^{n} x_{\sigma(i)}^{a_i}.
\]

Dividing by 2 completes the proof. □
Lemma 4. Let $G$ be a subgroup of $S_n$ that contains the transposition $\tau = (j,k)$ with $j < k$. Let $u = \left(\begin{array}{c} u_1 \\ \vdots \\ u_n \end{array}\right)$ be a nonnegative vector such that $u_j > u_k$ and let
\[
\rho = \frac{u_j + u_k}{2} \quad \text{and} \quad \Delta = \frac{u_j - u_k}{2}.
\]
Then
\[
0 < \Delta \leq \rho.
\]
Let
\[
0 \leq \delta < \Delta.
\]
Define the nonnegative vector $v = \left(\begin{array}{c} v_1 \\ \vdots \\ v_n \end{array}\right)$ by
\[
v_j = \rho + \delta \quad \text{and} \quad v_k = \rho - \delta
\]
and
\[
v_i = u_i \quad \text{if} \ i \neq j, k.
\]
If $x = \left(\begin{array}{c} x_1 \\ \vdots \\ x_n \end{array}\right) \in \mathbb{R}_{\geq 0}^n$ is a nonconstant vector such that $x_{\sigma(j)} \neq x_{\sigma(k)}$ for some $\sigma \in G$, then
\[
[x^v]_G < [x^u]_G.
\]
Proof. We have $v_i = u_i \geq 0$ for $i \neq j, k$ and
\[
0 \leq u_k = \rho - \Delta < \rho - \delta = v_k \leq v_j = \rho + \delta < \rho + \Delta = u_j
\]
and so the vector is $v$ is nonnegative.

Let $x$ be a nonconstant vector in $\mathbb{R}_{\geq 0}^n$. Applying Lemma 4, we have
\[
[x^u]_G = \frac{1}{2|G|} \sum_{\sigma \in G} \left( x_{\sigma(j)}^u x_{\sigma(k)}^u + x_{\sigma(j)}^u x_{\sigma(k)}^u \right) \prod_{i=1}^n x_{\sigma(i)}^{u_i}
\]
\[
= \frac{1}{2|G|} \sum_{\sigma \in G} \left( x_{\sigma(j)}^{\rho + \Delta} x_{\sigma(k)}^{\rho - \Delta} + x_{\sigma(j)}^{\rho - \Delta} x_{\sigma(k)}^{\rho + \Delta} \right) \prod_{i=1}^n x_{\sigma(i)}^{u_i}
\]
and
\[
[x^v]_G = \frac{1}{2|G|} \sum_{\sigma \in G} \left( x_{\sigma(j)}^v x_{\sigma(k)}^v + x_{\sigma(j)}^v x_{\sigma(k)}^v \right) \prod_{i=1}^n x_{\sigma(i)}^{u_i}
\]
\[
= \frac{1}{2|G|} \sum_{\sigma \in G} \left( x_{\sigma(j)}^{\rho + \delta} x_{\sigma(k)}^{\rho - \delta} + x_{\sigma(j)}^{\rho - \delta} x_{\sigma(k)}^{\rho + \delta} \right) \prod_{i=1}^n x_{\sigma(i)}^{u_i}.
\]
Therefore,
\[
[x^u]_G - [x^v]_G
\]
\[
= \frac{1}{2|G|} \sum_{\sigma \in G} \left( x_{\sigma(j)}^{\rho + \Delta} x_{\sigma(k)}^{\rho - \Delta} + x_{\sigma(j)}^{\rho - \Delta} x_{\sigma(k)}^{\rho + \Delta} - x_{\sigma(j)}^{\rho + \delta} x_{\sigma(k)}^{\rho - \delta} - x_{\sigma(j)}^{\rho - \delta} x_{\sigma(k)}^{\rho + \delta} \right) \prod_{i=1}^n x_{\sigma(i)}^{u_i}.
\]
If \( x_{\sigma(j)} = x_{\sigma(k)} \), then
\[
x_{\sigma(j)}^{\rho+\Delta} x_{\sigma(k)}^{\rho-\Delta} + x_{\sigma(j)}^{\rho-\Delta} x_{\sigma(k)}^{\rho+\Delta} - x_{\sigma(j)}^{\rho+\delta} x_{\sigma(k)}^{\rho-\delta} - x_{\sigma(j)}^{\rho-\delta} x_{\sigma(k)}^{\rho+\delta} = 0.
\]

By (5) of Corollary 2 if \( x_{\sigma(j)} \neq x_{\sigma(k)} \), then
\[
x_{\sigma(j)}^{\rho+\Delta} x_{\sigma(k)}^{\rho-\Delta} + x_{\sigma(j)}^{\rho-\Delta} x_{\sigma(k)}^{\rho+\Delta} - x_{\sigma(j)}^{\rho+\delta} x_{\sigma(k)}^{\rho-\delta} - x_{\sigma(j)}^{\rho-\delta} x_{\sigma(k)}^{\rho+\delta} > 0.
\]

Because \( x_{\sigma(j)} \neq x_{\sigma(k)} \) for some \( \sigma \in G \), we have \( |x^u_G| - |x^v_G| > 0 \). This completes the proof. \( \square \)

4. Muirhead’s Inequality

The vector \( \mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n \) is decreasing if
\[
a_1 \geq a_2 \geq \cdots \geq a_n.
\]

Associated to every vector \( \mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n \) is a unique decreasing vector \( \mathbf{a}^\downarrow = \begin{pmatrix} a_1^\downarrow \\ \vdots \\ a_n^\downarrow \end{pmatrix} \in \mathbb{R}^n \) obtained from \( \mathbf{a} \) by a rearrangement of coordinates. Thus, \( \mathbf{a}^\downarrow = \sigma(\mathbf{a}) \) for some \( \sigma \in S_n \).

For example, from the vector \( \mathbf{a} = \begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \end{pmatrix} \in \mathbb{R}^4 \) we obtain the decreasing vector
\( \mathbf{a}^\downarrow = \begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \end{pmatrix}. \)

The permutation
\[
\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} \in S_4
\]
satisfies \( \mathbf{a}^\downarrow = \sigma(\mathbf{a}) \).

Let \( \mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \) and \( \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \) be vectors in \( \mathbb{R}^n \), and let \( \mathbf{a}^\downarrow = \begin{pmatrix} a_1^\downarrow \\ \vdots \\ a_n^\downarrow \end{pmatrix} \) and \( \mathbf{b}^\downarrow = \begin{pmatrix} b_1^\downarrow \\ \vdots \\ b_n^\downarrow \end{pmatrix} \) be the corresponding decreasing vectors obtained by permutation of coordinates. The following definition is fundamental: The vector \( \mathbf{a} \) majorizes the vector \( \mathbf{b} \), denoted \( \mathbf{b} \prec \mathbf{a} \), if
\[
\sum_{i=1}^k b_i \leq \sum_{i=1}^k a_i \quad \text{for all } i \in \{1, \ldots, n-1\}
\]
and
\[
\sum_{i=1}^n b_i = \sum_{i=1}^n a_i.
\]

The vector \( \mathbf{a} \) strictly majorizes \( \mathbf{b} \) if \( \mathbf{b} \prec \mathbf{a} \) and \( \mathbf{b} \neq \mathbf{a} \).

The Hamming distance between vectors \( \mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \) and \( \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \) in \( \mathbb{R}^n \) is
\[
d_H(\mathbf{a}, \mathbf{b}) = \text{card}\{i \in \{1, \ldots, n\} : a_i \neq b_i\}.
\]
Lemma 5. Let \( u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \) and \( v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \) be vectors in \( \mathbb{R}^n_{\geq 0} \) such that \( v \prec u \) and \( d_H(u, v) = 2 \). Then
\[
[x^v]_{S_n} < [x^u]_{S_n}
\]
for all nonconstant vectors \( x \in \mathbb{R}^n_{>0} \).

Proof. By Lemma 2 we can assume that the vectors \( u \) and \( v \) are decreasing.

Hamming distance \( d_H(u, v) = 2 \) implies that there are unique integers \( j < k \) such that \( u_i = v_i \) for all \( i \neq j, k \) and \( 0 \leq u_k < v_k \leq v_j < u_j \) and \( v_j + v_k = u_j + u_k \).

The numbers
\[
\rho = \frac{u_j + u_k}{2} = \frac{v_j + v_k}{2}, \quad \Delta = \frac{u_j - u_k}{2}
\]
and
\[
\delta = \frac{v_j - u_j + u_k}{2} = \frac{v_j - v_k}{2}
\]
satisfy the inequality \( 0 \leq \delta < \Delta \leq \rho \).

We have
\[
\rho + \delta = \frac{u_j + u_k}{2} + \left( \frac{v_j - u_j + u_k}{2} \right) = v_j
\]
and
\[
\rho - \delta = \frac{v_j + v_k}{2} - \frac{v_j - v_k}{2} = v_k.
\]
Applying Lemma 4 completes the proof. □

Theorem 3. Let \( a \) and \( b \) be distinct vectors in \( \mathbb{R}^n \) such that \( b \prec a \). For some positive integer \( r \leq d_H(a, b) \), there is a sequence of decreasing vectors \( c_0, c_1, \ldots, c_r \in \mathbb{R}^n_{\geq 0} \) such that
\[
(10) \quad b = c_r \prec c_{r-1} \prec \cdots \prec c_1 \prec c_0 = a
\]
and
\[
(11) \quad d_H(c_{i-1}, c_i) = 2 \quad \text{for all } i \in \{1, \ldots, r\}.
\]

Proof. This is Theorem 4 in Nathanson [5]. □

We observe that if \( a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \) and \( b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \) and if \( \sum_{i=1}^n a_i = \sum_{i=1}^n b_i \), then, for every \( x > 0 \), the constant vector \( x = \begin{pmatrix} x \\ \vdots \\ x \end{pmatrix} \in \mathbb{R}_0^n \) satisfies
\[
[x^b]_{S_n} = [x^a]_{S_n} = x \sum_{i=1}^n a_i.
\]
In particular, if \( b < a \), then \( [x^b]_{S_n} = [x^a]_{S_n} \) for every constant vector \( x \). Muirhead’s inequality is a strict inequality for nonconstant vectors.

Theorem 4 (Muirhead’s inequality). Let \( a \) and \( b \) be distinct vectors in \( \mathbb{R}^n_{\geq 0} \). If \( b \prec a \), then
\[
[x^b]_{S_n} < [x^a]_{S_n}
\]
for every nonconstant vector \( x \in \mathbb{R}_{>0}^n \).
Proof. We give two proofs of this inequality. By Theorem 3, there are decreasing vectors \( c_1, \ldots, c_r \) and a strict majorization chain

\[
\mathbf{b} = c_r \prec c_{r-1} \prec \cdots \prec c_1 \prec c_0 = \mathbf{a}
\]

such that

\[
d_H(c_{i-1}, c_i) = 2 \quad \text{for all } i \in \{1, \ldots, r\}.
\]

Let \( \mathbf{x} \) be a nonconstant vector in \( \mathbb{R}^n_{>0} \). The symmetric group \( S_n \) contains all transpositions, and so, by Lemma 5 with \( u = c_{i-1} \) and \( v = c_i \),

\[
[x^{c_i}]_{S_n} < [x^{c_{i-1}}]_{S_n}
\]

for all \( i \in \{1, \ldots, r\} \). Therefore,

\[
[x^b]_{S_n} = [x^c]_{S_n} < [x^{c_{r-1}}]_{S_n} < \cdots < [x^{c_1}]_{S_n} < [x^{c_0}]_{S_n} = [x^a]_{S_n}.
\]

This completes the first proof.

The second proof of Theorem 4 uses only the arithmetic and geometric mean inequality.

Proof. Let \( (c_i)_{i=1}^n \) be a sequence of real numbers such that \( c_i \neq 0 \) for some \( i \) and

\[
\sum_{i=1}^n c_i = 0.
\]

Let \( \mathbf{x} = \left( \begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \right) \) be a nonconstant positive vector. For every pair of integers \( (i, j) \) with \( i, j \in \{1, \ldots, n\} \), there are \( (n-1)! \) permutations \( \sigma \in S_n \) with \( \sigma(i) = j \) and so there are \( (n-1)! \) permutations \( \sigma \) such that \( x^{c_i}_{\sigma(i)} = x^{c_j}_j \). Therefore,

\[
\prod_{\sigma \in S_n} x^{c_1}_{\sigma(1)} x^{c_2}_{\sigma(2)} \cdots x^{c_n}_{\sigma(n)} = \prod_{\sigma \in S_n} \prod_{i=1}^n x^{c_i}_{\sigma(i)} = \prod_{i=1}^n \prod_{\sigma \in S_n} x^{c_i}_{\sigma(i)}
\]

\[
= \prod_{i=1}^n \prod_{j=1}^n x^{(n-1)!c_i}_{j} = \prod_{j=1}^n \prod_{i=1}^n x^{(n-1)!c_i}_{j}
\]

\[
= \prod_{j=1}^n x^{(n-1)! \sum_{i=1}^n c_i} = 1.
\]

The arithmetic and geometric mean inequality gives

\[
1 = \left( \prod_{\sigma \in S_n} x^{c_1}_{\sigma(1)} x^{c_2}_{\sigma(2)} \cdots x^{c_n}_{\sigma(n)} \right)^{1/n!} < \frac{1}{n!} \sum_{\sigma \in S_n} x^{c_1}_{\sigma(1)} x^{c_2}_{\sigma(2)} \cdots x^{c_n}_{\sigma(n)}
\]

and so

\[
n! < \sum_{\sigma \in S_n} x^{c_1}_{\sigma(1)} x^{c_2}_{\sigma(2)} \cdots x^{c_n}_{\sigma(n)}.
\]

Equivalently,

\[
0 < \sum_{\sigma \in S_n} \left( x^{c_1}_{\sigma(1)} x^{c_2}_{\sigma(2)} \cdots x^{c_n}_{\sigma(n)} - 1 \right).
\]
The vector \( \mathbf{a} = \left( \begin{array}{c} a_1 \\ \vdots \\ a_n \end{array} \right) \) strictly majorizes the vector \( \mathbf{b} = \left( \begin{array}{c} b_1 \\ \vdots \\ b_n \end{array} \right) \) and so \( \mathbf{a} \neq \mathbf{b} \).

Let \( c_i = a_i - b_i \) for \( i \in \{1, \ldots, n\} \). Then

\[
\sum_{i=1}^{n} c_i = \sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i = 0
\]

and \( c_i \neq 0 \) for some \( i \in \{1, \ldots, n\} \). We have

\[
\sum_{\sigma \in S_n} x^{a_i}_{\sigma(i)} = \sum_{\sigma \in S_n} x^{b_i + c_i}_{\sigma(i)} = \sum_{\sigma \in S_n} x^{b_i}_{\sigma(i)} \prod_{i=1}^{n} x^{c_i}_{\sigma(i)}.
\]

Inequality (12) implies

\[
[x^{a}]_{S_n} - [x^{b}]_{S_n} = \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{i=1}^{n} x^{a_i}_{\sigma(i)} - \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{i=1}^{n} x^{b_i}_{\sigma(i)}
\]

\[
= \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{i=1}^{n} x^{b_i}_{\sigma(i)} \prod_{i=1}^{n} x^{c_i}_{\sigma(i)} - \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{i=1}^{n} x^{b_i}_{\sigma(i)}
\]

\[
= \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{i=1}^{n} x^{b_i}_{\sigma(i)} \left( \prod_{i=1}^{n} x^{c_i}_{\sigma(i)} - 1 \right)
\]

\[
> 0.
\]

This completes the proof. \( \square \)

Next we prove the converse of Muirhead’s inequality.

**Theorem 5.** Let \( \mathbf{a} = \left( \begin{array}{c} a_1 \\ \vdots \\ a_n \end{array} \right) \) and \( \mathbf{b} = \left( \begin{array}{c} b_1 \\ \vdots \\ b_n \end{array} \right) \) be vectors in \( \mathbb{R}^{n}_{\geq 0} \). If

\[
[x^{b}]_{S_n} \leq [x^{a}]_{S_n}
\]

for every vector \( x \in \mathbb{R}^{n}_{\geq 0} \), then \( \mathbf{b} \prec \mathbf{a} \).

**Proof.** For all \( x > 0 \), the constant vector \( \mathbf{x} = \left( \begin{array}{c} z \\ \vdots \\ z \end{array} \right) \) is positive and satisfies

\[
[x^{a}]_{S_n} = x^{\sum_{i=1}^{n} a_i}, \quad \text{and} \quad [x^{b}]_{S_n} = x^{\sum_{i=1}^{n} b_i}.
\]

Suppose that \( [x^{b}]_{S_n} \leq [x^{a}]_{S_n} \) for every positive vector \( x \). For all \( x > 1 \) we have

\[
x^{\sum_{i=1}^{n} b_i} \leq x^{\sum_{i=1}^{n} a_i}
\]

and so

\[
\sum_{i=1}^{n} b_i \leq \sum_{i=1}^{n} a_i.
\]

Similarly, \( 1/x > 0 \) and

\[
\frac{1}{x^{\sum_{i=1}^{n} b_i}} = \left( \frac{1}{x} \right)^{\sum_{i=1}^{n} b_i} \leq \left( \frac{1}{x} \right)^{\sum_{i=1}^{n} a_i} = \frac{1}{x^{\sum_{i=1}^{n} a_i}}.
\]
and so
\[ \sum_{i=1}^{n} b_i \geq \sum_{i=1}^{n} a_i. \]

Therefore,
\[ \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} a_i. \]

By Lemma 2, we can assume that the vectors \( \mathbf{a} \) and \( \mathbf{b} \) are decreasing. For \( k \in \{1, \ldots, n-1\} \), we define the variables
\[ x_i = \begin{cases} x & \text{if } i \in \{1, \ldots, k\} \\ 1 & \text{if } i \in \{k+1, \ldots, n\} \end{cases} \]

the vector
\[
\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_k \\ x_{k+1} \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x \\ \vdots \\ x \\ 1 \\ \vdots \\ 1 \end{pmatrix}
\]

and the polynomial
\[ [\mathbf{x}^a]_{S_n} = \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)}^{a_1} \cdots x_{\sigma(n)}^{a_n}. \]

The leading coefficient of the polynomial \([\mathbf{x}^a]_{S_n}\) is positive and, because the vector \( \mathbf{a} \) is decreasing, the degree of this polynomial is \( \sum_{i=1}^{k} a_i \). Similarly, the polynomial
\[ [\mathbf{x}^b]_{S_n} = \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)}^{b_1} \cdots x_{\sigma(n)}^{b_n} \]

has a positive leading coefficient and degree \( \sum_{i=1}^{k} b_i \). The inequality \([\mathbf{x}^b]_{S_n} \leq [\mathbf{x}^a]_{S_n}\) for all \( x > 0 \) implies that \( \sum_{i=1}^{k} b_i \leq \sum_{i=1}^{k} a_i \) for all \( k \in \{1, 2, \ldots, n\} \) and so \( \mathbf{b} \prec \mathbf{a} \). This completes the proof. \( \square \)

We use Muirhead’s inequality to prove the following result.

**Theorem 6.** Let \( (u_i)_{i=1}^{n} \) and \( (v_i)_{i=1}^{n} \) be decreasing sequences of positive numbers. If \( (u_i)_{i=1}^{n} \neq (v_i)_{i=1}^{n} \) and
\[ \prod_{i=1}^{j} v_i \leq \prod_{i=1}^{j} u_i \quad \text{for all } j \in \{1, \ldots, n\} \]

then
\[ \sum_{i=1}^{n} v_i < \sum_{i=1}^{n} u_i. \]

**Proof.** Let
\[ v_{n+1} = \min(v_n, u_n) \quad \text{and} \quad u_{n+1} = v_{n+1} \left( \frac{\prod_{i=1}^{n} v_i}{\prod_{i=1}^{n} u_i} \right). \]
We have \( v_n \geq v_{n+1} > 0 \) and so the sequence \( (v_i)_{i=1}^{n+1} \) is positive and decreasing. Inequality (13) with \( j = n \) implies that \( \prod_{i=1}^{n} u_i \geq \prod_{i=1}^{n} v_i > 0 \) and so
\[
0 < u_{n+1} = v_{n+1} \left( \frac{\prod_{i=1}^{n} v_i}{\prod_{i=1}^{n} u_i} \right) \leq v_{n+1} = \min(v_n, u_n) \leq u_n.
\]
Thus, the sequence \( (u_i)_{i=1}^{n+1} \) is also positive and decreasing. Moreover,
\[
\prod_{i=1}^{n+1} u_i = u_{n+1} \prod_{i=1}^{n} u_i = v_{n+1} \left( \frac{\prod_{i=1}^{n} v_i}{\prod_{i=1}^{n} u_i} \right) \prod_{i=1}^{n} u_i = \prod_{i=1}^{n+1} v_i.
\]

Let \( \lambda > 0 \). For all \( j \in \{1, \ldots, n\} \), we have
\[
\prod_{i=1}^{j} \lambda v_i \leq \prod_{i=1}^{j} \lambda u_i \quad \text{if and only if} \quad \prod_{i=1}^{j} v_i \leq \prod_{i=1}^{j} u_i
\]
and
\[
\sum_{i=1}^{j} \lambda v_i \leq \sum_{i=1}^{j} \lambda u_i \quad \text{if and only if} \quad \sum_{i=1}^{j} v_i \leq \sum_{i=1}^{j} u_i.
\]
Choosing \( \lambda \) sufficiently large, we can assume that \( v_i > 1 \) and \( u_i > 1 \) for all \( i \in \{1, \ldots, n, n+1\} \), and so
\[
a_i = \log u_i > 0 \quad \text{and} \quad b_i = \log v_i > 0
\]
for all \( i \in \{1, \ldots, n, n+1\} \). The sequences \( (b_i)_{i=1}^{n+1} \) and \( (a_i)_{i=1}^{n+1} \) are decreasing sequences of positive numbers. Moreover, \( (u_i)_{i=1}^{n+1} \neq (v_i)_{i=1}^{n+1} \) implies \( (a_i)_{i=1}^{n+1} \neq (b_i)_{i=1}^{n+1} \).

Let
\[
a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \\ a_{n+1} \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \\ b_{n+1} \end{pmatrix}.
\]

For all \( j \in \{1, \ldots, n\} \) we have
\[
\sum_{i=1}^{j} b_i = \sum_{i=1}^{j} \log v_i = \log \prod_{i=1}^{j} v_i \leq \log \prod_{i=1}^{j} u_i = \sum_{i=1}^{j} \log u_i = \sum_{i=1}^{j} a_i
\]
and
\[
\sum_{i=1}^{n+1} b_i = \sum_{i=1}^{n+1} \log v_i = \log \prod_{i=1}^{n+1} v_i = \log \prod_{i=1}^{n+1} u_i = \sum_{i=1}^{n+1} \log u_i = \sum_{i=1}^{n+1} a_i.
\]
Thus, the vector \( a \) strictly majorizes the vector \( b \).
By Muirhead’s inequality (Theorem 4), for every positive vector $x = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$, we have

$$\frac{1}{n!} \sum_{\sigma \in S_{n+1}} x_{\sigma(i)}^{b_i} = \frac{1}{n!} \sum_{\sigma \in S_{n+1}} x_{\sigma(i)}^{a_i} = [x^b]_{S_n} < [x^a]_{S_n} = \frac{1}{n!} \sum_{\sigma \in S_{n+1}} x_{\sigma(i)}^{a_i} = \frac{1}{n!} \sum_{\sigma \in S_{n+1}} x_{\sigma(i)}^{\log u_i}.$$

Let $x_1 = e$ and $x_i = 1$ for $i \in \{2, \ldots, n+1\}$. If $\sigma \in S_{n+1}$ and $\sigma(j) = 1$, then

$$\prod_{i=1}^{n+1} x_{\sigma(i)}^{b_i} \prod_{i=1}^{n+1} x_{\sigma(i)}^{u_i} = x_1^{\log v_j} = e^{\log v_j} = v_j.$$

There are $n!$ permutations $\sigma \in S_{n+1}$ such that $\sigma(j) = 1$, and so

$$n!v_j = \sum_{\sigma \in S_{n+1}, \sigma(j)=1} v_j.$$

Similarly,

$$n!u_j = \sum_{\sigma \in S_{n+1}, \sigma(j)=1} u_j.$$

Therefore,

$$n! \sum_{j=1}^{n+1} v_j = \sum_{j=1}^{n+1} \sum_{\sigma \in S_{n+1}, \sigma(j)=1} v_j = \sum_{j=1}^{n+1} \sum_{\sigma \in S_{n+1}, \sigma(j)=1} x_1^{b_j}$$

$$= \sum_{j=1}^{n+1} \sum_{\sigma \in S_{n+1}, \sigma(j)=1} \prod_{i=1}^{n+1} x_{\sigma(i)}^{b_i} = \sum_{\sigma \in S_{n+1}} \prod_{i=1}^{n+1} x_{\sigma(i)}^{b_i}$$

$$< \sum_{\sigma \in S_{n+1}} \prod_{i=1}^{n+1} x_{\sigma(i)}^{a_i} = \sum_{\sigma \in S_{n+1}} \prod_{i=1}^{n+1} x_{\sigma(i)}^{a_i}$$

$$= \sum_{j=1}^{n+1} \sum_{\sigma \in S_{n+1}, \sigma(j)=1} x_1^{a_j} = \sum_{j=1}^{n+1} \sum_{\sigma \in S_{n+1}, \sigma(j)=1} u_j$$

$$= n! \sum_{j=1}^{n+1} u_j.$$

The inequality $u_{n+1} \leq v_{n+1}$ implies

$$\sum_{i=1}^{n} v_i = \sum_{i=1}^{n+1} v_i - v_{n+1} < \sum_{i=1}^{n+1} u_i - u_{n+1} = \sum_{i=1}^{n} u_i.$$
This completes the proof. □

5. Rado’s inequality

Let $G$ be a subgroup of $S_n$ and let $a$ be a vector in $\mathbb{R}^n$. The $G$-permutohedron generated by $a$, denoted $K_G(a)$, is the convex hull of the finite set of vectors $\{\gamma(a) : \gamma \in G\}$. The $G$-permutohedron $K_G(a)$ is a compact convex subset of $\mathbb{R}^n$. If the vector $a$ is nonnegative, then $\gamma(a)$ is nonnegative for all $\gamma \in G$ and so every convex combination of the vectors $\gamma(a)$, that is, every vector in $K_G(a)$, is nonnegative.

We recall the following separation theorem from convexity theory.

**Theorem 7.** Let $K$ be a compact convex set in $\mathbb{R}^n$. If the vector $b \in \mathbb{R}^n$ is not in $K$, then the set $K$ and the vector $b$ are strictly separated by a hyperplane.

This means that there is a nonzero linear functional $H(x) = \sum_{i=1}^{n} u_i x_i$ and there are scalars $c$ and $\delta$ with $\delta > 0$ such that

$$H(x) \leq c \quad \text{for all } x \in K$$

and

$$H(b) \geq c + \delta.$$

We also use the general form of the arithmetic-geometric mean inequality.

**Theorem 8.** Let $G$ be a finite set. If $\{w_\gamma : \gamma \in G\}$ and $\{t_\gamma : \gamma \in G\}$ are sets of nonnegative numbers such that $\sum_{\gamma \in G} t_\gamma = 1$, then

$$\prod_{\gamma \in G} w_\gamma^{t_\gamma} \leq \sum_{\gamma \in G} t_\gamma w_\gamma.$$

Moreover,

$$\prod_{\gamma \in G} w_\gamma^{t_\gamma} = \sum_{\gamma \in G} t_\gamma w_\gamma,$$

if and only if $w_\gamma = w_\gamma'$ for all $\gamma, \gamma' \in G$.

The following result of Richard Rado extends Muirhead’s inequality to subgroups of $S_n$.

**Theorem 9 (Rado).** Let $G$ be a subgroup of the symmetric group $S_n$. Let $a$ be a nonnegative vector in $\mathbb{R}^n$ and let $K_G(a)$ be the $G$-permutohedron, that is, the convex hull of the set of vectors $\{\gamma(a) : \gamma \in G\}$. If $b \in K_G(a)$

then $b$ is nonnegative and

$$[x^b]_G \leq [x^a]_G$$

for all positive vectors $x \in \mathbb{R}^n$.

**Proof.** Let $a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}_{\geq 0}^n$ and $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in K_G(a)$. Then

$$b = \sum_{\gamma \in G} t_\gamma \gamma(a)$$
where \( t_\gamma \in [0, 1] \) for all \( \gamma \in G \) and \( \sum_{\gamma \in G} t_\gamma = 1 \). By [7], the \( i \)th component of the vector \( b \) is

\[
b_i = \sum_{\gamma \in G} t_\gamma (\gamma a)_i = \sum_{\gamma \in G} t_\gamma a_{\gamma^{-1}(i)}.
\]

Let \( x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \) be a positive vector. For every permutation \( \sigma \in G \), the arithmetic and geometric mean inequality (Theorem 8) gives

\[
\left( \prod_{\gamma \in G} \left( \prod_{i=1}^n x_{\sigma(i)}^{\gamma^{-1}(i)} \right)^{t_\gamma} \right)^{1/\gamma} \leq \prod_{\gamma \in G} \left( \sum_{\gamma \in G} t_\gamma \left( \prod_{i=1}^n x_{\sigma(i)}^{\gamma^{-1}(i)} \right)^{t_\gamma} \right)^{1/\gamma}
\]

and so

\[
[x^b]_G = \frac{1}{|G|} \sum_{\sigma \in G} \prod_{i=1}^n x_{\sigma(i)}^{b_i} = \frac{1}{|G|} \sum_{\sigma \in G} \prod_{i=1}^n x_{\sigma(i)}^{\sum_{\gamma \in G} t_\gamma a_{\gamma^{-1}(i)}}
\]

\[
= \frac{1}{|G|} \sum_{\sigma \in G} \prod_{i=1}^n x_{\sigma(i)}^{t_\gamma a_{\gamma^{-1}(i)}} = \frac{1}{|G|} \sum_{\sigma \in G} \prod_{i=1}^n x_{\sigma(i)}^{\sum_{\gamma \in G} t_\gamma a_{\gamma^{-1}(i)}}
\]

\[
\leq \frac{1}{|G|} \sum_{\sigma \in G} \prod_{i=1}^n x_{\sigma(i)}^{t_\gamma a_{\gamma^{-1}(i)}} = \frac{1}{|G|} \sum_{\sigma \in G} \prod_{i=1}^n x_{\sigma(i)}^{\sum_{\gamma \in G} t_\gamma a_{\gamma^{-1}(i)}}
\]

\[
= \frac{1}{|G|} \sum_{\sigma \in G} \prod_{j=1}^n x_{\sigma(j)}^{a_j} \left( \prod_{\gamma \in G} \left( \sum_{\gamma \in G} t_\gamma a_{\gamma^{-1}(i)} \right)^{1/\gamma} \right) = \frac{1}{|G|} \sum_{\sigma \in G} \prod_{j=1}^n x_{\sigma(j)}^{a_j}
\]

This completes the proof. \( \square \)

Next we prove the converse of Theorem 9.

**Theorem 10** (Rado). Let \( G \) be a subgroup of the symmetric group \( S_n \). Let \( a \) be a nonnegative vector in \( \mathbb{R}^n \) and let \( K_G(a) \) be the \( G \)-permutohedron. If \( b \) is a nonnegative vector in \( \mathbb{R}^n \) such that

\[
[x^b]_G \leq [x^a]_G
\]

for all positive vectors \( x \in \mathbb{R}^n \), then

\[
b \in K_G(a).
\]

**Proof.** Let \( a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \) and \( b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \). We shall prove that if \( b \notin K_G(a) \), then there is a positive vector \( x \in \mathbb{R}^n \) such that \( [x^b]_G > [x^a]_G \).

By Theorem 7, if \( b \notin K_G(a) \), then the compact convex set \( K_G(a) \) and the vector \( b \) are strictly separated by a hyperplane \( H \) and so there is a nonzero linear functional

\[
H(x) = \sum_{i=1}^n u_i x_i
\]
and scalars $c$ and $\delta$ with $\delta > 0$ such that
\[ H(x) \leq c \quad \text{for all } x \in K_G(a) \]
and
\[ H(b) \geq c + \delta. \]
For all $\gamma \in G$ we have
\[ \gamma^{-1}(a) = \begin{pmatrix} a_{\gamma(1)} \\ \vdots \\ a_{\gamma(n)} \end{pmatrix} \in K_G(a) \]
and so
\[ \sum_{i=1}^{n} u_i a_{\gamma^{-1}(i)} = H(\gamma(a)) \leq c \leq H(b) - \delta = \sum_{i=1}^{n} u_i b_i - \delta. \]
Let
\[ M > |G|^{1/\delta} > 1 \quad \text{and} \quad x_i = M^{u_i} \]
for all $i \in \{1, \ldots, n\}$. The vector $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is positive. The subgroup $G$ contains the identity permutation, and so
\[ M \sum_{i=1}^{n} u_i b_i \leq \sum_{\gamma \in G} M \sum_{i=1}^{n} u_i b_{\gamma(i)} = \sum_{\gamma \in G} \prod_{i=1}^{n} M^{u_i b_{\gamma(i)}} \]
\[ = \sum_{\gamma \in G} \prod_{i=1}^{n} x_i^{b_{\gamma(i)}} = |G| [x^b]_G. \]
We have $\gamma(a) \in K_G(a)$ and $H(\gamma(a)) \geq H(b) - \delta$. It follows that
\[ [x^a]_G = \frac{1}{|G|} \sum_{\gamma \in G} \prod_{i=1}^{n} x_i^{a_{\gamma^{-1}(i)}} = \frac{1}{|G|} \sum_{\gamma \in G} \prod_{i=1}^{n} M^{u_i a_{\gamma^{-1}(i)}} \]
\[ = \frac{1}{|G|} \sum_{\gamma \in G} M \sum_{i=1}^{n} u_i a_{\gamma^{-1}(i)} = \frac{1}{|G|} \sum_{\gamma \in G} M^{H(\gamma(a))} \]
\[ \leq \frac{1}{|G|} \sum_{\gamma \in G} M^{H(b)-\delta} = \frac{1}{|G|} M \sum_{i=1}^{n} u_i b_i - \delta \]
\[ = \frac{1}{M^\delta} \frac{1}{|G|} M \sum_{i=1}^{n} u_i b_i \leq \frac{1}{M^\delta} [x^b]_G. \]
This completes the proof. $\square$

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