Impact of Shear Forces on Variable Cross Section Shaft Stiffness

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Abstract. Influence of shear forces is presented in this paper for static analysis of a shaft with variable cross-section. A stiffness matrix method is used to evaluate displacement and reaction forces on the shaft in regard to the slope deflection equations. The Timoshenko beam theory is addressed according to the shear coefficient and the shear correction factor. Bending moments are calculated.

1. Introduction

The most common numerical analysis tool and widely used in software products to simulate loaded structures, spatial elements and systems is the finite element method (FEM). Combining many advantages, the method is preferred by scientists, engineers and mathematicians. The main advantage of FEM is used to study various model shapes with a different dimension number and to calculate stress and displacement of an element or a system of elements. Different material models with different characteristics can be studied. The FEM provides opportunity to be converted basic energy principles and calculate differential equations in a matrix form. Additional procedures for subsequent calculations allow a determination of integrated result at different points in the finite element shape. After processing the results, a detailed picture and a timeline study decision is obtained. An advantage is data evaluation and the ability to refined survey parameters to reach the user-defined level of accuracy. The FEM is basic tool to determined stress and displacement.

The classical beam bending theory (Euler – Bernoulli theory) [1] provided approximate calculations of stress and strain depending of a load type. The theory can suggest that the cross-section normal to the beam centroid axis remained normal (orthogonal) during and after the bending deformation (figure 1). In this case, the shear force deformations are ignored and may not participate in the mathematical model calculation of examined beams. Good results can be obtained for considerable beams length in comparison with cross section geometric parameters [2]. Calculations of short shafts deformations are a different case. When a shaft diameter value is approximated to its length, the shear forces are significant and play an important role in the determining shaft elasticity.

The impact of shear forces and the bending moment of short beams are taken from the Timoshenko model [3, 4]. He is proposing a shear coefficient $k$ as a criterion for compatibility of the actual shear stress and the beam deformation. $k$ is introduced by Timoshenko to account the curvature cross section with respect to the central axis (figure 1) [5-13]. Later the Cowper mathematical expressions for the shear coefficient is presented, regard to the Poisson’s ratio [7].
2. Stiffness matrix expression according to slope deflection method

Relationships of individual elements stiffness [14] are created by the stiffness matrix method to calculate a load and deformation in a system of elements. In applying the method, the system must be modeled as simple sections connected to each other by nodes [15]. The material hardness properties of the elements are used in the equations and characterizes behavior of the entire idealized structure. Unknown displacements and system loads can be determined by solving the equations (1, 2). The method is a base of most commercial software sources using the finite elements.

The basic form of the equations is [16]:

\[ F_i = k_i d_i + Q_i, \]  \hspace{1cm} (1)

where

- \( i \) – section number;
- \( F_i \) – vector of the section's characteristic forces, which are unknown internal forces;
- \( k_i \) – section stiffness matrix which characterizes the section's resistance against deformations;
- \( d_i \) – vector of the section's displacements or deformations characteristic;
- \( Q_i \) – vector of the section's forces characteristic caused by the external effects (such as the known forces and the temperature changes) applied to a member while \( d_i = 0 \).

For a system of elements connected with nodes (continuous beams), the equation (1) after integration has following observations:

\[ R = k_c d + R_0, \]  \hspace{1cm} (2)

where

- \( R \) – vector of the nodal forces, representing the external forces applied to the system's nodes;
- \( k_c \) – system stiffness matrix, which is established by assembling the section's stiffness matrices \( k_i \);
- \( d \) – vector of the system's nodal displacements that can define all possible deformed configurations of system subject to arbitrary the nodal forces \( R \);
- \( R_0 \) – vector of the equivalent nodal forces, representing all external effects other than the nodal forces which are already included in the preceding nodal force vector \( R \).

A simple section is shown in (figure 2)

\[
\begin{array}{c}
\text{M}_2, \theta_2 \\
\uparrow \\
\text{Q}_1, \psi_1 \\
\text{M}_4, \theta_4 \\
\downarrow \\
\text{Q}_3, \psi_3
\end{array}
\]

Figure 2. Simple idealized section

Where \( Q_1 \) – internal force acting on the left side of section; \( \psi_1 \) – deflection on the left side of section; \( M_2 \) – internal bending moment acting on the left side of section; \( \theta_2 \) – slope on the left side of section; \( Q_3 \) – internal force acting on the right side of section; \( \psi_3 \) – deflection on the right side of section; \( M_4 \) – internal bending moment acting on the right side of section; \( \theta_2 \) – slope on the right side of section.
A stiffness matrix of simple section is made on a base of possible deformations. Four options are considered, if the shear force and the bending moment are acting on the section (figure 3). Stiffness values are determined by the slope deflection equations.

Figure 3. Possible deformations

Due to the deformation of bending moment and the shear forces of simple section, the stiffness matrix becomes:

$$k_i = \begin{bmatrix}
12EI & 6EI & -12EI & 6EI \\
\frac{(1+\alpha)L^3}{6EI} & \frac{(1+\alpha)L^2}{(1+\alpha)L} & \frac{(1+\alpha)L^2}{(1+\alpha)L} & \frac{(1+\alpha)L^2}{(1+\alpha)L} \\
\frac{6EI}{(1+\alpha)L^2} & \frac{6EI}{(1+\alpha)L^2} & \frac{6EI}{(1+\alpha)L^2} & \frac{6EI}{(1+\alpha)L^2} \\
\frac{(1+\alpha)L^3}{6EI} & \frac{(1+\alpha)L^2}{(1+\alpha)L} & \frac{(1+\alpha)L^2}{(1+\alpha)L} & \frac{(1+\alpha)L^2}{(1+\alpha)L}
\end{bmatrix}$$

where

$$\alpha = \frac{12EI}{GA_tz^2} = 24(1 + \vartheta)\frac{A_A}{A_S}\left(\frac{r}{L}\right)^2$$

- shear correction factor; $E$ – Young's modulus; $G$ – Shear modulus; $I$ – moment of inertia; $L$ – length of section; $\vartheta$ – Poisson's ratio; $r$ – radius of gyration; $A$ – area of cross – section; $A_S = \frac{A}{k}$ – area of deformed cross – section of shear forces; $k = \frac{3}{4}$ – Timoshenko shear coefficient for circular cross – section [11, 17, 18].

More than two supports beam has a total stiffness matrix $k_C$ and must be structured as follows:

$$k_C = \begin{bmatrix}
[k_{uu}] & [k_{ur}] \\
[k_{ru}] & [k_{rr}]
\end{bmatrix}$$

where

- $k_{uu}$ – symmetric matrix of unknown system deformations;
- $k_{ru}$ – matrix of load effect over support reactions;
- $k_{ur}$ – transpose matrix of $k_{ru}$;
- $k_{rr}$ – symmetric matrix of support displacement effect over reactions forces.

$k_{ur}$ and $k_{rr}$ are used in structures with additional displacement of supports.
3. Reaction force and bending moment calculations of shaft with variable cross – section according to shear forces impact

Variable moment of inertia of a shaft is studied in the paper (figure 4).

![ Shaft with variable cross-section ](image)

**Figure 4.** Shaft with variable cross-section

Scheme of simple sections is introduced in (figure 5).

![ Simple sections of shaft ](image)

**Figure 5.** Simple sections of shaft

Number of joints in figure 5 are \( n_j = 5 \) and number of restraints are \( n_r = 3 (\theta_1, \theta_5, \theta_7) \).

Then degrees of freedom of the shaft is \( n = 2n_j - n_r = 2 \times 5 - 3 = 7 \) (\( \theta_2, \theta_4, \theta_6, \theta_8, \theta_{10}, \theta_3, \theta_9 \)).

Basic parameters are written in table 1 to start calculation of the shaft.

| Section | Joints | Possible deformations | Diameter (m) | Length (m) | Area of cross-section (m²) | Moment of inertia (m⁴) | Shear correction factor |
|---------|--------|-----------------------|--------------|------------|----------------------------|------------------------|------------------------|
| AB      | A B    | \( \theta_1 \) \( \theta_2 \) \( \theta_3 \) \( \theta_4 \) | 0.74         | 0.08       | 0.430084                   | 0.0147196              | 500.540625             |
| BC      | B C    | \( \theta_3 \) \( \theta_4 \) \( \theta_5 \) \( \theta_6 \) | 0.44         | 1.213      | 0.1520531                  | 0.0018398              | 0.7697321              |
| CD      | C D    | \( \theta_5 \) \( \theta_6 \) \( \theta_7 \) \( \theta_8 \) | 0.44         | 3.47       | 0.1520531                  | 0.0018398              | 0.0940594              |
| DE      | D E    | \( \theta_7 \) \( \theta_8 \) \( \theta_9 \) \( \theta_{10} \) | 0.34         | 1.11       | 0.090792                   | 0.000656               | 0.5488678              |

Total shaft stiffness matrix based on (3) is redistributed by (4) of possible displacements and has a form shown in figure 6. Where \( k_{uu} \) values are in orange box and \( k_{ru} \) – yellow box.
Internal reactions of the shaft are:

\[ I_{R_{AB}} = \begin{bmatrix} \frac{q_{AB}}{2} & \frac{q_{AB}^2}{12} \\ \frac{q_{AB}^2}{12} & \frac{q_{AB}^3}{24} \end{bmatrix} = \begin{bmatrix} 1324.81 \\ 17.66 \end{bmatrix} N, \]

\[ I_{R_{BC}} = \begin{bmatrix} \frac{q_{BC}}{2} & \frac{q_{BC}^2}{12} \\ \frac{q_{BC}^2}{12} & \frac{q_{BC}^3}{24} \end{bmatrix} = \begin{bmatrix} 7101.74 \\ 1435.74 \end{bmatrix} N, \]

\[ I_{R_{CD}} = \begin{bmatrix} \frac{q_{CD}}{2} & \frac{q_{CD}^2}{12} \\ \frac{q_{CD}^2}{12} & \frac{q_{CD}^3}{24} \end{bmatrix} = \begin{bmatrix} 20315.77 \\ 11749.29 \end{bmatrix} N, \]

\[ I_{R_{DE}} = \begin{bmatrix} \frac{q_{DE}}{2} & \frac{q_{DE}^2}{12} \\ \frac{q_{DE}^2}{12} & \frac{q_{DE}^3}{24} \end{bmatrix} = \begin{bmatrix} 3880.43 \\ 717.88 \end{bmatrix} N, \]

where \( q \) – a distributed load equal to the gravity force regard to the mass of shaft sections.

Slope and deflection of each joint on the shaft are pictured on figure 7.

Figure 6. Total stiffness matrix of shaft
Bending moments and shear forces are calculated for every possible joint displacement.

\[
J_{Lm} = \begin{bmatrix}
-17.66 \\
-1418.07 \\
-10313.56 \\
11031.41 \\
717.88 \\
-8426.54 \\
-3880.43
\end{bmatrix}
\text{Nm}
\]

Deformation vector \( \theta = k_{\text{ui}}^{-1} \cdot J_{Lm} \) is:

\[
\theta = \begin{bmatrix}
1.4429579 \times 10^{-6} \\
1.4425257 \times 10^{-6} \\
-1.2571074 \times 10^{-5} \\
2.4606417 \times 10^{-5} \\
1.2458872 \times 10^{-5} \\
1.2151947 \times 10^{-7} \\
1.5350089 \times 10^{-5}
\end{bmatrix}
\text{rad}
\]

\[
d_{AB} = \begin{bmatrix}
0 \\
1.4429579 \times 10^{-6} \\
1.2151947 \times 10^{-7} \\
1.4425257 \times 10^{-6}
\end{bmatrix}
\text{m, rad}
\]

\[
d_{BC} = \begin{bmatrix}
1.2151947 \times 10^{-7} \\
1.4425257 \times 10^{-6} \\
0 \\
-1.2571074 \times 10^{-5}
\end{bmatrix}
\text{m, rad}
\]

\[
d_{CD} = \begin{bmatrix}
0 \\
-1.2571074 \times 10^{-5} \\
0 \\
2.4606417 \times 10^{-5}
\end{bmatrix}
\text{m, rad}
\]

\[
d_{DE} = \begin{bmatrix}
0 \\
2.4606417 \times 10^{-5} \\
1.5350089 \times 10^{-5} \\
1.2458872 \times 10^{-5}
\end{bmatrix}
\text{m, rad}
\]

**Figure 7.** Slope and deflection on shaft

Joint reactions \( F_i = k_i d_i + Q_i \) are calculated and the values are:

\[
F_{AB} = \begin{bmatrix}
485.59 \\
0 \\
2164.02 \\
-67.14
\end{bmatrix}
\text{N, Nm; }
F_{BC} = \begin{bmatrix}
-2164.02 \\
67.14 \\
16367.49 \\
-11306.5
\end{bmatrix}
\text{Nm; Nm}
\]
\[ F_{CD} = \begin{bmatrix} 22332.84 \ Nm \\ 11306.5 \ N \\ 18298.71 \ Nm \\ -4307.27 \ Nm \end{bmatrix}, \quad F_{DE} = \begin{bmatrix} 7760.85 \ Nm \\ 4307.27 \ N \\ 0 \ Nm \\ 0 \ Nm \end{bmatrix} \]

Reaction forces and bending moments are:

\[ R_A = 485.59 \ N, \quad R_C = 38700.34 \ N, \quad R_D = 26059.56 \ N, \quad M_A = M_C = M_D = 0 \ Nm. \]

4. Conclusions

A matrix approach has been applied to investigate the impact of shear forces over the reaction force and the bending moment of shaft joints. A shear correction factor is used to calculate the exact values of the slope and deflection according to the total stiffness matrix of system. Significant shear effect is studied. The presented method can be used for different boundary conditions and for a verification of 3D shaft models loads.

5. Reference

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