The Newtonian Potential in Gravitational Cohomology

Richard Atkins
richard.atkins@twu.ca
Department of Mathematics
Trinity Western University
7600 Glover Road
Langley, BC, V2Y 1Y1 Canada

Abstract

We examine the Newtonian potential in gravitational cohomology. This is given by a symmetric, two-index tensor field, which satisfies the wave equation in empty space. Furthermore, the associated gravitational field strength, obtained by applying the coboundary operator to the potential, is constructed for the case of a stationary point mass and shown to give the classical result.

Key Words: bi-metric geometry; Liénard-Wiechert potential; coboundary operator; cohomology; cochain complex
1 Introduction

The quantization of electrodynamics was initiated in the 1920s through the efforts of Dirac, Heisenberg and Pauli. Fundamental to their approach was the description of the classical theory in terms of potentials, or 1-forms, whose coboundaries in the de Rham complex give the 2-form electromagnetic fields. The significance of this cohomological picture lies in the fact that it is the potentials that are to be quantized rather than the field strengths in the associated quantum field theory.

The dynamics of a free particle under the influence of a gravitational field is described by

\[ \ddot{x}^{\mu} = -\Gamma_{\nu\lambda}^{\mu} \dot{x}^{\nu} \dot{x}^{\lambda} \]

where \( \Gamma_{\nu\lambda}^{\mu} \) is the Levi-Civita connection of the Lorentzian spacetime metric. Comparing these geodesic equations to the equations of motion

\[ \ddot{x}^{\mu} = -k F_{\mu\nu} \dot{x}^{\nu} \]

of a charged particle in an electromagnetic field we see that the connection plays the role of the gravitational field strength. The question then arises whether a quantum formulation of gravitation also demands a characterization of the classical theory in terms of potentials, whose coboundaries give the connection. More generally, it may be asked whether spacetime can be represented within some cohomological framework, apart from concerns of quantization.

In [1], we develop a cochain complex associated to a manifold \( M \) endowed with a flat metric, whose cohomology is identified with the Čech cohomology of the sheaf of affine sections of \( M \) (cf. [2], [9], [11] and [12]). Moreover, it is shown that the Majumdar-Papapetrou spacetimes (cf. [3], [4] and [5]) as well as the Schwarzschild black hole may be expressed in terms of potentials. One may also envisage the inclusion of a curved metric \( g_{\mu\nu} \) so that the relevant fields are the flat background metric \( \eta_{\mu\nu} \), the spacetime metric \( g_{\mu\nu} \) and the gravitational potential \( A = A_{\mu\nu} \). The
field equations in empty space encompassing these three are

$$R_{\mu
u}(A) - \frac{1}{2}R(A)g_{\mu\nu} = 0$$

where $R_{\mu
u}(A)$ is the Ricci curvature of the connection $\Gamma^\mu_{\nu\lambda}(A)$, related to the potential $A$, and $R(A)$ is the scalar curvature obtained by contraction with $g$: $R(A) := R_{\mu\nu}(A)g^{\mu\nu}$. The theory thus takes the shape of a modification of the bi-metric approach to gravitation that enjoyed attention for many years (cf. [7]). It was in its mathematical simplicity that the appeal of bi-metric geometry lay but the addition of a flat structure appeared to tarnish the splendor of conventional general relativity and eventually it fell out of favour. However, as quantization of gravity remains one of the foremost outstanding problems in the natural sciences we reconsider bi-metric models, now within the context of cohomology.

In order to provide some justification for the course taken in [1] it is necessary to study some of the elementary properties of the complex and its correlation to reality. Accordingly, in this paper we look at the Newtonian potential $A_{\mu\nu}$ and derive some of its features. First, it is shown that $A_{\mu\nu}$ satisfies the wave equation in empty space. Then we consider the field strength defined by the Newtonian potential. It portrays a gravitational influence propagating at the speed of light and which reduces to the usual Newtonian description in the stationary limit, as required. This, in effect, reformulates the Newtonian theory in a manner that adopts general covariance and eschews action at a distance.

In the following section we review the definition and basic theorems of the gravitational cohomology presented in [1] in order to make the material herein self-contained. The cochain complex is built from symmetric-free $(n + 2)$-tensor fields

$$S_{\mu_1\ldots\mu_n\nu_1\nu_2}$$

which are skew-symmetric in the $\mu$ indices and symmetric in the $\nu$ indices. The gravitational potentials are representatives of such tensors for which $n = 0$ and the symmetric-free part of the field strengths are cocycles with $n = 1$. In Section 3
some elementary formulas are derived for the calculus of fields corresponding to a point source. The last section deals with the Newtonian potential. It satisfies physically reasonable properties and the associated field strength obtained by applying the coboundary operator of the cochain complex to the potential results in the Newtonian force field expressed in Lorentz covariant form.

2 Gravitational Cohomology

This section briefly reviews some of the developments in [1], to which the reader is referred for further elucidation and detail. We construct a cochain complex $(G^*(M), d)$ for a flat manifold, whose coboundary operator $d^{(1)}$ will define the gravitational field strength associated to a given potential. $d^{(1)}$ plays the analogous role in gravitation to the exterior derivative, acting on 1-form potentials, in electrodynamics.

Let $M$ be a manifold with a flat metric $\eta_{\mu\nu} = diag(+1, -1, -1, -1)$ and let $\nabla$ denote the Levi-Civita connection of $\eta_{\mu\nu}$. $\nabla$ operates on tensor fields $C = C_{\mu_1...\mu_n}$ as follows:

$$ (\nabla C)_{\mu_1...\mu_{n+1}} dx^{\mu_1} \otimes \cdots \otimes dx^{\mu_{n+1}} = dx^{\mu_1} \otimes \nabla_{\partial / \partial x^{\mu_1}} (C) $$

which may be written more compactly as

$$ \nabla_{\mu_1} C_{\mu_2...\mu_{n+1}} $$

We raise and lower indices on symmetric connections with the flat metric:

$$ \Gamma_{\mu\nu\lambda} := \eta_{\mu\sigma}\Gamma^\sigma_{\nu\lambda} $$

$\Gamma_{\mu\nu\lambda}$ may be decomposed into a symmetric part $\Gamma_{(\mu\nu\lambda)}$ and a symmetric-free part $F_{\mu\nu\lambda}$:

$$ \Gamma_{\mu\nu\lambda} = \Gamma_{(\mu\nu\lambda)} + F_{\mu\nu\lambda} $$

It is easy to see that the symmetric-free part satisfies

$$ \frac{1}{2} F_{(\mu\nu\lambda)} = F_{\mu\nu\lambda} + F_{\nu\lambda\mu} + F_{\lambda\mu\nu} = 0 $$
This motivates the following definition. For \( n \geq 2 \), let \( G^n(M) \) denote the \((n + 1)\)-tensor fields \( S_{\mu_1 \cdots \mu_{n+1}} \) on \( M \) which are

(1) symmetric in the two rightmost indices \( \mu_n \) and \( \mu_{n+1} \),

(2) skew-symmetric in the \( n - 1 \) leftmost indices \( \mu_1, \ldots, \mu_{n-1} \), and

(3) which satisfy the symmetric-free condition

\[
S_{\mu_1 \cdots \mu_{n+1}} + (-1)^n S_{\mu_2 \cdots \mu_{n+1} \mu_1} + S_{\mu_3 \cdots \mu_{n+1} \mu_1 \mu_2} + (-1)^n S_{\mu_4 \cdots \mu_{n+1} \mu_1 \mu_2 \mu_3} + \\
S_{\mu_5 \cdots \mu_{n+1} \mu_1 \mu_2 \mu_3 \mu_4} + \cdots + (-1)^n S_{\mu_{n+1} \mu_1 \cdots \mu_n} = 0
\]

\( G^0(M) \) shall designate the smooth functions on \( M \) and \( G^1(M) \), the symmetric tensor fields \( S_{\mu_1 \mu_2} \) in two indices. Skew-symmetrization of an \( n \)-tensor \( C = C_{\mu_1 \cdots \mu_n} \) is defined by

\[
C_{[\mu_1 \cdots \mu_n]} := \frac{1}{n!} \sum_{\sigma \in S_n} sg(\sigma) C_{\mu_{\sigma(1)} \cdots \mu_{\sigma(n)}}
\]

where \( S_n \) is the set of permutations \( \sigma \) on \( n \) letters. The exterior covariant derivative \( d \nabla \) is given by the formula

\[(d \nabla C)_{\mu_1 \cdots \mu_{n+1}} = \nabla_{[\mu_1} C_{\mu_2 \cdots \mu_n] \mu_{n+1}}\]

For \( n \geq 2 \), define \( \phi^{(n)} \) to be symmetrization of the two rightmost indices of an \((n + 1)\)-tensor:

\[
\phi^{(n)}(S)_{\mu_1 \cdots \mu_{n-1} \nu \lambda} := S_{\mu_1 \cdots \mu_{n-1} (\nu \lambda)} = \frac{1}{2} (S_{\mu_1 \cdots \mu_{n-1} \nu \lambda} + S_{\mu_1 \cdots \mu_{n-1} \lambda \nu})
\]

and let \( \psi^{(n)} \) be skew-symmetrization of the \( n \) leftmost indices of an \((n + 1)\)-tensor, up to a constant multiple:

\[
\psi^{(n)}(S)_{\mu_1 \cdots \mu_n \nu} := \frac{2n}{n + 1} S_{[\mu_1 \cdots \mu_n] \nu}
\]

Define \( d^{(0)} : G^0(M) \to G^1(M) \) by \( d^{(0)} = \nabla^2 \) and \( d^{(n)} : G^n(M) \to G^{n+1}(M) \) by

\[
d^{(n)} = \phi^{(n+1)} \circ d \nabla \circ \psi^{(n)}, \text{ for } n \geq 1.
\]

The cochain complex \((G^*(M), d)\) is

\[
0 \to G^0(M) \xrightarrow{d^{(0)}} G^1(M) \xrightarrow{d^{(1)}} G^2(M) \xrightarrow{d^{(2)}} G^3(M) \xrightarrow{d^{(3)}} \cdots
\]

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Since $\nabla$ is flat, $d^{(n+1)} \circ d^{(n)} = 0$.

For our purposes we are interested only in the map $d^{(1)} : G^1(M) \longrightarrow G^2(M)$, which defines the coboundary operator on gravitational potentials $A_{\mu\nu} \in G^1(M)$. This is given explicitly by

$$d^{(1)} A_{\mu\nu\lambda} = \frac{1}{2} \nabla_\mu A_{\nu\lambda} - \frac{1}{4} (\nabla_\nu A_{\mu\lambda} + \nabla_\lambda A_{\mu\nu})$$

and produces the symmetric-free part $F_{\mu\nu\lambda}$ of the lowered Christoffel symbol $\Gamma_{\mu\nu\lambda}$ of the connection.

The coboundary operators $d^{(n)}$ defined above may seem to be somewhat arbitrarily contrived. One partial validation of their form is that the symmetric-free component $F_{\mu\nu\lambda}$ of the lowered Christoffel symbols of the Levi-Civita connection of MP space-times as well as the Schwarzschild black hole may be represented as the coboundary $F_{\mu\nu\lambda} = d^{(1)} A_{\mu\nu\lambda}$ of a gravitational potential $A_{\mu\nu}$. That $(G^* (M), d)$ is a natural cochain complex associated to a flat manifold is justified, at least from a mathematical perspective, by the fact that the cohomology of the complex is isomorphic to the Čech cohomology of the sheaf Aff, of affine sections of $M$; the group of sections Aff$(U)$ over an open subset $U$ of $M$ is defined to be the kernel of the map $\nabla^2 : G^0(U) \longrightarrow G^1(U)$.

This paper seeks to establish further support for the complex $(G^* (M), d)$ in describing gravitation by investigating the behaviour of the Newtonian potential under $d^{(1)}$; if the complex expresses a feasible approach to gravity then we expect the coboundary (1) of the Newtonian potential to reduce to the conventional Newtonian force under stationary conditions. The remainder is devoted to this exploration.

### 3 Elementary Formulas

In this section we consider the fields related to a particle $Q$ moving through a flat background described by a four-manifold $M$ with a metric $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ and associated Levi-Civita connection $\nabla$. We shall work in flat coordinates $x = (t, \mathbf{x})$ on $M$ with respect to which covariant differentiation becomes partial differentiation.
It shall be supposed that the gravitational field propagates at the speed of light, which shall be set to unity by means of a suitable choice of units. The position of $Q$ as it moves through $M$ is described by a 4-vector $q^\mu(t) = (t, q(t))$. For an arbitrary point $x$ in spacetime we ask when and where would the emission of a gravitational influence from $Q$ be such that it would be felt at $x$. This leads to the definition of retarded time $t_r = t_r(x)$ defined by

$$ t - t_r(x) = |x - q(t_r(x))| $$

Thus, a gravitational signal emitted from $Q$ at position $(t_r(x), q(t_r(x)))$ will, in the course of time, propagate to $x$. Notice that $t_r(x)$ is a scalar field that assumes a value for each point $x$ in the manifold $M$. The gravitational field of $Q$ will be constructed from the following covariant fields, which depend upon at most second-order derivatives of $q^\mu$:

$$ C^\mu(x) := x^\mu - q^\mu(t_r(x)) $$

$$ U^\mu(x) := \frac{dq^\mu}{d\tau}_{t=t_r(x)} $$

and

$$ \dot{U}^\mu(x) := \frac{d^2q^\mu}{d\tau^2}_{t=t_r(x)} $$

Here, $\tau$ is proper time as measured by $Q$. Let us look more closely at these fields and their physical interpretation. $C^\mu(x) = (C^0(x), C(x))$ is the difference between the point $x$ and the 4-position of $Q$, when its gravitational influence, moving at the speed of light, would reach $x$. Clearly $C^\mu(x)$ is a lightlike vector for each $x \in M$:

$$ C^\mu(x)C_\mu(x) = 0 $$

and

$$ C^0(x) = t - t_r(x) = |x - q(t_r(x))| $$

$C^0(x)$ measures the time it takes for a light signal emitted from $Q$ at $(t_r(x), q(t_r(x)))$ to reach $x$ and $C(x)$ is the displacement vector from $q(t_r(x))$ to $x$. If $Q$ were stationary
at position \( q \) in the \((t, x)\)-coordinates then \( C^0(x) = |C(x)| \) would measure the distance \( r \) between \( x \) and \( Q \). \( U^\mu(x) = (U^0(x), U(x)) \) is the 4-velocity of \( Q \) at the moment a light signal from \( Q \) would reach \( x \). \( U^\mu(x) \) is not the conventional 4-velocity of a particle, which is defined as a function of proper time or some other parameter, but is in fact a vector field, defined for each \( x \in M \). \( \dot{U}^\mu(x) \) is the 4-acceleration of \( Q \) at the moment a signal from \( Q \) travelling at lightspeed would reach \( x \); again, this is a field and not the usual function of a single variable.

By forming contractions of \( C^\mu(x), U^\mu(x) \) and \( \dot{U}^\mu(x) \) with the metric \( \eta_{\mu\nu} \) we may obtain scalar fields, the most important of which are:

\[
X(x) := C^\mu(x)U_\mu(x) \quad \text{and} \quad Y(x) := C^\mu(x)\dot{U}_\mu(x)
\]

For \( Q \) stationary at \( q \), \( U_\mu = \eta_{0\mu} \) and \( X \) becomes the distance \( r \) to the source \( Q \):

\[
X(x) = C^0(x) = |x - q| = r \tag{10}
\]

The fields \( t_r, C^\mu, U^\mu, \dot{U}^\mu \) and \( X \) are all defined on \( M \) so they may be differentiated with respect to \( x^\mu \). Next, we seek formulas for these derivatives. We begin with \( t_r(x) \), from which the partial derivatives of the others will follow.

It will be convenient to use the relativistic scale factor \( \gamma \):

\[
\gamma(x) := U^0(x) = \frac{dt}{d\tau}|_{t=t_r(x)} = [1 - (\frac{d\mathbf{q}}{dt}(t_r(x)))^2]^{-1/2} \tag{11}
\]

Observe that

\[
\frac{d\mathbf{q}}{dt}(t_r(x)) = \frac{d\mathbf{q}}{d\tau}|_{t=t_r(x)} \frac{d\tau}{dt}(t_r(x)) = U(x)\gamma^{-1}(x) \tag{12}
\]

Differentiating (2) implicitly with respect to the time coordinate \( x^0 = t \) gives:

\[
1 - \frac{\partial t_r}{\partial t} = -\frac{\mathbf{x} - \mathbf{q}(t_r)}{|\mathbf{x} - \mathbf{q}(t_r)|} \cdot \frac{d\mathbf{q}}{dt}(t_r) \frac{\partial t_r}{\partial t}
\]

Hence,

\[
\frac{\partial t_r}{\partial t} = \frac{|\mathbf{x} - \mathbf{q}(t_r)|}{|\mathbf{x} - \mathbf{q}(t_r)| - (\mathbf{x} - \mathbf{q}(t_r)) \cdot \frac{d\mathbf{q}}{dt}(t_r)}
\]
By (3), (7), (11) and (12),
\[
\frac{\partial t_r}{\partial t} = \frac{C^0}{C^0 - C \cdot U} \gamma^{-1} = \frac{C^0}{(C^0 U^0 - C \cdot U)\gamma^{-1}}
\]
Definition (8) allows us to rewrite this expression more compactly:
\[
\frac{\partial t_r}{\partial t} = \gamma X^{-1} C_0
\tag{13}
\]
where we have lowered the index on \(C\).

Now consider the implicit partial derivative of (2) with respect to a spatial coordinate \(x^i, 1 \leq i \leq 3\). This yields
\[
-\frac{\partial t_r}{\partial x^i} = \frac{x - q(t_r)}{|x - q(t_r)|} \cdot (e_{(i)} - \frac{dq(t_r)}{dt} \frac{\partial t_r}{\partial x^i})
\]
where \(e_{(i)}\) denotes the \(i\)th standard basis vector in \(\mathbb{R}^3\). Solving algebraically we obtain
\[
\frac{\partial t_r}{\partial x^i} = -\frac{x^i - q(t_r)}{|x - q(t_r)|} \cdot (x - q(t_r)) \cdot \frac{dq(t_r)}{dt} \frac{\partial t_r}{\partial x^i}
\]
Applying (3), (7), (11) and (12) once again, this may be written
\[
\frac{\partial t_r}{\partial x^i} = -\frac{C^i}{(C^0 U^0 - C \cdot U)\gamma^{-1}}
\]
Lowering the index on \(C\) gives
\[
\frac{\partial t_r}{\partial x^i} = \gamma X^{-1} C_i
\tag{14}
\]
(13) and (14) may be combined into a single equation:
\[
t_{r,\mu} = \gamma X^{-1} C_\mu
\tag{15}
\]
Calculating the partial derivatives of the fields (3)-(5) and (8) is now simply a matter of invoking the definitions along with equation (15). For instance,
\[
\frac{\partial C^\mu(x)}{\partial x^\nu} = \frac{\partial x^\mu}{\partial x^\nu} - \frac{\partial q^\mu(t_r(x))}{\partial x^\nu}
\]
by (3)
\[
= \delta^\mu_\nu - \frac{dq^\mu}{dt} (t_r) \frac{\partial t_r}{\partial x^\nu}
\]
\[
= \delta^\mu_\nu - \frac{dq^\mu}{d\tau}_{|t=t_r(x)} \frac{d\tau}{dt} (t_r) \frac{\partial t_r}{\partial x^\nu}
\]
\[
= \delta^\mu_\nu - U^\mu \gamma^{-1} \gamma X^{-1} C_\nu
\tag{4}, (11) and (15)
\]
\[
= \delta^\mu_\nu - X^{-1} U^\mu C_\nu
\]
The other partial derivatives may be handled in a similarly straightforward, albeit tedious, manner and not wishing to weary the reader we content ourselves by merely stating the relevant results in the following theorem.

**Theorem 1**

\[
C_{\mu\nu} = \eta_{\mu\nu} - X^{-1}U_{\mu}C_{\nu} \\
U_{\mu,\nu} = X^{-1}\dot{U}_{\mu}C_{\nu} \\
\dot{U}_{\mu,\nu} = X^{-1}\ddot{U}_{\mu}C_{\nu} \quad \text{where} \quad \dot{U}_{\mu}(x) = \frac{d^3q_{\mu}}{d\tau^3}|_{t=t(x)} \\
X_{,\nu} = U_{\nu} + hC_{\nu} \quad \text{where} \quad h = X^{-1}(Y - 1) \\
\Box U_{\mu} = 2X^{-1}\dddot{U}_{\mu}
\]

4 The Newtonian Potential

The potential in Newtonian gravity is a scalar field; we must find its counterpart \( A_{\mu\nu} \) in \( G^1(M) \). It is expected to be constructed from covariant fields related to a point source \( Q \), which depend upon derivatives of at most first order in the position of \( Q \), as is the case for the Liénard-Wiechert potentials in electrodynamics. This leaves us with \( C^\mu, U^\mu \) and \( X \) as the basic building blocks for \( A_{\mu\nu} \). Furthermore, when \( Q \) is stationary the expression for \( A_{\mu\nu} \) must in some regard reflect the Newtonian form

\[
V = \frac{c}{r}
\]  

(16)

where \( c \) is a constant and \( r \) is distance from the source \( Q \). Without belabouring these heuristic arguments we suggest:

\[
A_{\mu\nu} := aX^{-1}U_{\mu}U_{\nu}
\]  

(17)

where the constant \( a \) shall be determined later. For stationary \( Q, U_{\mu} \) reduces to \( U_{\mu} = \eta_{0\mu} \) and \( X \) becomes \( r \), as observed in (10). So in this case the only non-zero component of \( A_{\mu\nu} \) would be

\[
A_{00} = \frac{a}{r}
\]
in accordance with (16).

Observe that \( X^{-1}U_\mu \) is the Liénard-Wiechert potential up to a constant multiple. In empty space it satisfies the wave equation

\[
\Box X^{-1}U_\mu = 0
\]  \hspace{1cm} (18)

Consider now the d’Alembertian operator acting on \( X^{-1}U_\mu \): 

\[
\Box X^{-1}U_\mu U_\nu = (\Box X^{-1}U)_\mu U_\nu + 2(X^{-1}U_\mu)^\sigma U_\nu,\sigma + X^{-1}U_\mu(\Box U)_\nu \\
= 2[-X^{-2}(U^\sigma + hC^\sigma)U_\mu + X^{-2}\dot{U}_\mu C^\sigma]X^{-1}\dot{U}_\nu C_\sigma + \\
X^{-1}U_\mu 2X^{-1}\dot{U}_\nu \\
= -2X^{-2}U_\mu \ddot{U}_\nu + 2X^{-2}U_\mu \dot{U}_\nu \\
= 0
\]

In this derivation we have used formulas from Theorem 1 as well as equations (6), (8) and (18). To summarize:

**Theorem 2** The Newtonian potential \( A_{\mu\nu} \) satisfies the wave equation

\[
\Box A_{\mu\nu} = 0
\]

in empty space.

Our final concern is the calculation of the gravitational field tensor derived from \( A_{\mu\nu} \). First consider the covariant derivative of the Newtonian potential

\[
\partial_\mu A_{\nu\lambda} = a(X^{-1}U_\nu U_\lambda)_\mu \\
= aX^{-2}[-(U_\mu + hC_\mu)U_\nu U_\lambda + C_\mu \dot{U}_\nu U_\lambda + C_\mu U_\nu \dot{U}_\lambda]
\]

Suppose henceforth that \( Q \) is stationary at the origin of the flat coordinate system: \( q = (0,0,0) \). Then

\[
U_\mu = \eta_{0\mu}; \quad \dot{U}_\mu = 0; \quad X = C_0 = r; \quad C_i = -x^i; \quad h = -r^{-1}
\]
By substituting these into the above expression for $\partial_{\mu}A_{\nu\lambda}$ we obtain

$$\partial_{\mu}A_{\nu\lambda} = \begin{cases} -\frac{ax_{\mu}}{r^3} & \text{if } \nu = \lambda = 0 \text{ and } \mu \neq 0 \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (19)$$

Recall that the coboundary of the Newtonian potential is defined by (1):

$$F_{\mu\nu\lambda} := d^{(1)}A_{\mu\nu\lambda} = \frac{1}{2}\partial_{\mu}A_{\nu\lambda} - \frac{1}{4}(\partial_{\nu}A_{\mu\lambda} + \partial_{\lambda}A_{\mu\nu})$$  \hspace{1cm} (20)$$

Consider a particle $P$ of sufficiently small mass that it does not affect the gravitational field produced by $Q$ in any substantial way. Its 4-position will be denoted $x^\mu(t) = (t, x(t))$ and $s$ shall designate its proper time. Assuming that the lowered Christoffel symbol has a vanishing symmetric part,

$$\Gamma^i_{\nu\lambda} = F^i_{\nu\lambda} = -F^i_{\nu\lambda}$$

for $1 \leq i \leq 3$. The equations of motion governing $P$ are then

$$\frac{d^2x^i}{ds^2} = -\Gamma^i_{\nu\lambda} \frac{dx^\nu}{ds} \frac{dx^\lambda}{ds} = F^i_{\nu\lambda} \frac{dx^\nu}{ds} \frac{dx^\lambda}{ds}$$

Placing the values of $\partial_{\mu}A_{\nu\lambda}$ from (19) into (20) results in the equations:

$$\frac{d^2x^i}{ds^2} = \frac{1}{2}\partial_{i}A_{\nu\lambda} \frac{dx^\nu}{ds} \frac{dx^\lambda}{ds} = -\frac{ax^i}{2r^3} \frac{dx^0}{ds} \frac{dx^0}{ds}$$

which for velocities of $P$ small compared to the speed of light becomes

$$\frac{d^2x^i}{dt^2} = -\frac{ax^i}{2r^3}$$

These are, in fact, the equations of motion for a mass in a Newtonian gravitational field if

$$a = 2GM$$

where $G$ is Newton’s constant and $M$ is the mass of $Q$. This is the desired outcome.

**Theorem 3** The potential

$$A_{\mu\nu} = 2GMX^{-1}U_{\mu}U_{\nu}$$

yields the Newtonian gravitational theory for a stationary particle of mass $M$. 

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We have required that the symmetric part $\Gamma_{(\mu\nu\lambda)}$ of the connection be zero; a non-vanishing symmetric part would amount to a modification of the Newtonian force law.
References

[1] Atkins, R., "Cohomology and MP Spacetimes." Preprint. http://arxiv.org/abs/0803.1673

[2] Bott R., Tu, L., "Differential Forms in Algebraic Topology." Springer 1982

[3] Majumdar, S.D., "A class of exact solutions of Einstein’s field equations." Phys. Rev 72, 390-398 (1947)

[4] Ortin, T., "Gravity and Strings." Cambridge University Press, 2004

[5] Papapetrou, A., "A static solution of the equations of the gravitational field for an arbitrary charge distribution." Proc. Royal Irish Acad. 51, 191-205 (1947)

[6] Poisson, E., "A Relativist’s Toolkit: The Mathematics of Black-Hole Mechanics." Cambridge University Press, 2004

[7] Rosen, N., "A bi-metric theory of gravitation." Gen. Rel. Grav. 4, 435 (1973)

[8] Schwarzschild, K., "Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie." Sitzungsber. Deutsch. Akad. Wiss. Berlin, Kl. Math.-Phys. Technik 189-196 (1916)

[9] Switzer, R.M., "Algebraic Topology: Homology and Homotopy." Springer-Verlag, 1975

[10] Wald, R.M., "General Relativity." University of Chicago Press, 1984

[11] Warner, F.W., "Foundations of Differentiable Manifolds and Lie Groups." Springer, 1983

[12] Weibel, C., "An introduction to homological algebra." Cambridge University Press, 1994