TAYLOR COEFFICIENTS OF INNER FUNCTIONS AND BEURLING’S THEOREM FOR THE SHIFT

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Abstract. By modifying an idea used by Kortram, we obtain a theorem of Schur type that characterizes the inner functions in terms of their Taylor coefficients. By Beurling’s theorem, this provides a sequential characterization of the shift-invariant subspaces of $\ell^2$.

Introduction and summary of results. A formal power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ represents an analytic function in the unit disk $\mathbb{D}$ if and only if $\limsup_{n \to \infty} |a_n|^{1/n} \leq 1$. Among all such functions, those which are bounded by one (i.e., map $\mathbb{D}$ into itself) can be characterized in terms of their Taylor coefficients $(a_n)_n$ by a classical theorem due to Schur [8]. This criterion is expressed by infinitely many conditions; see [5, p. 40, 180], [6] or [9, Theorem IV.25]. Among all such functions, an important subclass is formed by inner functions: those whose boundary values have modulus one almost everywhere on the unit circle. Basic facts about inner functions can be found in [4] or [5]. Taylor coefficients of inner functions have been studied in [1] and [7]. By using an approach analogous to that of Kortram [6], in this note we obtain a complete characterization of inner functions in terms of their Taylor coefficients. Naturally, like in Schur’s criterion, it is also given in terms of infinitely many conditions.

The question of describing the lattice of all closed non-trivial invariant subspaces of a given operator is fundamental and such descriptions are available for some classical operators. For the shift operator which transforms $f(z)$ into $zf(z)$, acting on the Hardy space $H^2$, Beurling’s celebrated theorem ([3], [4, Chapter 7], [5, Theorem II.7.1]) states that all of its invariant subspaces are of the form $\phi H^2$ for some inner function $\phi$. The importance of Beurling’s theorem stems for the fact that it gives a function-theoretic meaning to each invariant subspace. The space $H^2$ is usually identified with the complex space $\ell^2$ of all square-summable sequences and the shift with the operator $S$ on $\ell^2$ which acts as follows: $S(w_0, w_1, w_2, \ldots) = (0, w_1, w_2, \ldots)$. It was generally considered that no sequential description of such subspaces in $\ell^2$ would exist. It turns out that our characterization of inner functions...
does allow for a sequential description of invariant subspaces. Perhaps it is not very practical to check but nonetheless one can write down such a statement.

**Preliminaries.** Let $H^\infty$ denote the Banach space of bounded analytic functions in $\mathbb{D}$ equipped with the usual norm $\|f\|_{\infty} = \sup_{z \in \mathbb{D}} |f(z)|$. The Hardy space $H^2$ of the disk is the Hilbert space of all functions $f$ analytic in $\mathbb{D}$ for which

$$
\|f\|_2 = \lim_{r \to 1} \left( \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} \right)^{1/2} < \infty.
$$

Any such function has radial limits, $f(e^{i\theta}) = \lim_{r \to 1-} f(re^{i\theta})$, almost everywhere on the unit circle $\mathbb{T}$ with respect to the normalized arc length measure: $dm(\theta) = d\theta/(2\pi)$, and its norm can be recovered from the boundary values, as well as from the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in the disk, as follows:

$$
(1) \quad \|f\|_2 = \left( \int_{\mathbb{T}} |f|^2 dm \right)^{1/2} = \left( \sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2}.
$$

A function $\phi$ defined in $\mathbb{D}$ is said to be a pointwise multiplier of $H^2$ into itself if $\phi f \in H^2$ for all $f \in H^2$; in other words, if the multiplication operator $M_\phi$, defined by $M_\phi(f) = \phi f$, maps $H^2$ into itself. By standard pointwise estimates for the functions in $H^2$ and the Closed Graph Theorem, this automatically implies boundedness of the operator $M_\phi$. It is also known that this happens if and only if $\phi \in H^\infty$ and, furthermore, $M_\phi = \|\phi\|_{\infty}$. For the discussions of this, we refer the reader to [2] or [10], for example.

**A characterization of inner functions in terms of their Taylor coefficients.** As observed above, a function $\phi$ analytic in $\mathbb{D}$ is bounded by one if and only if the multiplication operator $M_\phi$ has norm at most one on $H^2$. This was the key to Kortram’s short proof [6] of Schur’s criterion. It turns out that this idea can also be adapted to the context of inner functions - with equalities and requiring slightly modified sums.

**Theorem 1.** Let $\phi$ be analytic in the unit disk, $\phi(z) = \sum_{n=0}^{\infty} \lambda_n z^n$. Then the following conditions are equivalent:

(a) $\phi$ is an inner function;

(b) $M_\phi$ is an isometric pointwise multiplier of $H^2$ into itself; in other words, $\|\phi f\|_2 = \|f\|_2$ for all $f \in H^2$;

(c) $$
\sum_{n=0}^{\infty} \left| \sum_{j=0}^{n} a_j \lambda_{n-j} \right|^2 = \sum_{n=0}^{\infty} |a_n|^2
$$
holds for all square-summable sequences $(a_n)_{n=1}^{\infty}$. 
Taylor coefficients of inner functions and Beurling’s theorem

That (a) is equivalent to (b) is contained in Theorem 2.1 from [2]. Here we give an indirect proof of this fact.

While it is readily checked that (c) \( \Rightarrow \) (d), the converse seems surprisingly non-trivial to prove (in spite of the density of the polynomials in \( H^2 \)) but this implication can simply be avoided, as will be done below.

Proof. It suffices to show that (a) \( \Rightarrow \) (b) \( \Rightarrow \) (c) \( \Rightarrow \) (d) \( \Rightarrow \) (a).

(a) \( \Rightarrow \) (b) Since \( |\phi| = 1 \) almost everywhere on \( T \), for every \( f \in H^2 \) we have

\[
\|\phi f\|_2^2 = \int_\mathbb{T} |\phi f|^2 dm = \int_\mathbb{T} |f|^2 dm = \|f\|_2^2.
\]

(b) \( \Rightarrow \) (c) Let \( f(z) = \sum_{n=0}^\infty a_nz^n \), where \( \|f\|_2^2 = \sum_{n=0}^\infty |a_n|^2 < \infty \). Just note that the \( n \)-th coefficient of \( \phi f \) is none other than \( \sum_{j=0}^n a_j \lambda_{n-j} \) by standard multiplication of two power series. The statement now follows from the assumption (b) and norm formula (1).

(c) \( \Rightarrow \) (d) This follows directly by substituting the stationary sequence \((a_0, a_1, \ldots, a_N, 0, 0, \ldots)\) into the equality in (c).

(d) \( \Rightarrow \) (a) Proving this statement is the crux of the proof. To this end, first substitute \((a_0, a_1, \ldots, a_{N-1}, a_N) = (0, 0, \ldots, 0, 1)\) in (d) to obtain

\[
\|\phi\|_2^2 = |\lambda_0|^2 + \sum_{n=N+1}^\infty |\lambda_{n-N}|^2 = |a_N|^2 = 1.
\]

Now let \( f \) be an arbitrary function in \( H^2 \) with \( f(z) = \sum_{n=0}^\infty a_nz^n \) in the disk. Apply equality (d) to the \((N+1)\)-tuple \((a_0, a_1, \ldots, a_N)\) of initial coefficients of \( f \) to deduce that

\[
\sum_{n=0}^N \left| \sum_{j=0}^n a_j \lambda_{n-j} \right|^2 \leq \sum_{n=0}^N |a_n|^2.
\]

Do this for every \( N \) and then let \( N \to \infty \). By (1), we get \( \|\phi f\|_2 \leq \|f\|_2 \). This means that the multiplication operator \( M_\phi \) has norm at most one and, thus, \( \|\phi\|_\infty \leq 1 \). Hence in the obvious double inequality:

\[
1 = \|\phi\|_2 \leq \|\phi\|_\infty \leq 1
\]
which follows from (1) and our assumptions on \( \phi \) equality must hold throughout, so \( |\phi| \) must have constant value one almost everywhere on the unit circle, which proves (a).

\[ \square \]

**A sequential interpretation of Beurling’s theorem.** In view of Beurling’s theorem, for every invariant subspace \( M \) of \( H^2 \) there is an inner function \( \phi \) such that for all \( g \in M \) we have \( g = \phi f \), and clearly \( \phi \in M \). Writing the power series expansions and using the above description of inner functions, we can immediately translate the statement of Theorem 1 into the language of \( \ell^2 \) sequences.

**Corollary 1.** A closed non-trivial invariant subspace \( M \) of \( \ell^2 \) is invariant for the shift operator if and only if there exists a fixed sequence \( (\lambda_n)_{n=0}^{\infty} \) in \( M \) which satisfies either of the conditions (c) or (d) of Theorem 1 (hence, both of them) and such that every sequence in \( M \) is of the form

\[
\left( \sum_{k=0}^{n} \lambda_k c_{n-k} \right)_{n=0}^{\infty}
\]

for some \( (c_n)_{n=0}^{\infty} \in \ell^2 \).

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