The Energy-Momentum tensor on low dimensional \textit{Spin}^c manifolds

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On a compact surface endowed with any Spin\textsuperscript{c} structure, we give a formula involving the Energy-Momentum tensor in terms of geometric quantities. A new proof of a Bär-type inequality for the eigenvalues of the Dirac operator is given. The round sphere \( S^2 \) with its canonical Spin\textsuperscript{c} structure satisfies the limiting case. Finally, we give a spinorial characterization of immersed surfaces in \( S^2 \times \mathbb{R} \) by solutions of the generalized Killing spinor equation associated with the induced Spin\textsuperscript{c} structure on \( S^2 \times \mathbb{R} \).

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1 Introduction

On a compact Spin surface, Th. Friedrich and E.C. Kim proved that any eigenvalue \( \lambda \) of the Dirac operator satisfies the equality [9, Thm. 4.5]:

\[
\lambda^2 = \frac{\pi \chi(M)}{\text{Area}(M)} + \frac{1}{\text{Area}(M)} \int_M |T^\psi|^2 v_g, \tag{1.1}
\]

where \( \chi(M) \) is the Euler-Poincaré characteristic of \( M \) and \( T^\psi \) is the field of quadratic forms called the Energy-Momentum tensor. It is given on the complement set of zeroes of the eigenspinor \( \psi \) by

\[
T^\psi(X, Y) = g(\ell^\psi(X), Y) = \frac{1}{2} \text{Re} \left( X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi, \frac{\psi}{|\psi|^2} \right),
\]

for every \( X, Y \in \Gamma(TM) \). Here \( \ell^\psi \) is the field of symmetric endomorphisms associated with the field of quadratic forms \( T^\psi \). We should point out that
since $\psi$ is an eigenspinor, the zero set is discret [3]. The proof of Equality (1.1) relies mainly on a local expression of the covariant derivative of $\psi$ and the use of the Schrödinger-Lichnerowicz formula. This equality has many direct consequences. First, since the trace of $\ell^{\psi}$ is equal to $\lambda$, we have by the Cauchy-Schwarz inequality that $|\ell^{\psi}|^2 \geqslant \frac{(\text{tr}(\ell^{\psi}))^2}{n} = \frac{\lambda^2}{n}$, where tr denotes the trace of $\ell^{\psi}$. Hence, Equality (1.1) implies the Bär inequality [2] given by

$$\lambda^2 \geqslant \lambda^2_1 := \frac{2\pi \chi(M)}{\text{Area}(M)}. \quad (1.2)$$

Moreover, from Equality (1.1), Th. Friedrich and E.C. Kim [9] deduced that $\int_M \det(T^{\psi}) v_g = \pi \chi(M)$, which gives an information on the Energy-Momentum tensor without knowing the eigenspinor nor the eigenvalue. Finally, for any closed surface $M$ in $\mathbb{R}^3$ of constant mean curvature $H$, the restriction to $M$ of a parallel spinor on $\mathbb{R}^3$ is a generalized Killing spinor of eigenvalue $-H$ with Energy-Momentum tensor equal to the Weingarten tensor $II$ (up to the factor $-\frac{1}{2}$) [21] and we have by (1.1)

$$H^2 = \frac{\pi \chi(M)}{\text{Area}(M)} + \frac{1}{4 \text{Area}(M)} \int_M |II|^2 v_g.$$

Indeed, given any surface $M$ carrying such a spinor field, Th. Friedrich [8] showed that the Energy-Momentum tensor associated with this spinor satisfies the Gauss-Codazzi equations and hence $M$ is locally immersed into $\mathbb{R}^3$.

Having a Spin$^c$ structure on manifolds is a weaker condition than having a Spin structure because every Spin manifold has a trivial Spin$^c$ structure. Additionally, any compact surface or any product of a compact surface with $\mathbb{R}$ has a Spin$^c$ structure carrying particular spinors. In the same spirit as in [14], when using a suitable conformal change, the second author [23] established a Bär-type inequality for the eigenvalues of the Dirac operator on a compact surface endowed with any Spin$^c$ structure. In fact, any eigenvalue $\lambda$ of the Dirac operator satisfies

$$\lambda^2 \geqslant \lambda^2_1 := \frac{2\pi \chi(M)}{\text{Area}(M)} - \frac{1}{\text{Area}(M)} \int_M |\Omega|^2 v_g, \quad (1.3)$$

where $i\Omega$ is the curvature form of the connection on the line bundle given by the Spin$^c$ structure. Equality is achieved if and only if the eigenspinor $\psi$ associated with the first eigenvalue $\lambda_1$ is a Killing Spin$^c$ spinor, i.e., for every $X \in \Gamma(TM)$ the eigenspinor $\psi$ satisfies

$$\begin{cases} \nabla_X \psi = -\frac{\Delta}{2} X \cdot \psi, \\ \Omega \cdot \psi = i|\Omega|\psi. \end{cases} \quad (1.4)$$

Here $X \cdot \psi$ denotes the Clifford multiplication and $\nabla$ the spinorial Levi-Civita connection [7].

Studying the Energy-Momentum tensor on a compact Riemannian Spin or Spin$^c$ manifolds has been done by many authors, since it is related to several geometric situations. Indeed, on compact Spin manifolds, J.P. Bourguignon and P. Gauduchon [2] proved that the Energy-Momentum tensor appears naturally
in the study of the variations of the spectrum of the Dirac operator. Th. Friedrich and E.C. Kim [10] obtained the Einstein-Dirac equation as the Euler-Lagrange equation of a certain functional. The second author extended these last two results to Spin$^c$ manifolds [24]. Even if it is not a computable geometric invariant, the Energy-Momentum tensor is, up to a constant, the second fundamental form of an isometric immersion into a Spin or Spin$^c$ manifold carrying a parallel spinor [21, 24]. For a better understanding of the tensor $q^\psi$ associated with a spinor field $\psi$, the first author [12] studied Riemannian flows and proved that, if the normal bundle carries a parallel spinor $\psi$, the tensor $q^\psi$ associated with $\psi$ (the restriction of $\psi$ to the flow) is the O’Neill tensor of the flow.

In this paper, we give a formula corresponding to (1.1) for any eigen-spinor $\psi$ of the square of the Dirac operator on compact surfaces endowed with any Spin$^c$ structure (see Theorem 3.1). It is motivated by the following two facts: First, when we consider eigenvalues of the square of the Dirac operator, another tensor field is of interest. It is the skew-symmetric tensor field $Q^\psi$ given by

$$Q^\psi(X,Y) = g(q^\psi(X),Y) = \frac{1}{2} \operatorname{Re} \left( X \cdot \nabla_Y \psi - Y \cdot \nabla_X \psi, \frac{\psi}{|\psi|^2} \right),$$

for all vector fields $X, Y \in \Gamma(TM)$. This tensor was studied by the first author in the context of Riemannian flows [12]. Second, we consider any compact surface $M$ immersed in $\mathbb{S}^2 \times \mathbb{R}$ where $\mathbb{S}^2$ is the round sphere equipped with a metric of curvature one. The Spin$^c$ structure on $\mathbb{S}^2 \times \mathbb{R}$, induced from the canonical one on $\mathbb{S}^2$ and the Spin structure on $\mathbb{R}$, admits a parallel spinor [22]. The restriction to $M$ of this Spin$^c$ structure is still a Spin$^c$ structure with a generalized Killing spinor [24].

In Section 2, we recall some basic facts on Spin$^c$ structures and the restrictions of these structures to hypersurfaces. In Section 3 and after giving a formula corresponding to (1.1) for any eigenspinor $\psi$ of the square of the Dirac operator, we deduce a formula for the integral of the determinant of $T^\psi + Q^\psi$ and we establish a new proof of the Bär-type inequality (1.3). In Section 4, we consider the 3-dimensional case and treat examples of hypersurfaces in $\mathbb{C}\mathbb{P}^2$. In the last section, we come back to the question of a spinorial characterisation of surfaces in $\mathbb{S}^2 \times \mathbb{R}$. Here we use a different approach than the one in [25]. In fact, we prove that given any surface $M$ carrying a generalized Killing spinor associated with a particular Spin$^c$ structure, the Energy-Momentum tensor satisfies the four compatibility equations stated by B. Daniel [6]. Thus there exists a local immersion of $M$ into $\mathbb{S}^2 \times \mathbb{R}.$

2 Preliminaries

In this section, we begin with some preliminaries concerning Spin$^c$ structures and the Dirac operator. Details can be found in [18], [20], [7], [23] and [24].

The Dirac operator on Spin$^c$ manifolds: Let $(M^n, g)$ be a Riemannian manifold of dimension $n \geq 2$ without boundary. We denote by $\text{SO}(n)$
the SO\(_n\)-principal bundle over \(M\) of positively oriented orthonormal frames. A Spin\(^{c}\) structure of \(M\) is a Spin\(_{n}\)-principal bundle (Spin\(^{c}\)\(M, π, M\)) and an S\(^1\)-principal bundle (S\(^1\)\(M, π, M\)) together with a double covering given by \(θ : \text{Spin}^cM \rightarrow \text{SOM} \times_M S^1M\) such that \(θ(u) = θ(a)\xi(a)\), for every \(u \in \text{Spin}^cM\) and \(a \in \text{Spin}^c\), where \(ξ\) is the 2-fold covering of Spin\(_n\) over SO\(_n\) \(\times S^1\). Let \(ΣM := \text{Spin}^cM \times πn, Σn\) be the associated spinor bundle where \(Σn = \mathbb{C}^2π\) and \(πn : \text{Spin}^c \rightarrow \text{End}(Σn)\) denotes the complex spinor representation. A section of \(ΣM\) will be called a spinor field. The spinor bundle \(ΣM\) is equipped with a natural Hermitian scalar product denoted by \((.,.)\). We define an \(L^2\)-scalar product \(ψ, φ = ∫M(ψ, φ)v_g\), for any spinors \(ψ\) and \(φ\). Additionally, any connection 1-form \(A : T(S^1M) \rightarrow i\mathbb{R}\) on \(S^1M\) and the connection 1-form \(ωM\) on SOM, induce a connection on the principal bundle \(\text{SOM} \times_M S^1M\), and hence a covariant derivative \(∇\) on \(Γ(ΣM)\). The curvature of \(A\) is an imaginary valued 2-form denoted by \(F_A = dA\), i.e., \(F_A = iω\), where \(ω\) is a real valued 2-form on \(S^1M\). We know that \(ω\) can be viewed as a real valued 2-form on \(M\). In this case \(iω\) is the curvature form of the associated line bundle \(L\). It is the complex line bundle associated with the S\(^1\)-principal bundle via the standard representation of the unit circle. For every spinor \(ψ\), the Dirac operator is locally defined by

\[
Dψ = \sum_{i=1}^{n} e_i \cdot ∇e_i ψ,
\]

where \((e_1, ..., e_n)\) is a local oriented orthonormal tangent frame and \(\cdot\) denotes the Clifford multiplication. The Dirac operator is an elliptic, self-adjoint operator with respect to the \(L^2\)-scalar product and verifies, for any spinor field \(ψ\), the Schrödinger-Lichnerowicz formula

\[
D^2ψ = ∇^*∇ψ + \frac{1}{4}Sψ + \frac{i}{2}Ω \cdot ψ
\]

where \(Ω\) is the extension of the Clifford multiplication to differential forms given by \((e_i \wedge e_j) \cdot ψ = e_i \cdot e_j \cdot ψ\). For any spinor \(ψ \in Γ(ΣM)\), we have

\[
(iΩ \cdot ψ, ψ) \geq -\frac{cn}{2}|Ω|_g|ψ|^2,
\]

where \(|Ω|_g\) is the norm of \(Ω\), with respect to \(g\) given by \(|Ω|_g^2 = ∑_{i<j}(Ω_{ij})^2\) in any orthonormal local frame and \(c_n = 2|Ω|^2\). Moreover, equality holds in (2.2) if and only if \(Ω \cdot ψ = i\frac{|Ω|}{2}|Ω|gψ\). Every Spin manifold has a trivial Spin\(^c\) structure. In fact, we choose the trivial line bundle with the trivial connection whose curvature \(iΩ\) is zero. Also every Kähler manifold \(M\) of complex dimension \(m\) has a canonical Spin\(^c\) structure. Let \(κ\) by the Kähler form defined by the complex structure \(J\), i.e. \(κ(X, Y) = g(JX, Y)\) for all vector fields \(X, Y \in Γ(TM)\). The complexified cotangent bundle

\[
T^*M \otimes \mathbb{C} = Λ^{1,0}M ⊕ Λ^{0,1}M
\]

decomposes into the \(±i\)-eigenbundles of the complex linear extension of the complex structure. Thus, the spinor bundle of the canonical Spin\(^c\) structure is given by

\[
ΣM = Λ^{0,*}M = ⊕_{r=0}^{m} Λ^{0,r}M,
\]

where \(Ω\) is the extension of the Clifford multiplication to differential forms given by \((e_i \wedge e_j) \cdot ψ = e_i \cdot e_j \cdot ψ\). For any spinor \(ψ \in Γ(ΣM)\), we have

\[
(iΩ \cdot ψ, ψ) \geq -\frac{cn}{2}|Ω|_g|ψ|^2,
\]

where \(|Ω|_g\) is the norm of \(Ω\), with respect to \(g\) given by \(|Ω|_g^2 = ∑_{i<j}(Ω_{ij})^2\) in any orthonormal local frame and \(c_n = 2|Ω|^2\). Moreover, equality holds in (2.2) if and only if \(Ω \cdot ψ = i\frac{|Ω|}{2}|Ω|gψ\). Every Spin manifold has a trivial Spin\(^c\) structure. In fact, we choose the trivial line bundle with the trivial connection whose curvature \(iΩ\) is zero. Also every Kähler manifold \(M\) of complex dimension \(m\) has a canonical Spin\(^c\) structure. Let \(κ\) by the Kähler form defined by the complex structure \(J\), i.e. \(κ(X, Y) = g(JX, Y)\) for all vector fields \(X, Y \in Γ(TM)\). The complexified cotangent bundle

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decomposes into the \(±i\)-eigenbundles of the complex linear extension of the complex structure. Thus, the spinor bundle of the canonical Spin\(^c\) structure is given by

\[
ΣM = Λ^{0,*}M = ⊕_{r=0}^{m} Λ^{0,r}M,
\]
where $\Lambda^{0,r}M = \Lambda^r(\Lambda^{0,1}M)$ is the bundle of $r$-forms of type $(0,1)$. The line bundle of this canonical Spin$^c$ structure is given by $L = (K_M)^{-1} = \Lambda^{m}(\Lambda^{0,1}M)$, where $K_M$ is the canonical bundle of $M$ \[7, 19\]. This line bundle $L$ has a canonical holomorphic connection induced from the Levi-Civita connection whose curvature form is given by $i\Omega = -i\rho$, where $\rho$ is the Ricci form given by $\rho(X,Y) = \text{Ric}(JX,Y)$. We point out that the canonical Spin$^c$ structure on every Kähler manifold carries a parallel spinor \[7, 22\].

**Spin$^c$ hypersurfaces and the Gauss formula:** Let $Z$ be an oriented $(n+1)$-dimensional Riemannian Spin$^c$ manifold and $M \subset Z$ be an oriented hypersurface. The manifold $M$ inherits a Spin$^c$ structure induced from the one on $Z$, and we have \[24\]

$$
\Sigma M \simeq \begin{cases} 
\Sigma Z_{|M} & \text{if } n \text{ is even}, \\
\Sigma^+ Z_{|M} & \text{if } n \text{ is odd}.
\end{cases}
$$

Moreover Clifford multiplication by a vector field $X$, tangent to $M$, is given by

$$X \bullet \varphi = (X \cdot \nu \cdot \psi)_{|M},$$

where $\psi \in \Gamma(\Sigma Z)$ (or $\psi \in \Gamma(\Sigma^+ Z)$ if $n$ is odd), $\varphi$ is the restriction of $\psi$ to $M$, $\cdot$ is the Clifford multiplication on $Z$, $\bullet$ that on $M$ and $\nu$ is the unit normal vector. The connection 1-form defined on the restricted $S^1$-principal bundle $(P_{S^1}M := P_{S^1}Z_{|M}, \pi, M)$, is given by $A = A^Z_{|M} : T(P_{S^1}M) = T(P_{S^1}Z)_{|M} \rightarrow i\mathbb{R}$. Then the curvature 2-form $i\Omega$ on the $S^1$-principal bundle $P_{S^1}M$ is given by $i\Omega = i\Omega^Z_{|M}$, which can be viewed as an imaginary 2-form on $M$ and hence as the curvature form of the line bundle $L^M$, the restriction of the line bundle $L^Z$ to $M$. For every $\psi \in \Gamma(\Sigma Z)$ ($\psi \in \Gamma(\Sigma^+ Z)$ if $n$ is odd), the real 2-forms $\Omega$ and $\Omega^Z$ are related by \[24\]

$$(\Omega^Z \cdot \psi)_{|M} = \Omega \bullet \varphi - (\nu \cdot \Omega^Z) \bullet \varphi.$$

We denote by $\nabla^{\Sigma Z}$ the spinorial Levi-Civita connection on $\Sigma Z$ and by $\nabla$ that on $\Sigma M$. For all $X \in \Gamma(TM)$, we have the spinorial Gauss formula \[24\]:

$$
(\nabla^{\Sigma Z}_X \psi)_{|M} = \nabla_X \psi + \frac{1}{2} H(X) \bullet \varphi,
$$

where $H$ denotes the Weingarten map of the hypersurface. Moreover, Let $D^Z$ and $D^M$ be the Dirac operators on $Z$ and $M$, after denoting by the same symbol any spinor and its restriction to $M$, we have

$$
\nu \cdot D^Z \varphi = \tilde{D} \varphi + \frac{n}{2} H \varphi - \nabla^{\Sigma Z}_\nu \varphi,
$$

where $H = \frac{1}{n} \text{tr}(II)$ denotes the mean curvature and $\tilde{D} = D^M \oplus (-D^M)$ if $n$ is odd.

### 3 The 2-dimensional case

In this section, we consider compact surfaces endowed with any Spin$^c$ structure. We have
Theorem 3.1 Let \((M^2, g)\) be a Riemannian manifold and \(\psi\) an eigenspinor of the square of the Dirac operator \(D^2\) with eigenvalue \(\lambda^2\) associated with any Spin\(^c\) structure. Then we have

\[
\lambda^2 = \frac{S}{4} + |T^\psi|^2 + |Q^\psi|^2 + \Delta f + |Y|^2 - 2Y(f) + \left(\frac{i}{2} \Omega \cdot \psi, \frac{\psi}{|\psi|^2}\right),
\]

where \(f\) is the real-valued function defined by \(f = \frac{1}{2} \ln |\psi|^2\) and \(Y\) is a vector field on \(TM\) given by \(g(Y, Z) = \frac{1}{|\psi|^2} \text{Re} \left( D\psi, Z \cdot \psi \right) \) for any \(Z \in \Gamma(TM)\).

Proof. Let \(\{e_1, e_2\}\) be an orthonormal frame of \(TM\). Since the spinor bundle \(\Sigma M\) is of real dimension 4, the set \(\{\frac{\psi}{|\psi|}, \frac{e_1 \psi}{|\psi|}, \frac{e_2 \psi}{|\psi|}, \frac{e_1 \cdot e_2 \psi}{|\psi|}\}\) is orthonormal with respect to the real product \(\text{Re} \left( \cdot, \cdot \right)\). The covariant derivative of \(\psi\) can be expressed in this frame as

\[
\nabla_X \psi = \delta(X)\psi + \alpha(X) \cdot \psi + \beta(X) e_1 \cdot e_2 \cdot \psi,
\]

for all vector fields \(X\), where \(\delta\) and \(\beta\) are 1-forms and \(\alpha\) is a \((1,1)\)-tensor field. Moreover, \(\beta, \delta\) and \(\alpha\) are uniquely determined by the spinor \(\psi\). In fact, taking the scalar product of \((3.1)\) respectively with \(\psi, e_1 \cdot \psi, e_2 \cdot \psi, e_1 \cdot e_2 \cdot \psi\), we get

\[
\alpha(X) = -\ell^\psi(X) + \nu^\psi(X) \quad \text{and} \quad \beta(X) = \frac{1}{|\psi|^2} \text{Re} \left( \nabla_X \psi, e_1 \cdot e_2 \cdot \psi \right).
\]

Using \((2.1)\), it follows that

\[
\lambda^2 = \frac{\Delta(|\psi|^2)}{2|\psi|^2} + |\alpha|^2 + |\beta|^2 + |\delta|^2 + \frac{1}{4} S + \left(\frac{i}{2} \Omega \cdot \psi, \frac{\psi}{|\psi|^2}\right).
\]

Now it remains to compute the term \(|\beta|^2\). We have

\[
|\beta|^2 = \frac{1}{|\psi|^4} \text{Re} \left( \nabla_{e_1 \cdot \psi}, e_1 \cdot e_2 \cdot \psi \right)^2 + \frac{1}{|\psi|^4} \text{Re} \left( \nabla_{e_2 \cdot \psi}, e_1 \cdot e_2 \cdot \psi \right)^2
\]

\[
= \frac{1}{|\psi|^4} \text{Re} \left( D\psi - e_2 \cdot \nabla_{e_2 \cdot \psi}, e_2 \cdot \psi \right)^2 + \frac{1}{|\psi|^4} \text{Re} \left( D\psi - e_1 \cdot \nabla_{e_1 \cdot \psi}, e_1 \cdot \psi \right)^2
\]

\[
= g(Y, e_1)^2 + g(Y, e_2)^2 + \left(\frac{d(|\psi|^2)}{|\psi|^2}\right)^2 - g(Y, \frac{d(|\psi|^2)}{|\psi|^2})
\]

\[
= |Y|^2 - 2Y(f) + \frac{|d(|\psi|^2)|^2}{4|\psi|^4},
\]

which gives the result by using the fact that \(\Delta f = \frac{\Delta(|\psi|^2)}{2|\psi|^2} + \frac{|d(|\psi|^2)|^2}{2|\psi|^4}\). \(\Box\)

Remark 3.2 Under the same conditions as Theorem \((3.1)\), if \(\psi\) is an eigenspinor of \(D\) with eigenvalue \(\lambda\), we get

\[
\lambda^2 = \frac{S}{4} + |T^\psi|^2 + \Delta f + \left(\frac{i}{2} \Omega \cdot \psi, \frac{\psi}{|\psi|^2}\right).
\]

In fact, in this case \(Y = 0\) and

\[
0 = \text{Re} \left( D\psi, e_1 \cdot e_2 \cdot \psi \right) = \text{Re} \left( e_1 \cdot \nabla_{e_1 \cdot \psi} + e_2 \cdot \nabla_{e_2 \cdot \psi}, e_1 \cdot e_2 \cdot \psi \right)
\]

\[
= \text{Re} \left( -e_2 \cdot \nabla_{e_1 \cdot \psi} + e_1 \cdot \nabla_{e_2 \cdot \psi}, \psi \right) = 2Q^\psi(e_1, e_2)|\psi|^2.
\]

This was proven by Friedrich and Kim in \((3)\) for a Spin structure on \(M\).
In the following, we will give an estimate for the integral \( \int_M \det(T^\psi + Q^\psi)v_g \) in terms of geometric quantities, which has the advantage that it does not depend on the eigenvalue \( \lambda \) nor on the eigenspinor \( \psi \). This is a generalization of the result of Friedrich and Kim in [9] for Spin structures.

**Theorem 3.3** Let \( M \) be a compact surface and \( \psi \) any eigenspinor of \( D^2 \) associated with eigenvalue \( \lambda^2 \). Then we have

\[
\int_M \det(T^\psi + Q^\psi)v_g \geq \frac{\pi \chi(M)}{2} - \frac{1}{4} \int_M |\Omega|v_g.
\]

Equality in (3.3) holds if and only if either \( \Omega \) is zero or has constant sign.

**Proof.** As in the previous theorem, the spinor \( D\psi \) can be expressed in the orthonormal frame of the spinor bundle. Thus the norm of \( D\psi \) is equal to

\[
|D\psi|^2 = \frac{1}{|\psi|^2} \text{Re} (D\psi, \psi)^2 + \frac{1}{|\psi|^2} \sum_{i=1}^2 \text{Re} (D\psi, e_i \cdot \psi)^2 + \frac{1}{|\psi|^2} \text{Re} (D\psi, e_1 \cdot e_2 \cdot \psi)^2
\]

\[
= (\text{tr} T^\psi)^2 |\psi|^2 + |Y|^2 |\psi|^2 + \frac{1}{|\psi|^2} \text{Re} (D\psi, e_1 \cdot e_2 \cdot \psi)^2,
\]

where we recall that the trace of \( T^\psi \) is equal to \( -\frac{1}{|\psi|^2} \text{Re} (D\psi, \psi) \). On the other hand, by (3.2) we have that \( \frac{1}{|\psi|^2} \text{Re} (D\psi, e_1 \cdot e_2 \cdot \psi)^2 = 2 |Q^\psi|^2 |\psi|^2 \). Thus Equation (3.4) reduces to

\[
\frac{|D\psi|^2}{|\psi|^2} = (\text{tr} T^\psi)^2 + |Y|^2 + 2 |Q^\psi|^2.
\]

Now with the use of the equality \( \text{Re} (D^2 \psi, \psi) = |D\psi|^2 - \text{div} \xi \), where \( \xi \) is the vector field given by \( \xi = |\psi|^2 Y \), we get

\[
\lambda^2 + \frac{1}{|\psi|^2} \text{div} \xi = (\text{tr} T^\psi)^2 + |Y|^2 + 2 |Q^\psi|^2.
\]

An easy computation leads to \( \frac{1}{|\psi|^2} \text{div} \xi = \text{div} Y + 2Y(f) \) where we recall that \( f = \frac{1}{2} \text{ln}(|\psi|^2) \). Hence substituting this formula into (3.5) and using Theorem 3.1 yields

\[
\frac{S}{4} + (\frac{i}{2} \Omega \cdot \psi, \frac{\psi}{|\psi|^2}) + \Delta f + \text{div} Y = (\text{tr} T^\psi)^2 + |Q^\psi|^2 - |T^\psi|^2 = 2 \det(T^\psi + Q^\psi).
\]

Finally integrating over \( M \) and using the Gauss-Bonnet formula, we deduce the required result with the help of Equation (2.2). Equality holds if and only if \( \Omega \cdot \psi = i|\Omega|\psi \). In the orthonormal frame \( \{e_1, e_2\} \), the 2-form \( \Omega \) can be written \( \Omega = \Omega_{12} e_1 \wedge e_2 \), where \( \Omega_{12} \) is a function defined on \( M \). Using the decomposition of \( \psi \) into positive and negative spinors \( \psi^+ \) and \( \psi^- \), we find that the equality is attained if and only if

\[
\Omega_{12} e_1 \cdot e_2 \cdot \psi^+ + \Omega_{12} e_1 \cdot e_2 \cdot \psi^- = i|\Omega_{12}|\psi^+ + i|\Omega_{12}|\psi^-,
\]

which is equivalent to say that,

\[
\Omega_{12} \psi^+ = -|\Omega_{12}|\psi^+ \quad \text{and} \quad \Omega_{12} \psi^- = |\Omega_{12}|\psi^-.
\]
Now if $\psi^+ \neq 0$ and $\psi^- \neq 0$, we get $\Omega = 0$. Otherwise, it has constant sign. In the last case, we get that $\int_M |\Omega|v_9 = 2\pi \chi(M)$, which means that the l.h.s. of this equality is a topological invariant. □

Next, we will give another proof of the Bär-type inequality (1.3) for the eigenvalues of any Spin$^c$ Dirac operator. The following theorem was proved by the second author in [23] using conformal deformation of the spinorial Levi-Civita connection.

**Theorem 3.4** Let $M$ be a compact surface. For any Spin$^c$ structure on $M$, any eigenvalue $\lambda$ of the Dirac operator $D$ to which is attached an eigenspinor $\psi$ satisfies

$$\lambda^2 \geq \frac{2\pi \chi(M)}{\text{Area}(M)} - \frac{1}{\text{Area}(M)} \int_M |\Omega|v_9.$$  \hspace{1cm} (3.6)

Equality holds if and only if the eigenspinor $\psi$ is a Spin$^c$ Killing spinor, i.e., it satisfies $\Omega \cdot \psi = i|\Omega|\psi$ and $\nabla_X \psi = -\frac{\lambda}{2} X \cdot \psi$ for any $X \in \Gamma(TM)$.

**Proof.** With the help of Remark (3.2), we have that

$$\lambda^2 = \frac{S^2}{4} + |T\psi|^2 + \Delta f + (\frac{i}{2} \Omega \cdot \psi, \frac{\psi}{|\psi|^2}).$$  \hspace{1cm} (3.7)

Substituting the Cauchy-Schwarz inequality, i.e. $|T\psi|^2 \geq \frac{\lambda^2}{2}$ and the estimate (2.2) into Equality (3.7), we easily deduce the result after integrating over $M$.

Now the equality in (3.6) holds if and only if the eigenspinor $\psi$ satisfies $\Omega \cdot \psi = i|\Omega|\psi$ and $|T\psi|^2 = \frac{\lambda^2}{2}$. Thus, the second equality is equivalent to say that $\ell^\psi(X) = \frac{\lambda}{2} X$ for all $X \in \Gamma(TM)$. Finally, a straightforward computation of the spinorial curvature of the spinor field $\psi$ gives in a local frame $\{e_1, e_2\}$ after using the fact $\beta = -(\ast \delta)$ that

$$\frac{1}{2} R_{1212} \cdot e_1 \cdot e_2 \cdot \psi = \left( \frac{\lambda^2}{2} + e_1(\delta(e_1)) + e_2(\delta(e_2)) \right) e_2 \cdot e_1 \cdot \psi - \lambda \delta(e_2) e_1 \cdot \psi + \lambda \delta(e_1) e_2 \cdot \psi + \left( e_1(\delta(e_2)) - e_2(\delta(e_1)) \right) \psi.$$

Thus the scalar product with $e_1 \cdot \psi$ and $e_2 \cdot \psi$ implies that $\delta = 0$. Finally, $\beta = 0$ and the eigenspinor $\psi$ is a Spin$^c$ Killing spinor. □

Now, we will give some examples where equality holds in (3.6) or in (3.3). Some applications of Theorem 3.4 are also given.

**Examples:**

1. Let $S^2$ be the round sphere equipped with the standard metric of curvature one. As a Kähler manifold, we endow the sphere with the canonical Spin$^c$ structure of curvature form equal to $i\Omega = -i\kappa$, where $\kappa$ is the Kähler 2-form. Hence, we have $|\Omega| = |\kappa| = 1$. Furthermore, we mentioned that for the canonical Spin$^c$ structure, the sphere carries parallel spinors, i.e., an eigenspinor associated with the eigenvalue $0$ of the Dirac operator $D$. Thus equality holds in (3.6). On the other hand, the equality in (3.3) also holds, since the sign of the curvature form $\Omega$ is constant.
2. Let \( f : M \to S^3 \) be an isometric immersion of a surface \( M^2 \) into the sphere equipped with its unique Spin structure and assume that the mean curvature \( H \) is constant. The restriction of a Killing spinor on \( S^3 \) to the surface \( M \) defines a spinor field \( \varphi \) solution of the following equation \[ \tag{3.8} \]

\[
\nabla_X \varphi = -\frac{1}{2} II(X) \cdot \varphi + \frac{1}{2} J(X) \cdot \varphi,
\]

where \( II \) denotes the second fundamental form of the surface and \( J \) is the complex structure of \( M \) given by the rotation of angle \( \frac{
\pi}{2} \) on \( TM \). It is easy to check that \( \varphi \) is an eigenspinor for \( D^2 \) associated with the eigenvalue \( H^2 + 1 \). Moreover \( D\varphi = H \varphi + e_1 \cdot e_2 \cdot \varphi \), so that \( Y = 0 \). Moreover the tensor \( T^g = \frac{1}{4} II \) and \( Q^g = \frac{1}{2} J \). Hence by Theorem 3.1 and since the norm of \( \varphi \) is constant, we obtain

\[
H^2 + \frac{1}{2} = \frac{1}{4} S + \frac{1}{4} |II|^2.
\]

3. On two-dimensional manifolds, we can define another Dirac operator associated with the complex structure \( J \) given by \( \bar{D} = \bar{D}e_1 \cdot \nabla_{e_1} + \bar{D}e_2 \cdot \nabla_{e_2} = e_2 \cdot \nabla_{e_1} - e_1 \cdot \nabla_{e_2} \). Since \( \bar{D} \) satisfies \( D^2 = (\bar{D})^2 \), all the above results are also true for the eigenvalues of \( \bar{D} \).

4. Let \( M^2 \) be a surface immersed in \( S^2 \times \mathbb{R} \). The product of the canonical Spin\(^c \) structure on \( S^2 \) and the unique Spin structure on \( \mathbb{R} \) define a Spin\(^c \) structure on \( S^2 \times \mathbb{R} \) carrying parallel spinors. Moreover, by the Schrödinger-Lichnerowicz formula, any parallel spinor \( \psi \) satisfies \( \Omega^{S^2 \times \mathbb{R}} \cdot \psi = i\psi \), where \( \Omega^{S^2 \times \mathbb{R}} \) is the curvature form of the auxiliary line bundle. Let \( \nu \) be a unit normal vector field of the surface. We then write \( \partial t = T + f\nu \) where \( T \) is a vector field on \( TM \) with \(|T|^2 + f^2 = 1 \). On the other hand, the vector field \( T \) splits into \( T = \nu_1 \) and \( h\partial t \) where \( \nu_1 \) is a vector field on the sphere. The scalar product of the first equation by \( T \) and the second one by \( \partial t \) gives \(|T|^2 = h \) which means that \( h = 1 - f^2 \). Hence the normal vector field \( \nu \) can be written as \( \nu = f\partial t - \frac{1}{f}\nu_1 \). As we mentioned before, the Spin\(^c \) structure on \( S^2 \times \mathbb{R} \) induces a Spin\(^c \) structure on \( M \) with induced auxiliary line bundle. Next, we want to prove that the curvature form of the auxiliary line bundle of \( M \) is equal to \( i\Omega(e_1, e_2) = -if \), where \( \{e_1, e_2\} \) denotes a local orthonormal frame on \( TM \). Since the spinor \( \psi \) is parallel, we have by \( \Omega^{S^2 \times \mathbb{R}} \cdot X \cdot \psi = i(X, \Omega^{S^2 \times \mathbb{R}}) \cdot \psi \). Therefore, we compute

\[
(\nu, \Omega^{S^2 \times \mathbb{R}}) \cdot \varphi = (\nu, \Omega^{S^2 \times \mathbb{R}}) \cdot \nu \cdot \psi|_M = iv \cdot \text{Ric}^{S^2 \times \mathbb{R}} \nu \cdot \psi|_M
\]

\[
= -\frac{1}{f} iv \cdot \nu_1 \cdot \psi|_M = iv \cdot (\nu - f\partial t) \psi|_M
\]

\[
= (-i\psi - if\nu \cdot \partial t \cdot \psi)|_M.
\]

Hence by Equation \( \tag{2.4} \), we get that \( \Omega \cdot \varphi = -i(f\nu \cdot \partial t \cdot \psi)|_M \). The scalar product of the last equality with \( e_1 \cdot e_2 \cdot \psi \) gives

\[
\Omega(e_1, e_2)|\psi|^2 = -f \text{Re} (iv \cdot \partial t \cdot \psi, e_1 \cdot e_2 \cdot \psi)|_M = -f \text{Re} (i\partial t \cdot \psi, \psi)|_M.
\]

We now compute the term \( i\partial t \cdot \psi \). For this, let \( \{e_1', J e_1'\} \) be a local orthonormal frame of the sphere \( S^2 \). The complex volume form acts as the
identity on the spinor bundle of $S^2 \times \mathbb{R}$, hence $\partial_t \cdot \psi = c'_1 \cdot Je'_1 \cdot \psi$. But we have
\[\Omega_{S^2 \times \mathbb{R}} \cdot \psi = -\rho \cdot \psi = -\kappa \cdot \psi = -c'_1 \cdot Je'_1 \cdot \psi.\]
Therefore, $i\partial_t \cdot \psi = \psi$. Thus we get $\Omega(e_1, e_2) = -\mathbf{f}$. Finally,
\[(i\Omega \cdot \varphi, \varphi) = f \text{Re} (\nu \cdot \partial_t \cdot \psi, \psi)|_M = -fg(\nu, \partial_t)|\varphi|^2 = -f^2|\varphi|^2.\]
Hence Equality in Theorem 3.1 is just
\[H^2 = \frac{S}{4} + \frac{1}{4} |II|^2 - \frac{1}{2} f^2.\]

4 The 3-dimensional case

In this section, we will treat the 3-dimensional case.

**Theorem 4.1** Let $(M^3, g)$ be an oriented Riemannian manifold. For any Spin$^c$ structure on $M$, any eigenvalue $\lambda$ of the Dirac operator to which is attached an eigenspinor $\psi$ satisfies
\[\lambda^2 \leq \frac{1}{\text{vol}(M, g)} \int_M (|T\psi|^2 + \frac{S}{4} + \frac{|\Omega|}{2}) \nu_g.\]
Equality holds if and only if the norm of $\psi$ is constant and $\Omega \cdot \psi = i|\Omega|\psi$.

**Proof.** As in the proof of Theorem 3.1, the set $\{\psi, \frac{\psi}{|\psi|}, \frac{i\psi}{|\psi|}, \frac{\psi}{|\psi|} \}$ is orthonormal with respect to the real product $\text{Re} (\cdot, \cdot)$. The covariant derivative of $\psi$ can be expressed in this frame as
\[\nabla_X \psi = \eta(X) \psi + \ell(X) \cdot \psi,\]
for all vector fields $X$, where $\eta$ is a 1-form and $\ell$ is a $(1, 1)$-tensor field. Moreover $\eta = \frac{d(|\psi|^2)}{2|\psi|^2}$ and $\ell(X) = -T\psi(X)$. Using (2.1), it follows that
\[\lambda^2 = \frac{\Delta (|\psi|^2)^2}{2|\psi|^4} + |T\psi|^2 + \frac{d(|\psi|^2)^2}{4|\psi|^4} + \frac{1}{4} S + \left(\frac{i}{2} \Omega \cdot \psi, \frac{\psi}{|\psi|^2}\right)\]
\[= \Delta f - \frac{|d(|\psi|^2)|^2}{2|\psi|^4} + |T\psi|^2 + \frac{1}{4} S + \left(\frac{i}{2} \Omega \cdot \psi, \frac{\psi}{|\psi|^2}\right).\]
By the Cauchy-Schwarz inequality, we have $\frac{1}{2}(i \Omega \cdot \psi, \frac{\psi}{|\psi|^2}) \leq \frac{1}{2} |\Omega|$. Integrating over $M$ and using the fact that $|d(|\psi|^2)|^2 \geq 0$, we get the result. \(\square\)

**Example 4.2** Let $M^3$ be a 3-dimensional Riemannian manifold immersed in $\mathbb{CP}^2$ with constant mean curvature $H$. Since $\mathbb{CP}^2$ is a Kähler manifold, we endow it with the canonical Spin$^c$ structure whose line bundle has curvature equal to $-3i\kappa$. Moreover, by the Schrödinger-Lichnerowicz formula we have that any parallel spinor $\psi$ satisfies $\Omega_{\mathbb{CP}^2} \cdot \psi = 6i\psi$. As in the previous example, we compute
\[\left(\nu, \Omega^2_{\mathbb{CP}^2}\right) \cdot \varphi = i (\nu \cdot \text{Ric}_{\mathbb{CP}^2} (\nu) \cdot \psi)|_M = -3i\varphi.\]
Finally, $\Omega \cdot \phi = 3i\phi$. Using Equation (2.6), we have that $-\frac{3}{2}H$ is an eigenvalue of $D$. Since the norm of $\phi$ is constant, equality holds in Theorem 4.1 and hence
\[ \frac{9}{4}H^2 + \frac{3}{2} = \frac{S}{4} + \frac{1}{4}||T||^2. \]

5 Characterization of surfaces in $S^2 \times \mathbb{R}$

In this section, we characterize the surfaces in $S^2 \times \mathbb{R}$ by solutions of the generalized Killing spinors equation which are restrictions of parallel spinors of the canonical Spin$^c$-structure on $S^2 \times \mathbb{R}$ (see also [25] for a different proof). First recall the compatibility equations for characterization of surfaces in $S^2 \times \mathbb{R}$ established by B. Daniel [6, Thm 3.3]:

**Theorem 5.1** Let $(M, g)$ be a simply connected Riemannian manifold of dimension 2, $A : TM \rightarrow TM$ a field of symmetric operator and $T$ a vector field on $TM$. We denote by $f$ a real valued function such that $f^2 + ||T||^2 = 1$. Assume that $M$ satisfies the Gauss-Codazzi equations, i.e. $G = \det A + f^2$ and
\[ d^\nabla A(X, Y) := (\nabla_X A)Y - (\nabla_Y A)X = f(g(Y, T)X - g(X, T)Y), \]
where $G$ is the gaussian curvature, and the following equations
\[ \nabla_X T = fA(X), \quad X(f) = -g(AX, T). \]

Then there exists an isometric immersion of $M$ into $S^2 \times \mathbb{R}$ such that the Weingarten operator is $A$ and $\partial t = T + f\nu$, where $\nu$ is the unit normal vector field to the surface $M$.

Now using this characterization theorem, we state our result:

**Theorem 5.2** Let $M$ be an oriented simply connected Riemannian manifold of dimension 2. Let $T$ be a vector field and denote by $f$ a real valued function such that $f^2 + ||T||^2 = 1$. Denote by $A$ a symmetric endomorphism field of $TM$. The following statements are equivalent:

1. There exists an isometric immersion of $M$ into $S^2 \times \mathbb{R}$ of Weingarten operator $A$ such that $\partial t = T + f\nu$, where $\nu$ is the unit normal vector field of the surface.

2. There exists a Spin$^c$ structure on $M$ whose line bundle has a connection of curvature given by $i\Theta = -if\kappa$, such that it carries a non-trivial solution $\phi$ of the generalized Killing spinor equation $\nabla_X \phi = -\frac{1}{2}AX \cdot \phi$, with $T \cdot \phi = -f\phi + \bar{\phi}$.

**Proof.** We begin with 1 $\Rightarrow$ 2. The existence of such a Spin$^c$ structure is assured by the restriction of the canonical one on $S^2 \times \mathbb{R}$. Moreover, using the spinorial Gauss formula [24, 25], any parallel spinor $\psi$ on $S^2 \times \mathbb{R}$ induces a generalized Killing spinor $\phi = \psi|_M$ with $A$ the Weingarten map of the surface $M$. Hence it remains
to show the relation $T \bullet \varphi = -f \varphi + \bar{\varphi}$. In fact, using that $\Omega^{S^2 \times R} \cdot \psi = i\psi$, we write in the frame $\{e_1, e_2, \nu\}$

$$\Omega^{S^2 \times R}(e_1, e_2)e_1 \cdot e_2 \cdot \psi + \Omega^{S^2 \times R}(e_1, \nu)e_1 \cdot \nu \cdot \psi + \Omega^{S^2 \times R}(e_2, \nu)e_2 \cdot \nu \cdot \psi = i\psi. \quad (5.1)$$

By the previous example in Section 3 we know that $\Omega^{S^2 \times R}(e_1, e_2) = -f$. For the other terms, we compute

$$\Omega^{S^2 \times R}(e_1, \nu) = \Omega^{S^2 \times R}(e_1, 1, \frac{1}{f} \partial_t - \frac{1}{f} T) = -\frac{1}{f} g(T, e_2) \Omega^{S^2 \times R}(e_1, e_2) = g(T, e_2),$$

where the term $\Omega^{S^2 \times R}(e_1, \partial_t)$ vanishes since we can split $e_1$ into a sum of vectors on the sphere and on $\mathbb{R}$. Similarly, we find that $\Omega^{S^2 \times R}(e_2, \nu) = -g(T, e_1)$. By substituting these values into (5.1) and taking Clifford multiplication with $e_1 \cdot e_2$, we get the desired property. For $2 \Rightarrow 1$, a straightforward computation for the spinorial curvature of the generalized Killing spinor $\varphi$ yields on a local frame $\{e_1, e_2\}$ of $TM$ that

$$(-G + \det A)e_1 \cdot e_2 \cdot \varphi = -(d^\nabla A)(e_1, e_2) \cdot \varphi + if\varphi. \quad (5.2)$$

In the following, we will prove that the spinor field $\theta := i\varphi - if\bar{\varphi} + JT \bullet \varphi$ is zero. For this, it is sufficient to prove that its norm vanishes. Indeed, we compute

$$|\theta|^2 = |\varphi|^2 + f^2|\bar{\varphi}|^2 + ||T||^2|\varphi|^2 - 2Re (i\varphi, if\bar{\varphi}) + 2Re (i\varphi, JT \bullet \varphi) \quad (5.3)$$

From the relation $T \bullet \varphi = -f\varphi + \bar{\varphi}$ we deduce that $Re (\varphi, \bar{\varphi}) = f|\varphi|^2$ and the equalities

$$g(T, e_1)|\varphi|^2 = Re (ie_2 \cdot \varphi, \varphi) \quad \text{and} \quad g(T, e_2)|\bar{\varphi}|^2 = -Re (ie_1 \cdot \varphi, \varphi).$$

Therefore, Equation (5.3) becomes

$$|\theta|^2 = 2|\varphi|^2 - 2f^2|\varphi|^2 + 2Re (i\varphi, JT \bullet \varphi)$$

$$= 2|\varphi|^2 - 2f^2|\varphi|^2 + 2g(JT, e_1)Re (i\varphi, e_1 \cdot \varphi) + 2g(JT, e_2)Re (i\varphi, e_2 \cdot \varphi)$$

$$= 2|\varphi|^2 - 2f^2|\varphi|^2 + 2g(JT, e_1)g(T, e_2)|\varphi|^2 - 2g(JT, e_2)g(T, e_1)|\varphi|^2$$

$$= 2|\varphi|^2 - 2f^2|\varphi|^2 - 2g(T, e_1)^2|\varphi|^2 - 2g(T, e_1)^2|\varphi|^2$$

$$= 2|\varphi|^2 - 2f^2|\varphi|^2 - 2||T||^2|\varphi|^2 = 0.$$

Thus, we deduce $if\varphi = -f^2e_1 \cdot e_2 \cdot \varphi - fJT \cdot \varphi$, where we use the fact that $\bar{\varphi} = ie_1 \cdot e_2 \cdot \varphi$. In this case, Equation (5.2) can be written as

$$(-G + \det A + f^2)e_1 \cdot e_2 \cdot \varphi = -(d^\nabla A)(e_1, e_2) + fJT \bullet \varphi.$$

This is equivalent to say that both terms $R_{1212} + \det A + f^2$ and $(d^\nabla A)(e_1, e_2) + fJT$ are equal to zero. In fact, these are the Gauss-Codazzi equations in Theorem 5.1. In order to obtain the two other equations, we simply compute the derivative of $T \cdot \varphi = -f\varphi + \bar{\varphi}$ in the direction of $X$ to get

$$\nabla_X T \cdot \varphi + T \cdot \nabla_X \varphi = \nabla_X T \cdot \varphi - \frac{1}{2} T \cdot A(X) \cdot \varphi$$

$$= -X(f)\varphi - f\nabla_X \varphi + \nabla_X \bar{\varphi}$$

$$= -X(f)\varphi + \frac{1}{2} fA X \cdot \varphi + \frac{1}{2} A X \cdot \varphi$$

$$= -X(f)\varphi + \frac{1}{2} fA X \cdot \varphi + \frac{1}{2} A X \cdot (T \bullet \varphi + f\varphi).$$
This reduces to $\nabla_X T \cdot \varphi + g(T, A(X)) \varphi = -X(f) \varphi + fA(X) \cdot \varphi$. Hence we obtain $X(f) = -g(A(X), T)$ and $\nabla_X T = fA(X)$ which finishes the proof. □

**Remark 5.3** The second condition in Theorem 5.2 is equivalent to the existence of a Spin$^c$ structure whose line bundle $L$ verifies $c_1(L) = \lceil \frac{1}{2} \pi f \cdot \varphi \rceil$ and $f \varphi$ is a closed 2-form. This Spin$^c$ structure carries a non-trivial solution $\varphi$ of the generalized Killing spinor equation $\nabla_X \varphi = -\frac{1}{2} AX \cdot \varphi$, with $T \cdot \varphi = -f \varphi + \bar{\varphi}$.

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