Quenched local central limit theorem for random walks in a time-dependent balanced random environment

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Abstract

We prove a quenched local central limit theorem for continuous-time random walks in \( \mathbb{Z}^d, d \geq 2 \), in a uniformly-elliptic time-dependent balanced random environment which is ergodic under space-time shifts. We also obtain Gaussian upper and lower bounds for quenched and (positive and negative) moment estimates of the transition probabilities and asymptotics of the discrete Green function.

1 Introduction

In this article we consider a random walk in a balanced uniformly-elliptic time-dependent random environment on \( \mathbb{Z}^d, d \geq 2 \).

For \( x, y \in \mathbb{Z}^d \), we write \( x \sim y \) if \( |x - y|_2 = 1 \). Denote by \( \mathcal{P} \) the set (of nearest-neighbor transition rates on \( \mathbb{Z}^d \))

\[
\mathcal{P} := \left\{ v : \mathbb{Z}^d \times \mathbb{Z}^d \to [0, \infty) \mid v(x, y) = 0 \text{ if } x \not\sim y \right\}.
\]

Equip \( \mathcal{P} \) with the the product topology and the corresponding Borel \( \sigma \)-field. We denote by \( \Omega \subset \mathcal{P}^\mathbb{R} \) the set of all measurable functions \( \omega : t \mapsto \omega_t \) from \( \mathbb{R} \) to \( \mathcal{P} \) and call every \( \omega \in \Omega \) a time-dependent environment. For \( \omega \in \Omega \), we define the parabolic difference operator

\[
\mathcal{L}_\omega u(x, t) = \sum_{y \sim x} \omega_t(x, y) (u(y, t) - u(x, t)) + \partial_t u(x, t)
\]

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for every bounded function $u : \mathbb{Z}^d \times \mathbb{R} \to \mathbb{R}$ which is differentiable in $t$. Let $(\tilde{X}_t)_{t \geq 0} = (X_t, T_t)_{t \geq 0}$ denote the continuous-time Markov chain on $\mathbb{Z}^d \times \mathbb{R}$ with generator $L_\omega$. Note that almost surely, $T_t = T_0 + t$. We say that $(X_t)_{t \geq 0}$ is a continuous-time random walk in the environment $\omega$ and denote by $P_\omega^{x,t}$ its law (called the quenched law) with initial state $(x,t) \in \mathbb{Z}^d \times \mathbb{R}$.

We equip $\Omega \subset \mathcal{P}^\mathbb{R}$ with the induced product topology and let $P$ be a probability measure on the Borel $\sigma$-field $\mathcal{B}(\Omega)$ of $\Omega$. An environment $\omega \in \Omega$ is said to be balanced if

$$\sum_y \omega_t(x,y)(y-x) = 0,$$

and uniformly elliptic if there is a constant $\kappa \in (0,1)$ such that

$$\kappa < \omega_t(x,y) < \frac{1}{\kappa} \quad \text{for all } t \in \mathbb{R}, x, y \in \mathbb{Z}^d \text{ with } x \sim y.$$

Let $\Omega_\kappa \subset \Omega$ denote the set of balanced and uniformly elliptic environments with ellipticity constant $\kappa \in (0,1)$. The measure $P$ is said to be balanced and uniformly elliptic if $P(\omega \in \Omega_\kappa) = 1$ for some $\kappa \in (0,1)$.

For each $(x,t) \in \mathbb{Z}^d \times \mathbb{R}$ we define the space-time shift $\theta_{x,t} : \Omega \to \Omega$ by

$$(\theta_{x,t}\omega)_s(y,z) := \omega_{s+t}(y+x,z+x).$$

We assume that the law $P$ of the environment is translation-invariant and ergodic under the space-time shifts $\{\theta_{x,t} : x \in \mathbb{Z}^d, t \geq 0\}$. I.e, $P(A) \in \{0,1\}$ for any $A \in \mathcal{B}(\Omega)$ such that $P(A \Delta \theta_{x,t}^{-1} A) = 0$ for all $x \in \mathbb{Z}^d \times [0,\infty)$.

Given $\omega$, the environmental process

$$\tilde{\omega}_t := \theta_{X_t,\omega}, \quad t \geq 0,$$

with initial state $\tilde{\omega}_0 = \omega$ is a Markov process on $\Omega$. With abuse of notation, we use $P_{\omega}^{0,0}$ to denote the quenched law of $(\tilde{\omega}_t)_{t \geq 0}$.

**Assumptions:** throughout this paper, we assume that $P$ is balanced, ergodic, and uniformly elliptic with ellipticity constant $\kappa > 0$.

We recall the quenched central limit theorem (QCLT) in [12].

**Theorem 1.** [12, Theorem 1.2] Under the above assumptions of $P$,

(a) there exists a unique invariant measure $Q$ for the process $(\tilde{\omega}_t)_{t \geq 0}$ such that $Q \ll P$ and $(\tilde{\omega}_t)_{t \geq 0}$ is an ergodic flow under $Q \times P_{\omega}^{0,0}$.

(b) (QCLT) For $P$-almost all $\omega$, $P_{\omega}^{0,0}$-almost surely, $(X_n/n)_{n \geq 1}$ converges weakly, as $n \to \infty$, to a Brownian motion with deterministic non-degenerate covariance matrix $\Sigma = \text{diag}\{2E_Q[\omega_0 e_i], i = 1,\ldots,d\}$.

In the special case where the environment is time-independent, i.e, $P(\omega_t = \omega_s \text{ for all } t, s \in \mathbb{R}) = 1$, we say that the environment is static.
Remark 2. For balanced random walks in a static, uniformly-elliptic, ergodic random environment on $\mathbb{Z}^d$, the QCLT has been first shown by Lawler [24], which is a discrete version of the result of Papanicolaou and Varadhan [26]. It is then generalized to static random environments with weaker ellipticity assumptions in [18, 7].

Remark 3. Write $\|f\|_{L^p(P)} := (E_P[|f|^p])^{1/p}$ for $p \in \mathbb{R}$. Let

$$\rho(\omega) := dQ/dP.$$ 

It is obtained in [12] that $\rho > 0$, $P$-almost surely. At the end of the proof of [12, Theorem 1.2], it is shown that $E_Q[g] \leq C\|g\|_{L^{d+1}(P)}$ for any bounded continuous function $g$, which implies

$$E_P[\rho^{(d+1)/d}] < \infty. \quad (2)$$

Moreover, one of our main results (see Theorem 15 below) shows that there exists $q = q(\kappa, d)$ such that the $L^{-q}(P)$ moment of $\rho$ is also bounded.

For $(x, t) \in \mathbb{Z}^d \times \mathbb{R}$, set

$$\rho_\omega(x, t) := \rho(\theta_t x, \omega).$$

Since $\Omega$ is equipped with a product $\sigma$-field, for any fixed $\omega \in \Omega$, the map $\mathbb{R} \to \Omega$ defined by $t \mapsto \theta_{0, t} \omega$ is measurable. Hence for almost-all $\omega$, the function $\rho_\omega(x, t)$ is measurable in $t$. Moreover, $\rho_\omega$ possesses the following properties. For $P$-almost all $\omega$,

(i) $\rho_\omega(x, t)\delta_x dt$ is an invariant measure for the process $\hat{X}_t$ under $P_\omega$;

(ii) $\rho_\omega(x, t) > 0$ is the unique density (with respect to $\delta_x dt$) for an invariant measure of $\hat{X}$ that satisfies $E_P[\rho_\omega(0, 0)] = 1$;

(iii) $\rho_\omega$ has a version which is absolutely continuous with respect to $t$ with

$$\partial_t \rho_\omega(x, t) = \sum_y \rho_\omega(y, t)\omega_t(y, x) \quad (3)$$

for almost every $t$, where $\omega_t(x, x) := -\sum_{y \sim x} \omega_t(x, y)$.

The proof of these properties can be found in [11, Appendix].

As a main result of our paper, we will present the following local limit theorem (LLT), which is a finer characterization of the local behavior of the random walk than the QCLT. Let

$$\hat{0} := (0, 0) \in \mathbb{Z}^d \times \mathbb{R}.$$ 

For $\hat{x} = (x, t), \hat{y} = (y, s) \in \mathbb{Z}^d \times \mathbb{R}$, $t \leq s$, define

$$p^\omega(\hat{x}, \hat{y}) := P^{t, \omega}(X_{s-t} = y), \quad q^\omega(\hat{x}, \hat{y}) = \frac{p^\omega(\hat{x}, \hat{y})}{\rho_\omega(y)}. \quad (4)$$
Theorem 4 (LLT). For \( \mathbb{P} \)-almost all \( \omega \) and any \( T > 0 \),
\[
\lim_{n \to \infty} \sup_{x \in \mathbb{R}^d, t > T} \left| n d q^\omega(\hat{0}; |nx|, n^2 t) - p_\omega^T(0, x) \right| = 0.
\]
Here \( p_\omega^T(0, x) = [(2\pi t)^d \det \Sigma]^{-1/2} \exp(-x^T \Sigma^{-1} x/2t) \) is the transition kernel of the Brownian motion with covariance matrix \( \Sigma \) and starting point \( 0 \), and \( |x| := ([x_1], \ldots, [x_d]) \in \mathbb{Z}^d \) for \( x \in \mathbb{R}^d \).

The proof of the LLT follows from Theorem 1 and a localization of the heat kernel \( q^\omega(\hat{0}, \cdot) \), an argument already implemented in [6] and [4] in the context of random conductance models. For this purpose, the regularity of \( \hat{x} \mapsto q^\omega(\hat{0}, \hat{x}) \) is essential. We use an analytical tool from classical PDE theory: the parabolic Harnack inequality (PHI) which yields not only Hölder continuity (cf. Corollary 24 below) of \( q^\omega(\hat{0}, \hat{x}) \) but also very sharp heat kernel estimates. Note that for fixed \( \hat{x} = (x, t) \), the function \( u(\hat{y}) = q^\omega(\hat{0}, \hat{x}) \) satisfies \( \mathcal{L}_\omega u = 0 \) in \( \mathbb{Z}^d \times (-\infty, t) \). However in our non-reversible model, we need to prove, instead of PHI for \( \mathcal{L}_\omega \), the PHI (Theorem 5) for the adjoint operator \( \mathcal{L}_\omega^* \) (defined below), since the heat kernel \( v_\omega(\hat{x}) := q^\omega(\hat{0}, \hat{x}) \) solves
\[
\mathcal{L}_\omega^* v(\hat{x}) := \sum_{y \sim x} \omega^*_t(x, y)(v(y, t) - v(\hat{x})) - \partial_t v(\hat{x}) = 0 \quad (5)
\]
for \( \hat{x} = (x, t) \in \mathbb{Z}^d \times (0, \infty) \), where
\[
\omega^*_t(x, y) := \frac{\rho_\omega(y, t) \omega_t(y, x)}{\rho_\omega(x, t)} \quad \text{for} \ x \sim y \in \mathbb{Z}^d.
\]
Note that \( \omega^* \) is not necessarily a balanced environment anymore.

For \( r > 0 \), \( x \in \mathbb{R}^d \), we let
\[
B_r(x) = \{ y \in \mathbb{Z}^d : |x - y| < r \}, \quad B_r = B_r(0)
\]
and define for \( \hat{x} = (x, t) \in \mathbb{R}^d \times \mathbb{R} \) the parabolic balls
\[
Q_r(\hat{x}) = B_r(x) \times [t, t + r^2], \quad Q_r = Q_r(\hat{0}) \quad (6)
\]
Throughout this paper, unless otherwise specified, \( C, c \) denote generic positive constants that depend only on \( (d, \kappa) \), and which may differ from line to line. If \( cB \leq A \leq CB \), we write
\[
A \asymp B.
\]

Our second main result is

Theorem 5 (PHI for \( \mathcal{L}_\omega^* \)). For \( \mathbb{P} \)-almost all \( \omega \), any non-negative solution \( v \) of the adjoint equation \( \mathcal{L}_\omega^* v = 0 \) in \( B_{2R} \times (0, 4R^2) \) satisfies
\[
\sup_{B_{2R} \times (R^2, 2R^2)} v \leq C \inf_{B_{R} \times (3R^2, 4R^2)} v.
\]
In PDE, the Harnack inequality for the adjoint of non-divergence form elliptic differential operators was first proved by Bauman [5], and was generalized to the parabolic setting by Escauriaza [14]. Our proof of Theorem 5 follows the main idea of [14].

Remark 6. For time discrete random walks in a static environment, Theorem 5 was obtained by Mustapha [25]. His argument follows basically [14], and uses the PHI [23, Theorem 4.4] of Kuo and Trudinger in the time discrete situation. Moreover, in the static case, the volume-doubling property of the invariant distribution, which is the essential part of the proof of Theorem 5, is much simpler, see [16]. In our dynamical setting, a parabolic volume-doubling property (Theorem 9) is required. To this end, we adapt ideas of Safonov-Yuan [27] and results in the references therein [13, 5, 17] into our discrete space setting.

The main challenge in proving Theorem 5 is that $L^\ast_\omega$ is neither balanced nor uniformly elliptic, and so the PHI for $L_\omega$ (Theorem 17) is not immediately applicable. This is the main difference with the random conductance model with symmetric jump rates where $\omega_t(x,y) = \omega_t(y,x) = \omega^\ast_t(x,y)$, and thus which PHI for $L_\omega$ is the same as PHI for $L^\ast_\omega$. See [1, 9, 10, 2, 19].

Let us explain the main idea for the proof of Theorem 5. An important observation is that solutions $v$ of $L^\ast_\omega$ can be expressed in terms of hitting probabilities of the time-reversed process, cf. Lemma 21 below. Thus to compare values of the adjoint solution, one only needs to estimate hitting probabilities of the original process that starts from the boundary. To this end, we will use a “boundary Harnack inequality” (Theorem 20) which compares $L_\omega$-harmonic functions near the boundary. We will also need a volume-doubling inequality for the invariant measure (Theorem 9) to control the change of probabilities due to time-reversal.

Recall the heat kernel $q^\omega$ in (4). For any $A \subset \mathbb{R}^d$ and $s \in \mathbb{R}$, let

$$\rho_\omega(A,s) = \sum_{x \in A \cap \mathbb{Z}^d} \rho_\omega(x,s).$$

We write the $\ell^2$-norm of $x \in \mathbb{R}^d$ as $|x| = |x|_2$. For $r \geq 0, t > 0$, define

$$h(r,t) = \frac{r^2}{t} + r \log(\frac{r}{t} \vee 1).$$

Note that $h(c_1 r, c_2 t) \asymp h(r,t)$ for constants $c_1, c_2 > 0$.

Our third main results are the following heat kernel estimates (HKE).

**Theorem 7 (HKE).** For $\mathbb{P}$-almost every $\omega$ and all $\hat{x} = (x,t) \in \mathbb{Z}^d \times (0,\infty)$,

$$\frac{C}{\rho_\omega(B_{\sqrt{t}}(y),s)} e^{-c \frac{|x|^2}{t}} \leq q^\omega(0,\hat{x}) \leq \frac{C}{\rho_\omega(B_{\sqrt{t}}(y),s)} e^{-c h(|x|,t)}$$

where $C > 0$.


for all $s \in [0, t]$ and $y$ with $|y| \leq |x| + c \sqrt{t}$. Moreover, recalling the definition of $L^p(\mathbb{P})$ in Remark 2, there exists $p = p(d, \kappa) > 0$ such that

$$
\| P^0_{\omega}(X_t = x) \|_{L^{(d+1)/d}(\mathbb{P})} \leq \frac{C}{(t+1)^{d/2}} e^{-c|x|t} \tag{9}
$$

and

$$
\| P^0_{\omega}(X_t = x) \|_{L^{-p}(\mathbb{P})} \geq \frac{c}{(t+1)^{d/2}} e^{-c|x|t^2} \tag{10}
$$

for all $(x, t) \in \mathbb{Z}^d \times (0, \infty)$. As a consequence, setting $G^\omega(0, x) = \int_0^\infty P^0_{\omega}(X_t = x)dt$, we have for $d \geq 3$ and $x \in \mathbb{Z}^d$,

$$
\| G^\omega(0, x) \|_{L^{(d+1)/2}(\mathbb{P})} \asymp \| G^\omega(0, x) \|_{L^{-p}(\mathbb{P})} \asymp (|x| + 1)^{2-d}. \tag{11}
$$

Note that for a general ergodic environment, the density $\rho_\omega$ does not have deterministic (positive) upper and lower bounds, thus one cannot expect deterministic quenched Gaussian bounds for $p^\omega(\hat{0}, \hat{x})$. However, our Theorem 7 shows that it has $L^{(d+1)/d}(\mathbb{P})$ and $L^{-q}(\mathbb{P})$ moment bounds. Furthermore, we can characterize asymptotics of the Green function of the RWRE. Recall the notations $\Sigma$ (in Theorem 1 (b)), $p^\Sigma_\omega$, $|x|$ (in Theorem 4), and $h$ in (7).

**Corollary 8.** The following statements are true for $\mathbb{P}$-almost every $\omega$.

(i) There exists $t_0(\omega) > 0$ such that for any $\hat{x} = (x, t) \in \mathbb{Z}^d \times (t_0, \infty)$,

$$
\frac{c}{t^{d/2}} e^{-\frac{|x|^2}{t}} \leq q^\omega(\hat{0}, \hat{x}) \leq \frac{C}{t^{d/2}} e^{-c|x|t}. \tag{12}
$$

As a consequence, the RWRE is recurrent when $d = 2$ and transient when $d \geq 3$.

(ii) When $d = 2$, for all $x \in \mathbb{R}^d \setminus \{0\}$,

$$
\lim_{n \to \infty} \frac{1}{\log n} \int_0^\infty \left[ q^\omega(\hat{0}; 0, t) - q^\omega(\hat{0}; |nx|, t) \right] dt = \frac{1}{\pi \sqrt{\det \Sigma}}. \tag{13}
$$

(iii) When $d \geq 3$, for all $x \in \mathbb{R}^d \setminus \{0\}$,

$$
\lim_{n \to \infty} n^{d-2} \int_0^\infty q^\omega(\hat{0}; |nx|, t) dt = \int_0^\infty p^\Sigma_{\omega}(0, x) dt. \tag{14}
$$

Similar results as Corollary 5(ii)(iii) are also obtained recently for the conductance model [4].

The organization of this paper is as follows. In Section 2 we prove a parabolic volume-doubling property and an $A_p$ inequality for $\rho_\omega$, and obtain a new proof of the PHI for $L_\omega$. In Section 3 we establish estimates of $L_\omega$-harmonic functions near the boundary, showing both the interior elliptic-type and boundary PHI’s. We prove the PHI for the adjoint operator (Theorem 3) in Section 4. Finally, with the adjoint PHI, we prove Theorems 4, 7, and Corollary 3 in Section 5. Section 6 contains probability estimates that are used in previous sections. Some estimates and standard arguments can be found in the Appendix of the longer arXiv version [11] of this paper.
2 A local volume-doubling property and its consequences

The purpose of this section is to obtain a parabolic volume-doubling property (Theorem 9) and a negative moment estimate (Theorem 15) for the density $\rho_\omega$. The proof relies crucially on a volume-doubling property for the hitting probabilities restricted in a finite ball (Theorem 10), which is an improved version of [27, Theorem 1.1] by Safonov and Yuan in the PDE setting.

As a by-product, we obtain a new proof of the classical PHI of Krylov and Safonov [22] in the lattice (Theorem 17). Our proof, which is of interest on its own, can be viewed as the parabolic version of Fabes and Stroock’s [16] proof in the elliptic PDE setting.

2.1 Volume-doubling properties

For a finite subgraph $D \subset \mathbb{Z}^d$, let

$$\partial D = \{ y \in \mathbb{Z}^d \setminus D : y \sim x \text{ for some } x \in D \}, \quad \bar{D} := D \cup \partial D.$$ 

For $\mathcal{D} \subset \mathbb{Z}^d \times \mathbb{R}$, define the parabolic boundary of $\mathcal{D}$ as

$$\partial^P \mathcal{D} := \{ (x, t) \notin \mathcal{D} : (B_{1+\epsilon}(x) \times (t-\epsilon, t)] \cap \mathcal{D} \neq \emptyset \text{ for all } \epsilon > 0 \}.$$ 

In the special case $\mathcal{D} = D \times [0, T)$ for some finite $D \subset \mathbb{Z}^d$, it is easily seen that $\partial^P \mathcal{D} = (\partial D \times [0, T)) \cup (D \times \{T\})$. See figure 1.

By the optional stopping theorem, for any $(x, t) \in \mathcal{D} \subset \mathbb{Z}^d \times \mathbb{R}$ and any bounded integrable function $u$ on $\mathcal{D} \cup \partial^P \mathcal{D}$,

$$u(x, t) = -E_{x, t}^{\mathcal{D}} \left[ \int_0^\tau L_\omega u(\hat{X}_r)dr \right] + E_{x, t}^{\mathcal{D}} [u(\hat{X}_\tau)], \quad (12)$$

where $\tau = \inf\{ r \geq 0 : (X_r, T_r) \notin \mathcal{D} \}$.

![Figure 1: The parabolic boundary of $D \times [0, T)$.](image-url)
Theorem 9. \( \mathbb{P} \)-almost surely, for every \( r \geq 1/2 \),

\[
\sup_{t:|t|^2 \leq r^2} \rho_\omega(B_{2r}, t) \leq C \rho_\omega(B_r, 0).
\]

In PDE setting, this type of estimate was first established by Fabes and Stroock [16] for adjoint solutions of non-divergence form elliptic operators, and then generalized by Escauriaza [14] to the parabolic case.

To obtain Theorem 9 a crucial estimate is a volume-doubling property (Theorem 10) for the hitting measure of the random walk, which we will prove by adapting some ideas of Safonov and Yuan [27, Theorem 1.1] in the PDE setting. Note that our proof of Theorem 10 relies on a probabilistic estimate (Lemma 27) rather than the Harnack inequality (Theorem 17).

For any \( A \subset \mathbb{Z}^d \), \( s \in \mathbb{R} \), define the stopping time

\[
\Delta(A, s) = \inf \{ t \geq 0 : \hat{X}_t \notin A \times (-\infty, s) \}. \tag{13}
\]

Theorem 10. Assume \( \omega \in \Omega_\kappa \). There exists \( k_0 = k_0(d, \kappa) \) such that for any \( k \geq k_0 \), \( m \geq 2 \), \( r, s > 0 \) and any \( y \in B_{k\sqrt{s}} \), we have

\[
P_{\omega}^{y,0}(X_{\Delta(B_{mk\sqrt{s}}, s)} \in B_{2r}) \leq C_k P_{\omega}^{y,0}(X_{\Delta(B_{mk\sqrt{s}}, s)} \in B_{r}).
\]

Here \( C_k \) depends only on \( (k, d, \kappa) \). In particular, for any \( k \geq 1 \), \( \|y\| \leq k\sqrt{s} \),

\[
P_{\omega}^{y,0}(X_s \in B_{2r}) \leq C_k P_{\omega}^{y,0}(X_s \in B_{r}).
\]

Proof of Theorem 10. Since \( B_1 = \{0\} \), we only consider \( r \geq 1/2 \). Fix \( s, r \).

For \( \rho \geq 0 \), \( k \geq k_0 \), define \( L_{k, \rho} = B_{k\rho} \times \{s - \rho^2\} \) and

\[
D_{k, \rho} = \bigcup_{R \leq \rho} L_{k, R} = \{(x, t) \in \mathbb{Z}^d \times (-\infty, s) : \|x\|/k \leq \sqrt{s - t} \leq \rho\}.
\]

Note that \( k_0 > 6 \) is a large constant to be determined. For any \( R \leq \rho \), by

![Figure 2: The shaded region is \( D_{k, \rho} \).](image)
Lemma 27 below, there exists $\alpha_k > 0$ depending on $(k, \kappa, d)$ such that

$$\min_{(x,t) \in L_{k,R}} P^x_t \left( X_{\Delta(B_{mkp},s)} \in B_r \right) \geq \left( \frac{x}{2k^2} \wedge \frac{1}{2} \right)^{\alpha_k}.$$ 

Let $\beta_k > 1$ be a large constant to be determined later. Then, letting

$$v_\rho(\hat{x}) = (8k)^{\alpha_k}(\beta_k + 1)P^x_\rho \left( X_{\Delta(B_{mkp},s)} \in B_r \right) - P^x_\rho \left( X_{\Delta(B_{mkp},s)} \in B_{2r} \right),$$

we get $\inf_{D_{k,R}} v_\rho \geq (\beta_k + 1)(\frac{4r}{R} \wedge 2k)^{\alpha_k} - 1$ for $0 < R \leq \rho$. In particular, $\inf_{D_{k,(4r)\wedge \rho}} v_\rho \geq \beta_k$ for $\rho \geq 0$. Set

$$R_\rho = \sup \{ R \in [0, m\rho] : \inf_{D_{k,R}} v_\rho \geq 0 \}.$$ 

Clearly, $R_\rho \geq (4r) \wedge \rho$. We will prove that

$$R_\rho \geq \rho \text{ for all } \rho > 0. \quad (14)$$

Assuming (14) fails, then $R_\rho < \rho$ for some $\rho > 4r$. We will show that this is impossible via contradiction. First, for such $\rho > 4r$, we claim that there exists a constant $\gamma = \gamma(d, \kappa) > 0$ such that

$$\min_{L_{1,R}} v_\rho \geq \beta_k \left( \frac{R}{r} \right)^{\gamma} \text{ for all } R \in [2r, R_\rho]. \quad (15)$$

By Lemma 27, $g(R,R/2) := \min_{x \in B_R} P^x_{t_0} \left( X_{\Delta(B_{2R,s-(R/2)^2})} \in B_{R/2} \right) \geq C$. Further, by the Markov property and that $R_\rho < \rho$, for $R \in [2r, R_\rho]$,

$$\min_{x \in B_R} P^x_{t_0} \left( X_{\Delta(D_{k,m\rho,s-(R/2^2)^2})} \in B_{R/2^n} \right) \geq g(R,R/2) \cdots g(R/2^{n-1}, R/2^n) \geq C^n.$$ 

Since $v_\rho(\hat{X}_t)$ is a martingale in $B_{mkp} \times (-\infty, s)$ and that $v_\rho \geq 0$ in $D_{k,R_\rho}$, choosing $n$ such that $R/2^n \leq r < R/2^{n-1}$, the above inequality yields

$$v_\rho(x, s - R^2) \geq P^x_{t_0} \left( X_{\Delta(D_{k,m\rho,s-(R/2^2)^2})} \in B_{R/2^n} \right) \inf_{D_{k,r}} v_\rho \geq C^n \beta_k \geq \beta_k \left( \frac{r}{R} \right)^c$$

for $R \in [2r, R_\rho)$ and $x \in B_R$. Display (15) is proved.

Next, we will show that for $R \in [2r, R_\rho)$,

$$f_\rho(R) := \sup_{C_{B,R} \times (s-R^2,s)} v^\rho_\rho \leq \left( \frac{r}{R} \right)^{-c \log_2 q_k}, \quad (16)$$

where $v^\rho_\rho = \max \{ 0, -v_\rho \}$, $q_k = 1 - \frac{2k}{r}$ and $c_1 > 0$ is a constant to be determined. Noting that $v^\rho_\rho = 0$ in $D_{k,R_\rho} \cup (B^r_{2r} \times \{ s \})$ and that $v^\rho_\rho(\hat{X}_t)$ is a sub-martingale, we know that $f_\rho(R)$ is a decreasing function for $R \in$
Further, for any \((x,t) \in \partial B_{kr} \times [s - R^2, s)\) with \(q_k R \in (\frac{2r}{k}, R_p)\), by the optional stopping lemma, in view of Theorem 25 below,

\[
v_{\rho}^*(x,t) \leq \mathbb{P}_\omega^x \{ X_a \in \partial B_{kr} \text{ for some } a \in [0, R^2] \} f(q_k R)
\]

\[
\leq \mathbb{P}_\omega^x \left( \sup_{0 \leq a \leq R^2} |X_a - x| \geq (1 - q_k)kR \right) f(q_k R)
\]

\[
\leq \frac{C}{k} R^{-c_1} e^{-c_1 R} f(q_k R) \leq f(q_k R)/2,
\]

if \(c_1\) is chosen to be big enough. So \(f_\rho(R) \leq f_\rho(q_k R)/2\). Let \(n \geq 0\) be the integer such that \(q_k^{n+1} R < r \leq q_k^n R\). We conclude that for \(R \in [2r, R_p)\),

\[
f_\rho(R) \leq 2^{-n} f_\rho(q_k^n R) \leq \left( \frac{r}{R} \right)^{-c/\log q_k} f_\rho(r).
\]

Inequality (16) then follows from the fact \(v_{\rho}^* \leq 1\).

Finally, if \(R_p < \rho\) for some \(\rho > 4r\), let \(\tau = \inf\{t \geq 0 : \hat{X}_t \notin (B_{mk\rho} \times (-\infty, s)) \} \). Since \(v_\rho = 0\) on \(\partial P (B_{mk\rho} \times (-\infty, s)) \) \(\{2r \times \{s\}\}\), the optional stopping lemma, for \(R \in [R_p, 2R_p)\) and \(x \in B_{kr}\),

\[
v_\rho(x,s-R^2)
\]

\[
= E_{\omega}^x \mathbb{P}_{\hat{X}_\tau} \{ X_t \in B_{kr} \times (s-R^2) \} \text{ or } X_t \in \partial B_{kr} \times (s-R^2), s) \}
\]

\[
\geq E_{\omega}^x \mathbb{P}_{\hat{X}_\tau} \{ X_{\Delta(B_{kr}\times(s-R^2))} \in B_{kr} \} \min_{L_{1,R^2}} v_\rho - f_\rho(R/2)
\]

\[
\geq \hat{A}_k \beta_k (\frac{2r}{R})^{\gamma - c/\log q_k},
\]

where \(A_k\) depends on \((k, \kappa, d)\). Taking \(k_0 > c_1\) to be big enough such that \(-c/\log q_k > \gamma\) for \(k \geq k_0\) and choosing \(\beta_k = A_k^{-1}\), the above inequality then implies \(\inf_{B_{kr}} v_{\omega} \geq 0\), which contradicts our definition of \(R_p\). Display (14) is proved, and therefore, \(\min_{x \in B_{kr}} v_{\omega}(x,0) \geq 0\). The theorem follows. \(\square\)

**Corollary 11.** Let \(\omega \in \Omega_{\kappa}\) and \(k_0\) as in Theorem 10. For any \(r > 0, k \geq k_0, m \geq 2, s > 0\) and \(y \in B_{k\sqrt{s}}\), we have

\[
\sup_{t \geq 0, |t-s| \leq r^2} P_{\omega}^{y,0}(X_{\Delta(B_{mk\sqrt{s}}) t} \in B_{2r}) \leq C_k P_{\omega}^{y,0}(X_{\Delta(B_{mk\sqrt{s}}) t} \in B_{r}),
\]

where \(C_k\) depends on \((k, \kappa, d)\). In particular, for any \(k \geq 1, |y| \leq k\sqrt{s}\),

\[
\sup_{t \geq 0, |t-s| \leq r^2} P_{\omega}^{y,0}(X_t \in B_{2r}) \leq C_k P_{\omega}^{y,0}(X_s \in B_r).
\]
Proof. It suffices to consider $r < \sqrt{\tau}$, because otherwise, by Lemma 27, the right side is bigger than a constant. When $t \in [0 \lor (s - r^2), s]$,

$$
\min_{y \in B_{k\sqrt{\tau}}} P^y_\omega (X_{\Delta(B_{mk\sqrt{\tau}}, s)} \in B_{4r})
\geq \min_{y \in B_{k\sqrt{\tau}}} P^y_\omega (X_{\Delta(B_{mk\sqrt{\tau}}, t)} \in B_{2r}) \min_{x \in B_{2r}} P^x_\omega (X_{s-t} \in B_{2r}(x))
\geq C \min_{y \in B_{k\sqrt{\tau}}} P^y_\omega (X_{\Delta(B_{mk\sqrt{\tau}}, t)} \in B_{2r}).
$$

By Theorem 10 we can replace $4r$ in the above inequality by $r$. When $t \in [s, s + r^2]$, for any $y \in B_{k\sqrt{\tau}}$,

$$
P^y_\omega (X_{\Delta(B_{mk\sqrt{\tau}}, t)} \in B_{2r})
\leq \sum_{n=0}^{\infty} \sum_{x \in [2^n r, 2^n r + 1])} P^y_\omega (X_{\Delta(B_{mk\sqrt{\tau}}, s)} = x) P^x_\omega (X_{t-s} \in B_{2r})
\leq C \sum_{n=0}^{\infty} P^y_\omega (X_{\Delta(B_{mk\sqrt{\tau}}, s)} \in B_{2n r}) (e^{-c2^n r} + e^{-c4^n r}).
$$

Observing that (cf. Theorem 10)

$$
P^y_\omega (X_{\Delta(B_{mk\sqrt{\tau}}, s)} \in B_{2n r}) \leq C^m P^y_\omega (X_{\Delta(B_{mk\sqrt{\tau}}, s)} \in B_r),
$$

our proof is complete. \qed

Proof of Theorem 11. Let $k_0 \geq 2$ be as in Theorem 10. Recall $\hat{\omega}, Q_r, \Delta$ in (1), (6), (13). For fixed $\xi \in \Omega_\kappa$, define a probability measure $Q_R = Q_R^\xi$ on $\{\theta_{s\xi} : \xi \in Q_R\}$ such that for any bounded measurable $f \in \mathbb{R}^\Omega$,

$$
E_{Q_R} [f] = \frac{1}{C_R} E^{\hat{\xi}, -R^2}_\xi \int_0^{\Delta(B_{2k_0 R}, R^2)} f(\tilde{\xi}s) \mathbb{1}_{\tilde{X}_s \in Q_R} ds,
$$

where $C_R$ is a renormalization constant such that $Q_R$ is a probability.

First, we claim that $C_R \asymp R^2$. Clearly, $C_R \leq 2R^2$. On the other hand,

$$
C_R = E^{\hat{\xi}, -R^2}_\xi \int_0^{\Delta(B_{2k_0 R}, R^2)} \mathbb{1}_{\tilde{X}_s \in Q_R} ds
\geq P^{\hat{\xi}, -R^2}_\xi (X_{\Delta(B_R, 0)} \in B_{R/2}) \min_{x \in B_{R/2}} E^{\hat{\xi}, 0}_\xi [\Delta(B_R, R^2)]
\geq C \min_{x \in B_{R/2}} E^{\hat{\xi}, 0}_\xi [\Delta(B_R, R^2)].
$$

Since $|X_t - X_0|^2 - \frac{d}{\kappa} t$ is a supermartingale, denoting $\tau = \Delta(B_R, R^2)$, we have $0 \geq E^{\hat{\xi}, 0}_\xi [|X_t - x|^2 - \frac{d}{\kappa} \tau]$. Hence for any $x \in B_{R/2}$,

$$
E^{\hat{\xi}, 0}_\xi [\tau] \geq c E^{\hat{\xi}, 0}_\xi [|X_t - x|^2] \geq C R^2 P^{\hat{\xi}, 0}_\xi (\tau < R^2),
$$

11
which implies \( E^x,0[\tau] \geq cR^2 \). Thus \( C_R \geq CR^2 \) and so \( C_R \propto R^2 \).

Next, since \( \Omega \) is pre-compact, by Prohorov’s theorem, there is a subsequence of \( \mathcal{Q}_R \) that converges weakly, as \( R \to \infty \), to a probability measure \( \hat{\mathcal{Q}} \) on \( \Omega \). We will show that \( \hat{\mathcal{Q}} \) is an invariant measure of the process \((\hat{\omega}_t)\). Indeed, let \( p_R = p_{R,\xi} \) denote the kernel \( P^x_t(\hat{x}; y, s) = P^x_t(\hat{X}_{\Delta[B_{2k_0},R]} = (y,s)) \). Then, letting \( \mathcal{L}f(\omega) = \sum \omega_0(0,\varepsilon)\|f(\theta,\varepsilon)\| + \partial_t f(\theta_0,\varepsilon)\|_{t=0} \) denote the generator of the process \((\hat{\omega}_t)\), and \( \hat{y} := (y,s) \), we have

\[
E_{\mathcal{Q}_R}[\mathcal{L}f(\omega)] = C_R^{-1} \sum_{y \in B_R} \int_0^{R^2} p_R(0,-R^2; \hat{y}) \mathcal{L}f(\theta_y \xi) d\sigma \tag{17}
\]

for \( f \in \text{dom}(\mathcal{L}) \), where \( \text{dom}(\mathcal{L}) \) denotes the domain of the generator \( \mathcal{L} \). Note that similar to \( \mu_\omega \), the function \( v(\hat{x}) = p_R(0,-R^2; \hat{x}) \) satisfies the equality \( \mathcal{L}^T v(\hat{x}) = 0 \) for \( \hat{x} \in B_{2R} \times (-R^2, R^2) \), where \( \mathcal{L}^T v(\hat{x}) = \sum_y v(y,t)\omega_t(y,x) - \partial_t v(x,t) \). Hence, using integration by parts,

\[
|\sum_{y \in B_R} \int_0^{R^2} p_R(0,-R^2; \hat{y}) \mathcal{L}f(\theta_y \xi) d\sigma| \\
\leq C \|f\|_\infty \int_0^{R^2} \sum_{y \in B_R \setminus B_R} p_R(0,-R^2; \hat{y}) d\sigma + 2\|f\|_\infty \tag{18}
\]

for all \( f \in \text{dom}(\mathcal{L}) \), where \( B_R = \{ x \in B_R : x \not\in \partial B_R \} \). Observe that

\[
u(\hat{x}) = \int_0^{R^2} \sum_{y \in B_R \setminus B_R} p_R(\hat{x}; \hat{y}) d\sigma = E^{\hat{x}}_0 \int_0^{\Delta(B_{2k_0} R, R^2)} 1_{B_R \setminus \partial B_R} d\tau
\]

satisfies \( \mathcal{L} \xi \nu(\hat{x}) = -1_{\hat{x} \in B_R \setminus \partial B_R} \) for \( \hat{x} \in \mathcal{Q} := B_{2k_0} \times [-R^2, R^2] \) and \( \nu|_{\partial \mathcal{Q}} = 0 \). By the parabolic maximum principle \[11\], Theorem A.3.1, we get \( u(0,-R^2) \leq CR^{2d+1}/(d+1) \). Hence, by \( [17], [18] \), and \( C_R \propto R^2 \),

\[
\lim_{R \to \infty} E_{\mathcal{Q}_R}[\mathcal{L}f] = 0 \quad \forall \text{ bounded function } f \in \text{dom}(\mathcal{L}),
\]

and so \( E_{\hat{\mathcal{Q}}}[\mathcal{L}f] = 0 \), which implies that \( \hat{\mathcal{Q}} \) is an invariant measure of \((\hat{\omega}_t)\).

Furthermore, we will show that \( \hat{\mathcal{Q}} \ll \mathcal{P} \). Notice that the function

\[
w(\hat{x}) := E^{\hat{x}}_0 \int_0^{\Delta(B_{2k_0} R, R^2)} f(\xi) 1_{\hat{x} \in \mathcal{Q}_R} d\tau
\]

satisfies \( \mathcal{L} \xi w(\hat{x}) = -f(\theta_y \xi) 1_{\hat{x} \in \mathcal{Q}_R} \) in \( \mathcal{Q} \) and \( w|_{\partial \mathcal{Q}} = 0 \). By \[11\] Theorem A.3.1, for any bounded measurable \( f \in \mathcal{R}^{\mathcal{Q}} \),

\[
E_{\mathcal{Q}_R}[f] \leq CR^{-2}w(0,-R^2) \leq C \left[ \int_0^{R^2} \sum_{y \in B_R} |f(\theta_y \xi)|^{d+1} d\tau / (R^{2d+1}) \right]^{1/(d+1)},
\]
which, by the multi-dimensional ergodic theorem, yields $E_Q[f] \leq C\|f\|_{L^d+1(P)}$ as we take $R \to \infty$. So $\tilde{Q} \ll \mathbb{P}$. By Theorem 11, $\tilde{Q} = Q$.

Finally, since $Q_R \Rightarrow \tilde{Q}$, for any bounded measurable $f \in \mathbb{R}^\Omega$,

$$E_\mathbb{P}[\rho_\omega(B_r,t)f] = \sum_{x \in B_r} E_Q[f(\theta_x,-\omega)]$$

$$= \lim_{R \to \infty} \sum_{x \in B_r, y \in B_R} \int_0^R P_\xi^{0,-R^2}(X_{\Delta(B_{2k_0R},s)} = y)f(\theta_{x+y,s-t}\xi)ds/C_R.$$  \hspace{1cm} (19)

Hence, for any measurable function $f \geq 0$, $|t| \leq r^2$, and $\mathbb{P}$-a.a. $\xi$,

$$E_\mathbb{P}[\rho_\omega(B_r,0)f] \geq \lim_{R \to \infty} \sum_{x \in B_{R-r}} \int_0^R P_\xi^{0,-R^2}(X_{\Delta(B_{2k_0R},s)} \in B_r(z))f(\theta_{z,s}\xi)ds/C_R$$

\textbf{Corollary 11.} \hspace{1cm} $C \lim_{R \to \infty} \sum_{x \in B_{R-r}} \int_0^R P_\xi^{0,-R^2}(X_{\Delta(B_{2k_0R},s+t)} \in B_{2r}(z))f(\theta_{z,s}\xi)ds/C_R$.

Since $f$ is arbitrary, the theorem follows. \hfill \Box

\textbf{Remark 12.} By Theorem 12, for any $r \geq 1$,

$$\frac{c}{r^2} \int_0^{r^2} \rho_\omega(B_r,s)ds \leq \rho_\omega(B_r,0) \leq \frac{C}{r^2} \int_0^{r^2} \rho_\omega(B_r,s)ds.$$

Hence, by the multi-dimensional ergodic theorem, for $\mathbb{P}$-almost every $\omega$,

$$c \leq \lim_{r \to \infty} \frac{1}{|B_r|} \rho_\omega(B_r,0) \leq \lim_{r \to \infty} \frac{1}{|B_r|} \rho_\omega(B_r,0) \leq C.$$ \hspace{1cm} (20)

\subsection{2.2 $A_p$ property and a new proof of the PHI for $L_\omega$}

We endow $\mathbb{Z}^d$ with the discrete topology and counting measure, and equip $\mathbb{Z}^d \times \mathbb{R}$ with the corresponding product topology and measure (where $\mathbb{R}$ has the usual topology and measure). For $\mathcal{D} \subset \mathbb{Z}^d \times \mathbb{R}$, let $|\mathcal{D}|$ be its measure, and denote the integration over $\mathcal{D}$ by $\int_\mathcal{D} f$. For instance,

$$\int_{B_R \times [0,T]} f = \sum_{x \in B_R} \int_0^T f(x,t)dt,$$

and $|\mathcal{D}| = \int_\mathcal{D} 1$. For $p > 0$, we define a norm

$$\|f\|_{\mathcal{D},p} := \left( \int_\mathcal{D} |f|^p / |\mathcal{D}| \right)^{1/p}.$$ \hspace{1cm} (23)
A function \( v \in \mathbb{R}^{2d} \times \mathbb{R} \) is called an adjoint solution of \( \mathcal{L}_\omega \) in \( \mathcal{D} = B_R \times [T_1, T_2] \) if \( \int_{\mathcal{D}} v \mathcal{L}_\omega \phi = 0 \) for any test function \( \phi(x, t) \in \mathbb{R}^{2d} \) that is supported on \( B_R \times (T_1, T_2) \) and smooth in \( t \).

For any function \( w \) defined on \( E \subset \mathbb{Z}^d \times \mathbb{R} \), we write \( w(E) := \int_E w \).

**Lemma 13.** Recall \( \| \cdot \|_{\mathcal{D}, p} \) in (23) and the parabolic balls \( Q_r \) in (10). Let \( \omega \in \Omega_r \). For any non-negative adjoint solution \( v \) of \( \mathcal{L}_\omega \) in \( Q_{2r} \), \( r > 0 \),

\[
\|v\|_{Q_r, (d+1)/d} \leq C \|v\|_{Q_{3r/2}, 1}.
\]

**Proof.** Denote the continuous balls of radius \( r \) by

\( O_r = \{ x \in \mathbb{R}^d : |x|_2 < r \} \) and \( O_r(y) = y + O_r, \quad y \in \mathbb{R}^d \). (24)

Let \( \phi_0 \geq 0 \) be a smooth (with respect to \( t \)) function supported on \( O_{3/2} \times [0, 9/4] \) with \( \phi_0|_{O_{3/2} \times [0, 1]} = 1 \) and set \( \phi(x, t) = \phi_0(x/r, t/r^2) \). Let \( f \) any non-negative smooth function supported on \( Q_r \) with \( \|f\|_{Q_r, d+1} = 1 \) and let \( u \in [0, \infty)^{2d} \times \mathbb{R} \) be supported on \( Q_{9r/5} \) with \( \mathcal{L}_\omega u = -f \) in \( Q_{9r/5} \). Since

\[
0 = \int v \mathcal{L}_\omega (\phi u) = \int v \phi \mathcal{L}_\omega u + \int vu \mathcal{L}_\omega \phi + \sum_{x,y} \int R v(x, t) \omega_t(x, y) \nabla_{x,y} u \nabla_{x,y} \phi dt,
\]

where \( \nabla_{x,y} u(\cdot, t) := u(x, t) - u(y, t) \) and (cf. (22)) \( \int = \int_{\mathbb{Z}^d \times \mathbb{R}} \), we get

\[
\int v \phi f = \int vu \mathcal{L}_\omega \phi + \sum_{x,y} \int R v(x, t) \omega_t(x, y) \nabla_{x,y} u \nabla_{x,y} \phi dt =: I + II.
\]

By the maximum principle ([11] Theorem A.3.1), \( u \leq Cr^2 \|f\|_{Q_r, d+1} \leq Cr^2 \). Thus, using \( |\mathcal{L}_\omega \phi| \leq C/r^2 \), we get \( ||I|| \leq C v(Q_{3r/2}) \). Further, noting that

\[
|\sum_{x,y} \int_R v(x, t) \omega_t(x, y)(\nabla_{x,y} u)^2 dt| = |\int v \mathcal{L}_\omega (u^2) - 2\int vu \mathcal{L}_\omega u| = 2|\int vu \mathcal{L}_\omega u| \leq Cr^2 \int vf,
\]

we have

\[
||II|| \leq \left( \sum_{x,y} \int_R v(x, t) \omega_t(x, y)(\nabla_{x,y} u)^2 dt \right)^{1/2} \left( \sum_{x,y} \int_R v(x, t) \omega_t(x, y)(\nabla_{x,y} u)^2 dt \right)^{1/2} \leq C v(Q_{3r/2})^{1/2}(\int vf)^{1/2}.
\]

Hence we obtain \( \int vf \leq \int v \phi f \leq C v(Q_{3r/2}) + C v(Q_{3r/2})^{1/2}(\int vf)^{1/2} \) and so \( v(Q_{3r/2}) \geq c \int vf \). The lemma follows by taking supremum over all \( f \) with \( \|f\|_{Q_r, d+1} = 1 \). \( \square \)
For \( \hat{x} = (x_1, \ldots, x_d, t) \), define \textit{parabolic cubes} with side-length \( r > 0 \) as

\[
K_r(\hat{x}) = \left( \prod_{i=1}^{d} (x_i - r, x_i + r) \cap \mathbb{Z}^d \right) \times \left[ t, t + r^2 \right), \quad K_r = K_r(\hat{0}).
\] (25)

We say that a function \( w \in \mathbb{R}^{2d \times \mathbb{R}} \) satisfies the \textit{reverse H"older inequality} \( RH_q (\mathcal{D}) \), \( 1 < q < \infty \), if for any parabolic subcube \( K \) of \( \mathcal{D} \),

\[
\|w\|_{K,q} \leq C \|w\|_{K,1}. \quad (RH_q)
\]

We say that \( w \) belongs to the \( A_p (\mathcal{D}) \) class (with \( A_p \) bound \( A \)), \( 1 < p < \infty \), if there exists \( A < \infty \) such that, for any parabolic subcube \( K \) of \( \mathcal{D} \),

\[
\|w\|_{K,1} \|1/w\|_{K,1/(p-1)} \leq A \quad (A_p)
\]

Recall the stopping time \( \Delta \) in (13). For \( R > 0, \hat{y} \in B_{2R} \times \mathbb{R} \), let

\[
g_R(\hat{y}; x, t) = P^y_\omega (X_{\Delta(B_{2R}, t)} = x). \quad (26)
\]

**Corollary 14.** Let \( \omega \in \Omega_\kappa, R > 0 \). Recall \( k_0 \) in Theorem 13.

(i) \( \rho_\omega \) satisfies \( RH_{d+1/d} (\mathbb{Z}^d \times \mathbb{R}) \).

(ii) For any \( y \in B_R \), \( v_y(\hat{x}) = g_R(y, 0; \hat{x}) \) satisfies

\[
RH_{d+1/d} (B_{R/2} \times [R^2/(2k_0^2), R^2/k_0^2]) .
\]

**Proof.** Note that \( \rho_\omega, v_y \) are adjoint solutions with volume-doubling properties Theorem 9 and Corollary 11. The corollary follows from Lemma 13.

**Theorem 15.** Let \( \omega, R, k_0, v_y \) be the same as in Corollary 14. There exist \( p = p(d, \kappa) > 1, A = A(d, \kappa) \) such that, for \( \mathbb{P} \)-a.e. \( \omega \),

(a) \( \rho_\omega \in A_p (\mathbb{Z}^d \times \mathbb{R}) \) with \( A_p \) bound \( A \). As a consequence,

\[
E_{\mathbb{P}}[\rho_\omega^{-1/(p-1)}] < \infty ;
\]

(b) For any \( y \in B_R \), \( v_y \) belongs to \( A_p (B_{R/2} \times [R^2/(2k_0^2), R^2/k_0^2]) \) with \( A_p \) bound \( A \). As a consequence, for any \( E \subset K \) where \( K \) is a parabolic subcube of \( B_{R/2} \times [R^2/(2k_0^2), R^2/k_0^2] \),

\[
\frac{g_R(y, 0; E)}{g_R(y, 0; K)} \geq C \left( \frac{|E|}{|K|} \right)^c . \quad (27)
\]

**Proof.** See [11, Section A.6].
Remark 16. The fact that $(RH)$ implies $(A_p)$ is a classical result in harmonic analysis. See e.g., [8, pg.246-249], [28, pg. 213-214]. In the elliptic non-divergence form PDE setting, the $A_p$ inequality for adjoint solutions was proved by Bauman [5], and estimate of form (27) was used by Fabes and Stroock [16] to obtain a short proof of the elliptic Harnack inequality.

In what follows, using (27), we will obtain a new proof of the parabolic Harnack inequality (Theorem 17). Our proof follows the ideas of [16]. Note that in our parabolic setting, the local volume-doubling property (Corollary 11) played a crucial role in the proof of (27).

Theorem 17 (PHI for $L_\omega$). Assume $\omega \in \Omega_\kappa$ and $\theta > 0$. Let $u$ be a non-negative function that satisfies $L_\omega u = 0$ in $B_R \times (0, \theta R^2)$. Then, for $0 < \theta_1 < \theta_2 < \theta_3 < \theta$, there exists a constant $C = C(\kappa, d, \theta_1, \theta_2, \theta_3, \theta)$ such that
\[
\sup_{B_{R/2} \times (\theta_2 R^2, \theta_3 R^2)} u \leq C \inf_{B_{R/2} \times [0, \theta R^2]} u.
\] (PHI)

We remark that in discrete time setting, (PHI) is obtained by Kuo and Trudinger for the so-called implicit form operators, see [23, (1.16)].

Proof of Theorem 17. Let $\ell_0 = 1/k^2_0$ and $\mathcal{D} = \{x : |x|_\infty < R/\sqrt{d}\} \times [\ell_0 R^2/2, \ell_0 R^2]$. We only prove a weaker version $\sup_{\mathcal{D}} u \leq C \min_{x \in B_R/\sqrt{d}} u(x, 0)$.

The theorem then follows by iteration. Indeed, assume $\min_{x \in B_R/\sqrt{d}} u(x, 0) = u(y, 0) = 1$ for $y \in B_{R/\sqrt{d}}$. Let $E_\lambda = \{\hat{x} \in D : u(\hat{x}) \geq \lambda\}$. By Lemma 27, $g_R(y, 0; \mathcal{D}) > CR^2$. Moreover, for $s \in [\ell_0 R^2/2, \ell_0 R^2]$, $1 = u(y, 0) \geq \lambda g_R(y, 0; E_\lambda \cap \{(x, t) : t = s\})$, and so
\[
1 \geq C \lambda g_R(y, 0; E_\lambda)/R^2 \geq C \lambda(|E_\lambda|/|\mathcal{D}|)^{\gamma/2}.
\]

Hence $|E_\lambda|/|\mathcal{D}| \leq C \lambda^{-\gamma}$ for some $\gamma > 0$. Therefore, for $0 < p < \gamma/2,$
\[
\|u\|_{\mathcal{D}, p} \leq \left[1 + \int_1^\infty \lambda^{p-1} |E_\lambda|/|\mathcal{D}| d\lambda\right]^{1/p} < C' = C' \min_{x \in B_R/\sqrt{d}} u(x, 0) < \infty.
\]

This inequality, together with [11] Theorem A.4.1, completes our proof. \qed

3 Estimates of solutions near the boundary

The purpose of this section is to establish estimates of $L_\omega$-harmonic functions near the parabolic boundary. For $x \in \mathbb{Z}^d$, $A \subset \mathbb{Z}^d$, let
\[
\text{dist}(x, A) := \min_{y \in A} |x - y|_1.
\]
3.1 An elliptic-type Harnack inequality

**Theorem 18** (Interior elliptic-type Harnack inequality). Assume \( \omega \in \Omega_\kappa \).

Let \( R \geq 2 \) and \( u \geq 0 \) satisfies

\[
\begin{aligned}
\mathcal{L}_\omega u &= 0 \quad \text{in } Q_R \\
u &= 0 \quad \text{in } \partial B_R \times [0, R^2).
\end{aligned}
\]

Then for \( 0 < \delta \leq \frac{1}{4} \), letting \( Q_\delta^R := B_{(1-\delta)R} \times [0, (1-\delta^2)R^2) \), there exists a constant \( C = C(d, \kappa, \delta) \) such that

\[
\sup_{Q_\delta^R} u \leq C \inf_{Q_\delta^R} u.
\]

Figure 3: The values of \( u \) are comparable inside the region \( Q_\delta^R \).

To prove Theorem 18, we need a so-called Carlson-type estimate. For parabolic differential operators in non-divergence form, this kind of estimate was first proved by Garofalo [17] (see also [15, Theorem 3.3]).

**Theorem 19.** Assume \( \omega \in \Omega_\kappa \), \( R > 2r > 0 \). Then for any function \( u \geq 0 \) that satisfies

\[
\begin{aligned}
\mathcal{L}_\omega u &= 0 \quad \text{in } (B_R \setminus B_{R-2r}) \times [0, 3r^2) \\
u &= 0 \quad \text{on } \partial B_R \times [r^2, 3r^2),
\end{aligned}
\]

with the convention \( \sup \emptyset = -\infty \), we have

\[
\sup_{(B_R \setminus B_{R-r}) \times [r^2, 3r^2)} u \leq C \min_{y \in \partial B_{R-r}} u(y, 0). \tag{28}
\]

**Proof.** Set \( \mathcal{D} = (B_R \setminus B_{R-2r}) \times [r^2, 3r^2) \). For \( \hat{x} = (x, t) \in \mathcal{D} \), let \( d_1(\hat{x}) = \sup \{ \rho \geq 0 : B_{\rho}(x) \subset B_R \setminus B_{R-2r} \} \geq 1 \).

First, we show that there exists \( \gamma = \gamma(d, \kappa) \) such that

\[
\sup_{\hat{x} \in \mathcal{D}} \left( d_1(\hat{x})/r \right)^\gamma u(\hat{x}) \leq C \min_{y \in \partial B_{R-r}} u(y, 0). \tag{29}
\]
Indeed, for any $\hat{x} = (x_1, x_2) \in \mathcal{O}$, we can find a sequence of $n \leq C \log(r/d_1(\hat{x}))$ balls with increasing radii $r_k := c2^k d_1(\hat{x})$:
\[
B_{r_1}(x_1) \subset B_{r_2}(x_2) \subset \cdots \subset B_{r_n}(x_n) \subset B_R \setminus \tilde{B}_{R-r}
\]
such that $x_1 = x$, $\text{dist}(x_n, \partial B_{R-r}) \leq r/2$, and $t - r_n^2 \geq r^2/2$. By Theorem 17
\[
u(x_1, t - r_1^2) \leq \cdots \leq C_n \nu(x_1, t - r^n_1) \leq C \left(\frac{r}{d_1(\hat{x})}\right)^c \min_{y \in \partial B_{R-r}} \nu(y, 0),
\]
where in the last inequality we applied Theorem 17 to a chain of parabolic balls with spatial centers at $\partial B_{R-r}$ and radius $cr$. Display (29) is proved.

Next, with $\gamma$ as in (29), letting $d_0(\hat{x}) = \sup(\rho \geq 0 : Q_\rho(\hat{x}) \subset (\mathbb{Z}^d \setminus \tilde{B}_{R-2r}) \times [r^2, 3r^2])$, we claim that
\[
\sup_{\hat{x} \in \mathcal{O}} d_0(\hat{x}) \gamma u(\hat{x}) \leq \epsilon^{-\gamma} \sup_{\hat{y} \in \mathcal{O}} d_1(\hat{y}) \gamma u(\hat{y}), \quad (30)
\]
where $\epsilon = \epsilon(d, \kappa) \in (0, 1/5)$ is to be determined. It suffices to show that $\sup_{\mathcal{O}} d_0^\gamma u$ is achieved at $\hat{x} \in \mathcal{O}$ with $cd_0(\hat{x}) \leq d_1(\hat{x})$. Indeed, if $cd_0(\hat{x}) > d_1(\hat{x})$, then $B_{2d_1(\hat{x})} \setminus B_R \neq \emptyset$, and for any $\hat{y} = (y, s) \in Q_{2d_1(\hat{x})}(\hat{x}) \cap \mathcal{O}$,
\[
d_0(\hat{x}) \leq d_0(\hat{y}) + |x - y| + |t - s|^{1/2} \leq d_0(\hat{y}) + 4d_1(\hat{x}) \leq d_0(\hat{y}) + 4\epsilon d_0(\hat{x})
\]
and so $d_0(\hat{x}) \leq (1 - 4\epsilon)^{-1} d_0(\hat{y})$. Moreover, by Corollary 28
\[
d_0(\hat{x}) \gamma u(\hat{x}) \leq [1 - P_{\omega}(X. \text{ exits } B_{2d_1(\hat{x})}(x) \cap B_R \text{ from } \partial B_R \text{ before time } d_1^2(\hat{x}))]
\times d_0(\hat{x}) \gamma \sup_{(B_{2d_1(\hat{x})}(x) \cap \partial B_R) \times [r^2, 3r^2]} u
\leq (1 - c_0)(1 - 4\epsilon)^{-\gamma} \sup_{\mathcal{O}} d_0^\gamma u
\]
for a constant $c_0 \in (0, 1)$. Thus, when $cd_0(\hat{x}) > d_1(\hat{x})$, choosing $\epsilon > 0$ so that $(1 - c_0)(1 - 4\epsilon)^{-\gamma} < 1 - \frac{c_0}{2}$, we get $d_0(\hat{x}) \gamma u(\hat{x}) < (1 - \frac{c_0}{2}) \sup_{\mathcal{O}} d_0^\gamma u$.

Display (30) is proved. Inequality (28) follows from (29) and (30).

Proof of Theorem 18. Since $u = 0$ on $\partial B_R \times [0, R^2)$,
\[
\sup_{Q_R^U} u \leq \sup_{B_{R \times \{R^2 - \frac{1}{4}(\delta R)^2\}}} u \leq C \sup_{B_{(1 - \delta)}R \times \{R^2 - \frac{1}{7}(\delta R)^2\}} u
\]
where we used Theorem 19 and Theorem 17 in the second inequality. □
3.2 A boundary Harnack inequality

For positive harmonic functions with zero values on the spatial boundary, the following boundary Harnack inequality compares values near the spatial boundary and values inside, with time coordinates appropriately shifted.

**Theorem 20** (Boundary PHI). Let \( R > 0 \). Suppose \( u \) is a nonnegative solution to \( \mathcal{L}_\omega u = 0 \) in \((B_{4R} \setminus B_{2R}) \times (-2R^2, 3R^2)\), and \( u|_{\partial B_{4R} \times \mathbb{R}} = 0 \). Then for any \( \hat{x} = (x,t) \in (B_{4R} \setminus B_{3R}) \times (-R^2, R^2) \), we have

\[
C \frac{\text{dist}(x, \partial B_{4R})}{R} \max_{y \in \partial B_{3R}} u(y, t+R^2) \leq u(\hat{x}) \leq C \frac{\text{dist}(x, \partial B_{4R})}{R} \min_{y \in \partial B_{3R}} u(y, t-R^2).
\]

Theorem 20 is a lattice version of [17, (3.9)]. In what follows we offer a probabilistic proof.

**Proof of Theorem 20**. Our proof uses the fact that \( u(\hat{X}_t) \) is a martingale before exiting the region \( \mathcal{D} := (B_{4R} \setminus B_{2R}) \times (-2R^2, 3R^2)\).

For the lower bound, let \( \tau_{3,4} := \inf\{s > 0 : X_s \notin B_{4R} \setminus \bar{B}_{3R}\} \). By the optional stopping lemma, \( u(\hat{x}) = E^x[u(\hat{X}_{\tau_{3,4} \wedge 0.5R^2})] \), and so

\[
u(\hat{x}) \geq P^{\omega,t}_{X} (\tau_{3,4} < R^2/2, X_{\tau_{3,4}} \in \partial B_{3R}) \inf_{\partial B_{3R} \times [t, t+0.5R^2]} u
\]

\[
\geq C \frac{\text{dist}(x, \partial B_{4R})}{R} \max_{y \in \partial B_{3R}} u(y, t+R^2)
\]

where in the last inequality we used Lemma 29 and applied Theorem 17 (to a chain of parabolic balls). The lower bound is obtained.

To obtain the upper bound, note that for \( \hat{x} \in (B_{4R} \setminus \bar{B}_{3R}) \times (-R^2, R^2) \),

\[
u(\hat{x}) \leq \left[ \max_{z \in B_{4R} \setminus \bar{B}_{3R}} u(z, t + R^2) + \max_{\partial B_{3R} \times (t, t+R^2)} u \right] P^{\omega,t}_{X} (X_{\tau_{3,4} \wedge 0.5R^2} \notin \partial B_{4R})
\]

\[
\leq C \left[ \max_{z \in B_{4R} \setminus \bar{B}_{3R}} u(z, t - \frac{R^2}{2}) + \max_{\partial B_{3R} \times (t, t+R^2)} u \right] P^{\omega,t}_{X} (X_{\tau_{3,4} \wedge 0.5R^2} \notin \partial B_{4R})
\]

\[
\leq C \min_{z \in \partial B_{3R}} u(z, t - R^2) \text{dist}(x, \partial B_{4R}) / R,
\]

where in the last inequality we applied Lemma 30 and used an iteration of the Harnack inequality (Theorem 17). \(\square\)

4 Proof of PHI for the adjoint operator (Theorem 5)

We define \( \hat{Y}_t = (Y_t, S_t) \) to be the continuous-time Markov chain on \( \mathbb{Z}^d \times \mathbb{R} \) with generator \( \mathcal{L}_\omega^* \). The process \( \hat{Y}_t \) can be interpreted as the time-reversal
of \( \hat{X}_t \). Denote by \( P_{\omega}^{y,s} \) the quenched law of \( \hat{Y} \), starting from \( \hat{Y}_0 = (y, s) \) and by \( E_{\omega}^{y,s} \) the corresponding expectation. Note that \( S_t = S_0 - t \).

For \( R > 0, \hat{x} = (x, t), \hat{y} = (y, s) \in B_R \times \mathbb{R} \) with \( s > t \), set

\[
p_{R}^{y}(\hat{x}; \hat{y}) = P_{\omega}^{x,y}(X_{s-t} = y, s-t < \tau_R(\hat{X})), \]
\[
p_{R}^{y}(\hat{y}; \hat{x}) = P_{\omega}^{y,x}(Y_{s-t} = x, s-t < \tau_R(\hat{Y})),
\]

where

\[
\tau_R(\hat{X}) := \inf\{ t \geq 0 : X_t \notin B_R \}
\]

and \( \tau_R(\hat{Y}) \) is defined similarly. Note that

\[
p_{R}^{y}(\hat{y}; \hat{x}) = \frac{\rho_{\omega}(\hat{x})}{\rho_{\omega}(\hat{y})} p_{R}^{y}(\hat{x}; \hat{y}).
\]

**Lemma 21.** For any \( \hat{y} = (y, s) \in B_R \times (0, \infty) \) and any non-negative solution \( v \) of \( L_{\omega}^{x} v = 0 \) in \( B_R \times (0, s] \),

\[
v(\hat{y}) = \sum_{x \in \partial B_R, z \in B_R, x \sim z} \int_0^s \frac{\rho_{\omega}(x, t)}{\rho_{\omega}(\hat{y})} \omega_t(z, x) p_{R}^{y}(z, t; \hat{y}) v(x, t) dt \]
\[
+ \sum_{x \in B_R} \frac{\rho_{\omega}(x, 0)}{\rho_{\omega}(\hat{y})} p_{R}^{y}(x, 0; \hat{y}) v(x, 0).
\]

**Proof.** Write the two summations in the lemma as I and II. Clearly, II = \( E_{\omega}^{y,s}[v(\hat{Y}_t)1_{\tau_R > s}] \). Since \( (v(\hat{Y}_t))_{t \geq 0} \) is a martingale, we have

\[
v(y, s) = E_{\omega}^{y,s}[v(\hat{Y}_t)1_{\tau_R \leq s}] + E_{\omega}^{y,s}[v(\hat{Y}_t)1_{\tau_R > s}].
\]

So it remains to show I = \( E_{\omega}^{y,s}[v(\hat{Y}_t)1_{\tau_R \leq s}] \). We claim that for \( x \in \partial B_R \),

\[
P_{\omega}^{y,s}(Y_{\tau_R} = x, \tau_R \in dt) = \sum_{z \in B_R, z \sim x} \frac{\rho_{\omega}(x, s-t)}{\rho_{\omega}(\hat{y})} \omega_{s-t}(z, x) p_{R}^{y}(z, s-t; \hat{y}) dt.
\]

Indeed, for \( h > 0 \) small enough, \( x \in \partial B_R \) and almost every \( t \in (0, s) \),

\[
P_{\omega}^{y,s}(Y_{\tau_R} = x, \tau_R \in (t-h, t+h))
\]
\[
= \sum_{z \in B_R, z \sim x} P_{\omega}^{y+s-t+h}(Y_{2h} = x) + o(h)
\]
\[
= \sum_{z \in B_R, z \sim x} p_{R}^{y}(\hat{y}; z, s-t) \int_{-h}^h \omega_{s-t+r}(z, x) dr + o(h).
\]

Dividing both sides by \( 2h \) and taking \( h \to 0 \), display (32) follows by Lebesgue’s differentiation theorem. Applying (32) to

\[
E_{\omega}^{y,s}[v(\hat{Y}_t)1_{\tau_R \leq s}] = \sum_{x \in \partial B_R} \int_0^s v(x, s-t) P_{\omega}^{y,s}(Y_{\tau_R} = x, \tau_R \in dt),
\]

we obtain I = \( E_{\omega}^{y,s}[v(\hat{Y}_t)1_{\tau_R \leq s}] \) with a change of variable. \( \square \)
For fixed $\hat{y} := (y, s) \in B_R \times \mathbb{R}$, set $u(\hat{x}) := p^{\nu}(\hat{x}, \hat{y})$. Then $L_\omega u = 0$ in $B_{2R} \times (-\infty, s) \cup (B_{2R} \setminus B_R) \times \mathbb{R}$ and $u(x, t) = 0$ when $x \in \partial B_{2R}$ or $t > s$.

By Theorem $20$ and Theorem $18$ for any $(x, t) \in B_{2R} \times (s - 4R^2, s - \frac{R^2}{2})$,

$$u(x, t) \asymp u(o, s - R^2) \text{dist}(x, \partial B_{2R})/R, \quad (33)$$

and, for any $(x, t) \in (B_{2R} \setminus B_{3R/2}) \times (s - 4R^2, s)$,

$$u(x, t) \leq Cu(o, s - R^2) \text{dist}(x, \partial B_{2R})/R. \quad (34)$$

**Lemma 22.** Let $v \geq 0$ satisfies $L_\omega^v v = 0$ in $B_{2R} \times (0, 4R^2]$, then for any $\hat{Y} = (\hat{y}, \hat{s}) \in B_R \times (3R^2, 4R^2]$ and $\bar{Y} = (\bar{y}, \bar{s}) \in B_R \times (R^2, 2R^2)$, we have

$$\frac{v(\hat{Y})}{v(\bar{Y})} \geq C \int_{0}^{R^2} \frac{\rho_\omega(\partial B_{2R}, t)}{\rho_\omega(\bar{Y})} dt + \sum_{x \in B_{2R}} \rho_\omega(x, 0) \text{dist}(x, \partial B_{2R}) \frac{v(x, s)}{R}. \quad (35)$$

**Proof.** Write $\hat{x} := (x, t)$ and set $\bar{u}(\hat{x}) := p^{\nu}(\hat{x}; \bar{Y})$, $u(\hat{x}) := p^{\nu}(\hat{x}; \hat{Y})$. By Lemma $21$ and $(33)$,

$$v(\hat{Y}) \geq C \sum_{x \in \partial B_{2R}, z \in B_{2R}, s < x} \int_{0}^{R^2} \frac{\rho_\omega(\hat{x})}{\rho_\omega(Y)} \bar{u}(z, t) v(\hat{x}) dt$$

$$+ \sum_{x \in B_{2R}} \rho_\omega(x, 0) \text{dist}(x, \partial B_{2R}) \frac{v(x, 0)}{R}. \quad (35)$$

Similarly, by Lemma $21$ and $(34)$, we have

$$v(\bar{Y}) \leq \bar{C} \sum_{x \in \partial B_{2R}} \int_{0}^{R^2} \rho_\omega(\hat{x}) v(\hat{x}) dt$$

$$+ \sum_{x \in B_{2R}} \rho_\omega(x, 0) \text{dist}(x, \partial B_{2R}) v(x, 0). \quad (36)$$

Combining $(35)$ and $(36)$, we get

$$\frac{v(\hat{Y})}{v(\bar{Y})} \geq C \frac{\bar{u}(\hat{y}, \hat{s} - R^2)/\bar{\rho}_\omega(\hat{Y})}{\bar{u}(o, \bar{s} - R^2)/\bar{\rho}_\omega(\bar{Y})}. \quad (37)$$

Next, taking $v = 1$, by Lemma $21$ and $(34)$,

$$1 = \sum_{x \in \partial B_{2R}, z \in B_{2R}, s < x} \int_{0}^{R^2} \rho_\omega(\hat{x}) \omega(z, x) \bar{u}(z, t) dt + \sum_{x \in B_{2R}} \rho_\omega(x, 0) \bar{u}(x, 0)$$

$$\leq C \bar{u}(\hat{y}, \hat{s} - R^2) \left[ \sum_{x \in \partial B_{2R}} \int_{0}^{R^2} \rho_\omega(\hat{x}) dt + \sum_{x \in B_{2R}} \rho_\omega(x, 0) \text{dist}(x, \partial B_{2R}) \right].$$
Similarly, by Lemma 21 and (33),

\[
1 \geq \frac{u(0, s - R^2)}{R \rho_\omega(Y)} \left[ \sum_{x \in \partial B_{2R}} \int_0^{2/2} \rho_\omega(\hat{x}) \, dt + \sum_{x \in B_{2R}} \rho_\omega(x, 0) \text{dist}(x, \partial B_R) \right].
\]

These inequalities, together with (37), yield the lemma.

**Remark 23.** It is clear that for static environments, the adjoint Harnack inequality (Theorem 5) follows immediately from Lemma 22. However, in time-dependent case, we need the parabolic volume-doubling property of \( \rho_\omega \).

**Proof of Theorem 5.** First, we will show that for all \( R > 0 \),

\[
\int_0^s \rho_\omega(\partial B_R, t) \, dt + \sum_{x \in B_R} \rho_\omega(x, 0) \text{dist}(x, \partial B_R)
\]

\[
\asymp \frac{1}{R} \int_0^s \rho_\omega(B_R, t) \, dt + \sum_{x \in B_R} \rho_\omega(x, s) \text{dist}(x, \partial B_R).
\]

Recall \( \tau_R \) at (31) and set \( g(x, t) = E^{x, t}_\omega[|\tau^R(x)|] \). Note that \( g(x, \cdot) = 0 \) for \( x \notin B \) and \( L_\omega g(x, t) = -1 \) if \( x \in B_R \). By (3), for any \( s > 0 \),

\[
0 = \sum_{x \in \mathbb{Z}^d} \int_0^s g(x, t) \left[ \sum_{y} \rho_\omega(y, t) \omega_l(y, x) - \partial_t \rho_\omega(x, t) \right] \, dt
\]

\[
= \sum_{x \in \partial B_R, y \in B_R} \int_0^s \rho(x, t) \omega_l(x, y) g(y, t) \, dt + \sum_{x \in B_R} g(x, 0) \rho(x, 0)
\]

\[
- \sum_{x \in B_R} \int_0^s \rho(x, t) \, dt - \sum_{x \in B_R} g(x, s) \rho(x, s).
\]

Moreover, since \( |X_t|^2 - \frac{dt}{\kappa} \) and \( |X_t|^2 - \kappa t \) are super- and sub-martingales,

\[
g(x, t) \asymp E^{x, t}_\omega[|X^t_{\tau_R}|^2 - |x|^2] \asymp R \text{dist}(x, \partial B_R) \quad \forall (x, t) \in B_R \times \mathbb{R}
\]

by the optional-stopping theorem. Display (38) then follows.

Combining (38) and Lemma 22 we obtain

\[
\frac{v(\hat{Y})}{v(Y)} \geq C \frac{\int_0^{R^2} \rho(B_{2R}, t) \, dt + R \sum_{x \in B_{2R}} \rho(x, R^2) \text{dist}(x, \partial B_{2R})}{\int_0^{4R^2} \rho(B_{2R}, t) \, dt + R \sum_{x \in B_{2R}} \rho(x, 4R^2) \text{dist}(x, \partial B_{2R})}.
\]

Finally, Theorem 5 follows by Theorem 9 and the above inequality.

**5 Proof of Theorem 7, Corollary 8 and Theorem 4**

**5.1 Proof of Theorem 7**

**Proof.** First, using Theorem 5 and standard arguments, we will prove 8. Recall that \( v(\hat{x}) := q^\omega(\hat{0}, \hat{x}) \) satisfies \( L_*^\omega v = 0 \) in \( \mathbb{Z}^d \times (0, \infty) \). By Theorem 34
for \( \hat{x} = (x, t) \in \mathbb{Z}^d \times (0, \infty) \), we have
\[
v(\hat{x}) \leq C \min_{y \in B_{\sqrt{T}}(x)} v(y, 3t) \text{ and so}
\]
\[
v(\hat{x}) \leq \frac{C}{\rho(B_{\sqrt{T}}(x), 3t)} \sum_{y \in B_{\sqrt{T}}(x)} \rho(y, 3t) v(y, 3t)
\]
\[
= \frac{C}{\rho(B_{\sqrt{T}}(x), 3t)} p^{0,0}_{\omega}(X_{3t} \in B_{\sqrt{T}}(x)) \leq C \exp[-c \rho(|x|, t)],
\]
where Corollary 26 is used in the last inequality. Moreover, for any \( s \in [0, t] \),
\[
|y| \leq |x| + c \sqrt{T},
\]
by Theorem 3 and iteration,
\[
\rho(B_{\sqrt{T}}(x), 3t) \geq C \rho(B_{\sqrt{T}/4}(x), s) \geq C \left( \frac{|y|}{\sqrt{T}} + 1 \right)^{-c} \rho(B_{\sqrt{T}/4}(y), s).
\]
Since \( \frac{|y|}{\sqrt{T}} + 1 \leq C e^{c \rho(|x|, t)} \) for any \( \epsilon > 0 \), the upper bound in (39) follows.

To obtain the lower bound in (39), by similar argument as above and
Theorem 5, \( v(\hat{x}) \geq C \max_{y \in B_{\sqrt{T}/2}(x)} v(y, t/4) \) for \( \hat{x} \in \mathbb{Z}^d \times (0, \infty) \), and so
\[
v(\hat{x}) \geq \frac{C}{\rho(B_{\sqrt{T}/2}(x), t/4)} p^{0,0}_{\omega}(X_{t/4} \in B_{\sqrt{T}/2}(x)). \tag{39}
\]
We claim that for any \( (y, s) \in \mathbb{Z}^d \times (0, \infty) \),
\[
p^{0,0}_{\omega}(X_{r} \in B_{\sqrt{T}}) \geq C e^{-c |y|^2 / s}. \tag{40}
\]
Indeed, the case \( |y|/\sqrt{T} \leq 3 \) follows from Lemma 27. When \( |y|/\sqrt{T} > 3 \), let
\[
n = \lfloor |y|^2 / s \rfloor.
\]
Set \( u(x, t) := p^\omega(x, t; B_{\sqrt{T}}, s) \). Then \( u \) is a \( \mathcal{L}_\omega \)-harmonic function on \( \mathbb{Z}^d \times (-\infty, s) \). Taking a sequence of points \( (y_i)_{i=1}^n \) such that \( y_0 = y, y_n = 0 \) and
\[
|y_i - y_{i+1}| \leq |y|/n, \text{ for } i = 0, \ldots, n - 1,
\]
\[
\text{Lemma } 27 \quad \min_{x \in B_{|y|/\sqrt{T}}(y_i)} u(x, \frac{|y|^2}{n^2} \geq \min_{x \in B_{|y|/\sqrt{T}}(y_{i+1})} u(x, \frac{(i+1)|y|^2}{n^2}) \quad \frac{C}{\rho(B_{\sqrt{T}/2}(x), t/4)} \geq C^n.
\]
Iteration then yields \( u(y, 0) \geq C^{n-1} \min_{x \in B_{|y|/\sqrt{T}}} u(x, \frac{|y|^2}{n}) \geq C^n \). Inequality 10 is proved. Then, by (39),
\[
v(x, t) \geq \frac{C}{\rho(B_{\sqrt{T}/2}(x), t/4)} e^{-c |x|^2 / t}.
\]
Moreover, by Theorem 9 we have for any \( s \in [0, t], |y| \leq |x|, \)
\[
\rho(B_{\sqrt{t/2}}(x), t/4) \leq C \rho(B_{\sqrt{t/2}}(x), s) \leq C\left(\frac{|x|}{t}\right)^c \rho(B_{\sqrt{t}}(y), s).
\]
The lower bound in (8) is proved.

Next, we will prove the moment bounds (9) and (10), which, by (8) and (20), are equivalent to showing that, for all \( \hat{\rho} \) and the volume-doubling property of \( Q \), where we used the Reverse Hölder inequality (Corollary 14(i)) in the last inequality. Recalling (22), inequality (9) then follows from the fact that
\[
|Q| \leq C_{\rho} (r \vee 1)^{-d},
\]
where \( Q \) is as defined in (15). Indeed, using the translation-invariance of \( \mathbb{P} \) and the volume-doubling property of \( \rho \), for \( q := (d+1)/d, \)
\[
\|\rho(\hat{0})/\rho(Q_r)\|_{L^{q}([\hat{y}])}^q \leq C/|Q_r|^q,
\]
where we used the Reverse Hölder inequality (Corollary 14(1)) in the last inequality. Recalling (22), inequality (9) then follows from the fact that
\[
|Q_r| = r^d \sum_{x \in B_r} 1 \approx r^2 (r \vee 1)^{-d}.
\]
To obtain (10), note that by translation invariance and \( \mathbb{P} \) and the volume-doubling property of \( \rho \), taking \( \epsilon \in (0, 1/(p-1)), \)
\[
\|\rho(Q_r)/\rho(\hat{0})\|_{L^{p'}}(\mathbb{P}) \leq \frac{C}{|Q_r|^p} \mathbb{E}_{\mathbb{P}} \left[ \int_{x \in Q_r} \frac{\rho(\hat{x})^\epsilon}{\rho(Q_r)^\epsilon} \right] \leq C|Q_r|^{-\epsilon},
\]
where we used the A\(_p\) inequality (Theorem 14(2)) of \( \rho \) in the last inequality. Therefore \( \|\rho(\hat{0})/\rho(Q_r)\|_{L^{p}([\hat{y}])} \geq Cr^2 (r \vee 1)^{-d} \) and (10) is proved.

Display (11) follows from (9), (10), and Minkowski’s integral inequality. \( \square \)

### 5.2 Proof of Theorem 4

As a standard consequence of the PHI for \( L^*_\omega \), we first state the following Hölder estimate. (See a proof in [11, Section A.2].)

**Corollary 24.** There exists \( \gamma = \gamma(d, \kappa) \in (0, 1] \) such that for \( \mathbb{P} \)-almost all \( \omega \), any non-negative solution \( u \) of \( L^*_\omega \) in \( B_R(x_0) \times (t_0 - R^2, t_0], R > 0, \) satisfies
\[
|u(\hat{x}) - u(\hat{y})| \leq C \left( \frac{r}{R} \right)^\gamma \sup_{B_r(x_0) \times (t_0 - R^2, t_0]} u
\]
for all \( \hat{x}, \hat{y} \in B_r(x_0) \times (t_0 - r^2, t_0] \) and \( r \in (0, R) \).

Recall \( q^\omega(\hat{y}, \hat{x}) \) in (3). For any \( \hat{x} = (x, t) \in \mathbb{R}^d \times \mathbb{R}, \) set
\[
v(\hat{x}) := q^\omega(\hat{0}; [x], t),
\]
where \(|x|\) is as in Theorem \([\text{4}]\). Note that \(L^\omega v = 0\) in \(Z^d \times (0, \infty)\). By Corollary \([\text{24}]\) and Theorem \([\text{7}]\) for any \(\hat{y} = (y, s) \in B_\sqrt\tau(x) \times \left(\frac{t}{2}, t\right)\),

\[
|v(\hat{x}) - v(\hat{y})| \leq C \left(\frac{|x - y| + \sqrt{t - s}}{\sqrt{t}}\right)^\gamma \sup_{B_\sqrt{\tau(x) \times \left(\frac{t}{2}, t\right)}} v \leq C \left(\frac{|x - y| + \sqrt{t - s}}{\sqrt{t}}\right)^\gamma t^{-d/2}
\]

(41)

when \(t > t_0(\omega)\) is big enough. Here in the last inequality we used Corollary \([\text{8(i)}]\) which is an immediate consequence of Theorem \([\text{7}]\) and \((\text{21})\).

Recall \(O_r\) in \((\text{24})\). For \(\hat{x} = (x, t) \in R^d \times R\), write \(\hat{x}^n = (\lfloor nx \rfloor, n^2 t)\).

To prove Theorem \([\text{4}]\) it suffices to show that for any \(K > T\),

\[
\lim_{n \to \infty} \sup_{\hat{x} \in O_K \times [T, K]} |n^d v(\hat{x}^n) - p_\tau^\Sigma(0, x)| = 0.
\]

(42)

Indeed, for any \(\epsilon > 0\), there exists \(K = K(T, \epsilon, d, \kappa) > 0\) such that, writing \(\mathcal{G} := (R^d \times [T, \infty)) \setminus (O_K \times [T, K])\), we have

\[
\lim_{n \to \infty} \sup_{\mathcal{G}} n^d v(\hat{x}^n) + p_\tau^\Sigma(0, x) \leq \epsilon.
\]

Hence Theorem \([\text{4}]\) follows provided that \((\text{42})\) is proved.

**Proof of Theorem \([\text{4}]\)** As we discussed in the above, it suffices to prove \((\text{42})\).

It suffices to consider the case \(T < K < 2T\). For any \(\epsilon > 0\),

\[
|n^d v(\hat{x}^n) - p_\tau^\Sigma(0, x)| \leq A(\hat{x}, \epsilon) + B_n(\hat{x}, \epsilon) + C_n(\hat{x}, \epsilon),
\]

(43)

where

\[
A(\hat{x}, \epsilon) = \left|\frac{\int_{\mathcal{G}} p_\tau^\Sigma(0, O_\epsilon(x))\,ds}{\epsilon^2 |O_\epsilon|} - p_\tau^\Sigma(0, O_\epsilon(x))\right|,
\]

\[
B_n(\hat{x}, \epsilon) = \left|\int_{t}^{t + \epsilon^2} \frac{P_\omega^0(X_n^2 \in nO_\epsilon(x)) - p_\tau^\Sigma(0, O_\epsilon(x))}{\epsilon^2 |O_\epsilon|}\,ds\right|,
\]

\[
C_n(\hat{x}, \epsilon) = \left|n^d v(\hat{x}^n) - \frac{\int_{t}^{t + \epsilon^2} P_\omega^0(X_n^2 \in nO_\epsilon(x))\,ds}{\epsilon^2 |O_\epsilon|}\right|.
\]

First, we will show that

\[
\lim_{n \to \infty} \sup_{\hat{x} \in O_K \times [T, K]} C_n(\hat{x}, \epsilon) = O(\epsilon^\gamma).
\]

(44)
To this end, note that there exists $N = N(T, \omega, d, \kappa)$ such that for $n \geq N$,

$$C_n(\hat{x}, \epsilon) \leq n^d v(\hat{x}^n) \left| 1 - \frac{\int_{t}^{t+\epsilon^2} \rho(\mathcal{O}_e(x), n^2 s) ds}{\epsilon^2 |\mathcal{O}_e|} \right| + \sum_{y \in n\mathcal{O}_e(x)} \int_{t}^{t+\epsilon^2} |v(y, n^2 s) - v(\hat{x}^n)| \rho(y, n^2 s) ds / (\epsilon^2 |\mathcal{O}_e|)$$

$$\leq CT^{-d/2} \left| 1 - \frac{\int_{t}^{t+\epsilon^2} \rho(\mathcal{O}_e(x), n^2 s) ds}{\epsilon^2 |\mathcal{O}_e|} \right|$$

$$+ CT^{-(\gamma+d)/2} \epsilon^{\gamma} \int_{t}^{t+\epsilon^2} \rho(\mathcal{O}_e(x), n^2 s) ds / (\epsilon^2 |\mathcal{O}_e|),$$

where in the second inequality we used Corollary [81] and (41). Further, by an ergodic theorem of Krengel and Pyke [21, Theorem 1] and (2),

$$\lim_{n \to \infty} \sup_{\hat{x} \in \mathcal{O}_K \times [T,K]} \left| 1 - \frac{\int_{t}^{t+\epsilon^2} \rho(\mathcal{O}_e(x), n^2 s) ds}{\epsilon^2 |\mathcal{O}_e|} \right| = 0. \quad (45)$$

Display (41) follows.

Next, for $\hat{x} = (x, t)$, by writing $B_n(\hat{x}, \epsilon)$ as

$$\left| \int_{t}^{t+\epsilon^2} P_{\omega}^0 (X_{n^2 s} \in n\mathcal{O}_e(x)) ds \right| - \left| \int_{t}^{t+\epsilon^2} P^0 (0, \mathcal{O}_e(x)) ds \right| =: |B_n^1(\hat{x}, \epsilon) - B^2(\hat{x}, \epsilon)|,$$

we will show that

$$\lim_{n \to \infty} \sup_{\hat{x} \in \mathcal{O}_K \times [T,K]} B_n(\hat{x}, \epsilon) = O(\epsilon^\gamma). \quad (46)$$

We claim that $B_n(\hat{x}, \epsilon)$ is approximately equicontinuous (with order $\epsilon^\gamma$). That is, there exist $N, \delta$ depending on $(\epsilon, \omega, d, \kappa, T, K)$ such that, whenever $n \geq N$ and $\hat{x}_1 = (x_1, t_1), \hat{x}_2 = (x_2, t_2) \in \mathcal{O}_K \times [T,K]$ satisfy $|\hat{x}_1 - \hat{x}_2| := |x_1 - x_2| + |t_1 - t_2| < \delta$, we have

$$|B_n(\hat{x}_1, \epsilon) - B_n(\hat{x}_2, \epsilon)| < C\epsilon^\gamma.$$

It suffices to show that $B_n^1(\hat{x}, \epsilon)$ is approximately equicontinuous. Indeed, by (41) and (41), when $n \geq N$ is large and $\hat{x}_1, \hat{x}_2 \in \mathcal{O}_K \times [T,K]$,

$$|B_n^1(\hat{x}_1, \epsilon) - B_n^1(\hat{x}_2, \epsilon)| \leq C_n(|\hat{x}_1, \epsilon| + C_n(|\hat{x}_2, \epsilon| + n^d |v(\hat{x}_1^n) - v(\hat{x}_2^n)|)$$

$$\leq C\epsilon^\gamma + C(|x_1 - x_2| + \sqrt{|t_1 - t_2|})^\gamma.$$

The approximate equicontinuity of $B_n^1(\hat{x}, \epsilon)$ follows. To prove (46), we choose a finite sequence $\{\hat{x}_i\}_{i=1}^M$ such that $\min_{1 \leq i \leq M} |\hat{x} - \hat{x}_i| < \delta$ for all $\hat{x} \in \mathcal{O}_K \times [T,K]$. Since $\lim_{n \to \infty} \max_{1 \leq i \leq M} B_n(\hat{x}_i) = 0$ by the QCLT (Theorem 1), display (46) follows by the approximate equicontinuity.

Clearly, $\lim_{n \to \infty} \sup_{\hat{x} \in \mathcal{O}_K \times [T,K]} A(\hat{x}, \epsilon) = 0$. This, together with (41) and (41), yields the uniform convergence of (43) by sending first $n \to \infty$ and then $\epsilon \to 0$. Our proof of Theorem 4 is complete.\[ \square \]
Proof of Corollary follows from Theorem and (21). and are consequences of Theorems and Their proofs, which are similar to [2, Theorem 1.14] and [4, Theorem 1.4], can be found in [11, A.6]. □

6 Auxiliary probability estimates

This section contains probability estimates that are useful in the rest of the paper. It does not rely on results in the previous sections, and can be read independently. Recall the definition of the function \( h(r, t) \) in (7).

**Theorem 25.** Assume \( \omega \in \Omega_\kappa \). Then for \( t > 0, r > 0 \),
\[
P_0^0(\sup_{0 \leq s \leq t} |X_s| > r) \leq C \exp(-ch(r, t)).
\]

**Proof.** Let \( x(i), i = 1, \ldots, d \), denotes the \( i \)-th coordinate of \( x \in \mathbb{R}^d \). It suffices to show that for \( i = 1, \ldots, d \),
\[
P_\omega^0(\sup_{0 \leq s \leq t} |X_s(i)| > r) \leq C \exp(-ch(r, t)).
\]

We will prove the statement for \( i = 1 \). Let \( \tilde{N}_t := \#\{0 \leq s \leq t : X_s(1) \neq X_{s-1}(1)\} \) be the number of jumps in the \( e_1 \) direction before time \( t \). Let \( (S_n) \) be the discrete time simple random walk on \( \mathbb{Z} \), then \( X_t(1) = S_{\tilde{N}_t} \). Note that \( \tilde{N}_t \) is stochastically dominated by a Poisson process \( N_t \) with rate \( c_0 := 2d/\kappa \), and so \( P_0^0(\sup_{0 \leq s \leq t} |X_s(1)| > r) \leq P(\max_{0 \leq m \leq \tilde{N}_t} |S_m| > r) \). Hence,
\[
P_\omega^0(\sup_{0 \leq s \leq t} |X_s(1)| > r) \leq P(N(t) \geq 2c_0(t \vee r)) + P(\max_{0 \leq m \leq 2c_0(t \vee r)} |S_m| > r)
\]
\[
\leq e^{-c(t \vee r)} + Ce^{-cr^2/(t \vee r)} \leq Ce^{-cr^2/(t \vee r)}.
\]

where we used Hoeffding’s inequality in the second inequality. On the other hand, since the random walk is in a discrete set \( \mathbb{Z} \), we have, for any \( \theta > 0 \),
\[
P_\omega^0(\sup_{0 \leq s \leq t} |X_s(1)| > r) \leq P(N(t) > r)
\]
\[
\leq E[\exp(\theta N(t) - \theta r)] = \exp[c_0 t(e^\theta - 1) - \theta r].
\]

When \( r \geq 9c_0^2 t \), taking \( \theta = \log(\frac{r}{c_0 t}) \), we get an upper bound \( \exp[-\frac{r}{4} \log(\frac{r}{t})] \). Hence, letting \( f(r, t) = \frac{r^2}{4t} 1_{r < 9c_0^2 t} + r \log(\frac{r}{t}) 1_{r \geq 9c_0^2 t} \), we obtain
\[
P_\omega^0(\sup_{0 \leq s \leq t} |X_s(1)| > r) \leq C \exp(-cf(r, t)).
\]

Since \( f(r, t) \sim \frac{r^2}{4t} + r \log(\frac{r}{t}) 1_{r \geq 9c_0^2 t} \propto h(r, t) \), our proof is complete. □
Corollary 26. Assume $\omega \in \Omega_\kappa$ and $\theta_2 > \theta_1 > 0$. There exist $C, c$ depending on $(d, \kappa, \theta_1, \theta_2)$ such that for $\theta \in (\theta_1, \theta_2)$, $(x, t) \in \mathbb{Z}^d \times (0, \infty)$,

$$P_{\omega}^{\sigma,0}(X_t \in B_{\sqrt{\theta t}}(x)) \leq C \exp[-c h(|x|, t)].$$

Proof. Since $h(0, t) = 0$, we only need to consider the case $x \neq 0$.

If $\theta t \leq 1$, then $P_{\omega}^{\sigma,0}(X_t \in B_{\sqrt{\theta t}}(x)) = P_{\omega}^{0}(X_t = x) \leq P_{\omega}^{0}(\sup_{0 \leq s \leq \theta t} |X_s| \geq |x|) \leq C \exp[-c h(|x|, t)]$ by Theorem 25.

If $\theta t > 1$ and $1 \leq |x| \leq 2\sqrt{\theta t}$, then $|x| \leq |x|^2 \leq 4\theta t$ and so $h(|x|, t) \asymp h(|x|, 4\theta t) \asymp \frac{|x|^2}{t}$. In particular, $h(|x|, t) \leq C|\frac{x}{t}|^2 \leq c$. Hence, trivially, $P_{\omega}^{0}(X_t \in B_{\sqrt{\theta t}}(x)) \leq 1 \leq C \exp(-c h(|x|, t)).$

It reminds to consider $|x| > 2\sqrt{\theta t} > 2$. In this case, by Theorem 25,

$$P_{\omega}^{0}(X_t \in B_{\sqrt{\theta t}}(x)) \leq P_{\omega}^{0}(\sup_{0 \leq s \leq \theta t} |X_s| \geq |x|/2) \leq C \exp(-c h(|x|, t)).$$

Lemma 27. Let $0 < \theta_1 < \theta_2$, $R > 0$ and $\omega \in \Omega_\kappa$. Recall the definition of the stopping time $\Delta$ in (13). There exists a constant $\alpha = \alpha(\kappa, d, \theta_1, \theta_2) \geq 1$ such that for any $s \in (\theta_1 R^2, \theta_2 R^2)$ and $\sigma > 0$

$$\min_{x \in B_R} P^{\sigma,0}_{\omega}(X_{\Delta(B_{2R}, s)} \in B_{\sigma R}) \geq \left(\frac{\sigma \wedge 1}{2}\right)^\alpha.$$

Proof. It suffices to consider the case $\sigma \in (0, 1)$ and $R \geq K_1$, where $K_1 = K_1(\theta_1, \theta_2, \kappa, d)$ is a large constant to be determined. Indeed, if $R < K_1$, then by uniform ellipticity, for any $x \in B_R$,

$$P^{\sigma,0}_{\omega}(X_{\Delta(B_{2R}, s)} \in B_{\sigma R}) \geq P^{\sigma,0}_{\omega}(X_s = 0, \Delta(B_{2R}, s) = s) > C(\kappa, d, \theta_1, \theta_2).$$

Further, for $R \geq K_1$, it suffices to consider the case $\sigma R \geq \sqrt{K_1}$. Indeed, assume the lemma is proved for $R \geq K_1$ and $\sigma R \geq \sqrt{K_1}$. Then, when $\sigma R < \sqrt{K_1}$ and $x \in B_R$, by uniform ellipticity,

$$P^{\sigma,0}_{\omega}(X_{\Delta(B_{2R}, s)} \in B_{\sigma R}) \geq P^{\sigma,0}_{\omega}(X_{\Delta(B_{2R}, s-K_1)} \in B_{\sqrt{K_1}}) \min_{y \in B_{\sqrt{K_1}}} P^{y,0}_{\omega}(X_{K_1} = 0, \Delta(B_{2R}, s) = K_1) \geq \left(\frac{\sqrt{K_1}}{2R}\right)^\alpha C(K_1, \kappa, d) \geq C(\frac{\sigma}{2})^\alpha.$$

Hence in what follows we only consider the case $R \geq K_1$ and $\sigma R \geq \sqrt{K_1}$.

For $(x, t) \in \mathbb{R}^d \times \mathbb{R}$, set

$$\psi_0(t) = 1 - \frac{1-\left(\frac{\sigma/2}{2}\right)^2}{s}, \quad \tilde{\psi}_1(x, t) = \psi_0 - \frac{|x|^2}{4R^2}, \quad \psi_1 = \tilde{\psi}_1 \vee 0,$$

and, for some large constant $q \geq 2$ to be chosen,

$$\psi(x, t) := \psi_1^q \psi_{\sigma q} - q, \quad w(x, t) = (\sigma/2)^{2q-4} \psi(x, t).$$
Let $U := \{ \hat{x} \in B_{2R} \times [0, s) : \psi_1(\hat{x}) > 0 \}$. We will show that for $\hat{x} \in U$, 

$$w(\hat{x}) \leq v(\hat{x}) := P_{\hat{x}}^s(x_\tau \in B_{\sigma R}).$$

Recall the parabolic boundary in page 7. We first show that $w$ satisfies

$$
\begin{cases}
    w|_{\partial P U} \leq 1_{x \in B_{\sigma R}, s = t} \\
    \min_{x \in B_R} w(x, 0) \geq \frac{1}{2} (\sigma/2)^{2q-4}, \\
    \mathcal{L}_\omega w \geq 0 \text{ in } U, \text{ for } q \text{ large.}
\end{cases}
$$

The first two properties in (47) are obvious. For the third property, note that

$$\partial_t \psi = R^{-2} \psi_0^{-1} \left( 1 - \frac{(\sigma/2)^2}{s/R^2} (q \psi_0) - 2 \psi_1 \right) \quad \text{in } U.$$

For any unit vector $e \in \mathbb{Z}^d$, let

$$\nabla^2_c u(x, t) := u(x + e, t) + u(x - e, t) - 2u(x, t).$$

When $\hat{x} \in U_1 := \{ (z, s) \in U : (y, s) \in U \text{ for all } y \sim z \}$, then $\nabla^2_c [\psi^2_1(\hat{x})] = \nabla^2_c [\tilde{\psi}^2_1(\hat{x})]$. When $\hat{x} = (x, t) \in U \setminus U_1$, then for some $|e| = 1$, either $(x + e, t)$ or $(x - e, t)$ is not in $U$. Say, $(x + e, t) \notin U$, then $x \cdot e \geq 1$ and $\exists \delta \in (0, 1)$ such that $\tilde{\psi}_1(x + \delta e, t) = 0$. In both cases, there exists $\delta \in (0, 1]$ such that

$$\begin{align*}
\nabla^2_c [\psi_1(x, t)^2] &= \tilde{\psi}^2_1(x + \delta e, t) + \tilde{\psi}^2_1(x - e, t) - 2\tilde{\psi}^2_1(x) \\
&= -\frac{1}{2R^2} \frac{1}{\psi_1^2} + \frac{1}{16R^4} - \frac{\psi_0^{1/2}}{2R^3}.
\end{align*}$$

The set $U$. The solid line is $\partial^P U$. 

Figure 4: The set $U$. The solid line is $\partial^P U$. 

Let $U := \{ \hat{x} \in B_{2R} \times [0, s) : \psi_1(\hat{x}) > 0 \}$.
In particular, \( \min x \). Hence, the optional-stopping theorem yields the third property in (47) is proved.

\[
R^2 \psi_0^{-1} \mathcal{L}_\omega \psi(\hat{x}) = R^2 \left( \sum_{i=1}^d \omega_i (x_i, x + e_i) \nabla^2 \omega_\sigma [\psi_1^2]/\psi_0 + \psi_0^{-1} \partial_t \psi \right)
\geq \frac{c|x|^2}{R^2 \psi_0} - C \xi - \frac{C}{R \psi_0^{1/2}} + \frac{1 - (\sigma/2)^2}{s/R^2} (q \xi - 2)\xi
\geq Cq \xi^2 - c_1 \xi + c_2 - c_3/R^{1/2},
\]
where in the last inequality we used the fact \( 1 \leq x \cdot e \leq |x| \leq 2R \psi_0^{1/2} \). Thus, letting \( \xi := \psi_1'/\psi_0 \in [0, 1] \), we have for \( \hat{x} = (x, t) \in \bar{U}, \)

\[
v(\hat{x}) - w(\hat{x}) \geq E^2_a \left[ v(\hat{X}_T) - w(\hat{X}_T) \right] \geq 0 \quad \text{for} \quad \hat{x} \in \bar{U}.
\]

In particular, \( \min_{x \in B_R} v(x, 0) \geq \max_{x \in B_R} w(x, 0) \geq (\sigma/2)^{2q - 4}/2. \)

**Corollary 28.** Assume that \( \omega \in \Omega_\kappa, \ R/2 > r > 1/2, \ \theta > 0. \ There \ exists \ c = c(d, \kappa, \theta) \in (0, 1) \ such \ that \ for \ any \ y \in \partial B_R \ with \ B_r(y) \cap B_R \neq \emptyset, \)

\[
\min_{x \in B_r(y) \cap B_R} P^{x,0}_\omega \left(X. \ \text{exits} \ B_{2r}(y) \cap B_R \ \text{from} \ \partial B_R \ \text{before time} \ \theta r^2 > c. \right)
\]

**Proof.** By uniform ellipticity, it suffices to prove the lemma for all \( r \geq 10. \)

Let \( z = \frac{y}{|y|} + y \in \mathbb{R}^d \). Note that \( B_r(y) \subset B_{5r/4}(z) \subset B_{3r/2}(z) \subset B_{2r}(y) \) and \( B_r(z) \subset \mathbb{Z}^d \setminus B_R. \) Recall \( \Delta \) in (13). Then, by Lemma 27

\[
\min_{x \in B_{r}(y) \cap B_R} P^{x,0}_\omega \left( X(\Delta(B_{2r}(y) \cap B_R, \theta r^2) \in \partial B_R \right)
\geq \min_{x \in B_{5r/4}(z)} P^{x,0}_\omega \left( X(\Delta(B_{3r/2}(z), \theta r^2) \in B_r(z) \right) \geq c(\theta, d, \kappa).
\]

The corollary is proved.

**Lemma 29.** Assume \( \omega \in \Omega_\kappa, \ \beta \in (0, 1). \ Let \ \tau_{\beta,1} = \tau_{\beta,1}(R) = \inf \{t \geq 0 : X_t \notin B_R \setminus B_{\beta R} \}. \ Then \ if \ y \in B_R \setminus B_{\beta R} \neq \emptyset \ and \ \theta > 0, \ we \ have

\[
P^{y,0}_\omega \left( X_{\tau_{\beta,1}} \in \partial B_{\beta R}, \tau_{\beta,1} \leq \theta R^2 \right) \geq C \text{dist}(y, \partial B_R) \frac{\text{dist}(y, \partial B_R)}{R},
\]

where \( C = C(\kappa, d, \beta, \theta). \)
Proof. It suffices to prove the lemma for $R > \alpha^2$, where $\alpha = \alpha(\kappa, d, \beta, \theta)$ is a large constant to be determined. We only need to consider $y$ with $\text{dist}(y, \partial(B_R \setminus \bar{B}_{\theta R})) \geq 2$ in which case $R - |y| \approx \text{dist}(y, \partial B_R)$.

For $\hat{x} = (x, t)$, let $g(\hat{x}) = \exp(-\frac{\alpha}{\theta R^2} |x|^2 - \frac{\alpha}{\theta R^2})$. Using the inequalities $e^a + e^{-a} \geq 2 + a^2$ and $e^a \geq 1 + a$, we get for $x \in B_R \setminus \bar{B}_{\theta R}, t \in \mathbb{R}$,

$$\mathcal{L}_x g(\hat{x}) = g(\hat{x}) \left( \sum_{i=1}^{d} \omega_i(x, x + e_i)[e^{-\frac{\alpha}{\theta R^2}(1 + 2x_i)} + e^{-\frac{\alpha}{\theta R^2}(1 - 2x_i)} - 2] - \frac{\alpha}{\theta R^2} \right)$$

$$\geq g(\hat{x}) \left( \sum_{i=1}^{d} \omega_i(x, x + e_i)[e^{-\alpha/2}(2 + 4\alpha^2x_i^2/R^4) - 2] - \frac{\alpha}{\theta R^2} \right)$$

$$\geq \frac{\alpha}{R^2} g(\hat{x})(c_0\beta^2 - C) > 0$$

if $a$ is chosen to be large enough. Hence $g(\hat{X}_i)$ is a submartingale for $t \leq \tau_{\beta,1}$.

Recall the definition of the stopping time $\Delta$ in (43). Let

$$v(\hat{x}) := \frac{g(\hat{x}) - e^{-\alpha}}{e^{-\alpha} - e^{-\alpha}} \quad \text{and} \quad u(\hat{x}) := P^\hat{x}_\omega(X_{\Delta(B_R \setminus \bar{B}_{\beta R} \setminus \bar{B}_{\theta R})} \in \partial B_{\theta R}).$$

Set $\mathcal{D} = (B_R \setminus \bar{B}_{\beta R}) \times [0, \theta R^2]$. Since $(u - v)|_{\partial \mathcal{D}} \geq 0$ and $u(\hat{X}_i)$ is a martingale in $\mathcal{D}$, by the optional-stopping theorem we conclude that $u \geq v$ in $\mathcal{D}$. In particular, $u(x, 0) \geq v(x, 0) \geq C(R^2 - |x|^2)/R^2$ for $x \in B_R \setminus \bar{B}_{\beta R}$. $\square$

Lemma 30. Let $\beta \in (0, 1)$, and let $\tau_{\beta,1}$ be as in Lemma 29. For $\theta > 0$, there exists a constant $C = C(\beta, \kappa, d, \theta)$ such that, if $x \in B_R \setminus \bar{B}_{\beta R} \neq \emptyset$,

$$P^x_{\omega}(X_{(\theta R^2)^{\tau_{\beta,1}}} \notin \partial B_R) \leq C \text{dist}(x, \partial B_R)/R.$$
which implies that \( h(\hat{X}_t)^{-k} \) is a submartingale inside the region \( \mathcal{D} \).

Next, set (Recall the stopping time \( \Delta \) in (13).)

\[
u(\hat{x}) = P_{\hat{x}} \omega(\hat{X}_{\Delta(B_R \setminus \bar{B}_R, \theta_R)} / \partial B_R).
\]

Then \( u(\hat{X}_t) + (2 - \beta^2)^k h(\hat{X}_t)^{-k} \) is a submartingale in \( \mathcal{D} \). Since

\[
\begin{align*}
\hbar^{-k} |_{x \in \partial B_R} &\leq (2 - 1 + 0)^{-k} = 1 \\
\hbar^{-k} |_{x \in \partial B_{\beta R}} &\leq (2 - \beta^2)^{-k} \\
\hbar^{-k} |_{t = \theta_R} &\leq (2 - 1 + 1)^{-k} \leq (2 - \beta^2)^{-k}
\end{align*}
\]

by the optional stopping theorem, we have for \( x \in B_R \setminus \bar{B}_{\beta R} \),

\[
u(x, 0) + (2 - \beta^2)^k h(x, 0)^{-k} \leq \sup_{\partial^n \mathcal{D}} [u + (2 - \beta^2)^k h^{-k}] \leq (2 - \beta^2)^k.
\]

Therefore, for any \( x \in B_R \setminus \bar{B}_{\beta R} \),

\[
u(x, 0) \leq (2 - \beta^2)^k (1 - h(x, 0)^{-k}) \\
\leq C(h(x, 0) - 1) = C[1 - |x|^2/(R + 1)^2] \\
\leq C \text{dist}(x, \partial B_R)/R.
\]

Our proof of Lemma 30 is complete. \( \square \)

\section{Appendix}

\subsection{Properties (i)-(iii) in Remark 3}

\textit{Proof.} (i) Since \( \hat{Q} \) is an invariant measure for \( (\hat{\omega}_t) \), we have for any bounded measurable function \( f \) on \( \Omega \), \( y \in \mathbb{Z}^d \), \( \hat{x} = (x, t) \), and \( s < t \),

\[
0 = E_{\hat{Q}} \mathbb{E}_{\hat{\omega}}^{0,0} [f(\theta_{-\hat{x}}(\hat{\omega}_{t-s})) - f(\theta_{-\hat{x}})] \\
= E_{\mathbb{P}} \left[ \rho(\omega) \sum_{y \in \mathbb{Z}^d} p^\omega(0, 0; x - y, t - s) [f(\theta_{-y, -s}\omega) - f(\theta_{-\hat{x}}\omega)] \right] \\
= E_{\mathbb{P}} \left[ f(\omega) \left[ \sum_{y \in \mathbb{Z}^d} \rho(\theta_{\hat{y}\omega}) \rho^\omega(\hat{y}, \hat{x}) - \rho(\theta_{\hat{x}\omega}) \right] \right],
\]

where \( \hat{y} = (y, s) \) and we used the translation-invariance of \( \mathbb{P} \) in the last equality. Moreover, by Fubini’s theorem, for any bounded compactly-supported continuous function \( \phi : \mathbb{R} \to \mathbb{R} \),

\[
E_{\mathbb{P}} \left[ f(\omega) \int_{-\infty}^{t} \phi(s) \left[ \sum_{y \in \mathbb{Z}^d} \rho(\theta_{\hat{y}\omega}) \rho^\omega(\hat{y}, \hat{x}) - \rho(\theta_{\hat{x}\omega}) \right] ds \right] = 0
\]

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Thus we have that \( P \)-almost surely, for any such test function \( \phi \) on \( \mathbb{R} \),

\[
\int_{-\infty}^{t} \phi(s) \left[ \sum_{y \in \mathbb{Z}^d} \rho_\omega(y) p^\omega(y, \hat{x}) - \rho_\omega(\hat{x}) \right] ds = 0,
\]

which (together with the translation-invariance of \( P \)) implies that \( P \)-almost surely, \( \rho_\omega(x, t) \delta_x dt \) is an invariant measure for the process \( (\hat{X}_t)_{t \geq 0} \).

(ii) We have \( \rho_\omega > 0 \) since the measures \( Q \) and \( P \) are equivalent. The uniqueness follows from the uniqueness of \( Q \) in [12, Theorem 1.2].

(iii) By (49) and Fubini’s theorem, we also have that \( P \)-almost surely, for any test function \( \phi(t) \) as in (i) and any \( h > 0, x \in \mathbb{Z}^d \),

\[
\int_{-\infty}^{\infty} \phi(t) \left[ \sum_{y \in \mathbb{Z}^d} \rho_\omega(y, t) (p^\omega(y, t; x, t+h) - \delta_x(y)) - (\rho_\omega(x, t+h) - \rho_\omega(\hat{x})) \right] dt = 0.
\]

Dividing both sides by \( h \) and letting \( h \to 0 \), we obtain (3) with \( \partial_t \rho_\omega \) replaced by the weak derivative. Note that the weak differentiability of \( \rho_\omega \) in \( t \) implies that it has an absolutely continuous (in \( t \)) version. Since \( \rho_\omega \) is only used as a density, we may always assume that \( P \)-almost surely, \( \rho_\omega(x, \cdot) \) is continuous and almost-everywhere differentiable in \( t \).

A.2 Proof of Corollary 24

Proof. Assume \((x_0, t_0) = (0, 0)\) and fix \( R > 0 \). Let \( R_k = 2^{-k} R \) and \( Q^k = B_{R_k} \times (-R_k^2, 0] \). Note that \( Q^{k+1} \subset Q^k \). For any bounded subset \( E \subset \mathbb{Z}^d \times \mathbb{R} \), denote \( \text{osc}_E u := \sup_E u - \inf_E u \). Set

\[
v_k := (u - \inf_{Q^k} u)/\text{osc}_{Q^k} u.
\]

Notice that \( \inf_{Q^k} v_k = 0, \sup_{Q^k} v_k = 1 \) and

\[
\text{osc}_{Q^{k+1}} u = \text{osc}_{Q^{k+1}} v_k \cdot \text{osc}_{Q^k} u.
\]

We claim that \( \text{osc}_{Q^{k+1}} v_k \leq 1 - \delta \) for some \( \delta = \delta(d, \kappa) \in (0, 1) \). Indeed, replacing \( v_k \) by \( 1 - v_k \) if necessary, we can assume \( \sup_{B_{R_{k+1}} \times (-\frac{3}{4} R_k^2, \frac{1}{2} R_k^2)} v_k \geq 1/2 \). By the PHI for \( L_\omega^* \) (Theorem 5),

\[
\inf_{Q^{k+1}} v_k \geq c \sup_{B_{R_{k+1}} \times (-\frac{3}{4} R_k^2, \frac{1}{2} R_k^2)} v_k \geq \frac{\delta}{2} := \delta \in (0, 1)
\]

and so \( \text{osc}_{Q^{k+1}} v_k \leq \sup_{Q^k} v_k - \inf_{Q^{k+1}} v_k \leq 1 - \delta \). The claim is proved. So

\[
\text{osc}_{Q^{k+1}} u \leq (1 - \delta) \text{osc}_{Q^k} u.
\]
If \( r > R/2 \), the Corollary is trivial. If \( r \leq R/2 \), we iterate the above inequality \( k = \lceil \log_2(R/r) \rceil \) times (so that \( Q^{k+1} \subset B_r \times (-r^2,0] \subset Q^k \)) to obtain
\[
\text{osc}_{B_r \times (-r^2,0]} u \leq \text{osc}_{Q^k} u \leq (1 - \delta)^k \text{osc}_{Q^0} u \leq (1 - \delta)^{-1}(r/R)^\gamma \text{osc}_{Q^0} u.
\]
where \( \gamma = -\log_2(1 - \delta) \). Our proof is complete.

\[\square\]

### A.3 Parabolic maximum principle

In what follows we will prove a maximum principle for parabolic difference operators under the discrete space and continuous time setting. For any \( D \subset B_R \times (0,T) \), \( \hat{x} := (x,t) \in D \) and \( u : \mathcal{D} \cup \partial \mathcal{D} \rightarrow \mathbb{R} \), define
\[
I_u(\hat{x}) := \{ p \in \mathbb{R}^d : u(x,t) - u(y,s) \geq p \cdot (x-y) \text{ for all } (y,s) \in D \cup \partial \mathcal{D} \text{ with } s > t \},
\]
\[
\Gamma = \Gamma(u,D) := \{ (x,t) \in D : I_u(x,t) \neq \emptyset \},
\]
\[
\Gamma^+ = \Gamma^+(u,D) = \{ \hat{x} \in \Gamma : R|p| < u(\hat{x}) - p \cdot x \text{ for some } p \in I_u(\hat{x}) \}. \quad (50)
\]

**Theorem A.3.1** (Maximum principle). Let \( \omega \in \Omega_n \). Recall \( f_{\mathcal{D}} \) in (22). Assume that \( \mathcal{D} \subset B_R \times (0,T) \) is an open subset of \( \mathbb{Z}^d \times \mathbb{R} \) for some \( R,T > 0 \). Let \( f \) be a measurable function on \( \mathcal{D} \) that solves \( L_a u \geq -f \) in \( \mathcal{D} \), we have
\[
\sup_{\mathcal{D}} u \leq \sup_{\partial \mathcal{D}} u + C R^{d/(d+1)} (\int_{\Gamma^+} |f|^{(d+1)/d})^{1/(d+1)}
\]

**Proof.** Without loss of generality, assume \( f \geq 0 \), \( \sup_{\partial \mathcal{D}} u = 0 \), and
\[
\sup_{\mathcal{D}} u := M > 0.
\]

Let
\[
\Lambda = \{ (\xi,h) \in \mathbb{R}^{d+1} : R|\xi| < h < M/2 \}.
\]
For \((x,t) \in \mathcal{D}\), define a set
\[
\chi(x,t) = \{ (p,u(x,t) - x \cdot p) : p \in I_u(x,t) \} \subset \mathbb{R}^{d+1}.
\]
First, we claim that
\[
\Lambda \subset \chi(\Gamma^+) := \bigcup_{(x,t) \in \Gamma^+} \chi(x,t). \quad (51)
\]
This will be proved by showing that for any \((\xi,h) \in \Lambda\), we have \((\xi,h) \in \chi(x_1,t_1)\) for some \((x_1,t_1) \in \Gamma^+\). Indeed, fix \((\xi,h) \in \Lambda\) and define
\[
\phi(x,t) := u(x,t) - \xi \cdot x - h.
\]
Since $\sup_{\mathcal{D}} \phi \geq M - |\xi| R - h > 0$, there exists $(x_0, t_0) \in \mathcal{D}$ with $\phi(x_0, t_0) > 0$. Now for any $x \in \mathbb{Z}^d$, set (with the convention $\sup \emptyset = -\infty$)

$$N_x = \sup \{ t : (x, t) \in \mathcal{D} \text{ and } \phi(x, t) \geq 0 \},$$

and let $(x_1, t_1)$ be such that

$$t_1 = N_{x_1} = \max_{x \in B_R} N_x \geq N_{x_0} \geq t_0.$$

By the continuity of $\phi$, we get $\phi(x_1, t_1) \geq 0$ and $(x_1, t_1) \in \mathcal{D} \cup \partial^P \mathcal{D}$. Since $\phi|_{\partial^P \mathcal{D}} < 0$, we have $(x_1, t_1) \in \mathcal{D}$. Moreover, since $\mathcal{D}$ is an open set, we can conclude that $\phi(x, t) = 0$ and $\phi(x_1, s) < 0$ for all $s > t_1$ with $(x_1, s) \in \mathcal{D} \cup \partial^P \mathcal{D}$. Hence $\xi \in I_u(x_1, t_1)$ and $u(x_1, t_1) - \xi \cdot x_1 = h > R|\xi|$, which implies that $(\xi, h) \in \chi(x_1, t_1)$ and $(x_1, t_1) \in \Gamma^+$. Display (51) is proved.

Next, setting

$$\chi(\Gamma^+, x) := \bigcup_{s(x, s) \in \Gamma^+} \chi(x, s),$$

we will show that

$$\text{Vol}_{d+1}(\chi(\Gamma^+, x)) \leq C \int_0^T (f(x, t)/\varepsilon)^{d+1} [u(x, t) \in \Gamma^+] dt, \tag{52}$$

where $\text{Vol}_{d+1}$ is the volume in $\mathbb{R}^{d+1}$. To this end, let $\tilde{\chi}(x, t) = I_u(x, t) \times \{u(x, t)\} \subset \mathbb{R}^{d+1}$. Noting that, for fixed $x$, the map $(y, s) \mapsto (y, s + y \cdot x)$ is volume preserving, we then have

$$\text{Vol}_{d+1}(\chi(\Gamma^+, x)) = \text{Vol}_{d+1}(\tilde{\chi}(\Gamma^+, x)) = \int_0^T (-\partial_t u) \text{Vol}_d(I_u(x, t))^1(x, t) \in \Gamma^+ dt. \tag{53}$$

For any fixed $p \in I(x, t)$, $(x, t) \in \Gamma^+$, set

$$w(y, s) = u(y, s) - p \cdot y.$$

Then $I_w(x, t) = I_u(x, t) + p$. Since $w(x, t) - w(x \pm e_i, t) \geq \mp q_i$ for any $q \in I_w(x, t)$, $i = 1, \ldots, d$, we have

$$\text{Vol}_d(I_u(x, t)) = \text{Vol}_d(I_w(x, t)) \leq \prod_{i=1}^d [2u(x, t) - u(x + e_i, t) - u(x - e_i, t)].$$

This inequality, together with (53), yields

$$\text{Vol}_{d+1}(\chi(\Gamma^+, x))$$

$$\leq -C \int_0^T \partial_t u \prod_{i=1}^d a_t(x, x + e_i)[2u(x, t) - u(x + e_i, t) - u(x - e_i, t)]1(x, t) \in \Gamma^+ dt$$

$$\leq C \int_0^T [-\mathcal{L} u(x, t)]^{d+1} 1(x, t) \in \Gamma^+ dt.$$

Display (52) is proved. Finally, by (51), (52) and $\text{Vol}_{d+1}(\Lambda) = CM_{d+1}/R^d$, we conclude that $M_{d+1}R^d \leq C \int_{\mathbb{R}^3} f^{d+1}$. The theorem follows immediately. □
A.4 Mean value inequality

Theorem A.4.1. Let $\theta > 0$ and $a \in \Omega_\kappa$. Recall $\| \cdot \|_{p, \rho}$ in (23). For any $\theta_1 \in (0, \theta)$, $\rho \in (0, 1)$ and $p > 0$, there exists $C = C(\kappa, d, p, \theta, \theta_1, \rho)$ such that for any function $u$ that solves $L_a u \geq 0$ in $\mathcal{D} = B_R \times [0, \theta R^2)$, we have

$$\sup_{B_{\rho R} \times [0, \theta_1 R^2)} u \leq C\|u^+\|_{p, \rho}.$$ 

Proof. Since $\|u^+\|_{p, \rho}$ is increasing in $p > 0$, it suffices to consider $p \in (0, 1)$. Let $\beta \geq 2$ be a constant to be determined, and set

$$\eta(x) := (1 - \frac{|x|^2}{R^2})^\beta 1_{x \in B_R}, \quad \zeta(t) := (1 - \frac{t}{\theta R^2})^\beta 1_{0 \leq t < \theta R^2},$$

and set $v = \eta u$, $\bar{v} = v \zeta$. Define an elliptic operator $L^E_a$ to be

$$L^E_a f(x, t) = \sum_{y:y \sim x} a_t(x, y) (f(y, t) - f(x, t)),$$

so that $L_a = L^E_a + \partial_t$. Note that $\bar{v}|_{\mathcal{D} \setminus \mathcal{G}} = 0$ and $\bar{v}(\hat{x}) > 0$ for $\hat{x} \in \Gamma^+(\bar{v}, \mathcal{D})$. (Recall the definition of $\Gamma^+$ above Theorem A.3.1.) By the same argument as in [18, displays (27), (28) and (29)], we have that on $\Gamma^+(\bar{v}, \mathcal{D})$, $u^+ = u$ and

$$L^E_a \bar{v} \geq \eta L^E_a u - C\beta \eta^{1-2/\beta} R^{-2} u^+.$$

Hence, for $X = (x, t) \in \Gamma^+(\bar{v}, \mathcal{D})$,

$$\mathcal{L}_a \bar{v} = \zeta L^E_a \bar{v} + \partial_t (\zeta \eta u)
\geq \zeta \eta L^E_a u - C\beta \eta^{1-2/\beta} R^{-2} u^+ + \eta u \partial_t \zeta + \zeta \eta \partial_t u
= \zeta \eta L_a u - C\beta \eta^{1-2/\beta} R^{-2} u^+ + \eta u \partial_t \zeta
\geq -C\beta \eta^{1-2/\beta} R^{-2} u^+ + \eta u \partial_t \zeta.$$

Noting that in $\mathcal{D}$, $\partial_t \zeta \geq -C\beta R^{-2} \zeta^{1-1/\beta} / \theta$ and $\zeta, \eta \in [0, 1]$, we have

$$\mathcal{L}_a \bar{v} \geq -C(\eta \zeta)^{1-2/\beta} R^{-2} u^+ \quad \text{in } \Gamma^+(\bar{v}, \mathcal{D}).$$

Applying Theorem A.3.1 to $\bar{v}$ and taking $\beta = 2(d + 1)/p$,

$$\sup_{\mathcal{D}} \bar{v} \leq C \| (\eta \zeta)^{1-2/\beta} u^+ / \epsilon \|_{\mathcal{D}, d+1}
\leq C(\sup_{\mathcal{D}} \bar{v})^{1-p/(d+1)} \| (u^+)^{p/(d+1)} / \epsilon \|_{\mathcal{D}, d+1}.$$

Since $\sup_{B_{\rho R} \times [0, \theta_1 R^2)} u \leq C \sup_{\mathcal{D}} \bar{v}$, the theorem follows. \qed

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A.5 Reverse Hölder implies $A_p$

Recall $|\mathcal{D}|, \int_{\mathcal{D}} \cdot \,\, \|\cdot\|_{2,p}$, and the parabolic cubes in $\mathcal{D}$, $\mathcal{P}$ and $\mathcal{P}^\infty$.

**Lemma A.5.1.** Let $K^0 \subset \mathbb{Z} \times \mathbb{R}$ be a parabolic cube with side-length $r > 0$. If a function $w > 0$ on $K^0$ satisfies $RH_q(K^0)$, $q > 1$, then

(i) $w \in A_p(K^0)$ for some $1 < p < \infty$;

(ii) $\frac{w(E)}{w(K)} \geq C(|E|/|K|)^{-c}$ for all $E \subset K$ where $K \neq \emptyset$ is a subcube of $K^0$.

**Proof.** First, we claim that there exist constants $\gamma, \delta \in (0,1)$ such that $w(E) > \gamma w(K)$ implies $|E| > \delta |K|$ for all $E \subset K$ where $K \neq \emptyset$ is a subcube of $K^0$. Indeed, this is a simple consequence of Hölder’s inequality:

$$\frac{1}{|K|} \int_K w_1 \leq \left( \frac{|E|}{|K|} \right)^{1/q'} \|w\|_{K,q} \leq C \left( \frac{|E|}{|K|} \right)^{1/q'} \frac{w(K)}{|K|},$$

where $q' = q/(q-1)$ denotes the conjugate of $q$.

Assume $K^0 = K_r$. Let $M_k(K_r), k > 1$ be the family of nonempty subcubes of $K_r$ of the form

$$\left( \prod_{i=1}^d \left( \frac{m_i}{2^k} r, \frac{1+m_i}{2^k} r \right) \cap \mathbb{Z}^d \right) \times \left( \frac{n_i}{4^k}, \frac{1+n_i}{4^k}, \frac{r}{4^k} \right)^2$$

where $m_i, n_i$ are integers. Elements in $M_k(K_r)$ are called $k$-level dyadic subcubes of $K_r$. Note that every $k$-level cube $K$ is contained in a unique $(k-1)$-level “parent” denoted by $K^{-1}$. Since the class $A_p$ is invariant under constant multiplication, we may assume that $w(K^0)/|K^0| = 1$.

Let $f := w^{-1}_K 1_{K^0}$ and define a maximal function

$$M_f(x) := \sup_{K \ni x} \frac{1}{w(K)} \sum_K |f|w,$$

where the supremum is taken over all dyadic subcubes $K$ of $K_0$. Consider the level sets

$$E_k = \{ x \in K^0 : M_f(x) > 2^{Nk} \}, \quad k = 0, 1, 2, \ldots$$

where $N$ is a big constant to be determined. Notice that by assumption, $E_0$ is comprised of dyadic subcubes strictly smaller than $K_0$. Since $w$ is volume-doubling, there exists a constant $c_0 > 0$ such that for any maximal dyadic subcube $K$ of $E_{k-1}$,

$$\int_K fw \leq \int_{K^{-1}} fw \leq 2^{N(k-1)} w(K^{-1}) \leq 2^{N(k-1)+c_0} w(K).$$

Moreover, for the same $K$, we have $2^{Nk} w(E_k \cap K) \leq \int_Kfw$ and so, by the inequality above, $w(E_k \cap K) \leq 2^{a_k-N} w(K)$. We now take $N$ to be large.

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enough that \( w(E_k \cap K) \leq (1 - \gamma)w(K) \) which implies \( |E_k \cap K| \leq (1 - \delta)|K| \). Summing over all such \( K \)'s, we have \( |E_k| \leq (1 - \delta)|E_{k-1}|, \ k \geq 1 \). Thus
\[
|E_k| \leq \delta^k|E_0| \leq \delta^k|K^0|, \ k = 0, \ldots
\]
and so, for \( p > 1 \) chosen so that \( p' = p/(p - 1) \) is sufficiently close to 1,
\[
\int_{K^0} f^{p'-1} = \int_{K^0 \cap \{x : M_f \leq 1\}} f^{p'-1} + \sum_{k=0}^{\infty} \int_{E_k \setminus E_{k+1}} f^{p'-1} \\
\leq |K^0| + \sum_{k=0}^{\infty} 2^{(p'-1)N(k+1)}\delta^k|K^0| \\
\leq C|K^0|.
\]
(i) is proved. (ii) then follows from Hölder’s inequality
\[
\frac{1}{w(K)} \int_E w^{-1}w \leq \left( \frac{1}{w(K)} \int_K w^{-p'}w \right)^{1/p'} \left( \frac{1}{w(K)} \int_K \mathbb{1}_{E}w \right)^{1/p}
\]
and the \( A_p \) inequality.

A.6 Proof of Corollary 8

Proof. (ii) For any \( \hat{x} = (x, t) \in \mathbb{R}^d \times [0, \infty) \) and \( \omega \in \Omega_\kappa \), set
\[
v(\hat{x}) = q^\omega(\hat{0}; [x], t) \quad \text{and} \quad a^\omega(x) := \int_0^\infty (v(0, t) - v(x, t))dt.
\]
When \( d = 2 \), it suffices to consider \( x \in \mathbb{B}_1 \setminus \{0\} \). We fix a small number \( \epsilon \in (0, 1) \) and split the integral \( a^\omega(nx) \) into four parts:
\[
a^\omega(nx) = \int_0^{n^\epsilon} + \int_{n^\epsilon}^{n^2} + \int_{n^2}^{\infty} =: I + II + III,
\]
where it is understood that the integrand is \( (v(0, t) - v(x, t)) \). dt.

First, we will show that \( \mathbb{P} \)-almost surely,
\[
\lim_{n \to \infty} \frac{|||}{\log n} \leq \epsilon. \tag{54}
\]
By Theorem 7, for any \( t \in (0, n^\epsilon) \), \( x \in \mathbb{Z}^2 \setminus \{0\} \) and all \( n \) large enough, \( v(nx, t) \leq C e^{-cn|x|}/\rho_\omega(B_{\sqrt{r}}, 0) \). Thus
\[
\int_0^{n^\epsilon} v(nx, t)dt \leq \frac{n^\epsilon}{\rho_\omega(0)} e^{-cn|x|}.
\]
By (ii), there exists \( t_0(\omega) > 0 \) such that for \( n \) big enough with \( n^\epsilon > t_0 \),
\[
\int_0^{n^\epsilon} v(0, t)dt \leq \frac{Ct_0}{\rho_\omega(0)} + \int_{t_0}^{n^\epsilon} C t dt \leq \frac{Ct_0}{\rho_\omega(0)} + C\epsilon \log n,
\]
and so, for \( p > 1 \) chosen so that \( p' = p/(p - 1) \) is sufficiently close to 1,
In the second step, we will show that (note that $2p^\Sigma_1(0,0) = 1/\pi \sqrt{\det \Sigma}$)

$$
\limsup_{n \to \infty} |II - 2p^\Sigma_1(0,0) \log n|/ \log n \leq C \epsilon, \quad \mathbb{P}\text{-a.s.} \quad (55)
$$

Indeed, by Theorem 4, there exists $C(\omega, \epsilon) > 0$ such that 

$$
|v(0, t) - p^\Sigma_1(0,0)| \leq \epsilon \quad \text{whenever } t \geq C(\omega, \epsilon).
$$

Now, taking $n$ large enough such that $n\epsilon > C(\omega, \epsilon)$,

$$
\left| \int_{n^2}^{n^2} v(0, t)dt - (2 - \epsilon)p^\Sigma_1(0,0) \log n \right| 
\leq \epsilon \int_{n^2}^{n^2} \frac{|v(0, t) - p^\Sigma_1(0,0)|}{t} dt 
\leq \epsilon \int_{n^2}^{n^2} \frac{dt}{t} < 2\epsilon \log n. \quad (56)
$$

On the other hand, for $t \geq n' > t_0(\omega)$, by (i), $v(nx, t) \leq \frac{C}{t}(e^{-c\nu|x|} + e^{-c\nu^2|x|^2/t})$. Thus

$$
\int_{n^2}^{n^2} v(nx, t)dt \leq \int_{n^2}^{n^2} \frac{C}{t}e^{-c\nu^2|x|^2} dt + \int_{n^2}^{n^2} \frac{C}{t} dt \leq C \epsilon \log n. \quad (57)
$$

Displays $(56)$ and $(57)$ imply $(55)$.

Finally, we will prove that for $\mathbb{P}$-almost every $\omega$,

$$
\limsup_{n \to \infty} |III|/ \log n = 0. \quad (58)
$$

Since $|x| < 1$, by (ii), for any $t \geq n^2 \geq t_0(\omega)$,

$$
|v(0, t) - v(nx, t)| \leq C \left( \frac{n}{\sqrt{t}} \right)^{\gamma} t^{-1}.
$$

Therefore, $\mathbb{P}$-almost surely, when $n^2 > t_0(\omega)$,

$$
\left| \int_{n^2}^{\infty} v(0, t) - v(nx, t)dt \right| \leq C n^\gamma \int_{n^2}^{\infty} \frac{1}{t^{\gamma/2 + 1}} dt \leq C.
$$

Display $(58)$ follows. Combining $(54)$, $(55)$ and $(58)$, we have for $d = 2$,

$$
\lim_{n \to \infty} \left| \frac{a^\omega(nx)}{\log n} - 2p^\Sigma_1(0,0) \right| \leq C \epsilon,
$$

Noting that $\epsilon > 0$ is arbitrary, we obtain Corollary 8(ii).

(iii) We fix a small constant $\epsilon \in (0, 1)$. Note that

$$
n^{d-2} \int_0^{\infty} q^\omega(0; |nx|, t)dt = \int_0^{\infty} n^d v(nx, n^2 s)ds.$$

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For any fixed $x \in \mathbb{R}^d$, write
\[
\int_0^\infty n^d v(nx, n^2 s)ds = \int_0^{n^{-\epsilon}} + \int_{n^{-\epsilon}}^\epsilon + \int_{\epsilon}^{1/\sqrt{\epsilon}} + \int_{1/\sqrt{\epsilon}}^\infty =: I + II + III + IV.
\]

First, by Theorem 7, for $s \in (0, n^{-\epsilon})$, we have $v(nx, n^2 s) \leq C e^{-cn^\epsilon |x|^2 / \rho_\omega(\hat{0})}$, hence
\[
\lim_{n \to \infty} I \leq C \lim_{n \to \infty} n^d e^{-cn^\epsilon |x|^2 / \rho_\omega(\hat{0})} = 0.
\]

Second, by (i), when $n$ is large enough, then for all $t \geq n^2 - \epsilon$, we have $v(nx, t) \leq C t^{-d/2} e^{-cn^2 |x|^2 / t}$. Hence
\[
\lim_{n \to \infty} II \leq \lim_{n \to \infty} C n^d \int_{n^{-\epsilon}}^\epsilon (n^2 s)^{-d/2} e^{-c|x|^2 / s} ds \leq C \epsilon.
\]

Moreover, by Theorem 4, there exists $N(\omega, \epsilon)$ such that for $n \geq N(\omega, \epsilon)$, we have $\sup_{|s| \geq \epsilon} |v(nx, n^2 s) - p_\Sigma^\omega(0, x)| \leq \epsilon$. Hence
\[
\left| \lim_{n \to \infty} III - \int_{\epsilon}^{1/\sqrt{\epsilon}} p_\Sigma^\omega(0, x) ds \right| \leq \sqrt{\epsilon}.
\]

Further, by (i), for $d \geq 3$,
\[
\lim_{n \to \infty} IV \leq C \int_{1/\sqrt{\epsilon}}^\infty \frac{n^d}{(n^2 s)^{d/2}} ds = C \epsilon^{(d-2)/4}.
\]

Finally, combining (59), (60), (61) and (62), we get
\[
\left| \lim_{n \to \infty} \int_0^\infty n^d v(nx, n^2 s)ds - \int_{\epsilon}^{1/\sqrt{\epsilon}} p_\Sigma^\omega(0, x)ds \right| \leq C \epsilon^{1/4}.
\]

Letting $\epsilon \to 0$, (iii) is proved. $\square$

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