Mean-field theory based on the Jacobi \( \mathfrak{hsp} := \text{semi-direct sum} \ \mathfrak{h}_N \ltimes \mathfrak{sp}(2N, \mathbb{R}) \) algebra of boson operators

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Dedicated to the Memory of Hideo Fukutome

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Abstract

In this paper, we give an expression for canonical transformation group with Grassmann variables, basing on the Jacobi \( \mathfrak{hsp} \) := semi-direct sum \( \mathfrak{h}_N \ltimes \mathfrak{sp}(2N, \mathbb{R}) \) algebra of boson operators. We assume a mean-field Hamiltonian (MFH) linear in the Jacobi generators. We diagonalize the boson MFH. We show a new aspect of eigenvalues of the MFH. An excitation energy arisen from additional SCF parameters has never been seen in the traditional boson MFT. We derive this excitation energy. We extend the Killing potential in the \( \text{Sp}(2N) U(N) \) coset space to the one in the \( \text{Sp}(2N+2) U(N+1) \) coset space and make clear the geometrical structure of Kähler manifold, a non-compact symmetric space \( \text{Sp}(2N+2) U(N+1) \). The Jacobi \( \mathfrak{hsp} \) transformation group is embedded into an \( \text{Sp}(2N+2) \) group and an \( \text{Sp}(2N+2) U(N+1) \) coset variable is introduced. Under such mathematical manipulations, extended bosonization of \( \text{Sp}(2N+2) \) Lie operators, vacuum function and differential forms for extended boson are presented by using integral representation of boson state on the \( \text{Sp}(2N+2) U(N+1) \) coset variables.

Keywords:
Jacobi algebra of boson operators;
Grassmann variable;
Mean-field Hamiltonian linear in Jacobi generators;
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1 Introduction

In nuclear and condensed matter physics, the time dependent Hartree-Bogoliubov (TDHB) theory \cite{1,2} has been regarded as a standard tool in many-body theoretical descriptions of superconducting fermion systems \cite{3,4}. An HB wavefunction (WF) for such systems represents bose condensate states of fermion pairs. Standing on the Lie-algebraic viewpoint, the pair operators of fermion with \( N \) single-particle states form the \( SO(2N) \) Lie algebra which contains the \( U(N) \) Lie algebra as a sub algebra. The \( SO(2N) \) and \( U(N) \) denote a special orthogonal group of \( 2N \)-dimension, say \( g \) and a unitary group of \( N \)-dimension. We can give an integral representation of a state vector on the group \( g \), exact coherent state representation (CS rep) of a fermion system \cite{5,6}. It makes possible global approach to such problems, e.g. see Ozaki \cite{7,8}. The canonical transformation of the fermion operators generated by the Lie operators in the \( SO(2N) \) Lie algebra arises the famous generalized Bogoliubov transformation for fermions.

For providing a general microscopic means for a unified self-consistent field (SCF) description for Bose and Fermi type collective excitations in fermion systems, Fukutome, Yamamura and one of the present authors (S. N.) proposed a new fermion many-body theory basing on the \( SO(2N+1) \) Lie algebra of fermion operators \cite{9,10}. The fermion creation-annihilation and pair operators form the Lie algebra of \( SO(2N+1) \) group. A representation of an \( SO(2N+1) \) group is derived by group theoretical extension of the \( SO(2N) \) fermion Bogoliubov transformation to a new canonical transformation group. The fermion Lie operators, when operated onto the integral representation of the \( SO(2N+1) \) WF, are mapped into the regular representation of the \( SO(2N+1) \) group and are given by the Schwinger-type boson rep \cite{11,12}. The boson images of all the Lie operators are also expressed by closed first order differential forms.

Along the above way, using Grassmann variables, we give a new mean-field theory (MFT), based on the \textbf{Jacobi hsp} := semi-direct sum \( h_N \ltimes sp(2N,\mathbb{R}) \) algebra (\textbf{Jacobi hsp} algebra) of boson operators, which is studied intensively by Berceanu \cite{13,14}. We take an Hamiltonian consisting of the generalized HB (GHB) MF Hamiltonian (MFH) and assume a linear MFH expressed in terms of the generators of the \textbf{Jacobi hsp} algebra. We diagonalize the boson MFH. A new aspect of eigenvalues of the MFH is shown. An excitation energy arises from additional SCF parameters has never been seen in the traditional boson MFT. We derive this excitation energy. We further extend the Killing potential in the \( Sp(2N) \) \( \ltimes \) \text{coset space to the Killing potential in the } \( Sp(2N+2) \) \( \ltimes \) \text{coset space. The extended Killing potential is equivalent with the generalized density matrix (GDM). Embedding the } \textbf{Jacobi hsp} \text{ group into an } Sp(2N+2) \text{ group and using } \frac{Sp(2N+2)}{U(N+1)} \text{ coset variables, we give an extended boson rep on a non-compact symmetric space } \frac{Sp(2N+2)}{U(N+1)}. \)

In §2, we give an expression for canonical transformation group with Grassmann variables, based on the \textbf{Jacobi hsp}. In §3, we give an inverse of the transformation. In §4, we give a boson MFH and its diagonalization and show a new aspect of eigenvalues of the MFH. In §5, we show that the additional SCF parameters are restricted by a new SCF condition. We study the MFH linear in the \textbf{Jacobi} generator. In §6, we investigate a coset space \( \frac{Sp(2N+2)}{U(N+1)} \) and make clear geometrical structure of Kähler manifold, a non-compact symmetric space \( \frac{Sp(2N+2)}{U(N+1)} \). In §7, a Killing potential in the coset space \( \frac{Sp(2N+2)}{U(N+1)} \) is given and its equivalence with the GDM is proved. Finally, in §8 we give discussions and summary. In Appendices, embedding of the \textbf{Jacobi hsp} group into an \( Sp(2N+2) \) group is made and introduction of \( \frac{Sp(2N+2)}{U(N+1)} \) coset variables is made. We give an extended bosonization procedure of the \( Sp(2N+2) \) Lie operators, vacuum function and differential forms for extended boson expressed in terms of the \( \frac{Sp(2N+2)}{U(N+1)} \) coset variables.
2 \, \textit{Jacobi }\mathfrak{hsp} \text{ algebra of boson operators and extension of boson Bogoliubov transformation}

We consider a boson system with \(N\) single-particle states. Let \(a_i\) and \(a_i^\dagger (i=1,\cdots,N)\) be the annihilation-creation operators satisfying the canonical commutation relation for the boson

\[
[a_i, a_j^\dagger] = \mathbb{I}\delta_{ij}, \quad [a_i, a_j] = 0, \quad [a_i^\dagger, a_j^\dagger] = 0, \quad \mathfrak{h} = \{a_i, a_i^\dagger, \mathbb{I}\},
\]

The \(\mathfrak{h}\) spans the Heisenberg algebra. The Roman indices \(i, j, \cdots\) denote the given \(N\) single-particle states. Following Gilmore [15], Zhang and Feng [16], the two-photon algebra is spanned by the following operators:

\[
\begin{align*}
a_i^\dagger a_j^\dagger &\equiv E_{ij}, \quad a_i^\dagger a_j + \frac{1}{2}\mathbb{I}\delta_{ij} \equiv E_{i,j}, \quad a_i a_j \equiv E_{ij}, \quad a_i^\dagger, a_j, \quad \mathbb{I}, \\
E_{i,j} &= E_{i,j}^\dagger, \quad E_{ij} = E_{ji}, \quad E_{ij} = E_{ij}^\dagger = E_{ji}.
\end{align*}
\]

They pointed out that these operators can be regarded as a graded algebra with grading \(d\),

\[
\begin{align*}
d = 2 &= (2, 2, 2) : \quad a_i^\dagger a_j^\dagger (E_{ei+ej}), \quad a_i^\dagger a_j + \frac{1}{2}\mathbb{I}\delta_{ij} (E_{ei-ej}, H_i if i = j), \quad a_i a_j (E_{ei-ej}), \\
d = 1 &= (1, 1) : \quad a_i^\dagger (E_{ei-eN+1}), \quad a_i (E_{ei-eN+1}), \\
d = 0 &= (0) : \quad \mathbb{I} (E_{-eN+1}),
\end{align*}
\]

whose algebra has dimension \((2N+1)\times(2N+1)\) and whose generators can be identified with a subset of generators of the \(\textit{Jacobi }\mathfrak{hsp}\) group. Due to the commutation relation (2.1), the commutation relations for the boson operators (2.3) in the \(\textit{Jacobi }\mathfrak{hsp}\) algebra are

\[
[E_{ij}, E_{kl}] = \delta_{kj} E_{il} - \delta_{il} E_{kj}, \quad (U(N) \text{ algebra})
\]

\[
[E_{ij}, E_{kl}] = -\delta_{il} E_{kj} - \delta_{jk} E_{il} - \delta_{ik} E_{jl} - \delta_{jl} E_{ik}, \quad (2.4)
\]

\[
[E_{ij}, E_{kl}] = -\delta_{il} E_{kj} - \delta_{jk} E_{il} - \delta_{ik} E_{jl} - \delta_{jl} E_{ik}, \quad (2.5)
\]

\[
[a_i, E_{kl}] = \delta_{ij} a_k, \quad [E_{kj}^\dagger, a_i^\dagger] = \delta_{ij} a_k^\dagger,
\]

\[
[a_i, E_{jk}] = 0, \quad [E_{kj}^\dagger, a_i^\dagger] = 0,
\]

\[
[a_i, E_{jk}] = \delta_{ij} a_k^\dagger + \delta_{ik} a_j^\dagger, \quad [E_{kj}^\dagger, a_i^\dagger] = \delta_{ij} a_k + \delta_{ik} a_j.
\]
This is proved by using the relations $a_i f(n_i) = f(n_i + 1) a_i$ and $a_i^\dagger f(n_i) = f(n_i - 1) a_i^\dagger$, $n_i = a_i a_i^\dagger$ (not summed for $i$) for any function $f(n_i)$ [23]. The operator $(-1)^n$ makes a crucial role to generate the generalized boson Bogoliubov transformation quite parallel to the generalized fermion Bogoliubov transformation, $SO(2N + 1)$ canonical transformation by Fukutome [24].

In order to realize a matrix representation, first we introduce a Berezin-type operator

$$\Theta \equiv a_i^\dagger \theta^B_i - a_i \theta^B_i [25],$$

a free-boson vacuum $|\rangle$ as $a_i |\rangle = 0$ and a transformation $U(G)$ defined by

$$U(G) \equiv e^{\Theta G} = U(G_X)U(F_G)U(F_\omega), \quad \Gamma \equiv a_i^\dagger \gamma_i a_j \left( \gamma^B_i = -\gamma^B_j, \right) \Lambda \equiv \frac{1}{2} \left( a_i^\dagger \lambda_i^B a_j^\dagger a_j \lambda_j^B a_i - a_i \lambda_i^B a_j^\dagger \lambda_j^B a_j a_i \right) \Lambda^{BT} = \Lambda^B.$$

The $U(G)$ acts on a boson state vector $|\Psi> \equiv \Gamma \exp(\gamma^B)$ to a function $\Psi(G) \equiv \int U(G) |\rangle <\Psi| U(G) dG$, shown in Appendix. The symbols $\dagger$, $\tau$ and overline denote hermitian-, transpose- and complex-conjugation of matrices and vectors.

Using the operator identity $e^{X} A e^{-X} = A + [X, A] + \frac{1}{2!} [X, [X, A]] + \cdots$, we obtain

$$e^\Gamma a_i e^{-\Gamma} = a_j \eta^B_{ji}, \quad e^\Gamma a_i^\dagger e^{-\Gamma} = a_j^\dagger u^B_{ji}, \quad u^B \equiv \exp(\gamma^B), \quad u^B \equiv u^B u^B \equiv 1_N,$$

where $u^B$ is a $U(N)$ matrix. The $U(N)$ matrix $u^B$ is identified with $u^B$ induced by $U(F_G)$. The transformation (2.9) is the well known Thouless transformation [26] for boson. We also obtain

$$e^\Lambda a_i e^{-\Lambda} = a_j [Ch(\lambda^B)_{ji}] - a_i^\dagger \left[ Sh(\lambda^B)_{ji} \right], \quad e^\Lambda a_i^\dagger e^{-\Lambda} = a_j^\dagger [Ch(\lambda^B)_{ji}] - a_j \left[ Sh(\lambda^B)_{ji} \right],$$

where the $Ch(\lambda^B)$ and $Sh(\lambda^B)$ are $N \times N$ matrices defined in terms of a symmetric matrix $\lambda^B$ as

$$\begin{align*}
Ch(\lambda^B) &\equiv \sum_{n=0}^{\infty} \frac{1}{(2n)!} \lambda^n \lambda^B \lambda^n, & Ch(\lambda^B) &\equiv Ch(\lambda^B), \\
Sh(\lambda^B) &\equiv \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \lambda^n \lambda^B \lambda^n, & Sh(\lambda^B) &\equiv Sh(\lambda^B).
\end{align*}$$

The matrices $Ch(\lambda^B)$ and $Sh(\lambda^B)$ have the properties analogous to the hyperbolic functions

$$\begin{align*}
Ch^T(\lambda^B) Ch(\lambda^B) - Sh^T(\lambda^B) Sh(\lambda^B) &= 1_N, \quad Ch(\lambda^B) Ch^T(\lambda^B) - Sh(\lambda^B) Sh^T(\lambda^B) = 1_N, \\
Sh^T(\lambda^B) Ch(\lambda^B) - Ch^T(\lambda^B) Sh(\lambda^B) &= 0, \quad Ch(\lambda^B) Sh^T(\lambda^B) - Sh(\lambda^B) Ch^T(\lambda^B) = 0.
\end{align*}$$

Let $[(b_j), (b_j^\dagger)]$ be the $N$-dimensional row vector. We define the $N \times N$ matrices $a=(a_{ij})$ and $b=(b_{ij})$ and the $2N \times 2N$ matrix $F$ identified with the $F$ in $U(F) = U(F_G) U(F_\omega)$ (2.8), $(F = F_G F_\omega)$, by

$$F = \begin{bmatrix} Ch(\lambda^B) & -Sh(\lambda^B) \\ -Sh^T(\lambda^B) & Ch^T(\lambda^B) \end{bmatrix} \begin{bmatrix} \pi & 0 \\ 0 & u \end{bmatrix} = \begin{bmatrix} a^B & b^B \\ b^T & c^B \end{bmatrix} = a^B a^B - b^B b^B = 1_N, \quad a^B a^B - c^B c^B = 1_N,$$

$$a^B a^B - b^B b^B = 1_N, \quad a^B a^B - c^B c^B = 1_N.$$

The $U(F)$ induces a linear transformation of the complex bases $a_i$ and $a_i^\dagger$, i.e., the generalized boson Bogoliubov transformation given by an $Sp(2N)$ matrix $F$ which is not unitary but satisfies

$$F^\dagger \tilde{I}_{2N} F = F \tilde{I}_{2N} F^\dagger = \tilde{I}_{2N}, \quad \tilde{I}_{2N} = \begin{bmatrix} 1_N & 0 \\ 0 & -1_N \end{bmatrix}, \quad \det F = 1,$$

and can be transformed to a real $Sp(2N, \mathbb{R})$ matrix by the transformation $S = VFV^{-1}, V \equiv \begin{bmatrix} \sqrt{2} \nu_N & \sqrt{2} \nu_1 \\ \nu_2 \nu_1 & \nu_2 \nu_1 \end{bmatrix}$. The $\Theta$ in (2.8) has $\theta^B_i$ and $\theta^B_i$ (anti-commuting Grassmann variables) [27] [28]. This $\Theta$ is essentially different from a $\Theta'$ suggested by Ozaki who uses no such variables [22]. Instead, we should require that the variables $\theta^B_i$ and $\theta^B_i$ even more anti-commute with $a_i$ and $a_i^\dagger$.\]
\[ \{ \theta^B_i, \theta^B_j \} = 0, \quad \{ \theta^B_i, \overline{\theta}^B_j \} = 0; \quad \{ \theta^B_i, a_j \} = 0, \quad \{ \theta^B_i, a_j^\dagger \} = 0; \quad \{ \overline{\theta}^B_i, a_j \} = 0, \quad \{ \overline{\theta}^B_i, a_j^\dagger \} = 0. \quad (2.15) \]

A possible quantization of fermion TDHB theory in terms of Grassmann variables based on the \( SO(2N+1) \) algebra has also been made by Yamamura [29]. Due to the relations (2.15) which lead to \( \Theta^2 = \overline{\theta}^B_i \theta^B_j \equiv \overline{\theta}^B \theta^B \) and using the formal expression for square root of \( \overline{\theta}^B \theta^B \), we have

\[
e^\Theta = Z^B + a_i^\dagger X_i^B - a_i X_i^B \equiv Z^B + \Theta^B \left( \Theta^B = -X_i^B \right), \quad (Z^B)^2 - \Theta^B = (Z^B)^2 - X_i^B X_i^B = 1,
\]

\[
Z^B = \cosh |\theta|, \quad X_i^B = \frac{\theta^B}{|\theta|} \sinh |\theta|, \quad |\theta| = \sqrt{\theta^B \theta^B}.
\]

(2.16)

The even-Grassmann variable \( \overline{\theta}^B \theta^B \) corresponds to the density of the probability of finding the violation of boson number conservation by odd numbers [9, 10] and relates to the occupation number. See Friedrichs [30] and Berezin [25]. Then, the use of the \( \overline{\theta}^B \theta^B \) meaning such the density and the occupation number is recognized to be appropriate. See Baez et al. [31]. The \( \dagger \) operation on both the \( a_i^\dagger \theta^B_i \) and \( a_i \overline{\theta}^B_i \) is made as the usual \( \dagger \) operation on both the \( a_i^\dagger \overline{\theta}^B_i \) and \( a_i^\dagger \theta^B_i \) in the case of the boson. The appearance of the hyperbolic functions \( \cosh |\theta| \) and \( \sinh |\theta| \) is due to the introduction of the anti-commuting Grassmann variables \( \theta^B_i \) and \( \overline{\theta}^B_i \) for which we demand that they commute with \((-1)^n\). From (2.1), (2.7), (2.15) and (2.16), thus we obtain

\[
e^\Theta \frac{1}{\sqrt{2}} (-1)^n e^{-\Theta} = \frac{1}{\sqrt{2}} (Z^B + \Theta X)(Z^B + \Theta X)(-1)^n
\]

\[
= \left\{ a_j \left( \Theta_j^B X_j^B \right) + a_j^\dagger \left( X_j^B X_j^B \right) + \frac{1}{\sqrt{2}} \left( 2 (Z^B)^2 - 1 \right) \right\} (-1)^n.
\]

(2.17)

which shows mixing of \( a_i(-1)^n \) and \( a_i^\dagger(-1)^n \). Then, we need to know transformation rules for both the \( e^\Theta a_i(-1)^n e^{-\Theta} \) and \( e^\Theta a_i^\dagger(-1)^n e^{-\Theta} \). For the purpose of realizing the rules, we notice the commutation relations \([a_i, Z^B + \Theta X] = -X_i^B + 2a_i \Theta X\) and \([a_i^\dagger, Z^B + \Theta X] = -\overline{X}^B_i + 2a_i^\dagger \Theta X\). They play essential roles. First the former transformation is exactly calculated as follows:

\[
e^\Theta a_i(-1)^n e^{-\Theta} = a_i(-1)^n + X_i^B (Z^B + \Theta X)(-1)^n
\]

\[
= \left\{ a_j \left( \delta_{ji} - \overline{X}_j^B X_i^B \right) + a_j^\dagger \left( X_j^B X_i^B \right) + \frac{1}{\sqrt{2}} \left( \sqrt{2} Z^B X_i^B \right) \right\} (-1)^n.
\]

(2.18)

Similarly, we compute the transformation rule for the latter one as

\[
e^\Theta a_i^\dagger(-1)^n e^{-\Theta} = a_i^\dagger(-1)^n + \overline{X}_i^B (Z^B + \Theta X)(-1)^n
\]

\[
= \left\{ a_j \left( -\overline{X}_j^B X_i^B \right) + a_j^\dagger \left( \delta_{ji} + X_j^B \overline{X}_i^B \right) + \frac{1}{\sqrt{2}} \left( \sqrt{2} Z^B \overline{X}_i^B \right) \right\} (-1)^n.
\]

(2.19)

Expressing (2.17), (2.18) and (2.19) in a lump, we can obtain

\[
e^\Theta \left[ a_i, a_i^\dagger, \frac{1}{\sqrt{2}} I \right] (-1)^n e^{-\Theta} = \left[ a_j, a_j^\dagger, \frac{1}{\sqrt{2}} I \right] (-1)^n G_{X_{ji}},
\]

(2.20)

where \( G_{X_{ji}} \) is defined as

\[
G_{X_{ji}} \overset{\text{def}}{=} \begin{bmatrix}
\delta_{ji} - \overline{X}_j^B X_i^B & -\overline{X}_j^B \overline{X}_i^B & -\sqrt{2} Z^B \overline{X}_j^B \\
X_j^B X_i^B & \delta_{ji} + X_j^B \overline{X}_i^B & \sqrt{2} Z^B X_j^B \\
\sqrt{2} Z^B X_i^B & \sqrt{2} Z^B \overline{X}_i^B & 2 (Z^B)^2 - 1
\end{bmatrix}.
\]

(2.21)
Let $G$ be a $(2N + 1) \times (2N + 1)$ matrix defined by

$$
G \equiv G_x \begin{bmatrix}
    a^B & \bar{b}^B & 0 \\
    b^B & \bar{a}^B & 0 \\
    0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
    a^B - X^B Y^B & \bar{b}^B - \bar{X}^B \bar{Y}^B & -\sqrt{2} Z^B \bar{X}^B \\
    b^B + X^B Y^B & \bar{a}^B + X^B \bar{Y}^B & \sqrt{2} Z^B X^B \\
    \sqrt{2} Z^B Y^B & \sqrt{2} \bar{Z}^B \bar{Y}^B & 2(Z^B)^2 - 1
\end{bmatrix}
$$

(2.22)

where $a^B$ and $b^B$ appear in $F$ (2.13) and $Y^B$ is a new row vector with the essential difference having a linear combination with minus sign in the fermion row vector $Y^F$ used in the $SO(2N+1)$ canonical transformation [9], which is given by

$$
Y_i^B = X^B a^B_j + \bar{X}^B b^B_j, \quad (Z^B)^2 - \bar{Y}^B Y_i^B = 1.
$$

(2.23)

Introducing new matrices $A^B$ and $B^B$ and a column vector $x^B$ and a row vector $y^B$ and a scalar $z^B$, the matrix $G$ is rewritten with the use of the Grassmann variables $X^B$ and $Y^B$ as

$$
G = \begin{bmatrix}
    A^B & B^B \\
    B^B & A^B \\
    y^B & z^B
\end{bmatrix}
\begin{bmatrix}
    a^B - X^B Y^B = a^B - \frac{x^B y^B}{2(1 + z^B)}, \\
    b^B + X^B Y^B = b^B + \frac{x^B y^B}{2(1 + z^B)}, \\
    x^B = 2Z^B X^B, \quad y^B = 2Z^B Y^B, \quad z^B = 2(Z^B)^2 - 1
\end{bmatrix}
$$

(2.24)

which is formally almost the same form as the $SO(2N+1)$ matrix [9] and a form suggested by Ozaki [22]. Here we have used the complex conjugation rule for products of $\bar{X}^B Y^B$ and $X^B Y^B$ for Grassmann variables $X^B$ and $Y^B$. This kind of rule is provided explicitly by Berezin [25]:

$$
\bar{X}^B Y^B = \bar{Y}^B X^B = -X^B \bar{Y}^B, \quad X^B \bar{Y}^B = \bar{Y}^B X^B = -X^B \bar{Y}^B.
$$

(2.25)

By the transformation $U(G) = e^{\Theta} e^{A^B} e^{B^B}$ for bosons $[a, a^\dagger, \frac{1}{\sqrt{2}}]$, thus we obtain

$$
U(G) \begin{bmatrix}
    a_i, \quad a_i^\dagger, \quad \frac{1}{\sqrt{2}}
\end{bmatrix} (-1)^n U^{-1}(G) = \begin{bmatrix}
    a_j, \quad a_j^\dagger, \quad \frac{1}{\sqrt{2}}
\end{bmatrix} (-1)^n \begin{bmatrix}
    A^B_{ji} & B^B_{ji} & \frac{x^B_{ji}}{\sqrt{2}} \\
    B^B_{ji} & A^B_{ji} & \frac{x^B_{ji}}{\sqrt{2}} \\
    \bar{y}^B_{ji} & \bar{y}^B_{ji} & z^B_{ji}
\end{bmatrix}.
$$

(2.26)

Let us introduce new bosons $[b, b^\dagger, \frac{1}{\sqrt{2}}]$ defined as $[b, b^\dagger, \frac{1}{\sqrt{2}}] \equiv U(G)[a, a^\dagger, \frac{1}{\sqrt{2}}] U^{-1}(G)$ and a new operator $\Theta_x$ defined as $\Theta_x \equiv a_i^\dagger x^B_i - a_i \bar{x}^B_i$ where $x^B_i$ and $\bar{x}^B_i$ are Grassmann variables. We derive the normalization condition governing the transformation parameters. For this aim, we further introduce new boson operators $\alpha_i \equiv a_i^\dagger A^B_{ji} + a^B_{ji} B^B_{ji}$ and $\alpha_i^\dagger \equiv a_i^\dagger B^B_{ji} + a^B_{ji} A^B_{ji}$. First we compute the commutators $\alpha_i$ and $\alpha_i^\dagger$ between $\Theta_x$. Using the anti-commutators (2.13) and the first equation of (2.23), the computations are made as follows:

$$
[\alpha_i, \Theta_x] = [a_j A^B_{ji} + a^B_{ji} B^B_{ji}, \quad a^B_{ji} x^B_i - a^B_{ji} \bar{x}^B_i] = 2\alpha_i \Theta_x - (x^B_i A^B_{ji} + \bar{x}^B_i B^B_{ji}).
$$

(2.27)

On the other hand, using the relations in (2.24), the last term of (2.27) is rewritten as

$$
x^B_j A^B_{ji} + \bar{x}^B_i B^B_{ji} = y^B_i + 2X^B_j X^B_j y^B_i = (2(Z^B)^2 - 1)y^B_i = z^B_i y^B_i.
$$

(2.28)
Then we get the commutation relation \([\alpha_i, \Theta_x] = 2\alpha_i \Theta_x - z^B y_i^B\). Using (2.35), \([\Theta_x, y_i^B] = 0\) and the relation \(\alpha_i \Theta_x = -\Theta_x \alpha_i + z^B y_i^B\), we have

\[
b_i = \left(\alpha_i + \frac{1}{2} y_i^B\right) (z^B - \Theta_x) = (z^B + \Theta_x) \left(\alpha_i - \frac{1}{2} y_i^B\right).
\]

(2.29)

While we have \([\alpha_i^\dagger, \Theta_x] = 2\alpha_i^\dagger \Theta_x - z^B y_i^B\), which is also derived in the similar way as the one made in (2.27). Using (2.35), \([\Theta_x, y_i^B] = 0\) and the relation \(\alpha_i^\dagger \Theta_x = -\Theta_x \alpha_i^\dagger + z^B y_i^B\), we also have

\[
b_i^\dagger = \left(\alpha_i^\dagger + \frac{1}{2} y_i^B\right) (z^B - \Theta_x) = (z^B + \Theta_x) \left(\alpha_i^\dagger - \frac{1}{2} y_i^B\right).
\]

(2.30)

To find the normalization conditions of \(A_{ij}^B, B_{ij}^B, x_i^B\) and \(y_i^B\) in (2.35) for the commutators \([b_i, b_j^\dagger]\) and \([b_i, b_j]\), we calculate the following commutators with the aid of (2.29) and (2.30):

\[
[b_i, b_j^\dagger] = \left(\alpha_i + \frac{1}{2} y_i^B\right) \left(\alpha_j - \frac{1}{2} y_j^B\right) - \left(\alpha_j + \frac{1}{2} y_j^B\right) \left(\alpha_i - \frac{1}{2} y_i^B\right).
\]

(2.31)

\[
= [\alpha_i, \alpha_j^\dagger] - \frac{1}{2} \{\alpha_i, y_j^B\} + \frac{1}{2} \{y_i^B, \alpha_j^\dagger\} - \frac{1}{4} (y_i^B y_j^B - y_j^B y_i^B) = A^{B\dagger}_{ik} A^B_{kj} - B^{B\dagger}_{ik} B^B_{kj} - \frac{1}{2} y_i^B y_j^B = \delta_{ij},
\]

[2.32]

\[
[b_i, b_j] = \left(\alpha_i + \frac{1}{2} y_i^B\right) \left(\alpha_j - \frac{1}{2} y_j^B\right) - \left(\alpha_j + \frac{1}{2} y_j^B\right) \left(\alpha_i - \frac{1}{2} y_i^B\right).
\]

(2.32)

\[
= [\alpha_i, \alpha_j] - \frac{1}{2} \{\alpha_i, y_j^B\} + \frac{1}{2} \{y_i^B, \alpha_j\} - \frac{1}{4} (y_i^B y_j^B - y_j^B y_i^B) = A^{B\dagger}_{ik} B^B_{kj} - B^{B\dagger}_{ik} A^B_{kj} - \frac{1}{2} y_i^B y_j^B = 0.
\]

Here we have used \((z^B)^2 - \Theta_x^2 = 1\). Then we get the following normalization conditions:

\[
\langle A^{B\dagger} B^B - B^{B\dagger} A^B \rangle_i = \delta_{ij} - \frac{1}{2} y_i^B y_j^B, \quad \langle A^{B\dagger} B^B - B^{B\dagger} A^B \rangle_i = \frac{1}{2} y_i^B y_j^B,
\]

(2.33)

\[
\langle x_i^B x^B \rangle_i = (z^B)^2 - x_i^B x^B = 1.
\]

By using the new bosons \(\alpha_i\) and \(\alpha_i^\dagger\) and noticing the third column of the matrix in (2.26), \(U(G) [a_i, a_i^\dagger, \frac{1}{\sqrt{2}}] U^{-1}(G)\), (at the region near \(z^B \approx 1\) and then \(x^B \approx 0\)), is approximated as

\[
U(G) [a_i, a_i^\dagger, \frac{1}{\sqrt{2}}] U^{-1}(G) = \begin{bmatrix}
\alpha_i + \frac{1}{2} y_i^B & (z^B - \Theta_x) & \frac{1}{2} (z^B + \Theta_x) \\
\frac{1}{2} y_i^B & (z^B + \Theta_x) & \frac{1}{2} (z^B - \Theta_x) \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{bmatrix}
\]

(2.34)

\[
= \begin{bmatrix}
\alpha_i + \frac{1}{2} y_i^B & \frac{1}{2} y_i^B & \frac{1}{2} y_i^B \\
\frac{1}{2} y_i^B & \frac{1}{2} y_i^B & \frac{1}{2} y_i^B \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{bmatrix} (z^B - \Theta_x) .
\]

The above approximation is justified by using \((z^B)^2 - \Theta_x^2 = 1\) and \(z^B \approx 1\) and then \(x^B \approx 0\). Finally, we obtain the approximate expression at \(z^B \approx 1\), i.e., \(x^B \approx 0\) for equation (2.26) as

\[
[b_i, b_i^\dagger, \frac{1}{\sqrt{2}}] = [a_j, a_j^\dagger, \frac{1}{\sqrt{2}}] (z^B - \Theta_x) .
\]

(2.35)
3 Inverse of boson Bogoliubov transformation

To derive an inverse transformation of (2.35), we introduce new boson operators $\beta_i^\dagger$ and $\beta_i$

$$
\beta_i = b_j A_{ji}^{B\dagger} + b_j^\dagger B_{ji}^{BT}, \quad \beta_i^\dagger = b_j B_{ji}^{B\dagger} + b_j^\dagger A_{ji}^{BT}. \tag{3.1}
$$

The inverse of the canonical transformation (2.35) is made in the similar way to (2.23) and (2.30). Using the expression (2.35), $[\Theta_y, x_i^B] = [\Theta_y, y_i^B] = 0$ and the relations $\beta_i \Theta_y = -\Theta_y \beta_i + z^B x_i^B$ and $\beta_i^\dagger \Theta_y = -\Theta_y \beta_i^\dagger + z^B x_i^B$, we also have

$$
a_i = \left(\beta_i - \frac{1}{2} x_i^B\right) (z^B + \Theta_y) = (z^B - \Theta_y) \left(\beta_i + \frac{1}{2} x_i^B\right), \quad \begin{cases} 
\beta_i - \frac{1}{2} \tau_i^B \\
\beta_i^\dagger - \frac{1}{2} \tau_i^B
\end{cases} (z^B + \Theta_y) = (z^B - \Theta_y) \left(\beta_i^\dagger + \frac{1}{2} \tau_i^B\right). \tag{3.2}
$$

To find also the normalization conditions of $A_{ij}^B$, $B_{ij}^B$, $x_i^B$ and $y_i^B$ in (2.35) for the commutators $[a_i, a_j^\dagger]$ and $[a_i, a_j]$, we calculate the following commutators with the aid of (3.2):

$$
\begin{align*}
[a_i, a_j^\dagger] &= [\beta_i, \beta_j^\dagger] + \frac{1}{2} \left(\beta_i, x_j^B\right) - \frac{1}{4} \left(\beta_i x_j^B - x_j^B \beta_i\right) = A_{ki}^B A_{kj}^{B\dagger} - B_{ki}^{BT} B_{kj}^{B\dagger} - \frac{1}{2} x_i^B x_j^B = \delta_{ij}; \\
[a_i, a_j] &= [\beta_i, \beta_j] + \frac{1}{2} \left(\beta_i, x_j^B\right) - \frac{1}{4} \left(\beta_i x_j^B - x_j^B \beta_i\right) = A_{ki}^B B_{kj}^{B\dagger} - B_{ki}^{BT} A_{kj}^{B\dagger} - \frac{1}{2} x_i^B x_j^B = 0.
\end{align*} \tag{3.3}
$$

Then we get the following normalization conditions:

$$
\begin{align*}
\left(A^B A^{B\dagger} - \bar{\Theta}^B B^{B\dagger} \right)_{ij} &= \delta_{ij} - \frac{1}{4} x_i^B x_j^B, \quad \text{(due to (2.23))} \\
\left(y^B A^{B\dagger} - \bar{\Theta}^B B^{B\dagger} \right)_{ij} &= \frac{1}{2} x_i^B x_j^B, \\
(y^B A^{B\dagger} + \bar{\Theta}^B B^{B\dagger})_{ij} &= z_i^B x_j^B, \quad (z^B)^2 - y^B y^{B\dagger} = 1.
\end{align*} \tag{3.4}
$$

By using another new bosons $\beta_i$ and $\beta_i^\dagger$, (3.1), $U(G) [b_i, b_i^\dagger, \frac{1}{\sqrt{2}}] U^{-1}(G)$ is approximated as

$$
U(G) \left[ b_i, b_i^\dagger, \frac{1}{\sqrt{2}} \right] U^{-1}(G)
= \left[ \left(\beta_i - \frac{1}{2} x_i^B\right) (z^B + \Theta_y), \left(\beta_i^\dagger - \frac{1}{2} \tau_i^B\right) (z^B + \Theta_y), \frac{1}{\sqrt{2}} (y_j^B + b_j y_j^B + z^B) \right] \\
= \left[ \beta_i - \frac{1}{2} x_i^B, \beta_i^\dagger + \frac{1}{2} \tau_i^B, \frac{1}{\sqrt{2}} \right] (z^B + \Theta_y) + [0, x_i^B (z^B + \Theta_y), \sqrt{2} b_i y_i^B]
\approx \left[ \beta_i - \frac{1}{2} x_i^B, \beta_i^\dagger + \frac{1}{2} \tau_i^B, \frac{1}{\sqrt{2}} \right] (z^B + \Theta_y), \quad (z^B \approx 1, x^B \approx 0, y^B \approx 0). \tag{3.5}
$$

Thus we have the inverse transformation for (2.35) as

$$
\left[ a_i, a_i^\dagger, \frac{1}{\sqrt{2}} \right] = \left[ b_j, b_j^\dagger, \frac{1}{\sqrt{2}} \right] G^\dagger (z^B + \Theta_y), \quad G^\dagger =
\begin{bmatrix}
A_{ji}^{B\dagger} & B_{ji}^{B\dagger} & \frac{y_j^B}{\sqrt{2}} \\
B_{ji}^{B\dagger} & A_{ji}^{BT} & \frac{y_j^B}{\sqrt{2}} \\
\frac{x_i^{BT}}{\sqrt{2}} & \frac{x_i^{B\dagger}}{\sqrt{2}} & z^B
\end{bmatrix}. \tag{3.6}
$$

The transformations (2.35) and (3.0) hold the orthogonality conditions but do not exactly:

$$
G^\dagger \tilde{T}_{2N+1} G = \tilde{T}_{2N+1}, \quad G \tilde{T}_{2N+1} G^\dagger = \tilde{T}_{2N+1}, \quad \tilde{T}_{2N+1} = \begin{bmatrix} \tilde{T}_{2N} & 0 \\
0 & 1 \end{bmatrix}, \quad (z^B \approx 1, x^B \approx 0, y^B \approx 0), \tag{3.7}
$$

though we omit the proof here because their explicit expressions become too long to write.
4 GHB mean-field Hamiltonian and its diagonalization

Let us consider the following Hamiltonian consisting of the generalized Hartree-Bogoliubov (GHB) mean-field Hamiltonian (MFH) for which we assume a linear MFH expressed in terms of the generators of the Jacobi hsp algebra, the Jacobi algebra, \(^{(2.3)}\) as follows:

\[
H_{\text{Jacobi hsp}} = iF^B E_j + \frac{1}{2} D^B E_{ij} - \frac{1}{2} D^B E_{ij} + iM^B a_i^\dagger - i\tilde{M}^B a_i
\]

\[
= \frac{1}{2} \begin{pmatrix} a, a^\dagger, \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1_N & 0 \\ 1_N & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathcal{F}^B \begin{pmatrix} a^\dagger, a, \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} a, a^\dagger, \frac{1}{\sqrt{2}} \end{pmatrix} \tilde{\mathcal{F}}^B \begin{pmatrix} a^\dagger \\ a, a^\dagger, \frac{1}{\sqrt{2}} \end{pmatrix}.
\]  

(4.1)

Here \(F^B, D^B\) and \(M^B\) are the SCF parameters satisfying the properties

\[
F^{B\dagger} = -F^B, \quad D^{B\dagger} = D^B, \quad M^{B\dagger} = [M^B, \cdots, M^B],
\]

(4.2)

and matrices \(\mathcal{F}^B\) and \(\tilde{\mathcal{F}}^B\) are given by

\[
\mathcal{F}^B = \begin{bmatrix} iF^B & iD^B & i\sqrt{2} M^B \\ -i\tilde{D}^B & -i\tilde{F}^B & -i\sqrt{2} \tilde{M}^B \\ -i\sqrt{2} M^B \dagger & i\sqrt{2} M^B \dagger & 0 \end{bmatrix},
\]

(4.3)

\[
\tilde{\mathcal{F}}^B = \begin{bmatrix} 0 & 1_N & 0 \\ 1_N & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathcal{F}^B \begin{bmatrix} 0 & 1_N & 0 \\ 1_N & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -i\tilde{F}^B & -i\tilde{D}^B & -i\sqrt{2} \tilde{M}^B \\ iD^B & iF^B & i\sqrt{2} M^B \\ i\sqrt{2} M^B \dagger & -i\sqrt{2} M^B \dagger & 0 \end{bmatrix}.
\]

The fermion GHB MFH is studied intensively by one of the present authors (S. N.) \(^{(32)}\)

Using the generalized Bogoliubov transformation \(^{(2.35)}\) and its inverse transformation \(^{(2.35)}\) and the operator relation \((z^B)^2 - \Theta_y^2 = 1\), we can diagonalize the MFH \(H_{\text{Jacobi hsp}}\) \(^{(4.1)}\) in the following form:

\[
H_{\text{Jacobi hsp}} = \frac{1}{2} \begin{pmatrix} b, b^\dagger, \frac{1}{\sqrt{2}} \end{pmatrix} \begin{bmatrix} G^B (z^B + \Theta_y) \tilde{G}^B (z^B - \Theta_y) G \end{bmatrix} \begin{pmatrix} b^\dagger \\ b, b^\dagger, \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} b, b^\dagger, \frac{1}{\sqrt{2}} \end{pmatrix} \tilde{G}^B G \begin{pmatrix} b^\dagger \\ b, b^\dagger, \frac{1}{\sqrt{2}} \end{pmatrix},
\]

(4.4)

\[
G^B \tilde{G}^B = \begin{bmatrix} E_{2N} \cdot I_{2N} & 0 \\ 0 & \epsilon \end{bmatrix} \equiv \tilde{E} \quad (\epsilon: \text{real number}), \quad E_{2N} = \begin{bmatrix} E_{\text{diag}}, E_{\text{diag}} \end{bmatrix}, \quad E_{\text{diag}} \equiv [E_1, \cdots, E_N],
\]

(4.5)

where \(E_i\) is a quasi-particle energy and \(\epsilon\) is another excitation energy which is not artificial but possibly exists mathematically. Then, we get \(H_{\text{Jacobi hsp}} = \sum_{i=1}^{N} \left( E_i b_i^\dagger b_i + \frac{1}{2} \epsilon \right) + \frac{1}{2} \epsilon \). Using \(^{(3.7)}, (4.5)\) and the second of \(^{(4.3)}\), we obtain the following eigenvalue equation:

\[
\mathcal{F}^B \begin{bmatrix} 0 & 1_N & 0 \\ 1_N & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} G = \begin{bmatrix} 0 & -1_N & 0 \\ 1_N & 0 & 0 \\ 0 & -1_N & 0 \end{bmatrix} \tilde{E},
\]

(4.6)

which is explicitly written by
\[
\begin{bmatrix}
  iF^B & iD^B & i\sqrt{2}M^B \\
  -iD^B & -iF^B & -i\sqrt{2}M^B \\
  -i\sqrt{2}M^B & i\sqrt{2}M^B & 0
\end{bmatrix}
\begin{bmatrix}
  b^B + \frac{1}{1+z^B} \frac{x^B y^B}{\sqrt{2}} \\
  a^B - \frac{1}{1+z^B} \frac{\overline{\tau}^B y^B}{\sqrt{2}} \\
  -a^B - \frac{1}{1+z^B} \frac{\overline{\tau}^B y^B}{\sqrt{2}}
\end{bmatrix}
\begin{bmatrix}
  \bar{a}^B + \frac{1}{1+z^B} \frac{x^B \overline{y}^B}{\sqrt{2}} \\
  b^B - \frac{1}{1+z^B} \frac{\overline{\tau}^B \overline{y}^B}{\sqrt{2}} \\
  -\overline{a}^B - \frac{1}{1+z^B} \frac{\overline{\tau}^B \overline{y}^B}{\sqrt{2}}
\end{bmatrix}
\begin{bmatrix}
  x^B \\
  y^B \\
  z^B
\end{bmatrix}
\]

from the first column in both sides of equations of which, we get the following set of equations:

\[
\begin{align*}
  iF^B \left( b^B + \frac{1}{1+z^B} \frac{x^B y^B}{\sqrt{2}} \right) + iD^B \left( a^B - \frac{1}{1+z^B} \frac{\overline{\tau}^B y^B}{\sqrt{2}} \right) + i\sqrt{2}M^B b^B &= 0 \\
  -iD^B \left( b^B + \frac{1}{1+z^B} \frac{x^B y^B}{\sqrt{2}} \right) - iF^B \left( a^B - \frac{1}{1+z^B} \frac{\overline{\tau}^B y^B}{\sqrt{2}} \right) - i\sqrt{2}M^B b^B &= 0 \\
  -i\sqrt{2}M^B a^B + i\sqrt{2}M^B b^B &= 0 \\
  -i\sqrt{2}M^B b^B + i\sqrt{2}M^B a^B &= 0
\end{align*}
\]  

Then we have

\[
\begin{align*}
  iF^B b^B + iD^B a^B &= -b^B e, & -iD^B b^B - iF^B a^B &= a^B e, \\
  \frac{1}{1+z^B} \left( F_B \frac{x^B}{\sqrt{2}} - D_B \frac{\tau_B}{\sqrt{2}} \right) \frac{y^B}{\sqrt{2}} + i\sqrt{2}M^B \frac{y^B}{\sqrt{2}} &= -b^B e - \frac{1}{1+z^B} \frac{x^B y^B}{\sqrt{2}} (e + \varepsilon), \\
  -\frac{1}{1+z^B} \left( D_B \frac{x^B}{\sqrt{2}} - F_B \frac{\tau_B}{\sqrt{2}} \right) \frac{y^B}{\sqrt{2}} - i\sqrt{2}M^B \frac{y^B}{\sqrt{2}} &= a^B e - \frac{1}{1+z^B} \frac{x^B y^B}{\sqrt{2}} (e + \varepsilon), \\
  -i\sqrt{2}M^B b^B + i\sqrt{2}M^B a^B &= \frac{y^B}{\sqrt{2}} e - \frac{1}{1+z^B} \left( \sqrt{2}M^B \frac{x^B}{\sqrt{2}} + \sqrt{2}M^B \frac{\tau_B}{\sqrt{2}} \right) \frac{y^B}{\sqrt{2}} = e.
\end{align*}
\]
Through the second column in both sides of equations of (4.7), we get the set of equations
\[
\begin{aligned}
&iF^B\left(\pi^B + \frac{1}{1+z^B} \frac{x^B y^B}{\sqrt{2} \sqrt{2}}\right) + iD^B\left(\tau^B - \frac{1}{1+z^B} \frac{x^B y^B}{\sqrt{2} \sqrt{2}}\right) + i\sqrt{2} M^B \frac{y^B}{\sqrt{2}} \\
&= iF^B \overline{a}^B + iD^B b^B + \frac{1}{1+z^B} i\left(F^B B^B - D^B \tau^B\right) \frac{y^B}{\sqrt{2}} + i\sqrt{2} M^B \frac{y^B}{\sqrt{2}} \\
&= \left(\overline{a}^B + \frac{1}{1+z^B} \frac{x^B y^B}{\sqrt{2} \sqrt{2}}\right) (e + \varepsilon) = \overline{a}^B e - \overline{a}^B \varepsilon + \frac{1}{1+z^B} \frac{x^B y^B}{\sqrt{2} \sqrt{2}} (e + \varepsilon),
\end{aligned}
\]
(4.10)

\[
\begin{aligned}
&-iD^B\left(\pi^B + \frac{1}{1+z^B} \frac{x^B y^B}{\sqrt{2} \sqrt{2}}\right) - iF^B\left(\tau^B - \frac{1}{1+z^B} \frac{x^B y^B}{\sqrt{2} \sqrt{2}}\right) - i\sqrt{2} M^B \frac{y^B}{\sqrt{2}} \\
&= -iD^B \pi^B - iF^B b^B - \frac{1}{1+z^B} i\left(D^B x^B - F^B \tau^B\right) \frac{y^B}{\sqrt{2}} - i\sqrt{2} M^B \frac{y^B}{\sqrt{2}} \\
&= - \left(\pi^B - \frac{1}{1+z^B} \frac{x^B y^B}{\sqrt{2} \sqrt{2}}\right) (e + \varepsilon) = -\pi^B e - \pi^B \varepsilon + \frac{1}{1+z^B} \frac{x^B y^B}{\sqrt{2} \sqrt{2}} (e + \varepsilon),
\end{aligned}
\]
(4.11)

from which we have
\[
\begin{aligned}
iF^B \pi^B + iD^B b^B &= \pi^B e, \\
\frac{1}{1+z^B} i\left(F^B x^B - D^B \tau^B\right) \frac{y^B}{\sqrt{2}} + i\sqrt{2} M^B \frac{y^B}{\sqrt{2}} &= \pi^B e + \frac{1}{1+z^B} \frac{x^B y^B}{\sqrt{2} \sqrt{2}} (e + \varepsilon),
\end{aligned}
\]
(4.12)

Through the third column in both sides of equations of (4.7), we get the set of equations
\[
\begin{aligned}
iF^B \frac{x^B}{\sqrt{2}} - iD^B \frac{x^B}{\sqrt{2}} + i\sqrt{2} z^B M^B &= -\frac{x^B}{\sqrt{2}} e, \\
-iF^B \frac{x^B}{\sqrt{2}} + iD^B \frac{x^B}{\sqrt{2}} - i\sqrt{2} z^B M^B &= -\frac{x^B}{\sqrt{2}} e,
\end{aligned}
\]
(4.13)

The first and second equations of (4.13) and (4.14) are set of the boson Bogoliubov equations with eigenvalue $e$ [4]. Note that inside round brackets the SCF parameters $F^B$, $D^B$ and $M^B$ are regarded as the Grassmann-like numbers, to ensure the conjugate complexity of the equations and also to assure the real number of the $e$. The additional eigenvalues, $\varepsilon$, however, become to be $\varepsilon = 0$ due to the last equations of (4.13) and (4.14). The division of $E$ into $e$ and $\varepsilon$ is a useful means to treat the MF eigenvalue equation and to make clear distinct computational steps.
5 SCF condition and MFH linear in \( \varphi \) generator

According to suggestion by Ozaki [22], multiplying the first of (4.9) by \( b_{B\dagger} \) and the first of (4.11) by \( a_{B\dagger} \) from the right, we obtain

\[
iFb_{B\dagger}b_{B\dagger} + iD_{B\dagger}a_{B\dagger}b_{B\dagger} = -b_{B\dagger}eb_{B\dagger}, \quad iF_{\dagger}a_{B\dagger} \cdot a_{B\dagger} + iD_{B\dagger}b_{B\dagger}a_{B\dagger} = \bar{a}_{B\dagger}e_{a_{B\dagger}}, \quad (5.1)
\]

and multiplying the first of (4.9) by \( a_{B\dagger} \) and the first of (4.11) by \( b_{B\dagger} \) from the right, we obtain

\[
iFb_{B\dagger}a_{B\dagger} + iD_{B\dagger}a_{B\dagger}b_{B\dagger} = -b_{B\dagger}ea_{B\dagger}, \quad iF_{\dagger}a_{B\dagger} \cdot a_{B\dagger} + iD_{B\dagger}b_{B\dagger}a_{B\dagger} = \bar{a}_{B\dagger}e_{b_{B\dagger}}. \quad (5.2)
\]

Subtracting the second of (5.1) from the first and the second of (5.2) from the first, we have

\[
iF = \bar{a}_{B\dagger}e_{a_{B\dagger}} + b_{B\dagger}e_{b_{B\dagger}} = -iF_{\dagger}, \quad iD_{B\dagger} = -\left( \bar{a}_{B\dagger}e_{a_{B\dagger}} + a_{B\dagger}e_{b_{B\dagger}} \right) = iD_{B\dagger}. \quad (5.3)
\]

Multiplying the first equation in the last line of (4.9) by \( b_{B\dagger} \) and the second equation by \( a_{B\dagger} \) from the right and using \( z_{B}y_{B} = x_{B}y_{B} \) \( A_{B} + x_{B}y_{B} = x_{B}y_{B} \), \( b_{B} - (1 - z_{B})y_{B} \), respectively, we have

\[
\begin{aligned}
&-i\sqrt{2}z_{B}M_{B\dagger}b_{B\dagger} + i\sqrt{2}z_{B}M_{B\dagger}a_{B\dagger}b_{B\dagger} = \frac{x_{B}}{\sqrt{2}}a_{B\dagger}eb_{B\dagger} + \frac{x_{B}}{\sqrt{2}}b_{B\dagger}eb_{B\dagger} + \left( 1 - z_{B} \right) \frac{y_{B}}{\sqrt{2}}eb_{B\dagger}, \\
&-i\sqrt{2}z_{B}M_{B\dagger}a_{B\dagger} + i\sqrt{2}z_{B}M_{B\dagger}b_{B\dagger}a_{B\dagger} = -\left( \frac{x_{B}}{\sqrt{2}}a_{B\dagger}ea_{B\dagger} + \frac{x_{B}}{\sqrt{2}}b_{B\dagger}e_{a_{B\dagger}} \right) + \left( 1 - z_{B} \right) \frac{y_{B}}{\sqrt{2}}a_{B\dagger}e_{a_{B\dagger}}.
\end{aligned} \quad (5.4)
\]

Subtracting the first equation of (5.4) from the second one and using (2.13) and (5.3), we get

\[
-\sqrt{2}z_{B}M_{B\dagger} = -\frac{x_{B}}{\sqrt{2}}(a_{B\dagger}ea_{B\dagger} + b_{B\dagger}eb_{B\dagger}) - \frac{x_{B}}{\sqrt{2}}(b_{B\dagger}ea_{B\dagger} + a_{B\dagger}eb_{B\dagger}) + (1 - z_{B}) \left\{ \frac{y_{B}}{\sqrt{2}}eb_{B\dagger} + \frac{y_{B}}{\sqrt{2}}e_{a_{B\dagger}} \right\}.
\]

\[
\frac{x_{B}}{\sqrt{2}} + iD_{B\dagger} + (1 - z_{B}) \left\{ \frac{y_{B}}{\sqrt{2}}eb_{B\dagger} + \frac{y_{B}}{\sqrt{2}}e_{a_{B\dagger}} \right\}.
\]

whose hermitian adjoint and complex conjugation become to be

\[
i\sqrt{2}z_{B}M_{B} = -iF_{B} \frac{x_{B}}{\sqrt{2}} + iD_{B} \frac{x_{B}}{\sqrt{2}} + (1 - z_{B}) \left\{ b_{B\dagger}ye_{B\dagger} + \bar{a}_{B\dagger}e_{y_{B\dagger}} \right\}, \quad (5.6)
\]

\[
-\sqrt{2}z_{B}M_{B} = -iF_{B} \frac{x_{B}}{\sqrt{2}} + iD_{B} \frac{x_{B}}{\sqrt{2}} + (1 - z_{B}) \left\{ b_{B\dagger}ye_{B\dagger} + \bar{a}_{B\dagger}e_{y_{B\dagger}} \right\}.
\]

\[
\text{Owing to the relations in the first line of (4.11) with } \epsilon = 0, \text{ equations (5.6) and (5.7) give the additional conditions}
\]

\[
b_{B\dagger}ey_{B\dagger} + \bar{a}_{B\dagger}e_{y_{B\dagger}} = 0, \quad \bar{a}_{B\dagger}e_{y_{B\dagger}} + a_{B\dagger}ey_{B\dagger} = 0.
\]

Multiplying (5.6) by \( F_{B} \) and (5.7) by \( D_{B} \) and adding them, we get

\[
\frac{x_{B}}{\sqrt{2}} = \left( F_{B}F_{B\dagger} + D_{B}D_{B\dagger} \right)^{-1} \left( F_{B} \sqrt{2}z_{B}M_{B} + D_{B} \sqrt{2}z_{B}M_{B} \right),
\]

\[
\frac{x_{B\dagger}}{\sqrt{2}} = \left( \sqrt{2}z_{B}M_{B\dagger}F_{B\dagger} + \sqrt{2}z_{B}M_{B\dagger}D_{B\dagger} \right) \left( F_{B}F_{B\dagger} + D_{B}D_{B\dagger} \right)^{-1}.
\]
On the other hand, combining the equations in the first and the second lines of (4.9) and those in the first line of (4.10) and using \( \varepsilon = 0 \), we have the following relations:

\[
\begin{aligned}
&i \left( 1 - \frac{z^B}{1 + z^B} \right) \sqrt{2} M^B \frac{y^B}{\sqrt{2}} = -i \left( 1 - \frac{z^B}{1 + z^B} \right) \frac{x^B y^B}{\sqrt{2}} e - \left( b^B + \frac{1}{1 + z^B} \frac{x^B y^B}{\sqrt{2}} \right) \varepsilon = - \frac{1}{1 + z^B} \frac{x^B y^B}{\sqrt{2}} e, \\
&i \left( 1 - \frac{z^B}{1 + z^B} \right) \sqrt{2} M^B \frac{y^B}{\sqrt{2}} = \frac{1}{1 + z^B} \frac{x^B y^B}{\sqrt{2}} e,
\end{aligned}
\]

and further combining the equations in the first and the second lines of (4.11) and those in the first line of (5.11) and using \( \varepsilon = 0 \), we have the following relations:

\[
i \sqrt{2} M^B \frac{y^B}{\sqrt{2}} = \frac{x^B y^B}{\sqrt{2}} e, \\
i \sqrt{2} M^B \frac{y^B}{\sqrt{2}} = - \frac{x^B y^B}{\sqrt{2}} e.
\]

The relations (5.8) and \( eb^{B-1} \pi^B = b^{B-1} \pi^B e \) give the consistent result with (5.6) and (5.7) as

\[
b^{B-1} e \frac{y^{B\dagger}}{\sqrt{2}} + \pi^B e \frac{y^{B\dagger}}{\sqrt{2}} = b^{B-1} \left( b^{B-1} e \frac{y^{B\dagger}}{\sqrt{2}} + \pi^B e \frac{y^{B\dagger}}{\sqrt{2}} \right) = 0, \\
\frac{b^B y^{B\dagger}}{\sqrt{2}} + \frac{a^B y^{B\dagger}}{\sqrt{2}} = 0.
\]

In (5.10), multiplying the equations in the first and second lines by \( \frac{y^{B\dagger}}{\sqrt{2}} \) and \( \frac{y^{B\dagger}}{\sqrt{2}} \) from the right and left, respectively, we obtain the equations

\[
\begin{aligned}
\frac{x^B y^B}{\sqrt{2}} &= \left( F B F B^{\dagger} + D B D^{\dagger} \right)^{-1} \left( F B^{\dagger} \frac{x^B y^B}{\sqrt{2}} + D B \frac{y^B}{\sqrt{2}} \right) e, \\
&= i z^B \left( F B F B^{\dagger} + D B D^{\dagger} \right)^{-1} \left( F B^{\dagger} \frac{x^B y^B}{\sqrt{2}} + D B \frac{y^B}{\sqrt{2}} \right) e, \\
\frac{y^B}{\sqrt{2}} &= \left( \frac{y^{B\dagger}}{\sqrt{2}} \sqrt{2} z^B M^B F^B + \sqrt{2} z^B M^B D^{\dagger} \right) \left( F B F B^{\dagger} + D B D^{\dagger} \right)^{-1} \frac{x^B y^B}{\sqrt{2}} e, \\
&= - i z^B \left( F B F B^{\dagger} + D B D^{\dagger} \right)^{-1} \frac{x^B y^B}{\sqrt{2}} e, \\
&= - i z^B \left( F B F B^{\dagger} + D B D^{\dagger} \right)^{-1} \frac{x^B y^B}{\sqrt{2}} e,
\end{aligned}
\]

where we have used the relations (5.11) and (5.12). Further multiplying the first and second equations in (5.14) by \( \frac{y^{B\dagger}}{\sqrt{2}} \) and \( \frac{y^{B\dagger}}{\sqrt{2}} \) from the right and left, respectively, we have

\[
\begin{aligned}
\frac{x^B y^B}{\sqrt{2}} &= - \frac{x^B}{\sqrt{2}} \frac{1 - (z^B)^2}{2} = - i z^B \left( F B F B^{\dagger} + D B D^{\dagger} \right)^{-1} \sqrt{2} z^B M^B \frac{y^B}{\sqrt{2}} e, \\
\frac{x^B y^B}{\sqrt{2}} &= - \frac{x^B}{\sqrt{2}} \frac{1 - (z^B)^2}{2} = - i z^B \left( F B F B^{\dagger} + D B D^{\dagger} \right)^{-1} \sqrt{2} z^B M^B \frac{y^B}{\sqrt{2}} e,
\end{aligned}
\]

Then, at last we could reach the following expressions for the two vectors \( \frac{x^B}{\sqrt{2}} \) and \( \frac{x^{B\dagger}}{\sqrt{2}} \):

\[
\begin{aligned}
\frac{x^B}{\sqrt{2}} &= - i \left( F B F B^{\dagger} + D B D^{\dagger} \right)^{-1} \frac{2 (z^B)^2}{1 - (z^B)^2} \sqrt{2} M^B \frac{y^B}{\sqrt{2}} e, \\
&= - i \frac{2 (z^B)^2}{1 - (z^B)^2} \left( F B F B^{\dagger} + D B D^{\dagger} \right)^{-1} \sqrt{2} M^B, \\
\frac{x^{B\dagger}}{\sqrt{2}} &= i \frac{2 (z^B)^2}{1 - (z^B)^2} \sqrt{2} M^B \left( F B F B^{\dagger} + D B D^{\dagger} \right)^{-1} \frac{y^B}{\sqrt{2}} e, \\
&= i \frac{2 (z^B)^2}{1 - (z^B)^2} \left( F B F B^{\dagger} + D B D^{\dagger} \right)^{-1} \sqrt{2} M^B e,
\end{aligned}
\]

\[
\begin{aligned}
\frac{y^B}{\sqrt{2}} &= i \frac{2 (z^B)^2}{1 - (z^B)^2} \sqrt{2} \frac{y^B}{\sqrt{2}} e, \\
&= i \frac{2 (z^B)^2}{1 - (z^B)^2} \sqrt{2} M^B \left( F B F B^{\dagger} + D B D^{\dagger} \right)^{-1} \frac{y^B}{\sqrt{2}} e.
\end{aligned}
\]
where \( \langle e \rangle \) is defined as \( \langle e \rangle = \frac{\langle U \rangle}{\sqrt{N}} \). Therefore, the \( \langle e \rangle \) stands for the averaged value of all the eigenvalue distribution. Thus, this is the first time that the final solutions for the vectors \( \frac{x^B}{\sqrt{2}} \) and \( \frac{x^{BT}}{\sqrt{2}} \) could be derived within the present framework of the \( \text{Jacobi hsp} \) MFT. The inner product of the vectors leads to the relation

\[
\frac{x^{BT} x^B}{2} = \frac{1 - (z^B)^2}{2} = \frac{4(z^B)^4}{(1 - (z^B)^2)^3} 2 \langle e \rangle >^2 M^{BT} \left( F^B F^{BT} + D^B D^{BT} \right)^{-2} M^B, \tag{5.17}
\]

which is the remarkable result in the \( \text{Jacobi hsp} \) MFT and is simply rewritten as

\[
\frac{16(z^B)^4}{(1 - (z^B)^2)^3} \langle e \rangle >^2 M^{BT} \left( F^B F^{BT} + D^B D^{BT} \right)^{-2} M^B = 1. \tag{5.18}
\]

The above relation shows that the magnitude of the additional SCF parameter \( M^B \) is inevitably restricted by the behavior of the SCF parameters \( F^B \) and \( D^B \) which should be governed by the condition \( F^B D^B - D^B F^B = 0 \). Remember that this condition is one of the crucial condition in \( \frac{\langle U \rangle}{\sqrt{N}} \) to derive \( \frac{\langle U \rangle}{\sqrt{N}} \) for \( \frac{x^B}{\sqrt{2}} \) and \( \frac{x^{BT}}{\sqrt{2}} \). Finally, substituting the solutions for \( \frac{x^B}{\sqrt{2}} \) and \( \frac{x^{BT}}{\sqrt{2}} \) which is derivable from the complex conjugation of the first equation of (5.16), into the equation in the last line of (4.12), we can determine \( \epsilon \), saying another excitation energy, approximately as

\[
\epsilon \approx \frac{4z^B}{1 - (z^B)^2} < e > M^{BT} \left( F^B F^{BT} + D^B D^{BT} \right)^{-1} M^B \tag{5.19}
\]

\[+ \frac{4z^B}{1 - (z^B)^2} < e > M^{B^T} \left( F^{B^T} F^B + D^B D^B \right)^{-1} M^B, \]

de to the property of the complex conjugation given in (4.12), \(-i\sqrt{2} M^{BT} \frac{x^B}{\sqrt{2}} = -i\sqrt{2} M^{B^T} \frac{x^B}{\sqrt{2}} \).

\( \epsilon \) means another type of excitation energy never been seen in the traditional boson MFT.

In this section, we have considered the boson GHB MFH \( H_{\text{Jacobi hsp}} \) (4.1). Half a century ago, in his famous textbook, Berezin already gave this type of Hamiltonian [25], i.e., \( H_{\text{Berezin}} = C_{ij} a_i^\dagger a_j + \frac{1}{2} A_{ij} a_i^\dagger a_j^\dagger + \frac{1}{2} A_{ij} a_i a_j + a_{ij}^\dagger f_i + f_i^\dagger a_i \). As is shown clearly from the structure of the \( H_{\text{Berezin}} \), we should emphasize that there exists the essential difference between \( H_{\text{Berezin}} \) and our Hamiltonian \( H_{\text{Jacobi hsp}} \). Recently, Berceanu, in his papers (13, 14), also considered a linear Hamiltonian in terms of the \( \text{Jacobi} \) generators with matrices of coefficient, \( \epsilon^0, \epsilon^+ \) and \( \epsilon^- \),

\[
H_{\text{Berezin}} = \epsilon_i a_i + \epsilon^0 a_i^\dagger + \epsilon^+ K_{ij}^0 + \epsilon^- K_{ij}^+ + \epsilon^- K_{ij}^-, \quad \left( [K_{ij}^0, K_{ij}^+], [K_{ij}^+, K_{ij}^-] \right) = \frac{1}{2} [E^i_j, E^j_i, E_{ij}], \tag{5.20}
\]

The Perelomov coherent state (CS) \( e_{z,w} \) [5] associated with the \( \text{Jacobi} \) algebra is defined as

\[
e_{z,w} = \exp(X) e_0, \quad X = \sum_{i=1}^N z_i a_i^\dagger + \sum_{i,j=1}^N w_{ij} K_{ij}^+, \quad (a_i e_0 = 0). \tag{5.21}
\]

We denote \( \{ w_{ij} \} \) as \( w \). Using the CS on the Siegel-\( \text{Jacobi} \) domain involving the Siegel-\( \text{Jacobi} \) ball or Siegel-\( \text{Jacobi} \) upper half plane [33, 34], a classical equation of motion arisen from the linear Hamiltonian (5.20) is given by a time-dependent differential equation, saying, the well-known Riccati equation,

\[
i\dot{w} = \epsilon^- + \frac{1}{2} (we^0 + e^0 w)^T + we^+ w, \quad i\dot{z} = \epsilon + w\bar{\epsilon} + \frac{1}{2} e^0 z + we^+ z. \tag{5.22}
\]

Berceanu has also found that a classical equation of motion on both non-compact and compact symmetric spaces arisen from the linear Hamiltonian is described by the matrix-type Riccati equation [35, 36]. On the other hand, we also have obtained several years ago the matrix-type Riccati-Hartree-Bogoliubov equation on the compact symmetric coset space \( SO(2N)/U(N) \) for a fermion system accompanying with a general two-body interaction [37].
6 Geometrical structure of $\frac{\text{Sp}(2N+2)}{\text{U}(N+1)}$ coset manifold

Let us introduce a $(N+1) \times (N+2)$ isometric matrix $U^T$ by

$$U^T = [B^T, A^T], \quad (A, B : \text{given by } (A.5)), \quad (6.1)$$

Along the direction taken by the present authors et al. [38, 39], here we also use the matrix elements of $U$ and $U^T$ as the coordinates on the manifold $Sp(2N+2)$ instead of the manifold $SO(2N+2)$. Even in the case of $Sp(2N+2)$, it turns out that a real line element can be defined by a hermitian metric tensor on the manifold. Under the transformation $U \rightarrow VU$ the metric is invariant. Then, the metric tensor defined on the manifold may become singular, due to the fact that one can use too many coordinates through the introduction of another matrix $V$.

According to Zumino [40], if $A^B$ is non-singular, we have relations governing $U^T U$ as

$$U^T (\tilde{I}_{2N+2}) U = A^{B}_{A} A^{B}_{A} B^{B}_{B} = A^{B}_{A} \{ 1_{N+1} - (B^{B}_{A} A^{B}_{A} - 1) \} A^B_{B} = A^B_{B} (1_{N+1} - Q^B_{Q} Q^B_{Q} A^B), \quad (6.2)$$

$$\ln \det U^T (\tilde{I}_{2N+2}) U = \ln \det U^T U = \ln \det (1_{N+1} - Q^B_{Q} Q^B) + \ln \det A^B + \ln \det A^B_i,$$

where we have used the $Sp(2N+2)$, $U(N+1)$ coset variable $Q^B$ given by (A.13) and the $I_{2N+2}$ of (A.5). If we take the matrix elements of $Q^B$ and $\bar{Q}^B$ as the coordinates on the $Sp(2N+2)$, $U(N+1)$ coset manifold, the real line element can be well defined by a hermitian metric tensor on the coset manifold as

$$ds^2 = G_{pq} \bar{r} s dQ^{pq} d\bar{Q}^{pq} (Q^{pq} = Q^{pq}_{Q}; \bar{Q}^{pq} = \bar{Q}^{pq}_Q; G_{pq} \bar{r} s = G_{s \bar{r} q \bar{s}}; ; (p, q, r, s) \text{ take over } 0, i, j). \quad (6.3)$$

The condition that a certain manifold is a Kähler manifold is that its complex structure is covariantly constant relative to the Riemann connection and that it has vanishing torsions:

$$G_{pq} \bar{r} s , tu = \frac{\partial G_{pq} \bar{r} s }{\partial Q^{tu}} = G_{tu} \bar{r} s , pq; \quad G_{pq} \bar{r} s , tu = \frac{\partial G_{pq} \bar{r} s }{\partial \bar{Q}^{tu}} = G_{pq} \bar{r} s , tu; \quad (6.4)$$

As in the case of $SO(2N+2)$ [38, 39], the hermitian metric tensor $G_{pq} \bar{r} s$ can be locally given through a real scalar function, the Kähler potential, which takes the well-known form

$$\mathcal{K}(Q^{B}, Q^B) = \ln \det (1_{N+1} - Q^{B} Q^B), \quad (6.5)$$

and the explicit expression for the components of the metric tensor is given as

$$G_{pq} \bar{r} s = \frac{\partial \mathcal{K}(Q^{B}, Q^B) }{\partial Q^{pq} \bar{r} s} = \left\{ 1_{N+1} - Q^{B} Q^B \right\}^{-1} \left\{ 1_{N+1} - Q^{B} Q^B \right\}^{-1} \partial_t (r \leftrightarrow s) - \partial_p (p \leftrightarrow q) + \partial_q (p \leftrightarrow q, r \leftrightarrow s). \quad (6.6)$$

Notice that the above function does not determine the Kähler potential $\mathcal{K}(Q^{B}, Q^B)$ uniquely since the metric tensor $G_{pq} \bar{r} s$ is invariant under transformations of the Kähler potential

$$\mathcal{K}(Q^{B}, Q^B) \rightarrow \mathcal{K}'(Q^{B}, Q^B) = \mathcal{K}(Q^{B}, Q^B) + \mathcal{F}(Q^B) + \overline{\mathcal{F}(Q^B)} \quad (6.7)$$

$\mathcal{F}(Q^B)$ and $\overline{\mathcal{F}(Q^B)}$ are analytic functions of $Q^B$ and $\bar{Q}^B$, respectively. In the case of the Kähler metric tensor, we have only the components of the metric connections with unmixed indices

$$\Gamma_{pq}^{tu} = G_{qw}^{uv} G_{wp}^{vw}, \quad \Gamma_{pq}^{tu} = G_{tu}^{vw} G_{vw}^{pq}, \quad \Gamma_{pq}^{tu} = \frac{(G^{-1})_{vw}}{w_t w_v}, \quad (6.8)$$

and also have only the components of the curvatures

$$R_{pq}^{tu} = G_{qw}^{uv} \Gamma_{pq}^{tu}^{vw}, \quad R_{pq}^{tu} = G_{tu}^{vw} \Gamma_{pq}^{tu}^{vw}, \quad R_{pq}^{tu} = R_{pq}^{tu} = \Gamma_{pq}^{tu}^{vw}, \quad \Gamma_{pq}^{tu}^{vw}.$$
7 Expression for $\frac{Sp(2N+2)}{U(N+1)}$ Killing potential and GDM

Along the direction taken in Refs. [38, 39], we consider an $Sp(2N+2)$ infinitesimal left transformation of an $Sp(2N+2)$ matrix $G$ to $G'$, $G'=(1_{2N+2}+\delta G)G$, by using the first equation of (B.3):

$$G' = \begin{bmatrix} 1_{N+1} + \delta A^B & \delta B^B \\ \delta B^B & 1_{N+1} + \delta A^\dagger \end{bmatrix} G = \begin{bmatrix} A^B + \delta A^B A^\dagger + \delta B^B B^\dagger & B^B + \delta A^B B^\dagger + \delta B^B A^\dagger \\ B^\dagger + \delta A^\dagger B^\dagger + \delta B^\dagger A^\dagger & A^\dagger + \delta A^\dagger A^\dagger + \delta B^\dagger B^\dagger \end{bmatrix}. \quad (7.1)$$

Let us define an $Sp(2N+2)$ coset variable $Q^{B'} (= B^B A^B - 1)$ in the $G'$ frame. With the aid of (7.1), the $Q^{B'}$ is calculated infinitesimally as

$$Q^{B'} = B^B A^B - 1 = (B^B + \delta A^B B^\dagger + \delta B^B A^\dagger) \left( A^B + \delta A^B A^\dagger + \delta B^B B^\dagger \right)^{-1} \quad (7.2)$$

$$= Q^B + \delta B^B - Q^B \delta A^B + \delta A^B Q^B - \delta B^B Q^B.$$

The Kähler metrics admit a set of holomorphic isometries, the Killing vectors, $R^{i[k]}(Q^B)$ and $\overline{R}^{\dagger[i]}(\overline{Q}^B) \ (i = 1, \cdots, \text{dim} G)$, which are the solution of the Killing equation $\delta G' = \mathcal{R}^{i[k]}(Q^B)$ and $\delta \overline{G}' = \overline{\mathcal{R}}^{\dagger[i]}(\overline{Q}^B)$ such that $G'(Q^B, \overline{Q}^B) = G(Q^B, \overline{Q}^B)$. The Killing equation (7.3) is the necessary and sufficient condition for an infinitesimal coordinate transformation

$$\delta Q^{B[k]} = (\delta B^B - \delta A^B T Q^B - Q^B \delta A^B - Q^B \delta B^B T Q^B) \delta k = \xi_i R^{i[k]}(Q^B), \quad \delta \overline{Q}^{B[k]} = \overline{\xi}_i \overline{R}^{\dagger[i]}(\overline{Q}^B). \quad (7.4)$$

The $\xi_i$ is infinitesimal and global parameter. Due to the Killing equation, Killing vectors $R^{i[k]}(Q^B)$ and $\overline{R}^{\dagger[i]}(\overline{Q}^B)$ can be written locally as the gradient of some real scalar function, namely Killing potential $\mathcal{M}^{i}(Q^B, \overline{Q}^B)$ such that

$$R^{i[k]}(Q^B) = -i \mathcal{M}^{i[k]}, \quad \overline{R}^{\dagger[i]}(\overline{Q}^B) = i \mathcal{M}^{\dagger[i][k]}. \quad (7.5)$$

According to van Holten [41] and Refs. [38, 39] and using the infinitesimal $Sp(2N+2)$ matrix $\delta G$ given by first of (B.3), the Killing potential $\mathcal{M}$ for the coset space $\frac{Sp(2N+2)}{U(N+1)}$ is written as

$$\mathcal{M}(\delta A, \delta B, \delta B^\dagger) = \text{Tr}(\delta G \tilde{\mathcal{M}}) = \text{tr}(\delta A M_{\delta A} + \delta B M_{\delta B} + \delta B^\dagger M_{\delta B^\dagger}),$$

$$\tilde{\mathcal{M}} = \begin{bmatrix} M_{\delta A} & M_{\delta B} \\ -M_{\delta A}^\dagger & M_{\delta B}^\dagger \end{bmatrix}, \quad M_{\delta A} = \tilde{M}_{\delta A} + (\tilde{M}_{\delta A})^\dagger, \quad M_{\delta B} = \tilde{M}_{\delta B}, \quad M_{\delta B^\dagger} = \tilde{M}_{\delta B^\dagger}. \quad (7.6)$$

Trace $\text{Tr}$ is taken over $(2N+2) \times (2N+2)$ matrices, while trace $\text{tr}$ is taken over $(N+1) \times (N+1)$ matrices. Let us introduce $(N+1)$-dimensional matrices $\mathcal{R}(Q^B, \delta G)$, $\mathcal{R}_{T}(Q^B, \delta G)$ and $\chi$ by

$$\mathcal{R}(Q^B, \delta G) = \delta B^B - \delta A^B T Q^B - Q^B \delta A^B - Q^B \delta B^B T Q^B, \quad \mathcal{R}_{T}(Q^B, \delta G) = -\delta A^B T - Q^B \delta B^B, \quad \chi = (1_{N+1} - Q^B Q^B) = \chi^\dagger. \quad (7.7)$$

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In \((7.4)\), putting \(\xi_i\) as \(\xi_i = 1\), we have \(\delta Q^B = R(Q^B, \delta G)\) which is just the Killing vector in the coset space \(Sp(2N+2)_{U(N+1)}\) and tr of holomorphic matrix-valued function \(R_T(Q^B, \delta G)\), i.e., \(\text{tr} R_T(Q^B, \delta G) = F(Q^B)\) is a holomorphic Kähler transformation. Then, the Killing potential \(\mathcal{M}\) is given as

\[
-i\mathcal{M} \left( Q^B, \overline{Q}^B; \delta G \right) = -\text{tr} \Delta \left( Q^B, \overline{Q}^B; \delta G \right),
\]

\[
\Delta \left( Q^B, \overline{Q}^B; \delta G \right) \overset{\text{def}}{=} R_T(Q^B, \delta G) - R(Q^B, \delta G) Q^{B\dagger} \chi = \left( Q^B \delta A^B Q^{B\dagger} - \delta A^{B\dagger} - \delta B^B Q^{B\dagger} - Q^B \delta B^{B\dagger} \right) \chi.
\]

From \((7.6)\) and \((7.8)\), we obtain

\[
-i\mathcal{M}_{\delta B} = -\mathcal{X} Q^B, \quad -i\mathcal{M}_{\delta B^i} = Q^{B^i} \mathcal{X}, \quad -i\mathcal{M}_{\delta A} = 1_{N+1} - 2\mathcal{X}. \quad (7.9)
\]

Using the expression for \(\tilde{\mathcal{M}}\), equation \((7.9)\), their components are written in the form

\[
-i\tilde{\mathcal{M}}_{\delta B} = -\mathcal{X} Q^B, \quad -i\tilde{\mathcal{M}}_{\delta B^i} = -Q^{B^i} \mathcal{X}, \quad -i\tilde{\mathcal{M}}_{\delta A} = -Q^{B^i} \mathcal{X} Q^B, \quad -i\tilde{\mathcal{M}}_{\delta \mathcal{A}^i} = -\mathcal{X}. \quad (7.10)
\]

As already shown for a fermion system \([38, 39]\), it is also easily verified that the result of \((7.9)\) satisfies the gradient of the real function \(\mathcal{M} \quad (7.3)\). Of course, setting \(r = 0\) in \(Q^B \quad (7.14)\), the Killing potential \(\mathcal{M}\) in the \(Sp(2N+2)_{U(N+1)}\) coset space leads the Killing potential \(\tilde{M}\) in the \(Sp(2N)_{U(N)}\) coset space obtained by van Holten et al. \([41]\) and also by the present authors et. al. \([38, 39]\).

To make clear the meaning of the Killing potential, using the \((2N+2) \times (N+1)\) isometric matrix \(U\), let us introduce the following \((2N+2) \times (2N+2)\) matrix:

\[
\mathcal{W} = U U^\dagger = \begin{bmatrix} \mathcal{R} & \mathcal{K} \\ \mathcal{K}^* & 1_{N+1} + \mathcal{R} \end{bmatrix}, \quad \mathcal{R} = B^B B^{B\dagger}, \quad \mathcal{K} = B^A A^{B\dagger}, \quad (7.11)
\]

which is hermitian on the \(Sp(2N+2)\) manifold. The \(\mathcal{W}\) is a natural extension of the generalized density matrix (GDM) for boson systems in the \(Sp(2N)\) CS rep to the \(Sp(2N+2)\) CS rep given in the textbook \([4]\). Since the matrices \(A^B\) and \(B^B\) are represented in terms of \(Q^B = (Q^B_{pq})\) as

\[
A^B = (1_{N+1} - Q^{B\dagger} Q^B)^{-\frac{1}{2}} U, \quad B^B = Q^B (1_{N+1} - Q^{B\dagger} Q^B)^{-\frac{1}{2}} U, \quad U \in U(N+1), \quad (7.12)
\]

then, we have

\[
\mathcal{R} = Q^B (1_{N+1} - Q^{B\dagger} Q^B)^{-\frac{1}{2}} Q^{B\dagger} = Q^B \chi Q^{B\dagger} = -1_{N+1} + \chi, \quad \mathcal{K} = Q^B (1_{N+1} - Q^{B\dagger} Q^B)^{-1} = \chi Q^B. \quad (7.13)
\]

Substituting \((7.13)\) into \((7.10)\), the Killing potential \(-i\tilde{\mathcal{M}}\) is expressed in terms of submatrices \(\mathcal{R}\) and \(\mathcal{K}\) of the GDM for boson systems \((7.11)\) as

\[
-i\tilde{\mathcal{M}} = \begin{bmatrix} -\mathcal{R} & -\mathcal{K} \\ -\mathcal{K} & -(1_{N+1} + \mathcal{R}) \end{bmatrix}, \quad (7.14)
\]

from which we finally obtain

\[
-i\mathcal{M} = \begin{bmatrix} \mathcal{R} & \mathcal{K} \\ \mathcal{K} & 1_{N+1} + \mathcal{R} \end{bmatrix}. \quad (7.15)
\]

To our great surprise, the expression for the Killing potential \((7.15)\) just becomes equivalent with the GDM for boson systems \((7.11)\).
8 Discussions and summary

Basing on the \textit{Jacobi hsp} algebra, the Perelomov CS $e_{z,w}$ \cite{5} is generated by the action of $\exp \{za^{\dagger} + \text{Tr}(wK^+)\}$ to $e_0$ with row vector $z$ and column vector $a^{\dagger}$. Such the CS makes important roles in the very broad range of classical and quantum mechanics, geometric quantization and current topics of quantum optics \cite{12}. Right up to now, Berceanu has investigated how the operators of the \textit{Jacobi} algebra admit the realization of differential forms in the Siegel-\textit{Jacobi} domain \cite{13} on which the Siegel-\textit{Jacobi} upper half plane $\mathcal{X}_N^J = (\mathcal{X}_N \times \mathbb{R}^{2N})$ \cite{33} is expressed as $\mathcal{X}_N := \{ Z \in M(N, \mathbb{C}) | Z = X + iY, X, Y \in M(N, \mathbb{R}), Y > 0, \ X^T = X, Y^T = Y \}$ and on which the Siegel-\textit{Jacobi} ball $\mathcal{D}_N^J (= \mathcal{D}_N \times \mathbb{C}^N)$ \cite{33} is given as $\mathcal{D}_N := \{ w \in M(N, \mathbb{C}) : w = w^T, \ I_N - wN > 0 \}$. The plane and ball are connected each other by analytic partial Cayley transform as follows:

partial Cayley transform \( \Phi : \mathcal{X}_N^J \rightarrow \mathcal{D}_N^J, \Phi(Z, u) = (w, z), \)

\[
\begin{align*}
\Phi &= (Z - iI_N) \frac{I_N}{Z + iI_N}, \\
z &= 2i \frac{I_N}{Z + iI_N} u,
\end{align*}
\]

inverse partial Cayley transform \( \Phi^{-1} : \mathcal{D}_N^J \rightarrow \mathcal{X}_N^J, \Phi^{-1}(w, z) = (Z, u), \)

\[
\begin{align*}
Z &= i \frac{I_N}{w - iI_N} (I_N + w), \\
u &= \frac{I_N}{w - iI_N} z.
\end{align*}
\]

The Siegel-\textit{Jacobi} domain is related to the \textit{Jacobi hsp} group by the Harish-Chandra embedding. For this embedding, see Satake’s book \cite{43}. Particularly, the CS based on the Siegel-\textit{Jacobi} ball $\mathcal{D}_N^J$ is connected with the Gausson (Gaussian pure state) proposed by Simon, Sudarshan and Mukunda \cite{44}. While, Rowe, Rosensteel and Gilmore also showed the simple explicit forms for the Cayley transform $w = (Z - iI_N) \frac{I_N}{Z + iI_N}$ and the inverse Cayley transform $Z = i \frac{I_N}{w - iI_N} (I_N + w)$ between Siegel upper half plane $S_N$ and Siegel unit disk $D_N$ \cite{45}. Rosensteel and Rowe further showed that the representation spaces of discrete series are given as Hilbert spaces of the holomorphic vector-valued functions on the Siegel upper half plane $S_N$ \cite{46, 47}. The $Sp(2N, \mathbb{R})$ vector CS is the holomorphic vector-valued functions on the Siegel unit disk $D_N$ \cite{45}.

In the present paper, we have proposed a new boson mean-field theory (MFT) basing on the \textit{Jacobi hsp} algebra, studied intensively by Berceanu \cite{13} and using the anti-commuting Grassmann variables. Embedding the \textit{Jacobi hsp} group into an $Sp(2N + 2)$ group and using the $Sp(2N+2)_{U(N+1)}$ coset variables, we have build the new boson MFT on the Kähler manifold of the non-compact symmetric space $Sp(2N+2)_{U(N+1)}$. We have given a linear boson MF Hamiltonian (MFH) expressed in terms of the operators of the \textit{Jacobi hsp} algebra and diagonalized the linear MFH. A new aspect of eigenvalues of the MFH has been made clear. A new excitation energy arisen from the additional SCF parameter has never been seen in the traditional boson MFT on the Kähler coset space $Sp(2N)_{U(N)}$. This kind of excitation energy has been first derived in this work. We also have constructed the Killing potential, which is the extension of Killing potential in the $Sp(2N)_{U(N)}$ coset space to that in the $Sp(2N+2)_{U(N+1)}$ coset space. To our great surprise, it turns out that the Killing potential is entirely equivalent with the generalized density matrix which is a powerful tool to approach difficult quantal-problems occurred in many boson systems \cite{4, 48}. 

18
Appendix

A Embedding of Jacobi hsp group into an Sp(2N+2) group

Referring to the fermion case proposed by Fukutome [24], we define the projection operators (OPs) $P_+$ and $P_-$ onto the sub spaces of even- and odd-boson numbers, respectively, by

$$P_{\pm} \overset{\text{def}}{=} \frac{1}{2}(1 \pm (-1)^n), \quad P_2^2 = P_{\pm}, \quad P_+ P_- = 0,$$

where $n$ is the boson-number OP and define the OPs with the spurious index 0:

$$E_0^0 \overset{\text{def}}{=} \frac{1}{2}(P_- - P_+), \quad E_i^0 \overset{\text{def}}{=} a_i^\dagger P_+ = P_+ a_i, \quad E_i^e \overset{\text{def}}{=} a_i P_+ = P_ a_i,$$

(A.1)

where each block-matrix is computed with aid of the relation

$$a_i = E_{i0}^0 + E_{i1}^0, \quad a_i^\dagger \overset{\text{def}}{=} -E_{i0}^0 + E_{i1}^0.$$

(A.2)

We introduce the indices $p, q, \cdots$ running over $N+1$ values $0, 1, \cdots, N$. Then the OPs of (A.4) and (A.2) can be denoted in a unified manner as $E_{pq}^0$, $E_{pq}^1$ and $E_{pq}^2$. They satisfy

$$E_{pq}^0 = E_{qp}^0, \quad E_{pq}^1 = E_{qp}^1, \quad E_{pq}^2 = E_{qp}^2, \quad (p, q = 0, 1, \cdots, N)$$

(A.3)

$$[E_{pq}^0, E_{rs}^0] = \delta_{qr} E_{ps}^0 - \delta_{ps} E_{qr}^0, \quad (U(N+1) \text{ algebra})$$

The above commutation relations in (A.4) are of the same form as (2.4) and (2.5).

The Jacobi hsp group is embedded into an Sp(2N+2) group. The embedding leads us to an unified formulation of the $Sp(2N+2)$ regular representation in which the paired and unpaired modes are treated in an equal way. Define $(N+1)\times(N+1)$ matrices $\mathbf{A}^B$ and $\mathbf{B}^B$ as

$$\mathbf{A}^B \equiv \begin{bmatrix} A^B & -\frac{y^B}{2} \\ y^B & 2 + z^B \end{bmatrix}, \quad \mathbf{B}^B \equiv \begin{bmatrix} B^B & \frac{x^B}{2} \\ y^B & 2 - z^B \end{bmatrix}, \quad \mathbf{A}^{BT} \mathbf{B}^B - \mathbf{B}^B \mathbf{A}^{BT} = 1_{N+1}, \quad \mathbf{A}^{BT} \mathbf{B}^B - \mathbf{B}^B \mathbf{A}^{BT} = 0,$$

(A.5)

$$y^B = x^{BT} a^B + x^B b^B.$$

Imposing the normalization of the matrix $G$ given in (2.22) and using the $I_{2N+2}$ defined similarly as the second of (2.14), then the $Sp(2N+2)$ matrix $G$ (not unitary) is represented as

$$G = \begin{bmatrix} A^B & \bar{B}^B \\ B^B & \bar{A}^B \end{bmatrix}, \quad G^\dagger I_{2N+2} G = G I_{2N+2} G^\dagger = \bar{I}_{2N+2}, \quad \bar{I}_{2N+2} \equiv \begin{bmatrix} 1_{N+1} & 0 \\ 0 & -1_{N+1} \end{bmatrix},$$

(A.6)

The matrix $G$ satisfies $\det G = 1$ as is proved easily below

$$\det G = \det (A^B - \bar{B}^B \bar{A}^B - B^B) \det \bar{A}^B = \det (A^B A^{B\dagger} - \bar{B}^B \bar{A}^B - B^B A^{B\dagger}) = 1.$$

(A.7)

Using the explicit expressions for $\mathbf{A}^B$ and $\mathbf{B}^B$ given in (A.5), $\mathbf{A}^{B\dagger} \mathbf{A}^B - \mathbf{B}^{B\dagger} \mathbf{B}^B$ is calculated as

$$\mathbf{A}^{B\dagger} \mathbf{A}^B - \mathbf{B}^{B\dagger} \mathbf{B}^B = \begin{bmatrix} A^{B\dagger} A^B - B^{B\dagger} B^B & -A^{B\dagger} \frac{x^B}{2} - B^{B\dagger} \frac{x^B}{2} + z^{B\dagger} y^B \\ -x^{B\dagger} A^B - x^{B\dagger} B^B + z^B y^B & x^{B\dagger} \frac{x^B}{2} - x^{B\dagger} \frac{x^B}{2} + z^B \frac{y^B}{2} \end{bmatrix},$$

(A.8)

where each block-matrix is computed with aid of the relation $y^B = x^{B\dagger} a^B + x^B b^B$ as follows:
$A^B \equiv A^B - B^B \equiv B^B$

$$A^B = 1_N - \frac{1}{2(1 + z^B)} (a^B \tau^B + b^B \tau^B) y^B - \frac{1}{2(1 + z^B)} y^B (x^B a^B + x^B b^B)$$

$$= 1_N - \frac{1}{1 + z^B} y^B \approx 1_N, \quad \frac{x^B}{2} - \frac{x^B}{2} + z^B \frac{y^B}{2} = - \frac{x^B}{2} \left( a^B - \frac{y^B}{2} \left( a^B - \frac{y^B}{2} \right) \right) - \frac{x^B}{2} \left( b^B + \frac{y^B}{2} \left( b^B + \frac{y^B}{2} \right) \right) + z^B \frac{y^B}{2} \quad (A.9)$$

$$= 1_N - \frac{1}{2(1 + z^B)} (x^B a^B + x^B b^B) + \frac{x^B y^B}{2} (4(1 + z^B)) + z^B \frac{y^B}{2} = - \left( 1 - z^B \right) \frac{y^B}{2} + \left( 1 - z^B \right) \frac{y^B}{2} = 0,$$

$$\frac{x^B y^B}{2} - \frac{x^B}{2} - \frac{x^B}{2} + z^B = \frac{1 - (z^B)^2}{2} + z^B \approx 1.$$
Extended boson realization of Sp(2N+2) Lie operators

Referring to the integral representation of a state vector in fermion space given by Fukutome [24], we consider a boson-state vector $|\Psi\rangle$ corresponding to a function $\Psi(\mathcal{G})$ in $\mathcal{G} \in Sp(2N+2)$:

$$|\Psi\rangle = \int U(\mathcal{G}) |0\rangle \otimes |\Psi d\mathcal{G} = \int U(\mathcal{G}) |0\rangle \otimes \Psi(\mathcal{G}) d\mathcal{G}. \quad (B.1)$$

The $\mathcal{G}$ is given by (A.5) and (A.6) and the $d\mathcal{G}$ is an invariant group integration. When an infinitesimal operator $I_\mathcal{G} + \delta \mathcal{G}$ and a corresponding infinitesimal unitary OP $U(1_{2N+2} + \delta \mathcal{G})$ is operated on $|\Psi\rangle$, using $U^{-1}(1_{2N+2} + \delta \mathcal{G}) = U(1_{2N+2} - \delta \mathcal{G})$, it transforms $|\Psi\rangle$ as

$$U(1_{2N+2} - \delta \mathcal{G}) |\Psi\rangle = (I_\mathcal{G} - \delta \mathcal{G}) |\Psi\rangle = \int U(\mathcal{G}) |0\rangle \otimes (1_{2N+2} + \delta \mathcal{G}) |\Psi(\mathcal{G}) d\mathcal{G}, \quad (B.2)$$

$$l_{2N+2} + \delta \mathcal{G} = \begin{bmatrix} 1_{N+1} + \delta A^B & \delta B^B \\ \delta B^B & 1_{N+1} + \delta A^B \end{bmatrix}, \quad \delta A^B = -\delta A^B, \quad \text{tr} \delta A^B = 0, \quad \delta B^B = \delta B^{BT}, \quad (B.3)$$

$$\delta \mathcal{G} = \delta A^B q E_{q}^p + \frac{1}{2} \left( \delta B^B_{pq} E_{qp} \right), \quad \delta \mathcal{G} = \delta A^B q \mathcal{E}_{q}^p + \frac{1}{2} \left( \delta B^B_{pq} \mathcal{E}_{qp} + \delta B^{BP}_{pq} \mathcal{E}_{pq} \right).$$

Equation (B.2) shows that the operation of $I_\mathcal{G} - \delta \mathcal{G}$ on the $|\Psi\rangle$ in the boson space corresponds to the left multiplication by $l_{2N+2} + \delta \mathcal{G}$ to variable $\mathcal{G}$ of function $\Psi(\mathcal{G})$. For a small parameter $\epsilon$, we obtain a representation on the $\Psi(\mathcal{G})$ as

$$\rho(\epsilon \delta \mathcal{G}) \Psi(\mathcal{G}) = \Psi(\epsilon \delta \mathcal{G}) = \Psi(\mathcal{G} + \epsilon \delta \mathcal{G} \mathcal{G}) = \Psi(\mathcal{G} + d\mathcal{G}), \quad (B.4)$$

which leads us to a relation $d\mathcal{G} = \epsilon \delta \mathcal{G} \mathcal{G}$. From this, we express it explicitly as,

$$d\mathcal{G} = \begin{bmatrix} dA^B & dB^B \\ dB^B & dA^B \end{bmatrix} = \epsilon \begin{bmatrix} \delta A^B A^B + \delta B^B B^B & \delta A^B B^B + \delta B^B A^B \\ \delta B^B A^B + \delta A^B B^B & \delta A^B A^B + \delta B^B B^B \end{bmatrix}, \quad (B.5)$$

$$dA^B = \epsilon \frac{\partial A^B}{\partial \epsilon} = \epsilon (\delta A^B A^B + \delta B^B B^B), \quad dB^B = \epsilon \frac{\partial B^B}{\partial \epsilon} = \epsilon (\delta B^B A^B + \delta A^B B^B).$$

A differential representation of $\rho(\delta \mathcal{G})$, $d\rho(\delta \mathcal{G})$, is given as

$$d\rho(\delta \mathcal{G}) \Psi(\mathcal{G}) = \left[ \frac{\partial A^B_p}{\partial \epsilon} \frac{\partial}{\partial A^B q} + \frac{\partial B^B_p}{\partial \epsilon} \frac{\partial}{\partial B^B q} + \frac{\partial A^B_p}{\partial \epsilon} \frac{\partial}{\partial A^B q} + \frac{\partial B^B_p}{\partial \epsilon} \frac{\partial}{\partial B^B q} \right] \Psi(\mathcal{G}). \quad (B.6)$$

Substituting (B.5) into (B.6), we can get explicit forms of the differential representation

$$d\rho(\delta \mathcal{G}) \Psi(\mathcal{G}) = \delta \mathcal{G} \Psi(\mathcal{G}), \quad (B.7)$$

where each OP in $\delta \mathcal{G}$ is expressed in a differential form as

$$\mathcal{E}^p_q = \mathcal{E}^{p_{qr}} \frac{\partial}{\partial \mathcal{E}^r_{qr}} - B^r_{qr} \frac{\partial}{\partial B^r_{qr}} - A^r_{q} \frac{\partial}{\partial A^r_{q}} \psi^r_p, \quad (B.8)$$

$$\mathcal{E}_{pq} = \mathcal{A}^r_{q} \frac{\partial}{\partial \mathcal{A}^r_{q}} + B^r_{qr} \frac{\partial}{\partial B^r_{qr}} + A^r_{p} \frac{\partial}{\partial A^r_{p}} \psi^r_q, \quad (B.9)$$

$$\mathcal{E}^{pq}_{r} = \mathcal{E}^{p}_{q} \mathcal{E}^{q}_{p} \mathcal{E}^{r}_{pq}, \quad \mathcal{E}^{p}_{q} = \mathcal{E}^{q}_{p} \mathcal{E}^{r}_{pq}, \quad \mathcal{E}^{p}_{q} = \mathcal{E}^{q}_{p}. \quad (B.10)$$
We define extended boson OPs $\mathcal{A}^{Bp}_{pq}$, $\overline{\mathcal{A}}^{Bp}_{pq}$, etc., from every variable $A^{Bp}_{pq}$, $\overline{A}^{Bp}_{pq}$, etc., as

$$
\mathcal{A}^B = \frac{1}{\sqrt{2}} \left( A^B + \frac{\partial}{\partial A^B} \right), \quad \mathcal{A}^{B\dagger} = \frac{1}{\sqrt{2}} \left( \overline{A}^B - \frac{\partial}{\partial \overline{A}^B} \right),
$$

$$
\overline{\mathcal{A}}^B = \frac{1}{\sqrt{2}} \left( \overline{A}^B + \frac{\partial}{\partial \overline{A}^B} \right), \quad \mathcal{A}^{B^T} = \frac{1}{\sqrt{2}} \left( A^B - \frac{\partial}{\partial \overline{A}^B} \right),
$$

\begin{equation}
\begin{aligned}
[\mathcal{A}^B, \mathcal{A}^{B\dagger}] = 1, \quad [\mathcal{A}^B, \mathcal{A}^{B^T}] = 1, \\
[\mathcal{A}^B, \overline{\mathcal{A}}^B] = [\mathcal{A}^{B\dagger}, \mathcal{A}^{B^T}] = [\mathcal{A}^{B^T}, \mathcal{A}^{B\dagger}] = 0,
\end{aligned}
\end{equation}

(B.9)

where $A^B$ is a complex variable. Similar definitions hold for $B^B$ in order to define the extended boson OPs $B^{B}_{pq}$, $\overline{B}^{B}_{pq}$, etc. By noting the relations

$$
\mathcal{A}^B \frac{\partial}{\partial \mathcal{A}^B} + \mathcal{A}^{B\dagger} = \mathcal{A}^{B\dagger} \mathcal{A}^B + \mathcal{A}^{B^T} \overline{\mathcal{A}}^B, \quad \overline{\mathcal{A}}^B \frac{\partial}{\partial \overline{\mathcal{A}}^B} + \mathcal{A}^{B^T} = \mathcal{A}^{B^T} \mathcal{A}^B + \mathcal{A}^{B\dagger} \overline{\mathcal{A}}^B,
$$

(B.10)

the differential OPs \[(B.8)\] can be converted into an extended boson OP rep

\begin{equation}
\begin{aligned}
\mathcal{E}^p_q - B^{B\dagger}_{pr} B^{B*}_{qr} - B^{B\dagger}_{qpr} \mathcal{A}^{B\dagger}_{r} + \mathcal{A}^{Bp}_{r} - A^{Bp}_{r} \overline{A}^{B}_{r} + B^{B\dagger}_{r} \mathcal{A}^{B\dagger}_{r} + B^{B}_{r} \mathcal{A}^{B}_{r}, \quad \mathcal{E}^{0}_{0} = 0,
\end{aligned}
\end{equation}

(B.11)

by using the notation $A^{Bp}_{r+\hat{N}}$, $B^{B\dagger}_{r+\hat{N}}$, and suffix $\hat{r}$ running from 0 to $\hat{N}$ and from $N+1$ to $2N$. Then we have the extended boson images of the boson $Sp(2N+2)$ Lie OPs as

\begin{equation}
\begin{aligned}
E^{i}_{j} = \mathcal{E}^{i}_{j} \mathcal{A}^{Bp}_{pr} \mathcal{A}^{Bp}_{jr} - \mathcal{A}^{Bp}_{pr} \mathcal{A}^{Bp}_{jr}, \quad \mathcal{E}^{0}_{0} = 0, \\
E^{i}_{ij} = \mathcal{E}^{i}_{ij} \mathcal{A}^{Bp}_{pr} \mathcal{A}^{Bp}_{jr} + \mathcal{A}^{Bp}_{pr} \mathcal{A}^{Bp}_{jr}, \quad \mathcal{E}^{0}_{0} = 0, \\
\alpha^{i}_{i} = \alpha^{i}_{i} + \mathcal{E}^{0}_{0} = -\mathcal{A}^{Bp}_{pr} \mathcal{A}^{Bp}_{pr} + \mathcal{A}^{Bp}_{pr} \mathcal{A}^{Bp}_{pr}, \\
= \sqrt{2} \left( \mathcal{Y}^{B(+)}(\mathcal{A}^{Bp}_{pr} \mathcal{A}^{Bp}_{pr}) + \mathcal{Y}^{B(-)}(\mathcal{A}^{Bp}_{pr} \mathcal{A}^{Bp}_{pr}) \right), \quad \mathcal{Y}^{B(+)}(\pm) \defeq \frac{1}{\sqrt{2}}(\mathcal{A}^{Bp}_{pr} \pm \mathcal{A}^{Bp}_{pr}).
\end{aligned}
\end{equation}

(B.12)

The representation for $\alpha^{i}_{i}$ in \[(B.12)\] involves, in addition to the original $A^{B}_{ij}$ and $B^{B}_{ij}$ bosons, their complex conjugate bosons. The complex conjugate bosons arise from the use of the matrix $\mathcal{G}$ as variables of representation. The $\mathcal{Y}^{B}_{\hat{r}}$ bosons also arise from extension of algebra from $Sp(2N)$ to $\mathcal{Jacobi} hsp$ algebra and embedding of the $\mathcal{Jacobi} hsp$ into $Sp(2N+2)$ algebra.

Using the relations

\begin{equation}
\begin{aligned}
\frac{\partial}{\partial A^{Bp}_{pq}} = (A^{B-1})_p^q \partial, \quad \frac{\partial}{\partial B^{Bp}_{pq}} = (A^{B-1})_s^p \partial, \quad \frac{\partial}{\partial \overline{A}^{Bp}_{pq}} = -(A^{B-1})_s^p \partial, \quad \frac{\partial}{\partial \overline{B}^{Bp}_{pq}} = -(A^{B-1})_s^p \partial,
\end{aligned}
\end{equation}

(B.13)

we get the relations which are valid when operated on functions on the right coset $Sp(2N+2)/SU(N+1)$

\begin{equation}
\begin{aligned}
\frac{\partial}{\partial B^{Bp}_{pq}} = \sum_{r<s} (A^{B-1})^q_r \frac{\partial}{\partial Q^{B}_{rp}}, \\
\frac{\partial}{\partial \overline{A}^{Bp}_{pq}} = -\sum_{s<r} Q^{B}_{rp} (A^{B-1})^q_r \frac{\partial}{\partial \overline{Q}^{B}_{rs}} = -i \frac{1}{2} (A^{B-1})^q_r \frac{\partial}{\partial Tr},
\end{aligned}
\end{equation}

(B.14)

from which later we can derive the expressions \[(D.1)\].
C Vacuum function for extended bosons

We show here that the function $\Phi_{00}(G)$ in $G \in Sp(2N+2)$ corresponds to the free-boson vacuum function in the physical boson space. Then the $\Phi_{00}(G)$ must satisfy the conditions

$$\left( \mathcal{E}^p_q + \frac{1}{2} \delta_{pq} \right) \Phi_{00}(G) = \mathcal{E}_{pq} \Phi_{00}(G) = 0, \quad \Phi_{00}(1_{2N+2}) = 1. \quad (C.1)$$

The vacuum function $\Phi_{00}(G)$ which satisfy $(C.1)$ is given by $\Phi_{00}(G) = \left[ \det(\mathcal{A}^B) \right]^\frac{1}{2}$, the proof of which is made easily as follows:

$$\left( \mathcal{E}^p_q + \frac{1}{2} \delta_{pq} \right) \left[ \det(\mathcal{A}^B) \right]^\frac{1}{2}$$

$$= \frac{1}{2} \delta_{pq} \left[ \det(\mathcal{A}^B) \right]^\frac{1}{2} + \left( B_{pr}^q - B_{pq}^r \frac{\partial}{\partial B_{qr}^p} - \mathcal{A}_{r}^{B_q} \frac{\partial}{\partial \mathcal{A}_{pr}^B} + A_{pr}^q \frac{\partial}{\partial A_{qr}^B} \right) \left[ \det(\mathcal{A}^B) \right]^\frac{1}{2}$$

$$= \frac{1}{2} \delta_{pq} \left[ \det(\mathcal{A}^B) \right]^\frac{1}{2} - \frac{1}{2} \left[ \det(\mathcal{A}^B) \right]^\frac{1}{2} (A_{r}^{B_q} A_{pr}^{B-1})_{qp} \det(\mathcal{A}^B) = 0, \quad (C.2)$$

$$\mathcal{E}_{pq} \left[ \det(\mathcal{A}^B) \right]^\frac{1}{2} = \left( \mathcal{A}_{r}^{B_p} + B_{qr}^q \frac{\partial}{\partial A_{pr}^B} + A_{r}^{B_q} \frac{\partial}{\partial \mathcal{A}_{pq}^B} + B_{pq}^p \frac{\partial}{\partial B_{qr}^B} \right) \left[ \det(\mathcal{A}^B) \right]^\frac{1}{2} = 0. \quad (C.3)$$

The vacuum functions $\Phi_{00}(G)$ in $G \in Sp(2N+2)$ group and $\Phi_{00}(g)$ in $g \in Sp(2N)$ group satisfy

$$\alpha_i \Phi_{00}(G) = \left( E_i^j + \frac{1}{2} \delta_{ij} \right) \Phi_{00}(G) = E_{ij} \Phi_{00}(G) = 0, \quad \Phi_{00}(1_{2N+2}) = 1, \quad (C.4)$$

$$\left( e_i^j + \frac{1}{2} \delta_{ij} \right) \Phi_{00}(g) = e_{ij} \Phi_{00}(g) = 0, \quad \Phi_{00}(1_{2N}) = 1. \quad (C.5)$$

By using the $Sp(2N+2)$ Lie OPs $E_{pq}$, expression for the $Sp(2N+2)$ WF $|G\rangle$ is given in a form quite similar to the $Sp(2N)$ WF $|g\rangle$ as

$$|G\rangle = \langle 0 | U(G) | 0 \rangle \exp \left( \frac{1}{2} \mathcal{Q}^B_{pq} E_{pq} \right) | 0 \rangle. \quad (C.6)$$

Equation $(C.6)$ has the property $U(G) | 0 \rangle = U(G) | 0 \rangle$. On the other hand, from $(A.12)$ we get

$$\det \mathcal{A}^B = \frac{1+z^B}{2} \det a^B, \quad \det \mathcal{B}^B = \left\{ \frac{1-z^B}{2} + \frac{1}{2(1+z^B)} \left( x^{BT} q^{B-1} x^B - x^{B^1} x^B \right) \right\} \det b^B = 0. \quad (C.7)$$

Then we obtain the vacuum function $\Phi_{00}(G)$ expressed in terms of the Kähler potential as

$$\Phi_{00}(G) = \Phi_{00}(G) = \sqrt{\frac{1+z^B}{2}} \left[ \det(\mathcal{A}^B) \right]^\frac{1}{2} = e^{-\frac{1}{2} \kappa(\mathcal{Q}^B, q^{B^1})}, \quad (C.8)$$

$$\Phi_{00}(G) = \Phi_{00}(G) = \sqrt{\frac{1+z^B}{2}} \left[ \det(\mathcal{A}^B) \right]^\frac{1}{2} = \sqrt{\frac{1+z^B}{2}} e^{-\frac{1}{2} \kappa(\mathcal{Q}^B, q^{B^1})}. \quad (C.9)$$
D Differential forms for extended bosons over $\frac{\text{Sp}(2N+2)}{\text{U}(N+1)}$ coset space

Using the differential formulas (B.14), the boson $\text{Sp}(2N+2)$ Lie OPs are mapped into the regular representation consisting of functions on the coset space $\frac{\text{Sp}(2N+2)}{\text{U}(N+1)}$. The extended boson images of the boson $\text{Sp}(2N+2)$ Lie OPs $E^p_q$ etc. are represented by the closed first order differential forms over the $\frac{\text{Sp}(2N+2)}{\text{U}(N+1)}$ coset space in terms of the $\frac{\text{Sp}(2N+2)}{\text{U}(N+1)}$ coset variables $Q^B_{pq}$ and the phase variable $\tau = \frac{1}{2} \ln \left[ \frac{\text{det}(\tau^B)}{\text{det}(A^B)} \right]$ of $\text{U}(N+1)$ identical with the one, $\tau = \frac{1}{2} \ln \left[ \frac{\text{det}(\tau^B)}{\text{det}(A^B)} \right]$ of $\text{U}(N)$ due to the first equation of (C.7), as

$$E^p_q = Q^B_{pr} \frac{\partial}{\partial Q^B_{qr}} + Q^B_{qr} \frac{\partial}{\partial Q^B_{pr}} + i \delta_{pq} \frac{\partial}{\partial \tau}, \quad E_{pq} = Q^B_{pr} \frac{\partial}{\partial Q^B_{rs}} + \frac{\partial}{\partial Q^B_{pr}} + i Q^B_{pq} \frac{\partial}{\partial \tau},$$

(D.1)

derivation of which is made in quite the same way as $\text{Sp}(2N)$ case. The extended boson images of the \textit{Jacobi} $\mathfrak{sp}$ OPs are given with the aid of those of the $\text{Sp}(2N+2)$ OPs. From (D.1), we get representation of the $\text{Sp}(2N+2)$ Lie OPs in terms of the variables $q^B_{ij}$ and $r^B_{il}$. They are very similar to those of the fermion $\text{SO}(2N+1)$ Lie OPs except that the hermitian adjoint of the OP image is obtained by the complex conjugate of the OP image with minus sign [50];

$$E^i_j = E^i_j = e^i_j + r^B_{ij} \frac{\partial}{\partial r^B_{ij}} + r^B_{ji} \frac{\partial}{\partial r^B_{ji}}, \quad E^0_i = \frac{\partial}{\partial \tau},$$

(D.2)

Contrastively, Berceanu, in [13], has given the differential action of the generators of the \textit{Jacobi} $\mathfrak{sp}$ group on the coherent state $e_{z,w}$ given by (5.21) in the following forms:

$$K^0_{ij} e_{z,w} = \left( k \delta_{ij} + \frac{z_j}{2} \frac{\partial}{\partial z_i} + w_{jl} \frac{\partial}{\partial w_{li}} \right) e_{z,w}, \quad (k \text{ : extremal weight}),$$

$$K^+_{ij} e_{z,w} = \frac{\partial}{\partial w_{ij}} e_{z,w},$$

$$K^-_{ij} e_{z,w} = \left( \frac{k}{2} w_{ij} + \frac{z_i z_j}{2} + \frac{1}{2} (z_i w_{lj} + z_j w_{il}) \frac{\partial}{\partial z_l} + w_{kj} w_{li} \frac{\partial}{\partial w_{kl}} \right) e_{z,w},$$

$$\alpha_{ij}^+ e_{z,w}, \quad \alpha_i e_{z,w} = \left( z_i + w_{il} \frac{\partial}{\partial z_l} \right) e_{z,w}. \quad (D.4)$$

The explicit forms of the expressions (D.4) have just a few similarity to those of the ones (D.2) and (D.3) obtained exactly using the coset space $\frac{\text{SO}(2N+2)}{\text{U}(N+1)}$. Contrary to the similarity, they have no exact property with respect to complex conjugacy in the operator-realization, as shown from the forms of the third and last Eqs in (D.4). Berceanu has further investigated how the \textit{Jacobi} $\mathfrak{sp}$ algebra admits what kind of realization of the differential OPs in the Siegel-\textit{Jacobi} space on both the Siegel-\textit{Jacobi} ball $D^N_\mathfrak{J}$ and the Siegel-\textit{Jacobi} upper half plane $X^N_\mathfrak{J}$ [33].
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References

[1] J. Bardeen, L.N. Cooper and J.R. Schrieffer, Theory of superconductivity, *Phys. Rev.* 108 (1957) 1175-1204.

[2] N.N. Bogoliubov, The compensation principle and the self-consistent field method, *Soviet Phys. -Uspekhi* 67 (1959) 236-254.

[3] P. Ring and P. Schuck, *The nuclear many-body problem*, Springer, Berlin, 1980.

[4] J.P. Blaizot and G. Ripka, *Quantum Theory of Finite Systems*, The MIT Press, Cambridge, Massachusetts, 1986.

[5] A. Perelomov, *Generalized Coherent States and Their Applications*, Springer-Verlag, 1986.

[6] A. Perelomov, Chiral models: Geometrical aspects, *Phys. Rep.* 146 (1987) 135-213, and references therein.

[7] M. Ozaki, Group theoretical analysis of the Hartree-Bogoliubov equation. I. General theory, *J. Math. Phys.* 26 (1985) 1514-1520.

[8] M. Ozaki, Group theoretical analysis of the Hartree-Bogoliubov equation. II. The case of the electronic system with triclinic lattice symmetry, *J. Math. Phys.* 26 (1985) 1521-1528.

[9] H. Fukutome, M. Yamamura and S. Nishiyama, A new fermion many-body theory based on the $SO(2N+1)$ Lie algebra of the fermion operators, *Progr. Theor. Phys.* 57 (1977) 1554-1571.

[10] H. Fukutome and S. Nishiyama, Time dependent $SO(2N+1)$ theory for unified description of bose and fermi type collective excitations, *Progr. Theor. Phys.* 72 (1984) 239-251.

[11] J. Schwinger, in *Quantum Theory of Angular Momentum*, eds. L. C. Biedenharn and H. van Dam (Academic Press, New Yor, 1965), p. 229-279.

[12] M. Yamamura and S. Nishiyama, An a priori quantized time-dependent Hartree-Bogoliubov theory. - A generalization of the Schwinger representation of quasi-spin to the fermion pair algebra -, *Progr. Theor. Phys.* 56 (1976) 124-134.

[13] S. Berceanu, A convenient coordinatization of Siegel-Jacobi manifolds, *Rev. Math. Phys.* 24 (2012) 1250024.

[14] S. Berceanu, *A holomorphic representation of the multidimensional Jacobi algebra* in the Proceedings of the Third Operator Algebras and Mathematical Physics Conference, Bucharest, Romania, August 10-1, 2005. vol.18. Apr 2006, 26 pages; [arXiv:math/0604381](https://arxiv.org/abs/math/0604381)

[15] R. Gilmore, *Lie Group, Lie Algebras and Some of Their Applications*, J.Wiley&Sons, 1974.

[16] Wei-Min Zhang, Da Husan Feng, R. Gilmore, Coherent states: Theory and some applications, *Rev. Mod. Phys.* 62 (1990) 867-928.

[17] M. Eichler and D. Zagier, *The Theory of Jacobi forms*, Progress in Mathematics, 55 (Birkhäuser, Boston, MA, 1985).
[18] R. Berndt and R. Schmidt, *Elements of the Representation Theory of the Jacobi Group*, Progress in Mathematics, 163 (Birkhäuser Verlag, 1998).

[19] D.J. Rowe, Coherent state theory of the noncompact symplectic group, *J. Math. Phys.* **25** (1984) 2662-2671.

[20] P. Kramer, On the Theory of Collective Motion in Nuclei. I. Classical Theory, *Ann. Phys. (N.Y.)* **141** (1982) 254-268.

[21] P. Kramer, On the Theory of Collective Motion in Nuclei. II. Quantum Theory, *Ann. Phys. (N.Y.)* **141** (1982) 269-282.

[22] M. Ozaki, private communication

[23] R.C. Hwa and J. Nuytsu, Group Embedding for the Harmonic Oscillator, *Phys. Rev.* **145** (1966) 1188-1195.

[24] H. Fukutome, The group theoretical structure of fermion many-body systems arising from the canonical anticommutation relation. I - *Lie algebras of fermion operators and exact generator coordinate representations of state vectors*, *Progr. Theor. Phys.* **65** (1981) 809-827.

[25] F.A. Berezin, *The Method of Second Quantization*, Academic Press, New York and London, 1966.

[26] D.J. Thouless, Stability Conditions and Nuclear Rotations in the Hartree-Fock Theory, *Nucl. Phys.* **21** (1960) 225-232.

[27] R. Casalbuoni, On the Quantization of Systems with Anticommuting Variables, *Il Nuovo Cimento* **33A** (1976) 115-125.

[28] L. Frappat, A. Sciarrino and P. Sorba, *Dictionary on Lie Algebras and Superalgebras*, Academic Press, New York and London, 2000.

[29] M. Yamamura, A Possible Quantization of Time-Dependent Hartree-Bogoliubov Theory Based on the $SO(2N+1)$ Algebra, *Progr. Theor. Phys.* **63** (1980) 486-497.

[30] K.O. Friedrichs, *Mathematical Aspects of the Quantum Theory of Fields*, Interscience Publisher, INC., New York, 1953.

[31] J.C. Baez, I.E. Segal and Z. Zhou, *Introduction to Algebraic and Constructive Quantum Field Theory*, Princeton University Press, Princeton, 1992.

[32] S. Nishiyama, Time Dependent Hartree-Bogoliubov Equation on the Coset Space $SO(2N+2)/U(N+1)$ and Quasi Anti-Commutation Relation Approximation, *Int. J. Mod. Phys. E7* (1998) 677-707.

[33] C.L. Siegel, Symplectic geometry. *Amer. J. Math.* **65** (1943) 1-86.

[34] C.L. Siegel, *Symplectic Geometry* (Academic Press, New York, 1964).
[35] S. Berceanu and L. Boutet de Monvel, Linear dynamical systems, coherent state manifolds, flows, and matrix Riccati equation, *J. Math. Phys.* **34** (1992) 2353-2371.

[36] S. Berceanu and A. Gheorghe, On equations of motion on compact Hermitian symmetric spaces, *J. Math. Phys.* **33** (1992) 998-1007.

[37] S. Nishiyama and J. da Providência, $\frac{SO(2N)}{U(N)}$ Riccati-Hartree-Bogoliubov equation based on the $SO(2N)$ Lie algebra of the fermion operators, *Int. J. Geom. Methods Mod. Phys.* **12** (2015) 1550035.

[38] S. Nishiyama, J. da Providência, C. Providência and F. Cordeiro, Extended supersymmetric sigma-model based on the $SO(2N+1)$ Lie algebra of the fermion operators, *Nucl. Phys. B* **802** (2008) 121-145.

[39] S. Nishiyama, J. da Providência, C. Providência and F. Cordeiro, Anomaly-free supersymmetric $\frac{SO(2N+2)}{U(N+1)}$ $\sigma$-model based on the $SO(2N+1)$ Lie algebra of the fermion operators, *J High Energy Phys.* **02** (2011) 093-1-093-30.

[40] B. Zumino, Supersymmetry and Kähler manifolds, *Phys. Lett. B* **87** (1979) 203-206.

[41] S. Groot Nibbelink, T.S. Nyawelo and J.W. van Holten, Construction and analysis of anomaly-free supersymmetric $\frac{SO(2N)}{U(N)}$ $\sigma$-models, *Nucl. Phys. B* **594** (2001) 441-476.

[42] S.T. Ali, J.-P. Antoine and J.-P. Gazeau, *Coherent states, wavelets, and generalizations* (Springer-Verlag, New York, 2000).

[43] I. Satake, *Algebraic Structures of Symmetric Domains*, Publ. Math. Soc. Japan, Vol. 14 (Princeton Univ. Press, New York, 1980).

[44] R. Simon, E.C.G. Sudarshan and N. Mukunda, Gaussian pure states in quantum mechanics and the symplectic group, *Phys. Rev. A* **37** (1988) 3028-3038.

[45] D.J. Rowe, G. Rosensteel and R. Gilmore, Vector coherent state representation theory, *J. Math. Phys.* **26** (1985) 2787-2791.

[46] G. Rosensteel and D.J. Rowe, The Discrete Series of $Sp(n, \mathbb{R})$, *Int. J. Theoret. Phys.* **16** (1977) 63-79.

[47] M. Kashiwara and M. Vergne, On the Segal-Shale-Weil Representations and Harmonic Polynomials, *Invent. Math.* **44** (1978) 1-47.

[48] S. Choi, V. Chernyak and S. Mukamel, Mechanical response functions of finite-temperature Bose-Einstein condensates, *Phys. Rev. A* **67** (2003) 043602.

[49] R.A. Brandt and O.W. Greenberg, Generalized Bose Operators in the Fock Space of a Single Bose Operators, *J. Math. Phys.* **10** (1969) 1168-1176.

[50] H. Fukutome, On the $SO(2N+1)$ regular representation of operators and wave functions of fermion many-body systems, *Progr. Theor. Phys.* **58** (1977) 1692-1708.