Extremal noncommutative black holes as dark matter furnaces

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Abstract
In this paper, we consider dark matter annihilation in the gravitational field of noncommutative black holes. Instead of a violent fate predicted in the usual Hawking radiation, we propose a thermal equilibrium state where a mildly burning black hole relic is fueled by dark matter accretion at the final stage of evaporation.

Keywords: primordial black hole, dark matter, noncommutative geometry

(Some figures may appear in colour only in the online journal)

1. Introduction

Dark matter contributes about 26.8% content of our Universe nowadays. Several pieces of evidence such as a spiral galaxy’s rotation curve suggest it is abundant in each galaxy and forms a dark matter halo that envelops the galactic disc. Consequently, dark matter particles (DMPs) may accrete and significantly affect various annihilation channels on the supermassive black hole at the center of each galaxy [1]. On the other hand, primordial black holes (PBHs) may be created in the very early Universe due to the gravitational collapse of quantum fluctuation and a certain amount of them survive to this day. Those tiny relics may possess a mass range from $10^{14}$ to $10^{23}$ kg and their possible role as a dark matter candidate have been widely discussed⁴. The fate of those tiny black holes, however, remains unclear for us unless a complete theory of quantum gravity is available. Nevertheless, the conventional thermal description of black holes is doomed to failure thanks to the unphysical prediction that the Hawking temperature is infinitely high for an infinitesimal mass. Among proposals to modify

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⁴ Readers are directed to [2, 3] for a review.
the fundamental properties of spacetime, the generalized uncertainty principle (GUP) or noncommutative geometry (NG), for instance, predicts that a black hole may stop evaporating at the Planck size. In particular, an NG inspired Schwarzschild black hole can only reach a sub-Planck temperature before it cools down to its extremal state, a remnant of Planck size \[4, 5\]. It is proposed that this extremal state is a candidate for dark matter \[6\]. In this paper, however, we consider a different scenario that dark matter annihilates in the gravitational field of noncommutative black holes. At the final stage of evaporation, we predict a thermal equilibrium state where a mildly burning black hole relic is fueled by dark matter accretion. Note that it has been discussed that the accretion of PBHs may be significant in the radiation-dominated era for some types of modified theories of gravity and the extension of black hole lifetime is also estimated \[7\]. In contrast, the PBH relics in our consideration only appear at a much later time, in which the effect of noncommutativity cannot be neglected in the process of Hawking radiation.

We remark that the effect of noncommutativity in the early Universe has been considered in various contexts; for example, via the density fluctuation and the cosmic microwave background (CMB) power spectrum \[8\]. Our study, on the other hand, aims to highlight a different aspect of NG cosmology, namely the stabilized primordial black hole relics in late time physics. Since any cosmic discovery of an imprint from Planck scale physics helps to build a complete theory of quantum gravity, it is worth investigating the warm relics predicted in our model.

This paper is organized as follows: in section 2, we review the fate of a noncommutative geometry inspired Schwarzschild (NCGS) black hole. In section 3, we calculate the dark matter distribution in the vicinity of an extremal NCGS black hole. In section 4, the accretion model for a polytropic type of dark matter is proposed. In section 5, we discuss the stable configuration of a near-extremal NCGS black hole. Lastly, we have comments and discussion in section 6.

### 2. The fate of Schwarzschild black holes in noncommutative space

For a noncommutative space, one expects its coordinates do not commute and satisfy the following relation:

\[
[x^\mu, x^\nu] = i \theta \epsilon^{\mu\nu}\rho.
\]

The formulation of coherent states in a complexified plane suggests that a position measurement gives a smearing Gaussian distribution instead of the delta function \[9\], that is

\[
\delta(\vec{r}) \rightarrow \frac{1}{(4\pi\theta)^{3/2}} \exp\left(-\frac{r^2}{4\theta}\right).
\]

Incorporating this smearing effect into mass distribution in usual General Relativity (GR), a NCGS black hole was constructed \[4\] and has the metric\(^5\)

\[
d{s}^2 = \left(1 - \frac{r_s(r)}{r}\right)dr^2 - \left(1 - \frac{r_s(r)}{r}\right)^{-1}d\rho^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2),
\]

\[
r_s(r) = \frac{4GM}{\sqrt{\pi} \gamma \left(\frac{3}{2}, \frac{r^2}{4\theta}\right)},
\]

where the lower incomplete Gamma function is defined as

\(^5\)We use \(c = \hbar = k_B = 1\) unit but keep \(G\) explicit throughout the paper.
\[ \gamma(s, x) \equiv \int_0^t dt \, t^{-1} e^{-t}. \]  

(4)

It is straightforward to show that the Schwarzschild radius \( r_g \to R_g \equiv 2GM \) is recovered as the commutative limit \( (\theta \to 0) \) is taken. The fact that a noncommutative black hole has two horizons when its mass is larger than a critical value \( M_c = 1.904 \sqrt{\theta/G} \) shows an interesting similarity to the nonextremal Reisner–Nordström black hole [10]. The thermal behavior of a NCGS black hole starts to gradually deviate from that of a Schwarzschild black hole when \( r_H \lesssim 6 \sqrt{\theta} \) with \( r_H \) being the radius of the horizon. The final stage of a Schwarzschild black hole can be violent and unpredictable according to Hawking’s relation \( T_H = 1/8\pi GM \). It is more likely that Einstein’s theory of general relativity is replaced by a UV finite theory of quantum gravity at the Planck scale. Though a fully comprehensive theory of quantum gravity is still unavailable, its effect on spacetime may still be captured by some effective theories. A hand-waving argument by GUP predicts that the evaporation can stop at a finite but relatively high temperature [11, 12]. In contrast, a NCGS black hole cools down and reaches its extremal state with a critical mass \( M = M_c \). In this paper, we argue that these cold relics may not stay completely quiet since they can trap the surrounding abundant dark matter particles and therefore become warmed up by the accretion.

3. Dark matter particle phase-space distribution

In this section, we would like to derive dark matter phase-space distribution in the vicinity of an extremal NCGS black hole. The distribution around the Schwarzschild black hole has been discussed in [1] and here we generalize it to the case of extremal NCGS black holes. We will assume those are non-interacting and non-relativistic DMPs and their speeds are estimated to be a few hundred kilometers per second in the Galaxies. The trajectory of a DMP with mass \( m \) influenced by the gravitational field of a NCGS black hole can be summarized in the following equations:

\[ t = \frac{E_0}{m} \int \frac{dr}{(1 - \frac{r_g}{r}) \sqrt{(\frac{g_{tt}}{m^2})^2 - (1 - \frac{r_g}{r})(1 + \frac{L^2}{mr^2})}}, \]

\[ \phi = \int \frac{L \, dr}{r^2 \sqrt{E_0^2 - (m^2 + \frac{L^2}{r^2})(1 - \frac{r_g}{r})}}, \]

(5)

where the conserved quantities such as total energy \( E_0 \simeq m \) and angular momentum \( L \) are defined as

\[ E_0 = \left( 1 - \frac{r_g}{r} \right) ml, \quad L = mr^2 \dot{\phi}. \]

(6)

We remark the dot derivative is with respect to proper time. The radial speed component \( v_r \) and tangent one \( v_t \) can be calculated from (5):

\[ v_r = \frac{-g_{rr}}{g_{tt}} \frac{dr}{dt} = \sqrt{\frac{r_g}{r} - \frac{\alpha^2}{r^2}(1 - \frac{r_g}{r})}, \]

\[ v_t = \frac{-g_{tt}}{g_{tt}} \frac{d\phi}{dr} = \frac{\alpha}{r} \sqrt{1 - \frac{r_g}{r}}. \]

(7)
where we have defined $\alpha \equiv L/m$ for convenience, and DMP is also assumed to be non-relativistic. Here we define flux as the number of particles crossing a sphere of fixed radius $r$ in unit proper time and the unit solid angle, that is
\[
d F = 4\pi r^2 N v \cos \theta d\tau d\Omega,
\]
where $N$ is the particle number density. After substituting
\[
\cos \theta d\Omega = \frac{\pi}{r g} \left( 1 - \frac{r g}{r} \right) d(\alpha^2),
\]
\[
v = \sqrt{v_r^2 + v_t^2} = \frac{r g}{r},
\]
one obtains
\[
d F = 4\pi^2 N \frac{(r - r g)^{3/2}}{\sqrt{r g}} d(\alpha^2) d\tau.
\]
Assume that our DMP detector locates at the far distance $r_\infty$ away from the black hole center where the space is asymptotically flat. Then this flux becomes
\[
d F_\infty = \pi \frac{n_\infty}{v_\infty} d(\alpha^2) d\tau,
\]
in which the number density per radius $n_\infty = \int d\Omega N$ and DMP speed $v_\infty$ found at $r_\infty$. To proceed, we will further assume the flux per angular momentum and per time $\frac{df}{d(\alpha^2) dt}$ remains constant at an arbitrary distance. This determines the DMP density distribution per solid angle
\[
N = \frac{n_\infty}{4\pi v_\infty} \frac{\sqrt{r g}}{(r - r g)^{3/2}},
\]
for each given $n_\infty$ and $v_\infty$ found at our detector, as plotted in figure 1. We remark that from (7) only those DMPs with the angular momentum parameter $\alpha < 2.408$ can reach the horizon and be caught by the extremal NCGS black hole.

4. Accretion versus radiation

Now one considers the model of dark matter accreted to the NCGS black hole. The accretion model was first derived in [13] and applied to many situations including supermassive black holes [14, 15]. We assume the DMP behaves like an ideal fluid, i.e. $T^{\mu\nu} = (P + E) u^\mu u^\nu - P g^{\mu\nu}$, and the accretion is spherically symmetric. The conservation of mass flux and energy–momentum flux gives
\[
J^k_{\,\,0} = 0 \longrightarrow N u^1 \sqrt{-g} = c_1; \quad (13)
\]
\[
T^k_{\,\,\mu} = 0 \longrightarrow (P + E) u^1 u_0 \sqrt{-g} = c_2, \quad (14)
\]
when $u^\mu = \frac{dx^\mu}{dt}$ is the four velocity, and $x^0 = t$ and $x^1 = r$. From now on, we denote $u^1 = u$ for simplicity. Then $u_\mu u^\mu = 1$ leads to $u_0 = \frac{1}{\sqrt{g_{00} + u^2}}$ for our metric. From these relations, we obtain

\[\text{As discussed in the previous section, DMPs of smaller angular momenta are more likely to reach the horizon. On top of that, according to (7), the tangential component of the velocity becomes smaller when they get closer to the horizon. Therefore the case with spherical symmetry can capture essential features though the realistic accretion should be axisymmetric.} \]
The differential relation between \( u \) and \( r \) is captured by the (solar) wind equation:

\[
\frac{du}{u} \left[ V^2 F(r, u) - u^2 \right] - \frac{d}{dr} \left[ \frac{r g'(r)}{r} - \frac{r}{2r} g(r) - 4V^2 F(r, u) \right] = 0,
\]

where

\[
F(r, u) = 1 - \frac{r_g(r)}{r} + u^2, \quad V^2 = \frac{d \log(P + E)}{d \log N} - 1.
\]

The critical point flow (\( u \) monotonically increases or decreases along the trajectory) occurs where both bracketed factors vanish simultaneously, namely

\[
u^2_c = \frac{1}{4} \left( \frac{r_g(r_c)}{r_c} - r_g'(r_c) \right), \quad V^2 \big|_{r_c} = \frac{u^2_c}{1 - 3u^2_c + r_g'(r_c)}.
\]

It is convenient to use the indicator of phase space density \( Q \) for a self-similar radial infalling DMP [16], where \( Q \propto r^{-\beta} \). A typical \( \beta \approx 1.87 \), for example, was found by [17] for a cluster-size halo. The indicator, which is conserved during expansion of the Universe, is defined as

\[
Q = \frac{m N}{(u^2)^{3/2}}
\]
for velocity dispersion $\langle u^2 \rangle$. To be general, let us consider a polytropic gas of index $1/(\gamma - 1)$ with total energy density and pressure as follows:

$$\mathcal{E} = mN + \frac{P}{\gamma - 1}, \quad P = \frac{1}{3}mN\langle u^2 \rangle. \quad (20)$$

One can further obtain the differential:

$$\frac{d(P + \mathcal{E})}{dN} = m + \frac{5\gamma}{9(\gamma - 1)}m\left(\frac{N}{Q_N}\right)^{2/3}, \quad (21)$$

for $Q_N \equiv Q/m$. After the Taylor expansion at large $Q_N$, equation (17) becomes

$$V^2 \simeq \frac{2\gamma}{9(\gamma - 1)}\left(\frac{N}{Q_N}\right)^{2/3} - \frac{2\gamma^2}{27(\gamma - 1)^2}\left(\frac{N}{Q_N}\right)^{4/3} + \cdots. \quad (22)$$

Far away from the black hole where $u_\infty \to 0$ for $r_\infty \gg r_g$, one obtains

$$\frac{c_2}{c_1} = m + \frac{\gamma}{3(\gamma - 1)}m\left(\frac{N}{Q_N}\right)^{2/3} \quad (23)$$

and

$$\left(\frac{N_c}{N_\infty}\right)^{4/3} \simeq \frac{2(\gamma - 1)}{\gamma}\langle u^2_\infty \rangle, \quad \langle u^2_c \rangle \simeq \left(\frac{2(\gamma - 1)}{\gamma}\langle u^2_\infty \rangle\right)^{1/2}. \quad (24)$$

We remark that $r'_g(r_c)$ is negligible since it becomes exponentially small for $r \gg \sqrt{\theta}$. This is justified if $r_c$ is somehow large compared to the location of the degenerate horizon, $r_c \simeq 3.02\sqrt{\theta}$. At last, $r_c$ is solved from the first relation of (18). Again, we assume that $r'_g(r_c)$ is negligible and $r_g(r_c)$ is taken as its asymptotic value $r_g(\infty) = R_g$, and the solution is found to be

$$r_c \simeq \frac{R_g}{4}\sqrt{\frac{\gamma}{2(\gamma - 1)\langle u^2_\infty \rangle}}. \quad (25)$$

The accretion rate becomes

$$\frac{dM_{bh}}{dt} = 4\pi r_c^2 \tilde{T}_0^4 = \frac{\pi}{4}r_g(r)^2Q. \quad (26)$$

under the assumption that $\langle u^2_\infty \rangle \ll 1$ and $r'_g(r_c) \ll 1$. Taking account of Hawking radiation, one obtains the radiation rate

$$\frac{dM_{bh}}{dt} = \frac{\pi}{4}R_g^2Q - (4\pi r_\text{H}^2)\epsilon\sigma T^4, \quad (27)$$

where the Stefan–Boltzmann constant $\sigma = \frac{\pi^2}{60}$ and $\epsilon = 1$ is assumed for a perfect blackbody. $r_\text{H}$ denotes the location of the outer horizon and is the largest root of equation $r_g(r_\text{H}) = r_\text{H}$. To the leading order calculation, we can approximate $r_\text{H} \simeq R_g$. This implies a unique temperature in the thermal equilibrium

$$T_{\text{rem}} = \frac{1}{2}\left(\frac{Q}{\sigma}\right)^{1/4}. \quad (28)$$

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7 We notice a mistake in [15] for the power in the ratio $N_c/N_\infty$ due to picking up a subleading term in Taylor’s expansion.
A careful treatment with the subleading correction is given in the next section. We have the following remarks: at first, the fact that equilibrium temperature (28) is independent of \( \theta \) implies that we simply obtained the solution of Schwarzschild counterpart. Namely the difference between a noncommutative black hole and an ordinary black hole is hardly observed at this equilibrium point. Secondly, a NCGS black hole of size \( 10^{-9} \) m immersed in the dark matter halo with typical phase-space density \( Q \approx 3.51 \times 10^{-9} M_\odot \text{pc}^{-3} \text{km}^{-3} \text{s}^{-1} \) can burn at a temperature \( T_{\text{rem}} \approx 1.17 \times 10^5 \text{K} \), which will look like a hot spot in the present CMB but difficult to be identified due to its small size. This is microscopic in the context of cosmology, but huge compared to the Planck size. The total mass of this NCGS black hole is \( M = 4.82 \times 10^{25} m_p = 1.05 \times 10^{18} \text{kg} \), or equivalent \( 10^{-12} M_\odot \). This black hole has a longer life time than the age of the Universe, and in the range of proposed primordial black holes as a dark matter candidate. Finally, a conventional Schwarzschild may also reach a thermal equilibrium with surrounding dark matter at the same temperature predicted in (28), when the radiation and accretion rate are the same. However, this equilibrium is unstable for the following reason: while the mass of a black hole decreases, the accretion rate also decreases. On the other hand, the Hawking temperature increases for its negative specific heat. In contrast, the temperature of a NCGS black hole may behave like a conventional black hole at large mass, but it starts to decrease after reaching the maximum temperature \( T_{\text{max}} \approx 0.015/\sqrt{\theta} \) and then cools down as a remnant. We then expect to have an additional equilibrium point for a NCGS black hole at a much smaller mass. From the discussion above, this equilibrium is expected to be stable against evaporation. The detailed analysis will be carried out in section 5, and it can be shown that the other equilibrium takes place at \( T \approx 10^7 \text{K} \), which is very close to its extremal state. Therefore at this stable equilibrium point, the Planck-size NCGS black hole has a much cooler temperature than its Schwarzschild companion. The total flux is also much smaller compared to the previous equilibrium case.

\section*{5. Another equilibrium point near the extremal one}

In the previous section, we considered the equilibrium configuration of a PBH and a dark matter halo through Hawking radiation and accretion. In this section, we look at an equilibrium point which is very close to the extremal point of NCBHs. We will keep \( r'(r_c) \) and also the difference between \( r_H \) and \( R_g \) (the gravitational radius for the total mass), which are neglected in the previous analysis.

While re-deriving (24), we find a correction,

\begin{equation}
T_H = T_P \frac{\ell_P}{4\pi R_g} = 1.17 \times 10^5 \text{K},
\end{equation}

By use of the Planck temperature \( T_P = 1.42 \times 10^{32} \text{K} \) and the Planck length \( \ell_P = 1.62 \times 10^{-35} \text{m} \), the size of the black hole is \( R_g = 9.64 \times 10^{33} \ell_P \approx 1.56 \times 10^{-9} \text{m} \).

The phase-space density of halo is well fitted by the relation [18]

\begin{equation}
Q \approx \frac{3.51 \times 10^{-9}}{M_{11}^{1.54}} M_\odot \text{pc}^{-3} \text{km}^{-3} \text{s}^{-1},
\end{equation}

for halo mass \( M_{11} \) in the units of \( 10^{11} M_\odot \).

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Note:

8 Now we can use the formula for conventional black holes. The equilibrium temperature is

\[ T_H = T_P \frac{\ell_P}{4\pi R_g} = 1.17 \times 10^5 \text{K}. \]

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for halo mass \( M_{11} \) in the units of \( 10^{11} M_\odot \).
\[
\left( \frac{u_c}{u_{\infty}} \right)^2 \simeq \frac{2(\gamma - 1)}{\gamma} \frac{1}{1 + r'_g(r_c)} \left( u_{\infty}^2 \right) + \frac{6(\gamma - 1)^2}{\gamma^2} \left( \frac{1}{1 + r'_g(r_c)} - 1 \right) \left( u_{\infty}^2 \right)^2. \tag{31}
\]

Now \( r_c \) is solved as a solution to the following relation,
\[
\begin{align*}
\frac{R_g}{2(4u_c^2 + r'_g(r_c))} & \left[ 1 + \sqrt{1 - \frac{8\theta r'_g(r_c)}{R_g^2} (4u_c^2 + r'_g(r_c))} \right] \\
& \simeq \frac{R_g}{4u_c^2 + r'_g(r_c)} \left[ 1 - \frac{2\theta r'_g(r_c)}{R_g^2} (4u_c^2 + r'_g(r_c)) \right].
\end{align*}
\tag{32}
\]

where we have kept the first order \( O(\theta) \) in the second line. When \( r_c \) is sufficiently large compared to \( 2\sqrt{\theta} \), we may regard \( r'_g(r_c) \ll O(\theta) \), and
\[
\begin{align*}
r_c & \simeq \frac{R_g}{4u_c^2} \left[ 1 + \frac{r'_g(r_c^{(0)})}{4u_c^2} \right] \tag{33}
\end{align*}
\]

where \( r_c^{(0)} = R_g/4u_c^2 \) is the leading order solution and independent of \( \theta \). As a result, the corrected accretion rate is given by
\[
\frac{dM_{bh}}{dt} = \frac{\pi}{4} R_g^2 Q \cdot \left( 1 + \frac{r'_g(r_c^{(0)})}{4u_c^2} \right)^2 \left( 1 + \frac{1}{3} \frac{\gamma}{\gamma - 1} \left( u_{\infty}^2 \right) \right). \tag{34}
\]

Now we are ready for correction to the Hawking radiation. The location of the horizon now receives correction as follows:
\[
r_H \simeq R_g - \frac{2\theta}{R_g} r'(R_g), \tag{35}
\]

and the corrected Hawking temperature \( T_H \) is given by [4]
\[
T_H = T_F \ell_p \left[ \frac{1}{4\pi} \frac{d\theta_0}{dr} \right]_{r=r_H} = \frac{1}{4\pi r_H} \left[ 1 - R_g r_H^2 \frac{r_H^2}{2\sqrt{\gamma} \theta^{1/2}} e^{-\lambda} \right]. \tag{36}
\]

One obtains a new balance condition for radiation
\[
\frac{dM_{bh}}{dt} = \frac{\pi}{4} R_g^2 Q \cdot \left( 1 + \frac{r'_g(r_c^{(0)})}{4u_c^2} \right)^2 \left( 1 + \frac{1}{3} \frac{\gamma}{\gamma - 1} \left( u_{\infty}^2 \right) \right) - 4\pi r_H^2 \ell c T_H^4
\]
\[
= 0,
\]

and the temperature at equilibrium becomes
\[
T_H = \frac{1}{2} \left( \frac{Q}{\sigma} \right)^{1/4} \sqrt{\frac{R_g}{r_H}} \left( 1 + \frac{r'_g(r_c^{(0)})}{4u_c^2} \right)^{1/2} \left( 1 + \frac{1}{3} \frac{\gamma}{\gamma - 1} \left( u_{\infty}^2 \right) \right)^{1/4}. \tag{38}
\]

To better appreciate those corrected equations mentioned above, one should seek a black hole solution closer to the extremal state. We recall that an extremal NCBH possesses the mass \( M_{\text{ext}} = 1.904 m_p \sqrt{\theta \gamma} \) and a degenerate horizon \( r_0 = 3.02 \sqrt{\theta} \). We now examine the case in which the total mass of the NCBH is very close to \( M_{\text{ext}} \). By setting \( r_H = r_0 + \varepsilon \), the Hawking
temperature is \( T_H = T_P \tilde{\theta}^{-1/2} \left( 0.0224 - 0.00744 \tilde{\varepsilon} + O(\tilde{\varepsilon}^2) \right) \) where we have introduced dimensionless parameters \( \tilde{\varepsilon} = \varepsilon / \sqrt{\tilde{\theta}} \) and \( \tilde{\theta} = \theta \xi_\ell^{-2} \). Note that the extremal limit \( r_H \to r_0 \) (\( \varepsilon \to 0 \)) and the commutative limit \( \theta \to 0 \) do not commute. This reflects the fact that extremal NCBH can only be found in NC geometry. Using the leading order part of \( T_H \) as well as dropping the correction term for the accretion rate, the equilibrium condition (37) becomes

\[
\frac{\pi}{4} R^2 \dot{Q} = 4\pi \sigma r_0^2 \tilde{\varepsilon}^4 \frac{1}{\tilde{\theta}} \times (0.0224)^4.
\]  

(39)

If we adopt the same typical \( Q \) value in the previous section, we find the expression for total mass

\[
M = 1.111 \times 10^{51} \frac{\tilde{\varepsilon}^2}{\sqrt{\tilde{\theta}}} m_p.
\]

(40)

Under our assumption, this mass is approximately the extremal mass \( M_{\text{ext}} \) and one can derive the relation \( \tilde{\varepsilon} \simeq 4.14 \times 10^{-25} \sqrt{\tilde{\theta}} \). and the temperature for stable equilibrium is \( T_H \simeq 0.927 \times 10^{-25} T_P = 1.31 \times 10^7 \) K.

We would like to emphasize that a thermally stable black hole relic cannot exist without noncommutativity. Otherwise, accretion would be impossible due to the furious circumference heated up by a usual Schwarzschild black hole of Planck size.

6. Discussion

So far, our simple model only discussed the equilibrium between accretion of the dark matter halo and the Hawking radiation. For the stable equilibrium, the mass and size of black hole are about Planck scale, while for the unstable equilibrium (namely for the conventional Schwarzschild black hole), the mass is about \( 10^{18} \) kg (\( 10^{-6} \) of the Earth’s mass or \( 10^{-4} \) of the mass of the Moon) and the size is of order \( 10^{-9} \) m. Though the gravitational field near the horizon is not so small (about \( 10^{13} \) stronger compared to typical neutron stars in the case of unstable black holes, \( 10^{60} \) for the stable case), their total gravitational field and the flux are still quite small due to small size. Black holes of Planck size would hardly form a dark matter halo around them due to their weak gravitational field, but the stable equilibrium is still possible when they happen to locate in a dense region of DMPs. As for the unstable equilibrium, the situation is more subtle; \( 10^{-12} \) of the mass of the Sun is too light to apply the standard formula of the phase-space density distribution of the dark matter halo as this is for heavy stellar objects. We may thus need to reexamine the applicability of the accretion formula and rectify it if necessary.

One can set a lower bound for \( \theta \) as follows; note that the highest temperature which can be reached by a noncommutative black hole is given by \( T_{\text{max}} = \frac{0.015}{\sqrt{\theta}} \). Assuming the wavelength of Hawking radiation is not shorter than the Planck length, one can estimate the lower bound as \( \sqrt{\tilde{\theta}} \geq 10l_p \) [4].

The production rate for primordial black holes were considered in [19], but they mainly focused on those heavier than \( 10^{15} \) g, as the light ones have already evaporated in the current Universe. Since noncommutativity can make these light black holes survive longer and turn into stable relics, one can also have an upper bound for possible \( \theta \) if part of those black holes, lighter than \( 10^{15} \) g, were considered in the computation of the production rate.

A recent study has suggested a possible connection between the cosmic infrared background (CIB) and primordial black holes [20]. Those near-IR fluctuation of wavelength 2–5 \( \mu \text{m} \)
could have been majorly contributed from objects with temperature $580-1450\,\text{K}$ according to Wien’s displacement law. Those objects could be ordinary primordial black holes with mass among $4-11 \times 10^{-11} M_{\odot}$. However, if some of those unidentified sources were from extremal black hole remnants, they should have a larger number density but much lighter individual mass. For instance, the radiation flux from an ordinary primordial black hole of mass $M_h$ is proportional to $(2GM_h)^2/(8\pi GM_h)^{-2}$, but if the same amount mass were made of extremal NCGS black holes at the same temperature, the flux $\propto (M_h/M_{\text{ext}})(2GM_{\text{ext}})^2/(8\pi GM_h)^{-4}$, which is much weaker than the former by a factor $M_{\text{ext}}/M_h \sim 10^{-26}$. The abundance of these extremal remnants may modify our resolutions toward some of the puzzles in modern cosmology. For instance, radiation from those warm remnants may speed up local reionization of hydrogens after the Big Bang. Those remnants may also help early galaxies formation by serving as seeds. Inspection of the abovementioned phenomenological application is in progress and will be reported in a separate paper.

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