Equivalent Conditions on Periodic Feedback Stabilization for Linear Periodic Evolution Equations

Gengsheng Wang*  Yashan Xu†

Abstract

This paper studies the periodic feedback stabilization for a class of linear $T$-periodic evolution equations. Several equivalent conditions on the linear periodic feedback stabilization are obtained. These conditions are related with the following subjects: the attainable subspace of the controlled evolution equation under consideration; the unstable subspace (of the evolution equation with the null control) provided by the Kato projection; the Poincaré map associated with the evolution equation with the null control; and two unique continuation properties for the dual equations on different time horizons $[0, T]$ and $[0, n_0 T]$ (where $n_0$ is the sum of algebraic multiplicities of distinct unstable eigenvalues of the Poincaré map). It is also proved that a $T$-periodic controlled evolution equation is linear $T$-periodic feedback stabilizable if and only if it is linear $T$-periodic feedback stabilizable with respect to a finite dimensional subspace. Some applications to heat equations with time-periodic potentials are presented.

Keywords. periodic evolution equations, periodic feedback stabilization, equivalent conditions, attainable subspaces, unique continuation properties, the Poincaré map, the Kato projection

2010 MSC. 34H15 49N20

1 Introduction

1.1 The problem and the motivation

Consider the following controlled evolution equation:

$$y'(t) + Ay(t) + B(t)y(t) = D(t)u(t) \text{ in } \mathbb{R}^+ \triangleq [0, \infty).$$

Here and throughout this paper, we make the following assumptions.

(H1) The operator $(-A)$, with its domain $\mathcal{D}(-A)$, generates a $C_0$ compact semigroup $\{S(t)\}_{t \geq 0}$ in a real Hilbert space $H$ (identified with its dual) with its norm and inner product denoted by $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$, respectively.

*School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, China. (wanggs62@yeah.net) The author was partially supported by the National Natural Science Foundation of China under grant 11161130003.

†School of Mathematical Sciences, Fudan University, KLMNS, Shanghai 200433, China. (yashanxu@fudan.edu.cn)
(H2) The operator-valued function $B(\cdot) \in L^1_{loc}(\mathbb{R}^+; \mathcal{L}(H))$ is $T$-periodic, i.e., $B(t + T) = B(t)$ for a.e. $t \in \mathbb{R}^+$, where $T > 0$ and $\mathcal{L}(H)$ denotes the space of all linear bounded operators on $H$.

(H3) The operator-valued function $D(\cdot) \in L^\infty(\mathbb{R}^+; \mathcal{L}(U,H))$ is $T$-periodic. Here $U$ is also a real Hilbert space (identified with its dual) with its norm and inner product denoted by $\| \cdot \|_U$ and $(\cdot, \cdot)_{U}$, respectively; and $\mathcal{L}(U,H)$ stands for the space of all linear bounded operators from $U$ to $H$. Controls $u(\cdot)$ are taken from the space $L^2(\mathbb{R}^+; U)$.

For each $h \in H$, $s \geq 0$ and $u(\cdot) \in L^2(\mathbb{R}^+; U)$, Equation (1.1) (over $[s, \infty)$) with the initial condition that $y(s) = h$ has a unique mild solution $y(\cdot; s, h, u) \in C([s, \infty); H)$. (See, for instance, Proposition 5.3 on Page 66 in [11].) The following definitions about the periodic feedback stabilization will be used throughout this paper:

- Equation (1.1) is said to be linear periodic feedback stabilizable (LPFS, for short) if there is a $T$-periodic $K(\cdot) \in L^\infty(\mathbb{R}^+; \mathcal{L}(H,U))$ such that the feedback equation
  \[
  y'(t) + Ay(t) + B(t)y(t) = D(t)K(t)y(t) \quad \text{in } \mathbb{R}^+ \tag{1.2}
  \]
  is exponentially stable, i.e., there are two positive constants $M$ and $\delta$ such that for each $h \in H$, the solution $y_K(\cdot; 0, h)$ to the equation (1.2) with the initial condition that $y(0) = h$ satisfies that $\|y_K(t; 0, h)\| \leq M e^{-\delta t} \|h\|$ for all $t \geq 0$. Any such a $K(\cdot)$ is called an LPFS law for Equation (1.1).

- Equation (1.1) is said to be LPFS with respect to a subspace $Z$ of $U$ if there is a $T$-periodic $K(\cdot) \in L^\infty(\mathbb{R}^+; \mathcal{L}(H,Z))$ such that the equation (1.2) is exponentially stable. Any such a $K(\cdot)$ is called an LPFS law for Equation (1.1) with respect to $Z$.

Let
\[
U^{FS} \triangleq \{ Z \mid Z \text{ is a subspace of } U \text{ s.t. Equation (1.1) is LPFS w.r.t. } Z \}. \tag{1.3}
\]

In this paper, we provide three criteria for judging whether a subspace $Z$ belongs to $U^{FS}$. We also show that if $U \in U^{FS}$, then there is a finite dimensional subspace $Z$ in $U^{FS}$. The aforementioned three criteria are related with the following subjects: the attainable subspace of (1.1); the unstable subspace (of (1.1) with the null control) provided by the Kato projection; the Poincaré map associated to (1.1) with the null control; and two unique continuation properties for the dual equations of (1.1) (with the null control) on different time horizons $[0, T]$ and $[0, n_0 T]$ (where $n_0$ is the sum of algebraic multiplicities of distinct unstable eigenvalues of the Poincaré map).

Among three criteria, the most important one is a geometric condition connecting the attainable set with the unstable subspace of the system (1.1); while the other two are analytic conditions related with the unique continuation of the dual equations of (1.1) over different time horizons and with initial data in different finite dimensional subspaces of $H$.

The motivation for this work is as follows. First, the equation (1.1) with the null control is exponentially stable if and only if the spectrum of the Poincaré map associated with the system is contained in the unit open ball of the complex plane. (This can be proved by the exactly
same way to show Corollary 7.2.4 on page 200, [8].) Thus, it is a natural problem to explore equivalent conditions on the periodic stabilization for a linear periodic controlled evolution system. Second, there are two important kinds of solutions for evolution equations: equilibrium and periodic solutions. The stabilization for equilibrium solutions of time-invariant systems has been extensively studied (see for instance [1], [5], [19], [20] and the references therein). However, the understanding on the periodic stabilization of periodic solutions for time-varying evolution systems is quite limited. (See [2], [13], [14] and [18]. Here, we would like to mention [3] which establishes a feedback law stabilizing a smooth non-stationary solutions, for instance, around a periodic trajectory, for Navier-Stokes equations.) Finally, when the system (1.1) is LPFS, it should be important and interesting to answer if there is a finite dimensional subspace $Z$ of $U$ such that (1.1) is LPFS w.r.t. $Z$, from perspectives of both mathematics and applied sciences.

### 1.2 Main results

Before stating our main results, we give some preliminaries in order:

**I) Notations**  We will use $\|\cdot\|$ to denote the usual norm of $\mathcal{L}(H)$ when there is no risk of causing any confusion. Given $L \in \mathcal{L}(X,Y)$ (where $X$ and $Y$ are two Hilbert spaces), we write $L^*$ for its adjoint operator. For $L \in \mathcal{L}(X) \triangleq \mathcal{L}(X,X)$, we denote by $\sigma(L)$ the spectrum of $L$. When $X$ is a real Hilbert space and $L \in \mathcal{L}(H)$, we denote by $X^C$ and $L^C$ their complexification, respectively, i.e., $X^C = X + iX$ and $L^C(\alpha + i\beta) = L\alpha + iL\beta$ for any $\alpha, \beta \in X$, where $i$ is the imaginary unit. We write $\mathbb{B}$ for the open unit ball in $\mathbb{C}$ and $\mathbb{B}(0,\delta)$ for the open ball in $\mathbb{C}$, centered at the origin and of radius $\delta > 0$. Denote by $\partial \mathbb{B}(0,\delta)$ the boundary of $\mathbb{B}(0,\delta)$.

**II) The Poincaré map**  Let $\{\Phi(t,s)\}_{0 \leq s \leq t < +\infty}$ be the evolution system generated by $(-A - B(\cdot))$. It follows from Lemma 5.6 in [11] (see Page 68, [11]) that $\Phi(t,s)$ is strongly continuous over $\{(t,s) \in \mathbb{R}^+ \times \mathbb{R}^+ | 0 \leq s \leq t < \infty\}$, and that

$$\Phi(t,s)h = S(t-s)h + \int_s^t S(t-r)B(r)\Phi(r,s)hdr, \quad \text{when } 0 \leq s \leq t < \infty \text{ and } h \in H. \quad (1.4)$$

By Proposition 5.7 on Page 69 in [11], for each $s \geq 0$, $h \in H$ and $u(\cdot) \in L^2(\mathbb{R}^+;U)$, it holds that

$$y(t; s, h, u) = \Phi(t,s)h + \int_s^t \Phi(t,r)D(r)u(r)dr, \quad s \in [t, \infty). \quad (1.5)$$

By the $T$-periodicity of $B(\cdot)$ and (1.4), one can easily check that $\Phi(\cdot, \cdot)$ is $T$-periodic, i.e.,

$$\Phi(t + T, s + T) = \Phi(t, s) \quad \text{for all } 0 \leq s \leq t < \infty. \quad (1.6)$$

Now, we introduce the following Poincaré map (see Page 197, [8]):

$$\mathcal{P}(t) \triangleq \Phi(t + T, t), \quad t \in \mathbb{R}^+. \quad (1.7)$$

It is proved that (see Lemma 2.1)

$$\sigma(\mathcal{P}(t)^C) \setminus \{0\} = \{\lambda_j\}_{j=1}^\infty \text{ for each } t \geq 0, \quad (1.8)$$
where $\lambda_j$, $j = 1, 2, \ldots$, are all distinct non-zero eigenvalues of the compact operator $\mathcal{P}(0)^C$ such that $\lim_{j \to \infty} |\lambda_j| = 0$. Thus, there is a unique $n \in \mathbb{N}$ such that

$$|\lambda_j| \geq 1, \quad j \in \{1, 2, \cdots, n\} \quad \text{and} \quad |\lambda_j| < 1, \quad j \in \{n + 1, n + 2, \cdots\}. \quad (1.9)$$

Set

$$\tilde{\delta} \triangleq \max\{|\lambda_j|, j > n\} < 1. \quad (1.10)$$

Let $l_j$ be the algebraic multiplicity of $\lambda_j$ for each $j \in \mathbb{N}$, and write

$$n_0 \triangleq l_1 + \cdots + l_n. \quad (1.11)$$

(iii) The Kato projection

Arbitrarily fix a $\delta \in (\tilde{\delta}, 1)$, where $\tilde{\delta}$ is given by (1.10). Let $\Gamma$ be the circle $\partial \mathbb{B}(0, \delta)$ with the clockwise direction in $\mathbb{C}^1$. We introduce the Kato projections (see [9]):

$$\hat{P}(t) = \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - \mathcal{P}(t)^C)^{-1} d\lambda, \quad t \geq 0. \quad (1.12)$$

It is proved that (see Lemma 2.2) for each $t \geq 0$, the operator $P(t)$, defined by

$$P(t) \triangleq \hat{P}(t) \big|_H$$

is a projection on $H$; $H = H_1(t) \bigoplus H_2(t) \triangleq \mathcal{P}(t)H \bigoplus (I - \mathcal{P}(t))H$ for each $t \geq 0$; both $H_1(t)$ and $H_2(t)$ are invariant subspaces of $\mathcal{P}(t)$; $\sigma(\mathcal{P}(t)^C|_{H_1(t)^C}) = \{\lambda_j\}_{j=1}^n$, $\sigma(\mathcal{P}(t)^C|_{H_2(t)^C}) \setminus \{0\} = \{\lambda_j\}_{j=n+1}^\infty$; and $\dim H_1(t) = n_0$. It is also shown that (see Lemma 2.2) $P(\cdot)$ is $T$-periodic. We simply write

$$H_1 \triangleq H_1(0), \quad H_2 \triangleq H_2(0), \quad P \triangleq P(0) \quad \text{and} \quad \mathcal{P} \triangleq \mathcal{P}(0). \quad (1.14)$$

The subspaces $H_1$ and $H_2$ are respectively called the unstable subspace and the stable subspace of Equation (1.1) with the null control. Each eigenvalue in $\{\lambda_j\}_{j=1}^n$ (or in $\{\lambda_j\}_{j=n+1}^\infty$) is called an unstable (or stable) eigenvalue of $\mathcal{P}^C$. Each eigenfunction of $\mathcal{P}^C$ corresponding to an unstable (or stable) eigenvalue is called an unstable (or stable) eigenfunction of $\mathcal{P}^C$.

(iv) Attainable subspaces

For each subspace $Z \subseteq U$, we let

$$V_k^Z \triangleq \left\{ \int_0^{kT} \Phi(kT, s)D(s)u(s)ds \bigg| u(\cdot) \in L^2(\mathbb{R}^+; Z) \right\} \quad \text{for all} \quad k \in \mathbb{N}. \quad (1.15)$$

The space $V_k^Z$ is called the attainable subspace of Equation (1.1) (over $(0, kT)$) w.r.t. $Z$. Let

$$\hat{V}_k^Z = PV_k^Z, \quad k \in \mathbb{N}, \quad (1.16)$$

where $P$ is given by (1.14).

Now the main results of this paper are presented by the following two theorems:

**Theorem 1.1** Let $P$, $\mathcal{P}$ and $H_j$ with $j = 1, 2$ be given by (1.14). Let $n_0$ be given by (1.11). Then, for each subspace $Z \subseteq U$, the following statements are equivalent:
(a) Equation (1.1) is LPFS with respect to \( Z \), i.e., \( Z \in \mathcal{U}^F \).
(b) The subspace \( Z \) satisfies
\[
\dot{V}^Z_{n_0} = H_1, \quad \text{where } \dot{V}^Z_{n_0} \text{ is given by (1.16).}
\] (1.17)
(c) The subspace \( Z \) satisfies
\[
\xi \in P^*H_1 \text{ and } (D(\cdot)|_Z)^* \Phi(n_0T, \cdot)^* \xi = 0 \text{ over } (0, n_0T) \Rightarrow \xi = 0.
\] (1.18)
(d) The subspace \( Z \) satisfies
\[
\mu \notin \mathbb{B}, \ \xi \in H^C, \ (\mu I - P^*C)\xi = 0, \ \text{and } (D(\cdot)|_Z)^*C \Phi(T, \cdot)^*C \xi = 0 \text{ over } (0, T) \Rightarrow \xi = 0.
\] (1.19)

**Theorem 1.2** Equation (1.1) is LPFS if and only if it is LPFS with respect to a finite dimensional subspace \( Z \) of \( U \).

It is worthwhile to make the following remarks:

- The key to show the above two theorems is to build up the equivalence \((a) \Leftrightarrow (b)\) in Theorem 1.1.

- The functions \( \Phi(n_0T, \cdot)^* \xi \) with \( \xi \in H \) and \( \Phi(T, \cdot)^*C \xi \) with \( \xi \in H^C \) are respectively the solutions to the following dual equations:

\[
\psi_1(t) - A^*\psi(t) - B(t)^*\psi(t) = 0 \text{ for a.e. } t \in (0, n_0T), \ \psi(n_0T) = \xi
\]
and
\[
\psi_1(t) - A^{*C}\psi(t) - B(t)^{*C}\psi(t) = 0 \text{ for a.e. } t \in (0, T), \ \psi(T) = \xi.
\]

Thus, the condition (c) in Theorem 1.1 presents a unique continuation property for solutions of the first dual equation with initial data in \( P^*H_1 \); while the condition (d) in Theorem 1.1 presents a unique continuation property for solutions of the second equation where initial data are unstable eigenfunctions of \( P^*C \).

- There have been studies, in the past, on equivalence conditions of periodic feedback stabilization for linear periodic evolution systems. In [13] and [18], the authors established an equivalent condition on stabilizability for linear time-periodic parabolic equations with open-loop controls. Their equivalence (see Theorem 3.1 in [13] and Proposition 3.1 in [18]) can be stated, under our framework, as follows: the condition (d) (in our Theorem 1.1 where \( Z = U \)) is equivalent to the statement that for any \( h \in H \), there is a control \( u^h(\cdot) \in C(\mathbb{R}^+; U) \), with \( \sup_{t \in \mathbb{R}^+} ||e^{\delta t}u^h(t)||_U \) bounded (where \( \delta \) is given by (1.10)), such that the solution \( y(\cdot; 0, h, u^h) \) is stable. Meanwhile, it was pointed out in [13] (see the paragraph before the last one in Section 1 in [13]) that when open-looped stabilization controls exist, one can construct a periodic feedback stabilization law through using a method provided in [17]. From this point of view, the equivalence \((a) \Leftrightarrow (d)\) in Theorem 1.1 is not new, though our way to approach the equivalence differs from those in [13] and [18] and our method to construct the stabilization feedback law is different from that in [17].
• To the best of our knowledge, both Theorem 1.2 and the equivalences: 
(a) ⇔ (b) and (a) ⇔ (c) in Theorem 1.1 appear to be new. It is worth mentioning that the equivalence (a) ⇔ (b) in Theorem 1.1 is an extension of a result in our previous paper [22] which studies the stabilization of finite-dimensional periodic systems.

• A byproduct of this study (see Proposition 3.3, and Remark 3.1) shows that when both $B(\cdot)$ and $D(\cdot)$ are time-invariant, linear time-period functions $K(\cdot)$ will not aid the linear stabilization of Equation (1.1), i.e., Equation (1.1) is linear $\hat{T}$-periodic feedback stabilizable for some $\hat{T} > 0$ if and only if Equation (1.1) is linear time-invariant feedback stabilizable. On the other hand, when Equation (1.1) is periodic time-varying, linear time-periodic $K(\cdot)$ do aid in the linear stabilization of this equation.

The rest of this paper is organized as follows. Section 2 provides some properties on Poincaré map, Kato projection and attainable subspaces. Section 3 studies the multi-periodic feedback stabilization. Section 4 proves Theorem 1.1 and 1.2. Section 5 presents some applications of the main theorems to internally controlled heat equations with time-periodic potentials.

2 Poincaré map, Kato projection and attainable subspaces

In this section, we will first present three lemmas (Lemma 2.2, 2.3, 2.4) which show certain properties on the Poincaré map and the Kato projection. (These lemmas are slightly different versions of the existing results. For the sake of the completeness of the paper, we provide their detailed proof in Appendix.) Then, we show certain properties on attainable subspaces $V_k^Z$ defined by (1.15).

The first one is another version of Lemma 7.2.2 in [8] where $B(\cdot)$ is assumed to be $T$-periodic and Hölder continuous (see Page 197, [8]); while in our case, $B(\cdot) \in L^1(\mathbb{R}^+; \mathcal{L}(H))$ is $T$-periodic.

Lemma 2.1 Let $\mathcal{P}(\cdot)$ be defined by (1.7). Then $\sigma(\mathcal{P}(t)^C) \setminus \{0\}$ is independent of $t \in \mathbb{R}^+$. Moreover, $\sigma(\mathcal{P}(t)^C) \setminus \{0\}$ consists entirely of distinct eigenvalues $\{\lambda_j\}_{j=1}^\infty$ (of $\mathcal{P}(0)$) with $\lim_{j \to \infty} |\lambda_j| = 0$.

The second one is another version of Theorem 7.2.3 in [8] where complex case is studied (see Page 198, [8]); while in following lemma, we consider the real case.

Lemma 2.2 Let $\mathcal{P}(\cdot)$ and $\mathcal{P}(\cdot)$ be defined by (1.7) and (1.13) respectively. Then each $P(t)$ (with $t \geq 0$) is a projection on $H$ such that

$$H = H_1(t) \bigoplus H_2(t),$$

where

$$H_1(t) \overset{\Delta}{=} P(t)H \quad \text{and} \quad H_2(t) \overset{\Delta}{=} (I - P(t))H.$$ (2.1)

Moreover, $P(\cdot)$, $H_1(\cdot)$ and $H_2(\cdot)$ have the following properties:

(a) $P(\cdot)$, $H_1(\cdot)$ and $H_2(\cdot)$ are $T$-periodic.

(b) For each $t \geq 0$, both $H_1(t)$ and $H_2(t)$ are invariant subspaces of $\mathcal{P}(t)$. 

6
(c) If \( \{\lambda_j\}_{j=1}^{\infty} \), \( n \) and \( n_0 \) are given by (1.8), (1.9) and (1.11), then
\[
\sigma(P(t)^C|_{H(t)c}) = \{\lambda_j\}_{j=1}^n, \quad \sigma(P(t)^C|_{H_2(t)c}) \setminus \{0\} = \{\lambda_j\}_{j=n+1}^{\infty}; \tag{2.3}
\]
\[
dim H_1(t) = n_0. \tag{2.4}
\]

(d) When \( 0 \leq s \leq t < \infty \), \( \Phi(t,s) \in \mathcal{L}(H_j(s), H_j(t)) \) with \( j = 1, 2 \).

(e) It holds that
\[
\Phi(t,s)P(s) = P(t)\Phi(t,s), \text{ when } 0 \leq s \leq t. \tag{2.5}
\]

(f) Let \( \tilde{\rho} \overset{\triangle}{=} (-\ln \delta)/T > 0 \) with \( \delta \) given by (1.10). For any \( \rho \in (0, \tilde{\rho}) \), there is a positive constant \( C_\rho \) such that
\[
\|\Phi(t,s)h_2\| \leq C_\rho e^{-\rho(t-s)}\|h_2\|, \text{ when } 0 \leq s \leq t < \infty \text{ and } h_2 \in H_2(s). \tag{2.6}
\]

The third one is essentially another version of Theorem 6.22 in [9] (see Page 184, [9]). To state it, we recall that \( P^* \) and \( P^* \) are the adjoint operators of \( P \) and \( P \). It is clear that
\[
\sigma(P^C) = \overline{\sigma(P^C)}. \tag{2.7}
\]

Since \( \sigma(P^C) \setminus \{0\} = \{\lambda_j\}_{j=1}^{\infty} \) (see (1.8)), it holds that \( \sigma(P^*C) \setminus \{0\} = \{\tilde{\lambda}_j\}_{j=1}^{\infty} \). Write \( \tilde{l}_j \) for the algebraic multiplicity of \( \tilde{\lambda}_j \) w.r.t. \( P^*C \). It is clear that
\[
\tilde{l}_j = l_j \text{ for all } j, \quad \text{and } \tilde{l}_1 + \cdots + \tilde{l}_n = n_0, \tag{2.8}
\]
where \( l_j \) is the algebraic multiplicity of \( \lambda_j \) w.r.t. \( P^C \); \( n \) and \( n_0 \) are given by (1.9) and (1.11) respectively. Let \( \Gamma \) be the circle used to defined \( \hat{P} \overset{\triangle}{=} \hat{P}(0) \) (see (1.12)). Define the Kato projection with respect to \( P^*C \) as follows:
\[
\hat{P} = \frac{1}{2\pi i} \int_{\Gamma}(\lambda I - P^*C)^{-1}d\lambda. \tag{2.9}
\]

From Theorem 6.17 on Page 178 in [9], it follows that
\[
H^C = \hat{H}_1 \bigoplus \hat{H}_2 \overset{\triangle}{=} \hat{P}H^C \bigoplus (I - \hat{P})H^C \tag{2.10}
\]
and
\[
\text{both } \hat{H}_1 \text{ and } \hat{H}_2 \text{ are invariant w.r.t. } P^*C. \tag{2.11}
\]

**Lemma 2.3** Let \( \hat{P} \) be defined by (2.9). Then, \( \hat{P} \overset{\triangle}{=} \hat{P}|_H \) is a projection on \( H \); \( H = \hat{H}_1 \bigoplus \hat{H}_2 \), where \( \hat{H}_1 \overset{\triangle}{=} \hat{P}H \) and \( \hat{H}_2 \overset{\triangle}{=} (I - \hat{P})H \); \( P^*H_1 \subseteq \hat{H}_1 \); \( \sigma(P^*C|_{\hat{H}_1^C}) = \{\lambda_j\}_{j=1}^{n} \) and \( \sigma(P^*C|_{\hat{H}_2^C}) \subseteq \mathbb{B} \); and \( \dim \hat{H}_1 = n_0 \). It further holds that
\[
\hat{P} = P^*, \tag{2.12}
\]
\[
\hat{H}_1 = P^*H = P^*H_1, \text{ where } H_1 \text{ is given by (1.14)}; \tag{2.13}
\]
\[
\xi \in \hat{H}_1^C, \text{ when } \mu \in \sigma(P^*C) \setminus \mathbb{B} \text{ and } (\mu I - P^*C)\xi = 0. \tag{2.14}
\]
Next, we will introduce certain properties on attainable subspaces $V_k^Z$, $k \in \mathbb{N}$. They will play important roles in the proof of our main theorems. We start with recalling (2.1) and (1.15). Since $H_1$ is invariant w.r.t. $\mathcal{P}$ (see Part (b) of Lemma 2.2), we can define $\mathcal{P}_1 : H_1 \rightarrow H_1$ by setting

$$\mathcal{P}_1 \triangleq \mathcal{P} |_{H_1}.$$  

(2.15)

By (2.3), it holds that

$$\sigma(\mathcal{P}_1) \cap \mathbb{N} = \emptyset.$$  

(2.16)

**Lemma 2.4** Let $\mathcal{P}_1$ and $n_0$ be given by (2.15) and (1.11), respectively. Suppose that $Z \subseteq U$ is a subspace with $V_k^Z$ and $\hat{V}_k^Z$ given by (1.15) and (1.16), respectively. Then for each $k \in \mathbb{N},$

$$V_k^Z = V_1^Z + \mathcal{P}V_1^Z + \cdots + \mathcal{P}^{k-1}V_1^Z; \quad \hat{V}_k^Z = \hat{V}_1^Z + \mathcal{P}_1\hat{V}_1^Z + \cdots + \mathcal{P}_{k-1}^1\hat{V}_1^Z. \tag{2.17}$$

Furthermore, $\mathcal{P}_1$ is invertible and it holds that

$$\hat{V}_k^Z = \hat{V}_{n_0}^Z; \quad \mathcal{P}_1\hat{V}_1^Z = \hat{V}_k^Z = \mathcal{P}_1^{-1}\hat{V}_1^Z,$$  

where

$$\hat{V}_k^Z \triangleq \bigcup_{k=1}^{\infty} \hat{V}_k^Z. \tag{2.18}$$

Proof. We begin with proving the first equality in (2.17) by the mathematical induction. Clearly, it stands when $k = 1$. Assume that it holds in the case when $k = k_0$ for some $k_0 \geq 1$, i.e.,

$$V_{k_0}^Z = V_1^Z + \mathcal{P}V_1^Z + \cdots + \mathcal{P}^{k_0-1}V_1^Z. \tag{2.20}$$

Because of (1.6) and (1.14), we have that $\Phi((k_0 + 1)T, T) = \Phi(T, 0)^{k_0} = \mathcal{P}^{k_0}$. This, along with (1.15), the $T$-periodicity of $D(\cdot)$ and (2.20), indicates that

$$V_{k_0+1}^Z = \left\{ \int_0^{(k_0+1)T} \Phi((k_0 + 1)T, s)D(s)u(s)ds \mid u(\cdot) \in L^2(\mathbb{R}^+; Z) \right\}$$

$$= \left\{ \Phi((k_0 + 1)T, T) \int_0^T \Phi(T, s)D(s)u(s)ds \mid u(\cdot) \in L^2(\mathbb{R}^+; Z) \right\}$$

$$+ \left\{ \int_0^{k_0T} \Phi(k_0T, s + T)D(s + T)u(s + T)ds \mid u(\cdot) \in L^2(\mathbb{R}^+; Z) \right\}$$

$$= \mathcal{P}^{k_0}V_1^Z + \left\{ \int_0^{k_0T} \Phi(k_0T, s)D(s)u(s + T)ds \mid u(\cdot) \in L^2(\mathbb{R}^+; Z) \right\}$$

$$= \mathcal{P}^{k_0}V_1^Z + V_{k_0}^Z = V_1^Z + \mathcal{P}V_1^Z + \cdots + \mathcal{P}^{k_0}V_1^Z,$$

which leads to the first equality in (2.17).

We next show the second equality in (2.17). By (1.14) and (2.5) with $t = T$ and $s = 0$, we have

$$\mathcal{P}P = PP.$$  

(2.21)

Since $P$ is a projection from $H$ onto $H_1$ (see Lemma 2.2), the second equality in (2.17) follows from the first one in (2.17), (1.16) and (2.21).
Then we show the first equality in (2.18). It follows respectively from (2.19) and (2.17) that

$$\hat{V}_{n_0}^Z \subseteq \hat{V}^Z$$ and $$\hat{V}_k^Z \subseteq \hat{V}_{n_0}^Z$$, when $$k \leq n_0$$. \hspace{1cm} (2.22)

Since $$\dim H_1 = n_0$$ (see (2.4)) and $$P_1 : H_1 \rightarrow H_1$$ (see (2.15)), according to the Hamilton-Cayley theorem, each $$P_j^1$$ with $$j \geq n_0$$ is a linear combination of $$\{ I, P_1^1, P_1^2, \ldots, P_1^{(n_0-1)} \}$$. This, along with the second equality in (2.17), indicates that

$$\hat{V}_k^Z = \sum_{j=0}^{k-1} P_j^1(\hat{V}_1^Z) \subseteq \sum_{j=0}^{n_0-1} P_j^1(\hat{V}_1^Z) = \hat{V}_{n_0}^Z$$, when $$k \geq n_0$$. \hspace{1cm} (2.23)

Now the first equality in (2.18) follows from (2.22) and (2.23).

Finally, we show the non-singularity of $$P_1$$ and the second equality in (2.18). By the first equality in (2.18) and the Hamilton-Cayley theorem, we see that

$$P_1 \hat{V}^Z = P_1 \hat{V}_{n_0}^Z = P_1 \sum_{j=0}^{n_0-1} P_j^1(\hat{V}_1^Z) = \sum_{j=1}^{n_0} P_j^1(\hat{V}_1^Z) \subseteq \sum_{j=0}^{n_0-1} P_j^1(\hat{V}_1^Z) = \hat{V}_{n_0}^Z,$$

from which, it follows that

$$P_1 \hat{V}^Z \subseteq \hat{V}^Z. \hspace{1cm} (2.24)$$

Because $$0 \notin \sigma(P_1^C)$$ (see (2.3) as well as (1.9)), and since the domain of $$P_1^C, H_1^C$$ is a finite dimensional subspace, the operator $$P_1^C$$ is invertible. Hence, the operator $$P_1$$ is also invertible. This implies that $$\dim(P_1 \hat{V}^Z) = \dim \hat{V}^Z$$, which, together with (2.24), yields that $$P_1 \hat{V}^Z = \hat{V}^Z$$. This completes the proof.

**Lemma 2.5** Let $$n_0$$ be given by (1.11). Then, for each subspace $$Z$$ of $$U$$, there is a finite dimension subspace of $$\hat{Z}$$ of $$Z$$ such that

$$\hat{V}_{n_0}^Z = \hat{V}_{n_0}^Z, \hspace{1cm} (2.25)$$

where $$\hat{V}_{n_0}^Z$$ and $$\hat{V}_{n_0}^Z$$ are defined by (1.16).

**Proof.** Let $$Z$$ be a subspace of $$U$$. Since $$\hat{V}_{n_0}^Z$$ is a subspace of $$H_1$$ and $$\dim H_1 = n_0 < \infty$$ (see (2.4), we can assume that $$\dim \hat{V}_{n_0}^Z = m \leq n_0$$. Write $$\{\xi_1, \ldots, \xi_m\}$$ for an orthonormal basis of $$\hat{V}_{n_0}^Z$$. By the definition of $$\hat{V}_{n_0}^Z$$ (see (1.16), as well as (1.15)), there are $$u_j(\cdot) \in L^2(\mathbb{R}^+; Z), j = 1, \ldots, m$$, such that

$$\int_0^{n_0 T} P\Phi(n_0 T, s) D(s) u_j(s) ds = \xi_j \hspace{0.5cm} \text{for all } j = 1, \ldots, m. \hspace{1cm} (2.26)$$

By the boundedness of $$\Phi(n_0 T, \cdot)$$ and $$D(\cdot)$$ over $$[0, n_0 T]$$, there is a constant $$C > 0$$ such that

$$\|P\Phi(n_0 T, \cdot) D(\cdot)\|_{L^\infty([0, n_0 T]; C(U; H))} \leq C. \hspace{1cm} (2.27)$$

Let $$\hat{\varepsilon} > 0$$ small enough to satisfy

$$(1 - \hat{\varepsilon})^2 - (2\hat{\varepsilon} + \hat{\varepsilon}^2)(m - 1) > 0. \hspace{1cm} (2.28)$$
By the definition of the Bochner integration (see [4]), there are simple functions
\[ v_j(\cdot) = \sum_{l=1}^{k_j} \chi_{E_{jl}}(\cdot)z_{jl} \text{ over } (0, n_0 T), \quad j = 1, \ldots, m, \] (2.29)
with \( z_{jl} \in Z, E_{jl} \) measurable sets in \((0, n_0 T)\) and \( \chi_{E_{jl}} \) the characteristic function of \( E_{jl} \), such that
\[ \int_0^{n_0 T} \| u_j(s) - v_j(s) \|_U ds \leq \varepsilon / C. \]
This, along with (2.27), yields that for each \( j \in \{1, \ldots, m\} \),
\[ \left\| \int_0^{n_0 T} \Phi(n_0 T, s)D(s)u_j(s)ds - \int_0^{n_0 T} \Phi(n_0 T, s)D(s)v_j(s)ds \right\| \leq \varepsilon. \] (2.30)
Let
\[ \eta_j = \int_0^{n_0 T} \Phi(n_0 T, s)D(s)v_j(s)ds, \quad j = 1, \ldots, m. \] (2.31)
By (1.16), (1.15), (2.29) and (2.31), we see that \( \eta_j \in \hat{V}^Z_{n_0} \) for all \( j = 1, \ldots, m \). Meanwhile, it follows from (2.30) that
\[ \| \eta_j - \xi_j \| \leq \varepsilon \quad \text{for all } \quad j \in \{1, \ldots, m\}. \] (2.32)
Now we claim that \( \{\eta_1, \ldots, \eta_m\} \) is a basis of \( \hat{V}^Z_{n_0} \). In fact, since \( \{\xi_1, \ldots, \xi_m\} \) is orthonormal, it follows from (2.32) that
\[ | \langle \eta_j, \eta_l \rangle | = | \langle \xi_j - \xi_l, \xi_l + (\eta_l - \xi_l) \rangle | \leq 2\varepsilon + \varepsilon^2 \quad \text{for all } \quad j, l \in \{1, \ldots, m\} \quad \text{with} \quad j \neq l. \]
From this and (2.32), one can directly check that
\begin{align*}
| \sum_{j=1}^{m} \alpha_j \eta_j, \sum_{l=1}^{m} \alpha_l \eta_l | & = \sum_{j=1}^{m} \alpha_j^2 \| \eta_j \|^2 + \sum_{j=1}^{m} \sum_{l \neq j} \alpha_l \alpha_j \langle \eta_j, \eta_l \rangle \\
& \geq \sum_{j=1}^{m} \alpha_j^2 (\| \xi_j \| - \| \xi_j - \eta_j \|)^2 - \sum_{j=1}^{m} \sum_{l \neq j} | \alpha_l \alpha_j | \cdot | \langle \eta_j, \eta_l \rangle | \\
& \geq \left( (1 - \varepsilon)^2 - (2\varepsilon + \varepsilon^2)(m - 1) \right) \sum_{j=1}^{m} \alpha_j^2, \quad \text{when } \alpha_1, \ldots, \alpha_m \in \mathbb{R}.
\end{align*}
This, along with (2.28), indicates that \( \alpha_1 = \cdots = \alpha_m = 0 \) whenever \( \sum_{j=1}^{m} \alpha_j \eta_j = 0 \). Namely, \( \{\eta_1, \ldots, \eta_m\} \) is linearly independent group in the subspace \( \hat{V}^Z_{n_0} \) which has the dimension \( m \). Hence, \( \{\eta_1, \ldots, \eta_m\} \) is a basis of \( \hat{V}^Z_{n_0} \).
Let
\[ \hat{Z} = \text{span} \{ z_{11}, \ldots, z_{1k_1}, z_{21}, \ldots, z_{2k_2}, \ldots, z_{m1}, \ldots, z_{mk_m} \}, \]
where \( z_{jl}, j = 1, \ldots, m, l = 1, \ldots, k_j, \) are given by (2.29). Clearly, \( \hat{Z} \) is a finite-dimensional subspace of \( Z \) and all \( v_j(\cdot), j = 1, \ldots, m, \) (given by (2.29)) belong to \( L^2(\mathbb{R}^+; \hat{Z}) \). These, along with (2.31), yield that all \( \eta_j, j = 1, \ldots, m, \) are in \( \hat{V}^Z_{n_0} \). Therefore, it holds that
\[ \hat{V}^Z_{n_0} \supseteq \hat{V}^Z_{n_0} \supseteq \text{span} \{ \eta_1, \ldots, \eta_m \} = \hat{V}^Z_{n_0}. \]
This leads to (2.25) and completes the proof. \( \square \)
3 The multi-periodic feedback stabilization

In this section, we will introduce three propositions. The first two propositions will be used in the proof of our main theorems. The last one is independent interesting. We begin with the following definitions:

- Equation (1.1) is said to be linear multi-periodic feedback stabilizable (LMPFS, for short) if there is a $kT$-periodic $K(\cdot) \in L^\infty(\mathbb{R}^+; \mathcal{L}(H,U))$ for some $k \in \mathbb{N}$ such that the equation (1.2) is exponentially stable. Any such a $K(\cdot)$ is called an LMPFS law for Equation (1.1).

- Equation (1.1) is said to be LMPFS with respect to a subspace $Z$ of $U$ if there is a $kT$-periodic $K(\cdot) \in L^\infty(\mathbb{R}^+; \mathcal{L}(H,Z))$ for some $k \in \mathbb{N}$ such that the equation (1.2) is exponentially stable. Any such a $K(\cdot)$ is called an LMPFS law for Equation (1.1) with respect to $Z$.

**Proposition 3.1** Let $n_0$ and $H_1$ be given by (1.11) and (1.14) respectively. Suppose that $Z \subseteq U$ is a finite dimensional subspace satisfying (1.17). Then, Equation (1.1) is LMPFS with respect to $Z$.

**Proof.** Let $Z \subseteq U$ satisfy (1.17). We organize the proof by several steps as follows.

**Step 1.** For any $h_1 \in H_1$, to construct control $u^{h_1}(\cdot) \in L^2(\mathbb{R}^+; Z)$ such that $Py(n_0T; 0, h_1, u^{h_1}) = 0$

Because $\dim H_1 = n_0$ (see (2.4)), we can set $\{\eta_1, \cdots, \eta_{n_0}\}$ to be an orthonormal basis of $H_1$. Define a linear map $\mathcal{F} : \mathbb{R}^{n_0} \to H_1$ by setting

$$\mathcal{F}(a) \triangleq \sum_{j=1}^{n_0} a_j \eta_j \triangleq (\eta_1, \cdots, \eta_{n_0})(a_1, \cdots, a_{n_0})^* \text{ for each } a = (a_1, \cdots, a_{n_0})^* \in \mathbb{R}^{n_0}. \quad (3.1)$$

Clearly, $\mathcal{F}$ is invertible and

$$\mathcal{F}^{-1}(h_1) = ((h_1, \eta_1), \cdots, (h_1, \eta_{n_0}))^*. \quad (3.2)$$

Since $Z$ is finite dimensional, we can assume that $\dim Z = m_0 < \infty$. Write $\{z_1, \cdots, z_{m_0}\}$ for an orthonormal basis of $Z$. Define a linear map $\mathcal{G} : L^2(\mathbb{R}^+; \mathbb{R}^{m_0}) \to L^2(\mathbb{R}^+; Z)$ by setting

$$\mathcal{G}(\beta)(t) \triangleq (z_1, \cdots, z_{m_0})\beta(t) \triangleq \sum_{j=1}^{m_0} \beta_j(t)z_j, \text{ a.e. } t \geq 0, \quad (3.3)$$

for each $\beta(\cdot) \triangleq (\beta_1(\cdot), \cdots, \beta_{m_0}(\cdot))^* \in L^2(\mathbb{R}^+; \mathbb{R}^{m_0})$ with $\beta_j(\cdot) \in L^2(\mathbb{R}^+; \mathbb{R})$. Clearly, $\mathcal{G}$ is invertible and it holds that

$$u(t) = \mathcal{G}^{-1}(u)(t) = (z_1, \cdots, z_{m_0})\mathcal{G}^{-1}(u)(t) \text{ for a.e. } t \geq 0, \text{ when } u(\cdot) \in L^2(\mathbb{R}^+; Z). \quad (3.4)$$

By the facts that $PP = PP$, $H_1$ is invariant with respect to $P$ and $P : H \to H_1$ is a projection (see Lemma 2.2), and by (2.15), we obtain that

$$PP^{m_0}h_1 = P^{m_0}h_1 = P_1^{m_0}h_1 \text{ for each } h_1 \in H_1.$$
From this, we see that

\[ P(y(n_0T; 0, h_1, u) = P\mathcal{P}^{n_0}h_1 + \int_0^{n_0T} P\Phi(n_0T, s)D(s)u(s)ds \]

\[ = \mathcal{P}^{n_0}h_1 + \int_0^{n_0T} P\Phi(n_0T, s)D(s)u(s)ds \quad \text{for all } h_1 \in H_1, u(\cdot) \in L^2(\mathbb{R}^+; Z). \]  \hfill (3.5)

Let \( \hat{A} \in \mathbb{R}^{n_x \times n_0} \) be the matrix of \( \mathcal{P}_1 \) under \( \{\eta_1, \ldots, \eta_{n_0}\} \), i.e., \( \mathcal{P}_1(\eta_1, \ldots, \eta_{n_0}) = (\eta_1, \ldots, \eta_{n_0})\hat{A} \). Then, it follows from (3.1) and (3.2) that

\[ \mathcal{P}_1^{n_0}h_1 = \mathcal{P}_1^{n_0}(\eta_1, \ldots, \eta_{n_0})F^{-1}(h_1) = (\eta_1, \ldots, \eta_{n_0})\hat{A}^{n_0}F^{-1}(h_1), \quad \text{when } h_1 \in H_1. \]  \hfill (3.6)

Since \( P(y(n_0T; 0, h_1, u) \in H_1 \), it follows by (3.1) and (3.2) that

\[ P(y(n_0T; 0, h_1, u) = (\eta_1, \ldots, \eta_{n_0})F^{-1}(P(y(n_0T; 0, h_1, u)) \quad \text{for all } h_1 \in H_1, u(\cdot) \in L^2(\mathbb{R}^+; Z). \]  \hfill (3.7)

Since \( D(\cdot) \in L^\infty(\mathbb{R}^+; \mathcal{L}(U; H)) \) and \( P\Phi(n_0T, s)D(s)(z_1, \ldots, z_{m_0}) \in (H_1)^{m_0} \) for a.e. \( s \geq 0 \), there is a unique \( \hat{B}(\cdot) \in L^\infty(\mathbb{R}^+; \mathbb{R}^{n_x \times m_0}) \) such that

\[ P\Phi(n_0T, s)D(s)(z_1, \ldots, z_{m_0}) = (\eta_1, \ldots, \eta_{n_0})\hat{B}(s) \quad \text{for a.e. } s \in \mathbb{R}^+. \]  \hfill (3.8)

Now, from (3.5), (3.6), (3.7), (3.4) and (3.8), we see that for each \( h_1 \in H_1 \) and \( u(\cdot) \in L^2(\mathbb{R}^+; Z) \),

\[ F^{-1}(P(y(n_0T; 0, h_1, u)) = \hat{A}^{n_0}F^{-1}(h_1) + \int_0^{n_0T} \hat{B}(s)G^{-1}(u(s))ds. \]  \hfill (3.9)

Meanwhile, it follows from (1.16), (1.15), (3.4), (3.8) and (3.3) that

\[ \hat{V}_{n_0}^Z = \left\{ (\eta_1, \ldots, \eta_{n_0}) \int_0^{n_0T} \hat{B}(s)\beta(s)ds \mid \beta(\cdot) \in L^2(\mathbb{R}^+; \mathbb{R}^{m_0}) \right\}. \]

Since \( Z \) satisfies (1.17), the above equality yields

\[ \text{span}\{\eta_1, \ldots, \eta_{n_0}\} = \left\{ (\eta_1, \ldots, \eta_{n_0}) \int_0^{n_0T} \hat{B}(s)\beta(s)ds \mid \beta(\cdot) \in L^2(\mathbb{R}^+; \mathbb{R}^{m_0}) \right\}, \]

which is equivalent to

\[ \left\{ \int_0^{n_0T} \hat{B}(s)\beta(s)ds \mid \beta(\cdot) \in L^2(\mathbb{R}^+; \mathbb{R}^{m_0}) \right\} = \mathbb{R}^{n_0}. \]  \hfill (3.10)

From (3.10), we see that the finite-dimensional controlled system \( x'(t) = \hat{B}(t)\beta(t), \ t \geq 0 \), (where \( x(\cdot) = (x_1(\cdot), \ldots, x_{m_0}(\cdot))^* \) is treated as a state and \( \beta(\cdot) \) is treated as a control) is exactly controllable.

Hence, the matrix \( \int_0^{n_0T} \hat{B}(s)\hat{B}(s)^*ds \) is positive definite (see, for instance, [21]).

Now for each \( h_1 \in H_1 \), we define \( \beta^{h_1}(\cdot) \in L^2(\mathbb{R}^+; \mathbb{R}^{m_0}) \) by setting

\[ \beta^{h_1}(t) = \begin{cases} -\hat{B}(t)^*\left( \int_0^{n_0T} \hat{B}(s)\hat{B}(s)^*ds \right)^{-1}\hat{A}^{n_0}F^{-1}(h_1) & \text{for a.e. } t \in [0, n_0T); \\ 0 & \text{for a.e. } t \in [n_0T, +\infty). \end{cases} \]  \hfill (3.11)
It is clear that
\[ \hat{A}^{n_0} \mathcal{F}^{-1}(h_1) + \int_0^{n_0T} \hat{B}(s)\hat{\beta}^{h_1}(s) ds = 0. \] (3.12)

Then, for each \( h_1 \in H_1 \), we construct a control
\[ u^{h_1}(\cdot) = G(\beta^{h_1})(\cdot) \text{ over } \mathbb{R}^+. \] (3.13)

By (3.3), \( u^{h_1}(\cdot) \in L^2(\mathbb{R}^+; Z) \). Meanwhile, it follows from (3.12) and (3.9) that
\[ \mathcal{F}^{-1}(P_y(n_0T; 0, h_1, u^{h_1})) = 0, \text{ when } h_1 \in H_1. \]

This, implies that
\[ P_y(n_0T; 0, h_1, u^{h_1}) = 0, \text{ when } h_1 \in H_1. \] (3.14)

**Step 2. To show the existence of an \( N_0 \in \mathbb{N} \) such that**
\[ \|y(NT; 0, h, \mathcal{L}(Ph))\| \leq \delta_0\|h\| \text{ for all } h \in H \text{ and } N \geq N_0, \] (3.15)
where \( \delta_0 \triangleq (1 + \delta)/2 \) with \( \delta \) given by (1.10).

Define an operator \( \mathcal{L} : H_1 \to L^2(\mathbb{R}^+; Z) \) by setting
\[ \mathcal{L}h_1(\cdot) = u^{h_1}(\cdot) \text{ for all } h_1 \in H_1, \] (3.16)
where \( u^{h_1}(\cdot) \) is given by (3.13). Several observations on \( \mathcal{L} \) are given in order. First, it is clear that \( \mathcal{L} \) is linear. Second, from (3.16), (3.13), (3.11) and the fact that \( \hat{B}(\cdot) \in L^\infty(\mathbb{R}^+; \mathbb{R}^{n_0 \times m_0}) \), we have
\[ \|\mathcal{L}\| \triangleq \|\mathcal{L}|_{H_1, L^2(\mathbb{R}^+; Z)}| \leq +\infty. \] (3.17)

Next, by (3.16), (3.13), (3.3) and (3.11), we see that
\[ \mathcal{L}h_1(\cdot) = 0 \text{ over } [n_0T, +\infty), \text{ when } h_1 \in H_1. \] (3.18)

Finally, it follows from (3.14) that
\[ P_y(n_0T; 0, h_1, \mathcal{L}h_1) = 0, \text{ when } h_1 \in H_1. \] (3.19)

Let \( \rho_0 = -\ln \delta_0/T \). Since \( \delta_0 \triangleq (1 + \delta)/2 \), we see that \( 0 < \rho_0 < -\ln \delta/T \triangleq \hat{\rho} \). Then, by Part (f) of Lemma 2.2, there is a constant \( C_{\rho_0} > 0 \) such that
\[ \|y(kT; 0, h_2, 0)\| = \|\Phi(kT, 0)h_2\| \leq C_{\rho_0} e^{-\rho_0 kT}\|h_2\| = C_{\rho_0} \delta_0^k\|h_2\|, \text{ when } k \in \mathbb{N}, h_2 \in H_2. \] (3.20)

We claim that there is a constant \( C > 0 \) such that
\[ \|y(NT; 0, h_1, \mathcal{L}h_1)\| \leq CC_{\rho_0} \delta_0^{N-n_0}\|h_1\|, \text{ when } h_1 \in H_1, N \geq n_0, \] (3.21)
where \( \mathcal{L} \) is given by (3.16). In fact, because
\[ \|y(n_0T; 0, h_1, \mathcal{L}h_1)\| \leq \|\Phi(n_0T, 0)h_1\| + \left\| \int_0^{n_0T} \Phi(n_0T, s)D(s)\mathcal{L}h_1(s)ds \right\| \text{ for each } h_1 \in H_1, \]
there is a constant $C > 0$ such that

$$\|y(n_0T;0,h_1,Lh_1)\| \leq C\|h_1\| \text{ for all } h_1 \in H_1. \tag{3.22}$$

Here, we used facts that $D(\cdot) \in L^\infty(\mathbb{R}^+;L(U;H))$ (see the assumption $(H_3)$) and $L$ is linear and bounded (see (3.17)). Meanwhile, it follows from (3.18), (1.6) and (1.5) with $u(\cdot) \equiv 0$ that when $N \geq n_0$ and $h_1 \in H_1$,

$$y(NT;0,h_1,Lh_1) = y(NT;n_0T,y(n_0T;0,h_1,Lh_1),Lh_1)$$

$$= \Phi(NT,n_0T)y(n_0T;0,h_1,Lh_1)$$

$$= \Phi((N-n_0)T,0)y(n_0T;0,h_1,Lh_1) = y((N-n_0)T;0,y(n_0T;0,h_1,Lh_1),0). \tag{3.23}$$

Because of (3.19) and (2.1)-(2.2) with $t = 0$, it holds that $y(n_0T;0,h_1,Lh_1) \in H_2$, when $h_1 \in H_1$. This along with (3.23), (3.20) and (3.22), leads to (3.21).

Let

$$N_0 = \max \left\{ \frac{\ln C_{\rho_0} + \ln (C\delta_0^{-n_0})}{\ln 1/\delta_0} + 2, n_0 \right\}. \tag{3.24}$$

(Here, $[r]$ with $r \in \mathbb{R}$ denotes the integer such that $r - 1 < [r] \leq r$.) Then, it follows from (3.21), (3.20) and (3.24) that

$$\|y(NT;0,h,L(Ph))\| \leq \|y(NT;0,Ph,L(Ph))\| + \|y(NT;0,(I-P)h,0)\|$$

$$\leq C_{\rho_0}\delta_0^N(C\delta_0^{-n_0}\|P\| + \|I-P\|)\|h\| \leq \delta_0\|h\|, \quad \text{when } N \geq N_0 \text{ and } h \in H. \tag{3.25}$$

This leads to (3.15).

**Step 3. To study a value function associated with a class of optimal control problems**

Given $N \in \mathbb{N}$ and $t \in [0,NT)$, $h \in H$ and $u(\cdot) \in L^2(0,NT;Z)$, consider the equation:

$$\begin{cases}
y'(s) + Ay(s) + B(s)y(s) = D(s)|_Z u(s) \text{ in } (t,NT), \\
y(t) = h,
\end{cases} \tag{3.25}$$

where $D(t)|_Z$ is the restriction of $D(t)$ on the subspace $Z$. Because of assumptions $(H_1)$-$\tilde{(H_3)}$, Equation (3.25) has a unique mild solution $y^Z_N(\cdot;t,h,u) \in C([0,NT];H)$ (see Proposition 5.3 on Page 66 in [11]). Clearly, $y^Z_N(\cdot;t,h,u) = y(\cdot;t,h,u)|_{[t,NT]}$. For each $\varepsilon > 0$, we define a cost functional $J^Z_{\varepsilon,N,t,h}(\cdot) : L^2(0,NT;Z) \to \mathbb{R}^+$ by setting

$$J^Z_{\varepsilon,N,t,h}(u) = \int_t^{NT} \varepsilon\|u(s)\|^2 h ds + \|y^Z_N(NT;t,h,u)\|^2, \quad u \in L^2(0,NT;Z). \tag{3.26}$$

Then, for each $N \in \mathbb{N}$, $\varepsilon > 0$, $t \in [0,NT]$ and $h \in H$, we define the optimal control problem

$$(P)_{\varepsilon,N,t,h} : \inf_{u \in L^2(0,NT;Z)} J^Z_{\varepsilon,N,t,h}(u).$$

This a classical linear quadratic optimal control problem (see Page 370 in [11]). For each $\varepsilon > 0$ and $N \in \mathbb{N}$, the value function associated with the above optimal control problems is

$$W^Z_{\varepsilon,N}(t,h) \triangleq \inf_{u \in L^2(0,NT;Z)} J^Z_{\varepsilon,N,t,h}(u), \quad t \in [0,NT] \text{ and } h \in H. \tag{3.27}$$
Let 
\[ \varepsilon_0 \triangleq (\delta_0 - \delta_0^2)/(\|L\|\|P\| + 1)^2, \] (3.28)
where \( \delta_0 \in (0, 1) \) and \( L \) are given by (3.15) and (3.16) respectively. Because of (3.17), it holds that \( 0 < \varepsilon_0 < +\infty \). We claim that
\[ W_{\varepsilon,N}^Z(0, h) \leq \delta_0\|h\|^2 \text{ for all } h \in H, \text{ when } N \geq N_0 \text{ and } \varepsilon \in (0, \varepsilon_0], \] (3.29)
where \( N_0 \) is given by (3.24). In fact, it follows from (3.28) that
\[ \varepsilon\|L(Ph)(\cdot)\|_{L^2(\mathbb{R}^+; Z)}^2 \leq \varepsilon_0\|L\|^2\|P\|^2\|h\|^2 \leq (\delta_0 - \delta_0^2)\|h\|^2, \text{ when } h \in H \text{ and } \varepsilon \in (0, \varepsilon_0]. \] (3.30)
By (3.16) and the fact that \( P \) is a projection from \( H \) to \( H_1 \) (see Lemma 2.2), we find
\[ L(Ph) \in L^2(\mathbb{R}^+; Z) \text{ for all } h \in H. \] (3.31)
Since \( y^Z_N(\cdot; t, h, u)|_{(0, NT)} = y(\cdot; t, h, u)|_{[0, NT]} \) for any \( u \in L^2(\mathbb{R}^+; Z) \), it holds that
\[ y^Z_N(NT; 0, h, L(Ph)|_{(0, NT)}) = y(NT; 0, h, L(Ph)). \]
This, together with (3.27), (3.26), (3.31), (3.30) and (3.15), indicates that
\[ W_{\varepsilon,N}^Z(0, h) \leq J_{\varepsilon,N,0,h}(L(Ph)|_{(0, NT)}) = \varepsilon \int_0^{NT} \|L(Ph)(s)\|^2 ds + \|y^Z_N(NT; 0, h, L(Ph))\|^2 \leq \delta_0\|h\|^2, \]
when \( N \geq N_0, \varepsilon \in (0, \varepsilon_0], h \in H, \) i.e., (3.29) stands.

**Step 4. To construct an NT-periodic \( K^Z_{\varepsilon,N}(\cdot) \in L^\infty(\mathbb{R}^+; L(H, Z)) \)**

Arbitrarily fix an \( \varepsilon \in (0, \varepsilon_0] \) and an \( N \geq N_0 \), where \( N_0 \) and \( \varepsilon_0 \) are given by (3.24) and (3.28) respectively. By the exactly same way to show Corollary 2.10 on Page 379 and Theorem 4.3 on Page 397 in [11], we can verify that
\[ W_{\varepsilon,N}^Z(t, h) = \langle Q_{\varepsilon,N}^Z(t)h, h \rangle, \text{ when } h \in H, \]
where \( Q_{\varepsilon,N}^Z(\cdot) \in C([0, NT]; L(H)) \) has the following properties: (i) for each \( t \geq 0 \), \( Q_{\varepsilon,N}^Z(t) \) is self-adjoint; (ii) it solves the Riccati integral equation:
\[ Q(t)h = \Phi(NT, t)^*\Phi(NT, t)h - \frac{1}{\varepsilon} \int_t^{NT} \Phi(s, t)^*Q(s)^*D(s)^*Q(s)\Phi(s, t)h ds, \] (3.32)
for all \( h \in H \) and \( t \in [0, NT) \).

Besides, it follows form (3.26) that
\[ 0 \leq \langle h, Q_{\varepsilon,N}^Z(t)h \rangle \leq J_{\varepsilon,N,t,h}(0) \leq \|\Phi(NT, t)\|^2\|h\|^2 \text{ for all } h \in H. \]

Define \( K^Z_{\varepsilon,N}(\cdot) : [0, NT) \to L(H; Z) \) by
\[ K^Z_{\varepsilon,N}(t) = -\frac{1}{\varepsilon} (D(s)|_Z)^*Q_{\varepsilon,N}^Z(t) \text{ for a.e. } t \in [0, NT). \] (3.33)
One can easily check that $K_{ε,N}^{Z}(\cdot) \in L^{\infty}(0, NT; \mathcal{L}(H; Z))$. From this and assumptions ($\mathcal{H}_1$)-($\mathcal{H}_3$), we see that (see Proposition 5.3 on Page 66 in [11]) the feedback equation:

$$
\begin{aligned}
\begin{cases}
    y'(s) + Ay(s) + B(s)y(s) = D(s)\big|_{Z} K_{ε,N}^{Z}(s)y(s) & \text{in } (0, NT), \\
y(0) = h \in H
\end{cases}
\end{aligned}
$$

has a unique mild solution $y_{ε,N}^{Z}(\cdot; 0, h) \in C([0, NT]; H)$. Let

$$
u_{ε,N,0,h}^{Z}(s) = K_{ε,N}^{Z}(s)y_{ε,N}^{Z}(s; 0, h) \quad \text{for a.e. } s \in (0, NT).$$

By the state feedback representation of optimal controls for linear quadratic control problems (see Section 3.4 in Chapter 9 in [11], in particular, (3.71) on Page 392 and (4.12) on Page 397 in [11]), we see that $u_{ε,N,0,h}^{Z}(\cdot)$ defined by (3.35) is the optimal control to $(P)_{ε,N,0,h}^{Z}$. This, along with (3.27), yields that

$$
W_{ε,N}^{Z}(0, h) = J_{ε,N,0,h}^{Z}(u_{ε,N,0,h}^{Z}), \quad \text{when } h \in H.
$$

By (3.25) with $t = 0$, (3.34) and (3.35), we see that $y_{ε,N}^{Z}(NT; 0, h) = y_{N}^{Z}(NT; 0, h, u_{ε,N,0,h}^{Z})$. From this, (3.26), (3.36) and (3.29), it follows that

$$
\|y_{ε,N}^{Z}(NT; 0, h)\|^2 \leq J_{ε,N,0,h}^{Z}(u_{ε,N,0,h}^{Z}) = W_{ε,N}^{Z}(0, h) \leq \delta_0\|h\|^2, \quad \text{when } h \in H.
$$

Now, we extend $NT$-periodically $K_{ε,N}^{Z}(\cdot)$ over $\mathbb{R}^{+}$ by setting

$$
K_{ε,N}^{Z}(t + kNT) = K_{ε,N}^{Z}(t) \quad \text{for all } t \in [0, NT), \quad k \in \mathbb{N}.
$$

Clearly, $K_{ε,N}^{Z}(\cdot) \in L^{\infty}(\mathbb{R}^{+}; \mathcal{L}(H; Z))$ is $NT$-periodic.

**Step 5.** To prove that when $ε \in (0, ε_0]$ and $N \geq N_0$ (where $N_0$ and $ε_0$ are given by (3.24) and (3.28), respectively), $K_{ε,N}^{Z}(\cdot)$ defined by (3.33) and (3.38) is an LMPFS law for Equation (1.1) with respect to $Z$

Consider the feedback equation:

$$
\begin{aligned}
\begin{cases}
    y'(s) + Ay(s) + B(s)y(s) = D(s)\big|_{Z} K_{ε,N}^{Z}(s)y(s) & \text{in } \mathbb{R}^{+}, \\
y(0) = h \in H
\end{cases}
\end{aligned}
$$

By assumptions ($\mathcal{H}_2$) and ($\mathcal{H}_3$), and by the fact that $K_{ε,N}^{Z}(\cdot) \in L^{\infty}(\mathbb{R}^{+}; \mathcal{L}(H; Z))$, we have

$$B(\cdot) - D(\cdot)\big|_{Z} K_{ε,N}^{Z}(\cdot) \in L^{1}_{loc}(\mathbb{R}^{+}; \mathcal{L}(H)).$$

Thus, for each $h \in H$, Equation (3.39) has a unique mild solution $y_{ε}^{Z}(\cdot; 0, h) \in C(\mathbb{R}^{+}; H)$ (see Proposition 5.3 on Page 66 in [11]). Clearly,

$$y_{ε}^{Z}(t; 0, h) = y_{ε,N}^{Z}(t; 0, h) \quad \text{for each } t \in [0, NT].$$

Write $\{φ_{ε,N}^{Z}(t, s)\}_{0 \leq s \leq t < \infty}$ for the evolution system generated by $-A - B(\cdot) + D(\cdot)\big|_{Z} K_{ε,N}^{Z}(\cdot)$. Then it holds that (see Proposition 5.7, Page 69, [11])

$$y_{ε,N}^{Z}(NT; 0, h) = y_{ε}^{Z}(NT; 0, h) = φ_{ε,N}^{Z}(NT, 0)h \quad \text{for all } h \in H.$$
This, along with (3.37) and the fact that \( \delta_0 < 1 \) (see (3.15)), yields
\[
\| \Phi_{\varepsilon,N}(NT,0) \| \leq \sqrt{\delta_0} < 1. \tag{3.40}
\]
Since \( B(\cdot) \) and \( D(\cdot) \) are \( T \)-periodic and \( K_{\varepsilon,N}^Z(\cdot) \) is \( NT \)-periodic, it follows that
\[
\Phi_{\varepsilon,N}(t + NT, s + NT) = \Phi_{\varepsilon,N}(t, s), \quad \text{when} \quad 0 \leq s \leq t < +\infty. \tag{3.41}
\]
By (3.41) and (3.40), one can easily shows that Equation (3.39) is exponentially stable. Hence, \( K_{\varepsilon,N}^Z(\cdot) \), with \( \varepsilon \in (0, \varepsilon_0] \) and \( N \geq N_0 \), is an LMPFS law for Equation (3.39). This completes the proof.

\textbf{Proposition 3.2} Let \( Z \) be a subspace of \( U \). Then, Equation (1.1) is LPFS with respect to \( Z \) if and only if it is LMPFS with respect to \( Z \).

\textbf{Proof.} It is clear that Equation (1.1) is LPFS w.r.t. \( Z \Rightarrow \) Equation (1.1) is LMPFS w.r.t. \( Z \). We will show the reverse. Suppose that Equation (1.1) is LMPFS with respect to \( Z \). Then there are an \( N \in \mathbb{N} \) and an \( NT \)-periodic \( \hat{K}_N^Z(\cdot) \in L^\infty(\mathbb{R}^+; \mathcal{L}(H; Z)) \) such that the feedback equation
\[
y'(s) + Ay(s) + B(s)y(s) = D(s)\Big|_Z \hat{K}_N^Z(s)y(s), \quad s \geq 0 \tag{3.42}
\]
is exponentially stable. From this, assumptions (H1)-(H3) and the fact that \( \hat{K}_N^Z(\cdot) \in L^\infty(\mathbb{R}^+; \mathcal{L}(H; Z)) \) is \( NT \)-periodic, one can easily verify that for each \( t \geq 0 \) and \( h \in H \), the solution \( \hat{y}_N^Z(\cdot; t, h) \) to the equation
\[
\begin{cases}
y'(s) + Ay(s) + B(s)y(s) = D(s)\Big|_Z \hat{K}_N^Z(s)y(s), & s \geq 0 \\
y(t) = h
\end{cases} \tag{3.43}
\]
satisfies
\[
\| \hat{y}_N^Z(s; t, h) \| \leq M_1 e^{-\delta_1(s-t)} \| h \|, \quad \text{when} \quad s \geq t \quad \text{and} \quad h \in H, \tag{3.44}
\]
where \( M_1 \) and \( \delta_1 \) are two positive constants independent of \( h \), \( t \) and \( s \). Write
\[
\hat{C}_0 \triangleq \| \hat{K}_N^Z(\cdot) \|_{L^\infty(\mathbb{R}^+; \mathcal{L}(H; Z))}. \tag{3.45}
\]
The rest of the proof is organized by two steps as follows.

\textit{Step 1. To study a value function associated with a class of optimal control problems}

For each \( t \geq 0 \) and \( h \in H \), we define an optimal control problem:
\[
(P)_{t,h}^Z : \inf_{u(\cdot) \in L^2(\mathbb{R}^+; Z)} \int_t^\infty \left( \| y(s; t, h, u) \|^2 + \| u(s) \|^2 \right) ds. \tag{3.46}
\]
Associated with \( \{ (P)_{t,h}^Z \}_{t \geq 0, h \in H} \), we define a value function \( W^Z(\cdot, \cdot) : \mathbb{R}^+ \times H \rightarrow \mathbb{R} \) by
\[
W^Z(t, h) = \inf_{u(\cdot) \in L^2(\mathbb{R}^+; Z)} \int_t^\infty \left( \| y(s; t, h, u) \|^2 + \| u(s) \|^2 \right) ds, \quad (t, h) \in \mathbb{R}^+ \times H. \tag{3.47}
\]
It is well defined. In fact, define a control \( u_{N,t,h}^Z(\cdot) \) by setting
\[
  u_{N,t,h}^Z(s) = \begin{cases} 
    K_N^Z(s)\hat{y}_N^Z(s; t, h) & \text{for a.e. } s \in [t, \infty), \\
    0 & \text{for a.e. } s \in [0, t).
  \end{cases}
\]

By (3.44), we see that \( u_{N,t,h}^Z(\cdot) \in L^2(\mathbb{R}^+; Z) \). Meanwhile, it is clear that \( \hat{y}_N^Z(\cdot; t, h) = y(\cdot; t, h, u_{N,t,h}^Z) \).

These, along with (3.44) and (3.45), yields that for each \( t \geq 0 \) and \( h \in H \),
\[
  0 \leq W^Z(t, h) \leq \int_t^\infty \left( \|\hat{y}_N^Z(s; t, h)\|^2 + \|u_{N,t,h}^Z(s)\|_U^2 \right) ds
  \leq \int_t^\infty (1 + \|\hat{K}_N^Z(s)\|_{\mathcal{L}(H,Z)}^2)\|\hat{y}_N^Z(s; t, h)\|^2 ds
  \leq (1 + \hat{C}_0^2)M_1^2 \|h\|^2.
\]
(3.48)

Thus \( W^Z(\cdot, \cdot) \) is well-defined. By (3.47), one can directly check that when \( h, g \in H, t \geq 0, c \in \mathbb{R}, \)
\[
  W^Z(t, ch) = e^2 W^Z(t, h) \quad \text{and} \quad W^Z(t, h + g) + W^Z(t, h - g) = 2W^Z(t, h) + 2W^Z(t, g).
\]

These, together with (3.48), imply that (see [10]) there is a unique \( Q^Z(\cdot) : \mathbb{R}^+ \to \mathcal{L}(H), \) with \( Q^Z(t) \)
self-adjoint for each \( t \geq 0 \), such that
\[
  W^Z(t, h) = \langle Q^Z(t)h, h \rangle \quad \text{for all } t \in \mathbb{R}^+ \text{ and } h \in H.
\]
(3.49)

This, together with (3.48), implies that
\[
  0 \leq Q^Z(t) \leq \frac{(1 + \hat{C}_0^2)M_1^2}{2\hat{\delta}_1} I \quad \text{for all } t \in \mathbb{R}^+.
\]

Meanwhile, from the \( T \)-periodicity of \( B(\cdot) \) and \( D(\cdot) \), one can easily derive the \( T \)-periodicity of \( W^Z(\cdot, h) \) for each \( h \in H \). Thus, by (3.49), \( Q^Z(\cdot) \) is \( T \)-periodic, i.e.,
\[
  Q^Z(t) = Q^Z(T + t) \quad \text{for all } t \in \mathbb{R}^+.
\]
(3.50)

Now we present other properties of \( Q^Z(\cdot) \). By the Bellman optimality principle (see Section 1, Chapter 6 in [11]), it holds that for any \( t \in [0, T] \) and \( h \in H \),
\[
  W^Z(t, h) = \inf_{u(\cdot) \in L^2(\mathbb{R}^+; Z)} \left\{ \int_t^T (\|y(s; t, h, u)\|^2 + \|u(s)\|_U^2) ds + W^Z(T, y(T; t, h, u)) \right\}.
\]
(3.51)

This, along with (3.49) and (3.50), yields that for any \( t \in [0, T] \) and \( h \in H \),
\[
  W^Z(t, h) = \inf_{u(\cdot) \in L^2(\mathbb{R}^+; Z)} \left\{ \int_t^T (\|y(s; t, h, u)\|^2 + \|u(s)\|_U^2) ds + \|Q^Z(0)\|^{1/2} y(T; t, h, u)\| \right\},
\]
(3.52)
i.e., \( W^Z(\cdot, \cdot)|_{[0, T] \times H} \) is the value function associated with the LQ problems \( (P)^{Z,T}_{t,h} \), \( t \in [0, T] \), \( h \in H \):
\[
  \inf_{v(\cdot) \in L^2(0,T; Z)} \left\{ \int_t^T (\|y^T(s; t, h, v)\|^2 + \|v(s)\|_U^2) ds + \langle y^T(T; t, h, v), Q^Z(0)y^T(T; t, h, v) \rangle \right\},
\]
(3.53)
where $y^T(:t,h,v)$ is the solution of Equation (1.1) (over $[t,T]$), with the initial condition that $y(t) = h$ and with the control $v(\cdot) \in L^2(0,T;Z)$. Furthermore, by the exactly same way to show Corollary 2.10 on Page 379 and Theorem 4.3 on Page 397 in [11], we can obtain that

$$W^Z(t,h) = \langle h, \hat{Q}(t)h \rangle, \quad \text{when } t \in [0,T] \text{ and } h \in H, \quad (3.54)$$

where $\hat{Q}(\cdot) \in C([0,T];\mathcal{L}(H))$, with $\hat{Q}(t)$ self-adjoint and non-negative for each $t \in [0,T]$, satisfies the following Riccati integral equation:

$$\hat{Q}(t) = \Phi(T,t)^*Q(0)\Phi(T,t)h + \int_t^T \Phi(s,t)^*\Phi(s,t)h ds$$

$$(3.55)$$

$$- \int_t^T \Phi(s,t)^*\hat{Q}(s)^*D(s)_{Z}(D(s)_{Z})^*\hat{Q}(s)\Phi(s,t)h ds \text{ for all } h \in H, t \in [0,T].$$

Because $Q(t)$ and $\hat{Q}(t)$ are selfadjoint, it follows from (3.49) and (3.54) that

$$Q^Z(\cdot)|_{0,T} = \hat{Q}(\cdot). \quad (3.56)$$

Step 2. To construct a $T$-periodic $\tilde{K}(\cdot) \in L^\infty(\mathbb{R}^+;\mathcal{L}(H;Z))$

Let $Q^Z(\cdot)$ be given by (3.49). We define $\tilde{K}(\cdot) \in L^\infty(\mathbb{R}^+;\mathcal{L}(H;Z))$ by setting

$$\tilde{K}(s) = - \left((D(s)_{Z})^*Q^Z(s)\right) \text{ for a.e. } s \in \mathbb{R}^+. \quad (3.57)$$

Since both $Q^Z(\cdot)$ and $D(\cdot)$ are $T$-periodic, so is $\tilde{K}(\cdot)$. Let $\tilde{y}_T(\cdot;0,g) \in C([0,T];H)$ be the mild solution to the feedback equation:

$$\begin{cases} 
    y'(s) + Ay(s) + B(s)y(s) = D(s)_{Z}\tilde{K}(s)y(s) & \text{for a.e. } s \in (0,T); \\
    y(0) = g \in H. 
\end{cases}$$

The existence and uniqueness of the solution to the above equation is ensured by of assumptions $(\mathcal{H}_1)$-$(\mathcal{H}_3)$ and the fact that $\tilde{K}(\cdot) \in L^\infty(\mathbb{R}^+;\mathcal{L}(H;Z))$ (see Proposition 5.3 on Page 66 in [11]). For each $g \in H$, we define

$$\tilde{u}_g(t) \triangleq \tilde{K}(t)\tilde{y}_T(t;0,g) \text{ for a.e. } t \in (0,T).$$

By the state feedback representation of optimal controls for linear quadratic control problems (see Section 3.4 in Chapter 9, in particular the formula (3.71) on Page 392 in [11]), it follows from (3.57) and (3.56) that $\tilde{u}_g(\cdot)$ is the optimal control to $(P)^{Z,T}_{0,g}$ (which is defined by (3.53)). By (3.52), (3.54) and (3.56), we see that for each $g \in H$,

$$\langle g, Q^Z(0)g \rangle = \int_0^T \|\tilde{y}_T(t;0,g)\|^2 + \|\tilde{u}_g(t)\|^2_{Z} dt + \langle \tilde{y}_T(T;0,g), Q^Z(0)\tilde{y}_T(T;0,g) \rangle. \quad (3.58)$$

Associated with each $h \in H$, we define three sequences $\{h_k\}_{k=0}^\infty \subseteq H$, $\{y^b_k(\cdot)\}_{k=1}^\infty \subseteq C([0,T];H)$ and $\{u^b_k(\cdot)\}_{k=1}^\infty \subseteq L^2(0,T;Z)$ by setting

$$h_0 \triangleq h, \quad h_k \triangleq \tilde{y}_T(T;0,h_{k-1}), \quad y^b_k(\cdot) \triangleq \tilde{y}_T(\cdot;0,h_{k-1}), \quad u^b_k(\cdot) \triangleq \tilde{u}_{h_{k-1}}(\cdot) \text{ for all } k \in \mathbb{N}, \quad (3.59)$$
Taking \( g = h_{k-1} \) in (3.58), we find that

\[
\langle h_{k-1}, Q^Z(0)h_{k-1} \rangle = \int_0^T \|y^h_k(t)\|^2 + \|u^h_k(t)\|^2 dt + \langle h_k, Q^Z(0)h_k \rangle \quad \text{for each } k \in \mathbb{N}. \tag{3.60}
\]

**Step 3.** To prove that \( \tilde{K} \) given by (3.57) is an LPFS law for Equation (1.1) with respect to \( Z \)

Consider the feedback equation:

\[
y'(t) + Ay(t) + B(t)y(t) = D(t)|_Z \tilde{K}(t)y(t) \quad \text{in } \mathbb{R}^+. \tag{3.61}
\]

Because of assumptions \((H_1)-(H_3)\) and the fact that \( \tilde{K}() \in L^\infty(\mathbb{R}^+; \mathcal{L}(H; Z)) \), corresponding to each initial condition that \( y(0) = h \in H \), Equation (3.61) has a unique mild solution \( y_{\tilde{K}}(\cdot; 0, h) \in C(\mathbb{R}^+; H) \) (see Proposition 5.3 on Page 66 in [11]). Let \( \{\tilde{K}(t, s)\}_{t \geq s \geq 0} \) for the evolution system generated by \(-A - B(\cdot) + D(\cdot)|_Z \tilde{K}(\cdot)\). Then \( y_{\tilde{K}}(t; 0, h) = \Phi_{\tilde{K}}(t, 0)h \) for each \( t \geq 0 \) and \( h \in H \) (see Proposition 5.7 on Page 69 in [11]). By the \( T \)-periodicity of \( B(\cdot), D(\cdot) \) and \( \tilde{K}(\cdot) \), and by (3.59), one can easily check that

\[
\Phi_{\tilde{K}}(t + T, s + T) = \Phi_{\tilde{K}}(t, s), \quad \text{when } t \geq s \geq 0. \tag{3.62}
\]

Meanwhile, it follows from the definition of \( y^h_k \) (see (3.59)) that given \( h \in H \),

\[
y_{\tilde{K}}(t; 0, h) = y^h_{[t/T]+1}(t - [t/T]T) \quad \text{for all } t \in \mathbb{R}^+. \tag{3.63}
\]

We first claim that

\[
\int_0^\infty \|\Phi_{\tilde{K}}(t, 0)h\|^2 dt \leq \|Q^Z(0)||h||^2, \quad \text{when } h \in H. \tag{3.64}
\]

Indeed, it follows from (3.63) that

\[
\int_0^\infty \|\Phi_{\tilde{K}}(t, 0)h\|^2 dt = \sum_{k=1}^\infty \int_{(k-1)T}^{kT} \|y_{\tilde{K}}(t, 0, h)\|^2 dt = \sum_{k=1}^\infty \int_{0}^{T} \|y^h_k(t)\|^2 dt. \tag{3.65}
\]

Meanwhile, by (3.60), we find that

\[
\sum_{k=1}^N \langle h_{k-1}, Q^Z(0)h_{k-1} \rangle = \sum_{k=1}^N \left[ \int_0^T \|y^h_k(t)\|^2 + \|u^h_k(t)\|^2 dt + \langle h_k, Q^Z(0)h_k \rangle \right], \quad N \in \mathbb{N}.
\]

Thus

\[
\sum_{k=1}^\infty \int_0^T \|y^h_k(t)\|^2 dt = \lim_{N \to \infty} \sum_{k=1}^N \int_0^T \|y^h_k(t)\|^2 dt
\]

\[
= \lim_{N \to \infty} \left[ \langle h, Q^Z(0)h \rangle - \sum_{k=1}^N \int_0^T \|u^h_k(t)\|^2 dt - \langle h_N, Q^Z(0)h_N \rangle \right]
\]

\[
\leq \langle h, Q^Z(0)h \rangle \leq \|Q^Z(0)||h||^2.
\]

This, along with (3.65), leads to (3.64).
Since $\bar{K}(\cdot) \in L^\infty(\mathbb{R}^+; \mathcal{L}(H;U))$ is $T$-periodic, in order to show that $\bar{K}(\cdot)$ is an LPFS law for Equation (1.1) with respect to $Z$, it suffices to prove that there are positive $M$ and $\delta$ such that $\|y_{\bar{K}}(t;0,h)\| \leq Me^{-\delta t}\|h\|$ for all $h \in H$. This will be done if one can show that $\hat{\delta} < 1$, where

$$\hat{\delta} \triangleq \lim_{k \to \infty} \|\Phi_{\bar{K}}(T,0)^k\|^{1/k}. \quad (3.66)$$

The reason is that $\Phi_{\bar{K}}(\cdot, \cdot)$ is $T$-periodic and $y_{\bar{K}}(t;0, h) = \Phi_{\bar{K}}(t,0)h$.

The rest is to prove that $\hat{\delta} < 1$. First we can assume that

$$\hat{\rho} \triangleq \ln(\hat{\delta} + 1)/T > 0, \quad (3.67)$$

for otherwise $\hat{\delta} = 0 < 1$. By (3.66) and (3.67), one can easily check that there is a constant $\hat{C}_1 > 0$ such that

$$\|\Phi_{\bar{K}}(kT,0)\| \leq \hat{C}_1e^{\hat{\rho}kT} \text{ for all } k \in \mathbb{N}. \quad (3.68)$$

Because of $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ and the fact that $\bar{K}(\cdot) \in L^\infty(\mathbb{R}^+; \mathcal{L}(H;U))$, one can easily check that $\{\|\Phi_{\bar{K}}(t,s)\|_{\mathcal{L}(H)}; 0 \leq s \leq t \leq T\}$ is bounded. Thus, we can write

$$\hat{C}_2 \triangleq \max_{0 \leq t_1 \leq t_2 \leq T} \|\Phi_{\bar{K}}(t_2,t_1)\| \in \mathbb{R}^+ \text{ and } \hat{C}_3 \triangleq \max\{\hat{C}_2, \hat{C}_1\hat{C}_2\} \in \mathbb{R}^+. \quad (3.69)$$

We claim that

$$\|\Phi_{\bar{K}}(t,s)\| \leq \hat{C}_3e^{\hat{\rho}(t-s)}, \text{ when } t \geq s \geq 0. \quad (3.70)$$

In fact, given $0 \leq s \leq t$, there are only two cases: (i) $[t/T] = [s/T]$ and (ii) $[t/T] \neq [s/T]$. In the first case, we have

$$0 \leq t - [s/T] \leq T \text{ and } 0 \leq s - [s/T] \leq T.$$ 

These, along with (3.62) and (3.69), yields

$$\|\Phi_{\bar{K}}(t,s)\| = \|\Phi_{\bar{K}}(t-[s/T]T, s-[s/T]T)\| \leq \hat{C}_2 \leq \hat{C}_3e^{\hat{\rho}(t-s)},$$

i.e., (3.70) holds for the first case. In the second case, we have

$$[t/T]T \geq [s/T]T + T \text{ and } ([t/T] - [s/T] - 1)T \leq t - s.$$ 

These, together with (3.62), (3.68) and (3.69), yields

$$\|\Phi_{\bar{K}}(t,s)\| \leq \|\Phi_{\bar{K}}(t, [t/T]T)\| \cdot \|\Phi_{\bar{K}}([t/T]T, [s/T]T + T)\| \cdot \|\Phi_{\bar{K}}([s/T]T + T, s)\| \leq \|\Phi_{\bar{K}}(t-[t/T]T, 0)\| \cdot \|\Phi_{\bar{K}}(([t/T] - [s/T] - 1)T, 0)\| \cdot \|\Phi_{\bar{K}}(T, s-[s/T]T)\| \leq \hat{C}_1\hat{C}_2^2e^{\hat{\rho}(t-s)} \leq \hat{C}_3e^{\hat{\rho}(t-s)},$$

i.e., (3.70) holds for the second case. In summary, we conclude that (3.70) stands.

Let

$$\hat{C}_4 \triangleq \max\{\max_{t \in [0,T]} \|\Phi_{\bar{K}}(t,0)\|, \hat{C}_3\sqrt{\|Q^0(0)\|2\hat{\rho}/(1-e^{-2\hat{\rho}T})}\} \quad \text{and} \quad \hat{C}_5 \triangleq \max\{\hat{C}_2, \hat{C}_4\hat{C}_2\}. \quad (3.71)$$
where \( \hat{C}_2 \) and \( \hat{C}_3 \) are given by (3.69), \( \hat{\rho} \) is given by (3.67). Then, we claim

\[
\| \Phi_{\bar{K}}(t, s) \| \leq \hat{C}_5, \quad \text{when } t \geq s \geq 0 \quad \text{(where } \hat{C}_5 \text{ is given by (3.71)).} \tag{3.72}
\]

For this purpose, we first show

\[
\| \Phi_{\bar{K}}(t, 0) \| \leq \hat{C}_4 \text{ for all } t \geq 0 \quad \text{(where } \hat{C}_4 \text{ is given by (3.71)).} \tag{3.73}
\]

In fact, by (3.70) and (3.64), we have

\[
\frac{1 - e^{-2\hat{\rho}t}}{2\hat{\rho}} \| \Phi_{\bar{K}}(t, 0)h \|^2 = \int_0^t e^{-2\hat{\rho}(t-r)}\| \Phi_{\bar{K}}(t, 0)h \|^2 dr \\
\leq \int_0^t e^{-2\hat{\rho}(t-r)}\| \Phi_{\bar{K}}(t, r) \|^2\| \Phi_{\bar{K}}(r, 0)h \|^2 dr \\
\leq \int_0^t e^{-2\hat{\rho}(t-r)}\| \Phi_{\bar{K}}(t, r) \|^2 \| \Phi_{\bar{K}}(r, 0)h \|^2 dr \\
< \hat{C}_3 \int_0^\infty \| \Phi_{\bar{K}}(r, 0)h \|^2 dr \leq \hat{C}_3 \| Q^Z(0) \| \| h \|^2,
\]

when \( h \in H \) and \( t \geq 0 \),

from which, it follows that

\[
\| \Phi_{\bar{K}}(t, 0) \| \leq \hat{C}_3 \sqrt{\| Q^Z(0) \| \| 2\hat{\rho}/(1 - e^{-2\hat{\rho}}) \}} \text{ for all } t > 0.
\]

This, along with (3.71), yields

\[
\sup_{t \in \mathbb{R}^+} \| \Phi_{\bar{K}}(t, 0) \| = \max \left\{ \max_{t \in [0, T]} \| \Phi_{\bar{K}}(t, 0) \|, \sup_{t \in [T, \infty)} \| \Phi_{\bar{K}}(t, 0) \| \right\}
\]

\[
\leq \max \left\{ \max_{t \in [0, T]} \| \Phi_{\bar{K}}(t, 0) \|, \hat{C}_3 \sqrt{\| Q^Z(0) \| \| 2\hat{\rho}/(1 - e^{-2\hat{\rho}}) \}} \right\} = \hat{C}_4,
\]

which leads to (3.73). Now, given \( 0 \leq s \leq t \), there are only two possibilities: (i) \( [t/T] = [s/T] \); and (ii) \( [t/T] > [s/T] \). In the first case, (3.72) follows from (3.69) and (3.71). In the second case, we have that \( t \geq ([s/T] + 1)T \geq s \). This, along with (3.62), (3.73), (3.69) and (3.71), indicates that

\[
\| \Phi_{\bar{K}}(t, s) \| \leq \| \Phi_{\bar{K}}(t, [s/T]T + T) \| \cdot \| \Phi_{\bar{K}}([s/T]T + T, s) \| \leq \hat{C}_4 \hat{C}_2 \leq \hat{C}_5.
\]

In summary, we conclude that (3.72) holds.

Finally, it follows from (3.72) and (3.64) that when \( t \geq 0 \) and \( h \in H \),

\[
t\| \Phi_{\bar{K}}(t, 0)h \|^2 = \int_0^t \| \Phi_{\bar{K}}(t, 0)h \|^2 ds \leq \int_0^t \| \Phi_{\bar{K}}(t, s) \|^2\| \Phi_{\bar{K}}(s, h) \|^2 ds \leq \hat{C}_5^2 \| Q^Z(0) \| \| h \|^2.
\]

This implies that

\[
\| \Phi_{\bar{K}}(t, 0) \| \leq \hat{C}_5 \sqrt{\| Q^Z(0) \| /t}, \quad \text{when } t > 0.
\]

Therefore, there is an \( N_0 \in \mathbb{N} \) such that \( \| \Phi_{\bar{K}}(N_0T, 0) \| < 1 \). This, together with (3.66) and (3.62), yields that

\[
\hat{\delta} = \lim_{k \to \infty} \| \Phi_{\bar{K}}(T, 0)^{N_0k} \|^{\frac{1}{N_0k}} = \lim_{k \to \infty} \| \Phi_{\bar{K}}(N_0T, 0)^k \|^{\frac{1}{N_0k}} \leq \| \Phi_{\bar{K}}(N_0T, 0) \|^{\frac{1}{N_0}} < 1.
\]

This completes the proof.
Proposition 3.3  When both $B(\cdot)$ and $D(\cdot)$ are time-invariant, i.e., $B(t) \equiv B$ and $D(t) \equiv D$ for all $t \geq 0$, the following statements are equivalent:

(i) Equation (1.1) is linear $\hat{T}$-periodic feedback stabilizable for some $\hat{T} > 0$.

(ii) Equation (1.1) is linear $\hat{T}$-periodic feedback stabilizable for any $\hat{T} > 0$.

(iii) Equation (1.1) is linear time-invariant feedback stabilizable.

Proof. It suffices to show that (i) $\Rightarrow$ (iii). For this purpose, we suppose that (i) holds. Let $N \in \mathbb{N}$ with $N \geq 2$ and let $T = \hat{T}/N$. Since $B(\cdot)$ and $D(\cdot)$ are time-invariant, Equation (1.1) is $T$-periodic. Because of (i), there is an $NT$-periodic $\hat{K}_N^U(\cdot) \in L^\infty(\mathbb{R}^+; \mathcal{L}(H; U))$ such that the feedback equation (3.42), where $Z = U$, is exponentially stable. Now, by the same way to show that Equation (1.1) is LMPFS $\Rightarrow$ Equation (1.1) is LPFS in the proof of Proposition 3.2 (where $Z = U$), we see that $\hat{K}(\cdot)$ given by (3.57), where $Z = U$, is a LPFS law for Equation (1.1). We claim that this $\hat{K}(\cdot)$ is time-invariant in the case that $B(\cdot)$ and $D(\cdot)$ are time-invariant. When this is done, $\hat{K}(\cdot) \equiv \hat{K} \in \mathcal{L}(H; U)$ is a feedback law for Equation (1.1), which leads to (iii).

The rest is to show that $\hat{K}(\cdot)$ is time-invariant. By the time-invariance of $D(\cdot)$, and by (3.49) and (3.57), when $Z = U$, it suffices to show the valued function $W^U(t, h)$, given by (3.47) with $Z = U$ is time-invariant. The later will be proved as follows. Since Equation (1.1) is time-invariant, we have that for each $t \in \mathbb{R}^+$, $h \in H$ and $u(\cdot) \in L^2(\mathbb{R}^+; U)$,

$$y(s; t, h, u) = y(s - t; 0, h, v) \text{ for all } s \geq t,$$

where $v(\cdot)$ is defined by $v(s) = u(s + t)$ for all $s \geq 0$. Hence, given $t \in \mathbb{R}^+$ and $h \in H$,

$$\int_t^\infty (\|y(s; t, h, u(s))\|^2 + \|u(s)\|_{L^2}^2) \, ds = \int_0^\infty (\|y(r; 0, h, u(r + t))\|^2 + \|u(r + t)\|^2) \, dr,$$

for all $u(\cdot) \in L^2(\mathbb{R}^+; U)$. Taking the infimum on the both sides of the above equation with respect to $u(\cdot) \in L^2(\mathbb{R}^+; U)$, we get that $W^U(t, h) = W^U(0, h)$, i.e., the value function $W^U(t, h)$ is time-invariant. This completes the proof. \hfill \Box

Remark 3.1 By Proposition 3.3, we see that linear time-periodic functions $K(\cdot)$ will not aid in the linear stabilization of Equation (1.1) when both $B(\cdot)$ and $D(\cdot)$ are time-invariant. On the other hand, when Equation (1.1) is $T$-periodically time-varying, linear time-periodic functions $K(\cdot)$ do aid in the linear stabilization of Equation (1.1). This can be seen from the following 2-periodic ordinary differential equation:

$$y'(t) = \sum_{j=1}^\infty [\chi_{[2j,2j+1)}(t) - \chi_{[2j+1,2j+2)}(t)] \ u(t).$$

For each $k \in \mathbb{R}$, consider the feedback equation

$$y'(t) = \sum_{j=1}^\infty [\chi_{[2j,2j+1)}(t) - \chi_{[2j+1,2j+2)}(t)] \ ky(t).$$
Clearly, the corresponding Poincaré map $\mathcal{P}_k \equiv 1$. Thus any linear time-invariant feedback equation is not exponentially stable. On the other hand, by a direct computation, one can easily check that the 2-periodic time-varying feedback law given by

$$k(t) = \sum_{j=1}^{\infty} \left[ \chi_{[2j,2j+1]}(t) + 2\chi_{[2j+1,2j+2]}(t) \right].$$

is an LPFS law.

## 4 The proof of Theorem 1.1 and Theorem 1.2

### Proof of Theorem 1.1. $(a) \iff (b)$: We first show that $(b) \Rightarrow (a)$. Suppose that a subspace $Z$ of $U$ satisfies (1.17). Then by Lemma 2.5, there is a finite dimension subspace $\hat{Z}$ of $Z$ such that $\hat{V}_{n_0} = H_1$. From this, we can apply Proposition 3.1 to get that Equation (1.1) is LMPFS with respect to $\hat{Z}$. This, along with Proposition 3.2, yields that Equation (1.1) is LPFS with respect to $\hat{Z}$. Because $\hat{Z}$ is a subspace of $Z$, Equation (1.1) is also LPFS with respect to $Z$, i.e., $(a)$ stands.

We next show that $(a) \Rightarrow (b)$. Seeking for a contradiction, we suppose that $Z \in U^{FS}$, but (1.17) does not hold. Then $\hat{V}_{n_0}$ would be a proper subspace of $H_1$. This, along with (2.18), yields that $\hat{V}$ is a proper subspace of $H_1$. Write $(\hat{V})^\perp$ for the orthogonal complement subspace of $\hat{V}$ in $H$. Then, one can directly check that $H_1 \cap (\hat{V})^\perp$ is the orthogonal complement subspace of $\hat{V}$ in $H_1$, i.e.,

$$H_1 \cap (\hat{V})^\perp \perp \hat{V}; \quad H_1 = \hat{V} \bigoplus (H_1 \cap (\hat{V})^\perp).$$

(4.1)

Since $\hat{V}$ is a proper subspace of $H_1$ and $\dim H_1 = n_0$ (see (2.4)), we have

$$n_0 \geq l \triangleq \dim \left( H_1 \cap (\hat{V})^\perp \right) \geq 1. \quad (4.2)$$

By (4.1) and (4.2), we can let $\{\eta_1, \ldots, \eta_{n_0}\}$ be a basis of $H_1$ such that $\{\eta_1, \ldots, \eta_l\}$ and $\{\eta_{l+1}, \ldots, \eta_{n_0}\}$ are bases of $H_1 \cap (\hat{V})^\perp$ and $\hat{V}$, respectively. By (2.18) in Lemma 2.4, $\hat{V}$ is an invariant subspace under $\mathcal{P}_1$. Thus there are matrices $A_1 \in \mathbb{R}^{l \times l}, A_2 \in \mathbb{R}^{(n_0-l) \times l}, A_3 \in \mathbb{R}^{(n_0-l) \times (n_0-l)}$ such that

$$\mathcal{P}_1 \left( \eta_1, \ldots, \eta_l \mid \eta_{l+1}, \ldots, \eta_{n_0} \right) = \begin{pmatrix} \eta_1, \ldots, \eta_l \mid \eta_{l+1}, \ldots, \eta_{n_0} \end{pmatrix} \begin{pmatrix} A_1 & 0_{l \times (n_0-l)} \\ A_2 & A_3 \end{pmatrix}.$$ 

(4.3)

Let $P_{11}$ be the orthogonal projection from $H_1$ onto $H_1 \cap (\hat{V})^\perp$. Define a linear bijection $\mathcal{J} : \mathbb{R}^l \to (H_1 \cap (\hat{V})^\perp)$ by setting

$$\mathcal{J}(\alpha) \overset{\Delta}{=} (\eta_1, \ldots, \eta_l) \alpha, \quad \alpha \in \mathbb{R}^l,$$

(4.4)

where $\alpha$ denotes the column vectors. By (4.3) and (4.4), we see that

$$P_{11} \mathcal{P}_1^k \mathcal{J}(\alpha) = P_{11} \mathcal{P}_1^k (\eta_1, \ldots, \eta_l) \alpha$$

$$= P_{11} (\eta_1, \ldots, \eta_l \mid \eta_{l+1}, \ldots, \eta_{n_0}) \begin{pmatrix} A_1 & 0_{l \times (n_0-l)} \\ A_2 & A_3 \end{pmatrix}^k \begin{pmatrix} \alpha \\ 0_{(n_0-l) \times 1} \end{pmatrix},$$

(4.5)

$$= (\eta_1, \ldots, \eta_l) A_1^k \alpha, \quad \text{when } \alpha \in \mathbb{R}^l \text{ and } k \in \mathbb{N}.$$
On the other hand, since $Z \in U^{FS}$, there is a $T$-periodic $K(\cdot) \in L^\infty(\R^+; \mathcal{L}(H; Z))$ such that Equation (1.2) is exponentially stable, which implies that

$$
\lim_{t \to +\infty} y_K(t; 0, h)) = 0, \quad \text{when } h \in H,
$$

where $y_K(\cdot; 0, h))$ denotes the solution of Equation (1.2) with the initial condition that $y(0) = h$. Let $u^h_K(t) = K(t)y_K(t; 0, h)$ for $t \geq 0$. The by Proposition 5.7 on Page 69 in [11], we have

$$
y_K(t; 0, h) = \Phi(t, 0)h + \int_0^t \Phi(t, s)D(s)u^h_K(s)ds, \quad \text{when } t \in \R^+ \text{ and } h \in H.
$$

From (4.7) and (1.15), it follows that

$$
P_{11} Py_K(kT; 0, h)) \in P_{11}P(\mathcal{P}^k h + V^Z_k) \text{ for all } h \in H \text{ and } k \in \mathbb{N}.
$$

Since $PV_k^Z \triangleq \hat{V}_k^Z \subseteq \hat{V}^Z$ for all $k \in \mathbb{N}$ (see (1.16) and (2.19)) and $P_{11}$ is the orthogonal projection from $H_1$ onto $H_1 \cap (\hat{V}^Z)^\perp$, and because of (4.1), we have

$$
P_{11} PV_k^Z \subset P_{11} \hat{V}^Z = \{0\}.
$$

This, along with (4.8) and the fact that $P\mathcal{P}^k = \mathcal{P}^k P$ for all $k \in \mathbb{N}$ (see Parts (a) and (e) in Lemma 2.2), indicates that

$$
P_{11} Py_K(kT; 0, h) = P_{11} P\mathcal{P}^k h = P_{11} \mathcal{P}^k Ph \text{ for all } h \in H \text{ and } k \in \mathbb{N}.
$$

Since $P : H \to H_1$ is a projection (see Lemma 2.2), it follows from (4.6) and (4.9) that

$$
\lim_{k \to +\infty} P_{11} \mathcal{P}^k h_1 = 0, \quad \text{when } h \in H_1.
$$

Now by (4.10), (4.5) and (2.15), we have that $\lim_{k \to +\infty} A^k_1 \alpha = 0$, when $\alpha \in \mathbb{R}^l$, from which, it follows that

$$
\sigma(A_1) \subset \mathbb{B} \quad \text{(the open unit ball in } \mathbb{C}^1\text{)}.
$$

By (4.3), it holds that $\sigma(A_1) \subset \sigma(\mathcal{P}_1)$. This, together with (4.11) and (2.16), leads to a contradiction. Hence, (a) $\implies$ (b). This completes the proof of (a) $\iff$ (b).

(b) $\iff$ (c): First of all, we introduce two complex adjoint equations as follows:

$$
\psi'(t) - A^*C \psi(t) - B(t)^*C \psi(t) = 0 \text{ in } (0, n_0 T), \quad \psi(n_0 T) \in H^C;
$$

$$
\psi'(t) - A^*C \psi(t) - B(t)^*C \psi(t) = 0 \text{ in } (0, T), \quad \psi(T) \in H^C.
$$

For each $\xi \in H^C$, Equation (4.12) (or (4.13)) with the initial condition that $\psi^\xi_{n_0}(n_0 T) = \xi$ (or $\psi^\xi(T) = \xi$) has a unique solution in $C[0, n_0 T]; H^C)$ (or $C([0, T]; H^C)$). We denote this solution by $\psi^\xi_{n_0}(\cdot)$ (or $\psi^\xi(\cdot)$). Clearly, when $\xi \in H$, $\psi^\xi_{n_0}(\cdot) \in C[0, n_0 T]; H)$ and $\psi^\xi(\cdot) \in C([0, T]; H)$ are accordingly the solutions of (4.12) and (4.13) where $A^C$ and $B(t)^C$ are replaced by $A$ and $B(t)$ respectively. One can easily check that

$$
\psi^\xi(0) = \mathcal{P}^*C \xi \quad \text{and} \quad \psi^\xi_{n_0}(0) = (\mathcal{P}^*C)^n_0 \xi \quad \text{for all } \xi \in H^C.
$$
By the $T$-periodicity of $B^\ast(\cdot)$, we see that for each $\xi \in H^C$,
\begin{equation}
\psi_{\eta_0}^\xi((k - 1)T + t) = \psi_{\eta_k}^\xi(t), \; t \in [0, T], \; k \in \{1, \ldots, n_0\}, \; \text{where} \; \xi_k \triangleq (P^*C)^{n_0-k} \xi. \quad (4.15)
\end{equation}

Now we carry out the proof of (b) $\iff$ (c) by several steps as follows.

**Step 1.** To prove that (1.17) is equivalent to the following condition:

\begin{equation}
\forall \; h \in H, \; \exists \; u^h(\cdot) \in L^2(\mathbb{R}^+; Z) \; s.t. \; Py(n_0T; 0, h, u^h) = 0, \; \text{where} \; P \; \text{is given by (1.14)}. \quad (4.16)
\end{equation}

Suppose that (1.17) holds. Then by (1.15), we have
\begin{equation}
Py(n_0T; 0, h, u^h) = P\Phi(n_0T, 0)h + P \int_0^{n_0T} \Phi(n_0T, t)D(t)u^h(t)dt = 0,
\end{equation}
which leads to (4.16).

Assume that (4.16) holds. Then for any $h \in H$, there exists $u^h(\cdot) \in L^2(\mathbb{R}^+; Z)$ such that
\begin{equation}
Py(n_0T; 0, h, u^h) = P\Phi(n_0T, 0)h = P \int_0^{n_0T} \Phi(n_0T, t)D(t)u^h(t)dt.
\end{equation}

But, we have
\begin{equation}
Py(n_0T; 0, h, u^h) = P\Phi(n_0T, 0)h + P \int_0^{n_0T} \Phi(n_0T, t)D(t)u^h(t)dt = 0,
\end{equation}
which leads to (4.16).

Assume that (4.16) holds. Then for any $h \in H$, there exists $u^h(\cdot) \in L^2(\mathbb{R}^+; Z)$ such that
\begin{equation}
Py(n_0T; 0, h, u^h) = P\Phi(n_0T, 0)h + P \int_0^{n_0T} \Phi(n_0T, t)D(t)u^h(t)dt = 0.
\end{equation}

Thus, we have
\begin{equation}
H_1 \supseteq \hat{V}_{\eta_0}^Z \triangleq P \left\{ \int_0^{n_0T} \Phi(n_0T, t)D(t)u(t)dt \mid u(\cdot) \in L^2(\mathbb{R}^+; Z) \right\}
\end{equation}
\begin{equation}
\supseteq P \left\{ \int_0^{n_0T} \Phi(n_0T, t)D(t)u^h(t)dt \mid h \in H \right\}
\end{equation}
\begin{equation}
= -P \left\{ \Phi(n_0T, 0)h \mid h \in H \right\} = P\mathcal{P}^{n_0} H.
\end{equation}

By the facts that $PP = PP$ (see (2.5)), $PH = H_1$ and $\mathcal{P}H_1 = \mathcal{P}_1H_1 = H_1$ (see (2.15) and lemma 2.4), we see that $P\mathcal{P}^{n_0} H = H_1$. This, together with (4.18), leads to (1.17).

**Step 2.** To show that $\xi \in P^*H_1$ and $\psi_{\eta_0}^\xi(0) = 0 \Rightarrow \xi = 0$

Recall Lemma 2.3. Because $\tilde{H}_1$ is an invariant subspace of $\mathcal{P}^*$, it follows from (4.14) that
\begin{equation}
\psi_{\eta_0}^\xi(0) = (\mathcal{P}^*)^{n_0}\xi = (\mathcal{P}^*|_{\tilde{H}_1})^{n_0}\xi \in \tilde{H}_1, \; \text{when} \; \xi \in \tilde{H}_1.
\end{equation}

By Lemma 2.3, we have
\begin{equation}
\sigma(\mathcal{P}^*|_{\tilde{H}_1}) \cap \mathcal{B} = \emptyset \; \text{and} \; \dim \tilde{H}_1 = n_0 < \infty.
\end{equation}

Thus, the map $(\mathcal{P}^*|_{\tilde{H}_1})^{n_0}$ is invertible from $\tilde{H}_1$ onto $\tilde{H}_1$. Then by (4.19), we see that $\xi = 0$ when $\xi \in \tilde{H}_1$ and $\psi_{\eta_0}^\xi(0) = 0$. This, together with (2.13), implies that $\xi = 0$ when $\xi \in P^*H_1$ and $\psi_{\eta_0}^\xi(0) = 0$.  

26
Step 3. To show that (4.16) \(\Rightarrow\) (1.18)

Clearly, when \(\eta, h \in H\) and \(u(\cdot) \in L^2(\mathbb{R}^+; Z)\),
\[
\langle \psi^\xi_{n_0}(0), h \rangle = \langle \eta, y(n_0 T; 0, h, u) \rangle - \int_0^{n_0 T} \langle (D(t)|_Z)^* \psi^\xi_{n_0}(t), u(t) \rangle dt.
\] (4.20)

Let \(\xi \in P^*H_1\) satisfy the conditions on the left side of (1.18). Then by (4.20) where \(\eta = \xi\) and \(\psi^\xi_{n_0}(t) = \Phi(n_0 T, t)^*\xi\), we find
\[
\langle \psi^\xi_{n_0}(0), h \rangle = \langle \xi, y(n_0 T; 0, h, u) \rangle, \text{ when } h \in H \text{ and } u(\cdot) \in L^2(\mathbb{R}^+; Z).
\] (4.21)

By (4.16), given \(h \in H\), there is a \(u^h(\cdot) \in L^2(\mathbb{R}^+; Z)\) such that
\[
Py(n_0 T; 0, h, u^h) = 0.
\] (4.22)

Since \(\xi \in P^*H_1\), there is \(g \in H_1\) such that \(\xi = P^*g\). This, along with (4.21) and (4.22), indicates
\[
\langle \psi^\xi_{n_0}(0), h \rangle = \langle \xi, y(n_0 T; 0, h, u^h) \rangle = \langle P^*g, y(n_0 T; 0, h, u^h) \rangle = \langle g, Py(n_0 T; 0, h, u^h) \rangle = 0, \forall h \in H.
\]

Hence, \(\psi^\xi_{n_0}(0) = 0\). Then by the result in Step 2, we have \(\xi = 0\), i.e., (1.18) holds.

Step 4. To show that (1.18) \(\Rightarrow\) (4.16)

Assume that (1.18) holds. Define two subspaces
\[
\Gamma \overset{\Delta}{=} \left\{ (D(\cdot)|_Z)^* \psi^\xi_{n_0}(\cdot) \mid \xi \in P^*H_1 \right\} \subseteq L^2(0, n_0 T; Z) \text{ and } \Gamma_0 \overset{\Delta}{=} \left\{ \psi^\xi_{n_0}(0) \mid \xi \in P^*H_1 \right\} \subseteq H.
\] (4.23)

Let
\[
\mathcal{L}_1 \left( (D(\cdot)|_Z)^* \psi^\xi_{n_0}(\cdot) \right) = \psi^\xi_{n_0}(0) \text{ for all } \xi \in P^*H_1.
\] (4.24)

By (1.18) and the result in Step 2, we see that \(\mathcal{L}_1 : \Gamma \mapsto \Gamma_0\) is well defined. Clearly, it is linear. Given \(h \in H\), define a linear functional \(\mathcal{F}^h\) on \(\Gamma\) by
\[
\mathcal{F}^h(\gamma) = \langle \mathcal{L}_1(\gamma), h \rangle \text{ for all } \gamma \in \Gamma.
\] (4.25)

Since \(\dim(P^*H_1) = \dim H_1 = n_0 < \infty\), it holds that \(\dim \Gamma < \infty\). Thus, \(\mathcal{F}^h \in \mathcal{L}(\Gamma; \mathbb{R})\). By the Hahn-Banach theorem, there is a \(\tilde{\mathcal{F}}^h \in \mathcal{L}(L^2(0, n_0 T; Z); \mathbb{R})\) such that
\[
\tilde{\mathcal{F}}^h(\gamma) = \mathcal{F}^h(\gamma) \text{ for all } \gamma \in \Gamma; \text{ and } \|\tilde{\mathcal{F}}^h\| = \|\mathcal{F}^h\|.
\] (4.26)

By making use of the Riesz representation theorem (see Page 59 in [6]), there exists a function \(u^h(\cdot) \in L^2(0, n_0 T; Z)\) such that
\[
\tilde{\mathcal{F}}^h(\gamma) = -\int_0^{n_0 T} \langle u^h(t), \gamma(t) \rangle dt \text{ for all } \gamma \in L^2(0, n_0 T; Z).
\] (4.27)

Now, since \(P^*H_1 = P^*H\) (see (2.13)), it follows from (4.24), (4.25), (4.26) and (4.27) that
\[
-\int_0^{n_0 T} \langle (D(t)|_Z)^* \psi^\eta_{n_0}(t), u^h(t) \rangle dt = \langle \psi^\eta_{n_0}(0), h \rangle \text{ for all } \eta \in H.
\]
Meanwhile, it follows by (4.20) that for each \( \eta \in H \),
\[
\langle \psi_{n_0}^\ast \eta(0), h \rangle = \langle P^\ast \eta, y(n_0T; 0, h, u^h) \rangle - \int_0^{n_0T} \langle (D(t)|_Z)^\ast \psi_{n_0}^\ast \eta(t), u^h(t) \rangle \, dt
\]
The above two equalities imply that
\[
\langle \eta, Py(n_0T; 0, h, u^h) \rangle = \langle P^\ast \eta, y(n_0T; 0, h, u^h) \rangle = 0 \quad \text{for all } \eta \in H,
\]
i.e., \( Py(n_0T; 0, h, u^h) = 0 \), which leads to (4.16).

From Step 1-Step 4, one can easily check that \((b) \Leftrightarrow (c)\).

\((c) \Leftrightarrow (d)\): We first show that \((c) \Rightarrow (d)\). Suppose that a subspace \( Z \) of \( U \) satisfies (1.18). Let \( \mu \) and \( \xi \) satisfy the conditions on the left side of (1.19) with the aforementioned \( Z \). Then by (2.14) (see Lemma 2.3), it holds that \( \xi \in \tilde{H}_1^C \). Hence, we can write \( \xi = \xi_1 + i\xi_2 \) with \( \xi_1, \xi_2 \in \tilde{H}_1 \). By (2.13), we have \( \xi_1, \xi_2 \in P^\ast H_1 \). By the last condition on the left side of (1.19), we have
\[
(D(t)|_Z)^\ast \psi_\xi(t) = 0 \quad \text{for a.e. } t \in (0, T).
\]
By (4.15) and the third condition on the left side in (1.19), it holds that
\[
\psi_{n_0}^\xi((k - 1)T + t) = \psi^{\mu_{n_0-k}}((k - 1)T + t) = \psi^{\mu_{n_0-k}}(t) \quad \text{for all } t \in [0, T], k = 1, \ldots, n_0.
\]
Notice that \( \psi_{n_0}^\xi(\cdot) = \psi_{n_0}^{\xi_1}(\cdot) + i\psi_{n_0}^{\xi_2}(\cdot) \). This, along with the above two equalities, yields that
\[
(D(\cdot)|_Z)^\ast \psi_{n_0}^{\xi_1}(\cdot) + i(D(\cdot)|_Z)^\ast \psi_{n_0}^{\xi_2}(\cdot) = (D(\cdot)|_Z)^\ast \psi_{n_0}^\xi(\cdot) = 0 \quad \text{over } (0, n_0T).
\]
Since \( \xi_1, \xi_2 \in P^\ast H_1 \), the above-mentioned equality, along with (1.18), leads to \( \xi_1 = \xi_2 = 0 \), i.e., \( \xi = 0 \). Hence, \( Z \) satisfies (1.19). Thus, \((c) \Rightarrow (d)\).

We next show that \((d) \Rightarrow (c)\). Suppose that a subspace \( Z \) satisfies (1.19). In order to show that \( Z \) satisfies (1.18), it suffices to prove
\[
\hat{\xi} \in (P^\ast H_1)^C \quad \text{and} \quad (D(\cdot)|_Z)^\ast \psi_{n_0}^\xi(\cdot) = 0 \quad \text{over } (0, n_0T) \Rightarrow \hat{\xi} = 0.
\] (4.28)
First of all, we notice that \( (P^\ast H_1)^C = \tilde{H}_1^C \) and \( \dim \tilde{H}_1^C = n_0 \) (see Lemma 2.3). In this step, we simply write
\[
Q \triangleq P^\ast|_{\tilde{H}_1^C} \in \mathcal{L}(\tilde{H}_1^C) \quad \text{and} \quad D_1(\cdot) \triangleq ((D(\cdot)|_Z)^\ast|_{(0, T)}) \in L^2(0, T; \mathcal{L}(H, Z)).
\]
By Lemma 2.3 and (2.8), we have that \( \sigma(Q) = \{ \bar{\lambda}_j \}_{j=1}^n \); \( l_j \) is the algebraic multiplicity of \( \bar{\lambda}_j \). Hence, \( \prod_{j=1}^n (\lambda - \bar{\lambda}_j)^{l_j} \) is the characteristic polynomial of \( Q \). Write \( \hat{l}_j \) for the geometric multiplicity of \( \bar{\lambda}_j \). Clearly, \( \hat{l}_j \leq l_j \) for all \( j \). Let \( \beta \triangleq \{ \beta_1, \ldots, \beta_{n_0} \} \) be a basis of \( (P^\ast H)^C = \tilde{H}_1^C \) such that
\[
Q(\beta_1, \ldots, \beta_{n_0}) = J(\beta_1, \ldots, \beta_{n_0}).
\] (4.29)
Here \( J \) is the Jordan matrix: \( \text{diag}\{J_{1i_1}, \ldots, J_{1i_1}, J_{2i_2}, \ldots, J_{ni_n}\} \) with

\[
J_{jk} = \begin{pmatrix}
\hat{\lambda}_j & 1 \\
\vdots & \ddots & 1 \\
\vdots \\
\hat{\lambda}_j
\end{pmatrix}
\]

a \( d_{jk} \times d_{jk} \) matrix,

where \( j = 1, \ldots, n \), \( k = 1, \ldots, \hat{i}_j \), and for each \( j \), \( \{d_{jk}\}_{k=1}^{\hat{i}_j} \) is decreasing. It is clear that \( \sum_{k=1}^{\hat{i}_j} d_{jk} = i_j \) for each \( j = 1, \ldots, n \), and \( \sum_{j=1}^{n} \sum_{k=1}^{\hat{i}_j} d_{jk} = n_0 \). We rewrite the basis \( \beta \) as

\[
\beta = \{\xi_{111}, \ldots, \xi_{1d_{i_1}}, \ldots, \xi_{1d_{i_1}d_{i_1}}, \ldots, \xi_{1d_{i_1}d_{i_1}} \}_{0}^{111}, \ldots, \xi_{n11d_{11}d_{n}}, \ldots, \xi_{n1d_{n1}}, \ldots, \xi_{n1d_{n1}d_{n1}} \}.
\]

Then by (4.29), one can easily check that for each \( j \in \{1, \ldots, n\} \) and \( k \in \{1, \ldots, \hat{i}_j\} \),

\[
(\hat{\lambda}_j I - \mathcal{Q})^q \xi_{jkr} = \begin{cases}
\xi_{jk(r-q)} & \text{when } r > q, \\
0 & \text{when } r \leq q.
\end{cases}
\]  

(4.30)

Now we assume \( \hat{\xi} \) satisfies the conditions on the left side of (4.28). Since \( \hat{\xi} \in (P^*H_1)^C = \tilde{H}_1^C \), there is a vector

\[
(C_{111}, \ldots, C_{11d_{11}}, C_{1d_{11}}, \ldots, C_{1d_{i_1}d_{i_1}}, \ldots, C_{n11}, \ldots, C_{n1d_{n1}}, C_{n1d_{n1}}, \ldots, C_{ni_nd_{ni_n}})^* \in \mathbb{C}^{n_0},
\]

such that

\[
\hat{\xi} = \sum_{j=1}^{n} \sum_{k=1}^{\hat{i}_j} \sum_{r=1}^{d_{jk}} C_{jkr} \xi_{jkr}.
\]  

(4.31)

From (4.14) and the second condition on the left side of (4.28), it follows that for each \( m \in \{0, \ldots, n_0 - 1\} \),

\[
D^1(\cdot)\psi_{n_0}(\cdot)|_{((n_0 - m - 1)T, (n_0 - m)T)} = 0 \quad \text{i.e.} \quad \sum_{j=1}^{n} \sum_{k=1}^{\hat{i}_j} \sum_{r=1}^{d_{jk}} C_{jkr} D^1(t)\psi^m(t) = 0 \quad \text{for a.e. } t \in (0, T),
\]

from which, we see that

\[
\sum_{j=1}^{n} \sum_{k=1}^{\hat{i}_j} \sum_{r=1}^{d_{jk}} C_{jkr} D^1(\cdot)\psi^g(\xi_{jkr}) = 0 \quad \text{over } (0, T)
\]  

(4.32)

for any polynomial \( g \) with degree(\( g \)) \( \leq n_0 - 1 \). Arbitrarily fix a \( \tilde{j} \in \{1, \ldots, n\} \). Let

\[
\mathbb{P}_{\tilde{j}}(\lambda) = \prod_{j=1, j \neq \tilde{j}}^{n} (\lambda - \hat{\lambda}_j)^{i_j}.
\]

29
By taking \( g(\lambda) = \lambda^m P_j(\lambda) \), with \( m = 0, 1, \ldots, \hat{l}_j - 1 \), in (4.32), we have
\[
\sum_{j=1}^{n} \sum_{k=1}^{l_j} \sum_{r=1}^{d_{jk}} C_{jk} D_1(\cdot) \psi^{Q^m P_j(Q) \xi_{jkr}}(\cdot) = 0 \quad \text{over} \ (0, T), \quad \text{when} \ m \in \{0, 1, \ldots, \hat{l}_j - 1\}.
\]

By (4.30), we see that
\[
P_j(Q) \xi_{jkr} = 0, \quad \text{when} \ j \in \{1, \ldots, n\}, j \neq \hat{j}, k \in \{1, \ldots, \hat{l}_{\hat{j}}\}, r \in \{1, \ldots, d_{jk}\}.
\]

The above two equalities imply that for each \( m \in \{0, 1, \ldots, \hat{l}_j - 1\}, \)
\[
\sum_{k=1}^{l_j} \sum_{r=1}^{d_{jk}} C_{jk} D_1(\cdot) \psi^{Q^m P_j(Q) \xi_{jkr}}(\cdot) = 0 \quad \text{over} \ (0, T),
\]
from which, it follows that
\[
\sum_{k=1}^{l_j} \sum_{r=1}^{d_{jk}} C_{jk} D_1(\cdot) \psi^{f(Q) P_j(Q) \xi_{jkr}}(\cdot) = 0 \quad \text{over} \ (0, T), \quad (4.33)
\]
for any polynomial \( f \) with degree(\( f \)) \( \leq \hat{l}_j - 1 \). Given \( m \in \{0, 1, 2, \ldots, \hat{l}_j - 1\} \), since \( P_j(\lambda) \) and \( (\lambda - \hat{\lambda}_j)^{m+1} \) are coprime, there are polynomials \( g_m^1(\lambda) \) and \( g_m^2(\lambda) \) with degree(\( g_m^1 \)) \( \leq m \) and degree(\( g_m^2 \)) \( \leq \) degree(\( P_j \)) \( - 1 \), respectively, such that
\[
g_m^1(\lambda) P_j(\lambda) + g_m^2(\lambda) (\lambda - \hat{\lambda}_j)^{m+1} \equiv 1,
\]
from which, we see that
\[
(\xi - \hat{\lambda}_j)^{l_j - m - 1} g_m^1(\xi) P_j(\xi) \xi_{jkr} + g_m^2(\xi) (\xi - \hat{\lambda}_j)^{l_j - m - 1} \xi_{jkr} \equiv (\xi - \hat{\lambda}_j)^{l_j - m - 1} \xi_{jkr}, \quad (4.34)
\]
for all \( m \in \{0, 1, \ldots, l_j - 1\} \), \( k \in \{1, 2, \ldots, \hat{l}_j\} \), and \( r \in \{1, 2, \ldots, d_{jk}\} \). By (4.30), we have
\[
(\xi - \hat{\lambda}_j)^{l_j} \xi_{jkr} = 0 \quad \text{for all} \ k \in \{1, 2, \ldots, \hat{l}_j\}, r \in \{1, 2, \ldots, d_{jk}\}.
\]

Taking \( f(\lambda) = (\lambda - \hat{\lambda}_j)^{l_j - m - 1} g_m^1(\lambda) \), with \( m = 0, \ldots, l_j - 1 \), in (4.33), using (4.34) and (4.35), we find
\[
\sum_{k=1}^{l_j} \sum_{r=1}^{d_{jk}} C_{jk} D_1(\cdot) \psi^{(\lambda - \hat{\lambda}_j)^m \xi_{jkr}}(\cdot) = 0 \quad \text{over} \ (0, T), \quad \text{for each} \ m \in \{0, 1, \ldots, \hat{l}_j - 1\}. \quad (4.36)
\]

Now we are on the position to show
\[
C_{jk} = 0 \quad \text{for all} \ k \in \{1, 2, \ldots, \hat{l}_j\}, r \in \{1, \ldots, d_{jk}\}, \quad (4.37)
\]
which leads to \( \hat{\xi} = 0 \) because of (4.31). For this purpose, we write
\[
K_m^j = \left\{ k \in \{1, 2, \ldots, \hat{l}_j\} \mid d_{jk} > m \right\}, \quad m = 0, 1, \ldots, \hat{l}_j - 1.
\]
One can easily check that (4.37) is equivalent to
\[ C_{\tilde{m}} \Delta \left\{ C_{jk\tilde{m}} : \hat{m} \in K_{j}^{\tilde{m}-1} \right\} = \{0\} \] (for all $\hat{m} \in \{1, \ldots, d_{j}^{\tilde{m}}\}$) \hspace{1cm} (4.38)

We will use the mathematical induction method with respect to $\hat{m}$ to prove (4.38). (Notice that $d_{jk}$ is decreasing with respect to $k$.) First of all, we let
\[ Q_{j}^{\tilde{m}}(\lambda) = \left( \hat{\lambda}_{j} - \lambda \right)^{\tilde{m}}_{m}, \quad m = 0, 1, \ldots, \hat{\ell}_{j} - 1, \] \hspace{1cm} (4.39)

In the case that $\hat{m} = d_{j}^{\tilde{m}}$, it follows from (4.39) and (4.30) that
\[ Q_{j}^{\tilde{m}-1}(Q)\xi_{jk\hat{m}} = (\hat{\lambda}_{j}I - Q)^{\tilde{m}-1} \xi_{jk\hat{m}} = \xi_{jk1}, \quad \text{when } k \in K_{j}^{\tilde{m}-1}, \]
and
\[ Q_{j}^{\tilde{m}-1}(Q)\xi_{jk0} = 0, \quad \text{when } k \in K_{j}^{\tilde{m}-1}, r < \hat{m}; \text{ or } k \notin K_{j}^{\tilde{m}-1}, r \in \{1, \ldots, d_{j}^{\tilde{m}}\}. \]

These, along with (4.36) (where $m = \hat{m} - 1$), imply that
\[ \sum_{k \in K_{j}^{\tilde{m}-1}} C_{jk\hat{m}}D_{1}(\cdot)\psi_{\xi_{jk1}}(\cdot) = 0 \quad \text{over } (0, T). \]

Let
\[ \tilde{\xi}_{\hat{m}} \Delta \sum_{k \in K_{j}^{\tilde{m}-1}} C_{jk\hat{m}}\xi_{jk1}, \quad \hat{m} = 1, \ldots, d_{j}^{\tilde{m}}. \]

Then, it holds that
\[ D_{1}(\cdot)\psi_{\tilde{\xi}_{\hat{m}}}(\cdot) = 0 \quad \text{over } (0, T). \] \hspace{1cm} (4.40)

Since for each $k \in \{1, \ldots, \hat{\ell}_{j}\}$, $\xi_{jk1}$ is an eigenfunction of $Q$ with respect to the eigenvalue $\hat{\lambda}_{j}$, it follows from the definition of $\tilde{\xi}_{\hat{m}}$ that $(\hat{\lambda}_{j}I - Q)\tilde{\xi}_{\hat{m}} = 0$. This, along with (4.40) and (1.19), yields that $\tilde{\xi}_{\hat{m}} = 0$, i.e., $\tilde{\xi}_{d_{j}^{\tilde{m}}1} = 0$, which leads to $C_{d_{j}^{\tilde{m}}1} = 0$ because of the linear independence of the group $\{\xi_{jk1}, \quad k \in K_{j}^{\tilde{m}-1}\}$. Hence, (4.38) holds when $\hat{m} = d_{j}^{\tilde{m}}$.

Suppose inductively that (4.38) holds when $\hat{m} + 1 \leq \hat{m} \leq d_{j}^{\tilde{m}}$ for some $\hat{m} \in \{1, \ldots, d_{j}^{\tilde{m}} - 1\}$, i.e.,
\[ C_{\hat{m}} = \{0\}, \quad \text{when } \hat{m} + 1 \leq \hat{m} \leq d_{j}^{\tilde{m}}. \] \hspace{1cm} (4.41)

We will show that (4.38) holds when $\hat{m} = \hat{\ell}_{j}$, i.e., $C_{\hat{m}} = \{0\}$. In fact, it follows from (4.30) that
\[ Q_{j}^{\hat{\ell}_{j}}(Q)\xi_{jk\hat{m}} = \begin{cases} \xi_{jk(r - \hat{m} + 1)}, & \text{when } k \in K_{j}^{\hat{m}-1}, \ r \geq \hat{m}, \\ 0, & \text{when } k \in K_{j}^{\hat{m}-1}, \ r < \hat{m}, \\ 0, & \text{when } k \notin K_{j}^{\hat{m}-1}, \ r \in \{1, \ldots, d_{j}^{\hat{m}}\}. \end{cases} \]

This, alone with (4.36) (where $m = \hat{m} - 1$), indicates that
\[ \sum_{k=1}^{\hat{\ell}_{j}} \sum_{r=1}^{d_{jk}^{\hat{m}}} C_{jk\hat{m}}D_{1}(\cdot)\psi_{Q_{j}^{\hat{m}-1}(Q)\xi_{jk\hat{m}}}(\cdot) = \sum_{k \in K_{j}^{\hat{m}-1}} \sum_{r=\hat{m}}^{d_{jk}^{\hat{m}}} C_{jk\hat{m}}D_{1}(\cdot)\psi_{\xi_{jk(r - \hat{m} + 1)}}(\cdot) = 0 \quad \text{over } (0, T). \]

31
Then, by (4.41), we have
\[ \sum_{k \in \hat{K}^{\hat{m}-1}} C_{jk\hat{m}} D_1(\cdot) \psi_{jk1}(\cdot) = 0 \text{ over } (0, T). \] (4.42)

Let
\[ \bar{\xi}_{\hat{m}} \overset{\Delta}{=} \sum_{k \in \hat{K}^{\hat{m}-1}} C_{jk\hat{m}} \xi_{jk1}. \]

Then, it follows from (4.42) that
\[ D_1(\cdot) \psi_{\bar{\xi}_{\hat{m}}}(\cdot) = 0 \text{ over } (0, T). \] (4.43)

Since for each \( k \in \{1, \ldots, \hat{l}_j\} \), \( \xi_{jk1} \) is an eigenfunction of \( Q \) with respect to the eigenvalue \( \bar{\lambda}_j \), it holds that \( (\bar{\lambda}_j I - Q) \bar{\xi}_{\hat{m}} = 0 \). This, along with (4.43) and (1.19), yields that \( \bar{\xi}_{\hat{m}} = 0 \). Hence, \( C_{\hat{m}} = \{0\} \) because of the linear independence of the group \( \{\xi_{jk1}, k \in \hat{K}^{\hat{m}-1}\} \). In summary, we conclude that \((d) \Rightarrow (c)\). This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Clearly, it suffice to show the only if part. Assume that Equation (1.1) is LPFS. By the equivalence of \((a)\) and \((b)\) in Theorem 1.1, it holds that
\[ \hat{V}_{n_0}^U = H_1. \] (4.44)

Meanwhile, according to Lemma 2.5, there is a finite dimensional subspace of \( \hat{Z} \) of \( U \), such that
\[ \hat{V}_{n_0}^{\hat{Z}} = \hat{V}_{n_0}^U. \] (4.45)

From (4.44) and (4.45), it follows that \( \hat{V}_{n_0}^{\hat{Z}} = H_1 \). This, along with the equivalence of \((a)\) and \((b)\) in Theorem 1.1, indicates that Equation (1.1) is LPFS with respect to \( \hat{Z} \). Hence, we complete the proof of Theorem 1.2.

5 Applications

In this section, we will present some applications of Theorem 1.1 to the internally controlled heat equations with time-periodic potential.

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^d \) \((d \geq 1)\) with a \( C^2 \)-smooth boundary \( \partial \Omega \). Write \( Q \overset{\triangle}{=} \Omega \times \mathbb{R}^+ \) and \( \sum \overset{\triangle}{=} \partial \Omega \times \mathbb{R}^+ \). Let \( \omega \subseteq \Omega \) be a non-empty open subset with its characteristic function \( \chi_\omega \).

Let \( T > 0 \) and \( a \in L^\infty(Q) \) be \( T \)-periodic (with respect to the time variable \( t \)), i.e., for a.e. \( t \in \mathbb{R}^+ \), \( a(\cdot, t) = a(\cdot, t + T) \) over \( \Omega \). One can easily check that the function \( a \) can be treated as a \( T \)-periodic function in \( L^1_{loc}(\mathbb{R}^+; L^2(\Omega)) \). Consider the following controlled heat equation:
\[ \begin{cases} \partial_t y(x, t) - \Delta y(x, t) + a(x, t)y(x, t) = \chi_\omega(x)u(x, t) & \text{in } Q, \\ y(x, t) = 0 & \text{on } \sum, \end{cases} \] (5.1)
where \( u \in L^2(\mathbb{R}^+; L^2(\Omega)) \). Given \( y_0 \in L^2(\Omega) \) and \( u \in L^2(\mathbb{R}^+; L^2(\Omega)) \), Equation (5.1) with the initial condition that \( y(x,0) = y_0(x) \) has a unique solution \( y(\cdot;0,y_0,u) \in C(\mathbb{R}^+; L^2(\Omega)) \).

Let \( H = U = L^2(\Omega) \) and \( A = -\Delta \) with \( \mathcal{D}(A) = H^1_0(\Omega) \cap H^2(\Omega) \). Define, for a.e. \( t \in \mathbb{R}^+ \), \( B(t) : H \rightarrow H \) by \( B(t)z(x) = a(x,t)z(x), \) \( x \in \Omega \), and \( D(t) : U \rightarrow H \) by \( D(t)v(x) = \chi_\omega(x)v(x), \) \( x \in \Omega \). Clearly, \( -A \) generates a compact semigroup on \( L^2(\Omega) \) and both \( B(\cdot) \in L^1_{\text{loc}}(\mathbb{R}^+; L^2(\Omega)) \) and \( D(\cdot) \in L^\infty(\mathbb{R}^+; \mathcal{L}(U;H)) \) are \( T \)-periodic. Thus, we can study Equation (5.1) under the framework (1.1). Write \( \{\Psi_a(t,s)\}_{0 \leq s \leq t} \) for the evolution system generated by \(-A - B(\cdot)\). We use notations \( n_0, P, H_j \) (with \( j = 1,2 \)), \( V_k^U \) and \( \hat{V}_k^Z \) (with \( k \in \mathbb{N} \)) to denote the same subjects as those introduced in section 1. We will use the different equivalent conditions in Theorem 1.1 to show that Equation (5.1) is LPFS.

**Corollary 5.1** Equation (5.1) is LPFS. Consequently, it is LPFS with respect to a finite dimensional subspace of \( L^2(\Omega) \).

**Proof.** We will provide two ways to show that Equation (5.1) is LPFS. We first use the equivalence \((a) \iff (c)\) in Theorem 1.1. In fact, \( \psi(\cdot) = \Psi_a(n_0T,\cdot)\ast \xi \) with \( \xi \in H \) is the solution to the equation:

\[
\begin{aligned}
\partial_t \psi(x,t) + \triangle \psi(x,t) - a(x,t)\psi(x,t) &= 0 \quad \text{in } \Omega \times (0,n_0T), \\
\psi(x,t) &= 0 \quad \text{on } \partial \Omega \times (0,n_0T), \\
\psi(x,n_0T) &= \xi(x) \quad \text{in } \Omega,
\end{aligned}
\]  

and it holds that

\[
D(t)\eta = \chi_\omega \eta \quad \text{for any } \eta \in H \text{ and } t \in [0,T].
\]  

These, along with the unique continuation property of parabolic equations established in [12] (see also [15] and [16]), leads to the condition \((c)\) in Theorem 1.1 for the current case. Then, according to the equivalence \((a) \iff (c)\) in Theorem 1.1, Equation (5.1) is LPFS.

We next use the equivalence \((a) \iff (b)\) in Theorem 1.1. Without loss of generality, we can assume that \( n_0 \geq 1 \), for otherwise Equation (5.1), with the null control \( u = 0 \), is stable. When \( n_0 \geq 1 \), we have \( H_1 \neq \{0\} \) and \( \|P\| > 0 \). Write \( \{\xi_1,\ldots,\xi_{n_0}\} \) for an orthonormal basis of \( H_1 \). By the approximate controllability of the heat equation (see [7]), \( V_1^U \) is dense in \( H \). Thus there are \( \eta_j, j = 1, \ldots, n_0 \), in \( V_1^U \) such that such that

\[
\|\eta_j - \xi_j\| \leq \frac{1}{16n_0\|P\|} \quad \text{for all } j = 1, \ldots, n_0.
\]  

Since \( P \) is a projection from \( H \) onto \( H_1 \), we have \( P\xi_j = \xi_j \) for all \( j = 1, \ldots, n_0 \). This, along with (5.4), yields that for each \( j \in \{1, \ldots, n_0\} \),

\[
\|P\eta_j\| \leq \|\xi_j\| + \|P\eta_j - \xi_j\| = \|\xi_j\| + \|P(\eta_j - \xi_j)\|
\]

\[
\leq \|\xi_j\| + \|P\|\|\eta_j - \xi_j\| \leq 1 + \frac{1}{16n_0};
\]

and

\[
\langle P\eta_j, \xi_j \rangle = \langle \xi_j + (P\eta_j - \xi_j), \xi_j \rangle = 1 + \langle P\eta_j - \xi_j, \xi_j \rangle
\]

\[
\geq 1 - \|P\eta_j - \xi_j\| \geq 1 - \|P\|\|\eta_j - \xi_j\| \geq 1 - \frac{1}{16n_0}.
\]
Suppose that $\mu$ satisfies (1.19), i.e.,

$$\mu \notin \mathbb{B}, \xi \in H^C, (\mu I - P^*P)\xi = 0, (D(\cdot)|_Z)^{*C}\Psi_a(T, \cdot)^{*C}\xi = 0 \text{ over } (0, T) \Rightarrow \xi = 0. \quad (5.10)$$

Suppose that $\mu$ and $\xi$ satisfy the conditions on the left side of (5.10). Write $\xi = \xi_1 + i\xi_2$ where $\xi_1, \xi_2 \in H$. Then, we have

$$(D(\cdot)|_Z)^{*C}\Psi_a(T, \cdot)^{*C}\xi_j = 0, \quad j = 1, 2. \quad (5.11)$$

Finally, according to Theorem 1.2, there is a finite-dimensional subspace $Z$ of $U$ such that Equation (5.1) is LPFS with respect to $Z$. This completes the proof.

**Corollary 5.2** Equation (5.1) is LPFS with respect to the subspace $P^*H$.

**Proof.** Let $Z = P^*H$. By the equivalence between (a) and (d) in Theorem 1.1, it suffices to show that $Z$ satisfies (1.19), i.e.,

$$\mu \notin \mathbb{B}, \xi \in H^C, (\mu I - P^*P)\xi = 0, (D(\cdot)|_Z)^{*C}\Psi_a(T, \cdot)^{*C}\xi = 0 \text{ over } (0, T) \Rightarrow \xi = 0. \quad (5.10)$$

From (5.7), (5.5) and (5.6), it follows that for each $j \in \{1, \ldots, n_0\}$,

$$\sum_{k \neq j} |\langle P\eta_j, \xi_k \rangle| \leq (n_0 - 1)^{1/2} \left( \sum_{k \neq j} |\langle P\eta_j, \xi_k \rangle|^2 \right)^{1/2}$$

$$\leq (n_0 - 1)^{1/2} (\|P\eta_j\|^2 - |\langle P\eta_j, \xi_j \rangle|^2)^{1/2}$$

$$\leq n_0^{1/2} \left( (1 + 1/(16n_0))^2 - (1 - 1/(16n_0))^2 \right)^{1/2} = 1/2.$$
Clearly, \( \psi_j(\cdot) = \Psi_a(T, \cdot) \xi_j \) (with \( j = 1, 2 \)) is the solution to the equation (5.2) where \( n_0T \) and \( \xi \) are replaced by \( T \) and \( \xi_j \) respectively. Since \( \Psi_a(T, \cdot) \xi_j \) is continuous on \([0, T]\) and \( D(t) \) is independent of \( t \), it follows from (5.11) that

\[
(D(0)|_z)^* \Psi_a(T, 0)^* \xi_j = 0, \quad j = 1, 2.
\] (5.12)

For each \( \eta \in H \), we have \( P^* \eta \in P^* H \). This, along with (5.3), yields

\[
\langle (D(0)|_z)^* \Psi_a(T, 0)^* \xi_j, P^* \eta \rangle = \langle \Psi_a(T, 0)^* \xi_j, (D(0)|_z)P^* \eta \rangle = \langle \Psi_a(T, 0)^* \xi_j, \chi_\omega P^* \eta \rangle = \langle P\chi_\omega \Psi_a(T, 0)^* \xi_j, \eta \rangle, \quad j = 1, 2.
\]

This, along with (5.12), implies that \( P\chi_\omega \Psi_a(T, 0)^* \xi_j = 0, \quad j = 1, 2 \), from which, we have

\[
\langle P^* \Psi_a(T, 0)^* \xi_j, \chi_\omega \Psi_a(T, 0)^* \xi_j \rangle = \langle \Psi_a(T, 0)^* \xi_j, P^* \chi_\omega \Psi_a(T, 0)^* \xi_j \rangle = 0, \quad j = 1, 2.
\] (5.13)

Two facts are as follows. First, it follows from (2.5) that

\[
P^* \Psi_a(T, 0)^* \xi_j = \Psi_a(T, 0)^* P^* \xi_j, \quad j = 1, 2.
\] (5.14)

Second, by (2.14), (2.13), and the first three conditions on the left side of (5.10), we have \( \xi \in \tilde{H}_1^C \). Since \( P^* = \tilde{P} \) and \( \tilde{P} \) is a projection from \( H \) to \( \tilde{H}_1 \) (see Lemma 2.3), we see that \( P^* : H \rightarrow \tilde{H}_1 \) is a projection. Hence, \( P^{*C} : H^C \rightarrow \tilde{H}_1^C \) is a projection. These two facts yields that \( P^{*C} \xi = \xi \), from which, it follows that \( P^* \xi_j = \xi_j, \quad j = 1, 2 \). This along with (5.13) and (5.14), indicates that \( \| \chi_\omega \Psi_a(T, 0)^* C \xi_j \| = 0 \), i.e., \( \chi_\omega \psi_j(T) = 0 \). By the unique continuation property of parabolic equations established in [12] (see also [15] and [16]), we find that \( \xi_j = 0, \quad j = 1, 2 \), which leads to \( \xi = 0 \). This completes the proof.

Finally, we will present a controlled heat equation which is not LPFS. Write \( \lambda_1 \) and \( \lambda_2 \) for the first and the second eigenvalues of the operator \(-\Delta\) with \( \mathcal{D}(-\Delta) = H_0^1(\Omega) \cap H^2(\Omega) \), respectively. Let \( \xi_j, \quad j = 1, 2 \), be an eigenfunction corresponding to \( \lambda_j \). Consider the following heat equation:

\[
\begin{cases}
\partial_t y(x, t) - \Delta y(x, t) - \lambda_2 y(x, t) = \langle u(t), \xi_1 \rangle \xi_1(x) & \text{in } Q, \\
y(x, t) = 0 & \text{on } \Sigma ,
\end{cases}
\] (5.15)

where \( u(\cdot) \in L^2(\mathbb{R}^+; L^2(\Omega)) \). By a direct calculation, one has that

\[ V_{n_0} = \text{span} \{ \xi_1 \} \quad \text{and} \quad H_1 \supset \text{span} \{ \xi_1, \xi_2 \} . \]

These, along with the equivalence \((a) \Leftrightarrow (b)\) in Theorem 1.1, indicates that (5.15) is not LPFS.

**Appendix**

*The proof of Lemma 2.1.* By the compactness of \( \{ S(t) \}_{t \geq 0} \), the assumption \((H_2)\) and (1.4), one can easily check that each \( \mathcal{P}(t) \), with \( t \geq 0 \), is compact. Hence, each \( \mathcal{P}(t)^C : H^C \rightarrow H^C \), with \( t \geq 0 \), is
also compact. Thus, for each $t \geq 0$, $\sigma(\mathcal{P}(t)^C) \setminus \{0\}$ consists of all nonzero eigenvalues $\{\lambda_j(t)\}_{j=1}^{\infty}$ (in $\mathbb{C}$) of $\mathcal{P}(t)^C$ such that $\lim_{j \to \infty} |\lambda_j(t)| = 0$.

We next show that $\{\lambda_j(t)\}_{j=1}^{\infty}$ is independent of $t$. For this purpose, we arbitrarily fix $s_1$ and $s_2$ with $0 \leq s_1 \leq s_2 + T$. Let $\lambda \in \mathbb{C}$ be a non-zero eigenfunction of $\mathcal{P}(s_1)^C$ and $\eta \in H^C$ be a corresponding eigenfunction, i.e.,

$$\mathcal{P}(s_1)^C \eta = \lambda \eta.$$  \hspace{1cm} (a.1)

Write $\lambda = \alpha_1 + i\alpha_2$ with $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\eta = \eta_1 + i\eta_2$ with $\eta_1, \eta_2 \in H$. By (a.1), we have

$$\mathcal{P}(s_1)\eta_1 = \alpha_1 \eta_1 - \alpha_2 \eta_2, \quad \mathcal{P}(s_1)\eta_2 = \alpha_2 \eta_1 + \alpha_1 \eta_2.$$ \hspace{1cm} (a.2)

From (a.2) and (1.6), one can easily check that $\mathcal{P}(s_2)^C(\Phi(s_2, s_1)^C \eta) = \lambda \Phi(s_2, s_1)^C \eta$. This implies that $\lambda$ is an eigenvalue of $\mathcal{P}(s_2)^C$ and $\Phi(s_2, s_1)^C \eta$ is a corresponding eigenfunction. Hence,

$$\sigma(\mathcal{P}(s_1)^C) \setminus \{0\} \subseteq \sigma(\mathcal{P}(s_2)^C) \setminus \{0\}.$$  \hspace{1cm} (a.3)

Similarly, we can show

$$\sigma(\mathcal{P}(s_2)^C) \setminus \{0\} \subseteq \sigma(\mathcal{P}(s_1 + T)^C) \setminus \{0\}.$$  \hspace{1cm} (a.4)

Then by the $T$-periodicity of $\mathcal{P}(\cdot)$, $\sigma(\mathcal{P}(t)^C) \setminus \{0\}$ is independent of $t$. This completes the proof.

\textit{The proof of Lemma 2.2.} First of all, we let

$$\hat{H}_1(t) \triangleq \hat{P}(t)H^C \quad \text{and} \quad \hat{H}_2(t) \triangleq (I - \hat{P}(t))H^C, \quad t \geq 0.$$ \hspace{1cm} (a.5)

From Theorem 6.17 on Page 178 in [9], it follows that when $t \geq 0$, both $\hat{H}_1(t)$ and $\hat{H}_2(t)$ are invariant w.r.t. $\mathcal{P}(t)^C$;

$$\hat{P}(t) : H^C(t) \to \hat{H}_1(t) \quad \text{is a projection;}$$ \hspace{1cm} (a.6)

$$H^C = \hat{H}_1(t) \bigoplus \hat{H}_2(t);$$ \hspace{1cm} (a.7)

and

$$\sigma(\mathcal{P}(t)^C|_{\hat{H}_1(t)}) = \{\lambda_j\}_{j=1}^{\infty} \quad \text{and} \quad \sigma(\mathcal{P}(t)^C|_{\hat{H}_2(t)}) \setminus \{0\} = \{\lambda_j\}_{j=n+1}^{\infty},$$ \hspace{1cm} (a.8)

where $\{\lambda_j\}_{j=1}^{\infty}$ and $n$ are given by (1.8) and (1.9) respectively.

Then we prove that the operator $P(t)$, with $t \geq 0$, is a linear operator from $H$ to $H$. For this purpose, it suffices to show that

$$\hat{P}(t)h \in H, \quad \text{when } h \in H \quad \text{and} \quad t \geq 0.$$ \hspace{1cm} (a.9)

The proof of (a.7) is as follows. By (1.12), it holds that

$$\hat{P}(t)h = \frac{-\delta}{2\pi} \int_{0}^{2\pi} (\delta e^{i\theta} I - \mathcal{P}(t)^C)^{-1} e^{i\theta} d\theta h, \quad \text{when } h \in H \quad \text{and} \quad t \geq 0.$$ \hspace{1cm} (a.10)

Write $F$ for the conjugate map from $H^C$ to $H^C$, i.e., $F(h + ig) = h - ig$ for any $h, g \in H$. We claim

$$F(\delta e^{i\theta} I - \mathcal{P}(t)^C)^{-1} e^{i\theta} h) = (\delta e^{-i\theta} I - \mathcal{P}(t)^C)^{-1} e^{-i\theta} h, \quad \text{for all } \theta \in [0, 2\pi], h \in H \quad \text{and} \quad t \geq 0.$$ \hspace{1cm} (a.11)
When (a.9) is proved, it follows from (a.8) and (a.9) that

\[ F(\hat{P}(t))h = \frac{-\delta}{2\pi} \int_{0}^{2\pi} (\delta e^{-i\theta} I - \mathcal{P}(t)^C)^{-1} e^{-i\theta} d\theta h \]

\[ = \frac{-\delta}{2\pi} \int_{0}^{2\pi} (\delta e^{i\theta} I - \mathcal{P}(t)^C)^{-1} e^{i\theta} d\theta h = \hat{P}(t)h \text{ for each } t \geq 0, h \in H, \]

Which leads to (a.7). Now we are on the position to show (a.9). Arbitrarily fix \( \theta \in [0, \pi], t \geq 0 \) and \( h \in H \). Write

\[ (\delta e^{i\theta} I - \mathcal{P}(t)^C)^{-1} e^{i\theta} h = g_1 + ig_2, \quad g_1, g_2 \in H. \quad \text{(a.10)} \]

It is clear that \((\delta e^{i\theta} I - \mathcal{P}(t)^C)(g_1 + ig_2) = e^{i\theta} h\), from which, one can directly check that

\[ (\delta e^{-i\theta} I - \mathcal{P}(t)^C)(g_1 - ig_2) = e^{-i\theta} h. \]

Hence, \((\delta e^{-i\theta} I - \mathcal{P}(t)^C)^{-1}(e^{-i\theta} h) = g_1 - ig_2 = F(g_1 + ig_2)\). This, along with (a.10), leads to (a.9).

Next we prove that \(P(t)\), with \( t \geq 0 \), is a projection on \( H \). Let \( H_1(t) \) and \( H_2(t) \), with \( t \geq 0 \), be defined by (2.2). Two observations are given in order:

\[ \hat{H}_1(t) \triangleq \hat{P}(t)H^C = \{ \hat{P}(t)(h_1 + ih_2) \mid h_1, h_2 \in H \} = \{ P(t)h_1 + iP(t)h_2 \mid h_1, h_2 \in H \} \]

\[ = P(t)H + iP(t)H \triangleq H_1(t) + iH_1(t) \triangleq H_1^C(t); \quad \text{(a.11)} \]

\[ \hat{H}_2(t) = H_2^C(t). \quad \text{(a.12)} \]

By (a.4) and (a.11), we see that \(\hat{P}(t)\) (with \( t \geq 0 \)) is a projection from \( H^C \) onto \( H_1(t)^C \). Thus, for each \( t \geq 0 \),

\[ P(t)(h_1 + h_2) = \hat{P}(t)(h_1 + h_2) = h_1, \text{ when } h_1 \in H_1(t), h_2 \in H_2(t), \]

i.e., \(P(t)\) is a projection from \( H \) onto \( H_1(t) \). Besides, (2.1) follows from (a.5), (a.11) and (a.12).

Finally, we will show properties (a)-(f) one by one.

The proof of (a): Since \(\mathcal{P}(\cdot)\) is \(T\)-periodic, so is \(\hat{P}(\cdot)\) (see (1.12)). This, along with (1.13), indicates the \(T\)-periodicity of \(P(\cdot)\). Then by (2.2), both \(H_1(\cdot)\) and \(H_2(\cdot)\) are \(T\)-periodic.

The proof of (b): Let \( t \geq 0 \). Since \(\hat{H}_1(t)\) and \(\hat{H}_2(t)\) are invariant w.r.t. \(\mathcal{P}(t)^C\), so are \(H_1(t)^C\) and \(H_2(t)^C\) (see (a.11) and (a.12)). Hence, \(H_1(t)\) and \(H_2(t)\) are invariant w.r.t. \(\mathcal{P}(t)\).

The proof of (c): (2.3) follows from (a.6), (a.11) and (a.12). Meanwhile, by (1.11) and (2.3), we see that \(\dim H_1(t)^C = n_0\), which leads to (2.4).

The proof of (d) and (e): Let \( 0 \leq s \leq t < \infty \). By (1.6), we have that \(\Phi(t, s)\mathcal{P}(s) = \mathcal{P}(t)\Phi(t, s)\). From this, one can directly verify that \(\Phi(t, s)^C \hat{P}(s) = \hat{P}(t)\Phi(t, s)^C\). This, along with (2.2), (1.13) and (a.7), indicates that

\[ \Phi(t, s)H_1(s) \subseteq P(t)H \triangleq H_1(t), \quad \text{(a.13)} \]

which leads to (e). Meanwhile, it follows from (a.13) that \(\Phi(t, s) \in \mathcal{L}(H_1(s), H_1(t))\). Similarly, one can show that \(\Phi(t, s) \in \mathcal{L}(H_2(s), H_2(t))\). Hence, (d) stands.

37
The proof of (f): Let \( \tilde{\rho} \triangleq (-\ln \tilde{\delta})/T > 0 \) with \( \tilde{\delta} \) given by (1.10). Because of (2.3), it follows from Theorem 4 on Page 212 in [23] that the spectral radius of \( I_{\mathbb{H}_2(0)^C} \) equals to \( \tilde{\delta} \). Thus, we have

\[
\tilde{\delta} = \lim_{k \to \infty} \| (P(0)^C|_{\mathbb{H}_2(0)^C})^k \| \frac{1}{k}.
\]

Now we arbitrarily fix a \( \rho \in (0, \tilde{\rho}) \) where \( \tilde{\rho} \) is given by (1.10). Then it holds that \( \tilde{\delta} \triangleq e^{-\tilde{\rho}T} < e^{-\rho T} \). Thus there is positive integer \( \tilde{N} \) such that \( \| (P(0)^C|_{\mathbb{H}_2(0)^C})^k \| < e^{-\rho kT} \) for all \( k \geq \tilde{N} \), which implies

\[
\| (P(0)|_{\mathbb{H}_2(0)})^k \| < e^{-\rho kT} \text{ for all } k \geq \tilde{N}.
\]

(a.14)

Notice that \( \Phi(\cdot, \cdot) \) is continuous from \( [0, T] \times [0, T] \) to \( \mathcal{L}(H) \) (see Lemma 5.6 on Page 68 in [11]). Thus, we can write

\[
C_1 \triangleq \max_{0 \leq t_1 \leq t_2 \leq T} \| \Phi(t_2, t_1) \| \in \mathbb{R}^+; \quad C_\rho \triangleq (C_1 + 1)^2 e^{3\rho T} \in \mathbb{R}^+.
\]

(a.15)

We are going to show that the above \( C_\rho \) satisfies (2.6). For this purpose, we let \( 0 \leq s \leq t < \infty \) and \( h_2 \in \mathbb{H}_2(s) \). For each \( r \in \mathbb{R}^+ \), we denote by \( [r] \) the integer such that \( r - 1 < [r] \leq r \). There are only two possibilities: (i) \( [t/T] = [s/T] \) and (ii) \( [t/T] \neq [s/T] \). In the first case, it follows from (1.6) and (a.15) that

\[
\| \Phi(t, s)h_2 \| = \| \Phi(t - [s/T] T, s - [s/T] T)h_2 \| \leq \| \Phi(t - [s/T] T, s)h_2 \| < (C_1 + 1)^2 e^{3\rho T} e^{-\rho(l-s)} ||h_2|| = C_\rho e^{-\rho(l-s)} ||h_2||,
\]

i.e., \( C_\rho \) satisfies (2.6) in the first case. In the second case, we have that \( [t/T] T \geq [s/T] T + T \); and it follows from (d) and (a) that \( \Phi([s/T] T + T, s)h_2 \in \mathbb{H}_2(0) \triangleq H_2 \). These, along with (1.6) and (a.14), indicate that

\[
\| \Phi(t, s)h_2 \| \leq \| \Phi(t, [t/T] T) \| \cdot \| (P(0)^{([t/T] - [s/T] - 1)} \Phi([s/T] T + T, s)h_2 \| \leq \| \Phi(t - [t/T] T, 0) \| \cdot e^{-\rho(T - [t/T] - [s/T] - 1)} \cdot \| \Phi(T, s - [s/T] T) \| ||h_2||.
\]

By this and (a.15), one can directly check that \( \| \Phi(t, s)h_2 \| \leq C_\rho e^{-\rho(l-s)} ||h_2|| \), i.e., \( C_\rho \) satisfies (2.6) in the second case. This shows (2.6) and completes the proof. \( \square \)

The proof of Lemma 2.3. By (2.7), (2.9), (2.8), (2.10) and (2.11), one can make use of the exactly same way utilized in the proof of Lemma 2.2 to verify all properties in Lemma 2.3, except for (2.12)-(2.14). Since \( (\lambda - \mathcal{P}_sC)^{-1} = ((\lambda - \mathcal{P}_sC)^{-1})^* \), (2.12) follows from (2.9), (1.12) and (1.13). Now, we prove (2.13). The first equation of (2.13) follows from the definition of \( \tilde{H}_1 \) and (2.12). It is clear that \( P^* H \supseteq P^* H_1 \). On the other hand, since \( P^* P h = 0 \Rightarrow \langle h, P^* P h \rangle = 0 \Rightarrow P h = 0 \), it holds that \( \mathcal{N}(P^* P) \subseteq \mathcal{N}(P) \). This, together with the fact that \( H_1 = P H \) (see (2.1) and (1.14)), yields

\[
P^* H_1 = P^* P H = \mathcal{R}(P^* P) = \mathcal{N}(P^* P) \cap \mathcal{N}(P) = \mathcal{R}(P^*) = P^* H.
\]

Therefore, (2.13) holds.
The proof of (2.14) is as follows. Since \( \mathcal{P}^{*C} \xi = \mu \xi \), we derive from (2.9) that
\[
\hat{P} \xi = \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - \mathcal{P}^{*C})^{-1} d\lambda \xi = \frac{1}{2\pi i} \int_{\Gamma} (\lambda - \mu)^{-1} d\lambda \xi = \xi.
\]
Hence, \( \xi \in \hat{P} H^C \). Meanwhile, by the definitions of \( \hat{P} \) and \( \tilde{H}_1 \), we find that
\[
\tilde{H}_1^C = (\hat{P}|_H)^C H^C = \hat{P} H^C.
\]
Thus, it holds that \( \xi \in \tilde{H}_1^C \). This completes the proof.

Acknowledgement  The authors would like to thank Professor Xu Zhang for his valuable suggestions on this paper.

References

[1] V. Barbu, Stabilization of Navier-Stokes Flows, Springer-Verlag, London, 2011.

[2] V. Barbu and G. Wang, Feedback stabilization of periodic solutions to nonlinear parabolic-like evolution systems, Indiana Univ. Math. J. 54 (2005), pp. 1521-1546.

[3] V. Barbu, S. S. Rodrigues and A. Shirikyan, Internal exponential stabilization to a nonstationary solution for 3D Navier-Stoke equations, SIAM J. Control Optim. 49, 4 (2011) 1454-1478.

[4] S. Bochner, Integration von Funktionen, deren Werte die Elemente eines Vectorraumes sind, Fundamenta Mathematicae 20 (1933) pp. 262-276.

[5] J. M. Coron, Control and nonlinearity, Mathematical Surveys and Monographs, 136. American Mathematical Society, Providence, RI, 2007.

[6] J. Diestel and J. J. Uhl, Jr., Vector Measures, Math. Surveys 15, AMS, Providence, RI, 1977.

[7] C. Fabre, J. P. Puel and E. Zuazua, Approximate controllability of the semilinear heat equation, Proc. Royal Soc. Edinburgh, 125A (1995), pp. 31-61.

[8] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Mathematics, vol. 840, Springer-Verlag, Berlin-New York, 1981.

[9] T. Kato, Perturbation Theory for Linear Operators, Die Grundlehren der mathematischen Wissenschaften, Band 132, Springer-Verlag, New York, 1966.

[10] S. Kurepa, On the quadratic functional, Acad. Serbe Sci. Publ. Inst. Math., 13 (1961), pp. 57-72.

[11] X. Li and J. Yong, Optimal Control Theory for Infinite-Dimensional Systems, Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA, 1995.
[12] F. H. Lin, A uniqueness theorem for parabolic equations, Comm. Pure Appl. Math. 43 (1990), pp. 127-136.

[13] A. Lunardi, Stabilizability of time-periodic parabolic equations, SIAM J. Control and Optim., 29 (1991), pp. 810-828.

[14] K. D. Phung and G. Wang, Quantitative uniqueness for time-periodic heat equation with potential and its applications. Differential Integral Equations, 19, 6 (2006), 627-668.

[15] K. D. Phung and G. Wang, Quantitative unique continuation for the semilinear heat equation in a convex domain. J. Funct. Anal, 259 (2010) pp. 1230-1247.

[16] K. D. Phung and G. Wang, An observability estimate for parabolic equations from a measurable set in time and its applications. J. Eur. Math. Soc., 15, 2 (2013) pp. 681-703.

[17] G. Da Prato and A. Ichikawa, Quadratic control for linear time-varying systems. SIAM J. Control and Optim., 28 (1990) pp. 359-381.

[18] G. Da Prato and A. Lunaridi, Floquent exponents and Stablitizability in Time-periodic parabolic systems, Appl. Math. Optim., 22 (1990) pp. 91-113.

[19] A. J. Pritchard and J. Zabczyk, Stability and stabilizability of infinite dimensional systems, SIAM Rev., 23 (1981), pp. 25-52.

[20] D. L. Russell, Controllability and stabilizability theory for linear partial differential equations: Recent progress and open problems, SIAM Rev., 20 (1978), pp. 639-739.

[21] E. D. Sontag, Mathematical Control Theory: Deterministic Finite-Dimensional Systems, Second edition, Texts in Applied Mathematics, 6, Springer-Verlag, New York, 1998.

[22] G. Wang and Y. Xu, Periodic Stabilization for Linear Time-Periodic Ordinary Differential Equations, submitted.

[23] K. Yosida, Functional analysis, Reprint of the sixth edition (1980), Classics in Mathematics, Springer-Verlag, Berlin, 1995.