Existence and uniqueness of solution of free boundary problems with partially degenerate diffusion

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Abstract. In this paper, we mainly introduce a general method to study the existence and uniqueness of solution of free boundary problems with partially degenerate diffusion.

Keywords: Partially degenerate diffusion; Free boundary problems; Existence-uniqueness.

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1 Introduction and main result

In recent years, the following free boundary problem with a partially degenerate diffusion

\[
\begin{align*}
  u_t &= f_1(t, x, u, v), & t > 0, \quad g(t) < x < h(t), \\
  v_t &= dv_{xx} + f_2(t, x, u, v), & t > 0, \quad g(t) < x < h(t), \\
  u(t, x) &= v(t, x) = 0, & t \geq 0, \quad x = g(t), \ h(t), \\
  g'(t) &= -\mu v_x(t, g(t)), \quad h'(t) = -\beta v_x(t, h(t)), & t \geq 0, \\
  u(0, x) &= u_0(x), \quad v(0, x) = v_0(x), & -h_0 \leq x \leq h_0, \\
  h(0) &= -g(0) = h_0,
\end{align*}
\]

\( (1.1) \)

has been studied by some authors to describe the nature of spreading and vanishing of multiple species, where \( d, \mu, \beta \) and \( h_0 \) are positive constants. In the problem \((1.1)\), the diffusion of species \( u \) is relatively faster than that of species \( v \), or species \( v \) has no diffusion, so the diffusion of \( v \) is omitted.

Wang and Cao \((6)\) studied the case \( f_1(t, x, u, v) = f_1(u, v) \) and \( \beta = \mu \), where \((f_1(u, v), f_2(u, v))\) has a cooperative structure and is controlled by a linear system. Ahn et al. \((1)\) investigated a man-environment-man epidemic model: \( f_1(t, x, u, v) = G(v) - au, \ f_2(t, x, u, v) = bu - cv \) and \( \beta = \mu \). Tarboush et al. \((5)\) discussed a West Nile virus model: \( f_1(t, x, u, v) = r_1(a - u)v - bu, \ f_2(t, x, u, v) = r_2(b - v)u - cv \) and \( \beta = \mu \). In the study of the local existence and uniqueness of solution, they used the different methods. In \((1)\), the function \( G \) satisfies

- \( G \in C^1([0, \infty)), \ G(0) = 0, \ G'(v) > 0, \ \frac{G(v)}{v} \) is decreasing and \( \lim_{v \to \infty} \frac{G(v)}{v} < ac/b. \)

In \((1.1)\), the curves \( x = g(t) \) and \( x = h(t) \) are the free boundaries to be determined together with \( u(t, x) \) and \( v(t, x) \).

The main aim of this paper is to give another rigorous proof of existence and uniqueness of solution. Denote \( C^{1-}([0, \infty)) \) be the Lipschitz continuous functions space. We assume that the initial functions \( u_0 \) and \( v_0 \) satisfy

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Theorem 1.1. Under the above assumptions, there exists a unique solution \((u_0, v_0)\) in \(C^{1,\frac{1}{2}}([-h_0, h_0]) \times W_p^2((-h_0, h_0))\) with \(p > 3\), \(u_0(\pm h_0) = v_0(\pm h_0) = 0\), \(u_0, v_0 > 0\) in \((-h_0, h_0)\), and \(v_0'(h_0) < 0, v_0'(-h_0) > 0\), and denote by \(L_0\) the Lipschitz constant of \(u_0\) in \(x\).

It is assumed that \((f_1, f_2)\) satisfies

1. \(f_1(t, x, 0, v) \geq 0\) for all \(v \geq 0\) and \(f_2(t, x, u, 0) \geq 0\) for all \(u \geq 0\). For any given \(\tau, l, k_1, k_2 > 0\), \(f_i(\cdot, 0, 0) \in L^\infty((0, \tau) \times (-l, l))\) and there exists a constant \(L(\tau, l, k_1, k_2) > 0\) such that

\[
|f_i(t, x, u_1, v_1) - f_i(t, x, u_2, v_2)| \leq L(\tau, l, k_1, k_2)(|u_1 - u_2| + |v_1 - v_2|), i = 1, 2
\]

for all \(t \in [0, \tau], x \in [-l, l], u_1, u_2 \in [0, k_1], v_1, v_2 \in [0, k_2]\);

2. \(f_i\) is locally Lipschitz continuous in \((x, t) \in \mathbb{R}, i.e.,\), for any given any given \(\tau, l, k_1, k_2 > 0\), there exists a constant \(L^*(\tau, l, k_1, k_2) > 0\) such that

\[
|f_i(t, x, u, v) - f_i(t, y, u, v)| \leq L^*(\tau, l, k_1, k_2)|x - y|, i = 1, 2
\]

for all \(t \in [0, \tau], x, y \in [-l, l], u \in [0, k_1], v \in [0, k_2]\).

It is easy to notice that the condition (I) implies \(f_i \in L^\infty((0, \tau) \times (-l, l) \times (0, k_1) \times (0, k_2))\) for any given \(\tau, l, k_1, k_2 > 0\).

The result concerns with the local existence and uniqueness.

Theorem 1.1. Under the above assumptions, there exists a \(T > 0\) such that the problem \((1.1)\) has a unique solution \((u, v, g, h)\) which is defined on \([0, T]\). Moreover,

\[
g, h \in C^{1,\frac{1}{2}}([0, T]), \quad g'(t) < 0, \quad h'(t) > 0 \quad \text{in} \quad [0, T],
\]

\[
u \in C^{1,\frac{1}{2}}(\bar{D}_{g,h}^T), \quad v \in W_p^{1,2}(D_{g,h}^T) \cap C^{1,\frac{1+\alpha}{2}}(\bar{D}_{g,h}^T), \quad u, v > 0 \quad \text{in} \quad D_{g,h}^T,
\]

where

\[
D_{g,h}^T = \{0 < t \leq T, \; g(t) < x < h(t)\};
\]

\(u \in C^{1,\frac{1}{2}}(\bar{D}_{g,h}^T)\) means that \(u\) is differentiable continuously in \(t \in [0, T]\) and is Lipschitz continuous in \(x \in [g(t), h(t)]\).

When \(f_1, f_2\) do not depend on \((t, x)\), i.e., \(f_1(t, x, u, v) = f_1(u, v), f_2(t, x, u, v) = f_2(u, v)\), we have the following global existence results.

Theorem 1.2. Let \((f_1, f_2)\) be quasi-monotone increasing for \(u, v \geq 0\). If the initial value problem

\[
\begin{cases}
\phi'(t) = f_1(\phi, \psi), \quad \psi'(t) = f_2(\phi, \psi), & t > 0, \\
\phi(0) = \max_{[-h_0, h_0]} u_0 > 0, \quad \psi(0) = \max_{[-h_0, h_0]} v_0 > 0
\end{cases}
\]

has a global solution \((\phi, \psi)\), then the unique solution \((u, v, g, h)\) of \((1.1)\) also exists globally.

Theorem 1.3. Assume that there exists \(k_0 > 0\) such that \(f_1(u, v) < 0\) for all \(u > k_0, v \geq 0\), and for the given \(\eta > 0\), there exists \(\Theta(\eta) > 0\) such that \(f_2(u, v) < 0\) for \(0 \leq u \leq \eta, v \geq \Theta(\eta)\), then the unique solution \((u, v, g, h)\) of \((1.1)\) exists globally.
Remark 1.1. (i) Conditions \( u(t, g(t)) = u(t, h(t)) = 0 \) in (1.1) look like boundary conditions of \( u \), but they do actually play the roles of initial conditions of \( u \) at points \( x = g(t) \) and \( x = h(t) \), respectively.

(ii) Our conclusions are applicable to the models investigated in [1, 5, 6], and assert that the solution exists globally (using Theorems 1.1 and 1.2 for the models in [1, 6], and Theorems 1.1 and 1.3 for the models in [5]).

2 Proofs of Theorems 1.1, 1.3

Proof of Theorem 1.1 The proof is divided into several steps.

Step 1: For \( 0 < T < \infty \), set

\[
A = \max_{[-h_0, h_0]} u_0 + 1, \quad B = \max_{[-h_0, h_0]} v_0 + 1, \quad \Pi_T = [0, T] \times [-2h_0, 2h_0], \quad \Delta_T = [0, T] \times [-1, 1],
\]

and denote \( L_1 = L(1, 2h_0, A, B) \),

\[
A = \{ d, h_0, \mu, \beta, A, B, \|v_0\|_{W_2^p((-h_0, h_0))}, v_0'(\pm h_0), \|f_2\|_{L^\infty(\Pi_1 \times (0, A) \times (0, B))}, L_1 \}.
\]

We say \( u \in C^{1-}_x(\Pi_T) \) if there is a constant \( L(u, T) \) such that

\[
|u(t, x_1) - u(t, x_2)| \leq L(u, T)|x_1 - x_2|, \quad \forall x_1, x_2 \in [-2h_0, 2h_0], t \in [0, T].
\]

Define

\[
X_{u_0}^T = \{ \phi \in C(\Pi_T) : \phi(0, x) = u_0(x), \ 0 \leq \phi \leq A \}.
\]

Chosen \( u \in X_{u_0}^T \cap C^{1-}_x(\Pi_1) \) and consider the following problem

\[
\begin{cases}
  v_t = dv_{xx} + f_2(t, x, u(t, x), v), & 0 < t \leq 1, \ g(t) < x < h(t), \\
  v(t, g(t)) = v(t, h(t)) = 0, & 0 \leq t \leq 1, \\
  g'(t) = -\mu v_x(t, g(t)), \ h'(t) = -\beta v_x(t, h(t)), & 0 \leq t \leq 1, \\
  v(0, x) = v_0(x), \ h(0) = -g(0) = h_0 > 0, & |x| \leq h_0.
\end{cases}
\]

Due to the properties of \( f_2 \) and \( v \), using the similar arguments in the proof of [8, Theorem 1.1] we can show that there exists \( 0 < T_0 \ll 1 \) such that (2.1) has a unique solution \((v, g, h)\) and satisfies

\[
g, h \in C^{1+\frac{\alpha}{2}}([0, T_0]), \quad v \in W_p^{1,2}(D_{g,h}^{T_0}) \cap C^{1+\frac{\alpha}{2},1+\alpha}_{\frac{\alpha}{2}}(D_{g,h}^{T_0})
\]

with \( 0 < \alpha < 1 - 3/p \), and

\[
\begin{cases}
  \|g, h\|_{C^{1+\frac{\alpha}{2}}([0, T_0])}, \|v\|_{W_p^{1,2}(D_{g,h}^{T_0})} \leq K, \ 0 < v \leq B \quad \text{in} \quad D_{g,h}^{T_0}, \\
  0 < -g'(t), h'(t) \leq K, \ |g(t)|, h(t) \leq 2h_0 \quad \text{on} \quad [0, T_0],
\end{cases}
\]

where \( T_0 \) and \( K \) depend only on \( \mathcal{A} \), \( \alpha \) and the Lipschitz constant \( L(u, 1) \) of \( u \).
Define \( \tilde{u}_0(x) = u_0(x) \) when \( |x| \leq h_0 \), and \( \tilde{u}_0(x) = 0 \) when \( |x| > h_0 \). Then \( \tilde{u}_0 \in C^{1-}([g(T_0), h(T_0)]) \) since \( u_0 \in C^{1-}([-h_0, h_0]) \). For the functions \( g(t), h(t) \) obtained above, it is easy to see that the inverse functions \( g^{-1}(x) \) and \( h^{-1}(x) \) exist for \( x \in [g(T_0), h(T_0)] \). We set

\[
    t_x = \begin{cases} 
        g^{-1}(x) & \text{if } x \in [g(T_0), -h_0), \\
        0 & \text{if } |x| \leq h_0, \\
        h^{-1}(x) & \text{if } x \in (h_0, h(T_0)],
    \end{cases}
\] (2.3)

which is Lipschitz continuous in \( x \). For the function \( v(t, x) \) obtained above, and every \( g(T_0) < x < h(T_0) \), we consider the following problem

\[
    \begin{cases}
        \tilde{u}_t = f_1(t, x, \tilde{u}, v(t, x)), & t_x < t \leq T_0, \\
        \tilde{u}(t_x, x) = \tilde{u}_0(x).
    \end{cases}
\] (2.4)

By the standard theory for ODE we can see that there exists \( 0 < T < T_0 \), which depends on \( L_1, A, B \) and \( K \), such that for all \( g(T) \leq x \leq h(T) \), \( \tilde{u}(t, x) \) is defined on \( [t_x, T] \) and so \( \tilde{u} \) is defined on \( \mathcal{D}_{g,h}^{T} \). Moreover, as the function of \( (t, x) \), we assert that \( \tilde{u} \in C^{1-}(\mathcal{D}_{g,h}^{T}) \), of which the Lipschitz constant will be calculated in the next step, and \( \tilde{u}(t, g(t)) = \tilde{u}(t, h(t)) = 0 \) for \( 0 \leq t \leq T \). Make the zero extension of \( \tilde{u} \) to \( [0, t_x] \) for every \( g(T) \leq x \leq h(T) \). Then \( \tilde{u} \in C^{1-}([0, T] \times [g(T), h(T)]) \).

**Step 2:** The estimate of the Lipschitz constant of \( \tilde{u} \) in \( x \). As \( g'(t) < 0, h'(t) > 0 \) on \( [0, T_0] \), there is a \( \sigma > 0 \) such that

\[
    |g'(t)| \geq \sigma, \quad |h'(t)| \geq \sigma, \quad \forall t \in [0, T_0].
\] (2.5)

Set \( F_1(s, x) = f_1(s, x, \tilde{u}(s, x), v(s, x)) \). It follows from the first equation of (2.4) that, for \( t_x < t \leq T \),

\[
    \tilde{u}(\tau, x) = \tilde{u}(t_x, x) + \int_{t_x}^{\tau} F_1(s, x)ds.
\]

For the given \( (t, x_1), (t, x_2) \in \Pi_T \). We divide the arguments into several cases

**Case 1:** \( (t, x_1), (t, x_2) \in \mathcal{D}_{g,h}^{T} \) with \( -h_0 \leq x_2 < x_1 \). Then \( t \geq t_{x_1} \geq t_{x_2} \geq 0 \). Thus we have, for any \( t_{x_1} \leq \tau \leq t \),

\[
    |\tilde{u}(\tau, x_1) - \tilde{u}(\tau, x_2)| \leq |\tilde{u}(t_{x_1}, x_1) - \tilde{u}(t_{x_2}, x_2)| + \int_{t_{x_2}}^{t_{x_1}} |F_1(s, x_2)|ds + \int_{t_{x_2}}^{\tau} |F_1(s, x_1) - F_1(s, x_2)|ds.
\]

By use of the conditions (I) and (II), it is easy to derive that

\[
    |F_1(s, x_1) - F_1(s, x_2)| \leq L_1(|\tilde{u}(s, x_1) - \tilde{u}(s, x_2)| + |v(s, x_1) - v(s, x_2)|) + L_1^*|x_1 - x_2|,
\]

where \( L_1 = L(1, 2h_0, A, B), L_1^* = L^*(1, 2h_0, A, B) \). It yields,

\[
    |\tilde{u}(\tau, x_1) - \tilde{u}(\tau, x_2)| \leq TL_1|\tilde{u}(\cdot, x_1) - \tilde{u}(\cdot, x_2)|_{C([t_{x_1}, \tau])} + |\tilde{u}(t_{x_1}, x_1) - \tilde{u}(t_{x_2}, x_2)|
    + L_1^*|x_1 - x_2| + C_1|t_{x_1} - t_{x_2}| + L_1 \int_{t_{x_1}}^{\tau} |v(s, x_1) - v(s, x_2)|ds
\] (2.6)
as $\tau \leq T \leq 1$, where $C_1 = \|f_1\|_{L^\infty(\Omega \times (0, T) \times (0, B))}$.

Noticing $\|v\|_{W^{1,2}_p(D_{g,h}^T)} \leq K$, we have $\|v_x\|_{L^\infty(D_{g,h}^T)} \leq C_2$ by the embedding theorem. Thus

$$\int_{t_1}^T \left| v(s, x_1) - v(s, x_2) \right| ds \leq T \|v_x\|_{L^\infty(D_{g,h}^T)} \left| x_1 - x_2 \right| \leq TC_2 \left| x_1 - x_2 \right| \leq C_2 \left| x_1 - x_2 \right|$$

as $\tau \leq T \leq 1$.

If $t_{x_1} > 0$, $t_{x_2} > 0$, then $\bar{u}(t_{x_1}, x_1) = \bar{u}(t_{x_2}, x_2) = 0$ and

$$|t_{x_1} - t_{x_2}| = \left| h^{-1}(x_1) - h^{-1}(x_2) \right| \leq \left( h^{-1} \right) \left| x_1 - x_2 \right| \leq \sigma^{-1} \left| x_1 - x_2 \right|.$$

If $t_{x_1} > 0$, $t_{x_2} = 0$, then $x_2 \in [-h_0, h_0]$, $x_1 > h_0$, $\bar{u}(t_{x_1}, x_1) = 0$. Let $L_0$ be the Lipschitz constant of $u_0$ in $x$. It then follows that

$$|t_{x_1} - t_{x_2}| = |h^{-1}(x_1) - h^{-1}(0)| = \sigma^{-1} |x_1 - h_0| \leq \sigma^{-1} |x_1 - x_2|,$$

$$|\bar{u}(t_{x_1}, x_1) - \bar{u}(t_{x_2}, x_2)| = |0 - u_0(x_2)| = |u_0(0) - u_0(x_2)| \leq L_0 |h_0 - x_2| \leq L_0 |x_1 - x_2|.$$

If $t_{x_1} = t_{x_2} = 0$, i.e., $x_1, x_2 \in [-h_0, h_0]$, then

$$|\bar{u}(t_{x_1}, x_1) - \bar{u}(t_{x_2}, x_2)| = |u_0(x_1) - u_0(x_2)| \leq L_0 |x_2 - x_1|.$$

Substituting these estimates into (2.6), we have

$$|\bar{u}(\tau, x_1) - \bar{u}(\tau, x_2)| \leq TL_1 \|\bar{u}(\cdot, x_1) - \bar{u}(\cdot, x_2)\|_{C([t_{x_1}, t])} + (L_0 + L_1^* + C_1 \sigma^{-1} + C_2 L_1) |x_1 - x_2|.$$

Take the maximum of $|\bar{u}(\tau, x_1) - \bar{u}(\tau, x_2)|$ in $[t_{x_1}, t]$ it yields

$$\|\bar{u}(\cdot, x_1) - \bar{u}(\cdot, x_2)\|_{C([t_{x_1}, t])} \leq TL_1 \|\bar{u}(\cdot, x_1) - \bar{u}(\cdot, x_2)\|_{C([t_{x_1}, t])} + (L_0 + L_1^* + C_1 \sigma^{-1} + C_2 L_1) |x_1 - x_2|.$$

Set $M = 2(L_0 + L_1^* + C_1 \sigma^{-1} + C_2 L_1)$. Then

$$|\bar{u}(t, x_1) - \bar{u}(t, x_2)| \leq \|\bar{u}(\cdot, x_1) - \bar{u}(\cdot, x_2)\|_{C([t_{x_1}, t])} \leq M |x_1 - x_2| \quad (2.7)$$

provided that $0 < T \leq \min\{1, \frac{1}{2L_1}\}$.

Case 2: $(t, x_1), (t, x_2) \in \overline{D_{g,h}^T}$ with $x_2 < x_1 \leq h_0$. Similar to the above, (2.7) holds.

Case 3: $(t, x_1), (t, x_2) \in \overline{D_{g,h}^T}$ with $x_2 < h_0 < h_0 < x_1$. Then

$$|\bar{u}(t, x_1) - \bar{u}(t, x_2)| \leq |\bar{u}(t, x_1) - \bar{u}(t, h_0)| + |\bar{u}(t, h_0) - \bar{u}(t, x_2)| \leq 2M |x_1 - x_2| \quad (2.8)$$

Case 4: $(t, x_1), (t, x_2) \not\in \overline{D_{g,h}^T}$. Then $\bar{u}(t, x_1) = \bar{u}(t, x_2) = 0$.

Case 5: $(t, x_1) \not\in \overline{D_{g,h}^T}$, $(t, x_2) \in \overline{D_{g,h}^T}$. We may assume that $x_1 > h(t)$. Thus $x_2 \leq h(t)$, $\bar{u}(t, x_1) = \bar{u}(t, h(t)) = 0$, and

$$|\bar{u}(t, x_1) - \bar{u}(t, x_2)| \leq |\bar{u}(t, h(t)) - \bar{u}(t, x_2)| \leq M |h(t) - x_2| \leq 2M |x_1 - x_2|.$$
In conclusion, the estimate (2.8) always holds provided that \(0 < T \leq \min\{1, \frac{1}{2T}\}\). Define

\[
\mathcal{Y}_{u_0}^T = \{ \phi \in C(\Pi_T) : \phi(0, x) = u_0(x), \ 0 \leq \phi \leq A, \ |\phi(t, x) - \phi(t, y)| \leq 2M|x - y| \}.
\]

Obviously, \(\mathcal{Y}_{u_0}^T\) is complete with the metric \(d(\phi_1, \phi_2) = \sup_{\Pi_T} |\phi_1 - \phi_2|\). For any given \(u \in \mathcal{Y}_{u_0}^T\), we extend \(u\) to \([T, 1] \times [-2h_0, 2h_0]\) by setting \(u(t, x) = u(T, x)\). Then \(u \in \mathcal{X}_{u_0}^1 \cap C_T^1(\Pi_1)\). Define a mapping \(\Gamma\) by

\[
\Gamma(u) = \tilde{u}.
\]

The above discussions show that \(\Gamma\) maps \(\mathcal{Y}_{u_0}^T\) into itself.

**Step 3**: We shall show that \(\Gamma\) is a contraction mapping in \(\mathcal{Y}_{u_0}^T\) for \(0 < T \ll 1\). Let \((v_i, g_i, h_i)\) be the unique solution of (2.1) with \(u = u_i\), \(i = 1, 2\), and define \(t_i^T\) by the manner (2.3) with \((g, h) = (g_i, h_i)\). Let \(\tilde{u}_i\) be the unique solution of (2.4) with \(t_x = t_i^T\), \(v = v_i\) and \(T_0 = T\). Then

\[
\tilde{u}_i(t, x) = \tilde{u}_i(t_i^T, x) + \int_{t_i^T}^t f_1(s, x, \tilde{u}_i, v_i)ds \quad \text{for} \quad x \in [g_i(T), h_i(T)].
\]

Set

\[
U = u_1 - u_2, \quad \tilde{U} = \tilde{u}_1 - \tilde{u}_2, \quad h = h_1 - h_2, \quad g = g_1 - g_2, \quad \Omega_T = D_{g_1, h_1}^T \cup D_{g_2, h_2}^T.
\]

The following arguments are inspired by those of [2, 3]. Make the zero extensions of \(\tilde{u}_i\) and \(v_i\) in \(([0, T] \times \mathbb{R}) \setminus D_{g_i, h_i}^T\). Fix \((t, x) \in \Omega_T\), we now estimate \(|\tilde{U}(t, x)|\) in all the possible cases.

**Case 1**: \(x \in (g_1(t), h_1(t)) \setminus (g_2(t), h_2(t))\). In such case, either \(g_1(t) < x \leq g_2(t)\) or \(h_2(t) \leq x < h_1(t)\), and \(\tilde{u}_1(t_x^2, x) = 0, \tilde{u}_2(t_x, x) = 0\). Thus we have

\[
|\tilde{U}(t, x)| = |\tilde{u}_1(t, x)| = \left| \int_{t_x^2}^t f_1(s, x, \tilde{u}_1, v_1)ds \right| \leq C_1|t - t_x^2|,
\]

where \(C_1 = \|f_1\|_{L^\infty([0, T] \times (0, A) \times (0, B))}\).

When \(h_2(t) \leq x < h_1(t)\), then \(0 < t_x^2 < t\) and \(h_1(t) > h_1(t_x^2) = x \geq h_2(t)\). Therefore,

\[
|\tilde{U}(t, x)| \leq C_1|t - t_x^2| = C_1|h_1^{-1}(h_1(t)) - h_1^{-1}(h_1(t_x^2))| \\
\leq C_1\|\phi_1\|_{L^\infty([0, T])}|h_1(t) - h_1(t_x^2)| \\
\leq C_1\sigma^{-1}|h_1(t) - h_1(t_x^2)| \\
\leq C_1\sigma^{-1}|h_1(t) - h_2(t)| \\
\leq C_1\sigma^{-1}|h_1(t)|_{C([0, T])},
\]

where \(\sigma > 0\) is determined by (2.5). When \(g_1(t) < x \leq g_2(t)\), we can analogously obtain

\[
|\tilde{U}(t, x)| = |\tilde{u}_1(t, x)| \leq C_1\sigma^{-1}\|g\|_{C([0, T])}.
\]

**Case 2**: \(x \in (g_2(t), h_2(t)) \setminus (g_1(t), h_1(t))\). Similar to Case 1 we have

\[
|\tilde{U}(t, x)| = |\tilde{u}_2(t, x)| \leq C_1\sigma^{-1}\|g, h\|_{C([0, T])}.
\]

**Case 3**: \(x \in (g_1(t), h_1(t)) \cap (g_2(t), h_2(t))\). If \(x \in [-h_0, h_0]\), then \(t_x^1 = t_x^2 = 0\) and \(\tilde{u}_1(t_x^1, x) = \tilde{u}_2(t_x^2, x) = \tilde{u}_0(x)\). Hence

\[
|\tilde{U}(t, x)| \leq \int_0^t |f_1(s, x, \tilde{u}_1, v_1) - f_1(s, x, \tilde{u}_2, v_2)|ds \leq TL_1 \left(\|\tilde{U}\|_{C(T)} + \|v_1 - v_2\|_{C(T)}\right).
\]
If \( x \in (g_1(t), h_1(t)) \cap (g_2(t), h_2(t)) \setminus [-h_0, h_0], \) we have \( t^1_x > 0, t^2_x > 0, \) \( \tilde{u}_1(t^1_x, x) = \tilde{u}_2(t^2_x, x) = 0. \) Without loss of generality we assume \( x > h_0 \) and \( t^2_x > t^1_x > 0. \) Then \( h_1(t^2_x) > h_1(t^1_x) = x = h_2(t^2_x), \) \( x \in (g_1(s), h_1(s)) \cap (g_2(s), h_2(s)) \) for all \( t^2_x < s \leq t \) and \( x \in (g_1(t^2_x), h_1(t^2_x)) \setminus (g_2(t^2_x), h_2(t^2_x)). \) Hence,

\[
|\tilde{U}(t^2_x, x)| = |\tilde{u}_1(t^2_x, x)| \leq C_1 \sigma^{-1}\|g, h\|_{C([0,T])}
\]

by the conclusion of Case 1. Integrating the differential equation of \( \tilde{u}_i \) from \( t^2_x \) to \( t \) we obtain

\[
\begin{align*}
\tilde{u}_1(t, x) &= \tilde{u}_1(t^2_x, x) + \int_{t^2_x}^{t} f_1(s, x, \tilde{u}_1, v_1) \, ds, \\
\tilde{u}_2(t, x) &= \int_{t^2_x}^{t} f_1(s, x, \tilde{u}_2, v_2) \, ds.
\end{align*}
\]

It follows that

\[
|\tilde{U}(t, x)| \leq \tilde{u}_1(t^2_x, x) + \int_{t^2_x}^{t} |f_1(s, x, \tilde{u}_1, v_1) - f_1(s, x, \tilde{u}_2, v_2)| \, ds
\]

\[
\leq |\tilde{U}(t^2_x, x)| + \int_{0}^{t} |f_1(s, x, \tilde{u}_1, v_1) - f_1(s, x, \tilde{u}_2, v_2)| \, ds
\]

\[
\leq C_1 \sigma^{-1}\|g, h\|_{C([0,T])} + TL_1 \left( \|\tilde{U}\|_{C(\bar{\Omega}_T)} + \|v_1 - v_2\|_{C(\bar{\Omega}_T)} \right).
\]

In conclusion,

\[
|\tilde{U}(t, x)| \leq C_1 \sigma^{-1}\|g, h\|_{C([0,T])} + TL_1 \left( \|\tilde{U}\|_{C(\bar{\Omega}_T)} + \|v_1 - v_2\|_{C(\bar{\Omega}_T)} \right). \tag{2.9}
\]

We will show in the following Step 4 that if \( 0 < T \ll 1 \) then there exists positive constant \( C \) such that

\[
\|g, h\|_{C^1([0,T])} \leq C \|U\|_{C(\bar{\Pi}_T)}, \quad \|v_1 - v_2\|_{C(\bar{\Omega}_T)} \leq C \|U\|_{C(\bar{\Pi}_T)}. \tag{2.10}
\]

Once this is done, notice that \( g(0) = h(0) = 0, \) the first inequality of (2.10) implies

\[
\|g, h\|_{C^1([0,T])} \leq T \|g, h\|_{C([0,T])} \leq TC \|U\|_{C(\bar{\Pi}_T)}.
\]

Then combing with (2.9), we have

\[
\|\tilde{U}\|_{C(\bar{\Pi}_T)} \leq \frac{1}{3} \|U\|_{C(\bar{\Pi}_T)} \quad \text{if} \quad 0 < T \ll 1.
\]

This demonstrate that \( \Gamma \) is a contraction mapping in \( Y^T_{\tilde{u}_0}. \) Thus, \( \Gamma \) has a unique fixed point \( u \) in \( Y^T_{\tilde{u}_0}. \) Let \((v, g, h)\) be the unique solution of (2.11) with such \( u. \) Then \((u, v, g, h)\) is a solution of (1.1) and it is the unique one provided \( u \in Y^T_{\tilde{u}_0}. \) Moreover, we can see that \( u \in C^{1+\alpha, \frac{1-\alpha}{2}}(\bar{D}_{g,h}^T) \) and \( v \in W^{1,2}_p(D^T_{g,h}). \) Thus \( v \in C^{1+\alpha, \frac{1-\alpha}{2}}(\bar{D}^T_{g,h}) \) by the embedding theorem as \( p > 3. \)

**Step 4:** Proof of (2.10). Its proof is similar to that of [3] Theorem 2.1: Step 4] and [7] Theorem 2.1: Step 3]. Before our statement, some preparations are needed. Let

\[
x_i(t, y) = \frac{1}{2}[(h_i(t) - g_i(t))y + h_i(t) + g_i(t)],
\]

\[
\xi_i(t, y) = \frac{2}{h_i(t) - g_i(t)}, \quad \zeta_i(t, y) = \frac{h_i'(t) + g_i'(t)}{h_i(t) - g_i(t)} + \frac{h_i'(t) - g_i'(t)}{h_i(t) - g_i(t)} y,
\]
\[ w_i(t, y) = u_i(t, x_i(t, y)), \quad z_i(t, y) = v_i(t, x_i(t, y)), \]

and
\[ f_i^2(t, y) = f_2(t, x_i(t, y), u(t, x_i(t, y)), v(t, x_i(t, y))) \]

for \( i = 1, 2 \). Then,
\[
\begin{aligned}
 z_{i,t} &= d \xi_i^2 z_{i,y} + \xi_i z_{i,y} + f_i^1(t, y), \quad 0 < t \leq T, \quad |y| < 1, \\
 z_i(t, \pm 1) &= 0, \quad 0 \leq t \leq T, \\
 z_i(0, y) &= v_0(h_{0y}) =: z_{i,0}(y), \quad |y| \leq 1.
\end{aligned}
\]

Recall (2.2), it follows that
\[
\|\xi_i\|_{L^\infty([0, T])} \leq \frac{1}{h_0}, \quad \|\xi_i\|_{L^\infty(\Delta_T)} \leq \frac{2K}{h_0}, \quad \|f_i^1\|_{L^\infty(\Delta_T)} \leq C_0. \tag{2.11}
\]

And by the \( L^p \) theory \( \|z_i\|_{W^{1,2}_p(\Delta_T)} \leq C_1' \). Using the arguments in the proof of [8] Theorem 1.1 we can obtain
\[
[z_i, z_{i,y}]_{C^{\infty}_0(\Delta_T)} \leq \overline{C}, \tag{2.12}
\]

where \( C_1 \) is independent of \( T^{-1} \). This implies
\[
\|z_{i,y}\|_{C(\Delta_T)} \leq \|z_{i,0}(y)\|_{C([-1, 1])} + \overline{C} T^{\frac{2}{n}} \leq \|z_{i,0}(y)\|_{C([-1, 1])} + \overline{C} := C_2'. \tag{2.13}
\]

Thanks to (2.2) and \( v_{i,x}(t, x) = z_{i,y}(t, y) \frac{2}{h(t) - g(t)} \), it yields
\[
\|v_{i,x}\|_{C(\overline{D}^2_{g, h})} \leq C_2'/h_0. \tag{2.14}
\]

On the other hand, \( z = z_1 - z_2 \) satisfy
\[
\begin{aligned}
 z_{t} - d \xi_1^2 z_{yy} - \xi_1 z_y - a(t, y)z &= d(\xi_1 - \xi_2)z_{2,y} + (\xi^2_1 - \xi^2_2)z_{2,y} \\
 &\quad + b(t, y)(w_1 - w_2) + c(t, y), \quad 0 < t \leq T, \quad |y| < 1, \\
 z(t, \pm 1) &= 0, \quad 0 \leq t \leq T, \\
 z(0, y) &= 0, \quad |y| \leq 1,
\end{aligned}
\tag{2.15}
\]

and \( g(t) = g_1(t) - g_2(t), \ h(t) = h_1(t) - h_2(t) \) satisfy
\[
\begin{aligned}
 g'(t) &= -\mu \xi_1(t) z_y(t, -1) - \mu (\xi_1(t) - \xi_2(t))z_{2,y}(t, -1), \quad 0 < t \leq T, \\
 h'(t) &= -\beta \xi_1(t) z_y(t, 1) - \beta (\xi_1(t) - \xi_2(t))z_{2,y}(t, 1), \quad 0 < t \leq T, \\
 g(0) &= h(0) = 0,
\end{aligned}
\]

where
\[
a(t, y) = \int_0^1 f_{2,v}^1(t, y, w_1, z_2 + (z_1 - z_2)t) \, dt,
\]
\[ b(t, y) = \int_0^1 f_{2,u}(t, y, w_2 + (w_1 - w_2)\tau, z_2) d\tau, \]
\[ c(t, y) = f_{2}^{2}(t, y, w_1, z_2) - f_{2}^{2}(t, y, w_1, z_2). \]

Clearly, \( \|a, b\|_{L^\infty(\Delta_T)} \leq L_1, \|c\|_{L^\infty(\Delta_T)} \leq L_1^* \). Due to (2.11), (2.13), applying the parabolic \( L^p \) theory to (2.15) we can obtain
\[
\|z\|_{W^{0,2}_p(\Delta_T)} \leq C_3(\|g, h\|_{C^1([0,T])} + \|w_1 - w_2\|_{C(\Delta_T)}),
\]
where \( C_3 \) depends on \( h_0, \mu, \beta, A, B \) and \( K \). The same as (2.12), we have
\[
[z]_{C^{0,\alpha}(\Delta_T)} + [z_y]_{C^{2,\alpha}(\Delta_T)} \leq C_4(\|g, h\|_{C^1([0,T])} + \|w_1 - w_2\|_{C(\Delta_T)}),
\]
where \( C_4 > 0 \) is independent of \( T^{-1} \). When \((t, y) \in \Delta_T\), we have
\[
|w_1(t, y) - w_2(t, y)| = |u_1(t, x_1(t, y)) - u_2(t, x_2(t, y))| \\
\leq |u_1(t, x_1(t, y)) - u_2(t, x_1(t, y))| + |u_2(t, x_1(t, y)) - u_2(t, x_2(t, y))| \\
\leq \|U\|_{C(\Pi_T)} + L(u, 1)|x_1(t, y) - x_2(t, y)| \\
\leq C_5(\|U\|_{C(\Pi_T)} + \|g, h\|_{C([0,T])}),
\]
where \( C_5 \) depends only on \( h_0 \) and the Lipschitz constant \( L(u, 1) \) of \( u \). Therefore,
\[
\|w_1 - w_2\|_{C(\Delta_T)} \leq C_5(\|U\|_{C(\Pi_T)} + \|g, h\|_{C([0,T])}).
\]

This combined with (2.16) asserts
\[
[z]_{C^{0,\alpha}(\Delta_T)} + [z_y]_{C^{2,\alpha}(\Delta_T)} \leq C_6(\|g, h\|_{C^1([0,T])} + \|U\|_{C(\Pi_T)}).
\]

Notice \( z_y(0, 1) = 0 \). The above estimate implies
\[
|z_y(t, 1)|_{C([0,T])} \leq C_6 T^\frac{\alpha}{2}(\|g, h\|_{C^1([0,T])} + \|U\|_{C(\Pi_T)}).
\]

As \( h(0) = g(0) = 0 \), it is easy to see that
\[
|h(t)| \leq T\|h'\|_{C([0,T])}, \quad |g(t)| \leq T\|g'\|_{C([0,T])}.
\]

Making use of (2.13) and (2.18) we have
\[
|h'(t) - h'_2(t)| = \beta|v_{1,x}(t, h_1(t)) - v_{2,x}(t, h_2(t))| \\
\leq \beta \left| \frac{2[z_{1,y}(t, 1) - z_{2,y}(t, 1)]}{h_1(t) - g_1(t)} + 2z_{2,y}(t, 1) \frac{g(t) - h(t)}{h_1(t) - g_1(t)} \right| |h(t) - h_1(t)||h_2(t) - g_2(t)| \\
\leq \beta \frac{1}{h_0}|z_y(t, 1)| + 2\beta |z_{2,y}(t, 1)| \frac{|h(t)| + |g(t)|}{4h_0^2} \\
\leq C_7 T^\frac{\alpha}{2}(\|g, h\|_{C^1([0,T])} + \|U\|_{C(\Pi_T)}).
\]

Therefore, by use of (2.19),
\[
\|h'\|_{C([0,T])} \leq C_8 T^\frac{\alpha}{2}(\|g', h'\|_{C([0,T])} + \|U\|_{C(\Pi_T)}).
\]
Similarly, we have
\[ \|g'||_{C([0,T])} \leq C'_{0} T^{\frac{3}{4}} \left( \|g', h'||_{C([0,T])} + \|U||_{C'(\Pi_{T})} \right). \]

Consequently, \( \|g', h'||_{C([0,T])} \leq C_{0} \|U||_{C'(\Pi_{T})} \) provided \( T \) small enough. Recalling (2.19) we get the first inequality of (2.10):
\[ \|g, h||_{C^{1}([0,T])} \leq C_{0} \|U||_{C'(\Pi_{T})}. \]  \hfill (2.20)

Moreover, as \( z(0, y) = 0 \), we have
\[ |z(t, y)| = |z(t, y) - z(0, y)| \leq \nu^{\frac{3}{2}} \|z||_{C^{\frac{1}{2}}} T^\nu \|g, h||_{C^{1}([0,T])} \|U||_{C'(\Pi_{T})} \leq \nu^{\frac{3}{2}} \|z||_{C^{\frac{1}{2}}} T^\nu \|g, h||_{C^{1}([0,T])} \|U||_{C'(\Pi_{T})}. \]

This combined with (2.17) allows us to derive
\[ \|z||_{C(\Delta T)} \leq \nu^{\frac{3}{2}} \|z||_{C^{\frac{1}{2}}} T^\nu \|g, h||_{C^{1}([0,T])} \|U||_{C'(\Pi_{T})} \leq C_{0} T^{\frac{3}{4}} \left( \|g, h||_{C^{1}([0,T])} + \|U||_{C'(\Pi_{T})} \right). \]  \hfill (2.21)

Now we estimate \( \|v_1 - v_2||_{C^{0}(\Omega T)} \). Fix \((t, x) \in \overline{\Omega}_{T} \) let
\[ y(t, x) = \frac{2x - g_1(t) - h_1(t)}{h_1(t) - g_1(t)}. \]

**Case 1:** \( x \in [g_{1}(t), h_{1}(t)] \cap [g_{2}(t), h_{2}(t)] \). Using (2.14), (2.20) and (2.21) respectively, we have
\[ |v_1(t, x) - v_2(t, x)| = |z_1(t, y_1) - z_2(t, y_2)| \leq |z_1(t, y_1) - z_2(t, y_1)| + |z_2(t, y_1) - z_2(t, y_2)| \leq \|z||_{C(\Delta T)} + \|z_2,y||_{C(\Delta T)} |y_1 - y_2| \leq \|z||_{C(\Delta T)} + \frac{2}{h_0} \|z_2,y||_{C(\Delta T)} \|g, h||_{C^{0}([0,T])} \leq C_{10} \|U||_{C'(\Pi_{T})}. \]  \hfill (2.22)

**Case 2:** \( x \in [g_{1}(t), h_{1}(t)] \setminus [g_{2}(t), h_{2}(t)] \). In this case \( v_2(t, x) = 0 \). Without loss of generality, we may think of \( x \in [g_{1}(t), g_{2}(t)] \) and \( g_{2}(t) \leq h_{1}(t) \). Take advantage of (2.14) and (2.22), it yields
\[ |v_1(t, x) - v_2(t, x)| = |v_1(t, x) - v_2(t, g_2(t))| \leq \|v_1,x||_{C(\Pi_{T}, h_1)} |g_1(t) - g_2(t)| + C_{10} \|U||_{C'(\Pi_{T})} \leq C_{11} \|U||_{C'(\Pi_{T})}. \]

**Case 3:** \( x \in [g_{2}(t), h_{2}(t)] \setminus [g_{1}(t), h_{1}(t)] \). Similar to Case 2, we still have
\[ |v_1(t, x) - v_2(t, x)| \leq C_{12} \|U||_{C'(\Pi_{T})}. \]

In conclusion,
\[ \|v_1 - v_2||_{C(\Omega T)} \leq C \|U||_{C'(\Pi_{T})} \]
if \( 0 < T \ll 1 \). The estimate (2.10) is proved.

**Step 5:** The uniqueness. Let \((\tilde{u}, \tilde{v}, \tilde{g}, \tilde{h})\) be any solution of (1.1). It is easy to see from Step 2 that \( \tilde{u} \in Y^{T}_{u_0} \) if \( 0 < T \ll 1 \). Thus \((\tilde{u}, \tilde{v}, \tilde{g}, \tilde{h}) = (u, v, g, h) \) and the proof is complete.
Proof of Theorem 1.2. Clearly, $\phi(t) > 0$, $\psi(t) > 0$ for all $t \geq 0$. Let $T_*$ be the maximal existence time of $(u,v,g,h)$. For any fixed $0 < T < T_*$, applying the comparison principle in the region $D_{g,h}^T$ we have

$$
\begin{cases}
    u(t,x) \leq \phi(t) \leq \max_{[0,T+1]} \phi(t) := M(T) & \text{on } \overline{D}_{g,h}^T, \\
v(t,x) \leq \psi(t) \leq \max_{[0,T+1]} \psi(t) := N(T) & \text{on } \overline{D}_{g,h}^T.
\end{cases}
$$

(2.23)

It is not hard to see that $v_x(t,h(t)) < 0$ by the Hopf boundary lemma for $0 < t < T$, which yields $h'(t) > 0$. Set

$$A = \sup_{[0,M(T)] \times [0,N(T)]} f_2(u,v).$$

Define a comparison function by

$$w(t,x) = N(T) \left[ 2K(h(t) - x) - K^2(h(t) - x)^2 \right]$$

for some appropriate positive constant $K > 1/h_0$ over the region

$$\Omega_T = \{(t,x) : 0 < t < T, \ h(t) - 1/K < x < h(t)\}.$$

First of all, one can easily compute that, for any $(t,x) \in \Omega_T$,

$$w_t = 2N(T)K[1 - K(h(t) - x)]h'(t) \geq 0, \quad -w_{xx} = 2N(T)K^2.$$

It follows that, if $K^2 \geq \frac{A}{2dN(T)}$, then

$$w_t - dw_{xx} \geq 2dN(T)K^2 \geq A \geq f_2(t,x,u) = v_t - dv_{xx} \quad \text{in } \Omega_T.$$

It is clear that

$$w(t,h(t) - K^{-1}) = N(T) \geq v(t,h(t) - K^{-1}), \quad w(t,h(t)) = 0 = v(t,h(t))$$

for all $0 < t < T$. Taking advantage of

$$v_0(x) = -\int_x^{h_0} v_0'(y)dy \leq -\min_{[0,h_0]} v_0'(x)(h_0 - x), \quad x \in [h_0 - K^{-1}, h_0],$$

$$w(0,x) \geq N(T)K(h_0 - x), \quad x \in [h_0 - K^{-1}, h_0],$$

we have that if

$$N(T)K \geq -\min_{[0,h_0]} v_0'(x)$$

then

$$v_0(x) \leq w(0,x) \quad \text{in } [h_0 - K^{-1}, h_0].$$

Applying the maximum principle to $w - v$ over $\Omega_T$ we deduce $w \geq v$ in $\Omega_T$. It then leads to $v_x(t,h(t)) \geq w_x(t,h(t)) = -2N(T)K$. Thus we have

$$h'(t) = -\beta v_x(t,h(t)) \leq 2\beta N(T)K = 2\beta \max\left\{ \frac{2}{h_0}, \sqrt{\frac{AN(T)}{2d}}, -\min_{[0,h_0]} v_0'(x) \right\}$$

(2.24)
for all $0 \leq t \leq T$. Similarly,

$$g'(t) \geq -2\mu \max \left\{ \frac{2}{h_0}, \sqrt{\frac{AN(T)}{2d}}, \max_{[-h_0,0]} v_0'(x) \right\}, \forall \ 0 \leq t \leq T. \quad (2.25)$$

Recalling the estimates (2.23)-(2.25) and using a similar method to the proof of [8, Theorem 1.2] we have $T_* = \infty$. \hfill \Box

**Proof of Theorem 1.3.** It is easy to see that

$$u(t, x) \leq \max_{[-h_0,h_0]} u_0 + k_0 := \eta, \quad v(t, x) \leq \max_{[-h_0,h_0]} v_0 + \Theta(\eta).$$

The remaining proof is the same as that of Theorem 1.2. \hfill \Box

**References**

[1] I. Ahn, S. Baek and Z. G. Lin, *The spreading fronts of an infective environment in a man-environment-man epidemic model*, Appl. Math. Modelling, 40 (2016), 7082-7101.

[2] Y. H. Du, M. X. Wang and M. Zhao, *Two species nonlocal diffusion systems with free boundaries*. arXiv:1907.04542v1.

[3] S. Y. Liu, H. M. Huang and M. X. Wang, *A free boundary problem for a prey-predator model with degenerate diffusion and predator-stage structure*, Discrete Cont. Dyn. Syst. B.. In press.

[4] L. Li, W. J. Sheng and M. X. Wang, *Systems with nonlocal vs. local diffusions and free boundaries*, J. Math. Anal. Appl.. https://doi.org/10.1016/j.jmaa.2019.123646.

[5] A. K. Tarboush, Z. G. Lin and M. Y. Zhang, *Spreading and vanishing in a West Nile virus model with expanding fronts*, Science China: Mathematics, 60(5) (2017), 841-860.

[6] J. Wang and J. F. Cao, *The spreading frontiers in partially degenerate reaction-diffusion systems*, Nonlinear Analysis, 122 (2015), 215-238.

[7] M. X. Wang and Y. Zhang, *Dynamics for a diffusive prey-predator model with different free boundaries*, J. Differential Equations, 264 (2018), 3527-3558.

[8] M. X. Wang, *Existence and uniqueness of solutions of free boundary problems in heterogeneous environments*, Discrete Cont. Dyn. Syst. B., 24(2) (2019), 415-421.