Black Hole Entropy and Dilatations

Viaggiu Stefano
Dipartimento di Matematica,
Università di Roma “Tor Vergata”,
Via della Ricerca Scientifica, 1, I-00133 Roma, Italy
E-mail: viaggiu@mat.uniroma2.it
(or: stefano.viaggiu@ax0rm1.roma1.infn.it)

13th October 2018

Abstract
Dilatations by means of a constant factor can be seen in a double way: as a simple change of units length or as a conformal mapping of the starting spacetime into a “stretched” one with the same units length. The numerical value of the black hole entropy depends on the interpretations made for the stretched manifold. Further, we study the possibility to choose an unusual “mass dependent” normalization for the timelike Killing vector for a Kerr black hole with and without a cosmic string.

1 Introduction
What does it happen to the black hole entropy when we perform a constant global stretching of the manifold by means of a conformal constant factor $\Omega$, i.e.

$$ds^2 = \Omega^2 ds^2, \quad \Omega > 0? \quad (1)$$

As a first step we consider the Kerr solution written in Boyer-Lindquist coordinates (see [1]):

$$ds^2 = \Sigma \left( d\theta^2 + \frac{dr^2}{\Delta} \right) + \left( r^2 + a^2 \right) \sin^2 \theta d\phi^2 - dt^2 +$$

$$+ \frac{2mr}{\Sigma} (dt + a\sin^2 \theta d\phi)^2, \quad \Sigma = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 + a^2 - 2mr, \quad (2)$$

where $m$ and $a$ are respectively the mass and the spin per unit mass of the source. For $a^2 < m^2$ the solution [2] describes a black hole with outer
horizon at \( r_+ = m + \sqrt{m^2 - a^2} \). In this paper we use geometrized units with \( G = c = \hbar = 1 \). Hence mass, time and energy are measured in units of length \( L \). In these units the Newtonian constant \( G \) and the light velocity \( c \) are dimensionless quantities and the fundamental constant \( \hbar \) has dimension \( L^2 \). The black hole entropy \( S \) is given by the well known Bekenstein-Hawking formula \[ S = \frac{A}{4 l_p^2} \] where \( A \) is the horizon surface area given by \[ A = \int_{r_+} \sqrt{g_{\theta\theta}} \sqrt{g_{\phi\phi}} d\phi \] and \( l_p = \sqrt{\frac{G\hbar c^3}{G \hbar c^3}} \) is the Planck length that in geometrized units has value \( l_p = 1 \). For the metric (2) we find \[ S = 2\pi m (m + \sqrt{m^2 - a^2}). \] (4)

Besides, for the spacetime (2), we define the black hole mass \( m \) with respect to the timelike Killing field \( \xi^\nu = (\frac{\partial}{\partial t})^\nu \), where \( \xi^\mu \xi_\mu = g_{tt} = \frac{a^2 \sin^2 \theta - \Delta}{\zeta} \).

Furthermore, “away from each mass source”, the metric can be recast, in Cartesian coordinates, in the standard form \[ ds^2 = dx^2 + dy^2 + dz^2 - dt^2 \]. If we have an asymptotically flat spacetime, the normalization of \( \xi^\nu \) at spatial infinity for the spacetime (2) can be defined in the usual way \[ (\xi^\nu \xi_\nu)_\infty = -1. \] (5)

We now consider the Kerr solution stretched by a factor \( \Omega^2 \), i.e.

\[
ds'^2 = \Omega^2 \left( r^2 + a^2 \cos^2 \theta \right) \left( d\theta^2 + \frac{dr^2}{r^2 + a^2 - 2mr} \right) - \Omega^2 dt^2 + \right.
\]
\[+ \Omega^2 \left( r^2 + a^2 \right) \sin^2 \theta d\phi^2 + \frac{2mr \Omega^2}{(r^2 + a^2 \cos^2 \theta)} \left( dt + a \sin^2 \theta d\phi \right)^2.\] (6)

What is the relation between the spacetimes (2) and (6)? This paper tries to overcome this question.

In section 2 we study the transformation (1) in relation to the entropy. In section 3 we discuss the normalization of the timelike Killing field. In section 4 the same discussion is made for a particular simple non asymptotically flat case: the Kerr black hole with a static cosmic string.

2 Dilatations in asymptotically flat spacetimes

As a first consideration note that \( \Omega \) is dimensionless. Further, the Einstein’s tensor \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \) is invariant under (1). Thus, the transformation
has only two possible interpretations:

**First interpretation**: passive point of view.

The transformation (1) is a simple change in units length. In other words, if in (2) we measure the length in meters and if $\Omega = 10^2$, then we measure time, mass and energy in (3) in centimeters. If we take $r' = \Omega r$, $t' = \Omega t$, $m' = \Omega m$, $a' = \Omega a$, the metric (6) becomes

$$ds^2 = (r'^2 + a'^2 \cos^2 \theta) \left( d\theta^2 + \frac{dr'^2}{r'^2 + a'^2 - 2m'r'} \right) - dt'^2 +$$

$$+ (r'^2 + a'^2) \sin^2 \theta d\phi^2 + \frac{2m'r'}{(r'^2 + a'^2 \cos^2 \theta)} [dt' + a' \sin^2 \theta d\phi]^2.$$  \hspace{1cm} (7)

The entropy formula $S'$ for the line element (7) gives $S' = S$ because $A' = \Omega^2 A$ and $l_p' = \Omega$. This means that the microscopic realizations of the black hole (2) are independent of the units used.

**Second interpretation**: active point of view.

The transformation (1) is a conformal mapping of the spacetime (2) into a “stretched” spacetime. Obviously, in this second case, $G = c = 1$ and also $h = 1$ ($l_p' = l_p = 1$) because the units length are unchanged. Therefore, the entropy formula gives $S' = \Omega^2 S$ and the counting of the microscopic states fails to be the same for spacetimes (2) and (3).

On the grounds of these interpretations, the entropy value depends on the active and passive point of view. Further, the weak limit of Einstein’s equations leads to the Poissonian equation $\nabla^2 U = 4\pi \rho$, where $U$ is the Newtonian potential and $\rho$ is the mass density of the source [4]. The only change caused by (1) in the weak limit is given by $U \rightarrow U_\Omega = U - ln\Omega$, that is a simple gauge transformation which cannot affect Poissonian equation. This can be of some interest for the so called Immirzi ambiguity [5, 6], that arises in the context of loop gravity [7, 8] by means of the Ashtekar formulation of general relativity, or for considerations involving a generic quantum gravitational theory [9]. In loop gravity the counting of quantum states for the black hole entropy $S$ is affected by an arbitrary multiplicative constant $\beta$ that cannot be fixed a priori by the quantization procedure [10] and by the low energy physics.

When we have a generic quantum theory, it is not a trivial question to ponder, in relation to the active point of view, which is between $ds^2$ and $\Omega^2 ds^2$ the classical reference spacetime of the quantum theory. In this context, it is expected that in loop quantum gravity the entropy is affected by an arbitrary multiplicative constant. In literature the analogy between scale
transformations and the Immirzi ambiguity has been taken in account in [1], but only in the case of the passive point of view.

Now, let us suppose that Ω is a function of the parameters characterizing the black hole, i.e. Ω = F(m, a), with the normalization (5). Adopting the passive point of view, the entropy \( S' \) for the spacetime (6) is

\[
S' = \frac{2\pi m'(m' + \sqrt{m'^2 - a'^2})}{F^2},
\]

(8)

where \( m' = Fm, a' = Fa \). Conversely, adopting the active point of view, the entropy \( S' \) becomes

\[
S' = 2\pi m'(m' + \sqrt{m'^2 - a'^2}).
\]

(9)

What does it happen if we choose another normalization for the Killing vector? In the next sections we will study this problem from the active point of view. The active point of view is the most interesting case because all the observers use the same units length, independently on the scale transformation (1).

3 Normalization of the timelike Killing vector for asymptotically flat spacetimes

In this section we use another normalization instead of (5) as, for example, for black holes in 2D dilaton gravity [12]. In this case, the usual normalization (5) does not lead to the energy conservation: to achieve energy conservation is necessary a “mass dependent” normalization. Following this point of view, we could use the normalization

\[
(\xi'^\nu \xi'_\nu)_{\infty} = -\Omega^2 = -F^2(m, a).
\]

(10)

With respect to (10) we define the mass of the black hole. In this case, if we “see” the spacetime (2) with the normalization (10), the parameters which are effectively measured are not \( m \) and \( a \), but rather the rescaled ones \( m', a' \), where \( m = m'F, a = a'F \). Note that, since \( \frac{a}{m} = \frac{a'}{m'} \), if we take \( F = F(a/m) \) then \( F(a/m) = F(a'/m') \). This fails if the function \( F \) depends on \( h \). For example, if \( F = ma \) (\( h = 1 \) in our units) then, in terms of \( m', a', F = \frac{1}{m'a'} \). This lack of symmetry when \( h \) is present in \( F \) has an interesting analogy with the cosmological constant \( \Lambda \). In fact, if \( \Lambda \) is present in Einstein’s equations, i.e. \( G_{\mu\nu} \rightarrow G_{\mu\nu} + \Lambda g_{\mu\nu} \), then the invariance of \( G_{\mu\nu} \) under (11) is broken. To regain invariance we must rescale \( \Lambda \) as \( \Lambda \rightarrow \Lambda' \). Remember that it is the
presence of $\hbar$ in the formula for the entropy $S$ that breaks the symmetry between the active and the passive point of view. Besides, the entropy of a black hole arises when quantum reasonings are taken in account (classically the black hole entropy is exactly zero) and the presence of $\hbar$ is an indication of such quantum reasonings. By analogy, this suggests that the origin of $\Lambda$ could be found in quantum theory.

Generally, the increasing or decreasing character of the entropy is related to the normalization chosen for $\xi^\nu$. In practice, it is the presence of the black hole that can justify the normalization (10). Conversely, if we have a Minkowskian spacetime, no objects present in the universe modify the flat geometry. Therefore, we have no grounds to choose an “objects dependent” normalization. In other words, in a Minkowskian spacetime, surfaces and volumes are “absolute” objects that do not depend on the state of the matter and thus we can define a “privileged reference” metric. The function $F$ presents in (10) is arbitrary: by varying the function $F$ we choose the scale at which we see a given spacetime. For example, in (13) one compute the logarithmic correction to the Bekenstein-Hawking formula in the formulation of “quantum geometry” of Ashtekar [14] and in certain string theories [15].

The modified entropy is

$$S = \frac{A}{4} - \frac{3}{2} \ln \frac{A}{4} + \cdots$$  \hspace{1cm} (11)

If we “see” the spacetime (2) with the normalization (10), then we find the scale at which the corrections arise: we must impose

$$S = 2\pi m(m + \sqrt{m^2 - a^2}) = 2\pi m'(m' + \sqrt{m'^2 - a'^2})F^2 =$$
$$= 2\pi m'(m' + \sqrt{m'^2 - a'^2}) - \frac{3}{2} \ln[2\pi m'(m' + \sqrt{m'^2 - a'^2})] + \cdots$$  \hspace{1cm} (12)

Hence, by posing $F^2 = \zeta$, we find

$$\ln \frac{\zeta}{S} = \frac{2}{3} S - \frac{2}{3} \frac{S}{\zeta}.$$  \hspace{1cm} (13)

Graphically, it is easy to see that equation (13) has not solutions for $0 < S < \frac{3}{2} - \frac{3}{2} \ln \frac{3}{2}$, one solution for $S = \frac{3}{2} - \frac{3}{2} \ln \frac{3}{2}$ and two solutions for $S > \frac{3}{2} - \frac{3}{2} \ln \frac{3}{2}$. When $S < 1$ we have microscopic black holes with mass $m < 1$ expressed in terms of the unit Planck length $l_p$. For black holes with mass larger than the Planck length we have two solutions, the first with $F > 1$ and the latter with $F < 1$. The leading correction to the entropy formula has been obtained.
by taking the “mass dependent” normalization (10). In other words, when the entropy for the spacetime (2), at the scale given by (10), is expressed in terms of the parameters measured, i.e. \( m', a' \), corrections arise, provided that the equation (13) is satisfied.

4 Non asymptotically flat case

We consider now the case of a non asymptotically flat spacetime by “preserving” the active point of view. It is a well known fact [16] that, in polar coordinates, the spacetime

\[
ds'^2 = d\rho^2 + dz^2 + (1 - 4\mu)^2 \rho^2 d\phi^2 - dt^2,
\]

when \( 0 < \mu < \frac{1}{4} \), is a solution of Einstein equations \( G_{\mu\nu} = 8\pi T_{\mu\nu} \) with \( T_{\mu\nu} = \mu \delta(x') \delta(y') \text{diag}(1,0,0,-1) \). The solution (14) represents a static string (cosmic string) with a mass distribution on the \( z \) axis, where \( \mu \) is the mass density of the source. The parameter \( B = 1 - 4\mu \) gives the topological defect (angle deficit) of the spacetime. It is also known [17] that in the limit \( \rho \to 0 \) the quantity

\[
\Delta \Phi(\rho) = 2\pi - \frac{\int_0^{2\pi} \sqrt{g_{\phi\phi}} d\phi}{\int_0^\rho \sqrt{g_{\rho\rho}} d\rho}
\]

is directly related to the energy density per unit length of the string (\( \Delta \Phi(0) = 8\pi \mu \)). If \( \Delta \Phi(0) = 0 \) the topological defect disappears. For the metric (14) we get \( \Delta \Phi(0) = 2\pi(1 - B) \) (\( B < 1 \)). The spacetime (14) is locally, but not globally, Minkowskian. Now, also the metric

\[
ds'^2 = \frac{1}{(1 - 4\mu)^2}(d\rho^2 + dz^2) + \rho^2 d\phi^2 - \frac{dt^2}{(1 - 4\mu)^2}
\]

is a solution of the same equations satisfied by the line element (14). Both solutions (14) and (16) are invariant under Lorentz boosts along \( z \) axis. These two solutions are joined by (1) with \( \Omega^2 = B^2 \), and the parameter \( \mu \) in both solutions is the mass density of the string source. This can also be understood from expression (15) invariant under a constant stretching of the line element. In fact the spacetimes (14) and (16) have the same physical interpretation and describe the same spacetime “seen” at different scales. Note that if we choose for \( B \) a value different from \( B = 1 - 4\mu \), for example \( \tilde{B} = \frac{\mu^2}{1 + 4\mu} \) (\( \tilde{B} < 1 \to 0 < \tilde{\mu} < \frac{1}{4} \)), the parameter \( \tilde{\mu} \) cannot be interpreted as the mass density of the string because we must always have
$$\Delta \Phi(0) = 2\pi(1 - \tilde{B}) = 8\pi \mu,$$ where $\mu$ is the true mass density of the string.

In practice, the parameter $\mu$ is dimensionless in our units and thus it is a scale invariant object under (1). In any case both metrics (14) and (16) are locally but not globally equivalent to the Minkowskian spacetime and both describe a static string with mass density $\mu$ on the $z$ axis.

A rotating black hole with a cosmic string of mass density $\mu$ along $z$ axis [18, 19, 20] in Boyer-Lindquist, according to the asymptotic form (16) with the same notation of equation (2), has the line element

$$ds^2 = \frac{\Sigma}{B^2} \left( d\theta^2 + \frac{dr^2}{\Delta} \right) + (r^2 + a^2)\sin^2\theta d\phi^2 - \frac{dt^2}{B^2} + \frac{2mr}{\Sigma} \left( \frac{dt}{B} + a\sin^2\theta d\phi \right)^2.$$ (17)

Thanks to (15) we obtain, for solution (17), $\Delta \Phi(0) = 2\pi(1 - B)$. The horizon surface area of black hole is

$$A = 4\pi r_+^2$$ with $r_+ = m + \sqrt{m^2 - a^2}$.

If we take $g_{\mu\nu} \rightarrow g_{\mu\nu}B^2$ we have:

$$ds'^2 = \frac{\Sigma}{B^2} \left( d\theta^2 + \frac{dr^2}{\Delta} \right) + B^2(r^2 + a^2)\sin^2\theta d\phi^2 + \frac{2mr}{\Sigma} \left( dt + Basin^2\theta d\phi \right)^2 - dt^2.$$ (18)

Formula (15) gives again $\Delta \Phi(0) = 2\pi(1 - B)$ and $r_+ = m + \sqrt{m^2 - a^2}$.

We can measure the mass density $\mu$ at $r = \infty$ and therefore, for the discussion above, the parameter $\mu$ is the mass density of the string for both metrics (17) and (18). We must choose a normalization for the timelike Killing vector with respect to which we define the black hole mass $m$. Generally, if we have a non asymptotically flat spacetime there is not an usual way to fix the normalization of $\xi^\nu$. In our case, we have to choose one of the two asymptotic forms (14) and (16). Let us assume that the asymptotic line element (14) is our “reference” metric. For the entropy $S'$ we have:

$$S' = 2\pi m(m + \sqrt{m^2 - a^2})(1 - 4\mu).$$ (19)

The entropy formula (19) is a decreasing function of the parameter $\mu$.

Now, by keeping the normalization (5), we consider the solution (17) with $ds'^2 = B^2 ds^2$. The entropy for this spacetime is

$$S(m, a, \mu) = \frac{2\pi m(m + \sqrt{m^2 - a^2})}{(1 - 4\mu)} = S(m', a', \mu) = 2\pi m'(m' + \sqrt{m'^2 - a'^2})(1 - 4\mu)$$ (20)
with \( m = m' B, a = a' B \), where \( m' \) and \( a' \) are respectively the mass and the spin density effectively measured in the spacetime (17). Thus, if we consider the function \( S \) for spacetime (17), with the normalization (5), as a function of \( m', a', \mu \), the entropy continues to be a decreasing function of the parameter \( \mu \). The objects physically relevant are the observable quantities that, for spacetime (17) with the normalization (5), are \( m' \) and \( a' \).

Note that, if we take the limit \( \mu \to \frac{1}{4} \), then \( S' \to 0 \). Moreover, in this limit, for the black hole temperature \( T_{BH} \) with \( T_{BH} = \frac{\partial m}{\partial S} \), we find that \( T_{BH} \to \infty \). In this limit the metric becomes singular. Cosmic strings with string tension \( \mu \) appear in the context of cosmological models in order to act as seeds for galaxy formation. It is interesting to note that the very hot limit with high mass density \( \mu \) corresponds to the zero entropy limit. This could mean that black holes may have been formed during phase transition when the matter has been extremely hot and dense. i.e. at the very early stage of the universe.

If we choose as asymptotical “reference” metric the expression (16), we can take

\[
(\xi^\nu \xi_\nu)_\infty = -B^{-2}.
\]  

(21)

With this choice the parameters \( m \) and \( a \) that appear in the line element (17) are now the physical one measured and thus entropy is

\[
S(m, a, \mu) = \frac{2\pi m (m + \sqrt{m^2 - a^2})}{1 - 4\mu}
\]  

(22)

that is an increasing function of the parameter \( \mu \). For the spacetime (18) the entropy, with the new normalization (21), becomes

\[
S'(m', a', \mu) = \frac{2\pi m' (m' + \sqrt{m'^2 - a'^2})}{1 - 4\mu}
\]  

(23)

with \( a' = Ba, m' = Bm \). Now the parameters \( m' \) and \( a' \) are respectively the mass and spin density of the black hole source measured in (18). Also in this case, the entropy is an increasing function of the parameter \( \mu \).

Generally, we can multiply solution (17) by \( B^\gamma \), where \( \gamma \) is any real constant, and choose the normalization of \( \xi^\nu \) according to the asymptotic metric so obtained. Therefore the black hole entropy with a cosmic string with respect to the “reference” metric so obtained becomes

\[
S = 2\pi m (m + \sqrt{m^2 - a^2}) B^{\gamma - 1}, \quad (\xi^\nu \xi_\nu)_\infty = -B^{\gamma - 2},
\]  

(24)

If we take \( \gamma = 1 \) in expression (24), then the entropy formula is independent on the parameter \( \mu \). Besides, if \( \gamma < 1 \) the high mass density limit \( \mu \to \frac{1}{4} \).
leads to $S \to \infty$ and $T_{BH} \to 0$, in contrast to the situation\[19\]. The choice of normalization for $\xi^\nu$, i.e. the choice of the “reference” metric in relation to the “active” point of view, is equivalent to choose an arbitrary scale energy with respect to which we “see” the other solutions joined by \[11\].

Taking the phraseology of ordinary quantum field theory, one can think at $m'$ as a kind of “interacting” mass and at $m$ as a “bare” one. In fact, the parameters present in the action of a ordinary quantum fields theory, as it happens in the standard electroweak interaction model, are not the ones measured in the effective theory: it is by means of the renormalization procedure that one can define the “interacting” parameters really observed; the “bare”, non “interacting”, parameters are meaningless from a physical point of view. In the renormalization group approach the mass is a physical parameter that depends on the scale under consideration.

Finally, the imposition $(\xi^\nu \xi_\nu)_{\infty} = -B^{\gamma-2}$ with $m' = mB^{(1-\gamma)}$ is not bizarre because the parameter $\mu$ appears also in the asymptotic line element and thus we can choose a scale, with respect to which we measure the mass $m'$, which takes into account such presence at spatial infinity. In fact, according to Mach (see \[21\][22]), the presence of the string can modify the inertia and the gravitational mass of a body. In this sense the presence of the string can justify a kind of “interacting” normalization for $\xi^\nu$ depending on the parameter $\mu$.

References

[1] Robert H.Boyer, R.W.Lindquist, Journal of Math.Phys., 8, 2 (1968).
[2] S.W. Hawking, Commun.Math. Phys, 25, 152 (1972).
[3] J.Bekenstein, Phys. Rev. D7, 949 (1973).
[4] G. Ferrarese, “Lezioni di Relativitá Generale”, Pitagora editrice Bologna (1994).
[5] Immirzi G., Nucl. Phys. Proc. Suppl. 57, 65 (1997).
[6] Immirzi G., Class. Quantum Grav. 14, L177 (1997).
[7] Ashtekar A, Baez J, Corichi A and Krasnov K, phys. Rev. Lett. 80, 904.
[8] Ashtekar A, “Lectures on Non-Perturbative Canonical Gravity”, ed. L Z Fang and R. Ruffini (Singapore: World Scientific 1991).
[9] Birrell N.D. and Davies P C W, “Quantum Fields in curved space”, ed. P V Landshoff et. (Cambridge University Press, 1986).

[10] C. Rovelli, gr-qc/9705059

[11] Luis J Garay and Guillermo A Mena Marugán, gr-qc/0304055

[12] J. Cruz, A. Fabbri and J. Navarro-Salas, gr-qc/9902084

[13] R. K. Kaul and P. Majumadar, phys. Rev. Lett. 84, 5255 (2000).

[14] R. K. Kaul and P. Majumadar, Phys. Lett. B 439, 267 (1998).

[15] S. Carlip, gr-qc/00050017.

[16] A. Wilenkin, Phys. Rep. 121, 263 (1885).

[17] J. Garriga, E. Verdaguer, Phys. Rev. D 36, 2250 (1987).

[18] Aliev A N and Gal’tsov D V 1988 Sov.Astron.Lett.14 48

[19] Gal’tsov D V and Masar E 1989, Class. Quantum Grav. 6 1313

[20] R. Bergamini, V. Stefano, gr-qc/0305035

[21] E. Mach, “Mechanics” (St. Petesburg, 1906)

[22] A. Einstein, “Autobiographical notes” (Illinois, 1949)