Quark spectra, topology and random matrix theory

Robert G. Edwards and Urs M. Heller
SCRI, Florida State University, Tallahassee, FL 32306-4130, USA

Joe Kiskis
Dept. of Physics, University of California, Davis, CA 95616

Rajamani Narayanan
Dept. of Physics, Bldg. 510A, Brookhaven National Laboratory, P. O. Box 5000, Upton, NY 11973

Quark spectra in QCD are linked to fundamental properties of the theory including the identification of pions as the Goldstone bosons of spontaneously broken chiral symmetry. The lattice Overlap-Dirac operator provides a nonperturbative, ultraviolet-regularized description of quarks with the correct chiral symmetry. Properties of the spectrum of this operator and their relation to random matrix theory are studied here. In particular, the predictions from chiral random matrix theory in topologically non-trivial gauge field sectors are tested for the first time.

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An important property of massless QCD is the spontaneous breaking of chiral symmetry. The associated Goldstone pions dominate the low-energy, finite-volume scaling behavior of the Dirac operator spectrum in the microscopic regime, defined by $1/\Lambda_{QCD} \ll L \ll 1/m_\pi$, with $L$ the linear extent of the system. This behavior, in turn, can be characterized by chiral random matrix theory (RMT), which lead to a revival of RMT, first used to understand the energy levels of nuclear matter. What enters into the RMT description of the low-energy, finite-volume scaling behavior are some symmetry properties of the Dirac operator and the sector of fixed topological charge under consideration. The RMT predictions are universal in the sense that only the symmetry properties, but not the form of the potential matters. Furthermore, the properties tested in this letter can be derived directly from the effective, finite-volume partition functions of QCD of Leutwyler and Smilga, without the detour through RMT, though RMT nicely and succinctly describes and classifies all these properties. The topological charge enters the RMT prediction via the number of fermionic zero modes, related to the topological charge through the index theorem. The symmetry properties of the Dirac operator fall into three classes, corresponding to the chiral orthogonal, unitary, and symplectic ensembles. Examples are fermions in the fundamental representation of gauge group SU(2) for the orthogonal ensemble, fermions in the fundamental representation of gauge group SU($N_c$) with $N_c \geq 3$ for the unitary ensemble, and fermions in the adjoint representation of gauge group SU($N_c$) for the symplectic ensemble.

The classification according to the three different chiral random matrix ensembles is intimately connected to the chiral symmetry properties of the fermions. A good non-perturbative regularization of QCD should therefore retain those chiral symmetry properties. Until recently such a regularization was not known. The next best thing were staggered fermions, which at least retained a reduced chiral-like symmetry on the lattice. Indeed, staggered fermions were used to verify predictions of chiral RMT, albeit with two important shortcomings: (i) staggered fermions in the fundamental representation of SU(2) have the symmetry properties of the symplectic ensemble, not the orthogonal ensemble as continuum fermions, while adjoint staggered fermions belong to the orthogonal ensemble, not the symplectic one. (ii) staggered fermions do not have exact zero modes at finite lattice spacing, even for topologically non-trivial gauge field backgrounds, and thus seem to probe only the $\nu = 0$ predictions of chiral random matrix theory.

The development of the overlap formalism for chiral fermions on the lattice recently lead to the massless Overlap-Dirac operator, a lattice regularization for vector-like gauge theories that retains the chiral properties of continuum fermions on the lattice. In particular, the continuum predictions of chiral random matrix theory should apply. Overlap fermions have exact zero modes in topologically non-trivial gauge field backgrounds, allowing, for the first time, verification of the RMT predictions in $\nu \neq 0$ sectors. The nice agreement we shall describe further validates the chiral RMT predictions on the one hand and strengthens the case for the usefulness of the Overlap regularization of massless fermions on the other hand.

The massless Overlap-Dirac operator is given by

$$D = \frac{1}{2} \left[ 1 + \gamma_5 \epsilon(\mathcal{H}_w(m)) \right]. \quad (1)$$

Here, $\gamma_5 \mathcal{H}_w(-m)$ is the usual Wilson-Dirac operator on the lattice and $\epsilon$ denotes the sign function. The mass $m$ has to be chosen to be positive and well above the critical mass for Wilson fermions but below the mass where the doublers become light on the lattice. In this letter, we will be interested in the low lying eigenvalues of $H = \gamma_5 D$, which is a hermitian operator. Relevant properties of this operator can be found in Ref. 4. We will use the
Ritz algorithm \[15\] applied to \(H^2\) to obtain the lowest few eigenvalues. The numerical algorithm involves the action of \(H\) on a vector and for this purpose one will have to use a representation of \(e^{iH_W(m)}\). We will use the rational approximation discussed in Ref. \[14,16\].

For fermions in the fundamental representation of gauge group SU(2), the Overlap-Dirac operator is real since the underlying Wilson-Dirac operator is real \[17\]. They are therefore expected to belong to the symplectic ensemble. We next turn to the distribution of the lowest eigenvalues of the “unfolded” level spacing \[2,20\]. The unfolding of the chiral unitary ensemble.

For fermions in the fundamental representation of SU(2), the Overlap-Dirac operator is doubly degenerate \[18\], and they are therefore expected to belong to the orthogonal ensemble.

For three or more colors the Overlap-Dirac operator is doubly degenerate \[18\], and they are therefore expected to belong to the symplectic ensemble.

The lattice size was \(4^4\) in all cases. The various level spacing distributions are shown in Fig. \[1\]. There is very clear evidence that the SU(2) and SU(3) ensembles with fermions in the fundamental representation fall into the orthogonal and unitary ensemble, respectively, and the SU(2) ensemble with fermions in the adjoint representation falls into the symplectic ensemble.

We will first present our results for the distribution of the “unfolded” level spacing \[14,22\]. The unfolding of the eigenvalues of \(H\) are done in the following manner. Let \(E_i^n\) label the non-zero positive eigenvalues of \(H\) with \(n\) labeling the configuration number and \(E_i^n > E_i^{n-1}\) for all \(i\). We are considering only the positive eigenvalues of \(H\), since the non-zero eigenvalues of \(H\) all come in positive/negative pairs \[14\]. For fermions in the adjoint representation of SU(\(N_c\)) the spectrum of \(H\) is doubly degenerate and we only keep half the spectrum, dropping the degeneracy. Unfolding proceeds by first sorting all the \(E_i^n\) in ascending order and associating the location \(N_i^n\) of \(E_i^n\) in the sorted list with \(E_i^n\). \(N_i^n\) is referred to as the unfolded spectrum and the level spacing is simply given by \((N_{i+1}^n - N_i^n)/N\), where \(N\) is the number of configurations. The distributions of the unfolded level spacing, \(s\), in RMT are well approximated by the various Wigner distributions \[3\].

\[
P(s) = \begin{cases} \frac{s}{2}\pi e^{-\frac{t}{2}s^2} & \text{orthogonal ensemble} \\ \frac{32}{7}\pi^2 e^{-\frac{t}{2}s^2} & \text{unitary ensemble} \\ \frac{262144}{29}\pi e^{-\frac{t}{2}s^2} & \text{symplectic ensemble} \end{cases}
\]

In our numerical simulations we computed the low lying spectrum of the Overlap-Dirac operator in the fundamental representation on pure gauge SU(2) configurations with \(\beta = 1.8\) as an example of the chiral orthogonal ensemble, on pure gauge SU(3) configurations with \(\beta = 5.1\) as an example of the chiral unitary ensemble, and in the adjoint representation on pure gauge SU(2) configurations with \(\beta = 2.0\). The lattice size was \(4^4\) in all cases. The various level spacing distributions are shown in Fig. \[1\]. There is very clear evidence that the SU(2) and SU(3) ensembles with fermions in the fundamental representation fall into the orthogonal and unitary ensemble, respectively, and the SU(2) ensemble with fermions in the adjoint representation falls into the symplectic ensemble.

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We next turn to the distribution of the lowest eigenvalue for the various ensembles. Chiral RMT predicts that these distributions are universal when they are classified according to the three ensembles and according to the number of exact zero modes \(\nu\) within each ensemble and then considered as functions of the rescaled variable \(z = \Sigma V \lambda_{\min}\). Here \(V\) is the volume and \(\Sigma\) is the infinite volume value of the chiral condensate \(\langle \bar{\psi} \psi \rangle\) determined up to an overall wave function normalization, which is dependent in part on the Wilson–Dirac mass \(m\). RMT gives for the distribution of the rescaled lowest eigenvalue for the orthogonal ensemble, expected to apply to the fermions in the fundamental representation of SU(2), in the \(\nu = 0\) and \(\nu = 1\) sector \[21\].

\[
P_{\min}(z) = \begin{cases} \frac{2z^2}{\pi} e^{-\frac{t}{2}z^2} & \text{if } \nu = 0 \\ \frac{4}{\pi} e^{-\frac{1}{2}z^2} & \text{if } \nu = 1 \end{cases}
\]

* Fermions in U(1) abelian gauge theory are another example of the unitary case. For numerical tests in two dimension, see \[20\].
FIG. 2. Plots of $P_{\text{min}}(z)$ versus $z$ for the various ensembles in the lowest two topological sectors. The curve in each plot is a fit to the prediction from random matrix theory with the best value for the chiral condensate.

For the unitary ensemble, expected to apply to the fermions in the fundamental representation of $SU(N_c)$ with $N_c \geq 3$, the RMT predictions are \[ P_{\text{min}}(z) = \begin{cases} \frac{3}{2} e^{-\frac{z^2}{2}} & \text{if } \nu = 0 \\ \frac{3}{2} I_2(z) e^{-\frac{z^2}{4}} & \text{if } \nu = 1 \\ \frac{3}{2} [I_2^2(z) - I_1(z)I_3(z)] e^{-\frac{z^2}{4}} & \text{if } \nu = 2 \end{cases} \] (4)

Finally, for the symplectic ensemble, expected to apply to the fermions in the adjoint representation, the RMT prediction is \[ P_{\text{min}}(z) = \begin{cases} \sqrt{2} z^{3/2} I_1(z) e^{-\frac{z^2}{4}} & \text{if } \nu = 0 \\ \frac{3}{2^{\nu+3}} e^{-\frac{z^2}{4}} T_{\nu}(z^2) & \text{if } \nu > 0 \end{cases} \] (5)

We are interested in $\nu = 1$ since the eigenvalues are doubly degenerate. A closed form expression is not known for $T_{\nu}(z^2)$. Instead, a rapidly converging series is based on partitions of integers, namely $T_{\nu}(x) = 1 + \sum_{d=1}^{\infty} a_d x^d$ where

\[ a_d = \sum_{|\kappa| = d \atop l(\kappa) \leq 2\nu + 1} \prod_{(i,j)} \frac{(2\nu + 2j - i)}{(2\nu + 2j - i + 4)} \]

We note the consistency of the values for $\Sigma$ obtained in the $\nu = 0$ and $\nu = 1$ sectors of each ensemble. Alternatively, we could have used the value of $\Sigma$ obtained in the $\nu = 0$ sector, to obtain a parameter free prediction for the distribution of the rescaled lowest eigenvalue in the $\nu = 1$ sector. Obviously, the predictions would have come out very well.

For the two ensembles with the fermions in the fundamental representation, we also found 81 (for $SU(2)$), and 147 (for $SU(3)$) configurations with two zero modes and 1 and 3 with three zero modes. For the orthogonal
ensemble, we are not aware of a prediction for $P_{\min}(z)$ in the $\nu = 2$ sector, while for the unitary ensemble our data, albeit with very limited statistics, agrees reasonable well with the parameter free prediction, eq. (4) with $\Sigma$ from Table I.

For fermions in the adjoint representation, we keep only one of each pair of doubly degenerate eigenvalues, so $\nu = 1$ is the sector where there are two exact zero modes. Such gauge field configurations cannot be assigned an integer topological charge since integer charges give rise to zero modes in multiples of four $24$, and we note there are a significant number of configurations with two zero modes as seen in Table I. The good agreement with the RMT prediction found in this case lends further support to the existence of configurations with fractional topological charge $8$.

In this letter we have tested the predictions of chiral random matrix theory using the Overlap-Dirac operator on pure lattice gauge field ensembles. We find full agreement with the unfolded level spacing distributions on all three ensembles. We also found that the distribution of the lowest eigenvalue in the different topological sectors fitted well with the predictions of chiral RMT, with compatible values for the chiral condensate from the different topological sectors. This is the first test of the influence of topology on the Dirac spectrum in the microscopic regime.

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**TABLE I.** The chiral condensate, $\Sigma$, from fits of the distribution of the lowest eigenvalue to the RMT predictions. The third column gives the Wilson-Dirac mass parameter used, the fourth the number of configurations, $N_\nu$, in each topological sector and the last the confidence level of the fit.

| Repr. | $\beta$ | $m$ | $\nu$ | $N_\nu$ | $\Sigma$ | $Q$ |
|-------|--------|-----|-------|--------|---------|-----|
| SU(2) fund. | 1.8 | 2.3 | 0 | 1293 | 0.2181(31) | 0.915 |
| SU(2) fund. | 1.8 | 2.3 | 1 | 1125 | 0.2155(37) | 0.205 |
| SU(3) fund. | 5.1 | 2.0 | 0 | 2714 | 0.0827(8) | 0.444 |
| SU(3) fund. | 5.1 | 2.0 | 1 | 2136 | 0.0829(6) | 0.656 |
| SU(2) adj. | 2.0 | 2.3 | 0 | 1251 | 0.2900(30) | 0.611 |
| SU(2) adj. | 2.0 | 2.3 | 1 | 254 | 0.2031(45) | 0.313 |

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